Computer model of the two-pinhole interference experiment using two-dimensional Gaussian wave-packets

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Abstract
The two-slit interference experiment has been modeled a number of times using Gaussian wave-packets and the Bohm–de Broglie causal interpretation. Here we consider the experiment with pinholes instead of slits and model the experiment in terms of two-dimensional Gaussian wave-packets and the Bohm–de Broglie causal interpretation.

Keywords: quantum mechanics, Bohm, de Broglie, causal interpretation, two-slit, computer models

1. Introduction

The first computer models of quantum systems based on the Bohm–de Broglie causal interpretation [1, 2] were developed by Dewdney in his PhD thesis (1983) [3]. These models include the two-slit experiment and scattering from square barriers and square wells. Some of the results appeared in earlier articles with Phillipides, Hiley (1979) [4] and with Hiley (1982) [5]. In later years, Dewdney developed a computer model of Rauch’s Neutron interferometer (1982) [6] and, with Kyprianidis and Holland, models of a spin measurement in a Stern–Gerlach experiment (1986) [7]. He also went on, with Kyprianidis and Holland, to develop computer models of spin superposition in neutron interferometry (1987) [8] and of Bohm’s spin version of the Einstein, Podolsky, Rosen experiment (EPR-experiment) (1987) [9]. A review of this work appears in a 1988 Nature article [10]. The computer models of spin were based on the 1955 Bohm–Schiller–Toimno causal interpretation of spin [11]. Home and Kaloyerou in 1989 reproduced the computer model of the two-slit interference experiment [12] in the context of arguing against Bohr’s principle of complementarity (BPC) [13].

Though computer models of the two-slit experiment with each slit modeled by a one-dimensional Gaussian wave-packet have existed for many years, the extension to pinholes has never been made. We thought, therefore, that it might be interesting to attempt such an extension by modeling each pinhole by a two-dimensional Gaussian wave-packet. Though no new conceptual results are expected, we thought it might be interesting to see if the trajectories, now in three dimensional space, retain the characteristic features of the two-slit case. We shall see that a quantum potential structure is produced which guides the particles to the bright fringes as in the two-slit case. With the pinholes along the $x$-axis, trajectories in the $xy$-plane (see figure 1) show the same interference behavior as in the two-slit case, while trajectories in the $zy$-plane show no interference.

In passing we mention three more recent articles concerning a trajectory approach to interference. The article by Sanz and Miret [14] provides a detailed analysis of interference resulting from the superposition of wave-packets both from the perspective of standard quantum mechanics and from the Bohm–de Broglie causal interpretation. Yang and Su [15] demonstrate interference as an average over complex trajectories with a view to showing the equivalence of trajectory based statistics and wavefunction based statistics.
Chou’s article [16] is particularly interesting. He develops a nonlinear equation which demonstrates the quantum–classical transition in wave-packet interference using a hydrodynamical model.

2. A little bit of history

Before proceeding with the main focus of this article, it may be worth pausing to consider the emergence of the causal interpretation and how it differs from the commonly accepted Copenhagen interpretation of quantum mechanics which consists of the Born probability rule and BPC [13].

It was 25 years after the birth of quantum mechanics in 1900, with Planck’s introduction of quantized absorption/emission of the electromagnetic field by atoms in order to resolve the blackbody radiation problem, that the first mathematical formulation of quantum mechanics appeared. This was Heisenberg’s matrix mechanics, introduced in the summer of 1925. This was followed shortly after, at the beginning of 1926, by the introduction by Schrödinger of his partial differential wave equation. The same year, 1926, Dirac, and independently Jordan, showed the equivalence of Heisenberg’s matrix mechanics and the Schrödinger equation with what has come to be called the Dirac–Jordan transformation theory. At this time matrix methods were not well known, whereas physicists and mathematicians were very familiar with differential and partial differential equations through the myriad of applications in classical physics (physics theories prior to the advent of quantum mechanics and the special theory of relativity) so attempts to obtain a physical interpretation of quantum mechanics were focused on the Schrödinger equation. There were two issues that physicists had to contend with: first, the solution of Schrödinger’s equation were, in general, complex functions as opposed to the solutions of classical equations, which were real functions. The use of complex functions in classical physics was a matter of mathematical convenience, not necessity, so that physical interpretations were based on real functions. The complex solutions of the Schrödinger equation are fundamental, but it was not at all clear how to interpret complex functions physically. Second, experiments revealed a dual particle-wave behavior of fundamental entities. Classically, light revealed only wave behavior, whereas the explanations of the photoelectric effect (1905) and the Compton effect (1923) suggested that light is composed of photon particles. Hence, light displayed wave behavior in some experiments but, in mutually exclusive experiments, displayed particle behavior. Similarly, subatomic particles such as electrons, originally considered to be particles, were found to display wave behavior. For example, in the 1927 Davission–Germer experiments, electrons displayed wave-behavior. Particle-wave duality was embodied in the 1923–24 de Broglie particle-wave relation. These difficulties motivated quite a number of interpretations to be put forward. Among the most prominent were Schrödinger’s electromagnetic interpretation, Madelung’s hydrodynamical model, de Broglie’s pilot-wave theory and his theory of the double solution, Born’s probability rule and BPC. In his theory of the double solution, de Broglie’s sought to model a particle by a soliton wave solution of a certain nonlinear equation such that the motion of the soliton wave is guided by solutions of the Schrödinger equation, hence the name theory of the double solution. At the time, de Broglie could not find the nonlinear equation he sought and so he developed his simpler pilot-wave theory as an interim interpretation of the quantum theory. The nonlinear equation sought by de Broglie has never been found.

Bohr first presented his principle of complementarity in his 1927 Como lecture2. A month later, October 1927, the Fifth Physical Congress of the Solvay Institute was held in Brussels. Though the official program was ‘Electrons and Photons’, the stage was set to consider the various interpretations of quantum mechanics. As with the Como

2 He delivered this lecture at the International Congress of Physics, commemorating the centenary of Alessandro Volta’s death, held in the Italian city of Como where Volta was born. It was attended by an illustrious group of leading physicists, among whom were Quirino Majorana, Neils Bohr, Max Born, Satyandra Bose, Louis de Broglie, Arthur Holly Compton, Enrico Fermi, Werner Heisenberg, Hendrik Antoon Lorentz, Wolfgang Pauli and Max Planck.
Congress, the Solvay Congress was attended by an auspicious group of the world’s leading physicists among whom were Einstein, Lorentz, Born, de Broglie, Dirac, Heisenberg, Pauli and Planck. de Broglie was the first speaker on the main subject and spoke on his simpler pilot wave theory (rather than his theory of the double solution). His proposed interpretation was met with severe criticism from Pauli in particular. de Broglie was unable to respond to Pauli’s objection, which concerned Fermi’s rigid rotator (an objection essentially to do with the measurement problem of quantum mechanics). de Broglie was so disturbed by the poor response to his ideas that following the congress he abandoned his

| Quantity | Definition | Value |
|----------|------------|-------|
| $b$ | Angle for equal amplitudes | $\pi/4$ |
| $\tilde{b}$ | Angle for unequal amplitudes | $n/3$ |
| $\tilde{R}_1$ | Amplitude of $\psi_1$ | $\cos^2(\tilde{b})$ |
| $\tilde{R}_2$ | Amplitude of $\psi_2$ | $\sin^2(\tilde{b})$ |
| $x_0$ | $x$-distance of the center of the pinhole from the origin | $5 \times 10^{-7}$ m |
| $z_0$ | $z$-distance of the center of the pinhole from the origin | 0 m |
| $h$ | Planck’s constant | $6.626 070 04 \times 10^{-34}$ Js |
| $\tilde{h}$ | Planck’s constant/2$\pi$ | $1.054 571 80 \times 10^{-34}$ Js |
| $m$ | Mass of electron | $9.109 383 56 \times 10^{-31}$ kg |
| $\alpha$ | $h/m$ | $0.000 115 767 64$ Js m$^{-1}$ |
| $k_x$ | Magnitude of $x$-wavenumber | $1.295 698 717 \times 10^{6}$ m$^{-1}$ |
| $k_y$ | Magnitude of $y$-wavenumber | $1.122 938 132 \times 10^{12}$ m$^{-1}$ |
| $k_z$ | Magnitude of $z$-wavenumber | 0 |
| $\nu_x$ | $x$-component of the velocity of the Gaussian wave-packet | $\alpha k_x = 150$ m s$^{-1}$ |
| $\nu_y$ | $y$-component of the velocity of the Gaussian wave-packet | $\alpha k_y = 1.3 \times 10^{8}$ m s$^{-1}$ |
| $\nu_z$ | $z$-component of the velocity of the Gaussian wave-packet | $\alpha k_z = 0$ m s$^{-1}$ |
| $\omega$ | Angular frequency $\omega$ | $\hbar (k_x^2 + k_y^2) / 2m$ |
| $\chi$ | Phase shift of $\psi_2$ | 0 |
| $\Delta x_{00} = \Delta z_{00}$ | Width of the $-x_0$ wave-packet | $\Delta x_{00} = 7 \times 10^{-8}$ m |
| $\Delta y_{00}$ | Width of the $+y_0$ wave-packet | $\Delta y_{00} = \Delta y_{00}$ |
| $\Delta x_{00} = \Delta z_{00}$ | Width of the $-x_0$ wave-packet for unequal pinhole widths | $\Delta x_{00} = 7 \times 10^{-8}$ m |
| $\Delta y_{00}$ | Width of the $+y_0$ wave-packet for unequal pinhole widths | $\Delta y_{00} = 2 \Delta y_{00}$ |
interpretations. His interest in his interpretations was revived many years later, after the publication of Bohm’s 1952 articles (to be described below). By the end of the congress the Copenhagen interpretation gained overwhelming acceptance. Following the 1932 von Neumann impossibility proof, which claimed that a hidden variable theory could not reproduce the statistical results of the quantum theory, the Copenhagen interpretation gained almost universal acceptance.

The Born probability rule is an essential interpretational element which links theory with experiment, whereas BPC is a philosophical backdrop not directly linked to the mathematical formalism. BPC was guided by two elements, the quantum postulate (which states that a measuring apparatus is connected to the system being measured by a quantum of action which is uncontrollable, unanalysable and unpredictable) and the duality (especially particle-wave duality) revealed by experiment. The quantum postulate led Bohr to conclude that the classical concept of a well defined state (i.e. all relevant variables well defined) could not be maintained in the quantum theory and hence, an experiment must be viewed as an unanalysable whole (with the inner workings beyond our conceptual gaze). The duality of experiment led Bohr to conclude that pairs of complementary, mutually exclusive classical concepts were needed to exhaust the description of nature. Though Bohr emphasized that such classical concepts are essential to aide thought and communicate the results of experiment, reality cannot be attached to such concepts. Thus, according to Bohr, an electron, say, is never a particle and never a wave. Rather, in one experiment the concept of a particle can be used to describe the experiment, but in another mutually exclusive experiment the wave concept may be used for its description, but reality should not be attached to either concept. The logical consequence of these tenets of complementarity is that a description of underlying physical reality is impossible.\(^3\)

BPC achieves consistency (up to a point, since the mutually exclusive concepts of particle and wave are fundamentally different to other mutually exclusive classical concepts which are canonically conjugate dynamical variables—see [17], section 2, for more details) at the very

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\(^3\) See [17] for a more detailed discussion of BPC.
Figure 6. The intensity in a two-pinhole interference experiment with equal widths and equal amplitudes modeled by two-dimensional Gaussian wave-packets with equal widths and equal amplitudes.
heavy price of giving up a description of underlying physical reality. Despite this, the Copenhagen interpretation remained the overwhelmingly dominant interpretation of the quantum theory. But, in 1952, Bohm published two Physical Review articles [1] entitled, ‘A suggested interpretation of the quantum theory in terms of ‘hidden’ variables I and II’ in which he introduced an interpretation of the quantum theory in terms of a single well-defined model. In this model electrons, neutrons, protons etc are always particles guided by two quantum fields (the R and S-fields) which codetermine each other (see the next section for a brief summary). Bohm thus produced an interpretation of quantum theory describing underlying physical reality, the very description supposedly impossible according to BPC and supposedly ruled out by the von Neumann 1932 impossibility proof. Actually, in article II, Bohm showed that von Neumann’s impossibility proof did not apply to his model. Later, Bell [18] demonstrated more generally that von Neumann’s proof applied only to local hidden variable theories but not to nonlocal theories such as Bohm’s. In article II, Bohm presented a solution of the so called ‘measurement problem’ of quantum mechanics based on his model which answered Pauli’s Fermi rigid rotator objection to de Broglie’s pilot-wave theory. Though, Bohm did know it at the time of writing of his articles, his model turned out to be a completed version of de Broglie’s pilot-wave theory. For this reason the model has come to be called the Bohm–de Broglie causal interpretation.

Bohm’s 1951 book on quantum mechanics [19] was based on lectures, which Bohm attended, given by Bohr at Princeton university. The book, therefore, was written from the Perspective of the Copenhagen interpretation. Yet, only a year after publication of this book, Bohm published his two celebrated Physical Review articles on the causal interpretation. One of the authors, Kaloyerou5, asked Bohm why he made this radical conceptual change. He answered, ‘I wrote the book to try to understand quantum mechanics, but after finishing the book I still did not understand quantum mechanics’. He went on to explain that the idea for the causal interpretation occurred to him whilst writing his chapter on the WKB approximation. His idea was to take the WKB solution of the Shrödinger equation, \( \psi = Re^{iS/h} \), as exact instead of using an approximation for \( S \).

Given the focus of this paper it may be obvious that the present authors favor the Bohm–de Broglie causal interpretation over BPC. This is because of the conclusion that a description of underlying physical reality is impossible, a conclusion that the present authors believe is a serious failing of BPC. Having said this, it is also important to emphasize that BPC is based on an insight of genius, the quantum postulate, without which a proper understanding of the quantum theory is not possible. The quantum postulate embodies the wholeness or interconnectedness of quantum mechanics, so beautifully demonstrated by the 1935 EPR experiment [20] (see also [19], ch 22, section 16, for a spin version of the EPR experiment), a fundamental feature which marks the break with classical mechanics.

3. The mathematical model

Phillipides et al [4] derived the Gaussian function they used to model each of the two slits in the two-slit experiment using Feynman’s path integral formulation. We, instead, have generalized the one-dimensional Gaussian solutions of the Schrödinger equation developed by Bohm in chapter three of his book [19] to two-dimensions. Phillipides et al [4] considered Young’s two-slit experiment for electrons and used the values of an actual Young’s two-slit experiment for electrons performed by Jönsson in 1961 [21]. We will also model the interference of electrons and use Jönsson’s values, except that we will vary slightly the distance between the pinholes and the detecting screen in order to obtain clearer interference or quantum potential plots. We will, in any case, give the values used for each case we consider.

The orientation of the axes and the position of the pinholes are shown in figures 1 and 2, respectively. The pinholes are represented by two dimensional Gaussian wave-packets \( \psi_1 \) and \( \psi_2 \) given by

\[
\psi_1(x, y, z, t) = A \tilde{R} \times \exp \left[ \frac{-(x + x_0 - v_x t)^2}{2\Delta x_n^2} \right] \times \exp \left[ \frac{-(z + z_0 - v_z t)^2}{2\Delta z_n^2} \right] \times \exp[ik_z(z + z_0)] \times \exp[ik_y(y)] \times \exp[\left(-i(\omega_1 + \omega_2)t\right)],
\]

\[
\psi_2(x, y, z, t) = A \tilde{R} \times \exp \left[ \frac{-(x - x_0 + v_x t)^2}{2\Delta x_n^2} \right] \times \exp \left[ \frac{-(z + z_0 - v_z t)^2}{2\Delta z_n^2} \right] \times \exp[ik_z(z + z_0)] \times \exp[ik_y(y)] \times \exp[\left(-i(\omega_1 + \omega_2)t\right)].
\]

Figure 7. A sequence of density plots (3 slices of a 3D-plot) of the intensity \( R^2 \) in a two-pinhole interference experiment with equal widths and equal amplitudes modeled by two-dimensional Gaussian wave-packets with equal widths and equal amplitudes.
Figure 8. The quantum potential in a two-pinhole interference experiment with equal widths and equal amplitudes modeled by two-dimensional Gaussian wave-packets with equal widths and equal amplitudes.
Gaussian wave-packets with equal widths and equal amplitudes. Figure 9. A sequence of density plots (3 slices of a 3D-plot) of the quantum potential \( Q \) in a two-pinhole interference experiment with equal widths and equal amplitudes modeled by two-dimensional Gaussian wave-packets with equal widths and equal amplitudes.

The various functions and constants used are given by:

\[
\begin{align*}
\bar{R}_0 &= \cos^2 \frac{\pi}{4}, \quad \bar{R}_2 = \sin^2 \frac{\pi}{4} \quad \text{(for equal amplitudes)}, \\
\alpha &= \frac{\hbar}{m}, \quad v_x = \frac{\hbar k_x}{m}, \quad \omega_x = \frac{\hbar k_x^2}{2m}, \quad v_z = \frac{\hbar k_z}{m}, \quad \omega_z = \frac{\hbar k_z^2}{2m}, \\
\Delta x_{00} &= \text{width of the } - x_0 \text{ wavepacket}, \quad \Delta z_{00} = \Delta x_{00} \quad \Delta x_{p0} = \text{width of the } + x_0 \text{ wavepacket}, \quad \Delta z_{p0} = \Delta x_{p0} \\
A_{\alpha}(t) &= A_{\alpha_0}(t)A_{\alpha}(t) = \left( \frac{2\pi}{\Delta x_{\alpha_0} + i\alpha t} \right)^{\frac{1}{2}} \left( \frac{2\pi}{\Delta z_{\alpha_0} + i\alpha t} \right)^{\frac{1}{2}} \\
&= \beta_{\alpha}(t)e^{i\theta_{\alpha}(t)}\beta_{\alpha}(t)e^{i\theta_{\alpha}(t)} = \beta_{\alpha}e^{2i\theta_{\alpha}}, \quad (3)
\end{align*}
\]

Further definitions and values of quantities used in the plots are given in table 1. Note that the Gaussian wave packets are functions of \( x \) and \( z \), while the \( y \)-behavior is represented by a plane wave. Plane waves are useful idealizations that are not realizable in practice. This leads to a computer model in which the intensity fields which codetermine one another, into the Schrödinger equation

\[
i\hbar \frac{\partial \psi}{\partial t} = \frac{\hbar^2}{2m} \nabla^2 \psi + V\psi,
\]

Our computer model is based on the Bohm–de Broglie causal interpretation and we refer the reader to Bohm’s original papers for details of the interpretation [1]. Here we will give only a very brief outline in order to introduce the elements we will need to develop the formulae and equations needed for the model. The interpretation is obtained by substituting \( \phi(x, y, z, t) = R(x, y, z, t) \exp(iS(x, y, z, t)/\hbar) \), where \( R \) and \( S \) are two fields which codetermine one another, into the Schrödinger equation

\[
\psi(x, y, z, t) = A_0 e^{i\theta_0} = \frac{2\pi}{\Delta x_{\alpha_0} + i\alpha t} \left( \frac{2\pi}{\Delta z_{\alpha_0} + i\alpha t} \right)^{\frac{1}{2}}
\]

where

\[
\begin{align*}
\beta_{\alpha}(t) &= \beta_{\alpha}(t) = \left( \frac{4\pi^2}{\Delta x_{\alpha_0} + \alpha^2t^2} \right)^{\frac{1}{2}}, \quad \beta_{\alpha}(t) = \left( \frac{4\pi^2}{\Delta x_{\alpha_0} + \alpha^2t^2} \right)^{\frac{1}{2}} \\
\theta_1 &= \theta_{ \alpha}(t) = \theta_{ \alpha}(t) = \frac{1}{2} \tan^{-1} \left( \frac{\alpha t}{\Delta x_{\alpha_0}} \right) + 2k\pi, \quad k = 0, 1, ..., \\
\beta_{\alpha}(t) &= \beta_{\alpha}(t) = \left( \frac{4\pi^2}{\Delta z_{\alpha_0} + \alpha^2t^2} \right)^{\frac{1}{2}}, \quad \beta_{\alpha}(t) = \left( \frac{4\pi^2}{\Delta z_{\alpha_0} + \alpha^2t^2} \right)^{\frac{1}{2}} \\
\theta_2 &= \theta_{ \alpha}(t) = \theta_{ \alpha}(t) = \frac{1}{2} \tan^{-1} \left( \frac{\alpha t}{\Delta z_{\alpha_0}} \right) + 2k\pi, \quad k = 0, 1, ..., \\
\Delta x_{\alpha_0}^2 = \Delta z_{\alpha_0}^2 &= \left( \Delta x_{\alpha_0}^2 + \alpha^2t^2 \right), \quad \Delta z_{\alpha_0}^2 = \Delta z_{\alpha_0}^2 = \left( \Delta x_{\alpha_0}^2 + \alpha^2t^2 \right). \\
\Delta x_{\alpha_0}^2 = \Delta z_{\alpha_0}^2 &= \left( \Delta x_{\alpha_0}^2 + \alpha^2t^2 \right), \quad \Delta z_{\alpha_0}^2 = \Delta z_{\alpha_0}^2 = \left( \Delta x_{\alpha_0}^2 + \alpha^2t^2 \right).
\end{align*}
\]

Figure 9. A sequence of density plots (3 slices of a 3D-plot) of the quantum potential \( Q \) in a two-pinhole interference experiment with equal widths and equal amplitudes.
where \( V = V(x, y, z, t) \). Differentiating and equating real and imaginary terms gives two equations. One is the usual continuity equation

\[
\frac{\partial R^2}{\partial t} + \nabla \cdot \left( R^2 \frac{\nabla S}{m} \right) = 0,
\]

which expresses the conservation of probability \( R^2 \). The other is a Hamilton–Jacobi type equation

\[
-\frac{\partial S}{\partial t} = \frac{(\nabla S)^2}{2m} + V + \left( -\frac{\hbar^2}{2m} \frac{\nabla^2 R}{R} \right).
\]

This differs from the classical Hamilton–Jacobi equation by the extra term

\[
Q = -\frac{\hbar^2}{2m} \frac{\nabla^2 R}{R},
\]

which Bohm called the quantum potential. The classical Hamilton–Jacobi equation describes the behavior of a particle with energy \( E \), momentum \( p \) and velocity \( v \) under the action of a potential \( V \), with the energy, momentum and velocity given by

\[
E = -\frac{\partial S}{\partial t}, \\
p = \nabla S, \\
v_p(F) = \frac{\nabla F}{m} = \frac{\nabla S}{m}.
\]

Bohm retain’s these definitions, while de Broglie focused on the third definition and called it the guidance formula. This allows quantum entities such as electrons, protons, neutrons etc (but not photons\(^6\)) to be viewed as particles (always) with energy, momentum and velocity given by (9). Particle trajectories are found by integrating \( v(\vec{r}) \) given in (9). The extra \( Q \) term produces quantum behavior such as the interference of particles (which is what we will model in this article). Strictly, since the \( R \) and \( S \)-fields codetermine one another, the \( S \)-field is as much responsible for quantum behavior as the \( R \)-field; the \( S \)-field through the guidance formula, and the \( R \)-field through the quantum potential.

As mentioned earlier, the Born probability rule is an essential interpretational element that links theory with experiment. As such it will remain a part of any interpretation of the quantum theory. This is certainly true for the causal interpretation, where probability enters because the initial positions of particles cannot be determined precisely. Instead, initial positions are given with a probability found from the usual probability density \( \left| \psi(x, y, z, t = 0) \right|^2 = R(x, y, z, t = 0) \). The results of the usual interpretation are identical with those of the causal interpretation as long as the following assumptions are satisfied:

1. The \( \psi \)-field satisfies Schrödinger’s equation.
2. Particle momentum is restricted to \( \vec{p} = \nabla S \).
3. Particle position at time \( t \) is given by the probability density \( \left| \psi(\vec{r}, t) \right|^2 \).

\(^6\) The causal interpretation based on the Schrödinger is obviously non-relativistic, but it is more than adequate for the description of the behavior of electrons, protons, neutrons, atoms etc, in a large range of circumstances. Photons, however, are not accurately described by the Schrödinger equation, but by quantum optics which is based on the second-quantized Maxwell equations. The causal interpretation of the electromagnetic field, and, more generally, of boson fields, models bosons, including the photon, as fields (see the articles listed in [22]).
To obtain the intensity, $Q$ and trajectories we must first find the $R$ and $S$-fields defined by $\psi = R e^{iS/\hbar}$ in terms of $R_1$, $R_2$, $S_1$ and $S_2$ defined by $\psi_1 = R_1 e^{iS_1/\hbar}$ and $\psi_2 = R_2 e^{iS_2/\hbar}$. This is done by equating the real and imaginary parts of $R e^{iS/\hbar}$ to those of $\psi_1$ given by equation (1), equating the real and imaginary parts of $R e^{iS/\hbar}$ to those of $\psi_2$ given by

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure11.png}
\caption{The intensity in a two-pinhole interference experiment with equal widths but unequal amplitudes modeled by two-dimensional Gaussian wave-packets with equal widths but unequal amplitudes.}
\end{figure}
Figure 12. A sequence of density plots (3 slices of a 3D-plot) of the intensity $R^2$ in a two-pinhole interference experiment with equal widths but unequal amplitudes modeled by two-dimensional Gaussian wave-packets with equal widths but unequal amplitudes.

equation (2) and by noting equation (3) for $A_1$ and equation (4) for $A_2$. We get

$$ R_1(x, z, t) = \beta_1 \beta_1 \exp \left[ -\frac{(x + x_0 - v_x t)^2}{2\Delta x_n^2} \right] \times \exp \left[ -\frac{(z + z_0 - v_z t)^2}{2\Delta z_n^2} \right], $$

$$ R_2(x, z, t) = \beta_2 \beta_2 \exp \left[ -\frac{(x - x_0 + v_x t)^2}{2\Delta x_p^2} \right] \times \exp \left[ -\frac{(z - z_0 + v_z t)^2}{2\Delta z_p^2} \right], $$

$$ S_1(x, y, z, t) = \frac{\hbar \alpha t \alpha t}{2\Delta x_1} + \frac{\hbar \alpha t \alpha t}{2\Delta z_1} + \hbar k_x (x + x_0) + \hbar k_y (z + z_0) + \hbar k_y \psi - \hbar (\omega_x + \omega_2) t + 2\hbar \theta_1, $$

$$ S_2(x, y, z, t) = \frac{\hbar \alpha t \alpha t}{2\Delta x_2} + \frac{\hbar \alpha t \alpha t}{2\Delta z_2} - \hbar k_x (x - x_0) - \hbar k_x (z - z_0) + \hbar k_y \psi - \hbar (\omega_x + \omega_2) t + \hbar \psi + 2\hbar \theta_2. $$

The intensity (probability density) is easily found from $|\psi|^2 = R^2$:

$$ R^2 = R_1^2 + R_2^2 + 2R_1 R_2 \cos \left( \frac{S_1 - S_2}{\hbar} \right). \tag{11} $$

The quantum potential $Q$ is found from:

$$ Q = \frac{\hbar^2}{2m} \nabla^2 R \frac{\nabla^2 R}{R} = \frac{\hbar^2}{2m} \left( \frac{\partial^2 R}{\partial x^2} + \frac{\partial^2 R}{\partial y^2} + \frac{\partial^2 R}{\partial z^2} \right) = Q_x + Q_y + Q_z, $$

where

$$ Q_x = -\frac{\hbar^2}{2mR} \frac{\partial^2 R}{\partial x^2} = \frac{\hbar^2}{8mR^2} \left( \frac{\partial R^2}{\partial x} \right)^2 - \frac{\hbar^2}{4mR^2} \frac{\partial^2 R^2}{\partial x^2}, \tag{12} $$

with similar formulae for $Q_y$ and $Q_z$. Substituting equation (11) into the formulae for $Q_x$ and $Q_y$ and differentiating gives $Q_y = 0$ and

$$ Q_x = -\frac{\hbar^2}{2mR^2} \left[ \frac{2(x + x_0 - v_x t)^2}{\Delta x_n^2} - \frac{1}{\Delta x_n^2} \right] $$

$$ + \frac{2(x - x_0 + v_x t)^2}{\Delta x_p^2} \cos(S_{12}) $$

$$ + 2 \frac{(x - x_0 + v_x t)(x - x_0 + v_x t)}{\Delta x_n^2 \Delta x_p^2} $$

$$ + \frac{(x - x_0 + v_x t)^2}{\Delta x_p^2} \cos(S_{12}) $$

$$ - \frac{\hbar^2 R_1 R_2}{2mR^2} \left[ \frac{2(x + x_0 - v_x t)^2}{\Delta x_n^2} + \frac{(x - x_0 + v_x t)^2}{\Delta x_p^2} \right] $$

$$ - \frac{\hbar^2 R_1 R_2}{2mR^2} \left[ \frac{(x - x_0 - v_x t)^2}{\Delta x_n^2} + \frac{(x + x_0 + v_x t)^2}{\Delta x_p^2} \right] $$

$$ - S_{12} \sin(S_{12}) - S_{12} \sin(S_{12}) - S_{12} \sin(S_{12}) - S_{12} \sin(S_{12}). \tag{13} $$

where

$$ S_{12} = S_1 - S_2 $$

$$ S_{12} = \frac{\alpha t (x + x_0 - v_x t)}{\Delta x_{n1}} - \frac{\alpha t (x - x_0 + v_x t)}{\Delta x_{p1}} + 2k_x $$

$$ S_{12} = \frac{\alpha t (x - x_0 + v_x t)}{\Delta x_{n1}} - \frac{\alpha t (x - x_0 + v_x t)}{\Delta x_{p1}}. $$

The formulae for $Q_x$ is identical to that of $Q_x$, except that $x$ is replaced by $z$ everywhere it appears.

The trajectories, as we have said, are found by integrating equation (9). Therefore, to find the trajectories we do not need to find $S$, only its derivatives with respect to $x, y, z$. This can be done using the formula

$$ \nabla S = \frac{\hbar}{2i} \left( \frac{\nabla \psi}{\psi} - \frac{\nabla \psi^*}{\psi^*} \right). \tag{14} $$
Figure 13. The quantum potential in a two-pinhole interference experiment with equal widths but unequal amplitudes modeled by two-dimensional Gaussian wave-packets with equal widths but unequal amplitudes.
We get
\[ \frac{\partial S}{\partial y} = \hbar k_y, \]  

and
\[ \frac{\partial S}{\partial x} = \frac{\hbar}{R^2} \left[ \frac{x}{\Delta x^2_p \Delta x^2_{p'}} R_x R_x \sin(S_{1z}) \left( \Delta x_y^2 - \Delta x^2_{p'} \right) 
+ \frac{\alpha x}{\Delta x_n^2 \Delta x_{n'}^2} \left[ R_x^2 \Delta x_{p'}^2 + R_y^2 \Delta x_n^2 + R_z R_2 \cos(S_{1z}) \right] 
\times \left( \Delta x_{p'}^2 + \Delta x_{n'}^2 \right) 
- \frac{(x_0 - v_x t)}{\Delta x_n^2 \Delta x_{n'}^2} R_x R_x \sin(S_{1z}) \left( \Delta x_y^2 + \Delta x_n^2 \right) 
+ \frac{(\alpha x_0 - \alpha t^2 v_x)}{\Delta x_n^2 \Delta x_{n'}^2} \left[ R_x^2 \Delta x_{p'}^2 - R_y^2 \Delta x_n^2 \right] 
+ R_z R_2 \cos(S_{1z}) \left( \Delta x_{p'}^2 - \Delta x_{n'}^2 \right) 
+ k_x \left( R_t^2 - R_t^2 \right). \]  

The z-derivative $\frac{\partial S}{\partial z}$ is identical to $\frac{\partial S}{\partial x}$, except that $x$ is everywhere replaced by $z$. From equation (9),
\[ v_p = v_{p_x} \hat{i} + v_{p_y} \hat{j} + v_{p_z} \hat{k} = \frac{dx}{dt} \hat{i} + \frac{dy}{dt} \hat{j} + \frac{dz}{dt} \hat{k} 
= \frac{1}{m} \left( \frac{\partial S}{\partial x} \hat{i} + \frac{\partial S}{\partial y} \hat{j} + \frac{\partial S}{\partial z} \hat{k} \right). \]

we see that to obtain the electron trajectories $\vec{r}(t) = x(t) \hat{i} + y(t) \hat{j} + z(t) \hat{k}$, we must solve the following differential equations with various initial conditions:
\[ \frac{dx(t)}{dt} = \frac{1}{m} \frac{\partial S}{\partial x}, \]  

\[ \frac{dy(t)}{dt} = \frac{1}{m} \frac{\partial S}{\partial y}, \]  

\[ \frac{dz(t)}{dt} = \frac{1}{m} \frac{\partial S}{\partial z}. \]

Note that the components of the particle velocity $v_p$ are different from the velocities of the wave packets $v_x$, $v_y$, and $v_z$. Equation (19) can be solved immediately to give $y(t) = \hbar k_y t$. Equations (18) and (20) are coupled nonlinear differential equations. These were solved numerically using a Fortran program we wrote based on an adapted fourth-order Runge–Kutta algorithm [23] with fixed step size. This completes the various elements of the mathematical model. In the following section we show the various plots.

4. The computer plots

For the sake of comparison, we first reproduce plots of the intensity, $Q$ and trajectories for the two-slit experiment modeled by one-dimensional Gaussian wave-packets. These are shown in figures 3–5.

The quantities used for the two-pinhole experiment plots are given in table 1. Note that the slit width used by Jönsson is $2 \times 10^{-7}$ m. The width of the Gaussian wavepackets $\psi_1$ and $\psi_2$ are defined at half their amplitude. We have chosen the widths of $\psi_1$ and $\psi_2$ to be $\Delta x_0 = \Delta z_0 = \Delta x_0 = \Delta z_0 = 7 \times 10^{-8}$ m so that the width of the base of $\psi_1$ and $\psi_2$ approximately corresponds to $2 \times 10^{-7}$ m. For unequal pinhole widths $\Delta x_0 = \Delta z_0 = 2 \Delta x_0$. We will consider three configurations: (1) equal pinhole widths and equal amplitudes, (2) equal pinhole widths and unequal amplitudes and (3) unequal pinhole widths and equal amplitudes.

In the two-dimensional case, i.e. pinhole case, the intensity $R^2$ and quantum potential $Q$ are functions of four variables $x$, $y$, $z$, and $t$. To produce plots we note that because the $y$-behavior is represented by a plane-wave, $Q_y = 0$, while $Q_x$ and $Q_z$ depend only on $x$, $z$ and $t$. Similarly, the intensity does not depend on $y$. This means that the values of $R^2$ and $Q$ in the $x$-$y$ plane at a given instant of time are the same from $y = -\infty$ to $y = +\infty$, as mentioned in section 3. At a later instant, the form of $R^2$ and $Q$ change instantaneously from $y = -\infty$ to $y = +\infty$. This unphysical behavior is due to the use of the plane-wave idealization to represent the $y$-behavior. A more realistic picture would be to also use a Gaussian in the $y$-direction. However, as we shall see, the model produces a realistic picture of particle trajectories which depend on $x$, $y$, $z$, $t$. Since the quantum potential and intensity change in time, the electron ‘sees’ evolving values of these quantities. All the plots below show what the electron ‘sees’ at a particular instant of time $t$ and a particular position $x$, $y$, $z$.

To graphically represent $R^2$ and $Q$ we proceeded two ways. First, we produced animations of $R^2$ and $Q$. We produced animations of six frames, so that the sequence of frames is short enough to be reproduced in this article. In any case, our computer did not have enough memory to produce animations of more than six frames. The animations are
produced in the $xz$-plane and show the form of the intensity and quantum potential that the electron ‘sees’ at each instant of time as it moves along its trajectory. Second, we produced animations of density plots in the $xz$-plane. These results are presented by placing three two-dimensional $xz$-slices (three frames of the animation) along the $t$-axis, i.e., we pick out three slices of a fully three dimensional density plot.

4.1. Computer plots for equal pinhole widths and equal amplitudes

The animation sequence for the intensity $R^2$ for equal widths and equal amplitudes (EWEA) is shown in figure 6. The animation ranges are $x = -3.5 \times 10^{-6}$ to $3.5 \times 10^{-6}$ m, $z = -3.5 \times 10^{-6}$ to $3.5 \times 10^{-6}$ m, $R^2 = (0-1.8) \times 10^{15}$ J m$^{-2}$ s$^{-1}$ and $t = (0-1.5) \times 10^{-9}$ s. The plots show the time evolution of the intensity in the $xz$-plane. The first frame shows the Gaussian peaks at the two pinholes. Frames two and three show the spreading Gaussian packets beginning to overlap and also show the beginning of the formation of interference fringes. Frames four to five show the time evolution of distinct interference fringes. Frame six shows the intensity distribution at time $t = 1.5 \times 10^{-9}$ s which corresponds to a pinhole screen to detecting screen separation of $y = 0.195$ m given that the electron velocity in the $y$-direction is $v_y = 1.3 \times 10^8$ m s$^{-1}$. Our pinhole screen and detecting separation therefore differs from that in Jönsson’s experiment which was $0.35$ m corresponding a time evolution of $t = (0-2.6923) \times 10^{-9}$ s. We chose this time in order to show the beginnings of the overlap of the Gaussian wave packets. Using the Jönsson time of $t = (0-2.6923) \times 10^{-9}$ s resulted in a clear interference pattern in the second frame, missing out the early overlap.

We can make an approximate calculation of the visibility of the central fringe by taking readings from ‘face-on’ plots, i.e., plots with the $xz$-plane in the plane of the paper (not shown here). Readings can be taken from the plots shown by taking due consideration of the orientation, but even then, readings are less accurate than with face-on plots. Similarly, to calculate the visibility of the interference fringes for the case of unequal amplitudes and for the case of unequal widths, readings are taken from face-on plots not included here. The visibility of the central fringe for the EWEA case is:

$$V_{EWEA} = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}} = \frac{10 \times 10^{10} - 0}{10 \times 10^{10} + 0} = 1.$$  

A sequence of density plots (3 slices of a 3D-plot) of the intensity $R^2$ for EWEA is shown in figure 7. The plot ranges are $x = -4 \times 10^{-6}$ to $4 \times 10^{-6}$ m, $z = -4 \times 10^{-6}$ to $4 \times 10^{-6}$ m and $t = (0-1.5) \times 10^{-9}$ s. The first $xz$-slice shows the high intensity emerging from the two pinholes, the middle $xz$-slice shows the beginning of the formation of interference as the Gaussian packets begin to overlap, while the final $xz$-slice shows a fully formed interference pattern.

The animation sequence for the quantum potential $Q$ for EQEA is shown in figure 8. The animation ranges are $x = -3.5 \times 10^{-6}$ to $3.5 \times 10^{-6}$ m, $z = -3.5 \times 10^{-6}$ to $3.5 \times 10^{-6}$ m, $Q = -1 \times 10^{-23}$ to $1 \times 10^{-23}$ J and $t = (0-2.6923) \times 10^{-9}$ s. This time we used the same pinhole to detecting screen separation, $0.35$ m, (corresponding to a time of flight of $t = 2.6923 \times 10^{-9}$ s) as Jönsson, since this resulted in a clear plateau-valley formation in the final frame. The first frame shows that the quantum potential is restricted to the width of the two pinholes. The second frame shows the beginning of the formation of quantum potential plateaus and valleys corresponding to the beginning of the overlap of the Gaussian wave packets. Subsequent frames show the continued widening of the plateaus and the deepening of the valleys. The final frame, as mentioned, shows clear plateau and valley formation. The gradient of the quantum potential

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**Figure 15.** The trajectories in a two pinhole interference experiment with equal widths but unequal amplitudes modeled by two-dimensional Gaussian wave-packets with equal widths but unequal amplitudes.
gives rise to a quantum force. Where the gradient is zero, as on the flat plateaus, the quantum force is zero and electrons progress along their trajectory to a bright fringe on the detecting screen unhindered. At the edges of the plateaus the quantum potential slopes steeply down to the valleys. The steep gradient of these slopes gives rise to a large quantum force that pushes particles with trajectories along these slopes to adjacent plateaus, after which they proceed unhindered to a bright fringe on the detecting screen. In this way, the quantum potential guides the electrons to the bright fringes and

Figure 16. The intensity in a two-pinhole interference experiment with unequal widths but equal amplitudes modeled by two-dimensional Gaussian wave-packets with unequal widths but equal amplitudes.
prevents electrons reaching the dark fringes. Note though, as mentioned earlier, that since the $R$ and $S$-fields codetermine each other, the $S$-field can also be said to guide the electrons to the bright fringes.

A sequence of density plots (3 slices of a 3D-plot) of the quantum potential for EWEA is shown in figure 9. The plot ranges are $x = -3 \times 10^{-6}$ to $3 \times 10^{-6}$ m, $z = -3 \times 10^{-6}$ to $3 \times 10^{-6}$ m and $t = (0-1.5) \times 10^{-9}$ s. When producing the density animation we found that part of the image in the $t = 0$ s frame was missing, hence, we have left out this frame, beginning instead with the $t = 3 \times 10^{-10}$ s frame. The reason for the missing image is not clear, but is most likely due to the density plotting algorithm not handling difficult numbers very well, unlike the 3D plotting algorithm. The first slice shows the beginning of the overlap of the Gaussian packets and the beginning of plateau and valley formation. The middle slice shows the more developed plateaus and valleys, while the final slice shows distinct plateaus and valleys. The wide bright blue bands indicate the quantum potential plateaus where the quantum force is zero. The narrower dark bands show the quantum potential sloping down to the valleys, slopes were electrons experience a strong quantum force. The darker the bands the steeper the quantum potential slopes.

The trajectories for equal widths and equal amplitudes is shown in figure 10. The trajectory ranges are $x = -3.5 \times 10^{-6}$ to $3.5 \times 10^{-6}$ m, $z = -3.5 \times 10^{-6}$ to $3.5 \times 10^{-6}$ m and $t = (0-1.5) \times 10^{-9}$ s. Though the axes are labelled at the edges, the plots correspond to axes with their origin placed centrally between the pinholes. We have chosen to label the axis at the edges of the plot frame in order to show the trajectories clearly. In a real experiment, the initial position of the electrons can lie anywhere within the pinholes. But, to clearly show the behavior of the trajectories we have chosen square initial positions within each pinhole. It is clear, that interference occurs only along the $x$-direction; there is no interference along the $z$-direction. We also see clearly how the quantum potential (and $S$-field) guides the electron trajectories to the bright fringes. Electrons whose trajectories lie within the quantum potential plateaus, therefore experiencing no quantum force, move along straight trajectories to the bright fringes. Electrons whose trajectories lie along the quantum potential slopes are pushed by the quantum force to an adjacent plateau, thereafter proceeding along straight trajectories to the bright fringes.

4.2. Computer plots for equal pinhole widths and unequal amplitudes

The animation sequence for the intensity $R^2$ for equal widths and unequal amplitudes (EWUA) is shown in figure 11. The animation ranges are $x = -3.5 \times 10^{-6}$ to $3.5 \times 10^{-6}$ m, $z = -3.5 \times 10^{-6}$ to $3.5 \times 10^{-6}$ m, $R^2 = (0-1.8) \times 10^{11}$ J m$^{-2}$ s$^{-1}$ and $t = (0-1.5) \times 10^{-9}$ s. As indicated in table 1, for the case EWUA the angle $b = \frac{\pi}{2}$. This results in an increase in the intensity through the $+x_0$-pinhole from $\frac{2}{3}$ to $\frac{1}{3}$ and a reduction in the intensity at the $-x_0$-pinhole from $\frac{1}{3}$ to $\frac{2}{3}$. From figure 11, we see that as the Gaussian wave-packets begin to combine to form a single peak envelope with interference fringes beginning to form, the intensity peak is shifted toward the larger intensity $+x_0$-pinhole. This shift becomes less pronounced, almost disappearing, as the interference fringes become more distinct as in the last $t = 1.5 \times 10^{-9}$ s frame. Comparing the $t = 1.5 \times 10^{-9}$ frame for EWUA with the corresponding intensity frame for EWEA we can see visually that fringe visibility is reduced. We can confirm this visual observation by calculating the visibility of the central fringe:

$$V_{\text{EWUA}} = \frac{l_{\text{max}} - l_{\text{min}}}{l_{\text{max}} + l_{\text{min}}} = \frac{8 \times 10^{10} - 2 \times 10^{10}}{8 \times 10^{10} + 2 \times 10^{10}} = 0.6.$$  

Clearly, the visibility is lower for the EWUA case.

A sequence of density plots (3 slices of a 3D-plot) of the intensity $R^2$ for EWUA is shown in figure 12. The plot ranges are $x = -4 \times 10^{-6}$ to $4 \times 10^{-6}$ m, $z = -4 \times 10^{-6}$ to $4 \times 10^{-6}$ m and $t = (0-1.5) \times 10^{-9}$ s. Again, comparing with the EWUA case, we see that the intensity is reduced by noticing that the dark bands are not as distinct as for the EWEA case.

The animation sequence for the quantum potential $Q$ for EWUA is shown in figure 13. The animation ranges are $x = -3.5 \times 10^{-6}$ to $3.5 \times 10^{-6}$ m, $z = -3.5 \times 10^{-6}$ to $3.5 \times 10^{-6}$ m, $Q = -2 \times 10^{-25}$ to $4 \times 10^{-25}$ J and $t = (0-1.5) \times 10^{-9}$ s. The quantum potential in the early frames, perhaps unexpectedly, peaks on the side of the lower intensity $-x_0$-pinhole. This behavior is most pronounced in frames two and three. As the peaks and valleys become more pronounced, the envelope peak spreads and flattens as shown in frames 5 and 6. However, the valleys on the side of the lower intensity $-x_0$-pinhole are deeper. Correspondingly, the gradient of the quantum potential sloping down to the deeper valleys is greater giving rise to a stronger quantum force. This results in the formation of more distinct fringes on the side of the lower intensity pinhole. This feature is hardly visible in

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Figure 17. A sequence of density plots (3 slices of a 3D-plot) of the intensity $R^2$ in a two-pinhole interference experiment with unequal widths and equal amplitudes modeled by two-dimensional Gaussian wave-packets with unequal widths but equal amplitudes.
either the intensity animation frames, figure 11, or in the intensity density plots, figure 12. However, as we shall see below, the trajectory plots, figure 15, shows this feature more clearly.

A sequence of density plots (3 slices of a 3D-plot) of the quantum potential for EWUA is shown in figure 14. The plot ranges are $x = -3 \times 10^{-6}$ to $3 \times 10^{-6}$ m, $z = -3 \times 10^{-6}$ to $3 \times 10^{-6}$ m and $t = (0–1.5) \times 10^{-9}$ s. As for the EWEA...
case, the first $t = 0$ s slice is not shown, this time, because the image is badly formed. Instead, we begin with the $t = 3 \times 10^8$ s slice. This slice clearly shows that the deeper valleys, indicated by blacker bands, are on the side of lower intensity $-x_0$-pinhole. As above, the bright blue bands represent regions where the quantum potential gradient is either zero or very small, giving rise to either a zero or small quantum force. In the final $t = 1.5 \times 10^9$ s slice, the peaks and valleys even out, though a slight bias to deeper valleys on the $-x_0$ side is still discernible.

The trajectories for EWUA are shown in figure 15. The trajectory ranges are $x = -4.5 \times 10^{-6}$ to $4.5 \times 10^{-6}$ m, $z = -4.5 \times 10^{-6}$ to $4.5 \times 10^{-6}$ m and $t = (0-1.5) \times 10^{-9}$ s. We notice that some electron trajectories reach what were the dark regions for the case of EWUEA. This indicates the reduction of fringe visibility that we saw above in the intensity plots for this case. We also notice that this reduced intensity is less pronounced on the side of the lower intensity $-x_0$-pinhole, so that interference fringes on this side are more distinct, a feature we noted above for the quantum potential for this case. The overall reduction in visibility is clear to see.

**4.3. Computer plots for unequal pinhole widths and equal amplitudes**

The animation sequence for the intensity $R^2$ for unequal widths and equal amplitudes is shown in figure 16. The animation ranges are $x = -3.5 \times 10^{-6}$ to $3.5 \times 10^{-6}$ m, $z = -3.5 \times 10^{-6}$ to $3.5 \times 10^{-6}$ m, $R^2 = (0-1.8) \times 10^{11}$ J m$^{-2}$ s$^{-1}$ and $t = (0-1.5) \times 10^{-9}$ s. From table 1, we note that the width of the $+x_0$ Gaussian wave-packet is twice that of the $-x_0$ Gaussian wave-packet. The first frame clearly shows the narrower $-x_0$ wave-packet. The narrower wave-packet spreads more rapidly than a wider packet, as seen in the second frame. The more rapid spread of the narrower wave-packet results in the wave-packets beginning to overlap on the $+x$-side, as is shown in the second frame. As the wave-packets spread, the interference pattern becomes ever more distinct. Though becoming a little more symmetrical about $x = 0$, the fringe pattern is shifted toward the $+x$-side, with the fringes on the $+x$-side being slightly more pronounced.

This feature is seen more clearly in the intensity density plots, which we will describe next. The visibility of the central fringe for this case is:

$$V_{UWEA} = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}} = \frac{1.6 \times 10^{11} - 0.1 \times 10^{11}}{1.6 \times 10^{11} + 0.1 \times 10^{11}} = 0.88.$$ We see that the visibility is less than in the EWEA case, but greater than in the EWUA case.

We may note that the peak intensity for the case UWEA (see figure 16) is greater than the peak intensities for the cases EWEA (see figure 6) and EWUA (see figure 11). This is because the normalization factor for the wavefunction $\psi$ is different for each case, with that for the case UWEA being the least, resulting in the largest peak intensity. However, the peak intensity is not the indicator of fringe visibility so that the fringe visibility for the case UWEA, calculated by the usual formula, is less than for the case EWEA, as expected.

A sequence of density plots (3 slices of a 3D-plot) of the intensity $R^2$ for unequal widths and equal amplitudes is shown in figure 17. The plot ranges are $x = -4 \times 10^{-6}$ to $4 \times 10^{-6}$ m, $z = -4 \times 10^{-6}$ to $4 \times 10^{-6}$ m and $t = (0-1.5) \times 10^{-9}$ s. Again, the first slice clearly shows the difference in the size of the pinholes, while the second slice shows interference fringes beginning to form on the $+x$-side. The final slice shows that the interference pattern develops into a more symmetric form, though still shifted more to the $+x$-side and slightly more pronounced on this side. The dark bands are less distinct than in the EWEA case, reflecting the reduction in fringe visibility for this case.

The animation sequence for the quantum potential $Q$ for unequal widths and equal amplitudes is shown in figure 18. The animation ranges are $x = -3.5 \times 10^{-6}$ to $3.5 \times 10^{-6}$ m, $z = -3.5 \times 10^{-6}$ to $3.5 \times 10^{-6}$ m, $Q = -2 \times 10^{-25}$ to $4 \times 10^{-25}$ J and $t = 0$ to $1.5 \times 10^{-9}$ s. As for the interference animation, the first frame shows the difference in size of the pinholes, while the second frame shows the more rapid spread of the narrower wave-packet. The overlap of the wave-packets, as for the intensity, begins on the $+x$-side, as does the formation of plateaus and valleys. Subsequent frames show the skewed formation to the $+x$-side of plateaus and valleys. The shift to the $+x$-side is maintained even in the last frame, even though the pattern looks more symmetrical. This can be seen by noticing that in the last frame there are three quantum potential peaks on the $+x$-side, compared to two peaks on the $-x$-side.

A sequence of density plots (3 slices of a 3D-plot) of the quantum potential for unequal widths and equal amplitudes is shown in figure 19. The plot ranges are $x = -3 \times 10^{-6}$ to $3 \times 10^{-6}$ m, $z = -3 \times 10^{-6}$ to $3 \times 10^{-6}$ m and $t = (0-1.5) \times 10^{-9}$ s. As with the quantum potential density plots for the EWUA case, the image in the first slice is problematic. This time, the image is complete but it does not reflect the two-pinhole structure. As before, this is probably because the density plotting algorithm does not handle difficult numbers very well. The middle slice shows the early formation of plateaus and valleys skewed to the $+x$-side, with the plateau regions narrower than the valley regions. In the final slice, the plateau regions become wider than the valley.
regions, but are less distinct as compared to the EWEA case, or even as compared to the EWUA case. Despite this, the fringe visibility is greater than for the EWUA case, though of course, less than for EWEA case, as we saw above. Again, the shift of quantum potential peaks to the $+x$-side can be seen.

The trajectories for unequal widths and equal amplitudes is shown in figure 20. The trajectory ranges are $x = -4 \times 10^{-6}$ to $4 \times 10^{-6}$ m, $z = -4 \times 10^{-6}$ to $4 \times 10^{-6}$ m and $t = (0-1.5) \times 10^{-9}$ s. The electron trajectories clearly show the rapid spread of the narrower $-x_0$-Gaussian wave-packet. It can also be seen that electron trajectories from the $-x_0$-pinhole are more evenly spread on the detecting screen parallel to the $x$-axis than for the EWEA case, indicating a reduced interference pattern. The electron trajectories from the $+x_0$-pinhole, spread much less. Though only one bright and one dark fringe is shown on the $+x$-side, they appear more distinct than on the $-x$-side, reflecting the shift of the interference fringes to the $+x$-side.

5. Conclusion

We have seen that the behavior of the intensity, quantum potential and electron trajectories is similar to that for the two-slit experiment modeled by one-dimensional Gaussian wave-packets. In particular, the distinctive kinked behavior of the electron trajectories is seen in directions parallel to the $x$-axis. We saw in addition, as could possibly be guessed at the outset, that there is no interference in the vertical $z$-direction. We also saw the expected reduction in interference for the cases of unequal amplitudes and unequal widths. The reduction in interference is interpreted, as in the classical case, in terms of wave profiles with reduced coherence. This is a far more intuitive explanation for the reduction in fringe visibility (and in my view more appealing) than the common interpretation based on the Wootters–Zureck version of complementarity [24], where the reduction of interference for the cases of unequal amplitudes and unequal widths is attributed to partial particle behavior and partial wave behavior. The partial particle behavior is attributed to the increase in knowledge of the electrons path in the sense that an electron is more likely to pass through the larger pinhole or the pinhole with the larger intensity. In [25] and [17] we argued that the Wootters–Zureck version of complementarity as commonly interpreted actually contradicts BPC. In [17] we also indicated that by reference to two future, mutually exclusive experimental arrangements, an interpretation of the Wootters–Zureck version of complementarity consistent with BPC can be achieved.

Using the weak measurement protocol introduced by Aharanov, Albert and Vaidman (see [26] for a brief overview and further references), Kocsis et al reproduced experimentally Bohm’s trajectories in a two-slit interference experiment [27]. We might guess that it would not be difficult to modify the experiment slightly to reproduce the electron trajectories calculated here for the case of unequal widths and the case of unequal amplitudes.

Another experiment that might be interesting to model is the neutron partial absorption experiment of Summhammer et al [28, 29]. In this experiment the interference in the output channel of a neutron interferometer is measured for two different arrangements. In one arrangement an absorber is placed in one arm of the interferometer, while in the other arrangement a rotating toothed wheel (chopper) is placed in one arm.
such that the same proportion of the neutron beam in that arm is removed. It was observed that though interference is reduced as compared to the equal intensity beam case, the degree of reduction is different for the two cases. Summhammer et al [28] explain the difference in terms of loss of information. They argue that though the intensity of one of the neutron beams is reduced by the same amount, the degree of information lost is different for the two cases: the loss in the chopper case (deterministic) is greater than the information loss for the absorber (stochastic) case. Instead, Kaloyerou and Brown [29] argue that the difference in the degree of interference is due, simply, to the difference in the coherence of the two wave profiles, just as in classical theory. The latter explanation can be supported by a computer model of each of the two arrangements. The absorber is easily modeled by reducing the amplitude of one of the beams just as was done in the EWUA case in this article. On the other hand, the chopper arrangement can be modeled by representing the chopper arm neutron beam by a series of narrow wavepackets separated according to the rotation speed of the chopper, while the neutron beam in the other arm can be represented by a larger width single wavepacket.

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