How to calculate spectra of Rabi and related models?

Andrzej J. Maciejewski
J. Kepler Institute of Astronomy, University of Zielona Góra, Licealna 9, PL-65-417 Zielona Góra, Poland.

Maria Przybylska
Institute of Physics, University of Zielona Góra, Licealna 9, 65-417 Zielona Góra, Poland

Tomasz Stachowiak
Center for Theoretical Physics PAS, Al. Lotnikow 32/46, 02-668 Warsaw, Poland

(Dated: May 3, 2014)

We show how to use properly the Bargmann space of entire functions in the analysis of the Rabi model. We are able to correct a serious error in recent papers on the topic and develop a corrected method of finding the spectrum applicable also to other, more general systems.

PACS numbers: 03.65.Ge,02.30.Ik,42.50.Pq
Keywords: Rabi model

Several papers devoted to determination of the spectrum of the Rabi model have appeared recently, see e.g. [1,2] and references therein for a description of various approaches to this problem. Let us recall that the Rabi model describes interaction of a two-level atom with a single harmonic mode of the electromagnetic field. In the Bargmann-Fock representation this model is described by the following system of two differential equations

\[
\begin{align*}
(z + \lambda) \frac{d\psi_1}{dz} &= (E - \lambda z) \psi_1 - \mu \psi_2, \\
(z - \lambda) \frac{d\psi_2}{dz} &= (E + \lambda z) \psi_2 - \mu \psi_1,
\end{align*}
\]

(1)

where $E$ is the energy, $\lambda$ is the atom-field coupling constant, and $2\mu$ is the atomic level separation. In this representation, two component wave function $\psi = (\psi_1, \psi_2)$ is an element of Hilbert space $\mathcal{H}^2 = \mathcal{H} \times \mathcal{H}$, where $\mathcal{H}$ is the Bargmann-Fock Hilbert space of entire functions of one variable $z \in \mathbb{C}$.

The scalar product in $\mathcal{H}$ is given by

\[
\langle f, g \rangle = \frac{1}{\pi} \int_C \overline{f(z)} g(z) e^{-|z|^2} d(\Re(z)) d(\Im(z)).
\]

An entire function $f(z)$ belongs to $\mathcal{H}$ if it has the proper growth at infinity, for details, see [3]. Thus, energy $E$ belongs to the spectrum of the problem, if and only if for this value of $E$ equations (1) have entire solution $\psi = (\psi_1, \psi_2)$ with the proper behaviour at infinity. However, for the considered system all entire solutions belong to $\mathcal{H}^2$, see [2] for an explanation.

Equations (1) have two singular regular points at $z = \lambda$ and $z = -\lambda$, while infinity is an irregular singular point. The system has a $\mathbb{Z}_2$ symmetry. It is invariant with respect to the involution $\tau: \mathcal{H}^2 \to \mathcal{H}^2$ given by $\tau(\psi_1, \psi_2)(z) = (\psi_2(-z), \psi_1(-z))$. In other words, if $(\psi_1(z), \psi_2(z))$ is a solution of this system, then also $(\psi_2(-z), \psi_1(-z))$ is its solution. We say that a solution $\psi = (\psi_1, \psi_2)$ of (1) has parity $\sigma \in \{-1, +1\}$ if $\tau(\psi(z)) = \pm \psi(z)$.

In [1] it was pointed out that the the above mentioned symmetry plays the crucial role in determination of its spectrum. Later, in [3], it was shown that one can determine the spectrum of the Rabi model without any references to its $\mathbb{Z}_2$ symmetry. However, the method used in [3] has limited application as it is based on a result valid only for third order recurrence relations, see [3].

Braak’s approach is very interesting because it gives a chance for generalisation beyond third order recurrence relations. However in its original formulation it contains a flaw which invalidates its results concerning the spectrum of the standard Rabi model.

The aim of our paper is to correct results of [1] and to present our direct method which can be used to study of systems more complicated than the Rabi model.

We applied our method to study a generalised Rabi model with broken $\mathbb{Z}_2$ symmetry introduced in [1]. In the Bargmann representation it is described by the following equations

\[
\begin{align*}
(z + \lambda) \frac{d\psi_1}{dz} &= (E - \epsilon - \lambda z) \psi_1 - \mu \psi_2, \\
(z - \lambda) \frac{d\psi_2}{dz} &= (E + \epsilon + \lambda z) \psi_2 - \mu \psi_1,
\end{align*}
\]

(2)

For $\epsilon = 0$ it coincides with the Rabi model [1]. We denote by $p := (E, \lambda, \mu, \epsilon)$ parameters of this model. Instead of $E$ we will use also the parameter $x := E + \lambda^2$. Let us assume that $x$ is not a non-negative integer. Then the spectrum of the problem coincides with zeros of one function $w(p)$ defined in the following way. Let

\[
w(p; y) := H_1(y)H_2'(y) - H_1'(y)H_2(y),
\]

(3)

where

\[
H_1(y) := \text{HeunC}(a_0; y),
\]

(4)
and
\[ H_2(y) := \text{HeunC}(a_1; 1 - y), \]
are the confluent Heun functions, see [3], with parameters\(^\text{11}\)\(a_0 := (\alpha, \beta, \gamma, \delta, \eta)\), and \(a_1 := (-\alpha, \gamma, \beta, -\delta, \delta + \eta)\) in terms of \((x, \lambda, \mu, \epsilon)\) these parameters are given by \(\alpha = 4\lambda^2, \beta = -x + \epsilon, \gamma = -1 - x - \epsilon, \delta = 2(1 - 2\epsilon)\lambda^2, \) \(2\eta = 1 - 2\mu^2 + (1 + x)(x - 4\lambda^2) + (1 + 4\lambda^2 - \epsilon^2).\) \(\text{(6)}\)

Then, \(W(p) := w(p; 1/2)\).

Here we just recall the basic steps of the procedure which was formulated for the Rabi model [10]. It is fully described in [1].

Let us introduce new variables \(y := z + \lambda, \varphi_i(y) = e^{-\lambda y}\psi_i(y - \lambda), \) for \(i = 1, 2.\) It is obvious that functions \(\psi_i(z)\) are entire if and only if functions \(\varphi_i(y)\) are entire. In the new variables system [10] reads
\[
\begin{align*}
\frac{d\varphi_1}{dy} &= x\varphi_1 - \mu\varphi_2, \\
(y - 2\lambda)\frac{d\varphi_2}{dy} &= (x - 4\lambda^2 + 2\lambda y)\varphi_2 - \mu\varphi_1,
\end{align*}
\(\text{(7)}\)

Let us assume that for a given \(p = (x, \lambda, \mu)\), system [10] has an entire solution \(\varphi(y) = (\varphi_1(y), \varphi_2(y))\), where
\[
\varphi_1(y) = \sum_{n=0}^{\infty} \Phi_n(p)y^n, \quad \varphi_2(y) = \sum_{n=0}^{\infty} \Phi_n(p)y^n. \quad \text{(8)}
\]

To proceed further we assume also that \(x \not\in \mathbb{Z}\), since the existence of solutions for integer \(x\) has long been established, see [3]. Then we insert the expression for \(\varphi_2(y)\) into the first equation [10], and integrate it. We obtain
\[
\tilde{\Phi}_n(p) = \frac{\mu}{x - n}\Phi_n(p), \quad \text{for} \quad n \in \mathbb{N}. \quad \text{(9)}
\]

Function \(\Phi_n(y)\) can be determined recursively. In this way we obtain a solution \(\varphi(y)\), which is locally holomorphic at \(y = 0\). Thus, we have a solution \(\psi(z)\) which is locally holomorphic at \(z = -\lambda\). However, we are looking for an entire solution. In order to find conditions which guarantee that \(\psi(z)\) is entire we can assume, without loss of generality that it has a given parity \(\sigma\). Thus putting \(\psi_{\sigma}(z) = (\psi_{1,\sigma}(z), \psi_{2,\sigma}(z))\) we have \(\psi_{1,\sigma}(z) = \sigma\psi_{2,\sigma}(-z)\) for all \(z \in \mathbb{C}\). Hence, function
\[
G_{\sigma}(z, p) := \psi_{2,\sigma}(-z) - \sigma\psi_{1,\sigma}(z), \quad \text{(10)}
\]

must vanish identically for all \(z \in \mathbb{C}\). In [3], Braak claims that

1. if \(G_{\sigma}(z, p) = 0\) for a fixed but arbitrary \(z\) belonging to the intersection of domains of definiteness of \(\psi_{1,\sigma}(z)\) and \(\psi_{2,\sigma}(-z)\), then it vanishes identically. That is, if for a given \(p\) there exists \(z_* \in D(-\lambda, 2|\lambda|) \cap D(\lambda, 2|\lambda|)\) such that \(G_{\sigma}(z_*, p) = 0\), then \(G_{\sigma}(z, p) = 0\) for an arbitrary \(z \in \mathbb{C}\). Here \(D(z_0, r) \subset \mathbb{C}\) is an open disc of radius \(r\) and the centre at \(z_0\).

2. \(x\) belongs to the spectrum of the problem if and only if there exists \(z_* \in D(-\lambda, 2|\lambda|) \cap D(\lambda, 2|\lambda|)\) such that \(G_{\sigma}(z_*, p) = 0\), for a certain \(p = (x, \lambda, \mu)\). Both of these statements are incorrect. There exist entire functions, e.g., \(\sin(z)\), which have infinite number of zeros but they are not zero functions. So, the first claim needs at least some justification. We show that in the considered context it is not valid. To see this it is enough to make a contour graph of function \(G_{\sigma}(z, p)\) on the plane \((x, z)\), see Fig. [11]. One can notice that there exist \(x\) such that \(G_{\sigma}(z, p) = 0\) identically in \(z\), as well as that there exist isolated zeros of \(G_{\sigma}(z, p)\). An example is presented in Fig. [11]. This also shows that the second claim of Braak is not correct.

We notice that the function \(g(z) := G_{\sigma}(z, p)\), as a function of \(z\), satisfies a second order linear differential equation. In fact, it can be checked directly that
\[
(z^2 - \lambda^2)g'' + [z(1 - 2x - 3\lambda^2) - \lambda]g' + [\lambda z(1 - \lambda z) + (x - \lambda^2)^2 - x^2 - \mu^2]g = 0. \quad \text{(11)}
\]

So, fixing values \(G_{\sigma}(z, p)\) and \(G_{\sigma}'(z, p)\) at a given \(z\), we determine \(G_{\sigma}(z, p)\) uniquely. Hence, the spectrum of the Rabi problem in Braak's settings is determined by two transcendental equations \(G_{\sigma}(z_*, p) = 0\) and \(G_{\sigma}'(z_*, p) = 0\), not just one \(G_{\sigma}(z_*, p) = 0\), for a certain \(z_*\), as is claimed.
The radius of convergence of this series is not smaller than that choice \( x < 30 \), all zeros of \( G_\sigma(0, p) \) belong to the spectrum.

To proceed we have to make the following technical assumption. Namely, we require that

\[
U_i := D(s_i, r_i) \cap D(s_{i+1}, r_{i+1}) \neq \emptyset, \tag{14}
\]

for \( i = 1, \ldots, m - 1 \). If none of \( \sigma_i \) is a non-negative integer, then, up to a multiplicative constant, series \((13)\) are just Taylor expansions of entire solution \( \varphi(z) \) at points \( s_i \). This is why local solutions \( \varphi_i(z) \) and \( \varphi_{i+1}(z) \) have to be proportional for all \( z \in U_i \) where both are well defined. Hence, for a certain nonzero \( \alpha_i \in \mathbb{C} \), the function \( F_i(z) := \varphi_i(z) - \alpha_i \varphi(z) \) vanishes for all \( z \in U_i \), so it vanishes identically. But \( F_i(z) \), as a linear combination of solutions of equation \((12)\), its solution. So, \( F_i(z) \) vanishes identically if and only if \( F_i(z_s) = 0 \), and \( F_i'(z_s) = 0 \), for an arbitrary \( z_s \in U_i \). Eliminating \( \alpha_i \) from equations \( F_i(z_s) = 0 \) and \( F_i'(z_s) = 0 \), we obtain the following criterion. If, with the given assumptions, equation \((12)\) has a non-zero entire solution, then

\[
W_i(z_i) := \det \begin{bmatrix} \varphi_i(z_i) & \varphi_{i+1}(z_i) \\ \varphi'_i(z_i) & \varphi'_{i+1}(z_i) \end{bmatrix} = 0, \tag{15}
\]

for a certain \( z_i \in U_i \), and \( i = 1, \ldots, m - 1 \). These \((m - 1)\) conditions guarantee that \( \varphi(z) \) is an entire function. If infinity is a regular singular point, they guarantee also that \( \varphi(z) \) belongs to the Bargmann space.

Formulation of this method with relaxed assumptions and its version valid for systems of differential equations will be published in \([10]\).

Let us consider the \( \mathbb{Z}_2 \) symmetry-broken Rabi model \((2)\). An elimination of \( \psi_2(z) \) from this system gives one second order equation for \( \varphi(z) := \psi_1(z) \). This equation has the form \((12)\) with

\[
p(z) = -\frac{\lambda + 2\epsilon \lambda + z(-1 + 2E + 2\lambda^2)}{z^2 - \lambda^2},
\]

\[
q(z) = -\frac{e^2 - 2E^2 + 2\epsilon \lambda + \lambda(z - (1 + z\lambda)) + \mu^2}{z^2 - \lambda^2}. \tag{16}\]

Then, we introduce variables \( y \) and \( v(y) \) setting \( z = \lambda(2y - 1) \), and

\[
v(y) := \exp(-2\lambda^2 y)\psi_1(\lambda(2y - 1)). \tag{17}\]

By direct computation we obtain that \( v(y) \) satisfies the following equation

\[
v'' + \left( \frac{\alpha + \beta + 1}{y} + \frac{\gamma + 1}{y - 1} \right) v' + \left( \frac{\bar{\mu}}{y} + \frac{\bar{\nu}}{y - 1} \right) v = 0, \tag{18}\]

where

\[
\bar{\mu} = \frac{1}{2}(\alpha - \beta - \gamma + \alpha \beta - \beta \gamma - \eta),
\]

\[
\bar{\nu} = \frac{1}{2}(\alpha + \beta + \gamma + \alpha \gamma + \beta \gamma + \delta + \eta).
\]
and parameters $a_0 = (\alpha, \beta, \gamma, \delta, \eta)$ were defined earlier, see [3]. Equation (18) is the Heun confluent equation. Its local holomorphic solution near $y = 0$ is given by the Heun confluent function $v_0(y) =$ HeunC$(a_0; y)$. Solution holomorphic near $y = 1$ is given by $v_1(y) =$ HeunC$(a_1, 1-y)$, where $a_1 := (-\alpha, \gamma, \beta, -\delta, \delta + \eta)$. Hence, local solutions $\varphi_{\pm}(z)$ holomorphic near singular points $z = \mp \lambda$ are given by

$$\varphi_{-}(z) := \exp[\lambda z + \lambda^2] \text{HeunC}(a_0; (1 + z/\lambda)/2)$$  \hspace{1cm} (19)

and

$$\varphi_{+}(z) := \exp[\lambda z + \lambda^2] \text{HeunC}(a_1; (1 - z/\lambda)/2).$$  \hspace{1cm} (20)

These local solutions are given by expansion of one entire function, and hence their Wronskian

$$W(p) := \varphi_{+}(z_s)\varphi'_{-}(z_s) - \varphi'_{+}(z_s)\varphi_{-}(z_s),$$  \hspace{1cm} (21)

must vanish at arbitrary $z_s$. The graph of the Wronskian $W(p)$ as a function of energy for the Rabi model and the generalised Rabi model are shown in Fig. 3 and Fig. 4, respectively. The resulting spectrum as seen in Fig. 5 reproduces that of [1], with the benefit of not including possible isolated zeroes of $G_+$.  

This research has been partially supported by grant No. DEC-2011/02/A/ST1/00208 of National Science Centre of Poland.

![FIG. 3. Graph of Wronskian $W(p)$ for $p := (x, \lambda, \mu, \epsilon) = (x, 7/10, 4/10, 0)$.](image3.png)

![FIG. 4. Graph of Wronskian $W(p)$ for $p := (E, \lambda, \mu, \epsilon) = (E, 7/10, 4/10, 2/10)$.](image4.png)

FIG. 5. Spectrum of generalised Rabi model for $\mu = 0.7$, and $\epsilon = 0.2$. 

[1] D. Braak, Phys. Rev. Lett. 107, 100401 (2011).
[2] D. Braak, ArXiv e-prints (2012), arXiv:1205.3439 [quant-ph].
[3] A. Moroz, ArXiv e-prints (2012), arXiv:1205.3139 [quant-ph].
[4] V. Bargmann, Comm. Pure Appl. Math. 14, 187 (1961).
[5] S. Schweber, Annals of Physics 41, 205 (1967).
[6] W. Gautschi, SIAM Rev. 9, 24 (1967).
[7] A. Ronveaux, *Heun’s Differential Equations* (Oxford University Press, Oxford, 1995).
[8] D. Braak, “Supplement material to paper ”integrability of the Rabi model,” http://prl.aps.org/supplemental/PRL/v107/i10/e100401 (2011).
[9] M. Kuś and M. Lewenstein, J. Phys. A 19, 305 (1986).
[10] M. Kuś, A. J. Maciejewski, M. Przybylska, and T. Stachowiak, (2012), in preparation.