Robust utility maximisation under proportional transaction costs for càdlàg price processes

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Abstract
We consider robust utility maximisation in continuous-time financial markets with proportional transaction costs under model uncertainty. For this purpose, we work in the framework of Chau and Rásonyi [8], where robustness is achieved by maximising the worst-case expected utility over a possibly uncountable class of models that are all given on the same underlying filtered probability space with incomplete filtration. In this setting, we give sufficient conditions for the existence of an optimal trading strategy extending the result for utility functions on the positive half-line of Chau and Rásonyi [8] from continuous to general strictly positive càdlàg price processes. This result allows us to provide a positive answer to an open question pointed out in Chau and Rásonyi [8], and shows that the embedding into a countable product space is not essential.

Key words: Utility maximisation, proportional transaction costs, model uncertainty, incomplete filtrations.

1 Introduction
Maximising the expected utility from terminal wealth is a classical problem in Mathematical Finance and Financial Economics. Recently, there has been a lot of interest in robust utility maximisation under model uncertainty, where one maximises the worst-case expected utility over a class of models. The motivation for this model-independent view is that the resulting trading strategies are less sensitive to changes of the underlying model and in this sense more robust to model misspecification.

In this paper, we consider robust utility maximisation under proportional transaction costs in the framework of model uncertainty of Chau and Rásonyi [8]. Here, the worst-case expected utility over a possibly uncountable class of models, that are all given on the

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same underlying filtered probability space with incomplete filtration, is maximised. In this
setting, we extend the existence result of Chau and Rásonyi [8] for utility functions on the
positive half-line from continuous to general strictly positive càdlàg price processes. This
work answers an open question in Chau and Rásonyi [8]. Without frictions, a rich amount
of examples of discrete-time models for this setting of model uncertainty has been proposed
in Rásonyi and Meireles-Rodrigues [32]. Our results cover the corresponding models under
proportional transaction costs and allow to consider both discrete- and continuous-time
models in a unified framework.

As already explained in Chau and Rásonyi [8], most of the literature typically param-
eterises model uncertainty by a family $\mathcal{P}$ of probability measures that are given on some
underlying canonical probability space (see, e.g., Biagini and Pınar [2], Lin and Riedel [24]
and Neufeld and Nutz [29]). This means that the dynamics of the risky asset is given by a
fixed process and model uncertainty is described by a family of different distributions of this
process. The discrete-time case with unbounded utility functions defined on $(0, \infty)$ has been
solved in Blanchard and Carassus [4]. In the context of transaction costs, the discrete-time
case with transaction costs and exponential utility preferences has been treated in Deng,
Tan, and Yu [15]. For diffusion models, drift uncertainty can be modelled by considering a
family of absolutely continuous measures that are dominated by a single measure $P^*$ (see e.g.,
Quenez [31] and Schied [36]), while the case of volatility uncertainty requires an uncountable
family of singular measures (see e.g., Denis and Kervarec [16]).

In contrast to the approach above, Chau and Rásonyi [8] propose a setup of model
uncertainty, where different stock price processes are considered. That is, they suggest to
work on a fixed filtered probability space, and to use a family of stochastic processes $S^\theta$
indexed by $\theta$ in a non-empty set $\Theta$ to describe model uncertainty. From a mathematical
point of view, the main advantage of this setup is that no topological or measurability
assumptions are need to be put on the set of parameters $\Theta$ representing the different models.
In contrast, the parametrisation via a family of measures $\mathcal{P}$ incorporates technical issues such
as the treatment of null events and filtration completion (see e.g., Biagini et al. [3], Bouchard
and Nutz [8] and Nutz [30]). However, since typical examples consider an uncountable set
of models $\Theta$, one cannot complete the filtration with respect to the null sets arising from
any price process $S^\theta$ and has to work with incomplete filtrations. Working with filtrations
under “unusual” conditions, that is, without the usual condition of completeness, then brings
its own challenges. However, as pointed out in Jacod [20], we may still pass to the usual
conditions from time to time, at the cost of working with $P$-indistinguishable equivalence
classes. This additional difficulty is in contrast to Chau and Rásonyi [8], who still assume the
filtration to be complete. Besides this fact, we refer to the original paper of Chau and Rásonyi
[8] for a more detailed comparison between these two approaches to model uncertainty.

Within their setting of model uncertainty, Chau and Rásonyi [8] observe that, similarly
as in the case of a single model in Guasoni [19], it is more suitable to optimise directly
over trading strategies, and hence stochastic processes, rather than over terminal wealths
given by random variables as in the classical case with only one price process. For this
purpose, they exploit that, for continuous price processes, it is sufficient to model trading
strategies by càdlàg finite variation processes under proportional transaction costs. The key
insight is that càdlàg finite variation processes can be identified with their values along the
rational numbers and hence objects are taking values in the countable product of complete
metric spaces. However, this approach does not avoid the measurability problem arising from working with uncountably many models simultaneously, if one does not assume the filtration to be complete.

While càdlàg strategies are sufficient to obtain the optimal strategies for continuous price processes under proportional transaction costs, this fact is no longer true for price processes with jumps; see Example 4.1 in Czichowsky and Schachermayer [10]. For càdlàg price processes, it matters, whether one is trading immediately before, just at, or immediately after a jump. Therefore, trading strategies have to be modelled by general predictable finite variation processes that can have left and right discontinuities and can no longer be identified with their values along the rationals. To overcome this difficulty, we use a version of Helly’s Theorem of Campi and Schachermayer [7]. The latter allows us to obtain a sequence of trading strategies that converges to a càdlàg modification of a suitable limit process at all time points except for the discontinuity points of that modified limit process. The key is now that we can show that the set, where the convergence can fail, is the same for all models. Hence, it can be exhausted by countably many stopping times. The stopping times can be obtained by either passing to the usual conditions or by using a suitable version of the Debut Theorem for filtrations that are not complete. This insight allows us to achieve the convergence at these points as well by a diagonal sequence argument. Mathematically speaking, while Chau and Rásonyi [8] work with the topology of $P$-a.s. convergence along all rationales on the set of càdlàg finite variation processes, we work with the topology of convergence in probability at all $[0,T]$-valued stopping times. Somewhat surprisingly our result shows that the embedding into a countable product metric space indexed by the rationals as used in Chau and Rásonyi [8] is not essential.

We assume that our utility functions have a reasonable asymptotic elasticity as in the classical single model framework of Kramkov and Schachermayer [22]. This assumption allows us to obtain the optimal trading strategy by directly optimising in the primal problem of maximising expected utility from terminal wealth, and we do not need specific properties of the dual problem. Therefore, we only need the existence of (locally) consistent price systems for one level of transaction costs $\lambda' \in (0, \lambda)$ rather than for all $\lambda' \in (0, \lambda)$ as in Chau and Rásonyi [8]; see Remark 4.5 of Chau and Rásonyi [8] and Lemma 4.3 below.

In the framework of model uncertainty of Chau and Rásonyi [8], a super-replication theorem has been recently established in Chau et al. [9]. For model uncertainty with uncountably many probability measures on the same probability space, Bartl et al. [1] derived a duality result, in the spirit to the one of Kramkov and Schachermayer [22], for the case of a single model, for utility maximisation from terminal wealth without transaction costs.

The paper is organised as follows. We introduce the setting and formulate the problem in Section 2. Our main result is stated and explained in Section 3. The proof of the main result is covered in Section 4. For better readability, some proofs and explanations are deferred to Appendix A.

## 2 Formulation of the problem

We consider a financial market consisting of one riskless asset and one risky asset. The riskless asset has constant price 1. The dynamics of the risky asset is uncertain. For this purpose,
let $\Theta$ be a non-empty set and consider a family of strictly positive adapted càdlàg processes $(S^\theta_t)_{0 \leq t \leq T}$, for each $\theta \in \Theta$, on some underlying filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$. No further conditions are imposed on $\Theta$. By passing to the right-continuous version, we can assume without loss of generality that the filtration $\mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$ is right continuous. However, the filtration does not necessarily need to be complete. We denote by $\mathbb{F}^P$ the $P$-completion of $\mathbb{F}$, and accordingly $\mathbb{F}^P$ the usual augmentation of $\mathbb{F}$ by adjoining all the $P$-negligible sets to $\mathcal{F}_0$. In this framework, model uncertainty is then incorporated by considering simultaneously all models $(S^\theta_t)_{0 \leq t \leq T}$, for $\theta \in \Theta$, as possible evolutions for the price process of the risky asset, so that the set $\Theta$ provides the parametrisation of model uncertainty.

To illustrate why this particular setting of model uncertainty is useful, Chau and Rásonyi [8] provide some interesting examples of well-known models in finance literature (see Examples 2.2, 2.4 and 2.5 therein). For instance, in Example 2.2, they describe a robust version of the Black-Scholes model, where the parametrisation of model uncertainty is a subset of $\mathbb{R}^2$, which describes potential values for the drift and volatility term of the geometric Brownian motion that models the price process. To see why such a parametrisation of robustness is useful and interesting in our extended setup that includes càdlàg price processes, we provide some more examples.

**Example 2.1.** In the robust Merton jump diffusion model (cf., Merton [27]), the price of the risky asset satisfies the stochastic differential equation

$$
\mathrm{d}\delta_t^\theta(\mu, \sigma, \nu) = S_t^\theta((\mu - \nu k)\mathrm{d}t + \sigma \mathrm{d}W_t + \mathrm{d}X_t), \quad S_0^\theta = s_0 > 0,
$$

where $\mu$ and $\sigma$ are constants, $W$ is a standard Brownian motion, and $X$ is an independent compound Poisson process with intensity $\nu > 0$, whose jump size distribution has expected value $k$. In particular, the process $X = (X_t)_{t \geq 0}$ satisfies

$$
X_t := \sum_{i=1}^{N_t} Y_i, \quad t \geq 0,
$$

(2.1)

where $N = (N_t)_{t \geq 0}$ is a Poisson process with intensity $\nu$, the random variables $Y_i$, $i \in \mathbb{N}$, are independent and identically distributed with mean $\mathbb{E}[Y_i] = k$, and which are independent of the process $N$. The uncertainty is then modelled by

$$
\Theta = \{ \theta = (\mu, \sigma, \nu) \in \mathbb{R}^3 : \underline{\mu} \leq \mu \leq \overline{\mu}, \underline{\sigma} \leq \sigma \leq \overline{\sigma}, \underline{\nu} \leq \nu \leq \overline{\nu} \},
$$

where $\underline{\mu} \leq \overline{\mu}$, $0 < \underline{\sigma} \leq \overline{\sigma}$ and $0 < \underline{\nu} \leq \overline{\nu}$ are given constants. The classical, non-robust Merton jump diffusion model corresponds to the case where $\mu = \overline{\mu}$, $\sigma = \overline{\sigma}$ and $\nu = \overline{\nu}$. In fact, if we set $\nu = 0$, we obtain the non-robust Black-Scholes model as described in Example 2.2 in [8].

On the one hand, Example 2.1 shows the importance of considering càdlàg price processes, as very general models may include jump terms to describe the “abnormal” variations in the price, as Merton [27] describes it. These variations are due to the arrival of important new information that has more than a marginal effect on the price. On the other hand, we may see from Example 2.1 how this particular way of parameterising model uncertainty can be useful in statistical estimation and model calibration, and therefore relevant for practical applications.
Example 2.2. The compound Poisson process can be used to describe more abstract models with jumps. In particular, we may include the parameters of the jump size distribution in the parametrisation of model uncertainty. For instance, we can define the price process as

$$\log(S_t^{(\theta_1, \ldots, \theta_k, \nu)} / S_0^{(\theta_1, \ldots, \theta_k, \nu)}) = X_t^{1,(\theta_1, \ldots, \theta_k, \nu)} - X_t^{2,(\theta_1, \ldots, \theta_k, \nu)}, \quad k \in \mathbb{N},$$

where $X_t^{1,(\theta_1, \ldots, \theta_k, \nu)}$ and $X_t^{2,(\theta_1, \ldots, \theta_k, \nu)}$ are two independent copies of the compound Poisson process satisfying equation (2.1) (see Section 3 in [18] for details), and where the random variables $Y_i, i \in \mathbb{N}$, are distributed with specific parameters $\theta_1, \ldots, \theta_k$, and where the Poisson process $N$ has intensity $\nu > 0$. For instance, we could take $Y_i = |Z_i|$, where $Z_i \sim \mathcal{N}(0, \sigma^2)$, so that $\theta_1 = \sigma^2$. Another example is to take $Y_i \sim \text{Exp}(\alpha)$, so that $\theta_1 = \alpha$.

Example 2.3. The previous example is a specific case of a larger class of models (see Geman et al. [18]), where the price process is a time-changed Brownian motion. The underlying idea uses the result that every semimartingale can be written as a time-changed Brownian motion on some adequately defined probability space (see, e.g., Monroe [28]). If the time change was continuous, then under fairly general conditions it could be represented as an Itô process. Since time is increasing, this continuity assumption would force us to set the volatility term of the time change process to zero, which means that the time change is locally deterministic. If we relate the time change to the information flow embedded in the price, where there is local uncertainty about these flows, we must assume that the time change is a purely discontinuous process. Hence, the price process itself will be purely discontinuous too.

For instance, in Example 2.2 if we choose

$$\log(S_t^{(\sigma, \nu)} / S_0^{(\sigma, \nu)}) = X_t^{1,(\sigma, \nu)} - X_t^{2,(\sigma, \nu)}, \quad X_t^{j,(\sigma, \nu)} = \sum_{i=1}^{N_i^j} Y_i^j, \quad j \in \{1, 2\},$$

where $N_1$ and $N_2$ are independent Poisson processes with intensity $\nu > 0$, and where $Y_i^j, i \in \mathbb{N}$ and $j \in \{1, 2\}$, are independent random variables with a half-normal distribution (i.e., $Y_i^j = |Z_i|$ with $Z_i \sim \mathcal{N}(0, \sigma^2)$), the time change is given by $T(t) = N_1^1 + N_2^2$, so that

$$\log(S_t^{(\sigma, \nu)} / S_0^{(\sigma, \nu)}) = \sigma W_{T(t)} = \sigma W_{N_1^1 + N_2^2}, \quad t \geq 0,$$

where $W = (W_t)_{t \geq 0}$ is a standard Brownian motion. For details of this derivation, we refer to Section 3.2 in [18]. In such a model, the parametrisation of model uncertainty is of the form $\Theta = \{\theta = (\sigma, \nu) \in \mathbb{R}^2 : 0 < \sigma \leq \sigma, 0 < \nu \leq \nu \leq \nu\}$. Another idea is to use the same price structure as above, where the log-price is the difference of two independent pure jump processes, but instead of using the compound Poisson process, we may use a Gamma process. In particular, we let

$$\log(S_t^{(\mu_1, \mu_2, \nu_2)} / S_0^{(\mu_1, \mu_2, \nu_2)}) = U_t^{1,(\mu_1, \nu_1)} - U_t^{2,(\mu_2, \nu_2)}, \quad U_t^{j,(\mu_j, \nu_j)} = \frac{\nu_j}{\mu_j} \gamma_j \left( \frac{\mu_j^2}{\nu_j} t \right), \quad j \in \{1, 2\},$$

where $\gamma_1$ and $\gamma_2$ are two independent standard Gamma processes, so that $\gamma_j(t) \sim \text{Gamma}(t, 1)$ for $j \in \{1, 2\}$, and therefore $U_t^{j,(\mu_j, \nu_j)} \sim \text{Gamma} \left( \frac{\mu_j^2}{\nu_j}, \frac{\mu_j}{\nu_j} \right)$. More precisely, the increments
it turns out that the log-price process is of the form $U_{t+h}^{j,(\mu_j,\nu_j)} - U_t^{j,(\mu_j,\nu_j)}$, for any $h > 0$, have density

$$f_{U_{t+h}^{j,(\mu_j,\nu_j)} - U_t^{j,(\mu_j,\nu_j)}}(x) = \left( \frac{\mu_j^2 h}{\nu_1} \right) x^{\frac{\mu_j^2 h}{\nu_1} - 1} \exp \left( -\frac{\mu_j^2 h}{\nu_1} x \right) \Gamma \left( \frac{\mu_j^2 h}{\nu_1} \right), \quad x > 0,$$

so that $U_{t+h}^{j,(\mu_j,\nu_j)} - U_t^{j,(\mu_j,\nu_j)}$ has mean $\mu_j t$ and variance $\nu_j t$. The parameters $\mu_j$ and $\nu_j$ are thus called mean and variance rates, respectively. Under suitable assumptions on the parameters, it turns out that the log-price process is of the form $\alpha \gamma_3(t) + \beta W_\gamma(t)$, where $\alpha$ and $\beta$ are constants, $\gamma_3(t)$ is a linear combination of the gamma processes $\gamma_1(t)$ and $\gamma_2(t)$, and where $W = (W_t)_{t \geq 0}$ is a standard Brownian motion. For details we refer to Section 3.4 in [18] (see also [22] for more information on the variance Gamma process). With respect to the parametrisation of model uncertainty, we here use

$$\Theta = \{ \theta = (\mu_1, \nu_1, \mu_2, \nu_2) \in \mathbb{R}^4 : 0 < \mu_j \leq \mu_j \leq \mu_j, 0 < \nu_j \leq \nu_j, j \in \{1, 2\} \}.$$

Similar models using this particular approach of time change also include general subordinators (see [18] for details). These models are relevant because they are capable of modelling the local uncertainty of the information flow that influences the price of the risky asset, and therefore require jump processes. On the other hand, the described models are tailor made for statistical inference and model calibration, and therefore contain an important component for practical applications.

In each model, we assume that the risky asset $S^\theta$ can be traded under proportional transaction costs $\lambda \in (0, 1)$. That is, an agent can buy the risky asset at the higher ask price $S^\theta$ but can only sell it at the lower bid price $(1 - \lambda)S^\theta$. The riskless asset can be traded without transaction costs.

In the spirit of Jacod [20], we say that an $\mathbb{R}_+ \cup \{\infty\}$-valued process is increasing if all trajectories are increasing. We note that an increasing process does not necessarily start at zero. We then define the set $\mathcal{V}^+(\mathbb{F}, P)$ of all equivalence classes of processes, with respect to the relation $P$-indistinguishable admitting a version $H$, which is $\mathbb{F}$-adapted, increasing and satisfies $H_t < \infty$, $P$-a.s., for all $t \in \mathbb{R}_+$. We then define $\mathcal{V}(\mathbb{F}, P) = \mathcal{V}^+(\mathbb{F}, P) - \mathcal{V}^+(\mathbb{F}, P)$ as the set of differences of two elements from $\mathcal{V}^+(\mathbb{F}, P)$. These are exactly the equivalence classes of $\mathbb{F}$-adapted processes, whose trajectories are $P$-a.s. of finite variation on compacts. If $H \in \mathcal{V}(\mathbb{F}, P)$, we denote the total variation of $H$ by $|H|$. We denote the $\mathbb{F}$-predictable processes by $\mathcal{P}(\mathbb{F})$. The set $\mathcal{P}(\mathbb{F}) \cap \mathcal{V}(\mathbb{F}, P)$ consists of classes of processes from $\mathcal{V}(\mathbb{F}, P)$ with at least one $\mathcal{P}(\mathbb{F})$-measurable version. Accordingly, we define the sets $\mathcal{V}(\mathbb{F}^P, P), \mathcal{P}(\mathbb{F}^P)$, etc., by replacing $\mathbb{F}$ with $\mathbb{F}^P$.

Trading strategies are then modelled by $\mathbb{R}^2$-valued processes $H = (H^0_t, H^1_t)_{0 \leq t \leq T}$, where $H^0$ and $H^1$ are elements of $\mathcal{P}(\mathbb{F}) \cap \mathcal{V}(\mathbb{F}, P)$. In particular, $H^0$ (respectively $H^1$) is a process of finite variation that is $P$-indistinguishable from a predictable process. For each process $H \in \mathcal{V}(\mathbb{F}, P)$, there exists a unique decomposition $H = H^1 + H^1$, called Jordan-Hahn decomposition, where $H^1$ and $H^1$ are elements of $\mathcal{V}^+(\mathbb{F}, P)$, and such that $|H| = H^1 + H^1$. If $H \in \mathcal{P}(\mathbb{F}) \cap \mathcal{V}(\mathbb{F}, P)$, then also $H^1, H^1 \in \mathcal{P}(\mathbb{F}) \cap \mathcal{V}^+(\mathbb{F}, P)$ (see Lemma 1.35 in [20]).

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Two processes $X$ and $Y$ are called $P$-indistinguishable if the set $\{X \neq Y\}$ is $P$-evanescent, i.e., the set $N = \{\omega \in \Omega : \exists t \in \mathbb{R}_+, \text{ s.t. } (\omega, t) \in \{X \neq Y\}\}$ is $P$-negligible, that is $N \in \mathcal{F}^P$ and $P[N] = 0$. 

6
Moreover, we define $\Delta H_t := H_t - H_{t-}$, and $\Delta_+ H_t := H_{t+} - H_t$, where $H_t := \lim_{s \downarrow t} H_s$ and $H_{t+} := \lim_{s \uparrow t} H_s$ denote the left and right limits, respectively, and the right-continuous processes

$$H^d_t := \sum_{s \leq t} \Delta H_s, \quad \text{and} \quad H^{d,+}_t := \sum_{s \leq t} \Delta_+ H_s.$$ 

Finally, we define the continuous part $H^c$ of $H$ by $H^c_t := H_t - H^d_t - H^{d,+}_t$.

For a fixed model $\theta \in \Theta$, a strategy is called self-financing under transaction costs $\lambda$, if

$$\int_s^t dH^0_u \leq -\int_s^t S^\theta_u dH^{1,+}_u + \int_s^t (1-\lambda)S^\theta_u dH^{1,-}_u \quad (2.2)$$

for all $0 \leq s < t \leq T$, where the integrals

$$\int_s^t S_u dH^{1,+}_u := \int_s^t S_u dH^{1,+}_u + \sum_{s < u \leq t} S_u \Delta H^{1,+}_u + \sum_{s \leq u < t} S_u \Delta_+ H^{1,+}_u,$$

$$\int_s^t (1-\lambda)S_u dH^{1,-}_u := \int_s^t (1-\lambda)S_u dH^{1,-}_u + \sum_{s < u \leq t} (1-\lambda)S_u \Delta H^{1,-}_u + \sum_{s \leq u < t} (1-\lambda)S_u \Delta_+ H^{1,-}_u$$

can be defined pathwise by using Riemann-Stieltjes integrals. For details on the above integrals, we refer to Section 7 in [11]. In fact, the integrals $\int_s^t dH^0_u$, $\int_s^t S_u dH^{1,+}_u$ as well as $\int_s^t (1-\lambda)S_u dH^{1,-}_u$ represent a class of processes up to $P$-indistinguishability, which is in line with our setup, as $H^0$ and $H^1$ already represent an equivalence class of processes (see Section 1.d in [20]). Here it is also worth noting that $\mathcal{V}(\mathbb{F}, P) = \mathcal{V}(\mathbb{F}^P, P)$. The self-financing condition (2.2) then states that purchases and sales of the risky asset are accounted for in the riskless position:

$$dH^0_t \leq -S^\theta_t dH^{1,+}_t + (1-\lambda)S^\theta_t dH^{1,-}_t, \quad 0 \leq t \leq T, \quad (2.3)$$

$$\Delta H^0_t \leq -S^\theta_t \Delta H^{1,+}_t + (1-\lambda)S^\theta_t \Delta H^{1,-}_t, \quad 0 \leq t \leq T, \quad (2.4)$$

$$\Delta_+ H^0_t \leq -S^\theta_t \Delta_+ H^{1,+}_t + (1-\lambda)S^\theta_t \Delta_+ H^{1,-}_t, \quad 0 \leq t \leq T. \quad (2.5)$$

More precisely, we require that the processes $H^0 = H^{0,+} - H^{0,-}$ and $H^1 = H^{1,+} - H^{1,-}$ have increments that satisfy $dH^0_t \leq (1-\lambda)S^\theta_t dH^{1,+}_t$ and $dH^0_t \geq S^\theta_t dH^{1,-}_t$, which eventually leads to the self-financing condition (2.2), or in differential form to equations (2.3), (2.4) and (2.5).

For a fixed model $\theta \in \Theta$, a self-financing strategy $H$ is admissible under transaction costs $\lambda$, if its liquidation value $V^{\text{liq}}_t(\theta, H)$ satisfies

$$V^{\text{liq}}_t(\theta, H) := H^+_t + (H^-_t)^+(1-\lambda)S^\theta_t - (H^-_t)^-S^\theta_t \geq 0, \quad \text{a.s.,} \quad (2.6)$$

for all $t \in [0, T]$. For $x > 0$ and a fixed model $\theta \in \Theta$, we denote by $\mathcal{H}^\theta(x)$ the set of all self-financing, admissible trading strategies under transaction costs $\lambda$, starting with initial endowment $(H^0_0, H^1_0) = (x, 0)$.

In order to get towards a model-independent setup, that is, we want to consider self-financing trading strategies that are admissible for all models $\theta \in \Theta$, we pass to a dominating
pair \((H^0, H^1)\) for each trading strategy \(H \in \mathcal{H}^\theta(x)\) where equality holds true in (2.2). This way we only have to specify one of the holdings, e.g., the number of stocks \(H^1\). For a fixed model \(\theta \in \Theta\) and \(x > 0\), we thus define the set

\[
\mathcal{A}^\theta(x) := \{H^1 \in \mathcal{P}(\mathbb{R}) \cap \mathcal{V}(\mathbb{R}, P) : (H^0, H^1) \in \mathcal{H}^\theta(x), \ dH_t^0 = -S_t^\theta dH_t^{1,+} + (1 - \lambda)S_t^\theta dH_t^{1,-}\}.
\]

This definition is in line with the set of admissible trading strategies considered in the case of one single model, that is, \(\Theta = \{\theta\}\), as discussed in [10]. We will also refer to this situation as the non-robust case. Moreover, by letting \(\mathcal{A}(x) := \bigcap_{\theta \in \Theta} \mathcal{A}^\theta(x)\), we obtain the analogue of the set of model-independent admissible trading strategies given in [8].

Note that \(H^1 \in \mathcal{A}(x)\) does no longer depend on \(\theta\). However, the holdings in the bond \(H^0\) still depend on \(\theta\). We will use the notation \(H^{0,\theta}\) to indicate this dependence. In particular, we define \(H^{0,\theta} := x + H_{t}\uparrow^{0,\theta} - H_{t}\downarrow^{0,\theta}\) with

\[
H_{t}\uparrow^{0,\theta} := \int_0^t S_u^\theta dH_u^{1,+}, \quad \text{and} \quad H_{t}\downarrow^{0,\theta} := \int_0^t (1 - \lambda)S_u^\theta dH_u^{1,-}.
\]  

(2.7)

Moreover, we write \(V^\text{liq}_t(\theta, H^1)\) to indicate that \(H^0\) in (2.6) is defined via (2.7). We also notice that the mapping \(H^1 \mapsto V^\text{liq}_t(\theta, H^1)\) is concave. To see this statement, we first note that \(x^+ = \max(x, 0)\) and \(x^- = \min(-x, 0)\) are convex functions. By definition (see, e.g., equation (5.1) in Section X.5 in [17]) we have

\[
H_t^{1,-} = \lim_{n \to \infty} \sum_{k=1}^n (H_{tk/n} - H_{tk/n-1}/n)^-,
\]

which implies that \((\alpha H_t^1 + (1 - \alpha)\tilde{H}_t^1)^+ \leq \alpha H_t^{1,+} + (1 - \alpha)\tilde{H}_t^{1,+}\) for all \(\alpha \in [0, 1]\). Now, observe that

\[
V^\text{liq}_t(\theta, H^1) = -\lambda \int_0^t S_u^\theta dH_u^{1,+} - \int_0^t S_u^\theta dH_u^{1,-} + H_t^1 S_t^\theta - (H_t^1)^+ \lambda S_t^\theta,
\]

which together with the fact that \(S^\theta\) is strictly positive implies that \(V^\text{liq}_t(\theta, H^1)\) is concave in \(H^1\).

Now, we have everything in place to formulate the optimisation problem. For this cause, we consider an investor whose preferences are modelled by a standard utility function \(U : (0, \infty) \to \mathbb{R}\). For a given initial capital \(x > 0\), the investor wants to maximise the expected utility of terminal wealth with respect to the worst-case scenario of all possible models. This means that the investor wants to find the optimal strategy \(\hat{H}^1 \in \mathcal{A}(x)\) that maximises

\[
\inf_{\theta \in \Theta} E\left[U\left(V^\text{liq}_T(\theta, H^1)\right)\right].
\]

The value function of this primal optimisation problem is denoted by

\[
u(x) := \sup_{H^1 \in \mathcal{A}(x)} \inf_{\theta \in \Theta} E\left[U\left(V^\text{liq}_T(\theta, H^1)\right)\right].
\]

(2.8)

In the sequel, we answer the question whether, and under which assumptions, the robust primal problem (2.8) admits a solution.

\(^2\)That is a strictly concave, non-decreasing and continuously differentiable function satisfying the Inada conditions \(U'(0) = \lim_{x \to 0} U'(x) = \infty\) and \(U'(+\infty) = \lim_{x \to +\infty} U'(x) = 0\).
3 Main result

In the frictionless case, the Fundamental Theorem of Asset Pricing states that the no-arbitrage condition is equivalent to the property that the price process admits an equivalent local martingale measure (see Delbaen and Schachermayer [12]). In the setting of transaction costs, the notion of **consistent price systems** plays a role analogous to the notion of equivalent martingale measures in the frictionless case.

**Definition 3.1.** Fix $0 < \lambda < 1$ and $\theta \in \Theta$. A strictly positive adapted càdlàg process $S^\theta$ satisfies the condition (CPS$^\lambda$) of having a $\lambda$-**consistent price system**, if there exists a pair of processes $Z^\theta = (Z^0_t, Z^1_t)_{0 \leq t \leq T}$, consisting of a density process $Z^{0,\theta} = (Z^{0,\theta}_t)_{0 \leq t \leq T}$ of an equivalent local martingale measure $Q^\theta \approx P$ for a price process $\tilde{S}^\theta = (\tilde{S}^\theta_t)_{0 \leq t \leq T}$ evolving in the bid-ask spread $[(1-\lambda)S^\theta, S^\theta]$, and $Z^{1,\theta} = Z^{0,\theta}_t \tilde{S}^\theta$. In particular, $\tilde{S}^\theta$ satisfies

$$
(1-\lambda)S^\theta_t \leq \tilde{S}^\theta_t \leq S^\theta_t, \quad 0 \leq t \leq T. \tag{3.1}
$$

We further say that $S^\theta$ satisfies (CPS$^\lambda$) **locally**, if there exists a strictly positive stochastic process $Z^\theta = (Z^{0,\theta}, Z^{1,\theta})$ and a localising sequence $(\tau_n)_{n \geq 0}$ of stopping times, such that $(Z^{0,\theta})^{\tau_n}$ is a $\lambda$-consistent prices system for the stopped process $(\tilde{S}^\theta)^{\tau_n}$ for each $n \geq 0$. We denote the space of all such processes by $Z^\theta$ and $Z^{\theta,\text{loc}}$, respectively.

We impose the existence of consistent price systems for every model $\theta \in \Theta$.

**Assumption 3.2.** For each $\theta \in \Theta$ and for some $0 < \lambda' < \lambda$, the price process $S^\theta$ satisfies (CPS$^{\lambda'}$) locally.

In the non-robust setting, i.e., for a fixed model $\theta \in \Theta$ and $x > 0$, we define

$$
C^\theta(x) := \{g \in L^0_+(\Omega, \mathcal{F}, P) : g \leq V^{\text{liq}}_T(\theta, H), \text{ for some } H \in \mathcal{H}^\theta(x)\}.
$$

This is the set of terminal positions $g$ that one can superreplicate with an admissible trading strategy $H$ and initial endowment $x$. Since we are not interested in an analysis of the dual problem on the level of stochastic processes, we can define the dual variables simply on the level of random variables by setting

$$
D^\theta(y) := \{h \in L^0_+(\Omega, \mathcal{F}, P) : \mathbb{E}[gh] \leq y \text{ for all } g \in C^\theta(1)\}, \quad \text{for all } y > 0. \tag{3.2}
$$

Note that $\{yZ^\theta_f : (Z^0, Z^1) \in Z^{\theta,\text{loc}}\} \subseteq D^\theta(y)$ so that $D^\theta(y) \neq \emptyset$. Moreover, in the non-robust case it is possible to identify a suitable set of stochastic processes as dual variables, so-called **optional strong supermartingale deflators**, that yields $D^\theta(y)$ as terminal values of such processes. We refer to Czichowsky and Schachermayer [10] for the duality theory for portfolio optimisation under transaction costs on the level of stochastic processes in the non-robust setup.

Using the sets $C^\theta(x)$ and $D^\theta(y)$ allows us to define the non-robust value functions. In particular, the primal and dual value functions for the $\theta$-model are given by

$$
u^\theta(x) := \sup_{f \in C^\theta(x)} \mathbb{E}[U(f)], \quad \text{and} \quad j^\theta(y) := \inf_{h \in D^\theta(y)} \mathbb{E}[J(h)],
$$

where $J(y) := \sup_{x > 0} (U(x) - xy), y > 0$, is the Legendre transform of $-U(-x)$. For our purpose, we need the following assumption.
Assumption 3.3. The asymptotic elasticity of $U$ is strictly less than one, that is,

$$\text{AE}(U) := \limsup_{x \to \infty} \frac{xU'(x)}{U(x)} < 1,$$

and for each model $\theta \in \Theta$, the primal value function $u^\theta(x)$ is finite for some $x > 0$ and hence all $x > 0$ by concavity of $u^\theta(x)$.

In the original paper of Chau and Rásonyi [8], instead of Assumption 3.3, they use the assumption that $j^\theta(y)$, $y > 0$, is finite for all models $\theta \in \Theta$ (see Assumption 3.5 in [8]). However, note that Assumption 3.2 and Assumption 3.3 imply Assumption 3.5 in [8] (see Theorem 2 and the subsequent Note 2 in [23] together with Theorem 2.2 in [22] and Theorem 3.2 in [10]). Moreover, Assumption 3.2 is standard and Assumption 3.3 is satisfied for most popular utility functions, like logarithmic and power utility.

Remark 3.4. Under Assumptions 3.2 and 3.3, we obtain by Theorem 3.2 in [10] that

$$\lim_{x \to \infty} u^\theta(x) = 0, \quad \forall \theta \in \Theta,$$

which may be restated as

$$\lim_{x \to \infty} \frac{u^\theta(x)}{x} = 0, \quad \forall \theta \in \Theta.$$

As explained below Theorem 3.2 of Czichowsky and Schachermayer [10], the condition that $S^\theta = (S^\theta_t)_{0 \leq t \leq T}$ satisfies (CPS$^\lambda$) locally for all $0 < \lambda' < \lambda$ is only necessary to argue that there is no “duality gap” in part (4) of Theorem 3.2 of Czichowsky and Schachermayer [10]. Since we do not require this conclusion for our results, we can use the weaker assumption that $S^\theta = (S^\theta_t)_{0 \leq t \leq T}$ satisfies (CPS$^\lambda$) locally for some $0 < \lambda' < \lambda$.

The following theorem is the main result of this paper. It extends Theorem 3.6 in [8] for utility functions defined on the positive half-line from continuous price processes to the general price processes with jumps.

Theorem 3.5. Let $x > 0$. Under Assumptions 3.2 and 3.3 the robust utility maximisation problem (2.8) admits a solution, i.e., there is $\hat{H}^1 \in \mathcal{A}(x)$ satisfying

$$u(x) = \inf_{\theta \in \Theta} \mathbb{E}[U(V_{T}^{\text{ln}}(\theta, \hat{H}^1))].$$

When $U$ is bounded from above, the same conclusion holds assuming only that there exists (at least) one $\theta' \in \Theta$ for which (CPS$^\lambda$) locally holds true for some $0 < \lambda' < \lambda$.

4 Proof of the main theorem

To prove Theorem 3.5, we need the following four results. The first result states that for a fixed model $\theta$, the value process with respect to a consistent price system $(\hat{S}^\theta, Q^\theta)$ is an optional strong supermartingale under $Q^\theta$. Optional strong supermartingales have been introduced by Mertens [26] as a generalisation of the notion of a càdlàg supermartingale. We recall the definition (see Definition 1 of Appendix I in [14]).
Definition 4.1. An optional process $X = (X_t)_{t \geq 0}$ is an optional strong supermartingale, if:

1. For every bounded stopping time $\tau$, $X_\tau$ is integrable.
2. For every pair of bounded stopping times $\sigma$ and $\tau$, such that $\sigma \leq \tau$, we have
   $$E[X_\tau | F_\sigma] \leq X_\sigma, \quad P\text{-a.s.}$$

For further discussion of optional strong supermartingales we refer to Appendix I in Dellacherie and Meyer [14].

Proposition 4.2. For a fixed model $\theta \in \Theta$ with price process $S^\theta = (S^\theta_t)_{0 \leq t \leq T}$, transaction costs $0 < \lambda < 1$, and an admissible self-financing trading strategy $H \in \mathcal{H}^\theta(x)$, $x > 0$, as above, suppose that $(Z^{0,\theta}, Z^{0,\theta} \tilde{S}^\theta) \in \mathcal{Z}_0^{\theta}$ is a locally consistent price system under transaction costs $\lambda$. Then the process $\tilde{V}(\theta, H) = (V_t(\theta, H))_{0 \leq t \leq T}$, defined by
   $$\tilde{V}_t(\theta, H) := H_t^0 + H_t^1 \tilde{S}^\theta_t, \quad 0 \leq t \leq T,$$
satisfies $\tilde{V}_t(\theta, H) \geq V^{\mathrm{liq}}_t(\theta, H)$ almost surely, and $Z^{0,\theta}(H^0 + H^1 \tilde{S}^\theta)$ is an optional strong supermartingale.

In the setting where the filtration $\mathbb{F}$ satisfies the usual conditions, the proof of Proposition 4.2 is given in [34] (see Proposition 2). In the current setup, where $\mathbb{F}$ is right continuous but not complete, we may still follow the lines of the proof given in [34], but with special care on some technicalities. In particular, we first obtain the result for $\mathbb{F}^P$ by using the aforementioned result, and then pass to an $\mathbb{F}$-optional version by using Lemma 7 of Appendix I in [14]. Note that this argument is in line with our understanding of $\tilde{V}_t(\theta, H)$ being an optional process, as $H^0$ and $H^1$ already are $P$-indistinguishable from predictable processes. Since $\mathbb{F} \subseteq \mathbb{F}^P$, Property (1) of Definition 4.1 follows immediately, while Property (2) follows by using the tower property of conditional expectations.

It is worth noting that the proof of Proposition 2 in [34] requires the usual conditions, because it uses Theorem IV.117 in [13]. It is still possible though to argue similarly as in [34], by using instead Theorem B together with Remark E in [14] (see complements to chapter IV) and a corollary thereof, instead of Theorem IV.117 in [13]. To conclude the proof, one would also use Mertens decomposition (see Theorem 20 of Appendix I in [14]), as it is done in [34]. However, the result of Proposition 4.2 would not change (see also Remark 21 of Appendix I in [14]).

As a second result, we also need the following bipolar relation for the non-robust models.

Lemma 4.3. Fix $x, y > 0$. Suppose that $S^\theta$ satisfies (CPS$^{\lambda'}$) locally for some $\lambda' \in (0, \lambda)$. Then, a random variable $g \in \mathcal{L}_x^P$ satisfies $g \in \mathcal{C}^\theta(x)$ if and only if $E[gh] \leq xy$ for all $h \in \mathcal{D}^\theta(y)$ for some $y > 0$.

The third result we need extends Proposition 3.4 in [7] to a more general, model-independent view. In particular, while the result from Proposition 3.4 in [7] holds for predictable processes, our result in Proposition 4.4 is true for processes that are $P$-indistinguishable from an $\mathbb{F}$-predictable process but not predictable itself. The need for such a result stems from the more general assumption that the filtration $\mathbb{F}$ is not complete in our setup.
Proposition 4.4. Let $(H^{1,n})_{n \in \mathbb{N}} \subseteq \mathcal{A}(x)$ for $x > 0$. Assume that there is $\theta \in \Theta$ so that $S^\theta$ satisfies (CPS$^{\lambda'}$) locally, for some $0 < \lambda' < \lambda$. Then there exist processes $H^{1,\uparrow} \text{ and } H^{1,\downarrow}$, both in $\mathcal{P}(\mathbb{F}) \cap \mathcal{V}^+(\mathbb{F}, P)$, with $H^1 = H^{1,\uparrow} - H^{1,\downarrow} \in \mathcal{A}(x)$ and a sequence of convex combinations $(\widetilde{H}^{1,n,\uparrow}, \widetilde{H}^{1,n,\downarrow})_{n \in \mathbb{N}} \subseteq \text{conv}((H^{1,n,\uparrow}, H^{1,n,\downarrow}), (H^{1,n+1,\uparrow}, H^{1,n+1,\downarrow}), \ldots)$, such that $(\widetilde{H}^{1,n,\uparrow}, \widetilde{H}^{1,n,\downarrow})$ converges for almost every $\omega$ for every $t \in [0, T]$ to $(H^{1,\uparrow}, H^{1,\downarrow})$, i.e.,

$$P[(\widetilde{H}^{1,n,\uparrow}, \widetilde{H}^{1,n,\downarrow}) \to (H^{1,\uparrow}, H^{1,\downarrow}), \ \forall t \in [0, T)] = 1.$$ 

The proofs of Lemma 4.3 and Proposition 4.4 are given in Appendix A. In the proof of Proposition 4.4 we will also need the following result. It shows that the pointwise convergence of integrators of finite variation is sufficient for the convergence of the integrals of a fixed càdlàg function. While this result is not true for the standard Riemann-Stieltjes integrals, it is true for our notion of the integral that is motivated by self-financing trading in financial markets (given below (2.2)). The reason for this fact is that for trades immediately before a predictable stopping time, the price paid is the left limit of the price process. The proof of Lemma 4.5 is also given in Appendix A.

Lemma 4.5. Let $x > 0$ and consider the sequence $(H^{1,n})_{n \in \mathbb{N}} \subseteq \mathcal{A}(x)$ with canonical decomposition $H^{1,n} = H^{1,n,\uparrow} - H^{1,n,\downarrow}$. Moreover, assume that $H^1 = H^{1,\uparrow} - H^{1,\downarrow}$ is a process in $\mathcal{P}(\mathbb{F}) \cap \mathcal{V}(\mathbb{F}, P)$ with $H^{1,\uparrow}, H^{1,\downarrow} \in \mathcal{P}(\mathbb{F}) \cap \mathcal{V}^+(\mathbb{F}, P)$, such that

$$P[H^{1,n,\uparrow}_t \to H^{1,\uparrow}_t, \ \forall t \in [0, T)] = 1, \ \text{ and } \ P[H^{1,n,\downarrow}_t \to H^{1,\downarrow}_t, \ \forall t \in [0, T)] = 1. \quad (4.1)$$

Then, for all $\theta \in \Theta$, the sequence $(H^{0,\theta,n})_{n \in \mathbb{N}}$ with $H^{0,\theta,n} = x + H^{0,\theta,n,\uparrow} - H^{0,\theta,n,\downarrow}$ defined via (2.7), i.e.,

$$H^{0,\theta,n,\downarrow}_t = \int_0^t S^\theta_u dH^{1,n,\downarrow}_u, \ \text{ and } \ H^{0,\theta,n,\uparrow}_t = \int_0^t (1 - \lambda) S^\theta_u dH^{1,n,\uparrow}_u, \quad (4.2)$$

converges for almost every $\omega$ for every $t \in [0, T]$ to $H^{0,\theta} = x - \int_0^t S^\theta_u dH^{1,\uparrow}_u + \int_0^t (1 - \lambda) S^\theta_u dH^{1,\downarrow}_u$. In particular, $H^{0,\theta,\uparrow}$ and $H^{0,\theta,\downarrow}$ are elements of $\mathcal{P}(\mathbb{F}) \cap \mathcal{V}(\mathbb{F}, P)$ satisfying

$$P[H^{0,\theta,\uparrow}_t \to H^{0,\theta,\uparrow}_t, \ \forall t \in [0, T)] = 1, \ \text{ and } \ P[H^{0,\theta,\downarrow}_t \to H^{0,\theta,\downarrow}_t, \ \forall t \in [0, T)] = 1, \quad (4.3)$$

and therefore $H^1 \in \mathcal{A}(x)$.

We are now ready to prove our main result.

Proof of Theorem 3.3. Let $(H^{1,n})_{n \in \mathbb{N}} \subseteq \mathcal{A}(x)$ be a maximising sequence, i.e.,

$$\inf_{\theta \in \Theta} E[U(V_T^{\text{liq}}(\theta, H^{1,n}))] \uparrow u(x), \ \text{ as } n \to \infty.$$ 

By Proposition 4.4 there is a sequence $(\widetilde{H}^{1,n})_{n \in \mathbb{N}} \subseteq \text{conv}(H^{1,n}, H^{1,n+1}, \ldots)$ and $\widehat{H}^1 \in \mathcal{A}(x)$, such that, for almost every $\omega \in \Omega$, we have that $\widehat{H}^{1,n}$ converges for every $t \in [0, T]$ to $\widehat{H}^1$. Since the utility function is concave and increasing, and because $H^1 \Rightarrow V_T^{\text{liq}}(\theta, H^1)$ is concave, we obtain that $(\widehat{H}^{1,n})_{n \in \mathbb{N}}$ is also a maximising sequence, because

$$\inf_{\theta \in \Theta} E[U(V_T^{\text{liq}}(\theta, \widehat{H}^{1,n}))] \geq \inf_{\theta \in \Theta} E[U(V_T^{\text{liq}}(\theta, H^{1,n}))] \to u(x), \ \text{ as } n \to \infty.$$
We claim that $\hat{H}^1$ is an optimal solution to (2.8). For this purpose, we denote by $U^+$ and $U^-$ the positive and negative parts of the function $U$. From Fatou’s lemma, we deduce that
\[
\liminf_{n \to \infty} E[U^-(V_{T}^\text{liq}(\theta, \hat{H}^{1,n}))] \geq E[U^-(V_{T}^\text{liq}(\theta, \hat{H}^1))],
\]
for each model $\theta \in \Theta$. The optimality of $\hat{H}$ will follow if we show that
\[
\lim_{n \to \infty} E[U^+(V_{T}^\text{liq}(\theta, \hat{H}^{1,n}))] = E[U^+(V_{T}^\text{liq}(\theta, \hat{H}^1))],
\]
(4.4)
for each model $\theta \in \Theta$. If $U(\infty) \leq 0$, then there is nothing to prove. So we assume that $U(\infty) > 0$.

We claim that the sequence $(U^+(V_{T}^\text{liq}(\theta, \hat{H}^{1,n})))_{n \in \mathbb{N}}$ is uniformly integrable for each $\theta \in \Theta$. This assertion is equivalent to the validity of (4.4). Suppose by contradiction that the sequence is not uniformly integrable for some $\theta$. Then, using the characterization of uniform integrability given in [17], Theorem VI.16, and passing if necessary to a subsequence still denoted by $(\hat{H}^{1,n})_{n \in \mathbb{N}}$, we can find a constant $\alpha > 0$ and disjoint sets $A_n \in \mathcal{F}$, $n \in \mathbb{N}$, such that
\[
E[U^+(V_{T}^\text{liq}(\theta, \hat{H}^{1,n}))] \mathbb{1}_{A_n} \geq \alpha \quad \text{for } n \geq 1.
\]
We define the sequence of random variables $(h^n)_{n \in \mathbb{N}}$ by
\[
h^n := x_0 + \sum_{k=1}^{n} V_{T}^\text{liq}(\theta, \hat{H}^{1,k}) \mathbb{1}_{A_k},
\]
where $x_0 = \inf \{x > 0 : U(x) \geq 0\}$. It follows immediately that
\[
E[U(h^n)] \geq \sum_{k=1}^{n} E[U^+(V_{T}^\text{liq}(\theta, \hat{H}^{1,k}))] \mathbb{1}_{A_k} \geq n \alpha.
\]
On the other hand, for any $f \in \mathcal{D}^\theta(1)$ we have by Proposition 4.2 that
\[
E[h^n f] \leq x_0 + \sum_{k=1}^{n} E[V_{T}^\text{liq}(\theta, \hat{H}^{1,k}) f] \leq x_0 + nx.
\]
Hence, by Lemma 4.3 we obtain $h^n \in C^\theta(x_0 + nx)$. Therefore, we have
\[
\limsup_{x \to \infty} \frac{u^\theta(x)}{x} \geq \limsup_{n \to \infty} \frac{E[U(h^n)]}{x_0 + nx} \geq \limsup_{n \to \infty} \frac{n \alpha}{x_0 + nx} = \frac{\alpha}{x} > 0.
\]
By Remark 3.4, this strict inequality is a contradiction to our Assumptions 3.2 and 3.3. As a result, $(U^+(V_{T}^\text{liq}(\theta, \hat{H}^{1,n})))_{n \in \mathbb{N}}$ indeed is a uniformly integrable sequence for each $\theta \in \Theta$.

Since for almost every $\omega \in \Omega$, we have that $\hat{H}_t^{1,n} \to \hat{H}_t^1$ for every $t \in [0, T]$, we get that $V_{T}^\text{liq}(\theta, \hat{H}_t^{1,n}) \to V_{T}^\text{liq}(\theta, \hat{H}_t^1)$ almost surely for each $\theta \in \Theta$ by Lemma 4.3. Therefore, Fatou’s lemma and uniform integrability imply
\[
\limsup_{n \to \infty} \left( \inf_{\theta \in \Theta} E[U(V_{T}^\text{liq}(\theta, \hat{H}_t^{1,n}))] \right) \leq \inf_{\theta \in \Theta} \left( \limsup_{n \to \infty} E[U(V_{T}^\text{liq}(\theta, \hat{H}_t^{1,n}))] \right)
\leq \inf_{\theta \in \Theta} E[U(V_{T}^\text{liq}(\theta, \hat{H}_t^1))],
\]
which proves that \( \tilde{H}^1 \) is an optimal solution to (2.8).

In the case where \( U \) is bounded from above, we use Fatou’s lemma to get
\[
\limsup_{n \to \infty} \mathbb{E}[U^+(V_T^{\text{liq}}(\theta, \tilde{H}^{1,n}))] \leq \mathbb{E}[U^+(V_T^{\text{liq}}(\theta, \tilde{H}^1))]
\]
instead of (4.4). Hence, the existence of (at least) one \( \theta' \in \Theta \) so that \( S^{\theta'} \) satisfies (CPS\( ^X \)) locally for some \( 0 < \lambda' < \lambda \) is enough to obtain the optimal strategy and to conclude the proof. \( \square \)

### Appendix A Proofs for Section 4

We begin with the proof of Lemma 4.3.

**Proof of Lemma 4.3.** Under the assumption that \( S^{\theta} = (S^{\theta}_t)_{0 \leq t \leq T} \) satisfies (CPS\( ^X \)) locally for some \( 0 < \lambda' < \lambda \), it follows as in the proof of Lemma A.1 of Czichowsky and Schachermayer [10] that \( \tilde{C}^{\theta}(x) \) is a closed, convex and solid subset of \( L^1_+(P) \). Note that, as explained in the proof of Lemma A.1 of Czichowsky and Schachermayer [10], the latter assertion only needs the condition that \( S^{\theta} = (S^{\theta}_t)_{0 \leq t \leq T} \) satisfies (CPS\( ^X \)) locally for some \( 0 < \lambda' < \lambda \) and not for all \( 0 < \lambda' < \lambda \). As pointed out the condition that \( S^{\theta} = (S^{\theta}_t)_{0 \leq t \leq T} \) satisfies (CPS\( ^X \)) locally for all \( 0 < \lambda' < \lambda \) is only necessary to argue that there is no “duality gap” in part (4) of Theorem 3.2 of Czichowsky and Schachermayer [10]. Because we do not require this conclusion for our results, we can use the weaker assumption that \( S^{\theta} = (S^{\theta}_t)_{0 \leq t \leq T} \) satisfies (CPS\( ^X \)) locally for some \( 0 < \lambda' < \lambda \) and only need to argue slightly different for the bipolarity as follows. As we defined \( \mathcal{D}^{\theta}(y) \) as the polar of \( \tilde{C}^{\theta}(x) \) in \( L^1_+(P) \) in (3.2), the bipolar theorem in \( L^1_+(P) \) of Brannath and Schachermayer [6] (see Theorem 1.3) yields that \( g \in L^1_+(P) \) satisfies \( \tilde{C}^{\theta}(x) \) if and only if \( \mathbb{E}[gh] \leq xy \) for all \( h \in \mathcal{D}^{\theta}(y) \) for some \( y > 0 \). \( \square \)

We continue with a result that was originally proven in Campi and Schachermayer [7] in the setting of Kabanov and Safarian [21]. It provides an a posteriori, quantitative control on the size of the total variation of admissible trading strategies. We use a slightly adjusted version of this result to cover model independent trading strategies. Our proof is mainly the same as the one of Lemma 3.1 in [33] (see also [35], Lemma 4.10). We also refer to Remark 3.2 in [33] which comes together with the following result.

**Proposition A.1.** Let \( x > 0 \). For some \( \theta' \in \Theta \) and for some \( 0 < \lambda' < \lambda \), assume that \( S^{\theta'} \) satisfies (CPS\( ^X \)) locally. Then, for any strategy \( H^1 \in \mathcal{A}(x) \) with canonical decomposition \( H^1 = H^{1,\uparrow} + H^{1,\downarrow} \), the elements \( H^{1,\uparrow}_T \) and \( H^{1,\downarrow}_T \) as well as their convex combinations are bounded in \( L^0(\Omega, \mathcal{F}, P) \).

**Proof.** Fix \( 0 < \lambda' < \lambda \). For some \( \theta' \in \Theta \) there is by assumption a probability measure \( Q^{\theta'} \approx P \) and a local \( Q^{\theta'} \)-martingale \( \tilde{S}^{\theta'} = (\tilde{S}^{\theta'}_t)_{0 \leq t \leq T} \) satisfying (3.1). By stopping, we may assume that \( \tilde{S}^{\theta'} \) is a true martingale.

We consider a strategy \( H^1 \in \mathcal{A}(x) \), for some \( x > 0 \). Without loss of generality we assume that \( H^1_t = 0 \), i.e., that the position is liquidated at time \( T \). For each \( \theta \in \Theta \), using (2.7), we obtain the holdings of the bond \( H^{1,\theta}_t = x + H^{0,\theta,\uparrow}_t - H^{0,\theta,\downarrow}_t \) via \( dH^{0,\theta,\uparrow}_t = (1 - \lambda)S^{\theta}_t dH^{1,\uparrow}_t \) and
\[ \text{d}H_t^{0,\theta,\downarrow} = S_t^0 \text{d}H_t^{1,\uparrow} \] with \( H_0^{0,\theta,\uparrow} = H_0^{0,\theta,\downarrow} = 0 \). Now, working with \( \theta' \) as introduced before, we first show that

\[
E_{Q^{\theta'}}[H_T^{0,\theta',\uparrow}] \leq \frac{x}{\lambda - \lambda'}.
\] (A.1)

For this purpose, define the process \( H' = ((H^{0,\theta'})', (H^{1})') \) by

\[
H_t' = ((H^{0,\theta'})_t', (H^{1})_t') = \left( H_t^{0,\theta'} + \frac{\lambda - \lambda'}{1 - \lambda} H_t^{0,\theta',\uparrow}, H_t^{1} \right), \quad 0 \leq t \leq T.
\]

This process is a self-financing trading strategy under transaction costs \( \lambda' \): indeed, whenever we have \( \text{d}H_t^{0,\theta'} > 0 \) such that \( \text{d}H_t^{0,\theta'} = \text{d}H_t^{0,\theta',\uparrow} \), the agent sells some units of stock and receives \( \text{d}H_t^{0,\theta',\uparrow} = (1 - \lambda)S_t^{0'} \text{d}H_t^{1,\uparrow} \) (resp., \( (1 - \lambda')S_t^{0'} \text{d}H_t^{1,\uparrow} = \frac{1 - \lambda}{1 - \lambda'} \text{d}H_t^{0,\theta',\uparrow} \)) many bonds under transaction costs \( \lambda \) (resp., \( \lambda' \)). The difference between these two terms is \( \frac{\lambda - \lambda'}{1 - \lambda} \text{d}H_t^{0,\theta',\uparrow} \); this difference is the amount by which the \( \lambda' \)-agent does better than the \( \lambda \)-agent. It is also clear that \( ((H^{0,\theta'})', (H^{1})') \) under transaction costs \( \lambda' \) still is an \( x \)-admissible strategy for the model \( \theta' \), i.e., \( H' \in \mathcal{H}(x) \).

By Proposition 4.2 the process \( \tilde{V}(\theta', H') = (\tilde{V}_t(\theta', H'))_{0 \leq t \leq T} \) defined by

\[
\tilde{V}_t(\theta', H') = (H^{0,\theta'})_t' + (H^{1})_t S_t^{0'} = H_t^{0,\theta'} + \frac{\lambda - \lambda'}{1 - \lambda} H_t^{0,\theta',\uparrow} + H_t^{1} S_t^{0'}
\]

is an optional strong \( Q^{\theta'} \)-supermartingale. We thus get

\[
E_{Q^{\theta'}}[\tilde{V}_T(\theta', H')] = E_{Q^{\theta'}}[H_T^{0,\theta'} + H_T^{1} S_T^{0'}] + \frac{\lambda - \lambda'}{1 - \lambda} E_{Q^{\theta'}}[H_T^{0,\theta',\uparrow}] \leq x.
\]

By admissibility of \( H^1 \), we have \( H_T^{0,\theta'} + H_T^{1} S_T^{0'} \geq 0 \), and thus we have shown (A.1).

To obtain control on \( H_T^{0,\theta',\uparrow} \) too, we note that \( H_T^{0,\theta'} \geq -x \), since \( H_T^1 = 0 \). So we have

\[
H_T^{0,\theta',\uparrow} \leq x + H_T^{0,\theta',\uparrow}. \quad \text{Therefore, we obtain the following estimate for the total variation} \quad H_T^{0,\theta',\uparrow} + H_T^{0,\theta',\uparrow} \quad \text{of} \quad H^{0,\theta'}:
\]

\[
E_{Q^{\theta'}}[H_T^{0,\theta',\uparrow} + H_T^{0,\theta',\uparrow}] \leq x \left( 1 + \frac{2}{\lambda - \lambda'} \right).
\] (A.2)

Now we transfer the \( L^1(Q^{\theta'})-\)estimate in (A.2) to an \( L^0(P) \)-estimate. For \( \varepsilon > 0 \), there exists \( \delta_{\theta'} > 0 \), so that for \( A \in \mathcal{F} \) with \( Q^{\theta'}[A] < \delta_{\theta'} \), we have \( P[A] < \frac{\varepsilon}{2} \). Letting \( C^{0,\theta'} := \frac{x}{\delta_{\theta'}} (1 + \frac{2}{\lambda - \lambda'}) \) and applying Markov’s inequality to (A.2), we get

\[
P[H_T^{0,\theta',\uparrow} + H_T^{0,\theta',\uparrow} \geq C^{0,\theta'}] < \frac{\varepsilon}{2},
\] (A.3)

which is the desired \( L^0(P) \)-estimate. At this point we remark that (A.1) implies that the convex hull of the functions \( H_T^{0,\theta',\uparrow} \) is bounded in \( L^1(Q^{\theta'}) \) and (A.2) yields the same for \( H_T^{0,\theta',\downarrow} \). So by the above reasoning we obtain that also the convex combinations of \( H_T^{0,\theta',\uparrow} \) and \( H_T^{0,\theta',\downarrow} \) remain bounded in \( L^0(P) \).
As before, it follows from (2.7) that \( dH_t^{0, \theta, \downarrow} = S_t^{\theta'} dH_t^{1, \uparrow} \), which can be rewritten as
\[
dH_t^{1, \uparrow} = \frac{dH_t^{0, \theta, \downarrow}}{S_t^{\theta'}}. \tag{A.4}
\]

By assumption, the trajectories of \( S^{\theta'} \) are strictly positive. In fact, we even have for almost all trajectories \((S_t^{\theta'}(\omega))_{0 \leq t \leq T}\), that \( \inf_{0 \leq t \leq T} S_t^{\theta'}(\omega) \) is strictly positive. Indeed, \( \tilde{S}_{T}^{\theta'} \) being a \( Q^{\theta'} \)-martingale with \( \tilde{S}_{T}^{\theta'} > 0 \) almost surely satisfies that \( \inf_{0 \leq t \leq T} \tilde{S}_{t}^{\theta'} \) is \( Q^{\theta'} \)-a.s. and therefore \( P \)-a.s. strictly positive. In particular, for \( \varepsilon > 0 \) we may find \( \delta_{\theta'} > 0 \) such that
\[
P\left( \inf_{0 \leq t \leq T} S_t^{\theta'} < \delta_{\theta'} \right) < \frac{\varepsilon}{2}. \tag{A.5}
\]

Taking \( \eta_{\theta'} := \min(\delta_{\theta'}, \delta_{\theta'}') \) and letting \( C^{1, \theta} := \frac{x}{\sqrt{\theta'}}(1 + \frac{2}{\lambda - x}) \), we obtain from (A.3), (A.4) and (A.5) that
\[
P\left[ H_T^{1, \uparrow} \geq C^{1, \theta} \right] \leq P\left[ \inf_{0 \leq t \leq T} S_t^{\theta'} < \delta_{\theta'} \right] + P\left[ H_T^{0, \theta, \downarrow} \geq C^{0, \theta} \right] < \varepsilon. \tag{A.6}
\]

The first inequality in (A.6) follows from
\[
P\left[ H_T^{1, \uparrow} \geq C^{1, \theta} \right] \leq P\left[ H_T^{0, \theta, \downarrow} \geq \left( \inf_{0 \leq t \leq T} S_t^{\theta'} \right) C^{1, \theta} \right] \leq P\left[ \inf_{0 \leq t \leq T} S_t^{\theta'} < \delta_{\theta'} \right] + P\left[ H_T^{0, \theta, \downarrow} \geq \left( \inf_{0 \leq t \leq T} S_t^{\theta'} \right) C^{1, \theta} \right] \cap \left\{ \inf_{0 \leq t \leq T} S_t^{\theta'} \geq \delta_{\theta'} \right\},
\]
which uses (A.4) (note that \( H_0^{0, \theta, \downarrow} = H_0^{0, \theta, \downarrow} = 0 \) by definition) together with
\[
P\left[ \left\{ H_T^{0, \theta, \downarrow} \geq \left( \inf_{0 \leq t \leq T} S_t^{\theta'} \right) C^{1, \theta} \right\} \cap \left\{ \inf_{0 \leq t \leq T} S_t^{\theta'} \geq \delta_{\theta'} \right\} \right] \leq P\left[ H_T^{0, \theta, \downarrow} \geq \delta_{\theta'} C^{1, \theta} \right] \leq P\left[ H_T^{0, \theta, \downarrow} \geq C^{0, \theta} \right] \leq P\left[ H_T^{0, \theta, \downarrow} + H_T^{0, \theta, \downarrow} \geq C^{0, \theta} \right],
\]
where we use that \( \eta_{\theta'}^2 \leq \delta_{\theta'} \delta_{\theta'}' \), which implies \( \delta_{\theta'} C^{1, \theta} \geq C^{0, \theta} \). The second inequality in (A.6) then follows from (A.3) and (A.5).

To control the term \( H_T^{1, \uparrow} \), we observe that \( H_T^{1, \uparrow} - H_T^{1, \downarrow} = H_T = 0 \). Therefore, we may use the estimate (A.6) of \( H_T^{1, \uparrow} \) to also control \( H_T^{1 \downarrow} \). Moreover, we note that (A.4) also holds for convex combinations of \( H^{1, \uparrow} \). Indeed, for another strategy \( \tilde{H}^1 \in \mathcal{A}(x) \) and \( \alpha \in [0, 1] \) we have
\[
S_t^{\theta'} d((1 - \alpha)H_t^{1, \uparrow} + \alpha \tilde{H}_t^{1, \downarrow}) = (1 - \alpha)S_t^{\theta'} dH_t^{1, \uparrow} + \alpha S_t^{\theta'} d\tilde{H}_t^{1, \uparrow} = (1 - \alpha)dH_t^{0, \theta', \downarrow} + \alpha d\tilde{H}_t^{0, \theta', \downarrow},
\]
so that dividing by \( S_t^{\theta'} \) yields
\[
d((1 - \alpha)H_t^{1, \uparrow} + \alpha \tilde{H}_t^{1, \downarrow}) = \frac{d((1 - \alpha)H_t^{0, \theta', \downarrow} + \alpha \tilde{H}_t^{0, \theta', \downarrow})}{S_t^{\theta'}}.
\]

Since (A.3) also holds for convex combinations of \( H_T^{0, \theta', \downarrow} \) and \( H_T^{0, \theta', \downarrow} \), we obtain that also the convex combinations of \( H_T^{1, \uparrow} \) and \( H_T^{1, \downarrow} \) remain bounded in \( L^0(P) \).
Next, we establish the proof of Lemma 4.5.

**Proof of Lemma 4.5.** Let $\theta \in \Theta$ and fix an arbitrary $\varepsilon > 0$. Also fix $\omega \in \Omega$ so that $S^\theta(\omega)$ has càdlàg trajectories and such that $H^{1,n,\uparrow}_t(\omega) \rightarrow H^{1,\uparrow}_t(\omega)$ for all $t \in [0,T]$. Since the function $S^\theta(\omega) : [0,T] \rightarrow \mathbb{R}$ is càdlàg, there can only be finitely many times $0 \leq \tau_1 < \ldots < \tau_k \leq T$ such that $|\Delta S^\theta_{\tau_i}(\omega)| \geq \varepsilon$. Indeed, if there were infinitely many time points with jumps of size larger than $\varepsilon$, the jump times would have a cluster point in the compact set $[0,T]$, leading to a contradiction to the existence of right or left limits of $S^\theta_t(\omega)$ at every $t \in [0,T]$.

Therefore, setting

$$S^\theta_{t}(\omega) := S^\theta_t(\omega) - \sum_{i=1}^{k} \Delta S^\theta_{\tau_i}(\omega) 1_{[\tau_i, T]}(t), \quad t \in [0,T],$$

gives a càdlàg function with $|\Delta S^\theta_{t}(\omega)| \leq \varepsilon$ for all $t \in [0,T]$. Then, there are finitely many times $\sigma_0 = 0 < \sigma_1 < \ldots < \sigma_{m-1} < T$ such that $|S^\theta_{\sigma_{i+1}}(\omega) - S^\theta_{\sigma_i}(\omega)| > \varepsilon$, because $S^\theta_{\sigma_i}(\omega) : [0,T] \rightarrow \mathbb{R}$ is càdlàg. Indeed, if there were infinitely many such time points $(\sigma_i)_{i \in I}$ for some infinite index set $I$, the times $(\sigma_i)_{i \in I}$ would have a cluster point in the compact set $[0,T]$ leading to a contradiction to the existence of right or left limits of $S^\theta_t(\omega)$ at every $t \in [0,T]$. Because the jumps of $S^\theta_{\sigma_i}(\omega)$ are bounded by $\varepsilon$, we obtain $|S^\theta_{\sigma_i}(\omega) - S^\theta_{\sigma_i}(\omega)| \leq 2\varepsilon$ for all $t \in [\sigma_i, \sigma_{i+1})$ for $i = 0, \ldots, m-1$ with $\sigma_m := T$. Therefore, the step functions $S^\theta_{\sigma_i}(\omega)$ given by

$$S^\theta_{t}(\omega) := S^\theta_{t}(\omega) 1_{[0,1]}(t) + \sum_{i=1}^{m} S^\theta_{\sigma_{i-1}}(\omega) 1_{(\sigma_{i-1}, \sigma_i]}(t), \quad t \in [0,T],$$

satisfy $|S^\theta_{t}(\omega) - S^\theta_{t}(\omega)| \leq 2\varepsilon$ for all $t \in [0,T]$, which implies

$$\left| \int_0^t (S^\theta_u(\omega) - S^\theta_u(\omega)) dH^{1,\uparrow,n}(\omega) \right| \leq 2\varepsilon H^{1,\uparrow,n}(\omega),$$

and

$$\left| \int_0^t (S^\theta_u(\omega) - S^\theta_u(\omega)) dH^{1,\uparrow}(\omega) \right| \leq 2\varepsilon H^{1,\uparrow}(\omega).$$

Moreover, because of our definition of the stochastic integral at left jumps of the integrator given below the self-financing condition (2.7), we have that

$$\int_0^t S^\theta_{u,m}(\omega)dH^{1,n,\uparrow}_u(\omega) = \sum_{i=1}^{m} \int_{\sigma_{i-1}+\lambda \uparrow}^{\sigma_{i}+\lambda \uparrow} S^\theta_{\sigma_{i-1}}(\omega)dH^{1,n,\uparrow}_u(\omega) + \sum_{i=1}^{m} S^\theta_{\sigma_{i-1}}(\omega) \Delta_+ H^{1,n,\uparrow}_{\sigma_{i-1}+\lambda \uparrow}(\omega)$$

$$= \sum_{i=1}^{m} S^\theta_{\sigma_{i-1}}(\omega)(H^{1,n,\uparrow}_{\sigma_{i}+\lambda \uparrow}(\omega) - H^{1,n,\uparrow}_{\sigma_{i-1}+\lambda \uparrow}(\omega)) + \Delta_+ H^{1,n,\uparrow}_{\sigma_{i-1}+\lambda \uparrow}(\omega)$$

$$= \sum_{i=1}^{m} S^\theta_{\sigma_{i-1}}(\omega)(H^{1,n,\uparrow}_{\sigma_{i}+\lambda \uparrow}(\omega) - H^{1,n,\uparrow}_{\sigma_{i-1}+\lambda \uparrow}(\omega)).$$
where we use that $\Delta_z H_{\sigma^1 \land t}^{1,n}(\omega) = H_{\sigma^1 \land t}^{1,n}(\omega) - H_{\sigma^1 \land t}^{1,n}(\omega)$, by definition. Similarly, we obtain

$$
\int_0^t S_{\theta,\varepsilon}^{m}(\omega)dH_{u}^{1,n}(\omega) = \sum_{i=1}^{m} S_{\theta,\varepsilon}^{i}(\omega)(H_{\sigma^i \land t}^{1,n}(\omega) - H_{\sigma^i \land t}^{1,n}(\omega)),
$$

so that

$$
\int_0^t S_{\theta,\varepsilon}^{m}(\omega)dH_{u}^{1,n}(\omega) \to \int_0^t S_{\theta,\varepsilon}^{m}(\omega)dH_{u}^{1}(\omega), \quad n \to \infty,
$$
as $H_{u}^{1,n}(\omega) \to H_{u}^{1}(\omega)$ for all $t \in [0,T]$. In particular, for $n \in \mathbb{N}$ large enough, we have

$$
\left| \int_0^t S_{\theta,\varepsilon}^{m}(\omega)dH_{u}^{1,n}(\omega) - \int_0^t S_{\theta,\varepsilon}^{m}(\omega)dH_{u}^{1}(\omega) \right|
\leq \left| \int_0^t (S_{\theta,\varepsilon}^{m}(\omega) - S_{\theta,\varepsilon}^{m}(\omega))dH_{u}^{1,n}(\omega) \right| + \left| \int_0^t (S_{\theta,\varepsilon}^{m}(\omega) - S_{\theta,\varepsilon}^{m}(\omega))dH_{u}^{1}(\omega) \right|
+ \left| \int_0^t S_{\theta,\varepsilon}^{m}(\omega)dH_{u}^{1,n}(\omega) - \int_0^t S_{\theta,\varepsilon}^{m}(\omega)dH_{u}^{1}(\omega) \right|
\leq 4\varepsilon(H_{u}^{1}(\omega) + \varepsilon) + \varepsilon.
$$

On the other hand, the finite sum $\sum_{i=1}^{k} \Delta S_{\tau_i}^{\theta}(\omega)1_{[\tau_i,T]}(t)$, where the $\tau_i$’s are the times where $|\Delta S_{\tau_i}^{\theta}(\omega)| \geq \varepsilon$ for $i = 1, \ldots, k$, satisfies

$$
\int_0^t \left( \sum_{i=1}^{k} \Delta S_{\tau_i}^{\theta}(\omega)1_{[\tau_i,T]}(u) \right)dH_{u}^{1,n}(\omega) = \sum_{i=1}^{k} \Delta S_{\tau_i}^{\theta}(\omega)(H_{t}^{1,n}(\omega) - H_{\tau_i \land t}^{1,n}(\omega)), \quad (A.7)
$$
and

$$
\int_0^t \left( \sum_{i=1}^{k} \Delta S_{\tau_i}^{\theta}(\omega)1_{[\tau_i,T]}(u) \right)dH_{u}^{1}(\omega) = \sum_{i=1}^{k} \Delta S_{\tau_i}^{\theta}(\omega)(H_{t}^{1}(\omega) - H_{\tau_i \land t}^{1}(\omega)). \quad (A.8)
$$

Here, we again exploit our definition of the stochastic integral at left jumps of the integrator given below the self-financing condition $[2,2]$. Since $H_{u}^{1,n}(\omega) \to H_{u}^{1}(\omega)$ for all $t \in [0,T]$, we thus have that

$$
\int_0^t \left( \sum_{i=1}^{k} \Delta S_{\tau_i}^{\theta}(\omega)1_{[\tau_i,T]}(u) \right)dH_{u}^{1,n}(\omega) \to \int_0^t \left( \sum_{i=1}^{k} \Delta S_{\tau_i}^{\theta}(\omega)1_{[\tau_i,T]}(u) \right)dH_{u}^{1}(\omega),
$$
as $n \to \infty$, by $(A.7)$ and $(A.8)$. Together with the above, it thus follows that

$$
\int_0^t S_{\theta}^{\theta}(\omega)dH_{u}^{1,n}(\omega) \to \int_0^t S_{\theta}^{\theta}(\omega)dH_{u}^{1}(\omega), \quad n \to \infty,
$$
for every $t \in [0,T]$, since $\varepsilon > 0$ was arbitrary.
Repeating the same argument as above for $H^{1,n,\lambda}$, $H^{1,\lambda}$ and \((1-\lambda)S^\theta\), we also obtain
\[
\int_0^t S^\theta_u(\omega)dH^{1,n,\lambda}_u(\omega) \to \int_0^t S^\theta_u(\omega)dH^{1,\lambda}_u(\omega), \quad n \to \infty.
\]
Since $\omega$ can be chosen arbitrarily from a set with probability 1, this proves (4.3).

In order to prove that $H^1 \in A(x)$, we first note that all the $H^{0,\theta,n,\lambda}$ and $H^{0,\theta,\lambda}$ as defined in (4.2) are elements of $\mathcal{P}(\mathbb{F}) \cap \mathcal{V}^+ (\mathbb{F}, P)$. This consequence follows from our definition of the stochastic integral and the fact that $H^{1,n,\lambda}$ as well as $H^{1,\lambda}$ are elements of $\mathcal{P}(\mathbb{F}) \cap \mathcal{V}^+ (\mathbb{F}, P)$ (see also Remark 1.31 in [20]). The same argument then holds true for $H^{0,\theta,\lambda}$ and $H^{0,\theta}$. It remains to check that $V_{\text{liq}}(\theta, H^1)$ satisfies the admissibility condition (2.6) for all $\theta \in \Theta$. By assumption, the processes $(H^{1,n})_{n \in \mathbb{N}}$ are $x$-admissible for all $\theta \in \Theta$. Hence, by (4.1) and (4.3), we get for every $t \in [0, T]$ and for each $\theta \in \Theta$ that $V^{\text{liq}}(\theta, H^{1,n}) \to V^{\text{liq}}(\theta, H^1)$ almost surely, by the continuity of the liquidation function (2.6) with respect to $(H^{1,\theta}, H^1)$, so that admissibility condition (2.6) passes to the limit $H^1$. We thus have $H^1 \in A(x)$ and this concludes the proof.

To prove Proposition 4.4, we also use the following well-known variant of Komlós’ theorem (see [12], Lemma A1.1).

**Lemma A.2.** Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of $\mathbb{R}_+$-valued, random variables on a probability space $(\Omega, \mathcal{F}, P)$. There is a sequence $g_n \in \text{conv}(f_n, f_{n+1}, \ldots)$ of convex combinations that converges almost surely to a $[0, \infty]$-valued function $g$. If $\text{conv}(f_n, f_{n+1}, \ldots)$ is bounded in $L^0(\Omega, \mathcal{F}, P)$, then $g$ is finite almost surely.

We further need the next simple fact about the measurability of limits of measurable functions (see, e.g., Lemma 3.5 in [37]).

**Lemma A.3.** Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of measurable functions on a measure space $(\Omega, \mathcal{F})$. Then, $\liminf_{n \to \infty} f_n$ and $\limsup_{n \to \infty} f_n$ are $[-\infty, \infty]$-valued measurable functions and the set $F := \{\omega \in \Omega : f_n(\omega) \text{ converges to a limit in } \mathbb{R}\}$ is $\mathcal{F}$-measurable and given by
\[
F = \left\{\omega \in \Omega : \liminf_{n \to \infty} f_n(\omega) = \limsup_{n \to \infty} f_n(\omega) \in \mathbb{R}\right\}.
\]
Moreover, if $\liminf_{n \to \infty} f_n$ and $\limsup_{n \to \infty} f_n$ are measurable with respect to a sub-$\sigma$-algebra $\mathcal{G} \subseteq \mathcal{F}$, then $F \in \mathcal{G}$.

We are now ready to prove Proposition 4.4. Here, the main difference to the initial paper by Chau and Rásonyi [8] (see Lemma A.1) is the treatment of jump times. For this cause, we revisit the proof of [7], Proposition 3.4. Our key insight here is that we can show that the set, where the convergence can fail, is the same for all models. It can again be exhausted by countably many stopping times. However, since the filtration $\mathbb{F}$ is not assumed to be complete, the treatment of jump times needs special care. In particular, we need to pass from $\mathbb{F}$ to $\mathbb{F}^P$, in order to apply Theorem IV.117 [13]. As a consequence, the limit process is also only $P$-indistinguishable from an $\mathbb{F}$-predictable process and not predictable itself.

\(^3\)Note that, since we are already considering the liquidation value in [28], we do not need to assume that $H_T^1 = 0$. Therefore, we also do not need to assume that $F_T = F_{T^{-}}$ and $S^\theta_T = S^\theta_{T^{-}}$ as in Remark 4.2 in [3].
Proof of Proposition 4.4: Fix \( x > 0 \) and let \( (H^{1,n})_{n \in \mathbb{N}} \subseteq \mathcal{A}(x) \) be a sequence of admissible, self-financing strategies. In particular, \( H^{1,n} \in \mathcal{P}(\mathbb{F}) \cap \mathcal{V}(\mathbb{F}, P) \) is a finite variation process that is \( P \)-indistinguishable from an \( \mathbb{F} \)-predictable process, and \( V^{\text{lin}}(\theta, H^{1,n}) \) satisfies (2.6) for all \( \theta \in \Theta \). As above, we decompose these processes canonically as \( H^{1,n} = H^{1,n}_t - H^{1,n}_s \), with \( H^{1,n}_t \) and \( H^{1,n}_s \) both being elements of \( \mathcal{P}(\mathbb{F}) \cap \mathcal{V}(\mathbb{F}, P) \). By Proposition A.1, we know that \( (H^{1,n}_t)_{n \in \mathbb{N}} \) and \( (H^{1,n}_s)_{n \in \mathbb{N}} \) as well as their convex combinations are bounded in \( L^0(\Omega, \mathcal{F}, P) \). Hence, let \( D := (\{0, T\} \cap \mathbb{Q}) \cup \{T\} \) and use Lemma A.2 together with a diagonalisation procedure to obtain sequences of convex weights \( \alpha_n \) such that

\[
\tilde{H}^{1,n}_t = \sum_{j \geq 1} \alpha_n^j H^{1,n+j-1}_t, \quad \tilde{H}^{1,n}_s = \sum_{j \geq 1} \alpha_n^j H^{1,n+j-1}_s, \quad t \in D,
\]

there exist \( \mathcal{F}_t \)-measurable random variables \( \tilde{H}^{1}_t \) and \( \tilde{H}^{1}_s \), such that

\[
\tilde{H}^{1,n}_t \to \tilde{H}^{1}_t, \quad \tilde{H}^{1,n}_s \to \tilde{H}^{1}_s, \quad \forall t \in D, \quad (A.9)
\]

almost surely. We denote by \( \bar{\Omega}_0 \) the event where (A.9) holds true so that \( P[\bar{\Omega}_0] = 1 \). Observe now that \( q \mapsto \tilde{H}^{1}_q(\omega) \) is non-negative and non-decreasing over \( D \) for all \( \omega \in \bar{\Omega}_0 \). Now, the \( \mathbb{F} \)-stopping time \( \sigma \), defined as

\[
\sigma := \inf \left\{ r \in \mathbb{Q} : \sup_{q \in [r, q, D]} \tilde{H}^{1}_q = \infty \right\},
\]

satisfies \( P[\sigma = \infty] = 1 \). Hence, by letting \( \bar{\Omega}_0 = \bar{\Omega}_0 \cap \{ \sigma = \infty \} \) such that \( P[\bar{\Omega}_0] = 1 \), we may define

\[
\tilde{H}^{1}_t(\omega) = \lim_{q \downarrow t, q \in \mathbb{Q}} \tilde{H}^{1}_q(\omega), \quad t \in [0, T), \quad \omega \in \bar{\Omega}_0, \quad (A.10)
\]

and \( \tilde{H}^{1}_T(\omega) = \tilde{H}^{1}_T(\omega), \omega \in \bar{\Omega}_0 \). On the other hand, if \( \omega \notin \bar{\Omega}_0 \), we set \( \tilde{H}^{1}_t(\omega) = 0 \), for all \( t \in [0, T] \). Note that by Lemma A.3 and the right continuity of the filtration, the process \( \tilde{H}^{1}_t \) obtained in this way is right continuous and \( \mathbb{F}^P \)-adapted and hence \( \mathbb{F}^P \)-optional. Indeed, for fixed \( t \in [0, T] \), we have that \( \lim_{q \downarrow t, q \in \mathbb{Q}} \tilde{H}^{1}_q \) and \( \limsup_{q \downarrow t, q \in \mathbb{Q}} \tilde{H}^{1}_q \) are \( \mathcal{F}_t \)-measurable by the right-continuity of the filtration. By Lemma A.3, we have

\[
F := \left\{ \omega \in \Omega : \tilde{H}^{1}_t(\omega) = \lim_{q \downarrow t, q \in \mathbb{Q}} \tilde{H}^{1}_q(\omega) \right\}
\]

and hence \( P[F] = 1 \), since \( \bar{\Omega}_0 \subseteq F \) and \( P[\bar{\Omega}_0] = 1 \). However, by letting \( \tilde{H}^{1}_t = 0 \) outside of \( \bar{\Omega}_0 \), we need to adjoin all the \( P \)-null sets to \( \mathbb{F} \), and thus we only have that \( \tilde{H}^{1}_t \) is \( \mathbb{F}^P \)-adapted.

We now claim that if \( (\omega, t) \in \bar{\Omega}_0 \times (0, T) \) is such that \( t \) is a continuity point of the function \( s \mapsto \tilde{H}^{1,s}_t(\omega) \), then \( \tilde{H}^{1,n,s}_t(\omega) \to \tilde{H}^{1}_t(\omega) \). Indeed, for \( \varepsilon > 0 \) let \( q_1 < t < q_2 \) be rational numbers such that \( \tilde{H}^{1,s}_{q_2}(\omega) - \tilde{H}^{1,s}_{q_1}(\omega) < \varepsilon \). From (A.9), there exists \( N = N(\omega) \in \mathbb{N} \) such that

\[
|\tilde{H}^{1,n,s}_{q_1}(\omega) - \tilde{H}^{1}_{q_1}(\omega)| < \varepsilon, \quad |\tilde{H}^{1,n,s}_{q_2}(\omega) - \tilde{H}^{1}_{q_2}(\omega)| < \varepsilon, \quad \forall n \geq N.
\]
We then estimate, for all \( n \geq N \),
\[
\left| \widetilde{H}_{q_2}^{1,n,\uparrow}(\omega) - \widetilde{H}_{q_1}^{1,n,\uparrow}(\omega) \right| \leq \left| \widetilde{H}_{q_2}^{1,n,\uparrow}(\omega) - \widetilde{H}_{q_2}^{1,\uparrow}(\omega) \right| + \left| \widetilde{H}_{q_2}^{1,\uparrow}(\omega) - \widetilde{H}_{q_1}^{1,\uparrow}(\omega) \right| \\
+ \left| \widetilde{H}_{q_1}^{1,\uparrow}(\omega) - \widetilde{H}_{q_1}^{1,n,\uparrow}(\omega) \right| \\
< 3\varepsilon.
\]

Therefore, using monotonicity of \( \widetilde{H}^{1,n,\uparrow} \), we obtain for all \( n \geq N(\omega) \) that
\[
\left| \widetilde{H}_t^{1,n,\uparrow}(\omega) - \widetilde{H}_t^{1,\uparrow}(\omega) \right| \leq \left| \widetilde{H}_t^{1,n,\uparrow}(\omega) - \widetilde{H}_{q_2}^{1,n,\uparrow}(\omega) \right| + \left| \widetilde{H}_{q_2}^{1,n,\uparrow}(\omega) - \widetilde{H}_{q_2}^{1,\uparrow}(\omega) \right| \\
+ \left| \widetilde{H}_{q_2}^{1,\uparrow}(\omega) - \widetilde{H}_{q_1}^{1,\uparrow}(\omega) \right| \\
< 5\varepsilon.
\]

For \( t = T \), the convergence of \( \widetilde{H}_T^{1,n,\uparrow}(\omega) \rightarrow \widetilde{H}_T^{1,\uparrow}(\omega) \) follows from (A.9) and the identity \( \widetilde{H}_T^{1,\uparrow} = \widetilde{H}_T^{1,n,\uparrow} \) on \( \bar{\Omega}_0 \).

The process \( \widetilde{H}_t^{1,\uparrow} \) is not yet the desired limit because we still have to ensure the convergence at the jumps times of \( \widetilde{H}_t^{1,\uparrow} \). Since \( \widetilde{H}_t^{1,\uparrow} \) is right continuous and \( \mathbb{F}^P \)-adapted, there exists a sequence \( (\tilde{\tau}_k)_{k \in \mathbb{N}} \) of \([0,T]\) of \( \mathbb{F}^P \)-stopping times exhausting the jumps of the process \( \widetilde{H}_t^{1,\uparrow} \). This argument uses Theorem IV.117 [13] and the fact that \( \mathbb{F}^P \) is complete. By Theorem IV.59 [13], there also exists a sequence \( (\tilde{\tau}_k)_{k \in \mathbb{N}} \) of \( \mathbb{F} \)-stopping times, satisfying \( \tilde{\tau}_k = \tilde{\tau}_k \), \( P \)-a.s., for all \( k \in \mathbb{N} \). Hence, by passing once more to convex combinations, we may also assume that \( (\widetilde{H}_{\tilde{\tau}_k}^{1,n,\uparrow}) \) converges almost surely on \( \{\tilde{\tau}_k \leq T\} \) for every \( k \in \mathbb{N} \). We can therefore set
\[
\hat{\Omega}_0 := \{ \omega \in \Omega_0 : \widetilde{H}_{\tilde{\tau}_k(\omega)}^{1,n,\uparrow}(\omega) \text{ converges to a limit in } \mathbb{R} \text{ for all } k \} \tag{A.11}
\]
and still have \( P[\hat{\Omega}_0] = 1 \). In particular, we again apply Lemma [A.2] together with a diagonalisation procedure, but this time on the already constructed sequence \( (\widetilde{H}_t^{1,n,\uparrow}) \) and for the countably many stopping times \( (\tau_k)_{k \in \mathbb{N}} \). By still denoting this sequence as \( (\widetilde{H}_t^{1,n,\uparrow}) \), this construction yields a subset \( \hat{\Omega}_0 \subseteq \hat{\Omega}_0 \) of full measure, where the convergence of \( \widetilde{H}_t^{1,n,\uparrow} \) to a limit in \( \mathbb{R} \) holds true for all \( t \in [0,T] \). Indeed, for \( \omega \in \hat{\Omega}_0 \), the convergence in (A.11) together with (A.10) implies that \( \widetilde{H}_{\tilde{\tau}_k}^{1,n,\uparrow}(\omega) \) converges to a limit in \( \mathbb{R} \) for all \( t \in [0,T] \). Finally, we define \( H_t^{1,\uparrow} \) by setting \( H_t^{1,\uparrow}(\omega) = \lim_{n \to \infty} \widetilde{H}_t^{1,n,\uparrow}(\omega) \) on the set
\[
G := \{ (\omega,t) \in \Omega \times [0,T] : \widetilde{H}_{t}^{1,n,\uparrow}(\omega) \text{ converges to a limit in } \mathbb{R} \},
\]
and \( H_t^{1,\uparrow}(\omega) = 0 \) on \( G^c \). This procedure yields an \( \mathbb{F}^P \)-predictable process \( H_t^{1,\uparrow} \) by Lemma [A.3] since the processes \( \widetilde{H}_t^{1,n,\uparrow} \) are \( P \)-indistinguishable from an \( \mathbb{F} \)- and therefore \( \mathbb{F}^P \)-predictable process, and because \( \mathbb{F}^P \) is complete. Moreover, since for \( \omega \in \hat{\Omega}_0 \) we have that \( \widetilde{H}_t^{1,n,\uparrow}(\omega) \)
converges for all $t \in [0, T]$, and the mapping $t \mapsto \tilde{H}_{t}^{1,n,\uparrow}(\omega)$ is non-decreasing for all $n$, we have that $t \mapsto H_{t}^{1,\uparrow}(\omega)$ is non-decreasing for all $\omega \in \Omega_0 \subseteq \Hat{\Omega}_0$ with $P[\Omega_0] = 1$. In particular, we have that $H^{1,\uparrow} \in \mathcal{P}(\mathbb{F}) \cap \mathcal{V}^{+}(\mathbb{F}, P)$. Hence, by Proposition 1.1.1 [20] (see also Theorem IV.78 in [13]) and the fact that $\mathcal{V}(\mathbb{F}, P) = \mathcal{V}(\mathbb{F}, \mathbb{P})$ (see Remark 0.37 in [20]), we finally obtain that $H^{1,\uparrow} \in \mathcal{P}(\mathbb{F}) \cap \mathcal{V}^{+}(\mathbb{F}, \mathbb{P})$.

The case $H^{1,\downarrow}$ is treated analogously. In particular, we obtain two processes, $H^{1,\uparrow}$ and $H^{1,\downarrow}$, both $P$-indistinguishable from an increasing, $\mathbb{F}$-predictable process bounded in $L^0$, such that

$$P[\tilde{H}_{t}^{1,n,\uparrow} \rightarrow H_{t}^{1,\uparrow}, \forall t \in [0, T]] = 1,$$

and

$$P[\tilde{H}_{t}^{1,n,\downarrow} \rightarrow H_{t}^{1,\downarrow}, \forall t \in [0, T]] = 1.$$ 

To conclude the proof, define the process $H^1$ as $H^1_0 = (H^1_0)_{0 \leq t \leq T} = (H^1_{t}^{\uparrow} - H^1_{t}^{\downarrow})_{0 \leq t \leq T}$. Since $H^{1,\uparrow}$ and $H^{1,\downarrow}$ both are elements of $\mathcal{P}(\mathbb{F}) \cap \mathcal{V}^{+}(\mathbb{F}, \mathbb{P})$, the process $H^1$ is in $\mathcal{P}(\mathbb{F}) \cap \mathcal{V}(\mathbb{F}, \mathbb{P})$. It remains to check that $H^1 \in A(x)$. By construction, the processes $(\tilde{H}^{1,n})_{n \in \mathbb{N}}$, defined via the decomposition $\tilde{H}^{1,n} = \tilde{H}^{1,n,\uparrow} - \tilde{H}^{1,n,\downarrow}$, are $x$-admissible for all $\theta \in \Theta$. In particular, the process $H^1 \in \mathcal{P}(\mathbb{F}) \cap \mathcal{V}(\mathbb{F}, \mathbb{P})$ and the sequence $(\tilde{H}^{1,n})_{n \in \mathbb{N}} \subseteq A(x)$ satisfy (4.1), so Lemma 4.5 implies that $H^1 \in A(x)$. This finishes the proof.

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