Independent subsets of powers of paths, and Fibonacci cubes

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Abstract

We provide a formula for the number of edges of the Hasse diagram of the independent subsets of the $h$th power of a path ordered by inclusion. For $h = 1$ such a value is the number of edges of a Fibonacci cube. We show that, in general, the number of edges of the diagram is obtained by convolution of a Fibonacci-like sequence with itself.

Keywords: Independent subset, path, power of graph, Fibonacci cube.

1 Introduction

For a graph $G$ we denote by $V(G)$ the set of its vertices, and by $E(G)$ the set of its edges.

\textbf{Definition 1.1} For $n, h \geq 0$, the $h$-power of a path, denoted by $P_n^{(h)}$, is a graph with $n$ vertices $v_1, v_2, \ldots, v_n$ such that, for $1 \leq i, j \leq n$, $i \neq j$, $(v_i, v_j) \in E(P_n^{(h)})$ if and only if $|j - i| \leq h$.

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Thus, for instance, $P_n^{(0)}$ is the graph made of $n$ isolated nodes, and $P_n^{(1)}$ is the path with $n$ vertices.

**Definition 1.2** An independent subset of a graph $G$ is a subset of $V(G)$ not containing adjacent vertices.

**Notation.** (i) We denote by $p_n^{(h)}$ the number of independent subsets of $P_n^{(h)}$.

(ii) We denote by $H_n^{(h)}$ the Hasse diagram of the poset of independent subsets of $P_n^{(h)}$ ordered by inclusion, and by $H_n^{(h)}$ the number of edges of $H_n^{(h)}$.

In this work we evaluate $p_n^{(h)}$, and $H_n^{(h)}$. Our main result (Theorem 3.4) is that, for $n, h \geq 0$, the sequence $H_n^{(h)}$ is obtained by convolving the sequence $1, \ldots, 1, p_n^{(h)}, p_n^{(h)}_1, p_n^{(h)}_2, \ldots$ with itself.

Clearly, $H_n^{(0)}$ is the $n$-dimensional cube. Thus, on one hand, our work generalizes the known formula $n2^{n-1}$ for the number of edges of the Boolean lattice with $n$ atoms, obtained by the convolution of the sequence $(2^n)_{n \geq 0}$ with itself. From a different perspective, this work could be seen as yet another generalization of the notion of Fibonacci cube. Indeed, observe that every independent subset $S$ of $P_n^{(h)}$ can be represented by a binary string $b_1b_2\ldots b_n$, where, for $i = 1, \ldots, n$, $b_i = 1$ if and only if $v_i \in S$. More specifically, each independent subset of $P_n^{(h)}$ is associated with a binary string of length $n$ such that the distance between any two 1’s of the string is greater than $h$. For $h = 1$ the binary strings associated with independent subsets of $P_n^{(h)}$ are Fibonacci strings of order $n$, and the Hasse diagram of the set of all such strings ordered bitwise is a Fibonacci cube of order $n$ (see [5,7]). Fibonacci cubes were introduced as an interconnection scheme for multicomputers in [3], and their combinatorial structure has been further investigated, e.g. in [6,7]. Several generalizations of the notion of Fibonacci cubes has been proposed (see, e.g., [4,5]). As far as we now, our generalization, described in terms of independent subsets of powers of paths ordered by inclusion, is a new one.

## 2 The independent subsets of powers of paths

We denote by $p_{n,k}^{(h)}$ the number of independent $k$-subsets of $P_n^{(h)}$.

**Lemma 2.1** For $n, h, k \geq 0$, $p_{n,k}^{(h)} = \binom{n-hk+h}{k}$.

**Proof.** See [2, Theorem 1], and [1], where we establish a bijection between independent $k$-subset of $P_n^{(h)}$ and $k$-subsets of a set with $(n-hk+h)$ elements.\qed
For $n, h \geq 0$, the number of all independent subsets of $P_n^{(h)}$ is

$$p_n^{(h)} = \sum_{k=0}^{\lceil n/(h+1) \rceil} p_{n,k}^{(h)} = \sum_{k=0}^{\lceil n/(h+1) \rceil} \binom{n-hk+h}{k}.$$ 

**Remark 2.2** Denote by $F_n$ the $n$th element of the Fibonacci sequence $F_1 = 1$, $F_2 = 1$, and $F_i = F_{i-1} + F_{i-2}$, for $i > 2$. Then, $p_n^{(1)} = F_{n+2}$.

**Lemma 2.3** For $n, h \geq 0$,

$$p_n^{(h)} = \begin{cases} n + 1 & \text{if } n \leq h + 1, \\ p_{n-1}^{(h)} + p_{n-h-1}^{(h)} & \text{if } n > h + 1. \end{cases}$$

**Proof.** See the first part of [2, Proof of Theorem 1], or [1]. \hfill \Box

### 3 The poset of independent subsets of powers of paths

Figure 1 shows a few Hasse diagrams $H_n^{(h)}$. Notice that, as mentioned in the introduction, for each $n$, $H_n^{(1)}$ is a Fibonacci cube.

![Hasse diagrams](image)

Fig. 1. Some $H_n^{(h)}$.

Since in $H_n^{(h)}$ each non-empty independent $k$-subset covers exactly $k$ independent $(k-1)$-subsets, we can write

$$H_n^{(h)} = \sum_{k=1}^{\lceil n/(h+1) \rceil} kp_{n,k}^{(h)} = \sum_{k=1}^{\lceil n/(h+1) \rceil} k \binom{n-hk+h}{k}. \quad (1)$$

Let now $T_{k,i}^{(n,h)}$ be the number of independent $k$-subsets of $P_n^{(h)}$ containing the vertex $v_i$, and let, for $h, k \geq 0$, $n \in \mathbb{Z}$, $p_{n,k}^{(h)} = \begin{cases} p_0^{(h)} & \text{if } n < 0, \\ p_n^{(h)} & \text{if } n \geq 0. \end{cases}$

**Lemma 3.1** For $n, h, k \geq 0$, and $1 \leq i \leq n$,

$$T_{k,i}^{(n,h)} = \sum_{r=0}^{k-1} p_{i-h-1,r}^{(h)} p_{n-i-h,k-1-r}^{(h)}.$$

**Proof.** No independent subset of $P_n^{(h)}$ containing $v_i$ contains any of the elements $v_{i-h}, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{i+h}$. Let $r$ and $s$ be non-negative integers whose
sum is $k-1$. Each independent $k$-subset of $P_n^{(h)}$ containing $v_i$ can be obtained by adding $v_i$ to a $(k-1)$-subset $R \cup S$ such that

(a) $R \subseteq \{v_1, \ldots, v_{i-h-1}\}$ is an independent $r$-subset of $P_n^{(h)}$;

(b) $S \subseteq \{v_{i+h+1}, \ldots, v_n\}$ is an independent $s$-subset of $P_n^{(h)}$.

Vice versa, one can obtain each of this pairs of subsets by removing $v_i$ from an independent $k$-subset of $P_n^{(h)}$ containing $v_i$. Thus, $T_{k,i}^{(n,h)}$ is obtained by counting independently the subsets of type (a) and (b). Noting that the subsets of type (b) are in bijection with the independent $s$-subsets of $P_{n-i-h}$, the lemma is proved. □

In order to obtain our main result, we prepare a lemma.

**Lemma 3.2** For positive $n$,

$$H_n^{(h)} = \sum_{k=1}^{\lceil n/(h+1) \rceil} \sum_{i=1}^{n} T_{k,i}^{(n,h)}.$$ 

**Proof.** The inner sum counts the number of $k$-subsets exactly $k$ times, one for each element of the subset. That is, $\sum_{i=1}^{n} T_{k,i}^{(n,h)} = kp_{n,k}^{(h)}$. The lemma follows directly from Equation (1). □

Next we introduce a family of Fibonacci-like sequences.

**Definition 3.3** For $h \geq 0$, and $n \geq 1$, the $h$-Fibonacci sequence $F^{(h)} = \{F_n^{(h)}\}_{n \geq 1}$ is the sequence whose elements are

$$F_n^{(h)} = \begin{cases} 1 & \text{if } n \leq h + 1, \\ F_{n-1}^{(h)} + F_{n-h-1}^{(h)} & \text{if } n > h + 1. \end{cases}$$

From Lemma 2.3, and setting for $h \geq 0$, and $n \in \mathbb{Z}$, $p_{n}^{(h)} = \begin{cases} p_{0}^{(h)} & \text{if } n < 0, \\ p_{n}^{(h)} & \text{if } n \geq 0, \end{cases}$ we have that,

$$F_i^{(h)} = p_{i-h-1}^{(h)} , \text{ for each } i \geq 1 .$$ (2)

Thus, we can write $F^{(h)} = 1, \ldots, 1, p_{0}^{(h)}, p_{1}^{(h)}, p_{2}^{(h)}, \ldots$. In the following, we use the discrete convolution operation $\ast$, as follows.

$$(F^{(h)} \ast F^{(h)}) (n) = \sum_{i=1}^{n} F_i^{(h)} F_{n-i+1}^{(h)} .$$ (3)
Theorem 3.4 For \( n, h \geq 0 \), the following holds.

\[
H_n^{(h)} = \left( \mathcal{F}^{(h)} \ast \mathcal{F}^{(h)} \right)(n).
\]

Proof. The sum \( \sum_{k=1}^{\lceil n/(h+1) \rceil} T_{k,i}^{(n,h)} \) counts the number of independent subsets of \( P_n^{(h)} \) containing \( v_i \). We can also obtain such a value by counting the independent subsets of both \( \{v_1, \ldots, v_{i-h-1}\} \), and \( \{v_{i+h+1}, \ldots, v_n\} \). Thus, we have:

\[
\sum_{k=1}^{\lceil n/(h+1) \rceil} T_{k,i}^{(n,h)} = p_{i-h-1}^{(h)} p_{n-h-i}^{(h)}.
\]

Using Lemma 3.2 we can write

\[
H_n^{(h)} = \sum_{k=1}^{\lceil n/(h+1) \rceil} \sum_{i=1}^{n} T_{k,i}^{(n,h)} = \sum_{i=1}^{n} \sum_{k=1}^{\lceil n/(h+1) \rceil} T_{k,i}^{(n,h)} = \sum_{i=1}^{n} p_{i-h-1}^{(h)} p_{n-h-i}^{(h)}.
\]

By Equation (2) we have \( \sum_{i=1}^{n} p_{i-h-1}^{(h)} p_{n-h-i}^{(h)} = \sum_{i=1}^{n} F_i^{(h)} F_{n-i+1}^{(h)} \). By (3), the theorem is proved. \( \square \)

Further properties of coefficients \( H_n^{(h)} \), and \( p_n^{(h)} \) are discussed in [1]. Moreover, in [1] we investigate the case of powers of cycles, and its connection with Lucas cubes.

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