Covering of sparse subgraphs and Packing of rigid subgraphs

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Abstract
A graph $G$ is sparse if for every $X \subseteq V(G)$ with $|X| \geq 2$, the number of edges induced by $X$ is at most $2|X| - 3$. We prove a characterization of graphs that can be decomposed to sparse subgraphs, which extends the well-known result of Nash-Williams on decomposing graphs into forests. As an application, we show that if $\gamma_2(G) \leq k + 1$ and $k + 1 \leq l \leq 2k + 2$, then $G$ decomposes into $l$ forests and $2k + 2 - l$ subgraphs with maximum degree at most $(2|V(G)| - 5)/3$, which strengthens the Weaker Nine Dragon Tree Conjecture [13] when $\gamma(G)$ is close to $2k + 2$.

We also provide partition conditions for a graph with edge-disjoint spanning rigid subgraphs and spanning trees. In particular, we prove that every $[3k + l, k]$-partition-connected graph with multiplicity at most $k$ contains edge-disjoint $k$ spanning rigid subgraphs and $l$ spanning trees. As a corollary, every $(6k + 2l, 2k)$-connected graph with multiplicity at most $k$ contains edge-disjoint $k$ spanning rigid subgraphs and $l$ spanning trees, which is a multigraph version of a theorem of Cheriyan, Durand de Gevigney and Szigeti [3].

Key words: Sparse graph, rigid graph, rigidity matroid, spanning tree

1 Introduction
We consider undirected graphs with possible multiple edges but no loops. The multiplicity of a graph is the maximum number of multiple edges between any pair of vertices. Let $G$ be a graph. For a subset $X \subseteq V(G)$, $G[X]$ denotes the subgraph of $G$ induced by $X$. For a subset $F \subseteq E(G)$, $G[F]$ is the subgraph of $G$ induced by $F$, while $G(F)$ denotes the spanning subgraph of $G$ with edge set $F$. For any partition $\pi$ of $V(G)$, $e_G(\pi)$ denotes the number of edges of $G$ whose ends lie in two different parts of $\pi$. A part of $\pi$ is trivial if the part consists of a single vertex. A partition is nontrivial if it contains no trivial parts.

The following theorem of Nash-Williams and Tutte characterizes the graph with edge-disjoint spanning trees.

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**Theorem 1.1** (Nash-Williams [15] and Tutte [18]). Let \( l \geq 0 \) be an integer. A graph \( G \) has \( l \) edge-disjoint spanning trees if and only if for any partition \( \pi \) of \( V(G) \), \( e_G(\pi) \geq l(|\pi| - 1) \).

**Corollary 1.2.** Every \( 2l \)-edge-connected graph contains \( l \) edge-disjoint spanning trees.

The packing of spanning trees was then extended to the packing of 2-connected subgraphs by [3, 7, 8, 12]. Actually they provided stronger results by packing spanning rigid subgraphs.

For a subset \( X \subseteq V(G) \), let \( i_G(X) \) (or simply \( i(X) \) if \( G \) can be understood from the context) denote the number of edges in \( G[X] \). A graph \( G \) is sparse if \( i_G(X) \leq 2|X| - 3 \) for every \( X \subseteq V(G) \) with \( |X| \geq 2 \). If in addition \( |E(G)| = 2|V(G)| - 3 \), then \( G \) is minimally rigid. By definition, any sparse graph is simple. A graph \( G \) is rigid if \( G \) contains a spanning minimally rigid subgraph. It is not hard to see that every rigid graph \( G \) with \( |V(G)| \geq 3 \) is 2-connected. More about rigid graphs will be given in next section, or see [3, 7, 8, 12].

**Theorem 1.3** (Lovász and Yemini [12]). Every \( 6 \)-connected graph is rigid.

**Theorem 1.4** (Jordán [8]). Every \( 6k \)-connected graph contains \( k \) edge-disjoint spanning rigid subgraphs, and thus contains \( k \) edge-disjoint spanning \( 2 \)-connected subgraphs.

For positive integers \( p \) and \( q \), \( G \) is \( (p, q) \)-connected if \( |V(G)| > p/q \) and \( G - X \) is \( (p - q|X|) \)-edge-connected for every \( X \subseteq V(G) \). By definition, \( p \)-edge-connectedness is equivalent to \( (p, p) \)-connectedness. Every \( p \)-connected graph contains a \( (p, 1) \)-connected simple spanning subgraph and \( (p, 1) \)-connectedness implies \( (p, q) \)-connectedness. The \( (p, q) \)-connectedness was introduced in [9] and has turned out to be a useful concept in graph connectivity, see [1].

**Theorem 1.5** (Jackson and Jordán [7]). Every \( (6, 2) \)-connected simple graph is rigid.

**Theorem 1.6** (Cheriyan, Durand de Gevigney and Szigeti [3]). Let \( k \geq 1 \) and \( l \geq 0 \) be integers. Every \( (6k + 2l, 2k) \)-connected simple graph contains edge-disjoint \( k \) spanning rigid subgraphs and \( l \) spanning trees, and thus contains edge-disjoint \( k \) spanning \( 2 \)-connected subgraphs and \( l \) spanning trees.

As a dual problem of packing, covering of subgraphs also attracts much attention. Nash-Williams published the following result, characterizing graphs that can be decomposed to \( l \) forests.

**Theorem 1.7** (Nash-Williams [16]). Let \( l \geq 0 \) be an integer. A connected graph \( G \) can be decomposed to \( l \) forests if and only if for any nonempty subset \( X \subseteq V(G) \), \( i_G(X) \leq l(|X| - 1) \).

Motivated by the above results, we investigate the covering of sparse subgraphs. We have the following characterization (Theorem 1.8), which extends Theorem 1.7. As an application of Theorem 1.8, we will investigate degree bounded forest covering in Section 4, which strengthens the Weaker Nine Dragon Tree Conjecture of [13] when \( \epsilon \) is large (see Theorem 1.2).
Theorem 1.8. A connected graph $G$ can be decomposed into $k$ sparse subgraphs if and only if for any subset $X \subseteq V(G)$ with $|X| \geq 2$, $i_G(X) \leq k(2|X| - 3)$.

We also provide partition conditions for the packing of spanning trees and rigid subgraphs. Actually in [3], the proof of Theorem 1.6 implies a stronger result that for every $(6k + 2l, 2k)$-connected simple graph $G$ and $F \subseteq E(G)$ with $|F| \leq 3k + l$, $G - F$ contains edge-disjoint $k$ spanning rigid subgraphs and $l$ spanning trees. This motivates us to find a “tight” sufficient condition. In Theorem 1.9 we discover a sufficient partition condition for the packing of spanning trees and rigid subgraphs, which is an improvement of Theorem 1.6.

Let $Z \subseteq V(G)$ and $\pi = \{V_1, V_2, \cdots, V_t\}$ be a partition of $V(G - Z)$. The adjacent number $n_Z(\pi)$ of $\pi$ with respect to $Z$ is $\sum_{1 \leq i \leq t} |Z_i|$ where $Z_i$ is the set of vertices in $Z$ that are adjacent to $V_i$ for $1 \leq i \leq t$.

Theorem 1.9. Let $k \geq 1$ and $l \geq 0$ be integers. Let $G$ be a graph with multiplicity at most $k$ and $Z \subseteq V(G)$. If for any partition $\pi$ of $V(G - Z)$ with $n_0$ trivial parts and adjacent number $n_Z$, $e_{G - Z}(\pi) \geq (3k + l)(|\pi| - 1) - kn_0 - k\pi_Z$, then $G$ contains edge-disjoint $k$ spanning rigid subgraphs and $l$ spanning trees.

Let $p$ and $q$ be positive integers. A graph $G$ is $p$-partition-connected if for any partition $\pi$ of $V(G)$, $e_G(\pi) \geq p(|\pi| - 1)$. By Theorem 1.1 $G$ is $p$-partition-connected if and only if $G$ has $p$ edge-disjoint spanning trees. A graph $G$ is $[p, q]$-partition-connected if $|V(G)| > p/q$ and for any subset $Z \subset V(G)$ and any partition $\pi$ of $V(G - Z)$, $e_{G - Z}(\pi) \geq p(|\pi| - 1) - qn_Z(\pi)$. Partition-connectedness has been shown to play an important role in combinatorial optimization, see [5].

Remark 1. Any $[p, q]$-partition-connected graph is $(p, 2q)$-connected.

Proof. When $|\pi| = 2$, we have $n_Z(\pi) \leq 2|X|$, and it follows by definition. \hfill \Box

Remark 2. Any $(2p, 2q)$-connected graph is $[p, q]$-partition-connected.

Proof. It suffices to show for any subset $Z \subset V(G)$ and any partition $\pi$ of $V(G - Z)$, $e_{G - Z}(\pi) \geq p(|\pi| - 1) - qn_Z(\pi)$. Let $\pi = \{V_1, V_2, \cdots, V_t\}$ and $Z_i$ be the set of vertices in $Z$ that are adjacent to $V_i$ for $1 \leq i \leq t$. Then $n_Z(\pi) = \sum_{1 \leq i \leq t} |Z_i|$. Let $d(V_i)$ be the number of edges of $G - Z$ that has exactly one end in $V_i$ for $1 \leq i \leq t$. Since $G$ is $(2p, 2q)$-connected, $d(V_i) \geq 2p - 2q|Z_i|$. Thus $e_{G - Z}(\pi) = \frac{1}{2} \sum_{1 \leq i \leq t} d(V_i) \geq p(|\pi| - 1) - qn_Z(\pi)$. \hfill \Box

By Theorem 1.9 we have the following corollaries, which are improvements of Theorem 1.6.

Corollary 1.10. Let $k \geq 1$ and $l \geq 0$ be integers. Every $[3k + l, k]$-partition-connected graph with multiplicity at most $k$ contains edge-disjoint $k$ spanning rigid subgraphs and $l$ spanning trees, and thus contains edge-disjoint $k$ spanning 2-connected subgraphs and $l$ spanning trees.
Corollary 1.11. Let $k \geq 1$ and $l \geq 0$ be integers. Every $(6k + 2l, 2k)$-connected graph with multiplicity at most $k$ contains edge-disjoint $k$ spanning rigid subgraphs and $l$ spanning trees, and thus contains edge-disjoint $k$ spanning $2$-connected subgraphs and $l$ spanning trees.

Theorem 1.12 presents a necessary partition condition. As corollaries, we obtain some properties of rigid graphs.

Theorem 1.12. Let $k \geq 0$ and $l \geq 0$ be integers. If a graph $G$ contains edge-disjoint $k$ spanning rigid subgraphs and $l$ spanning trees, then for any partition $\pi$ of $V(G)$ with $n_0$ trivial parts, $e_G(\pi) \geq (3k + l)(|\pi| - 1) - kn_0$.

A graph $G$ is essentially $p$-edge-connected if $e_G(\pi) \geq p$ for every nontrivial partition $\pi$ of $V(G)$ with $|\pi| = 2$.

Corollary 1.13. Every rigid graph is essentially $3$-edge-connected.

Proof. It follows by Theorem 1.12 when $k = 1, l = 0, n_0 = 0$ and $|\pi| = 2$. \hfill $\square$

Corollary 1.14. Every rigid graph $G$ with $|E(G)| \geq 2(|V(G)| - 1)$ has $2$ edge-disjoint spanning trees.

Proof. By Theorem 1.11 it suffices to show that for any partition $\pi$ of $V(G)$, $e_G(\pi) \geq 2(|\pi| - 1)$. If $|\pi| = |V(G)|$, then $e_G(\pi) = |E(G)| \geq 2(|V(G)| - 1) = 2(|\pi| - 1)$. Thus we may assume that $|\pi| < |V(G)|$. Then $n_0 \leq |\pi| - 1$, where $n_0$ is the number of trivial parts of $\pi$. By Theorem 1.12 $e_G(\pi) \geq 3(|\pi| - 1) - n_0 \geq 2(|\pi| - 1)$. \hfill $\square$

We may also point out that the proof of Theorem 1.6 in [3] relies on a variation of the rank function of rigid matroids which requires $G$ is simple (see (2) in next section), while our proofs are similar to [3] but can be applied to multigraphs.

2 Preliminaries

In this section, we present some basic results on rigid graphs and rigidity matroids.

Suppose that $G = (V, E)$ is a graph with $|V(G)| = n$. Let $\mathcal{F}$ be the collection of all edge subsets each of which induces a forest. Then $\mathcal{F}$ forms all independent sets of a matroid on ground set $E$. The circuit matroid $\mathcal{M}(G)$ of $G$ is the matroid $(E, \mathcal{F})$. The rank function of $\mathcal{M}(G)$ is given by $r_{\mathcal{M}}(F) = n - c(F)$, where $c(F)$ denotes the number of components of $G(F)$.

For any subset $X \subseteq V$ and $F \subseteq E$, $E_F(X)$ and $i_F(X)$ denotes the set and the number of edges of $F$ in $G[X]$, respectively. A subset $S \subseteq E$ is sparse if $i_S(X) \leq 2|X| - 3$ for all $X \subseteq V$ with $|X| \geq 2$. Let $\mathcal{S}$ be the collection of all sparse sets of $G$. Then $\mathcal{S}$ forms all independent sets of a
matroid on ground set $E$. The matroid $(E, S)$ is the rigidity matroid of $G$, denoted by $R(G)$. By Lovász and Yemini [12], the rank function of $R(G)$ is

$$r_R(F) = \min \left\{ \sum_{X \in G} \left(2|X| - 3\right) \right\},$$

where the minimum is taken over all collections $G$ of subset $X \subseteq V$ such that $\{E_F(X)|X \in G\}$ partitions $F$. Each $X \in G$ induces a rigid subgraph of $G(F)$ (see [3] or the proof of Lemma 2.4 in [6]). By definition, a graph $G$ is rigid if and only if the rank of $R(G)$ is $2|V(G)| - 3$. When $G$ is simple, a variation of (1) is given in [3] as

$$r_R(F) = \min \left\{ \sum_{X \in \mathcal{X}} \left(2|X| - 3\right) + |F - H| \right\},$$

where the minimum is taken over all subset $H \subseteq F$ and all collections $\mathcal{X}$ of subset $X \subseteq V$ such that $\{E_F(X)|X \in \mathcal{X}\}$ partitions $H$ and each element of $\mathcal{X}$ induces a rigid subgraph of $G[H]$ of size at least 3.

As in [3], $N_{k,l}(G)$ is the matroid on ground set $E$ obtained by taking matroid union of $k$ copies of the rigidity matroids $R(G)$ and $l$ copies of circuit matroids $M(G)$. By a theorem of Edmonds on the rank of matroid union [4], the rank of $N_{k,l}(G)$ is

$$r_{k,l}(E) = \min_{F \subseteq E} \{kr_R(F) + lr_M(F) + |E - F|\}.$$  

Thus $r_{k,l}(E) \leq kr_R(E) + lr_M(E) = k(2n - 3) + l(n - 1)$.

### 3 Proofs of main results

In this section, we prove Theorem 1.8, 1.9 and 1.12.

**Proof of Theorem 1.8.** Suppose that $G$ decomposes into $k$ spanning sparse subgraphs and $l$ spanning forests. By the definition of sparse graphs, for any subset $X \subseteq V(G)$ with $|X| \geq 2$, $i_G(X) \leq k(2|X| - 3)$, which proves the necessity.

To prove the sufficiency, assume that for any subset $X \subseteq V(G)$ with $|X| \geq 2$, $i_G(X) \leq k(2|X| - 3)$. It suffices to show the rank of $N_{k,0}(G)$, $r_{k,0}(E) \geq |E|$.

Let $F \subseteq E$ be a set that minimizes the right side of (3) when $l = 0$, then

$$r_{k,0}(E) = kr_R(F) + |E - F|.$$  

By (1), there exists a collection $G$ of subset $X \subseteq V$ such that $\{E_F(X)|X \in G\}$ partitions $F$ and

$$r_R(F) = \sum_{X \in G} (2|X| - 3).$$
Then $|F| = \sum_{X \in g} i_F(X) \leq \sum_{X \in g} k(2|X| - 3) = k \sum_{X \in g} (2|X| - 3) = krR(F)$. Thus $|E| = \sum_{X \in g} |X| + |E - F| \leq krR(F) + |E - F| = r_{k,0}(E)$, which implies that $E$ is an independent set of $\mathcal{N}_{k,0}(G)$. This completes the proof.

**Proof of Theorem 1.9** It suffices to show that the rank of $\mathcal{N}_{k,l}(G)$ is

$$r_{k,l}(E) = k(2n - 3) + l(n - 1).$$

Choose $F \subseteq E$ to be a set with smallest size that minimizes the right side of (3), then

$$r_{k,l}(E) = krR(F) + lr_M(F) + |E - F|. \quad (4)$$

By (1), there exists a collection $\mathcal{X}$ of subset $X \subseteq V$ such that $\{E_F(X)|X \in \mathcal{X}\}$ partitions $F$ and

$$r_R(F) = \sum_{X \in \mathcal{X}} (2|X| - 3). \quad (5)$$

**Claim 1.** For each $X \in \mathcal{X}$, $|X| \geq 3$.

If not, then let $\mathcal{X}'$ denote the collection of $X \in \mathcal{X}$ with $|X| = 2$. Then $r_R(F) = \sum_{X \in \mathcal{X} - \mathcal{X}'} (2|X| - 3) + \sum_{X \in \mathcal{X}'} (2|X| - 3) = \sum_{X \in \mathcal{X} - \mathcal{X}'} (2|X| - 3) + |\mathcal{X}'|$. Let $H \subset F$ be the set of edges by deleting all edges induced by each $X$ with $|X| = 2$. Then $\mathcal{X} - \mathcal{X}'$ is the collection of $X \subseteq V$ that partition $H$. By (1), $r_R(H) \leq \sum_{X \in \mathcal{X} - \mathcal{X}'} (2|X| - 3)$. As the multiplicity of $G$ is at most $k$, $|F - H| \leq k|\mathcal{X}'|$. Thus $krR(H) + lr_M(H) + |E - H| \leq k \sum_{X \in \mathcal{X} - \mathcal{X}'} (2|X| - 3) + lr_M(F) + |E - F| + |F - H| \leq k \sum_{X \in \mathcal{X} - \mathcal{X}'} (2|X| - 3) + lr_M(F) + |E - F| + k|\mathcal{X}'| = krR(F) + lr_M(F) + |E - F|$, which is contrary to the minimality of $F$. This completes the proof of the claim.

Let $|V(G[F])| = n_1$ and $n_2 = n - n_1$. Then there are $n_2$ isolated vertices in $G(F)$. For each $X \in \mathcal{X}$, define $X_B = X \cap (\cup_{X \neq Y \in \mathcal{X}'} Y)$ and $X_I = X - X_B$. Let $\mathcal{I}_X = \{X \in \mathcal{X} : X_I \neq \emptyset\}$. As each $X \in \mathcal{X}$ induces a connected subgraph of $G(F)$, it is not hard to see

$$c(F) \leq |\mathcal{I}_X| + n_2. \quad (6)$$

Since $\mathcal{X}$ covers $F$ and thus covers all vertices of $G[F]$, each vertex of $X_B$ lies in at least two different $X \in \mathcal{X}$ and each $X_I$ is in a single $X$, we have $\sum_{X \in \mathcal{X}} |X_B| + 2 \sum_{X \in \mathcal{I}_X} |X_I| \geq 2n_1$, which implies

$$\sum_{X \in \mathcal{X}} |X| + \sum_{X \in \mathcal{I}_X} |X_I| \geq 2n_1. \quad (7)$$

Now we will use the partition condition to show a lower bound of $|E - F|$. Let $Z = \cup_{X \in \mathcal{X}} X_B$. Then $\{X_I : X \in \mathcal{I}_X\}$ together with all isolated vertices of $G(F)$ form a partition $\pi$ of $G - Z$ with $n_2$ trivial parts and $|\pi| = |\mathcal{I}_X| + n_2$. The adjacent number of $\pi$ is $n_Z(\pi) = \sum_{X \in \mathcal{I}_X} |X_B|$. Thus

$$|E - F| \geq c_{G - Z}(\pi) \geq (3k + l)(|\pi| - 1) - kn_2 - kn_Z$$

$$= k \sum_{X \in \mathcal{I}_X} (3 - |X_B|) + 2kn_2 - 3k + l(|\mathcal{I}_X| + n_2 - 1) \quad (8)$$
As r_{k,l}(E) \leq k(2n-3) + l(n-1), it turns out that r_{k,l}(E) = k(2n-3) + l(n-1).

Proof of Theorem 1.12. Let S be a spanning subgraph of G that consists of edge-disjoint k spanning minimally rigid subgraphs and l spanning trees. By definition, |E(S)| = k(2n-3) + l(n-1), where n = |V(G)|. Let \( \pi = \{V_1, V_2, \cdots, V_t\} \) be a partition of V(G) such that V_i is nontrivial for \( 1 \leq i \leq t \) and trivial for \( t+1 \leq i \leq t+n_0 \). Thus \( \sum_{1 \leq i \leq t} |V_i| = n - n_0 \) and \( t = |\pi| - n_0 \). For \( 1 \leq i \leq t \), \( |E(S[V_i])| \leq k(2|V_i| - 3) + l(|V_i| - 1) \). Then

\[
\begin{align*}
\gamma(G) & \geq \gamma(S) = |E(S)| - \sum_{1 \leq i \leq t} |E(S[V_i])| \\
& \geq k(2n-3) + l(n-1) - \sum_{1 \leq i \leq t} (k(2|V_i| - 3) + l(|V_i| - 1)) \\
& = k(2n-3) + l(n-1) - k(2n - 2n_0 - 3t) - l(n - n_0 - t) \\
& = k(2n_0 + 3t - 3) + l(n_0 + t - 1) \\
& = k(2n_0 + 3|\pi| - 3n_0 - 3) + l(n_0 + |\pi| - n_0 - 1) \\
& = (3k + l)(|\pi| - 1) - kn_0.
\end{align*}
\]

4 Application on degree bounded forest covering

In this section, we emphasis an application of Theorem 1.8 on degree bounded forest covering. Theorem 1.12 is the main result.

For a graph G, the fractional arboricity \( \gamma(G) \) of G is defined as

\[
\gamma(G) = \max_{X \subseteq V(G)} \frac{i_G(X)}{|X| - 1},
\]

where \( i_G(X) \) is the fractional index of \( X \).
whenever the denominate is nonzero. This notation was introduced by Payan [17] and was generalized to matroids by Catlin et al. [2]. The well-known theorem (Theorem [17]) of Nash-Williams on forest covering indicates that $G$ decomposes to $\lceil \gamma(G) \rceil$ forests. When $\gamma(G) = k + \epsilon$ with $0 < \epsilon < 1$, Nash-williams’s theorem tells us $G$ decomposes to $k + 1$ forests but does not give any information on different $\epsilon$ values. Towards this observation, Montassier et al. [13] posed the following Nine Dragon Tree Conjecture stating that the maximum degree of one of the forests should be bounded by a function of $\epsilon$. They also have a Weaker NDT Conjecture if the degree bounded forest is replaced by a degree bounded subgraph.

Conjecture 4.1 (NDT Conjecture [13]). If $\gamma(G) = k + \epsilon$ with $0 < \epsilon < 1$, then $G$ decomposes into $k + 1$ forests, one of which has maximum degree at most $\lceil \frac{(k+1)\epsilon}{1-\epsilon} \rceil$.

Conjecture 4.2 (Weaker NDT Conjecture [13]). If $\gamma(G) = k + \epsilon$ with $0 < \epsilon < 1$, then $G$ decomposes into $k$ forests and a subgraph with maximum degree at most $\lceil \frac{(k+1)\epsilon}{1-\epsilon} \rceil$.

More about the NDT Conjecture, please see [10, 11, 13]. The Weaker NDT Conjecture is also of interests by itself, and it has applications in bounding the game chromatic number [14]. Many partial results have been obtained (See [10] for a survey of previous results). In particular, the Weaker NDT Conjecture was confirmed when $\epsilon \geq 1/2$ in [10]. Actually in [10], the authors provided a stronger result by weakening the sufficient condition, see Theorem 4.1. Around the same time, [11] proved that the maximum degree of the subgraph is at most $\lceil \frac{k+1}{2} \rceil$, which almost solves the weaker conjecture.

Theorem 4.1 ([10]). For $d \geq k + 1$, if $(k+1)(k+d)|X| - (k + d + 1)i_G(X) - k^2 \geq 0$ for every nonempty subset $X \subseteq V(G)$, then $G$ decomposes into $k$ forests and a subgraph with maximum degree at most $d$.

Notice that if $\epsilon$ is large (close to 1), the Weaker NDT Conjecture states that the maximum degree of the subgraph is bounded by a very large number, which seems give no information about the subgraph. Motivated by this observation, we are interested in the upper bound of the maximum degree of the subgraph in general.

For a graph $G$, $\gamma_2(G)$ of $G$ is defined as

$$\gamma_2(G) = \max_{X \subseteq V(G)} \frac{i_G(X)}{2|X| - 3},$$

whenever the denominate is nonzero. We have the following result. Theorem 4.2 strengthens the Weaker NDT Conjecture when $\epsilon$ is large.

Theorem 4.2. Let $k, l \geq 0$ be integers with $k + 1 \leq l \leq 2k + 2$. If $\gamma_2(G) \leq k + 1$, then $G$ decomposes into $l$ forests and $2k + 2 - l$ subgraphs with maximum degree at most $(2|V(G)| - 5)/3$. 
We need the following lemmas to prove Theorem 4.2.

**Lemma 4.3.** Any sparse graph $G$ decomposes into a forest and a subgraph with maximum degree at most $(2|V(G)| - 5)/3$.

**Proof.** It suffices to show that any sparse graph satisfies the condition $(k + 1)(k + d)|X| - (k + d + 1)i_G(X) - k^2 \geq 0$ when $k = 1$ and $d = (2|V(G)| - 5)/3$ in Theorem 1.1. By definition, for any sparse graph $G$ and $X \subseteq V(G)$ with $|X| \geq 2$, $i_G(X) \leq 2|X| - 3$. Then $2(1 + d)|X| - (d + 2)i_G(X) - 1 \geq 2(1 + d)|X| - (d + 2)(2|X| - 3) - 1 = 3d - 2|X| + 5 \geq 0$, completing the proof.

**Lemma 4.4.** Any sparse graph decomposes into two forests.

**Proof.** The lemma follows easily from Theorem 1.7 and from the definition of sparse graphs.

**Proof of Theorem 4.2.** As $\gamma_2(G) \leq k + 1$, we have $i_G(X) \leq (k + 1)(2|X| - 3)$. By Theorem 1.8 $G$ decomposes into $k + 1$ sparse subgraphs. By Lemma 4.3 $l - k - 1$ sparse subgraphs decompose into $2l - 2k - 2$ forests. By Lemma 4.3, the other $2k + 2 - l$ sparse subgraphs decompose into $2k + 2 - l$ forests and $2k + 2 - l$ subgraphs with maximum degree at most $(2|V(G)| - 5)/3$. Thus $G$ can decompose into $l$ forests and $2k + 2 - l$ subgraphs with maximum degree at most $(2|V(G)| - 5)/3$.

5 Closing thought

In this section, we pose several problems. First, motivated by Theorem 1.7, 1.9 and 1.12, we have the following problem.

**Problem 1.** Find a partition condition to characterize graphs with edge-disjoint $k$ spanning rigid subgraphs and $l$ spanning trees.

As an analogue of Nine Dragon Tree problem, we pose the following problem for sparse graphs.

**Problem 2.** Find a minimum integer $f(k, \epsilon)$ such that if $\gamma_2(G) = k + \epsilon$ with $0 < \epsilon < 1$, then $G$ decomposes into $k + 1$ sparse subgraphs, one of which has maximum degree at most $f(k, \epsilon)$.

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