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Fuzzy filtering-based fault detection for a class of discrete-time conic-type nonlinear systems

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Abstract
The authors investigate the problem of fuzzy fault detection filter (FFDF) design for a class of discrete-time conic-type nonlinear systems. By applying Takagi–Sugeno fuzzy models, the conic-type dynamic FFDF system is established. Then, utilizing the Lyapunov function method to find a sufficient condition which ensures that the conic-type dynamic FFDF system is asymptotically stable. After that, using linear matrix inequalities techniques, the FFDF design problem is transformed into an optimization algorithm. Finally, the simulation results demonstrate that the designed FFDF is effective for detecting the faults.

1 | INTRODUCTION

In recent years, due to the increasing requirements for security and stability in modern industrial systems, the fault detection (FD) problem of the dynamic systems has been widely studied by many scholars [1–3]. In the existing FD results, the model-based FD [4–6] is a more effective one. Within the FD procedure, the residual signal can be obtained by designing a measurable output estimator and construct an evaluation function to judge whether the faults occur. Then, a predefined threshold has to be selected. If the evaluation function exceeds the given threshold, a fault alarm will be generated. As a kind of model-dependent FD methods, the $H_\infty$ filtering-based scheme is widely used. In the $H_\infty$ filtering formulation, an $H_\infty$ performance index is designed that makes the fuzzy fault detection filter (FFDF) system be sensitive to faults and robust to external disturbances. In fact, the $H_\infty$ filtering problem has been widely adopted to detect the faults for uncertain systems [7,8], discrete systems [9,10], nonlinear systems [11,12], etc.

On the other hand, by employing a group of fuzzy IF-THEN rules, the Takagi–Sugeno (T–S) fuzzy models were first presented in [13]. By this linearized method, T–S fuzzy models are employed to denote the local linear relations associated with the nonlinear systems [14]. Applying the fuzzy membership functions, the corresponding T–S fuzzy model is established and a reasonable framework for representing the nonlinear systems by a range of linear models is given. Based on this framework, the robust control theory and technology is applied in linear systems to the design of complex nonlinear systems. Hence, many significant consequences about the T–S fuzzy models have been published, such as [15–18] and the references therein. In [19], through switching the fuzzy model, the relaxed stability criterion of T–S fuzzy control systems was studied. Considering the discontinuous measurements in T–S fuzzy systems, the FD scheme was designed [20]. In [21], by applying the fuzzy model to linearize nonlinear systems, the FD problem of nonlinear Markov jump systems was investigated.

As a special kind of nonlinearities, the so-called conic-type nonlinear dynamics have attracted a great deal of attention recently. Practically, a wide variety of engineering nonlinearities may be converted into conic-type nonlinearities, such as, Lipschitz nonlinearities, locally sinusoidal nonlinearities, diodes and amplifiers with dead zone nonlinearities, etc. In [22], the problem of robust $H_\infty$ control for conic nonlinear stochastic jump systems with partially unknown transition probabilities was investigated. In [23], a second-order conic-type programming was studied to deal with disparate applications in power systems, such as operation and expansion planning. In [24], some significant works of the finite-time sliding mode control strategy on a class of conic-type nonlinear system was realised. As far as we know, the FD problems for conic-type nonlinear
systems have yet been comprehensively investigated. It prompts us to study this topic.

This study considers the design problem of FFDF for a class of discrete-time conic-type nonlinear systems by $H_{\infty}$ filtering-based method. Initially, the conic-type FFDF system is constructed based on T-S fuzzy models. Then, an optimal FFDF needs to be designed that satisfies the $H_{\infty}$ filtering index to minimise the difference between the designed FFDF and the reference model. Moreover, by applying linear matrix inequalities (LMIs) techniques [25] and the Lyapunov function method, sufficient conditions for the existence of the FFDF are verified. Finally, the FFDF design problem is transformed into an optimisation algorithm. The effectiveness of the designed method is illustrated by simulation examples.

To better understand the proposal in this paper, the following flow chart is designed to describe the process of FD in Figure 1.

In Figure 1, the residual signal $r(k)$ of the dynamic FFDF system is generated by $y(k)$ and $r_f(k)$. Then the $H_{\infty}$ filtering problem is converted to minimise the difference between the reference model and the FFDF to be designed. Besides, an appropriate threshold and an evaluation function have to be selected. Once the value of residual evaluation function exceeds the pre-defined threshold, an alarm of faults is generated.

The symbols presented in this study are listed in Table 1.

### 2 | PROBLEM FORMULATION

Consider a class of discrete-time conic-type nonlinear systems formulated by:

\[
\begin{align*}
    x(k+1) &= \Psi(x(k), d(k), f(k)) + Bu(k), \\
    y(k) &= Cx(k) + Dd(k) + D_f f(k),
\end{align*}
\]

where $x(k) \in \mathbb{R}^n$ is the state, $y(k) \in \mathbb{R}^m$ is the measured output, $u(k) \in \mathbb{R}^r$ is the controlled input, $d(t) \in L_2^2[0, \infty]$ is the external disturbance and $f(k) \in L_2^2[0, \infty]$ is the fault signal. Nonlinear function $\Psi(x(k), d(k), f(k))$ satisfies the following conic sector description: $2\|\Psi(x(k), d(k), f(k)) - [Ax(k) + B_d d(k) + B_f f(k)]\| \leq \|Gx(k) + Ed(k) + Hf(k)\|$

Remark 1 The conic-type nonlinear function $\Psi(x(k), d(k), f(k))$ lies in an $n$-dimensional hypersphere whose centre is a linear system described by $Ax(k) + B_d d(k) + B_f f(k)$, and whose radius is bounded by another linear system $Gx(k) + Ed(k) + Hf(k)$.

Further, conic-type nonlinear systems (1) can be described by T-S fuzzy models as stated by authors and are as follows: System Rule $i$:

If $\varpi_i(k) = F_{i1}, \varpi_2(k) = F_{i2}, ..., \varpi_S(k) = F_{iS}$ Then

\[
\begin{align*}
    x(k+1) &= A_{i} x(k) + B_{di} d(k) + B_{fi} f(k), \\
    y(k) &= C_{i} x(k) + D_{di} d(k) + D_{fi} f(k),
\end{align*}
\]

where $A_{i}, B_{di}, B_{fi}, C_{i}, D_{di}, D_{fi}$ are known constant matrices, $\varpi_i(k) = [\varpi_{i1}(k), \varpi_{i2}(k), ..., \varpi_{iS}(k)]$ is the premise variable. $F_{is}, i = 1, 2, ..., S$ represent the fuzzy sets and $R$ is the number of IF-THEN rules.

**Assumption 1** $g(k)$ is a nonlinear function which satisfies $g(k) = \Psi(x(k), d(k), f(k)) - [Ax(k) + B_d d(k) + B_f f(k)]$. Then, we have:

\[
g^T(k)g(k) \leq (G_{i} x(k) + E_{i} d(k) + H_{i} f(k))^T \times (G_{i} x(k) + E_{i} d(k) + H_{i} f(k)),
\]

where $G_{i}, E_{i}, H_{i}$ are known constant matrices with appropriate dimensions.

**TABLE 1** The notations

| Notation | Denotes |
|----------|---------|
| $\mathbb{R}^n$ | $n$-dimensional Euclidean space |
| $\mathbb{R}^{n \times m}$ | $n \times m$ real matrix |
| $I$ | Unit matrix |
| $0$ | Zero matrix |
| $A^{-1}$ | Matrix inverse |
| $A^T$ | Matrix transpose |
| $\text{diag}[A B]$ | Block-diagonal matrix of $A$ and $B$ |
| $*$ | Symmetric matrix |
| $P > (\leq, \geq)$ | Positive (negative, non-negative, non-positive) |
| $\leq 0$ | -Definite matrix |
The overall conic-type fuzzy model is given by:

\[
\begin{align*}
    x(k + 1) &= \sum_{i=1}^{R} b_i(\omega(k))[A_i x(k) + B_d d(k)] \\
    &\quad + B_f f(k) + B_i u(k) + g(k), \\
    y(k) &= \sum_{i=1}^{R} b_i(\omega(k))[C_i x(k) + D_d d(k) + D_f f(k)],
\end{align*}
\]

where

\[
b_i(\omega(k)) = \frac{\prod_{i=1}^{R} \mu_i(\omega_i(k))}{\sum_{i=1}^{R} \prod_{i=1}^{R} \mu_i(\omega_i(k))},
\]

in which \(\mu_i(\omega_i(k))\) is the grade of the membership of \(\omega_i(k)\). In addition, the fuzzy basis function satisfies:

\[
b_i(\omega(k)) \geq 0, \quad \sum_{i=1}^{R} b_i(\omega(k)) = 1, \quad i = 1, 2, ..., R.
\]

Further, we are interested in designing the following fuzzy filter:

Filter Rule \(i\):

If \(\omega_1(k) = F_{1i}, \omega_2(k) = F_{2i}, \ldots,\) and \(\omega_R(k) = F_{Ri}\), then

\[
\begin{align*}
    \bar{x}(k + 1) &= A_{F_i} \bar{x}(k) + B_{F_i} y(k), \\
    r_{ij}(k) &= C_{F_i} \bar{x}(k) + D_{F_i} y(k).
\end{align*}
\]

The global T-S FFDF model can be constructed as:

\[
\begin{align*}
    \bar{x}(k + 1) &= \sum_{i=1}^{R} b_i(\omega(k))[A_{F_i} \bar{x}(k) + B_{F_i} y(k)], \\
    r_{ij}(k) &= \sum_{i=1}^{R} b_i(\omega(k))[C_{F_i} \bar{x}(k) + D_{F_i} y(k)],
\end{align*}
\]

where \(\bar{x}(k) \in \mathbb{R}^n\) is the filter state and \(r(k) \in \mathbb{R}^m\) is the filter output. \(A_{F_i}, B_{F_i}, C_{F_i}, D_{F_i}, i = 1, 2, ..., R\) are FFDF parameters to be obtained. Thus, the conic-type dynamic FFDF system can be presented as:

\[
\begin{align*}
    \bar{x}(k + 1) &= \widetilde{A}(b_u) \bar{x}(k) + \widetilde{B}(b_u) \bar{w}(k) + \bar{g}(k), \\
    r(k) &= \widetilde{C}(b_u) \bar{x}(k) + \widetilde{D}(b_u) \bar{w}(k),
\end{align*}
\]

where

\[
\begin{align*}
    \bar{x}(k) &= [x_T(k) \ x_T(k)]^T, \\
    \bar{g}(k) &= [g_T(k) \ 0]^T, \\
    \bar{w}(k) &= [u_T(k) \ d_T(k) \ f_T(k)]^T, \\
    r(k) &= y(k) - r_f(k) \quad \text{and}
\end{align*}
\]

\[
\begin{align*}
    \widetilde{A}(b_u) &= \sum_{i=1}^{R} b_i(\omega(k)) \sum_{i=1}^{R} b_i(\omega(k)) \begin{bmatrix} A_i & 0 \\ B_{Fi} C_i & A_{Fi} \end{bmatrix}, \\
    \widetilde{B}(b_u) &= \sum_{i=1}^{R} b_i(\omega(k)) \sum_{i=1}^{R} b_i(\omega(k)) \begin{bmatrix} B_i & B_{Fi} \\ 0 & B_{Fi} D_{Fi} \end{bmatrix}, \\
    \widetilde{C}(b_u) &= \sum_{i=1}^{R} b_i(\omega(k)) \sum_{i=1}^{R} b_i(\omega(k)) \begin{bmatrix} C_i & -C_{Fi} \end{bmatrix}, \\
    \widetilde{D}(b_u) &= \sum_{i=1}^{R} b_i(\omega(k)) \sum_{i=1}^{R} b_i(\omega(k)) \begin{bmatrix} D_{di} - D_{Fi} D_{di} \\ 0 \\ D_{di} - D_{Fi} D_{di} \end{bmatrix}.
\end{align*}
\]

Referring to [26, 29], the FFDF design problem analysed can be expressed as the \(H_{\infty}\) filtering problem. Therefore, the aim of FD is how to find the appropriate FFDF parameters, such that the conic-type dynamic FFDF system (9) satisfies the following two objectives:

(a) It shows asymptotically stable when \(\bar{w}(k) \equiv 0\).

(b) Given a scalar \(\eta > 0\), under zero initial condition, it satisfies the following \(H_{\infty}\) performance index:

\[
\|r(k)\|_2 \leq \eta \|\bar{w}(k)\|_2,
\]

where

\[
\|r(k)\|_2 = \sqrt{\sum_{k=0}^{\infty} r(k) r(k)}, \quad \|\bar{w}(k)\|_2 = \sqrt{\sum_{k=0}^{\infty} \bar{w}(k) \bar{w}(k)}.
\]

Before conducting the research, we need to propose some important Lemmas for the latter developments.

**Lemma 1** [27] Let \(X\) and \(Y\) be real matrices of appropriate dimensions. For a given scalar \(\mu > 0\), and vectors \(x, y \in \mathbb{R}^n\), we have:

\[
2x^T X^T Y y \leq \mu^{-1} x^T X^T X x + \mu y^T Y^T Y y.
\]

**Lemma 2** [28] Let \(\Omega_1\) and \(\Omega_3\) be real matrices of appropriate dimensions. After that, for any matrix \(\Omega_2\) satisfying \(\Omega_2^T \Omega_2 \leq I\) and given a scalar \(\nu > 0\), we have:

\[
\Omega_1 \Omega_2 \Omega_3 + (\Omega_1 \Omega_2 \Omega_3)^T \leq \nu^{-1} \Omega_1 \Omega_1^T + \nu \Omega_3^T \Omega_3.
\]

### 3 MAIN RESULTS

**Theorem 1** For a given scalar \(\eta > 0\), the conic-type dynamic FFDF system (9) is asymptotically stable and satisfies
the given $H_{\infty}$ performance index (10), if there exist positive scalars $\mu_1$, $\mu_2$, $\xi$, positive-definite symmetric matrices $P_1$, $P_2$ and matrices $Y_a, Z_a$ such that

$$\Phi_i < 0, \ i = 1, 2, \ldots R,$$

$$\Phi_is + \Phi_i < 0, \ i < s, \ i, s = 1, 2, \ldots R,$$

where

$$\Phi_i = \begin{bmatrix} P_1 & P_2 \\ \ast & \Pi_4 \end{bmatrix}.$$

$$\Pi_1 = \begin{bmatrix} \Theta_1 & -P_2 & 0 & \Theta_2 & \Theta_3 & A_t^TP_1 + C^T_tZ_t \\ * & -P_2 & 0 & 0 & 0 & Y_t^T \\ * & * & -\eta I & 0 & 0 & B_t^TP_1 \\ * & * & * & \Theta_4 & \Theta_5 & B^T_{di}P_1 + D^T_{di}Z_t \\ * & * & * & * & \Theta_6 & B^T_{fi}P_1 + D^T_{fi}Z_t \\ * & * & * & * & * & -P_1 \end{bmatrix},$$

$$\Pi_2 = \begin{bmatrix} \Theta_7 & C^T_t - C^T_tD^T_{fi} & A^T_tP_1 + C^T_tZ_t \\ Y_t^T & -C^T_{fi} & Y_t^T \\ B^T_{fi}P_2 & 0 & 0 \\ \Theta_8 & D^T_{di} - D^T_{di}D^T_{fi} & 0 \\ \Theta_9 & D^T_{fi} - D^T_{fi}D^T_{fi} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\Pi_3 = \begin{bmatrix} \Theta_{10} & 0 & 0 & 0 \\ Y_t^T & 0 & 0 \\ 0 & B^T_{di}P_1 + D^T_{di}Z_t & B^T_{di}P_2 + D^T_{di}Z_t \\ 0 & B^T_{fi}P_1 & B^T_{fi}P_2 \\ 0 & B^T_{fi}P_1 + D^T_{fi}Z_t & B^T_{fi}P_1 + D^T_{fi}Z_t \\ 0 & 0 & 0 \end{bmatrix},$$

$$\Pi_4 = \begin{bmatrix} -P_2 & 0 & 0 & 0 & 0 \\ * & -I & 0 & 0 & 0 \\ * & * & -\mu_1I & 0 & 0 \\ * & * & * & -\mu_1I & 0 \\ * & * & * & * & -\mu_2I \end{bmatrix},$$

with

$$\Theta_1 = -P_1 + (\xi + \mu_1 + \mu_2)G^T_tG_t,$$

$$\Theta_2 = (\xi + \mu_1 + \mu_2)G^T_tE_t,$$

$$\Theta_3 = (\xi + \mu_1 + \mu_2)E^T_tH_t,$$

$$\Theta_4 = -\eta I + (\xi + \mu_1 + \mu_2)E^T_tE_t,$$

$$\Theta_5 = (\xi + \mu_1 + \mu_2)E^T_tH_t,$$

$$\Theta_6 = -\eta I + (\xi + \mu_1 + \mu_2)H^T_tH_t,$$

$$\Theta_7 = A^T_tP_2 + C^T_tZ_t,$$

$$\Theta_9 = B^T_{fi}P_2 + D^T_{fi}Z_t,$$

$$\Theta_{10} = A^T_tP_2 + C^T_tZ_t.$$

Moreover, the FFDF gains are given by

$$\Theta_i = B^T_{fi}P_2 + D^T_{fi}Z_t,$$

$$\Theta_{10} = A^T_tP_2 + C^T_tZ_t.$$

Proof: Choose a Lyapunov function as:

$$V(\bar{x}(k)) = \bar{x}^T(k)\mathcal{P}\bar{x}(k),$$

where $\mathcal{P}$ is a positive-definite symmetric matrix and the difference of $V(\bar{x}(k))$ can be obtained:

$$\Delta V(\bar{x}(k)) = V(\bar{x}(k + 1)) - V(\bar{x}(k))$$

$$= \bar{x}^T(k + 1)\mathcal{P}\bar{x}(k + 1) - \bar{x}^T(k)\mathcal{P}\bar{x}(k)$$

$$= (A(h_1)\bar{x}(k) + \bar{B}(h_2)\bar{w}(k))^T$$

$$\times \mathcal{P}(A(h_1)\bar{x}(k) + \bar{B}(h_2)\bar{w}(k))$$

$$+ 2\bar{g}^T(k)\mathcal{P}\bar{A}(h_1)\bar{x}(k) + \bar{g}(k)\mathcal{P}\bar{g}(k)$$

$$+ 2\bar{g}^T(k)\mathcal{P}\bar{B}(h_2)\bar{w}(k) - \bar{x}^T(k)\mathcal{P}\bar{x}(k).$$

For convenient analysis, let $\mathcal{P} = \begin{bmatrix} P_1 & P_2 \\ \ast & P_2 \end{bmatrix}$ and assume $0 < \mathcal{P} \leq \xi I$, where $\xi$ is a positive scalar. According to Lemma 1, we have:

$$2\bar{g}^T(k)\mathcal{P}\bar{A}(h_1)\bar{x}(k) \leq$$

$$\mu_1^{-1}\bar{x}^T(k)\bar{A}^T(h_1)\mathcal{P}\bar{A}(h_1)\bar{x}(k) + \mu_1\bar{g}^T(k)\bar{g}(k),$$

$$2\bar{g}^T(k)\mathcal{P}\bar{B}(h_2)\bar{w}(k) \leq$$

$$\mu_2^{-1}\bar{w}^T(k)\bar{B}^T(h_2)\mathcal{P}\bar{B}(h_2)\bar{w}(k) + \mu_2\bar{g}^T(k)\bar{g}(k),$$

$$\bar{g}^T(k)\mathcal{P}\bar{g}(k) \leq \xi \bar{g}^T(k)\bar{g}(k).$$

Submitting inequalities (17) into Equation (16), the following equation can be obtained:

$$\Delta V(\bar{x}(k)) = V(\bar{x}(k + 1)) - V(\bar{x}(k))$$

$$\leq \bar{x}^T(k)\bar{A}^T(h_1)\mathcal{P}\bar{A}(h_1)\bar{x}(k)$$

$$+ \bar{x}^T(k)\bar{A}^T(h_1)\mathcal{P}\bar{B}(h_2)\bar{w}(k)$$

$$+ \bar{w}^T(k)\bar{B}^T(h_2)\mathcal{P}\bar{B}(h_2)\bar{w}(k)$$

$$+ \mu_1^{-1}\bar{x}^T(k)\bar{A}^T(h_1)\mathcal{P}\bar{A}(h_1)\bar{x}(k)$$

$$+ \mu_1\bar{g}^T(k)\bar{g}(k) + \mu_2^{-1}\bar{w}^T(k)\bar{B}^T(h_2)\bar{w}(k)$$

$$+ \mu_2\bar{g}^T(k)\bar{g}(k) + \mu_2^{-1}\bar{g}^T(k)\bar{g}(k) + \xi \bar{g}^T(k)\bar{g}(k).$$

(18)
Under zero initial condition, the conic-type dynamic FFDF system (9) satisfies the $H_\infty$ performance index:

$$J = \sum_{0}^{\infty} [r^T(k)r(k) - \eta^2\bar{\omega}^T(k)\bar{\omega}(k) + \Delta V(k)] < 0.$$  \hspace{1cm} (19)

Recalling to the inequality condition shown in (3), inequality (19) can be rewritten as:

$$J = J + J',$$  \hspace{1cm} (20)

where

$$J = \Xi^T \Xi,$$

$$J' = 2\bar{g}^T(k)\mathcal{P}A(h_a)\bar{x}(k) + 2\bar{g}^T(k)\mathcal{P}B(h_a)\bar{\omega}(k) + \bar{g}(k)\mathcal{P}\bar{g}(k) < 0,$$

with

$$\Xi = [\bar{x}(k) \bar{\omega}^T(k)]^T,$$

$$\mathcal{P} = \left[\begin{array}{ccc} A_{(h_a)} & 0 & 0 \\ B_{Fi}C_i & F_i & A_{Fi} \end{array}\right]^T,$$

$$\bar{A}(h_a) = \left[\begin{array}{ccc} A_i & 0 & 0 \\ B_{Fi}C_i & F_i & A_{Fi} \end{array}\right],$$

$$\bar{B}(h_a) = \left[\begin{array}{ccc} B_i & B_{di} & B_{fi} \\ 0 & B_{Fi}D_{fi} & B_{Fi}D_{di} \end{array}\right],$$

$$\bar{C}(h_a) = \left[\begin{array}{ccc} C_i - D_{Fi}C_i & -C_{Fi} \end{array}\right],$$

$$\bar{D}(h_a) = \left[\begin{array}{ccc} 0 & D_{di} - D_{Fi}D_{di} & D_{fs} - D_{Fi}D_{fs} \end{array}\right].$$

Applying schur complements, the following is obtained:

$$\mathcal{P} = \left[\begin{array}{ccc} -\bar{A}(h_a) & \bar{\omega}^T(h_a) & \mathcal{C}^T(h_a) \\ * & -\eta^2I & \bar{B}^T(h_a) \bar{D}^T(h_a) \\ * & * & -\mathcal{P}^{-1} \end{array}\right].$$  \hspace{1cm} (21)

Using diag{I, I, $\mathcal{P}$, I} to pre- and post-multiply matrix $\Phi_{hi}$, we have:

$$\mathcal{P} = \left[\begin{array}{ccc} -\bar{A}(h_a) & \bar{\omega}^T(h_a) & \mathcal{C}^T(h_a) \\ * & -\eta^2I & \bar{B}^T(h_a) \bar{D}^T(h_a) \\ * & * & -\mathcal{P}^0 \end{array}\right].$$  \hspace{1cm} (22)

Then, substituting inequalities (17) and (22) into Equation (20), using schur complements and letting $Y_i = P_iA_{Fi}$, $Z_i = P_iB_{Fi}$, we obtain:

$$\Xi^T \left[\sum_{i=1}^{R} b_i \sum_{i=1}^{R} b_i \Phi_{hi} \right] \Xi < 0.$$  \hspace{1cm} (23)

It can be shown that inequality (23) is similar to the following condition:

$$\sum_{i=1}^{R} b_i^2 \Phi_{hi} + \sum_{i=1}^{R} b_i \left\{\sum_{i=1}^{R} b_i \left[\Phi_{hi} + \Phi_{hi} \right]\right\} < 0,$$  \hspace{1cm} (24)

which will lead to LMIs (13) and (14). The proof is complete.

**Theorem 2** To get an optimized fault detection performance lever against the external disturbance, the FFDF performance lever $\eta^2$ can decrease to the minimum feasible value if LMIs (13) and (14) are satisfied. The optimisation problem that is considered by the authors here can be represented by:

$$\min_{\delta} \delta,$$

s.t. LMIs (13, 14) with $\delta = \eta^2$.  \hspace{1cm} (25)

Remark 2 In order to solve the nonlinear problem conveniently, we give the range of matrix $\mathcal{P}$.

### 4 THRESHOLDS COMPUTATION

For the purpose of detecting the faults sensitively, a suitable threshold $J_{th}$ and an evaluation function $J(r)$ have to be set. Here, the threshold $J_{th}$ is defined as:

$$J_{th} = \sup_{d(k) \in L_2, f(k) = 0, k = k_0} \sum_{k=k_0}^{k_0+\rho} r^T(k)r(k).$$  \hspace{1cm} (26)

The evaluation function $J(r)$ is determined by:

$$J(r) = \sum_{k=k_0}^{k_0+\rho} r^T(k)r(k),$$  \hspace{1cm} (27)

where $k_0$ represents the initial time of the simulation and $\rho$ denotes the time steps. Thus, the following logical relationship is used to detect the faults:

$$\begin{align*}
J(r) > J_{th} & \rightarrow \text{alarm of fault,} \\
J(r) \leq J_{th} & \rightarrow \text{no fault.}
\end{align*}$$  \hspace{1cm} (28)
5 | NUMERICAL EXAMPLES

Example 1 Consider a class of conic-type nonlinear systems represented by:
\[
A_1 = \begin{bmatrix} -0.6 & 0.25 \\ 0.12 & -0.3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.46 & 0.46 \\ 0.1 & -0.3 \end{bmatrix},
\]
\[
B_1 = B_2 = \begin{bmatrix} 0.5 \\ 1.2 \end{bmatrix}, \quad B_{d1} = B_{d2} = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix},
\]
\[
B_{f1} = B_{f2} = \begin{bmatrix} -0.4 \\ 0.8 \end{bmatrix}, \quad D_{d1} = D_{d2} = \begin{bmatrix} 0.5 \end{bmatrix},
\]
\[
C_1 = C_2 = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \quad D_{f1} = D_{f2} = 0.5,
\]
\[
G_1 = G_2 = \begin{bmatrix} -0.0016 & 0 \\ 0 & 0 \end{bmatrix},
\]
\[
E_1 = E_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},
\]
\[
H_1 = H_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]
\[
g(k) = \begin{bmatrix} -0.00062(|x_1| + 1) - |x_1 - 1| \end{bmatrix}
\]

By solving the main results in Theorems 1 and 2, the performance level \( \eta = 2.4646 \) is obtained. Therefore, the relevant FFDF gain matrices can be obtained as
\[
A_{F1} = \begin{bmatrix} -0.1431 & 0.0328 \\ 0.0341 & -0.1260 \end{bmatrix}, \quad B_{F1} = \begin{bmatrix} 0.9235 \\ -1.1627 \end{bmatrix},
\]
\[
C_{F1} = \begin{bmatrix} -0.0085 \\ -0.0108 \end{bmatrix}, \quad D_{F1} = 0.9661,
\]
\[
A_{F2} = \begin{bmatrix} -0.1025 \\ 0.0928 \end{bmatrix}, \quad B_{F2} = \begin{bmatrix} 0.3897 \\ -0.2020 \end{bmatrix},
\]
\[
C_{F2} = \begin{bmatrix} 0.0002 \\ -0.0154 \end{bmatrix}, \quad D_{F2} = 0.9852.
\]

In order to verify the effectiveness of the designed FFDF, it is assumed that the external disturbance \( d(k) \) is a random noise within \([-1, 1]\). The fault signal \( f(k) \) is the sine wave which is given by:
\[
f(k) = \begin{cases} 0 & 0 < k < 8 \\ 2 \sin 2k & 8 \leq k \leq 20 \\ 0 & 20 < k < 25 \end{cases}
\]

The fault signal \( f(k) \) is shown in Figure 2 and the residual signal \( r(k) \) generated by the presented method is shown in Figure 3.

Figure 4 depicts the response of the residual function \( J(r) \) to the fault case and the fault-free case. The threshold is selected as \( J_{b} = \sup_{d(k) \in L_2} \sum_{k=0}^{25} r^T(k)r(k) = 1.61 \), based on the simulation results, we can get that \( f(r) = \sum_{k=0}^{10} r^T(k)r(k) = 1.65 > J_{b} \). Therefore, the appeared fault will be detected in two time steps after it has occurred.

Example 2 Consider a tunnel diode-circuit system established in [30]. Let \( x_1(t) = v_C(t), \ x_2(t) = i_L(t), \) where \( v_C(t) \) is the capacitor voltage and \( i_L(t) \) is the inductance current. Then, the tunnel diode circuit system is described by:
\[
\begin{align*}
C_{x_1}(t) &= -0.002x_1(t) - 0.01x_1^3 + x_2(t) \\
L_{x_2}(t) &= -x_1(t) - Rx_2(t) + d(t) \\
\gamma(t) &= 5x(t) + d(t)
\end{align*}
\]

where \( x(t) = [x_1^T(t) \ x_2^T(t)]^T \) is the state variable, \( d(t) \) and \( \gamma(t) \) are the external disturbance and the measured output,
respectively. \( S = [1 \ 0] \) is the sensor matrix. In the circuit model, the relevant parameters are presented as follows: \( C=200 \) mF, \( L=1 \) H and \( R = 1 \)Ω. Consider the nonlinearities in the tunnel diode circuit system satisfying the conic sector description. The sampling time is selected as \( T = 0.05 \) s. Then the conic-type nonlinear FFDF system can be described by:

\[
\begin{align*}
\dot{x}(k+1) &= \sum_{i=1}^{R} b_i(m(k))[A_i x(k) + B_i d(k)] + g(k), \\
y(k) &= \sum_{i=1}^{R} b_i(m(k))[C_i x(k) + D_i d(k)].
\end{align*}
\]

Assuming \( m(k) \in [-2 \ 2] \) and appropriate membership function is selected as:

\[
\begin{align*}
b_1(m(k)) &= 1 - \frac{1}{4} m^2(k), \\
b_2(m(k)) &= \frac{1}{4} m^2(k).
\end{align*}
\]

The parameter matrices are given by

\[
A_1 = \begin{bmatrix} 0.0995 & 0.25 \\ -0.05 & 0.95 \end{bmatrix}, A_2 = \begin{bmatrix} 0.9095 & 0.25 \\ -0.05 & 0.95 \end{bmatrix},
\]

\[
B_1 = B_2 = \begin{bmatrix} 0 \\ 0.05 \end{bmatrix},
\]

\[
C_1 = C_2 = [1 \ 0],
\]

\[
D_1 = D_2 = 1,
\]

\[
g(k) = \begin{bmatrix} -0.00062(|x_1| + 1) - |x_1 - 1| \\ 0 \end{bmatrix}.
\]

Solving the optimization problem (25), we can get \( \eta = 4.2908 \) and the FFDF gain matrices as:

\[
A_{F1} = \begin{bmatrix} 0.0053 & 0.0462 \\ -0.0586 & -0.2048 \end{bmatrix}, B_{F1} = \begin{bmatrix} -0.2017 \\ -0.6958 \end{bmatrix},
\]

\[
C_{F1} = [-0.0385 -0.0021], D_{F1} = 0.9411,
\]

\[
A_{F2} = \begin{bmatrix} -0.0083 & 0.0373 \\ -0.0898 & 0.1921 \end{bmatrix}, B_{F2} = \begin{bmatrix} -0.3188 \\ -1.0198 \end{bmatrix},
\]

\[
C_{F2} = [-0.0349 -0.0003], D_{F2} = 0.9946.
\]

Select the same disturbance signal \( d(k) \) as Example 1 and the fault signal \( f(k) \) as the square wave which is given by:

\[
f(k) = \begin{cases} 20 \text{ for } 1 \leq k \leq 20 \\ 0 \text{ otherwise} \end{cases}
\]

The residual signal \( r(k) \) is determined in Figure 5 and the residual function \( J(r) \) generated by the presented method is shown in Figure 6.

Figure 6 describes the response of the residual function \( J(r) \) to the fault case and the fault-free case. Selecting the threshold \( J_{th} = \sup_{d(k) \in L_2} J_F(r(k)) = 2.5 \), from the simulation results, \( J(r) = \sum_{k=0}^{10} r^T(k) r(k) = 2.8 > J_{th} \) is obtained. Thus, the fault will be detected in two time steps.

Remark 3 In [31], the dynamic observer approach was employed to deal with the FD problem. However, \( H_{\infty} \) filtering scheme is adopted to find a proper
compromise that achieves robustness to external disturbances and shows sensitivity to faults. Comparing with these two results, it can be seen that our approach has a fast detection speed. Besides, it is worth pointing out that the proposed approach in [31] is not fit for this experiment because of the conic-type nonlinearities in the systems. In this paper, the conic-type nonlinearity has been linearized by the designed method. In addition, an applicable experiment of a tunnel diode-circuit system is applied to verify the capability of the presented method.

6 CONCLUSIONS

The authors, by using the T-S fuzzy models, investigated the FFDF design problem for a class of conic-type nonlinear systems. Based on the Lyapunov function method and LMIs techniques, sufficient conditions on the existence of the FFDF are verified and the FFDF design problem is transformed into an optimisation algorithm. Numerical examples show the feasibility of the designed scheme. The authors pay more attention to FD in their analysis, not the isolation and estimation. Further, the authors have planned to study the fault isolation and fault estimation problems for nonlinear systems with time-varying actuator faults and the faults governed by stochastic dynamics in the future.

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