Information flow and optimal protocol for Maxwell’s demon single electron pump

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We study the entropy and information flow in a Maxwell demon device based on a single-electron transistor with controlled gate potentials. We construct the protocols for measuring the charge states and manipulating the gate voltages which minimizes irreversibility for (i) constant input power from the environment or (ii) given energy gain. Charge measurement is modeled by a series of detector readouts for time-dependent gate potentials, and the amount of information obtained is determined.

The protocols optimize irreversibility that arises due to (i) enlargement of the configuration space on opening the barriers, and (ii) finite rate of operation. These optimal protocols are general and apply to all systems where barriers between different regions can be manipulated.

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I. INTRODUCTION

Thermodynamic properties of driven small systems, where fluctuations play an important role, are a focus of interest, see [1, 2] for reviews. It has been shown that some mesoscopic systems can implement the so-called Maxwell’s demon (MD) process [3] in which the information of an observer is used to convert the energy of thermal fluctuations into free energy without performing work on the system [3, 5]. This process would violate the second law of thermodynamics if the information about the system could be obtained and deleted without expenditure of work or dissipation of heat. Landauer’s principle [2, 8], which equates the erasure of information to generation of heat, restores the second law. Among various mesoscopic systems, single-electron tunneling (SET) devices are particularly promising for studies of nanoscale thermodynamics. They manipulate individual electrons in systems of metallic tunnel junctions [9]. Being efficiently controlled in experiment, they allow one to design and implement various architectures. Recently it was demonstrated that a single-electron pump, monitored by a charge detector able to resolve individual electrons, can be adapted to act as MD [10]. To act efficiently, the detection should be very fast and error-free. This can be achieved by lowering the electron tunneling rates in the pump, in particular, by using hybrid normal-metal–insulator–superconductor junctions [11, 12].

Here we analyze thermodynamics and information flow in a model device close to that proposed in [6, 7, 10, 13]: a SET monitored by a tunnel junction with closed-loop feedback manipulating the time-dependent tunneling rates across its junctions. We first discuss the amount of information obtained on the system, and thereby the amount of heat necessarily dissipated in the erasure process. The analysis shows that the process is irreversible for two reasons: (i) irreversible expansion of the available configuration space at the opening of a barrier or after a measurement, and (ii) the finite rate of operation. An irreversible process means that some free energy is converted into heat instead of mechanical work. We propose the way to minimize this lost work by implementing a protocol that uses reversible expansion of configuration space similar to that suggested in Ref. [13]. The main idea is to enlarge the configuration space in a situation where in equilibrium the probability of occupying the extra configuration space is small. With this protocol, the lost work could be reduced to zero at infinite operation time. However, the power extracted from the device under infinitely slow operation implied in Ref. [12] would also vanish and make the entire machine meaningless. Here we construct an optimal finite-time protocol for both the measurement process and the gate manipulation. We consider the detection process as a series of successive charge measurements separated with time interval $\tau$ (see, e.g., ch. 7 of review [1]). Our idea is to identify $\tau$ as the response time of the detector and to relate the manipulation and measurement processes in an optimal way that minimizes the lost work under two different conditions of (i) given heat flux into the system or (ii) given energy gain. The paper is organized as follows: In Sec. II we present the model for the MD device. The thermodynamics and information flow in this model is discussed in Sec. III and the optimal protocol which minimizes the entropy production in Sec. IV. A short discussion of the results is given in Sec. V.

II. MODEL

Consider a single particle, which can be in one of the states: in the initial $E_0$ or the final state $E_M$ on the right or left lead, respectively, or on one of the intermediate islands with energies $E_i$, $i = 1, \ldots, M - 1$, where $E_M > E_{M-1} > \ldots > E_1 > E_0$. The islands are separated from each other and from the leads by potential barriers. Manipulating with these barriers one can open or close them. Closed barrier implies zero transparency,
while open barrier still has a finite transmission probability to ensure that the states on the leads and on the island are well defined. The barriers are treated as sliding doors, so that they can move without requiring work. In other words, we focus on the thermodynamic processes in the device itself, without taking into account the energy dissipation in the control unit and measurement device used for detecting a particle on an intermediate island. The system is in contact with a thermal reservoir at temperature $T$ (we measure temperature in energy units, $k_B = 1$). The model can be, in principle, implemented in a normal-metal–insulator–superconductor (NIS) single-electron pump such as discussed in [10].

We study the process when the system passes through the following states:

(1) Initially the particle is in the right lead and the gate between the right lead and the neighboring island is open

```
O  O  O  O
```

(2) The particle jumps to the island and is detected by the measurement device

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O  O  O  O
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(3) On detection we switch the gates so that the particle cannot jump back to the right lead

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O  O  O  O
```

(4) The particle jumps from the first island to the second island and our measuring device registers that the first island is empty

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O  O  O  O
```

(5) On detection we switch the gates so that the particle cannot jump back to the first island, and so on.

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O  O  O  O
```

Since the steps (3)-(5) are qualitatively similar to (1)-(3), we will only consider the first step, that is, the transition from the state 0 to the state 1. During the first process the energy of the particle was increased by $E_1 - E_0 = V$ and this energy was taken from thermal fluctuations. The transitions between the right lead and the first island when the barrier is open are described by rates $\Gamma_{01}$ and $\Gamma_{10}$, satisfying the detailed balance relation $\Gamma_{10}/\Gamma_{01} = e^{V/T}$.

The measurement establishes when the particle jumps onto the first island; it consists of a series of fast measurements at intervals $\tau$. This can be realistic when there is a pulsed detector which is sensitive to the presence or absence of a particle only during short measurement periods. Alternatively, we can consider a continuously working detector, but with a certain response time $\tau$. An ideally continuous process corresponds to $\tau \to 0$.

### III. THERMODYNAMICS AND INFORMATION FLOW

#### A. Extracted power, information and dissipated heat

After the particle is detected on the first island, the energy $E_1 - E_0 = V$ is available to do mechanical work. Therefore, the average power which can be extracted, is the ratio $V/\Theta$ where $\Theta$ is the average time per step. The probabilities $p_0 = 1 - p_1$ and $p_1$ to find the system in state 0 and 1, respectively, satisfy the master equations

\[
\dot{p}_0 = -\Gamma p_0 + \Gamma_{10}, \quad \dot{p}_1 = -\Gamma p_1 + \Gamma_{01}
\]

where $\Gamma = \Gamma_{01} + \Gamma_{10}$. The solution of Eq. (1) with initial conditions $p_0(0) = 1$, $p_1(0) = 0$, gives the probability $p_\tau = p_1(\tau)$ for detecting the particle in state 1 at time $\tau$. The probability of detecting the particle in state 1 at measurement $n$ and not before is $s_n = (1 - p_\tau)^{n-1} p_\tau$. The average number of trials, $\langle N \rangle$, is then

\[
\langle N \rangle = \sum_{n=1}^{\infty} ns_n = \frac{1}{p_\tau} = \frac{\Gamma}{\Gamma_{01}(1 - e^{-\Gamma\tau})}, \quad \Theta = \langle N \rangle \tau.
\]

How much information is obtained in this process? Each measurement has two outcomes, either to give the same result as the previous measurement (with probability $1 - p_\tau$) or to change to the opposite result (with probability $p_\tau$). These are independent for each measurement, and therefore the information per measurement is

\[
S_1 = -p_\tau \ln p_\tau - (1 - p_\tau) \ln(1 - p_\tau).
\]

We assume the measuring device to be error free. Otherwise the information content should be characterized by the mutual entropy [10]. Since $p_\tau = \langle N \rangle^{-1}$ and we need $\langle N \rangle$ repeated measurements before a particle is detected on the first island we get in the limit $\langle N \rangle \gg 1$ that $S = \langle N \rangle S_1 \approx \ln\langle N \rangle$. This is the number of binary digits needed to store $\langle N \rangle$ times $\ln 2$, the information per bit. Equation (2) yields $S \to \infty$ for continuous measurement, $\tau \to 0$, see also [14]. At $\tau \to \infty$, $S$ reaches its equilibrium value tending to $V/T$ at large $V$. In order to return the system, including the measurement device to the initial state, the information gained during measurement must be deleted. According to Landauer’s principle, this must lead to a dissipation of heat to the environment. Using, Eqs. (2) and (3) and the fact that detailed balance gives

\[
\Gamma_{01}/\Gamma = \frac{1}{1 + e^{V/T}}
\]

we get that the work required in deleting the information is

\[
W_{\text{delete}} = TS = T \ln \left( 1 + e^{V/T} \right) - T \ln (1 - e^{-\Gamma\tau}) - T e^{V/T} + e^{-\Gamma\tau} \ln \frac{e^{V/T} + e^{-\Gamma\tau}}{1 - e^{-\Gamma\tau}}.
\]
The arguments of the logarithms in the last two terms are both less than 1, and the two logarithms are therefore both negative. The two last terms give then positive contributions, and we have

$$W_{\text{delete}} > T \ln \left(1 + e^{V/T}\right) > V = \Delta U.$$  

Here $\Delta U$ is the change in internal energy of the system. This is available for converting to mechanical work at the end of the process. The work needed to delete the information is always greater than the work we could extract using this information. Only in the limit $\langle N \rangle \gg 1$ (or $V \gg T$) and $\Gamma \tau \gg 1$ do we get that

$$W_{\text{delete}} = T \ln \langle N \rangle = T \ln \left(1 + e^{\Delta E/T}\right) \approx V.$$  

Thus in the above limiting case the dissipated heat is the same as the work, which can be extracted. In other cases is seems that our device is operating not optimally. This would mean that some parts of the process are irreversible and generate net entropy in the surroundings.

\section*{B. Entropy production in irreversible expansion.}

Where does the entropy production occur? Every time we measure and find the particle not on the first island we know that it is on the right lead (including before the first measurement). But we do not use this information to get energy, we open the barrier (or keep it open). This is analogous to the free expansion of a gas, which is an irreversible process, leading to net increase of the total entropy. Indeed, at the beginning of the cycle, or after a measurement which did not detect a transition to state 1, the entropy is 0. Then we let the system evolve with the barrier open, and the entropy will increase. After the time $\tau$, when the next measurement is performed, it has the value given by (3). This is an irreversible process, implying that we could arrive at the same final state in a different way, which would allow us to extract some work $W_{\text{ex}}$ during the transition process, in addition to the increased internal energy $\Delta U = V$, which is available after the transition took place. According to the Landauer principle, if the information entropy in the end of the MD operation is deleted to the same heat bath, the full available work becomes non-positive.

In the limit of slow operation we can find the work $W_{\text{ex}}$ which could in principle be extracted during the transition process. Imagine it somehow performed reversibly. That would mean to go through some process that starts with the particle in the right lead (state 1) and ends with the particle statistically distributed between the right lead and the first island.

In this process, energy in the form of heat $Q = T \Delta S$ (where $\Delta S$ is the change of entropy of the system) would be taken from the thermal reservoir and work $W_1$ be extracted. In the limit of infinitely slow operation the probability $p_1$ in the final state is given by the thermal equilibrium values

$$p_1 = \frac{1}{e^v + 1}$$

where $v = V/T$.

The work that could be extracted is $W_1 = T \Delta S - \Delta U$ where

$$\Delta U = \frac{T v}{e^v + 1}$$

is the average increase in internal energy. Substituting

$$T \Delta S = -\frac{T}{e^v + 1} \left(\ln \frac{1}{e^v + 1} + e^v \ln \frac{1}{e^v + 1}\right)$$

we obtain

$$-\frac{W_1}{T} = \ln \frac{1}{e^v + 1} = \ln (1 - p_1) = \ln \left[1 - \frac{1}{\langle N \rangle}\right].$$  

Equation (8) gives the increase in entropy when opening the barrier once or keeping it open when we know that the particle is not on the island. In total, therefore, the work which could be extracted in a reversible transition is

$$W_{\text{ex}} = -T \langle N \rangle \ln \left[1 - \frac{1}{\langle N \rangle}\right].$$  

The information that we got but did not use must still be deleted from the memory, and the total work spent on deleting the memory is

$$W_{\text{delete}} = -T \langle N \rangle \left[p_r \ln p_r + (1 - p_r) \ln (1 - p_r)\right]$$

$$= T \ln \langle N \rangle - T (\langle N \rangle - 1) \ln \left[1 - \frac{1}{\langle N \rangle}\right].$$  

Here we have taken into account that $p_r = 1/\langle N \rangle$. Combining Eqs. (7) and (8) we can express the difference between the work in deleting and the extracted work as

$$W_{\text{delete}} - (W_{\text{ex}} + \Delta U) = T \ln (\langle N \rangle - 1) - V.$$  

Here we recall that the extracted energy $\Delta U = V$ is equal to the energy in the final state when the particle is detected on the first island. Using Eq. (2) we get

$$W_{\text{delete}} - (W_{\text{ex}} + \Delta U) = T \ln \left(\frac{1 + e^{-\Gamma \tau} e^{-V/T}}{1 - e^{-\Gamma \tau}}\right).$$  

The above expression shows explicitly that when $\Gamma \tau \gg 1$ we get $W_{\text{delete}} = W_{\text{ex}} + \Delta U$.

The limit $\Gamma \tau \gg 1$ corresponds to slow operation of the device, so it is similar to the usual requirement of quasi-static operation for reversible processes. It should be noted that the extracted work was calculated in the limit of quasistatic operation, so that for finite operation time, $\tau$, we would have a smaller amount of work that could be extracted because some entropy would be
created. This means that in Eq. (11), $W_{ex}$, which is the work that is wasted because of our protocol, should be less, since at finite operation rate also any other protocol would be suboptimal. What is the maximal amount of work which can be extracted in a finite time? Or equivalently, what is the minimal entropy production? This question will be answered in the following section.

**IV. OPTIMAL PROTOCOL**

We implement the idea of reversible expansion of the configuration space \([13]\) in the following way. (a) We start form the configuration when the particle is on the right lead and the barrier is closed. (b) Then we quickly lift the potential of the first island to a high value $V_0 \gg T$. (c) The potential of the left side is slowly moved down. At this stage the barrier gradually opens and transitions between the states can take place. (d) Lowering of the potential continues until time $\tau$ when the next measurement is due. At this time, some energy $V(\tau)$ is reached. (e) If the measurement shows absence of the particle, the voltage is quickly increased again, and the process is repeated. If the particle has been detected, a similar process is started at the adjacent grain. The raise of the potential at stage (b) occurs faster than the transition time $\Gamma^{-1}$ at stages (c) and (d) but slower than the relaxation in the heat bath, which is the fastest time in our system. Lowering of the barrier (or the measurement time $\tau$) is slower than $\Gamma^{-1}$.

Let us denote $E_i(t)$ the energy of state $i$ as function of time. The protocol described above has $E_0(t) = 0$ and $E_1(t) = V(t)$. Let $p_i(t)$ be the probabilities to find the particle in state $i$. These satisfy the master equation \([11]\) with the rates $\Gamma_{ij}$ now depending on the difference $V(t) = E_1(t) - E_0(t)$ and, therefore, on time. The extracted work during time $\tau$ is

$$W_{ex} = - \sum_i \int_0^\tau dt \, p_i(t) \dot{E}_i .$$

The average internal energy change is

$$\Delta U = \sum_i [p_i(\tau)E_i(\tau) - p_i(0)E_i(0)] .$$

The heat transfer is then

$$Q = \Delta U + W_{ex} = \sum_i \int_0^\tau dt \, \dot{p}_i E_i(t) .$$

Since the thermal bath is never brought out of equilibrium due to fast relaxation, the change in entropy of the environment is $\Delta S_{env} = -Q/T$. The entropy of the system is $S = - \sum_i p_i \ln p_i$. As in \([17, 18]\) we write the change in entropy as an integral

$$\Delta S = - \sum_i \int_0^\tau dt \frac{dp_i}{dt} \ln p_i .$$

This is the information entropy stored in the measuring device. It has to be deleted in the end to reset the device. Being interested here in optimizing the losses in course of extracting the work, we do not consider losses in the process of deleting the information, which should also be done in an optimal way. In general, Eq. (11) defines $\Delta S$ as a *functional* of the operation protocol $E_i(t)$. We need also to solve the master equation \([11]\). This is complicated by the fact the transition rates $\Gamma_{ij}$ depend on the energy difference $V(t)$ between the two sites maintaining detailed balance. By specifying the time dependence $V(t)$ we then get time-dependent $\Gamma_{ij}(t)$. The sum of the rates is $\Gamma(t) = (1 + e^{V(t)/T}) \Gamma_{01}(t)$. The master equation

$$\frac{dp_1}{dt} = -\Gamma(t)p_1 + \Gamma_{01}(t) \tag{12}$$

can now be integrated for any known dependence $V(t)$. With the initial condition $p_1(0) = 0$ we get

$$p_1(t) = \int_0^t dt'' e^{t''/\Gamma(t'')} \Gamma_{01}(t'') . \tag{13}$$

As in \([19]\), to simplify the following calculations we choose $\Gamma$ to be independent of $V$:

$$\Gamma \equiv \gamma_0 \quad \Gamma_{01}(t) = \gamma_0 \left(1 + e^{V(t)/T}\right)^{-1}$$

which does not affect the results qualitatively.

**A. Optimized entropy production for fixed heat flux.**

We now return to Eq. (11) and want to optimize it in the sense of finding the operation time $\tau$ and protocol $V(t)$, which will minimize the entropy production. It is clear that if this is done without any constraints, the entropy production can be made arbitrary small by choosing $\tau$ and $V_0$ large enough. This would mean that we are in the quasistatic regime discussed above. In this case the produced power is zero since we get a finite amount of energy in an infinite time. A more instructive situation would be to minimize the entropy production rate at a constant heat flux. If we imagine the Maxwell demon device as part of a heat engine, with the reservoir supplying the energy to the particle as a high temperature reservoir and the reservoir at which we delete the memory of the demon as a low temperature reservoir, this would correspond to maximizing the output power at a given input heat.

1. **Derivation of the entropy production functional and the integral equation for the optimal protocol**

The power is now defined as the average heat per cycle extracted from the reservoir, $P = Q/\tau$. The total
entropy is \( \Delta S_{tot} = \Delta S + \Delta S_{env} \) and the rate of entropy production is then
\[
\frac{\Delta S_{tot}}{u_0} = \frac{\Delta S}{u_0} - \frac{Q}{u_0 T} = \frac{\Delta S}{u_0} - P_0.
\]

where we introduce the dimensionless variables
\( u = \gamma_0 t, \ u_0 = \gamma_0 \tau, \ v(u) = V(t)/T, \ P_0 = \mathcal{P}/\gamma_0 T. \) (15)

Since \( P_0 \) is to be held constant, it is sufficient to minimize \( \Delta S/u_0 \). Denoting \( p = p_1 \) we obtain:
\[
\frac{\Delta S}{u_0} = -\frac{1}{u_0} \int_0^{u_0} du \left[ \ln \frac{dp}{du} + \ln(1-p) + \frac{d(1-p)}{du} \right]
\]
\[
= -\frac{1}{u_0} \int_0^{u_0} du \ln \left( \frac{p[v]}{1-p[v]} \right) \frac{dp[v]}{du}
\]
where from (13)
\[
p[v] = \int_0^u du' \frac{e^{u' - u}}{e^{v(u')} + 1}.
\]

This should be minimized subject to the constraint that the power \( P_0 \) is a given constant. Using Eq. (13) and the master equation (12) we find that it can be expressed as
\[
P_0 = \frac{1}{u_0} \int_0^{u_0} du \left[ -p[v] + \frac{1}{e^{v(u)} + 1} \right] v(u).
\]

Introducing the Lagrange multiplier \( \lambda \) this means that we have to minimize the functional
\[
I = \frac{1}{u_0} \int_0^{u_0} du \left( \lambda v - \ln \frac{p[v]}{1-p[v]} \right) \left( -p + \frac{1}{e^v + 1} \right)
\]
with the function \( v(u) \) and the time \( u_0 \) as variables.

The functional \( I \) is of the form
\[
I[v] = \frac{1}{u_0} \int_0^{u_0} du L(v, p[v])
\]
with the function
\[
L(v, p[v]) = \left( \lambda v - \ln \frac{p[v]}{1-p[v]} \right) \left( -p + \frac{1}{e^v + 1} \right)
\]
depending on \( v \) both directly and indirectly through \( p[v] \) which is a functional of \( v \) as given by Eq. (13). The Euler-Lagrange equation takes the form
\[
\frac{\partial L}{\partial v} - \int_u^{u_0} du' \frac{e^{u' - u}}{(e^{v(u')} + 1)^2} e^{v'} \left. \frac{\partial L}{\partial p} \right|_{u'} = 0.
\]

Calculating the derivatives, we get the Euler-Lagrange equation as
\[
\lambda \left[ v + \left( p - \frac{1}{e^v + 1} \right) \frac{(e^v + 1)^2}{e^v} - e^u \int_u^{u_0} du' e^{-u'} \right]
\]
\[
= \ln \left[ \frac{p}{1-p} - e^u \int_u^{u_0} du' e^{-u'} \left( \frac{1}{p'(1-p')} \right) \left( p' - \frac{1}{e^{v'} + 1} \right) + \ln \left( \frac{p'}{1-p'} \right) \right]
\]

where \( p' = p[v(u')] \). This equation is probably impossible to solve analytically and to proceed we consider operation of the device which is sufficiently slow so that the probability \( p \) never is far from the thermal equilibrium value.

2. Lowest order correction to the quasistatic solution

If we consider slow operation the deviation from the quasistatic solution will be small. That is, we write \( p = p_a + p_b \) where
\[
p_a = \frac{1}{e^v + 1}
\]
and the Master equation (12) is \( \dot{p} = -p + p_a \). For later use it is convenient to introduce the more general Master equation
\[
\dot{p} = -g(v)(p - p_a)
\]
where \( g(v) \) is some function of \( v \) and the present case corresponds to \( g(v) = 1 \). Neglecting \( p_b \) compared to \( p_a \) we get
\[
p_0 = -\frac{1}{g(v)} \dot{p}_a = -\frac{1}{g(v)} \frac{dp}{dv} \dot{v} = \frac{\dot{v} e^v}{g(v)(e^v + 1)^2}.
\]

Inserting this into the Lagrangian (20) we get
\[
L(v, \dot{v}) = \alpha(v) \dot{v}^2 + \beta(v) \dot{v}
\]
with
\[
\alpha(v) = \frac{e^v}{g(v)(e^v + 1)^2} = \frac{1}{4g(v) \cosh^2 v/2},
\]
\[
\beta(v) = \frac{(\lambda + 1) v g(v) \alpha(v)}{g(v)(e^v + 1)^2} = \frac{-v e^v}{\lambda}.
\]

The Euler-Lagrange equation (22) can be integrated to give
\[
\dot{v} = \frac{A}{\sqrt{\alpha(v)}}
\]
where \( A \) is a constant of integration. A second integration gives
\[
F(v) = \int_{v_0}^v dv \sqrt{\alpha(v)} = Au
\]
where \( v_0 = v(0) \) which at the moment is unspecified.

Since the values of \( v \) at the endpoints are not fixed we get the additional conditions
\[
\left. \frac{\partial L}{\partial v} \right|_0 = \left. \frac{\partial L}{\partial v} \right|_{u_0} = 0.
\]
Since
\[ \partial L / \partial \dot{v} = 2\alpha(v) \dot{v} + \beta(v) = \alpha(v) [2\dot{v} - (\lambda + 1)vg(v)] = 0 \]
the equality \( \partial L / \partial \dot{v} = 0 \) can be met either at \( \alpha(v) = 0 \)
or at \( \dot{v} = \frac{1}{2}(\lambda + 1)vg(v) \). \( \alpha(v) = 0 \) implies that \( v = \infty \)
and we guess that this is the proper solution at \( u = 0 \), so that we have \( v_0 \equiv v(0) = \infty \). At the final time, \( u_0 \),
we assume that \( v_u = v(u_0) \) is finite which means that we must have
\[ \dot{v}(u_0) = \frac{1}{2}(\lambda + 1)v_ug_u \]
where \( v_u = v(u_0) \) and \( g_u = g(v_u) \). Using (28) we get
\[ A = \frac{1}{2}(\lambda + 1)v_ug_u \sqrt{\alpha} = \frac{v_\lambda(\lambda + 1)}{4 \cosh v_u/2}. \]
(30)
The equation (17) for the constraint takes the form
\[ u_0P_0 = - \int_0^{u_0} du p_u = \int_{v_u}^{\infty} dv ve^v = K(v_u). \]
(31)
Finally, the action (19) should also be stationary with respect to variation of the operation time \( u_0 \). The condition \( \partial I / \partial u_0 = 0 \) can be rewritten as
\[ L(v_u, \dot{v}_u) = \frac{1}{u_0} \int_0^{u_0} du L(v, \dot{v}). \]
(32)
Using (28) we find
\[ L = \alpha(v) \dot{v}^2 + \beta(v) \dot{v} = A^2 + \dot{v}. \]
Using (27) and (31) we find
\[ \frac{1}{u_0} \int_0^{u_0} du \beta(v) \dot{v} = (\lambda + 1)P_0. \]
Equation (32) is then
\[ -v_u p_0(u_u) = P_0. \]
Using (24), (28) and (31) we get
\[ Au_0 = - \frac{2K(v_u)}{v_u} \cosh \left( \frac{v_u}{2} \right). \]
(33)
From (29) we have \( F(v_u) = Au_0 \), and we get an equation in terms of \( v_u \) only
\[ F(v_u) = - \frac{2K(v_u)}{v_u} \cosh \left( \frac{v_u}{2} \right). \]
(34)
This equation must in general be solved numerically to give \( v_u \). It is interesting to note that \( P_0 \) does not enter the equation, which means that \( v_u \) does not depend on \( P_0 \). All other quantities are expressed in terms of the solution \( v_u \):\[
\frac{K(v_u)}{P_0}, \quad A = - \frac{2P_0}{v_u} \cosh \left( \frac{v_u}{2} \right), \quad \lambda = - \frac{8P_0}{v_u^2} \cosh^2 \left( \frac{v_u}{2} \right) - 1.
\]
We can also calculate the rate of entropy generation as a function of the power \( P_0 \). We have Eq. (14) and using (26) with \( \lambda = 0 \) we find
\[ \frac{\Delta S}{u_0} = \frac{1}{u_0} \int_0^{u_0} \frac{du}{\int L(v, \dot{v})} \big|_{\lambda = 0} = A^2 + P_0. \]
Therefore
\[ \frac{\Delta S_{tot}}{u_0} = A^2 + \frac{4 \cosh^2(v_u/2)}{P_0^2}. \]
The rate of entropy production is quadratic in \( P_0 \). In the case \( g(v) = 1 \) we get
\[ F(v) = - \int_v^\infty \frac{e^{v/2}}{e^v + 1} = - \text{arccot}(\sinh \left( \frac{v}{2} \right)) \]
and
\[ K(v_u) = \int_{v_u}^{\infty} \frac{dv ve^v}{(e^v + 1)^2} = \ln (\cosh v_u) - \frac{v_u}{1 + e^{-v_u}}. \]
Equation (34) then becomes
\[ \arccot \left( \frac{\sinh \left( \frac{v_u}{2} \right)}{v_u/2} \right) = \frac{\cosh(v_u/2)}{v_u/2} \left[ \ln (\cosh(v_u) - \frac{v_u}{1 + e^{-v_u}}. \right] \]
This equation can be solved numerically giving \( v_u = 1.3256 \). From this we get \( K = 0.5138 \) and
\[ \lambda = -6.8630P_0 - 1 \quad \text{and} \quad A = -1.8524P_0. \]
(35)
The optimal protocol is plotted in Fig. 1 (left) for various values of \( P_0 \).

The entropy production per measurement interval is \( \Delta S_{tot}/\tau = 3.43\gamma_0P_0^2 \). The extracted work is \( \Delta W_{ex}/\tau = 0.46\mathcal{P} \) neglecting the quadratic term. The average operational time \( \Theta = 2.447/T \mathcal{P} \). Our results hold for slow processes, \( u_0 \gg 1 \) or \( \mathcal{P} \ll T \gamma_0 \).

![FIG. 1: (Color online) Plot of the optimal protocol \( v(u) \) for different output power \( P_0 \). The optimal period, \( u_0 \equiv \gamma_0 \), between measurements is \( \approx 0.51/P_0 \), the final value of \( v \) is \( \approx 1.33 \).](image-url)
B. Leads with many states and particles

Let us now consider a more realistic model where the leads have a band of states with different energies, and we have many particles in the lead as illustrated in Fig. 2. We make the following assumptions:

1. Detailed balance: \( \Gamma_{0i} = \Gamma_{i0} e^{(V-E_i)/T} \).

2. \( \Gamma_{0i} + \Gamma_{i0} = \Gamma_0 \) where \( \Gamma_0 \) is a constant independent of \( V \) and \( E_i \). Together with the detailed balance this implies

\[
\Gamma_{i0} = \frac{\Gamma_0}{1 + e^{(V-E_i)/T}}, \quad \Gamma_{0i} = \frac{\Gamma_0}{1 + e^{-(V-E_i)/T}}. \tag{36}
\]

3. Internal processes in the lead are fast so that the lead is always in equilibrium, which means that the probability of finding the level at \( E_i \) occupied is

\[
f(E_i) = \frac{1}{e^{E_i/T} + 1}. \tag{37}
\]

The chemical potential \( \mu = 0 \).

4. The density of states in the lead is constant:

\( g(E_i) = g_0 \)

Under these assumptions, the entropy of the leads is constant. If \( p \) denotes the probability that an electron is found on the island, the change in entropy is

\[
\Delta S = -p \ln p - (1-p) \ln(1-p),
\]

The master equation

\[
\dot{p} = \sum_i \Gamma_{i0} f(E_i) (1 - p) - \sum_i \Gamma_{0i} (1 - f(E_i)) p
\]

where

\[
\Gamma_{12}(V) = \int_{-\infty}^{\infty} \frac{g_0 dE}{1 + e^{(V-E)/T}(e^{E/T} + 1)} = \frac{g_0 V}{e^{V/T} - 1},
\]

\[
\Gamma_{21}(V) = \frac{\Gamma_0 g_0 V}{e^{-V/T} - 1} = \Gamma_{12}(V)e^{V/T}. \tag{39}
\]

\[
\text{FIG. 2: The more realistic model with a single level on the first island (left) and a band with many levels in the metallic lead (right). The Fermi level is chosen as } \mu = 0 \text{ and the rates for transitions between the island and level } i \text{ are } \Gamma_{0i} \text{ and } \Gamma_{i0}.\]

Denoting \( \gamma_0 = 2T\Gamma_0 g_0 \) and introducing dimensionless variables as in \( (15) \) we get

\[
\Gamma = \Gamma_{12} + \Gamma_{21} = \gamma_0 \frac{v}{2} \coth \frac{v}{2} \text{ and } \Gamma_{12} = \frac{\Gamma}{1 + e^v}.
\]

This is still a model. For some microscopic mechanisms \( \Gamma \propto E^p \) where \( p \) is some number. Then some extra power of \( v \) can appear in the expression for \( \Gamma \).

In the limit of slow operation we find that the Master equation \( (12) \) is

\[
\dot{p} = \frac{dp}{du} = -
\]

which is of the general form \( (23) \) with

\[
g(v) = \frac{v}{2} \coth \frac{v}{2}. \tag{41}
\]

This means that the more realistic model with the leads modeled as metallic bands has the same features as the simplified model discussed in Sec. VA. According to Eq. \( (29) \) we get

\[
F(v) = -2 \int_v^\infty d\epsilon \frac{e^{\epsilon/2}(e^\epsilon - 1)}{e^{(e^\epsilon + 1)^2}} \tag{42}
\]

which cannot be solved in closed form. From Eq. \( (31) \) we have

\[
K(v_u) = \frac{1}{2\cosh^2 v_u/2} \tag{43}
\]

and Eq. \( (33) \) gives \( v_u = 1.5076 \).

C. Optimal protocol for a fixed energy gain.

To raise the system up by a given energy \( V \) using one island, we need to minimize the entropy production keeping the final energy fixed, \( v_u = V/T \). For a given time between the measurements, \( \tau \), this will determine the required heat flux and the entropy production. As can be seen from Eqs. \( (14), (19) \), the total entropy production in this case is determined by the functional \( I[v] \) taken at \( \lambda = -1 \). The solution of the Euler-Lagrange equation is again given by Eq. \( (29) \) [see Fig. 1]. The entropy production and the required heat flux are easily calculated in the same way as before:

\[
P_0 = K(v_u)/u_0, \quad \Delta S_{tot}/u_0 = \int [F(v_u)]^2/dv_u^2 \]

where \( F(v_u) \) and \( K(v_u) \) are determined by Eqs. \( (20) \) and \( (30) \), see Fig. 3. The measurement interval has to be longer than \( \gamma_0 \), i.e., \( u_0 \gg 1 \). For \( v_u \lesssim 1 \), the heat flux is \( P \ll T \gamma_0 \). The average operational time needed to complete the transition, \( \Theta = \tau/p(v_u) = \tau(e^{v_u} + 1) \), grows exponentially for high energy gain \( V \gg T \). To optimize the total operational time, it is thus favorable to divide the total energy interval \( \Delta E_{fin} \) into \( M \) steps such that each step has the height \( V \sim T \). The device thus will have \( M - 1 \) island with the
FIG. 3: (Color online) Normalized heat flux $u_0 P_0$, the entropy production $u_0 \Delta S_{\text{tot}}$, and extracted work $W_{\text{ex}}/T$ as functions of the given energy gain.

The total operational time proportional to $M$ instead of being exponential. Minimizing $\Theta = \tau M \left( e^{\Delta E_{\text{fin}}/MT} + 1 \right)$ we find that the optimal number of steps is the closest integer to $M = 1.28 \Delta E_{\text{fin}}/T$. The extracted work is

$$W_{\text{ex}}/\tau = (T/\tau) [K(v_u) - v_u p(v_u)];$$

the ratio $W_{\text{ex}}/T$ is shown in Fig. 3. Since $K(0) = \ln 2$ we find that, for zero energy gain $v_u \to 0$, the extracted work assumes the value $W_{\text{ex}} = P \tau = T \ln 2$ as for the symmetric Szilárd engine.

V. DISCUSSION AND CONCLUSION

This simplified model of a MD device allows us to make several general conclusions applicable to a wider range of situations. It is clear that detection, feedback, and erasure of information should all be optimized. The position of the particle is a degree of freedom with thermal fluctuations. The detector registers these fluctuations and transfers the information into some non-fluctuating degree of freedom (or fluctuating on a very long time scale). We observe that the rate $1/\tau$ at which such a degree of freedom reads out the fluctuating quantity should not be larger than the rate $\Gamma$ of transitions between the states of the fluctuating degree of freedom. If $\Gamma \tau \ll 1$ the readout does not produce sufficient new information, while this information still has to be deleted. We have also seen that an optimal protocol is needed to minimize the irreversible entropy production in the course of opening the gates between the parts of the device, thereby enlarging the available configuration space. We construct such protocols for a given input power or a given energy gain. Our conclusions are relevant to many different devices, unrelated to the MD, where an optimized protocol is needed to reduce the production of entropy, i.e., the dissipated heat. The entropy production rate at small heat flux is proportional to the heat flux squared. Therefore, the ratio of the entropy production to the heat flux vanishes at small heat fluxes when the device operates reversibly.

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