LINEAR DIFFERENTIAL OPERATORS WITH POLYNOMIAL COEFFICIENTS
GENERATING GENERALISED SYLVESTER-KAC MATRICES

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Abstract. A method of generating differential operators is used to solve the spectral problem for a generalisation of the Sylvester-Kac matrix. As a by-product, we find a linear differential operator with polynomial coefficients of the first order that has a finite sequence of polynomial eigenfunctions generalising the operator considered by M. Kac. In addition, we explain spectral properties of two related tridiagonal matrices whose shape differ from our generalisation.

Keywords: Sylvester-Kac matrix, Eigenvalues, Eigenvectors, Linear differential operators.

MSC2010: Primary 15A18, 47B36; Secondary 34A05, 34L10, 34L05.

1. Sylvester-Kac-type matrices: historical remarks and applications

A matrix of the form

\[
K_N := \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & 0 & 0 \\
N & 0 & 2 & \ldots & 0 & 0 & 0 \\
0 & N - 1 & 0 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & N - 1 & 0 \\
0 & 0 & 0 & \ldots & 2 & 0 & N \\
0 & 0 & 0 & \ldots & 0 & 1 & 0
\end{pmatrix}
\] (1.1)

is called the Sylvester-Kac matrix. First time, it appeared in an extremely short paper by J. Silvester [35] in 1854. Sylvester gave its characteristic polynomial without a proof. According to T. Muir’s fundamental work on the history of determinants, the first proof of Sylvester’s claim was provided by F. Mazza in 1866 [28, p. 442].

In XX century, the Sylvester matrix got a new life and many applications as well as the second name, the Kac matrix. M. Kac [23] being not aware of Sylvester’s work found the spectrum of the matrix (1.1) and its eigenvectors by the method of generating functions. Later on, this matrix and its certain generalisations appeared in many publications. It was rediscovered many times by many authors and by different approaches, see [32, 10, 38, 12]. O. Taussky and J. Todd [36] gave an account of various linear algebra approaches to the study of the Sylvester-Kac matrix and its generalisation.

Also, matrix (1.1) and its generalisations found applications in such areas as orthogonal polynomials [2], linear algebra [22, 20, 9, 3], physics [1, 15], graph theory [4], numerical analysis [10, 14, 29], statistics [12, 13], statistical mechanics [23, 34, 19], biogeography [21] etc., see, e.g., [15] for more references.

The papers [2, 20, 9, 7, 15] study spectral properties of various Sylvester-Kac-type matrices. The present paper revisits and generalises some of their results by using a different approach. In fact, R. Askey [2] adopted the orthogonal polynomial approach and dealt with the Krawtchouk polynomials to prove some his results we cover here. O. Holtz used matrix block-triangularisation to obtain the same results as R. Askey. W. Chu [7] employed the so-called left eigenvector method to find eigenvalues of the matrix we consider here. However, he did not find its eigenvectors. Finally, the authors of the works [9, 15] guessed their results and proved that their guess is correct by direct substitutions. Our approach is more constructive.

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Indeed, we consider a linear differential operator of the first order with polynomial coefficients. Its specialisation with an infinite sequence of polynomial solutions may be transformed into another operator of a similar kind that has an infinite sequence of rational eigenfunctions – some of which (finitely many) are polynomials. As a result, we obtain a linear differential operator with polynomial coefficients having a finite sequence of polynomial eigenfunctions. M. Kac [23] came to a particular case of such an operator by the method of generating functions starting from the Sylvester-Kac matrix. In turn, our starting point is the differential operator, and we arrive at a generalisation of the the Sylvester-Kac matrix. In fact, being restricted to the space \( \mathbb{C}_N[z] \) of all complex polynomials of degree at most \( N \), our operator becomes finite-dimensional, and its matrix representation is a generalised Sylvester-Kac matrix. In this way, we obtain eigenvalues and eigenvectors of this matrix.

The paper is organised as follows. Section 2 is devoted to a first-order linear differential operator with polynomial coefficients. We find its eigenvalues and the corresponding eigenfunctions: they turn to be rational functions including a prescribed number of polynomials. In Section 3 we represent the restriction of the aforementioned differential operator to the space \( \mathbb{C}_N[z] \) of all complex polynomials of degree at most \( N \), \( N \in \mathbb{N} \), as finite tridiagonal matrix. Then we determine the eigenvalues and eigenvectors of that matrix, which is a generalisation of the Sylvester-Kac matrix (1.1) depending on 4 parameters. Section 4 deals with particular cases of our matrix. We show that our results cover and generalise the results of some previous publications. In Section 5, we discuss some future applications of our approach. Finally, Appendix A provides two distinct simple proofs of the main results of [16, 17] concerned with matrices similar to (1.1). The first proof shows that these two publications, in fact, rediscovered certain properties of persymmetric matrices for the particular case of the Sylvester-Kac matrix. The other proof establishes a connection to tridiagonal matrices related to the Hahn polynomials; it additionally produces formulae for the left and right eigenvectors of the matrices studied in [16, 17].

2. Spectral problem for differential operators with polynomial coefficients

Consider the differential operator

\[
Lu(x) = x \frac{du(x)}{dx}
\]  

acting in the space \( \mathcal{S} \) of all formal power series of the form

\[
\sum_{m=-\infty}^{+\infty} a_mx^m, \quad a_k \in \mathbb{C}.
\]  

(2.2)

It is easy to check that the eigenvalue problem

\[
Lu = \lambda u, \quad u \in \mathcal{S},
\]  

(2.3)

has the following solutions

\[
\lambda_j = j, \quad u_j(x) = x^j, \quad j = 0, \pm 1, \pm 2, \ldots.
\]  

(2.4)

Note that for \( j \geq 0 \), the eigenfunctions \( u_j(x) \) are polynomials, while for \( j < 0 \) they are rational functions with a unique pole of order \( -j \) at the origin.

The operator \( L \) is a particular (singular) case of a more general operator of the form

\[
\mathcal{L}u(z) = (a + bz + cz^2) \frac{du(z)}{dz} + hzu(z),
\]  

(2.5)

where \( a, b, c, h \in \mathbb{C} \). However, it turns out that the eigenvalues and eigenfunctions of \( \mathcal{L} \) in the space of formal power series (2.2) can be found for certain \( h \) by changing variables in the eigenvalue problem (2.3).

Indeed, let us consider the eigenvalue problem (2.3) and make the following change of the variable

\[
x := \frac{\alpha + \beta t}{\gamma + \delta t}, \quad \alpha \delta - \beta \gamma \neq 0,
\]  

(2.6)

\(^{1}\)Other results of [16, 17] are substantially generalised in [11].
At the same time, given a fixed integer $N \geq 1$ we also change the function $u$ by introducing a new function

$$w(t) := (\gamma + \delta t)^N u(x), \quad (2.7)$$

so that

$$u(x) = \frac{w(t)}{(\gamma + \delta t)^N}.$$ 

This gives us

$$\frac{d}{dx} \frac{x}{dx} = -\frac{(\alpha + \beta t)(\gamma + \delta t)}{\alpha \delta - \beta \gamma} \cdot \frac{d}{dt} \left[ \frac{w(t)}{(\gamma + \delta t)^N} \right] = -\frac{\alpha + \beta t}{\alpha \delta - \beta \gamma} \cdot \frac{(\gamma + \delta t) \frac{dw(t)}{dt} - N \delta w(t)}{(\gamma + \delta t)^N}. \quad (2.10)$$

Consequently, the problem (2.3) transforms into a new eigenvalue problem

$$\mathcal{L}_N w = \mu w, \quad w \in \mathcal{S}, \quad N \in \mathbb{N}, \quad (2.8)$$

where

$$\mathcal{L}_N w(t) = (\alpha + \beta t)(\gamma + \delta t) \frac{dw(t)}{dt} - \beta \delta N tw(t), \quad N \in \mathbb{N}, \quad (2.9)$$

and

$$\mu = \alpha \delta N - \lambda D \quad \text{with} \quad D = \alpha \delta - \beta \gamma \neq 0.$$ 

Now from (2.4), (2.6), and (2.7) we obtain that the solutions of the eigenvalue problem (2.8)–(2.9) are the following rational functions

$$w_j(t) = (\alpha + \beta t)^j (\gamma + \delta t)^{N-j}, \quad j \in \mathbb{Z}, \quad (2.10)$$

corresponding to the eigenvalues

$$\mu_j = \alpha \delta N - D j, \quad j \in \mathbb{Z}, \quad \text{with} \quad D = \alpha \delta - \beta \gamma \neq 0.$$ 

**Remark 2.1.** The formula (2.10) shows that for $j = 0, 1, \ldots, N$, the eigenvalue problem (2.8) has polynomial eigenfunctions $w_j$. All other eigenfunctions of (2.8) are rational functions.

### 3. Spectral problem for generalised Sylvester-Kac matrix

Let $\mathbb{C}_N[z], \ N \in \mathbb{N}$, be the set of all polynomials with complex coefficients of degree at most $N$. It is well known that $\mathbb{C}_N[z]$ is an $(N+1)$-dimensional space isomorphic to the space $\mathbb{C}^{N+1}$.

The operator $L$ defined in (2.1) being restricted to $\mathbb{C}_N[z]$ has exactly $N+1$ polynomial eigenfunctions in the space $\mathbb{C}_N[z]$ for any $N \in \mathbb{N}$. Remark 2.1 says that the operator $\mathcal{L}_N$ defined in (2.9) also has exactly $N+1$ eigenpolynomials. Therefore, we can restrict this operator to $\mathbb{C}_N[z]$, and, in this space, $\mathcal{L}_N$ has exactly $N+1$ distinct eigenvalues and the correspondent polynomial eigenfunctions.

Let

$$\mathcal{A}_N = \mathcal{L}_N \bigg|_{\mathbb{C}_N[z]} \quad (3.1)$$

From (2.9), it follows that if

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_N z^N \in \mathbb{C}_N[z], \quad (3.2)$$

then

$$(\alpha + \beta z)(\gamma + \delta z) \frac{dp(z)}{dz} - N \beta \delta p(z) = [N a_N (\alpha \delta + \beta \gamma) - \beta \delta \cdot a_{N-1}] z^N + O(z^{N-1}) \quad \text{as} \quad z \to \infty,$$

so $\mathcal{L}_N p \in \mathbb{C}_N[z]$ for any $p \in \mathbb{C}_N[z]$. Thus, we have

$$\mathcal{A}_N : \mathbb{C}_N[z] \to \mathbb{C}_N[z].$$

Consequently, $\mathcal{A}_N$ is a finite-dimensional operator, and the eigenvalue problem

$$\mathcal{A}_N v = \mu v$$
has exactly $N+1$ linearly independent polynomial eigenfunctions
\[ w_j(z) = (\alpha + \beta z)^j(\gamma + \delta z)^{N-j}, \quad j = 0, 1, \ldots, N, \tag{3.3} \]
corresponding to the eigenvalues
\[ \mu_j = \alpha\delta N - D_j, \quad j = 0, 1, \ldots, N, \quad \text{with} \quad D = \alpha\delta - \beta\gamma \neq 0. \tag{3.4} \]

On the other hand, the operator $A_N$ can be represented as an $(N+1) \times (N+1)$ matrix. Namely, for the polynomial $p$ defined by (3.2), let us consider the (column) vector $v = (a_0, a_1, \ldots, a_N)^T$ of its coefficients (here "$T$" stands for the transpose). Then there exists a matrix $J_N$ such that
\[ J_N v = (b_0, b_1, \ldots, b_N)^T \]
is the vector of the coefficients of the polynomial $A_N p$. From (2.9), (3.1), and (3.2), one gets
\begin{align*}
    b_0 &= \alpha \gamma \cdot a_1, \\
    b_1 &= -N\beta\delta \cdot a_0 + (\alpha\gamma + \beta\delta)a_1 + 2\alpha\gamma \cdot a_2, \\
    b_k &= -(N-k+1)\beta\delta \cdot a_{k-1} + k(\alpha\gamma + \beta\delta)a_k + (k+1)\alpha\gamma \cdot a_{k+1}, \\
    b_{N-1} &= -2\beta\delta \cdot a_{N-2} + (N-1)(\alpha\gamma + \beta\delta)a_{N-1} + N\alpha\gamma \cdot a_N, \\
    b_N &= -\beta\delta \cdot a_{N-1} + N(\alpha\gamma + \beta\delta)a_N.
\end{align*}

Thus, the matrix
\[
    J_N = \begin{pmatrix}
    0 & \alpha \gamma & 0 & \ldots & 0 & 0 & 0 \\
    -N\beta\delta & \alpha\delta + \beta\gamma & 2\alpha\gamma & \ldots & 0 & 0 & 0 \\
    0 & -(N-1)\beta\delta & 2(\alpha\delta + \beta\gamma) & \ldots & 0 & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
    0 & 0 & 0 & \ldots & (N-2)(\alpha\delta + \beta\gamma) & (N-1)\alpha\gamma & 0 \\
    0 & 0 & 0 & \ldots & -2\beta\delta & (N-1)(\alpha\delta + \beta\gamma) & N\alpha\gamma \\
    0 & 0 & 0 & \ldots & 0 & -\beta\delta & N(\alpha\delta + \beta\gamma)
    \end{pmatrix}
\tag{3.5}
\]
is the matrix representation of the operator $A_N$ in $\mathbb{C}^{N+1}$ in the canonical basis. Consequently, $J_N$ has the eigenvalues (3.4), and the correspondent eigenvectors are the vectors of the coefficients of the polynomials (3.3). We therefore arrive at the following theorem.

**Theorem 3.1.** Under the conditions that $\alpha, \beta, \gamma, \delta \neq 0$ and $\alpha\delta \neq \beta\gamma$, the eigenvalues of the matrix $J_N$ defined by (3.5) are
\[ \mu_j = \alpha\delta(N-j) + \beta\gamma \cdot j, \quad j = 0, 1, \ldots, N, \tag{3.6} \]
and $v_j = (v_{0j}, v_{1j}, \ldots, v_{Nj})^T$ is the eigenvector corresponding to $\mu_j$, where
\[ v_{kj} = \sum_{i=0}^{\min(k,j)} \binom{j}{i} \binom{N-j}{k-i} \binom{\delta}{i} \binom{\beta}{\alpha} \cdot k-i, \quad k = 0, 1, \ldots, N. \tag{3.7} \]

**Proof.** The formula (3.6) follows from (3.4). The formula (3.7) follows from the fact that the eigenpolynomials (3.3) of the operator $A_N$ can be represented in the form
\[ w_j(z) = (\alpha + \beta z)^j(\gamma + \delta z)^{N-j} = \alpha^j \gamma^{N-j} \sum_{i=0}^{j} \sum_{m=0}^{N-j} \binom{j}{i} \binom{N-j}{m} \binom{\beta}{\alpha} \cdot \binom{\delta}{\gamma} \cdot z^{i+m}, \]
which after a change of the summation index turns into
\[ 
    \frac{w_j(z)}{\alpha^j \gamma^{N-j}} = \sum_{k=0}^{N} \sum_{i=0}^{\min(k,j)} \binom{j}{i} \binom{N-j}{k-i} \binom{\delta}{i} \binom{\beta}{\alpha} \cdot z^k = \sum_{k=0}^{N} v_{kj} z^k.
\]
\[ \square \]
Remark 3.2. The case when at least one of the numbers \( \alpha, \beta, \gamma, \delta \) equals zero (with \( \alpha \delta - \beta \gamma \neq 0 \)) is not very interesting from the matrix point of view, since the matrix (3.5) is triangular in this case.

Regarding the differential operator \( L_N \) defined in (2.9), for \( \beta = 0 \) or \( \delta = 0 \) it degenerates (up to a linear change of the variable) to the operator \( L \) of the form (2.1). The case \( \alpha = 0 \) or \( \gamma = 0 \) with \( \beta \delta \neq 0 \) can be transformed by a linear change of the variable into the generic case when none of the numbers \( \alpha, \beta, \gamma, \delta \) in the operator \( L_N \) is zero.

Remark 3.3. If \( D = \alpha \delta - \beta \gamma = 0 \), we cannot use the linear-fractional transform as in (2.6). The operator \( L_N \) defined by (2.9) then has cases depending on whether \( \beta \delta = 0 \) or not.

If \( \beta \delta = 0 \), then (unless \( L_N \) is trivial) the condition \( D = 0 \) implies\(^2\) that \( \beta = \delta = 0 \), and hence

\[
L_N w(z) = \alpha \gamma \frac{dw(z)}{dz}.
\]

Here the only eigenpolynomial is \( w_0(z) \equiv 1 \), and the corresponding eigenvalue is \( \mu_0 = 0 \). In this case, the matrix of the operator \( A_N \) has one nontrivial diagonal, namely the superdiagonal; the unique eigenvalue \( \mu_0 \) of \( A_N \) is of algebraic multiplicity \( N + 1 \) and of geometric multiplicity 1.

If \( \beta \delta \neq 0 \), then \((\gamma + \delta z) = \frac{\alpha}{\beta}(\alpha + \beta z)\), and hence

\[
L_N w(z) = \frac{\delta}{\beta}(\alpha + \beta z)^2 \frac{dw(z)}{dz} - \beta \delta N zw(z).
\]

So, on letting \( t = (\alpha + \beta z) \) and \( p(t) = w(z) \) the eigenproblem \( L_N w(z) = \mu w(z) \) transforms into

\[
\delta t^2 \frac{dp(t)}{dt} - \delta Ntp(t) = (\mu - \alpha \delta N)p(t).
\]

An examination of the coefficients of this equality and the highest and lowest powers of \( t \) shows that it may only hold when \( \deg p = N \), and only when \( \mu = \alpha \delta N \). However, these two restrictions imply that \( p(t) = t^N \) up to a normalisation. Accordingly, the only eigenpolynomial of \( L_N \) in this case is \( w_0(z) = (\alpha + \beta z)^N \), which corresponds to the eigenvalue \( \mu_0 = \alpha \delta N \). The matrix of the operator \( A_N \) also has a unique eigenvalue \( \mu_0 \) of algebraic multiplicity \( N + 1 \) and of geometric multiplicity 1. The characteristic polynomial of \( J_N \) for the specific case \( \alpha = -\beta = -1/2 \) and \( \gamma = -\delta = 1 \) was found by L. Painvin in 1858, see [28, p. 434].

4. **PARTICULAR CASES**

In this section, we consider particular cases of the matrix \( J_N \) defined in (3.5).

Given \( a, b, c \in \mathbb{C}, \quad c \neq 0 \), let us set

\[
\alpha := \frac{b - \sqrt{D}}{4c}, \quad \beta := \frac{1}{2}, \quad \gamma := b + \sqrt{D}, \quad \delta := 2c, \quad \text{where} \quad D = b^2 - 4ac. \quad (4.1)
\]

Then the matrix (3.5) gets the form

\[
B_N(a, b, c) = \begin{pmatrix}
0 & a & 0 & \ldots & 0 & 0 & 0 \\
-Nc & b & 2a & \ldots & 0 & 0 & 0 \\
0 & -(N - 1)c & 2b & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & (N - 2)b & (N - 1)a & 0 \\
0 & 0 & 0 & \ldots & -2c & (N - 1)b & Na \\
0 & 0 & 0 & \ldots & 0 & -c & Nb
\end{pmatrix}. \quad (4.2)
\]

It represents the differential operator

\[
L_{a,b,c}u(z) = (a + bz + cz^2)\frac{dw(z)}{dz} - Nczu(z) \quad (4.3)
\]

restricted to \( \mathbb{C}_N[z] \).

\(^2\)Otherwise, the leading polynomial coefficient of the operator (2.9) becomes zero.
Remark 4.1. In (4.3) we additionally suppose that \( a \neq 0 \), since the case \( a = 0 \) can be transformed into the generic case \( (a \neq 0) \) by a linear change of the variable \( z \).

The expressions (3.6)–(3.7) and (4.1) imply that the matrix (4.2) has the following eigenvalues:

\[
\lambda_j = j \cdot \frac{b + \sqrt{b^2 - 4ac}}{2} + (N - j) \cdot \frac{b - \sqrt{b^2 - 4ac}}{2}, \quad j = 0, 1, \ldots, N, \tag{4.4}
\]

and the correspondent eigenvectors \( v_j = (v_{0j}, v_{1j}, \ldots, v_{Nj})^T \) are given by

\[
v_{kj} = \left( \frac{2c}{b + \sqrt{D}} \right)^k \cdot \sum_{i=0}^{\min(k,j)} \binom{j}{i} \binom{N - j}{k - i} \left( \frac{b + \sqrt{D}}{b - \sqrt{D}} \right)^i, \quad j = 0, 1, \ldots, N, \tag{4.5}
\]

where \( D \) is defined in (4.1).

The (rational) eigenfunctions of the operator (4.3) in the space \( S \) corresponding the eigenvalues (4.4) for \( j \in \mathbb{Z} \) are the following

\[
Q_j(z) = \frac{(2c)^N}{(\sqrt{D} - b)^j (\sqrt{D} + b)^{N-j}} \left( z - \frac{\sqrt{D} - b}{2c} \right)^j \left( z + \frac{\sqrt{D} + b}{2c} \right)^{N-j}, \quad j \in \mathbb{Z}. \tag{4.6}
\]

Let us list some particular cases of the matrix \( B_N(a, b, c) \) considered considered in literature.

1) The case \( b = 0, a = -c = 1 \) or \( \alpha = \beta = \gamma = 1, \delta = -1 \), corresponds to the Sylvester-Kac matrix [23, 9, 2, 36, 20, 28, 32, 38].

2) According to T. Muir [28, p. 434], the case \( b = 1, a + c = 1 \) or \( \alpha \delta + \beta \gamma = \alpha \gamma + \beta \delta = 1 \) was first considered by L. Painvin in 1858 for eigenvalues (see also [2, 20]). W. Chu and X. Wang [9] found eigenvectors for this matrix.

3) The case \( a = 1 - p, b = 2p - 1, c = -p \) or \( \alpha \delta + \beta \gamma = 2p - 1 = -2(\alpha \gamma + \beta \delta) \) (up to a transposition and a shift of eigenvalues) is related to the Krawtchouk polynomials [2, 20]. The corresponding eigenvectors were found in [9].

4) The eigenvalues and eigenvectors for the case \( b = -(c + a) \) or \( \alpha \delta + \beta \gamma = -(\alpha \gamma + \beta \delta) \) (up to a shift of eigenvalues) were found in [15]. This case covers the case 3). Note that the characteristic polynomial of this matrix (up to a diagonal shift) was found by T. Muir [27, § 576].

5) The eigenvalues of the matrix (4.2) for arbitrary \( a, b, \) and \( c \) were found in [7]. The eigenvectors (4.5) of the matrix \( B_N(a, b, c) \) are new.

As we mentioned in Section 1, all techniques in the aforementioned works are different from the one used here. Thus, we generalise the results of the works [2, 20, 9, 7, 15] in a simple and a unified way.

Note that in the degenerated case \( b^2 = 4ac \) (i.e. \( D = 0 \)) the matrix \( B_N(a, b, c) \) has a unique eigenvalue with exactly one eigenvector. In this case, the operator (4.3) restricted to \( \mathbb{C}_N[z] \) also has only one eigenvalue with a unique polynomial eigenfunction for every fixed \( N \in \mathbb{N} \), cf. Remark 3.3.

5. Discussion

The method applied in the present paper can be used to find the eigenvalues and eigenvectors of the tridiagonal matrix whose entries are the recurrence relation coefficients for the Hahn polynomials. It was noticed by A. Kovačec [26] that the spectrum of this matrix was conjectured by E. Schrödinger in [33]. A. Kovačec gave a proof of Schrödinger’s conjecture [26]. However, R. Askey [2] and O. Holtz [20] proved\(^3\) this conjecture much earlier, while W. Chu and X. Wang [9] found eigenvectors of the corresponding matrix. R. Oste and J. Van der Jeugt [31, 30] recovered the results of R. Askey and O. Holtz with their own original method, and gave an orthogonal polynomial interpretation of the results of W. Chu and X. Wang.

\(^3\)In two different ways distinguished from Kovačec’s one.
Our approach allows us to find the eigenvalues and eigenvectors of the Schrödinger matrix and to solve the generalised eigenvalue problem for a pair of linear differential operators in a very simple and more constructive manner. We believe that the results \cite{5, 8, 37, 31, 30} can also be improved by a similar approach, but this study is to be a subject for another paper.

There is a number of related problems where our approach does not give an immediate result, although other techniques prove to be very efficient. Appendix A of this work employs two distinct methods to explain why the spectra of two astonishingly simple tridiagonal matrices distinct from (1.1) are integer. These matrices were studied in works \cite{16, 17}, but we believe that our Appendix A provides deeper understanding of their properties.

Finally, some other matrices related to the Sylvester-Kac matrix (1.1) are studied in \cite{24, 25}. Our work \cite{11} not only finds spectra of those matrices in a straightforward way, but also determines their eigenvectors. In fact, \cite{11} expresses solution to the eigenvalue problem for a general tridiagonal matrix with 2-periodic main diagonal via the spectral data of the same matrix, in which the main diagonal is put to zero.

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APPENDIX A. CONCERNING TWO SPECIAL MATRICES STUDIED BY DA FONSECA ET AL.

The authors of \cite{16} calculate the eigenvalues of the matrix\(^4\)

\[
G_N = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & 0 & 0 \\
2N + 2 & 0 & 2 & \ldots & 0 & 0 & 0 \\
0 & 2N + 1 & 0 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & N - 1 & 0 \\
0 & 0 & 0 & \ldots & N + 4 & 0 & N \\
0 & 0 & 0 & \ldots & 0 & N + 3 & 0
\end{pmatrix}
\]  

(A.1)

via a smart choice of the basis and an induction in size of the matrix. In an analogous way, the work \cite{17} deals with the matrix \(H_N = \frac{1}{2}S_N\), where

\[
S_N = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & 0 & 0 \\
2N & 0 & 2 & \ldots & 0 & 0 & 0 \\
0 & 2N - 1 & 0 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & N - 1 & 0 \\
0 & 0 & 0 & \ldots & N + 2 & 0 & 2N \\
0 & 0 & 0 & \ldots & 0 & N + 1 & 0
\end{pmatrix}
\]  

(A.2)

The matrices \(G_N\) and \(S_N\) are the \((N + 1) \times (N + 1)\) leading principal submatrices of the Sylvester-Kac matrix (1.1): of \(K_{2N+2}\) and \(K_{2N}\), respectively, – except that the \((N, N + 1)\)th entry of \(S_N\) is doubled.

It turns out \cite{16, 17} that the spectra\(^5\) of these matrices are integer:

\[
\sigma(S_N) = \sigma(G_N) = \{2(2j - N)\}_{j=0}^{N}.
\]  

(A.3)

However, the method of \cite{16, 17} provides little understanding of the phenomenon. There is another reasoning relying on the properties of persymmetric Jacobi matrices, which is in our opinion both shorter and deeper. A matrix is called persymmetric if all its entries are symmetric with respect to the anti-diagonal.

\(^4\)We notice that this matrix is a particular case of the one appeared in the work of A. Caley \cite{6}, see also \cite[p. 429]{28} and \cite[p. 355]{36}.

\(^5\)Here, \(\sigma(A)\) stands for the spectrum of a matrix \(A\). We remind the reader that in \cite{17} the authors deal with \(H_N = \frac{1}{2}S_N\).
Let $D_N$ denote the $(N+1) \times (N+1)$ diagonal matrix of the factorials, $D_N = \|j! \delta_{i,j}\|_{i,j=0}^N$. Here $\delta_{i,j}$ stands for the Kronecker delta symbol. Direct computation shows that the product $D_{2N+2}K_{2N+2}D_{2N+2}^{-1}$ is a persymmetric matrix with a unit superdiagonal. Moreover, its $(N+1) \times (N+1)$ leading principal submatrix is the matrix $D_N G_N D_N^{-1}$ whose spectrum coincides with the spectrum of $G_N$. At the same time, [18, eq. (15), Lemma 3.3] implies that the spectrum of $D_N G_N D_N^{-1}$ contains every second spectral point of $D_{2N+2}K_{2N+2}D_{2N+2}^{-1}$, or more specifically that $\sigma(G_N) = \sigma(D_N G_N D_N^{-1})$ satisfies (A.3).

Now, let $I_N$ denote the $(N+1) \times (N+1)$ identity matrix. Consider two polynomials

$$\Omega_1(x) = \det(xI_N - G_N) \quad \text{and} \quad \Omega_0(x) = \frac{\det(xI_{2N+2} - K_{2N+2})}{\Omega_1(x)}.$$ 

From [18, eq. (15), Lemma 3.3] we additionally have

$$\det(xI_{N+1} - M_{N+1}) = \frac{\Omega_0(x) + x\Omega_1(x)}{2},$$

where $M_{N+1}$ is the $(N+2) \times (N+2)$ leading principal submatrix of $K_{2N+2}$. Now, using linearity of $\det(xI_{N+1} - S_{N+1})$ with respect to the last column we obtain:

$$\det(xI_{N+1} - S_{N+1}) = 2 \det(xI_{N+1} - M_{N+1}) - x\Omega_1(x) = \Omega_0(x) + x\Omega_1(x) - x\Omega_1(x) = \Omega_0(x),$$

which yields (A.3) for $\sigma(S_N)$. Thus, the main results of [16, 17] rediscover some properties of persymmetric Jacobi matrices for the particular case of the Sylvester-Kac matrix.

On the other hand, the main results of [16, 17] may also be derived from the results of the works [20, 2, 9]. Indeed, let us consider the matrix of the three-term recurrence relations for the Hahn polynomials with $\alpha = \beta$ (see [2, formula (4.9)]):

$$C_N(\alpha) = \begin{pmatrix}
  b_0 & a_0 & 0 & \cdots & 0 & 0 \\
  c_1 & b_1 & a_1 & \cdots & 0 & 0 \\
  0 & c_2 & b_2 & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & b_{N-1} & a_{N-1} \\
  0 & 0 & 0 & \cdots & c_N & b_N
\end{pmatrix},$$

where

$$a_0 = \frac{N}{2}, \quad a_i = \frac{(i + 2\alpha + 1)(N - i)}{2(2i + 2\alpha + 1)}, \quad i = 1, \ldots, N - 1,$$

$$b_i = \frac{N}{2}, \quad i = 0, 1, \ldots, N,$$

$$c_i = \frac{i(i + 2\alpha + N + 1)}{2(2i + 2\alpha + 1)}, \quad i = 1, \ldots, N,$$

As is shown in [2, 20], the spectrum of the matrix $C_N(\alpha)$ does not depend on $\alpha$ and has the form

$$\sigma(C_N(\alpha)) = \{0, 1, \ldots, N\}. \quad \text{(A.4)}$$

With $S_N$ defined by (A.2), the aforementioned matrix $H_N = \frac{1}{2}S_N$ is related to the matrix $C_N(-\frac{1}{2})$ as follows:

$$H_N^T = E_N \left[2C_N \left(-\frac{1}{2}\right) - N \cdot I_N \right] E_N^{-1}, \quad \text{where} \quad E_N = \|\delta_{i,N-j}\|_{i,j=0}^N. \quad \text{(A.5)}$$

In [9, Theorem 5.1], the authors found the eigenvectors for the matrix of the three-term recurrence relations corresponding to the Racah polynomials, which turn into the (right) eigenvectors of $C_N(\alpha)$ on putting $\alpha = \beta$ and $\gamma \to \infty$. According to (A.5), reversing the order of entries in these vectors yields the transposed left eigenvectors of the matrix $H_N$. In other words, from (A.4)–(A.5) and [9, Theorem 5.1] for $u_j H_N = \lambda_j u_j$ we directly obtain

$$\lambda_j = 2j - N, \quad j = 0, 1, \ldots, N, \quad \text{(A.6)}$$
with the entries of the row-vector \( u_j = (u_{0j}, u_{1j}, \ldots, u_{Nj}) \) given by the following formula\(^6\)

\[
  u_{kj} = \sum_{i=0}^{\min\{N-k,j\}} (-1)^{k+i} \binom{N-k}{i} \binom{N-i}{j-i} \cdot \frac{(N-k)_i}{(1/2)_i}, \quad k, j = 0, 1, \ldots, N,
\]

where \((a)_i = a(a+1) \cdots (a+i-1)\). Moreover, there is a simple formula expressing the right eigenvectors of a tridiagonal matrix via its left eigenvectors, see e.g. [11]. Thus, the entries of the right eigenvector \( v_j = (v_{0j}, v_{1j}, \ldots, v_{Nj})^T \) of \( H_N \) corresponding to the \( j \)th eigenvalue \((A.6)\), \( j = 0, 1, \ldots, N \), have the form

\[
  v_{kj} = \left( \frac{2N}{k} \right) \min\{N-k,j\} \sum_{i=0}^{\min\{N-k,j\}} (-1)^{k+i} \binom{N-k}{i} \binom{N-i}{j-i} \cdot \frac{(N-k)_i}{(1/2)_i}, \quad k, j = 0, 1, \ldots, N-1,
\]

\[
  v_{Nj} = \frac{1}{2} \left( \frac{2N}{N} \right) \binom{N}{j}.
\]

The matrix \( G_N \) defined in \((A.1)\) is related to the matrix \( C_N \left( \frac{1}{2} \right) \) in a similar manner, namely,

\[
  G_N^T = R_N \left[ 4C_N \left( \frac{1}{2} \right) - 2NI_N \right] R_N^{-1}
\]

with \( R_N = \|(i+1)\delta_{i,N-j}\|_{i,j=0}^N \). Now from \((A.4)\) and \((A.7)\) we obtain that the eigenvalues of \( G_N \) are

\[
  \lambda_j = 2(2j-N), \quad j = 0, 1, \ldots, N.
\]

On letting \( \alpha = \beta = \frac{1}{2} \) and \( \gamma \to \infty \) in \([9, \text{Theorem 5.1}]\), we see that for \( j = 0, 1, \ldots, N \) the left eigenvector \( u_j = (u_{0j}, u_{1j}, \ldots, u_{Nj}) \) of \( G_N \) corresponding to the eigenvalue \( \lambda_j \) has the following entries:

\[
  u_{kj} = \sum_{i=0}^{\min\{N-k,j\}} (-1)^{k+i} \binom{N-k}{i} \binom{N-i}{j-i} \cdot \frac{(N+1-k)_i+1}{(3/2)_i}, \quad k, j = 0, 1, \ldots, N.
\]

Consequently, for \( k, j = 0, 1, \ldots, N \) the \( k \)th entry of the right eigenvector \( v_j = (v_{0j}, v_{1j}, \ldots, v_{Nj})^T \) corresponding to the \( j \)th eigenvalue \((A.8)\) has the form

\[
  v_{kj} = \left( \frac{2N+2}{k} \right) \sum_{i=0}^{\min\{N-k,j\}} (-1)^{k+i} \binom{N-k}{i} \binom{N-i}{j-i} \cdot \frac{(N+1-k)_i+1}{(3/2)_i}.
\]

Thus, the main results of the works \([16, 17]\) on the spectra of the matrices \( G_N \) and \( H_N \) also follow from the results of the works \([18]\) or \([2, 20]\), while the formulae for the entries of the left and right eigenvectors of \( G_N \) and \( H_N \) were actually found in \([9]\).

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\(^6\)We put \( \alpha = \beta = -\frac{1}{2} \) and \( \gamma \to \infty \) in \([9, \text{Theorem 5.1}]\) and multiply the matrix of the right eigenvectors of \( C \left( \frac{1}{2} \right) \) by the matrix \( E_N \).
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