We consider the nonlinear transverse magnetic moment that arises in the Meissner state of superconductors with a strongly anisotropic order parameter. We compute this magnetic moment as a function of applied field and geometry, assuming d-wave pairing, for realistic samples, finite in all three dimensions, of high temperature superconducting materials. Return currents, shape effects, and the anisotropy of the penetration depth tensor are all included. We numerically solve the nonlinear Maxwell-London equations for a finite system. Results are discussed in terms of the relevant parameters. The effect, which is a probe of the order parameter symmetry in the bulk, not just the surface, of the sample should be readily measurable if pairing is in a d-wave state. Failure to observe it would set a lower bound to the s-wave component.

I. INTRODUCTION

The question of the pairing state of high temperature superconductors (HTSC’s) continues to baffle researchers in the field. A recent review of many of the experimental results closes after over eighty pages with the tentative conclusion that the only state compatible with all experiments would have to exhibit two separate transitions, in contrast with the single transition invariably observed in HTSC’s. Worse, even different experiments performed on the very same single crystal sample seem to lead to contradictory conclusions. This situation makes it particularly desirable to find good probes that may as unambiguously as possible discern the structure of the order parameter (OP) function.

One such probe is afforded by the nonlinear Maxwell-London electrodynamics of exotic pairing states in superconductors in the Meissner state. It was pointed out four years ago in the context of a two-dimensional model (HTSC’s are nearly all highly anisotropic, layered materials) that the presence of order parameter lines of nodes on a cylindrical Fermi surface would lead to observable nonlinear effects in the electrodynamic properties. These effects were shown to be potentially capable of yielding a signature for the structure of the order parameter function, and therefore for the nodal structure of the gap itself. The most obvious of them is the presence of a transverse magnetic moment \( \mathbf{m}_\perp \) perpendicular to the direction of an applied magnetic field \( \mathbf{H} \) lying in the \( x - y \) (a - b) plane (the z axis is taken to be along the c crystallographic direction, perpendicular to the stack of planes in the lattice structure). This transverse magnetic moment has an angular dependence, as the sample is rotated about the z axis, reflecting directly the periodicity of the energy spectrum, that is, of the square of the order parameter, on the azimuthal angle \( \phi \). This signal would be particularly prominent if the order parameter has nodes or very deep minima as a function of angle. The transverse moment can be measured directly or inferred from the torque it produces.

Electrodynamic effects are of particular interest because they probe the superconducting properties over a scale given by the field penetration depth, while many other experimental methods, such as tunneling and Josephson junctions, probe only over the scale of the correlation length, which in HTSC’s is over two orders of magnitude smaller. Thus, electrodynamics probes the bulk properties of the superconducting material, not merely the surface. Some of the apparent conflicts among experimental results may be resolved if the order parameter symmetries in the bulk and at the surface differ.

For these reasons, investigations of the transverse magnetic moment effect have continued. For the model of a “slab” shaped sample, infinite in the \( x - y \) directions and of fixed thickness, and a cylindrical Fermi surface, the equations involved in the nonlinear Maxwell-London electrodynamics have been solved numerically for the experimentally relevant ranges of applied magnetic field and temperature. At \( T = 0 \), for the same geometry and assumptions, a perturbative, but accurate, analytic solution has been obtained. In Ref., results were obtained for both a pure d-wave state and a mixture of s and d waves. The amplitude of the transverse magnetization at low temperatures was found to increase as approximately the second power of the applied field, and, since one is dealing with an effect arising from a finite penetration depth, as the surface area of the sample. The results indicated that, for a pure d-wave state, \( \mathbf{m}_\perp \) should be detectable by a SQUID magnetometer in single crystal samples, even though the applied field is limited, by the need to keep the sample in the Meissner regime, to the field of first flux penetration.
The results of only one experiment seeking to find the transverse magnetization effect have been published. No evidence of a transverse magnetic moment having the proper periodicity was obtained, but the nature of the data and of its analysis was such that the absence of evidence could not be conclusively taken to be evidence of absence. The resolution of the measurements was, over most of the field range studied, comparable to the signal measured. At the higher fields used the signal was below, but only by a factor of two, the theoretically expected results for a pure d-wave state, based on the infinite slab assumption discussed in the above paragraph, scaled to the finite size of the sample. Although a subsequent analysis of the field dependence of the experimental signal was also supportive of the inference that no nodes were observed, a completely definitive statement about the existence of nodes could not be made because of theoretical and experimental uncertainties.

Although this paper deals with the theoretical questions it is useful to briefly mention here the main difficulty associated with the experiments: except in the ideal world of theorists where all samples, even when finite, are spheres or other highly symmetric bodies, there always is in any superconductor, a transverse magnetic moment because of demagnetization effects. For the rectangular samples used in the experiments these effects are much larger than the nonlinear contribution. Of course, their main periodicity, \( \pi \), is twice that associated \((\pi/2)\) with the d-wave nonlinear effects. Fourier analysis of the data, as done in Ref. 8 does filter out the main spurious demagnetization signal, but its \(\pi/2\) harmonic still will be confounded with the nonlinear signal and, although it can partly be sorted out because of its different, linear, magnetic field dependence, the noise introduced in the original signal by the Fourier transforms and subtractions takes a heavy toll. The work in this paper is motivated in large part by experiments now being planned on samples having approximately a flat disk shape that minimizes the demagnetization effect for an applied field lying in the \( a - b \) plane. It seems to us that reaching conclusions from experimental data taken on samples lacking rotational symmetry would be difficult.

The theoretical uncertainties are related to the use in the work discussed above of two approximations: first, the use of an infinite slab geometry neglects the contribution to \( m_{\perp} \) of return currents flowing along the sides of the sample, parallel to the \( z \)-axis. Furthermore, the penetration depth for those currents, \( \lambda_c \), is much larger than that in the \( a-b \) plane, \( \lambda_{ab} \). Since \( m_{\perp} \) increases with penetration depth, the question of how the larger \( \lambda_c \) must be included in the analysis arises. The effect of \( \lambda_c \) was included in the analysis of the data, but in a purely heuristic way, which assumed that its influence was rather strong. Although this assumption was relaxed in a subsequent reanalysis, the whole question remains a major obstacle in reaching conclusions from experimental evidence. A finite \( \lambda_c \) implies also abandoning the notion of a cylindrical Fermi surface. Further, the very neglect of finite size effects is suspect: they have been shown to affect the analysis of penetration depth experiments in rather large samples. The uncertainties related to these complications in analyzing experimental data in the context of the geometric approximations of Ref. 8 are about as large as the experimental uncertainties and it might be argued that, in combination, are about as large as the discrepancy between theory and experiment.

In this paper we therefore undertake the examination of Maxwell-London electrodynamics, including nonlinear contributions, for finite samples. We include in the study the effects of \( \lambda_c \), which affect the electromagnetic response even at linear order in the field, and we specifically consider geometries relevant to the experimental systems currently been studied. Because of these realistic assumptions, not only is an analytic solution unattainable even at zero temperature, but the numerical work involved is quite considerable. Our results confirm that a transverse magnetic moment should be observable in obtainable single crystal samples, if the bulk pairing is in a d-wave state. The effect discussed in this paper is a fingerprint for the existence of an OP with d-wave symmetry. Its experimental observation in suitable samples, would constitute very hard to refute evidence for d-wave pairing in the bulk. Failure to observe this effect would at the very least put a lower bound on the existence of an s-wave component.

In the next Section we present the equations solved and we discuss the methods we use. In Section III, we present our results and predictions for experimental outcomes, assuming a d-wave order parameter is present. Our conclusions are given in the last Section.

II. METHODS

A. Maxwell-London electrodynamics

The equations of superconducting electrodynamics are in textbooks and need no rederivation. We will merely introduce here our notation, and briefly recall a couple of easy to overlook points. For the linear case, a particularly clear discussion of the complications that occur when the penetration depth tensor is anisotropic is in Chapter 3 of Ref. 9.

Outside the sample the current is \( j = 0 \) and therefore in the steady state the field \( H \) satisfies the Maxwell equations \( \nabla \cdot H = 0 \) and \( \nabla \times H = 0 \). One can therefore write \( H \) in terms of a scalar potential \( \Phi \) satisfying the Laplace equation:
\[ H = -\nabla \Phi \] (2.1a)

\[ \nabla^2 \Phi = 0 \] (2.1b)

In the interior of the sample one must consider three fields: \( H \), \( j \) and the “superfluid velocity field” \( v \) defined as:

\[ v = \frac{\nabla \chi}{2} + \frac{e}{c} A, \] (2.2)

where \( \chi \) is the phase of the superconducting order parameter, \( A \) the vector potential, and \( e \) the proton charge. We set \( \hbar = k_B = 1 \). Eq. (2.2) is equivalent to Eq. (5.99) in Ref. 9. The field \( v \) has dimensions of momentum, not velocity, but the factors associated with the effective mass and its anisotropies are more conveniently dealt with by placing them elsewhere. The three equations that one requires for these three fields are first, the second London equation obtained by taking the curl of (2.2):

\[ \nabla \times v = \frac{e}{c} H \] (2.3)

second, Ampère’s law:

\[ \nabla \times H = \frac{4\pi}{c} j \] (2.4)

and finally, a constitutive equation relating \( j \) and \( v \) which is discussed in general in the next subsection. For the usual linear case this relation is of course simply:

\[ j = -e\tilde{\rho}v \] (2.5)

where \( \tilde{\rho} \) is the superfluid density tensor. The nonlinear contribution is discussed below (see Eq. (2.12)). \( \tilde{\rho} \) is related to the penetration depth tensor \( \tilde{\Lambda} \), whose components in the diagonal representation are the square of the London penetration depths, by the relation:

\[ \tilde{\rho} = \frac{c^2}{4\pi e^2} \tilde{\Lambda}^{-1} \] (2.6)

When these tensors are proportional to the identity, then one can combine Eqs. (2.3) and (2.4) and find that any one of the three fields considered satisfies the vector Helmholtz equation. But this is not true when \( \tilde{\rho} \) is anisotropic. One still has, however, the completely general equation:

\[ \nabla \times \nabla \times v = \frac{4\pi e}{c^2} j(v) \] (2.7)

which is valid whatever the relation \( j(v) \) might be. Eq. (2.7) will be the basic equation we will consider here.

These equations must be solved with appropriate boundary conditions. These are: first, at infinity (that is, very far away from the finite sample) \( H \) must reduce to the applied field. Second, deep inside the sample all fields must vanish. Finally, at the interface \( H \) must be continuous \( \ref{12,13} \), and the component of \( j \) normal to the interface must vanish.

Once the currents \( j \) inside the sample are known, the magnetic moment can be obtained by integration:

\[ m = \frac{1}{2c} \int d\mathbf{r} \times j(\mathbf{r}) \] (2.8)

It is convenient to rewrite this equation in terms of surface integrals. Using Eq. (2.4) and formulas for vector calculus \( \ref{14} \) we derive:

\[ m = \frac{1}{8\pi} \int_S d^2 S \ \mathbf{n} \cdot (\mathbf{r} \times (\mathbf{H} \cdot \mathbf{n})) + \frac{1}{8\pi} \int_S d^2 S \ \mathbf{r} \times (\mathbf{H} \cdot \mathbf{n}) 
- \frac{1}{8\pi} \int_S d^2 S \ \mathbf{H} \times \mathbf{n} \times \mathbf{v} \] (2.9)

Integration is performed over the sample surface \( S \), \( \mathbf{r} \) is the position vector for a point on the surface \( S \), and \( \mathbf{n} \) is the unit normal pointing outwards.

In the linear regime, and for the case which is experimentally relevant here where one has a small but finite value of the ratio \( \lambda/d \) between a typical penetration depth \( \lambda \) and a characteristic sample dimension \( d \), the components of \( m \) can generically be written in the form \( m = m_0 (1 - \alpha(\lambda/d) + O(\lambda/d)^2) \). Examples for values of the positive constant \( \alpha \) are given in textbooks \( \ref{16,17} \). One has, then, a reduction in the magnetic moment due to current penetration in the material. The nonlinear effects may be conveniently viewed for our purposes as anisotropic, field-dependent corrections to the values of the \( \alpha \) coefficients.

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B. The supercurrent

To obtain the supercurrent response one has to specify the full constitutive relation $j(v)$ as discussed above. The linear contribution has been given in Eqns. (2.5) and (2.6). The anisotropy in $\Lambda$ is related to that of the Fermi surface and can be related to the effective mass tensor. The anisotropy of the order parameter considered in this work would not, if the Fermi surface were isotropic, lead to any anisotropies in the penetration depth, at linear order in the magnetic field. Thus we have:

$$\hat{\lambda} = \frac{c^2 \hat{m}}{4\pi ne^2} \quad (2.10)$$

where $\hat{m}$ is the effective mass tensor and $n$ the carrier density.

The nonlinear corrections to $j(v)$ were first discussed by Bardeen in the context of an $s$-wave superconductor. In that case, at $T = 0$, the current-velocity relation is linear for velocities less than a critical velocity $v_c$ defined below. Nonlinear corrections to the current-velocity relation arise from the thermal population of quasiparticles. These corrections are small (cubic in the ratio $v/v_c$) and in the regime of larger flow velocities vortex nucleation may occur before these effects become important. On the other hand, in unconventional superconductors, particularly those with nodes in the OP, nonlinear corrections are significantly larger and qualitatively different than in a superconductor with an $s$-wave excitation gap. The region on the Fermi surface (FS) close to the nodes in the gap provides enhanced nonlinear response due to the higher population of quasiparticles. There the energy required to break electron pairs will be reduced and will vanish along the direction of the lines of nodes in the gap. Thus nonlinear corrections in the presence of nodes in the OP are nonvanishing even at $T = 0$ and the nonlinear behavior exhibits anisotropy with respect to the relative angle between applied magnetic field and the lines of nodes of the gap function.

For the purpose of computing $m_1$ for a crystallographically strongly anisotropic HTSC, in the geometry considered here, we are concerned with the angular dependence of the OP in the plane of the applied field, the $a - b$ plane. In most of the numerical calculations performed in this work, we consider an order parameter of the pure $d$-wave form:

$$\Delta = \Delta_0 \sin(2\phi) \quad (2.11)$$

where $\phi$ is the azimuthal angle referred to a node and $\Delta_0$ the amplitude. The periodicity of $m_1$, which equals that of the energy, is therefore $\pi/2$. It would be possible, as we shall see, to generalize the calculations to other $\phi$ dependence for the OP if necessary. Possible dependence of $\Delta$ on the $c$ direction is very uncertain because of multilayering effects, but we believe it is unlikely to be important. This follows from the above physical considerations, and from the brief discussion of results for alternative forms in the next Section and Appendix A. On the other hand, we must properly take into account the strong anisotropy of $\Lambda$ in the $c$ direction, which is related to that of the Fermi surface.

The assumption of isotropy in the $a - b$ plane deserves further discussion, since anisotropies in the in-plane penetration depth in HTSC’s are known to exist. Anisotropy in $\lambda_{ab}$ will lead to a contribution to $m_1$ of periodicity $\pi$ (instead of $\pi/2$ for the nonlinear signal), and (see the paragraph below (2.3)) down by a factor of $\sim \lambda/d$ (the ratio of a typical penetration depth to a typical sample dimension) from the longitudinal, linear magnetic moment. This contribution will be linear, not quadratic, in the field. Hence, from an experimental point of view, it is in effect a small correction to the effective demagnetization factor which is, although to a small extent, present even in macroscopically symmetric samples. Thus, Fourier analysis of the experimental signal and examination of the field dependence of the harmonics would separate this from the nonlinear effect. As for the theoretical results discussed here it is sufficient to interpret our symbol $\lambda_{ab}$ for the $a - b$ plane penetration depth ($\lambda_c$ is that along the $c$-direction) as the geometric mean of $\lambda_a$ and $\lambda_b$. Finally, if because of orthorhombic anisotropy the nodes are not precisely at $\pi/2$, this can be accounted for by adding a small constant to (2.11) and therefore the $\pi/2$ Fourier component of the response would be little affected.

For an order parameter with nodes, it is known that the details of the FS shape are not important for the nonlinear properties we deal with here: only the nodes and their symmetry matter. Thus, our chief concern about the FS is to describe the anisotropy in $\Lambda$. For this purpose, we have used in our calculations an ellipsoidal (of revolution) Fermi surface characterized by effective masses $m_{ab}$ (in the $a - b$ plane) and $m_c$, as in Eq. (A1). We introduce the ratio $\delta \equiv (m_c/m_{ab})$. We define a speed $v_f$ in terms of the Fermi energy $\xi_f$ as $v_f^2 = 2\xi_f/m_{ab}$ which facilitates comparison with two-dimensional results.

The general expression for $j(v)$, including nonlinear terms, can be written as:

$$j = -eN_f \int d^2s n(s) v_f [\mathbf{v}_f \cdot \mathbf{v}] + 2 \int_0^\infty d\xi f(E(\xi) + v_f \cdot \mathbf{v}) \quad (2.12)$$

where $N_f$ is the total density of states at the Fermi Level, $n(s)$ is the density of states at point $s$ at the Fermi surface, normalized to unity, $v_f(s)$ is the $s$-dependent Fermi velocity, $f$ the Fermi function, and $E = \sqrt{\xi^2 + |\Delta(s)|^2}$. 

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In general, the integrals in Eq. (2.12) can be evaluated only numerically, as was done in 2-d in Ref. [4]. However it was shown there that at the temperatures (about two Kelvin) where the experiments are performed, one is very near the zero temperature limit. Therefore we confine ourselves in this work to this limit, where, within the assumptions discussed above, it is possible to derive an analytic expression for \(j(v)\), valid in the limit where \(\delta\) is large. Having an analytic expression for \(j(v)\) makes the subsequent substantial numerical work easier. In a magnetic field (hence \(v \neq 0\)) even at \(T = 0\) there exists a region with \(|\Delta(s)| + v f \cdot v < 0\) where it is possible to have a quasiparticle population. One can then perform, as in Ref. [3] an approximate but accurate integration of the general formula Eq. (2.12), to obtain the nonlinear corrections at zero temperature (details are given in Appendix A). The resulting \(j(v)\) relation is valid under the assumptions for the FS and the OP discussed above, and for any three dimensional strongly anisotropic superconductor (independent of sample geometry). Introducing cartesian axes \(x',y'\) fixed to the crystal along the nodal directions the expression we obtain is:

\[
j_{x',y'} = -e \rho_a v_{x',y'} (1 - \frac{9 \pi}{64 v_c} |v_{x',y'}|) \tag{2.13a}
\]

\[
j_z = -e \rho_c v_z (1 - \frac{3 \pi}{32 v_c} \frac{v_z^2}{v_{x'}^2 + v_{y'}^2}) \tag{2.13b}
\]

where \(v_c \equiv \Delta_0/v_f\). The components of the superfluid density tensor are \(\rho_{ab} = \frac{1}{2} N_j v_f^2\). \(\rho_c = \rho_{ab}/\delta\), in the \(a - b\) plane and along the \(c\) crystallographic axis respectively. Even for YBCO reported measured values for \(\delta = m_c/m_{ab} = (\lambda_e/\lambda_{ab})^2\) are quite large, between 15 and 50. Omitted terms in the above relation are those involving higher powers of \(v/v_c \ll 1\) (for the regime of experimental interest in the Meissner state) or supressed by the small parameter \(\delta^{-1}\).

### C. Geometry

In this subsection we discuss the sample geometry considered in the calculation of the supercurrent response in the Meissner state. The dependence of measured quantities such as the local magnetic field, actual current distribution and magnetic moment on sample shape is one of the issues of particular interest in this work, as explained in the Introduction.

The experimentally relevant regime is that of small but finite ratio of penetration depths to typical sample dimension. As mentioned below Eqn. (2.9), the geometric dependence of the effect under investigation in this regime, in is proportional to these ratios. The coefficient of this finite size term must be accurately calculated. A “locally flat” approximation (assuming a purely exponential decay for the fields away from the surface) is not sufficient for this purpose, as one can check even in the linear case from the exact solution for a sphere. [2] It is therefore necessary to solve the complete boundary value problem, that is, to find the solution to the equations (2.3) and (2.7) outside and inside of the sample. This is in principle a numerically difficult undertaking, since a priori it involves solving a system of partial differential equations with nonlinear terms in the entire three dimensional space, but with currents being confined to a very small region where great precision is required. Although sophisticated variable grid methods could perhaps be found to take care of these complications, the relatively symmetric shape of the required experimental samples allows for a simpler approach.

Single crystals of HTSC materials are typically flat, much thinner in the \(c\)-direction than in the \(a - b\) plane. The magnetic field, we recall, is applied parallel to this plane along the \(x\) direction. The geometry and coordinate system we consider are sketched in Fig. [1]. Because of the complications associated with demagnetization factors, as described in the Introduction, experiments must be performed on single crystals of a highly symmetric shape. This is achieved by laser cutting, or shaving the crystals so that their cross section in the direction perpendicular to the \(z\) axis is circular. Their shape is, therefore, roughly a disk, thinning towards the edges because of mechanical disintegration associated with the cutting. They have smooth edges and can therefore be described as ellipsoids of revolution. This geometry has also a considerable computational advantage: the form of the solution outside the sample in the limit \(\Lambda = 0\) is known exactly since the potential \(\Phi\) (see (2.1)) satisfies trivial Neumann boundary conditions at the interface and can be found by electrostatic analogy. The solution contains a single parameter which is simply related to \(m_0\), the value of the longitudinal magnetic moment in the zero penetration limit. When the penetration depth is finite, the longitudinal moment does change, but its correct value can be determined from the boundary conditions and the solution inside through an iteration process as described in the next Section. But, as important as these simplifications are, they only go so far: the fundamental equation (2.7) is not separable in spheroidal coordinates.

We therefore consider a superconductor in the shape of a flat ellipsoid of revolution (an oblate spheroid) with the axis of revolution along the \(z\)-axis. Its major and minor semiaxes are denoted by \(A\) and \(C\) respectively, and we have
$A > C$ for actual samples. We take (see Fig. 1) a coordinate system fixed to the direction of the magnetic field, with its z-axis parallel to the c crystallographic direction of the superconductor (and parallel to the C semiaxis of an ellipsoid). The field is applied along the x-axis, and we picture the experiment as being performed by rotating the crystal about the z-axis. As the crystal is rotated the axes $x - y$ remain fixed in space, and should not be confused with the previously introduced $x' - y'$ coordinates, affixed to the crystal structure.

Even with the nonseparability of the equations, it is still convenient from the point of view of fulfilling the interface boundary conditions, to introduce oblate spheroidal coordinates $\alpha, \beta, \varphi$. In the definition we use they are related to cartesian coordinates by the transformation:

$$x = f \cosh \alpha \sin \beta \cos \varphi$$  \hspace{1cm} (2.14a)  
$$y = f \cosh \alpha \sin \beta \sin \varphi$$  \hspace{1cm} (2.14b)  
$$z = f \sinh \alpha \cos \beta$$  \hspace{1cm} (2.14c)  

where $0 \leq \alpha < \infty, 0 \leq \beta \leq \pi, 0 \leq \varphi \leq 2\pi$, and $f$ is a focal length scale factor. The surface of an ellipsoid corresponds to a given value of $\alpha$.

For an ellipsoid of revolution (about its z axis) in the presence of uniform magnetic field $H_a$ applied along the direction that we denote as the x axis the magnetic potential in the outside region for $\lambda = 0$ has the form $\Phi = -H_a x (1 + g(\sinh \alpha))$, where the first term is the potential for the uniform applied field $H_a$ and the gradient of the second term is the field generated by the superconducting ellipsoid. Writing out in detail the function $g$ one has:

$$\Phi = -H_a f P_1^1(i \sinh \alpha) P^1_1(\cos \beta) \cos \varphi + A_1 f Q_1^1(i \sinh \alpha) P^1_1(\cos \beta) \cos \varphi$$  \hspace{1cm} (2.15)  

where $P^1_1$ and $Q^1_1$ are associated Legendre functions of the first and second kind respectively. The parameter $A_1$ is determined from the boundary conditions. It is proportional to $m\parallel$, the longitudinal magnetic moment:

$$m\parallel = \frac{2}{3} f^3 A_1$$  \hspace{1cm} (2.16)  

Its value at $\lambda = 0$ is:

$$A_1(0) = \frac{-H_a \sinh \alpha_0}{1 + 1/\cosh^2 \alpha_0 - \sinh \alpha_0 \arctan(1/\sinh \alpha_0)}$$  \hspace{1cm} (2.17)  

where $\alpha = \alpha_0$ is the value at the surface of the ellipsoid. Eqns. (2.17) and (2.16) can be combined to yield the usual expression for $m\parallel$, the magnetic penetration depth, purely longitudinal, magnetic moment, in terms of the ellipsoidal demagnetization factors. At finite $\Lambda$ the value for $A_1$ is obtained from the iterative procedure discussed in the next subsection. In the spherical limit $\sinh \alpha \to \infty$ we also recover the standard result.

### III. RESULTS

#### A. Numerical procedure

Let us discuss now the iterative procedure we use to solve (2.17), while avoiding to have to numerically solve the field equations in all space. For clarity, let us focus first on the case where the nonlinear effects are neglected but the penetration depths are finite. This has two effects on the solution outside: first, the value of $A_1$ (or of $m\parallel$) deviates from the value given by (2.17). Secondly, the field acquires higher order multipoles, i.e. the field potential acquires additional terms:

$$\Phi = \Phi_a + A_1 f Q_1^1(i \sinh \alpha) P^1_1(\cos \beta) \cos \varphi + \sum_{n \geq 1} A_n f Q_n^1(i \sinh \alpha) P_n^1(\cos \beta) \cos \varphi$$  \hspace{1cm} (3.1)  

where $\Phi_a$ is the potential corresponding to the applied field (the first term on the right side of (2.15)) and the gradient of the rest is the field created outside the sample by the current distribution in it. The terms with $n > 1$ vanish at $\Lambda = 0$ (and also in the spherical case if $\Lambda$ is isotropic.) The fields generated by $A_1$ are not purely dipolar, since they have ellipsoidal symmetry. However, the dipole moment is determined by this parameter only. At the
first step of the iteration we solve the linear version of (2.3) for the current field, assuming that the outside field is given by its \( \lambda = 0 \) limit, i.e. the gradient of (2.13) with (2.17). Parameter counting shows that in order to do so we must give up one boundary condition, and accordingly we temporarily sacrifice the continuity of the “radial” field (\( \hat{n} \) component). From the resulting current distribution we compute \( \mathbf{m} \) through (2.3). This is of course not the same as the input moment. We then replace this obtained value in the external potential (through (2.10)) and repeat the procedure until the moment generated by the computed currents equals the input value. The iteration is considerably simplified by observing that several terms in (2.9) vanish explicitly when \( \tilde{\Lambda} = 0 \) and by making use of the fact that the penetration depths are small compared to the sample dimensions. Once the iteration is concluded, one finds that \( H_\alpha \) is continuous except in a very narrow band near the equator corresponding to a symmetry higher than dipolar. This can then be eliminated by adding small values for higher order \( A_{\alpha} \)’s to the expansion (3.1), but symmetry considerations and examination of Eq. (2.9) show that these additions do not affect the already determined value of the sample magnetic moment. It is easy and very instructive to verify analytically that this procedure recovers the known result for the \( \lambda \) dependent magnetic moment of a sphere with an isotropic \( \Lambda \).

The same procedure is used with the nonlinear terms, with only two important differences: first, to the field outside one must add a transverse dipole, i.e. a contribution of the form of the last term in (2.13) rotated 90°:

\[
\Phi_\perp = A_{11} fQ^1_1(i \sinh \alpha)P^1_1(\cos \beta) \sin \varphi
\]

(3.2)

where \( \frac{2}{3} f^3 A_{11} = m_\perp \). Of course this term does not exist when the penetration depths vanish, since the nonlinear effects are absent unless the field can penetrate the sample. In principle one should also consider higher order multipole terms, as in (3.1) but we have found that any such terms are below the precision level of our numerical results. Both components of the moment \( \mathbf{m} \) are determined through the vector relation (2.3). The other important difference is the obvious one of using (2.13) for the \( j(v) \) relation. There are also practical differences, however, since in the linear equations the variable \( \varphi \) can be separated out while in the full nonlinear case all three coordinates are coupled.

In the actual solution of the equations we use a relaxation method. We proceed in two steps: first we solve the linear problem, which involves only two variables, since then the \( \varphi \) dependence of all quantities can be determined analytically. The iterated solution for that problem is then used as the initial guess in the full three-variable nonlinear problem.

We discretize the differential equations expressed in ellipsoidal (oblate spheroidal) coordinates, on a three dimensional grid. An obvious advantage of the ellipsoidal grid is that it simplifies consideration of boundary conditions at the surface of the ellipsoid, given by equation (2.3). Numerically, it is more intricate to consider boundary conditions that involve derivatives, and accuracy is increased if the grid points are also boundary points.

The discretization procedure we have used has an estimated error quadratic in the spacing between the grid points. All quantities involved in equations (2.7) have definite parity with respect to the exchange \( z \to -z \) and it is sufficient to solve them only for one half of the sample. Accuracy is predominantly governed by the spacing between grid points along the \( \hat{\alpha} \) direction. In the actual numerical solution (for half the sample), we consider ellipsoids of different shapes (different \( C/A \)) and sizes (different \( A \)). It is appropriate to increase \( n_\alpha \) proportionally to \( C/A \). Denoting by \( n_\alpha \) the number of grid points along the \( \hat{\alpha} \) direction, the smallest \( n_\alpha \) used was 100, and the largest 800, spaced in the region of nonvanishing currents. The number of grid points along the \( \hat{\beta} \) and \( \hat{\varphi} \) directions was \( n_\beta = 50 \) and \( n_\varphi = 30 \) respectively. Increasing these numbers by a factor of two gave only effects below the numerical accuracy attained, which is about two significant figures, an error much smaller that the uncertainty arising from the imprecise knowledge of the experimental values of the input parameters.

As one of the checks on the accuracy of the algorithm we use, it is instructive to consider the case of an isotropic spherical sample, (both the sample and the Fermi surface are spheres, \( \delta = 1 \)) in the linear regime, where the analytical solution is known. To ensure that we tested the same algorithm, we treated the sphere as the spherical limit of ellipsoidal coordinates, which corresponds to \( \sinh \alpha \to \infty \). We used \( \sinh \alpha = 1000 \), equivalent to eccentricity \( e = \sqrt{1 - (C/A)^2} = 0.001 \) with the grid given by \( n_\alpha = 200 \), \( n_\beta = 50 \) and \( n_\varphi = 30 \) grid points. We solved equations (2.7) in a spherical shell of thickness 10\( \lambda \) and studied both the magnetic moment and the current distributions. The accuracy for the current on any grid point corresponded to four significant figures for the region where currents are important. The magnetic moment calculated both from (2.3) and from the surface integrals (2.3) agrees with the exact result, including the correct finite penetration depth correction.

We have also checked that our results for the longitudinal magnetic moment extrapolated to the zero penetration depth limit agree with the known analytic result for ellipsoids. We have also verified that the magnetic moment calculated from (2.8) agrees from that found from the surface integrals (2.3). The latter procedure is, however, much more convenient.
B. Numerical results and discussion

In performing the calculations and describing the results, it is convenient to introduce dimensionless quantities. Because of the shape of the samples, we use $\lambda_{ab}$ as the unit of length. We then define the dimensionless fields: $\mathbf{V}$, $\mathbf{J}$, and $h$:

$$\mathbf{V} = \frac{\mathbf{v}}{v_{c}}, \quad h = \frac{H}{H_{0}}, \quad \mathbf{J} = \frac{cH_{0}}{4\pi\lambda_{ab}} \mathbf{j},$$

where we have introduced a characteristic magnetic field $H_{0}$ as:

$$H_{0} = \frac{\phi_{0}}{\pi^{2}\lambda_{ab}\xi_{0}}$$

where $\phi_{0}$ is the flux quantum and $\xi_{0} = v_{f}/\pi\Delta_{0}$ is the in-plane superconducting coherence length. The definition of $\xi_{0}$ involves precisely the same numerical factors as that used in Ref. 2. The required equations are easily rewritten in terms of these quantities. The boundary conditions for the velocity field in (2.7) are now:

$$\nabla \times \mathbf{V} = h \mid_{\alpha = \alpha_{0}}$$

where the right hand side is the external dimensionless field at the surface of the ellipsoid and from now on the derivatives are with respect to dimensionless length. The remaining boundary condition that there is no normal component of current at the surface is readily obtained from Eqs. (2.13). The relation between $j$ and $v$ in equations (2.13) then becomes:

$$(\nabla \times \nabla \times \mathbf{V})_{x',y'} = -V_{x',y'}(1 - \frac{9\pi}{64}|V_{x',y'}|)$$

$$(\delta \nabla \times \nabla \times \mathbf{V})_{z} = -V_{z}(1 - \frac{3\pi}{32}V_{x}^{2} + V_{y'}^{2})$$

where we recall $\delta = (\lambda_{c}/\lambda_{ab})^{2} = m_{c}/m_{ab}$. Equations (3.3) are transformed to ellipsoidal coordinates, as defined above. Expressions for the superfluid density tensor and the differential operator $\nabla \times \nabla \times \mathbf{V}$, in ellipsoidal coordinates are included in Appendix B.

In Figures 2 and 3 we show some of the results for the currents. These figures illustrate some of the physics, as well as the quality of the numerics. In Fig. 2 we show the current $j_{z}$ going along the $z$ direction in the $x - y$ plane as a function of distance from the surface of the sample starting at the point with cartesian coordinates $(0, A, 0)$. This current is overwhelmingly determined by the usual linear response to the field. One can clearly see that its decay as a function of depth from the surface is governed by the $\lambda_{c}$ penetration depth, as expected from the geometry. The component $j_{x}$ along the applied field, on the other hand, arises exclusively from nonlinear effects: symmetry considerations show that it vanishes in the linear limit. Its overall scale is down by a large factor (basically the ratio of longitudinal and transverse moments). One sees that even though the overall decay of $j_{x}$ is determined by the scale $\lambda_{ab}$, its behavior is very far from exponential: it changes sign as a function of depth. This can be readily understood: at the positions plotted, close to the center of the the top of the ellipsoid, $j_{x}$ is approximately proportional (from Ampère’s law) to the derivative with respect to $z$ of the anomalous component of the field, $H_{y}$. This field component nearly vanishes at the surface (it would vanish for a slab) and decreases exponentially at depths larger than $\lambda_{ab}$. Hence, at this position in the sample, $H_{y}$ has an extremum as a function of depth, and its derivative with respect to $z$ must change sign at some point, as we find. This shows the delicate intricacy of the nonlinear current patterns inside the material.

Before proceeding with the detailed discussion of the dependence of our results on the relevant physical quantitites (i.e. size, shape, applied field, and penetration depths) we illustrate their general scope by describing our prediction for the transverse magnetic moment of a possible HTSC superconducting sample, assumed to be in a pure d-wave pairing state. This is done in Fig. 4. We show there results for $m_{\perp}$ as a function of applied field. The quantity shown is the maximum value of $m_{\perp}$ as the crystal is rotated. It is assumed that the sample is an ellipsoid with $A = 2$ mm and $C = 0.1$ mm. Material parameters are taken to be $\xi_{0} = 20 \AA$, $\lambda_{ab} = 1800 \AA$, and $\lambda_{c} = 9000 \AA$. The characteristic field $H_{0}$ would be about 5800 gauss. One can see that the magnetic moments involved are readily accessible to
measurement. Predictions for samples of other sizes, shapes and material parameters can be conveniently extracted from the information given below.

We have obtained results for the transverse magnetic moment for a wide range of the experimentally accessible values of the appropriate dimensionless parameters, which as we shall see, can be taken to be $H_a/H_0$, $\lambda_c/\lambda_{ab}$, $\lambda_{ab}/A$, and $C/A$, the aspect ratio of the ellipsoid of revolution. We did not consider in our study the “thin film” situation where the sample is so small or so thin that its relevant dimensions are comparable to or smaller than the corresponding penetration depth. This case would be of no interest since the nonlinear effects then are vanishingly small, and it is excluded from the analysis that follows.

We begin our general discussion of the results by performing some dimensional analysis. The quantity $4\pi m_\perp$ has dimensions of magnetic field times volume. The expression

$$Q = \frac{4\pi m_\perp}{H_0 V}$$  \hspace{1cm} (3.7)

where $V$ is the sample volume, is therefore dimensionless. Even though $Q$ is suitable for some purposes, it is more convenient to analyze the dependence of the results on sample size in a different way. The reason is that $4\pi m_\perp$ does not scale with the sample volume but as its surface area. The coefficient of proportionality between transverse moment and area depends on the sample shape and one can express this dependence through the aspect ratio $A/C$. Since the area of an ellipsoid of revolution is $\pi A^2$ times a function of $A/C$, it is easier for the purpose of giving results in a form more accessible to experimentalists, to scale explicitly results for ellipsoids of the same aspect ratio (i.e. the same shape) by a factor of the “disk” area $S \equiv \pi A^2$. The third length that goes into the volume factor in the units of $m_\perp$ is a penetration depth. Since we are dealing with rather large $A/C$ ratios, with currents predominantly in the $a-b$ plane, it is advantageous for our purposes to take this length into account by writing out a factor of $\lambda_{ab}$ explicitly. Thus we put, as a first step:

$$4\pi m_\perp(\psi) = \gamma(\frac{A}{C}, \frac{\lambda_c}{\lambda_{ab}}, H)S\lambda_{ab}f(\psi)$$  \hspace{1cm} (3.8)

Here and hereafter we denote the magnitude of the applied field simply by $H$, rather than $H_a$, as there is no longer a possibility of confusion. It is not surprising in view of the above dimensional analysis that we find $\gamma$ to be a rather weak function of its first two arguments and independent of $S$ and of $\lambda_{ab}/A$. The angular dependence of $m_\perp$ is given by the function $f(\psi)$ in terms of the angle $\psi$ between the applied field and a node. We normalize $f(\psi)$ so that its value is unity at its maximum.

Our results for the angular dependence are shown in Figure 3. The points shown are the values obtained from our numerical calculation. The error bars indicate the numerical uncertainty. The solid line represents the analytic result for the two dimensional calculations in the slab case, normalized in the same way. We see from the Figure that the shape of $f(\psi)$ is, within numerical uncertainty, the same as for the flat case, where, with the same normalization, $f(\psi) = 3\sqrt{3}\sin\psi\cos\psi(\cos\psi - \sin\psi)$, (for $0 < \psi < \pi/2$). This is important, because the $\pi/2$ Fourier coefficient of $f(\psi)$, as analytically calculated from the above expression, is unity to three significant figures. We can therefore identify here the coefficient of $f(\psi)$ in (3.8) with the Fourier amplitude $m_\perp$ as introduced earlier in the paper. The results shown in this Figure were obtained at $H/H_0 = 0.1$, $\delta = 16$, and $A/C = 19$, but similar results are obtained in all other cases studied.

As can already be seen in Figure 3, the coefficient $\gamma$ depends strongly on its third argument, the applied field. We have studied the field range $0 < H/H_0 < 0.2$ at 0.05 intervals. We expect that the nonlinear $m_\perp$ is proportional to $H^2$, as in the slab case at zero temperature. We therefore conclude that a very convenient way of writing our results for the amplitude is:

$$4\pi m_\perp = M(\frac{A}{C}, \frac{\lambda_c}{\lambda_{ab}}, \frac{H}{H_0})H\lambda_{ab}Sf(\psi).$$  \hspace{1cm} (3.9)

Equation (3.9) implies explicitly that $m_\perp/H$ is a function of field only through the ratio $H/H_0$. We have verified that, as expected, $M$ is independent of the field. This can be seen in Fig. 4, where we plot the quantity $G \equiv 10^44\pi m_\perp/(H\lambda_{ab}S)$ vs $H$. We see that our results, represented by the symbols with error bars, are on a straight line, which is indicated by a best fit. Except for the field, the parameter values are the same as in the previous Figure. As in the case of the angular dependence, this qualitative result holds in all cases studied. The slope of the best fit straight line in plots such as that in this figure is used to extract $M$.

The geometric aspect ratios we have considered in the nonlinear case range from $A/C = 7$ up to 19 (that is from $\sin\alpha = 0.14433$ to 0.05270). Comparison of the results with largest eccentricity to previous work on the slab case illustrates the effect of the return currents. This comparison cannot be made by taking $C \to 0$ because we must have
\( C \gg \lambda_{ab} \). The central portion of a flat ellipsoid can be identified as a “slab”. It is simpler to make the comparison in terms of \( Q \) defined in (3.7). It follows from our dimensional analysis that:

\[
Q = \frac{\tilde{\gamma} H}{H_0} \frac{\lambda}{2C} f(\psi)
\]

(3.10)

The corresponding result for the slab is of the same form, and the coefficient may be extracted from the \( T = 0 \) results of Ref. [5]. When all the relevant factors, such as the different definition of \( H_0 \), the normalization of \( f(\psi) \), and the setting of the parameter \( \mu \) (slope of the OP near a node) to \( \mu = 2 \) are taken into account, one finds that the slab value is \( \tilde{\gamma} = 0.056 \). We obtain \( \tilde{\gamma} = 0.092 \) for our flattest ellipsoid at \( \delta = 16 \), a number only weakly dependent on \( \delta \). Hence, the presence of return currents enhances the nonlinear effects but, as hinted by the recent reanalysis of the experimental data, the enhancement is about three times smaller, in typical situations, than the \( \delta \) dependent factor postulated in Ref. [8].

We next show the dependence of our results on the remaining parameters, \( A/C \) and \( \lambda_c/\lambda_{ab} \). As explained above, our results are most useful to experimentalists if given in terms of the scaled dimensionless amplitude \( M \). We therefore summarize the dependence of this quantity on the mentioned length ratios in Table I. This table can be used, in conjunction with Eq. (2.9), to compute the theoretical predictions, when planning experiments or comparing data and theory. The rows are the relevant material parameter, given as the square of the penetration depth ratio, and the columns are the crystal shape. For crystals that are not quite ellipsoidal, one should choose a \( C/A \) ratio in the table so that the surface to volume ratio of the ellipsoid agrees with that of the crystal. As the results vary slowly across the rows and columns of the Table, interpolation and reasonable extrapolation are obviously feasible. The value of \( \chi \) in the Table must be multiplied by the square of the applied field and by the cross sectional area of the sample in the \( a-b \) plane, then divided by \( H_0 \) (which is again a material-dependent parameter), to obtain the expected value of the magnetic moment amplitude to be observed. The Table covers anisotropy parameter \( \delta \) values up to 50.

The dependence of \( M \) on its arguments is weak, less important than the unavoidable uncertainty in the values of the experimental input parameters such as those that go into e.g. the determination of \( H_0 \). The slow dependence of \( M \) on \( \delta \) can be roughly understood as follows: the portion of the current loop which is along the \( c \) direction does of course contribute to the magnetic moment. However, provided, as in the Table, that \( A \gg \lambda_c > \lambda_{ab} \) the exact value of \( \lambda_c \) does not matter very much, since (as seen in Fig. 3) the decay of the nonlinear current is governed primarily by \( \lambda_{ab} \). The nonlinear moment is then limited by the smaller of the penetration depths. The slight downtrend with \( \delta \) may be due to changes in the complicated current patterns discussed in connection with Fig. 3. The slight decrease with decreasing eccentricity is likely to be due to the influence of the \( j_z \) component (dominated by linear effects) in a thicker crystal. The Table indicates that flatter crystals are actually preferable to harder to obtain thick ones. Again, however, a simple explanation in intuitive terms is hard to come by, since these results must be related to the intricacies of the nonlinear current patterns.

Finally we consider the sensitivity of our results to the unknown dependence of the order parameter on the crystallographic \( c \) direction. In Appendix A, we have calculated the coefficients of the nonlinear terms in the current (as in Eq. (2.13)) for the simplest form of the appropriate 3d OP. We have verified that an increase occurs in the quantity \( M \) but it is negated by a compensating increase in \( H_0 \), where then one should replace \( \xi_0 \) by some average of in and out of plane correlation lengths. We conclude that the influence of any such dependence can be neglected as compared with the uncertainty in the material parameters.

### IV. CONCLUSIONS

The detailed calculations for realistic samples performed here strongly confirm that the transverse magnetization effect should be readily observed in available crystals if the order parameter has nodes of the form expected for d-wave OP symmetry. From our results, see e.g. Fig. 4, it follows that the transverse magnetic moment generated in a typical sample of about 2 mm diameter and 0.1 mm thickness (the precise thickness not being very important) should be \( 4\pi m_T \approx 10^{-8} \) gauss cm\(^3\). These numbers are above what can be readily measured by standard experimental techniques: the rather high uncertainty levels quoted in past experiments arise from the geometrical problems described in the Introduction and also from easy to overcome uncertainties in the angular positioning the sample. We have seen that the return currents in the \( z \)-direction produce an enhancement of the effect, although not nearly as large as that estimated in Ref. [5]. The effect of the boundary conditions and the finite size on the result is somewhat intricate and, perhaps not surprisingly, we have not been able to find a clear physical picture in terms of some nonlinear generalization of the demagnetization factors. Although our results for the influence of sample geometry and anisotropy in penetration depth were not amenable to simple summing up either, the tabulated values of the quantity \( M \) should facilitate experimental design and interpretation.
As briefly indicated in Appendix A, (see below Eq. (A1)) it is easy at \( T = 0 \) to take into account different shapes of the d-wave order parameter function close to the nodes. Our calculations can also be extended to other symmetries of the OP, such as an \( s + d \) or \( s + id \) state, or to take into account dependence of the OP in the \( c \)-direction. In the worst case, even at finite temperatures, the functional \( j(\mathbf{v}) \) could be, numerically evaluated and used in the calculations, although possibly at considerable cost in computer time, by means of a lookup table and interpolation scheme.

We have not considered in this work the effect of impurities. This is unnecessary as it was shown in Ref. 4 that the impurity concentration required to noticeably decrease the nonlinear Meissner effect at low temperatures is such that it would clearly reduce the transition temperature of the material. This result will not depend on the geometry of the sample. On the other hand, we have considered here only the case where the order parameter has nodes, not just dips. The nonlinear effects are in fact extremely sensible to the presence of nodes and \( m_{\perp} \) will decrease substantially, in finite as in infinite samples\(^3\) if an s-wave component eliminates the nodes. The magnitude of this decrease can be gauged from the infinite slab work\(^4\) since, again, it should not be excessively sensitive to return currents or sample shape. Therefore, a negative experimental result could be used to put a lower bound on the amount of s-wave component present.

The methods discussed here can be extended to other sample shapes, to finite temperatures, and to the computation of other measurable effects that arise from the same nonlinear phenomena.

ACKNOWLEDGMENTS

We thank A. Bhattacharya, A.M. Goldman, J. Buan and D. Grupp for many enlightening conversations concerning the experimental implications of our work, and B.P. Stojković for reading a draft of this work. We also thank B. Bayman and J. Sauls for discussions. I. Z. acknowledges support from the Foster Wheeler and Stanwood Johnston Memorial Fellowships.

APPENDIX A: CURRENTS

We derive here the expression for \( j(\mathbf{v}) \) at \( T = 0 \) starting from equation (2.12). Our procedure is a generalization, valid in the case of interest where the anisotropy \( \delta \) is large, of the two dimensional derivation\(^5\). We consider an ellipsoidal Fermi surface:

\[
\epsilon_f = \frac{k_x^2 + k_y^2}{2m_{ab}} + \frac{k_z^2}{2m_c} = \frac{1}{2} m_{ab} v_f^2
\]

(A1)

where we introduce \( v_f \) as in Section II. To establish our notation we briefly review the first, linear term in (2.12):

\[
 j = -eN_f \int d^2 s \, n(s) v_f (v_f \cdot \mathbf{v})
\]

(A2)

which is due to the condensate contribution to the current. For its evaluation one uses the convenient standard textbook method of rescaling \( k_z \) by a factor of \( \delta^{-1/2} \). The components of \( v_f \) are then given by:

\[
(v_f)_x' = v_f \sin \theta \cos \phi \quad (A3a)
\]

\[
(v_f)_y' = v_f \sin \theta \sin \phi \quad (A3b)
\]

\[
(v_f)_z' = \frac{v_f}{\sqrt{\delta}} \cos \theta \quad (A3c)
\]

where \((\theta, \phi)\), \((\phi \) measured from the \( x' \) axis) are the spherical angles of a vector with components \( k_x, k_y, k_z/\delta^{1/2} \). The scalar product in (A2) can then be written as:

\[
 v_f \cdot \mathbf{v} = v_f v_{x'} \sin \theta \cos \phi + v_{y'} \sin \theta \sin \phi + \frac{v_{z'}}{\sqrt{\delta}} \cos \theta \quad (A4)
\]

The integration can be performed by replacing \( \int d^2 s n(s) \) by \( \int_{\Omega} d\phi \, d\theta \, \sin \theta / (4\pi) \) which yields
\[ j_{x',y'} = -e \rho_{ab} v_{x',y'} \]  
\[ j_z = -e \rho_c v_z \]  

With \( \rho_{ab} = \frac{1}{4} N_f v_f^2 \), \( \rho_c = \rho_{ab}/\delta \), as quoted in Section II, (we recall that \( N_f \) is the total density of states, for both spins). Thus, as is well known, the effect of the ellipsoidal Fermi surface is to rescale \( \rho_c \) by a factor of \( 1/\delta \) relative to \( \rho_{ab} \).

The nonlinear corrections are contained in the second term of (2.12), which is due to the quasiparticle backflow. We denote this term by \( j_{qp} \). At \( T=0 \), one can perform the \( \xi \) integration:

\[ j_{qp} = -2eN_f \int_{\Omega} d\phi d\theta \sin \theta/(4\pi) \mathbf{v}_f \Theta(-\mathbf{v}_f \cdot \mathbf{v} - |\Delta(\phi)|) \sqrt{(\mathbf{v}_f \cdot \mathbf{v})^2 - |\Delta(\phi)|^2} \]  

where \( \Delta(\phi) \) is given by (2.11). In the Meissner regime, \( |\mathbf{v}_f \mathbf{v}| \ll \Delta_0 \) and the contribution to the quasiparticles arises from narrow wedges along the nodal regions, approximately described by the azimuthal angle \( \phi \lesssim (v/v_c) \ll 1 \). A superfluid velocity \( \mathbf{v} \) will give rise to backflow currents because of its components along nodal regions, separated by an azimuthal angle \( \pi/2 \). In two dimensions, \( \mathbf{v} \) can be uniquely decomposed into two "jets" \( v_1 \hat{x}' \) and \( v_2 \hat{y}' \) along two nodal directions. In the present three-dimensional case, where the anisotropy \( \delta \) is large, one can proceed in a similar way. We decompose \( \mathbf{v} \) into the sum of two jets each directed along a nodal direction and tilted by the same angle \( \omega \) with respect to the \( z \)-axis. Thus we write \( \mathbf{v} = v_x \hat{x}' + v_y \hat{y}' + v_z \hat{z} = v_1 + v_2 \). with:

\[ v_1 = v_x \hat{x}' + \cot \omega |v_x| \hat{z} \]  
\[ v_2 = v_y \hat{y}' + \cot \omega |v_y| \hat{z} \]  
\[ \cot \omega = \frac{v_z}{|v_x'| + |v_y'|} \]  

The decomposition thus specified is unique, and it ensures that the component \( v_z \) is distributed along \( x'-z \) and \( y'-z \) planes proportionally to the corresponding projections of \( \mathbf{v} \) along the \( \hat{x}' \) and \( \hat{y}' \) directions. It is easy to see however, that any other decomposition of \( v_z \) would lead to the same results for the nonlinear currents derived below, except for higher order corrections in \( \delta^{-1} \) which shall be neglected in any case.

The total phase space contributing to the quasiparticle part of the current can be obtained by considering the effects of \( v_1 \) and \( v_2 \) separately. Let us consider the effect of \( v_1 \) in (A7). Quasiparticle excitations are allowed in the region described by \( \phi \leq \phi_c \), where \( (\mathbf{v}_f \cdot v_1)^2 = (\Delta_0 \sin \phi_c)^2 \) and

\[ \phi_c^2 = \frac{\mathbf{v}_f^2 v_x^2}{4\Delta_0^2} (\sin^2 \theta + \frac{2}{\sqrt{\delta}} \sin \theta \cos \theta \cot \omega \frac{|v_x|}{v_x'} + \frac{1}{\delta} \cos^2 \theta \cot^2 \omega) \]  

where we have approximated the order parameter in the nodal region by \( \Delta(\phi) \approx \Delta_0 2\phi \). It would be easy to write instead \( \Delta(\phi) \approx \Delta_0 \mu \phi \), with \( \mu \) being a free parameter representing the slope of the OP function near the node, as was done in Ref. [15]. This can be viewed as modifying the characteristic field \( H_0 \) (Eq. [1.4]) by a factor of \( \mu/2 \).

The integrals involved in the calculation of the nonlinear contribution to the relation \( j(\mathbf{v}) \) due to the \( v_1 \) are then of the form

\[ I_{x'} = 2\Delta_0 \int_0^\pi d\theta \sin^2 \theta \int_{-\phi_c}^{\phi_c} d\phi \ v_f \sqrt{\phi_c^2 - \phi^2} \]  
\[ I_{z1} = 2\Delta_0 \int_0^\pi d\theta \sin \theta \cos \theta \frac{1}{\sqrt{\delta}} \int_{-\phi_c}^{\phi_c} d\phi \ v_f \sqrt{\phi_c^2 - \phi^2} \]  

After integration over the angles \( (\theta, \phi) \) we get:

\[ I_{x'} = \frac{3\pi^2}{32} \mathbf{v}_f \mathbf{v}_c (1 + \frac{\cot^2 \omega}{3\delta}) \]  

\[ I_{z1} = \frac{3\pi^2}{32} \mathbf{v}_f \mathbf{v}_c (1 + \frac{\cot^2 \omega}{3\delta}) \]
\[ I_{z1} = \frac{\pi^2}{16} \frac{1}{\delta} \frac{v_c}{v_r} (v_r^2, \cot \omega) \]  

(A15)

The contribution to the quasiparticle current due to \( v_2 \), is calculated in a precisely similar manner and it gives analogous expressions for \( I_{y'} \) and \( I_{z2} \). Combining the contributions due to \( v_1, v_2 \) and omitting, as indicated above, subleading terms of order \( 1/\delta \) in Eq. (A14) we get:

\[ j_{qp \; x', y'} = eN_f v_f^2 \frac{3\pi}{64} \frac{|v_{x', y'}|}{v_c} \]  

(A16)

\[ j_{qp \; z} = eN_f v_f^2 \frac{\pi}{32} \frac{1}{\delta} \frac{v_z^2}{v_c} (v_r^2, \cot \omega + v_y^2 \cot \omega) \]  

(A17)

In the above expression we substitute \( \cot \omega \) given in (A10) and recover the nonlinear part of Eq. (2.43). If, as an alternative, we consider an OP function of the naive 3-d form \( I \propto (k_x^2 - k_y^2) \):

\[ \Delta(\theta, \phi) = \Delta_1 \sin^2(\theta) \sin(2\phi) \]  

(A18)

then all integrals can still be done and one obtains instead:

\[ j_{qp \; x', y'} = eN_f v_f^2 \frac{\pi}{16} \frac{|v_{x', y'}|}{v_c} \]  

(A19)

\[ j_{qp \; z} = eN_f v_f^2 \frac{\pi}{8} \frac{1}{\delta} \frac{v_z^2}{v_c} (v_r^2, \cot \omega + v_y^2 \cot \omega) \]  

(A20)

where \( v_c \) is now defined in terms of \( \Delta_1 \). The important nonlinear coefficient in (A19) is a factor of \( 4/3 \) larger than that in (A16). The results for \( M \) increase by the same factor. However, in Eq. (3.4) the quantity \( \xi_0 \) should be replaced by some average such as \( (\xi_{0a}^2 \xi_{0c})^{1/2} \) which is smaller by about the same amount.

**APPENDIX B: QUANTITIES IN SPHEROIDAL COORDINATES**

The superfluid density tensor, given in the obvious cartesian coordinates by a diagonal tensor with components \( \rho_{ab}, \rho_{ab}, \rho_{cc} \), is converted to oblate spheroidal coordinates by performing the appropriate transformation. One obtains:

\[
\tilde{\rho} = \rho_{ab} \begin{pmatrix}
\sinh^2 \alpha \sin^2 \beta + \delta^{-1} \cosh \alpha \cos^2 \beta & (1-\delta^{-1}) \sinh \alpha \cosh \beta \sin \cos \beta & 0 \\
\sinh^2 \alpha + \cos^2 \beta & -\delta^{-1} \sinh^2 \alpha \sin \beta + \cos^2 \beta & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

To solve equations (2.7) the expression \( \nabla \times \nabla \times \mathbf{v} \) should be transformed to oblate spheroidal coordinates:

\[ f^2(\nabla \times \nabla \times \mathbf{v})_a = a_0 \partial_{\beta \beta} v_a + a_1 \partial_{\beta} v_a + a_2 \partial_{\phi \phi} v_a + a_3 v_a + a_4 \partial_{\alpha \beta} v_\beta + a_5 \partial_{\alpha} v_\beta + a_6 \partial_{\beta} v_\beta + a_7 v_\beta + a_8 \partial_{\phi} v_\phi + a_9 \partial_{\phi} v_\phi \]  

(B1)

\[ f^2(\nabla \times \nabla \times \mathbf{v})_\beta = b_0 \partial_{\alpha \beta} v_\alpha + b_1 \partial_{\alpha} v_\alpha + b_2 \partial_{\beta} v_\alpha + b_3 v_\alpha + b_4 \partial_{\beta} v_\beta + b_5 \partial_{\alpha} v_\beta + b_6 \partial_{\phi} v_\beta + b_7 v_\beta + b_8 \partial_{\phi} v_\phi + b_9 \partial_{\phi} v_\phi \]  

(B2)

\[ f^2(\nabla \times \nabla \times \mathbf{v})_\phi = p_0 \partial_{\alpha \phi} v_\alpha + p_1 \partial_{\phi} v_\alpha + p_2 \partial_{\beta \phi} v_\beta + p_3 \partial_{\phi} v_\beta + p_4 \partial_{\alpha \phi} v_\phi + p_5 \partial_{\beta \phi} v_\phi + p_6 \partial_{\alpha} v_\phi + p_7 \partial_{\phi} v_\phi + p_8 \partial_{\phi} v_\phi \]  

(B3)

we recall that \( f \) is a focal length scale factor. The coefficients \( a_i, b_i, p_i \) are given by (using the abbreviations \( t \equiv \sinh \alpha, u \equiv \cosh \alpha, s \equiv \sin \beta, c \equiv \cos \beta, w \equiv (t^2 + c^2)^{1/2} \)):
\[ a_0 = -a_4 = b_0 = b_4 = p_4 = p_5 = -\frac{1}{u^2} \]
\[ a_1 = -\frac{w^2}{u^2} a_5 = p_7 = \frac{w^2}{u^2} p_3 = -\frac{c}{u^2} \]
\[ a_2 = b_6 = -p_8 = -\frac{1}{u^2} \]
\[ a_3 = \frac{2u^2 - t^2 + 3t^2}{u^6} \]
\[ a_6 = \frac{1}{u^6} \]
\[ a_7 = \frac{tu(3t^2 - 2c^2)}{u^6} \]
\[ a_8 = b_8 = p_0 = p_2 = \frac{w}{u} a_9 = \frac{1}{c} b_9 = \frac{1}{u w s} \]
\[ b_1 = -\frac{a_6}{u^6} \]
\[ b_2 = \frac{w^2}{u^2} b_1 = -\frac{1}{u^6} \]
\[ b_3 = \frac{tu(3+2t^2 - c^2)}{u^2} \]
\[ b_5 = p_6 = \frac{2t}{u^4} \]
\[ b_7 = \frac{2t^2 - c^2 - 3t^2}{u^6} \]

The right hand side of (2.7) has to be transformed to spheroidal co-ordinates and added to the expressions above.

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28. One could use the eccentricity instead, but \(C/A\) is more immediately accessible.
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FIG. 1. Geometry considered in this paper. The sample is represented by an oblate ellipsoid of revolution. The left half of this figure shows the top view, and the right half a side view. The x, y and z directions are fixed in space. The field is applied along the x axis, as indicated, while m⊥ is along the y axis. The sample is rotated about the z direction. The long and short semiaxes values are called A and C in the text, respectively.

FIG. 2. The cartesian component of the current, jz, as a function of distance from the surface, beginning a the point with cartesian coordinates x = 0, y = A, z = 0. The quantity plotted is the current normalized to its value at the surface. D is distance in units of λab: D ≡ (A − y)/λab. This current is dominated by linear effects in the field. One can see that its decay is governed by λc, which is λc = 4λab for the example plotted here, where we have used H/H0 = 0.1 and A/C = 7.

FIG. 3. The linear and nonlinear components of the current as a function of distance from the surface beginning at the point x = y = 0, z = C. The dotted line is jy normalized to its value at the surface. It is predominantly linear and its decay is governed by λab. The solid line represents the nonlinear component jx along the field direction, also normalized to its own, much smaller, surface value. Its nontrivial behavior is discussed in the text. D ≡ (C − z)/λab and all other parameters are as in the previous Figure.

FIG. 4. Predictions of our theory for a typical HTSC single crystal sample. The size, shape, and material parameters are indicated in the text. The quantity plotted is the maximum value of 4πm⊥, (in units of 10^{-6} gauss cm^3) as the sample is rotated, as a function of applied field in gauss.

FIG. 5. The angular dependence of the transverse magnetization, normalized to unity at its maximum. The points are our numerical results at several angles. The solid line represents the analytic result for the angular dependence in the purely two-dimensional case, normalized in the same way. Within error bars the angular dependence is the same, although the amplitude changes.

FIG. 6. Dependence of the transverse moment amplitude on the applied field. The quantity plotted vs dimensionless field H/H0, is G ≡ 10^{2}4πm⊥/(HλabS) (see Eq. (3.9).) The symbols are our numerical results and the dashed line the best linear fit. The linear dependence of G on H means that M is independent of field, and hence that m⊥ is quadratic in H.

TABLE I. The dimensionless quantity M, defined by Eq. (3.4), as a function of the material parameter δ and of sample shape.

| δ = (λc/λab)^2 | A/C=19 | A/C=10 | A/C=7 |
|---------------|-------|-------|-------|
| 16            | 0.061 | 0.060 | 0.059 |
| 25            | 0.058 | 0.057 | 0.056 |
| 36            | 0.057 | 0.054 | 0.052 |
| 50            | 0.055 | 0.051 | 0.049 |