Yukun Song* and Fengming Liu

Existence of solutions for a shear thickening fluid-particle system with non-Newtonian potential

https://doi.org/10.1515/math-2018-0064
Received February 15, 2018; accepted May 6, 2018.

Abstract: This paper is concerned with a compressible shear thickening fluid-particle interaction model for the evolution of particles dispersed in a viscous non-Newtonian fluid. Taking the influence of non-Newtonian gravitational potential into consideration, the existence and uniqueness of strong solutions are established.

Keywords: Existence, Strong solutions, Compressible, Non-Newtonian fluid

MSC: 76A05, 76A10

1 Introduction

We consider a compressible non-Newtonian fluid-particle interaction model which reads as follows

\[
\begin{cases}
\rho_t + (\rho u)_x = 0, \\
(\rho u)_t + (\rho u^2 + \rho \Psi_x - \lambda (u_x^2 + \mu_1) \right)^{\frac{p+2}{2}} u_x)_x + (P + \eta) = -\eta \Psi_x, \quad (x, t) \in \Omega T \\
(\Psi^q - \psi)_x = 4\pi g (\rho - \frac{1}{\int_\Omega \rho d\xi}), \\
\eta_t + [\eta (u - \psi)]_x = \eta_{xx}
\end{cases}
\]

with the initial and boundary conditions

\[
\begin{cases}
(\rho, u, \eta)|_{t=0} = (\rho_0, u_0, \eta_0), \\
|u|_{\partial \Omega} = \psi|_{\partial \Omega} = 0, \\
\eta_{x} + [\eta (u - \psi)]_x = 0
\end{cases}
\]

and the no-flux condition for the density of particles

\[
(\eta_{x} + \eta \psi_x)|_{\partial \Omega} = 0, \quad t \in [0, T].
\]

where $\rho, u, \eta, P(\rho) = a \rho^\gamma$ denote the fluid density, velocity, the density of particle in the mixture and pressure respectively, $\Psi$ denotes the non-Newtonian gravitational potential and the given function $\psi(x)$ denotes the external potential. $a > 0, \gamma > 1, \mu_1 > 0, p > 2, 1 < q < 2, \lambda > 0$ is the viscosity coefficient and $\beta = 0$ is a constant. $\Omega$ is a one-dimensional bounded interval, for simplicity we only consider $\Omega = (0, 1), \Omega_T = \Omega \times [0, T]$.

In fact, there are extensive studies concerning the theory of strong and weak solutions for the multi-dimensional fluid-particle interaction models for the newtonian case. In [1], Carrillo et al. discussed the
global existence and asymptotic behavior of the weak solutions for a fluid-particle interaction model. Subsequently, Fang et al. [2] obtained the global classical solution in dimension one. In dimension three, Ballew and Trivisa [3, 4] established the global existence of weak solutions and the existence of weakly dissipative solutions under reasonable physical assumptions on the initial data. In addition, Constantin and Masmoudi [5] obtained the global existence of weak solutions for a coupled incompressible fluid-particle interaction model in 2D case followed the spirit of reference [6].

The non-Newtonian fluid is an important type of fluid because of its immense applications in many fields of engineering fluid mechanics such as inks, paints, jet fuels etc., and biological fluids such as blood (see [7]). Many researchers turned to the study of this type of fluid under different conditions both theoretically and experimentally. For details, we refer the readers to [8-12] and the references therein. To our knowledge, there seems to be a very few mathematical results for the case of the fluid-interaction model systems with non-Newtonian gravitational potential. There are still no existence results to problem (1)-(3) when $p > 2$, $1 < q < 2$ which describes that the motion of the compressible viscous isotropic gas flow is driven by a non-Newtonian gravitational force.

We are interested in the existence and uniqueness of strong solutions on a one dimensional bounded domain. The strong nonlinearity of (1) bring us new difficulties in getting the upper bound of $\rho$ and the method used in [2] is not suitable for us. Motivated by the work of Cho et al. [13, 14] on Navier-Stokes equations, we establish local existence and uniqueness of strong solutions by the iteration techniques.

Throughout the paper we assume that $a = \lambda = 1$. In the following sections, we will use simplified notations for standard Sobolev spaces and Bochner spaces, such as $L^p = L^p(\Omega), H^1_0 = H^1_0(\Omega), C(0, T; H^1) = C(0, T; H^1(\Omega))$.

We state the definition of strong solution as follows:

**Definition 1.1.** The $(\rho, u, \phi, \eta)$ is called a strong solution to the initial boundary value problem (1)-(3), if the following conditions are satisfied:

(i) \[ \rho \in L^\infty(0, T^*; H^1(\Omega)), u \in L^\infty(0, T^*; W^{1,p}_0(\Omega) \cap H^2(\Omega)), \]
\[ \eta \in L^\infty(0, T^*; H^2(\Omega)), \psi \in L^\infty(0, T^*; H^2(\Omega)), \rho_t \in L^\infty(0, T^*; L^2(\Omega)), \]
\[ u_t \in L^2(0, T^*; H^1_0(\Omega)), \psi_t \in L^\infty(0, T^*; H^1(\Omega)), \sqrt{\rho}u_t \in L^\infty(0, T^*; L^2(\Omega)), \]
\[ \eta_t \in L^\infty(0, T^*; L^2(\Omega)), ((u_x^2 + \mu_1) \nabla^2 u_x)_x \in L^2(0, T^*; L^2(\Omega)) \]

(ii) For all $\varphi \in L^\infty(0, T^*; H^1(\Omega)), \varphi_t \in L^\infty(0, T^*; L^2(\Omega))$, for a.e. $t \in (0, T)$, we have
\[ \int_\Omega \rho \varphi(x, t) \, dx - \int_0^t \int_\Omega (\rho \varphi_t + \rho u \varphi_x)(x, s) \, dx \, ds = \int_\Omega \rho_0 \varphi(x, 0) \, dx \tag{4} \]

(iii) For all $\phi \in L^\infty(0, T^*; W^{1,p}_0(\Omega) \cap H^2(\Omega)), \phi_t \in L^2(0, T^*; H^1_0(\Omega))$, for a.e. $t \in (0, T)$, we have
\[ \int_\Omega \rho u \phi(x, t) \, dx - \int_0^t \int_\Omega \{ \mu u_x^2 - \rho \psi_x \phi - \lambda(u_x^2 + \mu_1) \nabla^2 u_x \}
+ (P + \eta) \phi_x + \eta \psi_x \phi(x, s) \, dx \, ds = \int_\Omega \rho_0 u_0 \phi(x, 0) \, dx \tag{5} \]

(iv) For all $\vartheta \in L^\infty(0, T^*; H^2(\Omega)), \vartheta_t \in L^\infty(0, T^*; H^1(\Omega))$, for a.e. $t \in (0, T)$, we have
\[ - \int_0^t \int_\Omega |\psi_x|^2 \vartheta_x \, dx \, ds = \int_0^t \int_\Omega 4\pi g(\rho - \frac{1}{|\Omega|}) \vartheta(x, s) \, dx \, ds \tag{6} \]

(v) For all $\psi \in L^\infty(0, T^*; H^2(\Omega)), \psi_t \in L^\infty(0, T^*; L^2(\Omega))$, for a.e. $t \in (0, T)$, we have
\[ \int_\Omega \eta \psi(x, t) \, dx - \int_0^t \int_\Omega \eta (u - \Phi_x - \eta x) \psi_x(x, s) \, dx \, ds = \int_\Omega \eta_0 \psi(x, 0) \, dx \tag{7} \]
1.1 Main results

**Theorem 1.2.** Let $\mu_1 > 0$ be a positive constant and $\Phi \in C^2(\Omega)$, and assume that the initial data $(\rho_0, u_0, \eta_0)$ satisfy the following conditions

$$0 \leq \rho_0 \in H^1(\Omega), u_0 \in H^1_0(\Omega) \cap H^2(\Omega), \eta_0 \in H^2(\Omega)$$

and the compatibility condition

$$\left[ \left( \frac{\partial}{\partial t} + \mu_1 \right) \frac{\partial \rho}{\partial t} \right] + (P(\rho_0) + \eta_0) \frac{\partial \rho}{\partial x} + \eta_0 \Phi_x = \rho_0^{-\frac{1}{2}} (g + \beta \frac{\partial \eta}{\partial x}),$$

for some $g \in L^2(\Omega)$. Then there exist a $T_* \in (0, +\infty)$ and a unique strong solution $(\rho, u, \eta)$ to (1)-(3) such that

$$\begin{align*}
\rho &\in L^\infty(0, T_*; H^1(\Omega)), \\
u &\in L^\infty(0, T_*; W_0^{1,p}(\Omega) \cap H^2(\Omega)), \\
\eta &\in L^\infty(0, T_*; H^2(\Omega)), \\
\Psi &\in L^\infty(0, T_*; H^2(\Omega)), \\
\sqrt{\rho}u_t &\in L^\infty(0, T_*; L^2(\Omega)), \\
((u_x^2 + \mu_1) \frac{\partial}{\partial t} u_x)_x &\in L^2(0, T_*; L^2(\Omega)).
\end{align*}$$

2 A priori Estimates for Smooth Solutions

In this section, we will prove the local existence of strong solutions. By virtue of the continuity equation (1), we deduce the conservation of mass

$$\int_\Omega \rho(t)dx = \int_\Omega \rho_0 dx = m_0, \quad (t > 0, m_0 > 0).$$

Provided that $(\rho, u, \eta)$ is a smooth solution of (1)-(3) and $\rho_0 \geq \delta$, where $0 < \delta < 1$ is a positive number. We denote by $M_0 = 1 + \mu_1 + \mu_1^{-1} + |\rho_0|_{H^1} + |g|_{L^1}$, and introduce an auxiliary function

$$Z(t) = \sup_{0 \leq s \leq t} \left( 1 + |u(s)|_{W_0^{1,p}} + |\rho(s)|_{H^1} + |\eta(s)|_{L^2} + |\eta(s)|_{H^2} + |\sqrt{\rho}u_t(s)|_{L^1} \right).$$

Then we estimate each term of $Z(t)$ in terms of some integrals of $Z(t)$, apply arguments of Gronwall-type and thus prove that $Z(t)$ is locally bounded.

2.1 Estimate for $|u|_{W_0^{1,p}}$

By using (1), we rewrite the (1) as

$$\rho u_t + \rho u u_x + \rho \psi_x - \left( u_x^2 + \mu_1 \right) \frac{\partial}{\partial t} u_x + (P + \eta) x = -\eta \Phi_x.$$  

(10)

Multiplying (10) by $u_t$, integrating (by parts) over $\Omega_T$, we have

$$\begin{align*}
\int_\Omega u_t^2 dx + \int_\Omega (u_x^2 + \mu_1) \frac{\partial}{\partial t} u_x u_t dx ds &= \int_\Omega \left( \rho u u_x + \rho \psi_x + P + \eta \right) u_t dx ds.
\end{align*}$$

(11)

We deal with each term as follows:

$$\int_\Omega (u_x^2 + \mu_1) \frac{\partial}{\partial t} u_x u_t dx = \frac{1}{2} \int_\Omega \left( u_x^2 + \mu_1 \right) \frac{\partial}{\partial t} (u_x^2) dx = \frac{1}{2} \frac{d}{dt} \int_\Omega \left( u_x^2 + \mu_1 \right) \frac{\partial}{\partial t} (u_x^2) dx$$

$$\int_\mu^\infty (s + \mu_1) \frac{\partial}{\partial t} ds = \int_\mu^\infty \frac{u_t^2}{2} dt = \frac{2}{p} \int_\mu^\infty \left( (u_x^2 + \mu_1) \frac{\partial}{\partial t} u_x \right) dt \geq \frac{2}{p} |u_t|^p - \frac{2}{p} \mu_1^\frac{p}{2}$$
Substituting the above into (11), we obtain

\[ P_t = -\gamma P u_x - P_s u \]  
(12)

\[- \iint_{\Omega_t} (\eta_x + \eta \Phi_x) u_t \text{d}x \text{d}s = \frac{d}{dt} \iint_{\Omega_t} P u_x \text{d}x \text{d}s - \iint_{\Omega_t} P_t u_x \text{d}x \text{d}s.
\]

Since from (1), we get

\[ P_t = -\gamma P u_x - P_s u \]

\[- \iint_{\Omega_t} (\eta_x + \eta \Phi_x) u_t \text{d}x \text{d}s = \frac{d}{dt} \iint_{\Omega_t} (\eta_x + \eta \Phi_x) u \text{d}x \text{d}s - \iint_{\Omega_t} (\eta_x + \eta \Phi_x) u \text{d}x \text{d}s
\]

\[- \iint_{\Omega_t} (\eta_x + \eta \Phi_x) u \text{d}x \text{d}s = \iint_{\Omega_t} \eta_x (u_x - \Phi_x u) \text{d}x \text{d}s
\]

\[ = - \iint_{\Omega_t} [\eta_x - \eta(u - \Phi_x)](u_x - \Phi_x u) \text{d}x \text{d}s.
\]

Substituting the above into (11), we obtain

\[
\int_0^t \left[ \sqrt{\rho} u_t(s) \right]_0^L ds + \frac{1}{p} \int_0^t \left| u_x(t) \right|^p \text{d}x \\
\leq C + \int_\Omega \left| P u_x \right| \text{d}x + \iint_{\Omega_t} (|\rho u u_x| + |\rho \psi_x u| + |\gamma P u_x| + |P_s u u_x|) \text{d}x \text{d}s
\]

\[
+ \iint_{\Omega_t} \left( |\eta_x u_{xx}x| + |\eta_x \Phi_x u_x| + |\eta_x \Phi_{xx} u| + |\eta u_{xx}x| + |\eta u^2 \Phi_x x| + |\eta \psi_x u_x| + |\eta \psi_x \Phi_x u| + |\eta \psi_x^2 u_x| \right) \text{d}x \text{d}s.
\]

Using Young’s inequality, we obtain

\[
\int_0^t \left[ \sqrt{\rho} u_t(s) \right]_0^L ds + \left| u_x(t) \right|^p \text{d}x \\
\leq C + \int_\Omega \left| P u_x \right| \text{d}x + \iint_{\Omega_t} (|\rho u u_x| + |\rho \psi_x u| + |\gamma P u_x| + |P_s u u_x|) \text{d}x \text{d}s
\]

\[
+ \iint_{\Omega_t} \left( |\eta_x u_{xx}x| + |\eta_x \Phi_x u_x| + |\eta_x \Phi_{xx} u| + |\eta u_{xx}x| + |\eta u^2 \Phi_x x| + |\eta \psi_x u_x| + |\eta \psi_x \Phi_x u| + |\eta \psi_x^2 u_x| \right) \text{d}x \text{d}s + C \int_0^t \left[ \right]_\Omega^p
\]

On the other hand, multiplying (1) by \( \psi \) and integrating over \( \Omega \), we get

\[
\int_\Omega |\psi_x|^q \text{d}x = - \int_\Omega (|\psi_x|^{q-2} \psi_x) \phi \text{d}x = -4\pi g \left( \int_\Omega \rho \psi \text{d}x - m_0 \int_\Omega \psi \text{d}x \right)
\]

\[ \leq 8\pi g m_0 |\psi_x| \leq 8\pi g m_0 |\psi_x| \leq \frac{1}{q} |\phi_x|^q + \frac{1}{p} (8\pi g m_0)^p
\]

Then we have

\[
\int_\Omega |\psi_x|^q \text{d}x \leq C(m_0), \quad 1 < q < 2.
\]

Differentiating (1) with respect to \( x \), multiplying it by \( \psi_x \) and integrating over \( \Omega \), we have

\[
\int_\Omega (|\psi_x|^{q-2} \psi_x) \phi \text{d}x = -4\pi g \int_\Omega \rho \psi_x \text{d}x.
\]

By virtue of

\[
\int_\Omega (|\psi_x|^{q-2} \psi_x) \phi \text{d}x = \int_\Omega (|\psi_x|^{q-2} [(q-2) \psi_x^2 + \psi_x^2]) \phi \text{d}x
\]
\[
\geq C \int_{\Omega} |\psi_x|^q |\psi_{xx}|^2 \, dx \geq C |\psi_{xx}|^q \, dx
\]

and
\[
-4\pi g \int_{\Omega} \rho u \psi \, dx \leq C \int_{\Omega} |\rho u| \, dx \leq C |\rho u|^p_{L^p} + C |\rho u|^q_{L^q} \leq C |\rho u|^p_{L^p} + C(s) |\psi_{xx}|^q_{L^q}.
\]

Therefore,
\[
|\psi_{xx}|_{L^q} \leq CZ^2(t).
\] (14)

We deal with the term of \(|u_{xx}|_{L^2}\). Notice that
\[
\left| \left( (u_x^2 + \mu \frac{p-1}{2} u_x) \right)_x \right| \leq \mu \frac{p-1}{2} |u_{xx}|.
\]

Then
\[
|u_{xx}| \leq C |\rho u_t + \rho \mu u_x + \rho \psi_x + (P + \eta)_x + \eta \Phi_x|.
\]

Taking the above inequality by \(L^2\) norm, we get
\[
|u_{xx}|_{L^2} \leq C |\rho u_t + \rho \mu u_x + \rho \psi_x + (P + \eta)_x + \eta \Phi_x|_{L^2} \leq C \left( \frac{1}{\rho} \int_{0}^{\rho} |\sqrt{\rho} \mu_t|_{L^\infty} + |\rho|_{L^\infty} |u|_{L^2} + |\rho|_{L^2} |\psi_{xx}|_{L^2} + |P_s|_{L^2} + |\eta|_{L^2} + |\eta|_{L^2} \right) \Phi_x_{L^2}.
\]

Hence, we deduce that
\[
|u_{xx}|_{L^2} \leq CZ^{\max(\frac{p}{2} + 1, 3)}(t).
\] (15)

Moreover, using (1)1, we have
\[
|P(t)|_{L^2}^p \leq \int_{\Omega} |P(t)|^2 \, dx = \int_{\Omega} |P(0)|^2 \, dx + \int_{0}^{t} \frac{1}{2} \left( \int_{\Omega} P(s)^2 \, dx \right) \, ds \leq \int_{\Omega} |P(0)|^2 \, dx + 2 \int_{0}^{t} \int_{\Omega} P(s)^2 \, dx \, ds \leq C + C \int_{0}^{t} |P_t(t)|_{L^\infty}^2 \, dt \leq C + C \int_{0}^{t} Z^{p+1}(s) \, ds.
\] (16)

Combining (13)-(16), yields
\[
\int_{0}^{t} |\sqrt{\rho} u_t(s)|_{L^2}^2 \, ds + |u_x(t)|_{L^2}^p \leq C(1 + \int_{0}^{t} Z^{\max(\frac{p}{2}, 2\gamma + 3)}(s) \, ds),
\] (17)

where \(C\) is a positive constant, depending only on \(M_0\).

### 2.2 Estimate for \(|\rho|_{H^1}\)

From (1)3, taking it by \(L^2\) norm, we get
\[
|\eta_{xx}|_{L^2} \leq |\eta + (u - \Phi_x)|_{L^2}.
\]
Multiplying (1) by $\rho$, integrating over $\Omega$, we have

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\rho|^2 dx + \int_{\Omega} (\rho u)_x \rho dx = 0.$$  

Integrating by parts, using Sobolev inequality, we deduce that

$$\frac{d}{dt} [\rho(t)]_{L^2}^2 \leq \int |u_x| |\rho|^2 dx \leq |u_{xx}| |\rho|^2_{L^2}.  \tag{19}$$

Differentiating (1) with respect to $x$, and multiplying it by $\rho_x$, integrating over $\Omega$, using Sobolev inequality, we have

$$\frac{d}{dt} \int_{\Omega} |\rho_x|^2 dx = -\int_{\Omega} \left[ \frac{3}{2} u_x (\rho_x)^2 + \rho \rho_x u_{xx} \right](t) dx \leq C[|u_{xx}|_{L^2} + |\rho|_{L^2} |\rho_x|_{L^2}] \leq C|\rho|^2_{H^1} |u_{xx}|_{L^2}.  \tag{20}$$

From (19) and (20), by Gronwall’s inequality, it follows that

$$\sup_{0 \leq t \leq T} \rho(t)_{H^1}^2 \leq |\rho_0|_{H^1}^2 \exp \left( C \int_0^T |u_{xx}|_{L^2} ds \right) \leq C \exp \left( C \int_0^T Z^\max(\ell = 1, 3)(s) ds \right).  \tag{21}$$

Besides, by using (1), we can also get the following estimates:

$$|\rho_x(t)|_{L^2} \leq |\rho_x(t)|_{L^2} |u(t)|_{L^\infty} + |\rho(t)|_{L^\infty} |u_x(t)|_{L^2} \leq CZ^2(t).  \tag{22}$$

### 2.3 Estimate for $|\eta|_{L^2}$ and $|\eta|_{H^1}$

Multiplying (1) by $\eta$, integrating the resulting equation over $\Omega_T$, using the boundary conditions (3), Young’s inequality, we have

$$\int_0^T |\eta_x(s)|_{L^2}^2 ds + \frac{1}{2} |\eta(t)|_{L^2}^2 \leq \int \int (|\eta u_{\eta x}| + |\eta \phi_x \eta_x|) dx ds \leq \frac{1}{4} \int_0^T |\eta_x(s)|_{L^2}^2 ds + C \int_0^T |u_{xx}|_{L^2} |\eta|_{H^1}^2 ds + C \int_0^T |\eta|_{H^1}^2 + C \int_0^T Z^4(t) ds. \tag{23}$$

Multiplying (1) by $\eta_t$, integrating (by parts) over $\Omega_T$, using the boundary conditions (3), Young’s inequality, we have

$$\int_0^T |\eta_x(s)|_{L^2}^2 ds + \frac{1}{2} |\eta_x(t)|_{L^2}^2 \leq \int \int |\eta(u - \phi_x)\eta| dx ds \leq \frac{1}{4} \int_0^T |\eta_x(s)|_{L^2}^2 ds + C \int_0^T |\eta|_{H^1}^2 |u_{xx}|_{L^2} ds + C \int_0^T |\eta|_{H^1}^2 ds + C$$
Differentiating (1) with respect to $t$, multiplying the resulting equation by $\eta$, integrating (by parts) over $\Omega_t$, we get

$$
\int_0^t |\eta_t(s)|^2_L \, ds + \frac{1}{2} |\eta(t)|^2_L = \int_{\Omega_t} (\eta(\mathbf{u} - \mathbf{v}_x)) \eta_t \, d\mathbf{x} ds
$$

$$
\leq C + \int_{\Omega_t} (|\eta u\eta_t| + |\eta \Phi_x \eta_t| + |\eta u_t\eta| + |\eta u_{tt}\eta|) \, d\mathbf{x} ds
$$

$$
\leq C(1 + \int_0^t (|\eta|_{L^2}^2 + |\eta_t|_{L^2}^2 + |\eta u_t|_{L^2}^2 + |\eta u_{tt}|_{L^2}^2) \, d\mathbf{x})
$$

$$
+ \frac{1}{2} \int_0^t |\eta_t|^2_L + \frac{1}{2} \int_0^t |u_{tt}|^2_L 
$$

$$
\leq C(1 + \int_0^t Z^{2\gamma+6}(s) \, ds).
$$

Combining (23)-(25), we get

$$
|\eta|^2_H + |\eta|^2_L + \int_0^t (|\eta|_{L^2}^2 + |\eta_t|_{L^2}^2 + |\eta u_t|_{L^2}^2) \, ds \leq C(1 + \int_0^t Z^{2\gamma+6}(s) \, ds).
$$

### 2.4 Estimate for $|\sqrt{\rho}u_t|_{L^2}$

Differentiating equation (10) with respect to $t$, multiplying the result equation by $u_t$, and integrating it over $\Omega$ with respect to $x$, we have

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |u_t|^2 \, dx + \int_{\Omega} \left[ (u_x^2 + \mu_1) \frac{\partial}{\partial x} u_t \right] u_{xt} \, dx
$$

$$
= \int_{\Omega} \left[ (\rho u)_x u_t^2 + uu_{xt} u_t + \Psi_x u_t - \rho u_t u_t^2 - \rho \psi_{xt} u_t - (P + \eta)_t u_{xt} - \eta \Phi_x u_t \right] \, dx.
$$

Note that

$$
\left[ (u_x^2 + \mu_1) \frac{\partial}{\partial x} u_t \right] u_{xt} = (u_t^2 + \mu_1) \frac{\partial}{\partial x} (P - 1)(u_x^2 + \mu_1) u_t^2 \geq \mu_1 \frac{\partial}{\partial x} u_t^2.
$$

Combining (12), (27) can be rewritten into

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |u_t|^2 \, dx + \int_{\Omega} |u_{xt}|^2 \, dx
$$

$$
\leq 2 \int_{\Omega} \rho |u| |u_t| |u_{xt}| \, dx + \int_{\Omega} \rho |u| |u_x| |u_t| \, dx + \int_{\Omega} \rho |u|^2 |u_{xx}| |u_t| \, dx
$$

$$
+ \int_{\Omega} \rho |u|^2 |u_{xx}| |u_{xt}| \, dx + \int_{\Omega} \rho |u| |\psi_{xx}| |u_t| \, dx + \int_{\Omega} \rho |u| |\psi_{x}| |u_{xt}| \, dx
$$

$$
+ \int_{\Omega} \rho |u_{xx}| |u_t|^2 \, dx + \int_{\Omega} |P| |u_{xx}| |u_{xt}| \, dx + \int_{\Omega} |P_x| |u| |u_{xt}| \, dx
$$

$$
+ \int_{\Omega} |\eta| |u_{xt}| \, dx + \int_{\Omega} |\eta| |\psi_{x}| |u_t| \, dx + \int_{\Omega} \rho |\psi_{xt}| |u_t| \, dx = \sum_{j=1}^{12} I_j.
$$

(28)
By using Sobolev inequality, Hölder inequality and Young’s inequality, (14), (15), we estimate each term of $I_j$ as follows

$$I_1 = 2 \int_{\Omega} \rho |\Psi_x| |u| |x| |\text{d}x| \leq 2 |\rho|^{\frac{1}{2}} |\Psi_x| |\sqrt{\rho} u| |x| |\text{d}x| \leq C Z^5(t) + \frac{1}{7} |x| |\text{d}x|^2,$$

$$I_2 = \int_{\Omega} \rho |\Psi_x| |u| |\text{d}x| \leq |\rho|^{\frac{1}{2}} |\Psi_x| |\sqrt{\rho} u| |\text{d}x| \leq CZ^3(t),$$

$$I_3 = \int_{\Omega} \rho |\Psi_x| |u| |\text{d}x| \leq |\rho|^{\frac{1}{2}} |\Psi_x| |\sqrt{\rho} u| |\text{d}x| \leq CZ^{\max\{\frac{5}{4}, 3\}}(t),$$

$$I_4 = \int_{\Omega} \rho |\Psi_x| |u| |\text{d}x| \leq |\rho|^{\frac{1}{2}} |\Psi_x| |\sqrt{\rho} u| |\text{d}x| \leq CZ^3(t) + \frac{1}{7} |x| |\text{d}x|^2,$$

$$I_5 = \int_{\Omega} \rho |\Psi_x| |u| |\text{d}x| \leq |\rho|^{\frac{1}{2}} |\Psi_x| |\sqrt{\rho} u| |\text{d}x| \leq CZ^{\frac{4}{5}}(t) + \frac{1}{7} |x| |\text{d}x|^2,$$

$$I_6 = \int_{\Omega} \rho |\Psi_x| |u| |\text{d}x| \leq |\rho|^{\frac{1}{2}} |\Psi_x| |\sqrt{\rho} u| |\text{d}x| \leq CZ^{\frac{4}{5}}(t) + \frac{1}{7} |x| |\text{d}x|^2,$$

$$I_7 = \int_{\Omega} |\Psi_x| |u| |\text{d}x| \leq |\rho|^{\frac{1}{2}} |\Psi_x| |\sqrt{\rho} u| |\text{d}x| \leq CZ^{\max\{\frac{5}{4}, 3\}, 5}(t),$$

$$I_8 = \int_{\Omega} \gamma |P| |\Psi_x| |u| |\text{d}x| \leq C |P| |\Psi_x| |x| |\text{d}x| \leq CZ^{2\gamma + 2}(t) + \frac{1}{7} |x| |\text{d}x|^2,$$

$$I_9 = \int_{\Omega} |P_x| |\Psi_x| |u| |\text{d}x| \leq |P_x| |x| |\text{d}x| \leq CZ^{\gamma + 2}(t) + \frac{1}{7} |x| |\text{d}x|^2,$$

$$I_{10} = \int_{\Omega} |\nabla P| |\Psi_x| |u| |\text{d}x| \leq |\nabla P| |x| |\text{d}x| \leq CZ^2(t) + \frac{1}{7} |x| |\text{d}x|^2,$$

$$I_{11} = \int_{\Omega} |\nabla \Psi| |u| |\text{d}x| \leq |\nabla \Psi| |x| |\text{d}x| \leq CZ^2(t) + \frac{1}{7} |x| |\text{d}x|^2,$$

$$I_{12} = \int_{\Omega} \beta |\Psi| |u| |\text{d}x| \leq |\beta|^{\frac{1}{2}} |\Psi| |\sqrt{x} u| |\text{d}x| \leq C \frac{1}{7} |x| |\text{d}x|^2,$$

where $C$ is a positive constant, depending only on $M_0$.

Next, we deal with the term $|\Psi_x|_{L^2}$ of $I_{12}$. Differentiating (13) with respect to $t$, multiplying it by $\Psi_t$, integrating over $\Omega$ and using Young’s inequality, we obtain

$$\int_{\Omega} (|\Psi_x| |q-2| \Psi_x) \Psi_x \text{d}x = -4\pi g \int_{\Omega} \rho \Psi_t \text{d}x.$$

By virtue of

$$\int_{\Omega} (|\Psi_x| |q-2| \Psi_x) \Psi_x \text{d}x = \int_{\Omega} (|\Psi_x| |q-4| (q-2) \Psi_x^2 + \Psi_x^2) \Psi_x \text{d}x$$

$$\geq C \int_{\Omega} |\Psi_x| |q-2| \Psi_x^2 \text{d}x \geq C |\Psi_x|^2 |\Psi_x|_{L^2}^2$$

and

$$-4\pi g \int_{\Omega} \rho \Psi_t \text{d}x \leq C \int_{\Omega} |\rho| |\Psi_t| \text{d}x \leq C |\rho|_{L^2}^2 + C |\Psi_t|_{L^2}^2 \leq C |\rho|_{L^2}^2 + C(\varepsilon) |\Psi_x|_{L^2}^2,$$

then $|\Psi_x|_{L^2} \leq CZ^{\frac{n(n-1)}{4}}(t)$. Therefore,

$$I_{12} = \int_{\Omega} \rho |\Psi_x| |u_t| |\text{d}x| \leq C |\rho|^{\frac{1}{2}} |\Psi_x| |\sqrt{\rho} u_t| |\text{d}x| \leq CZ^{\frac{n(n-1)}{4}}(t).$$
Substituting $I_j (j = 1, 2, \ldots, 12)$ into (28), and integrating over $(\tau, t) \in (0, T)$ on the time variable, we have

$$
\sqrt{\rho} u_t(t)_{L^2}^2 + \int_0^t \| u_x(t) \|_{L^2}^2 \, ds \leq \sqrt{\rho} u_t(\tau)_{L^2}^2 + C \int_\tau^T Z_{\max}(\frac{\rho}{\eta} + \eta\Phi_x) \, ds.
$$

(29)

To obtain the estimate of $|\sqrt{\rho} u_t(t)_{L^2}|^2$, we need to estimate $\lim_{t \to 0} |\sqrt{\rho} u_t(t)_{L^2}|^2$. Multiplying (10) by $u_t$ and integrating over $\Omega$, we have

$$
\int_\Omega \rho |u_t|^2 \, dx \leq 2 \int_\Omega \left( \rho |u|^2 |u_x|^2 + \rho |\Psi_x|^2 + \rho^{-1} \right) - \left[ (u_x^2 + \mu_1) \frac{\rho}{\eta} u_{xx} \right] + (P + \eta) + \eta \Phi_x^2 \, dx.
$$

According to the smoothness of $(\rho, u, \eta)$, we obtain

$$
\lim_{\tau \to 0} \int_\Omega \left( \rho |u|^2 |u_x|^2 + \rho |\Psi_x|^2 + \rho^{-1} \right) - \left[ (u_x^2 + \mu_1) \frac{\rho}{\eta} u_{xx} \right] + (P + \eta) + \eta \Phi_x^2 \, dx
$$

$$
= \int_\Omega \left( \rho |u_0|^2 |u_{0x}|^2 + \rho_0 |\Psi_x|^2 + \rho_0^{-1} \right) - \left[ (u_{0x}^2 + \mu_1) \frac{\rho_0}{\eta} u_{0xx} \right] + (P_0 + \eta_0) + \eta_0 \Phi_{0x}^2 \, dx
$$

$$
\leq \rho_0 |\Omega| \| u_0 \|_{H^1}^2 \| u_{0x} \|^2 + \rho_0 \| \Psi_{0x} \|^2 + \rho_0^{-1} + \left( \eta_0 \Phi_{0x}^2 \right) \leq C.
$$

Therefore, taking a limit on $\tau$ in (29), as $\tau \to 0$, we conclude that

$$
|\sqrt{\rho} u_t(t)_{L^2}|^2 + \int_0^t \| u_x(t) \|_{L^2}^2 \, ds \leq C(1 + \int_0^T Z_{\max}(\frac{\rho}{\eta} + \eta\Phi_x) \, ds),
$$

(30)

where $C$ is a positive constant, depending only on $M_0$.

Combining the estimates of (15), (18), (21), (22), (17), (26), (30) and the definition of $Z(t)$, we conclude that

$$
Z(t) \leq \tilde{C} \exp(\tilde{C} \int_0^T Z_{\max}(\frac{\rho}{\eta} + \eta\Phi_x) \, ds),
$$

(31)

where $\tilde{C}, \hat{C}$ are positive constants, depending only on $M_0$. This means that there exist a time $T_1 > 0$ and a constant $\hat{C} > 0$, such that

$$
\text{ess sup}_{0 \leq t \leq T_1} \left( |\rho|_{H^1} + |u|_{W^{2, \eta} H^2} + |\eta|_{H^2} + |\eta|_{L^2} + |\sqrt{\rho} u_t|_{L^2} + |\rho|_{L^2} \right)
$$

$$
+ \int_0^{T_1} \left( |\sqrt{\rho} u_t|_{L^2}^2 + \| u_x \|_{L^2}^2 + \| u_{xx} \|_{L^2} + \| u_{xxx} \|_{L^2} \right) \, ds \leq \hat{C}.
$$

(32)

### 3 Proof of the Main Theorem

In this section, our proof will be based on the usual iteration argument and some ideas developed in [13, 14]. Precisely, we construct the approximate solutions, by using the iterative scheme, inductively, as follows: first define $u^0 = 0$ and assuming that $u^{k-1}$ was defined for $k \geq 1$, let $\rho_k, u^k, \eta_k$ be the unique smooth solution to the following problems:

$$
\begin{align*}
\rho_k + \rho u^k u^{k-1} + \rho_k u_x^{k-1} &= 0 \\
\rho_k u_k^k + \rho u^{k-1} u_x^k + \rho_k \eta^k &= \left( (u_x^k)^2 + \mu_1 \right) \frac{\rho_k}{\eta_k} u_{xx}^k + P^k + \eta^k \Phi_x^k \\
\rho_k \left( |\Psi_x^k|^2 \Phi_x^k \right)_{xx} &= 4\pi \left( \rho_k - m_0 \right) \\
\eta^k + \left( \eta^k (u^{k-1} - \Phi_x) \right)_{xx} &= \eta_{xx}^k
\end{align*}
$$
Multiplying (34) by \( \rho^k \), we have

\[
\rho^k u^k \big|_{t=0} = (\rho_0, u_0, \eta_0)
\]

\[
u^k \big|_{\partial \Omega} = (\eta_0 + \eta^k \phi_x) |_{\partial \Omega} = 0
\]

with the process, the nonlinear coupled system has been deduced into a sequence of decoupled problems and each problem admits a smooth solution. And the following estimates hold

\[
\begin{align*}
\text{ess sup}_{0 \leq t \leq T_1} \left( \rho^k |_{H^1} + |u^k|_{W_0^{1,\sigma}} + |\eta^k|_{H^2} + |\sqrt{\rho^k} u^k|_{L^2} + |\rho^k|_{L^2} \right) \\
+ \int_0^{T_1} \left( |\sqrt{\rho^k} u^k|_{L^2}^2 + |u^k_x|_{L^2}^2 + |\eta^k_x|_{L^2}^2 + |\eta^k_x|_{L^2}^2 \right) ds \leq C,
\end{align*}
\]

(33)

where \( C \) is a generic constant depending only on \( M_0 \), but independent of \( k \).

In addition, we first find \( \rho^k \) from the initial problem

\[
\rho^k + u^{k-1} \rho_x + u^{k-1} \rho = 0, \quad \text{and} \quad \rho^k |_{t=0} = \rho_0
\]

with smooth function \( u^{k-1} \), obviously, there is a unique solution \( \rho^k \) to the above problem and also by a standard argument, we could obtain that

\[
\rho^k (x, t) \geq \delta \exp \left[ - \int_0^t |u^{k-1}(x, s)|_{L^\infty} ds \right] > 0, \text{for all } t \in (0, T_1).
\]

Next, we have to prove that the approximate solution \( (\rho^k, u^k, \eta^k) \) converges to a solution to the original problem (1) in a strong sense. To this end, let us define

\[
\tilde{\rho}^{k+1} = \rho^{k+1} - \rho^k, \quad \tilde{u}^{k+1} = u^{k+1} - u^k, \quad \tilde{\eta}^{k+1} = \eta^{k+1} - \eta^k,
\]

then we can verify that the functions \( \tilde{\rho}^{k+1}, \tilde{u}^{k+1}, \tilde{\eta}^{k+1} \) satisfy the system of equations

\[
\begin{align*}
\tilde{\rho}^{k+1} + (\tilde{\rho}^{k+1} u^k)_x + (\tilde{\rho}^{k+1} u^k)_x &= 0 \\
\rho^{k+1} u^{k+1} + \rho^k u^k u^{k+1} - \left( (u_x^{k+1})^2 + \mu_1 \right) \frac{\partial}{\partial x} u^{k+1} x - \left( (u_x^k)^2 + \mu_1 \right) \frac{\partial}{\partial x} u^k = 0
\end{align*}
\]

(34)

(35)

(36)

(37)

Multiplying (34) by \( \rho^{k+1} \), integrating over \( \Omega \) and using Young’s inequality, we obtain

\[
\begin{align*}
\frac{d}{dt} |\rho^{k+1}|_{L^2}^2 &\leq C |\rho^{k+1}|_{L^2}^2 |u^k_{xL} + \rho^k |_{H^1} |u^k_x|_{L^2} |\rho^{k+1}|_{L^2} \\
&\leq C |u^k_{xL}|_{L^2} |\rho^{k+1}|_{L^2} + C |\rho^k |_{H^1} |\rho^{k+1}|_{L^2} + C |u^k_{xL}|_{L^2} \\
&\leq C |\rho^{k+1}|_{L^2}^2 + C |u^k_{xL}|_{L^2}^2,
\end{align*}
\]

(38)

where \( C \) is a positive constant, depending on \( M_0 \) and \( \zeta \) for all \( t < T_1 \) and \( k \geq 1 \).

Multiplying (35) by \( \tilde{u}^{k+1} \), integrating over \( \Omega \) and using Young’s inequality, we obtain

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_\Omega \rho^{k+1} |\tilde{u}^{k+1}|^2 dx + \int_\Omega \left( (u_x^{k+1})^2 + \mu_1 \right) \frac{\partial}{\partial x} u^{k+1} x - \left( (u_x^k)^2 + \mu_1 \right) \frac{\partial}{\partial x} u^k \tilde{u}^{k+1} dx &\leq C \left( |(\rho^k)^2|_{L^2} |u^k_x| + |\tilde{u}^{k+1}| + |\psi^{k+1}| |\tilde{u}^{k+1}| + |\tilde{P}^{k+1} - \tilde{P}^k| |\tilde{u}^{k+1}| + |\rho^k |u^k_x||\tilde{u}^{k+1}| \right) \\
&+ |\rho^k |\psi^{k+1} |\tilde{u}^{k+1}| + |\tilde{u}^{k+1} |\tilde{u}^{k+1} | + |\tilde{\eta}^{k+1} |\phi_x|\tilde{u}^{k+1}|)
\end{align*}
\]

(39)
Let 

\[ \sigma(s) = (s^2 + \mu_1)^{\frac{\nu}{2}} s, \]

then

\[ \sigma'(s) = \left( (s^2 + \mu_1)^{\frac{\nu}{2}} s \right)' = (s^2 + \mu_1)^{\frac{\nu}{2}} \left( (p - 1)s^2 + \mu_1 \right) \geq \mu_1^{\frac{\nu}{2}}. \]

We estimate the second term of (39) as follows

\[
\int \left( \left( (u^{k+1}_x)^2 + \mu_1 \right)^{\frac{\nu}{2}} u^{k+1}_x \right)_x - \left( \left( (u^k_x)^2 + \mu_1 \right)^{\frac{\nu}{2}} u^k_x \right)_x \right) \, dx
\]

\[
= \int \frac{1}{\mu_1} \left( \theta u^{k+1}_x + (1 - \theta) u^k_x \right) \, d\theta |\bar{\psi}^{k+1}_x|^2 \, dx \geq \mu_1^{\frac{\nu}{2}} \int |\bar{\psi}^{k+1}_x|^2 \, dx. \quad (40)
\]

Similarly, multiplying (36) by \( \bar{\psi}^{k+1}_x \), integrating over \( \Omega \), we get

\[
\int \left( \left( (\bar{\psi}^{k+1}_x)^2 q - \bar{\psi}^{k+1}_x \right)_x - \left( \left( (\bar{\psi}^k_x)^2 q - \bar{\psi}^k_x \right)_x \right) \right) \, dx = 4\pi \int \rho^{k+1} \bar{\psi}^{k+1}_x \, dx,
\]

since

\[
\int \left[ \left( (\bar{\psi}^{k+1}_x)^2 q - \bar{\psi}^{k+1}_x \right)_x - \left( \left( (\bar{\psi}^k_x)^2 q - \bar{\psi}^k_x \right)_x \right) \right] \, dx
\]

\[
= (q - 1) \int \frac{1}{\mu_1} \left( \theta \bar{\psi}^{k+1}_x + (1 - \theta) \bar{\psi}^k_x \right) \, d\theta |\bar{\psi}^{k+1}_x|^2 \, dx
\]

and

\[
\int \frac{1}{\mu_1} \left( \theta \bar{\psi}^{k+1}_x + (1 - \theta) \bar{\psi}^k_x \right) \, d\theta |\bar{\psi}^{k+1}_x|^2 \, dx = \int \frac{1}{\mu_1} \left( \theta \bar{\psi}^{k+1}_x + (1 - \theta) \bar{\psi}^k_x \right)^2 - q \, d\theta
\]

\[
\geq \int \frac{1}{\mu_1} \left( \left( \left( (\bar{\psi}^{k+1}_x)^2 q - \bar{\psi}^{k+1}_x \right)_x \right) \right) \, dx \geq \frac{1}{\mu_1} \int \left( \bar{\psi}^{k+1}_x \right)^2 \, dx.
\]

Then

\[
\int \left( \left( \bar{\psi}^{k+1}_x \right)^2 q - \bar{\psi}^{k+1}_x \right)_x \, dx \geq \frac{1}{\mu_1} \int \left( \bar{\psi}^{k+1}_x \right)^2 \, dx.
\]

That means (41) turns into

\[
\int \left( \bar{\psi}^{k+1}_x \right)^2 \, dx \leq C |\rho^{k+1}_x|_{L^2}^2. \quad (42)
\]

Substituting (40) and (42) into (39), using Young’s inequality, yields

\[
\frac{d}{dt} \int \rho^{k+1} |u^{k+1}_x|^2 \, dx + \int |\bar{\psi}^{k+1}_x|^2 \, dx
\]

\[
\leq C |\rho^{k+1}_x|_{L^2} |u^k_x|_{L^2} |\bar{\psi}^{k+1}_x|_{L^2} + |\rho^{k+1}_x|_{L^2} |u^k_x|_{L^2} |\bar{\psi}^{k+1}_x|_{L^2} + |\rho^{k+1}_x|_{L^2} |\bar{\psi}^{k+1}_x|_{L^2} |\bar{\psi}^{k+1}_x|_{L^2}
\]

\[
+ |p^{k+1}_x|_{L^2} |\bar{\psi}^{k+1}_x|_{L^2} + |\rho^{k+1}_x|_{L^2} |\bar{\psi}^{k+1}_x|_{L^2} + |\rho^{k+1}_x|_{L^2} |\bar{\psi}^{k+1}_x|_{L^2}
\]

\[
\leq B(t) |\rho^{k+1}_x|_{L^2} + C |\bar{\psi}^{k+1}_x|_{L^2} + |\rho^{k+1}_x|_{L^2} + |\bar{\psi}^{k+1}_x|_{L^2},
\]

where \( B(t) = C(1 + |u^k_x(t)|_{L^2}^2) \), for all \( t \leq T_1 \) and \( k \geq 1 \). Using (33) we derive

\[
\int_0^t B(t) \, ds \leq C + Ct.
\]
Collecting (38), (43) and (44), we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left| \bar{\rho}^{k+1} \right|^2 dx + \int_{\Omega} \left| u^{k+1}_{\bar{\eta}} \right|^2 dx
\leq \left| \eta^{k+1}_\Omega \right| \left| u^k - \Phi_x \right|_{L^2(H^{1/2}(\Omega))} + \int_{\Omega} \left( \left| \bar{\eta}^k \right| \eta^{k+1}_\Omega \right) dx
\leq \left| \eta^{k+1}_\Omega \right| \left| u^k - \Phi_x \right|_{L^2(H^{1/2}(\Omega))} + \left| \eta^{k+1}_\Omega \right| \left| u^k - \Phi_x \right|_{L^2(H^{1/2}(\Omega))}
\leq C \left| \eta^{k+1}_\Omega \right| \left| u^k - \Phi_x \right|_{L^2(H^{1/2}(\Omega))} + \left| \eta^{k+1}_\Omega \right| \left| u^k - \Phi_x \right|_{L^2(H^{1/2}(\Omega))}.
\]

(44)

Collecting (38), (43) and (44), we obtain

\[
\frac{d}{dt} \left( \left| \bar{\rho}^{k+1}(t) \right|^2_{L^2} + \left| \sqrt{\rho^{k+1} \bar{u}^{k+1}(t)} \right|^2_{L^2} + \left| \bar{\eta}^{k+1}(t) \right|^2_{L^2} \right) + \left| \bar{u}^{k+1}(t) \right|^2_{L^2} + \left| \eta^{k+1}_\Omega \right| \left| u^k - \Phi_x \right|_{L^2(H^{1/2}(\Omega))}
\leq E(\zeta) \left( \left| \bar{\rho}^{k+1}(t) \right|^2_{L^2} + C \left| \sqrt{\rho^{k+1} \bar{u}^{k+1}} \right|_{L^2}^2 + C \left| \eta^{k+1}_\Omega \right|^2_{L^2} + \left| \eta^{k+1}_\Omega \right| \left| u^k - \Phi_x \right|_{L^2(H^{1/2}(\Omega))} \right),
\]

(45)

where \( E(\zeta) \) depends only on \( B, C, \) and \( C_\zeta \), for all \( t \leq T_1 \) and \( k \geq 1 \). Using (33), we have

\[
\int_0^t E(\zeta) ds \leq C + C_\zeta t.
\]

Integrating (45) over \((0, t) \subset (0, T_1)\) with respect to \( t \), using Gronwall’s inequality, we have

\[
\left| \bar{\rho}^{k+1}(t) \right|^2_{L^2} + \left| \sqrt{\rho^{k+1} \bar{u}^{k+1}(t)} \right|^2_{L^2} + \left| \bar{\eta}^{k+1}(t) \right|^2_{L^2} + \int_0^t \left| \bar{u}^{k+1}(t) \right|^2_{L^2} ds + \int_0^t \left| \eta^{k+1}_\Omega \right| \left| u^k - \Phi_x \right|_{L^2(H^{1/2}(\Omega))} ds
\leq C \exp(C_\zeta t) \int_0^t \left( \left| \sqrt{\rho^{k+1} \bar{u}^{k+1}}(s) \right|^2_{L^2} + \left| \eta^{k+1}_\Omega \right| \left| u^k - \Phi_x \right|_{L^2(H^{1/2}(\Omega))} \right) ds.
\]

(46)

From the above recursive relation, choose \( \zeta > 0 \) and \( 0 < T_* < T_1 \) such that \( C \exp(C_\zeta T_*) < \frac{1}{2} \), using Gronwall’s inequality, we deduce that

\[
\sum_{k=1}^K \left( \sup_{0 < t \leq T_*} \left( \left| \bar{\rho}^{k+1}(t) \right|^2_{L^2} + \left| \sqrt{\rho^{k+1} \bar{u}^{k+1}(t)} \right|^2_{L^2} + \left| \bar{\eta}^{k+1}(t) \right|^2_{L^2} \right) + \int_0^{T_*} \left| \bar{u}^{k+1}(t) \right|^2_{L^2} + \int_0^{T_*} \left| \eta^{k+1}_\Omega \right| \left| u^k - \Phi_x \right|_{L^2(H^{1/2}(\Omega))} dt \right) < C.
\]

(47)

Since all of the constants do not depend on \( \delta \), as \( k \to \infty \), we conclude that sequence \(( \rho^k, u^k, \eta^k)\) converges to a limit \(( \rho^\delta, u^\delta, \eta^\delta)\) in the following convergence

\[
\rho \to \rho^\delta \quad \text{in} \quad L^\infty(0, T_*; L^2(\Omega)),
\]

(48)

\[
u \to u^\delta \quad \text{in} \quad L^\infty(0, T_*; L^2(\Omega)) \cap L^2(0, T_*; H^1(\Omega)),
\]

(49)

\[
\eta \to \eta^\delta \quad \text{in} \quad L^\infty(0, T_*; L^2(\Omega)) \cap L^2(0, T_*; H^1(\Omega)),
\]

(50)

and there also holds

\[
\text{ess sup}_{0 < t \leq T_1} \left( \left| \rho^\delta \right|_{H^2} + \left| u^\delta \right|_{W^{1,2}} \rho^{\delta} + \left| \eta^\delta \right|_{H^2} + \left| \eta^\delta \right|_{L^2} + \sqrt{\rho^\delta} \left| u^\delta \right|_{L^2} + \left| \rho^\delta \right|_{L^2} \right)
+ \int_0^{T_1} \left( \left| \sqrt{\rho^\delta} \left| u^\delta \right|^2_{L^2} + \left| \eta^\delta \right|^2_{L^2} + \left| \rho^\delta \right|^2_{L^2} \right) ds \leq C.
\]

(51)

For each \( \delta > 0 \), let \( \rho^\delta_0 = \rho^\delta - \rho_0 + \delta J_0 \) is a mollifier on \( \Omega \), and \( u^\delta_0 \in H^1_0(\Omega) \cap H^2(\Omega) \) is a smooth solution of the boundary value problem

\[
\begin{cases}
-(\mu_1 + \frac{\mu_2}{2} \left| u_{\alpha x}^\delta \right|^2)_{xx} + P(\rho^\delta_0 + \eta_0^\delta) + \eta_0^\delta \Phi_x = \left( \rho^\delta_0 \right)^{1/2} \left( g^\delta + \beta \Phi_x \right), \\
u_0^\delta(0) = u_0^\delta(1) = 0,
\end{cases}
\]

(52)
where $g^i \in C^{\infty}$ and satisfies $|g^i|_{L^2} \leq |g|_{L^2}$, $\lim_{\delta \to 0^+} |g^i - g|_{L^2} = 0$.

We deduce that $(\rho^\delta, u^\delta, \eta^\delta)$ is a solution of the following initial boundary problem

\[
\begin{cases}
\rho_t + (\rho u)_x = 0, \\
(\rho u)_t + (\rho u^2)_x + \rho \Phi_x - \lambda \left( (u^2_x + \mu_1) \frac{u_{x}^2}{u_{x}} \right)_x + (P + \eta) x = -\gamma \Phi_x, \\
(\Phi^2_x)_{xx} = 4 \gamma g(\rho - \frac{1}{|\Omega|} \int \rho dx), \\
\eta + (u(\eta - \Phi_x))_x = \eta_{xx}, \\
(\rho, u, \eta)_{|x=0} = (\rho_0^\delta, u_0^\delta, \eta_0^\delta), \\
u_{|x=\delta} = \psi_{|x=\delta} = (\eta_{x} + \eta \Phi_x)_{|x=\delta} = 0,
\end{cases}
\]

where $\rho_0^\delta \geq \delta, \ p > 2, \ 1 < q < 2$.

By the proof of Lemma 2.3 in [11], there exists a subsequence $\{u_0^\delta\}$ of $\{u_0^\delta\}$, as $\delta_j \to 0^+$, $u_0^\delta \to u_0$ in $H^1_0(\Omega) \cap H^2(\Omega), -(u_0^{\delta_i})_{|x=\delta} \to -(u_0^{\delta_i})_{|x=\delta}$ in $L^2(\Omega)$, Hence, $u_0$ satisfies the compatibility condition (8) of Theorem 1.2. By virtue of the lower semi-continuity of various norms, we deduce that $(\rho, u, \eta)$ satisfies the following uniform estimate

\[
\text{ess sup}_{0 \leq t \leq T_i} (|\rho|_{H^1} + |u|_{W^{2,q}_{x,t} \cap H^1} + |\eta|_{H^2} + |\eta|_{L^2} + |\sqrt{\rho} u|_{L^2} + |\rho|_{L^2})
\]

\[
+ \int_0^r (|\sqrt{\rho} u|^2_{L^2} + |u_x|^2_{L^2} + |\eta|^2_{L^2} + |\eta\Phi|^2_{L^2} + |\eta_{x}|^2_{L^2}) \, ds \leq C,
\]

where $C$ is a positive constant, depending only on $M_0$.

The uniqueness of solution can be obtained by the same method as the above proof of convergence, we omit the details here. This completes the proof.

**Acknowledgement:** The authors would like to thank the anonymous referees for their valuable suggestions and comments which improved the presentation of the paper. This work was supported by the National Natural Science Foundation of China (Nos.11526105; 11572146), the funds of education department of Liaoning Province (Nos.JQL201715411; JQL201715409).

**References**

[1] Carrillo J. A., Karper T., Trivisa K., On the dynamics of a fluid-particle model: the bubbling regime, Nonlinear Analysis, 2011, 74, 2778-2801.

[2] Fang D. Y., Zi R. Z., Zhang T., Global classical large solutions to a 1D fluid-particle interaction model: The bubbling regime, J. Math. Phys. 2012, 53, 033706.

[3] Ballew J., Trivisa K., Suitable weak solutions and low stratification singular limit for a fluid particle interaction model. Quart. Appl. Math., 2012, 70, 469-494.

[4] Ballew J., Trivisa K., Weakly dissipative solutions and weak-strong uniqueness for the Navier-Stokes-Smoluchowski system. Nonlinear Analysis, 2013, 91, 1-19.

[5] Constantin P., Masmoudi N., Global well-posedness for a Smoluchowski equation coupled with Navier-Stokes equations in 2D, Commun. Math. Phys., 2008, 278, 179-191.

[6] Chemin J. Y. and Masmoudi N., About lifespan of regular solutions of equations related to viscoelastic fluids, SIAM J. Math. Anal., 2001, 33, 84-112.

[7] Chhabra R. P., Bubbles, Drops, and Particles in Non-Newtonian Fluids, 2nd ed. (Taylor & Francis, New York, 2007).

[8] Ladyzhenskaya O. A., New equations for the description of viscous incompressible fluids and solvability in the large of the boundary value problems for them. In Boundary Value Problems of Mathematical Physics, vol. V, Amer. Math. Soc., Providence, RI (1970).

[9] Málek J., Nečas J., Rokyta M., Růžička M., Weak and Measure-Valued Solutions to Evolutionary PDEs, Chapman and Hall, New York (1996).

[10] Mamontov A. E., Global regularity estimates for multidimensional equations of compressible non-Newtonian fluids, Ann. Univ. Ferrara-Sez. VII-Sc. Mat, 2000, 139-160.
[11] Yuan H. J. and Xu X. J., Existence and uniqueness of solutions for a class of non-Newtonian fluids with singularity and vacuum, J. Differential Equations, 2008, 245, 2871-2916.

[12] Rozanova O., Nonexistence results for a compressible non-Newtonian fluid with magnetic effects in the whole space, J. Math. Anal. Appl., 2010, 371, 190-194.

[13] Cho Y., Choe H., Kim H., Unique solvability of the initial boundary value problems for compressible viscous fluids, J. Math. Pures Appl. 2004, 83, 243-275.

[14] Cho Y., Kim H., Existence results for viscous polytropic fluids with vacuum, J. Differential Equations, 2006, 228, 377-411.