Lepton masses and mixings in a 331 model with $A_4$ flavour symmetry.

A. E. Cárcamo Hernández$^a$ and R. Martinez$^b$

$^a$Universidad Técnica Federico Santa María and Centro Científico-Tecnológico de Valparaíso
Caseilla 110-V, Valparaíso, Chile.
$^b$Universidad Nacional de Colombia, Departamento de Física, Ciudad Universitaria, Bogotá D.C., Colombia.

(Dated: January 26, 2015)

We propose a model based on the $SU(3)_C \otimes SU(3)_L \otimes U(1)_X$ gauge group with an extra $A_4 \otimes Z_2 \otimes Z_2 \otimes Z_{14}$ flavor symmetry, which successfully accounts for the lepton masses and mixing. The observed charged lepton mass hierarchy is originated from the $Z_{14}$ symmetry, which is broken at very high scale by the $SU(3)_L$ scalar singlet $\sigma$, charged under this symmetry. The small active neutrino masses are generated via a inverse seesaw mechanism. The obtained neutrino mixing parameters and neutrino mass squared splittings for both normal and inverted hierarchies, are in good agreement with the neutrino oscillation experimental data. The model predicts CP conservation in neutrino oscillations.

I. INTRODUCTION

Despite the great success of the Standard Model (SM), recently confirmed by the discovery of the $\sim 126$ GeV Higgs boson by LHC experiments$^{1-4}$, there are many aspects not yet explained such as the origin of the fermion mass and mixing hierarchy as well as the mechanism responsible for stabilizing the electroweak scale$^{3,6}$. This discovery of the Higgs scalar field allows to consider extensions of the SM with additional scalar fields that can be useful to explain the existence of Dark Matter$^7$.

The Standard Model is a theory with many phenomenological achievements. However in the Yukawa sector there are many parameters related with the fermion masses with no clear dynamical origin. Because of this reason, it is important to study realistic models that allow to set up relations among all these parameters of the Yukawa sector. Discrete flavor symmetries allow to establish ansatz that explain the flavor problem, for recent reviews see Refs. $^{8-10}$. These discrete flavour symmetries may be crucial in building models of fermion mixing that address the flavor problem. Non abelian discrete flavor symmetries arise in string theories due to the discrete features of the fix points$^{8-10}$. These discrete flavour symmetries may be crucial in building models of fermion mixing that address the flavor problem.

Besides that, another of the greatest mysteries in particle physics is the existence of three fermion families at low energies. The quark mixing angles are small whereas the leptonic mixing angles are large. Models based on the gauge symmetry $SU(3)_C \otimes SU(3)_L \otimes U(1)_X$ have the feature of being vectorlike with three families of fermions and are therefore anomaly free$^{12,14}$. When the electric charge is defined in the linear combination of the $SU(3)_L$ generators $T_3$ and $T_8$, it is a free parameter, independent of the anomalies ($\beta$). The choice of this parameter defines the charge of the exotic particles. Choosing $\beta = -\frac{1}{\sqrt{3}}$, the third component of the weak lepton triplet is a neutral field $\nu^c_R$, which allows to build the Dirac matrix with the usual field $\nu_L$ of the weak doublet. If one introduces a sterile neutrino $N_R$ in the model, then it is possible to generate light neutrino masses via inverse seesaw mechanism. The 3-3-1 models with $\beta = -\frac{1}{\sqrt{3}}$ have the advantage of providing an alternative framework to generate neutrino masses, where the neutrino spectrum includes the light active sub-eV scale neutrinos as well as sterile neutrinos which could be dark matter candidates if they are light enough or candidates for detection at the LHC, if their masses are at the TeV scale. This interesting feature make the 3-3-1 models very interesting since if the TeV scale sterile neutrinos are found at the LHC, these models can be very strong candidates for unraveling the mechanism responsible for electroweak symmetry breaking.

Neutrino oscillation experiments$^{6,17,21}$ indicate that there are at least two massive active neutrinos and at most one massless active neutrino. In the mass eigenstates, it is necessary for the solar neutrinos oscillations that $\delta m^2_{sun} = m^2_{21} = m^2_2 - m^2_1$ where $m^2_2 - m^2_1 > 0$. For the atmospheric neutrinos oscillations it is required that

---

$^a$Electronic address: antonio.carcamo@usm.cl
$^b$Electronic address: remartinezm@unal.edu.co
\[
\delta m_{\text{atm}}^2 = m_{13}^2 - m_{11}^2 = m_{12}^2 - m_{13}^2 \text{, where the difference can be positive (normal hierarchy) or negative (inverted hierarchy). Neutrino oscillations do not give information neither on the absolute value of the neutrino mass nor on the Majorana or Dirac nature of the neutrino. However there are neutrino mass bounds arising from cosmology}^{22}, \text{tritio beta decay}^{23} \text{and double beta decay}^{24-31}.
\]

The neutrino masses and mixings are known from neutrino oscillations, which depend on the squared neutrino mass differences and not on the absolute value of the neutrino masses. The global fits of the available data from the Daya Bay\cite{17}, T2K\cite{18}, MINOS\cite{19}, Double CHOOZ\cite{20} and RENO\cite{21} neutrino oscillation experiments, constrain the neutrino mass squared splittings and mixing parameters \cite{32}. The current neutrino data on neutrino mixing parameters can be very well accommodated in the approximated tribimaximal mixing matrix,

\[
U_{\text{TB}} = \begin{pmatrix}
\frac{\sqrt{2}}{2} & \frac{1}{\sqrt{2}} & 0 \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

which is consistent with two large mixing angles and one very small mixing angle of order zero. Specifically, the mixing angles predicted by the tribimaximal mixing matrix satisfy \((\sin^2 2\theta_{12})_{\text{TB}} = \frac{1}{3}\), \((\sin^2 2\theta_{23})_{\text{TB}} = \frac{1}{2}\), and \((\sin^2 \theta_{13})_{\text{TB}} = 0\). However, the 331 model is not able to generate the tribimaximal matrix structure. Because of this reason, discrete symmetry groups \cite{33,34} that act on the fermion families are imposed with the aim to generate ansatz that reproduce these matrices. One of the most promising discrete flavor groups is \(A_4\) since it is the smallest symmetry with one three-dimensional and three distinct one-dimensional irreducible representations, where the three families of fermions can be accommodated rather naturally. Another approach to describe the fermion mass and mixing pattern consists in postulating particular mass matrix textures (see Ref \cite{35} for some works considering textures). Besides that, models with Multi-Higgs sectors, Grand Unification, Extradimensions and Superstrings as well as with horizontal symmetries have been proposed in the literature \cite{36,37,38,39,40,41} to explain the observed pattern of fermion masses and mixings.

In this paper we propose a version of the \(SU(3)_C \times SU(3)_L \times U(1)_X\) model with an additional flavor symmetry group \(A_4 \otimes Z_2 \otimes Z'_2 \otimes Z_{14}\) and an extended scalar sector needed in order to reproduce the specific patterns of mass matrices in the lepton sector that successfully account for lepton masses and mixings. The particular role of each additional scalar field and the corresponding particle assignments under the symmetry group of the model are explained in details in Sec. \(\Pi\). The model we are building with the aforementioned discrete symmetries, preserves the content of particles of the minimal 331 model, but we add additional very heavy scalar fields with quantum numbers that allow to build Yukawa terms invariant under the local and discrete groups. This generates the right textures that explain the neutrino mass squared splittings and neutrino mixing parameters measured in neutrino oscillations experiments. Our model successfully describes the prevailing pattern of the SM lepton masses and mixing.

The content of this paper is organized as follows. In Sec. \(\Pi\) we outline the proposed model. In Sec. \(\Pi\) we discuss lepton masses and mixings and show our corresponding results. Finally in Sec. \(\Pi\) we state our conclusions. In appendix \(A\) we present a brief description of the \(A_4\) group.

\section{The Model}

We extend the \(SU(3)_C \otimes SU(3)_L \otimes U(1)_X\) group of the minimal 331 model by adding an extra flavor symmetry group \(A_4 \otimes Z_2 \otimes Z'_2 \otimes Z_{14}\), in such a way that the full symmetry \(G\) experiences a three-step spontaneous breaking, as follows:

\[
G = SU(3)_C \otimes SU(3)_L \otimes U(1)_X \otimes A_4 \otimes Z_2 \otimes Z'_2 \otimes Z_{14} \xrightarrow{\Lambda_{\text{int}}} SU(3)_C \otimes SU(3)_L \otimes U(1)_X \otimes Z_2^{v_n} \xrightarrow{\Lambda_{\text{int}}} SU(3)_C \otimes SU(3)_L \otimes U(1)_Q \text{,}
\]

where the different symmetry breaking scales satisfy the following hierarchy \(v_n, v_{\rho} \ll v_{\chi} \ll \Lambda_{\text{int}}\).

We define the electric charge in our 331 model in terms of the \(SU(3)\) generators and the identity, as follows:

\[
Q = T_3 - \frac{1}{\sqrt{3}}T_8 + XI,
\]

with \(I = \text{Diag}(1,1,1)\), \(T_3 = \frac{1}{2}\text{Diag}(1,-1,0)\) and \(T_8 = (\frac{1}{2\sqrt{3}})\text{Diag}(1,1,-2)\).
The anomaly cancellation of $SU(3)_L$ requires that the two families of quarks be accommodated in $3^*$ irreducible representations (irreps). Besides that, the number of $3^*$ irreducible representations is six, as follows from the quark colors. We accomodate the other family of quarks into a 3 irreducible representation. Furthermore, we have six 3 irreps taking into account the three families of leptons. Thus, the $SU(3)_L$ representations are vector like and free of anomalies. Having anomaly free $U(1)_X$ representations requires that the quantum numbers for the fermion families be assigned in such a way that the combination of the $U(1)_X$ representations with other gauge sectors cancels anomalies. Consequently, to avoid anomalies, the fermions have to be accommodated into the following ($SU(3)_C, SU(3)_L, U(1)_X$) left- and right-handed representations:

$$Q^1_L = \left( \frac{D^1_{L^2}}{-U^1_{L^2}} \right)_L : (3, 3^*, 0), \quad Q^3_L = \left( \frac{U^3}{D^3} \right)_L : (3, 3, 1/3), \quad L^{1,2,3}_L = \left( \begin{array}{c} e^{1,2,3}_{L^2} \\ \psi^{1,2,3}_{L^2} \end{array} \right)_L : (1, 3, -1/3), \quad (5)$$

where $U^i_L$ and $D^i_L$ for $i = 1, 2, 3$ are three up- and down-type quark components in the flavor basis, while $e^1_L$ and $e^3_L$ ($e_L, \mu_L, \tau_L$) are the neutral and charged lepton families. The right-handed fermions are assigned as $SU(3)_L$ singlets having $U(1)_X$ quantum numbers equal to their electric charges. Furthermore, the fermion spectrum of the model includes as heavy fermions: a single flavor quark $T$ with electric charge $2/3$, two flavor quarks $J^1,2$ with charge $-1/3$, three neutral Majorana leptons ($\nu^{1,2,3}_{L} \chi$) and three right-handed Majorana leptons $N^{1,2,3}_R$ (see Ref. [42] for a recent discussion about neutrino masses via double and inverse see-saw mechanism for a 331 model).

The 331 models extend the scalar sector of the SM into three $3'$ irreps of $SU(3)_L$, where one triplet $\chi$ acquires a vacuum expectation value (VEV) at the TeV scale, $v_\chi$, breaking the $SU(3)_L \times U(1)_X$ symmetry down to the $SU(2)_L \times U(1)_Y$ electroweak group of the SM and then giving masses to the non SM fermions and gauge bosons; and two light triplet fields $\eta$ and $\rho$ that get VEVs $v_\eta$ and $v_\rho$, respectively, at the electroweak scale thus generating the mass for the fermion and gauge sector of the SM. We enlarge the scalar sector of the minimal 331 model by introducing ten $SU(3)_L$ scalar singlets, namely, $\xi_j, \zeta_j, S_j$ and $\sigma$ ($j = 1, 2, 3$).

The scalars of our model are accommodated into the following $[SU(3)_L, U(1)_X]$ representations:

$$\chi = \left( \frac{\chi^1_{\xi^1} + \xi \xi_{\chi}}{\sqrt{2}(v_{\chi} + \xi \chi)} \right)_L : (3, -1/3), \quad \xi_j : (1, 0), \quad j = 1, 2, 3,$$

$$\rho = \left( \frac{\rho^1_{\nu^1} + \xi \rho_{\rho}}{\sqrt{2}(v_{\rho} + \xi \rho)} \right)_L : (3, 2/3), \quad \zeta_j : (1, 0), \quad j = 1, 2, 3,$$

$$\eta = \left( \frac{\eta^1_{\eta^1} + \xi \eta_{\eta}}{\sqrt{2}(v_{\eta} + \xi \eta)} \right)_L : (3, -1/3), \quad S_j : (1, 0), \quad \sigma \sim (1, 0), \quad j = 1, 2, 3. \quad (7)$$

The scalar fields are grouped into triplet and trivial singlet representations of $A_4$. The scalar fields of our model have the following assignments under $A_4 \otimes Z_2 \otimes Z_2' \otimes Z_{14}$:

$$\eta \sim (1, -1, 1, 1), \quad \rho \sim (1, -1, 1, 1), \quad \chi \sim (1, 1, 1, 1),$$

$$\xi \sim (3,1, -1, 1), \quad \zeta \sim (3, -1, 1, 1), \quad S \sim (3, 1, 1, 1), \quad \sigma \sim (1, 1, 1, e^{-4\pi i}), \quad (8)$$

whereas the leptons transform under $A_4 \otimes Z_2 \otimes Z_2' \otimes Z_{14}$ as:

$$L_L \sim (3, 1, 1, 1), \quad e_R \sim (1, -1, -1, 1), \quad \mu_R \sim (1', -1, -1, e^{i\frac{4\pi}{3}}),$$

$$\tau_R \sim (1''', -1, -1, e^{4\pi i}), \quad N_R \sim (3, 1, 1, 1). \quad (9)$$

Here the numbers in boldface are dimensions of the $A_4$ irreducible representations. Note that left handed leptons are unified into a $A_4$ triplet representation 3, whereas the right handed charged leptons are assigned into different $A_4$ singlets, i.e, 1, $1'$ and $1''$. Furthermore, the right handed Majorana neutrinos are unified into a $A_4$ triplet representation.
With the above particle content, the following Yukawa terms for the lepton sector arise:

\[- \mathcal{L}_Y^{(L)} = h_{\text{pec}} (T_L \rho \xi)_1 v_R \frac{\sigma_7^2}{\Lambda} + h_{\rho \mu}^L (T_L \rho \xi)_1 \mu_R \frac{\sigma^4}{3 \Lambda} + h_{\rho \tau}^L (T_L \rho \xi)_1 \tau_R \frac{\sigma^2}{\Lambda} + h_{\xi}^L (T_L \chi N_R)_1 + \frac{1}{2} m_N (N_R N_R^C)_1 + h_N (N_R N_R^C)_3 S\]

\[
+ h_{\rho \xi} \epsilon_{abc} (T_L^a (L_L^b)_1)_{3s} \rho \xi \frac{\zeta}{\Lambda} + H.c.,
\]

where the dimensionless couplings \( h_{\text{pec}}, h_{\rho \mu}, h_{\rho \tau}, h_{\xi}, h_N \) and \( h_{\rho} \) are \( O(1) \) parameters.

To describe the pattern of fermion masses, one needs to postulate particular Yukawa textures. A candidate for generating specific Yukawa textures is the \( A_4 \otimes Z_2 \otimes Z_2' \otimes Z_{14} \) flavor symmetry that can successfully account for lepton masses and mixings. The inclusion of the \( A_4 \) discrete group reduces the number of parameters in the Yukawa and scalar sector of the \( SU(3)_C \otimes SU(3)_L \otimes U(1)_X \) model making it more predictive. We choose \( A_4 \) since it is the smallest discrete group with a three-dimensional irreducible representation and 3 distinct one-dimensional irreducible representations, which allows to naturally accommodate the three fermion families. We unify the left-handed leptons in the \( A_4 \) triplet representation and the right-handed leptons are assigned to \( A_4 \) singlets. The \( Z_2 \otimes Z_2' \) symmetry separates the \( A_4 \) scalar triplets participating in the Yukawa interactions for charged leptons from those ones participating in the neutrino Yukawa interactions. The \( Z_{14} \) symmetry shapes the charged lepton mass hierarchy that yields the observed pattern of charged lepton masses. In the following we will explain the reason for employing the \( Z_{14} \) symmetry in our model. Note that in the charged lepton sector, the five dimensional Yukawa operators \( \frac{1}{\Lambda} (T_L \rho \xi)_1 \epsilon_R, \frac{1}{\Lambda} (T_L \rho \xi)_1 \mu_R \) and \( \frac{1}{\Lambda} (T_L \rho \xi)_1 \tau_R \) are \( A_4 \) invariant but do not preserve the \( Z_{14} \) symmetry, since in these operators only the right handed charged lepton fields have nontrivial \( Z_{14} \) charges. To explain the smallness of the electron mass, without tuning its corresponding Yukawa coupling, we need to build a twelve dimensional Yukawa term. The simplest way to build that twelve dimensional Yukawa term is by including a \( \frac{\sigma_7}{\Lambda} \) insertion on it, crucial to get the required \( \lambda^8 \) supression (where \( \lambda = 0.225 \) is one of the Wolfenstein parameters) needed to naturally explain the smallness of the electron mass. As follows from the properties of the \( Z_{14} \) groups, \( Z_{14} \) is the lowest cyclic symmetry that allows to build the aforementioned twelve dimensional charged lepton Yukawa term.

Regarding the VEV pattern for the \( SU(3)_L \) singlet scalar fields, we make the following assumption:

\[
\langle \xi \rangle = \frac{v_\xi}{\sqrt{3}} (1, 1, 1), \quad \langle \zeta \rangle = \frac{v_\zeta}{\sqrt{2}} (1, 0, e^{-i \phi}), \quad \langle S \rangle = \frac{v_S}{\sqrt{3}} (1, 1, -e^{i \phi}),
\]

which is crucial to get a predictive model that successfully reproduces the experimental values of the physical observables in the lepton sector.

Furthermore we assume that these \( SU(3)_L \) scalar singlets get VEVs at a scale \( \Lambda_{\text{int}} \) much larger than \( v_\chi \) (which is of the order of the TeV scale), with the exception of \( \xi \) and \( S_j \) (\( j = 1, 2, 3 \)), which get VEVs much smaller than the electroweak symmetry breaking scale \( v = 246 \) GeV. The VEVs of the \( \xi \) and \( S \) scalar singlets break the \( SU(3)_C \otimes SU(3)_L \otimes U(1)_X \otimes A_4 \otimes Z_2 \otimes Z_2' \otimes Z_{14} \) symmetry down to \( SU(3)_C \otimes SU(3)_L \otimes U(1)_X \otimes Z_2 \) at the scale \( \Lambda_{\text{int}} \).

Besides that, in order to naturally explain the charged lepton mass hierarchy taking into account that it arises from the \( Z_{14} \) symmetry, we set the VEVs of the \( SU(3)_L \) singlet scalar fields \( \xi \) and \( S \), as follows:

\[
v_\xi = v_\sigma = \Lambda_{\text{int}} = \lambda \Lambda,
\]

where \( \lambda = 0.225 \) is one of the parameters of the Wolfenstein parametrization and \( \Lambda \) the cutoff of our model. Furthermore, we assume that the \( A_4 \) scalar triplets participating in the neutrino Yukawa interactions have VEVs much smaller than the electroweak symmetry breaking scale. Therefore, we have the following hierarchy among the VEVs of the scalar fields in our model:

\[
v_S << v_\zeta << v_\rho << v_\eta \sim v << v_\chi << \Lambda_{\text{int}}.
\]
III. LEPTON MASSES AND MIXINGS

From Eq. (10) and taking into account that the VEV pattern of the $SU(3)_L$ singlet scalar fields satisfies Eq. (11) with the nonvanishing VEVs set to be equal to $\lambda \Lambda$ (being $\Lambda$ the cutoff of our model) as indicated by Eq. (12), we find that the charged lepton mass matrix is given by:

$$M_l = V_{lL}^\dagger \text{diag}(m_e, m_\mu, m_\tau),$$

$$V_{lL} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad \omega = e^{i\pi/3},$$

where the charged lepton masses are:

$$m_e = \frac{h_{\nu e}^{(L)} \lambda \sqrt{v}}{\sqrt{2}}, \quad m_\mu = \frac{h_{\nu \mu}^{(L)} \lambda \sqrt{v}}{\sqrt{2}}, \quad m_\tau = \frac{h_{\nu \tau}^{(L)} \lambda \sqrt{v}}{\sqrt{2}}.$$

Since $v_\rho \approx v = 246$ GeV, we have that charged lepton masses are linked with the scale of electroweak symmetry breaking through their power dependence on the Wolfenstein parameter $\lambda = 0.225$, with $O(1)$ coefficients.

Regarding the neutrino sector, we can write the neutrino mass terms as:

$$-\mathcal{L}_{\text{mass}}^{(\nu)} = \frac{1}{2} \begin{pmatrix} \nu_L \\ \nu_R \\ N_R \end{pmatrix} M_\nu \begin{pmatrix} \nu_L \\ \nu_R \\ N_R \end{pmatrix}^\dagger + H.c.,$$

where the neutrino mass matrix is constrained from the $A_4$ flavor symmetry and has the following form:

$$M_\nu = \begin{pmatrix} 0_{3 \times 3} & D_M & 0_{3 \times 3} \\ M_D^T & 0_{3 \times 3} & M_X \\ 0_{3 \times 3} & M_X & M_R \end{pmatrix},$$

and the submatrices are given by:

$$M_D = \frac{h_{\nu e} \sqrt{v}}{2\Lambda} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -e^{i\phi} \\ -1 & 0 & e^{i\phi} \end{pmatrix}, \quad M_R = \begin{pmatrix} m_N & -h N \sqrt{v} e^{-i\phi} & h N \sqrt{v} \\ -h N \sqrt{v} e^{i\phi} & m_N & h N \sqrt{v} \\ h N \sqrt{v}/\sqrt{3} & h N \sqrt{v}/\sqrt{3} & m_N \end{pmatrix}, \quad M_X = \frac{h_{\nu e} \sqrt{v}}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$

Here, for the sake of simplicity, we assume that the Majorana neutrinos have very small masses satisfying $m_N \sim v_S << v_\tau \ll v$, implying that the small active neutrino masses are generated via an inverse seesaw mechanism.

As shown in detail in Ref. [43], the full rotation matrix is approximatively given by:

$$U = \begin{pmatrix} V_\nu & B_3 U_X & B_2 U_R \\ -B_3 U_X^{\dagger} V_\nu^{\dagger} & (1-S) U_X^{\dagger} & (1+S) U_R^{\dagger} \\ \sqrt{2} S U_X^{\dagger} & \sqrt{2} U_X & \sqrt{2} U_R \end{pmatrix},$$

where

$$S = -\frac{1}{2\sqrt{2}h_X^{(L)} \sqrt{v}}, \quad B_2 \simeq B_3 \simeq \frac{1}{h_X^{(L)} \sqrt{v}} M_D^\ast,$$

and the physical neutrino mass matrices are:

$$M_\nu^{(1)} = M_D \left( M_X^T \right)^{-1} M_N M_X^{-1} M_D^T, \quad M_\nu^{(2)} = -M_X + \frac{1}{2} M_R, \quad M_\nu^{(3)} = M_X^T + \frac{1}{2} M_R,$$

where $M_\nu^{(1)}$ corresponds to the active neutrino mass matrix whereas $M_\nu^{(2)}$ and $M_\nu^{(3)}$ are the exotic Dirac neutrino mass matrices. Note that the physical eigenstates include three active neutrinos and six exotic neutrinos. The exotic neutrinos are pseudo-Dirac, with masses $\sim \pm M_X^T$ and a small splitting $M_R$. Furthermore, $V_\nu$, $U_R$ and $U_X$ are the rotation matrices which diagonalize $M_\nu^{(1)}$, $M_\nu^{(2)}$ and $M_\nu^{(3)}$, respectively [43].
From Eq. (21) it follows that the light active neutrino mass matrix is given by:

\[
M^{(1)}_\nu = -\frac{\hbar\nu^2\nu^2}{2hX_L^2} \begin{pmatrix}
m_N & m_N e^{i\phi} + \frac{2h}{\sqrt{3}} v_s e^{i\phi} + m_N & m_N e^{2i\phi} \\
m_N e^{i\phi} & 0 & m_N e^{i\phi} \\
0 & m_N e^{i\phi} & 0
\end{pmatrix} = \begin{pmatrix}
A & 0 & Ae^{i\phi} \\
0 & Be^{i\phi} & 0 \\
0 & 0 & Ae^{2i\phi}
\end{pmatrix},
\]

The neutrino mass matrix given in Eq. (23) only depends on three effective parameters: \(A, B\) and \(\phi\). These effective parameters include the dependence on the various model parameters. It is noteworthy that \(A\) and \(B\) are supressed by inverse powers of the high energy cutoff \(\Lambda\) of our model.

The squared light neutrino mass matrix \(M^{(1)}_\nu \left(M^{(1)}_\nu\right)^\dagger\) is diagonalized by a unitary rotation matrix \(V_\nu\), according to:

\[
V_\nu^\dagger M^{(1)}_\nu \left(M^{(1)}_\nu\right)^\dagger V_\nu = \begin{pmatrix}
m_1^2 & 0 & 0 \\
0 & m_2^2 & 0 \\
0 & 0 & m_3^2
\end{pmatrix}, \quad \text{with} \quad V_\nu = \begin{pmatrix}
\cos \theta & 0 & \sin \theta e^{-i\phi} \\
0 & 1 & 0 \\
-\sin \theta e^{i\phi} & 0 & \cos \theta
\end{pmatrix}, \quad \theta = \pm \frac{\pi}{4},
\]

where the upper sign corresponds to normal \((\theta = +\pi/4)\) and the lower one to inverted \((\theta = -\pi/4)\) hierarchy, respectively. The light active neutrino masses for the normal (NH) and inverted (IH) mass hierarchies are given by:

\[
\text{NH} : \quad \theta = +\frac{\pi}{4}; \quad m_{\nu_1} = 0, \quad m_{\nu_2} = B, \quad m_{\nu_3} = 2|A|,
\]

\[
\text{IH} : \quad \theta = -\frac{\pi}{4}; \quad m_{\nu_1} = 2|A|, \quad m_{\nu_2} = B, \quad m_{\nu_3} = 0.
\]

We also find that the PMNS leptonic mixing matrix is given by:

\[
U = V_{\nu L}^\dagger V_\nu = \begin{pmatrix}
\frac{\cos \theta}{\sqrt{3}} - \frac{e^{i\phi} \sin \theta}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{\cos \theta + e^{-i\phi} \sin \theta}{\sqrt{3}} \\
\frac{\cos \theta}{\sqrt{3}} - \frac{e^{i\phi} \sin \theta}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{e^{-i\phi} \cos \theta}{\sqrt{3}} \\
\frac{\cos \theta}{\sqrt{3}} - \frac{e^{i\phi} \sin \theta}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{e^{-i\phi} \sin \theta}{\sqrt{3}}
\end{pmatrix}.
\]

It is worth commenting that the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix depends only on the parameter \(\phi\), while the neutrino mass squared splittings are controlled by parameters \(A\) and \(B\).

The standard parametrization of the leptonic mixing matrix implies that the lepton mixing angles satisfy [3]:

\[
\sin^2 \theta_{12} = \frac{|U_{e2}|^2}{1 - |U_{e3}|^2} = \frac{1}{2} \mp \cos \phi, \quad \sin^2 \theta_{13} = |U_{e3}|^2 = \frac{1}{3}(1 \pm \cos \phi),
\]

\[
\sin^2 \theta_{23} = \frac{|U_{\mu 3}|^2}{1 - |U_{e3}|^2} = \frac{2 \pm (\cos \phi + \sqrt{3} \sin \phi)}{4 \pm 2 \cos \phi}.
\]

The resulting PMNS matrix \(U\) reduces to the tribimaximal mixing matrix \(T\) in the limit \(\phi = 0\) and \(\phi = \pi\) for the inverted and normal hierarchies of the neutrino mass spectrum, respectively. Let’s note that the lepton mixing angles are controlled by a single parameter \(\phi\), whereas the neutrino mass squared splittings only depend on the parameters \(A\) and \(B\).

The Jarlskog invariant and the CP violating phase are [3]:

\[
J = \text{Im} \left( U_{e1} U_{\mu 2} U_{e2}^* U_{\mu 1}^* \right) = -\frac{1}{6\sqrt{3}} \cos 2\theta, \quad \sin \delta = \frac{8J}{\cos \theta_{13} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13}}.
\]

Taking into account that \(\theta = \pm \frac{\pi}{4}\), our model predicts \(J = 0\) and \(\delta = 0\), which results in CP conservation in neutrino oscillations.
masses and mixings. The small active neutrino masses are generated via an inverse seesaw mechanism. In this scenario, the spectrum arises from Eqs. (26), (25) and the definition $\Delta m^2_{ij} = m^2_i - m^2_j$. The best fit values of $\Delta m^2_{ij}$ have been taken from Tables I and II for the normal and inverted mass hierarchies, respectively.

We vary the model parameter $\phi$ in Eq. (28) to fit the leptonic mixing parameters $\sin^2 \theta_{ij}$ to the experimental values reported in Tables I and II. We obtain the following best fit result:

\[
\text{NH : } m_{\nu_1} = 0, \quad m_{\nu_2} = B = \sqrt{\Delta m^2_{21}} \approx 9 \text{meV}, \quad m_{\nu_3} = 2 |A| = \sqrt{\Delta m^2_{31}} \approx 51 \text{meV};
\]

\[
\text{IH : } m_{\nu_2} = B = \sqrt{\Delta m^2_{21} + \Delta m^2_{13}} \approx 50 \text{meV}, \quad m_{\nu_1} = 2 |A| = \sqrt{\Delta m^2_{13}} \approx 49 \text{meV}, \quad m_{\nu_3} = 0,
\]

as resulting from Eqs. (26), (25) and the definition $\Delta m^2_{ij} = m^2_i - m^2_j$. The best fit values of $\Delta m^2_{ij}$ have been taken from Tables I and II for the normal and inverted mass hierarchies, respectively.

We vary the model parameter $\phi$ in Eq. (28) to fit the leptonic mixing parameters $\sin^2 \theta_{ij}$ to the experimental values reported in Tables I and II. We obtain the following best fit result:

\[
\text{NH : } \phi = -0.877 \pi, \quad \sin^2 \theta_{12} \approx 0.34, \quad \sin^2 \theta_{23} \approx 0.61, \quad \sin^2 \theta_{13} \approx 0.0246;
\]

\[
\text{IH : } \phi = 0.12 \pi, \quad \sin^2 \theta_{12} \approx 0.34, \quad \sin^2 \theta_{23} \approx 0.6, \quad \sin^2 \theta_{13} \approx 0.025.
\]

From the comparison of Eqs. (33), (32) with Tables I and II it follows that $\sin^2 \theta_{13}$ and $\sin^2 \theta_{23}$ are in excellent agreement with the experimental data, for both normal and inverted mass hierarchies, whereas $\sin^2 \theta_{12}$ is deviated $2\sigma$ away from its best fit values. This shows that the physical observables in the lepton sector obtained in our model are consistent with the experimental data. Furthermore, as previously mentioned, our model predicts CP conservation in the lepton sector.

| Parameter | $\Delta m_{21}(10^{-5}\text{eV}^2)$ | $\Delta m_{13}(10^{-3}\text{eV}^2)$ | $(\sin^2 \theta_{12})_{\exp}$ | $(\sin^2 \theta_{23})_{\exp}$ | $(\sin^2 \theta_{13})_{\exp}$ |
|-----------|-------------------------------|-------------------------------|-----------------|-----------------|-----------------|
| Best fit  | 7.62                          | 2.55                          | 0.320           | 0.613           | 0.0246          |
| 1σ range  | 7.43 - 7.81                   | 2.46 - 2.61                   | 0.303 - 0.336   | 0.573 - 0.635   | 0.019 - 0.0275  |
| 2σ range  | 7.27 - 8.01                   | 2.38 - 2.68                   | 0.29 - 0.35     | 0.38 - 0.66     | 0.019 - 0.030   |
| 3σ range  | 7.12 - 8.20                   | 2.31 - 2.74                   | 0.27 - 0.37     | 0.36 - 0.68     | 0.017 - 0.033   |

Table I: Range for experimental values of neutrino mass squared splittings and leptonic mixing parameters taken from Ref. [32] for the case of normal hierarchy.

| Parameter | $\Delta m_{21}(10^{-5}\text{eV}^2)$ | $\Delta m_{13}(10^{-3}\text{eV}^2)$ | $(\sin^2 \theta_{12})_{\exp}$ | $(\sin^2 \theta_{23})_{\exp}$ | $(\sin^2 \theta_{13})_{\exp}$ |
|-----------|-------------------------------|-------------------------------|-----------------|-----------------|-----------------|
| Best fit  | 7.62                          | 2.43                          | 0.320           | 0.600           | 0.0250          |
| 1σ range  | 7.43 - 7.81                   | 2.37 - 2.50                   | 0.303 - 0.336   | 0.569 - 0.626   | 0.0223 - 0.0276 |
| 2σ range  | 7.27 - 8.01                   | 2.29 - 2.58                   | 0.29 - 0.35     | 0.39 - 0.65     | 0.020 - 0.030   |
| 3σ range  | 7.12 - 8.20                   | 2.21 - 2.64                   | 0.27 - 0.37     | 0.37 - 0.67     | 0.017 - 0.033   |

Table II: Range for experimental values of neutrino mass squared splittings and leptonic mixing parameters taken from Ref. [32] for the case of inverted hierarchy.

IV. CONCLUSIONS

In this paper we proposed an extension of the minimal 331 model with $\beta = -\frac{1}{\sqrt{3}}$, where the symmetry group is extended to be $SU(3)_C \otimes SU(3)_L \otimes U(1)_X \otimes A_4 \otimes Z_2 \otimes Z_3 \otimes Z_{14}$. Our model successfully accounts for the lepton masses and mixings. The $A_4$, $Z_2$ and $Z_3$ symmetries allow to reduce the number of parameters in the lepton Yukawa terms, increasing the predictivity power of the model. The observed charged lepton mass hierarchy is a consequence of the $Z_{14}$ symmetry breaking at a very high scale, by the $SU(3)_L$ scalar singlet $\sigma$, charged under this symmetry. Assuming that the Majorana neutrinos are very light with very small masses much smaller than the Dirac neutrino masses, the small active neutrino masses are generated via a inverse seesaw mechanism. In this scenario, the spectrum
of neutrinos includes very light active neutrinos and TeV scale pseudo Dirac nearly degenerate sterile neutrinos. Our model has only three effective parameters in the neutrino sector, which allows us to successfully reproduce the three neutrino mixing parameters and two neutrino mass squared splittings for both normal and inverted mass hierarchies. Our model predicts CP conservation in neutrino oscillations.

Acknowledgments

A.E.C.H was supported by Fondecyt (Chile), Grant No. 11130115 and by DGIP internal Grant No. 111458. R.M. was supported by COLCIENCIAS.

Appendix A: The product rules for $A_4$

The $A_4$ group has one three-dimensional $3$ and three distinct one-dimensional $1$, $1'$ and $1''$ irreducible representations, satisfying the following product rules:

$$3 \otimes 3 = 3_s \oplus 3_a \oplus 1 \oplus 1' \oplus 1'',$$  \hspace{1cm} (A1)  

$$1 \otimes 1 = 1, \quad 1' \otimes 1'' = 1, \quad 1'' \otimes 1' = 1', \quad 1'' \otimes 1'' = 1',$$  \hspace{1cm} (A2)

Considering $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ as the basis vectors for two $A_4$-triplets $3$, the following relations are fulfilled:

$$(3 \otimes 3)_1 = x_1 y_1 + x_2 y_2 + x_3 y_3,$$  \hspace{1cm} (A3)  

$$(3 \otimes 3)_3 = (x_2 y_3 + x_3 y_2, x_1 y_3, x_1 y_2 + x_2 y_1), \quad (3 \otimes 3)'_1 = x_1 y_1 + \omega x_2 y_2 + \omega^2 x_3 y_3,$$  \hspace{1cm} (A4)  

$$(3 \otimes 3)'_a = (x_2 y_3 - x_3 y_2, x_1 y_3 - x_1 y_2, x_1 y_2 - x_2 y_1), \quad (3 \otimes 3)'_2 = x_1 y_1 + \omega^2 x_2 y_2 + \omega x_3 y_3,$$  \hspace{1cm} (A5)

where $\omega = e^{i\frac{2\pi}{3}}$. The representation $1$ is trivial, while the non-trivial $1'$ and $1''$ are complex conjugate to each other. Some reviews of discrete symmetries in particle physics are found in Refs. [8–10, 44].
ton, C. M. Ho and T. W. Kephart, Phys. Rev. D 89, 027701 (2014) [arXiv:1305.4402 [hep-ph]].

[38] H. Fritzsch, Phys. Lett. B70, 436 (1977); H. Fritzsch, Phys. Lett. B73, 317 (1978); H. Fritzsch, Nucl. Phys. B155, 189 (1979); T.P. Cheng and M. Sher, Phys. Rev. D35, 3484 (1987); H. Fritzsch and J. Planck, Phys. Lett. B 237, 451 (1990); D. s. Du and Z. z. Xing, Phys. Rev. D 48, 2349 (1993); K. Matsuda and H. Nishiura, Phys. Rev. D74, 033014 (2006); A. E. Cárcamo Hernández, R. Martinez and J. A. Rodriguez, Eur. Phys. J. C50, 935 (2007); A. E. Cárcamo Hernández, R. Martinez and J. A. Rodríguez, AIP Conf. Proc. 1026 (2008) 272 H. Okada and K. Yagyu, arXiv:1405.2368 [hep-ph], H. Okada and K. Yagyu, Phys. Rev. D 89 053008 (2014) [arXiv:1311.4340 [hep-ph]]; A. E. Cárcamo Hernández and I. d. M. Varzielas, arXiv:1410.2481 [hep-ph].

[39] R. Barbieri, G. R. Dvali, A. Strumia, Z. Berezhiani and L. J. Hall, Nucl. Phys. B 432, 49 (1994) [arXiv:hep-ph/9405428]; Z. Berezhiani, Phys. Lett. B 355, 481 (1995) [arXiv:hep-ph/9503366]; A. E. Cárcamo Hernández and Rakibur Rahman [arXiv:hep-ph/1007.0447].

[40] B. A. Dobrescu, Phys. Lett. B 461, 99 (1999) [arXiv:hep-ph/9812349]; G. Altarelli and F. Feruglio, Nucl. Phys. B 720 64 (2005) [hep-ph/0504165]; A. E. Cárcamo Hernández, Claudio. O. Dib, Nicolás Neill H and Alfonso R. Zerwekh, JHEP 1202 (2012) 132 [arXiv:hep-ph/1201.0878].

[41] K. S. Babu and R. N. Mohapatra, Phys. Rev. Lett. 74, 2418 (1995) [arXiv:hep-ph/9410326].

[42] L. E. Ibanez and G. G. Ross, Phys. Lett. B 332, 100 (1994); P. Binetruy and P. Ramond, Phys. Lett. B 350, 49 (1995); Y. Nir, Phys. Lett. B 354, 107 (1995); V. Jain and R. Shrock, Phys. Lett. B 352, 83 (1995); E. Dudas, S. Pokorski and C. A. Savoy, Phys. Lett. B 356, 45 (1995); B 369, 255 (1996).

[43] M.E. Catano, R. Martinez and F. Ochoa, Phys. Rev. D 86, 073015 (2012) [arXiv:1206.1966 [hep-ph]]; A. E. Cárcamo Hernández, M.E. Cataño and R. Martinez, Phys. Rev. D 90, 073001 (2014) [arXiv:1407.5217 [hep-ph]].

[44] P. Ramond, Group Theory: A Physicist’s Survey, Cambridge University Press, UK (2010).