Chiral Symmetry in Dirac Equation and its Effects on Neutrino Masses and Dark Matter

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Abstract

Chiral symmetry is included into the Dirac equation using the irreducible representations of the Poincaré group. The symmetry introduces the chiral angle that specifies the chiral basis. It is shown that the correct identification of these basis allows explaining small masses of neutrinos and predicting a new candidate for Dark Matter massive particle.

1 Introduction

The work of Wigner [1] was among the first to highlight the crucial role of group theory in Quantum Field Theory (QFT). He classified the irreducible representations (irreps) of the Poincaré group and identified an elementary particle with an unitary irrep of the group [2]; the explication of single particle states as the irreps of the Poincaré group has done much to validate the physical foundations of modern particle physics (e.g., [3]). As elucidated in previous works [4-7], such group theoretical notions may be utilized to define a fundamental theory as satisfying the following principles: the principle of Poincaré and gauge invariance, the principle of locality, and the principle of least action.

Among the fundamental equations of QFT, the Dirac equation [8] plays a special role because it describes fermionic fields, which include quarks and leptons of the Standard Model of particle physics [3]. The original Dirac method to obtain this equation [9] is often replaced in QFT textbooks by giving the required Lagrangian but without any explanation of its first principle origin (e.g., [10]). Other methods to derive the Dirac equation include the Lorentz transformations of four-component spinors [3] and the Bargmann-Wigner approach [4] that is based on the Poincaré group of the Minkowski metric of QFT. The general structure of the Poincaré group is $\mathcal{P} = SO(3, 1) \otimes s T(3 + 1)$, where
\(SO(3,1)\) is a non-invariant Lorentz group of rotations and boosts and \(T(3+1)\) an invariant subgroup of spacetime translations, and this structure includes reversal of parity and time [2].

In this paper, we present a novel method that uses the irreps of \(T(3+1)\) to derive the Dirac equation with chiral symmetry. There are two main aims of this paper, namely, to derive the Dirac equation with chiral symmetry and discuss its far reaching physical implications. Throughout this paper, we refer to our new equation as the Dirac equation with chiral symmetry (DECS) to make distinction from the commonly used name 'generalized Dirac equation' in many published papers.

Different generalizations of the Dirac equation (DE) were previously presented and, in general, they can be divided into two categories: those that derived the equation and those that just add \textit{ad hoc} terms to it. More specifically, some obtained equations were used to either unify leptons and quarks [11-13], or account for different masses of three generations of elementary particles [14-16], or extend the mass term to include \textit{ad hoc} a pseudoscalar mass [17]. In other generalizations, the DE was derived for distances comparable to the Planck length [18], or with the external magnetic field included [19], or even for higher integer and half-integer spins [20], which requires combining the Dirac [8] and Klein-Gordon [21,22] equations. Another generalization of the DE involved changing a phase factor in four-component spinors, which allowed for different masses in the equation [23].

The Dirac equation with chiral symmetry derived in this paper is new and its physical implications are far reaching. First, our derivation of the DECS demonstrates that this equation can be obtained from the eigenvalue equation that represents the condition required by the four-component spinors to transform as one of the irreps of the Poincaré group \(P\) extended by parity. Second, the eigenvalue equation allows deriving either the DE or the DECS. Third, the DE is obtained by factorization of the Klein-Gordon equation if, and only if, a specific choice of chiral basis is selected. Fourth, as compared to the DE, there is an extra mass term in the DECS and its properties allow identifying it with pseudo-scalar mass.

Our formal derivation of the pseudo-scalar mass term and relating it directly to chiral symmetry gives physical justification for the existence of this term and allows us to discuss its physical implications; this makes our approach so different from the previously \textit{ad hoc} addition of pseudo-scalar mass to the DE without neither justifying its physical presence nor its origin [17]. Our obtained results demonstrate that this pseudo-scalar mass in the DECS, its relationship to chiral symmetry and the resulting pseudo-scalar Higgs can be used to explain smallness of neutrino masses [24,25], and also properties Dark Matter (DM) particles [26,27].
The paper is organized as follows: the eigenvalue equation for bispinors are given in Section 2; the fundamental equation for bispinors with chiral symmetry is derived in Section 3; physical implications of the obtained results are discussed in Section 4; and conclusions are presented in Section 5.

2 Eigenvalue equation for bispinors

The condition that a scalar wavefunction $\phi$ transforms as one of the irreps of $T(3+1) \subset P$ is given by the following eigenvalue equation \[ i\partial_\mu \phi = k_\mu \phi , \] where $k_\mu$ labels the irreps. To generalize this result to Dirac spinors, called also bispinors, we follow Wigner \[1,2\], who proved that the proper irreps of spin-1/2 elementary particles are the four-component bispinors $\psi$ for which the eigenvalue equation given by Eq. (1) becomes

\[ iA_\mu \partial_\mu \psi = A_\mu k_\mu \psi , \]

where $A_\mu$ is an arbitrary constant matrix of $4 \times 4$. Defining $X_\mu = -iA_\mu$ and $Y = A_\mu k_\mu$, we obtain

\[ (X_\mu \partial_\mu + Y)\psi = 0 , \]

with $X_\mu$ and $Y$ to be determined. This is the general condition for the bispinors to transform as one of the irreps of $P$ and this condition will be now used to derive the DECS for the four-component bispinors given by

\[ \psi = \begin{bmatrix} \chi_L \\ \chi_R \end{bmatrix} \] \[ (4) \]

where $\chi_L$ and $\chi_R$ are two component of bispinors. The necessity of coupling these spinors is understood mathematically as accommodating the sign ambiguity introduced in the construction of the isomorphism between boosts in SO(3,1) and those in SU(2). As a result of this, we find $\chi_L$ and $\chi_R$ to transform identically under rotations, but oppositely under boosts \[3\]. We may thus write our Lorentz transformation for the bispinors as

\[ \Lambda = \begin{bmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{bmatrix} = \begin{bmatrix} \exp \left( i\tilde{\sigma} \cdot (\tilde{\theta} - i\tilde{\phi}) \right) & 0 \\ 0 & \exp \left( i\tilde{\sigma} \cdot (\tilde{\theta} + i\tilde{\phi}) \right) \end{bmatrix} \]

where $\tilde{\theta}$ and $\tilde{\phi}$ parameterize our rotations and boosts, respectively, and are related to the transformations of four vectors via the four-by-generators $\tilde{J}$ and

\[ (5) \]
such that
\[ \hat{\Lambda} = \exp \left( i \bar{J} \cdot \bar{\theta} + i \bar{K} \cdot \bar{\phi} \right) . \]

### 3 Fundamental equation for bispinors with chirial symmetry

To determine forms of the matrices \( X^\mu \) and \( Y \), the Lorentz transformation must be applied to the RHS and LHS of Eqs. (1) to (3), and this yields
\[
\left( (\Lambda^{-1} \hat{\Lambda}^\nu X^\mu \Lambda) \partial_\nu + (\Lambda^{-1} Y \Lambda) \right) \psi = 0 .
\]
(6)

This leads to the necessary conditions for invariance
\[
\hat{\Lambda}^\nu X^\mu = \Lambda X^\nu \Lambda^{-1} \quad \text{and} \quad Y = \Lambda Y \Lambda^{-1} .
\]
(7)

Solving these, we find the most general form of our matrix coefficients written in block form

\[
X^\mu = \begin{bmatrix}
0 & x_R (\sigma^0 \delta^\mu_0 + \sigma^k \delta^\mu_k) \\
x_L (\sigma^0 \delta^\mu_0 - \sigma^k \delta^\mu_k) & 0
\end{bmatrix}
\]
(8)

\[
Y = \begin{bmatrix}
y_L \sigma^0 & 0 \\
0 & y_R \sigma^0
\end{bmatrix}
\]
(9)

where \( x_R, x_L, y_R, \) and \( y_L \) are free parameters.

Taking the Dirac \( \gamma \) matrices in the Weyl basis

\[
\gamma^0 = \begin{bmatrix}
0 & \sigma^0 \\
\sigma^0 & 0
\end{bmatrix}, \quad \gamma^k = \begin{bmatrix}
0 & \sigma^k \\
\sigma^k & 0
\end{bmatrix}
\]
(10)

and identifying the chiral projection operators as

\[
P_L = \begin{bmatrix}
\sigma^0 & 0 \\
0 & 0
\end{bmatrix}, \quad P_R = \begin{bmatrix}
0 & 0 \\
0 & \sigma^0
\end{bmatrix}
\]
(11)

we obtain
\[
\left( (x_L P_R + x_R P_L) \gamma^\mu \partial_\mu + (y_L P_L + y_R P_R) \right) \psi = 0 .
\]
(12)
Now, under the assumption that \( x_L \) and \( x_R \) are nonzero we are free to multiply from left with \( i(x_L P_R + x_R P_L)^{-1} \), and obtain
\[
\left( i\gamma^\mu \partial_\mu + i \left( \frac{y_L}{x_R} P_L + \frac{y_R}{x_L} P_R \right) \right) \psi = 0,
\]
which shows that there are only two independent degrees of freedom in the derived equation. To identify the physical basis of these degrees, we observe that Eq. (13) gives the following squared-Hamiltonian
\[
\mathcal{H}^2 \psi = \left( \partial^k \partial_k - \frac{y_L y_R}{x_L x_R} \right) \psi,
\]
and thus we find the emergence of the propagation mass term
\[
m \equiv \pm i \sqrt{\frac{y_L y_R}{x_L x_R}}.
\]
The restriction of the square of Eq. (15) to positive real numbers is equivalent to the physical restriction of Einstein energy-momentum relationship. Our remaining degree of freedom may be identified with the choice of a chiral basis. Let us define the chiral angle as
\[
\alpha \equiv -\frac{i}{2} \ln \left( \mp \sqrt{\frac{x_L y_L}{x_R y_R}} \right).
\]
Thus, the final compact form of our Dirac equation with chiral symmetry is
\[
(i\gamma^\mu \partial_\mu - m e^{-2i\alpha\gamma^5}) \psi = 0.
\]
Since this equation is Poincaré invariant, it is the fundamental equation of physics. This equation reduces to the original Dirac equation when \( \alpha = 0 \).

4 Chiral symmetry and pseudoscalar mass

A consequence of Eq. (17) is made explicit by proffering an alternative, suggestive parameterization. Let us define
\[
M \equiv -\frac{i}{2} \left( \frac{y_R}{x_L} + \frac{y_L}{x_R} \right) = m \cos 2\alpha, \quad \widetilde{M} \equiv -\frac{i}{2} \left( \frac{y_R}{x_L} - \frac{y_L}{x_R} \right) = -im \sin 2\alpha,
\]
so that Eq. (17) becomes
\[
(i\gamma^\mu \partial_\mu - M - \widetilde{M} \gamma^5) \psi = 0.
\]
This form admits the simultaneous validity of both fundamental scalar and fundamental pseudoscalar mass terms. To better understand these mass terms, consider a global chiral transformation given by

$$\psi \rightarrow \psi' = e^{i\gamma^5 \beta} \psi.$$  \hspace{1cm} (20)

The Lagrangian of Eq. (20) may be written as

$$\mathcal{L} = i\bar{\psi} \gamma^\mu \partial_\mu \psi - \bar{\psi}(M + \tilde{M}\gamma^5)\psi,$$  \hspace{1cm} (21)

and under the transformation of (20) this becomes

$$\mathcal{L}' = i\bar{\psi} \gamma^\mu \partial_\mu \psi - \bar{\psi}(M + \tilde{M}\gamma^5)e^{-2i\beta\gamma^5} \psi,$$  \hspace{1cm} (22)

so we the effect of (20) is equivalent to the rotation of our mass parameters

$$\begin{bmatrix} M' \\ i\tilde{M}' \end{bmatrix} = \begin{bmatrix} \cos 2\beta & -\sin 2\beta \\ \sin 2\beta & \cos 2\beta \end{bmatrix} \begin{bmatrix} M \\ i\tilde{M} \end{bmatrix} = \begin{bmatrix} m \cos[2(\alpha + \beta)] \\ m \sin[2(\alpha + \beta)] \end{bmatrix}.$$  \hspace{1cm} (23)

This is precisely the transformation required to leave invariant the propagation mass in Eq. (17). Note that for a single field, we may always apply a chiral rotation so as to reorient our chiral axes and force the pseudoscalar mass to vanish. However, this is not a valid procedure in the case of multipartite systems whose constituent fields obey Eq. (17) for different chiral angles. Therefore, we conclude that it is only the nonequivalence of chiral basis that gives rise to physically observable effects.

We found that the chiral rotation of a massive field is equivalent to an alternative choice of chiral basis. This, in turn, is an alternative factorization of the Klein-Gordon equation (i.e. the Einstein energy relationship). These nonstandard factorizations necessarily redistribute the fraction of the mass available to the field between the left- and right-chiral components. Our work presented in this paper significantly differs from the previous studies of both chiral symmtery and generalization of Dirac equation.

Our main result is first-principle proof of the necessity of specifying the chiral basis in which the fundamental induced representations of the Poincaré group reside. This is not a mere mathematical triviality, but rather the necessary consequence of considering fundamental physical symmetries in flat space-time and its anticipated physical implications that are now presented and discussed.
5 Physical implications

5.1 Yukawa-generated masses

As demonstrated above, together $m$ and $\alpha$ define the infinite parameter space in which all first-order Poincare-invariant equations for massive spin-1/2 particles must reside. The simplest and most physically-motivated selection process for the determination of $m$ and $\alpha$ is obtained by the interpretation of scalar and pseudoscalar mass terms as having their origin in Yukawa couplings to real scalar fields with non-vanishing vacuum expectation values (VEVs).

Treating these fields independently and expanding them about the ground state, we may write the relevant contributions of these coupling to the Lagrangian as

$$\mathcal{L}_Y = -\lambda_1 \bar{\psi}\phi_1 \psi - \lambda_2 \bar{\psi}\phi_2 \gamma^5 \psi$$
$$\approx -\frac{\lambda_1 v_1}{\sqrt{2}} \bar{\psi}\psi - \frac{\lambda_2 v_2}{\sqrt{2}} \bar{\psi}\gamma^5 \psi ,$$ (24)

where the $v_1$ and $v_2$ are the VEVs of the two fields $\phi_1$ and $\phi_2$, which are treated here as independent. It is easy to restrict this model to encompass a single field with a complicated set of coupling parameters. The coupling constants $\lambda_1$, $\lambda_2$ and VEVs are related to our parameters $m$ and $\alpha$ via

$$m = \sqrt{\frac{\lambda_1^2 v_1^2 - \lambda_2^2 v_2^2}{2}}$$
$$\alpha = \frac{i}{4} \ln \left( \frac{\lambda_1 v_1 + \lambda_2 v_2}{\lambda_1 v_1 - \lambda_2 v_2} \right) ,$$ (25)

which allows both $\alpha$ and $m$ to be completely fixed by the VEVs of the fields $\phi_1$ and $\phi_2$ and their respective coupling constants.

5.2 Neutrino masses

In the context of Eq. (24), the case of neutrino fields takes on an interesting dimension by offering a potential solution to an open problem. In the Standard Model [3], right-chiral leptons form singlets that preclude the existence of right-chiral neutrinos. However, if this condition is enforced, then the neutrino field $\nu$ becomes

$$\frac{1}{2}(1 - \gamma^5)\nu = \nu$$
$$\frac{1}{2}(1 + \gamma^5)\nu = 0 .$$ (26)
Combining these constraints with Eq. (24), we find
\[ \mathcal{L}_Y = -\frac{1}{2\sqrt{2}} \tilde{\nu}(\lambda_1 v_1 - \lambda_2 v_2)\nu . \]  
(27)

Thus, the neutrino mass in the Dirac Lagrangian appears as
\[ m_{\nu} = \frac{1}{2\sqrt{2}}(\lambda_1 v_1 - \lambda_2 v_2) . \]  
(28)

The obtained results demonstrate that the mass of a left-chiral field generated by coupling to the Standard Model (SM) Higgs field may be suppressed by a non-zero real pseudoscalar coupling to an external scalar field with non-vanishing VEV. This is clearly of potential relevance when considering why the observed masses of neutrinos vary from those of the other fermions by orders of magnitudes. Many different explanations have been offered to account for this discrepancy [24,25], but the mechanism presented here is particularly appealing as it suggests the smallness of the neutrino masses is the necessary consequence of the non-existence of right-chiral neutrinos; more detailed calculations of this phenomenon will be presented elsewhere.

Let us point out that the most immediate testable consequence of additional neutrino-scalar couplings is the necessary modification of elastic scattering cross sections. While in the absence of interference phenomena it is likely for modifications to the SM predictions to be sub-leading, evidences of deviation from SM predictions have already been obtained by the LSND [28] and MiniBooNE [29] collaborations. So far these anomalies have refuted explanation. However, additional precision measurements being preformed by the MicroBooNE collaboration and to-be-performed by the DUNE collaboration will provide further constraints and may–given long enough livetimes–obtain constraints sufficient to definitively probe the validity of the model proposed.

5.3 Dark matter candidate

Another attractive aspect of this mass-generating mechanism is the natural inclusion of a dark matter candidate. The evidence for the existence of particle dark matter is strongly supported by the astronomical observations of galactic rotational curves and stability of clusters of galaxies [31,32] and while there have been numerous theories postulated different elementary particles [27,33,34], so far the existence of those particles have not yet been verified experimentally [35-37]. As possible candidates, weakly interacting massive particles (WIMPs), supersymmetric (SUSY) particles like neutralinos [33], axions [38], and extremly light bosonic particles (ELBPs) suggested by [39,40] and showed to be physically unacceptable by [41,42].
All evidence suggests dark matter couples to ordinary matter primarily through mass-mass terms, ostensibly gravitational [43,44]. All other couplings are, at most, sub-leading contributions to DM interactions. Thus, the existence of a chiral mass may offer insight into these observations in several ways. First, it is possible that pseudoscalar mass contribution to the gravitational field differs from those of scalar mass terms; an in-depth investigation of that possibility is beyond the scope of this paper. Second, the results obtained in this work make also plausible the existence of a massive $\phi_2$ particle that couples minimally and exclusively to chirally-asymmetric fields (neutrinos) and thereby satisfies many of the required characteristics of dark matter: a minimal non-gravitational coupling with a restricted phase space for decay resulting in a long-lived particle [45]. The contribution of this postulated field to gravitational interactions depends on the mass and abundance of the postulated particle, both of which cannot be determined by the presented theory.

As a last remark, we note a distinguishing characteristic of this model is the wide range of allowable inertial (and thus gravitational) masses consistent with Higgs-portal and direct-detection observations. For massive DM particles, we must amend the aforementioned exclusivity of interactions to include scalar as well as chiral-asymmetric couplings. Then, we must allow the mass of the postulated particle to be generated by couplings to both $\phi_1$ (nominally the standard model Higgs) and itself. This method of mass generation allows for enhancement of the inertial mass and admits a mass parameter that differs non-trivially from its Higgs-coupling parameter. Thus, the proposed model may plausibly span the intermediate parameter space inaccessible to both ELBP and WIMP models. Intriguingly, a non-zero reciprocal coupling between $\phi_1$ and $\phi_2$ may have important implications for the renormalization of the Higgs self-energy in the absence of supersymmetric models.

6 Conclusions

In this paper, a group-theoretical derivation of the most general Poincaré-invariant Dirac equation with chiral symmetry, which introduces pseudo-scalar mass to the equation, is presented. The derivation is based on the eigenvalue equation that represents the condition required by the four-component spinors to transform as one of the irreps of the Poincaré group. It is shown that the chiral rotation of a massive field is equivalent to a choice of chiral basis and that this choice must be specified in order to factorize the Klein-Gordon equation and derive the original Dirac equation.

The derived Dirac equation with chiral symmetry has two additional degrees of freedom, which are identified with the propagation mass ($m$) and the chiral angle ($\alpha$). Being a physical parameter, $\alpha$ specifies the chiral basis of the spinor
solutions and therefore its value must be physically justified. The existence of the pseudo-scalar mass generating Higgs-like couplings is an interesting phenomenon as this pseudo-scalar Higgs when combined with the suppression of right-chiral neutrinos offers a natural mechanism for producing anomalously small left-chiral neutrino masses. Moreover, this pseudo-scalar Higgs field satisfies many of the requirements for a DM candidate, whose existence can be verified experimentally.

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