The continuous Anderson hamiltonian in dimension two

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November 26, 2015

Abstract

We define the Anderson hamiltonian on the two dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$. This operator is formally defined as $H := -\Delta + \xi$ where $\Delta$ is the Laplacian operator and where $\xi$ belongs to a general class of singular potential which includes the Gaussian white noise distribution. We use the notion of paracontrolled distribution as introduced by Gubinelli, Imkeller and Perkowski in [14]. We are able to define the Schrödinger operator $H$ as an unbounded self-adjoint operator on $L^2(\mathbb{T}^2)$ and we prove that its real spectrum is discrete with no accumulation points for a general class of singular potential $\xi$. We also establish that the spectrum is a continuous function of a sort of enhancement $\Xi(\xi)$ of the potential $\xi$. As an application, we prove that a correctly renormalized smooth approximations $H_\varepsilon := -\Delta + \xi_\varepsilon + c_\varepsilon$ (where $\xi_\varepsilon$ is a smooth mollification of the Gaussian white noise $\xi$ and $c_\varepsilon$ an explicit diverging renormalization constant) converge in the sense of the resolvent towards the singular operator $H$. In the case of a Gaussian white noise $\xi$, we obtain exponential tail bounds for the minimal eigenvalue (sometimes called ground state) of the operator $H$ as well as its order of magnitude $\log L$ when the operator is considered on a large box $\mathbb{T}_L := \mathbb{R}^2/(L\mathbb{Z}^2)$ with $L \to \infty$.

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1 Introduction and main results

The aim of this paper is to define and study the spectral statistics of the so called Anderson hamiltonian which is a random linear operator on the torus $T^d := \mathbb{R}^d / \mathbb{Z}^d$, formally defined as

$$\mathcal{H} = -\Delta + \xi$$

where $\Delta$ is the Laplacian operator with periodic boundary conditions and $\xi$ a real white noise distribution on $T^d$, i.e. a centered Gaussian random field with covariance function given by

$$\mathbb{E}[\xi(x)\xi(y)] = \delta(x - y)$$

where $\delta$ is the Dirac delta distribution. We are interested in the two dimensional case $d = 2$ for which the operator $\mathcal{H}$ is ill-defined, as we shall see. The Anderson hamiltonian has been defined for $d = 1$ in [10] and we briefly review the main spectral properties in this case (see section 2) to be compared with the properties we establish for the two dimensional case (see below).

The Parabolic Anderson model (PAM) is at the heart of an active research area both in mathematics and theoretical physics. This model refers to the (linear) Cauchy problem

$$\partial_t u(t, x) - \Delta u(t, x) = u(t, x)\xi(x), \quad u(0, x) = u_0(x) \quad (2)$$

for $x \in \mathbb{T}^2$ and where the function $u_0 \in L^2(\mathbb{T}^2)$. The PAM has connections with questions on random motions in random potential, directed polymers, trapping of random paths, branching processes in random medium, Anderson localization, etc. Of course, the solution $u$ to the Parabolic Anderson equation may be written for $x \in \mathbb{T}^2$ as a function of the operator $\mathcal{H}$ as

$$u(t, x) = \exp(-t\mathcal{H})u_0(x) := \sum_{n=0}^{+\infty} \exp(-t\Lambda_n)\langle e_n, u_0 \rangle_{L^2(\mathbb{T}^2)} e_n(x) , \quad (3)$$

provided one is able to define the operator $\mathcal{H}$ and prove that it has a discrete real spectrum $(\Lambda_n) \subset \mathbb{R}^N$ and associated orthonormal eigenvectors $(e_n)_{n \in \mathbb{N}}$ in $L^2(\mathbb{T}^2)$. This fact is not trivial at all in dimension two where the operator $\mathcal{H}$ is ill-defined due to the irregularity of the white noise distribution $\xi$.

The equation (2) and the associated operator $\mathcal{H}$ have been investigated in many papers before for general dimension $d$ in a discrete setting. In this case, the Laplacian is discrete on a grid (for example $\mathbb{Z}^2$) with a fixed mesh size and the white noise $\xi$ is a sequence of independent and identically distributed (i.i.d.) random variables indexed by $\mathbb{Z}^2$. The operator $\mathcal{H}$ is first defined on a finite box of $\mathbb{Z}^d$ with some boundary conditions (either periodic, Dirichlet or Neumann). The main challenge is then to consider the case of a large volume and establish the limiting properties of the model when the box size tends to infinity. In our continuous setting, a similar situation holds. We work in a finite volume where the Anderson hamiltonian is restricted to the two dimensional torus (our results may easily be extended to other boundary conditions such as Dirichlet or Neumann). Our main results provide the construction of the Anderson hamiltonian and we also establish that this operator displays a discrete real spectrum with an orthonormal family of eigenvectors in $L^2(\mathbb{T}^2)$. We are also able to give partial results on the limiting statistics of its ground state (i.e. the minimal eigenvalue) of the Anderson hamiltonian in the limit of a large volume, considering the growing family of torus $\mathbb{T}^d_L := \mathbb{R}^2/(L^{-1}\mathbb{Z}^2)$ for $L \to +\infty$.

One should not confuse our continuous setting with a finite volume with the infinite volume case of the discrete setting. In particular, an important conjecture is that the limiting spectrum (when the volume tends to infinity) in the discrete setting contains only isolated (pure) points in $\mathbb{R}$ (counting multiplicity) associated to localized eigenvectors. Equivalently, the solution of the parabolic Anderson equation (2) on $\mathbb{R}^2$ is expected to be intermittent i.e. with a support that is localized on a few isolated islands that are far apart from each other,
for large time (see [12, 13, 19, 20] or the forthcoming book of W. König [22] for a state of the art review on the discrete setting). In our continuous setting (the mesh size is 0), we do prove that the spectrum is discrete when the phase space has a finite volume but this situation does not correspond to the previous one (discrete setting with infinite volume for the phase space). Regarding the limit of infinite volume in our continuous setting, we conjecture that the spectrum becomes continuous (with a limiting density) when the volume tends to infinity, in contrast with the conjecture made in the discrete setting for the infinite volume with a finite mesh size.

As mentioned before, the main difficulty lies in handling the singularity of the white noise distribution $\xi$. If $\xi_\varepsilon$ is a smooth function, the definition of the operator $H_\varepsilon := -\Delta + \xi_\varepsilon$ is elementary and it is classical to prove using the spectral theory for operators with compact resolvent that it is a self-adjoint operator with a discrete spectrum $(\Lambda_\varepsilon^n)_{n \in \mathbb{N}}$ and orthonormal eigenfunctions $(\varepsilon^n_i)_{n \in \mathbb{N}}$.

With a rough potential $\xi$ such as the two-dimensional white noise, the situation is much more delicate as one has to make sense of $Hf := -\Delta f + \xi f$ for a sufficiently large class of functions $f$ to contain the eigenvectors of the operator $H$. The eigenfunctions of $H$ ought to have Hölder regularity $1^-$ (they barely fail to be differentiable) and for such functions, the product $\xi f$ is not well defined. This is the classical problem which motivated Itô’s theory of stochastic integrals. Powerful tools to make sense of such products have recently been provided by paracontrolled distributions introduced in [14] by Gubinelli, Imkeller and Perkowsky. Our approach relies on their results. The theory of regularity structures developed by M. Hairer in [15] in 2013 could also have been used for our purpose.

Our construction of the operator involves two steps: in the first one, which is purely analytic, we construct the Schrödinger operator $H$ for a general class of rough potentials $\xi$ living in a space of Hölder distributions. The important point in this deterministic construction is that the knowledge of the distribution $\xi$ is actually not sufficient to define the operator $H$ (as an unbounded operator of $L^2(\mathbb{T}^2)$). We will in fact need another piece of information $\Xi_2$ which is (roughly speaking) the ill-defined part of the product $\xi(1 - \Delta)^{-1}\xi$ (this is explained in more details below). The operator $H$ is then defined on an explicit domain $\mathcal{D}_\Xi \subset L^2(\mathbb{T}^2, \mathbb{R})$ of functions $f : \mathbb{T}^2 \to \mathbb{R}$ which depends on an enhancement $\Xi := (\xi, \Xi_2)$ of the rough distribution $\xi$ containing the additional information $\Xi_2$, necessary to make sense of the ill-defined product $\xi f$ between two distributions. In the second part of our work (see section 5), we show that the Gaussian white noise fits in the analytic framework developed in section 4.1. More precisely, we prove that one can construct $\Xi_2$ in a robust way via smooth approximation techniques. Note that this part is somehow purely stochastic and relies on classical stochastic analysis techniques.

We first give the results obtained in the deterministic setting where $\xi$ is a general rough distribution living in a Sobolev space with index $\alpha < -1$ defined as

$$H^\alpha(\mathbb{T}^2, \mathbb{R}) := \{ f \in \mathcal{S}'(\mathbb{T}^2, \mathbb{R}) : \sum_{k \in \mathbb{Z}^2_L} (1 + |k|^2)^\alpha |\hat{f}(k)|^2 < +\infty \}$$

where $\mathbb{T}^2_L := \mathbb{R}/(L^{-1}\mathbb{Z}^2), \mathcal{S}'(\mathbb{T}^2_L, \mathbb{R})$ is the Schwartz space of tempered distributions and where for $k \in \mathbb{Z}^2_L, \hat{f}(k)$ denotes the $k$-th Fourier coefficient of the distribution $f$,

$$\hat{f}(k) := \langle f, L^{-1} \exp(-i2\pi \langle k, \cdot \rangle) \rangle = \frac{1}{L} \int_{\mathbb{T}^2_L} \exp(-i2\pi \langle k, x \rangle) f(x) \, dx.$$

The Fourier transform will sometimes be denoted $\mathcal{F}$ so that $\mathcal{F} f(k) = \hat{f}(k)$ for $k \in \mathbb{Z}^2_L$ and $f \in \mathcal{S}'$. Let also denote by $\mathcal{C}^\alpha$ the Hölder-Besov space (see below in section 3 for a reminder of the definitions of those spaces).

The following Theorem describes the results obtained in the first analytical part of our work. Because we are interested in the limiting spectral properties of the operator $H$ when considered on a Torus or domain with large volume, we will enunciate this Theorem for the Torus $\mathbb{T}^2_L := \mathbb{R}^2/(L^{-1}\mathbb{Z}^2)$ of size $L$. We note that the bounds we obtain are uniform in $L$. This property shall be useful later on, for the asymptotic study of the spectrum of $H$ in the limit of large volume $L \to \infty$. 

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Theorem 1.1. Let $\alpha \in (-\frac{4}{3}, -1)$. Then, there exists a Banach space $\mathcal{F}^\alpha(T^2_\mathcal{L}) \subset \mathcal{C}^\alpha(T^2_\mathcal{L}) \times \mathcal{C}^{2\alpha+2}(T^2_\mathcal{L})$ such that for all $\Xi = (\xi, \xi_2) \in \mathcal{F}^\alpha$, there exists a Hilbert space $\mathcal{D}_\Xi \subset L^2(T^2_\mathcal{L})$ (which is dense in $L^2(T^2_\mathcal{L})$) and a unique self-adjoint operator $\mathcal{H}(\Xi) : \mathcal{D}_\Xi \rightarrow L^2(T^2_\mathcal{L})$ with the following properties:

1. If $\xi$ is a smooth function, then we can choose $\Xi_2$ such that:
   $\mathcal{D}(\xi, \xi_2 + c) = H^2(T^2_\mathcal{L})$, $\mathcal{H}(\Xi)f = -\Delta f + f(\xi + c)$
   for all $f \in H^2(T^2_\mathcal{L})$ and $c \in \mathbb{R}$.

2. The spectrum $(\Lambda_n(\Xi))_{n \in \mathbb{N}^\tau}$ of $\mathcal{H}(\Xi)$ is real, discrete without any accumulation point and satisfy $\Lambda_n(\Xi) \rightarrow +\infty$ when $n \rightarrow \infty$,
   $$\Lambda_1(\Xi) \leq \Lambda_2(\Xi) \leq \ldots \leq \Lambda_n(\Xi)$$
   and $\dim(\Lambda_n(\Xi) - \mathcal{H}(\Xi)) < +\infty$. Moreover, $L^2(T^2_\mathcal{L}) = \bigoplus_n \ker(\Lambda_n(\Xi) - \mathcal{H}(\Xi))$.

3. The eigenvalues $(\Lambda_n)_{n \in \mathbb{N}}$ are solution of a min-max principle (see Lemma 4.26 for a more precise statement).

4. For each $n \in \mathbb{N}$, the map $\Xi \rightarrow \Lambda_n(\Xi)$ is locally-Lipschitz. More precisely, there exists two positive constants $C$ and $M$ which do not depend on $L$ such that, for all $\alpha \in (-4/3, -1)$, $\gamma < \alpha + 2$, $n \in \mathbb{N}$, $\Xi, \Xi_2 \in \mathcal{F}^\alpha$, $|\Lambda_n(\Xi) - \Lambda_n(\Xi_2)| \leq Cn \left(1 + n \frac{2\gamma - \alpha}{\alpha + 2} + (1 + \Lambda_n(0))^{2\gamma}\right)^2 \|\Xi - \Xi_2\|_{\mathcal{F}^\alpha} (1 + \|\Xi_2\|_{\mathcal{F}^\alpha} + \|\Xi\|_{\mathcal{F}^\alpha})^M$
   where $\Lambda_n(0)$ is the $n$-lowest eigenvalue of the Laplacian operator $-\Delta$.

5. For all $a \in \mathbb{R} \setminus \{\Lambda_n(\Xi), n \geq 1\}$, the resolvent map $\Xi \rightarrow \mathcal{G}_a(\Xi) = (a + \mathcal{H}(\Xi))^{-1}$ is locally Lipschitz.

Let us emphasize that the Gaussian white noise on the Torus satisfies the assumption of Theorem 1.1 since for any $\alpha < -1$, we have $\xi \in \mathcal{C}^\alpha$ almost surely.

The conclusions of Theorem 1.1 follow from the spectral Theorem applied to the resolvent operator $\mathcal{G}_a := (a + \mathcal{H})^{-1}$ which is shown to exist for $a$ sufficiently large (with a fixed point argument) and to be a compact self-adjoint operator.

We see at least two interesting applications of Theorem 1.1. The first one concerns the Parabolic Anderson model (2) in two dimension considered on the Torus. The operator $\mathcal{H}$ is simply the hamiltonian associated to the linear stochastic partial differential equation (2) and Theorem 1.1 leads to the spectral decomposition (3) of the solution $u(x, t)$ of (2) which was first constructed in [14].

The second application we see is about the Schrödinger equation
   $$\partial_t u = i (\Delta u - \xi u), \quad u(x, 0) = u_0(x)$$
considered with periodic boundary conditions and where $u_0 \in L^2(T^2, \mathbb{R})$. The solution of this singular stochastic partial differential equation (SPDE) has not been constructed so far (to the best of our knowledge). This is due to the fact that the imaginary factor $i$ kills the Schauder’s estimate which is usually available in the presence of a Laplacian term in a singular SPDE. Theorem 1.1 provides again a spectral decomposition of the solution $u(x, t)$ to the Schrödinger equation (4)
   $$u(x, t) = \sum_{n=1}^{+\infty} \exp(-i\Lambda_n t) \langle e_n, u_0 \rangle_{L^2} e_n(x).$$

Our construction is straightforward to extend on a more general domain with Dirichlet or Neumann boundary conditions (see also [5] where the authors study singular PDEs on general domains).
Remark 1.2. Let us notice that the well posedness of the parabolic Anderson equation (2) which was proven in [14] implies the second point of Theorem 1.1. Indeed, as pointed out in [17], the global well posedness result of the Parabolic Anderson equation ensures that the heat kernel $K_t u_0 := u(t, \cdot)$ is a compact self-adjoint operator of $L^2(T^d_2)$ which satisfies $\mathcal{H}_t \mathcal{H}_s = \mathcal{H}_{t+s}$. Therefore, we can define the operator $\mathcal{H}$ as

$$\mathcal{H} := \frac{1}{\varepsilon} \log \mathcal{H}_t$$

where the logarithm is understood in the sense of functional analysis. From a classical spectral analysis result (see [9]), it is well known that the compactness of $\mathcal{H}_t$ implies the compactness of the resolvent of $\mathcal{H}$ so that the spectral theorem applies. Our work in this paper can be seen as the converse of this approach in the sense that we define the operator $\mathcal{H}$ on an explicit domain and we recover the well posedness result by taking $u(t, \cdot) = e^{-t \mathcal{H}} f$. Our construction for the operator has the advantage of being more explicit.

Remark 1.3. Let us emphasize that Theorem 1.1 can be generalized to the $d$ dimensional Torus. Moreover, the reader who is familiar with the theory of regularity structure could be improved to $\alpha > -\frac{4}{3}$ could be improved to $\alpha > -2$. This extension would allow one to handle even rougher potentials $\xi$ such as the Gaussian white noise on the three dimensional torus.

As mentioned previously, the two-dimensional Gaussian white noise $\xi$ satisfies the assumptions of Theorem 1.1. In particular, the operator $\mathcal{H}$ has in this case a discrete real spectrum which is continuous with respect to the enhanced Gaussian white noise $(\xi, \Xi_2^\alpha)$, as explained below. Theorem 1.1 shall also permit one to obtain a smooth approximation result for the operator $\mathcal{H}$ associated to the Gaussian white noise $\xi$. We now explain this approximation result in more details. If $\hat{\theta}_\varepsilon = e^{-2 \theta(\xi)}$ is an approximation of the identity and $\xi_\varepsilon = \xi * \hat{\theta}_\varepsilon$ is a mollification of the Gaussian white noise $\xi$, then the operator

$$\mathcal{H}_\varepsilon := -\Delta + \xi_\varepsilon$$

is an unbounded operator of $L^2(T^d_2)$ whose domain is the Sobolev space $H^2(T^d_2)$. Its resolvent $G^n_\alpha := (a + \mathcal{H}_\varepsilon)^{-1} : L^2 \rightarrow H^2$, which is well defined for a large enough, is compact and we can apply the spectral theorem. Note that we know from Theorem 1.1 that there is a choice of $\Xi_2^\alpha$ such that $\mathcal{H}_\varepsilon = \mathcal{H}(\xi_\varepsilon, \Xi_2^\alpha)$. Our approximation result can be enunciated as follows.

Theorem 1.4. Let $\alpha < -1$, $\xi$ be a Gaussian white noise, $\xi_\varepsilon := \xi * \hat{\theta}_\varepsilon$ be a smooth mollification of $\xi \in C^\alpha$ and $\Xi_2^\alpha$ as given in Theorem 1.1 such that $\mathcal{H}(\xi_\varepsilon, \Xi_2^\alpha) = \mathcal{H}_\varepsilon$. Then, there exists $\Xi^{\alpha n}_2 = (\xi, \Xi_2^{\alpha n}) \in \mathcal{H}^\alpha(T^d_2)$ and a constant $c_\varepsilon := c_\varepsilon(\theta) \rightarrow +\infty$ as $\varepsilon \rightarrow 0$ such that the following convergence holds

$$(\xi_\varepsilon, \Xi_2^\alpha + c_\varepsilon) \quad \overset{\varepsilon \rightarrow 0}{\longrightarrow} \quad (\xi, \Xi^{\alpha n}_2)$$

in $L^p(\Omega, C^\alpha \times C^{2\alpha+2})$ for all $p > 0$ and almost surely in $C^\alpha \times C^{2\alpha+2}$. Moreover, the limiting distribution $\Xi$ does not depend on the mollification function $\theta_\varepsilon$ and the normalizing constant $c_\varepsilon$ has the following asymptotic expansion

$$c_\varepsilon = \frac{1}{2\pi} \log(\frac{1}{\varepsilon}) + O(1), \quad (5)$$

where $O(1)$ refers to any fixed constant, independent of $\varepsilon$.

Remark 1.5. Note that the asymptotic expansion (5) is universal at the leading order when $\varepsilon \rightarrow 0$, in the sense that the largest term does not depend on the mollification $\theta_\varepsilon$ used for the regularization. The second term $O(1)$ is however not universal and one can actually choose any constant in $\mathbb{R}$ so that the Theorem 1.4 remains valid. As a consequence, the limiting distribution $\Xi_2$ is unique up to an additive constant. Theorem 1.4 was first proved in [14, 15] where the authors obtain the well posedness of the parabolic Anderson equation on the two dimensional torus. The present version is a slight modification of their result.
Endowed with Theorems 1.1 and 1.4, we are now able to define the Schrödinger operator $\mathcal{H}$ associated to the Gaussian white noise potential $\xi$ simply by setting

$$\mathcal{H} := \mathcal{H}(\Xi_{wn}).$$

(6)

We now establish the convergence in the sense of the resolvent (convergence of the spectrum) of the smooth approximations $\mathcal{H}_\varepsilon + c_\varepsilon$ as defined above in Theorem 1.4 towards the operator $\mathcal{H}$, so that the definition 6 makes sense. The following Theorem is the second main result of our paper.

**Theorem 1.6.** With the same notations as in Theorem 1.4, we denote by

$$\Lambda_1^\varepsilon \leq \Lambda_2^\varepsilon \leq \Lambda_3^\varepsilon \leq \cdots$$

the eigenvalues of the operator $\mathcal{H}_\varepsilon$. Then, for any $n \in \mathbb{N}$, almost surely,

$$\Lambda_n^\varepsilon + c_\varepsilon \to_{\varepsilon \to 0} \Lambda_n(\Xi_{wn}),$$

where $(\Lambda_n(\Xi_{wn}))_{n \in \mathbb{N}}$ denotes the discrete set of the eigenvalues of $\mathcal{H}(\Xi_{wn})$.

So far we have constructed the operator $\mathcal{H}(\Xi_{wn})$ associated to the two-dimensional Gaussian white noise $\xi$ and establish the convergence of smooth approximations. We are now interested in the limiting spectral statistics of the operator $\mathcal{H}(\Xi_{wn})$ when the volume of the torus, denoted as $L$ above, tends to $+\infty$. In the limit of a very large torus, it is expected that the eigenfunctions associated to the lowest eigenvalues will be localized (this property is known as the Anderson localization) as can be observed in the picture of the eigenfunction associated to the bottom eigenvalue in Fig. 2 for $L = 10$ (this picture is obtained from a numerical simulation and diagonalization of the discretized operator on a grid with small mesh size). We provide a picture of the first eigenfunction in Fig. 1 with $L = 1$ for comparison and to illustrate the effect of the Gaussian white noise.

Instead of the localization, a weaker result is to prove the convergence of the bottom eigenvalues in their scaling region as $L \to \infty$ towards a Poisson point process. Even in the one dimensional case (described below in section 2), this conjecture remains to be proved (see [8] for a discussion on this conjecture and [2] for a related model where this convergence is proved).

In the two-dimensional case, we obtain in this paper only partial results in this direction. Mainly, we are able to give an upper-bound on the asymptotic order of the ground state in the limit of large volume $L \to \infty$. In the one-dimensional case, McKean [23] established the convergence in law of the ground state towards a Gumbel distribution, the asymptotic order of the minimal eigenvalue being (up to a multiplicative factor) $-\log(L)^{2/3}$.

**Theorem 1.7.** For any $n \in \mathbb{N}$ and $p \geq 1$,

$$\sup_{L > 0} \mathbb{E} \left[ \left| \frac{\Lambda_n(\Xi_{wn})}{\log L} \right|^p \right] < +\infty.$$

(7)

Besides, there exist two positive constants $C_1$ and $C_2$ such that for any $x < 0$, we have

$$e^{C_2 x} \leq \mathbb{P}(\Lambda_1(\Xi_{wn}) \leq x) \leq e^{C_1 x}.$$

(8)

**Remark 1.8.** The estimate (7) proves that the asymptotic order of the ground state is larger or equal to $-\log L$ when $L \to \infty$ (the ground state can not be much below a multiple of $-\log L$ as $L \to +\infty$). The tail estimates for the ground state $\Lambda_1(\Xi_{wn})$ hold for any $L > 0$ but unfortunately, we do not have a good control on the constants $C_1$ and $C_2$ in $L$, so that we are not able to extract further information on the asymptotic order of the ground state. We think that this question deserves to be investigated in more details.
Figure 1: (Color online). Sample graph of the first eigenfunction associated to the minimal eigenvalue of the operator $H$ with Dirichlet boundary conditions and on a square of size $L = 1$. We also display the first eigenfunction of the Laplacian operator $-\Delta$ to illustrate the effect of the noise.
Let us close this section with a brief remark about the three dimensional case.

**Remark 1.9.** As pointed out before, our work can be generalized to the case of the three dimensional Gaussian white noise which lives in the space $\mathcal{C}^{-\frac{1}{2}} - 3\mathbb{T}^3$. According to the well-posedness result of the Parabolic Anderson equation stated in [17], the renormalization constant takes the form

$$c_\varepsilon = \frac{a}{\varepsilon} + c \log(\frac{1}{\varepsilon}) + O(1)$$

when $\varepsilon \to 0$ and where $a$ and $b$ are two constants. Moreover, as we shall see in Section 5.2, the growth and tail estimates for the ground state are proved with a scaling argument which should work also in general dimension $d \leq 3$. We expect the following estimates to hold in dimension $d$,

$$\exp(-C_2(-x)^{2-d}) \leq \mathbb{P}(\Lambda_1(\Xi^{wn}) \leq x) \leq \exp(-C_1(-x)^{2-d})$$

for any $x < 0$ and

$$\sup_L \mathbb{E} \left[ \frac{\Lambda_n(\Xi^{wn})}{(\log L)^{\frac{2}{2+d}}} \right]^p < \infty$$

for all $p \geq 1$.

**Acknowledgements:** We are very grateful to Professor M. Hairer for pointing out that the space of rough distribution is trivial in our setting and for his numerous comments which have permitted us to improve this manuscript. We also would like to thank Professors P. K. Friz and N. Perkowski for the numerous discussions and fruitful advices. KC is funded by the RTG 1845 and a large part of this work was carried out while K.C was employed by the T.U Berlin university. Both authors were supported by the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013) / ERC grant of Professor P.K.Friz. agreement nr. 258237.
2 The Anderson hamiltonian on a finite interval

A rigorous definition of the random operator \( \mathcal{H} \) in dimension \( d = 1 \) was first given in 1977 by Fukushima and Nakao in [10]. Although no precise formulation of the eigenvalues problem was available before this paper, the random spectrum of \( \mathcal{H} \) was first studied in the physics literature by Frisch and Lloyd back in 1960 [11] and also by Halperin in 1965 [18]. A rigorous approach due to McKean to compute the limiting distribution of the ground state when the operator \( \mathcal{H} \) is considered on a long box can be found in [23].

Let us briefly recall the construction [10] of the operator \( \mathcal{H} \) and the main results on the ground state distribution due to McKean [23].

The authors of [10] work with Dirichlet boundary conditions and define the stochastic linear operator

\[
\mathcal{H} := -\frac{d^2}{dx^2} + B'(x)
\]

on the space of functions

\[
\mathcal{D} := \{ f \in H^1([0, L], \mathbb{R}) : f(0) = f(L) = 0 \}.
\]

Their construction can be extended to other boundary conditions along the same lines. If \( f \in H^1([0, L], \mathbb{R}) \), the product \( B'(x)f(x) \) is defined as the Schwarz derivative of a continuous function using integration by part: for any \( x \in [0, L] \),

\[
B'(x)f(x) := \frac{d}{dx} \left[ f(x)B(x) - \int_0^x f'(y)B(y)dy \right].
\]

At this point, the operator \( \mathcal{H} \) is defined on the domain \( \mathcal{D} \) and takes values in the space of distribution. The eigenvalue problem can now be defined: we say that \( (\lambda, f_\lambda) \in \mathbb{R} \times \mathcal{D} \) is an eigenvalue/eigenfunction pair if

\[
-f''_\lambda(x) + B'(x)f_\lambda(x) = \lambda f_\lambda(x)
\]

This equality can be integrated with respect to \( x \) and rewritten in its equivalent integrated form

\[
f_\lambda'(x) - f_\lambda'(0) = f_\lambda(x)B(x) - \int_0^x f_\lambda(y)B(y)dy - \lambda \int_0^x f_\lambda(y)dy
\]

for any \( x \in [0, L] \). From (10), we see that \( f_\lambda' \) has the same regularity as the Brownian motion \( B(x) \) and therefore has Hölder regularity \( 1/2 - \varepsilon \) for any \( \varepsilon > 0 \). We can conclude that the eigenfunction \( f_\lambda \) itself has Hölder regularity \( 3/2 - \varepsilon \) and in particular belongs to the space \( H^1 \), which proves that it is indeed sufficient to define \( \mathcal{H} f \) for \( f \in H^1 \) (the space \( H^1 \) contains the eigenvectors of \( \mathcal{H} \)).

With this definition of the stochastic linear operator \( \mathcal{H} \), the authors of [10] prove the following Theorem.

**Theorem 2.1** (Fukushima, Nakao (1977)). Let \( L > 0 \). The random spectrum of \( \mathcal{H} \), when considered with Dirichlet boundary conditions on the finite box \([0, L]\), has a well defined \( k \)-th lowest element \( \Lambda_k \). Furthermore, almost surely,

\[
\Lambda_1 < \Lambda_2 < \Lambda_3 < \cdots
\]

The eigenvectors \( (f^*_n)_n \) are Hölder \( 3/2 - \varepsilon \) and form an orthonormal basis of \( L^2 \).

The proof follows the classical lines of the spectral Theorem with a minimization of the associated quadratic form. Let us outline this proof here for completeness.
The quadratic form associated to the operator $\mathcal{H}$ is well defined for $f \in H^1$ and satisfies
\[
\langle f, \mathcal{H} f \rangle_{L^2} = \int_0^L f'(x)^2 \, dx - 2 \int_0^L f'(x) f(x) B(x) \, dx.
\]

Before going into the optimization procedure, we first need to establish upper and lower bounds for the quadratic form. We denote by $M$ the supremum value of the Brownian path on the interval $[0, L]$,
\[
M := \sup_{x \in [0, L]} B(x).
\]

It is easy to check that, almost surely,
\[
\langle f, \mathcal{H} f \rangle \leq ||f'||_{L^2}^2 + \frac{M^2}{2} (||f'||_{L^2}^2 + ||f||_{L^2}^2).
\]

The lower bound is slightly more involved: Using Cauchy-Schwarz inequality and minimizing a quadratic form in $(||f||_{L^2}, ||f'||_{L^2})$, we obtain
\[
\langle f, \mathcal{H} f \rangle + (M^2 + 1) ||f||_{L^2}^2 \geq \frac{1}{M^2 + 3} (||f'||_{L^2}^2 + ||f||_{L^2}^2) \geq 0.
\] (11)

We now set
\[
\Lambda_1 := \inf_{f \in H^1, ||f||_{L^2} = 1} \langle f, \mathcal{H} f \rangle > -\infty,
\]
and consider a sequence $(f_n)_{n \in \mathbb{N}}$ of functions in $H^1$ such that $||f_n||_{L^2} = 1$ and $\langle f_n, \mathcal{H} f_n \rangle \to \Lambda_1$. It is plain to check from the lower bound (11) that almost surely
\[
\sup_{n \in \mathbb{N}} ||f_n'||_{L^2} < +\infty.
\] (12)

This uniform control on the $L^2$ norms of the derivatives $f_n'$ is crucial as it permits us to prove the existence of a limit point $f^*_1 \in H^1$ such that along a subsequence

- $f_n \to f^*_1$ uniformly in $C^0$ (using the Ascoli-Arzela Theorem),
- $f_n \to f^*_1$ in $L^2$,
- $f_n \to f^*_1$ weakly in $H^1$ (using the Banach-Alaoglu Theorem).

Therefore, $||f^*_1||_{L^2} = 1$ and we can prove passing to the limit along the subsequence that
\[
\langle f^*_1, \mathcal{H} f^*_1 \rangle = \Lambda_1.
\]

To check that $\mathcal{H} f^*_1 = \Lambda_1 f^*_1$ in the sense of distributions, it suffices to write that the derivative of the quadratic form in the direction of any smooth function $\varphi$ is zero (as $f^*_1$ is a minimizer):
\[
\frac{d}{d\varepsilon} \left. \left\langle (f^*_1 + \varepsilon \varphi), \mathcal{H} (f^*_1 + \varepsilon \varphi) \right\rangle \right|_{\varepsilon = 0} = 0.
\]

For the next eigenvalues/eigenvectors, we restrict to the orthogonal complementary of Vect$(f^*_1)$ to obtain following the same method the second eigenfunction $f^*_2$ such that $\mathcal{H} f^*_2 = \Lambda_2 f^*_2$, $\langle f^*_2, f^*_1 \rangle_{L^2} = 0$ and $||f^*_2||_{L^2} = 1$. 

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We iterate this argument to obtain the existence of an orthonormal family of eigenfunctions \((f_n^\star)\) respectively associated to eigenvalues \((\Lambda_n)\) which satisfy
\[
\Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \cdots
\]
We have not shown yet that the eigenvalues have multiplicity one. This fact will actually follow from the forthcoming characterization of the law of the eigenvalues in term of a family of interacting diffusions valid in the special case of Dirichlet boundary conditions. We believe the eigenvalues are also simple with Periodic boundary conditions although (to the best of our knowledge) no proof of this fact is available at the time of writing this paper.\(^1\)

The main idea (which was used in many papers \([2, 8, 11, 18, 25, 26]\)) is to use the Riccati transform to rewrite the eigenfunction differential equation \(\mathcal{H} f_\lambda = \lambda f_\lambda\) (see also \((9)\)) as a first order stochastic differential equation. More precisely, if one sets \(X_\lambda(x) := f_\lambda^\prime(x)/f_\lambda(x)\), the second degree equation \((9)\) is mapped to the stochastic differential equation
\[
dX_\lambda(x) = -(\lambda + X_\lambda(x)^2) \, dt + dB(x) .
\]
The initial condition imposed on \(f_\lambda\) translates as an initial condition on \(X_\lambda\). With Dirichlet boundary conditions for \(f_\lambda\), one has
\[
X_\lambda(0) = +\infty .
\]
The trajectory of \(X_\lambda\) determines the trajectory of \(f_\lambda\) up to a normalization factor (the equation on \(f_\lambda\) is linear). The function \(X_\lambda\) is associated to an eigenfunction (so that at the same time \(\lambda\) is an eigenvalue) if and only if it the diffusion blows up to \(-\infty\) precisely at the end point \(t = L\), i.e. \(X_\lambda(L) = -\infty\). under Dirichlet boundary condition.

Using this observation, we can easily deduce a characterization of the joint law of the eigenvalues in term of the family of coupled \(^2\) diffusions \(\{X_\lambda, \lambda \in \mathbb{R}\}\) such that \((13)\) and \((14)\) hold for all \(\lambda \in \mathbb{R}\). The idea is that the zeros of the eigenfunction \(f_\lambda\) for \(\lambda = \Lambda_k, k \in \mathbb{N}\) correspond to the zeros of the associated diffusion process \(X_\lambda\) and that the trajectories of \(X_\lambda\) is a monotonic function of \(\lambda\). Tuning the value of \(\lambda\) permits one to find the eigenvalues, which correspond to the values of \(\lambda\) for which the diffusion \(X_\lambda\) explodes precisely at the end point \(x = L\). We do not need to explicit the characterization of the joint law for the purpose of this section. Let us just state two simple properties regarding the law of the eigenvalues:

- The distribution of the minimal eigenvalue \(\Lambda_0 = \Lambda_0(L)\) is characterized in terms of the family of diffusions \(X_\lambda\) as
\[
\mathbb{P}[\Lambda_0 \leq \lambda] = \mathbb{P}[X_\lambda \text{ blows up before time } L] .
\]

- The number of \(\mathcal{H}\)-eigenvalues below \(\lambda\) is equal to the number of explosions of the diffusion \(X_\lambda\) before time \(L\).

Using this characterization, McKean \([23]\) proves the following convergence in law after a careful analysis of the explosion time of the diffusion \(X_\lambda\) for a fixed value of \(\lambda\).

**Proposition 2.2 (McKean (1994)).** In the limit \(L \to \infty\), the fluctuations of the minimal eigenvalue \(\Lambda_1\) of \(\mathcal{H}\) are governed by the Gumbel distribution. More precisely, we have the following convergence in law as \(L \to \infty\),
\[
-2 \cdot 3^{1/3} (\ln L)^{1/3} \left[ \Lambda_1 + \left( \frac{3}{8} \ln \frac{L}{\pi} \right)^{2/3} \right] \Rightarrow e^{-x} \exp(-e^{-x}) \, dx .
\]

\(^1\)Note that the Laplacian operator has eigenvalues with multiplicity strictly greater than one when considered under Periodic boundary conditions. We believe that adding the Gaussian white noise will separate the multiple eigenvalues into several simple eigenvalues.

\(^2\)They are all driven by the same Brownian motion \((B(x))_{x \geq 0}\).
Remark 2.3. Let us finish this section by pointing out the main difference between the one dimensional construction of the operator and the two dimensional one. The construction presented in this section is crucially related to the fact that we can multiply the white noise $\xi := dB$ by any function in $H^1$. This allows one to define the quadratic form $\langle \mathcal{K} f, f \rangle$ by duality on the space $H^1$ and to prove that it is lower semi-bounded. In dimension 2, this picture is blurred, indeed in that case, the white noise can only be multiplied by function in $\cap_{\varepsilon} H^{1+\varepsilon}$ and one can think to take $H^{1+\varepsilon}$ as a domain of $\mathcal{K}$. Unfortunately, this approach is not consistent in the sense that the quadratic form $\langle \mathcal{K} f, f \rangle$ is not lower semi-bounded any more. As we will see in the next section, the general idea to overcome this problem is to define the operator on a Hilbert space of irregular functions for which the most irregular part of the product $f \xi$ is compensated by the term $-\Delta f$ so that $\mathcal{K} f \in L^2$.

3 Besov spaces and Bony paraproducts

We recall the definitions of the Bony paraproducts, the Besov and Sobolev spaces and collect the two main results that we will be useful throughout the paper regarding the products between two Schwartz distributions and the effect of differentiation on the regularity of the distributions. We work on the two dimensional torus $\mathbb{T}^2 := \mathbb{R}^2/(L^{-1} \mathbb{Z})^2$ with diameter $L$ (see [3] for a review on this subject). For any $f$ in the Schwartz space $\mathcal{S}’(\mathbb{T}^2, \mathbb{R})$ of tempered distributions on $\mathbb{T}^2$, the Fourier transform of $f$ will be denoted $\hat{f} : \mathbb{T}^2 \rightarrow \mathbb{C}$ (or sometimes $\mathcal{F} f$) and is defined for $k \in \mathbb{Z}^2$ by

$$\hat{f}(k) := \langle f, L^{-1} \exp(i2\pi \langle k, \cdot \rangle) \rangle = \frac{1}{L} \int_{\mathbb{T}^2} f(x) \exp(-i2\pi \langle k, x \rangle) dx.$$

Recall that for any $f \in L^2(\mathbb{T}^2, \mathbb{R})$ and $x \in \mathbb{T}^2$, we have

$$f(x) = \frac{1}{L} \sum_{k \in \mathbb{Z}^2} \hat{f}(k) \exp(i2\pi \langle k, x \rangle). \quad (15)$$

The Sobolev space $H^\alpha(\mathbb{T}^2, \mathbb{R})$ with index $\alpha \in \mathbb{R}$ is defined as

$$H^\alpha(\mathbb{T}^2, \mathbb{R}) := \{ f \in \mathcal{S}’(\mathbb{T}^2, \mathbb{R}) : \sum_{k \in \mathbb{Z}^2} (1 + |k|^2)^\alpha |\hat{f}(k)|^2 < +\infty \}.$$

Before recalling the definition of the Besov spaces, we first need to introduce the Littlewood-Paley blocks which permit us to decompose a distribution $f$ into an infinite series of smooth functions. We denote by $\chi$ and $\rho$ two nonnegative smooth and compactly supported radial functions $\mathbb{R}^2 \rightarrow \mathbb{C}$ such that

1. The support of $\chi$ is contained in a ball $\{ x \in \mathbb{R}^2 : |x| \leq R \}$ and the support of $\rho$ is contained in an annulus $\{ x \in \mathbb{R}^2 : a \leq |x| \leq b \}$;

2. For all $\xi \in \mathbb{R}^2$, $\chi(\xi) + \sum_{j \geq 0} \rho(2^{-j} \xi) = 1$;

3. For $j \geq 0$, $\chi(\rho(2^{-j}) = 0$ and $\rho(2^{-i}) \rho(2^{-j}) = 0$ for $|i - j| \geq 1$.

The Littlewood-Paley blocks $(\Delta_j)_{j \geq -1}$ associated to a tempered distribution $f \in \mathcal{S}’(\mathbb{T}^2, \mathbb{R})$ are defined by

$$\mathcal{F}(\Delta_{-1} f) = \chi \mathcal{F} f \quad \text{and for } j \geq 0, \quad \mathcal{F}(\Delta_j f) = \rho(2^{-j}) \mathcal{F} f.$$

\textsuperscript{3}This decomposition is more convenient than the Fourier decomposition (15)

\textsuperscript{4}The existence of two such functions in insured by [3, Proposition 2.10]
Note that, for $f \in \mathcal{S}'(T^2_L, \mathbb{R})$, the Littlewood-Paley blocks $(\Delta_j f)_{j \geq -1}$ define smooth functions (their Fourier transform have compact supports). We also set, for $f \in \mathcal{S}'$ and $j \geq -1$,

$$S_j f := \sum_{i=-1}^{j-1} \Delta_i f$$

and note that $S_j f$ converges weakly to $f$ when $j \to \infty$.

The Besov space with parameters $p, q \in \mathbb{R}_+, \alpha \in \mathbb{R}$ can now be defined as

$$B^\alpha_{p,q}(T^2_L, \mathbb{R}) := \left\{ u \in \mathcal{S}'(T^2_L, \mathbb{R}); \ ||u||_{B^\alpha_{p,q}} = \left( \sum_{j \geq -1} 2^{j\alpha q} ||\Delta_j u||_{L^p}^q \right)^{1/q} < +\infty \right\}. \quad (16)$$

We also define the Besov $\alpha$-Hölder space

$$\mathcal{C}^\alpha := B^\alpha_{\infty,\infty}$$

which is naturally equipped with the norm $||f||_{\mathcal{C}^\alpha} := ||f||_{B^\alpha_{\infty,\infty}} = \sup_{j \geq -1} 2^{j\alpha} ||\Delta_j f||_{L^\infty}$. Note also that the Sobolev space $H^\alpha$ coincides with $B^\alpha_{2,2}$.

We now consider the product between two distributions $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$. At least formally, we can decompose the product $fg$ as

$$fg = f \prec g + f \circ g + f \succ g$$

where

$$f \prec g := \sum_{j \geq -1} \sum_{i=-1}^{j-2} \Delta_i f \Delta_j g, \quad f \circ g := \sum_{j \geq -1} \sum_{i=-1}^{j-2} \Delta_i g \Delta_j f$$

are usually referred as the paraproduct terms whereas

$$f \succ g := \sum_{j \geq -1} \sum_{|i-j| \leq 1} \Delta_i f \Delta_j g$$

is called the resonating term. The paraproduct terms are always well defined whatever the values of $\alpha$ and $\beta$. The resonating term is well defined if and only if $\alpha + \beta > 0$. This is reminiscent to the well known fact that one can not generically form the product of two distributions: the two regularities must compensate one another in such a way that the sum is strictly positive. These (deterministic) facts can be summarized as in the following proposition where we give estimates on the regularities of the paraproducts and resonating terms.

**Proposition 3.1 (Bony estimates).** Let $\alpha, \beta \in \mathbb{R}$. We have the following upper bounds:

1. If $f \in L^2$ and $g \in \mathcal{C}^\beta$, then

$$||f \prec g||_{H^{\beta - \delta}} \leq C_{\delta,\beta} ||f||_{L^2} ||g||_{\mathcal{C}^\beta}.$$

   for all $\delta > 0$

2. if $f \in H^\alpha$ and $g \in L^\infty$ then

$$||f \succ g||_{H^\alpha} \leq C_{\alpha,\beta} ||f||_{H^\alpha} ||g||_{\mathcal{C}^\beta}.$$
3. If \( \alpha < 0 \), \( f \in H^\alpha \) and \( g \in \mathcal{C}^\beta \), then
\[
\| f \prec g \|_{H^\alpha + \beta} \leq C_{\alpha,\beta} \| f \|_{H^\alpha} \| g \|_{\mathcal{C}^\beta}.
\]

4. If \( g \in \mathcal{C}^\beta \) and \( f \in H^\alpha \) for \( \beta < 0 \) then
\[
\| f \succ g \|_{H^\alpha + \beta} \leq C_{\alpha,\beta} \| f \|_{H^\alpha} \| g \|_{\mathcal{C}^\beta}.
\]

5. If \( \alpha + \beta > 0 \) and \( f \in H^\alpha \) and \( g \in \mathcal{C}^\beta \), then
\[
\| f \circ g \|_{H^\alpha + \beta} \leq C_{\alpha,\beta} \| f \|_{H^\alpha} \| g \|_{\mathcal{C}^\beta}.
\]

(17)

where \( C_{\alpha,\beta} \) is a finite positive constant which does not depend on the size of the Torus.

**Remark 3.2.** We will use extensively the first, the fourth and the last estimates of this Proposition in the Section 4.1 to construct our operator while the second and the third will be used in the proof of the commutation Lemma 4.3.

We end this section by describing the action of the Fourier multipliers on the Besov spaces. Those ”multiplications” in the Fourier space correspond to differentiations (respectively “integrations”) of the distributions in the Besov spaces and the following proposition quantifies the loss (resp. gain) of regularity obtained by differentiating (resp. “integrating”) distributions.

**Proposition 3.3** (Schauder estimate). Let \( \alpha, n \in \mathbb{R} \) and \( \sigma : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R} \) be an infinitely differentiable function such that \( |D^k \sigma(x)| \leq C(1 + |x|)^{-n-k} \) for all \( x \in \mathbb{R}^2 \). For \( f \in H^\alpha \) (respectively \( \mathcal{C}^\alpha \)), we define the distribution \( \sigma(D)f \) obtained from applying the differentiation operator \( \sigma(D) \) to \( f \) as
\[
\sigma(D)f := \mathcal{F}^{-1}(\sigma \mathcal{F} f).
\]

Then, \( \sigma(D)f \in H^{\alpha+n} \) (respectively \( \mathcal{C}^{\alpha+n} \)) and
\[
\| \sigma(D)f \|_{H^{\alpha+n}} \leq C_{\alpha,n} C \| f \|_{H^\alpha}.
\]

and the same bound hold for the Hölder space \( \mathcal{C}^{\alpha} \). Moreover the constant \( C_{\alpha,n} \) does not depend on \( L \).

Further important technical results related to Besov spaces and Bony paraproducts (which will be used in the following) are gathered in the appendix.

4 Schrödinger operator with singular potential

4.1 Definition on a space of paracontrolled distributions

Our goal in this section is to define the linear operator \( \mathcal{H} \) defined in (1). More precisely, starting from a test function \( f \in L^2(\mathbb{T}_L^2, \mathbb{R}) \), we would like to define the function \( g \) such that
\[
g = \mathcal{H}f = -\Delta f + \xi f.
\]

This operation is ill defined for a generic element \( f \in L^2(\mathbb{T}_L^2, \mathbb{R}) \) because of the product \( \xi f \) which makes sense only if \( f \) is sufficiently smooth. Indeed, we know from Young’s integration theory that the product \( fg \) between two distributions \( f \) and \( g \) with respective Hölder-Besov regularities \( \alpha \) and \( \beta \) (see Appendix A for a reminder on Besov-Hölder spaces) is well defined if and only if the sum \( \alpha + \beta > 0 \).
Here, we need to define the linear operator $\mathcal{H}$ on a functional space large enough to contain the eigenfunctions of $\mathcal{H}$. The eigenvalue problem associated to the operator $\mathcal{H}$ simply writes

$$\mathcal{H} f_\lambda = \lambda f_\lambda$$

or in the more explicit form

$$(1 - \Delta) f_\lambda = -f_\lambda \xi + (\lambda + 1) f_\lambda$$

where we seek for an eigenfunction $f_\lambda$ associated to the eigenvalue $\lambda$ such that (18) holds together with the boundary conditions under consideration.

Let us fix $\alpha < -1$ such that the white noise $\xi$, which may be seen as a random distribution, belongs to the Besov-Hölder space $C^\alpha$ (we know that $\xi \in C^{-1-\varepsilon}$ almost surely for any $\varepsilon > 0$).

Then, if $f$ satisfies equation (18), we expect from standard (heuristic) arguments the regularity of $f$ to be $\alpha + 2$ (thanks to the additional regularity induced by the Laplacian), which barely prevents us from defining the product $\xi f$ using standard Young integration (this is the classical problem which motivated Itô’s theory of stochastic integrals). Powerful tools to make sense of such products in the present stochastic context have been recently developed by Gubinelli, Imkeller and Perkowski in [14]. Another alternative more general theory has been independently developed by Hairer in [15], the so called theory of regularity structures, which permits one to tackle the same problems, with applications to singular stochastic partial differential equations, in a broader context. This later theory has received much attention recently.

For our study, we shall use the Fourier approach introduced in [14] as it is somehow more basic and does not require a large background of harmonica analysis. Working in Sobolev spaces will turn out to be crucial for us because they are Hilbert spaces.

We come back to the eigenvalue problem (18) which may be rewritten, using Fourier’s multipliers and thanks to the Bony paraproduct decomposition (see again Appendix A for a reminder on this), as

$$f = f \prec \sigma(D) \xi - f^2,$$

$$f^2 = \sigma(D)((\lambda + 1)f + f \circ \xi + f \triangleright \xi) + \sigma(D)(f \prec \xi) - f \prec \sigma(D)\xi$$

where $D$ is the differentiation operator (see Appendix A) and $\sigma(k) = -(1 + |k|^2)^{-1}$ for $k \in \mathbb{Z}_L^2$.

Thanks to a formal analysis of (19), we expect $f \in H^{\alpha+2}$ if $\xi \in C^\alpha$. Indeed, auto consistently, if $f \in H^{\alpha+2}$ and $\xi \in C^\alpha \subseteq H^\alpha$, then we know using the Schauder’s estimate Proposition 3.3 that the regularity of $(\lambda + 1)f + f \circ \xi + f \triangleright \xi \in H^{2\alpha+2}$ increases by 2 when multiplied by $\sigma(D)$ and also that $\sigma(D)(f \prec \xi)$ and $\sigma(D)\xi \in H^{2\alpha+4}$, so that finally $f^2 \in H^{2\alpha+4}$. Those heuristic remarks motivate, following [14], the next definition which shall permit us to make sense of the resonating term $f \circ \xi$ (yet ill defined) and define the operator $\mathcal{H}$ on the space of paracontrolled distribution $\mathcal{D}_\xi^\gamma$ introduced.

**Definition 4.1.** Let $\alpha < -1$ and $\xi \in C^\alpha(\mathbb{T}_L^2)$. For $\gamma \leq \alpha + 2$, we define the space of distributions which are paracontrolled by $\sigma(D)\xi$, i.e.

$$\mathcal{D}_\xi^\gamma = \left\{ f \in H^\gamma(\mathbb{T}_L^2), \quad f^\sharp := f - f \prec \sigma(D)\xi \in H^{2\gamma}\right\}.$$  \hfill (20)

The space $\mathcal{D}_\xi^\gamma$ equipped with the scalar product $(\cdot, \cdot)_{\mathcal{D}_\xi^\gamma}$, defined for $f, g \in \mathcal{D}_\xi^\gamma$, by

$$(f, g)_{\mathcal{D}_\xi^\gamma} = (f, g)_{H^\gamma} + (f^\sharp, g^\sharp)_{H^{2\gamma}}.$$ 

is a Hilbert space.

**Remark 4.2.** The Hilbert space $\mathcal{D}_\xi^\gamma, (\cdot, \cdot)_{\mathcal{D}_\xi^\gamma}$ is continuously embedded in $L^2(\mathbb{T}_L^2)$.
Now we claim that we can give a meaning to the resonating term $f \circ \xi$ for any $f \in \mathcal{D}_\xi^\gamma$ with $\gamma \leq \alpha + 2$ in a “robust way”, provided we enhance (consistently) the information contained in the white noise $\xi$ (see below).

Using (19), we can decompose $f \circ \xi$ as

$$f \circ \xi = (f \prec \sigma(D)\xi) \circ \xi + f^2 \circ \xi$$

$$= f(\xi \circ \sigma(D)\xi) + \mathcal{R}(f, \sigma(D)\xi, \xi) + f^2 \circ \xi \quad \text{(21)}$$

where

$$\mathcal{R}(f, \sigma(D)\xi, \xi) = (f \prec \sigma(D)\xi) \circ \xi - f(\xi \circ \sigma(D)\xi).$$

If $f \in \mathcal{D}_\xi^\gamma$ with $\gamma \leq \alpha + 2$, then $f^2 \in H^{2\gamma}$ and the Bony estimate (see Proposition 3.1 Eq. (17)) insures that $f^2 \circ \xi$ is well defined with regularity $2\gamma + \alpha > 0$.

The key result of [14] yields that the trilinear operator $(f, g, h) \rightarrow \mathcal{R}(f, g, h)$, which is well defined for smooth test functions $f, g, h$, can be continuously extended to any product space $\mathcal{E}^\alpha \times \mathcal{E}^\beta \times \mathcal{E}^\gamma$ provided that $\alpha + \beta + \gamma > 0$. We need to modify slightly this so called commutation Lemma [14, Proposition 4.7] because we work with $f \in H^{\alpha+2}$ instead of $\mathcal{E}^{\alpha+2}$. This commutation Lemma is a crucial tool as it permits us to define the resonating term $f \circ \xi$ for a general $f \in \mathcal{D}_\xi^\gamma$ and $\xi \in \mathcal{E}^\alpha$ thanks to the knowledge of $\xi \circ \sigma(D)\xi = \Xi_2 \in \mathcal{E}^{2\alpha+2}$.

**Proposition 4.3.** Given $\alpha \in (0, 1)$, $\beta, \gamma \in \mathbb{R}$ such that $\beta + \gamma < 0$ and $\alpha + \beta + \gamma > 0$, the following trilinear operator $\mathcal{R}$ defined for any smooth functions $f, g, h$ by

$$\mathcal{R}(f, g, h) := (f \prec g) \circ h - f(g \circ h)$$

can be extended continuously to the product space $H^\alpha \times \mathcal{E}^\beta \times \mathcal{E}^\gamma$. Moreover, we have the following bound

$$||\mathcal{R}(f, g, h)||_{H^{\alpha+\beta+\gamma-\delta}} \lesssim ||f||_{H^\alpha} ||g||_{\mathcal{E}^\beta} ||h||_{\mathcal{E}^\gamma}$$

for all $f \in H^\alpha$, $g \in \mathcal{E}^\beta$ and $h \in \mathcal{E}^\gamma$, and every $\delta > 0$ where the last bound is uniform in $L$.

**Remark 4.4.** Let us point out that a general version of this commutation Lemma was proved recently in [24] for the Besov space $\mathcal{B}^\alpha_{2,\infty}$. However the result presented in [24] does not cover our special case but with a slight modification of the proof we are able to proof the needed result (see in the Appendix A for the proof).

Applying Proposition 4.3 in our case with $f \in H^{\alpha+2}$, $g = \sigma(D)\xi \in \mathcal{E}^{\alpha+2}$ and $h = \xi \in \mathcal{E}^\alpha$ for $\alpha < -1$, we deduce that $\mathcal{R}(f, \sigma(D)\xi, \xi) \in H^{3\alpha+4}$ is well defined almost surely.

Now we still have to define the resonating term $\xi \circ \sigma(D)\xi$ which appears in (21). This is where the enhancement of $\xi \in \mathcal{E}^\alpha$ is needed. Actually from the equation (21) we can see the resonating term as a continuous functional of $(\xi, \xi \circ \sigma(D)\xi)$ which push us to introduce the following definition.

**Definition 4.5.** Let $\alpha < -1$ and $\mathcal{E}^\alpha = \mathcal{E}^\alpha \times \mathcal{E}^{2\alpha+2}$. Then, the space of rough distributions $\mathcal{R}^\alpha$ is defined as the closure of the set

$$\{(\xi, \xi \circ \sigma(D)\xi + c), \quad \xi \in C^{\infty}(T^2_L), \quad c \in \mathbb{R}\} \quad \text{(22)}$$

for the topology of the Banach space $\mathcal{E}^\alpha$. A generic element of $\mathcal{R}^\alpha$ will be denoted by $\Xi = (\Xi_1, \Xi_2)$. If $\xi \in \mathcal{E}^\alpha$ is such that $\xi = \Xi$, we say that $\Xi$ is an enhancement (or lift) of $\xi$.

**Remark 4.6.** One should notice that if $\xi_\varepsilon$ is a mollification of a two dimensional white noise $\xi$ then the expectation of the regularized resonating term $\xi_\varepsilon \circ \sigma(D)\xi_\varepsilon$ blows up when $\varepsilon \to 0$ as

$$\mathbb{E}[\xi_\varepsilon \circ \sigma(D)\xi_\varepsilon] \sim -\log(\varepsilon).$$
Therefore, there is no hope to define $\xi \circ \sigma(D)\xi$ as the limit of $\xi_{\varepsilon} \circ \sigma(D)\xi_{\varepsilon}$ when $\varepsilon \to 0$. One should subtract the diverging expectation so that $\xi_{\varepsilon} \circ \sigma(D)\xi_{\varepsilon} - \mathbb{E}[\xi_{\varepsilon} \circ \sigma(D)\xi_{\varepsilon}]$ converges, as we shall see in section 5. This is precisely the reason why we need to introduce the constants $c \in \mathbb{R}$ to form the closure of the set of smooth functions introduced in (22).

**Remark 4.7.** Let us point that in general the space of rough distribution have a complicated algebraic structure however in our special case $\mathcal{X}^\alpha$ turn out to be the closure of the couple of smooth function in the space $\mathcal{E}^\alpha$ which of course a linear space. See Lemma A.7 for the exact statement.

Endowed with this definition, we can now define the resonating term $f \circ \xi$ simply by postulating the value of $\xi \circ \sigma(D)\xi = \Xi_2$ from the enhancement $\Xi$ of $\xi$. We have the following Proposition.

**Proposition 4.8.** Let $-4/3 < \alpha < -1$ and $-2/3 < \gamma \leq \alpha + 2$. Denote by $\Xi = (\xi, \Xi_2) \in \mathcal{X}^\alpha$ an enhancement of $\xi \in \mathcal{E}^\alpha$ and let $f \in \mathcal{D}_\xi^2$. We can now define $f \circ \xi$ as

$$f \circ \xi = f\Xi_2 + \mathcal{R}(f, \sigma(D)\xi, \xi) + f^\sharp \circ \xi.$$  \hfill (23)

We have the following bound

$$||f \circ \xi||_{H^{2\alpha+2}} \lesssim ||f||_{\mathcal{D}_\xi^2} ||\Xi||_{\mathcal{E}^\alpha}(1 + ||\Xi||_{\mathcal{E}^\alpha}).$$  \hfill (24)

**Proof.** If $f \in \mathcal{D}_\xi^2$ and $\Xi_2 \in \mathcal{E}^{2\alpha+2}$, we know from Proposition 3.1 by Bony that, if $\gamma + 2\alpha + 2 > 0$, then the product $f\Xi_2$ is well defined with regularity $\min(\gamma, 2\alpha + 2, \gamma + 2\alpha + 2) = 2\alpha + 2$ (under our assumptions on $\alpha$ and $\gamma$ i.e. $f\Xi_2 \in H^{2\alpha+2}$. Moreover, we have

$$||f\Xi_2||_{H^{2\alpha+2}} \lesssim ||f||_{H^\gamma}||\Xi_2||_{\mathcal{E}^{2\alpha+2}}.$$  \hfill (25)

From Proposition 4.3, the second term $\mathcal{R}(f, \sigma(D)\xi, \xi)$ is also well defined if $\gamma + 2\alpha + 2 > 0$ and $\mathcal{R}(f, \sigma(D)\xi, \xi) \in H^{\gamma+2\alpha+2}$. Using in addition the Schauder’s estimate Proposition 3.3, we obtain the following bound

$$||\mathcal{R}(f, \sigma(D)\xi, \xi)||_{H^{\gamma+2\alpha+2}} \lesssim ||f||_{H^\gamma}||\sigma(D)\xi||_{\mathcal{E}^{\alpha+2}}||\xi||_{\mathcal{E}^\alpha} \lesssim ||f||_{H^\gamma}||\xi||_{\mathcal{E}^\alpha}^2.$$  \hfill (26)

A sufficient condition for the existence of a positive number $\gamma > 0$ such that $\gamma \leq \alpha + 2$ and $\gamma + 2\alpha + 2 > 0$ is $-4/3 < \alpha < -1$. Under this condition, it is sufficient for us to pick any $\gamma$ such that $2/3 < \gamma \leq \alpha + 2$.

The resonating term $f^\sharp \circ \xi$ is also well defined because $f^\sharp \in H^{2\gamma}$, $\xi \in \mathcal{E}^\alpha$ with $\alpha + 2\gamma > 0$ and we have thanks to Bony’s Proposition 3.1 the upper bound

$$||f^\sharp \circ \xi||_{H^{2\gamma+\alpha}} \lesssim ||\xi||_{\mathcal{E}^\alpha}||f^\sharp||_{H^{2\gamma}}.$$  \hfill (27)

Gathering the three upper bounds (25), (26) and (27), we obtain (24). 

\[\blacksquare\]

The bound (24) on the norm of the resonating term implies the following crucial continuity approximation of $f \circ \xi$ with smooth functions.

**Corollary 4.9.** Let $-4/3 < \alpha < -1$, $2/3 < \gamma < \alpha + 2$ and $\Xi := (\xi, \Xi_2) \in \mathcal{X}^\alpha$ and $\Xi^\varepsilon := (\xi_\varepsilon, \xi_\varepsilon \circ \sigma(D)\xi_\varepsilon - c_\varepsilon), \varepsilon > 0, c_\varepsilon \in \mathbb{R}$ a family of smooth functions such that

$$\Xi^\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} \Xi \quad \text{in} \quad \mathcal{E}^\alpha.$$
Let also \( f_\epsilon \) be a smooth approximation of \( f \in \mathcal{D}_{\xi}^\gamma \) such that
\[
||f - f_\epsilon||_{H^\gamma} + ||f_\epsilon^2 - f^2||_{H^{2\gamma}} \xrightarrow{\epsilon \to 0} 0
\]
where \( f_\epsilon^2 \) is the smooth function defined from \( f_\epsilon \) and \( \xi_\epsilon \) as \( f_\epsilon^2 := f_\epsilon - f_\epsilon \prec \sigma(D)\xi_\epsilon \).

Then, we have the following continuity approximation of the resonating term \( f \circ \xi \) by smooth functions
\[
||f_\epsilon \circ \xi_\epsilon + c_\epsilon f - f \circ \xi||_{H^{2\alpha+2}} \xrightarrow{\epsilon \to 0} 0.
\]

**Proof.** Using the bilinearity of \( (f, \xi) \to f \circ \xi \) and the trilinearity of \( \mathcal{R} \), we easily check that
\[
||f_\epsilon \circ \xi_\epsilon + c_\epsilon f - f \circ \xi||_{H^{2\alpha+2}} \lesssim (||f||_{H^\gamma} + ||f_\epsilon^2 - f^2||_{H^{2\gamma}})(1 + ||\Xi||_{\sigma^\alpha})^2 + ||\Xi_\epsilon - \Xi||_{\sigma^\alpha}(1 + ||\Xi||_{\sigma^\alpha}) ||f||_{\mathcal{H}^\gamma_\xi},
\]
which yields the result. \( \square \)

We are finally ready to define the linear operator \( \mathcal{H} \) on the space \( \mathcal{D}_{\xi} \) of paracontrolled distribution.

**Definition 4.10.** Let \( \alpha \in (-4/3, -1) \), \( -\alpha < \gamma < \alpha + 2 \) and \( \Xi = (\xi, \xi_2) \in \mathcal{R}^{\alpha} \). We introduce the linear operator \( \mathcal{H} : \mathcal{D}_{\xi}^\gamma \to H^{\gamma-2} \) such that for \( f \in \mathcal{D}_{\xi}^\gamma \),
\[
\mathcal{H} f := -\Delta f + f \xi
\]
where \( f \xi \) is defined through the Bony decomposition
\[
f \xi := f \prec \xi + f \circ \xi + f \succ \xi
\]
where the resonating term \( f \circ \xi \in H^{2\alpha+2} \) is defined thanks to Proposition 4.8. Then \( \mathcal{H} \) can be seen as an unbounded operator on \( H^{\gamma-2}(\mathbb{T}_L^2) \) with domain \( \mathcal{D}_{\xi}^\gamma \).

**Remark 4.11.** Note that the operator \( \mathcal{H} \) as introduced in Definition 4.10 depends on the enhancement \( \Xi = (\xi, \xi_2) \in \mathcal{R}^{\alpha}(\mathbb{T}_L) \).

As explained in the heuristic discussion which motivated Definition 20, we expect the eigenfunctions of the linear operator \( \mathcal{H} \) (which will later be shown to be self-adjoint) to belong to the spaces \( \mathcal{D}_{\xi}^\gamma \) for \( \gamma < \alpha + 2 \). Those eigenfunctions will form an orthonormal basis of \( L^2(\mathbb{T}_L^2) \) and it is therefore natural to expect the spaces \( \mathcal{D}_{\xi}^\gamma \) for \( \gamma < \alpha + 2 \) to be dense in \( L^2(\mathbb{T}_L^2) \). This is the content of the following Lemma.

**Lemma 4.12.** Let \( \alpha < -1, \xi \in \mathcal{C}^\alpha \) and \( \frac{2}{3} < \gamma < \alpha + 2 \). Then, the space of paracontrolled distributions \( \mathcal{D}_{\xi}^\gamma \) is dense in \( L^2(\mathbb{T}_L^2, \mathbb{R}) \).

**Proof.** It is sufficient to prove that \( \mathcal{D}_{\xi}^\gamma \) is dense in \( C^\infty(\mathbb{T}_L^2) \). Let \( g \in C^\infty(\mathbb{T}_L^2) \), define the Fourier multiplier \( \sigma_\alpha(k) := \frac{1}{1 + a + |k|} \) for \( a > 0 \) and consider the map \( \Gamma : H^\gamma \to H^\gamma \) defined as:
\[
\Gamma(f) = \sigma_\alpha(D)(f \prec \xi) + g.
\]
The idea is to first prove that the map \( \Gamma \) admits a fixed point \( f_a \in \mathcal{D}_{\xi}^\gamma \). It is then straightforward to deduce that \( f_a \to g \) in \( L^2(\mathbb{T}_L^2, \mathbb{R}) \) from the fact that \( \sigma_\alpha(D)\xi \to 0 \) in \( \mathcal{C}^{\alpha-\delta} \) when \( a \to \infty \) for all \( \delta > 0 \). We can bound the Fourier multiplier \( \sigma_\alpha \): for any \( k \in \mathbb{Z}_L^2, r \in \mathbb{N}^2 \) and \( \theta \in [0, 1],
\[
|\partial^r \sigma_\alpha(k)| \lesssim \frac{1}{a^{1-\theta}(|k| + 1)^{2\theta + |r|}}.
\](28)
Using (28), the Bony estimate (3.1) and the Schauder’s inequality Proposition 3.3, we obtain

\[
\|\Gamma(f_1) - \Gamma(f_2)\|_{H^\gamma} \lesssim a^{(\gamma-(a+2))/2}\|\xi\|_{\mathcal{E}^\alpha}\|f_1 - f_2\|_{H^\gamma}
\]

(29)

We now fix \(a\) large enough such that \(a^{(\gamma-(a+2))/2}\|\xi\|_{\mathcal{E}^\alpha} < 1/2\). For such an \(a\), the map \(\Gamma\) is a contraction and therefore admits a unique fixed point \(f_a\). With the same arguments as above, we easily check that \(\sup_{a \geq 0} \|f_a\|_{H^\gamma} \lesssim \|g\|_{H^\gamma}\) and we eventually obtain

\[
\|f_a - g\|_{H^\gamma} \lesssim a^{\frac{1}{2}(\gamma-(a+2))}\|g\|_{H^\gamma}\|\xi\|_{\mathcal{E}^\alpha}
\]

which permits us to conclude that \(f_a \to g\) in \(H^\gamma\) when \(a\) goes to infinity and thus in \(L^2(\mathbb{T}_L^2, \mathbb{R})\).

We still have to prove that \(f_a \in \mathcal{D}_\xi^\gamma\). By definition, \(f_n \in H^\gamma\) and we just have to check that \(f_a - f_a \prec \sigma(D)\xi \in H^{2\gamma}\). We can decompose this function as

\[
f_a - f_a \prec \sigma(D)\xi = \sigma_a(D)(f_a \prec \xi) - f_a \prec \sigma_a(D)\xi + f_a \prec (\sigma_a(D) - \sigma(D))\xi + g.
\]

From Proposition 3.3, we know that the second term \((\sigma_a(D)(f_a \prec \xi) - f_a \prec \sigma_a(D)\xi) \in H^{2\gamma}\) with the following upper-bound

\[
\|\sigma_a(D)(f_a \prec \xi) - f_a \prec \sigma_a(D)\xi\|_{H^{2\gamma}} \lesssim \|f_n\|_{H^\gamma}\|\xi\|_{\mathcal{E}^\alpha}
\]

For the last term, we need to bound the derivatives of the Fourier multiplier \(\sigma_a(k) - \sigma(k) = -\frac{a}{(|k|+1)}(\alpha+|k|^2)^4\): \(|\partial^m(\sigma_a - \sigma)(k)| \lesssim a (1 + |k|)^{-4-|m|}\)

(30)

for \(k \in \mathbb{Z}_L^2\) and \(m \in \mathbb{N}^2\), where we have used the standard multi-index notation: if \(m = (m_1, m_2), \partial^m = \partial^{m_1}\partial^{m_2}\). Then, thanks to the Schauder’s estimate Proposition 3.3, we know that the regularity increases by four when applying the operator \(\sigma_a(D) - \sigma(D)\) so that \((\sigma_a(D) - \sigma(D))\xi \in \mathcal{E}^{(\alpha+4)}\) with the following upper-bound

\[
\|f_a \prec (\sigma_a(D) - \sigma(D))\xi\|_{H^{(\alpha+4)}} \lesssim \|f_a\|_{H^\gamma}\|\xi\|_{\mathcal{E}^{\alpha}}
\]

Gathering the above arguments, we can conclude that \(f_a \in \mathcal{D}_\xi^\gamma\) and the Lemma is proved. \(\Box\)

We now construct the resolvent operator \(G_a : L^2 \to \mathcal{D}_\xi^\gamma\) and establish that it is bounded operator for \(a > 0\) sufficiently large. The following Proposition basically proves that the (punctual) spectrum of \(\mathcal{H}\) is almost surely bounded from below (in the sense that it has an almost surely finite lowest element). The resolvent operator \(G_a\) shall play a crucial role in the next section to define the spectrum of \(\mathcal{H}\).

**Proposition 4.13.** Let \(-4/3 < \alpha < -1, 2/3 < \gamma < \alpha + 2\), \(\rho \in (\gamma - \frac{a+2}{2}, 1 + \frac{a}{2})\) and \(\Xi = (\xi, \xi_2) \in \mathcal{E}^\alpha\). Then, there exists \(A := A(||\Xi||_{\mathcal{E}^\alpha})\) such that for all \(a \geq A\) and \(g \in H^{2\gamma-2}(\mathbb{T}_L^2)\), the equation

\[
(\mathcal{H} + a)f = g
\]

admits a unique solution \(f_a \in \mathcal{D}_\xi^\gamma\). In addition, the maps \(G_a : g \in L^2(\mathbb{T}_L^2) \mapsto G_a g = f_a \in \mathcal{D}_\xi^\gamma, a \geq A\) is uniformly bounded: For any \(g \in H^{-\delta}\) with \(\delta \in [0, 2 - 2\gamma]\) and \(a \geq A\),

\[
\|G_a g\|_{\mathcal{D}_\xi^\gamma} \lesssim a^{-1+\gamma+\frac{\delta}{2}}\|g\|_{H^{-\delta}}
\]

(31)

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Proof. Our proof is based on a fixed point argument. For $A > 0$, we introduce the following auxiliary Banach space\footnote{\(C([A, +\infty), H^\gamma(T^2))\) denotes the space of continuous functions \([A, +\infty) \to H^\gamma(T^2)\).}

\[
\mathcal{D}_\xi^{\gamma,\rho,A} = \left\{ (f_a, f'_a)_{a \geq A} \in C([A, +\infty), H^\gamma)^2; \quad \|(f, f')\|_{\mathcal{D}_\xi^{\gamma,\rho,A}} < +\infty \right\}
\]

with:

\[
\|(f, f')\|_{\mathcal{D}_\xi^{\gamma,\rho,A}} = \sup_{a \geq A} \|f'_a\|_{H^\gamma} + \sup_{a \geq A} \frac{\|f^2_a\|_{H^{2\gamma}}}{a^\rho} + \sup_{a \geq A} ||f_a||_{H^\gamma}.
\]

For $f \in \mathcal{D}_\xi^{\gamma,\rho,A}$ and $(\xi, \Xi_2) \in \mathcal{D}^{\alpha}$, we can define the product $f_a \cdot \xi$ for any $a \geq A$ as

\[
f_a \cdot \xi := f_a \prec \xi + f_a \circ \xi + f_a \triangleright \xi, \quad f_a \circ \xi := f'_a \triangleright \Xi_2 + D(f'_a, \sigma(D)\xi, \xi) + f_a \circ \xi. \quad (32)
\]

Let us introduce the map $\mathcal{M}$ defined for any $(f, f') \in \mathcal{D}_\xi^{\gamma,\rho,A}$ as

\[
\mathcal{M}(f, f') := (M(f, f'), f), \quad M(f, f')_a := \sigma_a(D)(f_a \circ \xi - g).
\]

where $\sigma_a(k) := -1/(a + |k|^2)$ for $a > 2$. It is sufficient to prove that the map $\mathcal{M}$ admits a unique fixed point in the space $\mathcal{D}_\xi^{\gamma,\rho,A}$.

We first prove that $\mathcal{M}(\mathcal{D}_\xi^{\gamma,\rho,A}) \subset \mathcal{D}_\xi^{\gamma,\rho,A}$ by checking that $M(f, f')_a \in H^\gamma$ and that if $(f, f') \in \mathcal{D}_\xi^{\gamma,\rho,A}$, then we have $M(f, f')_a := M(f, f') - f \prec \sigma(D)\xi \in H^{2\gamma}$ (using the notations introduced, we have $M(f, f')_a = f$).

We have the following decomposition

\[
M(f, f')_a = \sigma_a(D)(f_a \prec \xi) + \sigma_a(D)(f_a \circ \xi + \xi \prec f_a) - \sigma_a(D)g.
\]

Using the fact that $|\partial^m \sigma_a(k)| \leq a^{\theta - 1}|(k| + 1)^{-2\theta - |m|}$ for any $m \in \mathbb{Z}^2$, we use again the Schauder and Bony estimates which give the following upper-bounds

\[
\|\sigma_a(D)(f_a \prec \xi)\|_{H^\gamma} \leq a^{(\gamma - (\alpha + 2)/2)}\|f_a\|_{H^\gamma}\|\xi\|_{\Phi^\alpha},
\]

\[
\|\sigma_a(D)(f_a \circ \xi + f_a \triangleright \xi)\|_{H^\gamma} \leq a^{-1/\alpha/2}(\|f_a\|_{H^\gamma}\|\xi\|_{\Phi^\alpha} + \|f_a \circ \xi\|_{H^{2\alpha + 2}}),
\]

\[
\|\sigma_a(D)g\|_{H^\gamma} \leq a^{1/2 - 1}\|g\|_{L^2}.
\]

To bound $\|f_a \circ \xi\|_{H^{2\alpha + 2}}$, we write $f_a \circ \xi = f_a \circ \xi - f_a^\sharp \circ \xi + f_a^\sharp \circ \xi$ and, using also (32),

\[
\|f_a \circ \xi - f_a^\sharp \circ \xi\|_{H^{2\alpha + 2}} \lesssim \left\|f'_a\right\|_{H^\gamma} (\|\Xi_2\|_{\Phi^{2\alpha + 2}} + \|\xi\|^2_{\Phi^\alpha}). \quad (33)
\]

and

\[
\|f_a^\sharp \circ \xi\|_{H^{2\gamma + \alpha}} \lesssim \left\|f'_a\right\|_{H^{2\gamma}} \left\|\xi\right\|_{\Phi^\alpha} \lesssim a^{\theta} ||(f, f')||_{\mathcal{D}_\xi^{\gamma,\rho,A}} \|\xi\|_{\Phi^\alpha}. \quad (34)
\]

Gathering those upper-bounds, we obtain

\[
\|M(f, f')_a\|_{H^\gamma} \lesssim a^{\max\left(\frac{\gamma - (\alpha + 2)}{2}, \frac{\gamma - (\alpha + 2)}{2}, \frac{\gamma - (\alpha)}{2}\right)} \|(f, f')\|_{\mathcal{D}_\xi^{\gamma,\rho,A}} \left(\|\xi\|_{\Phi^\alpha} + \|\Xi_2\|_{\Phi^{2\alpha + 2}} + a^{1/2 - 1}\|g\|_{L^2}\right). \quad (35)
\]
We now prove that $M(f, f')^2 := M(f, f') - f \prec \sigma(D)\xi \in H^{2\gamma}$. For this, we decompose $M(f, f')^2$ as follows for $a \geq A$.

$$M(f, f')_a = \sigma_a(D)(f_a \prec \xi) - f_a \prec \sigma_a(D)\xi + f_a \prec (\sigma_a - \sigma)(D)\xi + \sigma_a(D)(f_a \circ \xi - f_a^2 \circ \xi) + \sigma_a(D)(f_a^2 \circ \xi) + \sigma_a(D)(f \prec \xi) - \sigma_a(D)g.$$ 

Using the Schauder estimate recalled in Proposition 3.3, we can easily see that

$$\|\sigma_a(D)(f_a \prec \xi) - f_a \prec \sigma_a(D)\xi\|_{H^{2\gamma}} \lesssim a^{-\alpha/2} \|f_a\|_{H^\gamma} \|\xi\|_{\mathcal{E}^{\alpha}}.$$ 

For the last term, we use the fact that $|\partial^m(\sigma_a - \sigma)(k)| \lesssim a^\theta |k|^{-2\theta - |m|}$ for any $m \in \mathbb{N}^2$ and the Schauder estimate to obtain

$$\|f_a \prec (\sigma_a - \sigma)(D)\xi\|_{H^{2\gamma}} \lesssim a^{-\alpha/2} \|f_a\|_{H^\gamma} \|\xi\|_{\mathcal{E}^{\alpha}}.$$

The Schauder estimate and the bound $|\partial^m\sigma_a(k)| \leq a^\theta |k|^{-2\theta - |m|}$ valid for any $m \in \mathbb{N}^2$ permits us to obtain the following upper bounds (using (33) and (34))

$$\|\sigma_a(D)(f_a \prec \xi)\|_{H^{2\gamma}} \lesssim a^{-\alpha/2} \|f_a\|_{H^\gamma} \|\xi\|_{\mathcal{E}^{\alpha}},$$

$$\|\sigma_a(D)(f_a \circ \xi - f_a^2 \circ \xi)\|_{H^{2\gamma}} \lesssim a^{-\alpha/2} \|f_a^2\|_{H^\gamma} \left(\|\xi\|_{\mathcal{E}^{2\alpha}} + \|\xi\|^2_{\mathcal{E}^{\alpha}}\right),$$

$$\|\sigma_a(D)(f_a^2 \circ \xi)\|_{H^{2\gamma}} \lesssim a^{-\alpha/2} \|f_a^2\|_{H^\gamma} \|\xi\|_{\mathcal{E}^{\alpha}}.$$

We deduce (using also the inequality $\|f_a^2\|_{H^{2\gamma}} \leq a^\alpha \|(f, f')\|_{\mathcal{E}^{\gamma,\rho, A}}$) that for all $a > 1$,

$$a^{-\alpha} \|M(f, f')_a\|_{H^{2\gamma}} \lesssim \|(f, f')\|_{\mathcal{E}^{\gamma,\rho, A}} \left(a^\gamma (a+2) (\|\xi\|_{\mathcal{E}^{2\alpha}} + \|\xi\|^2_{\mathcal{E}^{\alpha}}) + a^\max(\gamma - \frac{a+2}{2} - \rho - 1) |\xi| \|\mathcal{E}^{\alpha} \right) + a^{\gamma - 1 - \rho} \|g\|_{L^2}. $$

Gathering the above inequalities, we can finally establish the following upper-bound for any $a \geq A$,

$$\|M(f, f')\|_{\mathcal{E}^{\gamma,\rho, A}} \lesssim A^{-\theta} \|(f, f')\|_{\mathcal{E}^{\gamma,\rho, A}} (\|\xi\|_{\mathcal{E}^{\alpha}} + \|\xi\|^2_{\mathcal{E}^{\alpha}} + \|\Xi\|_{\mathcal{E}^{2\alpha}}) + \sup_{a \geq A} \|f_a\|_{H^\gamma} + A^{\gamma/2 - 1} \|g\|_{L^2} \quad (36) $$

where $\theta := \min\left(\frac{\alpha + 2 - \gamma}{2}, 1 + \frac{\alpha}{2} - \rho, \rho - \gamma + \frac{\alpha + 2}{2}\right) > 0$ (under the constraints imposed on the parameters) and conclude that $M(f, f') \in \mathcal{D}^{\rho, A}$.

Using the linearity of $(f, f') \mapsto M(f, f')$, we get, for any $(f, f'), (h, h') \in \mathcal{D}^{\gamma,\rho, A}$, the inequality

$$\|M(f, f') - M(h, h')\|_{\mathcal{E}^{\gamma,\rho, A}} \lesssim A^{-\theta} \|(f, f') - (h, h')\|_{\mathcal{E}^{\gamma,\rho, A}} (\|\xi\|_{\mathcal{E}^{\alpha}} + \|\xi\|^2_{\mathcal{E}^{\alpha}} + \|\Xi\|_{\mathcal{E}^{2\alpha}}) \quad (37)$$

The function $M$ fails to be contracting. This issue can be circumvented by iterating $M$ one more time: Using (35) and (37), we see that $M^2(f, f') := (M(M(f, f'), f), M(f, f'))$ satisfies

$$\|M^2(f, f') - M^2(h, h')\|_{\mathcal{E}^{\gamma,\rho, A}} \lesssim A^{-\theta} \|(f, f') - (h, h')\|_{\mathcal{E}^{\gamma,\rho, A}} (\|\xi\|_{\mathcal{E}^{\alpha}} + \|\xi\|^2_{\mathcal{E}^{\alpha}} + \|\Xi\|_{\mathcal{E}^{2\alpha}})^2 \quad \text{sup} \quad a \geq A \|M(f, f')_a - M(h, h')_a\|_{H^\gamma}$$

$$\lesssim A^{-\theta} \|(f, f') - (h, h')\|_{\mathcal{E}^{\gamma,\rho, A}} (1 + \|\xi\|_{\mathcal{E}^{\alpha}} + \|\xi\|^2_{\mathcal{E}^{\alpha}} + \|\Xi\|_{\mathcal{E}^{2\alpha}})^2.$$
If $A$ is large enough so that $A^{-\theta}(1 + ||\xi||_{\varrho, a} + ||\xi||_{\varrho, a}^2 + ||\Xi||_{\varrho, a}^2 + ||\Xi||_{\varrho, a}^{2+2})^2 \ll 1$, the map $M^2 : \tilde{G}^{\gamma, \varrho, A}_\xi \to \tilde{G}^{\gamma, \varrho, A}_\xi$ is a contraction. The fixed point Theorem insures that the map $M$ admits a unique fixed point $(f, f') \in \tilde{G}^{\gamma, \varrho}$. Observing that $f' = f$, we deduce that for all $a \geq A$, $f_a \in \mathcal{D}^{\gamma}_{\xi}$ and Eq. (36) shows that for any $a \geq A$,

$$||G_a f||_{\varrho, a}^2 = ||f_a||_{\varrho, a}^2 \lesssim a^\gamma - 1 ||g||_{L^2}^2,$$

so that the map $G_a$ is bounded.

**Remark 4.14.** Proposition 4.13 implies that the spectrum of $\mathcal{H}$ is contained in the interval $(-A(\Xi), +\infty)$. Moreover using the resolvent identity $G_a - G_b = (a-b)G_a G_b$ gives us by induction that

$$||G_a f||_{\varrho, a}^2 \lesssim \exp(C(A(\Xi) - a)) ||f||_{L^2}$$

for all point $a < A(\Xi)$ which is not contained in the spectrum and where $C$ is a positive constant.

The maps $G_a, a \geq A(\Xi)$ are constructed through a fixed point procedure. We can deduce continuity bounds with respect to the rough distribution $(\xi, \Xi) \in \mathcal{D}^{\alpha}$ (noise) for those maps.

**Lemma 4.15.** Let $\alpha \in (-4/3, -1)$, $\gamma \in (\frac{\beta}{2}, \alpha + 2]$. Then there exist two constants $C, \varrho > 0$ such that for all $\Xi = (\xi, \Xi_2) \in \mathcal{D}^{\alpha}$ and $a \geq C(1 + ||\Xi|| + ||\Xi||^2)^{\frac{2}{3}}$, we have the following continuity bound

$$||G_a(\Xi)g - G_a(\tilde{\Xi})g||_{H^\gamma} \lesssim ||g||_{L^2} ||\Xi - \tilde{\Xi}||_{\varrho, a} (1 + ||\Xi||_{\varrho, a} + ||\Xi||_{\varrho, a}^2)^2$$

where $G_a(\Xi) : L^2 \to \mathcal{D}^{\gamma}_{\xi}$ (respectively $G_a(\tilde{\Xi}) : L^2 \to \mathcal{D}^{\gamma}_{\xi}$) is the resolvent operator associated to the rough distribution $\Xi \in \mathcal{D}^{\alpha}$ (resp. $\tilde{\Xi}$) as constructed in Proposition 4.13.

**Proof.** From Proposition 4.13, the operator $G_a^{\Xi}$ is well defined for $a \geq A(\Xi)$ and satisfies

$$||G_a^{\Xi}g||_{H^\gamma} \lesssim a^{\gamma/2} ||g||_{L^2}.$$ 

Let $a \geq A(\Xi)_{\varrho, a}$ and, to simplify notations, set $f_a = G_a^{\Xi}g$ (resp. $\tilde{f}_a := G_a(\tilde{\Xi})g$). Using the relations $f_a = \sigma_a(D)(f_a \xi - g)$ satisfied by $f_a$ and $\tilde{f}_a$, we deduce that

$$f_a - \tilde{f}_a = \sigma_a(D)(f_a \xi - f_a \xi + f_a \xi - f_a \xi + f_a \Xi_2 - f_a \Xi_2 + f_a \xi - f_a \xi) + \sigma_a(D)(\mathcal{R}(f_a, \xi, \sigma(D)\xi) - \mathcal{R}(\tilde{f}_a, \xi, \sigma(D)\xi)).$$

and

$$f_a^2 - \tilde{f}_a^2 = \sigma_a(D)(f_a \xi - f_a \xi + f_a \Xi_2 - f_a \Xi_2 + f_a \xi - f_a \xi) + \sigma_a(D)(\mathcal{R}(f_a, \xi, \sigma(D)\xi) - \mathcal{R}(\tilde{f}_a, \xi, \sigma(D)\xi)) + C_a(f_a, \xi) - C_a(\tilde{f}_a, \xi) + f_a \prec (\sigma_a - \sigma)(f_a \xi - \tilde{f}_a \xi).$$

where $C_a(f_a, \xi) = \sigma_a(D)(f_a \xi - f_a \xi) - f_a \prec \sigma_a(D)\xi$. Therefore using the same argument as in the proof of Proposition 4.13 (based on the Schauder, Bony and commutator estimates) and the bilinearity of $\mathcal{G}_a$, we obtain

$$a^{-\rho} ||f_a^2 - \tilde{f}_a^2||_{H^\gamma} \lesssim a^{-\left(1 + \alpha/2\right)} \left( a^{-\rho} ||f_a^2 - \tilde{f}_a^2||_{H^\gamma} ||\xi||_{\varrho, a} + a^{-\rho} ||f_a^2||_{H^2} ||\xi||_{\varrho, a} - ||f_a||_{H^2} ||\xi||_{\varrho, a} - ||\xi||_{\varrho, a} \right) + a^{-\min\left(\frac{\beta + \alpha}{2}, \frac{\beta + \alpha}{2} + \beta + \gamma, \beta + \gamma + 1, \gamma + 1\right)} ||f_a - \tilde{f}_a||_{H^\gamma} (||\xi||_{\varrho, a} + ||\xi||_{\varrho, a}^2 + ||\Xi_2||_{\varrho, a}^2) + a^{-\min\left(\frac{\beta + \alpha}{2}, \frac{\beta + \alpha}{2} + \gamma, \beta + \gamma + 1, \beta + \gamma + 1\right)} ||f_a||_{H^\gamma} (||\xi - \tilde{\xi}||_{\varrho, a}(1 + ||\xi||_{\varrho, a}) + ||\tilde{\xi}||_{\varrho, a} + ||\Xi_2 - \Xi_2||_{\varrho, a}^2).$$

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Thus it suffices to bound the term \( a^{-\rho}||f_a - \tilde{f}_a||_{H^{2\gamma}} \leq a^{\gamma/2-1}||g||_{L^2}||\xi - \tilde{\xi}||_{\mathcal{C}^\alpha} \) \( + a^{-\min\left(\frac{2+\alpha-\gamma}{2} + \rho, \rho - \gamma + \frac{\alpha+2}{2}\right)}||f_a - \tilde{f}_a||_{H^{2\gamma}}||\Xi||_{\mathcal{C}^\alpha} (1 + ||\xi||_{\mathcal{C}^\alpha}) \)
\( + a^{-\min\left(\frac{2+\alpha-\gamma}{2} + \rho, \rho - \gamma + \frac{\alpha+2}{2}\right)}||g||_{H^\gamma} ||\xi - \tilde{\xi}||_{\mathcal{C}^\alpha} (1 + ||\xi||_{\mathcal{C}^\alpha} + ||\tilde{\xi}||_{\mathcal{C}^\alpha}) + ||\Xi_2 - \tilde{\Xi}_2||_{\mathcal{C}^{2\alpha+2}} \).

Thus it suffices to bound the term \( ||f_a - \tilde{f}_a||_{H^\gamma} \), which can be treated easily with the same argument and we finally obtain
\[
\| f_a - \tilde{f}_a \|_{H^\gamma} \lesssim a^{-\min(1+\alpha/2, \frac{\alpha+2-\gamma}{2})} \| f_a - \tilde{f}_a \|_{H^\gamma} (\| \xi \|_{\mathcal{C}^\alpha} + \| \xi \|_{\mathcal{C}^\alpha} + \| \tilde{\xi} \|_{\mathcal{C}^\alpha} + \| \Sigma_2 \|_{\mathcal{C}^{2\alpha+2}} ) \\
+ a^{-\min(1+\alpha/2, \frac{\alpha+2-\gamma}{2})} \bigl(1+\gamma\bigr) \| \tilde{f}_a \|_{H^\gamma} .
\]

It is now plain to deduce that for \( a \) large enough
\[
\| f_a - \tilde{f}_a \|_{H^\gamma} \lesssim a^{-\min(1+\alpha/2, \frac{\alpha+2-\gamma}{2})} \bigl(1+\gamma\bigr) \| \tilde{f}_a \|_{H^\gamma}
\]
and then since \( \| f_a^2 - \tilde{f}_a \|_{H^{2\gamma}} \) is controlled by \( \| f_a - \tilde{f}_a \|_{H^\gamma} \), the needed result follows. \( \square \)

**Remark 4.16.** Another important remark we shall use later is that the resolvent \( \mathcal{G}_a \) allows us to describe the space \( \mathcal{D}_2^\gamma \), indeed if \( \alpha, \gamma, \Xi \) and \( A(\Xi) \) satisfy the assumptions of Proposition 4.13, then for all \( a \geq A \) we have that \( \mathcal{G}_a H^{2\gamma-2} = \mathcal{D}_2^\gamma \).

### 4.2 Restriction on a space of strongly paracontrolled distribution

So far, we have constructed the operator \( \mathcal{H} \) on the space \( \mathcal{D}_2^\gamma \) with values in the Sobolev space \( H^{2\gamma-2} \). The space \( \mathcal{D}_2^\gamma \) depends on the realization of the noise \( \xi \) and our construction also uses an enhancement \( (\xi, \Xi_2) \in \mathcal{E}^\alpha (\Xi_2 \) is an additional input) of the noise \( \xi \in \mathcal{C}^\alpha \). At this point, the function \( g := \mathcal{H} f \in H^{2\gamma-2} \) is a distribution with regularity \( \gamma - 2 \leq \alpha \).

In this subsection, we define a smaller space \( \mathcal{D}_2^\gamma \) (continuously embedded in \( \mathcal{D}_2^\gamma \)) such that, when restricted to the subspace \( \mathcal{D}_2^\gamma \), the operator \( \mathcal{H} \), as constructed in Definition 4.10, takes values in \( L^2(T_{\Sigma}^2) \). We shall also establish that the resolvent \( \mathcal{G}_a : L^2 \to \mathcal{D}_2^\gamma \) is a bounded (continuous) operator.

Before giving the definition of the subspace \( \mathcal{D}_2^\gamma \), let us analyze the regularity of the eigenvectors of \( \mathcal{H} \) such that \( \mathcal{H} f = \lambda f \) for some \( \lambda \in \mathbb{R} \).

First, if \( f \in \mathcal{D}_2^\gamma \) with \( 2/3 < \gamma < \alpha + 2 \), it is easy to see that
\[
-\Delta f = -\Delta f^2 - 2 \nabla f \prec \nabla (\sigma(D)\xi) + (1 - \Delta) f \prec \sigma(D)\xi - f \prec \xi +
\]
where \( \nabla \) is the gradient. It follows that
\[
\mathcal{H} f = -\Delta f^2 - 2 \nabla f \prec \nabla (\sigma(D)\xi) + (1 - \Delta) f \prec \sigma(D)\xi + f \Xi_2 + \mathcal{R}(f, \xi, \sigma(D)\xi) + f^2 \circ \xi + f \succ \xi .
\]
Checking the regularity of each term in this latter expression, we see that \( \mathcal{H} f \in H^{2\gamma-2} \). We note that \( 2\gamma - 2 \in (-2/3,0) \) if \( 2/3 < \gamma < \alpha + 2 \).
Now, coming back to the eigenvalue problem, we see that, if \( f \in \mathcal{D}_\xi^{\gamma} \) (for \( \gamma < \alpha + 2 \)) is an eigenvector of the operator \( \mathcal{H} \), then the associated remainder \( f^\flat \) satisfies

\[
(1 - \Delta) f^\flat = \lambda f + 2 \nabla f - \nabla(\sigma(D)\xi) - (1 - \Delta) f \prec \sigma(D)\xi - f \succ \xi - f\Xi_2 - \mathcal{R}(f, \xi, \sigma(D)\xi) - f^\flat \circ \xi + f^\flat.
\]

Rewriting this equality with Fourier multipliers, we get

\[
\text{Lemma 4.19. Let } B \in \mathcal{D}_\xi^{\alpha} \text{ be a paracontrolled distribution, then for } \xi, \sigma(D)\xi \in \mathcal{D}_\xi^{\alpha}, \text{ we define the bilinear form }
\]

\[
B : \langle f, \xi \rangle \in \mathcal{D}_\xi^{\alpha}, \quad \sigma(D)(2 \nabla f - \nabla(\sigma(D)\xi)) - (1 - \Delta) f \prec \sigma(D)\xi - f \succ \xi - f\Xi_2 \in H^{2\gamma}.
\]

and where the remainder term \( f^\flat \in H^{3\gamma} \).

This discussion motivates the following definition for the subspace \( \mathcal{D}_\Xi^\gamma \subseteq \mathcal{D}_\xi^{\alpha} \) which is such that \( \mathcal{D}_\Xi^\gamma \) contains the eigenvectors of \( \mathcal{H} \).

**Definition 4.17.** Let \( -4/3 < \alpha < -1, \ -2 < \gamma \leq \alpha + 2 \) and \( \Xi = (\xi, \Xi_2) \in \mathcal{X}^\alpha \), we define the space of strong paracontrolled distribution

\[
\mathcal{D}_\Xi^\gamma := \left\{ f \in H^{\gamma}; \quad f^\flat := f - f \prec \sigma(D)\xi - B(f, \Xi) \in H^2 \right\}
\]

equipped with the following scalar product

\[
\langle f, g \rangle_{\mathcal{D}_\Xi^\gamma} = \langle f, g \rangle_{H^{\gamma}} + \langle f^\flat, g^\flat \rangle_{H^2}.
\]

**Remark 4.18.** Of course, the Hilbert space \( \mathcal{D}_\Xi^\gamma \) is continuously embedded in \( \mathcal{D}_\xi^{\alpha} \) as \( f^\flat := f - f \prec \sigma(D)\xi = B(f, \Xi) + f^\flat \in H^{2\gamma} \) for \( f \in \mathcal{D}_\Xi^\gamma \). It is also clear that actually \( \mathcal{D}_\Xi^\gamma \subseteq H^{\alpha+2} \) so that the spaces \( \mathcal{D}_\Xi^\gamma, \gamma \in [2/3; \alpha + 2] \) are all equal. We drop the super script \( \gamma \) for the space

\[
\mathcal{D}_\Xi := \left\{ f \in H^{\alpha+2}; \quad f^\flat := f - f \prec \sigma(D)\xi - B(f, \Xi) \in H^2 \right\}.
\]

If \( \xi \) is a smooth function, then \( \mathcal{D}_\Xi = H^2 \).

The following technical estimates on the bilinear form \( B \) will be useful in the sequel.

**Lemma 4.19.** Let \( -4/3 < \alpha < -1, 2/3 < \gamma \leq \alpha + 2 \). For \( \alpha \geq 2 \), we define the bilinear form \( B_a \) such that for \( (f, \Xi) \in H^\gamma \times \mathcal{X}^\alpha \),

\[
B_a(f, \Xi) := \sigma_a(D)(2 \nabla f - \nabla(\sigma(D)\xi)) - (1 - \Delta) f \prec \sigma(D)\xi - f \succ \xi - f\Xi_2
\]

where for \( \alpha > 2 \), \( \sigma_a(k) := \frac{1}{a + |k|^2} \). Then we have the following estimates

\[
\|B_a(f, \Xi)\|_{H^{2\gamma}} \lesssim a^{-\frac{2 - \gamma + 2\alpha}{2}}, \quad \|f\|_{H^{\gamma}}, \quad \|\Xi\|_{\mathcal{D}^\alpha}, \quad \|(B - B_a)(f, \Xi)\|_{H^{2\gamma+2}} \lesssim a\|f\|_{H^{\gamma}}, \quad \|\Xi\|_{\mathcal{D}^\alpha}.
\]

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Proof. Towards the first estimate, it suffices to notice that, for any \( \theta \in (0, 1) \) and \( m \in \mathbb{N}^2 \), \( |\partial^m a(k)| \lesssim a^{-(1-\theta)(|k|+1)^{-20-|m|}} \) and the result follows from the Schauder estimate Proposition 3.3 applied for the particular value \( \theta = (\gamma - \alpha)/2 \).

The second inequality is proved with the same method recalling Eq. (30)

\[
|\partial^m (\sigma - \alpha)(k)| \lesssim a \,(|k|+1)^{-4-|m|}
\]

for all \( m \in \mathbb{N}^2 \) and using the Schauder estimate again. \( \square \)

We can finally prove that, when restricted to the space \( \mathcal{D}_2 \) of strongly paracontrolled distributions, the linear operator \( \mathcal{H} \) as defined in Definition 4.10, takes values in \( L^2 \).

**Proposition 4.20.** If \( f \in \mathcal{D}_2 \), then \( \mathcal{H} f \in L^2(T^2) \).

**Proof.** If \( f \in \mathcal{D}_2 \), we have

\[
\mathcal{H} f = (1 - \Delta)(f \prec \sigma(D)\xi) + (1 - \Delta)B(f, \Xi) - \Delta f^\flat - f \prec \sigma(D)\xi - B(f, \Xi) + \xi f \\
= -\Delta f^\flat - f \prec \sigma(D)\xi - B(f, \Xi) + \mathcal{R}(f, \xi, \sigma(D)\xi) + (B(f, \Xi) + f^\flat) \circ \xi \in H^{3a+4} \subset L^2(T^2) .
\]

In the following Proposition, we prove that any element \( f \in \mathcal{D}_2 \) can be approximated with smooth functions \((f_\varepsilon)_{\varepsilon>0} \) in \( H^2 \).

**Proposition 4.21.** Let \(-4/3 < \alpha < -1, -2/3 < \gamma < \alpha + 2 \) and \( \Xi := (\xi, \tilde{\Xi})_2, \tilde{\Xi} = (\hat{\xi}, \widetilde{\Xi})_2 \in \mathcal{R}^\alpha \) Then, for any \( f \in \mathcal{D}_2^\alpha \), there exists a function \( g \in \mathcal{D}_2^\alpha \) such that

\[
||f - g||_{H^\gamma} + ||f^\flat - g^\flat||_{H^2} \lesssim ||f||_{H^{\gamma}(1 + ||\tilde{\Xi}||_{\mathcal{E}^\alpha})}||\Xi - \tilde{\Xi}||_{\mathcal{E}^\alpha}
\]

where \( g^\flat := g - g \prec \sigma(D)\tilde{\xi} - B(f, \tilde{\Xi}) \). In particular, if \( (\xi_\varepsilon, c_\varepsilon) \in C^\infty(T^2) \times \mathbb{R} \) is a sequence such that \( (\xi_\varepsilon, \xi_\varepsilon \circ \sigma(D)\xi_\varepsilon - c_\varepsilon) \) converges to \( \Xi \) in \( \mathcal{E}^\alpha \) as \( \varepsilon \to 0 \), then there exists a sequence \((f_\varepsilon)_{\varepsilon>0} \) in \( H^2 \) such that

\[
||f_\varepsilon - f||_{H^\gamma} + ||f^\flat_\varepsilon - f^\flat||_{H^2} \to 0 \quad \varepsilon \to 0
\]

with \( f^\flat_\varepsilon := f_\varepsilon - f_\varepsilon \prec \sigma(D)\xi_\varepsilon - B(f_\varepsilon, (\xi_\varepsilon, (\xi_\varepsilon \circ \sigma(D)\xi_\varepsilon - c_\varepsilon)) \).

**Proof.** Let \( f \in \mathcal{D}_2 \) and define the map \( \Gamma : H^\gamma \to H^\gamma \) such that

\[
\Gamma(g) := g \prec \sigma_a(D)\tilde{\xi} + B_a(g, \tilde{\Xi}) + f - f \prec \sigma_a(D)\xi - B_a(f, \Xi)
\]

where for \( a > 2, \sigma_a(k) := \frac{1}{a+|k|^2} \) and

\[
B_a(g, \tilde{\Xi}) = \sigma_a(D)(2\nabla g \prec \nabla(D)\tilde{\xi}) + \Delta g \prec \sigma(D)\tilde{\xi} - g \prec \tilde{\xi} - g\tilde{\Xi}_2, \\
B_a(f, \Xi) = \sigma_a(D)(2\nabla f \prec \nabla(D)\xi) + \Delta f \prec \sigma(D)\xi - f \prec \xi - f\Xi_2 .
\]

Using bilinearity and the Schauder’s estimate Proposition 3.3, we have the following bound for any \( g_1, g_2 \in H^\gamma \),

\[
||\Gamma(g_1) - \Gamma(g_2)||_{H^\gamma} \lesssim a^{(\gamma-a-2)/2}(1 + ||\tilde{\Xi}||_{\mathcal{E}^\alpha})||g_1 - g_2||_{H^\gamma} \lesssim a^{(\gamma-a-2)/2}(1 + ||\tilde{\Xi}||_{\mathcal{E}^\alpha})||g_1 - g_2||_{H^\gamma} .
\]
Moreover, the operator $H$

**Definition 4.10.** We introduce the two linear operators

$$
\begin{align*}
\sigma (k) - \sigma (a) &= \frac{a}{(\alpha + 1) (\alpha + k^2)}\quad \text{and Eq. (30), it is easy to check that}
\end{align*}
$$

$$(\sigma (D) - \sigma (a)) (2 \partial f < \partial (\sigma (D) \xi) + \Delta f < \sigma (D) \xi - f > \xi - f \varepsilon_2 ) \in H^{2+2}.$$

Therefore, using also the definition of $f \in \mathcal{D}_\varepsilon$ which implies that $f^0 \in H^2$, we deduce that $g \in \mathcal{D}_\varepsilon$.

Now, using the Bony and Schauder's Propositions 3.1 and 3.3, we obtain

$$
\|f - g\|_{H^\gamma} \lesssim \|f - g\|_{H^\gamma} + \|f - \sigma (D) (\hat{\xi} - \xi)\|_{H^{\gamma}}
$$

By definition of the space $X$ and Eq. (30), it is easy to check that

$$
\|f - g\|_{H^\gamma} \lesssim \|f - g\|_{H^\gamma} + \|f - \sigma (D) (\hat{\xi} - \xi)\|_{H^{\gamma}}
$$

**Fixing $\alpha$ sufficiently large, we have the bound**

$$
\|f - g\|_{H^\gamma} \lesssim \|f - g\|_{H^\gamma} + \|f - \sigma (D) (\hat{\xi} - \xi)\|_{H^{\gamma}}
$$

Observing that:

$$
\begin{align*}
&\frac{\sigma (D) - \sigma (a)}{D} \xi - f < \sigma (D) (\hat{\xi} - \xi) + B (f, \Xi) + B (f, \Xi) - B (f, \Xi)
\end{align*}
$$

Recalling that $(\sigma (D) - \sigma (D)) \xi \in \mathcal{P}^{\alpha+4}$, we can finally prove that:

$$
\|f^0 - g^0\|_{H^2} \lesssim \|f - g\|_{H^\gamma} + \|f - g\|_{H^\gamma} + \|f - \sigma (D) (\hat{\xi} - \xi)\|_{H^{\gamma}} \lesssim \|f - g\|_{H^\gamma} + \|f - \sigma (D) (\hat{\xi} - \xi)\|_{H^{\gamma}}
$$

**Proposition 4.22.** Let $-4/3 < \alpha < -1$, $2/3 < \gamma < \alpha + 2$, $\Xi = (\xi, \zeta_2)$, $\Xi = (\hat{\xi}, \hat{\zeta}_2) \in \mathcal{D}_\alpha$. Following Definition 4.10, we introduce the two linear operators $\mathcal{H} : \mathcal{D}_\varepsilon \to \mathcal{D}_\varepsilon$, $g \in \mathcal{D}_\varepsilon$. We have the following continuity upper-bound for any $f \in \mathcal{D}_\varepsilon, g \in \mathcal{D}_\varepsilon$.

$$
\|\mathcal{H} (\Xi) f - \mathcal{H} (\Xi) g\|_{L^2} \lesssim \left( \|f - g\|_{H^\gamma} + \|f^0 - g^0\|_{H^2} + \|\Xi - \hat{\Xi}\|_{\mathcal{P}^{\alpha}} \right)
$$

$$
\times (1 + \|\hat{\Xi}\|_{\mathcal{P}^{\alpha}})^2 \times (1 + \|g\|_{H^{\alpha+2}} + \|g^0\|_{H^2}).
$$

Moreover, the operator $\mathcal{H} : \mathcal{D}_\varepsilon \to L^2 (\mathcal{D}_\varepsilon)$ is symmetric in $L^2$ in the sense that, for any $f, g \in \mathcal{D}_\varepsilon$, we have

$$
\langle \mathcal{H} f, g \rangle_{L^2} = \langle f, \mathcal{H} g \rangle_{L^2}.
$$

**Proof.** The estimate (41) follows from the expansion obtained in (40) for $\mathcal{H}^1 f$ and $\mathcal{H}^2 g$, from the bilinearity of the maps $\mathcal{B}$ and $B$, from Proposition 4.3 (continuity bound for $\mathcal{B}$) and Proposition 4.19 (continuity bound for $B$) and from the Schauder’s and Bony’s estimates Propositions 3.3 and 3.1 for paraproduts.

**Towards the symmetry of the operator $\mathcal{H}$ defined on the domain $\mathcal{D}_\varepsilon$ for $\Xi \in \mathcal{D}^{\alpha}$, we introduce a family of smooth functions $\Xi_{\varepsilon} := (\xi_\varepsilon, \xi_\varepsilon \circ \sigma (D) (\hat{\xi} - \xi_\varepsilon), \varepsilon > 0$ such that $\Xi_{\varepsilon}$ converges to $\Xi$ in $\mathcal{D}^{\alpha}$ (this family exists by definition of the space $\mathcal{D}^{\alpha}$). Then, we know from Proposition 4.21 that there exists a family of functions $f_{\varepsilon}, \varepsilon > 0$ in $H^2$ such that

$$
\|f - f_{\varepsilon}\|_{H^\gamma} + \|f^0 - f_{\varepsilon}^0\|_{H^2} \rightarrow 0
$$

(42)
for any $2/3 < \gamma < \alpha + 2$ where $f_\varepsilon^\alpha := f_\varepsilon - f_\varepsilon \propto \sigma(D)\xi_\varepsilon - B(f, \Xi_\varepsilon)$.

Thanks to the smoothness of $\xi_\varepsilon$, it is straightforward to define the operator $\mathcal{H}_\varepsilon := -\Delta + \xi_\varepsilon$ on the domain $H^2$.

From the estimates (41) and (42), we deduce that $\mathcal{H}_\varepsilon f_\varepsilon$ converges towards $\mathcal{H} f$ in $L^2$ for any $f \in \mathcal{D}_\Xi$. If $g \in \mathcal{D}_\Xi$ and $(g_\varepsilon)_{\varepsilon > 0} \in H^2$ are such that $\mathcal{H}_\varepsilon g_\varepsilon \to \mathcal{H} g$ in $L^2$, it is plain to see that
\[
\langle \mathcal{H}_\varepsilon f_\varepsilon, g_\varepsilon \rangle_{L^2} = \langle f_\varepsilon, \mathcal{H} g_\varepsilon \rangle_{L^2}.
\]
We obtain the symmetry of $\mathcal{H}$ by sending $\varepsilon \to 0$ in the last equality.

We now come back to the study of the resolvent operator $\mathcal{G}_a$ by establishing that $\mathcal{G}_a$ takes values in the space of strongly paracontrolled distributions $\mathcal{D}_\Xi$. We also prove that the resolvent $\mathcal{G}_a$ is a self-adjoint and compact operator $L^2 \to L^2$. Proposition 4.23 will be used later to establish the main properties of the spectrum of the operator $\mathcal{H}$.

**Proposition 4.23.** Let $-4/3 < \alpha < -1$, $2/3 < \gamma < \alpha + 2$ and $A := A(\|\Xi\|_{X^{\alpha}})$ as introduced in Proposition 4.13.

Then, for all $a \geq A$, the operator $\mathcal{H} + a : \mathcal{D}_\Xi \to L^2$ is invertible with inverse $\mathcal{G}_a : L^2 \to \mathcal{D}_\Xi$. In addition, the operator $\mathcal{G}_a : L^2 \to L^2$ is bounded, self-adjoint and compact.

**Proof.** Let $g \in L^2$ and set $f_a := \mathcal{G}_a g \in \mathcal{D}_\Xi^\gamma$. By definition, $(\mathcal{H} + a)f_a = g$ and it is easy to check that
\[
(1 - \Delta)f_a^\alpha = g - a f_a + 2\nabla f_a \propto \nabla (\sigma(D)\xi) + \Delta f_a \propto \sigma(D)\xi - f_a \propto \xi - f_a \Xi - \mathcal{R}(f_a, \xi, \sigma(D)\xi) - f_a^\alpha \circ \xi + f_a^\sharp.
\]
It follows that
\[
f_a^\alpha := f_a^\alpha - B(f_a, \Xi) = \sigma(D)(g - a f_a - \mathcal{R}(f_a, \xi, \sigma(D)\xi) - f_a^\sharp \circ \xi + f_a^\sharp - f_a - \sigma(D)\xi) \in H^2 \tag{43}
\]
so that $f_a \in \mathcal{D}_\Xi$.

We now prove that the map $\mathcal{G}_a : L^2 \to \mathcal{D}_\Xi^\gamma$ is bounded. We have
\[
\|\mathcal{G}_a g\|_{\mathcal{D}_\Xi^\gamma} = \|\mathcal{G}_a g\|_{H^2} + \|(\mathcal{G}_a g)^\gamma\|_{H^2}
\leq \|\mathcal{G}_a g\|_{\mathcal{D}_\Xi^2} + \|f_a^\alpha\|_{H^2}
\leq a^{\gamma/2 - 1}\|g\|_{L^2} + \|f_a^\gamma\|_{H^2}
\]
where we have used the boundedness of $\mathcal{G}_a : L^2 \to \mathcal{D}_\Xi^2$ (see (31)). It suffices to upper-bound the $H^2$ norm of the remainder $f_a^\alpha$. Using (43), we have
\[
\|f_a^\alpha\|_{H^2} \lesssim (a\|f_a\|_{H^2} + \|f_a^\sharp\|_{H^2}) (1 + \|\xi\|_{X^{\alpha}})^2 + \|g\|_{L^2}
\lesssim a\|f_a\|_{\mathcal{D}_\Xi^2} (1 + \|\xi\|_{X^{\alpha}})^2 + \|g\|_{L^2}
\lesssim (1 + a^\gamma/2 (1 + \|\xi\|_{X^{\alpha}})^2) \|g\|_{L^2}
\]
and we can finally deduce that for $a > 2$,
\[
\|\mathcal{G}_a g\|_{\mathcal{D}_\Xi^2} \lesssim \left( a^{\gamma/2 - 1} + a^{\gamma/2} (1 + \|\xi\|_{X^{\alpha}})^2 \right) \|g\|_{L^2}.
\]

The fact that the resolvent operator $\mathcal{G}_a : L^2 \to L^2$ is self-adjoint follows from the symmetry of $\mathcal{H}$. The resolvent operator $\mathcal{G}_a : L^2 \to L^2$ is the composition of the bounded operator $\mathcal{G}_a : L^2 \to H^\gamma$ ($\mathcal{G}_a : L^2 \to \mathcal{D}_\Xi^2$ is bounded and $\|\cdot\|_{H^\gamma} \leq \|\cdot\|_{\mathcal{D}_\Xi^2}$ with the compact injection operator $i : H^\gamma \to L^2$ (this fact follows from Rellich-Kondrachov Theorem, see Appendix A.6 for a reminder) and therefore is a compact operator. \qed
Remark 4.24. Before proceeding with the study of the operator $\mathcal{H}$, let us point out that the space of strong paracontrolled distributions $\mathcal{D}_\Xi$ is dense in the space $\mathcal{D}_\gamma^\xi$. Indeed, we have $\mathcal{D}_\gamma^\xi = G_a H^{2\gamma-2}$ and $\mathcal{D}_\Xi = G_a L^2$ and the result follows from the fact that $H^{2\gamma-2}$ is dense in $L^2$.

A simple application of the spectral Theorem to the operator $G_a$ yields the following properties for the spectrum of the operator $H$.

Corollary 4.25. Let $\alpha \in (-4/3, -1)$ and $\Xi = (\xi, \Xi_2) \in \mathcal{X}^\alpha$. Then the operator $H$ is self-adjoint and has a discrete spectrum (only pure points) $(\Lambda_n)_{n \geq 1}$ which is such that $\Lambda_1 \leq \Lambda_2 \leq \cdots \leq \Lambda_n \leq \cdots$ with no accumulation points except in $+\infty$. In addition, we have

$$L^2 = \bigoplus_{n \in \mathbb{N}} \text{Ker} (\Lambda_n - \mathcal{H})$$

and for any $n$, the dimension of the subspace $\text{Ker} (\Lambda_n - \mathcal{H})$ is finite. The eigenvalues $\{\Lambda_n, n \in \mathbb{N}\}$ satisfy the following minimax principle

$$\Lambda_n = \inf_{F} \sup_{\|f\|_{L^2} = 1} \langle \mathcal{H}(\Xi)f, f \rangle_{L^2}$$

where $F$ ranges over all $n$-dimensional subspaces of $\mathcal{D}_\Xi$.

The min-max principle (44) is a very useful variational tool to study the eigenvalues. however the fact that the supremum is taken on the linear space $\mathcal{D}_\Xi$ can give a rise to a very complicated computation. As pointed out previously the operator $\mathcal{H}$ can be defined on the more simpler space $\mathcal{D}_\gamma^\xi$, the problem then is that the operator has value in the space $H^{2\gamma-2}(\mathbb{T}_L^2)$ and not in $L^2(\mathbb{T}_L^2)$. Fortunately this fact turns out to be sufficient to make sense of the quadratic form $\langle \mathcal{H}(\Xi)f, f \rangle$ by duality. Namely if $f \in \mathcal{D}_\gamma^\xi$ and $\gamma > 2/3$ then the following bound :

$$|\langle \mathcal{H}(\Xi)f, f \rangle| \leq \|\mathcal{H}(\Xi)f\|_{H^{2\gamma-2}} \|f\|_{H^\gamma} \lesssim \|f\|_{\mathcal{D}_\gamma^\xi} \|\mathcal{H}(\Xi)f\|_{H^\gamma} (1 + \|\xi\|_\mathcal{X}^\alpha + \|\xi\|_\mathcal{X}^{2\alpha} + \|\Xi_2\|_\mathcal{X}^{2\alpha+2})$$

hold. Now the point is that we can replace the space $\mathcal{D}_\Xi$ by $\mathcal{D}_\gamma^\xi$ in the min-max equation 44. More precisely we have the following statement:

Lemma 4.26. [min-max principle] Given $2/3 < \gamma < \alpha + 2$ we have the following eigenvalue representation :

$$\Lambda_n(\Xi) = \inf_{F \subseteq \mathcal{D}_\gamma^\xi, \text{dim}(F) = n} \sup_{\|f\|_{L^2} = 1} \langle \mathcal{H}(\Xi)f, f \rangle_{L^2}$$

Proof. Since $\mathcal{D}_\Xi \subseteq \mathcal{D}_\gamma^\xi$ we have trivially the following inequality

$$\Lambda_n(\Xi) \geq \inf_{F \subseteq \mathcal{D}_\gamma^\xi, \text{dim}(F) = n} \sup_{\|f\|_{L^2} = 1} \langle \mathcal{H}(\Xi)f, f \rangle_{L^2}$$

The other inequality is obtained simply by the fact that the space $\mathcal{D}_\Xi$ is dense in $\mathcal{D}_\gamma^\xi$. 

Now we will show that the eigenvalue are continuous as function of $\Xi$. For that we will need the following result :
Proposition 4.27. Given $\alpha \in (-4/3, -1)$, $-\alpha/2 < \gamma < \alpha + 2$, $n \in \mathbb{N}$ and $\Xi = (\xi, \Xi) \in X^{\alpha}$ and denoting by $\Lambda_1(0) \leq \Lambda_2(0) \leq ... \leq \Lambda_n(0)$ the $n$ first eigenvalues of $-\Delta$ repeated with their multiplicity. Then for all $C > 0$ there exists a free family of vector $f_1, f_2, ..., f_n \in \mathcal{D}_\xi$ such that:

1. $\frac{1}{2} \leq \|f_i\|_{L^2} \leq 2$
2. $|\langle f_i, f_j \rangle_{L^2}| \leq \frac{C}{2n}$ for $i \neq j$
3. $\|f_i\|_{\mathcal{D}_\xi} \lesssim_{\alpha, \gamma} (1 + \Lambda_n(0))^{2\gamma} + n^{\frac{2n-\alpha}{2+\alpha}}\|\xi\|_{\mathcal{D}_\alpha}$

And therefore from the min-max principle gives us the following bound

$$\Lambda_n(\Xi) \lesssim n((1 + \Lambda_n(0))^{2\gamma} + n^{\frac{2n-\alpha}{2+\alpha}}\|\xi\|_{\mathcal{D}_\alpha})^2 (1 + \|\Xi\|_{X^{\alpha}})^2 \tag{45}$$

hold for all $n$.

Proof. Let $e_1(0), e_2(0), ..., e_n(0)$ an orthonormal family of eigenvector respectively associated to the $n$ first eigenvalues of $-\Delta$ then as in the proof of the Lemma 4.12 we prove that for $a^{-1} \|\xi\|_{\mathcal{D}_\alpha}^{1/2}$ small enough the map $\Gamma_i : L^2 \to L^2$ defined by

$$\Gamma_i(f) = f - \sigma_a(D)\xi + e_i(0)$$

admit a unique fix point $f_i$ which satisfy $\|f_i\|_{L^2} \leq 2\|e_i(0)\|_{L^2} = 2$. Now we will show that the family $f_1, f_2, ..., f_n$ is free for $a$ large enough. Indeed let us assume that

$$f_1 = \sum_{i=2}^{n} \kappa_i f_i$$

using that $f_i$ is a fixed point of the map $\Gamma_i$ we get easily the following equation

$$(f_1 - \sum_{i=2}^{n} \kappa_i f_i) - \sigma_a(D)\xi = \sum_{i=1}^{n} \kappa_i e_i(0) - e_1(0)$$

As usual from Bony and Schauder estimate we get

$$\|\sum_{i=2}^{n} \kappa_i e_i(0) - e_1(0)\|_{L^2} \lesssim a^{-(1+\alpha/2)}\|\sum_{i=2}^{n} \kappa_i f_i\|_{L^2} \|\xi\|_{\mathcal{D}_\alpha}$$

$$\lesssim \sqrt{n}a^{-(1+\alpha/2)}(1 + \sum_i |\kappa_i|^2)\|\xi\|_{\mathcal{D}_\alpha}$$

on the other side we have that

$$\|\sum_{i=2}^{n} \kappa_i e_i(0) - e_1(0)\|_{L^2}^2 = 1 + \sum_i |\kappa_i|^2$$

which is impossible if we choose $a$ such that $a^{-1} n^{\frac{1}{2+\alpha}}\|\xi\|_{\mathcal{D}_\alpha}^{1/2}$ is small enough. Now if we take $i \neq j$ is easy to see that

$$|\langle f_i, f_j \rangle_{L^2}| \lesssim \|f_i - e_i(0)\| + \|f_j - e_j(0)\|_{L^2}$$
where we have used the Cauchy-Schwartz inequality and the orthogonality between $e_i(0)$ and $e_j(0)$. Once again Bony estimate and the fact that $\|f_i\|_{L^2} \leq 2$ gives:

$$\|f_i - e_i(0)\|_{L^2} \lesssim a^{-(1 + \frac{\alpha}{2})} \|\xi\|_{\mathcal{X}^{\alpha}}$$

which for $a^{-1} n^{\frac{2}{1+\alpha}} \|\xi\|_{\mathcal{X}^{\alpha}}^2$ small enough (Depending on $C$) ensure that

$$|\langle f_i, f_j \rangle_{L^2}| \leq C \frac{4}{4n}$$

Now we have to control the norm of our vector in the space $\mathcal{D}^\gamma_{\xi}$. To do that let us start by observing that $f_i^\sharp = f \prec (\sigma_a(D) - \sigma(D))\xi + e_i(0)$ then Bony and Schauder estimates allow us to get the two following bounds:

$$\|f_i^\sharp\|_{H^{2\gamma}} \lesssim a^{2\gamma - \alpha/2} \|\xi\|_{\mathcal{X}^{\alpha}} + (1 + \Lambda_i(0))^{2\gamma}$$

and

$$\|f_i\|_{H^\gamma} \lesssim \|\xi\|_{\mathcal{X}^{\alpha}} + (1 + \Lambda_i(0))^{\gamma}$$

Which gives the needed estimate for $\|f\|_{\mathcal{D}^\gamma_{\xi}}$ if we take $a = Kn^{\frac{2}{1+\alpha}} \|\xi\|_{\mathcal{X}^{\alpha}}^2$ for $K > 0$ a sufficiently large constant. Now we will prove the bound (45). Using the fact that span$\{f_1, f_2, \ldots, f_n\}$ is a $n$- dimensional sub-space of $\mathcal{D}^\gamma_{\xi}$ and the min-max principle we get easily that:

$$\Lambda_n(\Xi) \leq \sup_{f \in \text{span}\{f_1, f_2, \ldots, f_n\}} \|f\|_{L^2} \lesssim \sup_{f \in \text{span}\{f_1, f_2, \ldots, f_n\}} \|f\|_{L^2}^2$$

Let $f$ such that

$$f = \sum_{i=1}^{n} \kappa_i f_i$$

then

$$\|f\|_{\mathcal{D}^\gamma_{\xi}}^2 \lesssim \max_i \|f_i\|_{\mathcal{D}^\gamma_{\xi}}^2 \left(\sum_{i=1}^{n} |\kappa_i|^2\right)^2$$

on the other side we have

$$\|f\|_{L^2}^2 = \sum_{i=1}^{n} |\kappa_i|^2 \|f_i\|_{L^2}^2 + \sum_{i \neq j} \kappa_i \kappa_j \langle f_i, f_j \rangle_{L^2}$$

which can be lower-bounded using the two firsts properties of the function $f_i$. Namely

$$\|f\|_{L^2} \geq \frac{1}{2} \sum |\kappa_i|^2 - \frac{1}{4n} \left(\sum |\kappa_i|^2\right)^2 \geq \frac{1}{4n} \left(\sum |\kappa_i|^2\right)^2$$

and finally we can conclude that

$$\|f\|_{\mathcal{D}^\gamma_{\xi}}^2 \lesssim \max_i \|f_i\|_{\mathcal{D}^\gamma_{\xi}}^2$$

which by the third property of the function $f_i$ complete the proof.
Proposition 4.28. Given $\alpha \in (-4/3, -1)$, $\Xi \in \mathcal{X}^\alpha$ and let us denote by $(\Lambda_n(\Xi))_{n \geq 1}$ the set of the eigenvalue of the operator $\mathcal{H} = \mathcal{H}(\Xi)$. Then for all $n$ $\geq 1$ the map $\Lambda_n : \mathcal{X}^\alpha \to \mathbb{R}$ is locally Lipschitz. More precisely it exist $M > 0$ such that for all $n \in \mathbb{N}$, $\exists \Xi \in \mathcal{X}^\alpha$ and $-\alpha/2 < \gamma < \alpha + 2$

$$|\Lambda_n(\Xi) - \Lambda_n(\tilde{\Xi})| \lesssim_{\gamma, \alpha} n \|\Xi - \tilde{\Xi}\|_{\mathcal{X}^\alpha} (1 + \|\Xi\|_{\mathcal{X}^\alpha} + \|\tilde{\Xi}\|_{\mathcal{X}^\alpha})^M (1 + n \frac{2\gamma - \alpha}{\alpha + 2} + (1 + \Lambda_n(0))^{2\gamma}$$

Where the last bound is uniform on $L$.

Proof. The min-max principle for the resolvent $G_a$ gives that

$$\left| \frac{1}{\Lambda_n(\Xi) + a} - \frac{1}{\Lambda_n(\tilde{\Xi}) + a} \right| \leq \|G_a(\Xi) - G_a(\tilde{\Xi})\|_{L(L^2, L^2)}$$

where $L(L^2, L^2)$ is the space of bounded operator on $L^2$ equipped on his usual norm. Then using the bound given in the Lemma 4.15 we can easily deduce that

$$|\Lambda_n(\Xi) - \Lambda_n(\tilde{\Xi})| \lesssim \|\Xi - \tilde{\Xi}\|_{\mathcal{X}^\alpha} (1 + \|\Xi\|_{\mathcal{X}^\alpha} + \|\tilde{\Xi}\|_{\mathcal{X}^\alpha})^2 (\Lambda_n(\Xi) + a)(\Lambda_n(\tilde{\Xi}) + a)$$

for all $a \geq (1 + \|\Xi\|_{\mathcal{X}^\alpha} + \|\tilde{\Xi}\|_{\mathcal{X}^\alpha})^{\frac{1}{\gamma}}$ with $\gamma$ as in the Lemma 4.15 it is sufficient to use the bound (45). \(\square\)

5 Renormalization for the Anderson Hamiltonian

5.1 Renormalization for the white noise potential

For $\alpha < -1$, we consider a smooth approximation $\xi_\varepsilon$ of the white noise $\xi \in \mathcal{C}^\alpha$ and establish the convergence in $\mathcal{X}^\alpha$ of the mollified family $(\xi_\varepsilon, \xi_\varepsilon \circ \sigma(D)\xi_\varepsilon + c_\varepsilon)$ for some diverging constants $(c_\varepsilon)_{\varepsilon > 0}$ towards some $(\xi, \Xi^{wn}) \in \mathcal{X}^\alpha$. The following Theorem is a cornerstone of our study as it permits us to handle concretely the relevant case of the white noise and use the approximation theory developed in the previous sections to this case.

Theorem 5.1. Let $\alpha < -1$ and $\xi \in \mathcal{C}^\alpha$ a white-noise on the two dimensional torus $\mathbb{T}_L^2$. We consider $\xi_\varepsilon$ a smooth approximation of the white noise $\xi$ defined as

$$\xi_\varepsilon(x) := \sum_{k \in \mathbb{Z}^2_L} \theta(\varepsilon |k|) \xi(k) \frac{1}{L} \exp(i2\pi \langle k, x \rangle)$$

where $\theta$ is a smooth function on $\mathbb{R} \setminus \{0\}$ with compact support, such that $\lim_{x \to 0} \theta(x) = 1$. Then, there exists a diverging sequence of constants $(c_\varepsilon)_{\varepsilon > 0} = (c_\varepsilon(\theta))_{\varepsilon > 0}$ and a limit point $\Xi^{wn} = (\xi, \Xi^{wn}) \in \mathcal{X}^\alpha$ such that the following convergence

$$\Xi^{\varepsilon} := (\xi_\varepsilon, \xi_\varepsilon \circ \sigma(D)\xi_\varepsilon + c_\varepsilon) \overset{\varepsilon \to 0}{\longrightarrow} \Xi^{wn}$$

holds in $L^p(\Omega, \mathcal{E}^\alpha)$ for all $p > 1$ and almost-surely in $\mathcal{E}^\alpha$. Moreover $\Xi^{wn}$ is independent on the choice of $\theta$.

Remark 5.2. For simplicity we will denote by $\Xi$ (instead of $\Xi^{wn}$) the rough distribution associate to the white noise.

Proof. We first prove the convergence of $\xi_\varepsilon$ in the space $\mathcal{C}^\alpha$ for any $\alpha < -1$. For $i \geq -1$ and $x \in \mathbb{T}_L^2$, we have

$$\Delta_i(\xi_\varepsilon - \xi)(x) = \sum_{k \in \mathbb{Z}^2_L} \rho(2^{-i} |k|) \xi(k)(\theta(\varepsilon |k|) - 1) \frac{1}{L} \exp(i2\pi \langle k, x \rangle)$$

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where the $\hat{\xi}(k), k \in \mathbb{Z}^2_L$ form a family of independent and identically distributed complex Gaussian variables such that, for any $k, \ell \in \mathbb{Z}^2_L$,

$$\hat{\xi}(-k) = \overline{\hat{\xi}(k)},$$

$$\mathbb{E}[\hat{\xi}(k)\hat{\xi}(\ell)] = \delta(k + \ell).$$

We deduce the following upper-bound valid up to a constant independent of $i \geq -1$ and for any $\delta > 0, x \in \mathbb{T}_L^2$,

$$\mathbb{E}[|\Delta_i(\xi - \xi)(x)|^2] \lesssim \sum_{k \in \mathbb{Z}^2_L} |\theta(\varepsilon |k|) - 1|^2 |2^{-\delta} |k||^2$$

$$\lesssim 2^{(2+\delta)n} \left( L^{-2} \sum_{k \in \mathbb{Z}^2_L} |\theta(\varepsilon |k|) - 1|^2 \right)$$

where we have used the fact that the function $\rho$ is supported on a compact annulus for the second line. Before proceeding with the computation let us observe that by Riemann-sum approximation we have the following bound

$$L^{-2} \sum_{k \in \mathbb{Z}^2_L} |\theta(\varepsilon |k|) - 1|^2 \lesssim \int_{\mathbb{R}^2} |\theta(\varepsilon |y|) - 1|^2 \frac{dy}{(1 + |y|)^{2+\delta}},$$

which show in particularly that our bound is uniform in $L$. Integrating over $x \in \mathbb{T}_L^2$, we obtain an upper-bound for the $L^2$ norm of $\Delta_i(\xi - \xi)$ which can be generalized for any $p > 1$ thanks to the Gaussian hypercontractivity property [21]

$$\mathbb{E}[||\Delta_i(\xi - \xi)(x)||^p_{L^p}] \lesssim \int_{\mathbb{T}_L^2} \mathbb{E}[|\Delta_i(\xi - \xi)(x)|^2]^\frac{p}{2} dx$$

$$\lesssim_p L^2 2^{ip(1+\delta/2)} \left( L^{-2} \sum_{k \in \mathbb{Z}^2_L} |\theta(\varepsilon |k|) - 1|^2 \right)^{p/2}.$$

Multiplying both sides of this inequality by $2^{-ip(1+\delta)}$ and summing over $i \geq -1$ gives

$$\mathbb{E}[||\xi - \xi||^p_{p \delta_{-1-\frac{\delta}{2}}}] \lesssim_p L^2 \left( L^{-2} \sum_{k \in \mathbb{Z}^2_L} \frac{|\theta(\varepsilon |k|) - 1|^2}{1 + |k|^{2+\delta}} \right)^{p/2}.$$

From the embedding property A.1 of the Besov spaces, we obtain

$$\mathbb{E}[||\xi - \xi||^p_{p \delta_{-1-\frac{\delta}{2}-\frac{\delta}{2}}}] \lesssim_p L^2 \left( L^{-2} \sum_{k \in \mathbb{Z}^2_L} \frac{|\theta(\varepsilon |k|) - 1|^2}{1 + |k|^{2+\delta}} \right)^{p/2}.$$

For any $\delta > 0$, the bounded convergence Theorem permits us to show that the right hand side of the last equation converges to zero and we can finally conclude that $\xi$ converge to $\xi$ in $L^p(\Omega, \mathcal{G}^\alpha)$ for any $\alpha < -1$.

Towards the almost sure convergence of $\xi$, the same arguments apply to $\xi - \xi$ instead of $\xi - \xi$ and we get

$$\mathbb{E}[||\xi - \xi'||^p_{p \delta_{-1-\frac{\delta}{2}-\frac{\delta}{2}}}] \lesssim L^2 \left( L^{-2} \sum_{k \in \mathbb{Z}^2_L} \frac{|\theta(\varepsilon |k|) - \theta(\varepsilon' |k|)|^2}{1 + |k|^{2+\delta}} \right)^{p/2}$$

6The sum in the first line contains only the terms with indices $|k| \sim 2^i$. 32
Using the fact that $|\theta(\varepsilon|k|) - \theta(\varepsilon'|k|)| \lesssim_p |\varepsilon - \varepsilon'| |k|^{\alpha}$, we obtain that for any $0 < \eta < \delta$

$$
\mathbb{E} \left[ ||\xi_\varepsilon - \xi_\varepsilon'||_p^{\frac{p}{p-1}} \right] \lesssim |\varepsilon - \varepsilon'|^\eta/2.
$$

This proves the convergence of $(\xi_\varepsilon)_{\varepsilon > 0}$ in $L^p(\Omega, \mathcal{F}^{\alpha})$ for any $\alpha < -1$ and $p > 0$.

Towards the almost sure convergence, the Kolmogorov criterion applied for $p > 2/\eta$ permits us to show that there exists $\kappa \in (0, \frac{\eta}{2})$ such that for $p$ large enough, we have almost surely,

$$
||\xi_\varepsilon - \xi_\varepsilon'||_p^{\frac{p}{p-1}} \lesssim |\varepsilon - \varepsilon'|^\kappa.
$$

The sequence $(\xi_\varepsilon)_{\varepsilon > 0}$ is therefore almost surely a Cauchy sequence in the Banach space $\mathcal{F}^{-1-\frac{\kappa}{2}-\frac{2}{p}}$, where $p$ (resp. $\delta$) is arbitrarily large (resp. small). We can conclude that $(\xi_\varepsilon)_{\varepsilon > 0}$ converges almost surely in $\mathcal{F}^{\alpha}(T^2_L)$ for any $\alpha < -1$.

Now we turn to the second chaos term which may be expanded as

$$
\xi_\varepsilon \circ \sigma(D)\xi_\varepsilon(x) = L^{-2} \sum_{k \in \mathbb{Z}^2_L, \ell \in \mathbb{Z}^2_L, |i-j| \leq 1} \rho(2^{-i}|k|) \rho(2^{-j}|\ell|) \theta(\varepsilon|k|) \frac{\theta(\varepsilon|\ell|)}{1 + |\ell|^2} \hat{\xi}(k) \hat{\xi}(\ell) e_{k+\ell}(x).
$$

Taking the expectation in this last formula yields

$$
-\mathbb{E}[\xi_\varepsilon \circ \sigma(D)\xi_\varepsilon(x)] = L^{-2} \sum_{k \in \mathbb{Z}^2_L, |i-j| \leq 1} \rho(2^{-i}|k|) \rho(2^{-j}|k|) \frac{|\theta(\varepsilon|k|)|^2}{1 + |k|^2} = \sum_{k \in \mathbb{Z}^2_L} \frac{|\theta(\varepsilon|k|)|^2}{1 + |k|^2}
$$

so that it is sufficient to set

$$
c_\varepsilon := L^{-2} \sum_{k \in \mathbb{Z}^2_L} \frac{|\theta(\varepsilon|k|)|^2}{1 + |k|^2}.
$$

One can easily check that for any function $\theta$, which is smooth on $\mathbb{R} \setminus \{0\}$ and such that $\theta(x) \to 1$ when $x \to 0$, we have

$$
c_\varepsilon = \frac{1}{2\pi} \log(\frac{1}{\varepsilon}) + O(1).
$$

For $x \in T^2_L$, we set

$$
\Xi_\varepsilon(x) := \xi_\varepsilon \circ \sigma(D)\xi_\varepsilon(x) + c_\varepsilon.
$$

We will now prove that the sequence $(\Xi_\varepsilon)_{\varepsilon > 0}$ is a Cauchy convergent sequence in the Banach space $\mathcal{F}^{\alpha+2}(T^2_L)$ for any $\alpha < -1$. We have

$$
\Xi_\varepsilon(x) = L^{-2} \sum_{k \in \mathbb{Z}^2_L, \ell \in \mathbb{Z}^2_L, |i-j| \leq 1} \rho(2^{-i}|k|) \rho(2^{-j}|\ell|) \frac{\theta(\varepsilon|k|)}{1 + |k|^2} \hat{\xi}(k) \hat{\xi}(\ell) - \delta(k + \ell) e_{k+\ell}(x).
$$

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We want to bound the second moments of the Littlewood-Paley blocks of the difference $\Xi_2^n - \Xi_2'^n$ for $\varepsilon, \varepsilon' > 0$:
For any $q \geq -1$,
\[
\mathbb{E}[|\Delta_q(\Xi_2^n - \Xi_2'^n)(x)|^2] = L^{-4} \sum_{k \in \mathbb{Z}_L^2, \ell \in \mathbb{Z}_L^2, |i_1 - j_1| \leq 1, |i_2 - j_2| \leq 1} \rho(2^{-q}|k + \ell|)^2 |\Pi_{m=1,2} \rho(2^{-i_m}|k|)\rho(2^{-j_m}|\ell|)| \frac{|\theta(\varepsilon|k|)\theta(\varepsilon|\ell|) - \theta(\varepsilon'|k|)\theta(\varepsilon'|\ell|)|^2}{(1 + |\ell|)^2}.
\]
\[
+ L^{-4} \sum_{k \in \mathbb{Z}_L^2, \ell \in \mathbb{Z}_L^2 \setminus \{0\}, |i_1 - j_1| \leq 1, |i_2 - j_2| \leq 1} \rho(2^{-q}|k + \ell|)^2 |\Pi_{m=1,2} \rho(2^{-i_m}|k|)\rho(2^{-j_m}|\ell|)| \frac{|\theta(\varepsilon|k|)\theta(\varepsilon|\ell|) - \theta(\varepsilon'|k|)\theta(\varepsilon'|\ell|)|^2}{(1 + |\ell|)^2(1 + |\ell|^2)}.
\]
Due to the fact that $|i_1 - j_1| \leq 1$ and that the support of $\rho$ is a fixed annulus, we can deduce that the indices of the non zero terms of the two sums appearing in the last formula are such that
\[
|k| \lesssim |\ell| \lesssim |k|
\]
where the undisplayed multiplicative constants in those inequalities do not depend on $i_1, j_1, i_2, j_2$. This implies that the two sums are of the same order in the sense that one can bound any of the two sums with the other one. Using the fact that $\rho$ is bounded, we can bound the sum over $i_1, j_1, i_2, j_2$ by a constant and we are eventually reduced to estimate the quantity
\[
\sum_{k \in \mathbb{Z}_L^2, \ell \in \mathbb{Z}_L^2 \setminus \{0\}, |k| \leq |\ell| \leq A|k|} \rho(2^{-q}|k + \ell|)^2 |\theta(\varepsilon|k|)\theta(\varepsilon|\ell|) - \theta(\varepsilon'|k|)\theta(\varepsilon'|\ell|)|^2
\]
for some given constants $a, A > 0$ (independent on $L$). We set $n = k + \ell$ and note that if $a|k| \leq |\ell| \leq A|k|$, then $n \leq |k| + |\ell| \leq (a^{-1} + 1)|\ell|$. We deduce that, up to a multiplicative constant and denoting by $\delta > 0$ a fixed (small enough) parameter, we can bound (46) by
\[
\left( \sum_{n \in \mathbb{Z}_L^2} \rho(2^{-q}|n|)^2 \right) \sup_{n \in \mathbb{Z}_L^2} \left( \sum_{k \in \mathbb{Z}_L^2, \ell \in \mathbb{Z}_L^2 : k + \ell = n, a|k| \leq |\ell| \leq A|k|} \frac{|\theta(\varepsilon|k|)\theta(\varepsilon|\ell|) - \theta(\varepsilon'|k|)\theta(\varepsilon'|\ell|)|^2}{1 + |\ell|^{2+\delta}} \right) \lesssim 2^{\delta} \sup_{n \in \mathbb{Z}_L^2} \left( \sum_{k \in \mathbb{Z}_L^2, \ell \in \mathbb{Z}_L^2 : k + \ell = n, a|k| \leq |\ell| \leq A|k|} \frac{|\theta(\varepsilon|k|)\theta(\varepsilon|\ell|) - \theta(\varepsilon'|k|)\theta(\varepsilon'|\ell|)|^2}{|\ell|^{2+\delta}} \right)
\]
where we have used the fact that $n \to \rho(2^{-q}|n|)$ is supported in a ball (in fact an annulus) of $\mathbb{Z}_L^2$ with radius $2^n$. Using the inequality $|\theta(\varepsilon|k|)\theta(\varepsilon|\ell|) - \theta(\varepsilon'|k|)\theta(\varepsilon'|\ell|)| \lesssim |\varepsilon - \varepsilon'| |k|^\eta + |\ell|^\eta$ valid for $\eta > 0$ small enough, we easily check that for $0 < \eta < \delta/2$,
\[
L^{-4} \sup_{n \in \mathbb{Z}_L^2} \left( \sum_{k \in \mathbb{Z}_L^2, \ell \in \mathbb{Z}_L^2 : k + \ell = n, a|k| \leq |\ell| \leq A|k|} \frac{|\theta(\varepsilon|k|)\theta(\varepsilon|\ell|) - \theta(\varepsilon'|k|)\theta(\varepsilon'|\ell|)|^2}{1 + |\ell|^{2+\delta}} \right) \lesssim |\varepsilon - \varepsilon'|^{2\eta}.
\]
Where the last bound is uniform in $L$. Gathering the above inequalities, we finally obtain the second moment estimate, valid with $0 < \eta < \delta/2$,

$$\mathbb{E} \left[ |\Delta_{q} (\Xi_{2}^{\epsilon} - \Xi_{2}^{\epsilon}) (x) |^2 \right] \lesssim 2^{q \eta} |\epsilon - \epsilon'|^{2 \eta}.$$  

where this bound is uniform in $L$. Using as before the Gaussian hypercontractivity and the Besov embedding arguments, we deduce the following upper-bound

$$\mathbb{E} \left[ \|\Xi_{2}^{\epsilon} - \Xi_{2}^{\epsilon'}\|^p_{\mathcal{G}^{2 \eta}(-2/p)} \right] \lesssim L^2 |\epsilon - \epsilon'|^{p \eta}$$

The convergence of the sequence $(\Xi_{2}^{\epsilon})_{\epsilon > 0}$ in $L^p(\Omega, \mathcal{G}^{2 \alpha+2}(\mathbb{T}_L^2))$ for $\alpha < -1$ and $p > 0$ is proved.

The Kolmogorov criterion permits us to conclude that the sequence $(\Xi_{2}^{\epsilon})_{\epsilon > 0}$ converges almost surely in the space $\mathcal{G}^{2 \alpha+2}(\mathbb{T}_L^2)$ for any $\alpha < -1$. $\square$

### 5.2 Growth of the eigenvalue

In this section we are interested to quantify the growth of the eigenvalue when $L$ the size of the Torus become large. Of course to do that the first step is to control the growth of the rough distribution associate to the white noise when $L$ become large. Namely we have the following result

**Lemma 5.3.** Given $L \in \mathbb{N}^*$, $\alpha < -1$ and $\xi$ a white noise on the two dimensional torus $\mathbb{T}_L^2$ of size $L$ then there exists two finite constant $C \geq 2$ and $\lambda$ such that

$$\sup_{L} L^{-C} \mathbb{E} \left[ \exp(\lambda \mathbb{E} \left[ \|\xi\|^2_{\mathcal{G}^{0}} + \|\Xi_{2}\|_{\mathcal{G}^{2\alpha+2}} \right] \right] < +\infty$$

and therefore

$$A_\alpha = \sup_{L} L^{-C} \mathbb{E} \left[ \frac{\|\xi\|^2_{\mathcal{G}^{0}(\mathbb{T}_L^2)}}{\log(L)} + \sup_{L} \frac{\|\Xi_{2}\|_{\mathcal{G}^{2\alpha+2}(\mathbb{T}_L^2)}}{\log(L)} \right] < +\infty$$

almost surely. Moreover $\mathbb{E} [\exp(h A_\alpha)] < +\infty$ for a sufficiently small constant $h > 0$.

**Proof.** Let us recall that the white noise have the following representation

$$\xi = \sum_{k \in \mathbb{Z}_L^2} g_k e_k^L$$

with $e_k^L(x) = \frac{1}{L} \exp(2i\pi \langle k, x \rangle)$ with $g_k$ is an i.i.d sequence of Gaussian random variable which satisfy that $g_{-k} = \overline{g_k}$. Then the same computation as in the Theorem 5.1 allow us to get that

$$\mathbb{E} [\|\Delta_{q} \xi(x) \|^p] \leq \mathbb{E} [\|\mathcal{N}(0, 1) \|^p] (L^{-2} \sum_{k \in \mathbb{Z}_L^2} |\theta(2-\eta|k|)|^{2/p} \mathbb{E} [\|\mathcal{N}(0, 1) \|^p]^{2q/p}$$

with $\mathcal{N}(0, 1)$ is centered Gaussian random variable with unit variance. Integrating over $x$ and taking the sum on $q$ gives :

$$\sup_{L} L^{-2} \mathbb{E} [\|\xi\|^p_{\mathcal{G}^{2\alpha-1-\alpha}}] \lesssim \mathbb{E} [\|\mathcal{N}(0, 1) \|^p]$$

for $p$ such that $\frac{2}{p} \leq \frac{\alpha}{2}$ and where we have used the Besov embedding.

$$\mathbb{E} [e^{\lambda \|\xi\|^2_{\mathcal{G}^{2\alpha-1-\alpha}}}] = \sum_{r=0}^{\frac{\alpha}{2}} \frac{\lambda^r}{r!} \mathbb{E} [\|\xi\|^{2r}_{\mathcal{G}^{2\alpha-1-\alpha}}] + \sum_{r=\frac{\alpha}{2}}^{\frac{\alpha}{2}} \frac{\lambda^r}{r!} \mathbb{E} [\|\xi\|^{2r}_{\mathcal{G}^{2\alpha-1-\alpha}}].$$
Using the inequality (48) the second sum of this last equation can be bounded in the following way:
\[
\sum_{r > \frac{8}{\kappa}} \frac{\lambda^r}{r!} E[|\theta|^{2r}] \lesssim L^2 \sum_{r > \frac{8}{\kappa}} \frac{\lambda^r}{r!} E[|\mathcal{N}(0, 1)|^{2r}] = L^2 E[\exp(\lambda|\mathcal{N}(0, 1)|^2)] < +\infty
\]
under the condition that \( \lambda \) is small enough. Since the first sum is a finite sum we bound each term of it using the Jensen inequality:
\[
\sum_{r=0}^{\frac{8}{\kappa}} \frac{\lambda^r}{r!} E[|\theta|^{2r}] \lesssim L^2 \sum_{r=0}^{\frac{8}{\kappa}} \frac{\lambda^r}{r!} E[|\mathcal{N}(0, 1)|^{2r}]^\frac{\kappa}{2} \lesssim L^2
\]
from which we can conclude that:
\[
\sup_L L^{-(\max(2, \frac{8}{\kappa}))} E[\exp(\lambda|\theta|^{2\alpha(2L)})] < \infty
\]
for \( \alpha = -1 - \kappa \). Therefore using Fubini theorem gives:
\[
E \left[ \sum_{L} \frac{1}{L^a} \exp(\lambda|\theta|^{2\alpha(2L)}) \right] < \infty
\]
for \( a > \max(2, \frac{8}{\kappa}) + 1 \), which in particular prove that:
\[
||\theta||^{2\alpha(2L)} \lesssim \lambda a \sqrt{\log(L) + \log A}
\]
with \( A = \sum_{L} \frac{1}{L^a} \exp(\lambda|\theta|^{2\alpha(2L)}) \) is an integrable random variable. To complete our proof, we just have to establish the same estimate for \( ||\Xi||_{2^\alpha} \) for which the same computation as in Theorem 5.1 allows us to get the following upper bounds
\[
\sup_L E[|\mathcal{D}_q \Xi(x)|^p] \lesssim 2^{q\kappa}
\]
for all \( \kappa > 0 \) small enough. Now since \( \mathcal{D}_q \Xi \) is in the second chaos of the white noise \( \xi \) the Gaussian hypercontractivity tell us that
\[
E[|\mathcal{D}_q \Xi(x)|^p] \lesssim A_p E[|\mathcal{D}_q \Xi(x)|^\frac{p}{2}] \lesssim A_p 2^{qsp/2}
\]
for all \( \delta > 0 \) and where \( A_p = E[|\mathcal{N}(0, 1)|^{2p}] \). Then repeating the argument used to control the growth of the white noise allows us to obtain the following integrability result
\[
E \left[ \sum_{L} \frac{1}{L^2} \exp(\lambda|\Xi||^{2\alpha-\delta}) \right] < \infty
\]
which finishes the proof.

Now a crucial observation is that the eigenvalues satisfy a rescaling property. Indeed let \( V \) a smooth \( L \)-periodic potential, \( \tilde{\Lambda}_n(V) \) is the \( n \)-lowest eigenvalue of \(-\Delta + V\) and \( e_n(V) \) an eigenvector associate to \( \tilde{\Lambda}_n(V) \). Then for \( r > 0 \) we can see that the function \( e'_{n}(x) = e_{n}(V)(rx) \) is an eigenvector of the operator \(-\Delta + r^2 V(r\cdot)\) with eigenvalue \( r^2 \tilde{\Lambda}_n(V) \) and therefore
\[
\tilde{\Lambda}_n(V) = \frac{1}{r^2} \tilde{\Lambda}_n(r^2 V(r\cdot))
\]
where $\hat{\Lambda}_n(V(r))$ is the $n$-lowest eigenvalue of $-\Delta + V(r \cdot)$ seen as an operator of $\mathbb{T}^2_{-1}$. As the reader can guess we want to extend the identity to the irregular setting for that we observe that the identity (5.2) can be reformulated in the following manner

$$\Lambda_n(V, V \circ \sigma(D) V + c) = \frac{1}{r^2} \Lambda_n(r^2 V(r \cdot), r^4 V(r \cdot) \circ \sigma(D)(V(r \cdot)) + r^2 c)$$

for every $c \in \mathbb{R}$. Since this eigenvalue identity holds for any smooth function $V$ it can be extend to the case of the white noise easily. Indeed let $\xi$ the white noise on $\mathbb{T}^2_{\mathbb{R}}$ and $\xi = \theta(\varepsilon | D) \xi$. From the fact that $\xi$ is smooth and $\xi(r \cdot) = \theta(\varepsilon | D) \xi(r \cdot) = (\theta(\varepsilon | D) \xi(r \cdot))(\cdot) = (\xi(r \cdot))^2$ we get immediately the following relation

$$\Lambda_n(\xi \xi \circ \sigma(D) \xi \xi + c) = \frac{1}{r^2} \Lambda_n(r^2 \xi \xi (r \cdot), r^4 \xi (r \cdot) \circ \sigma(D)(\xi \xi (r \cdot)) + r^2 c)$$

$$= \frac{1}{r^2} \Lambda_n(r^2 (\xi \xi (r \cdot))^2, r^4 (\xi (r \cdot))^2 \circ \sigma(D)((\xi \xi (r \cdot))^2) + r^2 c)$$

where we recall that $c_{\varepsilon} = -\mathbb{E}[\xi \xi \circ \sigma(D) \xi \xi]$ is the diverging constant given in th Theorem 5.1. Now the point is to take the limit when $\varepsilon$ goes to zero in this equation, indeed the continuity of $\Lambda_n$ and the convergence result of the Theorem 5.1 tell us that the left hand side of this equality converge to $\Lambda_n(\Xi)$. On the other hand is easy to see that $\xi_{\varepsilon} := r \xi r \cdot$ is a white noise on $\mathbb{T}^2_{\mathbb{R}}$ and theretofore we have that

$$(\xi(r \cdot))^2 \circ \sigma(D)((\xi(r \cdot))^2) + \frac{1}{r^2} \xi_{\varepsilon} = \frac{1}{r^2} (\xi_{\varepsilon}^2 \circ \sigma(D) \xi_{\varepsilon}^2 + \tilde{c} \xi)$$

converge almost surly in $\mathbb{S}^\alpha(\mathbb{T}^2_{\mathbb{R}})$ to $\frac{1}{r^2} \tilde{\xi}_{\varepsilon}^2$ where $\tilde{\xi}_{\varepsilon}$ is the rough distribution associate to $\xi_{\varepsilon}$ and $\tilde{c} = -\mathbb{E}[\xi_{\varepsilon}^2 \circ \sigma(D) \xi_{\varepsilon}^2]$. Of course this imply in particularly that

$$r^4 (\xi(r \cdot))^2 \circ \sigma(D)((\xi(r \cdot))^2) + + r^2 \tilde{c}$$

converge to $r^2 \tilde{\xi}_{\varepsilon}^2$. To handle the right side of the equation (5.2) we start by observing that

$$\Lambda_n(r^2 (\xi \xi (r \cdot))^2, r^4 (\xi (r \cdot))^2 \circ \sigma(D)((\xi \xi (r \cdot))^2) + r^2 c_{\varepsilon} = \Lambda_n(r \xi_{\varepsilon}^2 r^2, r^2 \xi_{\varepsilon}^2 \circ \sigma(D) \xi_{\varepsilon}^2 + r^2 \tilde{c} \xi) + r^2 (c_{\varepsilon} - \tilde{c} \xi)$$

At this point the continuity of the map $\Lambda_n$ imply that the first term appearing in the right hand side of this equation converge when $\varepsilon$ goes to 0 toward $\Lambda_n(r \xi_{\varepsilon} r^2, r^2 \tilde{\xi}_{\varepsilon}^2)$. Now it remain to control the difference between the diverging constant $c_{\varepsilon} - \tilde{c} \xi$ which can be written explicitly :

$$c_{\varepsilon} - \tilde{c} \xi = \frac{1}{L^2} \sum_{2^k L} \frac{|\theta(\varepsilon | k)|^2}{1 + |k|^2} - \frac{r^2}{L^2} \sum_{2^k L} \frac{|\theta(\varepsilon^{-1} | k)|^2}{1 + |k|^2} = \frac{1 - r^2}{L^2} \sum_{2^k L} \frac{|\theta(\varepsilon | k)|^2}{(1 + |k|^2)(1 + r^2 |k|^2)}$$

which by dominate convergence goes to

$$m_{r,L} = \frac{1 - r^2}{L^2} \sum_{2^k L} \frac{1}{(1 + |k|^2)(1 + r^2 |k|^2)} < \infty$$

when $\varepsilon$ goes to zero. Therefore we can conclude that

$$\Lambda_n(\Xi) = \frac{1}{r^2} (\Lambda_n(r \xi_{\varepsilon} r^2, \tilde{\xi}_{\varepsilon}^2) + r^2 m_{r,L})$$

(49)
Before proceeding with our computation let us observe that Riemann-sum approximation gives the following inequality

\[
m_{L,r} \lesssim L^{-2}(1 - r^2)(1 + \sum_{n \geq 1} \frac{n}{(1 + L^2)(1 + r^2 L^2)}) \lesssim \frac{1 - r^2}{L^2} + (1 - r^2) \int_0^{+\infty} \frac{\rho d\rho}{(1 + \rho^2)(1 + r^2 \rho^2)} = \frac{1 - r^2}{L^2} + (1 - r^2) \log(\frac{1}{r})
\]

where we have assumed \( r \in (0, 1) \). Now let us come back to our original problem which is to bound the eigenvalue \( \Lambda_n(\Xi) \) for that we will use the scaling property 49 with \( r = \frac{1}{\sqrt{\log L}} \). Namely we have that

\[
\frac{1}{\log(L)} \Lambda_n(\Xi) = \Lambda_n(\tilde{\Xi}^r) + \frac{1}{\log L} m_{L,r}
\]

where

\[
\tilde{\Xi}^r = (\frac{\tilde{\xi}}{\sqrt{\log L}}, \frac{\tilde{\Xi}^r_2}{\log L})
\]

and we recall that \( \tilde{\xi} \) is a white noise on \( T^2_{L \sqrt{\log L}} \). Using the growth estimate for the eigenvalue given in Proposition 4.28 allow us to get that

\[
\mathbb{E}\left[\left|\frac{1}{\log(L)} \Lambda_n(\Xi)\right|^p\right] \lesssim \left(\frac{1}{\log L} m_{L \sqrt{\log L}}\right)^p + \mathbb{E}[|\Lambda_n(\tilde{\Xi}^r)|^p] \\
\lesssim 1 + \Lambda_n(0)^p + n^p \mathbb{E}\left[\left\|\Xi^r\right\|_{\mathcal{F}^{2\alpha}(T^2_{L \sqrt{\log L}})}^p \left(1 + \left\|\Xi^r\right\|_{\mathcal{F}^{2\alpha}(T^2_{L \sqrt{\log L}})}\right)^p M\right] \left(1 + n^{\frac{27-\alpha}{2+\gamma}} + (1 + \Lambda_n(0))^2\right)^{2p}
\]

(50)

for all \( p > 1 \) and where we have used that \( \frac{1}{\log L} m_{L \sqrt{\log L}} \lesssim 1 \). On the other hand Lemma 5.3 give us the following bound:

\[
\mathbb{E}\left[(\left\|\tilde{\xi}\right\|^2_{\mathcal{F}^{2\alpha}(T^2_{L \sqrt{\log L}})} + \left\|\tilde{\Xi}^2\right\|_{\mathcal{F}^{2\alpha}(T^2_{L \sqrt{\log L}})}\right]^p \lesssim \left(\log(L \sqrt{\log L})\right)^p \lesssim (\log(L))^p.
\]

which yield that

\[
\sup_L \mathbb{E}\left[\left\|\Xi^r\right\|^p_{\mathcal{F}^{\alpha}} \right] < +\infty.
\]

Therefore the bound 50 allow us to conclude that:

\[
\sup_L \mathbb{E}\left[\frac{1}{\log(L)} \Lambda_n(\Xi)\right|^p < +\infty
\]

5.3 Tail estimate for the minimal eigenvalue

In this section we will assume that \( L = 1 \) and we are interested to study the tail estimate for the minimal eigenvalue \( \Lambda_1 \), namely we have the following result

**Proposition 5.4.** Let \( \Lambda_1 \) the first eigenvalue of the operator \( \mathcal{H}(\Xi) \) where \( \Xi \) is the rough distribution associated. Then there exist \( C_1, C_2 > 0 \) and such that :

\[
e^{C_1 x} \leq \mathbb{P}(\Lambda_1 \leq x) \leq e^{C_2 x}
\]

when \( x \to -\infty \)
Proof. Upper bound:

As pointed out previously the following relation
\[ r^2 \Lambda_1(\Xi) \equiv \Lambda_1(\tilde{\Xi}^r) + r^2 m_{1,r} \]
hold in law for every \( r > 0 \), where \( \tilde{\Xi}^r = (r \tilde{\xi}, r^2 \tilde{\xi}_2) \) with \( \tilde{\xi} \) is a white noise on \( T^2 \) and \( \tilde{\xi}_2 \) the associate rough distribution. Of course this relation oblivious imply the following equality
\[ P(r^2 \Lambda_1(\Xi) \leq -1) = P(\Lambda_1(\tilde{\Xi}^r)) \leq -1 - r^2 m_{1,r} \]
Since \( r^2 m_{1,r} \to r^0 \) is easy to see that
\[ P(\Lambda_1(\tilde{\Xi}^r) \leq -\frac{3}{2}) \leq P(\Lambda_1(\tilde{\Xi}^r)) \leq -1 - r^2 m_{1,r} \leq P(\Lambda_1(\tilde{\Xi}^r)) \leq -\frac{1}{2} \]
for \( r \) small enough. Using the continuity estimate for the ground state given in the Proposition 4.28 we can see that the event \( \{ (\Lambda_1(\tilde{\Xi}^r)) \leq -\frac{1}{2} \} \) is contained in \( \{ \| \hat{\Xi}^r \|_{x \alpha} (1 + \| \hat{\Xi}^r \|_{x \alpha})^M \geq C \} \) for a deterministic constant \( C > 0 \) and for \( \alpha < -1 \). Thus we have
\[ P(r^2 \Lambda_1(\Xi) \leq -1) \leq P(\| \hat{\Xi}^r \|_{x \alpha} (1 + \| \hat{\Xi}^r \|_{x \alpha})^M \geq C) \]
splitting the right hand side according of this equation to the event \( \{ \| \Xi \|_{x \alpha} \geq 1 \} \) and it complementary we can finally bound our probability by
\[ P(r^2 \Lambda_1(\Xi) \leq -\frac{1}{2}) \leq P(\| \hat{\Xi}^r \|_{x \alpha} \geq 1) + P(\| \hat{\Xi}^r \|_{x \alpha} \geq C2^{-M}) \]
Now it suffice to observe that
\[ P(\| \Xi \|_{x \alpha} \geq 1) \leq P(r\| \bar{\xi}^r \|_{x \alpha} \geq \frac{1}{2}) + P(r^2 \| \bar{\Xi}_2 \|_{x \alpha + 2} \geq \frac{1}{2}) \leq r^{-\theta} e^{\frac{-2\lambda}{r^2}} \sup \lambda (E|e^{\lambda \| \bar{\xi}^r \|_{x \alpha}}| + E|e^{\lambda \| \bar{\Xi}_2 \|_{x \alpha + 2}}|) \]
for \( \lambda > 0 \) small enough, where we have used the Markov inequality, then choosing \( \theta \) according to the Lemma 5.3 allow us to get the needed upper bound. The term \( P(\| \Xi \|_{x \alpha} \geq C2^{-M}) \) can be treated in the same way and of course if take \( x = -\frac{1}{r^2} \) this bound can be reformulated in the following way
\[ P(\Lambda_1(\Xi) \leq x) \leq e^{2\lambda x} \]
Lower bound:

Given \( c < 0 \), a subset \( S \subset T^2 \) which have size 1 (ie: \( |S| = 1 \)), \( f \) a smooth function on \( T^2 \) with support contained in \( S \) and such that \( f_S f^2 = 1 \), \( b = -\| \nabla f \|_{L^2}^2 + \frac{c}{2} \) and \( h(x) = b\|_S \) where \( \|_S \) is the characteristic function of the set \( S \). Then is easy to see from the min-max principle that:
\[ \Lambda_1(h, h \circ \sigma(D) h) \leq \| \nabla f \|_{L^2}^2 + b \int_S f^2 = \frac{c}{2} \]
and before proceeding with proof let us remark that \( b \) does not depend on \( r \). The continuity bound of the eigenvalue gives us that:
\[ \Lambda_1(\tilde{\Xi}^r) \leq \frac{c}{2} + C(r \| \bar{\xi} - h \|_{x \alpha} + r^2 \| \bar{\Xi}_2 - h \circ \sigma(D) h \|_{x \alpha + 2}) (1 + \| \Xi \|_{x \alpha} + \| h \|_{L^\infty})^M \]
for a deterministic constant $C$ (which depend only on $c$). Now if $(\|r\tilde{\xi} - h\|_{\mathcal{G}^\alpha} + \|r^2\tilde{\Xi}_2 - h_N \circ \sigma(D)h_N\|_{\mathcal{G}^{2n+2}}) \leq \delta$ for a fixed $\delta$ which satisfy $C\delta(\delta + 1)^M \leq \frac{1}{4}$. Then

$$\Lambda_1(\tilde{\Xi}^r) \leq \frac{c}{4}$$

therefore if $c = -6$ the event $\{\|r\tilde{\xi} - h\|_{\mathcal{G}^\alpha} + \|r^2\tilde{\Xi}_2 - h \circ \sigma(D)h\|_{\mathcal{G}^{2n+2}} \leq \delta\}$ is contained in $\{\Lambda_0(\tilde{\Xi}^r) \leq -\frac{3}{2}\}$ and this immediately implies that

$$\mathbb{P}(\Lambda_0(\tilde{\Xi}^r) \leq -\frac{3}{2}) \geq \mathbb{P}\left(\|r\tilde{\xi} - h\|_{\mathcal{G}^\alpha} \leq \frac{\delta}{2}; \|r^2\tilde{\Xi}_2 - h \circ \sigma(D)h\|_{\mathcal{G}^{2n+2}} \leq \frac{\delta}{2}\right)$$

(51)

To get a lower bound for the right hand side of this inequality we will use the Cameron-Martin theorem, indeed:

$$\mathbb{P}\left(\|r\tilde{\xi} - h_N\|_{\mathcal{G}^\alpha} \leq \frac{\delta}{2}; \|r^2\tilde{\Xi}_2 - h \circ \sigma(D)h\|_{\mathcal{G}^{2n+2}} \leq \frac{\delta}{2}\right) = \exp(-r^{-2}\|h\|_{L^2})\mathbb{E}[\exp(r^{-1}\tilde{\xi}(h))\mathbb{I}_{A^r}]$$

$$= \exp(-\frac{\|h\|_{L^2}^2}{r^2})\mathbb{E}[\exp(r^{-1}\tilde{\xi}(h))|A^r]\mathbb{P}(A^r)$$

(52)

with $A^r = \left\{\omega; \|r\tilde{\xi}(\omega)\|_{\mathcal{G}^\alpha} \leq \frac{\delta}{2}; \|r^2\tilde{\Xi}_2(\omega + r^{-1}h) - h \circ \sigma(D)h\|_{\mathcal{G}^{2n+2}} \leq \frac{\delta}{2}\right\}$ and where we have used the Jensen inequality to obtain the last lower bound. Finally to get our lower bound it suffice to control the probability that the event $A^r$ happen for $r$ large enough. On the other hand we observe that:

$$\tilde{\Xi}_2(\omega + r^{-1}h) = \tilde{\Xi}_2(\omega) + r^{-1}\tilde{\xi}(\omega) \circ \sigma(D)h + r^{-1}\sigma(D)\tilde{\xi}(\omega) \circ h$$

which gives the following representation

$$A^r = \left\{r\|\tilde{\xi}\|_{\mathcal{G}^\alpha} \leq \frac{\delta}{2}; \|r^2\tilde{\Xi}_2 + r\tilde{\xi} \circ \sigma(D)h + rh \circ \sigma(D)\tilde{\xi}\|_{\mathcal{G}^{2n+2}} \leq \frac{\delta}{2}\right\}.$$

At this point the Lemma 5.3 tell us that

$$r\|\tilde{\xi}\|_{\mathcal{G}^\alpha} + r^2\|\tilde{\Xi}_2\|_{\mathcal{G}^{2n+2}} \to r^0$$

almost surely. On the other side using the fact that $\|h\|_{L^\infty((0,\frac{1}{2})^2)} \leq b$ gives us that

$$r\|\tilde{\xi} \circ \sigma(D)h\|_{\mathcal{G}^{2n+2}} \leq r\|\tilde{\xi} \circ \sigma(D)h\|_{H^{2n+3}} \leq r\|\tilde{\xi}\|_{\mathcal{G}^\alpha}\|h\|_{L^\infty} \to r^0 0$$

almost surely. Same type of estimate show that $r\|\sigma(D)\tilde{\xi} \circ h\|_{\mathcal{G}^{2n+2}}$ vanish when $r$ goes to 0. All this convergence imply in particular that $\mathbb{P}(A^r) \to r^0 1$ which combined with the two bound (52) and (51) allow us to get:

$$\mathbb{P}(\Lambda_1(\tilde{\Xi}^r) \leq -\frac{3}{2}) \geq \frac{1}{2}\exp(-\frac{b^2}{r^2})$$

for all $r$ small enough, which is the needed lower bound due to the fact that $\mathbb{P}(\Lambda_0(\Xi) \leq -\frac{1}{r^2}) \geq \mathbb{P}(\Lambda_0(\tilde{\Xi}^r) \leq -\frac{3}{2})$.
A Other useful results on Besov spaces and Bony paraproducts

The following Besov embedding property is used in the proof of the convergence of the mollified white noise (see Theorem 5.1).

**Proposition A.1.** Let $1 \leq p_1 \leq p_2 \leq +\infty$ and $1 \leq q_1 \leq q_2 \leq +\infty$. For all $s \in \mathbb{R}$, the space $\mathcal{B}_{p_1,q_1}^s$ is continuously embedded in $\mathcal{B}_{p_2,q_2}^{s-d(\frac{1}{p_1} - \frac{1}{p_2})}$. In particular, we have

$$||u||_{\mathcal{B}_{p_1,q_1}^{s-\frac{d}{p}}} \lesssim ||u||_{\mathcal{B}_{p_2,q_2}^s}.$$ 

We will need also the following useful extension of the Schauder estimate.

**Proposition A.2.** Let $f \in H^\alpha$, $g \in \mathcal{C}^\beta$ with $\alpha \in (0,1)$, $\beta \in \mathbb{R}$ and $\sigma : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ an infinitely differentiable function such that $|D^k \sigma(x)| \lesssim |x|^{-n-k}$. Set

$$\mathcal{C}(f,g) := \sigma(D)(f \prec g) - f \prec \sigma(D)g.$$ 

Then,

$$||\mathcal{C}(f,g)||_{H^{\alpha+\beta+n-\delta}} \lesssim ||f||_{H^\alpha}||g||_{\mathcal{C}^\beta}.$$ 

for all $\delta > 0$

**Remark A.3.** A proof of this Lemma is contained in the first version of [14] (or in [6]) where $f$ is in the Besov-Hölder space $\mathcal{C}^\alpha$, however a slight modification show that the same result is true for Sobolev space.

Before proving Proposition 4.3, we give an elementary commutation Lemma.

**Lemma A.4.** Let $\alpha \in (0,1)$, $f \in H^\alpha$ and $g \in L^\infty$. Then,

$$||\Delta_j(fg) - f\Delta_j g||_{L^2} \lesssim 2^{-j\alpha}||f||_{H^\alpha}||g||_{L^\infty}.$$ 

**Proof.** For $y \in \mathbb{T}^2$, we introduce the inverse Fourier transform

$$\theta_j(y) := \sum_{k \in \mathbb{Z}^2} \rho(2^{-j}|k|) \exp(i2\pi \langle k, y \rangle)$$ (53)

of the function $\rho(2^{-j} \cdot)$ introduced to define the Littlewood-Paley blocks. For $x \in \mathbb{T}^2$, we have by definition

$$|\Delta_j(fg)(x) - f(x)\Delta_j g(x)|^2 = \left| \int_{\mathbb{T}^2} \theta_j(x-y)g(y)(f(y) - f(x))dy \right|^2 \lesssim ||g||_{L^\infty}^2 \left( \int_{\mathbb{T}^2} |\theta_j(x-y)|^2 |x-y|^{2\alpha+2}dy \right) \int_{\mathbb{T}^2} \left| \frac{f(y) - f(x)}{|y-x|^{2\alpha+2}} \right|^2 dy.$$ (54)

From (53), we see that $\theta_j(y)$ concentrates in a ball of radius $2^{-j}$ and is of order $2^{2j}$ for $j$ large. Therefore, we deduce that

$$\int_{\mathbb{T}^2} |\theta_j(x-y)|^2 |x-y|^{2\alpha+2}dy \lesssim 2^{-2j\alpha}.$$ 

We eventually obtain

$$||\Delta_j(fg) - f\Delta_j g||_{L^2} \lesssim 2^{-j\alpha}||g||_{L^\infty} \left( \int_{\mathbb{T}^2 \times \mathbb{T}^2} \left| \frac{f(y) - f(x)}{|y-x|^{2\alpha+2}} \right|^2 dxdy \right)^{\frac{1}{2}}$$

which permits us to conclude thanks to the equivalent definition of the Sobolev space $H^\alpha$. 

\[ \square \]
We now give a simple consequence of the previous Lemma.

**Lemma A.5.** Let \( \alpha \in (0, 1) \), \( f \in H^\alpha \) and \( g \in \mathcal{C}^\beta \) and set

\[
R_j(f, g) := \Delta_j(f \prec g) - f \Delta_j g.
\]

Then

\[
\|R_j(f, g)\|_{L^2} \lesssim 2^{-j(\alpha + \beta)}\|f\|_{H^\alpha}\|g\|_{\mathcal{C}^\beta}.
\]

**Proof.** We have by definition that

\[
\Delta_i(f \prec g) = \sum_{j: j \sim i} \Delta_i(f \prec \Delta_j g) = \sum_{j: j \sim i} f \Delta_j \Delta_i g + \sum_{j: j \sim i} \Delta_i(f \Delta_j g) - f \Delta_i \Delta_j g - \sum_{j: j \sim i} \Delta_i(f \Delta_j g + f \circ \Delta_j g)
\]

So that the first sum over \( g \) can be chosen such that \( \sum_{j: j \sim i} \Delta_i \Delta_j g = \Delta_i g \). Then the Lemma A.4 take care of the second sum of this equation and the paraproduct estimate gives the needed bound for the the last term. \( \square \)

**Proof of Proposition 4.3.** Let us write that

\[
(f \prec g) \circ h = \sum_{[i-j] \leq 1, k} 1_{k \leq j} \Delta_i(\Delta_k f \prec g) \Delta_j h = \sum_{i \sim j, k} 1_{k \leq j} \Delta_k f \Delta_i g \Delta_j h + \sum_{i \sim j, k i} R_i(\Delta_k f, g) \Delta_j h
\]

\[
= f(g \circ h) + \sum_{i \sim j, k \leq 2N} \Delta_k f \Delta_i g \Delta_j h + \sum_{i \sim j, k i} R_i(\Delta_k f, g) \Delta_j h \tag{55}
\]

Now let us remark that for fixed \( k \) the sum \( \sum_{i \sim j} 1_{i \leq k - N} \Delta_k f \Delta_i g \Delta_j h \) is supported in a ball \( 2^k B \) then is suffice to setimate the \( L^2 \) norm

\[
2^{k\alpha} \left\| \sum_{i \sim j} 1_{i \sim k - N} \Delta_k f \Delta_i g \Delta_j h \right\|_{L^2} \lesssim (2^{k\alpha})\|\Delta_k f\|_{\alpha}\|g\|_{\beta}\|h\|_\gamma \sum_{i \sim j, k \leq 2N} 2^{-i(\beta + \gamma)}
\]

therefore using the fact that \( \beta + \gamma < 0 \) we obtain the needed bound for this term. Now we remark that for fixed \( j \) the sum \( \sum_{i \sim j, k \leq i} R_i(\Delta_k f, g) \Delta_j h \) is localized in a ball of size \( 2^j \) then estimating the sum

\[
\left\| \sum_{i \sim j, k \leq i} R_i(\Delta_k f, g) \Delta_j h \right\|_{L^2} \lesssim \left\| \sum_{k \leq j} \Delta_k f\|_{\alpha}\|g\|_{\beta}\|h\|_\gamma 2^{-j\gamma} \sum_{j \sim i} 2^{-i(\alpha + \beta)}
\]

which gives the needed estimates. \( \square \)

We end the appendix by recalling a version of the Rellich-Kondrachov Theorem

**Lemma A.6.** Let \( \gamma < \beta \). Then, the injection \( i : H^\beta(\mathbb{T}_L^2) \to H^\gamma(\mathbb{T}_L^2) \) is compact.

**Proof.** Let \( (f_n)_{n \in \mathbb{N}} \) be a bounded sequence in \( H^\beta \) and set \( M := \sup_n \|f_n\|_{H^\beta} \). It is well known that there exists a subsequence of \( (f_n) \) (still denoted for simplicity by \( f_n \)) which converges in \( H^\beta(\mathbb{T}_L^2) \) to some \( f \in H^\beta(\mathbb{T}_L^2) \), or equivalently such that the Fourier coefficients converge, i.e. \( \lim_n \hat{f}_n(k) = \hat{f}(k) \) for all \( k \in \mathbb{Z}_L^2 \). It is easy to prove that \( f \in H^\beta \) using the Fatou Lemma

\[
\sum_{k \in \mathbb{Z}_L^2} (1 + |k|)^{2\beta} |\hat{f}(k)|^2 \leq \lim \inf_n \sum_{k \in \mathbb{Z}_L^2} (1 + |k|)^{2\beta} |\hat{f}_n(k)|^2 \leq M^2.
\]
Let us now prove that \((f_n)\) converges to \(f\) in the space \(H^7(\mathbb{T}_L^2)\). We have
\[
\|f_n - f\|_{H^7} \leq \sum_{|k| \leq N} (1 + |k|)^{2\gamma} |\hat{f}_n(k) - \hat{f}(k)|^2 + \sum_{|k| > N} (1 + |k|)^{2(\gamma - \beta)} (1 + |k|)^{2\beta} |\hat{f}_n(k) - \hat{f}(k)|^2
\]
\[
\leq \sum_{|k| \leq N} (1 + |k|)^{2\gamma} |\hat{f}_n(k) - \hat{f}(k)|^2 + 2N^{2(\gamma - \beta)} M^2.
\]
The convergence in \(H^7\) follows from this latter inequality.

Now let us end by giving a more simplest description of the space \(\mathcal{X}^\alpha\)

**Lemma A.7.** Given \(\alpha < -1\) and let denote by \(\mathcal{C}^{0,\alpha}\) (respectively \(\mathcal{C}^{0,2\alpha+2}\)) the closure of the space of infinitely differentiable function in the space \(\mathcal{C}^\alpha\) (respectively \(\mathcal{C}^{2\alpha+2}\)) then the following identity set
\[
\mathcal{X}^\alpha = \mathcal{C}^{0,\alpha} \times \mathcal{C}^{0,2\alpha+2}
\]

**Proof.** To prove our equality set is sufficient to show that \(\mathcal{X}^\alpha\) contain the space \(\{0\} \times \mathcal{C}^\infty\). Let \(X^N(x) = 2^N \cos(2^N \pi \langle z, x \rangle)\) for \(x \in [0, 1]^2\). It was proved in [7] that
1. \(\|X^N\|_{\mathcal{C}^\alpha} \to N \to +\infty 0\)
2. \(\|X^N \circ \sigma(D)X^N + 1\|_{\mathcal{C}^{2\alpha+2}} \to N \to +\infty 0\)

for all \(\alpha < -1\). Let \(V \in \mathcal{C}^\infty(\mathbb{T}^2)\) and \(X^{N,V} \equiv V X^N\), then is easy to see that
\[
\|X^{N,V}\|_{\mathcal{C}^\alpha} \lesssim \|V\|_{\mathcal{C}^\beta} \|X^N\|_{\mathcal{C}^\alpha} \to N \to +\infty 0
\]
where \(\beta > -\alpha\). Now we claim that \(X^{N,V} \circ \sigma(D)X^{N,V} \to -V^2\) in \(\mathcal{C}^{2\alpha+2}\). Indeed from the Bony estimate we can see that \(\|V \circ X^N\|_{\mathcal{C}^{\alpha+\beta}} + \|V \triangleright X^N\|_{\mathcal{C}^{\alpha+\beta}} \to 0\) for all \(\beta > -\alpha\) which in particularly imply that
\[
(X^N \circ V + X^N \triangleright V) \circ \sigma(D)X^{N,V} \to N \to +\infty 0
\]
in \(\mathcal{C}^{2\alpha+2}\). On the other side Schauder estimate allow us to see that
\[
X^{N,V} \circ \sigma(D)(X^N \circ V + X^N \triangleright V) \to N \to +\infty 0
\]
in \(\mathcal{C}^{2\alpha+2}\). Then we can conclude that
\[
\|X^{N,V} \circ \sigma(D)X^{N,V} - (V \triangleright X^N) \circ \sigma(D)(V \triangleright X^N)\|_{\mathcal{C}^{2\alpha+2}} \to N \to +\infty 0
\]
Moreover from the Proposition A.2 is easy to show that
\[
\||\sigma(D)(V \triangleright X^N) - V \triangleright \sigma(D)X^N||_{\mathcal{C}^{2\alpha+2}} \lesssim \|V\|_{\mathcal{C}^\beta} \|X^N\|_{\mathcal{C}^\alpha} \to N \to +\infty 0
\]
Therefore the proof of our convergence result is reduced to the study of
\[
(V \triangleright X^N) \circ (V \triangleright \sigma(D)X^N)
\]
which can be handled by using the commutation Lemma 4.3. Indeed we have the following expansion
\[
(V \triangleright X^N) \circ (V \triangleright \sigma(D)X^N) = V^2X^{N+1} \circ \sigma(D)X^N + \mathcal{R}(V, X^N, V \triangleright \sigma(D)X^N)
+ V \mathcal{R}(V, X^N, \sigma(D)X^N)
\]
which converge to \(-V^2\) in the space \(\mathcal{C}^{2\alpha+2}\). Then in particularly we have proved that \((0, c - V^2) \in \mathcal{X}^\alpha\) for every smooth function \(V\) and every \(c \in \mathbb{R}\) which finishes the proof. \(\square\)
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