Higher Schur-multiplicator of a Finite Abelian Group

by

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Abstract

In this paper we obtain an explicit formula for the higher Schur-multiplicator of an arbitrary finite abelian group with respect to the variety of nilpotent groups of class at most $c \geq 1$.

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1. Introduction and Preliminaries

In 1907 I. Schur [7] proved that the Schur-multiplicator of a direct product of two finite groups is isomorphic to the direct sum of the Schur-multiplicators of the direct factors and the tensor product of the two groups abelianized. (see also J. Wiegold [8].) If

$$G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots \oplus \mathbb{Z}_{n_k}$$

is a finite abelian group, where \(n_{i+1}|n_i\), for all \(1 \leq i \leq k - 1\), then using the above fact one obtains the Schur-multiplicator of \(G\) as follows (see [4]):

$$M(G) \cong \mathbb{Z}_{n_2} \oplus \mathbb{Z}_{n_3}^{(2)} \oplus \ldots \oplus \mathbb{Z}_{n_k}^{(k-1)}$$

where \(\mathbb{Z}_{n}^{(m)}\) denotes the direct product of \(m\) copies of the cyclic group \(\mathbb{Z}_n\).

Now, in this paper, a similar result will be presented for the higher Schur-multiplicator of an arbitrary finite abelian group with respect to the variety of nilpotent groups of class at most \(c \geq 1\), \(\mathcal{N}_c\), say. (see [5] for the notation.)

Let \(G\) be any group with a free presentation

$$1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1$$

where \(F\) is a free group. Then, following Heinz Hopf [3],

$$\frac{R \cap F'}{[R, F]}$$
is isomorphic to the Schur-multiplicator of $G$, denoted by $M(G)$. Now, the higher Schur-multiplicator of $G$ with respect to the variety of nilpotent groups of class at most $c \geq 1$, is defined to be

$$\mathcal{N}_cM(G) = \frac{R \cap \gamma_{c+1}(F)}{[R, \gamma F]}$$

where $\gamma_{c+1}(F)$ is the $(c + 1)$st-term of the lower central series and $[R, \gamma F]$ denotes $[R, F, \cdots, F]_c$, see [5 or 6] for further details.

Let $H_i = \langle x_i^{r_i} \rangle \cong \mathbb{Z}_{r_i}$, $i = 1, 2, \ldots, t$, $r_i \in \mathbb{N}$ be cyclic groups of order $r_i$, $1 \leq i \leq t$, $r_i \geq 0$ and let

$$1 \rightarrow R_i = \langle x_i^{r_i} \rangle \xrightarrow{\pi_i} F_i = \langle x_i \rangle \rightarrow H_i \rightarrow 1$$

be the free presentation for $H_i$, where $1 \leq i \leq t$. Also, let

$$G = \prod_{i=1}^{t} H_i = \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \cdots \times \mathbb{Z}_{r_t}$$

be the direct product of cyclic groups $\mathbb{Z}_{r_i}$'s. Then

$$1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1$$

is the free presentation for $G$, where

$$F = \prod_{i=1}^{t} F_i = \langle x_1, \ldots, x_t \rangle \quad \text{and} \quad R = \langle x_i^{r_i}, \gamma_2(F) \cap [F_i]^* | i = 1, \ldots, t \rangle^F, $$

where $\prod_{i=1}^{t} F_i$ is the free product of $F_i$'s, $i = 1, 2, \ldots, t$, and $[F_i]^*$ is the normal closure of some commutator subgroups of the free product, defined
as follows:

\[ [F_i]^* = \langle [F_i, F_j] \mid 1 \leq i, j \leq t, i \neq j \rangle^F \]

Since \( F_i \)'s are cyclic, we have \( \gamma_2(F) \subseteq [F_i]^* \). Hence

\[ R = \langle x_i^{\sigma_2}, \gamma_2(F) \mid i = 1, \ldots, t \rangle^F. \]

Now, put \( S = \langle x_1^{\sigma_1}, \ldots, x_t^{\sigma_t} \rangle^F \) and for all \( m \geq 1 \), define \( \rho_1(S) = S \), \( \rho_m(S) = [S, m^{-1}F] \), inductively. This yields the central series

\[ S = \rho_1(S) \supseteq \rho_2(S) \supseteq \rho_3(S) \supseteq \ldots \supseteq \rho_m(S) \supseteq \ldots. \]

Thus we have

\[ R = S\gamma_2(F) \quad \text{and} \quad [R, mF] = \rho_{m+1}(S)\gamma_{m+2}(F) \quad (\ast) \]

Let \( F = \prod_{i=1}^t F_i \) be the free product of \( F_1, F_2, \ldots, F_t \). We define a basic commutator subgroup \( B(F_1, F_2, \ldots, F_t)_s \) of weight \( s \) \((s \in \mathbb{N})\) on \( t \) free groups \( F_1, F_2, \ldots, F_t \), as follows:

We first order the subgroups \( F_1, F_2, \ldots, F_t \) by setting \( F_i < F_j \) if \( i < j \). Then \( B(F_1, F_2, \ldots, F_t)_s \) is the subgroup generated by all the basic commutators of weight \( s \) on \( t \) letters \( x_1, x_2, \ldots, x_t \), where \( x \in F_i \) for all \( 1 \leq i \leq t \). For the definition of basic commutators see M.Hall [1].

Note that here we have slightly modified the definition of basic commutator subgroups from M.R.R.Moghaddam [6].

Now, let \( T(H_1, H_2, \ldots, H_t)_s \) denote the summation of all the tensor products corresponding to the basic commutator subgroups \( B(F_1, F_2, \ldots, F_t)_s \).
where

\[ 1 \rightarrow R_i \rightarrow F_i \xrightarrow{\pi_i} H_i \rightarrow 1 \]

is the free presentation for \( H_i \), \( i = 1, 2, \ldots, t \). More precisely, if \([F_j, F_i, \ldots]\), with any bracketing, is a basic commutator subgroup of weight \( s \) in \( F_i \)'s, then the "corresponding" tensor product will be

\[ (H_j \otimes H_i \otimes \ldots) \]

bracketed in the same way. (Note that \( H_i \)'s are abelian groups.)

Similarly, the element \([x_j, x_i, \ldots]\) of the commutator subgroup \( B(F_1, F_2, \ldots, F_t)_s \), with any bracketing, corresponds to the element of the tensor product

\[ (\pi_j x_j \otimes \pi_i x_i \otimes \ldots) \]

bracketed in the same way, where \( x_k \) is the generator of \( F_k \).

We keep this notation throughout the rest of the paper, and it will be used without further reference.

2. The Main Results

Lemma 2.1

Let \( G \) be a finite abelian group, then by the previous notation for all \( c \geq 1 \),

\[ N_c M(G) \cong \frac{\gamma_{c+1}(F)}{\rho_{c+1}(S)\gamma_{c+2}(F)} \].

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Proof.

Let $1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1$ be a free presentation for $G$, then by the definition and using the fact that $\gamma_{c+1}(F)$ is contained in $S\gamma_2(F) = R$, and $(*)$,

$$
\mathcal{N}_c M(G) \cong \frac{R \cap \gamma_{c+1}(F)}{[R, cF]}
\cong \frac{\gamma_{c+1}(F)}{[R, cF]}
\cong \frac{\gamma_{c+1}(F)}{\rho_{c+1}(S)\gamma_{c+2}(F)}.
$$

$\square$

Now, we are in a position to prove the following important theorem.

**Theorem 2.2**

Consider the above assumption and notation. Then, for all $c \geq 1$,

$$
\frac{\gamma_{c+1}(F)}{\rho_{c+1}(S)\gamma_{c+2}(F)} \cong T(H_1, H_2, \ldots, H_t)_{c+1}.
$$

Proof.

Since $F_i$’s are infinite cyclic groups, using Hall’s Theorem [2] we obtain

$$
B(F_1, F_2, \ldots, F_t)_{c+1} = \gamma_{c+1}(F) \mod \gamma_{c+2}(F).
$$

By the notation of previous section, there exists an obvious relation between $B(F_1, F_2, \ldots, F_t)_{c+1}$ and $T(H_1, H_2, \ldots, H_t)_{c+1}$. Also the commutators of weight $c + 1$ in $B(F_1, F_2, \ldots, F_t)_{c+1} \mod \gamma_{c+2}(F)$ are multilinear. Now
one may construct a homomorphism \( \mu \) from \( \gamma_{c+1}(F)/\rho_{c+1}(S)\gamma_{c+2}(F) \) into the abelian group \( T(H_1, H_2, \ldots, H_t)_{c+1} \) given by

\[
\prod_{\text{wt.} c+1} [f, g, \ldots] \rho_{c+1}(S)\gamma_{c+2}(F) \mapsto \sum (\pi_j f \pi_i g, \ldots)
\]

with any bracketing “corresponding” element .

We know that \( \gamma_{c+1}(F)/\gamma_{c+2}(F) \) is the free abelian group on the basic commutators of weight \( c + 1 \) on \( t \) letters , (by P.Hall [2]) . Now we define the same mapping, as the above correspondence, on the set of basic commutators of weight \( c + 1 \) on \( t \) letters into the abelian group \( T(H_1, H_2, \ldots, H_t)_{c+1} \) . Then by the universal property of free abelian groups it can be extended to a homomorphism \( \phi \), say, from \( \gamma_{c+1}(F)/\gamma_{c+2}(F) \) into \( T(H_1, H_2, \ldots, H_t)_{c+1} \) .

To show that \( \phi \) induces the homomorphism \( \mu \) from \( \gamma_{c+1}(F)/\rho_{c+1}(S)\gamma_{c+2}(F) \) into \( T(H_1, H_2, \ldots, H_t)_{c+1} \), we must prove that \( \rho_{c+1}(S) \) is mapped onto zero under \( \phi \). But this is obvious, since

\[
[x_i, f, g, \ldots]^{r_i} \equiv [x_i, f, g, \ldots]^{r_i} \pmod{\gamma_{c+2}(F))}
\]

and

\[
[x_i, f, g, \ldots]^{r_i} \xrightarrow{\phi} r_i(\pi_i x \otimes \pi_j f \otimes \pi_k g \otimes \ldots)
\]

\[
= (r_i \pi_i x \otimes \pi_j f \otimes \pi_k g \otimes \ldots) = 0 .
\]

Hence \( \mu \) is the required homomorphism, which is also onto.
Conversely, by using the universal property of tensor product, we can define $\lambda$ to be the homomorphism from $T(H_1, H_2, \ldots, H_t)_{c+1}$ into $\gamma_{c+1}(F)/\rho_{c+1}(S)\gamma_{c+2}(F)$ given by

$$
\sum_{(c+1) \text{ times}} (h \otimes k \otimes \ldots) \longmapsto \prod_{\text{wt.c+1}} [f, g, \ldots] \rho_{c+1}(S)\gamma_{c+2}(F),
$$

with any bracketing bracketed the same way

where for $h \in H_i$, $k \in H_j$, ..., we pick $f \in F_i, g \in F_j, \ldots$, such that $\pi_i f = h$, $\pi_j g = k$, ... Clearly, this is a well-defined map since the commutators on the right hand side are multilinear. One can easily see that $\lambda$ is an epimorphism.

Now the result follows, since $\mu\lambda$ and $\lambda \mu$ are the identity maps on $T(H_1, H_2, \ldots, H_t)_{c+1}$ and $\gamma_{c+1}(F)/\rho_{c+1}(S)\gamma_{c+2}(F)$, respectively. □

In some aspect, the following theorem is a generalization of I.Schur [7,4], J.Wiegold [8], and M.R.R.Moghaddam [6], where its proof follows from Lemma 2.1 and Theorem 2.2.

**Theorem 2.3**

Let $\prod_{i=1}^{t} H_i$ be the direct product of finite cyclic groups. Then by the above notation, the higher Schur-multiplicator of $G$ is as follows:

$$\mathcal{N}_c M(\prod_{i=1}^{t} H_i) \cong T(H_1, H_2, \ldots, H_t)_{c+1}.$$

Now we are ready to give an explicit formula for the higher Schur-multiplicator of a finite abelian group with respect to the variety of nilpotent groups of
class at most \( c \geq 1 \), \( N_c \).

Let \( G \) be an arbitrary finite abelian group, then by the fundamental theorem of finitely generated abelian groups, \( G \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots \oplus \mathbb{Z}_{n_k} \), where \( n_{i+1} | n_i \) for all \( 1 \leq i \leq k - 1 \) and \( k \geq 2 \). \( \mathbb{Z}_n^{(m)} \) will denote the direct product of \( m \) copies of the cyclic group \( \mathbb{Z}_n \). Then with the above assumption, we obtain the following theorem, which is a vast generalization of I. Schur (see [4 or 7]).

**Theorem 2.4**

Let \( G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots \oplus \mathbb{Z}_{n_k} \) be a finite abelian group. Then, for all \( c \geq 1 \), the higher Schur-multiplier of \( G \) is

\[
N_c M(G) \cong \mathbb{Z}_{n_2}^{(b_2)} \oplus \mathbb{Z}_{n_3}^{(b_3 - b_2)} \oplus \ldots \oplus \mathbb{Z}_{n_k}^{(b_k - b_{k-1})},
\]

where \( b_i \) is the number of basic commutators of weight \( c + 1 \) on \( i \) letters.

**Proof.**

Clearly \( \mathbb{Z}_m \otimes \mathbb{Z}_n \cong \mathbb{Z}_{(m,n)} \) for all \( m, n \in \mathbb{N} \), where \( (m,n) \) is the greatest common divisor of \( m \) and \( n \). Hence

\[
\mathbb{Z}_{m_1} \otimes \mathbb{Z}_{m_2} \otimes \ldots \otimes \mathbb{Z}_{m_k} \cong \mathbb{Z}_{m_k},
\]

when \( m_{i+1} | m_i \) for all \( 1 \leq i \leq k - 1 \). Thus

\[
T(\mathbb{Z}_{n_1}, \mathbb{Z}_{n_2})_{c+1} \cong \mathbb{Z}_{n_2}.
\]

Now, by induction hypothesis assume

\[
T(\mathbb{Z}_{n_1}, \mathbb{Z}_{n_2}, \ldots, \mathbb{Z}_{n_{k-1}})_{c+1} \cong \mathbb{Z}_{n_2}^{(b_2)} \oplus \mathbb{Z}_{n_3}^{(b_3 - b_2)} \oplus \ldots \oplus \mathbb{Z}_{n_{k-1}}^{(b_{k-1} - b_{k-2})}.
\]
Then we have

\[ T(\mathbf{Z}_{n_1}, \mathbf{Z}_{n_2}, \ldots, \mathbf{Z}_{n_k})_{c+1} = T(\mathbf{Z}_{n_1}, \mathbf{Z}_{n_2}, \ldots, \mathbf{Z}_{n_{k-1}})_{c+1} \oplus L, \]

where \( L \) is the summation of all those tensor products of \( \mathbf{Z}_{n_1}, \mathbf{Z}_{n_2}, \ldots, \mathbf{Z}_{n_k} \) corresponding to the basic commutators of weight \( c + 1 \) on \( k \) letters which involve \( \mathbf{Z}_{n_k} \). Since \( n_k | n_i \) for all \( 1 \leq i \leq k - 1 \), all those tensor products are isomorphic to \( \mathbf{Z}_{n_k} \). So \( L \) is the direct product of \( (b_k - b_{k-1}) \) copies of \( \mathbf{Z}_{n_k} \). Hence the result follows, by induction. \( \square \)

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