Pattern formation in growing sandpiles

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Abstract – Adding grains at a single site on a flat substrate in the Abelian sandpile models produces beautiful complex patterns. We study in detail the pattern produced by adding grains on a two-dimensional square lattice with directed edges (each site has two arrows directed inward and two outward), starting with a periodic background with half the sites occupied. The model shows proportionate growth and the size of the pattern formed by adding $N$ grains scales as $\sqrt{N}$. We give exact characterization of the asymptotic pattern, in terms of the position and shape of different features of the pattern.

Introduction. – While real sand, poured at one point on a flat substrate, produces a rather simple pyramidal shape, very beautiful, but complex patterns are produced this way in theoretical models of sandpiles, like the Abelian sandpile model (see fig. 1). In general, a detailed and exact mathematical characterization of such patterns has not been possible so far\(^1\). This is what we do in this paper, for the specific case of patterns produced on two directed lattices, starting from a simple periodic background.

The reason for interest in this problem is twofold. Firstly, these are examples of complex patterns that are obtained from simple deterministic evolution rules, which are analytically tractable, and hence understanding them should also help in studying the more general problem. Secondly, these are the simplest examples that show nontrivial spatial patterns, and proportionate growth. In the animal kingdom, a young animal, typically, grows in size with time, with different parts of the body growing roughly at the same rate. Proportionate growth requires some coordination and communication between different parts. While there are many models of growing objects studied in physics literature so far, e.g. the Eden model, diffusion-limited aggregation, invasion-percolation etc. [3], all of these are mainly models of aggregation, where growth occurs by accretion on the surface of the object, and inner parts do not evolve significantly. Also, in biology, one needs to ensure a supply of “food” to different parts of the body for the growth to occur. The same nutrients are used, but get converted into different types of tissues in different regions. Our simple mathematical model captures these important qualitative features, even though the actual mechanism of growth is much more complicated in the biological world.

Sandpile models were originally introduced in physics in the context of self-organized criticality, where the main
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Fig. 2: The directed square lattices studied in this paper: (a) the F-lattice, (b) the Manhattan lattice.

interest has been the power law tail in the distribution of avalanche sizes [4]. A special class of these, which are analytically tractable, is called Abelian Sandpile Models (ASM) [5]. In this paper our interest is different. We study the asymptotic pattern produced by adding \( N \) grains of sand at a single site in a two-dimensional Abelian sandpile model starting from a periodic background, and allowing the system to relax. For large \( N \), the pattern produced shows proportionate growth, with the linear dimension of all the features scaling as \( \sqrt{N} \).

There has been some earlier work on growing sandpiles, in particular [6]. Other papers have mostly dealt with determining the asymptotic shape of the boundary of the growing cluster [7,8]. The limiting shape has been determined in the related rotor-router model, and in the model of divisible sandpiles with multiple sites of addition [9]. Other spatial configurations in the ASM models, like the identity [10], or the stable state produced from special unstable states also show complex internal self-similar structures [11]. In particular, the identity configuration on the F-lattice has recently been shown to have spatial structure similar to what we study here [12].

**Definition of the model.** – The standard square lattice produces a rather complicated pattern (fig. 1), and it has not been possible to characterize it so far. In this paper, we consider two variations, assigning orientations to the edges of the lattice, as shown in fig. 2(a) and fig. 2(b). The initial state was chosen to be a periodic checkerboard arrangement of sites with heights 0 and 1. The asymptotic pattern produced in the two cases turns out to be the same, and is shown in fig. 3 and fig. 4. The pattern is somewhat simpler than in fig. 1, which makes its study easier. Taking some qualitative features of the observed pattern (e.g. only two types of patches are present, and they are all 3- or 4-sided polygons) as input, we show how one can get a complete and quantitative characterization of the pattern. We also show that the asymptotic pattern has an unexpected exact 8-fold rotational symmetry, and determine the exact coordinates of all the boundaries in it. We discuss some other cases, where exactly the same pattern is obtained.

In fig. 2, each bond of the lattice is directed with two in-arrows, and two out-arrows at each vertex. The ASM on these lattices is defined by the toppling rule: a site \( (x,y) \) is unstable if the number of grains at the site \( z_{x,y} \geq 2 \), and then transfers one grain each in the direction of its outward arrows. We start with an initial configuration in which \( z_{x,y} = 1 \), for sites with \( (x+y) = \) even, and 0, otherwise.

We used a lattice large enough so that none of the avalanches, started from the origin, reaches the boundary. Using the Abelian property, we add all \( N \)-particles in the beginning, and relax the configuration to get the final pattern. The result of adding \( N = 5 \times 10^4 \) particles on the F-lattice is shown in fig. 3. The pattern formed on the Manhattan lattice is indistinguishable from this at large scales except that the thin lines of 1’s forming two

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triangles outside the octagon are rotated by 45° (fig. 4). Since the lattices are different, this is quite intriguing. Specially there is no obvious lattice symmetry, and it is easily checked that patterns produced for small N are quite different (fig. 5).

Characterizing asymptotic pattern: general theory. – It was already noted in [6] that for N large, the asymptotic pattern is made of union of distinct regions, called “patches”, where inside each patch the heights are periodic in space. There are few defect-lines, which move with N, and can also be seen in fig. 1 and fig. 3. But these can be ignored in discussing the asymptotic pattern. Formally, we can characterize the asymptotic pattern in terms of rescaled coordinates, \( x = x/\sqrt{N} \), \( y = y/\sqrt{N} \) and the density function \( \rho(\xi, \eta) \) which gives the local density of grains in the pattern in a small rectangle of size \( \Delta \xi \), \( \Delta \eta \) about the point \((\xi, \eta)\), with \( N^{-1/2} < \Delta \xi, \Delta \eta \ll 1 \). Then, within each patch, \( \rho(\xi, \eta) \) takes a constant rational value.

Let \( T_N(x, y) \) be the number of toppling at site \((x, y)\) when N-particles are added at the origin, and the configuration is relaxed. We define the scaling limit of this function by

\[
\phi(\xi, \eta) = \lim_{N \to \infty} \frac{1}{2N} T_N([\sqrt{N} \xi], [\sqrt{N} \eta]),
\]

where the floor function \([x]\) is the largest integer \( \leq x \). From the conservation of sand grains, it is easily seen that \( \phi(\xi, \eta) \) is related to the density function \( \rho(\xi, \eta) \) by

\[
\left( \frac{\delta^2}{\delta \xi^2} + \frac{\delta^2}{\delta \eta^2} \right) \phi(\xi, \eta) = \Delta \rho(\xi, \eta) - \delta(\xi) \delta(\eta),
\]

where the excess density \( \Delta \rho(\xi, \eta) \) is the difference between \( \rho(\xi, \eta) \) and the initial density \( \rho_0(\xi, \eta) \). In an electrostatic analogy, we can think of \( \Delta \rho(\xi, \eta) \) as a real charge density, and \( \phi(\xi, \eta) \) as the corresponding electrostatic potential.

The key observation that allows us to determine the asymptotic pattern is that in each patch of constant \( \Delta \rho(\xi, \eta) \), \( \phi(\xi, \eta) \) is a quadratic function of \( \xi \) and \( \eta \). This was already noted in [6]. We indicate the proof here. In each patch the function \( \phi(\xi, \eta) \) is Taylor expandable around any point inside the patch. Consider any term of order \( \geq 3 \) in the expansion, for example the term \( \sim (\Delta \xi)^3 \). This can only arise due to a term \( \sim (\Delta x)^3/\sqrt{N} \) in \( T(x, y) \). Then the integer function \( T(x, y) \) will change discontinuously at intervals of \( \Delta x \sim \mathcal{O}(N^{1/6}) \) leading to infinitely many defect-lines in the asymptotic pattern. However there are no such features in fig. 1 or fig. 3. Therefore inside a patch of constant \( \Delta \rho(\xi, \eta) \), \( \phi(\xi, \eta) \) can at most be quadratic in \( \xi \) and \( \eta \). In each periodic patch the toppling function \( T(x, y) \) is a sum of two terms: a patch that is a simple quadratic function of \( x \) and \( y \), and a periodic part. The periodic part averages to zero, and does not contribute to the coarse-grained function \( \phi(\xi, \eta) \) (see footnote 2).

Now consider two neighboring periodic patches \( P \) and \( P' \) with mean densities \( \rho \) and \( \rho' \), respectively. Let the quadratic toppling function be \( Q(\xi, \eta) \) and \( Q'(\xi, \eta) \) in these patches. Then the boundary between the patches is given by the equation \( Q(\xi, \eta) = Q'(\xi, \eta) \). As the derivatives of \( \phi \) are also continuous across the boundary, the boundary between two periodic patches must be a straight line, and

\[
Q'(\xi, \eta) = Q(\xi, \eta) = \frac{1}{2}(\rho' - \rho)^2 \theta_i, \quad (3)
\]

where \( \theta_i \) is the perpendicular distance of \((\xi, \eta)\) from the boundary. We can start with a periodic patch \( P \), and go to another patch \( P' \) by more than one path. Since the final quadratic function at \( P' \) should be the same whichever path we take, this imposes consistency conditions which restrict the allowed values of slopes of the boundaries. Consider a point \( z_0 \) where \( n \) periodic patches meet, with \( n > 2 \) (fig. 6). If the \( j \)-th boundary at this point makes an angle \( \theta_j \) with the \( x \)-axis, and the density of the patch in the wedge \( \theta_j \leq \theta \leq \theta_{j+1} \) is \( \rho_{j+1} \) (fig. 6), then the condition that the net change in the quadratic form is zero if we go around \( z_0 \) once, reduces to the following condition:

\[
\sum_{j=1}^{n} (\rho_{j+1} - \rho_j) e^{2i\theta_j} = 0, \quad (4)
\]

Footnote 2: In some patterns, with other backgrounds (not discussed here) there are regions of finite fractional area which show aperiodic height patterns. These cases are harder to analyze.
with \( \rho_{n+1} = \rho_1 \). For \( n = 3 \), with \( \rho_1 \neq \rho_2 \neq \rho_3 \), this equation has only trivial solutions with \( \theta_j \) equal to 0 or \( \pi \) for all \( j \). Hence, only \( n \geq 4 \) are allowed.

**Determination of the potential function.** – We now discuss how the exact function \( \rho(\xi, \eta) \) can be determined for our problem. We note that in fig. 3, there are no aperiodic patches, only two types of periodic patches, where \( \rho(\xi, \eta) \) only take values 1 or 1/2. Also, the slopes of the boundaries between patches only take values 0, \( \pm 1 \) or \( \infty \). The patches are typically dart shaped quadrilaterals, and some triangles. These simplifications, not present in fig. 1, make possible a full characterization of the pattern in fig. 3.

We start by determining the exact asymptotic size of the pattern. We note from fig. 3 that the boundary of the pattern is an octagon (we shall prove later that this is a regular octagon). In fact, there are four lines of 1’s outside the octagon. But these have zero areal density in the limit \( N \to \infty \), and do not contribute to \( \rho(\xi, \eta) \). We will ignore these in the following discussion.

Let \( B \) be the minimum boundary square containing all \( (\xi, \eta) \) that have a non-zero charge density \( \Delta \rho(\xi, \eta) \). We observe that \( B \) can be considered as a union of disjoint smaller squares, each of which is divided by a diagonal into two parts where \( \Delta \rho(\xi, \eta) \) takes values 1/2 and 0 (fig. 7). This is seen to be true for the outer layer patches. Towards the center, the squares are not so well resolved. Assuming that this construction remains true all the way to the center, in the limit of large \( N \), the mean density of negative charge in the bounding square = 1/4. Given that the total amount of negative charge is \( -1 \), the area of the bounding square should be 4. Hence, the boundaries of the minimum bounding square are

\[
|\xi| = 1, \quad |\eta| = 1.
\]

Let \( N_b \) be the minimum number of particles that have to be added so that at least one site at \( y = b \) topples. We find that for \( b = 10, 50, 100, \) and 300, \( \sqrt{N_b} = 10.770, 49.436, 98.894 \) and 297.798. This is consistent with eq. (5).

We now describe the topological structure of the pattern. We note that the patches become smaller, and there are more of them in number, as we move towards the center. One can use a coordinate transformation \( r' = 1/r^2 \), \( \theta' = \theta \) to avoid this overcrowding (fig. 8). We can now draw the adjacency graph (fig. 9(a)) of the pattern, where each vertex denotes a patch, and a bond between the vertices is drawn if the vertices share a common boundary. It is convenient to think of the triangular patches in the pattern as degenerate quadrilaterals, with one side of length zero. Then we see that the adjacency graph is planar with each vertex of degree four, except a single vertex of coordination number eight corresponding to the exterior of the pattern. The graph has the structure of a square lattice wedge of wedge angle \( 4\pi \). The square lattice structure of the adjacency graph is seen more clearly, if rather than \( 1/r^2 \) transformation, the transformation used is \( z' = 1/z^2 \) (this has been used earlier in [6]), where \( z = \xi + i\eta \), and view it in the complex \( z' \)-plane. Thus, one can equivalently represent the graph as a square grid on a Riemann surface of two sheets (fig. 9(b)).

We now use the qualitative information obtained from the adjacency matrix of the observed pattern, to obtain
quantitative prediction of the exact coordinates of all the patches. Consider an arbitrary patch \( P \), having an excess density 1/2. The potential function in the patch is a quadratic function of \((\xi, \eta)\) and we parametrize it as

\[
\phi_P(\xi, \eta) = \frac{1}{8}(m_P + 1)\xi^2 + \frac{1}{4}n_P \xi \eta + \frac{1}{8}(1 - m_P)\eta^2 + d_P \xi + e_P \eta + f_P. \tag{6}
\]

The potential function in a patch \( P \) having zero excess density will be parametrized as

\[
\phi_P(\xi, \eta) = \frac{1}{8}(m_P^* + 1)\xi^2 + \frac{1}{4}n_P^* \xi \eta + \frac{1}{8}(1 - m_P^*)\eta^2 + d_P^* \xi + e_P^* \eta + f_P^*. \tag{7}
\]

Now consider two neighboring patches \( P \) and \( P' \) with excess densities 1/2 and 0, respectively. Then using the matching condition (eq. (3)), it is easy to show that if the boundary between them is a horizontal line \( \eta = \eta_P \), we must have

\[
m_P = m_P + 1, \quad n_P = n_P, \quad d_P = d_P, \quad e_P = e_P + \eta_P/2, \quad f_P = f_P - \eta_P^2/4. \tag{8}
\]

There are similar conditions for other boundaries. These result a coupled set of linear equations for the coefficients \( \{m_P, n_P, d_P, e_P, f_P\} \). The equations for \( m_P \) and \( n_P \) do not involve other variables. In the outermost patch, clearly \( \phi(\xi, \eta) = 0 \), and for this patch both \( m \) and \( n \) are zero. It follows that \( m_P \) and \( n_P \) are integers, equal to the Cartesian coordinates of the vertex corresponding to the patch \( P \) in the discretized Riemann surface in fig. 9(b). In the following, we denote a patch by integers \((m, n)\), and write the corresponding coefficients \( d_P, e_P, f_P \) as \( d_{m,n}, e_{m,n} \) and \( f_{m,n} \). With this convention, the matching conditions in eq. (8) can be rewritten as

\[
d_{m+1,n} = d_{m,n}, \quad e_{m+1,n} = -e_{m,n} = \eta_{m,n}/2, (m + n) \text{ odd}. \tag{9}
\]

Using similar matching conditions for the boundary of patch \((m, n)\) with slope \(\pm 1\), we get the conditions

\[
d_{m,n+1} = d_{m,n}, \quad e_{m,n+1} = -e_{m,n} = \eta_{m,n}/2, (m + n) \text{ odd}, \tag{10}
\]

We can eliminate the variables \(d_{m,n}\) and \(e_{m,n}\) with \((m + n)\) even using eq. (9) and eq. (10). Then the equations become

\[
e_{m+1,n} - e_{m,n} = \eta_{m,n}/2, \tag{11}
\]

\[
d_{m-1,n} - d_{m,n} = \xi_{m,n}/2, \tag{12}
\]

\[
d_{m-1,n} - d_{m,n} = e_{m+1,n-1} - e_{m,n}, \tag{13}
\]

\[
d_{m-1,n} = e_{m,n+1} - e_{m,n}. \tag{14}
\]

It is convenient to introduce the complex variables \( z = \xi + i\eta, M = m + in \) and \( D = d + ie \). In these variables we can write eq. (6) as

\[
\phi(z) = \frac{1}{8}z^n + \frac{1}{8}\text{Re}(z^2M + iz) + f, \tag{15}
\]

On the \((m, n)\) lattice, with \((m + n)\) odd, the natural basis vectors are \((1, 1)\) and \((-1, -1)\). We define the finite difference operators \( \Delta_{\pm \alpha} \) and \( \Delta_{\pm \beta} \) by

\[
\Delta_{\pm \alpha} f(z) = f(z \pm \alpha) - f(z), \quad \Delta_{\pm \beta} f(z) = f(z \pm \beta) - f(z). \tag{16}
\]

Then eqs. (13), (14) can be written as

\[
\Delta_{\pm \alpha} d = \Delta_{\pm \beta} e, \tag{17}
\]

These equations are the discrete analog of the familiar Cauchy-Riemann conditions connecting the partial derivatives of real and imaginary parts of an analytic function where the role of the analytic function is played by \( D = d + ie \).

From eq. (13) and eq. (14), it is easy to deduce that \( D \) satisfies the discrete Laplace’s equation

\[
[\Delta_{\alpha} \Delta_{\beta} + \Delta_{\beta} \Delta_{-\beta}] D = 0. \tag{18}
\]

If \( m \) and \( n \) are large, the corresponding patch is near the origin \((|\xi| + |\eta|) \) small), and where the leading behavior of \( \phi(\xi, \eta) \) is given by \( \phi(\xi, \eta) \sim -1/2\log(\xi^2 + \eta^2) \). Consider a point \( z_0 \), such that at \( z_0 \)

\[
\partial^2 \phi/\partial \xi^2 \approx m/4; \quad \partial^2 \phi/\partial \xi \partial \eta \approx n/4. \tag{19}
\]

Then, \( z_0 \) would be expected to lie in the patch labeled by \((m, n)\). This gives \( z_0 \approx \pm(\pi M/2)^{-1/2} \). Then, setting \( \partial \phi/\partial z \) equal to \( D/2 \) gives us

\[
D_{m,n} \approx \pm \frac{1}{\sqrt{2\pi}} \sqrt{m + in}. \tag{20}
\]

Equation (18), subjected to the behavior at large \(|m| + |n|\) given by eq. (20) on the \( 4\pi \)-wedge graph (for each value of \((m, n)\), \( D_{m,n} \) has two values) has a unique solution. Clearly the solution has eightfold rotational symmetry about the origin in the \((m, n)\) space. This implies that

\[
D_{m,n} = i^{1/2} D_{m,n}; \quad \text{for all} \quad (m, n). \tag{21}
\]

Given \( D_{m,n} \), its real and imaginary parts determine \( d_{m,n} \) and \( e_{m,n} \), and using eq. (11), (12) we determine the exact positions of all the patch corners. The exact eightfold rotational symmetry of the adjacency graph of the pattern, and the fact that \( D \) satisfies eq. (21) on the adjacency graph together imply the eightfold rotational symmetry of all the distances in the pattern.

Note that for the usual square lattice, the solution of eq. (18) is the well-known 2-dimensional lattice Greens function, that is explicitly calculable for any finite \((m, n)\), and is a simple polynomial of \(1/\pi \) with rational coefficients [13]. However, for our case of the two sheeted Riemann surface, we have not been able to find a closed-form formula for \( D_{m,n} \). But the solution can be
determined numerically to very good precision by solving it on a finite grid \(-L \leq m, n \leq L\) with the condition in eq. (20) imposed exactly at the boundary. We determined \(d_{m,n}\) and \(e_{m,n}\) numerically for \(L = 100, 200, 400\), and extrapolated our results for \(L \to \infty\). We find \(d_{1,0} = 0.5000\) and \(d_{2,1} = 0.6464\), in perfect agreement with the exact theoretical values \(1/2\) and \(1 - 1/2\sqrt{2}\), respectively.

**Extensions.** – Our arguments above can be extended to other two-dimensional lattices, so long as there are only two allowed values of \(\Delta \rho\). As mentioned already, and we do not know why, this seems to happen for the Manhattan lattice (fig. 4), for initial density \(1/2\). Also, this happens on the F-lattice, with a periodic background pattern with initial density 5/8 (\(z_{i,j} = 1\) if \(i + j\) even, or \((i, j)\) congruent to \((0, 1)\) or \((2, 3)\) mod 4) (fig. 10). In some other cases, like the F-lattice, with initially all sites empty, the pattern is very similar, but there are some non periodic patches in the outermost ring (fig. 11). Since the behavior of \(\phi(\xi, \eta)\) in such patches is not known, the equations for \(D_{m,n}\) do not close in this case.

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