On twin observables in entangled mixed states

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Abstract. It is pointed out that every mixed-state statistical operator is, up to a normalization constant, a super state vector in the Hilbert space of linear Hilbert-Schmidt operators acting in the state space of the quantum system. Hence, the well understood Schmidt canonical expansion of ordinary state vectors can be carried over to mixed states. In particular, it can be utilized for evaluating all the twins, i.e., the opposite-subsystem observables the measurement of one of which is, on account of entanglement, ipso facto also a measurement of the other. This is illustrated in full detail in the case of the Horodecki two spin-one-half-particle states with maximally disordered subsystems.

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1. Introduction

Recently twin observables, which make possible distant measurement, i.e., distant orthogonal state decomposition on account of entanglement, have been extended from the pure state case \([1]\) to the mixed states \([2]\). The importance of twins for entanglement studies, in particular for quantum communication and quantum information theory, was pointed out.

Statistical operators (physically: quantum states) are elements of the Hilbert-Schmidt space of operators or supervectors, in which the scalar product is defined by

\[
(A,B) \equiv \text{Tr} A^\dagger B.
\]

Hence, in many cases the very well understood twin theory in pure states can be formally carried over to the mixed state case. This is illustrated by an example.

We make use of the Schmidt canonical expansion of an arbitrary state vector \(|\Phi\rangle_{12}\) of a composite system \([1]\). It is expressed in terms of its reduced statistical operators (subsystem states) \(\rho_1 (\equiv \text{Tr}_2 |\Phi\rangle_{12}\langle\Phi|_{12})\) and \(\rho_2\) (defined symmetrically), and their spectral forms

\[
\rho_1 = \sum_i r_i |i\rangle_1 \langle i|_1, \quad \rho_2 = \sum_i r_i |i\rangle_2 \langle i|_2, \quad \forall i : \quad r_i > 0.
\]  

(1a, b)

(Note that the positive spectra are always equal for pure states.) Further, the mentioned expansion utilizes the (antiunitary) correlation operator \(U_a\), which maps the range \(\mathcal{R}(\rho_1)\) onto the range \(\mathcal{R}(\rho_2)\). (Note that they are always equally dimensional for pure states.) The correlation operator is determined by \(|\Phi\rangle_{12}\), and, in conjunction with \(\rho_1\), it determines back \(|\Phi\rangle_{12}\). The Schmidt canonical expansion reads:

\[
|\Phi\rangle_{12} = \sum_i r_i^{1/2} |i\rangle_1 \otimes (U_a |i\rangle_1)_2.
\]  

(2)

The normalized vectors \(|i\rangle_2\) in (1b) may (and need not) be equal to \(U_a |i\rangle_1\).

For further use it is important to note that all the mentioned canonical entities of \(|\Phi\rangle_{12}\) are simply read off relation (2), because the Schmidt canonical expansion is any biorthogonal expansion in state vectors with positive expansion coefficients.

Hermitian twin operators (twin observables) or simply twins \(A_1\) and \(A_2\) are opposite-subsystem observables characterized by the algebraic relation

\[
A_1 |\Phi\rangle_{12} = A_2 |\Phi\rangle_{12},
\]  

(3)

where \(A_1\) stands for \(A_1 \otimes I_2\), and \(I_2\) stands for the second subsystem identity operator, etc. It was shown that a Hermitian operator \(A_1\) is a twin if and only if it commutes with \(\rho_1\). The corresponding twin \(A_2\) then satisfies

\[
A_2 = U_a A_1 U_a^{-1} Q_2 + A_2 Q_2^\perp,
\]  

(4)

where \(Q_2\) is the range projector of \(\rho_2\), and \(Q_2^\perp\), its orthocomplementary projector, projects onto the null space of \(\rho_2\).
One should note that, on account of symmetry, also \([A_2, \rho_2] = 0\), hence both the range and the null space of \(\rho_2\) are invariant for \(A_2\). Further, the second term on the RHS of (4) (the component in the null space) is completely arbitrary and immaterial for the twin property (3), because it acts as zero on \(|\Phi\rangle_{12}\). (Naturally, the symmetric claim holds true for \(A_1\) and \(\rho_1\).)

2. MDS states

Now we turn to the example that is, for illustrative purposes, investigated in this study, i.e., to states (statistical operators) \(\rho\) in \(C^2 \otimes C^2\). We say that \(\rho\) is an MDS state (one with maximally disordered subsystems or rather subsystem states) if \(\rho_1 = (1/2)I_1\) and \(\rho_2 = (1/2)I_2\). R. and M. Horodecki have shown [3] that for every MDS state there exist unitary subsystem operators \(U_1\) and \(U_2\) such that

\[
\left(U_1 \otimes U_2\right)\rho\left(U_1^\dagger \otimes U_2^\dagger\right) = \frac{1}{4}\left(I \otimes I + \sum_{i=1}^{3} t_i \sigma_i \otimes \sigma_i\right) \equiv T,
\]

where \(\sigma_i, i = 1, 2, 3\), are the well known Pauli matrices \(\sigma_x, \sigma_y\) and \(\sigma_z\); and it is seen from their place in the expression if they are meant for the first or for the second spin-one-half particle.

Further, they have shown that the operator \(T\) is a statistical operator (a quantum state) if and only if the vector \(\vec{t}\) from \(\mathbb{R}^3\) the components of which appear in (5) is not outside the tetrahedron determined as the set of all mixtures of the four pure Bell states:

\[
|\psi^1_2\rangle \equiv (1/2)^{1/2}\left(|+\rangle |+\rangle + |+\rangle |-\rangle - |-\rangle |+\rangle\right), \quad |\psi^3_0\rangle \equiv (1/2)^{1/2}\left(|+\rangle |-\rangle + |-\rangle |+\rangle\right),
\]

where \(|+\rangle\) are the spin-up and the spin-down state vectors respectively.

It is straightforward to see that the three nonsinglet Bell states \(|\psi_s\rangle, s = 1, 2, 3\), when written in the form (5), are given by \(t_s = -1\), and the other two components of \(\vec{t}\) equal to +1. The singlet state \(|\psi_0\rangle\) is in the form (5) determined by all three components of \(\vec{t}\) being equal to \(-1\).

It is also easy to see that for all mixtures one has

\[-1 \leq t_i \leq +1 \quad i = 1, 2, 3.\]

This is a necessary, but not a sufficient condition for \(T\) being a state. In other words, the tetrahedron is embedded in a cube, in which there are also nonphysical \(\vec{t}\). In view of the LHS of (5), we call \(T\) that belong to the tetrahedron: generating MDS states.

What we want to find out is: Which of the MDS states have nontrivial twins? For those that do have, we want to find the set of all nontrivial pairs of twins.

The importance of the question, as it was stated, lies in the fact that nontrivial twins and only they make possible distant orthogonal state decomposition (measurement of \(A_2\) without interaction, on account of entanglement) by a measurement (of \(A_1\)) on the nearby subsystem.
It is sufficient to find the generating MDS states $T$ with nontrivial twins, because the validity of

$$A_1 T = A_2 T$$

obviously implies

$$
\left( U_1 A_1 U_1^\dagger \right) \left( U_1 U_2 T U_1^\dagger U_2^\dagger \right) = \left( U_2 A_2 U_2^\dagger \right) \left( U_1 U_2 T U_1^\dagger U_2^\dagger \right),
$$

i. e., if the generating MDS states have nontrivial twins, then also the generated MDS states do have nontrivial twins, and they are immediately obtained.

As far as the pure generating MDS states (the Bell states) are concerned, the first-particle reduced statistical operator $\rho_1$ is equal to $(1/2) I_1$, all nontrivial Hermitian operators $A_1$ commute with it, hence, they are twins. To evaluate the corresponding twin $A_2$, one has to read off the antilinear correlation operator $U_a$ from (6) having in mind (2), and then utilize (4). For the best known Bell state, the singlet state $|\psi_0\rangle$, e. g., $U_a$ takes $|+\rangle$ into $|-\rangle$, and $|-\rangle$ into $(-|+\rangle)$ (cf (6)). If

$$A_1 = \alpha_{++} |+\rangle\langle+| + \alpha_{--} |-\rangle\langle-| + \alpha_{+-} |+\rangle\langle-| + (\alpha_{-+})^* |-\rangle\langle+|,$$

$$\alpha_{++}, \alpha_{--} \in \mathbb{R}, \quad \alpha_{+-} \in \mathbb{C},$$

then the twin $A_2$ has the form:

$$A_2 = \alpha_{--} |+\rangle\langle+| + \alpha_{++} |-\rangle\langle-| - \alpha_{+-} |+\rangle\langle-| - (\alpha_{-+})^* |-\rangle\langle+| .$$

Now we turn to the mixtures of Bell states in our search for nontrivial twins.

3. Mixtures of Bell states

Viewing statistical operators as super vectors, and utilizing (redundantly, but for the sake of better overview) the ket notation for super state vectors (i. e., Hilbert-Schmidt operators as normalized super vectors), one can rewrite the generating vectors $T$ given by (5) as a biorthogonal expansion with positive expansion coefficients:

$$|T||T|^{-1}\rangle_{12} = (1 + \sum_{i=1}^{3} t_i^2)^{-1/2} \left( \left| (1/2)^{1/2} I \right|_{1} \otimes \left| (1/2)^{1/2} I \right|_{2} + \right. + \\
\left. \sum_{i=1}^{3} |t_i| \left| (1/2)^{1/2} \sigma_i \right|_{1} \otimes \left| sg(t_i)(1/2)^{1/2} \sigma_i \right|_{2} \right)$$

(7)

(”sg” denotes the sign), i. e., as a (super state vector) Schmidt canonical expansion.

One can read off (7) the following canonical entities of the super state vector $|T||T|^{-1}\rangle_{12}:

The first-subsystem reduced statistical super operator $\hat{\rho}_1$ has the characteristic super state vectors $\{| (1/2)^{1/2} I \rangle_{1}, \left| (1/2)^{1/2} \sigma_i \right|_{1} : i = 1, 2, 3 \}$; the second subsystem reduced statistical super operator $\hat{\rho}_2$ has the characteristic state vectors
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\{ | (1/2)^{1/2} I_2 \}, | s g(t_i)(1/2)^{1/2} \sigma_i \rangle_2 : i = 1, 2, 3 \}; and the common spectrum of \( \hat{\rho}_1 \) and \( \hat{\rho}_2 \) is \( \{ R_0 \equiv (1 + \sum_{i=1}^{3} t_i^2)^{-1}, R_i \equiv R_0 t_i^2 : i = 1, 2, 3 \} \). Finally, the antiunitary correlation super operator \( \hat{U}_a \) maps the enumerated characteristic state vectors of \( \hat{\rho}_1 \) into the correspondingly ordered ones of \( \hat{\rho}_2 \).

4. Nontrivial MDS twins

Every super operator \( \hat{A}_1 \) that commutes with \( \hat{\rho}_1 \), i.e., for which every characteristic subspace of the latter is invariant (and no other super operator), has a twin super operator \( \hat{A}_2 \). But we are interested only in those pairs \((\hat{A}_1, \hat{A}_2)\), both super operators, which are, what may be called, *multiplicative* ones, i.e., which have the form

\[ \hat{A}_1 \rho_{12} = A_1 \rho_{12}, \quad \hat{A}_2 \rho_{12} = A_2 \rho_{12}, \]

where \( A_p, p = 1, 2 \), are ordinary (subsystem) operators. It is easy to see that a multiplicative super operator is *Hermitian* (in the Hilbert-Schmidt space of supervectors) if so is the ordinary operator (in the usual sense) that determines it.

The basic result of this study is given in the following two theorems:

*Theorem 1.* Mixed generating MDS states have nontrivial twins if and only if they are mixtures of two Bell states (binary mixtures).

*Theorem 2.* A) Let us take a binary mixture of two Bell states both distinct from the singlet one, and let \( T_i \equiv | \psi_i \rangle \langle \psi_i | \) (cf (6)) be the nonsinglet Bell state that does not participate in the mixture. Then the nontrivial twins are:

\[ A_1 \equiv \alpha I_1 + \beta \sigma_i^{(1)}, \quad A_2 \equiv \alpha I_2 + \beta \sigma_i^{(2)}, \quad \alpha, \beta \in \mathbb{R}, \quad \beta \neq 0, \]

where the suffix on \( \sigma_i \) refers to the corresponding tensor factor space.

B) In case of a binary mixture of the singlet state with another Bell state, say \( T_i \equiv | \psi_i \rangle \langle \psi_i | \) (cf (6)), the twins are:

\[ A_1 \equiv \alpha I_1 + \beta \sigma_i^{(1)}, \quad A_2 \equiv \alpha I_2 - \beta \sigma_i^{(2)}, \quad \alpha, \beta \in \mathbb{R}, \quad \beta \neq 0. \]

Proof of the two theorems and of some subsidiary results is given in the Appendix.

All the binary generating MDS mixtures \( T^{(2)} \) appear to be nonseparable. But this actually is not so.

It is shown elsewhere [4] that a necessary and sufficient condition for a *separable* composite-system state to have *nontrivial twins* is that the terms can be grouped into *biorthogonal* groups of terms (as many groups as there are detectable characteristic values of the twin Hermitian operator).

According to Theorem 1, all binary generating MDS states \( T^{(2)} \) do have nontrivial twins. On the other hand, it is easily seen that *only two* of these states satisfy the
mentioned condition for separable states. As one can easily ascertain making use of (6), they are:

\[
\frac{1}{2} \left( (|+\rangle \langle +| \otimes |+\rangle \langle +|) + (-|\rangle \langle -| \otimes |+\rangle \langle +|) \right) = \\
\frac{1}{2} \left( |\psi_1\rangle \langle \psi_1| + |\psi_2\rangle \langle \psi_2| \right),
\]

(10a)

and

\[
\frac{1}{2} \left( (|+\rangle \langle +| \otimes |-\rangle \langle -|) + (-|\rangle \langle -| \otimes |+\rangle \langle +|) \right) = \\
\frac{1}{2} \left( |\psi_0\rangle \langle \psi_0| + |\psi_3\rangle \langle \psi_3| \right).
\]

(10b)

The proof of Theorem 2 (given in the Appendix) is only of methodological significance: it illustrates a method how to evaluate nontrivial twins. In our case of binary mixtures \(T^{(2)}\), another method gives a simpler evaluation.

**Theorem 3.** A pair of opposite-subsystem observables \((A_1, A_2)\) are twins for a composite-system mixture

\[\rho = \sum_n w_n |\Psi_n\rangle \langle \Psi_n|\]

if and only if they are simultaneously twins for each of the pure term states.

(This is one of the results of a previous article [2], section 3., C2.) Proceeding as outlined in the Introduction, it is straightforward to evaluate the twins in the operator basis consisting of the four supervectors \(|+_\rangle \langle +|\). But for comparison with the results (8) and (9) obtained by the Schmidt canonical expansion method, to which this article is devoted, we do this in a little bit more difficult way using the form (7) for the Bell states (see their description beneath (7)).

We can read off the antiunitary correlation superoperator \(\hat{U}_a\) from the mentioned form (7) of the Bell state. As it was stated before, every first-subsystem observable \(A_1 \equiv \alpha I_1 + \sum_{i=1}^{3} \beta_i \sigma_i^{(1)}\), \((\alpha, \beta_i \in \mathbb{R}, \ i = 1, 2, 3)\), is a twin. The corresponding second-subsystem twins for the Bell states are:

\[
T_1 : \quad A_2 \equiv \alpha I_2 - \beta_1 \sigma_1^{(2)} + \beta_2 \sigma_2^{(2)} + \beta_3 \sigma_3^{(2)};
\]

\[
T_2 : \quad A_2 \equiv \alpha I_2 - \beta_1 \sigma_1^{(2)} - \beta_2 \sigma_2^{(2)} + \beta_3 \sigma_3^{(2)};
\]

\[
T_3 : \quad A_2 \equiv \alpha I_2 + \beta_1 \sigma_1^{(2)} + \beta_2 \sigma_2^{(2)} - \beta_3 \sigma_3^{(2)};
\]

\[
T_0 : \quad A_2 \equiv \alpha I_2 - \beta_1 \sigma_1^{(2)} - \beta_2 \sigma_2^{(2)} - \beta_3 \sigma_3^{(2)}.
\]

Now, in view of the position of the minus sign in \(A_2\), evidently, utilizing \(m \neq i \neq j \neq m\ \ i, j, m \in \{1, 2, 3\}\), and \(0 < w < 1\), the simultaneous twins are:

\[
wT_j + (1-w)T_m : \quad A_2 \equiv \alpha + \beta_i \sigma_i;
\]

\[
wT_0 + (1-w)T_i : \quad A_2 \equiv \alpha - \beta_i \sigma_i;
\]

and \(A_2\) is, of course, the twin of \(A_1 \equiv \alpha + \beta_i \sigma_i\).

In this way proof of (8) and (9) is obtained.
5. Appendix

Since we are going to prove the theorems making use of (7), first we must be able to recognize the binary mixtures $T^{(2)}$ on the Horodecki tetrahedron.

Proposition A.1. One has a binary mixture $T^{(2)}$ if and only if precisely one of the three $|t_i|$ values in (7) equals 1.

A) If $t_i = +1$, $|t_{i+1}|, |t_{i+2}| < 1$ (where the three values $\{1, 2, 3\}$ of $i$ are meant cyclically), then the mixture is of two Bell states both distinct from the singlet state. If $T_i$ is the nonsinglet Bell state that does not participate in the mixture, one has $t_{i+2} = -t_{i+1}$. Finally, the binary mixture $T^{(2)}$ in question is

$$T^{(2)} = \left[ (1 - t_{i+1})/2 \right] T_{i+1} + \left[ (1 - t_{i+2})/2 \right] T_{i+2}. \quad (A1)$$

B) If $t_i = -1$, $|t_{i+1}|, |t_{i+2}| < 1$ (in the cyclic sense), then one deals with a mixture of two states: the singlet state and another Bell state $T_i$. One has $t_{i+1} = t_{i+2}$, and the binary mixture $T^{(2)}$ in question is

$$T^{(2)} = \left[ (1 + t_{i+1})/2 \right] T_i + \left[ (1 - t_{i+1})/2 \right] T_0. \quad (A2)$$

Both in the cases (A) and (B), $t_{i+1}$ can be any number in the interval $-1 \leq t_{i+1} \leq +1$; equivalently, one can have any point on the corresponding border of the Horodecki tetrahedron (the vertices excluded).

For proof a few subsidiary results are required.

Lemma A.1. If among the four numbers $\{1, |t_i| : i = 1, 2, 3\}$ appearing in the form (7) of the generating MDS state $T$ there is one distinct from the rest, then $T$ has no nontrivial twins.

Proof. As clearly follows from the above stated spectrum of $\hat{\rho}_1$, the mentioned "one number distinct from the rest" corresponds to a nondegenerate characteristic value. Assuming that $A_1$ is a twin, it is a multiplicative superoperator reducing in each characteristic subspace of $\hat{\rho}_1$. (This is equivalent to commutation with $\hat{\rho}_1$.)

a) Let us take the case when $|t_i| < 1, \ i = 1, 2, 3$. Then the first characteristic value of $\hat{\rho}_1$ is nondegenerate, and the corresponding characteristic super state vector has to be invariant (up to a constant):

$$A_1(1/2)^{1/2}I_1 = \alpha(1/2)^{1/2}I_1,$$

i.e., $A_1 = \alpha$, and the twin is trivial.

b) Let $|t_i|$ for some value of $i$ be distinct from the other three numbers. Then the corresponding characteristic super state vector $\sigma_i(1/2)^{1/2}$ must be invariant (up to a constant):

$$A_1\sigma_i(1/2)^{1/2} = \alpha\sigma_i(1/2)^{1/2},$$
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which, upon multiplication with $\sigma_i$ from the right, implies $A_1 = \alpha$ again.

\[8\]

Corollary A.1. If a generating MDS state $T$ has nontrivial twins, then for at least one value of $i$: $|t_i| = 1$.

Proof is obvious from Lemma A.1.

\[\square\]

Lemma A.2. Expressing a generating MDS state $T$ written in the form (7) in terms of the statistical weights with respect to the Bell states $\{T_k \equiv |\psi_k\rangle\langle \psi_k|: k = 0, 1, \ldots, 3\}$ (cf (6)), one has:

\[T = \sum_{k=0}^{3} w_k T_k = (1/4) \left[ I \otimes I + (-w_1 + w_2 + w_3 - w_0)\sigma_1 \otimes \sigma_1 + (w_1 - w_2 + w_3 - w_0)\sigma_2 \otimes \sigma_2 + (w_1 + w_2 - w_3 - w_0)\sigma_3 \otimes \sigma_3 \right], \quad (A3)\]

where

\[\forall k: \quad w_k \in [0, 1], \quad k = 0, 1, 2, 3; \quad \sum_{k=0}^{3} w_k = 1.\]

Proof is straightforward substituting the Bell states in (7) (cf (6) and beneath it).

\[\square\]

Lemma A.3. If one has $|t_i| = 1$, $i = 1, 2, 3$, for a generating MDS state $T$ in the form (7), then it is a Bell state.

Proof. Each $t_i$ has two sign possibilities; altogether there are $2^3 = 8$ possibilities. A straightforward analysis of each of these, taking into account Lemma A.2 and $\sum_{k=0}^{3} w_k = 1$, shows that 4 possibilities do not give states. These are: $\{sg(t_i) = + : i = 1, 2, 3\}$, $\{+ - -\}$, $\{- + -\}$, and $\{- - +\}$. The remaining four sign possibilities give the four Bell states:

\[\{- + +\} : T_1; \quad \{+ - +\} : T_2; \quad \{+ + -\} : T_3; \quad \{- - -\} : T_0.\]

Proof of claim (A) in Proposition A.1. Since it is clear from (A3) that the $t_i$ as functions of $w_k$ are symmetric (in the sense of the cycle $\{1, 2, 3\}$), it is sufficient to take $i = 1$. Then

\[-w_1 + w_2 + w_3 - w_0 = 1, \quad \text{and} \quad \sum_{k=0}^{3} w_k = 1.\]

This gives $w_2 + w_3 = 1$, $w_1 = w_0 = 0$, and $t_2 = w_3 - w_2 = -t_3$. Hence, $w_2 = (1 - t_2)/2$ and $w_3 = (1 + t_2)/2$ as claimed. Since $0 < w_1, w_0 < 1$, the claimed intervals for $t_2$ and $t_3$ follow.

\[\square\]
Proof of claim (B) of the Proposition. It runs in full analogy with the proof for case (A).

Proof of the main claim of the Proposition. It is easy to see that the proofs of claims (A) and (B) of the Proposition go through also for \(|t_{i+1}^t\) or \(|t_{i+2}^t|\) equalling one. Hence, one cannot have \(|t_i| = 1\) for precisely two values of \(i\). If it is so for one value, then either it is so for all three values - and one has a pure Bell state, or it is so for precisely one value of \(i\), then we have a binary mixture.

Proof of Theorem 2. We now assume that for one value of \(i\), \(|t_i| = 1\), and that the other two components of \(\bar{t}\) in (7) are by modulus less than one. Then it is sufficient and necessary for an observable \(A_1\) that defines a superoperator \(\hat{A}_1\) by multiplication (we write this as \(\hat{A}_1 \equiv (A_1 \bullet)\)) to have a superoperator twin \(A_2\) (that is not necessarily multiplicative as \(\hat{A}_1\)) that it reduces in the two-dimensional supervector subspace spanned by \(I_1\) and \(\sigma_i^{(1)}\). If we write \(A_1 = \alpha I_1 + \sum_{j=1}^{3} \beta_j \sigma_j^{(1)}\) \((\alpha, \beta_j \in \mathbb{R})\), and multiply with this from the left \(\sigma_i^{(1)}\), it turns out that the condition amounts to \(\beta_j = 0\), \(j \neq i\). The symmetrical argument gives the symmetrical result. Thus the multiplicative superoperators defined by \(A_1\) and, separately, by \(A_2\) do have superoperator twins if and only if they are of the form

\[
A_1 = \alpha I_1 + \beta \sigma_i^{(1)}, \quad A_2 = \gamma I_2 + \delta \sigma_i^{(2)},
\]

where \(\alpha, \beta, \gamma, \delta \in \mathbb{R}\).

The mentioned operators are twins of each other if and only if

\[
(A_2 \bullet) = \hat{U}_a (A_1 \bullet) \hat{U}_a^{-1}.
\]

Now we find out the necessary and sufficient conditions when (A5) is valid for the operators given by (A4). Since both sides of (A5) are linear operators, we apply them to the basis of supervectors \(\{I_2, \sigma_i^{(2)} : i = 1, 2, 3\}\):

\[
(A_2 \bullet) I_2 = \gamma I_2 + \delta \sigma_i^{(2)};
\]

\[
(\hat{U}_a (A_1 \bullet) \hat{U}_a^{-1}) I_2 = \hat{U}_a (\alpha I_1 + \beta \sigma_i^{(1)}) = \alpha I_2 + sg(t_i) \beta \sigma_i^{(2)}.
\]

Thus, we obtain the condition

\[
\gamma = \alpha, \quad \delta = sg(t_i) \beta.
\]

Utilizing the well known relation

\[
\sigma_i \sigma_j = \delta_{ij} I + \sum_{m=1}^{3} i \epsilon_{ijm} \sigma_m,
\]

we, further, have

\[
(A_2 \bullet) \sigma_j^{(2)} = (\gamma I_2 + \delta \sigma_i^{(2)}) \sigma_j^{(2)} = \gamma \sigma_j^{(2)} + \delta (\delta_{ij} I_2 + \sum_m i \epsilon_{ijm} \sigma_m^{(2)}).
\]
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\[
\left( \hat{U}_a(A_1 \bullet \hat{U}_a^{-1}) \right) \sigma_j^{(2)} = \text{sg}(t_j) \hat{U}_a(\alpha I_1 + \beta \sigma_i^{(1)}) \sigma_j^{(1)} = \\
\text{sg}(t_j) \hat{U}_a \left( \alpha \sigma_j^{(1)} + \beta (\delta_{ij} I_1 + \sum_m i \epsilon_{ijm} \sigma_m^{(1)}) \right) = \\
\text{sg}(t_j) \left( \alpha \text{sg}(t_j) \sigma_j^{(2)} + \beta (\delta_{ij} I_2 - \sum_m i \epsilon_{ijm} \text{sg}(t_m) \sigma_m^{(2)}) \right).
\]

For \( i = j \) we obtain the condition \( \gamma = \alpha \), and \( \delta = \text{sg}(t_i) \beta \), and, for \( j \neq i \), in addition: \( \delta = -\text{sg}(t_j) \text{sg}(t_m) \beta \). Since \( i \neq m \neq j \), we know from the Proposition that, irrespective of \( \text{sg}(t_i) \), one has \(-\text{sg}(t_j) \text{sg}(t_m) = \text{sg}(t_i)\). Hence, we actually obtain the condition expressed by (8) and (9).

The claim in Theorem 1 that binary mixtures \( T^{(2)} \) do have nontrivial twins is an immediate consequence of Theorem 2.

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