Completion by perturbations

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Abstract
Any non-complete orthonormal system in a Hilbert space can be transformed into a basis by small perturbations.

1. Introduction. In 2006 Harold Shapiro posed the following problem: does there exist a compact family of functions that provides an orthonormal basis in $L_2(\mathbb{R})$ by appropriate translates?

Recall that a set of translates of a single function is never a Riesz basis (Olson, Zalik [5]). Moreover, a finite union of such systems cannot be even a frame (Christensen, Deng, Heil [3]).

Nevertheless, the answer to Shapiro’s question turns out to be positive: in [4] we constructed a compact family $\{\phi_n(t)\}$ of functions in $L_2(\mathbb{R})$ such that the set $\{\phi_n(t-n)\}$ is an orthonormal basis.

In the present note we use that technique in a more general situation.

Let $H$ be a separable Hilbert space. The standard way to make a basis out of a non-complete system of orthogonal vectors is adding the appropriate quantity of missing vectors.

Our goal here is to show another approach to the problem of completion: one can get a basis just by a gentle perturbation of the original vectors.

Theorem. Let $\{\chi_n\}, n \in \mathbb{N}$, be an orthonormal system of vectors in $H$. Then there exists an orthonormal basis $\{\psi_n\}$ such that

$$\|\psi_n - \chi_n\| \longrightarrow_ {n \to \infty} 0.$$

In addition, all the norms can be made smaller than a given $\epsilon > 0$. 

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2. Proof of Theorem. We introduce the following \((n + 1) \times (n + 1)\) matrices:

\[
A_n = \begin{pmatrix}
1 - \gamma_n & -\gamma_n & \cdots & -\gamma_n & \frac{1}{\sqrt{2n}} \\
-\gamma_n & 1 - \gamma_n & \cdots & -\gamma_n & \frac{1}{\sqrt{2n}} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-\gamma_n & -\gamma_n & \cdots & 1 - \gamma_n & \frac{1}{\sqrt{2n}} \\
-\frac{1}{\sqrt{2n}} & -\frac{1}{\sqrt{2n}} & \cdots & -\frac{1}{\sqrt{2n}} & \frac{1}{\sqrt{2}}
\end{pmatrix},
\]

where \(\gamma_n = \frac{1}{(2 + \sqrt{2})n}\). It is easy to check that the matrices are orthogonal. Their key feature is that the upper left \(n \times n\) submatrix is close to the identity matrix while the lower right element is essentially between 0 and 1.

Our construction below is inspired by Bourgain’s paper [2]. We will use the following Lemma.

**Lemma.** Let \(\Psi = \{\psi_n\}\) be a set in a Hilbert space. For some \(\alpha < 1\), suppose there is a set \(\Gamma\), dense in the unit sphere, such that every \(g \in \Gamma\) can be approximated, with an error less than \(\alpha\), by a linear combination of vectors \(\psi_n\). Then \(\Psi\) is complete.

**Proof.** If \(\Psi\) were not complete, then there would be a vector \(f, \|f\| = 1\), orthogonal to span(\(\Psi\)). Take \(g \in \Gamma\) with \(\|f - g\| < 1 - \alpha\), and find \(\psi \in \text{span}(\Psi)\) with \(\|g - \psi\| < \alpha\). Then \(\|f - \psi\| < 1\) contradicts the choice of \(f\). \(\square\)

Fix a sequence of integers \(0 < n_1 < n_2 < \ldots\), and split the original system \(\{\chi_n\}\) into blocks consisting of the corresponding number of vectors:

\[
\chi_1^{(1)} = \chi_1, \chi_2^{(1)} = \chi_2, \ldots, \chi_{n_1}^{(1)} = \chi_{n_1};
\]

\[
\chi_1^{(2)} = \chi_{n_1 + 1}, \chi_2^{(2)} = \chi_{n_1 + 2}, \ldots, \chi_{n_2}^{(2)} = \chi_{n_1 + n_2};
\]

etc. Denote the linear span of the \(k\)-th block by \(L^{(k)}\). The subspaces \(L^{(k)}\) are pairwise orthogonal. Define also \(L_k = L^{(k)} \oplus L^{(k+1)} \oplus \ldots\). The orthogonal complement of \(L_1\), call it \(L_0\), is non-empty, otherwise \(\{\chi_n\}\) would be a basis.

Let now \(\Gamma = \{g_k\}, k \in \mathbb{N}\), be a dense set in the unit sphere of \(H\), with an additional property: \(g_k \perp L_k \forall k\). Such a set exists since \(\bigcap_k L_k = \{\bar{0}\}\), i.e. \(\bigcup_k L_k^\perp\) is dense in \(H\).

The desired orthonormal basis will be built up inductively.

**Step 1.** Take \(n = n_1\), and apply the matrix \(A_n\) to the orthonormal set of vectors \(\chi_1^{(1)}, \chi_2^{(1)}, \ldots, \chi_n^{(1)}\), and \(g^{(1)} = g_1\):

\[
A_n \begin{pmatrix}
\chi_1^{(1)} \\
\vdots \\
\chi_n^{(1)} \\
g^{(1)}
\end{pmatrix} = \begin{pmatrix}
\psi_1^{(1)} \\
\vdots \\
\psi_n^{(1)} \\
h^{(1)}
\end{pmatrix}.
\]
The obtained vectors $\psi^{(1)}_1, \psi^{(1)}_2, \ldots, \psi^{(1)}_n$, and $h^{(1)}$ are also orthonormal. We have:

$$
\left\| \psi^{(1)}_j - \chi^{(1)}_j \right\|^2 = \left\| (1 - \gamma_n) \chi^{(1)}_j - \gamma_n \sum_{i \neq j} \chi^{(1)}_i + \frac{1}{\sqrt{2n}} g^{(1)} - \chi^{(1)}_j \right\|^2 
$$

$$
= \left\| \frac{1}{\sqrt{2n}} g^{(1)} - \gamma_n \sum_{i=1}^n \chi^{(1)}_i \right\|^2 = \frac{1}{2n} + \gamma_n^2 n = \frac{1}{2n} \left( 1 + \frac{1}{3 + 2\sqrt{2}} \right) < \frac{1}{n} .
$$

Since $A^{-1}_n = A_n^T$ and, consequently, $g^{(1)} = \frac{1}{\sqrt{2n}} \sum_{i=1}^n \psi^{(1)}_i + \frac{1}{\sqrt{2}} h^{(1)}$, we also get

$$
\left\| \sum_{i=1}^n \frac{1}{\sqrt{2n}} \psi^{(1)}_i - g_1 \right\|^2 = \frac{1}{2} .
$$

**Step 2.** Take now $n = n_2$. By definition, the vector $g_2 \in L_2^1 = L_0 \oplus L^{(1)}$. The dimension of this subspace is greater than $n_1$, so we may assume that $g_2$ does not belong to the linear span of $\{\psi^{(1)}_1, \ldots, \psi^{(1)}_{n_1}\}$:

$$
g_2 = \sum_{i=1}^{n_1} \langle g_2, \psi^{(1)}_i \rangle \psi^{(1)}_i + \lambda g^{(2)} ,
$$

where $g^{(2)}$ is orthogonal to all $\{\psi^{(1)}_i\}$, $\|g^{(2)}\| = 1$, and $0 < \lambda \leq 1$. Apply the operator $A_n$ to the orthonormal set of vectors $\chi^{(2)}_1, \ldots, \chi^{(2)}_n$, and $g^{(2)}$:

$$
A_n \begin{pmatrix}
\chi^{(2)}_1 \\
\vdots \\
\chi^{(2)}_n \\
g^{(2)}
\end{pmatrix} = \begin{pmatrix}
\psi^{(2)}_1 \\
\vdots \\
\psi^{(2)}_n \\
h^{(2)}
\end{pmatrix} .
$$

The obtained vectors $\psi^{(2)}_1, \ldots, \psi^{(2)}_n$, and $h^{(2)}$ are again orthonormal. Besides, $\psi^{(2)}_j \perp \psi^{(1)}_i \forall i, j$. Arguing exactly as in Step 1, we have

$$
\left\| \psi^{(2)}_j - \chi^{(2)}_j \right\|^2 < \frac{1}{n} ;
$$

$$
\left\| \sum_{i=1}^n \frac{1}{\sqrt{2n}} \psi^{(2)}_i - g^{(2)} \right\|^2 = \frac{1}{2} .
$$

Due to the decomposition of $g_2$, this equality implies, for some numbers $\lambda^{(k)}_i$, that

$$
\left\| \sum_{k=1}^{n_k} \sum_{i=1}^{\lambda^{(k)}_i} \psi^{(k)}_i - g_2 \right\|^2 \leq \frac{1}{2} .
$$
**Step 3 and so forth** are similar to Step 2, just $g^{(k)}$ should be taken orthogonal to all $\psi_j^{(1)}, \ldots, \psi_j^{(k-1)}$, not only to $\psi_j^{(k-1)}$. Finally, we obtain an orthonormal system of functions $\{\psi_j^{(k)}\}$, $j = 1, \ldots, n_k$, $k \in \mathbb{N}$, such that

$$\|\psi_j^{(k)} - \chi_j^{(k)}\| < \frac{1}{\sqrt{n_k}},$$

and their linear combinations approximate every $g_k$ up to $\frac{1}{\sqrt{2}}$. By the lemma, the system is complete. Denote it, in the sequential numbering, by $\Psi = \{\psi_j\}$. We have

$$\|\psi_n - \chi_n\| \longrightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

In addition, taking $n_1 > \frac{1}{\epsilon^2}$, we get the estimate $\|\psi_n - \chi_n\| < \epsilon \quad \forall n$. 

**3. Remark.** The proof provides the optimal speed of convergence of the perturbed vectors to the original ones, in the following sense: choosing the numbers $\{n_k\}$ growing exponentially, one can reach

$$\|\psi_n - \chi_n\| = O\left(\frac{1}{\sqrt{n}}\right)$$

for $n \rightarrow \infty$. On the other hand, the norms must satisfy the equality

$$\sum_n \|\psi_n - \chi_n\|^2 = \infty,$$

otherwise, according to a well-known theorem of N. Bari [1], the orthonormal systems $\{\psi_n\}$ and $\{\chi_n\}$ are complete or non-complete simultaneously.

**Conflict of interest** Author declare that there is no conflict of interest.

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