On Integrable Models Related to the $osp(1,2)$ Gaudin Algebra

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December 1993

ABSTRACT We define the $osp(1,2)$ Gaudin algebra and consider integrable models described by it. The models include the $osp(1,2)$ Gaudin magnet and the Dicke model related to it. Detailed discussion of the simplest cases of these models is presented. The effect of the presence of fermions on the separation of variables is indicated.

I. INTRODUCTION

In 1976 [1] (see also [2]) M. Gaudin introduced a quantum mechanical model of $N$ particles with $su(2)$ or $su(1,1)$ spin, ever since known as a Gaudin model. The model is described by a set of $N$ hamiltonians

$$H_i = \sum_{j \neq i} \frac{S_i \cdot S_j}{\epsilon_i - \epsilon_j}, \quad i = 1, \ldots, N,$$

where $\epsilon_1 > \epsilon_2 > \ldots > \epsilon_N$ are free parameters and $S^i$ is the $su(2)$ or $su(1,1)$ spin of the $i$-th particle, i.e.

$$[S^i_z, S^j_{\pm}] = \pm \delta_{ij} S^i_{\pm}, \quad [S^i_+, S^j_-] = 2\delta_{ij} S^i_z. \quad (2)$$

Here $[\ , \ ]_-$ denotes the commutator. The total spin $S^2 = \sum_i (S^i)^2$ and $S_z = \sum_i S^i_z$ are remaining constants of motion. Gaudin showed that the hamiltonians (1) commute with
each other and constructed their common eigenvectors by the coordinate Bethe Ansatz. In [3] the Gaudin model was generalised to any semisimple Lie algebra.

Because of its simplicity, the Gaudin model has been used as a testing ground for the ideas such as the functional Bethe Ansatz and the general procedure of separation of variables [4]. Its connection to the Neumann model [5], revealed in [6], allows one to generalise separation of variables for some types of coordinates on Riemannian manifolds. It has also applications in quantum optics [7].

In [4] the Gaudin model was solved in the framework of the algebraic Bethe Ansatz. It was also shown there [4] that the model is governed by a Yang-Baxter algebra, called a Gaudin algebra, with commutation relations linear in the generators and determined by a classical $r$-matrix. It is to be stressed these features are present in the model despite its quantum mechanical nature. In fact, the Gaudin model is one of a large class of models, with such an algebraic nature, so that study of Gaudin algebras becomes an important issue.

To construct a Gaudin algebra we need two ingredients: the classical $r$-matrix and quantum $L$-operator. By the former we mean an $M^2 \times M^2$ matrix, of which the components are functions satisfying the classical Yang-Baxter equation

$$\left[r_{12}(\lambda - \mu), r_{13}(\lambda - \nu)\right]_+ + \left[r_{12}(\lambda - \mu), r_{23}(\mu - \nu)\right]_+ + \left[r_{13}(\lambda - \nu), r_{23}(\mu - \nu)\right]_+ = 0. \quad (3)$$

The subscripts in (3) indicate the vector spaces in $\mathbb{R}^M \otimes \mathbb{R}^M \otimes \mathbb{R}^M$ on which $r$ acts non-trivially, and $\lambda, \mu, \nu$ are spectral parameters. The quantum $L$-operator is an $M \times M$ matrix, of which the components are operator valued functions, such that

$$\left[L_1(\lambda), L_2(\mu)\right]_- = -[r(\lambda - \mu), L_1(\lambda) + L_2(\mu)]_- \quad (4)$$

where $L_1(\lambda) = L(\lambda) \otimes 1$ and $L_2(\mu) = 1 \otimes L(\mu)$. The infinite dimensional algebra generated by the components of $L(\lambda)$ is called a Gaudin algebra. If a Gaudin algebra describes the Gaudin system corresponding to Lie algebra $\mathcal{G}$, then we term this kind of a Gaudin algebra a $\mathcal{G}$ Gaudin algebra. To any simple Lie algebra one can associate different types of Gaudin algebras, corresponding to different types of $r$-matrices [8]. For instance we can consider rational, trigonometric or elliptic Gaudin algebras. In what follows we restrict ourselves to the rational case. Using language just introduced we can say that the model (1) is described by the rational $su(2)$ or $su(1,1)$ Gaudin algebra.
It is easy to extend the notion of a Gaudin algebra to superalgebras. In this case, both \( r(\lambda) \) and \( L(\lambda) \) have a supermatrix structure however. Hence the tensor product in (3) and (4) must be understood in a \( \mathbb{Z}_2 \)-graded sense. Precisely, Eq.(4) can be written as

\[
L(\lambda) \otimes L(\mu) - (-1)^{\partial L(\lambda)\partial L(\mu)} L(\mu) \otimes L(\lambda)
\]

\[
= L(\lambda) r(1)(\lambda - \mu) \otimes r(2)(\lambda - \mu) + (-1)^{\partial L(\mu)\partial r(1)} r(1)(\lambda - \mu) \otimes L(\mu) r(2)(\lambda - \mu)
\]

\[
- (-1)^{\partial L(\lambda)\partial r(2)} r(1)(\lambda - \mu) L(\lambda) \otimes r(2)(\lambda - \mu) - r(1)(\lambda - \mu) \otimes r(2)(\lambda - \mu) L(\mu),
\]

where we have used the notation \( r(\lambda - \mu) = r(1)(\lambda - \mu) \otimes r(2)(\lambda - \mu) \) and e.g. \( \partial L(\mu) \) denotes the parity of an appropriate element of \( L(\mu) \).

In the present paper we describe models which can be solved using \( osp(1,2) \) Gaudin algebra. In particular we generalise the Gaudin model by considering a system of \( N \) hamiltonians

\[
H_i = \sum_{j \neq i} \frac{S_i^j \cdot S^j + V^j_i V^j_i - V^j_i V^j_\pm}{\epsilon_i - \epsilon_j}, \quad i = 1, \ldots, N,
\]

(5)

where the components of \( S^j \), satisfy the algebra (2) and

\[
[S^i_+, V^j_\pm]_- = \pm \frac{1}{2} \delta_{ij} V^j_i, \quad [S^i_-, V^j_\pm]_- = 0, \quad [S^i_-, V^j_\pm]_- = \delta_{ij} V^j_i
\]

\[
[V^i_\pm, V^j_\pm]_+ = \mp \frac{1}{2} \delta_{ij} S^j_\pm, \quad [V^i_+, V^j_+]_+ = \pm \frac{1}{2} \delta_{ij} S^j_+
\]

(6)

where \([ , ]_-\) denotes the anticommutator.

The paper is organised as follows. In Section II we construct the \( osp(1,2) \) Gaudin algebra and we define a generating function for integrals of motion of systems described by this algebra. In Section III we construct the spectrum of the generating function and we derive the Bethe equations. In Section IV we use the \( osp(1,2) \) Gaudin algebra to solve the \( osp(1,2) \) Gaudin model. We show that the hamiltonians (5) commute with each other and construct explicitly their complete spectrum, treating the two particle (\( N=2 \)) special case as an example. In section five we discuss the \( osp(1,2) \) Dicke model. This is a natural \( osp(1,2) \) generalisation of the ordinary Dicke model governed by an \( su(1,1) \) Gaudin algebra, a generalisation which is realised by coupling a system of boson and fermion oscillators to the Gaudin model itself. We prove that the spectrum obtained for it by the algebraic Bethe Ansatz of section three is complete. Then we focus our attention on the special (\( N=1 \)) case in which there is only one particle in the underlying Gaudin model. In an oscillator representation of \( osp(1,2) \), this case describes a system with
two oscillators with frequency ratio 2:1 together with one ordinary and one Majorana fermionic oscillators. We discuss the non-separability of the Schrödinger equation for this case.

II. STRUCTURE OF THE \( osp(1, 2) \) GAUDIN ALGEBRA

Here we define the \( osp(1, 2) \) Gaudin algebra. To do it we first define a classical \( r \)-matrix and a quantum \( L \)-operator. The \( r \)-matrix is constructed out of the quadratic Casimir of \( osp(1, 2) \) in a standard way \(^4\),

\[
r(\lambda) = \frac{2}{\lambda} (S_z \otimes S_z + \frac{1}{2} (S_+ \otimes S_- + S_- \otimes S_+) + V_+ \otimes V_- - V_- \otimes V_+), \quad \lambda \in \mathbb{R},
\]

written in the fundamental representation of \( osp(1, 2) \) (unitary in the compact case, non-unitary in the non-compact case),

\[
S_z = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad S_+ = S_- = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V_+ = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad V_- = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\]

Here and in what follows we use the convention, in which a matrix element \( a_{ij} \) is even (odd) if \( i + j \) is even (odd). This convention proves more suitable for our purposes, than the standard one, in which a supermatrix is written in a canonical block form (cf. \( \text{(10)} \)). Inserting \( \text{(8)} \) into \( \text{(7)} \) we obtain an explicit expression for \( r(\lambda) \)

\[
r(\lambda) = \frac{1}{2\lambda} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.
\]

\( \text{(9)} \)
Next we define the quantum $L$-operator
\[
L(\lambda) \equiv \begin{pmatrix} A(\lambda) & V_-(\lambda) & B(\lambda) \\ -V_+(\lambda) & 0 & V_-(\lambda) \\ C(\lambda) & V_+(\lambda) & -A(\lambda) \end{pmatrix}.
\]
(10)

The components of $L(\lambda)$ generate the $osp(1,2)$ Gaudin algebra. We assume that they are non-singular except for a finite number of distinct values of $\lambda$, namely $\epsilon_1, \ldots, \epsilon_N$, such that $\epsilon_1 > \ldots > \epsilon_N$.

Using definitions (9), (10) and relation (4) we obtain the complete set of relations defining the (rational) $osp(1,2)$ Gaudin algebra:

\[
\begin{align*}
[A(\lambda), A(\mu)]_- &= [B(\lambda), B(\mu)]_- = [C(\lambda), C(\mu)]_- = 0, \\
[B(\lambda), V_-(\mu)]_- &= [C(\lambda), V_+(\mu)]_- = 0, \\
[A(\lambda), C(\mu)]_- &= -\frac{C(\lambda) - C(\mu)}{\lambda - \mu}, \\
[A(\lambda), B(\mu)]_- &= \frac{B(\lambda) - B(\mu)}{\lambda - \mu}, \\
[A(\lambda), V_\pm(\mu)]_- &= \mp \frac{1}{2} \frac{V_\pm(\lambda) - V_\pm(\mu)}{\lambda - \mu}, \\
[B(\lambda), C(\mu)]_- &= 2 \frac{A(\lambda) - A(\mu)}{\lambda - \mu}, \\
[V_-(\lambda), V_+(\mu)]_- &= \frac{1}{2} \frac{A(\lambda) - A(\mu)}{\lambda - \mu}, \\
[B(\lambda), V_+(\mu)]_- &= \frac{V_-(\lambda) - V_-(\mu)}{\lambda - \mu}, \\
[C(\lambda), V_-\mu)]_- &= \frac{V_+(\lambda) - V_+(\mu)}{\lambda - \mu}, \\
[V_-(\lambda), V_-\mu)]_- &= -\frac{1}{2} \frac{B(\lambda) - B(\mu)}{\lambda - \mu}, \\
[V_+(\lambda), V_+(\mu)]_+ &= \frac{1}{2} \frac{C(\lambda) - C(\mu)}{\lambda - \mu}.
\end{align*}
\]
(11)

In the $osp(1,2)$ Lie superalgebra there is a clear relation between $V_\pm$ and $S_\pm$, i.e. $4V_\pm^2 = \mp S_\pm$. Therefore the universal enveloping algebra $U(osp(1,2))$ is generated by $V_\pm$ and $S_z$ only and the highest weight representations are constructed by the actions of a single raising operator $V_+$ say. In the Gaudin algebra the similar relation can be read off the last two of Eqs. (11). When $\mu$ approaches $\lambda$ we obtain

\[
B'(\lambda) = 4V_\pm^2(\lambda), \quad C'(\lambda) = -4V_\pm^2(\lambda),
\]
(12)
where prime denotes differentiation with respect to $\lambda$. We learn from Eqs. (12) that construction of highest weight modules of $osp(1,2)$ Gaudin algebra involves both $V_+(\lambda)$ and $C(\lambda)$ operators. This also makes the Bethe Ansatz construction of the next section more complicated, as compared to the $su(1,1)$ case, in which there is only one raising operator.

From the point of view of integrable models described by the $osp(1,2)$ Gaudin algebra, it is important to consider a generating function

$$t(\lambda) \equiv \frac{1}{2} \text{str} L^2(\lambda) = A^2(\lambda) + \frac{1}{2} [B(\lambda), C(\lambda)]_+ + [V_+(\lambda), V_-(\lambda)]_-.$$  

(13)

This definition implies

$$[t(\lambda), t(\mu)]_- = 0,$$  

(14)

so that the function $t(\lambda)$ generates a family of commuting hamiltonians. Solution of an integrable model is equivalent to the construction for it of the spectrum of $t(\lambda)$. This can be carried out in the framework of the algebraic Bethe Ansatz, as will be discussed in the next section.

III. THE SPECTRUM OF $t(\lambda)$

Now we construct the spectrum of the generating function $t(\lambda)$. We look for the highest weight representations of the Gaudin algebra (11) with a vacuum $|0\rangle$ defined by

$$V_-(\lambda)|0\rangle = B(\lambda)|0\rangle = 0, \quad A(\lambda)|0\rangle = \alpha(\lambda)|0\rangle.$$  

(15)

Using definition (13) of $t(\lambda)$ it is easy to find that

$$t(\lambda)|0\rangle = (\alpha^2(\lambda) + \frac{1}{2} \alpha'(\lambda))|0\rangle$$  

$$\equiv \tau_0(\lambda)|0\rangle,$$  

(16)

where the prime denotes differentiation with respect to $\lambda$. Other eigenstates of $t(\lambda)$ are then generated from the vacuum $|0\rangle$ by the repeated actions of $V_+(\lambda)$ and $C(\lambda)$. Accordingly we look for an eigenstate $|\mu_1, \ldots, \mu_n\rangle$ of $t(\lambda)$ of the form

$$|\mu_1, \ldots, \mu_n\rangle = V_+(\mu_1, \ldots, \mu_n)|0\rangle.$$  

(17)
Here $V_+(\mu_1, \ldots, \mu_n)$ is constructed from $V_+(\mu_i)$ and $C(\mu_i)$, $i = 1, \ldots, n$. Demanding that $|\mu_1, \ldots, \mu_n\rangle$ be an eigenstate of $t(\lambda)$ we determine the form of $V_+(\mu_1, \ldots, \mu_n)$ and the numbers $\mu_1, \ldots, \mu_n$. The corresponding eigenvalue of $t(\lambda)$ will be denoted by $\tau(\lambda; \mu_1, \ldots, \mu_n)$. The explicit construction of $V_+(\mu_1, \ldots, \mu_n)$ is rather involved. We list operators $V_+(\mu_1, \ldots, \mu_n)$ for the first few excitations at the end of the section. Nevertheless it is possible to find the eigenvalues $\tau(\lambda; \mu_1, \ldots, \mu_n)$ as well as numbers $\mu_1, \ldots, \mu_n$ without performing a fully explicit construction of $V_+(\mu_1, \ldots, \mu_n)$.

The operator $V_+(\mu_1, \ldots, \mu_n)$ may be decomposed as follows

$$V_+(\mu_1, \ldots, \mu_n) = V_+(\mu_1) \cdots V_+(\mu_n) + W(\mu_1, \ldots, \mu_n),$$

where $W(\mu_1, \ldots, \mu_n)$ involves at least one operator $C(\mu_i)$, i.e.

$$W(\mu_1, \ldots, \mu_n) = \sum_{\sigma \in S_n} \gamma_\sigma(\mu_1, \ldots, \mu_n) V_+(\mu_{\sigma_1}) \cdots V_+(\mu_{\sigma_{n-2}}) C(\mu_{\sigma_{n-1}}) + \cdots,$$

where $\gamma_\sigma(\mu_1, \ldots, \mu_n)$ are number coefficients. It is crucial to observe that $W(\mu_1, \ldots, \mu_n)$ cannot produce terms proportional to $V_+(\mu_1) \cdots V_+(\mu_n)$, when commuted with $t(\lambda)$. Likewise $W(\mu_1, \ldots, \mu_n)$ itself cannot generate an eigenvector of $t(\lambda)$. Since (17) defines an eigenvector of $t(\lambda)$, the eigenvalue $\tau(\lambda; \mu_1, \ldots, \mu_n)$ has to be equal to the coefficient of $V_+(\mu_1) \cdots V_+(\mu_n)$, when acted upon by $t(\lambda)$. We compute

$$[t(\lambda), V_+(\mu_1) \cdots V_+(\mu_n)]_- = V_+(\mu_1) \cdots V_+(\mu_n) \left( \sum_{i=1}^n \frac{1}{\lambda - \mu_i} A(\lambda) + \frac{1}{2} \sum_{i<j} \frac{1}{(\lambda - \mu_i)(\lambda - \mu_j)} \right)$$

$$+ \text{(terms involving at least one } C(\mu_i)) .$$

From this we immediately see that

$$\tau(\lambda; \mu_1, \ldots, \mu_n) = \sum_{i=1}^n \frac{1}{\lambda - \mu_i} \alpha(\lambda) + \frac{1}{2} \sum_{i<j} \frac{1}{(\lambda - \mu_i)(\lambda - \mu_j)} + \tau_0(\lambda).$$

(20)

The numbers $\mu_i$, $i = 1, \ldots, n$ are not yet specified. They can be determined as follows [1]. We observe that the operator $t(\lambda)$ is non-singular for any $\lambda \neq \epsilon_i$, $i = 1, \ldots, N$. In particular, $\mu_i$, $i = 1, \ldots, n$ are regular points of $t(\lambda)$. This implies that $\tau(\lambda; \mu_1, \ldots, \mu_n)$ should be non-singular at $\lambda = \mu_i$, $i = 1, \ldots, n$. The conditions for non-singularity in (20) at $\lambda = \mu_i$, $i = 1, \ldots, n$ read

$$\alpha(\mu_i) = \frac{1}{2} \sum_{j \neq i} \frac{1}{\mu_i - \mu_j}, \quad i = 1, \ldots, n.$$
These are the Bethe equations for the $osp(1, 2)$ Gaudin algebra. Taking (21) into account we can write (20) as

$$\tau(\lambda; \mu_1, \ldots, \mu_n) = \tau_0(\lambda) + \sum_{i=1}^{n} \frac{\alpha(\lambda) - \alpha(\mu_i)}{\lambda - \mu_i}. \quad (22)$$

We can also write (20) in the form

$$\tau(\lambda; \mu_1, \ldots, \mu_n) = (\chi(\lambda; \mu_1, \ldots, \mu_n) + \alpha(\lambda))^2 + \frac{1}{2} \frac{d}{d\lambda}(\chi(\lambda; \mu_1, \ldots, \mu_n) + \alpha(\lambda)), \quad (23)$$

where

$$\chi(\lambda; \mu_1, \ldots, \mu_n) = \frac{1}{2} \frac{d}{d\lambda} \frac{q'_n(\lambda)}{q_n(\lambda)}, \quad q_n(\lambda) = \prod_{i=1}^{n} (\lambda - \mu_i).$$

Eq. (23), rearranged as a differential equation for $q_n(\lambda)$, reads

$$q''_n(\lambda) + 4\alpha(\lambda)q'_n(\lambda) + 4(\tau_0(\lambda) - \tau(\lambda; \mu_1, \ldots, \mu_n))q_n(\lambda) = 0, \quad (24)$$

so that (20) and (21) are equivalent to the differential equation (24). This equation plays an important role in the analysis of the completeness of the spectrum of $t(\lambda)$ constructed by the Bethe Ansatz. We illustrate this in the next section.

We would like to end this section with a few comments on the structure of operators $V_+(\mu_1, \ldots, \mu_n)$. Since each $\mu_i$ corresponds to a fermionic degree of freedom, the function $V_+(\mu_1, \ldots, \mu_n)$ is totally antisymmetric. The first three operators $V_+(\mu_1, \ldots, \mu_n)$ come out as

$$V_+(\mu_1), \quad V_+(\mu_1)V_+(\mu_2) - \frac{1}{4} \frac{C(\mu_1) + C(\mu_2)}{\mu_1 - \mu_2},$$

$$V_+(\mu_1)V_+(\mu_2)V_+(\mu_3) - \frac{1}{8} \sum_{i,j,k=1}^{3} \epsilon_{ijk} V_+(\mu_i) \frac{C(\mu_j) + C(\mu_k)}{\mu_j - \mu_k}. \quad (25)$$

Here $V_+(\mu_1) \cdots V_+(\mu_n)$ denotes the antisymmetric product

$$V_+(\mu_1) \cdots V_+(\mu_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma)V_+(\mu_{\sigma(1)}) \cdots V_+(\mu_{\sigma(n)}).$$

It can be checked directly that these operators generate eigenvectors of $t(\lambda)$, provided that conditions (21) are satisfied.

Finally we notice that if there is no pseudovacuum (15), but there is a pseudovacuum $V_+(\lambda)|0\rangle = C(\lambda)|0\rangle = 0$, $A(\lambda)|0\rangle = \bar{\alpha}(\lambda)|0\rangle$ instead, then the whole of the construction
can be repeated with $V_+(\lambda)$ replaced by $-V_-(\lambda)$, $C(\lambda)$ replaced by $-B(\lambda)$ and $\alpha(\lambda)$ replaced by $-\tilde{\alpha}(\lambda)$.

IV. THE $osp(1,2)$ GAUDIN MODEL

Here we discuss the model defined by the hamiltonians (5). We show that hamiltonians (5) are in involution and construct their common spectrum. We also prove, this spectrum is complete. Finally we derive the spectrum explicitly for the simplest non-trivial case of the two-particle system. We begin with a proof that the model is integrable.

We consider the $L$-operator

$$L(\lambda) = \sum_{i=1}^{N} \frac{1}{\lambda - \epsilon_i} \begin{pmatrix} S_i^z & V_i^+ & -S_i^- \\ -V_i^+ & 0 & V_i^- \\ -S_i^+ & V_i^- & -S_i^z \end{pmatrix}.$$  

(25)

Using the commutation rules (2) and (6), we can directly check that $L(\lambda)$ generates the $osp(1,2)$ Gaudin algebra. Writing generators of the $osp(1,2)$ Gaudin algebra in the explicit form (25), we easily find that the generating function (13) takes the following form

$$t(\lambda) = \sum_{i=1}^{N} \frac{(S_i^z)^2}{(\lambda - \epsilon_i)^2} + \sum_{i \neq j} \frac{S_i \cdot S_j + V_i^+V_j^+ - V_i^-V_j^-}{(\lambda - \epsilon_i)(\lambda - \epsilon_j)}.$$  

(26)

Hamiltonians (5) are identified with

$$H_i = \text{res}_{\lambda = \epsilon_i} t(\lambda)$$  

(27)

and thanks to Eq.(14) they commute with each other. They also satisfy $\sum_i H_i = 0$, so that (27) gives $N - 1$ independent hamiltonians. It is obvious that $S^2$ commutes with all $H_i$. Now it remains to show that $S_z$ commutes with $t(\lambda)$ and hence with each of $H_i$’s. The easiest way to do it is first to observe that the algebra (11) will not change if we transform $A(\lambda) \rightarrow A(\lambda) + g, \quad g \in \mathbb{R}$.

This transformation induces the transformation

$$t(\lambda) \rightarrow t(\lambda) + g^2 + 2A(\lambda) = \tilde{t}(\lambda).$$

Obviously $\tilde{t}(\lambda)$ generates a family of commuting hamiltonians and

$$\tilde{H}_i = \text{res}_{\lambda = \epsilon_i} \tilde{t}(\lambda) = H_i + 2gS_i^z.$$ 

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Now $\sum_i \tilde{H}_i = 2gS_z$ and from $[\tilde{H}_i, \tilde{H}_j]_- = 0$ we get $[S_z, H_i]_- = 0$ as claimed.

From now on we focus on the non-compact case, in which $osp(1,2)$ contains $su(1,1)$ as a subalgebra. We show, how to construct the complete spectrum of $t(\lambda)$ (23). We define the numbers $s_i > 0$, $i = 1,\ldots,N$ by

$$S_z^i |0\rangle = s_i |0\rangle, \quad i = 1,\ldots,N.$$ 

With these definitions

$$\alpha(\lambda) = \sum_{i=1}^{N} \frac{s_i}{\lambda - \epsilon_i}$$

and the Bethe equations (21) take the form

$$\sum_{i=1}^{N} \frac{s_i}{\lambda - \epsilon_i} = -\frac{1}{2} \sum_{j \neq i}^{n} \frac{1}{\mu_i - \mu_j}, \quad i = 1,\ldots,n.$$ 

They can be identified with those corresponding to $su(1,1)$ Gaudin model by the rescaling $s_i \to \frac{1}{2} s_i$. Therefore, as in the $su(1,1)$ case, the differential equation (24) becomes the Lamé equation, intensively studied in the context of orthogonal polynomials [12]. The result, which we need here is the Heine-Stieltjes theorem [12] asserting that there are

$$C_{N-1}^m = \binom{n + N - 2}{n}$$

polynomial solutions to (24), corresponding to $C_{N-1}^m$ different choices of $\tau(\lambda; \mu_1,\ldots,\mu_n)$. Hence the Bethe Ansatz gives $C_{N-1}^m$ eigenstates of $t(\lambda)$. On the other hand the Hilbert space of the system can be identified with $H_1 \otimes \cdots \otimes H_N$, where $H_i$, $i = 1,\ldots,N$ is a Hilbert space of representation of $osp(1,2)$ corresponding to the $i$-th particle. All eigenstates of $t(\lambda)$ are constructed as linear combinations of basic states in $H_1 \otimes \cdots \otimes H_N$, the latter obtained by the repeated actions of $V_+^i$’s, $i = 1,\ldots,N$ on the vacua $|0\rangle_i$. The number of all possible eigenstates of $t(\lambda)$ at the $n$-th level equals the number of all possible distributions of $n$ numbers between $N$ intervals and hence reads

$$C_N^m = \binom{n + N - 1}{n}.$$ 

This means that still $C_N^m - C_{N-1}^m = C_{N}^{m-1}$ states need to be constructed. Using an argument similar to the one that lead to $[t(\lambda), S_z]_- = 0$, we can show that $[t(\lambda), V_+]_- = 0$, where $V_+ = \sum_{i=1}^{N} V_+^i$. The remaining states at the $n$-th level are obtained by the action of $V_+$ on each state of the $n - 1$-th level.
Since $V_+$ commutes with $t(\lambda)$, it is clear that at the $n$-th level the full spectrum of $t(\lambda)$ consists of
\[
\tau_0(\lambda), \ \tau(\lambda; \mu_{1}^{(1)}), \ \ldots, \ \tau(\lambda; \mu_{1}^{(n)}, \ldots, \mu_{n}^{(n)}),
\]
corresponding to the states
\[
(V_+)^n|0\rangle, \ (V_+)^{n-1}V_+(\mu_{1}^{(1)})|0\rangle, \ \ldots, \ V_+(\mu_{1}^{(n)}, \ldots, \mu_{n}^{(n)})|0\rangle.
\]

To illustrate, how the theory of orthogonal polynomials can be used to solve the $osp(1,2)$ Gaudin model, we look at the case $N = 2$, $\epsilon_1 = -\epsilon_2 = \epsilon$, $\epsilon > 0$, $s_1 = s_2 = s$. Without the loss of generality we can set $\epsilon = 1$. We construct the spectrum of $t(\lambda)$ explicitly for this simple model. The Lamé equation (24) can be now written
\[
(1 - \lambda^2)q''_n(\lambda) - 8s\lambda q'_n(\lambda) - 4(\tau_0(\lambda) - \tau(\lambda; \mu_1, \ldots, \mu_n))(1 - \lambda^2)q_n(\lambda) = 0,
\]
where $q_n(\lambda)$ is a polynomial of degree $n$. Eq. (28) has a polynomial solution of degree $n$, provided that
\[
\tau(\lambda; \mu_1, \ldots, \mu_n) = \tau_0(\lambda) + \frac{n(n + 8s - 1)}{8s\lambda} \alpha(\lambda).
\]
If (29) is satisfied, then
\[
q_n(\lambda) = P_n^{(4s - 1, 4s - 1)}(\lambda),
\]
where $P_n^{(4s - 1, 4s - 1)}(\lambda)$ denotes the Jacobi polynomial. The numbers $\mu_i, i = 1, \ldots, n$ are then zeros of the Jacobi polynomial [31]. The full spectrum of $t(\lambda)$ at the $n$-th excitation reads
\[
\{\tau_0(\lambda) + \frac{k(k + 8s - 1)}{8s\lambda} \alpha(\lambda); \ k = 0, 1, \ldots, n\}.
\]

The $osp(1,2)$ algebra can be represented in a Fock space of the harmonic oscillator as follows [13]
\[
S^+_i = \frac{1}{4}[a_i, a_i^+]_+, \quad S^-_i = -\frac{1}{2}a_i^2, \quad S^+_i = \frac{1}{2}(a_i^+)^2
\]
\[
V^+_i = \frac{c_i a_i}{\sqrt{8}}, \quad V^-_i = -\frac{c_i a_i}{\sqrt{8}}, \quad i = 1, 2,
\]

\[
\text{(31)}
\]
where $a_i, a_i^\dagger$ are destruction and creation operators, $[a_i, a_j^\dagger]_- = \delta_{ij}$ and $c_i^2 = 1, c_1 c_2 = -c_2 c_1$. In this case $s = \frac{1}{4}$ and the Jacobi polynomials (30) become the Legendre polynomials $P_n(\lambda)$, and the spectrum of $t(\lambda)$ reads

$$\left\{ \frac{k(k+1)(\lambda^2-1)-1}{4(\lambda^2-1)^2}, \ k = 0, \ldots, n \right\},$$

so that the eigenvalues of hamiltonian $H_1$ say, at the $n$-th level are $k(k+1)/8, k = 0, 1, \ldots, n$.

**V. THE DICKE MODEL**

Here we present another model which satisfies the $osp(1,2)$ Gaudin algebra and hence can be solved in the way described in Section III. This is the model of a bosonic and a fermionic oscillator coupled to the $osp(1,2)$ Gaudin system. Let $[b, b^\dagger]_- = 1$ be a bosonic oscillator and let $[f, f^\dagger]_+ = 1$ be a fermionic oscillator. It is easy to check that the operators

$$A(\lambda) = -\frac{1}{2} \lambda + A^0(\lambda), \quad B(\lambda) = b + B^0(\lambda), \quad C(\lambda) = -b^\dagger + C^0(\lambda)$$

$$V_-(\lambda) = -\frac{1}{2} f + V^-_0(\lambda), \quad V_+(\lambda) = \frac{1}{2} f^\dagger + V^+_0(\lambda)$$

satisfy the algebra (11) provided that $A^0(\lambda), B^0(\lambda), C^0(\lambda), V^-_0(\lambda), V^+_0(\lambda)$ satisfy it and $b, f$ oscillators (anti-)commute with them. We assume that $A^0(\lambda), B^0(\lambda), C^0(\lambda), V^-_0(\lambda), V^+_0(\lambda)$ correspond to the N-particle $osp(1,2)$ Gaudin model and we show that the Bethe Ansatz in this case gives all the eigenstates of $t(\lambda)$. In other words we show that the Bethe equations (21), which read explicitly in this case

$$-\frac{1}{2} \mu_i + \sum_{k=1}^N \frac{s_k}{\mu_i - \epsilon_k} = -\frac{1}{2} \sum_{j=1}^n \frac{1}{\mu_i - \mu_j}, \quad i = 1, \ldots, n,$$

have precisely $C_{N+1}^n = \binom{n+N}{n}$ solutions. Using the classical method of Stieltjes (cf. [12]) we consider the function

$$Z(\mu_1, \ldots, \mu_n) = e^{-\frac{1}{2} \sum_{i=1}^n \mu_i^2} \prod_{k=1, \ldots, N \atop j=1, \ldots, n} \left[ \mu_j - \epsilon_k \right]^{2s_k} \prod_{r<s} \left| \mu_r - \mu_s \right|,$$

which can be defined in $C_{N+1}^n$ regions in $\mathbb{R}^n$ corresponding to all possible distributions of $n$ variables $\mu_i$ between $N + 1$ intervals, $(-\infty, \epsilon_1], [\epsilon_1, \epsilon_2], \ldots, [\epsilon_n, +\infty)$. Let us fix
one of these regions. It is clear that at the boundary of this region \( Z \) vanishes, and since \( Z \geq 0 \), it has a maximum inside the region. At this maximum \( \mu_r \neq \mu_s \), if \( r \neq s \), and \( \frac{\partial Z}{\partial \mu_i} = 0 \), \( i = 1, \ldots, n \), i.e. Eqs. (33) hold. Since there are \( C_{N+1}^n \) regions, there are \( C_{N+1}^n \) different maxima, hence (33) has at least \( C_{N+1}^n \) real solutions. Knowing that there can be at most \( C_{N+1}^n \) solutions to (33) we have also proved that there are precisely \( C_{N+1}^n \) maxima of (34).

Now we look at the simplest non-trivial case of the Dicke model. We assume that \( N = 1 \), \( \epsilon_1 = 0 \), and we write \( osp(1, 2) \) generators in the oscillator representation (31). In the purely bosonic case this model corresponds to the system of two oscillators with the frequency ratio 2 : 1. In the \( osp(1, 2) \) case we have the model of two such oscillators with, in addition, an ordinary and one Majorana fermionic oscillators. The generating function \( t(\lambda) \) takes the form

\[
t(\lambda) = \frac{1}{4} \lambda^2 - \frac{1}{2} H - \frac{1}{2} G \lambda^{-1} - \frac{1}{16} \lambda^{-2},
\]

where

\[
H = \frac{1}{2} [a, a^\dagger]_+ + [b, b^\dagger]_+ - \frac{1}{2} [f, f^\dagger]_-,
\]

\[
G = b a^2 + b^\dagger a^2 - \frac{1}{\sqrt{2}} f c a^\dagger + \frac{1}{\sqrt{2}} f^\dagger c a.
\]

The use of (22) gives immediately the eigenvalues of \( t(\lambda) \) in the form

\[
\tau(\lambda; \mu_1, \ldots, \mu_n) = \frac{1}{4} \lambda^2 - \frac{1}{16} \lambda^{-2} - \frac{1}{2} (n + 1) - \left( \frac{1}{4} \sum_i \frac{1}{\mu_i} \right) \lambda^{-1}.
\]

Therefore the hamiltonians \( H \) and \( G \) have the eigenvalues

\[
E_n = n + 1, \quad g_n = \frac{1}{2} \sum_i \frac{1}{\mu_i}
\]

respectively. The eigenvalues \( g_n \) need more discussion. From the symmetries of the function \( Z(\mu_1, \ldots, \mu_n) \) (34), we immediately deduce that if \( \mu_i \) is a solution to (33) so is \( -\mu_i \), hence the spectrum of \( G \) is closed under the change of sign. In addition it contains 0 for \( n \) even. So the spectrum of \( G \) is fully characterised by the positive eigenvalues. On the other hand \( g_n \) can be derived directly from the differential equation (24), which now takes the form

\[
\lambda q_n''(\lambda) - (2 \lambda^2 - 1) q_n'(\lambda) + 2(n \lambda + g_n) q_n(\lambda) = 0.
\]
Here we have used the explicit form (38) of the eigenvalue $E_n$ of $H$. Making the ansatz, $q_n(\lambda) = \sum_{k=0}^{n} a_k \lambda^k$, $a_n = 1$ we can derive the recurrence relations for $a_k$,

$$a_{n-1} = -g_n$$

$$(n - k)^2 a_{n-k} + 2(k + 2)a_{n-k-2} + 2g_na_{n-k-1} = 0, \quad k = 0, 1, \ldots, n - 1. \quad (40)$$

Solving (40) for $a_{n-1}$ we find that the eigenvalues of $G$ are $\pm \sqrt{\frac{2}{2}}$ and $0, \pm \sqrt{3}$ respectively.

We can represent bosonic oscillators $a$, $b$ in terms of the canonical coordinates, $x_i$, $p_i$, $i = 1, 2$, $[x_i, p_j]_-= i \delta_{ij}$, as follows

$$a = \frac{1}{\sqrt{2}}(x_2 + ip_2), \quad b = x_1 + \frac{i}{2} p_1. \quad (41)$$

We can also represent fermions $f$, $c$ in terms of the Pauli matrices $\sigma_i$,

$$c = \sigma_3, \quad f = \frac{1}{2}(\sigma_1 - i \sigma_2). \quad (42)$$

In this representation, the hamiltonians (43) take the form

$$H = \frac{1}{2} \begin{pmatrix} p_1^2 + p_2^2 + 4x_1^2 + x_2^2 + 1 & 0 \\ 0 & p_1^2 + p_2^2 + 4x_1^2 + x_2^2 - 1 \end{pmatrix},$$

$$G = \frac{1}{2} \begin{pmatrix} \quad -2x_1p_2^2 + p_1[x_2, p_2]_+ + 2x_1x_2^2 & -x_2 - ip_2 \\ -x_2 + ip_2 & -2x_1p_2^2 + p_1[x_2, p_2]_+ + 2x_1x_2^2 \end{pmatrix}. \quad (43)$$

It is known [14] that, in the purely bosonic case, Schrödinger equations corresponding to $G$ and $H$ can be separated in parabolic coordinates $y_1$, $y_2$,

$$2x_1 = y_1^2 - y_2^2, \quad x_2 = y_1y_2. \quad (44)$$

It is also known that these are the only coordinates other than Cartesian’s, in which the bosonic system separates. We can try to implement this procedure in the $osp(1,2)$ case. First we write hamiltonians (43) in parabolic coordinates (14),

$$H = \frac{1}{2(y_1^2 + y_2^2)} \begin{pmatrix} -\frac{\partial^2}{\partial y_1^2} + y_1^6 + y_1^7 - \frac{\partial^2}{\partial y_2^2} + y_2^6 + y_2^7 & 0 \\ 0 & -\frac{\partial^2}{\partial y_1^2} + y_1^6 - y_1^7 - \frac{\partial^2}{\partial y_2^2} + y_2^6 - y_2^7 \end{pmatrix},$$

$$G = \frac{y_1^2y_2^2}{2(y_1^2 + y_2^2)} \begin{pmatrix} \quad \frac{1}{y_1^2} \frac{\partial^2}{\partial y_1^2} - y_1^4 - \frac{1}{y_1^2} \frac{\partial^2}{\partial y_2^2} + y_1^4 & -\frac{1}{y_1y_2} \left( \frac{1}{y_1^2} \frac{\partial}{\partial y_1} + y_1^2 + \frac{1}{y_2^2} \frac{\partial}{\partial y_2} + y_2^2 \right) \\ -\frac{1}{y_1y_2} \left( \frac{1}{y_1^2} \frac{\partial}{\partial y_1} + y_1^2 + \frac{1}{y_2^2} \frac{\partial}{\partial y_2} - y_2^2 \right) & \quad \frac{1}{y_2^2} \frac{\partial^2}{\partial y_2^2} - y_2^4 - \frac{1}{y_2^2} \frac{\partial^2}{\partial y_1^2} + y_2^4 \end{pmatrix}. \quad (45)$$
We look for vector wave functions
\[\Psi_n(y_1, y_2) = \begin{pmatrix} \Psi^+_n(y_1, y_2) \\ \Psi^-_n(y_1, y_2) \end{pmatrix},\]
which solve the Schrödinger equations
\[H\Psi_n = E_n\Psi_n, \quad G\Psi_n = g_n\Psi_n. \quad (46)\]
It can be easily seen that the first of Eqs.(46) can be solved, assuming that \(\Psi_n(y_1, y_2)\) separates, i.e.
\[\Psi_n(y_1, y_2) = \begin{pmatrix} \Phi^+_n(y_1)\tilde{\Phi}^+_n(y_2) \\ \Phi^-_n(y_1)\tilde{\Phi}^-_n(y_2) \end{pmatrix}. \quad (47)\]
In this case the first of Eqs.(46) leads to the four 1-dimensional equations
\[-\Phi^+_n(y_1) - (2n + 2 \pm 1)y_1^2 - c_\pm \Phi^+_n = 0,\]
\[-\tilde{\Phi}^+_n(y_2) - (2n + 2 \pm 1)y_2^2 + c_\pm \tilde{\Phi}^+_n = 0, \quad (48)\]
where \(c_\pm\) are constants. The equations for \(\Phi^-_n\) are the same as for \(\Phi^+_{n+1}\), hence we need to consider only equations for \(\Phi^+_{n+1}\) say. Eqs.(48) are precisely identical with the equations one obtains in the purely bosonic case. From this we immediately know, that all independent solutions to the first of Eqs.(46) can have the separated form (47). Inserting (47) into the second of Eqs.(46) we immediately see that if \(\Phi^\pm_n, \tilde{\Phi}^\pm_n\) satisfy (48), they cannot satisfy the Schrödinger equation for \(G\). One has to combine different solutions of (48) to obtain the function \(\Psi_n(y_1, y_2)\). Knowing that the parabolic coordinates (44) are the only ones, for which separation of variables in the first of Eqs.(46) is possible, we immediately deduce that we cannot separate (44) changing only the position variables.

One can nevertheless expect that the separation of (46) is possible, but a more general canonical transformation of the Cartesian variables has to be considered. An indication comes from the analysis of the bosonic case. In that case, the zeros of corresponding functions \(\Phi_n\) can be identified with the zeros of the polynomial \(q_n\), satisfying the bosonic version of Eq.(39). It is therefore reasonable to hope that there are coordinates, in which Schrödinger equations (44) give such a natural meaning to Eq.(39).

VI. ACKNOWLEDGEMENTS

T. Brzeziński would like to thank St. John’s College, Cambridge for a Benefactors’ Scholarship. His work is also supported by the grant KBN 2 0218 91 01.


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