THE POINCARÉ-BENDIXSON THEORY FOR CERTAIN SEMI-FLOWS
IN HILBERT SPACES

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Abstract. We study semi-flows satisfying a certain squeezing condition with respect to
the quadratic form of a bounded self-adjoint operator acting in a Hilbert space. Under
certain compactness assumptions from our previous results it follows that there exists
an invariant topological manifold that attracts all compact trajectories. In the case
of a two-dimensional manifold we obtain an analog of the Poincaré-Bendixson theorem
on the trichotomy of ω-limit sets. Moreover, we obtain the conditions of existence of
an orbitally stable periodic trajectory. We present applications of our results to study
certain nonlinear systems of delay equations and reaction-diffusion systems. For these the
required operator is obtained as a solution to certain operator inequalities with the use
of the Yakubovich-Likharnikov frequency theorem for $C_0$-semigroups and its properties
are established from the Lyapunov inequality and dichotomy of the linear part.

1. Introduction

We start with the precise statements of our main results.

Let $H$ be a separable Hilbert space and $\varphi^t: H \to H$ be a semiflow on $H$, i.e.
1) $\varphi^0(u) = u$ for all $u \in H$
2) $\varphi^{t+s}(u) = \varphi^t(\varphi^s(u))$ for all $u \in H$ and $t, s \geq 0$.
3) The map $\mathbb{R}_+ \times H \to H$ defined as $(t, u) \mapsto \varphi^t(u)$ is continuous.

For the sake of brevity we denote the flow by $\varphi$. Our main conditions imposed on $\varphi$ are
introduced as follows.

(H1) There is a continuous linear operator $P: H \to H$, self-adjoint ($P = P^*$) such that
$H$ splits into the direct sum of orthogonal $P$-invariant subspaces $H^+$ and $H^-$, i.e.
$H = H^+ \oplus H^-$, such that $P|_{H^-} < 0$ and $P|_{H^+} > 0$.

(H2) We have dim $H^- = j < \infty$.

(H3) For $V(u) := (Pu, u)$ and some numbers $\delta > 0$, $\nu > 0$ we have

\[ e^{2\nu s}V(\varphi^s(u) - \varphi^s(v)) - e^{2\nu l}V(\varphi^l(u) - \varphi^l(v)) \leq -\delta \int_l^s e^{2\nu s}|\varphi^s(u) - \varphi^s(v)|^2 ds, \quad (1.1) \]

for every $u, v \in H$ and $0 \leq l < s$.

Remark 1. Let $H$ be a real Hilbert space. The set $\mathcal{K} := \{(Pu, u) \leq 0\}$ is called a $j$-
dimensional quadratic cone in $H$. From (H3) we have that the semi-flow is monotone
w. r. t. the cone $\mathcal{K}$. Namely, if $u - v \in \mathcal{K}$ then $\varphi^t(u) - \varphi^t(v) \in \mathcal{K}$ for all $t \geq 0$. For $j = 1$
There exists the quadratic cone $K$ is a union of two convex closed cones $K^+$ and $-K^+$. In this case (H3) gives the monotonicity of $\varphi$ w. r. t. the partial order given by $K^+$. Such kind of monotonicity is considered in the classical theory of monotone dynamical systems [8]. For $j > 1$ there is no such order (due to the lack of convexity). However, the cone $K$ (a cone of rank $j$ in the terminology of [24]) defines a pseudo-order, which leads to certain consequences for the semi-flow. Semi-flows, which are monotone w. r. t. such high-rank cones, were studied in finite-dimensional spaces by L. A. Sanchez [24] and in the context of Banach spaces by L. Feng et al. [11]. In particular, in these works analogs of the Poincaré-Bendixson theorem were obtained. However, it seems that the stability results, which we present below, along with some other topological consequences cannot be deduced from the abstract monotonicity used in [23, 11]. Not to mention for this paper the starting point is our previous result [1], which holds for cocycles (non-autonomous dynamical systems) and, for example, leads to an extension of the Massera convergence theorem (with certain stability results) for periodic cocycles. Below we will see that (H3) naturally arises for the quadratic cones obtained from certain operator inequalities.

We also make use of certain compactness assumptions.

(COM1) The operator $P$ from (H1) is compact.

(COM2) There exists $t_{com} > 0$ such that the map $\varphi^{t_{com}} : \mathbb{H} \rightarrow \mathbb{H}$ is compact.

We have to mention that we do not know any example, in which (COM1) holds and (COM2) does not. However, certain delay equations satisfy (COM2), but cannot satisfy (COM1) (see [2]). Anyway, both assumptions lead to the compactness of bounded semi-orbits.

For a point $u_0 \in \mathbb{H}$ we denote its positive semi-orbit by $\gamma^+(u_0)$, i. e. $\gamma^+(u_0) = \bigcup_{t \geq 0} \varphi^t(u_0)$, and we denote its $\omega$-limit set by $\omega(u_0) := \bigcap_{s \geq 0} \bigcup_{t \geq s} \varphi^t(u_0)$ [9]. A complete trajectory is a continuous map $u : \mathbb{R} \rightarrow \mathbb{H}$ such that the equality $u(t + s) = \varphi^s(u(t))$ holds for all $t \geq 0$ and $s \in \mathbb{R}$. In this case we say that $u(\cdot)$ is passing through $u(0)$. If there is a unique complete trajectory $u(\cdot)$ passing through $u_0$ we also consider its $\alpha$-limit set $\alpha(u_0) := \bigcap_{t \leq 0} \bigcup_{s \leq t} u(s)$, its negative semi-orbit $\gamma^-(u_0) := \bigcup_{t \leq 0} u(t)$ and its complete orbit $\gamma(u_0) := \gamma^+(u_0) \cup \gamma^-(u_0)$. We will sometimes call the $\omega$-limit set of a point $u_0$ the $\omega$-limit of its orbit $\gamma^+(u_0)$ or its semi-trajectory $t \mapsto \varphi^t(u_0)$.

One of the main results is the following theorem.

**Theorem 1.** Suppose that the semiflow $\varphi$ satisfies (H1), (H2) with $j = 2$, (H3) and one of (COM1) or (COM2); then the $\omega$-limit set $\omega(u_0)$ of any point $u_0 \in \mathbb{H}$ with a bounded positive semi-orbit is one of the following:

- (T1) A stationary point;
- (T2) A periodic orbit;
- (T3) A union of some set of stationary points $\mathcal{N}$ and a set of complete orbits whose $\alpha$- and $\omega$-limit sets lie in $\mathcal{N}$.

**Corollary 1.1.** Within the hypotheses of Theorem 1 the trichotomy (T1), (T2), (T3) holds for the $\alpha$-limit set of any complete trajectory bounded in the past.

**Corollary 1.2.** Within the hypotheses of Theorem 1 any isolated orbitally stable periodic orbit is asymptotically orbitally stable.

Recall that a stationary point $v_0 \in \mathbb{H}$ is called Lyapunov stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\varphi^t(v) - v_0| < \varepsilon$ for all $t \geq 0$ provided that $|v - v_0| < \delta$.
Let $\mathcal{D}$ be some set containing a stationary point $v_0$. We say that $\mathcal{D}$ is a $k$-dimensional local unstable set for $v_0$ if

(U1) $\mathcal{D}$ is a homeomorphic image of some open $k$-dimensional cube;
(U2) For every point $w_0 \in \mathcal{D}$ there is a unique complete trajectory $\gamma(w)$ with $\gamma(0) = w_0$ and $\gamma(t) \to v_0$ as $t \to -\infty$;
(U3) For every $\varepsilon > 0$ there is $\delta > 0$ such that if $|w_0 - v_0| < \delta$ and $w_0 \in \mathcal{D}$ then $|\gamma(t) - v_0| < \varepsilon$ for all $t \leq 0$.

We call a stationary point $v_0 \in \mathbb{H}$ terminal if either it is Lyapunov stable or there is a 2-dimensional unstable set for $v_0$. In the latter case we call $v_0$ an unstable terminal point. The role of terminal points is in the following.

**Corollary 1.3.** Within the hypotheses of Theorem_4 suppose that $\omega(u_0)$ contains a terminal point $v_0$. Then $\omega(u_0) = \{v_0\}$.

If a Banach space $E$ is continuously embedded into $\mathbb{H}$ we identify the elements of $E$ and their images under the embedding.

(EMB) There is a continuously embedded Banach space $E \subset \mathbb{H}$ such that

(A) The space $E$ is positively invariant w. r. t. $\varphi$ and $\varphi^t : E \to E$, $t \geq 0$, is a semi-flow on $E$.
(B) The space $E$ contains all complete orbits of the flow $\varphi$ in $\mathbb{H}$.
(C) There exists $t_{emb} \geq 0$ such that the map $\varphi^{t_{emb}} : E \to E$ takes each set $B \subset E$, which is compact in $\mathbb{H}$, into a set, which is compact in $E$.

In applications to delay equations, which are considered in a proper Hilbert space $\mathbb{H}$, the role of $E$ is played by the space of continuous functions. From [29] it follows that the flow $\varphi$ in $E$ agrees (=coincides) with the classical one [13]. In this case the number $t_{emb}$ from (C) is the same as $t_{com}$ in (COM2). In certain parabolic problems $E$ is a proper space in the scale of Hilbert spaces given by powers of the Laplacian and $\mathbb{H}$ is a suitable $L_2$ space. Moreover, in both cases, we may have the property $\varphi^{t_{emb}}(\mathbb{H}) \subset E$ that implies the equivalence of certain convergence and stability properties (see Lemma_3 w. r. t. norms of $E$ and $\mathbb{H}$. Note also that (EMB) is always satisfied for $E := \mathbb{H}$.

Now suppose (EMB) is satisfied for some Banach space $E$. We call a subset $A \subset E$ an attractor w. r. t. (EMB) if $A$ is closed in $E$, bounded in $\mathbb{H}$ and there exists a set $A \subset U_A \subset E$, which is open in $E$, such that for any $u_0 \in U_A$ the positive semi-orbit $\gamma^+(u_0)$ is compact in $\mathbb{H}$ and $\omega(u_0) \subset A$. We call any such set $U_A$ a neighborhood of the attractor $A$.

**Theorem 2.** Let the semiflow $\varphi$ satisfy (H1), (H2) with $j = 2$, (H3) and one of (COM1) or (COM2). Suppose there is an attractor $A$ w. r. t. (EMB) that either contains no stationary points or the only stationary points in $A$ are unstable terminal points. Then the set $A$ contains at least one orbitally stable periodic orbit.

**Remark 2.** In applications, the monotonicity property given in (H3) is linked with the Lipschitz property of nonlinearities in the system. Sometimes we are interested only in the dynamics on a certain invariant set $S$ (for example, $S$ may be the cone of non-negative functions). Outside $S$ the nonlinearities may not posses proper Lipschitz constants and therefore (H3) can fail to hold globally. Often it is possible to redefine the nonlinearities outside $S$ in such a way that the system with new nonlinearities will satisfy (H3) globally. Since the dynamics of both systems coincide on $S$ we can apply above theorems to the second system to derive results for the first one, but on the set $S$ (see Sections 5 and 6). Thus, for practical purposes we do not need to consider any local versions of (H3), which
can greatly complicate the proofs (or require additional assumptions) as can be seen from [27, 25, 26]. We also note that the construction of invariant regions is possible with the use of the frequency-domain theorem [15].

With the above theorems we present here an attempt to give a unified approach for main results of [27, 26, 25] and to make them clearer. The consideration of (H1), (H2) and (H3) instead of many a priori estimates depending on the equation (as it is done in the mentioned papers) makes it possible to study the problem in the abstract context of semi-flows in Hilbert spaces. The squeezing-like condition in (H3) has several topological conclusions (such as Corollary 3.1 or Lemma 5 below), which allows us to treat the Poincaré-Bendixson theory in a purely topological context. In this direction we also use the topological results of O. Hájek [12], who showed the existence of transversals for flows on 2-dimensional manifolds.

The presented unification leads to the possibility of wider applications, which may concern parabolic problems with nonlinear boundary control (see [1]) and parabolic problems with delay. The use of the frequency theorem [20, 21, 22] sometimes lead to the sharper estimations of regions in the space of parameters than those were obtained by R. A. Smith (see Remarks 5 and 7 for the comparison of Smith’s frequency condition with the usual one). Since applications of the frequency theorem requires the checking of stabilizability, it is not clear whether all the results from, for example, [26] are contained in our approach. But in concrete examples given in Sections 5 and 6 which are taken from [27, 26], this property is satisfied. Another obstacle arises when we apply the frequency theorem for delay-equations with the measurement operator that is unbounded in $L^2$ (see Section 5). We hope that our results will stimulate developments of frequency-domain methods.

A historical background on the development of the Poincaré-Bendixson theorem from the classical smooth version up to semi-flows on the plane is given by K. Ciesielski [10]. A review of several works extending the Poincaré-Bendixson theory for certain high-dimensional ODEs is contained in the paper of Li B. [19]. Another review for ODEs is done by Burkin I. [7], who especially treats the works of R. A. Smith and their connection with the frequency-domain methods (see also [18]). In [27, 25, 26] a comparison of the presented results with many others in the field of delay and parabolic equations is given by R. A. Smith. For applications of (H3) with the use of frequency-domain methods to study almost periodic solutions (and, in particular, their dimension-like properties) of almost periodic ODEs see [3, 4, 5, 17].

This paper is organized as follows. In Section 2 we expose several auxiliary facts concerned with the existence of a certain topological invariant manifold, establish its properties and state a theorem on the existence of transversals for flows on two-dimensional manifolds. In Section 3 we prove Theorem 1 and the corresponding corollaries. Section 4 is devoted to the proof of Theorem 1. In Section 5 we consider applications of Theorems 1 and 2 to a system of delay equations. In Section 6 the theorems are applied to certain reaction-diffusion equations.

2. Preliminaries

2.1. Amenable set $\mathfrak{A}$. Let (H1), (H2) and (H3) be satisfied. A complete trajectory $u(\cdot)$ of the semi-flow $\varphi$ is called amenable if

$$\int_{-\infty}^0 e^{2\nu s} |u(s)|^2 ds < \infty. \quad (2.1)$$

\footnote{This condition can be weakened in several cases [23].}
Define $\mathfrak{A}$ as the set of all $u_0 \in H$ such that there exists an amenable trajectory passing through $u_0$. We call the set $\mathfrak{A}$ the amenable set. Lemma 1 from [1] gives us the following property.

**Lemma 1.** Let $u(\cdot)$ and $v(\cdot)$ be two amenable trajectories; then $V(u(t) - v(t)) \leq 0$ for all $t \in \mathbb{R}$.

Let $\Pi: H \to H^-$ be the orthogonal projector onto $H^-$. An immediate corollary of Theorem 1 from [1] is the following.

**Theorem 3.** Suppose $(H1), (H2), (H3)$ are satisfied and $\mathfrak{A} \neq \emptyset$; then $\Pi: \mathfrak{A} \to \Pi \mathfrak{A} \subset H^-$ is a homeomorphism. Moreover, if one of $(COM1)$ or $(COM2)$ holds then $\Pi \mathfrak{A} = H^-.$

If the hypotheses of Theorem 3 hold we define the map $\Phi: H^- \to \mathfrak{A}$ by the relation $\Pi \Phi(\zeta) = \zeta$ for all $\zeta \in H^-$. By Theorem 3 the map $\Phi$ is a homeomorphism.

**Corollary 3.1.** Suppose that $(H1), (H2), (H3)$ are satisfied and $\Pi \mathfrak{A} = H^-;$ then

(A1) $\mathfrak{A}$ is an invariant $j$-dimensional topological manifold, i.e., $\varphi^t(\mathfrak{A}) = \mathfrak{A}$ for all $t \geq 0$.

(A2) Any map $\varphi^t, t \geq 0,$ is continuously invertible on $\mathfrak{A}$ and $\varphi^t|_{\mathfrak{A}}, t \in \mathbb{R}$, is a dynamical system (=flow) on $\mathfrak{A}$.

(A3) For any $u_0 \in H$ with a compact semi-orbit we have

$$|\varphi^t(u_0) - \Phi(\Pi \varphi^t(u_0))| \to 0 \text{ as } t \to +\infty. \quad (2.2)$$

**Proof.** The map $\Phi$ defines the structure of a $j$-dimensional topological manifold on $\mathfrak{A}$. The invariance now follows from the definition of $\mathfrak{A}$. Thus, (A1) is satisfied.

From (H3) with $r = 0$ for two points $u(0), v(0) \in \mathfrak{A}$ it follows that

$$-V(u(0) - v(0)) \geq \int_{-\infty}^{0} e^{2\nu s}|u(s) - v(s)|^2 ds. \quad (2.3)$$

From this it is clear that any amenable trajectory passing through a given point is unique and, therefore, $\varphi^t: \mathfrak{A} \to \mathfrak{A}$ is bijective. To prove the continuity suppose that $t_k \to t_0 \leq 0$ and $u_k(0) \to v(0) \in \mathfrak{A}$ as $k \to \infty$, but for some $\delta > 0$ we have $|\varphi^{t_k}(u_k(0)) - \varphi^{t_0}(v(0))| \geq \delta > 0$ for all $k = 1, 2, \ldots$. Then from (2.3) we have that $\int_{-\infty}^{0} e^{2\nu s}|u_k(s) - v(s)|^2 ds \to 0$ as $k \to +\infty$. In particular, there is $s_0 < t_0$ and a subsequence such that $u_{k_m}(s_0) \to v(s_0)$ as $m \to \infty$ and, consequently, $u_{k_m}(t) \to v(t)$ uniformly in $t \in [s_0, 1]$ and $\varphi^{t_{k_m}}(u_{k_m}(0)) = u_{k_m}(t_{k_m}) \to v(t_0) = \varphi^{t_0}(v(0))$ as $m \to \infty$ that leads to a contradiction. Thus (A2) is proved.

The convergence in (A3) is a consequence of the continuity of $\Phi$ (see Proposition 4 from [1]).

**Lemma 2.** Within the conditions of Corollary 3.1 let $\text{(EMB)}$ be satisfied with some Banach space $E$; then $\mathfrak{A} \subset E$ and the topologies of $E$ and $H$ coincide on $\mathfrak{A}$.

**Proof.** The inclusion $\mathfrak{A} \subset E$ follows from the property (B) in (EMB) and the definition of $\mathfrak{A}$. To show the coincidence of topologies it is sufficient to show that $\Pi: \mathfrak{A} \to H^-$ is a homeomorphism when $\mathfrak{A}$ is endowed with the topology of $E$. From Theorem 3 and since the embedding $E \subset H$ is continuous it follows that $\Pi$ is a continuous bijection. Let us check that $\Pi^{-1}: H^- \to \mathfrak{A}$ is continuous. Suppose that a sequence $\zeta_k \in H^-$, $k = 1, 2, \ldots$, converges to some $\zeta \in H^-$ as $k \to \infty$. Let $u_k, u \in \mathfrak{A}$ be the unique points with $\Pi u_k = \zeta_k$ and $\Pi u = \zeta$. By Theorem 3 we have $u_k \to u$ in $H$ and, in virtue of (A2), $\varphi^{-t_{k_m}}(u_k) \to \varphi^{-t_{k_m}}(u)$ in $H$ as $k \to \infty$. Suppose that $u_k \not\to u$ in $E$. So, there are $\delta > 0$ and a subsequence (keep the same index) such that $\|u_k - u\|_E \geq \delta$. Since $\varphi^{-t_{k_m}}(u_k)$ converges in $H$, it is precompact
in $\mathbb{H}$ and, by (C), its image after the time $t_{emb}$, that is the sequence $u_k$, is precompact in $E$. Since the embedding $E \subset \mathbb{H}$ is continuous and $u_k \to u$ in $\mathbb{H}$, any of its limit points (that is not empty due to the mentioned precompactness) in $E$ must coincide with $u$. This contradicts to $\|u_k - u\|_E \geq \delta$. Therefore, $u_k \to u$ in $E$ and $\Pi$ is a homeomorphism.  

**Remark 3.** Within the conditions of Corollary 3.1 and Lemma 2 we see that properties (U1), (U2) and (U3) of a stationary point having a $k$-dimensional local unstable set are equivalent to analogous properties, which can be formulated using the norm of $E$.

**Lemma 3.** Within the conditions of Corollary 3.1 let (EMB) be satisfied with some Banach space $E$. Assume that $\varphi^{emb}(\mathbb{H}) \subset E$. Then

1. (S1) For a set $A \subset E$, which is compact in $\mathbb{H}$, and $u_0 \in \mathbb{H}$ the properties

   1) dist$_{\mathbb{H}}(\varphi^t(u_0), A) \to 0$ as $t \to +\infty$.

   2) dist$_E(\varphi^t(u_0), A) \to 0$ as $t \to +\infty$.

   are equivalent.

2. (S2) A periodic orbit is orbitally stable w. r. t. $\varphi$ iff it is orbitally stable w. r. t. $\varphi|_E$.

3. (S3) A stationary point is Lyapunov stable w. r. t. $\varphi$ iff it is Lyapunov stable w. r. t. $\varphi|_E$.

**Proof.** Note from (EMB) and $\varphi^{emb}(\mathbb{H}) \subset E$ we have that if $u_k \to u$ in $\mathbb{H}$ then $\varphi^t(u_k) \to \varphi^t(u)$ in $E$ as $k \to \infty$ for every $t \geq 2t_{emb}$. From this the statements of (S1), (S2) and (S3) can be proved by assuming the contrary.  

2.2. Transversals of flows at non-stationary points. Let $\xi^t: \mathcal{X} \to \mathcal{X}$, where $t \in \mathbb{R}$, be a flow on a complete metric space $\mathcal{X}$. Suppose $\varepsilon > 0$ and a point $x_0 \in \mathcal{X}$ are given. A subset $S \subset \mathcal{X}$ is called an $\varepsilon$-section of $\xi$ at $x_0$ if $x_0 \in S$, the set $\mathcal{U} = \mathcal{U}(\varepsilon, S) := \bigcup_{t \in [-\varepsilon, \varepsilon]} \xi^t(S)$ is a topological neighbourhood$^2$ of $x_0$ in $\mathcal{X}$ and for every $y \in \mathcal{U}$ there exists a unique point $y_0 \in S$ and a unique time moment $t \in [-\varepsilon, \varepsilon]$ such that $\xi^t(y_0) = y$.

Now suppose $\mathcal{X}$ is a two-dimensional manifold. A set $T \subset \mathcal{X}$ is called an $\varepsilon$-transversal of $\xi$ at $x_0$ if it is simultaneously an $\varepsilon$-section and a homeomorphic image of a closed segment, for which $x_0$ corresponds to some of its interior points. We call $T$ simply a transversal of $\xi$ if for some $\varepsilon > 0$ it is an $\varepsilon$-transversal of $\xi$ at some point $x_0$.

The following theorem is due to O. Hájek (see Chapter VII, Corollary 2.6 in [12]).

**Theorem 4.** Let $\xi$ be a flow on a two-dimensional manifold $\mathcal{X}$; then for any non-stationary point $x_0$ and all sufficiently small $\varepsilon > 0$ there exists an $\varepsilon$-transversal $T$ of $\xi$ at $x_0$.

Let $T$ be a $\varepsilon$-transversal of $\xi$ at $x_0$. It is clear that any closed connected subset of $T$ is also a $\varepsilon$-transversal of $\xi$. Without loss of generality, we may assume that $T$ is given by a homeomorphism $h: [-1, 1] \to T$ with $h(0) = x_0$. In this case the map $[-\varepsilon, \varepsilon] \times [-1, 1] \to \mathcal{U} = \mathcal{U}(\varepsilon, T)$ defined as $(t, s) \mapsto \varphi^t(h(s))$ gives a homeomorphism of the cube $[-\varepsilon, \varepsilon] \times [-1, 1]$ onto its image. By the Brouwer theorem on invariance of domain, the interior $(-\varepsilon, \varepsilon) \times (-1, 1)$ is mapped onto an open subset of $\mathcal{X}$.

3. Structure of $\omega$-limit sets

In this section we suppose that (H1), (H2) with $j = 2$, (H3) are satisfied and $\Pi \mathcal{X} = \mathbb{H}^-$. Then from Corollary 3.1 we get that $\mathcal{X}$ is an invariant 2-dimensional manifold homeomorphic to $\mathbb{H}^-$ and $\varphi^t$ induces a flow on $\mathcal{X}$. By $\xi^t: \mathbb{H}^- \to \mathbb{H}^-$, where $t \in \mathbb{R}$, we denote the flow on

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$^2$That is $x_0$ belongs to the interior of $\mathcal{U}$. 

Lemma 4. Let $T$ be an $\varepsilon$-transversal of $\xi$ and $v_0 \in \omega(u_0)$. Then for the corresponding complete trajectory $v(t)$ with $v(0) = v_0$ the trajectory $\Pi v(t)$ crosses $T$ in at most one point.

Proof. Supposing the contrary, we obtain two moments of time $t_1, t_2$ with $t_1 < t_2$ such that $v(t_1) \in T$, $v(t_2) \in T$ and $v(t) \notin T$ for all $t \in (t_1, t_2)$. Consider the curve $\Gamma$ given by the part of the trajectory $v(t)$ for $t \in [t_1, t_2]$ and the part of the transversal $T$ between $v(t_1)$ and $v(t_2)$, which we denote by $T_{t_1}^{t_2}$. Clearly, $\Gamma$ is a simple closed curve and, by the Jordan curve theorem, $\Gamma$ divides $\mathfrak{A}$, which is a homeomorphic image of the plane, into two parts: the bounded one and the unbounded one. From this there are only two possible cases, in which $\varphi^t$ maps $T_{t_1}^{t_2}$ into the bounded part of the plane or into the unbounded one (see Fig. 1).

Consider two points $v_{in} = v(t_{in})$ and $v_{out} = v(t_{out})$ corresponding to two moments of time $t_{in} > t_2$ and $t_{out} < t_1$ such that $v_{in}$ belongs to the bounded part and $v_{out}$ belongs to the unbounded part of $\mathfrak{A}$. By definition, we have $v_{in}, v_{out} \in \omega(u_0)$. Let $r > 0$ be given such that the open balls of radii $r$ centred at $v_{in}$ and $v_{out}$ respectively do not intersect with the $r$-neighbourhood of $\Gamma$.

Let $\mathcal{K} \subset \mathfrak{H}$ be any compact set containing the semi-trajectory $\varphi^t(u_0)$, $t \geq 0$, the set $\Phi(\Pi \varphi^t(u_0))$, $t \geq 0$, and the set $T_{t_1}^{t_2}$. In particular, $\omega(u_0) \subset \mathcal{K}$. Put $T := t_{in} - t_{out}$. By the continuity of the semi-flow $\varphi^t$, $t \geq 0$, there is $\delta > 0$ such that $|\varphi^t(v_1) - \varphi^t(v_2)| < r$ for all $t \in [0, T]$ provided that $v_1, v_2 \in \mathcal{K}$ and $|v_1 - v_2| < \delta$.

In virtue of ($A_3$) there is $t_3 > 0$ such that $|\varphi^t(u_0) - \Phi(\Pi \varphi^t(u_0))| < \delta/2$ holds for all $t \geq t_3$. Since $v_{in}, v_{out} \in \omega(u_0)$ there are two moments of time $t_{in}^{(r)} > t_3$ and $t_{out}^{(r)} > t_3$ such that $|\varphi^{t_{in}^{(r)}}(u_0) - v_{in}| < r$, $|\varphi^{t_{out}^{(r)}}(u_0) - v_{out}| < r$, $|\Phi(\Pi \varphi^{t_{in}^{(r)}}(u_0)) - v_{in}| < r$ and $|\Phi(\Pi \varphi^{t_{out}^{(r)}}(u_0)) - v_{out}| < r$. Since the trajectory of $u_0$ is continuous there must be a moment of time $t_0 \in (t_{in}^{(r)}, t_{out}^{(r)})$ with $\Phi(\Pi \varphi^{t_0}(u_0)) \in \Gamma$. Denote $\Gamma_{t_1}^{t_2} := \{ \varphi^t(v_0) \mid t \in [t_1, t_2] \}$ Suppose $\Phi(\Pi \varphi^{t_0}(u_0)) \in \mathcal{O}_{\delta/2}(\Gamma_{t_1}^{t_2})$, i.e. $|\Phi(\Pi \varphi^{t}(u_0)) - \varphi^t(v_0)| < \delta/2$ for some $t \in [t_1, t_2]$. From this it follows that for $v_1 := \varphi^{t_0}(u_0)$ and $v_2 := \varphi^{t_0}(v_0)$ we have $|v_1 - v_2| < \delta$, $v_1, v_2 \in \mathcal{K}$ and, consequently, $|\varphi^{t_0}(v_1) - \varphi^{t_0}(v_2)| < r$ for $t \in [0, T]$. Thus for all $t_0 \geq t_0$ each time we have $\varphi^{t_0}(u_0) \in \mathcal{O}_{\delta/2}(\Gamma_{t_1}^{t_2})$ the point $\varphi^{t_0+t}(u_0)$ remains in $\mathcal{O}_{\epsilon}(\Gamma_{t_1}^{t_2})$ and, moreover, $\varphi^{t_0+t}(u_0) \in \mathcal{O}_{\epsilon}(v_{in})$ for some $t' \in [0, T]$. So, for $t \geq t_0$ the curve $\Phi(\Pi \varphi^{t_0}(u_0))$ cannot reach $\mathcal{O}_{\epsilon}(v_{out})$ crossing $\Gamma \cap \mathcal{O}_{\delta/2}(\Gamma_{t_1}^{t_2})$.

Consider the remaining part of the $\varepsilon$-transversal, i.e. $S_{t_1}^{t_2} := T_{t_1}^{t_2} \setminus \mathcal{O}_{\delta/2}(\Gamma_{t_1}^{t_2})$. There is $r > d > 0$ such that $\mathcal{O}_d(\varphi^t(S_{t_1}^{t_2}))$ lies in the bounded part of the plane and do not intersect $\Gamma$. For some $\delta_1 > 0$ we have $|\varphi^t(v_1) - \varphi^t(v_2)| < d$ for all $t \in [0, \varepsilon]$ provided that $v_1, v_2 \in \mathcal{K}$ and $|v_1 - v_2| < \delta_1$. Let $t_4 > 0$ be such that $|\varphi^t(u_0) - \Phi(\Pi \varphi^t(u_0))| < \delta_1$ for all $t \geq t_4$.

Thus, if for some $t_0 \geq \max\{t_3, t_4\}$ we have $\varphi^{t_0}(u_0) \in \mathcal{O}_{\epsilon}(v_{in})$ then $\varphi^t(u_0) \notin \mathcal{O}_{\epsilon}(v_{out})$ for all $t \geq t_0$ that leads to a contradiction with $v_{out} \in \omega(u_0)$. \qed
Lemma 5. Let \( v_1, v_2 \in \mathbb{H} \) be two periodic points with distinct orbits \( \gamma(v_1) \) and \( \gamma(v_2) \) respectively; then there exists \( \delta = \delta(v_1, v_2) > 0 \) such that for any \( u_0 \in \mathbb{H} \) with \( \text{dist}(u_0, \gamma(v_1)) < \delta \) we have \( \Pi \varphi^t(u_0) \not\in \Pi \gamma(v_2) \) for any \( t \geq 0 \).

Proof. Suppose the contrary, i. e. that for every \( \delta > 0 \) there exists a point \( u_\delta \in \mathbb{H} \) such that \( \text{dist}(u_\delta, \gamma(v_1)) < \delta \) and there is \( t_\delta > 0 \) such that \( \Pi \varphi^{t_\delta}(u_\delta) \in \Pi \gamma(v_2) \). Let \( v_{1,\delta} \in \gamma(v_1) \) and \( v_{2,\delta} \in \gamma(v_2) \) be such that \( |u_\delta - v_{1,\delta}| < \delta \) and \( \Pi \varphi^{t_\delta}(u_\delta) = \Pi v_{2,\delta} \). Let \( \delta = \delta_k \), where \( k = 1, 2, \ldots \), be a sequence tending to zero such that \( v_{1,\delta_k} \to v_1 \in \gamma(v_1), v_{2,\delta_k} \to v_2 \in \gamma(v_2) \) and \( \varphi^{-t_\delta}(v_{2,\delta_k}) \to \bar{v}_2 \in \gamma(v_2) \) as \( k \to \infty \). From (H3) with \( r = t_\delta, l = 0, u = u_\delta \) and \( v = \varphi^{-t_\delta}(v_{2,\delta_k}) \) we get

\[
-e^{2 \nu t_\delta} V(\varphi^{t_\delta}(u_\delta) - \varphi^{t_\delta}(\varphi^{-t_\delta}(v_{2,\delta_k}))) + V(u_\delta - \varphi^{-t_\delta}(v_{2,\delta_k})) \geq 0
\]

(3.1)

Since \( \Pi \varphi^{t_\delta}(u_\delta) = \Pi v_{2,\delta} \), the first term in the left-hand side of (3.1) is non-positive. Thus, we get

\[
V(u_\delta - \varphi^{-t_\delta}(v_{2,\delta_k})) \geq \delta \int_0^{t_\delta} |\varphi^s(u_\delta) - \varphi^s(\varphi^{-t_\delta}(v_{2,\delta_k}))|^2 ds.
\]

(3.2)

Our purpose is to show that the term in the left-hand side is negative for \( \delta = \delta_k \) with sufficiently large \( k \) that will lead to a contradiction. But this follows from the choice of \( \delta_k \) and Lemma 1 since as \( k \to \infty \) we have

\[
V(u_{\delta_k} - \varphi^{-t_{\delta_k}}(v_{2,\delta_k})) \to V(v_1, \bar{v}_2) < 0.
\]

(3.3)

Thus the lemma is proved. \( \square \)

Lemma 6. If \( v_0 \in \omega(u_0) \) is a periodic point then \( \omega(u_0) = \gamma(v_0) \).
Proof. To prove the statement we suppose the contrary, i.e. that the set \( \omega(u_0) \setminus \gamma(v_0) \) is non-empty. Since \( \omega(u_0) \) is connected there exists a point \( \gamma \in \gamma(v_0) \) that is non-isolated from \( \omega(u_0) \setminus \gamma(v_0) \). Let \( T \subset A \) be an \( \varepsilon \)-transversal of \( \varphi|_A \) at \( \gamma \) and put \( U = \bigcup_{t \in [-\varepsilon, \varepsilon]} \varphi^t(T) \). Let us show that if \( \bar{v} \in A \) is sufficiently close to \( \gamma \) then \( \bar{v} \) is a periodic point. Indeed, any \( \bar{v} \in U \) has its trajectory crossing \( T \) at least once. If \( \bar{v} \) is close enough to \( \gamma \) then it must return to \( U \) after the period of \( \gamma \) and by Lemma 4 it must cross \( T \) at the same point and, consequently, \( \bar{v} \) is a periodic point. Since \( \gamma \) is non-isolated from \( \omega(u_0) \setminus \gamma(v_0) \) it is a limit of a sequence of distinct periodic points from \( \omega(u_0) \). From this periodic points \( \bar{v}, \nu_{\text{sep}} \in \omega(u_0) \) can be chosen such that \( \gamma(\nu_{\text{sep}}) \) separates \( \gamma(v_0) \) and \( \gamma(\bar{v}) \). By Lemma 5 there is \( \delta = \delta(\bar{v}, \nu_{\text{sep}}) \) such that for \( v \in A \) and \( \text{dist}(v, \gamma(v_0)) < \delta \) then \( \Pi \varphi^t(v) \notin \Pi \gamma(\nu_{\text{sep}}) \) for all \( t \geq 0 \). Since \( \nu_{\text{sep}} \), \( \bar{v} \in \omega(u_0) \) there must be moments of time \( t_0 > 0 \) and \( t > t_0 \) such that \( \text{dist}(\varphi^{t_0}(u_0), \gamma(v_0)) < \delta \) and \( \varphi^t(u_0) \in \mathcal{O}(\bar{v}) \) with \( r > 0 \) sufficiently small. But this gives a moment of time \( t' \in (t_0, t) \) with \( \Pi \varphi^{t'}(u_0) \in \Pi \gamma(\nu_{\text{sep}}) \) that contradicts to the previously established property. The lemma is proved.

Proof of Theorem 7. Let \( v_0 \in \omega(u_0) \) be a non-stationary point. Let us show that in this case either \( \alpha(v_0) \) and \( \omega(v_0) \) consist of stationary points or \( \omega(u_0) \) is a periodic orbit. If there is a non-stationary point \( \bar{v} \) in any of these sets then the trajectory of \( v_0 \) must intersect a transversal at \( \bar{v} \) infinitely many times. By Lemma 6 all these intersections coincide and the point \( \bar{v} \) must be periodic. From Lemma 6 it follows that \( \omega(u_0) = \gamma(\bar{v}) \).

So, either there is at least one stationary point in \( \omega(u_0) \) or \( \omega(u_0) \) is a periodic orbit \((T_2)\) is realized. In the first case either the stationary point is the only point in \( \omega(u_0) \) \((T_1)\) is realized or we have \((T_3)\).

Proof of Corollary 7.1. Due to \((A_1)\) and \((A_2)\) the statement can be deduced from the Poincaré-Bendixson theorem for flows on the plane as in [12]. However, the key lemmas after obvious modifications and much simpler proofs can be applied to show that the arguing as in the proof of Theorem 4 still holds.

Proof of Corollary 7.2. Let \( \gamma_0 \) be an isolated Lyapunov stable periodic orbit. There is \( \varepsilon > 0 \) and \( \delta > 0 \) such that \( \varphi^t(v) \in \mathcal{O}_\varepsilon(\gamma_0) \) for all \( t \geq 0 \) provided that \( v \in \mathcal{O}_\delta(\gamma_0) \) and the closure of \( \mathcal{O}_\delta(\gamma_0) \) does not contain other periodic orbits and stationary points. Therefore, any \( v \in \mathcal{O}_\delta(\gamma_0) \) has a bounded (and, consequently, compact) positive semi-orbit and its \( \omega \)-limit set in virtue of Theorem 4 must coincide with \( \gamma_0 \). This shows the asymptotic orbital stability of \( \gamma_0 \).

Let \( v_0 \) be a stationary point having a \( k \)-dimensional local unstable set \( D \). Under our assumptions it is clear that we always have \( D \subset A \). Therefore, any local unstable set has dimension \( k \leq 2 \).

Proof of Corollary 7.3. If \( v_0 \) is Lyapunov stable the statement is obvious. Let us consider the case where there is a 2-dimensional unstable set \( D \) for \( v_0 \). By \((U_1)\) the set \( D \) is an open neighborhood of \( v_0 \) in \( A \). To prove the statement it is sufficient to show that \( v_0 \) is the only stationary point in \( \omega(u_0) \). Indeed, if \( v_0 \) is the only stationary point and there is a non-stationary point \( \bar{v} \in \omega(u_0) \) then by Theorem 4 it must be homoclinic to \( v_0 \) that contradicts to \((U_3)\).

Now suppose there is another stationary point \( w_0 \in \omega(u_0) \). Let \( r > 0 \) be a number and consider the closed ball of radius \( r \) in \( A \) centered at \( v_0 \), which we denote by \( C \), and the open ball of radius \( r/2 \) centered at \( v_0 \) in \( A \), which we denote by \( B \). We assume that \( r \) is chosen
such that $\mathcal{C}$ is contained in $\mathcal{D}$ and therefore do not intersect with $w_0$. From (U2), (U3) and the compactness of $\mathcal{C}$ we can find a number $T > 0$ such that if $\bar{v} \in \mathcal{C}$ then $\varphi'(\bar{v}) \in \mathcal{B}$ for all $t \leq -T$. From this it follows that

$$\text{if } \bar{v} \in \partial \mathcal{B} \text{ then } \varphi'(\bar{v}) \in \partial \mathcal{C} \text{ for some } \bar{t} \in (0, T).$$

Let $d > 0$ be such that $\mathcal{O}_d(\partial \mathcal{C}) \cap \mathcal{B} = \emptyset$. Consider a compact set $\mathcal{K}$ containing $\varphi'(u_0), t \geq 0$, and $\Phi(\Pi \varphi'(u_0)), t \geq 0$. Then there exists $\delta > 0$ such that $|\varphi'(v_1) - \varphi'(v_2)| < d$ for all $t \in [0, T]$ provided that $|v_1 - v_2| < \delta$ and $v_1, v_2 \in \mathcal{K}$. Using (A3) consider $t \delta > 0$ such that $|\varphi'(u_0) - \Phi(\Pi \varphi'(u_0))| < \delta$ for all $t \geq t \delta$. From this and (3.4) it follows that any time we have $\Phi(\Pi \varphi'(u_0)) \in \partial \mathcal{B}$ for some $t_0 \geq t \delta$ there is $t \in (0, T)$ such that $|\varphi'(u_0) - v_0| \geq r/2$. Since $w_0 \in \omega(u_0)$ then there must be a time $t' \geq t \delta$ when $\Phi(\Pi \varphi'(u_0)) \notin \mathcal{C}$. Thus, for $t \geq t'$ the trajectory $\varphi'(u_0)$ cannot remain close to $v_0$ for the time intervals larger than $T$. This contradicts the fact that $v_0 \in \omega(u_0)$ and $v_0$ is stationary. So, $v_0$ is the only stationary point in $\omega(u_0)$ and with the above remark the lemma is proved.

\[\square\]

4. Orbital stability

In this part we suppose that in addition to the hypotheses of Section 3 we have also one of the compactness assumptions (COM1) or (COM2) satisfied. In particular, this implies that any bounded semi-trajectory is compact (see 2).

Recall that a periodic orbit $\gamma_0$ is called orbitally stable if for for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\varphi'(v) \in \mathcal{O}_\varepsilon(\gamma_0)$ for all $t \geq 0$ provided that $v \in \mathcal{O}_\delta(\gamma_0)$. In our context to study orbital stability of periodic orbits it is convenient to introduce the following definition.

A periodic orbit $\gamma_0$ is called amenable stable if it is orbitally stable as a periodic orbit of the flow $\varphi$ restricted to $\mathfrak{A}$, i. e. for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\varphi'(v) \in \mathcal{O}_\varepsilon(\gamma_0)$ for all $t \geq 0$ provided that $v \in \mathcal{O}_\delta(\gamma_0) \cap \mathfrak{A}$.

The following lemma is a generalization of Theorem 3 from [27].

Lemma 7. Suppose a periodic orbit $\gamma_0$ is amenable stable; then it is orbitally stable.

Proof. We will obtain a contradiction by assuming that the amenable stable periodic orbit $\gamma_0$ is not orbitally stable. Let $\delta_k > 0$, $k = 1, 2, \ldots,$ be a sequence tending to zero. For all sufficiently small $\varepsilon > 0$ there exists a point $w_\varepsilon \in \mathcal{O}_{\delta_k}(\gamma_0)$ such that $w_\varepsilon(t) := \varphi'(w_\varepsilon(t)) \in \mathcal{O}_\varepsilon(\gamma_0)$ for all $t \in [0, t_k]$ and $\text{dist}(w_\varepsilon(t_k), \gamma_0) = \varepsilon$. Since $\delta_k \to 0$, we must have $t_k \to +\infty$ as $k \to \infty$. Put $w_\varepsilon(t) := w_\varepsilon(t + t_k)$ for $t \geq -t_k$. Using the boundedness of $w_\varepsilon(t)$ for $t \in (-\infty, 0]$ and (COM1) or (COM2) we can obtain a subsequence (we keep the same index $k$), which converges to some amenable trajectory $v_\varepsilon(\cdot)$ as $k \to \infty$ (see [4]). For $v_\varepsilon(\cdot)$ we have the properties

\begin{align*}
(\ast) \quad & \text{dist}(v_\varepsilon(t), \gamma_0) \leq \varepsilon \text{ for all } t \in (-\infty, 0) ; \\
(\ast\ast) \quad & \text{dist}(v_\varepsilon(0), \gamma_0) = \varepsilon .
\end{align*}

If $\varepsilon$ is sufficiently small then the closure of $\mathcal{O}_\varepsilon(\gamma_0)$ does not contain stationary points. From this, $\ast$ and Corollary 1.1 it follows that $\alpha(v_\varepsilon(\cdot))$ must be a periodic trajectory $\gamma_\varepsilon$. Moreover, since $\gamma_0$ is amenable stable and $\ast\ast$ holds, we must have $\gamma_\varepsilon \neq \gamma_0$. Thus, $\gamma_0$ is a non-isolated periodic orbit. From this it follows that for some $\varepsilon_2 > \varepsilon_1 > 0$ and corresponding orbits $\gamma_1 := \gamma_{\varepsilon_1}$ and $\gamma_2 := \gamma_{\varepsilon_2}$ we have the property that $\gamma_1$ separates $\gamma_0$ and $\gamma_2$ on $\mathfrak{A}$. Let $\delta > 0$ be given by Lemma 5 applied to $\gamma_0$ and $\gamma_1$, i. e. if $v \in \mathcal{O}_\delta(\gamma_0)$ then $\Pi \varphi'(v) \notin \Pi \gamma_1$ for all $t \geq 0$. From $\ast\ast$ it is clear that $\gamma_1$ separates $v_\varepsilon(0)$ and $\gamma_0$ and, consequently, $\gamma_1$ separates $v_{\varepsilon_2}(0)$ and $\gamma_0$ for all sufficiently large $k$. Since $v_{\varepsilon_2}(-t_k) \in \mathcal{O}_{\delta_k}(\gamma_0)$, $\gamma_1$ separates $v_{\varepsilon_2}(0)$ and
For a periodic orbit $\gamma$ in $\mathfrak{A}$ by $\mathcal{G}_\gamma$ we denote its interior (i.e. the bounded component of $\mathfrak{A} \setminus \gamma$), which is well-defined by the Jordan curve theorem. Let $\gamma_1$ and $\gamma_2$ be two periodic orbits in $\mathfrak{A}$ we write $\gamma_1 \preceq \gamma_2$ iff $\mathcal{G}_{\gamma_1} \subset \mathcal{G}_{\gamma_2}$. Clearly, the relation $\preceq$ defines a partial order on the set of periodic orbits.

To describe the amenable stability the following concepts is useful. A periodic orbit $\gamma$ is called externally (resp. internally) stable if either $\gamma$ is a limit of periodic orbits $\gamma_k \neq \gamma$, $k = 1, 2, \ldots$, with $\gamma \leq \gamma_k$ (resp. $\gamma_k \leq \gamma$) or there is a point $v_0 \in \mathfrak{A} \setminus \text{Cl}\mathcal{G}_\gamma$ (resp. $v_0 \in \mathcal{G}_\gamma$) with $\omega(v_0) = \gamma$. The following lemma is obvious.

**Lemma 8.** A periodic orbit $\gamma$ which is both externally and internally stable is amenable stable.

Suppose $\mathcal{A}$ is an attractor w. r. t. (EMB) satisfying the conditions of Theorem 2. In virtue of Lemma 2 without loss of generality we may assume that $\mathcal{A}$ is a compact subset of $\mathfrak{A}$. Moreover, any its neighborhood $\mathcal{U}_\mathcal{A}$ has the property that $\mathcal{U}_\mathcal{A} \cap \mathfrak{A}$ is an open subset of $\mathfrak{A}$.

**Remark 4.** Until the end of this section we will work only in the topology of $\mathfrak{A}$.

Let $\text{Per}(\mathcal{A})$ denote the set of all periodic points in $\mathcal{A}$. At above we defined a partial order $\preceq$ on the set $\text{Per}(\mathcal{A})$.

**Lemma 9.** The set $\text{Per}(\mathcal{A})$ is non-empty and closed.

**Proof.** Since unstable terminal points are separated from each other and cannot lie in $\omega$-limit sets of other amenable trajectories, by Lemma 2 there exist non-stationary points in $\mathcal{U}_\mathcal{A} \cap \mathfrak{A}$, which by the properties of $\mathcal{U}_\mathcal{A}$ must lie in $\mathcal{A}$ and, consequently, due to Theorem 1 must be attracted by some periodic trajectories. Thus, the set $\text{Per}(\mathcal{A})$ is not empty.

Now suppose $v_0 \in \mathcal{A}$ is a limit of a sequence $v_k \in \text{Per}(\mathcal{A})$, $k = 1, 2, \ldots$, of periodic points. As above, from Theorem 1 for some periodic trajectory $\gamma_0$ we must have that $\omega(v_0) = \gamma_0$ and $\gamma_0 \subset \text{Per}(\mathcal{A})$. Let $\bar{v} \in \gamma_0$ be any point and for some sufficiently small $\varepsilon_0 > 0$ let $\mathcal{T} \subset \mathfrak{A}$ be an $\varepsilon_0$-transversal of $\varphi$ restricted to $\mathfrak{A}$ at $\bar{v}$. For every $t \in \mathbb{R}$ such that $\varphi^t(v_0) \in \mathcal{U}(\varepsilon_0, \mathcal{T})$ of $\bar{v}$ we must have an intersection with $\mathcal{T}$ for some $t_0 \in (t - \varepsilon_0, t + \varepsilon_0)$. Therefore, there are infinitely many and arbitrary large times $t_0$ such that $\varphi^{t_0}(v_0)$ belongs to $\mathcal{T}$. Let $\sigma > 0$ be a period of $\gamma_0$ and consider $\varepsilon > 0$ such that $\mathcal{O}_\varepsilon(\bar{v}) \cap \mathfrak{A} \subset \mathcal{U}$. Let $\delta > 0$ be such that $|\varphi^t(v) - \varphi^t(\bar{v})| < \varepsilon$ for $t \in [0, \sigma]$ provided that $v \in \mathcal{O}_\delta(\bar{v}) \cap \mathfrak{A}$. Suppose that for some times $0 < t_1 < t_2$ with $\varphi^{t_1}(v_0) \in \mathcal{O}_\delta(\bar{v})$ and $\varphi^{t_2}(v_0) \in \mathcal{O}_\delta(\bar{v})$ we have two distinct intersections with $\mathcal{T}$. Since $\varphi^{t_2}(v_0) \in \text{Cl}\text{Per}(\mathcal{A})$, a periodic orbit $\tilde{\gamma}$ can be chosen such that $\gamma_0 \preceq \tilde{\gamma}$ and either $\tilde{\gamma}$ separates $\varphi^{t_1}(v_0)$ and $\varphi^{t_2}(v_0)$ or $\tilde{\gamma}$ separates $\varphi^{t_2}$ (or $\varphi^{t_1}$) and $\gamma_0$. In both cases we derive a contradiction. Therefore, all intersections of $\mathcal{T}$ are at the same point and consequently, $v_0$ is a periodic point. □

Let $\Gamma(\mathcal{A})$ denote the set of all periodic orbits in $\mathcal{A}$.

**Lemma 10.** Every chain $\mathcal{C}$ in $\Gamma$ has an upper bound $\gamma^+$ and a lower bound $\gamma^-$.

**Proof.** To construct an upper bound we consider the set $\mathcal{G}^+ := \bigcup_{\gamma \in \mathcal{G}} \mathcal{G}_\gamma$. Since each of $\mathcal{G}_\gamma$’s is invariant and uniformly bounded, the set $\mathcal{G}^+$ is invariant and bounded. As a consequence, its boundary $\partial \mathcal{G}^+$ is a non-empty invariant compact set of $\mathfrak{A}$. It is easy to see that every point in $\partial \mathcal{G}^+$ is a limit of periodic points and, by Lemma 2, $\partial \mathcal{G}^+$ consists of periodic points.
Let $\gamma^+ \subset \partial G^+$ be a periodic orbit. By the previous argument, $\gamma^+$ is a limit of some sequence of $\gamma_k \in C$, $k = 1, 2, \ldots$. From this it follows that $G^+$ contains a point in the interior of $\gamma^+$ and since $G^+$ is connected it must lie in this interior part. Therefore $\gamma \leq \gamma^+$ for every $\gamma \in C$ that is required.

Now we consider the set $G_\gamma = \bigcap_{0 \leq k \leq n} G_{\gamma_k}$. Clearly, $G_\gamma$ is a compact invariant set, which consists of periodic points. Let $\gamma^- \subset \partial G_\gamma$ be a periodic orbit. By a similar as above argument, $\gamma^- \leq \gamma$ for every $\gamma \in C$ and, consequently, $\gamma^-$ is a lower bound. 

\textbf{Proof of Theorem 2.} By Lemma 10 and Zorn’s lemma the set $\Gamma(\mathcal{A})$ contains a maximal element $\gamma_{\text{max}}$, i. e. there is no $\gamma \in \Gamma(\mathcal{A})$, $\gamma \neq \gamma_{\text{max}}$, such that $\gamma_{\text{max}} \leq \gamma$. Let us show that there exists at least one externally stable periodic orbit. If $\gamma_{\text{max}}$ is not externally stable then by Lemma 2 there is a point $v_0 \in (\mathcal{A} \setminus \text{Cl} G_{\gamma_{\text{max}}}) \cap \mathcal{U}_A$, such that $\alpha(v_0) = \gamma_{\text{max}}$ and $\omega(v_0) \neq \gamma^+$. Since $\omega(v_0) \subset A$ and the stationary points in $A$ are unstable terminal, the set $\omega(v_0)$ does not contain any stationary point and, consequently, by Theorem 7 it is a periodic orbit $\gamma \neq \gamma_{\text{max}}$. Since $\alpha(v_0) = \gamma_{\text{max}}$ the point $v_0$ lies in the exterior of $\gamma$, i. e. in $\mathcal{A} \setminus \text{Cl} G_{\gamma}$. Therefore, $\gamma$ is externally stable.

Let $\Gamma^{\text{ext}}(\mathcal{A})$ denote the set of all externally stable periodic orbits in $\mathcal{A}$. By the above considerations, the set $\Gamma^{\text{ext}}(\mathcal{A})$ is not empty. Let $C^{\text{ext}}$ be a chain in $\Gamma^{\text{ext}}(\mathcal{A})$. By Lemma 10 there exists a minimal element $\gamma_{\text{min}} \in \Gamma(\mathcal{A})$. Moreover, by the construction of $\gamma_{\text{min}}$ in the proof we have that either $\gamma_{\text{min}} \in C^{\text{ext}}$ or $\gamma_{\text{min}}$ is a limit of a sequence of periodic orbits $\gamma_k \in C^{\text{ext}}$ with $\gamma_{\text{min}} \leq \gamma_k$ and $\gamma_k \neq \gamma_{\text{min}}$, $k = 1, 2, \ldots$. In any of these cases we have $\gamma_{\text{min}} \in \Gamma^{\text{ext}}(\mathcal{A})$. By Zorn’s lemma, there exists a minimal element $\gamma_{\text{min}}^{\text{ext}} \in \Gamma^{\text{ext}}(\mathcal{A})$, i. e. there is no $\gamma^{\text{ext}} \in \Gamma^{\text{ext}}(\mathcal{A})$ such that $\gamma^{\text{ext}} \leq \gamma_{\text{min}}^{\text{ext}}$ and $\gamma^{\text{ext}} \neq \gamma_{\text{min}}^{\text{ext}}$.

Let us show that $\gamma_{\text{min}}^{\text{ext}}$ is internally stable. If $\gamma_{\text{min}}^{\text{ext}}$ is not a limit of a sequence of periodic orbits $\gamma_k$ with $\gamma_k \leq \gamma_{\text{min}}^{\text{ext}}$ and $\gamma_k \neq \gamma_{\text{min}}^{\text{ext}}$, $k = 1, 2, \ldots$, then by Lemma 2 there is $\delta > 0$ such that $\mathcal{U}^{\text{int}} := \mathcal{O}_{\delta}(\gamma_{\text{min}}^{\text{ext}}) \cap G_{\gamma_{\text{min}}^{\text{ext}}}^{\text{int}}$ lies in $\mathcal{U}_A$ and does not contain any periodic or stationary points. Then any point $v_0 \in \mathcal{U}^{\text{int}}$ must satisfy $\omega(v_0) = \gamma_{\text{ext}}^{\text{int}}$. Indeed, since all stationary points in $\mathcal{A}$ are unstable terminal, $\omega(v_0)$ must be a periodic orbit $\gamma$ with $\gamma \subset \text{Cl} G_{\gamma_{\text{min}}^{\text{ext}}}$. If $\gamma \neq \gamma_{\text{min}}^{\text{ext}}$ then $v_0 \in \mathcal{A} \setminus \text{Cl} G_{\gamma}$ and, consequently, $\gamma$ is an externally stable orbit with $\gamma \leq \gamma_{\text{min}}^{\text{ext}}$, that contradicts to the minimality of $\gamma_{\text{min}}^{\text{ext}}$. So, the externally stable orbit $\gamma_{\text{min}}^{\text{ext}}$ is also internally stable and, by Lemmas 8 and 7 it is orbitally stable. The proof is finished. 

5. DELAYED GOODWIN EQUATIONS

In this section we consider applications of the obtained results to the following system of delay-differential equations

$$
\dot{x}_1(t) = g \left( \int_{-\tau}^{0} g(s)x_n(t - s) ds \right) - \lambda x_1(t),
$$

$$
\dot{x}_2(t) = x_1(t) - \lambda x_2(t),
$$

$$
\cdots
$$

$$
\dot{x}_n(t) = x_{n-1}(t) - \lambda x_n(t).
$$

(5.1)

Here $\tau, \lambda > 0$ are positive constants and $g: \mathbb{R} \to \mathbb{R}$ is a continuous scalar-valued function, which is continuously differentiable in $(0, +\infty)$ and satisfies $0 > g'(u) \geq -\kappa_0$ for some
\( \kappa_0 > 0 \) and \( v \in (0, +\infty) \). Moreover, \( g(v) \to 0 \) as \( v \to +\infty \). The function \( \rho \) is a non-negative continuous\(^3\) which is not identically zero.

In \cite{27} such systems (for \( n = 3 \) and \( n = 4 \)) were studied by R. A. Smith. In what follows we will show that his results (based on a huge number of a priori estimates) follows from our abstract approach (but at the current time only for the bounded in \( L_2 \) measurements, which approximates \( \delta_{-\tau} \)) and, moreover, we will obtain more sharper regions in the space of parameters \( (\mu, \tau, \kappa_0) \) of system (5.1) for which the presented theory can be applied. For simplicity, we consider only the case \( n = 3 \).

System (5.1) can be used as a model for certain biochemical reactions concerned with the protein synthesis. In this case the quantities \( x_j \) represent concentrations of certain chemicals and therefore satisfy \( x_j \geq 0 \). Since \( g > 0 \) and \( g(v) \geq 0 \) for \( v \geq 0 \), from (5.1) it is clear that the cone of non-negative functions \( C^+([\tau, 0]; \mathbb{R}^n) \) is invariant w. r. t. the solutions of (5.1). By the Lipschitz property of \( g \) in \( C^+ \) system (5.1) generates a flow in \( C^+ \). In order to apply our results to study system (5.1) we redefine the nonlinearity \( g \) outside of \( C^+ \). Namely, let \( \hat{g}: \mathbb{R} \to \mathbb{R}^3 \) be defined as \( \hat{g}(v) := g(v) \) for \( v > 0 \) and \( \hat{g}(v) := g(0) \) for \( v \leq 0 \). Clearly, system (5.1), where \( g \) is replaced by \( \hat{g} \), does not change in the cone \( C^+ \).

So, we will apply our theory to study the system

\[
\begin{align*}
\dot{x}_1(t) &= \hat{g} \left( \int_{-\tau}^{0} g(s)x_3(t-s)ds \right) - \lambda x_1(t), \\
\dot{x}_2(t) &= x_1(t) - \lambda x_2(t), \\
\dot{x}_3(t) &= x_2(t) - \lambda x_3(t),
\end{align*}
\]  
(5.2)

where \( \hat{g} \) satisfy for all \( v_1, v_2 \in \mathbb{R} \), \( v_1 \neq v_2 \) the inequality

\[
-\kappa_0 \leq \frac{\hat{g}(v_2) - \hat{g}(v_1)}{v_2 - v_1} \leq 0
\]  
(5.3)

and \( \hat{g}'(v) < 0 \) for \( v > 0 \).

At first, we should write (5.2) as an evolution equation in a proper Hilbert space \( \mathbb{H} \). In our case it is the space \( \mathbb{H} = \mathbb{R} \times L_2(-\tau, 0; \mathbb{R}^3) \). Consider the unbounded linear operator \( A: \mathcal{D}(A) \to \mathbb{H} \) defined as

\[
(x, \psi) \mapsto \left( A_0 x, \frac{\partial}{\partial s} \psi \right),
\]  
(5.4)

where \( (x, \psi) \in \mathcal{D}(A) := \{(x, \psi) \in \mathbb{H} \mid \psi(0) = x \text{ and } \psi \in W^{1,2}(-\tau, 0; \mathbb{R}^3)\} \) and the matrix \( A_0 \) is the matrix of system (5.2) for \( n = 3 \), i. e.

\[
A_0 := \begin{bmatrix}
-\lambda & 0 & 0 \\
1 & -\lambda & 0 \\
0 & 1 & -\lambda
\end{bmatrix}.
\]  
(5.5)

Next, we put \( \Xi := \mathbb{R} \) and consider the operator \( B \in \mathcal{L}(\Xi, \mathbb{H}) \) defined as \( B\xi := ((\xi, 0, 0), 0) \).

The operator \( C \in \mathcal{L}(\mathbb{H}, \Xi) \) is defined as \( C(x, \varphi) := \int_{-\tau}^{0} g(s)\pi_3(\varphi(s))ds \). Here \( \pi_3: \mathbb{R}^3 \to \mathbb{R} \) is defined as \( \pi_3(x_1, x_2, x_3) := x_3 \). Now system (5.2) can be considered as the following evolution equation in \( \mathbb{H} \):

\[
\dot{u} = Au + B\hat{g}(Cu)
\]  
(5.6)

\(^3\)It is more natural to consider \( \rho = \delta_{-\tau} \), where \( \delta_{-\tau} \) is the delta function at \( -\tau \). However, we choose \( \rho \) as a continuous function to make the applications of the frequency theorem from [20] possible. For this it is required that the form in (5.8) must be continuous on the entire space \( \mathbb{H} \). The author believes that there must exist versions of the frequency theorem that allows considerations of the measurements like \( \rho = \delta_{-\tau} \).
From the results of [29] and the Lipschitz property of $\hat{g}$ it follows that (5.6) defines a semi-flow $\varphi$ in $\mathbb{H}$.

Let $C^+$ denote the cone of non-negative functions in $\mathbb{H}$, i.e., the set of all $(x, \psi) \in \mathbb{H}$ such that $x_j \geq 0$ and $\psi_j(s) \geq 0$ for almost all $s \in [-\tau, 0]$ and $j = 1, 2, 3$. Clearly, $C^+$ is a closed subset of $\mathbb{H}$.

**Lemma 11.** The set $C^+$ is positively invariant w.r.t. the flow $\varphi$, i.e., $\varphi^t(C^+) \subset C^+$ for all $t \geq 0$.

**Proof.** From (5.2) and since $\hat{g}(\xi) \geq 0$ for $\xi \geq 0$, the invariance is clear for the classical solutions, i.e., with initial conditions $u_0 \in E$. Since $C^+$ is closed this invariance can be extended for $u_0 \in \mathbb{H}$. \hfill $\square$

Now we put $E := C([-\tau, 0]; \mathbb{R}^3)$. It is clear that $E$ is densely and continuously embedded into $\mathbb{H}$ by the map $\psi \mapsto (\psi(0), \psi)$. The flow $\varphi$ coincides in $\mathbb{H}$ with the flow generated by the classical solutions of (5.1) and $\varphi^\tau(\mathbb{H}) \subset E$ (see [29]). From Proposition 5.6 in [29], Section 3.6 in [13] and the Lipschitz property of $\hat{g}$ it follows that (EMB) and (COM2) are satisfied with $t_{\text{emb}} = t_{\text{com}} = 2\tau$ and $\varphi^{t_{\text{emb}}}(\mathbb{H}) \subset E$.

Below we shall consider different forms of (5.6) as a control system in the Lur’e form. Namely, for $\rho > 0$ we consider

$$\dot{u} = (A - \rho BC)u + B(g(Cu) + \rho Cu) = A_{\rho} + B(g_{\rho}(Cu)).$$  \hfill (5.7)

With the nonlinearity $g_\rho$ we associate the quadratic form in $\mathbb{H}$:

$$F_{\rho}(u, \xi) = F_{\rho}(x, \psi) = (\xi - (\kappa_0 - \rho)(Cu)(\rho Cu - \xi))$$  \hfill (5.8)

and its Hermitian extension $F_{\rho}^C$ defined for $u \in \mathbb{H}^C, \xi \in \Xi^C$ as

$$F_{\rho}^C(u, \xi) = \Re [(\xi + (\kappa_0 - \rho)(Cu))^*(\rho Cu - \xi)].$$  \hfill (5.9)

From (5.3) it is clear that $F_{\rho}$ for $0 \leq \rho \leq \kappa_0$ satisfies the important inequality

(DQ1) $F_{\rho}(u_1 - u_2, \xi_1 - \xi_2) \geq 0$ for all $u_1, u_2 \in \mathbb{H}$ and $\xi_1 = g_{\rho}(Cu_1), \xi_2 = g_{\rho}(Cu_2)$.

Let $W_{\rho}(p)$ denote the transfer function of the triple $(A_{\rho}, B, C)$, i.e. $W_{\rho}(p) := C^*(A_{\rho}^C - pI^C)^{-1}B^C$, defined for $p \notin \sigma(A_{\rho})$. Since in our case $W_{\rho}(p): \mathbb{C} \to \mathbb{C}$ is a linear operator, it can be identified with a complex number. Straightforward calculations show that

$$W_{\rho}(p) = -\frac{\int_{-\tau}^{0} \theta(s)e^{ps}ds}{(\lambda + p)^3 + \rho \int_{-\tau}^{0} \theta(s)e^{ps}ds}. \hfill (5.10)$$

Note that the denominator in (5.10) vanishes if $p \in \sigma(A_{\rho})$. If we put $\theta \equiv \delta_{-\tau}$ then (5.10) coincides with the function

$$W(p) := -\frac{1}{(\lambda + p)^2e^{p\tau} + \rho}, \hfill (5.11)$$

which was used in [27]. We choose $\varrho_n \equiv n \cdot \chi_{[-\tau, -\tau + 1/n]}$, where $\chi_S$ denotes the characteristic function of $S$.

**Theorem 5.** Suppose there exists $\rho \in (0, \kappa_0)$ such that

(DF1) $\rho^3 e^{\lambda \tau} < 84.2$.

(DF2) $\Re [(1 + (\kappa_0 - \rho)W(i\omega - \lambda))^*(1 + \rho W(i\omega - \lambda))] > 0$ for all $\omega \in \mathbb{R}$.

Then for all sufficiently large $n$ there exists a bounded self-adjoint operator $P_n: \mathbb{H} \to \mathbb{H}$ such that (H1), (H2) with $j = 2$ and (H3) are satisfied for the semi-flow $\varphi$ generated by (5.6) with $\varphi := \varphi_n = n \cdot \chi_{[-\tau, -\tau + 1/n]}$. In particular, $\varphi$ satisfies the hypotheses of Theorem 4.
Proof. Let us choose \( \varepsilon \in [0,1] \) and consider the family of equations

\[
(\lambda + p)^3 + \rho \int_0^1 \varrho_n(s)e^{\varepsilon ts}ds = 0.
\]

As in \([27]\) from (DF1) we get that (5.12) has no roots with \( \text{Re} \, p = -\lambda \) provided \( n \) is sufficiently large. Moreover, it is clear that for every \( \varepsilon > 0 \) equation (5.12) has only finite number of roots in the half-plane \( \text{Re} \, p > -\lambda \) and these roots lie in a bounded set. From this it follows that the roots of (5.12) depend continuously (to be more precise, upper semi-continuously) on \( \varepsilon \) and, consequently, (5.12) with \( \varepsilon = 1 \) has the same number of roots in the half-plane \( \text{Re} \, p > -\lambda \) as (5.12) with \( \varepsilon = 0 \), i. e. \( (\lambda + p)^3 + \rho = 0 \). Clearly, the latter equation has exactly two simple roots with \( \text{Re} \, p > -\lambda \).

So, the operator \( A_\rho + \lambda I \) has exactly two simple eigenvalues with \( \text{Re} \, p > 0 \) and the remained part of the spectrum is separated from the imaginary axis. This also holds for the operator \( A_\rho + \nu I \), where \( \nu = \lambda - \delta \) with a sufficiently small \( \delta > 0 \). Let us check that the pair \( (A_\rho + \nu I, B) \) is exponentially stabilizable. Consider \( C_\rho : \mathbb{H} \rightarrow \Xi \) defined as \( C(x, \varphi) = \rho C(x, \phi) \). Straightforward calculations show that the operator \( A_\rho + B C_\rho \) has all eigenvalues with negative real part and thus the stabilizability property is satisfied. From (DF2) it follows that the inequality

\[
\text{Re} [(1 + (\kappa_0 - \rho)W_\rho(\omega - \nu))^*(1 + \rho W_\rho(\omega - \nu))] > 0 \quad \text{for all } \omega \in \mathbb{R}
\]

is satisfied provided that \( n \) is sufficiently large and \( \delta > 0 \) is sufficiently small. \( ^4 \)

So, if \( n \) and \( \delta > 0 \) are chosen properly, Theorem 3 from \([20]\) gives us the existence of a bounded self-adjoint operator \( P_\rho : \mathbb{H} \rightarrow \mathbb{H} \) such that

\[
((A_\rho + \nu I)u + B\xi, P_\nu u) + F_\rho(u, \xi) \leq -\delta_V |u|^2 + |\xi|^2, \quad \text{for all } u \in \mathcal{D}(A_\rho), \xi \in \Xi.
\]

Putting \( \xi = 0 \) in (5.14) and using \( F_\rho(u, 0) \geq 0 \) we get

\[
((A_\rho + \nu I)u, P_\nu u) \leq -\delta_V |u|^2.
\]

From (5.15) and the spectral properties of \( A_\rho + \nu I \) established above we get (see Appendix B in \([1]\)) that the operator \( P_\rho \) satisfies (H1) and (H2) with \( j = 2 \).

Now in (5.14) we put \( u = u_1 - u_2, \xi = g_\rho(u_1) - g_\rho(u_2) \) and use (DQ1) to get

\[
((A + \nu I)(u_1 - u_2) + B(\hat{g}(u_1) - \hat{g}(u_2)), P(u_1 - u_2)) \leq -\delta_V |u_1 - u_2|^2.
\]

Consider \( V_n(u) := (Pu, u) \). Since for \( u \in \mathcal{D}(A) \) we have strong solutions\( ^5 \) to (5.6), we get

\[
\frac{d}{dt} [\varphi(u_1 - u_2)] \leq -\delta_V \varepsilon^{2\nu t} |\varphi(u_1) - \varphi(u_2)|^2,
\]

for any \( u_1, u_2 \in \mathcal{D}(A) \) and almost all \( t \in [l, r] \) with given \( 0 \leq l < r \). Integrating (5.17) on every interval \([l, r]\) and extending the obtained inequality for \( u_1, u_2 \in \mathbb{H} \) by continuity, we get (H3). Thus, to the flow \( \varphi \) we can apply Theorem \([1]\). \( \square \)

\( ^4 \)From (5.10) with \( \varrho = \varrho_0 \), it is clear that \( W_\rho(p) \rightarrow 0 \) as \( p \rightarrow \infty \) uniformly in sufficiently large \( n \). Therefore, the inequality in (5.13) is uniformly (in \( n \)) satisfied for \( \omega \) outside of a proper compact interval \( \mathcal{I} \). Since the functions \( W_\rho(p) \) converges to \( W(p) \) uniformly in \( p \) from compact sets, from (DF2) we get (5.13) satisfied for all \( \omega \in \mathcal{I} \) provided \( n \) is sufficiently large and \( \delta \) is sufficiently small.

\( ^5 \)A strong solution to (5.6) is a continuous function \( u : (c, d) \rightarrow \mathbb{H} \) that is absolutely continuous on every compact sub-interval \([a, b] \subset (c, d)\) and satisfies (5.6) for almost all \( t \in [a, b] \). See \([29]\).
Remark 5. In [27] instead of conditions (DF1) and (DF2) it was used the condition
\[ k_0 \tau^3 e^{-\lambda \tau} < 84.2 \]  
and \( \rho \) is chosen to satisfy \( \frac{1}{2} \kappa_0 < \rho < \frac{1}{4} 84.2 \tau^{-3} e^{-\lambda \tau} \) (and (DF1) in particular). It is also shown that for these \( \rho \)'s we have the inequality
\[ |W(-\lambda + i\omega)| < \rho^{-1} \]  
for all \( \omega \in \mathbb{R} \).

Clearly, the stability of \( u \) is determined by the roots of the equation
\[ (\lambda + p)^3 - g'(\eta_0) \int_{-\tau}^{0} g_n(s)e^{\rho s}ds = 0. \]  
Let \( \mu > 0 \) and consider
\[ (\lambda + p)^3 + \mu \int_{-\tau}^{0} g_n(s)e^{\rho s}ds = 0. \]  
In particular, the number of roots with \( \text{Re} \ p > 0 \) do not exceed 2. Since the roots of (5.22) in the half-plane \( \text{Re} \ p > -\lambda \) are uniformly bounded in sufficiently large \( n \) and \( \kappa \) from compact intervals, they converge to the roots of
\[ (\lambda + p)^3 + \mu e^{-\tau p}ds = 0 \]  
as \( n \to \infty \). The above argumentation used for (5.12) shows that (5.24) has exactly two roots with \( \text{Re} \ p > -\lambda \) provided that \( \mu \tau^3 e^{-\lambda \tau} < (\pi/2)^4 \). In [27] it is shown that (5.24) has a purely imaginary root of the form \( p = i \lambda \tan(\theta) \) if \( \mu = \mu_k = (-1)^{k+1}(\lambda \sec(\theta_k))^{-3} \) holds for some \( k = 0, 1, 2, \ldots \), where \( \theta_k \) satisfies \( \tau \lambda \tan(\theta_k) = k\pi - 3\theta_k \) and \( 0 < \theta_k < \pi/2 \). For all such \( \mu_k \) with \( k \neq 0 \) there are exactly two of such roots corresponding to \( \theta_k \) and \( -\theta_k \). Since we are interested only in \( \mu > 0 \) we consider the sequence \( 0 < \mu_1 < \mu_3 < \mu_5 < \ldots \). For \( \mu \in [0, \mu_1) \) equation (5.24) has no roots with \( \text{Re} \ p > 0 \) since this is so for \( \mu = 0 \). Analogously,
if \( \mu \in (\mu_1, \mu_3) \) there are exactly two roots with \( \text{Re} \, p > 0 \). For \( \mu \in (\mu_3, \mu_5) \) there are 4 roots with \( \text{Re} \, p > 0 \). However, within the conditions of the theorem this means that \( -g'(\eta_0) < \mu_3 \). 

Thus, as in [27] if \( -g'(\eta_0) \in [0, (\lambda \sec \theta_1)^3) \) the stationary point \( u_0 \) is Lyapunov stable, and if \( -g'(\eta_0) > (\lambda \sec \theta_1)^3 \) it is an unstable terminal point for \( \varphi \) (see Theorem 1.1, p. 230 in [13]). The first assertion of the theorem now follows from Corollary 1.3 since \( u_0 \) is the only stationary point \( \mathcal{C}^+ \) and it is terminal due to (5.21). The second statement follows from Theorem 6 applied to \( A \) and \( \mathcal{U}_A \) defined above since \( u_0 \) is an unstable terminal point. \( \square \)

Theorem 6 leads to analogous consequences for the dynamics of (5.2) in the cone \( \mathcal{C}^{+,0} \) and all the convergence and stability properties hold in the uniform norm due to Lemma 3.

6. Reaction-Diffusion Equations

Let \( \Omega \subset \mathbb{R}^d, d \leq 3, \) be a bounded domain having a \( C^2 \)-smooth boundary \( \partial \Omega \). We consider the following parabolic equation in \( \Omega \):

\[
\begin{aligned}
  u_t(t, x) &= A \Delta u(t, x) + f(u(t, x)), \quad t > 0, \ x \in \Omega \\
  \frac{\partial u}{\partial n}(t, x) + a(x)u(t, x) &= 0, \ x \in \partial \Omega, \ t > 0.
\end{aligned}
\]

Here \( A \) is a constant symmetric \( n \times n \)-matrix, \( \Delta \) is the Laplace operator, \( f \in C^2(\mathbb{R}^n) \) and \( a(x) \geq 0 \) with \( a \in C^2(\partial \Omega) \).

Let \{ \psi_k \}, \( k = 0, 1, \ldots, \) be the orthonormal system of functions in \( \mathbb{H} := L_2(\Omega; \mathbb{R}^d) \) corresponding to the eigenvalues \( \lambda_k \) of the boundary value problem in (6.2) for the scalar equation \( \Delta \psi + \lambda \psi = 0 \), which are arranged in increasing order \( \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \). For \( \phi \in L_2(\Omega; \mathbb{R}^d) \) denote by \( \phi_k \) its Fourier coefficient w. r. t. this system. Let \( \mathbb{H}_s, s \in (0, 1], \) be given as

\[
\mathbb{H}_s := \left\{ \phi \in \mathbb{H} \left| \sum_{k=1}^{\infty} (1 + \lambda_k)^{2s}|\phi_k|^2 < +\infty \right. \right\}. \tag{6.3}
\]

Clearly, \( \mathbb{H}_0 = \mathbb{H} \). It is well-known (see [9]) that the inclusion \( \mathbb{H}_{s_1} \subset \mathbb{H}_{s_2} \) is compact provided that \( s_1 > s_2 \). We suppose that \( s_0 \in (\frac{1}{2}, 1) \) is given and \( f : \mathbb{R}^n \to \mathbb{R}^n \) is globally Lipschitz.

**Lemma 12.** Under the above assumptions the problem (6.1), (6.2) generates a semi-flow in \( \mathbb{H} \), which satisfies (COM2) with any \( t_{\text{com}} > 0 \). Moreover, (EMB) satisfied with \( E := \mathbb{H}_{s_0} \), any \( t_{\text{emb}} > 0 \) and \( \varphi_{t_{\text{emb}}} (H) \subset E \).

**Proof.** As in [25] if it follows that for every \( u_0 \in \mathbb{H}_{s_0} \) the map \( \varphi^t(u_0) := u(t, \cdot) \in \mathbb{H} \), where \( u(t, \cdot) \) is a strong solution \( \square \) for \( t > 0 \), defines a semi-flow in \( \mathbb{H}_{s_0} \). Moreover, from Lemma 3.1 in [20] for any \( u_1, u_2 \in \mathbb{H}_{s_0} \) we have (\( \| \cdot \| \) denotes the norm in \( \mathbb{H} \))

\[
|\varphi^t(u_1) - \varphi^t(u_2)| \leq e^{\kappa t}|u_1 - u_2|, \tag{6.4}
\]

where \( \kappa > 0 \) is some constant. From (6.4) and the density of \( \mathbb{H}_{s_0} \) the flow \( \varphi \) can be extended to the entire \( \mathbb{H} \) by continuity. Moreover, from (6.4) and Lemma 3.3 in [20] we get that

---

1. By Lemma 3 and Remark 3 properties of terminal points can be deduced from the classical theory.
2. The strict inequality \( \lambda_0 < \lambda_1 \) is proved in [20].
3. The space \( \mathbb{H}_s \) becomes a Hilbert space when it is endowed with the usual scalar product \( \left( \phi, \eta \right)_s := \sum_{k=1}^{\infty} (1 + \lambda_k)^{2s} \phi_k \eta_k \) for \( \phi, \eta \in \mathbb{H}_s \).
4. That is, \( u, u_t, \nabla u, \Delta u \) are continuous in \((0, +\infty) \times \Omega\).
Since the inclusion $H \subset H_s^0$ for any $t > 0$ and $\varphi^t$ takes bounded sets from $H$ into bounded sets in $H_{s_0}$, the map $\varphi^t: H \to H$ is compact for all $t > 0$. □

6.1. An example. As a special case of (6.1) we consider the system [25, 26]

$$
\begin{align*}
\dot{y}(t, x) &= d_1 \triangle y(t, x) + c_{12} z(t, x) - c_{11} y(t, x), \quad x \in \Omega, \ t > 0, \\
\dot{z}(t, x) &= d_2 \triangle z(t, x) - c_{21} y(t, x) + g(z(t, x)), \quad x \in \Omega, \ t > 0,
\end{align*}
$$

(6.5)
satisfying the boundary condition (6.2). Here $g: \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function; $d_1, d_2, c_{11}, c_{12}, c_{21}$ are real parameters with $d_1, d_2 > 0$. We suppose that

$$
c_{12}c_{21} > \begin{cases}
c_{11} + \lambda_0 d_1 |\mu|, & \text{when } c_{11} + \lambda_0 d_1 \leq 0, \\
(c_{11} + \lambda_0 d_1) \max\{0, (\lambda_1 - \lambda_0) d_2 - \mu\}, & \text{when } c_{11} + \lambda_0 d_1 > 0,
\end{cases}
$$

(6.6)

where $\mu := c_{11} + \frac{1}{2}(\lambda_1 + \lambda_0)d_1$. In [26] it is shown that if $g(0) = 0$ and (6.6) is satisfied then $(y, z) = (0, 0)$ is the only stationary point (= time-invariant solution) of (6.5). In general, it is not easy to determine stability properties of stationary points in non-linear parabolic problems. However, this is possible for space-invariant stationary points such as $(0, 0)$ in our case.

We put $\mathbb{H} := L_2(\Omega) \times L_2(\Omega), \mathbb{X} := L_2(\Omega)$. For $\rho > 0$ we consider the operator $A_\rho: D(A) \to \mathbb{H}$ with $D(A_\rho) = W^{2,2}(\Omega) \times W^{2,2}(\Omega)$ defined as

$$
\begin{bmatrix} y \\ z \end{bmatrix} \mapsto \begin{bmatrix} d_1 \triangle y + c_{12} z - c_{11} y \\ d_2 \triangle z - c_{21} y - \rho z \end{bmatrix}.
$$

(6.7)

We define the linear bounded operators $C: \mathbb{H} \to \mathbb{X}$ by $C(y, z) := z$ and $B: \mathbb{X} \to \mathbb{H}$ as $B\xi := (0, \xi)$. System (6.5) can be written in the following form:

$$
\dot{u} = A_\rho u + B g_\rho(Cu),
$$

(6.8)

where $g_\rho(z) = g(z) + \rho z$.

Remark 6. It can be checked that the pair $(A_\rho + \nu I, B)$ is exponentially stabilizable for any $\rho, \nu \in \mathbb{R}$ if $d_1, d_2 > 0$. Consider the operator $C_{\rho, \nu}: \mathbb{H} \to \mathbb{X}$ defined as $(y, z) \mapsto \delta_1 y + \delta_2 z$. Straightforward calculations show that the numbers $\delta_1$ and $\delta_2$ can be chosen such that the spectrum of $A_\rho + \nu I + BC_{\rho, \nu}$ will lie to the left of the imaginary axis.

The transfer operator of the triple $(A_\rho, B, C)$, i.e. $W_\rho(p) := C^C(A_\rho^C - pI)^{-1} B^C$, is defined for $p \notin \sigma(A_\rho)$ and it can be represented for $\xi \in \mathbb{X}^C$ as

$$
W_\rho(p)\xi = -\sum_{k=0}^{\infty} W_\rho^{(k)}(p)\xi_k \psi_k,
$$

(6.9)

where $W_\rho^{(k)}(p) := (0, 1) \cdot (pI + \lambda_k A - K_\rho)^{-1} \cdot (0, 1)^T$ (i.e. $W_\rho^{(k)}(p)$ is the bottom right element of the matrix in the middle), where

$$
A = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \quad \text{and} \quad K_\rho := \begin{bmatrix} -c_{11} & c_{12} \\ -c_{21} & \rho \end{bmatrix}
$$

(6.10)

For $\rho \geq 0$ we consider two numbers $\mu_{1, \rho}, \mu_{2, \rho}$ such that $\mu_{1, \rho} \leq g_\rho'(z) \leq \mu_{2, \rho}$ for all $z \in \mathbb{R}$, and the quadratic form in $(u, \xi) \in \mathbb{H} \times \mathbb{X}$

$$
F_\rho(u, \xi) = F(y, z, \xi) := \int_{\Omega} (\xi(x) - \mu_{1, \rho} z(x))(\mu_{2, \rho} z(x) - \xi(s)) dx
$$

(6.11)
and its Hermitian extension for \((u, \xi) \in \mathbb{H}^\mathbb{C} \times \Xi^\mathbb{C}\)

\[
F^\mathbb{C}_\rho(u, \xi) = \operatorname{Re} \int_\Omega [(\xi(x) - \mu_1 u) z(x)]^*(\mu_2 z(x) - \xi(s))dx. \tag{6.12}
\]

From (6.9) we get that

\[
F^\mathbb{C}_\rho((-A_\rho - pI)^{-1}B, \xi, \xi) = -\sum_{k=0}^{\infty} |\xi_k|^2 \operatorname{Re} \left[ \left(1 - \mu_1 W_{\rho}^{(k)}(p)\right)^* \left(1 - \mu_2 W_{\rho}^{(k)}(p)\right) \right]. \tag{6.13}
\]

\((\text{PF})_{\nu, \rho}\) There are \(\delta > 0, \nu > 0\) and \(\rho > 0\) such that

\[
\operatorname{Re} \left[ \left(1 - \mu_1 W_{\rho}^{(k)}(p)\right)^* \left(1 - \mu_2 W_{\rho}^{(k)}(p)\right) \right] \geq \delta \tag{6.14}
\]

for all \(k = 0, 1, \ldots\) and \(p = -\nu + i\omega\), where \(\omega \in \mathbb{R}\).

**Remark 7.** Put

\[
L_{\rho,\nu} := \sup_{\omega \in \mathbb{R}} \left| W_{\rho}^{(k)}(-\nu + i\omega) \right|. \tag{6.15}
\]

Then it is clear that \((\text{PF})_{\nu, \rho}\) will be satisfied if \(\max\{|\mu_1|, |\mu_2|\} < L_{\rho,\nu}^{-1}\). This is the condition that was used by Smith [25, 26].

Consider the semi-flow \(\varphi\) in \(\mathbb{H}\), generated by (6.5), (6.2). We have the following analog of Theorem 8.1 from [26] and Theorem 6 from [25].

**Theorem 7.** Let (6.6) be satisfied with \(d_1, d_2, \mu > 0\). Suppose that \(g(0) = 0, g'(0) \neq c_1 + \lambda_0 (d_1 + d_2)\) and

\[
\lambda_0 d_2 - \mu < \inf_{z \in \mathbb{R}} g'(z) \quad \text{and} \quad \sup_{z \in \mathbb{R}} g'(z) < \lambda_1 d_2 - \mu. \tag{6.16}
\]

Then the \(\omega\)-limit set of any point with a bounded positive semi-orbit is either the stationary point \((0, 0)\) or a periodic orbit. Moreover, if

\[
\lim_{|z| \to \infty} \frac{g(z)}{z} = l \quad \text{and} \quad l < c_1 + \lambda_0 (d_1 + d_2) < g'(0) \tag{6.17}
\]

then all positive semi-orbits are bounded and there is at least one orbitally stable periodic orbit.

**Proof.** \(\nu := \mu\), where \(\mu = c_1 + \frac{1}{2} (\lambda_1 + \lambda_0) d_1\) is from (6.6), and \(\rho := c_1 + \frac{1}{2} (\lambda_1 + \lambda_0) (d_1 - d_2)\). In the proof of Theorem 8.1 from [26] it is shown that \((\text{PF})_{\nu, \rho}\) is satisfied with some \(-\mu_1 \rho = \mu_2 \rho < \frac{1}{2} (\lambda_1 - \lambda_0) d_2\) (see Remark 7). From Remark 6 and Theorem 3 from [26] applied to the pair \((A_\rho + \nu I, B)\) and the form \(F_\rho\) with the above defined \(\mu_1, \mu_2\), we get a bounded self-adjoint operator \(P: \mathbb{H} \to \mathbb{H}\) and a number \(\delta > 0\) such that

\[
((A_\rho + \nu I)u + B\xi, Pu) + F_\rho(u, \xi) \leq -\delta \left(|u|^2 + |\xi|^2\right), \quad \text{for } u \in D(A), \xi \in \Xi. \tag{6.18}
\]

Note that \(F_\rho(u, 0) \geq 0\) since we chose \(\mu_1 \rho \leq 0 \leq \mu_2 \rho\). Putting \(\xi = 0\) in (6.18) we get

\[
((A_\rho + \nu I)u, Pu) \leq -\delta |u|^2, \quad \text{for } u \in D(A). \tag{6.19}
\]

In [26] it is also shown that the operator \(A_\rho + \nu I\) has exactly two eigenvalues with \(\operatorname{Re} \lambda > 0\) and no eigenvalues with \(\operatorname{Re} \lambda = 0\). Thus (see, Appendix B in [1]), the operator \(P\) has the negative space \(\mathbb{H}^-\) of dimension 2 and trivial neutral subspace, i.e. \(\mathbb{H}^0 = \operatorname{Ker} P = \{0\}\). Analogously to the proof of Theorem 5 we get the conditions \((\text{H1}), (\text{H2})\) with \(j = 2\) and \((\text{H3})\) satisfied for the semi-flow \(\varphi\) in \(\mathbb{H}\).
As in [26], the conditions \( g(0) = 0 \) and \( (6.6) \) implies that \((0, 0)\) is the only stationary point. If \( g'(0) \neq 0 \) it follows from Theorem 1 that \( (6.6) \) is an unstable terminal point provided that \( g'(0) > c_{11} + \lambda_0 (d_1 + d_2) \). Therefore the first conclusion of the theorem follows from Lemma 12, Theorem 1 and Corollary 1.3.

Finally, \((6.17)\) implies that the solutions of \((6.5)\) are ultimately bounded. This means that for a proper \( R > 0 \) the set \( \mathcal{A} := \{ u \in \mathbb{H}_{s_0} \mid ||u||_{s_0} \leq R \} \) is an unstable terminal point provided that \( g'(0) > c_{11} + \lambda_0 (d_1 + d_2) \). Therefore the second conclusion of the theorem follows from Lemma 12 and Theorem 2.

By Lemma 3 any convergence in Theorem 7 holds in the norm of \( \mathbb{H}_{s_0} = \mathbb{E} \).

6.2. The FitzHugh-Nagumo systems. A special case of \((6.5)\) is the FitzHugh-Nagumo system (see [28], p. 99-100, [26])

\[
\begin{align*}
\dot{y}(t, x) &= d_0 \Delta y(t, x) + c_{12} z(t, x) - c_{11} y(t, x), \quad x \in \Omega, t > 0, \\
\dot{z}(t, x) &= d_0 \Delta z(t, x) - c_{21} y(t, x) - z(t, x)(z(t, x) + \kappa)(z(t, x) - 1), \quad x \in \Omega, t > 0.
\end{align*}
\]

System \((6.20)\) represents a simplified model for the Hodgkin-Huxley nerve equations. As in [26], the conditions \( c_{11}, c_{12}, c_{21} > 0 \) that however may be not of biological interest. The unboundedness of the nonlinearity can be avoided by constructing an appropriate invariant region as follows.

Note that since \( s_0 \in (\frac{3}{4}, 1) \), \( \partial \Omega \) is smooth and \( \Omega \subset \mathbb{R}^d \) with \( d \leq 3 \), the space \( \mathbb{H}_{s_0} \) is continuously embedded into the space of continuous functions \( C(\Omega) \). Therefore, for \( r > 0 \) the set

\[
\mathcal{S}_r := \{(y, z) \in \mathbb{H}_{s_0} \mid c_{21} y^2(x) + c_{12} z^2(x) < c_{12} r^2 \text{ for all } x \in \Omega \}
\]

is well-defined. The following lemma is proved in [26] (see Theorem 8.2 therein).

**Lemma 13.** Let \( d, c_{11}, c_{12}, c_{21} > 0 \). Then the set \( \mathcal{S}_r \) is positively invariant w. r. t. to the solutions generated by \((6.20)\) in \( \mathbb{H}_{s_0} \) provided that

\[
r > \mu_0 := (2c_{11})^{-1/2} \left( c_{11} + \frac{1}{2} \right) (1 + \kappa^2). \tag{6.22}
\]

Let \( g(z) := -z(z + \kappa)(z - 1) \). Then for \( z \in [-\mu_0, \mu_0] \) we have

\[
\kappa^2 + \kappa + 1 - (3\mu_0 + |\kappa - 1|)^2 \leq 3g'(z) \leq \kappa^2 + \kappa + 1 \tag{6.23}
\]

Let \( \dot{g} : \mathbb{R} \to \mathbb{R} \) be a twice continuously differentiable function such that \( \dot{g}(z) = g(z) \) for \( -\mu_0 \leq z \leq \mu_0 \), \( \dot{g}(z) < 0 \) for \( |z| > \mu_0 \) and \( z^{-1} g(z) \to 0 \) as \( |z| \to +\infty \). The consideration of \((6.20)\) with the nonlinearity changed to \( \dot{g} \) and Theorem 7 leads to the following theorem.

**Theorem 8.** Suppose \( c_{11}, c_{12}, c_{21}, \kappa > 0 \) and

\[
\frac{1}{2} (\lambda_1 - \lambda_0) d_0 > \max \left\{ q, \frac{1}{3} (3\mu_0 + |\kappa - 1|)^2 - q \right\}, \tag{6.24}
\]

\[
c_{12} c_{21} > (c_{11} + \lambda_0 d_0) \max \left\{ 0, \frac{1}{2} (\lambda_1 - 3\lambda_0) d_0 - c_{11} \right\}, \tag{6.25}
\]

where \( q = \frac{1}{3} (\kappa^2 + \kappa + 1) + c_{11} \). If \( c_{11} + 2\lambda_0 d_0 \neq \kappa \) then every solution of \((6.20)\) with initial value in \( \text{Cl}(\mathcal{S}_{\mu_0}) \) (the closure is taken in \( \mathbb{H}_{s_0} \)) converges either to the stationary point \((0, 0)\) or to a periodic solution in \( \text{Cl}(\mathcal{S}_{\mu_0}) \). Moreover, if

\[
c_{11} + 2\lambda_0 d_0 < \kappa, \tag{6.26}
\]

then \((6.20)\) has at least one orbitally stable periodic solution in \( \text{Cl}(\mathcal{S}_{\mu_0}) \).
The proof repeats the same arguing as in the proof of Theorem 8.2 from [26].

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