Main Physical Aspects of the Mathematical Conception of Energy in Thermodynamics

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Abstract

We consider the main physical notions and phenomena described by the author in his mathematical theory of thermodynamics. The new mathematical model yields the equation of state for a wide class of classical gases consisting of non-polar molecules provided that the spinodal, the critical isochore and the second virial coefficient are given. As an example, the spinodal, the critical isochore and the second virial coefficient are taken from the Van-der-Waals model. For this specific example, the isotherms constructed on the basis of the author’s model are compared to the Van-der-Waals isotherms, obtained from completely different considerations.

Keywords: Van-der-Waals model; compressibility factor; spinodal; number of degrees of freedom; admissible size of clusters; quasi-statical process.

1 Introduction

In the present paper, we present the foundations of the new mathematical model of thermodynamics for the values of the energy for which the molecules are in the pre-plasma state, in particular as pressure \( P \) is near-zero.

The new mathematical model constructed by the author in the cycle of papers [1]–[2], differs somewhat from the commonly accepted model of phenomenological thermodynamics and allows us to construct the equation of state for a wider class of classical gases.

We mainly study the metastable states in the case of gases and in the case of fluids up to the critical isochore \( \rho = \rho_c \). The liquid isotherms are divided into two parts: the region with temperatures near the critical temperature \( T_c \) and the region of the “hard” liquid whose isotherms pass through the point of zero pressure.

Further, the distributions in the Van-der-Waals model are compared to the distributions appearing in the author’s new model for classical gases with given critical temperatures, in particular, with Van-der-Waals critical temperature. The precision with which the isotherms constructed from the author’s theory coincide with the Van-der-Waals isotherms turns out to be less than the accepted precision in experimental studies. The probability of such a coincidence being accidental is “infinitely small”.

All the distributions of energy levels for classical gas considered in the present paper have been established mathematically, see [1]–[3] and references therein) and have met with approval (see, for example, the review [4], published in the journal “Probability Theory and
its Applications” in connection with the State Prize of the Russian Federation in Science and Technology granted to the author in 2013 (for developing the mathematical foundations of modern thermodynamics).

2 Van-der-Waals Normalization and the Law of Corresponding States

The correct choice of units of measurement for different gases allowed Van-der-Waals to compare the parameters of the gases to dimensionless quantities, thus obtaining the famous “law of corresponding states”.

For every gas, there exists a critical temperature $T_c$ and a critical pressure $P_c$ such that if the temperature or the pressure is greater than their critical values, then gas and liquid can no longer be distinguished. This state of matter is known as a fluid.

The Van-der-Waals “normalization” consists in taking the ratio of the parameters by their critical value, so that we consider the reduced temperature $T_r = T/T_c$ and the reduced pressure $P_r = P/P_c$. In these coordinates, the diagrams of the state of various gases resemble each other – this a manifestation of the law of corresponding states.

The equation of state for the Van-der-Waals gas in dimensionless variables has the form:

\[(P_r + \alpha \rho^2)(1 - \tilde{\alpha} \rho) = \rho T_r,\]

where $\alpha = 3^3/2^6$, $\tilde{\alpha} = 1/2^3$, and the dimensionless concentration (density) $\rho = N/V$ is chosen so that $\rho_c = 8/3$ in the critical point.

In thermodynamics, the dimension of the internal energy $E$ is equal to the dimension of the product $P V$ of the pressure by the volume. On the other hand, temperature plays the role of mean energy $E/N$ of the gas (or of the liquid). For that reason, when the dimension of temperature is given in energy units, the parameter $Z = P/\rho T$, known as the compressibility factor, is dimensionless. For the equation of state of the gas or liquid, the most natural diagram is the $P_r-Z$ diagram, in which the dimensionless pressure $P_r$ is plotted along the $x$-axis and the dimensionless parameter $Z$, along the $y$-axis. A diagram with axes $P_r, Z$ is called a Hougen–Watson diagram. In the Van-der-Waals model, the critical value of the compressibility factor is

\[Z_c = \frac{P_c V_c}{N T_c} = \frac{3}{8}.\]

In Figs. 2–11 the Van-der-Waals isotherms are shown in a $P_r-Z$ Hougen–Watson diagram, for which we used the normalization $P_r = P/P_c$, $T_r = T/T_c$, where $T_c$ is the critical temperature.

3 On the Number of Degrees of Freedom and the Partition Theory of Integers

In the physics literature, the notion of number of degrees of freedom is used for ideal gases, in which particles do not interact. In probability, the number of degrees of freedom is also considered independently of any interaction.

When physicists speak of the number of degrees of freedom, they usually have in mind the number of degrees of freedom of a single molecule. Normally, a one-atom molecule will have 3 degrees of freedom, a two-atom molecule, 5. In a gas, the molecules move with
Figure 1: Van-der-Waals isotherms on the $P_r$-$Z$ diagram. On the plot $P = P_r$, $T = T_r$. The curve shown by the fat hashed line is the critical isotherm $T_r = 1$. The dotted line is the spinodal, the thin hashed line is the binodal.

Figure 2: The $P_r$-$Z$ diagram for the Van-der-Waals equation with $P = P_r$.

different speeds and different energies; although their mean energy – temperature – is the same, their individual energies may be different, and the number of degrees of freedom of different molecules can also differ. Thus a two-atom molecule with very high energy may have a number of degrees of freedom greater than than 5, and this will affect the mean value (average over all molecules) of the number of degrees of freedom of the given gas. This mean value is called the collective number of degrees of freedom of the gas, and is a fractional rather than a whole number.

A one-atom molecule has 3 degrees of freedom, a two-atom molecule, 5. Two-atom molecules are regarded in [5] as molecules of ideal gas. Nevertheless, saying that the number of degrees of freedom of a molecule is 5 means that we are implementing an exclusion rule: we are saying that one of the degrees of freedom is excluded. More precisely, we regard the molecule as a dumbbell, and exclude its oscillations as a rod. However, if the temperature increases, these “rod” oscillations will take place and so, for sufficiently high temperatures (energies), the molecules will have a greater fractional number of degrees of freedom.

In the author’s model, the number of degrees of freedom is an important independent parameter, so that the notion of exclusion need not be defined by means of some internal
considerations.

Although the collective number of degrees of freedom of a classical gas can be determined by the initial interaction between its particles, in the probabilistic description of the generalized ideal gas (a notion that we present in Section 5), this parameter is introduced independently.

If nuclear forces are taken into consideration, it can be shown that a wide class of two-atom molecules have 5 degrees of freedom, but this is difficult to establish. And so we prefer to stipulate axiomatically, as in probability theory, that on a given energy level there can be only one or, say, no more than $K$ particles.

Another important consideration is that we do not take into account the number of particles and the volume separately, we consider their ratio – density (concentration).

The consideration of density alone leads to the following important consequence in thermodynamics: density does not depend on the numeration of particles contained in the given volume. Whatever the numeration of the particles, the density remains the same. It is commonly believed (in particular, it is stated in the book [7]) that classical particles differ from quantum particles in that they can be numbered and then the motion of any individual particle can be followed by keeping track of its number. This is correct, but if the behavior of a multi-particle system is described by equations involving density, e.g. via a probability distribution, then such a description does not depend on the numbering of particles.

Therefore, all the results from quantum mechanics that follow from the remarkable fact that the solution does not depend on numeration can be carried over to the description of any classical multi-particle system by means of equations containing density.

As an illustrative example, we consider the purchase of 1 kilogram of granulated sugar. If this amount of sugar is measured by calculated the grains (and thus the grains must be first numerated), then it is obvious that this process takes a lot of time. Assume that the buyer has a fixed time for the measurement $t_{mes}$ which is equal to one hour and there is no balance to weight the sugar. Then the buyer will measure the granulated sugar by using some vessels. The results of weighting and of measuring by vessels does not change if two grains in the measured volume interchange their places. This means that (1) the particle identity principle holds for the sugar grains and (2) the sum of particles is independent of their location, i.e., the arithmetic property is satisfied [9]. In particular, this implies that one can use a Bose–Einstein-type distribution in this "classical" situation.

The exclusions that we impose will lead us to a more general distribution in which we stipulate that no more than $K$ particles are on the same energy level. The natural number $K$ will be called the maximal occupation number or maximal admissible size of clusters.

The corresponding energy distribution, often called parastatistical [10]-[11], will be described in the next sections. To specific and generalize the parastatistical distributions, we shall use the above-listed items (1) and (2).

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1Sometimes, instead of the number of degrees of freedom, such notions as free will or freedom of choice and so on are considered (see, e.g. [6]).

2In particular, the most important thing in the chapter on identical particles in [7] is the method of secondary quantization introduced by Dirac and neatly explained in mathematical terms by Fock (the Fock space, the creation and annihilation operators and so on). Since density does not depend on numeration, the method of secondary quantization had been carried over to classical multi-particle systems described by equations depending on density. This was done by Schoenberg in 1953–54. Later Schoenberg’s approach was generalized in the author’s joint article with O.Yu. Shvedov [8].
Remark 1. In the Landau–Lifshits textbook [5], two relations are given as the basic equations

\[ \sum_j N_j = N, \]
\[ \sum_j \varepsilon_j N_j = M, \]

where \( M \) is the energy, \( N \) is the number of particles and \( \{ \varepsilon_j \} \) is a discrete family of energy levels (see [12], [13]).

The values \( \varepsilon_j \) in Eq. (2) are related to the existing interaction potential for the particles. It turns out that if we use the interaction with collective number of degrees of freedom equal to \( D \), then we must put\(^3\)

\[ \varepsilon_j = \text{const} \cdot j^{D/2}, \]

i.e.,

\[ M = \sum_j \text{const} \cdot j^{D/2} N_j. \]

This statement is a theorem proved in the author’s paper [14].

The first person who noticed the unexpected relationship between the partition problem in number theory (in particular the famous discovery of Ramanujan) and nuclear disintegration, was Niels Bohr, the founder of quantum theory (see [15]).

It is well known that Bohr’s most famous opponent was the great Albert Einstein himself. The polemic between Bohr and Einstein, during which Bohr gave convincing answers to all of Einstein’s counterarguments, only fortified Bohr’s position and helped him in developing his quantum approach. In a commentary to one of Einstein’s letters, Max Born wrote: “... Einstein’s attitude to quantum mechanics was a heavy blow to me, he refuted the theory without any argumentation, only referring to his ‘inner voice’...” [16]. It was only after the experimental corroboration of the disintegration of heavy Lithium [17] and Uranium-235 that Einstein did support, albeit reluctantly, the A-bomb project.

The Bohr–Kalckar paper [18] of 1937 was the first work in which the coincidence of “thermodynamical formulas” with the number-theoretic formulas for partitions of integers into sums was discovered. Here is a quotation from that paper. “Under the simplifying assumption that each level is the combination of a certain number of quantities assuming nearly equidistant values, one can easily calculate the density of nuclear levels under high perturbations. Denote by \( p(M) \) the number of possible ways of presenting a positive integer as a sum of smaller positive integers. An asymptotic formula for \( p(M) \) was obtained by G. Hardy and S. Ramanujan. For large values of \( M \) this formula may be approximately written in the form

\[ p(M) \sim \frac{1}{4\sqrt{3M}} e^{\pi \sqrt{\frac{2}{3} M}}. \]

Let us choose \( 2 \cdot 10^5 \text{eV} \) for the unit of energy, which approximately corresponds to the common distance between the lowest levels of the heavier nuclei. For the number of partitions for which the perturbation energy of \( 8 \cdot 10^6 \text{eV} \) will be obtained, we will then find \( p(M) \sim 2 \cdot 10^4 \). This means that the mean distance between levels approximately equals \( 10 \text{eV} \), which roughly corresponds to the densities of distribution of levels calculated from the collisions of slow neutrons. ⟨...⟩ The formulas for the density of nuclear levels obtained by analogy with thermodynamics practically coincide, at least in the exponential dependence on the total

\[^3\]The Landau–Lifshits book gives a more general system of equations, later specified in [22], [23]:

\[ \sum_j \varepsilon_j N_j = E, \varepsilon_j = j^{2/D}. \] For Landau \( M = E \) is energy.
energy of excited nuclei, with the formula for $p(M)$ if one understands the number $M$ as the measure of total energy expressed as the difference of energies between the lowest levels taken for the unit of measure” (see pp. 337–338 in the Russian translation [18]).

Bohr’s and Kalckar’s considerations about the distance between lowest energy levels is correct, because for the Schrödinger equation, the equations of the lower levels in the potential trough are quadratic.

Note that the probability of the coincidences indicated by Bohr and Kalckar being accidental is practically zero.

Later Bohr proposed using Uranium-235 as an example of the nucleus of an atom, since it was most appropriate for his construction and used it to explain how to generate a nuclear reaction [19].

The historian and archivist of the British Committee for Atomic Energy Margaret Going, on the basis of her study of the opinions of authoritative physicists, wrote: “... The work of English and American scientists on the authoritative physicists...” Thus we may conclude that the creation of the A-bomb was based on the fantastic discovery of Ramanujan.

An essential role in establishing the new world outlook in physics was played by the numerous volumes of the famous treatise by Landau and Lifshits, in particular, [5] and [20]. In the book [5] in Section 40 “Nonequilibrium ideal gas,” the authors present the system of equations (2) and (3)–(4), on which their further exposition is based. Both $N$ and $N_j$ in these equations are integers. And so equations (2) coincide with the Diophantine equations for partitions in number theory. In the subsequent sections of the book [5] the main equations of thermodynamics are obtained without appealing to the so-called three main principles of thermodynamics, which appear in all textbooks on thermodynamics, but not in [5].

Bohr’s paper [18] generated only a trickle of mathematical and physical papers, in which the connection between the statistics of Bose–Einstein and Fermi–Dirac and the Ramanujan formula was studied in more detail [22]–[24]. Thus papers on this topic appeared in the mathematical journal “Mathematical Proceedings of the Cambridge Philosophical Society.” It was shown that if one takes into account the connections with partition theory, the Ramanujan formula and the Hardy–Ramanujan theorem in number theory, then it becomes clear that, in the case when there are repeated summands in the partition, the leading term of the Ramanujan formula coincides with the entropy of the Bose–Einstein statistics, whereas in the case when there are no repeated summands, this term coincides with the Fermi–Dirac entropy (bosons and fermions – no other particles are observed). Apparently, one of the last papers in this series is [24]. It contains a bibliography of the topic. But overall it seems that contemporary physicists did not pay attention to that series of papers and forgot that they are indebted to Ramanujan for the remarkable revolution in the scientific world outlook.

It follows from Bohr’s paper that his model of the nucleus of an atom does not involve the interaction of particles in the form of attraction. When Bohr visited the USSR in 1961, he demonstrated a simple model of a nucleus consisting of little balls in a cup. Several balls were placed in a cup, and then another little ball, endowed with a certain energy, was slipped into the cup. Were the cup empty, the new ball would have slipped out of the cup, but it shared some of its energy with the other balls and stayed in the cup. This is due to the fact that all the balls were in a common potential field. Thus Bohr regarded the nucleus as consisting of particles that do not attract each other.

Bohr’s model of the nucleus is a model without attraction of molecules, and so is Frenkel’s model of liquid (liquid drop model). Such a model of the nucleus appears to contradict the physical viewpoint as well as the commonsense (or naive) point of view. Nevertheless, it adequately describes nuclear fission, and Frenkel’s model adequately describes the behavior
of liquids. The Frenkel model contains "holes" but does not involve mutual attraction of molecules.

Incidentally, Frenkel subtly complained about the difficulties of overcoming traditional viewpoints, using the word "we," i.e. "physicists." He writes: "We easily get used to uniform and steady things, we stop noticing them. What is habitual seems understandable, but we don’t understand the new and unusual, it seems unnatural and obscure (. . .). Essentially, we never really understand, we can only get used to" [25, p. 63]. If we follow Bohr, we see that in the thermodynamics that he talks about (see above), there is no attraction between molecules, but there is a common potential, in particular the Earth’s gravitational attraction. Molecules can collide, as they do in ideal gas, but there is no mutual attraction.

In Bohr’s paper mentioned above, he noted the connection between nuclear fission and the theory of partitions in number theory. On the other hand, he proposed the drop liquid model of the nucleus. This leads us to the idea that the theory liquids is also connected to partition theory. In that connection Bohr writes only about the analogy with thermodynamics, and so, like his pupil Landau, this means that they did not have in mind the old thermodynamics based on the “three fundamental principles,” but the thermodynamics understood in the framework of the new world outlook based on Bohr’s quantum theory postulates and partition theory.

The author, following the way traced out by Bohr and Landau, has developed a mathematical approach based on number theory problems and quantum theory. In the author’s model there is no attraction between particles but there is a common potential, in particular, the potential due to the Earth’s gravitation and its rotation.

The author has shown that his approach to thermodynamics yields the same formulas as partition theory with logarithmic precision (which means the formulas coincide from the point of view of tropical arithmetic). The notion of logarithmic accuracy was introduced in Vol. X of the Landau–Lifshits textbooks [20] on p. 211. It is defined as follows: logarithmic accuracy of $M$ means that we find $M$ not up to $o(M)$, but only up to $o(\ln(M))$.

Before passing to the formulation of the main formulas and results, we recall the notions of tropical arithmetic and logarithmic accuracy.

The arithmetics of a system may be decimal or binary. The well-known notion of integer part of a real number, \textit{entier}, is denoted by square brackets: $[a]$ stands for the maximum integer that does not exceed a given real number $a$. It is this integer that is usually retained in human memory. For example, although the number 2.99 is close to 3, our memory retains its integer part 2; this peculiarity is used by marketing people when assigning prices to commodities in shops.

From the viewpoint of arithmetics, it is quite natural to discard the fractional part for sufficiently large $a$. If the numbers are as large as is common for macroscopic systems, it is convenient to use a generalization of this notion for large numbers. By $[a]_{10}$ we denote decimal arithmetics, where $[\,]_{10}$ means the same entier for rationals but with respect to 10. For example, $[15]_{10} = 10$, $[90]_{10} = 10$, $[105]_{10} = 100$, and $[6 \cdot 10^{23}]_{10} = 10^{23}$.

For the sum, we have $[10^{15} + 10^{14}]_{10} = 10^{15}$; i.e., the sum equals max($10^{15}, 10^{14}$). For the product, $10^{15} \cdot 10^{14} = 10^{29}$; i.e., the product is equal to the number of zeros in a multidigit number.

Similar rules hold for a system considered in the binary arithmetics $[a]_2$.

The arithmetics thus constructed not only corresponds to the (max, $+$)-algebra but also takes into account precision and neglects $c$-numbers, just as it happens in the Maslov-
Litvinov dequantization. Hence we can say that this arithmetics is “tropical.”

For the natural logarithm, we deal with a special situation. Here we must use the great formula due to the great Ramanujan and the natural logarithm of the solution given by this formula. By considering the integer parts \([\log_e p(M)]\), we obtain a partition of \(p(M)\) into similar subsets of integers.

Note that the use of the natural logarithm is natural if the formulas use the polylogarithm, because the polylogarithm emerges from the Stirling formula, and the Stirling formula contains Euler’s number \(e\).

Thus, in addition to the arithmetics described by the entiers \([\cdot]_{10}\) and \([\cdot]_{2}\), we introduce the arithmetics \([\cdot]_{e}\) on the basis of partition theory.

**Definition 1.** We say that two numbers coincide with logarithmic accuracy if the values of their normal arithmetics \([\cdot]_{e}\) coincide. In other words, \(\ln A \approx \ln B\) means that \(\ln A = \ln B + o(\ln B)\).

Since this arithmetics puts all elements coinciding with each other with “logarithmic accuracy” into the same equivalence class, we see that convergent series are equivalent to polynomials. It follows that, to each series \(\Phi\), we can assign the degree of the corresponding polynomial. We denote this number by \(n(\Phi)\). If one can assign an enveloping series to a divergent series, then the minimum number of the enveloping series naturally determines the degree \(n(\Phi)\) of the corresponding polynomial.

Now we can briefly present a classical problem in partition theory of integers renewed by the inclusion of the number zero and negative entropy (negentropy).

First, consider some examples.

**Example 1.** If we partition the number 5 into 2 summands, zero summands not allowed, then we obtain \(p_N(M) = 2, 5=4+1=3+2\), where \(N\) is the number of summands, \(N = 1, 2, \ldots, M\), and \(p_N(M)\) is the number of partitions of \(M\) into \(N\) summands. There are two possible partitions in this example for \(N = 2\).

**Example 2.** Let us allow zero summands in partitions of \(M\) into \(N\) summands. Then \(p_N(5) = 3\) for \(N = 2\) in the preceding example, \(5=5+0=4+1=3+2\). All in all, there are three possible partitions in this case.

When partitioning 5 into 3 with zeros allowed, the \(p_N(M)\) partitions without zeros, \(5 = 3 + 1 + 1 = 2 + 2 + 1 = \ldots\), are supplemented with partitions containing zeros, \(4 + 1 + 0 = 3 + 2 + 0 = 5 + 0 + 0\). With zeros taken into account, all preceding partitions are repeated; i.e., we obtain the sum of all partitions without zeros.

We introduce the following notation:

\[
\bar{p}_s(M) = \sum_{i=1}^s p_i(M).
\]

**Proposition 1.** The number of partitions of \(M\) into \(N\) summands, zero summands allowed, coincides with \(\bar{p}_s(M)\) for \(s = N\).

**Definition 2.** Let \(q_N(M)\) be the number of partitions of \(M\) into \(N\) distinct summands, and let

\[
q(M) = \sum_{N} q_N(M).
\]

By definition, the partition 3+1+1 of 5 into 3 terms is excluded from \(q_N(M)\). All the zeros are excluded as well.

Let us continue the partition \(\bar{p}_N(M)\) by the numbers \(q_N(M)\). This continuation is continuous, because for \(M = 1\) there are two partitions, \(M = 1 = 1 + 0\), taken into account in \(\bar{p}_N(M)\) and one partition, \(q_N(1) = 1\), taken into account in \(q_N(M)\).
We define natural entropy as the natural logarithm of $\bar{p}_N(M)$ and natural negentropy as the natural logarithm of $q_N(M)$. The passage through the point $M = 1$ is the passage from positive natural entropy into negative natural negentropy $\log_e(q_N(M))$.

Since $p_1(M) = 1$ and $p_M(M) = 1$, it follows that there exist maxima between $N = 1$ and $N = M$. Of these maxima, take the maximum value $\bar{N}$.

Since $p_N(M)$ decreases for $N > \bar{N}$ and $\bar{p}_N(M)$ is nondecreasing, we see that the increase in $\bar{p}_N(M)$ can only be due to the increase in the number of zeros.

4 The Equation of State

In the simplest version of thermodynamics, one considers the conjugate extensive-intensive pairs pressure-volume, temperature-entropy, chemical potential-number of particles.

One of the main notions of thermodynamics is the equation of state. In 6-dimensional phase space, where the intensive thermodynamical variables $P$, $T$, and $-\mu$ play the role of coordinates and the corresponding extensive variables $V$, $-S$, and $N$ play the role of momenta, the equation of state is described by a 3-dimensional surface on which several additional identities, corresponding to the so-called thermodynamical potentials, hold. This results in the fact that this 3-dimensional surface is a Lagrangian manifold\(^5\) and the thermodynamical potential corresponds to action in mechanics.

If one does not consider the number of particles and the volume separately, and only considers their ratio, i.e., density, then one variable turns out to be redundant, and we can consider 4-dimensional phase space and the 2-dimensional Lagrangian surface.

If the volume $V$ is given, we can use a single thermodynamical potential $\Omega$ that can be expressed as follows:

$$d\Omega = (-S)\,dT + N\,d(-\mu), \quad \Omega = -PV. \tag{5}$$

Besides, since the internal energy is $M = \text{const} \cdot PV$, it is customary in thermodynamical diagrams to plot the pressure $P$ along the abscissa axis, as this was done in the Hougen–Watson diagram.

On the $P_r$-$Z$ diagram, the spinodal is defined as the curve which is the locus of all points where the tangents to the isotherms are perpendicular to the $P_r$ axis (see Figures\(^2\) and\(^1\)).

The mathematical model constructed by the author in the papers\(^1\)–\(^2\), which uses the density “effect” mentioned in Section 3 above, shows that

1) if the spinodal in the gaseous region for any gas with non-polarized molecules is given, then all the gas isotherms can be constructed;

2) if we know the slope of the isotherm for $P_r = 0$ on the $P_r$-$Z$ diagram (this is equivalent to knowing the value of the second virial coefficient\(^36\)) as well as the critical isochore for any fluid (the supercritical state for any collection of non-polarized molecules), then we can construct all the isotherms of the fluid up to the critical isochore;

3) if we know the liquid binodal for any gas consisting of non-polarized molecules, and also know the pressure $P_r$ and the density $\rho$ on the line $Z = 1$ (the Zeno line), then we can construct the liquid isotherms passing through the point $P_r = 0$, $Z = 0$ for so called “hard” liquid consisting of non-polarized molecules on the interval $0 \leq Z \leq 1$.

\(^5\)In the data base MathSciNet (American Mathematical Society) the query to MathJax on Publications results for “Anywhere = (Maslov index) OR Anywhere (Maslov class)” in a list of 575 papers published before 2013, where the notions “Lagrangian submanifolds”\(^29\), “Maslov index”, “Maslov class” are developed and extended. See also\(^30\)–\(^31\) and so on.
For an arbitrary liquid, by the Temperley temperature we mean the minimal temperature of the set of liquid isotherms passing through the point \( P_r = 0, Z = 0 \). This temperature (corresponding to \( K = \infty \)), calculated by the physicist Temperley for the Van-der-Waals gas, equals \( 3^3/2^5 T_c \) [35].

As proved by the author in [1]–[2], the corresponding distribution can be expressed in terms of the polylogarithm:

\[
\text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} = z + \frac{z^2}{2^s} + \frac{z^3}{3^s} + \ldots
\] (6)

This distribution yields the following expressions for the thermodynamical potential \( \Omega \), the density (concentration) \( \rho \) and the reduced pressure \( P_r \):

\[
\Omega = -\Lambda(\gamma, K) T_r^{2+\gamma} \left\{ \text{Li}_{2+\gamma}(a) - \frac{1}{(K+1)^{1+\gamma}} \text{Li}_{2+\gamma}(a^{K+1}) \right\}, \quad \gamma = \gamma(T_r).
\] (7)

\[
\rho(T_r, \gamma, K) = \frac{\Lambda(\gamma, K)}{\zeta(2 + \gamma c)} T_r^{1+\gamma} \left( \text{Li}_{1+\gamma}(a) - \frac{1}{(K+1)^\gamma} \text{Li}_{1+\gamma}(a^{K+1}) \right),
\] (8)

\[
P_r(T_r, \gamma, K) = \frac{\Lambda(\gamma, K)}{\zeta(2 + \gamma c)} T_r^{2+\gamma} \left( \text{Li}_{2+\gamma}(a) - \frac{1}{(K+1)^{1+\gamma}} \text{Li}_{2+\gamma}(a^{K+1}) \right),
\] (9)

where \( K \) is the maximal occupation number, \( a = e^{\mu/T_r} \) (\( a \) is called the activity), \( \gamma = D/2 - 1 \), \( D \) is the collective number of degrees of freedom, \( \Lambda(\gamma, K) \) is a normalizing constant.

For each type of gas, the critical parameters \( T_c, P_c, \rho_c \) are determined experimentally. Substituting these values into (8)–(9), we can determine the parameters \( \gamma = \gamma_c \) and \( K = K_c \). The parameter \( \gamma_c \) is the number of collective degrees of freedom \( D_c = 2\gamma_c + 2 \) of the critical isotherm. By \( K_c \) we denote the admissible size of clusters (maximal occupation number) at the energy level of the critical point.

Using (7) and the fact that

\[
M = -\mu(\partial \Omega)/(\partial \mu) - T(\partial \Omega)/(\partial T) + \Omega,
\]

where \( \Omega = -PV \), we derive \( M = (D/2)PV \).

## 5 Entropy and Negentropy

Above we introduced the notion

Let \( l \) be the number of particles on a given energy level. If we impose the restriction that each energy level can host at most \( K \) particles, \( l \leq K \), then we arrive at the Gentile statistics, also known as parastatistics, and we have the following relation, which generalizes the Bose–Einstein statistics as \( K \to \infty \) and the Fermi–Dirac statistics for \( K = 1 \):

\[
N = \frac{\Lambda}{\Gamma(\gamma + 1)} \int_0^{\infty} \left( \frac{1}{(e^{\xi/T - \mu/T} - 1)} - \frac{K + 1}{K^{(K+1)(\xi/T - \mu/T) - 1}} \right) \xi^\gamma d\xi,
\] (10)

\[
M = \frac{\Lambda}{\Gamma(\gamma + 2)} \int_0^{\infty} \left( \frac{1}{(e^{\xi/T - \mu/T} - 1)} - \frac{K + 1}{K^{(K+1)(\xi/T - \mu/T) - 1}} \right) \xi^{\gamma+1} d\xi,
\] (11)

\[6\text{Note that for } z = 1 \text{ and } \text{Re}(s) > 1, \text{Li}_s(1) = \zeta(1) \text{ is Riemann’s zeta function.}\]

\[7\text{The separation of the clusters of molecules near the flat surface of a liquid for } T = T_c \text{ can be observed experimentally.}\]
where $T$ is the temperature, $\gamma = D/2 - 1$, and $\Lambda$ is the parameter to be determined in number theory. In quantum mechanics, it depends in a well-known way on the mass and the Planck constant [15].

Indeed, if $K = 1$, then (without restrictions on the value of the chemical potential $\mu$) we obtain the Fermi distribution

$$N|_{K=1} = \frac{\Lambda}{\Gamma(\gamma + 1)} \int_0^\infty \frac{1}{\exp\{\xi/T\} + 1} \xi^\gamma d\xi. \quad (12)$$

For $K = \infty$, the formula coincides with the usual Bose–Einstein distribution. For $\mu = 0$ and $\gamma = 0$ the solution has a singularity; it becomes infinite.

To regularize the integral (10) for $K = \infty$, since $l \leq N$, we apply formula (10) with $K = N$. From (10), we obtain the relation

$$N = \Lambda \int_0^\infty \left( \frac{1}{\exp\{\xi/T\} - 1} - \frac{N + 1}{\exp\{(N + 1)\xi/T\} - 1} \right) d\xi. \quad (13)$$

Since the relations

$$N = -\Lambda T \log(1 - a), \quad M = \Lambda T^2 \text{dilog}(1 - a), \quad (14)$$

where $\text{dilog}(1 - a) = Li_2(a)$, hold for the Bose–Einstein distribution with $a < 1$ in the two-dimensional case, we start by finding $\Lambda$.

Since the Ramanujan formula for $\bar{p}_M(M)$ gives the first term of the expansion as $M \to \infty$ in the form

$$\log_e \bar{p}_M(M) \simeq 2 \sqrt{M \text{dilog}(0)}, \quad (15)$$

and the first term of the expansion of the entropy is of the form (see [22, formula 20])

$$S \simeq 2\Lambda \sqrt{M \text{dilog}(0)}, \quad (16)$$

we see that $\Lambda = 1$.

Let $N = N_c$ be the solution of Eq. (13). Consider the value of the integral in (13) (with the same integrand) taken from $\delta$ to $\infty$ and then pass to the limit as $\delta \to 0$. After making the change $\beta x = \xi$ in the first term and $\beta(N_c + 1)x = \xi$ in the second term, where $\beta = 1/T$, we obtain

$$N_c = \frac{1}{\beta} \int_{\delta\beta}^\infty \frac{d\xi}{e^{\xi} - 1} - \int_{\delta\beta(N_c + 1)}^\infty \frac{d\xi}{e^{\xi} - 1} + O(\delta) = \frac{1}{\beta} \int_{\delta\beta}^{\delta\beta(N_c + 1)} \frac{d\xi}{e^{\xi} - 1} + O(\delta) \quad (17)$$

$$\sim \frac{1}{\beta} \int_{\delta\beta}^{\delta\beta(N_c + 1)} \frac{d\xi}{\xi} + O(\delta) = \frac{1}{\beta} \{\ln(\delta\beta(N_c + 1)) - \ln(\delta\beta)\} + O(\delta) = \frac{1}{\beta} \ln(N_c + 1) + O(\delta). \quad (18)$$

On the other hand, after neglecting the second summand in the integrand of (11) for large $M$ and making the change $\beta x = \xi$, we can write

$$\frac{1}{\beta^2} \int_0^\infty \frac{\xi d\xi}{e^{\xi} - 1} \approx M. \quad (19)$$

After passing to the limit as $\delta \to 0$ and expressing $\beta$, from (19), we obtain

$$N_c = \sqrt{\frac{M}{\text{dilog}(0)}} \log(N_c + 1). \quad (20)$$
The ideal entropy is the same for the Fermi and Bose particles (see [22, Eq. (13a) as well as Eqs. (20) and (27)]).

Hardy and Ramanujan derived the formula

$$p(M) = \frac{1}{(4\sqrt{3})M} e^{\pi \sqrt{\frac{M}{3}}}.$$  \hspace{1cm} (21)

The asymptotic formula for \(q(M)\) has the form [26]

$$q(M) = \frac{1}{(4 \cdot 3^{1/4})M^{3/4}} e^{\pi \sqrt{M/3}}.$$  \hspace{1cm} (22)

If a fermion nucleus emits a single neutron, it becomes a boson, and vice versa. Thus, the energy density flattens out under the successive emission of neutrons according to Bohr’s concept [18].

By (21) and (22), boson fission overtakes fermion fission as \(M \to \infty\). For a boson not to become a fermion when emitting neutrons, it must emit them in pairs, which may bring about fission of the nucleus. For the Bose–Einstein distribution (20), the critical value of \(N_c\) satisfies the implicit relation

$$N_c = \sqrt{\frac{M}{\zeta(2)}} \ln(N_c + 1).$$  \hspace{1cm} (23)

Therefore,

$$M = \zeta(2) \left( \frac{N_c}{\ln N_c} \right)^2.$$  \hspace{1cm} (24)

As we see, this value corresponds to the limit value of activity \(a = 1\) (i.e., the chemical potential \(\mu = 0\)). In this case, we use the Gentile statistics to obtain self-consistent equations. Similarly, for the Fermi system, this limit values corresponds to \(a \to -\infty\) so that

$$M = \frac{N_c(N_c - 1)}{2} + \zeta(2) \left( \frac{N_c}{\ln N_c} \right)^2.$$  \hspace{1cm} (25)

These relations can be derived using the Bose and Fermi statistics and the Gentile statistics.

By the Bohr–Kalckar “correspondence principle” between the physical notion of nucleus and number theory, we can transfer these relations to the above-cited construction of number theory with the zeros taken into account [15], [27]. We have

$$p \left( \frac{\zeta(2)}{\ln N_c} \right)^2 = q \left( \frac{N_c(N_c - 1)}{2} + \zeta(2) \left( \frac{N_c}{\ln N_c} \right)^2 \right).$$  \hspace{1cm} (26)

This formula allows us to determine the point \(N_c\) such that, for \(N > N_c\), the entropy of the branch with repeated terms and zeros becomes greater than the entropy of the branch without repetitions and zeros.

The mesoscopic values in number theory correspond to dimension (number of degrees of freedom) equal to 2, i.e. in the region, where the parameter \(k\) varies from 0 to 90. Let us find these values \(N_c\) from the relation

$$M = T^2 \zeta(2) \left( 1 - \frac{1}{N_c + 1} \right),$$  \hspace{1cm} (27)

where \(T\) can be found from the relation [28]

$$N_c = T \ln(N_c + 1).$$  \hspace{1cm} (28)
Figure 3: The dependence of the entropy $S$ on $N$ for $M = 70$. The upper curve is obtained when zero summands are allowed, the lower one, when they are not. The vertical line indicates the value $N = N_c$ calculated according to formula (23). The exact value of the first point of the phase transition is the maximum of the lower curve.

These relations provide an exact dependence of $N_c$ on $M$. The graph for $M = 70$ is shown in Fig. 3. For $M = 70$, the value of $N_c$ is 20. The value of the entropy $S$ is 15. The entropy does not practically vary up to $M = 70$, i.e., up to $N = M$.

The point of maximum of the lower curve in Fig. 3 for number theory is determined precisely. This is the value $N_{\text{max}}$ for which the number $p_N(M)$ attains its maximum.

Now let us return to thermodynamics.

6 The Notion of Generalized Ideal Gas

We pass to the definition of the new classical ideal gas.

Note that in the description of the behavior of multi-particle systems it is possible that the initial conditions involve interactions. Consider the following analogy. There is a battery of canons. The gunners position the canons and interact with each other in the process. At some moment the command “Fire!” is heard (it means “all interactions (all conversations) must stop”). At the initial situation interactions took place, but once the cannon balls are fired, they fly without interacting with each other. However elastic collisions between the balls are possible.

This model can be regarded as a model of ideal gas in the traditional understanding only if the initial data is given at time “minus infinity”.

A generalized classical ideal gas is a classical multi-particle system, where interactions cannot occur only inside certain open regions of the diagram, although at the initial moment of time or on the boundary of a region interactions between particles are allowed. In the region itself, only elastic collisions between particles are possible.

Why is it important to specify the open regions where no interaction (more precisely, no attraction) occurs? Obviously, if one must rapidly mix ball bearings of different sizes, it is easier to do this if the balls are not magnetized. The solution of this problem has several practical applications. Thus the speed with which two gases mix influences the driving force of jet engines: the higher the speed, the higher the driving force. If there is no attraction between molecules of a mixture of gases moving in a pipeline, the probability of the appearance of liquid bottlenecks is considerably less.

Thus the parameters of the Van-der-Waals gas coincide closely enough with the parameters of the new (generalized) gas in the following open regions of the parameters: 1) the region from the critical isotherm ($T_r = 1$) to the critical isochore ($\rho = \rho_c$); 2) the region
outside the gas spinodal; 3) the region of hard liquid (below the Temperley temperature, see Section 7).

7 Comparison of the Generalized Ideal Gas with the Van-der-Waals model

The main practical results that follow from the author’s theory are related to metastable states. In that sense, the main notion of the thermodynamics described here is not the binodal: the curve on the diagram that indicates the stable phase transition “vapor-to-liquid” (the gas binodal) and the phase transition “liquid-to-vapor” (the liquid binodal). The main notion is the spinodal – the curve on the diagram that delimits the regions of stable and metastable states (a more precise definition of the spinodal was given above). It is practically impossible to reach values of the thermodynamical parameters on the spinodal experimentally, because that requires “infinite time”.

From relations (8)–(9) as $K \to \infty$ we obtain the following equations:

$$P_r = \frac{\Lambda(\gamma) T_r^{2+\gamma} Li_{2+\gamma}(a)}{\zeta(2 + \gamma_c)}, \quad a = e^{\mu/T_r}, \quad \Lambda(\gamma) = \Lambda(\gamma, \infty) = \lim_{K \to \infty} \Lambda(\gamma, K), \quad (29)$$

$$\rho = \frac{\Lambda(\gamma) T_r^{1+\gamma} Li_{1+\gamma}(a)}{\zeta(2 + \gamma_c)}. \quad (30)$$

For $a = 1$, we obtain the spinodal defined above. This fact was proved by the author in [34]. The constants $\Lambda$ and $\gamma$ can be found for $a = 1$ ($\mu = 0$) from the coincidence of the values of $P_r$ and $\rho$ from (29) and (30) with the values of $P_{sp}$ and $\rho_{sp}$ on the Van-der-Waals spinodal.

In Figures 4 and 5, the isotherms of the Van-der-Waals model are compared with those of the author’s model on Hougen–Watson diagrams.

![Figure 4](image)

**Figure 4:** On this diagram the pressure is $P = P_r$. Coincidence of the Van-der-Waals isotherms (solid lines) with the isotherms from the author’s model (dashed lines). The following isotherms are shown $T_r = 0.1$, $T_r = 0.2$, $T_r = 0.3$, $T_r = 0.4$, $T_r = 0.5$, $T_r = 0.6$, $T_r = 0.7$, $T_r = 0.8$, $T_r = 0.9$, $T_r = 1$. The value of $\gamma = \gamma(T_r)$ in equations (29)–(30) is determined from the relations on the Van-der-Waals spinodal $Z = Z_{sp}$, $P = P_{sp}$ for $a = 1$.

In Figure 5 it is shown how the isotherms of the ideal gas (in the new sense) obtained from (29)–(30), differ from the isotherms of the Van-der-Waals model. The figure shows that
the isotherms practically coincide except at points near critical values. In regions with less than critical parameter values, the coincidence is up to 0.006%, while at points near critical values the error is of the order of 0.01%. Note the important point $T_{c}$. At that point, the maximal discrepancy between the isotherms is observed. It is of the order of $\sim 0.01$.

For isotherms corresponding to temperatures less than the Temperley temperature ("cluster-free liquids"), we obtain a very precise coincidence between the isotherms constructed according to the distribution (29)–(30) and the isotherms coming from the Van-der-Waals model. The temperature $T_{\text{Temperly}}$ specifies a boundary above which the number $K$ (maximal occupation number) becomes infinite.

8 Analytical number theory and the energy of transition of the Bose gas to the Fermi gas

Niels Bohr [18] notes an important relationship between the number theory and the entropy, which arises in his concept of the nucleus. After that paper, several physicists, especially Hindu physicists [22]–[24], studied this problem and calculated the entropy for the Bose–Einstein gas and the entropy corresponding to the Ramanujan formula. These studies have been carried out up to now.

We show that, in the two-dimensional case, the formulas that were written for the Bose–Einstein distribution in the book [5] coincide with the formulas of analytical number theory.

Indeed, assume that there is a decomposition

$$M = a_{1} + \cdots + a_{N}$$

of a number $M$ into $N$ terms. By $N_{j}$ we denote the number of terms exactly equal to the number $j$ in the right-hand side of this decomposition.

Then the total number of terms is $\sum j N_{j}$, and this number is equal to $N$, because we known that there is $N$ terms at all. Further, the sum of terms equal to $j$ is equal to $jN_{j}$, because there are $N_{j}$ of them and the sum of all terms is then obtained by summing these expressions over $j$, i.e., $\sum j N_{j}$, and it is equal to $M$. Namely,

$$\sum_{i=1}^{\infty} N_{i} = N, \quad \sum_{i=1}^{\infty} iN_{i} = M. \quad (31)$$

Let us consider an example of the well-known Erdős theorem in the number theory, i.e., the solution of ancient problem called “partitio numerorum” in Latin. This problem deals
with an integer $M$ which is decomposed into $N$ terms, for example, $M = 5$, $N = 2$:

$$5 = 1 + 4 = 2 + 3 = 0 + 5,$$

which gives three versions $\mathcal{M}$ of the solution of this problem: $\mathcal{M} = 3$.

If $M = 10^{23}$, $N = 1$, then there is only one version: $\mathcal{M} = 1$. If $M = 10^{23}$ and $N = 10^{23}$, then there is also only one version, i.e., the sum of units, $\mathcal{M} = 1$.

Obviously, for a fixed $M$, there exists a number $\tilde{N}$ for which the number of decomposition versions $\mathcal{M}$ is maximal (in general, this number is not unique). The number $\log_2 \mathcal{M}$ is called the Hartley entropy. At the point where it attains its maximum, the entropy is also maximal. The chemical potential equal to zero corresponds to this point. The Erdős formula determines the maximal number of solutions of the decomposition function $\tilde{N}$ and has the form

$$\tilde{N} = \beta^{-1} M^{1/2} \log M + \alpha M^{1/2} + o(M^{1/2}), \quad \beta = \pi \sqrt{2/3},$$

where the coefficient $\alpha$ is determined by the formula $\beta/2 = e^{-\alpha \beta/2}$.

Erdős obtained his result only up to $o(\sqrt{M})$ because of the nonuniqueness of the above-mentioned maximum and the ambiguity of the number of these maxima.

Remark 2. The classical number theory deals with the space $\mathbb{Z}$ of integers. Generalizations to the space $\mathbb{R}^+$ were obtained in the works which were surveyed in detail by A.G. Postnikov in the book [45]. This approach agrees well with the concept of statistical physics, where some of the variables belong to $\mathbb{R}$, and hence the integers such as the number of particles $N$ and the parastatistical number $k$ (restricting the number of particles at one energy level), together with various numbers belonging to the set $\mathbb{R}$, loose their meaning of integer numbers.

Erdős considered the case where the number $N$ is fixed. We shall consider the sum of all decomposition versions less than or equal to $N$. Obviously, in this case, the maximal entropy is also between $N$ and zero. The formula is of the same order as in the Erdős, but the answer is significantly different.

Let us consider this question in more detail.

In the book [5], the mass $m$ and the Planck constant $\hbar$ are introduced as parameters. In number theory, one can assume that all these constants and the volume $V$ are equal to 1. To compare the formulas of number theory and of statistical distributions, we introduce the notation of several quantities which are equal to 1 in number theory.

We introduce a general parameter $\Phi$ which correlates with the notation used in Landau and Lifshits book [5]

$$\Phi = \lambda^{2(\gamma+1)} V T^{\gamma+1} = V (\lambda^2 T)^{\gamma+1},$$

where $V$ is the volume, $T$ is the temperature, and $\lambda = \sqrt{2\pi m \hbar^2}$.

The following formulas hold for the Bose–Einstein distribution in the case of $D$ degrees of freedom [5, 26]:

$$E = \Phi T (\gamma + 1) \text{Li}_{2+\gamma}(a), \quad N = \Phi \text{Li}_{1+\gamma}(a).$$

Here and below, $\gamma = D/2 - 1$, $T$ is the temperature, $a = e^{\mu/T}$ is the activity, $\mu$ is the chemical potential.

The following formulas hold for the Fermi–Dirac distribution:

$$E = -\Phi T (\gamma + 1) \text{Li}_{2+\gamma}(-a), \quad N = -\Phi \text{Li}_{1+\gamma}(-a).$$

We see that formulas (33) and (34) differ in sign, and hence the activity $a$ passes through the point $a = 0$. 

9 Boson–Fermion Transition in Mesoscopy

Our goal is to determine the energy of transition of Bose particles to Fermi particles. For this, it suffices to study the transition of \((\ref{33})\) into \((\ref{34})\) through the point \(a = 0\). If we want to extend the interval of the jump from the Bose distribution to the Fermi distribution, then it is necessary to use the parastatistics or the Gentile statistics \([10]\).

In the case of parastatistics, we have relations, where the first term in parentheses gives the distribution for Bose particles, and the second term, the parastatistical correction:

\[
E = \Phi T(\gamma + 1)(\text{Li}_{2+\gamma}(a) - \frac{1}{(k + 1)^{\gamma+1}}\text{Li}_{2+\gamma}(a^{k+1})), \tag{35}
\]

\[
N = \Phi(\text{Li}_{1+\gamma}(a) - \frac{1}{(k + 1)^{\gamma}}\text{Li}_{1+\gamma}(a^{k+1})). \tag{36}
\]

We let \(N_i\) denote the number of particles at the \(i\)th energy level. In the case of parastatistics, there can be at most \(k\) particles at each energy level. By the usual definitions, the Fermi case is realized for \(k = 1\), and the Bose case, for \(k = \infty\). But by formulas \((\ref{31})\), it is obvious that \(N_i \leq N\) for the Bose system. Therefore, \(k \leq N\) for the Bose system. This implies that the maximal value of \(k\) is equal to \(N\), but not to infinity. This logical conclusion significantly changes the formulas. In particular, there arise new equations for \(N = k\):

\[
E = \Phi T(\gamma + 1)(\text{Li}_{2+\gamma}(a) - \frac{1}{(N + 1)^{\gamma+1}}\text{Li}_{2+\gamma}(a^{N+1})), \tag{37}
\]

\[
N = \Phi(\text{Li}_{1+\gamma}(a) - \frac{1}{(N + 1)^{\gamma}}\text{Li}_{1+\gamma}(a^{N+1})). \tag{38}
\]

The maximal number of particles at the energy level \(N_c\) in the system occurs at the caustic point\(8\) \(a = 1\). We denote this value of \(N = k\) by \(N_c\). The value of \(N_c\) is calculated by using the integral representation has the form

\[
N_c = \Phi \beta \int_0^\infty \left( \frac{1}{\exp\left(\frac{\xi}{\beta}\right) - 1} - \frac{N_c + 1}{\exp\left(\frac{(N_c + 1)\xi}{\beta}\right) - 1} \right) d\xi, \quad \beta = 1/T. \tag{39}
\]

Thus, we obtained a self-consistent equation wit the parameter \(N_c\) in both the left and right sides.

Since the expression \(\text{Li}_{1+\gamma}(1)\) is meaningful only for \(\gamma > 0\), it is necessary to consider the cases \(\gamma = 0\) and \(\gamma < 0\) separately.

9.1 Case \(\gamma = 0\)

Let us consider the value of the integral in \((\ref{13})\) (with the same integrand) taken from \(\delta\) to \(\infty\) and then pass to the limit as \(\delta \to 0\). After making the change \(\beta x = \xi\) in the first term and \(\beta(N_c + 1)x = \xi\) in the second term, where \(\beta = 1/T\), we obtain

\[
N_c = \Phi \int_\infty^{\delta(N_c+1)} \frac{d\xi}{\xi} - \Phi \int_\delta^{\infty} \frac{d\xi}{\xi} + O(\delta) = \Phi \int_\delta^{\delta(N_c+1)} \frac{d\xi}{\xi} + O(\delta) \tag{40}
\]

\[
\sim \Phi \int_\delta^{\delta(N_c+1)} \frac{d\xi}{\xi} + O(\delta) = \Phi \{\ln(\delta + \delta(N_c + 1)) - \ln(\delta + \delta)\} + O(\delta) = \Phi \ln(N_c + 1) + O(\delta). \tag{41}
\]

---

8In thermodynamics, this caustic is called a spinodal.
In the three-dimensional case, the corresponding formula for $N$ becomes (see [50])

$$N = \lambda \left\{ \int_0^\infty \frac{p^2 dp}{\exp \left\{ \frac{p^2}{2mT} \right\}} - (k + 1) \int_0^\infty \frac{p^2 dp}{\exp \left\{ (k + 1) \frac{p^2}{2mT} \right\}} - 1 \right\}. \quad (42)$$

After passing to the limit as $\delta \to 0$ we obtain

$$N_c = \Phi \log(N_c + 1). \quad (43)$$

### 9.2 Case $\gamma < 0$

Let us consider the case $\gamma < \gamma_0 < 0$, i.e., $D < D_0 < 2$.

The following lemma holds for $\gamma < \gamma_0 < 0$.

**Lemma 1.** Consider the integral

$$N = B \int_0^\infty \frac{1}{e^{\beta x - \beta \mu} - 1} - \frac{k}{e^{k(\beta x - \beta \mu) - 1}})x^\gamma dx, \quad (44)$$

where $-1 < \gamma < \gamma_0 < 0$ and $B > 0$, $k > 0$ are constants.

Then

$$N = -\frac{B}{\beta^{\gamma+1}} c_{\beta, \mu, \gamma} + \frac{B k^{-\gamma}}{\beta^{\gamma+1}} c_{k, \beta, \mu, \gamma}, \quad (45)$$

where

$$c_{\mu, \gamma} = \int_0^\infty \left( \frac{1}{\xi - \mu} - \frac{1}{e\xi - 1} \right) \xi^\gamma d\xi. \quad (46)$$

By Lemma 1, the equation for $N_c$ becomes

$$N_c = \Phi C(\gamma) \left( -1 + (N_c + 1)^{-\gamma} \right), \quad (47)$$

where

$$C(\gamma) = \frac{1}{\Gamma(\gamma + 1)} \int_0^\infty \left( \frac{1}{\xi} - \frac{1}{e\xi - 1} \right) \xi^\gamma d\xi. \quad (48)$$

As $N_c \to \infty$, we can neglect the term $-1$ in formula (47) (under the condition $\gamma < \gamma_0$), and the unity compared to $N_c$, which implies

$$N_c = \Phi C(\gamma) N_c^{-\gamma}, \quad (49)$$

and finally, we obtain the expression

$$N_c = \left( \Phi C(\gamma) \right)^{1/(\gamma+1)} = T \left( \frac{\sqrt{2\pi m}}{2\pi h} \right)^{2(\gamma+1)} V C(\gamma)^{1/(\gamma+1)}, \quad (50)$$

and finally, we obtain the expression

$$N_c = T \left( VC(\gamma) \right)^{1/(\gamma+1)} \left( \frac{\sqrt{2\pi m}}{2\pi h} \right)^2. \quad (51)$$

For a problem of the number theory, it was shown in [51] that $\Phi = T^{1+\gamma}$, and the expression for the temperature in terms of the energy $M$ was obtained by the formula

$$T = \left( \frac{M}{\xi^{(2+\gamma)}} \right)^{1/(2+\gamma)}.$$
Figure 6: Graph of the dependence $N(a)$ for $W = 1000$, $W = V(\lambda^2 T)^{\gamma+1}$, where $\lambda$ is a parameter depending on the mass and $\gamma = 0$. The dotted curve corresponds to $N = -1/\log(a)$. The solid line corresponds to $N = 0$.

This implies

$$N_c = \left(\frac{M}{\zeta(2+\gamma)}\right)^{1/(2+\gamma)} C(\gamma)^{1/(\gamma+1)}.$$  \hspace{1cm} (52)

The following thermodynamical relation for the pressure $P$ is known:

$$P = \frac{E}{(\gamma + 1)V}.$$  \hspace{1cm} (53)

Let us consider another important dimensionless quantity, namely, the compressibility factor defined by the formula $Z = PV/NT$. It follows from (53) that $PV$ can be expressed in terms of $E$, and hence the compressibility factor can be expressed by using the polylogarithm. In the Bose case, the compressibility factor becomes

$$Z|_{\text{Bose}} = \frac{\text{Li}_{2+\gamma}(a)}{\text{Li}_{1+\gamma}(a)}.$$  \hspace{1cm} (54)

and for the Fermi case,

$$Z|_{\text{Fermi}} = \frac{\text{Li}_{2+\gamma}(-a)}{\text{Li}_{1+\gamma}(-a)}.$$  \hspace{1cm} (55)

In the Gentile statistics with parameter $k$, we have the following expression for the compressibility factor:

$$Z|_{k} = \frac{\text{Li}_{2+\gamma}(a) - \frac{1}{(k+1)^{\gamma+1}} \text{Li}_{2+\gamma}(a^{k+1})}{\text{Li}_{1+\gamma}(a) - \frac{1}{(k+1)^{\gamma}} \text{Li}_{1+\gamma}(a^{k+1})}.$$  \hspace{1cm} (56)
9.3 Case of small $N$

As was pointed out above, the author obtained a self-consistent equation, where $N$ is an unknown quantity. We are interested in the point at which the number of Bose particles $N$ is equal to 0, i.e., the point at which the Bose particles disappear. Figure 6 illustrates an example of this point. We stress that the value of the activity $a$ does not vanish at this point, but significantly depends on the function $\Phi$ and the parameter $\gamma$.

Let us consider the mesoscopic case. We expand the right-hand side of Eq. (71) for $\alpha = 1$ in a power series in $N \rightarrow 0$. Using the identity for the polylogarithm

$$z \frac{\partial \text{Li}_s(z)}{\partial z} = \text{Li}_{s-1}(z), \quad (57)$$

we expand obtain the following expansion in the Taylor series in a small $N$:

$$\frac{1}{(N+1)^\gamma} = 1 - \gamma N + \frac{1}{2} (\gamma^2 + \gamma) N^2 + O(N^3 \log^3(a)),$$

$$\text{Li}_{1+\gamma}(a^{N+1}) = \text{Li}_{1+\gamma}(a) + N \log(a) \text{Li}_{\gamma}(a) + \frac{1}{2} N^2 \log^2(a) \text{Li}_{-1}(a) + O(N^3 \log^3(a)).$$

This implies

$$\text{Li}_{1+\gamma}(a) - \frac{1}{(N+1)^\gamma} \text{Li}_{1+\gamma}(a^{N+1}) = N(\gamma \text{Li}_{1+\gamma}(a) - \log(a) \text{Li}_{\gamma}(a))$$

$$+ \frac{1}{2} N^2 (\gamma^2 (-\text{Li}_{1+\gamma}(a)) - \gamma \text{Li}_{1+\gamma}(a) - \log^2(a) \text{Li}_{-1}(a) + 2\gamma \log(a) \text{Li}_{\gamma}(a)) + O(N^3 \log^3(a)). \quad (58)$$

Then the expansion of the number $N$ becomes (up to $o(N^2 \log^2(a))$):

$$\frac{N}{V} = (\lambda^2 T)^{\gamma+1} [N(\gamma \text{Li}_{1+\gamma}(a) - \log(a) \text{Li}_{\gamma}(a))$$

$$+ \frac{1}{2} N^2 (\gamma^2 (-\text{Li}_{1+\gamma}(a)) - \gamma \text{Li}_{1+\gamma}(a) - \log^2(a) \text{Li}_{-1}(a) + 2\gamma \log(a) \text{Li}_{\gamma}(a))]. \quad (59)$$

Dividing Eq. (59) by $N$, we obtain

$$\frac{1}{V} = (\lambda^2 T)^{\gamma+1} [(\gamma \text{Li}_{1+\gamma}(a) - \log(a) \text{Li}_{\gamma}(a))$$

$$+ \frac{1}{2} N (-\gamma^2 + \gamma) \text{Li}_{1+\gamma}(a) - \log^2(a) \text{Li}_{-1}(a) + 2\gamma \log(a) \text{Li}_{\gamma}(a))]. \quad (60)$$

This implies

$$-1 + V(\lambda^2 T)^{1+\gamma} (\gamma \text{Li}_{1+\gamma}(a) - \log(a) \text{Li}_{\gamma}(a))$$

$$+ \frac{V(\lambda^2 T)^{1+\gamma}}{2} N (-\gamma^2 + \gamma) \text{Li}_{1+\gamma}(a) - \log^2(a) \text{Li}_{-1}(a) + 2\gamma \log(a) \text{Li}_{\gamma}(a)) = 0, \quad (61)$$

and we obtain

$$N = -2 \frac{-1 + V(\lambda^2 T)^{1+\gamma} (\gamma \text{Li}_{1+\gamma}(a) - \log(a) \text{Li}_{\gamma}(a))}{V(\lambda^2 T)^{1+\gamma} (-\gamma^2 + \gamma) \text{Li}_{1+\gamma}(a) - \log^2(a) \text{Li}_{-1}(a) + 2\gamma \log(a) \text{Li}_{\gamma}(a))}, \quad (62)$$

or

$$N = 2 \frac{\gamma \Phi \text{Li}_{1+\gamma}(a) - \Phi \log(a) \text{Li}_{\gamma}(a) - 1}{\Phi (\log^2(a) \text{Li}_{-1}(a) + \gamma ((\gamma + 1) \text{Li}_{1+\gamma}(a) - 2 \log(a) \text{Li}_{\gamma}(a)))}. \quad (63)$$
With regard to the asymptotics as $N \to 0$, we transform the equation for the energy $E$ as follows:

$$E = V(\lambda^2 T)^{\gamma+2}(\gamma + 1)(\text{Li}_{2+\gamma}(a) - \frac{1}{(N+1)^{1+\gamma}} \text{Li}_{2+\gamma}(a^{N+1}))$$

$$= VN(\lambda^2 T)^{\gamma+2}(\gamma + 1)[(1 + \gamma) \text{Li}_{2+\gamma}(a) - \log(a) \text{Li}_{1+\gamma}(a)] + o(N^2 \log^2(a)).$$

(64)

Remark 3. We have considered two repeated limits. The first case is

$$\lim a \lim N.$$ 

In this case, $N \ll a$, and we obtain a certain value $a_0$ for which the number of particles of the Bose distribution is $N = 0$.

In the second case, $N$ is fixed and $a \ll N$. As a result, the term corresponding to the Gentile parastatistics is zero.

Let $a_0$ be the value of $a$ at which $N = 0$ (see Fig. 6). From Eq. (63) we derive the following equation for $a_0$:

$$\gamma \Phi \text{Li}_{\gamma+1}(a_0) - \Phi \log(a_0) \text{Li}_\gamma(a_0) - 1 = 0.$$ (65)

For small $a_0$, we have the asymptotic formula

$$\Phi = -\frac{1}{a_0 \ln a_0}$$ (66)

or

$$\Phi = \frac{1}{a_0 \ln \Phi}.$$ (67)

10 Transition of the Helium-6 boson to the Helium-5 fermion

Following the author’s concept related to the abstract analytical number theory [45], one can mathematically calculate the transition of Bose particles to Fermi particles, at least in the two-dimensional case. Based on this concept, the boson branch of the decomposition of the number $M$ into terms (with possible repetition of terms) turns into the fermion branch of the decomposition (without repeated terms). It follows from the continuity of such a transition that there exists a point of transition from the boson branch into the fermion branch according to the number of terms $N$.

The specific volume $V$ (the area in the two-dimensional case) was determined in the number theory in [46] as $N = V \text{Li}_1(a)$ for the distribution of the Bose gas. Here $N/V$ is the density. For the volume $V$, the energy can be written as $E = V \text{Li}_2(a)$.

When the activity $a$ changes the sign, the boson branch (with repetition of terms in the decomposition of the number) in the number theory turns into the fermion branch (without repetition of the terms in the decomposition). One succeeds in passing from the number theory to the case of small dimension, i.e., to any number of degrees of freedom greater or smaller than 2. In this case, one succeeds in calculating the coefficient of $1/\log a$, which permits determining the value of the energy required for the Bose gas to go over into the Fermi gas (for example, Helium-4 into Helium-3) for a given volume and a given temperature.
10.1 Quantization of activity and energy of Gödel, Maltsev and Ershov numbering. Calculation of the hidden parameter for the microscopy

In this and other works, the author considered the “hidden parameter” \( t_{\text{meas}} \), which is not hidden in the sense of the Einstein–Podolsky–Rosen paradox (EPR). The parameter introduced by the author is quite natural and open. In the assertion on the identity of particles in the works of Landau and Lifshits \([7]\) which has been many times cited by the author in his papers, this parameter is veiled as the “time moment” at which the numeration of particles is attained. This citation states the following: “one can imagine that the particles contained in a given physical system are ‘renumbered’ at a certain time moment” (p. 252). The time required to numerate the particles is precisely the additional parameter introduced by the author in \([27]\), \([52]\)–\([53]\) and which is discussed in the present paper. This time depends on the algorithm used to numerate the particles. In turn, the time of implementation of the algorithm depends on the calculating person and his device. Thus, this parameter is not hidden but is a somewhat veiled. It can be determined exactly only under a lot of additional conditions.

In the paragraph of the book \([7]\) cited above, Landau and Lifshits also speak about other time moments, namely, if “one further watches the motion of each of the particles in its trajectory, then the particles can be identified at any time moment (italics added by VM).”

The time moments form a discrete set of points. If the intervals between these points are much less than the veiled parameter, then the observer sees the classical picture of the neutron (wave packet) revolution around the Helium-4 nucleus independently of the life time of the Helium-5 fermion.

We consider a gas, i.e., a sufficiently many particles, each of which is a boson. If we consider specific characteristics of the gas (specific energy, specific volume, etc.), then we can assume that these specific quantities are related to the microscopy, i.e., to the situation of a single nucleus with several nucleons.

Assume that the hidden parameter \( t_{\text{meas}} \) (i.e., the time given to the experimenter for observation) is of the order of \( 10^{-11} \), and the life time of a boson is of the order of \( 10^{-13} \). Our assumption is not strict, but since the value of time of the order of \( 10^{-11} \) is much greater than the value of the order of \( 10^{-13} \), and hence much greater than the time during which the experimenter observed 20 boson revolutions around the nucleus, we can assume that this value of time is approximately equal to the hidden parameter in the microscopy in the sense explained above. If we have one nucleus and each nucleon is a boson, then for the bosons in this approximation (in terms of approximate specific energies), we can calculate the sum of specific energies for all bosons. This total energy coincides with the energy of microparticles, i.e. for all nucleons in the order sufficient for us.

Indeed, we are interested not in the energy but in the hidden parameter \( t_{\text{meas}} \). With regard to the above, we can state the hidden parameter is revealed with a sufficient accuracy, i.e., with a logarithmic accuracy.

In the present paper, we propose a method for determining the energy of fermions and bosons in the case of a gas, i.e., in the case where there are very many particles. For the specific energy, we can add the specific values of separate particles and obtain the microscopic sum of energies of separate nucleons. Since the hidden parameter is calculated approximately (with a logarithmic accuracy), we can justify the method proposed for calculations.

In \([18]\), it is similarly proposed to sum the microscopic energies of particles to calculate the binding energy of the whole nucleus. This permits calculating the value of the hidden parameter.
Let us consider the specific energy of a gas consisting of particles obeying the Bose–Einstein statistics for \( a = 1 \):

\[
E_{\text{spec}} = T(\gamma + 1) \frac{\zeta(2 + \gamma)}{\zeta(1 + \gamma)}.
\]  
(68)

This energy corresponds to the macroscopy, while the specific binding energy of the nucleus \( \epsilon \), i.e., the energy per 1 nucleon, corresponds to the microscopy. The equality of these energies permits calculating the parameter \( \beta = \frac{1}{T} \):

\[
\beta = (\gamma + 1) \frac{\zeta(2 + \gamma)}{\epsilon \zeta(1 + \gamma)}.
\]  
(69)

The time \( t_0 \) equal to the interval between the moments of measurement can be calculated by the formula

\[
t_0 = \beta \hbar = (\gamma + 1) \hbar \frac{\zeta(2 + \gamma)}{\epsilon \zeta(1 + \gamma)},
\]  
(70)

where \( \hbar \) is the Planck constant.

For Helium-5, the specific binding energy is equal to 5.481 MeV, the life time of the nucleus is \( 1.01 \times 10^{-21} \) s, and the time \( t_0 = 5.2 \times 10^{-22} \) s for \( \gamma = 3.5 \). For Helium-6, the specific binding energy is equal to 4.878 MeV, the life time of the nucleus is 1.16 d, and the time \( t_0 = 5.9 \times 10^{-22} \) s for \( \gamma = 3.5 \).

The dependence \( t_0(\gamma) \) is depicted in Fig. 7. In particular, for \( \gamma = 0.5 \), the life time for Helium-5 is equal to \( 9.25 \times 10^{-23} \) s, and for Helium-6, to \( 1.04 \times 10^{-22} \) s.

![Figure 7: Dependence of the time \( t_0(\gamma) \) given by formula (70). The upper curve corresponds to Helium-6, the lower, to Helium-5.](image)

We assume that to distinguish one revolution of a neutron around the nucleus, the experimenter has to take at least 10 pictures in this time period. Thus, to distinguish 20 revolutions, it is necessary to take 200 during the time \( 200t_0 \). In the case \( \gamma = 3.5 \) for Helium-5, this time is equal to \( t_0 = 1.05 \times 10^{-19} \) s, which significantly exceeds the time given for the experiment, i.e., the value of the hidden parameter.

Thus if the life time of a particle is less that the time given for the experimental observation (i.e., less that the hidden parameter), then the experimenter can distinguish this particle from the others. But if the life time of a particle is greater than the observation time, then the the experimenter cannot distinguish the particles. But during the time given to him, he can observe some separate moments or stages of the particle life, for example, 20 revolutions of the particle around the nucleus. But these separate moments of the particle life do not allow him to distinguish this particle from the other particles, i.e. to identify it,
In other words, our concept is based on the following three times: the time given for the experiment (hidden parameter), the life time of a particle, and the real time of events in the life of a particle which is less than the time of its whole life and which is observed by the experimenter.

Since we distinguish the hidden parameter in the quantum mechanics and the hidden parameter in classical statistical physics, it is necessary, more precisely than usually in the physical literature, to separate the quantum quantities and the classical quantities which disappear as $\hbar \to 0$. For example, in the literature, the light polarization is related to the wave optics, despite the fact that, as is known, the polarization does not disappear as the frequency tends to infinity and hence remains in the geometric optics.

Similarly, the spin does not disappear in the classical limit $\hbar \to 0$ and is preserved in the classical transport equation \cite{29}, \cite{54}. The hidden parameter for the spin is not related to the quantum hidden parameter $\Delta t = \hbar / \Delta E$.

The WKB method of transition from the quantum mechanics in classical mechanics leads to the following two classical equations: the Hamilton–Jacobi equation and the transport equation. Both of these equations describe the classical mechanics. The Hamilton–Jacobi equation contains a term which determines the interaction of particles in the Newton equation. The interaction between the spin and the magnetic field is contained in the transport equation, and hence can be explained in classical mechanics. The question of how to describe the interaction between the terms in the transport equation if there is no interaction in the Hamilton–Jacobi equation was posed by Anosov and by the author at the beginning of the 1960s. At present, this problem is related to the Bell inequality and hence to the hidden parameters arising in this case \cite{55}. The author agrees with Bohm in that the main difficulty in revealing the EPR paradox reduces to a problem related to the spin and polarization. But as follows from the above, this by no means does not concern the problem of the quantum hidden parameter considered in this section. The difficulty is in the problem of spin, and it reduces to the classical mechanics. Precisely in the same way, we can neglect the Newtonian attraction.

The experiment with rotation of a boson around the nucleus revealed the following. On the one hand, the experimenter allowed an increase in the hidden parameter to such an extent that the wave packet corresponding to the nucleon began to differ from its twin. This implies that the nucleon started to behave not as a quantum particle, which is located at a separate Bohr energy level, but as a classical particle, which rotates as the Rutherford electron.

This fact already shows that, even for small energies, the nucleon does behave according to the Bohr rule related to separate orbits. Does this mean that there is a WKB approximation? The transition to the classics was previously explained only by the WKB approximation. But the experiments allowed one to calculate the hidden parameter and confirmed the fact that the hidden parameter is large so that it permits observing the behavior of a nucleon separately from its twin behavior. By the hidden parameter law, the nucleon behaves like a wave packet independently of its twin, and hence it makes revolutions.

Thus, this well-known experimental fact can be explained by using the hidden parameter, which permits considering the classical mechanics and the quantum mechanics from the common point of view. This unifies the two sciences into a comprehensive whole and exactly shows when we must pass from the quantum theory to the classical theory without using the WKB method. This is a principally different method for such a transition as compared to the WKB method. In what follows, we show the converse; namely, the quantum problem, in turn, gives important results in classical theory of thermodynamics without the semiclassical transition.
10.2 Mesoscopic case

We first consider the mesoscopic case (the number of particles is less than $10^6$). In [28], the author obtained self-consistent equations relating the Gentile statistics to the Bose–Einstein statistics and the Fermi–Dirac statistics:

$$N = VT^{\gamma+1}(\text{Li}_{1+\gamma}(a) - \frac{1}{(N^\alpha + 1)^\gamma} \text{Li}_{1+\gamma}(a^{N^\alpha+1})), \quad (71)$$

$$M = VT^{\gamma+2}(\text{Li}_{2+\gamma}(a) - \frac{1}{(N^\alpha + 1)^{1+\gamma}} \text{Li}_{2+\gamma}(a^{N^\alpha+1})), \quad (72)$$

where $M = \Omega$ is the potential, $\Omega = -VP$, $P$ is the pressure, $V$ is the volume, $T$ is the temperature, $a$ is the activity, and $D = 2\gamma + 2$ is the number of degrees of freedom. The value $\alpha$ ranges from 1 to zero.

The case where we let $N \log(a) \to 0$ in macroscopic equations (71)–(72) belongs to the mesoscopic physics. In the thermodynamics, this situation arises near the point at which the activity $a$ changes its sign, i.e., near the point of transition of the Bose–Einstein distribution into the Fermi–Dirac distribution (for $\gamma \leq 0$).

The notation $\Phi = V(\lambda^2 T)^{\gamma+1}$, where $\lambda$ is a parameter depending on the mass, was introduced above. In the number theory, $\gamma = 0$ and $\Phi = T$.

We apply the Gentile statistics (parastatistics) and self-consistently relate it to the statistics of bosons and fermions in the mesoscopic physics.

Let us consider the case where $N_i$ is the number of holes. This means that we assume that $N_i$ is a negative number. Then

$$\sum (\ -N_i) = -N, \quad -\sum \varepsilon_i N_i = -M. \quad (73)$$

Thus, the numbers $-N$ and $-M$ are also negative. The multiplication of both relation by $-1$ does not change anything as compared to the case where $N_i$ is positive. This means that, in the formulas of the Gentile statistics, we can replace the numbers $N$ and $M$ by their absolute values. We thus continue the Gentile statistics to the negative $N_i$, i.e., to the case of holes.

10.3 Model of the Bohr nucleus

By the Bohr model, the nucleons do not interact inside the shell of the nucleus (there is no attraction between them). They form a set of colliding balls. Indeed, the last experiments showed that the nucleons are attracted only at distances less than or equal to their radii.

But this fact is also an approximation. Indeed, by the Schrödinger equation, the nucleon is a wave packet. Therefore, it spreads with a $\delta$-like structure, and hence there is a small interaction $\varepsilon V(x - y)$ between the nucleons. Here $\varepsilon$ is a small parameter, $x$ corresponds of a nucleon, $y$ corresponds to another nucleon, and $V(x - y)$ is the interaction potential. For $\gamma > 0$, the number of degrees of freedom depends on the relation between this parameter and the Planck constant as follows:

(1) first, the maximal number of degrees of freedom of nucleons is equal to $6n - 5$, where $n$ is the number of nucleons ([5, § 44]);

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9This transition was studied in detail by the author in the theory of decompositions of rational numbers; in particular, see [14], where the authors sews the boson and fermion branches. In this case, there arises a mesoscopy between the values $N = 0$ and $N = 1/\log(a)$ according to the concept of the abstract analytic number theory (a detailed bibliography in the analytic number theory can be found in the book [45]).
This energy is very small, but the fermion separation occurs in a macroscopic volume. This can be calculated by determining the value of the energy required for the boson decay.

What happens with this fermion? At what distance has it moved from the nucleus shell? disappeared splitting into two fermions and one of the two fermions must somehow disappear.

there do not exist any bosons. This it is the transition time at which the boson has already disappeared. We are interested in the problem of specific energy jump which occurs in the transition from the Fermi system to the Bose system.

We consider the energy jump in the transition of the boson gas to the fermion gas, (in particular, of the Helium-6 gas to the Helium-5 gas), and calculate this energy of spin jump as a specific energy.

The following representation is known for the polylogarithm:

\[ \text{Li}_s(a) = \sum_{i=1}^{\infty} \frac{a^i}{i^s}. \] (74)

Substituting this representation into (54)–(56), one can obtain the following expansions of the compressibility factor:

\[ Z|_{\text{Fermi}} = 1 + a2^{-\gamma-2} - a^22^{-2\gamma-3\gamma-2} (22^\gamma+4 - 3\gamma+2) - a^32^{-3\gamma-4\gamma-2} (7 2^\gamma+2 - 3\gamma+2 - 2^\gamma3^\gamma+3) - a^42^{-4\gamma-5\gamma-3\gamma-5\gamma-2} (2^\gamma+73^\gamma+3 - 2^\gamma+65\gamma+2 - 3^\gamma+3^\gamma+2 + 2^\gamma+33^\gamma+15\gamma+3 - 2^\gamma3^\gamma+35^\gamma+3) + O(a^5), \] (75)

\[ Z|_{\text{Bose}} = 1 - a2^{-\gamma-2} - a^22^{-2\gamma-3\gamma-2} (22^\gamma+4 - 3\gamma+2) + a^32^{-3\gamma-4\gamma-2} (7 2^\gamma+2 - 3\gamma+2 - 2^\gamma3^\gamma+3) - a^42^{-4\gamma-5\gamma-3\gamma-5\gamma-2} (2^\gamma+73^\gamma+3 - 2^\gamma+65\gamma+2 - 3^\gamma+3^\gamma+2 + 2^\gamma+33^\gamma+15\gamma+3 - 2^\gamma3^\gamma+35^\gamma+3) + O(a^5). \] (76)
Figure 8: Dependence of the compressibility factor $Z$ on the activity $a$ in the two-dimensional case. $\Phi = 100$, $\gamma = 0$. The upper curve corresponds to the Fermi system. The lower curve corresponds to the exact Bose system. The dots on the curves correspond to integer $N$.

The compressibility factor $Z$ multiplied by the temperature $T$ is the specific energy. To obtain the jump of the specific energy, it suffices to consider the jump of the compressibility factor from the Fermi system to the Bose system:

$$\Delta Z(a) = Z|_{\text{Fermi}} - Z|_{\text{Bose}} = \frac{\text{Li}_{2+\gamma}(-a)}{\text{Li}_{1+\gamma}(-a)} - \frac{\text{Li}_{2+\gamma}(a)}{\text{Li}_{1+\gamma}(a)}. \quad (77)$$

The jump of the compressibility factor for small $a$ can be expressed as (see Fig. 8)

$$\Delta Z(a) = Z|_{\text{Fermi}} - Z|_{\text{Bose}} = \frac{a}{2^{\gamma+1}} + O(a^2), \quad (78)$$

and the jump of the specific energy $E_{\text{spec}}$ has the form

$$\Delta E_{\text{spec}}(a) = T(\gamma + 1)\Delta Z(a) = T(\gamma + 1)\frac{a}{2^{\gamma+1}} + O(a^2). \quad (79)$$

For $a = a_0$, we obtain

$$\Delta Z(a_0) = \frac{a_0}{2^{\gamma+1}} = -\frac{1}{2^{\gamma+1}\Phi \ln a_0}. \quad (80)$$

Figure 8 shows the dependence of the compressibility factor $Z$ on the activity $a$ in the case $\gamma = 0$.

In the case $\gamma > 0$, the difference of the energies of the whole system of particles (measured in units of $T$) does not exceed

$$\Delta E = \frac{\gamma + 1}{2^{\gamma+1} \ln \Phi} \text{Li}_{1+\gamma}(1)\left(1 - \frac{1}{(N_c + 1)\gamma}\right). \quad (81)$$
and in the case \( \gamma = 0 \), does not exceed
\[
\Delta E = \frac{1}{2 \ln \Phi} \log(N_c + 1),
\] (82)

where \( N_c \) is calculated by formula (39).

For \( \gamma < 0 \), the jump of the energy can be expressed as
\[
\Delta E = \frac{\gamma + 1}{2^{\gamma+1} \Phi \ln \Phi} (\Phi C(\gamma))^{1/(\gamma+1)} = \frac{\gamma + 1}{2^{\gamma+1} \ln \Phi} C(\gamma)^{1/(\gamma+1)} \Phi^{-\frac{\gamma}{\gamma+1}}.
\] (83)

As is shown in [51], the jump of the specific energy \( E_{\text{spec}} \) has the form
\[
\Delta E_{\text{spec}}(a_0) = T(\gamma + 1)\Delta Z(a) = T(\gamma + 1)\frac{a_0}{2^{\gamma+1}} + O(a_0^2).
\] (84)

### 10.5 Continuation of the self-consistent equation

We note that the number of degrees of freedom \( \gamma \) need not be an integer, because it depends on the temperature value and is the averaging over the number of particles each of which can generally have its own number of degrees of freedom. As was shown above, the cases \( \gamma > 0 \) and \( \gamma < 0 \) differ significantly. The two-dimensional case associated with number theory corresponds to the point \( \gamma = 0 \).

In the two-dimensional cases, there are two important points. One of them is the point of solution of the self-consistent equation obtained above in (39). The other point was obtained by Erdös who used the remarkable Ramanujan formula. One can say that the point of self-consistent equation is closer to negative values of \( \gamma \) and the Erdös point is closer to positive values of \( \gamma \), but both of these points correspond to the case \( \gamma = 0 \).

The jump from the solution of self-consistent equation to the solution of the Erdös formula will be continued for the values \( \gamma > 0 \) so as to obtain the Erdös formula for \( \gamma = 0 \). In this case, we use both the Gentile statistics and the Bose–Einstein statistics.

It follows from the Bose–Einstein distribution that
\[
T = \left( \frac{M}{\zeta(\gamma + 2)} \right)^{1/(\gamma+2)},
\] (85)

where \( \zeta(\cdot) \) is the Riemann zeta function.

By \( \tilde{N} \) we denote the value of the number of particles which make contributions to the energy, i.e., \( N = N_0 + \tilde{N} \), where \( N_0 \) is the number of particles at the zero energy level (Bose condensate) for \( \gamma > 0 \).

The relation for \( \tilde{N} \) in the case \( \gamma > 0 \) has the form
\[
\tilde{N} = \left( \frac{M}{\zeta(\gamma + 2)} \right)^{1/(\gamma+2)} \ln \left\{ \left( \frac{M}{\zeta(\gamma + 2)} \right)^{1/(\gamma+2)} \right\}.
\] (86)

For \( \gamma < 0 \), formula (80) holds (has the same form) for both small \( N \) and large \( N \). For \( \gamma > 0 \), formula (80) splits, i.e., for large \( \tilde{N} \), this is a continuation of the Erdös formula, and for small \( \tilde{N} \), this is a continuation of general formula (80).

Thus, the self-consistent equations can directly be continued for \( O(a^2) \) and \( \gamma > 0 \) and determines the point at which Bose particles become Fermi particles.

Since the theory of nucleus deals, as a rule, with the three-dimensional case, precisely the formula of the self-consistent equation for small \( \tilde{N} \) gives the desired transition which occurs for sufficiently small values of the activity \( a \) (see Fig. 2 in [51]).
11 Relationship to classical thermodynamics. The boundaries of the ideal gas which is an analog of the Bose gas

We see that the one-dimensional situation is a special one. To obtain the classical thermodynamics, we must, for each gas, perform experiments which would give critical lines on the Hougen–Watson diagram. We call them references lines.

Let us distinguish three reference lines. One of them is a spinodal, which is a caustic in a certain sense. Another line is the critical isochore $\rho = \rho_c$ for different $T > T_c$ on which a gas fluid transforms to a liquid fluid. The author calls them light and heavy fluids, because the light fluid is still attracted by the Earth gravity.

The third line corresponds to $P = 0$, i.e., this line separates a liquid under pressure from a liquid under the extended pressure. This line characterizes specific properties which permit transitions from a liquid to an amorphous solid [57].

We believe that, in the region between these lines, which were determined experimentally, the gas is noninteracting, i.e., it is an analog of the Bose gas. We can draw an analogy with shooting of a battery of guns. At the initial time, there is an interaction between the artillerymen located on the same reference line, but the missiles (balls) themselves fly already without any mutual interaction but elastic collisions are admissible [58].

As was already said, these reference lines must experimentally be determined for each specific gas. Then, for each specific gas, we obtain the correct thermodynamics. In particular, if these lines correspond to the Van-der-Waals gas, then the other parameters of the gas coincide with the parameters of the Van-der-Waals gas. In other words, if we take the reference lines from the Van-der-Waals gas equation and consider particles without interaction, then we can obtain the total Van-der-Waals gas model without using the Van-der-Waals gas forces.

The number of degrees of freedom of the lines is much less than the number of degrees of the plane and other geometric shapes. This plays the most significant role in the “heap antinomy”. The author developed an approach related to this problem but, in addition, it should be stressed that the transition of a liquid into glass, the so-called vitrification, is a phase transition of the first kind despite the fact that neither the density nor the number of particles not the temperature are changed. Only the number of degrees of freedom and the energy are changed. It is clear that the glass energy (and hence the pressure) are less than the energy of the particles obeying the mean free time laws and other similar laws. In the case under study, we pass from unnumbered particles (in the liquid) to numbered particles (in the glass). Despite the fact that the glass is not a crystal, the particles can still be numbered, because although the particles in glass are in a chaotic state, they do not move. They can be renumbered and calculated, as the author showed in his previous papers.

Thus, we see that, for equal basic parameters of the gas (temperature, density, volume, number of particles, etc.), one can observe a phase transition due to the energy jump. The vitrification energy is less than the liquid energy, and hence there is a jump of pressure. Actually, this transition is related only to the fact of transition from a heap to a set whose elements can be numbered. This is a philosophic question which had been considered many times by the author when he explained the phase transition of the first kind which occurs in the vitrification. This treatment of the phase transition permits explaining very important phase transitions such as the transition into amorphous metal (manufacturing of damask steel and armor) and the processes of freezing of human organs from the deceased for their transplantation to the living people.
Figure 9: Dependence of the number of particles $N$ on $\gamma$ for $T = 100$ constructed by formula (87).

### 12 Relationship between number theory and the Van-der-Waals model

Let us consider the problem of relationship between the Van-der-Waals model and number theory.

#### 12.1 Dependence of the pressure and the number of particles on $\gamma$

As was already noted, the value of the maximal number of particle at the same energy level $N_c$ is calculated by formula (39).

By Lemma 1, relation (45) holds. Following the reasoning in Sec. 9.2 and using formulas (47)–(51), we obtain the relation

$$N_c = TC(\gamma)^{1/(\gamma+1)}. \quad (87)$$

This implies that the Bose gas energy can be written as

$$M = \Phi T \text{Li}_{2+\gamma}(a) = T^{2+\gamma} \left( \frac{\sqrt{2\pi m}}{2\pi \hbar} \right)^{2(\gamma+1)} V \text{Li}_{2+\gamma}(a). \quad (88)$$

In number theory, i.e., in the case $\Phi = T^{1+\gamma}$, for $a = 1$ we obtain

$$M = T^{2+\gamma} \zeta(2 + \gamma) \quad (89)$$

Figures 9–10 illustrate the dependence of $N$ and $M$ on $\gamma$.

#### 12.2 Comparison with the Van-der-Waals model

For the Van-der-Waals gas, the dependence of the pressure on the temperature $P_{sp}(T)$ on the negative spinodal is known. If this pressures is equated with the negative pressure of the Bose gas for $\lambda = 1$ at the point $a = 1$, then one can derive the dependence $\gamma(T)$ (see Fig. 11)

$$P_{sp}(T) = -T^{2+\gamma} \zeta(2 + \gamma). \quad (90)$$
Figure 10: Dependence of the energy $M$ on $\gamma$ for $T = 100$ constructed by formula (89).

Figure 11: Dependence of $\gamma$ on $T$ constructed by formula (90).

Figure 12: Dependence of $Z$ on $a$ for the Bose–Einstein model for different fixed $\gamma$. The parameter $\gamma$ varies from $-1$ with $1$ with step $0.2$ (upwards).
Figure 13: Dependence of $Z$ on $P$ for the Bose–Einstein model for different fixed $\gamma$. The parameter $\gamma$ varies near the point $Z = 1$ from $-1$ to $0$ with step $0.2$ (upwards).

In the Bose–Einstein model, we have $Z(a, \gamma) = \frac{Li_{2+\gamma}(a)}{Li_{1+\gamma}(a)}$. As $\gamma \leq 0$, the compressibility factor $Z(1, \gamma)$ vanishes, which is illustrated by the graphs 12, 13.

Figure 12 shows a rather good correspondence to the Van-der-Waals model for $|\gamma| \geq 1/2$: the parameter $\gamma$ varies a little with the temperature $T$ ($\gamma = -1/2$ is the one-dimensional case). And as in the Van-der-Waals model, there is no transition into solid. The transition of an amorphous solid occurs for $|\gamma| > 0.8$.

12.3 Main solid line of the temperature-dependence of the number of degrees of freedom

These are many methods for approximating of a finite ordered set by a smooth curve. In this section, we present a method based on calculating the leading term of an asymptotics. This method in the case where the number of particles $N$ tends to infinity was described in detail in [59].

For the Van-der-Waals gas and an experimentally calculated spinodal of the gas under study, the dependence $Z_{sp}(xT)$ on the negative spinodal is known. If we equate this expression with the value $Z$ in the Bose–Einstein model, then we obtain the dependence $\gamma(T)$:

$$Z_{sp}(xT) = \frac{T^\gamma \zeta(2 + \gamma)}{(C(\gamma))^{1/(\gamma+1)}},$$

(91)

where $C(\gamma)$ is determined by relation (48) (for more details, see [59]).

Thus, one can pass from discrete values of $\gamma$ to the continuous (solid) curve $\gamma(T)$; see Fig. 14.

The passage to the continuous case can be illustrated by an example of rough approximation. Consider two particles connected by a string as a dumbbell. Such particles have five degrees of freedom rather than six. But if the temperature increases, then the string also begins to play a certain role, namely, it starts to vibrate and produce an additional degree of freedom.
Figure 14: Main solid line of the dependent of $\gamma$ on $T$ given by formula (91).

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