Generating Solutions via Sigma-Models

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We review recent development of solution-generating techniques for four and five-dimensional Einstein equations coupled to vector and scalar fields. This includes $D = 4$ Einstein-Maxwell-dilaton-axion theory with multiple vector fields, $D = 5$ Einstein-Maxwell gravity with the Chern-Simons term (minimal five-dimensional supergravity), and some other models which attracted attention in connection with black rings. The method is based on reduction to three-dimensional gravity coupled sigma-models with symmetric target space. Our recent results open a way to construct the general charged black rings in five-dimensional supergravity possibly coupled to vector multiplets.

§1. Introduction

To find exact solutions of the Einstein equations is a formidable task already in four dimensions, not to say about higher-dimensional gravity/supergravity theories. The system is non-separable, so usually one chooses a certain ansatz for the metric to obtain a truncated system suitable for integration. The choice of the ansatz is based on the assumption that the desired solution possess some isometry group. Assumption of symmetry means that all the unknown functions depend on a smaller number of variables than the initial dimension $D$ of the space-time. This leads to dimensional reduction of Einstein equations to a lower-dimensional system involving additional vector and scalar fields. An especially fruitful direction is based on the assumption of an Abelian isometry group of the rank $D - 3$. This gives rise to the toroidal reduction to three dimensions, where vector fields (and higher rank antisymmetric forms, if present) can be dualized to scalars, and one obtains three-dimensional gravity coupled to the set of purely scalar fields. The remarkable feature of vacuum gravity in any dimension as well as the bosonic sectors of ungauged supergravities is that these scalar fields form sigma-models on coset spaces. The residual of the diffeomorphism group acting in the reduced dimensions together with symmetries of the vector multiplets combine into an enhanced duality group of global symmetries, often termed as “hidden symmetries”. The existence of hidden symmetries gives rise to a variety of tools like generation of new solutions from known ones, construction the BPS solutions as null geodesics of the target spaces, and derivation of two-dimensional integrable systems.

Hidden symmetry $SL(2, R)$ of the four-dimensional vacuum Einstein gravity reduced to three dimensions was discovered by Ehlers, the corresponding twodimensional integrable systems were explored in a number of papers (see for a review). The group $SL(2, R)$ is always present as a part of hidden symmetries of more complicated systems, so it remains an ingredient of generating techniques for them too. The next in complexity is the four-dimensional Einstein-Maxwell theory.
(or bosonic part of $D = 4$, $N = 2$ supergravity). A novel feature is presence of the Harrison transformations which generate charged solutions from neutral ones.\textsuperscript{8)–12)\textsuperscript{b}} Vacuum gravity in five and higher dimensions was studied along the same lines in 13) and 14). Later it was realized that various higher-dimensional supergravities also lead to the three-dimensional gravity coupled sigma-models with symmetric target spaces.\textsuperscript{15)–18)\textsuperscript{b}} A very convenient dimensional reduction scheme was elaborated in 19) (see also 20)) for the toroidal reduction of maximal supergravities originating from eleven-dimensional supergravity. It allows to obtain the Cartan-Weyl root system of the hidden symmetry group directly in terms of the so-called dilaton vectors — the coefficients of the dilatonic exponents emerging in the dimensional reduction.

Three-dimensional sigma models resulting from supergravities are quite complicated generically, so the possibility of their application for solution generation is rather limited. Meanwhile some physically interesting truncated systems were systems other than vacuum and $D = 4$ electrovacuum were discovered which are relatively simple and lead to manageable models. One such system is the $D = 4$ Einstein-Maxwell-dilaton-axion (EMDA) gravity which is a truncated version of the $N = 4$ supergravity (or toroidally compactified heterotic string theory).\textsuperscript{21)–24)\textsuperscript{b}} It contains one Maxwell field and two massless scalar fields (dilaton and axion) and leads in three dimensions to the sigma-model with the global symmetry $SO(2, 3) \sim Sp(4, R)$. It admits a representation in terms of $4 \times 4$ matrices which can be split into the $2 \times 2$ blocks. It also allows for a Kähler representation in complex variables similar to the Ernst formulation of vacuum and electrovacuum gravity. This representation can be generalized to EMDA with an arbitrary number of vector fields.\textsuperscript{25)\textsuperscript{b}} Especially interesting is the case of two vector fields giving rise to a quaternion analog of the Ernst potential.\textsuperscript{26)\textsuperscript{b}}

Recent interest to black rings (for a review see 27)) stimulated investigation of hidden symmetries of five-dimensional gravity coupled to vector fields. Somewhat unexpectedly, five-dimensional Einstein-Maxwell gravity does not possess non-trivial hidden symmetries (the isometry group of the target space of the corresponding three-dimensional sigma-model is solvable) which can generate charges. This explains why there is no analytic solutions for charged rotating black holes in this theory. However there exist more complicated five-dimensional gravity theories with vector fields which lead to non-trivial hidden symmetries containing charging (Harrison) transformations. One of them corresponding to dimensional reduction of six-dimensional vacuum gravity to five dimensions contains a Kaluza-Klein vector field and a dilaton with the hidden symmetry $SL(4, R)$. A matrix representation for this theory similar to that of EMDA was constructed in 29). The second model is five-dimensional minimal supergravity whose bosonic sector is the Einstein-Maxwell theory with the Chern-Simons term.\textsuperscript{30), 31)\textsuperscript{b}} The presence of the latter leads to a non-trivial hidden symmetry realized by an exceptional group $G_2(2)$ (the non-compact version of $G_2$).\textsuperscript{32), 33)}\textsuperscript{b} A detailed study of the corresponding three-dimensional sigma-model was performed in 34) (see also 35)). A more general five-dimensional model containing three vector and two independent scalar fields arising in some truncated toroidal reduction of $D = 11$ supergravity is often invoked in the discussion of black
rings. As we have shown recently, the hidden symmetry in this case is SO(4, 4).

\section{D = 4}

\subsection{Ehlers group SL(2,R)\footnote{1}}

Consider vacuum Einstein equations $R_{\mu\nu} = 0$. For stationary metrics admitting a time-like Killing vector field, $\mathcal{L}_K g_{\mu\nu} = 0$, $K = \partial_t$, the line element can be presented as

$$ds^2 = -e^\xi (dt + \omega_i dx^i)^2 + e^{-\xi} h_{ij} dx^i dx^j,$$

where the scalar $\xi$, the three-vector $\omega_i$ and the three-dimensional metric $h_{ij}$ depend only on spatial coordinates $x^i$. Then the equations of motion coincide with those of the three-dimensional gravitating non-linear sigma-model

$$S_\sigma = \int \left[R_3(h) - G_{AB}(\Phi) \partial_i \Phi^A \partial_j \Phi^B h^{ij}\right] \sqrt{h} d^3x,$$

with two scalar fields $\Phi^A = (\xi, \chi)$, where the twist potential $\chi$ is related to the one-form $\omega_i$ by the dualization equation $d\chi = - e^{2\xi} \star d\omega$, and the target space is a coset space $SL(2,R)/SO(1,1)$, the corresponding metric being

$$dl^2 = G_{AB} d\Phi^A d\Phi^B = \frac{1}{2} \left(d\xi^2 + e^{-2\xi} d\chi^2\right).$$

Now the initial four-metric $g_{\mu\nu}$ is presented by the three-metric $h_{ij}$ and two “matter” fields $\xi, \phi$, the field equations being invariant under the Ehlers group $SL(2,R)$ acting transitively on the target space as an isometry: $\xi, \chi \rightarrow \xi', \chi'$. So starting with some solution $g_{\mu\nu} = \{h_{ij}, \xi, \chi\}$ one can find another set $\xi', \chi'$. Dualizing back $\chi'$ to $\omega_i'$ one obtains a new 4D solution with the same three-metric $g_{\mu\nu} = \{h_{ij}, \xi', \omega_i'\}$. Three $SL(2,R)$ transformations consist of

\begin{itemize}
  \item[i)] twist shift (gauge) $\xi \rightarrow \xi + \lambda_g$,
  \item[ii)] scaling $\xi \rightarrow \xi + \lambda_s$, $\chi \rightarrow e^{\lambda_s} \chi$,
  \item[iii)] proper Ehlers transformation $(\chi - i e^\xi)^{-1} \rightarrow (\chi - i e^\xi)^{-1} + \lambda_E$,
\end{itemize}

the last one generating the Taub-NUT metric from Schwarzschild.

Similarly, vacuum gravity in D dimensions has hidden symmetry $SL(D-2,R)$.

\subsection{Einstein-Maxwell}

The sigma-model representation can be derived for 4D gravity coupled to a massless vector field $S = -\frac{1}{16\pi} \int \left(R + F^2\right) \sqrt{-g} \; d^4x$. With the assumption of stationarity, the $\mu = i$ component of the Bianchi identity $\partial_\nu (\sqrt{-g} F^{\mu\nu}) = 0$ is satisfied introducing the electric potential $F_{i0} = \partial_i v$, while the $\mu = i$ component of the Maxwell equation $\partial_\nu (\sqrt{-g} F^{\mu\nu}) = 0$ is solved by introducing the magnetic potential $u$: $F^{ij} = \frac{e^\xi}{\sqrt{h}} \epsilon^{ijk} \partial_k u$. The remaining components of $F^{\mu\nu}$ can be expressed in terms of $v$ and $u$, so a vector field is reduced to two scalars. The resulting 3D sigma-model is especially simple in the static case $\omega_i = \chi = 0$. Then the full Einstein-Maxwell system splits into independent electric and magnetic sectors. In the electric case one obtains $SL(2,R)/SO(2)$ sigma-model with the target metric $dl^2 = \frac{1}{2} \left(d\xi^2 - e^{-2\xi} dv^2\right)$, in the magnetic case $dl^2 = \frac{1}{2} \left(d\xi^2 - e^{2\xi} du^2\right)$. Thus we deal again with the $SL(2,R)$
transformations acting on $\xi$, $v$ or $\xi$, $u$. The last one, $iii)$, now is the Harrison transformation generating charged solutions from uncharged ones. Taking the Schwarzschild solution as a seed, one can construct the Reissner-Nordström solution.

For a general stationary EM system one gets the four-dimensional target space with the signature (++–)

$$dl^2 = \frac{1}{2} \left[ d\xi^2 + e^{-2\xi}(d\chi + v du - udv)^2 - e^{-\xi}(dv^2 + du^2) \right], \quad (2.4)$$

which is the coset space $SU(2,1)/S(U(1) \times U(1))$. Complex Ernst potentials\(^2,9\) $\Phi = \frac{1}{\sqrt{2}} (v + i u)$, $\mathcal{E} = e^\xi + i \chi + \Phi \Phi^*$ realize a non-linear representation of $SU(2,1)$, whose action consists of three gauge (shifts of $\chi$, $v$, $u$), an electric-magnetic rotation, a scaling, two charging Harrison transformations

$$\Phi \rightarrow \frac{\Phi + c \mathcal{E}}{1 - 2c^* \Phi - |c|^2 \mathcal{E}}, \quad \mathcal{E} \rightarrow \frac{\mathcal{E}}{1 - 2c^* \Phi - |c|^2 \mathcal{E}}, \quad (2.5)$$

and Ehlers transformation $\Phi \rightarrow (1 + i \gamma \mathcal{E})^{-1}$, $\mathcal{E} \rightarrow (1 + i \gamma \mathcal{E})^{-1}$ where complex $c$ and real $\gamma$ are parameters.

### 2.3. EM-dilaton

Adding a dilaton with an arbitrary coupling

$$S = \frac{1}{16\pi} \int \left( -R + 2(\partial \phi)^2 - e^{-2\phi} F^2 \right) \sqrt{-g} \, d^4 x, \quad (2.6)$$

one obtains the target space

$$dl^2 = \frac{1}{2} \left[ d\xi^2 + e^{-2\xi}(d\chi + v du - udv)^2 + 2d\phi^2 - e^{-\xi}(e^{-2\phi} dv^2 + e^{2\phi} du^2) \right]. \quad (2.7)$$

It is a symmetric space only for $\alpha^2 = 0, 3,12$) In the first case we have EM + a decoupled scalar, in the second — a compactified 5D vacuum gravity (Kaluza-Klein), in which case the isometry group is 8-parametric $SL(3,R)$. For other $\alpha$ the isometry group is a 5-parametric solvable group not containing Harrison-Ehlers transformations.

It is worth noting, that contrary to vacuum gravity, the Einstein-Maxwell theory in $D > 4$ does not possess semi-simple hidden symmetries.

### 2.4. EM-dilaton-axion

The previous model has only a discrete electric-magnetic duality. Adding an axion $\kappa$ we obtain the EMDA model (a truncated $N = 4$ supergravity) which possess a continuous $SL(2,R)$ electric-magnetic symmetry

$$S = \frac{1}{16\pi} \int \left\{ -R + 2\partial_\mu \phi \partial^\mu \phi + \frac{1}{2} e^{4\phi} \partial_\mu \kappa \partial^\mu \kappa - e^{-2\phi} F_{\mu\nu} F^{\mu\nu} - \kappa F_{\mu\nu} \tilde{F}^{\mu\nu} \right\} \sqrt{-g} \, d^4 x, \quad (2.8)$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\tau} F_{\lambda\tau}$, $F = dA$. The resulting target space is the 6-dimensional coset $Sp(4,R)/(SO(2) \times SO(1,2)):\(^{21)-24}\)

$$dl^2 = \frac{1}{2} \left[ d\xi^2 + e^{-2\xi}(d\chi + v du - udv)^2 + e^{4\phi} d\kappa^2 \right] + 2d\phi^2 - e^{-\xi} \left[ e^{-2\phi} dv^2 + e^{2\phi} (du - \kappa dv)^2 \right]. \quad (2.9)$$
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Its isometry group $Sp(4, R) \sim SO(3, 2)$ consists of three gauge, one scale, three $SL(2, R)$ S-duality, two Harrison and Ehlers transformations (altogether 10). All isometry transformations are known algebraically.

§3. Matrix methods

To find symmetry transformations explicitly one can use an appropriate matrix representative $M = M(\Phi)$ of the coset, so that the target space metric is presented as

$$dl^2 = G_{AB} d\Phi^A d\Phi^B = -k \text{Tr} (dM dM^{-1}).$$

(3.1)

One-parametric subgroups $G_\lambda = e^{\lambda K}$ corresponding to Killing vector $K$ ($\bar{K}$ being the corresponding matrix generator) act on $M$ as $M' = G_\lambda^{-1} M G_\lambda$. Then the transformation of the potentials can be read off from the relation $M'(\Phi) = M(\Phi')$.

For EMDA, $M$ is given by a symmetric symplectic 4X4 matrix:\textsuperscript{23)

$$M = \begin{pmatrix} P^{-1} & Q^{-1} \\ Q P & P Q^{-1} \end{pmatrix},$$

(3.2)

where $P$ and $Q$ are the real symmetric 2 x 2 matrices

$$P = -e^{-2\phi} \begin{pmatrix} v^2 - e^{2\phi + \xi} & v \\ v & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} vw - \chi & w \\ w & -\kappa \end{pmatrix},$$

(3.3)

$w = u - \kappa v$. Further simplification consists in presenting ten $Sp(4, R)$ isometries as $SL(2, R)$ transformation acting on 2 x 2 complex matrices $Z = P + iQ$

$$Z = \begin{pmatrix} E & \Phi \\ \Phi & -z \end{pmatrix}, \quad Z = \kappa + i e^{-2\phi}, \quad \Phi = u - z v,$$

(3.4)

where the quantities $\Phi, E$ can be regarded as the EMDA Ernst-type complex potentials. Isometry transformations consist of a shift and a “scaling” of $Z$, and a shift of 1/Z:

$$Z \to Z + B, \quad B = \begin{pmatrix} g & m \\ m & b \end{pmatrix}, \quad Z \to A^T Z A, \quad A = \begin{pmatrix} e^s & h_e \\ h_e & -e \end{pmatrix},$$

(3.5)

$$\frac{1}{Z} \to \frac{1}{Z} + C, \quad C = \begin{pmatrix} c_E & h_m \\ h_m & c \end{pmatrix}.$$  

(3.6)

Real parameters correspond to: $g$ — a gravitational gauge, $e, m$ — an electric and a magnetic gauge; $a, b, c - SL(2, R)$ S-duality, $h_e, h_m$ — electric and magnetic Harrison transformations, $c_E$ — Ehlers transformation. $SL(2, R)$ S-duality was the symmetry of the 4D theory, acting on the axidilaton $z$ and the Maxwell field, now it is incorporated into 3D U-duality group $Sp(4, R)$.

Further simplification comes from the fact that the target space (2.9) admits Kähler complex parametrization $z_\alpha, \alpha = 0, 1, 2$ with $z_0 = E - z, z_1 = u - v \Phi, z_2 = E + z$. The Kähler target space metric is constructed from the Kähler potential:

$$G_{\alpha \beta} = \partial_\alpha \partial_\beta K(z^\alpha, \bar{z}^\beta), \quad K = -\ln V, \quad V = \eta_{\alpha \beta} \text{Im} z^\alpha \text{Im} z^\beta = e^{\xi - 2\phi},$$

(3.7)

where $\eta_{\alpha \beta} = \text{diag}(-1, 1, 1)$. This construction can be directly generalized to the case of an arbitrary number $p$ of vector fields introducing additional complex coordinates $z_n = u_n - v_n \Phi, n = 1, \ldots, p$. Remarkably, the Kähler potential remains the same $V = e^{\xi - 2\phi}$ being expressed in terms of the real quantities,\textsuperscript{25) though its realization in complex variables now involves the metric $\eta_{\alpha \beta} = \text{diag}(-1, 1, \ldots, 1)$.
diag$(-1,1,...,1)$, $\alpha, \beta = 0, 1,..., p + 1$. The target space of EMDA with $p$ vector fields is the coset $SO(2,2+p)/(SO(2) \times SO(p,2))$, its matrix representation is realized by $(p+4) \times (p+4)$ matrices.

The case $p = 2$ is exceptional: due to isomorphism $SO(2,4) \sim SU(2,2)$, the $4 \times 4$ representation is possible instead of the expected $6 \times 6$. It is generated by the $2 \times 2$ complex matrix $Z$ in the same way as before, but now with a generic $Z$ incorporating four independent complex potentials:

$$Z = \begin{pmatrix} E & \Phi_1 - i\Phi_2 \\ \Phi_1 + i\Phi_2 & -z \end{pmatrix}.$$  

Moreover, the target space metric can be directly expressed through $Z$ as follows:

$$dt^2 = -2\text{Tr} \left\{dZ (Z^\dagger - Z)^{-1} dZ^\dagger (Z^\dagger - Z)^{-1} \right\},$$

while the symmetry transformations are again (3.5, 3.6). This realizes the quaternion $SL(2,Q)$ representation of $SU(2,2)$. Complex quaternion coordinates can be read from an expansion of $Z$ in terms of Pauli matrices $Z = z^0 I_2 + z^a \sigma_a$.

§4. $D = 5$

4.1. Black rings

Recent interest to exact solutions in five-dimensional gravity is related to possibility of topologically non-spherical stationary asymptotically flat black holes, with the $S^1 \times S^2$ topology of the event horizon. While in 4D the uniqueness theorems imply that the most general stationary black hole is Kerr for vacuum and Kerr-Newman for electrovacuum (both with $S^2$ horizon topology), in 5D it is not so. It is possible that black holes with the horizon topology $S^1 \times S^{D-3}$ exist for any $D$. The most general non-singular 5D black ring is a three-parametric solution endiwen with a mass and two independent rotation parameters. The most general charged black ring in 5D gravity coupled to a single Maxwell and possibly scalar fields should be five-parametric, with an electric charge and a magnetic dipole moment as additional parameters. Such solution is still unknown. Another interesting model includes three vector and two independent scalar fields, within this theory a nine-parametric solution should exist (also unknown).

This stimulates further investigation of 5D theories which lead to 3D sigma-models on symmetric spaces. Somewhat unexpectedly, pure Einstein-Maxwell theory in 5D generates only trivial (solvable) hidden symmetry algebra. Two other theories which possess non-trivial hidden symmetries are discussed below.

4.2. 5D EM-dilaton (KK)

Compactifying 6D vacuum gravity on a circle, one obtains 5D Einstein-Maxwell-dilaton theory with the Kaluza-Klein dilaton coupling $\alpha^2 = 8/3$: 

$$S_5 = \int d^5x \sqrt{-g_5} \left\{ \right. R_5 - \frac{1}{2} (\partial \phi)^2 - \frac{e^{-\alpha \phi}}{12} H^2 \left. \right\},$$

$$S_5 = \int d^5x \sqrt{-g_5} \left\{ R_5 - \frac{1}{2} (\partial \phi)^2 - \frac{e^{-\alpha \phi}}{12} H^2 \right\},$$

(4.1)
where \( \phi \) is the dilaton and \( H = dB \) is an antisymmetric three-form dual to the KK vector in 5D. This model obviously have the \( SL(4, R) \) hidden symmetry in 3D. It is instructive to present it in a form similar to that of 4D EMDA.\(^{29}\) In fact, in 4D this action contains two vector and three scalar fields, differing from EMDA with two vector fields by presence of an additional scalar \( \psi \), as well as by different interaction structure. The target space of the corresponding 3D sigma-model has the line element

\[
dl^2 = \frac{1}{2} \left( d\xi^2 + e^{-2\xi} \left[ d\chi + \frac{1}{2} (v_a du_a - u_a dv_a) \right]^2 \right) + \frac{1}{2} d\phi^2 + \frac{1}{4} d\psi^2 + \frac{1}{2} e^{2\phi} d\kappa^2 - \frac{e^{-\xi}}{2} \left[ e^{\psi-\phi} dv_1^2 + e^{-\psi+\phi} (du_1 - \kappa dv_2)^2 + e^{-\psi-\phi} (dv_2)^2 + e^{\psi+\phi} (du_2 - \kappa dv_1)^2 \right], \tag{4.2}
\]

where \( a = 1, 2 \). This is the metric of the symmetric space \( SL(4, R)/SO(2, 2) \). As the coset representatives one can choose the symmetric \( SL(4, R) \) matrix

\[
M = \begin{pmatrix}
P_1^{-1} & P_1^{-1}Q \\
P_1^{-1} & P_2 + Q^T P_1^{-1}Q
\end{pmatrix}, \tag{4.3}
\]

where \( Q \) is a real \( 2 \times 2 \) matrix and \( P_1, P_2 \) are symmetric matrices with the same determinant. This matrix can be regarded as a generalization of the EMDA \( Sp(4, R) \) matrix to which it reduces for \( P_1 = P_2 \) and \( Q^T = Q \). Now we have

\[
P_1 = e^{\psi/2} \begin{pmatrix}
e^{\xi-\psi} - (v_1)^2 e^{-\phi} & -v_1 e^{-\phi} \\
-v_1 e^{-\phi} & e^{-\phi}
\end{pmatrix},
\]

\[
P_2 = e^{-\psi/2} \begin{pmatrix}
e^{\xi+\psi} - (v_2)^2 e^{-\phi} & -v_2 e^{-\phi} \\
-v_2 e^{-\phi} & e^{-\phi}
\end{pmatrix}, \quad Q = \begin{pmatrix}
\frac{1}{2} \mu - \chi & u_2 - \kappa v_1 \\
u_1 - \kappa v_2 & -\kappa
\end{pmatrix},
\]

where \( \mu = v_1 (u_1 - \kappa v_2) + v_2 (u_2 - \kappa v_1) \). Transformations \( M \rightarrow {G^T} M G \) with constant \( G \in SL(4, R) \) can be presented by its action on \( 2 \times 2 \) matrices \( P_1, P_2, Q \) similarly to the above quaternion form of \( SU(2, 2) \).\(^{29}\)

4.3. 5D minimal supergravity

Another symmetric model is minimal 5D supergravity containing in addition to the Maxwell term also the Chern-Simons term. This makes the theory to possess enough hidden symmetry, the 3D U-duality being the non-compact form of the exceptional group \( G_2 \). Five-dimensional minimal supergravity contains a graviton, two \( N = 2 \) symplectic-Majorana gravitini (equivalent to a single Dirac gravitino), and a \( U(1) \) gauge field. The bosonic part of the Lagrangian is very similar to that of \( D = 11 \) supergravity:

\[
S_5 = \frac{1}{16\pi G_5} \left[ \int d^5 x \sqrt{-g} \left( R - \frac{1}{4} F^2 \right) - \frac{1}{3\sqrt{3}} \int F \wedge F \wedge A \right],
\]

where \( F = dA \). This action can be obtained as a suitably truncated Calabi-Yau compactification of \( D = 11 \) supergravity. Its reduction to 3D leads to the sigma model on a symmetric space \( G_{2(2)}/(SL(2, R) \times SL(2, R)) \).\(^{32,33}\) The corresponding
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generating technique was developed in 34). The TS metric reads ($v_1, u_1$ being KK, $v_2, u_2$ — Maxwell potentials)

$$dl^2 = \frac{1}{2} \left\{ d\xi^2 + e^{-2\xi}(d\chi + v_1 du_1 + v_2 du_2)^2 + 3 \left( d\phi^2 + e^{2\phi} d\kappa^2 \right) \\
- e^{-\xi} \left[ e^{-\phi} \left( dv_2 + \sqrt{3}\kappa dv_1 \right)^2 + e^{-3\phi} dv_1^2 + e^{3\phi} \left[ du_1 + \kappa^3 dv_1 - \sqrt{3}\kappa (du_2 - \kappa dv_2) \right]^2 \\
+ e^\phi \left( du_2 - 2\kappa dv_2 - \sqrt{3}\kappa^2 dv_1 \right)^2 \right\} \right\}.$$ 

This is the metric on the coset $G_{2(2)}/(SL(2, R) \times SL(2, R))$. Note that the number of variables is 8, the same as for 4D EMDA with two vector fields. However, the isometry group is 14-parametric (15 parametric for EMDA). The matrix representation can be given either in terms of 7 $\times$ 7 or 8 $\times$ 8 matrices corresponding to embedding of $G_{2(2)}$ into $SO(3, 4)$ or $SO(4, 4)$:

$$M = \left( \begin{array}{ccc} A & B & \sqrt{2}U \\
B^T & C & \sqrt{2}V \\
\sqrt{2}U^T & \sqrt{2}V^T & S \end{array} \right),$$

(4.4)

where $A, B, C$ are 3 $\times$ 3 matrices and $U, V$ are 3-columns, whose explicit form is given in 34). This matrix realizes a noncompact coset $G_{2(2)}/(SL(2, R) \times SL(2, R))$. In spite of being rather complicated, this representation opens a way to construct new solutions acting on this matrix by one-parametric subgroups of $G_{2(2)}$ which are easily found by exponentiating the generators. The fourteen transformations include five gauge, one scale, three S-duality, four Harrison and Ehlers transformations. A charged rotating 5D black hole with two rotation parameters was generated in 34) from the solution $^{28)}$ as a seed. It is plagued with a conical singularity. To get a regular charged doubly rotating solution one should start with a more general doubly rotating vacuum ring solution $^{37)}$ with non-compensated conical singularity.

4.4. 5D SUGRA with vector multiplets

More general models of this type containing multiple $U(1)$ vector fields give rise to black rings with many electric and dipole charges. One popular model leading to three-charge black rings contains three vector and three scalar fields subject to a constraint:

$$S = \frac{1}{16\pi G_5} \int \left( R_5 \ast 1 - \frac{1}{2} G_{IJ} \left( dX^I \wedge \ast dX^J - F^I \wedge F^J \right) - \frac{\delta_{IJK}}{6} F^I \wedge F^J \wedge A^K \right),$$

(4.5)

where $G_{IJ} = \text{diag} \left( (X^1)^{-2}, (X^2)^{-2}, (X^3)^{-2} \right), \ F_I = dA_I, \ I, J, K = 1, 2, 3,$ and $\delta_{IJK} = 1$, if $I, J, K$ is a permutation of 1,2,3, and zero otherwise. 3D reduction of this theory leads to the sigma model on the homogeneous space $SO(4, 4)/SO(4) \times SO(4)$ or $SO(4, 4)/SO(2, 2) \times SO(2, 2)$ depending on the signature of the three-space. A $8 \times 8$ matrix representation which can be used for solution generating purposes was
constructed.\textsuperscript{36}) It opens a way to find the nine-parametric charged black ring in five dimensions. An identification of the three vector fields $A^I$ with the corresponding contraction the scalars $X^I$ returns us to the $G_{2(2)}/(SL(2, R) \times SL(2, R))$ sigma model of the previous section.

\section*{§5. Conclusions}

We have reviewed the main ideas of solution generating techniques based on dimensional reduction to three-dimensions and presented some new sigma-models which can be used to construct charged black holes and black rings in five dimensions. These models can be further reduced to two dimensions giving new integrable systems based on $G_{2(2)}$ and $SO(4, 4)$ groups.

\section*{Acknowledgements}

We would like to thank Yukawa Institute for Theoretical Physics and Nara Women’s University for hospitality and support during the ICGA8 and the GC workshop. Useful discussions with M. Sasaki and K.-I. Maeda, R. Kallosh, Y.M. Cho and S. Odintsov are gratefully acknowledged. We also thank G. Clement and C.-M. Chen for collaboration. The work was done within the RFBR project 08-02-01398-a.

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