Effective Laplacian for bilayer coupling

S. Emre Tuna∗

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Abstract

Networks under bilayer coupling topology are studied. A reduction approach is proposed where a single Laplacian matrix (called the effective Laplacian) is constructed out of the pair of Laplacian matrices, each describing a separate coupling layer. The presented construction generalizes the parallel sum of matrices. For a network of identical LC tanks under bilayer RL coupling, it is shown that the oscillators asymptotically synchronize if and only if the complex-valued effective Laplacian has a single eigenvalue on the imaginary axis.

1 Introduction

In studying the collective behavior of coupled units electrical networks provide us with excellent instances [6, 3]. Most of those examples (when, say, the units are two-terminal devices) are modeled in terms of a monolayer coupling structure, under the implicit assumption that there exists a (ground) node in the network at which one terminal of every unit meets, see Fig. (a). Such a framework is reasonable to employ when the devices are physically close to each other, which lets one approximate the lines connecting separate units as perfect conductors. Otherwise, that is, when the geographical distance between units renders the line impedances nonnegligible, the more general bilayer coupling structure shown in Fig. (b) inevitably emerges.

![Network Diagrams](image)

Figure 1: (a) Electrical network with monolayer coupling. (b) Electrical network with bilayer coupling.

The network of Fig. (a) under monolayer coupling has only half as many nodes as that of Fig. (b), where the coupling is bilayer. With this come some significant notational and analytical conveniences. Moreover, there is a rich collection of tools and results developed for the analysis of networks where the coupling manifests itself through a single layer [8, 10, 4, 9], which may not be directly applicable in the bilayer case. Therefore, if possible, it is desirable to reduce (in some meaningful way) a pair of coupling topologies (each representing a layer in a bilayer framework) to a single topology. This makes how to perform such reduction? an interesting question, to which we provide an answer in this paper.

∗The author is with Department of Electrical and Electronics Engineering, Middle East Technical University, 06800 Ankara, Turkey. Email: etuna@metu.edu.tr
Our answer will be a generalization of the following simple example. Consider the assembly of three units coupled via linear time invariant (LTI) resistors as shown in Fig. 2a. Let \( v = [v_1 \ v_2 \ v_3]^T \) and \( i = [i_1 \ i_2 \ i_3]^T \) be the associated voltage and current vectors, respectively. Clearly, the bilayer coupling of Fig. 2a is equivalent to the monolayer one shown in Fig. 2b; where by “equivalent” we mean “the units in Fig. 2a are subject to the same \( i-v \) constraint as those in Fig. 2b.” This constraint reads \( i = \Lambda v \) with

\[
\Lambda = \begin{bmatrix}
1 & -1 & 0 \\
-1 & 3 & -2 \\
0 & -2 & 2
\end{bmatrix}.
\]

We note that the Laplacian matrix \( \Lambda \) completely characterizes the monolayer coupling of Fig. 2b. Likewise, the top and bottom layers of coupling in Fig. 2a can be represented by their own separate Laplacians as

\[
L_{\text{top}} = \begin{bmatrix}
1 & -1 & 0 \\
-1 & 4 & -3 \\
0 & -3 & 3
\end{bmatrix} \quad \text{and} \quad L_{\text{bot}} = \begin{bmatrix}
\infty & -\infty & 0 \\
-\infty & \infty + 6 & -6 \\
0 & -6 & 6
\end{bmatrix}
\]

where the symbol “\( \infty \)” in \( L_{\text{bot}} \) stands for the short circuit connection of the bottom terminals of the first and second units in Fig. 2a. Observe that the two coupling layers (in Fig. 2a) are in series connection because the current \( i \) that leaves the bottom layer enters the top layer, i.e., it is the same current that visits both layers. Therefore the matrix \( \Lambda \) can be interpreted as the effective Laplacian for the series connection of the pair \( (L_{\text{top}}, L_{\text{bot}}) \). Henceforth we will express this last statement compactly by writing

\[ \Lambda = L_{\text{top}} : L_{\text{bot}}. \]

In the first part of the paper we study the mapping \( (X, Y) \mapsto X : Y \) and establish its certain properties relevant to network analysis. Moreover, we obtain an explicit expression that can be used to compute the effective Laplacian in a general setting, where the matrices \( X, Y \) are allowed to have entries that are infinite. For the special case where both Laplacians \( X \) and \( Y \) have only finite entries, it turns out that the general expression boils down to \( X : Y = X(X + Y)^+Y \), where \( (\cdot)^+ \) indicates the pseudoinverse. This identity explains why we use the colon (:) notation, which indicates the parallel sum of matrices [1]. In the second part, we present an application of the effective Laplacian in oscillator synchronization [5, 7]. The result we establish concerns the setup where an array of identical harmonic oscillators (LC tanks) are sandwiched between two layers of RL coupling, i.e., the type of coupling where the connectors are passive LTI resistors and inductors. (An illustration of this setup is produced in Fig. 5). We show that the oscillators asymptotically synchronize if and only if the (complex-valued) effective Laplacian (for the bilayer RL coupling) has a single eigenvalue on the imaginary axis.

Possible contributions of this paper are intended to be in two places. First. The presented effective Laplacian construction makes a generalization of the parallel sum, which has previously been studied under the assumptions that the matrices involved are Hermitian positive semidefinite and that they have finite entries; see, for instance, [2, 13]. Here we make neither of those assumptions. Despite the frequent pattern where the real-world instances make special cases of mathematical generalities, it is curious that here it is the other way around: a real-world instance (bilayer coupling with short circuit connections) leads the way to generalization. Second. We make a first step toward a systematic approach
(which involves studying the spectral properties of the effective Laplacian) for understanding the joint tendencies of electrical oscillators under bilayer coupling. To the best of our knowledge, this is a novelty, for the mature literature on synchronization of harmonic oscillators (see, e.g., [11, 12, 16, 14]) does not seem to provide one with off-the-shelf tools to determine the asymptotic behavior of coupled LC tanks in the absence of a ground node.

2 Effective Laplacian

Our goal in this section is to obtain a closed-form expression for the effective Laplacian representing a pair of coupling layers in series connection. Let us introduce some notation first. The vector of all ones is denoted by \( \mathbf{1}_q \in \mathbb{R}^q \), the identity matrix by \( I \in \mathbb{R}^{q \times q} \). The unit vector \( e_k \in \mathbb{R}^q \) denotes the \( k \)th column of the identity matrix. A \( \text{finite} \) matrix has no infinite entries. Each row of an \text{index matrix} \( E \in \mathbb{R}^{q \times p} \) has exactly one nonzero entry and that entry is 1.

Let \( C \) be a coupling (layer) with \( q \) nodes and \( L \) be its \( q \times q \) Laplacian matrix. Recall that the short circuit connections throughout \( C \) generate in \( L \) some entries that are infinite. Now, even though \( L \) is an accurate description of \( C \), for analysis and computation it is not convenient to work with when it is not finite. Let us therefore propose an equivalent representation of \( C \) that is more manageable. The simple procedure to obtain this representation is as follows: (i) merge the nodes of \( C \) that are connected by short circuit in order to obtain the new coupling \( C_{\text{new}} \) that is free of short circuits and has (fewer) \( p \) nodes; (ii) relabel the nodes of \( C_{\text{new}} \) and construct the corresponding (finite) Laplacian \( G \in \mathbb{C}^{p \times p} \); and (iii) keep record of the labeling in the index matrix \( E \in \mathbb{R}^{q \times p} \) whose \( r \)th column reads \( e_{k_1} + e_{k_2} + \ldots + e_{k_s} \) if the \( r \)th node of \( C_{\text{new}} \) is obtained by merging the nodes of \( C \) with indices \( k_1, k_2, \ldots, k_s \). To illustrate this procedure, consider the coupling \( C \) with \( q = 6 \) nodes given in Fig. 3a. After merging the nodes of \( C \) that are short-circuited, the new coupling \( C_{\text{new}} \) emerges which has \( p = 3 \) nodes, see Fig. 3b. The corresponding Laplacian \( G \) and the index matrix \( E \) reads

\[
G = \begin{bmatrix}
  8 & -5 & -3 \\
 -5 & 7 & -2 \\
 -3 & -2 & 5
\end{bmatrix}, \quad E = \begin{bmatrix}
  0 & 1 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  0 & 1 & 0 \\
  1 & 0 & 0
\end{bmatrix}.
\]

To recapitulate, a coupling \( C \) can be equally well represented either by its Laplacian \( L \) or by some \((G, E)\) pair, obtained via the procedure described above. We will write

\[
(G, E) \in \text{merge } L
\]

when referring to this equivalence. Note that (1) implies \((PTGP, EP) \in \text{merge } L\) for any permutation matrix \( P \in \mathbb{R}^{p \times p} \). Also, for finite \( L \) we can write \((L, I) \in \text{merge } L\). For two pairs \((G, E) \in \text{merge } L_1 \) and \((H, F) \in \text{merge } L_2 \), the expression \((G, E) : (H, F)\) shall denote the effective Laplacian \( L_1 : L_2 \) (for some size \( L_1, L_2 \)).

**Definition 1** A \( q \)-coupling is a pair \((G, E)\) such that \( E \in \mathbb{R}^{q \times p} \) is an index matrix and \( G \in \mathbb{C}^{p \times p} \) satisfies: \( G = G^T \) (it is symmetric); \( \text{Re}(G) \geq 0 \) and \( \text{Im}(G) \geq 0 \) (both its real and imaginary parts are separately positive semidefinite); and \( G \mathbf{1}_p = 0 \) (the vector \( \mathbf{1}_p \) belongs to its null space).

Consider now an electrical network of \( q \) units under bilayer coupling. Let the \( q \)-couplings \((G, E)\) and \((H, F)\) (with \( E \in \mathbb{R}^{q \times p_1} \) and \( F \in \mathbb{R}^{q \times p_2} \)) represent the top and bottom layer connections, respectively; see Fig. 4. The top layer node voltages are denoted by \( x_1, x_2, \ldots, x_q \in \mathbb{C} \) and the bottom layer node voltages by \( y_1, y_2, \ldots, y_q \in \mathbb{C} \) with respect to some (arbitrary) common reference. The current and voltage of the \( k \)th unit are denoted by \( i_k \in \mathbb{C} \) and \( v_k = x_k - y_k \), respectively. Let us construct the vectors \( x = [x_1 \ x_2 \ \cdots \ x_q]^T, \ y = [y_1 \ y_2 \ \cdots \ y_q]^T, \ i = [i_1 \ i_2 \ \cdots \ i_q]^T, \) and \( v = x - y \). Note that we have to have \( x \in \text{range } E \) and \( y \in \text{range } F \). Let therefore the vectors \( a \in \mathbb{C}^{p_1} \) and \( b \in \mathbb{C}^{p_2} \) be such that \( x = Ea \) and \( y = Fb \). We want the set of possible voltages \( v \) to be the entire space \( \mathbb{C}^q \). Hence we let
Figure 3: (a) Coupling $\mathcal{C}$ with short circuit connections. (b) Coupling $\mathcal{C}_{\text{new}}$ after node merging, free of short circuit connections.

rank $|EF| = q$. Taking $G$ and $H$ as node admittance matrices yields $Ga = E^T i$ and $Hb = -F^T i$. Also, we have $Eu - Fb = v$. Combining these three identities, we obtain the equation

$$\begin{bmatrix}
G & 0 & -E^T \\
0 & H & F^T \\
E & -F & 0
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
i
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
v
\end{bmatrix}
$$

which is worthy of analysis.

Figure 4: Network under bilayer coupling. The top layer coupling is represented by the pair $(G, E)$ and the bottom layer by $(H, F)$. The variables $x_1, x_2, \ldots, x_q$ and $y_1, y_2, \ldots, y_q$ denote the node voltages with respect to some common reference.

**Theorem 1** Let the pairs $(G, E)$ and $(H, F)$ (with $E \in \mathbb{R}^{q \times p_1}$ and $F \in \mathbb{R}^{q \times p_2}$) be $q$-couplings and rank $|EF| = q$. Consider the equation

$$\begin{bmatrix}
G & 0 & -E^T \\
0 & H & F^T \\
E & -F & 0
\end{bmatrix}
\begin{bmatrix}
A \\
B \\
\Lambda
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
I
\end{bmatrix}
$$

where $A \in \mathbb{C}^{p_1 \times q}$, $B \in \mathbb{C}^{p_2 \times q}$, and $\Lambda \in \mathbb{C}^{q \times q}$. Eq. (3) admits a solution $(A, B, \Lambda)$ with a unique $\Lambda$. Moreover, the following hold.

1. $\Lambda = \Lambda^T$.
2. Each eigenvalue $\lambda$ of $\Lambda$ satisfies $\text{Re}(\lambda) \geq 0$ and $\text{Im}(\lambda) \geq 0$.
3. $\Lambda 1_q = 0$.
4. If both $G$ and $H$ are real then $\Lambda$ is real and $\Lambda \geq 0$.  


Proof. Existence. A solution \((A, B, \Lambda)\) exists for (3) if and only if range \(M \subset \text{range } N\), which is equivalent to

\[
\text{null } M^* \subset \text{null } N^*.
\] (4)

To establish (4), suppose otherwise, that is, \(M^* \nsubset \text{null } N^*\). This implies that we can find a nonzero vector \(\eta\) satisfying \(M^* \eta = 0\) and \(N^* \eta \neq 0\). Let us partition this vector as \(\eta = [\eta_1^T \eta_2^T \eta_3^T]^T\) where \(\eta_1 \in \mathbb{C}^p, \eta_2 \in \mathbb{C}^q,\) and \(\eta_3 \in \mathbb{C}^r\). Expanding \(M^* \eta = 0\) gives us

\[
\begin{align*}
G^* \eta_1 + E^T \eta_3 &= 0 \quad \text{(5)} \\
H^* \eta_2 - F^T \eta_3 &= 0 \quad \text{(6)} \\
F \eta_2 - E \eta_1 &= 0
\end{align*}
\]

through which we observe

\[
\eta_1^* G^* \eta_1 + \eta_2^* H^* \eta_2 = \eta_1^* (G^* \eta_1 + E^T \eta_3) + \eta_2^* (H^* \eta_2 - F^T \eta_3) + (F \eta_2 - E \eta_1)^* \eta_3 = 0.
\]

Therefore

\[
\left(\eta_1^* [\text{Re}(G)] \eta_1 + \eta_2^* [\text{Re}(H)] \eta_2\right) - j \left(\eta_1^* [\text{Im}(G)] \eta_1 + \eta_2^* [\text{Im}(H)] \eta_2\right) = \eta_1^* G^* \eta_1 + \eta_2^* H^* \eta_2 = 0.
\]

Since the four matrices \(\text{Re}(G), \text{Re}(H), \text{Im}(G), \text{Im}(H)\) are all symmetric positive semidefinite, we have to have \(G^* \eta_1 = 0\) and \(H^* \eta_2 = 0\). Then (5) and (6) yield \(E^T \eta_3 = 0\) and \(F^T \eta_3 = 0\). This allows us to write \(\eta_3^* [E \ F] = 0\), which means \(\eta_3 = 0\) because \(\text{rank } [E \ F] = q\). But \(\eta_3 = 0\) contradicts \(N^* \eta \neq 0\).

Uniqueness & symmetry. Let the triples \((A_1, B_1, \Lambda_1)\) and \((A_2, B_2, \Lambda_2)\) both satisfy (3). We can write

\[
\begin{align*}
\Lambda_1 &= [0 \ 0 \ I] \left[ \begin{array}{c}
-A_1 \\
-B_1 \\
A_1
\end{array} \right] \\
&= \left[ A_1^T \ B_2^T \ A_2^T \right] \left[ \begin{array}{ccc}
G & 0 & E^T \\
0 & H & -F^T \\
-E & F & 0
\end{array} \right] \left[ \begin{array}{c}
-A_1 \\
-B_1 \\
A_1
\end{array} \right] \\
&= \left[ A_1^T \ B_2^T \ A_2^T \right] \left[ \begin{array}{ccc}
0 \\
E^T \\
0
\end{array} \right] \\
&= \Lambda_2^T.
\end{align*}
\] (7)

The choice \((A_1, B_1, \Lambda_1) = (A_2, B_2, \Lambda_2)\) gives us at once the symmetry \(\Lambda_1 = \Lambda_2^T\). Then, thanks to this symmetry, (4) implies the uniqueness \(\Lambda_1 = \Lambda_2\).

Eigenvalues. Let \(\lambda \in \mathbb{C}\) be an eigenvalue of \(\Lambda\) and \(v \in \mathbb{C}^q\) be the corresponding unit eigenvector, i.e., \(\Lambda v = \lambda v\) and \(v^* v = 1\). Multiplying both sides of eq. (3) by \(v\) and letting \(a = Av, b = Bv\) we obtain

\[
\begin{align*}
Ga - \lambda E^T v &= 0 \\
Hb + \lambda F^T v &= 0 \\
E a - F b &= v.
\end{align*}
\]

Using these identities we can write

\[
\begin{align*}
\lambda &= \lambda v^* v \\
&= \lambda (E a - F b)^* v + a^* (Ga - \lambda E^T v) + b^* (Hb + \lambda F^T v) \\
&= a^* Ga + b^* Hb \\
&= \left(a^* [\text{Re}(G)] a + b^* [\text{Re}(H)] b\right) + j \left(a^* [\text{Im}(G)] a + b^* [\text{Im}(H)] b\right).
\end{align*}
\]
Since the four matrices $\text{Re}(G)$, $\text{Re}(H)$, $\text{Im}(G)$, $\text{Im}(H)$ are all symmetric positive semidefinite, it follows that $\text{Re}(\lambda) \geq 0$ and $\text{Im}(\lambda) \geq 0$.

**Null space.** Using eq. (3), the symmetry of $\Lambda$, the identities $H \mathbf{1}_{p_2} = 0$ and $F \mathbf{1}_{p_2} = \mathbf{1}_q$ we can write

$$\Lambda \mathbf{1}_q = \begin{bmatrix} A^T & B^T & \Lambda \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \mathbf{1}_q \end{bmatrix} = \begin{bmatrix} G & 0 & E^T \\ 0 & H & -F^T \\ -E & F & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1_{p_2} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1_{p_2} \\ 1_{p_2} & 0 \end{bmatrix}$$

$$= 0.$$

**Positive semidefiniteness.** Let $G$ and $H$ both be real. Then $G \geq 0$ and $H \geq 0$ follow from Definition 1. Also, now that the matrix $M$ is real, a real solution $(A, B, \Lambda)$ exists for (3). Then $\Lambda \in \mathbb{R}^{q \times q}$ by uniqueness. To show that $\Lambda \geq 0$ let us write (for real $A, B$)

$$\Lambda = \begin{bmatrix} A^T & B^T & \Lambda \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} = \begin{bmatrix} G & 0 & -E^T \\ 0 & H & F^T \\ E & -F & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ \Lambda \end{bmatrix}$$

$$= A^T GA + B^T HB + \Lambda^T EA - \Lambda^T F \Lambda + \Lambda^T F B + B^T F^T \Lambda.$$

Now, since $\Lambda$ is symmetric, the skew-symmetric term on the right-hand side must vanish. This yields $\Lambda = A^T GA + B^T HB$ whence the positive semidefiniteness follows. □

Let us revisit the network under bilayer coupling (shown in Fig. 4) where the $i-v$ relation the coupled units are subject to is embedded in (2). In the light of Theorem 1 it is now clear that this relation has to be $i = \Lambda v$ where the matrix $\Lambda$ uniquely satisfies (3) or, what amounts to the same thing, $\Lambda = N^T M^+ N$. This motivates:

**Definition 2** For a given pair $((G, E), (H, F))$ of $q$-couplings with rank $[E F] = q$, the effective Laplacian $(G, E) : (H, F) \in \mathbb{C}^{q \times q}$ is defined as

$$(G, E) : (H, F) = [0 \ 0 \ I] \begin{bmatrix} G & 0 & -E^T \\ 0 & H & F^T \\ E & -F & 0 \end{bmatrix}^+ \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}$$

and satisfies the properties 1-4 listed in Theorem 1.

We end this section with two straightforward observations.

**Remark 1** $(G, E) : (H, F) = (H, F) : (G, E)$.

**Remark 2** $(G, I) : (H, I) = G(G + H)^+ H$.

### 3 Synchronization of LC tanks

In the second part of the paper we study the synchronization of coupled harmonic oscillators through the spectral properties of the effective Laplacian. A somewhat similar approach is reported in [15] for a
different setup. Consider now a network of \( q \) identical LC tanks under bilayer RL coupling; see Fig. 5 for an example. Each unit obeys

\[
cv_k + \ell^{-1}v_k + u_k = 0, \quad k = 1, 2, \ldots, q
\]

where \( v_k \in \mathbb{R} \) is the voltage of the \( k \)th oscillator, \( u_k \in \mathbb{R} \) is the time derivative of its current, i.e., \( u_k = \frac{d}{dt}i_k \), and the parameters \( \ell, c > 0 \) are the associated inductance and capacitance, respectively. Treating each set of terminals joined by short circuit connectors as a single node, let us let \( a_1, a_2, \ldots, a_p \in \mathbb{R} \) denote the top-layer node voltages and \( b_1, b_2, \ldots, b_p \in \mathbb{R} \) the bottom-layer node voltages (with respect to some common reference). Construct the vectors \( a = [a_1 \ a_2 \ \cdots \ a_p]^T \), \( b = [b_1 \ b_2 \ \cdots \ b_p]^T \), and \( v = [v_1 \ v_2 \ \cdots \ v_q]^T \). These three vectors are related to one another through the relation \( Ea - Fb = v \), where \( E \in \mathbb{R}^{q \times p} \) and \( F \in \mathbb{R}^{q \times p} \) are the the index matrices for the top and bottom layers, respectively.

Now, the constraints imposed on the oscillators by the top layer can be written as

\[
\sum_{r=1}^{p_1} g_{sr}^R (\dot{a}_s - \dot{a}_r) + \sum_{r=1}^{p_1} g_{sr}^R (a_s - a_r) - \sum_{k=1}^{q} e_{ks} u_k = 0, \quad s = 1, 2, \ldots, p_1
\]

where the scalar \( g_{sr}^R = g_{rs}^R \geq 0 \) (measured in \( \Omega^{-1} \)) denotes the conductance of the resistor between the \( s \)th and \( r \)th nodes of the top layer. Likewise, \( g_{sr}^L = g_{rs}^L \geq 0 \) (measured in \( \text{H}^{-1} \)) is the reciprocal of the inductance of the inductor between the \( s \)th and \( r \)th nodes. (We let \( g_{rr}^R = 0 \) and \( g_{rr}^L = 0 \).)

As for \( e_{ks} \in \{0, 1\} \), it is the \( k \)th entry of the index matrix \( E \). Similarly, the bottom layer generates

\[
\sum_{r=1}^{p_2} h_{sr}^R (\dot{b}_s - \dot{b}_r) + \sum_{r=1}^{p_2} h_{sr}^L (b_s - b_r) + \sum_{k=1}^{q} f_{ks} u_k = 0, \quad s = 1, 2, \ldots, p_2
\]

where the scalars \( h_{sr}^R = h_{rs}^R \geq 0 \) characterize the resistive coupling, \( h_{sr}^L = h_{rs}^L \geq 0 \) the inductive coupling. (We let \( h_{rr}^R = 0 \) and \( h_{rr}^L = 0 \).) And \( f_{ks} \in \{0, 1\} \) is the \( k \)th entry of \( F \).

![Figure 5: Network of identical LC tanks under bilayer RL coupling.](image)

Let us express the overall dynamics of the network succinctly. Define the Laplacian \( G_{BR} \in \mathbb{R}^{p_1 \times p_1} \) as

\[
G_{BR} = \begin{bmatrix}
\sum_{r=1} g_{1r}^R & -g_{12}^R & \cdots & -g_{1p_1}^R \\
-g_{21}^R & \sum_{r=1} g_{2r}^R & \cdots & -g_{2p_1}^R \\
\vdots & \vdots & \ddots & \vdots \\
-g_{p_11}^R & -g_{p_12}^R & \cdots & \sum_{r=1} g_{p_1r}^R
\end{bmatrix}.
\]
The Laplacians \( G_{BL} \in \mathbb{R}^{p_1 \times p_1}, H_{BR} \in \mathbb{R}^{p_2 \times p_2}, H_{BL} \in \mathbb{R}^{p_2 \times p_2} \) are constructed similarly. Note that these four matrices are all symmetric positive semidefinite and have zero row sum. Under this new notation, the top-layer coupling is now represented by the triple \((G_{BR}, G_{BL}, E)\) and the bottom-layer coupling by \((H_{BR}, H_{BL}, F)\). For instance, for the network in Fig. 5 these parameters read

\[
G_{BR} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad G_{BL} = \begin{bmatrix} \alpha + 4 & -4 & -\alpha \\ -4 & 5 & -1 \\ -\alpha & -1 & \alpha + 1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
H_{BR} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{bmatrix}, \quad H_{BL} = \begin{bmatrix} 8 & -5 & -3 \\ -5 & 5 & 0 \\ -3 & 0 & 3 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

The entire dynamics can now be compactly written as

\[
\begin{align*}
v' + \omega_0^2 v + u &= 0 \\
G_{BR}a + G_{BL}a - ETu &= 0 \\
H_{BR}b + H_{BL}b + FTu &= 0 \\
Ea - Fb &= v
\end{align*}
\]

where we let \( c = 1 \) (without loss of generality), \( \omega_0 = 1/\sqrt{c} \), and \( u = [u_1 \ u_2 \ \cdots \ u_q]^T \). We are interested in the case where the set of possible voltages \( v \) is the entire space \( \mathbb{R}^q \). Hence we assume \( \text{rank} \ [E \ F] = q \). An interesting question here suggests itself. Under what conditions do the coupled oscillators \((S)\) synchronize, namely, the individual trajectories \( v_1(t), v_2(t), \ldots, v_q(t) \) satisfy \( |v_m(t) - v_k(t)| \to 0 \) as \( t \to \infty \) for all \( (m, k) \) and all initial conditions? Below is a possible answer.

**Theorem 2** The coupled oscillators \((S)\) synchronize if and only if the effective Laplacian

\[
\Lambda = (G_{BR} + jG_{BL}, E) : (H_{BR} + jH_{BL}, F)
\]

has a single eigenvalue on the imaginary axis.

We prove this theorem in two steps.

**Lemma 1** The oscillators \((S)\) fail to synchronize if and only if there exist \( \omega \in \mathbb{R}, \ a \in \mathbb{C}^{p_1}, \ b \in \mathbb{C}^{p_2}, \ \bar{a} \in \mathbb{C}^q, \ \text{and} \ \bar{v} \in \mathbb{C}^q \setminus \text{span} \{1_q\} \) satisfying

\[
\begin{align*}
-\omega^2 \bar{v} + \omega_0^2 \bar{v} + \bar{u} &= 0 \\
G_{BL}\bar{a} - ET\bar{u} &= 0 \\
G_{BR}\bar{a} &= 0 \\
H_{BL}\bar{b} + FT\bar{u} &= 0 \\
H_{BR}\bar{v} &= 0 \\
E\bar{a} - F\bar{b} &= \bar{v}.
\end{align*}
\]

**Proof.** The network \((S)\) comprises LTI passive components. Therefore it is evident physically that the solutions have to be bounded. To demonstrate that let us construct the nonnegative function

\[
W = \frac{1}{2} \left( v^T \dot{v} + \omega_0^2 v^T v + a^T G_{BL} a + b^T H_{BL} b \right).
\]

Combining \((10)\) and \((S)\) yields \( \dot{W} = -a^T G_{BR} \dot{a} - b^T H_{BR} \dot{b} \). Since \( G_{BR}, H_{BR} \geq 0 \) we have \( \dot{W} \leq 0 \). Hence \( W(t) \) remains bounded and with it, so must \( v(t) \). Since \( v(t) \) is produced by an LTI system, it enjoys the form \( v(t) = \sum_k \text{Re} (\pi_k(t) e^{\lambda_k t}) \) where \( \pi_k(t) \) are polynomials with vector coefficients and \( \lambda_k \in \mathbb{C} \) distinct. The boundedness then implies \( \text{Re}(\lambda_k) \leq 0 \) for all \( k \). Also, \( \pi_k(t) \) must be of degree zero (i.e., constant)
when \( \text{Re}(\lambda_k) = 0 \). Let us further add that since we are interested in asymptotic synchronization, the terms with \( \text{Re}(\lambda_k) < 0 \) can be safely ignored, for they vanish as \( t \to \infty \). Suppose now, for certain initial conditions, the oscillators \( \mathbf{\infty} \) fail to synchronize. From the arguments above it follows that we should be able to find a solution \( \mathbf{v}(t) = \mathbf{Re}(\bar{v}(e^{j\omega t})) \) with \( \omega \in \mathbb{R} \) and \( \bar{v} \in \mathbb{C}^q \) satisfying \( \mathbf{\infty} \) together with the node voltages \( a(t) = \mathbf{Re}(\bar{a}(e^{j\omega t})) \) and \( b(t) = \mathbf{Re}(\bar{b}(e^{j\omega t})) \) for some \( \bar{a} \in \mathbb{C}^{p_1} \) and \( \bar{b} \in \mathbb{C}^{p_2} \). Substituting this particular triple \( (\mathbf{v}(t), a(t), b(t)) \) into \( \mathbf{\infty} \) and letting \( \bar{u} = (\omega^2 - \omega_0^2)\bar{v} \) we obtain

\[
\begin{align}
-\omega^2\bar{v} + \omega_0^2\bar{v} + \bar{u} &= 0 \quad (11a) \\
j\omega \bar{G}_{RR}\bar{a} + \bar{G}_{BL}\bar{a} - ET\bar{u} &= 0 \quad (11b) \\
j\omega \bar{H}_{RR}\bar{b} + \bar{H}_{BL}\bar{b} + FT\bar{u} &= 0 \quad (11c) \\
E\bar{a} - F\bar{b} &= \bar{v}. \quad (11d)
\end{align}
\]

Using \( \mathbf{\infty} \) and letting (without loss of generality) \( \bar{v}^*\bar{v} = 1 \) we can write

\[
\begin{align}
\omega^2 - \omega_0^2 &= \bar{v}^*\bar{u} \\
&= (E\bar{a} - F\bar{b})^*\bar{u} \\
&= \bar{a}^*ET\bar{u} - \bar{b}^*FT\bar{u} \\
&= \bar{a}^*(j\omega G_{RR}\bar{a} + G_{BL}\bar{a}) + \bar{b}^*(j\omega H_{RR}\bar{b} + H_{BL}\bar{b}) \\
&= \bar{a}^*G_{BL}\bar{a} + \bar{b}^*H_{BL}\bar{b} + j\omega(\bar{a}^*G_{RR}\bar{a} + \bar{b}^*H_{RR}\bar{b}).
\end{align}
\]

Since the right-hand side has to be real and the matrices \( G_{RR}, H_{RR}, G_{BL}, H_{BL} \) are all symmetric positive semidefinite, we have \( \omega^2 - \omega_0^2 = \bar{a}^*G_{BL}\bar{a} + \bar{b}^*H_{BL}\bar{b} \geq 0 \) and \( j\omega(\bar{a}^*G_{RR}\bar{a} + \bar{b}^*H_{RR}\bar{b}) = 0 \). Observe that \( \omega \) has to be nonzero. Therefore \( G_{RR}\bar{a} = 0 \) and \( H_{RR}\bar{b} = 0 \). Combining this with \( \mathbf{\infty} \) gives us \( \mathbf{\infty} \).

To show the other direction suppose \( \mathbf{\infty} \) is satisfied by some choice of parameters \( \omega, \bar{v}, \bar{a}, \bar{b}, \bar{u} \) with \( \bar{v} \notin \text{span}\{1_q\} \). Clearly, those parameters also satisfy \( \mathbf{\infty} \). Without loss of generality let \( \bar{v}^*\bar{v} = 1 \). We observe that \( \omega \neq 0 \) because \( \omega^2 - \omega_0^2 = \bar{v}^*\bar{u} = (E\bar{a} - F\bar{b})^*\bar{u} = \bar{a}^*ET\bar{u} - \bar{b}^*FT\bar{u} = \bar{a}^*G_{BL}\bar{a} + \bar{b}^*H_{BL}\bar{b} \geq 0 \). Let us now construct the functions \( \mathbf{v}(t) = \mathbf{Re}(\bar{v}(e^{j\omega t})) = [v_1(t) \ v_2(t) \ \cdots \ v_q(t)]^T \), \( a(t) = \mathbf{Re}(\bar{a}(e^{j\omega t})) \), and \( b(t) = \mathbf{Re}(\bar{b}(e^{j\omega t})) \), which satisfy \( \mathbf{\infty} \) because \( \mathbf{\infty} \) holds true. Then, since \( \bar{v} \notin \text{span}\{1_q\} \) and \( \omega \neq 0 \), we have to have \( |v_m(t) - v_k(t)| \neq 0 \) for some pair of indices \( (m, k) \).

**Lemma 2** There exist \( \omega \in \mathbb{R}, \bar{a} \in \mathbb{C}^{p_1}, \bar{b} \in \mathbb{C}^{p_2}, \bar{u} \in \mathbb{C}^q \), and \( \bar{v} \in \mathbb{C}^q \setminus \text{span}\{1_q\} \) satisfying \( \mathbf{\infty} \) if and only if \( \Lambda \) has two or more eigenvalues on the imaginary axis.

**Proof.** By Theorem \( \mathbf{\infty} \) we have \( \Lambda 1_q = 0 \). Therefore \( \lambda_1 = 0 \) is an eigenvalue of \( \Lambda \) with the eigenvector \( 1_q \). Suppose now this eigenvalue at the origin is not the only eigenvalue on the imaginary axis. That is, there exists a second eigenvalue \( \lambda_2 = j\mu \) with \( \mu \in \mathbb{R} \). (We note that \( \mu \geq 0 \) by Theorem \( \mathbf{\infty} \).) This implies there exists a unit eigenvector \( \bar{v} \notin \text{span}\{1_q\} \) satisfying \( \Lambda \bar{v} = j\mu \bar{v} \). This is obvious if \( \lambda_2 \neq 0 \). To see that it is still true even if the eigenvalue at the origin is repeated (i.e., \( \lambda_2 = 0 \)) suppose otherwise. That is, \( 1_q \) is the only eigenvector for the repeated eigenvalue at the origin. This requires that there exists a generalized eigenvector \( \mathbf{\infty} \) satisfying \( \Lambda \mathbf{w} = 1_q \). But then the symmetry \( \Lambda = \Lambda^T \) produces the contradiction \( 0 = (\Lambda 1_q)^T \mathbf{w} = 1_q^T \Lambda 1_q = q \). Consider now \( \mathbf{\infty} \) with \( G = G_{RR} + jG_{BL} \) and \( H = H_{RR} + jH_{BL} \), which is satisfied for some \( A \in \mathbb{C}^{p_1 \times q} \) and \( B \in \mathbb{C}^{p_2 \times q} \). Let \( \bar{a} = A\bar{v} \) and \( \bar{b} = B\bar{v} \). Multiplying both sides of \( \mathbf{\infty} \) by \( \bar{v} \) yields

\[
\begin{bmatrix}
G & 0 & -ET \\
0 & H & FT \\
E & -F & 0
\end{bmatrix}
\begin{bmatrix}
\bar{a} \\
\bar{b} \\
\mu \bar{v}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\bar{v}
\end{bmatrix}
\]

whence (letting \( \bar{u} = \mu \bar{v} \)) we extract

\[
\begin{align}
G_{RR}\bar{a} + j(G_{BL}\bar{a} - ET\bar{u}) &= 0 \quad (12a) \\
H_{RR}\bar{b} + j(H_{BL}\bar{b} + FT\bar{u}) &= 0 \quad (12b) \\
E\bar{a} - F\bar{b} &= \bar{v}. \quad (12c)
\end{align}
\]
Using these identities and \( \bar{v}^* \bar{v} = 1 \) we can write
\[
\begin{align*}
\bar{a}^* G_{\text{BR}} \bar{a} + \bar{b}^* H_{\text{BL}} \bar{b} &= -j \bar{a}^* (G_{\text{BL}} \bar{a} - E^T \bar{a}) - j \bar{b}^* (H_{\text{BL}} \bar{b} + F^T \bar{a}) \\
&= -j (\bar{a}^* G_{\text{BL}} \bar{a} + \bar{b}^* H_{\text{BL}} \bar{b} - (E \bar{a} - F \bar{b})^* \bar{a}) \\
&= -j (\bar{a}^* G_{\text{BL}} \bar{a} + \bar{b}^* H_{\text{BL}} \bar{b} - \bar{v}^* \bar{u}) \\
&= -j (\bar{a}^* G_{\text{BL}} \bar{a} + \bar{b}^* H_{\text{BL}} \bar{b} - \mu).
\end{align*}
\]

Recall that the matrices \( G_{\text{BR}}, H_{\text{BR}}, G_{\text{BL}}, H_{\text{BL}} \) are all symmetric positive semidefinite and \( \mu \) is real. This implies that the left-hand side of the equation is purely real whereas the right-hand side is purely imaginary. This is only possible if the both sides are zero. This at once gives us
\[
\begin{align*}
G_{\text{BR}} \bar{a} &= 0, \\
H_{\text{BL}} \bar{b} &= 0. \\
\end{align*}
\]
(13a)
(13b)

Finally, defining \( \omega = \sqrt{\omega_0^2 + \mu} \) we can rewrite \( \bar{u} = \mu \bar{v} \) as
\[
-\omega^2 \bar{v} + \omega_0^2 \bar{v} + \bar{u} = 0.
\]
(14)

Combining (12), (13), (14) then yields (9).

Now we show the other direction. Suppose (9) holds for some \( \omega \in \mathbb{R}, \bar{a} \in \mathbb{C}^{p_1}, \bar{b} \in \mathbb{C}^{p_2}, \bar{u} \in \mathbb{C}^q \), and \( \bar{v} \in \mathbb{C}^q \setminus \text{span} \{1_q\} \). Defining the real number \( \mu = \omega^2 - \omega_0^2 \) and resorting once again to the shortcuts \( G = G_{\text{BR}} + jG_{\text{BL}} \) and \( H = H_{\text{BR}} + jH_{\text{BL}} \) we can mold (9) into
\[
\begin{bmatrix}
G & 0 & E^T \\
0 & H & -F^T \\
-E & F & 0
\end{bmatrix}
\begin{bmatrix}
-\bar{a} \\
-\bar{b} \\
j \mu \bar{v}
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \bar{v}.
\]

Choose some \( A \in \mathbb{C}^{p_1 \times q} \) and \( B \in \mathbb{C}^{p_2 \times q} \) satisfying (3). Using the symmetries of \( G, H, \Lambda \) we can write
\[
\begin{align*}
\Lambda \bar{v} &= \begin{bmatrix} A^T & B^T \end{bmatrix} \begin{bmatrix}
0 & I \\
0 & I
\end{bmatrix} \bar{v} \\
&= \begin{bmatrix} A^T & B^T \end{bmatrix} \begin{bmatrix}
G & 0 & E^T \\
0 & H & -F^T \\
-E & F & 0
\end{bmatrix}
\begin{bmatrix}
-\bar{a} \\
-\bar{b} \\
j \mu \bar{v}
\end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \end{bmatrix}
\begin{bmatrix}
-\bar{a} \\
-\bar{b} \\
j \mu \bar{v}
\end{bmatrix} \\
&= j \mu \bar{v}.
\end{align*}
\]

Recall \( \Lambda 1_q = 0 \). Then \( \Lambda \bar{v} = j \mu \bar{v} \) implies \( \Lambda \) has at least two eigenvalues on the imaginary axis because \( \bar{v} \notin \text{span} \{1_q\} \) and \( \mu \) is real.

\[\blacksquare\]

**Proof of Theorem 2** Combine Lemmas 1, 2

Consider the network of coupled LC tanks shown in Fig. 3. For the parameter choice \( \alpha = 1 \) the eigenvalues of the effective Laplacian \( \Lambda \in \mathbb{C}^{4 \times 4} \) can be computed to be \( \{\lambda_1 = 0, \lambda_2 = 0.5795 + j1.8886, \lambda_3 = 0.6283 + j4.1990, \lambda_4 = 1.4393 + j11.3242\} \). Since \( \lambda_1 = 0 \) is the only eigenvalue on the imaginary axis, by Theorem 2 we can say that the oscillators will asymptotically synchronize when \( \alpha = 1 \). For \( \alpha = 4 \), however, the eigenvalues read \( \lambda_1 = 0, \lambda_2 = j6, \lambda_3 = 1.1989 + j11.3818, \lambda_4 = 1.3931 + j2.3622 \). This time there are two eigenvalues on the imaginary axis, namely, \( \lambda_1 = 0 \) and \( \lambda_2 = j6 \). Therefore the oscillators are not guaranteed to synchronize for this case. This example tells us that synchronization cannot be determined merely by the structure of the coupling. In other words, without the actual parameter values, knowing only which oscillator is connected to which and by what type of connector is in general not sufficient to make definite conclusions about the collective behavior of the oscillators.
4 Conclusion

In this paper we studied networks under bilayer coupling, where each layer is represented by a separate Laplacian matrix. For this framework we first showed that the overall coupling can be described by a single Laplacian, called the effective Laplacian, whose construction enjoyed a certain generalization of the parallel sum of matrices. Then we used this matrix in the analysis of asymptotic synchronization in a network of harmonic oscillators under bilayer coupling. There we discovered that the oscillators synchronize if and only if the effective Laplacian has a single eigenvalue on the imaginary axis.

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