QUANTUM TRACES FOR SLₙ(ℂ): THE CASE n = 3

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ABSTRACT. We generalize Bonahon–Wong’s SL₂(ℂ)-quantum trace map to the setting of SLₙ(ℂ). More precisely, given a non-zero complex parameter \( q = e^{2\pi i \hbar} \), we associate to each isotopy class of framed oriented links \( K \) in a thickened punctured surface \( \mathcal{S} \times (0,1) \) a Laurent polynomial \( \text{Tr}_q^\lambda(K) = \text{Tr}_q^\lambda(K)(X^q) \) in \( q \)-deformations \( X^q \) of the Fock–Goncharov \( \lambda \)-coordinates for higher Teichmüller space. This construction depends on a choice \( \lambda \) of ideal triangulation of the surface \( \mathcal{S} \). Along the way, we propose a definition for a SLₙ(ℂ)-version of this invariant.

1. INTRODUCTION

For a finitely generated group \( \Gamma \) and a suitable Lie group \( G \), a primary object of study in low-dimensional geometry and topology is the \( G \)-character variety

\[
\mathcal{R}_G(\Gamma) = \{ \rho : \Gamma \to G \} / \!/ G,
\]

consisting of group homomorphisms \( \rho \) from \( \Gamma \) to \( G \), considered up to conjugation. Here, the quotient is taken in the algebraic geometric sense of geometric invariant theory [MFK94]. Character varieties can be explored using a wide variety of mathematical skill sets. Some examples include the Higgs bundle approach of Hitchin [Hit92], the dynamics approach of Labourie [Lab06], and the representation theory approach of Fock–Goncharov [FG06b].

In the case where the group \( \Gamma = \pi_1(\mathcal{S}) \) is the fundamental group of a punctured surface \( \mathcal{S} \) of finite topological type with negative Euler characteristic, and where the Lie group \( G = \text{SL}_n(\mathbb{C}) \) is the special linear group, we are interested in studying a relationship between two competing deformation quantizations of the character variety \( \mathcal{R}_{\text{SL}_n(\mathbb{C})(\pi_1(\mathcal{S}))} \), which we denote simply by \( \mathcal{R}_{\text{SL}_n(\mathbb{C})(\mathcal{S})} \). Here, a deformation quantization \( \{ R_q \} \) of a Poisson
space $R$ is a family of non-commutative algebras $R^q$ parametrized by a non-zero complex parameter $q = e^{2\pi i h}$, such that the lack of commutativity in $R^q$ is infinitesimally measured in the semi-classical limit $\hbar \to 0$ by the Poisson bracket of the space $R$. In the case where $R = R_{SL_n(C)}(\mathcal{G})$ is the character variety, the bracket is provided by the Goldman Poisson structure on $R_{SL_n(C)}(\mathcal{G})$ [Gol84, Gol86, BG93].

The first quantization of the character variety is the $SL_n(C)$-skein algebra $S^q_{SL_n(C)}(\mathcal{G})$ of the surface $\mathcal{G}$; see [Tur89, Wit89, Prz91, BFKB99, Kup96, Sik05, CKM14]. The skein algebra is motivated by the classical algebraic geometric approach to studying the character variety $R_{SL_n(C)}(\mathcal{G})$ by means of its commutative algebra of regular functions $C[R_{SL_n(C)}(\mathcal{G})]$. An example of a regular function is the trace function $\text{Tr}_\gamma : R_{SL_n(C)}(\mathcal{G}) \to C$ associated to a closed curve $\gamma \in \pi_1(\mathcal{G})$ sending a representation $\rho : \pi_1(\mathcal{G}) \to SL_n(C)$ to the trace $\text{Tr}(\rho(\gamma)) \in C$ of the matrix $\rho(\gamma) \in SL_n(C)$. A theorem of classical invariant theory, due to Procesi [Pro76], implies that the trace functions $\text{Tr}_\gamma$ generate the algebra of functions $C[R_{SL_n(C)}(\mathcal{G})]$ as an algebra. According to the philosophy of Turaev and Witten, quantizations of the character variety should be of a 3-dimensional nature. Indeed, elements of the skein algebra $S^q_{SL_n(C)}(\mathcal{G})$ are represented by (formal linear combinations of) knots (or links) $K$ in the thickened surface $\mathcal{G} \times (0, 1)$. The skein algebra $S^q_{SL_n(C)}(\mathcal{G})$ has the advantage of being natural, but is difficult to work with in practice.

The second quantization of the $SL_n(C)$-character variety is the Fock–Goncharov quantum space $T^q_{SL_n(C)}(\mathcal{G})$; see [CF99, Kas98, FG09]. At the classical level, Fock–Goncharov [FG06b] introduced a framed version $R_{PSL_n(C)}(\mathcal{G})_{fr}$ (also called the $X$-moduli space) of the $PSL_n(C)$-character variety, which, roughly speaking, consists of representations $\rho : \pi_1(\mathcal{G}) \to PSL_n(C)$ equipped with additional linear algebraic data attached to the punctures of $\mathcal{G}$. Associated to each ideal triangulation $\lambda$ of the punctured surface $\mathcal{G}$ is a $\lambda$-coordinate chart $U_\lambda$ for $R_{PSL_n(C)}(\mathcal{G})_{fr}$ parametrized by $N$ non-zero complex coordinates $X_1, X_2, \ldots, X_N$ where the integer $N$ depends only on the topology of the surface $\mathcal{G}$ and the rank of the Lie group $SL_n(C)$. When written in terms of these coordinates $X_i$ the trace functions $\text{Tr}_\gamma$ on the character variety take the form of Laurent polynomials $\widetilde{\text{Tr}}_\gamma(X_i^{1/n})$ in $n$-roots of the $X_i$ (a
subtlety being that $\widetilde{\text{Tr}}_\gamma(X_i^{1/n})$ depends on the regular homotopy class of $\gamma$, represented by immersed curves, rather than the homotopy class of $\gamma$). At the quantum level, there are $q$-deformed versions $X_i^q$ of these coordinates, which no longer commute but $q$-commute with each other, according to the underlying Poisson structure. The quantized character variety $T^q_n(\mathcal{S})$ is obtained by gluing together quantum tori $T^q_n(\sigma)$, including one for each triangulation $\sigma = \lambda$ consisting of Laurent polynomials in the quantized Fock–Goncharov coordinates $X_i^q$.

The quantum character variety $T^q_n(\mathcal{S})$ has the advantage of being easier to work with than the skein algebra $S^q_n(\mathcal{S})$, but is less intrinsic.

We seek $q$-deformed versions $\widetilde{\text{Tr}}^q_\gamma$ of the trace functions, associating to a closed curve $\gamma$ a Laurent polynomial in the quantized Fock–Goncharov coordinates with respect to a fixed triangulation $\lambda$. Turaev and Witten’s philosophy leads us from the 2-dimensional setting of curves $\gamma$ on the surface $\mathcal{S}$ to the 3-dimensional setting of knots $K$ in the thickened surface $\mathcal{S} \times (0,1)$. More precisely:

**Conjecture 1 (SL$_n(\mathbb{C})$-quantum trace map).** Fix a $(2 \ast n^2)$-root $\omega^{1/2} = q^{1/(2 \ast n^2)} \in \mathbb{C} - \{0\}$.

For each ideal triangulation $\lambda$ of the punctured surface $\mathcal{S}$ (with empty boundary, $\partial \mathcal{S} = \emptyset$), there exists an injective algebra homomorphism

$$\text{Tr}^\omega_\lambda : S^q_n(\mathcal{S}) \hookrightarrow T^\omega_n(\lambda),$$

such that if $\omega^{1/2} = 1$, then for every blackboard-framed oriented knot $K$ in the thickened surface $\mathcal{S} \times (0,1)$ projecting to an immersed closed curve $\gamma$ on the surface $\mathcal{S}$,

$$\text{Tr}^1_\lambda(K) = \widetilde{\text{Tr}}_\gamma(X_i^{1/n}).$$

This last equation says that the Fock–Goncharov classical trace polynomial associated to the curve $\gamma$ is recovered in the classical limit. Moreover, the SL$_n(\mathbb{C})$-quantum trace map should be natural, appropriately interpreted, with respect to the choice of triangulation $\lambda$; see [Kim21].
Conjecture 1 is due to Chekhov–Fock [Foc97, CF00] in the case $n = 2$, and was proved 'by hand' in that case by Bonahon–Wong [BW11]. One of the motivations of the present work is to develop a conceptual understanding of their construction.

We prove the following, slightly weaker, version of Conjecture 1 in the case $n = 3$:

**Theorem 2** (Theorem 22, $\text{SL}_3(\mathbb{C})$-quantum trace polynomials). Fix a $(2 \times 3^2)$-root $\omega^{1/2} = q^{1/(2 \times 3^2)} = q^{1/18} \in \mathbb{C} - \{0\}$. For each ideal triangulation $\lambda$ of the punctured surface $\mathcal{S}$ (with possibly non-empty boundary), there exists a function

$$\text{Tr}_\omega^\lambda : \{\text{isotopy classes of (stated) framed oriented links } K \text{ in } \mathcal{S} \times (0, 1)\} \to \mathcal{T}_3(\omega)(\lambda)$$

such that if $\omega^{1/2} = 1$, then for every blackboard-framed oriented link $K$ whose components $K_1, K_2, \ldots, K_\ell$ project to immersed closed oriented curves $\gamma_1, \gamma_2, \ldots, \gamma_\ell$ in $\mathcal{S}$,

$$\text{Tr}_1(K) = \prod_{j=1}^\ell \tilde{\text{Tr}}_{\gamma_j}(X_1^{1/3}).$$

Moreover, this invariant satisfies the $q$-evaluated HOMFLYPT relation [FYH+85, PT87], as well as the unknot and framing skein relations, for $n = 3$; see Figures 1, 2, 3. In the figures, $[n]_q = (q^n - q^{-n})/(q - q^{-1})$ is the quantum integer, and $\zeta_n = (-1)^{n-1}q^{(1-n^2)/n}$ is (essentially) the (co)ribbon element of the quantum special linear group $\text{SL}_n^q$ (see Appendix A). $\square$

![Figure 1. HOMFLYPT skein relation.](image1)

![Figure 2. Unknot skein relation.](image2)
In particular, the isotopy invariance property of Theorem 2 can be thought of as the main step toward proving Conjecture 1 in the case \( n = 3 \).

Theorem 2 was originally proved as part of [Dou20]. Our proof is also ‘by hand’, generalizing the strategy of [BW11], and relies on computer assistance for some of the calculations; see Appendix B.

Along the way, we propose a definition for a \( SL_n(\mathbb{C}) \)-version of the quantum trace polynomials; see §5.3. That this construction is well-defined for general \( n \) is expected to follow from recent related work [CS23], which shares some overlap with [Dou21]; see Remark 37.

The solution of [BW11] in the \( n = 2 \) case is implicitly related to the theory of the quantum group \( U_q(sl_2) \) or, more precisely, of its Hopf dual \( SL^q_2 \); see for instance [Kas95]. For general \( n \), we make this relationship more explicit; see §3.4 and Appendix A.

In addition to the HOMFLYPT relation, the skein algebra \( S^q_n(\mathcal{S}) \) has other relations, best expressed as identities among certain \( n \)-valent graphs \( W \) in \( \mathcal{S} \times (0,1) \) called webs [Kup96, Sik01, CKM14]. It is therefore desirable to extend the definition of the quantum trace polynomials from links \( K \) to webs \( W \). Building on our construction for links, Kim [Kim20, Kim21] defined a \( SL_3(\mathbb{C}) \)-quantum trace map for webs, which is natural with respect to the choice of ideal triangulation \( \lambda \). Combined with Kim’s work, the results of [DS24, DS20] lead to a proof of the injectivity property of Conjecture 1 in the case \( n = 3 \), which is closely related to the study of linear bases of skein algebras; see [DS24, §9.3].

As another application, Kim [Kim20] constructed a \( SL_3(\mathbb{C}) \)-quantum Fock–Goncharov duality map [FG09] (of the bangle, rather than bracelet, form [Thu14]), generalizing much
of the $n = 2$ solution of \cite{AK17}; see also \cite{All19, CKKO20}. For other related studies in the \( \text{SL}_3(\mathbb{C}) \)-setting, see \cite{Hig23, ES22, LY23}.

Lé–Yu \cite{LY23} constructed a \( \text{SL}_n(\mathbb{C}) \)-quantum trace map for webs, agreeing at the level of links with the definition proposed in this paper. Their construction fits into a theory of \( \text{SL}_n(\mathbb{C}) \)-stated skein algebras \cite{Lê18, CL22, LS21}.

Quantum traces also appear in the context of spectral networks \cite{Gab17, NY20}. Empirical computations performed together with A. Neitzke (see \cite{NY22}) suggest that, at least for simple curves, the \( \text{SL}_3(\mathbb{C}) \)-quantum trace map defined in this paper agrees with that constructed in \cite{NY22}; see also \cite{KLS23} in the case \( n = 2 \).

The quantum trace map is a tool for studying the representation theory of skein algebras \cite{BW16, FKBL19}, relevant to topological quantum field theories \cite{Wit89, BHMV95}.

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2. Classical trace polynomials for \( \text{SL}_n \)

In this section, we will associate a Laurent polynomial \( \widetilde{\text{Tr}}_{\gamma}(X_1^{1/n}) \) in commuting formal \( n \)-roots \( X_1^{\pm 1/n}, X_2^{\pm 1/n}, \ldots, X_N^{\pm 1/n} \) to each immersed oriented closed curve \( \gamma \) transverse to a fixed ideal triangulation \( \lambda \) of a punctured surface \( \mathcal{S} \), where \( N \) depends only on the topology of \( \mathcal{S} \) and the rank of the Lie group \( \text{SL}_n(\mathbb{C}) \).

2.1. Topological setup. Let \( \mathcal{S} \) be an oriented punctured surface of finite topological type, namely \( \mathcal{S} \) is obtained by removing a finite subset \( P \), called the set of punctures, from a compact oriented surface \( \overline{\mathcal{S}} \). In particular, note that \( \mathcal{S} \) may have non-empty boundary,
\( \partial \mathcal{S} \neq \emptyset \). We require that there is at least one puncture, that each component of \( \partial \mathcal{S} \) is punctured (that is, intersects \( P \)), and that the Euler characteristic \( \chi(\mathcal{S}) \) of the punctured surface \( \mathcal{S} \) satisfies \( \chi(\mathcal{S}) < d/2 \) where \( d \) is the number of components of \( \partial \mathcal{S} \). Note that each component of \( \partial \mathcal{S} \) is an ideal arc. These topological conditions guarantee the existence of an ideal triangulation \( \lambda \) of the punctured surface \( \mathcal{S} \), namely a triangulation \( \lambda \) of the surface \( \overline{\mathcal{S}} \) whose vertex set is equal to the set of punctures \( P \). See Figure 4 for some examples of ideal triangulations. The ideal triangulation \( \lambda \) consists of \( \epsilon = -3\chi(\mathcal{S}) + 2d \) edges \( E \) and \( \tau = -2\chi(\mathcal{S}) + d \) triangles \( T \).

For simplicity, we always assume that \( \lambda \) does not contain any self-folded triangles. Consequently, each triangle \( T \) of \( \lambda \) has three distinct edges. Such an ideal triangulation \( \lambda \) always exists (except when \( \mathcal{S} \) is a disk with one internal puncture and one puncture on the boundary). Our results should generalize to allow for self-folded triangles, requiring only minor adjustments (one would need to modify Definition 17—compare [BW11, §2.1], which makes use of the Weyl quantum ordering (§3.1.3)—but the main definition, Definition 35, is un-changed).

\[
\Theta_n \overset{\text{def}}{=} \left\{ (a, b, c) \in \mathbb{Z}^3; a, b, c \geq 0, a + b + c = n \right\},
\]

as shown in Figure 5. The interior of the \( n \)-discrete triangle is

\[
\text{int}(\Theta_n) \overset{\text{def}}{=} \left\{ (a, b, c) \in \mathbb{Z}^3; a, b, c > 0, a + b + c = n \right\}.
\]

### Figure 4. Ideal triangulations \((\partial \mathcal{S} = \emptyset)\).
2.3. Dotted ideal triangulations. Let the punctured surface $\mathcal{S}$ be equipped with an ideal triangulation $\lambda$, and let $N = \epsilon(n - 1) + \tau(n - 1)(n - 2)/2$; see §2.1.

The associated dotted ideal triangulation consists of $\lambda$ together with $N$ distinct black dots attached to the edges $E$ and triangles $\mathcal{T}$ of $\lambda$, where there are $n - 1$ edge-dots attached to each edge $E$ and $(n - 1)(n - 2)/2$ triangle-dots attached to each triangle $\mathcal{T}$ (punctures, that is, triangle vertices, are always drawn as white dots). For each triangle $\mathcal{T}$ including its three boundary edges $E_1$, $E_2$, $E_3$, these dots are arranged as the vertices of the discrete $n$-triangle $\Theta_n$ (minus its three corner vertices) overlaid on top of the ideal triangle $\mathcal{T}$; see Figures 6 and 7. We talk about boundary-dots or interior-dots depending on whether the dots are on the boundary of interior of the surface.

Given a triangle $\mathcal{T}$ of $\lambda$, which acquires an orientation from the orientation of $\mathcal{S}$, and given an edge $E$ of $\mathcal{T}$, it makes sense to say that an edge-dot on $E$ is to the left or to the right of another edge-dot on $E$ as viewed from the triangle $\mathcal{T}$; see Figure 6b.

We always assume that we have chosen an ordering for the $N$ dots lying on the dotted ideal triangulation $\lambda$, so we can talk about the $i$-th dot, $i = 1, 2, \ldots, N$.

2.4. Classical polynomial algebra. Let the punctured surface $\mathcal{S}$ be equipped with a dotted ideal triangulation $\lambda$.

Definition 3. The classical polynomial algebra $\mathcal{T}_n^1(\lambda) = \mathbb{C}[X_1^{\pm 1/n}, X_2^{\pm 1/n}, \ldots, X_N^{\pm 1/n}]$ associated to the dotted ideal triangulation $\lambda$ is the commutative algebra of Laurent polynomials generated by formal $n$-roots $X_i^{1/n}$ and their inverses. We think of the generator $X_i^{1/n}$ as
associated to the $i$-th dot lying on $\lambda$. As for dots, see §2.3 we speak of *edge-* and *triangle-*generators as well as *boundary-* and *interior-*generators. Elements \( X_i^{\pm 1} = (X_i^{\pm 1/n})^n \) of \( \mathcal{T}_n^l(\lambda) \) are called *coordinates*. We often indicate edge-coordinates with the letter \( Z \) instead of \( X \).

**Remark 4.** The algebraic coordinates \( X_i \) in the classical polynomial algebra correspond to Fock–Goncharov’s geometric coordinates for the framed \( \text{PSL}_n(C) \)-character variety \( \mathcal{R}_{\text{PSL}_n(C)}(\mathcal{G})_{fr} \); see §1. As a caveat, in the classical geometric setting the Fock–Goncharov coordinates \( X_i \) are associated only to the interior-dots (not to the boundary-dots), while in the quantum algebraic setting there are coordinates \( X_i \) associated to the boundary-dots as well.

In the language of cluster algebras [FZ02], these boundary-variables are also called *frozen variables* [FG06a, FG06b, FWZ16]. From the classical geometric point of view, one can think of the frozen boundary-coordinates as having the potential to become ‘actual’ un-frozen interior-coordinates if the surface-with-boundary \( \mathcal{G} \) were included inside the interior of a larger surface \( \mathcal{G}' \). At the quantum algebraic level, the inclusion of boundary-coordinates is an essential step in order to observe the local quantum properties; see, for instance, Theorem [16].

In the case of \( \text{SL}_2(\mathbb{R}) \) or \( \text{SL}_2(\mathbb{C}) \), the Fock–Goncharov coordinates \( X_i \) coincide with the shear or shear-bend coordinates for Teichmüller space due to Thurston [Thu97]; see
for more details. There is also a geometric interpretation of Fock–Goncharov’s coordinates in the case $n = 3$, where the coordinates parametrize convex projective structures on the surface $\Sigma$; see [FG07b, CTT20].

2.5. **Elementary edge and triangle matrices.** Let $M_n(\mathcal{T}_n^1(\lambda))$ denote the algebra of $n \times n$ matrices with coefficients in the commutative classical polynomial algebra $\mathcal{T}_n^1(\lambda) = \mathbb{C}[X_1^{\pm 1/n}, X_2^{\pm 1/n}, \ldots, X_N^{\pm 1/n}]$; see Definition 9. Let the special linear group $\text{SL}_n(\mathcal{T}_n^1(\lambda))$ be the subset of $M_n(\mathcal{T}_n^1(\lambda))$ consisting of the matrices with determinant equal to 1.

Let $Z = X_i^{\pm 1}$ be an edge-coordinate in the classical polynomial algebra $\mathcal{T}_n^1(\lambda)$. For $j = 1, 2, \ldots, n - 1$ define the $j$-th elementary edge matrix $S_{\text{edge}}^j(Z) \in \text{SL}_n(\mathcal{T}_n^1(\lambda))$ by

$$S_{\text{edge}}^j(Z) \overset{\text{def}}{=} Z^{-j/n} \begin{pmatrix} Z & \cdots & \cdots & Z \\ Z & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ Z & \cdots & \cdots & Z \end{pmatrix} \in \text{SL}_n(\mathcal{T}_n^1(\lambda)) \ (Z \text{ appears } j \text{ times}).$$

Note the normalizing factor $Z^{-j/n}$ multiplying the matrix on the left (or on the right). Similarly, for any triangle-coordinate $X = X_i^{\pm 1}$ in $\mathcal{T}_n^1(\lambda)$ and for any index $j = 1, 2, \ldots, n - 1$ define the $j$-th left elementary triangle matrix $S_{\text{left}}^j(X) \in \text{SL}_n(\mathcal{T}_n^1(\lambda))$ by

$$S_{\text{left}}^j(X) \overset{\text{def}}{=} X^{-(j-1)/n} \begin{pmatrix} X & \cdots & \cdots & X \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ X & \cdots & \cdots & X \end{pmatrix} \in \text{SL}_n(\mathcal{T}_n^1(\lambda)) \ (X \text{ appears } j - 1 \text{ times}),$$

and define the $j$-th right elementary triangle matrix $S_{\text{right}}^j(X) \in \text{SL}_n(\mathcal{T}_n^1(\lambda))$ by

$$S_{\text{right}}^j(X) \overset{\text{def}}{=} X^{(j-1)/n} \begin{pmatrix} 1 & \cdots & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 \end{pmatrix} \in \text{SL}_n(\mathcal{T}_n^1(\lambda)) \ (X \text{ appears } j - 1 \text{ times}).$$

Note that $S_{\text{left}}^1(X)$ and $S_{\text{right}}^1(X)$ do not actually involve the variable $X$, so we will denote these matrices by $S_{\text{left}}^1$ and $S_{\text{right}}^1$, respectively.
Remark 5. In the theory of Fock–Goncharov, these elementary matrices for edges and triangles are called *snake-move matrices*. Each such matrix is the coordinate transformation matrix passing between a pair of compatibly normalized projective coordinate systems associated to a pair of adjacent *snakes*. For a framed local system in \( \mathcal{R}_{\text{PSL}(\mathbb{C})}(\mathcal{G})_{\text{fr}} \) with coordinates \( X_i \), computing the monodromy of the local system around a curve \( \gamma \) amounts to multiplying together a sequence of snake-move matrices along the direction of the curve. For more details, see [FG06b, §9], [GMN14, Appendix A], and [Don20, Chapter 2].

2.6. Local monodromy matrices. Let \( \Sigma \) be a dotted ideal triangle, which we think of as sitting inside a larger dotted ideal triangulation \( \lambda \); see Figure 6. We assign \( n \times n \) matrices with coefficients in the classical polynomial algebra \( \mathcal{I}_n(\lambda) = \mathbb{C}\left[X_1^{\pm 1/n}, X_2^{\pm 1/n}, \ldots, X_N^{\pm 1/n}\right] \) to various ‘short’ oriented arcs lying on the surface \( \mathcal{G} \).

Recall that we think of the \( n \)-discrete triangle \( \Theta_n \) (§2.2) as overlaid on top of the triangle \( \Sigma \), so that there is a one-to-one correspondence between coordinates \( X_i = X_{abc} \) in \( \mathcal{I}_n(\Sigma) \subseteq \mathcal{I}_n(\lambda) \) and vertices \((a, b, c)\) in \( \Theta_n - \{(n, 0, 0), (0, n, 0), (0, 0, n)\} \) (in other words, in the \( n \)-discrete triangle \( \Theta_n \) minus its three corner vertices). Note that \( X_{abc} \) is a triangle-coordinate if and only if \((a, b, c)\) is an interior point \((a, b, c) \in \text{int}(\Theta_n)\), otherwise \( X_{abc} \) is an edge-coordinate.

We will use the following notational convention. Given an arbitrary family \( M_i \) of \( n \times n \) matrices, put

\[
\prod_{i=M}^{N} M_i \overset{\text{def}}{=} M_M M_{M+1} \cdots M_N, \quad \prod_{i=N+1}^{M} M_i \overset{\text{def}}{=} 1 \quad (M \leq N),
\]

\[
\prod_{i=N}^{M} M_i \overset{\text{def}}{=} M_N M_{N-1} \cdots M_M, \quad \prod_{i=M-1}^{N} M_i \overset{\text{def}}{=} 1 \quad (M \leq N).
\]

First, consider a *left-moving arc* \( \gamma \), as shown in Figure 8. We assume \( \gamma \) has no kinks (Figures 3a and 3b). Let \( X = (X_i) \) be a vector consisting of the triangle-coordinates \( X_i \) (Definition 3). Define the associated *left matrix* \( M^{\text{left}}(X) \) in \( \text{SL}_n(\mathcal{I}_n(\lambda)) \) by

\[
M^{\text{left}}(X) \overset{\text{def}}{=} \prod_{i=n-1}^{1} \left( S_i^{\text{left}} \prod_{j=2}^{i} S_j^{\text{left}} \left( X_{(j-1)(n-i)(i-j+1)} \right) \right) \in \text{SL}_n(\mathcal{I}_n(\lambda)),
\]
where the matrix $S_j^\text{left}(X_{abc})$ is the $j$-th left elementary triangle matrix; see §2.5 (The dots colored red in the figure correspond to the coordinates appearing in the expression of the matrix associated to the curve.)

Next, consider a \textit{right-moving arc} $\bar{\gamma}$, as shown in Figure 9. We assume $\bar{\gamma}$ has no kinks. We define the associated \textit{right matrix} $M^\text{right}(X)$ in $\text{SL}_n(\mathcal{T}_n^1(\lambda))$ by

$$M^\text{right}(X) \overset{\text{def}}{=} \prod_{i=n-1}^1 \left( S_1^{\text{right}} \prod_{j=2}^i S_j^{\text{right}} (X_{(i-j+1)(n-i)(j-1)}) \right) \in \text{SL}_n(\mathcal{T}_n^1(\lambda)),$$

where the matrix $S_j^{\text{right}}(X_{abc})$ is the $j$-th right elementary triangle matrix; see §2.5.

Next, consider an \textit{edge-crossing arc} $\bar{\gamma}$, as shown on the left hand or right hand side of Figure 10. Let $Z_j$, $j = 1, 2, \ldots, n - 1$, be the $j$-th edge-coordinate (Definition 3), measured from right to left as seen by the triangle out of which the arc is moving. Let $Z = (Z_j)$ be a vector consisting of the $Z_j$’s. Define the associated \textit{edge matrix} $M^\text{edge}(Z)$ in $\text{SL}_n(\mathcal{T}_n^1(\lambda))$ by

$$M^\text{edge}(Z) \overset{\text{def}}{=} \prod_{j=1}^{n-1} S_j^{\text{edge}}(Z_j) \in \text{SL}_n(\mathcal{T}_n^1(\lambda)),$$

where the matrix $S_j^{\text{edge}}(Z_j)$ is the $j$-th elementary edge matrix; see §2.5. Note that if the orientation of the edge-crossing arc is reversed, then the edge matrix changes by permuting the coordinates $Z_j$ by $Z_j \leftrightarrow Z_{n-j}$; see Figure 10.

Observe that the order in which the elementary matrices $S_j$ are multiplied does not matter in the formula for the edge matrix $M^\text{edge}(Z)$, since they are diagonal, but does matter in the formulas for the triangle matrices $M^\text{left}(X)$ and $M^\text{right}(X)$.

Lastly, define the \textit{clockwise U-turn matrix} $U$ in $\text{SL}_n(\mathbb{C})$ by

$$U \overset{\text{def}}{=} \begin{pmatrix} (-1)^{n-1} & \cdots & \cdots \cr -1 & \ddots & \cdots \cr \cdots & \cdots & -1 \end{pmatrix} \in \text{SL}_n(\mathbb{C}).$$

We associate to a \textit{clockwise U-turn arc} (resp. \textit{counterclockwise U-turn arc}) $\bar{\gamma}$, as shown on the left hand (resp. right hand) side of Figure 11, the U-turn matrix $U$ (resp. transpose $U^T$
of the U-turn matrix). Again, we have assumed that $\gamma$ has no kinks. Note that $U^T = -U$ (resp. $= U$) when $n$ is even (resp. odd).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure8.png}
\caption{Left matrix $(n = 5)$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure9.png}
\caption{Right matrix $(n = 5)$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure10.png}
\caption{Edge matrices $(n = 4)$.}
\end{figure}

2.7. **Definition of the SL$_n$-classical trace polynomials.** Let $\gamma$ be an immersed oriented closed curve in the surface $\mathcal{S}$ such that $\gamma$ is transverse to the ideal triangulation $\lambda$. We want to calculate the classical trace polynomial $\widetilde{\text{Tr}}_\gamma(X_i^{1/n})$ in $\mathcal{T}_n(\lambda) = \mathbb{C}[X_1^{\pm 1/n}, X_2^{\pm 1/n}, \ldots, X_N^{\pm 1/n}]$.
associated to the immersed curve $\gamma$. We say that the polynomial $\widetilde{\text{Tr}}_\gamma(X^{1/n}_i)$ is obtained from a ‘state sum’, ‘local-to-global’, or ‘transfer matrices’ construction.

More precisely, as we travel along the curve $\gamma$ according to its orientation, assume $\gamma$ crosses edges $E_{jk}$ for $k = 1, 2, \ldots, K$ in that order, and assume $\gamma$ crosses triangles $\mathfrak{T}_{ik}$ for $k = 1, 2, \ldots, K$ in that order. As the curve $\gamma$ crosses the edge $E_{jk}$, moving out of the triangle $\mathfrak{T}_{ik-1}$ into the triangle $\mathfrak{T}_{ik}$, this defines an edge-crossing arc $\gamma_{jk}$; see \S 2.6 and Figure 10. Put $Z_{jk} = ((Z_{jk})_{j'})$ and put

$$M_{jk}^\text{edge} = M^\text{edge}(Z_{jk}) \in \text{SL}_n(\lambda),$$

the associated edge matrix, where the $(Z_{jk})_{j'}$’s are the $j' = 1, \ldots, n - 1$ edge-coordinates attached to the edge $E_{jk}$ measured from right to left as seen from $\mathfrak{T}_{ik-1}$. As $\gamma$ traverses the triangle $\mathfrak{T}_{ik}$ between two edges $E_{jk}$ and $E_{jk+1}$, it does one of three things:

- the curve $\gamma$ turns left, ending on $E_{jk+1} \neq E_{jk}$, see Figure 8
- or $\gamma$ turns right, ending on $E_{jk+1} \neq E_{jk}$, see Figure 9
- or $\gamma$ does a U-turn, thereby returning to the same edge $E_{jk+1} = E_{jk}$, see Figure 11.

We also keep track of the following winding information: for the first and second items above, the number $t_k$ of full turns to the right that the curve $\gamma$ makes while traversing the triangle $\mathfrak{T}_{ik}$; and for the third item above, the number $2t_k + 1$ of half turns to the right that the curve $\gamma$ makes before coming back to the same edge $E_{jk}$. Note $t_k \in \mathbb{Z}$. Note that the turning integer $t_k$ associated to the curve $\gamma$ as it traverses the triangle $\mathfrak{T}_{ik}$ will only be relevant when $n$ is even. Let the $(X_{ik})_{i'}$’s be the $i' = 1, \ldots, (n - 1)(n - 2)/2$ triangle-coordinates attached

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure11.png}
\caption{U-turn matrices ($n = 3$).}
\end{figure}
to the triangle $\mathcal{T}_i$, and put $X_{ik} = ((X_{ik})')$; see Figure 8. Let $\gamma'$ be the curve obtained by ‘pulling tight’ the kinks of $\gamma$. (In particular, $\gamma$ and $\gamma'$ are homotopic, but not, in general, regularly homotopic.) Corresponding to the three items above:

- the curve $\gamma'$ turns left, defining a left-moving arc $\gamma'$ and an associated left matrix $M'_{ik} = M_{\text{left}}(X_{ik})$, see §2.6 and Figure 8
- or the curve $\gamma'$ turns right, defining a right-moving arc $\gamma'$ and an associated right matrix $M'_{ik} = M_{\text{right}}(X_{ik})$, see §2.6 and Figure 9
- or the curve $\gamma'$ does a clockwise or counterclockwise U-turn, thereby returning to the same edge $E_{jk+1} = E_{jk}$ and defining a U-turn arc $\gamma'$, see §2.6 and Figure 11.

In the first two cases, where $\gamma'$ is either left- or right-moving, put

$$M_{ik} \overset{\text{def}}{=} (-1)^{(n-1)t_k} M'_{ik} \in \text{SL}_n(\mathcal{T}_n^1(\lambda)),$$

and in the third case, where $\gamma'$ is a U-turn, put

$$M_{ik} \overset{\text{def}}{=} (-1)^{(n-1)t_k} U \in \text{SL}_n(\mathbb{C}),$$

where $U$ is the U-turn matrix defined in §2.6. Note, in the third case, that this is consistent with what was said in §2.6 where, in the case $\gamma$ has no kinks, $\gamma' = \gamma$ is associated to $U$ (resp. $U^T = (-1)^{n-1}U$) if $\gamma$ travels clockwise hence $t_k = 0$ (resp. counterclockwise hence $t_k = -1$).

**Definition 6.** The SL$_n$-classical trace polynomial $\widetilde{\text{Tr}}_{\gamma}(X_1^{1/n}) \in \mathcal{T}_n^1(\lambda)$ associated to the immersed oriented closed curve $\gamma$, transverse to the ideal triangulation $\lambda$, is defined by

$$\widetilde{\text{Tr}}_{\gamma}(X_1^{1/n}) \overset{\text{def}}{=} \text{Tr}(M_{j_1}^{\text{edge}} M_{i_1} M_{j_2}^{\text{edge}} M_{i_2} \cdots M_{j_k}^{\text{edge}} M_{i_k}) \in \mathcal{T}_n^1(\lambda) = \mathbb{C}[X_1^{\pm 1/n}, \ldots, X_N^{\pm 1/n}],$$

where on the right hand side we have taken the usual matrix trace. Note that this is independent of where one starts along the curve $\gamma$, by the conjugation invariance of the trace.
2.8. Relation to Fock–Goncharov theory. In this subsection, we assume (for simplicity) the surface \( \mathcal{S} \) has empty boundary, \( \partial \mathcal{S} = \emptyset \). A complete flag \( E \in \text{Flag}(\mathbb{C}^n) \) is a maximal nested sequence of distinct sub-spaces \( \{0\} = E^{(0)} \subsetneq E^{(1)} \subsetneq E^{(2)} \subsetneq \cdots \subsetneq E^{(n)} = \mathbb{C}^n \). The group \( \text{PSL}_n(\mathbb{C}) \) acts on the set of complete flags by matrix multiplication.

In [FG06b], Fock–Goncharov define the moduli space of framed local systems, also called the \( \mathcal{X} \)-moduli space, whose complex points have been denoted in this paper by \( \mathcal{R}_{\text{PSL}_n(\mathbb{C})}(\mathcal{S})_{\text{fr}} \). A framed local system over \( \mathbb{C} \) is, roughly speaking, a pair \( (\rho, \xi) \) where \( \rho : \pi_1(\mathcal{S}) \to \text{PSL}_n(\mathbb{C}) \) is a group homomorphism and \( \xi \) assigns to each puncture \( p \) a complete flag \( \xi(p) \) invariant under the monodromy of any peripheral curve \( \gamma_p \) around \( p \), that is, \( \rho(\gamma_p)(\xi(p)) = \xi(p) \in \text{Flag}(\mathbb{C}^n) \). Framed local systems are considered up to equivalence under the action of \( \text{PSL}_n(\mathbb{C}) \), which in particular acts on representations \( \rho \) by conjugation.

Fock–Goncharov associate to each ideal triangulation \( \lambda \) of \( \mathcal{S} \) a coordinate chart \( U_\lambda \) for the moduli space \( \mathcal{R}_{\text{PSL}_n(\mathbb{C})}(\mathcal{S})_{\text{fr}} \) parametrized by non-zero complex coordinates \( X_i \). Here, \( N \) is the number of dots associated to the ideal triangulation \( \lambda \); see \S 2.3, 2.4. In particular, choosing \( \lambda \) and \( N \)-many coordinates \( X_i \) determines a representation \( \rho = \rho(X_i) \) up to conjugation. Since \( \rho \) is valued in \( \text{PSL}_n(\mathbb{C}) \), the trace \( \text{Tr}(\rho(\gamma)) \) for any \( \gamma \in \pi_1(\mathcal{S}) \) is well-defined only up to multiplication by a \( n \)-root of unity.

**Theorem 7** ([FG06b], \( \text{SL}_n \)-classical trace polynomials). Fix an ideal triangulation \( \lambda \) of the punctured surface \( \mathcal{S} \). Let \( \rho = \rho(X_i) \) be a representation \( \rho : \pi_1(\mathcal{S}) \to \text{PSL}_n(\mathbb{C}) \) given by Fock–Goncharov coordinates \( X_i \) as above. Moreover, choose arbitrary \( n \)-roots \( X_i^{1/n} \in \mathbb{C} - \{0\} \). Then for each immersed oriented closed curve \( \gamma \) in \( \mathcal{S} \) transverse to the ideal triangulation \( \lambda \), the trace \( \text{Tr}(\rho(\gamma)) \) equals the evaluated classical trace polynomial \( \tilde{T}_\gamma(X_i^{1/n}) \in \mathbb{C} \) (Definition 6) up to multiplication by a \( n \)-root of unity. \( \square \)

3. Quantum matrices for \( \text{SL}_n \)

In this section, we will define quantum versions of the classical polynomial algebra and the classical local monodromy matrices, and relate them to the quantum special linear group...
Throughout, let $q \in \mathbb{C} - \{0\}$ and $\omega = q^{1/n^2}$ be a $n^2$-root of $q$. Technically, also choose $\omega^{1/2}$.

### 3.1. Quantum tori, matrix algebras, and the Weyl quantum ordering.

#### 3.1.1. Quantum tori. For a natural number $N' > 0$, let $P$ (for ‘Poisson’) be an integer $N' \times N'$ anti-symmetric matrix.

**Definition 8.** The quantum torus (with $n$-roots) $T_\omega(P)$ associated to $P$ is the quotient of the free algebra $\mathbb{C}\{X_1^{1/n}, X_1^{-1/n}, \ldots, X_{N'}^{1/n}, X_{N'}^{-1/n}\}$ in the indeterminates $X^{\pm 1/n}_i$ by the two-sided ideal generated by the relations

$$X_i^{m_i/n} X_j^{m_j/n} = \omega^{P_{ij} m_i m_j} X_j^{m_j/n} X_i^{m_i/n} (m_i, m_j \in \mathbb{Z}), \quad X_i^{m/n} X_i^{-m/n} = X_i^{-m/n} X_i^{m/n} = 1 (m \in \mathbb{Z}).$$

Put $X^{\pm 1}_i = (X^{\pm 1/n}_i)^n$. We refer to the $X^{\pm 1/n}_i$ as generators, and the $X_i$ as quantum coordinates, or just coordinates. Define the subset of fractions

$$\mathbb{Z}/n \overset{\text{def}}{=} \{m/n; m \in \mathbb{Z}\} \subseteq \mathbb{Q}.$$

Written in terms of the coordinates $X_i$ and the fractions $r \in \mathbb{Z}/n$, the relations above become

$$X_i^{r_i} X_j^{r_j} = q^{P_{ij} r_i r_j} X_j^{r_j} X_i^{r_i} (r_i, r_j \in \mathbb{Z}/n), \quad X_i^r X_i^{-r} = X_i^{-r} X_i^r = 1 (r \in \mathbb{Z}/n).$$

#### 3.1.2. Matrix algebras.

**Definition 9.** Let $\mathcal{J}$ be a, possibly non-commutative, algebra, and let $n'$ be a positive integer. The matrix algebra with coefficients in $\mathcal{J}$, denoted $M_{n'}(\mathcal{J})$, is the complex vector space of $n' \times n'$ matrices, equipped with the usual multiplicative structure. Specifically, the product $MN$ of two matrices $M$ and $N$ is defined entry-wise by

$$(MN)_{ij} \overset{\text{def}}{=} \sum_{k=1}^{n'} M_{ik} N_{kj} \in \mathcal{J} \quad (1 \leq i, j \leq n').$$
As usual, the entry $M_{ij}$ of a matrix $M$ is the entry in the $i$-th row and $j$-th column. Note that the order of $M_{ik}$ and $N_{kj}$ in the above equation matters since these elements might not commute in $T$.

3.1.3. **Weyl quantum ordering.** If $T$ is a quantum torus (§3.1.1), then there is a linear map

$$[-]: \mathbb{C}\{X_1^{\pm 1/n}, \ldots, X_N^{\pm 1/n}\} \to T,$$

from the free algebra to $T$, called the **Weyl quantum ordering**, defined by the property that a word $X_{i_1}^{r_1}X_{i_2}^{r_2}\cdots X_{i_k}^{r_k}$ for $r_a \in \mathbb{Z}/n$ (note $i_a$ may equal $i_b$ if $a \neq b$) is mapped to

$$[X_{i_1}^{r_1}X_{i_2}^{r_2}\cdots X_{i_k}^{r_k}] \overset{\text{def}}{=} \left(q^{-\frac{1}{2}}\sum_{1 \leq a < b \leq k} P_{i_a i_b} r_a r_b \right) X_{i_1}^{r_1}X_{i_2}^{r_2}\cdots X_{i_k}^{r_k} \in T.$$

Also, the empty word is mapped to 1. Note the Weyl ordering $[-]$ depends on the choice of $\omega^{1/2}$; see the beginning of §3. The Weyl ordering is specially designed to satisfy the symmetry

$$[X_{i_1}^{r_1}\cdots X_{i_k}^{r_k}] = [X_{i_{\sigma(1)}}^{r_{\sigma(1)}}\cdots X_{i_{\sigma(k)}}^{r_{\sigma(k)}}] \in T,$$

for every permutation $\sigma$ of $\{1, \ldots, k\}$. Also, $[X_i^{1/n}X_i^{-1/n}] = 1$. Let

$$[-]: \mathbb{C}[X_1^{\pm 1/n}, \ldots, X_N^{\pm 1/n}] \to T,$$

be the induced linear map from the commutative Laurent polynomial algebra to $T$. This determines a linear map of matrix algebras

$$[-]: M_n'(\mathbb{C}[X_1^{\pm 1/n}, \ldots, X_N^{\pm 1/n}]) \to M_n'(T), \quad [M]_{ij} \overset{\text{def}}{=} [M_{ij}] \in T.$$

3.2. **Fock–Goncharov quantum torus for a triangle.** Let $\Gamma(\Theta_n)$ denote the set of corner vertices $\Gamma(\Theta_n) = \{(n, 0, 0), (0, n, 0), (0, 0, n)\}$ of the discrete triangle $\Theta_n$; see §2.2.

Define a function

$$P: (\Theta_n - \Gamma(\Theta_n)) \times (\Theta_n - \Gamma(\Theta_n)) \to \{-2, -1, 0, 1, 2\},$$
using the *quiver* with vertex set $\Theta_n - \Gamma(\Theta_n)$ illustrated in Figure 12. The function $P$ is defined by sending the ordered tuple $(v_1, v_2)$ of vertices of $\Theta_n - \Gamma(\Theta_n)$ to $2$ (resp. $-2$) if there is a solid arrow pointing from $v_1$ to $v_2$ (resp. $v_2$ to $v_1$), to $1$ (resp. $-1$) if there is a dotted arrow pointing from $v_1$ to $v_2$ (resp. $v_2$ to $v_1$), and to $0$ if there is no arrow connecting $v_1$ and $v_2$. Note that internal arrows are solid, and boundary arrows are dotted. By labeling the vertices of $\Theta_n - \Gamma(\Theta_n)$ by their coordinates $(a, b, c)$ we may think of the function $P$ as an $N \times N$ anti-symmetric matrix $P = (P_{abc,a'b'c'})$ called the Poisson matrix associated to the quiver. Here, $N = 3(n-1) + (n-1)(n-2)/2$; see §2.3.

**Definition 10.** The Fock–Goncharov quantum torus $T_\omega^c(\mathfrak{T})$, also denoted $\mathbb{C}[X_{\pm1/n}^{a_0}, \ldots, X_N^{\pm1/n}]^\omega$, associated to the triangle $\mathfrak{T}$ is defined to be the quantum torus $T_\omega^c(P)$ determined by the $N \times N$ Poisson matrix $P$, with generators $X_i^{\pm1/n} = X_{abc}^{\pm1/n}$ for all $(a, b, c) \in \Theta_n - \Gamma(\Theta_n)$. Note that when $q = \omega = 1$ this recovers the classical polynomial algebra $T_1^c(\mathfrak{T})$ for $\mathfrak{T}$; see §2.4.

As a notational convention, for $j = 1, 2, \ldots, n-1$ we write $Z_j^{\pm1/n}$ (resp. $Z_j'^{\pm1/n}$ and $Z_j''^{\pm1/n}$) in place of $X_j^{\pm1/n}_{0(n-j)}$ (resp. $X_j^{\pm1/n}_{(n-j)0}$ and $X_{0j}^{\pm1/n}_{(n-j)}$); see Figure 13. So, triangle-coordinates will be denoted $X_i = X_{abc}$ for $(a, b, c) \in \text{Int}(\Theta_n)$ while edge-coordinates will be denoted $Z_j, Z_j', Z_j''$.

**Remark 11.** Intuitively speaking, we think of the $Z$-coordinates as quantizations of ‘half’ of their corresponding classical edge-coordinates. This is because the ‘other half’ of each coordinate lives in an adjacent triangle. When viewed inside an ideal triangulation $\lambda$, an edge $E$ of $\lambda$ ‘splits’ this classical edge-coordinate into its two ‘quantum halves’. Compare Figure 16.

### 3.3. Quantum left and right matrices.

Although the general (global) definition of the $\text{SL}_n$-quantum trace polynomials (§5.3) is somewhat technical (requiring that one keep track of the ordering of the non-commuting quantum torus variables), the extension of the local monodromy matrices (§2.6) to the quantum setting is more straightforward, just using the Weyl quantum ordering (§3.1.3) to symmetrize the variables.

### 3.3.1. Weyl quantum ordering for the Fock–Goncharov quantum torus.

Let $\mathfrak{T} = T_\omega^c(\mathfrak{T})$ be the Fock–Goncharov quantum torus (§3.2). Then the Weyl ordering $[-]$ of §3.1.3 gives a
map

\[ [-] : M_n(\mathcal{T}_n^1(\Sigma)) \to M_\omega(\mathcal{T}_n^\omega(\Sigma)), \]

where we have used the identification \( \mathcal{T}_n^1(\Sigma) \cong \mathbb{C}[X_1^{\pm 1/n}, X_2^{\pm 1/n}, \ldots, X_N^{\pm 1/n}] \) discussed in §3.2.

\[ \theta_n - \Gamma(\theta_n) \]

Figure 12. Quiver defining the Fock–Goncharov quantum torus \((n = 5)\).

3.3.2. Quantum left and right matrices. Let \( \Sigma \) be a triangle. An extended left-moving arc \( \gamma \) is similar to a left-moving arc, from §2.6, except that it extends all the way to the two distinct edges of the triangle \( \Sigma \); see Figure 13. We think of an extended left-moving arc \( \gamma \) as the concatenation of ‘half’ of an edge-crossing arc \( \gamma_1^{1/2} \) together with a left-moving arc \( \gamma_2 \) together with another half of an edge-crossing arc \( \gamma_3^{1/2} \), as indicated in Figure 13; compare Remark 11. We refer to these halves of edge-crossing arcs as half-edge-crossing arcs. Similarly, we define extended right-moving arcs \( \gamma \).

Defined as in §2.6 are left matrices \( M^\text{left}(X) \), right matrices \( M^\text{right}(X) \), and edge matrices \( M^\text{edge}(Z) \) in \( \text{SL}_n(\mathcal{T}_n^1(\Sigma)) \) associated to non-extended left-moving arcs (Figure 8), non-extended right-moving arcs (Figure 9), and half-edge-crossing arcs, respectively.

Definition 12. Put vectors \( X = (X_i) \), \( Z = (Z_j) \), \( Z' = (Z'_j) \), and \( Z'' = (Z''_j) \) as in Figure 13. To an extended left-moving arc \( \gamma \), as in Figure 13, we associate a quantum left matrix \( L^\omega \) in \( M_\omega(\mathcal{T}_n^\omega(\Sigma)) \) by the formula

\[ L^\omega \overset{\text{def}}{=} L^\omega(Z, X, Z') \overset{\text{def}}{=} [M^\text{edge}(Z)M^\text{left}(X)M^\text{edge}(Z')] \in M_\omega(\mathcal{T}_n^\omega(\Sigma)), \]

where we have applied the Weyl quantum ordering \([\cdot] \) discussed in §3.3.1 to the product \( M^\text{edge}(Z)M^\text{left}(X)M^\text{edge}(Z') \) of classical matrices in \( M_n(\mathcal{T}_n^1(\Sigma)) \) (actually, in \( \text{SL}_n(\mathcal{T}_n^1(\Sigma)) \)). This just means that we apply the Weyl ordering to each entry of the classical matrix.
Similarly, to an extended right-moving arc $\gamma$, as in Figure 13, we associate a quantum right matrix $R^\omega$ in $M_n(\mathcal{T}_n^\omega(\mathcal{T}))$ by the formula

$$R^\omega \overset{\text{def}}{=} R^\omega(Z, X, Z'') \overset{\text{def}}{=} [M^\text{edge}(Z)M^\text{right}(X)M^\text{edge}(Z'')] \in M_n(\mathcal{T}_n^\omega(\mathcal{T))).$$

Figure 13. Quantum left and right matrices ($n = 5$).

3.4. Quantum $SL_n$ and its points: first result. (For a more theoretical discussion about quantum $SL_n$, see Appendix A) Let $\mathcal{T}$ be a, possibly non-commutative, algebra.

**Definition 13.** A $2 \times 2$ matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $M_2(\mathcal{T})$ is a $\mathcal{T}$-point of the quantum matrix algebra $M_2^q$, denoted $M \in M_2^q(\mathcal{T}) \subseteq M_2(\mathcal{T})$, if

\begin{align*}
(*): \quad ba &= qab, \quad dc = qcd, \quad ca = qac, \quad db = qbd, \quad bc = cb, \quad da - ad = (q - q^{-1})bc \in \mathcal{T}.
\end{align*}

A matrix $M \in M_2(\mathcal{T})$ is a $\mathcal{T}$-point of the quantum special linear group $SL_2^q$, denoted $M \in SL_2^q(\mathcal{T}) \subseteq M_2^q(\mathcal{T}) \subseteq M_2(\mathcal{T})$, if $M \in M_2^q(\mathcal{T})$ and the quantum determinant

$$\text{Det}^q(M) \overset{\text{def}}{=} ad - q^{-1}bc = 1 \in \mathcal{T}.$$

These notions are also defined for $n \times n$ matrices, as follows.

**Definition 14.** A matrix $M \in M_n(\mathcal{T})$ is a $\mathcal{T}$-point of the quantum matrix algebra $M_n^q$, denoted $M \in M_n^q(\mathcal{T}) \subseteq M_n(\mathcal{T})$, if every $2 \times 2$ submatrix $\begin{pmatrix} M_{ik} & M_{im} \\ M_{jk} & M_{jm} \end{pmatrix}$ of $M$ is a $\mathcal{T}$-point of $M_2^q$. 
That is,

\[ M_{im} M_{ik} = q M_{ik} M_{im}, \quad M_{jm} M_{jk} = q M_{jk} M_{jm}, \quad M_{mj} M_{ik} = q M_{ik} M_{mj}, \quad M_{jm} M_{im} = q M_{im} M_{jm}, \]

\[ M_{im} M_{jk} = M_{jk} M_{im}, \quad M_{jm} M_{ik} - M_{ik} M_{jm} = (q - q^{-1}) M_{im} M_{jk}, \]

for all \( i < j \) and \( k < m \), where \( 1 \leq i, j, k, m \leq n \). A matrix \( M \in M_n(\mathcal{J}) \) is a \( \mathcal{T} \)-point of the quantum special linear group \( SL^q_n \), denoted \( M \in SL^q_n(\mathcal{J}) \subseteq M_n(\mathcal{J}) \), if both \( M \in M^q_n(\mathcal{J}) \) and \( \text{Det}^q(M) = 1 \). Here, the quantum determinant \( \text{Det}^q(M) \in \mathcal{T} \) of a matrix \( M \in M_n(\mathcal{J}) \) is

\[ \text{Det}^q(M) \overset{\text{def}}{=} \sum_{\sigma \in S_n} (-q^{-1})^{\ell(\sigma)} M_{1\sigma(1)} M_{2\sigma(2)} \cdots M_{n\sigma(n)} \in \mathcal{T}, \]

where the length \( \ell(\sigma) \) of the permutation \( \sigma \in S_n \) is the minimum number of factors appearing in a decomposition of \( \sigma \) as a product of adjacent transpositions \((i, i + 1)\); see, for example, [BG02 Chapter I.2].

**Remark 15.** Note that the definitions satisfy the property that if a \( \mathcal{T} \)-point \( M \in M^q_n(\mathcal{J}) \subseteq M_n(\mathcal{J}) \) is a triangular matrix, then the diagonal entries \( M_{ii} \in \mathcal{T} \) commute, and \( \text{Det}^q(M) = \prod_i M_{ii} \in \mathcal{T} \).

Note also that the subsets \( M^q_n(\mathcal{J}) \subseteq M_n(\mathcal{J}) \) and \( SL^q_n(\mathcal{J}) \subseteq M_n(\mathcal{J}) \) are generally not closed under matrix multiplication.

Take \( \mathcal{T} = \mathcal{T}_n^\omega(\mathfrak{T}) \) to be the Fock–Goncharov quantum torus for the triangle \( \mathfrak{T} \), as defined in §3.2 Let \( L^\omega \) and \( R^\omega \) in \( M_n(\mathcal{T}_n^\omega(\mathfrak{T})) \) be the quantum left and right matrices, respectively, as defined in Definition [12]. In a companion paper, we prove:

**Theorem 16 ([Dou21], SL\(_n\)-quantum matrices).** The quantum left and right matrices,

\[ L^\omega = L^\omega(Z, X, Z') \quad \text{and} \quad R^\omega = R^\omega(Z, X, Z'') \in M_n(\mathcal{T}_n^\omega(\mathfrak{T})), \]

are \( \mathcal{T}_n^\omega(\mathfrak{T}) \)-points of the quantum special linear group \( SL^q_n \). That is, \( L^\omega, R^\omega \in SL^q_n(\mathcal{T}_n^\omega(\mathfrak{T})) \subseteq M_n(\mathcal{T}_n^\omega(\mathfrak{T})) \). □
The proof uses a quantum version of Fock–Goncharov snakes; see Remark 5. See also Remark 37 for recent related work.

3.5. Examples.

3.5.1. SL₃ example. Consider the case \( n = 3 \); see Figure 14. On the right hand side, we show the quiver defining the commutation relations in the quantum torus \( \mathcal{T}_3(\mathfrak{T}) \), recalling Figure 12 and the definitions of §3.1.1 and 3.2. For instance, the following are some sample commutation relations:

\[
XZ' = q^2Z'X, \quad XW' = q^{-2}W'X, \quad ZW = qWZ, \quad ZW' = q^2W'Z.
\]

Then, the quantum left and right matrices are computed as

\[
L^\omega = \begin{pmatrix} \left[ W^{-\frac{1}{3}} Z^{-\frac{2}{3}} \left( \frac{WZ}{Z_1} \right) \left( \frac{X^{-\frac{1}{3}} (X_1 X^{-\frac{1}{3}})}{1 1} \right) \left( \frac{X^{-\frac{1}{3}} (X_1 X^{-\frac{1}{3}})}{1 1} \right) \right] Z^{-\frac{1}{3}} W^{-\frac{2}{3}} \left( \frac{ZW}{W_1} \right) \\
0 \\
0 \\
0 \\
\end{pmatrix},
\]

and

\[
R^\omega = \begin{pmatrix} \left[ W'^{-\frac{1}{3}} Z'^{-\frac{2}{3}} \left( \frac{W'Z'}{Z'_1} \right) \left( \frac{X'^{-\frac{1}{3}} (X'_1 X'^{-\frac{1}{3}})}{1 1} \right) \left( \frac{X'^{-\frac{1}{3}} (X'_1 X'^{-\frac{1}{3}})}{1 1} \right) \right] Z'^{-\frac{1}{3}} W'^{-\frac{2}{3}} \left( \frac{ZW}{W'_1} \right) \\
0 \\
0 \\
0 \\
\end{pmatrix},
\]

Theorem 16 says that these two matrices are elements of \( \text{SL}_3(\mathcal{T}_3^\omega(\mathfrak{T})) \). For instance, the entries \( a, b, c, d \) of the \( 2 \times 2 \) sub-matrix of \( L^\omega \),

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} L^{12}_{12} & L^{13}_{12} \\ L^{12}_{22} & L^{13}_{22} \end{pmatrix} = \begin{pmatrix} \left[ W^{\frac{3}{4}} Z^{\frac{1}{4}} X^{\frac{1}{4}} X^{-\frac{1}{4}} W^{-\frac{1}{4}} \right] & \left[ W^{\frac{3}{4}} Z^{\frac{1}{4}} X^{\frac{1}{4}} X^{-\frac{1}{4}} Z^{-\frac{1}{4}} W^{-\frac{1}{4}} \right] \\ \left[ W^{-\frac{1}{4}} Z^{\frac{1}{4}} X^{\frac{1}{4}} X^{-\frac{1}{4}} W^{-\frac{1}{4}} \right] & \left[ W^{-\frac{1}{4}} Z^{\frac{1}{4}} X^{\frac{1}{4}} X^{-\frac{1}{4}} Z^{-\frac{1}{4}} W^{-\frac{1}{4}} \right] \end{pmatrix},
\]

satisfy Equation (6). For a computer verification of this, see Appendix A. We also demonstrate in the appendix that Equation (6) is satisfied by the entries \( a, b, c, d \) of the \( 2 \times 2 \)
sub-matrix of $R$, 

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} R_{21}^{z} & R_{22}^{z} \\ R_{31}^{z} & R_{32}^{z} \end{pmatrix} = \begin{pmatrix} \left[ w'^{-\frac{1}{4}} z'^{-\frac{1}{4}} x^{\frac{1}{4}} z^{\frac{1}{4}} w^{\frac{1}{4}} \right] & \left[ w'^{-\frac{1}{4}} z'^{-\frac{1}{4}} x^{\frac{1}{4}} z^{\frac{1}{4}} w^{\frac{1}{4}} \right] \\ \left[ w'^{-\frac{1}{4}} z'^{-\frac{1}{4}} x^{\frac{1}{4}} z^{\frac{1}{4}} w^{\frac{1}{4}} \right] & \left[ w'^{-\frac{1}{4}} z'^{-\frac{1}{4}} x^{\frac{1}{4}} z^{\frac{1}{4}} w^{\frac{1}{4}} \right] + \left[ w'^{-\frac{1}{4}} z'^{-\frac{1}{4}} x^{\frac{1}{4}} z^{\frac{1}{4}} w^{\frac{1}{4}} \right] \end{pmatrix}.
\]

![Figure 14. Example in the case $n = 3$.](image)

3.5.2. SL4 example. Consider the case $n = 4$; see Figure 15. On the right hand side, we show the quiver defining the commutation relations in the quantum torus $T_{4}^{\omega}(\mathfrak{g})$, recalling Figure 12 and the definitions of §3.1.1 and 3.2. For instance, the following are some sample commutation relations:

\[
X_3 Z''_2 = q^2 X_3 Z''_2, \quad X_3 X_1 = q^{-2} X_1 X_3, \quad Z_3 Z_2 = q Z_2 Z_3, \quad Z_3 Z'_3 = q^2 Z'_3 Z_3.
\]

Then, the quantum left and right matrices are computed as

\[
L^{\omega} = \begin{bmatrix}
\left( X_3 \right)^{-\frac{1}{4}} Z_1^{-\frac{1}{4}} Z_2^{-\frac{3}{4}} Z_3^{-\frac{3}{4}} \\
\left( Z_1 Z_2 Z_3 \right)^{-\frac{1}{4}} \left( X_1 \right)^{\frac{1}{4}} \left( X_2 \right)^{\frac{3}{4}}
\end{bmatrix}
\]

and

\[
R^{\omega} = \begin{bmatrix}
\left( X_3 \right)^{\frac{1}{4}} Z_1^{\frac{1}{4}} Z_2^{\frac{3}{4}} Z_3^{\frac{3}{4}} \\
\left( X_1 \right)^{-\frac{1}{4}} \left( X_2 \right)^{-\frac{3}{4}} \left( X_3 \right)^{-\frac{3}{4}}
\end{bmatrix}
\]
Theorem 16 says that these two matrices are elements of $\text{SL}_4^2(\mathcal{T}_\lambda^1(\Sigma))$. For instance, the entries $a, b, c, d$ of the $2 \times 2$ sub-matrix (arranged as a $4 \times 1$ matrix) of $L^\omega$, \[
abla \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} L^\omega_{L_1} \\ L^\omega_{L_2} \\ L^\omega_{R_1} \\ L^\omega_{R_2} \end{pmatrix} = \]
\[
\begin{pmatrix} [z_3^{1/4} z_2^{1/4} z_1^{1/4} X_1^{3/4} X_2^{1/4} X_3^{1/4} X_4^{1/4}] + [z_3^{3/4} z_2^{1/4} z_1^{1/4} X_1^{3/4} X_2^{1/4} X_3^{1/4} X_4^{1/4}] + [z_3^{1/4} z_2^{3/4} z_1^{1/4} X_1^{3/4} X_2^{1/4} X_3^{1/4} X_4^{1/4}] + [z_3^{3/4} z_2^{1/4} z_1^{1/4} X_1^{3/4} X_2^{1/4} X_3^{1/4} X_4^{1/4}] \\
[z_3^{1/4} z_2^{3/4} z_1^{1/4} X_1^{3/4} X_2^{1/4} X_3^{1/4} X_4^{1/4}] + [z_3^{3/4} z_2^{1/4} z_1^{1/4} X_1^{3/4} X_2^{1/4} X_3^{1/4} X_4^{1/4}] + [z_3^{1/4} z_2^{3/4} z_1^{1/4} X_1^{3/4} X_2^{1/4} X_3^{1/4} X_4^{1/4}] + [z_3^{3/4} z_2^{1/4} z_1^{1/4} X_1^{3/4} X_2^{1/4} X_3^{1/4} X_4^{1/4}] \end{pmatrix}, \]

satisfy Equation (\ast). For a computer verification of this, see Appendix B. We also demonstrate in the appendix that Equation (\ast) is satisfied by the entries $a, b, c, d$ of the $2 \times 2$ sub-matrix (arranged as a $4 \times 1$ matrix) of $R^\omega$, \[
\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} R^\omega_{L_1} \\ R^\omega_{L_2} \\ R^\omega_{R_1} \\ R^\omega_{R_2} \end{pmatrix} = \]
\[
\begin{pmatrix} [z_3^{1/4} z_2^{1/4} z_1^{1/4} X_1^{3/4} X_2^{1/4} X_3^{1/4} X_4^{1/4}] + [z_3^{3/4} z_2^{1/4} z_1^{1/4} X_1^{3/4} X_2^{1/4} X_3^{1/4} X_4^{1/4}] + [z_3^{1/4} z_2^{3/4} z_1^{1/4} X_1^{3/4} X_2^{1/4} X_3^{1/4} X_4^{1/4}] + [z_3^{3/4} z_2^{1/4} z_1^{1/4} X_1^{3/4} X_2^{1/4} X_3^{1/4} X_4^{1/4}] \\
[z_3^{1/4} z_2^{3/4} z_1^{1/4} X_1^{3/4} X_2^{1/4} X_3^{1/4} X_4^{1/4}] + [z_3^{3/4} z_2^{1/4} z_1^{1/4} X_1^{3/4} X_2^{1/4} X_3^{1/4} X_4^{1/4}] + [z_3^{1/4} z_2^{3/4} z_1^{1/4} X_1^{3/4} X_2^{1/4} X_3^{1/4} X_4^{1/4}] + [z_3^{3/4} z_2^{1/4} z_1^{1/4} X_1^{3/4} X_2^{1/4} X_3^{1/4} X_4^{1/4}] \end{pmatrix}. \]

![Quantum left and right matrices](A)  ![Fock–Goncharov quiver](B) 

**Figure 15.** Example in the case $n = 4$.

3.6. **Quantum tori for surfaces.** For a dotted ideal triangulation $\lambda$ of $\mathcal{S}$, in §2.4 we defined the classical polynomial algebra $\mathcal{T}_n^1(\lambda) = \mathbb{C}[X_1^{1/n}, X_2^{1/n}, \ldots, X_N^{1/n}]$ where there is one generator $X_i^{1/n}$ associated to every dot on $\lambda$. In the case where $\mathcal{S} = \mathcal{F}$ is an ideal triangle, in §3.2 we deformed the classical polynomial algebra $\mathcal{T}_n^1(\mathcal{F})$ to a quantum torus $\mathcal{T}_n^\omega(\mathcal{F})$. We now generalize the quantum torus $\mathcal{T}_n^\omega(\mathcal{F})$ to a quantum torus $\mathcal{T}_n^\omega(\lambda)$ associated to the triangulated surface $(\mathcal{S}, \lambda)$ which deforms the classical polynomial algebra $\mathcal{T}_n^1(\lambda)$.

For each dotted triangle $\mathcal{F}$ of $\lambda$, associate a copy $\mathcal{F}\hat{\mathcal{F}}$ of $\mathcal{F}$, which is also a dotted triangle, such as that shown in Figure 6b. Note that the boundary $\partial \mathcal{F}\hat{\mathcal{F}}$ consists of three ideal edges.
The dotted ideal triangulation $\lambda$ can be reconstructed from the individual triangles $\widehat{\mathcal{T}}$ by supplying additional gluing data. To each dotted triangle $\widehat{\mathcal{T}}$ associate the Fock–Goncharov quantum torus $T^\omega_n(\mathcal{T})$ of the triangle $\widehat{\mathcal{T}}$, whose coordinates we will denote by $\widehat{X}$. Recall that a generator $\widehat{X}_{abc}^{\pm 1/n}$ of the quantum torus $T^\omega_n(\mathcal{T})$ is either a triangle-generator or an edge-generator. If $\widehat{X}_{abc}^{\pm 1/n}$ is an edge-generator, then there are two cases:

- the corresponding generator $X_i^{\pm 1/n}$ in the classical polynomial algebra $T^1_n(\lambda)$ for the glued surface $\mathcal{G}(\lambda)$ is a boundary-generator;
- the corresponding generator $X_i^{\pm 1/n}$ in $T^1_n(\lambda)$ is an interior-generator.

In the second case, the corresponding interior-generator $X_i^{\pm 1/n}$ in $T^1_n(\lambda)$ lies on an internal edge $E$ of the ideal triangulation $\lambda$. So, there exists a triangle $\mathcal{T}'$ adjacent to $\mathcal{T}$ along the edge $E$. Moreover, there exists a unique edge-generator $\widehat{X}_{a'b'c'}^{t\pm 1/n}$ in the quantum torus $T^\omega_n(\mathcal{T}')$ for the triangle $\widehat{\mathcal{T}}'$ that also corresponds to the interior-generator $X_i^{\pm 1/n}$ in $T^1_n(\lambda)$ lying on the internal edge $E$. Therefore, we may say that the two quantum generators $\widehat{X}_{abc}^{\pm 1/n}$ in $T^\omega_n(\mathcal{T})$ and $\widehat{X}_{a'b'c'}^{t\pm 1/n}$ in $T^\omega_n(\mathcal{T}')$ correspond to one another; see Figure 16.

**Definition 17.** The **Fock–Goncharov quantum torus** $T^\omega_n(\lambda)$ associated to the surface $\mathcal{G}$ equipped with the dotted ideal triangulation $\lambda$ is the sub-algebra,

$$T^\omega_n(\lambda) \subseteq \bigotimes_{\text{copies } \widehat{\mathcal{T}} \text{ of triangles } \mathcal{T} \text{ of } \lambda} T^\omega_n(\mathcal{T}),$$

of the tensor product of the Fock–Goncharov quantum tori $T^\omega_n(\mathcal{T})$ associated to the copies $\widehat{\mathcal{T}}$ of the dotted triangles $\mathcal{T}$ of the ideal triangulation $\lambda$, generated:

- by triangle-generators $\widehat{X}_{abc}^{\pm 1/n}$ in $T^\omega_n(\mathcal{T})$;
- by tensor products $\widehat{X}_{abc}^{\pm 1/n} \otimes \widehat{X}_{a'b'c'}^{t\pm 1/n}$ in $T^\omega_n(\mathcal{T}) \otimes T^\omega_n(\mathcal{T}')$ of corresponding edge-generators associated to a common internal edge $E$ lying between two triangles $\mathcal{T}$ and $\mathcal{T}'$ in $\lambda$;
- and by edge-generators $\widehat{X}_{abc}^{\pm 1/n}$ in $T^\omega_n(\mathcal{T})$ associated to boundary edges $E \subseteq \partial \mathcal{G}$ of $\lambda$.

In particular, when $q = \omega = 1$, the Fock–Goncharov quantum torus $T^1_n(\lambda)$ is naturally isomorphic to the classical polynomial algebra $\mathbb{C}[X_1^{\pm 1/n}, X_2^{\pm 1/n}, \ldots, X_N^{\pm 1/n}]$ (as indicated by
the notation). (Going forward, we will omit the ‘hat’ symbol in the notation, naturally identifying triangles $\mathfrak{T}$ with $\hat{\mathfrak{T}}$.)

**Remark 18.** An important difference between the local quantum tori $\mathcal{T}_n^\omega(\mathfrak{T})$ for the triangles $\mathfrak{T}$ and the global quantum torus $\mathcal{T}_n^\omega(\lambda)$ for the triangulated surface $(\mathcal{G}, \lambda)$ is that two edge-generators $X_{abc}^{\pm1/n}$ and $X_{ABC}^{\pm1/n}$ in $\mathcal{T}_n^\omega(\mathfrak{T})$ lying on the same boundary edge of $\mathfrak{T}$ may not commute, rather may $q$-commute, while the corresponding interior-generators $X_{abc}^{\pm1/n} \otimes X_{a'b'c'}^{\pm1/n}$ and $X_{ABC}^{\pm1/n} \otimes X_{A'B'C'}^{\pm1/n}$ in $\mathcal{T}_n^\omega(\lambda)$ always commute. This is because the orientations of the two triangles’ $\mathfrak{T}$ and $\mathfrak{T}'$ quivers go against each other (Figures 12, 16). Intuitively, the local $q$-commutation relations on the boundary are created upon ‘splitting the edge-coordinates in half’ at the quantum level (Remarks \[4][11\]). This phenomenon does not occur for SL$_2$ because there each edge carries only one coordinate.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig16}
\caption{Interior-generators as tensor products of local edge-generators, shown in the case $n = 3$.}
\end{figure}

4. Main theorem: quantum trace polynomials for SL$_3$

4.1. **Framed oriented links in thickened surfaces.** So far, we have been working in the 2-dimensional setting of the punctured surface $\mathcal{G}$. We now turn to the 3-dimensional setting of the thickened surface $\mathcal{G} \times (0, 1)$. We will follow [BW11 §3.1], the only difference being that we consider oriented links.

**Definition 19.** A **framed oriented link** $K$ in the thickened surface $\mathcal{G} \times (0, 1)$ is a compact oriented one-dimensional manifold, possibly-with-boundary, $K \subseteq \mathcal{G} \times (0, 1)$ that is embedded in $\mathcal{G} \times (0, 1)$ and is equipped with a framing (see below), satisfying the following properties:
• we have $\partial K = K \cap ((\partial \mathcal{S}) \times (0, 1))$;

• the framing at a boundary point of $K$ is vertical, meaning parallel to the $(0, 1)$ axis and pointing in the $1$ direction (or, in pictures, toward the eye of the reader);

• for each boundary component $k$ of $\mathcal{S}$, the finitely many points $(\partial K) \cap (k \times (0, 1))$ have distinct heights, meaning that the coordinates with respect to $(0, 1)$ are distinct.

Here, by a framing, we mean the choice of a smooth assignment along the link $K$ of unit vectors in the tangent spaces of $\mathcal{S} \times (0, 1)$ such that this vector field on $K$ is everywhere orthogonal to $K$. A framed oriented knot $K$ is a closed framed oriented link (namely, a framed oriented link with empty boundary $\partial K = \emptyset$) with one connected component. Two framed oriented links $K$ and $K'$ are isotopic if $K$ can be smoothly deformed to $K'$ through the class of framed oriented links. By possibly introducing kinks (Figures 3a and 3b), one can always isotope a framed link so that it has blackboard framing, meaning constant vertical framing in the $1$ direction (with respect to the $(0, 1)$ coordinate).

**Remark 20.** We display links in figures by their diagrams, namely their projections onto the surface $\mathcal{S} \cong \mathcal{S} \times \{1/2\}$ equipped with over/under crossing information. By convention, all link diagrams represent blackboard-framed links.

Instead of using the picture conventions of [BWT1 §3.5], in our diagrams we will indicate explicitly which points lying on a single boundary component $k \times (0, 1)$ of $\mathcal{S} \times (0, 1)$ are higher or lower with respect to the $(0, 1)$ direction. (Note, importantly, that two points of $\partial K$ on a single boundary component $k \times (0, 1)$ cannot exchange heights during an isotopy of the link.)

One can think of a framed link $K$ as a ‘ribbon’, namely an oriented annulus (that is, oriented as a surface, not to be confused with link orientations) embedded in $\mathcal{S} \times (0, 1)$ where the framing is perpendicular to the annulus and determined by the orientation.

4.2. Stated links.
Definition 21. A \((n-)\) \textit{stated framed oriented link} \((K, s)\) is a framed oriented link \(K\) equipped with a function
\[
s : \partial K \to \{1, 2, \ldots, n\},
\]
called the \textit{state}, assigning to each element of the boundary of the link a \textit{state-number} in \(\{1, 2, \ldots, n\}\) (we often confuse ‘state’ with these state-numbers). Note that a stated closed link is the same thing as a closed link. As for links, there is the corresponding notion of \textit{isotopy} of stated links.

Let \((K, s)\) be a stated framed oriented link in a triangulated surface \((\mathcal{S}, \lambda)\) obtained by gluing together two triangulated surfaces \((\mathcal{S}_1, \lambda_1)\) and \((\mathcal{S}_2, \lambda_2)\) along edges of the triangulations. Let \(K_1\) and \(K_2\) be the associated links in \(\mathcal{S}_1\) and \(\mathcal{S}_2\). We say states \(s_1\) and \(s_2\) for stated links \((K_1, s_1)\) and \((K_2, s_2)\) such that \(s_1\) and \(s_2\) agree with \(s\) on \(\partial K\) are \textit{compatible} if their values agree on the common boundaries of \(K_1\) and \(K_2\) (resulting from cutting \(K\)).

4.3. Main result. In this subsection, we restrict to the case \(n = 3\). Let the surface \(\mathcal{S}\) be equipped with a dotted ideal triangulation \(\lambda\). Recall the Fock–Goncharov quantum torus \(\mathcal{T}_3^\omega(\lambda) \subseteq \bigotimes_{\text{triangles } \mathcal{I}} \mathcal{T}_3^\omega(\mathcal{I})\) associated to this data; see Definition 17.

Note that if \(K \subseteq \mathcal{S} \times (0, 1)\) is a blackboard-framed oriented knot (meaning, in particular, that it is closed), and if \(\pi : \mathcal{S} \times (0, 1) \to \mathcal{S} \times \{1/2\} \cong \mathcal{S}\) is the natural projection, then, possibly after an arbitrarily small perturbation of the knot \(K\), we have that \(\gamma = \pi(K)\) is an immersed oriented closed curve in \(\mathcal{S}\), so we may consider the classical trace polynomial \(\overline{\text{Tr}}_\gamma(X_1^{1/3})\) in \(\mathcal{T}_3^1(\lambda) = \mathbb{C}[X_1^{\pm 1/3}, X_2^{\pm 1/3}, \ldots, X_N^{\pm 1/3}]\) associated to \(\gamma\); see Definition 6.

We now give a more detailed version of Theorem 2 from §1. Technically, our solution involves choosing a square root \(\omega^{1/2}\) of the parameter \(\omega\). Proofs will be given in §5, 6.

Theorem 22 (SL\(_3\)-quantum trace polynomials). Let \(q \in \mathbb{C} - \{0\}\) be a non-zero complex number, and let \(\omega = q^{1/3^2}\) be a 3\(^2\)-root of \(q\); choose also \(\omega^{1/2}\). There is a function
\[
\text{Tr}_\lambda^\omega : \{\text{stated framed oriented links } (K, s) \text{ in } \mathcal{S} \times (0, 1)\} \to \mathcal{T}_3^\omega(\lambda),
\]
satisfying the following properties:
the element $\text{Tr}_\lambda^\omega(K, s) \in \mathfrak{T}_3^\omega(\lambda)$ is invariant under isotopy of stated framed oriented links;

(B) the $\text{SL}_3$-HOMFLYPT skein relation (Figure 1 with $n = 3$) holds;

(C) the $\text{SL}_3$-quantum unknot and framing relations (Figures 2 and 3 with $n = 3$) hold.

**Complement 23.** Moreover, this invariant satisfies the following additional properties.

- **(Classical Trace Property)** Let $q = \omega = \omega^{1/2} = 1$ and let $K$ be a closed blackboard-framed oriented knot. Then,

  $$\text{Tr}_\lambda^1(K) = \text{Tr}_\gamma(X_1^{1/3}) \in \mathfrak{T}_3^1(\lambda),$$

  where $\gamma$ is the immersed oriented closed curve obtained by projecting $K$ to $\mathcal{G}$.

- **(Multiplication Property)** Let $(K, s) = (K_1, s_1) \cup (K_2, s_2) \cup \cdots \cup (K_\ell, s_\ell)$ be a stated framed oriented link, written as a disjoint union of links $K_j$. Assume in addition that $K_{j-1}$ lies entirely below $K_j$ in $\mathcal{G} \times (0, 1)$, with respect to the height coordinate. Then,

  $$\text{Tr}_\lambda^\omega(K, s) = \text{Tr}_\lambda^\omega(K_1, s_1) \text{Tr}_\lambda^\omega(K_2, s_2) \cdots \text{Tr}_\lambda^\omega(K_\ell, s_\ell) \in \mathfrak{T}_3^\omega(\lambda).$$

  Note that the order of multiplication matters, since $\mathfrak{T}_3^\omega(\lambda)$ is non-commutative.

- **(State Sum Property)** Let $(K, s)$ be a stated framed oriented link in a triangulated surface $(\mathcal{G}, \lambda)$ obtained by gluing together two triangulated surfaces $(\mathcal{G}_1, \lambda_1)$ and $(\mathcal{G}_2, \lambda_2)$. Let $K_1$ and $K_2$ be the associated links in $\mathcal{G}_1$ and $\mathcal{G}_2$. Then,

  $$\text{Tr}_\lambda^\omega(K, s) = \sum_{\text{compatible } s_1, s_2} \text{Tr}_\lambda^\omega(K_1, s_1) \otimes \text{Tr}_\lambda^\omega(K_2, s_2) \in \mathfrak{T}_3^\omega(\lambda).$$

### 5. Quantum Trace Polynomials for $\text{SL}_n$

The corresponding version of Theorem 22 and Complement 23 should hold in the case of $\text{SL}_n$, by replacing 3 with $n$ everywhere in the statement. In this section, we will construct the quantum trace map $\text{Tr}_\lambda^\omega$ for general $n$. However, we only give a proof that it is well-defined for $n = 3$. For concreteness, along the way we will give explicit formulas for the case $n = 3$.

When $n = 2$, our construction coincides with that in [BWT11]. In particular, our construction gives a way to think of their construction, which was defined for un-oriented links, in terms
of oriented links. Throughout, fix \( q \in \mathbb{C} - \{0\} \) and a \( n^2 \)-root \( \omega = q^{1/n^2} \in \mathbb{C} - \{0\} \) of \( q \). Technically, also choose \( \omega^{1/2} \).

5.1. **Matrix conventions.** We will need to display \( 3 \times 3 \) and \( 3^2 \times 3^2 \) matrices. Lower indices will indicate rows and upper indices will indicate columns. A \( 3 \times 3 \) matrix \( M = (M_i^j) \) will be displayed in the general form

\[
M = \begin{pmatrix}
M_1^1 & M_1^2 & M_1^3 \\
M_2^1 & M_2^2 & M_2^3 \\
M_3^1 & M_3^2 & M_3^3
\end{pmatrix}.
\]

A \( 3^2 \times 3^2 \) matrix \( M = (M_{ij}^{ik}) \) will be displayed in the general form

\[
M = \begin{pmatrix}
M_{11}^{11} & M_{11}^{12} & M_{11}^{13} & M_{11}^{21} & M_{11}^{22} & M_{11}^{23} & M_{11}^{31} & M_{11}^{32} & M_{11}^{33} \\
M_{12}^{11} & M_{12}^{12} & M_{12}^{13} & M_{12}^{21} & M_{12}^{22} & M_{12}^{23} & M_{12}^{31} & M_{12}^{32} & M_{12}^{33} \\
M_{13}^{11} & M_{13}^{12} & M_{13}^{13} & M_{13}^{21} & M_{13}^{22} & M_{13}^{23} & M_{13}^{31} & M_{13}^{32} & M_{13}^{33} \\
M_{21}^{11} & M_{21}^{12} & M_{21}^{13} & M_{21}^{21} & M_{21}^{22} & M_{21}^{23} & M_{21}^{31} & M_{21}^{32} & M_{21}^{33} \\
M_{22}^{11} & M_{22}^{12} & M_{22}^{13} & M_{22}^{21} & M_{22}^{22} & M_{22}^{23} & M_{22}^{31} & M_{22}^{32} & M_{22}^{33} \\
M_{23}^{11} & M_{23}^{12} & M_{23}^{13} & M_{23}^{21} & M_{23}^{22} & M_{23}^{23} & M_{23}^{31} & M_{23}^{32} & M_{23}^{33} \\
M_{31}^{11} & M_{31}^{12} & M_{31}^{13} & M_{31}^{21} & M_{31}^{22} & M_{31}^{23} & M_{31}^{31} & M_{31}^{32} & M_{31}^{33} \\
M_{32}^{11} & M_{32}^{12} & M_{32}^{13} & M_{32}^{21} & M_{32}^{22} & M_{32}^{23} & M_{32}^{31} & M_{32}^{32} & M_{32}^{33} \\
M_{33}^{11} & M_{33}^{12} & M_{33}^{13} & M_{33}^{21} & M_{33}^{22} & M_{33}^{23} & M_{33}^{31} & M_{33}^{32} & M_{33}^{33}
\end{pmatrix}.
\]

If \( V \) and \( W \) are finite-dimensional complex vector spaces with bases \( \{v_1, \ldots, v_m\} \) and \( \{w_1, \ldots, w_p\} \) and if \( T: V \to W \) is a linear map, we define the \( p \times m \) matrix \([T] \in M_{p,m}(\mathbb{C})\) associated to \( T \) and these bases of \( V \) and \( W \) by the property

\[
T(v_j) = \sum_{i=1}^{p} [T]_{ij}^{\cdot}w_i \quad (j = 1, 2, \ldots, m).
\]

5.2. **Biangle quantum trace map.** A biangle \( \mathcal{B} \) is a closed disk with two punctures on its boundary. Biangles do not admit ideal triangulations, so \( \mathcal{G} \) is never a biangle. However, we may still consider stated framed oriented links \((K, s)\) in the thickened biangle \( \mathcal{B} \times (0, 1) \) defined just as before. In this subsection, we will (implicitly) use the Reshetikhin–Turaev construction [RT90] to provide an analogue of Theorem 22 and Complement 23 for biangles, valued in the complex numbers \( \mathbb{C} \); see Appendix A for the explicit (and more conceptual) connection to [RT90]. Note that this subsection does not require the choice of square root \( \omega^{1/2} \).
Parametrize the thickened biangle $\mathcal{B} \times (0, 1) \cong [0, 1] \times \mathbb{R} \times (0, 1)$ such that, in Figure \textsuperscript{17a}, say, the first coordinate points along the page to the right, the second coordinate points along the page up, and the third coordinate points out of the page toward the eye of the reader. Note this parametrization is not canonical: there are two possibilities, related by ‘turning the biangle on its head’. The construction will be independent of this choice of parametrization (see the comments after Proposition \textsuperscript{30}).

In order to state the result, we first define some elementary matrices associated to certain local link diagrams, namely various U-turns and crossings.

5.2.1. U-turns. In Figures \textsuperscript{17} and \textsuperscript{18} we show the four possible U-turns, which are in particular stated framed oriented links with the blackboard framing. In agreement with our picture conventions (see Remark \textsuperscript{20}), the boundary point of the link that is labeled ‘Higher’ or ‘H’ is higher, namely has a greater coordinate with respect to the $(0, 1)$ direction, than the boundary point of the link that is labeled ‘Lower’ or ‘L’.

Definition 24.

- The SL\textsubscript{n}-coribbon element is

$$\zeta_n \overset{\text{def}}{=} (-1)^{n-1}q^{(1-n^2)/n} = (-1)^{n-1}\omega^{n(1-n^2)} \in \mathbb{C} - \{0\}.$$  

(Note the overbar notation in $\bar{\zeta}_n$ does not mean ‘complex conjugation’.) For the categorical context, see §A.3.2.

- Define the square root of the (signed) coribbon element by

$$\sigma_n \overset{\text{def}}{=} +q^{(1-n^2)/2n} = +\omega^{n(1-n^2)/2} \in \mathbb{C} - \{0\}.$$  

We require the parenthetical ‘signed’ because we can only say $\sigma_n^2 = (-1)^{n-1}\zeta_n$. Note that $n(1 - n^2)$ is always even, so here we do not need to choose a square root $\omega^{1/2}$. 


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- Define a $n \times n$ matrix $U^q$ over the complex numbers by

$$U^q \text{ def } = \sigma_n \begin{pmatrix}
(−1)^{n−1}q(1−n)/2 \\
-1 & q(n−5)/2 & \cdots \\
q(n−1)/2 & -1 & q(n−3)/2 & \cdots \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots 
\end{pmatrix} \in M_n(\mathbb{C}),$$

where it is implicit that we are putting $q = \omega^{n^2}$ as above, so $q^{(n−1−2k)/2}$ is defined.

Note that the common ratio between adjacent entries in the matrix is equal to $−q$.

Note also that putting $q = \omega = 1$ recovers the classical U-turn matrix $U$ from §2.6.

For example, in the case $n = 3$, the $3 \times 3$ matrix $U^q$ is

$$U^q = +q^{−4/3} \begin{pmatrix}
0 & 0 & +q^{−1} \\
0 & -1 & 0 \\
+q & 0 & 0 
\end{pmatrix} \in M_3(\mathbb{C}).$$

**Definition 25.** For each pair of states $s_1, s_2 \in \{1, 2, \ldots, n\}$, define four complex numbers

\[
\text{Tr}_B^\omega(UCW_{dec})_{s_2}^{s_1}, \text{Tr}_B^\omega(UCCW_{dec})_{s_2}^{s_1}, \text{Tr}_B^\omega(UCCW_{inc})_{s_2}^{s_1}, \text{Tr}_B^\omega(UCW_{inc})_{s_2}^{s_1} \in \mathbb{C},
\]

by the matrix equations

\[
(\text{Tr}_B^\omega(UCW_{dec})_{s_2}^{s_1}) \text{ def } = U^q \in M_n(\mathbb{C}), \quad (\text{Tr}_B^\omega(UCCW_{dec})_{s_2}^{s_1}) \text{ def } = (\zeta_n)^{-1}U^q \in M_n(\mathbb{C}),
\]

\[
(\text{Tr}_B^\omega(UCW_{inc})_{s_2}^{s_1}) \text{ def } = (U^q)^T \in M_n(\mathbb{C}), \quad (\text{Tr}_B^\omega(UCCW_{inc})_{s_2}^{s_1}) \text{ def } = (\zeta_n)^{-1}(U^q)^T \in M_n(\mathbb{C}),
\]

see §5.1. Here, the superscript $T$ indicates that we are taking the matrix transpose. For example, in the case $n = 3$, in the above formulas $(\zeta_3)^{-1} = +q^{+8/3}$.

**Remark 26.** When $n = 2$, these formulas agree with those in [BWll, Proposition 13.2.b] for the underlying un-oriented link, taking $\alpha = −\omega^{-5}$ and $\beta = \omega^{-1}$; see [BWll, Prop. 26].

5.2.2. Crossings. Shown in Figures 19 and 20 are the eight possible crossings, with black-board framing and the usual picture conventions as above.
Let $V$ be a $n$-dimensional complex vector space, and let $V^*$ be the complex vector space dual to $V$. Choose a linear basis $\{e^1, e^2, \ldots, e^n\}$ for $V$, and let $\{e_1^*, e_2^*, \ldots, e_n^*\}$ be the corresponding dual basis for $V^*$. Define four linear isomorphisms

\[
\begin{align*}
\bar{c}_{V,V} : V \otimes V &\rightarrow V \otimes V, \\
\bar{c}_{V^*,V^*} : V^* \otimes V^* &\rightarrow V^* \otimes V^*, \\
\bar{c}_{V^*,V} : V^* \otimes V &\rightarrow V \otimes V^*, \\
\bar{c}_{V,V^*} : V \otimes V^* &\rightarrow V^* \otimes V, 
\end{align*}
\]
by extending linearly the following assignments for tensor product basis elements

\[
\begin{align*}
\overline{\tau}_{V,V}(e^i \otimes e^j) & \overset{\text{def}}{=} q^{1/n} \begin{cases} 
q^{-1} e^i \otimes e^i, & i = j, \\
(q^{-1} - q) e^i \otimes e^j + e^j \otimes e^i, & i < j, \\
e^j \otimes e^i, & i > j,
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\overline{\tau}_{V^*,V^*}(e_i^* \otimes e_j^*) & \overset{\text{def}}{=} q^{1/n} \begin{cases} 
q^{-1} e_i^* \otimes e_i^*, & i = j, \\
(q^{-1} - q) e_i^* \otimes e_j^* + e_j^* \otimes e_i^*, & i > j, \\
e_j^* \otimes e_i^*, & i < j,
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\overline{\tau}_{V^*,V}(e_i^* \otimes e_j^*) & \overset{\text{def}}{=} q^{-1/n} \begin{cases} 
q e_i^* \otimes e_i^* + (q - q^{-1}) \sum_{1 \leq k < i} e_k^* \otimes e_k^*, & i = j, \\
e_j^* \otimes e_i^*, & i \neq j,
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\overline{\tau}_{V,V^*}(e^i \otimes e_j^*) & \overset{\text{def}}{=} q^{-1/n} \begin{cases} 
q e_i^* \otimes e_i^* + (q - q^{-1}) \sum_{i < k \leq n} q^{k-i} e_k^* \otimes e^k, & i = j, \\
e_j^* \otimes e^i, & i \neq j.
\end{cases}
\end{align*}
\]

Define bases \(\beta_{V,V}, \beta_{V^*,V^*}, \beta_{V^*,V},\) and \(\beta_{V,V^*}\) of \(V \otimes V, V^* \otimes V^*, V^* \otimes V,\) and \(V \otimes V^*\) by

\[
\begin{align*}
(\beta_{V,V})_{ij} & \overset{\text{def}}{=} e^i \otimes e^j, 
(\beta_{V^*,V^*})_{ij} & \overset{\text{def}}{=} (-q)^{n-i} e_{n-i}^* \otimes (-q)^{n-j} e_{n-j}^*, \\
(\beta_{V^*,V})_{ij} & \overset{\text{def}}{=} (-q)^{n-i} e_{n-i}^* \otimes e^j, 
(\beta_{V,V^*})_{ij} & \overset{\text{def}}{=} e^i \otimes (-q)^{n-j} e_{n-j}^*.
\end{align*}
\]

For example, when \(n = 2,\) the ordered basis \(\beta_{V,V}\) is \(\{e^1 \otimes e^1, e^1 \otimes e^2, e^2 \otimes e^1, e^2 \otimes e^2\} .\)

The following fact, a simple calculation from the above definitions, motivates the definitions of the matrices \(C^q_{\text{same}}\) and \(C^q_{\text{opp}}\) below (and will be used in Appendix \(\text{A}\)).

**Fact 27.** We have the following equalities of matrices

\[
C^q_{\text{same}} \overset{\text{def}}{=} [\overline{\tau}_{V,V}] = [\overline{\tau}_{V^*,V^*}] \in M_{n^2}(\mathbb{C}), 
C^q_{\text{opp}} \overset{\text{def}}{=} [\overline{\tau}_{V^*,V}] = [\overline{\tau}_{V,V^*}] \in M_{n^2}(\mathbb{C}),
\]

representing the linear isomorphisms \(\overline{\tau}_{V,V}, \overline{\tau}_{V^*,V^*}, \overline{\tau}_{V^*,V}\) and \(\overline{\tau}_{V,V^*}\) when expressed in terms of the bases \(\beta_{V,V}, \beta_{V^*,V^*}, \beta_{V^*,V}\) and \(\beta_{V,V^*}.\) Also, these matrices are symmetric. \(\square\)
For example, in the case \( n = 3 \), these two \( 3^2 \times 3^2 \) matrices \( C^q_{\text{same}} \) and \( C^q_{\text{opp}} \) are given by

\[
C^q_{\text{same}} = q^{+1/3} \begin{pmatrix}
q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & q^{-1} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q^{-1} & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & q^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-1}
\end{pmatrix} \in M_{3^2}(\mathbb{C}),
\]

\[
C^q_{\text{opp}} = q^{+2/3} \begin{pmatrix}
q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & q^{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-1}
\end{pmatrix} \in M_{3^2}(\mathbb{C}).
\]

See Remark 29 for a discussion of how Fact 27 relates to the quantum group \( \text{SL}_n^q \).

An observation is that, for general \( n \), when \( q = \omega = 1 \) then the two matrices \( C^1_{\text{same}} \) and \( C^1_{\text{opp}} \) are identical. For another property of these \( R \)-matrices, see §5.4.1.

**Definition 28.** For each quadruple of states \( s_1, s_2, s_3, s_4 \in \{1, 2, \ldots, n\} \), define eight complex numbers

\[
\begin{align*}
\text{Tr}_B^\omega(C_{\text{pos-same}}^{\text{over-to-lower}})_{s_1 s_2} &\equiv C^q_{\text{same}} \in M_n^2(\mathbb{C}), \\
\text{Tr}_B^\omega(C_{\text{neg-same}}^{\text{over-to-higher}})_{s_1 s_2} &\equiv C^q_{\text{same}}^{-1} \in M_n^2(\mathbb{C}), \\
\text{Tr}_B^\omega(C_{\text{pos-same}}^{\text{over-to-higher}})_{s_1 s_2} &\equiv C^q_{\text{same}} \in M_n^2(\mathbb{C}), \\
\text{Tr}_B^\omega(C_{\text{neg-same}}^{\text{over-to-lower}})_{s_1 s_2} &\equiv C^q_{\text{same}}^{-1} \in M_n^2(\mathbb{C}), \\
\text{Tr}_B^\omega(C_{\text{pos-opp}}^{\text{over-to-higher}})_{s_1 s_2} &\equiv C^q_{\text{opp}} \in M_n^2(\mathbb{C}), \\
\text{Tr}_B^\omega(C_{\text{neg-opp}}^{\text{over-to-lower}})_{s_1 s_2} &\equiv C^q_{\text{opp}}^{-1} \in M_n^2(\mathbb{C}), \\
\text{Tr}_B^\omega(C_{\text{pos-opp}}^{\text{over-to-lower}})_{s_1 s_2} &\equiv C^q_{\text{opp}} \in M_n^2(\mathbb{C}), \\
\text{Tr}_B^\omega(C_{\text{neg-opp}}^{\text{over-to-higher}})_{s_1 s_2} &\equiv C^q_{\text{opp}}^{-1} \in M_n^2(\mathbb{C}).
\end{align*}
\]

**Remark 29.** In the case \( n = 2 \), these formulas agree with those in [BW11 Lemma 22], for the underlying un-oriented link, taking \( A = \omega^{-2}, \alpha = -\omega^{-5} \) and \( \beta = \omega^{-1} \) (see [BW11 Proposition 26]). In particular, as another indication of the un-oriented nature of \( \text{SL}_2 \), when
\(n = 2\) the two matrices \(C^q_{\text{same}}\) and \(C^q_{\text{opp}}\) are identical for all \(q\) and \(\omega\) (we saw above that, for general \(n\), this is only true for \(q = \omega = 1\)). This can be explained conceptually as follows. For any \(n\), the vector spaces \(V\) and \(V^*\) can be given the structure of a right \(\text{SL}_n^q\)-comodule; see Appendix [A] When \(n = 2\), the linear isomorphism \(V \rightarrow V^*,\ e^1 \mapsto -qe^2_1,\ e^2 \mapsto e^*_1\) is an isomorphism of right \(\text{SL}_2^q\)-comodules, but this is not true for \(n > 2\). This is why, loosely speaking, the choices above for the bases \(\beta_{V,V},\ \beta_{V^*,V^*},\ \beta_{V^*,V}\) and \(\beta_{V,V^*}\) are ‘preferred’.

The linear isomorphisms \(\varphi_{V,V},\ \varphi_{V^*,V^*},\ \varphi_{V^*,V}\) and \(\varphi_{V,V^*}\) arise naturally as braidings in the ribbon category of finite-dimensional right \(\text{SL}_n^q\)-comodules, where the categorical coribbon element \(\varpi_{\text{SL}_n^q}\) is essentially given by Definition [24] see Appendix [A]. Possibly of interest, we have implicitly taken a ‘symmetric’ duality, which is more fitting for the current setting.

In the notation of [Kas95] (compare [Kas95 Chapter XIV.2, Example 1]), these symmetric dualities \(b_V\) and \(d_V\) are related to the usual ones by \(b_V = \nu b^K_{\text{Kassel}}\) and \(d_V = \nu^{-1}d^K_{\text{Kassel}}\), where \(\nu = q^{(1-n)/2n} = \sigma_n q^{(n-1)/2};\) see Appendix [A]. Note that \(\nu\) is the bottom left entry of \(U^q\), see Definition [24].

5.2.3. **Trivial strand.** Consider a single strand crossing from one boundary edge of the biangle to the other boundary edge, as shown in Figure [21] Note that the height of the strand with respect to the \((0,1)\) component does not play a role in this particular case.
This trivial strand corresponds to the $n \times n$ identity matrix. That is, define for each pair of states $s_1, s_2 \in \{1, 2, \ldots, n\}$ the complex number $\text{Tr}^{\omega}(I)^{s_2}_{s_1}$ by the matrix equation

\[(\text{Tr}^{\omega}(I)^{s_2}_{s_1}) \triangleq \text{Id}_n \in M_n(\mathbb{C}).\]

5.2.4. Kinks and the biangle quantum trace map. Up to this point, we have assigned complex numbers $\text{Tr}^{\omega}(K, s)$ to a handful of small stated blackboard-framed oriented links $(K, s)$ (but not their isotopy classes) in the parametrized (see the beginning of §5.2) thickened biangle $\mathcal{B} \times (0, 1)$. We now provide the general assignment.

Assume first that the blackboard-framed link $K$ has no kinks. For the moment, we also assume that higher points of $K \cap (\{0\} \times \mathbb{R} \times (0, 1))$ (resp. $K \cap (\{1\} \times \mathbb{R} \times (0, 1))$) have larger second coordinates with respect to the parametrization of $\mathcal{B} \times (0, 1) \cong [0, 1] \times \mathbb{R} \times (0, 1)$; see, for example, Figures 17, 18, 19, 20 (Recall, in particular, Remark 20). Fixing endpoints, isotope (without introducing kinks) $K$ into an arbitrary bridge position. This means that,
after isotopy, there exists a partition $0 = x_0 < x_1 < \cdots < x_p = 1$ of $[0, 1]$ such that $K \cap ([x_i, x_{i+1}] \times \mathbb{R} \times (0, 1)) = K_i = \cup_{\ell} K_{i,\ell}$ is a disjoint union of links $K_{i,\ell}$ satisfying:

- the higher points of $K_i \cap (\{x_i\} \times \mathbb{R} \times (0, 1))$ have a larger second coordinate;
- for each $i$, there is a single $\ell$ such that $K_{i,\ell}$ is either a U-turn (Figures 17, 18) or a crossing (Figures 19, 20), and the other $K_{i,\ell}$'s are trivial (Figure 21 and its inverse, with opposite orientation).

(Compare [BW11, §4, proof of Lemma 15].) For each $i = 0, 1, \ldots, p - 1$ and any state $s_i$ on $K_i$, define

$$
\text{Tr}^\omega_{\mathcal{B}}(K_i, s_i) \overset{\text{def}}{=} \prod_{\ell} \text{Tr}^\omega_{\mathcal{B}}(K_{i,\ell}, s_i|_{K_{i,\ell}}) \in \mathbb{C},
$$

see §5.2.1 5.2.2 5.2.3. We then define the number

$$
\text{Tr}^\omega_{\mathcal{B}}(K, s) \overset{\text{def}}{=} \sum_{\text{compatible } s_0, s_1, \ldots, s_{p-1}} \prod_{i=0,1,\ldots,p-1} \text{Tr}^\omega_{\mathcal{B}}(K_i, s_i) \in \mathbb{C}.
$$

Here, the states $s_i$ are compatible (Definition 21) if $s|_{K \cap (\{0\} \times \mathbb{R} \times (0, 1))} = s_0|_{K_0 \cap (\{0\} \times \mathbb{R} \times (0, 1))}$ and $s_i|_{K_i \cap (\{x_i\} \times \mathbb{R} \times (0, 1))} = s_{i+1}|_{K_{i+1} \cap (\{x_{i+1}\} \times \mathbb{R} \times (0, 1))}$ and $s_{p-1}|_{K_{p-1} \cap (\{1\} \times \mathbb{R} \times (0, 1))} = s|_{K \cap (\{1\} \times \mathbb{R} \times (0, 1))}.$

Define $\text{Tr}^\omega_{\mathcal{B}}(K, s) \in \mathbb{C}$ for a general stated blackboard-framed oriented link $(K, s)$, possibly with kinks (§4.1), as follows. By isotopy, sliding the link horizontally along the boundary of $\mathcal{B} \times (0, 1)$ while preserving blackboard framing throughout, we can arrange that the boundary $\partial K$ satisfies the ‘higher point, larger second coordinate’ condition assumed just above. (Note this sliding might introduce or remove kinks.) Let $K'$ be the blackboard-framed link without kinks obtained by removing the kinks of $K$ (more precisely, pulling tight by homotopy—not isotopy—the kinks in the un-framed link underlying $K$), and further isotoped into a bridge position as above. Then we have defined a complex number $\text{Tr}^\omega_{\mathcal{B}}(K', s) \in \mathbb{C}$. Define $\text{Tr}^\omega_{\mathcal{B}}(K, s) \in \mathbb{C}$ by modifying $\text{Tr}^\omega_{\mathcal{B}}(K', s)$ according to the un-kinking ‘skein relations’ shown in Figures 3a and 3b. (For example, in the case $n = 3$, $\zeta_3 = +q^{-8/3}$.) More precisely, for $P$ (resp. $N$) the number of positive (resp. negative) kinks of $K$,

$$
\text{Tr}^\omega_{\mathcal{B}}(K, s) \overset{\text{def}}{=} (\zeta_n)^{P-N} \text{Tr}^\omega_{\mathcal{B}}(K', s) \in \mathbb{C}.
$$
Proposition 30 (SL$_n$-biangle quantum trace map). Let $\mathcal{B} \times (0, 1)$ be the (non-parametrized) thickened biangle. Then the construction of the current section determines a function

$$\Tr_\mathcal{B}^\omega : \{\text{stated framed oriented links } (K, s) \text{ in } \mathcal{B} \times (0, 1)\} \to \mathbb{C},$$

satisfying the following properties:

(A) the number $\Tr_\mathcal{B}^\omega (K, s) \in \mathbb{C}$ is invariant under isotopy of stated framed oriented links;

(B) the SL$_n$-HOMFLYPT skein relation (Figure 1) holds;

(C) the SL$_n$-quantum unknot and framing relations (Figures 2 and 3) hold.

Moreover, this invariant satisfies the Multiplication Property

$$\Tr_\mathcal{B}^\omega (K, s) = \prod_{j=1}^\ell \Tr_\mathcal{B}^\omega (K_j, s_j) \in \mathbb{C},$$

in the case where $K$ is the disjoint union of links $K_j$ having mutually non-overlapping heights (note the order of multiplication is immaterial, in contrast to Complement 23), as well as the State Sum Property

$$\Tr_\mathcal{B}^\omega (K, s) = \sum_{\text{compatible } s_1, s_2} \Tr_{\mathcal{B}_1}^\omega (K_1, s_1)\Tr_{\mathcal{B}_2}^\omega (K_2, s_2) \in \mathbb{C},$$

where $K_1$ and $K_2$ are the links obtained by cutting the biangle $\mathcal{B}$ into two biangles $\mathcal{B}_1$ and $\mathcal{B}_2$.

For the proof of part (A) of the proposition, namely the isotropy invariance, we refer the reader to Appendix A which makes use of a ‘symmetric’ specialization of the Reshetikhin–Turaev invariant (see Figure 41). The skein relations, parts (B) and (C), are local calculations; see §5.4.1.

Note that (assuming isotopy invariance) the State Sum Property is immediate from the construction. It suffices to establish the Multiplication Property when $\ell = 2$ (assuming $K_1$ lies below $K_2$, say). This follows from the State Sum Property and the definition for the trivial strand (Figure 21) by choosing a parametrization of $\mathcal{B} \times (0, 1) \cong [0, 1] \times \mathbb{R} \times (0, 1)$, the partition $0 < 1/2 < 1$ of $[0, 1]$, and a position for the links such that $(K_1, s_1) \subseteq [0, 1] \times \mathbb{R} \times (0, 1)$. 

...
(0, 1/2) (resp. \((K_2, s_2) \subseteq [0, 1] \times \mathbb{R} \times (1/2, 1)\)) with only trivial strands in \([1/2, 1] \times \mathbb{R} \times (0, 1/2)\) (resp. \([0, 1/2] \times \mathbb{R} \times (1/2, 1)\)). (Compare \[BW11\] Lemma 19.)

We note, in particular, that the biangle quantum trace map is independent of the choice of parametrization of the biangle \(\mathfrak{B}\), discussed at the beginning of \S 5.2. Indeed, this property follows either from the properties of the symmetric specialization of the Reshetikhin–Turaev invariant (Appendix A), or directly from the symmetries of the matrices corresponding to the links displayed in Figures 17, 18, 19, 20, 21.

\textbf{Remark 31.} The Reshetikhin–Turaev invariant can be defined more generally for ribbon graphs, including so-called webs \[Kup96, Sik05\]. In \[BW11\], the \(\text{SL}_2\)-quantum trace is defined by splitting the edges of the ideal triangulation \(\lambda\) to form biangles and then “pushing all of the complexities of the link into the biangles,” \[BW11\] p.1596 leaving only flat arcs lying over the triangles. In order to construct the \(\text{SL}_n\)-quantum trace for webs, one can perform the same procedure, in particular pushing all of the vertices of the web into the biangles. Then, the Reshetikhin–Turaev invariant can be applied to the webs in the biangles and (as we will see in the next section) the Fock–Goncharov matrices can be associated to the arcs lying over the triangles. This is essentially the strategy employed in \[Kim20\] in the case \(n = 3\).

\textbf{5.3. Definition of the \(\text{SL}_n\)-quantum trace polynomials.} Our construction of the quantum trace map in the general \(n\) case will follow exactly the same procedure as explained in \[BW11\] \S 3.4-6 for the case \(n = 2\), where our Proposition 30 plays the role of Proposition 13 in \[BW11\] \S 4. It remains to discuss how Property (2)(a) of Theorem 11 in \[BW11\] \S 3.4, concerning the values of the quantum traces for arcs in triangles (see Remark 31), generalizes to our setting, which we have essentially already done. (In particular, we follow the ‘ordered lower to higher, multiply left to right’ convention of \[BW11\] for ordering the non-commutative variables associated to different arc components lying over a single triangle.) After this, the rest of the construction is identical to \[BW11\] \S 6, pp. 1600-1601], where the quantum trace for a general triangulated surface \((\mathcal{G}, \lambda)\) is defined as a state sum over the
triangles of the ideal triangulation $\lambda$ of $\mathcal{G}$. We proceed below to spell all of this out in greater detail. Now is the point where we require the choice of square root $\omega^{1/2}$; see Remark 33.

5.3.1. Arcs in a triangle. Generalizing Property (2)(a) of Theorem 11 in [BW11, §3.4] to the case of general $n$ is accomplished by using the quantum left and right matrices $L^\omega$ and $R^\omega$, with coefficients in the Fock–Goncharov quantum torus $T^\omega_n(T)$ for a triangle $\mathfrak{T}$ in the ideal triangulation $\lambda$, appearing earlier in Theorem 16.

Consider a single extended left-moving or right-moving arc crossing the triangle between two distinct boundary edges, such as those shown in Figure 13; see §3.3.2. For example, in the case $n = 3$, these extended left-moving and right-moving arcs are displayed in Figures 22a and 22b. Using the notation from these figures, the $n = 3$ quantum left and right matrices

$$L^\omega(W, Z, W', Z', X) = \begin{pmatrix}
D_{L^{-1/3}}^{-1/3} & 0 & 0 & D_{L^{-1/3}}^{-1/3} \\
0 & D_{L^{-1/3}}^{-1/3} & 0 & D_{L^{-1/3}}^{-1/3} \\
D_{L^{-1/3}}^{-1/3} & 0 & D_{L^{-1/3}}^{-1/3} & 0 \\
0 & D_{L^{-1/3}}^{-1/3} & 0 & D_{L^{-1/3}}^{-1/3}
\end{pmatrix} \in SL_3^q(T^\omega_3(\mathfrak{T})),$$

where $D_{L^{-1/3}}$ in $T^\omega_3(\mathfrak{T})$ is defined by

$$D_{L^{-1/3}} \equiv W^{-1/3}Z^{-2/3}X^{-1/3}Z'^{-1/3}W'^{-2/3} \in T^\omega_3(\mathfrak{T}),$$

and

$$R^\omega(W, Z, W', Z', X) = \begin{pmatrix}
D_{R^{-1/3}}^{-1/3} & 0 & 0 & D_{R^{-1/3}}^{-1/3} \\
0 & D_{R^{-1/3}}^{-1/3} & 0 & D_{R^{-1/3}}^{-1/3} \\
D_{R^{-1/3}}^{-1/3} & 0 & D_{R^{-1/3}}^{-1/3} & 0 \\
0 & D_{R^{-1/3}}^{-1/3} & 0 & D_{R^{-1/3}}^{-1/3}
\end{pmatrix} \in SL_3^q(T^\omega_3(\mathfrak{T})),$$

where $D_{R^{-1/3}}$ in $T^\omega_3(\mathfrak{T})$ is defined by

$$D_{R^{-1/3}} \equiv W'^{-1/3}Z'^{-2/3}X^{1/3}Z^{-1/3}W^{-2/3} \in T^\omega_3(\mathfrak{T}).$$

(This is the result of multiplying out the snake-move matrices in the case $n = 3$; compare §3.5.1.)
**Definition 32.** For general $n$, define for each pair of states $s_1, s_2 \in \{1, 2, \ldots, n\}$ two elements in the quantum torus $\mathcal{T}_n^\omega(\mathfrak{X})$

$$\text{Tr}_\mathcal{T}^\omega(L)_{s_2}^{s_1}, \ \text{Tr}_\mathcal{T}^\omega(R)_{s_2}^{s_1} \in \mathcal{T}_n^\omega(\mathfrak{X}),$$

by the matrix equations (see §5.1)

$$(\text{Tr}_\mathcal{T}^\omega(L)_{s_2}^{s_1}) \overset{\text{def}}{=} L^\omega \in \text{SL}_n^q(\mathcal{T}_n^\omega(\mathfrak{X})) \subseteq M_n(\mathcal{T}_n^\omega(\mathfrak{X})),

(\text{Tr}_\mathcal{T}^\omega(R)_{s_2}^{s_1}) \overset{\text{def}}{=} R^\omega \in \text{SL}_n^q(\mathcal{T}_n^\omega(\mathfrak{X})) \subseteq M_n(\mathcal{T}_n^\omega(\mathfrak{X})).$$

**Remark 33.** In the above matrices, we recall that the square brackets surrounding the monomials indicate that we are taking the Weyl quantum ordering, which depends on the quiver defining the $q$-commutation relations in the Fock–Goncharov quantum torus $\mathcal{T}_n^\omega(\mathfrak{X})$; see §3.1.3 and Figure 12. It is here that the choice of $\omega^{1/2}$ enters into the construction.

In Theorem 16, we saw that the quantum left and right matrices $L^\omega$ and $R^\omega$ are points of the quantum special linear group $\text{SL}_n^q$. Note that, in order for these matrices to satisfy even just the relations required to be in the quantum matrix algebra $M_n^q$, they had to be normalized by ‘dividing out’ their determinants. For example, the above $n = 3$ version of the matrix $L^\omega$ would not satisfy the $q$-commutation relations required to be a point of $M_3^q$ if we had instead put $D_L = 1$.

![Figure 22. Quantum left and right matrices for $n = 3$.](image)

**5.3.2. Good position of a link.** Fix an ideal triangulation $\lambda$ of $\mathfrak{S}$. Form the corresponding split ideal triangulation $\hat{\lambda}$ of $\mathfrak{S}$ by ‘splitting’ each edge $E$ of $\lambda$ into a biangle $\mathfrak{B}_E$. (Compare
For notational simplicity, we identify the triangles of $\lambda$ with the triangles of $\hat{\lambda}$. A framed link $K$ is said to be in good position with respect to the split ideal triangulation $\hat{\lambda}$ if:

- the link $K$ is transverse to $\hat{E} \times (0,1)$ for each edge $\hat{E}$ of $\hat{\lambda}$;
- for each triangle $\Xi_j$ of $\hat{\lambda}$, the intersection $K \cap (\Xi_j \times (0,1)) = K_j = \bigcup \ell K_{j,\ell}$ consists of a disjoint union of arcs $K_{j,\ell}$, each connecting distinct sides of $\Xi_j \times (0,1)$;
- the arcs $K_{j,\ell}$ are ‘flat’, in the sense that each arc has a constant height with respect to the vertical coordinate of $\Xi_j \times (0,1)$ and has the blackboard framing.

(Compare [BW11, Lemma 23].) In particular, when in good position, all of the ‘complexity’ of the link $K$ resides in the thickened biangles $\bigcup_i (\mathcal{B}_i \times (0,1))$. A good position move between framed oriented links $K_1$ and $K_2$ in good position is one of the oriented versions of the local moves depicted in Figures 15-19 in [BW11, §5]. In the present article, these moves are displayed in Figures 24, 25, 26, 27 (Move I); Figures 28, 29 (Move II); Figures 30, 31, 32, 33 (Move III); Figures 34, 35, 36, 37 (Move IV); and, Figure 38 (Move V).

The proof of the following fact is the same as the proof of the corresponding un-oriented version ([BW11, Lemma 24]).

**Fact 34.** Any framed oriented link $K$ has a good position with respect to the split ideal triangulation $\hat{\lambda}$. Any two isotopic framed oriented links $K_1$ and $K_2$ in good position are related by a sequence of good position moves and their inverses (and isotopies through framed oriented links in good position).
5.3.3. General case. Let \( K \) be any blackboard-framed oriented link (recall that a framed link can be isotoped to have the blackboard framing by possibly introducing kinks). By §5.3.2 we may assume that \( K \) is in good position with respect to the split ideal triangulation \( \hat{\lambda} \).

Let the biangles of \( \hat{\lambda} \) be denoted \( \mathfrak{B}_i \) for \( i = 1, 2, \ldots, p \) and let the triangles be denoted \( \mathfrak{T}_j \) for \( j = 1, 2, \ldots, m \). Put \( L_i = K \cap (\mathfrak{B}_i \times (0, 1)) \) and \( K_j = K \cap (\mathfrak{T}_j \times (0, 1)) \). By definition of good position, \( K_j = K_{j,1} \cup K_{j,2} \cup \cdots \cup K_{j,\ell_j} \) where each component \( K_{j,\ell} \) is a flat oriented arc connecting distinct sides of \( \mathfrak{T}_j \times (0, 1) \).

Choose indices such that \( K_{j,\ell} \) lies below \( K_{j,\ell+1} \) with respect to the height order of \( \mathfrak{T}_j \times (0, 1) \). For any state \( s_j \) on \( K_j \), by §5.3.1 the triangle quantum torus elements \( \text{Tr}^\omega_{\mathfrak{T}_j}(K_{j,\ell}, s_j|_{K_{j,\ell}}) \in \mathcal{T}_n(\mathfrak{T}_j) \) are defined for \( \ell = 1, 2, \ldots, \ell_j \). Assign such a quantum torus element to the stated link \((K_j, s_j)\) by

\[
\text{Tr}^\omega_{\mathfrak{T}_j}(K_j, s_j) \overset{\text{def}}{=} \text{Tr}^\omega_{\mathfrak{T}_j}(K_{j,1}, s_j|_{K_{j,1}}) \text{Tr}^\omega_{\mathfrak{T}_j}(K_{j,2}, s_j|_{K_{j,2}}) \cdots \text{Tr}^\omega_{\mathfrak{T}_j}(K_{j,\ell_j}, s_j|_{K_{j,\ell_j}}) \in \mathcal{T}_n(\mathfrak{T}_j).
\]

Note, importantly, the order in which the non-commuting elements \( \text{Tr}^\omega_{\mathfrak{T}_j}(K_{j,\ell}, s_j|_{K_{j,\ell}}) \) are multiplied, the convention being ‘ordered lower to higher, multiply left to right’. For any state \( t_i \) on \( L_i \), let the numbers \( \text{Tr}^\omega_{\mathfrak{B}_i}(L_i, t_i) \in \mathbb{C} \) be defined by Proposition 30.

**Definition 35.** Let \((K, s)\) be a stated blackboard-framed oriented link in \( \mathfrak{S} \times (0, 1) \) in good position with respect to the split ideal triangulation \( \hat{\lambda} \). The \( \text{SL}_n \)-quantum trace polynomial \( \text{Tr}^\omega_{\hat{\lambda}}(K, s) \) is defined by

\[
\text{Tr}^\omega_{\hat{\lambda}}(K, s) \overset{\text{def}}{=} \sum_{\text{compatible } t_1, t_2, \ldots, t_p, s_1, s_2, \ldots, s_m} \left( \prod_{i=1}^p \text{Tr}^\omega_{\mathfrak{B}_i}(L_i, t_i) \right) \left( \bigotimes_{j=1}^m \text{Tr}^\omega_{\mathfrak{T}_j}(K_j, s_j) \right) \in \bigotimes_{\text{triangles } \mathfrak{T}_j} \mathcal{T}_n(\mathfrak{T}_j),
\]

where the compatibility condition (Definition 21) for the states \( t_i \) and \( s_j \) with respect to the state \( s \) and the split triangulation \( \hat{\lambda} \) is analogous to that in §5.2.4 (Compare [BW11, §6, p. 1600-1601].) Note that the quantities \( \text{Tr}^\omega_{\mathfrak{T}_j}(K_j, s_j) \) commute in the tensor product, since they lie in different tensor factors \( \mathcal{T}_n(\mathfrak{T}_j) \subseteq \bigotimes_{\mathfrak{T}_j} \mathcal{T}_n(\mathfrak{T}_j) \).

This completes the construction of the \( \text{SL}_n \)-quantum trace map for links. One would still need to show it is well-defined, that is, independent of the choice of good position (equivalently, independent of isotopy); see §6 for a proof in the case \( n = 3 \).
5.4. Properties. Assuming isotopy invariance, we conclude this section with a few observations.

The above state sum definition of the quantum trace polynomial takes as input a stated framed oriented link \((K, s)\) and outputs an element of the tensor product \(\bigotimes_{i} \mathcal{T}_n^{\omega}(\Sigma)\). We had indicated earlier (Theorem 22) that the image should lie in the Fock–Goncharov quantum torus sub-algebra \(\mathcal{T}_n^{\omega}(\lambda) \subseteq \bigotimes_{i} \mathcal{T}_n^{\omega}(\Sigma)\); see §3.6. The following fact is justified by a straightforward analysis of the structure of the local U-turn, crossing, left, and right matrices. (Compare [BW11, Lemma 25]. See also [Kim20], where a stronger property is established.)

**Fact 36.** The quantum trace polynomial \(\text{Tr}_{\lambda}^\omega(K, s)\) is an element of \(\mathcal{T}_n^{\omega}(\lambda)\).

**Proof of (the general \(n\) version of) Complement 23.** The Classical Trace Property is by construction, comparing with the classical matrices of §2. The State Sum Property is immediate from the construction (assuming isotopy invariance). The Multiplication Property follows from the State Sum Property by the corresponding property for biangles (Proposition 30), together with the definitions of good position and the quantities \(\text{Tr}_{\lambda}^\omega(K_j, s_j) \in \mathcal{T}_n^{\omega}(\Sigma_j)\). (Compare [BW11] §6, p.1609.)

We remark that the quantum trace \(\text{Tr}_{\lambda}^\omega(K, s)\) of a stated framed oriented link \((K, s)\) can be thought of as a tensor having dimension equal to the number of boundary points \(p_i \in \partial K\) of the link, each associated to a state \(s_i\). If the states \(s_i\) are partitioned into two groups \(s_{i1}, \ldots, s_{i_k}\) and \(s_{j1}, \ldots, s_{jm}\), then the quantum trace of the link can be written as a matrix \((\text{Tr}_{\lambda}^\omega(K, s)_{s_{i1},\ldots,s_{i_k}}^{s_{j1},\ldots,s_{jm}})\) with coefficients in \(\mathcal{T}_n^{\omega}(\lambda)\); see §6 for examples.

5.4.1. Skein relations. We justify parts (B)-(C) in (the general \(n\) version of) Theorem 22.

The first skein relation is the well-known \((q\text{-evaluated})\) HOMFLYPT relation from knot theory [FYH+85, PT87]. The \(R\)-matrices for the quantum group \(\text{SL}_q^\omega\) satisfy this skein relation. For us, this relation appears with the normalization displayed in Figure 1. One can check from, say, Figures 19a, 19b, 21 together with the definitions of §5.2.2 and §5.2.3 that
the quantum trace map $\text{Tr}_\lambda^\omega$ satisfies this skein relation, translating to the matrix equation

$$q^{-1/n}C_{\text{same}}^q - q^{+1/n}(C_{\text{same}}^q)^{-1} = (q^{-1} - q)\text{Id}_{n^2} \in M_{n^2}(\mathbb{C}).$$

The second skein relation, coming from the U-turn ‘duality’ matrices (Figures 17 and 18), says that the contractible untwisted unknot $K$ evaluates to $(-1)^{n-1}$ times the quantum integer $[n]_q = (q^n - q^{-n})/(q - q^{-1}) = \sum_{k=1}^{n} q^{2k-n-1}$; see Figure 2. The third skein relation consists of the positive and negative framing relations; see Figure 3.

6. Isotopy invariance: proof of the main theorem

Proof of Theorem 22. Parts (B)-(C) were discussed above. In this section, we will establish part (A): the $\text{SL}_3$-quantum trace map is invariant under isotopy. It suffices to check the oriented good position moves; see §5.3.2. We do this ‘by hand’, using computer assistance. □

The more difficult moves are those of type (II) and (IV). Indeed, (I) can be computed directly from the definitions (although it is still somewhat non-trivial), (III) is essentially equivalent to Theorem 16, and (V) is equivalent to the kink-removing skein relations appearing in Figure 3. However, below we will justify moves (I), (III), and (V) as well.

Remark 37. In the general case of $\text{SL}_n$, a proof of essentially these same algebraic identities (including Theorem 16), which are equivalent to the local isotopy moves discussed in this section, is given in [CS23] (motivated by [SS19, SS17] and earlier by [FG06a, GSV09]) in the context of quantum integrable systems; see also [GS19]. Consequently, these works can be applied to finish the proof of the general $n$ version of Theorem 22.

6.1. Notation. Throughout this section, we will be considering a single triangle with 7 coordinates, denoted as in Figure 28 (Note that the coordinates we are currently labeling as $W_2, Z_2, W_3, Z_3, X$ were labeled, respectively, $W, Z, W', Z', X$ in §5.3.1) We define matrices $L^\omega(W_2, Z_2, W_3, Z_3, X)$ and $R^\omega(W_2, Z_2, W_3, Z_3, X)$ in $\text{SL}_3(\mathcal{T}_3^\omega(\Sigma)) \subseteq M_3(\mathcal{T}_3^\omega(\Sigma))$ by the same formulas as in §5.3.1. These matrices are considered as functions of the ordered 5-tuple...
(W_2, Z_2, W_3, Z_3, X). For example, we may also consider a matrix R^\omega(W_3, Z_3, W_1, Z_1, X) corresponding to the right turn in Figure 28.

6.2. Move (I). In Figure 24, we show one of the oriented versions of Move (I). Let K be the link on the left, and K' the link on the right. According to the definition of the quantum trace (§5.3) as a State Sum Formula, the equality expressing Move (I) can be interpreted as an equality of 3 \times 3 matrices. Specifically, the claim is that the matrix,

\[
\begin{pmatrix}
A_1 & 0 & 0 \\
D_1 & E_1 & 0 \\
G_1 & H_1 & I_1
\end{pmatrix} = R^\omega(W_2, Z_2, W_3, Z_3, X)
\]

is equal to the matrix

\[
\begin{pmatrix}
A_1 & b_1 & c_1 \\
0 & e_1 & f_1 \\
0 & 0 & i_1
\end{pmatrix}
\]

\[\left(\begin{array}{ccc}
q^{-1/3}A_1c_1-q^{-4/3}D_1b_1+q^{-7/3}G_1a_1 \\
q^{-4/3}D_1e_1 \\
q^{-7/3}A_1i_1
\end{array}\right) = \left(\begin{array}{ccc}
0 & 0 & +q^{-1} \\
0 & -1 & 0 \\
+q & 0 & 0
\end{array}\right) \left(\begin{array}{ccc}
A_1 & 0 & 0 \\
0 & E_1 & 0 \\
G_1 & H_1 & I_1
\end{array}\right)
\]

is equal to the matrix

\[
\begin{pmatrix}
0 & 0 & +q^{-1} \\
0 & -1 & 0 \\
+q & 0 & 0
\end{pmatrix}
\]

\[\left(\begin{array}{ccc}
A_1 & b_1 & c_1 \\
0 & e_1 & f_1 \\
0 & 0 & i_1
\end{array}\right) = (\operatorname{Tr}^\omega(K')^s_2) \in M_3(\mathcal{T}_{3}^w(\mathcal{X}))
\]

where we have used Figure 17a and the matrix (\operatorname{Tr}^\omega(U_{\text{dec}})^s_2) = U^q from §5.2.1 for the middle matrix, and where we have put (§6.1)

\[
\left(\begin{array}{ccc}
A_1 & 0 & 0 \\
D_1 & E_1 & 0 \\
G_1 & H_1 & I_1
\end{array}\right) \overset{\text{def}}{=} R^\omega(W_2, Z_2, W_3, Z_3, X), \quad \left(\begin{array}{ccc}
a_1 & b_1 & c_1 \\
0 & e_1 & f_1 \\
0 & 0 & i_1
\end{array}\right) \overset{\text{def}}{=} L^\omega(W_2, Z_2, W_3, Z_3, X) \in M_3(\mathcal{T}_{3}^w(\mathcal{X})).
\]

See Appendix B for a computer check of the above equality of 3 \times 3 matrices in M_3(\mathcal{T}_{3}^w(\mathcal{X})) representing this oriented Move (I) example. Also checked in Appendix B are the other three oriented versions of Move (I), whose equivalent matrix formulations are displayed in Figures 25, 26, 27. (Note that the reversal of order of the non-commuting variables in the case of Moves (I) and (I.c) is due to the ‘ordered lower to higher, multiply left to right’ rule; see §5.3.)
(I.b) \[ \left( \text{Tr}_\lambda^\omega(K)^{s_2}_{s_1} \right) = \left( \begin{array}{ccc} a_1 b_1 c_1 & 0 & 0 \\ 0 & e_1 f_1 & 0 \\ 0 & 0 & i_1 \end{array} \right) \left( \begin{array}{ccc} 0 & 0 & +q^{-1/3} \\ 0 & -q^{-4/3} & 0 \\ +q^{-7/3} & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} A_1 & 0 & 0 \\ D_1 & E_1 & 0 \\ G_1 & H_1 & I_1 \end{array} \right) = \\
= \left( \begin{array}{ccc} q^{-7/3} e_1 A_1 - q^{-4/3} b_1 D_1 + q^{-1/3} a_1 G_1 - q^{-4/3} b_1 E_1 + q^{-1/3} a_1 H_1 - q^{-1/3} a_1 I_1 \\ +q^{-7/3} f_1 A_1 - q^{-4/3} e_1 D_1 \\ +q^{-7/3} i_1 A_1 
\end{array} \right) \left( \begin{array}{ccc} 0 & 0 & +q^{-1/3} \\ 0 & -q^{-4/3} & 0 \\ +q^{-7/3} & 0 & 0 \end{array} \right) = \left( \text{Tr}_\lambda^\omega(K')^{s_2}_{s_1} \right) \in M_3(\mathcal{J}_3^\omega(\mathcal{I})). \]

(I.c) \[ \left( \text{Tr}_\lambda^\omega(K)^{s_2}_{s_1} \right) = \left( \begin{array}{ccc} A_1 & 0 & 0 \\ D_1 & E_1 & 0 \\ G_1 & H_1 & I_1 \end{array} \right) \left( \begin{array}{ccc} 0 & 0 & +q^{1/3} \\ 0 & -q^{4/3} & 0 \\ +q^{7/3} & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} a_1 b_1 c_1 & 0 & 0 \\ 0 & e_1 f_1 & 0 \\ 0 & 0 & i_1 \end{array} \right) = \\
= \left( \begin{array}{ccc} 0 & 0 & -q^{4/3} e_1 E_1 \\ 0 & -q^{4/3} e_1 f_1 \\ +q^{4/3} i_1 A_1 \n\end{array} \right) \left( \begin{array}{ccc} 0 & 0 & +q^{1/3} \\ 0 & -q^{4/3} & 0 \\ +q^{7/3} & 0 & 0 \end{array} \right) = \left( \text{Tr}_\lambda^\omega(K')^{s_2}_{s_1} \right) \in M_3(\mathcal{J}_3^\omega(\mathcal{I})). \]

(I.d) \[ \left( \text{Tr}_\lambda^\omega(K)^{s_2}_{s_1} \right) = \left( \begin{array}{ccc} A_1 & 0 & 0 \\ D_1 & E_1 & 0 \\ G_1 & H_1 & I_1 \end{array} \right) \left( \begin{array}{ccc} 0 & 0 & +q^{7/3} \\ 0 & -q^{4/3} & 0 \\ +q^{7/3} & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} a_1 b_1 c_1 & 0 & 0 \\ 0 & e_1 f_1 & 0 \\ 0 & 0 & i_1 \end{array} \right) = \\
= \left( \begin{array}{ccc} 0 & 0 & -q^{4/3} e_1 E_1 \\ 0 & -q^{4/3} e_1 f_1 \\ +q^{4/3} i_1 A_1 \n\end{array} \right) \left( \begin{array}{ccc} 0 & 0 & +q^{7/3} \\ 0 & -q^{4/3} & 0 \\ +q^{7/3} & 0 & 0 \end{array} \right) = \left( \text{Tr}_\lambda^\omega(K')^{s_2}_{s_1} \right) \in M_3(\mathcal{J}_3^\omega(\mathcal{I})). \]

\[ \begin{array}{c}
Z_2 \\
W_3 \\
W_2 \\
Z_3 \\
X
\end{array} \quad \begin{array}{c}
(I)
\end{array} \quad \begin{array}{c}
s_1 \\
\leftrightarrow
\end{array} \]

\textbf{Figure 24.} One of the oriented versions of Move (I).

6.3. **Move (II).** In Figure 28 we show one of the oriented versions of Move (II). Let \( K \) be the link on the left, and \( K' \) the link on the right. According to the definition of the quantum trace (§5.3) as a State Sum Formula, the equality expressing Move (II) can be interpreted
as an equality of $3 \times 3$ matrices. Specifically, the claim is that the matrix,

\[
(II) \quad (\text{Tr}_\lambda^\omega(K')_{s_1}^{s_2}) = \left(\begin{array}{ccc}
A_2 & 0 & 0 \\
D_2 & E_2 & 0 \\
G_2 & H_2 & I_2
\end{array}\right)_{q^{-4/3}} \left(\begin{array}{ccc}
0 & 0 & +q \\
0 & -1 & 0 \\
+q^{-1} & 0 & 0
\end{array}\right) \left(\begin{array}{ccc}
A_1 & 0 & 0 \\
D_1 & E_1 & 0 \\
G_1 & H_1 & I_1
\end{array}\right) = \\
\left(\begin{array}{ccc}
q^{-1/3}A_2G_1 & 0 & 0 \\
-q^{-4/3}E_2D_1 + q^{-1/3}D_2G_1 & q^{-1/3}A_2H_1 & q^{-1/3}A_2I_1 \\
q^{-7/3}I_2A_1 - q^{-4/3}E_2D_1 + q^{-1/3}D_2G_1 & q^{-4/3}H_2E_1 + q^{-1/3}G_2H_1 & q^{-1/3}I_2H_1
\end{array}\right) \in M_3(\mathcal{T}_3^\omega(\mathcal{S})),
\]

is equal to the matrix

\[
\left(\begin{array}{ccc}
a_3 & b_3 & c_3 \\
e_3 & f_3 & i_3
\end{array}\right) = (\text{Tr}_\lambda^\omega(K')_{s_1}^{s_2}) \in M_3(\mathcal{T}_3^\omega(\mathcal{S})),
\]

where we have used Figure 18a and the matrix \((\text{Tr}_\lambda^\omega(U_{\text{inc}}^{\text{cw}})_{s_1}^{s_2}) = (U^q)^T\) from §5.2.1 for the middle matrix, and where we have put (§6.1)

\[
\left(\begin{array}{ccc}
A_2 & 0 & 0 \\
D_2 & E_2 & 0 \\
G_2 & H_2 & I_2
\end{array}\right) \overset{\text{def}}{=} \mathbf{R}^\omega(W_3, Z_3, W_1, Z_1, X), \quad \left(\begin{array}{ccc}
A_1 & 0 & 0 \\
D_1 & E_1 & 0 \\
G_1 & H_1 & I_1
\end{array}\right) \overset{\text{def}}{=} \mathbf{R}^\omega(W_2, Z_2, W_3, Z_3, X),
\]

\[
\left(\begin{array}{ccc}
a_3 & b_3 & c_3 \\
e_3 & f_3 & i_3
\end{array}\right) \overset{\text{def}}{=} \mathbf{L}^\omega(W_1, Z_1, W_2, Z_2, X) \in M_3(\mathcal{T}_3^\omega(\mathcal{S})).
\]
See Appendix B for a computer check of the above equality of $3 \times 3$ matrices in $M_3(\mathbb{T}_3(\mathfrak{F}))$ representing this oriented Move (II) example. Also checked in Appendix B is the other oriented version of Move (II), whose equivalent matrix formulation is displayed in Figure 29.

\[(II.b) \quad (\text{Tr}^\omega_{\lambda}(K)_{s_1}^{s_2}) = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & e_1 & f_1 \\ 0 & 0 & i_1 \end{pmatrix} \begin{pmatrix} 0 & 0 & +q^{7/3} \\ 0 & -q^{-4/3} & 0 \\ +q^{-1/3} & 0 & 0 \end{pmatrix} \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & e_2 & f_2 \\ 0 & 0 & i_2 \end{pmatrix} =
\begin{pmatrix} q^{-1/3}a_2c_1 & q^{-1/3}b_2c_1-q^{-4/3}e_2b_1 & q^{-1/3}c_2c_1-q^{-4/3}f_2b_1+q^{-7/3}i_2a_1 \\ q^{-1/3}a_2f_1 & q^{-1/3}b_2f_1-q^{-4/3}e_2f_1 & q^{-1/3}c_2f_1-q^{-4/3}f_2e_1 \\ q^{-1/3}a_2i_1 & q^{-1/3}b_2i_1 & q^{-1/3}c_2i_1 \end{pmatrix}
\begin{pmatrix} A_1 & 0 & 0 \\ 0 & E_3 & 0 \\ G_3 & H_3 & I_3 \end{pmatrix} = (\text{Tr}^\omega_{\lambda}(K')_{s_1}^{s_2}) \in M_3(\mathbb{T}_3(\mathfrak{F})).\]

**Figure 28.** One of the oriented versions of Move (II).

**Figure 29.** Move (II.b).

6.4. **Move (III).** In Figure 30, we show one of the oriented versions of Move (III). Let $K$ be the link on the left, and $K'$ the link on the right. According to the definition of the quantum trace (§5.3) as a State Sum Formula, the equality expressing Move (III) can be interpreted

\[(III.b) \quad (\text{Tr}^\omega_{\lambda}(K)_{s_1}^{s_2}) = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & e_1 & f_1 \\ 0 & 0 & i_1 \end{pmatrix} \begin{pmatrix} 0 & 0 & +q^{7/3} \\ 0 & -q^{-4/3} & 0 \\ +q^{-1/3} & 0 & 0 \end{pmatrix} \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & e_2 & f_2 \\ 0 & 0 & i_2 \end{pmatrix} =
\begin{pmatrix} q^{-1/3}a_2c_1 & q^{-1/3}b_2c_1-q^{-4/3}e_2b_1 & q^{-1/3}c_2c_1-q^{-4/3}f_2b_1+q^{-7/3}i_2a_1 \\ q^{-1/3}a_2f_1 & q^{-1/3}b_2f_1-q^{-4/3}e_2f_1 & q^{-1/3}c_2f_1-q^{-4/3}f_2e_1 \\ q^{-1/3}a_2i_1 & q^{-1/3}b_2i_1 & q^{-1/3}c_2i_1 \end{pmatrix}
\begin{pmatrix} A_1 & 0 & 0 \\ 0 & E_3 & 0 \\ G_3 & H_3 & I_3 \end{pmatrix} = (\text{Tr}^\omega_{\lambda}(K')_{s_1}^{s_2}) \in M_3(\mathbb{T}_3(\mathfrak{F})).\]
as an equality of $3^2 \times 3^2$ matrices (§5.1). Specifically, the claim is that the matrix,

$$(III) \quad (\text{Tr}_X(K))_{s_3 s_4}^{s_2 s_1} = \begin{pmatrix}
  a_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  a_1 b_1 & a_1 e_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  a_1 c_1 & a_1 f_1 & a_1 i_1 & 0 & 0 & 0 & 0 & 0 \\
  b_1 a_1 & 0 & 0 & c_1 e_1 & 0 & 0 & 0 & 0 \\
  b_1 b_1 & b_1 e_1 & 0 & e_1 b_1 & e_1^2 & 0 & 0 & 0 \\
  b_1 c_1 & b_1 f_1 & b_1 i_1 & e_1 c_1 & e_1 f_1 & e_1 i_1 & 0 & 0 \\
  c_1 a_1 & 0 & 0 & f_1 b_1 & 0 & 0 & i_1 c_1 & i_1 e_1 \\
  c_1 b_1 & c_1 e_1 & f_1 c_1 & f_1 i_1 & i_1 c_1 & i_1 f_1 & i_1^2 & 0
\end{pmatrix} \in M_{3^2}(\mathcal{J}_3^2(\mathcal{I})),$$

is equal to the matrix

$$\begin{pmatrix}
  a_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  q b_1 a_1 & e_1 a_1 & 0 & (q^{-1}) (e_1 a_1 - a_1 e_1) & 0 & 0 & 0 & 0 \\
  q c_1 a_1 & f_1 a_1 & i_1 a_1 & (q^{-1}) (f_1 a_1 - a_1 f_1) & 0 & 0 & (q^{-1}) (i_1 a_1) & 0 \\
  q a_1 b_1 & 0 & 0 & -a_1 c_1 & 0 & 0 & 0 & 0 \\
  q c_1 b_1 & q^{-1} b_1 e_1 & 0 & q^{-1} b_1 e_1 (1 + (q^{-1}) e_1 b_1) & e_1^2 & 0 & 0 & 0 \\
  q a_1 c_1 & 0 & 0 & -a_1 f_1 & 0 & 0 & a_1 i_1 & 0 \\
  q b_1 c_1 & e_1 c_1 & 0 & b_1 f_1 (1 + (q^{-1}) e_1 c_1) & q f_1 e_1 & i_1 e_1 & (q^{-1}) (i_1 b_1) & (q^{-1}) (i_1 e_1) \\
  c_1^2 & q^{-1} f_1 c_1 & q^{-1} i_1 c_1 & q^{-1} c_1 f_1 (1 + (q^{-2}) f_1 c_1) & f_1^2 & q^{-1} i_1 f_1 & (1 + (q^{-2}) i_1 c_1) & (1 + (q^{-2}) i_1 f_1) i_1^2
\end{pmatrix}$$

$$= (\text{Tr}_X(K'))_{s_3 s_4}^{s_2 s_1} \in M_{3^2}(\mathcal{J}_3^2(\mathcal{I})),
$$

where we have used Figures 19b, 19a and the matrices $(\text{Tr}_X(K'))_{s_3 s_4}^{s_2 s_1} = (\text{C}_{\text{neg-same}})^{-1}$ and $(\text{Tr}_X(K'))_{s_3 s_4}^{s_2 s_1} = (\text{C}_{\text{pos-same}})$, respectively, from §5.2.2 as part of the computation for the matrix on the right, and where we have put (§6.1)

$$\begin{pmatrix}
  a_1 & b_1 & c_1 \\
  0 & e_1 & f_1 \\
  0 & 0 & i_1
\end{pmatrix} \overset{\text{def}}{=} \mathbf{L}^o(W_2, Z_2, W_3, Z_3, X) \in M_{3}(\mathcal{J}_3^2(\mathcal{I})).$$

See Appendix B for a computer check of the above equality of $3^2 \times 3^2$ matrices in $M_{3^2}(\mathcal{J}_3^2(\mathcal{I}))$ representing this oriented Move (III) example. In Figures 31, 32 and 33 we prove the remaining oriented versions of Move (III), in terms of the moves already established.

Figure 30. One of the oriented versions of Move (III).
6.5. Move (IV). In Figure 34 we show one of the oriented versions of Move (IV). Let \( K \) be the link on the left, and \( K' \) the link on the right. According to the definition of the quantum trace (§5.3) as a State Sum Formula, the equality expressing Move (IV) can be interpreted as an equality of \( 3^2 \times 3^2 \) matrices (§5.1). Specifically, the claim is that the matrix,

\[
(IV) \quad (\text{Tr}_\lambda^\omega(K))_{s_1 s_2} = M_{3^2}(T_3^c(\Sigma))
\]

is equal to the matrix

\[
q^{+1/3} \begin{pmatrix}
q^{-1}A_{2a_3} & 0 & 0 & q^{-1}A_{2b_3} & 0 & 0 & q^{-1}A_{2c_3} & 0 & 0 \\
D_{2a_3} & E_{2a_3} & D_{2a_3} + (q^{-1} - q)A_{2e_3} & E_{2b_3} & 0 & D_{2c_3} + (q^{-1} - q)A_{2f_3} & E_{2c_3} & 0 & 0 \\
G_{2a_3} & H_{2a_3} & I_{2a_3} & G_{2b_3} & H_{2b_3} & I_{2b_3} & G_{2c_3} + (q^{-1} - q)A_{2e_3} & H_{2c_3} & I_{2c_3} \\
0 & 0 & 0 & A_{2f_3} & 0 & 0 & A_{2f_3} & 0 & 0 \\
0 & 0 & 0 & q^{-1}D_{2e_3} & q^{-1}D_{2e_3} & 0 & q^{-1}D_{2f_3} & q^{-1}D_{2f_3} & 0 \\
0 & 0 & 0 & G_{2e_3} & H_{2e_3} & I_{2e_3} & G_{2f_3} + (q^{-1} - q)D_{2f_3} & H_{2f_3} + (q^{-1} - q)E_{2f_3} & I_{2f_3} \\
0 & 0 & 0 & 0 & 0 & 0 & A_{2i_3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & D_{2i_3} & E_{2i_3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & q^{-1}G_{2i_3} & q^{-1}H_{2i_3} & q^{-1}I_{2i_3}
\end{pmatrix}
\]

\[
= (\text{Tr}_\lambda^\omega(K'))_{s_1 s_2} \in M_{3^2}(T_3^c(\Sigma)),
\]
where we have used Figure 19c and the matrix $(T^{\omega \text{over-to-higher}}_{3^2}(C_{\text{pos-same}})_{s_3 s_4}) = C_{\text{same}}^q$ from §5.2.2 as part of the computation for the matrix on the right, and where we have put (§6.1)

$$
\begin{pmatrix}
A_2 & 0 & 0 \\
D_2 & E_2 & 0 \\
G_2 & H_2 & I_2
\end{pmatrix}
= \mathbf{R}^\omega (W_3, Z_3, W_1, Z_1, X), 
\begin{pmatrix}
a_3 & b_3 & c_3 \\
0 & c_1 & f_3 \\
0 & 0 & 1 \end{pmatrix}
= \mathbf{L}^\omega (W_1, Z_1, W_2, Z_2, X) \in M_3(\mathcal{T}^{\omega}_{3^2}(\mathfrak{I})).
$$

See Appendix B for a computer check of the above equality of $3^2 \times 3^2$ matrices in $M_{3^2}(\mathcal{T}^{\omega}_{3^2}(\mathfrak{I}))$ representing this oriented Move (IV) example. In Figures 35, 36, and 37 we prove the remaining oriented versions of Move (IV). (Note that Move (I.b') and Move (I.d'), used here, are proved in §6.7)

**Figure 34.** One of the oriented versions of Move (IV).

**Figure 35.** Move (IV.b) and its proof.

**Figure 36.** Move (IV.c) and its proof.

6.6. **Move (V).** See Figure 38. This move is implied by the kink-removing skein relations appearing in Figure 3; see §5.4.1.
6.7. **Auxiliary moves.** Move (I.b') and Move (I.d') were used to establish Moves (IV.c) and (IV.d). The proof of Move (I.b') is shown in Figure 39. Here \( \sim \) denotes an isotopy preserving good position. (See also the second to last paragraph of the proof of Lemma 24 in [BW11].) The proof of Move (I.d') is obtained from that of Move (I.b') by horizontal reflection.

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**Appendix A. Proof of Proposition 30**

This is an application of Theorem XIV.5.1 in [Kas95]. We will closely follow the definitions, notations, and conventions of [Kas95], informing otherwise. Essentially all of the following theory is standard. See, for instance, [Kas95, BG02, JS91, Maj95].

Our first goal is to define the ribbon category \( C_V \) of interest. In particular, this requires defining the objects \( V \), morphisms \( f : U \to V \), tensor products \( V \otimes W \), tensor unit \( I \), braiding morphisms \( c_{V,W} : V \otimes W \to W \otimes V \) (see Remark 38), dual objects \( V^* \), left duality morphisms \( b_V : I \to V \otimes V^* \) and \( d_V : V^* \otimes V \to I \), twist morphisms \( \theta_V : V \to V \), and right
duality morphisms $b'_V : I \to V^* \otimes V$ and $d'_V : V \otimes V^* \to I$. As above, fix $n \in \mathbb{Z}$, $n > 1$, as well as $q$ and $\omega = q^{1/n^2}$ in $\mathbb{C} - \{0\}$. This section does not require a choice of square root $\omega^{1/2}$; compare §5.2. All vector spaces are over $\mathbb{C}$.

A.1. Quantum special linear group $H = \text{SL}_n^q$. The quantum matrix algebra $M_n^q$ is the quotient of the free algebra in generators $(T^j_i)_{1 \leq i,j \leq n}$ by the relations

$$T^m_i T^k_i = q T^k_i T^m_i, \quad T^m_j T^m_i = q T^m_i T^m_j, \quad T^m_i T^k_j = T^k_j T^m_i, \quad T^m_i T^k_j - T^k_j T^m_i = (q - q^{-1}) T^m_i T^k_j,$$

for $i < j$ and $k < m$ (we think of lower indices indicating rows and upper indices columns). The quantum determinant $\text{Det}^q \in M_n^q$ is (compare §3.4)

$$\text{Det}^q \overset{\text{def}}{=} \sum_{\sigma \in S_n} (-q^{-1})^{\ell(\sigma)} T^{\sigma(1)}_1 T^{\sigma(2)}_2 \cdots T^{\sigma(n)}_n.$$

The quantum special linear group $\text{SL}_n^q$ is the quotient

$$H \overset{\text{def}}{=} \text{SL}_n^q \overset{\text{def}}{=} M_n^q / (\text{Det}^q - 1).$$

The algebra $H$ is a Hopf algebra, meaning it is equipped with linear maps

$$\mu_H : H \otimes H \to H, \quad \eta_H : \mathbb{C} \to H, \quad \Delta_H : H \to H \otimes H, \quad \epsilon_H : H \to \mathbb{C}, \quad S_H : H \to H,$$

namely the product, unit, coproduct, counit, and antipode. Specifically, if $\delta_{ij}$ denotes the Kronecker delta (equals 1 if $i = j$ and 0 else), then the coproduct $\Delta_H$, counit $\epsilon_H$, and antipode $S_H$ are defined by (abusing notation by using the same symbol for elements of $M_n^q$ as their images in $H$)

$$\Delta_H(T^j_i) = \sum_{k=1}^n T^k_i \otimes T^j_k, \quad \epsilon_H(T^j_i) = \delta_{ij}, \quad S_H(T^j_i) = (-q)^{j-i} A^j_i,$$

where the quantum minor $A^j_i$ is the quantum determinant of the subalgebra, isomorphic to $M^q_{n-1}$, of $M_n^q$ generated by the $T^\ell_k$ with $k \neq j$ and $\ell \neq i$. (We write simply $\mu_H(x \otimes y) = xy$ and $\eta_H(z) = z 1_H.$)
A.2. Braided tensor category $H$-Comod of right $H$-comodules.

A.2.1. Right $H$-comodules. A vector space $V$ is a right $H$-comodule if it is equipped with a linear map $\Delta_V : V \to V \otimes H$, namely a (right) coaction, satisfying certain properties. The tensor product $V \otimes W$ of two right $H$-comodules is a right $H$-comodule, with coaction

$$\Delta_{V \otimes W}(v \otimes w) \overset{\text{def}}{=} \sum_{(v),(w)} v_V \otimes w_W \otimes v_H w_H.$$ 

Here, we have used Sweedler’s notation for the coactions $\Delta_V$ and $\Delta_W$. The trivial right $H$-comodule $C$ has coaction $\Delta_C(z) = z \otimes 1_H$.

Let $H$-Comod denote the tensor category whose objects $V$ are right $H$-comodules, morphisms $f : U \to V$ are homomorphisms of right $H$-comodules, and with tensor products $V \otimes W$ and tensor unit $I = C$ as above.

A.2.2. Braidings. The bialgebra $H$ is cobraided, meaning it is equipped with linear maps $r_H : H \otimes H \to C$ and its inverse (with respect to the convolution operator $\star$ defined in §A.3.2) $\tau_H : H \otimes H \to C$, namely the universal $R$-forms. Specifically, if $E_i^j : H \to C$ is the linear map $E_i^j(T_k^l) = \delta_{ik}\delta_{jl}$, then

$$r_H \overset{\text{def}}{=} q^{-1/n} \left( \sum_{1 \leq i \neq j \leq n} E_i^i \otimes E_j^j + q \sum_{i=1}^n E_i^i \otimes E_i^i + (q - q^{-1}) \sum_{1 \leq i < j \leq n} E_i^i \otimes E_j^j \right),$$

$$\tau_H \overset{\text{def}}{=} q^{1/n} \left( \sum_{1 \leq i \neq j \leq n} E_i^i \otimes E_j^j + q^{-1} \sum_{i=1}^n E_i^i \otimes E_i^i + (q^{-1} - q) \sum_{1 \leq i < j \leq n} E_i^i \otimes E_j^j \right).$$

Consequently, the tensor category $H$-Comod of right $H$-comodules is braided, with braiding morphisms $c_{V,W} : V \otimes W \to W \otimes V$ and inverse braiding $\bar{c}_{V,W} : V \otimes W \to W \otimes V$ (so, $c_{V,W}^{-1} = \bar{c}_{W,V}$) defined by

$$c_{V,W}(v \otimes w) \overset{\text{def}}{=} \sum_{(v),(w)} (w_W \otimes v_V) r_H(v_H \otimes w_H), \quad \bar{c}_{V,W}(v \otimes w) \overset{\text{def}}{=} \sum_{(v),(w)} (w_W \otimes v_V) \tau_H(w_H \otimes v_H).$$

Remark 38. By symmetry, one can just as well take the inverse braidings $\bar{c}_{V,W}$ to be ‘the’ braidings for the category. For technical reasons, we will prefer this choice going forward.
(Note that, in order to compute with \( \tau_{V,W} \), the formulas,
\[
\bar{r}(xy \otimes z) = \sum (z) \bar{r}(y \otimes z') \bar{r}(x \otimes z''), \quad \bar{r}(x \otimes yz) = \sum (x) \bar{r}(x' \otimes y) \bar{r}(x'' \otimes z),
\]
can be helpful, here using Sweedler’s notation for the coproduct \( \Delta_H \).)

A.3. Ribbon sub-category \( H\text{-Comod}_f \) of finite-dimensional right \( H \)-comodules.

A.3.1. Left dualities. Let \( V \) be a right \( H \)-comodule of dimension \( N < \infty \). The dual space \( V^* = \text{Hom}_C(V, C) \) is a right \( H \)-comodule as follows. Choose a basis \( e^1, e^2, \ldots, e^N \) for \( V \) with corresponding dual basis \( e^*_1, e^*_2, \ldots, e^*_N \) for \( V^* \). Let \( h^j_i \in H \) for \( 1 \leq i, j \leq N \) satisfy
\[
\Delta_V(e^j) = \sum_{k=1}^N e^k \otimes h^j_k.
\]
The coaction \( \Delta_{V^*} : V^* \rightarrow V^* \otimes H \) is defined by
\[
\Delta_{V^*}(e^*_i) \overset{\text{def}}{=} \sum_{k=1}^N e^*_k \otimes S_H h^k_i.
\]

Let \( H\text{-Comod}_f \) denote the braided sub-category of \( H\text{-Comod} \) consisting of finite-dimensional right \( H \)-comodules. Then \( H\text{-Comod}_f \) has left duality, the dual objects \( V^* \) being defined as above, and with left duality morphisms \( b_V : C \rightarrow V \otimes V^* \) and \( d_V : V^* \otimes V \rightarrow C \) defined by
\[
b_V(1) \overset{\text{def}}{=} \nu \sum_{k=1}^N e^k \otimes e^*_k, \quad d_V(e^*_i \otimes e^j) \overset{\text{def}}{=} \nu^{-1} \delta_{ij}.
\]
Here, \( \nu \) is a fixed complex duality parameter.

**Remark 39.** One possible choice for the duality parameter is \( \nu = 1 \). However, it is better for our purposes to take \( \nu = q^{(1-n)/2n} = \sigma_n q^{(n-1)/2} \), where \( \sigma_n = q^{(1-n^2)/2n} \) is the square root of the (signed) coribbon element (Definition 24); see Lemma 41 and Remarks 29 and 42.

A.3.2. Twists. The following has been adapted to our purposes from [Kas95 Chapter XIV, Exercises 5-6].
The **convolution operator** $\star : H^* \otimes H^* \to H^*$ on the dual space $H^* = \text{Hom}_C(H, C)$ is defined by

\[
(f \star g)(x) \overset{\text{def}}{=} \sum_{(x)} f(x')g(x'').
\]

This operation makes $H^*$ into an algebra, with multiplicative unit $1_{H^*} = \epsilon_H$ the counit for $H$. Similarly, $\star$ operates on $(H \otimes H)^* = \text{Hom}_C(H \otimes H, C)$ by

\[
(r \star s)(x \otimes y) \overset{\text{def}}{=} \sum_{(x),(y)} r(x' \otimes y')s(x'' \otimes y'').
\]

The cobraided Hopf algebra $H$ is **coribbon**, meaning there exists an invertible central element $\zeta_H$ in $H^*$ such that

\[
\zeta_H \circ \mu_H = (r_H \circ \tau_{H,H}) \star r_H \star (\zeta_H \otimes \zeta_H), \quad \zeta_H(1_H) = 1, \quad \zeta_H \circ S_H = \zeta_H.
\]

Here, $\tau_{H,H} : H \otimes H \to H \otimes H$ is the swapping map $x \otimes y \mapsto y \otimes x$. Specifically, $\zeta_H \in H^*$ and its convolution inverse $\overline{\zeta}_H \in H^*$ are defined by

\[
\zeta_H(T^i_j) \overset{\text{def}}{=} \zeta_n \delta_{ij}, \quad \zeta_n \overset{\text{def}}{=} (-1)^{n-1}q^{(n^2-1)/n} \left( = (-1)^{n-1}\omega^{n(n^2-1)} \right),
\]

\[
\overline{\zeta}_H(T^i_j) \overset{\text{def}}{=} \overline{\zeta}_n \delta_{ij}, \quad \overline{\zeta}_n \overset{\text{def}}{=} (-1)^{n-1}q^{(1-n^2)/n} \left( = (-1)^{n-1}\omega^{n(1-n^2)} \right).
\]

Consequently, the braided category with left duality $H\text{-Comod}_f$ of finite-dimensional right $H$-comodules (with braidings $\overline{c}_{V,W}$, see Remark 38) is ribbon, with twist morphisms $\theta_V : V \to V$ defined by

\[
\theta_V(v) \overset{\text{def}}{=} \sum_{(v)} v_V \overline{\zeta}_H(v_H).
\]

**Remark 40.** Note that $\overline{\zeta}_n \in \mathbb{C} - \{0\}$ is what we previously called the ‘coribbon element’ in Definition 24; compare Remarks 29 and 39. We also refer to $\overline{\zeta}_H \in H^*$ as the **coribbon element**.

A.3.3. **Right dualities.** Moreover, the ribbon category $H\text{-Comod}_f$ of finite-dimensional right $H$-comodules (with braidings $\overline{c}_{V,W}$, see Remark 38) has right duality, with right duality
mappings $b'_V : \mathbb{C} \to V^* \otimes V$ and $d'_V : V \otimes V^* \to \mathbb{C}$ defined by

$$b'_V \overset{\text{def}}{=} (\text{id}_{V^*} \otimes \theta_V) \circ \tau_{V,V^*} \circ b_V, \quad d'_V \overset{\text{def}}{=} d_V \circ \tau_{V,V^*} \circ (\theta_V \otimes \text{id}_{V^*}).$$

A.4. **Ribbon sub-category** $C_V$ of $H$-Comod$_f$ coming from the quantum row-space.

A.4.1. **Quantum row-space.** The quantum row-space $A$ is the quotient of the free algebra in generators $(e^i)_{1 \leq j \leq n}$ by the relations $e^j e^i = q e^i e^j$ for $j > i$. The (infinite-dimensional) algebra $A$ is a right $H$-comodule (in fact, a right $H$-comodule-algebra), with coaction $\Delta_A : A \to A \otimes H$ defined by

$$\Delta_A(e^j) \overset{\text{def}}{=} \sum_{k=1}^n e^k \otimes T^j_k.$$

For each integer $d \geq 0$, let $V_d \subseteq A$ denote the (finite-dimensional) sub-space of $A$ consisting of homogeneous polynomials of degree $d$. Then $V_d$ is a right $H$-sub-comodule of $A$. For the remainder of this appendix, define $V \subseteq A$ by

$$V \overset{\text{def}}{=} V_1 = \text{span}_\mathbb{C}(e^1, e^2, \ldots, e^n).$$

Let $C_V$ denote the ribbon sub-category of $H$-Comod$_f$ generated by $V$. In particular, objects of $C_V$ are finite tensor products of $V$ and its dual $V^*$.

A.4.2. **Morphism formulas.** We will give explicit formulas for the braidings $\tau_{V,W}$ (see Remark 38), left dualities $b_V, d_V$, twists $\theta_V$, and right dualities $b'_V, d'_V$. Since we are working in the sub-category $C_V$, it suffices to compute the formulas for $V$ and $V^*$.

It can be shown that the braidings $\tau_{V,V}, \tau_{V^*,V^*}, \tau_{V,V^*}$ and $\tau_{V,V^*}$ are calculated by the same formulas as those provided in §5.2.2; see also Remark 29. The left dualities $b_V : \mathbb{C} \to V \otimes V^*$ and $d_V : V^* \otimes V \to \mathbb{C}$ are calculated by the formulas in §A.3.1, taking the symmetric duality parameter $\nu = q^{(1-n)/2n} = \tau_n q^{(n-1)/2}$; see Remark 39 and Lemma 41. The twists $\theta_V : V \to V$ and $\theta_{V^*} : V^* \to V^*$ can be computed by the formulas in §A.3.2 as

$$\theta_V(e^j) = \zeta_n e^j, \quad \theta_{V^*}(e^*_i) = \zeta_n e^*_i.$$
The right dualities $b'_V : \mathbb{C} \to V^* \otimes V$ and $d'_V : V \otimes V^* \to \mathbb{C}$ can be computed by the formulas in §A.3.3 as

$$b'_V(1) = (-1)^{n-1} \nu \sum_{k=1}^{n} q^{2k-n-1} e^*_k \otimes e^k, \quad d'_V(e^i \otimes e^*_j) = (-1)^{n-1} \nu^{-1} q^{n-2i+1} \delta_{ij}.$$

A.4.3. Matrix formulas. We will make an observation for the duality morphisms similar to Fact 27. This will require our choice of the symmetric duality parameter $\nu = \sigma n q^{(n-1)/2}$; see Remark 39.

Recall the matrix $U^q$ from §5.2.1 (See §5.1 for matrix conventions.) More generally, define for any scalar $\nu$ a matrix $U^q_\nu$ in $M_n(\mathbb{C})$ by

$$(U^q_\nu)^j_i \overset{\text{def}}{=} \nu(-q)^{i-n} \delta_{i,n-j+1}.$$

Note that the matrices $U^q = U^q_\nu$ agree for $\nu = \sigma n q^{(n-1)/2}$. Similarly, define $b_V(\nu)$, $d_V(\nu)$, $b'_V(\nu)$, and $d'_V(\nu)$ for any scalar $\nu$ to be the dualities defined in §A.3.1 and §A.3.3 (see also §A.4.2). Recall from §5.2.2 the ordered bases $\beta_{V^*V}$ and $\beta_{V\otimes V^*}$ of $V^* \otimes V$ and $V \otimes V^*$. When written in terms of the basis $\beta_{V^*V}$ (and the basis $\{1\}$ of $\mathbb{C}$), the left duality $b_V(\nu) : \mathbb{C} \to V \otimes V^*$ becomes a $n^2 \times 1$ matrix with coefficients $[b_V(\nu)]^1_{i,j}$. Similarly, when written in terms of these bases, the dualities $d_V(\nu) : V^* \otimes V \to \mathbb{C}$, $b'_V(\nu) : \mathbb{C} \to V^* \otimes V$, and $d'_V(\nu) : V \otimes V^* \to \mathbb{C}$ become $1 \times n^2$, $n^2 \times 1$, and $1 \times n^2$ matrices with coefficients $[d_V(\nu)]^1_{i,j}$, $[b'_V(\nu)]^1_{i,j}$, and $[d'_V(\nu)]^1_{i,j}$.

**Lemma 41** (defining property of the symmetric duality parameter $\nu$). The following equalities of matrix coefficients,

$$[b'_V(\nu)]^1_{i,j} = (U^q_\nu)^j_i, \quad [d'_V(\nu)]^1_{i,j} = ((\zeta_n)^{-1} U^q_\nu)^j_i, \quad [b_V(\nu)]^1_{i,j} = ((U^q_\nu)^T)^j_i, \quad [d_V(\nu)]^1_{i,j} = ((\zeta_n)^{-1} (U^q_\nu)^T)^j_i;$$

hold if and only if $\nu = \nu_\pm = \pm \sigma n q^{(n-1)/2}$, where $\sigma_n = q^{(1-n^2)/2n}$.
Proof. We display the case of \(d'_V(\nu)\). The other computations are similar. Put \(v_{ij} = (\beta_{V,V^*})_{ij} = e^i \otimes (-q)^{-j} e_{n-j+1}^*\). We calculate

\[
[d'_V(\nu)]_{ij}^{1} = d'_V(\nu)(v_{ij}) = (-q)^{-j}d'_V(\nu)(e^i \otimes e_{n-j+1}^*) = (-q)^{-j}(-1)^{n-1}\nu^{-1}q^{n-2i+1}\delta_{i,n-j+1} = (-1)^{j-1}q^{-j}\nu^{-1}q^{2j-n-1}\delta_{i,n-j+1} = (-1)^{j-1}q^{-j-1}\nu^{-1}\delta_{i,n-j+1},
\]
as well as

\[
((\overline{\zeta}_n)^{-1}U^q_{\nu})_{ij}^{1} = (-1)^{n-1}|\overline{\zeta}_n|^{-1}\nu(-q)^{-n}\delta_{j,n-i+1} = (-1)^{j-1}q^{1-n}|\overline{\zeta}_n|^{-1}\nu\delta_{i,n-j+1},
\]
where we define \(|\overline{\zeta}_n| = +q^{(1-n^2)/n}\). We gather

\[
[d'_V(\nu)]_{ij}^{1} = ((\overline{\zeta}_n)^{-1}U^q_{\nu})_{ij}^{1} \iff \nu^{-1} = q^{1-n}|\overline{\zeta}_n|^{-1}\nu \iff \pm\sigma_nq^{(n-1)/2} = \nu. \quad \square
\]

Remark 42. Compare the four analogous matrix equations in §5.2.1; see Figures 17 and 18. We will see the reason for the transposition of the indices \(i, j\) in the first two equalities of Lemma 41 when we discuss diagrammatics below. Essentially, this is because the tails of the U-turns in Figure 17, corresponding (see §A.5 and A.6) to the morphisms \(b'_V\) and \(d'_V\), are associated with the second tensor factor (measured from bottom to top). The opposite is true for the U-turns in Figure 18, corresponding to the morphisms \(b_V\) and \(d_V\).

The sign ambiguity in Lemma 41 is resolved by our need for the matrix \(U^q = U^q_{\nu}\). Why, in §5.2.1, was the matrix \(U^q\) preferred over \(-U^q = U^q_{-}\)? For instance, this was required for the quantum trace to satisfy the local isotopy Move (II); see Figure 28 and Equation (II). (Assuming we ask for the local monodromy matrices to have positive entries.)

Note that the first equation in Lemma 41 for \(b'_V(\nu)\), uniquely determines the value (including the sign) of the coribbon parameter \(\overline{\zeta}_n\) (see Remark 40), independent of \(\nu\).

A.5. Category \(\mathcal{R}\) of ribbons and the Reshetikhin–Turaev functor \(F_V : \mathcal{R} \to \mathcal{C}_V\).

A.5.1. Category of ribbons. The universal ribbon category \(\mathcal{R}\), also called the category of oriented ribbons, is defined exactly as in [Kas95, §XIV.5.1]. Roughly speaking, the objects are collections of oriented ribbon ends, and the morphisms are isotopy classes of oriented
ribs $L$ matching this end data. It is useful to, rather, think of ribbons as framed links, the link being the spine of the ribbon, and where the framing at a point on the link is normal to the tangent space of the ribbon at that point (in this way of thinking about the framing we differ from \[Kas95\], but it is immaterial mathematically, so long as we work with framed links rather than ribbons).

Ribbons live in the space $\mathbb{R}^2 \times [0, 1]$ having the usual $x, y, z$-coordinates. However, when drawing ribbon diagrams (which are, importantly, different from our previous pictures such as those appearing in Figures 17-21), the $x$-coordinate is drawn on the page horizontally right, the $z$-coordinate on the page vertically up, and the $y$-coordinate into the page. By definition, the ribbon ends lying on the same boundary plane in $\mathbb{R}^2 \times \{0, 1\}$ are required to have distinct $x$-coordinates (and isotopy preserves this property). The coordinates $x, y, z$ are called ribbon coordinates.

Ribbon diagrams always represent ribbons $L$ with the blackboard framing, meaning the constant framing in the $(0, -1, 0)$ direction, that is, out of the page toward the eye of the reader. Such a framing is always possible by introducing kinks into the ribbon. Here, a positive kink (Figure 3a) replaces a full right-handed twist, and a negative kink (Figure 3b) a full left-handed twist \[Kas95\] §X.8. Note that ribbon ends lying on $\mathbb{R}^2 \times \{0, 1\}$ are also required to have this blackboard framing.

A.5.2. Reshetikhin–Turaev functor. We will apply Theorem XIV.5.1 in \[Kas95\]. This says that there is a unique functor $F_V$ from the category $\mathcal{R}$ of ribbons to the ribbon category $\mathcal{C}_V$, which preserves the braiding, duality, and twist, and satisfies the property that a single downward-pointing (namely, negative $z$ direction) ribbon end (a distinguished object in $\mathcal{R}$) is mapped to $V$. (Consequently, upward-pointing ribbon ends are mapped to $V^*$.) In particular, $F_V$ provides an isotopy invariant of stated oriented ribbons $L$ (see below).

Diagrammatically speaking, we use exactly the same conventions for displaying morphisms in the category $\mathcal{C}_V$ as in \[Kas95\] Chapter XIV]. For example, in Figure 40 we show how the twist morphisms are displayed diagrammatically \[Kas95\] Chapters X.8 and XIV.5.1]; compare Figures 3a-3b as well as our calculations in §A.4.2 To help distinguish these
ribbon diagrams from our previous pictures, as in Figures 17-21, we put a white arrow on the boundary axis $\mathbb{R} \times \{0\} \times \{0\}$ indicating the positive $x$-direction.

\[
\theta_V = \zeta_n \quad \text{id}_V \\
\theta_{V^*} = \zeta_n \quad \text{id}_{V^*}
\]

**Figure 40.** Ribbon diagrams for twist morphisms.

A.6. **Alternative definition of the biangle quantum trace map.** We will prove part (A) of Proposition 30. The strategy is to use the Reshetikhin–Turaev functor $F_V$ to give an alternative definition of the quantum trace map $\text{Tr}_\mathcal{B}$ for a biangle $\mathcal{B}$, equivalent to the definition provided in §5.2. To do this, we need to be able to pass back and forth between the more symmetric topological setting of framed links $K$ in the thickened biangle $\mathcal{B} \times (0, 1)$ (which comes without any preferred parametrization—see the beginning of §5.2), and the less symmetric categorical setting of framed links $L$ in $\mathbb{R}^2 \times [0, 1]$ (where the parametrization matters—see, for example, Figure 40).

A.6.1. ‘Turning your head’. To pass between the two settings, we ‘turn our head’. That is, instead of viewing the thickened biangle $\mathcal{B} \times (0, 1)$ ‘from the top’ (as in Figures 17-21), instead we view it ‘from the side’. This can be done in two different ways, illustrated in Figure 41 (intuitively, from the perspective of Person G or that of Person D).

More precisely, let the thickened biangle $\mathcal{B} \times (0, 1) = [0, 1] \times \mathbb{R} \times (0, 1)$ have biangle coordinates $X, Y, Z$ with respect to a choice of parametrization $\mathcal{P}$ (intuitively, this choice is only to tell Person G and Person D where to stand, but the point is that where they stand
does not matter, as they will see the same answer). For example, in the left hand side of Figure 41 the $X$-coordinate is drawn on the page horizontally right, the $Y$-coordinate on the page vertically up, and the $Z$-coordinate out of the page toward the eye of the reader. Then, by Figure 41 and our discussion of ribbon coordinates in §A.5.1 if $x_G, y_G, z_G$ denote the ribbon coordinates from Person G’s perspective, the G-(ribbon) coordinate transformation $\varphi_G : \mathcal{B} \times (0, 1) \sim (0, 1) \times \mathbb{R} \times [0, 1] \hookrightarrow \mathbb{R}^2 \times [0, 1]$ is

$$\varphi_G(X, Y, Z) \overset{\text{def}}{=} (x_G, y_G, z_G) \overset{\text{def}}{=} (+Z, +Y, 1 - X).$$

Similarly, the D-(ribbon) coordinate transformation $\varphi_D : \mathcal{B} \times (0, 1) \sim (0, 1) \times \mathbb{R} \times [0, 1] \hookrightarrow \mathbb{R}^2 \times [0, 1]$ is

$$\varphi_D(X, Y, Z) \overset{\text{def}}{=} (x_D, y_D, z_D) \overset{\text{def}}{=} (+Z, -Y, +X).$$

Note that from either perspective the positive $x$-ribbon coordinate corresponds to the direction of increasing height in the thickened biangle $\mathcal{B} \times (0, 1)$.

Figure 41. Calculating the biangle quantum trace by ‘turning your head’. This can be done in two different ways, but if we choose the symmetric duality parameter $\nu = \sigma_n q^{(n-1)/2}$, depending on the square root of the (signed) coribbon element for the Hopf algebra $H = SL_n$, then the result is independent of which perspective, that of Person G or Person D, is taken; see Lemma 41.
A.6.2. Definition via the Reshetikhin–Turaev functor: topological setup. Fix a stated framed oriented link \((K, s)\) in the thickened biangle \(\mathcal{B} \times (0, 1);\) see §4.2. Recall that by definition the framing on the boundary \(\partial K\) points up in the vertical direction. That is, the framing vector is \((X, Y, Z) = (0, 0, 1)\) in biangle coordinates (§A.6.1); see the left hand side of Figure 41 where the ‘bullseyes’, indicating the tips of the framing vectors, point out of the page toward the eye of the reader. Recall that, by definition, elements of \(\partial K\) lying on the same boundary component of \(\mathcal{B} \times (0, 1)\) have distinct heights, namely, distinct Z-coordinates (Remark 20).

We now ‘turn our head’ as in §A.6.1, say from the perspective of Person G. More precisely, by applying the \(G\)-coordinate transformation \(\varphi_G\) we obtain a link \(L_G = \varphi_G(K)\) in \(\mathbb{R}^2 \times [0, 1]\). However, \(L_G\) is not yet a framed link, according to the definition in §A.5.1. Indeed, its framing vectors on the boundary \(\mathbb{R}^2 \times \{0, 1\}\) are all \((x_G, y_G, z_G) = (1, 0, 0)\) in ribbon coordinates. To fix this, we rotate each framing vector 90 degrees toward Person G, yielding an appropriate framing vector \((x_G, y_G, z_G) = (0, -1, 0)\). We call the resulting framed link again \(L_G\). Similarly, by this process we obtain a, possibly different, framed link \(L_D\) in \(\mathbb{R}^2 \times [0, 1]\) from Person D’s perspective; see Figure 41. We say that the new framed links \(L_G\) and \(L_D\) have been corrected.

Importantly, note that this process of correcting the framing may introduce a twist in the link. Indeed, as an example, on the right hand side of Figure 41 the framed link \(K\) acquires a right-handed twist from Person D’s perspective. On the other hand, there is no twist from Person G’s perspective.

Note also that the distinct Z-coordinates condition for the link boundary \(\partial K\) is consistent with the distinct \(x\)-coordinates condition for \(\partial L_G\) and \(\partial L_D\) (see §A.5.1).

A.6.3. Definition via the Reshetikhin–Turaev functor: algebraic setup. From Person G’s perspective, let \(N_G^i\) for \(i = 0, 1\) denote the number of points of the corrected framed link \(L_G\) lying on the boundary plane \(\mathbb{R}^2 \times \{i\}\). The framed link \(L_G\) comes with a pair of sequences \(V_G^i = ((V_G^i)_j)_{j=1,2,...,N_G^i}\) of vector spaces \((V_G^i)_j \in \{V, V^*\}\), where the sequence is ordered in the increasing \(x_G\)-direction; see §A.5.2. In other words, the sequences \(V_G^i\) come from evaluating the Reshetikhin–Turaev functor \(F_V\) on the domain and codomain objects of the framed link.
$L_G$ viewed as a morphism in the category of ribbons $\mathcal{R}$; see §A.3.1. Moreover, the evaluation of the functor $F_V$ on the framed link $L_G$ provides a linear map

$$F_V(L_G) : \bigotimes_{j=1,2,\ldots,N_G^0} (V_G^0)_j \to \bigotimes_{j=1,2,\ldots,N_G^1} (V_G^1)_j.$$ 

Similarly, define $N_D^i$ for $i = 0, 1$, sequences $(V_D^i)_{j=1,2,\ldots,N_D^i}$ of vector spaces, and a linear map $F_V(L_D)$ from Person D’s perspective.

For example, on the right hand side of Figure 41, we see $V_G^i = ((V_G^i)_1, (V_G^i)_2) = (V^*, V)$ and, as a degenerate case, $V_G^0 = (\mathbb{C})$. The corresponding linear map is $F_V(L_G) = b'_V : \mathbb{C} \to V^* \otimes V$ (see [Kas95, p.351]). On the other hand, from Person D’s perspective, $V_D^1 = (\mathbb{C})$ and $V_D^0 = ((V_D^0)_1, (V_D^0)_2) = (V, V^*)$ and the corresponding linear map is $F_V(L_D) = d_V \circ (\theta_V \otimes \text{id}_V) : V \otimes V^* \to \mathbb{C}$ (compare Figure 40).

Let $v_i \in V$ be the basis vector $v_i = e^i$. By abuse of notation, we also let $v_j \in V^*$ denote the covector $(-q)^{n-j} e_{n-j+1}^*$. Note that $\{v_i\}_{i=1,2,\ldots,n}$ provides an ordered basis for either $V$ or $V^*$. From Person G’s perspective, say, we may then consider the basis $\beta(V_G^i)$ for the tensor product $\bigotimes_{j=1,2,\ldots,N_G^i} (V_G^i)_j$ defined by

$$\beta(V_G^i)_{j_1j_2\ldots j_{N_G^i}} \overset{\text{def}}{=} v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_{N_G^i}} \left( j_1, j_2, \ldots, j_{N_G^i} \in \{1, 2, \ldots, n\} \right),$$

ordered as in §5.1. Similarly, from Person D’s perspective, we define a basis $\beta(V_D^i)$ for the tensor product $\bigotimes_{j=1,2,\ldots,N_D^i} (V_D^i)_j$. Note that this procedure recovers the familiar bases $\beta_{V,V}$, $\beta_{V^*,V^*}$, $\beta_{V^*,V}$, and $\beta_{V,V^*}$ for $V \otimes V$, $V^* \otimes V^*$, $V^* \otimes V$, and $V \otimes V^*$ used in §A.4.3 and §5.2.2.

A.6.4. Alternative definition of the SL$_n$-biangle quantum trace map via the Reshetikhin–Turaev functor. Recall that the framed link $K$ in the thickened biangle $\mathfrak{B} \times (0, 1)$ has in addition been equipped with a state $s$, meaning that to each point of the boundary $\partial K$ is associated a state-number in $\{1, 2, \ldots, n\}$. From Person G’s perspective, say, mimicking the sequences $V_G^i = ((V_G^i)_j)_j$ we obtain a pair of sequences of state-numbers $s_G^i = ((s_G^i)_j)_{j=1,2,\ldots,N_G^i}$ for $i = 0, 1$. Similarly, from Person D’s perspective, we obtain a pair of state sequences $s_D^i$. Note $s_G^0 = s_D^1$ and $s_G^1 = s_D^0$. 
For example, on the right hand side of Figure 41 we see $s_G^1 = ((s_G^1)_1, (s_G^1)_2) = (s_2, s_1)$ and $s_G^0 = \emptyset$ from Person G’s perspective. We also see $s_D^1 = \emptyset$ and $s_D^0 = ((s_D^0)_1, (s_D^0)_2) = (s_2, s_1)$ from Person D’s perspective. (Here and in Figure 41 the names of the states $s_1, s_2$ were arbitrary, taken from §5.2.1 and not intended to be ordered in any particular way.)

**Definition 43.** Choose one of the two parametrizations $\mathcal{P}$ of the thickened biangle $\mathcal{B} \times (0, 1)$, as in §A.6.1. Given a stated framed oriented link $(K, s)$ in the thickened biangle $\mathcal{B} \times (0, 1)$, from Person G’s perspective (Figure 41) define the complex number $\text{Tr}^\omega_{\mathcal{B}}(K, s) = \text{Tr}^\omega_{\mathcal{B}}(K, s)(\mathcal{P}) \in \mathbb{C}$, depending on the parametrization $\mathcal{P}$, by

$$
\text{Tr}^\omega_{\mathcal{B}}(K, s) \overset{\text{def}}{=} (\text{Tr}^\omega_{\mathcal{B}})_{\mathcal{G}}(K, s) = [F_V(L_G)]_{i_1 i_2 \cdots i_{N_G^0}}^{j_1 j_2 \cdots j_{N_G^1}}
$$

where $[F_V(L_G)]$ denotes the matrix of the linear map $F_V(L_G)$ written in the bases $\beta(V_G^0)$ and $\beta(V_G^1)$, and with the state-numbers $i_k = (s^1_G)_k$ and $j_k = (s^0_G)_k$. In addition, define the number (also depending on $\mathcal{P}$)

$$
(\text{Tr}^\omega_{\mathcal{B}})_{\mathcal{D}}(K, s) \overset{\text{def}}{=} [F_V(L_D)]_{i_1 i_2 \cdots i_{N_D^0}}^{j_1 j_2 \cdots j_{N_D^1}}
$$

from Person D’s perspective (also Figure 41), where $[F_V(L_D)]$ denotes the matrix of $F_V(L_D)$ written in the bases $\beta(V_D^0)$ and $\beta(V_D^1)$, and with the states $i_k = (s^1_D)_k$ and $j_k = (s^0_D)_k$.

**A.6.5. Finishing the proof of Proposition 30.**

**Lemma 44 (symmetry of the SL$_n$-biangle quantum trace map).** For a given parametrization $\mathcal{P}$ of $\mathcal{B} \times (0, 1)$, we have $(\text{Tr}^\omega_{\mathcal{B}})_{\mathcal{G}}(K, s) = (\text{Tr}^\omega_{\mathcal{B}})_{\mathcal{D}}(K, s)$ for any stated framed oriented link $(K, s)$.

**Proof.** As a first case, consider the stated link $(K, s)$ displayed in Figure 41 (and Figure 17a). In §A.6.3 we saw that $F_V(L_G) = b'_V : \mathbb{C} \to V^* \otimes V$ and $F_V(L_D) = d'_V \circ (\theta_V \otimes \text{id}_V) : V \otimes V^* \to \mathbb{C}$. By Lemma 41 and the formula for the twist $\theta_V$ appearing in §A.4.2 we compute

$$(\text{Tr}^\omega_{\mathcal{B}})_{\mathcal{G}}(K, s) = [F_V(L_G)]_{s_2 s_1}^{1} = [b'_V]_{s_2 s_1}^{1} = (U_q^{s_2})_{s_1},$$
and
\[(\mathsf{Tr}_B^\omega)_D(K, s) = [F_V(L_D)]_{s_2 s_1} = [d'_V \circ (\theta_V \otimes \text{id}_V)]_{s_2 s_1} = \zeta_n [d'_V]_{s_2 s_1} = \zeta_n (\zeta_n)^{-(n-1)/2} (U^q)_{s_2, s_1} = (U^q)_{s_2, s_1},\]
as desired. (Recall the topological discussion surrounding Figure 41, where it is important that we have chosen the symmetric duality parameter \(\nu = \sigma_n q^{(n-1)/2}\).) The proof when the stated link \((K, s)\) is one of the three other kinds of U-turns (see Figures 17b and 18) is similar (two of these U-turns acquire left-handed twists in \(L_D\)).

Next, consider the stated link \((K, s)\) displayed in Figure 19a. Note that no twists are introduced when either \(L_G\) or \(L_D\) are corrected, because their constituent arcs connect different boundary components of the thickened biangle. We compute
\[(\mathsf{Tr}_B^\omega)_G(K, s) = [F_V(L_G)]_{s_2 s_1} = [\bar{c}_{V, V}]_{s_1 s_2} = (\mathcal{C}_{\text{same}} q^1)_{s_2, s_1},\]
see [Kas95, p.341], Remark 38 and Fact 27 as well as
\[(\mathsf{Tr}_B^\omega)_D(K, s) = [F_V(L_D)]_{s_2 s_1} = [\bar{c}_{V, V^*}]_{s_1 s_2} = (\mathcal{C}_{\text{same}} q^0)_{s_2, s_1},\]
see [Kas95, p.348] and Fact 27. The proofs for \((K, s)\) as in Figures 19b, 19c and 19d are similar. For the other type of crossing, say, the stated link \((K, s)\) displayed in Figure 20a, we compute in the same way
\[(\mathsf{Tr}_B^\omega)_G(K, s) = [F_V(L_G)]_{s_2 s_1} = [\bar{c}_{V, V^*}]_{s_1 s_2} = (\mathcal{C}_{\text{opp}} q^0)_{s_2, s_1},\]
\[(\mathsf{Tr}_B^\omega)_D(K, s) = [F_V(L_D)]_{s_2 s_1} = [\bar{c}_{V, V^*}]_{s_1 s_2} = (\mathcal{C}_{\text{opp}} q^1)_{s_1 s_2} = (\mathsf{Tr}_B^\omega)_D(K, s).\]
The proofs for \((K, s)\) as in Figures 20b, 20c and 20d are similar, as well as for the trivial strand (possibly with twists), Figure 21.

The argument for a general stated link \((K, s)\) follows from the previous special cases, together with the fact that the Reshetikhin–Turaev functor \(F_V\) satisfies the State Sum Property (essentially by definition) and is an isotopy invariant. Put \(K\) into a bridge position (§5.2.4) with respect to a partition \(0 = X_0 < X_1 < \cdots < X_p = 1\) in biangle-coordinates, call the
resulting link $K'$. This determines bridge-positions for $L'_G$ and $L'_D$ with respect to partitions $0 = (z_G)_0 < (z_G)_1 < \cdots < (z_G)_p = 1$ and $0 = (z_D)_0 < (z_D)_1 < \cdots < (z_D)_p = 1$ in $G$- and $D$-coordinates. Let $((L'_G)_i, (s_G)_i)$ and $((L'_D)_i, (s_D)_i)$ be the corresponding restricted stated links in $(0, 1) \times \mathbb{R} \times [(z_G)_i, (z_G)_{i+1}]$ and $(0, 1) \times \mathbb{R} \times [(z_D)_i, (z_D)_{i+1}]$, where for the moment the states $(s_G)_i$ and $(s_D)_i$ are chosen arbitrarily on the internal boundaries $z_G = (z_G)_i$ and $z_D = (z_D)_i$ for $0 < i < p$. Note that each component of $(L'_G)_i$ or $(L'_D)_i$ is either a U-turn, crossing, or trivial strand (possibly with twists) (we are abusing the word ‘component’ here, since we are considering a crossing to be a single component). Also, upon correction of $L'_G$ and $L'_D$, no components of any $(L'_G)_i$ acquire twists, while all U-turn components (and no others) of the $(L'_D)_i$ acquire twists, which are right-handed (resp. left-handed) when the U-turn component of $(L'_D)_i$ has boundary with $z_D$-coordinate $(z_D)_i$ (resp. $(z_D)_{i+1}$), namely, is a cap (resp. cup). By the coordinate transformation $x_G = x_D$, $y_G = -y_D$, $z_G = 1 - z_D$, the link $(L'_G)_i$ is related to the link $(L'_D)_{p-1-i}$ as follows: (1) as un-oriented links, they are the same, except cups (resp. caps) of the former flip to become caps (resp. cups) of the latter (with twists as described just above); (2) the orientations of U-turns, thought of as pointing along the positive or negative $x$-direction, are preserved; (3) the orientations of same (resp. opposite) direction crossings are flipped (resp. preserved); and, (4) the orientations of trivial strands are flipped. Write $(s_G)_i = ((s_G)_i^0, (s_G)_i^1)$ (as in §A.6.4, note $(s_G)_0^0 = s_G^0$ and $(s_G)_{p-1}^1 = s_G^1$), and $((s_G)_i^0|((L'_G)_i))_k$ the restriction of these states to a component $((L'_G)_i)_k$ of $(L'_G)_i$, and similarly for Person D. If it happens that $((s_G)_i^0) = ((s_D)_{p-1-i})^1$ and $((s_G)_i^1) = ((s_D)_{p-1-i})^0$, then by the previous special cases we have

$$[F_V((L'_G)_i)]((s_G)_i^0)((s_G)_i^1) = \prod_k[F_V(((L'_G)_i)_k)]((s_G)_i^0|((L'_G)_i)_k) = \prod_k[F_V(((L'_D)_{p-1-i})_k)]((s_G)_i^0|((L'_D)_{p-1-i})_k)$$

$$= [F_V((L'_D)_{p-1-i})]((s_G)_i^1)^0 = [F_V((L'_D)_{p-1-i})]((s_D)_{p-1-i})^1.$$
Since \( s_G^0 = s_B^1 \) and \( s_G^1 = s_B^0 \) (§A.6.4), it follows by the State Sum Formula that

\[
[F_V(L'_G)]_{s_G^0} = \sum_{\text{compatible } (s_G)_i} [F_V((L'_G)_0)]_{s_G^0} [F_V((L'_G)_1)]_{(s_G)_i} [F_V((L'_G)_{p-1})]_{s_G^1}
\]

\[
= \sum_{\text{compatible } (s_G)_i} [F_V((L'_D)_{p-1})]_{s_G^0} [F_V((L'_D)_{p-2})]_{(s_G)_i} [F_V((L'_D)_{0})]_{s_G^1}
\]

\[
= \sum_{\text{compatible } (s_D)_i} [F_V((L'_D)_{p-1})]_{s_D^0} [F_V((L'_D)_{p-2})]_{(s_D)_i} [F_V((L'_D)_{0})]_{s_D^1} = [F_V(L'_D)]_{s_D^0}.
\]

Lastly, by isotopy invariance,

\[
(\text{Tr}^\omega)_{G}(K, s) = [F_V(L_G)]_{s_G^0} = [F_V(L'_G)]_{s_G^0} = [F_V(L'_D)]_{s_D^0} = [F_V(L_D)]_{s_D^0} = (\text{Tr}^\omega)_{D}(K, s). \quad \Box
\]

**Proof of Proposition 30** Let \( (\text{Tr}^\omega_{\mathcal{B}})(K, s)(\mathcal{P}) \) be defined as in Definition 43 depending on the parametrization \( \mathcal{P} \) of the thickened biangle \( \mathcal{B} \times (0, 1) \). We first argue that this is independent of the choice of \( \mathcal{P} \). Indeed, if \( \mathcal{P}' \) is the other parametrization, then

\[
(\text{Tr}^\omega_{\mathcal{B}})(K, s)(\mathcal{P}') = (\text{Tr}^\omega_{\mathcal{B}})(K, s)(\mathcal{P}) = (\text{Tr}^\omega_{\mathcal{B}})(K, s)(\mathcal{P}) = (\text{Tr}^\omega_{\mathcal{B}})(K, s)(\mathcal{P}) = (\text{Tr}^\omega_{\mathcal{B}})(K, s)(\mathcal{P}),
\]

where the second equation is immediate from the definition, and the third equation is by Lemma 44.

It is then straightforward to check that \( (\text{Tr}^\omega_{\mathcal{B}})(K, s) \), so defined via the Reshetikhin–Turaev functor \( F_V \), agrees with the definition given in §5.2 (this is, more or less, implicit in the proof of Lemma 44). In particular, since \( F_V \) is isotopy invariant, so is \( (\text{Tr}^\omega_{\mathcal{B}})(K, s) \), which is what remained to be proved. \( \Box \)

**Appendix B. Computer calculations**

In the Mathematica code appearing at the end of this article: section 2.1 verifies the claims for the local SL\(_1\) example of §3.5.2 section 2.2 verifies the claims for the local SL\(_3\) example of §3.5.1 section 3.2.1 verifies the claims for Moves (I), (I.b), (I.c), (I.d) of §6.2 section 3.2.2 verifies the claims for Moves (II), (II.b) of §6.3 section 3.2.3 verifies the claims for Move (III) of §6.4 and, section 3.2.4 verifies the claims for Move (IV) of §6.5.
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(* SECTION 1: DEFINITIONS NEEDED FOR QUANTUM CALCULATIONS *)

(* number of coordinates *)
a different definition is given in Section 3 *)
n = 12;

(* Poisson structure matrix *)
(* Z₃₁, Z₂₂, Z₁₃, Z₃₄, Z₂₅, Z₁₆, X₁₇, X₂₈, X₃₉, Z₃₁₀, Z₂₁₁, Z₁₁₂ *)

(* Encodes, e.g., Z₃₂₈·q₄(1)Z₂₈Z₃, Z₃₂₈·q₄(2)Z₃₁₃Z₃, Z₃₂₈·q₄(-1)Z₃₂₈Z₂ *)

(* the following definition of P is for Section 2; *)
a different definition is given in Section 3 *)
P = {{(0, 1, 0, 2, 0, 0, 0, -2, 0, 0, 0, 0),
(-1, 0, 1, 0, 0, 0, 0, -2, 2, 0, 0, 0, 0),
(0, -1, 0, 0, 0, 0, 0, -2, -2, 0, 0, 0, 0),
(0, 0, 0, 0, 0, 0, 0, -2, -2, 0, 0, 0, 0),
(0, 2, -2, 0, 0, 0, 0, -2, 2, 0, 0, -2, 2),
(2, -2, 0, -2, 2, 0, 0, -2, 0, 0, 0, 0, 0),
(0, 0, 0, 0, 0, -2, 2, -2, 0, -2, 2, 0),
(0, 0, 0, 0, -2, 0, 0, -2, -2, 0, -1, 0),
(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0),
(0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)}};

(* i[]: For encoding a {monic} monomial in the n variables *)
(* E.g., Z₃₁(1/2)·Z₂₈^(-1/2)·Z₁₃(1/4)·Z₃ is encoded by [i[1,1/2], i[2,-1/2], i[3,1/4], i[1,1]] *)
i[____] := (2)[[]] + (4)[[2]]; (* a[]: Weyl ordering {[],[]} coefficient for a monomial *)
(* E.g., since [Z₃₂₈Z₂] = q₄(-1/2)Z₃₂₈Z₂Z₂, then a[i[1,1], i[2,1]] = q₄(-1/2) *)
a[_____] := Flatten[w];

temp = Sum[Sum[-(1/2)·Im[temp[[1]]]·Im[temp[[2]]]·P[Re[temp[[1]]], Re[temp[[2]]]]],
{1, 1, Length[tempw]}, {1, 1, Length[tempw]}]; q^temp;

(* Test: *)
(a[i[1,1], i[2,1]] + i[2,1], i[1,1])] *)

(* b[]: Coefficient resulting from re-ordering a monomial according to the order 1, 2,..., n above *)
(* E.g., since Z₂₈Z₃ = q₄(-1)Z₃₂₈Z₂Z₂, then b[i[2,1], i[1,1]] = q₄(-1), but b[i[1,1], i[2,1]] = 1 *)
b[_____] := temp = 0;
tempw = Flatten[w];

Do[ Do[If[Re[tempw[[Length[tempw] - i1 + 1]]] > 0, tempw = tempw + Im[tempw[[Length[tempw] - i1 + 1]]]
Im[tempw[[i2]]]·P[Re[tempw[[Length[tempw] - i1 + 1]]], Re[tempw[[i2]]]]],
{i1, 1, Length[tempw] - i1 + 2}, {i2, 1, Length[tempw]}],
{i1, Length[tempw]}]; q^temp;

(* Test: *)
(b[i[2,1], i[1,1]] + i[2,1], i[1,1])
(b[i[1,1], i[2,1]] + i[2,1], i[1,1])
(b[i[2,1], i[1,1], i[2,1]])
(* c[]: Prints a possibly re-ordered monomial as if the variables commuted *)
(* E.g., prints either Z₃₂₈Z₂ or Z₂₈Z₃ when Z₂₈Z₃ is entered, based on Mathematica's whims *)
(* the following definition of c[] is for Section 2; *)
a different definition is given in Section 3 *)
c[_____] := temp = ConstantArray[0, n];
tempw = Flatten[w];

Do[tempw[[i1]]] = tempw[[i1]] + Im[tempw[[i1]]], {i1, Length[tempw]}];

(* the following depends on the ordering of the n variables *)
(* Z₃₁=1, Z₂₂=2, Z₁₃=3, Z₃₄=4, Z₂₅=5, Z₁₆=6, X₁₇=7, X₂₈=8, X₃₉=9, Z₃₁₀=10, Z₂₁₁=11, Z₁₁₂ *)

Z₃·temp[1] + Z₂·temp[2] + Z₁·temp[3] + Z₃·temp[4] + Z₂·temp[5] + Z₁·temp[6] +
X₁·temp[7] + X₂·temp[8] + X₃·temp[9] + Z₃·temp[10] + Z₂·temp[11] + Z₁·temp[12];

(* Test: *)
(* c[[i[2,1],i[1,1]]] *)
(* c[[i[1,1],i[2,1]]] *)
(* c[[i[2,1],i[1,1],i[2,1]]] *)
(* f[m1,m2]: Computes [m_1]*[m_2] = q^r m, where m should be viewed as if in the order 1,2,...,n, but, as for c[], Mathematica may not present the monomial in the proper order, so, for instance, if the output is q^(-1) Z_2*Z_3, the correct output is actually q^(-1) Z_3*Z_2 *)
f[x_, y_] := a[x]*a[y]*b(x, y)*c((x, y));
(* Test: *)
(* f[[i[2,1],i[1,1]]] *)
(* f[[i[1,1],i[2,1]]] *)
(* f[[i[2,1],i[1,1],i[2,1]]] *)
(* g[m], like f[m1,m2] but for only one monomial m *)
(* only used in Section 3 *)
g[x_] := a[x]*b[x]*c[x];

(* SECTION 2: QUANTUM LEFT AND RIGHT MATRICES *)

(* SECTION 2.1: n= *)

4: CHECKING COMMUTATION RELATIONS FOR QUANTIZED 2x2 SL4 SUB-MATRICES *)

(* Encoding 2x2 sub-matrix, e.g. a := [[a_1]]*[[a_2]]*[[a_3]] *)
(* Recall *)
(* 2x3=1,2x2=2,2x1=3,3x3=4,2x5=5,2x6=6,X2=8,X3=9,3x9=10,2x11=11,1x11=12 *)
a1 = [i[1,1/4], i[2,2/4], i[3,3/4], i[4,1/4], i[5, 2/4], i[6, -1/4], i[7, -1/4], i[8, -2/4], i[9, -1/4], i[10, 0], i[11, 0], i[12, 0]];
a2 = [i[1, 1/4], i[2, 2/4], i[3, 3/4], i[4, 1/4], i[5, -2/4], i[6, -1/4], i[7, -1/4], i[8, 2/4], i[9, -1/4], i[10, 0], i[11, 0], i[12, 0]];
a3 = [i[1, 1/4], i[2, 2/4], i[3, 3/4], i[4, 1/4], i[5, -2/4], i[6, -1/4], i[7, 3/4], i[8, 2/4], i[9, -1/4], i[10, 0], i[11, 0], i[12, 0]];
b1 = [i[1, 1/4], i[2, 2/4], i[3, 3/4], i[4, -3/4], i[5, -2/4], i[6, -1/4], i[7, -1/4], i[8, -2/4], i[9, -1/4], i[10, 0], i[11, 0], i[12, 0]];
c1 = [i[1, 1/4], i[2, 2/4], i[3, -1/4], i[4, 1/4], i[5, -2/4], i[6, -1/4], i[7, -1/4], i[8, -2/4], i[9, -1/4], i[10, 0], i[11, 0], i[12, 0]];
c2 = [i[1, 1/4], i[2, 2/4], i[3, -1/4], i[4, 1/4], i[5, -2/4], i[6, -1/4], i[7, -1/4], i[8, 2/4], i[9, -1/4], i[10, 0], i[11, 0], i[12, 0]];
d1 = [i[1, 1/4], i[2, 2/4], i[3, -1/4], i[4, -3/4], i[5, -2/4], i[6, -1/4], i[7, -1/4], i[8, -2/4], i[9, -1/4], i[10, 0], i[11, 0], i[12, 0]];

In[18]:= (* Checking relations *)
(* a da-ad = (q-q^(-1))bc *)
x = Expand[(f[d1, a1] + f[d1, a2] + f[d1, a3]) - (f[a1, d1] + f[a2, d1] + f[a3, d1])]
y = Expand[(q - q^(-1)) * (f[b1, c1] + f[b1, c2])]
x = y

Out[18]= \sqrt{Z_1} Z_2 \sqrt{Z_3} + \sqrt{q} \sqrt{Z_1} Z_2 \sqrt{Z_3}
q^{3/2} \sqrt{X_1} \sqrt{X_3} \sqrt{Z_1} \sqrt{Z_2} \sqrt{Z_3} + \sqrt{X_1} \sqrt{X_3} \sqrt{Z_1} \sqrt{Z_2} \sqrt{Z_3}
Out[19]= \sqrt{q} \sqrt{Z_1} Z_2 \sqrt{Z_3}
q^{3/2} \sqrt{Z_1} Z_2 \sqrt{Z_3}
\sqrt{X_1} \sqrt{X_3} \sqrt{Z_1} \sqrt{Z_2} \sqrt{Z_3}
Out[20]= True

In[21]:= (* bc = cb *)
x = Expand[(f[b1, c1] + f[b1, c2])]
y = Expand[(f[c1, b1] + f[c2, b1])]
x = y

Out[21]= \sqrt{Z_1} Z_2 \sqrt{Z_3}
q^{3/2} \sqrt{Z_1} Z_2 \sqrt{Z_3}
Out[22]= \sqrt{X_1} \sqrt{X_3} \sqrt{Z_1} \sqrt{Z_2} \sqrt{Z_3}
q^{3/2} \sqrt{Z_1} Z_2 \sqrt{Z_3}
Out[23]= True

In[24]:= (* ca = qac *)
x = Expand[(f[c1, a1] + f[c1, a2] + f[c1, a3] + f[c2, a1] + f[c2, a2] + f[c2, a3])]
y = Expand[q * (f[a1, c1] + f[a2, c1] + f[a3, c1] + f[a1, c2] + f[a2, c2] + f[a3, c2])]
x = y

Out[24]= \sqrt{Z_1} Z_2 \sqrt{Z_3} \sqrt{Z_3} + \sqrt{X_1} \sqrt{X_3} \sqrt{Z_1} \sqrt{Z_2} \sqrt{Z_3}
q^{3/2} \sqrt{Z_1} Z_2 \sqrt{Z_3} \sqrt{Z_3}
\sqrt{X_1} \sqrt{X_3} \sqrt{Z_1} \sqrt{Z_2} \sqrt{Z_3}
Out[25]= \sqrt{X_1} \sqrt{X_3} \sqrt{Z_1} \sqrt{Z_2} \sqrt{Z_3}
q^{3/2} \sqrt{X_1} \sqrt{X_3} \sqrt{Z_1} \sqrt{Z_2} \sqrt{Z_3}
\sqrt{X_1} \sqrt{X_3} \sqrt{Z_1} \sqrt{Z_2} \sqrt{Z_3}
Out[26]= True
\( (*) \ dc = qcd \)
\[
\begin{align*}
x &= \text{Expand}\left[ f[d1, c1] + f[d1, c2] \right] \\
y &= \text{Expand}\left[ q\left[ f[c1, d1] + f[c2, d1] \right] \right] \\
x &= y
\end{align*}
\]

\( \text{Out[27]} = \)
\[
\frac{Z2 \sqrt{Z3}}{\sqrt{X1} \sqrt{X3} \sqrt{Z1} \sqrt{Z2} \sqrt{Z3}} + \frac{q^2 Z2 \sqrt{Z3}}{\sqrt{X1} X2 \sqrt{X3} \sqrt{Z1} \sqrt{Z2} \sqrt{Z3}}
\]

\( \text{Out[28]} = \)
\[
\frac{Z2 \sqrt{Z3}}{\sqrt{X1} \sqrt{X3} \sqrt{Z1} \sqrt{Z2} \sqrt{Z3}} + \frac{q^2 Z2 \sqrt{Z3}}{\sqrt{X1} X2 \sqrt{X3} \sqrt{Z1} \sqrt{Z2} \sqrt{Z3}}
\]

\( \text{Out[29]} = \text{True} \)

\( (*) \ ba = qab \)
\[
\begin{align*}
x &= \text{Expand}\left[ f[b1, a1] + f[b1, a2] + f[b1, a3] \right] \\
y &= \text{Expand}\left[ q\left[ f[a1, b1] + f[a2, b1] + f[a3, b1] \right] \right] \\
x &= y
\end{align*}
\]

\( \text{Out[30]} = \)
\[
\frac{Z1^{3/2} Z2 \sqrt{Z3}}{\sqrt{X1} \sqrt{X3} \sqrt{Z1} \sqrt{Z2} \sqrt{Z3}} + \frac{X1 Z1^{3/2} Z2 \sqrt{Z3}}{\sqrt{X1} X2 \sqrt{X3} \sqrt{Z1} \sqrt{Z2} \sqrt{Z3}} + \frac{q^2 Z1^{3/2} Z2 \sqrt{Z3}}{\sqrt{X1} X2 \sqrt{X3} \sqrt{Z1} \sqrt{Z2} \sqrt{Z3}}
\]

\( \text{Out[31]} = \)
\[
\frac{Z1^{3/2} Z2 \sqrt{Z3}}{\sqrt{X1} \sqrt{X3} \sqrt{Z1} \sqrt{Z2} \sqrt{Z3}} + \frac{X1 Z1^{3/2} Z2 \sqrt{Z3}}{\sqrt{X1} X2 \sqrt{X3} \sqrt{Z1} \sqrt{Z2} \sqrt{Z3}} + \frac{q^2 Z1^{3/2} Z2 \sqrt{Z3}}{\sqrt{X1} X2 \sqrt{X3} \sqrt{Z1} \sqrt{Z2} \sqrt{Z3}}
\]

\( \text{Out[32]} = \text{True} \)

\( (*) \ db = qbd \)
\[
\begin{align*}
x &= \text{Expand}\left[ f[d1, b1] \right] \\
y &= \text{Expand}\left[ q\left[ f[b1, d1] \right] \right] \\
x &= y
\end{align*}
\]

\( \text{Out[33]} = \)
\[
\frac{q^2 Z1^{3/2} Z2 \sqrt{Z3}}{\sqrt{X1} X2 \sqrt{X3} \sqrt{Z1} \sqrt{Z2} \sqrt{Z3}} + \frac{q^2 Z1^{3/2} Z2 \sqrt{Z3}}{\sqrt{X1} X2 \sqrt{X3} \sqrt{Z1} \sqrt{Z2} \sqrt{Z3}}
\]

\( \text{Out[34]} = \)
\[
\frac{q^2 Z1^{3/2} Z2 \sqrt{Z3}}{\sqrt{X1} X2 \sqrt{X3} \sqrt{Z1} \sqrt{Z2} \sqrt{Z3}} + \frac{q^2 Z1^{3/2} Z2 \sqrt{Z3}}{\sqrt{X1} X2 \sqrt{X3} \sqrt{Z1} \sqrt{Z2} \sqrt{Z3}}
\]

\( \text{Out[35]} = \text{True} \)
(* SECTION 2.1.2: n=4: CHECKING COMMUTATION RELATIONS FOR QUANTIZED 2x2 SUB-MATRIX OF RIGHT SL4 MATRIX *)

(* Encoding 2x2 sub-matrix, e.g. d := [[d_1]], [[d_2]], [[d_3]] *)

(* Recall 1 *)

(* Z3=1, Z2=2, Z1=3, Z3=4, Z2=5, Z1=6, X1=7, X2=8, X3=9, Z3=10, Z2=11, Z1=12 *)

a1 = {\{1, 1/4\}, \{2, -1/2\}, \{3, -1/4\}, \{4, 0\}, \{5, 0\},
\{6, 0\}, \{8, 1/4\}, \{7, 1/2\}, \{9, 1/4\}, \{10, 1/4\}, \{11, 1/2\}, \{12, 3/4\}};

b1 = {\{1, 1/4\}, \{2, -1/2\}, \{3, -1/4\}, \{4, 0\}, \{5, 0\}, \{6, 0\}, \{8, 1/4\},
\{7, -1/2\}, \{9, 1/4\}, \{10, 1/4\}, \{11, 1/2\}, \{12, -1/4\}};

b2 = {\{1, 1/4\}, \{2, -1/2\}, \{3, -1/4\}, \{4, 0\}, \{5, 0\}, \{6, 0\}, \{8, 1/4\},
\{7, 1/2\}, \{9, 1/4\}, \{10, 1/4\}, \{11, 1/2\}, \{12, -1/4\}};

c1 = {\{1, -3/4\}, \{2, -1/2\}, \{3, -1/4\}, \{4, 0\}, \{5, 0\}, \{6, 0\}, \{8, 1/4\},
\{7, 1/2\}, \{9, 1/4\}, \{10, 1/4\}, \{11, 1/2\}, \{12, 3/4\}};

d1 = {\{1, -3/4\}, \{2, -1/2\}, \{3, -1/4\}, \{4, 0\}, \{5, 0\}, \{6, 0\}, \{8, -3/4\},
\{7, -1/2\}, \{9, 1/4\}, \{10, 1/4\}, \{11, 1/2\}, \{12, -1/4\}};

d2 = {\{1, -3/4\}, \{2, -1/2\}, \{3, -1/4\}, \{4, 0\}, \{5, 0\}, \{6, 0\}, \{8, 1/4\},
\{7, -1/2\}, \{9, 1/4\}, \{10, 1/4\}, \{11, 1/2\}, \{12, -1/4\}};

d3 = {\{1, -3/4\}, \{2, -1/2\}, \{3, -1/4\}, \{4, 0\}, \{5, 0\}, \{6, 0\}, \{8, 1/4\},
\{7, 1/2\}, \{9, 1/4\}, \{10, 1/4\}, \{11, 1/2\}, \{12, -1/4\}};

(* da-ad = (q-a^(-1))bc *)

x = Expand[{f[d1, a1] + f[d2, a1] + f[d3, a1]} - {f[a1, d1] + f[a1, d2] + f[a1, d3]}]
y = Expand[{q - a^(-1)] * {f[b1, c1] + f[b2, c1]}]
x = y

Out[43]= 0

Out[44]= True

In[45]:= (* bc = cb *)

x = Expand[{f[b1, c1] + f[b2, c1]}]
y = Expand[{f[c1, b1] + f[c1, b2]}]
x = y

Out[46]= 0

Out[47]= True
\(x = \text{Expand}\{f[c_1, a_1]\}\)
\(y = \text{Expand}\{q*f[a_1, c_1]\}\)
\(x = y\)
\(x_1 x_2 x_3 z_1^{3/2} z_2 z_3^{3/2}\)
\(\sqrt{q}\ z_1 z_2 \sqrt{z_3}\)
\(x_1 x_2 x_3 z_1^{3/2} z_2 z_3^{3/2}\)
\(\sqrt{q}\ z_1 z_2 \sqrt{z_3}\)
\(\text{True}\)

\(x = \text{Expand}\{f[d_1, c_1] + f[d_2, c_1] + f[d_3, c_1]\}\)
\(y = \text{Expand}\{q*f[c_1, d_1] + f[c_1, d_2] + f[c_1, d_3]\}\)
\(x = y\)
\(\sqrt{x_3} \sqrt{z_1} z_2 z_3^{3/2}\)
\(\sqrt{\sqrt{x_2} \sqrt{z_3} z_2 z_3^{3/2}}\)
\(\sqrt{x_1 x_2 x_3} \sqrt{z_1} z_2 z_3^{3/2}\)
\(\sqrt{\sqrt{x_1 x_2 x_3} \sqrt{z_1} z_2 z_3^{3/2}}\)
\(\text{True}\)

\(x = \text{Expand}\{f[b_1, a_1] + f[b_2, a_1]\}\)
\(y = \text{Expand}\{q*f[a_1, b_1] + f[a_1, b_2]\}\)
\(x = y\)
\(q^{3/2} \sqrt{x_2} \sqrt{x_3} \sqrt{z_1} z_2 z_3^{3/2}\)
\(q^{3/2} x_1 \sqrt{x_2} \sqrt{x_3} \sqrt{z_1} z_2 z_3^{3/2}\)
\(\sqrt{z_1} z_2\)
\(\text{True}\)

\(x = \text{Expand}\{f[d_1, b_1] + f[d_2, b_1] + f[d_3, b_1] + f[d_1, b_2] + f[d_2, b_2] + f[d_3, b_2]\}\)
\(y = \text{Expand}\{q*f[b_1, d_1] + f[b_1, d_2] + f[b_1, d_3] + f[b_2, d_1] + f[b_2, d_2] + f[b_2, d_3]\}\)
\(x = y\)
\(\sqrt{x_3} z_2 z_3^{3/2}\)
\(\sqrt{x_2} \sqrt{x_3} z_2 z_3^{3/2}\)
\(\sqrt{x_1 \sqrt{x_2} \sqrt{x_3} z_2 z_3^{3/2}}\)
\(q^{3/2} x_1 \sqrt{x_2} \sqrt{x_3} z_2 z_3^{3/2}\)
\(\sqrt{z_1} \sqrt{z_1} z_2 z_3^{3/2}\)
\(\text{True}\)
(* SECTION 2.2: n=
3: CHECKING COMMUTATION RELATIONS FOR QUANTIZED 2x2 SL3 SUB-MATRICES *)

(* see Section 1 *)

n = 5;
(* W=1,Z=2,X=3,Z=4,W=5 *)
P = {{0, -1, 2, 0, 0},
     {1, 0, -2, 0, 2},
     {-2, 0, 2, -2},
     {0, 0, -2, 0, 1},
     {0, -2, 2, -1, 0}};
C[w_... :=(temp = ConstantArray[0, n];
          tempw = Flatten[w];
          Do[temp[[Re[tempw[[i]]]]]] = temp[[Re[tempw[[i]]]]] + Im[tempw[[i]]], {i, Length[tempw]};
          (* W1=1,Z1=2,W2=3,Z2=4,W3=5,Z3=6,X=7 *)
          W^temp[[1]]*Z^temp[[2]]*X^temp[[3]]*Z^temp[[4]]*W^temp[[5]]);

(* SECTION 2.2.1: n=
3: CHECKING COMMUTATION RELATIONS FOR QUANTIZED 2x2 SUB-MATRIX OF LEFT SL3 MATRIX *)

(* Encoding 2x2 sub-matrix *)
(* Recall *)
(* W=1,Z=2,X=3,Z=4,W=5 *)
a1 = {{1, 2/3}, {1, 2/3}, {1, 2/3}, {1, 2/3}, {1, 2/3}};
a2 = {{1, 2/3}, {1, 2/3}, {1, 2/3}, {1, 2/3}, {1, 2/3}};
b1 = {{1, 2/3}, {1, 2/3}, {1, 2/3}, {1, 2/3}, {1, 2/3}};
b2 = {{1, 2/3}, {1, 2/3}, {1, 2/3}, {1, 2/3}, {1, 2/3}};
c1 = {{1, 2/3}, {1, 2/3}, {1, 2/3}, {1, 2/3}, {1, 2/3}};
c2 = {{1, 2/3}, {1, 2/3}, {1, 2/3}, {1, 2/3}, {1, 2/3}};
d1 = {{1, 2/3}, {1, 2/3}, {1, 2/3}, {1, 2/3}, {1, 2/3}};
d2 = {{1, 2/3}, {1, 2/3}, {1, 2/3}, {1, 2/3}, {1, 2/3}};

(* Checking relations *)
(* da-ad = (q-q^(-1))bc *)
x = Expand[(f[d1, a1] + f[d1, a2]) - (f[a1, d1] + f[a2, d1])]
y = Expand[(q - q^(-1)) * f[b1, c1]]
x = y

Out[71]= \[\frac{-q^{11/9} W_1^{1/3} Z_2^{1/3}}{q^{2/9} W_1^{1/3} X_2^{1/3} Z_2^{1/3}}\]

Out[72]= \[\frac{W_1^{1/3} Z_2^{1/3}}{q^{11/9} W_1^{1/3} X_2^{1/3} Z_2^{1/3}}\]

Out[73]= True

(* bc = cb *)
x = Expand[f[b1, c1]]
y = Expand[f[c1, b1]]
x = y

Out[74]= \[\frac{W_1^{1/3} Z_2^{1/3}}{q^{2/9} W_1^{1/3} X_2^{1/3} Z_2^{1/3}}\]

Out[75]= \[\frac{W_1^{1/3} X_2^{1/3} Z_2^{1/3}}{q^{2/9} W_1^{1/3} X_2^{1/3} Z_2^{1/3}}\]

Out[76]= True
\begin{verbatim}
(* ca = qac *)
\)
x = Expand[(f[c1, a1] + f[c1, a2])]
y = Expand[q*(f[a1, c1] + f[a2, c1])]
x = y
Out[77]= W^1/3 W^2/3 Z^2/3 q^17/18 W^4/3 W^2/3 X^1/3 Z^2/3
\)
q^13/18 X^2/3 Z^2/3
\)
Out[78]= W^1/3 W^2/3 Z^2/3 q^17/18 W^4/3 W^2/3 X^1/3 Z^2/3
\)
q^13/18 X^2/3 Z^2/3
\)
Out[79]= True

(* dc = qcd *)
\)
x = Expand[f[d1, c1]]
y = Expand[q*f[c1, d1]]
x = y
Out[80]= Z^2/3
\)
q^13/18 W^2/3 W^1/3 X^2/3 Z^2/3
\)
Out[81]= Z^2/3
\)
q^13/18 W^2/3 W^1/3 X^2/3 Z^2/3
\)
Out[82]= True

(* ba = qab *)
\)
x = Expand[(f[b1, a1] + f[b1, a2])]
y = Expand[q*f[a1, b1] + f[a2, b1])]
x = y
Out[83]= q^23/18 W^4/3 Z^2/3
\)
W^1/3 X^2/3 Z^2/3 q^17/18 W^4/3 X^1/3 Z^2/3
\)
Out[84]= q^23/18 W^4/3 Z^2/3
\)
W^1/3 X^2/3 Z^2/3 q^17/18 W^4/3 X^1/3 Z^2/3
\)
Out[85]= True

(* db = qbd *)
\)
x = Expand[f[d1, b1]]
y = Expand[q*f[b1, d1]]
x = y
Out[86]= q^23/18 W^4/3 Z^2/3
\)
W^1/3 X^2/3 Z^2/3
\)
Out[87]= q^23/18 W^4/3 Z^2/3
\)
W^1/3 X^2/3 Z^2/3
\)
Out[88]= True

(* SECTION 2.2.2: n= 3: CHECKING COMMUTATION RELATIONS FOR QUANTIZED 2x2 SUB-MATRIX OF RIGHT SL3 MATRIX *)
\end{verbatim}
(* Encoding 2x2 sub-matrix *)
(* Recall *)
(* W=1,Z=2,X=3,Z′=4,W′=5 *)
\[a_1 = \{i[1, 1/3], i[2, 2/3], i[3, 1/3], i[4, 1/3], i[5, -1/3]\};\]
\[b_1 = \{i[2, 1/3], i[2, 2/3], i[3, 1/3], i[4, 1/3], i[5, -1/3]\};\]
\[c_1 = \{i[3, 1], i[2, 2/3], i[3, 1/3], i[4, 2/3], i[5, -1/3]\};\]
\[d_1 = \{i[4, 1/3], i[2, 1/3], i[3, 1/3], i[4, 1/3], i[5, -1/3]\};\]
\[d_2 = \{i[4, 1/3], i[3, 2/3], i[3, 1/3], i[4, -2/3], i[5, -1/3]\};\]

(* Checking relations *)
(* da-ad = (q-q^(-1))bc *)
\[x = \text{Expand}[\{f[d_1, a_1] + f[d_2, a_1] - (f[a_1, d_1] + f[a_1, d_2])\}]\]
\[y = \text{Expand}[\{q - q^(-1)\} * f[b_1, c_1]\] \]
\[x = y\]

Out[89] = \[-W^{2/3} X^{2/3} Z^{1/3} + q^{7/9} W^{2/3} X^{2/3} Z^{1/3} + q^{11/9} W^{2/3} Z^{1/3}\]

Out[90] = \[-W^{2/3} X^{2/3} Z^{1/3} + q^{7/9} W^{2/3} X^{2/3} Z^{1/3} + q^{11/9} W^{2/3} Z^{1/3}\]

Out[91] = True

(* bc = cb *)
\[x = \text{Expand}[f[b_1, c_1]\]
\[y = \text{Expand}[f[c_1, b_1]\]
\[x = y\]

Out[92] = \[-W^{2/3} X^{2/3} Z^{1/3} + q^{2/9} W^{2/3} Z^{1/3}\]

Out[93] = \[-W^{2/3} X^{2/3} Z^{1/3} + q^{2/9} W^{2/3} Z^{1/3}\]

Out[94] = True

(* ca = qac *)
\[x = \text{Expand}[f[c_1, a_1]\]
\[y = \text{Expand}[q*f[a_1, c_1]\]
\[x = y\]

Out[95] = \[q^{35/18} W^{2/3} X^{2/3} Z^{4/3} + W^{2/3} Z^{1/3}\]

Out[96] = \[q^{35/18} W^{2/3} X^{2/3} Z^{4/3} + W^{2/3} Z^{1/3}\]

Out[97] = True
\( x = \text{Expand}[f[d1, c1] + f[d2, c1]] \)
\( y = \text{Expand}[q \cdot f[c1, d1] + f[c1, d2]] \)
\( x = y \)
\( \frac{q^{5/18} W^{2/3} Z^{1/3}}{W^{2/3} X^{1/3} Z^{2/3}} + \frac{q^{11/18} W^{2/3} Z^{1/3}}{W^{2/3} Z^{4/3}} \)
\( \frac{q^{5/18} W^{2/3} Z^{1/3}}{W^{2/3} X^{1/3} Z^{2/3}} + \frac{q^{11/18} W^{2/3} Z^{1/3}}{W^{2/3} Z^{4/3}} \)
\( \text{True} \)
\( x = \text{Expand}[f[b1, a1]] \)
\( y = \text{Expand}[q \cdot f[a1, b1]] \)
\( x = y \)
\( \frac{W^{2/3} X^{2/3} Z^{1/3} Y^{2/3}}{q^{1/15} W^{2/3}} \)
\( \frac{W^{2/3} X^{2/3} Z^{1/3} Y^{2/3}}{q^{1/15} W^{2/3}} \)
\( \text{True} \)
\( x = \text{Expand}[f[d1, b1] + f[d2, b1]] \)
\( y = \text{Expand}[q \cdot f[b1, d1] + f[b1, d2]] \)
\( x = y \)
\( \frac{q^{5/18} W^{2/3}}{W^{2/3} X^{1/3} Z^{2/3}} + \frac{W^{2/3} X^{2/3}}{q^{15/16} W^{2/3} Z^{2/3} Z^{1/3}} \)
\( \frac{q^{5/18} W^{2/3}}{W^{2/3} X^{1/3} Z^{2/3}} + \frac{W^{2/3} X^{2/3}}{q^{15/16} W^{2/3} Z^{2/3} Z^{1/3}} \)
\( \text{True} \)
\( \text{(* SECTION 3: MOVES (I)-(IV) *)} \)
\( \text{(see Section 1 *)} \)
\( n = 7; \)
\( \text{(W1\{l, Z1\{l, W2\{l, Z2\{l, W3\{l, Z3\{l, X\{l, i.e. going around in order clockwise *)} \)
\( P = \{(0, -1, 0, 0, 0, -2, Z), \)
(1, 0, 2, 0, 0, -2), \)
(0, -2, 0, -1, 0, 0, 2), \)
(0, 0, 1, 0, 2, 0, -2), \)
(0, 0, 0, -2, 0, -1, Z), \)
(2, 0, 0, 0, 1, 0, -2), \)
(-2, 2, -2, 2, -2, 2, 0)); \)
\( C[(_\_\_)] := \{\text{temp} = \text{ConstantArray}[0, n]; \)
\( \text{temp} = \text{Flatten}[w]; \)
\( \text{Do[ temp[\text{Re}[tempw[[i]]]]] = temp[[\text{Re}[tempw[[i]]]] + \text{Im}[tempw[[i]]], \text{[i, Length[tempw]]]]; \)
\( \text{(* W1\{l, Z1\{l, W2\{l, Z2\{l, W3\{l, Z3\{l, X\{l, *)} \)
\( W1^\text{temp[\{l]}}, Z1^\text{temp[\{l]}}, W2^\text{temp[\{l]}], Z2^\text{temp[\{l]}], W3^\text{temp[\{l]}], Z3^\text{temp[\{l]}], X^\text{temp[\{l]}]}; \)
(* SECTION 3.1: ENCODING THE THREE LEFT MATRICES (LOWER CASE LETTERS) AND THREE RIGHT MATRICES (CAPITAL LETTERS) *)

(* W1=1, W2=3, W3=5, W4=7 *)

(* going from 1-edge to 2-edge *)

a3 = \{i[1, 2/3], i[2, 1/3], i[3, 1/3], i[4, 2/3], i[7, 2/3]\};
b31 = \{i[1, 2/3], i[2, 1/3], i[3, 1/3], i[4, -1/3], i[7, -1/3]\};
b32 = \{i[1, 2/3], i[2, 1/3], i[3, 1/3], i[4, -1/3], i[7, 2/3]\};
c3 = \{i[1, 2/3], i[2, 1/3], i[3, -2/3], i[4, -1/3], i[7, -1/3]\};
e3 = \{i[1, -1/3], i[2, 1/3], i[3, 1/3], i[4, -1/3], i[7, -1/3]\};
f3 = \{i[1, -1/3], i[2, 1/3], i[3, -2/3], i[4, -1/3], i[7, -1/3]\};
i3 = \{i[1, -1/3], i[2, -2/3], i[3, -2/3], i[4, -1/3], i[7, -1/3]\};
A3 = \{i[1, 1/3], i[2, 2/3], i[3, 2/3], i[4, 1/3], i[7, 1/3]\};
D3 = \{i[1, 1/3], i[2, 2/3], i[3, -1/3], i[4, 1/3], i[7, 1/3]\};
E3 = \{i[1, 1/3], i[2, -1/3], i[3, -1/3], i[4, 1/3], i[7, 1/3]\};
G3 = \{i[1, 1/3], i[2, 2/3], i[3, -1/3], i[4, -2/3], i[7, 1/3]\};
H3 = \{i[1, 1/3], i[2, -1/3], i[3, -1/3], i[4, -2/3], i[7, -2/3]\};
H32 = \{i[1, 1/3], i[2, -1/3], i[3, -1/3], i[4, -2/3], i[7, 1/3]\};
i3 = \{i[1, -2/3], i[2, -2/3], i[3, -2/3], i[4, -2/3], i[7, -2/3]\};

(* going from 2-edge to 3-edge *)

a1 = \{i[3, 2/3], i[4, 1/3], i[5, 1/3], i[6, 2/3], i[7, 2/3]\};
b11 = \{i[3, 2/3], i[4, 1/3], i[5, 1/3], i[6, -1/3], i[7, -1/3]\};
b12 = \{i[3, 2/3], i[4, 1/3], i[5, 1/3], i[6, -1/3], i[7, 2/3]\};
c1 = \{i[3, 2/3], i[4, 1/3], i[5, -2/3], i[6, -1/3], i[7, -1/3]\};
e1 = \{i[3, -1/3], i[4, 1/3], i[5, 1/3], i[6, -1/3], i[7, -1/3]\};
f1 = \{i[3, -1/3], i[4, 1/3], i[5, -2/3], i[6, -1/3], i[7, -1/3]\};
i1 = \{i[3, -1/3], i[4, -2/3], i[5, -2/3], i[6, -1/3], i[7, -1/3]\};
A1 = \{i[3, 1/3], i[4, 2/3], i[5, 2/3], i[6, 1/3], i[7, 1/3]\};
D1 = \{i[3, 1/3], i[4, 2/3], i[5, -1/3], i[6, 1/3], i[7, 1/3]\};
E1 = \{i[3, 1/3], i[4, -1/3], i[5, -1/3], i[6, 1/3], i[7, 1/3]\};
G1 = \{i[3, 1/3], i[4, 2/3], i[5, -1/3], i[6, -2/3], i[7, 1/3]\};
H1 = \{i[3, 1/3], i[4, -1/3], i[5, -1/3], i[6, -2/3], i[7, 2/3]\};
H12 = \{i[3, 1/3], i[4, -1/3], i[5, -1/3], i[6, -2/3], i[7, 1/3]\};
i1 = \{i[3, -2/3], i[4, -1/3], i[5, -1/3], i[6, -2/3], i[7, -2/3]\};

(* going from 3-edge to 1-edge *)

a2 = \{i[5, 2/3], i[6, 1/3], i[1, 1/3], i[2, 2/3], i[7, 2/3]\};
b21 = \{i[5, 2/3], i[6, 1/3], i[1, 1/3], i[2, -1/3], i[7, -1/3]\};
b22 = \{i[5, 2/3], i[6, 1/3], i[1, 1/3], i[2, -1/3], i[7, 2/3]\};
c2 = \{i[5, 2/3], i[6, 1/3], i[1, -2/3], i[2, -1/3], i[7, -1/3]\};
e2 = \{i[5, -1/3], i[6, 1/3], i[1, 1/3], i[2, -1/3], i[7, -1/3]\};
f2 = \{i[5, -1/3], i[6, 1/3], i[1, -2/3], i[2, -1/3], i[7, -1/3]\};
i2 = \{i[5, -1/3], i[6, -2/3], i[1, -2/3], i[2, -1/3], i[7, -1/3]\};
A2 = \{i[5, 1/3], i[6, 2/3], i[1, 2/3], i[2, 1/3], i[7, 1/3]\};
D2 = \{i[5, 1/3], i[6, 2/3], i[1, -1/3], i[2, 1/3], i[7, 1/3]\};
E2 = \{i[5, 1/3], i[6, -1/3], i[1, -1/3], i[2, 1/3], i[7, 1/3]\};
G2 = \{i[5, 1/3], i[6, 2/3], i[1, -1/3], i[2, -2/3], i[7, 1/3]\};
H2 = \{i[5, 1/3], i[6, -1/3], i[1, -1/3], i[2, -2/3], i[7, -2/3]\};
H22 = \{i[5, 1/3], i[6, -1/3], i[1, -1/3], i[2, -2/3], i[7, 1/3]\};
i2 = \{i[5, -2/3], i[6, -1/3], i[1, -1/3], i[2, -2/3], i[7, -2/3]\};

(* SECTION 3.2 *)

(* SECTION 3.2.1: CHECK OF MOVES (I) *)

(* MOVE (I) *)
\[ x = \text{Expand}\left(q^{(-1/3)} \times f[A1, c1] - q^{(-4/3)} \times \{f[D1, b11] + f[D1, b12]\} + q^{(-7/3)} \times f[G1, a1]\right) \]
\[ y = \text{Expand}[0] \]
\[ x = y \]
\[ \text{Out[160]} = 0 \]
\[ \text{Out[161]} = 0 \]
\[ \text{Out[162]} = \text{True} \]

\[ x = \text{Expand}\left(q^{(-1/3)} \times f[A1, f1] - q^{(-4/3)} \times f[D1, e1]\right) \]
\[ y = \text{Expand}[0] \]
\[ x = y \]
\[ \text{Out[163]} = 0 \]
\[ \text{Out[164]} = 0 \]
\[ \text{Out[165]} = \text{True} \]

\[ x = \text{Expand}\left[q^{(-1/3)} \times f[A1, i1]\right] \]
\[ y = \text{Expand}\left[q^{(-1/3)}\right] \]
\[ x = y \]
\[ \text{Out[166]} = \frac{1}{q^{1/3}} \]
\[ \text{Out[167]} = \frac{1}{q^{1/3}} \]
\[ \text{Out[168]} = \text{True} \]

\[ x = \text{Expand}\left[-q^{(-4/3)} \times \{f[E1, b11] + f[E1, b12]\} + q^{(-7/3)} \times \{f[H11, a1] + f[H12, a1]\}\right] \]
\[ y = \text{Expand}[0] \]
\[ x = y \]
\[ \text{Out[169]} = 0 \]
\[ \text{Out[170]} = 0 \]
\[ \text{Out[171]} = \text{True} \]

\[ x = \text{Expand}\left[-q^{(-4/3)} \times f[E1, e1]\right] \]
\[ y = \text{Expand}\left[-q^{(-4/3)}\right] \]
\[ x = y \]
\[ \text{Out[172]} = \frac{1}{q^{4/3}} \]
\[ \text{Out[173]} = \frac{1}{q^{4/3}} \]
\[ \text{Out[174]} = \text{True} \]
\[
\begin{align*}
\text{In}[178] := & \quad (* 13 *) \\
& x = \text{Expand}\left[q^{(-7/3)} \ast f[I1, a1]\right] \\
& y = \text{Expand}\left[q^{(-7/3)}\right] \\
& x = y \\
\text{Out}[178] := & \quad \frac{1}{q^{1/3}} \\
\text{Out}[179] := & \quad \frac{1}{q^{1/3}} \\
\text{Out}[177] := & \quad \text{True} \\
\text{In}[178] := & \quad (* \text{ MOVE } (I.b) *) \\
\text{In}[179] := & \quad (* 11 *) \\
& x = \text{Expand}\left[q^{(-7/3)} \ast f[c1, A1] - q^{(-4/3)} \ast \{f[b11, D1] + f[b12, D1]\} \ast q^{(-1/3)} \ast f[a1, G1]\right] \\
& y = \text{Expand}[0] \\
& x = y \\
\text{Out}[179] := & \quad 0 \\
\text{Out}[180] := & \quad 0 \\
\text{Out}[181] := & \quad \text{True} \\
\text{In}[182] := & \quad (* 21 *) \\
& x = \text{Expand}\left[q^{(-7/3)} \ast f[i1, A1]\right] \\
& y = \text{Expand}[0] \\
& x = y \\
\text{Out}[182] := & \quad 0 \\
\text{Out}[183] := & \quad 0 \\
\text{Out}[184] := & \quad \text{True} \\
\text{In}[185] := & \quad (* 31 *) \\
& x = \text{Expand}\left[q^{(-7/3)} \ast f[i1, A1]\right] \\
& y = \text{Expand}\left[q^{(-7/3)}\right] \\
& x = y \\
\text{Out}[185] := & \quad \frac{1}{q^{1/3}} \\
\text{Out}[186] := & \quad \frac{1}{q^{1/3}} \\
\text{Out}[187] := & \quad \text{True} \\
\text{In}[188] := & \quad (* 12 *) \\
& x = \text{Expand}\left[-q^{(-4/3)} \ast \{f[b11, E1] + f[b12, E1]\} \ast q^{(-1/3)} \ast \{f[a1, H11] + f[a1, H12]\}\right] \\
& y = \text{Expand}[0] \\
& x = y \\
\text{Out}[188] := & \quad 0 \\
\text{Out}[189] := & \quad 0 \\
\text{Out}[190] := & \quad \text{True}
\end{align*}
\]
\begin{verbatim}
In[191]= (* 22 *)
    x = Expand[-q^(-4/3)*f[e1, E1]]
    y = Expand[-q^(-4/3)]
    x == y

Out[191]= \(-\frac{1}{q^{4/3}}\)

Out[192]= \(-\frac{1}{q^{4/3}}\)

Out[193]= True

In[194]= (* 13 *)
    x = Expand[q^(-1/3)*f[a1, I1]]
    y = Expand[q^(-1/3)]
    x == y

Out[194]= \(\frac{1}{q^{1/3}}\)

Out[195]= \(\frac{1}{q^{1/3}}\)

Out[196]= True

In[197]= (* MOVE (I.c) *)

In[198]= (* 31 *)
    x = Expand[q^(-7/3)*f[a1, I1]]
    y = Expand[q^(-7/3)]
    x == y

Out[198]= \(\frac{1}{q^{7/3}}\)

Out[199]= \(\frac{1}{q^{7/3}}\)

Out[200]= True

In[201]= (* 22 *)
    x = Expand[-q^(-4/3)*f[e1, E1]]
    y = Expand[-q^(-4/3)]
    x == y

Out[201]= \(-q^{4/3}\)

Out[202]= \(-q^{4/3}\)

Out[203]= True
\end{verbatim}
\[ x = \text{Expand}\left[q^{7/3} \cdot \left\{ f[b11, I1] + f[b12, I1] \right\} - q^{4/3} \cdot \left\{ f[e1, H11] + f[e1, H12] \right\} \right] \]
\[ y = \text{Expand}[0] \]
\[ x = y \]

Out[204]= 0

Out[205]= 0

Out[206]= True

\[ x = \text{Expand}\left[q^{1/3} \cdot f[i1, A1] \right] \]
\[ y = \text{Expand}\left[q^{1/3} \right] \]
\[ x = y \]

Out[207]= q^{1/3}

Out[208]= q^{1/3}

Out[209]= True

\[ x = \text{Expand}\left[-q^{4/3} \cdot f[f1, E1] + q^{1/3} \cdot f[i1, D1] \right] \]
\[ y = \text{Expand}[0] \]
\[ x = y \]

Out[210]= 0

Out[211]= 0

Out[212]= True

\[ x = \text{Expand}\left[q^{7/3} \cdot f[c1, I1] - q^{4/3} \cdot \left\{ f[f1, H11] + f[f1, H12] \right\} + q^{1/3} \cdot f[i1, G1] \right] \]
\[ y = \text{Expand}[0] \]
\[ x = y \]

Out[213]= 0

Out[214]= 0

Out[215]= True

\[ x = \text{Expand}\left[q^{1/3} \cdot f[I1, a1] \right] \]
\[ y = \text{Expand}\left[q^{1/3} \right] \]
\[ x = y \]

Out[217]= q^{1/3}

Out[218]= q^{1/3}

Out[219]= True
\( (* \ 22 \ *) \)

\[
\begin{align*}
\text{x} &= \text{Expand}\left[-q^{4/3} \cdot f[E1, e1]\right] \\
y &= \text{Expand}\left[-q^{4/3}\right] \\
x &= y
\end{align*}
\]

\text{Out}[220]= -q^{4/3}

\text{Out}[221]= -q^{4/3}

\text{Out}[222]= \text{True}

\( (* \ 32 \ *) \)

\[
\begin{align*}
\text{x} &= \text{Expand}\left[{q^{1/3} \cdot \left[f[I1, b11] + f[I1, b12]\right] - q^{4/3} \cdot \left[f[H11, e1] - f[H12, e1]\right]}\right] \\
y &= \text{Expand}[0] \\
x &= y
\end{align*}
\]

\text{Out}[223]= 0

\text{Out}[224]= 0

\text{Out}[225]= \text{True}

\( (* \ 13 \ *) \)

\[
\begin{align*}
\text{x} &= \text{Expand}\left[{q^{7/3} \cdot f[A1, i1]}\right] \\
y &= \text{Expand}[q^{7/3}] \\
x &= y
\end{align*}
\]

\text{Out}[226]= q^{7/3}

\text{Out}[227]= q^{7/3}

\text{Out}[228]= \text{True}

\( (* \ 23 \ *) \)

\[
\begin{align*}
\text{x} &= \text{Expand}\left[-q^{4/3} \cdot f[E1, f1] + q^{7/3} \cdot f[D1, i1]\right] \\
y &= \text{Expand}[0] \\
x &= y
\end{align*}
\]

\text{Out}[229]= 0

\text{Out}[230]= 0

\text{Out}[231]= \text{True}

\( (* \ 33 \ *) \)

\[
\begin{align*}
\text{x} &= \text{Expand}\left[{q^{1/3} \cdot \left[f[I1, c1] - q^{4/3} \cdot \left[f[H11, f1] + f[H12, f1]\right] + q^{7/3} \cdot f[G1, i1]\right]}\right] \\
y &= \text{Expand}[0] \\
x &= y
\end{align*}
\]

\text{Out}[232]= 0

\text{Out}[233]= 0

\text{Out}[234]= \text{True}

\( (* \ \text{SECTION 3.2.2: CHECK OF MOVES (II) } *) \)

\( (* \ \text{MOVE (II) } *) \)
\( x = \text{Expand}[(g[a3])] \)
\( y = \text{Expand}[(q^\frac{-1}{3}) \cdot f[A2, G1]] \)
\( x = y \)
\( q^{1/9} W^{2/3} W^{1/3} X^{2/3} Z^{1/3} Z^{2/3} \)
\( q^{1/9} W^{2/3} W^{1/3} X^{2/3} Z^{1/3} Z^{2/3} \)
\( \text{True} \)

\( x = \text{Expand}[(g[b31] + g[b32])] \)
\( y = \text{Expand}[(q^\frac{-1}{3}) \cdot (f[A2, H11] + f[A2, H12])] \)
\( x = y \)
\( q^{5/18} W^{2/3} W^{1/3} Z^{1/3} + \frac{W^{1/3} W^{2/3} X^{2/3} Z^{1/3}}{q^{13/18} Z^{1/3}} \)
\( q^{5/18} W^{2/3} W^{1/3} Z^{1/3} + \frac{W^{1/3} W^{2/3} X^{2/3} Z^{1/3}}{q^{13/18} Z^{1/3}} \)
\( \text{True} \)

\( x = \text{Expand}[(g[c3])] \)
\( y = \text{Expand}[(q^\frac{-1}{3}) \cdot f[A2, I1]] \)
\( x = y \)
\( q^{4/9} W^{2/3} Z^{1/3} + \frac{W^{2/3} X^{1/3} Z^{2/3}}{q^{13/18} Z^{1/3}} \)
\( q^{4/9} W^{2/3} Z^{1/3} + \frac{W^{2/3} X^{1/3} Z^{2/3}}{q^{13/18} Z^{1/3}} \)
\( \text{True} \)

\( x = \text{Expand}[(0)] \)
\( y = \text{Expand}[(q^\frac{-4}{3}) \cdot f[E2, D1] + q^\frac{-1}{3}) \cdot f[D2, G1])] \)
\( x = y \)
\( 0 \)
\( 0 \)
\( \text{True} \)
\begin{verbatim}
In[248]= (* 22 *)
   x = Expand[(g[e3])]
   y = Expand[(-q^(-4/3) * f[E2, E1]
     + q^(-1/3) * {f[D2, H11] + f[D2, H12]})]
   x = y

Out[248]= W2^{1/3} Z1^{1/3}

Out[249]= q^{2/9} W1^{1/3} X^{1/3} Z2^{1/3}

Out[250]= W2^{1/3} Z1^{1/3}

Out[251]= True

In[252]= (* 23 *)
   x = Expand[(g[f3])]
   y = Expand[(q^(-1/3) * f[D2, I1])]
   x = y

Out[252]= Z1^{1/3}

Out[253]= q^{1/18} W1^{1/3} W2^{2/3} X^{1/3} Z2^{1/3}

Out[254]= True

In[255]= (* 31 *)
   x = Expand[(0)]
   y = Expand[(q^(-7/3) * f[I2, A1]
     - q^(-4/3) * {f[H21, D1] + f[H22, D1]}
     + q^(-1/3) * {f[G2, G1]})]
   x = y

Out[255]= 0

Out[256]= 0

Out[257]= True

In[258]= (* 32 *)
   x = Expand[(0)]
   y = Expand[(-q^(-4/3) * {f[H21, E1] + f[H22, E1]}
     + q^(-1/3) * {f[G2, H11] + f[G2, H12]})]
   x = y

Out[258]= 0

Out[259]= 0

Out[260]= True
\end{verbatim}
\begin{verbatim}
(* 33 *)
In[261]:=
x = Expand[(g[[13]])]
y = Expand[(q^(-1/3)*f[G2, I1])]
x = y

Out[261]=
y = \frac{1}{q^{2/9} W_{1}^{1/3} W_{2}^{2/3} X_{1}^{1/3} Z_{1}^{2/3} Z_{2}^{1/3}}

Out[262]=
y = \frac{1}{q^{2/9} W_{1}^{1/3} W_{2}^{2/3} X_{1}^{1/3} Z_{1}^{2/3} Z_{2}^{1/3}}

Out[263]= True

(* Move (II.b) *)

(* 11 *)
In[264]:=
x = Expand[q^(-1/3)*f[a2, c1]]
y = Expand[g[A3]]
x = y

Out[265]=
y = \frac{W_{1}^{1/3} W_{2}^{2/3} X_{1}^{1/3} Z_{1}^{2/3} Z_{2}^{1/3}}{q^{2/9}}

Out[266]=
y = \frac{W_{1}^{1/3} W_{2}^{2/3} X_{1}^{1/3} Z_{1}^{2/3} Z_{2}^{1/3}}{q^{2/9}}

Out[267]= True

(* 21 *)
In[268]:=
x = Expand[q^(-1/3)*f[a2, f1]]
y = Expand[g[D3]]
x = y

Out[269]=
y = \frac{q^{11/18} W_{1}^{1/3} X_{1}^{1/3} Z_{1}^{2/3} Z_{2}^{1/3}}{W_{2}^{1/3}}

Out[268]=
y = \frac{q^{11/18} W_{1}^{1/3} X_{1}^{1/3} Z_{1}^{2/3} Z_{2}^{1/3}}{W_{2}^{1/3}}

Out[270]= True

(* 31 *)
In[271]:=
x = Expand[q^(-1/3)*f[a2, i1]]
y = Expand[g[G3]]
x = y

Out[272]=
y = \frac{q^{6/9} W_{1}^{1/3} X_{1}^{1/3} Z_{1}^{2/3}}{W_{2}^{1/3} Z_{2}^{2/3}}

Out[271]=
y = \frac{q^{6/9} W_{1}^{1/3} X_{1}^{1/3} Z_{1}^{2/3}}{W_{2}^{1/3} Z_{2}^{2/3}}

Out[273]= True
\end{verbatim}
\begin{verbatim}
\textbf{In[12]}:
\texttt{x = Expand[q^-1/3 * ( f[b21, c1] + f[b22, c1]) - q^-4/3 * ( f[e2, b11] + f[e2, b12]) ]}
\texttt{y = Expand[0]}
\textbf{Out[12]}:
\texttt{0}
\textbf{Out[13]}:
\texttt{True}
\textbf{In[22]}:
\texttt{x = Expand[ q^-1/3 * ( f[b21, f1] + f[b22, f1]) - q^-4/3 * f[e2, b11] + f[e2, b12]) ]}
\texttt{y = Expand[ g[E3] ]}
\textbf{Out[22]}:
\texttt{0}
\texttt{0}
\texttt{True}
\textbf{In[32]}:
\texttt{x = Expand[ q^-1/3 * ( f[c2, c1] + f[c2, b11] + f[e2, b12]) + q^-7/3 * f[i2, a1]) ]}
\texttt{y = Expand[0]}
\textbf{Out[32]}:
\texttt{0}
\texttt{0}
\texttt{True}
\textbf{In[42]}:
\texttt{x = Expand[ q^-1/3 * ( f[c2, f1] - q^-4/3 * f[e2, e1]) ]}
\texttt{y = Expand[0]}
\textbf{Out[42]}:
\texttt{0}
\texttt{0}
\texttt{True}
\end{verbatim}
\((\ast \ 33 \ \ast)\)

\[
x = \text{Expand}\left[q^\left(-\frac{1}{3}\right) \cdot f[c2, i1]\right]
\]

\[
y = \text{Expand}[g[I3]]
\]

\[
x = y
\]

\[
q^{1/9} \cdot W_{1^{2/3}} W_{2^{1/3}} X_{2^{1/3}} Z_{1^{1/3}} Z_{2^{2/3}}
\]

\[
q^{1/9} \cdot W_{1^{2/3}} W_{2^{1/3}} X_{2^{1/3}} Z_{1^{1/3}} Z_{2^{2/3}}
\]

\[
\text{True}
\]

\((\ast \ \text{SECTION 3.2.3: CHECK OF MOVE (III) EXAMPLE} \ \ast)\)

\((\ast \ 11/11 \ \ast)\)

\[
x = \text{Expand}[f[a1, a1]]
\]

\[
y = \text{Expand}[f[a1, a1]]
\]

\[
x = y
\]

\[
q^{4/9} W_{2^{4/3}} W_{3^{2/3}} X^{4/3} Z_{2^{2/3}} Z_{3^{4/3}}
\]

\[
q^{4/9} W_{2^{4/3}} W_{3^{2/3}} X^{4/3} Z_{2^{2/3}} Z_{3^{4/3}}
\]

\[
\text{True}
\]

\((\ast \ 12/11 \ \ast)\)

\[
x = \text{Expand}[f[a1, b1] \cdot f[a1, b12]]
\]

\[
y = \text{Expand}\left[q \cdot \left(f[b11, a1] + f[b12, a1]\right) + (1 - q^{^2}) \cdot \left(f[a1, b11] + f[a1, b12]\right)\right]
\]

\[
x = y
\]

\[
W_{2^{4/3}} W_{3^{2/3}} X_{1^{3/3}} Z_{2^{2/3}} Z_{3^{1/3}} + W_{2^{4/3}} W_{3^{2/3}} X_{4^{3/3}} Z_{2^{2/3}} Z_{3^{1/3}}
\]

\[
q^{13/18} + q^{11/18}
\]

\[
W_{2^{4/3}} W_{3^{2/3}} X_{1^{3/3}} Z_{2^{2/3}} Z_{3^{1/3}} + W_{2^{4/3}} W_{3^{2/3}} X_{4^{3/3}} Z_{2^{2/3}} Z_{3^{1/3}}
\]

\[
q^{13/18} + q^{11/18}
\]

\[
\text{True}
\]

\((\ast \ 13/11 \ \ast)\)

\[
x = \text{Expand}[f[a1, c1]]
\]

\[
y = \text{Expand}\left[q \cdot f[c1, a1] + (1 - q^{^2}) \cdot f[a1, c1]\right]
\]

\[
x = y
\]

\[
q^{1/9} W_{2^{4/3}} X_{1^{3/3}} Z_{2^{2/3}} Z_{3^{1/3}} + W_{3^{1/3}}
\]

\[
q^{1/9} W_{2^{4/3}} X_{1^{3/3}} Z_{2^{2/3}} Z_{3^{1/3}} + W_{3^{1/3}}
\]

\[
\text{True}
\]
\[
\begin{align*}
\text{In[301]} &= (* \frac{12}{12} *) \\
&\quad \text{x} = \text{Expand}[f[a1, e1]] \\
&\quad \text{y} = \text{Expand}[f[e1, a1]] \\
&\quad \text{x} = \text{y} \\
\text{Out[301]} &= \frac{W2^{1/3} W3^{2/3} X^{1/3} Z2^{2/3} Z3^{1/3}}{q^{2/9}} \\
\text{Out[302]} &= \frac{W2^{1/3} W3^{2/3} X^{1/3} Z2^{2/3} Z3^{1/3}}{q^{2/9}} \\
\text{Out[303]} &= \text{True} \\
\text{In[304]} &= (* \frac{13}{12} *) \\
&\quad \text{x} = \text{Expand}[f[a1, f1]] \\
&\quad \text{y} = \text{Expand}[f[f1, a1]] \\
&\quad \text{x} = \text{y} \\
\text{Out[304]} &= \frac{q^{11/18} W2^{1/3} X^{1/3} Z2^{2/3} Z3^{1/3}}{W3^{1/3}} \\
\text{Out[305]} &= \frac{q^{11/18} W2^{1/3} X^{1/3} Z2^{2/3} Z3^{1/3}}{W3^{1/3}} \\
\text{Out[306]} &= \text{True} \\
\text{In[307]} &= (* \frac{13}{13} *) \\
&\quad \text{x} = \text{Expand}[f[a1, i1]] \\
&\quad \text{y} = \text{Expand}[f[i1, a1]] \\
&\quad \text{x} = \text{y} \\
\text{Out[307]} &= \frac{W2^{1/3} X^{1/3} Z3^{1/3}}{q^{2/9} W3^{1/3} Z2^{1/3}} \\
\text{Out[308]} &= \frac{W2^{1/3} X^{1/3} Z3^{1/3}}{q^{2/9} W3^{1/3} Z2^{1/3}} \\
\text{Out[309]} &= \text{True} \\
\text{In[310]} &= (* \frac{12}{21} *) \\
&\quad \text{x} = \text{Expand}[0] \\
&\quad \text{y} = \text{Expand}[\text{(q - q}^{\text{-1})} \times (f[e1, a1] - f[a1, e1])] \\
&\quad \text{x} = \text{y} \\
\text{Out[310]} &= 0 \\
\text{Out[311]} &= 0 \\
\text{Out[312]} &= \text{True} \\
\text{In[313]} &= (* \frac{13}{21} *) \\
&\quad \text{x} = \text{Expand}[0] \\
&\quad \text{y} = \text{Expand}[\text{(q - q}^{\text{-1})} \times (f[f1, a1] - f[a1, f1])] \\
&\quad \text{x} = \text{y} \\
\text{Out[313]} &= 0 \\
\text{Out[314]} &= 0 \\
\text{Out[315]} &= \text{True}
\end{align*}
\]
\begin{verbatim}
In[310]:= (* 13/31 *)
x = Expand[0]
y = Expand[(q - q^(-1)) \ast (f[i1, a1] - f[a1, i1])]
x = y
Out[310]= 0
Out[311]= 0
Out[312]= True

In[319]:= (* 21/11 *)
x = Expand[f[b11, a1] + f[b12, a1]]
y = Expand[q \ast (f[a1, b11] + f[a1, b12])]
x = y
Out[319]= \frac{q^{1/18} W^{4/3} Z^{2/3} X^{1/3} Z^{2/3} Z^{3/3}}{q^{13/18}}
Out[320]= \frac{q^{1/18} W^{4/3} W^{2/3} X^{1/3} Z^{2/3} Z^{3/3}}{q^{13/18}}
Out[321]= True

In[322]:= (* 22/11 *)
x = Expand[f[b11, b11] + f[b12, b11] + f[b11, b12] + f[b12, b12]]
y = Expand[f[b11, b11] + f[b12, b11] + f[b11, b12] + f[b12, b12]]
x = y
Out[322]= \frac{q^{19/9} W^{6/3} W^{2/3} Z^{2/3} Z^{2/3}}{X^{2/3} Z^{3/3}} + \frac{W^{4/3} W^{3/3} X^{1/3} Z^{2/3}}{q^{17/9} Z^{3/3}} + \frac{q^{1/9} W^{4/3} W^{3/3} X^{1/3} Z^{2/3} Z^{2/3}}{Z^{3/3}} + \frac{W^{4/3} W^{3/3} X^{4/3} Z^{2/3}}{q^{26/9} Z^{3/3}}
Out[323]= \frac{q^{19/9} W^{6/3} W^{2/3} Z^{2/3} Z^{2/3}}{X^{2/3} Z^{3/3}} + \frac{W^{4/3} W^{3/3} X^{1/3} Z^{2/3}}{q^{17/9} Z^{3/3}} + \frac{q^{1/9} W^{4/3} W^{3/3} X^{1/3} Z^{2/3} Z^{2/3}}{Z^{3/3}} + \frac{W^{4/3} W^{3/3} X^{4/3} Z^{2/3}}{q^{26/9} Z^{3/3}}
Out[324]= True

In[325]:= (* 23/11 *)
x = Expand[f[b11, c1] + f[b12, c1]]
y = Expand[q \ast (f[c1, b11] + f[c1, b12]) + (1 - q^2) \ast (f[b11, c1] + f[b12, c1])]
x = y
Out[325]= \frac{q^{17/18} W^{4/3} Z^{2/3}}{W^{3/3} X^{1/3} Z^{2/3}} + \frac{W^{4/3} X^{1/3} Z^{2/3}}{W^{3/3} X^{2/3} Z^{3/3}} + \frac{q^{1/18} W^{3/3} Z^{2/3}}{Q^{1/18} W^{3/3} Z^{3/3}}
Out[326]= True
Out[327]= True
\end{verbatim}
\[\text{In}[328] = \text{Expand}\left[f[b_{11}, e_{1}] + f[b_{12}, e_{1}]\right] \]
\[x = y\]

\[\text{Out}[328] = x = y\]

\[\begin{align*}
W_{2}^{1/3}W_{3}^{2/3}Z_{2}^{2/3} + q^{7/18}W_{3}^{2/3}X_{1}^{1/3}Z_{2}^{2/3} \\
&\quad + q^{25/18}Z_{3}^{2/3}
\end{align*}\]

\[\text{Out}[329] = \frac{W_{2}^{1/3}W_{3}^{2/3}Z_{2}^{2/3}}{q^{7/18}X_{2}^{1/3}Z_{3}^{2/3}} + \frac{W_{2}^{1/3}W_{3}^{2/3}X_{1}^{1/3}Z_{2}^{2/3}}{q^{25/18}Z_{3}^{2/3}}\]

\[\text{Out}[330] = \text{True}\]

\[\text{In}[331] = \text{Expand}\left[f[b_{11}, f_{1}] + f[b_{12}, f_{1}]\right] \]
\[x = y\]

\[\text{Out}[331] = x = y\]

\[\begin{align*}
\frac{W_{2}^{1/3}Z_{2}^{2/3}}{q^{8/9}W_{3}^{1/3}X_{2}^{1/3}Z_{3}^{2/3}} &+ \frac{q^{4/9}W_{1}^{1/3}Z_{2}^{2/3}}{W_{3}^{1/3}Z_{3}^{2/3}} \\
&+ \frac{W_{2}^{1/3}Z_{2}^{2/3}}{q^{8/9}W_{3}^{1/3}X_{1}^{1/3}Z_{3}^{2/3}} + \frac{q^{4/9}W_{1}^{1/3}Z_{2}^{2/3}}{W_{3}^{1/3}Z_{3}^{2/3}}
\end{align*}\]

\[\text{Out}[332] = \text{True}\]

\[\text{In}[334] = \text{Expand}\left[f[b_{11}, i_{1}] + f[b_{12}, i_{1}]\right] \]
\[x = y\]

\[\text{Out}[334] = x = y\]

\[\begin{align*}
q^{11/18}W_{2}^{1/3} &+ \frac{W_{3}^{1/3}X_{2}^{1/3}Z_{1}^{1/3}Z_{3}^{2/3}}{q^{7/18}W_{3}^{1/3}Z_{1}^{1/3}Z_{3}^{2/3}} \\
&+ \frac{q^{11/18}W_{2}^{1/3}}{W_{3}^{1/3}X_{2}^{1/3}Z_{1}^{1/3}Z_{3}^{2/3}} + \frac{W_{3}^{1/3}X_{2}^{1/3}Z_{1}^{1/3}Z_{3}^{2/3}}{q^{7/18}W_{3}^{1/3}Z_{1}^{1/3}Z_{3}^{2/3}}
\end{align*}\]

\[\text{Out}[335] = \text{True}\]

\[\text{In}[337] = \text{Expand}\left[f[e_{1}, a_{1}]\right] \]
\[x = y\]

\[\text{Out}[337] = x = y\]

\[\begin{align*}
W_{2}^{1/3}W_{3}^{2/3}X_{1}^{1/3}Z_{2}^{2/3} &+ \frac{Z_{3}^{1/3}}{q^{2/9}} \\
&+ \frac{W_{2}^{1/3}W_{3}^{2/3}X_{1}^{1/3}Z_{2}^{2/3}Z_{3}^{1/3}}{q^{2/9}}
\end{align*}\]

\[\text{Out}[338] = \text{True}\]

\[\text{Out}[339] = \text{True}\]
\( x = \text{Expand}\left[ f[e_1, b_{11}] + f[e_1, b_{12}] \right] \)

\( y = \text{Expand}\left[ q^\left(-1\right) \ast \left(f[b_{11}, e_1] \ast f[b_{12}, e_1] + (1 - q^\left(-2\right)) \ast \left(f[e_1, b_{11}] + f[e_1, b_{12}]\right)\right) \right] \)

\( x = y \)

\( \frac{q^{11/18} W^{2/3} W^{3/3} Z^{2/3}}{X^{2/3} Z^{3/3}} \)

\( \frac{W^{2/3} W^{3/3} X^{1/3} Z^{2/3}}{q^{7/18} Z^{3/3}} \)

\( \frac{q^{11/18} W^{2/3} W^{3/3} Z^{2/3}}{X^{2/3} Z^{3/3}} \)

\( \frac{W^{2/3} W^{3/3} X^{1/3} Z^{2/3}}{q^{7/18} Z^{3/3}} \)

\( \text{Out[341]} = \text{True} \)

\( \text{Out[342]} = \text{True} \)

\( \text{Out[343]} = \text{True} \)

\( \text{Out[344]} = \text{True} \)

\( \text{Out[345]} = \text{True} \)

\( \text{Out[346]} = \text{True} \)

\( \text{Out[347]} = \text{True} \)

\( \text{Out[348]} = \text{True} \)

\( \text{Out[349]} = \text{True} \)

\( \text{Out[350]} = \text{True} \)

\( \text{Out[351]} = \text{True} \)
\( \text{(23/23)} \)
\( x = \text{Expand}[f[e_1, i_1]] \)
\( y = \text{Expand}[f[i_1, e_1]] \)
\( x = y \)
\( q^{1/9} \frac{W^{2/3} W^{3/3} X^{2/3} Z^{1/3} Z^{2/3}}{W^{2/3} W^{3/3} X^{2/3} Z^{1/3} Z^{2/3}} \)
\( q^{1/9} \frac{W^{2/3} W^{3/3} X^{2/3} Z^{1/3} Z^{2/3}}{W^{2/3} W^{3/3} X^{2/3} Z^{1/3} Z^{2/3}} \)
\( \text{True} \)

\( \text{(23/31)} \)
\( x = \text{Expand}[0] \)
\( y = \text{Expand}[(q - q^((-1))) \ast ((f[i_1, b_1] + f[i_1, b_1]) - (f[b_1, i_1] + f[b_1, i_1]))] \)
\( x = y \)
\( 0 \)
\( 0 \)
\( \text{True} \)

\( \text{(23/32)} \)
\( x = \text{Expand}[0] \)
\( y = \text{Expand}[(q - q^((-1))) \ast ((f[i_1, e_1] - (f[e_1, i_1]))] \)
\( x = y \)
\( 0 \)
\( 0 \)
\( \text{True} \)

\( \text{(31/11)} \)
\( x = \text{Expand}[f[c_1, a_1]] \)
\( y = \text{Expand}[q \ast f[a_1, c_1]] \)
\( x = y \)
\( q^{1/9} \frac{W^{2/3} W^{3/3} X^{2/3} Z^{1/3} Z^{2/3}}{W^{3/3}} \)
\( q^{1/9} \frac{W^{2/3} W^{3/3} X^{2/3} Z^{1/3} Z^{2/3}}{W^{3/3}} \)
\( \text{True} \)

\( \text{(32/11)} \)
\( x = \text{Expand}[f[c_1, b_1] + f[c_1, b_1]] \)
\( y = \text{Expand}[q \ast (f[b_1, c_1] + f[b_1, c_1])] \)
\( x = y \)
\( q^{35/18} \frac{W^{2/3} X^{1/3} Z^{2/3} Z^{2/3}}{W^{3/3} X^{1/3} Z^{2/3} Z^{2/3}} \)
\( q^{17/18} \frac{W^{2/3} X^{1/3} Z^{2/3} Z^{2/3}}{W^{1/3} X^{1/3} Z^{2/3} Z^{2/3}} \)
\( q^{35/18} \frac{W^{2/3} X^{1/3} Z^{2/3} Z^{2/3}}{W^{3/3} X^{1/3} Z^{2/3} Z^{2/3}} \)
\( q^{17/18} \frac{W^{2/3} X^{1/3} Z^{2/3} Z^{2/3}}{W^{1/3} X^{1/3} Z^{2/3} Z^{2/3}} \)
\( \text{True} \)
\( x = \text{Expand}\left[f[c1, c1]\right] \)
\( y = \text{Expand}\left[f[c1, c1]\right] \)
\( x = y \)
\[ q^{16/9} W^{4/3} Z^{2/3} \]
\[ W^{4/3} X^{2/3} Z^{2/3} \]
\[ q^{16/9} W^{4/3} Z^{2/3} \]
\[ W^{4/3} X^{2/3} Z^{2/3} \]
\( \text{Out[369]} = \text{True} \)

\( x = \text{Expand}\left[f[c1, e1]\right] \)
\( y = \text{Expand}\left[f[e1, c1]\right] \)
\( x = y \)
\[ q^{6/9} W^{1/3} Z^{2/3} \]
\[ W^{1/3} X^{2/3} Z^{2/3} \]
\[ q^{6/9} W^{1/3} Z^{2/3} \]
\[ W^{1/3} X^{2/3} Z^{2/3} \]
\( \text{Out[371]} = \text{True} \)

\( x = \text{Expand}\left[f[c1, f1]\right] \)
\( y = \text{Expand}\left[q^4 (-1) \times f[c1, c1]\right] \)
\( x = y \)
\[ q^{5/18} W^{1/3} Z^{2/3} \]
\[ W^{4/3} X^{2/3} Z^{2/3} \]
\[ q^{5/18} W^{1/3} Z^{2/3} \]
\[ W^{4/3} X^{2/3} Z^{2/3} \]
\( \text{Out[375]} = \text{True} \)

\( x = \text{Expand}\left[f[c1, i1]\right] \)
\( y = \text{Expand}\left[q^4 (-1) \times f[i1, c1]\right] \)
\( x = y \)
\[ W^{1/3} \]
\[ q^{5/9} W^{3/3} X^{2/3} Z^{2/3} \]
\[ W^{1/3} \]
\[ q^{5/9} W^{3/3} X^{2/3} Z^{2/3} \]
\( \text{Out[380]} = \text{True} \)
\begin{align*}
\text{In[370]} &= \text{(31/21*)} \\
&= x = \text{Expand}\left[f[f_1, a_1]\right] \\
&= y = \text{Expand}\left[f[a_1, f_1]\right] \\
&= x = y \\
\text{Out[370]} &= \frac{q^{11/18} W_2^{1/3} X_1^{1/3} Z_2^{2/3} Z_3^{1/3}}{W_3^{1/3}} \\
\text{Out[380]} &= \frac{q^{11/18} W_2^{1/3} X_1^{1/3} Z_2^{2/3} Z_3^{1/3}}{W_3^{1/3}} \\
\text{Out[381]} &= \text{True} \\
\text{In[382]} &= \text{(32/21*)} \\
&= x = \text{Expand}\left[f[f_1, b_{11}] + f[f_1, b_{12}]\right] \\
&= y = \text{Expand}\left[(f[b_{11}, f_1] + f[b_{12}, f_1]) + (q - q^4 (-1)) \ast (f[e_1, c_1])\right] \\
&= x = y \\
\text{Out[382]} &= \frac{q^{13/9} W_2^{1/3} Z_2^{2/3}}{W_3^{1/3} X_1^{1/3} Z_2^{2/3}} - \frac{q^{13/9} W_2^{1/3} Z_2^{2/3}}{W_3^{1/3} Z_3^{2/3}} \\
\text{Out[383]} &= \frac{q^{13/9} W_2^{1/3} Z_2^{2/3}}{W_3^{1/3} X_1^{1/3} Z_2^{2/3}} - \frac{q^{13/9} W_2^{1/3} Z_2^{2/3}}{W_3^{1/3} Z_3^{2/3}} \\
\text{Out[384]} &= \text{True} \\
\text{In[385]} &= \text{(33/21*)} \\
&= x = \text{Expand}\left[f[f_1, c_1]\right] \\
&= y = \text{Expand}\left[q^4 (-1) \ast f[c_1, f_1] + (1 - q^4 (-2)) \ast f[f_1, c_1]\right] \\
&= x = y \\
\text{Out[385]} &= \frac{q^{23/18} W_2^{1/3} Z_2^{2/3}}{W_3^{1/3} X_1^{1/3} Z_2^{2/3}} \\
\text{Out[386]} &= \frac{q^{23/18} W_2^{1/3} Z_2^{2/3}}{W_3^{1/3} X_1^{1/3} Z_2^{2/3}} \\
\text{Out[387]} &= \text{True} \\
\text{In[388]} &= \text{(32/22*)} \\
&= x = \text{Expand}\left[f[f_1, e_1]\right] \\
&= y = \text{Expand}\left[q \ast f[e_1, f_1]\right] \\
&= x = y \\
\text{Out[388]} &= \frac{Z_2^{2/3}}{q^{11/18} W_2^{1/3} W_3^{1/3} X_1^{1/3} Z_2^{2/3}} \\
\text{Out[389]} &= \frac{Z_2^{2/3}}{q^{11/18} W_2^{1/3} W_3^{1/3} X_1^{1/3} Z_2^{2/3}} \\
\text{Out[390]} &= \text{True}\end{align*}
\*

\( x = \text{Expand}\left[f\left[f_1, f_1\right]\right] \)
\( y = \text{Expand}\left[f\left[f_1, f_1\right]\right] \)
\( x = y \)
\[
\frac{Z_2^{2/3}}{q^{2/9} W_2^{2/3} W_3^{4/3} X^{2/3} Z_3^{2/3}}
\]
\[
\frac{Z_2^{2/3}}{q^{2/9} W_2^{2/3} W_3^{4/3} X^{2/3} Z_3^{2/3}}
\]
\( \text{Out[392]}= \text{True} \)

\*

\( x = \text{Expand}\left[q^{(-1)} f\left[i_1, f_1\right]\right] \)
\( y = \text{Expand}\left[q^{(-1)} f\left[i_1, f_1\right]\right] \)
\( x = y \)
\[
\frac{1}{q^{19/18} W_2^{2/3} W_3^{4/3} X^{2/3} Z_2^{1/3} Z_3^{2/3}}
\]
\[
\frac{1}{q^{19/18} W_2^{2/3} W_3^{4/3} X^{2/3} Z_2^{1/3} Z_3^{2/3}}
\]
\( \text{Out[396]}= \text{True} \)

\*

\( x = \text{Expand}\left[f\left[i_1, a_1\right]\right] \)
\( y = \text{Expand}\left[f\left[a_1, i_1\right]\right] \)
\( x = y \)
\[
W_2^{1/3} X^{1/3} Z_3^{1/3}
\]
\[
q^{2/9} W_3^{1/3} Z_2^{1/3}
\]
\( \text{Out[399]}= \text{True} \)

\*

\( x = \text{Expand}\left[f\left[i_1, b_11\right] + f\left[i_1, b_12\right]\right] \)
\( y = \text{Expand}\left[f\left[b_11, i_1\right] + f\left[b_12, i_1\right]\right] \)
\( x = y \)
\[
\frac{q^{11/18} W_2^{1/3}}{W_3^{1/3} X^{2/3} Z_2^{1/3} Z_3^{2/3}} + \frac{W_2^{1/3} X^{1/3}}{q^{17/18} W_3^{1/3} Z_2^{1/3} Z_3^{2/3}}
\]
\[
\frac{q^{11/18} W_2^{1/3}}{W_3^{1/3} X^{2/3} Z_2^{1/3} Z_3^{2/3}} + \frac{W_2^{1/3} X^{1/3}}{q^{17/18} W_3^{1/3} Z_2^{1/3} Z_3^{2/3}}
\]
\( \text{Out[401]}= \text{True} \)}
\[ (*) \frac{33}{31} \cdot \]
\[ x = \text{Expand}\left[f[i1, c1]\right] \]
\[ y = \text{Expand}\left[q^(-1) \cdot f[c1, i1] + (1 - q^(-2)) \cdot f[i1, c1]\right] \]
\[ x = y \]
\[ \text{Out[403]} = \begin{array}{l}
q^{4/9} W2^{1/3} \\
W3^{4/3} X^{2/3} Z^{2/3} Z^{2/3}
\end{array} \]
\[ \text{Out[404]} = \begin{array}{l}
q^{4/9} W2^{1/3} \\
W3^{4/3} X^{2/3} Z^{2/3} Z^{2/3}
\end{array} \]
\[ \text{Out[405]} = \text{True} \]
\[ (*) \frac{32}{32} \cdot \]
\[ x = \text{Expand}\left[f[i1, e1]\right] \]
\[ y = \text{Expand}\left[f[e1, i1]\right] \]
\[ x = y \]
\[ \text{Out[406]} = \begin{array}{l}
q^{1/9} \\
W2^{2/3} W3^{1/3} X^{2/3} Z^{2/3} Z^{2/3}
\end{array} \]
\[ \text{Out[407]} = \begin{array}{l}
q^{1/9} \\
W2^{2/3} W3^{1/3} X^{2/3} Z^{2/3} Z^{2/3}
\end{array} \]
\[ \text{Out[408]} = \text{True} \]
\[ (*) \frac{33}{32} \cdot \]
\[ x = \text{Expand}\left[f[i1, f1]\right] \]
\[ y = \text{Expand}\left[q^(-1) \cdot f[f1, i1] + (1 - q^(-2)) \cdot f[i1, f1]\right] \]
\[ x = y \]
\[ \text{Out[409]} = \begin{array}{l}
1 \\
q^{1/18} W2^{2/3} W3^{4/3} X^{2/3} Z^{2/3} Z^{2/3}
\end{array} \]
\[ \text{Out[410]} = \begin{array}{l}
1 \\
q^{1/18} W2^{2/3} W3^{4/3} X^{2/3} Z^{2/3} Z^{2/3}
\end{array} \]
\[ \text{Out[411]} = \text{True} \]
\[ (*) \frac{33}{33} \cdot \]
\[ x = \text{Expand}\left[f[i1, i1]\right] \]
\[ y = \text{Expand}\left[f[i1, i1]\right] \]
\[ x = y \]
\[ \text{Out[412]} = \begin{array}{l}
1 \\
q^{8/9} W2^{2/3} W3^{4/3} X^{2/3} Z^{2/3} Z^{2/3}
\end{array} \]
\[ \text{Out[413]} = \begin{array}{l}
1 \\
q^{8/9} W2^{2/3} W3^{4/3} X^{2/3} Z^{2/3} Z^{2/3}
\end{array} \]
\[ \text{Out[414]} = \text{True} \]
(* SECTION 3.2.4: CHECK OF MOVE (IV) EXAMPLE *)

(* 11/11 *)
\[ x = \text{Expand}\left[ \left( f[a3, A2] \right) \right] \]
\[ y = \text{Expand}\left[ q^4 (1/3) \ast \left( q^4 (-1) \ast f[A2, a3] \right) \right] \]
\[ x = y \]
\[ \text{Out[410]} = q^{7/9} w_1^{21/3} w_2^{1/3} w_3^{1/3} x z_{12/3} z_{22/3} z_{32/3} \]
\[ \text{Out[410]} = q^{7/9} w_1^{21/3} w_2^{1/3} w_3^{1/3} x z_{12/3} z_{22/3} z_{32/3} \]
\[ \text{Out[417]} = \text{True} \]

(* 12/11 *)
\[ x = \text{Expand}\left[ \left( f[a3, D2] \right) \right] \]
\[ y = \text{Expand}\left[ q^4 (1/3) \ast \left( f[D2, a3] \right) \right] \]
\[ x = y \]
\[ \text{Out[418]} = q^{23/18} w_1^{1/3} w_2^{1/3} w_3^{1/3} x z_{12/3} z_{22/3} z_{32/3} \]
\[ \text{Out[419]} = q^{23/18} w_1^{1/3} w_2^{1/3} w_3^{1/3} x z_{12/3} z_{22/3} z_{32/3} \]
\[ \text{Out[420]} = \text{True} \]

(* 13/11 *)
\[ x = \text{Expand}\left[ \left( f[a3, G2] \right) \right] \]
\[ y = \text{Expand}\left[ q^4 (1/3) \ast \left( f[G2, a3] \right) \right] \]
\[ x = y \]
\[ \text{Out[421]} = q^{6/9} w_1^{1/3} w_2^{1/3} w_3^{1/3} x z_{22/3} z_{32/3} z_{11/3} \]
\[ \text{Out[422]} = q^{6/9} w_1^{1/3} w_2^{1/3} w_3^{1/3} x z_{22/3} z_{32/3} z_{11/3} \]
\[ \text{Out[423]} = \text{True} \]

(* 12/12 *)
\[ x = \text{Expand}\left[ \left( f[a3, E2] \right) \right] \]
\[ y = \text{Expand}\left[ q^4 (1/3) \ast \left( f[E2, a3] \right) \right] \]
\[ x = y \]
\[ \text{Out[424]} = w_1^{1/3} w_2^{1/3} w_3^{1/3} x z_{12/3} z_{22/3} \]
\[ \text{Out[425]} = w_1^{1/3} w_2^{1/3} w_3^{1/3} x z_{12/3} z_{22/3} \]
\[ \text{Out[426]} = \text{True} \]

(* 13/12 *)
\[ x = \text{Expand}\left[ \left( f[a3, H21] \ast f[a3, H22] \right) \right] \]
\[ y = \text{Expand}\left[ q^4 (1/3) \ast \left( f[H21, a3] \ast f[H22, a3] \right) \right] \]
\[ x = y \]
\[ \text{Out[427]} = \frac{w_1^{1/3} w_2^{1/3} w_3^{1/3} z_{22/3}}{q^{11/18} z_{11/3} z_{31/3}} \]
\[ \text{Out[428]} = \frac{w_1^{1/3} w_2^{1/3} w_3^{1/3} z_{22/3}}{q^{11/18} z_{11/3} z_{31/3}} \]
\[ \text{Out[429]} = \text{True} \]
\( x = \text{Expand} \left[ \{ f[a3, I2] \} \right] \)

\( y = \text{Expand} \left[ q \left( 1/3 \right) \ast \{ f[I2, a3] \} \right] \)

\( x = y \)

\( \frac{q^{7/9} W_1^{1/3} W_2^{1/3} Z_1^{1/3} Z_3^{1/3}}{W_3^{2/3}} \)

\( \frac{q^{7/9} W_1^{1/3} W_2^{1/3} Z_1^{1/3} Z_3^{1/3}}{W_3^{2/3}} \)

\( True \)

\( x = \text{Expand} \left[ \{ f[b31, A2] + f[b32, A2] \} \right] \)

\( y = \text{Expand} \left[ q \left( 1/3 \right) \ast \{ q \left( -1 \right) \ast \{ f[A2, b31] + f[A2, b32] \} \} \right] \)

\( x = y \)

\( \frac{q^{17/18} W_1^{4/3} W_2^{1/3} W_3^{1/3} X Z_1^{1/3} Z_3^{1/3}}{Z_2^{1/3}} + \frac{W_1^{4/3} W_2^{1/3} W_3^{1/3} X Z_1^{1/3} Z_3^{1/3}}{Z_2^{1/3}} \)

\( \frac{q^{17/18} W_1^{4/3} W_2^{1/3} W_3^{1/3} X Z_1^{1/3} Z_3^{1/3}}{Z_2^{1/3}} + \frac{q^{1/18} Z_2^{1/3}}{Z_2^{1/3}} \)

\( True \)

\( x = \text{Expand} \left[ \{ f[b31, D2] + f[b32, D2] \} \right] \)

\( y = \text{Expand} \left[ q \left( 1/3 \right) \ast \{ f[D2, b31] + f[D2, b32] \} \ast \{ q \left( -1 \right) \ast q \ast f[A2, e3] \} \right] \)

\( x = y \)

\( \frac{W_1^{1/3} W_2^{1/3} W_3^{1/3} X Z_1^{1/3} Z_3^{1/3}}{q^{6/9} Z_2^{1/3}} + \frac{W_1^{4/3} W_2^{1/3} W_3^{1/3} X Z_1^{1/3} Z_3^{1/3}}{Z_2^{1/3}} \)

\( \frac{W_1^{1/3} W_2^{1/3} W_3^{1/3} X Z_1^{1/3} Z_3^{1/3}}{q^{6/9} Z_2^{1/3}} + \frac{W_1^{4/3} W_2^{1/3} W_3^{1/3} X Z_1^{1/3} Z_3^{1/3}}{Z_2^{1/3}} \)

\( True \)

\( x = \text{Expand} \left[ \{ f[b31, G2] + f[b32, G2] \} \right] \)

\( y = \text{Expand} \left[ q \left( 1/3 \right) \ast \{ f[G2, b31] + f[G2, b32] \} \right] \)

\( x = y \)

\( \frac{q^{11/18} W_1^{1/3} W_2^{1/3} W_3^{1/3} Z_1^{1/3} Z_3^{1/3}}{Z_2^{1/3}} + \frac{W_1^{4/3} W_2^{1/3} W_3^{1/3} X Z_1^{1/3} Z_3^{1/3}}{Z_2^{1/3}} \)

\( \frac{q^{11/18} W_1^{1/3} W_2^{1/3} W_3^{1/3} Z_1^{1/3} Z_3^{1/3}}{Z_2^{1/3}} + \frac{W_1^{4/3} W_2^{1/3} W_3^{1/3} X Z_1^{1/3} Z_3^{1/3}}{Z_2^{1/3}} \)

\( True \)
\texttt{In[442]=} (* 21/21 *)
\texttt{x = Expand[\{f[e3, A2]\}]}\par
\texttt{y = Expand[q^\{1/3\} * \{f[A2, e3]\}]}\par
\texttt{x = y} \par
\texttt{Out[442]= q^{4/9} W^{1/3} W^{1/3} Z^{1/3} Z^{1/3} Z^{2/3} Z^{1/3}}\par
\texttt{Out[443]= q^{4/9} W^{1/3} W^{1/3} Z^{1/3} Z^{1/3} Z^{2/3} Z^{1/3}}\par
\texttt{Out[444]= True} \par
\texttt{In[445]=} (* 22/21 *)
\texttt{x = Expand[\{f[e3, D2]\}]}\par
\texttt{y = Expand[q^\{1/3\} * \{f[A2, e3]\}]}\par
\texttt{x = y} \par
\texttt{Out[445]= W^{2/3} W^{1/3} Z^{1/3} Z^{1/3} Z^{1/3} Z^{2/3} q^{19/18} W^{1/3} Z^{1/3}}\par
\texttt{Out[446]= W^{2/3} W^{1/3} Z^{1/3} Z^{1/3} Z^{1/3} Z^{1/3} q^{19/18} W^{1/3} Z^{1/3}}\par
\texttt{Out[447]= True} \par
\texttt{In[448]=} (* 23/21 *)
\texttt{x = Expand[\{f[e3, G2]\}]}\par
\texttt{y = Expand[q^\{1/3\} * \{f[A2, e3]\}]}\par
\texttt{x = y} \par
\texttt{Out[448]= q^{1/9} W^{2/3} W^{1/3} Z^{2/3}} \par
\texttt{Out[449]= q^{1/9} W^{2/3} W^{1/3} Z^{2/3}} \par
\texttt{Out[450]= True} \par
\texttt{In[451]=} (* 12/22 *)
\texttt{x = Expand[\{f[b31, E2] + f[b32, E2]\}]}\par
\texttt{y = Expand[q^\{1/3\} * \{f[E2, b31] + f[E2, b32]\}]}\par
\texttt{x = y} \par
\texttt{Out[451]= W^{1/3} W^{1/3} W^{1/3} Z^{1/3} Z^{1/3} Z^{1/3} W^{1/3} W^{1/3} W^{1/3} X Z^{2/3} q^{1/18} Z^{1/3} Z^{1/3} Z^{1/3}} \par
\texttt{Out[452]= W^{1/3} W^{1/3} W^{1/3} Z^{1/3} Z^{1/3} Z^{1/3} W^{1/3} W^{1/3} W^{1/3} X Z^{2/3} q^{1/18} Z^{1/3} Z^{1/3} Z^{1/3}} \par
\texttt{Out[453]= True}
\[ x = \text{Expand}\left[ f[b31, H21] + f[b32, H21] + f[b31, H22] + f[b32, H22] \right] \]
\[ y = \text{Expand}\left[ q^{(1/3)} \times f[H21, b31] + f[H22, b31] + f[H21, b32] + f[H22, b32] \right] \]
\[ x = y \]
\[ \frac{W_{1^{1/3}} W_{2^{1/3}} W_{3^{1/3}}}{q^{18/9} Z_{1^{1/3}} Z_{2^{1/3}} Z_{3^{1/3}}} + \frac{q^{18/9} W_{1^{1/3}} W_{2^{1/3}} W_{3^{1/3}} X}{Z_{1^{1/3}} Z_{2^{1/3}} Z_{3^{1/3}}} + \frac{W_{1^{1/3}} W_{2^{1/3}} W_{3^{1/3}}}{X Z_{1^{1/3}} Z_{2^{1/3}} Z_{3^{1/3}}} \]
\[ \frac{W_{1^{1/3}} W_{2^{1/3}} W_{3^{1/3}}}{q^{18/9} Z_{1^{1/3}} Z_{2^{1/3}} Z_{3^{1/3}}} + \frac{q^{18/9} W_{1^{1/3}} W_{2^{1/3}} W_{3^{1/3}}}{X Z_{1^{1/3}} Z_{2^{1/3}} Z_{3^{1/3}}} + \frac{W_{1^{1/3}} W_{2^{1/3}} W_{3^{1/3}}}{X Z_{1^{1/3}} Z_{2^{1/3}} Z_{3^{1/3}}} \]
\[ \text{True} \]
\[(\ast \frac{23}{23} \ast)\]
\[
x = \text{Expand}\left[f(e3, I2)\right]
\]
\[
y = \text{Expand}\left[q^{(1/3)} \ast \{f[I2, e3]\}\right]
\]
\[
x = y
\]
\[
\frac{q^{4/9} W2^{1/3}}{W1^{2/3} W3^{2/3} X Z1^{1/3} Z3^{2/3}}
\]
\[
\frac{q^{8/9} W2^{1/3}}{W1^{2/3} W3^{2/3} X Z1^{1/3} Z3^{2/3} Z3^{1/3}}
\]
\[
\text{True}
\]

\[(\ast \frac{11}{31} \ast)\]
\[
x = \text{Expand}\left[f(c3, A2)\right]
\]
\[
y = \text{Expand}\left[q^{(1/3)} \ast \{q^{(-1)} \ast f[A2, c3]\}\right]
\]
\[
x = y
\]
\[
\frac{q^{16/9} W1^{2/3} W3^{1/3} Z1^{2/3} Z3^{2/3}}{W2^{2/3} Z2^{1/3}}
\]
\[
\frac{q^{16/9} W1^{2/3} W3^{1/3} Z1^{2/3} Z3^{2/3}}{W2^{2/3} Z2^{1/3}}
\]
\[
\text{True}
\]

\[(\ast \frac{12}{31} \ast)\]
\[
x = \text{Expand}\left[f(c3, D2)\right]
\]
\[
y = \text{Expand}\left[q^{(1/3)} \ast \{f[D2, c3] \ast (q^{(-1)} - q) \ast f[A2, f3]\}\right]
\]
\[
x = y
\]
\[
\frac{q^{5/18} W1^{2/3} W3^{1/3} Z1^{2/3} Z3^{2/3}}{W2^{2/3} Z2^{1/3}}
\]
\[
\frac{q^{5/18} W1^{2/3} W3^{1/3} Z1^{2/3} Z3^{2/3}}{W2^{2/3} Z2^{1/3}}
\]
\[
\text{True}
\]

\[(\ast \frac{13}{31} \ast)\]
\[
x = \text{Expand}\left[f(c3, G2)\right]
\]
\[
y = \text{Expand}\left[q^{(1/3)} \ast \{f[G2, c3] \ast (q^{(-1)} - q) \ast f[A2, f3]\}\right]
\]
\[
x = y
\]
\[
\frac{W1^{1/3} W3^{1/3} Z3^{2/3}}{q^{5/9} W2^{2/3} Z1^{1/3} Z2^{1/3}}
\]
\[
\frac{W1^{1/3} W3^{1/3} Z3^{2/3}}{q^{5/9} W2^{2/3} Z1^{1/3} Z2^{1/3}}
\]
\[
\text{True}
\]
\begin{verbatim}
In[478]:= (* 21/31 *)
   x = Expand[(f[f3, A2])]
   y = Expand[q^(1/3) * (f[A2, f3])]
   x = y
Out[478]= q^{23/18} W^{1/3} W^{1/3} Z^{2/3} Z^{2/3}
   W^{2/3} Z^{1/3}
Out[479]= q^{23/18} W^{1/3} W^{1/3} Z^{2/3} Z^{2/3}
   W^{2/3} Z^{1/3}
Out[480]= True

In[481]:= (* 22/31 *)
   x = Expand[(f[f3, D2])]
   y = Expand[q^(1/3) * (q^(-1) * f[D2, f3])]
   x = y
Out[481]= W^{3/3} Z^{2/3} Z^{2/3}
   q^{9/2} W^{2/3} W^{2/3} Z^{1/3}
Out[482]= W^{3/3} Z^{2/3} Z^{2/3}
   q^{9/2} W^{2/3} W^{2/3} Z^{1/3}
Out[483]= True

In[484]:= (* 23/31 *)
   x = Expand[(f[f3, G2])]
   y = Expand[q^(1/3) * (f[G2, f3] + (q^(-1) - q) * f[D2, f3])]
   x = y
Out[484]= W^{3/3} Z^{3/3}
   q^{19/18} W^{2/3} W^{2/3} Z^{1/3} Z^{1/3}
Out[485]= W^{3/3} Z^{3/3}
   q^{19/18} W^{2/3} W^{2/3} Z^{1/3} Z^{1/3}
Out[486]= True

In[487]:= (* 31/31 *)
   x = Expand[(f[f13, A2])]
   y = Expand[q^(1/3) * (f[A2, f13])]
   x = y
Out[487]= q^{4/9} W^{1/3} W^{1/3} Z^{2/3} Z^{2/3}
   W^{2/3} Z^{1/3} Z^{1/3}
Out[488]= q^{4/9} W^{1/3} W^{1/3} Z^{2/3} Z^{2/3}
   W^{2/3} Z^{1/3} Z^{1/3}
Out[489]= True
\end{verbatim}
\[
\begin{align*}
\text{In[480]} & = (32/31*) \\
& \quad \text{x = Expand}\left[\left\{f\left[i3, D2\right]\right\}\right] \\
& \quad \text{y = Expand}\left[\left(q^{(1/3)} * \left\{f\left[D2, i3\right]\right\}\right)\right] \\
& \quad \text{x = y} \\
\text{Out[480]} & = W3^{1/3} Z3^{2/3} \\
\text{Out[481]} & = q^{1/18} W1^{2/3} W2^{2/3} Z1^{1/3} Z2^{1/3} \\
\text{Out[482]} & = W3^{1/3} Z3^{2/3} \\
\text{Out[483]} & = q^{1/18} W1^{2/3} W2^{2/3} Z1^{1/3} Z2^{1/3} \\
\text{Out[484]} & = W3^{1/3} Z3^{2/3} \\
\text{Out[485]} & = q^{1/18} W1^{2/3} W2^{2/3} Z1^{1/3} Z2^{1/3} \\
\text{Out[486]} & = W3^{1/3} Z3^{2/3} \\
\text{Out[487]} & = q^{1/18} W1^{2/3} W2^{2/3} Z1^{1/3} Z2^{1/3} \\
\text{Out[488]} & = W3^{1/3} Z3^{2/3} \\
\text{Out[489]} & = W3^{1/3} Z3^{2/3} \\
\text{Out[490]} & = W3^{1/3} Z3^{2/3} \\
\text{Out[491]} & = W3^{1/3} Z3^{2/3} \\
\text{Out[492]} & = W3^{1/3} Z3^{2/3} \\
\text{Out[493]} & = W3^{1/3} Z3^{2/3} \\
\text{Out[494]} & = W3^{1/3} Z3^{2/3} \\
\text{Out[495]} & = W3^{1/3} Z3^{2/3} \\
\text{Out[496]} & = W3^{1/3} Z3^{2/3} \\
\text{Out[497]} & = W3^{1/3} Z3^{2/3} \\
\text{Out[498]} & = W3^{1/3} Z3^{2/3} \\
\text{Out[499]} & = W3^{1/3} Z3^{2/3} \\
\text{Out[500]} & = W3^{1/3} Z3^{2/3} \\
\text{Out[501]} & = \text{True}
\end{align*}
\]
\(x = \text{Expand}\left[\{f[f3, E2]\}\right]\)
\(y = \text{Expand}\left[q^\left(1/3\right) \ast \{q^\left(-1\right) \ast f[E2, f3]\}\right]\)
\(x = y\)
\(\frac{q^{5/18} W3^{1/3} Z1^{2/3}}{W1^{2/3} Z2^{1/3} Z3^{1/3}} + \frac{q^{5/18} W3^{1/3} Z1^{2/3}}{W1^{2/3} Z2^{1/3} Z3^{1/3}}\)
\(\text{True}\)

\(x = \text{Expand}\left[\{f[f3, H21] + f[f3, H22]\}\right]\)
\(y = \text{Expand}\left[q^\left(1/3\right) \ast \{f[H21, f3] + f[H22, f3] + (q^\left(-1\right) - q) \ast f[E2, f3]\}\right]\)
\(x = y\)
\(\frac{W3^{1/3}}{q^{6/9} W3^{1/3} W1^{2/3} Z1^{1/3} Z2^{1/3} Z3^{1/3}} + \frac{W3^{1/3}}{q^{6/9} W3^{1/3} W1^{2/3} Z1^{1/3} Z2^{1/3} Z3^{1/3}} + \frac{W3^{1/3}}{q^{6/9} W3^{1/3} W1^{2/3} Z1^{1/3} Z2^{1/3} Z3^{1/3}} + \frac{W3^{1/3}}{q^{6/9} W3^{1/3} W1^{2/3} Z1^{1/3} Z2^{1/3} Z3^{1/3}}\)
\(\text{True}\)

\(x = \text{Expand}\left[\{f[f3, E2]\}\right]\)
\(y = \text{Expand}\left[q^\left(1/3\right) \ast \{f[E2, f3]\}\right]\)
\(x = y\)
\(\frac{W3^{1/3}}{q^{6/9} W3^{1/3} W1^{2/3} Z1^{1/3} Z2^{1/3} Z3^{1/3}} + \frac{W3^{1/3}}{q^{6/9} W3^{1/3} W1^{2/3} Z1^{1/3} Z2^{1/3} Z3^{1/3}}\)
\(\text{True}\)

\(x = \text{Expand}\left[\{f[f3, H21] + f[f3, H22]\}\right]\)
\(y = \text{Expand}\left[q^\left(1/3\right) \ast \{f[H21, f3] + f[H22, f3] \ast (q^\left(-1\right) - q) \ast f[E2, f3]\}\right]\)
\(x = y\)
\(\frac{W3^{1/3}}{q^{11/18} W3^{1/3} W1^{2/3} Z1^{4/3} Z2^{1/3} Z3^{1/3}} + \frac{W3^{1/3}}{q^{11/18} W3^{1/3} W1^{2/3} Z1^{4/3} Z2^{1/3} Z3^{1/3}} + \frac{W3^{1/3}}{q^{11/18} W3^{1/3} W1^{2/3} Z1^{4/3} Z2^{1/3} Z3^{1/3}} + \frac{W3^{1/3}}{q^{11/18} W3^{1/3} W1^{2/3} Z1^{4/3} Z2^{1/3} Z3^{1/3}}\)
\(\text{True}\)
\begin{verbatim}
In[514]:= (* 13/33 *)
x = Expand[f[c3, I2]]
y = Expand[q^(1/3) * f[I2, c3]]
x = y

Out[514]= W1^{1/3}
q^{2/9} W2^{2/3} W3^{2/3} X Z1^{1/3} Z2^{1/3} Z3^{1/3}

Out[515]= W1^{1/3}
q^{2/9} W2^{2/3} W3^{2/3} X Z1^{1/3} Z2^{1/3} Z3^{1/3}

Out[516]= True

In[517]:= (* 23/33 *)
x = Expand[f[f3, I2]]
y = Expand[q^(1/3) * f[I2, f3]]
x = y

Out[517]= \frac{1}{q^{13/18} W1^{2/3} W2^{2/3} W3^{2/3} X Z1^{1/3} Z2^{1/3} Z3^{1/3}}

Out[518]= \frac{1}{q^{13/18} W1^{2/3} W2^{2/3} W3^{2/3} X Z1^{1/3} Z2^{1/3} Z3^{1/3}}

Out[519]= True

In[520]:= (* 33/33 *)
x = Expand[f[i3, I2]]
y = Expand[q^(1/3) * (q^(-1) * f[I2, i3])]
x = y

Out[520]= \frac{1}{q^{6/9} W1^{2/3} W2^{2/3} W3^{2/3} X Z1^{6/3} Z2^{1/3} Z3^{1/3}}

Out[521]= \frac{1}{q^{6/9} W1^{2/3} W2^{2/3} W3^{2/3} X Z1^{6/3} Z2^{1/3} Z3^{1/3}}

Out[522]= True
\end{verbatim}