Regularization of multi-soliton form factors in sine-Gordon model

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Abstract
A general and systematic regularization is developed for the exact solitonic form factors of exponential operators in the (1+1)-dimensional sine-Gordon model by analytical continuation of their integral representations. The procedure is implemented in Mathematica. Test results are shown for four- and six-soliton form factors.

1 Introduction
Form factors (matrix elements of local operators) are important quantities in quantum field theories. It is a remarkable feature of certain two-dimensional field theories (integrable models), that their S-matrices can be obtained exactly in the framework of factorized scattering theory [1, 2]. Furthermore, in integrable models there is a rather restrictive set of equations satisfied by the form factors (that is the form factor axioms [3, 2]), which makes it possible in many cases to obtain them exactly as well. For instance, in the sine-Gordon model all form factors of exponential operators are known [3, 2]. The spectrum of the sine-Gordon model consists of a soliton-antisoliton doublet and their bound states, called “breathers”. While the breather-breather form factors can be given explicitly (see e.g. [4]), the solitonic ones, in general, are only known in terms of some highly non-trivial integral representations. In addition, the integrals converge in a limited domain of the parameters. In this paper we give a regularization procedure to calculate the solitonic form factors in the sine-Gordon model for arbitrary choice of the parameters. The regularized multi-soliton form factors could then be used to obtain correlation functions of direct physical interest, e.g. in condensed matter physics [6].

The outline of the paper is as follows. In Section 2 the sine-Gordon model along with its exact form factors are reviewed and based on [4] integral representations for the form factors of exponential operators are given. Section 3 is devoted to the analysis of a certain function which appears in the integral representations. Giving this function’s asymptotic series and identifying its poles make it possible to analytically continue the integral representations. In Section 4 explicit formulae are provided for the four-soliton form factors. Section 5 is devoted to discussion of test results, while Section 6 is left for conclusions and outlook.

2 Form factors in the sine-Gordon model

2.1 Definitions, S-matrix and the form factor axioms
The sine-Gordon model is defined by the classical Lagrangian

\[ L = \int_{-\infty}^{\infty} dx \left( \frac{1}{2} \partial_\nu \phi \partial^\nu \phi + \frac{m_0^2}{\beta^2} \cos (\beta \phi) \right) \]  \hspace{1cm} (2.1)

Define the parameter

\[ \xi = \frac{\beta^2}{1 - \beta^2} \]  \hspace{1cm} (2.2)

which is relevant in the low-energy description of the theory. The spectrum of the quantum theory contains the soliton-antisoliton doublet and their bound states, the “breathers”. The number of breather states \((B_1, B_2, ..., B_N)\) is bounded, there are \(N = \left\lfloor \frac{1}{\xi} \right\rfloor\) of them. For our purposes it is enough to consider only the solitonic particles of the spectrum, indexed in the following with \(\pm\) (soliton-antisoliton).

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2.1.1 S-matrix

The S-matrix for the soliton-antisoliton doublet reads

\[
\begin{pmatrix}
S & S_T & S_R \\
S_T & S & S_R \\
S_R & S_R & S
\end{pmatrix},
\]

(2.3)

with the non-zero elements

\[
S_{++}^{++}(\Theta) = S_{--}^{--}(\Theta) = S(\Theta),
\]

(2.4)

\[
S_{+-}^{+-}(\Theta) = S_{-+}^{-+}(\Theta) = S_T(\Theta) = S(\Theta) \frac{\sinh^{\Theta}}{\sinh^{\xi}},
\]

(2.5)

\[
S_{++}^{++}(\Theta) = S_{-+}^{-+}(\Theta) = S_R(\Theta) = S(\Theta) \frac{\sinh^{\Theta}}{\sinh^{\xi}},
\]

(2.6)

where

\[
S(\Theta) = (-1)^N \prod_{k=1}^{N} \frac{ik\pi\xi + \Theta}{ik\pi\xi - \Theta},
\]

\[
\times \exp\left[ -i \int_{0}^{\infty} \frac{dt}{t} \sin(\Theta t) \right]
\]

\[
\times \frac{2 \sinh^{\pi(1-\xi)t} e^{-N\pi\xi t} + (e^{-N\pi\xi t} - 1) \left( e^{\frac{\pi(1-\xi)t}{2}} + e^{-\frac{\pi(1-\xi)t}{2}} \right)}{2 \sinh^{\frac{\pi\xi}{2}}} ,
\]

(2.7)

which is independent of the integer \( N \), however the integral converges in a larger domain of \( \mathbb{C} \) for \( N > 0 \).

2.1.2 Form factors

Consider the matrix elements

\[
\mathcal{F}_{a_1,\ldots,a_n}^{b_1,\ldots,b_m} (\Theta_1,\ldots,\Theta_m | \Theta_1,\ldots,\Theta_n) = \text{out} \langle A_{b_m} (\Theta_n) \ldots A_{b_1} (\Theta_1) | O | A_{a_1} (\Theta_1) \ldots A_{a_n} (\Theta_n) \rangle_{in}
\]

(2.8)

of the local, hermitian operator \( O \) between asymptotic states. The form factors are defined by

\[
\mathcal{F}_{a_1,\ldots,a_n} (\Theta_1,\ldots,\Theta_n) = \langle 0 | O | A_{a_1} (\Theta_1) \ldots A_{a_n} (\Theta_n) \rangle_{in},
\]

(2.9)

as the matrix elements of the operator between the vacuum and an \( n \)-particle state. Crossing symmetry implies

\[
\mathcal{F}_{a_1,\ldots,a_n}^{b_1,\ldots,b_m} (\Theta_1,\ldots,\Theta_m | \Theta_1,\ldots,\Theta_n) = \mathcal{F}_{a_1,\ldots,a_n} (-b_1,\ldots,-b_m | \Theta_1,\ldots,\Theta_n, \Theta_1' + i\pi, \ldots, \Theta_m' + i\pi),
\]

(2.10)

which is understood as an analytic continuation. The form factors can be reconstructed based on the following axioms.

1. Analyticity and the physical matrix elements. \( \mathcal{F}_{a_1,\ldots,a_n} (\Theta_1,\ldots,\Theta_n) \) is analytic in the variables \( \Theta_i - \Theta_j \) inside the physical strip \( 0 < \text{Im} \Theta < 2\pi \) except for simple poles. It is the physical matrix element when all \( \Theta_i \) are real and ordered as \( \Theta_1 < \Theta_2 < \ldots < \Theta_n \).

2. Relativistic invariance. The form factors satisfy

\[
\mathcal{F}_{a_1,\ldots,a_n} (\Theta_1 + z,\ldots,\Theta_n + z) = e^{zS(O)} \mathcal{F}_{a_1,\ldots,a_n} (\Theta_1,\ldots,\Theta_n),
\]

(2.11)

where \( S(O) \) is the spin of the operator \( O \).

3. Watson’s theorem. The following symmetry properties are satisfied

\[
\mathcal{F}_{a_1,\ldots,a_{j+1},\ldots,a_n} (\Theta_1,\ldots,\Theta_j, \Theta_{j+1},\ldots,\Theta_n) = S_{a_{j+1},a_j}^{a_j,a_{j+1}} (\Theta_{j+1} - \Theta_j) \mathcal{F}_{a_1,\ldots,a_{j+1-1},\ldots,a_n} (\Theta_1,\ldots,\Theta_{j+1-1}, \Theta_{j+1},\ldots,\Theta_n),
\]

(2.12)

\[
\mathcal{F}_{a_1,\ldots,a_n} (\Theta_1,\ldots,\Theta_n + 2\pi i) = e^{2\pi i \omega(O,\Psi)} \mathcal{F}_{a_1,\ldots,a_n} (\Theta_n,\Theta_1,\ldots,\Theta_{n-1}),
\]

(2.13)

where the latter is understood as an analytic continuation and \( \omega(O,\Psi) \) is the mutual non-locality index of the operator \( O \) and \( \Psi \), the “elementary” field, upon which the whole operator product algebra can be constructed.
where $\sum$ can be represented by $\langle\langle \gamma \rangle\rangle$. Since $2.2$ Integral representations of multi-soliton form factors

In [4] it is proposed that the $2n$-particle form factors of the exponential operator $e^{ia\varphi}$ in the sine-Gordon model can be represented by

$$F_{\sigma_1\ldots\sigma_{2n}}(\Theta_1, \ldots ,\Theta_{2n}) = \langle 0|e^{ia\varphi}|A_{\sigma_{2n}}(\Theta_{2n}) \ldots A_{\sigma_1}(\Theta_1)\rangle$$

$$= \mathcal{G}_a(\langle Z_{\sigma_{2n}}(\Theta_{2n}) \ldots Z_{\sigma_1}(\Theta_1)\rangle) = \mathcal{G}_a F_{\sigma_1\ldots\sigma_{2n}}(\Theta_1, \ldots ,\Theta_{2n})$$

(2.15)

where $\sum_{i=1}^{2n} \sigma_i = 0$ because of charge conservation and $\mathcal{G}_a$ is the vacuum expectation value of the exponential operator $\mathcal{H} \mathcal{H}$. The operators $Z_\pm(\Theta)$ are defined by

$$Z_+(\Theta) = \sqrt{i\frac{C_2}{4C_1}} e^{\frac{a\varphi}{\pi}} e^{i\phi(\Theta)},$$

$$Z_-(\Theta) = \sqrt{i\frac{C_2}{4C_1}} e^{-\frac{a\varphi}{\pi}} \left\{ e^{\frac{a\varphi}{\pi}} \int_{C^+} \frac{d\gamma}{2\pi e^{\frac{a\varphi}{\pi}}} e^{(\gamma-\Theta)e^{-i\phi(\Theta)}} e^{i\phi(\Theta)} - e^{-\frac{a\varphi}{\pi}} \int_{C^-} \frac{d\gamma}{2\pi e^{\frac{a\varphi}{\pi}}} e^{(\gamma-\Theta)e^{-i\phi(\Theta)}} e^{-i\phi(\Theta)} \right\}.$$ 

(2.17)

Since $\phi(\Theta)$ and $\bar{\phi}(\gamma)$ are free fields the averaging $\langle\ldots\rangle$ is performed by the multiplicative Wick’s theorem, using

$$\langle e^{i\phi(\Theta_2)}e^{i\bar{\phi}(\Theta_1)}\rangle = G(\Theta_1 - \Theta_2),$$

(2.18)

$$\langle e^{i\bar{\phi}(\Theta_2)}e^{-i\phi(\Theta_1)}\rangle = W(\Theta_1 - \Theta_2) = \frac{1}{G(\Theta_1 - \Theta_2 - \frac{\pi i}{\xi}) G(\Theta_1 - \Theta_2 + \frac{\pi i}{\xi})},$$

(2.19)

$$\langle e^{-i\bar{\phi}(\Theta_2)}e^{-i\phi(\Theta_1)}\rangle = \bar{G}(\Theta_1 - \Theta_2) = \frac{1}{W(\Theta_1 - \Theta_2 - \frac{\pi i}{\xi}) W(\Theta_1 - \Theta_2 + \frac{\pi i}{\xi})}.$$ 

(2.20)

The appearing functions and constants are as follows.

$$G(\Theta) = iC_1 \sinh\left(\frac{\Theta}{2}\right) \exp\left\{ \int_0^{\infty} \frac{dt \sinh^2 t}{t} \frac{t}{\sin(2t) \cosh(t) \sinh(t)} \right\},$$

$$W(\Theta) = -\frac{2}{\cosh(\Theta)} \exp\left\{ -2 \int_0^{\infty} \frac{dt \sinh^2 t}{t} \frac{t}{\sinh(2t) \sinh(t)} \right\},$$

(2.21)

$$\bar{G}(\Theta) = -\frac{C_2}{4\xi \sin\left(\frac{\Theta + i\pi}{\xi}\right)} \sinh(\Theta),$$

$$C_1 = \exp\left\{ -\int_0^{\infty} \frac{dt \sinh^2 (\frac{\xi t}{2})}{\sinh(2t) \cosh(t) \sinh(t)} t \right\} = G(-i\pi),$$

(2.22)

$$C_2 = \exp\left\{ 4 \int_0^{\infty} \frac{dt \sinh^2 (\frac{\xi t}{2})}{\sinh(2t) \sinh(t)} t \right\} = \frac{4}{W \left( \frac{\pi i}{\xi} \right) \xi \sin \left( \frac{\pi}{\xi} \right)^2}.$$ 

(2.25)
The integration contours appearing in the consequent expressions for the form factors are such that the "principal poles" of the $W$-functions are always between the contour and the real line. (We define the "principal pole" of $W(\Theta)$ as the pole located at $\Theta = -\frac{i\pi}{2}$.)

For the two-particle form factors it is only necessary to evaluate Eq. (2.15) for two $Z$ operators. Let $A = -\left(\frac{1}{3} + \frac{2\xi}{3}\right)$, then the result is

$$
\langle (Z_+ (\Theta_2) Z_- (\Theta_1)) \rangle = \frac{iC_2}{4c_1} e^{\frac{\pi}{2}(\Theta_2 - \Theta_1)} G(\Theta_1 - \Theta_2) e^{-A\Theta_1}
$$

$$
\times \left\{ e^{\frac{\pi}{2}B} \int \frac{d\gamma}{2\pi} e^{A\gamma} W(\gamma - \Theta_2) W(\Theta_1 - \gamma) - e^{-\frac{\pi}{2}B} \int \frac{d\gamma}{2\pi} e^{A\gamma} W(\gamma - \Theta_2) W(\gamma - \Theta_1) \right\},
$$

(2.26)

$$
\langle (Z_- (\Theta_2) Z_+ (\Theta_1)) \rangle = \frac{iC_2}{4c_1} e^{\frac{\pi}{2}(\Theta_1 - \Theta_2)} G(\Theta_1 - \Theta_2) e^{-A\Theta_2}
$$

$$
\times \left\{ e^{\frac{\pi}{2}B} \int \frac{d\gamma}{2\pi} e^{A\gamma} W(\Theta_2 - \gamma) W(\Theta_1 - \gamma) - e^{-\frac{\pi}{2}B} \int \frac{d\gamma}{2\pi} e^{A\gamma} W(\gamma - \Theta_2) W(\Theta_1 - \gamma) \right\},
$$

(2.27)

The four-particle form factors can also be obtained through evaluating Eq. (2.15) with the result

$$
\langle (Z_{\sigma_4} (\Theta_4) Z_{\sigma_3} (\Theta_3) Z_{\sigma_2} (\Theta_2) Z_{\sigma_1} (\Theta_1)) \rangle = \frac{\xi C_3}{1024 \pi^4 c_1^3} G(\Theta_4 - \Theta_3) G(\Theta_2 - \Theta_4) G(\Theta_1 - \Theta_3)
$$

$$
\times G(\Theta_2 - \Theta_3) G(\Theta_1 - \Theta_4) G(\Theta_1 - \Theta_2) J_{\sigma_1 \sigma_2 \sigma_3 \sigma_4},
$$

(2.28)

where

$$
J_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} = e^{\frac{\pi}{2} \sum_{i=1}^{4} \sigma_i} e^{-A \sum_{i=1}^{4} \Theta_i} I_{\sigma_1 \sigma_2 \sigma_3 \sigma_4}
$$

(2.29)

and $I_{\sigma_1 \sigma_2 \sigma_3 \sigma_4}$’s are given by

$$
I_{+++} = e^{\frac{i\pi}{4}} \mathbf{P}((I_{22}, I_{31}) - P(I_{22}, I_{40}) - P(I_{31}, I_{31}) + e^{-\frac{i\pi}{4}} P(I_{31}, I_{40}),
$$

(2.30)

$$
I_{++-} = e^{\frac{i\pi}{4}} \mathbf{P}((I_{13}, I_{31}) - P(I_{22}, I_{31}) - P(I_{13}, I_{40}) + e^{-\frac{i\pi}{4}} P(I_{22}, I_{40}),
$$

(2.31)

$$
I_{+-+} = e^{\frac{i\pi}{4}} \mathbf{P}((I_{04}, I_{31}) - P(I_{13}, I_{31}) - P(I_{04}, I_{40}) + e^{-\frac{i\pi}{4}} P(I_{13}, I_{40}),
$$

(2.32)

$$
I_{++-} = e^{\frac{i\pi}{4}} \mathbf{P}((I_{04}, I_{13}) - P(I_{13}, I_{13}) - P(I_{04}, I_{22}) + e^{-\frac{i\pi}{4}} P(I_{13}, I_{22}),
$$

(2.33)

$$
I_{+-+} = e^{\frac{i\pi}{4}} \mathbf{P}((I_{04}, I_{22}) - P(I_{13}, I_{22}) - P(I_{04}, I_{31}) + e^{-\frac{i\pi}{4}} P(I_{13}, I_{31}),
$$

(2.34)

$$
I_{++-} = e^{\frac{i\pi}{4}} \mathbf{P}((I_{13}, I_{22}) - P(I_{22}, I_{22}) - P(I_{13}, I_{31}) + e^{-\frac{i\pi}{4}} P(I_{22}, I_{31}).
$$

(2.35)

The integrals $I_{ij}$ have four components, $I_{ij, k} k = 1, \ldots, 4$ and the operation $\mathbf{P}$ is defined by

$$
\mathbf{P}(a, b) = e^{\frac{i\pi}{4}} (a_1 b_1 - a_2 b_2) - e^{-\frac{i\pi}{4}} (a_3 b_3 - a_4 b_4).
$$

(2.36)

$I_{ij, k}$’s read

$$
I_{04, k} = \int e^{(A + \alpha_1)x} W(\Theta_4 - x) W(\Theta_3 - x) W(\Theta_2 - x) W(\Theta_1 - x) dx,
$$

(2.37)

$$
I_{13, k} = \int e^{(A + \alpha_1)x} W(x - \Theta_4) W(\Theta_3 - x) W(\Theta_2 - x) W(\Theta_1 - x) dx,
$$

(2.38)

$$
I_{22, k} = \int e^{(A + \alpha_1)x} W(x - \Theta_4) W(x - \Theta_3) W(\Theta_2 - x) W(\Theta_1 - x) dx,
$$

(2.39)

$$
I_{31, k} = \int e^{(A + \alpha_1)x} W(x - \Theta_4) W(x - \Theta_3) W(x - \Theta_2) W(\Theta_1 - x) dx,
$$

(2.40)

$$
I_{40, k} = \int e^{(A + \alpha_1)x} W(x - \Theta_4) W(x - \Theta_3) W(x - \Theta_2) W(x - \Theta_1) dx
$$

(2.41)

with $\alpha_1 = -1 - \frac{1}{3}, \alpha_2 = 1 - \frac{2}{3}, \alpha_3 = -1 + \frac{1}{3}, \alpha_4 = +1 + \frac{1}{3}$ coming from writing $G(x)$ as the sum of four exponentials; the contours are as before.
In general, the $2n$-particle form factor is realized as

$$
\langle \prod_{i=1}^{n} Z_{+} (\Theta_{i+n}) \prod_{i=1}^{n} Z_{-} (\Theta_{i}) \rangle = \left( \frac{iC_{2}}{8\pi C_{1}} \right)^{n} e^{\frac{5}{2} \sum_{i=1}^{n} (\Theta_{i+n} - \Theta_{i})} e^{-A \sum_{j=1}^{n} \Theta_{j}} \prod_{j>\ell} G (\Theta_{j} - \Theta_{\ell})
$$

$$
\times \int \left\{ \prod_{i=1}^{n} d\gamma_{i} e^{A \gamma_{i}} \left( e^{\frac{5}{W} \gamma_{i}} W (\Theta_{i} - \gamma_{i}) - e^{-\frac{5}{W} \gamma_{i}} W (\gamma_{i} - \Theta_{i}) \right) \right\} \prod_{j>\ell} G (\gamma_{j} - \gamma_{\ell})
$$

$$
(2.42)
$$

The last product gives the numerical factor $\left( -\frac{C_{2}}{16} \right)^{\frac{n(n-1)}{2}}$ and the sum of $4^{\frac{n(n-1)}{2}}$ exponentials containing $\gamma_{i}$'s. All in all, we have $(n+1)$ combinations of the $W$-functions, which must be integrated over with some exponential factors. Note that the exponential factors do not alter the structure (e.g. the poles) of the integrands. The other kinds of $2n$-particle form factors can be obtained e.g. through the symmetry properties of form factors (Watson’s theorem).

The problem with such integrals is that they diverge for either

$$
\text{Re } a > \beta + \frac{\beta}{2}
$$

or

$$
\text{Re } a < -\frac{1}{\beta} + \beta + \frac{\beta}{2}.
$$

(2.43)

(2.44)

For such choices of $a$ the integrands have essential singularities at $\text{Re } x \to \pm \infty$. In the next section we prove that $W (x)$ has an asymptotic series in exponentials of $x$, therefore the divergent integrals can always be analytically continued to obtain a finite result. Our strategy is to first deform the integration contours to the real line, then extract the divergent terms of the integrands in the form of exponentials and give their contributions exactly by the analytic continuation rules

$$
\int_{-\infty}^{0} \exp (\alpha x) dx \equiv \frac{1}{\alpha}, \quad \alpha \in \mathbb{C},
$$

$$
\int_{0}^{+\infty} \exp (\alpha x) dx \equiv -\frac{1}{\alpha}, \quad \alpha \in \mathbb{C}.
$$

(2.45)

(2.46)

Then if the integral is expected to be analytic and $\int_{0}^{\infty} f (x) dx$ exists we have

$$
\int_{-\infty}^{\infty} f (x) dx \equiv \sum_{i} \frac{a_{i}}{\alpha_{i}} + \int_{-\infty}^{0} f (x) - [f] (x) dx + \int_{0}^{\infty} f (x) dx
$$

(2.47)

for some $f (t)$ admitting an asymptotic expansion in exponentials,

$$
f (x) = \sum_{\alpha_{i} > 0} a_{i} e^{\alpha_{i} x} + O (e^{\alpha_{i} x}) \equiv [f] (x) + O (e^{\alpha_{i} x}), \quad x \to -\infty, \quad \alpha_{+} > 0.
$$

(2.48)

The previous equation defines the function $[f] (x)$. The case when $\int_{0}^{\infty} f (x) dx = \infty$ is similar.

It should be noted that for some combination of the parameters, the analytic continuation may still produce an infinite result, that is in the case $\alpha_{i} = 0$ for some $i$. This happens e.g. for the integral $I_{22.1}$ when $\frac{2}{3} = \frac{1}{2}$. These infinities, however, must and indeed do cancel out from our end results, the form factors, therefore the $\alpha_{i} = 0$ terms in the asymptotic series should be omitted before making the analytic continuation prescribed in (2.47).

3 Analysis of the $W$-function

3.1 Asymptotic series

The function $W (x)$ is given by (2.22). The asymptotic series of $\cosh (x)^{-1}$ reads as

$$
\cosh (x)^{-1} = 2e^{-sx} \left( 1 - e^{-2sx} + e^{-4sx} + \ldots \right), \quad \text{Re } x \to s \cdot \infty,
$$

(3.1)
where \( s \equiv \text{sign} \Re x \) was introduced for convenience. The exponent of the remaining part of \( W(x) \) can be rewritten as

\[
\int_0^\infty dt \frac{\sinh(\xi - 1) t}{\sinh(\xi t) \sinh(2t)} \left( 1 - \cosh(2t) \cos \left( \frac{2tx}{\pi} \right) + i \sinh(2t) \sin \left( \frac{2tx}{\pi} \right) \right) .
\]  

(3.2)

Differentiate the previous formula with respect to \( x \) and obtain

\[
\frac{2}{\pi} \int_0^\infty dt \frac{\sinh(\xi - 1) t}{\sinh(\xi t) \sinh(2t)} \left( \cosh(2t) \sin \left( \frac{2tx}{\pi} \right) + i \sinh(2t) \cos \left( \frac{2tx}{\pi} \right) \right) = \frac{i}{\pi} \left( \int_{-\infty}^\infty \frac{\sinh(\xi - 1) t}{\sinh(\xi t)} (1 - \coth(2t))e^{itx} dt \right) .
\]  

(3.3)

The asymptotic series of the Fourier integrals was first obtained by \([8]\), where an analog to Watson’s lemma for Laplace transforms was discussed. Given a function \( q(t) \) with the asymptotic series near \( t = 0 \):

\[
q(t) = \sum_{n=0}^\infty b_n t^{n+\lambda-1}
\]  

(3.5)

with some \( 0 < \lambda \leq 1 \), the asymptotic series of

\[
F(x) = \int_0^\infty q(t)e^{itx} dt
\]  

(3.6)

as \( x \to \infty \) is given by

\[
F(x) = \sum_{n=0}^\infty b_n e^{\mp \xi \pi (n+\lambda) \Gamma (n+\lambda) x^{-n-\lambda} + O \left( e^{-\mu x} \right)} , \quad x \to \infty ,
\]  

(3.7)

where \( O (e^{-\mu x}) \) denotes corrections “beyond all orders”, i.e. exponentially small terms \( (\mu > 0) \). Applying this construction to the integrals occurring in Eq. (3.3) we get

\[
\frac{2}{\pi} \int_0^\infty dt \frac{\sinh(\xi - 1) t}{\sinh(\xi t) \coth(2t)} \sin \left( \frac{2tx}{\pi} \right) = \frac{2 \xi - 1}{2 \xi} \lim_{\lambda \to 0} \text{Im} \left( e^{\mp \frac{\pi \xi}{\lambda}} \Gamma (\lambda) + O \left( e^{-\mu x} \right) \right) \]  

(3.8)

and

\[
\frac{2}{\pi} \int_0^\infty dt \frac{\sinh(\xi - 1) t}{\sinh(\xi t) \cos \left( \frac{2tx}{\pi} \right) } = O \left( e^{-\mu x} \right)
\]  

(3.9)

considering that \( q(t) \) is odd in the first integral and even in the second which implies that all but the first term of the first integral disappears because of the \( \text{Im} / \text{Re} \) operation, respectively. For simplicity it was assumed, that \( x \) is real. It is easy to check our statements remain true if this condition is relaxed.

To find the exponentially small terms the integral is evaluated by the residue theorem which yields only the residues times \( 2\pi i \) since the integrands are of small enough order on the half circles \( C^+_R = \{ z : \pm \text{Im} z > 0, |z| = R \} \) as \( R \to \infty \) (where \( C^-_R \) is associated with \( \Re x > 0 \) while \( C^+_R \) with \( \Re x < 0 \)). The result is

\[
\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \cot \left( \frac{\pi k (2k+1)}{2} \right) e^{-(2k+1)x} + \sum_{k=1}^{\infty} \frac{(-1)^k}{k} e^{-2kx} - \frac{2}{\xi} \sum_{k=1}^{\infty} \sin \left( \frac{\pi k}{\xi} \right) \cot \left( \frac{2\pi k}{\xi} \right) e^{-\frac{\pi k}{\xi} x} + \frac{2 \xi}{\pi} \sum_{k=1}^{\infty} (-1)^{k+1} \sin \left( \frac{(\xi - 1) \pi k}{\xi} \right) e^{-\frac{\pi k}{\xi} x} .
\]  

(3.10)

When integrated with respect to \( x \) one gets the exponent of the \( W \)-function:

\[
\frac{\xi - 1}{2 \xi} x + \sum_{k=1}^{\infty} \frac{1}{k} \sin \left( \frac{\pi k}{\xi} \right) \left[ \cot \left( \frac{2\pi k}{\xi} \right) - i s \right] e^{-\frac{\pi k}{\xi} x} - \sum_{k=1}^{\infty} \frac{1}{k} \left\{ \left( \begin{array}{c} k+1 \cot \left( \frac{\pi k}{2} \right) \\ k \end{array} \right) , \quad \text{for odd } k \right\} e^{-kx} + C_s .
\]  

The integration constant, \( C_s \) is determined from the relation

\[
G(x) = \frac{1}{W \left( x + \frac{i \pi}{2} \right) W \left( x - \frac{i \pi}{2} \right)}
\]  

(3.11)

and the explicit form of the function \( G(x) \), that is
\[ G(x) = -\frac{C_2}{4} \xi \sinh \left( \frac{\Theta + i\pi}{\xi} \right) \sinh(\Theta), \]

with the result

\[ -4e^{C_x} = \frac{4i}{\sqrt{C_2}} e^{-\frac{i\pi}{2}x}. \]

Now we are ready to give the asymptotic expansion of \( W(x) \):

\[ W(x) = \frac{4i}{\sqrt{C_2}} e^{-\frac{i\pi}{2}x} \left( \sum_{l=0}^{\infty} (-1)^l e^{-2lx} \right) \prod_{k=1}^{\infty} \left( \sum_{l=0}^{\infty} \frac{a_k}{l!} e^{-\frac{2\pi k}{x}lx} \right) \left( \sum_{l=0}^{\infty} \frac{b_k}{l!} e^{-l\xi lx} \right), \quad \text{Re} \ x \to s \cdot \infty, \]

where the coefficients depend only on \( \xi \) and \( s \equiv \text{sign}(\text{Re} \ x) \) and are expressed as

\[ a_k = \frac{1}{k} \left[ \frac{\cos \left( \frac{2\pi k}{\xi} \right)}{2 \cos \left( \frac{\pi k}{\xi} \right)} - i \sin \left( \frac{\pi k}{\xi} \right) \right] \]

and

\[ b_k = -\frac{1}{k} \left\{ i^{k+1} \cot \left( \frac{\pi k}{2} \right) \right\} \quad \text{for odd } k, \]

\[ b_k = -\frac{1}{k} \left\{ i^k \right\} \quad \text{for even } k. \]

Note, that, strictly speaking, our expansion is limited to the case of irrational \( \xi \) parameters since otherwise the coefficients \( a_k \) and \( b_k \) always become infinite for some \( k \). However, such infinities can be shown to cancel out.

Let \( \xi = \frac{n_1}{n_2} \), where \( n_1 \) and \( n_2 \) are relative primes. We have singular \( a_k \)'s whenever \( \frac{2kn_2}{n_1} = \text{odd} \), which immediately implies \( n_1 = \text{even} \) and \( n_2 = \text{odd} \) and the \( N \)th singular \( a \)-coefficient is indexed by \( \frac{Nn_2}{2} \), where \( N \) is necessarily odd. On the other hand \( b_l \) is singular if \( \frac{n_2}{2l} = \text{even} \) while \( l \) is odd, implying again \( n_1 = \text{even} \) and \( n_2 = \text{odd} \) and the \( N \)th singular \( b \)-coefficient is indexed by \( Nn_2 \). Now in both cases the \( N \)th singular term contribute terms of order \( e^{-Nn_2x} \) to the exponent of \( W(x) \). All that remains is to show that the \( N \)th diverging coefficients cancel each other. To see this let \( \xi = \frac{n_1}{n_2} (1 + \varepsilon) \) or equivalently

\[ n_1 \to n_1 (1 + \varepsilon), \]

\[ n_2 \to n_2 (1 - \varepsilon), \]

resulting in

\[ \text{Re} \ a_{Nn_1/2} = \frac{1}{Nn_1} \frac{\cos (\pi Nn_2/2)}{\cos (\pi Nn_2/2)} \to -\frac{1}{Nn_1} \frac{\cos (\pi Nn_2 (1 - \varepsilon)/2)}{\cos (\pi Nn_2 (1 - \varepsilon)/2)}, \]

\[ b_{Nn_2} = -\frac{1}{Nn_2} \frac{\cos (\pi Nn_1/2)}{\sin (\pi Nn_1/2)} \to -\frac{1}{Nn_2} \frac{\cos (\pi Nn_1 (1 + \varepsilon)/2)}{\sin (\pi Nn_1 (1 + \varepsilon)/2)} \]

which if expanded in \( \varepsilon \) yield

\[ \text{Re} \ a_{Nn_1/2} = b_{Nn_2} + O(\varepsilon) = (-1)^{\frac{n_2+1}{2}} \frac{4}{\pi^2 n_1 n_2} \frac{1}{\varepsilon} + O(\varepsilon), \]

which agrees for \( \varepsilon \to 0 \). Because of the cancellation the following rules can be formulated for rational \( \xi \)'s:

\[ a_k = -\frac{is}{k} \quad \text{for odd } \frac{2k}{\xi}, \]

\[ b_k = 0 \quad \text{for } k \text{ odd and } k\xi = \text{even}. \]

### 3.2 Poles

With the asymptotic expansion at hand the divergences of the integral representations can be readily remedied. However there is another issue with the integrals containing \( W \)-functions. In fact, \( W(x) \) has a number of poles on the line \( \text{Re} \ x = 0 \). When the integration contour is fixed (which is the desired scenario), poles can cross it and one needs to analytically continue the result by adding the residue contributions of the crossing poles. In
the followings we determine the poles of \( W(x) \). The poles of \( W(x) \) are easily extracted from the identity (2.24), but also follows from a similar representation of \( G(x) \), given in \([10]\).

\[
W(x) = -\frac{2}{\cosh x} \prod_{k=1}^{N} \frac{\Gamma \left( 1 + \frac{2k - 2 + i\epsilon}{\xi} \right) \Gamma \left( 1 + \frac{2k - 2 - i\epsilon}{\xi} \right) \Gamma \left( \frac{2k - 1 + 4\xi}{\xi} \right)}{\Gamma \left( 1 + \frac{2k - 2 + i\epsilon}{\xi} \right)^2 \Gamma \left( 2k - 3 - i\epsilon \right) \Gamma \left( \frac{2k - 3 - 4\xi}{\xi} \right)} 
\times \exp \left\{ -2 \int_{0}^{\infty} \frac{dt e^{-4Nt} \sinh^2 t (1 - i\xi) \sinh t (\xi - 1)}{\sinh 2t \sinh \xi t} \right\}.
\]

(3.24)

They originate from the poles of the gamma functions and the roots of \( \cosh x \). It is apparent that \( \text{Re} \, x = 0 \) for every pole. We are interested in the poles of \( W(x - x_0) \) which cross the real line (or the original integration contour) when \( \xi \) is decreased. The previous equation yields two infinite series of poles given by

\[
\begin{align*}
\xi + 2k - \frac{5}{2} + \frac{i \xi_{1,k,n}}{\pi} &= -n \xi, \\
\xi + 2k - \frac{1}{2} - \frac{i \xi_{2,k,n}}{\pi} &= -n \xi,
\end{align*}
\]

\( n \) being a non-negative integer and \( k \) being a positive number. We have the following estimates for the series of poles:

\[
\begin{align*}
\text{Im} \, x_{1,k,n} &> \frac{(4k - 5)\pi}{2}, \\
\text{Im} \, x_{2,k,n} &< \frac{(1 - 4k)\pi}{2}.
\end{align*}
\]

(3.27)

(3.28)

For \( |\text{Im} \, x_0| \leq \frac{3}{2}\pi \) the only poles that can cross the real line are

\[
x_n \equiv x_{1,1,n} = i\pi \left( n\xi - \frac{1}{2} \right), \quad n = 1, 2, \ldots.
\]

(3.29)

and no poles can cross the original contour, which intersects the \( \text{Im} \, x = 0 \) line at \( x - x_0 = -\frac{\pi}{2\sqrt{2}} - \xi, \xi \to 0 \). With reference to the form factors, note that because of Watson’s theorem it is enough to give a calculation method when all the rapidities satisfy \( |\text{Im} \, \Theta_i| \leq \pi \). Thus it is not necessary to analyze further the \( \xi \)-dependent poles of \( W(x - x_0) \), that is covering the case \( |\text{Im} \, x_0| > \pi \).

4 Implementation of the four-soliton form factor formula

As an example of the machinery outlined in Section 2 in this section the implementation of the four-soliton form factors is discussed. Implementing Eq. (2.28) is non-trivial only in the calculation of the integrals \( I_{ij,k} \).

First, by Cauchy’s theorem we deform the integration contour to the real line. For this we need to identify the poles between the real line and the original contour, which consists of the the principal poles and (possibly) several \( \xi \)-dependent poles (given by Eq. (3.29)) of \( W \)-functions. The principal poles give the following contributions to \( I_{22,k} \):

\[
\begin{align*}
P_{1,k} &= -\frac{4}{\pi \sqrt{C_2}} e^{(A + \alpha_b)(\Theta_1 + \frac{i\pi}{2})} W \left( \Theta_1 - \Theta_4 + \frac{i\pi}{2} \right) W \left( \Theta_1 - \Theta_3 + \frac{i\pi}{2} \right) W \left( \Theta_2 - \Theta_1 - \frac{i\pi}{2} \right), \\
P_{2,k} &= -\frac{4}{\pi \sqrt{C_2}} e^{(A + \alpha_b)(\Theta_2 + \frac{i\pi}{2})} W \left( \Theta_2 - \Theta_4 + \frac{i\pi}{2} \right) W \left( \Theta_2 - \Theta_3 + \frac{i\pi}{2} \right) W \left( \Theta_1 - \Theta_2 - \frac{i\pi}{2} \right), \\
P_{3,k} &= -\frac{4}{\pi \sqrt{C_2}} e^{(A + \alpha_b)(\Theta_3 - \frac{i\pi}{2})} W \left( \Theta_3 - \Theta_4 - \frac{i\pi}{2} \right) W \left( \Theta_2 - \Theta_3 + \frac{i\pi}{2} \right) W \left( \Theta_1 - \Theta_3 + \frac{i\pi}{2} \right), \\
P_{4,k} &= -\frac{4}{\pi \sqrt{C_2}} e^{(A + \alpha_b)(\Theta_4 - \frac{i\pi}{2})} W \left( \Theta_1 - \Theta_3 - \frac{i\pi}{2} \right) W \left( \Theta_2 - \Theta_4 + \frac{i\pi}{2} \right) W \left( \Theta_1 - \Theta_4 + \frac{i\pi}{2} \right),
\end{align*}
\]

if \( \text{Im} \, \Theta_1 > -\frac{\pi}{2}, \text{Im} \, \Theta_2 > -\frac{\pi}{2}, \text{Im} \, \Theta_3 < +\frac{\pi}{2}, \text{Im} \, \Theta_4 < +\frac{\pi}{2} \), respectively. The \( \xi \)-dependent poles yield
\[
X_{1,k} = \sum_{n=1}^{N_1} 2\pi ir_n e^{(A+\alpha_k)(\Theta_1-x_n)} W(\Theta_1-\Theta_4-x_n) W(\Theta_1-\Theta_3-x_n) W(\Theta_2-\Theta_1+x_n) \tag{4.5}
\]
\[
X_{2,k} = \sum_{n=1}^{N_2} 2\pi ir_n e^{(A+\alpha_k)(\Theta_2-x_n)} W(\Theta_2-\Theta_4-x_n) W(\Theta_2-\Theta_3-x_n) W(\Theta_1-\Theta_2+x_n) \tag{4.6}
\]
\[
X_{3,k} = \sum_{n=1}^{N_3} 2\pi ir_n e^{(A+\alpha_k)(\Theta_3+x_n)} W(\Theta_3-\Theta_4+x_n) W(\Theta_2-\Theta_3-x_n) W(\Theta_1-\Theta_3-x_n) \tag{4.7}
\]
\[
X_{4,k} = \sum_{n=1}^{N_4} 2\pi ir_n e^{(A+\alpha_k)(\Theta_4+x_n)} W(\Theta_4-\Theta_3+x_n) W(\Theta_2-\Theta_4-x_n) W(\Theta_1-\Theta_4-x_n) \tag{4.8}
\]

where \(N_{1,2} = [(\text{Im } \Theta_{1,2}/\pi + 1/2)/\xi]\) and \(N_{3,4} = [(-\text{Im } \Theta_{3,4}/\pi + 1/2)/\xi]\), \(x_n\) are defined by Eq. (3.29) and \(r_n\) is the residue of \(W(x)\) at \(x_n\), calculated numerically by the definition:

\[
r_n = \lim_{x \rightarrow x_n} (x - x_n) W(x). \tag{4.9}
\]

Second, the integrals, \(I_{ij,k}\) with the deformed contours are evaluated by the analytic continuation formula (2.37). In conclusion one gets

\[
I_{22,k} = \sum_{i} \frac{a_i}{\alpha_i} + \int_{-\infty}^{0} (A(x) - [A(x)]) dx + \int_{0}^{\infty} A(x) dx + \sum_{n=1}^{4} (P_{n,k} + X_{n,k}), \tag{4.10}
\]

with

\[
A(x) = \exp [(A + \alpha_k) x] W(x - \Theta_4) W(x - \Theta_3) W(\Theta_2 - x) W(\Theta_1 - x)
\]
\[
= [A] + O(e^{a_+x}) = \sum_{\alpha_i < 0} a_i e^{\alpha_i x} + O(e^{a_+x}), \quad \text{Re } x \rightarrow -\infty. \tag{4.11}
\]

Upon generalization to the 2\(n\)-soliton form factors the only non-trivial component of this procedure is the determination of the residues picked up when deforming the contour. For an integrand

\[
A(x) = e^{Bx} \prod_{i=1}^{N} W(s_i (x - \Theta_i)), \quad s_i = \pm 1 \tag{4.13}
\]

we have

\[
\int_{C} A(x) dx = \int_{-\infty}^{\infty} A(x) dx + P + X, \tag{4.14}
\]

with the residue contributions

\[
P = -\frac{4}{\pi \sqrt{c_2}} \sum_{i=1}^{N} \Theta_i \left[ -s_i \text{Im } \Theta_i + \frac{\pi}{2} \right] e^{B(\Theta_i-s_i r_n)} \prod_{j \neq i} W\left( s_j \left[ \Theta_i - \Theta_j - s_i \frac{i\pi}{2} \right] \right), \tag{4.15}
\]

and

\[
X = \sum_{i=1}^{N} \sum_{n=1}^{N_i} 2\pi ir_n e^{B(\Theta_i+s_i r_n)} \prod_{j \neq i} W\left( s_j [\Theta_i - \Theta_j + s_i r_n] \right), \quad N_i = \left\lfloor \frac{\pi - 2s_i \text{Im } \Theta_i}{2\pi \xi} \right\rfloor. \tag{4.16}
\]

To conclude this section, we give some details of the Mathematica [11] package SGFF. M. After the above, only one element remains that is not straightforward in the implementation: the calculation of the asymptotic series of the product of several \(W\)-functions. Simple products of the series quickly produce an intractable number of terms, most of which are inaccurate (higher order terms, to which further orders in the series of the constituent functions would contribute). Our solution makes use of 2-by-\(n\) matrices containing the coefficients and the exponents of the terms in the asymptotic series. We have the following key procedures in the package SGFF. M.

- PRO [\(\mathfrak{A}, \mathfrak{B}\)] calculates the 2-by-\(n\) matrix corresponding to the product of asymptotic series \(\mathfrak{A}, \mathfrak{B}\) including orders only with accurate coefficients.
- SHI [\(\mathfrak{A}, a\)] generates the 2-by-\(n\) matrix corresponding to the asymptotic series of \(f(x + a)\) from that of \(f(x)\) (i.e. \(\mathfrak{A}\)).
• $A\text{Int} [\mathcal{A}]$ gives the integral of the asymptotic series corresponding to $\mathcal{A}$ on the negative half-line.

• $A\text{Syfun} [\mathcal{A}, x]$ yields the value of the asymptotic series corresponding to $\mathcal{A}$ at $x$.

In terms of these procedures $I_{22,k}$ is calculated as

$$I_{22,k} = A\text{Int} [\mathcal{A}] + \int_{-\infty}^{0} (A(x) - A\text{Syfun} [\mathcal{A}, x]) \, dx + \int_{0}^{\infty} A(x) \, dx + \sum_{n=1}^{4} (P_{n,k} + X_{n,k}),$$

with $A(x)$ as before, and

$$\mathcal{A} = \text{PRO} (\text{PRO} (\text{PRO} [\text{SHI} \mathcal{W}^{+}, -\Theta_{1}], \text{SHI} \mathcal{W}^{+}, -\Theta_{3}], \text{SHI} \mathcal{W}, -\Theta_{3}], \text{SHI} \mathcal{W}, -\Theta_{3}], \mathcal{W}),$$

$\mathcal{W}$ being the asymptotic series of $W(x)$ for $\text{Re} \, x \geq 0$. The remaining integrals are performed by the routine \texttt{NIntegrate}.

From a practical point of view one should note, that the second term in (4.17) can be numerically unstable. On the other hand, provided that the truncated asymptotic series is a good enough approximation of $A(x)$ for $x < 0$ the contribution of this term can be neglected altogether. Therefore, we omitted this term from our code and supposed that the input rapidities are big enough for this to cause no harm. This can be assumed safely, since Lorentz invariance implies that the rapidities can be shifted by an arbitrary real number.

Considering now the general case of $2n$-particle form factors, we give the asymptotic series of

$$A(x) = e^{\beta x} \prod_{i=1}^{N} W(s_{i} (x - \Theta_{i})), \quad (4.19)$$

diverging for $x \to -\infty$, as

$$\mathcal{A} = \text{PRO} \prod_{i=1}^{N} [\text{SHI} \mathcal{W}_{a,i} (\mathcal{W}), -\Theta_{i}], \quad \mathcal{W}_{a,i} (\mathcal{W}) = \begin{cases} \mathcal{W}^{+}, & s_{i} = +1 \\ \mathcal{W}^{-}, & s_{i} = -1 \end{cases}$$

The main functions available in \texttt{SGFF.M} are to calculate the two-, four- and six-soliton form factors.

• $\text{FF2}[\Theta_{1}, \Theta_{2}]$ gives the two-particle form factors

$$\{F_{+}(-\Theta_{1}, -\Theta_{2}), F_{-}(\Theta_{1}, -\Theta_{2})\}$$

where $\Theta_{1}$ and $\Theta_{2}$ are arrays of the same length with elements $\Theta_{a,i} (a = 1, 2, i = 1, 2, \ldots N)$ rapidities where the two-soliton form factors are to be evaluated.

• $\text{FF4}[\Theta_{1}, \Theta_{2}, \Theta_{3}, \Theta_{4}]$ gives the four-particle form factors

$$\{F_{++}(\Theta_{1}, \Theta_{2}, \Theta_{3}, \Theta_{4}), F_{--}(\Theta_{1}, \Theta_{2}, \Theta_{3}, \Theta_{4}), F_{++}(\Theta_{1}, \Theta_{2}, \Theta_{3}, \Theta_{4}), F_{--}(\Theta_{1}, \Theta_{2}, \Theta_{3}, \Theta_{4})\}.$$  

• $\text{FF6}[\Theta_{1}, \Theta_{2}, \Theta_{3}, \Theta_{4}, \Theta_{5}, \Theta_{6}]$ gives the six-particle form factors

$$\{F_{+++}(\Theta_{1}, \Theta_{2}, \Theta_{3}, \Theta_{4}, \Theta_{5}, \Theta_{6}), F_{---}(\Theta_{1}, \Theta_{2}, \Theta_{3}, \Theta_{4}, \Theta_{5}, \Theta_{6}), F_{+++}(\Theta_{1}, \Theta_{2}, \Theta_{3}, \Theta_{4}, \Theta_{5}, \Theta_{6}), F_{---}(\Theta_{1}, \Theta_{2}, \Theta_{3}, \Theta_{4}, \Theta_{5}, \Theta_{6}), F_{+++}(\Theta_{1}, \Theta_{2}, \Theta_{3}, \Theta_{4}, \Theta_{5}, \Theta_{6}), F_{---}(\Theta_{1}, \Theta_{2}, \Theta_{3}, \Theta_{4}, \Theta_{5}, \Theta_{6})\}.$$  

• $\text{FF2p}[\Theta_{1}, \Theta_{2}]$ gives the two-particle form factors for physical rapidities, i.e. ones with imaginary parts of $\pm \pi$.

• $\text{FF4p}[\Theta_{1}, \Theta_{2}, \Theta_{3}, \Theta_{4}]$ gives the four-particle form factors for physical rapidities.

• $\text{FF6p}[\Theta_{1}, \Theta_{2}, \Theta_{3}, \Theta_{4}, \Theta_{5}, \Theta_{6}]$ gives the six-particle form factors for physical rapidities.

Note that accurate results can only be expected when all rapidities have big enough positive real parts and imaginary part in the interval $[-\pi, \pi]$. The functions calculating form factors only at physical rapidities are considerably faster compared to the general ones if the form factors are needed in more than one points; they calculate the necessary $W$-function values for the integrals only once as part of the initialization.

The parameters that can be specified in \texttt{SGFF.M} are the following, which can be edited e.g. in Mathematica before loading the package.
• $\xi$ is the IR parameter $(5.2)$.
• $\alpha\over\beta$ is the ratio of the parameter $\alpha$ appearing in the operator $O = e^{i\alpha\varphi}$ and the UV parameter $\beta$ of the Lagrangian.
• $\NN$ is the regularization parameter for the $G-$ and $W-$functions denoted by $N$ in the formula $(5.24)$.
• $\Na$ is the maximum number of terms treated in the individual asymptotic series in the formula $(5.11)$.
• $\Nu$ is the number of interpolation points used to calculate the integrands of type $(4.19)$. When evaluating the form factors at general rapidities, mainly $\Nu$ determines the time of evaluation. However, it is this parameter that influences the accuracy the most, as well. A safe choice is $\Nu = 2000$.
• $\epsilon$ is a technical parameter for the calculation of the residues $(4.10)$, $\epsilon = x - x_n$.
• $aa$ and $bb$ are the lower and upper bounds of the integrals of the type $\int_0^\infty A(x)dx$ in $(4.17)$.

5 Tests

The four-particle form factor $F_{---+}$ was checked against the free fermion point result (omitting the vacuum expectation value)

$$\langle\langle Z_+ (\Theta_1) Z_+ (\Theta_2) Z_- (\Theta_3) Z_- (\Theta_4)\rangle\rangle = \sin^2\left(\sqrt{2\pi a}\right)e^{\sqrt{2\pi a}(\Theta_1 + \Theta_2 - \Theta_3 - \Theta_4)} \times \frac{\sinh\left(\frac{\Theta_1 - \Theta_2}{2}\right) \sinh\left(\frac{\Theta_3 - \Theta_4}{2}\right)}{\cosh\left(\frac{\Theta_1 + \Theta_2}{2}\right) \cosh\left(\frac{\Theta_3 + \Theta_4}{2}\right) \cosh\left(\frac{\Theta_1 + \Theta_3}{2}\right) \cosh\left(\frac{\Theta_2 + \Theta_4}{2}\right)}.$$  \hspace{1cm} (5.1)

We do not show test results for this formula since our calculations agreed with the exact results to the machine precision (of 15 digits).

Also, we investigated whether the numerically obtained form factors satisfy the form factor axioms. In the four-particle case the equation

$$F_{\sigma_1\sigma_2\sigma_3\sigma_4} (\Theta_1 + z, \Theta_2 + z, \Theta_3 + z, \Theta_4 + z) = F_{\sigma_1\sigma_2\sigma_3\sigma_4} (\Theta_1, \Theta_2, \Theta_3, \Theta_4)$$  \hspace{1cm} (5.2)

must hold, which was checked. Watson’s theorem is another axiom, e.g. in the form

$$F_{---+} (\Theta_1, \Theta_2, \Theta_3, \Theta_4) = S_{--} (\Theta_3 - \Theta_2) F_{---+} (\Theta_1, \Theta_3, \Theta_2, \Theta_4) + S_{+-} (\Theta_3 - \Theta_2) F_{-++-} (\Theta_1, \Theta_3, \Theta_2, \Theta_4),$$  \hspace{1cm} (5.3)

and

$$F_{---+} (\Theta_1, \Theta_2, \Theta_3, \Theta_4 + 2\pi i) = e^{2\pi i\omega} F_{---+} (\Theta_1, \Theta_1, \Theta_2, \Theta_3),$$  \hspace{1cm} (5.4)

$\omega = \frac{\pi}{4}$ being the mutual non-locality index.

The residues of the kinematical poles of the four-particle form factors were also checked by:

$$i \lim_{\Theta_1 \to \Theta_2 - i\pi} \frac{1}{\Theta_4 - \Theta_2 - i\pi} F_{---+} (\Theta_1, \Theta_2, \Theta_3, \Theta_4) = F_+ (\Theta_1 - \Theta_3) \left[ S_{+-} (\Theta_3 - \Theta_2) - e^{2\pi i\omega} S_{--} (\Theta_2 - \Theta_1) \right].$$  \hspace{1cm} (5.5)

Testing the kinematical poles is especially important: for the cases when the two-soliton form factor is known explicitly (e.g. for half-integer $\frac{a}{\beta}$), equation $(5.5)$ gives the only check that is independent of numerical integrals and their analytic continuations. E.g. for $\frac{a}{\beta} = 1$ the two-particle form factors are known to be $[4]

$$F_{\frac{a}{\beta}}^{\frac{a}{\beta}} (\Theta) = \frac{G(\Theta)}{G(-i\pi)} \cot\left(\frac{\pi \xi}{2}\right) \frac{4i \cosh\left(\frac{\Theta}{\beta}\right) e^{\frac{\Theta + i\pi}{\beta}}}{\xi \sinh\left(\frac{\Theta + i\pi}{\xi}\right)}.$$  \hspace{1cm} (5.6)

In Tables 1 and 2 we listed test results for the four-particle form factors. One can see that magnitude of the error varies greatly for different scenarios. This is because we work in fixed precision (double precision) and the integrals appearing in the formulas can assume values of very different magnitudes and rounding errors can get magnified.

We also implemented the 6-particle form factors. The numerical results agreed with the exact results for the free fermion point, where the 6-particle form factor $F_{---+++}$ reads

$$-i\sin^3\left(\sqrt{2\pi a}\right)e^{\sqrt{2\pi a}(\Theta_1 + \Theta_2 + \Theta_3 - \Theta_4 - \Theta_5 - \Theta_6)} \times \frac{\sinh\left(\frac{\Theta_1 - \Theta_2}{2}\right) \sinh\left(\frac{\Theta_3 - \Theta_4}{2}\right) \sinh\left(\frac{\Theta_5 - \Theta_6}{2}\right)}{\cosh\left(\frac{\Theta_1 + \Theta_2}{2}\right) \cosh\left(\frac{\Theta_3 + \Theta_4}{2}\right) \cosh\left(\frac{\Theta_1 + \Theta_3}{2}\right) \cosh\left(\frac{\Theta_2 + \Theta_4}{2}\right) \cosh\left(\frac{\Theta_1 + \Theta_5}{2}\right) \cosh\left(\frac{\Theta_6 - \Theta_3}{2}\right) \cosh\left(\frac{\Theta_4 - \Theta_5}{2}\right) \cosh\left(\frac{\Theta_2 + \Theta_6}{2}\right) \cosh\left(\frac{\Theta_4 + \Theta_5}{2}\right) \cosh\left(\frac{\Theta_3 + \Theta_6}{2}\right) \cosh\left(\frac{\Theta_5 + \Theta_6}{2}\right).$$  \hspace{1cm} (5.7)
In the four-soliton case. The rapidities were chosen to be $\Theta_1 = 7.6$, $\Theta_2 = 7$, $\Theta_3 = 7.2$, and $\Theta_4 = 6 - i\pi$. In all the tests $\frac{\beta}{\Theta} = \frac{1}{2}$ was set.

| $\xi$ | LHS | RHS |
|-------|-----|-----|
| 2.23  | 0.45330 - 1.4092i | 0.45336 - 1.4093i |
| 0.34  | 0.00089 - 0.051i  | 0.00091 - 0.049i  |
| 2.23  | 0.453360 - 1.4093198i | 0.453358 - 1.4093196i |
| 0.34  | 0.0009063 - 0.04937438i | 0.0009065 - 0.04937441i |
| 2.23  | -0.04255089122137 + 0.03246926430660i | -0.04255089122139 + 0.03246926430663i |
| 0.34  | -0.043292833089 + 0.0219194033i | -0.043292833083 + 0.0219194037i |

Table 1: Comparison of the LHS’s and RHS’s of the form factor axioms (5.2) in the four-soliton case. The rapidities were chosen to be $\Theta_1 = 7.6$, $\Theta_2 = 7$, $\Theta_3 = 7.2$, and $\Theta_4 = 6 - i\pi$. In all the tests $\frac{\beta}{\Theta} = \frac{1}{2}$ was set.

| $\xi$ | LHS | RHS |
|-------|-----|-----|
| 2.23  | 0.8211182 + 0.7147548i | 0.8211175 + 0.7147545i |
| 1.17  | -0.2812726 + 0.0213804i | -0.2812724 + 0.0213801i |
| 0.34  | -0.4726029 - 0.6620907i | -0.4726070 - 0.6620917i |

Table 2: Comparison of residues of four-particle form factors with exact results (5.5). We took $\frac{\beta}{\Theta} = 1$ and for the rapidities $\Theta_1 = 7.6$, $\Theta_2 = 7$, $\Theta_3 = 7.2$, and $\Theta_4 = 7 + 10^{-8} + i\pi$.

In addition, the following kinematic pole equation was tested:

$$i \lim_{\Theta_6 \to \Theta_3 + i\pi} (\Theta_6 - \Theta_3 - i\pi) F_{-+--+} (\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \Theta_6) =$$

$$F_{-+--+} (\Theta_1, \Theta_2, \Theta_4, \Theta_5) \left[ S_{+}^{--} (\Theta_5 - \Theta_3) S_{+}^{--} (\Theta_4 - \Theta_3) - e^{2\pi i\omega} S_{--}^{+-} (\Theta_4 - \Theta_1) S_{--}^{+-} (\Theta_3 - \Theta_2) \right].$$

Watson’s theorem requires e.g.

$$F_{-+--+} (\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5, \Theta_6) = S_{+}^{--} (\Theta_4 - \Theta_3) F_{-+--+} (\Theta_1, \Theta_2, \Theta_4, \Theta_3, \Theta_5, \Theta_6)$$

$$+ S_{+}^{--} (\Theta_4 - \Theta_3) F_{-+--+} (\Theta_1, \Theta_2, \Theta_4, \Theta_3, \Theta_5, \Theta_6).$$

Test results for the six-particle form factors are listed in Table 3.

It interesting to note, that the use of multi-soliton form factors extends to the calculation of soliton-breather and breather-breather form factors by virtue of the bound state pole axiom [3]. In case of higher breather-breather form factors, using soliton-antisoliton form factors can be preferable: e.g. to calculate the $B_n - B_m$ form factor one needs to evaluate either the $(n + m) - B_1$- or the four-soliton form factors.

### 6 Conclusions and outlook

We established a method to obtain the multi-soliton form factors numerically in the $(1+1)$-dimensional sine-Gordon model. The form factors are known in terms of integral representations, whose domains of convergence were extended by analytical continuation. In order to do this we needed the asymptotic series of the $W$-function. Detailed formulae were only shown for the four-soliton form factors, however the number of treatable particles is not limited by the procedure. Test results obtained by the code provided for the four- and six-soliton form factors were shown.

Based on the formalism developed for finite volume form factors in [12, 13, 14], a program to investigate the sine-Gordon form factors is currently underway [10], which can now be extended to multi-soliton states [15].

The present formalism is also expected to be relevant to boundary form factors [16], for which finite size corrections have been developed in [17] and applied to sine-Gordon theory in [18]. Future applications will also include the calculation of finite temperature correlation functions based on the formalism developed in [19, 20, 21].

Table 3: Comparison of the LHS’s and RHS’s of the form factor axioms in the six-soliton case. For $\xi = 2.1$, the rapidities were chosen to be $\Theta_1 = 2.1$, $\Theta_2 = 1.9$, $\Theta_3 = 6$, $\Theta_4 = 5.9$, $\Theta_5 = 1.2$, $\Theta_6 = 5.5 + i\pi$, while for $\xi = 0.34$, $\Theta_1 = 2.1$, $\Theta_2 = 1.9$, $\Theta_3 = 5.9$, $\Theta_4 = 1.2$, $\Theta_5 = 6$, $\Theta_6 = 5.90001 + i\pi$ was taken. In all the tests $\frac{\beta}{\Theta} = \frac{1}{2}$ was set.
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