AN ALGORITHM TO FIND MAXIMUM AREA POLYGONS CIRCUMSCRIBED ABOUT A CONVEX POLYGON

MARKUS AUSSERHOFER, SUSANNA DANN, ZSOLT LÁNGI AND GÉZA TÓTH

Abstract. A convex polygon \(Q\) is circumscribed about a convex polygon \(P\) if every vertex of \(P\) lies on at least one side of \(Q\). We present an algorithm for finding a maximum area convex polygon circumscribed about any given convex \(n\)-gon in \(O(n^3)\) time. As an application, we disprove a conjecture of Farris. Moreover, for the special case of regular \(n\)-gons we find an explicit solution.

1. Introduction

The algorithmic aspects of finding convex polygons under geometric constraints with some extremal property have been studied for a long time. We list just a few examples. Boyce et al. [9] dealt with the problem of finding maximum area or perimeter convex \(k\)-gons with vertices in a given set of \(n\) points in the plane. Eppstein et al. [11] presented an algorithm that finds minimum area convex \(k\)-gons with vertices in a given set of \(n\) points in the plane. Minimum area triangles [14, 17, 8] or more generally, convex \(k\)-gons [10, 3], enclosing a convex \(n\)-gon with \(k < n\) were studied in several papers. Other variants, where area is replaced by another geometric quantity, were also investigated, see e.g. [15]. Maximum area convex polygons in a given simple polygon were examined, e.g. in [6, 16]. Algorithms to find polygons with a minimal number of vertices, nested between two given convex polygons, were presented in [8]. The authors of [18] examined among other questions the problem of placing the largest homothetic copy of a convex polygon in another convex polygon. For more information on geometry-related algorithmic questions, see [1].

Definition 1. Let \(P \subset \mathbb{R}^2\) be a convex \(n\)-gon. If \(Q\) is a convex polygon that contains \(P\) and each vertex of \(P\) is on the boundary of \(Q\), then we say that \(Q\) is circumscribed about \(P\). Set

\[A(P) = \sup\{\text{area}(Q) : \text{Q circumscribed about } P\},\]

if it exists. If \(\text{area}(Q) = A(P)\) and \(Q\) is circumscribed about \(P\), then \(Q\) is a maximum area polygon circumscribed about \(P\).

Note that any side of a polygon \(Q\) circumscribed about \(P\) contains at most two vertices of \(P\), and thus, it has at least \(\frac{n}{2}\) sides. Such a polygon may have arbitrarily many sides that do not contain any vertex of \(P\). Nevertheless, it is not hard to see that if \(Q\) is a maximum area polygon circumscribed about \(P\), then every side of \(Q\) contains at least one vertex of \(P\), and hence, it has at most \(n\) sides. Furthermore, \(A(P)\) is finite if and only if the sum of any two consecutive angles of \(P\) is greater than \(\pi\). Indeed, if this property holds, then the area of any polygon circumscribed about \(P\) is less than the sum of the area of \(P\) and the areas of all triangles bounded by three consecutive sidelines of \(P\). On the other hand, if \(P\) has two consecutive angles whose sum is at most \(\pi\), then there is a point \(q\) arbitrarily far from \(P\) such that \(\text{conv}(P \cup \{q\})\) is a convex \((n+1)\)-gon. Since this polygon is circumscribed about \(P\) and it can have arbitrarily large area, in this case \(A(P) = \infty\). In particular, this means that if \(A(P) < \infty\), then \(P\) has at least five vertices.

Given any convex polygon \(P\), our aim is to find a maximum area convex polygon circumscribed about \(P\). We investigate the properties of these polygons and present an algorithm to find them. Our results can be used to bound an integral of a positive convex function. As an application, we bound an integral of the Lorenz curve and disprove a conjecture of Farris about the Gini index in statistics.

2020 Mathematics Subject Classification. 52A38 and 52B60 and 68W01 and 62H12.

Key words and phrases. circumscribed polygon, area, Gini index.
This paper is organized as follows: In Section 2 we establish some geometric properties of maximum area polygons circumscribed about a convex \( n \)-gon. In Section 3 we present an algorithm with \( O(n^3) \) running time that finds \( A(P) \) and the maximum area polygons circumscribed about \( P \). Suppose that \( Q \) is circumscribed about \( P \). Let \( S_1, \ldots, S_n \) be sides of \( P \) in counterclockwise order. We say that \( S_i \) is “used” by \( Q \) if it is on the boundary of \( Q \), and “not used” otherwise. We can assign a sequence from \( \{U, N\}^n \) to \( Q \) such that the \( i \)th term is \( U \) if \( S_i \) is used and \( N \) otherwise. In Section 4 we investigate the following problem: which sequences can be assigned to a maximum area circumscribed polygon, for some \( P \). We give a complete solution to this problem. In particular, we correct an error that appeared in a previous, published version of this manuscript, based on a faulty construction in the proof of Theorem 4. In Section 5 we describe an application of our method to statistics, and disprove a conjecture of Farris in [12]. Finally, in Section 6 we collect our additional remarks and propose some open problems.

Throughout this paper, \( P \subset \mathbb{R}^2 \) denotes a convex \( n \)-gon, \( n \geq 5 \), and the sum of any two consecutive angles of \( P \) is greater than \( \pi \). The vertices of \( P \) are denoted by \( p_1, p_2, \ldots, p_n \), in counterclockwise order. We extend the indices to all integers so that indices are understood mod \( n \); that is, we let \( p_i = p_j \) if \( i \equiv j \pmod{n} \). For any \( i \), we denote the side \( p_i p_{i+1} \) of \( P \) by \( S_i \). By \( T_i \) we denote the triangle bounded by \( S_i \) and the lines through \( S_{i-1} \) and \( S_{i+1} \); it is called the \( i \)th external triangle. Clearly, if \( Q \) is circumscribed about \( P \), then every vertex of \( Q \) lies in an external triangle of \( P \). Note that if three consecutive vertices of \( Q \) lie on the same line, then removing the middle vertex does not change the area of \( Q \). Thus, without loss of generality, in our investigation we deal only with circumscribed polygons without an angle equal to \( \pi \). This implies, in particular, that if a side of \( P \) is used (i.e. it is contained in the boundary of the circumscribed polygon \( Q \)), then it is contained in a side of \( Q \). We denote the boundary of \( Q \) by \( \text{bd}(Q) \), and for any point \( p \in \mathbb{R}^2 \) and set \( S \subset \mathbb{R}^2 \), we call the rotated copy of \( S \) about \( p \) by \( \pi \) the reflection of \( S \) about \( p \).

2. Geometric properties of maximum area circumscribed polygons

**Theorem 1.** For any \( i, j \) with \( i \leq k \leq i + n \), let \( Q \) be a convex polygon circumscribed about \( P \) of maximal area containing \( S_k, S_{k+1}, \ldots, S_{i+n} \) on its boundary: that is, these edges of \( P \) are used by \( Q \). Then for every \( j = i + 2, i + 3, \ldots, k - 1 \) either

(a) \( p_j \) is the midpoint of the side of \( Q \) containing it,

or

(b) \( S_{j-1} \) or \( S_j \) lies on \( \text{bd}(Q) \).

**Proof.** Assume that neither \( S_{j-1} \) nor \( S_j \) lies on \( \text{bd}(Q) \). This clearly implies that \( p_j \) belongs to exactly one side of \( Q \), we denote it by \( V \). We show that in this case \( p_j \) is the midpoint of \( V \). Let \( V_+ \) (resp. \( V_- \)) be the side of \( Q \) immediately before (resp. before) \( V \) in the counterclockwise order, let \( q_+ = V \cap V_+ \) and \( q_- = V \cap V_- \), and let \( L, L_+, L_- \) be lines through \( V, V_+ \) and \( V_- \), respectively. Suppose that \( p_k \) is not the midpoint of \( V \). We can assume without the loss of generality that \( |p_j q_+| > |p_k q_-| \). Rotate \( L \) about \( p_k \) by a very small angle \( \alpha \) in the clockwise direction, denote the resulting line by \( L' \) and its intersection with \( L_+ \) and \( L_- \) by \( q'_+ \) and \( q'_- \), respectively. Let \( T_+ \) be the triangle determined by \( L_+, L, \) and \( L' \), and let \( T_- \) be the triangle determined by \( L_-, L, \) and \( L' \). Since \( |p_j q_+| > |p_j q_-| \), \( \text{area}(T_+) = |p_j q_+| \alpha + O(\alpha^2) \), and \( \text{area}(T_-) = |p_j q_-| \alpha + O(\alpha^2) \). Thus, if \( \alpha \) is very small, then \( \text{area}(T_+) > \text{area}(T_-) \). Another possible argument is that since \( |p_j q_+| > |p_j q_-| \) and \( \alpha \) is very small, we have that \( |p_j q'_+| > |p_j q'_-| \), implying that the reflection of \( T_+ \) about \( p_k \) contains \( T_- \) (cf. Figure 1). Thus modifying \( Q \) by replacing \( L \) by \( L' \) would increase its area. This implies that if \( Q \) has the maximum area, then \( p_j \) is the midpoint of \( V \). \( \square \)

**Remark 1.** Using the same proof, it is clear that Theorem 1 holds also for any maximum area polygon \( Q \) not restricted to use any side of \( P \).

**Definition 2.** Let \( 1 \leq k \leq n \) and let \( q_0, q_1, \ldots, q_k \) be points in the plane. We say that the polygonal curve \( C = q_0 q_1 \cdots q_k \) satisfies the midpoint property for the index \( i \), if for \( j = 1, 2, \ldots, k \), the vertex \( p_{i+j} \) of \( P \) is the midpoint of \( q_{j-1} q_j \).
follows that we denote by index $i$, then we say that $C$ satisfies the midpoint property.

Theorem 2. We have the following:

1) If $n$ is odd, there is exactly one closed polygonal curve satisfying the midpoint property.

2) If $n$ is even and $\sum_{k=1}^{n} (-1)^k p_k \neq 0$, then there is no closed polygonal curve satisfying the midpoint property.

3) If $n$ is even and $\sum_{k=1}^{n} (-1)^k p_k = 0$, then for every $q \in \mathbb{R}^2$, there is exactly one closed polygonal curve $C$ satisfying the midpoint property such that $q$ is the common endpoint of the two sides of $C$ containing $p_1$ and $p_n$. In addition, the absolute value of the signed area of $C$ is independent of $q$.

Proof. Let $C = q_0 q_1 \cdots q_n$, $q_0 = q_n$ be a closed polygonal curve satisfying the midpoint property for the index 0. For every $k$, $q_k$ is the reflection of $q_{k-1}$ about $p_k$. Thus setting $q := q_0$, we have $q_1 = 2p_1 - q$, $q_2 = 2p_2 - 2p_1 + q$, or in general, $q_k = 2\sum_{j=1}^{k} (-1)^{k-j} p_j + (-1)^k q$. In particular, since $C$ is closed, we obtain $q = 2\sum_{k=1}^{n} (-1)^{n-k} p_k + (-1)^n q$. If $n$ is odd, it follows that $q = \sum_{k=1}^{n} (-1)^{n-k} p_k$, proving (2.1). If $n$ is even, it follows that $\sum_{k=1}^{n} (-1)^{n-k} p_k = \sum_{k=1}^{n} (-1)^k p_k = 0$, implying (2.2) and the first part of (2.3).

Next we show that the signed area of $C$, also denoted by area($C$), is independent of $q$. For any $u, v \in \mathbb{R}^2$, we denote by $|u, v|$ the determinant of the $2 \times 2$ matrix with $u$ and $v$ as its columns. Since for every $k$, $|q_{k-1}, q_k| = |q_{k-1}, q_k - q_k + q_k| = 2|q_{k-1}, p_k|$, we obtain that

$$\text{area}(C) = \frac{1}{2} \sum_{k=1}^{n} |q_{k-1}, q_k| = \sum_{k=1}^{n} |q_{k-1}, p_k| = \sum_{k=1}^{n} 2 \sum_{j=1}^{k-1} (-1)^{k-1-j} p_j + (-1)^{k-1} q, p_k.$$ 

Thus for some function $f = f(p_1, p_2, \ldots, p_n)$, we have

$$\text{area}(C) = f(p_1, p_2, \ldots, p_n) + \sum_{k=1}^{n} |(-1)^{k-1} q, p_k| = f(p_1, p_2, \ldots, p_n) - q, \sum_{k=1}^{n} (-1)^k p_k,$$

which is independent of $q$, since $\sum_{k=1}^{n} (-1)^k p_k = 0$. \hfill \Box

The following variant of Theorem 2 can also be proved. We omit the proof since it is based on exactly the same calculations as Theorem 2.
Theorem 3. Let $1 \leq k < n - 1$. Let $C$ denote the family of polygonal curves $C = q_0q_1 \ldots q_k$ satisfying the midpoint property for the index $i$ such that $q_0$ lies on the line $L_{i-1}$ through $S_{i-1}$ and $q_k$ lies on the line $L_{i+k+1}$ through $S_{i+k+1}$.

1. If $L_{i-1}$ and $L_{i+k+1}$ are not parallel, then $C$ has exactly one element.
2. If $L_{i-1}$ and $L_{i+k+1}$ are parallel and $L_{i+k+1} \neq 2 \sum_j^{k} (-1)^{k-j}p_{i+j} + (-1)^k L_{i-1}$, then $C = \emptyset$.
3. If $L_{i-1}$ and $L_{i+k+1}$ are parallel and $L_{i+k+1} = 2 \sum_j^{k} (-1)^{k-j}p_{i+j} + (-1)^k L_{i-1}$, then for every $q_0 \in L_{i-1}$ there is exactly one polygonal curve $C \in C$ that starts at $q_0$. Furthermore, the signed area enclosed by $p_{i-1}q_0 \cup C \cup q_k, p_{i+k+2} \cup \left( \bigcup_{j=i+k+2}^{n+2} S_j \right)$ is independent of the choice of $q_0$.

Remark 2.

- The unique starting point $q$ in (2.1) can be found in $O(n)$-time by computing the quantity $q = \sum_{k=1}^{n} (-1)^{n-k}p_k$.
- In (2.3) the region for all possible starting points $q$, resulting in a convex solution, can be found in $O(n \log n)$ steps. Indeed, the polygonal curve satisfying the midpoint property is convex if only if each vertex lies in the corresponding external triangle $T_i$ of $P$. Each of these conditions gives three linear constraints on the starting point $q$. The constraints can be obtained in $O(n)$ steps, and the intersection of these $3n$ halfplanes can be computed in $O(n \log n)$ time [4].
- If there exists a convex solution $Q$ in (2.3), then there is a convex solution $Q$ that contains a side of $P$. Indeed, if $q$ is on the boundary of the feasible region, then for some $j$, $q_j$ lies on a sideline of $P$, which yields that $Q$ contains a side of $P$.

In (2.1) the unique starting point $q_0$, the corresponding solution $C = q_0q_1 \ldots q_k$, its convexity properties and its area, can be found in $O(k)$ steps, using the fact that $q_0$ is the intersection of $L_{i-1}$ with $2 \sum_j^{k} (-1)^{j-l}p_{i+j} + (-1)^k L_{i+k+1}$, and subsequently reflecting $q_0$ about $p_{i+1}, p_{i+2}, \ldots, p_{i+k}$.

3. An algorithm to find the maximum area circumscribed polygons

For any $i, j$ with $i < j \leq i + n$, we define $Q_{ij}$ to be a maximum area convex polygon circumscribed about $P$ with the property that the sides $S_j, S_{j+1}, \ldots, S_{i+n} = S_i$ lie on the boundary of $Q_{ij}$. Let $A_{ij} = A_{ij}(P)$ be the area of some $Q_{ij}$. Note that in the case $j = i + n$, a polygon $Q_{i(i+n)}$ is restricted to contain the side $S_{i+n} = S_i$ in its boundary. We extend this definition to any ordered pair of integers $(i, j)$ by taking indices modulo $n$.

We present a recursive algorithm which computes $A_{ij}$ for all $i < j \leq i + n$. It also finds $A(P)$ and the maximum area circumscribed polygons about $P$.

It follows from the definition that for $j = i + 1$, $Q_{ij} = P$. For $j = i + 2$, we add an external triangle $T_{i+1}$ to $P$. Now let $2 < k \leq n$ and suppose that we already know the value of $A_{i', j'}(P)$ for every $i', j'$ with $i' < j' < i' + k$. Let $j = i + k$. Consider a polygon $Q$ circumscribed about $P$ such that the sides $S_j, S_{j+1}, \ldots, S_{i+n}$ lie on the boundary of $Q$. We distinguish between $k$ types of such polygons.

Type (0): $bd(Q)$ does not contain any of the sides $S_{i+1}, S_{i+2}, \ldots, S_{j-1}$.

Type (a): $bd(Q)$ contains the side $S_{i+\alpha}$, for some $1 \leq \alpha \leq k - 1$.

Note that $Q$ can have several types, except for Type (0), which excludes the other types. We find the maximum area of a circumscribed convex polygon of each type separately.

Type (0): By Theorem 1 for any convex polygon $Q$ of maximum area, each of the vertices $p_{i+2}, \ldots, p_{j-1}$ has to be the midpoint of the corresponding side of $Q$. Whether the sides $S_i$ and $S_j$ are parallel or no, the existence of $Q$ and its area can be found in $O(k)$-time. This follows from Theorem 3 and Remark 2.

Type (a): By Theorem 1 an optimal solution $Q_{ij}$ is a union of some - by assumption already known - $Q_{i,i+\alpha}$ and $Q_{i+\alpha,j}$. It contains the sides $S_j, S_{j+1}, \ldots, S_{i+n}$ and $S_{i+\alpha}$, between $S_{i+n}$ and $S_{i+\alpha}$ it has the same vertices and sides as $Q_{i,i+\alpha}$, and between $S_{i+\alpha}$ and $S_j$ it has the same vertices and sides as $Q_{i+\alpha,j}$. Its area is $A_{ij} = A_{i,i+\alpha} + A_{i+\alpha,j} - \text{area}(P)$. By construction, the convexity of $Q_{i,i+\alpha}$ and $Q_{i+\alpha,j}$ implies that $Q_{ij}$ is convex as well. Since $Q_{ij}$ can be of any Type (a), this step requires $O(k)$-time.
For each fixed \( k \), starting with \( k = 3 \), we execute the above procedure for all \( 1 \leq i \leq n \). Then we increase the value of \( k \) by one and repeat all steps until \( k = n \). We obtain the values of \( A_{ij}(P) \) for all \( i, j \) with \( i < j \leq i + n \). This is done in \( O(n^3) \)-time. Indeed, let \( k \) be fixed, \( 3 \leq k \leq n \). Only the case \( j = i + k \) is unknown. \( Q \) can be of any type, by above all types require \( O(k) \)-time. Executing this for all \( i, 1 \leq i \leq n \), requires \( O(kn) \)-time. Now we have \( A_{ij} \) for \( i < j \leq i + k, 1 \leq i \leq n \) with \( A_{is} \leq A_{il} \) for \( s < l \). Hence it remains to take the maximum of \( A_{i(i+k)} \) over \( i \), which requires \( O(n) \)-time. Thus for a fixed \( k \), the algorithm needs \( O(kn) \)-time. Summing over \( k \), we obtain the claimed \( O(n^3) \)-time.

Once we have \( A_{i(i+n)}(P) \) for all \( i \), we can calculate the maximum area \( A_{21}(P) \) of a convex polygon circumscribed about \( P \) containing at least one side of \( P \). \( A_{21}(P) = \max \{ A_{i(i+n)} : 1 \leq i \leq n \} \).

Denote by \( A' = A'(P) \) the maximum area of a convex polygon circumscribed about \( P \) containing none of the sides of \( P \). By Remark 2 for even \( n \), if there exists a convex solution containing none of the sides of \( P \), then there is a solution containing one side of \( P \) with the same area, hence \( A' \leq A_{21} \). However, if we would like to list all maximum area polygons circumscribed about \( P \), we need to execute this step also in case \( n \) is even. For an odd \( n \), the existence of a convex solution, the solutions and their area can be computed in \( O(n) \)-time, see (2.1) and Remark 2. For an even \( n \), all convex solutions can be found in \( O(n \log n) \)-time, see (2.3) and Remark 2.

Finally, \( A(P) = \max \{ A'(P), A_{21}(P) \} \), so we get the final answer in \( O(n^3) \) time. It is clear that we can keep track of the best circumscribed polygons of different types throughout the algorithm. Hence, in addition to \( A(P) \), we also get the polygons \( Q \) realizing it.

4. COMBINATORIAL PROPERTIES OF MAXIMUM AREA CIRCUMSCRIBED POLYGONS

Let \( Q \) be a convex polygon circumscribed about \( P \). Recall that a side of \( P \) is called \emph{used}, if it lies on \( \text{bd}(Q) \), and \emph{not used} otherwise. Thus to any such \( Q \), we associate an \( n \)-element \emph{characteristic sequence} \( s = s(P, Q) \) of Us and Ns in such a way that the \( i \)-th element of this sequence is U if \( S_i \) is used and N otherwise. In particular, for each value of \( i \), \( s \) determines whether the condition (a) or (b) of Theorem 4 is satisfied for \( p_i \).

Our aim is to determine which sequences \( s \in \{U, N\}^n \) are \emph{realizable}; that is, which sequences can appear for \( s \) as a characteristic sequence for some \( P \) and a maximum area convex polygon \( Q \) circumscribed about \( P \).

In the previous version of this paper [4] we had an almost complete characterization. However, very recently, N. Bonmel [7] pointed out that one of the technical statements (part (ii) in the proof of Theorem 4) would contradict an old result of Zaremba [19]. Indeed, that statement turned out to be false. Here we present a corrected version of Theorem 4 which gives a complete characterization. We formulate the theorem for \emph{cyclic sequences}, in which the indices of the elements are meant mod \( n \); that is, in which we regard the first and the last elements as consecutive. We denote the family of cyclic sequences consisting of \( n \) Ns and Us by \( \{U, N\}^n \).

**Theorem 4.** (a) Let \( s \in \{U, N\}^k \). It is a (contiguous) subsequence of a realizable characteristic sequence \( s' \in \{U, N\}^n \) for some \( n \geq k \) if and only if \( s \) does not contain three consecutive Us.

(b) A sequence \( s \in \{U, N\}^n \) with \( n \geq 5 \) is realizable if and only if the following holds.

(i) \( s \) does not contain three consecutive Us,

(ii) \( s \) contains at least two disjoint subsequences of Ns separated by some Us,

(iii) \( s \) is not of the form \( UNUN \ldots N \), and not one of \( UNUNN \) and \( UNNUNN \).

**Proof.** For any \( x, y, z \in \mathbb{R}^2 \), we denote the triangle \( \text{conv} \{x, y, z\} \) by \([x, y, z]\) and its area by \( A(x, y, z) \).

First observe that no realizable sequence contains three consecutive Us. Indeed, in this case adding to \( Q \) the triangle bounded by these three sidelines strictly increases the area of \( Q \) while preserving its convexity. This proves one direction of the statement in (a). The opposite direction follows from the statement in (b).

Suppose that we have a convex polygonal curve \( \Gamma_i = p_0 p_1 \ldots p_i \) and two halflines, \( L \) and \( L' \), starting at \( p_0 \) and \( p_i \), respectively.

For a polygonal curve \( \Delta_i = q_1 q_2 \ldots q_i \) property (*) is the following:
property (i): \( p_0q_1q_2\ldots q_ip_i \) is a convex polygon, \( q_1 \in L, q_i \in L' \) and \( p_j \in q_jq_{j+1} \) for \( 1 \leq j \leq i-1 \).

The rest of proof is based on the following four technical statements about realizing a long sequence of \( Ns \) as a subsequence.

(i) For any \( i \geq 1 \) there exists a convex polygonal curve \( \Gamma_i = p_0p_1\ldots p_i \) and two half-lines, \( L \) and \( L' \), starting at \( p_0 \) and \( p_i \), respectively, such that the triangle \( T_i \), bounded by \( L, L' \) and \( p_0p_i \), contains \( \Gamma \) and the following property is satisfied.

Let \( \Delta_i = q_1q_2\ldots q_i \) be a convex polygonal curve with property (i) that maximizes the area of \( p_0q_1q_2\ldots q_ip_i \). Then \( q_1q_2\ldots q_i \) satisfies the midpoint property; that is, \( p_j \) is the midpoint of \( q_jq_{j+1} \) for \( 1 \leq j \leq i-1 \) (cf. Figure 2).

(ii) For any \( i \geq 2 \) there exists a convex polygonal curve \( \Gamma_i = p_0p_1\ldots p_i \) and parallel half-lines \( L, L' \) starting at \( p_0 \) and \( p_i \), respectively, such that \( p_0p_1\ldots p_i \subset \text{conv}(L \cup L') \), and the following property is satisfied.

There is a polygonal curve \( \Delta_i = q_1q_2\ldots q_i \) with property (i) which satisfies the midpoint property; that is, \( p_j \) is the midpoint of \( q_jq_{j+1} \) for \( 1 \leq j \leq i-1 \), and \( \Delta_i \) maximizes the area of \( p_0q_1q_2\ldots q_ip_i \) among the polygonal curves with property (i) (see Figure 3).

(iii) For any \( i \geq 3 \) there exists a convex polygonal curve \( \Gamma_i = p_0p_1\ldots p_i \) and parallel half-lines \( L, L' \) starting at \( p_0 \) and \( p_i \), respectively, such that \( p_0p_1\ldots p_i \subset \text{conv}(L \cup L') \), and the following property is satisfied.

There is a polygonal curve \( \Delta_i = q_1q_2\ldots q_i \) with property (i) which satisfies the midpoint property; that is, \( p_j \) is the midpoint of \( q_jq_{j+1} \) for \( 1 \leq j \leq i-1 \), and \( \Delta_i \) maximizes the area of \( p_0q_1q_2\ldots q_ip_i \) among the polygonal curves with property (i). Furthermore, there is no polygonal curve \( \Delta_i = q_1q_2\ldots q_i \) of property (i) that maximizes the area of \( p_0q_1q_2\ldots q_ip_i \), and \( q_i = p_i \).

(iv) For any \( i \geq 4 \) there exists a convex polygonal curve \( \Gamma_i = p_0p_1\ldots p_i \) and parallel half-lines \( L, L' \) starting at \( p_0 \) and \( p_i \), respectively, such that \( p_0p_1\ldots p_i \subset \text{conv}(L \cup L') \), and the following property is satisfied.

There is a polygonal curve \( \Delta_i = q_1q_2\ldots q_i \) with property (i) which satisfies the midpoint property; that is, \( p_j \) is the midpoint of \( q_jq_{j+1} \) for \( 1 \leq j \leq i-1 \), and \( \Delta_i \) maximizes the area of \( p_0q_1q_2\ldots q_ip_i \) among the polygonal curves with property (i). There is no polygonal curve \( \Delta_i = q_1q_2\ldots q_i \) of property (i) that maximizes the area of \( p_0q_1q_2\ldots q_ip_i \), and \( p_0 = q_1 \) or \( q_i = p_i \).

First, we prove (i). Since our statement is trivial if \( i = 1 \) or \( i = 2 \), we prove it first for \( i = 3 \). Consider an isosceles triangle \([p_1, p_2, y_2]\) with base \( p_1p_2 \). Let \( x_2 \) be an arbitrary interior point of \( p_1p_2 \), and \( q_2 \) be an arbitrary
point of $y_2 x_2$, very close to $y_2$. Reflect the points $y_2, q_2$ and $x_2$ about $p_1$ to obtain the points $p_0, q_1, y_1$, respectively. Reflect also the points $y_2, q_2$ and $x_2$ about $p_2$ to obtain $p'_3, q_3$ and $y'_3$, respectively (see Figure 4). Then we clearly have $A(p_1, p_2, y_2) = A(p_0, p_1, y_1) + A(p_2, p'_3, y'_3) = A(p_0, p_1, q_1) + A(p_1, p_2, q_2) + A(p_2, p'_3, q_3)$. Now slightly rotate the line of $p'_3 y'_3$ about $q_3$ so that it intersects $p_2 y'_3$ at an interior point $y'_3$. Let $p_3$ be the intersection point of this rotated line and the line of $p_2 p'_3$. By the idea of the proof of Theorem 1 we have $A(y_3, p_0, q_1) + A(y'_3, p_2, q_2) < A(p_1, p_2, y_2) < \sum_{j=1}^{3} A(p_{j-1}, q_j, p_j)$. Furthermore, if $q_2$ is sufficiently close to $y_2$, then for any $q'_1 \in p_0 y_1$ and $q'_2 \in y_2 p_2$, if $p_1$ is the midpoint of $q'_1 q'_2$, then $A(p_0, q'_1, p_1) + A(p_1, q'_2, p_2) < \sum_{j=1}^{3} A(p_{j-1}, q_j, p_j)$, and a similar statement holds if we choose points $q'_2 \in p_1 y_2$ and $q'_3 \in p_3 y_3$ in the same way. Set $\Gamma_3 = p_0 p_1 p_2 p_3$, let $L$ be the half-line containing $p_0 q_1$ and starting at $p_0$, and let $L'$ be the half-line containing $p_3 q_3$ and starting at $p_3$. Then the conditions of (i) are satisfied.

Now, we show how to modify this construction for larger values of $i$. We start with the configuration in the last paragraph. Let $\Delta_3$ be the curve that satisfies the conditions in (i). Rotate the line through $y_3 p_3$ around $q_3$ by a very small angle such that the rotated line intersects $p_2 p_3$ at an interior point $p'_3$. Let this line intersect the line through $p_2 y_3$ at $y''_3$. Reflect the points $y''_3, q_3, p'_3$ about $p_3$ to obtain the points $p_4, q_4, y'_4$, respectively.

Figure 3. An illustration of (ii) for the case $i = 4$

Figure 4. An illustration of the proof of (i) for the case $i = 3$
Note that as the angle of rotation tends to zero, for the convex polygonal curve $\Delta_4 = q_1q_2q_3q_4$ satisfying the conditions, the initial part $q_1q_2q_3$ will get arbitrarily close to $\Delta_3$. Since for a very small rotation angle $A(p_3, q_4, p_4) > 0$, $\Delta_4 = q_1q_2q_3q_4$ does not use any of the sides, so it satisfies the conditions. We can proceed similarly and extend our construction for any $i$.

Next, we prove (ii). Let $p_0 = (0, 0)$, $r_1 = (0, 1)$ and $p_1 = (1, 1)$. Define $r_2$ and $m_2$ as the reflections of $p_0$ and $r_1$ about $p_1$, respectively; that is, $r_2 = 2p_1 - p_0 = (2, 2)$ and $m_2 = 2p_1 - r_1 = (2, 1)$. Define $p_2$ as the reflection $p_2 = 2m_2 - p_1 = (3, 1)$ of $p_1$ about $m_2$. For $3 \leq j \leq i$, let $m_j$ and $r_j$ be the reflections of $r_{j-1}$ resp. $m_{j-1}$ about $p_{j-1}$, and for $3 \leq j \leq i - 1$ let $p_j$ be the reflection of $p_{j-1}$ about $m_j$. Finally, we let $p_i = m_i$ and $L'$ as the half-line starting at $p_i$ and passing through $r_i$ (see Figure 5).

Let $q_0q_1\ldots q_i$ be a polygonal curve with property (s), that is, $Q_i = pq_0q_1\ldots q_ip_i$ is convex, $q_0 \in L$, $q_i \in L'$, and $p_j \in q_jq_{j+1}$, $1 \leq j \leq i - 1$. Assume that the area of $pq_0q_1\ldots q_ip_i$ is maximum under these conditions.

We allow the angles of this polygon to be equal to $\pi$, or in other words, some sides of $pq_0q_1\ldots q_ip_i$ might be used. Our key observation is that for any $1 \leq j \leq i - 2$, we have

$$2|p_{j+1}p_j| = |p_{j+1}p_{j+2}| = 2|p_jr_{j+1}|.

First, we show that $Q_i$ does not contain two consecutive sides of $pq_0q_1\ldots q_ip_i$. Indeed, suppose for contradiction that $Q_i$ contains the sides $p_{j-1}p_j$ and $p_jp_{j+1}$ for some $1 \leq j \leq i - 1$. Then we have $q_j = p_j$ or $q_{j+1} = p_j$. Suppose that $q_j = p_j$, the other case is analogous. Then $2 \leq j$ and $q_{j-1} \in r_{j-1}p_{j-1}$. Define the convex polygon $Q_i^*$ by replacing the vertices $q_{j-1}, q_j$ of $Q_i$ by the points $q_{j-1}^*, q_j^*$, respectively, as follows: we slide $q_j$ a little bit on the line of $p_jp_{j+1}$ to the direction $r_{j-1}$ to obtain the point $q_j^*$. Then we choose the point $q_{j-1}^*$ at the intersection of $q_{j-1}q_{j-2}$ and the line through $q_j^*p_{j-1}$. Then, by (1), the area of $Q_i$ is strictly smaller than that of $Q_i^*$, which shows the contradiction.

Now we consider the case that $Q_i$ uses a side $p_{j-2}p_j$ for some $3 \leq j \leq i - 2$, but it does not use the sides $p_{j-2}p_{j-1}$ and $p_{j}p_{j+1}$. Then we may assume that $q_j$ lies in the interior of $p_{j-1}p_j$, and we also have $q_{j-1} \in r_{j-1}p_{j-1}$ and $q_{j+1} \in p_{j}r_{j+1}$. As in the previous paragraph, we modify $Q_i$ to obtain a convex polygon $Q_i^*$. Replace the points $q_{j-1}, q_j, q_{j+1}$ by points $q_{j-1}^*, q_j^*, q_{j+1}^*$, respectively, defined as follows: we slightly move $q_j$ vertically upward to obtain the point $q_j^*$. Let $q_{j-1}^*$ be the intersection of $q_{j-1}q_{j-2}$ and the line through $q_j^*p_{j-1}$, and let $q_{j+1}^*$ be the intersection of $q_{j+1}q_{j+2}$ and the line through $q_j^*p_j$. Note that if $q_{j-1} \neq r_{j-1}$ or $q_{j+1} \neq r_{j+1}$, then by (1) we have that the area of $Q_i^*$ is strictly greater than that of $Q_i$. Thus, by the choice of $Q_i$, we have $q_{j-1} = r_{j-1}$
and \( q_{j+1} = r_{j+1} \). From this, we obtain that \( Q_i \) uses the sides \( p_{j-3}p_{j-2} \) and \( p_{j+1}p_{j+2} \) as well. A straightforward modification of our argument yields a similar statement if \( j = 1, 2, i - 1, i \), implying that either \( Q'_i \) uses no side \( p_{j-1}p_j \), or it uses every second side.

Our considerations allow us to characterize the polygons \( Q_i \) of maximum area. Indeed, by Theorem 1 these are obtained by picking an arbitrary point \( q_i \in p_0r_1 \), and we define \( q_j \) for \( 2 \leq j \leq i \) subsequently as the reflected copy of \( q_{j-1} \) about \( p_{j-1} \). We note that then \( q_i \in p_ir_i \), and \( Q_i \) uses no side if and only if \( q_1 \) lies in the interior of \( p_0r_1 \).

To prove (iii) and (iv), we slightly modify the construction in (ii). First we prove (iii). Here, after defining \( L \) and the points \( m_j, r_j, p_j \) as in the previous example, with a little abuse of notation, we relabel the points \( p_i, r_{i-1} \) as \( p'_i \) and \( r'_{i-1} \), respectively. We set \( p_i = p'_i - (0, \epsilon) \) for some small value \( \epsilon > 0 \), and define \( r_{i-1} \) as the intersection point of the lines through \( p_{i-1}p_i \) and \( p_{i-3}p_{i-2} \). Finally, we set \( L' = p_i + L \) (see Figure 6). Applying a consideration as in the original construction we obtain the following:

- \( p_{i-1}p_i \) is not used;
- for \( j = 1, 2, \ldots, i - 3, i - 1 \), if \( p_{j-1}p_j \) is used, then \( p_{j-3}p_{j-2} \) and \( p_{j+1}p_{j+2} \) are used, if they exist;
- if \( p_{i-3}p_{i-2} \) is used, then we have \( q_{i-1} \in r'_{i-1}r_{i-1} \).

We also observe that if \( p_{i-3}p_{i-2} \) is used and \( q'_{i-1} \neq r'_{i-1} \), then slightly rotating the sideline of \( Q_i \) through \( p_{i-1} \) in counterclockwise direction increases the area of \( Q_i \), which leads to a contradiction. Thus, the property that \( p_{i-3}p_{i-2} \) is used implies that \( q_{i-1} = r'_{i-1} \), yielding also \( q_{i-3} = r_{i-3} \) and the property that \( p_{i-5}p_{i-4} \) is used, if it exists. By these properties, we can characterize the polygons \( Q_i \) of maximal area as in the previous case: we pick an arbitrary point \( q_1 \in p_0r_1 \), and define \( q_j \) for \( 2 \leq j \leq i \) subsequently as the reflected copy of \( q_j-1 \) about \( p_j-1 \). Nevertheless, in this case \( q_i \in p'_i r_i \), therefore, \( q_i \neq p_i \).

Finally, to prove (iv) we can apply the modification in the proof of (iii) not only for \( p_i \) and \( r_{i-1} \), but also for \( p_0 \) and \( r_2 \) in the same way.

**Remark 3.** We note that using similar arguments, one can show the following: For any polygonal curve \( \Gamma_i = p_0p_1p_2p_3 \) and parallel half-lines \( L, L' \) starting at \( p_0 \) and \( p_3 \), respectively, so that the polygon \( p_0p_1 \ldots p_i \) is convex and is contained in the convex hull of \( L, L' \), if there is a convex polygon \( p_0q_1q_2q_3p_3 \) of maximal area under the constraints \( q_0 \in L, q_3 \in L', p_1 \in q_1q_2, p_2 \in q_2q_3 \) that does not use the sides \( p_0p_1, p_1p_2, p_2p_3 \), then there is
a convex polygon \( p_0q_1'q_2'q_3'p_3 \) of maximal area satisfying the same contraints which uses at least one of the sides \( p_0p_1, p_2p_3 \).

Next, we show how part (b) of Theorem 4 follows from the previous constructions. Decompose \( s \) into \( k \) consecutive subsequences \( n_1, n_2, \ldots, n_k \) consisting only of \( Ns \) that are separated by either \( U \) or by \( UU \).

If \( k \geq 3 \), we use only the curve \( \Gamma_i \) from (i). Let \( \bar{P}_k = r_0r_1 \ldots r_k \), where \( r_0 = r_k \), be a regular \( k\)-gon. Sides \( r_{i+1}r_{i+1} \) of \( \bar{P}_k \) will be called old sides. Now we add one or two very small sides, called new sides at each vertex of \( \bar{P}_k \), according to the number of \( Us \) separating \( n_{m-1} \) and \( n_m \). When we add one new side at \( r_i \), let it have the same angle with the two consecutive old sides. When we add two, we allow them to have almost the same angle with the two consecutive old sides such that one of them is much shorter than the other one. Let \( \bar{P} \) denote the resulting polygon. Let \( \bar{T}_m \) be the external triangle bounded by the \( m \)th old side and the lines of the adjacent new sides. If \( n_m \) consists of \( i \) \( Ns \), let \( h_m \) be the affine transformation such that the area of \( \text{conv}(\bar{T}_i) \) is strictly less than area(\( \bar{T}_i \)).

Consider the case that \( k = 2 \) and the two subsequences consisting only of \( Ns \), of lengths \( n_1 \) and \( n_2 \), respectively, are separated by \( U \) and \( UU \). First, consider the case that one of \( n_1, n_2 \), say \( n_1 \), is at least 4, and \( n_2 \geq 2 \). Let \( pp \) and \( p'p' \) be two parallel, sufficiently small segments of length \( \delta \), satisfying \( p - \bar{p} = \bar{p}' - \bar{p} \). Let \( S \) be the infinite strip bounded by the lines through \( pp \) and \( p'p' \). Let \( h \) be the affine transformation satisfying \( h(\text{conv}(S)) = R \), where \( L \) and \( L' \) are the half-lines in (iv), with \( i = 1 \). Similarly, we define \( h \) as the affine transformation satisfying \( h(\text{conv}(L \cup L')) = R \), where \( L \) and \( L' \) are the half-lines in (ii) with \( i = 2 \). Let \( P(\delta) \) be the polygon bounded by \( h(\Gamma_{n_1}), h(\Gamma_{n_2}), pp \) and \( p'p' \).

We show that if \( \delta \) is sufficiently small, the convex hull \( Q(\delta) \) of \( h(\Delta_{n_1}) \) and \( h(\Delta_{n_2}) \) is a maximal area polygon circumscribed about \( P \). Indeed, consider a maximal area polygon \( Q'(\delta) \) circumscribed about \( P \). Suppose for contradiction that for some \( \delta_k \to 0 \), \( Q'(\delta_k) \) does not use one of \( pp \) and \( p'p' \), say \( pp \). Then the limit of the area of \( Q' \cap R \) is strictly smaller than the area of \( \text{conv}(h(\Delta_{n_1}) \cup pp) \), and the limit of the area of \( Q' \cap \bar{R} \) is at most the area of \( \text{conv}(h(\Delta_{n_2}) \cup pp) \). Since the areas of the external triangles of \( P(\delta) \) containing \( pp \) and \( p'p' \) tend to zero as \( \delta \to 0 \), we obtain that the limit of area(\( Q'(\delta) \)) is strictly less than area(\( Q \)); a contradiction. Thus, if \( \delta \) is sufficiently small, \( Q'(\delta) \) uses both \( pp \) and \( p'p' \).

If \( n_1 = n_2 = 3 \), we can use a slight modification of the above construction.

Now we show that no other cyclic sequence can be realized.

For the case \( k = 1 \), we apply the following result of Zaremba [19]. Let \( P = p_1p_2 \ldots p_n \) be a convex polygon, \( Q \) maximum area a convex polygon circumscribed about \( P \). Suppose that \( j < i \), \( p_jp_{j+1} \) and \( p_iP_{i+1} \) are used, but no other side is used between them. Then the turning angle between \( p_jp_{j+1} \) and \( p_iP_{i+1} \), that is, the clockwise angle determined by vectors \( p_jp_{j+1} \) and \( p_iP_{i+1} \), is at most \( \pi \).

If \( k = 1 \), then there is only one \( U \), or two consecutive \( Us \). But the total turning angle of a convex polygon \( P \) at all but one of its vertices is strictly greater than \( \pi \), so this sequence is not realizable.

We are left with the cases \( s = UNNUNN, UNNNUNNN \). We show that \( UNNUNN \) is not realizable, as in the other case a simplified variant of our argument can be applied. Suppose for contradiction that there is a convex polygon \( P = p_1p_2 \ldots p_7 \) and a maximum area circumscribed polygon \( Q \) that uses only the sides \( p_2p_1 \) and \( p_4p_5 \). Then, by the result of Zaremba, \( p_1p_7 \) and \( p_4p_6 \) are parallel. Let the lines through these two segments be \( L \) and \( L' \), respectively. Let \( Q = q_1q_2q_3q_4q_5 \), where \( p_1p_7 \subseteq q_1q_5 \subseteq L \) and \( p_4p_6 \subseteq q_3q_4 \subseteq L' \). Then, by Remark 3 there are points \( q'_1q'_2q'_3 \) with \( q'_1 \in L, p_2 \in q'_2q'_3, p_4 \in q'_2q'_3 \) and \( q'_3 \in L' \) such that area\( (p_1q_1q_2q_3p_4) = \text{area}(p_1q_1q_2q_3p_4) \), and \( q'_1 = p_1 \) or \( q'_3 = p_3 \). Without loss of generality, we may assume that \( q'_1 = p_1 \). On the other hand, our conditions imply that \( p_6 \) is the midpoint of \( q_4q_5 \), and thus, if \( q'_4 \) denotes the intersection of \( L \) with the line through \( p_6p_7 \),
then \( \text{area}(p_5q_4q_5p_7) = \text{area}(p_5q_4'q_5p_7) \). Thus, \( Q' = p_1q_2q_3'q_4p_7 \) is a maximal area polygon circumscribed about \( P \). On the other hand, \( Q' \) uses three consecutive sides of \( P \), namely \( p_0p_7, p_7p_1, p_1p_2 \), a contradiction. □

**Remark 4.** Let \( P \) be a regular \( n \)-gon with unit circumradius, where \( n \geq 5 \). Then an elementary (but tedious) computation yields the following.

1. If \( 2|n \), then \( A(P) = n \tan \frac{\pi}{n} + \frac{n}{2} \frac{\sin \frac{2\pi}{n}}{\cos \frac{2\pi}{n}} \tan \frac{2\pi}{n} \), and the sequence assigned to a maximum area polygon circumscribed about \( P \) is \( UNUN \ldots UN \).

2. If \( 4|(n-1) \), then \( A(P) = n \tan \frac{\pi}{n} + \frac{n}{4} \frac{\sin \frac{2\pi}{n}}{\cos \frac{2\pi}{n}} \left( \tan \frac{2\pi}{n} - \tan \frac{3\pi}{2n} \right) \), and the sequence assigned to a maximum area polygon circumscribed about \( P \) is \( U N N \ldots N U U N \ldots N U \).

3. If \( 4|(n+1) \), then \( A(P) = n \tan \frac{\pi}{n} + \frac{n}{4} \frac{\sin \frac{2\pi}{n}}{\cos \frac{2\pi}{n}} \left( \tan \frac{2\pi}{n} - \tan \frac{\pi}{2n} \right) \), and the sequence assigned to a maximum area polygon circumscribed about \( P \) is \( U N N \ldots N U U N \ldots N U \).

5. An application to statistics

The above content has a connection to the **Gini index**, a measure originally used in economics, statistics, and nowadays being used in many applications, see [12] for a very nice introduction to this subject.

In economics, the **Lorenz curve** is a representation of the distribution of wealth, income, or some other parameter. For a population of size \( n \), with values (say, wealth) \( x_i \) in increasing order, \( F_i = i/n, S_i = \sum_{j=1}^{i} x_j \), and \( L_i = S_i/S_n \). Then the function \( L(F) : F_i \rightarrow L_i \) is the Lorenz curve of the given distribution. That is, \( L_i \) is the relative share of the poorest \( i/n \) part of the population from the total wealth.

In general, for \( 0 \leq \alpha \leq 1 \), let \( x_\alpha \) denote the \( \alpha \)-quantile of a distribution, that is, exactly \( \alpha \) portion of the population has wealth less than \( x_\alpha \). Let \( \bar{x} \) be the mean of the distribution. Then the Lorenz curve is the function \( L(p) := \frac{1}{p} \int_{0}^{p} x_\alpha \, d\alpha \) for \( 0 \leq p \leq 1 \). Clearly, \( L(0) = 0 \) and \( L(1) = 1 \) and \( L \) is always a convex function. \( L(p) = p \) for every \( p \) iff everybody has exactly the same wealth.

The functional

\[
G(L) = 2 \int_{0}^{1} (p - L(p)) \, dp
\]

is called the Gini coefficient. It measures the relative area between a neutral scenario and the observed scenario (See figure 7). (More precisely, twice the area between a neutral scenario and the observed scenario divided by the area under the curve for the neutral scenario.) It is 0 in case of “perfect equality” (everybody has the same wealth) and (almost) 1 in case of “perfect inequality” (one person has all the wealth).

\[\begin{align*}
\text{L}(p) & \quad \text{neutral line} \\
\text{L}(p) & \quad \frac{1}{2} \text{Gini index} \\
\end{align*}\]

\[\begin{align*}
\text{L}(p) & \quad \text{Lorenz curve} \\
\text{L}(p) & \quad \text{P} \\
\end{align*}\]

**Figure 7.** Gini index from a known (left) and unknown (right) Lorenz curve
We defined the Lorenz curve as a continuous curve. In practice often only points on the Lorenz curve are known.

**Remark 5.** Data for every individual is often not available and only data for groups is accessible. From that data only points of the Lorenz curve can be reconstructed.

**Remark 6.** In credit modeling, banks group their clients in $n$ rating groups. After twelve months they see which clients could not pay back their loans and the Lorenz curve is then taken as the percentage of defaults in the worst $i$ groups. A high Gini coefficient indicates that the bank succeeded in discriminating safe clients from dangerous clients.

For both cases above, the real Gini coefficient is not known and upper and lower bounds are of interest. By the convexity of a Lorenz curve the best lower bound is attained by the polygonal curve obtained by connecting the known points on the Lorenz curve. On the other hand, whereas it is easy to see that any maximal area convex curve must be piecewise linear, it does not seem easy to find the best upper bound corresponding to any given point set. A conjecture related to this problem, made by Farris [12], states that the maximal value is attained at a convex polygonal curve with the property that each side of it lies on a sideline of the polygonal curve connecting the given points, or, using our terminology, no sequence associated to any polygonal curve contains consecutive $N$s.

Our results imply that Farris’ conjecture does not hold. As a specific counterexample, we may take the part of a regular $n$-gon $P_n$ with $8 | (n - 4)$, centered at $(0, 1)$, with a vertex at $(0, 0)$ and contained in the unit square $[0, 1]^2$ (cf. Figure 8). From the computations proving Remark 4, it is easy to see that in this case the optimal circumscribed polygonal curve does not use any sides of $P_n$. Equivalently, using the idea of the proof of Theorem 3, we may construct counterexamples assigned to ‘almost all’ sequences of $U$s and $N$s.

Moreover, our algorithm from Section 3 provides an efficient way to find the best upper bound for the Gini index for any given points of the Lorenz curve.

6. **Remarks and Questions**

We strongly suspect that the $O(n^3)$ running time of our algorithm in Section 3 is far from optimal. On the other hand, the best lower bound we can prove is linear, which is trivial.
Theorem 5. Let \( P \) be a convex polytope in Euclidean \( d \)-space, and let \( Q \) be a convex polytope such that every \( k \)-face of \( P \) lies on the boundary of \( Q \). If \( Q \) has maximum volume among such polytopes, then for every facet \( F \) of \( Q \), \( P \) contains the center of gravity of \( F \).

Proof. Assume for contradiction that the center of gravity \( q \) of \( F \) does not belong to \( P' = P \cap F \). Then there is a \((d - 2)\)-dimensional affine subspace \( L \) in \( F \) that separates \( P' \) and \( q \), but for which \( P' \cap L \neq \emptyset \) and \( q \notin L \). Let \( r_{L,\phi} \) denote the rotation of \( \mathbb{R}^d \) about \( L \) with angle \( \phi \), such that for sufficiently small \( \phi > 0 \), \( r_{L,\phi}(F) \) intersects \( P \). If it exists, let \( Q_\phi \) denote the convex polytope, obtained by replacing the supporting halfspace \( H \) of \( Q \) determined by \( F \) by \( r_{L,\phi}(H) \). Observe that the derivative of \( \text{vol}_d(Q_\phi) \) is proportional to the torque of \( F \) with respect to \( L \). Nevertheless, since \( L \) separates \( q \) and \( F' \), this torque is positive, which means that \( Q \) has no maximum volume.

It is an interesting question to ask how well the area of a convex polygon \( P \) can be approximated by the area of a maximum area circumscribed polygon \( Q \). Clearly, \( \frac{\text{area}(Q)}{\text{area}(P)} \) can be arbitrarily large. This happens, for example, if the sum of two consecutive angles of \( P \) is only slightly larger than \( \pi \). On the other hand, \( \frac{\text{area}(Q)}{\text{area}(P)} > 1 \) is satisfied for every convex polygon \( P \). The following proposition shows that this ratio can be arbitrarily close to one as well.

Proposition 1. Let \( n \geq 6 \). Then, for every \( \varepsilon > 0 \) there is a convex \( n \)-gon \( P \) such that for any maximum area polygon \( Q \) circumscribed about \( P \), we have \( \frac{\text{area}(Q)}{\text{area}(P)} < 1 + \varepsilon \).

Proof. Let \( p_1, p_2, \ldots, p_n \) be the vertices of \( P \) in counterclockwise order. For \( i = 1, 2, \ldots, n \), let \( T_i \) denote the external triangle that belongs to the side \( p_ip_{i+1} \). We show the existence of a convex \( n \)-gon \( P \) such that \( \sum_{i=1}^n \text{area}(T_i) \leq \varepsilon \), and \( \text{area}(P) \geq 1 \). Since \( Q \subset P \cup (\bigcup_{i=1}^n T_i) \), this will clearly imply our statement.

Let \( p_1, p_2 \) and \( p_3 \) be the vertices of a triangle of unit area, in counterclockwise order. We choose the vertex \( p_4 \) in such a way that \( p_4 \) is sufficiently close to \( p_3 \), and \( \text{area}(T_2) < \frac{\varepsilon}{4} \). We choose \( p_n \) similarly, close to \( p_1 \) and satisfying \( \text{area}(T_1) < \frac{\varepsilon}{4} \). Note that if \( p_4 \) and \( p_n \) are sufficiently close to \( p_3 \) and \( p_1 \), respectively, then the sum of the areas of the two triangles, one bounded by the lines through \( p_2p_3 \), \( p_3p_4 \) and \( p_4p_n \), and the other one bounded by the lines through \( p_4p_n \), \( p_n p_1 \) and \( p_1 p_2 \), is less than \( \frac{\varepsilon}{4} \) (cf. Figure 9). Now if we put the remaining vertices sufficiently close to the segment \( p_4p_n \), then \( \sum_{i=1}^n \text{area}(T_i) < \varepsilon \).

Acknowledgements

Markus Ausserhofer acknowledges support through FWF-project Y782. Susanna Dann thanks Oberwolfach Research Institute for Mathematics for its hospitality and support, where part of this project was carried out. Zsolt Lángi was supported by the National Research, Development and Innovation Office, NKFIH, K-119670, and the János Bolyai Research Scholarship of the Hungarian Academy of Sciences. Support of grant BME FIKP-VÍZ by EMMI is kindly acknowledged. Géza Tóth was supported by the National Research, Development and Innovation Office, NKFIH, K-111827, and his work is connected to the scientific program of the "Development
Figure 9. Construction of a convex polygon $P$ such that the area of a maximum area polygon circumscribed about $P$ is ‘close’ to $\text{area}(P)$.

of quality-oriented and harmonized R+D+I strategy and functional model at BME” project, supported by the New Hungary Development Plan (Project ID: TÁMOP-4.2.1/B-09/1/KMR-2010-0002).

We express our sincere gratitude to Nicolas Bonneel, who directed our attention to the contradiction between the result of Zaremba in [19] and our result in [4].

References

[1] P.K. Agarwal and M. Sharir, Algorithmic techniques for geometric optimization, Computer science today, 234–253, Lecture Notes in Comput. Sci. 1000, Springer, Berlin, 1995.
[2] A. Aggarwal, H. Booth, J. O’Rourke and S. Suri, Finding minimal convex nested polygons, Inform. and Comput. 83 (1989), no. 1, 98–110.
[3] A. Aggarwal, J.S. Chang and C.K. Yap, Minimum area circumscribing polygons, The Visual Computer 1 (1985), no. 2, 112–117.
[4] M. Ausserhofer, S. Dann, Z. Lángi and G. Tóth, An algorithm to find maximum area polygons circumscribed about a convex polygon, Discrete Appl. Math. 255 (2019), 98-108.
[5] M. de Berg, O. Cheong, M. van Kreveld, M. Overmars: Computational Geometry, Algorithms and Applications, Third Edition, Springer-Verlag, Berlin, Heidelberg, 2008.
[6] G. Barequet and V. Rogol, Maximizing the area of an axially symmetric polygon inscribed in a simple polygon, Computers & Graphics 31 (2003), no. 1, 127–136.
[7] N. Bonneel, personal communication, 2023.
[8] P. Bose and J.L. De Carufel, Minimum-area enclosing triangle with a fixed angle, Comput. Geom. 47 (2014), no. 1, 90–109.
[9] J.E. Boyce, D.P. Dobkin, R.L. Drysdale III and L.J. Guibas, Finding extremal polygons, SIAM J. Comput. 14 (1985), no. 1, 134–147.
[10] D. Dori and M. Ben-Bassat, Circumscribing a convex polygon by a polygon of fewer sides with minimal area addition, Comput. Vision Graphics Image Process. 24 (1983), no. 2, 131–159.
[11] D. Eppstein, M. Overmars, G. Rote and G. Woeginger, Finding minimum area k-gons, Discrete Comput. Geom. 7 (1992), no. 1, 45–58.
[12] F.A. Farris, The Gini index and measures of inequality, Amer. Math. Monthly 117 (2010), no. 10, 851–864.
[13] R.T. Jantzen and K. Volpert, On the mathematics of income inequality: splitting the Gini index in two Amer. Math. Monthly 119 (2012), no. 10, 824–837.
[14] V. Klee and M. Laskowski, Finding the smallest triangles containing a given convex polygon, J. Algorithms 6 (1985), no. 3, 359–375.
[15] J.S.B. Mitchell, and V. Polishchuk, Minimum-perimeter enclosures, Inform. Process. Lett. 107 (2008), no. 3-4, 120–124.
[16] R. Molano, P.G. Rodríguez, A. Caro and M. L. Durán, Finding the largest area rectangle of arbitrary orientation in a closed contour, Appl. Math. Comput. 218 (2012), no. 19, 9866–9874.
[17] J. O’Rourke, A. Aggarwal, S. Maddila and M. Baldwin, *An optimal algorithm for finding minimal enclosing triangles*, J. Algorithms 7 (1986), no. 2, 258–269.

[18] M. Sharir and S. Toledo, *Extremal polygon containment problems*, Comput. Geom. 4 (1994), no. 2, 99–118.

[19] S.K. Zaremba, *Computing the isotropic discrepancy of point sets in two dimensions*, Discrete Math. 11 (1975), 79–92.

Markus Ausserhofer, University of Vienna, Oskar-Morgenstern-Platz 1, 1010 Vienna, Austria

*Email address:* markus.ausserhofer@hotmail.com

Susanna Dann, Institute of Discrete Mathematics and Geometry, Vienna University of Technology, Wiedner Hauptstrasse 8-10, 1040 Vienna, Austria

*Email address:* susanna.dann@tuwien.ac.at

Zsolt Lángi, Dept. of Geometry, Budapest University of Technology and Economics, Egyházmegyatermelési Ut 1., Hungary, 1111, Research Group of Morphodynamics, Hungarian Academy of Sciences, supported by the National Research, Development and Innovation Office, NKFI, K-119670

*Email address:* zlangi@math.bme.hu

Géza Tóth, Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Reáltanoda u. 13-15., 1053 Budapest, Hungary, supported by the National Research, Development and Innovation Office, NKFIH, K-111827.

*Email address:* toth.geza@renyi.mta.hu