The transitional behavior of quantum Gaussian memory channels

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We address the question of optimality of entangled input states in quantum Gaussian memory channels. For a class of such channels, which can be traced back to the memoryless setting, we state a criterion which relates the optimality of entangled inputs to the symmetry properties of the channels’ action. Several examples of channel models belonging to this class are discussed.

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I. INTRODUCTION

In communication theory, it is generally believed that memory effects improve the information transfer capabilities of a communication line \cite{1}. Memory effects can be introduced both by the presence of correlations in the noise affecting different channel uses (inputs), and by interference among theme.

In the quantum communication scenario the problem of determining the optimal ensemble of input states, depending on the memory, naturally arises. An optimal ensemble is the most robust under the action of the noisy channel, leading to the highest transmission rates. In particular, for the classical information transmission through quantum memory channels, the possibility of discriminating between the optimality of separable or entangled input states would be useful.

Then, by the transitional behavior it is intended the possibility to single out two “phases” in the channel capacity. One for which the optional input states are entangled among different channel uses, and the other for which the optional input states are separable. To model the memory effects it is customary to introduce a memory parameter, which quantifies the amount of correlations in the quantum channel. The transition between the two “phases” may happen at a finite value of the memory parameter, implying that separable input states are optimal in the presence of small correlations, or at zero value of the memory parameter, implying that separable states are optimal only in the memoryless limit. Several models showing such an effect have been proposed for discrete quantum memory channels \cite{2} as well as for continuous ones \cite{3}. However, the majority of these works restrict their analysis to few channel uses (thus not relying on capacity arguments) and above all do not provide general criterion to characterize the transitional behavior. Here we present necessary conditions to have the transitional behavior for a class of quantum Gaussian memory channels. Actually we show that such behavior is intimately related to the symmetry properties of the channels’ action. This is possible thanks to the introduction of a technique which allows us to unravel the memory effects.

The article is organized as follows. In Sec. II we review the basic tools for working out quantum Gaussian memory channels, and to the problem of evaluating the channel capacity. In Sec. III we consider the problem of computing the Holevo function for a Gaussian memory channel, and introduce a class of Gaussian memory channels for which this problem can be traced back to the memoryless setting. In Sec. IV we enunciate a criterion for the transitional behavior and we relate the optimality of entangled inputs to the symmetry properties of the channels’ action. In Sec. V illuminating examples are presented. Finally, Sec. VI is for concluding remarks.

II. GAUSSIAN STATES AND GAUSSIAN CHANNELS

Gaussian quantum states and Gaussian quantum channels are defined in the context of continuous variable quantum systems. Here we consider the case of a continuous variable quantum system consisting of \( n \) identical quantum harmonic oscillators (for a complete presentation of the subject see for instance \cite{4}). The \( n \) quantum harmonic oscillators are associated with a set of \( 2n \) canonical operators \( \hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2, \ldots, \hat{q}_n, \hat{p}_n \), satisfying canonical commutation relations \( [\hat{q}_k, \hat{p}_k] = i\hbar \delta_{kk} \) (here and in what follows we assume \( \hbar = 1 \)).

The state of the \( n \) modes can be described, using the formalism of density operator, as a certain density \( \hat{\rho}_n \) defined in the \( n \)-mode Fock space. However, we find it more convenient to work in the Wigner function representation. Let us recall that the Wigner function associated to a density \( \hat{\rho}_n \) is defined as

\[
W_n(q, p) = \int d^n y \langle \hat{\rho}_n | \hat{q} + y \rangle e^{2i\pi y p} | \hat{p} + y \rangle
\]

where we have introduced the numerical vectors \( q := (q_1, \ldots, q_n) \), \( p := (p_1, \ldots, p_n) \), \( y := (y_1, \ldots, y_n) \in \mathbb{R}^n \), and we have denoted

\[
| q \pm y \rangle := \bigotimes_{k=1}^n | q_k \pm y_k \rangle
\]

the joint (generalized) eigenstates of the ‘position’ operators \( \{ \hat{q}_k \}_{k=1, \ldots, n} \), i.e. \( \hat{q}_k | q \pm y \rangle = (q_k \pm y_k) | q \pm y \rangle \), for all \( k = 1, \ldots, n \).

By definition, Gaussian states are those described by a Gaussian Wigner function. In the \( n \)-mode scenario, the Wigner function of a Gaussian state is a multivariate Gaussian function:

\[
W_n(x) = \frac{\exp \left[ -\frac{1}{2} (x - m)^T V^{-1} (x - m) \right]}{(2\pi)^n \sqrt{\det(V)}},
\]
where we have introduced the numerical vector \( x := (q_1, p_1, q_2, p_2, \ldots, q_n, p_n)^T \in \mathbb{R}^{2n} \). The Wigner function of an \( n \)-mode Gaussian state is hence completely described by the vector of the first moments

\[
\mathbf{m} = \langle x \rangle = \int x W_n(x) d^n x ,
\]

and by the \( 2n \times 2n \) covariance matrix (CM)

\[
V = \langle (x - \mathbf{m})(x - \mathbf{m})^T \rangle = \int (x - \mathbf{m})(x - \mathbf{m})^T W_n(x) d^n x .
\]

Finally, let us notice that certain conditions have to be imposed on the CM, to ensure that the Wigner function describes a *bona fide* quantum state. Indeed, the Heisenberg principle imposes the condition

\[
V - i\Omega \geq 0,
\]

where

\[
\Omega = \bigoplus_{k=1}^n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

is the \( 2n \times 2n \) matrix representing the \( n \)-mode symplectic form.

We will also consider an equivalent representation defined by a different ordering of the canonical variables, expressed by the numerical vector \( \bar{x} = (q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n)^T \), with the corresponding vector of first moments \( \bar{\mathbf{m}} \) and the CM

\[
\bar{V} = \langle (\bar{x} - \bar{\mathbf{m}})(\bar{x} - \bar{\mathbf{m}})^T \rangle .
\]

In this representation, the symplectic form is represented by the matrix

\[
\tilde{\Omega} = \begin{pmatrix} \mathbb{0} & \mathbb{I}_n \\ -\mathbb{I}_n & \mathbb{0} \end{pmatrix},
\]

where \( \mathbb{0} \) denotes the null matrix, and \( \mathbb{I}_n \) the identity matrix of size \( n \).

A Gaussian quantum channel acting on \( n \) bosonic modes (in short, an \( n \)-mode Gaussian channel) is by definition a channel mapping Gaussian states into Gaussian states. As a consequence, its action on Gaussian states is completely characterized by the rule of transformations of the vector of first moments and of the CM. One can show (see, e.g., [7]) that a Gaussian channel transforms the pair \( (\mathbf{m}, V) \) (vector of first moment, CM) as follows:

\[
(\mathbf{m}, V) \mapsto (X \mathbf{m} + \mathbf{d}, X V X^T + Y).
\]

Where \( \mathbf{d} \in \mathbb{R}^{2n} \) is a displacement vector, and \( X, Y \) are two \( 2n \times 2n \) matrices. In order to represent a *bona fide* quantum channel, these matrices have to obey the inequalities

\[
Y + iX\Omega X^T - i\Omega \geq 0, \quad Y \geq 0,
\]

In conclusion, a Gaussian channel is characterized by the triad \( (\mathbf{d}, X, Y) \), satisfying Eq. (11).

Given a pair of Gaussian channels: \( \Phi \) with associated triad \( (\mathbf{d}, X, Y) \), and \( \Phi' \) with triad \( (\mathbf{d}', X', Y') \), the composition of the two channels \( \Phi' \circ \Phi \) is associated to the triad \( (X' \mathbf{d} + \mathbf{d}', X'X X'X^T + Y') \).

In the family of Gaussian channels, a special sub-family is those of unitary Gaussian channels. Unitary Gaussian channels are characterized by the conditions \( Y = 0 \), and

\[
X\Omega X^T = \Omega.
\]

The last equation characterizes linear symplectic transformations, where the matrix \( X \) is symplectic. In conclusion, a Gaussian unitary transformation is characterized by the triad \( (\mathbf{d}, X, Y) \), where \( \mathbf{d} \) is generic, \( X \) is symplectic, and \( Y = 0 \). In the following we consider the subgroup of Gaussian unitary transformations preserving the total number of excitations, such maps are represented by matrices \( X \) which are both symplectic and orthogonal. Using the representation (8), it is possible to show (see, e.g., [4] and the references therein) that those matrices are of the form

\[
\hat{X} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix},
\]

where \( A, B \) are real matrices of size \( n \) such that the matrix \( A + iB \) is unitary.

### A. Gaussian memory channels, normal forms

For any integer \( n \), \( n \) uses of a Gaussian channel transform \( n \) input bosonic modes into \( n \) output bosonic modes. The action of a Gaussian quantum channel is hence described by a sequence of Gaussian channels \( \Phi_n \), acting on \( n \) bosonic modes, which is in turn associated to the sequence of triads \( (\mathbf{d}_n, X_n, Y_n) \).

A very special case is that of the memoryless channels, for which \( \Phi_n = \Phi_1^{\otimes n} \) is the direct product of \( n \) identical one-mode Gaussian channels. Each of these identical one-mode Gaussian channels is characterized by a triad \( (\mathbf{d}_1, X_1, Y_1) \). Hence, a memoryless channel is characterized by a sequence \( (\mathbf{d}_n, X_n, Y_n) = (\bigoplus_{k=1}^n \mathbf{d}_1, \bigoplus_{k=1}^n X_1, \bigoplus_{k=1}^n Y_1) \). Notice that \( \mathbf{d}_1 = (d_q, d_p)^T \), and we have denoted \( \bigoplus_{k=1}^n \mathbf{d}_1 := (d_q, d_p, d_q, d_p, \ldots, d_q, d_p)^T \in \mathbb{R}^{2n} \). Also notice that \( X_1, Y_1 \) are \( 2 \times 2 \) matrices, and the direct sums \( \bigoplus_{k=1}^n X_1, \bigoplus_{k=1}^n Y_1 \) are \( 2n \times 2n \) block-diagonal matrices.

Let us now consider the case of a quantum channel with memory. We call a channel with memory, or simply a memory channel, any channel which is not memoryless. Making no assumption on additional structures that might be present (e.g. causality, invariance under time translations), we can only say that the associated sequence of \( n \)-mode Gaussian channels satisfies \( \Phi_n \neq \Phi_1^{\otimes n} \), and the associated sequence of triads is \( (\mathbf{d}_n, X_n, Y_n) \neq (\bigoplus_{k=1}^n \mathbf{d}_1, \bigoplus_{k=1}^n X_1, \bigoplus_{k=1}^n Y_1) \).

The problem of finding normal forms for \( n \)-mode Gaussian channels was considered in [2, 8], the case of one-mode channel was considered in [9]. Normal forms are equivalence classes of \( n \)-mode Gaussian channels, up to (Gaussian) unitary equivalence. Hence, given a pair of \( n \)-mode Gaussian
channels $\Phi_n$, $\Phi'_n$, they are equivalent if there are Gaussian unitary transformations $E_n$, $D_n$ such that $\Phi'_n = D_n \circ \Phi_n \circ E_n$ ($\circ$ denotes the composition of channels). As the chosen notation suggests, we want to look at $E_n$, $D_n$ respectively as unitary encoding and decoding transformations, $E_n$ anticipating and $D_n$ following the action of the quantum channel. We will hence write $(d_n, X_n, Y_n) \simeq (d'_n, X'_n, Y'_n)$ if the two Gaussian quantum channels are equivalent in the sense declared above. 

The first thing to be noticed is that $(d_n, X_n, Y_n) \simeq (0, X_n, Y_n)$. This is a well known result, a consequence of the fact that the displacement vector can always be eliminated by applying a proper $n$-mode displacement operator (which is a Gaussian unitary transformation) at the encoding, or decoding, stage. For this reason, in what follows we only consider $n$-mode Gaussian channels of the form $(0, X_n, Y_n)$. Let us consider a quantum channel $\Phi_n$ with the triad $(0, X_n, Y_n)$, an encoding $E_n$ with triad $(0, E_n, 0)$, and decoding $D_n$ with triad $(0, D_n, 0)$. The application of the encoding and decoding units leads to the dressed channel $\Phi'_n$ associated to the triad $(0, D_n X_n E_n, D_n Y_n D_n^T)$. 

As was shown in [7], an $n$-mode Gaussian channel is always unitary equivalent to a $n$-mode channel in a normal form, i.e. $(0, X_n, Y_n) \simeq (0, X'_n, Y'_n)$, where

$$X'_n = \left[ \begin{array}{c} p \ X_2^{(h)} \\ n \ X_{k}^{(i)} \end{array} \right] \bigoplus \left[ \begin{array}{c} n \ X_1^{(i)} \end{array} \right]$$

(14)

is the direct sum of $p$ two-mode $(4 \times 4)$ matrices $X_2^{(h)}$, and $n - 2p$ one-mode $(2 \times 2)$ matrices $X_1^{(i)}$. In other words, by applying suitable encoding and decoding units, the matrix $X_n$ is reduced to the direct sum of two-mode and one-mode terms. However, it is important to notice that the matrix $Y'_n$ cannot be in general jointly reduced to the same form.

### B. Classical capacity of Gaussian channels

A quantum channel can be used to transmit classical information by encoding a classical stochastic continuous variable $Z$, distributed according to a probability density distribution $p_Z$, into a set of quantum states $\hat{\rho}_Z$. 

In the case of a memoryless quantum channel $\Phi_n = \Phi_{1^n}$, the maximum rate at which classical information can be reliably sent through the quantum channel is given by the regularized limit [10]

$$C = \lim_{n \to \infty} \frac{1}{n} \chi(\Phi_{1^n}) ,$$

(15)

where the Holevo function $\chi$ evaluated on $n$ channel uses is

$$\chi(\Phi_{1^n}) = \max_{\{\hat{\rho}_Z, p_Z\}} \left\{ S \left[ \Phi_{1^n} \left( \int dZ p_Z \hat{\rho}_Z \right) \right] - \int dZ p_Z S \left[ \Phi_{1^n}(\hat{\rho}_Z) \right] \right\} ,$$

(16)

where $S$ denotes the von Neumann entropy. If the von Neumann entropy is expressed in qubits, then the classical capacity of the quantum channel in measured in bits per channel use. The computation of the memoryless channel capacity is based on the optimization over all input ensembles, including those made of states which are entangled among different channel uses. On the other hand, if the input states are restricted to ensembles of separable states, one obtains the so-called one-shot capacity

$$C_1 = \chi(\Phi_1) .$$

(17)

Clearly the one-shot capacity is a lower bound on the memoryless channel capacity. If the two quantities coincide, the Holevo function is said to be additive. Additivity of the Holevo function dramatically simplifies the problem of evaluating the memoryless channel capacity. Even though the Holevo function has been shown to be additive for several relevant channels, e.g. the lossy channel in the framework of Gaussian channels [11], this property does not hold in general [12]. Non-additivity of the Holevo function implies that the optimal ensembles of input states, i.e. the most robust to noise, are entangled among different channel uses: a phenomenon which has no counterpart in the classical theory of information.

Moving to the case of quantum channels with memory, characterized by the inequality $\Phi_n \neq \Phi_{1^n}$, one could be tempted to generalize the formula in Eq. (15) and write

$$C \simeq \lim_{n \to \infty} \frac{1}{n} \chi(\Phi_n) ,$$

(18)

with

$$\chi(\Phi_n) = \max_{\{\hat{\rho}_Z, p_Z\}} \left\{ S \left[ \Phi_n \left( \int dZ p_Z \hat{\rho}_Z \right) \right] - \int dZ p_Z S \left[ \Phi_n(\hat{\rho}_Z) \right] \right\} .$$

(19)

Indeed, it is possible to show [13] that the right hand side of Eq. (18) is in general only an upper bound for the classical capacity of the memory channel. On the other hand, it has been proven that this quantity coincides with the memory channel capacity for the class of so-called forgetful channels [14]. Those channels have the property that correlations among channels uses decay exponentially. Moreover, they exhibit a causal structure and are invariant under time translation.

The problem of computing the classical capacity is extremely hard. Indeed, if one cannot rely on the additivity property, one has to evaluate the regularized limit of the Holevo information, whose complexity increases exponentially in $n$. Moreover, for the case of bosonic channels, the relevant Hilbert space is infinite dimensional even for a single channel use, i.e. $n = 1$. Clearly, an infinite dimensional Hilbert space could carry an infinite amount of classical bits. Hence, to avoid unphysical results, one is led to introduce a physically motivated constraint, and to exploit the concept of constrained capacity. A typical choice in the framework of bosonic Gaussian channels is to impose a constraint on the maximal input energy per channel use, hence one introduces
the constrained Holevo function

\[
\chi^N(\Phi_n) = \max_{\rho_{Z,Z'}} \left\{ S\left[\Phi_n \left(\int dZ \rho_Z \rho_{Z'}\right)\right] - \int dZ \rho_Z \left[\Phi_n \left(\rho_{Z}\right)\right] \right\} \geq N + \frac{1}{2},
\]

where we have assumed unit frequency for the bosonic oscillators, and \( N \) represents the maximum number of excitations per mode on average. One can conjecture that the maximum is reached in correspondence with a Gaussian ensemble. Indeed, the optimality of the Gaussian ensemble has been proven for the lossy channel [11] and conjectured for other families of bosonic Gaussian channels [15].

Here, we estimate the Holevo function when restricted to Gaussian encoding [16]. We consider Gaussian encoding defined as follows. For \( n \) uses of the quantum channel, we fix a reference \( n \)-mode Gaussian state, with zero mean, described by the Wigner function

\[
W(x) = \frac{\exp\left[-\frac{1}{2} x^T V_{\text{in}}^{-1} x\right]}{(2\pi)^n \sqrt{\det(V_{\text{in}})}}.
\]

A classical variable \( m \in \mathbb{R}^{2n} \) is hence encoded by applying a displacement operation on the reference state, thus obtaining

\[
W_m(x) = \frac{\exp\left[-\frac{1}{2} (x - m)^T V_{\text{in}}^{-1} (x - m)\right]}{(2\pi)^n \sqrt{\det(V_{\text{in}})}}.
\]

We assume the stochastic variable \( m \) to be itself distributed according to the Gaussian probability density distribution with zero mean:

\[
\rho_m = \frac{\exp\left[-\frac{1}{2} m^T V_{\text{in}}^{-1} m\right]}{(2\pi)^n \sqrt{\det(V_{\text{in}})}}.
\]

By linearity of the relation (1), the corresponding ensemble state

\[
\int d^{2n} m \rho_m W_m(x)
\]

is itself described by a Gaussian Wigner function, i.e.

\[
\int d^{2n} m \rho_m W_m(x) = \frac{\exp\left[-\frac{1}{2} x^T (V_{\text{in}} + V_{\text{c}})^{-1} x\right]}{(2\pi)^n \sqrt{\det(V_{\text{in}} + V_{\text{c}})}}.
\]

The restriction to Gaussian states, which are mapped into Gaussian states by Gaussian channels, dramatically simplifies the problem, since the complexity of Gaussian states is polynomial in the number of modes \( n \). Moreover, the von Neumann entropy can be calculated in terms of the symplectic eigenvalues of the CM.

The symplectic eigenvalues of an \( n \)-mode CM \( V \) are defined as follows. Notice that the matrix \( V \Omega \), where \( \Omega \) is the symplectic form introduced in Eq. (7), has \( 2n \) purely imaginary eigenvalues \( \{\pm iv_k\}_{k=1,...,n} \), where the \( n \) real numbers \( \{v_k\}_{k=1,...,n} \) are the symplectic eigenvalues of the CM \( V \). From the uncertainty relations, expressed by Eq. (6), it follows that the symplectic eigenvalues satisfy the inequalities

\[
v_k \geq 1/2,
\]

which are saturated by pure Gaussian states [4].

The von Neumann entropy of a Gaussian state \( \rho \), characterized by a CM \( V \), is given by the formula:

\[
S[\rho] = \sum_{k=1}^{n} g(v_k - \frac{1}{2}),
\]

where we have introduced the function \( g \) defined by

\[
g(x) = (x + 1) \log_2 (x + 1) - x \log_2 (x).
\]

Finally, we notice that the input energy constraint in Eq. (19) can be written in terms of the CM as follows:

\[
\frac{\text{tr}(V_{\text{in}} + V_{\text{c}})}{2n} \leq N + \frac{1}{2}.
\]

III. THE HOLEVO FUNCTION FOR GAUSSIAN ENSEMBLES

For \( n \) uses of the quantum channel, the constrained Holevo function, when restricted over Gaussian ensembles, reads

\[
\chi_G^N(\Phi_n) = \max_{V_{\text{in}}, V_{\text{c}}} \frac{1}{n} \left\{ \Sigma \left[X_n (V_{\text{in}} + V_{\text{c}}) X_n^T + Y_n\right] - \Sigma \left[\frac{\text{tr}(V_{\text{in}} + V_{\text{c}})}{2n}\right] \right\} \leq N + \frac{1}{2},
\]

where the optimization is over the CMs \( V_{\text{in}}, V_{\text{c}} \) satisfying the energy constraints.

In the case of a memoryless Gaussian channel, by restricting on Gaussian input states which are separable among different channel uses, we get to the one-shot capacity:

\[
\chi_G^N(\Phi_1) = \max_{V_{\text{in}}, V_{\text{c}}} \left\{ \Sigma \left[\frac{\text{tr}(V_{\text{in}} + V_{\text{c}})}{2}\right] \leq N + \frac{1}{2} \right\}.
\]

In general, for a Gaussian memory channel, characterized by the sequence of triads \( (d, X_n, Y_n) \), the optimization of the \( n \)-use Holevo function in Eq. (30) cannot be reduced to the one-use case in Eq. (7). That is a consequence of the the normal form for an \( n \)-mode Gaussian channel in Eq. (14), which
in general is not in the form the product of \( n \) independent one-mode Gaussian channels. However, we can still identify a class of Gaussian memory channels such that, for any \( n \), there are Gaussian unitary encoding and decoding transformations \( \mathcal{E}_n = (0, E_n, 0) \), \( \mathcal{D}_n = (0, D_n, 0) \), such that

\[
\mathcal{D}_n \circ \Phi_n \circ \mathcal{E}_n = \bigotimes_{k=1}^{n} \Phi_{(k)}^{(1)},
\]

(32)

where \( \{ \Phi_{(k)}^{(1)} \}_{k=1,\ldots,n} \) is a collection of \( n \) (not necessarily identical) one-mode Gaussian channels. In other words, the memory channels belonging to that class factorize in terms of the collective input and output variables defined by the encoding and decoding transformations. Hence we introduce the following

**Definition 1 (memory unraveling)** A bosonic Gaussian memory channel, characterized by a sequence \((0, X_n, Y_n)\), can be unraveled if there is a sequence of encoding Gaussian unitaries \((0, E_n, 0)\), and a sequence of decoding Gaussian unitaries \((0, D_n, 0)\), such that, for any \( n \),

\[
D_nX_nE_n = \bigotimes_{k=1}^{n} X_{1,n}^{(k)},
\]

(33)

\[
D_nY_nD_n^T = \bigotimes_{k=1}^{n} Y_{1,n}^{(k)}.
\]

(34)

Moreover, we require that the encoding unitary preserves the form of the energy constraint.

Since the Holevo function is invariant under unitary transformations, the transformation (32) preserves the Holevo function, i.e.

\[
\chi_{NG} [\Phi_n] = \chi_{NG} \left[ \bigotimes_{k=1}^{n} \Phi_{(k)}^{(1)} \right].
\]

(35)

Moreover, since the encoding Gaussian unitary preserves the form of the energy constraint, we have

\[
\text{tr} \left[ E_n (V_{in} + V_c) E_n^T \right] = \text{tr} \left[ V_{in} + V_c \right],
\]

(36)

which is satisfied if \( E_n^T E_n = 1 \), i.e. if the matrix \( E_n \) is orthogonal. Symplectic matrices that are also orthogonal constitute a subgroup whose elements are characterized by the form in Eq. (33).

For a memory channel that can be unraveled, it is natural to restrict to Gaussian input ensembles such that

\[
E_nV_{in}E_n^T = \bigotimes_{k=1,\ldots,n} V_{1,in}^{(k)},
\]

(37)

and

\[
E_nV_cE_n^T = \bigotimes_{k=1,\ldots,n} V_{1,c}^{(k)}.
\]

(38)

Using this ansatz, the calculation of the Holevo function for \( n \) uses of the memory channel reduces to the one-mode case:

\[
\chi_{NG}^n [\Phi_n] = \frac{1}{n} \max_{\{N_k\}} \left[ \sum_k N_k \right] \max_{n=1}^{N_k} \left[ \sum_{y=1}^{n} \left( x_{y}^{(k)}, y_{y}^{(k)} \right) \right]
\]

\[
\left\{ \sum_{y=1}^{n} X_{1,n}^{(k),(y)} (V_{1,in}^{(y)} + V_{1,c}^{(y)}) X_{1,n}^{(k),(y)^T} + Y_{1,n}^{(k),(y)} \right\}
\]

\[
- \sum_{y=1}^{n} X_{1,n}^{(k),(y)} V_{1,in}^{(y)} X_{1,n}^{(k),(y)^T} + Y_{1,n}^{(k),(y)} \right\}
\]

\[
\frac{\text{tr} \left( V_{1,in}^{(k)} + V_{1,c}^{(k)} \right)}{2} \leq N_k + 1/2,
\]

(39)

where we have rewritten the input energy constraint in two steps

\[
\left\{ \text{tr} \left( V_{1,in}^{(k)} + V_{1,c}^{(k)} \right) /2 \leq N_k + 1/2, \right. \sum_{k=1}^{n} N_k/n = N, \right.
\]

(40)

and the maximization is over both the CMs \( V_{1,in}^{(k)} \), \( V_{1,c}^{(k)} \), and over the positive integers \( N_k \) under the constraint \( \sum_k N_k/n = N \).

In conclusion, for quantum memory channels that can be unraveled, the calculation of the Holevo function has been reduced to the case of independent, but not identical, one-mode channels, each characterized by the matrices \( X_{1,n}^{(k),(y)}, Y_{1,n}^{(k),(y)} \) and a maximal number \( N_k \) of excitations per mode. The only ingredient that mixes the one-mode channels is the constraint on the total number of excitations.

A. Examples

The properties defining the class of Gaussian memory channels that can be unraveled are rather peculiar, however several relevant models of Gaussian channels belong to this class. In this section we review some examples of Gaussian memory channels that can (or cannot) be unraveled.

**Lossy Bosonic memory channel.** We refer to the general model of lossy bosonic Gaussian channel with memory that has been introduced in [17]. Upon \( n \) uses of the channel, \( n \) input modes are mixed with a corresponding set of \( n \) environmental modes at a beam splitter with transmissivity \( \eta \). In the Heisenberg picture the canonical field operators transform as

\[
\hat{q}_k \to \sqrt{\eta} \hat{q}_k + \sqrt{1-\eta} \hat{Q}_k,
\]

(41)

\[
\hat{p}_k \to \sqrt{\eta} \hat{p}_k + \sqrt{1-\eta} \hat{P}_k,
\]

(42)

where \( \{ \hat{Q}_k, \hat{P}_k \}_{k=1,\ldots,n} \) are the field operators of the environmental modes. The environmental modes are in a correlated Gaussian state, which is characterized by a CM \( V_{\text{env}} \) with non-vanishing off-diagonal terms coupling different modes. The memory channel is associated with the matrices \( X_n = \sqrt{\eta} V_n \), \( Y_n = (1-\eta) V_{\text{env}} \). Memory effects appear in the channel if the \( n \)-mode environment is in a non-factorized state. A factorized state is characterized by a block-diagonal \( n \)-mode
CM, i.e. $V_{\text{env}} = \bigoplus_{k=1}^{n} v_k$. Remarkable examples of non-factorized states are the multimode entangled states, which belong to the family of multimode squeezed states. To address the problem of memory unraveling, we notice that the CM $V_{\text{env}}$ can be diagonalized by a $2n \times 2n$ orthogonal matrix $O$, i.e. $OV_{\text{env}}O^T = \bigoplus_{k=1}^{n} v_k$. Hence one could identify $D_n := O$, $E_n := O^T$. Notice that the orthogonal matrix preserves the trace, hence it preserves the energy constraint. However, a Gaussian unitary is represented with a symplectic matrix, hence the given orthogonal matrix represents a physical transformation only if it is also symplectic. In conclusion, a lossy bosonic memory channel can be unraveled if the CM of the environment can be diagonalized by a linear transformation which is both symplectic and orthogonal. A characterization of this class of environment CMs is presented in [13]: pure Gaussian states and squeezed thermal states belong to this class.

**Additive noise channel.** For this class of channels, the field operators transforms according to the Heisenberg picture map:

$$\hat{q}_k \rightarrow \hat{q}_k + t_k,$$

$$\hat{p}_k \rightarrow \hat{p}_k + u_k,$$

where $t_k$, $u_k$ are classical stochastic variables. The channel is Gaussian if the noise variables are Gaussian distributed, with a CM $V_{\text{cl}}$. The subscript ‘cl’ refers to the fact that $V_{\text{cl}}$ is a classical CM, i.e. it is only subject to the conditions of being symmetric and positive semi-definite. Differently from the quantum CM, it needs not obey the Heisenberg uncertainty relations as expressed by Eq. (6). Memory effects arise when the $V_{\text{cl}}$ has nonvanishing off-diagonal terms coupling different channel uses. The matrices associated to $n$ uses of the channel are $X_n = I_{2n}$, and $Y_n = V_{\text{cl}}$. The memory channel can hence be unraveled if there is a $2n \times 2n$ matrix $S$, being both symplectic and orthogonal, such that $S_n V_{\text{cl}} S_n^T = \bigoplus_{k=1}^{n} v_k$. The encoding and decoding Gaussian unitaries which unravel the memory channel are hence chosen as $D_n := S_n$, $E_n := S_n^T$. Notice that since $S_n$ is orthogonal, $S_n^T = S_n^{-1}$. Examples of additive noise channels with memory that can be unraveled were studied in [19], [20]. The conditions on the CM $V_{\text{cl}}$ can be easily obtained as it is done for the lossy channel in [18], with the only difference that $V_{\text{cl}}$ needs not obey the uncertainty relations.

**Inter-symbol interference channels.** In this family of quantum channels memory effects come from the fact that the signals at different channel inputs do interfere at the channel output, while in the previous examples they are caused by noise correlations. For such a case, one can write the Heisenberg picture transformations acting on the field operators as

$$\hat{q}_k \rightarrow \sum_h \left\{ M_{kh}^{(q)} \hat{q}_h + M_{kh}^{(qp)} \hat{p}_h + N_{kh}^{(qQ)} \hat{Q}_h + N_{kh}^{(qP)} \hat{P}_h \right\},$$

$$\hat{p}_k \rightarrow \sum_h \left\{ M_{kh}^{(pp)} \hat{q}_h + M_{kh}^{(qp)} \hat{p}_h + N_{kh}^{(pQ)} \hat{Q}_h + N_{kh}^{(pP)} \hat{P}_h \right\},$$

where the operators $\{ \hat{Q}_h, \hat{P}_h \}$ correspond to environmental modes, and the matrices $M_{kh}^{(qq)}$, $M_{kh}^{(qp)}$, $\ldots$, $N_{kh}^{(qQ)}$, $N_{kh}^{(pP)}$ satisfy proper conditions [4]. To describe this channel, we work in the representation [8]. In this representation, assuming that the environmental modes are in a Gaussian state characterized by the CM $V_{\text{env}}$, the matrices associated with the memory channel are

$$\hat{X}_n = \left( \begin{array}{cc} M_{pq}^{(qq)} & M_{pq}^{(qp)} \\ M_{pp}^{(qp)} & M_{pp}^{(pp)} \end{array} \right),$$

and

$$\hat{Y}_n = \left( \begin{array}{cc} N_{pq}^{(qQ)} & N_{pq}^{(qP)} \\ N_{pp}^{(qQ)} & N_{pp}^{(pP)} \end{array} \right) V_{\text{env}} \left( \begin{array}{cc} N_{pq}^{(qQ)} & N_{pq}^{(qP)} \\ N_{pp}^{(qQ)} & N_{pp}^{(pP)} \end{array} \right)^T.$$  

An instance of this kind of model was considered in [21], in which $M_{pq}^{(qq)} = M_{pq}^{(pp)} = M_{pq}^{(pp)} = 0$, $N_{pq}^{(qQ)} = N_{pq}^{(qP)}$, $N_{pq}^{(pQ)} = N_{pq}^{(pP)} = 0$, and the environment is in the vacuum state, i.e. $V_{\text{env}} = I_{2n}/2$. As is shown in [21], such a channel can be unraveled by performing the singular value decomposition of the matrix $X_n$.

**IV. OPTIMIZATION UNDER SYMMETRIES**

In the cases of both memoryless and memory quantum channels, one can pose the question of finding the optimal input ensemble, i.e. the most robust one under the action of the noisy channel. In the memoryless setting, this is related to the issue of additivity of the Holevo information: if the Holevo information is additive the optimal input ensemble constitutes of states which are separable among different channel uses; otherwise, for channels with a non-additive Holevo information the optimal input ensemble is made of entangled states. For memory channels it has been observed, in the cases of both discrete [2] and continuous [3] variables, that entangled input states may be optimal to maximize the Holevo information. In models for quantum channels with memory, is customary to introduce a memory parameter, used to quantify the memory in the channel, which vanishes in the memoryless limit. It may happen that the optimal input ensemble constitutes of entangled states when the memory parameter is above a certain threshold. In this case one says that the memory channel exhibits a transitional behavior. In the case of discrete variables, several models exhibit a finite value of the memory threshold [2], while for continuous-variable models the threshold value may vanish [3, 20], i.e. entangled input states are optimal even for arbitrary small, but not zero, values of the memory parameter.

Restricting to the case of bosonic Gaussian channels that can be unraveled, here we introduce a criterion to decide whether the Holevo information, restricted on Gaussian input ensembles and under input energy constraint, is optimized by separable input states. We formulate the following:

**Criterion 1** Given a Gaussian memory channel, represented by the sequence $(0, X_n, Y_n)$, and provided that it can be unraveled, a necessary condition for the optimality of entangled input states is the non invariance under phase rotation.

**Proof of Criterion 1** For a Gaussian memory channel that can be unravelled, it holds $(0, X_n, Y_n) \sim
(0, \bigoplus_{k=1}^{n} X^{(k),n}, \bigoplus_{k=1}^{n} Y^{(k),n})$. Let us assume, by contradiction, that the one-mode channels $(0, X^{(k),n}, Y^{(k),n})$ are invariant under phase rotation, i.e.

\[
R_1(\theta) X^{(k),n} R_1(\theta)^T = X^{(k),n}, \quad (47)
\]

\[
R_1(\theta) Y^{(k),n} R_1(\theta)^T = Y^{(k),n}, \quad (48)
\]

where the $2 \times 2$ matrix $R_1(\theta)$ represents a phase rotation

\[
R_1(\theta) := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad (49)
\]

Matrices that have this symmetry are scalar, i.e.

\[
X^{(k),n} = \begin{pmatrix} x^{(k),n} & 0 \\ 0 & x^{(k),n} \end{pmatrix}, \quad (50)
\]

\[
Y^{(k),n} = \begin{pmatrix} y^{(k),n} & 0 \\ 0 & y^{(k),n} \end{pmatrix}. \quad (51)
\]

It follows that the solution of the optimization problem (39) is also invariant under phase rotation and is given by the matrices:

\[
V_{1,\text{in}}^{(k)} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad V_{1,\text{c}}^{(k)} = \begin{pmatrix} N_k & 0 \\ 0 & N_k \end{pmatrix}, \quad (52)
\]

where the optimal values of the parameters $\{N_k\}_{k=1,\ldots,n}$ are obtained from the maximization problem

\[
\chi_n = \max_{\{N_k\}} \frac{1}{n} \sum_{k=1}^{n} \left\{ g \left[ (x^{(k),n})^2 (N_k + 1/2) + (y^{(k),n} - 1/2) \right] 
- g \left[ (x^{(k),n})^2 (1/2) + (y^{(k),n} - 1/2) \right] \right\}, \quad (53)
\]

where the maximum is under the constraint $\sum_{k=1}^{n} N_k = N$.

Let us notice that the matrix $V_{1,\text{in}}^{(k)}$ in (52) represents the CM of coherent states [4]. Then, the matrix $\bigoplus_{k=1}^{n} V_{1}^{(k),n}$ represents the CM of the optimal $n$-mode input state of the dressed memory channel, which includes the encoding unitary transformation. The actual optimal input state is obtained from it by undoing the encoding transformation, i.e.

\[
V_{\text{in}}^{\text{opt}} = E_n \left[ \bigoplus_{k=1}^{n} V_{1}^{(k),n} \right] E_n^T. \quad (54)
\]

However, since the encoding matrix $E_n$ is orthogonal (and, of course symplectic), it follows that

\[
V_{\text{in}}^{\text{opt}} = \bigoplus_{k=1}^{n} \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad (55)
\]

i.e. the optimal Gaussian inputs are coherent states, which are separable among different channel uses.

\[\square\]

**Remark 1** If the decoding symplectic matrix $D_n$ is also orthogonal, the conditions (47), (48) can be equivalently formulated as follows:

\[
R_n(\theta) X_n R_n(\theta)^T = X_n, \quad (56)
\]

\[
R_n(\theta) Y_n R_n(\theta)^T = Y_n, \quad (57)
\]

where the matrix $R_n(\theta) = \bigoplus_{k=1}^{n} R_1(\theta)$ represents a global phase rotation on the $n$ modes.

The equivalence can be readily proved by working in the representation [8]. It is a consequence of the fact that the subgroup of symplectic and orthogonal matrices [having the form as in Eq. (13)] commutes with phase rotations, which are represented by matrices of the following form:

\[
\tilde{R}_n = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}. \quad (58)
\]

In other words, if both the encoding and decoding symplectic matrices are orthogonal, the symmetry under phase rotation can be checked directly on the matrices $X_n, Y_n$.

Furthermore, being the $D_n$, $E_n$ elements of Lie groups, they reduce to identity when the group’s parameters reduce to zero. Since the latter would characterize the degree of memory, we may argue that the transition can only occur at zero value of the memory parameters. This is in contrast to what happens in discrete quantum memory channels.

As a consequence of the Criterion 1, entangled Gaussian codewords may be necessary for optimizing the Holevo information (39) only if the rotational invariance is broken. Notice however that this is a necessary but not sufficient condition. Below we are going to present examples for all possible cases.

\[\text{V. EXAMPLES}\]

Let us first consider the case of an additive noise channel. The definition of the channel and its basic properties are briefly recalled in Sec. IIIA. Let us recall that for $n$ uses of the channel we have $X_n = E_{2n}$, and $Y_n = V_{1}$. Since we are considering channels which can be unraveled, we assume the existence of an orthogonal and symplectic matrix $S_n$, such that $S_n V_{1} S_n^T = \bigoplus_{k=1}^{n} v_k$, with $D_n = S_n$, $E_n = S_n^T$. In this case, the symmetry condition is verified if the matrix $V_{1}$ is symmetric under phase rotation (remark [1]). Two models of Gaussian memory channels with Markovian correlated noise were studied and characterized in [19, 20]. Using the representation [8], the noise CM in [19] has the following form

\[
\tilde{V}_{1} = \begin{pmatrix} \bigvee & \bigodot & \bigvee \\ \bigodot & \bigvee \bigvee \end{pmatrix}, \quad (59)
\]

which is clearly symmetric under phase rotations (58), hence the optimal Gaussian inputs are separable for this model. On the contrary, the noise CM for the model studied in [20] has the form

\[
\tilde{V}_{1} = \begin{pmatrix} \bigvee & \bigodot & \bigvee' \\ \bigodot & \bigvee \bigvee' \end{pmatrix}, \quad (60)
\]
with $\mathcal{V} \neq \mathcal{V}'$. Such a matrix is not symmetric under phase rotations $^{58}$ and, as shown in $^{20}$, the optimal input states, when restricted to Gaussian states, are entangled.

The case of lossy bosonic memory channel $^{17}$ is analogous to the additive channel. Its basic properties are reviewed in Sec. IIIA in this case $X_n = \sqrt{\eta} e_{2n}$, and $Y_n = (1 - \eta) V_{\text{env}}$. A model of memory channel belonging to this family has been studied in $^{18}$. Using the representation $^{9}$, the environmental CM $V_{\text{env}}$ has the form

$$\tilde{V}_{\text{env}} = \left( T + \frac{1}{2} \right) \begin{pmatrix} e^{M s} & 0 \\ 0 & e^{-M s} \end{pmatrix}, \quad (61)$$

where $M$ is a symmetric matrix of size $n$, and $T$, $s$ are two positive parameters. The parameter $s$ quantifies the amount of memory in the channel: For $s = 0$ the environmental state is an uncorrelated thermal state, while it is entangled for $s \neq 0$. For all $s \neq 0$, the matrix $Y_n$ is not symmetric under phase rotation and, as shown in $^{18}$, the optimal Gaussian input states are entangled among different channel uses.

The general case of an inter-symbol interference channel is recalled in Sec. IIIA. An example of such a channel was studied and characterized in $^{21}$, where

$$\tilde{X}_n = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}, \quad (62)$$

and

$$\tilde{Y}_n = \frac{1}{2} \begin{pmatrix} N N^T & 0 \\ 0 & N N^T \end{pmatrix}. \quad (63)$$

The channel is invariant under phase rotation, thus, as shown in $^{21}$, the optimal input states are separable. To conclude, we notice that there are models of Gaussian memory channel for which the optimal Gaussian input states are separable, even though the channel is not symmetric under phase rotation. A channel model presenting this feature can be obtained by choosing the environment to be in a state of the form

$$\tilde{V}_{\text{env}} = \begin{pmatrix} e^{s I_n} & 0 \\ 0 & e^{-s I_n} \end{pmatrix}, \quad (64)$$

implying

$$\tilde{Y}_n = \frac{1}{2} \begin{pmatrix} e^{s N N^T} & 0 \\ 0 & e^{-s N N^T} \end{pmatrix}. \quad (65)$$

VI. CONCLUSION

In this article we have provided a unified framework for some recent results about the performance of quantum Gaussian memory channels. We have focused in particular on the entanglement of optimal input states and we have related this issue to the symmetry properties of the channel. More specifically, we have shown that entangled Gaussian code-words might be necessary for optimizing the Holevo information only if the rotational invariance is broken by the channel’s action. Similar considerations were also done in $^{19, 20, 22}$ for specific channel models. Moreover, for a Gaussian memory channel that can be unveled, we may argue that the transition from the optimality of separable states to the optimality of entangled states may only occur for vanishing value of the memory parameter $^{18}20$. This is in contrast to what happens in discrete quantum memory channels $^{2}$. However, while there investigations have only involved very few channel uses, here the analysis has been carried out for arbitrary number of channel uses by resorting to a mathematical machinery called memory unraveling. That allowed us to trace the Gaussian memory channel back to a memoryless one. Several examples have been discussed concerning memory unraveling as well as transitional behavior.

We think that the presented results shed light on the mechanisms and the structure of the correlations that lead to an enhancement of the channel performance with entanglement, although a complete characterization of memory channels transition features is still far away.

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