A unified quantum SO(3) invariant for rational homology 3-spheres

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Abstract: Given a rational homology 3-sphere M with $|H_1(M)| = b$ and a link L inside M, colored by odd numbers, we construct a unified invariant $I_{M,L}$ belonging to a modification of the Habiro ring where $b$ is inverted. Our unified invariant dominates the whole set of the SO(3) Witten-Reshetikhin-Turaev invariants of the pair $(M,L)$. If $b=1$ and $L = \emptyset$, $I_M$ coincides with Habiro’s invariant of integral homology 3-spheres. For $b > 1$, the unified invariant defined by the third author is determined by $I_M$. Important applications are the new Ohtsuki series (perturbative expansions of $I_{M,L}$) dominating quantum SO(3) invariants at roots of unity whose order is not a power of a prime. These series are not known to be determined by the LMO invariant.

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A UNIFIED QUANTUM SO(3) INVARIANT
FOR RATIONAL HOMOLOGY 3–SPHERES

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ABSTRACT. Given a rational homology 3–sphere $M$ with $|H_1(M,\mathbb{Z})| = b$, we construct a unified invariant $I_M$ belonging to a modification of the Habiro ring where $b$ is inverted. Our unified invariant dominates the whole set of the $SU(3)$ Witten–Reshetikhin–Turaev invariants of $M$. If $b = 1$, $I_M$ coincides with Habiro’s invariant of integral homology 3–spheres. For $b > 1$, the unified invariant defined by the third author is a special case of $I_M$. One of the applications are the new Ohtsuki series (perturbative expansions of $I_M$ at roots of unity) dominating all quantum $SU(3)$ invariants.

INTRODUCTION

Background. The $SU(2)$ Witten–Reshetikhin–Turaev (WRT) invariant is defined for any closed oriented 3–manifold $M$ and any root of unity $\xi$ [21]. Kirby and Melvin [10] introduced the $SO(3)$ version of the invariant $\tau_M(\xi) \in \mathbb{Q}(\xi)$ for roots of unity $\xi$ of odd order. If the order of $\xi$ is prime, then by the results of Murakami [19] (also Masbaum–Roberts [18]), $\tau_M(\xi)$ is an algebraic integer. This integrality result was the starting point for the construction of finite type 3–manifold invariants, Ohtsuki series [22], integral TQFTs, representations of the mapping class group over $\mathbb{Z}$, and categorification of quantum 3–manifold invariants [9]. The proofs in [19] and [18] depend heavily on the arithmetics of $\mathbb{Z}[\xi]$ for a root of unity $\xi$ of prime order and do not extend to other roots of unity.

Is it true that $\tau_M(\xi)$ is always an algebraic integer (belongs to $\mathbb{Z}[\xi]$), even when the order of $\xi$ is not a prime? The positive answer to this question was given first for integral homology spheres by Habiro [6], and then for arbitrary 3–manifolds by the first and third author [3], in connection with the study of “strong integrality”.

What Habiro proved for integral homology 3–spheres is actually much stronger than integrality. For any integral homology 3–sphere $M$, Habiro [6] constructed a unified invariant $J_M$ whose evaluation at any root of unity coincides with the value of the Witten–Reshetikhin–Turaev invariant at that root. Habiro’s unified invariant $J_M$ is an element of the following ring (Habiro’s ring)

$$\mathbb{Z}[[q]] := \lim_{k \to \infty} \mathbb{Z}[q]_{((q; q)_k)}$$

where $(q; q)_k = \prod_{j=1}^{k} (1 - q^j)$.

Every element $f(q) \in \mathbb{Z}[[q]]$ can be written as an infinite sum

$$f(q) = \sum_{k \geq 0} f_k(q) (1 - q)(1 - q^2)...(1 - q^k),$$

with $f_k(q) \in \mathbb{Z}[q]$. When $q = \xi$, a root of unity, only a finite number of terms on the right hand side are not zero, hence the right hand side gives a well–defined value, called the evaluation $ev_\xi(f(q))$.

Since $f_k(q) \in \mathbb{Z}[q]$, $ev_\xi(f(q)) \in \mathbb{Z}[\xi]$ is an algebraic integer. The fact that the unified invariant belongs to $\mathbb{Z}[[q]]$ is stronger than just integrality of $\tau_M(\xi)$. We will refer to it as “strong” integrality.

The Habiro ring has beautiful arithmetic properties. Every element $f(q) \in \mathbb{Z}[[q]]$ can be considered as a function whose domain is the set of roots of unity. Moreover, there is a natural Taylor series for $f$ at every root of unity. Two elements $f, g \in \mathbb{Z}[[q]]$ are the same if and only if their Taylor series at a root of unity coincide. In addition, each function $f(q) \in \mathbb{Z}[[q]]$ is totally determined by its values at, say, infinitely many roots of order $3^n$, $n \in \mathbb{N}$. Due to these properties the Habiro ring is also called a ring of “analytic functions at roots of unity” [6]. Thus belonging to $\mathbb{Z}[[q]]$ means that
the collection of the SO(3) WRT invariants is far from a random collection of algebraic integers; together they form a nice function.

Perturbative expansion at 1 of WRT invariants for rational homology 3–spheres was first constructed by Ohtsuki in the case when the order of the quantum parameter \( \xi \) is prime \([21]\). General properties of the Habiro ring imply that for any integral homology 3–sphere \( M \), the Taylor expansion of the unified invariant \( J_M \) at \( q = 1 \) coincides with the Ohtsuki series and dominates WRT invariants of \( M \) at all roots of unity (not only of prime order).

To generalize Habiro’s results to rational homology 3–spheres, new ideas and techniques are required. Strong integrality of quantum invariants for rational homology 3–spheres was studied in \([12]\) and \([3]\). Among other things, in \([3]\), it was proved that for any 3–manifold \( M \) (not necessary a rational homology 3–sphere), the SO(3) WRT invariant \( \tau_M(\xi) \) is always an algebraic integer, i.e. \( \tau_M(\xi) \in \mathbb{Z}[\xi] \) with no restriction on the order of \( \xi \) at all. There we used a (2–nd order) Laplace transform method \([2]\) and a difficult identity of Andrews \([1]\) in \( q \)–calculus, generalizing those of Rogers–Ramanujan.

Thus, although we have had integrality of all SO(3) WRT invariants, we still lacked a “strong integrality” for the case when \((r, b) \neq 1\). This is the main object of this paper.

In this paper we will generalize Habiro’s construction of the unified invariant to all rational homology 3–spheres. Our new unified invariant \( I_M \) dominates SO(3) WRT invariants also in the case when the order \( r \) of the quantum parameter is coprime with \( b \). In \([3]\), it was proved that for any 3–manifold \( M \) (not necessary a rational homology 3–sphere), the SO(3) WRT invariant \( \tau_M(\xi) \) is always an algebraic integer, i.e. \( \tau_M(\xi) \in \mathbb{Z}[\xi] \) with no restriction on the order of \( \xi \) at all. There we used a (2–nd order) Laplace transform method \([2]\) and a difficult identity of Andrews \([1]\) in \( q \)–calculus, generalizing those of Rogers–Ramanujan.

Results. The WRT or quantum SO(3) invariant \( \tau_{M,L}(\xi) \) is defined for a pair of a closed 3–manifold \( M \) and a link \( L \) in it, with link components colored by integers. Here \( \xi \) is a root of unity of odd order. We will recall the definitions in Section \([1]\).

Suppose \( M \) is a rational homology 3–sphere, i.e. \( |H_1(M, \mathbb{Z})| := \text{card } H_1(M, \mathbb{Z}) < \infty \). There is a unique decomposition \( H_1(M, \mathbb{Z}) = \bigoplus b_i \mathbb{Z}/b_i \mathbb{Z} \), where each \( b_i \) is a prime power. We renormalize the SO(3) WRT invariant of the pair \((M, L)\) as follows:

\[
\tau^r_{M,L}(\xi) = \frac{\tau_{M,L}(\xi)}{\prod_i \tau_{L(b_i,1)}(\xi)} ,
\]

where \( L(b, a) \) denotes the \((b, a)\) lens space. We will see that \( \tau_{L(b,1)}(\xi) \) is always nonzero.

For any positive integer \( b \), we define the cyclotomic completion ring \( R_b \) to be

\[
R_b := \lim_{k} \mathbb{Z}[\frac{1}{b}]^{((q;q^2)_k)}, \quad \text{where} \quad (q;q^2)_k = (1 - q)(1 - q^3)\ldots(1 - q^{2k-1}).
\]

For any \( f(q) \in R_b \) and a root of unity \( \xi \) of odd order, the evaluation \( \text{ev}_\xi(f(q)) := f(\xi) \) is well–defined. Similarly, we put

\[
S_b := \lim_{k} \mathbb{Z}[\frac{1}{b}]^{((q;q)_k)}. \quad \text{Here the evaluation at any root of unity is well–defined. For odd } b, \text{ there is a natural embedding } S_b \hookrightarrow R_b, \text{ see Section 4.}
\]

Let us denote by \( M_b \) the set of rational homology 3–spheres such that \( |H_1(M, \mathbb{Z})| \) divides \( b^n \) for some \( n \). The main result of this paper is the following.

Theorem 1. Suppose the components of a framed oriented link \( L \subset M \) have odd colors, and \( M \in M_b \). Then there exists an invariant \( I_{M,L} \in R_b \), such that for any root of unity \( \xi \) of odd order

\[
\text{ev}_\xi(I_{M,L}) = \tau^r_{M,L}(\xi).
\]

In addition, if \( b \) is odd, then \( I_{M,L} \in S_b \).
For $b = 1$ and $L$ the empty link, $I_{M,L}$ coincides with Habiro’s unified invariant $J_M$.

The proof of Theorem 1 uses the Laplace transform method and Andrew’s identity. We also construct a Frobenius type isomorphism to get rid of the formal fractional power of $q$ that appeared in [12, 3]. The rings $R_b$ and $S_b$ have properties similar to those of the Habiro ring. An element $f(q) \in R_b$ is totally determined by the values at many infinite sets of roots of unity (see Section 3), one special case is the following.

**Proposition 2.** Let $p$ be an odd prime not dividing $b$ and $T$ the set of all integers of the form $p^kb'$, $k \in \mathbb{N}$ and $b'$ an odd divisor of $b^n$ for some $n$. Any element $f(q) \in R_b$, and hence also $\{r_M(\xi)\}$, is totally determined by the values at roots of unity with orders in $T$.

The Ohtsuki series [21, 14], originally defined through some arithmetic congruence property of the $SO(3)$ invariant, can be identified with the Taylor expansion of $I_M$ at $q = 1$ [9, 12]. We will also investigate the Taylor expansions of $I_M$ at roots of unity and show that these Taylor expansions satisfy congruence relations similar to the original definition of the Ohtsuki series, see Section 4.

**Plan of the paper.** In Section 1 we recall known results and definitions. In the next section we explain the strategy of our proof of Theorem 1. In Sections 3 and 5 we develop properties of cyclotomic completions of polynomial rings. New Ohtsuki series are discussed in Section 4. The unified invariant of lens spaces, needed for the diagonalization, is defined in Section 6. The main technical result of the paper based on Andrew’s identity is proved in Section 7.

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### 1. Quantum (WRT) invariants

#### 1.1. Notations and conventions.

We will consider $q^{1/4}$ as a free parameter. Let

$$\{n\} = q^{n/2} - q^{-n/2}, \quad \{n\}! = \prod_{i=1}^{n} \{i\}, \quad [n] = \frac{\{n\}}{\{1\}}, \quad \left[\begin{array}{c} n \\ k \end{array}\right] = \frac{\{n\}!}{\{k\}!\{n-k\}!}.$$

We denote the set $\{1, 2, 3, \ldots\}$ by $\mathbb{N}$. We also use the following notation from $q$–calculus:

$$(x; q)_n := \prod_{j=1}^{n} (1 - xq^{j-1}).$$

Throughout this paper, $\xi$ will be a primitive root of unity of odd order $r$ and $e_n := \exp(2\pi i/n)$.

All 3–manifolds in this paper are supposed to be closed and oriented. Every link in a 3–manifold is framed, oriented, and has components ordered.

In this paper, $L \cup L'$ denotes a framed link in $S^3$ with disjoint sublinks $L$ and $L'$, with $m$ and $l$ components, respectively. Surgery along the framed link $L$ transforms $(S^3, L')$ into $(M, L')$. We use the same notation $L'$ to denote the link in $S^3$ and the corresponding one in $M$.

#### 1.2. The colored Jones polynomial.

Suppose $L$ is a framed, oriented link in $S^3$ with $m$ ordered components. For positive integers $n_1, \ldots, n_m$, called the colors of $L$, one can define the quantum invariant $J_L(n_1, \ldots, n_m) \in \mathbb{Z}[q^{\pm 1/4}]$, known as the colored Jones polynomial of $L$ (see e.g. [24, 17]). Let us recall here a few well–known formulas. For the unknot $U$ with 0 framing one has

$$J_U(n) = [n].\quad (3)$$

If $L_1$ is obtained from $L$ by increasing the framing of the $i$–th component by 1, then

$$J_{L_1}(n_1, \ldots, n_m) = q^{(n_i^2-1)/4} J_L(n_1, \ldots, n_m).\quad (4)$$

If all the colors $n_i$ are odd, then $J_L(n_1, \ldots, n_m) \in \mathbb{Z}[q^{\pm 1}]$. 


1.3. Evaluation and Gauss sums. We first define, for each root of unity $\xi$ of odd order $r$, the evaluation map $ev_\xi$, which replaces $q$ by $\xi$. Suppose $f \in \mathbb{Q}[q^{\pm 1/d}]$, where $d$ is coprime with $r$. There exists an integer $d_*$, unique modulo $r$, such that $(\xi^{d_*})^d = \xi$. Then we define

$$ev_\xi f := f|_{q^{1/d}=\xi^{d_*}}.$$ 

Suppose $f(q; n_1, \ldots, n_m)$ is a function of variables $q^{1/d}$ and integers $n_1, \ldots, n_m$. In quantum topology, the following sum plays an important role

$$\sum_{n_i} f := \sum_{0 < n_i < 2r, n_i \text{ odd}} ev_\xi f(q; n_1, \ldots, n_m)$$

where in the sum all the $n_i$ run over the set of odd numbers between 0 and $2r$.

A variation $\gamma_b(\xi)$ of the Gauss sum is defined by

$$\gamma_b(\xi) := \sum_n q^{b n^2} \xi^n.$$ 

It is known that, for odd $r$, $|\gamma_b(\xi)|$ is never 0.

1.4. Definition of the WRT invariant. Suppose the components of $L'$ are colored by fixed integers $j_1, \ldots, j_i$. Let

$$F_{L\cup L'}(\xi) := \sum_{n_i} \left\{ J_{L\cup L'}(n_1, \ldots, n_m, j_1, \ldots, j_i) \prod_{i=1}^m \xi^{[n_i]} \right\}.$$ 

An important special case is when $L = U^b$, the unknot with framing $b \neq 0$, and $L' = \emptyset$. In that case $F_{L\cup}(\xi)$ can be calculated using the Gauss sum and is nonzero, see Section 4 below.

Let $\sigma_+$ (resp. $\sigma_-$) be the number of positive (negative) eigenvalues of the linking matrix of $L$. Then the quantum $SO(3)$ invariant of the pair $(M, L')$ is defined by (see e.g. [10, 24])

$$\tau_{M,L'}(\xi) = \frac{F_{L\cup L'}(\xi)}{(F_{U^+}(\xi))^{\sigma_+} (F_{U^-}(\xi))^{\sigma_-}}.$$ 

The invariant $\tau_{M,L'}(\xi)$ is multiplicative with respect to the connected sum.

For example, the $SO(3)$ invariant of the lens space $L(b, 1)$, obtained by surgery along $U^b$, is

$$\tau_{L(b, 1)}(\xi) = \frac{F_{U^b}(\xi)}{F_{U^m(1)}(\xi)}.$$ 

were $\text{sn}(b)$ is the sign of the integer $b$.

Let us focus on the special case when the linking matrix of $L$ is diagonal, with $b_1, b_2, \ldots, b_m$ on the diagonal. Then $H_1(M, \mathbb{Z}) = \oplus_{i=1}^m \mathbb{Z}/|b_i|$, and

$$\sigma_+ = \text{card} \{i \mid b_i > 0\}, \quad \sigma_- = \text{card} \{i \mid b_i < 0\}.$$ 

Thus from the definitions (5), (6) and (11) we have

$$\tau'_{M,L'}(\xi) = \left( \prod_{j=1}^m \tau'_{L(b_j, 1)}(\xi) \right) \frac{F_{L\cup L'}(\xi)}{\prod_{j=1}^m F_{U^{b_j}}(\xi)},$$

with

$$\tau'_{L(b_j, 1)}(\xi) = \frac{\tau_{L(b_j, 1)}(\xi)}{\tau_{L(b_j, 1)}(\xi)}.$$ 

1.5. Habiro’s cyclotomic expansion of the colored Jones polynomial. Recall that $L$ and $L'$ have $m$ and $l$ components, respectively. Let us color $L'$ by fixed $J = (j_1, \ldots, j_i)$ and vary the colors $n = (n_1, \ldots, n_m)$ of $L$.

For non-negative integers $n, k$ we define

$$A(n, k) := \frac{\prod_{i=0}^k \left(q^n + q^{-n} - q^i - q^{-i}\right)}{(1 - q) (q^{n+1}; q)_{k+1}}.$$
For $k = (k_1, \ldots, k_m)$ let
\[ A(n, k) := \prod_{j=1}^{m} A(n_j, k_j). \]

Note that $A(n, k) = 0$ if $k_j \geq n_j$ for some index $j$. Also
\[ A(n, 0) = q^{-1} J_L(n)^2. \]

Theorem 4. (Laplace transform) The following is the main technical result of the paper. A proof will be given in Section 2.1.

Theorem 4. Suppose $b = \pm 1$ or $b = \pm p^l$ where $p$ is a prime and $l$ positive. For any non-negative integer $k$, there exists an element $Q_{b,k} \in \mathcal{R}_b$, such that for every root $\xi$ of odd order $r$ one has
\[ \sum_{n} \xi^n q^{\frac{n-1}{2}} A(n, k) \quad \frac{F_{L,b}(\xi)}{F_{L,b}(\xi)} = ev_{\xi}(Q_{b,k}). \]

In addition, if $b$ is odd, $Q_{b,k} \in \mathcal{S}_b$. 

2. Strategy of the proof of the main theorem

Here we give the proof of Theorem 3 using technical results that will be proved later.

As before, $L \cup L'$ is a framed link in $S^3$ with disjoint sublinks $L$ and $L'$, with $m$ and $l$ components, respectively. Assume that $L'$ is colored by fixed $j = (j_1, \ldots, j_l)$, with $j_i$'s odd. Surgery along the framed link $L$ transforms $(S^3, L')$ into $(M, L')$. We will define $I_{M,L'} \in \mathcal{R}_b$, such that
\[ \tau_{M,L'}^\prime(\xi) = ev_{\xi}(I_{M,L'}) \]
for any root of unity $\xi$ of odd order. This unified invariant is multiplicative with respect to the connected sum.

The following observation is important. By Proposition 2, there is at most one element $f(q) \in \mathcal{R}_b$ such that for every root $\xi$ of odd order one has
\[ \tau_{M,L}^\prime(\xi) = ev_{\xi}(f(\xi)). \]

That is, if we can find such an element, it is unique, and we put $I_{M,L'} := f(q)$.

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\[ \sum_{n} \xi^n q^{\frac{n-1}{2}} A(n, k) \quad \frac{F_{L,b}(\xi)}{F_{L,b}(\xi)} = ev_{\xi}(Q_{b,k}). \]

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\[ \sum_{n} \xi^n q^{\frac{n-1}{2}} A(n, k) \quad \frac{F_{L,b}(\xi)}{F_{L,b}(\xi)} = ev_{\xi}(Q_{b,k}). \]

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2.1. Laplace transform. The following is the main technical result of the paper. A proof will be given in Section 2.1.
2.2. Definition of the unified invariant: diagonal case. Suppose that the linking number between any two components of $L$ is 0, and the framing on components of $L$ are $b_i = \pm p_i^{k_i}$ for $i = 1, \ldots, m$, where each $p_i$ is prime or 1. Let us denote the link $L$ with all framings switched to zero by $L_0$.

Using [10], taking into account the framings $b_i$'s, we have

$$J_{L;\cup L'}(n,j) \prod_{i=1}^{m} |n_i| = \sum_{k \geq 0} C_{L;\cup L'}(k,j) \prod_{i=1}^{m} q^{b_i^{k_i-1}} A(n_i, k_i).$$

By the definition of $F_{L;\cup L'}$, we have

$$F_{L;\cup L'}(\xi) = \sum_{k \geq 0} ev_{\xi}(C_{L;\cup L'}(k,j)) \prod_{i=1}^{m} q^{b_i^{k_i-1}} A(n_i, k_i).$$

From [7] and Theorem 4 we get

$$\tau'_{M,L'}(\xi) = ev_{\xi} \left( \prod_{i=1}^{m} I_{L(b_i,1)} \sum_{k} C_{L;\cup L'}(k,j) \prod_{i=1}^{m} Q_{b_i,k_i} \right),$$

where the unified invariant of the lens space $I_{L(b_i,1)} \in R_b$, with $ev_{\xi}(I_{L(b_i,1)}) = \tau'_{L(b_i,1)}(\xi)$, exists by Lemma 6 below. Thus if we define

$$I_{(M,L')} := \prod_{i=1}^{m} I_{L(b_i,1)} \sum_{k} C_{L;\cup L'}(k,j) \prod_{i=1}^{m} Q_{b_i,k_i},$$

then [11] is satisfied. By Theorem 3 $C_{L;\cup L'}(k,j)$ is divisible by $(q^{k+1};q)_{k+1}/(1-q)$, which is divisible by $(q;q)_k$, where $k = max k_i$. It follows that $I_{(M,L')} \in R_b$. In addition, if $b$ is odd, then $I_{(M,L')} \in S_b$.

2.3. Diagonalization using lens spaces. The general case reduces to the diagonal case by the well-known trick of diagonalization using lens spaces. We say that $M$ is diagonal if it can be obtained from $S^3$ by surgery along a framed link $L$ with diagonal linking matrix, where the diagonal entries are of the form $\pm p^\xi$ with $p = 0, 1$ or a prime. The following lemma was proved in [12] Proposition 3.2 (a)].

**Lemma 5.** For every rational homology sphere $M$, there are lens spaces $L(b_i, a_i)$ such that the connected sum of $M$ and these lens spaces is diagonal. Moreover, each $b_i$ is a prime power divisor of $|H_1(M, Z)|$.

To define the unified invariant for a general rational homology sphere $M$, one first adds to $M$ lens spaces to get a diagonal $M'$, for which the unified invariant $I_{M'}$ had been defined in Subsection 2.2. Then define $I_M$ as the quotient of $I_{M'}$ by the unified invariants of the lens spaces. But unlike the simpler case of [12], the unified invariant of lens spaces are not invertible in general. To overcome this difficulty we insert knots in lens spaces and split the unified invariant into different components. This will be explained in the remaining part of this section.

2.4. Splitting of the invariant. Suppose $p$ is a prime divisor of $b$, then it’s clear that $R_p \subset R_b$.

In Section 3 we will see that there is a decomposition

$$R_b = R_b^{p,0} \times R_b^{p,0},$$

with canonical projections $\pi_0^p : R_b \rightarrow R_b^{p,0}$ and $\pi_0^p : R_b \rightarrow R_b^{p,0}$. If $f \in R_b^{p,0}$ then $ev_{\xi}(f)$ can be defined when the order of $\xi$ is coprime with $p$; and in this case $ev_{\xi}(g) = ev_{\xi}(\pi_0^p(g))$ for every $g \in R_b$.

On the other hand, if $f \in R_b^{p,0}$ then $ev_{\xi}(f)$ can be defined when the order of $\xi$ is divisible by $p$, and one has $ev_{\xi}(g) = ev_{\xi}(\pi_0^p(g))$ for every $g \in R_b$.

It also follows from the definition that $R_b^{p,\epsilon} \subset R_b^{p,\epsilon}$ for $\epsilon = 0$ or $0$.

For $S_b$, there exists a completely analogous decomposition. For any odd divisor $p$ of $b$, an element $x \in R_b$ (or $S_b$) determines and is totally determined by the pair $(\pi_0^p(x), \pi_0^p(x))$. If $p = 2$ divides $b$, then for any $x \in R_b$, $x = \pi_0^p(x)$.
To define $I_M$ we will define $I_M^0 = \pi^n_0(I_M)$ and $I_M^0 = \pi^n_0(I_M)$. The first part $I_M^0 = \pi^n_0(I_M)$, when $b = p$, was defined in [12] (up to normalization), where the third author considered the case when the order of roots of unity is coprime with $b$. We will give a self-contained definition of $I_M^0$, and show that it is coincident (up to normalization) with the one introduced in [12].

2.5. Lens spaces. Suppose $b, a, d$ are integers with $(b, a) = 1$ and $b \neq 0$. Let $M(b, a; d)$ be the pair of lens space $L(b, a)$ and a knot $K$ inside, colored by $d$, as described in Figure 1. Among these pairs we want to single out some whose quantum invariants are invertible.

For $\varepsilon \in \{0, 1\}$, let $M^\varepsilon(b, a) := M(b, a; d(\varepsilon))$, where $d(0) := 1$ and $d(0)$ is the smallest odd positive integer such that $ad(0) \equiv 1$ (mod $b$). Note that if $a = 1$, $d(0) = d(0) = 1$.

It is known that if the color of a link component is 1, then the component can be removed from the link without affecting the value of quantum invariants. Hence

$$\tau_{M(a, 1)} = \tau_{L(a, 0)}.$$

Lemma 6. Suppose $b = p^j$ is a prime power. For $\varepsilon \in \{0, 1\}$, there exists an invertible invariant $I_{M^\varepsilon(b, a)}^\varepsilon \in \mathcal{R}_{p, \varepsilon}^+$ such that

$$\tau_{M^\varepsilon(b, a)}(\xi) = ev(\xi) \left(I_{M^\varepsilon(b, a)}^\varepsilon\right)$$

where $\varepsilon = 0$ if the order of $\xi$ is not divisible by $p$, and $\varepsilon = 0$ otherwise. Moreover, if $p$ is odd, then $I_M^\varepsilon(b, a)$ belongs to and is invertible in $\mathcal{S}_{p, \varepsilon}^+$.

2.6. Definition of the unified invariant: general case. Now suppose $(M, L')$ is an arbitrary pair of a rational homology 3-sphere with a link $L'$ in it colored by odd numbers $j_1, \ldots, j_t$. Let $L(b_i, a_i)$ for $i = 1, \ldots, m$ be the lens spaces of Lemma 5. We use induction on $m$. If $m = 0$, then $M$ is diagonal and $I_{M, L'}$ has been defined in Subsection 2.4.

Since $(M, L') \# M(b_i, a_i; d)$ becomes diagonal after adding $m - 1$ lens spaces, the unified invariant of $(M, L') \# M(b_i, a_i; d)$ can be defined by induction, for any odd integer $d$. In particular, one can define $I_{M^\varepsilon}$, where $M^\varepsilon := (M, L') \# M^\varepsilon(b_i, a_i)$. Here $\varepsilon = 0$ or $\varepsilon = 0$ and $b_1$ is a power of a prime $p$ dividing $b$. It follows that the components $\pi_2^\varepsilon(I_{M^\varepsilon}) \in \mathcal{R}_{p, \varepsilon}^+$ are defined.

By Lemma 6, $I_{M^\varepsilon(b_i, a_i)}^\varepsilon$ is defined and invertible. Now we put

$$I_{M, L'}^\varepsilon := I_{M^\varepsilon}^\varepsilon \times (I_{M^\varepsilon(b_i, a_i)})^{-1}.$$

It is easy to see that $I_{M, L'} := (I_{M, L'}^\varepsilon, I_{M, L'}^\varepsilon)$ satisfies [10]. This completes the construction of $I_{M, L'}$. It remains to prove Lemma 5 and Theorem 4.

3. Cyclotomic completions of polynomial rings

In this section we adapt the results of Habiro on cyclotomic completions of polynomial rings [7] to our rings.

3.1. On cyclotomic polynomial. Recall that $e_n := \exp(2\pi i/n)$ and denote by $\Phi_n(q)$ the cyclotomic polynomial

$$\Phi_n(q) = \prod_{\substack{j,b \in \mathbb{Z}^+ \setminus \{0\}}} (q - e_n^j).$$
The degree of $\Phi_n(q) \in \mathbb{Z}[q]$ is given by the Euler function $\varphi(n)$. Suppose $p$ is a prime and $n$ an integer. Then (see e.g. [20])

\[
\Phi_n(q^p) = \begin{cases} 
\Phi_{np}(q) & \text{if } p \mid n \\
\Phi_{np}(q)\Phi_n(q) & \text{if } p \nmid n.
\end{cases}
\]

It follows that $\Phi_n(q^p)$ is always divisible by $\Phi_{np}(q)$.

The ideal of $\mathbb{Z}[q]$ generated by $\Phi_n(q)$ and $\Phi_m(q)$ is well–known, see e.g. [12] Lemma 5.4:

**Lemma 7.** a) If $m/n \neq p^e$ for any prime $p$ and an integer $e \neq 0$, then $(\Phi_n) + (\Phi_m) = (1)$ in $\mathbb{Z}[q]$.

b) If $m/n = p^e$ for a prime $p$ and some integer $e \neq 0$, then $(\Phi_n) + (\Phi_m) = (1)$ in $\mathbb{Z}[1/p][q]$.

Note that in a commutative ring $R$, if $(x) + (y) = (1)$ if and only if $x$ is invertible in $R/(y)$. Also $(x) + (y) = (1)$ implies $(x^k) + (y^l) = (1)$ for any integers $k, l \geq 1$.

**Lemma 8.** Suppose $r$ is coprime with $p$, and $x, y \in \mathbb{Q}[e_r]$ such that $x^k = y^l$ for some $k > 1$ which is a power of $p$. Then $x = y$.

**Proof.** Then $x/y$ is a root of $1$ of order a power of $p$. If $x/y \neq 1$, then this means $\mathbb{Q}[e_r]$ contains a primitive $p$–th root of $1$, or $e_p \in \mathbb{Q}[e_r]$, which is impossible. \hfill $\square$

3.2. Habiro’s results. Let us summarize some of Habiro’s results on cyclotomic completions of polynomial rings [7]. Let $R$ be a commutative integral domain of characteristic zero and $R[q]$ the polynomial ring over $R$. For any $S \subset \mathbb{N}$, Habiro defined the $S$–cyclotomic completion ring $R[q]^S$ as follows:

\[
R[q]^S := \lim_{\longrightarrow} \frac{R[q]}{(f(q))}.
\]

where $\Phi_S^*$ denotes the multiplicative set in $\mathbb{Z}[q]$ generated by $\Phi_S = \{\Phi_n(q) \mid n \in S\}$ and directed with respect to the divisibility relation.

For example, since the sequence $(q; q)_n$, $n \in \mathbb{N}$, is cofinal to $\Phi_{\mathbb{N}}^*$, we have

\[
\mathbb{Z}[q] \simeq \mathbb{Z}[q]^\mathbb{N}.
\]

Note that if $S$ is finite, then $R[q]^S$ is identified with the $(\prod_{S} \Phi_S)$–adic completion of $R[q]$. In particular,

\[
R[q]^{(1)} \simeq R[[q - 1]], \quad R[q]^{(2)} \simeq R[[q + 1]].
\]

Suppose $S' \subset S$, then $\Phi_{S'} \subset \Phi_S$, hence there is a natural map

\[
\rho_{S,S'}^R : R[q]^S \rightarrow R[q]^{S'}.
\]

Recall important results concerning $R[q]^{S}$ from [7]. Two positive integers $n, n'$ are called adjacent if $n'/n = p^e$ with a nonzero $e \in \mathbb{Z}$ and a prime $p$, such that the ring $R$ is $p$–adically separated, i.e. $\cap_{e=1}^{\infty} (p^n) = 0$ in $R$. A set of positive integers is $R$–connected if for any two distinct elements $n, n'$ there is a sequence $n = n_1, n_2, \ldots, n_{k-1}, n_k = n'$ in the set, such that any two consecutive numbers of this sequence are adjacent. Theorem 4.2 of [7] says that if $S$ is $R$–connected, then for any subset $S' \subset S$, the natural map $\rho_{S,S'}^R : R[q]^S \rightarrow R[q]^{S'}$ is an embedding.

If $\zeta$ is a root of unity of order in $S$, then for every $f(q) \in R[q]^S$ the evaluation $\text{ev}_\zeta(f(q)) \in R[\zeta]$ can be defined by sending $q \rightarrow \zeta$. For a set $\Xi$ of roots of unity whose orders form a subset $T \subset S$, one defines the evaluation

\[
\text{ev}_\Xi : R[q]^S \rightarrow \prod_{\zeta \in \Xi} R[\zeta].
\]

Theorem 6.1 of [7] shows that if $R \subset \mathbb{Q}$, $S$ is $R$–connected, and there exists $n \in S$ that is adjacent to infinitely many elements in $T$, then $\text{ev}_\Xi$ is injective.
3.3. Taylor expansion. Fix a natural number $n$, then we have
\[ R[q]^n = \lim_{k \to \infty} R[q]/((\Phi_n^k(q))). \]
Suppose $\mathbb{Z} \subset R \subset \mathbb{Q}$, then the natural algebra homomorphism
\[ h : R[q]/((\Phi_n^k(q))) \to R[e_n][q]/((q - e_n)^k) \]
is injective, by Proposition 13 below. Taking the inverse limit, we see that there is a natural injective algebra homomorphism
\[ h : R[q]^n \to R[e_n][q]/((q - e_n)). \]
Suppose $n \in S$. Combining $h$ and $\rho_{S,\{n\}} : R[q]^S \to R[q]^n$, we get an algebra map
\[ t_n : R[q]^S \to R[e_n][q]/((q - e_n)). \]
If $f \in R[q]^S$, then $t_n(f)$ is called the Taylor expansion of $f$ at $e_n$.

3.4. Splitting of $S_p$ and evaluation. For every integer $a$, we put $\mathbb{N}_a := \{n \in \mathbb{N} \mid (a, n) = 1\}$.
Suppose $p$ is a prime. Analogously to (8), we have
\[ S_p \simeq \mathbb{Z}[1/p][q]^{\mathbb{N}}. \]
Observe that $\mathbb{N}$ is not $\mathbb{Z}[1/p]$–connected. In fact one has $\mathbb{N} = \prod_{j=0}^{\infty} p^j \mathbb{N}_p$; and each $p^j \mathbb{N}_p$ is $\mathbb{Z}[1/p]$–connected. Let us define
\[ S_{p,j} := \mathbb{Z}[1/p][q]^{p^j \mathbb{N}_p}. \]
Note that for every $f \in S_p$, the evaluation $\text{ev}_\xi(f)$ can be defined for every root $\xi$ of unity. For $f \in S_{p,j}$, the evaluation $\text{ev}_\xi(f)$ can be defined when $\xi$ is a root of unity of order in $p^j \mathbb{N}_p$.

**Proposition 9.** For every prime $p$ one has
\[ S_p \simeq \prod_{j=0}^{\infty} S_{p,j}. \]

**Proof.** Suppose $n_i \in p^j \mathbb{N}_p$ for $i = 1, \ldots, m$, with distinct $j_i$'s. Then $n_i/n_s$, with $i \neq s$, is either not a power of a prime or a non–zero power of $p$, hence by Lemma 4 (and the remark right after Lemma 4), for any positive integers $k_1, \ldots, k_m$, we have
\[ (\Phi_{n_i}^{k_i}) + (\Phi_{n_j}^{k_j}) = (1) \quad \text{in } \mathbb{Z}[1/p][q]. \]
By the Chinese remainder theorem, we have
\[ \mathbb{Z}\left[\frac{1}{p}\right][q]/\langle \prod_{i=1}^{m} \Phi_{n_i}^{k_i}\rangle \simeq \prod_{i=1}^{m} \mathbb{Z}\left[\frac{1}{p}\right][q]/(\Phi_{n_i}^{k_i}). \]
Taking the inverse limit, we get (14). \[ \square \]

Let $\pi_j : S_p \to S_{p,j}$ denote the projection onto the $j$–th component in the above decomposition.

**Lemma 10.** Suppose $\xi$ is a root of unity of order $r = p^i p^{r'}$, with $(r', p) = 1$. Then for any $x \in S_p$, one has
\[ \text{ev}_\xi(x) = \text{ev}_\xi(\pi_j(x)). \]
If $i \neq j$ then $\text{ev}_\xi(\pi_i(x)) = 0$.

**Proof.** Note that $\text{ev}_\xi(x)$ is the image of $x$ under the projection $S_p \to S_p/(\Phi_r(q)) = \mathbb{Z}[1/p][q]$. It remains to notice that $S_{p,i}/(\Phi_r(q)) = 0$ if $i \neq j$. \[ \square \]
3.5. Splitting of $S_b$. Suppose $p$ is a prime divisor of $b$. Let
\[ S_b^{p,0} := \mathbb{Z}[1/b][q]^{Sp} \quad \text{and} \quad S_b^{0,0} := \mathbb{Z}[1/b][q]^{p0}. \]

We have similarly
\[ S_b = S_b^{p,0} \times S_b^{0,0} \]
with canonical projections $\pi^0_b : S_b \to S_b^{p,0}$ and $\pi^0_b : S_b \to S_b^{0,0}$. Note that if $b = p$, then $S_p^{0,0} = S_{p,0}$ and $S_p^{p,0} = \prod_{j>0} S_{p,j}$.

Suppose $f \in S_b$. If $\xi$ is a root of unity of order coprime with $p$, then $\text{ev}_\xi(f) = \text{ev}_\xi(\pi^0_b(f))$. Similarly, if the order of $\xi$ is divisible by $p$, then $\text{ev}_\xi(f) = \text{ev}_\xi(\pi^p_b(f))$.

3.6. Properties of the ring $\mathcal{R}_b$. For any $b \in \mathbb{N}$, we have
\[ \mathcal{R}_b \simeq \mathbb{Z}[1/b][q]^{N_2} \]

since the sequence $(q; q^k)_k, k \in \mathbb{N}$, is cofinal to $\Phi_{N_2}$. Here $N_2$ is the set of all odd numbers.

Let $(p_i | i = 1, \ldots, m)$ be the set of all distinct odd prime divisors of $b$. For $n = (n_1, \ldots, n_m)$, a tuple of numbers $n_i \in \mathbb{N}$, let $p^n = \prod_i p_i^{n_i}$. Let $S_n := \mathbb{Z}[1/b][q]^{N_2}$. Then $N_2 = \Pi_n S_n$. Moreover, for $a \in S_{n', a' \in S_{n'}}$, we have $(\Phi_a(q), \Phi_{a'}(q)) = (1)$ in $\mathbb{Z}[1/b]$ if $n \neq n'$. In addition, each $S_n$ is $\mathbb{Z}[1/b]$-connected. An argument similar to that for Equation (14) gives us
\[ \mathcal{R}_b \simeq \prod_n \mathbb{Z}[1/b][q]^{S_n}. \]

In particular, $\mathcal{R}_b^{p,0} := \mathbb{Z}[1/b][q]^{N_2p}$, and $\mathcal{R}_b^{0,0} := \mathbb{Z}[1/b][q]^{N_20}$ for any $1 \leq i \leq m$. If $2 | b$, then $\mathcal{R}_b^{2,0}$ coincides with $\mathcal{R}_b$.

Let $T$ be an infinite set of powers of an odd prime not dividing $b$ and $P$ be an infinite set of odd primes not dividing $b$.

Proposition 11. With the above notations, one has the following.

(a) For any $l \in S_n$, the Taylor map $t_l : \mathbb{Z}[1/b][q]^{S_n} \to \mathbb{Z}[1/b][e_l][[q - e_l]]$ is injective.

(b) Suppose $f, g \in \mathbb{Z}[1/b][q]^{S_n}$ such that $\text{ev}_\xi(f) = \text{ev}_\xi(g)$ for any root of unity $\xi$ with $\text{ord}(\xi) \in p^n P$, then $f = g$. The same holds true if $p^n T$ is replaced by $p^n P$.

(c) For odd $b$, the natural homomorphism $\rho_{\mathcal{R}_b,N_2} : \mathcal{R}_b \to \mathcal{R}_b$ is injective. If $2 | b$, then the natural homomorphism $\rho_{\mathcal{R}_b,N_2} : \mathcal{R}_b \to \mathcal{R}_b$ is an isomorphism.

Proof. a) Since each $S_n$ is $\mathbb{Z}[1/b]$-connected in Habiro sense, by [7] Theorem 4.2, for any $l \in S_n$
\[ (15) \]
\[ \rho_{S_n, l} : \mathbb{Z}[1/b][q]^{S_n} \to \mathbb{Z}[1/b][q]^{\{l\}} \]
is injective. Hence $t_l = h \circ \rho_{S_n, l}$ is injective too.

b) Since both sets contain infinitely many numbers adjacent to $p^n$, the claim follows from Theorem 6.1 in [7].

c) Note that for odd $b$
\[ S_b \simeq \prod_n \mathbb{Z}[1/b][q]^{S_n} \]
where $\mathbb{Z}_n := p^n N_b$. Further observe that $S_n$ is $\mathbb{Z}[1/b]$-connected if $b$ is odd. Then by [7] Theorem 4.2] the map
\[ \mathbb{Z}[1/b][q]^{S_n} \to \mathbb{Z}[1/b][q]^{S_n} \]
is an embedding. If $2 | b$, then $S_b^{0,0} := \mathbb{Z}[1/b][q]^{N_2} \simeq \mathcal{R}_b$. \qed

Assuming Theorem [1] Proposition [11] (b) implies Proposition [2]
4. On the Ohtsuki series at roots of unity

The Ohtsuki series was defined for SO(3) invariants by Ohtsuki [21] and extended to all other Lie algebras by the third author [13, 14].

In the works [21, 13, 14], it was proved that the sequence of quantum invariants at \( e_r \), where \( p \) runs through the set of primes, obeys some congruence properties that allow to define uniquely the coefficients of the Ohtsuki series. The proof of the existence of such congruence relations is difficult. In [6], Habiro proved that Ohtsuki series coincide with the Taylor expansion of the unified invariant at \( q = 1 \) in the case of integral homology spheres; this result was generalized to rational homology spheres by the third author [12].

Here we prove that the sequence of SO(3) invariant at the \( pr \)-th root \( e_re_p \), where \( r \) is a fixed odd number and \( p \) runs the set of primes, obeys some congruence properties that allow to define uniquely the coefficients of the “Ohtsuki series” at \( e_r \), which is coincident with the Taylor expansion at \( e_r \).

4.1. Extension of \( \mathbb{Z}[1/b][e_r] \). Fix an odd positive integer \( r \). Assume \( p \) is a prime bigger than \( b \) and \( r \). The cyclotomic rings \( \mathbb{Z}[1/b][e_r] \) and \( \mathbb{Z}[1/b][e_r] \) are extensions of \( \mathbb{Z}[1/b] \) of degree \( \varphi(rp) = \varphi(r)\varphi(p) \) and \( \varphi(r) \), respectively. Hence \( \mathbb{Z}[1/b][e_r] \) is an extension of \( \mathbb{Z}[1/b][e_r] \) of degree \( \varphi(p) = p - 1 \). Actually, it is easy to see that for

\[
\phi : \mathbb{Z}[1/b,e_r][q]/(f_p(q)) \rightarrow \mathbb{Z}[1/b][e_{rp}],
\]

the map

\[
\phi : \mathbb{Z}[1/b,e_r][q]/(f_p(q)) \rightarrow \mathbb{Z}[1/b][e_{pr}],
\]

which sends \( q \) to \( e_pe_r \), is an isomorphism. We put \( x = q - e_r \) and get

\[
\mathbb{Z}[1/b][e_{pr}] \simeq \mathbb{Z}[1/b,e_r][x]/(f_p(x + e_r)).
\]

Note that

\[
f_p(x + e_r) = \sum_{n=0}^{p-1} \binom{p}{n+1} x^n e_r^{p-n-1}
\]

is a monic polynomial in \( x \) of degree \( p - 1 \), and the coefficient of \( x^n \) in \( f_p(x + e_r) \) is divisible by \( p \) if \( n \leq p - 2 \).

4.2. Arithmetic expansion of \( \tau'_M \). Suppose \( M \) is a rational homology 3-sphere with \( |H_1(M,\mathbb{Z})| = b \). By Theorem [1] for any root of unity \( \xi \) of order \( pr \)

\[
\tau'_M(\xi) \in \mathbb{Z}[1/b][e_{pr}] \simeq \mathbb{Z}[1/b,e_r][x]/(f_p(x + e_r)).
\]

Hence we can write

\[
\tau'_M(e_re_p) = \sum_{n=0}^{p-2} a_{p,n} x^n
\]

where \( a_{p,n} \in \mathbb{Z}[1/b,e_r] \). The following proposition shows that the coefficients \( a_{p,n} \) stabilize as \( p \to \infty \).

**Proposition 12.** Suppose \( M \) is a rational homology 3-sphere with \( |H_1(M,\mathbb{Z})| = b \), and \( r \) an odd positive integer. For every non-negative integer \( n \), there exists a unique invariant \( a_n = a_n(M) \in \mathbb{Z}[1/b,e_r] \) such that for every prime \( p > \max(b,r) \), we have

\[
a_n \equiv a_{p,n} \pmod{p} \quad \text{in} \quad \mathbb{Z}[1/b,e_r] \quad \text{for} \quad 0 \leq n \leq p - 2.
\]

Moreover, the formal series \( \sum_{n=0}^{\infty} a_n(q - e_r)^n \) is equal to the Taylor expansion of the unified invariant \( I_M \) at \( e_r \).

**Proof.** The uniqueness of \( a_n \) follows from the easy fact that if \( a \in \mathbb{Z}[1/b,e_r] \) is divisible by infinitely many rational primes \( p \), then \( a = 0 \).

Assume Theorem [1] holds. We define \( a_n \) to be the coefficient of \((q - e_r)^n \) in the Taylor series of \( I_M \) at \( e_r \), and will show that Equation (18) holds true.
Recall that $x = q - e_r$. The diagram
\[
\begin{array}{c}
\mathbb{Z}[\frac{1}{q}]^\mathbb{N}_2 \quad \mathbb{Z}[\frac{1}{q}, e_r][q]^\mathbb{N}_2 \quad \mathbb{Z}[\frac{1}{q}, e_r][x] \\
\downarrow \quad q - e_r, e_r \quad (f_p(q)) \quad (f_p(x + e_r)) \\
\mathbb{Z}[\frac{1}{q}, e_r][x]/(f_p(q)) \quad \mathbb{Z}[\frac{1}{q}, e_r][x]/(f_p(x + e_r))
\end{array}
\]
is commutative. Here the middle and the right vertical maps are the quotient maps by the corresponding ideals. Note that $I_M$ is in the upper left corner ring, its Taylor series is the image in the upper right corner ring, while the evaluation (17) is in the lower middle ring. Using the commutativity at the lower right corner ring, we see that
\[
\sum_{n=0}^{p-2} a_{p,n}x^n = \sum_{n=0}^{\infty} a_n x^n \pmod{f_p(x + e_r)} \quad \text{in} \quad \mathbb{Z}[1/b, e_r][x].
\]
Since the coefficients of $f_p(x + e_r)$ up to degree $p - 2$ are divisible by $p$, we get the congruence (18).

**Remark 13.** Proposition 12 when $r = 1$ was the main result of Ohtsuki [21], which leads to the development of the theory of finite type invariant and the LMO invariant.

When $(r, b) = 1$, then Taylor series at $e_r$ determines and is determined by the Ohtsuki series. But when, say, $r$ is a divisor of $b$, a priori the two Taylor series, one at $e_r$ and the other at 1, are independent. We suspect that the Taylor series at $e_r$, with $r \mid b$, corresponds to a new type of LMO invariant.

5. Frobenius maps

The proof of Theorem 3 and hence of the main theorem, uses the Laplace transform method. The aim of this section is to show that the image of the Laplace transform, defined in Section 7, belongs to $\mathcal{R}_b$, i.e. that certain roots of $q$ exist in $\mathcal{R}_b$.

5.1. On the module $\mathbb{Z}[q]/(\Phi^k_n(q))$. Since cyclotomic completions are built from modules like $\mathbb{Z}[q]/(\Phi^k_n(q))$, we first consider these modules. Fix $n, k \geq 1$. Let
\[
E := \mathbb{Z}[q]/(\Phi^k_n(q)), \quad \text{and} \quad G := \mathbb{Z}[e_n][x]/(x^k).
\]
The following is probably well-known.

**Proposition 14.** a) Both $E$ and $G$ are free $\mathbb{Z}$–module of the same rank $k\varphi(n)$.

b) The algebra map $h : \mathbb{Z}[q] \to \mathbb{Z}[e_n][x]$ defined by
\[
h(q) = e_n + x
\]
descends to a well–defined algebra homomorphism, also denoted by $h$, from $E$ to $G$. Moreover, the algebra homomorphism $h : E \to G$ is injective.

**Proof.** a) Since $\Phi^k_n(q)$ is a monic polynomial in $q$ of degree $k\varphi(n)$, it is clear that
\[
E = \mathbb{Z}[q]/(\Phi^k_n(q))
\]
is a free $\mathbb{Z}$–module of rank $k\varphi(n)$. Since $G = \mathbb{Z}[e_n] \otimes \mathbb{Z}[x]/(x^k)$, we see that $G$ is free over $\mathbb{Z}$ of rank $k\varphi(n)$.

b) To prove that $h$ descends to a map $E \to G$, one needs to verify that $h(\Phi^k_n(q)) = 0$. Note that
\[
h(\Phi^k_n(q)) = \Phi^k_n(x + e_n) = \prod_{(j,n)=1} (x + e_n - e_n^j)^k.
\]
When $j = 1$, the factor is $x^k$, which is 0 in $\mathbb{Z}[e_n][x]/(x^k)$. Hence $h(\Phi^k_n(q)) = 0$.

Now we prove that $h$ is injective. Let $f(q) \in \mathbb{Z}[q]$. Suppose $h(f(q)) = 0$, or $f(x + e_n) = 0$ in $\mathbb{Z}[e_n][x]/(x^k)$. It follows that $f(x + e_n)$ is divisible by $x^k$; or that $f(x)$ is divisible by $(x - e_n)^k$. Since $f$ is a polynomial with coefficients in $\mathbb{Z}$, it follows that $f(x)$ is divisible by all Galois conjugates $(x - e_n^j)^k$ with $(j,n) = 1$. Then $f$ is divisible by $\Phi^k_n(q)$. In other words, $f = 0$ in $E = \mathbb{Z}[q]/(\Phi^k_n(q))$. \qed
5.2. A Frobenius homomorphism. We use $E$ and $G$ of the previous subsection. Let $p$ be a prime such that $(p, n) = 1$. Recall that when $(n, p) = 1$, then $\Phi_n(q^p) = \Phi_{np}(q)\Phi_n(q)$ is divisible by $\Phi_n(q)$. Therefore the algebra map $F_p : \mathbb{Z}[q] \to \mathbb{Z}[q]$, defined by $F_p(q) = q^p$, descends to a well-defined algebra map, also denoted by $F_p$, from $E$ to $E$. We want to understand the image $F_p(E)$.

**Proposition 15.** The image $F_p(E)$ is a free $\mathbb{Z}$-submodule of $E$ of maximal rank, i.e. $\text{rk}(F_p(E)) = \text{rk}(E)$. Moreover, the index of $F_p(E)$ in $E$ is $p^{k(k−1)\varphi(n)/2}$.

**Proof.** Using Proposition 14 we identify $E$ with its image $h(E)$ in $G$.

Let $\tilde{F}_p : G \to G$ be the $\mathbb{Z}$-algebra homomorphism defined by $\tilde{F}_p(e_n) = e_n^p$. Note that $\tilde{F}_p(x) = px^{p−1} + O(x^p)$, hence $\tilde{F}_p(x^k) = 0$. It is easy to see that $\tilde{F}_p$ is a well-defined algebra homomorphism, and that $\tilde{F}_p$ restricted to $E$ is exactly $F_p$. Since $E$ is a lattice of maximal rank in $G \otimes \mathbb{Q}$, it follows that the index of $F_p$ is exactly the determinant of $\tilde{F}_p$, acting on $G \otimes \mathbb{Q}$.

A basis of $G$ is $e^1_n, x^1$, with $(j, n) = 1, 0 < j < n$ and $j = 0$, and $0 \leq l < k$. Note that

$$\tilde{F}_p(e^1_n, x^1) = p^l e^1_n x^l + O(x^{l+1}).$$

Since $(p, n) = 1$, the set $e^p_n$, with $(j, n) = 1$ is the same as the set $e^1_n$, with $(j, n) = 1$. Let $f_1 : G \to G$ be the $\mathbb{Z}$-linear map defined by $f_1(x) = e^1_n x^l$. Since $f_1$ permutes the basis elements, its determinant is $\pm 1$. Let $f_2 : G \to G$ be the $\mathbb{Z}$-linear map defined by $f_2(x) = e^1_n x^l$. The determinant of $f_2$ is again $\pm 1$. This is because, for any fixed $l$, $f_2$ restricts to the automorphism of $\mathbb{Z}[e_n]$ sending $a$ to $e^p_n a$, each of these maps has a well-defined inverse: $a \mapsto e^p_n a$. Now $f_1 f_2 \tilde{F}_p(e^1_n, x^1) = p^l e^1_n x^l + O(x^{l+1})$ can be described by an upper triangular matrix with $p^l$s on the diagonal; its determinant is equal to $p^{k(k−1)\varphi(n)/2}$. \hfill \Box

From the proposition we see that if $p$ is invertible, then the index is equal to 1, hence we have

**Proposition 16.** For any $n$ coprime with $p$ and $k \in \mathbb{N}$, the Frobenius homomorphism $F_p : \mathbb{Z}[1/p][q]/(\Phi^k_n(q)) \to \mathbb{Z}[1/p][q]/(\Phi^k_n(q))$, defined by $F_p(q) = q^p$, is an isomorphism.

5.3. Frobenius endomorphism of $S_{p,0}$. For finitely many $n_i \in \mathbb{N}_p$ and $k_i \in \mathbb{N}$, the Frobenius endomorphism

$$F_p : \mathbb{Z}[1/p][q]/\left(\prod_i \Phi^k_{n_i}(q)\right) \to \mathbb{Z}[1/p][q]/\left(\prod_i \Phi^k_{n_i}(q)\right)$$

sending $q$ to $q^p$, is again well-defined. Taking the inverse limit, we get an algebra endomorphism

$$F_p : \mathbb{Z}[1/p][q]^{\mathbb{N}_p} \to \mathbb{Z}[1/p][q]^{\mathbb{N}_p}.$$ 

**Theorem 17.** The Frobenius endomorphism $F_p : \mathbb{Z}[1/p][q]^{\mathbb{N}_p} \to \mathbb{Z}[1/p][q]^{\mathbb{N}_p}$, sending $q$ to $q^p$, is an isomorphism.

**Proof.** For finitely many $n_i \in \mathbb{N}_p$ and $k_i \in \mathbb{N}$, consider the natural algebra homomorphism

$$J : \mathbb{Z}[1/p][q]/\left(\prod_i \Phi^k_{n_i}(q)\right) \to \prod_i \mathbb{Z}[1/p][q]/\left(\Phi^k_{n_i}(q)\right).$$

This map is injective, because in the unique factorization domain $\mathbb{Z}[1/p][q]$, one has

$$(\Phi_{n_1}(q)^{k_1} \ldots \Phi_{n_s}(q)^{k_s}) = \prod_{j=1}^{s} \Phi_{n_j}(q)^{k_j}.$$ 

Since the Frobenius homomorphism commutes with $J$ and is an isomorphism on the target of $J$ by Proposition 16, it is an isomorphism on the domain of $J$. Taking the inverse limit, we get the claim. \hfill \Box
5.4. Existence of $p$–th root of $q$ in $S_{p,0}$.

**Lemma 18.** For any subset $T \subset \mathbb{N}_q$, the ring $\mathbb{Q}[q]^T$ does not contain any $p$–th root of unity except for $1$, i.e. if $y^p = 1$, $y \in \mathbb{Q}[q]^T$, then $y = 1$.

**Proof.** It suffices to show that for any $n_1, n_2 \ldots n_m \in T$, the ring $\mathbb{Q}[q]/(\Phi_{n_1}^{k}, \ldots \Phi_{n_m}^{k})$ does not contain a $p$–th root of $1$ except for $1$. Using the Chinese remainder theorem, it suffices to consider the case where $m = 1$.

The ring $\mathbb{Q}[q]/(\Phi_{n}^{k}(q))$ is isomorphic to $\mathbb{Q}[e_n][x]/(x^k)$, by Proposition 13. If

$$y = \sum_{j=0}^{k-1} a_j x^j, \quad a_j \in \mathbb{Q}[e_n]$$

satisfies $y^p = 1$, then it follows that $a_0^p = 1$. Since $(n, p) = 1$, $\mathbb{Q}[e_n]$ does not contain $e_p$ and we must have $a_0 = 1$. One can easily see that $a_1 = \ldots = a_{k-1} = 0$. Thus $y = 1$.

In contrast with Lemma 18 we have

**Proposition 19.** For any $k \in \mathbb{N}$ and any subset $T \subset \mathbb{N}_p$, the ring $\mathbb{Z}[1/p][q]^T$ contains a unique $p^k$–th root of $q$, which is invertible in $\mathbb{Z}[1/p][q]^T$.

**Proof.** Let us first consider the case $T = \mathbb{N}_p$. Since $F_p$ is an isomorphism by Theorem 17, we can define

$$q^{1/p^k} := F_{p}^{−k}(q) \in S_{p,0}.$$

Since our ring does not contain $p$–th roots of unity, this root of $q$ is unique. Further observe that $q^{-1} \in S_{p,0}$. In fact, $q^{-1} = \sum_{n} q^n(q; q)_n \in \mathbb{Z}[q]^T$. We define

$$q^{-1/p^k} := F_{p}^{−k}(q^{-1}) \in S_{p,0}.$$

In the general case of $T \subset \mathbb{N}_p$, we use the natural map $\mathbb{Z}[1/p][q]^{\mathbb{N}_p} \hookrightarrow \mathbb{Z}[1/p][q]^T$.

**Relation with 12.** By Proposition 10, $S_{b,0}$ is isomorphic to the ring $A_{b}^{N_b} := \mathbb{Z}[1/b][q^{1/b}]^{N_b}$ used in 12. Furthermore, our invariant $\pi_0 M$ and the one defined in 12 belong to $S_{b,0}$. This follows from Theorem 11 for $b$ odd, and from Proposition 11(c) for $b$ even. Finally, the invariant defined in 12 for $M$ divided by the invariant of $\#_L(q^{k_i}, 1)$ (which is invertible in $S_{b,0}$ [12 Subsection 4.1]) coincides with $\pi_0 M$ up to factor $q^{1/b}$ by Theorem 11 [12 Theorem 3] and Proposition 11(b).

5.5. Another Frobenius homomorphism. We define another Frobenius type algebra homomorphism. The difference of the two types of Frobenius homomorphisms is in the target spaces of these homomorphisms.

Suppose $m$ is a positive integer. Define the algebra homomorphism

$$G_m : R[q]^T \rightarrow R[q]^{mT} \quad \text{by} \quad G_m(q) = q^m.$$ 

Since $\Phi_{m_T}(q)$ always divides $\Phi_{m}(q^m)$, $G_m$ is well-defined.

5.6. Realization of $q^{a^2/b}$ in $S_p$. Suppose $b = \pm p^j$ and $a$ is an integer. Let $B_{p,j} = G_{p,j}(S_{p,0})$.

Note that $B_{p,j} \subset S_{p,j}$. By Proposition 15 there is a unique $b$–th root of $q$ in $S_{p,0}$; we denote it by $x_{b,0}$. We define an element $z_{b,a} \in S_{p}$ as follows.

If $b \mid a$, let $z_{b,a} := q^{a^2/b} \in S_{p}$.

If $b \nmid a$, then $z_{b,a} \in S_{p}$ is defined by specifying its projections $\pi_{j}(z_{b,a}) := z_{b,a,j} \in S_{p,j}$ as follows. Suppose $a = p^e c$, with $(c, p) = 1$. Then $s < l$. For $j > s$ let $z_{b,a,j} := 0$. For $0 \leq j \leq s$ let

$$z_{b,a,j} := [G_{p}(x_{b,0})]^{a^2/p^j} = [G_{p}(x_{b,0})]^{e p^{2−j}} \in B_{p,j} \subset S_{p,j}.$$

Similarly, for $b = \pm p^j$ we define an element $x_{b} \in S_{p}$ as follows. We put $\pi_0(x_{b}) := x_{b,0}$. For $j < l$, $\pi_{j}(x_{b}) := [G_{p}(x_{b,0})]^{a^b}$. If $j \geq l$, $\pi_{j}(x_{b}) := q^b$. Notice that for $c = (b, p^j)$ we have

$$\pi_{j}(x_{b}) = z_{b,c,j}.$$
Proposition 20. Suppose \( \xi \) is a root of 1 of order \( r = cr' \), where \( c = (r, b) \). Then

\[
ev_{\xi}(z_{b,a}) = \begin{cases} 0 & \text{if } c \nmid a \\ ((\xi^c)^{a^2}b') & \text{if } a = ca_1,
\end{cases}
\]

where \( b' \) is the unique element in \( \mathbb{Z}/r'\mathbb{Z} \) such that \( b'(b/c) \equiv 1 \pmod{r'} \). Moreover,

\[
ev_{\xi}(x_b) = ((\xi^c)^{b'}) .
\]

Proof. Let us compute \( \ev_{\xi}(z_{b,a}) \). The case of \( \ev_{\xi}(x_b) \) is completely analogous.

If \( b \mid a \), then \( c \mid a \), and the proof is obvious.

Suppose \( b \nmid a \). Let \( a = p^e \) and \( c = p^i \). Then \( s < l \). Recall that \( z_{b,a} = \sum_{j=0}^\infty z_{b,a,j} \). By Lemma [10]

\[
ev_{\xi}(z_{b,a}) = \ev_{\xi}(z_{b,a,i}).
\]

If \( c \nmid a \), then \( i > s \). By definition, \( z_{b,a,i} = 0 \), hence the statement holds true.

It remains the case \( c \mid a \), or \( i \leq s \). Note that \( \zeta = \xi^c \) is a primitive root of order \( r' \) and \((p, r') = 1 \).

Since \( z_{b,a,i} \in B_{p,i} \),

\[
ev_{\xi}(z_{b,a,i}) \in \mathbb{Z}[1/p][\zeta].
\]

From the definition of \( z_{b,a,i} \) it follows that \( (z_{b,a,i})^{b'/c} = (q^e)^{a^2/c^2} \), hence after evaluation we have

\[
[\ev_{\xi}(z_{b,a,i})]^{b'/c} = (\zeta)^{a^2}.
\]

Note also that

\[
[(\xi^c)^{a^2}b']^{b'/c} = (\zeta)^{a^2}.
\]

From Lemma [3] we conclude \( (\ev_{\xi}(z_{b,a,i})) = (\zeta^c)^{a^2}b' \). \( \square \)

6. Invariant of lens spaces

The purpose of this section is to prove Lemma [3]

6.1. Invariants of lens spaces. Let us compute the \( \text{SO}(3) \) invariant of the lens space \( M(b, a; d) \).

Recall that \( M(b, a; d) \) is the lens space \( L(b, a) \) together with a knot \( K \) inside colored by \( d \) (see Figure 1). Since \((b, a) = 1 \), there exist \( a^* \) and \( b^* \), such that \( aa^* + bb^* = 1 \).

We denote by \( \left( \frac{c}{b} \right) \) the Jacobi symbol and by \( s(a, b) \) the Dedekind sum (see e.g. [11]).

Proposition 21. Suppose \( c = (b, r) \) is a divisor of \((d - a^*)\). Then

\[
\tau_{M(b, a; d)}(\xi) = (-1)^{a+1} (-1)^{a+1} \left( \frac{a}{r} \right) \times \ev_{\xi} \left( q^{s(1,b) - 3s(n(b)s(a,b))} q^{(a-1)^2 + 2s(n(a,d-b))} \right) \ev_{\xi} \left( 1 - q^{-s(n(a,d-b))} \right)
\]

where \( \chi(c) = 1 \) if \( c = 1 \) and is zero otherwise. If \( c \nmid (a^* \pm d) \), \( \tau_{M(b, a; d)}(\xi) = 0 \).

In particular, it follows that \( \tau_{L(b, a)}(\xi) = 0 \) if \( c \nmid a^* \pm 1 \).

Proof. Let \( b \) and \( a \) be coprime. We consider first the case where \( b, a > 0 \). Since two lens spaces \( L(b, a_1) \) and \( L(b, a_2) \) are homeomorphic if \( a_1 = a_2 \pmod{b} \), we can assume \( a < b \). Let \( b/a \) be given by a continued fraction

\[
\frac{b}{a} = m_n - \frac{1}{m_{n-1} - \frac{1}{\ldots - \frac{1}{m_2 - \frac{1}{m_1}}}}.
\]

Using the Lagrange identity

\[
a - \frac{1}{b} = (a - 1) + \frac{1}{1 + \frac{1}{b - 1}}
\]

we can assume \( m_i \geq 2 \) for all \( i \).
The $\tau_{M(b,a,d)}(\xi)$ can be computed in the same way as the invariant $\xi_7(L(b,a), A)$ in [16], after replacing $A^4$ by $\xi^{1/2}$. Notice that $A$ and $\xi_7$ are both $r$-th roots of unity, because $(4, r) = 1$. Replacing the $b/a$-framed unknot in Figure 1 by a Hopf chain (as e.g. in Lemma 3.1 of [3]), we have
\[ F_{L\cup K}(\xi, d) = \sum_{j_1} \prod_{i=1}^n q^{m_i} q^{-1} \prod_{i=1}^{n-1} [j_i,j_{i+1}] \cdot [j_n]d[j_1] = \frac{S_n(d)}{\{1\}^{n+1}} \cdot \xi^{\frac{1}{r} \sum_{i=1}^m n_i} \]
where
\[ S_n(d) = \sum_{j} q^{\sum_i m_i} (q^{\frac{d}{r}} - q^{-\frac{d}{r}})(q^{\frac{i+1}{2}} - q^{-\frac{i+1}{2}}) \cdots (q^{\frac{d-1}{r}n} - q^{-\frac{d-1}{r}n}) (q^{\frac{d}{r}} - q^{-\frac{d}{r}}) \].

Choose $a^*$ and $b^*$, such that $bb^* + aa^* = 1$ with $0 < a^* < b$. When $a = 1$ we have $1^* = 1$ and $b^* = 0$. Using Lemmas 4.12 and 4.20 of [16] (and replacing $c$, $\xi$, $d$, $c_n$ by $c$, $\xi_7$, $d$, $c_n$), we get
\[ S_n(d) = (-2)^n (\sqrt{\xi_7(r)})^n \sqrt{\xi_7(c)} \left( \frac{a}{r} \right) \left( \frac{d}{r} \right) \left( \frac{b}{r} \right) \sum_{\pm} \chi^{\pm}(d) \xi^{-\frac{2}{r} \pm \frac{1}{r} \frac{a^*}{b^*} (d \mp a^*) + \frac{1}{r} \frac{d^*}{b^*} \}
where $\chi^{\pm}(d) = \pm 1$ if $c$ divides $d \mp a^*$ and is zero otherwise. Further $\epsilon(x) = 1$ if $x \equiv 1 \pmod{4}$ and $\epsilon(x) = I$ if $x \equiv 3 \pmod{4}$. This implies the second claim of the lemma.

Note that when $c = 1$, both $\chi^{\pm}(d)$ are nonzero. If $c > 1$ and $c \mid (d - a^*)$, $\chi^{\pm}(d) = 1$, but $\chi^{+}(d) = 0$. Indeed, for $c$ dividing $d - a^*$, $c \mid (d + a^*)$ if and only if $c \mid a^*$ which is impossible, because $c \mid b$ but $(b, a^*) = 1$.

Inserting the last formula into the definition [5] we get
\[ \tau_{M(b,a,d)}(\xi) = \frac{S_n(d)}{\xi^{1/2} - \xi^{-1/2}} \left( -2\xi^{-3/4} \sum_{j=1}^r \xi^{\frac{d}{r}} \right) \xi^{-\frac{1}{r} \sum_{i=1}^m n_i} \]
where we used that $\sigma_+ = n$ and $\sigma_- = 0$ (compare [11, p. 243]). From $\sum_{j=1}^r \xi^{\frac{d}{r}j} = \epsilon(r)\sqrt{r}$, we obtain
\[ \tau_{M(b,a,d)}(\xi) = \left( -1 \right)^{\frac{(r-1)(r-2)}{4}} \epsilon(c) \left( \frac{a'}{r} \right) \sqrt{c} \epsilon \epsilon \xi \left( q^{\frac{3n-\sum m_i}{4} - \frac{a^*-a}{4} - \frac{d^*}{4} + \frac{1}{4}} \right) \left( 1 - q^{-\chi(c)d/b} \right) \]
where $b = cb'$ and $r = cr'$.

Dividing the formula for $\tau_{M(b,a,d)}(\xi)$ by the formula for $\tau_{L(b,1)}(\xi)$ we get
\[ \tau'_{M(b,a,d)}(\xi) = \left( \frac{a}{c} \right) \left( \frac{b'}{r} \right) \sqrt{c} \epsilon \epsilon \xi \left( q^{\frac{3n-\sum m_i}{4} - \frac{a^*-a}{4} - \frac{d^*}{4} + \frac{1}{4}} \right) \left( 1 - q^{-d/b} \right) \frac{\chi(c)}{1 - q^{-1/b}} \]
Further observe, that by using $aa^* + bb^* = 1$, we get
\[ \frac{-a(d-a^2)}{4b} = \frac{b(a^* + 2d)}{4b} = \frac{a + a^*}{4b} + \frac{a(1-d^2) + 2d}{4b} \]
and with the following formulas for the Dedekind sum (compare [11, Theorem 1.12])
\[ -3s(a, b) = \frac{3n-\sum m_i}{4} - \frac{a + a^*}{4b} , \quad -3s(b, 1) = \frac{3 - b}{4} - \frac{1}{2b} \]
we can bring the previous result into the following form
\[ \tau'_{M(b,a,d)}(\xi) = \left( \frac{a}{c} \right) \left( \frac{b}{r} \right) \sqrt{c} \epsilon \epsilon \xi \left( q^{3s(1,b)-3s(a,b)+\frac{a^*-a+2d}{4b}} \right) \left( 1 - q^{-d/b} \right) \frac{\chi(c)}{1 - q^{-1/b}} \]
This implies the result for $0 < a < b$.
To compute $\tau_{M(-b,a,d)}(\xi)$, observe that $\tau_{M(b,-a,d)} = \tau_{M(-b,a,d)}$ is equal to the complex conjugate of $\tau_{M(b,a,d)}$. The ratio
\[ \tau'_{M(-b,a,d)}(\xi) = \frac{\tau_{M(b,a,d)}(\xi)}{\tau_{L(1,1)}(\xi)} \]

1There are misprints in Lemma 4.21: $q^* \pm n$ should be replaced by $q^* + n$ for $n = 1, 2$. 

can be computed analogously. Using $\epsilon(c) = (1)^{c+1} \epsilon(c)$, we have for $a, b > 0$
\[
\tau'_{M(-b,a,d)}(\xi) = (-1)^{\frac{a+b}{2}} \left( \frac{a}{c} \right) \text{ev}_\xi \left( q^{\frac{3n(a+1)+m_{a+b}+1}{2}(1-\tau) \left( 1 - q^{d/b} \right)^{\frac{a+b}{2}} \frac{1}{1-q^{-1/b}} \chi(c)} \right).
\]
Using $s(a, b) = s(a, -b) = -s(-a, b)$, we get the result.

\[\square\]

**Example.** For $b > 0$, we have
\[
\tau'_{L(-b,1)}(\xi) = (-1)^{\frac{a+b}{2}} \chi(c) \xi^{\frac{a+b}{2}} \left( \frac{a}{c} \right).
\]

To simplify the invariant of lens spaces we will use the Rademacher function $\phi(A) \in \mathbb{Z}$ for any
\[
A = \begin{pmatrix} a & x \\ b & d \end{pmatrix} \in PSL(2, \mathbb{Z})
\]
(see e.g. [11, 23]). If $b \neq 0$, then:
\[
\phi(A) = \frac{a + d}{b} - 12 \text{sn}(b) s(a, b).
\]
For example, for $b > 0$, $\phi\left( \begin{pmatrix} 1 \\ b \\ 0 \\ 1 \end{pmatrix} \right) = 3 - b$.

**Corollary 22.** Suppose $A = \begin{pmatrix} a & x \\ b & d \end{pmatrix} \in PSL(2, \mathbb{Z})$ with $0 < a < b$, and $\xi$ is a root of unity of odd order $r$ such that $c = (b, r) > 1$. Then
\[
\tau'_{M^0(b, a)}(\xi) = \left( \frac{a}{c} \right) \text{ev}_\xi (q^u),
\]
where
\[
u = \frac{1}{4} (\phi(A) + b - 3 - xd)
\]

Proof. Recall that $M^0(b, a) := M(b, a; d(0))$ with $d(0)$ being the smallest odd positive integer satisfying $d(0) a \equiv 1 \pmod{b}$. From [19], the statement of Corollary 22 holds true for the matrix $A$, where $d = a^*$ and $x = -b^*$. Here $\phi(A) = 3n - \sum m_i$.

It remains to prove that it holds for the matrix $A_s = AS^*$, where
\[
S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]

We denote the lens space with the knot inside corresponding to the matrix $A_s$ by $M(A_s)$. In this case $d' = a^* + sb$, $x' = -b^* + sa$. Note that $\phi(A_s) = \phi(A) + \phi(S^*) = \phi(A) + s$, see [23]. According to Proposition 21 or [19],
\[
\tau'_{M(A_s)} = \left( \frac{a}{c} \right) \text{ev}_\xi (q^u)
\]
where
\[
4u_s = \phi(A) + (b - 3) - as^2b + b^*(a^* + 2sb) = \phi(A) + (b - 3) - s^2ab + b^*a^* + sbb^* + s(1 - aa^*) \quad \text{since } 1 - aa^* = bb^* = (\phi(A) + s) + (b - 3) - (a^* + sb)(sa - b^*) = \phi(A_s) + (b - 3) - x'd'.
\]

\[\square\]
6.2. Proof of Lemma \([2]\) Assume \(b = p^l\) and \(p\) is prime. Recall that for \(\varepsilon \in \{0, 1\}\), we defined \(M^\varepsilon(b, a) := M(b, a; d(\varepsilon))\), where \(d(0) = 1\) and \(d(0)\) is the smallest odd positive integer such that \(ad(0) \equiv 1 \pmod{b}\). First observe that such \(d(\varepsilon)\) always exists. Indeed, if \(p\) is odd, we can achieve this by adding \(b\), otherwise the inverse of any odd number modulo 2 is odd.

Since \(a\) is defined modulo \(b\), it is sufficient to consider the case \(0 < a < b\). Let us further assume that \(a\) is a quadratic non–residue modulo \(p\), otherwise the proof is completely analogous and even simpler, since \(\left(\frac{a}{b}\right) = 1\) in that case.

We define the unified invariant \(I_{M^\varepsilon(b, a)} \in \mathcal{R}_b\) as follows. If \(p \neq 2\), then \(I_{M^\varepsilon(b, a)} \in \mathcal{S}_p\) is defined by specifying its projections

\[
\pi_j I_{M^\varepsilon(b, a)} = \begin{cases} 
q^{3s(1,0) - 3s(a,b)} & \text{if } j = 0 \\
(-1)^j q^n & \text{if } 0 < j < l \\
(-1)^j q^n & \text{if } j \geq l.
\end{cases}
\]

If \(p = 2\), then only \(\pi_0 I_{M(2,s,a)} \in \mathcal{S}_{2,0} = \mathcal{R}_2\) is non–zero and it is equal to \(q^{3s(1,2) - 3s(a,2)}\). By Proposition \([21]\) for any \(\xi\) of order \(k\) coprime with \(p\), \(ev_\xi(I_{M^\varepsilon(b, a)}) = \tau_{M^\varepsilon(b, a)}(\xi)\). If \(\xi = p^j k\) with \(j > 0\), then \(ev_\xi(I_{M^\varepsilon(b, a)}) = \tau_{M^\varepsilon(b, a)}(\xi)\) by Corollary \([22]\). It remains to show that \(I_{M^\varepsilon(b, a)}\) is invertible.

Note that \((3s(1, b) - 3s(a, b)) \in \mathbb{Z}\). Since the \(b\)–th root of \(q\) belongs to \(\mathcal{R}_{b,0}, \pi_0 I_{M^\varepsilon(b, a)} \in \mathcal{R}_{b,0}\) is invertible.

It remains to consider the case when \(b\) is odd. We claim that for odd \(b\), \(u \in \mathbb{Z}\) and hence \(\pi_j I_{M^\varepsilon(b, a)}\) is invertible for \(j > 0\). Indeed, by Proposition \([21]\) we have

\[u = \frac{a(1 - d^2) + 2d - 2}{4b} + 3s(1, b) - 3s(a, b).\]

When \(d\) is odd, we have \(4 \mid (1 - d^2)\), and \(4 \mid (2d - 2)\). It follows that \(u \in \frac{1}{4}\mathbb{Z}\). But we know that \(u \in \frac{1}{4}\mathbb{Z}\). Hence \(u \in \mathbb{Z}\).

7. LAPLACE TRANSFORM

This section is devoted to the proof of Theorem \([3]\) by using Andrew’s identity. Throughout this section, let \(p\) be a prime or \(p = 1\), and \(b = \pm p^l\).

7.1. Definition. The Laplace transform is a \(\mathbb{Z}[q^{\pm 1}]\)–linear map defined by

\[\mathcal{L}_b : \mathbb{Z}[z^{\pm 1}, q^{\pm 1}] \rightarrow \mathcal{S}_p, \quad z^a \mapsto z_{b,a}.\]

In particular, we put \(\mathcal{L}_{b,j} := \pi_j \circ \mathcal{L}_b\) and have \(\mathcal{L}_{b,j}(z^a) = z_{b,a,j} \in \mathcal{S}_{p,j}\).

Further, for any \(f \in \mathbb{Z}[z^{\pm 1}, q^{\pm 1}]\) we define

\[\hat{f} := f|_{z = q^n} \in \mathbb{Z}[q^{kn}, q^{\pm 1}].\]

Lemma 23. Suppose \(f \in \mathbb{Z}[z^{\pm 1}, q^{\pm 1}]\). Then for a root of unity \(\xi\) of odd order \(r\),

\[\sum_{n} \xi^n q^{\frac{1}{r} \frac{n^2}{2}} \hat{f} = \gamma_b(\xi) ev_\xi(\mathcal{L}_{-b}(f)).\]

Proof. It is sufficient to consider the case \(f = z^a\). Then, by the same arguments as in the proof of \([3]\) Lemma 1.3, we have

\[
\sum_{n} \xi^n q^{\frac{1}{r} \frac{n^2}{2}} q^{na} = \begin{cases} 
0 & \text{if } c \nmid a \\
(\xi^{-a} q^{-1})^{\gamma_b(\xi)} & \text{if } a = ca_1,
\end{cases}
\]

with the notation as in Proposition \([20]\). The result follows now from Proposition \([20]\). \(\square\)
7.2. Proof of Theorem 4  
Recall that
\[ A(n,k) = \prod_{i=0}^{k} (q^n + q^{-n} - q^i - q^{-i}). \]

We have to show that there exists an element \( Q_{b,k} \in R_b \) such that for every root of unity \( \xi \) of odd order \( r \) one has
\[ \sum_{\xi} q^{\frac{2z^2-1}{4}} A(n,k) = ev_{\xi}(Q_{b,k}). \]

Applying Lemma 24 to \( F_{U^n}(\xi) = \sum_{\xi} q^{\frac{2z^2-1}{4}}[n]^2 \), we get for \( c = (b,r) \)

\[ F_{U^n}(\xi) = 2\gamma_b(\xi) ev_{\xi} \left( \frac{(1 - q^{-1/b})\chi(c)}{(1 - q^{-1})(1 - q)} \right), \]

where as in Proposition 21 \( \chi(c) = 1 \) if \( c = 1 \) and is zero otherwise. We will prove that for odd \( p \) and any number \( j \geq 0 \) there exists an element \( Q_b(q_1, x, j) \in S_{p,j} \) such that

\[ \pi_{j} Q_{b,k} := \frac{1 - q^{-1}}{(1 - q^{-1/b})\chi(p)^j} Q_b(q^{-sn(b)}, x, j). \]

We split the proof into two parts. In the first part we will show that there exists an element \( Q_b(q_1, x, j) \) such that Equality 24 holds. In the second part we show that \( Q_b(q_1, x, j) \in S_{p,j} \).

Part 1, \( b \) odd case. Assume \( b = \pm p^i \) with \( p \neq 2 \). We split the proof into several lemmas.

Lemma 24. For \( x_{b,j} := \pi_{j}(x_b) \) and \( c = (b,p^i) \),
\[ \mathcal{L}_{b,j} \left( \prod_{i=0}^{k} (z + z^{-1} - q^i - q^{-i}) \right) = 2 (-1)^{k+1} \left[ \frac{2k+1}{k} \right] S_{b,j}(k,q), \]
where
\[ S_{b,j}(k,q) := 1 + \sum_{n=1}^{\infty} \frac{q^{(k+1)cn} (q^{-k-1}; q)_{cn} (1 + q^{cn}) x^{n^2}}{(q^{k+1}; q)_{cn}}. \]

Observe that for \( n > \frac{k+1}{c} \) the term \( (q^{-k-1}; q)_{cn} \) is zero and therefore the sum in (25) is finite.

**Proof.** Since \( \mathcal{L}_b \) is invariant under \( z \to z^{-1} \) one has
\[ \mathcal{L}_b \left( \prod_{i=0}^{k} (z + z^{-1} - q^i - q^{-i}) \right) = -2 \mathcal{L}_b(z^{-k}(zq^{-k}; q)_{2k+1}), \]
and the \( q \)-binomial theorem (e.g. see [4], II.3) gives
\[ z^{-k}(zq^{-k}; q)_{2k+1} = (-1)^k \sum_{i=-k}^{k+1} \left[ \frac{2k+1}{k+i} \right] z^i. \]

Notice that \( \mathcal{L}_{b,j}(z^a) \neq 0 \) if and only if \( c \mid a \). Applying \( \mathcal{L}_{b,j} \) to the RHS of (24), only the terms with \( c \mid i \) survive and therefore
\[ \mathcal{L}_{b,j} \left( z^{-k}(zq^{-k}; q)_{2k+1} \right) = (-1)^k \sum_{n=\left[ \frac{k+1}{c} \right]}^{\left[ \frac{(k+1)c}{c} \right]} \left[ \frac{2k+1}{k + cn} \right] z_{b,cn;j}. \]

Seprating the case \( n = 0 \) and combining positive and negative \( n \) this is equal to
\[ (-1)^k \left[ \frac{2k+1}{k} \right] + (-1)^k \sum_{n=\left[ \frac{k+1}{c} \right]}^{\left[ \frac{(k+1)c}{c} \right]} \left[ \frac{2k+1}{k + cn} \right] + \left[ \frac{2k+1}{k - cn} \right] z_{b,cn;j}, \]
where we use the convention that \( \left[ \begin{array}{c} x \\ -1 \end{array} \right] \) is put to be zero for positive \( x \). Further,
\[
\begin{bmatrix} 2k+1 \\ k+cn \end{bmatrix} + \begin{bmatrix} 2k+1 \\ k-cn \end{bmatrix} = \frac{\{k+1\}}{2(k+2)} \begin{bmatrix} 2k+2 \\ k+cn+1 \end{bmatrix} \left( q^{n/2} + q^{-cn/2} \right)
\]
and
\[
\begin{bmatrix} 2k+2 \\ k+cn+1 \end{bmatrix} \begin{bmatrix} 2k+1 \\ k \end{bmatrix}^{-1} = (-1)^{cn} q^{(k+1)cn} + \frac{\{q^{-k-1}; q\}^n}{(q^{k+2}; q)_n}.
\]
Using \( z_{b,cn} = (z_{b,c})^n = x_{b,c} \) we get the result.

To define \( Q_k(q, x, \omega) \) we will need Andrew’s identity (3.43) of [1]:
\[
\sum_{n \geq 0} (-1)^n \alpha_n t^{-\frac{\alpha n + 1}{2} + m + Nn} \frac{(t^{-N})}{(t^{N+1})_n} \prod_{l=1}^s \beta_l^{n_l} \gamma_l^{n_l-1} \left( \begin{bmatrix} b_l \end{bmatrix}_n \right)_n =
\]
\[
\frac{(t)_{N+1}(\text{re}_c)_{N+1}}{(t)_N(\text{re}_c)_N} \sum_{n_1 \geq \cdots \geq n \geq 0} \beta_n t^{n_1} (t^{-N} \beta_l)_n \prod_{l=1}^s \beta_l^{n_l} = \frac{(t)_{N+1}(\text{re}_c)_{N+1}}{(t)_N(\text{re}_c)_N} \sum_{n_1 \geq \cdots \geq n \geq 0} \beta_n t^{n_1} (t^{-N} \beta_l)_n \prod_{l=1}^s \beta_l^{n_l}.
\]
Here and in what follows we use the notation \( (a)_n := (a; t)_n \). The special Bailey pair \( (\alpha_n, \beta_n) \) is chosen as follows
\[
\begin{align*}
\alpha_n &= 1, & \alpha_0 &= (-1)^n \frac{\alpha n + 1}{2}, \\
\beta_n &= 1, & \beta_0 &= 0 \text{ for } n \geq 1.
\end{align*}
\]

**Lemma 25.** \( S_{b,j}(k, q) \) is equal to the LHS of Andrew’s identity with the parameters fixed below.

**Proof.** Since
\[
S_{b,j}(k, q) = S_{-b,j}(k, q^{-1}),
\]
it is enough to look at the case when \( b > 0 \). Define \( b' := \frac{b}{c} \) and let \( \omega \) be a \( b' \)-th primitive root of unity. For simplicity, put \( N := k + 1 \) and \( t := x_{b,j} \). Using the following identities
\[
(q^t; q)_n = \prod_{l=0}^{c-1} (q^{l+n}, q^n)_n,
\]
\[
(q^t; q)_n = \prod_{l=0}^{c-1} (\omega^{l+n}, q^n)_n,
\]
where the later is true due to \( t^{b'} = x_{b,j}^{b'} = q^{b'} \) for all \( j \), and choosing a \( c \)-th root of \( t \) denoted by \( t^\omega \) we can see that
\[
S_{b,j}(k, q) = 1 + \sum_{n_1 \geq \cdots \geq n \geq 0} \prod_{l=0}^{c-1} (\omega^{l+n}, q^n)_n (1 + t^{b'} n^{b'+1} N n).
\]

Now we choose the parameters for Andrew’s identity as follows. We put \( a := \frac{c-1}{2}, d := \frac{c-1}{2} \) and \( m := \left\lfloor \frac{N}{2} \right\rfloor \). For \( l \in \{1, \ldots, c-1\} \) there exist unique \( u_l, v_l \in \{0, \ldots, c-1\} \) such that \( u_l = N + l \mod c \) and \( v_l = N - l \mod c \). Note that \( v_l = u_{c-l} \). We define \( U_l := -\frac{N + u_l}{c} \) and \( V_l := -\frac{N + v_l}{c} \). Then \( U_l, V_l \in \frac{1}{c} \mathbb{Z} \) but \( U_l + V_l \in \mathbb{Z} \). We define
\[
\begin{align*}
b_l &= t^{U_l}, & c_l &= t^{V_l} \quad \text{for } l = 1, \ldots, a, \\
b_{a+l} &= \omega^l t^{-m}, & c_{a+l} &= \omega^{-l} t^{-m} \quad \text{for } i = 1, \ldots, d, \\
b_{a+l+d+i} &= \omega^l t^{U_l}, & c_{a+l+d+i} &= \omega^{-l} t^{V_l} \quad \text{for } i = 1, \ldots, d \text{ and } l = 1, \ldots, c-1, \\
b_{a+l+d+i} &= -\omega^l t^l, & c_{a+l+d+i} &= -\omega^{-l} t^l \quad \text{for } i = 1, \ldots, d, \\
b_{a+l+d+i} &= t^{-m}, & c_{a+l+d+i} &= t^{-m}, \\
b_{a+l+d+i} &\rightarrow \infty, & c_{a+l+d+i} &\rightarrow \infty.
\end{align*}
\]
where \( g = a + cd \) and \( s = (c+1)k+b+1 \).

We now calculate the LHS of Andrew’s identity. Using the notation
\[
(\omega^{l+n} t^x)_n = (\omega^x)_n (\omega^{-l} t^x)_n
\]
and the identities
\[
\lim_{c \to \infty} \frac{(c)_n}{c^n} = (-1)^n t^{\frac{n(n-1)}{2}} \quad \text{and} \quad \lim_{c \to \infty} \frac{t}{c} = 1
\]
we get
\[
\text{LHS} = 1 + \sum_{n \geq 1} t^{n(n-1+s+N-y)} \left( 1 + t^m \right) \frac{(t^{-N})_n}{(t^{m+1})_n} \prod_{i=1}^a \frac{(U_i)_n (V_i)_n}{(1-U_i)_n (1-V_i)_n} \prod_{i=1}^d \frac{(\omega^{\pm i} t^{-m})_n}{(\omega^{\pm i} t^{-1+m})_n}
\]
\[
\prod_{i=1}^{c-1} \frac{(\omega^{\pm i} t)_n}{(\omega^{\pm i} t^{1-i})_n (\omega^{\pm i} t^{1+m})_n}
\]
where
\[
y := \sum_{i=1}^a (U_i + V_i) + \sum_{i=1}^d \sum_{k=1}^{c-1} (U_i + V_i) - m(2d+1) + 2d + 1 + N.
\]
Since \(\sum_{i=1}^a (U_i + V_i) = 2 \sum_{i=1}^a (U_i + V_i) = 2(-N + m + \frac{c-1}{2})\) and \(2d + 1 = b'\), we have
\[
n - 1 + s + N - y = n + Nb'.
\]
Further,
\[
\prod_{i=1}^d \frac{(-\omega^{\pm i} t)_n}{(-\omega^{\pm i} t^m)_n} = \prod_{i=1}^{b'-1} \frac{1 + \omega^i t_n}{1 + \omega^x} = \frac{1 + \omega^{b'n}}{1 + t^n}
\]
and
\[
\prod_{i=1}^a \frac{(t^i U_i)_n (t^i V_i)_n}{(t^{1-i} U_i)_n (t^{1-i} V_i)_n} \cdot \prod_{i=1}^d \frac{(-\omega^{\pm i} t^{-m})_n}{(-\omega^{\pm i} t^{m+1})_n} \prod_{i=1}^{c-1} \frac{(\omega^{\pm i} t^{-1})_n (\omega^{\pm i} t^{1+m})_n}{(\omega^{\pm i} t^{1-1})_n (\omega^{\pm i} t^{1+m+1})_n}
\]
\[
= \prod_{i=0}^{b'-1} \frac{(\omega^i t^{N+i})_n}{(\omega^i t^{N+i+1})_n}.
\]
Taking all the results together, we see that the LHS is equal to \(S_{b,c}(k,q)\).

Let us now calculate the RHS of Andrew's identity with parameters chosen as above. For simplicity, we put \(\delta_j := n_{j+1} - n_j\). Then the RHS is given by
\[
\text{RHS} = (t)_N \sum_{n_s \geq \cdots \geq n_2 \geq n_1 = 0} \frac{t^n \cdot (t^{-N})_{n_s} (b_s)_{n_s} (c_s)_{n_s}}{(t^{m+1})_{n_s} (l^{-1+n_s})_{n_s}} \cdot \prod_{i=1}^a \frac{(U_i)_n (V_i)_n (l^{1-U_i}_V)_n}{(l^{1-V_i})_n (l^{1-U_i})_n} \cdot \prod_{i=1}^d \frac{(-\omega^{\pm i} t^{-m})_{n_{a+i}} (2m+1)_{n_{a+i}} (-\omega^{\pm i} t)_{n_{a+i}} (t^{-1})_{n_{a+i}}}{(-\omega^{\pm i} t)_{n_{a+i}} (t^{-1})_{n_{a+i}}}
\]
\[
\prod_{i=1}^{c-1} \frac{(\omega^i t U_i)_n (\omega^{-i} t V_i)_n (l^{1-U_i-V_i})_{n_{a+i}}}{(l^{1-U_i})_n (l^{1-V_i})_{n_{a+i}}} \prod_{i=1}^{c-1} \frac{(-\omega^{\pm i} t^{-1})_{n_{a+i}} (\omega^{\pm i} t^{1+m})_{n_{a+i+1}}}{(-\omega^{\pm i} t^{1-m})_{n_{a+i+1}}}
\]
where
\[
x := \sum_{i=1}^a (1 - U_i - V_i) n_i + \sum_{i=1}^d (2m + 1) n_{a+i}
\]
\[
+ \sum_{i=1}^d \sum_{l=1}^{c-1} (1 - U_i - V_i) n_{a+ld+i} - \sum_{i=1}^d n_{a+i} + (m - N) n_{a-1} + n_s.
\]
For \(c = 1\) or \(d = 0\), we use the convention that empty products are set to be 1 and empty sums are set to be zero.

Let us now have a closer look at the RHS. Notice, that
\[
\lim_{b_s, c_s \to \infty} \frac{(b_s)_{n_s} (c_s)_{n_s}}{(l^{-N})_{b_s c_s}} = (-1)^{n_s} t^{\frac{n_s(n_s-1)}{2}} l^{N n_s}.
\]
The term \((t^{-1})_{g_{y+i}}\) is zero unless \(\delta_{y+i} \in \{0, 1\}\). Therefore, we get
\[
\prod_{i=1}^{c-1} \left( -\omega^{\pm i} t \right)_{n_{y+i+1}} = \prod_{i=1}^{d} (1 + \omega^{\pm i} t_{n_{y+i}})^{1-\delta_{y+i}}.
\]
Due to the term \((t^{-m})_{n_{x}}\), we have \(n_{x} \leq m\) and therefore \(n_{i} \leq m\) for all \(i\). Multiplying the numerator and denominator of each term of the RHS by
\[
\prod_{l=1}^{d} \left( t^{1-U_{i}+n_{x}} \right)_{m-n_{x}} \prod_{l=1}^{d} \left( t^{1-V_{i}+n_{x}} \right)_{m-n_{x}}
\]
\[
\prod_{i=1}^{d} \left( \omega^{i} t^{m+1+n_{x}} \right)_{m-n_{x}}
\]
gives the denominator \(\prod_{l=1}^{d} \left( (t^{m+1}+y_{i})_{m} \right)_{m} \). This is equal to
\[
\prod_{l=1}^{d} \left( (t^{m+1})_{m} \right)_{m}
\]
Further,
\[
(1)_{N+n_{x}} = (t)_{N+n_{x}} = (t)_{m+N+n_{x}}.
\]
The term \((t^{-N+m})_{\delta_{s-1}}\) is zero unless \(\delta_{s-1} \leq N - m\) and therefore
\[
(t^{m+1})_{N-m+n_{x}} = (t^{m+1+n_{x}})_{N-m-\delta_{s-1}}.
\]
Taking the above calculations into account, we get
\[
(27) \quad \text{RHS} = \frac{(t; t)_{2m}}{(q^{N+1}; q)_{cm}} \cdot T_{k}(q, t)
\]
where
\[
T_{k}(q, t) := \sum_{n_{x} \geq \cdots \geq n_{y} \geq 0} (-1)^{n_{x}} x^{x'} \cdot (t^{-m})_{n_{x}} \cdot (t^{m+1+n_{x}})_{N-m-\delta_{s-1}} \cdot \frac{(t^{m+1})_{y_{i}}_{l}}{\prod_{l=1}^{d} (t^{m+1})_{y_{i}}_{l}}
\]
and \(x' := x + \sum_{l=1}^{c-1} \omega^{\pm i} t_{n_{x}} \).

We now define the element \(Q_{k}(q, x, j)\) by
\[
Q_{k}(q, x, j) := \left( (-1)^{k+1} q^{k(k+1)/2} \right)^{\frac{t^{m+1}(x^{b_{j}}; x_{b_{j}})_{2m}}{(q; q)_{N+cm}}} T_{k}(q, x_{b_{j}}).
\]

By Lemmas 24 and 25, Equation (27) and the following Lemma 26, we see that this element satisfies Equation (24).

**Lemma 26.** The following formula holds.
\[
(-1)^{k+1} \left( \frac{2k+1}{k} \right) (q^{k+1}; q)_{k+1} = (-1)^{k+1} q^{k(k+1)/2} \frac{(q; q)^{-k(k+1)/2}}{(q^{-1}; q^{-1})_{k+1}}
\]
Proof. This is an easy calculation using
\[(q^{k+1};q)_{k+1} = (-1)^{k+1}q^{(3k^2/4)+k+1} \{2k+1\}!/(k!)^2.\]

\[\square\]

**Part 1, b even case.** Let \(b = \pm 2^t\). We have to prove Equality (28) only for \(j = 0\), i.e.

to show
\[\frac{1}{(q^{k+1};q)_{k+1}} L_{2q,0} \left( \prod_{i=0}^{k} (z + z^{-1} - q^{i} - q^{-i}) \right) = 2 Q_k(q^{s^n(b)}, x_b, 0).\]

The calculation works similar to the odd case. Note that we have \(c = 1\) here. This case was already done in [3] and [12]. Since their approaches are slightly different and for the sake of completeness, we will give the parameters for Andrew’s identity and the formula for \(Q_k(q, x_b, 0)\) nevertheless.

We put \(t := x_b, 0, d := \frac{b}{2} - 1\), \(w\) a \(b\)-th root of unity and choose a primitive square root \(\nu\) of \(w\).

Define the parameters of Andrew’s identity by
\[
\begin{align*}
q, t := & \omega^i t^{N}, & c_i := & \omega^{-i} t^{N}, \quad \text{for } i = 1, \ldots, d, \\
c_{d+i} := & -\nu^{2i-1} t, & c_{d+i} := & -\nu^{-2i-1} t, \quad \text{for } i = 1, \ldots, d + 1, \\
b_0 := & t^{N}, & c_{s-1} := & t^{N+1}, \\
b_{2s-1} := & t^{N}, & c_s := & \infty, \\
b_{s-1} := & t^{N}, & c_s := & \infty,
\end{align*}
\]

where \(s = b + 2\). Now we can define the element
\[
Q_k(q, x_b, 0) := \left( (-1)^{k+1} q^{\frac{k+1}{2}} \right)^{s+n} \left( \frac{q^{k+1}}{q^{k+1}} \right)^{1-s+n} \left( \frac{(x_b,q)_0}{(x_b,0)} \right)^{2N} \frac{1}{(q^{s+n}(b), x_b, 0)_N T_k(q, x_b, 0)}
\]

where
\[
T_k(q, t) := \sum_{n=1, N-n=0}^{\infty} (-1)^{n+1} t^{x''}, \cdot \prod_{i=1}^{d} (t^{2N+1+n_i})^{s_i}, \cdot \prod_{i=1}^{d+1} (t^{-1})^{s_{d+i}}, \cdot (t^{N+1})^{s_{d+i}} \cdot (t^{N+1})^{s_{d+i}}
\]

and
\[x'' := \sum_{i=1}^{d} (2N+1)n_i - \sum_{i=1}^{d+1} n_{d+i} + \frac{n_{s-1}(n_{s-1}+1)}{2} + (N+1)(n_b + n_{s-1}).\]

We use the notation \((a; b)_{-1} := (1 - ab)^{-1}\).

**Part 2.** We have to show that \(Q_k(q, x, j) \in \mathcal{S}_{p,j}\), where \(j \in \mathbb{N} \cup \{0\}\) if \(p\) is odd, and \(j = 0\) for \(p = 2\). The following two lemmas do the proof.

**Lemma 27.** For \(t = x_b, j\),
\[T_k(q, t) \in \mathbb{Z}[q^{\pm 1}, t^{\pm 1}].\]

**Proof.** Let us first look at the case \(b\) odd and positive. Since for \(a \neq 0\), \((t^a)_n\) is always divisible by \((t)_n\), it is easy to see that the denominator of each term of \(T_k(q, t)\) divides its numerator. Therefore we proved that \(T_k(q, t) \in \mathbb{Z}[q^{k+1};\nu]\).

Since
\[
S_{b, j}(k, q) = \frac{(t; t)_m}{(q^{k+1}; q)_m} \cdot T_k(q, t),
\]

there are \(f_0, g_0 \in \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]\) such that \(T_k(q, t) = \frac{f_0}{g_0}\). This implies that \(T_k(q, t) \in \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]\) since \(f_0\) and \(g_0\) do not depend on \(\omega\) and the \(c\)-th root of \(t\).

The proofs for the even and the negative case work similar.  

\[\square\]
Lemma 28. For \( t = x_{b,j} \),

\[
\frac{(t; t)_{2m}}{(q; q)_{N + cm}((-t; t)N)} \in S_{p,j}
\]

where \( \lambda = 1 \) and \( j = 0 \) if \( p = 2 \), and \( \lambda = 0 \) and \( j \in \mathbb{N} \cup \{0\} \) otherwise.

Proof. Notice that

\[
(q; q)_{N + cm} = (\eta q)_{N + cm}(q'; q')_{2m},
\]

where we use the notation

\[
(q'; q)_n := \prod_{c \mid (a+j)} (1 - q^{a+j}).
\]

We have to show that

\[
\frac{(q'; q')_{2m}}{(t; t)_{2m}} \cdot \frac{1}{(q; q)_{N + cm}((-t; t)N)}
\]

is invertible in \( \mathbb{Z}[1/p][q] \) modulo any ideal \( (f) = (\prod_a \Phi_a^n(q)) \) where \( n \) runs through a subset of \( p^n \mathbb{N}_p \). Recall that in a commutative ring \( A \), an element \( a \) is invertible in \( A/(d) \) if and only if \( (a + (d)) = (1) \). If \( (a + (d)) = (1) \) and \( (a + (c)) = (1) \), multiplying together we get \( (a + (dc)) = (1) \). Hence, it is enough to consider \( f = \Phi_p^n(q) \) with \( (n, p) = 1 \). For any \( X \in \mathbb{N} \), we have

\[
(\eta q) = \prod_{i=1}^{X} \prod_{c \mid i} \Phi_d(q),
\]

\[
(-t; t) = \frac{(t^2; t^2)_X}{(t; t)_X} \prod_{i=1}^{X} \Phi_2 d(t)
\]

\[
\left(\frac{q'; q'}{t; t}\right)_X = \frac{(t'; t')_X}{(t; t)_X} \prod_{i=1}^{X} \prod_{d 
mid i} \Phi_d(t)
\]

for \( b' = b/c \). Recall that \( (\Phi_r(q), \Phi_a(q)) = (1) \) in \( \mathbb{Z}[1/p][q] \) if either \( r/a \) is not a power of a prime or a power of \( p \). For \( r = p^n a \), one of the conditions is always satisfied. Hence (29), (30), and (31) are invertible in \( S_{p,j} \). If \( b = c \) or \( b' = 1 \), (31) does not contribute. For \( c < b \), notice that \( q \) is a \( cn \)-th primitive root of unity in \( \mathbb{Z}[1/p][q]/(\Phi_2^n(q)) = \mathbb{Z}[1/p][e_{cn}] \). Therefore \( t^k = q^c \) is an \( n \)-th primitive root of unity. Since \( (n, b') = 1 \), \( t \) must be a primitive \( n \)-th root of unity in \( \mathbb{Z}[1/p][e_{cn}] \), too, and hence \( \Phi_n(t) = 0 \) in that ring. Since for \( i \) with \( (i, p) > 1 \), \((\Phi_i(t), \Phi_n(t)) = (1) \) in \( \mathbb{Z}[1/p][t] \), we have \( \Phi_i(t) \) is invertible in \( \mathbb{Z}[1/p][e_{cn}] \), and therefore (31) is invertible, too. \( \square \)

Appendix

Proof of Theorem 3. Since we will modify the proof of Habiro, here we will use the notations of [6]. Habiro defined a new basis \( \tilde{F}_{k}^r \), \( k = 0, 1, 2, \ldots \), for the Grothendieck ring of finite-dimensional \( sl_2 \)-modules, where

\[
\tilde{F}_{k}^r := \frac{q^{k(k-1)/2}}{\{k\}} \prod_{i=0}^{k-1} \left(V_1 - q^{2i+1/2} - q^{-(2i+1)/2}\right).
\]

Put \( \tilde{F}_{k} := \{\tilde{F}_{k,j}^r, \ldots, \tilde{F}_{k,j}^r\} \).

Lemma 29. We have

\[
C_{L \cup L'}(k, j) = J_{L \cup L'}(\tilde{F}_{k}^r) \prod_{i} (-1)^{k_i} q^{(k_i+1)(k_i+2)/2}.
\]

Proof. For any links \( L \) and \( L' \), using Lemma 6.1 in [6]

\[
J_{L \cup L'}(n, j) = \sum_{0 \leq k_i \leq n_i - 1} J_{L \cup L'}(\tilde{F}_{k}^r) q^{-\frac{k(k-1)}{2}} \sum_{i=1}^{m} q^{-\frac{k_i(k_i-1)}{2}} \left[n_i + k_i \atop 2k_i + 1\right] \{k_i\}!
\]
Applying
\[
\left[ \frac{n+k}{2k+1} \right] \{k\}!\{n\} = (-1)^k q^{1/2} q^{(3k+2)(k+1)/4} \prod_{j=0}^k (q^n + q^{-n} - q^j - q^{-j}) \over (1-q)(q^{k+1}; q)_{k+1}
\]
we get the result.

Hence, to prove Theorem 3 it is enough to show the following.

**Proposition 30.** Suppose \( L \cup L' \) is a link in \( S^3 \), such that \( L \) is an algebraically split, \( 0 \)-framed one. Assume that \( L' \) is colored by \( \mathbf{j} \), a tuple of odd numbers. Then, we have
\[
J_{L \cup L'}(\hat{P}_{k, j}) \in \frac{(q^{k+1}; q)_{k+1}}{1-q} \mathbb{Z}[q^{\pm 1}],
\]
where \( k = \max\{k_1, \ldots, k_m\} \).

**Proof.** We first recall Habiro’s setting. We denote by \( U_h = U_h(sl_2) \) the \( h \)-adically complete \( \mathbb{Q}[[h]] \)-algebra, topologically generated by \( H, E \) and \( F \), satisfying the relations
\[
HE - EH = 2E, \quad HF - FH = -2F, \quad EF - FE = K - K^{-1} \quad \frac{v}{v-1},
\]
where we set \( K = \nu^H = e^{\frac{2\pi i}{v}} \). Further, \( U_q(\mathcal{U}_q^{ev}) \) denotes the subalgebra of \( U_h \) freely generated over \( \mathbb{Z}[q^{\pm 1}] \) by \( \hat{F}^{(i)} K^i e^k \) (\( \hat{F}^{(i)} K^j e^k \), respectively) for \( i, k \geq 0, j \in \mathbb{Z} \), where
\[
\hat{F}^{(n)} = \frac{F^{n}K^{n}}{v^{(n-1)/2} [n]!} \quad \text{and} \quad e = (v - v^{-1})E.
\]

On \( \mathcal{U}_q^{ev} \), Habiro introduced the filtration \( \mathcal{F}_n(\mathcal{U}_q^{ev}) \), which is spanned by \( (\hat{F}^{(k)} K^k) K^j e^l \) over \( \mathbb{Z}[v, v^{-1}] \), and the completion
\[
\hat{U}_q^{ev} = \lim_{\rightarrow} \frac{\mathcal{U}_q^{ev}}{\mathcal{F}_n(\mathcal{U}_q^{ev})}.
\]

Further, he denoted by \( \hat{U}_q^{ev} \) the image of the map \( \hat{U}_q^{ev} \to U_h \), the completion of \( \mathcal{U}_q^{ev} \) in \( U_h \).

**Lemma 31.** For odd \( k \) and \( x \in \mathcal{U}_q \), we have
\[
(1 \otimes \text{tr}_{q^{-1}}^V)(e^{\hat{F}^{(H \otimes H)}})(1 \otimes x) \in \mathcal{U}_q^{ev}
\]

**Proof.** For fixed \( k \) and \( x \), we can find a basis \( e_1, \ldots, e_k \) of \( V_{k-1} \), such that
\[
H = \begin{pmatrix}
    & & & & k-3 & & & \\
    & & & & & 0 & & & \\
    & & & & & & \ddots & & \\
    & & & & & & & & 1-k
\end{pmatrix}.
\]

Therefore,
\[
(1 \otimes \text{tr}_{q^{-1}}^V)(e^{\hat{F}^{(H \otimes H)}})(1 \otimes x) = \sum_m \frac{1}{m!} \left( \frac{h}{2} \right)^m H^m \otimes \text{tr}^{V_{k-1}}(K^{-1}H^m x) = \sum_i \sum_m \frac{1}{m!} \left( \frac{h}{2} \right)^m H^m \otimes \langle e_i, K^{-1}H^m x e_i \rangle
\]
is only nonzero if \( x \) contains summands of the form \( \hat{F}^{(i)} K^j e^l \) with \( l \leq k \). In the last case, we have
\[
\sum_m \frac{1}{m!} \left( \frac{h}{2} \right)^m H^m \otimes \langle e_i, K^{-1}H^m \hat{F}^{(i)} K^j e^l \rangle = \sum_m \frac{1}{m!} \left( \frac{h}{2} \right)^m H^m \otimes \langle e_i, K^{-1} \hat{F}^{(i)} K^j e^l H^m e_i \rangle
\]
\[
= \sum_m \frac{1}{m!} \left( \frac{h}{2} \right)^m (2e_i)^m H^m \otimes \langle e_i, K^{-1} \hat{F}^{(i)} K^j e^l \rangle = \sum_m K^{2e_i} \otimes \langle e_i, K^{-1} \hat{F}^{(i)} K^j e^l \rangle \in \mathcal{U}_q^{ev}
\]
where \( 2e_i := k - 2i + 1 \) is even because \( k \) is odd. \( \square \)
Now, to show that
\[ J_{L \cup L'}(\bar{P}_k^j, j) \in \frac{(q^{k+1}; q)_{k+1}}{1 - q} \mathbb{Z}[q^{\pm 1}], \]
let us open the first component of \( L \cup L' \) and denote the resulting \((1, 1)-\)tangle by \( T \). The universal invariant \( J_T \) of \( T \) is defined in Section 4 of [6]. We claim that
\[ (1 \otimes \cdots \otimes 1 \otimes \text{tr}_q^{V_{ij} - 1} \otimes \cdots \otimes \text{tr}_q^{V_{ij} - 1}) J_T \in \mathcal{U}^\infty_q. \]

If \( \text{lk}(L_i, L'_j) = 0 \) for all \( i, j \), the result follows from Theorem 4.1 of Habiro [6], since \( q^{rac{1}{m_k-1}} \in \mathbb{Z}[q] \) if \( j \) is odd. Assume \( \text{lk}(L_i, L'_j) = 1 \), then according to the definition, \( J_T \) is an infinite sum of terms
\[ e^{h \sum_{i,j} \frac{\text{tr}_q^{n_i} H_{ij}}{m_i} y_1 \otimes \cdots \otimes y_{m_i+j}} \]
where \( y_i \in \mathcal{U}_q \) for all \( i \) and \( H_{ij} \) is defined to be \( 1 \otimes \cdots \otimes H \otimes 1 \otimes \cdots \otimes 1 \) with everywhere a 1 except \( H \) at the \( i \)-th and the \( j \)-th position. By Lemma [51] if \( k_j \) is odd, we have
\[ (1 \otimes \cdots \otimes \text{tr}_q^{V_{ij} - 1} \otimes \cdots \otimes 1)(e^{h \sum_{i} \frac{\text{tr}_q^{n_i} H_{ij}}{m_i} y_{m_i+j} \otimes \cdots \otimes 1}) \in \mathcal{U}_q^\infty. \]
This proves the claim. The rest is analogous to the proof of Theorem 8.2 in [6].

\[
\square
\]

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