POSTULATION OF DISJOINT UNIONS OF LINES AND A MULTIPLE POINT, II

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Abstract. We study the postulation of a general union $X \subset \mathbb{P}^3$ of one $m$-point $mP$ and $t$ disjoint lines. We prove that it has the expected Hilbert function, proving a conjecture by E. Carlini, M. V. Catalisano and A. V. Geramita.

1. Introduction

A scheme $X \subset \mathbb{P}^r$ is said to have maximal rank if for all integers $t > 0$ the restriction map $H^0(O_{\mathbb{P}^r}(t)) \to H^0(X, O_X(t))$ is either injective or surjective, i.e. if either $h^0(I_X(t)) = 0$ or $h^1(I_X(t)) = 0$, i.e. if $X$ imposes the “expected” number of conditions to the vector space of all homogeneous degree $t$ polynomials in $r + 1$ variables. R. Hartshorne and A. Hirschowitz proved that for all integers $t > 0$ and $r \geq 3$ a general union $X \subset \mathbb{P}^r$ of $t$ general lines has maximal rank. E. Carlini, M. V. Catalisano and A. V. Geramita considered several cases in which we allow unions of linear spaces with certain multiplicities [2], [3], [4]). We recall that for each $P \in \mathbb{P}^r$ the $m$-point $mP$ of $\mathbb{P}^r$ is the closed subscheme of $\mathbb{P}^r$ with $(I_P)^m$ as its ideal sheaf. E. Carlini, M. V. Catalisano and A. V. Geramita proved that for all $r \geq 4, m > 0$ and $d > 0$ a general union of an $m$-point and $d$ disjoint lines has maximal rank [4]). In the case $r = 3$ they proved that there are some exceptional cases (the one with $2 \leq d \leq m$ and $t = m$); in [4] the failure of maximal rank for these cases is exactly described, i.e. all positive integers $h^0(I_X(t))$ and $h^1(I_X(t))$ are computed [4, Theorem 4.2, part (ii)]. They conjectured in [4] that these are the only exceptional cases and proved the conjecture in some cases (e.g. if $m = 2$ by [4, Theorem 4.2, part (i)(e)]. In [4] their conjecture was proved when $m = 3$ and an asymptotic result was proved for arbitrary $m$ ([4, Propositions 1 and 2]). In this paper we prove their conjecture in the case $m = 3$, i.e. we prove the following result.

Theorem 1. Fix integers $m \geq 2$, $t > 0$ and $d > 0$. If $2 \leq d \leq m$, then assume $t \geq m + 1$. Let $Y \subset \mathbb{P}^3$ be a general union of $d$ lines. Then either $h^1(I_{mP \cup Y}(t)) = 0$ or $h^0(I_{mP \cup Y}(t)) = 0$.

A crucial step of the proof is contained in [4] Theorem 4.2, part (i)(c)]: the proof of the case $d = m + 2$ and $t = m + 1$. Let $Y \subset \mathbb{P}^3$ be a general union of $m + 2$ lines. They proved that $h^i(I_{mP \cup Y}(m + 1)) = 0$, $i = 0, 1$. After [7] and [6] it is well-known that if certain crucial curves or unions of curves and points, say $X_1$ and

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X₂, have \( h^i(\mathcal{I}_{X_2}(t_0)) = 0, i = 0, 1, \) and \( h^i(\mathcal{I}_{X_2}(t_0 + 1)) = 0, i = 0, 1, \) then it should be easy to control the postulation of all curves of degree \( \geq \deg(X_2) \) with respect to all forms of degree \( \geq t_0 + 2 \). In our case by [4] Theorem 4.2, part (i)(c) we may take \( X_1 = mP \cup Y \) with \( \deg(Y) = m + 2 \). The key part of the proof is the construction of a good \( X_2 \) for \( t_0 + 1 = m + 2 \) and then to control the cases \( t = t_0 + 3 \) and \( t = t_0 + 4 \).

We work over an algebraically closed field \( \mathbb{K} \). As far as we understand none of our quotations of [4] require the characteristic zero assumption made in [4].

2. Preliminaries

For any integer \( d > 0 \) let \( L(d) \) be the set of all unions \( Y \subset \mathbb{P}^3 \) of \( d \) disjoint lines. For any \( P \in \mathbb{P}^3 \) set \( L(P, d) := \{ Y \in L(d) : P \notin Y \} \). If \( P \) is a smooth point of a scheme \( T \) let \( \{ mP, T \} \) be the closed subscheme of \( T \) with \( (\mathcal{I}_{P,T})^m \) as its ideal sheaf. We write \( mP \) instead of \( \{ mP, \mathbb{P}^3 \} \). For any positive-dimensional \( A \subset \mathbb{P}^3 \) and any smooth point \( O \) of \( A \) a tangent vector of \( A \) with \( O \) as its support is a degree \( 2 \) connected zero-dimensional scheme \( v \subset A \) such that \( \deg(v) = 2 \) and \( v_{\text{red}} = \{ O \} \).

Let \( F \subset \mathbb{P}^3 \) be an surface. Set \( t := \deg(F) \). For each closed subscheme \( Z \subset \mathbb{P}^3 \) let \( \text{Res}_F(Z) \) denote the residual scheme of \( Z \) with respect to \( F \), i.e. the closed subscheme of \( \mathbb{P}^3 \) with \( \mathcal{I}_Z : \mathcal{I}_F \) as its ideal sheaf. If \( Z \) is reduced, then \( \text{Res}_F(Z) \) is the union of the irreducible components of \( Z \) not contained in \( F \). Now assume \( Z = mP \) for some \( m > 0 \) and some \( P \in \mathbb{P}^3 \). If \( P \notin F \), then \( \text{Res}_F(mP) = mP \). If \( P \) is a smooth point of \( F \), then \( \text{Res}_F(mP) = (m - 1)P \) (with the convention \( 0P = \emptyset \)).

For any integer \( x \geq t \) we have an exact sequence

\[
0 \to \mathcal{I}_{\text{Res}_F(Z)}(x - t) \to \mathcal{I}_Z(x) \to \mathcal{I}_{Z \cap F,F}(x) \to 0
\]

Hence

- \( h^0(\mathcal{I}_Z(x)) \leq h^0(\mathcal{I}_{\text{Res}_F(Z)}(x - t)) + h^0(F, \mathcal{I}_{Z \cap F,F}(x)) \);
- \( h^1(\mathcal{I}_Z(x)) \leq h^1(\mathcal{I}_{\text{Res}_F(Z)}(x - t)) + h^1(F, \mathcal{I}_{Z \cap F,F}(x)) \).

As in [2], [4] Lemma 3.3] and [4] we will call “the Castelnuovo’s inequality” any of these two inequalities. If \( F \) is either a plane or a smooth quadric, \( D \) is an effective divisor of \( F \) and \( Z \subset F \) is a closed subscheme of \( F \), \( \text{Res}_D(Z) \) is the closed subscheme of \( F \) with \( \mathcal{I}_{Z,F} : \mathcal{I}_{D,F} \) as its ideal sheaf (of course, \( \mathcal{I}_{D,F} \cong O_F(-D) \) as abstract line bundles on \( F \)). We also have the corresponding Castelnuovo’s exact sequence of \( \text{Res}_D \) and the associated Castelnuovo’s inequalities.

Set \( 0P := \emptyset \). We use the convention that \( \binom{t}{3} = 0 \) if \( -2 \leq t \leq 2 \) and \( \binom{t+2}{2} = 0 \) if \( -1 \leq t \leq 1 \). We have \( \deg(0P) = 0 = \binom{2}{3} \). For all integers \( m \geq 0 \) and \( k \geq 0 \) define the integers \( a_{m,k} \) and \( b_{m,k} \) by the relations

\[
(1) \quad \binom{m+1}{3} + (k+1)a_{m,k} + b_{m,k} = \binom{k+3}{3}, 0 \leq b_{m,k} \leq k
\]

If \( k \geq 2 \) from [11] for \( k, k-2, k-1 \) and \( m - 1 \) we get

\[
(2) \quad 2a_{m,k-2} + (k+1)(a_{m,k} - a_{m,k-2}) + b_{m,k} - b_{m,k+1} = (k+1)^2
\]

Taking the difference of (2) with \( k = m + 2 \) and the same equation with \( (m', k') = (m-1, m+1) \) and using that \( \binom{m+2}{3} - \binom{m+1}{3} = \binom{m+1}{2} \) and \( \binom{m+4}{2} - \binom{m+1}{2} = 3m + 6 \), we get

\[
(3) \quad a_{m-1,m+1} + (m+3)(a_{m,m+2} - a_{m-1,m+2}) + b_{m,m+2} - b_{m-1,m+1} = 3m + 6
\]
for all \( m > 0 \). Taking \( k = m + 2 \) in \((1)\) we get

\[
(m + 3)a_{m,m+2} + b_{m,m+2} = (3m^2 + 15m + 30)/2
\]

**Remark 1.** We have \( b_{m,m+1} = 0 \) and \( a_{m,m+1} = m + 2 \) for all \( m \). From \((4)\) we get that if \( m \) is even, then \( a_{m,m+2} = 3m/2 + 3 \) and \( b_{m,m+2} = 1 \), while if \( m \) is odd, then \( a_{m,m+2} = 3m/2 + 5/2 \) and \( b_{m,m+2} = m/2 + 5/2 \). Hence for all \( m \geq 3 \) we have \( a_{m-1,m+1} > m \), \( a_{m,m+2} = a_{m-1,m+1} + 2 \) if \( m \) is even and \( a_{m,m+2} = a_{m-1,m+1} + 1 \) if \( m \) is odd. We have \( (m+6) - (m+2) = 2m^2 + 12m + 10 \) and hence \( a_{m,m+3} = 2m + 4 \) and \( b_{m,m+3} = 4 \) for all \( m \geq 1 \), \( a_{0,3} = 6 \), \( b_{0,3} = 2 \). We have \( (m+7) - (m+2) = (5m^2 + 35m + 70)/2 \). If \( m \) is even and \( m \geq 6 \), then \( a_{m,m+4} = 5m/2 + 5 \) and \( b_{m,m+4} = 10 \). If \( m \in \{2, 4\} \), then \( a_{m,m+4} = 5m/2 + 6 \) and \( b_{m,m+4} = 5 - m \). If \( m \) is odd and \( m \geq 17 \), then \( a_{m,m+4} = 5m/2 + 9/2 \) and \( b_{m,m+4} = (m + 25)/2 \). If \( m \in \{3, 5, 7, 9, 11, 13, 15\} \), then \( a_{m,m+4} = 5m/2 + 11/2 \) and \( b_{m,m+4} = (15 - m)/2 \).

For all positive integers \( m, d \) the critical value of the pair \((m, d)\) is the minimal integer \( k \geq m \) such that \( (m^2/3) + (k+1)d \leq (k+2)/3 \). Let \( W \subset \mathbb{P}^3 \) be a union of \( d \) disjoint lines with \( P \notin W \). The scheme \( mP \cup W \) has maximal rank if and only if \( h^0(I_{mP \cup W}(k-1)) = 0 \) and \( h^1(I_{mP \cup W}(k)) = 0 \), where \( k \) is the critical value of \((m, d)\). Using \((2)\) it is easy to check that for a fixed integer \( m > 0 \) the sequence \( a_{m,k} \) is strictly increasing for all \( k \geq m - 1 \) (we have \( a_{m,m-1} = 0 \)). The integer \( k \) is the critical value of the pair \((m, d)\) if and only if \( a_{m,k-1} < d \leq a_{m,k} \).

3. Assertions \( B(m) \), \( R(m) \) and \( H_{m,k} \)

For every odd positive integer \( m \) we define Assertion \( B(m) \) in the following way.

**Assertion \( B(m) \), \( m \geq 1 \), \( m \) odd:** There is a 7-ple \((Y, L, R, S, O, H, v)\) with the following properties:

1. \( H \) is a plane containing \( P, L \) and \( R \) are lines of \( H, L \neq R, P \notin L \cup R \), and \( \{O\} := L \cap R \);
2. \( Y \) is a union of \( a_{m,m+2} \) disjoint lines, \( P \notin Y \) and \( Y \cap H \) is finite;
3. \( S \subset H \cap Y \), \( \sharp(S) = b_{m,m+2} - 2 \);
4. \( v \) is a disjoint union of \( b_{m,m+2} - 2 \) tangent vectors of \( \mathbb{P}^3 \), each of them with a point of \( S \) as its support;
5. \( \sharp(S \cap L) = (m+3)/4 \), \( \sharp(S \cap R) = (m+3)/4 \) and \( L \cap R \cap S = \emptyset \);
6. \( h^1(I_{mP \cup Y \cup L \cup O}(m+2)) = 0 \).

Take \((Y, L, R, S, O, H, v)\) satisfying the second, third and fourth of the conditions of \( B(m) \). We have \( h^0(O_{mP \cup Y \cup L \cup O}(m + 2)) = (m^2)/3 - 1 \) and hence \( h^1(I_{mP \cup Y \cup L \cup O}(m + 2)) = h^0(I_{mP \cup Y \cup L \cup O}(m + 2)) - 1 \).

For every even integer \( m \geq 2 \) we define Assertion \( B(m) \) in the following way.

**Assertion \( B(m) \), \( m \geq 2 \), \( m \) even:** There is a quadruple \((Y, L, R, H)\) with the following properties:

1. \( H \) is a plane containing \( P, L \) and \( R \) are lines of \( H, L \neq R, P \notin L \cup R \);
2. \( Y \) is a union of \( a_{m,m+2} \) disjoint lines, \( P \notin Y \) and \( Y \cap H \) is finite;
3. \( \sharp((Y \cap H) \cap L) = (m+2)/4 \), \( \sharp((Y \cap H) \cap R) = (m+2)/4 \);
4. \( h^1(I_{mP \cup Y}(m + 2)) = 0 \).

The last condition of \( B(m) \), \( m \) even, is equivalent to \( h^0(I_{mP \cup Y}(m + 2)) = 1 \).

**Lemma 1.** \( B(m) \) is true for all \( m \geq 2 \).
Proof. We first prove $B(2)$. Let $Y \subset \mathbb{P}^3$ be a general union of 6 lines (hence $P \notin Y$). By [4] part (i)(c) of Theorem 4.2 we have $h^1(I_{2P,Y}(4)) = 0$. Let $H \subset \mathbb{P}^3$ be a general plane though $P$. Moving $Y$ we see that may assume that no 3 of the points of $(Y \cap H) \cup \{P\}$ are collinear.

Now assume $m \geq 3$ and that $B(m-1)$ is true.

(a) In this step we assume that $m$ is odd. Take $(Y, L, R, S, H)$ satisfying $B(m-1)$. We have $h^1(I_{(m-1)P,Y}(m+1)) = 1$. Let $D \subset H$ be a general line. Let $v \subset H$ be a union of tangent vectors of $H$ with $S$ as its support, but no tangent vector being a tangent vector of $L \cup R$. We first check that $h^1(I_{(mP,Y)\cup D,Y}(O)(m+2)) = 0$. Since $\text{Res}_H(mP,Y \cup D \cup v \cup \{O\}) = (m-1)P \cup Y$ and $h^1(I_{(m-1)P,Y}(m+1)) = 0$, it is sufficient to prove that $h^1(H, I_{(mP,Y)\cap H} \cup D \cup v \cup \{O\})(m+2)) = 0$, i.e. $h^1(H, I_{(mP,Y)\cap H} \cup v \cup \{O\})(m+1)) = 0$. The scheme $((mP,Y) \cap H) \cup v \cup \{O\}$ is a general union of $(mP,H)$; the scheme $v \cup \{O\}$ and $a_{m-1,m+1} - (m+1)/2$ general points of $H$. Hence it has degree $(m+1)/2+3(m+1)/2+3-(m+1)/2+1 = (m+3)/2$. We deform $D$ in a flat family of lines outside $H$ (we may do it even fixing either the point of $D \cap L$ or the point of $D \cap R$). For general $v$ it is easy to check that $h^1(H, I_{(mP,H)\cup D} \cup v \cup \{O\})(m+1)) = 0$ (order the points of $S$ and then add the corresponding connected component $v_i$ of $v$ following the ordering first with the point $P_i$ of $S$ general in a component of $L \cup R$ and then with $v_i$ general among the tangent vectors of $H$ with $P_i$ as its support; at each point use that $h^0(H, O_{(mP,H)}(m)) = m+1$ and that if $P_i \in L_i$, then $I_{(mP,H)\cup (2P_i,L_i)}(m+1)] = [I_{(mP,H)}(m)]$. Since $Y\cap H \cup S$ is general in $H$, we get $h^1(H, I_{(mP,Y)\cap H} \cup v \cup \{O\})(m+1)) = 0$.

(b) In this step we assume that $m$ is even. Take $(Y, L, R, S, O, H, v)$ satisfying $B(m-1)$. Let $w \subset \mathbb{P}^3$ be a general tangent vector with $O$ as its support. The scheme $Y \cup L \cup R \cup w \cup v$ is a flat limit of a family of disjoint unions of $a_{m,m+2}$ lines (i.e. there are a flat family $\{Y_i\}_{i \in \Gamma}$, an integral affine curve, $o \in \Gamma$, $Y_o = Y \cup L \cup R \cup w \cup v$ such that $Y \subset Y_i$, say $Y_i = Y \cup L_i \cup R_i$ for all $t$ with $\{L_i\}$, and $\{R_i\}$ flat families with $L_o = L$ and $L_o = L$, and either $L_i \cap L \neq \emptyset$ for all $t$, $R_i \cap R \neq \emptyset$ for all $t$ or $R_i \cap R \neq \emptyset$ for all $t$ and $L_i \cap L = \emptyset$ for all $t \neq o$ (case $m \equiv 2 \pmod{4}$) or $R_i \cap R \neq \emptyset$ for all $t$ and $L_i \cap L = \emptyset$ for all $t \neq o$ (case $m \equiv 0 \pmod{4}$). We may take as the new set $S$ the set $S \cup (L_i \cap R_i) \cap (L \cup R)$ for a general $t \in \Gamma$. By the semicontinuity theorem for cohomology ([3] III.12.8) it is sufficient to prove that $h^1(I_{(mP,Y)\cup L \cup R \cup w}(m+2)) = 0$. Since $(mP,Y \cup L \cup R \cup w) \cap H = (mP,H) \cup L \cup R \cup (Y \cap H) \cup S)$ and $\text{Res}_H(mP,Y \cup L \cup R \cup v \cup w) = (m-1)P \cup Y \cup v \cup \{O\}$, it is sufficient to prove that $h^1(H, I_{(mP,H)} \cup L \cup R \cup (Y \cap H) \cup S)(m+2)) = 0$, i.e. $h^1((H, I_{(mP,H)}) \cup L \cup R \cup (Y \cap H) \cup S)(m)) = 0$. This is true, because $(Y \cap H) \cup S$ is general in $H$ and $i((Y \cap H) \cup S)(m/2 + 1 - m/2 = m + 1) = h^0(H, I_{(mP,H)}(m))$.

Remark 2. Fix $(m, d)$ with critical value $m+1$ and degree $d \geq m+1$. Let $W \subset \mathbb{P}^m$ be a general union of $d$ lines. Since $a_{m,m+1} = m + 2$, we have $m + 1 \leq d \leq m + 2$. By [4] part (i)(c) of Theorem 4.2] the scheme $mP \cup W$ has maximal rank.

Lemma 2. Fix an integer $d \leq a_{m,m+2}$ and let $X \subset \mathbb{P}^3$ be a general union of $d$ lines. Then $h^1(I_{mP,X}(m+2)) = 0$.

Proof. This statement is obvious if $m = 1$ by [6]. Assume $m \geq 2$. It is sufficient to find a disjoint union $W$ of $d$ lines such that $P \notin W$ and $h^1(I_{mP,W}(m+2)) = 0$. Take a solution of $B(m)$ and call $Y$ the curve in it. Take as $W$ the union of $d$ of the lines of $Y$. \qed
For all odd integer $m \geq 3$ let $R(m)$ denote the following assertion:

**Assertion** $R(m)$, $m$ odd, $r \geq 3$: There exists a quintuple $(Y,S,D,H,v)$ with the following properties:

1. $Y \subset \mathbb{P}^3$ is a disjoint union of $3m/2 + 5/2$ lines, $P \not\subset Y$, $H$ is a plane containing $P$, $D \subset H$ is a smooth conic such that $P \not\subset D$ and $S := (Y \cap H) \cap D$ has cardinality $m/2 + 5/2$;
2. $v \subset \mathbb{P}^3$ is a disjoint union of tangent vectors of $\mathbb{P}^3$ with $v_{\text{red}} = S$; no connected component of $v$ is contained in $Y$;
3. $h^i(I_{mP\cup Y\cup v}(m + 2)) = 0$.

**Lemma 3.** $R(m)$ is true for all odd integers $m \geq 3$.

**Proof.** Take $(Y,L,R,H)$ satisfying $B(m-1)$. We have $\mathfrak{g}(Y \cap (L \cup R)) = (m+1)/2$, $h^1(I_{(m-1)P\cup Y}(m+1)) = 0$ and $h^0(I_{(m-1)P\cup Y}(m+1)) = 1$. Since $h^0(I_{(m-2)P\cup Y}(m)) = 0$, $P \in H$, and $Y \cap (R \cup L) \neq Y \cap H$, there is $o \in L \cup R$ not in the base locus of $|I_{(m-1)P\cup Y}(m+1)|$ and hence $h^i(I_{(m-1)P\cup Y\cup o}(m+1)) = 0$, $i = 0,1$. We may deform $(Y,L \cup R,o)$ to $(Y',C,o')$, where $C \subset H$ is a smooth conic, $P \not\subset C$, $o' \in C \cap Y$, $\mathfrak{g}(Y' \cap C) = (m+1)/2$ and $h^0(I_{(m-1)P\cup Y\cup o'}(m+1)) = 0$. We may take as $o$ a general point of $C$. Let $o''$ be another general point of $C$ and call $T$ the line spanned by $o'$ and $o''$ (alternatively, take a general line $T \subset H$ and set $\{o',o''\} := C \cap T$). Let $w \in H$ be a general union of tangent vectors of $H$, each of them supported by a different point of $Y \cap (L \cup R)$. Let $v' \subset \mathbb{P}^3$ be a general tangent vector of $\mathbb{P}^3$ with $o'$ as its support (hence $\text{Res}_H(v') = \{o'\}$). Let $v'' \subset H$ be a general tangent vector of $H$ with $v'' \text{red} = \{o''\}$. Since $v'' \subset H$, we have $\text{Res}_H(v'') = \emptyset$. Since $v''$ is general, it is not tangent to $T$ and hence $\text{Res}_T(v'') = \{o''\}$. Take $Y' := Y \cup T$, $v := w \cup v' \cup v''$ and $S := Y \cap (L \cup R) \cup \{o',o''\}$. We want to check that the quintuple $(Y',S,C,H,v)$ satisfies $R(m)$. The scheme $v$ is a union of tangent vectors, one for each point of $S$. We have $\mathfrak{g}(S) = (m+1)/2 + 2 = (m+5)/2$. The set $S$ is contained in the smooth conic $D$. It is sufficient to check that $h^i(I_{mP\cup Y\cup v}(m+2)) = 0$, $i = 0,1$. Since $\text{Res}_H(mP \cup Y \cup v) = (m-1)P \cup Y \cup \{o'\}$, $\text{Res}_H(I_{(m-1)P\cup Y\cup o}(m+1)) = 0$, $i = 0,1$, $o' \in D$, $(mP \cup Y \cup v) \cap H = (m-1)P \cup Y \cup v \cup (Y \cap H) \cap (L \cup R)$ and $\text{Res}_H(I_{mP \cup H} \cup T \cup v'' \cup (Y \cap H) \cup w = (mP,H) \cup (Y \cap H) \cup w \cup \{o''\}$, it is sufficient to prove that $h^i(H,I_{(mP,H)\cup T\cup v''\cup (Y \cap H)\cup w}(m+1)) = 0$. We have $\text{deg}\{mP,H\} \cup (Y \cap H) \cup w \cup \{o''\} = (m+1)^2 + 3m/2 + 3/2 + (m+1)/2 + 1 = (m+3)^2$. Use again that $h^i(H,I_{(mP,H)\cup T\cup v''\cup (Y \cap H)\cup w}(m+1)) = 0$ (as in part (a) of the proof of Lemma 10) and that $Y \cap H \setminus v_{\text{red}}$ is general in $H$. \hfill \Box

**Lemma 4.** Fix an integer $d > a_{m,m+2}$ and let $X \subset \mathbb{P}^3$ be a general union of $d$ lines. Then $h^0(I_{mP\cup X}(m+2)) = 0$.

**Proof.** It is sufficient to prove the lemma when $d = a_{m,m+2}$. First assume that $m$ is even. Take a solution $(Y,L,R,H)$ of $B(m)$. Since $h^0(I_{mP\cup Y}(m+2)) = 1$, we have $h^0(I_{mP\cup Y\cup D}(m+2)) = 0$ for any line $D$ through a general point of $\mathbb{P}^3$.

Now assume that $m$ is odd. Let $W \subset \mathbb{P}^3$ be a general union of $a_{m-1,m+1}$ lines. Since $B(m-1)$ is true, we have $h^1(I_{(m-1)P\cup W}(m+1)) = 0$ and hence $h^0(I_{(m-1)P\cup W\cup o}(m+1)) = 0$ for a general $o \in \mathbb{P}^3$. Let $M \subset \mathbb{P}^3$ be a general plane containing $\{P,o\}$. Let $L',R' \subset M$ be two general lines through $o$. It is sufficient to prove that $h^0(I_{mP\cup W \cup L' \cup R' \cup 2o}(m+2)) = 0$. Since $\text{Res}_M(mP \cup W \cup L' \cup R' \cup 2o) = (m-1)P \cup W \cup \{o\}$, it is sufficient to prove that $h^0(H,I_{(mP,H)\cup (W \cap H)\cup L' \cup R'}(m+2)) = 0$.
2)) = 0, i.e. $h^0(H, \mathcal{I}_{(mP,H) \cup (W \cap H)}(m)) = 0$. Since $W \cap H$ is a general union of $a_{m-1,m+1} > m$ points of $H$, we have $h^0(H, \mathcal{I}_{(mP,H) \cup (W \cap H)}(m)) = 0$. □

Consider the following statement:

**Assertion** $H_{m,k}$, $m > 0$, $k \geq m + 2$: There exist a quintuple $(Y,Q,S,v,E)$ with the following properties:

1. $Y \in L(P,a_{m,k})$, $Q$ is a smooth quadric surface intersecting transversally $Y$, $P \not\in Q$;
2. $S \subseteq Y \cap Q$, $\sharp(S) = b_{m,k}$ and $v \subset \mathbb{P}^3$ is a disjoint union of tangent vectors with $v_{\text{red}} = S$ and no connected component of $v$ contained in $Y$;
3. $E \subset Q$ is a disjoint union of $\lfloor b_{m,k}/2 \rfloor$ lines, $S \subset E$ and each component of $E$ contains at most two points;
4. $h^i(\mathcal{I}_{mP \cup Y \cup v}(k)) = 0$, $i = 0, 1$.

Take $(Y,Q,S,v,E)$ satisfying the first two conditions of the definition of $H_{m,k}$.

We have $h^i(\mathcal{O}_{mP \cup Y \cup v}(k)) = \binom{k+3}{3}$ and hence $h^0(\mathcal{I}_{mP \cup Y \cup v}(k)) = h^1(\mathcal{I}_{mP \cup Y \cup v}(k))$.

Now assume that $(Y,Q,S,v,E)$ satisfies the third condition of the definition of $H_{m,k}$. If $b_{m,k}$ is even, then each line of $S$ contains exactly two points of $S$. If $b_{m,k}$ is odd, then $\sharp(S \cap L) = 2$ for $(b_{m,k} - 1)/2$ of the components of $E$, while $\sharp(S \cap L) = 1$ for the other component.

From now on $Q \subset \mathbb{P}^3$ is a smooth quadric surface such that $P \not\in Q$.

**Lemma 5.** $H_{m,m+3}$ is true for all $m > 0$.

**Proof.** We have $a_{m,m+1} = m + 2$, $b_{m,m+1} = 0$, $a_{m,m+3} = 2m + 4$ and $b_{m,m+3} = 4$ (Remark 1). Let $Y \subset \mathbb{P}^3$ be a general union of $m + 2$ lines. By [4] Part (i)(c) of Theorem 4.2 we have $h^i(\mathcal{I}_Y(m + 1)) = 0$, $i = 0, 1$. For a general $Y$ we may assume that $Y \cap Q$ is formed by $2m + 4$ general points of $Q$. Let $F \subset Q$ be a general union of $m + 2$ lines of type $(0,1)$. Fix $S_1 \subset Y \cap Q$ such that $\sharp(S_1) = 2$. Let $E' \subset Q$ be the union of the lines of type $(1,0)$ containing a point of $S_1$. Fix $S_2 \subset E' \cap F$ such that $\sharp(S_2 \cap L) = 1$ for each component $L$ of $E'$ and that no component of $F$ contains two points of $S_2$. Set $S := S_1 \cup S_2$ and call $v \subset Q$ a general union of tangent vectors of $Q$ with $S$ as its support. We claim that $h^i(\mathcal{I}_{mP \cup Y \cup F \cup v}(m + 3)) = 0$.

Since \text{Res}_Q(mP \cup Y \cup F \cup v) = mP \cup Y$, $Q \cap (mP \cup Y \cup F \cup v) = F \cup v \cup ((Y \cap Q) \setminus S_1)$, \text{Res}_F(F \cup v \cup ((Y \cap Q) \setminus S_1) = ((Y \cap Q) \setminus S_1) \cup v_{\text{red}} = Y \cap Q \cup S_2$ and $h^i(\mathcal{I}_{mP \cup Y}(m + 1)) = 0$, $i = 0, 1$, to prove the claim it is sufficient to prove that $h^1(Q, \mathcal{I}_{((Y \cap Q) \setminus S_1) \cup v}(m + 3,1)) = 0$, $i = 0, 1$. We have $\deg(((Y \cap Q) \setminus S_1) \cup S)2m + 6$. Hence it is sufficient to use that $h^1(Q, \mathcal{I}_S(m + 3,1)) = 0$ ($S$ is the union of two degree 2 schemes on two different lines of type $(1,0)$) and $(Y \cap Q) \setminus S_1$ is general in $Q$. □

**Lemma 6.** $H_{m,m+4}$ is true for all $m \geq 2$.

**Proof.** The proof depends on the parity of $m$.

(a) First assume that $m$ is even. We have $a_{m,m+2} = 3m/2 + 3$ and $b_{m,m+2} = 1$.

(a1) Assume for the moment $m \geq 6$ and hence $a_{m,m+4} = 5m/2 + 5$ and $b_{m,m+4} = 10$ (Remark 1). Let $Y \subset \mathbb{P}^3$ be a general union of $a_{m,m+2} = 3m/2 + 3$ lines. Since $B(m)$ is true (Lemma 1) and $b_{m,m+2} = 1$, we have $h^1(\mathcal{I}_{mP \cup Y}(m + 2)) = 0$ and $h^0(\mathcal{I}_{mP \cup Y}(m + 2)) = 1$. The last equality implies $h^0(\mathcal{I}_{mP \cup Y}(m + 1)) = 0$.

Let $T \subset \mathbb{P}^3$ be the only surface of degree $m + 2$ containing $mP \cup Y$. Fix a system $x_0, x_1, x_2, x_3$ of homogeneous coordinates and let $f(x_0, x_1, x_2, x_3)$ be a degree $m + 2$ homogeneous equation of $T$. In characteristic zero we have $\frac{\partial f}{\partial x_i} \neq 0$ for at least
one index $i$. Since $\partial f / \partial x_i \neq 0$ and $h^0(\mathcal{I}_{m,P,Y}(m+1)) = 0$, we have $\partial f / \partial x_i|mP \cup Y \neq 0$, i.e. $mP \cup Y \not\subseteq \text{Sing}(T)$, i.e. $Y \not\subseteq \text{Sing}(T)$. In characteristic $p > 0$ we need to prove the existence of $Y \in L(P, 3m+2)$ such that $h^0(\mathcal{I}_{m,P,Y}(m+2)) = 1$ and at least one component of $Y$ is not contained in the singular locus of the only degree $m$ hypersurface containing $mP \cup Y$. We get this using the proof that $B(m-1)$ implies $B(m)$ when $m$ is even (part (b) of the proof of Lemma 1). Take $(Y, L, R, S, O, H, v)$ satisfying $B(m-1)$ and use that $h^0(H, \mathcal{I}_{(m,P,H) \cup 2L}, 2R(m+2)) = 0$.

Since $Y \not\subseteq \text{Sing}(T)$, there is $O \in Y$ and a tangent vector $w$ to $\mathbb{P}^3$ with $w \not\subseteq T$. Hence $h^0(\mathcal{I}_{m,P,Y \cup w}(m+2)) = 0$ and $h^1(\mathcal{I}_{m,P,Y \cup w}(m+2)) = 0$. Take as $Q$ a general quadric surface through $O$. We have $P \not\subseteq Q$ and $\text{Res}_Q(mP \cup Y \cup w) = mP \cup Y \cup w$. Moving the lines of $Y$ among the unions of $a_{m,m+2}$ disjoint lines of $\mathbb{P}^3$, one of them containing $O$, we may assume that $(Y \cap Q) \setminus \{O\}$ is a general union of $3m+5$ points of $Q$. Let $F \subset Q$ be a union of $m+2$ distinct lines of type $(0,1)$ of $Q$ with $\{O\} = Y \cap F$. Fix $S_1 \subset (Y \cap Q) \setminus \{O\}$ with $\sharp(S_1) = 5$. Let $E \subset Q$ be the union of the $5$ lines of type $(1,0)$ of $Q$ containing one point of $S_1$. Take $S_2 \subset E \cap F$ such that each line of $E$ contains exactly one point of $S_2$ and each line of $F$ contains at most one point of $S_2$. Set $S := S_1 \cup S_2$. Let $v \subset Q$ be a general union of tangent vectors of $Q$ with $v_{\text{red}} = S$. As in the proof of Lemma 3 it is sufficient to prove that $h^i(\mathcal{I}_{m,P,Y \cup w,F \cup v}(m+4)) = 0$, $i = 0,1$. Since $h^1(\mathcal{I}_{m,P,Y \cup w}(m+2)) = 0$, $i = 0,1$, it is sufficient to prove that $h^i(Q, \mathcal{I}_{(Y \cap Q) \cup F \cup v}(m+4)) = 0$, i.e. $h^i(Q, \mathcal{I}_{(Y \cap Q) \cup v}(m+4,2)) = 0$, $i = 0,1$. We have $\deg((Y \cap Q) \cup v) = 20 + (3m+6) - 10$. Since $v$ is general, $\deg(v) = 20$ and $S$ is general with the only restriction that $5$ lines of type $(1,0)$ of $Q$ contain each two points of $S$, we have $h^i(\mathcal{I}_v(m+4,2)) = 0$. Since $Y \cap Q \setminus S_1$ is general in $Q$, we get $h^i(Q, \mathcal{I}_{(Y \cap Q) \cup v}(m+4,2)) = 0$, $i = 0,1$.

(a2) Now assume $m \in \{2,4\}$. We have $a_{m,m+4} = 5m/2 + 6$ and $b_{m,m+4} = 5 - m$. Now $F$ is a union of $m+3$ lines, $\sharp(S_1) = 2 - m/2$, $\deg(E) = 3 - m/2$ and $\sharp(S_2) = 3 - m/2$.

(b) Now assume that $m$ is odd.

(b1) Assume $m \geq 17$. We have $a_{m,m+2} = 3m/2 + 5/2$ and $b_{m,m+2} = (m+5)/2$. Since $m \geq 17$, we have $a_{m,m+4} = 5m/2 + 9/2$ and $b_{m,m+4} = (m+25)/2$ (Remark 1). Take a solution $(Y, S, D, H, v)$ of $R(m)$ (Lemma 4). Let $Q \subset \mathbb{P}^3$ be a general quadric surface containing $D$. Since $P \not\subseteq D$, then $P \not\subseteq Q$. The quadric $Q$ is smooth and it intersects transversally $Q$. By the semicontinuity theorem for cohomology (3.12.8) for general $v$ we may also assume that no connected component of $v$ is contained in $Q$, i.e. that $(mP \cup Y \cup v) = Y \cap Q$ (as schemes) and that $\text{Res}_Q(mP \cup Y \cup v) = mP \cup Y \cup v$. We may move each component of $Y$ keeping fixed its point in $D$. Hence we may assume that $Y \cap (Q \setminus D)$ is a general subset of $Q$ with cardinality $2a_{m,m+2} - b_{m,m+2} = m + 2 \geq (m+5)/2 = b_{m,m+2}$. Let $F \subset Q$ be a disjoint union of $m+2$ lines of type $(0,1)$ with the only restriction that $F \cap Y = Y \cap D$ (it exists, because $m+2 \geq b_{m,m+2}$). Fix $S_1 \subset (Y \cap Q) \setminus S$ such that $\sharp(S_1) = |b_{m,m+4}|/2$ (it exists because $2a_{m,m+2} = 3m + 5 \geq (m+5)/2 + m \geq b_{m,m+4} + |b_{m,m+4}|/2$). Let $E_1$ be the union of the lines of type $(1,0)$ of $Q$ containing one point of $S_1$. If $b_{m,m+4}$ is even, then set $E' := E$. If $b_{m,m+4}$ is odd, then let $E'$ be the union of $E_1$ and a general line of type $(1,0)$ of $Q$. Let $S_2 \subset E' \cap E''$ be the union of one point for each component of $E'$, with the restriction that $S_2 \cap S_1 = \emptyset$ and that each point of $S_2$ is contained in a different line of $E''$; we may find such a set $S_2$, because $E'' \cap S_1 = \emptyset$ and
deg($E''$) = $a_{m,m+4} - a_{m,m+2} = m + 2 \geq \lceil b_{m,m+4}/2 \rceil$. Let $v' \subset Q$ be a general union of $b_{m,m+4}$ tangent vectors of $Q$ with $v'_{\text{red}} = S'$. Since $v'$ is general, no connected component of $v'$ is contained in $E''$ (hence $\text{Res}_{E''}((Y \cap Q) \cup v') = Y \cap (Q \setminus E'') \cup S' = ((Y \cap Q) \setminus S) \cup S'$).

(b2) Assume $m \in \{3, 5, 7, 9, 11, 13, 15\}$. We have $a_{m,m+4} - a_{m,m+2} = m + 3$ and $b_{m,m+4} = (15 - m)/2$. We make the construction of step (b1) with $\deg(F) = m + 3$, $\sharp(S_1) = \lfloor (15 - m)/4 \rfloor$, and $\deg(E) = \sharp(S_2) = \lfloor (15 - m)/4 \rfloor$.

Lemma 7. For all integers $k \geq m + 3$ we have $a_{m,k} - a_{m,k-2} \geq a_{0,k} - a_{0,k-2} - 1 \geq \lceil k/2 \rceil$.

Proof. From (2) and the same equation for $m = 0$ we get
\[ 2a_{m,k-2} + (k + 1)(a_{m,k} - a_{m,k-2}) + b_{m,k} - b_{m,k-2} = 2a_{0,k} + (k + 1)(a_{0,k} - a_{0,k-2}) + b_{0,k} - b_{0,k-2} \]

We have $b_{0,x} = 0$ if $x \equiv 0, 1 \pmod{3}$ and $b_{0,x} = (x + 1)/3$ if $x \equiv 2 \pmod{3}$. The definitions of the integers $a_{m,k}$ and $a_{0,k}$ give $a_{m,k} \leq a_{0,k}$, proving the lemma.

Lemma 8. $H_{m,k}$ is true for all $k \geq m + 3$.

Proof. By Lemmas 5 and 6 we may assume $k \geq m + 5$ and that $H_{m,k-2}$ is true. Fix a solution $(Y, Q, S, v, E)$ of $H_{m,k-2}$. Deforming if necessary each line of $Y$ we may assume that $(Q \cap Y) \setminus S$ is a general subset of $Q$. Taking instead of $v$ a union of general tangent vectors of $\mathbb{P}^3$ with the points of $S$ as their support we may assume that no connected component of $v$ is contained in $Q$. Therefore $\text{Res}_{Q}(Y \cup v) = Y \cup v$ and $(Y \cup v) \cap Q = Y \cap Q$ (as schemes). Call (0, 1) the ruling of $Q$ containing $E$ (any ruling of $Q$ if $b_{m,k-2} = 0$ and hence $E = \emptyset$). Lemma 7 gives $a_{m,k} - a_{m,k-2} \geq \deg(E)$. Let $F \subset Q$ be a general union of $a_{m,k} - a_{m,k-2} - \lfloor b_{m,k-2}/2 \rfloor$ lines of type $(0, 1)$ of $Q$. Set $E'' := E \cup F$.

Claim 1: If $k \geq m + 5$, then $2a_{m,k-2} \geq k + 2 - k/2$.

Proof of Claim 1: We have $2(k - 1)a_{m,k-2} + 2b_{m,k-2} = 2(k + 1)a_{m,k-2} + 2b_{m,k-2} = 2(k + 1)a_{m,k-2} + 2b_{m,k-2} \geq k - 2$. Set $\psi(k, m) := 2(k + 1)a_{m,k-2} + 2b_{m,k-2} \geq k + 2$. It is sufficient to prove that $\psi(k, m) \geq 0$ for all $k \geq m + 5$. We have $\psi(m + 5, m) = (m + 5)(m + 7)/2 - (m + 2)(m + 1)/2 - m + 6 \geq 0$ and $\psi(k + 1, m) \geq \psi(k, m)$ for all $k \geq m + 5$.

Fix $S_1 \subseteq (Y \cap Q) \setminus S$ such that $\sharp(S_1) = \lfloor b_{m,k} \rfloor$ (it exists by Claim 1 and the inequalities $b_{m,k-2} \leq k - 2$, $b_{m,k} \leq k$). Let $E_1 \subset Q$ be the union of the lines of type $(1, 0)$ of $Q$ containing one point of $S_1$. If $b_{m,k}$ is even, then set $E' := E$. If $b_{m,k}$ is odd, then let $E'$ be the union of $E_1$ and a general line of type $(1, 0)$ of $Q$. Let $S_2 \subset E' \cap E''$ be the union of one point for each component of $E'$, with the restriction that $S_2 \cap S_1 = \emptyset$ and that each point of $S_2$ is contained in a different line of $E''$; we may find such a set $S_2$, because $E'' \cap S_1 = \emptyset$ and $\deg(E'') = a_{m,k} - a_{m,k-2} \geq \lfloor b_{m,k}/2 \rfloor$ (Lemma 7). Set $S' := S_1 \cup S_2$. Let $v' \subset Q$ be a general union of $b_{m,k}$ tangent vectors of $Q$ with $v'_{\text{red}} = S'$. Since $v'$ is general, no connected component of $v'$ is contained in $E''$ (hence $\text{Res}_{E''}((Y \cap Q) \cup v') = Y \cap (Q \setminus E'') \cup S' = ((Y \cap Q) \setminus S) \cup S'$).

Claim 2: We claim that $h^i(T_{m,P_{Y \cup v} \cup E'' \cup v'}(k)) = 0$, $i = 0, 1$.

Proof of Claim 2: Since $\text{Res}_{Q}(mP_{Y \cup v} \cup E'' \cup v') = mP_{Y \cup v} \cup v \cup E' \cup v' = (Y \cap Q) \cup E'' \cup v'$ and $h^i(T_{m,P_{Y \cup v} \cup E'' \cup v'}(k - 2)) = 0$, $i = 0, 1$, it is sufficient to prove that $h^i(Q, T_{(Y \cap Q) \cup E'' \cup v'}(k)) = 0$, $i = 0, 1$, i.e. $h^i(Q, T_{(Y \cap Q) \setminus S}^{(k - 2)}) = 0$, $i = 0, 1$.
Since Res connected component of $P$ lines, none of them containing $P$ tangent vectors of $d$ is sufficient to do the case $L$. Let $Q$ be a general union of $k$ lines of $Y,Q,S,v,E$ that satisfies $m,k$. We may deform $Y∪v∪E''$ to a family of members of $L(P,a_{m,k})$ containing the points of $S'$ and whose general member, $Y''$, intersects transversally $Q$, because each line of $E''$ contains at most one point of $Q$. The quintuple $(Y'',Q,S',v',E')$ satisfies $H_{m,k}$.

Proof of Theorem 7. Fix positive integers $m,d$ with critical value $k$. Hence $a_{m,k−1} < d ≤ a_{m,k}$. See Remark 2 and Lemma 2 for the cases $k = m,m + 1,m + 2$. Hence we may assume $k ≥ m + 3$ and that the theorem is true for the integers $d$ such that $(m,d)$ has critical value $< k$. Since $L(P,d)$ is irreducible, it is sufficient to prove the existence of $A,B ∈ L(P,d)$ such that $h^0(\mathcal{I}_{mP∪A,B}(k−1)) = 0$ and $h^1(\mathcal{I}_{mP∪A}(k)) = 0$. Let $Q ∈ P^3$ be a smooth quadric surface such that $P ⊄ Q$.

(a) In this step we prove the existence of $A$. Since any element of $L(P,d)$ is a union of some of the connected components of an element of $L(P,a_{m,k})$, it is sufficient to do the case $d = a_{m,k}$. Fix a solution $(Y,Q,S,v,E)$ of $H_{m,k−2}$. Deforming if necessary each line of $Y$ we may assume that $(Q ∩ Y) \setminus S$ is a general subset of $Q$. Taking instead of $v$ a union of general tangent vectors of $P^3$ with the points of $S$ as their support we may assume that no connected component of $v$ is contained in $Q$. Therefore $\text{Res}_Q(Y ∪ v) = Y ∪ v$ and $(Y ∪ v) ∩ Q = Y ∩ Q$ (as schemes). Call $(0,1)$ the ruling of $Q$ containing $E$ (any ruling of $Q$ if $b_{m,k−2} = 0$ and hence $E = ∅$). Lemma 7 gives $a_{m,k} − a_{m,k−2} ≥ \deg(E)$. Let $F ⊂ Q$ be a general union of $a_{m,k} − a_{m,k−2} − \lceil b_{m,k−2}/2 \rceil$ lines of type $(0,1)$ of $Q$. The scheme $Y ∪ E ∪ F ∪ v$ is a flat limit of a family of disjoint unions of $a_{m,k}$ lines, none of them containing $P$. By the semicontinuity theorem for cohomology to prove the existence of $A$ it is sufficient to prove that $h^1(\mathcal{I}_{mP∪Y∪E∪F∪v}(k)) = 0$. We proved a more difficult vanishing in the proof of Lemma 5 (copy it without $v'$).

(b) In this step we prove the existence of $B$. By Lemma 4 we may assume $k−1 ≥ m + 4$, i.e. $k−3 ≥ m + 2$. Fix a solution $(Y,Q,S,v,E)$ of $H_{m,k−3}$. Deforming if necessary each line of $Y$ we may assume that $(Q ∩ Y) \setminus S$ is a general subset of $Q$. Taking instead of $v$ a union of general tangent vectors of $P^3$ with the points of $S$ as their support we may assume that no connected component of $v$ is contained in $Q$. Therefore $\text{Res}_Q(Y ∪ v) = Y ∪ v$ and $(Y ∪ v) ∩ Q = Y ∩ Q$ (as schemes). Call $(0,1)$ the ruling of $Q$ containing $E$ (any ruling of $Q$ if $b_{m,k−3} = 0$ and hence $E = ∅$). Lemma 7 gives $a_{m,k−1} − a_{m,k−3} ≥ \deg(E)$. Let $F ⊂ Q$ be a general union of $a_{m,k} − a_{m,k−3} − \lceil b_{m,k−2}/2 \rceil + 1$ lines of type $(0,1)$ of $Q$. The scheme $Y ∪ E ∪ F ∪ v$ is a flat limit of a family of disjoint unions of $a_{m,k−1} + 1$ lines, none of them containing $P$. By the semicontinuity theorem for cohomology to prove the existence of $B$ it is sufficient to prove that $h^0(\mathcal{I}_{mP∪Y∪E∪F∪v}(k−1)) = 0$. Since $\text{Res}_Q(mP∪Y∪E∪F∪v) = mP∪Y∪E∪v$ and $Q ∩ (mP∪Y∪E∪v) = (Y ∩ Q)∪E∪F$, it is sufficient to prove that $h^0(Q,\mathcal{I}_{F∪E∪(Y∩Q)}(k−1)) = 0$, i.e. that $h^0(Q,\mathcal{I}_{Y∩Q}S(k−1,k−a_{m,k−1}+a_{m,k−3}−2)) = 0$. Since $(Y ∩ Q) \setminus S$ is general in...
Q, it is sufficient to prove that \( z(Y \cap Q) - z(S) \geq k(k - a_{m,k-1} + a_{m,k-3} - 1) \). By (2) for the integer \( k' := k - 1 \) we have \( z(Y \cap Q) - z(S) = k(k - a_{m,k-1} + a_{m,k-3} - 1) + k - b_{m,k-1} > k(k - a_{m,k-1} + a_{m,k-3} - 1) \).

(b2) Now assume \( k = m + 4 \). We modify the proof of \( H_{m,m+3} \) (Lemma 5). We have \( a_{m,m+1} = m + 2 \), \( b_{m,m+1} = 0 \), \( a_{m,m+3} = 2m + 4 \) and \( b_{m,m+3} = 4 \) (Remark 1). Let \( Y \subset P^3 \) be a general union of \( m + 2 \) lines. By [4] Part (i)(c) of Theorem 4.2 we have \( h^0(I_Y(m+1)) = 0 \), \( i = 0, 1 \). For a general \( Y \) we may assume that \( Y \cap Q \) is formed by \( 2m + 4 \) general points of \( Q \). Let \( F \subset Q \) be a general union of \( m + 3 \) lines of type \( (0,1) \). Use \( Y \cup F \). Since \( Y \cap Q \) is a general subset of \( Q \) with cardinality \( 2m + 4 \), we have \( h^0(Q, I_{Q \cap Y}(m+3,0)) = 0 \). We also have \( Y \cap F = \emptyset \) and hence \( h^0(Q, I_{(Y \cap Q) \cup F}(m+3)) = 0 \).

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