Abstract

We study the problem of finding functions, defined within and on an ellipse, whose Laplacian is -1 and which satisfy a homogeneous Robin boundary condition on the ellipse. The parameter in the Robin condition is denoted by $\beta$. The general solution and various asymptotic approximations are obtained. To find the general solution, the boundary value problem is formulated in elliptic cylindrical coordinates. A Fourier series solution is then derived. The integral of the solution over the ellipse, denoted by $Q$, is a quantity of interest in some physical applications. The dependence of $Q$ on $\beta$ and the ellipse’s geometry is found. Finding asymptotics directly from the pde formulations is easier than from our series solution. We use the asymptotic approximations to $Q$ as checks on the series solution. Several other inequalities are also used to check the solution.

It is intended that this arXiv preprint will be referenced by the journal version, which will be submitted soon, as the arXiv contains material, e.g. codes for calculating $Q$, not in the much shorter journal version.

Maple codes used in deriving or checking results in this paper are in the process of being tidied prior to being made available via links given at https://sites.google.com/site/keadyPerthUnis/papers.html
1 Introduction

1.1 The pde problem

We seek the solution of

\[-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 1 \quad \text{in } \Omega, \quad (1.1)\]

\[u + \beta \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega. \quad (1.2)\]

In the context of flows, this boundary condition is called a *slip boundary condition* or *Navier’s boundary condition*; in the wider mathematical literature it is called a *Robin boundary condition*. The \( \beta = 0 \) case is, in the fluid mechanics context, called a ‘no-slip boundary condition’: in elasticity the \( \beta = 0 \) case is called ‘the elastic torsion problem’. A functional of interest, in the context of flows, is the volume flow rate

\[Q := \int_\Omega u. \quad (1.3)\]

We remark that the same pde problem arises in contexts other than slip flow, for example, heat-flow with ‘Newton’s law of cooling’, e.g. [16, 24]. While there are other applications of this pde problem we will, on occasions, use the fluid flow terminology for \( u, Q \) and \( \beta \).

Except in this subsection (and a very small number of introductory subsections, e.g. §6.1) where we allow \( \Omega \) to be more general, we will be treating \( \Omega \) as the elliptical domains within ellipses described by

\[\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \quad \text{with } a \geq b. \quad (1.4)\]

In this paper, we derive a Fourier series form of the analytical solution of the problem and provide many checks on it.

Some recent papers treat the problem in the context of slip flows in elliptic microchannels, e.g. [4, 6, 7]. (Again in the context of flows in microchannels, time-dependent and other aspects are treated in various papers. [11, 12, 29, 30] study pulsatile flows. Papers [5, 33] are relevant to studies of the decay of transients and starting flows. Other physical effects are studied in [9, 13]. The steady flow case is, for each of these, a special case when a parameter is set to zero.) We make considerable effort to check our results. For example,
there are a number of rigorously proved bounds. An old variational result, a lower bound on $Q$ is equation (4.4) of [16],
\[
Q(\beta) \geq Q(\beta = 0) + \frac{\beta |\Omega|^2}{|\partial \Omega|} \quad \text{for general } \Omega,
\]
\[
= \frac{\pi a^3 b^3}{4(a^2 + b^2)} + \frac{\pi^2 \beta a^2 b^2}{|\partial \Omega|} \quad \text{for ellipse } \Omega. \quad (1.5)
\]
Inequality (1.5) and the more general form preceding it are very easy to establish: see §4. Inequality (1.5) becomes an equality when the ellipse is a circle, i.e. $b = a$. The left-hand term in the right-hand side of inequality (1.5) gives the dominant term in the asymptotic expansion of $Q$ as $\beta$ tends to 0. The right-hand term in inequality (1.5) gives the dominant term in the asymptotic expansion of $Q$ as $\beta$ tends to infinity. However (except when $b = a$), in both asymptotic limits, the other term is not the next term in the asymptotic expansion.

We denote the area by $|\Omega|$ (though in code it is sometimes denoted by $A$). The perimeter $|\partial \Omega|$ is sometimes denoted by $L$. For an ellipse, the formula for $L$ is given in equation (6.1).

A much more recent inequality, a generalization of the St Venant Inequality (see [26]), is given in [2] (with related work in [1]):
\[
Q \leq Q_{\odot}(\beta) := \frac{\pi (a b)^{3/2}}{8} \left(\sqrt{ab} + 4\beta\right). \quad (1.6)
\]
In words: amongst all ellipses with a given area that which has the greatest $Q$ is the circular disk. See equation (2.2). (This and other isoperimetric inequalities are reviewed, in the context of microchannels in [17, 18].)

1.2 The structure of this paper

The rest of this paper is organized as follows.

- In §2 we summarize the very well-known elementary solutions, polynomial in Cartesian coordinates, for the simplest special cases against which to check our general solution. Each of them forms the lowest order term in asymptotics developed later in this paper.

- In §3 the boundary value problem is formulated in elliptic cylindrical coordinates, and then the series solution is derived.
• The next sections provide many checks. To reduce the number of parameters, in these sections, we take \( b = 1/a \), so the area of the ellipse is \( \pi \).

  – In §4 we present a variational formulation and associated bounds.
  – In §5 we present asymptotics for the situation where the ellipse is nearly circular.
  – In §6 we give asymptotics for \( \beta \) small.
  – In §7 we study asymptotics for \( \beta \) large.

• Finally, a conclusion is given in §8.

• Appendix A treats various geometric matters. Appendix B has some notes on elliptic integrals. These arise in the Fourier series for a functions \( g \) and \( \hat{g} \) used at several sections of the paper, notably §3 and §7. Appendix C collects information on these functions primarily in connection with possible further work mentioned in §8. Appendix D gives some numerical examples, its first parts comparing our numerics with some previously published examples. We conclude with an attempt to suggest numbers which might arise in connection with possible applications involving blood flow.

2 Some simple explicit solutions:
   \( u \) quadratic polynomial in \( x, y \)

2.1 Circular cross-section with \( \beta \geq 0 \)

When \( \Omega \) is circular, radius \( a \), in polar coordinates, \( r = \sqrt{x^2 + y^2} \), the solution is

\[
    u_{\beta_\circ} = u = \frac{1}{4} (a^2 - r^2) + \frac{\beta a}{2}.
\]

(2.1)

In the context of fluid flows, when \( \beta = 0 \) this is Poiseuille flow.

\[
    Q_{\beta_\circ} = Q(\beta) = \frac{\pi a^3}{8} (a + 4\beta).
\]

(2.2)

See also [21]§331, p586.

In later sections the circle will have radius \( a = 1 \), area \( \pi \).
2.2 Elliptic cross-section with $\beta = 0$

In this subsection (and in sections §4 to §7) we consider ellipses with $b = 1/a$ and $a \geq 1$. The $\beta = 0$ solution dates back at least to St Venant. In polar coordinates with the origin at the centroid of the ellipse the velocity $u_\epsilon$ is, in Cartesian coordinates, a quadratic polynomial in $x$ and $y$ while, in polar coordinates, it is

$$u_0 = \frac{1}{4} \left( \sqrt{1 - \epsilon^2} - r^2 + \epsilon r^2 \cos(2\theta) \right).$$

Here

$$\epsilon = \frac{a^2 - a^{-2}}{a^2 + a^{-2}}.$$

In Cartesian coordinates

$$u_0 = \frac{1 - (x/a)^2 - (y/b)^2}{2/a^2 + 2/b^2}. \quad (2.3)$$

Also

$$Q_0 = Q(\text{ellipse, } \beta = 0) = \frac{\pi}{4(a^2 + a^{-2})} = \frac{\pi}{8} \sqrt{1 - \epsilon^2}. \quad (2.4)$$

Also relevant is $e$ the eccentricity as defined in equation (A.3).

Figure 1: $\beta = 0$: plot of $Q_{\text{ratio}} = (Q_\odot - Q(e))/Q_\odot$ against eccentricity, $e$. 

5
There are various interesting alternative expressions for $Q$. The ellipse’s moments of inertia are

$$I_{xx} = \frac{\pi a^2}{4}, \quad I_{yy} = \frac{\pi}{4a^2}$$

so

$$Q = J := \frac{I_{xx}I_{yy}}{I_{xx} + I_{yy}}.$$

See [26] p112. For any domain (and $\beta = 0$), it is known that $Q \leq J$. 

6
3 Fourier series solution

3.1 Using elliptic coordinates

Formulae when \( a \geq b > 0 \)

For the geometry of elliptic domains specified as in equation (1.4) with, as always, \( a > b \), it is convenient here to use elliptic coordinates \((\eta, \psi)\) which are related to the rectangular coordinates (as in \([4]\) equation (18)) by

\[
x = c \cosh(\eta) \cos(\psi), \quad y = c \sinh(\eta) \sin(\psi),
\]

(3.1)

where \( 0 \leq \eta \leq \infty \), \( 0 \leq \psi \leq 2\pi \), and \( c \) and \(-c\) are two common foci of the ellipse. The origin of coordinates in the Cartesian system (the centre of our ellipse) is at \( \eta = 0, \psi = \pi/2 \). The upper side of the positive semi-major axis is \( \eta = 0, 0 < \psi < \pi/2 \); the lower side is \( \eta = 0, 0 > \psi > -\pi/2 \). Let \( a \) and \( b \) denote lengths of the semi-axes of the ellipse and \( a > b > 0 \). Except in this subsection, and elsewhere when so specified, we take \( b = 1/a \). Returning to the general situation,

\[
a = c \cosh(\eta) \quad \text{and} \quad b = c \sinh(\eta), \quad \text{and} \quad c = \sqrt{a^2 - b^2}.
\]

Defining \( \eta_0 \) (as in \([4]\) equation (19)) by

\[
\eta_0 = \ln \frac{1 + b/a}{\sqrt{1 - (b/a)^2}} = \arctanh \left( \frac{b}{a} \right),
\]

the boundary of the ellipse can be represented by

\[
x = c \cosh(\eta_0) \cos(\psi), \quad y = \sinh(\eta_0) \sin(\psi),
\]

so that

\[
c = \frac{a}{\cosh \eta_0} = \frac{b}{\sinh \eta_0}.
\]

The eccentricity is

\[
e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \tanh^2(\eta_0)} = \frac{1}{\cosh(\eta_0)}.
\]

(Nearly circular ellipses have \( \eta_0 \) large and \( c \) near 0.)
The Jacobian $J$ of the transformation from $(x,y)$-coordinates to $(\eta,\psi)$ coordinates is

$$J(\eta,\psi) = c^2 \left( \cosh^2(\eta) - \cos^2(\psi) \right) = \frac{c^2}{2} \left( \cosh(2\eta) - \cos(2\psi) \right).$$

This is needed in the derivation of equation (3.32). The Jacobian will be used in other places in the paper, e.g. §7.2.

**Formulae when $b = 1/a$**

In all major calculations henceforth we scale distances so that we have $b = 1/a$. Formulae used later include the following:

$$c^2 = \frac{2}{\sinh(2\eta_0)} = a^2 - \frac{1}{a^2} = \frac{e^2}{\sqrt{1 - e^2}},$$

(3.2)

$$\tanh(\eta_0) = \frac{1}{a^2},$$

(3.3)

$$\cosh(\eta_0) = \frac{1}{e},$$

(3.4)

$$e^2 = 1 - \frac{1}{a^4}.$$  

(3.5)

3.2 The pde and boundary conditions

The pde (1.1) is recast in these coordinates as

$$\frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 u}{\partial \psi^2} = -c^2 (\cosh^2(\eta) - \cos^2(\psi)),

= -\frac{c^2}{2} (\cosh(2\eta) - \cos(2\psi)).$$

(3.6)

We will solve the pde (3.6) subject to boundary conditions (3.7), (3.8) and (3.9). The solution is assumed to be symmetric about the mid-plane for $\psi = 0, \frac{\pi}{2}$. Therefore, symmetric boundary condition is applied, i.e.,

$$\frac{\partial u}{\partial \psi} = 0 \text{ on } \psi = 0, \frac{\pi}{2}. \quad (3.7)$$

In addition, we set

$$\frac{\partial u}{\partial \eta} = 0 \text{ on } \eta = 0. \quad (3.8)$$
The boundary condition on the ellipse, i.e. at those points when $\eta = \eta_0$ is

$$u(\eta_0, \psi) + \frac{\beta}{c \sqrt{\cosh^2 \eta_0 - \cos^2 \psi}} \frac{\partial u}{\partial \eta}(\eta_0, \psi) = 0. \quad (3.9)$$

Define

$$g(\psi) = \frac{\cosh(\eta_0)}{(\cosh^2(\eta_0) - \cos^2(\psi))^{1/2}}. \quad (3.10)$$

Boundary condition (3.9) becomes

$$u(\eta_0, \psi) + \frac{\beta}{c \cosh(\eta_0)} g(\psi) \frac{\partial u}{\partial \eta}(\eta_0, \psi) = 0. \quad (3.11)$$

In summary, the problem in elliptic coordinates is given as the following boundary value problem (BVP).

**BVP:** Find $u(\eta, \psi)$ such that the above governing equation (3.6) and associated boundary conditions (3.7), (3.8) and (3.11) are satisfied.

### 3.3 The Fourier series for $g$

**Ellipses in general**

The definition of $g$ from (3.10) can be re-cast in several ways:

$$g(\psi) = \left(1 - \frac{\cos^2(\psi)}{\cosh^2(\eta_0)}\right)^{-1/2},$$

$$= C g_*(\psi) \text{ with } g_*(\psi) = \left(1 - \frac{\cos(2\psi)}{\cosh(2\eta_0)}\right)^{-1/2} \text{ and } C = \left(1 + \frac{\cosh(2\eta_0)}{\cosh(2\eta_0)}\right)^{1/2}.$$

In an appendix we will use a more economic notation using

$$q = \cosh^2(\eta_0), \quad (3.12)$$

and the form for $g_*$ is consistent with the occurrence, in Fourier coefficients, of polynomials in $\cosh(2\eta_0) = 2q - 1$.

The Fourier series for $g$ is used. Define

$$g_n = \frac{2}{\pi} \int_0^\pi g(\psi) \cos(2n\psi) \, d\psi,$$
so
\[ g(\psi) = \frac{g_0}{2} + \sum_{n=1}^{\infty} g_n \cos(2n\psi). \]

In various approximations the first few terms in the Fourier series are used. For these define
\[ \text{EllipticE}_0 = \text{EllipticE}\left(\frac{1}{\cosh(\eta_0)}\right) \quad \text{and} \quad \text{EllipticK}_0 = \text{EllipticK}\left(\frac{1}{\cosh(\eta_0)}\right). \]

We have
\[ g_0 = \frac{4}{\pi} \text{EllipticK}_0, \quad (3.14) \]
\[ g_1 = -\frac{8 \cosh(\eta_0)^2}{\pi} \text{EllipticE}_0 + \frac{4(2 \cosh(\eta_0)^2 - 1)}{\pi} \text{EllipticK}_0. \quad (3.15) \]

All the terms \( g_n \) are of the form
\[ g_n = \frac{4}{\pi} (E_n \text{EllipticE}_0 + K_n \text{EllipticK}_0), \]
where \( E_n \) and \( K_n \) are polynomials of degree \( n \) in \( \cosh(\eta_0)^2 \) with rational number coefficients.

Should future work require more terms, we remark that a three term recurrence relation exists to determine the polynomials \( E_n \) and \( K_n \). See the appendix. In the notation of equation (3.12), the \( K_n \) sequence of polynomials starts with
\[ K_0 = 1, \quad K_1 = 2q - 1 = q_2. \]

The \( E_n \) sequence of polynomials starts with
\[ E_0 = 0, \quad E_1 = 2q = q_2 - 1. \]

There are also other representations: see the Appendix. There are several very elementary facts which have some use. These include that
\[ \cos(2k\psi) = \text{ChebyshevT}(\cos(2\psi)). \]

Also, the Taylor series
\[ (1 - X)^{-1/2} = \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2} X^k; \]
can be applied to the preceding expressions for \( g \) to obtain series which are sums of powers of \( \cos(\psi)^2 \) or of \( \cos(2\psi) \). To date, our only use of these has been in checking the asymptotic approximations, at small eccentricity, to the Fourier coefficients of \( g \) and of \( 1/g \).

The reciprocal of \( g \) also occurs in later calculations (and also in [4] equation (61) though they use a Taylor series rather than Fourier series used here).

**Approximating the Fourier series for nearly circular ellipses**

Nearly circular ellipses have \( \eta_0 \) large. Then the lowest approximation is

\[
g(\psi) \sim g_\epsilon(\psi) = 1 + \frac{\cos^2(\psi)}{2 \cosh^2(\eta_0)} = 1 + \frac{1}{4 \cosh^2(\eta_0)} + \frac{\cos(2\psi)}{4 \cosh^2(\eta_0)},
\]

and

\[
\frac{1}{g(\psi)} \sim 1 - \frac{1}{4 \cosh^2(\eta_0)} - \frac{\cos(2\psi)}{4 \cosh^2(\eta_0)}.
\] (3.16)

We will use the next approximation after this, and will use \( e = 1/\cosh(\eta_0) \):

\[
\frac{1}{g(\psi)} \sim 1 - \frac{1}{4} e^2 - \frac{3}{64} e^4 - \left( \frac{1}{4} e^2 + \frac{1}{16} e^4 \right) \cos(2\psi) - \frac{1}{64} e^4 \cos(4\psi).
\] (3.17)

We have, for \( e \) small, the following Fourier coefficients of \( \hat{g} = 1/g \):

\[
\hat{g}_0 \sim 2 - \frac{1}{2} e^2 - \frac{3}{32} e^4
\] (3.18)

\[
\hat{g}_1 \sim -\frac{1}{4} e^2 - \frac{1}{16} e^4
\] (3.19)

\[
\hat{g}_2 \sim -\frac{1}{64} e^4
\] (3.20)

(We will need these in the treatment of \( u_\infty \). See equation (7.1).)

**3.4 Completing the Fourier series solution**

The general solution of equation (3.6) can be expressed in the form of

\[
u(\eta, \psi) = v_h + v_p,
\] (3.21)
where $v_h$ and $v_p$ denote the homogeneous and the particular solutions of the pde (3.6) subject to the boundary conditions (3.7), (3.8) and (3.11).

For the particular solution $v_p$, from the right hand side of the equation (3.6), we can expressed $v_p$ in the form of

$$v_p = \hat{g}_1 \cosh(2\eta) + \hat{g}_2 \sinh(2\eta) + \hat{g}_3 \cos(2\psi) + \hat{g}_4 \sin(2\psi),$$  

(3.22)

which gives

$$\frac{\partial^2 v_p}{\partial \eta^2} = 2\hat{g}_1 \sinh(2\eta) + 2\hat{g}_2 \cosh(2\eta);$$  

(3.23)

$$\frac{\partial^2 v_p}{\partial \psi^2} = -2\hat{g}_3 \sin(2\psi) + 2\hat{g}_4 \cos(2\psi).$$  

(3.24)

Substituting $v_p$ into the pde (3.6) and comparing the coefficients of $\cosh(2\eta)$ and $\cos(2\psi)$ yields

$$v_p = -\frac{c^2}{8} \left( \cosh(2\eta) + \cos(2\psi) \right) \left( -\frac{1}{4} (x^2 + y^2) \right).$$  

(3.25)

By separation of variables, the solution $v_h$ of the Laplace equation subject to above boundary conditions (3.7) and (3.8) is

$$v_h = \sum_{n=0}^{\infty} \frac{A_n}{2} \cosh(2n\eta) \cos(2n\psi).$$  

(3.26)

Hence, the general solution can be expressed in the form of

$$u(\eta, \psi) = v_h + v_p = \sum_{n=0}^{\infty} \frac{A_n}{2} \cosh(2n\eta) \cos(2n\psi) - \frac{c^2}{8} \left( \cosh(2\eta) + \cos(2\psi) \right),$$  

(3.27)

where term $A_n$ can be determined by using the boundary condition (3.11).

We then obtain

$$\sum_{n=0}^{\infty} \frac{A_n}{2} \cosh(2n\eta_0) \cos(2n\psi) - \frac{c^2}{8} \left( \cosh(2\eta_0) + \cos(2\psi) \right) + \frac{\beta g(\psi)}{c \cosh(\eta_0)}$$

$$\sum_{n=0}^{\infty} nA_n \sinh(2n\eta_0) \cos(2n\psi) - \frac{c^2 \beta g(\psi)}{4c \cosh(\eta_0)} \sinh(2n\eta_0) = 0.$$  

(3.28)

or

$$\sum_{n=0}^{\infty} \frac{A_n}{2} \left( \cosh(2n\eta_0) + \frac{2n\beta g(\psi)}{c \cosh(\eta_0)} \sinh(2n\eta_0) \right) \cos(2n\psi)$$

$$= \frac{c^2}{8} (\cosh(2\eta_0) + \cos(2\psi)) + \frac{c^2 \beta g(\psi)}{4c \cosh(\eta_0)} \sinh(2\eta_0) = f(\psi).$$  

(3.29)
As $g$ is an even function so is $f$ and $h_n$ where

$$h_n(\psi) = \cosh(2n\eta_0) + \frac{2n\beta g(\psi)}{c \cosh(\eta_0)} \sinh(2n\eta_0).$$

(The Fourier coefficients of $f$ and of $h_n$ are very simply related to the Fourier coefficients of $g$.) We then have

$$\frac{1}{2\pi} \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} A_n h_n(\psi) \cos(2n\psi) \cos(2m\psi) d\psi = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(\psi) \cos(2m\psi) d\psi,$$

or

$$\sum_{n=1}^{\infty} A_n \int_{-\pi}^{\pi} h_n(\psi) \cos(2n\psi) \cos(2m\psi) d\psi = -2 \int_{-\pi}^{\pi} f(\psi) \cos(2m\psi) d\psi.$$

Hence if we truncate the summation at $n_{\text{max}}$ we have a system of linear equations for the $A_n$ for $n$ up to $n_{\text{max}}$. Using the $A_n$ so found, we obtain an approximation to the Laplace solution $v_h$ and then steady-state fluid flow, $u(\eta, \psi) = v_h + v_p$ in the elliptical channel. Finally, the volume flow rate of fluid passing through an elliptical cross sectional area, $Q$, can determined by

$$Q = c^2 \int_{0}^{\eta_0} \int_{0}^{2\pi} u(\eta, s) (\cosh^2(\eta) - \cos^2(s)) \, d\eta ds.$$  \hspace{1cm} (3.32)

### 3.5 $Q$ from the Fourier series of $v_h$

We will substitute $u = v_p + v_h$ with $v_h$ given by its Fourier series into the formula for $Q$ given in (3.32). Note that all the Fourier components with $n > 1$ contribute nothing to the integral. First

$$Q_p = c^2 \int_{0}^{\eta_0} \int_{0}^{2\pi} v_p \left( \cosh^2(\eta) - \cos^2(s) \right) d\eta ds,$$

$$= -\frac{1}{4} I_2 = -\frac{1}{4} \frac{\pi}{4} \left( a^2 + \frac{1}{a^2} \right) = -\frac{\pi}{16} \frac{2 - e^2}{\sqrt{1 - e^2}},$$
$I_2$ being the polar area moment of inertia about the centroid. On using $c^2 = 2/\sinh(2\eta_0)$,

$$ Q = Q_p + c^2 \int_0^{2\pi} \int_0^{\eta_0} (A_0 + A_1 \cosh(2\eta) \cos(2s)) (\cosh^2(\eta) - \cos^2(s)) \, d\eta ds,$$

$$ = -\frac{\pi \cosh(2\eta_0)}{8 \sinh(2\eta_0)} + \pi A_0 - \frac{1}{2} \pi A_1. \quad (3.33) $$
4 Results from a variational formulation

4.1 The general setting, and a simple first case

Variational formulations are available. For example, in [16] equation (4.1), the functional $J$ is defined as

\[ J(v) = A(v) - \frac{1}{\beta} B(v) \]

where $A(v) = \int_\Omega (2v - |\nabla v|^2)$, $B(v) = \int_{\partial\Omega} v^2$.  \hfill (4.1)

For any simply-connected domain with sufficiently smooth boundary, the maximiser $u$ of this functional satisfies equations (1.1,1.2). Furthermore, $J(u)$ is $Q = Q(u)$ where

\[ Q(v) = \int_\Omega v. \]

Thus we can find lower bounds for $Q$ by evaluating $J(v)$ for particular choices of $v$. For domains like ellipses symmetric about both the $x$-axis and $y$-axis one nice choice for $v$ are the quadratic functions

(i) $v = c_0 + c_2 \left( \frac{x}{a} \right)^2 + a^2 y^2$,

and, more generally (and with a different $c_0$)

(ii) $v = c_0 + c_{xx} x^2 + c_{yy} y^2$.

(Case (ii) is motivated by the likelihood that for a range of $\beta$ and a range of $a$ the level curves of $u$ look to be like ellipses. See, for example, Figure 4 of [32] and our own contour plots shown in Appendix D.)

We briefly return to general domains. An application of the Divergence Theorem gives

\[ \text{if } -\Delta v = 1 \text{ then } J(v) - Q(v) = \int_{\partial\Omega} v(v + \beta \frac{\partial v}{\partial n}). \]  \hfill (4.2)

Case (i)

We begin with case (i). While the beginning of the study with case (ii) will allow $\Omega$ to be more general than merely an ellipse, for case (i) we restrict to the ellipse. The final result, lower bound, will give the same result as we
reported in \[1.1\] from equation (4.4) of \[16\]. Write \(J\) or \(J(c_0, c_2)\) for the value of \(J\) evaluated at the quadratic function (i)

\[
J = 2c_0A + 2c_2 \left( \frac{1}{a^2}(1 - \frac{2c_2}{a^2})I_{XX} + a^2(1 - 2a^2c_2)I_{YY} \right) - (c_0 + c_2)^2\frac{L}{\beta}.
\]

Finding the gradient of \(J\) with respect to \([c_0, c_2]\) we find the maximum of \(J\) will occur when the gradient is 0, i.e.

\[
Lc_0 + Lc_2 = \pi\beta,
\]

\[
2Lc_0 + c_2 \left( 2L + \pi\beta\left(a^2 + \frac{1}{a^2}\right) \right) = \pi\beta.
\]

This is readily solved, and on integrating, yields the result given before at inequality \[1.5\].

It is straightforward to show that the quadratic \(u\) which is the variational winner in Case (i) is such that the result in (4.2) can be applied to establish \(J(u) = Q(u)\) (a fact which can be established by other means). The application of (4.2) is facilitated by the fact that the quadratic \(u = u_0 + \beta|\Omega|/|\partial\Omega|\) is constant on \(\partial\Omega\).

**Case (ii), an introduction**

Next we begin to treat case (ii). Write \(J\) or \(J(c_0, c_{xx}, c_{yy})\) for the value of \(J\) evaluated at the quadratic function (ii).

\[
J = 2 \left( c_0A + c_{xx}(1 - 2c_{xx})I_{xx} + c_{yy}(1 - 2c_{yy})I_{yy} \right) - \frac{1}{\beta} \left( c_0^2L + 2c_0c_{xx}i_{xx} + 2c_0c_{yy}i_{yy} + 2c_{xx}c_{yy}i_{xxyy} + c_{xx}^2i_{xxxx} + c_{yy}^2i_{yyyy} \right).
\]

where \(A\) is the area, \(L\) the perimeter and the \(I\) are area moments and \(i\) boundary moments

\[
I_{xx} = \int_{\Omega} x^2, \quad I_{yy} = \int_{\Omega} y^2,
\]

\[
i_{xx} = \int_{\partial\Omega} x^2, \quad i_{yy} = \int_{\partial\Omega} y^2, \quad i_{xxyy} = \int_{\partial\Omega} x^2y^2, \quad i_{xxxx} = \int_{\partial\Omega} x^4, \quad i_{yyyy} = \int_{\partial\Omega} y^4.
\]

Define also

\[
Q_q = c_0A + c_{xx}I_{xx} + c_{yy}I_{yy}, \quad (4.3)
\]
the subscript $q$ reminding us of the quadratic approximation to the velocity field. The gradient of $J$, here denoted $g$ is

$$
g = -\frac{2}{\beta} \left( \begin{array}{c} c_0 L + c_{xx} i_{xx} + c_{yy} i_{yy} - \beta A \\
 c_0 i_{xx} + c_{xx} (4 \beta I_{xx} + i_{xxxx}) + c_{yy} i_{xyyy} - \beta I_{xx} \\
 c_0 i_{yy} + c_{xx} i_{xxyy} + c_{yy} (4 \beta I_{yy} + i_{yyyy}) - \beta I_{yy} \end{array} \right). \tag{4.4}
$$

We remark that

$$2(J - Q_q) = g[1] c_0 + g[2] c_{xx} + g[3] c_{yy}. \tag{4.4}
$$

This gradient $g$ is zero when

$$
\begin{align*}
c_0 L + c_{xx} i_{xx} + c_{yy} i_{yy} &= \beta A, \\
c_0 i_{xx} + c_{xx} (4 \beta I_{xx} + i_{xxxx}) + c_{yy} i_{xyyy} &= \beta I_{xx}, \\
c_0 i_{yy} + c_{xx} i_{xxyy} + c_{yy} (4 \beta I_{yy} + i_{yyyy}) &= \beta I_{yy}.
\end{align*}
$$

Up until now the $c_0$, $c_{xx}$, $c_{yy}$ have been general. Henceforth we use the same letters to denote the $c$ obtained by solving $g = 0$, the gradient of $J$ equals zero. We remark that after finding $c_0$, $c_{xx}$, $c_{yy}$ there are two methods of estimating $Q$. On the one hand, we can estimate $Q$ from

$$Q_c := c_0 A + c_{xx} I_{xx} + c_{yy} I_{yy}, \tag{4.5}$$

while, on the other hand, we have the lower bound

$$J_c := J(c_0, c_{xx}, c_{yy}) \leq Q.$$

From identity (4.4) we have $Q_c = J_c$.

For general $\Omega$ inequalities on moments (such as some in references in [15]) may lead to lower bounds for $Q$ in terms of simpler geometric quantities. Also for some domains, polygons for example, the various moments could be calculated. In general the restriction of the test functions to quadratics isn’t likely to give very tight lower bounds though there may be exceptions. Rectangles at large $\beta$ might be one such as $u_\infty$ (see [16] or equations (7.4)) is, for rectangles, quadratic.

For an ellipse with $a > 1$ (and $b = 1/a$) the area moments are elementary, but the perimeter and other boundary moments involve elliptic integrals:
we present these in the next subsection. For a circle, all the moments are elementary. For a circle of radius 1, the moments are:

\[
A = \pi, \quad I_{xx} = \frac{\pi}{4}, \quad I_{yy} = \frac{\pi}{4},
\]

\[
L = 2\pi, \quad i_{xx} = \pi = i_{yy}, \quad i_{xxxx} = \frac{3\pi}{4} = i_{yyyy}, \quad i_{xxyy} = \frac{\pi}{4}.
\]

For the circle, the variational winner has \(c_0, c_{xx} = c_{yy}\) giving the exact solution previously presented in equation (2.1): both \(Q_c\) and \(J_c\) evaluate to the formula for \(Q\) given in equation (2.2).

A note of caution concerning the behaviour when \(\beta = 0\) is appropriate here. If one naively sets \(\beta = 0\) in the system of equations \(\text{grad}(J) = 0\) the right-hand sides are all zero. The solution with all the \(c\) equal to zero inconvertibly gives a lower bound on \(Q\): the lower bound being zero. Of course if \(\beta = 0\) the term \(B(v)/\beta\) is the difficulty for functions \(v\) not vanishing identically over the whole boundary. As a quadratic function vanishing on the boundary provides the minimizer for ellipses with \(\beta = 0\) it seems appropriate, at this stage, to return to the specifics for an ellipse.

### 4.2 Variational methods for the ellipse, Case (ii) continued

For the ellipse we can write \(i_{xx}\) and \(i_{yy}\) in terms of the polar second moment about the centre, \(i_2 = i_{xx} + i_{yy}\), and the perimeter \(L\) Similarly the 4-th moments can all be written in terms of the polar fourth moment about the centre, \(i_4\) and \(i_2\) and \(L\). Each of \(L, i_2\) and \(i_4\) can be integrated in terms of elliptic integrals.

The variational winner over the quadratics gives lower bounds on \(Q\) as in the code below:

```maple
# FILE varlGenQfnmpl.txt
Qu0fn0:= proc(a,beta) # simplest as in KM93
  local Q0, ecc, L;
  Q0:= Pi/(4*(1/a^2 + a^2));
  ecc:= sqrt(1 - 1/a^4);
  L:= 4*a*EllipticE(ecc);
end proc;
```

18
Q0 + beta*Pi^2/L
end proc:

Qu0fn:= proc(a,beta) # quadratic test functions
local ecc, Eecc, Kecc, dQu0s, nQu0s;
ecc:= sqrt(1 - 1/a^4);
Eecc:= EllipticE(ecc);
Kecc:= EllipticK(ecc);
dQu0s:= 64*a*(4*a^12-15*a^8+4-15*a^4)*Eecc^2+(512*a*(a^8+2*a^4+1)*Kecc+720*a^2*Pi*beta*(1-2*a^4+a^8))*Eecc-320*a*(a^4+1)*Kecc^2;
nQu0s:= 16*a^3*(-19*a^4+4*a^8+4)*Pi*Eecc^2+(128*a^3*(a^4+1)*Pi*Kecc+3*Pi^2*beta*(-55*a^4-55*a^8+23+23*a^12))*Eecc-80*a^3*Pi*Kecc^2+24*Pi^2*beta*(7+7*a^8-6*a^4)*Kecc+180*Pi^3*beta^2*a*(1-2*a^4+a^8);
nQu0s/dQu0s
end proc:

Qfn:= proc(a,beta) # both the above
local Q0, ecc, Eecc, Kecc, QKM93, nQu0, dQu0beta0, dQu0;
Q0:= Pi/(4*(1/a^2 + a^2));
ecc:= sqrt(1 - 1/a^4);
Eecc:= EllipticE(ecc);
Kecc:= EllipticK(ecc);
QKM93:= Q0 + beta*Pi^2/(4*a*Eecc);
nQu0:= (4*(a^4+1)*Kecc+(a^8-10*a^4+1)*Eecc)^2;
dQu0beta0:= (4*a^8-19*a^4+4)*Eecc^2+8*(a^4+1)*Eecc*Kecc-5*Kecc^2;
dQu0:= 45*a^2*Pi*beta*(a^4-1)^2*(a^4+1)*Eecc+ 4*a*(a^4+1)^2*dQu0beta0;
[QKM93, QKM93+ 5*Pi^2*beta*nQu0/(16*Eecc*dQu0)]
end proc:
5 \ Q for nearly circular ellipses

5.1 \ Q at $\beta = 0$

Now, at $\beta = 0$, the explicit solution as given in §2 yields asymptotics for $\epsilon \to 0$: 

\[ Q = \frac{\pi}{4(a^2 + a^{-2})} = \frac{\pi}{8} \sqrt{1 - \epsilon^2}, \]

\[ \sim \frac{\pi}{8} \left( 1 - 2(a - 1)^2 + O((a - 1)^3) \right) \text{ as } a \to 1, \]

\[ \sim \frac{\pi}{8} \left( 1 - \frac{1}{2} \epsilon^2 + O(\epsilon^3) \right) \text{ as } \epsilon \to 0, \quad (5.1) \]

5.2 \ Nearly circular ellipses with $\beta \geq 0$

The explicit solution available for when $\beta = 0$ is presented above, and is useful to check against our asymptotics for the solution when $\beta \geq 0$. When $\epsilon \to 0$, equation (2.4) agrees with the asymptotics given earlier in equation (5.1).

Concerning the ellipse when $\beta > 0$, when $\epsilon$ is small, $u$ can be approximated in the form

\[ u = \frac{1}{4} \left( 1 - r^2 \right) + \frac{\beta}{2} + \epsilon^2 t_{02} + \epsilon t_{11} r^2 \cos(2\theta) + \epsilon^2 t_{22} r^4 \cos(4\theta). \quad (5.2) \]

Details of the perturbation analysis are in [17]. The result of the perturbation analysis is

\[ t_{11} = \frac{1}{4} \frac{1 + \beta}{1 + 2\beta}, \]

\[ t_{02} = -\frac{1}{32} \frac{4 + 5\beta + 6\beta^2}{1 + 2\beta} = -\frac{1}{32} \left( 1 + 3\beta + \frac{3}{1 + 2\beta} \right), \]

\[ t_{22} = -\frac{1}{32} \frac{\beta(1 - 2\beta)}{(1 + 4\beta)(1 + 2\beta)}. \]

Integrating $u$ with these parameters over the ellipse gives the expansion for $Q(\text{ellipse})$:

\[ Q(\text{ellipse}) \sim \frac{\pi}{8} (1 + 4\beta) + q_1 \epsilon^2 \quad (5.3) \]

\[ q_1 = -\frac{\pi}{16} (1 - 8t_{11} - 16t_{02}) = -\frac{\pi}{16} \left( 1 + \frac{\beta}{2(2\beta + 1)} \right). \quad (5.4) \]
Maple code for the near-circular approximation is as follows:

```maple
# FILE nearCircApproxQfnmpl.txt
nearCircApproxQfn:= proc(a,beta)
  local eps,q1;
  eps:= (a^2-1/a^2)/(a^2+1/a^2);
  q1:= -(Pi/16)*(1+beta*(1+6*beta)/(2*(2*beta+1)));
  Pi*(1+4*beta)/8 +q1*eps^2
end proc;

Near circular, $\beta$ small

When $\beta$ is small but non-zero, asymptotics given in §6.2 check: see equation (6.3).

Near circular, $\beta$ large

When we first take $\epsilon$ small, then let $\beta$ tend to infinity we have the approximation

$$Q \sim \frac{\pi}{2} \beta + \frac{\pi}{8} - \frac{3\pi}{32} \beta \epsilon^2 - \frac{\pi}{32} \epsilon^2.$$  

This is used as a check in §7.2.
**Numerical results**

Table 5.2 compares the previous asymptotics (column A) with variational lower bound (column V) and with the Fourier series solution (column F).

### \( a = 5/4 \)

| \( \beta \) | A         | V         | F         |
|---------|-----------|-----------|-----------|
| 1/64    | 0.3825119944 | 0.3807427556 | 0.380740330 |
| 1/16    | 0.4551128646 | 0.4530404184 | 0.4530434724 |
| 1/4     | 0.7437766793 | 0.7406928885 | 0.7406944966 |
| 1       | 1.888863782  | 1.882275712  | 1.882277797  |
| 4       | 6.450075877  | 6.429038519  | 6.429053564  |
| 16      | 24.68100694  | 24.60087273  | 24.60089638  |
| 64      | 97.59955268  | 97.28234644  | 97.28237296  |

### \( a = 17/16 \)

| \( \beta \) | A         | V         | F         |
|---------|-----------|-----------|-----------|
| 1/64    | 0.4143605368 | 0.413493072  | 0.413493804 |
| 1/16    | 0.4879061200 | 0.4878930227 | 0.4878930894 |
| 1/4     | 0.7819440804 | 0.7819247200 | 0.7819248140 |
| 1       | 1.957301880  | 1.957260968  | 1.957261120  |
| 4       | 6.657144997  | 6.657014853  | 6.657015098  |
| 16      | 25.45536249  | 25.45486728  | 25.45486742  |
| 64      | 100.6478027  | 100.6458341  | 100.6458432  |

### \( a = 65/64 \)

| \( \beta \) | A         | V         | F         |
|---------|-----------|-----------|-----------|
| 1/64    | 0.4170525380 | 0.4170525625 | 0.4170524896 |
| 1/16    | 0.4906779729 | 0.4906782880 | 0.490655446  |
| 1/4     | 0.7851701838 | 0.7851700411 | 0.8350137728  |
| 1       | 1.963086617  | 1.963085377  | 1.963085489  |
| 4       | 6.674647535  | 6.674645198  | 6.675381808  |
| 16      | 25.52081497  | 25.52081128  | 25.51131048  |
| 64      | 100.9054563  | 100.9054466  | 100.9054478  |
A simpler, but less accurate, variational approach

Figure 2: Plot of $Q_{\text{ratio}}$ at various values of $\beta$ for small eccentricity.

The lower bound shown in Figure 3 comes from a trial function which is just the $\beta = 0$ quadratic in Cartesians plus the (best) constant. Of course this is exact for eccentricity zero. Of course we do better with a more general quadratic but the integrals are more ugly.
Figure 3: \([5.2]\) Plot of small \(\beta\) approximation term \(Q_1\) and, just below it, the variational lower bound.
6 $\beta$ small

6.1 General $\Omega$

The asymptotics are described in [16]. The problem is

$$-\Delta u = 1 \quad \text{in} \; \Omega,$$
$$\beta \frac{\partial u}{\partial n} + u = 0 \quad \text{on} \; \partial \Omega.$$

When $\beta$ is small, the solution $u_\beta$ is asymptotically

$$u_\beta \sim u_{\beta=0} + \sum_{j=1}^{n} \beta^j u_j,$$

with each $u_j$ ($j \geq 1$) harmonic (and independent of $\beta$). Our interest is in $Q$, here denoted $Q_\beta$,

$$Q_\beta = \int_\Omega u_\beta,$$

so

$$Q_\beta \sim Q_0 + \beta Q_1$$

where $Q_1 = \int_\Omega u_1$.

and the integral of $u_1$ over $\Omega$ can be found explicitly (at least up to a single-variable integral) even when $u_1$ cannot be found explicitly. Now

$$-\Delta u_1 = 0 \quad \text{in} \; \Omega,$$
$$\beta u_1 = -\beta \frac{\partial u_0}{\partial n} \quad \text{on} \; \partial \Omega.$$

The Divergence Theorem gives

$$Q_1 = \int_\Omega u_1 = \int_\Omega (u_0 \Delta u_1 - u_1 \Delta u_0) = \int_\Omega \text{div} (u_0 \nabla u_1 - u_1 \nabla u_0),$$

$$= \int_{\partial \Omega} \left( u_0 \frac{\partial u_1}{\partial n} - u_1 \frac{\partial u_0}{\partial n} \right) = \int_{\partial \Omega} \left( \frac{\partial u_0}{\partial n} \right)^2 ds.$$

We remark

$$\int_{\partial \Omega} ds = |\partial \Omega|, \quad \int_{\partial \Omega} \left( \frac{\partial u_0}{\partial n} \right) ds = -|\Omega|.$$

This is useful in the maple code for checking. It also yields, via Cauchy-Schwarz,

$$|\Omega|^2 \leq |\partial \Omega| Q_1,$$

an inequality which becomes more accurate the closer the ellipse is to a circle.
6.2 Ω an ellipse

In general, and in the notation of the preceding subsection,

\[ Q_1 = \int_{\partial \Omega} \left( \frac{\partial u_0}{\partial n} \right)^2 ds = \int_{\partial \Omega} |\nabla u_0|^2 ds. \]

For an ellipse, with here a temporary reminder with the ‘e’, \( u_0 = u_{0e} \) as given in §2, thus

\[ Q_{1e} = \int_{\partial \Omega} |\nabla u_{0e}|^2 ds. \]

Although the calculation can be done in polar coordinates, perhaps Cartesians lead to simpler integrals:

\[ |\nabla u_{0e}|^2 = \frac{a^{-4}x^2 + a^4y^2}{(a^2 + a^2)^2}. \]

The Cartesian equation for that part of the ellipse in the first quadrant is

\[ Y(x) = \frac{1}{a} \sqrt{1 - \left(\frac{x}{a}\right)^2}, \quad \text{for } 0 < x < a. \]

The perimeter is

\[
|\partial \Omega| = 4 \int_0^a \sqrt{1 + Y'(x)^2} \, dx,
\]

\[ = 4a \text{EllipticE} \left( \frac{\sqrt{a^4 - 1}}{a^2} \right) \quad \text{for } a \geq 1, \]

\[ = 4a \text{EllipticE}(e) \quad \text{for } a \geq 1, \quad (6.1) \]

where \( e \) is the eccentricity as defined in equation (3.5).

We now calculate \( Q_1 \), beginning with

\[ |\nabla u_{0e}|^2 = \frac{a^2 - a^{-2}(a^2 - a^{-2})x^2}{(a^{-2} + a^2)^2} \quad \text{on } y = Y(x). \]

On integrating this \( Q_1 \) is given by

\[ Q_1 = \frac{4}{3} \frac{a^3}{(1 + a^4)^2} (2(1 + a^4)\text{EllipticE}(e) - \text{EllipticK}(e)) \quad \text{for } a \geq 1. \quad (6.2) \]
In maple code this is

```maple
# FILE QsmallBetampl.txt
betaSmallQfn:= proc(a,beta)
    local ecc, Q0, Q1;
    ecc:=sqrt(1-1/a^4);
    Q0:= Pi/(4*(a^2 +1/a^2));
    Q1:= (4/3)*(a^3/((1+a^4)^2))*(2*(1+a^4)*EllipticE(ecc)-EllipticK(ecc));
    Q0+beta*Q1
end proc;
```

**β small, near circular**

The asymptotic expansion for nearly circular ellipse, $a$ near 1, has

$$Q_1 \sim \frac{\pi}{2} - \frac{\pi}{8}(a - 1)^2 \quad \text{for} \quad a \to 1. \quad (6.3)$$

This agrees with the earlier expansion given in equations (5.3) and (5.4) where, for $\beta$ small,

$$Q \sim \frac{\pi}{8}(1 + 4\beta) - \frac{\pi}{4}(1 + \frac{\beta}{2})(a - 1)^2 \quad \text{for} \quad a \to 1, \beta \to 0.$$ 

**Numerical results**

We can compare with the numerical values given by the lower-bound variational winner amongst quadratics.
\begin{table}
\centering
\begin{tabular}{lcccc}
\hline
\( \beta = 1/4 \) & \( a \) & \( A \) & \( V \) & \( F \) \\
\hline
65/64 & .7851858133 & .7851702110 & .8350137728 \\
17/16 & .7821607512 & .7819248090 & .7819248140 \\
5/4 & .7432200201 & .7406928855 & .7406944966 \\
2 & .4953609194 & .4909982744 & .4910897996 \\
4 & .2147581278 & .2139371008 & .2142092926 \\
16 & .4473378672e-1 & .4463047672e-1 & .4473097574e-1 \\
\hline
\( \beta = 1/16 \) & \( a \) & \( A \) & \( V \) & \( F \) \\
\hline
65/64 & .4906792259 & .4906780838 & .4909655446 \\
17/16 & .4879127252 & .4878930761 & .4878930894 \\
5/4 & .4532504638 & .4530404192 & .4530434724 \\
2 & .2624399058 & .2620398525 & .2621170972 \\
4 & .9036181975e-1 & .9023065471e-1 & .9032360830e-1 \\
16 & .1348438275e-1 & .1345885965e-1 & .1348419675e-1 \\
\hline
\( \beta = 1/64 \) & \( a \) & \( A \) & \( V \) & \( F \) \\
\hline
65/64 & .4170525792 & .4170526698 & .4170524896 \\
17/16 & .4143507188 & .4143493767 & .4143493804 \\
5/4 & .3807580746 & .3807427560 & .3807440330 \\
2 & .2042096524 & .2041611381 & .2041881644 \\
4 & .5926274272e-1 & .5923472697e-1 & .5926025316e-1 \\
16 & .5672031762e-2 & .5665670014e-2 & .5671955380e-2 \\
\end{tabular}
\end{table}
6.3 \( u_1 \) for \( \beta \) small: Fourier series?

We begin by recalling (3.25)

\[
v_p = -\frac{c^2}{8} (\cosh(2\eta) + \cos(2\psi)) = -\frac{1}{4}(x^2 + y^2).
\]

Several items relating to \( v_p \) are used, in particular its normal derivative at the boundary, the \( \eta \)-derivative being closely related.

The diagonal matrix \( M_0 \) has as its \((n,n)\) entry

\[ m_{0,0} = 2\pi, \quad \text{and for } n \geq 1 \quad m_{0,n} = \pi \cosh(2n\eta_0). \]

The first subscript, here 0, reminds us that the entries come from \( M_0 \). The vector \( F_0 \) has nonzero entries only at \( n = 0 \) and \( n = 1 \) when

\[ f_{0,0} = \frac{c^2\pi}{4} \cosh(2\eta_0) \quad \text{and} \quad f_{0,1} = \frac{c^2\pi}{8}. \]

Again, the first subscript, here 0, reminds us that the entries come from \( f_0 \).

The symmetric matrix \( M_1 \) is full with entries involving \text{EllipticK} and \text{EllipticE}. The vector \( f_1 \), related to the Fourier series of \( g \), similarly involves \text{EllipticK} and \text{EllipticE}.

For \( \beta \) positive but small, we seek a representation of the form

\[ u \sim u_0 + \beta u_1 \quad \text{for} \quad \beta \to 0. \]

Our main interest is checking with an asymptotic result obtained using Cartesian coordinates:

\[ Q \sim Q_0 + \beta Q_1 \quad \text{for} \quad \beta \to 0. \]

Our calculation in Cartesian coordinates was completed a year earlier than the elliptic coordinate work here.

\( \beta = 0 \) re-visited

As one small check on the equations determining the \( A_n \), we note that when \( \beta = 0 \) the equations are very easy to solve. The matrix \( M = M_0 \) is diagonal. The only nonzero \( A_n \) are those corresponding to \( n = 0 \) and \( n = 1 \).
When $\beta = 0$, and subscripting with the first 0 in a pair merely reminding us that the values are for $\beta = 0$, $u_0 = v_p + A_{0,0} + A_{0,1} \cosh(2\eta) \cos(2\psi)$ and the equation $M_0 A = f_0$ gives

$$A_{0,0} = \frac{f_0}{m_{0,0}} = \frac{c^2 \cosh(2\eta_0)}{8} = \frac{\cosh(2\eta_0)}{4 \sinh(2\eta_0)}$$
$$A_{0,1} = \frac{f_1}{m_{0,1}} = \frac{c^2}{8 \cosh(2\eta_0)} = \frac{1}{4 \cosh(2\eta_0) \sinh(2\eta_0)}.$$

As we have already calculated $Q_0$ from other methods, the calculation here merely serves as a check on $A_{0,0}$ and $A_{0,1}$.

$\beta > 0$ small

Consider now approximations when $\beta$ is small. In §[6] we find

$$Q \sim Q_0 + \beta Q_1 \quad \text{as} \quad \beta \to 0,$$

with $Q_0$ as in equation (2.4). The approach began with assuming

$$u \sim u_0 + \beta u_1 \quad \text{as} \quad \beta \to 0,$$

but did not require finding $u_1$. If one wished to find $u_1$, a Fourier series approach as in this section might be appropriate. Approximate the vector $A$ of Fourier coefficients by

$$A \sim A_0 + \beta A_1,$$

where the vector $A_0$ is that already calculated in the preceding subsection concerning $\beta = 0$. Then we require, to order $\beta$,

$$(M_0 + \beta M_1)(A_0 + \beta A_1) = (f_0 + \beta f_1),$$

i.e., on approximating to order $\beta$,

$$M_0 A_1 = f_1 - M_1 A_0.$$

Equivalently, using, as noted previously, that $M_0$ is diagonal,

$$m_{0,0} A_{1,0} = f_{1,0} - m_{1,0,0} A_{0,0} - m_{1,0,1} A_{0,1},$$
$$m_{0,1} A_{1,1} = f_{1,1} - m_{1,1,0} A_{0,0} - m_{1,1,1} A_{0,1},$$
$$\vdots = \vdots$$
$$m_{0,n} A_{1,n} = f_{1,n} - m_{1,n,0} A_{0,0} - m_{1,n,1} A_{0,1}$$
Again the first subscript denotes the appropriate order in $\beta$.

While the calculations are routine, it may be that the solution $A_1$ is ugly, and, in any event, the human effort at keeping all the details correct is considerable. We suspect that, at this stage in the study and use of microchannels, the greater accuracy for $u$ is not needed. A calculation which would provide an additional check is to finding $A_{1,0}$ and $A_{1,1}$ which is all that is needed to check against the result for $Q_1$ given in equation (6.2).
7 \( \beta \) large

7.1 The dominant term

For general \( \Omega \)

\[
Q(\beta) \sim \beta \frac{|\Omega|^2}{|\partial \Omega|} \quad \text{for} \quad \beta \to \infty.
\]

For our ellipses of area \( \pi \) the perimeter is given by equation (6.1). Hence

\[
Q(\beta) \sim \beta \frac{\pi^2}{4a \text{EllipticE}(e)} \quad \text{for} \quad \beta \to \infty.
\]

Maple code for this is:

\[
\text{betaLargeQfn:= (a,beta) -> beta*Pi^2/(4*a*EllipticE(sqrt(1-1/a^4)))};
\]

\( \beta \) large, nearly circular ellipses

Further asymptotic approximation of that above for nearly circular ellipses is:

\[
Q(\beta) \sim \beta \left( \frac{\pi}{2} - \frac{3\pi}{8} (a - 1)^2 + O((a - 1)^3) \right) \quad \text{for} \quad \beta \to \infty, \quad a \to 1.
\]

This may be compared with the asymptotics found in equations (5.3, 5.4) further approximated for \( \beta \) large. Then \( q_1 \sim -\pi\beta/32 \) so

\[
Q \sim \frac{\pi\beta}{2} + q_1 e^2 \sim \frac{\pi\beta}{2} - \frac{3\pi\beta}{32} e^2 \sim \frac{\pi\beta}{2} - \frac{3\pi\beta}{8} (a - 1)^2,
\]

which agrees with the result of the preceding paragraph.

We have yet to find higher order approximations.

The variational approximation using quadratic test functions agrees with the lowest order term. This first term \( O(\beta) \) is, at fixed \( \beta \), a decreasing function of \( a \) on \( a > 1 \). Note that the quadratic lower bound in column V is a better approximation than \( \beta |\Omega|^2 / |\partial \Omega| \) which is less than the lower bound.
Numerical results

| $\beta = 4$ | A     | V     | F     |
|------------|-------|-------|-------|
| 65/64      | 6.282052744 | 6.674647024 | 6.675381808 |
| 17/16      | 6.265911941  | 6.657014849  | 6.657015098 |
| 5/4        | 6.056752387  | 6.429038511  | 6.429053564 |
| 2          | 4.602060688  | 4.869625399  | 4.870154656 |
| 4          | 2.449872080  | 2.627235185  | 2.627747294 |
| 16         | .6168200091  | .6676825774  | .6691074764 |

| $\beta = 16$ | A     | V     | F     |
|--------------|-------|-------|-------|
| 65/64        | 25.12821097 | 25.52081287 | 25.51131048 |
| 17/16        | 25.06364776  | 25.45486727  | 25.45486742 |
| 5/4          | 24.22700955  | 24.60087275  | 24.60089638 |
| 2            | 18.40824275  | 18.69310262  | 18.69427704 |
| 4            | 9.799488320  | 10.09964112  | 10.10542728 |
| 16           | 2.467280036  | 2.656743920  | 2.661123514 |

| $\beta = 64$ | A     | V     | F     |
|--------------|-------|-------|-------|
| 65/64        | 100.5128439 | 100.9054480 | 100.9054478 |
| 17/16        | 100.2545911  | 100.6458431  | 100.6458432 |
| 5/4          | 96.90803820  | 97.28234641  | 97.28237296 |
| 2            | 73.63297100  | 73.92334640  | 73.92489450 |
| 4            | 39.19795328  | 39.57699513  | 39.59760516 |
| 16           | 9.869120146  | 10.54995058  | 10.55945235 |
7.2 The next term, and $u_\infty$, for ellipses in general

We seek the term $\Sigma_\infty$ (independent of $\beta$) in the asymptotic expansion

$$Q \sim \frac{\beta |\Omega|^2}{|\partial \Omega|} + \Sigma_\infty \quad \text{for} \quad \beta \to \infty.$$ 

The notation here is, as in [16],

$$\Sigma_\infty = \int_{\Omega} u_\infty,$$

with $u_\infty$ satisfying equations (7.4).

The approach here is to seek a Fourier series solution for $u_\infty$, investigate its contour plot, and after this to integrate $u_\infty$ thereby finding the next term in the expansion for $Q$. The integration task only requires the constant ($n = 0$) and the next ($n = 1$) Fourier coefficients of $u_\infty$. The tasks have been completed for nearly circular ellipses, but the general case, specifically determining the $n = 0$ constant coefficient of $u_\infty$, remains ‘work in progress’.

We recast the elliptic coordinates Fourier series approach so that one uses the Fourier series of $1/g$ rather than $g$.

**Fourier series of $\hat{g} = 1/g$**

Write the Fourier series of $1/g$ as

$$\frac{1}{g(s)} = \frac{\hat{g}_0}{2} + \sum_{n=1}^\infty \hat{g}_n \cos(2n\pi s). \quad (7.1)$$

As with the earlier calculation of the Fourier series of $g$, the coefficients $\hat{g}_n$ can be found in terms of EllipticE and EllipticK functions. In particular, with the notation of [3.13], we have

$$|\partial \Omega| = 4c \cosh(\eta_0) \text{EllipticE}_0,$$

$$\hat{g}_0 = \frac{4}{\pi} \text{EllipticE}_0 = \frac{1}{\pi c \cosh(\eta_0)} |\partial \Omega|, \quad (7.2)$$

$$\hat{g}_1 = \frac{4}{3\pi} (- \cosh(2\eta_0) \text{EllipticE}_0 + (\cosh(2\eta_0) - 1) \text{EllipticK}_0), \quad (7.3)$$

and all the higher $\hat{g}_n$ can also be expressed similarly in terms of $\text{EllipticE}_0$ and $\text{EllipticK}_0$:

$$\hat{g}_n = \frac{4}{\pi} \left( \hat{E}_n \text{EllipticE}_0 + \hat{K}_n \text{EllipticK}_0 \right),$$

34
where \( \hat{E}_n \) and \( \hat{K}_n \) are polynomials of degree \( n \) in \( q = \cosh(\eta_0)^2 \) with rational number coefficients.

Again there is a three-term recurrence relation exists to determine the polynomials \( \hat{E}_n \) and \( \hat{K}_n \). In the notation of equation (3.12), the \( \hat{K}_n \) sequence of polynomials starts with

\[
\hat{K}_0 = 0, \quad \hat{K}_1 = \frac{2}{3} q = \frac{1}{3} (q_2 - 1).
\]

The \( \hat{E}_n \) sequence of polynomials starts with

\[
\hat{E}_0 = 1, \quad \hat{E}_1 = -\frac{1}{3} (2q - 1) = -\frac{1}{3} q_2.
\]

See Appendix C and the recurrence (C.4).

**Calculating \( u_\infty \) (at least up to an additive constant)**

As in [16], \( u \sim \beta|\Omega|/|\partial\Omega| + u_\infty \) where

\[
- \Delta u_\infty = 1 \quad \text{and} \quad \frac{\partial u_\infty}{\partial n} = -\frac{|\Omega|}{|\partial\Omega|}, \quad \int_{\partial\Omega} u_\infty = 0. \tag{7.4}
\]

Set

\[
u_\infty = v_\infty + v_p,
\]

with \( v_p \) as in (3.25) before. Once again the harmonic function \( v_\infty \) can be represented as a Fourier series

\[
v_\infty = \sum_{n=0}^{\infty} V_n \cosh(2n\eta) \cos(2n\psi). \tag{7.5}
\]

To determine \( V_n \) for \( n \geq 1 \) we will need

\[
\frac{\partial v_\infty}{\partial \eta} (\eta_0, \psi) = \sum_{n=1}^{\infty} 2n V_n \sinh(2n\eta_0) \cos(2n\psi) \quad \text{on} \eta = \eta_0.
\]

The Neumann boundary condition is

\[
\frac{g(\psi)}{c \cosh(\eta_0)} \frac{\partial u_\infty}{\partial \eta} (\eta_0, \psi) = \frac{1}{c \sqrt{\cosh^2 \eta_0 - \cos^2 \psi}} \frac{\partial u_\infty}{\partial \eta} (\eta_0, \psi) = -\frac{|\Omega|}{|\partial\Omega|}.
\]
In terms of the harmonic function $v_\infty$ this is

$$\frac{\partial v_\infty}{\partial \eta} = -\frac{\partial v_p}{\partial \eta} - \frac{|\Omega|}{|\partial \Omega|} \frac{c \cosh(\eta_0)}{g(\psi)}$$
on $\eta = \eta_0$. \hspace{1cm} (7.6)

Now one finds

$$\frac{\partial v_p}{\partial \eta} = -\frac{1}{2}$$
on $\eta = \eta_0$. \hspace{1cm} (7.7)

(In Cartesian coordinates we have $\nabla v_p \cdot \nabla u_0$ is constant around $\partial \Omega$, but we seek $u_\infty$ such that $\nabla u_\infty \cdot \nabla u_0$ is a constant multiple of $|\nabla u_0|$ there.) The integral around the boundary of the normal derivative of a harmonic function must be 0, and this accords with the value of $\hat{g}_0$, the constant term in the Fourier series of $1/g$ to ensure this as combining (7.7) and

$$\frac{|\Omega|}{|\partial \Omega|} c \cosh(\eta_0) \hat{g}_0 = \frac{\pi}{2\pi|\partial \Omega|} = \frac{1}{2}$$

in (7.6) we have the required result.

From the Fourier series of $1/g(\psi)$ we can find all the Fourier coefficients (7.5), except the constant term, in the series for $v_\infty$. In particular the term $V_1$ is found from (7.6) (on using (7.7), contributing 0): \hspace{1cm}

$$2V_1 \sinh(2\eta_0) = 0 - \frac{|\Omega|}{|\partial \Omega|} c \cosh(\eta_0) \hat{g}_1 = -\frac{\pi}{4} \hat{g}_1 \text{ EllipticE}_0.$$ \hspace{1cm} (7.8)

Hence, \hspace{1cm}

$$V_1 = -\frac{\pi}{8} \text{ EllipticE}_0 \sinh(2\eta_0).$$ \hspace{1cm} (7.10)

Similarly, \hspace{1cm}

$$V_2 = -\frac{\pi}{16} \text{ EllipticE}_0 \sinh(4\eta_0),$$ \hspace{1cm} (7.9)

and, more generally, for $n > 0$, \hspace{1cm}

$$V_n = -\frac{\pi}{8n} \text{ EllipticE}_0 \sinh(2n\eta_0).$$ \hspace{1cm} (7.10)

**Contour plots for $u_\infty$**

As we have all the Fourier coefficients $V_n$ except $V_0$ we can produce contour plots. We show one of these, that for $a = 2$. 

36
The contourplot (at $a = 2$) is visually indistinguishable from that produced from the quadratic found using the $c_0$, $c_X$, $c_Y$ values found in §4.2. Except when $\eta$ is small, the $V_n$ will decrease rapidly with $n$, and the plots are essentially those from truncating the Fourier series so that it only includes the $V_1$ term. The numerical solutions given in Figure 5 are, for $\beta \geq 4$, very much like that of Figure 4.

Figure 4: $a = 2$. Contours of $u_{\infty}$
Finding $V_0$ and $Q$?

The calculation of $V_0$ remains. We will outline how it can be found as an infinite series, but shall only complete the details for nearly circular ellipses. The condition that the boundary integral of $u_\infty$ is zero remains to be used. Now, with, as usual, $s$ for arc length around the boundary, $\eta = \eta_0$,

$$(\frac{ds}{d\psi})^2 = (\frac{dx}{d\psi})^2 + (\frac{dy}{d\psi})^2 = c^2 \left( \cosh(\eta_0)^2 - \cos(\psi)^2 \right) = J(\eta_0, \psi).$$

The square root of the Jacobian is related to the function $g$:

$$\sqrt{J(\eta_0, \psi)} = \frac{c \cosh(\eta_0)}{g(\psi)} = \frac{a}{g(\psi)}.$$

We need to satisfy

$$0 = \int_{\partial\Omega} u_\infty = 4 \int_{0}^{\pi/2} (v_p + v_\infty) \sqrt{J(\eta_0, \psi)} d\psi.$$

The integral $I_{p,\theta}$ around the boundary of $v_p$ is $-i_2/4$ where $i_2$ is the boundary moment of inertia and can also be evaluated directly,

$$I_{p,\theta} = 4 \int_{0}^{\pi/2} v_p \sqrt{J(\eta_0, \psi)} d\psi = 4 \int_{0}^{\pi/2} v_p \hat{g}(\psi) d\psi,$$

$$= -\frac{a^3}{3} \left( (2 - e^2) \text{EllipticE}_0 + (1 - e^2) \text{EllipticK}_0 \right).$$

When evaluating the boundary integral of $v_\infty$, the square root brings in all the $V_n$.

- It may be that there are useful Mean Value results applying to ellipses, and harmonic functions in them, of

$$\int_{\partial\Omega} v_\infty - v_\infty(0) |\partial\Omega| ?$$

It is, of course, zero for a circle. In general it won’t be zero, as, for example,

$$\int_{\partial\Omega} (x^2 - y^2) = i_X - i_Y \neq 0 \text{ when } a > 1.$$

Nevertheless, the quantity can be related to an integral along the major axis of the ellipse, and [27] is given as a reference in papers by Symeonidis.
The integral
\[ I_{\infty, \partial} = 4 \int_{0}^{\pi/2} v_\infty \sqrt{J(\eta_0, \psi)} \, d\psi = 4a \int_{0}^{\pi/2} v_\infty \hat{g}(\psi) \, d\psi, \]
can be evaluated, with \( V_0 \) left as an unknown, using Parseval’s Theorem. We could find \( V_0 \) if we could evaluate
\[ \sum_{j=1}^{\infty} \frac{\hat{g}_n^2}{n \sinh(2n\eta_0)}. \]
At each fixed \( a > 1 \) (or equivalently \( \eta_0 \) or eccentricity \( e \)) this would be a simple float numerical task. However, we have not yet found a simple formula for \( V_0 \) and hence \( \Sigma_\infty \) as simple as that we have for \( Q_1 \), in equation (6.2).

Finally to determine \( \Sigma_\infty \) using \( u_\infty = v_p + v_\infty \) and recalling that we will only need, as in (3.33), the \( n = 0 \) and \( n = 1 \) Fourier coefficients of \( v_\infty \):
\[ \Sigma_\infty = -\frac{\pi \cosh(2\eta_0)}{8 \sinh(2\eta_0)} + \pi V_0 - \frac{1}{2} \pi V_1. \quad (7.11) \]
Combining this with the dominant term we have
\[ Q \sim \frac{\beta |\Omega|^2}{|\partial \Omega|} + \Sigma_\infty \quad \text{as } \beta \to \infty. \]

\( \beta \) **large for nearly circular ellipses**

Recall that the eccentricity \( e = 1 / \cosh(\eta_0) \). From equation (3.16)
\[ \hat{g}_0 = 1 - \frac{e^2}{4}, \quad \hat{g}_1 = -\frac{e^2}{4}. \]
Also
\[ \text{EllipticE}_0 \sim \frac{\pi}{2} - \frac{\pi}{8} e^2 - \frac{3\pi}{128} e^4, \quad \text{EllipticK}_0 \sim \frac{\pi}{2} + \frac{\pi}{8} e^2 + \frac{9\pi}{128} e^4, \]
and
\[ a = \left( \frac{1}{1 - e^2} \right)^{1/4} \sim 1 + \frac{1}{4} e^2 + \frac{5}{32} e^4, \quad c = a e, \]

39
and

$$|\partial \Omega| = 4a \text{EllipticE}_0 \sim 2\pi + \frac{3\pi}{32} e^4.$$  

(The terms in $e^4$ in the series for EllipticE$_0$ and $a$ are only there to check against the series for $|\partial \Omega|$.) We also note

$$\sinh(2\eta_0) = \frac{1}{e} \sqrt{\frac{1}{e^2} - 1} \sim \frac{2}{e^2} - 1 - \frac{1}{4} e^2.$$  

The Fourier coefficients of $v_\infty$ are needed. From equation (7.8)

$$V_1 = -\frac{\pi \hat{g}_1}{8\text{EllipticE}_0 \sinh(2\eta_0)} \sim \frac{1}{32} e^4.$$  

Comparison with earlier near-circular results

Because, for nearly circular ellipses, the asymptotics for all the $V_n$, $n \geq 1$ are uncomplicated, and the terms get rapidly smaller in $n$, we can find $V_0$. Maple code gives, for $\beta \to \infty$ and $e \to 0$,

$$Q \sim \left( \frac{1}{2} \pi - \frac{3}{128} \pi e^4 \right) \beta + \frac{1}{8} \pi - \frac{1}{128} \pi e^4 - \frac{1}{256} \frac{\pi e^4}{\beta}.$$  

Another quick look at the $\epsilon$ small solution of §5.2, in $(\eta, \psi)$ coordinates, produces the same result.

### 7.3 $u_\infty$ for other domains

It seems to rarely happen that both $u_0$ and $u_\infty$ have simple explicit formulae representing them. An exception is the equilateral triangle: see [16, 24]. And, of course, the circular disk has $u_0 = u_\infty$ and is the only domain for which there is equality.

For our ellipse, $u_0$ is simply, in Cartesians, a quadratic polynomial, or equivalently has a Fourier series (in elliptic coordinates) with just 2 terms, while $u_\infty$ has a Fourier series with all terms nonzero.

The situation with a rectangle is as follows. There is a elaborate Fourier series in Cartesian coordinates for $u_0$. However, $u_\infty$ is simply a quadratic in the Cartesian coordinates. Indeed our function $u_0$, given in equation (2.3), is $u_\infty$ for the rectangle $(-a^2, a^2) \times (-b^2, b^2)$. The methods in §4.1 would give good lower bounds for $Q$(rectangle) when $\beta$ is large. (For large $\beta$ asymptotics for a rectangle, see [18].)
8 Conclusion and Open Problems

The solution to the Robin boundary problem has been approached from several directions: Fourier series, variational bounds, and asymptotic approximations.

There are several questions that remain.

1. What can be said about asymptotics when the eccentricity tends to 1? The small $\beta$ and the large $\beta$ approximations yield some information but more should be possible. (We remark that at large $\beta$, small eccentricity will cause the Fourier coefficients $V_n$ to decay more slowly.) Matched asymptotic approximations may be appropriate.

2. Can $u_\infty$ be found explicitly, or at least its integral over $\Omega$, $\Sigma_\infty$? Given that there are three-term recurrence relations for the Fourier coefficients of both $g$ and $1/g$, might there be a recurrence relation for the coefficients $V_n$? One might combine equation (7.10) with items from Appendix C.

3. In the asymptotics for $\beta$ tending to zero, we found how $Q$ changed, but can one find tidy formulae giving the departure of $u$ from $u_0$?

4. Might it be possible to make better use of the Fourier series for $g$ and $1/g$?

In connection with items 2 and 3 above, we remark that there is a ‘Poisson Integral Formula’ for solving the Dirichlet problem for Laplace’s equation in an ellipse. This follows from the conformal map between ellipse and disk: see [20], p177. See also [27 25 23 28].
Appendix: Geometry of ellipses

A.1 General ellipses

A.1.1 Polar coordinates

Consider the ellipse \( \frac{x^2}{a^2} + ay^2 \leq 1 \). In polar coordinates relative to the centre the boundary curve is

\[
r = \frac{1}{\sqrt{\cos(\theta)^2 + a^2 \sin(\theta)^2}} = \sqrt{\frac{2}{a^2 + a^{-2} - (a^2 - a^{-2}) \cos(2\theta)}}
\]

\[
= r \left( \frac{\pi}{4} \right) \left( 1 - \frac{a^2 - a^{-2}}{a^2 + a^{-2}} \cos(2\theta) \right)^{-1/2}
\]

where \( r \left( \frac{\pi}{4} \right) = \sqrt{\frac{2}{a^2 + a^{-2}}} \) \( (A.1) \)

\[
= \sqrt{\frac{1 + \tan^2(\theta)}{a^{-2} + a^2 \tan^2(\theta)}}. \quad (A.2)
\]

In our computations we take \( a \geq 1 \).

Without any assumptions on \( a \), the coefficients in the Fourier series for \( r(\theta) \) involve elliptic integrals. Our interest in some later sections will be in \( a \) near 1. However \( (a - 1) \) might not be the best perturbation parameter. We have also used, in computations,

\[
\epsilon = \frac{a^2 - a^{-2}}{a^2 + a^{-2}} \sim 2(a - 1) \quad \text{as} \quad a \rightarrow 1.
\]

The ellipse is

\[
r^2 = \frac{\sqrt{1 - \epsilon^2}}{1 - \epsilon \cos(2\theta)}.
\]

We have also used (as have others, e.g. [5]), the eccentricity, as in equation (3.5),

\[
e = \sqrt{1 - \frac{1}{a^4}} \sim 2(a - 1)^{1/2} \quad \text{as} \quad a \rightarrow 1. \quad (A.3)
\]

As both \( \epsilon^2 \) and \( \epsilon \) are relatively similar low-order rational functions of \( a^2 \) one readily finds that

\[
\frac{\epsilon}{\epsilon^2} = \frac{1}{2} (1 + \epsilon).
\]
We remark that the polar equation in terms of the eccentricity is

\[ r = \frac{1}{a \sqrt{1 - (e \cos(\theta))^2}} = \frac{g(\theta)}{a}, \]

Including an \( a \) dependence in \( r(a, \theta) \) we remark that \( r(1/a, \theta) = r(a, \theta + \pi/2) \) and \( \epsilon(1/a) = -\epsilon(a) \). The binomial expansions

\[
\frac{r(0)}{r(\pi/4)} = (1 - \epsilon)^{-1/2} = 1 + \sum_{k=1}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2} \epsilon^k,
\]

\[
\frac{r(\theta)}{r(\pi/4)} = (1 - \epsilon \cos(2\theta))^{-1/2} = 1 + \sum_{k=1}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2} \cos(2\theta)^k \epsilon^k, \tag{A.4}
\]

may be useful in finding higher terms in the perturbation expansions (for \( a \) near 1) of some domain functionals. The symmetries of the ellipse explain the form of the expansion in (5.2):

- It is symmetric about \( \theta = 0 \), hence only cosine terms.
- It is symmetric about \( \theta = \pi/2 \) and hence only the even order cosine terms \( \cos(2m\theta) \).
- When \( \epsilon \) is replaced by \( -\epsilon \) and \( \theta \) by \( \theta + \pi/2 \) the expression is unchanged and hence the form of the polynomial coefficients in \( \epsilon \) forming the Fourier coefficients. For \( m \) is odd, only odd powers of \( \epsilon \) appear: for \( m \) is even, only even powers of \( \epsilon \) appear.

We see these symmetries in connection the solutions of our pde problem. Returning to the study of the boundary curve, we also need the expansion for \( r(\pi/4) \):

\[
a^2 = \sqrt{\frac{1 + \epsilon}{1 - \epsilon}}, \quad \frac{2}{r(\pi/4)} = a^2 + a^{-2} = \frac{2}{\sqrt{1 - \epsilon^2}}, \quad r\left(\frac{\pi}{4}\right) = (1 - \epsilon^2)^{1/4}.
\]

There are a few items of undergraduate calculus that are used. Let \( s \)
denote arclength measured around the curve. Then

\[ \frac{ds}{d\theta} = \sqrt{r(\theta)^2 + \left(\frac{dr(\theta)}{d\theta}\right)^2} \]

in general,

\[ = 2a \sqrt{ \frac{(1 + a^8) - (a^8 - 1) \cos(2\theta)}{((1 + a^4) - (a^4 - 1) \cos(2\theta))^3} } \quad \text{for an ellipse,} \]

\[ = 2a \sqrt{ \frac{(1 + \epsilon)(1 + \epsilon^2 - 2\epsilon \cos(2\theta))}{(1 - \epsilon \cos(2\theta))^3} } \quad \text{for an ellipse.} \]

We remark that, for the ellipse the normal can be found from the gradient of the elastic torsion function, \( \nabla u_{\Omega e} = \nabla u(\text{ellipse}, \beta = 0) \).

A.1.2 The usual parametric description of an ellipse

The boundary of the ellipse can be described by

\[ x = a \cos(\psi), \quad y = \frac{1}{a} \sin(\psi). \quad (A.5) \]

(The relation between the parameter \( \psi \) and the polar angle \( \theta \) is \( \tan(\theta) = \tan(\psi)/a^2 \).)

\[ \left( \frac{ds}{d\psi} \right)^2 = \left( \frac{dx}{d\psi} \right)^2 + \left( \frac{dy}{d\psi} \right)^2, \]

\[ = a^2 \sin(\psi)^2 + \frac{1}{a^2} \cos(\psi)^2, \]

\[ = a^2 \left( 1 - \epsilon^2 \cos(\psi)^2 \right). \]

From this the perimeter is calculated:

\[ |\partial \Omega| = 4a \int_{0}^{\pi/2} \sqrt{1 - (\epsilon \cos(\hat{\psi}))^2} \, d\psi, \]

\[ = 4a \int_{0}^{\pi/2} \sqrt{1 - (\epsilon \sin(\hat{\psi}))^2} \, d\hat{\psi}, \]

\[ = 4a \text{EllipticE}(\epsilon). \]
The support function, $h$

There are several equivalent definitions of the support function $h$ which we consider as a function of points on the boundary. Thus at the point associated with the value $\psi$ in definition (A.5)

$$h = \frac{1}{a\sqrt{1 + e^2 \cos^2(\psi)}}.$$

As noted in [19], the curvature is $h^3$. Once again, the functions $g$ and $\hat{g}$ make an appearance.

A.1.3 Elliptic coordinates

The elliptic coordinates are related to Cartesians by

$$x = c \cosh(\eta) \cos(\psi), \quad y = c \sinh(\eta) \sin(\psi),$$

and we will set the parameter $c$ by

$$c = \sqrt{a^2 - a^{-2}}.$$

The boundary of the ellipse can be represented, with fixed $\eta_0$,

$$x = c \cosh(\eta_0) \cos(\psi), \quad y = c \sinh(\eta_0) \sin(\psi).$$

In this

$$a = c \cosh(\eta_0), \quad a^{-1} = c \sinh(\eta_0),$$

so that

$$\tanh(\eta_0) = a^{-2}, \quad (c^2 = a^2 - a^{-2}), \quad \epsilon = \frac{1}{\cosh(2\eta_0)}.$$

Nearly circular ellipses will have $\eta_0$ large.

The perimeter is calculated:

$$|\partial \Omega| = 4 \int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\psi}\right)^2 + \left(\frac{dy}{d\psi}\right)^2} d\psi,$$

$$= 4c \int_0^{\pi/2} \sqrt{\cosh(\eta_0)^2 - \cos(\psi)^2} d\psi,$$

$$= 4c \cosh(\eta_0) \text{EllipticE} \left( \frac{1}{\cosh(\eta_0)} \right).$$
A.2 Nearly circular ellipses, polar coordinates

The first few terms in the expansion of \( r(\theta) \) are

\[
r(\theta) = 1 + \frac{1}{2} \cos(2\theta) \epsilon + \left( \frac{3}{16} \cos(4\theta) - \frac{1}{16} \right) \epsilon^2 + O(\epsilon^3).
\]

Alternatively we can consider asymptotics as \( a \to 1 \). Then \( \rho = r_{\text{ellipse}} - 1 \) satisfies

\[
\rho(\theta) \sim (a - 1) \cos(2\theta) - \left( \frac{1}{4} + \frac{1}{2} \cos(2\theta) - \frac{3}{4} \cos(4\theta) \right) (a - 1)^2 + O((a - 1)^3)
\]

\[
= -\frac{1}{4} (a - 1)^2 + \left( (a - 1) - \frac{1}{2} (a - 1)^2 \right) \cos(2\theta) + \frac{3}{4} (a - 1)^2 \cos(4\theta) + O((a - 1)^3).
\]

(A.6)

All other Fourier coefficients are of sufficiently small order that they are not needed in the calculations in this paper.

For calculations extending the use of higher order terms in the Fourier series (A.6), one might need formulae like

\[
x^{2n} = 2^{1-2n} \left( \frac{1}{2} \binom{2n}{n} + \sum_{j=1}^{n} \binom{2n}{n-j} T_{2j}(x) \right),
\]

and, at some future date, we may for elliptical \( \Omega \) use this but have yet to do so.

A.3 Moments of inertia

A.3.1 General \( \Omega \)

Area moments

The polar moment of inertia, taking the origin at the centroid, is

\[
I_c = \int_{\Omega} \left( (x - x_c)^2 + (y - y_c)^2 \right) d\Omega,
\]

where \( z_c = (x_c, y_c) \) is the centroid of \( \Omega \). When the boundaries are given in polar coordinates, this is

\[
I_c = \frac{1}{4} \int_{0}^{2\pi} r(\theta)^4 d\theta.
\]
Higher order moments arise in connection with calculations based on polynomial test functions in the variational approach of §4.

**Boundary moments**
See §4.

### A.3.2 Ellipse

**Area moments**
For our disk and ellipse these are

\[ I_c(\text{disk}) = \frac{\pi}{2} a^4, \quad I_c(\text{ellipse}) = \frac{\pi}{4} (a^2 + a^{-2}) = \frac{\pi}{2 \sqrt{1 - \epsilon^2}}. \quad (A.7) \]

The asymptotics below check with the entries in the table of [26] treating domain functionals for nearly circular domains:

\[
\frac{2I_c(\text{ellipse})}{\pi} \sim 1 + 2(a - 1)^2 \quad \text{as } a \to 1, \\
\left( \frac{2I_c(\text{ellipse})}{\pi} \right)^{1/4} \sim 1 + \frac{1}{2} (a - 1)^2 + o(a - 1)^2 \quad \text{as } a \to 1, \\
\sim 1 + a_0 + \frac{3}{4} a_2 + o(a - 1)^2 \quad \text{as } a \to 1.
\]

**Boundary moments**
For doubly-symmetric domains like our ellipse, our notation in §4 is

\[ i_{2n} = \int_{\partial \Omega} r^{2n} ds. \]

For the ellipse these can be evaluated in terms of elliptic integrals.

### A.4 Ellipse geometry: miscellaneous

The modulus of asymmetry for an ellipse is calculated in [8]. The calculation involves calculating \( \hat{g}_0 \).

In the case \( \beta = 0 \) there are improvements to the St Venant inequality in terms of the modulus of asymmetry. (For the St Venant inequality, see [26], and, for ellipses, trivially the \( \beta = 0 \) case in inequality (1.6).)
Appendix: Elliptic integrals

The functions denoted by EllipticE and EllipticK in this paper are, in this appendix, written as $E(k)$ and $K(k)$. Our notation follows that of many (but not all!) authors, in particular \cite{22} §3.8 and \cite{10}:

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2(\theta)} \, d\theta,$$

$$K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2(\theta)}} \, d\theta.$$

For $0 \leq z < 1$

$$K(k) = \int_0^1 \frac{1}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} \, dt,$$

$$E(k) = \int_0^1 \sqrt{\frac{1 - k^2 t^2}{1 - t^2}} \, dt.$$

The Cauchy-Schwarz inequality yields

$$\frac{\pi^2}{4} \leq K(k) E(k).$$

The functions can be written as hypergeometric functions:

$$K(k) = \frac{\pi}{2} \, _2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right),$$

$$E(k) = \frac{\pi}{2} \, _2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; k^2\right).$$

In showing, in Appendix C equation (C.11), that two different representations of $\hat{g}_n$ are equivalent, we needed to tell maple that, as in \cite{10} 8.126, with $k' = \sqrt{1 - k^2}$,

$$K\left(\frac{1 - k'}{1 + k'}\right) = \frac{1 + k'}{2} K(k),$$

$$E\left(\frac{1 - k'}{1 + k'}\right) = \frac{1}{1 + k'} (E(k) + k' K(k)).$$

(The first of these is given, in a different notation, on functions.wolfram.com.) In our application $k = 1/\sqrt{q} = e$. 48
C Further properties of $g$ and $1/g$

We denote the reciprocal of $g$ by $\hat{g}$:

$$\hat{g}(\psi) = \frac{1}{g(\psi)} = \sqrt{1 - \frac{\cos^2(\psi)}{q}}.$$ 

As described earlier, the Fourier cosine coefficients of $g$ are denoted $g_n$, and those of $\hat{g}$ by $\hat{g}_n$.

There are various simple relationships between $g$ and $\hat{g}$ which lead to relationships between their Fourier coefficients.

• Since

$$\hat{g}(\psi) = \left(1 - \frac{\cos^2(\psi)}{q}\right) g(\psi), \quad \frac{1}{2q} ((2q - 1) - \cos(2\psi)) g(\psi),$$

we have by multiplying both sides by $\cos(2n\psi)$, using cosine formulae, and integrating

$$4q\hat{g}_n = 2(2q - 1)g_n - g_{n+1} - g_{n-1}. \quad (C.1)$$

• Since

$$2q \frac{d\hat{g}(\psi)}{dq} + \hat{g}(\psi) - g(\psi) = 0,$$

the Fourier coefficients $g_n$ can be found from those of $\hat{g}$:

$$g_n = \hat{g}_n + 2q \frac{d\hat{g}_n}{dq} = 2\sqrt{q} \frac{d}{dq} (\sqrt{q}\hat{g}_n). \quad (C.2)$$

Define also

$$q_2 = 2q - 1.$$

There are three-term recurrence relations between the coefficients:

$$(n - \frac{1}{2})g_n = 2q_2(n - 1) g_{n-1} - (n - \frac{3}{2}) g_{n-2}, \quad (C.3)$$

$$(n + \frac{1}{2})\hat{g}_n = 2q_2(n - 1) \hat{g}_{n-1} - (n - \frac{5}{2}) \hat{g}_{n-2}. \quad (C.4)$$
The derivation is treated below, where we derive the following equivalent to (C.3):

\[(n + \frac{1}{2})g_{n+1} = 2q_2n g_n - (n - \frac{1}{2}) g_{n-1}.\]  
(C.5)

Corresponding to (C.4) we have

\[(n + \frac{3}{2}) \hat{g}_{n+1} = 2q_2n \hat{g}_n - (n - \frac{3}{2}) \hat{g}_{n-1}.\]  
(C.6)

As before, with the argument \(1/\sqrt{q}\) of the elliptic integrals and the notation as before,

\[g_n = \frac{4}{\pi} \left( E_n \text{EllipticE}_0 + K_n \text{EllipticK}_0 \right), \quad \text{(C.7)}\]
\[\hat{g}_n = \frac{4}{\pi} \left( \hat{E}_n \text{EllipticE}_0 + \hat{K}_n \text{EllipticK}_0 \right), \quad \text{(C.8)}\]

with the \(E_n, K_n, \hat{E}_n, \hat{K}_n\) polynomials in \(q_2\) of degree \(n\). The polynomials \(E_n, K_n\) satisfy the recurrence (C.3); the polynomials \(\hat{E}_n, \hat{K}_n\) satisfy the recurrence (C.4). The starting values for the iterations to determine the polynomials are given, for the \(g_n\) iteration by (3.14), (3.15) and for the \(\hat{g}_n\) iteration by (7.2), (7.3). Maple’s rsolve will solve the special cases when \(q_2 = 0\) and \(q_2 = 1\). When \(q_2 = 1\) there are constant solutions \((g_n = 1\) for all \(n\)), and reduction of order gives the other.

We have yet to determine if the polynomial sequences solving the recurrences are any of the many named families of polynomials. Both recurrences are of the form

\[(n + \alpha)u_{n+1} = 2q_2nu_n - (n - \alpha)u_{n-1}.\]

The \(g_n\) recurrence has \(\alpha = 1/2\); the \(\hat{g}_n\) recurrence has \(\alpha = 3/2\). At other values of \(\alpha\) the recurrence can be solved. At \(\alpha = 0\) the \(n\) cancels and the recurrence is constant coefficient, so solvable by \(u_n = r^n\) for appropriate \(r\). Furthermore a solution polynomial in \(q_2\) is \(u_n(\alpha = 0) = T_n(q_2)\), where \(T_n\) is the Chebyshev polynomial. Chebyshev polynomials also arise in solving the recurrence when \(\alpha = 1\): the solutions are \(u_n(\alpha = 1) = T_n(q_2)/n\).

A posting on stackexchange considers related Fourier series, those when the \(\cos^2(\psi)\) term in the definitions of \(g_n\) and \(\hat{g}_n\) is replaced by \(\sin^2(\psi)\). The \(n\)-th coefficient in the stackexchange series is just \((-1)^n\) times ours.
stackexchange posting suggests there might be reasonably neat expressions for generating functions,

\[ G(X) = g_0 + g_1 X + \sum_{n=2}^{\infty} g_n X^n, \quad \hat{G}(X) = \hat{g}_0 + \hat{g}_1 X + \sum_{n=2}^{\infty} \hat{g}_n X^n, \]

and that these functions satisfy first order linear differential equations. The differential equation satisfied by \( G \) is

\[ \mathcal{L}G = (X - 2q_2 X^2 + X^3) \frac{dG}{dX} - \frac{1}{2} (1 - X^2)G = \frac{1}{2} (-g_0 + g_1 X). \]

\( X = 0 \) is a singular point. The homogeneous de \( \mathcal{L}G_h = 0 \) is solved by a function \( G_h \) with \( G_h(0) = 0 \),

\[ G_h(X) = \sqrt{\frac{X}{1 - 2q_2 X + X^2}}. \]

We would be content with representations of \( G \) valid for \( q_2 > 1 \) and \( 0 \leq X < 1/(2q_2) \). Applying the ‘variation of parameters’ formula to the nonhomogenous de above, using \( G_h \), leads to a messy particular solution in terms of elliptic integrals. The corresponding de for \( \hat{G} \) is:

\[ \hat{\mathcal{L}}\hat{G} = (X - 2q_2 X^2 + X^3) \frac{d\hat{G}}{dX} + \frac{1}{2} (1 - X^2)\hat{G} = \frac{1}{2} (\hat{g}_0 + 3\hat{g}_1 X). \]

\( X = 0 \) is a singular point. The homogeneous de \( \mathcal{L}\hat{G}_h = 0 \) is solved by a function \( \hat{G}_h \) with \( \hat{G}_h(0) = 0 \),

\[ \hat{G}_h(X) = \sqrt{\frac{1 - 2q_2 X + X^2}{X}}, \]

(which is the reciprocal of \( G_h \)). The relations \((C.1,C.2)\) lead to relations between \( G \) and \( \hat{G} \), for example

\[ G = 2\sqrt{q} \frac{d}{dq} (\sqrt{q}\hat{G}). \]

One can derive a differential equation for \( \hat{G}(q) \) involving derivatives with respect to \( q \). It may be that if one starts from a closed form for \( \hat{G}(1) \), i.e. at \( q = 1 \), solving the initial value problem might be useful.
The method to establish (C.3), (C.4) involves a further relation between the coefficients $g_n$, $\hat{g}_n$ of the two series, and is suggested in a posting by Jack D’Aurizio on stackexchange. See math.stackexchange.com/questions/930003/fourier-series-of-sqrt1-k2-sin2t

We have

\[ g_{m+1} = (2q - 1)g_m + (4m - 2)q \hat{g}_m, \]  
\[ \text{(C.9)} \]

and also, exactly as in D’Aurizio’s post,

\[ 8mq\hat{g}_m = g_{m+1} - g_{m-1}. \]  
\[ \text{(C.10)} \]

Eliminating $\hat{g}_m$ between equations (C.9), (C.10) we find the recurrence (C.5) with $n$ there replaced by $m$.

It may be that, for $q^2 > 1$, all the $g_m$ are positive, and the sequence is decreasing.

We have $\hat{g}_0 > 0$ and it may be that, for $q^2 > 1$ and $m \geq 1$ that the terms $\hat{g}_m$ are all negative and form an increasing sequence.

Also presented on stackexchange is a hypergeometric formula for $\hat{g}_n$. See also [3] at Lemma 1 which gives a slightly different representation. Define

\[ \lambda = \frac{1}{\sqrt{q} + \sqrt{q-1}} = \sqrt{q} - \sqrt{q-1}, \]

from which

\[ \lambda^2 = \frac{1}{q^2 + \sqrt{q^2 - 1}}. \]

We have

\[ \hat{g}_n = -\frac{2}{\lambda\sqrt{q}} \frac{(2n)!}{(2n-1)2^{2n+1}(n!)^2} \left(\lambda^2\right)^n \binom{-1/2}{n+1} \frac{\Gamma(n+1)\Gamma(5/2)}{\Gamma(n+5/2)}, \]  
\[ \text{(C.11)} \]

(Quick check, $\hat{g}_0 > 0$ and, for $n > 0$, $\hat{g}_n < 0$, though we only have numeric evidence for this.) It may be possible to use this to find a second linearly independent solution to the recurrence (C.4). We would also like to find solutions to this recurrence which are polynomial in $q^2$.

There are many equivalent forms of recurrence relation. For example, defining the $a_n$ sequence via

\[ \hat{g}_{n+1} = a_{n+1} \left(2q^2\right)^n \frac{\Gamma(n+1)\Gamma(5/2)}{\Gamma(n+5/2)}, \]
we have, for \( n > 1 \)

\[
a_{n+1} = a_n + \frac{(n - 3/2)(n + 1/2)}{4 q_2^2 n(n - 1)} a_{n-1}.
\]

While \( \hat{g}_n \) might, for some initial conditions be polynomial in \( q_2 \) degree \( n - 1 \), the \( a_n \) are no longer polynomial, merely rational. Also, we cannot set \( n = 1 \) in the last equation.

Having introduced the \( g_n \) and \( \hat{g}_n \), we might note that boundary moments can be expressed in terms of them. For example, as

\[
r^2 = x^2 + y^2 = a^2 (1 - e^2) \left( 1 - \frac{e^2}{2} + \frac{e^2}{2} \cos(2\psi) \right),
\]

we have

\[
i_2 = a \int_{-\pi}^{\pi} r^2 \hat{g}(\psi) \, d\psi
\]

\[
= a^3 (1 - e^2) \left( 1 - \frac{e^2}{2} \hat{g}_0 + \frac{e^2}{2} \hat{g}_1 \right).
\]

There may be useful recurrence relations involving the \( i_n \).

\section{Result Validation and Numerical Examples}

\subsection{Slip flow in Elliptic Channel with \( a = 2, \ b = \frac{1}{2} \)}

Figure 5 gives, at left, contour plots of \( u \) for \( \beta = \frac{1}{4}, \ \frac{1}{16} \) and \( \frac{1}{64} \), and, at right, those for \( \beta = 4, \ 16, \ 64 \). The contours are very similar to those in Figure 4 of [32]. It is noted that when \( \beta > 0 \) on the boundary (as well as in the interior) \( u > 0 \). At any given fixed position \( z \) on the boundary, \( u(z) \) increases when \( \beta \) value increases.
Figure 5: $a = 2$. Contour plot of axial velocity on the ellipse for six different $\beta$ values.
D.2 Comparing the results with Ritz Method for Slip Flow

Here we compare with results given in [34].

| $\lambda$ | $c$ | $a = \frac{1}{\sqrt{c}}$ | $\beta = \frac{1}{\sqrt{c}}$ | A | V | F | Ritz method |
|---|---|---|---|---|---|---|---|
| .25 | 2.000000 | .200000 | .027080 | .026893 | .0268994 | .0268994 |
| 0.1 | .5 | 1.414213 | .141421 | .131628 | .131224 | .1312286 | .1312286 |
| .75 | 1.154700 | .115470 | .313497 | .313320 | .3133204 | .3133204 |
| .25 | 2.000000 | .400000 | .042611 | .041996 | .0419990 | .0419990 |
| 0.2 | .5 | 1.414213 | .282843 | .184716 | .183372 | .1833741 | .1833741 |
| .75 | 1.154700 | .230940 | .414937 | .414338 | .4143388 | .4143388 |
| .25 | 2.000000 | 1.000000 | .071907 | .086515 | .0865169 | .0865167 |
| 0.5 | .5 | 1.414214 | .707107 | .343981 | .338284 | .3382847 | .3382847 |
| .75 | 1.154700 | .577350 | .719257 | .716698 | .7166986 | .7166986 |
| .25 | 2.000000 | 2.000000 | .143814 | .159568 | .1595802 | .1595797 |
| 1.0 | .5 | 1.414214 | 1.414214 | .509349 | .594535 | .5945397 | .5945397 |
| .75 | 1.154701 | 1.154701 | 1.004665 | 1.219750 | 1.2197505 | 1.2197505 |
| .25 | 2.000000 | 4.000000 | .287629 | .304352 | .3043838 | .3043838 |
| 2.0 | .5 | 1.414214 | 2.828427 | 1.018698 | 1.105106 | 1.1051201 | 1.1051201 |
| .75 | 1.154700 | 2.309401 | 2.009330 | 2.224974 | 2.2249748 | 2.2249748 |
| .25 | 2.000000 | 10.000000 | .719072 | .736625 | .7366869 | .7366851 |
| 5.0 | .5 | 1.414214 | 7.071068 | 2.546745 | 2.634130 | 2.6341541 | 2.6341541 |
| .75 | 1.154701 | 5.773503 | 5.023326 | 5.239416 | 5.2394175 | 5.2394175 |
| .25 | 2.000000 | 20.000000 | 1.438144 | 1.456036 | 1.4561139 | 1.4561106 |
| 10 | .5 | 1.414214 | 14.142136 | 5.093491 | 5.181257 | 5.1812861 | 5.1812861 |
| .75 | 1.154701 | 11.547005 | 10.046652 | 10.262916 | 10.2629186 | 10.2629186 |

D.3 Blood flow problem

Most of the veins and arteries in our bodies can be taken as having circular cross-section. However in places where the vein or artery has to go through a region which is squeezed in by muscle or bone one might expect the cross-section of the vein or artery to depart somewhat from circular. In this appendix, we take fluid density, $\rho$, of 1.05 g/mL and fluid viscosity, $\mu$, of 0.04 Poise. See Table 2 in [14]. The geometry of elliptical cross-section of the channel is described by coordinates $(x, y)$ with

$$x = \bar{r}(\psi) \cos(\psi); \quad y = \bar{r}(\psi) \sin(\psi),$$

55
Figure 6: The cross sections of elliptical shapes with the same length of minor axis $b = 4.9 \, \mu m$. and various lengths of major axis $a = 6.6, 7.1, 7.6, 8.1$ and $8.5 \, \mu m$.

where

$$\tau(\psi) = \gamma k (1 + \kappa(k, \psi)), \quad 0 \leq \psi \leq 2\pi$$

and

$$\kappa = -\frac{1}{4}(k-1)^2 + (k-1 - \frac{1}{2}(k-1)^2) \cos(2\psi) + \frac{3}{4}(k-1)^2 \cos(k\psi).$$

The tube ellipticity $\varepsilon = \sqrt{1 - b^2/a^2}$ with the lengths of the half-axes $a = \bar{r}(0) \cos(0)$ and $b = \bar{r}(\pi/2) \sin(\pi/2)$ is determined by setting $\gamma = 0.005$ and $k = 1.15$. Thus, the tube has the ellipticity of 0.6720 with $a = 6.6 \, \mu m,$ $b = 4.9 \, \mu m,$ $\eta_0 = 0.9518$ and $c = 4.4 \, \mu m.$ The boundary of the elliptical cross-section is described by

$$\partial\Omega : (x, y) = (c \cosh \eta_0 \cos \psi, c \sinh \eta_0 \sin \psi).$$

The investigate the influence of elliptical shapes on constant pressure-driven flow of fluid, various sizes of the major axis $a$ are chosen to be vary between 6.6 $\mu m.$ to 8.6 $\mu m.$ while the length of minor axis is fixed as $b = 4.9 \, \mu m.$

To demonstrate the impact of the slip length $\ell = \beta$ on the constant pressure-driven flow of fluid, values of slip length $\ell$ are chosen to be vary from 0.0 to 50 $\mu m$. The results show that the slip length has a direct influence on the axial velocity. Larger slip length provides higher velocity as shown in Figures 7, 8 and 9.
Figure 7: Axial velocity profiles in x and y directions obtained from the model with no slip lengths: $\ell = 0.0$, and various shapes of cross section.

Figure 8: Axial velocity profiles obtained from the model with $a = 6.6 \ \mu m$. and $b := 4.9 \ \mu m$. 
Figure 9: Axial velocity profiles in $x$ and $y$ directions obtained from the model having $a = 6.5 \, \mu m$ and $b = 4.9 \, \mu m$ and various slip lengths $\ell(\mu m.)$ of 0.0 (solid line), 6.25 (dotted line), 25 (dashed line), 37.5 (dash-dotted line) and 50 (long dashed line).
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