Invariant integration theory on non-compact quantum spaces: Quantum \((n, 1)\)-matrix ball

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Abstract
An operator theoretic approach to invariant integration theory on non-compact quantum spaces is introduced on the example of the quantum \((n, 1)\)-matrix ball \(O_q(\text{Mat}_{n,1})\). In order to prove the existence of an invariant integral, operator algebras are associated to \(O_q(\text{Mat}_{n,1})\) which allow an interpretation as “rapidly decreasing” functions and as functions with compact support on the quantum \((n, 1)\)-matrix ball. It is shown that the invariant integral is given by a generalization of the quantum trace. If an operator representation of a first order differential calculus over the quantum space is known, then it can be extended to the operator algebras of integrable functions. Hilbert space representations of \(O_q(\text{Mat}_{n,1})\) are investigated and classified. Some topological aspects concerning Hilbert space representations are discussed.

Keywords: invariant integration, quantum groups, operator algebras.
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1 Introduction

The development of quantum mechanics at the beginning of the past century resulted in the discovery that nuclear physics is governed by non-commutative quantities. Recently, there have been made various suggestions that spacetime may be described by non-commutative structures at Planck scale. Within this approach, quantum groups might play a fundamental role. They can be viewed as \(q\)-deformations of a classical Lie group or Lie algebra and allow thus an interpretation as generalized symmetries. At the present stage, the theory is still in the beginning. Before constructing physical models, one has to establish the mathematical foundations—most important, the machineries of differential and integral calculus.

In this paper, we deal with integral calculus on non-compact quantum spaces. The integration theory on compact quantum groups is well established and was mainly developed by S. L. Woronowicz [16]. He proved the existence of a unique normalized invariant functional (Haar functional) on compact quantum groups. If one turns to the study of non-compact quantum groups or non-compact quantum spaces, one faces new
difficulties which do not occur in the compact case. For instance, we do not expect that there exists a normalized invariant functional on the polynomial algebra of the quantum space. The situation is analogous to the classical theory of locally compact spaces, where one can only integrate functions which vanish sufficiently rapidly at infinity.

Our aim is to define appropriate classes of quantized integrable functions for non-compact $q$-deformed manifolds. The ideas are similar to those in [13], where a space of finite functions was associated to the the quantum disc. However, our treatment will make this construction more general and will allow us to consider a wider class of integrable functions. Furthermore, the invariant integral turns out to be a generalization of the well-known quantum trace—an observation that provides us with a rather natural proof of its invariance.

Starting point of our approach will be what we call an operator expansion of the action. Suppose we are given a Hopf *-algebra $\mathcal{U}$ and a $\mathcal{U}$-module *-algebra $\mathcal{X}$ with action $\triangleright$. Let $\pi : \mathcal{X} \to \mathcal{L}^+(D)$ be a *-representation. (Precise definitions will be given below.) If for any $Z \in \mathcal{U}$ there exists a finite number of operators $L_i, R_i \in \mathcal{L}^+(D)$ such that

$$\pi(Z \triangleright x) = \sum_i L_i \pi(x) R_i, \quad x \in \mathcal{X},$$

then we say that we have an operator expansion of the action. Obviously, it is sufficient to know the operators $L_i, R_i$ for the generators of $\mathcal{U}$. The operators $L_i, R_i$ are not unique as it can be seen by replacing $L_i$ and $R_i$ by $(-L_i)$ and $(-R_i)$.

Let us briefly outline our method of introducing an invariant integration theory on non-compact quantum spaces. Assume that $\mathfrak{g}$ is a finite-dimensional complex semi-simple Lie algebra. Let $\mathcal{U}_q(\mathfrak{g})$ denote the corresponding quantized universal enveloping algebra. With the adjoint action $\text{ad}_q(X)(Y) := X_{(1)} Y S(X_{(2)})$, $\mathcal{U}_q(\mathfrak{g})$ becomes a $\mathcal{U}_q(\mathfrak{g})$-module (⋆)-algebra. It is a well-known fact that, for finite dimensional representations $\rho$ of $\mathcal{U}_q(\mathfrak{g})$, the quantum trace formula $\text{Tr}_q(X) := \text{Tr} \rho(X K_{2q}^{-1})$, $X \in \mathcal{U}_q(\mathfrak{g})$, defines an $\text{ad}_q$-invariant linear functional on $\mathcal{U}_q(\mathfrak{g})$. Here, the element $K_{2q} \in \mathcal{U}_q(\mathfrak{g})$ is taken such that $K_{2q}^{-1} X K_{2q} = S^2(X)$.

Now consider a $\mathcal{U}_q(\mathfrak{g})$-module *-algebra $\mathcal{X}$ and a *-representation $\pi : \mathcal{X} \to \mathcal{L}^+(D)$. In our examples, the operator expansion (1) of the $\mathcal{U}_q(\mathfrak{g})$-action on $\mathcal{X}$ will resemble the adjoint action. Furthermore, it can be extended to the *-algebra $\mathcal{L}^+(D)$ turning $\mathcal{L}^+(D)$ into a $\mathcal{U}_q(\mathfrak{g})$-module *-algebra. The quantum trace formula suggests that we can try to define an invariant integral by replacing $K_{2q}$ by the operator that realizes the operator expansion of $K_{2q}$ and taking the trace on the Hilbert space $\mathcal{H} = D$. Since we deal with unbounded operators, this can only be done for an appropriate class of operators, say $\mathfrak{B}$.

First of all, the generalized quantum trace should be well defined. Next, we wish that $\mathfrak{B}$ is a $\mathcal{U}_q(\mathfrak{g})$-module *-algebra. This means that $\mathfrak{B}$ should be stable under the action defined by the operator expansion. If we choose $\mathfrak{B}$ such that the closures of its elements are of trace class and that multiplying the elements of $\mathfrak{B}$ by any operator appearing in the operator expansion yields an element of $\mathfrak{B}$, then $\mathfrak{B}$ is certainly stable under the action of $\mathcal{U}_q(\mathfrak{g})$ on $\mathcal{L}^+(D)$ and the generalized quantum trace is well defined on $\mathfrak{B}$. Our intention is to interpret $\mathfrak{B}$ as the rapidly decreasing functions on a $q$-deformed manifold. For this reason, we suppose additionally that $\mathfrak{B}$ is stable under multiplication by elements of $\mathcal{X}$.
Clearly, the assumptions on $B$ are satisfied by the $*$-algebra of finite rank operators $F$ in $L^+(D)$. The elements of $F$ are considered as functions with finite support on the $q$-deformed manifold.

If we think of $U_q(g)$ as generalized differential operators, then we can think of $B$ and $F$ as infinitely differentiable functions since both algebras are stable under the action of $U_q(g)$.

The algebras $B$ and $F$ were mainly introduced in order to develop an invariant integration theory on $q$-deformed manifolds. Nevertheless, our approach also allows to include differential calculi. By means of an operator representation of a first order differential calculus over $X$, one can build a differential calculus over the operator algebras $B$ and $F$. In this case, we view the differential calculus over $B$ and $F$ as an extension of the differential calculus over $X$.

There is another notable feature of our approach. The algebras $X$ (more exactly, $\pi(X)$), $B$, and $F$ are subalgebras of $L^+(D)$. In particular, they are subspaces of the topological space $L(D,D^+)$. Therefore we can view this algebras as topological spaces in a rather natural way. As a consequence, it makes sense to discuss topological concepts such as continuity, density, etc.

In this paper, we treat the quantum $(n,1)$-matrix ball $O_q(Mat_{n,1})$ as a $U_q(su_{n,1})$-module $*$-algebra [12]. Since our approach to invariant integration theory is based on Hilbert space representations, we shall also study $*$-representations of $O_q(Mat_{n,1})$.

When $n = 1$, $O_q(Mat_{n,1})$ is referred to as quantum disc $O_q(U)$ [13]. As the algebraic relations and the $*$-representations of $O_q(U)$ are comparatively simple, it will serve as a guiding example in order to motivate and illustrate our ideas and, therefore, we shall discuss it in a greater detail.

2 Preliminaries

2.1 Algebraic preliminaries

Throughout this paper, $q$ stands for a real number such that $0 < q < 1$, and we abbreviate $\lambda = q - q^{-1}$.

Let $U$ be a Hopf algebra. The comultiplication, the counit, and the antipode of a Hopf algebra are denoted by $\Delta$, $\varepsilon$, and $S$, respectively. For the comultiplication $\Delta$, we employ the Sweedler notation: $\Delta(x) = x(1) \otimes x(2)$. The main objects of our investigation are $U$-module algebras. An algebra $X$ is called a left $U$-module algebra if $X$ is a left $U$-module with action $\triangleright$ satisfying

$$f \triangleright (xy) = (f(1) \triangleright x)(f(2) \triangleright y), \quad x,y \in X, \quad f \in U.$$  \hspace{1cm} (2)

For an algebra $X$ with unit 1, we additionally require

$$f \triangleright 1 = \varepsilon(f) 1, \quad f \in U.$$  \hspace{1cm} (3)

Let $X$ be a $*$-algebra and $U$ a Hopf $*$-algebra. Then $X$ is said to be a left $U$-module $*$-algebra if $X$ is a left $U$-module algebra such that the following compatibility condition holds

$$(f \triangleright x)^* = S(f)^* \triangleright x^*, \quad x \in X, \quad f \in U.$$  \hspace{1cm} (4)
By an invariant integral we mean a linear functional $h$ on $\mathcal{X}$ such that

$$h(f \cdot x) = \varepsilon(f) h(x), \quad x \in \mathcal{X}, \ f \in \mathcal{U}.$$ (5)

Synonymously, we refer to it as $\mathcal{U}$-invariant.

A first order differential calculus (abbreviated as FODC) over an algebra $\mathcal{X}$ is a pair $(\Gamma, d)$, where $\Gamma$ is an $\mathcal{X}$-bimodule and $d : \mathcal{X} \to \Gamma$ a linear mapping, such that

$$d(xy) = x \cdot dy + dx \cdot y, \quad x, y \in \mathcal{X}, \quad \Gamma = \text{Lin}\{ x \cdot dy ; \ x, y, z \in \mathcal{X} \}.$$ (\Gamma, d) is called a first order differential $*$-calculus over a $*$-algebra $\mathcal{X}$ if the complex vector space $\Gamma$ carries an involution $*$ such that

$$(x \cdot dy \cdot z)^* = z^* \cdot d(y^*) \cdot x^*, \quad x, y, z \in \mathcal{X}.$$ 

Let $(a_{ij})_{i,j=1}^n$ be the Cartan matrix of $\text{sl}(n+1, \mathbb{C})$, that is, $a_{jj} = 2$ for $j = 1, \ldots, n$, $a_{j,j+1} = a_{j+1,j} = -1$ for $j = 1, \ldots, n-1$ and $a_{ij} = 0$ otherwise. The Hopf algebra $\mathcal{U}_q(\text{sl}_{n+1})$ is generated by $K_j, K_j^{-1}, E_j, F_j, j = 1, \ldots, n$, subjects to the relations

$$K_i K_j = K_j K_i, \quad K_j^{-1} K_j = 1, \quad K_i E_j = q^{a_{ij}} E_j K_i, \quad K_i F_j = q^{-a_{ij}} F_j K_i,$$ (6)

$$E_i E_j - E_j E_i = 0, \quad i \neq j \pm 1, \quad E_i^2 E_j \pm 1 - (q + q^{-1}) E_j E_{j \pm 1} E_j + E_{j \pm 1} E_j^2 = 0,$$ (7)

$$F_i F_j - F_j F_i = 0, \quad i \neq j \pm 1, \quad F_i^2 F_j \pm 1 - (q + q^{-1}) F_j F_{j \pm 1} F_j + F_{j \pm 1} F_j^2 = 0,$$ (8)

$$E_i F_j - E_j F_i = 0, \quad i \neq j, \quad E_j F_j - F_j E_j = \lambda^{-1} (K_j - K_j^{-1}), \quad j = 1, \ldots, n.$$ (9)

The comultiplication $\Delta$, counit $\varepsilon$, and antipode $S$ are given by

$$\Delta(E_j) = E_j \otimes 1 + K_j \otimes E_j, \quad \Delta(F_j) = F_j \otimes K_j^{-1} + 1 \otimes F_j, \quad \Delta(K_j) = K_j \otimes K_j,$$ (10)

$$\varepsilon(K_j) = \varepsilon(K_j^{-1}) = 1, \quad \varepsilon(E_j) = \varepsilon(F_j) = 0,$$ (11)

$$S(K_j) = K_j^{-1}, \quad S(E_j) = -K_j^{-1} E_j, \quad S(F_j) = -F_j K_j.$$ (12)

Consider the involution on $\mathcal{U}_q(\text{sl}_{n+1})$ which is determined by

$$K_i^* = K_i, \quad E_j^* = E_j F_j, \quad F_j^* = E_j K_j^{-1}, \quad j \neq n, \quad E_n^* = -K_n F_n, \quad F_n^* = -E_n K_n^{-1}.$$ (13)

The corresponding Hopf $*$-algebra is denoted by $\mathcal{U}_q(\text{su}_{n+1})$.

If $n = 1$, we write $K, K^{-1}, E, F$ rather than $K_1, K_1^{-1}, E_1, F_1$. These generators satisfy the following relations:

$$KK^{-1} = K^{-1} K = 1, \quad KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F.$$ (14)
\[ EF - FE = \lambda^{-1}(K - K^{-1}). \]

The involution on \( U_q(su_{1,1}) \) is given by
\[ K^* = K, \quad E^* = -KF, \quad F^* = -EK^{-1}. \]

If \( n > 1 \), then \( K_j, K_j^{-1}, E_j, F_j, j = 1, \ldots, n - 1 \) with relations (6)–(13) generate the Hopf *-algebra \( U_q(su_n) \).

### 2.2 Operator theoretic preliminaries

We shall use the letters \( \mathcal{H} \) and \( \mathcal{K} \) to denote complex Hilbert spaces. If \( I \) is an at most countable index set and \( \mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i \), where \( \mathcal{H}_i = \mathcal{K} \) for all \( i \in I \), we denote by \( \eta_i \) the vector of \( \mathcal{H} \) which has the element \( \eta \in \mathcal{K} \) as its \( i \)-th component and zero otherwise. It is understood that \( \eta_i = 0 \) whenever \( i \notin I \).

If \( T \) is an (unbounded) operator on \( \mathcal{H} \), we denote by \( D(T) \), \( \sigma(T) \), \( \mathcal{T} \), and \( T^* \) the domain, the spectrum, the closure, and the adjoint of \( T \), respectively. A self-adjoint operator \( A \) is called strictly positive if \( A \geq 0 \) and \( \ker A = \{0\} \). We write \( \sigma(A) \subseteq [a, b] \) if \( \sigma(A) \subseteq [a, b] \) and \( a \) is not an eigenvalue of \( A \). By definition, two self-adjoint operators strongly commute if their spectral projections mutually commute.

Let \( D \) be a dense subspace of \( \mathcal{H} \). Then the vector space
\[
\mathcal{L}^+(D) := \{ x \in \text{End}(D) ; D \subset D(x^*) , x^*D \subset D \}
\]
is a unital *-algebra of closeable operators with the involution \( x \mapsto x^+ := x^*[D] \) and the operator product as its multiplication. Since it should cause no confusion, we shall continue to write \( x^+ \) in place of \( x^+ \). Unital *-subalgebras of \( \mathcal{L}^+(D) \) are called \( O^* \)-algebras.

Two *-subalgebras of \( \mathcal{L}^+(D) \) which are not \( O^* \)-algebras will be of particular interest: The *-algebra of all finite rank operators
\[
\mathfrak{F}(D) := \{ x \in \mathcal{L}^+(D) ; \bar{x} \text{ is bounded, } \dim(\bar{x}\mathcal{H}) < \infty , \bar{x}\mathcal{H} \subset D , \bar{x}^*\mathcal{H} \subset D \} \tag{17}
\]
and, given an \( O^* \)-algebra \( \mathfrak{A} \),
\[
\mathbb{B}_1(\mathfrak{A}) := \{ t \in \mathcal{L}^+(D) ; \bar{t}\mathcal{H} \subset D , \bar{t}^*\mathcal{H} \subset D , \bar{a}b \text{ is of trace class for all } a, b \in \mathfrak{A} \}. \tag{18}
\]

It follows from [8] Lemma 5.1.4] that \( \mathbb{B}_1(\mathfrak{A}) \) is a *-subalgebra of \( \mathcal{L}^+(D) \). Obviously, we have \( \mathfrak{F}(D) \subset \mathbb{B}_1(\mathfrak{A}) \) and \( 1 \notin \mathbb{B}_1(\mathfrak{A}) \) if \( \dim(\mathcal{H}) = \infty \). An operator \( A \in \mathfrak{F}(D) \) can be written as \( A = \sum_{i=1}^n \alpha_i e_i \otimes f_i \), where \( n \in \mathbb{N} \), \( \alpha_i \in \mathbb{C} \), \( f_i, e_i \in D \), and \( (e_i \otimes f_i)(x) := f_i(x)e_i \) for \( x \in D \).

Assume that \( \mathfrak{A} \) is an \( O^* \)-algebra on a dense domain \( D_{\mathfrak{A}} \). A natural choice for a topology on \( D_{\mathfrak{A}} \) is the graph topology \( t_{\mathfrak{A}} \) generated by the family of semi-norms
\[
\{ || \cdot ||_a \}_{a \in \mathfrak{A}} , \quad || \varphi ||_a := || a\varphi || , \quad \varphi \in D_{\mathfrak{A}}. \tag{19}
\]

\( \mathfrak{A} \) is called closed if the locally convex space \( D_{\mathfrak{A}} \) is complete. The closure \( \bar{\mathfrak{A}} \) of \( \mathfrak{A} \) is defined by
\[
D_{\bar{\mathfrak{A}}} := \bigcap_{a \in \mathfrak{A}} D(\bar{a}), \quad \bar{\mathfrak{A}} := \{ \bar{a} | D_{\bar{\mathfrak{A}}} : a \in \mathfrak{A} \}. \tag{20}
\]
By [6] Lemma 2.2.9, $D_\mathfrak{A}$ is complete.

We say that $\mathfrak{A}$ is a commutatively dominated $O^*$-algebra on the Frechet domain $D_\mathfrak{A}$ if it satisfies the following assumptions (which are consequences from the definitions given in [6]). There exist a self-adjoint operator $A$ on $\mathcal{H}$ and a sequence of Borel measurable real-valued functions $r_n$, $n \in \mathbb{N}$, such that $1 \leq r_1(t)$, $r_n(t)^2 \leq r_{n+1}(t)$, $r_n(A)[D_\mathfrak{A} \in \mathfrak{A}$, and $D_\mathfrak{A} = \cap_{n \in \mathbb{N}} D(r_n(A))$.

Let $D_\mathfrak{A}$ denote the strong dual of the locally convex space $D_\mathfrak{A}$. Then the conjugate space $D^{*\mathfrak{A}}_\mathfrak{A}$ is the topological space $D^*_\mathfrak{A}$ with the addition defined as before and the multiplication replaced by $\alpha \cdot f := \overline{\alpha}f$, $\alpha \in \mathbb{C}$, $f \in D^{*\mathfrak{A}}_\mathfrak{A}$. For $f \in D^{*\mathfrak{A}}_\mathfrak{A}$ and $\varphi \in D_\mathfrak{A}$, we shall write $\langle f, \varphi \rangle$ rather than $f(\varphi)$. The vector space of all continuous linear operators mapping $D_\mathfrak{A}$ into $D^{*\mathfrak{A}}_\mathfrak{A}$ is denoted by $\mathcal{L}(D_\mathfrak{A}, D^{*\mathfrak{A}}_\mathfrak{A})$. We assign to $\mathcal{L}(D_\mathfrak{A}, D^{*\mathfrak{A}}_\mathfrak{A})$ the bounded topology $\tau_0$ generated by the system of semi-norms

$$\{ p_M : M \subset D_\mathfrak{A}, \text{ bounded} \}, \quad p_M(A) := \sup_{\varphi, \psi \in M} |\langle A\varphi, \psi \rangle|, \quad A \in \mathfrak{A}.$$  

Notice that $\mathfrak{A} \subset \mathcal{L}(D_\mathfrak{A}, D^{*\mathfrak{A}}_\mathfrak{A})$ for any $O^*$-algebra $\mathfrak{A}$. Furthermore, it is known that $\mathcal{L}^+(D_\mathfrak{A}) \subset \mathcal{L}(D_\mathfrak{A}, D^{*\mathfrak{A}}_\mathfrak{A})$ if $D_\mathfrak{A}$ is a Frechet space.

By a *-representation $\pi$ of a *-algebra $\mathfrak{A}$ on a domain $D$ we mean a *-homomorphism $\pi : \mathfrak{A} \to \mathcal{L}^+(D)$. For notational simplicity, we usually suppress the representation and write $x$ instead of $\pi(x)$ when no confusion can arise. If each decomposition $\pi = \pi_1 \oplus \pi_2$ of $\pi$ as direct sum of *-representations $\pi_1$ and $\pi_2$ implies that $\pi_1 = 0$ or $\pi_2 = 0$, then $\pi$ is said to be irreducible.

Given a *-representation $\pi$, it follows from [6] Proposition 8.1.12] that the mapping

$$\overline{\pi} : \mathfrak{A} \to \mathcal{L}^+(D(\overline{\pi})), \quad \overline{\pi}(a) := \overline{\pi(a)}|D(\overline{\pi}),$$

defines a *-representation on $D(\overline{\pi}) := \cap_{a \in \mathfrak{A}} D(\pi(a))$. $\overline{\pi}$ is called the closure of $\pi$ and $\pi$ is said to be closed if $\overline{\pi} = \pi$.

If we consider *-representations of *-algebras, we shall restrict ourself to representations which are in a certain sense "well behaved". This means that we shall impose some regularity conditions on the (in general) unbounded operators under consideration. Such *-representations will be called admissible. The requirements will strongly depend on the situation. Therefore there is no general definition of "admissible". For further discussion on "well behaved" representations, see [10],[11].

Suppose that $\mathcal{X}$ is a *-algebra and $\pi : \mathcal{X} \to \mathcal{L}^+(D)$ a *-representation. Each symmetric operator $C \in \mathcal{L}^+(D)$ gives rise to a first order differential *-calculus $(\Gamma_{\pi,C}, d_{\pi,C})$ over $\mathcal{X}$ defined by

$$\Gamma_{\pi,C} := \text{Lin}\{ \pi(x)(C\pi(y) - \pi(y)C)\pi(z) : x, y, z \in \mathcal{X} \} \quad \text{and} \quad (21)$$

$$d_{\pi,C} : \mathcal{X} \to \Gamma_{\pi,C}, \quad d_{\pi,C}(x) := i(C\pi(x) - \pi(x)C), \quad x \in \mathcal{X}, \quad (22)$$

where $i$ denotes the imaginary unit (see [6]). Let $(\Gamma, d)$ be a first order differential *-calculus over $\mathcal{X}$. Then $(\Gamma_{\pi,C}, d_{\pi,C})$ is called a commutator representation of $(\Gamma, d)$, if there exists a linear mapping $\rho : \Gamma \to \Gamma_{\pi,C}$ such that $\rho(x \cdot dy \cdot z) = \pi(x)d_{\pi,C}(y)\pi(z)$ and $\rho(\gamma^*) = \rho(\gamma)^*$ for all $x, y, z \in \mathcal{X}, \gamma \in \Gamma$.

We close this subsection by stating three auxiliary lemmas.
Lemma 2.1 Let $A$ be a self-adjoint operator and let $w$ be an unitary operator on a Hilbert space $\mathcal{H}$ such that
\begin{equation}
qwA \subseteq Aw. \tag{23}
\end{equation}
i. The spectral projections of $A$ corresponding to $(-\infty, 0)$, $\{0\}$, and $(0, \infty)$ commute with $w$.

ii. Suppose additionally that $A$ is strictly positive. Then there exists a self-adjoint operator $A_0$ on a Hilbert space $\mathcal{H}_0$ with $\sigma(A_0) \subseteq (q, 1]$ such that, up to unitary equivalence, $\mathcal{H} = \bigoplus_{n=-\infty}^{\infty} \mathcal{H}_n$, $\mathcal{H}_n = \mathcal{H}_0$, and
\begin{equation}
A_0 \eta_n = q^n A_0 \eta_n, \quad w \eta_n = \eta_{n+1},
\end{equation}
where $\eta \in \mathcal{H}_0$ and $n \in \mathbb{Z}$.

Proof. (i): Let $e(\mu)$ denote the spectral projections of $A$. Since $w$ is unitary, it implies that $A = qwAw^*$ and hence $e(q\mu) = we(\mu)w^*$. This proves (i).

(ii): Let $\mathcal{H}_n := e((q^{n+1}, q^n))\mathcal{H}$ and $A_0 := A[\mathcal{H}_n, n \in \mathbb{Z}$. Since $A$ is strictly positive, $\mathcal{H} = \bigoplus_{n=-\infty}^{\infty} \mathcal{H}_n$. Now $e((q^{n+1}, q^n)) = we((q^n, q^{n-1}))w^*$ yields $w\mathcal{H}_n = \mathcal{H}_{n+1}$. Up to unitary equivalence, we can assume that $\mathcal{H}_n = \mathcal{H}_0$ and $w \eta_n = \eta_{n+1}$ for $\eta \in \mathcal{H}_0$. Moreover, $A_0 \eta_n = q^n w^* Aw^{n+1} \eta_n = q^n w^* A_0 \eta_0 = q^n A_0 \eta_n$. \hfill \Box

Lemma 2.2 Let $A$ be a self-adjoint operator and let $w$ be a linear isometry on a Hilbert space $\mathcal{H}$ such that
\begin{equation}
swA \subseteq Aw \tag{24}
\end{equation}
for some fixed positive real number $s \neq 1$. Suppose that $A$ has an eigenvalue $\lambda$ such that the eigenspace $\mathcal{H}_0 := \ker (A - \lambda)$ coincides with $w^*$. Then the eigenspace $\mathcal{H}_n := \ker (A - s^n \lambda)$ coincides with $w^*\mathcal{H}_0$ for each $n \in \mathbb{N}$.

Proof. Taking adjoints in (24) gives $s^{-1}w^*A \subseteq A w^*$. Let $n \in \mathbb{N}_0$, $\varphi \in \mathcal{H}_n$, and $\psi \in \mathcal{H}_{n+1}$. Then $Aw\varphi = swA\varphi = s^{n+1}\lambda w\varphi$ and $Aw^*\psi = s^{-1}w^*A\psi = s^n\lambda w^*\psi$. Hence $w\mathcal{H}_n \subset \mathcal{H}_{n+1}$ and $w^* \mathcal{H}_{n+1} \subset \mathcal{H}_n$. Since $\mathcal{H}_{n+1} \perp \mathcal{H}_0$, we have $ww^*\psi = \psi$. This together with $w^*w = 1$ implies that $w|\mathcal{H}_n$ is a bijective mapping from $\mathcal{H}_n$ onto $\mathcal{H}_{n+1}$ with inverse $w^*|\mathcal{H}_{n+1}$. \hfill \Box

Lemma 2.3 Let $\epsilon \in \{\pm 1\}$. Assume that $x$ is a closed, densely defined operator on a Hilbert space $\mathcal{H}$. Then we have $D(xx^*) = D(x^*x)$ and the relation
\begin{equation}
xx^* - q^2 x^* x = \epsilon (1 - q^2) \tag{25}
\end{equation}
holds if and only if $x$ is unitarily equivalent to an orthogonal direct sum of operators of the following form.

$\epsilon = 1$:
\begin{enumerate}
\item[(I)] $x \eta_n = (1 - q^{2n})^{1/2} \eta_{n-1}$ on the Hilbert space $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$, $\mathcal{H}_n = \mathcal{H}_0$.
\item[(II)] $A x \eta_n = (1 + q^{2n}A)^{1/2} \eta_{n-1}$ on $\mathcal{H} = \bigoplus_{n=-\infty}^{\infty} \mathcal{H}_n$, $\mathcal{H}_n = \mathcal{H}_0$, where $A$ is a self-adjoint operator on $\mathcal{H}_0$ such that $\sigma(A) \subseteq (q^2, 1]$.
\end{enumerate}
(III) \( u \ x = u \), where \( u \) is a unitary operator on \( \mathcal{H} \).

\[ \epsilon = -1:\]

\[ x\eta_n = (q^{-2n} - 1)^{1/2} \eta_{n+1} \text{ on the Hilbert space } \mathcal{H} = \oplus_{n=1}^{\infty} \mathcal{H}_n, \mathcal{H}_n = \mathcal{H}_1. \]

**Proof.** Direct calculations show that the operators described in Lemma 2.3 satisfy (24).

Suppose now we are given an operator \( x \) satisfying the assumptions of the lemma. Let \( e(\mu) \) denote the spectral projections of the self-adjoint operator \( Q = \epsilon - x^*x \). For \( \varphi \in D(Q^2) = D((x^*x)^2) \), it follows from (25) that

\[ Qx^*\varphi = x^*(\epsilon - xx^*)\varphi = x^*(\epsilon - q^2x^*x - \epsilon(1 - q^2))\varphi = q^2x^*Q\varphi, \]

\[ xQ\varphi = (\epsilon - xx^*)x\varphi = (\epsilon - q^2x^*x - \epsilon(1 - q^2))x\varphi = q^2Qx\varphi. \]

The cases \( \epsilon = 1 \) and \( \epsilon = -1 \) will be analyzed separately.

\( \epsilon = 1: \) Let \( x^* = ua \) be the polar decomposition of \( x^* \). Note that

\[ a^2 = xx^* = 1 - q^2 + q^2x^*x = 1 - q^2Q \geq 1 - q^2, \]

which implies, in particular, that \( \ker a = \ker u = 0 \), so \( u \) is an isometry. Inserting \( \varphi = a^{-1}\psi \) in (26), where \( \psi \in D(Q^{3/2}) \), one obtains \( Qu\psi = q^2uaQa^{-1}\psi = q^2uQ\psi \). Since \( D(Q^{3/2}) \) is a core for \( Q \), it follows that \( q^2uQ \subseteq Qu \). By taking adjoints, one also gets \( u^*Q \subseteq q^2Qu^* \). Furthermore, \( \varphi \in \ker x = \ker x^*x = \ker u^* \) if and only if \((Q - 1)\varphi = 0 \). If \( \ker u^* \neq \{0\} \), Lemma 2.4 implies that \( \mathcal{K} := \oplus_{n=0}^{\infty} \mathcal{H}_n \)

where \( \mathcal{H}_n = \ker (Q - q^2) \), is a reducing subspace for \( u \) and \( Q \). Moreover, \( x|\mathcal{K} = (1 - q^2)Q^{1/2}u^*|\mathcal{K} \) is unitarily equivalent to an operator of the form (I).

It suffices now to prove the assertion under the additional assumption that \( \ker u^* = \{0\} \). By Lemma 2.4(i), we can treat the cases where \( Q \) is strictly positive, zero, or strictly negative separately.

If \( Q \) were strictly positive, then it would be unbounded by Lemma 2.4(ii), which contradicts (25). Hence we can discard this case. If \( Q = 0 \), then \( x = u^* \) is unitarily equivalent to an operator of the form (III). When \( Q \) is strictly negative, Lemma 2.4(ii) applied to the relation \( q^2u(-Q) \subseteq (-Q)u \) shows that \( x = (1 - q^2Q)^{1/2}u^* \) is unitarily equivalent to an operator of the form (II).

\( \epsilon = -1: \) In this case, we use the polar decomposition \( x = vb \) of \( x \). From

\[ b^2 = x^*x = -1 - Q = q^{-2}(xx^* + 1 - q^2) \geq q^{-2} - 1, \]

it follows that \( \ker b = \ker v = \{0\} \) so that \( v \) is an isometry. Using (27) and arguing as above, one obtains \( q^{-2}vQ \subseteq Qv \) and \( q^2v^*Q \subseteq Qv^* \). Note that, in the present case, \( Q \leq -q^{-2} \) by (29). Therefore \( \ker v^* \neq \{0\} \) since otherwise Lemma 2.1 would imply that 0 belongs to the spectrum of \( Q \). Now \( \varphi \in \ker v^* = \ker x^*x = \ker xx^* \) if and only if \( Q\varphi = (1 - q^2x^*x)\varphi = (q^{-2}(1 - q^2))\varphi = q^{-2}\varphi \). From Lemma 2.2 it follows that \( \mathcal{K} := \oplus_{n=1}^{\infty} \mathcal{H}_n \)

where \( \mathcal{H}_n = \ker (Q + q^{-2n}) \), is a reducing subspace for \( v \) and \( Q \). In particular, \( x|\mathcal{K} = (1 - Q)^{1/2}|\mathcal{K} \) is unitarily equivalent to an operator of the form stated in the lemma. Finally, we conclude that \( \mathcal{H} = \mathcal{K} \) since the restriction of \( v^* \)

to a nonzero orthogonal complement of \( \mathcal{K} \) would be injective, which is impossible as noted before.

\[ \square \]
Remark. For \( \epsilon = 1 \), a characterization of irreducible representations of \( \mathfrak{o}_5 \) can be found in \( \text{[7]} \) as a special case of the results therein. For \( \epsilon = -1 \), the irreducible representations of \( \mathfrak{o}_5 \) were obtained in \( \text{[3]} \) by assuming in the proof that \( x^* x \) has eigenvectors.

3 Quantum disc

3.1 Invariant integration on the quantum disc

The quantum disc \( \mathcal{O}_q(U) \) is defined as the *-algebra generated by \( z \) and \( z^* \) with relation

\[
z^* z - q^2 z z^* = 1 - q^2.
\]

By \( \text{[8]} \), it is obvious that \( \mathcal{O}_q(U) = \text{Lin}\{z^n z^m; n, m \in \mathbb{N}\} \). Set

\[
y := 1 - zz^*.
\]

Then \( y = y^* \) and

\[
y z = q^2 z y, \quad y z^* = q^{-2} z^* y.
\]

From \( z z^* = 1 - y, z^* z = 1 - q^2 y \), and \( \text{[8]} \), we deduce

\[
z^n z^m = (y; q^{-2})_n, \quad z^* z^n = (q^2 y; q^2)_n,
\]

where \( (t; q)_0 := 1 \) and \( (t; q)_n := \prod_{k=0}^{n-1} (1 - q^k t), n \in \mathbb{N}. \) In particular, each element \( f \in \mathcal{O}_q(U) \) can be written as

\[
f = \sum_{n=0}^{N} z^n p_n(y) + \sum_{n=1}^{M} p_n(y) z^* z^n, \quad N, M \in \mathbb{N},
\]

with polynomials \( p_n \) in \( y \).

The left action \( \triangleright \) which turns \( \mathcal{O}_q(U) \) into a \( \mathcal{U}_q(\mathfrak{su}_{1,1}) \)-module *-algebra can be found in \( \text{[13, 14]} \) or \( \text{[4]} \). On generators, it takes the form

\[
K^{\pm} z = q^{\pm} z, \quad E \triangleright z = -q^{1/2} z^2, \quad F \triangleright z = q^{1/2},
\]

\[
K^{\pm} z^* = q^{\mp} z^*, \quad E \triangleright z^* = q^{-3/2} z^2, \quad F \triangleright z^* = -q^{5/2} z^2.
\]

Remind our notational conventions regarding representations. For instance, if \( \pi : \mathcal{O}_q(U) \to \mathcal{L}^+(D) \) is a representation, we write \( f \) instead of \( \pi(f) \) and \( X \triangleright f \) in instead of \( \pi(X \triangleright f) \), where \( f \in \mathcal{O}_q(U), X \in \mathcal{U}_q(\mathfrak{su}_{1,1}) \). The key observation of this subsection is the following simple operator expansion.

Lemma 3.1 Let \( \pi : \mathcal{O}_q(U) \to \mathcal{L}^+(D) \) be a *-representation of \( \mathcal{O}_q(U) \) such that \( y^{-1} \) belongs to \( \mathcal{L}^+(D) \). Set \( A := q^{-1/2} \lambda^{-1} z \) and \( B := -y^{-1} A^* \). Then the formulas

\[
K \triangleright f = y f y^{-1}, \quad K^{-1} \triangleright f = y^{-1} f y, \quad (37)
\]

\[
E \triangleright f = Af - y f y^{-1} A, \quad (38)
\]

\[
F \triangleright f = B f y - q^2 f y B \quad (39)
\]
define an operator expansion of the action $\triangleright$, where $f \in \mathcal{O}_q(U)$. The same formulas applied to $f \in \mathcal{L}^+(D)$ turn the $\mathcal{O}^*$-algebra $\mathcal{L}^+(D)$ into a $\mathcal{U}_q(\mathfrak{su}_{1,1})$-module $\ast$-algebra.

**Proof.** We take Equations (37)–(39) as definition and show that the action $\triangleright$ defined in this way turns $\mathcal{L}^+(D)$ into a $\mathcal{U}_q(\mathfrak{su}_{1,1})$-module $\ast$-algebra. To verify that $\triangleright$ is well defined, we use the commutation relations

$$ yA = q^2 Ay, \quad yB = q^{-2} By, \quad AB - BA = -\lambda^{-1} y^{-1} y $$

which are easily obtained by applying (30) and (32). Let $f \in \mathcal{L}^+(D)$. It follows that

$$ KE \triangleright f = y(Af - yfy^{-1}A)y^{-1} = q^2(Afy - y^2 fy^{-2}A) = q^2 EK \triangleright f $$

and

$$ (EF - FE) \triangleright f = ABfy + yfBA - BAFy - yfAB $$

$$ = (AB - BA)fy + yf(AB - BA) $$

$$ = \lambda^{-1}(yfy - y^{-1} fy) = \lambda^{-1}(K - K^{-1}) \triangleright f. $$

The other relations of (14) are treated in the same way, so we conclude that the action is well defined.

We continue by verifying (2)–(4). Since the action is associative, it is sufficient to prove (2)–(4) for the generators $E$, $F$, $K$, and $K^{-1}$. Let $f, g \in \mathcal{L}^+(D)$. Then

$$ K^{\pm 1} \triangleright (fg) = y^{\pm 1} fgy^{\mp 1} = y^{\pm 1} fy^{\mp 1} y^{\mp 1} gy^{\mp 1} = (K^{\pm 1} \triangleright f)(K^{\pm 1} \triangleright g). $$

Furthermore,

$$ (E \triangleright f)g + (K \triangleright f)(E \triangleright g) = (Af - yfy^{-1}A)g + yfy^{-1}(Ag - ygy^{-1}A) $$

$$ = Afy - yfy^{-1}A = E \triangleright (fg) $$

and, analogously, $(F \triangleright f)(K^{-1} \triangleright g) + f(F \triangleright g) = F \triangleright (fg)$.

Clearly, $K^{\pm 1} \triangleright 1 = (K^{\pm 1} \triangleright 1)$, $E \triangleright 1 = A - yy^{-1}A = 0 = E(1)$, and, similarly, $F \triangleright 1 = 0 = F(1)$. Equation (41) reads for $K^{\pm 1}$

$$ (K^{\pm 1} \triangleright f)^* = (y^{\pm 1} fy^{\mp 1})^* = y^{\mp 1} f^* y^{\mp 1} = K^{\mp 1} \triangleright f^* = S(K^{\pm 1}) \triangleright f^*,$$

and, for $E$, we have

$$(E \triangleright f)^* = f^* A^* - A^* y^{-1} f^* y = -f^* yB + q^{-2} Bf^* y = q^{-2} Ff^* = S(E)^* \triangleright f^* \quad (41)$$

since $S(E)^* = KFK^{-1} = q^{-2} F$. Replacing in (41) $f$ by $q^2 f^*$ and applying the involution gives $(F \triangleright f)^* = q^2 E \triangleright f^* = S(F)^* \triangleright f^*$, where we used $S(F)^* = q^2 E$. Summarizing, we have shown that the action $\triangleright$ defined by (37)–(39) equips $\mathcal{L}^+(D)$ with the structure of a $\mathcal{U}_q(\mathfrak{su}_{1,1})$-module $\ast$-algebra.

It remains to prove that (37)–(39) define an operator expansion of the action $\triangleright$ given by (35) and (36). Since $\pi(\mathcal{O}_q(U))$ is a $\ast$-subalgebra of the $\mathcal{U}_q(\mathfrak{su}_{1,1})$-module $\ast$-algebra $\mathcal{L}^+(D)$, it is sufficient to verify (37)–(39) for the generators of $\mathcal{U}_q(\mathfrak{su}_{1,1})$ and $\mathcal{O}_q(U)$. 

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(see Equation (2)). From the definition of $A$ and $y$, it follows by using (40) and (36) that

$$K_{\pm 1}z = y_{\pm 1}z^\mp 1 = q_{\pm 1}z, \quad K_{\pm 1}z^* = y_{\pm 1}z^*y^{\mp 1} = q_{\mp 1}z^*,$$

(42)

$$E\circ z = A z - y z y^{-1} A = q_{-1/2}A^{-1}(z^2 - q^2z^2) = -q_{1/2}z^2,$$

(43)

$$E\circ z^* = A z^* - y z^* y^{-1} A = q_{-5/2}A^{-1}(q^2zz^* - z^*z) = q^{-3/2}$$

(44)

and, similarly, $F\circ z = q^{1/2}$, $F\circ z^* = -q^{5/2}z^*2$. This completes the proof. □

Recall that the left adjoint action $\text{ad}_L(a)(b) := a_{(1)}bS(a_{(2)})$, $a,b \in \mathcal{U}_q(\mathfrak{su}_{1,1})$, turns $\mathcal{U}_q(\mathfrak{su}_{1,1})$ into a $\mathcal{U}_q(\mathfrak{su}_{1,1})$-module *-algebra. For the generators $E$, $F$, and $K$, we obtain $\text{ad}_L(E)(b) = Eb - KbK^{-1}E$, $\text{ad}_L(F)(b) = FbK - q^2bKF$, and $\text{ad}_L(K)(b) = K b K^{-1}$. There is an obvious formal coincidence of this formulas with (37)–(39) but $A$, $B$, and $y$ do not satisfy the relations of $E$, $F$, and $K$ because the last equation of (40) differs from (15).

We mentioned that for a finite dimensional representation $\rho$ of $\mathcal{U}_q(\mathfrak{su}_{1,1})$ the quantum trace

$$\text{Tr}_q \alpha := \text{Tr} \rho(\alpha K^{-1})$$

defines an invariant integral on $\mathcal{U}_q(\mathfrak{su}_{1,1})$ (see [5 Proposition 7.1.14]). The proof does not involve the whole set of relations of $\mathcal{U}_q(\mathfrak{su}_{1,1})$ but the trace property and the relation $K^{-1}fK = S^2(f)$ for all $f \in \mathcal{U}_q(\mathfrak{su}_{1,1})$. The last relation reads on generators as $K^{-1}KK = K$, $K^{-1}EK = q^{-2}E$, $K^{-1}FK = q^2F$ and these equations are also satisfied if we replace $K$ by $y$, $E$ by $A$, and $F$ by $B$.

The main result of this section, achieved in Proposition 3.2 below, is a generalization of the quantum trace formula to the operator algebras $\mathcal{B}_1(\mathfrak{A})$ and $\mathcal{F}(D)$ from Subsection 2.2 by using the above observations. Notice that we cannot have a normalized invariant integral on $\mathcal{O}_q(U)$; if there were an invariant integral $h$ on $\mathcal{O}_q(U)$ satisfying $h(1) = 1$, then we would obtain

$$1 = h(1) = q^{-1/2}h(F\circ z) = q^{-1/2}\varepsilon(F)h(z),$$

(45)

a contradiction since $\varepsilon(F) = 0$.

**Proposition 3.2** Suppose that $\pi : \mathcal{O}_q(U) \to L^+(D)$ is a *-representation of $\mathcal{O}_q(U)$ such that $y^{-1} \in L^+(D)$. Let $\mathfrak{A}$ be the $O^*$-algebra generated by the operators $z, z^*$, and $y^{-1}$. Then the *-algebras $\mathcal{F}(D)$ and $\mathcal{B}_1(\mathfrak{A})$ defined in (17) and (18), respectively, are $\mathcal{U}_q(\mathfrak{su}_{1,1})$-module *-algebras, where the action is given by (37)–(39). The linear functional

$$h(g) := c \text{Tr} g y^{-1}, \quad c \in \mathbb{R},$$

(46)

defines an invariant integral on both $\mathcal{F}(D)$ and $\mathcal{B}_1(\mathfrak{A})$.

**Proof.** Obviously, by the definition of $\mathcal{F}(D)$ and $\mathcal{B}_1(\mathfrak{A})$, we have $afb \in \mathcal{F}(D)$ and $agb \in \mathcal{B}_1(\mathfrak{A})$ for all $f \in \mathcal{F}(D)$, $g \in \mathcal{B}_1(\mathfrak{A})$, $a,b \in \mathfrak{A}$, so both algebras are stable under the action of $\mathcal{U}_q(\mathfrak{su}_{1,1})$. By Lemma 3.1, this action turns $\mathcal{F}(D)$ and $\mathcal{B}_1(\mathfrak{A})$ into $\mathcal{U}_q(\mathfrak{su}_{1,1})$-module *-algebras.
The proof of the invariance of \( h \) uses the trace property \( \text{Tr} \, abg = \text{Tr} \, gba = \text{Tr} \, bag \) which holds for all \( g \in \mathcal{B}_1(\mathfrak{A}) \) and all \( a, b \in \mathfrak{A} \) (see [3]). Since the action is associative and \( \varepsilon \) a homomorphism, we only have to prove the invariance of \( h \) for generators. Let \( g \in \mathcal{B}_1(\mathfrak{A}) \). Clearly,
\[
\begin{align*}
    h(K_{\pm 1}^y g) &= \text{Tr} \, y^{\pm 1} g y^{1} y^{-1} = \text{Tr} \, g y^{-1} = \varepsilon(K_{\pm 1}) h(g), \\
    h(E \circ g) &= \text{Tr} \, (A g y^{-1} - y g y^{-1} A) = \text{Tr} \, A g y^{-1} - \text{Tr} \, A g y^{-1} = 0 = \varepsilon(E) h(g).
\end{align*}
\]
Using the second relation of (40), we compute
\[
h(F \circ g) = \text{Tr} \, (B g - q^2 g y B y^{-1}) = \text{Tr} \, g B - \text{Tr} \, g B = 0 = \varepsilon(B) h(g).
\]
Hence \( h \) defines an invariant integral on \( \mathcal{B}_1(\mathfrak{A}) \). It is obvious that the restriction of \( h \) to \( \mathcal{F}(D) \) gives an invariant integral on \( \mathcal{F}(D) \).

Commonly, the algebra \( \mathcal{O}_q(\mathfrak{U}) \) is considered as the polynomial functions on the quantum disc. Observe that \( agb \in \mathcal{B}_1(\mathfrak{A}) \) for all \( g \in \mathcal{B}_1(\mathfrak{A}) \) and all polynomial functions \( a, b \in \mathcal{O}_q(\mathfrak{U}) \). Notice, furthermore, that the action of \( E \) and \( F \) satisfies a “twisted” Leibniz rule. If we think of \( \mathcal{U}_q(sl_{1,1}) \) as an algebra of “generalized differential operators”, then we can think of \( \mathcal{B}_1(\mathfrak{A}) \) as the algebra of infinitely differentiable functions which vanish sufficiently rapidly at “infinity” and of \( \mathcal{F}(D) \) as the infinitely differentiable functions with compact support.

### 3.2 Topological aspects of \(*\)-representations

This subsection is concerned with some topological aspects of the representations of \( \mathcal{O}_q(\mathfrak{U}) \). The representations of \( \mathcal{O}_q(\mathfrak{U}) \) are given by Lemma 2.3. Here we restate Lemma 2.3 by considering only irreducible \(*\)-representations and specifying the domain on which the operators act. As we require that \( y^{-1} \) exists, we exclude the case \((\mathfrak{III})_y\) in which \( y = 0 \). Let \( \{ \eta_j \}_{j \in J} \) denote the canonical basis in the Hilbert space \( \mathcal{H} = l_2(J) \), where \( J = \mathbb{N}_0 \) or \( J = \mathbb{Z} \).

(I) The operators \( z, z^* \), and \( y \) act on \( D := \text{Lin} \{ \eta_n; n \in \mathbb{N}_0 \} \) by
\[
    z \eta_n = \lambda_{n+1} \eta_{n+1}, \quad z^* \eta_n = \lambda_n \eta_{n-1}, \quad y \eta_n = q^{2n} \eta_n.
\]

(II) Let \( \alpha \in [0, 1) \). The actions of \( z, z^* \), and \( y \) on \( D := \text{Lin} \{ \eta_n; n \in \mathbb{Z} \} \) are given by
\[
    z \eta_n = \lambda_{\alpha, n+1} \eta_{n+1}, \quad z^* \eta_n = \lambda_{\alpha, n} \eta_{n-1}, \quad y \eta_n = q^{2(\alpha+n)} \eta_n,
\]
where \( \lambda_n = (1 - q^{2n})^{1/2} \) and \( \lambda_{\alpha, n} = (1 + q^{2(\alpha+n)})^{1/2} \). Obviously, \( y^{-1} \in \mathcal{L}^+(D) \) in both cases.

Let \( \mathfrak{A} \) be the \( \mathcal{O}\)-algebra defined in Proposition 3.2. If we equip \( D \) with the graph topology \( \tau_\mathfrak{A} \), \( D \) is not complete. The situation becomes better if we pass to the closure of \( \mathfrak{A} \). Since this can always be done, there is no loss of generality in assuming \( \mathfrak{A} \) to be closed, that is, \( \mathcal{D}_\mathfrak{A} := \cap_{a \in \mathfrak{A}} D(a) \) (see Equation 20). Some topological facts concerning \( \mathfrak{A} \) and \( \mathcal{L}^+(\mathcal{D}_\mathfrak{A}) \) are collected in the following lemma and the next proposition.
Lemma 3.3 Suppose that we are given an irreducible $*$-representation of type (I) or (II)$_{\omega}$ and the $O^*$-algebra $\mathcal{A}$ from Proposition 3.2 is closed.

i. $\mathcal{A}$ is a commutatively dominated $O^*$-algebra on a Frechet domain.

ii. $D_\mathcal{A}$ is nuclear, in particular, $D_\mathcal{A}$ is a Frechet–Montel space.

Proof. (i): The operator $y$ is essentially self-adjoint on $D_\mathcal{A}$ and so is

$$T := 1 + y^2 + y^{-2}.$$  

Let $\varphi \in D_\mathcal{A}$. A standard argument shows that, for each polynomial $p(y, y^{-1})$, there exist $k \in \mathbb{N}$ such that $||p(y, y^{-1})\varphi|| \leq ||T^k\varphi||$. By using (32), we get the estimates

$$||z^n p(y, y^{-1})\varphi|| \leq (||\bar{p}(y, y^{-1})(q^2 y; q^2) \varphi(\varphi)||)^{1/2} \leq ||T'\varphi||,$$

$$||z^n p(y, y^{-1})\varphi|| \leq (||\bar{p}(y, y^{-1})(y; q^2) \varphi(\varphi)||)^{1/2} \leq ||T''\varphi||$$

for some $l', l'' \in \mathbb{N}$. Since $T \geq 2$ and $T^k \leq T^n$ for $k \leq m$, we can find for each finite sequence $k_1, \ldots, k_N \in \mathbb{N}$ a $k_0 \in \mathbb{N}$ such that $\sum_{k=1}^N ||T^{k_j}\varphi|| \leq ||T^{k_0}\varphi||$. By (33), (34), and the definition of $\mathcal{A}$, it follows that each $f \in \mathcal{A}$ can be written as $f = \sum_{n=0}^M z^n p_n(y, y^{-1}) + \sum_{m=1}^N z^n p_m(y, y^{-1})$. From the foregoing, we conclude that there exist $m \in \mathbb{N}$ such that $||f\varphi|| \leq ||T^m\varphi||$, consequently $|| \cdot ||_f \leq || \cdot ||_T$. This shows that the family $\{|\cdot||_T\} \in \mathbb{N}$ generates the graph topology and $D_\mathcal{A} = \cap_{k \in \mathbb{N}} D(T^{2k})$, which proves (i).

(ii): By (i), the graph topology is metrizable. It follows from [8, Proposition 2.2.9 and Corollary 2.3.2.(ii)] that $D_\mathcal{A}$ is a reflexive Frechet space, in particular, $D_\mathcal{A}$ is barreled. To see that $D_\mathcal{A}$ is nuclear, consider $E_n := (D_\mathcal{A}, ||\cdot||_{T^n})$, where the closure of $D_\mathcal{A}$ is taken in the norm $||\cdot||_{T^n}$, and the embeddings $\iota_{n+1} : E_{n+1} \to E_n$, where $\iota_{n+1}$ denotes the identity on $E_{n+1}$, $n \in \mathbb{N}$. It is easy to see that the operator $T^{-1} : \mathcal{H} \to \mathcal{H}$ is a Hilbert–Schmidt operator and that the canonical basis $\{e_j\}_{j \in J}$, where $J = \mathbb{N}_0$ in case (I) and $J = \mathbb{Z}$ in case (II), is a complete set of eigenvectors. The set $\{f_j^n\}_{j \in J}$, $f_j^n = ||T^n e_j||^{-1} e_j$ constitutes an orthonormal basis in $E_n$, and we have

$$\sum_{j \in J} ||\iota_{n+1}(f_j^{n+1})||_{T^n}^2 = \sum_{j \in J} ||T^n f_j^{n+1}||^2 = \sum_{j \in J} ||T^n (||T^n e_j||^{-1} e_j)||^2$$

$$= \sum_{j \in J} ||T^{-1} e_j||^2 < \infty$$

which shows that $\iota_{n+1}$ is a Hilbert–Schmidt operator. From this, we conclude that $D_\mathcal{A}$ is a nuclear space since the family $\{|\cdot||_{T^n}\}_{n \in \mathbb{N}}$ of Hilbert semi-norms generates the topology on $D_\mathcal{A}$. As each nuclear space is a Schwartz space and as each barreled Schwartz space is a Montel space, $D_\mathcal{A}$ is a Montel space. \hfill $\square$

Proposition 3.4 Suppose we are given an irreducible $*$-representation of type (I) or (II)$_{\omega}$. Assume that the $O^*$-algebra $\mathcal{A}$ from Proposition 3.2 is closed.

i. $\mathcal{F}(D_\mathcal{A})$ is dense in $\mathcal{L}(D_\mathcal{A}, D_\mathcal{A})$ with respect to the bounded topology $\tau_b$. 

ii. The $U_q(\text{su}_{1,1})$-action on $L^+(D_\mathfrak{A})$ is continuous with respect to $\tau_\beta$.

**Proof.** (i) follows immediately from Lemma 3.3(ii) and [8] Theorem 3.4.5.

(ii): Let $x \in L^+(D_\mathfrak{A})$ and $a, b \in \mathfrak{A}$. According to [8] Proposition 3.4(ii), the multiplication $x \mapsto axb$ is continuous. By Lemma 3.1 the action of $U_q(\text{su}_{1,1})$ is given by a finite linear combination of such expressions, hence it is continuous. 

The algebra $\mathbb{F}(D)$ is the linear span of operators $\eta_n \otimes \eta_m$, where $n, m \in \mathbb{N}_0$ for the type (I) representation and $n, m \in \mathbb{Z}$ for type (II) representations. Since $D \subset D_\mathfrak{A}$, we can consider $\mathbb{F}(D)$ as a subalgebra of $\mathbb{F}(D_\mathfrak{A})$ and, moreover, as a $U_q(\text{su}_{1,1})$-module $*$-algebra. The interest in $\mathbb{F}(D)$ stems from the fact that the operators $\eta_n \otimes \eta_m$ are more suitable for calculations. With a little extra effort, we can deduce from Proposition 3.4 that the linear span of this operators is dense in $L(D_\mathfrak{A}, D_\mathfrak{A}^+)$.

**Corollary 3.5** $\mathbb{F}(D)$ is dense in $L(D_\mathfrak{A}, D_\mathfrak{A}^+)$ with respect to the bounded topology $\tau_\beta$.

**Proof.** In view of Proposition 3.4(i), it is sufficient to show that $\mathbb{F}(D_\mathfrak{A})$ lies in the closure of $\mathbb{F}(D)$. With $\bar{T}$ defined in (47), consider the set of Borel measurable functions

$$
\mathcal{R} := \{ r : \sigma(\bar{T}) \to [0, \infty) ; \sup_{t \in \sigma(\bar{T})} r(t) t^{2n} < \infty \}.
$$

It follows from Lemma 3.3(i) and [6] Proposition 3.4 that the family of semi-norms

$$
\{ || \cdot ||_{r \in \mathcal{R}}, \quad ||a||_r := ||r(\bar{T})a r(\bar{T})||, \quad a \in L(D_\mathfrak{A}, D_\mathfrak{A}^+)\text{,}
$$

(the norm $|| \cdot ||$ being the operator norm in $L(\mathcal{H})$) generates the topology $\tau_\beta$.

Let $\varphi, \psi \in D_\mathfrak{A}$. Notice that $||r(\bar{T})(\varphi \otimes \psi)r(\bar{T})|| \leq ||r(\bar{T})||^2 ||\varphi|| ||\psi||$. With $\alpha_n, \beta_n \in \mathbb{C}$, write $\varphi = \sum_{n \in J} \alpha_n \eta_n$, $\psi = \sum_{n \in J} \beta_n \eta_n$, where $J = \mathbb{N}_0$ or $J = \mathbb{Z}$ according to the type of representation considered. For $k \in \mathbb{N}$, set $\varphi_k := \sum_{|n| \leq k} \alpha_n \eta_n$ and $\psi_k := \sum_{|n| \leq k} \beta_n \eta_n$. Clearly, $\varphi_k, \psi_k \in \mathbb{F}(D)$. Now

$$
||\varphi \otimes \psi - \varphi_k \otimes \psi_k||_r = ||r(\bar{T})(\varphi \otimes \psi - \varphi_k \otimes \psi_k)r(\bar{T})||
\leq ||r(\bar{T})||^2 ||\varphi - \varphi_k|| ||\psi|| + ||r(\bar{T})||^2 ||\psi_k|| ||\psi - \psi_k|| \to 0
$$
as $k \to \infty$ for all $r \in \mathcal{R}$, hence $\varphi \otimes \psi$ lies in the closure of $\mathbb{F}(D)$. Since $\mathbb{F}(D_\mathfrak{A})$ is the linear span of operators $\varphi \otimes \psi$, the assertion follows. 

Proposition 3.3 and Corollary 3.3 show how $\mathbb{F}(D)$ and $\mathbb{F}(D_\mathfrak{A})$ are related to the image of $\mathcal{O}_q(U)$ in $L^+(D_\mathfrak{A})$: By density and continuity, $\mathbb{F}(D)$ and $\mathbb{F}(D_\mathfrak{A})$ carry the whole information about the action of $U_q(\text{su}_{1,1})$ on $L^+(D_\mathfrak{A}) \subset L(D_\mathfrak{A}, D_\mathfrak{A}^+)$ and, in particular, on $\mathcal{O}_q(U) \subset L(D_\mathfrak{A}, D_\mathfrak{A}^+)$. 

It would be desirable to have also the converse statement, that is, to obtain the action on $\mathbb{F}(D)$ (or $\mathbb{F}(D_\mathfrak{A})$) by taking the closure of $\mathcal{O}_q(U)$ in $L(D_\mathfrak{A}, D_\mathfrak{A}^+)$. Unfortunately, this is not possible. From [8] Theorem 4.5.4, it follows that $\tau_\beta$ coincides with the finest locally convex topology on $\mathfrak{A}$. Since $\mathfrak{A}$ is closed with respect to the finest locally convex topology, it is closed with respect to $\tau_\beta$. 

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For $\mathcal{F}(D)$ to be in the closure of $\mathcal{O}_q(U)$, we can consider a different locally convex topology on $D$. Let $D'$ be the vector space of all formal series $\sum_{j \in J} \alpha_j \eta_j$, where $J = \mathbb{N}$ or $J = \mathbb{Z}$. There exists a dual pairing $\langle \cdot, \cdot \rangle$ of $D'$ and $D$ given by

$$\langle \sum_{j \in J} \alpha_j \eta_j, \sum_{|n| \leq n_0} \beta_n \eta_n \rangle = \sum_{|n| \leq n_0} \alpha_n \beta_n.$$ 

We equip $D$ and $D'$ with the weak topologies arising from this dual pairing. To $\mathcal{L}(D, D')$, the vector space of all continuous linear mappings from $D$ into $D'$, we assign the operator weak topology $\tau_{ow}$, that is, the topology generated by the family of semi-norms

$$\{ p_{\varphi, \psi} \varphi, \psi \in D \}, \quad p_{\varphi, \psi}(a) := \left| \langle a \varphi, \psi \rangle \right|, \quad a \in \mathcal{L}(D, D').$$

Then $\mathcal{O}_q(U)$ is dense in $\mathcal{L}(D, D')$ with respect to $\tau_{ow}$ and the action of $U_q(su_{1,1})$ on $\mathcal{L}(D, D')$ defined by (37)–(39) is continuous. This is essentially the method for constructing the space $D(U_q)'(= \mathcal{L}(D, D'))$ of distributions on the quantum disc as performed in [13]. The topological space $D(U_q)$ of finite functions on the quantum disc defined in [13] is homeomorphic to $\mathcal{F}(D)$ with the operator weak topology $\tau_{ow}$.

We now give another description of $\mathcal{F}(D)$.

**Lemma 3.6** Let $\mathcal{F}(\sigma(\bar{y}))$ be the set of (Borel measurable) functions on $\sigma(\bar{y})$ with finite support, that is,

$$\mathcal{F}(\sigma(\bar{y})) = \{ \psi : \sigma(\bar{y}) \to \mathbb{C}; \# \{ t \in \sigma(\bar{y}); \psi(t) \neq 0 \} < \infty \}.$$ 

Each $f \in \mathcal{F}(D)$ can be written as

$$f = \sum_{n=0}^{N} z^n \psi_n(\bar{y}) + \sum_{n=1}^{M} \psi_{-n}(\bar{y}) z^{*n}, \quad N, M \in \mathbb{N},$$

where $\psi_k \in \mathcal{F}(\sigma(\bar{y}))$, $k = -M, \ldots, N$.

Conversely, if $\psi_k \in \mathcal{F}(\sigma(\bar{y}))$, then $\sum_{n=0}^{N} z^n \psi_n(\bar{y}) + \sum_{n=1}^{M} \psi_{-n}(\bar{y}) z^{*n} \in \mathcal{F}(D)$.

**Proof.** To see this, consider the functions

$$\delta_k(t) := \begin{cases} 1 & : \text{ for } t = q^{2k} \\ 0 & : \text{ for } t \neq q^{2k} \end{cases}$$

if we are given a type (I) representation, and

$$\delta_k(t) := \begin{cases} 1 & : \text{ for } t = -q^{2\alpha + 2k} \\ 0 & : \text{ for } t \neq -q^{2\alpha + 2k} \end{cases}$$

if we are given a representation of type $(II)_\alpha$. Notice that $\delta_k(\bar{y})$ is the projection on $\mathcal{H}$ with range $\mathbb{C} \eta_k$, that is, $\delta_k(\bar{y}) = \eta_k \otimes \eta_k$. 

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Each \( \psi_n \in \mathcal{F}(\sigma(y)) \) can be written as a finite sum \( \sum_k \psi_{n,k} \delta_k(t) \), where \( \psi_{n,k} = \psi_n(q^{2k}) \) for the type \((I)\) representation and \( \psi_{n,k} = \psi_n(-q^{2\alpha+2k}) \) for type \((II)_\alpha\) representations. Furthermore, we have

\[
\begin{aligned}
\psi_n(y) &= z^n(\eta_k \otimes \eta_k) = z^n \eta_k \otimes \eta_k \in \mathcal{F}(D), \\
\delta_k(y)z^{*n} &= (\eta_k \otimes \eta_k)z^{*n} = \eta_k \otimes (z^n \eta_k) \in \mathcal{F}(D),
\end{aligned}
\]

hence \( \sum_{n=0}^N \psi_n(y) + \sum_{n=1}^M \psi_{-n}(y) z^{*n} \in \mathcal{F}(D) \) whenever \( \psi_n \in \mathcal{F}(\sigma(y)) \), \( n = -M, \ldots, N \).

On the other hand, for \( k \leq n \), we can write

\[
\begin{aligned}
\eta_n \otimes \eta_k &= \gamma_{n,k}(z^{n-k}) \otimes \eta_k = \gamma_{n,k}z^{n-k} \delta_k(y), \\
\eta_k \otimes \eta_n &= \gamma_{n,k} \eta_k \otimes (z^{n-k}) = \gamma_{n,k} \delta_k(y) z^{*n-k},
\end{aligned}
\]

where \( \gamma_{n,k} = (q^{2(k+1)}; q^{2k+1})^{-1/2} \) and \( \gamma_{n,k} = (-q^{2(\alpha+k+1)}; q^{2k+1})^{-1/2} \) for the representations of type \((I)\) and type \((II)_\alpha\), respectively. Hence any linear combination of \( \eta_m \otimes \eta_n \) is equivalent to a linear combination of \( z^n \delta_k(y) \) and \( \delta_k(y) z^{*n} \). Summing over equal powers of \( z \) and \( z^* \) yields coefficients of \( z^n \) and \( z^{*n} \) of the form \( \sum_k \psi_{n,k} \delta_k(y) \), \( \psi_{n,k} \in \mathbb{C} \), and the functions \( \sum_k \psi_{n,k} \delta_k(t) \) belong to \( \mathcal{F}(\sigma(y)) \) since all sums are finite. \( \square \)

A similar result can be obtained by considering the following set of \((\text{Borel measurable})\) functions

\[
S(\sigma(y)) := \{ \psi : \sigma(y) \to \mathbb{C} ; \sup_{t \in \sigma(y)} |t^k \psi(t)| < \infty \text{ for all } k \in \mathbb{Z} \}.
\]

**Lemma 3.7** The element \( f = \sum_{n=0}^N z^n \psi_n(y) + \sum_{n=1}^M \psi_{-n}(y) z^{*n}, \psi_n \in S(\sigma(y)) \), \( N, M \in \mathbb{N} \), belongs to \( \mathcal{B}_1(\mathfrak{A}) \). The operators \( \psi(y), \psi \in S(\sigma(y)) \), satisfy on \( D_\mathfrak{A} \) the commutation rules

\[
z \psi(y) = \psi(q^2 y) z, \quad z^* \psi(y) = \psi(q^{-2} y) z^*.
\]  

(48)

The linear space

\[
S(D) := \left\{ \sum_{n=0}^N z^n \psi_n(y) + \sum_{n=1}^M \psi_{-n}(y) z^{*n} ; \psi_k \in S(\sigma(y)) \text{ for all } -M \leq k \leq N \right\}
\]

forms a \( \mathcal{U}_\phi(\text{su}_{1,1}) \)-module \(*\)-subalgebra of \( \mathcal{L}^+(D_\mathfrak{A}) \).

**Proof.** By definition of \( \mathcal{B}_1(\mathfrak{A}) \), \( a \psi b \in \mathcal{B}_1(\mathfrak{A}) \) for all \( a, b \in \mathfrak{A} \) whenever \( \psi \in \mathcal{B}_1(\mathfrak{A}) \). Fix \( a \in \mathfrak{A} \). From the proof of Lemma 3.3, we know that \( \{ || \cdot ||_{T^n} \}_{n \in \mathbb{N}} \), \( T = 1 + y^2 + y^{-2} \), generates the graph topology on \( D_\mathfrak{A} \), so there exist \( n_a \in \mathbb{N} \) such that \( ||a \varphi|| \leq ||T^{n_a} \varphi|| \) for all \( \varphi \in D_\mathfrak{A} \). Consequently, \( ||a T^{-n_a} \varphi|| \leq ||\varphi|| \), hence \( a T^{-n_a} \) and \( T^{-n_a} a^* \) are bounded. The operators \( \psi_m(y) T^m, \psi_n \in S(\sigma(y)) \), \( m \in \mathbb{N} \), are
bounded by the definition of $S(\sigma(\bar{y}))$, and $\bar{T}^{-1}$ is of trace class. From this facts, we conclude that
\[
\alpha \psi_n(\bar{y}) b = a \bar{T}^{-n_b} \psi_n(\bar{y}) \bar{T}^{n_a+n_b+1} T^{-1} \bar{T}^{-n_b} b
\]
is of trace class. This shows that the operator $f$ from Lemma 3.8 belongs to $B_1(\mathfrak{A})$.

The commutation relations (32) are satisfied if we restrict the operators to $D \subset D_{\mathfrak{A}}$. Consider the $O^*$-algebra generated by the elements $\psi(\bar{y})[D, \psi \in S(\sigma(\bar{y})), \text{ and } a[D, a \in \mathfrak{A}$. Since the operators $\psi(\bar{y})$ are bounded, the closure of this algebra is contained in $L^+(D_3)$. Taking the closure of an $O^*$-algebra does not change the commutation relations, hence Equation (48) holds.

Recall that $\mathfrak{A}$ is the linear span of operators $z^n p_n(y, y^{-1})$ and $p_{-n}(y, y^{-1}) z^s n$, where $p_n(y, y^{-1})$ and $p_{-n}(y, y^{-1})$ are polynomials in $y$ and $y^{-1}$. Notice, furthermore, that $p(t, t^{-1}) \psi(t) \in S(\sigma(\bar{y}))$ for all $\psi(t) \in S(\sigma(\bar{y}))$ and all polynomials $p(t, t^{-1})$. Now it follows from (32), (33), (48), and the definition of $S(D)$ that $S(D)$ is stable under the $U_q(su_{1,1})$-action defined in Lemma 3.1. Similarly, using (32), (33), (48), and the definition of $S(D)$, it is easy to check that $S(D)$ forms a $*$-algebra. Therefore, by Lemma 3.1, $S(D)$ is a $U_q(su_{1,1})$-module $*$-algebra.

The description of $F(D)$ and $S(D)$ by functions $\psi : \sigma(\bar{y}) \to \mathbb{C}$ suggests that we can consider the elements of $F(D)$ and $S(D)$ as infinitely differentiable functions on the quantum disc with compact support and which are rapidly decreasing, respectively. Notice that $F(D) \neq F(D_{\mathfrak{A}})$ (e.g., $\eta \otimes \eta \notin F(D)$ for $\eta = \sum_{n=0}^{\infty} \alpha_n \eta_n \in D_{\mathfrak{A}}$ if an infinite number of $\alpha_n$ are non-zero), and $S(\sigma(\bar{y})) \neq B_1(\mathfrak{A})$ (e.g., $f = \sum_{k=0}^{\infty} \exp(-\bar{y}^2) \delta_k(\bar{y}) z^s k \in B_1(\mathfrak{A})$, $f \notin S(D)$).

Clearly, $F(D) \subset S(D)$. On $S(D)$, the invariant integral can be expressed nicely in terms of the Jackson integral. The Jackson integral is defined by
\[
\int_0^1 \varphi(t) d_q t = (1-q) \sum_{k=0}^{\infty} \varphi(q^k) q^k \quad \text{and} \quad \int_0^\infty \varphi(t) d_q t = (1-q) \sum_{k=-\infty}^{\infty} \varphi(q^k) q^k.
\]

**Proposition 3.8** Suppose that $\psi = \sum_{n=0}^{N} z^n \psi_n(\bar{y}) + \psi_0(\bar{y}) + \sum_{n=1}^{M} \psi_{-n}(\bar{y}) z^{-n} \in S(D)$. Let $h$ denote the invariant integral defined in Proposition 3.2. For irreducible type (I) representations, we have
\[
h(\psi) = c(1-q^2)^{-1} \int_0^1 \psi_0(t) t^{-2} d_q t,
\]
and, for irreducible type (II) representations, we have
\[
h(\psi) = cq^{-2\alpha}(1-q^2)^{-1} \int_0^\infty \psi_0(-q^{2\alpha} t) t^{-2} d_q t.
\]
Proof. Since \( \langle \eta_k, z^n \psi_n(y) y^{-1} \eta_k \rangle = \langle \eta_k, \psi_n(y) z^n y^{-1} \eta_k \rangle = 0 \) for all \( n \neq 0 \), we obtain
\[
h(\psi) = c \text{Tr} \overline{\psi y^{-1}} = c \sum_{k=0}^{\infty} \langle \eta_k, \psi_0(y) y^{-1} \eta_k \rangle = c \sum_{k=0}^{\infty} \psi_0(q^{2k}) q^{-2k}
\]
for the type (I) representation and
\[
h(\psi) = c \text{Tr} \overline{\psi y^{-1}} = c \sum_{k=-\infty}^{\infty} \langle \eta_k, \psi_0(y) y^{-1} \eta_k \rangle = c \sum_{k=-\infty}^{\infty} \psi_0(-q^{2\alpha} q^{2k}) q^{-2(\alpha+k)}
\]
for type (II)\( \alpha \) representations.

3.3 Application: differential calculus

The bimodule structure of a first order differential \(*\)-calculus \((\Gamma, d)\) over \( O_q(U) \) has been described in [13] and [9]. The commutation relations are given by
\[
d zz = q^2 z dz, \quad dz z^* = q^{-2} z^* dz, \quad dz^* z = q^2 z dz^*, \quad dz^* z^* = q^{-2} z^* dz^*.
\]
Our aim is to extend this FODC to the classes of integrable functions on the quantum disc defined in Subsection 3.2. To this end, we use a commutator representation of the FODC. A faithful commutator representation of the above differential calculus can be found in [9] and is obtained as follows. Given a \(*\)-representation \( \pi \) of \( O_q(U) \) from Subsection 3.2, consider the direct sum \( \rho := \pi \oplus \pi \) on \( D \oplus D \subset H \oplus H \) and set
\[
C := (1 - q^2)^{-1} \begin{pmatrix} 0 & \pi(z) \\ \pi(z^*) & 0 \end{pmatrix}.
\]
Then the differential mapping \( d_{\rho,C} \) defined in (22) is given by
\[
d_{\rho,C}(f) = i[C, \rho(f)] = (1 - q^2)^{-1} i \begin{pmatrix} 0 & \pi(z f - f z) \\ \pi(z^* f - f z^*) & 0 \end{pmatrix}, \; f \in O_q(U).
\]
Clearly, \( C \in \mathcal{L}^+(D \oplus D) \), so we can extend \( d_{\rho,C} \) to \( \mathcal{L}^+(D \oplus D) \), that is,
\[
d_{\rho,C}(x) := i[C, x], \quad x \in \mathcal{L}^+(D \oplus D).
\]
The same formula applies to any \(*\)-subalgebra of \( \mathcal{L}^+(D \oplus D) \). Notice that we can consider \( \mathcal{L}^+(D) \) as a \(*\)-subalgebra of \( \mathcal{L}^+(D \oplus D) \) by identifying \( A \in \mathcal{L}^+(D) \) with the operator \( A \oplus A \) acting on \( D \oplus D \). In particular, the algebras \( \mathbb{F}(D) \) and \( B_1(\mathbb{H}) \) from Proposition 3.2 become \(*\)-subalgebras of \( \mathcal{L}^+(D \oplus D) \). In this way, we obtain a FODC over these algebras.
For $z$ and $z^*$, we have
\[
\text{d}_{\rho,C}(z) = i \left( \begin{array}{cc} 0 & \sigma(y) \\ \sigma(y) & 0 \end{array} \right), \quad \text{d}_{\rho,C}(z^*) = i \left( \begin{array}{cc} 0 & -\sigma(y) \\ -\sigma(y) & 0 \end{array} \right).
\]

For functions $\psi(\bar{y})$, the differential mapping $\text{d}_{\rho,C}$ can be expressed in terms of the $q$-differential operator $D_q$ defined by $D_q f(x) = (x - q x)^{-1} (f(x) - f(q x))$. It follows from
\[
(1 - q^2)^{-1} (z \psi(\bar{y}) - \psi(\bar{y}) z) = z y (y - q^2 y)^{-1} (\psi(\bar{y}) - \psi(q^2 \bar{y})) = z D_q \psi(\bar{y}) y,
\]
\[
(1 - q^2)^{-1} (z^* \psi(\bar{y}) - \psi(\bar{y}) z^*) = y (y - q^2 y)^{-1} (\psi(q^2 \bar{y}) - \psi(\bar{y})) z^* = -q^{-2} D_q \psi(\bar{y}) z y
\]
that
\[
\text{d}_{\rho,C}(\psi(\bar{y})) = -i \rho(z) D_q \psi(\bar{y}) \text{d}_{\rho,C}(z) - i q^{-2} D_q \psi(\bar{y}) \rho(z^*) \text{d}_{\rho,C}(z^*).
\]

In particular, the “$\delta$-distributions” $\delta_k(\bar{y})$ are differentiable.

## 4 Quantum $(n, 1)$-matrix ball

### 4.1 Algebraic relations

Let $n \in \mathbb{N}$ and $q \in (0, 1)$. We denote by $\mathcal{O}_q(\text{Mat}_{n,1})$ the *-algebra generated by $z_1, \ldots, z_n, z_1^*, \ldots, z_n^*$ obeying the relations
\[
z_k z_l = q z_l z_k, \quad k < l, \quad z_l^* z_k = q z_k z_l^*, \quad k \neq l,
\]
\[
z_k z_k = q^2 z_k z_k^* - (1 - q^2) \sum_{j=k+1}^{n} z_j z_j^* + (1 - q^2), \quad k < n,
\]
\[
z_n z_n = q^2 z_n z_n^* + (1 - q^2).
\]

Equations (49)–(52) are called twisted canonical commutation relations \[7\] and $\mathcal{O}_q(\text{Mat}_{n,1})$ is also known as $q$-Weyl algebra \[5\]. Here we consider it as a special case of the quantum matrix balls introduced in \[12\] because the $\mathcal{U}_q(\text{su}_{n,1})$-action on $\mathcal{O}_q(\text{Mat}_{n,1})$ defined below is taken from the latter.

The following hermitian elements $Q_k$ will play a crucial role throughout this section. Set
\[
Q_k := 1 - \sum_{j=k}^{n} z_j z_j^*, \quad k \leq n, \quad Q_{n+1} := 1.
\]

Equations \[51\], \[52\], and \[53\] imply immediately
\[
z_k z_k - q^2 z_k z_k^* = (1 - q^2) Q_{k+1}, \quad z_k^* z_k - z_k z_k^* = (1 - q^2) Q_k.
\]

Taking the difference of the first with the second and of the first with $q^2$ times the second equation gives
\[
z_k z_k^* = Q_{k+1} - Q_k, \quad z_k^* z_k = Q_{k+1} - q^2 Q_k.
\]
Furthermore, one easily shows by using Equations (49)–(53) that
\[ Q_k z_j = z_j Q_k, \quad j < k, \quad Q_k z_j = q^j z_j Q_k, \quad j \geq k, \]  
(56)
\[ Q_k z_j^* = z_j^* Q_k, \quad j < k, \quad Q_k z_j^* = q^{-j} z_j^* Q_k, \quad j \geq k. \]  
(57)
As a consequence,
\[ Q_k Q_l = Q_l Q_k, \quad \text{for all} \quad k, l \leq n + 1. \]  
(58)

For \( I = (i_1, \ldots, i_n) \in \mathbb{N}_0^n \), \( J = (j_1, \ldots, j_n) \in \mathbb{N}_0^n \), set \( z^I := z_1^{i_1} \cdots z_n^{i_n} \), \( z^J := z_1^{j_1} \cdots z_n^{j_n} \) and define \( I \cdot J = (i_1 j_1, \ldots, i_n j_n) \in \mathbb{N}_0^n \). We write 0 instead of \( (0, \ldots, 0) \). It follows from (56)–(58) together with the defining relations (49)–(52) that each \( f \in \mathcal{O}_q(\text{Mat}_{n,1}) \) can be expressed as a finite sum
\[ f = \sum_{I \cdot J = 0} z^I p_{I,J}(Q_1, \ldots, Q_n) z^J, \]  
(59)
with polynomials \( p_{I,J}(Q_1, \ldots, Q_n) \) in \( Q_1, \ldots, Q_n \).

The \( \mathcal{U}_q(\text{su}_{n,1}) \)-action \( \cdot \) on \( \mathcal{O}_q(\text{Mat}_{n,1}) \) which turns \( \mathcal{O}_q(\text{Mat}_{n,1}) \) into a \( \mathcal{U}_q(\text{su}_{n,1}) \)-module *-algebra is given by the following formulas (12).

\[
\begin{align*}
    &j \neq n: & E_j z_{j+1} &= q^{-1/2} z_j, & E_j z_k &= 0, & k \neq j + 1, \\
    & & E_j^* z_j &= -q^{-3/2} z_j^*, & E_j^* z_k &= 0, & k \neq j, \\
    & & F_j z_j &= q^{1/2} z_{j+1}, & F_j z_k &= 0, & k \neq j, \\
    & & F_j^* z_{j+1} &= -q^{3/2} z_j^*, & F_j^* z_k &= 0, & k \neq j + 1, \\
    & & K_j z_j &= q z_j, & K_j z_{j+1} &= q^{-1} z_{j+1}, & K_j z_k &= z_k, & k \neq j, j + 1, \\
    & & K_j^* z_j^* &= q^{-1} z_j, & K_j^* z_{j+1}^* &= q z_{j+1}^*, & K_j^* z_k^* &= z_k^*, & k \neq j, j + 1, \\
    &n: & E_n z_n &= -q^{1/2} z_n^2, & k < n: & E_n z_k &= -q^{1/2} z_n z_k, \\
    & & E_n^* z_n &= q^{-3/2} z_n, & E_n^* z_k &= 0, \\
    & & F_n z_n &= q^{1/2}, & F_n^* z_k &= 0, \\
    & & F_n^* z_n^* &= -q^{1/2} z_n^2, & F_n^* z_k^* &= -q^{1/2} z_n^2 z_k^*, \\
    & & K_n z_n &= q^2 z_n, & K_n^* z_k &= z_k, \\
    & & K_n^* z_n &= q^{-2} z_n^*, & K_n^* z_k &= -q^{-2} z_n^* z_k.
\end{align*}
\]

If \( n = 1 \), we recover the relations of the quantum disc. For \( n > 1 \), we obtain by omitting the elements \( K_n, K_n^{-1}, E_n, \) and \( F_n \) a \( \mathcal{U}_q(\text{su}_{n}) \)-action on \( \mathcal{O}_q(\text{Mat}_{n,1}) \) such that \( \mathcal{O}_q(\text{Mat}_{n,1}) \) becomes a \( \mathcal{U}_q(\text{su}_{n}) \)-module *-algebra. Notice that, by Equation (12) and (10), it is sufficient to describe the action on generators.

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4.2 Representations of the *-algebra $\mathcal{O}_q(\text{Mat}_{n,1})$

Irreducible *-representations of the twisted canonical commutation relations have been classified in [1] under the condition that $1 - Q_1$ is essentially self-adjoint. In this subsection, we study admissible *-representations of the twisted canonical commutation relations without requiring the representation to be irreducible.

Remind our notational conventions from Subsection 2.2 regarding direct sums of a Hilbert space $\mathcal{K}$. Let $A$ be a self-adjoint operator on $\mathcal{K}$ such that $\sigma(A) \subseteq (q^2, 1]$. Then the expression $\mu_j(A)$, $j \in \mathbb{Z}$, stands for the operator $\mu_j(A) = (1 + q^{-2j}A)^{1/2}$. We shall also abbreviate $\lambda_j = (1 - q^{2j})^{1/2}$ and $\beta_j = (q^{-2j} - 1)^{1/2}$ for $j \in \mathbb{N}_0$.

**Proposition 4.1** Assume that $m, k, l \in \mathbb{N}_0$ such that $m + l + k = n$. Let $\mathcal{K}$ denote a Hilbert space. Set

$$\mathcal{H} := \bigoplus_{i_1,...,i_{n-1}} \mathcal{H}_{i_1,...,i_{n-1}} = \bigoplus_{i_k=-\infty}^\infty \bigoplus_{i_{k-1},...,i_1=1}^{\infty} \mathcal{H}_{i_k,...,i_1},$$

where $\mathcal{H}_{i_n,...,i_1} = \mathcal{K}$, and

$$D := \text{Lin}\{\eta_{i_n,...,i_1} : \eta \in \mathcal{K}, i_n,...,i_{n-m+1} \in \mathbb{N}_0, i_k \in \mathbb{Z}, i_{k-1},...,i_1 \in \mathbb{N}\}.$$  

(For $l > 0$, we retain the notation $\eta_{i_n,...,i_1}$ and do not write $\eta_{i_n,...,i_{n-1}}, i_{n-1},...,i_1$.) Consider the operators $z_1, ..., z_n$ acting on $D$ by

$$(m,0,k): \quad
\begin{align*}
 z_j \eta_{i_n,...,i_1} &= q^{i_j+1+...+i_1} \lambda_{j+1} \eta_{i_n,...,i_j+1,...,i_1}, \quad \text{if } k < j \leq n, \\
 z_k \eta_{i_n,...,i_1} &= q^{i_k+1+...+i_1} \mu_{k-1}(A^2) \eta_{i_n,...,i_{k-1},...,i_1}, \\
 z_j \eta_{i_n,...,i_1} &= q^{-(i_j+1+...+i_k)+(i_k+1+...+i_1)} \beta_{k-1} A \eta_{i_n,...,i_{k-1},...,i_1}, \quad \text{if } 1 \leq j < k,
\end{align*}$$

and, for $l > 0$,

$$(m,l,k): \quad
\begin{align*}
 z_j \eta_{i_n,...,i_1} &= q^{i_j+1+...+i_1} \lambda_{j+1} \eta_{i_n,...,i_j+1,...,i_1}, \quad \text{if } n-m < j \leq n, \\
 z_{n-m} \eta_{i_n,...,i_1} &= q^{i_{n-m+1}+...+i_1} \eta_{i_n,...,i_{n-m-1},...,i_1}, \\
 z_j &= 0, \quad \text{if } k < j < n-m, \\
 z_k \eta_{i_n,...,i_1} &= q^{i_k+1+...+i_1} A \eta_{i_n,...,i_{k-1},...,i_1}, \\
 z_j \eta_{i_n,...,i_1} &= q^{-(i_j+1+...+i_k)+(i_k+1+...+i_1)} \beta_{k-1} A \eta_{i_n,...,i_{k-1},...,i_1}, \quad \text{if } 1 \leq j < k.
\end{align*}$$

(If $k = 0$, then the indices $i_1,...,i_k$ are omitted; similarly, if $m = 0$, then the indices $i_{n-m+1},...,i_n$ are omitted.) In both series, $A$ denotes a self-adjoint operator acting on the Hilbert space $\mathcal{K}$ such that $\sigma(A) \subseteq (q, 1]$. In the series $(m,l,k)$, $l > 0$, $v$ is a unitary operator on $\mathcal{K}$ such that $Av = vA$.

Then the operators $z_1, ..., z_n$ define a *-representation of $\mathcal{O}_q(\text{Mat}_{n,1})$, where the action of $z_j^*$, $j = 1, ..., n$, is obtained by restricting the adjoint of $z_j$ to $D$. Representations belonging to different series $(m,k,l)$ or to different operators $A$ and $v$ are not unitarily equivalent. A representation of this series is irreducible if and only if $\mathcal{K} = \mathbb{C}$. In this case, $v$ is a complex number of modulus one and $A \in (q, 1]$. Only the representations $(m,0,k)$ are faithful.
Proof. Direct calculations show that the formulas given in Proposition 4.1 define a *-representation of $O_q(M_n)$. Clearly, if a *-representation of these series is irreducible, then $A$ and $v$ must be complex numbers and $K = \mathbb{C}$. The converse statement was shown in [5]. That the representations $(m, 0, k)$ are faithful is proved by showing that for each $x \in O_q(M_n)$, $x \neq 0$, there exist $\eta_{i_n}, \ldots, \eta_{i_1} \in \mathcal{H}$ such that the matrix element $\langle \eta_{i_n}, \ldots, \eta_{i_1} | x \eta_{j_n}, \ldots, \eta_{j_1} \rangle$ is non-zero. The vectors can easily be found by writing $x$ in the standard form and observing that $z_j, z_j'$ act as shift operators. We omit the details. The other assertions of the proposition are obvious. □

Remarks. The operators $Q_j$ are given by

\begin{align*}
Q_j \eta_{i_n, \ldots, i_1} &= q^{2(i_j + \cdots + i_n)} \eta_{i_n, \ldots, i_1}, \quad \text{if } n - m < j \leq n, \quad (60) \\
Q_j &\equiv 0, \quad \text{if } k < j \leq n - m, \quad (61) \\
Q_j \eta_{i_n, \ldots, i_1} &= -q^{-2(i_j + \cdots + i_n) + 2(i_{n-m+1} + \cdots + i_n)} A^2 \eta_{i_n, \ldots, i_1}, \quad \text{if } 1 \leq j \leq k. \quad (62)
\end{align*}

The numbers $m, l, k \in \mathbb{N}_0$ correspond to the signs of the operators $Q_j$, that is, we have $Q_n \geq \ldots \geq Q_{n-m+1} > 0$ if $m > 0$, $Q_{n-m} = \ldots = Q_{k+1} = 0$ if $l > 0$, and $0 > Q_k \geq \ldots \geq Q_1$ if $k > 0$. The only bounded representations are the series $(m, l, 0)$.

We now give a constructive method for finding “admissible” *-representations of $O_q(\mathbb{C}^n)$. In view of [54–58], the assumptions on admissible *-representations of the *-algebra $O_q(\mathbb{C}^n)$ will include the following two conditions: First, the closures of the operators $Q_k$, $k = 1, \ldots, n$, are self-adjoint and strongly commute. Second, $\varphi(Q_k)z_j \subset z_j \varphi(Q_k)$, $j < k$, and $\varphi(Q_k)z_j \subset z_j \varphi(q^2Q_k)$, $j \geq k$, for all complex functions $\varphi$ which are measurable with respect to the spectral measure of $Q_k$ and which have at most polynomial growth. In the course of the argumentation, we shall impose further regularity conditions on the operators. The outcome will precisely be the series of Proposition 4.1. So, if one takes as admissible *-representations those which satisfy all regularity conditions, then Proposition 4.1 states that any admissible *-representation of $O_q(\mathbb{C}^n)$ is a direct sum of *-representation which are determined by the formulas of the series $(m, l, k)$, $m + l + k = n$. The argumentation is based on a reduction procedure.

Observe that $z_n$ satisfies the relation of the quantum disc $O_q(\bar{U})$. The “admissible” representations of this relation are given by Lemma 2.3 and correspond to the cases $(1, 0, 0), (0, 1, 0)$, and $(0, 0, 1)$.

Now let $0 < m < n$. Suppose that we are given a *-representation of $O_q(\mathbb{C}^n)$ such that the operators $z_n, \ldots, z_{n-m+1}$ act on $\mathcal{H} = \bigoplus_{i_n, \ldots, i_{n-m+1} = 0}^\infty \mathcal{H}_{i_n, \ldots, i_{n-m+1}}$ by the formulas of the series $(m, 0, 0)$, where all $\mathcal{H}_{i_n, \ldots, i_{n-m+1}}$ are equal to a given Hilbert space, say $\mathcal{H}_{0,0,0}$. Fix $f_n, \ldots, f_{n-m+1} \in \mathbb{N}_0$. Since the representation is assumed to be admissible, it follows from $z_{n-m}Q_j = Q_j z_{n-m}, n - m < j \leq n$, and 60 that the operator $z_{n-m}$ maps the Hilbert spaces

\begin{align*}
\mathcal{H}(f_n) &:= \text{Lin} \{ \eta_{i_n} \in \mathcal{H} : i_n = f_n \}, \\
\mathcal{H}(f_n, f_{n-1}) &:= \text{Lin} \{ \eta_{i_n, i_1} \in \mathcal{H} : i_n = f_n, \ i_n + i_{n-1} = f_{n-1} \}, \ldots, \\
\mathcal{H}(f_n, \ldots, f_{n-m+1}) &:= \text{Lin} \{ \eta_{i_n, i_1} \in \mathcal{H} : i_n = f_n, \ i_n + \ldots + i_{n-m+1} = f_{n-m+1} \}
\end{align*}
into itself. But the \( m \) equations \( i_{n} = f_{n}, \ldots, i_{n} + \ldots + i_{n+m+1} = f_{n+m+1} \) determine uniquely the numbers \( i_{n}, \ldots, i_{n+m+1} \), therefore \( z_{n-m} \) maps each \( \mathcal{H}_{i_{n}, \ldots, i_{n+m+1}} \) into itself. Write

\[
z_{n-m} \xi_{n, \ldots, i_{n+m+1}} = Z_{i_{n}, \ldots, i_{n+m+1}} \eta_{n, \ldots, i_{n+m+1}}
\]

with operators \( Z_{i_{n}, \ldots, i_{n+m+1}} \) acting on \( \mathcal{H}_{0, \ldots, 0} \). Applying \( z_{n-m} \xi_{j} = q \xi_{n-m}, n-m < j \leq n, \) to vectors \( \eta_{n, \ldots, i_{n+m+1}} \) gives

\[
q^{i_{j+1} + \ldots + i_{n}} \lambda_{j+1} Z_{i_{n}, \ldots, i_{j+1} \ldots i_{n+m+1}} \eta_{n, \ldots, i_{j+1} \ldots i_{n+m+1}} =
\]

\[
q^{i_{j+1} + \ldots + i_{n}} \lambda_{j+1} Z_{i_{n}, \ldots, i_{j+1} \ldots i_{n+m+1}} \eta_{n, \ldots, i_{j+1} \ldots i_{n+m+1}},
\]

hence \( Z_{i_{n}, \ldots, i_{j+1} \ldots i_{n+m+1}} = q Z_{i_{n}, \ldots, i_{j+1} \ldots i_{n+m+1}} \). From this, we conclude

\[
Z_{i_{n}, \ldots, i_{j+1} \ldots i_{n+m+1}} = q^{i_{j+1} + \ldots + i_{n}} Z_{0, \ldots, 0}.
\]

On \( \mathcal{H}_{0, \ldots, 0} \), the relation \( z_{n-m} \xi_{n-m} = q z_{n-m} z_{n-m} = (1 - q^{2}) Q_{n-m+1} \) gives

\[
Z_{0, \ldots, 0} Z_{0, \ldots, 0} - q^{2} Z_{0, \ldots, 0} Z_{0, \ldots, 0} = 1 - q^{2}.
\]

Here and subsequently, we suppose that operators satisfying this relation also satisfy the assumptions of Lemma 2.13. By a slight reformulation of Lemma 2.13, we get the following three series of *-representations of the last equation:

i. \( \mathcal{H}_{0, \ldots, 0} = \bigoplus_{n=-\infty}^{\infty} \mathcal{H}_{0, \ldots, 0, i_{n-m}} \), \( \mathcal{H}_{0, \ldots, 0, i_{n-m}} = \mathcal{H}_{0, \ldots, 0, 0} \),

\[
Z_{0, \ldots, 0} \xi_{i_{n-m}} = \lambda_{i_{n-m}+1} \xi_{i_{n-m}+1};
\]

ii. \( Z_{0, \ldots, 0} = v \), where \( v \) is a unitary operator on \( \mathcal{H}_{0, \ldots, 0} \);

iii. \( \mathcal{H}_{0, \ldots, 0} = \bigoplus_{n=-\infty}^{\infty} \mathcal{H}_{0, \ldots, 0, i_{n-m}} \), \( \mathcal{H}_{0, \ldots, 0, i_{n-m}} = \mathcal{H}_{0, \ldots, 0, 0} \),

\[
Z_{0, \ldots, 0} \xi_{i_{n-m}} = \mu_{i_{n-m}+1} (A^{2}) \xi_{i_{n-m}+1};
\]

where \( A \) is a self-adjoint operator on \( \mathcal{H}_{0, \ldots, 0, 0} \) such that \( \sigma(A) \subseteq (q^{2}, 1] \).

Inserting these formulas into the representation \( (m, 0, 0) \) shows that the cases (i), (ii), and (iii) correspond to a representation of the operators \( z_{n}, \ldots, z_{n-m} \) of the series \( (m, 1, 0) \), \( (m, 0, 1) \), and \( (m, 0, 1) \), respectively.

Next, let \( m, l \in \mathbb{N} \) such that \( m+1 < n \) and \( l > 0 \). Set \( k = n-m-l \). Suppose that the operators \( z_{n}, \ldots, z_{k+1} \) act on the Hilbert space \( \mathcal{H} = \bigoplus_{n=-\infty}^{\infty} \mathcal{H}_{i_{n}, \ldots, i_{n+m+1}} \) by the formulas of the series \( (m, l, 0) \). As in the case \( (m, 0, 0) \), we conclude from \( z_{k} Q_{j} = Q_{j} z_{k}, k < j \leq n \), that \( z_{k} \) acts on \( \mathcal{H}_{i_{n}, \ldots, i_{n-m+1}} \) by

\[
z_{k} \xi_{i_{n}, \ldots, i_{n-m+1}} = q^{i_{n-m+1} + \ldots + i_{n}} Z_{0, \ldots, 0} \xi_{i_{n}, \ldots, i_{n-m+1}},
\]

where \( Z_{0, \ldots, 0} \) is an operator acting on \( \mathcal{H}_{0, \ldots, 0} \). As \( Q_{k} = Q_{k+1} - z_{k} z_{k}^{*} \) and \( Q_{k+1} = 0 \) by (55) and (61), we have \( Q_{k} \leq 0 \). The assumptions on admissible *-representations imply that \( \ker Q_{k} \) is reducing. Thus we can consider the cases \( Q_{k} = 0 \) and \( Q_{k} < 0 \) separately.
First let $Q_k = 0$. Then $z_k z_k^* = Q_{k+1} - Q_k = 0$, hence $z_k = z_k^* = 0$. Consequently, we have obtained a representation of the type $(m, l + 1, 0)$.

Now assume that $Q_k < 0$. Then, by (5.4), $z_k^* z_k - q^2 z_k z_k^* = 0$ and $z_k z_k - z_k z_k^* = (1 - q^2)Q_k$. Inserting the first equation into the second one gives $z_k z_k^* = -Q_k$, hence $|z_k^*| = |Q_k|^{1/2}$. Evaluating $z_k z_k - q^2 z_k z_k^*$ on $H_{0...0}$ yields $Z_{0...0}^* Z_{0...0} - q^2 Z_{0...0} Z_{0...0} = 0$. Let $Z_{0...0} = U|Z_{0...0}|$ be the polar decomposition of the closed operator $Z_{0...0}$. Since $\ker z_k^* = \ker z_k z_k^* = \ker Q_k = \{0\}$ and $\ker z_k = \ker z_k z_k^* = \ker z_k z_k^* = \{0\}$, $U$ is unitary. From $Z_{0...0}^* Z_{0...0} - q^2 Z_{0...0} Z_{0...0} = 0$, it follows that $|Z_{0...0}|^2 = q^2 U|Z_{0...0}|^2 U^*$ and therefore $|Z_{0...0}|U = qU|Z_{0...0}|$. The representations of this relation are given by Lemma 2.1. It states that $H_{0...0} = \oplus_{i_k = -\infty}^{\infty} H_{0...0,i_k}$, $H_{0...0,i_k} = H_{0...0,0}$, and the operators $U$ and $|Z_{0...0}|$ act as

$$U \zeta_{i_k} = \zeta_{i_k-1}, \quad |Z_{0...0}| \zeta_{i_k} = q^{-i_k} A \zeta_{i_k},$$

where $A$ is a self-adjoint operator on $H_{0...0,0}$ such that $\sigma(A) \subseteq (q, 1)$.

We have not yet considered the relations $z_k z_{n-m} = q z_{n-m} z_k$ and $z_k^* z_{n-m} = q z_{n-m}^* z_k$. On $H_{0...0}$, this leads to $Z_{0...0} v = q v Z_{0...0}$ and $Z_{0...0}^* v = q v Z_{0...0}^*$. Thus $Z_{0...0}^* Z_{0...0} v = q^2 Z_{0...0} Z_{0...0} v$, or, since $v$ is unitary, $|Z_{0...0}|v = |Z_{0...0}|$. This implies $U^* v |Z_{0...0}| = |Z_{0...0}| U^* v$, hence the unitary operator $U^* v$ leaves each space $H_{0...0,i_k}$ invariant. Therefore there exist unitary operators $v_{i_k}$ on $H_{0...0,i_k}$ such that $U^* v_{i_k} = v_{i_k} \zeta_{i_k}$, hence $v_{i_k} = v_{i_k} \zeta_{i_k-1}$. From

$$v U |Z_{0...0}| = v Z_{0...0} = q^{-1} Z_{0...0} v = q^{-1} U|Z_{0...0}|v = U v |Z_{0...0}|,$$

we conclude $v U = U v$ since $\ker |Z_{0...0}| = \{0\}$. This implies

$$v_{i_k} \zeta_{i_k} = U v_{i_k} \zeta_{i_k} = v U \zeta_{i_k} = v_{i_k+1} \zeta_{i_k},$$

thus $v_{i_k+1} = v_{i_k}$, consequently $v_{i_k} = v_0$ for all $i_k \in Z$. Evaluating $|Z_{0...0}|v = q v |Z_{0...0}|$ for vectors $\zeta_{i_k}$ shows $v_0 A = A v_0$. This determines the actions of $z_n, \ldots, z_k$, completely. Comparing the result with the action of the operators $z_n, \ldots, z_k$ from the proposition shows that we have obtained a representation of the type $(m, l, 1)$.

We finally turn to a representation of the type $(m, l, k)$, where $m + l + k < n$ and $k > 0$. Set $s := n - (m + l + k)$. Suppose that the operators $z_n, \ldots, z_{s+1}$ act on a Hilbert space $H = \oplus_{i_k = -\infty}^{\infty} H_{0...i_k+1} \oplus_{i_k = -\infty}^{\infty} \oplus_{i_k = -\infty}^{\infty} H_{i_k+1...i_{s+1}}$ by the formulas given in the proposition. Similarly to the case $(m, 0, 0)$, we conclude from $z_s Q_j = q z_s, s < j \leq n$, that $z_s$ maps each $H_{i_k+1...i_{s+1}}$ into itself. Write

$$z_s \eta_{i_k+1...i_{s+1}} = Z_{i_k+1...i_{s+1}} \eta_{i_k+1...i_{s+1}}.$$  

On applying $z_s Q_{s+1} = Q_{s+1} z_s$ to a vector $\eta_{i_k+1...i_{s+1}}$, it follows from Equation (6.2) that $Z_{i_k+1...i_{s+1}} A^2 = A^2 Z_{i_k+1...i_{s+1}}$. Thus $Z_{i_k+1...i_{s+1}, i_{s+1}} A^2 = q z_s z_s, s > j$, on vectors $\eta_{n...i_{s+1}},$ we see that $Z_{i_k+1...i_{s+1}} A^2 = q Z_{i_k+1...i_{s+1}} A^2$ because the representation is assumed to be admissible. By using this relations and evaluating $z_s z_j = q z_s z_j, s > j$, on vectors $\eta_{n...i_{s+1}}$, we see that $Z_{i_k+1...i_{s+1}} = q Z_{i_k+1...i_{s+1}}$ and $Z_{i_k+1...i_{s+1}} = q Z_{i_k+1...i_{s+1}}$ if $s < j \leq k$ and $n - m < j \leq n$, respectively. Thus we can write

$$Z_{i_k+1...i_{s+1}} = q^{-i_k+1...i_{s+1}+i_{n-m+1}+...+i_n} Z_{0...0}.$$

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When \( l \neq 0 \), this formula together with \( z_0 z_{n-m} = q z_{n-m} z_0 \) implies \( Z_{0..0} v = v Z_{0..0} \).

The relation \( z_0^* z_0 - q^2 z_0 z_0^* = (1 - q^2)qz_{s+1} \) reads on \( \mathcal{H}_{0..0} \) as
\[
Z_{0..0} Z_{0..0} - q^2 Z_{0..0} Z_{0..0} = -(1 - q^2)A^2. 
\]

Setting \( X_0 := Z_{0..0} A^{-1} = A^{-1} Z_{0..0} \) and replacing \( Z_{0..0} \) by \( X_0 A \), we obtain
\[
X_0^* X_0 - q^2 X_0 X_0^* = -(1 - q^2). 
\]

By Lemma 23 the admissible \(*\)-representations of this relation can be described in the following way:
\[
\mathcal{H}_{0..0} = \oplus_{i=1}^{\infty} \mathcal{H}_{0..0..i}, \quad \mathcal{H}_{0..0..i} = \mathcal{H}_{0..0..i+1}, \quad \text{and} \quad X_0 \text{ acts by} \\
X_0 \zeta_i = \beta_{i-1} \zeta_{i-1}.
\]

Since \( X_0^* X_0 A = AX_0^* X_0 \), \( A \) leaves each Hilbert space \( \mathcal{H}_{0..0..i} \) invariant. From \( X_0 A = AX_0 \), it follows that the restrictions of \( A \) to \( \mathcal{H}_{0..0..i} \) and \( \mathcal{H}_{0..0..i-1} \) coincide, hence \( A \) acts on \( \mathcal{H}_{0..0..l} \) by \( A \zeta_l = A \zeta_{l-1} \), where \( A \) is a self-adjoint operator on \( \mathcal{H}_{0..0..l} \) such that \( \sigma(A) \subset (q, 1] \). Inserting the expression for \( X_0 \) into \( Z_{0..0} = X_0 A \) shows that \( \zeta_l \) acts on \( \mathcal{H}_{i_{n-1} i_{n-1+1} i_{n-1}} \) by
\[
z_0 \zeta_{i_{n-1} i_{n-1+1} i_{n-1}} = q^{-(i_{n-1} + i_{n-1} + \ldots + i_k)} \beta_{i-1} A \zeta_{i_{n-1} i_{n-1+1} i_{n-1}}.
\]

Recall that \( v Z_{0..0} = Z_{0..0} v \) when \( l \neq 0 \). Moreover, \( v Z_{0..0}^* = Z_{0..0} v \) since \( X_0^* v = v X_0 \) and \( X_0^* X_0 v = v X_0^* X_0 \). Therefore it follows by the same reasoning as for \( A \) that \( v \) acts on \( \mathcal{H}_{0..0..l} \) by \( v \zeta_l = v_0 \zeta_l \), where \( v_0 \) is a unitary operator on \( \mathcal{H}_{0..0..l} \). From \( v A = A v \), we conclude \( v_0 A_0 = A_0 v_0 \). Inserting \( A_0 \) and \( v_0 \) (when \( l \neq 0 \)) into the expressions for \( \zeta_n, \ldots, \zeta_s \), we obtain for both \( l = 0 \) and \( l > 0 \) a representation of the list \((m, l, k + 1)\). This completes the reduction procedure.

### 4.3 Invariant integration on the quantum \((n, 1)\)-matrix ball

Throughout this subsection, we assume that we are given an admissible \(*\)-representation \( \pi : \mathcal{O}_q(\text{Mat}_{n,1}) \to \mathcal{L}^+ (D) \) of the series \((m, 0, k)\) such that \(|Q_j|^{-1/2} \in \mathcal{L}^+ (D)\) for all \( j = 1, \ldots, n \). Set \( \epsilon_j = 1 \) if \( j > k \) and \( \epsilon_j = -1 \) if \( j \leq k \). Then we have \( Q_j = \epsilon_j |Q_j| \).

To develop an invariant integration theory on the quantum \((n, 1)\)-matrix ball, we proceed as in Subsection 3.1. The crucial step is to find an operator expansion of the action. To begin, we prove some useful operator relations.

**Lemma 4.2** Define
\[
l < n : \quad \rho_l = |Q_l|^{1/2} |Q_{l+1}|^{-1} |Q_{l+2}|^{1/2}, \quad \rho_n = |Q_1|^{1/2} |Q_n|^{1/2}, \quad (63) \\
A_l = -q^{-5/2} \lambda^{-1} Q_{l+1}^{-1} z_{l+1}^* z_l, \quad A_n = q^{-1/2} \lambda^{-1} z_n, \quad (64) \\
B_l = \rho_l^{-1} A_l^*, \quad B_n = -\rho_n^{-1} A_n^* \quad (65)
\]

The operators \( \rho_l, A_l, \) and \( B_l \) satisfy the following commutation relations:
\[
\rho_i \rho_j = \rho_j \rho_i, \quad \rho_j^{-1} \rho_j = \rho_j \rho_j^{-1} = 1, \quad \rho_l A_j = q^{a_{ij}} A_j \rho_l, \quad \rho_l B_j = q^{-a_{ij}} B_j \rho_l, \quad (66)
\]

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\[ A_i A_j - A_j A_i = 0, \quad i \neq j \pm 1, \quad A_i^2 A_{j \pm 1} - (q + q^{-1}) A_j A_{j \pm 1} A_i + A_{j \pm 1} A_i^2 = 0, \quad (67) \]

\[ B_i B_j - B_j B_i = 0, \quad i \neq j \pm 1, \quad B_i^2 B_{j \pm 1} - (q + q^{-1}) B_j B_{j \pm 1} B_i + B_{j \pm 1} B_i^2 = 0, \quad (68) \]

\[ A_i B_j - A_j B_i = 0, \quad i \neq j, \quad A_i B_j - B_j A_i = \lambda^{-1}(\epsilon_{j+2} \epsilon_j \rho_j - \rho_j^{-1}) , \quad j < n, \quad (69) \]

\[ A_n B_n - B_n A_n = -\lambda^{-1} \rho_n^{-1}, \quad (70) \]

where \( (a_{ij})_{i,j=1}^n \) denotes the Cartan matrix of \( \text{sl}(n+1, \mathbb{C}) \).

**Proof.** Since the representation is assumed to be admissible, we conclude from \( (56) \) and \( (57) \) that

\[ |Q|^{1/2} z_j = z_j |Q|^{1/2}, \quad |Q|^{1/2} z_j^* = z_j^* |Q|^{1/2}, \quad j < l, \quad (71) \]

\[ |Q|^{1/2} z_j = q z_j |Q|^{1/2}, \quad |Q|^{1/2} z_j^* = q^{-1} z_j^* |Q|^{1/2}, \quad j \geq l. \quad (72) \]

Now \( (56) \) follows immediately from \( (58) \), \( (71) \), and \( (72) \). The first equations of \( (67) - (69) \) are easily shown by repeated application of the commutation rules in \( O_q(\text{Mat}_{n,1}) \) and Equations \( (11) \) and \( (12) \). We continue with the second equation of \( (67) \) and \( (70) \). Using \( (58) \), we compute

\[ \begin{align*}
A_i B_l - B_l A_i & = q^{-5} \lambda^{-2} \rho_l^{-1} (q^2 Q_{l+1}^{-1} z_l^* z_l z_{l+1} Q_{l+1}^{-1} z_l^* z_{l+1} Q_{l+1}^{-1} - z_l^* z_{l+1} Q_{l+1}^{-2} z_l^* z_{l+1} z_l) \\
& = q^{-1} \lambda^{-2} \rho_l^{-1} Q_{l+1}^{-2} ((Q_{l+2} - q^2 Q_{l+1}))(Q_{l+1} - Q_l) \\
& \quad - (Q_{l+2} - Q_{l+1})(Q_{l+1} - q^2 Q_l) \\
& = \lambda^{-1} \rho_l^{-1} (Q_{l+2} Q_{l+1}^{-2} Q_l - 1) = \lambda^{-1}(\epsilon_{l+2} \epsilon_l \rho_l - \rho_l^{-1}), \\
A_n B_n - B_n A_n & = -q^{-1} \lambda^{-1} \rho_n^{-1}(q^2 z_n z_n^* - z_n^* z_n) = -\lambda^{-1} \rho_n^{-1}.
\end{align*} \]

Next, we claim that

\[ \begin{align*}
A_{l-1} A_l & = q A_l A_{l-1} - q^{-3} \lambda^{-1} Q_l^{-1} z_{l+1} z_l - 1 < l < n, \quad (73) \\
A_{n-1} A_n & = q A_n A_{n-1} + q^{-1} \lambda^{-1} Q_n^{-1} z_{n-1}.
\end{align*} \quad (74) \]

Indeed, inserting the definition of \( A_l \) and applying \( (49) \), \( (50) \), \( (54) \), \( (56) \), and \( (57) \), one obtains

\[ \begin{align*}
A_{l-1} A_l & = q^{-5} \lambda^{-2} Q_l^{-1} z_{l+1} z_l - 1 Q_l^{-1} z_{l+1} z_l \\
& = q^{-6} \lambda^{-2} Q_{l+1} Q_l^{-1} z_{l+1} (q^2 z_l^* - (1 - q^2) Q_l z_{l+1} z_{l-1} \\
& = q A_l A_{l-1} - q^{-3} \lambda^{-1} Q_l^{-1} z_{l+1} z_{l-1}.
\end{align*} \]

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which proves (73). Equation (74) is proved similarly. Let \( l < n \). Multiplying (75) by \(-q^{-1}A_l\) from the left and by \( A_l\) from the right and summing both results yields (77) with the minus sign since \( A_lQ_l^{-1}z_{l+1}^1z_{l-1}^1 = qQ_l^{-1}z_{l+1}^1z_{l-1}^1A_l\). Equation (77) with the plus sign is obtained similarly by computing \( A_{l-1}^{-1}Q_l^{-1}z_{l+1}^1z_{l-1}^1A_{l-1}\) using \( A_{l-1}Q_l^{-1}z_{l+1}^1z_{l-1}^1 = q^{-1}Q_l^{-1}z_{l+1}^1z_{l-1}^1A_{l-1}\) and replacing \( l \) by \( l+1 \) for \( l < n \). The same steps applied to (74) yield (77) for \( l = n \) and \( l+1 = n \) since also \( A_nQ_n^{-1}z_{n+1}^1Q_n^{-1}z_{n-1}^1A_n \) and \( A_{n-1}Q_n^{-1}z_{n+1}^1Q_n^{-1}z_{n-1}^1A_{n-1}\). The second equations of (68) follow from the second equations of (67) by applying the involution and multiplying by \( \rho_j^{-2}Q_q^{\pm 1}\).

\[ \]

\textbf{Remark.} By (70), the operators \( A_l, B_l, \) and \( \rho_l \) do not satisfy the defining relations of \( U_q(\mathfrak{su}_{n,1}) \). If \( n > 1 \), then we get only for the series \((n,0,0)\) a \( * \)-representation of \( U_q(\mathfrak{su}_{n}) \) by assigning \( K_l \) to \( \rho_j, E_j \) to \( A_j, \) and \( F_j \) to \( B_j, j < n \).

To see this, observe that we must have \( \epsilon_{j+2} = \epsilon_j \) by (59). But \( \epsilon_{n+1} = 1 \) since \( Q_{n+1} = 1 \), and \( \epsilon_n = 1 \) since \( \epsilon_n|Q_n| = Q_{n-1} + z_{n-1}^1z_{n-1}^1 > 0 \) by (56) (cf. the remarks after Proposition 3.1), so \( \epsilon_n = \ldots = \epsilon_1 = 1 \).

Although Equations (66)–(70) do not yield a representation of \( U_q(\mathfrak{su}_{n,1}) \), the analogy to (6)–(9) is obvious, so it is natural to try to define an operator expansion of the action by imitating the adjoint action. That this can be done is the assertion of Lemma 4.3 (after Proposition 4.1), so

\[ \]

\textbf{Lemma 4.3} With the operators \( \rho_l, A_l, \) and \( B_l \) defined in Lemma 4.2 set

\[ K_j \triangleright f = \rho_j f \rho_j^{-1}, \quad K_j^{-1} \triangleright f = \rho_j^{-1} f \rho_j, \]

\[ E_j \triangleright f = A_j f - \rho_j f \rho_j^{-1} A_j, \]

\[ F_j \triangleright f = B_j \rho_j - q^2 f \rho_j B_j \]

for \( j = 1, \ldots, n \). Then Equations (75)–(77) applied to \( f \in \mathcal{O}_q(\mathfrak{Mat}_{n,1}) \) define an operator expansion of the action \( \triangleright \) on \( \mathcal{O}_q(\mathfrak{Mat}_{n,1}) \). The same formulas applied to \( f \in \mathcal{L}^+(D) \) turn the \( \mathcal{O}_q^* \)-algebra \( \mathcal{L}^+(D) \) into a \( U_q(\mathfrak{su}_{n,1}) \)-module \( * \)-algebra.

\textbf{Proof.} The lemma is proved by direct verifications. We start by showing that \( \mathcal{L}^+(D) \) with the \( U_q(\mathfrak{su}_{n,1}) \)-action defined by (75)–(77) becomes a \( U_q(\mathfrak{su}_{n,1}) \)-module \( * \)-algebra. That the action satisfies (4)–(4) is readily seen if we replace in the proof of Lemma 3.1 \( y_j^\pm 1 \) by \( \rho_j^\pm 1, A \) by \( A_j, \) and \( B \) by \( B_j \). By using Lemma 4.2 it is easy to check that the action is consistent with (6) and the first relations of (7)–(9). For example, (66) and (67) give

\[ (K_i E_j) \triangleright f = \rho_i(A_j f - \rho_j f \rho_j^{-1} A_j) \rho_i^{-1} = q^{\alpha_{ij}} (A_j \rho_i f \rho_i^{-1} - \rho_j \rho_i f \rho_i^{-1} \rho_j^{-1} A_j) = (q^{\alpha_{ij}} E_j K_i) \triangleright f, \]

\[ (E_i E_j) \triangleright f = A_j (A_i f - \rho_j f \rho_j^{-1} A_j) - \rho_i (A_i f - \rho_j f \rho_j^{-1} A_j) \rho_i^{-1} A_i = A_j (A_i f - \rho_i f \rho_i^{-1} A_i) - \rho_j (A_i f - \rho_i f \rho_i^{-1} A_i) \rho_j^{-1} A_j = (E_i E_j) \triangleright f \]

\[ \]

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for all \( f \in \mathcal{L}^+(D) \), \( i,j,l = 1, \ldots, n \), \( l \neq j \pm 1 \). As in the proof of Lemma 3.1, we have

\[
(E_j F_j - F_j E_j)\cdot f = A_j B_j f \rho_j + \rho_j f B_j A_j - B_j A_j f \rho_j - \rho_j f A_j B_j = (A_j B_j - B_j A_j) f \rho_j - \rho_j f (A_j B_j - B_j A_j).
\]

Inserting (69) if \( j < n \) and (70) if \( j = n \) shows that the action is consistent with the second equation of (7).

We continue with the second equation of (7). A straightforward calculation shows that

\[
(E_j E_{j+1})\cdot f = A_j^2 A_{j+1} f - \rho_j^2 \rho_{j+1} f \rho_j^{-2} \rho_{j+1}^{-1} A_j A_{j+1}^2
\]

\[
- (q + q^{-1}) A_j A_{j+1} \rho_j f \rho_j^{-1} A_j A_{j+1} + A_j \rho_j^2 f \rho_j^{-2} A_j
\]

\[
- A_j^2 \rho_j \rho_{j+1}^2 f \rho_j^{-1} A_{j+1} + (q + q^{-1}) A_j \rho_j \rho_{j+1} f \rho_j^{-1} \rho_{j+1}^{-1} A_j A_{j+1},
\]

\[
(E_j E_{j+1} E_j)\cdot f = A_j A_{j+1} A_j f - \rho_j^2 \rho_{j+1} f \rho_j^{-2} \rho_{j+1}^{-1} A_j A_{j+1} A_j
\]

\[
- q A_j A_{j+1} \rho_j f \rho_j^{-1} A_j - A_j A_{j+1} \rho_j f \rho_j^{-1} A_j
\]

\[
+ q A_j \rho_j^2 f \rho_j^{-2} A_j^2 - q^{-1} A_j^2 \rho_j f \rho_j^{-1} \rho_{j+1}^{-1} A_{j+1}
\]

\[
+ A_j \rho_j \rho_{j+1} \rho_j^2 f \rho_j^{-1} \rho_{j+1}^{-1} A_j A_{j+1} + q^{-1} A_j \rho_j \rho_{j+1} f \rho_j^{-1} \rho_{j+1}^{-1} A_j A_{j+1},
\]

where we repeatedly used (65). Taking the sum (78) - (q + q^{-1}) \cdot (79) + (80) gives 0 since the sums over the first and the second summands vanish by (67) and the other summands cancel. The last result implies also the second relation of (8) since \( X \cdot f = (S(X)^* \cdot f)^* \) for all \( X \in U_q(sl(\mathfrak{n}^+)) \) and \( S(F_j)^* = (-1)^{\rho_j^2 q^2} E_j \).

It remains to prove that (75) - (77) define an operator expansion of the action. That (75) yields the action of \( R^{\pm 1} \) on \( z \) and \( z^* \) is easily verified by using (71) and (72). Let \( l < n \). Applying (49), (50), (71), and (72), we get \( \rho_l \rho_{l-1} A_l = A_{l+1} A_l \) and \( \rho_l \rho_{l-1} = A_{l+1} A_l \) whenever \( j \notin \{l, l+1\} \), hence \( E_l z_j = E_l z_j^* = 0 \). Similarly, \( \rho_l \rho_{l-1} A_l = q z_l A_l = A_l z_l \) and \( \rho_l \rho_{l-1} A_l = q^{-1} z_l A_l = A_l z_l^* \), so \( E_l z_l = E_l z_l^* = 0 \). Equation (76) applied to \( z_{l+1} \) and \( z_l^* \) gives

\[
E_l z_{l+1} = A_l z_{l+1} - q^{-1} z_{l+1} A_l = -q^{-3/2} \lambda^{-1} Q_{l+1}^{-1} (z_l z_{l+1}^* - z_{l+1}^* z_l) z_l
\]

\[
= q^{-1/2} z_l z_{l+1},
\]

\[
E_l z_l^* = A_l z_l^* - q^{-1} z_l^* A_l = -q^{-5/2} \lambda^{-1} z_{l+1} (q^2 z_l z_l^* - z_l^* z_l) Q_{l+1}^{-1} = -q^{3/2} z_l^* z_{l+1},
\]

where we used (54). For \( E_n \), we obtain

\[
E_n z_j = q^{-1/2} \lambda^{-1} (z_n z_j - q z_j z_n) = -q^{1/2} z_n z_j,
\]

\[
E_n z_j^* = q^{-1/2} \lambda^{-1} (z_n^* z_j - q^{-1} z_j^* z_n) = 0,
\]

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if \( j < n \). The action of \( E_n \) on \( z_n \) and \( z_n^* \) is calculated analogously to (43) and (44).

We have thus proved that (76) is consistent with the action of \( E_j, j = 1, \ldots, n, \) on \( \mathcal{O}_q(\text{Mat}_{n,1}) \). The corresponding result for \( F_j \) follows from this by using \( F_j \cdot f = -(1)^h \gamma \cdot (E_j \cdot f^*)^* \).

Let \( \omega_1, \ldots, \omega_n \) be the simple roots of the Lie algebra \( \mathfrak{sl}_{n+1} \). For \( \gamma = \sum_{j=1}^n p_j \omega_j \), we write \( K_\gamma = K_1^{p_1} \cdots K_n^{p_n} \). Recall that, for a finite dimensional representation \( \sigma \) of \( \mathcal{U}_q(\text{su}_{n,1}) \), the quantum trace

\[
\text{Tr}_{q,L} a := \text{Tr} \sigma(aK_2^{-1})
\]

defines an invariant integral on \( \mathcal{U}_q(\text{su}_{n,1}) \), where \( \omega \) denotes the half-sum of all positive roots (see [5 Proposition 7.14]). \( K_2 \) is chosen such that \( XK_2 = K_2X^2(X) \) for all \( X \in \mathcal{U}_q(\text{su}_{n,1}) \). In Subsection 5.1 we replaced \( K (= K_2) \) by \( y \) and proved the existence of invariant integrals on appropriate classes of functions. Our aim is to generalize this result to \( \mathcal{O}_q(\text{Mat}_{n,1}) \).

The half-sum of positive roots is given by \( \omega = \frac{1}{2} \sum_{j=1}^n l(n - l + 1)\omega_j \). Consider \( \Gamma := \prod_{l=1}^n \rho_l^{-l(n-l+1)} \). Inserting the definition of \( \rho_l \) gives

\[
\Gamma = |Q_1|^{-n} |Q_2| \cdots |Q_n|, \quad n > 1, \quad \Gamma = |Q_1|^{-1}, \quad n = 1, \quad (81)
\]

since \( -\frac{1}{2}(l-1)(n-l+2) + l(n-l+1) - \frac{1}{2}(l+1)(n-l) = 1 \) for \( 1 < l \leq n \). The operator \( |Q_1| \) appears in the definition of \( \Gamma \) twice, in \( \rho_1^n \) and \( \rho_1^{-n} \), in each factor to the power \(-n/2\). For \( n = 1 \), Equation (81) is trivial (cf. Equation (63)). The following proposition shows that \( \Gamma \) enables us to define a generalization of the quantum trace.

Notice that \( z_n, z_n^*, K_1^{\pm 1}, E_n, \) and \( F_n \) satisfy the relations of the quantum disc, in particular, Equation (43) applies. Therefore we cannot have a normalized invariant integral on \( \mathcal{O}_q(\text{Mat}_{n,1}) \).

**Proposition 4.4** Let \( \mathfrak{A} \) be the \( O^* \)-algebra generated by the operators \( z_j, z_j^*, |Q_j|^{1/2}, \) and \( |Q_j|^{-1/2}, j = 1, \ldots, n \). Then the \( * \)-algebras \( \mathbb{F}(D) \) and \( \mathbb{B}_1(\mathfrak{A}) \) defined in (75) and (18), respectively, are \( \mathcal{U}_q(\text{su}_{n,1}) \)-module \( * \)-algebras, where the action is given by (75) - (77). The linear functional

\[
h(f) := c \text{Tr} \overline{f \Gamma}, \quad c \in \mathbb{R}, \quad (82)
\]
defines an invariant integral on both \( \mathbb{F}(D) \) and \( \mathbb{B}_1(\mathfrak{A}) \).

**Proof.** From the definition of \( \mathbb{F}(D) \) and \( \mathbb{B}_1(\mathfrak{A}) \), it is obvious that both algebras are stable under the \( \mathcal{U}_q(\text{su}_{n,1}) \)-action defined by (75) - (77), in particular, by Lemma 4.3 they are \( \mathcal{U}_q(\text{su}_{n,1}) \)-module \( * \)-algebras.

We proceed as in the proof of Proposition 5.2 and show the invariance of \( h \) for generators by using the trace property \( \text{Tr} \, a \, g \, b \, h = \text{Tr} \, g \, b \, a \, h \) for all \( g \in \mathbb{B}_1(\mathfrak{A}), \) \( a, b \in \mathfrak{A} \). Let \( g \in \mathbb{B}_1(\mathfrak{A}) \). Clearly, \( \rho_l \) commutes with \( \Gamma \), hence

\[
h(K_1^{\pm 1} g) = \text{Tr} \rho_l^{-1} g \rho_l \Gamma = \text{Tr} \, g \Gamma = \varepsilon(K_1^{\pm 1}) h(g)
\]
It follows from the definition of $\Gamma$ and from (66) that $A_l \Gamma = q^2 \Gamma A_l$ for all $l$ since

$$-(l-1)(n-l+2) + 2l(n-l+1) - (l+1)(n-l) = 2.$$ 

Hence $\rho_t^{-1} A_l \Gamma = \Gamma A_l \rho_t^{-1}$ and therefore

$$h(E_l \rho_t g) = \text{Tr} \left( A_l \rho_t g \Gamma - \rho_t g \rho_t^{-1} A_l \Gamma \right) = \text{Tr} A_l \rho_t g \Gamma - \text{Tr} A_l g \Gamma = 0 = \varepsilon(E_l) h(g).$$

Applying the involution to $A_l \Gamma = q^2 \Gamma A_l$ shows that $\Gamma B_l = q^2 B_l \Gamma$. Thus

$$h(F \rho_t g) = \text{Tr} \left( B_l \rho_t g \Gamma - q^2 \rho_t g \rho_t^{-1} B_l \Gamma \right) = \text{Tr} B_l \rho_t g \Gamma - \text{Tr} B_l g \Gamma = 0 = \varepsilon(F_l) h(g).$$

This completes the proof. \hfill \Box

Remark. As in Subsection 3.1 we consider $B_1(\mathfrak{g})$ as the algebra of infinitely differentiable functions which vanish sufficiently rapidly at “infinity” and $F(D)$ as the infinitely differentiable functions with compact support.

### 4.4 Topological aspects of $*$-representations

In this subsection, we shall restrict ourselves to irreducible $*$-representations of the series $(\mathfrak{g}, 0, k)$. Let $D$ denote the linear space defined in Proposition 4.1. Then the operators $|Q_j|^{-1/2}, 1 \leq j \leq n$, belong to $L^+(D)$ and the $O^*$-algebra $\mathfrak{g}$ of Proposition 4.2 is well defined. As in Subsection 3.2 we prefer for topological reasons to work with closed $O^*$-algebras. In particular, we suppose that the $*$-representation is given on the domain $D_\mathfrak{g} = \bigcap_{a \in \mathfrak{g}} D(a)$. It turns out that the topological properties of the (closed) $O^*$-algebra $\mathfrak{g}$ are very similar to that of Subsection 3.2.

**Lemma 4.5**

i. $\mathfrak{g}$ is a commutatively dominated $O^*$-algebra on a Frechet domain.

ii. $D_\mathfrak{g}$ is nuclear, in particular, $D_\mathfrak{g}$ is a Frechet–Montel space.

**Proof.** The operator

$$T := 1 + Q_1^2 + \ldots + Q_n^2 + Q_1^{-2} + \ldots + Q_n^{-2}$$

is essentially self-adjoint on $D_\mathfrak{g}$, and $T > 2$. Let $\varphi \in D_\mathfrak{g}$. As in the proof of Lemma 3.3 we conclude from a standard argument that, for each polynomial $p = p(|Q_1|^{1/2}, \ldots, |Q_n|^{1/2}, |Q_1|^{-1/2}, \ldots, |Q_n|^{-1/2})$, there exist $k \in \mathbb{N}$ such that $\|p \varphi\| \leq \|T^k \varphi\|$. Furthermore, for each finite sequence $k_1, \ldots, k_N \in \mathbb{N}$ and real numbers $\gamma_1, \ldots, \gamma_N \in (0, \infty)$, we find $k_0 \in \mathbb{N}$ such that $\sum_{j=1}^{N} \gamma_j \|T^{k_j} \varphi\| \leq \|T^{k_0} \varphi\|$. Let $p$ be as above and let $I, J \in \mathbb{N}^n$ such that $I \cdot J = 0$. By (49)–(52) and (53)–(58), $(z^I \rho_z^{*J})^*(z^I \rho_z^{*J})$ is a polynomial in $|Q_j|^{1/2}, |Q_j|^{-1/2}$, $j = 1, \ldots, n$, say $\tilde{p}$. Thus there exist $k \in \mathbb{N}$ such that

$$\|z^I \rho_z^{*J} \varphi\| = \langle \tilde{p} \varphi, \varphi \rangle^{1/2} \leq \langle \|\tilde{p} \varphi\| \|\varphi\| \rangle^{1/2} \leq \|T^k \varphi\|x. $$

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From the definition of $\mathfrak{A}$, $\mathfrak{S}$, $\mathfrak{I}$, and $\mathfrak{J}$, it follows that each $f \in \mathfrak{A}$ can be written as $f = \sum_{I,J=0} z^I p_I z^J$, where $p_I$ are polynomials in $|Q_j|^{|1/2|, |Q_j|^{-1/2}}$, $j = 1, \ldots, n$. From the preceding arguments, we conclude that there exist $m \in \mathbb{N}$ such that $\|f \varphi\| \leq \|T^m \varphi\|$ for all $\varphi \in D_{\mathfrak{A}}$, therefore $\|f\| \leq \|T\| T^m$. This implies that the family $\{\|f\| T^m\}_{k \in \mathbb{N}}$ generates the graph topology on $D_{\mathfrak{A}}$ and $D_{\mathfrak{A}} = \cap_{k \in \mathbb{N}} D(T^{2k}))$ which proves (i).

Note that the proof of Lemma 3.3 ii) is based on the observation that the operator $\tilde{T}^{-1}$ is a Hilbert–Schmidt operator. One easily checks that this holds also for the operator $T$ defined in (83). Now the rest of the proof runs completely analogous to that of Lemma 5.3.

Proposition 4.6

i. $\mathcal{F}(D_{\mathfrak{A}})$ is dense in $\mathcal{L}(D_{\mathfrak{A}}, D^+_\mathfrak{A})$ with respect to the bounded topology $\tau_b$.

ii. The $\mathcal{U}_q(\text{su}_{n,1})$-action on $\mathcal{L}^+(D_{\mathfrak{A}})$ is continuous with respect to $\tau_b$.

The proof of Proposition 4.6 is completely analogous to that of Proposition 3.4.

Corollary 4.7 Let $\mathcal{F}(D)$ denote the $\mathcal{O}^*$-algebra of finite rank operators on $D$ defined in (17). Then $\mathcal{F}(D)$ is a $\mathcal{U}_q(\text{su}_{n,1})$-module $\ast$-subalgebra of $\mathcal{F}(D_{\mathfrak{A}})$ and $\mathcal{F}(D)$ is dense in $\mathcal{L}(D_{\mathfrak{A}}, D_{\mathfrak{A}}^+)$.

Proof. Since $D \subset D_{\mathfrak{A}}$, we can consider $\mathcal{F}(D)$ as a $\ast$-subalgebra of $\mathcal{F}(D_{\mathfrak{A}})$. It follows from Proposition 4.1 that $\mathcal{F}(D)$ is stable under the $\mathcal{U}_q(\text{su}_{n,1})$-action defined in Lemma 4.3, in particular, it is a $\mathcal{U}_q(\text{su}_{n,1})$-module $\ast$-algebra. The density of $\mathcal{F}(D)$ in $\mathcal{L}(D_{\mathfrak{A}}, D_{\mathfrak{A}}^+)$ can be proved in exactly the same way as in Corollary 3.5.

Recall that the self-adjoint operators $\bar{Q}_j$, $j = 1, \ldots, n$, strongly commute. Set

$$\mathfrak{M} := \sigma(\bar{Q}_1) \times \ldots \times \sigma(\bar{Q}_n).$$

By the spectral theorem of self-adjoint operators, we can assign to each (Borel measurable) function $\psi: \mathfrak{M} \to \mathbb{C}$ an operator $\psi(\bar{Q}_1, \ldots, \bar{Q}_n)$ such that

$$\psi(\bar{Q}_1, \ldots, \bar{Q}_n) \eta_{i_1 \ldots i_n} = \psi(t_{i_1}, \ldots, t_{i_n}) \eta_{i_1 \ldots i_n},$$

where $t_{i_j} = q^{2(i_1 + \ldots + i_n + \alpha)}$ for $j > k$, $t_{i_j} = -q^{-2(i_1 + \ldots + i_n) + 2(i_{k+1} + \ldots + i_n + \alpha)}$ for $j \leq k$, and $A = q^{2\alpha}$. (A denotes the operator appearing in the type $(m, 0, k)$ representations for $k > 0$. If $k = 0$, set $\alpha = 0$.) Define

$$S(\mathfrak{M}) = \{ \psi: \mathfrak{M} \to \mathbb{C}; \sup_{(t_1, \ldots, t_n) \in \mathfrak{M}} |t_1^{s_1} \cdots t_n^{s_n} \psi(t_1, \ldots, t_n)| < \infty \text{ for all } s_1, \ldots, s_n \in \mathbb{Z} \}$$

and

$$S(D) = \{ \sum_{I,J=0} z^I \psi_{IJ}(\bar{Q}_1, \ldots, \bar{Q}_n) z^{-J}; \psi_{IJ} \in S(\mathfrak{M}), \# \{\psi_{IJ} \neq 0\} < \infty \}.$$
Lemma 4.8  With the action defined in Lemma 4.3, $S(D)$ becomes a $U_q(su_{n,1})$-module $\ast$-subalgebra of $B_1(\mathfrak{g})$. The operators $z_j$, $z_j^*$, $j = 1, \ldots, n$, and $\psi(Q_1, \ldots, Q_n)$, $\psi \in S(\mathfrak{g})$, satisfy the following commutation rules

$$
\psi(Q_1, \ldots, Q_j, Q_{j+1}, \ldots, Q_n)z_j = z_j\psi(q^2Q_1, \ldots, q^2Q_j, Q_{j+1}, \ldots, Q_n)
$$

$$
z_j^*\psi(Q_1, \ldots, Q_j, Q_{j+1}, \ldots, Q_n) = \psi(q^2Q_1, \ldots, q^2Q_j, Q_{j+1}, \ldots, Q_n)z_j^*.
$$

The proof of Lemma 4.8 differs from that of Lemma 3.7 only in notation, the argumentation to establish the result remains the same.

Since $F(D) \subset F(D_3)$ and $S(D) \subset B_1(\mathfrak{g})$, we can consider $F(D)$ and $B_1(\mathfrak{g})$ as algebras of infinite differentiable functions with compact support and which are rapidly decreasing, respectively. It is not difficult to see that $F(D)$ is the set of all $\sum_{I,J=0}^{\infty} z^I \psi_{IJ}(Q_1, \ldots, Q_n)z^J \in S(D)$, where the functions $\psi_{IJ} \in S(\mathfrak{g})$ have finite support. On $S(D)$, we have the following explicit formula of the invariant integral.

Proposition 4.9  Set

$$
\mathcal{M}_0 := \sigma(\mathfrak{g}) \setminus \{0\} \times \cdots \times \sigma(\mathfrak{g}) \setminus \{0\}.
$$

Assume that $f = \sum_{I,J=0}^{\infty} z^I \psi_{IJ}(Q_1, \ldots, Q_n)z^J \in S(D)$. Then the invariant integral $h$ defined in Proposition 4.4 is given by

$$
h(f) = c \sum_{(t_1, \ldots, t_n) \in \mathcal{M}_0} \psi_0(t_1, \ldots, t_n) |t_1|^{-n} |t_2| \cdots |t_n|.
$$

(If $n = 1$, then $t_2, \ldots, t_n$ are omitted.)

Proof. Recall that $h(f) = c \text{Tr} \overline{f}$, where $\Gamma$ is given by \ref{81}. If $I \neq (0, \ldots, 0)$ or $J \neq (0, \ldots, 0)$, then $\langle \eta_{n_1-i_1}, z^I \psi_{IJ}(Q_1, \ldots, Q_n)z^J \eta_{n_2-i_2} \rangle = 0$ since, by Proposition 4.3, $\psi_{IJ}(Q_1, \ldots, Q_n)$ and $\Gamma$ are diagonal and $z^I$ and $z^J$ act as shift operator on $\mathcal{H}$. Hence only $\psi_0(Q_1, \ldots, Q_n) \Gamma$ contributes to the trace.

For each tuple $(t_1, \ldots, t_n) \in \mathcal{M}_0$, there exists exactly one tuple $(i_1, \ldots, i_n)$ such that $\eta_{n, i_1} \in D$ and $Q_j\eta_{n, i_1} = t_j\eta_{n, i_1}$, $j = 1, \ldots, n$. This can be seen inductively; $Q_0$ determines $i_1$ uniquely, and if $i_n, \ldots, i_{n-k+1}$ are fixed, then $Q_{n-k}$ determines uniquely $i_{n-k}$ (see the remark after Proposition 4.4). Since the vectors $\eta_{n, i_1}$ constitute an orthonormal basis of eigenvectors of the $Q_j$'s, and since $\Gamma$ is given by $\Gamma = |Q_1|^n |Q_2|^{-1} \cdots |Q_n|^{-1}$ for $n > 1$, $\Gamma = |Q_1|$ for $n = 1$, the assertion follows.белый
Notice that the proof of Proposition 4.4 uses only the commutation relations of $A$ with $A_i$, $B_i$, and $\rho_j$, $i = 1, \ldots, n$. The crucial observation is that $Q_1$ commutes with $A_j$, $B_j$, and $\rho_j$, $j = 1, \ldots, n - 1$. Therefore the commutation relations used in proving the invariance of $h$ remain unchanged if we multiply $\Gamma$ by $Q_1^{n+1}$. Furthermore, $\Gamma Q_1^{n+1}$ is of trace class. This suggests that $h(f) := c \text{ Tr } f \Gamma Q_1^{n+1}$ defines a $\mathcal{U}_q(\mathfrak{su}_n)$-invariant integral on $\mathcal{O}_q(\text{Mat}_{n,1})$. The only difficulty is that the definitions of $A_j$, $B_j$, and $\rho_j^{\pm1}$ involve the unbounded operators $Q_1^{\pm1}$, therefore we cannot freely apply the trace property in proving the invariance of $h$. Nevertheless, a modified proof will establish the result.

**Proposition 4.10** Let $n > 1$ and set $c := \prod_{k=1}^n (1 - q^{2k})^{-1}$. Suppose we are given an irreducible *-representation of $\mathcal{O}_q(\text{Mat}_{n,1})$ of type $(n,0,0)$. Then the linear functional

$$h(f) := c \text{ Tr } f \Gamma Q_1^{n+1} = c \text{ Tr } f Q_1 \cdots Q_n, \quad f \in \mathcal{O}_q(\text{Mat}_{n,1}),$$

defines a normalized $\mathcal{U}_q(\mathfrak{su}_n)$-invariant integral on $\mathcal{O}_q(\text{Mat}_{n,1})$.

**Proof.** First note that the vectors $\eta_{i_n \ldots i_1}, i_1, \ldots, i_n \in \mathbb{N}_0$, form a complete set of eigenvectors of the positive operator $Q_1$ with corresponding eigenvalues $q^{\sum_{i=1}^n |i|}$. As $\sum_{i=1}^n q^{\sum_{i=1}^n |i|} < \infty$, $Q_1$ is of trace class. This implies that $f \Gamma Q_1^{n+1} = f Q_1 \cdots Q_n$ is of trace class for all $f \in \mathcal{O}_q(\text{Mat}_{n,1})$ since the representations of the series $(n,0,0)$ are bounded. Therefore $h$ is well defined. An easy calculation shows that $h(1) = 1$.

As in the proof of Proposition 3.4, it suffices to verify the invariance of $h$ for the generators of $\mathcal{U}_q(\text{Mat}_{n,1})$. Recall that $\mathcal{O}_q(\text{Mat}_{n,1})$ is the linear span of the elements $z^I p_{IJ} z^J$, where $I, J \in \mathbb{N}_0^n$, $I \cdot J = 0$, and $p_{IJ}$ is a polynomial in $Q_i, i = 1, \ldots, n$. If $I \neq 0$ or $J \neq 0$, then

$$0 = \langle \eta_{i_n \ldots i_1}, \rho_j^{\pm1} z^I p_{IJ} z^J \rho_j^{\mp1} \Gamma Q_1^{n+1} \eta_{i_n \ldots i_1} \rangle = \langle \eta_{i_n \ldots i_1}, z^I p_{IJ} z^J \Gamma Q_1^{n+1} \eta_{i_n \ldots i_1} \rangle$$

by the same arguments as in the proof of Proposition 4.9. Hence

$$h(K_j^{\pm1} z^I p_{IJ} z^J) = \varepsilon(K_j^{\pm1}) h(z^I p_{IJ} z^J) = 0.$$  

If $I = J = 0$, then

$$K_j^{\pm1} p_{IJ} = \rho_j^{\pm1} p_{IJ} \rho_j^{\mp1} = p_{IJ},$$

thus $h(K_j^{\pm1} p_{IJ}) = h(p_{IJ}) = \varepsilon(K_j^{\pm1}) h(p_{IJ})$. This proves the invariance of $h$ with respect to $K_j^{\pm1}$, $j = 1, \ldots, n - 1$.

Recall that $A_k = -q^{-5/2} \lambda^{-1} Q_{k+1}^{\pm1} z_k z_k$, $k < n$. If $I \neq (0,0,\ldots,0)$ or $J \neq (0,1,\ldots,0)$ with 1 in the $(k+1)$th and $k$th positions, respectively, then we have similarly to Equation (85)

$$0 = \langle \eta_{i_n \ldots i_1}, -A_k z^I p_{IJ} z^J \Gamma Q_1^{n+1} \eta_{i_n \ldots i_1} \rangle = \langle \eta_{i_n \ldots i_1}, \rho_k z^I p_{IJ} z^J \rho_k^{-1} A_k \Gamma Q_1^{n+1} \eta_{i_n \ldots i_1} \rangle.$$

Thus $h(E_k z^I p_{IJ} z^J) = \varepsilon(E_k) h(z^I p_{IJ} z^J) = 0$. 

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Now let $p$ denote an arbitrary polynomial in $Q_i$, $i = 1, \ldots, n$. Then, by the definition of $\Gamma$ and repeated application of the commutation rules of $Q_i$ with $z_j$ and $z_j^*$, we obtain
\[
Q_k^{-1} z_{k+1} z_k z_{k+1} p z_k \Gamma Q_1^{n+1} = z_{k+1} z_k z_{k+1} p z_k Q_1 \cdots Q_k Q_{k+2} \cdots Q_n,
\]
\[
\rho_k z_{k+1} p z_k \rho_{k+1}^{-1} Q_k^{-1} z_{k+1} z_k \Gamma Q_1^{n+1} = z_{k+1} p z_k Q_1 \cdots Q_k Q_{k+2} \cdots Q_n z_{k+1} z_k.
\]
All operators on the right hand sides are bounded and $Q_1$ is of trace class, in particular, the trace property applies. Therefore, the difference of the traces of the right hand sides vanishes. Hence
\[
h(E_k \triangleright (z_{k+1} p z_k^*)) = c \text{Tr} (A_k z_{k+1} p z_k^* - \rho_k z_{k+1} p z_k^* \rho_{k+1}^{-1} A_k) \Gamma Q_1^{n+1} = 0
\]
\[
= \varepsilon(E_k) h(z_{k+1} p z_k^*)
\]
which establishes the invariance of $h$ with respect to $E_k$, $k = 1, \ldots, n-1$.

To verify that $h$ is invariant with respect to $F_k$, $k = 1, \ldots, n-1$, notice that $h(f^*) = h(f)$ for all $f \in \mathcal{O}_q(\text{Mat}_{n,1})$ since the operator $\Gamma Q_1^{n+1}$ is self-adjoint. Thus, by (4) and the preceding,
\[
h(F_k \triangleright f) = h(S(F_k)^* \triangleright f^*) = -q^2 h(E_k \triangleright f^*) = 0 = \varepsilon(F_k) h(f)
\]
for all $f \in \mathcal{O}_q(\text{Mat}_{n,1})$.

**Corollary 4.11** Let $f = \sum_{j=0}^n \sum_{j=0}^n z^j p_{ij} (Q_1, \ldots, Q_n) z^{*j} \in \mathcal{O}_q(\text{Mat}_{n,1})$. Then the invariant integral $h$ defined in Proposition 4.10 is given by
\[
h(f) = c \prod_{j_1, \ldots, j_n} p_{00}(q^{j_1}, \ldots, q^{j_n}) q^{j_1} \cdots q^{j_n}.
\]

**Proof.** Taking into account that $\mathfrak{N}_0 = \{(q^{j_1}, \ldots, q^{j_n}) : j_1, \ldots, j_n \in \mathbb{N}_0\}$ for representations of the series $(n, 0, 0)$, Corollary 4.11 is verified by an obvious modification of the proof of Proposition 4.9. 

### 5 Concluding remarks

In general, the definition of quantum groups and quantum spaces is completely algebraic. However, our definition of integrable functions involves operator algebras. The discussion in this paper shows that operator algebras form a natural setting for the study of non-compact quantum spaces. For example, Hilbert space representations provide us with the powerful tool of spectral theory which allows to define functions of self-adjoint operators. We emphasize that different representations will lead to different algebras of integrable functions. If one accepts that representations carry information about the underlying quantum space (for instance, by considering the spectrum of self-adjoint operators), then representations can be used to distinguish between $q$-deformed manifolds which are isomorphic on purely algebraic level.

The crucial step of our approach was to find an operator expansion of the action. At first sight it seems a serious drawback that no direct method was given to obtain an
operator expansion of the action. This problem can be removed by considering cross
product algebras. Inside the cross product algebra, the action can be expressed by alge-
braic relations. Representations of cross product algebras lead therefore to an operator
expansion of the action. Moreover, the operator expansion is given by the adjoint ac-
tion so that our ideas concerning invariant integration theory apply \[15\]. Hilbert space
representations of some cross product algebras can be found in \[11\] and \[15\].

References

[1] S.J. Bhatt, A. Inoue, and K.-D. Kürsten. Well-behaved unbounded operator rep-
resentations and unbounded C*-seminorms. *to appear in J. Math. Soc. Japan.*, 2003.
[2] S.J. Bhatt, A. Inoue, and H. Ogi. Unbounded C*-seminorms and unbounded C*-
spectral algebras. *J. Operator Theory*, 45:53–80, 2001.
[3] M. Chaichian, H. Grosse, and P. Presnajder. Unitary representations of the q-
oscillator algebra. *J. Phys. A: Math. Gen.*, 27:2045–2051, 1994.
[4] S. Klimek and A. Lesniewski. A two-parameter quantum deformation of the unit
disc. *J. Funct. Anal.*, 155:1–23, 1993.
[5] A. Klimyk and K. Schmüdgen. *Quantum Groups and Their Representations*. Texts and Monographs in Physics. Springer-Verlag, Heidelberg, 1997.
[6] K.-D. Kürsten. On commutatively dominated Op*-algebras with Fréchet do-
 mains. *J. Math. Anal. Appl.*, 157:506–526, 1991.
[7] W. Pusz and S.L. Woronowicz. Twisted second quantization. *Rep. Math. Phys.*, 27:231–257, 1989.
[8] K. Schmüdgen. *Unbounded Operator Algebras and Representation Theory*. Akademie-Verlag, Berlin, 1990.
[9] K. Schmüdgen. Commutator representations of differential calculi on the quan-
tum group SU_q(2). *J. Geom. Phys.*, 31:241–264, 1999.
[10] K. Schmüdgen. On well-behaved unbounded representations of *-algebras. *J.
Operator Theory*, 48:487–502, 2002.
[11] K. Schmüdgen and E. Wagner. Hilbert space representations of cross product
algebras. *J. Funct. Anal.*, 200:451–493, 2003.
[12] D. Shklyarov, S. Sinel’shchikov, and L.L. Vaksman. Quantum matrix balls: Dif-
ferential and integral calculi. *e-print: math.QA/9905035* 1999.
[13] D. Shklyarov, S. Sinel’shchikov, and L.L. Vaksman. On function theory in quan-
tum disc: Integral representations. *e-print: math.QA/9808015* 1998.
[14] S. Sinel’šchikov and L.L. Vaksman. On q-analogues of bounded symmetric domains and dolbeault complexes. In *Mathematical Physics, Analysis and Geometry*, volume 1, pages 75–100. Kluwer Academic Publishers, 1998.

[15] E. Wagner. Hilbert space representations of some quantum algebras. Universität Leipzig, 2002. Dissertation.

[16] S.L. Woronowicz. Compact matrix pseudogroups. *Commun. Math. Phys.*, 111:613–665, 1987.