Eigenvalue Bounds for Schrödinger Operators with a Homogeneous Magnetic Field

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Received: 13 January 2011 / Revised: 10 May 2011 / Accepted: 10 May 2011
Published online: 9 June 2011 – © The Author(s) 2011

Abstract. We prove Lieb-Thirring inequalities for Schrödinger operators with a homogeneous magnetic field in two and three space dimensions. The inequalities bound sums of eigenvalues by a semi-classical approximation which depends on the strength of the magnetic field, and hence quantifies the diamagnetic behavior of the system. For a harmonic oscillator in a homogenous magnetic field, we obtain the sharp constants in the inequalities.

Mathematics Subject Classification (2000). Primary 35P15; Secondary 35J10, 81Q10.

Keywords. Schrödinger operator, Lieb-Thirring inequalities, magnetic field.

1. Introduction and Main Result

Lieb-Thirring inequalities [15] provide bounds on the sum of negative eigenvalues of Schrödinger operators in terms of a phase space integral. In this paper, we are interested in two-dimensional Schrödinger operators $H_B + V$ with a homogenous magnetic field of strength $B > 0$. Here

$$H_B = \left(-i \frac{\partial}{\partial x_1} + \frac{B x_2}{2}\right)^2 + \left(-i \frac{\partial}{\partial x_2} - \frac{B x_1}{2}\right)^2$$

is the Landau Hamiltonian in $L^2(\mathbb{R}^2)$ and $V$ is a real-valued function. The Lieb-Thirring inequality states that

$$\text{Tr} \left( (H_B + V)_{-} \right) \leq r_2 (2\pi)^{-2} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left( |p|^2 + V(x) \right)_- \, dx \, dp$$

(1)

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with the (currently best, but presumably non-optimal) constant \( r_2 = \pi / \sqrt{3} \) from [3]. Physically, the left side is (minus) the energy of a system of non-interacting fermions in an external potential \( V \) and an external, homogeneous magnetic field of strength \( B \), whereas the right side is \(-r_2\) times a semi-classical approximation to that energy.

Physically, one expects the system to show a diamagnetic behavior, that is, to have a higher energy in the presence of a magnetic field. This is however not reflected in (1), which has a right hand side independent of \( B \). We refer to [6] for further references and a survey over this problem. Our goal in this letter is to obtain a bound similar to (1), but with a more refined semi-classical approximation which takes \( B \) into account. The approximation we propose is:

\[
\frac{B}{2\pi} \sum_{m=0}^\infty \int_{\mathbb{R}^2} ((2m+1)B + V(x))_- \, dx.
\]

(2)

This quantity reflects the diamagnetic behavior since,

\[
\frac{B}{2\pi} \sum_{m=0}^\infty \int_{\mathbb{R}^2} ((2m+1)B + V(x))_- \, dx \leq \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (|p|^2 + V(x))_- \, dx \, dp
\]

(3)

for every \( V \). Inequality (3) follows (even before the \( x \)-integration) from an easy convexity inequality (see Lemma 12 below). We also note that when \( B \to 0 \), by a Riemann sum argument, the quantity (2) approaches

\[
(4\pi)^{-1} \int_0^\infty dE \int_{\mathbb{R}^2} (E + V(x))_- \, dx = (2\pi)^{-2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left( |p|^2 + V(x) \right)_- \, dx \, dp,
\]

(4)

which is the ‘usual’ phase space integral.

While the right side of (1) (up to the constant \( r_2 \)) has the correct limiting behavior when a small parameter \( \hbar \) is introduced, it is not useful in the coupled limit \( B \to \infty \) and \( \hbar \to 0 \). This limit is physically relevant, for instance, in the study of neutron stars [13]. The magnetic quantity (2) reproduces the correct behavior in this regime. It is remarkable that this asymptotic profile is, indeed, a uniform, non-asymptotic bound. This is implicitly contained in [14] who use, however, only an approximation of (2). Our first result is:

**THEOREM 1.** For any \( B > 0 \) and any \( V \) on \( \mathbb{R}^2 \) one has,

\[
\text{Tr} \left( H_B + V \right)_- \leq \rho_2 \frac{B}{2\pi} \sum_{m=0}^\infty \int_{\mathbb{R}^2} ((2m+1)B + V(x))_- \, dx
\]

(5)

with \( \rho_2 = 3 \).
Hence, up to the moderate increase from \( r_2 = \pi/\sqrt{3} \approx 1.81 \) to \( \rho_2 = 3 \), we have found a magnetic analogue of (1) which reflects the desired diamagnetic behavior (3). An important ingredient in our proof is a method developed recently by Rumin [16] to derive kinetic energy inequalities; see Section 2.1.

Similarly as in the non-magnetic case, one might ask for the optimal value of the constant \( \rho_2 \). By the semi-classical result mentioned above one necessarily has \( \rho_2 \geq 1 \). A first result in this direction was obtained in [7] (extending previous work of [4]), where it was shown that if one takes \( V \) to be constant on a set of finite measure and plus infinity otherwise, then (5) holds with \( \rho_2 = 1 \). Our second main result is an analogous optimal bound for a harmonic oscillator.

**THEOREM 2.** For any \( B > 0, \omega_1 > 0, \omega_2 > 0 \) and \( \mu > 0 \), inequality (5) holds with \( \rho_2 = 1 \) for \( V(x) = \omega_1^2 x_1^2 + \omega_2^2 x_2^2 - \mu \).

In particular, letting \( B \to 0 \) and using the limit in (4) we recover the known bounds in the non-magnetic case from [2,11]. Even though the eigenvalues of a harmonic oscillator in a homogeneous magnetic field are explicitly known (Lemma 10), the proof of Theorem 2 relies on a delicate property of a subclass of convex functions (Lemma 14) which, we feel, could be useful even beyond the context of this paper.

**Moments of eigenvalues.** Using some by now standard techniques we derive a few consequences of Theorems 1 and 2. First, following Aizenman and Lieb [1] one can replace \( V \) by \( V - \mu \) in (5) and integrate with respect to \( \mu \) to obtain that for any \( \gamma \geq 1 \)

\[
\text{Tr} \left( H_B + V \right)^\gamma \leq \rho_2^2 \frac{B}{2\pi} \sum_{m=0}^{\infty} \int (2m+1)B + V(x))^{\gamma} \ dx, \tag{6}
\]

where \( \rho_2 = 3 \) for general \( V \) and \( \rho_2 = 1 \) for \( V(x) = \omega_1^2 x_1^2 + \omega_2^2 x_2^2 - \mu \). The restriction \( \gamma \geq 1 \) is necessary, since one easily checks that for \( 0 \leq \gamma < 1 \) there is no constant \( \rho_2 \) such that (6) holds for all potentials \( V \). Restricting ourselves to the quadratic case we shall show in Section 3.4

**PROPOSITION 3.** For any \( 0 \leq \gamma < 1 \) there are \( B > 0, \mu > 0 \) and \( \omega_1 = \omega_2 \) such that for \( V(x) = \omega_1^2 x_1^2 + \omega_2^2 x_2^2 - \mu \) one has;

\[
\text{Tr} \left( H_B + V \right)^\gamma > \frac{B}{2\pi} \sum_{m=0}^{\infty} \int (2m+1)B + V(x))^{\gamma} \ dx. \tag{7}
\]

In particular, this shows that (6) does not hold with \( \rho_2 = 1 \) if \( 0 \leq \gamma < 1 \), not even for a quadratic potential. Our counterexample in Proposition 3 appears in the limit
$\omega_j/B \to 0$ (with $\mu/B = 3$ fixed). Another counterexample can be obtained in the $B \to 0$ limit from the (non-magnetic) counterexample of Helffer-Robert [8] and the fact that for potentials of the special form $V(x) = \omega_1^2 x_1^2 + \omega_2^2 x_2^2 - \mu$ one has

$$\frac{B}{2\pi} \sum_{m=0}^{\infty} \int_{\mathbb{R}^2} ((2m+1)B + V(x))_+ \, dx \leq \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} (|p|^2 + V(x))_+ \, dx \, dp$$

for all $\gamma \geq 0$ [and not only for $\gamma \geq 1$, as in (3)].

**Three dimensions.** Next, we shall show that our bounds for $d = 2$ can be applied to deduce analogous bounds for $d = 3$. This argument is in the spirit of the lifting argument from [10–12]. We denote by $\hat{H}_B = H_B - \frac{\partial^2}{\partial x_3^2}$ the Landau Hamiltonian in $L^2(\mathbb{R}^3)$.

**Corollary 4.** For any $B > 0$ and any $\hat{V}$ on $\mathbb{R}^3$, one has

$$\text{Tr}(\hat{H}_B + \hat{V})_- \leq \rho_3 \frac{B}{(2\pi)^2} \int_{\mathbb{R}^3 \times \mathbb{R}} ((2m+1)B + p_3^2 + \hat{V}(x))_- \, dx \, dp_3 \tag{8}$$

with $\rho_3 = \sqrt{3}\pi$.

**Proof.** From the operator-valued Lieb-Thirring inequality of [3] we know that

$$\text{Tr}(\hat{H}_B + \hat{V})_- \leq \frac{\pi}{\sqrt{3}} \int_{\mathbb{R} \times \mathbb{R}} \text{Tr}_{L^2(\mathbb{R}^2)}(H_B + p_3^2 + \hat{V}(\cdot, x_3))_- \, dx_3 \, dp_3.$$ 

Inequality (8) is therefore a consequence of Theorem 1. \qed

For the harmonic oscillator we have:

**Corollary 5.** For any $B > 0$, $\omega_1 > 0$, $\omega_2 > 0$, $\omega_3 > 0$ and $\mu > 0$, inequality (8) holds with $\rho_3 = 1$ for $\hat{V}(x) = \omega_1^2 x_1^2 + \omega_2^2 x_2^2 + \omega_3^2 x_3^2 - \mu$.

**Proof.** We denote by $E_j$ the eigenvalues of the one-dimensional harmonic oscillator $H = -\frac{\partial^2}{\partial x_3^2} + \omega_3^2 x_3^2$. Then, since $\hat{H}_B + \hat{V} = (H_B + V) \otimes I + I \otimes H$ with $V(x_1, x_2) = \omega_1^2 x_1^2 + \omega_2^2 x_2^2$, we have

$$\text{Tr}_{L^2(\mathbb{R}^3)}(\hat{H}_B + \hat{V})_- = \sum_j \text{Tr}_{L^2(\mathbb{R}^2)}(H_B + V + E_j - \mu)_-.$$
According to Theorem 2 (which trivially holds for \( \mu \leq 0 \) as well), this is bounded from above by

\[
\frac{B}{2\pi} \sum_{j} \sum_{m=0}^{\infty} \int \left((2m+1)B + V(x_1, x_2) + E_j - \mu\right)_- \, dx_1 \, dx_2
\]

\[
= \frac{B}{2\pi} \sum_{m=0}^{\infty} \int \text{Tr}_{L^2(\mathbb{R})} \left((H + (2m+1)B + V(x_1, x_2) - \mu\right)_- \, dx_1 \, dx_2.
\]

Next, we shall use that \( H \) satisfies a Lieb-Thirring inequality with semi-classical constant \([2,11]\), that is, for any \( \Lambda \in \mathbb{R} \),

\[
\text{Tr}_{L^2(\mathbb{R})}(H - \Lambda)_- \leq \frac{1}{2\pi} \int_{\mathbb{R} \times \mathbb{R}} \left(p_3^2 + \omega_3 x_3^2 - \Lambda\right)_- \, dx_3 \, dp_3.
\]

(This can also be seen from Lemma 12 and recalling the explicit form of the eigenvalues of \( H \).) It follows that for every fixed \((x_1, x_2)\)

\[
\text{Tr}_{L^2(\mathbb{R})}(H + (2m+1)B + V(x_1, x_2) - \mu)_-
\]

\[
\leq \frac{1}{2\pi} \int_{\mathbb{R} \times \mathbb{R}} \left(p_3^2 + \omega_3 x_3^2 + (2m+1)B + V(x_1, x_2) - \mu\right)_- \, dx_3 \, dp_3,
\]

which proves the claimed bound.

\[\square\]

Remark 6. The previous proof shows that (8) with \( \rho_3 = 1 \) is valid for more general potentials \( \hat{V}(x) = V(x_1, x_2) + v(x_3) \), where \( V(x_1, x_2) = \omega_1^2 x_1^2 + \omega_2^2 x_2^2 \) and where \( v \) is such that \( \text{Tr}_{L^2(\mathbb{R})}(-\frac{d^2}{dx_3^2} + v(x_3) - \Lambda)_- \leq \frac{1}{2\pi} \int_{\mathbb{R} \times \mathbb{R}} (p_3^2 + v(x_3) - \Lambda)_- \, dx_3 \, dp_3 \) for all \( \Lambda \).

A similar argument as in the proofs of Corollaries 4 and 5 (based on the operator-valued Lieb-Thirring inequalities of \([9,12]\)) shows that for general \( \hat{V} \) one has

\[
\text{Tr}(\hat{H}_B + \hat{V})_\gamma \leq \rho_{3,\gamma} \frac{B}{(2\pi)^2} \sum_{m=0}^{\infty} \int_{\mathbb{R}^3} \int \left((2m+1)B + p_3^2 + \hat{V}(x)\right)_\gamma^\gamma \, dx \, dp_3
\]

with \( \rho_{3,\gamma} = 6 \) if \( \gamma \geq 1/2 \), with \( \rho_{3,\gamma} = \pi \sqrt{3} \) if \( \gamma \geq 1 \) and with \( \rho_{3,\gamma} = 3 \) if \( \gamma \geq 3/2 \). Moreover, in the special case of \( \hat{V}(x) = \omega_1^2 x_1^2 + \omega_2^2 x_2^2 + \omega_3^2 x_3^2 - \mu \), (9) holds with \( \rho_3 = 1 \) for \( \gamma \geq 1 \) and with \( \rho_{3,\gamma} = 2(\gamma/(\gamma + 1)) \) for \( 0 \leq \gamma < 1 \). The latter follows from the fact \([7]\) that

\[
\text{Tr}_{L^2(\mathbb{R})}(\frac{-d^2}{dx_3^2} + \omega_3^2 x_3^2 - \Lambda)_\gamma \leq 2 \left(\frac{\gamma}{\gamma + 1}\right)^\gamma \frac{1}{2\pi} \int_{\mathbb{R}^3} (p_3^2 + \omega_3^2 x_3^2 - \Lambda)_\gamma \, dx_3 \, dp_3.
\]
2. Proof of Theorem 1

2.1. A kinetic energy inequality

We define a piecewise affine function \( j : [0, \infty) \to [0, \infty) \) by

\[
j(\rho) = \frac{B^2}{2\pi} (L^2 + (2L + 1)r) \quad \text{if} \quad \rho = \frac{B}{2\pi} (L + r), \quad L \in \mathbb{N}_0, \ r \in [0, 1).
\]

We note that \( j \) is continuous, increasing and convex. One has \( j(\rho) = B\rho \) if \( \rho \leq B/(2\pi) \) and \( j(\rho) \sim 2\pi\rho^2 \) if \( \rho \gg B \). The connection between this function and the right side of (5) will become clearer in the next section.

**THEOREM 7.** Let \( 0 \leq \gamma \leq 1 \) be a density matrix on \( L^2(\mathbb{R}^2) \) with finite kinetic energy. Then

\[
\text{Tr} \ H_B \gamma \geq 3 \int_{\mathbb{R}^2} j(\rho_\gamma(x)/3) \, dx,
\]

where \( \rho_\gamma(x) = \gamma(x, x) \).

It is easy to see that \( 3 \, j(\rho/3) \geq (1/3) \, j(\rho) \) for all \( \rho \geq 0 \), and therefore we also have

\[
\text{Tr} \ H_B \gamma \geq (1/3) \int_{\mathbb{R}^2} j(\rho_\gamma(x)) \, dx.
\]

**Proof.** The first part of our proof follows the method introduced by Rumin [16]. We define \( j_R : [0, \infty) \to [0, \infty) \) by

\[
j_R(\rho) = B\rho + 2B \sum_{k=1}^{\infty} \left( \sqrt{\rho} - \sqrt{\frac{Bk}{2\pi}} \right)^2.
\]

We note that \( j_R \) is differentiable and convex, \( j_R(\rho) = B\rho \) if \( \rho \leq B/(2\pi) \) and \( j_R(\rho) \sim 2\pi\rho^2/3 \) if \( \rho \gg B \). We shall first show that

\[
\text{Tr} \ H_B \gamma \geq \int_{\mathbb{R}^2} j_R(\rho_\gamma(x)) \, dx. \quad (10)
\]

In the second part of our proof (see Lemma 8) we show that \( j_R(\rho) \geq 3 \, j(\rho/3) \) for all \( \rho \geq 0 \).

For the proof of (10) we write:

\[
\text{Tr} \ H_B \gamma = \int_0^\infty \text{Tr} (P^E \gamma) \, dE = \int_0^\infty \int_{\mathbb{R}^2} \rho_\gamma^E(x) \, dE \, dx. \quad (11)
\]
where \( P^E \) is the spectral projection of \( H_B \) corresponding to the interval \([E, \infty)\) and where \( \rho^E_\gamma(x) = (P^E \gamma P^E)(x, x) \). It is well known that,

\[
(1 - P^E)(x, x) = \frac{B}{2\pi} \# \{ m \in \mathbb{N}_0 : (2m + 1)B < E \}.
\]

The same clever use of the triangle inequality as in [16] leads to the pointwise lower bound:

\[
\rho^E_\gamma(x) \geq \left( \sqrt{\rho_\gamma(x)} - \sqrt{\frac{B}{2\pi}} \# \{ m \in \mathbb{N}_0 : (2m + 1)B < E \} \right)^2.
\]

Indeed, if \( \| \cdot \|_2 \) denotes the Hilbert-Schmidt norm, we have for any set \( \Omega \) of finite measure

\[
\sqrt{\int_\Omega \rho_\gamma(x) \, dx} \leq \| \gamma^{1/2} P_E \chi_\Omega \|_2 + \| \gamma^{1/2}(1 - P_E) \chi_\Omega \|_2
\]

\[
\leq \| \gamma^{1/2} P_E \chi_\Omega \|_2 + \| (1 - P_E)^{1/2} \chi_\Omega \|_2
\]

\[
= \sqrt{\int_\Omega \rho^E_\gamma(x) \, dx} + \sqrt{\frac{B}{2\pi}} \# \{ m \in \mathbb{N}_0 : (2m + 1)B < E \}.
\]

Choosing \( \Omega \) to be a ball around any fixed point \( x \) we obtain, as the radius of this ball tends to zero, that

\[
\sqrt{\rho_\gamma(x)} \leq \sqrt{\rho^E_\gamma(x)} + \sqrt{\frac{B}{2\pi}} \# \{ m \in \mathbb{N}_0 : (2m + 1)B < E \}.
\]

This implies the claimed bound (12).

Inserting bound (12) in (11) we obtain

\[
\text{Tr } H_B \gamma \geq \int_{\mathbb{R}^2} \left( \int_0^B \rho_\gamma(x) \, dE + \sum_{k=1}^{\infty} \int_{(2k-1)B}^{(2k+1)B} \left( \sqrt{\rho_\gamma(x)} - \sqrt{\frac{Bk}{2\pi}} \right)^2 \, dE \right) \, dx
\]

\[
= \int_{\mathbb{R}^2} j_R(\rho_\gamma(x)) \, dx.
\]

This completes the proof of (10) and also, by Lemma 8 below, the proof of the theorem.

\[\square\]

**Lemma 8.** \( j_R(\rho) \geq 3 \, j(\rho/3) \) for all \( \rho \geq 0 \).

**Proof.** We are going to prove that

\[
j_R(3\rho) \geq 3 \, j(\rho).
\]

(13)
Note that this is an equality for \( \rho \leq B/(6\pi) \). Moreover, since the left side of (13) is convex and the right side linear for \( \rho \leq B/(2\pi) \), we conclude that (13) holds for all \( \rho \leq B/(2\pi) \).

Henceforth, we shall assume that \( \rho \geq B/(2\pi) \) and we write \( 3\rho = (B/2\pi)(K + s) \) for some integer \( K \geq 3 \) and some \( s \in [0, 1) \). If \( K = 3L + m \) with \( L \in \mathbb{N} \) and \( m \in \{0, 1, 2\} \), then the lemma says that

\[
K + s + 2 \sum_{k=1}^{K} (\sqrt{K + s} - \sqrt{k})^2 \geq 3(L^2 + \frac{1}{3}(2L + 1)(m + s)).
\]

We expand the square on the left side and insert \( L = (K - m)/3 \) on the right side. This shows that the assertion is equivalent to

\[
K + s + 2K(K + s) - 4\sqrt{K + s} \sum_{k=1}^{K} \sqrt{k} + K(K + 1) \geq \frac{1}{3}K^2 + \frac{2}{3}Ks + s + R,
\]

for \( K \in \mathbb{N} \) and \( s \in [0, 1) \), where \( R = -\frac{1}{3}m^2 - \frac{2}{3}ms + m. \) Since the inequality has to be true for any \( m \in \{0, 1, 2\} \), we can replace \( R \) by its maximum over these \( m \) (with fixed \( s \)), that is, by \((2/3)(1-s)\). Thus, (13) is equivalent to

\[
4K^2 + (3 + 2s)K - 6\sqrt{K + s} \sum_{k=1}^{K} \sqrt{k} - 1 + s \geq 0.
\]  

(14)

By the concavity of the square root we have

\[
\frac{\sqrt{k} + \sqrt{k+1}}{2} \leq \int_{k}^{k+1} \sqrt{t} \, dt.
\]

Summing this from \( k=1 \) to \( k=K-1 \) we get

\[
\sum_{k=1}^{K} \sqrt{k} \leq \int_{1}^{K} \sqrt{t} \, dt + \frac{1 + \sqrt{K}}{2} = \frac{2K^{3/2}}{3} + \frac{K^{1/2}}{2} - \frac{1}{6}.
\]

This shows that

\[
4K^2 + (3 + 2s)K - 6\sqrt{K + s} \sum_{k=1}^{K} \sqrt{k} \geq 4K^2 + (3 + 2s)K - \sqrt{K(K + s)}(4K + 3) + \sqrt{K + s}
\]

\[
= \frac{sK((4s - 12)K - 9)}{4K^2 + (3 + 2s)K + \sqrt{K(K + s)}(4K + 3)} + \sqrt{K + s}.
\]

In the quotient on the right side we estimate the numerator from below by \(-3sK(4K + 3)\) and the denominator from below by \(4K^2 + 3K + K(4K + 3) = \)
2K(4K + 3). Thus, the quotient is bounded from below by \(-3s/2\), and since \(K \geq 3\) we conclude that

\[
4K^2 + (3 + 2s)K - 6\sqrt{K + s} \sum_{k=1}^{K} \sqrt{k - 1 + s} \geq \sqrt{K + s} - 1 - \frac{s}{2} > 0.
\]

This proves (14) and completes the proof of the lemma. \(\square\)

### 2.2. Proof of Theorem 1

In this section, we are going to deduce Theorem 1 from Theorem 7. We define

\[
p(v) := -\frac{B}{2\pi} \sum_{m=0}^{\infty} ((2m + 1)B + v)_-
\]

for \(v \in \mathbb{R}\). This is a concave, increasing and non-positive function. The key observation is that this \(p\) is the Legendre transform of the function \(-j\) from the previous subsection, that is,

\[
p(v) = \inf_{\rho \geq 0} (j(\rho) + v\rho).
\]

This can be verified by elementary computations.

In order to prove Theorem 1 we apply Theorem 7 to get the estimate

\[
\text{Tr} (HB + V) \gamma \geq \int_{\mathbb{R}^2} \left( 3j(\rho_\gamma(x)/3) + V(x)\rho_\gamma(x) \right) dx
\]

for any \(0 \leq \gamma \leq 1\). According to (15) this is bounded from below by \(3 \int_{\mathbb{R}^2} p(V(x)) dx\). For \(\gamma\) equal to the projection corresponding to the negative spectrum of \(HB + V\) we obtain the assertion of Theorem 1.

**Remark 9.** Similar arguments show that Theorem 7 can be deduced from Theorem 1. Indeed, since \(j\) is convex it is its double Legendre transform. By (15) we obtain

\[
j(\rho) = \sup_{v \in \mathbb{R}} (p(v) - v\rho).
\]

By the variational principle and Theorem 1 we can estimate for any \(0 \leq \gamma \leq 1\) and any \(V\)

\[
\text{Tr} H_B \gamma \geq -\text{Tr} (HB + V)_- - \int_{\mathbb{R}^2} V(x)\rho_\gamma(x) dx \geq \int_{\mathbb{R}^2} \left( 3p(V(x)) - V(x)\rho_\gamma(x) \right) dx.
\]

Because of (16) we can choose the function \(V\) such that the right side is equal to \(3 \int_{\mathbb{R}^2} j(\rho_\gamma(x)/3) dx\), and this shows Theorem 7.
3. Proof of Theorem 2

3.1. The spectrum of $H_B + V$

The explicit form of the eigenvalues of $H_B + \omega^2 |x|^2$ was discovered in [5]. We include an alternative derivation of this result, which is also valid in the non-radial case.

**Lemma 10.** For any $B > 0$ and $\omega_1, \omega_2 > 0$ the operator $H_B + \omega_1^2 x_1^2 + \omega_2^2 x_2^2$ has discrete spectrum and its eigenvalues, including multiplicities, are given by

$$B \left( a_+ \left( \frac{\omega_1}{B}, \frac{\omega_2}{B} \right) (2k + 1) + a_- \left( \frac{\omega_1}{B}, \frac{\omega_2}{B} \right) (2l + 1) \right), \quad k, l \in \mathbb{N}_0,$$

where

$$a_{\pm}(\sigma_1, \sigma_2) = \sqrt{\frac{1}{2} \left( 1 + \sigma_1^2 + \sigma_2^2 \pm \sqrt{(1 + \sigma_1^2 + \sigma_2^2)^2 - 4\sigma_1^2 \sigma_2^2} \right)}.$$  \hspace{1cm} (17)

**Remark 11.** It will be important for our analysis below that

$$a_-(\sigma) a_+(\sigma) = \sigma_1 \sigma_2,$$

which is easily checked.

**Proof.** By means of the gauge transform $e^{-iB x_1 x_2/2}$ we see that $H_B + V$ is unitarily equivalent to the operator

$$-\frac{\partial^2}{\partial x_1^2} + \left(-i \frac{\partial}{\partial x_2} - B x_1 \right)^2 + \omega_1^2 x_1^2 + \omega_2^2 x_2^2,$$

which, in turn, by a partial Fourier transform with respect to $x_2$, is unitarily equivalent to

$$-\frac{\partial^2}{\partial x_1^2} + (x_2 - B x_1)^2 + \omega_1^2 x_1^2 - \omega_2^2 \frac{\partial^2}{\partial x_2^2}.$$

After scaling $x_2 \mapsto \omega_2 x_2$ this becomes the non-radial harmonic oscillator $-\Delta + x' A x$ with the matrix

$$A = \begin{pmatrix} B^2 + \omega_1^2 & -B \omega_2 \\ -B \omega_2 & \omega_2^2 \end{pmatrix}.$$  \hspace{1cm}

The eigenvalues of $A$ are $B^2 a_+(\omega_1/B, \omega_2/B)^2$ and $B^2 a_- (\omega_1/B, \omega_2/B)^2$. Using the eigenvectors of $A$ as basis in $\mathbb{R}^2$, we obtain a direct sum of two one-dimensional harmonic oscillators with frequencies $B a_+$ and $B a_-$, respectively. This leads to the stated form of the eigenvalues. \hfill \Box
According to Lemma 10 and a simple computation, (5) with $\rho_2 = 1$ is equivalent to

$$\sum_{k,l \geq 0} \left( \mu - B a_+ \left( \frac{\omega_1}{B}, \frac{\omega_2}{B} \right) (2k + 1) - B a_- \left( \frac{\omega_1}{B}, \frac{\omega_2}{B} \right) (2l + 1) \right)_+ \leq \frac{B}{4\omega_1 \omega_2} \sum_{m \geq 0} (\mu - (2m + 1)B)_+^2,$$

with $a_\pm$ given by (17). Setting $\Lambda = \mu / B$, $\sigma_j = \omega_j / B$ and $a_\pm = a_\pm(\sigma)$ and substituting (18) we can rewrite the desired inequality as

$$\sum_{k,l \geq 0} (\Lambda - a_+ (2k + 1) - a_- (2l + 1))_+ \leq \frac{1}{4a_- a_+} \sum_{m \geq 0} (\Lambda - (2m + 1))_+^2,$$

and this is what we shall prove.

3.2. TWO INEQUALITIES FOR CONVEX FUNCTIONS

For the proof of (19) we shall need

**Lemma 12.** Let $\phi$ be a non-negative convex function on $(0, \infty)$ such that $\int_0^\infty \phi(t) \, dt$ exists. Then

$$\sum_{k=0}^\infty \phi(k + \frac{1}{2}) \leq \int_0^\infty \phi(t) \, dt.$$

**Proof.** Indeed, by the mean-value property of convex functions $\phi(k + \frac{1}{2}) \leq \int_k^{k+1} \phi(t) \, dt$ for each $k$. Now sum over $k$. \hfill $\Box$

**Remark 13.** The proof also shows that $\sum_{k=0}^{K-1} \phi(k + \frac{1}{2}) \leq \int_0^K \phi(t) \, dt$ for each integer $K$. This observation will be useful later.

The inequality from Lemma 12 is sufficient to prove a sharp Lieb-Thirring inequality in the non-magnetic case, but for the proof of our Theorem 2 we need a more subtle fact about convex functions. We note that by the previous lemma $h \sum_{k=0}^\infty \phi(h(k + \frac{1}{2})) \leq \int_0^\infty \phi(t) \, dt$ for any $h > 0$. Moreover, $h \sum_{k=0}^\infty \phi(h(k + \frac{1}{2})) \to \int_0^\infty \phi(t) \, dt$ as $h \to 0$ by the definition of the Riemann integral. The key for proving our sharp result is that, for a certain subclass of convex functions, this limit is approached monotonically. More precisely, one has

**Lemma 14.** Let $\phi$ be a non-negative convex function on $(0, \infty)$ such that $\int_0^\infty \phi(t) \, dt$ exists. Assume that $\phi$ is differentiable and that $\phi'$ is concave. Then the sum

$$\sum_{k=0}^\infty \phi(k + \frac{1}{2}) \leq \int_0^\infty \phi(t) \, dt.$$
\[ h \sum_{k=0}^{\infty} \phi \left( h \left( k + \frac{1}{2} \right) \right) \]

is decreasing in the parameter \( h > 0 \).

We emphasize that without assumptions on \( \phi' \) the inequality

\[ \sum_{k=0}^{\infty} \phi \left( k + \frac{1}{2} \right) \leq h \sum_{k=0}^{\infty} \phi \left( h \left( k + \frac{1}{2} \right) \right) \]

is not true for all \( h < 1 \). Indeed, take for instance \( \phi(t) = (1-t)_+ \) and \( h \geq 2/3 \).

In the proof of this lemma we shall make use of the following well-known fact about convex functions: If \( \psi \) is a non-negative convex function on \((0, \infty)\) such that \( \int_0^\infty \psi(t) \, dt \) exists, then \( \psi(t) = \int_0^\infty (T - t)_+ \, d\mu(T) \) for some non-negative measure \( \mu \). Indeed, it is known that the left-sided derivative \( \partial_- \psi \) exists everywhere on \((0, \infty)\) and satisfies \( \psi(b) - \psi(a) = \int_a^b \partial_- \psi(t) \, dt \) for \( 0 < a < b < \infty \). Moreover, \( \partial_- \psi \) is increasing and left-continuous, and therefore there is a non-negative measure \( \mu \) such that \( \partial_- \psi(b) - \partial_- \psi(a) = \mu([a, b)) \). Since \( \lim_{t \to \infty} \psi(t) = \lim_{t \to \infty} \partial_- \psi(t) = 0 \), we have by Fubini's theorem

\[ \psi(t) = -\int_t^\infty \partial_- \psi(a) \, da = \int_t^\infty \left( \int_a^\infty \chi_{[a, \infty)}(T) \, d\mu(T) \right) \, da = \int_0^\infty (T - t)_+ \, d\mu(T), \]

as claimed.

**Proof.** According to the fact recalled above (applied to \( \psi = -\phi' \)) we have \( \phi(t) = \int_0^\infty (T - t)_+ \, d\mu(T) \) for a non-negative measure \( \mu \). Hence it suffices to prove the lemma for \( \phi(t) = (T - t)_+^2 \) with \( T > 0 \). We have to prove that

\[ \sum_{k=0}^{\infty} (\phi(h(k + \frac{1}{2})) + h(k + \frac{1}{2}) \phi'(h(k + \frac{1}{2}))) \leq 0, \]

which for our \( \phi \) reads

\[ \sum_{k=0}^{\infty} ((S - 2k - 1)_+^2 - 2(2k + 1)(S - 2k - 1)_+) \leq 0, \]

with \( S = 2T / h \). Choose \( K \in \mathbb{N}_0 \) such that \( 2K - 1 \leq S < 2K + 1 \). Then the left side above equals

\[ \sum_{k=0}^{K-1} ((S - 2k - 1)_+^2 - 2(2k + 1)(S - 2k - 1)) = \sum_{k=0}^{K-1} (S^2 - 4S(2k + 1) + 3(2k + 1)^2) = K(S^2 - 4SK + (2K - 1)(2K + 1)) = K(S - 2K + 1)(S - 2K - 1). \]

This is clearly non-positive for \( 2K - 1 \leq S < 2K + 1 \), thus proving the claim. \( \square \)
3.3. PROOF OF THEOREM 2

We have to prove (19). By Lemma 12 for any \( k \)
\[
\sum_{l \geq 0} (\Lambda - a_+ (2k + 1) - a_- (2l + 1))_+ \leq \int_0^\infty (\Lambda - a_+ (2k + 1) - 2a_- t)_+ \, dt
\]
\[
= \frac{1}{4a_-} (\Lambda - a_+ (2k + 1))^2_+.
\]
A simple computation shows that \( a_+ = a_+(\sigma) \geq 1 \), and hence by Lemma 14
\[
a_+ \sum_{k \geq 0} (\Lambda - a_+ (2k + 1))^2_+ \leq \sum_{k \geq 0} (\Lambda - (2k + 1))^2_+.
\]
The previous two inequalities imply the desired (19).

3.4. PROOF OF PROPOSITION 3

Given \( 0 \leq \gamma < 1 \), we want to find \( \omega_1 = \omega_2 \) and \( B \) such that the reverse inequality (7) holds. We may assume \( \gamma > 0 \) in the following. (The case \( \gamma = 0 \) can be treated similarly, or one may use the argument of Aizenman and Lieb mentioned in the introduction to conclude that a counterexample for \( \gamma = \gamma_0 \) implies one for all \( \gamma < \gamma_0 \).)

By the same computation that lead to (19) we see that (7) can be written as:
\[
\sum_{k,l \geq 0} (\Lambda - a_+ (2k + 1) - a_- (2l + 1))^{\gamma}_+ > \frac{1}{2(\gamma + 1)a_- \sum_{m \geq 0} (\Lambda - (2m + 1))^{\gamma + 1}_+}
\]
with \( \Lambda = \mu / B, \sigma_j = \omega_j / B \) and \( a_\pm = a_\pm(\sigma) \). We will let \( \omega_1 = \omega_2 \) and use the notation \( t = \sigma^2 \). One can show that \( a_+ = 1 + t + O(t^2) \) and \( a_- = t + O(t^2) \) as \( t \to 0^+ \).

We now choose \( \Lambda = 3 \) and recall that \( a_+ = a_+(\sigma) \geq 1 \). This gives us the inequality
\[
2(\gamma + 1)a_- a_+ \sum_{l \geq 0} (3 - a_+ - a_- (2l + 1))^{\gamma}_+ - 2^{\gamma + 1} > 0,
\]
which may be written as
\[
(\gamma + 1)a_-^{\gamma + 1} a_+ \sum_{l \geq 0} (x - l)^{\gamma}_+ - 1 > 0
\]
with \( x = (3 - a_+ - a_-) / (2a_-) \). Since \( x = t^{-1}(1 + O(t)) \) as \( t \to 0^+ \), we may choose \( \sigma \) so that \( x \) is an integer. In this case we may use the concavity of \( \gamma^\gamma \) and Remark 13 to bound
\[
\sum_{l \geq 0} (x - l)^{\gamma}_+ = \sum_{l=1}^x l^{\gamma} \geq \int_{1/2}^{x+1/2} t^{\gamma} \, dt = \frac{1}{\gamma + 1} ((x + 1/2)^{\gamma + 1} - (1/2)^{\gamma + 1}).
\]
This shows that
\[
(y + 1)a_+^{\gamma + 1} a_+ \sum_{l \geq 0} (x - l)^\gamma \geq a_+ ((a_- x + a_-/2)^{\gamma + 1} - (a_-/2)^{\gamma + 1})
\]
\[
= a_+ (((3 - a_-)/2)^{\gamma + 1} - (a_-/2)^{\gamma + 1})
\]
\[
= (1 + t + O(t^2))((1 - t/2 + O(t^2))^{\gamma + 1} + O(t^{\gamma + 1})
\]
\[
= 1 + \frac{1 - \gamma}{2} t + O(t^{\gamma + 1})
\]

Since this is strictly larger than 1 for sufficiently small \(t\), we have proved our claim.

Acknowledgements

R. Olofsson acknowledges support through the Swedish Research Council.

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