The Hopf Algebra Structure of Connes and Kreimer in Epstein-Glaser Renormalization

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Abstract

We show how the Hopf algebra structure of renormalization discovered by Kreimer can be found in the Epstein-Glaser framework without using an analogue of the forest formula of Zimmermann.

1 Introduction

Epstein-Glaser renormalization is the mathematical formulation of what is done in renormalization theory. It is formulated in position space and its applications up to now are of more abstract nature than explicit calculations, which are up to now predominantly made in momentum space. The combinatorics of momentum space renormalization is described by Bogoliubov’s recursion formula or equivalently by the forest formula of Zimmermann. Recently Kreimer discovered a Hopf algebra structure in the combinatorics of the subtraction procedure in momentum space. Connes and Kreimer worked out the connection of this algebra to the Riemann-Hilbert problem [CoKr2] and to noncommutative geometry [CoKr1]. For a survey of the present status and the literature in this field see [Kre2]. In this work we derive the structure of this Hopf algebra in Epstein Glaser renormalization in a completely different way than Bondía and Lazzarini [GBLa] recently did. A short version of the derivation given here is presented in [Pin2].

The main difficulty to transfer the work of Connes and Kreimer into the Epstein-Glaser framework is that the recursive definition of $T$-products gives no insight into the combinatorics. Furthermore there is a different definition of subdiagrams in position space. Bondía and Lazzarini [GBLa] discussed the Hopf algebra structure in
Epstein Glaser renormalization, but they adapted a formulation of the forest formula of Zimmermann with the following form:

\[
R_{\Gamma} f(\Gamma) = \left[ 1 + \sum_{F} \prod_{\gamma \in F} (-S_{\gamma}) \right] f(\Gamma),
\]

where the sum is over all nonempty sets whose elements are proper, divergent (and may be connected) subgraphs of \( \Gamma \). The problem with this formula is that \( R_{\Gamma} \) is not an operator defined on the space of distributions. Especially it cannot be adjoint to give an operator on the testfunction \( f(\Gamma) \) is smeared with.

Starting point of our derivation is the structure of finite renormalizations proved in [Pin1]. We show that the divergencies of the numerical distributions in Epstein-Glaser renormalization possess a Hopf algebra structure analogously to the structure described in [CoKr2]. Therefore we describe in the first section the Hopf algebra of graphs and subgraphs, which is different from the one in [CoKr2] because of the different definition of subgraphs. Then we recall the structure of finite renormalizations and show that the singularities of the numerical distributions obey the same equation as the antipode of the Hopf algebra of the corresponding algebra of graphs.

## 2 The Hopf Algebra of Graphs

We take the algebra \( \mathcal{A} \) of graphs with multiplication and addition as in [CoKr2], but with connected instead of one particle irreducible graphs: \( \mathcal{A} \) has a basis labeled by all Feynman graphs \( \Gamma \) which are disjoint unions of connected graphs:

\[
\Gamma = \bigcup_{j=1}^{n} \Gamma_{j}
\]

(2)

The empty graph \( \Gamma = \emptyset \) is the unit element \( I \) of the algebra. The product in \( \mathcal{A} \) is bilinear and defined on the basis by the operation of disjoint union:

\[
m : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}
m(\Gamma_{1}, \Gamma_{2}) = \Gamma_{1} \cdot \Gamma_{2} = \Gamma_{1} \cup \Gamma_{2}
\]

(3)

This means that the product of two graphs is a graph consisting of this two disconnected subgraphs. For the definition of a coproduct we need the notion of a subgraph in Epstein-Glaser renormalization. In a graph \( \Gamma \) with \( n \) vertices labeled with the set of indices \( J = \)
each subset $I$ of $J$ defines a subgraph $\gamma_I$ consisting of all vertices labeled by $I$ and all lines joining them.

In momentum space, used in [CoKr2], a subgraph is defined in contrast by the lines belonging to it. All vertices which are endpoints of lines of a subgraph belong to the subgraph. So in momentum space the set of subgraphs is larger because not all lines between two vertices of the graph have to belong to the subgraph.

If the set $I$ defining a subgraph consists of only one element, the subgraph has only one point and is defined as the empty graph $\mathbb{I}$.

Every partition $P$ of $J$ defines a set of subgraphs, more precisely if $P = \{O_1, \ldots, O_k\}$ the subgraphs are those belonging to the subsets $O_i$. We use the following notation:

$$\gamma_P = \{\gamma_{O_1}, \ldots, \gamma_{O_k}\}. \tag{4}$$

$\Gamma \setminus \gamma_P$ is obtained from $\Gamma$ by shrinking its nonempty connected subgraphs belonging to $\gamma_P$ to a point.

Following [CoKr2] we restrict ourselves to a scalar massive theory. In this case we have two two-line vertices according to the terms $\Phi^2$ and $\partial_\mu \Phi \partial^\mu \Phi$ in the Lagrangian.

Now we define a coproduct on $\mathcal{A}$. For a connected nonempty graph $\Gamma$ with $|J|$ vertices we define:

$$c : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$$

$$c(\Gamma) = \Gamma \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma + \sum'_{\substack{P \in \text{Part}\, J \\text{1} \neq P \neq |J|}} \gamma_P^{(i)} \otimes \Gamma \setminus \gamma_P^{(i)}. \tag{5}$$

where the sum $\sum'$ is over all partitions with subgraphs $\gamma_{O_i}$ where $\gamma_{O_i}$ is the empty graph or a graph with a superficial divergence (in $\Phi^4$ theory all graphs with only two or four external lines). The multiindex $i$ has one value for each connected component of $\gamma_P$. This value is 0 or 1 for a component with two external lines and 0 for all other components, corresponding to the two different two-line vertices.

In the special case $\Gamma = \emptyset$ we define

$$c(\emptyset) = \mathbb{I} \otimes \mathbb{I}. \tag{6}$$

Since this definition varies from the one given in [CoKr2] only by the definition of subgraphs, the proof of the coassociativity is analogous to that of [CoKr2], if we substitute 1PI graphs by connected graphs.

A counit is defined by

$$\bar{e}(\Gamma) = 1 \text{ for } \Gamma = \emptyset$$

$$\bar{e}(\Gamma) = 0 \text{ for } \Gamma \neq \emptyset \tag{7}$$

The defining equation for the antipode $s : \mathcal{A} \to \mathcal{A}$ is
\[ m(s \otimes \text{id})c(\Gamma) = c(\Gamma) \cdot \mathbb{I}. \] (8)

(s is an antihomomorphism of algebras). From this equation we obtain a recursion formula for the antipode:

\[ s(\Gamma) = -\Gamma - \sum'_{P \in \text{Par}(J), 1 \neq |P| \neq n} s(\gamma_P^{(i)}) \cdot \Gamma \setminus \gamma_P^{(i)} \] (9)

for \( \gamma \neq \mathbb{I} \) and

\[ s(\mathbb{I}) = \mathbb{I}, \] (10)

where \( s(\gamma_P^{(i)}) = \prod_{O_j \in P} s(\gamma_{O_j}^{(i)}) \), where \( (i) \) is the part of the multiindex \( (i) \) belonging to \( O_j \).

We now show that we can derive an equation of the same form for the singularities of the numerical distributions belonging to these graphs.

3 Structure of Epstein-Glaser Renormalization

The main objects studied in Epstein-Glaser renormalization are \( T \)-products. Usually \( T \)-products are defined as multilinear functions on Wick monomials of quantized fields [BrFr]. Here we use another definition of \( T \)-products, which was first introduced in [Boa]. Let \( \mathcal{A} \) be a commutative algebra generated by the so-called classical symbolical fields \( \phi_i \) and their derivatives. We regard

\[ \mathcal{D}(\mathbb{R}^4, \mathcal{A}) \ni f = \sum_i g_i \phi_i, \] (11)

where the sum is a finite sum over elements \( \phi_i \) of the algebra \( \mathcal{A} \). An element \( f \) is then given by its coefficients \( g_i \in C_0^\infty(\mathbb{R}^4) \).

The time-ordered product (\( T \)-product) is a family of maps \( T_n, n \in \mathbb{N} \), called \( T_n \)-products. They are functions from \( (\mathcal{D}(\mathbb{R}^4, \mathcal{A}))^\otimes n \) into the set of operators on the Hilbert space \( \mathcal{H} \) with the following properties:

1. \( T_0 = 1 \)

\[ T_1(f) = \sum_i : \phi_i(g_i) : \quad \forall \ f \in \mathcal{D}(\mathbb{R}^4, \mathcal{A}), \]

where the sum is taken over all generators of \( \mathcal{A} \). Each local field is the image of an element \( f \in \mathcal{D}(\mathbb{R}^4, \mathcal{A}) \) under \( T_1 \).
2. Symmetry in the arguments:

\[ T_n (f_1, \ldots, f_n) = T_n (f_{\pi_1}, \ldots, f_{\pi_n}) \quad \forall \pi \in S_n \quad \forall f_i \in D (\mathbb{R}^4, A), i = 1, \ldots n, \]

where \( S_n \) is the set of all permutations of \( n \) elements.

3. The factorization property:

\[ T_n (f_1, \ldots, f_n) = T_i (f_1, \ldots, f_i) T_{n-i} (f_{i+1}, \ldots, f_n) \quad (12) \]

if \( (\text{supp} f_1 \cup \ldots \cup \text{supp} f_i) \gtrsim (\text{supp} f_{i+1} \cup \ldots \cup \text{supp} f_n) \) and \( f_i \in D (\mathbb{R}^4, A) \) \( \forall i \).

The \( T \)-products are not uniquely determined by these properties, so there are different \( T \)-products. Popineau and Stora \([PoSt]\) proved that there exist finite renormalizations between two different \( T \)-products. We know from \([Pin1]\) that the combinatorics in Epstein-Glaser renormalization are hidden in the structure of finite renormalizations (differences between two different \( T \)-products). For two different \( T \)-products denoted by \( T \) and \( \hat{T} \) there exist functions \( \Delta_n : D (\mathbb{R}^{4n}, A^n) \to D (\mathbb{R}^4, A) \) with \( \text{supp} \Delta_n \subset D_n \) and

\[
\hat{T}_n \left( \bigotimes_{j \in J} f_j \right) = \sum_{P \in \text{Part}(J)} T_{|P|} \left[ \bigotimes_{O_i \in P} \Delta_{|O_i|} \left( \bigotimes_{j \in O_i} f_j \right) \right].
\]

(13)

In the proof of this equation given in \([Pin1]\), the \( \Delta_n \) are constructed inductively by the following formula:

\[
T_i \left( \Delta_n \left( \bigotimes_{j \in J} f_j \right) \right) = \hat{T}_n \left( \bigotimes_{j \in J} f_j \right) - \sum_{P \in \text{Part}(J) \mid |P| > 1} T_{|P|} \left[ \bigotimes_{O_i \in P} \Delta_{|O_i|} \left( \bigotimes_{j \in O_i} f_j \right) \right].
\]

(14)

A \( T_n \)-product is a sum of terms each of which can be associated to the contribution of a special Feynman graph with \( n \) vertices. We now only regard the contribution of \( T_n \) to a special Feynman graph \( \gamma \) with \( n \) vertices, denoted by \( T^\gamma_n \). With the notion of subgraphs explained in the previous section we obtain from (13):

\[
\hat{T}^\gamma_n \left( \bigotimes_{j \in J} f_j \right) = T^\gamma_n \left( \bigotimes_{j \in J} f_j \right) + \sum_{P \in \text{Part}(J) \mid |P| < n} T_{|P|}^{\gamma \setminus P} \left[ \bigotimes_{O_i \in P} \Delta_{|O_i|}^{\gamma \setminus P} \left( \bigotimes_{j \in O_i} f_j \right) \right]
\]

(15)

where \( \Delta^\gamma \) is defined inductively by
\[ T_1 \left( \Delta_n^\gamma \left( \bigotimes_{j \in J} f_j \right) \right) = \hat{T}_n^\gamma \left( \bigotimes_{j \in J} f_j \right) - \sum_{P \in \text{Part}(J), \mid P \mid > 1} T_{[P]}^\gamma \left[ \bigotimes_{O \in P} \Delta_{[O]}^\gamma \left( \bigotimes_{j \in O} f_j \right) \right]. \]  

(16)

Comparing the recursive formula of renormalization of Bogoliubov (which can be found in [Zim1] for example) with (15) we see that both renormalizations consist of a sum of terms, each belonging to a set of superficially divergent parts of \( \Gamma \). The superficially convergent parts do not contribute to the sum in (15) because in this case \( \Delta = 0 \), and we can replace the sum \( \sum \) by the sum \( \sum' \) which is over all partitions with empty subgraphs or subgraphs with a superficial degree of divergence.

In contrast to Bogoliubov’s formula equation (15) is formulated for finite renormalizations.

In the following we omit the lower indices of the \( T \)-products and their arguments \( f \) and obtain with \( \Delta^\gamma_P = \prod_{O_j \in P} \Delta^\gamma_{[O]} \):

\[ \hat{T}^\Gamma = T^\Gamma + \Delta^\Gamma + \sum'_{P \in \text{Part}(J), \mid P \mid < n} T_{[P]}^\gamma \left[ \Delta^\gamma_P \right]. \]  

(17)

We now restrict ourselves to the case of \( \Phi^4 \) theory. The graphs are built of one kind of line, two kinds of two line vertices and one kind of four line vertex. A generalization to other theories is obvious but more complicated to write down. In [Pin1] it is shown that if \( \gamma \) is a diagram with four external legs, \( \Delta^\gamma \) has the form

\[ \Delta_n^\gamma((g \phi^4)^{\otimes n}) = C^{(n)} g^n(v) \phi^4, \]  

(18)

where the constants \( C^{(n)} \) are distributions (more precise products of Feynman propagators) smeared with testfunctions which are derivatives of functions \( w \) needed in the subtraction procedure. In the case where \( \gamma \) is the empty graph we have \( \Delta^\gamma(f) = f \) and if \( \gamma \) has two external lines we have

\[ \Delta_n^\gamma = A^{(n)} g^n(v) \phi^2 + B^{(n)} g^n(v) \partial^\mu \phi \partial^\mu \phi, \]  

(19)

where \( A^{(n)} \) and \( B^{(n)} \) have the same form as \( C^{(n)} \).

Now we formulate equation (17) on the level of numerical distributions, this means we omit the Wick products and the smearing functions. According to the values of \( \Delta^\gamma_P \) we denote with \( \delta^{\gamma(i)}_P = \prod_{O_j \in P} \delta^{\gamma(i)}_{[O]} \) the numerical distribution belonging to \( \Delta^\gamma_P = \prod_{O_j \in P} \Delta^{\gamma(i)}_{[O_j]} \). As in the previous chapter the multiindex \( (i) \) has a value for each connected component of \( P \) and \( (i_j) \) is the subset of \( (i) \) belonging to the connected components of \( O_j \). For a connected component of \( P \) with two external legs the value
can be 0 or 1, if the value is 0 we take in the product the numerical distribution of the coefficient $A^{(n)}$ and if the value is 1 we take the numerical distribution of the coefficient $B^{(n)}$. With $t^\gamma$ we denote the numerical distributions belonging to the graphs $\gamma$. We obtain

$$t^\Gamma = t^\Gamma + \delta t^\Gamma + \sum_{P \in Part(J)} t^{\Gamma \setminus \gamma_P^{(i)}} \cdot \delta \gamma_P^{(i)}.$$  \hfill (20)

We now assume that $t$ are the regularized but unrenormalized distributions and $\hat{t}$ are the regularized and renormalized distributions. Then $\delta(t)$ describes the renormalization of the superficial divergence.

Taking only the parts of (20) which are divergent when the regularization is removed, we obtain:

$$\delta \Gamma = -t^\Gamma - \sum_{P \in Part(J)} \delta \gamma_P^{(i)} t^{\Gamma \setminus \gamma_P^{(i)}},$$  \hfill (21)

which has to be interpreted as the structure of singularities in renormalization. This recursive equation for the singularity of $\delta^\Gamma$ has exactly the same structure as equation (9) for the antipode (one has to identify a graph with the singularity of its numerical distribution).

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