A REDUCED HDG METHOD FOR THE STOKES EQUATIONS

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ABSTRACT. In this paper, we propose and analyze a hybridized discontinuous Galerkin (HDG) method with reduced stabilization for the Stokes equations. The reduced stabilization enables us to reduce the number of facet unknowns and improve the computational efficiency of the method. We provide optimal error estimates in an energy and $L^2$ norms. It is shown that the reduced method with the lowest-order approximation is closely related to the nonconforming Crouzeix-Raviart finite element method. We also prove that the solution of the reduced method converges to the nonconforming Gauss-Legendre finite element solution as a stabilization parameter $\tau$ tends to infinity and that the convergence rate is $O(\tau^{-1})$.

1. INTRODUCTION

The aim of this paper is to propose and analyze a reduced hybridized discontinuous Galerkin (HDG) method for the Stokes equations with no-slip boundary condition:

$-\Delta u + \nabla p = f$ in $\Omega,$

$\text{div} u = 0$ in $\Omega,$

$u = 0$ on $\partial\Omega,$

where $\Omega \subset \mathbb{R}^d (d=2)$ is a convex polygonal domain and $f \in L^2(\Omega) := [L^2(\Omega)]^d$ is a given function. For the Stokes problem, various HDG methods were already proposed and studied [8, 11, 13, 22, 12, 10, 9, 14, 21, 18]. We also refer to [15] for an overview. The method we investigate in this paper is the HDG-IP method proposed by Egger and Waluga in [21]. The HDG-IP method is based on the gradient-velocity-pressure formulation of the Stokes equations. We remark that the HDG method of the local discontinuous Galerkin type [22, 12] is close to the HDG-IP with a slight difference of a numerical flux.

A reduced stabilization was introduced to the DG methods and analyzed for elliptic problems [6, 7]. In [4], Becker et al. studied the reduced DG method for the Stokes equations and analyzed the limit case as a stabilization parameter tends to infinity. In [21], Lehrenfeld proposed a reduced HDG method for the Poisson equation and the Stokes equations. In the reduced method, two piecewise polynomials of degree $k$ and $k-1$ are used to approximate element and hybrid unknowns, respectively, unlike the standard HDG method. As a result, the number of degrees of freedom is reduced compared to the standard method, which is the main advantage of the reduced method. Lehrenfeld also remarked that the convergence rate of the method is optimal, however, error analysis was not presented. In [23], for the Poisson problem, the author provided the optimal error estimates and
showed the lowest-order reduced HDG method is closely related to the nonconforming Crouzeix-Raviart finite element method.

In this paper, we provide optimal error estimates of the reduced method for the Stokes problem. Since we need to use a weaker energy norm in our analysis, it is necessary to modify the error analysis of the standard method. We note that the main difficulties can be overcome by the techniques used in the author’s previous work [23] and the discrete inf-sup condition proved by Egger and Waluga [18]. We also show a relation between the reduced method and the Gauss-Legendre element (see [5, 24] for example). It is proved that the hybrid part of velocity and the pressure of the lowest-order reduced HDG method coincides with those of the nonconforming Crouzeix-Raviart finite element solution. In the limit case as the stabilization parameter $\tau$ tends to infinity, the solution of the reduced method converges to that of the nonconforming Gauss-Legendre method. The convergence rate is estimated to be $O(\tau^{-1})$. This result is inspired by [4, Theorem 3], however, our proof is completely different and novel.

The rest of this paper is organized as follows. Section 2 is devoted to the preliminaries. In section 3, we introduce a reduced stabilization and present a reduced HDG method. In section 4, we provide a priori error estimates in an energy and $L^2$ norms. In section 5, some relations between the nonconforming Gauss-Legendre method and the reduced method are shown. In section 6, numerical results are presented to confirm our theoretical results. Finally, in section 7, we conclude the paper with some remarks.

2. Preliminaries

2.1. Meshes and function spaces. Let $\{T_h\}_h$ be a family of shape-regular triangulations of $\Omega$ and define $\Gamma_h = \bigcup_{K \in T_h} \partial K$. Let $\mathcal{E}_h$ be the set of all edges in $T_h$. The mesh size of $T_h$ is denoted by $h$, namely $h := \max_{K \in T_h} h_K$, where $h_K = \text{diam} K$. The length of an edge $e \in \mathcal{E}_h$ is denoted by $h_e$.

We use the usual Lebesgue and Sobolev spaces; $L^2(\Omega)$, $L^2(\Gamma_h)$ and $H^m(\Omega)$, and also $L^2_0(\Omega) = \{ q \in L^2(\Omega) : \int_{\Omega} q \, dx = 0 \}$. We introduce piecewise Sobolev spaces $H^m(T_h) = \{ v \in L^2(\Omega) : v|_K \in H^m(K) \ \forall K \in T_h \}$. For vector-valued function spaces, we write them in bold, such as $L^2(\Omega) = [L^2(\Omega)]^d$ and $H^m(\Omega) = [H^m(\Omega)]^d$. The usual $L^2$ inner product is denoted by $(\cdot , \cdot)$. Let us define the piecewise inner product by

$$
(u,v)_{T_h} = \sum_{K \in T_h} \int_K uv \, dx, \quad (u,v)_{\partial T_h} = \sum_{K \in T_h} \int_{\partial K} uv \, ds.
$$

Let $P_k(T_h)$ and $P_k(\mathcal{E}_h)$ denote the space of element-wise and edge-wise polynomials of degree $k$, respectively. We employ $V_h^k = P_k(T_h)$, $V_h^{k-1} = P_{k-1}(\mathcal{E}_h) \cap \{ \tilde{v} \in L^2(\Gamma_h) : \tilde{v} = 0 \text{ on } \partial \Omega \}$ and $Q_h^{k-1} = P_{k-1}(T_h) \cap L^2_0(\Omega)$ as finite element spaces, which we call $P_k - P_{k-1}$ approximation. The $L^2$-projection from $\prod_{K \in T_h} L^2(\partial K)$ onto $\prod_{K \in T_h} P_{k-1}(\partial K)$ is denoted by $P_{k-1}$, and $I$ stands for the identity operator.

2.2. Norms and seminorms. As usual, we use the Sobolev norms $|v| = |v|_{H^m(\Omega)}$ and $\|v\|_{m,D} = \|v\|_{H^m(\Omega)}$ for a domain $D$. The $L^2$-norm is denoted by $\|v\| = \|v\|_{0,D} = \|v\|_{L^2(D)}$. The energy norms are defined as follows: for $(v, \tilde{v}) \in H^2(T_h) \times
\[ L^2(\Gamma_h), \]

\begin{align}
(2.1) \quad \| (\mathbf{v}, \mathbf{\tilde{v}}) \|^2 &= |\mathbf{v}|^2_{1,h} + |\mathbf{v}|^2_{2,h} + |(\mathbf{v}, \mathbf{\tilde{v}})|^2_j, \\
(2.2) \quad \| (\mathbf{v}, \mathbf{\tilde{v}}) \|^2 &= |\mathbf{v}|^2_{1,h} + |\mathbf{v}|^2_{2,h} + |(\mathbf{v}, \mathbf{\tilde{v}})|^2_j, \\
(2.3) \quad \| (\mathbf{v}, \mathbf{\tilde{v}}) \|^2 &= |\mathbf{v}|^2_{1,h} + |(\mathbf{v}, \mathbf{\tilde{v}})|^2_j, \\
(2.4) \quad \| (\mathbf{v}, \mathbf{\tilde{v}}) \|^2_{h,r} &= |\mathbf{v}|^2_{1,h} + |(\mathbf{v}, \mathbf{\tilde{v}})|^2_j, \\
(2.5) \quad |(\mathbf{v}, \mathbf{\tilde{v}})|^2_j &= \sum_{K \in T_h} \sum_{e \in \partial K} \frac{1}{h_e} \| P_{k-1}(\mathbf{\tilde{v}} - \mathbf{v}) \|^2_{0,e}, \\
(2.6) \quad |(\mathbf{v}, \mathbf{\tilde{v}})|^2_{1,r} &= \tau |(\mathbf{v}, \mathbf{\tilde{v}})|^2_j \\
(2.7) \quad |\mathbf{v}|^2_{1,h} &= \sum_{K \in T_h} |\mathbf{v}|^2_j, \\
(2.8) \quad |\mathbf{v}|^2_{2,h} &= \sum_{K \in T_h} h^K_2 |\mathbf{v}|^2_{2,K}.
\end{align}

The symbol \( \tau \) is a stabilization parameter which will be defined in section 3. The parameter-free energy norms \( \| \cdot \| \) and \( \| \cdot \|_h \) are used to analyze the convergence rate with respect to the mesh size \( h \). We need the parameter-dependent energy norms in order to analyze the proposed method when \( \tau \to \infty \). We also use the stronger \( L^2 \) norm

\[ \| q \|^2_h = \| q \|^2 + \sum_{K \in T_h} h^K_2 |q|_{1,K}^2. \]

By the inverse inequality [1], we see that the two energy norms are equivalent to each other on \( V_h \times \tilde{V}_h \), i.e.,

\begin{align}
(2.9) \quad \|(\mathbf{v}_h, \mathbf{\tilde{v}}_h)\|_h \leq \|(\mathbf{v}_h, \mathbf{\tilde{v}}_h)\| \leq C\|(\mathbf{v}_h, \mathbf{\tilde{v}}_h)\|_h, \\
(2.10) \quad \|(\mathbf{v}_h, \mathbf{\tilde{v}}_h)\|_{h,r} \leq \|(\mathbf{v}_h, \mathbf{\tilde{v}}_h)\|_r \leq C\|(\mathbf{v}_h, \mathbf{\tilde{v}}_h)\|_{h,r}
\end{align}

for some constant \( C > 0 \) independent of \( h \). Similarly, it holds that \( \|q_h\| \leq \|q_h\|_h \leq C\|q_h\| \) for all \( q_h \in Q_h^{k-1} \). In the following, the symbol \( C \) will stand for a generic constant independent of the mesh size \( h \) and the stabilization parameter \( \tau \).

### 2.3. Approximation property

The approximation property in the energy norm holds as well as in the standard energy norm.

**Theorem 1.** If \( \psi \in H^{k+1}(\Omega) \) and \( \pi \in H^k(\Omega) \), then we have

\begin{align}
(2.11) \quad \inf_{(\mathbf{v}_h, \mathbf{\tilde{v}}_h) \in V_h \times \tilde{V}_h} \| (\psi - \mathbf{v}_h, \mathbf{\tilde{v}}_h) \|_h &\leq C h^k \| \psi \|_{H^{k+1}(\Omega)}, \\
(2.12) \quad \inf_{q_h \in Q_h^{k-1}} \| \pi - q_h \| &\leq C h^k \| \pi \|_{H^k(\Omega)}.
\end{align}

**Proof.** We refer to [23]. \( \square \)

### 3. A reduced HDG method

In this section, we present a reduced HDG method based on the HDG method proposed by Egger and Waluga in [18]. By taking the \( L^2 \)-projection onto the polynomial space of lower degree by one in the stabilization term of the standard
the boundedness of \\
holds: 
\[ \inf-
\text{sup condition of Theorem 3.} \]

Let 
\[ \text{Consistency.} \]

Due to the symmetricity of 
\[ (4.2) \]

\[ \text{Proof.} \quad \square \]

\[ (3.1a) \quad a_h(u_h, \tilde{u}_h; v_h, \tilde{v}_h) + b_h(v_h, \tilde{v}_h; p_h) = (f, v_h)_\Omega \quad \forall (v_h, \tilde{v}_h) \in V_h^k \times \hat{V}_h^{k-1}, \]

\[ b_h(u_h, \tilde{u}_h; q_h) = 0 \quad \forall q_h \in Q_h^{k-1}, \]

where the bilinear forms are given by 
\[ a_h(u_h, \tilde{u}_h; v_h, \tilde{v}_h) = (\nabla u_h, \nabla v_h)_{\Omega} + \langle \partial_n u_h, \tilde{v}_h - v_h \rangle_{\partial \Omega} + \langle \partial_n v_h, \tilde{u}_h - u_h \rangle_{\partial \Omega} + \langle \tau h \nu^{-1} \Pi_{k-1}(\tilde{u}_h - u_h), \Pi_{k-1}(\tilde{v}_h - v_h) \rangle_{\partial \Omega}, \]

\[ b_h(v_h, \tilde{v}_h; p_h) = -(\nabla v_h, p_h)_{\Omega} - (\tilde{v}_h - v_h, p_h n)_{\partial \Omega}. \]

Here \( \tau \) is a stabilization parameter assumed to be greater than or equal to one and sufficiently large. We recall that \( \Pi_{k-1} \), which is defined in section 2.1., is the \( L^2 \)-projection onto the edge-wise polynomial space of degree \( k-1 \).

**Remark 2.** In the two-dimensional case, we can easily implement the reduced method by using a reduced-order quadrature formula in the computations of the reduced stabilization term, see [23] Lemma 5 for details.

4. Error Analysis

In this section, we provide the optimal error estimates of the method in both the energy and \( L^2 \) norms. To do that, we first show the consistency of the method, the boundedness of \( a_h \) and \( b_h \), and the coercivity of \( a_h \). In addition, the discrete inf-sup condition of \( b_h \) is proved based on the results of [19].

4.1. Consistency. We state the consistency and adjoint consistency of the method.

**Theorem 3.** Let \( (u, p) \) be the exact solution of the Stokes equations (1.1). Then we have

\[ \begin{align*}
(4.1) \quad a_h(u, u|\Gamma_h; v_h, \tilde{v}_h) + b_h(v_h, \tilde{v}_h; p_h) = (f, v_h)_\Omega \quad \forall (v_h, \tilde{v}_h) \in V_h^k \times \hat{V}_h^{k-1}, \\
b_h(u, u|\Gamma_h; q_h) = 0 \quad \forall q_h \in Q_h^{k-1}.
\end{align*} \]

**Proof.** Since \( u - u|\Gamma_h = 0 \) on \( \Gamma_h \), we can easily see that the consistency (4.1) holds. \( \square \)

Let \( (u_h, \tilde{u}_h, p_h) \in V_h^k \times \hat{V}_h^{k-1} \times Q_h^{k-1} \) be the solution of the method (3.3). From the consistency, the Galerkin orthogonality follows immediately:

\[ \begin{align*}
(4.2) \quad a_h(u - u_h, u|\Gamma_h - \tilde{u}_h; v_h, \tilde{v}_h) + b_h(v_h, \tilde{v}_h; p - p_h) = 0 \quad \forall (v_h, \tilde{v}_h) \in V_h^k \times \hat{V}_h^{k-1}, \\
b_h(u - u_h, u|\Gamma_h - \tilde{u}_h; q_h) = 0 \quad \forall q_h \in Q_h^{k-1}.
\end{align*} \]

Due to the symmetricity of \( a_h \), we readily see that the adjoint consistency also holds:

\[ \begin{align*}
(4.3) \quad a_h(v_h, \tilde{v}_h; u, u|\Gamma_h) + b_h(v_h, \tilde{v}_h; p) = (f, v_h)_\Omega \quad \forall (v_h, \tilde{v}_h) \in V_h^k \times \hat{V}_h^{k-1}, \\
b_h(u, u|\Gamma_h; q_h) = 0 \quad \forall q_h \in Q_h^{k-1}.
\end{align*} \]
4.2. Boundedness and coercivity. We first prove the boundedness of \( a_h \) and \( b_h \).

**Theorem 4.** Let \((\xi, \tilde{\xi}) = (w + w_h, w|_{\Gamma_h} + \tilde{w}_h)\) and \((\eta, \tilde{\eta}) = (v + v_h, v|_{\Gamma_h} + \tilde{v}_h)\), where \((w_h, \tilde{w}_h), (v_h, \tilde{v}_h) \in V_h^k \times \tilde{V}_h^{k-1}\) and \(w, v \in H_0^1(\Omega)\). Then there exists a constant \( C > 0 \) independent of \( h \) and \( \tau \) such that

\[
|a_h(\xi, \tilde{\xi}; \eta, \tilde{\eta})| \leq C\tau \| (\xi, \tilde{\xi}) \| \| (\eta, \tilde{\eta}) \|.
\]

With respect to the parameter-dependent energy norm, we have

\[
|a_h(\xi, \tilde{\xi}; \eta, \tilde{\eta})| \leq C\| (\xi, \tilde{\xi}) \|_r \| (\eta, \tilde{\eta}) \|_r.
\]

**Proof.** By the Schwarz inequality, the first term of \( a_h \) is bounded as

\[
| (\nabla \xi, \nabla \eta)_T_h | \leq |\xi|_{1, h} |\eta|_{1, h}.
\]

We estimate the second term. Note that

\[
\langle \partial_n \xi, \hat{\eta} - \eta \rangle_{\partial T_h} = \langle \partial_n \xi, P_{k-1}(\hat{\eta} - \eta) \rangle_{\partial T_h} + \langle \partial_n \xi, (1 - P_{k-1})(\hat{\eta} - \eta) \rangle_{\partial T_h} = \langle \partial_n \xi, P_{k-1}(\hat{\eta} - \eta) \rangle_{\partial T_h} - \langle \partial_n \xi, (1 - P_{k-1})v_h \rangle_{\partial T_h}.
\]

Since \( \langle \partial_n w, (1 - P_{k-1})v \rangle_{\partial T_h} = \langle \partial_n w_h, (1 - P_{k-1})v \rangle_{\partial T_h} = 0 \), it follows that

\[
\langle \partial_n \xi, (1 - P_{k-1})v \rangle_{\partial T_h} = 0.
\]

Using this, we deduce that \( \langle \partial_n \xi, (1 - P_{k-1})v_h \rangle_{\partial T_h} = \langle \partial_n \xi, (1 - P_{k-1})\eta \rangle_{\partial T_h} \). Then we have

\[
|\langle \partial_n \xi, \hat{\eta} - \eta \rangle_{\partial T_h}| = |\langle \partial_n \xi, P_{k-1}(\hat{\eta} - \eta) \rangle_{\partial T_h} - \langle \partial_n \xi, (1 - P_{k-1})\eta \rangle_{\partial T_h}| \\
\leq C \max \{1, \tau^{-1/2}\} \| (\xi \eta)^2 h + h^2 \| (\xi \eta)^2 \| \| (\eta, \hat{\eta}) \|_{j, r} + |\eta|_{1, h}^2 \|_{1/2} \\\n\leq C \| (\xi, \tilde{\xi}) \|_r \| (\eta, \tilde{\eta}) \|_r,
\]

where we have used the trace inequality and the following estimate (see [23] for the proof)

\[
|\langle \eta, \eta \rangle|_{j} \leq C|\eta|_{1, h}^2.
\]

The stabilization term is bounded as

\[
|\langle \tau h^{-1} P_{k-1}(\hat{\xi} - \xi), \nabla (\hat{\eta} - \eta) \rangle_{\partial T_h}| \leq C \| (\hat{\xi}, \hat{\eta}) \|_{j, r} \| (\eta, \hat{\eta}) \|_{j, r}.
\]

From (4.7), (4.8) and (4.10), we obtain the boundedness (4.5). With a slight modification, we can also show that (4.4). \(\square\)

**Theorem 5** (Boundedness of \( b_h \)). Let \((\xi, \tilde{\xi})\) be the same as in Lemma 2 and \( r = q + q_h \) with \( q \in H^1(\Gamma_h) \cap L_0^2(\Omega) \) and \( q_h \in Q_{k-1}^k \). Then there exists a constant \( C > 0 \) independent \( h \) and \( \tau \) such that

\[
|b_h(\xi, \tilde{\xi}; r)| \leq C \| (\xi, \tilde{\xi}) \| \| r \|_h.
\]

In particular, in the case of \( r = q_h \in Q_{k-1}^k \), we have

\[
|b_h(\xi, \tilde{\xi}; q_h)| \leq C \| (\xi, \tilde{\xi}) \| \| q_h \|.
\]
Proof. By the Schwarz inequality, we have
\begin{equation}
|(\text{div} \xi, r)_{T_h}| \leq |\xi|_{1,h} \| r \|.
\end{equation}
Next, we estimate the second term of the bilinear form \( b_h \). In a similar manner of \((4.7)\), we have
\begin{equation}
(\hat{\xi} - \xi, r\eta)_{\partial T_h} = (P_{k-1}(\hat{\xi} - \xi), r\eta)_{\partial T_h} + ((1 - P_{k-1})(\hat{\xi} - \xi), r\eta)_{\partial T_h}
= (P_{k-1}(\hat{\xi} - \xi), r\eta)_{\partial T_h} - ((1 - P_{k-1})w_h, r\eta)_{\partial T_h}.
\end{equation}
Since \(((1 - P_{k-1})w_h, r\eta)_{\partial T_h} = 0\), we get
\begin{equation}
(1 - P_{k-1}w_h, r\eta)_{\partial T_h} = ((1 - P_{k-1})(w + w_h), r\eta)_{\partial T_h} = ((1 - P_{k-1})\xi, r\eta)_{\partial T_h}.
\end{equation}
Hence
\begin{align*}
|\langle \hat{\xi} - \xi, r\eta \rangle_{\partial T_h}| &= |\langle P_{k-1}(\hat{\xi} - \xi) - (1 - P_{k-1})\xi, r\eta \rangle_{\partial T_h}| \\
&\leq C(\| (\xi, \hat{\xi}) \|^2 + |\xi|_{1,h}^{2})^{1/2}\| r \|_{h},
\end{align*}
where we have used the trace inequality and \((4.9)\). Consequently, we obtain the inequality \((4.11)\). From the inverse inequality, \((4.12)\) follows immediately. \( \square \)

**Theorem 6** (Coercivity). Assume that \( \tau \) is sufficiently large. There exists a constant \( C > 0 \) independent of \( h \) and \( \tau \) such that
\begin{equation}
a_h(v_h, \hat{v}_h; v_h, \hat{v}_h) \geq C\| (v_h, \hat{v}_h) \|^2 \quad \forall (v_h, \hat{v}_h) \in V_h^k \times \hat{V}_h^{k-1}.
\end{equation}
In particular, we have
\begin{equation}
a_h(v_h, \hat{v}_h; v_h, \hat{v}_h) \geq C\| (v_h, \hat{v}_h) \|^2 \quad \forall (v_h, \hat{v}_h) \in V_h^k \times \hat{V}_h^{k-1}.
\end{equation}
Proof. We note that
\begin{equation}
(\partial_n v_h, \hat{v}_h - v_h)_{\partial T_h} = (\partial_n v_h, P_{k-1}(\hat{v}_h - v_h))_{\partial T_h}.
\end{equation}
Then it follows that
\begin{equation}
a_h(v_h, \hat{v}_h; v_h, \hat{v}_h) \geq \| v_h \|_{1,\tau}^2 - 2\| \partial_n v_h, P_{k-1}(\hat{v}_h - v_h) \|_{\partial T_h} + \| (v_h, \hat{v}_h) \|_{1,\tau}.
\end{equation}
By the trace and inverse inequalities and Young’s inequality, we have
\begin{equation}
2\| \partial_n v_h, P_{k-1}(\hat{v}_h - v_h) \|_{\partial T_h} \leq C \left( \varepsilon \| v_h \|_{1,\tau}^2 + \varepsilon^{-1} \tau^{-1} \right) \| (v_h, \hat{v}_h) \|_{1,\tau}^2
\end{equation}
for any \( \varepsilon > 0 \). From \((4.18)\) and \((4.19)\), it follows that
\begin{equation}
a_h(v_h, \hat{v}_h; v_h, \hat{v}_h) \geq (1 - C\varepsilon) \| v_h \|_{1,\tau}^2 + (1 - C\varepsilon^{-1} \tau^{-1}) \| (v_h, \hat{v}_h) \|_{1,\tau}.
\end{equation}
We can take \( \varepsilon = \tau^{-1/2} \) and deduce that, by assuming \( \tau \geq 4C^2 \),
\begin{equation}
a_h(v_h, \hat{v}_h; v_h, \hat{v}_h) \geq \left( 1 - C\tau^{-1/2} \right) \| (v_h, \hat{v}_h) \|_{h,\tau}^2 \geq \frac{1}{2} \| (v_h, \hat{v}_h) \|_{h,\tau}^2.
\end{equation}
By the inverse inequality, we have \( \| (v_h, \hat{v}_h) \|_{\tau} \leq \frac{C}{\tau} \| (v_h, \hat{v}_h) \|_{h,\tau} \) for some positive constant \( C' \), which completes the proof. \( \square \)
4.3. The discrete inf-sup condition. To prove the discrete inf-sup condition of the bilinear form $b_h$, we introduce a Fortin operator. The main idea and techniques for constructing the Fortin operator are entirely based on [18]. The global $L^2$-projection operators $\Pi^K_k : H^1_0(\Omega) \to V^K_h$ and $\hat{\Pi}_h^k : H^1(\Omega) \to \hat{V}_h^k$ are defined by

$$(\Pi^K_k v)|_K = \Pi^K_k(v|_K) \text{ for } K \in T_h,$$

$$(\hat{\Pi}_h^k v)|_e = \hat{\Pi}_h^k(v|_e) \text{ for } e \in E_h,$$

where $\Pi^K_k$ and $\hat{\Pi}_h^k$ are the $L^2$-projections onto $P_k(K)$ and $P_h(e)$, respectively. We define the Fortin operator by

$$(\Pi^K_h, \hat{\Pi}_h^{k-1}) : H^1_0(\Omega) \to V_h^k \times \hat{V}_h^{k-1}.$$

In the following, we show this operator satisfies the Fortin properties.

**Theorem 7.** For all $v \in H^1_0(\Omega)$, we have

$$b_h(\Pi^K_k v, \hat{\Pi}_h^{k-1} v; q_h) = (\text{div} v, q_h)_{\Omega} \quad \forall q_h \in Q_h^{k-1}. \tag{4.21}$$

Moreover, there exists a constant $C > 0$ such that

$$\| (\Pi^K_k v, \hat{\Pi}_h^{k-1} v) \|_h \leq C|v|_{1,h} \quad \forall v \in H^1_0(\Omega). \tag{4.22}$$

**Proof.** First, we prove (4.21). Using the Green formula and the property of $L^2$ projection, we have

$$b_h(\Pi^K_k v, \hat{\Pi}_h^{k-1} v; q_h) = -(\text{div} \Pi^K_k v, q_h)_{\Omega} - \langle \hat{\Pi}_h^{k-1} v - \Pi^K_k v, q_h n \rangle_{\partial \Omega}$$

$$= (\Pi^K_k v, \nabla q_h)_{\Omega} - \langle \hat{\Pi}_h^{k-1} v - \Pi^K_k v, q_h n \rangle_{\partial \Omega}$$

$$= (v, \nabla q_h)_{\Omega} - \langle v, q_h n \rangle_{\partial \Omega}$$

$$= -(\text{div} v, q_h)_{\Omega}. \tag{4.23}$$

Next, we prove (4.22). Note that $|\Pi^K_k v|_{1,h} \leq |v|_{1,h}$ and

$$h^{-1/2}\|P_{k-1}(\hat{\Pi}_h^{k-1} v - \Pi^K_k v)\|_{0,e} \leq h^{-1/2}\|v - \Pi^K_k v\|_{0,e} \leq C|v|_{1,h}. \tag{4.24}$$

From (4.23) and (4.24), it follows that

$$\| (\Pi^K_k v, \hat{\Pi}_h^{k-1} v) \|^2_h = |\Pi^K_k v|^2_{1,h} + |(\Pi^K_k v, \hat{\Pi}_h^{k-1} v)|^2 \leq C|v|^2_{1,h}. \tag{4.25}$$

The proof is completed. \hfill $\Box$

By using the above results, we can prove the discrete inf-sup condition for the bilinear form $b_h$.

**Theorem 8** (Discrete inf-sup condition). There exists a constant $\beta > 0$ independent of $h$ such that

$$\sup_{(v_h, \hat{v}_h) \in V_h^k \times \hat{V}_h^{k-1}} \frac{b_h(v_h, \hat{v}_h; q_h)}{\| (v_h, \hat{v}_h) \|} \geq \beta \| q_h \| \quad \forall q_h \in Q_h^{k-1}. \tag{4.26}$$

**Proof.** It is well-known that the continuous inf-sup condition holds: there exists $\beta' > 0$ such that

$$\sup_{v \in V_h(\Omega)} \frac{(\text{div} v, q)}{|v|_{1,h}} \geq \beta' \| q \| \quad \forall q \in Q. \tag{4.27}$$
For all \( q_h \in Q_h \subset L_0^2(\Omega) \), we have
\[
\sup_{(v_h, \tilde{v}_h) \in \mathbf{V}_h \times \tilde{\mathbf{V}}_h} \frac{b_h(v_h, \tilde{v}_h; q_h)}{\|(v_h, \tilde{v}_h)\|_h} \geq \sup_{v \in H^k(\Omega)} \frac{b_h(\Pi_h^k v, \Pi_h^{k-1} v; q_h)}{\|(\text{div} v, q_h)\|_\Omega} 
= \sup_{v \in H^k(\Omega)} \frac{\|(\Pi_h^k v, \Pi_h^{k-1} v)\|_h}{\|(\text{div} v, q_h)\|_\Omega} \geq C \sup_{v \in H^k(\Omega)} \frac{\|(\text{div} v, q_h)\|_\Omega}{\|v\|_{1,h}} \geq \beta' \|q_h\|.
\]

The proof is completed. \(\Box\)

4.4. **A priori error estimates.** We prove optimal error estimates by using the results in the previous section. In this section, the stabilization parameter \( \tau \) is fixed to be a sufficiently large value.

**Theorem 9** (Energy-norm error estimate). Let \((u, p) \in H^1_0(\Omega) \times L_0^2(\Omega)\) be the exact solution of the Stokes equations (1.1), and let \((u_h, \tilde{u}_h, p_h) \in \mathbf{V}_h \times \tilde{\mathbf{V}}_h \times Q_h \) be the solution of the method (4.1). Then we have
\[
\|u - u_h, u|_{\Gamma_h} - \tilde{u}_h\| + \|p - p_h\|
\leq C \left( \inf_{(v_h, \tilde{v}_h) \in \mathbf{V}_h \times \tilde{\mathbf{V}}_h} \|(u - v_h, u|_{\Gamma_h} - \tilde{v}_h)\| + \inf_{q_h \in Q_h} \|q_h\| \right).
\]

If \((u, p) \in H^{k+1}(\Omega) \times H^k(\Omega)\), we obtain
\[
\|u - u_h, u|_{\Gamma_h} - \tilde{u}_h\| + \|p - p_h\| \leq C h^k \left( \|u\|_{H^{k+1}(\Omega)} + \|p\|_{H^k(\Omega)} \right).
\]

**Proof.** Let \(v_h \in \mathbf{V}_h\), \(\tilde{v}_h \in \tilde{\mathbf{V}}_h\) and \(r_h \in Q_h\) be arbitrary, and set \(\eta_h = u - v_h, \tilde{\eta}_h = \tilde{u}_h - \tilde{v}_h\) and \(\delta_h = p - q_h\). Then we have
\[
a_h(\eta_h, \tilde{\eta}_h; w_h, \tilde{w}_h) + b_h(w_h, \tilde{w}_h; \delta_h) = F(w_h, \tilde{w}_h) \quad \forall (w_h, \tilde{w}_h) \in \mathbf{V}_h \times \tilde{\mathbf{V}}_h,
\]
\[
b_h(\eta_h, \tilde{\eta}_h; r_h) = G(r_h), \forall r_h \in Q_h,
\]
where \(F : \mathbf{V}_h \times \tilde{\mathbf{V}}_h \to \mathbb{R}\) and \(G : Q_h \to \mathbb{R}\) are defined by
\[
F(w_h, \tilde{w}_h) = a_h(u - v_h, u|_{\Gamma_h} - \tilde{v}_h; w_h, \tilde{w}_h) + b_h(w_h, \tilde{w}_h; p - q_h),
\]
\[
G(r_h) = b_h(u - v_h, u|_{\Gamma_h} - \tilde{v}_h, r_h).
\]

By the boundedness of \(a_h\) and \(b_h\), we can estimate the dual norms of \(F\) and \(G\) as follows:
\[
\|F\| = \sup_{(w_h, \tilde{w}_h) \in \mathbf{V}_h \times \tilde{\mathbf{V}}_h} \frac{|F(w_h, \tilde{w}_h)|}{\|(w_h, \tilde{w}_h)\|} \leq C \|(u - v_h, u|_{\Gamma_h} - \tilde{v}_h)\| + \|p - q_h\|,
\]
\[
\|G\| = \sup_{r_h \in Q_h} \frac{|G(r_h)|}{\|r_h\|} \leq C \|(u - v_h, u|_{\Gamma_h} - \tilde{v}_h)\|.
\]
By the coercivity \(4.15\), we have
\[
C \|(\eta_h, \bar{\eta}_h)\|^2 \leq a_h(\eta_h, \bar{\eta}_h; \eta_h, \bar{\eta}_h)
\]
(4.32)
\[
= F(\eta_h, \bar{\eta}_h) - G(\delta_h)
\]
\[
\leq \|F\|(\eta_h, \bar{\eta}_h) + \|G\|\|\delta_h\|.
\]
By the discrete inf-sup condition, we deduce that
\[
\beta \|\delta_h\| \leq \sup_{(w_h, \bar{w}_h) \in V_h^k \times \bar{V}_h^{k-1}} \frac{b(w_h, \bar{w}_h; \delta_h)}{\|(w_h, \bar{w}_h)\|}
\]
(4.33)
\[
\leq \sup_{(w_h, \bar{w}_h) \in V_h^k \times \bar{V}_h^{k-1}} \frac{F(w_h, \bar{w}_h) - a_h(\eta_h, \bar{\eta}_h; w_h, \bar{w}_h)}{\|(w_h, \bar{w}_h)\|}
\]
\[
\leq \|F\| + C\|(\eta_h, \bar{\eta}_h)\|.
\]
From \(4.32\) and \(4.33\), it follows that
\[
\|\eta_h, \bar{\eta}_h\| + \|\delta_h\| \leq C (\|u - v_h, u|_{\Gamma_h} - \bar{v}_h\| + \|p - q_h\|).
\]
By the triangle inequality, we obtain the estimate \(4.23\). In addition, using the approximation property, we see that \(4.29\) holds.

We can prove the \(L^2\) error estimate of optimal order by the Aubin-Nitsche duality argument.

**Theorem 10** \((L^2\)-error estimate). Let the notation be the same as in Theorem \(3\). If \((u, p) \in H^{k+1}(\Omega) \times H^k(\Omega)\), then we have
\[
\|u - u_h\| \leq C h^{k+1} \left( |u|_{H^{k+1}(\Omega)} + |p|_{H^k(\Omega)} \right).
\]
(4.34)

**Proof.** We consider the following adjoint problem: find \((\psi, \pi) \in (H^2(\Omega) \cap H^1_0(\Omega)) \times (H^1(\Omega) \cap L^2_0(\Omega))\) such that
\[
-\Delta \psi + \nabla \pi = u - u_h,
\]
\[
\text{div} \psi = 0.
\]
Note that \(|\psi|_{H^2(\Omega)} + |\pi|_{H^1(\Omega)} \leq C \|u - u_h\|\). From the adjoint consistency \(4.33\), the solution of the problem satisfies
\[
a_h(v_h, \bar{v}_h; \psi, \psi|_{\Gamma_h}) + b_h(v_h, \bar{v}_h; \pi) = (u - u_h, v_h)_{\Omega}\quad \forall (v_h, \bar{v}_h) \in V_h^k \times \bar{V}_h^{k-1},
\]
\[
b_h(\psi, \psi|_{\Gamma_h}; \pi_h) = 0\quad \forall \pi_h \in Q_h^{k-1}.
\]
Let \((\psi_h, \pi_h) \in V_h^k \times \bar{V}_h^{k-1} \times Q_h^{k-1}\) be an approximation to \((\psi, \psi|_{\Gamma_h}, \pi)\) satisfying
\[
\|(\psi - \psi_h, \psi|_{\Gamma_h} - \bar{\psi}_h)\| \leq Ch |\psi|_{H^2(\Omega)},\quad \|\pi - \pi_h\| \leq Ch |\pi|_{H^1(\Omega)}.
\]
Taking \(v_h = u - u_h\) and \(\bar{v}_h = u|_{\Gamma_h} - \bar{u}_h\) in \(4.33\),
\[
\|u - u_h\|^2 = a_h(u - u_h, u_h|_{\Gamma_h} - \bar{u}_h; \psi, \psi|_{\Gamma_h}) + b_h(u - u_h, u_h|_{\Gamma_h} - \bar{u}_h; \pi)
\]
\[
= a_h(u - u_h, u_h|_{\Gamma_h} - \bar{u}_h; \psi - \psi_h, \psi|_{\Gamma_h} - \bar{\psi}_h)
\]
\[
+ b_h(u - u_h, u_h|_{\Gamma_h} - \bar{u}_h; \pi - \pi_h)
\]
\[
\leq C \|u - u_h, u_h|_{\Gamma_h} - \bar{u}_h\| \|(\psi - \psi_h, \psi|_{\Gamma_h} - \bar{\psi}_h)\| + \|\pi - \pi_h\|
\]
\[
\leq Ch^{k+1} \left( |u|_{H^{k+1}(\Omega)} + |p|_{H^k(\Omega)} \right) \|u - u_h\|.
\]
Thus we obtain the assertion. \(\square\)
5. Relations with the nonconforming Gauss-Legendre finite element method

5.1. The Gauss-Legendre element. The approximation space of the \( k \)-th Gauss-Legendre element for velocity is defined by

\[
\tilde{V}_h^k = \{ \mathbf{v}_h \in V_h^k : [P_{k-1}\mathbf{v}_h] = 0 \},
\]

where \([\cdot]\) is a jump operator (see \cite{2} for example). This space is known as the Crouzeix-Raviart \cite{17}(\( k = 1 \)), Fortin-Soulie\cite{19}(\( k = 2 \)) or Crouzeix-Falk \cite{10}(\( k = 3 \)) element. Note that \( \tilde{v}_h \in \tilde{V}_h^k \) is continuous at the \( k \)-th order Gauss-Legendre points, and thereby \( P_{k-1}\tilde{v}_h \) is single-valued on \( \Gamma_h \).

The nonconforming Gauss-Legendre finite element method reads as: find \((u_h^*, p_h^*) \in \tilde{V}_h^k \times Q_h^{k-1}\) such that

\[
\begin{align*}
(\nabla u_h^*, \nabla \tilde{v}_h)_{\Omega_h} - (\text{div} \tilde{v}_h, p_h^*)_{\Gamma_h} &= (f, \tilde{v}_h)_{\Omega} \quad \forall \mathbf{v}_h \in \tilde{V}_h^k, \\
(\text{div} u_h^*, q_h)_{\Gamma_h} &= 0 \quad \forall q_h \in Q_h^{k-1}.
\end{align*}
\]

The nonconforming method is well-posed for \( k = 1, 2, 3 \). For \( k \geq 4 \), it is the case under some assumption on a mesh, see \cite{3} Lemma 3.1]. We here assume that the method \((5.2)\) is well-posed for simplicity. In our analysis, we will use the following inf-sup condition for the Gauss-Legendre element, see also \cite{3}.

**Theorem 11** (Discrete inf-sup condition for the Gauss-Legendre element). There exists a constant \( \beta > 0 \) such that

\[
\beta \|q_h\| \leq \sup_{\tilde{v}_h \in \tilde{V}_h^k} \frac{(\text{div} \tilde{v}_h, p_h)_{\Gamma_h}}{|\tilde{v}_h|_{1,h}} \quad \forall q_h \in Q_h^{k-1}.
\]

5.2. A relation between the lowest-order reduced HDG method and the Crouzeix-Raviart element. It is known that there are relations between HDG methods and the conforming or nonconforming finite element methods for Poisson's equation. The hybrid part the solution of the embedded discontinuous Galerkin(EDG) method \cite{20} with the continuous \( P_1 \) element is identical to the conforming finite element solution on \( \Gamma_h \). In \cite{22}, it was proved that the hybrid part of the solution of the reduced HDG method using the discontinuous \( P_1-P_0 \) approximation coincides with the nonconforming Crouzeix-Raviart finite element method at the mid-points of edges. In this section, we discover a relation between the reduced method with the \( P_1-P_0 \) approximation and the nonconforming Crouzeix-Raviart finite element method.

The Crouzeix-Raviart interpolation operator for scalar-valued functions, \( \Pi_h^* : L^2(\Gamma_h) \to V_h^k \), is defined as

\[
\int_e \Pi_h^* \hat{v}ds = \int_e \hat{v}ds \quad \forall \hat{v} \in E_h.
\]

For a vector-valued function \( \hat{\mathbf{v}} = (\hat{v}_1, \cdots, \hat{v}_d)^T \in L^2(\Gamma_h) \), we set

\[
\Pi_h^* \hat{\mathbf{v}} := (\Pi_h^* \hat{v}_1, \cdots, \Pi_h^* \hat{v}_d)^T.
\]

We are now in a position to prove the relation.

**Theorem 12.** Let \((u_h, \tilde{u}_h, p_h) \in V_h^1 \times \tilde{V}_h^0 \times Q_h^0\) be the solution of the reduced method \( (3.1) \) with \( k = 1 \) and \((u_h^*, p_h^*) \in \tilde{V}_h^1 \times Q_h^0\) be the solution of the nonconforming
Crouzeix-Raviart finite element method. Then we have
\[ \Pi_h^* \hat{u}_h = u_h^*, \quad p_h = p_h^*. \]

**Proof.** Since \( \Pi_h^* \hat{v}_h \) is a piecewise linear polynomial, it follows that, by the Green formula,
\[
(\partial_n \Pi_h^* \hat{v}_h, \hat{u}_h)_{\partial T_h} = (\partial_n \Pi_h^* \hat{v}_h, \Pi_h^* \hat{u}_h)_{\partial T_h} = (\nabla \Pi_h^* \hat{v}_h, \nabla \Pi_h^* \hat{u}_h)_{T_h}.
\]
Choosing \( v_h = \Pi_h^* \hat{v}_h \) in (3.1a) and noting that \( \hat{v}_h - \Pi_h^* \hat{v}_h = 0 \) at the mid-points of edges, we have
\[
(\nabla \Pi_h^* \hat{u}_h, \nabla \Pi_h^* \hat{v}_h)_{T_h} - (\text{div} \Pi_h^* \hat{u}_h, \text{div} \Pi_h^* \hat{v}_h)_{T_h} = (f, \Pi_h^* \hat{v}_h)_{\Omega} \quad \forall \hat{v}_h \in \hat{V}_h^0.
\]

The equation (3.1b) can be rewritten as
\[ -(\text{div} u_h, q_h)_{T_h} - (\hat{u}_h - u_h, q_h)_{\partial T_h} = -(\hat{u}_h, q_h)_{\partial T_h} = -\nabla^h \hat{u}_h, q_h)_{\partial T_h} = -\nabla^h \hat{u}_h, q_h)_{\partial T_h}, \]
From (5.5) and (5.6), we obtain the following equations to determine \( \hat{u}_h \):
\[
(\nabla \Pi_h^* \hat{u}_h, \nabla \Pi_h^* \hat{v}_h)_{T_h} - (\text{div} \Pi_h^* \hat{u}_h, \text{div} \Pi_h^* \hat{v}_h)_{T_h} = (f, \Pi_h^* \hat{v}_h)_{\Omega} \quad \forall \hat{v}_h \in \hat{V}_h^0,
\]
\[ (\text{div} \Pi_h^* \hat{u}_h, q_h)_{T_h} = 0 \quad \forall q_h \in Q_h^0, \]
which is nothing but (5.2) in the case of \( k = 1 \). Therefore we have \( \Pi_h^* \hat{u}_h = u_h^* \) and \( p_h = p_h^* \).

**Remark 13.** From the definition of the Crouzeix-Raviart interpolation, we see that
\[
\int_e \hat{u}_h ds = \int_e \Pi_h^* \hat{u}_h ds = \int_e u_h^* ds,
\]
which implies \( \hat{u}_h \) and \( u_h^* \) are equal at the mid-point of \( e \in E_h \).

### 5.3. The limit case as \( \tau \) tends to infinity

We will show that, as the stabilization parameter \( \tau \) tends to infinity, the solution of the reduced method converges to the nonconforming Gauss-Legendre finite element solution with rate \( O(\tau^{-1}) \). To do that, we first prove the following key lemma.

**Theorem 14.** There exists a constant \( C > 0 \) such that, for any \((v_h, \hat{v}_h) \in V_h^k \times \hat{V}_h^{k-1}\),
\[
\inf_{\hat{v}_h \in V_h^k} \| (v_h - \hat{w}_h, \hat{v}_h - P_{k-1} \hat{w}_h) \|_h \leq C \|(v_h, \hat{v}_h)\|_j.
\]

**Proof.** Let us define \( G(\hat{V}_h^k) = \{ (\hat{w}_h, P_{k-1} \hat{w}_h) : \hat{w}_h \in \hat{V}_h^k \} \) which is a closed subspace of \( V_h^k \times \hat{V}_h^{k-1} \). We consider the quotient space \( V_h^k \times \hat{V}_h^{k-1}/G(\hat{V}_h^k) \), where the quotient norm is given by
\[
\| (v_h, \hat{v}_h) \|_{V_h^k \times \hat{V}_h^{k-1}/G(\hat{V}_h^k)} = \inf_{\hat{v}_h \in V_h^k} \| (v_h - \hat{w}_h, \hat{v}_h - P_{k-1} \hat{w}_h) \|_h.
\]

We proved that \( \|(v_h, \hat{v}_h)\|_j \) is also a norm on \( V_h^k \times \hat{V}_h^{k-1}/G(\hat{V}_h^k) \). Suppose that \( \|(v_h, \hat{v}_h)\|_j = 0 \), then we readily have \( v_h \in V_h^k \) and \( \hat{v}_h = P_{k-1} v_h \). Hence it follows that \( (v_h, \hat{v}_h) \in G(\hat{V}_h^k) \), which implies that \( (v_h, \hat{v}_h) \) is the zero element of \( V_h^k \times \hat{V}_h^{k-1}/G(\hat{V}_h^k) \). Since any two norms on a finite-dimensional space are equivalent to each other, we conclude the assertion. \( \square \)
Let us denote \((u_h^*, \tilde{u}_h^*, p_h^*)\) the solution of the reduced method \((3.1)\) with the stabilization parameter \(\tau\). By using Lemma \([14]\) it can be proved that \(|(u_h^*, \tilde{u}_h^*)|_{j}\) converges to zero as \(\tau \to \infty\) with rate \(O(\tau^{-1})\).

**Theorem 15.** If \(\tau\) is sufficiently large, we have
\[
|(u_h^*, \tilde{u}_h^*)|_{j} \leq C\tau^{-1}\|f\|.
\]

**Proof.** Let \(\tilde{v}_h \in \tilde{V}_h^k\) be arbitrary. Since \(P_{k-1}(\tilde{v}_h - P_{k-1}\tilde{v}_h)\) vanishes on \(\Gamma_h\), we have
\[
|(u_h^*, \tilde{u}_h^*)|_{j, \tau} = |(u_h^* - \tilde{v}_h, \tilde{u}_h^* - P_{k-1}\tilde{v}_h)|_{j, \tau}
\leq \|(u_h^* - \tilde{v}_h, \tilde{u}_h^* - P_{k-1}\tilde{v}_h)\|_{\tau}.
\]
Therefore,
\[
|(u_h^*, \tilde{u}_h^*)|_{j, \tau} \leq \inf_{\tilde{w}_h \in V_h^k} \|(u_h^* - \tilde{w}_h, \tilde{u}_h^* - P_{k-1}\tilde{w}_h)\|_{\tau}.
\]
By the coercivity of \(a_h\) and the Schwarz inequality, we have
\[
C_c\|(u_h^* - \tilde{w}_h, \tilde{u}_h^* - P_{k-1}\tilde{w}_h)\|_{\tau}^{2} \leq a_h(u_h^* - \tilde{w}_h, \tilde{u}_h^* - P_{k-1}\tilde{w}_h; u_h^* - \tilde{w}_h, \tilde{u}_h^* - P_{k-1}\tilde{w}_h)
= (f, u_h^* - \tilde{v}_h)
\leq \|f\|\|u_h^* - \tilde{v}_h\|.
\]
Note that the energy norm is stronger than the \(L^2\) norm (see \([23]\) for the proof):
\[
\|z_h\| \leq C\|(z_h, \tilde{z}_h)\|_h \quad \forall (z_h, \tilde{z}_h) \in V_h^k \times \tilde{V}_h^k.
\]
From \((5.11), (5.12), (5.13)\) and Lemma \([14]\) it follows that
\[
|(u_h^*, \tilde{u}_h^*)|_{j, \tau} \leq \inf_{\tilde{v}_h \in V_h^k} \|(u_h^* - \tilde{v}_h, \tilde{u}_h^* - P_{k-1}\tilde{v}_h)\|_{\tau}^{2}
\leq C\|f\| \inf_{\tilde{v}_h \in V_h^k} \|(u_h^* - \tilde{v}_h, \tilde{u}_h^* - P_{k-1}\tilde{v}_h)\|_h
\leq C\|f\|\|(u_h^*, \tilde{u}_h^*)|_{j}.
\]
The proof is completed. \(\square\)

In the following theorem, we prove that \((u_h^*, p_h^*)\) converges to \((u_h^*, p_h^*)\) with the same rate as \(|(u_h^*, \tilde{u}_h^*)|_{j}\), namely \(O(\tau^{-1})\).

**Theorem 16.** If \(\tau\) is sufficiently large, we have
\[
|u_h^* - u_h^*|_{1,h} + \|p_h^* - p_h^*\| \leq C\tau^{-1}\|f\|.
\]

**Proof.** Choosing \(v_h = \tilde{v}_h \in \tilde{V}_h^k\) and \(p_h = P_{k-1}\tilde{v}_h\) in \((3.1)\), we have
\[
(\nabla u_h^*, \nabla \tilde{v}_h)_{\Omega} + \langle \partial_{n}(u_h^*, \tilde{u}_h^*) \rangle_{\partial \Omega} + (\text{div} \tilde{v}_h, p_h^*)_{\Omega} = (f, \tilde{v}_h)_{\Omega} \quad \forall \tilde{v}_h \in \tilde{V}_h^k,
\]
\[
(\text{div} u_h^*, q_h)_\Omega + \langle \tilde{u}_h^* - u_h^*, q_h n \rangle_{\partial \Omega} = 0 \quad \forall q_h \in Q_h^{k-1}.
\]
Let us denote \(\xi_h^* = u_h^* - u_h^*, \xi_h^* = P_{k-1}u_h^* - \tilde{u}_h^*\) and \(\delta_h^* = p_h^* - p_h^*.\) Subtracting \((5.16)\) from \((5.2)\) leads to
\[
(\nabla \xi_h^*, \nabla \tilde{v}_h)_{\Omega} - \langle \partial_{n}(\tilde{v}_h, \tilde{u}_h^*) \rangle_{\partial \Omega} - (\text{div} \tilde{v}_h, \delta_h^*)_{\Omega} = 0 \quad \forall \tilde{v}_h \in \tilde{V}_h^k,
\]
\[
(\text{div} \xi_h^*, q_h)_\Omega + \langle \tilde{u}_h^* - u_h^*, q_h n \rangle_{\partial \Omega} = 0 \quad \forall q_h \in Q_h^{k-1}.
\]
Let $\tilde{w}_h \in \bar{V}_h^k$ be arbitrary, then we have
\[
(\nabla \xi_h, \nabla \xi_h) = (\nabla (u_h^* - \bar{w}_h), \nabla \xi_h) + (\nabla (\bar{w}_h - u_h^*), \nabla \xi_h) \\
= (\partial_n (\bar{w}_h - u_h^*), \tilde{u}_h - u_h^*)_{\partial T_h} + (\text{div}(\bar{w}_h - u_h^*), \delta_h^k)_{T_h} \\
+ (\nabla (\bar{w}_h - u_h^*), \nabla \xi_h) \\
\leq C \left( (u_h^*, \tilde{u}_h^*) |_T \right) \| |(\bar{w}_h - u_h^*, P_{k-1} \bar{w}_h - \tilde{u}_h^*) \|_h.
\]

Taking the infimum with respect to $\bar{w}_h$ and using Lemma 14 we have
\[
(5.18) \quad \left| \xi_h^k \right|_{1,h} \leq C \left( (u_h^*, \tilde{u}_h^*) |_T \right) + \| \delta_h^k \| + \| \xi_h^k \|_{1,h} \inf_{\bar{w}_h \in V_h^k} \| (\bar{w}_h - u_h^*, P_{k-1} \bar{w}_h - \tilde{u}_h^*) \|_h \\
\leq C \left( (u_h^*, \tilde{u}_h^*) |_T \right) \| \delta_h^k \| + \| \xi_h^k \|_{1,h} |(u_h^*, \tilde{u}_h^*)|_T.
\]

On the other hand, by the inf-sup condition for the Gauss-Legendre element, we have
\[
|\hat{\beta}| \| \delta_h^k \| \leq \sup_{\bar{v}_h \in V_h^k} \frac{(\text{div}(\bar{v}_h), \delta_h^k)_{T_h}}{||v_h||_{1,h}} \\
(5.19) \quad = \sup_{\bar{v}_h \in V_h^k} \frac{(\nabla \xi_h, \nabla \bar{v}_h)_{T_h} - (\partial_n \bar{v}_h, \tilde{u}_h - u_h^*)_{\partial T_h}}{||v_h||_{1,h}} \\
\leq |\xi_h^k|_{1,h} + C |(u_h^*, \tilde{u}_h^*)|_T.
\]

Combining (5.18) with (5.19) gives us
\[
|\xi_h^k|_{1,h} \leq C \left( (u_h^*, \tilde{u}_h^*) |_T \right) + \| \delta_h^k \| + \| \xi_h^k \|_{1,h} |(u_h^*, \tilde{u}_h^*)|_T.
\]

By Young’s inequality, we get $|\xi_h^k|_{1,h} \leq C |(u_h^*, \tilde{u}_h^*)|_T$. Using (5.19) again, we have $\| \delta_h^k \| \leq C |(u_h^*, \tilde{u}_h^*)|_T$. By Theorem 14 we obtain the assertion. \qed

**Remark 17.** The convergence rate $O(\tau^{-1})$ provided in Theorems 13 and 10 is best possible. Suppose that there exists $\varepsilon > 0$ such that $|(|(u_h^*, \tilde{u}_h^*)|_T = O(\tau^{-\varepsilon})$ holds. Since $\tau h_c^{-1} \|P_{k-1}(\tilde{u}_h - u_h^*)\|_{0,c} = O(\tau^{-\varepsilon})$, letting $\tau \to \infty$ in (3.3), we have
\[
(5.21a) \quad (\nabla u_h^*, \nabla v_h)_{T_h} + (\partial_n u_h^*, \tilde{v}_h - v_h)_{\partial T_h} + b_h(v_h, \tilde{v}_h; p_h^*) = (f, v_h)_{\Omega} \quad \forall (v_h, \tilde{v}_h) \in V_h^k \times \bar{V}_h^{k-1},
\]
\[
(5.21b) \quad (\text{div} u_h^*, q_h)_{T_h} = 0 \quad \forall q_h \in Q_h^{k-1}.
\]

This means that $(u_h^*, P_{k-1} u_h^*, p_h^*)$ is also a solution of (3.3). Therefore, $(u_h^*, P_{k-1} u_h^*, p_h^*) = (u_h^*, \tilde{u}_h^*, p_h^*)$ for any $\tau$ large enough, which implies that $|(|(u_h^*, \tilde{u}_h^*)|_T = 0$. It occurs rarely, for example, when the exact solution $(u, p)$ is belonging to $V_h^k \times Q_h^{k-1}$.

6. Numerical results

We consider the case of $\Omega = (0, 1)^2$ and
\[
f(x, y) = (4\pi^2 \sin(2\pi y), 4\pi^2 \sin(2\pi x)(-1 + 4\cos(2\pi y))^T
\]
as a test problem. The source term is chosen so that the exact solution $(u, p)$ is
\[
u(x, y) = (2 \sin^2(\pi x) \sin(\pi y), -2 \sin(\pi x) \sin^2(\pi y))^T,
\]
\[
p(x, y) = 4\pi \sin(2\pi x) \sin(2\pi y).
\]

We carry out numerical computations to examine the convergence rates of the reduced method. We employ unstructured triangular meshes and the $P_k-P_{k-1}$ approximations for $k = 1, 2, 3$. The convergence history is shown at Table 1 and
the diagrams are displayed in Figure 1. From these results, we observe that the convergence orders are optimal in all the cases, which agrees with Theorems 9 and 10. In Figure 2, the reduced HDG method is compared to the standard method in terms of computational efficiency. The $x$-axis denotes the number of degrees of freedom of the hybrid variable $\hat{u}_h$. The errors of the reduced and standard methods are drawn by solid and dashed lines, respectively. It can be seen that the reduced method is indeed superior to the standard one in these cases.

Table 1. Convergence history of the reduced method.

| Degree | Mesh size | $\|u - u_h\|$ Error | Order | $\|\nabla (u - u_h)\|$ Error | Order | $\|p - p_h\|$ Error | Order |
|--------|-----------|-------------------|-------|-----------------------------|-------|-------------------|-------|
| 1      | 0.2634    | 1.789E-01 3.672E+00 | 2.07  | 9.340E-01 1.01 6.054E-01 | 1.03  | 2.925E-01 1.15  |
| 2      | 0.1414    | 1.639E-03 2.97  | 2.11  | 1.259E-01 2.06  | 2.16  | 6.859E-03 2.20  |
| 3      | 0.0701    | 1.815E-04 3.13  | 2.17  | 2.768E-02 2.16  | 2.16  | 4.310E-04 3.14  |

7. Concluding remarks

It was shown that the reduced stabilization can reduce a computational cost and the optimal convergence rates are achieved for the Poisson equation in [23] and the

![Figure 1](image.png)

**Figure 1.** Convergence diagrams of the reduced method: $L^2$-error of velocity(left), $H^1$-error of velocity(center) and $L^2$-error of pressure(right).
Stokes problems in the present paper. We conclude that the reduced stabilization is effective for purely diffusive problems. Unfortunately, as remarked in [21], the convergence rate of the reduced method is sub-optimal for convective problems.

In section 5.3, we proved the solution of the reduced method converges to the nonconforming Gauss-Legendre finite element solution. The arguments used therein can be applied to more general cases. For example, in a similar fashion, we can prove that the solution of the standard HDG method converges to some finite element solution when a stabilization parameter goes to infinity.

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