Accurate finite-volume high-order compact interpolation with non-orthogonal and non-uniform grids for computational aeroacoustics

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Abstract. In recent years, with the development of supercomputers, the need for engineering applications of computational aeroacoustics is increasing. For industrial applications, the finite-volume compact interpolation is often applied as a discretization method. However, the original finite-volume compact interpolation can be applied only for calculation grids which are orthogonal and uniform, because the original finite-volume compact scheme is formulated assuming that calculation grids are orthogonal and uniform. Indeed, calculation grids for the computational aeroacoustics cannot be limited to orthogonal and uniform. Therefore, the finite-volume compact scheme for arbitrary calculation grids is required. In this paper, the finite-volume NOGS compact scheme is established in three dimensions to be suited for calculations using arbitrary grids. The interpolation error tests were conducted using a number of non-orthogonal and non-uniform grids to verify the NOGS compact scheme. From these interpolation tests, it is revealed that the NOGS compact scheme can reduce the interpolation error compared to the original compact scheme. Therefore, it is predicted that the reduction of the interpolation error leads to accurate evaluation of the numerical flux and that the calculation results are improved. Also, the large eddy simulations around a circular cylinder were conducted to verify the influence of the interpolation error reduction on the flow field and on the sound field.

1. Introduction

In recent years, with the development of supercomputers, the need for engineering applications of computational aeroacoustics is increasing. Computational aeroacoustics make it possible to understand the generation, propagation and radiation of sound waves even without the expensive experimental equipment. However, computational aeroacoustics require stricter calculation schemes than computational fluid dynamics [1], because it is necessary to calculate unsteady broadband sound waves and also because low-dissipative and low-dispersive characteristics are indispensable [2].

For the computational aeroacoustics, the finite-volume method and the finite-difference method are mainly applied as a discretization method. For industrial applications, the finite-volume method is often preferred because it satisfies the governing conservation laws of the fluid mechanics [6] and can be applied for complicated grids. With regard to the finite-difference method, although it is easy to increase accuracy [5], it is difficult to calculate under complicated grids. Therefore, the finite-difference method is not preferred for industrial applications.
For the finite-volume method, the interpolation of interface-averaged field values using known cell-averaged values is necessary to calculate the numerical flux. For the computational aeroacoustics, the compact scheme is often applied as a convection scheme for the interface-averaged values interpolation [3]. It is because the compact scheme has both low-dispersive and low-dissipative characteristics while having the same accuracy as the standard explicit methods of the same order of accuracy. However, for the finite-volume method, the compact scheme can be applied only for calculation grids which are orthogonal and uniform, because the original finite-volume compact interpolation is formulated assuming that calculation grids are orthogonal and uniform. Indeed, calculation grids for complex geometry such as aircraft engine components are no longer orthogonal or uniform.

Efforts have been done to solve it. However, one can find that abundant literature about it is always in the finite-difference context [4]. Therefore, to solve it, Fosso et al. proposed the improved finite-volume compact interpolation scheme for arbitrary structured grids [2]. It is made by directly accounting for multi-dimensional derivatives involving in the Taylor series expansion of the function to interpolate. Also, it is revealed that this improved compact scheme is well suited for highly complicated grids. However, it is established only in two dimensions. Therefore, the establishment of this improved finite-volume compact interpolation scheme in three dimensions is required. The objective in this paper is the establishment of the improved compact scheme suited for non-orthogonal and non-uniform grids in three dimensions, which is called as Non-orthogonal grid system (NOGS) compact scheme. Also, the validation of the NOGS compact scheme is required.

2. The original compact scheme

The original compact coefficients are determined assuming that the calculation grids are orthogonal and uniform in one dimensions. The one-dimensional grid line is shown in figure 2.1.

\[ \alpha \bar{u}_{i-1/2} + \bar{u}_{i+1/2} + \beta \bar{u}_{i+3/2} = \sum_{l=-n}^{n} a_l \bar{u}_{i+l} \]  \hspace{1cm} (2.1)

In this research, the 6th-order compact interpolation is discussed. Therefore, equation (2.1) is reduced to equation (2.2). However, on the block boundary cell interface, equation (2.1) can be replaced by equation (2.3), because the equation has to be explicit, that is, the 4th-order cell center explicit scheme is applied on the block boundary cell interface.

\[ \alpha \bar{u}_{i-1/2} + \bar{u}_{i+1/2} + \beta \bar{u}_{i+3/2} = \sum_{l=-1}^{z} a_l \bar{u}_{i+l} \]  \hspace{1cm} (2.2)

\[ \bar{u}_{i+1/2} = \sum_{l=-1}^{z} a_l \bar{u}_{i+l} \]  \hspace{1cm} (2.3)

\( \alpha, \beta, a_l \) are called as compact coefficients which are determined from the grid geometry. One has to invert the tridiagonal matrix, which is made from equations (2.2) and (2.3), to obtain the interfaced-

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**Figure 2.1.** One-dimensional grid line.

where \( \bar{u} \) denotes the cell-averaged value and \( \bar{u} \) denotes the interface-averaged field value. Also, integer numbers denote the cell center and fractions denote the cell interface. For the compact interpolation, the interface-averaged field values are approximated by cell-averaged values. The following equation (2.1) is written including the interface-averaged values on the interfaces \( (i - 1/2) \) and \( (i + 3/2) \) implicitly.
averaged field values. It is known that the tridiagonal matrix can be inverted at low cost. To determine the compact coefficients, the Taylor series expansion is executed to the field values as equation (2.4). The Taylor series expansion is 6th-order accuracy in \( x \) direction to obtain 6 equations about the compact coefficients.

\[
u(x) = \sum_{m=0}^{m_0} \frac{1}{m!} (x - x_0)^m \frac{d^m u_0}{dx^m}
\]

where \( x_0 \) is the position of \( \bar{u}_{i+1/2} \). One can obtain 6 equations for each derivative term from equations (2.2) to (2.4). Also, one can determine all compact coefficients assuming that the calculation grids are orthogonal and uniform. Finally, the original compact scheme is expressed as following equation (2.5) on the inner cell interface and equation (2.6) on the block boundary cell interface.

\[
\frac{1}{3} \bar{u}_{i-1/2} + \frac{1}{3} \bar{u}_{i+1/2} + \frac{1}{3} \bar{u}_{i+3/2} = \frac{1}{36} \bar{u}_{i-1} + \frac{29}{36} \bar{u}_{i} + \frac{29}{36} \bar{u}_{i+1} + \frac{1}{36} \bar{u}_{i+3/2}
\]

(2.5)

\[
\bar{u}_{N+1/2} = -\frac{1}{12} \bar{u}_{N-1} + \frac{7}{12} \bar{u}_{N} + \frac{7}{12} \bar{u}_{N+1} - \frac{1}{12} \bar{u}_{N+2}
\]

(2.6)

The compact coefficients are constant and they do not depend on the cell geometry.

3. The NOGS compact scheme

The NOGS compact scheme is formulated in three dimensions. In this respect, the formulation is a little different from the original compact scheme in addition to the NOGS handling. Also, on the block boundary cell interface, the 4th-order upwind scheme is applied to increase the stability instead of the 4th-order cell center explicit scheme.

3.1. Formulation on the inner cell interface

The NOGS compact coefficients are determined in three dimensions. The interface-averaged field values are approximated by cell-averaged values. The following equation (3.1) is written like equation (2.1) including the interface-averaged values on the interfaces \((i - 1/2,j,k)\) and \((i + 3/2,j,k)\) implicitly.

\[
a_1 \bar{u}_{i-1/2,j,k} + \bar{u}_{i+1/2,j,k} + \beta \bar{u}_{i+3/2,j,k} = \sum_{l=-m}^{l=n} \sum_{p=-q}^{l=p} \sum_{s=-t}^{s=m} a_{l,p,s} \bar{u}_{i+l,j+p,k+s}
\]

(3.1)

4 main interface-averaged field values and 8 supplemental interface-averaged field values are used in this research. Therefore, equation (3.1) is reduced to equation (3.2) on the inner cell interface.

\[
a_1 \bar{u}_{i-1/2,j,k} + \bar{u}_{i+1/2,j,k} + \beta \bar{u}_{i+3/2,j,k} = \sum_{l=-1}^{l=0} \sum_{p=0}^{p=0} \sum_{s=0}^{s=0} a_{l,p,s} \bar{u}_{i+l,j+p,k+s} + \sum_{l=1}^{l=1} \sum_{p=1}^{p=1} \sum_{s=1}^{s=1} a_{l,p,s} \bar{u}_{i+l,j+p,k+s}
\]

(3.2)

3.2. Formulation on the block boundary cell interface

Equation (3.2) cannot be used on the block boundary cell interface, because there is no adjacent cell interface. Therefore, the 4th-order upwind scheme is applied on the block boundary cell interface as expressed in equations (3.3) and (3.4). It is well known that the present upwind scheme has stable characteristics.

\[
\bar{u}_{3/2,j,k} = \sum_{l=-1}^{l=2} a_{l} \bar{u}_{i+l,j,k}
\]

(3.3)

\[
\alpha \bar{u}_{N-1/2,j,k} + \bar{u}_{N+1/2,j,k} = \sum_{l=-1}^{l=1} a_{l} \bar{u}_{N+l,j,k}
\]

(3.4)

However, when the upwind scheme is applied on the block boundary cell interface, the interpolation values on each boundary cell interface is different, because the evaluation of the approximation of the interpolation values is different. It breaks the governing conservation laws of the fluid mechanics. Therefore, the Roe average is applied on the block boundary cell interface to evaluate the consistent numerical flux in the adjacent block. It means that the consistent numerical flux is calculated uniquely.
from the interpolated interface-averaged field values and that the present scheme satisfies the governing conservation laws.

3.3. Calculation method of the Compact coefficients
There are 14 compact coefficients to be determined from equations (3.2) to (3.4). To determine all compact coefficients, at least 14 linear equations are necessary. The Taylor series expansion is executed to the field values as expressed in equation (3.5). In order to reduce cost and to account for the relevant direction, the Taylor series expansion is 6th-order accuracy in x direction and 4th-order accuracy in other directions.

\[
u(x, y, z) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{1}{m! n! p!} (x - x_0)^m (y - y_0)^n (z - z_0)^p \frac{d^{m+n+p} u_0}{dx^m dy^n dz^p} \tag{3.5}
\]

where \((x_0, y_0, z_0)\) is the position of \(\tilde{u}_{i+1/2,j,k}\). Also, for the complicated grids, the orthogonal frame \((x, y)\) is not always suitable for the Taylor series expansion. In a previous research, Fosso et al. proposed to use the non-orthogonal frame \(\left(x', y''\right)\) as shown in figure 3.1 [2]. Therefore, the non-orthogonal frame is used for the Taylor series expansion also in this research. One can obtain 22 equations for each derivative term from equations (3.2) to (3.5). The Taylor series expansion derivatives are listed in table 1.

![Figure 3.1. Non-orthogonal local frame [2].](image)

| Index of row | Derivatives | Total number |
|--------------|-------------|--------------|
| x-direction  | \(\partial^5, \partial x, \partial x^2, \partial x^3, \partial x^4, \partial x^5\) | 6            |
| 1st-order    | \(\partial y, \partial z\)        | 2            |
| 2nd-order    | \(\partial y^2, \partial x \partial y, \partial x \partial z, \partial y \partial z\) | 5            |
| 3rd-order    | \(\partial y^3, \partial z^3, \partial x^2 \partial y, \partial x \partial y^2, \partial x^2 \partial z, \partial x \partial z^2, \partial y^2 \partial z, \partial y \partial z^2, \partial x \partial y \partial z\) | 9            |

The number of the compact coefficients, which should be determined, is different from the number of the equations. Therefore, the least square method is applied to determine the compact coefficients uniquely. The determined compact coefficients are constant and they depend on the cell geometry. One
has to invert the tridiagonal matrix which is made from equations (3.2) to (3.4) to obtain the interface-averaged field values.

4. Interpolation error verification
In the compact interpolation, the interface-averaged field values are approximated by cell-averaged values. In this section, the pure interpolation accuracy is verified.

4.1. Verification method
The interpolated interface-averaged values are compared to the exact solutions. The initial cell-averaged values are given by sine function as expressed in equation (4.1).

\[ \bar{u}_{i,j,k} = \frac{1}{V_{i,j,k}} \int_{V_{i,j,k}} \sin(2\pi kx) \, dV \]  

where \( k \) is the non-dimensional wave number, \( V_{i,j,k} \) is the volume of the cell. The interface-averaged value \( \bar{u}_{i+1/2,j,k} \) is approximated by cell-averaged values. On the other hand, the exact interface-averaged value \( \bar{u}_{\text{exact};i+1/2,j,k} \) is given by equation (4.2) theoretically.

\[ \bar{u}_{\text{exact};i+1/2,j,k} = \frac{1}{S_{i+1/2,j,k}} \int_{S_{i+1/2,j,k}} \sin(2\pi kx) \, dS \]  

where \( S_{i+1/2,j,k} \) is the area of the interface. The interpolation error between the interpolated interface-averaged value and the exact interface-averaged value is calculated as equation (4.3).

\[ e = |\bar{u}_{\text{exact}} - \bar{u}_{\text{interpolated}}| \]  

where, \( e \) is the interpolation error. The interpolation error can be calculated for every non-dimensional wave number. In this research, the non-dimensional wave number is from 0.25 to 4.0. The complicated grids which are used in the interpolation error verification are shown in figures 4.1 and 4.2. The wavy grid is non-orthogonal and non-uniform. On the other hand, the stretch grid is orthogonal and non-uniform.

4.2. Results and discussion
The interpolation error verification is executed for both wavy grids and stretch grids. The results are shown in figures 4.3 and 4.4.

The results for wavy grids are shown in figure 4.3. For wavy grids, two kind of grids are used. The upper figure shows the result for the wavy grid, whose wave amplitude is 4% of the length of one side. The lower figure shows the result for the wavy grid, whose wave amplitude is 8% of the length of one side. For both wavy grids, it is revealed that the NOGS compact scheme can reduce the interpolation error.

![Figure 4.1. The wavy grid.](image1)

![Figure 4.2. The stretch grid.](image2)
error compared to the original compact scheme, that is, the NOGS compact scheme is effective for wavy grids with any non-dimensional wave number.

The results for stretch grids are shown in figure 4.4. Also for stretch grids, two kind of grids are used. The upper figure shows the result for the stretch grid, in which adjacent cells get 0.8 times smaller as the position goes far from the origin. The lower figure shows the result for the stretch grid, in which adjacent cells get 0.9 times smaller. For both stretch grids, it is revealed that the NOGS compact scheme can reduce the interpolation error compared to the original compact scheme especially for small non-dimensional wave number. For large non-dimensional wave number, the NOGS compact scheme is not effective, because the interpolation is difficult near the origin, where the cell size is large compared to the non-dimensional wave number.

![Figure 4.3. Interpolation error for wavy grids.](image1)

![Figure 4.4. Interpolation error for stretch grids.](image2)

5. The large eddy simulations around a circular cylinder
It is revealed that the NOGS compact scheme can reduce the interpolation error in the previous section. The large eddy simulations around a circular cylinder are executed to estimate the influence of the interpolation error reduction on the flow field and the sound field. The flow past a circular cylinder at Mach number of 0.2 and Reynolds number based on the cylinder diameter of $Re = 3.9 \times 10^3$ is simulated to compare the flow field. The flow past a circular cylinder at Mach number of 0.21 and Reynolds number of $Re = 4.8 \times 10^4$ is simulated to compare the sound field. For the approximation of the far field sound pressure level, the Ffowcs Williams-Hawkings method [16] is applied.

5.1. Calculation method
The governing equations are the Navier-Stokes equations. The fluid is compressible air with a constant specific heat ratio, and external forces such as gravity are neglected. The flow solver is the UPACS-LES developed by JAXA, Japan Aerospace Exploration Agency, which uses the cell-centered finite volume method. The calculation method is listed in table 2 in detail.

| Table 2. Calculation method. |
|-------------------------------|
| CFD solver | UPACS-LES |
| SGS model | Implicit LES |
| Spatial discretization | Cell center type finite volume method |
| Convection term | 6th-order original and NOGS compact scheme |
| Viscous term | 2nd-order central difference scheme |
| Time integration | 3rd-order Runge-Kutta method |


The compact scheme sometimes causes the numerical oscillation. Therefore, the 10th-order compact filter is applied to reduce the high frequency numerical oscillation. The compact filter coefficient is 0.49.

5.2. Calculation grids
Two O-type grids are prepared for each calculation case. The calculation grids are shown in figures 5.1 and 5.2. The grid mesh system is the structured multi-block grids. The total cells are about 1.5M cells for each grid. The non-dimensional distance $y^+$, which is defined as the height of the first mesh cell off the wall, is 1.0 to ensure accurate simulation of the flow field. The domain size is $(50 \times 50 \times \pi)$ in non-dimensional ($x, y, z$) for $Re = 3.9 \times 10^3$, $(50 \times 50 \times 0.5)$ for $Re = 4.8 \times 10^4$. The boundary conditions are given as follows. At the outer boundary, far-field subsonic boundary condition is prescribed. The no-slip condition is invoked on the cylinder surface. At the spanwise boundary, periodicity is applied.

5.3. Results and discussion
The mean flow statistics at $Re = 3.9 \times 10^3$ from the current simulations, together with the results of the experimental studies [7] are shown in figures 5.3 and 5.4. Figure 5.3 shows that the pressure coefficient on the cylinder surface is same even if different compact schemes are applied to the convection term. The mean flow field converges to one solution by continuing the calculation. Therefore, the pressure coefficient is almost same with different compact schemes.

Figure 5.4 shows the streamwise velocity on the center line in the wake of a circular cylinder at $Re = 3.9 \times 10^3$. The calculation results have a difference with the experimental result. The calculation results have the minimum streamwise velocity around $X/D = 2.0$, where $D$ is the diameter of a circular cylinder. On the other hand, the experimental result has the minimum streamwise velocity around $X/D = 1.0$. It is caused by the overestimated longer shear layer. It is reported that the prediction of the length of the shear layer is difficult [7]. The difference between the result of the original compact scheme and the result of the NOGS compact scheme is seen in the wake domain near the circular cylinder. In this domain, Mach number is small and shed vortex is so complicated. Also, the averaged calculation result is difficult to converge. Therefore, it is possible that the influence of the discretization of the convection term on the flow field is large even in the mean flow field where the flow is complex. The NOGS compact scheme can calculate more accurate numerical flux. Thus, the calculation result with the NOGS compact scheme is more reliable.

Figure 5.1. Calculation grid for $Re = 3.9 \times 10^3$.

Figure 5.2. Calculation grid for $Re = 4.8 \times 10^4$. 
Figure 5.3. The pressure coefficient on the cylinder surface at Re = 3.9 × 10^3. One can understand that the result is same even if the different compact schemes are applied to the convection term.

Figure 5.4. The streamwise velocity on the center line in the wake of a circular cylinder at Re = 3.9 × 10^3. The difference of the compact schemes is seen in the wake near the circular cylinder, where the flow field is complex.

The large eddy simulations around a circular cylinder at Re = 4.8 × 10^4 are conducted to estimate the difference of the sound field. The acoustic measurements are conducted in the far field at (15.5, 185, 0) in non-dimensional (x, y, z) from the circular cylinder center [9]. The observation point in the experiment is outside of the calculation grid. Therefore, the sound pressure fluctuation is approximated by the Ffowcs Williams-Hawkings method [16]. The results of calculations at Re = 4.8 × 10^3 from the current simulations, together with the results of the experimental results [9] are shown in figures 5.5 and 5.6.

Figure 5.5 shows the sound pressure fluctuation at the observation point at Re = 4.8 × 10^4. The pressure amplitude is almost same. However, the fluctuating period is a little different. The result of the NOGS compact scheme has a longer fluctuating period compared to the result of the original compact scheme.

Figure 5.6 shows the sound pressure spectrum at the observation point. It is known that for the Karman vortex around a circular cylinder at Re = 3.0 × 10^2 to 3.0 × 10^5, Strouhal number has a constant value. Both calculation results realize this constant Strouhal number and the sound pressure level. In high and low frequency domain, spectrum is quite different with different compact schemes. This suggests that the discretization method of the convection term has a large influence on the random
behavior of the large and low frequency eddies predicted. However, it is difficult to realize the superiority of the NOGS compact scheme.

Figure 5.5. Sound pressure fluctuation.

Figure 5.6. SPL in the frequency domain.

6. Conclusion
The NOGS compact scheme is established in three dimensions in this paper. From the interpolation error verification, it is proven that the NOGS compact scheme can reduce the interpolation error for any non-dimensional wave number with wavy grids, and especially for small wave number with stretch grids compared to the original compact scheme. It means that the NOGS compact scheme can calculate accurate numerical flux for arbitrary wavy and stretch grids. The large eddy simulations around a circular cylinder at $Re = 3.9 \times 10^3$ show the influence of a discretization method of the convection term on the mean flow field, where Mach number is small and shed vortex is complex. Also, from the large eddy simulations around a circular cylinder at $Re = 4.8 \times 10^4$, it is revealed that the calculated sound pressure level is quite different in low or high frequency spectrum.

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