PERTURBING PLA

GADY KOZMA AND ALEXANDER OLEVSKII

ABSTRACT. We proved earlier that every measurable function on the circle, after a uniformly small perturbation, can be written as a power series (i.e. a series of exponentials with positive frequencies), which converges almost everywhere. Here we show that this result is basically sharp: the perturbation cannot be made smooth or even Hölder. We discuss also a similar problem for perturbations with lacunary spectrum.

1. INTRODUCTION

1.1. Functions representable by analytic sums. Let a power series converge almost everywhere on the circle $T$ to a function $g$:

$$g(t) = \sum_{n \geq 0} c(n)e^{int}$$

It follows from the Privalov uniqueness theorem, that any $g$ may have at most one such decomposition. An analogy with the classical Riemannian theory suggests that $c(n)$ are the Fourier coefficients, whenever $g$ is integrable.

Quite surprisingly, this is not the case: a few years ago we constructed an $L^2$-function $g$ on $T$ which admits the representation (1) but

$$\sum |c(n)|^2 = \infty.$$ 

Later we proved that such a function even can be smooth.

The space of functions $g$ which admit an “analytic” representation (1) we named PLA. The classic PLA-part of $L^2(T)$ is the set of functions whose Fourier series contains exponentials with non-negative frequencies only, namely the Hardy space $H^2$. This set is “small”, in particular it is nowhere dense. In contrast the “non-classic” part is dense. Moreover, the following equality is true:

$$L^0 = PLA + C(T)$$

which means that every measurable finite function $f$ can be decomposed as a sum

$$f = g + h$$

Both authors partially supported by their respective Israel Science Foundation grants.
where \( g \in \text{PLA} \) and \( h \) is continuous. Further, one can replace \( C(T) \) with \( U(T) \), the space of uniformly convergent Fourier series, and one can require from \( h \) to have arbitrarily small norm (the norm in \( U(T) \) being the supremum of the modulus of the partial sums of the Fourier expansion). The described results are proved in [KO.06, KO.07].

Our first result is that the equality (2) is close to best possible: one can not replace the second summand by a space of functions which possess any smoothness, like Hölder or Sobolev one. We state the result in the following form. Given a sequence \( \omega = \{\omega(n)\}, 0 < \omega(n) \nearrow \infty \), denote:

\[
\mathcal{H}_\omega = \left\{ h : \sum |\hat{h}(n)|^2 \omega^2(n) < \infty \right\}
\]  

(4)

**Theorem 1.** For any \( \omega \) the sum \( \text{PLA} + \mathcal{H}_\omega \) does not cover neither \( L^0(T) \), nor even the Wiener algebra \( A(T) = \hat{l}_1(Z) \).

This theorem will be proved in §3.

1.2. Menshov spectra revisited. The classic Menshov representation theorem (1940), see [B64, §XV.2] says that every function \( f \in L^0(T) \) admits representation by a trigonometric series which converges a.e.:

\[
f = \sum_{n=-\infty}^{\infty} c(n)e^{int}.
\]  

(5)

This representation is non-unique, as follows from another remarkable result of Menshov’s proved much earlier (1916): there is a non-trivial trigonometric series which converges to zero almost everywhere. Menshov’s construction for the representation reveals the non-uniqueness phenomenon in a stronger form: one can avoid in (5) using any finite and even some infinite sets of harmonics. This leads to the following definition, see [KO.01]:

**Definition.** A sequence \( \Lambda \subset \mathbb{Z} \) is called a Menshov spectrum if every function \( f \in L^0(T) \) can be decomposed to a series (5) in which only frequencies from \( \Lambda \) may appear with non-zero amplitudes.

In this terminology, Menshov’s theorem states that \( \mathbb{Z} \) is a Menshov spectrum. There are many results that show that Menshov spectra could be quite sparse. For example Arutyunyan [A85] showed that any symmetric set which contains arbitrarily long intervals is a Menshov spectrum. In other words, the set

\[
\bigcup_{n=1}^{\infty} [a_n, a_n + n] \cup [-a_n - n, -a_n]
\]
is a Menshov spectrum, no matter how fast do the $a_n$ grow. Of course, such sets can be extremely sparse. Here we wish to compare to the following sparseness result, taken from [KO.01]:

*Given a sequence*

$$\omega(k) = o(1)$$  \hspace{1cm} (6)

*one can construct a sequence $\lambda(k) \in \mathbb{Z}^+$ with $\lambda(k+1)/\lambda(k) > 1 + \omega(k)$, such that $\Lambda = \{ \pm l(k) \}$ is a Menshov spectrum.*

The condition (6) is sharp: a Menshov spectrum cannot be lacunary in Hadamard sense. Further, the symmetry condition is also essential. Indeed, Privalov’s uniqueness theorem implies that the set $\mathbb{Z}^+$ is not a Menshov spectrum. See [KO.01] for details on all these claims.

One may now ask: how many negative frequencies one should add to $\mathbb{Z}^+$ in order to get a Menshov spectrum? According to the theorem above an extra set with gaps of any sub-exponential growth could be sufficient. Our second result is that this result is close to the best possible one: super-exponential growth is not sufficient.

**Theorem 2.** Let $Q := \{ q(k) \} \subset \mathbb{Z}^+$ satisfy the condition

$$\frac{q(k+1)}{q(k)} \to \infty.$$  \hspace{1cm} (7)

*Then the set $\Lambda = \mathbb{Z}^+ \cup \{-Q\}$ is not a Menshov spectrum.*

Theorem 2 can be reformulated in the language of theorem 1. Let

$$\mathcal{L}_Q = \{ f \in L^1 : \hat{f}(n) = 0 \ \forall n \notin Q \}.$$

Then

**Theorem 2’.** With the same $Q$ as in theorem 2, PLA $+$ $\mathcal{L}_Q \neq L^0$.

The equivalence of theorems 2 and 2’ follows by taking the Menshov representation of $f$ and making the positive part into a PLA function and the negative part into an $\mathcal{L}_Q$ function. This requires Plessner’s theorem and some standard facts on lacunary trigonometric series — we fill these details in §4.

The formulation of theorem 2’ leads to a natural generalisation. Can one find a function $f \notin \text{PLA} + \mathcal{L}_Q$ for all superexponential $Q$ simultaneously? We present a weakened version of this

**Theorem 3.** For a function $\ell(n) \to \infty$ there is a function $f$ such that $f \notin \text{PLA} + \mathcal{L}_Q$ for any $Q$ satisfying

$$\frac{q(k+1)}{q(k)} > \ell(q(k)).$$
We remark that, as in theorem 1, the $f$ of theorem 3 may be taken to be in the Weiner algebra $A(T)$.

Theorem 3 is clearly stronger than theorem 2', and hence also from theorem 2. On the other hand, the proof is also more technical. Hence we first prove theorem 2 in §4, and only afterwards give the proof of theorem 3 in §5.

2. LEMMAS

In this section we introduce some notation and lemmas which will be used for the proof of all three theorems. For a PLA-function $g$ the coefficients $c_n$ in the expansion (1) are unique, so we will denote them by $\hat{g}(n)$. Below we denote by $g$ any PLA-function with $\hat{g}(0) = 0$. We will use the following notations

$$
Q_t := \text{conv}\{\{e^{it}\} \cup \{z : |z| < \frac{1}{2}\}\}.
$$

This $Q_t$ is often called the Privalov ice-cream cone at $e^{it}$. We always denote by $E$ a measurable subset of $T$; by $|E|$ its Lebesgue measure. By $\|\cdot\|_2$ we denote the norm in $L^2(T)$.

**Lemma 1.** If $g^*(t) = A$ then $|G(z)| \leq 3A$ for all $z \in Q_t$.

**Proof.** Without loss of generality one may assume $t = 0$. We now apply Abel’s summation formula to get, for any $|z| < 1$,

$$
G(z) = \sum_{n=0}^{\infty} \hat{g}(n) z^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \hat{g}(k) (z^n - z^{n+1})
$$

so

$$
|G(z)| \leq A \sum_{n=0}^{\infty} |z^n - z^{n+1}| = \frac{A|1-z|}{1-|z|}
$$

but in $Q_0$ one has $|1-z|/(1-|z|) \leq 3$, with the maximum achieved at $z = -\frac{1}{2}$ (the exact value of the constant 3 will play no role in what follows). \(\square\)

**Lemma 2.** There is some universal constant $c_1 > 0$ such that for every $K > 0$ there is a number $\epsilon = \epsilon(K)$ such that if $\|1 + g\|_2 < \epsilon$

then

$$
|\{t : g^*(t) > K\}| > c_1.
$$
We remark that in fact $c_1$ may be taken to be $1/2$ or any number smaller than 1, but this requires an extra argument that we prefer to skip. The dependency between $\varepsilon$ and $K$ will turn out to be $\varepsilon \approx K^{-0.0001}$ at the price of making $c_1$ smaller (we will not need all these in this paper).

**Proof.** The proof is a simple variation on the proof of Privalov’s uniqueness theorem [K80, §D.III]. Let $A > 1$ be some sufficiently large parameter to be fixed later, and let $E \subset [0, 2\pi]$ be the set of $t$ satisfying the following two requirements

$$
|1 + g(t)| < A\varepsilon \quad \forall t \in E
$$

$$
|g^*(t)| \leq K \quad \forall t \in E.
$$

Assume by contradiction that $\{t : g^*(t) > K\} \leq c_1$. Then we may assume that

$$
|E| > 1 - \frac{1}{A^2} - c_1
$$

since Markov’s inequality gives us

$$
|\{t : |1 + g(t)| \geq A\varepsilon\}| < \frac{1}{A^2},
$$

(the fact that the power is 2 will play no role in the argument).

Next, recall that $G$ is the “extension” of $g$ into the disk $\{|z| \leq 1\}$ defined by (8) whenever the sum converges, which is on all of $\{|z| < 1\}$ and almost everywhere on $\{|z| = 1\}$, since $g$ is in PLA. By Abel’s theorem, if $\sum \hat{g}(n)e^{int}$ converges then $G(z) \to G(e^{it})$ when $z$ converges to $e^{it}$ non-tangentially [Z68, §3.14]. Assume therefore, without loss of generality, that the convergence $G(z) \to G(e^{it})$ is uniform on $E$ and that $E$ is closed (if it is not, use Egoroff’s theorem to find an $E' \subset E$ satisfying the requirement and having large measure, $|E'| > 1 - A^{-2} - c_1$).

Examine the Privalov domain over $E$, namely

$$
P = \bigcup_{t \in E} Q_t.
$$

See figure 1 which demonstrates a Privalov domain for a Cantor set. By the above, $G$ is continuous on $P$. Next examine the function $\ell := \log |1 + G|$. It is subharmonic on $P$, and continuous on $\overline{P}$ (in the sense that allows the value $-\infty$). Therefore

$$
\ell(0) \leq \int_{\partial P} \ell(z) d\Omega(z)
$$

where $\Omega$ is the harmonic measure of $P$ from 0 (which is clearly a point of $P$). For background on the harmonic measure (and especially its construction using Brownian motion) see the book [B95].
To use (10) we first note that $G(0) = 0$ so $\ell(0) = 0$. Now examine the boundary of $P$. We write $\partial P = E \cup I$. On $E$ we have $\ell \leq \log A\varepsilon$. On $I$ we apply lemma 1 to see that $\ell \leq \log 3K$. Finally we need to estimate the harmonic measure $\Omega$ of $I$ in the domain $P$. Every interval $J$ in the complement of $E$ corresponds to a piece $J'$ of $I$ — usually to just two straight lines from the edges of $e^iJ$, but sometimes also to a piece of $\{|z| = \frac{1}{2}\}$. One $J$ and $J'$ of the second kind (i.e. with a piece of $\{|z| = \frac{1}{2}\}$) are noted in figure 1. Either way, a straightforward calculation shows that the probability that Brownian motion starting from 0 hits $J'$ before leaving the disk is $\leq C|J|$ and hence

$$\Omega(I) = \sum \Omega(J') \leq C \sum |J| = C|E^c| < C(A^{-2} + c_1).$$

Define therefore

$$A = 2C^{-1/2}, \quad c_1 = \frac{1}{4C},$$

and get that $\Omega(I) < \frac{1}{2}$ and hence that $\Omega(E) > \frac{1}{2}$. With the estimates above for $\ell$ and (10) we get

$$0 = \ell(0) \leq \Omega(E) \log A\varepsilon + \Omega(I) \log 3K < \frac{1}{2} \log A\varepsilon + \log 3K$$

which leads to a contradiction if only $\varepsilon$ is sufficiently small. □

Remark. It might be worthwhile to compare this lemma to lemma 2.6 in [N93], which is also proved by Privalov’s approach.
3. **Proof of Theorem 1.**

Let $\{b(k)\}$ ($k = 1, 2, \ldots$) be a fast decreasing sequence of positive numbers, such that

$$\sum_{j > k} b(j) = o \left( b(k) \varepsilon \left( \frac{k}{b(k)} \right) \right) \quad \forall k$$

(11)

where $\varepsilon(K)$ are from Lemma 2.

Recall the sequence $\omega$ going to infinity from the statement of the theorem. Given $\omega$, choose a fast increasing sequence $\{n(k)\}$ of integers so that

$$b(k) \varepsilon \left( \frac{k}{b(k)} \right) \omega(n(k)) \to \infty$$

(12)

Set

$$f(t) := \sum_{k > 0} b(k)e^{-in(k)t}.$$ 

We claim that the function $f$ does not belong to PLA$+\mathcal{H}_\omega$. Take therefore any $h \in \mathcal{H}_\omega$ and let $q := f - h$. We need to show that $q \not\in$ PLA. Fix a (large) number $N$. Denote

$$f'(N; t) = \sum_{k < N} b(k)e^{-in(k)t}$$

$$f''(N; t) = \sum_{k > N} b(k)e^{-in(k)t}$$

so

$$f = f' + b(N)e^{-in(N)t} + f''.$$ 

Similarly, let

$$h = h' + h''$$

where

$$h' := \sum_{|n| < n(N)} \hat{h}(n)e^{int}.$$ 

Clearly

$$|f''(t)| \leq \sum_{j > k} b(j)$$

and

$$\|h''\|_2 < \|h\|_{\mathcal{H}_\omega} / \omega(n(N)).$$

(13)
Examine now the function \( g := \frac{1}{b(N)}(f' - h' - q)e^{in(N)t} \). It is in PLA since \( q \in PLA \) (we argue by contradiction here) and \( e^{in(N)t}(f' - h') \) is an analytic polynomial. In other words

\[
\hat{g}(k) = \frac{1}{b(N)} \left( \hat{f}'(k - n(N)) - \hat{h}'(k - n(N)) - \frac{e^n}{\hat{q}}(k - n(N)) \right)
\]

and \( \hat{g}(0) = 0 \). Hence we may apply lemma 2. For the \( L^2 \) norm we can write

\[
\|1 + g\|_2 = \left\| \frac{e^{in(N)t}}{b(N)} \left( b(N)e^{-in(N)t} + f' - h' - q \right) \right\|_2 = \frac{1}{b(N)} \| - f'' + h'' \|_2 \leq \frac{1}{b(N)} \left( o \left( b(N)e \left( \frac{N}{b(N)} \right) \right) + \frac{||h||_{\mathcal{H}_w}}{\omega(n(N))} \right)
\]

By (11) and (13)

\[
\leq o \left( e \left( \frac{N}{b(N)} \right) \right)
\]

(where the \( o \) is allowed to depend on \( ||h||_{\mathcal{H}_w} \)). So for \( N \) sufficiently large the \( o \) is smaller than 1, and the lemma gives that

\[
\left| \left\{ t : g^*(t) > \frac{N}{b(N)} \right\} \right| > c_1.
\]

At this point we only need to go back from \( g \) to \( q \), so we need to estimate the contributions of \( f' \) and \( h' \). \( f' \) is straightforward as

\[
\sup_k \left| \sum_{j<k} \hat{f}'e^{in(N)t}(j)e^{ijt} \right| \leq \sum_k |\hat{f}(k)| < C.
\]

For \( h' \) we use Carleson’s theorem [C66, L04] for both \( h^+ \) and \( h^- \) defined by

\[
h^+ = \sum_{n \geq 0} \hat{h}(n)e^{int} \quad h^- = \sum_{n < 0} \hat{h}(n)e^{int}
\]

and get that both expansions converge almost everywhere. This gives a set \( E \) with \( |E^c| \leq \frac{1}{2}c_1 \) such that

\[
\left| \sum_{n=0}^k \hat{h}(n)e^{int} \right| \leq C \quad \left| \sum_{n=-k}^{-1} \hat{h}(n)e^{int} \right| \leq C \quad \forall t \in E, \forall k.
\]
For \( h' e^{in(N)t} \), the analogous sum is bounded by either a sum of two terms from (16), or by a difference of two, and in both cases we get

\[
\left| \sum_{j<k} h' e^{in(N)t} (j) e^{iit} \right| \leq 2C. \tag{17}
\]

This proves the theorem: since \( g^* \) is large (14) and \( f' \) and \( h' \) are bounded (15), (17), we get

\[ q^* \geq N - C \]

on a set of measure \( > \frac{1}{2}c_1 \). Since \( N \) was arbitrary and \( C \) depends only on \( h \), this proves that \( q \notin \text{PLA} \). \( \square \)

**Question.** Does \( \text{PLA} + A(\mathbb{T}) \) cover \( C(\mathbb{T}) \)?

### 4. Proof of Theorem 2

Let us start by showing that theorem 2 is equivalent to theorem 2' (this will also aid in its proof). For this we need two classical results

(i) A lacunary trigonometric sum converges almost everywhere if and only if it is in \( L^2 \). A function in \( L^1 \) with a lacunary Fourier expansion is in \( L^2 \). See e.g. \cite{Z68} §5.6. Here lacunary means in Hadamard sense, i.e. \( q(k + 1)/q(k) > 1 + c \).

(ii) If a trigonometric series converges pointwise on a set \( E \), then both its positive and negative parts converge almost everywhere on \( E \). This result is due to Plessner \cite{P25}. A careful treatment can be found in \cite{B64}, §VIII.23, volume 2, page 151.

To see that theorem 2 implies theorem 2' note that a function \( f \) which proves that \( \Lambda \) is not a Menshov spectrum also cannot be in \( \text{PLA} + \mathcal{L}_Q \) as a decomposition

\[ f = g + h, \text{ } g \in \text{PLA}, \text{ } h \in \mathcal{L}_Q \]

carries over to a representation

\[ f(t) = \sum_{n=0}^{\infty} \hat{g}(n)e^{int} + \sum_{n=-\infty}^{0} \hat{h}(n)e^{int} \]

which converges almost everywhere since both its parts converge almost everywhere: the \( g \) part by definition of PLA and the \( h \) part because of (i).

Vice versa, assume by contradiction that \( \Lambda \) is a Menshov spectrum. Than every \( f \) has a representation as a sum \( \sum_{n \in \Lambda} c(n)e^{int} \) converging almost everywhere. But by Plessner’s theorem the positive part converges a.e., so its limit, call it \( g \), is a PLA function. Also the negative part converges a.e. so call its limit \( h \). By the other direction of (i), \( h \in L^2 \) and hence in \( \mathcal{L}_Q \). We get \( f = g + h \) with \( g \in \text{PLA} \) and \( h \in \mathcal{L}_Q \) so, since \( f \) was arbitrary, \( \text{PLA} + \mathcal{L}_Q = L^0 \). This shows that theorem 2' implies theorem 2, so they are equivalent. \( \square \)
Satisfied that theorem 2 and 2’ are equivalent, we start their proof. The first step is the following simple lemma.

**Lemma 3.** Let $Q$ satisfy (7). Then there is a number $\alpha, \frac{1}{3} < \alpha < \frac{2}{3}$ such that
\[
\{\alpha q(k)\} = o(1)
\] (18)
where $\{x\}$ denotes the fractional part of $x$.

We remark that in §5 we will need that the estimate of $\{\alpha q\}$ can be done uniformly in the superexponential growth of the $Q$, namely,
\[
\{\alpha q(k)\} \leq C \max_{l \geq k} \left\{ \frac{q(l)}{q(l+1)} \right\}.
\] (19)

Also the restriction $\alpha \in (\frac{1}{3}, \frac{2}{3})$ is only used in §5, here $\alpha$ can be taken anywhere in $(0,1)$.

**Proof.** We may assume without loss of generality that $q(k+1)/q(k) > 2$ for all $k$ (for $k = 1$ we assume $q(1) > 2$). Set
\[
\alpha = \frac{1}{3} + \sum_{k=1}^{\infty} \frac{\gamma(k)}{q(k)}
\] (20)
where the numbers $0 < \gamma(k) \leq 1$ are to be defined. Assuming they are already defined for $k \leq n$, we denote by $a(n)$ the $n^{th}$ partial sums of the series (20) and set $\gamma(n+1) := 1 - \{a(n)q(n+1)\}$ which implies that $q(n+1)a(n+1)$ is integer. Continuing this process we get $\alpha$.

Now, for every $n > 1$:
\[
\alpha q(n) = a(n)q(n) + q(n) \sum_{k>n} \frac{\gamma(k)}{q(k)}.
\]

As already explained, $a(n)q(n)$ is an integer. The second term is $\leq q(n) \sum_{k>n} 1/q(k)$, which is $o(1)$ due to (7). This gives (18) and also the uniform estimate (19) remarked upon after the lemma. □

**Step 1.** With the lemma proved we can start the proof of theorem 2. Fix numbers $d(n) > 0$ decreasing so fast that
\[
\sum_{n > N} d^2(n) < \frac{d(N)^2}{N^2} e^2 \left( \frac{N}{d(N)} \right),
\] (21)
where the function $\varepsilon(n)$ was defined in lemma 2. Since this expression will repeat a lot, we will denote it for short by $\varepsilon_N$, 

$$\varepsilon_N := \frac{d(N)}{N} \varepsilon \left( \frac{N}{d(N)} \right),$$

(22)

so $\sum_{n>N} d^2(n) < \varepsilon_N^2$.

**Step 2.** Next use lemma 3 to find a number $\beta \in (0, 2\pi)$ such that

$$e^{i\beta q(k)} \to 1.$$  

(23)

**Step 3.** With $\beta$ defined, one can find $\nu(N)$ such that the following two properties hold,

$$|1 - e^{i\beta q(k)}| < \varepsilon_N \quad \forall k \text{ such that } q(k) > \nu(N)$$  

(24)

$$|1 - e^{i\beta \nu(N)}| > 1.$$  

(25)

These properties can be satisfied simultaneously because (24) is satisfied whenever $\nu(N)$ is sufficiently large, while (25) is satisfied on a sequence converging to $\infty$. We now define

$$f(t) = \sum_{N=1}^{\infty} d(N) e^{-i\nu(N)t}.$$  

(26)

This is the required function. As an aside we remark that it is in the Wiener algebra, but it might be highly non-smooth as we have no control over the relation between $d(N)$ and $\nu(N)$.

**Step 4.** Recall now the discussion in the beginning of this section. We claim that $f$ is a function demonstrating that $\Lambda = \{-Q\} \cup \mathbb{Z}^+$ is not a Menshov spectrum, i.e. that $f$ has no expansion

$$f(t) = \sum_n c(n) e^{int} \quad n \notin \Lambda \implies c(n) = 0$$  

(27)

which converges almost everywhere. Assume therefore by contradiction that an expansion (27) exists. Due to Plessner’s theorem we know that $\sum_{n<0} c(n) e^{int}$ converges (to some value), and since the negative part is lacunary we must have $\sum_{n<0} |c(n)|^2 < \infty$.

Somewhat similarly to the proof of theorem 1, we will now subtract the Fourier expansion of $f$ and the non-standard one (27) and get a null series i.e. a trigonometric series converging to zero almost everywhere. Namely, define

$$\gamma(n) = c(n) - \hat{f}(n) = c(n) - \begin{cases} 
  d(N) & n = -\nu(N) \\
  0 & \text{otherwise}
\end{cases}$$
and get that
\[ \sum_{n=-\infty}^{\infty} \gamma(n)e^{int} = 0 \quad \text{for almost every } t. \]

The crucial step is to examine \( f(t + \beta) - f(t) \) and the corresponding null series
\[ \sum_{n=-\infty}^{\infty} \gamma(n)(e^{int} - 1)e^{int} = 0 \quad \text{for almost every } t. \quad (28) \]

As in the remark after \((27)\), the positive and negative parts of \((28)\) converge almost everywhere (not necessarily to zero).

**Step 5.** We will need some estimates for the \( L^2 \) norm of the “tails” of \((28)\) so let us state them now: for every \( N \),
\[ \sum_{n<-\nu(N)} |\gamma(n)(e^{i\beta n} - 1)|^2 \leq C\varepsilon_N^2. \quad (29) \]

Here and below \( C \) may depend on \( \sum_{n<0} |c(n)|^2 \) (but not on \( N \)).

**Proof of \((29)\).** \( \gamma(n) \) can be non-zero only if \( n = -q(k) \) or if \( n = -\nu(k) \). In the first case we have
\[ |e^{i\beta n} - 1| = |e^{-i\beta q(k)} - 1| = |e^{i\beta q(k)} - 1| < \varepsilon_N \quad (24) \]
(recall that we are looking at \( n < -\nu(N) \) so \((24)\) applies). All in all this gives
\[ \sum_{k:q(k)>\nu(N)} |c(-q(k))(e^{i\beta q(k)} - 1)|^2 < \varepsilon_N^2 \sum_{n<0} |c(n)|^2 \]
which we agreed to denote by \( C\varepsilon_N^2 \). The second kind of non-zero \( n \) is \( -\nu(k) \) and for this we simply use the definition of the \( d(k) \), \((21)\) and of \( f \), \((26)\), and get
\[ \sum_{k>N} d(k)^2 |e^{i\beta \nu(k)} - 1|^2 \leq 4 \sum_{k>N} d(k)^2 \leq 4\varepsilon_N^2. \quad (21) \]

Taking these two estimates together gives
\[ \sum_{n<-\nu(N)} |\gamma(n)(e^{i\beta n} - 1)|^2 = \sum_{n<-\nu(N)} |(c(n) + \hat{f}(n))(e^{i\beta n} - 1)|^2 \leq \]
\[ \leq \sum_{n<-\nu(N)} (2|c(n)|^2 + 2|\hat{f}(n)|^2) \cdot |e^{i\beta n} - 1|^2 \]
By the above
\[ \leq \varepsilon_N^2 \cdot (2C + 8) \]
as needed. \( \square \)
**Step 6.** We now proceed as in the proof of theorem 1 i.e. we wish to apply lemma 2 for some PLA function related to the null-series (28). We shift the null-series (28) by \( \nu(N) \) and divide it by \( d(N)(e^{-iv(N)\beta} - 1) \). We get

\[
q(t) : = \frac{1}{d(N)(e^{-iv(N)\beta} - 1)} \sum_{n > -\nu(N)} \gamma(n)(e^{i\beta n} - 1)e^{i(n+\nu(N))t} \tag{30}
\]

\[
= 1 + \frac{1}{d(N)(e^{-iv(N)\beta} - 1)} \sum_{n < -\nu(N)} \gamma(n)(e^{i\beta n} - 1)e^{i(n+\nu(N))t}.
\]

In other words, the first line is the “PLA expansion” of \( q \) and the second is the Fourier expansion. In particular \( q \) is a PLA function with \( \hat{q}(0) = 0 \). By (29),

\[
||1 - q||_2 \leq \frac{C\epsilon_N}{d(N)|e^{-iv(N)\beta} - 1|}.
\]

By requirement (25), \( |e^{-iv(N)\beta} - 1| > 1 \), and with the definition of \( \epsilon_N \) we get

\[
||1 - q||_2 \leq \frac{C}{N} \epsilon \left( \frac{N}{d(N)} \right).
\]

Hence for \( N > C \) we may apply lemma 2 (to \( -q \), but \( q^* = (-q)^* \)) and get

\[
\left| \left\{ t : q^*(t) > \frac{N}{d(N)} \right\} \right| > c_1.
\]

Recalling that the PLA expansion of \( q \) is (30) we get a set of measure \( > c_1 \) where

\[
\sup_k \left| \sum_{n = -\nu(N)}^k \gamma(n)(e^{i\beta n} - 1)e^{i(n+\nu(N))t} \right| > d(N)|e^{-iv(N)\beta} - 1|\frac{N}{d(N)} > N. \tag{31}
\]

**Step 7.** We only need to change the lower bound in the sum. But clearly

\[
\sum_{n = -\nu(N)}^0 |\gamma(n)(e^{i\beta n} - 1)|^2 \leq C
\]

(if you want you can deduce this from (29) with the \( N \) there being 0). Using Markov’s inequality gives that the corresponding function cannot be large on a set of large measure:

\[
\left| \left\{ t : \sum_{n = -\nu(N)}^0 \gamma(n)(e^{i\beta n} - 1)e^{int} > C \right\} \right| < \frac{1}{2}c_1.
\]
We subtract this from (31) and get a set of measure \( \frac{1}{2} c_1 \) where

\[
\sup_k \left| \sum_{n=0}^{k} \gamma(n) (e^{i\beta n} - 1)e^{int} \right| > N - C.
\]

Since \( N \) was arbitrary, we get a set of measure \( \frac{1}{2} c_1 \) where

\[
\sup_k \left| \sum_{n=0}^{k} \gamma(n) (e^{i\beta n} - 1)e^{int} \right| = \infty.
\]

But this is exactly the positive part of (28). This is a contradiction since it was supposed to converge almost everywhere. This finishes the proof of theorem 2. \( \square \)

**Conjecture.** Probably theorem 2, and perhaps even theorem 3, hold for \( Q \) lacunary in Hadamard sense.

There is another version of this problem. Let us introduce the concept of “Privalov spectrum”. We say that \( \Lambda \) is a Privalov spectrum if

\[
\sum_{n \in \Lambda} c(n)e^{int} = 0 \quad \forall t \in E, \quad |E| > 0 \implies c(n) \equiv 0.
\]

Clearly, a Menshov spectrum can never be a Privalov spectrum. The trick of shifting by \( \beta \) employed above is useful also for this problem. For example, \( \Lambda = \{-2^n\}_{n=1}^{\infty} \cup \mathbb{Z}^+ \) is a Privalov spectrum. To see this, it is enough to shift by \( \beta = 2^{-k} \) with \( k \) sufficiently large so as to satisfy \( E \cap (E + \beta) \neq \emptyset \), and this reduces the result to the original Privalov theorem.

Thus a natural variation on the conjecture is: how sparse must \( Q \) be in order to ensure that \( -Q \cup \mathbb{Z}^+ \) is a Privalov spectrum? This problem was considered by F. Nazarov in the early 90s in an unpublished work (private communication). The trick of shifting can be used to show that if \( Q \) is very fast increasing, then \( -Q \cup \mathbb{Z}^+ \) is Privalov, but it seems that not under the condition (7) of superexponential growth. Faster growth of \( Q \) is necessary.

Another interesting generalization is to ask whether removing a superexponential sequence from a Menshov spectrum leaves one with a Menshov spectrum. Let us remark that a theorem of Talalyan \([T69]\) shows that removing a single element from a Menshov spectrum will always result in a new Menshov spectrum.

5. PROOF OF THEOREM 3

The \( f \) demonstrating theorem 3 cannot be exactly as in the proof of theorem 2, as that \( f \) was lacunary itself! Hence it is itself in some \( L_{-Q} \), without the need to add any PLA function. It turns out that one can construct an \( f \) demonstrating
theorem 3 and very close to lacunary. We will construct an \( f \not\in PLA + \mathcal{L}_Q \) for any \( Q \) which is a sum of extremely lacunary couples of consecutive harmonics. The role of \( \beta \) (the value you shift by in the proof) in the theorem also changes — it has to be chosen after \( f \) is already known, so \( f \) cannot depend on it.

**Step 1.** To start the proof of theorem 3, we fix numbers \( d(n) > 0 \) decreasing very fast. The precise condition will not make much sense now, so please do not dwell on it: it will become clearer in later stages of the proof. Precisely we define

\[
\varepsilon \varepsilon_N := \frac{d(N)^2}{N^2} \varepsilon \left( \frac{N}{d(N)} \right) \varepsilon \left( \frac{N^2}{d(N)^2} \varepsilon^{-1} \left( \frac{N}{d(N)} \right) \right),
\]

and then require \( d(n) \) to satisfy

\[
\sum_{n > N} d^2(n) < \varepsilon \varepsilon_N^2.
\]

Comparing to (21) we see that instead of using the function \( \varepsilon(n) \) from lemma 2 once, as we did in (21), here we need to iterate it. This is the reason for the notation \( \varepsilon \varepsilon_N \).

**Step 2.** The choice of \( \nu \) now cannot depend on \( \beta \) as it is not yet known — it will instead depend on \( \ell \), the rate at which \( q(k + 1)/q(k) \) goes to infinity. Precisely, for every \( N \) find a \( \nu(N) \) such that

\[
\ell(\nu(N)) > \frac{1}{\varepsilon \varepsilon_N}
\]

where \( \ell \) is from the statement of theorem 3. We assume at this point that \( \ell \) is increasing, which we may, without loss of generality.

**Step 3.** With these we may define our function \( f \),

\[
f(t) := \sum_{n=1}^{\infty} d(n) \left[ e^{-i(\nu(n)-1)t} + e^{-i\nu(n)t} \right].
\]

**Step 4.** We now need to show that \( f \not\in PLA + \mathcal{L}_Q \), for any \( Q \). Assume to the contrary that \( f = g + h \) with \( g \in PLA \) and \( h \in \mathcal{L}_Q \) for some \( Q \) with \( q(k + 1)/q(k) > \ell(q(k)) \). As in the proof of theorem 2 we denote by \( c(n) \) the coefficients of this “non-standard expansion” of \( f \), i.e. \( c(n) = \tilde{g}(n) \) for \( n \geq 0 \) and \( c(n) = \tilde{h}(n) \) for \( n < 0 \). Again we get a null series by subtracting the Fourier expansion of \( f \) and the non-standard one. Namely, define

\[
\gamma(n) = c(n) - \begin{cases} 
d(k) & n = -\nu(k) + 1 \text{ or } n = -\nu(k) \\
0 & \text{otherwise}
\end{cases}
\]
and get that
\[ \sum_{n=-\infty}^{\infty} \gamma(n)e^{int} = 0 \quad \text{for almost every } t. \]
Next apply lemma 3 (and the remark following it) to find a number \( \beta \in \left(\frac{2\pi}{3}, \frac{4\pi}{3}\right) \) such that
\[ |e^{i\beta q(k)} - 1| < \frac{C}{\ell(q(k))} \quad \forall k. \tag{36} \]
(the \( C \) has two sources: the first is (19) and the second is the inequality \( |e^{2\pi it} - 1| \leq C\{t\} \)). As before, the crucial step is to examine \( f(t + \beta) - f(t) \) and the corresponding null series
\[ \sum_{n=-\infty}^{\infty} \gamma(n)(e^{in\beta} - 1)e^{int} = 0 \quad \text{for almost every } t. \tag{37} \]
Again this series not only converges to 0 symmetrically, also its positive part \( \sum_{n=0}^{\infty} \) and its negative part converge individually, almost everywhere, for the same reasons as before.

**Step 5.** We will need estimates for the \( L^2 \) norm of the tails of (37), analogous to those of (29). Precisely,
\[ \sum_{n<-\nu(N)} \gamma(n)(e^{i\beta n} - 1)^2 \leq C\varepsilon^2 N. \tag{38} \]
The proof is practically the same as that of (29), but we include it for the convenience of the reader.

**Proof.** \( c(n) \) can be non-zero only if \( n = -q(k) \) or if \( n = -\nu(k) + 1 \) or \(-\nu(k) \). In the first case we have
\[ |e^{i\beta n} - 1| = |e^{-i\beta q(k)} - 1| = |e^{i\beta q(k)} - 1| < \frac{C}{\ell(q(k))}. \tag{36} \]
Now, we are looking at \( n < -\nu(N) \) so by the definition of \( \nu(N) \), (34),
\[ \ell(q(k)) \geq \ell(\nu(N)) \quad \text{so} \quad \frac{1}{\varepsilon N}. \tag{34} \]
All in all this gives
\[ \sum_{l: q(l) > \nu(k)} |c(-q(l))(e^{i\beta q(l)} - 1)|^2 < C\varepsilon^2 N. \]
The second kind of non-zero $n$ is $-\nu(k) + 1$ and $-\nu(k)$ and for this we simply use the definition of the $d(k)$, (33) and of $f$, (35), and get

$$\sum_{k>N} d(k)^2 \left( |e^{i\beta n(k)} - 1|^2 + |e^{i\beta (n(k)+1)} - 1|^2 \right) \leq 8 \sum_{k>N} d(k)^2 \leq 8\varepsilon e^2_N.$$  

Taking these two estimates together gives

$$\sum_{n<\nu(N)} |\gamma(n)(e^{i\beta n} - 1)|^2 = \sum_{n<\nu(N)} |(c(n) + \hat{f}(n))(e^{i\beta n} - 1)|^2 \leq$$

$$\leq \sum_{n<\nu(N)} (2|c(n)|^2 + 2|\hat{f}(n)|^2) \cdot |e^{i\beta n} - 1|^2$$

By the above \( \leq \varepsilon e^2_N \cdot (2C + 16). \)

**Step 6.** We now proceed as in the proof of theorem 1 i.e. we wish to apply lemma 2 for some PLA function related to the null-series (37). Fix some $N$ large and examine $e^{-i(\nu(N)-1)\beta} - 1$ and $e^{-iv(N)\beta} - 1$. Since $\beta \in \left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$, it is not possible for both numbers to be small. We therefore examine two cases:

(i) $|e^{-i(\nu(N)-1)\beta} - 1| > (d(N)/N)e(N/d(N)).$

(ii) $|e^{-i(\nu(N))\beta} - 1| \leq (d(N)/N)e(N/d(N)).$ This implies that $|e^{-i(\nu(N)-1)\beta} - 1| > \epsilon$.

Let us start with the first case (the other is similar but slightly simpler). We shift the null-series (37) by $\nu(N)$ and get a new null-series whose positive part is the PLA expansion of some PLA function, and whose negative part is its Fourier expansion. Namely, define

$$q(t) := \frac{1}{d(N)|e^{-iv(N)\beta} - 1|} \sum_{n>\nu(N)} \gamma(n)(e^{i\beta n} - 1)e^{i(n+\nu(N))t} \quad (39)$$

$$= 1 + \frac{1}{d(N)|e^{-iv(N)\beta} - 1|} \sum_{n<-\nu(N)} \gamma(n)(e^{i\beta n} - 1)e^{i(n+\nu(N))t}.$$  

(he term “1” in the second line requires that $N$ be sufficiently large because it requires that $\nu(N) \not\in Q$. But this follows from our assumption (i) since if $\nu(N) = q(k)$ then $|e^{i\beta q(k)} - 1| < C\varepsilon e^2_N$ which contradicts (i) for $N > C$).

Now, by (38),

$$\sum_{n<-\nu(N)} |\gamma(n)(e^{i\beta n} - 1)|^2 \leq C\varepsilon e^2_N.$$  

Hence

$$||q - 1||_2 \leq \frac{C\varepsilon e_N}{d(N)|e^{-iv(N)\beta} - 1|} \quad \square.$$
and since we assumed $|e^{-iv(N)\beta} - 1| > (d(N)/N)\varepsilon(N/d(N))$ we get,

$$||q - 1||_2 \leq \frac{CN\varepsilon N}{d(N)^2\varepsilon(N/d(N))}.$$  

Recalling the definition of $\varepsilon_n$ (32),

$$||q - 1||_2 \leq \frac{C}{N}\varepsilon\left(\frac{N^2}{d(N)^2}\varepsilon^{-1}\left(\frac{N}{d(N)}\right)\right)$$

and if $N$ is sufficiently large the fraction is $< 1$ and we can apply lemma 2. We get

$$\left|\left\{ t : q^*(t) > \frac{N^2}{d(N)^2}\varepsilon^{-1}\left(\frac{N}{d(N)}\right)\right\}\right| > c_1.$$  

Recalling that the PLA expansion of $q$ is given by (39) we get that there is a set of measure $> c_1$ where

$$\sup_k \left| \sum_{n=1-v(N)}^k \gamma(n)(e^{i\beta n} - 1)e^{i(n+v(N))t} \right| >$$

$$> d(N)|e^{-iv(N)\beta} - 1|\frac{N^2}{d(N)^2}\varepsilon^{-1}\left(\frac{N}{d(N)}\right) > N \quad (40)$$

where the second inequality again uses our assumption (i). We can replace in (40) the $e^{i(n+v(N))t}$ by simply $e^{int}$ as it does not change the absolute value of the expression. Finally to change the limit of the sum to 0 we note that clearly

$$\sum_{n=1-v(N)}^0 |c(n)(e^{i\beta n} - 1)|^2 \leq C.$$  

This we may subtract from estimate (40) and get that on a set of measure $> \frac{1}{2}c_1$,

$$\sup_k \left| \sum_{n=0}^k \gamma(n)(e^{i\beta n} - 1)e^{int} \right| > N - C \quad (41)$$

and we are done with case (i).

**Step 7.** We are left with case (ii) which is very similar, except that instead of shifting by $v(N)$ we shift by $v(N) - 1$. The main reason to read this step is to see why we needed to define $\varepsilon_1 N$ by iterating $\varepsilon$ twice. As in case (i) for $N$ sufficiently large we would have $v(N) - 1 \notin \mathbb{Q}$ so $\gamma(-v(N) + 1) = d(N)$. This gives, instead
of (39),

\[
q(t) := \frac{1}{d(N)|e^{-i(v(N)-1)\beta} - 1|} \sum_{n>1-v(N)} \gamma(n)(e^{i\beta n} - 1)e^{i(n+v(N)-1)t}
\]

\[
= 1 + \frac{1}{d(N)|e^{-i(v(N)-1)\beta} - 1|} \sum_{n<1-v(N)} \gamma(n)(e^{i\beta n} - 1)e^{i(n+v(N)-1)t}.
\]

The argument that \( ||q - 1||_2 \) is small is similar. We have

\[
\sum_{n<1-v(N)} |\gamma(n)(e^{i\beta n} - 1)|^2 \leq C\epsilon^2 + \frac{d(N)^2}{N^2}\epsilon^2 \left( \frac{N}{d(N)} \right)
\]

where the extra term is the one corresponding to \( n = -v(N) \) and is estimated by our assumption (ii). The extra term is the dominant one, so we may write

\[
\sum_{n<1-v(N)} |\gamma(n)(e^{i\beta n} - 1)|^2 \leq C\frac{d(N)^2}{N^2}\epsilon^2 \left( \frac{N}{d(N)} \right).
\]

Since \( |e^{-i(v(N)-1)\beta} - 1| > c \) by our assumption, we get

\[
||q - 1||_2 \leq \frac{1}{c d(N)} \cdot C\frac{d(N)\epsilon}{N} \left( \frac{N}{d(N)} \right)
\]

so again for \( N \) sufficiently large we may apply lemma 2 and get

\[
\left| \left\{ t : q^*(t) > \frac{N}{d(N)} \right\} \right| > c_1
\]

the same argument as in the previous case then shows that on a set of measure \( > c_1 \),

\[
\sup_k \left| \sum_{n=2-v(N)}^k \gamma(n)(e^{i\beta n} - 1)e^{int} \right| > cN
\]

and again on a set of measure \( > \frac{1}{2}c_1 \),

\[
\sup_k \left| \sum_{n=0}^k \gamma(n)(e^{i\beta n} - 1)e^{int} \right| > cN - C. \quad (42)
\]

As our conclusion (41) for case (i) is stronger, we in fact get that (42) holds regardless of whether case (i) or case (ii) held.
Since $N$ was arbitrary, we see that on a set of measure $> \frac{1}{2} c_1$ (the upper limit of the sets from (42)),

$$\sup_k \left| \sum_{n=0}^{k} \gamma(n)(e^{i\theta n} - 1)e^{int} \right| = \infty.$$ 

In contradiction to our assumption after (37). Theorem 3 is thus proved. \hfill \Box

**REFERENCES**

[A85] F. G. Arutyunyan, Представление измеримых функций многих переменных кратными тригонометрическими рядами [Russian: Representation of measurable functions of several variables by multiple trigonometric series]. Mat. Sbornik 126(168):2 (1985), 267–285. Available at: mathnet.ru. English translation in: Math. USSR Sbornik 54:1 (1986), 259–277. Available at: iop.org

[B64] Nina K. Bary, Тригонометрические ряды. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow 1961. English translation in: A treatise on trigonometric series. Authorized translation by Margaret F. Mullins. A Pergamon Press Book. The Macmillan Co., New York 1964.

[B95] Richard F. Bass, Probabilistic techniques in analysis. Probability and its Applications (New York). Springer-Verlag, New York, 1995.

[C66] Lennart Carleson, On convergence and growth of partial sums of Fourier series. Acta Math. 116 (1966), 135–157. Available at: springerlink.com

[K80] Paul Koosis, Introduction to $H_p$ spaces. With an appendix on Wolff’s proof of the corona theorem. London Mathematical Society Lecture Note Series, 40. Cambridge University Press, Cambridge-New York, 1980.

[KO.01] Gady Kozma and Alexander Olevskii, Menshov representation spectra. J. Anal. Math. 84 (2001), 361–393. Available at: springerlink.com, arXiv:math/0510616

[KO.06] Gady Kozma and Alexander Olevskii, Analytic representation of functions and a new quasi-analyticity threshold. Annals of Math. 164:3 (2006), 1033–1064. princeton.edu

[KO.07] Gady Kozma and Alexander Olevskii, Is PLA large? Bull. Lond. Math. Soc. 39:2 (2007), 173–180. Available at: oxfordjournals.org, arXiv:math/0510130

[L04] Michael T. Lacey, Carleson’s theorem: proof, complements, variations. Publ. Mat. 48:2 (2004), 251–307. Available at: uab.es

[N93] Fedor L. Nazarov, Локальные оценки экспоненциальных полиномов и их приложения к неравенствам типа принципа неопределенности [Russian: Local estimates for exponential polynomials and their applications to inequalities of the uncertainty principle type]. Algebra i Analiz 5:4 (1993), 3–66. Available at: mathnet.ru. English translation in: St. Petersburg Math. J. 5:4 (1994), 663–717. Available at: msu.edu/~fedja

[P25] Abraham Plessner, Über Konvergenz von trigonometrischen Reihen [German: On convergence of trigonometric series]. J. Reine Angew. Math. 155 (1925), 15–25. Available at: digizeitschriften.de

[T69] A. A. Talalyan, О существовании нуль-рядов по некоторым системам функций [Russian: The existence of null series in certain systems of functions]. Mat. Zametki 5:1 (1969), 3–12. Available at: mathnet.ru. English translation in: Math. Notes 5:1 (1969), 3–9. Available at: springerlink.com
[Z68] Antoni Zygmund, *Trigonometric series*. Second edition, reprinted with corrections and some additions Cambridge University Press, London-New York 1968.