Synchronization transition in ensemble of coupled phase oscillators with coherence-induced phase correction

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(Dated: April 13, 2015)

We study a synchronization phenomenon in the self-correcting population of noisy phase oscillators with randomly distributed natural frequencies. In our model each oscillator stochastically switches its phase to the ensemble-averaged value ψ at a typical rate which is linearly proportional to the degree of coherence r. The system exhibits a continuous phase transition to collective synchronization similar to classical Kuramoto model. Based on the self-consistent arguments and linear stability analysis of the incoherent state we derive analytically the threshold value kc of coupling constant corresponding to the onset of partially synchronized state. Just above the transition point the linear scaling law r ∝ k − kc is found. We also show that nonlinear relation between rate of phase correction and order parameter leads to non-trivial transition between incoherence and synchrony.

To illustrate our results, numerical simulations have been performed for large population of phase oscillators with proposed type of coupling. The present model could become useful for explaining cooperative phenomena in communities of oscillatory units and in designing self-correcting systems with well-controlled dynamical behaviour.

The collective synchronization of oscillatory units is observed in many complex physical, chemical, biological and sociological systems with different origins of rhythmic activity. Depending on the context the term "oscillatory unit" may refer to neurons [1], cardiac pacemaker cells [2], flashing fireflies [3], chirping crickets [4], nano-mechanical resonators [5], lasers [6] and Josephson junctions [7], applauding spectators [8]. Understanding of the crucial role of mutual coupling in the synchronization phenomena goes back to Huygens, who described the synchronization of pendulum-clocks hanging on the same wooden beam [9]. Various types of interaction between oscillating elements have been proposed and investigated: pulse coupling [10–12], continuous coupling through the phase difference in Kuramoto-like models [13, 14], coupling through the collective output in a system of two-mode stochastic oscillators [8], coupling through the transition rate in a model of discrete phase units [13]. Here we present a new model of coupled oscillators exhibiting collective synchrony due to stochastic mechanism of coherence-induced phase correction.

Consider a diverse population of N noisy oscillators which are solely characterized by the phase variables \( \varphi_i \in [-\pi, \pi] \). We will use the Kuramoto’s order parameter [13, 16]

\[
r e^{i\psi} = \frac{1}{N} \sum_{j=1}^{N} e^{i\varphi_j},
\]

(1)

to describe the collective rhythm produced by the whole population. The magnitude r of this complex parameter measures the degree of phase coherence, while \( \psi \) is the average phase. In the absence of coupling the dynamics of the ith oscillator is given by

\[
\partial_t \varphi_i = \Omega_i + \xi_i,
\]

(2)

where the intrinsic frequency \( \Omega_i \) is randomly chosen from some probability density \( g(\Omega) \) and \( \xi_i(t) \) is the Gaussian white noise with zero mean and pair correlator \( \langle \xi_i(t_1)\xi_j(t_2) \rangle = 2D\delta_{ij}\delta(t_1-t_2) \). We then obtain the following Fokker-Plank equation [17]

\[
\partial_t n = D\partial_\varphi^2 n - \Omega \partial_\varphi n,
\]

(3)

where function \( n(\varphi, \Omega, t) \) is the one-oscillator probability density. Apparently, \( n \) is nonnegative, 2π-periodic in \( \varphi \), and satisfies the normalization condition \( \int_{-\pi}^{\pi} n(\varphi, \Omega, t) d\varphi = 1 \). The order parameter \( |n(\varphi, \Omega, t)| \) can be written as

\[
r e^{i\psi} = \int d\Omega g(\Omega) \int_{-\pi}^{\pi} e^{i\varphi} n(\varphi, \Omega, t) d\varphi.
\]

(4)

Any solution of Eq. 4 relaxes to the steady-state distribution \( n(\varphi, \Omega) = 1/2\pi \) corresponding to zero coherence r. Predictably, the diverse population of non-interacting noisy units behaves incoherently.

Now let us turn to analysis of coupled oscillators. In our model the coupling manifests itself as a tendency of each oscillator to adjust its phase to the ensemble-averaged value \( \psi \) through episodic phase shifts. Namely, the phase of the ith oscillator evolves accordingly to equation 2 (as if there is no coupling at all), but sometimes the stochastic correction events shift the phase to \( \psi \). The random time points at which these events occur are treated as statistically independent for different oscillators. In what follows we assume that time \( \tau \) between consecutive phase shifts has a Poisson distribution \( \alpha e^{-\alpha \tau} \) with parameter \( \alpha \) representing an average rate of phase correction per oscillator. A crucial assumption of our model is the coherence-dependent correction rate: the more pronounced the coherence r of the current state of population, the more frequently the oscillator switches its phase to \( \psi \). First, we focus on the case of linear proportionality between correction rate and the magnitude.
of the order parameter. In continuum limit $N \to \infty$ we write the following equation for the probability density $n(\varphi, \Omega, t)$

$$\partial_t n = D \partial_{\varphi}^2 n - \Omega \partial_{\varphi} n - \alpha n + \alpha \delta(\varphi - \psi),$$

(5)

where the correction rate is given by

$$\alpha(t) = kr(t) = k \left| \int_{-\infty}^{+\infty} d\Omega g(\Omega) \int_{-\pi}^{\pi} e^{i\varphi} n(\varphi, \Omega, t) d\varphi \right|,$$

(6)

and $k \geq 0$ is the interaction constant. The structure of the equation [3] is quite transparent: the first and second terms in the right hand side correspond to a standard diffusion and uniform rotation at natural frequency (see Eq. 3), while the third and fourth terms are related to stochastic phase correction and represent a negative probability flux out of each point $\varphi$ and a corresponding positive probability flux into $\varphi = \psi$. Taking into account the $2\pi$-periodicity of function $n(\varphi)$ it is easy to show the conservation of total probability, $\int_{-\pi}^{\pi} n(\varphi, \Omega, t) d\varphi = 1$.

Note that Eq. (6) may be thought of as the Master equation for diffusion in the presence of stochastic resetting (see e.g. [18]) with $\psi$ and $\alpha$ playing a role of resetting position and resetting rate respectively.

Apparently, the proportionality law [14] sets up a positive feedback so that as the population becomes more coherent, $r$ grows and so the resetting rate $\alpha$ increases, which tends to synchronize the oscillators even stronger. A central question is whether there is a synchronization transition in this system. The answer is positive, below we demonstrate that for sufficiently large coupling constant $k$ the phase correction compensates the destructive effects of noises and frequency’s diversity and the partially synchronized state emerges.

Let us consider the case of unimodal frequency distribution $g(\Omega)$ which is symmetric about its mean value $\Omega_0$. In the spirit of original Kuramoto’s analysis [13] we seek the solution, where $r$ is constant and $\psi(t)$ rotates at frequency $\Omega_0$. Passing into the rotating frame $\varphi \to \varphi + \Omega t$ one can set $\psi = 0$ without loss of generality. The solution of the stationary equation

$$D \partial_{\varphi}^2 n - \omega \partial_{\varphi} n - \alpha n + \alpha \delta(\varphi) = 0,$$

(7)

with periodic condition $n(-\pi, \omega) = n(\pi, \omega)$ is

$$n(\varphi, \omega) = \begin{cases} A_+ \exp[\gamma_1 \varphi] + B_+ \exp[\gamma_2 \varphi], & 0 \leq \varphi \leq \pi, \\ A_- \exp[\gamma_1 \varphi] + B_- \exp[\gamma_2 \varphi], & -\pi < \varphi \leq 0, \end{cases}$$

(8)

where

$$\omega = \Omega - \Omega_0, \quad \gamma_{1,2} = \frac{\omega \pm \sqrt{\omega^2 + 4\alpha D}}{2D},$$

(9)

and the coefficients $A_\pm$ and $B_\pm$ are given by [24]. For $\alpha \neq 0$ this stationary distribution is nonequilibrium since phase correction produces ongoing circulation of probability.

We can now evaluate the order parameter [11] by using [8]. Since $\psi = 0$ and $\alpha = kr$, the self-consistency condition reduces to

$$r = 16rkD^2(kr + D) \int_{-\infty}^{+\infty} d\omega g(\Omega_0 + \omega) \frac{g(\Omega_0 + \omega)}{(4D^2 + (\omega + \sqrt{\omega^2 + 4\alpha D})^2)(4D^2 + (\omega - \sqrt{\omega^2 + 4\alpha D})^2)}.$$

(10)

This equation has always the trivial solution $r = 0$ corresponding to incoherence $n = 1/2\pi$. Setting $r \to +0$ we find the critical coupling strength

$$k_c = 1\left| \int_{-\infty}^{+\infty} d\omega \frac{g(\Omega_0 + \omega)}{\omega^2 + D^2} \right|^{-1},$$

(11)

corresponding to onset of partially synchronized state with non-zero order parameter. Remarkably, formula (11) has the same structure as the well-known result for critical coupling in the noisy Kuramoto model [19]. At the same time, the models differ by the critical behaviour of the order parameter. An expansion of the right-hand side of Eq. (10) in powers of $kr$ yields the linear scaling law $r \propto k - k_c$ in the vicinity of transition. In contrast, in classical Kuramoto model the order parameter of the bifurcating branch obeys the square-root law $r \propto \sqrt{k - k_c}$ near threshold.

The critical value [11] is confirmed by the linear stability analysis. Let us consider a small perturbation of the uniform incoherent state

$$n(\varphi, \omega, t) = \frac{1}{2\pi} + \varepsilon \rho(\varphi, \omega, t),$$

(12)

where $\varepsilon \ll 1$. At lowest (linear) order in $\varepsilon$ the evolution of perturbation is governed by equation

$$\partial_t \rho = D \partial_{\varphi}^2 \rho - \omega \partial_{\varphi} \rho - \frac{kr}{2\pi} + k \rho \delta(\varphi - \psi).$$

(13)

To proceed we write a $2\pi$-periodic function $\rho$ as a superposition of Fourier harmonics

$$\rho(\varphi, \omega, t) = \sum_{m=1}^{\infty} \epsilon_m(\omega, t)e^{im\varphi} + c.c.$$

(14)
Note that normalization condition automatically provides $c_0 = c_0^* = 0$. Only the first harmonic contributes to the order parameter:

$$r e^{i\psi} = 2\pi \int_{-\infty}^{+\infty} d\omega g(\omega + \Omega_0) c_1^*(\omega, t).$$  \hfill (15)

When (14) and (15) are substituted into (13), we obtain the system of equations for amplitudes $c_m$ and $c_m^*$.

$$\partial_t c_1 = -(D + i\omega)c_1 + k \int d\nu g(\nu + \Omega_0)c_1(\nu, t),$$  \hfill (16)

and the same closed equation describes also evolution of $c_1^*(\omega, t)$.

Let $c_1(\omega, t) = a(\omega) e^{\lambda t}$, then we obtain $\hat{L} a = \lambda a$, where

$$\hat{L} = -(D + i\omega) + k \int d\nu g(\nu + \Omega_0).$$

The spectrum of the operator $\hat{L}$ was constructed in [20] in the context of noisy Kuramoto model [19]. They show that the continuous part of spectrum lies in left half-plane ($\Re \lambda = -D$) for any $k$, thus corresponding modes never cause instability. In contrast, the location of discrete spectrum depends strongly on interaction constant $k$. When the coupling constant $k$ exceeds a threshold [11], the incoherent state becomes unstable thanks to the discrete eigenvalue $\lambda > 0$.

Thus, for $k > k_c$, the incoherence is linearly unstable to perturbation involving first order harmonic. To prove the linear stability below this threshold we should consider also the dynamics of high order harmonics

$$\partial_t c_m = -(m^2 D + im\omega)c_m + kr(t)e^{-im\psi(t)}.$$  \hfill (17)

This equation may be solved easily in terms of $r(t)$, $\psi(t)$ and the initial condition $c_m(\omega, 0)$. The result is

$$c_m(\omega, t) = c_m(\omega, 0)e^{-(m^2 D + im\omega)t} +$$

$$+ ke^{-(m^2 D + im\omega)t} \int_{0}^{t} dt' e^{(m^2 D + im\omega)t'}r(t')e^{-im\psi(t')}.$$  \hfill (18)

The first term in the rhs rapidly dies out, while the behaviour of the second term depends on the order parameter. The latter is completely determined by first harmonic through Eq. (15), thus we substitute $r(t)e^{i\psi(t)} = r(0)e^{i\psi(0) + \lambda t}$, where the eigenvalue $\lambda$ belongs to the spectrum described above. This leads us to the following estimate

$$\left| e^{-(m^2 D + im\omega)t} \int_{0}^{t} dt' e^{(m^2 D + im\omega)t'}r(t')e^{-im\psi(t')} \right| \leq r(0)e^{-m^2 D t} \int_{0}^{t} dt' e^{m^2 D t'} + \Re \lambda \int_{0}^{t} dt' e^{m^2 D t'} = \frac{r(0)}{m^2 D + \Re \lambda} (e^{\Re \lambda t} - e^{-m^2 D t}).$$  \hfill (19)

The denominator in (19) never vanishes since $m > 1$ and $\Re \lambda \geq -D$ for any $k$ [20]. At $k < k_c$ the spectrum $\lambda$ lies in the left half-plane, therefore $c_m(t)$ decays to zero.

More direct analysis of the global dynamics is possible when the diversity of natural frequencies is negligible, $g(\Omega) = \delta(\Omega - \Omega_0)$. Using Eq. (15) one can derive a closed-form equations associated with angular and radial motions of the order parameter $re^{i\psi} = \int_{-\pi}^{+\pi} e^{i\varphi} n(\varphi, t)d\varphi$

$$\dot{\psi} = \Omega_0,$$  \hfill (20)

$$\dot{r} = -Dr - \alpha r + \alpha.$$  \hfill (21)

The equation (20) suggests that the average phase just uniformly rotates at frequency $\Omega_0$. Substituting $\alpha = kr$
we find easily the solution of amplitude equation (21)
\[ r(t) = \frac{(k - D)e^{(k - D)t}r_0}{k - D - k(1 - e^{(k - D)t})r_0} \] (22)
where \( r_0 = r(0) \) is an initial condition. For \( k < D \) the coherence \( r \) always decays to zero as \( t \to \infty \), but for \( k > D \) incoherent state loses asymptotic stability and one obtains another attracting point \( r = (k - D)/k \) corresponding to partially synchronized state. The critical value \( k_c = D \) is consistent with (11) provided that \( g(\Omega) = \delta(\Omega - \Omega_0) \).

Now let us describe our numerical scheme. We use temporal discretization of Eq. (20) with time step \( \Delta t \) so that \( \varphi^n_i \) is the approximation of \( \varphi_i(n\Delta t) \), where \( n = 0, 1, 2, \ldots \). Then, the complex order parameter is \( r_n e^{i\psi_n} = (1/N) \sum_{i=1}^{N} e^{i\psi_i} \). The random forces \( \xi_i \) are modeled by the telegraph processes whose correlation times are equal to time step duration \( \Delta t \). Specifically, the values of \( \xi_i \) inside the \( n \)th step are chosen to be independent random constants \( \xi_i^n \) with identical normal distributions. To ensure a given value of the diffusion coefficient \( D \), one should choose \( \langle \xi_i^n \rangle = 2D/\Delta t \). At each time step we generate the set of \( N \) random numbers \( \tau_1^n, \tau_2^n, \ldots, \tau_N^n \) having Poisson distribution \( \alpha_n e^{-\alpha_n \tau} \) with \( \alpha_n = kr_n \). If \( \tau^n_i > \Delta t \), we switch the phase of \( i \)th oscillator to the current ensemble-averaged value \( \psi_n \). Thus, the rules for the evolution of the system are as follows

\[
\varphi_i^{n+1} = \begin{cases} 
\varphi_i^n + (\Omega_i + \xi_i^n) \Delta t, & \text{if } \tau_i^n > \Delta t, \\
\psi_i^n + (\Omega_i + \xi_i^n)(\Delta t - \tau_i^n), & \text{if } \tau_i^n \leq \Delta t.
\end{cases}
\] (23)

In numerics the initial phases of \( N = 1000 \) oscillators were uniformly distributed over the interval \([\pi, \pi]\) and the discrete time step \( \Delta t = 0.001 \) was used. The diffusivity was \( D = 1 \) and the natural frequencies \( \Omega_i \) were chosen from normal distribution \( g(\Omega) \) with expected value \( \Omega_0 = 0 \) and standard deviation \( \sigma_\Omega = 0.5 \). Figure 1 represents numerical results for the time-averaged coherence \( r \) as a function of coupling constant \( k \) in comparison with theoretical curve for infinite-\( N \) system. It can be seen that degree of phase coherence remains close to zero until \( k \) reaches a critical value \( k_c \approx 1.3 \), above which \( r \) rapidly increases towards its asymptotic value of unity.

Our model of collective synchronization, Eqs. (20) and (21), postulates the linear relation between correction rate \( \alpha \) and amplitude of the order parameter \( r \). One may expect more complex and interesting collective behaviour in system with non-linear dependence \( \alpha = \alpha(r) \). Let us consider as an illustration the quadratic model \( \alpha = kr^2 \) focusing on the case of identical oscillators. Then, the amplitude equation (21) reads: \( \dot{r} = -Dr - kr^3 + kr^2 \).

At \( k < 4D \) the only stationary solution is \( r = 0 \) and \( r(t) \) decays to zero as \( t \to \infty \) for any initial condition \( r(0) = r_0 \). However, for \( k > 4D \) there are three fixed points: \( r = 0, (1 \pm \sqrt{1 - 4D/k})/2 \). Thus, two partially synchronized branches bifurcate discontinuously from \( r = 1/2 \) at \( k_c = 4D \). The unstable fixed point (\( 1 - \sqrt{1 - 4D/k})/2 \) plays a role of border between the regions of stability of unsynchronized and synchronized states. The initial condition \( r_0 < (1 - \sqrt{1 - 4D/k})/2 \) relaxes to \( r = 0 \) as \( t \to \infty \), while \( r_0 > (1 - \sqrt{1 - 4D/k})/2 \) at long-times tends to \( r = (1 + \sqrt{1 - 4D/k})/2 \). This conclusion is confirmed by the direct numerical simulation of large population of oscillators, see Fig. 2.

FIG. 2. Temporal evolution of coherence \( r(t) \) in population of \( N = 1000 \) identical oscillators with quadratic phase-correction rate, \( \alpha = kr^2 \). The diffusion coefficient is \( D = 1 \) and the coupling strength is \( k = 5 > k_c \). Different curves stand for different initial configurations of phases. The upper and the lower dashed lines correspond to stable and unstable partially synchronized states of infinite-\( N \) population respectively.

To summarize, in this paper we proposed a model of coupled phase oscillators which undergo the coherence-induced phase correction. The model contains three sources of disorder: randomly distributed natural frequencies of oscillators, noise forces and independent stochastic correction processes. There is no continuous attraction between phases of oscillators, but from time to time correction events adjust the phases so as to facilitate phase coherence. We do not specify the mechanism responsible for these events just postulating phenomenologically the proportionality between the rate of phase correction and the coherence of the current state of population. Our analytical and numerical results revealed that as the coupling constant is increased beyond a certain threshold the partially synchronized state emerges. For nonlinear dependence of correction rate on the degree of phase coherence the synchronization transition is discontinuous and population exhibits multistability of global activity. Note, that we have restricted ourselves to the case of "constructive" phase correction which attempts to mutually synchronize the oscillators.
are many practical situations in which the formation of a coherent state in ensemble of oscillatory units is unacceptable and should be avoided [21]. One may expect that "distructive" phase correction (i.e. the phase shifts to $\psi + \pi$ rather than to $\psi$) is a good strategy to break the undesired synchronization of Kuramoto-like coupled oscillators. In future studies, it is interesting also to examine the generalization of our model on the case when the correction rate is expressed through the set of generalized order parameters [14].

I. APPENDIX

The coefficients in the expression (23) are

$$A_+ = \frac{\alpha}{(e^{2\pi \gamma_1} - 1) \sqrt{\omega^2 + 4\alpha D}}, \quad B_+ = -\frac{\alpha}{(e^{2\pi \gamma_2} - 1) \sqrt{\omega^2 + 4\alpha D}}$$

$$A_- = \frac{\alpha e^{2\pi \gamma_1}}{(e^{2\pi \gamma_1} - 1) \sqrt{\omega^2 + 4\alpha D}}, \quad B_- = -\frac{\alpha e^{2\pi \gamma_2}}{(e^{2\pi \gamma_2} - 1) \sqrt{\omega^2 + 4\alpha D}}$$

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