Cosmic Statistics of Statistics

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ABSTRACT
The errors on statistics measured in finite galaxy catalogs are exhaustively investigated. The theory of errors on factorial moments by Szapudi & Colombi (1996) is applied to cumulants via a series expansion method. All results are subsequently extended to the weakly non-linear regime. Together with previous investigations this yields an analytic theory of the errors for moments and connected moments of counts in cells from highly nonlinear to weakly nonlinear scales. For nonlinear functions of unbiased estimators, such as the cumulants, the phenomenon of cosmic bias is identified and computed. Since it is subdued by the cosmic errors in the range of applicability of the theory, correction for it is inconsequential. In addition, the method of Colombi, Szapudi & Szalay (1998) concerning sampling effects is generalized, adapting the theory for inhomogeneous galaxy catalogs. While previous work focused on the variance only, the present article calculates the cross-correlations between moments and connected moments as well for a statistically complete description. The final analytic formulae representing the full theory are explicit but somewhat complicated. Therefore as a companion to this paper we supply a FORTRAN program capable of calculating the described quantities numerically. An important special case is the evaluation of the errors on the two-point correlation function, for which this should be more accurate than any method put forward previously. This tool will be immensely useful in the future both for assessing the precision of measurements from existing catalogs, as well as aiding the design of new galaxy surveys. To illustrate the applicability of the results and to explore the numerical aspects of the theory qualitatively and quantitatively, the errors and cross-correlations are predicted under a wide range of assumptions for the future Sloan Digital Sky Survey. The principal results concerning the cumulants \( \xi, Q_3 \) and, \( Q_4 \), is that the relative error is expected to be smaller than 3, 5, and 15 percent, respectively, in the scale range of \( 1h^{-1}\text{Mpc} - 10h^{-1}\text{Mpc} \); the cosmic bias will be negligible.

Key words: keywords large scale structure of the universe – galaxies: clustering – methods: numerical – methods: statistical

1 INTRODUCTION

According to theories of cosmological structure formation small initial fluctuations grew by gravitational amplification. In the last decade, higher order statistics emerged as an important tool to test both the Gaussianity of initial conditions and the gravitational amplification process. These tests are a priori possible in the perturbation theory (PT) regime where many predictions have been obtained by now (see Juszkiewicz & Bouchet 1995, Bernardeau 1996b for recent short reviews), or in the nonlinear regime. In both cases, they can potentially alleviate the ambiguity of the galaxy two-point correlation function when light does not trace mass (biasing), thereby shedding light on cosmology as well as the physics of galaxy formation (e.g. Fry & Gaztañaga 1993; Gaztañaga & Frieman 1994; Szapudi 1998b).

A tight control of the errors is crucial for the interpretation of higher order measurements from galaxy catalogs. A sufficiently general and reliable knowledge of the expected errors is all the more timely as new galaxy surveys will come online in the near future. Building on the groundwork described in two previous papers, Szapudi, & Colombi (1996, hereafter SC), and Colombi, Szapudi & Szalay (1998, hereafter CSS), the aim of this article is to formulate a coherent analytic theory for the errors of moments and connected moments of counts in cells in all scale regimes for possibly inhomogeneous galaxy surveys.

There has been several explorations in the past con-
centrating mainly on the errors of the two-point correlation function in real and Fourier space (e.g., Peebles 1980; Kaiser 1986; Landy & Szalay 1993; Feldman, Kaiser & Peacock 1994; Hamilton 1993; Hamilton 1997a, 1997b; Scoccimarro, Zaldarriaga & Hui 1999) or the $N$-point correlation function (Szapudi & Szalay 1998). The analytic calculation of the error on the void probability function is described in Colombi, Bouchet & Schaeffer (1995). As moments of counts in cells have been the most successful descriptors of higher order statistics so far, SC set out to formulate the general theory of variances related to counts in cells in a finite number of galaxies tracing the underlying continuous function ($\hat{C}_v$). The analytic calculation of the error on the void probability function is described in Colombi, Bouchet & Schaeffer (1995). As moments of counts in cells have been the most successful descriptors of higher order statistics so far, SC set out to formulate the general theory of variances related to counts in cells in a finite galaxy catalog. Explicit, analytic formulæ were determined for estimating cosmic errors of the factorial moments. The main underlying assumptions were the locally Poissonian approximation, and the hierarchical ansatz for the higher order correlations. The first consists of neglecting correlations among parts of overlapping cells, while the latter is known to be an excellent approximation in existing galaxy catalog (e.g., Groth & Peebles 1977; Fry & Peebles 1978; Sharp, Bonometto & Lucchin 1984; Szapudi, Szalay & Boshčan 1997; Meiksin, Szapudi & Szalay 1992; Bouchet et al. 1993; Szapudi et al. 1995; Szapudi & Szalay 1997) and in $N$-body simulations in the highly non-linear regime (e.g., Efstathiou et al. 1988; Bouchet et al. 1991; Bouchet & Hernquist 1992; Fry, Melott & Shandarin 1993; Bromley 1994; Lucchin et al. 1994; Colombi, Bouchet & Schaeffer 1994; Colombi, Bouchet & Hernquist 1996; Munshi et al. 1999a; Szapudi et al. 1999d). CSS applied the previously developed theory and investigated the effects of variable sampling and thereby extended the results for inhomogeneous galaxy surveys. An exhaustive description of the previous calculations would be superfluous here since all details can be found in SC. Some of the main concepts and the general framework, however, is summarized next.

Careful examination of the generating functions and their expansions yields a unique classification of the errors according to their origin and an approximate separation between them. Part of the uncertainty on counts in cells is due to the finite number of sampling cells, $C$. It is termed measurement error and it is proportional to $1/\sqrt{C}$; therefore it can be rendered arbitrarily small. The algorithm of Szapudi (1998a) achieves the limit of $C \to \infty$ in practice, i.e., the measurement errors are absent.

The rest of the variance, termed cosmic error, is inherent to the galaxy catalog and cannot be substantially improved upon except for extending the survey itself. It splits further into a trichotomy of finite volume errors, arising from the fluctuations on scales larger than the survey, edge effects, from the uneven weights given to galaxies in relation to survey geometry, and discreteness effects, due to the finite number of galaxies tracing the underlying continuous random field. To leading order in $v/V$, these three effects are approximately disjoint and the corresponding relative errors are proportional to $[\xi(\hat{L})]^{1/2}$, $(\xi v/V)^{1/2}$, and $[v/(\sqrt{V})]^{1/2}$, respectively; $\xi(\hat{L})$ is the integral of the correlation function (with some restrictions) over the whole survey area, $\xi$ is the average correlation function in a cell, $\sqrt{V}$ is the average count in a cell, $k$ is the order of the statistic, and $v$ and $V$ are the volumes of the cell and the survey, respectively. Only the discreteness error depends on the number of particles, and it disappears in the continuum limit. The separation of these effects is only approximate, and depends on the leading order nature of the calculation. Next to leading order contributions are presented elsewhere (Colombi et al. 1999a).

There are further refinements and qualifications to the above summarized theory. Edge effects, usually dominant on large scales, can be corrected for to some extent (Landy & Szalay 1993). Such a correction is always equivalent to a virtual extension of the survey, thus it is controversial as often pointed out by “fractalists”. A fraction of discreteness effects depends on the geometry of the survey thus can be termed as edge-discreteness effect (Szapudi & Szalay 1998). Finally, finite volume effects overlap slightly with edge effects, even though the appropriate splitting of the corresponding integral yields an approximate separation.

The present work generalizes the previous calculations for many useful statistics, such as the connected moments or cumulants of the probability distribution of counts in cells, and extends the validity of the theory into the weakly non-linear regime by dropping the hierarchical assumption. Moreover, cross correlation matrices for moments and connected moments are computed as well for statistical completeness. To facilitate the practical application of this somewhat complicated but fully explicit and analytic theory, we supply FORTRAN programs to evaluate all the (co)variances of moments and cumulants. This should diminish the efforts needed to assess the accuracy of counts in cells measurements in present and future galaxy catalogs, such as the Sloan Digital Sky Survey (SDSS) and the two degree Field Survey, as well as in simulations. In addition, design of future galaxy catalogs should be optimized in light of the expected errors for different alternatives. To demonstrate the practicality of our approach, the theory is illustrated throughout this paper by calculating the cosmic errors, cross-correlations, and biases for all relevant statistics related to count-in-cells in the future SDSS. It is worth to emphasize that our technique can be used to obtain the errors on the two-point correlation function with more accurate results than any previous method.

The next Section describes the general theory of non-linear error propagation including the resulting bias and the calculation of (co)variances, with extension of the analysis to the weakly non-linear regime. Sect. 3 presents practical results for the SDSS survey: the expected errors, biases and cross-correlation of factorial moments and cumulants up to fourth order are given for a wide variety of clustering models. Finally, Sect. 4 summarizes and discusses the results. In addition, Appendix A illustrates the theory with explicit formulæ too cumbersome to be included in the main text. Appendix B compares in detail our predictions for the cosmic bias on cumulants with the recent results of Hui & Gaztañaga (1998, hereafter HG).

## 2 THEORY

In this section we present the theory of cosmic errors on the quantities of interest, cumulants (or connected moments) $\bar{Q}_N$ and $Q_N$ of the probability distribution function of the cosmic density. The central issue addressed here is the propagation of errors from the factorials moments $F_k$ to the cumulants, the latter being non-linear combinations of the former. For this sake in Sect. 2.1 we present the theory of error prop-
agitation in a general setting for functions of correlated random variables. Sect. 2.2 applies this formalism to factorial moments and cumulants, taking advantage of the theory of cosmic errors on factorial moments by SC. Finally, Sect. 2.3 discusses the specific models of clustering employed for numerical demonstration of the theory, including generalization of the original framework for PT.

2.1 General Error Propagation and Bias

Let us assume that \( f(x) \) is constructed from unbiased measurements of a set of random variables \( \{x_k\} \) with known errors and cross-correlations. For measurements of a statistical quantity \( x_k \), a different notation (such as \( \bar{x}_k \ldots \)) could be introduced for added precision. However, such notation is dispensed of since it would only clutter the formulae without adding anything of importance. If the measurements \( \{x_k\} \) are sufficiently close to the ensemble average, \( \langle x_k \rangle \), it is meaningful to expand \( f \) around the mean value

\[
f(x) = f(\langle x \rangle) + \frac{\partial f}{\partial x_k} \delta x_k + \frac{1}{2} \frac{\partial^2 f}{\partial x_k \partial x_l} \delta x_k \delta x_l + \ldots \mathcal{O}(\delta x^3),
\]

where \( \delta x_k = x_k - \langle x_k \rangle \), and the Einstein convention was used. It is fruitful to evaluate the variance and bias of \( f \), and the cross-correlation of two such functions \( f, g \), up to second order precision. The resulting theory will be reasonably accurate as long as the variances and correlations of the underlying statistics are sufficiently small, i.e. \( \langle \delta x_k \delta x_l \rangle / \langle x_k \rangle \langle x_l \rangle \ll 1 \). Taking the ensemble average of the above equation yields the average of \( f \) in a finite survey

\[
\langle f \rangle = f(\langle x \rangle) + \frac{1}{2} \frac{\partial^2 f}{\partial x_k \partial x_l} \langle \delta x_k \delta x_l \rangle + \ldots \mathcal{O}(\delta x^3).
\]

According to this equation \( f(x) \) is a biased estimator of \( f(\langle x \rangle) \) (see also HG). More precisely, if \( x \) is an unbiased estimator, the (relative) bias on \( f(x) \) can be defined as

\[
b_f = \frac{\langle f(x) \rangle - f(x)}{f(x)}.
\]

To second order, an unbiased estimator can be constructed from the formula. The bias is the result of the non-linear construction of \( f \) from unbiased measurements \( x \). As the survey becomes larger the errors decrease, \( \langle \delta x_k \delta x_l \rangle \to 0 \), and, in agreement with intuition, \( f \) becomes less and less biased.

Similarly the covariance of two functions \( f \) and \( g \) can be evaluated,

\[
\text{Cov}(f, g) = \langle \delta f \delta g \rangle = \frac{\partial f}{\partial x_k} \frac{\partial g}{\partial x_l} \langle \delta x_k \delta x_l \rangle + \mathcal{O}(\delta x^3),
\]

where \( \delta X = X - \langle X \rangle \). The variance of a function \( f \) is simply \( \langle \Delta f \rangle^2 \equiv \text{Cov}(f, f) \), and the relative error

\[
\sigma_f = \sqrt{\text{Cov}(f, f) / \langle f \rangle} = \frac{\Delta f}{\langle f \rangle}.
\]

This is the general form of the widely quoted “error propagation” formula with correlated errors.

For a set of (possibly biased) statistics \( f = \{f_k\}_{k=1,K} \), the covariance matrix is defined as \( \mathcal{C}_f = \text{Cov}(f, f) \), which is in turn crucial for maximum likelihood analyses. For reference, the appropriate likelihood function in the Gaussian limit is (the logarithm of)

\[
\mathcal{L}(f) = \frac{1}{\sqrt{2\pi}^K \text{Det}(\mathcal{C})} \exp \left[ -\frac{1}{2} \mathcal{C}^{-1} \delta f \right].
\]

where \( \text{Det}(\mathcal{C}) \) and \( \mathcal{C}^{-1} \) are the determinant and inverse of the covariance matrix.

The range of applicability of the previous equations merits some comments. The most obvious condition is that the relative (co)variance \( \mathcal{C} \), is \( \sigma_f \ll 1 \), otherwise the Taylor expansion diverges. From equations (6), (7) and (8), the bias is of order \( b_f = \mathcal{O}(\sigma_f^2) \). Clearly, there is a meaningful regime

\[
b_f \ll \sigma_f \ll 1,
\]

where the theory is certainly valid. In practice \( b_f \ll \sigma_f \ll 1 \) can happen, contradicting, however, the condition that \( b_f \ll \sigma_f^2 \). This is a sign of cancellations in the coefficients, and in that case higher order expansions would be necessary to obtain the leading order results.

2.2 Cosmic Errors and Cross-Correlations on Cumulants and Factorial Moments

For the present applications of the above formulae, the average count \( N \), the variance \( \xi \), and the cumulants \( Q_3 \) and \( Q_4 \) are substituted for \( \{f_k\} \). As shown below, each of these can be expressed in terms of the factorial moments \( F_k \) (identified with \( x_k \)). For further reference we first recall basic definitions, then we formulate the theory of errors of SC for factorial moments.

The variance of count in cells is the average of the correlation function in a cell

\[
\xi \equiv \int \frac{d^3 r_1}{v} \frac{d^3 r_2}{v} \xi(r_1, r_2).
\]

The cumulants of higher order are geometrical averages of the \( N \)-point correlations functions

\[
Q_N \equiv \frac{1}{N^{N-2} \xi^N} \int \xi_N(r_1, \ldots, r_N) \frac{d^3 r_1}{v} \ldots \frac{d^3 r_N}{v},
\]

and by definition \( Q_1 \equiv Q_2 \equiv 1 \). Another widespread notation exists in the literature for \( Q_N \)

\[
S_N \equiv N^{N-2} Q_N,
\]

where the \( N^{N-2} \) factor corresponds to the number of trees that connect \( N \) points.

The connected moments are non-linear functions of the factorial moments (see Szapudi & Szalay 1993), e.g.,

\[
N = F_1
\]

\[
\xi = \frac{F_2}{F_1} - 1
\]

\[
Q_3 = \frac{F_3 - 3F_1 F_2 + 2F_1^3}{3(F_2 - F_1)^2}
\]

\[
Q_4 = \frac{F_4^2 - 4F_2 F_3 - 3F_2^2 + 12F_2 F_1^2 - 6F_1^2}{16(F_2 - F_1)^2}
\]

where
Factorial moments are estimated in an unbiased fashion, bias affecting the cumulants is due to non-linear construction. Both the errors and biases of cumulants can be deduced from the errors and cross-correlations of the factorial moments, \( \text{Cov}(F_k, F_l) = \langle \delta F_k \delta F_l \rangle \), through the series expansions \( E \) and \( F \) if the variances are sufficiently small.

The diagonal term, \( \text{Cov}(F_k, F_k) \), was evaluated by SC under the hierarchical and local Poisson behavior assumptions. For the present generalizations i) the hierarchical assumption has to be discarded (see next subsection), ii) the \( k \neq l \) cross terms need to be evaluated as well. The cross-correlations of the factorial moments are obtained through a completely analogous if cumbersome calculation as described in SC. The basic steps are outlined next.

To evaluate the cross-correlations the full error generating function of the factorial moments, which contains the measurement errors and the cosmic errors, should be expanded (SC),

\[
\text{Cov}(F_k, F_l) = \left[ \frac{\partial}{\partial x} \right]^k \left[ \frac{\partial}{\partial y} \right]^l E^{C,V}(x + 1, y + 1)_{x=y=0}, \tag{16}
\]

where

\[
E^{C,V}(x, y) = \left( 1 - \frac{1}{C} \right) E^{\infty,V}(x, y) + E^{C,\infty}(x, y). \tag{17}
\]

For completeness, the measurement errors are generated by \( F \),

\[
E^{C,\infty}(x, y) = \frac{P(xy) - P(x)P(y)}{C}, \tag{18}
\]

where \( P(x) \) is the generating function of the distribution of counts in cells; \( F_k = (d/dx)^k P(x + 1)|_{x=0} \). The measurement errors can always be eliminated with large or infinite number of sampling cells employed in state of the art measurement algorithms (Szapudi 1998a, Szapudi et al. 1999b). Therefore the limit \( C \to \infty \) is taken, i.e. the number of sampling cells tends to infinity, and measurement errors shall not be mentioned further.

The surviving part of the generating function is

\[
E^{\infty,V}(x, y) = \langle P(x)P(y) \rangle - \langle P(x) \rangle \langle P(y) \rangle \tag{19}
\]

with

\[
P(x)P(y) = \frac{1}{V^2} \int d^D r_1 d^D r_2 P(x, y) = \int_0^\omega + \int_{\omega_0}^\infty, \tag{20}
\]

where \( D \) is the dimension of the survey, \( V \) is the volume covered by cells included in the catalog and \( P(x, y) \) is the generating function of bicounts for cells separated by a distance \( |r_1 - r_2| \). Throughout the paper three-dimensional geometry is assumed. The above equation yields both cosmic errors and cross-correlations. The calculation is facilitated by separating the double integral according to whether cells corresponding to coordinates overlap (o) or not (no). Details can be found in SC where \( k = l \) terms were evaluated.

The contribution to the cosmic errors from disjoint cells corresponds to the finite volume errors, obtained from Taylor expanding the bivariate generating function of counts in cells, as shown below.

The contribution from overlapping cells corresponds to the edge and discreteness effects. Its evaluation is somewhat tedious, involving a numerical integration after the expansion of the generating function. Nevertheless there are no further complications compared to the diagonal case of SC. The locally Poissonian assumption allows a major simplification of the calculation: only the monovariate generating function is integrated instead of the significantly more complicated trivariate function.

### 2.3 Generating functions and models

The original calculations of SC were based on a successful model for the highly non-linear regime, the hierarchical tree assumption. This assumption has never been fully demonstrated although some hints for it has been given recently (e.g., Scoccimarro & Frieman 1998). Since the coherent infall on large scales introduces an angle dependence in the perturbation theory kernels (e.g., Goroff et al. 1986), this approximation breaks down in the weakly non-linear regime.

This necessitated a generalization of the previously used assumptions for this article. The resulting new generating functions accommodate most models currently used, such as the Ansatz by Szapudi & Szalay (1993), denoted by SS and the one by Bernardeau & Schaeffer (1992), denoted by BeS, perturbation theory (PT), and extended perturbation theory (Colombi et al. 1997), hereafter EPT.

The other simplifying assumption of SC, the local Poisson Ansatz, is kept for the present calculations. To eliminate it would require major modification in the numerical method, due to the trivariate generating function. Fortunately all indications point to the extreme accuracy of this assumption for error calculations, although for cross correlations it becomes increasingly questionable as the difference of orders, \( |k - l| \), increases (Colombi et al. 1999).

Since the models and the method of calculation are described by SC in sufficient detail, only the new features arising from the present general setting are pointed out next.

As described in § 2.2, the calculation of errors requires the knowledge of the monovariate and the bivariate generating functions for the counts.

The monovariate generating function remains formally unchanged compared to SC, since the original form (White 1979; Balian & Schaeffer 1989; Szapudi & Szalay 1993) is completely general,

\[
P(x) = \exp \left\{ \sum_{N=1}^{\infty} (x - 1)^N \Gamma_N Q_N \right\}, \tag{23}
\]

with

\[
\Gamma_N = \frac{N^{N-2}}{N!} N^N \xi^{N-1}. \tag{24}
\]

However, various assumptions about \( \xi \) and \( Q_N \) are different for each model. These can be obtained either from measurements or phenomenology in the case of the SS and BeS models, or from the form of the primordial power spectrum for PT. PT has specific rules to relate \( Q_N \) to the local derivatives of the power spectrum (Juszkiewicz, Bouchet & Colombi 1993; Bernardeau 1994a,b), e.g.,

\[
Q_3 = \frac{34}{21} \frac{n + 3}{3}, \tag{25}
\]

\[
Q_4 = \frac{7589}{2846} \frac{31(n + 3)}{24} + \frac{7(n + 3)^2}{48}, \tag{26}
\]

with \( n = -3 - d \log \xi / d \log t \). From here on higher order derivatives \( \gamma_j = d^j \xi / (d \log t)^j \) are neglected in the calculation of \( Q_N \), \( N \geq 4 \) (Bernardeau 1994b). This is an accurate
This is an integral of the $N + M$-point correlation function over two separate cells. The normalization corresponds to the number of possible trees in each cell multiplied with possible non-loop connections between the cells multiplied with the appropriate power of the average correlation function. Thus the $Q_{NM}$’s become unity when the authors refer to the bivariate generating function for the $Q_{NM}$’s, as implied by Szapudi & Szalay (1997). The version used in this work, which is evaluated empirically (Schaeffer & Bernardeau 1997; Bernardeau & Schaeffer 1998; Szapudi et al. 1999d). This idea can be generalized to the bivariate distribution in several ways, as proposed by Bernardeau & Schaeffer (1999) for a more detailed discussion of this model. For instance, $Q_{12}^{\text{BeS}} = Q_3$, $Q_{13}^{\text{BeS}} = 4/3Q_4 - 1/3Q_3$, $Q_{22}^{\text{BeS}} = Q_5^{\text{PT}}$. (32)

Interestingly, the SS and BeS models are identical when $Q_N = 1$ for all $N$. Since in practice, the $Q_N$’s depart from unity only weakly, the difference between the two models is usually insignificant, despite the formal dissimilarity between them.

When calculations are done in PT framework the properties $Q_{NM}^{\text{BeS}}$ and $Q_{NM}^{\text{SS}}$ are also naturally obtained (Bernardeau 1996a), $Q_{NM}^{\text{PT}} = Q_{NM}^{\text{PT}1} Q_{NM}^{\text{PT}2}$.

The evaluation of the lowest non-trivial orders yields (Fry 1984; Bernardeau 1996),

\begin{align}
&Q_{12}^{\text{PT}} = 34/21 - n + 3/6,
&Q_{13}^{\text{PT}} = 11710/3969 - 61(n + 3)/63 + 2(n + 3)^2/27,
&Q_{22}^{\text{PT}} = 1156/441 - 34(n + 3)/63 + (n + 3)^2/36. 
\end{align}

where $\gamma_2 = d^2\zeta/(d \log \ell)^2$ term in the second equation above is neglected as previously.

Note that, in the weakly nonlinear regime where the $Q_N$’s are given by equations (22) and (23), SS and BeS models give factors $Q_{NM}$ of same order as the correct result (34). In fact, the BeS model agrees exactly with PT for $n = -3$.

PT as a model can be extended throughout the non-linear regime as well. In the resulting theory, EPT (Colombi et al. 1997), the form of the $Q_N$’s is still taken from PT; e.g., equation (22) can be extended into the non-linear regime. Then $n$, formerly the slope of the power spectrum, becomes a formal fitting parameter, denoted with $n_{\text{eff}}$. It was found empirically in simulations and galaxy data that all higher order $Q_N$ can be described fairly accurately with a single $n_{\text{eff}}$ parameter (Colombi et al. 1997; Szapudi, Meiksin & Nichol 1998; Szapudi et al. 1999d). This idea can be generalized to the bivariate distribution in several ways, as proposed by Szapudi & Szalay (1997). The version used in this work, denoted by $E^2$PT, consists of taking the same $n_{\text{eff}}$ for the $Q_{NM}$’s in equations (23) as for the $Q_N$’s.

The new assumptions for the generating function are sufficiently general to incorporate most conceivable models, notably perturbation theory and its variants. Fortunately the changes do not incur many complications for the error calculations compared to that of SC. The overlapping part of the integral in equation (19) depends on the unchanged monovariate distribution. This calculation, the most complicated and CPU consuming component of the technique, was performed by SC. Here only the appropriate values of $\xi$ and the $Q_N$’s had to be substituted into the analytic results. The missing cross-correlations of the overlapping part were computed in an exactly analogous fashion as previ-
Table 1. The standard CDM model used by CSS (CDM1) and the four CDM variants proposed by the Virgo Consortium (CDM2,3,4,5). The notations are the same as in Jenkins et al. (1998).

| Model   | Ω₀  | Λ   | h   | Γ   | σₜ  |
|---------|-----|-----|-----|-----|-----|
| CDM1    | 1.0 | 0.0 | 0.5 | 0.50| 1.00|
| CDM2    | 0.3 | 0.0 | 0.7 | 0.21| 0.85|
| CDM3    | 0.3 | 0.7 | 0.7 | 0.21| 0.90|
| CDM4    | 1.0 | 0.0 | 0.5 | 0.21| 0.51|
| CDM5    | 1.0 | 0.0 | 0.5 | 0.50| 0.51|

3 APPLICATION: SDSS-LIKE SURVEYS

The results were applied to calculate the expected errors, cross-correlations, and biases for SDSS-like galaxy catalogs as defined in detail in CSS. The SDSS is a magnitude limited survey, up to third order. Cross-correlations are given in Appendix A for factorial moments for reasonable alternatives of higher order clustering, especially in the highly non-linear regime. The most successful model of all for error calculations is CDM1. E²PT was used as a default unless otherwise noted.

For the sake of conciseness, the technical information on figures is contained in the captions only and the physical results are explained in the main text with the least possible overlap. The more conventional procedure of duplicating in figures is contained in the captions only and the physical results are explained in the main text with the least possible overlap. The more conventional procedure of duplicating in figures is contained in the captions only and the physical results are explained in the main text with the least possible overlap.

3.1 Cosmic Errors and Bias

Figure 1 shows the expected errors on the factorial moments in SDSS-like surveys for various models and contributions. The estimator for the factorial moments proposed by CSS is assumed,

$$\hat{F}_0^C \equiv \frac{1}{C} \sum_{i=1}^{C} \left( \frac{N_i}{k} \overline{\omega_{\ell,k}(r_i)} \right) \right)^2,$$

where C is the (very large) number of sampling cells thrown at positions rᵢ, ωᵢ is the selection function, and the weight ωᵢ is determined to minimize the variance of the estimator. As shown by CSS, the weights can be optimized by numerically solving an integro-differential equation, while the approximate solution is ω ∝ 1/σ², with σ representing the full errors of the given statistic. The above optimal weight is assumed for most curves. (See the figure caption for details). In general, i) the different models SS, BeS, and E²PT yield almost same results, ii) the dependence on the point function causes a spread reaching a factor of 5 on certain scales almost independently of order, iii) different reasonable assumptions for the underlying QN’s generate significant spread which, depending both on order and scale, can reach up to an order of magnitude. The assumptions for the QN’s, however, allowed a quite generous variation taking into account the typical difference between weakly non-linear and highly non-linear regime in CDM-type simulations. Uniform weighting scheme boosts the errors on small scales considerably compared to the optimal weights introduced by CSS except for F₁ where there is no significant difference. In most of the relevant dynamic range, 1h⁻¹ Mpc ≤ ℓ ≤ 50h⁻¹ Mpc, edge effects are dominating the errors. For any realistic survey, the geometry is expected to be more complex because of the cut out holes caused by bright stars, cosmic rays, etc. This could significantly boost the edge effects compared to the calculations presented here. Discreteness effects are important for very small scales ℓ ≤ 1h⁻¹ Mpc and uniform weights only. Optimal weights render discreteness and finite volume effects on a par in this regime.

Figure 2 is analogous to Figure 1 for the connected moments. In contrast with the factorial moments, i) finite volume error is completely negligible compared to the other contributions, and for orders N > 2 it is strongly dependent on the models, SS ≫ BeS ≫ E²PT, ii) the dependence on the two point correlations is less pronounced, iii) the dependence on higher order clustering appears to be less sensitive to order.

Figure 3 recapitulates the results of Figs. 1-2 by comparing the errors on measurements of factorial moments with connected moments. For small scales ℓ ≲ 7 – 10h⁻¹ Mpc the cumulants fare much better than factorial moments; one reason is the suppression of finite volume effects. Note especially the large difference between Q₃ and F₃. Interestingly, Q₃ has small errors, within a factor of two ΔQ₃/F₃, and there is a range in which ΔQ₃/F₃ ≲ ΔF₃/F₃. The edge effects for the cumulants are greatly boosted on large scales compared to the factorial moments. However, this has to be interpreted
Figure 1. Prediction of the cosmic error on factorial moments, $\sigma_{F_k} = \Delta F_k / F_k$, for $k \leq 4$. The first column shows the cosmic error from disjoint cells for different models SS (long-dashes), BeS (dot-dashes), $E^2$PT (long dashes with dots), and also from overlapping cells (dashes). The indistinguishable solid curves display the total error for each model. All the above assumes optimal radial sampling weight $\omega^{CSS}$, while the dotted line was computed with uniform weight for comparison. The second and third column demonstrate the robustness of the results with respect to variation of the two-point correlation function in the different CDM models (respectively solid, dots, dashes, long dashes and dot-dashes for CDM1,2,3,4,5), and the choice of the spectral index for $E^2$PT (solid for $n = -2.5$, dots for $n = -9$ and dashes for $n = -1$), respectively. In the first column $n = -2.5$ and CDM1 was used. Second column has $n = -2.5$ with $E^2$PT, the third column has CDM1 with $E^2$PT. Note that for the first and third columns the errors for $F_1$ are independent of higher order statistics, therefore the different models superpose.

cautiously since those scales are close to the limit of applicability of the theory according to equation (36).

Figure 4 compares the magnitude of the cosmic bias to the cosmic error for cumulants. As expected from theoretical prejudice, the cosmic bias is by orders of magnitude smaller than the cosmic error in the regime where the perturbative approach is applicable, i.e. $\ell \lesssim 10 h^{-1}$ Mpc for the SDSS. On larger scales the bias calculation apparently becomes unstable. Thus Figure 4 re-confirms the correctness of equation (36) as a guidance for the validity of the theory.

3.2 Cosmic Cross-Correlations

Figure 5 displays the cross-correlation coefficients

$$\delta_{kl} = \frac{\langle \delta F_k \delta F_l \rangle}{\Delta F_k \Delta F_l}$$  (36)
Figure 2. Same as Fig. 1 for connected moments, i.e. for $\xi$ and $S_N = Q_N N^{N-2}$. The curves are only plotted when the expansion in equation (4) yields positive results for the cosmic error.

Figure 3. Comparison of the cosmic errors for the factorial and connected moments. CDM1 was assumed for the two-point correlation function and $E^2$PT with $n = -2.5$ for higher order statistics. Solid, dotted, dash, and long dash lines correspond to orders 1 through 4, respectively. Of each pair of curves with the same line-types the one turning up on large scales relates to the cumulant. The right stopping point of the long dash curve for $S_4 = 16 Q_4$ was determined similarly to Figure 2.

In addition to the previous comments, the following observations can be made from Fig. 5: i) similarly to the cosmic errors, the different models SS, BeS, and $E^2$PT yield almost exactly the same cross-correlations, ii) optimal weighting naturally increases the cross-correlation, especially on small scales, and when the weights are selected to be optimal for the higher order of the two statistics, iii) the effects of the choice of the two-point correlation function are considerable,
Figure 5. Cosmic cross-correlation coefficients of the factorial moments. The individual columns correspond to cross terms $\delta_{kl}$ for pairs of indices $(k, l) = (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)$, respectively. Homogeneous weights were used for all panels, except for the second row. Except for row five, $n = -2.5$ is assumed for higher order statistics. Except for the first row, $E^2_{PT}$ is used. Finally, except for the fourth row, CDM1 is the underlying cosmology.

The first row of panels compares various contributions within the framework of SS (long-dashes), BeS (dot-dashes), $E^2_{PT}$ (long dashes with dots). The difference between the three models is negligible. The resulting three groups of curves in increasing order at $\ell = 8h^{-1}$ Mpc correspond to the finite volume, overlapping (i.e. discreteness+edge effects), and the total contributions, respectively. Note that the full cosmic error was used in the denominator for each curve to preserve additivity. This explains the residual dependence of the overlapping contributions on the model.

The second row is analogous to the first one but examines the dependence on the optimal weights. The uniform weights (solid) are compared to the optimal weights for orders $k, l$ (dashes), where $k < l$.

The third row illustrates dilution effects. The full sampling is shown by solid lines while the effects of 10 times dilution are displayed by dots. The curves in increasing order at $\ell = 8h^{-1}$ Mpc again correspond to the finite volume, overlapping (i.e. discreteness+edge effects), and the total contributions, respectively.

The fourth row displays how total contributions are affected by the choice of the two-point correlation functions in different variants CDM1 through CDM5 (in the same order, solid, dots, dashes, long dashes, dot-dashes), respectively.

The fifth row shows the changes on the cross-correlations due to varying the higher order statistics via changing the spectral index in the framework of $E^2_{PT}$, $n = -1$ (dashes) $n = -2.5$ (solid), and $n = -9$ (dots).
while the results are robust against variations of higher order statistics.

Figure 6 displays the correlation coefficients $\delta_{kl}$ for the cumulants. The figure is exactly analogous to Figure 5. Similar conclusions can be drawn as previously; we only point out the differences: i) the perturbative nature of our method limits the domain of applicability of the results, ii) finite volume contributions are appreciably weaker than for factorial moments, as already established for the cosmic errors, iii) the dependence on the underlying clustering is complicated to interpret because of the different ratio natures of the various statistics involved; this is explained in more detail below.

Figure 7 illustrates the principal results for cross-correlations in the SDSS. The factorial moments are always positively correlated. The correlations depend on the difference of orders $|k - l|$, the larger the difference the smaller the correlation coefficient, in agreement with intuition. It is worth noticing that the correlations exhibit approximately the same magnitude and scale dependence for the same value of $|k - l|$, i.e. increase from small scales $\ell \lesssim 1h^{-1}$ Mpc to a plateau at larger scales. On small scales the correlations are diluted by discreteness.

The behavior of the cross-correlations for the cumulants is more difficult to interpret. There are three classes of cu-
Figure 6. Same as Figure 5, with the orders $N = 1, 2, 3, 4$ corresponding to the average count $F_1 = \overline{N}$, $\xi$, $S_3 = 3Q_3$, and $S_4 = 16Q_4$, respectively. There are some differences, however, which are listed next. The range of the $y$ axis is changed to $[-1, 1]$. The sequence of the various contributions in the three upper rows is different from that of Figure 5, expect for the first column. (In the third row of this column the cross-correlations are approximately zero for the diluted case, and the order is slightly different but unimportant). The rest of the columns in the three upper rows have approximately zero finite volume contributions to the correlation coefficient. Thus the finite volume effect is easily identifiable as a straight line, while the other curves all superpose and they correspond to the overlapping and total contributions.

The right end point of the curves is chosen according to equation (7), replacing “≪” with “≤”. This condition is not exact, the sharp downturn on many panels suggests that a realistic limit is around $10h^{-1}\text{Mpc}$.

mulants: $\overline{N}$ (order 1), $\xi$ (order 2), and $Q_N$ (order $N$) each with slightly different normalization for historical and practical reasons: $\xi$ scales with $\overline{N}^{-2}$, and the $Q_N$’s likewise with $\overline{N}^{(N-1)}$. Thus one has to interpret separately the correlations between $\overline{N}$ and $\xi$, $\overline{N}$ and $Q_N$’s, $\xi$ and $Q_N$’s, and finally between the $Q_N$’s themselves. The latter are the simplest to understand: they have similar positive correlations to the factorial moments, as expected. The rest of the correlations are fairly weak, in agreement with intuition when the dif-

† Thus the first two classes have only one member each.
The correlations for \( N \) and \( \xi \), and for \( \xi \) and \( Q_3 \) are smaller than for factorial moments of the same order. This is due to the ratio nature of \( \xi \) and \( Q_3 \) which suppresses the correlations somewhat. As mentioned earlier, the perturbative nature of our method limits the validity of our results above \( 10 h^{-1} \text{Mpc} \) for the SDSS-like surveys. Also, there are some small negative correlations which should not be over-interpreted. At the present level of accuracy only the weakness of correlations can be established.

### 4 SUMMARY AND DISCUSSION

This article formulated the theory of errors on quantities related to counts in cells, focusing especially on cumulants and factorial moments. A universal, analytic method based on Taylor expansion approach was devised to calculate explicitly the cosmic error, the cosmic bias, and the cosmic cross-correlations for virtually any statistics derived from counts in cells. There are always three contributions to these quantities [SC]: finite volume, edge, and discreteness effects. The principal results are the following:

(i) Cosmic errors: SC have computed the cosmic errors on factorial moments for two particular cases of the hierarchical model. CSS have extended the results for inhomogeneous catalogs and for optimal weighting. These previous...
calculations have been generalized for cumulants, and for PT; explicit analytic results for the factorial moments are given in Appendix A. The cosmic error depends on the bivariate distributions, for which EPT had to be generalized. The new Ansatz is termed E\(^2\)PT, and explained in detail in Colombi et al. (1999b). For the SDSS it is predicted that the cumulants fare better than the factorial moments on scales \(\ell \lesssim 10h^{-1}\) Mpc. On large scales the situation is reversed due to the enhanced sensitivity of the connected moments to edge effects. For the particular example of the SDSS, however, this regime is outside the validity of our perturbative method. In the scale range of \(1h^{-1}\) Mpc – \(10h^{-1}\) Mpc the expected errors are smaller than 3 \% for \(\xi\), 4 \% for \(Q_3\), and 15 \% for \(Q_4\). For reference, the errors determined by CSS for factorial moments of order \(k = 2, 3\), and 4 were 1 – 2\%, 3 – 5\%, and, 5 – 10\%, respectively, in the regime \(1h^{-1}\) Mpc \(\lesssim \ell \lesssim 50h^{-1}\) Mpc. A detailed investigation in a range of reasonable models shows that the estimates are robust within a factor of \(\sim 2\).

Note that according to equation (8) \(\xi\) is a linear functional of \(\xi\), the two point correlation function. In fact, if \(\xi\) is a power-law of index \(\gamma\) the two are proportional to each other \(\xi \propto \xi\). For a linear functional, the error propagation is expected to be especially simple: the errors on \(\xi\) should be a linear function of the errors on \(\xi\). Of course, this statement is only approximate, because its validity depends on the nature of the estimators used to measure \(\xi\) and \(\xi\). For a power-law correlation function, we conjecture that the approximation

\[
\sigma_{\xi} \simeq \sigma_{\xi}
\]

holds for the relative cosmic error. There might be some difference at large scales, where edge effect dominate and can be at least partly corrected for estimators of \(\xi\) (e.g., Ripley 1988; Landy & Szalay 1993; Szapudi & Szalay 1998) but not for standard estimators of \(\xi\) (e.g., CSS). In that regime, it is therefore expected that \(\sigma_{\xi} \lesssim \sigma_{\xi}\). Nevertheless, approximation (37) should be more accurate for estimating the errors on the two point correlation function than the methods prevailing in the literature, especially the meaningless bootstrap method.

(ii) Cosmic bias: an estimator is biased if its ensemble average is different from the true value. This is typical when non-linear functions of unbiased estimators are constructed, such as \(\xi\), and the \(Q_N\)’s. For such statistics a perturbative expansion can be used to determine the bias \(b\). A simple but important consequence is that \(b = \mathcal{O}(\sigma^2)\), where \(\sigma\) is the relative cosmic error. As a result the cosmic bias is negligible compared to the cosmic error in the perturbative regime. A necessary, and in practice sufficient (Colombi et al. 1999b), criterion for the validity of the series expansion is that \(b \ll \sigma \ll 1\). For the SDSS the cosmic bias is predicted to be negligible on scales \(\lesssim 50h^{-1}\) Mpc for \(\xi\), and \(\lesssim 10h^{-1}\) Mpc.
for higher order statistics. Explicit formulae are given for $\xi$ and $bQ_{\gamma}$ in Appendix B.

(iii) Cosmic cross-correlations: they generalize the concept of the cosmic error by considering the full correlation matrix of the statistics. Correlations between indicators influence the constraining power of measurements on theories. The calculation for the cross-correlations of the factorial moments is exactly analogous to that of the cosmic error presented by SC. Explicit analytic results are given in Appendix A. Together with the results of SC this completes the theory of the full cosmic cross-correlation matrix and forms the basis of subsequent calculations concerning the errors of any quantity related to counts in cells, such as the cumulants.

While the following results were established in a concrete example, i.e. a suit of SDSS like surveys, we conjecture that they are quite generic. In agreement with intuition factorial moments of close orders appear to exhibit stronger correlations than those of far orders. The results are more complex for cumulants, although the $Q_{N}$'s behave similarly to factorial moments. Interestingly, the correlations between $\xi$ and $N$, and between $Q_{N}$'s are weaker than for factorial moments of the same order. Optimal weighting naturally augments correlations, and discrete effects likewise reduce them. These results depend significantly on the clustering properties of the underlying distribution of galaxies, although the qualitative features are robust.

The theoretical calculations of this paper were confronted with measurements in a state of the art large $\sigma$CDM simulation (Colombi et al. 1999); the results are previewed next.

The detailed investigations suggest that the theory of errors presented in this article is fairly accurate, especially in the weakly non-linear regime, where a few percent precision was achieved for the factorial moments. In the highly non-linear regime it appears that the approximate nature of the models for bivariate distribution translates into a slight overestimation of the errors, perhaps by a factor of two in the worst case. The situation will be improved in the future, if more realistic models are constructed for the bivariate counts.

The predicted cross-correlations for the factorial moments describe the qualitative features of the measurements quite well, however, the details are less precise than for the errors. When the difference of orders $|k - l| = 1$, the theory is about 20% accurate, while it gradually loses precision, up to about 50% in the worst case, as the difference of orders increases. This behavior suggests that the underlying locally Poisson assumption becomes less precise. An attempt to improve on this would introduce encumbering complications because of the necessity of the trivariate generating function, and is left for future research.

The present results complement the investigations of SC, and their generalization by CSS for inhomogeneous catalogs. Together they constitute the statistically complete description of the errors whenever the Gaussian approximation for the cosmic distribution of events is sufficiently accurate. This is true when the cosmic errors are small (Szapudi et al. 1999d), an essential result for likelihood analyses. Applications of the theory of cross-correlations are discussed elsewhere (Szapudi, Colombi & Bernardeau 1999d; Bouchet, Colombi & Szapudi 1999).

While the investigations presented in this article are sufficiently accurate for any foreseeable practical application and included all crucial effects and contributions, there are some minor points which were not mentioned thus far:

(i) Galaxy bias (not to be confused with the cosmic bias): light might not trace mass, thus the statistical properties of galaxies might be different from those of the dark matter. Theories and models relying only on dark matter dynamics such as PT and EPT might miss some important aspects of the galaxy distribution. However, current measurements in two and three dimensional galaxy catalogs suggest that the models used here such as SS, BeS, and even EPT, yield fairly realistic description (e.g., Gaztañaga 1994; Szapudi, Meiksin & Nichol 1996). To be complete, however, one should in principle include the effects of bias in the theory.

(ii) Redshift distortions: they arise from the peculiar velocities of galaxies in three dimensional catalogs. Their effect on the statistics is well known. The two-point correlation function and the amplitude of the $Q_{N}$'s decreases in the highly non-linear regime, while in the weakly non-linear regime only the normalization of the two-point correlation function is affected significantly (e.g., Matsubara & Suto 1994; Hivon et al. 1995; Szapudi et al. 1999b). The extent to which redshift distortions alter clustering is thus well within the range of variations considered previously.

(iii) Cosmological parameters: the dependence of the $Q_{N}$ coefficients on cosmological parameters is extremely weak. This has been explicitly shown in PT (Bouchet et al. 1992; Bernardeau 1994a; Hivon et al. 1995), and it is expected to carry over to the nonlinear regime as well (Nusser & Colberg 1998; Scoccimarro & Frieman 1998; Szapudi et al. 1999b). This point is investigated elsewhere (Bernardeau, Colombi & Szapudi 1999).

(v) Edge effects: so far the calculations were performed to leading order in $v/V$, and the results are independent of the geometry of the catalog. This is sufficiently precise approximation for compact surveys such as the SDSS. However, for more complicated survey geometries, such as the 2dF or the VIRLOS survey, the computations can be improved by taking into account higher order terms. The next to leading order term depends on the perimeter (surface) of the survey. This is studied in Colombi et al. (1999).

(vi) Full description: for maximum likelihood analyses with multi-scale measurements there is one more step needed to complete the statistical description. The cross-correlation matrix should be calculated between statistics estimated on different scales. This calculation is a trivial although somewhat tedious generalization of the previous considerations. It is left for future work.

(vii) Cosmic bias: these results are in contrast with that of Hui & Gaztañaga (1998, HG). The reasons for the difference are that a) they neglected discreteness effects, which could be significant on small scales for cumulants $Q_{N}$, b) although their calculation in principle includes edge effects dominant on large scales, they finally neglected them (how-
ever, see the discussion in Appendix B), c) they did not realize that $b = O(a^2)$ in the perturbative regime. Outside of the domain of validity, this condition naturally breaks as the measurement of HG suggests. However, to estimate the cosmic bias and the cosmic error they use only 10 realizations of the local universe. In the Virgo Hubble Volume simulation with 4096 realizations, Colombi et al. (1999b) find that the cosmic bias is always dominated by the cosmic errors. Moreover, according to Szapudi et al. (1999c), the cosmic distribution function, the probability distribution function of measurements, shows significant skewness. This is a source of effective bias for only one realization, i.e. our local universe; see Colombi et al. (1999b) and Szapudi et al. (1999c) for a detailed discussion. HG have proposed an Ansatz for scales beyond the validity of Taylor expansion in the theory. This recipe, however, neglects edge effects, which constitute the dominant contribution on large scales, except for $\bar{\xi}$ (see Appendix B); the apparent agreement of their Ansatz with measurements appears to be a coincidence. Nevertheless, their calculations, if sufficiently tested and gauged with $N$-body experiments, may be still used to estimate the cosmic bias. A detailed comparison of their analytic results for $\bar{\xi}$ with ours is contained in the Appendix B.

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APPENDIX A: THE COSMIC ERROR AND CROSS-CORRELATIONS FOR FACTORIAL MOMENTS

This section complements the analytic results for the cosmic errors obtained in SC with explicit formulae for the cross-correlations. These together with the previous results establish the full cosmic cross-correlation matrix, which underlies all error calculations for statistics related to counts in cells.

For the sake of conciseness and simplicity the following notation is introduced for the cosmic cross-correlation matrix

$$\Delta_{kl} \equiv \text{Cov}(F_k, F_l) = \langle \delta F_k \delta F_l \rangle.$$  \hfill (A1)

Note that $\Delta_{kk} = (\delta F_k)^2$ is the cosmic error. $\Delta_{kl}$ has three contributions

$$\Delta_{kl} = \Delta_{kl}^F + \Delta_{kl}^E + \Delta_{kl}^D,$$  \hfill (A2)

where $\Delta_{kl}^F$, $\Delta_{kl}^E$, and $\Delta_{kl}^D$ are the finite volume, edge and discreteness effect contributions, respectively. SC computed $\Delta_{kl}^F$, $k \leq 4$, and presented the analytic results for $k < 3$, within the framework of the SS and the BeS models. Assuming local Poissonian behavior, and a power-law $r^{-\gamma}$ for $\xi(r)$ on scales $r \leq 2\ell$, they also calculated the discreteness and edge effect contributions, $\Delta_{kl}^D$ and $\Delta_{kl}^E$ for $k \leq 4$ with explicit formulation for $k \leq 3$. All computations were performed to leading order in $v/V$, where $v$ and $V$ are the cell and the sample volume, respectively. The aim of this Appendix is to present the extension of their results for PT (and EPT) for the finite volume contribution (Appendix A.1), and for cross-correlations $k < l \leq 3$ (Appendix A.2). Note that, as in SC, all the calculations were performed up to fourth, but the results are only printed to third order. A FORTRAN program can be obtained from the authors for computing numerically the cosmic errors, cross-correlations and biases for factorial moments and cumulants.

A1 The Finite Volume Error for Factorial Moments in PT and E²PT framework

The bivariate generating function for counts in cells employed in SC had to be generalized to incorporate PT. This generalization can be used for most other models, including SS and BeS. The explicit results from this formalism are presented next:

$$\Delta_{11}^F = N^2 \xi(\hat{L}),$$  \hfill (A3)

$$\Delta_{22}^F = 4N^2 \xi(\hat{L}) \left( 1 + 2\xi Q_{12} + \xi^2 Q_{22} \right),$$  \hfill (A4)

$$\Delta_{33}^F = 9N^2 \xi(\hat{L}) \left( 1 + 2\xi + 4\xi Q_{12} + 4\xi^2 Q_{12} + 6\xi^2 Q_{13} + 6\xi Q_{13} + 4\xi^2 Q_{22} + 12\xi^3 Q_{23} + 9\xi^3 Q_{33} \right).$$  \hfill (A5)

The quantity $\xi(\hat{L})$ is roughly the average of the two-point correlation function over the survey volume:

$$\xi(\hat{L}) \equiv \frac{1}{V/2} \int_{|r_1 - r_2| \geq 2\ell} d^3r_1 d^3r_2 \xi(|r_1 - r_2|).$$  \hfill (A6)

To leading order in $v/V$ this integral reads (Colombi et al. 1999a)

$$\xi(\hat{L}) \simeq \xi_0(\hat{L}) - \frac{8\pi^2}{V} \xi(2\ell),$$  \hfill (A7)

with

$$\xi_0(\hat{L}) = \frac{1}{V/2} \int_{\psi} d^3r_1 d^3r_2 \xi(|r_1 - r_2|),$$  \hfill (A8)

$$\xi_1(\ell) = \frac{1}{v} \int_{r \leq \ell} 4\pi r^2 dr \xi(r).$$  \hfill (A9)

For most practical cases, the term proportional to $\xi_0(2\ell)$ can be neglected and the integral can be performed on the sample volume $V$ instead of the volume covered by the cells included in the catalog, $\hat{V}$: $\xi(\hat{L}) \simeq \xi_0(\hat{L})$. If kept, the correction $8\pi^2 \xi_0(2\ell)/V$, which can be viewed as an “edge-finite volume effect”, yields usually a small correction compared to the edge effect errors (see Colombi et al. 1999a,b for practical examples).

In the PT framework, the cumulants factorize $Q_{kl} = Q_{k1} Q_{l1}$. Each $Q_{k1}$ depends on logarithmic derivatives $\gamma_j = -n_j - 3$ of the (linear) variance, $\xi$, with respect to scale $\ell$ (Bernardeau 1996a). Note that in the E²PT framework, the nonlinear variance $\xi$ is taken. The parameter $\gamma_1$ is adjusted such that $S_3 = 3Q_3 = 34/7 + \gamma_1$ fits the measured, nonlinear skewness. Higher order statistics and bivariate statistics are then derived from PT expressions with this value of $\gamma_1$ (and $\gamma_0 = 0, j \geq 2$). A detailed numerical investigation of E²PT for the cosmic errors can be found elsewhere (Colombi et al. 1999b).

The above results can represent the SS model as well by replacing $Q_{kl}$ with $Q_{k1} Q_{l1}$. In the BeS framework, similarly as in PT, the relation $Q_{kl} = Q_{k1} Q_{l1}$ holds. In that case the $Q_{k1}$ can be computed explicitly from the vertex generating function as combinations of $Q_1$, $\ell \leq k + 1$ (See BeS and SC for details). Corresponding analytic expressions of the finite volume error can be found in SC.

Note finally that for the BeS and PT models, because of the factorization properties (31) and (33), we have
\[ \frac{\Delta^p_{12}}{N^{-1+2} \xi(L)} = \frac{\Delta^p_{11}}{N^{-1+2} \xi(L)} + \frac{\Delta^p_{12}}{N^{-1+2} \xi(L)} \]  

(A10)

**A2  The Cosmic Cross-Correlations for Factorial Moments**

The explicit formulae of the cosmic cross-correlations presented next complete the cosmic cross-correlation matrix. They provide the full statistical description to second order and can be used both for maximum likelihood analysis, and for calculating the cross-correlation matrix of any estimator related to factorial moments with the method presented in the main text.

\[ \Delta^p_{12} = 2N^p \bar{\xi}(L) \left( 1 + \bar{\xi} Q_{12} \right), \]  

(A11)

\[ \Delta^p_{13} = N^p \bar{\xi}^2 \left( 8.525 + 11.42 \bar{\xi} Q_3 \right), \]  

(A12)

\[ \Delta^p_{12} = N^p \bar{\xi} \left( 2.0 + 1.478 \bar{\xi} \right), \]  

(A13)

\[ \Delta^p_{13} = 3N^p \bar{\xi}(L) \left( 1 + \bar{\xi} + 2 \bar{\xi} Q_{12} + 3 \bar{\xi}^2 Q_{13} \right), \]  

(A14)

\[ \Delta^p_{13} = N^p \bar{\xi} \left( 9.05 + 11.42 \bar{\xi} + 21.67 \bar{\xi} Q_3 + 42.24 \bar{\xi}^2 Q_4 \right), \]  

(A15)

\[ \Delta^p_{13} = N^p \bar{\xi} \left( 3.0 + 6.653 \bar{\xi} + 4.949 \bar{\xi}^2 Q_3 \right), \]  

(A16)

\[ \Delta^p_{21} = 6N^p \bar{\xi}(L) \left( 1 + \bar{\xi} + 3 \bar{\xi} Q_{12} + 3 \bar{\xi}^2 Q_{13} + 2 \bar{\xi} Q_{12} + 2 \bar{\xi}^2 Q_{22} + 3 \bar{\xi}^3 Q_{23} \right), \]  

(A17)

\[ \Delta^p_{21} = N^p \bar{\xi} \left( 23.08 + 33.00 \bar{\xi} + 90.17 \bar{\xi} Q_3 + 55.19 \bar{\xi}^2 Q_3 + 211.2 \bar{\xi}^2 Q_4 + 229.9 \bar{\xi}^3 Q_5 \right), \]  

(A18)

\[ \Delta^p_{21} = N^p \bar{\xi} \left( 1.943 + 6 \bar{\xi} + 4.522 \bar{\xi} + 26.61 \bar{\xi} + 9.898 \bar{\xi}^2 + 3.531 \bar{\xi}^3 Q_3 + 39.59 \bar{\xi}^4 Q_3 + 39.53 \bar{\xi}^5 Q_4 \right). \]  

(A19)

In the above equations the edge and discreteness effect contribution was calculated from a locally Poisson Ansatz. On scales smaller than twice the cell size the two-point correlation function is assumed to be a power law \( \xi(r) \propto r^{-\gamma} \) with \( \gamma = 1.8 \). Detailed investigation of SC showed that variations of \( \gamma \) affect insignificantly the coefficients in the above equations. Therefore these equations are valid even when \( \xi \) departs weakly from a strict power-law.

**APPENDIX B: THE COSMIC BIAS: COMPARISON WITH HG**

**B1  The cosmic bias on \( \bar{\xi} \): detailed analysis**

Within the theoretical framework of this article, the cosmic bias on \( \bar{\xi} \) can be expressed in terms of \( \Delta_{kl} \) (defined in Appendix A):

\[ b_{\bar{\xi}} = \frac{F_2}{\overline{\xi} F_1^2} \left( 3 \delta_{11} - 2 \delta_{12} \right). \]  

(B1)

with

\[ \delta_{kl} \equiv \frac{\Delta_{kl}}{F_k F_l}. \]  

(B2)

Using the analytic results in Appendix A and assuming \( \mathcal{E}^2 \mathcal{P} \), the cosmic bias can be written to leading order in \( v/V \) as

\[ b_{\bar{\xi}} = b_D + b_E + b_F, \]  

(B3)

where the discreteness, edge, and finite volume effects are, respectively,

\[ b_D \approx \left( \frac{-1}{\xi} + 0.04 \right) \frac{v}{N V}, \]  

(B4)

\[ b_E \approx \left( \frac{-16.5}{\xi} + 16.5 - 18.5 Q_3 \right) \frac{\bar{\xi} v}{V}. \]  

(B5)

\[ b_F \approx \left( \frac{-1}{\xi} + 3 - 2 Q_{12} \right) \frac{\bar{\xi}(L)}{V}. \]  

(B6)

The result of HG is the following

\[ b_{\bar{\xi}} = \left( \frac{-1}{\xi} + 3 - 2 Q_{12} \right) \frac{\bar{\xi}^2}{V}. \]  

(B7)
with
\[
\xi_2' \equiv \frac{1}{V^2} \int d^3r_1 d^3r_2 \tilde{\xi}(|r_1 - r_2|),
\]
(B8)

\[
\tilde{\xi}(r) = \int_{v_1, v_2} d^3x_1 d^3x_2 \xi(|x_1 - x_2|).
\]
(B9)

The integral in the above equation is performed over two cells with volumes \(v_1, v_2\) separated by distance \(r\). Thus the calculation of HG drops discreteness effects, claiming that they can be neglected since \(v/(NV) = 1/N_L\) is small. In contrast, SC have shown that terms proportional to \(1/N_L\) are dominating the cosmic error on small scales. This may be true in principle for the cosmic bias as well. Equation (B4), however, shows that discreteness effects are indeed negligible, unless \(\xi \sim 1/N_L\). Note that the same argument is invalid for higher order cumulants such as \(Q_3\) and \(Q_4\); there discreteness effects can induce a significant contribution to bias, particularly on small scales (see the example below).

The calculation of HG includes edge effects through the integral \(\tilde{\xi}_2\) over the volume \(\tilde{V}\) covered by cells included in the catalog. Following SC one can split integral (B8) into two contributions according to whether the cells overlap or not
\[
\tilde{\xi}_2 = \frac{1}{V^2} \left[ \int_{|r_1 - r_2| > 2\ell} + \int_{|r_1 - r_2| \leq 2\ell} \right].
\]
(B10)

While it would be superfluous here to enter into details of this somewhat tedious calculation, it is clear, as in SC, that the overlapping term will typically yield a contribution \(b_\ell\) proportional to \(\xi/v/V\). On the other hand, disjoint cells contribute approximately of order \(\tilde{\xi}(L) = \tilde{\xi}_0(L) - 8(c/V)\tilde{\xi}_1(2\ell)\). [This reasoning is valid to leading order in \(v/V\). Higher order corrections proportional to the perimeter of the survey must be taken into account for more accuracy (Colombi et al. 1999b).] Since the correction proportional to \(\tilde{\xi}_1\) might exactly compensate for the term \(b_\ell\) introduced by overlapping cells, HG argue that \(\tilde{\xi}_2 \sim \tilde{\xi}_0(L)\), suggesting exact cancellation. Our calculations based on local Poisson approximation indeed show that \(b_\ell/\tilde{\xi}(L)\) is of same order of \(b_\ell/[(8c/V)\tilde{\xi}_1(2\ell)]\) for the particular case of \(\tilde{\xi}\). This result does not hold, however, for cumulants of higher order, where edge effects are dominant on large scales. At this level of accuracy our calculation becomes approximate as well mainly because of the local Poisson assumption (Colombi et al. 1999b), therefore it is impossible to evaluate the residual edge effects for \(\tilde{\xi}\) in this framework.

B2 The cosmic bias on higher order statistics

A simple algebraic calculation of the cosmic bias on \(Q_3 = S_3/3\) yields
\[
b_{Q_3} = b_3 \xi_3 - 3b_2 \xi_2 + 2\delta_{21} + 3\delta_{22},
\]
(B11)

with
\[
b_3 \xi_3 = \frac{F_3}{\xi_3 F_1^2} (6\delta_{11} - 3\delta_{13}) - 3\frac{F_2}{\xi_3 F_1^2} (3\delta_{11} - 2\delta_{12}).
\]
(B12)

Explicit writing of the discreteness contribution in equation (B11), although trivial, would go beyond the scope of this paper. To illustrate that it is not negligible, numerical results are given next. For \(\ell = 1h^{-1}\) Mpc, \(b_\ell \simeq -5 \times 10^{-5}\), \(b_{Q_3} = -2 \times 10^{-4}\) in the standard SDSS-like catalog of CSS. After a dilution by a factor 100 (which means that the catalog would still contain \(\sim 8000\) objects, e.g. CSS), these terms become \(b_\ell = -4 \times 10^{-5}\), a small change as expected, and \(b_{Q_3} = -0.2\), a change by three orders of magnitude. This means that discreteness effects can have a significant contribution to the bias on small scales, in contrast with the claims of HG. The accuracy of this statement is limited by the local Poisson assumption, which is, however, increasingly more precise as the the sample becomes more and more diluted.