Existence of optimizers of the Stein-Weiss inequalities on Carnot groups

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Abstract

This paper proves existence of optimizers of the Stein-Weiss inequalities on Carnot groups under some conditions. The adjustment of Lions’ concentration compactness principles to Carnot groups plays an important role in our proof. Unlike known treatment to the Hardy-Littlewood-Sobolev inequality on Heisenberg group, our arguments relate to the powers of the weight functions.

Keywords: Carnot group; Stein-Weiss inequality; maximizing sequence; concentration compactness principle.

MSC2010: 35R03; 39B62; 49J45.

1 Introduction

1.1 Classic Hardy-Littlewood-Sobolev inequality and Stein-Weiss inequality

The well known Hardy-Littlewood-Sobolev inequality on \( \mathbb{R}^N \) (short for HLS inequality) is of the form

\[
\left| \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^{N+\lambda}} \, dx \, dy \right| \leq C_{r,s,N} \|f\|_r \|g\|_s,
\]

where \( 1 < r, s < \infty \), \( 0 < \lambda < N \) and \( \frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} = 2 \). \( C_{r,s,N} \) is a positive constant independent of \( f \) and \( g \), \( \|f\|_p \) denotes the \( L^p(\mathbb{R}^N) \) norm of \( f \).

\[\text{MSC2010: 35R03; 39B62; 49J45.}\]
This inequality was first proved by Hardy and Littlewood [5, 6] in $\mathbb{R}^1$ and extended by Sobolev [15] to $\mathbb{R}^N$. Lieb [11] classified all the optimizers of (1.1) on $\mathbb{R}^N$ and obtained the sharp constant of this inequality in the special case $r = s = 2N/(2N - \lambda)$. Existence of optimizers of (1.1) was also investigated by Lions [13], which is an application of the concentration compactness principle.

The weighted HLS inequality, i.e. Stein-Weiss inequality, derived by Stein and Weiss [7] on $\mathbb{R}^N$ which reads

$$\left| \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{f(x)g(y)}{|x|^{\alpha}|y|^{\beta}|x-y|^\lambda} dxdy \right| \leq C_{\alpha, \beta, r, \lambda, N} \|f\|_r \|g\|_s,$$  \hspace{1cm} (1.2)

where $1 < r, s < \infty, 0 < \lambda < N, \alpha + \beta \geq 0, \lambda + \alpha + \beta \leq N, \alpha < N/r'$ ($r' = r/(r-1)$), and $\beta < N/s'$ ($s' = s/(s-1)$), such that $\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} = 2$, $C_{\alpha, \beta, r, \lambda, N}$ is a positive constant independent of $f, g$.

Recently, Han, Lu and Zhu [7] gave two classes of Stein-Weiss inequalities on the Heisenberg group and claimed that these inequalities hold in stratified groups. The authors [9] built the Stein-Weiss type inequalities on Carnot groups, which supports the opinion of Han, Lu and Zhu. Readers can also see [10] for results concerned.

Han and Niu [8] derived existence of optimizers of the Hardy-Sobolev inequalities on the H-type group, which applied a generalization of Lions’ concentration compactness principles. On the other hand, Han [4] furnished a proof of existence of optimizers of the HLS inequality on the Heisenberg group in which the concentration compactness principle plays an important role too. These inspire us to consider the related problems on the Stein-Weiss inequalities on Carnot groups.

### 1.2 Structure of Carnot group and the Stein-Weiss inequality

We begin by describing Carnot group. For more information, we refer to [2, 3, 14]. A Carnot group $G$ of step $r$ is a simply connected nilpotent Lie group such that its Lie algebra $\mathfrak{g}$ admits a stratification

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \ldots \oplus V_r = \oplus_{l=1}^r V_l,$$

in which $[V_1, V_l] = V_{l+1}$ ($l = 1, 2, \ldots, r-1$) and $[V_1, V_r] = \{0\}$.

Denoting $m_l = \dim V_l$, we fix on $G$ a system of coordinates $u = (z_1, z_2, \ldots, z_r), z_l \in \mathbb{R}^{m_l}$. Every Carnot group $G$ is naturally equipped with a family of non-isotropic dilations

$$\delta_r(u) = (rz_1, r^2z_2, \ldots, r^rz_r), \forall u \in G, \forall r > 0,$$

the homogeneous dimension of $G$ is given by $Q = \sum_{l=1}^r lm_l$. We denote by $du$ a fixed bi-invariant Haar measure on $G$. One easily sees $(d \circ \delta_r)(u) = r^Q du$. The group law given
by Baker-Campbell-Hausdorff formula is
\[ uv = u + v + \sum_{1 \leq l, k \leq r} Z_{l,k}(u, v), \quad \forall u, v \in G, \]
where each \( Z_{l,k}(u, v) \) is a fixed linear combination of iterated commutators containing \( l \) times \( u \) and \( k \) times \( v \). The homogenous norm of \( u \) on \( G \) is defined by
\[ |u| = \left( \sum_{j=1}^r |z_j|^{2r^j} \right)^{\frac{1}{2r}}, \]
where \( |z_j| \) denotes the Euclidean distance from \( z_j \in \mathbb{R}^m \) to the origin in \( \mathbb{R}^m \). Such homogenous norm on \( G \) can be used to define a pseudo-distance \( d(u, v) = |u^{-1}v| \) on \( G \).
Denote the pseudo-ball of radius \( r \) centered at \( u \) by \( B(u, r) = \{ v \in G | d(u, v) < r \} \), and the pseudo-ball centered at the origin by \( B_r \) or \( \{ |u| < r \} \).

The Stein-Weiss type inequality on the Carnot group \( G \) states as below (see [9] or [10]).

Let \( 1 < r, s < \infty, 0 \leq \lambda < Q, \alpha + \beta > 0, \alpha < Q/r' \), \((r' = r/(r-1))\), \( \lambda + \alpha + \beta \leq Q \), and \( \beta < Q/s' \), \((s' = s/(s-1))\), such that \( \frac{1}{s} + \frac{\lambda + \alpha + \beta}{r} = 2 \), then there exists a positive constant \( C_{\alpha, \beta, r, \lambda, G} \) independent of \( f, g \) such that
\[ \left| \int_{G \times G} \frac{f(u)g(v)}{|u|^{\alpha} |u^{-1}v|^{\lambda} |v|^{\beta}} dudv \right| \leq C_{\alpha, \beta, r, \lambda, G} \|f\|_r \|g\|_s, \quad (1.3) \]
where \( u = (z_1, z_2, \ldots, z_r), v = (z'_1, z'_2, \ldots, z'_r) \in G \).

1.3 Main results

The aim of this paper is to observe existence of optimizers of (1.3). By the dual argument (see [7]), it is easy to get an alternative version of (1.3):

Let \( 1 < p \leq q < \infty, 0 < \lambda < Q, \alpha + \beta \geq 0, \alpha < Q/q \), and \( \beta < Q/p' \), such that \( \frac{1}{p} = \frac{1}{p} + \frac{\lambda + \alpha + \beta}{q} - 1 \), then
\[ \|Sg\|_q \leq C \|g\|_p, \quad (1.4) \]
where \( p = s, q = r', Sg(u) := \int_G \frac{g(v)}{|u|^{\alpha} |u^{-1}v|^{\lambda} |v|^{\beta}} dv \), \( C = C_{\alpha, \beta, p, \lambda, G} \) is a positive constant independent of \( g \).

Obviously, if we find an optimizer of (1.4), then we obtain an optimizer of (1.3). But since (1.3) has the weight function \( |u|^{\alpha} \) and \( |v|^{\beta} \), so we should consider the range of \( \alpha \) and \( \beta \), which is different from [4].

For \( 1 < p \leq q < \infty, 0 < \lambda < Q, \alpha + \beta \geq 0 \), we shall assume that
\[ \beta < \frac{Q}{p'}, \alpha \leq 0, \]
\[ -\frac{Q}{p} < \beta < \frac{Q}{p'}, 0 < \alpha < \min\{\frac{2n-\lambda}{2n+2}\}, \]
\[ \frac{1}{q} = \frac{1}{p} + \frac{1}{s} + \alpha + \beta \]
\[ \text{such that} \]
\[ 1 = q = \frac{1}{p} + \lambda + \alpha + \beta \]
\[ \text{under the constraint} \]
\[ \|f\|_p = 1. \]

The main result of this paper is:

**Theorem 1.1.** Let \( \{f_j\} \) be maximizing sequence of (1.5), then there exists \( \{u_j\} \in G \) and \( \{d_j\} \subset \mathbb{R}_+ \) such that the following new maximizing sequence \( \{h_j\} \):
\[
h_j(u) := \frac{1}{d_j^{\frac{Q}{p}}} f_j(u),
\]
is relatively compact in \( L^p(G) \), and the limitation of its convergent subsequence is an optimizer of (1.5).

We will prove Theorem 1.1 by adjusting the concentration compactness principles on the Euclidean space by Lions [12, 13] to one on the Carnot group \( G \). If \( G \) is replaced with the Heisenberg group \( \mathbb{H}^n \), then we have

**Corollary 1.1.** Let \( 1 < r, s < \infty, 0 < \lambda < 2n + 2, 0 \leq \alpha + \beta < 2n + 2 - \lambda \) and conditions below hold
\[ (H_1') \beta < \frac{(2n + 2)}{s'}, \alpha \leq 0, \]
\[ (H_2') -\left(\frac{2n + 2}{s}\right) < \beta < \left(\frac{2n + 2}{s'}\right), 0 < \alpha < \min\{\frac{2n+2-\lambda}{2n+2}, \beta\}, \frac{Q}{r'} \]
such that \( \frac{1}{r} + \frac{1}{s} + \frac{\lambda + \alpha + \beta}{Q} = 2 \), then there exists an optimizer of the Stein-Weiss inequality on \( \mathbb{H}^n \).

Notice that the condition (H1’) in Corollary 1.1 contains the special case \( \alpha = \beta = 0 \), so the conclusion in Corollary 1.1 generalizes results in [4].

## 2 Concentration compactness principles on Carnot groups

### 2.1 The first concentration compactness principle

We state a lemma on the Carnot group \( G \) which is actually true in general measure spaces due to Brézis and Lieb (see [11]).

**Lemma 2.1.** Let \( 0 < p < \infty, \{f_j\} \in L^p(G) \) satisfy \( f_j \rightarrow f \text{ a.e.} \), then
\[
\lim_{j \to \infty} \int_G |f_j(u)|^p - |f(u) - f_j(u)|^p - |f(u)|^p \, du = 0.
\]
Let us introduce the first concentration compactness principle on $G$. The principle on the Heisenberg group was given by Han in [4]. The original version can see Lions [12].

**Lemma 2.2.** Let $\rho_j = |f_j|^p du$ be a nonnegative Haar measure on $G$ with $\int_G \rho_j = 1$, then there exists a subsequence of $\{\rho_j\}$ (still denoted by $\{\rho_j\}$) such that one of the following holds:

1. For all $R > 0$, we have
   \[
   \lim_{j \to \infty} \left( \sup_{u \in G} \int_{B(u,R)} \rho_j \right) = 0.
   \]

2. There exists $\{u_j\} \subset G$ such that for each $\varepsilon > 0$ small enough, we can find $R_0 > 0$ with
   \[
   \int_{B(u_j,R_0)} \rho_j > 1 - \varepsilon, \forall j \in \mathbb{N}.
   \]

3. There exists $0 < k < 1$ such that for each $\varepsilon > 0$ small enough, we can find $R_0 > 0$ and $\{u_j\} \subset G$ such that given any $R \geq R_0$, there exist $\rho_1^j$ and $\rho_2^j$ satisfying
   \[
   \begin{align*}
   & (a) \rho_1^j + \rho_2^j = \rho_j, \\
   & (b) \text{supp}(\rho_2^j) \subset (B(u_j,R))^c, \\
   & (c) \limsup_{j \to \infty} \left( |k - \int_G \rho_1^j| + |(1-k) - \int_G \rho_2^j| \right) \leq \varepsilon.
   \end{align*}
   \]

Its proof is omitted, since it is similar to [4] without any new difficult except replacing the Heisenberg group $\mathbb{H}^n$ by the Carnot group $G$.

Now let us define the Levy concentration function for $\rho_j$ on $G$ by

\[
P_j(R) = \sup_{u \in G} \int_{B(u,R)} \rho_j, \text{ for any } R \in [0,\infty].
\]

It is obvious that $P_j \in BV[0,\infty]$ is nonnegative and non-decreasing with

\[
P_j(0) = 0, P_j(\infty) = 1, \text{ for any } j \in \mathbb{N}.
\]

Therefore, we can take a nonnegative and non-decreasing function $P \in BV[0,\infty]$ such that $P$ is a limit of some subsequence of $\{P_j\}$ (still denoted the subsequence by $\{P_j\}$):

\[
\lim_{j \to \infty} P_j(R) = P(R), \text{ for any } R \in [0,\infty).
\]

Denote

\[
k = \lim_{R \to \infty} P(R),
\]

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thus $0 \leq k \leq 1$. The case (1) of Lemma 2.2 holds if $k = 0$; the case (2) holds if $k = 1$ and the case (3) holds for $0 < k < 1$.

Let $\{f_j\}$ be a maximizing sequence of (1.5) satisfying $\|f_j\|_p = 1$. Lemma 2.2 ensures that one of the three cases must happen. Using dilations in $G$ and choosing $d_j$ large enough, we can make a new maximizing sequence (still denoted by $\{f_j\}$) such that

$$P_j(1) = \sup_{u \in G \cdot B(u)} \rho_j = \frac{1}{2}.$$ 

Then the case (1) of lemma 2.2 cannot occur.

The following result for $\alpha$ and $\beta$ such that (1.4) holds is needed.

**Lemma 2.3.** Let $\{f_j\} \subset L^p(G)$ be a maximizing sequence of (1.5) satisfying $\|f_j\|_p = 1$, then (3) of Lemma 2.2 cannot occur.

**Proof:** If the case (3) in Lemma 2.2 occurs, then there exist $0 < k < 1$ and a subsequence of $\{f_j\}$ (still denoted by $\{f_j\}$) such that for each $\varepsilon > 0$, one can find $R_0 > 0$ and $\{u_j\} \subset G$ such that for any $R \geq R_0$,

$$\|f_j \chi_{B(R)}\|_p^p = k + O(\varepsilon),$$

$$\|f_j \chi_{(B(R))^C}\|_p^p = 1 - k + O(\varepsilon).$$

Without loss of generality, we may assume $u_j = 0, j \in N$ since (1.5) is translation-invariant. For any $u \in G$, choose $i \geq i(\varepsilon, |u|)$ such that $i|u| > R_0$ and let $R = i|u|$. We observe that $|u| \leq \frac{1}{i} |v|$ for all $v \in (B(R))^C$, then

$$|u^{-1}v| \geq |v| - |u| \geq \frac{i - 1}{i} |v|.$$ 

A direct calculation gives

$$\left| S(f_j)(u) - S(f_j \chi_{B(R)})(u) \right|$$

$$= \left| S(f_j \chi_{(B(R))^C})(u) \right|$$

$$= \left| \int_G (f_j \chi_{(B(R))^C})(v) dv \right|$$

$$\leq \left[ \int_{(B(R))^C} |f_j(v)|^p dv \right]^{1/p} \left[ \int_{(B(R))^C} \left| u - \alpha u^{-1}v \right|^{-\lambda} \left| v - \beta v^\prime \right|^{-\beta} dv^\prime \right]^{1/p^\prime}$$

$$\leq \left[ \int_{(B(R))^C} |f_j(v)|^p dv \right]^{1/p} \left[ \int_{(B(R))^C} \left| u - \alpha u^{-1}v \right|^{-\lambda} \left| v - \beta v^\prime \right|^{-\beta} dv^\prime \right]^{1/p^\prime}.$$
\[
\leq C |u|^{-\alpha} \left( \frac{i}{i-1} \right)^{\lambda} \left( \int_{(B(R))^c} |v|^{-\lambda p' - \beta p'} dv \right)^{1/p'} \\
\leq C \left( \frac{i}{i-1} \right)^{\lambda} |u|^{-\alpha} \left( \int_{|v|}^\infty r^{-\lambda p' - \beta p' - 1} dr \right)^{1/p'} \\
\leq C \left( \frac{i}{i-1} \right)^{\lambda} |u|^{-\alpha} \left( \frac{1}{\lambda p' + \beta p' - Q} \right)^{1/p'} (i|u|)^{(Q - \lambda p' - \beta p')/p'},
\]
in which \( C \) depends only on \( G \).

Since \( \frac{1}{q} + \frac{1}{p'} = \frac{\lambda + \beta + \alpha}{Q} \), it follows \( Q/p' - \lambda - \beta = \alpha - \frac{Q}{q} < 0 \) and

\[ S(f_j \chi_{B(R)}) \xrightarrow{a.e.} S(f_j), \text{ as } i \to \infty. \]

By applying lemma 2.1, we have

\[
\|S(f_j)\|_q^q = \|S(f_j \chi_{B(R)})\|_q^q + \|S(f_j \chi_{(B(R))^c})\|_q^q + o(1), \text{ as } i \to \infty. \tag{2.1}
\]

Since \( \{f_j\} \) maximizes (1.5), it implies that the left hand side of (2.1) goes to \( C_0^q \) as \( j \to \infty \) for a large \( i \), while the right hand side of (2.1) satisfies

\[
\|S(f_j \chi_{B(R)})\|_q^q + \|S(f_j \chi_{(B(R))^c})\|_q^q + o(1) \\
\leq C_0^q \|f_j \chi_{B(R)}\|_p^q + C_0^q \|f_j \chi_{(B(R))^c}\|_p^q + o(1) \\
\leq C_0^q (k + O(\varepsilon))^{\frac{q}{p}} + C_0^q (1 - k + O(\varepsilon))^{\frac{q}{p}} + o(1) \\
\leq C_0^q \left( k^{\frac{q}{p}} + (1 - k)^{\frac{q}{p}} \right) + O(\varepsilon) + o(1) < C_0^q,
\]

which is a contradiction. \( \square \)

### 2.2 The convergent subsequence of maximizing sequence

Let \( \{f_j\} \) be a maximizing sequence of (1.5), we see from the argument in previous subsection that there exists \( \{u_j\} \subset G \) such that for \( R \) large enough

\[
\int_{B(u_j, R)} |f_j|^p \geq 1 - \varepsilon(R).
\]

Translating \( f_j(v) \) into \( f_j(u_j v) \), we make a new maximizing sequence \( \{f_j\} \) satisfying

\[
\int_{B(R)} |f_j|^p \geq 1 - \varepsilon(R). \tag{2.2}
\]
Now let us prove that we can take a convergent subsequence of the maximizing sequence of (1.5) by using (2.2).

**Lemma 2.4.** Let \( \{ f_j \} \subset L^p(G) \) be a maximizing sequence of (1.5) satisfying \( \| f_j \|_p = 1 \) and (2.2). Assume that \( f_j \rightharpoonup f \) weakly in \( L^p(G) \), then there exists a subsequence of \( \{ f_j \} \) (still denoted by \( \{ f_j \} \)) such that

\[
S(f_j) \xrightarrow{a.e.} S(f).
\]

**Proof:** We show \( S(f_j) \rightharpoonup S(f) \) in measure to ensure existence of a point-wisely convergent subsequence of \( \{ f_j \} \). A direct computation yields

\[
\left| \left| S(f_j) \chi_{(B(M))^c} \right| \right|_q \quad \leq \quad \left| \left| S(f_j \chi_{B(R)} \chi_{(B(M))^c} \right| \right|_q + \left| \left| S(f_j \chi_{B(R)} \chi_{(B(M))^c} \right| \right|_q
\]

\[
\leq \quad \left| \left| S(f_j \chi_{B(R)} \chi_{(B(M))^c} \right| \right|_q + C_0 \left| \left| f_j \chi_{B(R)} \chi_{(B(M))^c} \right| \right|_p
\]

\[
\leq \quad \left| \left| S(f_j \chi_{B(R)} \chi_{(B(M))^c} \right| \right|_q + \varepsilon(R).
\]

Notice that for \( M > R, v \in B(R), u \in (B(M))^c \), it follows \( |u^{-1}v| \geq |u| - R \). We apply Minkowski’s integral inequality to get

\[
\left| \left| S(f_j \chi_{B(R)} \chi_{(B(M))^c} \right| \right|_q
\]

\[
= \quad \left( \int_{|u| > M} \left| \int_{|v| \leq R} \left| \frac{f_j(v)}{|u|^{\alpha} |u^{-1}v|^{\lambda} |v|^{\beta}} \right|^q dv \right|^q du \right)^{1/q},
\]

\[
\leq \quad \left( \int_{|u| > M} \left| \frac{1}{|u|^{\alpha q} (|u| - R)^{\lambda q}} \right|^q du \right)^{1/q} \left( \int_{|v| \leq R} \left| \frac{f_j(v)}{|v|^{\beta}} \right|^q dv \right)^{1/q}
\]

\[
\leq \quad C \left( \frac{1}{(1-R/M)^{\lambda q}} \right)^{1/q} \left( \int_{M}^{+\infty} \left| \int_{R}^{+\infty} \frac{r^{Q-1-\alpha q - \lambda q}}{(1-R/r)^{\lambda q}} dr \right| dv \right)^{1/q}
\]

\[
\left( \int_{|v| \leq R} \left| f_j(v) \right|^{t'} dv \right)^{1/t'} \left( \int_{|v| \leq R} \left| \frac{1}{|v|^{\beta}} \right| dv \right)^{1/t},
\]

(2.5)

where \( t' \) denote the conjugate index of \( t \) such that \( t' < p, t < Q/\beta \).

Since \( Q - \alpha q - \lambda q = q (\beta - Q/p') < 0 \) and \( f_j \in L^p(G) \subset L^p_{loc}(G) \), for every fixed \( R, M \gg R \), we have

\[
\left| \left| S(f_j) \chi_{(B(M))^c} \right| \right|_q \leq \varepsilon(R).
\]

(2.6)
Similarly, one has

\[ \left\| S(f) \chi_{(B(M))}^c \right\|_q \leq \varepsilon(R). \]  

Thus, for every \( k > 0 \), it implies

\[ \left\{ \left| S(f_j) - S(f) \right| \geq 15k \right\} \]
\[ \leq \left\{ \left| S(f_j) - S(f) \chi_{B(M)} \right| \geq 5k \right\} + \left\{ \left| S(f) \chi_{B(M)} - S(f) \chi_{B(M)} \right| \geq 5k \right\} + \left\{ \left| S(f) \chi_{B(M)} - S(f) \right| \geq 5k \right\} \]
\[ \leq 2 \left[ \frac{\varepsilon(R)}{5k} \right]^q + \left\{ \left| S(f_j) - S(f) \right| \geq 5k \right\} \cap B(M). \]  

Now the remainder is to estimate \( \left\{ \left| S(f_j) - S(f) \right| \geq 5k \right\} \cap B(M) \). Noting

\[ \left\{ \left| S(f_j) - S(f) \right| \geq 5k \right\} \cap B(M) \]
\[ \leq \left\{ \left| S(f_j) - S(f) \chi_{B(R')} \right| \geq k \right\} + \left\{ \left| S(f) \chi_{B(R')} - S^q(f \chi_{B(R')}) \right| \geq k \right\} \cap B(M) \]
\[ + \left\{ \left| S^q(f \chi_{B(R')}) - \chi_{B(M)} \right| \geq k \right\} + \left\{ \left| S(f) \chi_{B(R')}(u) - S(f) \right| \geq k \right\} \]
\[ := J_1 + J_2 + J_3 + J_4 + J_5, \]  

where

\[ S^q(f)(u) = \int_{(B(u, \eta))^c} \frac{f(v)}{|u|^\alpha |u^{-1}v|^\lambda |v|^\beta} dv, R' > 0, \]

we estimate \( J_1, J_2, J_3, J_4 \) and \( J_5 \) respectively.

First notice that

\[ S^q(f_j \chi_{B(R')})(u) \to S^q(f \chi_{B(R')})(u), \forall u \in G. \]

Because \( |u|^{-\alpha} |u^{-1}v|^{-\lambda} |v|^{-\beta} \chi_{B(R')} \chi_{(B(u, \eta))^c} \in L^{p'}(G) \), we observe

\[ J_3 = \left\{ \left| S^q(f \chi_{B(R')}) - S^q(f \chi_{B(R')}) \right| \geq k \right\} \cap B(M) = o(1), \text{ as } j \to \infty. \]

Since \( f_j \to f \) weakly in \( L^p(G) \), we get

\[ \left\| f \chi_{(B(R'))^c} \right\|_p^p \leq \liminf_{j \to \infty} \left\| f_j \chi_{(B(R'))^c} \right\|_p^p \leq \varepsilon(R'), \]

it yields

\[ J_1 \leq \frac{1}{kq} \left\| S(f_j) - S(f) \chi_{B(R')} \right\|_q^{q} \leq \frac{C_0}{kq} \left\| f_j \chi_{(B(R'))^c} \right\|_p^q \leq \left( \frac{\varepsilon(R')}{k} \right)^q, \]  

\[ J_5 \leq \frac{1}{kq} \left\| S(f) - S(f \chi_{B(R')}) \right\|_q^{q} \leq \frac{C_0}{kq} \left\| f \chi_{(B(R'))^c} \right\|_p^q \leq \left( \frac{\varepsilon(R')}{k} \right)^q. \]
To compute $J_2$ and $J_4$, we claim the following two statements:

1) When (H1) holds, there is $m_1 \in (1, Q/\lambda)$ such that for a fixed $R'$,
\[ \| S(f_j \chi_{B(R')}) - S^n(f_j \chi_{B(R')}) \|_{m_2} \leq O(\eta), \eta \to 0. \] (2.14)

2) When (H2) holds, there exists $m_2 \in (1, Q/(\lambda + \alpha))$ such that for a fixed $R'$,
\[ \| S(f_j \chi_{B(R')}) - S^n(f_j \chi_{B(R')}) \|_{m_2} \leq O(\eta), \eta \to 0. \] (2.15)

We first maintain them and give their proofs latter. Choosing $m = \chi_{(-\infty, 0]}(\alpha)_m + \chi_{(0, +\infty)}(\alpha)_m$, one has for the condition (H1) or (H2),
\[ J_2 \leq \| S(f_j \chi_{B(R')}) - S^n(f_j \chi_{B(R')}) \|_m \leq O(\eta). \] (2.16)

Similarly it follows for $f$,
\[ J_4 \leq \| S(f \chi_{B(R')}) - S^n(f \chi_{B(R')}) \|_m \leq O(\eta). \] (2.17)

Substitute (2.11)-(2.13),(2.16) and (2.17) into (2.9) we have
\[ \| \{ |S(f_j) - S(f)| \geq 5k \} \cap B(M) \| \leq 2 \left[ \frac{\varepsilon(R')}{k} \right]^q + 2 \left[ \frac{O(\eta)}{k} \right]^m + o(1). \]

It shows that \{ $S(f_j)$ \} is convergent in measure, and then \{ $f_j$ \} is convergent in measure by properly choosing $\varepsilon, R, R'$ and $\eta$.

**Proof of (2.14) and (2.15).** Concretely, we need to prove (2.14) under the condition $\beta < Q/p'$ and $\alpha \leq 0$, and (2.15) under the condition $-Q/p < \beta < Q/p'$ and $0 < \alpha < \min\{ (\frac{Q-\beta}{Q}) (\frac{Q}{p'} - \beta), Q/q \}$.

To prove (2.14), we choose $m_1 \in (1, Q/\lambda)$ and apply Minkowski’s integral inequality,
\[ \| S(f_j \chi_{B(R')}) - S^n(f_j \chi_{B(R')}) \|_{m_1} \]
\[ = \left( \int_G \left| \int_{B(u, \eta)} \frac{f_j(v) \chi_{B(R')}(v)}{|u| |u-1v|^\lambda |v|^{\beta}} \, dv \right|^{m_1} \, du \right)^{1/m_1} \]
\[ \leq \int_G \left( \int_{B(v, \eta)} \left| \frac{f_j(v) \chi_{B(R')}(v)}{|u| |u-1v|^\lambda |v|^{\beta}} \right|^{m_1} \, dv \right)^{1/m_1} \, du \]
\[ = \int_G \left| \int_{B(v, \eta)} \frac{1}{|v|^{\beta}} \, dv \right|^{1/m_1} \, du \]
\[ \leq C(R', p, n)(\eta + R')^{-\alpha} \eta^{(Q-\lambda m_1)/m_1} \int_{B(R')} \frac{|f_j(v)|}{|v|^{\beta}} \, dv, \]
where $\beta < Q/p'$, $\lambda m_1 < Q$. Therefore for a fixed $R'$, as $\eta \to 0$, we get (2.14).

As to (2.15), we see

$$\frac{Q/p' - \beta}{(Q/p' - \beta) - \alpha} < \frac{Q}{\lambda}, (\lambda + \alpha) < Q,$$

and choose $m_2$ such that $1 < m_2 < Q/(\lambda + \alpha)$ and $1 < m_2 < \frac{Q}{\lambda} \cdot \frac{(Q/p' - \beta) - \alpha}{Q/p' - \beta}$. From $m_2 < Q/(\lambda + \alpha)$ we get

$$Q/(Q - \lambda m_2) < Q/\alpha m_2.$$

Notice that $m_2 < \frac{Q}{\lambda} \cdot \frac{(Q/p' - \beta) - \alpha}{Q/p' - \beta}$ implies that

$$\frac{Q}{Q - \lambda m_2} < \frac{1}{\alpha} \cdot \frac{Q - \lambda}{p' - \beta}.$$

Thus, we can take $l$ such that $Q/(Q - \lambda m_2) < l < \min\{Q/\alpha m_2, \frac{1}{\alpha} \cdot \frac{Q - \lambda}{p' - \beta}\}$. It yields

$$\alpha m_2 l < Q, \lambda m_2 l' < Q, \beta + \alpha l < Q/p',$$

in which $l' = l/(l - 1)$.

Applying Minkowski’s integral inequality to get

\[
\begin{align*}
&\left\| S(f_j \chi_{B(R')}) - S^n(f_j \chi_{B(R')}) \right\|_{m_2} \\
= &\left( \int_G \left( \int_{B(u, \eta)} \frac{f_j(v) \chi_{B(R')}(v)}{u^{|\alpha|} u^{-1} v^{|\alpha^2|} v^{|\beta|}} dv \right)^{m_2} \frac{1}{u^{m_2}} du \right)^{1/m_2} \\
\leq &\int_G \left( \int_{B(v, \eta)} \left| \frac{f_j(v) \chi_{B(R')}(v)}{u^{|\alpha|} u^{-1} v^{|\alpha^2|} v^{|\beta|}} \right|^{m_2} \frac{1}{u^{m_2}} du \right)^{1/m_2} dv \\
= &\int_G \left( \frac{f_j(v) \chi_{B(R')}(v)}{|v|^\beta} \right) \left( \int_{B(v, \eta)} \frac{1}{u^{|\alpha|} u^{-1} v^{|\alpha^2|} v^{|\beta|}} du \right)^{1/m_2} dv \\
\leq &\int_G \left( \frac{f_j(v) \chi_{B(R')}(v)}{|v|^\beta} \right) \left( \int_{B(v, \eta)} \frac{1}{l^{|\alpha|} u^{|\alpha^2|} l u^{|\beta|}} + \frac{1}{l^{|\alpha|} u^{-1} v^{|\alpha^2|} v^{|\beta|} u^{|\beta|}} du \right)^{1/m_2} dv \\
\leq &\int_G \left( \frac{f_j(v) \chi_{B(R')}(v)}{|v|^\beta} \right) \left( \int_{B(v, \eta)} \frac{1}{l^{|\alpha|} u^{|\alpha^2|} u^{|\beta|}} du \right)^{1/m_2} dv + \int_G \left( \frac{f_j(v) \chi_{B(R')}(v)}{|v|^\beta} \right) \left( \int_{B(v, \eta)} \frac{1}{l^{|\alpha|} u^{|\alpha^2|} l u^{|\beta|}} du \right)^{1/m_2} dv \\
:= &I_1 + I_2. 
\end{align*}
\]
To estimate $I_1$, noting $\alpha m_2 l < Q$, $|u|^{-\alpha m_2 l} \in L_{loc}(G)$, and applying Lebesgue's integral theorem, we have that as $\eta \to 0$,

$$\frac{1}{|B(v, \eta)|} \int_{B(v, \eta)} \frac{1}{|u|^{\alpha m_2 l}} du = \frac{1}{|v|^{\alpha m_2 l}} + O(\eta^{m_2}).$$

Thus

$$I_1 \lesssim \int_G \frac{|f_j(v)\chi_{B(R')}(v)|}{|v|^{\beta}} \left( \frac{C \eta^Q}{|B(v, \eta)|} \int_{B(v, \eta)} \frac{1}{|u|^{\alpha m_2 l}} du \right)^{1/m_2} dv$$

$$\lesssim C \eta^{Q/m_2} \int_G \frac{|f_j(v)\chi_{B(R')}(v)|}{|v|^{\beta}} \left( \frac{1}{|B(v, \eta)|} \int_{B(v, \eta)} \frac{1}{|u|^{\alpha m_2 l}} du \right)^{1/m_2} dv$$

$$\lesssim C \eta^{Q/m_2} \int_{B(R')} \frac{|f_j(v)|}{|v|^{\beta + \alpha l}} dv + O(\eta).$$

Since $f_j \in L^p$, $\beta + \alpha l < Q/p'$, it follows

$$I_1 = O(\eta). \quad (2.19)$$

Since $\beta < Q/p'$, $\lambda m_2 l' < Q$, we have that as $\eta \to 0$,

$$I_2 \lesssim \left( \int_{B(R')} |f_j(v)|^{p} dv \right)^{1/p} \left( \int_{B(R')} \frac{1}{|v|^{\beta p'}} dv \right)^{1/p'} \left( \int_{B(\eta)} \frac{1}{|u|^{\lambda m_2 l'}} du \right)^{1/m_2}$$

$$\lesssim O(\eta). \quad (2.20)$$

Combining with (2.18)-(2.20), it yields (2.15). □

### 2.3 The second concentration compactness principle

Now we have found a weakly convergent subsequence of the maximizing sequence of (1.5). To complete the proof of Theorem 1.1, it wants to prove that this subsequence converges strongly. For the purpose, we need the second concentration compactness principle on $G$, which is a special case of known results in measure spaces, see [13]. Here it is a description in $G$.

**Lemma 2.5.** Let $f_j \rightharpoonup f$ weakly in $L^p(G)$, $S(f_j) \rightharpoonup S(f)$ weakly in $L^p(G)$. Assume that (2.2) and (2.6) hold, and the nonnegative measures $\mu_j \to \mu$ weakly in $L(G)$, $|S(f_j)|^{q} du \to \nu$ in $L(G)$. Then, there exist two at most countable families $\{u_j\} \subset G, \{k_j\} \subset (0, \infty)$ such
that

\[
v = |S(f)|^q du + \sum_j C_0 k_j^{q/p} \delta_{u_j},
\]

\[
\mu \geq |f|^p du + \sum_j k_j \delta_{u_j},
\]
in which \(\delta_{u_j}\) is the Dirac measure at \(u_j\).

3 Proof of Theorem 1.1

Proof: Let \(\{f_j\}\) be the maximizing sequence in (1.5) satisfying (2.2), \(f_j \to f\) weakly in \(L^p(G)\) and condition (2.3) hold. We will show \(\|f\|_p = 1\). Let us notice that \(\mu(G) = 1\), \(\nu(G) = C_0^q\). If \(\|f\|_p^p = k < 1\), then

\[
\sum_j k_j \leq \mu(G) - \|f\|_p^p = 1 - k.
\]

Therefore,

\[
\nu(G) = \|S(f)\|_q^q + \sum_j C_0^q k_j^{q/p} \\
\leq C_0^q \|f\|_p^p + C_0^q (\sum_j k_j)^{q/p} \\
\leq C_0^q k_0^{q/p} + C_0^q (\sum_j k_j)^{q/p} \\
\leq C_0^q k_0^{q/p} + C_0^q (1 - k)^{q/p} \\
< C_0^q,
\]

which contradicts with the fact that \(\nu(G) = C_0^q\) and we complete the proof of Theorem 1.1. \(\Box\)

4 Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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