POISSON STRUCTURES OF NEAR-SYMPLECTIC MANIFOLDS AND THEIR COHOMOLOGY

PANAGIOTIS BATAKIDIS AND RAMÓN VERA

ABSTRACT. We connect Poisson and near-symplectic geometry by showing that there is an almost regular Poisson structure induced by a near-symplectic form \( \omega \) when its singular locus is a symplectic mapping torus. This condition is automatically satisfied on any near-symplectic 4-manifold. The Poisson structure \( \pi \) is of maximal rank \( 2n \) and it drops its rank by 4 on a degeneracy set that coincides with the singular locus of the near-symplectic form. We then compute its Poisson cohomology in dimension 4. The cohomology spaces are finite dimensional and depend on the modular class. We conclude with a comment on the interaction between the Poisson structure \( \pi \) and an overtwisted contact structure.

1. Introduction

It is well known that symplectic and Poisson structures are naturally related. A symplectic form on a smooth manifold determines a regular Poisson structure, whose symplectic leaf is the whole manifold. Relaxing the non-degeneracy condition of a symplectic form leads to a closed 2-form that is symplectic away from its degeneracy locus, i.e it is singular with respect to the non-degeneracy. However, this does not automatically imply that there is an induced Poisson structure as in the symplectic case. In this work we study this problem in relation to a near-symplectic form, a type of such singular symplectic structure. This is a closed 2-form \( \omega \) on a smooth \( 2n \)-manifold \( M \) that is positively non-degenerate outside a codimension-3 submanifold, where the rank of \( \omega \) drops by 4. The idea of looking at near-symplectic forms goes back to Taubes in relationship to \( J \)-holomorphic curves, Seiberg-Witten, and Gromov invariants [25, 27]. In dimension 4, these objects are equivalent to self-dual harmonic forms vanishing on circles for a generic metric [12, 27]. Near-symplectic forms have also been studied in the context of broken Lefschetz fibrations starting with the work of Auroux, Donaldson, and Katzarkov [2]. Deformations of these fibrations [15] and connections to overtwisted contact structures [12] have also been considered in the near-symplectic context. Here, we prove the existence of Poisson structures on near-symplectic manifolds, and characterize them in terms of their Poisson cohomology. Our first main result is the following.

Theorem 1.1. Let \((M, \omega)\) be a \(2n\)-dimensional closed near-symplectic manifold. Assume that the singular locus \( Z_\omega \subset M \) of the 2-form is a symplectic mapping torus. Denote by \( NZ_\omega \) its normal bundle, which splits into a line bundle \( L_1^- \) and a rank-2 bundle \( L_2^+ \), i.e \( NZ_\omega \cong L_1^- \oplus L_2^+ \). Then there is a a singular Poisson structure of maximal rank \( 2n \) on \( M \), such that the degeneracy locus of \( \pi^{n-1} \) contains \( Z_\omega \).

2010 Mathematics Subject Classification. Primary: 53D17, 57R17, 17B63. Secondary: 16E45, 17B56, 57M50.

Key words and phrases. near-symplectic forms, Poisson cohomology, harmonic self-dual 2-forms, Poisson algebra, smooth 4–manifolds, almost regular Poisson structure.
Propositions 3.1, 3.3 and 3.5 construct the Poisson structure. Afterwards, we continue by calculating its Poisson cohomology in dimension 4.

Poisson cohomology was introduced by Lichnerowicz in 1977 [16]. It is an important invariant of Poisson geometry, as it reveals features about deformations, normal forms, derivations, and other characteristics of a Poisson structure. In general it is hard to calculate, one of the reasons being that the complex used to define it is elliptic only at the points where the Poisson bivector is non-degenerate. In many cases it is infinite-dimensional, and it is unknown for many types of Poisson structures. For instance, it is known for Lie-Poisson structures of semi-simple and compact type, yet it is unknown for semi-simple but non-compact. The linear Poisson structure constructed in this work and whose Poisson cohomology in dimension 4 is computed, is neither semi-simple, nor compact.

Recently, Poisson cohomology has served as a valuable tool to understand certain singular Poisson structures. For example, it was essential in the work of Radko [23] in order to classify topologically stable Poisson structures on smooth, compact, oriented, surfaces. These structures were later generalized under the name of log or b-symplectic structures. The Poisson cohomology of b-symplectic structures was determined in the work of Guillemin, Miranda, and Pires [11], and Marcut and Osorno-Torres [19, 20].

The results of Theorem 1.1 motivate us to define the notion of near-positive Poisson bivectors on a 4–manifold $M$ (Section 4.1). By near-positive we mean a singular Poisson bivector $\pi$ of maximal rank 4 such that $\pi$ vanishes transversally on a 2-dimensional subspace $D_\pi \subset M$ and $\pi^2 > 0$ on $M \setminus D_\pi$. By Theorem 1.1 these Poisson structures exist on any 4–manifold $M$ admitting a near-symplectic form, that is, any smooth, oriented $M$ with second positive Betti number satisfying $b_+^2(M) > 0$ [14, 12].

In Section 4 we calculate the Poisson cohomology of the structure of Theorem 1.1 in dimension 4. The Poisson cohomology presents an analogous behaviour with log-symplectic structures: It is a mixture of the cohomology coming from the ambient manifold and the one of its degeneracy locus. However, the analogy does not extend to the de Rham cohomology. For the Poisson structure treated here, the part associated to the degeneracy locus is not its de Rham cohomology. Our second main result shows this characterization.

**Theorem 1.2.** Let $(M, \omega)$ be a near-symplectic 4–manifold with a near-positive Poisson structure $\pi$. Denote by $k$ the total number of circles in the singular locus $Z_\omega$. The Poisson cohomology of $(M, \pi)$ is given by

\[
\begin{align*}
H^0_\pi(M, \mathbb{R}) &\cong \mathbb{R} \cong \text{span}(1) \\
H^1_\pi(M, \mathbb{R}) &\cong \mathbb{R}^{2k} \oplus H^1_{\text{dR}}(M) \cong \bigoplus_{r=1}^k \text{span} \left\langle Y^\Omega_r(\pi), \partial L^1_r \right\rangle \oplus H^1_{\text{dR}}(M) \\
H^2_\pi(M, \mathbb{R}) &\cong \mathbb{R}^k \oplus H^2_{\text{dR}}(M) \cong \bigoplus_{r=1}^k \text{span} \left\langle Y^\Omega_r(\pi) \wedge \partial L^1_r \right\rangle \oplus H^2_{\text{dR}}(M) \\
H^3_\pi(M, \mathbb{R}) &\cong H^3_{\text{dR}}(M, \mathbb{R}) \\
H^4_\pi(M, \mathbb{R}) &\cong H^4_{\text{dR}}(M, \mathbb{R})
\end{align*}
\]
where the generators $Y_1^2(\pi), \ldots, Y_k^2(\pi)$ of $H^1_\pi(M, \mathbb{R})$ correspond to the modular vector field of $\pi$ at each component of the singular locus $Z_\omega$, and $\partial L^1_1, \ldots, \partial L^1_k$ are vector fields on the tubular neighbourhood of each component of $Z_\omega$.

Section 4 contains the proof of Theorem 1.2. We start by computing Poisson cohomology with formal coefficients in Proposition 4.7. A key observation comes from the action of Hamiltonian vector fields on polynomial functions with respect to a certain notion of degree. We follow with Poisson cohomology with smooth coefficients in Proposition 4.10. Section 4.4 finishes the proof by showing how to pass to smooth global cohomology on $M$.

We conclude with a note about Poisson and contact structures in a near-symplectic 4-manifold in connection to Theorem 1.2. It is known that in dimension 4 there is an overtwisted contact structure in the boundary of the tubular neighbourhood of the singular locus of a near-symplectic form $[12, 7]$. In Section 5 we make some observations regarding the orbits of the Reeb vector field in relation to the modular class and the action of the anchor map on the contact form.

The near-positive Poisson structure allows us to consider degeneracies in the rank of a Poisson structure $\pi$ which are different from regular and log-symplectic structures. Let us explain this briefly: Let $M$ be a smooth oriented 4-manifold and $\pi \in \Gamma(\Lambda^2 TM)$ a Poisson bivector. In terms of distinct degeneracies in the rank of $\pi$, we have that at any point $p \in M$, $\pi$ can have rank 4, 2, or 0 along symplectic leaves, so one has the following cases:

$(i)$ $\text{Rank}(\pi_p) = 4$, where $\pi^2(p) \neq 0$
$(ii)$ $\text{Rank}(\pi_p) = 2$, where $\pi^2(p) = 0$, but $\pi(p) \neq 0$
$(iii)$ $\text{Rank}(\pi_p) = 0$.

Regular Poisson structures are those with constant rank at all points of $M$. On one end of the spectrum we find symplectic manifolds, which determine a regular Poisson bivector satisfying condition $(i)$ everywhere. On the other end, a trivial Poisson structure corresponds to case $(iii)$. If a Poisson structure is singular, there can be a combination of the three cases in the list, at different points of the manifold. For instance, log-symplectic structures are those equipped with a Poisson bivector $\pi$ on an even dimensional manifold $M$ such that $\pi^n$ is transverse to the zero section in $\Lambda^{2n} TM$. In dimension 4, they capture cases $(i)$ and $(ii)$; the rank of $\pi$ is maximal except at a codimension-1 submanifold, where $\pi^4$ vanishes transversally. The near-positive structure that we introduce is an example for cases $(i)$ and $(iii)$.

In a forthcoming paper, we will compute the Poisson cohomology of a broken Lefschetz fibration using the associated Poisson structure constructed in [6]. This structure is a combination of the cases $(ii)$ and $(iii)$ in the previous list of possible degeneracies on a 4-manifold. Together with the Poisson cohomology computed in [11] and this paper, one will then have available Poisson cohomology computations for large classes of singular Poisson structures on 4-manifolds.

Acknowledgements We warmly thank Pedro Frejlich and Ralph Klaasse for their comments and feedback on the draft of this work. We are also very grateful to Aissa Wade for fruitful discussions and interest in this work. Our thanks extend also to Viktor Fromm, Luis García-Naranjo, Alexei Novikov, Tim Perutz and Pablo Suárez-Serrato.
2. Preliminaries

2.1. Poisson Geometry and Cohomology

We recall some basic facts about Poisson geometry, referring the reader to e.g. [13] for details. Let $M$ be a smooth manifold and $C^\infty(M)$ be the sheaf of smooth $\mathbb{R}$-valued functions on $M$. A Poisson structure on $M$ is a Lie bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$ obeying the Leibniz rule $\{fg,h\} = f\{g,h\} + g\{f,h\}$. Let $\mathfrak{X}^p(M) = \Gamma(\Lambda^p TM)$ be the space of $p$-vector fields on $M$ and $\{\cdot,\cdot\}_{SN} : \mathfrak{X}^p(M) \times \mathfrak{X}^q(M) \to \mathfrak{X}^{p+q-1}(M)$ the Schouten-Nijenhuis bracket. A Poisson structure on $M$ can be equivalently described by a bivector field $\pi \in \mathfrak{X}^2(M)$, called the Poisson bivector, satisfying $[\pi,\pi]_{SN} = 0$. In local coordinates $\{x_1, \ldots, x_n\}$, a Poisson bivector is determined by an antisymmetric matrix $\pi^{i,j}$, written explicitly as $\pi = \sum_{1 \leq i < j \leq n} \pi^{i,j} \partial_i \wedge \partial_j$. The pair $(M,\pi)$ is called a Poisson manifold. We henceforth assume a Poisson manifold $(M,\pi)$ and establish some notation.

Interior contraction with $\pi$ defines a vector bundle homomorphism, which on the spaces of sections reads $\pi^\sharp : \Omega^1(M) \to \mathfrak{X}^1(M)$, and is given pointwise by $\pi^\sharp_p(\alpha_p) = \pi_p(\alpha_p, \cdot)$. It is called the anchor map. This map extends to a $C^\infty(M)$-linear homomorphism

\[
\pi^\sharp : \Omega^*(M) \longrightarrow \mathfrak{X}^*(M),
\]

which we denote again by $\pi^\sharp$ for simplicity.

A vector field $X$ is said to be a Poisson vector field, if $L_X \pi = 0$. Additionally, the vector field $X_f = \pi^\sharp(df)$ is called the Hamiltonian vector field of the Hamiltonian function $f \in C^\infty(M)$. One can check directly that every Hamiltonian vector field is Poisson.

Due to the Poisson condition on $\pi$, the operator

\[
d_\pi : \mathfrak{X}^*(M) \to \mathfrak{X}^{*+1}(M), \quad X \mapsto d_\pi(X) = [\pi, X]_{SN}
\]

is a differential of the exterior algebra $\mathfrak{X}(M) = \oplus_k \mathfrak{X}^k(M)$ leading to the following.

**Definition 2.1.** The pair $(\mathfrak{X}(M), d_\pi)$ is called the Lichnerowicz-Poisson cochain complex, and

\[
H^k_\pi(M) := \frac{\ker \left(d_\pi^k : \mathfrak{X}^k(M) \to \mathfrak{X}^{k+1}(M)\right)}{\operatorname{Im} \left(d_\pi^{k-1} : \mathfrak{X}^{k-1}(M) \to \mathfrak{X}^k(M)\right)},
\]

are called the Poisson cohomology spaces of $(M,\pi)$.

For our purposes we recall the interpretation of the lower Poisson cohomology groups:

\[
\begin{align*}
H^0_\pi(M) &= \{\text{Casimir functions}\} \\
H^1_\pi(M) &= \begin{cases} 
\{\text{Poisson vector fields}\} \\
\{\text{Hamiltonian vector fields}\}
\end{cases} \\
H^2_\pi(M) &= \begin{cases} 
\{\text{infinitesimal deformations of } \pi\} \\
\{\text{trivial deformations of } \pi\}
\end{cases} \\
H^3_\pi(M) &= \{\text{obstructions to formal deformations of } \pi\}.
\end{align*}
\]
The map (1) is a chain map and defines a homomorphism of graded Lie algebras (3) $$\tilde{\pi}^2 : H^*_\text{dr}(M) \to H^*_\pi(M).$$

In general, \(\tilde{\pi}^2\) is neither injective nor surjective, however if \((M, \omega)\) is symplectic with associated Poisson structure \(\pi_\omega\), its Poisson cohomology is known, as \(\tilde{\pi}^2\) is an isomorphism:
$$H^*_\text{dr}(M) \simeq H^*_\pi(M),$$
and \([\pi^2(\omega)] = [\pi_\omega]\). Another well studied case emerges from results of Lu [17], Ginzburg and Weinstein [9]. If \(g\) is a compact Lie algebra and \(W\) the Lie-Poisson structure on \(g^*\), one has
$$H^k_W(g^*) = H^k_{\text{Lie}}(g^*) \otimes \text{Cas}(g^*, W),$$
where \(H^k_{\text{Lie}}(g^*)\) is the Lie algebra cohomology of \(g\) and \(\text{Cas}(g^*, W)\) denotes the space of Casimirs of \((g^*, W)\).

The second cohomology group allows one to characterize certain Poisson structures as exact. For any Poisson structure, \(\pi\) is said to be exact if the fundamental cohomology class vanishes, i.e. \([\pi] = 0\). The first cohomology encompasses another distinctive object of a Poisson structure, the modular class. To define it, consider an orientable Poisson manifold with positive oriented volume form \(\Omega\). The mapping
$$Y^\Omega : C^\infty(M) \to C^\infty(M)$$
defined by
$$\mathcal{L}_X \Omega = (Y^\Omega f)\Omega$$
is a Poisson vector field. The vector field \(Y^\Omega\) is known as the modular vector field with respect to \(\Omega\). One can check directly that \(Y^\Omega = 0\) if and only if \(\Omega\) is invariant by the flows of all the Hamiltonian vector fields on \((M, \pi)\). On the other hand, for another choice \(\Omega' = g \cdot \Omega, g \in C^\infty(M)\), the vector fields \(Y^\Omega'\) and \(Y^\Omega\) differ by a Hamiltonian vector field and thus there is a canonically defined Poisson cohomology class \([Y^\Omega]\) called the modular class of \((M, \pi)\). Modular vector fields and modular classes are defined for non-orientable Poisson manifolds using densities.

2.2. Near-symplectic forms

Since we are interested in the connection between Poisson and near-symplectic geometry, we briefly recall some facts about near-symplectic structures. We refer the reader to [2, 21, 25, 27, 30] and the references within for a detailed exposition on these structures.

Let \(M\) be a smooth, oriented manifold of dimension \(2n\). Consider a 2-form \(\omega \in \Omega^2(M)\) with the property of being near-positive everywhere, that is \(\omega^n \geq 0\). For \(p \in M\), let \(K_p = \{v \in T_p M \mid \omega_p(v, \cdot) = 0\}\) be the kernel of \(\omega\) at a point. If \(\omega\) is symplectic, \(K_p\) is trivial. The collection of fibrewise kernels constitutes the kernel \(K := \ker(\omega) \subset TM\) of the 2-form.

Consider \(\omega\) as a section \(\omega : M \to \Lambda^2 T^* M\). As any smooth map between manifolds, we can consider the differential on tangent spaces \(\nabla_{\omega_p} : T_p M \to \Lambda^2 T^*_p M\). In the context of near-symplectic 4-manifolds this map is known in the literature as the intrinsic gradient. Denote by \(\nabla_{\omega} : TM \to \Lambda^2 T^* M\) the derivative of \(\omega : M \to \Lambda^2 T^* M\), and by \(\tilde{\omega} : \Lambda^2 T^* M \to \Lambda^2 K^*\) the map induced by the dual of the
inclusion \( i: K \hookrightarrow TM \). Let \( \nabla_\omega|_K: K \to \Lambda^2 T^* M \) be the restriction to \( K \). Define the map \( D_K: K \to \Lambda^2 K^* \) as the composition \( \tilde{i} \circ \nabla_\omega|_K \).

On any 4-dimensional vector space \( V \) the wedge product \( q: \Lambda^2 V^* \otimes \Lambda^2 V^* \to \Lambda^4 V^* \) defines a quadratic form of signature \((3,3)\) on \( \Lambda^2 V^* \), giving a decomposition \( \Lambda^2 V^* = \Lambda^2_+ V^* \oplus \Lambda^2_- V^* \), where \( \Lambda^2_+ V^* = \{ \alpha \in \Lambda^2 V^* \mid \alpha \wedge \alpha \geq 0 \} \) and \( \Lambda^2_- V^* = \{ \alpha \in \Lambda^2 V^* \mid \alpha \wedge \alpha \leq 0 \} \). Going back to our original setting, if \( \dim(K_p) = 4 \), then \( \Lambda^2 K^* = \Lambda^2_+ K^* \oplus \Lambda^2_- K^* \). If \( \omega \) is near-positive, that is \( \omega^n \geq 0 \), then \( \dim(\text{Im}(D_K)) \leq 3 \) since the image of \( D_K: K \to \Lambda^2 K^* \) is a positive semi-definite subspace of \( \Lambda^2_+ K^* \).

This can be summarized by the following lemma.

**Lemma 2.2.** [30] If \( \omega \) is near-positive and \( \dim K_p = 4 \) then \( \text{Rank}(D_K) \leq 3 \).

**Definition 2.3.** A near-symplectic form on an oriented manifold \( M^{2n} \) is a closed, near-positive 2-form \( \omega \) such that at every point \( p \in M \) either

(i) \( K_p = 0 \), i.e. \( \omega_p \) is symplectic, or

(ii) \( \dim(K_p) = 4 \) and \( \text{Rank}(D_K) = 3 \).

Its singular locus, \( Z_\omega = \{ p \in M \mid \omega_p^{-1} = 0 \} \), is the submanifold of \( M \) consisting of the points where (ii) holds.

Symplectic manifolds are those satisfying condition (i) everywhere. In order to study singular symplectic structures we look also at condition (ii) with \( Z_\omega \) having non-empty interior. Since we are considering near-positive forms, \( \omega^n \geq 0 \), the condition \( \text{Rank}(D_K) = 3 \) gives an identification of the image of \( D_K \) with the positive bundle of self-dual forms in \( K \), i.e. \( \text{Im}(D_K) \cong \Lambda^2_+ K^* \subset \Lambda^2 K^* \). Moreover, \( Z_\omega \) is a submanifold of \( \dim(Z_\omega) = 2n - 3 \) as the following lemma shows.

**Lemma 2.4.** [30] The singular locus \( Z_\omega \) of a near-symplectic form is a codimension-3 smooth submanifold of \( M \).

Definition 2.3 is standard in the 4-dimensional case, where the intrinsic formulation is due to Donaldson. A near-symplectic 4-manifold \( (M^4, \omega) \) is equipped with a closed 2-form \( \omega \) that is symplectic on \( M^4 \setminus Z_\omega \) and vanishes at \( Z_\omega \). If \( M^4 \) is closed, then \( Z_\omega = \{ p \in M^4 \mid \omega_p = 0 \} \) consists of a collection of embedded circles (see Theorem 2.6 below). It is possible to modify the number of zero components, but it has been shown that \( Z_\omega \) is always non-empty unless the underlying manifold is symplectic [26, Section 5]. Furthermore, \( \omega^2 \geq 0 \) and there is no point \( p \in M^4 \) where \( \text{Rank}(\omega_p) = 2 \). At a degenerate point \( p \in M^4 \), the kernel \( K_p \) is \( T_p^* M^4 \), and the map \( D_K \) corresponds to the so called intrinsic gradient \( \nabla_\omega: T_p M^4 \to \Lambda^2 T^*_p M^4 \), which preserves the rank condition with \( \text{Rank}(\nabla_\omega) = 3 \).

**Remark 2.5.** Near-symplectic forms are related to self-dual harmonic 2-forms for some Riemannian metric. This equivalent formulation appears in the work of different authors [14, 12, 25, 26, 27] and can be seen by the following statement.

**Theorem 2.6.** [25, Thm. 4] [2, Prop. 1] Let \( M^4 \) be a smooth, oriented 4-manifold. For a near-symplectic form \( \omega \) on \( M^4 \), there is a Riemannian metric \( g \) on \( M^4 \) such that \( \omega \) is self-dual harmonic with respect to \( g \). Conversely, if \( M^4 \) is compact and \( b_2^+ (M^4) \geq 1 \), then for a generic Riemannian metric \( g \) there is a closed, self-dual harmonic form \( \omega \), that vanishes transversally as a section of \( \Lambda^2 T^* M^4 \) and defines a near-symplectic structure. The zero set of \( \omega \) is a finite, disjoint union of embedded circles.
Remark 2.7. Let $(M, \omega)$ be near-symplectic with $\dim(M) = 2n > 4$ and let $\iota : Z_\omega \hookrightarrow M$ be the inclusion of its $(2n - 3)$-dimensional singular locus $Z_\omega = \{ p \in M \ | \ \omega_p^{n-1} = 0 \}$. The 2-form $\omega_Z := \iota^* \omega$ is closed and has positive constant rank $2n - 4$, hence $(\omega_Z)^{n-2}_p \neq 0$ for all $p \in Z_\omega$. Recall that a presymplectic structure is a pair $(W, \Omega)$, $W$ being a smooth manifold of dimension $2k + r$ and $\Omega$ a closed 2-form of rank $2k$ on $W$. Thus, a near-symplectic form $\omega$ defines a presymplectic structure $(Z_\omega, \omega_Z)$.

Given a presymplectic manifold $(W, \Omega)$, let $V$ be its null distribution given by the collection of null or vertical subspaces $V_p = \{ v \in T_p W \ | \ \Omega_p(v, \cdot) = 0 \}$. There is a one-to-one correspondence between $\Omega$-compatible Poisson structures and horizontal subbundles $\mathcal{H} \subset TW$ such that $TW = V \oplus \mathcal{H}$. Such a structure is Poisson if and only if $\mathcal{H}$ is integrable [29].

In our setting, the null distribution $\varepsilon := \ker(\omega_Z)$ is a line subbundle of $TZ_\omega$. Since $\omega_Z$ is of maximal constant rank on $Z_\omega$, $\varepsilon$ is regular, thus integrable. The horizontal distribution $\mathcal{H}$ is equipped with a symplectic structure defined by $\omega_Z$, and assuming that $\mathcal{H}$ is integrable then $\pi_Z = \omega_Z^{-1}$ defines a Poisson structure. The previous decomposition on $TZ_\omega$ with respect to the presymplectic structure $(Z_\omega, \omega_Z)$ is

$$TZ_\omega = \varepsilon \oplus \mathcal{H}.$$ 

Due to the regularity of $\varepsilon$ there is a flat connection for which we can choose a trivialization and a vector field $X = \frac{\partial}{\partial \theta} \in \Gamma(\varepsilon)$.

We now recall the local expression of a near-symplectic form. Thinking of $\Lambda^2 K^*$ as $\Lambda^2 \mathbb{R}^4$ we have a splitting in two rank-3 subbundles $\Lambda^2 K^*$. With coordinates $(\theta, x_1, x_2, x_3)$, two bases of these bundles are given by the following elements

\begin{align*}
(4) & \quad \Lambda^2_1 K^* : \\
& \quad \beta_0 = d\theta \wedge dx_1 + dx_2 \wedge dx_3 \\
& \quad \beta_2 = d\theta \wedge dx_2 + dx_1 \wedge dx_3 \\
& \quad \beta_3 = d\theta \wedge dx_3 + dx_1 \wedge dx_2 \\
& \quad \beta_4 = d\theta \wedge dx_1 - dx_2 \wedge dx_3 \\
& \quad \beta_5 = d\theta \wedge dx_2 + dx_1 \wedge dx_3 \\
& \quad \beta_6 = d\theta \wedge dx_3 - dx_1 \wedge dx_2.
\end{align*}

A Darboux-type theorem for near-symplectic forms tells us that around a point $p \in Z_\omega \subset M$, there is coordinate chart $(U, (\mathbf{q}, \mathbf{p}, \theta, \mathbf{x}))$ such that

\begin{align*}
(5) & \quad \omega = \omega_Z + x_1\beta_1 - 2x_2\beta_2 + x_3\beta_3, \\
& \quad \omega = \sum_{i=1}^{n-2} dq_i \wedge dp_i. \text{ With respect to this model, } Z_\omega \text{ is given by } \{ x_1 = x_2 = x_3 = 0 \}.
\end{align*}

Remark 2.8. A near-symplectic form has the property of splitting the normal bundle $NZ_\omega$ of its singular locus $Z_\omega \subset M$ into two subbundles, a rank-1 bundle $L^1_-$ and a rank-2 bundle $L^1_+$. This can be seen through a self-adjoint, trace-free automorphism $F : NZ_\omega \to NZ_\omega$ constructed through the geometric information from $\omega$. Its representative matrix is symmetric, traceless, and has three eigenvalues, two positive and one negative (for more details see [12, sections 3 & 4], [21, sec. 2.3], [27, sec. 2c], [30, sec. 4B]). The negative and positive eigensubspaces draw the corresponding bundles $L^1_-$ and $L^1_+$. These properties are independent on the choice of the metric $g$. There can be many conformal classes $[g]$ for which $\omega$ is self-dual, yet they are all the same along $Z_\omega$ because the $\nabla \omega$ identifies the normal bundle.
$NZ_{\omega}$ with the bundle $\Lambda^2_+ K^*$ of self dual forms at each point of $Z_{\omega}$. Therefore, a near-symplectic form $\omega$ determines a canonical embedding of the intrinsic normal bundle $NZ_{\omega}$ as a subbundle of $TM|_{Z_{\omega}}$ complementary to $TZ_{\omega}$ [21, 27]. This is summarized for later use in the following Lemma.

**Lemma 2.9.** [21, 27, 30] Let $(M, \omega)$ be a near-symplectic manifold with singular locus $Z_{\omega}$. The normal bundle $NZ_{\omega}$ of $Z_{\omega}$ splits into a line bundle $L^1_{\omega}$ and a rank 2 bundle $L^2_{\omega}$, i.e $NZ_{\omega} \simeq L^1_{\omega} \oplus L^2_{\omega}$.

**Remark 2.10.** A comment regarding orientations. Let $M$ be a $2n$-dimensional manifold and $Z \subset M$ a submanifold of codimension $k$. Consider a 2-form $\omega$ in $M$, with the property that $\omega^n|_{M\setminus Z} > 0$. It follows from a standard algebraic topological argument that $M$ is oriented if $\text{codim}(Z) = k \geq 2$. In particular, a near-symplectic form guarantees the orientability of $M$.

Recall that from above the isomorphism $NZ_{\omega} \simeq \Lambda^2 K^*$ induced by $\nabla\omega|_K$. Since $\Lambda^2 K^*$ is oriented by the orientation of $M$, one obtains an orientation on $NZ_{\omega}$ with the declaration that $\nabla\omega|_K$ is orientation reversing. This orientation of $NZ_{\omega}$ induces one on $Z_{\omega}$ by adopting the convention $TM = TZ_{\omega} \oplus NZ_{\omega}$ [27].

A near-symplectic manifold $M$ is naturally related to a broken Lefschetz fibration (blf). A blf is a submersion $f: M \to B$ to a codimension 2 base with indefinite fold singularities $\Gamma$ and Lefschetz singularities $C$. The singular sets $\Gamma, C \subset M$ are submanifolds of codimension 3 and 4 respectively.

**Example 2.11.** Under a suitable cohomology condition on $H^2_{\text{dR}}(M)$, given a blf over a symplectic base, the total space $M$ can be equipped with a near-symplectic structure with $Z_{\omega} = \Gamma$. If $\dim(M) = 4$, the converse is also true, i.e. given a near-symplectic form one can build a blf on $(M, \omega)$. Hence, examples of near-symplectic manifolds arise from broken Lefschetz fibrations, as well as from symplectic fibrations as the next example shows.

**Example 2.12.** Let $g: M^4 \to S^2$ be a compact symplectic fibration with symplectic total space $M^4$, and let $(V^4, \omega_V)$ be a closed, near-symplectic, 4-manifold with a broken Lefschetz fibration $f: V^4 \to S^2$. Construct the pullback bundle $W$ with induced maps $\tilde{f}: W \to M^4$, and $\tilde{g}: W \to V^4$. The total space $W$ is a 6-dimensional manifold, and carries a near-symplectic form $\omega_W$ induced by $\tilde{g}: W \to V^4$. The singular locus $Z_{\omega}$ is a surface bundle over $S^1$ with $\omega_Z = \sigma_F$, where $\sigma_F$ is the symplectic form of the fibre.

Another prototypical example without using blfs is the following product manifold.

**Example 2.13.** Let $N = Z_{\omega} = ((Q, \omega_Q) \times [0, 1]) / \sim$ denote a symplectic mapping torus, where $(Q, \omega)$ is a symplectic manifold, $\phi: Q \to Q$ a symplectomorphism, and the equivalence relation determined by $(x, 0) \sim (\phi(x), 1)$. Since $N$ fibres over $S^1$, there is a nowhere vanishing closed 1-form $\beta \in \Omega^1(N)$. Consider a closed, connected, orientable, smooth 3-manifold $Y$, and let $\alpha \in \Omega^1(Y)$ be a closed 1-form with indefinite (i.e. no maximum nor minimum) Morse singular points. By a theorem of Calabi [5], there is a metric such that $\alpha$ is harmonic. Set $M = N \times Y$ and define the 2-form $\omega \in \Omega^2(M)$ by

$$\omega = \beta \wedge \alpha + \bar{\omega} + (\ast_Y \alpha),$$
where $\ast_Y$ denotes the Hodge $\ast$-operator. This 2-form is near-symplectic on $M$ and its singular locus is $Z_\omega = N \times \text{Crit}(\omega)$.

### 2.3. Euler-like vector fields and Tubular Neighbourhoods

In this section we recall some notions on Euler-like vector fields and tubular neighbourhoods based on [4]. Let $Z \subset M$ be a smooth submanifold and denote by $N\!Z = \nu(M, Z) = TM|_Z/TZ$ the normal bundle of $Z$. Let also $p: \nu(M, Z) \rightarrow Z$, $i: Z \rightarrow M$ be the projection and inclusion maps.

For a vector bundle $F \rightarrow Z$, the normal bundle relative to the zero section is $\nu(F, Z) = F$. The normal bundle of $TM$ relative to $TZ$ is canonically isomorphic to the tangent bundle of the normal bundle. In particular, the normal and the tangent functors commute, and there is a canonical isomorphism $\nu(TM, TZ) \cong T_\nu(M, Z)$. A smooth map of pairs $\psi: (M_1, N_1) \rightarrow (M_2, N_2)$ taking $M_1$ to $M_2$, and $N_1$ to $N_2$, induces a map on normal bundles $\nu(\psi): \nu(M_1, N_1) \rightarrow \nu(M_2, N_2)$. For instance, take a vector field $X \in \mathfrak{X}(M)$ tangent to a submanifold $Z$. View $X$ as a section $M \rightarrow TM$. The condition of $X$ being tangent to $Z$ means that it takes $Z$ to the submanifold $TZ$, i.e. $(M, Z) \rightarrow (TM, TZ)$. Applying the normal functor, one obtains $\nu(X): \nu(M, Z) \rightarrow \nu(TM, TZ) = T\nu(M, N)$. In this way, for a vector field tangent to $Z$, one can obtain a vector field on the normal bundle, called the linear approximation. A linear approximation is then a coordinate-free way of defining a tensor field, including Poisson bivectors and other multivector fields.

**Definition 2.14.** [4, Def. 2.6] Let $Z \subset M$ be a submanifold and $E \in \mathfrak{X}(\nu(M, Z))$ an Euler vector field. A vector field $R \in \mathfrak{X}(M)$ is called Euler-like if $R$ is complete, $R|_Z = 0$, with linear approximation being the Euler vector field i.e $\nu(R) = E$.

Linear approximations serve in the following definition of tubular neighbourhoods.

**Definition 2.15.** [4, Def. 2.3] A tubular neighbourhood embedding for $Z \subset M$ is an embedding of the normal bundle $\psi: (\nu(M, Z), Z) \rightarrow (M, Z)$ such that: (i) it takes the zero section of $\nu(M, Z)$ to $Z$, and (ii) its linear approximation is the identity map, i.e. $\nu(\psi) = \text{id}$.

There is a direct connection between Euler-like vector fields and tubular neighbourhood embeddings. If $E$ is the Euler vector field on the normal bundle, then any tubular neighbourhood embedding carries $E$ to an Euler-like vector field defined in a neighborhood of $Z$ in $M$.

**Proposition 2.16.** [4, Prop. 2.7] Let $Z \subset M$ be a submanifold and $E \in \mathfrak{X}(\nu(M, Z))$ an Euler vector field. Any $R \in \mathfrak{X}(M)$ Euler-like along $Z$, determines a unique tubular neighbourhood embedding $\psi: \nu(M, Z) \rightarrow M$ with $\psi_*E = R$.

In particular, Euler-like vector fields are always linearizable [4, Lemma 2.4].

### 3. Induced singular Poisson structures

In this section we show that a near-symplectic manifold of any dimension $2n \geq 4$ induces an almost regular Poisson structure. The next two propositions construct the Poisson structure of Theorem 1.1 in dimension 4.
3.1. Poisson structures in near-symplectic Manifolds

**Proposition 3.1.** Let \((M, \omega_M)\) be a closed near-symplectic 4-manifold. Denote by \(Z_\omega\) the singular locus of the 2-form and by \(U_Z \subset M\) a tubular neighbourhood of \(Z_\omega\). There is a Poisson structure \(\pi_U\) on \(U_Z \subset M\) such that the vanishing locus of \(\pi_U\) contains \(Z_\omega\).

**Proof.** Assume that \(Z_\omega\) has only one connected component, that is only one circle \(S^1\). Let \(X \in \mathfrak{X}(S^1)\) be the unit tangent vector field so that \(X = \frac{\partial}{\partial \theta}\). Recall that given a near-symplectic form, the normal bundle splits into \(NZ_\omega = L_1^1 \oplus L_2^2\), a rank 1-bundle \(L_1^1\) and a rank 2-bundle \(L_2^2\). We use this splitting property coming from \(\omega\) to construct a Poisson structure on the tubular neighbourhood of \(Z_\omega\).

Let \(\mathcal{E}_-, \mathcal{E}_+\) be Euler vector fields on \(L_1^1\) and \(L_2^2\) respectively. In bundle coordinates, with \((x_2) \in L_1^1\), \((x_1, x_3) \in L_2^2\) they are expressed as

\[
\mathcal{E}_- = x_2 \frac{\partial}{\partial x_2}, \quad \mathcal{E}_+ = x_1 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_3}.
\]

In particular, \(\mathcal{E}_-|_{L_1^1} = 0\), \(\mathcal{E}_+|_{L_1^1} = 0\), and \(\mathcal{E}_+|_{Z_\omega} = 0\). Define on \(NZ_\omega\) the bivector field

\[
P = X \wedge \mathcal{E}_+ = x_1 \frac{\partial}{\partial \theta} \wedge \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial \theta} \wedge \frac{\partial}{\partial x_3}.
\]

Let \(\psi: NZ_\omega \to M\) be Euler-like vector field \(R\). Denote by \(U_Z = \psi(NZ_\omega)\) the tubular neighbourhood of \(Z_\omega\) in \(M\). Push forward \(P\) into \(M\) via \(\psi\) to define a bivector field \(\eta := \psi_\ast P = X \wedge R_+\) on \(U_Z \subset M\), where \(R_+ \in \mathfrak{X}(M)\) is Euler-like on \(U_Z\) with \(\psi_\ast(\mathcal{E}_+) = R_+\).

Recall that given a near-symplectic form on a closed 4-manifold, there is a metric \(g\) such that \(\omega\) is self-dual and vanishes on a collection of circles. In dimension 4, we have \(K = TM\). Let \(*\) be the Hodge operator with respect to this \(g\) such that \(*\omega = \omega\). Using the orientation given by the volume form \(\omega^2\) (Rem. 2.10), one can define a Hodge duality isomorphism from the exterior algebra of the cotangent bundle to the one of the tangent bundle, thus obtaining a transformation of bivector fields, \(*: \Lambda^2 TM \to \Lambda^2 TM\). This Hodge operator is defined with respect to the volume form and a metric that makes \(\omega\) self-dual. The construction is independent on the particular choice of \(\omega\) and \(g\), since given any near-symplectic form, we can find a Riemannian metric \(g\) such that \(\omega\) is a self-dual harmonic 2-form vanishing on a 1-submanifold of \(M\) (see Thm. 2.6).

Define the following bivector field on \(U_Z\)

\[
\pi_U = \eta + *\eta = X \wedge R_+ + *(X \wedge R_+).
\]

For a sufficiently small neighbourhood around \(Z\), the linear model of \(\pi_U\) is given by

\[
\pi_U = x_1 \left( \frac{\partial}{\partial \theta} \wedge \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \right) + x_3 \left( \frac{\partial}{\partial \theta} \wedge \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \right).
\]

This bivector vanishes on \(\psi(L_1^1)\), which includes the singular locus of \(\omega\). A calculation shows that it satisfies the Poisson condition \([\pi_U, \pi_U]_{SN} = 0\), and \(\pi_U^2 \geq 0\). \(\square\)
To globalize the Poisson structure to $M$, we connect the 2-form dual to $\pi_U$ to the near-symplectic form with a deformation path of near-symplectic forms. For clarity we denote the near-symplectic form on $M$ as $\omega_M \in \Omega^2(M)$. The argument that we implement is due to Karl Luttinger and Carlos Simpson, who refer to this phenomena as the birth/flight, as it perturbs a self-dual 2-form with degeneracy on a plane to a self-dual 2-form with degeneracy on a circle. Their work [18] had been known and used in the literature, for example in [26, 21, 15]. Years later, Taubes and Perutz provided independent proofs of the theorems of Luttinger and Simpson [28, 21]. For completion we present the part relevant to this work here. The 2-form dual to the Poisson bivector is given by

$$\omega_U := \pi_U^{-1} = \frac{1}{(x_1^2 + x_3^2)} (x_1 (d\theta \wedge dx_1 + dx_2 \wedge dx_3) + x_3 (d\theta \wedge dx_3 + dx_1 \wedge dx_2)),$$

This 2-form is symplectic outside the degeneracy locus of $\pi_U$, where it is singular. Consider the 2-form

$$\hat{\omega} = x_1 (d\theta \wedge dx_1 + dx_2 \wedge dx_3) - x_3 (d\theta \wedge dx_3 + dx_1 \wedge dx_2),$$

which is closed and non-degenerate outside the singularity set of $\omega_U$. The 2-forms $\omega_U$ and $\hat{\omega}$ lie in the same de Rham cohomology class as $\omega_U - \hat{\omega} = d\kappa$ with $\kappa = \left(\frac{x_1^2 - x_3^2}{2} - \frac{1}{2} \ln(x_1^2 + x_3^2)\right) d\theta + \left(x_1 x_3 + \arctan(\frac{x_1}{x_3})\right) dx_2$. Consider the 1-parameter family

$$\omega_r = r \cdot \omega_U + (1 - r) \cdot \hat{\omega}$$

with $r \in [0, 1]$. This family is generated by a linear combination of the basis elements $\beta_1$ and $\beta_2$ of $\Lambda^2_+$ (see 4). Outside $D_\pi = \{x_1 = x_3 = 0\}$ this family of 2-forms is symplectic for each $r$. Moreover, the rank of the gradient remains constant, i.e. $\text{Rank}(\nabla \omega_U) = \text{Rank}(\nabla \hat{\omega}) = 2$, and so thus the singular locus. As $[\omega_U] = [\hat{\omega}] \in H^2(U_Z \setminus D_\pi)$ there is an isotopy $\rho_s : (U_Z \setminus D_\pi) \times \mathbb{R} \to U_Z \setminus D_\pi$ that such that $\rho_1^\ast \omega_U = \hat{\omega}$. Next we use $\hat{\omega}$ and apply the birth/flight perturbation to obtain a near-symplectic form. Consider the 2-parameter family of 2-forms

$$\omega(\epsilon, t) = x_1 (d\theta \wedge dx_1 + dx_2 \wedge dx_3 + \epsilon (d\theta \wedge dx_3 + x_2 dx_2 \wedge dx_3))$$

$$- x_3 (dx_1 \wedge dx_2 + d\theta \wedge dx_3 + \epsilon(x_2 dx_2 \wedge dx_3 - d\theta \wedge dx_3))$$

$$+ \frac{\epsilon}{2} (\theta^2 + x_2^2 - t) (d\theta \wedge dx_2 - dx_1 \wedge dx_3).$$

This path is a linear combination of the following three elements

$$\omega_1 = d\theta \wedge dx_1 + dx_2 \wedge dx_3 + \epsilon (d\theta \wedge dx_3 + x_2 dx_2 \wedge dx_3)$$

$$\omega_2 = dx_1 \wedge dx_2 + d\theta \wedge dx_3 + \epsilon(x_2 dx_2 \wedge dx_3 - d\theta \wedge dx_3)$$

$$\omega_3 = d\theta \wedge dx_2 - dx_1 \wedge dx_3.$$ 

The forms $\omega_1, \omega_2, \omega_3$ are a small perturbation of the frame of self-dual forms $\Lambda^2_+ T^* M$. For a sufficiently small $\epsilon$ and $t \in [-\delta, \delta]$ they span a smooth wedge-positive rank-3 subbundle $\Lambda^2_+ T^* M$ that varies smoothly in $t$.

The family $\omega(\epsilon, t)$ has different degeneracy loci depending on the values of the parameters. The parameter $\epsilon$ is responsible for generating the extra basis element, and for a sufficiently small real value it keeps $\omega(\epsilon, t)$ non-degenerate outside the singular locus. Since we are interested in near-positive forms, it makes sense that $\epsilon$ only takes non-negative values. Note also that $\omega(0, 0) = \hat{\omega}$.  

Fix a sufficiently small $\epsilon > 0$, and take $t \in [-\delta, \delta] \subset \mathbb{R}$. For $t < 0$ the degeneracy locus is empty, hence each form $\omega(\epsilon, t)$ is non-degenerate. At $t = 0$, the degeneracy locus become a point, with the special case $\omega(0, 0)$, where it is a plane. For $t > 0$, the degeneracy locus for each $\omega(\epsilon, t)$ is circle. These observations follow directly from the wedge square $\omega^{2}(\epsilon, t)$.

For $\epsilon > 0, t > 0$, the family $\omega(\epsilon, t)$ has the following features: it has a 4-dimensional kernel $K$ spanned by $\langle \partial_{\theta}, \partial_{x_{1}}, \partial_{x_{2}}, \partial_{\epsilon} \rangle$, it vanishes along a circle, and $\text{Rank}(\nabla \omega(\epsilon, t)) = 3$. Furthermore, for each path element one keeps the splitting of $NZ_{\omega} = L^{2}_{\epsilon} \oplus L^{\perp}_{\epsilon}$. Hence, away from the vanishing locus, for each $\epsilon, t > 0$, there is a diffeomorphism $\varphi: U_{Z} \to U_{Z}$ such that $\varphi^{*} \omega_{M} = \omega(\epsilon, \delta)$, and which is a homeomorphism on $Z_{\omega}$ [30, Thm. 1.2]. One could even apply $\varphi$ to bring $\omega(\epsilon, \delta)$ in Darboux-type form as in (5). We summarize the previous exposition in the following lemma.

**Lemma 3.2.** For a sufficiently small $\epsilon, \delta \in \mathbb{R}$ and a sufficiently small neighbourhood the 2-parameter family $\omega(\epsilon, t)$ has the following properties:

- $\omega(\epsilon, t)$ is nondegenerate for $t < 0$ with $Z_{\omega} = \emptyset$,
- $\omega(0, 0) = \omega$ has degeneracy on a plane
- $\omega(\epsilon, 0)$ is degenerate with $Z_{\omega} = \{\text{pt}\}$,
- $\omega(\epsilon, t)$ for $t > 0$ are near-symplectic with $Z_{\omega} = S^{1}$, and is near-symplectomorphic to $\omega_{M}$.

Consequently, the 2-parameter family $\omega(\epsilon, t)$ allows one to deform $\hat{\omega} = \omega(0, 0)$ to $\omega_{M}$. Consider the tubular neighbourhoods $U_{Z} \subset U' \subset U''$ of $Z_{\omega}$ in $M$. On $U_{Z}$ set $\pi = \pi_{U}$, on the intersection $(U'' \cap U') \setminus U_{Z}$ apply the deformation given by the 2-parameter family to perturb $\rho_{t}^{*}\omega_{U} = \hat{\omega} = \omega(0, 0)$ to $\omega(\epsilon, t)$ for $\epsilon > 0, t > 0$. This extends smoothly to $U_{Z}$ and to all $M$ since $\omega(\epsilon, t)$ is symplectomorphic to $\omega_{M}$ outside $Z_{\omega}$. Thus we get the following proposition.

**Proposition 3.3.** Let $(M, \omega)$ be a closed near-symplectic 4-manifold. The Poisson structure $\pi_{U}$ of Proposition 3.1 extends to a Poisson structure on $M$.

This concludes the construction of $\pi$ on $M$ in case $L^{\perp}_{\epsilon}$ is oriented. Recall that the decomposition $NZ_{\omega} \simeq L^{1}_{\epsilon} \oplus L^{2}_{\epsilon}$ has two possible splittings, as the line bundle $L^{\perp}_{\epsilon}$ can be oriented or not. In particular, on a tubular neighbourhood of a component of $Z_{\omega}$ we have a splitting of $S^{1} \times D^{3} \to S^{1}$ into a $D^{2}$-bundle and a $D^{1}$-bundle over $S^{1}$. These are classified by homotopy classes from $S^{1}$ into $\mathbb{R}P^{2}$. Since $\pi_{1}(\mathbb{R}P^{2}) = \mathbb{Z}/2\mathbb{Z}$, there are two possible splittings.

We have handled the oriented case, and it remains to be checked that the model $\pi_{U}$ is also valid on the non-trivial splitting on $U_{Z} = S^{1} \times D^{3}$ for the non-oriented case. This is shown in the next Lemma.

**Lemma 3.4.** The bivector field $\pi_{U}$ of equation (8) is Poisson on the two homotopy classes of splittings of $S^{1} \times D^{3} \to S^{1}$ over each component of $Z_{\omega}$.

**Proof.** The non-oriented model is given by the quotient of $S^{1} \times D^{3}$ by an involution reversing the orientation on both summands of the splitting [12]. Explicitly, it is written as
(11) \[ \iota: S^1 \times D^3 \to S^1 \times D^3 \]

\[(\theta, x_1, x_2, x_3) \mapsto (\theta + \pi, -x_1, x_2, -x_3).\]

We just need to check that if the normal bundle is non-orientable, the local model (8) still provides a Poisson structure. From the action of \( \iota \) we obtain

\[
\iota_* \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta}, \quad \iota_* \frac{\partial}{\partial x_1} = -\frac{\partial}{\partial x_1}, \quad \iota_* \frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_2}, \quad \iota_* \frac{\partial}{\partial x_3} = -\frac{\partial}{\partial x_3}.
\]

Thus, \( \iota_* \pi = \pi \) and the involution \( \iota \) is a Poisson map for \( \pi \).

**Proposition 3.5.** Let \((M, \omega)\) be a near-symplectic manifold of \( \dim(M) = 2n \) with singular locus being a symplectic mapping torus \( Z_\omega = ((Q, \omega_Q) \times [0, 1]) / \sim \). There is a Poisson structure on \( M \) such that \( \pi^{n-1} \) vanishes on \( Z_\omega \).

**Proof.** We extend the construction of proposition 3.1 by first defining a Poisson bivector \( \pi_U \) on the tubular neighbourhood \( U_Z \) as in equation (7), and then adding a symplectic Poisson structure on \( Z_\omega \). Since \( Z_\omega \) fibres over \( S^1 \) and \( \varepsilon \) is an integrable line bundle, there is a non-vanishing section \( X \in \Gamma(L^1) \). The kernel \( K := \varepsilon \oplus NZ_\omega \subset TM \) of \( \omega \) splits as \( K = \varepsilon \oplus L^1 \oplus L^2 \) due to the splitting of \( NZ_\omega \). Thus, we have an Euler vector field \( \mathcal{E}_+ \) on \( L^2 \). By definition the kernel \( K \subset TM \) is a rank 4 bundle, and since \( Z_\omega \) is a mapping torus, we can look at self-dual forms on \( \Lambda^2 K^* \) vanishing on circles. Fix a metric \( g_K \) on \( K \) such that \( \omega \) is self-dual with respect to \( g_K \) on \( \Lambda^2 K^* \). Using the orientation given by the volume form \( \omega^n \), we can obtain a transformation of bivector fields, \( *g_K : \Lambda^2 K \to \Lambda^2 K \).

Since the 2-form \( \omega_Q \) descends to the quotient and is well-defined and symplectic on \( Z_\omega \), the horizontal distribution \( \mathcal{H} \subset TZ_\omega \) is involutive. Thus, the bivector field \( \pi_Z := (\omega)^{-1}(\omega_Q) \) defines a symplectic Poisson structure on \( Z_\omega \), where \( (\omega)^{-1} \) is the inverse of \( (\omega) : TM \to T^* M \). This Poisson bivector has the property that \( \pi^{n-1}_Z \neq 0 \), \( \pi^{n-1}_Z = 0 \). On the tubular neighbourhood \( U_Z \) define the bivector field

\[
\pi_U = X \wedge R_+ + *g_K (X \wedge R_+) + \pi_Z.
\]

The assumption on the topology of \( Z_\omega \) allows us to apply the previous deformation argument of involving \( \omega_r \) and the 2-parameter family \( \omega(\epsilon, t) \) to extend the Poisson bivector field \( \pi_U \) to \( M \).

**Remark 3.6.** We briefly comment on a log-symplectic structure on the line bundle \( L^1 \). A log-symplectic manifold \( M \), also known as \( b \)-symplectic, is a smooth even-dimensional manifold \( M \) equipped with a Poisson bivector \( \pi \) whose Pfaffian \( \pi^n \) vanishes transversally on a codimension 1-submanifold \( D_\pi \). This is a Poisson hypersurface foliated by symplectic leaves where \( \pi^{n-1} \neq 0 \). Of particular interest to our setting are log-symplectic structures on line bundles [10].

A Poisson vector bundle over a Poisson manifold is a vector bundle equipped with a flat Poisson connection [10, Def. 1.5]. On any real line bundle \( E \) one can find a flat connection \( \nabla \), and a Poisson flat connection can be expressed as \( \partial = \pi \circ \nabla + V \).
for a Poisson vector field $V$. Let $(D, \pi_D)$ be a $(2k-1)$-dimensional Poisson manifold of rank $(2k-2)$. The multivector field $\chi = V \wedge \pi_D^{k-1}$ is called the residue of $(E, \partial)$, and is independent of the choice of $\nabla$. A Poisson line bundle with non-vanishing residue $\chi \in \Gamma(\Lambda^{2k-1}TD)$ over such a Poisson manifold $(D, \pi_D)$ admits a log-symplectic structure with degeneracy locus $D$ [10, Prop. 1.9]. Back to our context, assume the conditions of Theorem 1.1. We have then that the horizontal distribution $H \subset TZ$ of the presymplectic form $\omega$ is involutive and that $\omega$ fibres over $S^1$. Then $\pi_Z = (\omega_Z)^{-1}$ is Poisson, and there is a non-vanishing vector field $V$ on the null distribution of $\omega_Z$ complementary to $\mathcal{H}$. The residue is $\chi = V \wedge \pi_Z^{k-1}$.

**Remark 3.7.** The Poisson structure on a near-symplectic manifold that we constructed in Proposition 3.5, belongs to the class of almost regular Poisson structures [1] since it is generically symplectic. Almost regular Poisson structures include regular Poisson and log-symplectic structures among others. The structure induced by a near-symplectic form is neither regular, nor log-symplectic.

4. Poisson Cohomology on 4-manifolds.

In this section we compute the Poisson cohomology of the Poisson structure described in Proposition 3.1 on a smooth 4-manifold. For computational reasons we relabel the variable $\theta$ as $x_0$, the local model of such Poisson bivector on the tubular neighbourhood $U_Z$ is

$$\pi = x_1(\partial_0 \wedge \partial_1 + \partial_2 \wedge \partial_3) + x_3(\partial_0 \wedge \partial_3 + \partial_1 \wedge \partial_2).$$

Since $\text{rank}(\pi) = 4$ there are no nonconstant Casimirs. Our results show that the Poisson cohomology spaces vanish except for $k-$vector fields with constant coefficients. Furthermore, we get that the modular field $\frac{\partial}{\partial x_0}$ has a nontrivial cohomology class, while $[\pi] = 0$. For simplicity, we will make use of the reduced notation $\partial_i := \frac{\partial}{\partial x_i}$. To begin, we introduce the notion of near-positive Poisson bivector. This notion is independent of the cohomology results, yet we present it as a motivating idea of a Poisson structure analogous to a near-symplectic form.

4.1. Near-positive Poisson bivectors

Let $M$ be a smooth, oriented 4-manifold. We consider Poisson bivectors $\pi$ on $M$ that are near-positive, that is $\pi^2 \geq 0$, and such that $\pi$ has maximal rank outside a submanifold $D_\pi$ of $M$ where it vanishes transversally. In contrast to log-symplectic manifolds, where the transversality condition is in $\Lambda^3 TM$, here the condition is in $\Lambda^2 TM$.

Recall that on a 4-dimensional vector space the wedge-product

$$\wedge: \Lambda^2 \mathbb{R}^4 \otimes \Lambda^2 \mathbb{R}^4 \rightarrow \Lambda^4 \mathbb{R}^4 \simeq \mathbb{R}$$

defines a quadratic form of signature $(3, 3)$ on the exterior algebra $\Lambda^2 \mathbb{R}^4$. Thus, we have a decomposition $\Lambda^2 \mathbb{R}^4 = \Lambda^2_+ \mathbb{R}^4 \oplus \Lambda^2_- \mathbb{R}^4$ into two rank-3 bundles. On $M$, the positive subspace $\Lambda^2_+ T_p M$ consists of bivectors $\chi$, such that $\chi^2(\text{vol}) \geq 0$. The elements in $\Lambda^2_- T_p M$ are those such that $\chi^2(\text{vol}) \leq 0$. Using coordinates $(x_0, x_1, x_2, x_3)$
on $\mathbb{R}^4$, a basis of the spaces described above is given by

$$
\Lambda^2_+ \mathbb{R}^4 : \\
\chi_1 = \partial_0 \wedge \partial_1 + \partial_2 \wedge \partial_3 \\
\chi_2 = \partial_0 \wedge \partial_2 - \partial_1 \wedge \partial_3 \\
\chi_3 = \partial_0 \wedge \partial_3 + \partial_1 \wedge \partial_2 \\
\chi_4 = \partial_0 \wedge \partial_1 - \partial_2 \wedge \partial_3 \\
\chi_5 = \partial_0 \wedge \partial_2 + \partial_1 \wedge \partial_3 \\
\chi_6 = \partial_0 \wedge \partial_3 - \partial_1 \wedge \partial_2.
$$

Regard $\pi$ as a section of $\Lambda^2 TM$. Let $U \subset M$ be a neighbourhood of a point $p \in M$ where $\pi_p = 0$. Consider the covariant derivative of $\pi$, namely $\nabla \pi : \mathcal{X}(M) \to \Gamma(\Lambda^2 TM)$, $v \mapsto \nabla_v \pi$. Since $p$ is a zero of a smooth section of a bundle we have a derivative $(\nabla_v \pi)(p) \in \Lambda^2 T_p M$ in the direction of $v \in T_p M$. The restriction of $\nabla \pi$ at $p$ is $\nabla \pi_p : T_p M \to \Lambda^2 T_p M$, which map we call **intrinsic gradient** following the convention in near-symplectic geometry [21, 27]. Expanding $\pi$ with a Taylor series on $U$ about $p = 0$, satisfying $\pi(0) = 0$, we have that $\pi(t \cdot v) = \pi(0) + t \cdot \nabla_v \omega(0) + O(t^2) = t \cdot \nabla_v \omega(0) + O(t^2)$. Thus if $\pi^2 \geq 0$, then $(\nabla_v \pi_p)^2 \geq 0, \forall v \in T_p U$.

Observe that the dimension of the image of $\nabla \pi_p$ can be at most 3 due to the non-negative condition on $\pi^2$, thus it is a subset of the positive subbundle $\Lambda^2_+ \mathbb{R}^4$. Hence, at a point $p \in M$, where $\pi_p = 0$, the rank of $\nabla \pi_p$, seen as a linear map $\mathbb{R}^4 \to \mathbb{R}^6$, can be at most 3, and one can set a transversality condition on $\Lambda^2 TM$ by fixing $\text{Rank}(\nabla \pi_p)$. As Lemma 4.2 shows, this rank condition implies that the singular locus $D_\pi$ is a submanifold of $M$. Next, we will study the behaviour of $\pi$ along distinct singularities determined by the dimension of the image of $\nabla \pi_p$.

**Definition 4.1.** A bivector $\pi \in \Gamma(\Lambda^2 TM)$ on an oriented 4-manifold $M$ is said to be **near-positive** if

- $\pi^2 \geq 0$,
- at each point $p \in M$ we have that either $\pi^2 > 0$ or $\pi_p = 0$, and
- at all points where $\pi$ vanishes, the intrinsic gradient $\nabla \pi_p : T_p M \to \Lambda^2 T_p M$ is constant.

If additionally, $[\pi, \pi]_{SN} = 0$, we say that $\pi$ is a **near-positive Poisson bivector**.

**Lemma 4.2.** Let $\pi \in \Gamma(\Lambda^2 TM)$ be a near-positive bivector with singular locus $D_\pi = \{ p \in M \mid \pi_p = 0 \}$. Assume that $\text{Rank}(\nabla \pi_p) = r$ is constant for all points $p \in D_\pi$, where $\nabla \pi_p : T_p M \to \Lambda^2 T_p M$, and $r \in \{1, 2, 3\}$. Then $D_\pi$ is a $(4 - r)$-submanifold of $M$.

**Proof.** We proceed with a similar argument as in [21]. Let $p \in D_\pi$ and consider a 4-ball $B$ around $p$. Set $k = \dim(\Lambda^2 T_p M) - \dim(\text{Im}(\nabla \pi_p))$. Consider a $k$-bundle $E^k$ complementary to the image of $\nabla \pi_p$, where $k = 3, 4,$ or 5 and regard $\pi$ as a local section of $\Lambda^2 TB$. One has a natural $p : \Lambda^2 TB \to \Lambda^2 TB/E^k$ and a section $\bar{\pi} : B \to \Lambda^2 TB/E^k$ defined by $\bar{\pi} := p \circ \pi$.

Recall that $\Lambda^2 TB$ is a rank 6 bundle, thus the quotient $\Lambda^2 TB/E^k$ is of dimension 3, 2, or 1 depending on the value of $k$. Regarding $\nabla \bar{\pi} : TB \to \Lambda^2 TB/E^k$ as the differential of $\bar{\pi}$, one can see that this is map is surjective, since it is of maximal rank on its codomain due to the assumption on the rank of $\nabla \pi$. Hence, $\nabla \bar{\pi}$ is a submersion. The near-positive condition implies that $\text{Im}(\pi)$ is a subspace of $\Lambda^2_+ TB$ and so it includes the zero element, thus 0 is a regular value of $\bar{\pi}$. Let $v \in T_p B$
and consider a point $\pi(p) \in \text{Im}(\pi) \cap E^k$ such that $\nabla_{\pi_p}(v) \in E^k$. Since $E^k$ has been chosen to be the space complementary to $\text{Im}(\nabla_{\pi_p})$, the condition on the rank implies that $\text{Im}(\pi) \cap E^k = 0$ transversally in $\Lambda^2 TB$. Thus $\pi^{-1}(0)$ is a submanifold of $M$ of dimension $4 - (6 - k)$ as claimed. \hfill $\square$

**Example 4.3.** Consider the phase space $(\mathbb{R}^4, (q_1, p_1, q_2, p_2))$. The bivector
$$
\pi = p_1 (\partial_{q_1} \wedge \partial_{p_1} + \partial_{q_2} \wedge \partial_{p_2}) + p_2 (\partial_{q_1} \wedge \partial_{p_2} + \partial_{p_1} \wedge \partial_{q_2})
$$
is near-positive Poisson with singular locus $D_\pi = \{p_1 = p_2 = 0\} \cong \mathbb{R}^2 \times 0$. We have that $\pi^2 = (p_1^2 + p_2^2)(\text{vol}) \geq 0$. By looking at the matrix of partial derivatives $J_\pi$ coming from the linearization of $\nabla \pi$ we can see that $\text{Rank}(\nabla \pi) = 2$ at the singular points.

When $\dim(M) = 4$, the Poisson structure constructed in Proposition 3.1 is an example of a near-positive Poisson structure with $\text{Rank}(\nabla \pi_p) = 2$, at all points $p \in D_\pi$. As a consequence of Theorem 1.1 and Proposition 3.1 we obtain the following statement.

**Corollary 4.4.** Let $(M, \omega)$ be a near-symplectic 4-manifold. There is a near-positive Poisson structure on $M$.

### 4.2. The Poisson coboundary operator.

We start by writing down the equations of the Poisson coboundary operator (2). The basic Hamiltonian vector fields of (13) are given by

(14) \hspace{1cm} \pi^i(dx_0) = -x_1 \partial_1 - x_3 \partial_3,

(15) \hspace{1cm} \pi^i(dx_1) = x_1 \partial_0 - x_3 \partial_2,

(16) \hspace{1cm} \pi^i(dx_2) = x_3 \partial_1 - x_1 \partial_3,

(17) \hspace{1cm} \pi^i(dx_3) = x_3 \partial_0 + x_1 \partial_2.

For simplicity in notation, we set $X_k := \pi^i(dx_k)$, and then one may rewrite the Poisson bivector as

$$
\pi = \frac{1}{2} \sum_{k=0}^{3} X_k \wedge \partial_k.
$$

Set $d = [.] : \mathfrak{X}^* \to \mathfrak{X}^{*+1}$ to be the Poisson coboundary operator. For $f \in C^\infty(\mathbb{R}^4)$, it is then

(18) \hspace{1cm} d^0(f) = -\sum_{k=0}^{3} X_k(f) \partial_k.

Let $Y = \sum_{k=0}^{3} f_k \partial_k \in \mathfrak{X}^1$, and $s$ be the index completing the triplet $\{1, 2, 3\}$ once $i < j$ are chosen. Then

$$
d^1(Y) = \sum_{k=1}^{3} [X_k(f_0) - X_0(f_k) - \frac{1 - (-1)^k}{2} f_k] \partial_{0k}
$$

(19) \hspace{1cm} + \sum_{i<j=1}^{3} [X_j(f_i) - X_i(f_j) - \frac{1 - (-1)^{i+j}}{2} f_s] \partial_{ij}.

For easiness in our upcoming computations, we write $d^1(Y)$ in its expanded form
\[
d^1(Y) = [X_1(f_0) - X_0(f_1) - f_1] \partial_{01} \\
+ [X_2(f_0) - X_0(f_2)] \partial_{02} \\
+ [X_3(f_0) - X_0(f_3) - f_3] \partial_{03} \\
+ [X_2(f_1) - X_1(f_2) - f_2] \partial_{12} \\
+ [X_3(f_1) - X_1(f_3)] \partial_{13} \\
+ [X_3(f_2) - X_2(f_3) - f_1] \partial_{23},
\]
where we used the notation $\partial_{ij} := \partial_i \wedge \partial_j$ for $i < j$, and write a bivector field as
\[
W = \sum_{i=0}^{3} f_{ij} \partial_{ij} \in \mathfrak{X}^2.
\]
For $\partial_{ijk} := \partial_i \wedge \partial_j \wedge \partial_k$, $i < j < k$, the coboundary operator on $\mathfrak{X}^2$ is
\[
d^2(W) = [-X_0(f_{12}) + X_1(f_{02}) - X_2(f_{01}) - f_{12} + f_{03}] \partial_{012} \\
+ [-X_0(f_{13}) + X_1(f_{03}) - X_2(f_{01}) - 2f_{13}] \partial_{013} \\
+ [-X_0(f_{23}) + X_2(f_{03}) - X_3(f_{02}) + f_{01} - f_{23}] \partial_{023} \\
+ [-X_1(f_{23}) + X_2(f_{13}) - X_3(f_{12})] \partial_{123}.
\]
Finally, let $Z = \sum_{i=0}^{3} f_{ijk} \partial_{ijk} \in \mathfrak{X}^3$. Then
\[
d^3(Z) = [X_3(f_{012}) - X_2(f_{013}) + X_1(f_{023}) - X_0(f_{123}) - 2f_{123}] \partial_{0123}.
\]

4.3. Smooth cohomology.

Let $V_i = \mathbb{R}[[x_0, x_1, x_2, x_3]]$ be the space of homogeneous polynomials of degree $i$, $r_i := \dim(V_i)$ and $\mathfrak{X}^m_i$ be the space of $m$-vector fields on $U$ whose coefficients are elements of $V_i$. Let also $V_{\text{formal}} = \mathbb{R}[[x_0, x_1, x_2, x_3]]$ and $\mathfrak{X}^m_{\text{formal}}$ be the space of $m$-vector fields with coefficients from $V_{\text{formal}}$. Restricting $\partial$ to $\mathfrak{X}^m_{\text{formal}}$, since $\partial$ is linear, the operator can be further decomposed as $\partial^m = \sum_i \partial^m_i : \mathfrak{X}^m_i \rightarrow \mathfrak{X}^{m+1}_i$.

With the notation for polyvector fields and their coefficient functions as in equations (18), (19), (21), and (22), the operators $\partial^*_i$ are identified as the maps in the following sequence representing the coefficients in the complex $(\mathfrak{X}^m_*, \partial^*_i)$:
\[
0 \rightarrow V_i \xrightarrow{\partial^0_i} V_i \xrightarrow{\partial^1_i} V_i \xrightarrow{\partial^2_i} V_i \xrightarrow{\partial^3_i} V_i \rightarrow 0
\]
and more precisely
\[
\begin{align*}
0 & \rightarrow (f_0, f_1, f_2, f_3) \\
& \xrightarrow{\partial^0_i} (f_{01}, f_{02}, f_{03}, f_{12}, f_{13}, f_{23}) \\
& \xrightarrow{\partial^1_i} (f_{012}, f_{013}, f_{023}, f_{123}) \\
& \rightarrow f_{0123}.
\end{align*}
\]
Each $\Psi \in X^*_\text{formal}$ will then be a cocycle if and only if each of its homogeneous components is itself a cocycle. Respectively, $\Psi$ will be a coboundary if and only if each of its homogeneous components is itself a coboundary.

**Definition 4.5.** Let $\deg_{x_1x_3}$ denote the sum of degrees in the $(x_1, x_3)$-coordinates of a monomial in $V_i$, that is, $\deg_{x_1x_3}(x_0^{k_0} x_1^{k_1} x_2^{k_2} x_3^{k_3}) = k_1 + k_3$ and $\sum_{s=0}^3 k_s = i$.

**Remark 4.6.** If $\deg_{x_1x_3}(f) = c \in \mathbb{N}_0$, the action of Hamiltonian vector fields is related to this degree as follows:

\[ (25) \quad X_0(f) = -cf, \quad \deg_{x_1x_3}(X_0(f)) = c, \quad \deg_{x_1x_3}(X_1(f)) = c + 1, \quad \deg_{x_1x_3}(X_2(f)) = c, \quad \deg_{x_1x_3}(X_3(f)) = c + 1. \]

In the sequence, we denote by $H^*_\text{formal}(U_Z, \pi)$ the cohomology of the cochain complex $(X^*_\text{formal}; d^*)$ and $H^m(\pi, \pi)$ will denote the $m$-th Poisson cohomology group with coefficients from $V_i$. The next proposition computes the formal cohomology of (13) in the case where $Z_\omega$ has only one component. The general case with $n$ components will be addressed in section 4.4.

**Proposition 4.7.** Let $(M, \omega)$ be a near-symplectic 4–manifold. Consider the tubular neighbourhood $(U_Z, \pi)$ of the singular locus $Z_\omega$ equipped with the Poisson bivector (13). Assume $Z_\omega$ has only one component. Then

\[
\begin{align*}
H^0_{\text{formal}}(U_Z, \pi) &\cong \mathbb{R} \cong \langle 1 \rangle, \\
H^1_{\text{formal}}(U_Z, \pi) &\cong \mathbb{R}^2 \cong \langle \partial_0, \partial_2 \rangle, \\
H^2_{\text{formal}}(U_Z, \pi) &\cong \mathbb{R} \cong \langle \partial_0 \wedge \partial_2 \rangle, \\
H^3_{\text{formal}}(U_Z, \pi) &\cong \langle 0 \rangle \\
H^4_{\text{formal}}(U_Z, \pi) &\cong \langle 0 \rangle.
\end{align*}
\]

**Proof.** We will show that all cohomology groups vanish when the coefficient functions in (24) are homogeneous polynomials of fixed degree $i > 0$, in which case (23) becomes a short exact sequence. The cohomology with constant coefficient functions will then be computed at the end of the proof. This will yield the proposition; as the operators $d^*$ are linear, one can replace $V_i$ by $V\text{formal}$, the algebra of formal power series equipped with (13).

We henceforth restrict (23) to some fixed $i > 0$.

$H^0_i(U_Z, \pi)$.
Since $\text{Rank}(\pi) = 4$, there are no non-constant Casimirs, so $\ker(d^0_i) = 0$ and $\text{Im}(d^0_i) \cong V_i$, $\dim(\text{Im}(d^0_i)) = r_i$.

$H^1_i(U_Z, \pi)$.
The image of $d^0_i$ is spanned by vector fields of the following forms:

\[
\begin{align*}
d^0_0(x_0^{k_0}) x_1^{k_1} x_2^{k_2} x_3^{k_3} &= k_0 x_0^{k_0-1} x_1^{k_1} x_2^{k_2} x_3^{k_3} X_0 =: A, \\
d^0_1(x_1^{k_1}) x_0^{k_0} x_2^{k_2} x_3^{k_3} &= k_1 x_0^{k_0} x_1^{k_1-1} x_2^{k_2} x_3^{k_3} X_1 =: B, \\
d^0_2(x_2^{k_2}) x_0^{k_0} x_1^{k_1} x_3^{k_3} &= k_2 x_0^{k_0} x_1^{k_1} x_2^{k_2-1} x_3^{k_3} X_2 =: C, \\
d^0_3(x_3^{k_3}) x_0^{k_0} x_1^{k_1} x_2^{k_2} &= k_3 x_0^{k_0} x_1^{k_1} x_2^{k_2} x_3^{k_3-1} X_3 =: D.
\end{align*}
\]
with $\sum_{s=0}^{3} k_s = i$. We will show that any $X \in \text{Ker}(d^1_0)$ is written as a linear combination of vector fields of the form $A, B, C, D$.

Let $Y = f_0 \partial_0 + f_1 \partial_1 + f_2 \partial_2 + f_3 \partial_3 \in \text{Ker}(d^1_0)$, i.e $f_k \in V_i$. Without loss of generality, assume that $f_0$ is a monomial with coefficient equal to 1, so let $f_0 = x_0^{k_0} x_1^{k_1} x_2^{k_2} x_3^{k_3}$ and assume $\deg_{x_1,x_3}(f_0) = c \neq 0$. The vanishing of the coefficient of $\partial_{02}$ in (20) together with (25) imply that $f_0$ and $f_2$ share the same $\deg_{x_1,x_3}$ and in particular,

$$X_2(f_0) = -cf_2.$$  

On the other hand, by the vanishing of the coefficient of $\partial_{01}$ in (20) together with (25), one gets

$$X_1(f_0) = (1 - \deg_{x_1,x_3}(f_1))f_1 = -cf_1.$$  

Applying the same argument for the coefficient of $\partial_{03}$ we get

$$X_3(f_0) = -cf_3.$$  

Given that $k_3 = c - k_1, k_2 = i - c - k_0, k_1 = i - c - k_0, a$ direct computation of the formulas $f_j = -\frac{1}{c} X_j(f_0)$ gives

$$f_1 = -\frac{k_0}{c} x_0^{k_0-1} x_1^{k_1+1} x_2^{i-c-k_0} x_3^{c-k_1},$$  

$$f_2 = -\frac{k_1}{c} x_0^{k_0} x_1^{k_1-1} x_2^{i-c-k_0} x_3^{c-k_1},$$  

$$f_3 = -\frac{k_0}{c} x_0^{k_0-1} x_1^{k_1} x_2^{i-c-k_0} x_3^{c-k_1+1} - \frac{i - c - k_0}{c} x_0^{k_0} x_1^{k_1+1} x_2^{i-c-k_0} x_3^{c-k_1}.$$  

Splitting the coefficient of $f_0$ as $1 = \frac{c-k_1}{c} + \frac{k_1}{c}$, $Y = f_0 \partial_0 + f_1 \partial_1 + f_2 \partial_2 + f_3 \partial_3$ is now written as

$$Y = \left( \frac{k_0}{c} x_0^{k_0-1} x_1^{k_1} x_2^{i-c-k_0} x_3^{c-k_1} \right) X_0 + \left( \frac{i - c - k_0}{c} x_0^{k_0} x_1^{k_1+1} x_2^{i-c-k_0} x_3^{c-k_1} \right) X_2 + \left( \frac{k_1}{c} x_0^{k_0} x_1^{k_1-1} x_2^{i-c-k_0} x_3^{c-k_1} \right) X_1 + \left( \frac{i - c - k_0}{c} x_0^{k_0} x_1^{k_1+1} x_2^{i-c-k_0} x_3^{c-k_1+1} \right) X_3$$  

which is in $\text{Im}(d^0)$.

If $c = 0$, i.e if $f_0$ does not depend on $x_1, x_3$, then $X_0(f_0) = X_2(f_0) = X_0(f_2) = X_3(f_2) = 0$. Vanishing the coefficients of the bivectors in (20) and using a similar computation as before, one gets that $Y$ is written as a linear combination of vector fields of type $A$ and $C$. We have thus proved that $\bigoplus_{i > 0} H^1_i(U_Z, \pi) = 0$.

$H^4_i(U_Z, \pi)$.

Since $X_0(f_{123}) = \deg_{x_1,x_3}(f_{123})f_{123}$, taking $f_{012} = f_{013} = f_{123} = 0$ at the coefficient of $\partial_{0123}$ at (22), we get that $\dim(\text{Im}(d^1_i)) = r_i$. Since $\dim(\text{Ker}(d^1_i)) = r_i$, we have that $H^4_i(U_Z, \pi) = 0$ and $\bigoplus_{i > 0} H^4_i(U_Z, \pi) = 0$. 


$H^2_i(U, \pi)$.  

By the previous computations we have shown that $\dim(\ker(d^1_i)) = r_i$ and so $\dim(\text{Im}(d^1_i)) = 3r_i$. To prove that $H^2_i(U, \pi) = 0$, it is enough to prove that $\dim(\ker(d^2_i)) = 3r_i$. We do this by first examining the degree $\deg_{x_1x_3}$ in the equations defining $\ker(d^2_i)$, that is, vanishing the coefficients of $\partial_{ijk}$ in (21).

Let $W = \sum_{i<j=1}^3 f_{ij} \partial_{ij} \in \ker(d^2_i)$, $\deg_{x_1x_3}(f_{01}) = c \neq 1$ and $\deg_{x_1x_3}(f_{13}) = c'$. Recall that $X_0(f_{13}) = -c' f_{13}$ and that for a degree $\deg_{x_1x_3}$ homogeneous polynomial $f$, it is

$$\deg_{x_1x_3}(X_1(f)) = \deg_{x_1x_3}(X_3(f)) = \deg_{x_1x_3}(f) + 1.$$ 

The coefficient of $\partial_{013}$ in (21) is $(c' - 2)f_{13} + X_1(f_{03}) - X_3(f_{01})$ and in order for it to vanish, one gets that necessarily

$$c + 1 = c' \quad \text{and} \quad \deg_{x_1x_3}(f_{03}) = c.$$ 

Then we turn to the coefficient of $\partial_{012}$. By the previous argument, $\deg_{x_1x_3}(X_2(f_{01})) = \deg_{x_1x_3}(f_{01}) = c$. Suppose then that the other two terms are of different degree $\deg_{x_1x_3}$, so let

$$\deg_{x_1x_3}(f_{12}) = c'' \quad \text{and} \quad c'' = \deg_{x_1x_3}(X_1(f_{02})) = \deg_{x_1x_3}(f_{02}) + 1.$$ 

Vanishing the coefficient of $\partial_{023}$ one first gets the known fact $\deg_{x_1x_3}(f_{01}) = \deg_{x_1x_3}(X_2(f_{03})) = c$. Also, degree-wise, the other terms must satisfy

$$\deg_{x_1x_3}(f_{23}) = \deg_{x_1x_3}(X_3(f_{02})) = \deg_{x_1x_3}(f_{02}) + 1.$$ 

By the assumption (30), this constant is equal to $c''$.

We thus have the following sets of equations with respect to the degree $\deg_{x_1x_3}$ of the coefficient functions $f_{ij}$ of a $W \in \ker(d^2_i)$ :

$$\deg_{x_1x_3}(f_{23}) = \deg_{x_1x_3}(f_{12}) = \deg_{x_1x_3}(f_{02}) + 1 = c'',$$

$$\deg_{x_1x_3}(f_{13}) = \deg_{x_1x_3}(f_{01}) + 1 = \deg_{x_1x_3}(f_{03}) + 1 = c + 1.$$ 

Vanishing the coefficient of $\partial_{23}$ we then get that $c = c''$. Equations (31) then become

$$\deg_{x_1x_3}(f_{01}) = \deg_{x_1x_3}(f_{03}) = \deg_{x_1x_3}(f_{23}) = \deg_{x_1x_3}(f_{12}) = c,$$

$$\deg_{x_1x_3}(f_{02}) = c - 1, \quad \deg_{x_1x_3}(f_{13}) = c + 1.$$ 

Now set again the coefficients of $\partial_{012}, \partial_{013}, \partial_{023}$ in (21) to be equal to 0. Solving respectively for $f_{12}, f_{13}, f_{23}$ and with the help of (32), we get

$$f_{12} = \frac{1}{c - 1} \left[ X_2(f_{01}) - X_1(f_{02}) - f_{03} \right],$$

$$f_{13} = \frac{1}{c - 1} \left[ X_3(f_{01}) - X_1(f_{03}) \right],$$

$$f_{23} = \frac{1}{c - 1} \left[ X_3(f_{02}) - X_2(f_{03}) - f_{01} \right].$$

Replacing $f_{12}, f_{13}, f_{23}$ in the coefficient of $\partial_{123}$ in (21), one has
\[ X_1(f_{23}) - X_2(f_{13}) + X_3(f_{12}) = [X_1, X_3](f_{02}) - X_1(f_{01}) - [X_1, X_2](f_{03}) - [X_2, X_3](f_{01}) - X_3(f_{03}) = 0, \]

since \( \{x_2, x_3\} = -x_1, \{x_1, x_3\} = 0, \{x_1, x_2\} = -x_3. \)

Thus \( \dim(\ker(d^3_i)) = 3r_i \), and since \( \dim(\text{Im}(d^3_i)) = 3r_i \), we get that \( H^2_i(U, \pi) = 0 \) for \( i > 0 \).

To cover the remaining case, suppose \( c = 1 \). Then
\[
\deg_{x_1, x_3}(f_{01}) = \deg_{x_2, x_3}(f_{03}) = \deg_{x_1, x_3}(f_{23}) = \deg_{x_1, x_3}(f_{12}) = 1,
\]
\[ \deg_{x_1, x_3}(f_{02}) = 0, \quad \text{and} \quad \deg_{x_2, x_3}(f_{13}) = 2. \]

If \( W \in \ker(d^3_i) \), the equations satisfied by the \( f_{ij} \) are
\[
X_1(f_{02}) - X_2(f_{01}) + f_{03} = 0
\]
\[ X_1(f_{03}) - X_3(f_{01}) = 0
\]
\[ X_2(f_{03}) - X_3(f_{02}) + f_{01} = 0
\]
\[ X_1(f_{23}) - X_2(f_{13}) + X_3(f_{12}) = 0. \]

Observe that \( V_1 = f_{01}\partial_1 + f_{02}\partial_2 + f_{03}\partial_3 \in \ker(d^3_i) \). By the vanishing of the first cohomology group, there is a polynomial \( f \in \mathbb{R}^{[x_0, x_1, x_2, x_3]} \) with \( \deg_{x_1, x_3}(f) = 0 \) such that \( d^3_i(f) = V_1 \).

On the other hand, since \( \deg_{x_2, x_3}(f_{23}) = \deg_{x_1, x_3}(f_{12}) = 1 \) and \( \deg_{x_1, x_3}(f_{13}) = 2 \), the coefficient \( f_{123} \) of \( \partial_{123} \) in the image of \( d^3_i \) has \( \deg_{x_1, x_3}(f_{123}) = 2 \), and so \( V_2 = f_{123}\partial_{012} + f_{13}\partial_{013} + f_{23}\partial_{023} \in \ker(d^3_i) \). Thus also in the case \( c = 1 \), it is \( \dim \ker(d^3_i) = 3r_i \) and so \( H^2_i(U, \pi) = 0, \forall i > 0 \).

We have proved that all cohomology groups vanish when we consider homogeneous polynomials of fixed degree \( i > 0 \) as coefficients. Thus the formal cohomology is equal to the cohomology of \( X^*_0 \), the complex with constant coefficients. For \( H^1(U, \pi) \), it suffices to set all \( f_p \) in (19) to be constant, yielding
\[
d^1(\partial_1) = -\partial_{01} - \partial_{13}, \quad d^1(\partial_2) = -\partial_{02} - \partial_{12}. \]

Doing the same in (21) gives the claim for \( H^2_{\text{formal}}(U, \pi) \).

**Definition 4.8.** Define a function \( f \in C^\infty(U) \) to be flat if all its derivatives and the function itself vanish along the singular locus \( \{x_1 = x_3 = 0\} \) of (13).

**Remark 4.9.** Let \( X^\bullet_{\text{flat}}(U), X^\bullet_{\text{formal}}(U) \) and \( X^\bullet_{\text{smooth}}(U) \) be the multivector fields with flat, formal and smooth coefficients respectively. By a theorem of E. Borel, the sequence
\[
0 \longrightarrow X^\bullet_{\text{flat}}(U, \pi) \longrightarrow X^\bullet_{\text{smooth}}(U, \pi) \longrightarrow X^\bullet_{\text{formal}}(U, \pi) \longrightarrow 0
\]

is exact. This shows that the cohomology of Proposition 4.7 is actually smooth in \( x_0, x_2 \).

We now compute the smooth Poisson cohomology using an idea of Ginzburg [8].
Proposition 4.10. The smooth Poisson cohomology of the Poisson bivector (13) on $U_Z$ is given in Proposition 4.7.

Proof. Because of Remark 4.9, it suffices to show that the flat cohomology $H^*_{\flat}(U_Z, \pi)$ vanishes. Extend $\pi^\sharp$ to the chain map $\wedge^* \pi^\sharp : (\Omega^*(U_Z), d_{dR}) \longrightarrow (\mathfrak{x}^*(U_Z), d)$ and then consider the restriction to forms with flat coefficients $\wedge^* \pi^\sharp_{\flat} : \Omega^*_{\flat}(U_Z) \longrightarrow \mathfrak{x}^*_{\flat}(U_Z)$.

Away from the singular locus, $\pi^\sharp_{\flat}$ is an isomorphism. Indeed,

$$\pi^\sharp_{\flat} \left( \sum_{i=0}^{3} f_i dx_i \right) = 0 \Leftrightarrow \begin{cases} x_1 f_1 + x_3 f_3 = 0, \\ -x_1 f_0 + x_3 f_2 = 0, \\ -x_3 f_1 + x_1 f_3 = 0, \\ -x_3 f_0 - x_1 f_2 = 0. \end{cases}$$

Using Taylor series in 4 dimensions, the first and third equations above, imply that outside the singular locus, $f_1 = f_3 = 0$. Similarly the second and fourth equations imply that $f_0 = f_2 = 0$ and so $\pi^\sharp_{\flat}$ is injective. On the other hand, if $\sum_{i=0}^{3} g_i \partial_i$ is in the image of $\pi^\sharp_{\flat}$, then there is always a flat preimage $\sum_{i=0}^{3} f_i dx_i$ with

$$f_0 = \frac{-x_1 g_1 - x_3 g_3}{x_1^2 + x_3^2}, \quad f_1 = \frac{x_1 g_0 - x_3 g_2}{x_1^2 + x_3^2}, \quad f_2 = \frac{x_3 g_1 - x_1 g_3}{x_1^2 + x_3^2}, \quad f_3 = \frac{x_3 g_0 + x_1 g_2}{x_1^2 + x_3^2}.$$

Finally, the cohomology class of $Y \in \mathfrak{x}^\times_{\text{smooth}}(U_Z, \pi)$ written as a convergent Taylor series in a neighbourhood of the singular locus is 0, if and only if each $i-$homogeneous term of the Taylor series is itself a coboundary. \hfill \square

Remark 4.11. In terms of deformation quantization, the linearity of (13) implies that one has control on the polynomial degree of each term in the $\ast \ast$-product corresponding to $\pi$. As shown in [3] for the more general case of weight homogeneous Poisson structures, if $f, g$ are polynomials of weight $k$ and $n$ respectively, the $i-$th term $B_i(f, g)$ in the Taylor series defining the $\ast \ast$-product will be of weight $k + n - \overline{\omega}(\pi)$ where $\overline{\omega}(\pi)$ is the weight of the given Poisson structure. Here it’s easy to see that for the weight vector $\overline{\omega} = (1, 1, 1, 1)$, it is $\overline{\omega}(\pi) = -1$. However a global existence theorem for $\ast \ast$-products over these singular spaces is more complicated because of the singularities. With respect to near-symplectic manifolds, a reasonable approach would be through Fedosov’s deformation quantization and the use of Whitney functions [22] which are already used in the proof of Proposition 4.10.

4.4. Global Cohomology

This section contains the last step to prove Theorem 1.2. Our goal is to describe how to pass from the smooth semi-global cohomology on the tubular neighbourhood $U_Z$ to the global cohomology on all $M$. We follow a similar argument as Radko [23] and Roytenberg [24], and start by assuming that $Z_{\omega}$ has only one component, i.e one singular circle.

Proof of Theorem 1.2 Consider an open cover $\mathcal{D}$ of $(M, \omega)$. Let $V_1 = \mathcal{D} \setminus Z_{\omega}$. Let again $\pi^\sharp : H^*_{\text{dR}}(M) \rightarrow H^*_{\text{dR}}(M)$ the homomorphism on cohomology induced by the anchor map, i.e (3). On $V_1$, the 2-form $\omega$ is symplectic, thus it induces a symplectic Poisson bivector. Denote by $U_1$ the tubular neighbourhood of the single component
in $Z_\omega$. In this situation the Poisson cohomologies of $V_1$ and $U_1 \cap V_1$ are isomorphic to their corresponding de Rham cohomologies,
\begin{equation}
H^*_\text{dr}(V_1) \cong H^*_\pi(V_1) , \quad H^*_\text{dr}(U_1 \cap V_1) \cong H^*_\pi(U_1 \cap V_1).
\end{equation}
Observe that $U_1 \cap V_1 = (S^1 \times D^2) \setminus S^1$ is diffeomorphic to $I \times S^1 \times S^2$ and homotopy equivalent to $S^1 \times S^2$. For a fixed $(t) \in S^1 = [0, 2\pi]/\sim$, the product $\{t\} \times D^3 \setminus \{pt\}$ retracts to $S^2$. Since all self-diffeomorphisms of $D^3 \setminus \{pt\}$ are isotopic to the identity, gluing $\{0\} \times D^3 \setminus \{pt\}$ back to $\{2\pi\} \times D^3 \setminus \{pt\}$ results to $S^1 \times D^3 \setminus \{pt\}$ which retracts to $S^1 \times S^2$. Thus,
\[
H^*_\pi(U_1 \cap V_1) \cong H^*_\text{dr}(S^1 \times S^2).
\]

Associated to the cover given by $U_1$ and $V_1$ there is a short exact Mayer-Vietoris sequence at the level of multivector fields and differential forms
\[
0 \to \mathcal{X}^*(M) \to \mathcal{X}^*(U_1) \oplus \mathcal{X}^*(V_1) \to \mathcal{X}^*(U_1 \cap V_1) \to 0
\]
and
\[
0 \to \Omega^*(M) \to \Omega^*(U_1) \oplus \Omega^*(V_1) \to \Omega^*(U_1 \cap V_1) \to 0.
\]
Each of them leads to a long exact sequence at the level of Poisson and de Rham cohomology respectively
\begin{equation}
0 \to H^0_\pi(M) \to H^0_\pi(U_1) \oplus H^0_\pi(V_1) \to H^0_\pi(U_1 \cap V_1) \to 0
\end{equation}
and
\begin{equation}
0 \to H^0_\text{dr}(M) \to H^0_\text{dr}(U_1) \oplus H^0_\text{dr}(V_1) \to H^0_\text{dr}(U_1 \cap V_1) \to 0.
\end{equation}
We start by describing the de Rham cohomology of $V_1$, since $H^*_\pi(V_1) \cong H^*_\text{dr}(V_1)$ and this cohomology will be useful in subsequent calculations. Since $M$ is connected, $M \setminus S^1$ remains connected and $H^0_\text{dr}(V_1) \cong \mathbb{R}$. Removing an embedded circle from $M$ amounts to attaching a 1-handle $h$ to $M$. This implies that $H^1_\text{dr}(V_1)$ is given by the direct sum $H^1_\text{dr}(M) \oplus H^1_\text{dr}(h)$. Then, $H^1_\text{dr}(V_1) \cong H^1_\text{dr}(M) \oplus \langle d\theta \rangle$, where $d\theta$ comes from the fundamental class of $S^1$ of the handle attached. For the top cohomology group, observe that since $M$ is oriented (see remark 2.10), the fundamental cycle is no longer in $V_1$, thus $H^4_\text{dr}(V_1) \cong 0$. From a simple calculation on (38), it follows first that $H^2_\text{dr}(V_1) \cong H^2_\text{dr}(M) \oplus \langle \omega_S \rangle$, where $\omega_S$ is the area form of $S^2$ in $U_1 \cap V_1$, and secondly that $H^3_\text{dr}(V_1) \cong H^3_\text{dr}(M)$.\]
\( H^0(M) \)

Observe that the first row of (37) is exact. On the symplectic region containing \( V_1 \), a Casimir function is constant, thus by continuity it must be constant everywhere. Hence \( H^0(M) \cong \mathbb{R} \cong \text{span}(1) \).

\( H^1(M) \)

**Case: \( n = 1 \) component**

Assume that \( Z_\omega \) consists of only one circle, and denote this component by \( \zeta \). Exactness of (37) leads to \( H^1_\pi(M) = \text{Im} \left( \rho^0 \right) \oplus \text{ker} (\beta^1) \) and similarly \( H^1_{\text{dR}}(M) = \text{Im} \left( \rho^0 \right) \oplus \text{ker}(b^1) \). Since \( Z_\omega \) has only one component, \( V_1 \) and \( U_1 \cap V_1 \) are symplectic, hence

\[
H^1_\pi(V_1) = \hat{\pi}^\sharp \left( H^1_{\text{dR}}(V_1) \right), \quad H^1_\pi(U_1 \cap V_1) = \hat{\pi}^\sharp \left( H^1_{\text{dR}}(U_1 \cap V_1) \right).
\]

Since the first row of (37) is short exact we have that \( \text{Im}(\rho^0) = 0 \). To determine the kernel of

\[
\beta^1: H^1_\pi(U_1) \oplus H^1_\pi(V_1) \to H^1_\pi(U_1 \cap V_1), \quad (\lambda \mid U_1, \nu \mid V_1) \mapsto (\lambda - \nu \mid U_1 \cap V_1,
\]

recall that

\[
H^1_{\text{dR}}(V_1) = H^1_{\text{dR}}(M) \oplus \langle d\theta \rangle, \quad \text{and} \quad H^1_{\text{dR}}(U_1 \cap V_1) = H^1_{\text{dR}}(S^1 \times S^2) = \text{span}(d\theta).
\]

Propositions 4.7 and 4.10 described the Poisson cohomology of the tubular neighbourhood of \( Z_\omega \). The first cohomology group is generated by the class of the modular vector field \( \partial_0 \) and a vector field \( \partial_2 \). To simplify the notation, we relabel the variables \( \theta := x_0 \), and \( \lambda := x_2 \) so that the Poisson vector fields are denoted by

\[
\partial_{\theta} := \partial_0 \quad \text{and} \quad \partial_{\lambda} := \partial_2,
\]

and

\[
H^1_\pi(U_1) \cong \text{span}(\partial_{\theta}, \partial_{\lambda}) \cong \mathbb{R}^2.
\]

Then we can write

\[
\beta^1: \text{span}(\partial_{\theta}, \partial_{\lambda}) \oplus \hat{\pi}^\sharp \left( H^1_{\text{dR}}(M) \oplus \text{span}(d\theta) \right) \to \hat{\pi}^\sharp \left( \text{span}(d\theta) \right).
\]

The image of the anchor map \( \hat{\pi}^\sharp \) on de Rham classes as determined by equations (14)–(17), implies that

\[
(39) \quad \text{ker}(\beta^1) = \text{span}(\partial_{\theta}, \partial_{\lambda}) \oplus \hat{\pi}^\sharp \left( H^1_{\text{dR}}(M) \right)
\]

and thus

\[
(40) \quad H^1_\pi(M) \cong \text{span}(\partial_{\theta}, \partial_{\lambda}) \oplus H^1_{\text{dR}}(M).
\]

**Case: \( n \) components**

Suppose that \( Z_\omega \) contains \( n \) components \( \{\zeta_1, \ldots, \zeta_n\} \). We extend our previous argument inductively as in [23]. Choose an open cover \( V_0 = M \), and set \( V_i = V_{i-1} \setminus \zeta_i \) for \( i \in \{1, \ldots, n\} \). The sequence (37) can now be read on each row as

\[
\cdots \to H^k_\pi(V_{i-1}) \xrightarrow{\alpha^k} H^k_\pi(U_i) \oplus H^k_\pi(V_i) \xrightarrow{\beta^k} H^k_\pi(U_i \cap V_i) \to \cdots
\]

and we can use a similar notation for the long exact sequence of the de Rham complex. The regions \( V_n \) and \( U_i \cap V_i \) are symplectic for each \( i \). Thus one has

\[
H^\bullet_{\text{dR}}(V_n) \cong H^\bullet_\pi(V_n) \quad \text{and} \quad H^\bullet_{\text{dR}}(U_i \cap V_i) \cong H^\bullet_\pi(U_i \cap V_i).
\]

We also have that \( H^1_\pi(U_i) \cong \text{span}(\partial_{\theta}, \partial_{\lambda})_i \), and \( H^1_{\text{dR}}(V_i) \cong H^1_{\text{dR}}(V_i \cap \pi(V_i)) \). Recall that \( H^1_\pi(V_{i-1}) = \)
This argument extends also to the general case when there are \( n \) circles \( \{ \zeta_1, \ldots, \zeta_n \} \) in \( Z_\omega \); for every circle \( \zeta_i \), the Poisson and de Rham cohomology groups of \( V_i \), \( U_i \cap V_i \), and \( U_i \) are isomorphic, the latter following from Propositions 4.7, 4.10 since \( H_n^\omega(U_i) \cong 0 \cong H_{\text{dR}}^3(U_i) \).

\( H_n^4(M) \)

We can apply the same line of reasoning as for the previous cohomology group, since around every component \( \zeta_i \in Z_\omega \), we have \( H_{\text{dR}}^4(U_i) \cong 0 \), and by Proposition 4.7 we obtain \( H_n^4(U_i) \cong 0 \). Similarly, this holds for \( H_n^2(V) \cong \hat{\pi}^2(H_{\text{dR}}^3(U \cap V)) \) and \( H_n^2(U_1 \cap V) \cong \hat{\pi}^2(H_{\text{dR}}^3(U_1 \cap V)) \), thus

\[
\hat{\pi}^1(J_n^2(M)) \cong H_{\text{dR}}^4(M).
\]

\( H_n^2(M) \)

Case: \( n = 1 \) component

The second cohomology space is

\[
H_n^2(M) = \text{Im}(\rho^1) \oplus \ker(\beta^2).
\]

From (39) we know by exactness that \( \ker(\rho^1) \cong \mathbb{R} \) and hence \( \text{Im}(\rho^1) = 0 \). On the symplectic regions one has \( H_n^2(V_1) \cong \hat{\pi}^2(H_{\text{dR}}^3(V_1) \oplus \text{span}(\omega_{S^2}) \), and \( H_n^2(U_1 \cap V_1) \cong \hat{\pi}^1 \).
span(ωS2). Moreover, by Propositions 4.7, 4.10, it is \( H_3^2(U_2) = \text{span}(\partial_\theta \wedge \partial_\lambda) \). Then we can write
\[
\beta^2 : \text{span}(\partial_\theta \wedge \partial_\lambda) \oplus \pi^* (H^2_{\text{dR}}(M) \oplus \text{span}(\omega_{S^2})) \to \pi^* (\text{span}(\omega_{S^2})) .
\]
Since \( H^2_\pi(M) \cong H^3_{\text{dR}}(M) \cong H^3_{\text{dR}}(V_1) \), we have that \( \alpha^3 \) is injective and then by exactness \( \ker(\beta^2) \cong \mathbb{R} \). Thus
\[
\ker(\beta^2) = \text{span}(\partial_\theta \wedge \partial_\lambda) \oplus \pi^* (H^2_{\text{dR}}(M))
\]
and
\[
H^2_\pi(M) \cong \text{span}(\partial_\theta \wedge \partial_\lambda) \oplus H^2_{\text{dR}}(M).
\]

**Case: \( n \) components**

Extend this argument for \( n \) number of components \( \{\zeta_1, \ldots, \zeta_n\} \) in \( Z_\omega \) as in the case of \( H^1_\pi(M) \). After choosing an open cover \( V_0 = M \) and setting \( V_i = V_{i-1} \setminus \zeta_i \), we obtain
\[
H^2_\pi(V_n) \cong \pi^* (H^2_{\text{dR}}(V_n))
\]
\[
H^2_\pi(V_{n-1}) \cong \text{span}(\partial_{\theta_n} \wedge \partial_{\lambda_n}) \oplus \pi^* (H^2_{\text{dR}}(V_n))
\]
\[
H^2_\pi(V_{n-2}) \cong \text{span}(\partial_{\theta_{n-1}} \wedge \partial_{\lambda_{n-1}}) \oplus \text{span}(\partial_{\theta_n} \wedge \partial_{\lambda_n}) \oplus \pi^* (H^2_{\text{dR}}(V_{n-1}))
\]
\[
\vdots
\]
\[
H^2_\pi(M) \cong \bigoplus_{k=1}^n \text{span}(\partial_{\theta_k} \wedge \partial_{\lambda_k}) \oplus H^2_{\text{dR}}(M). \quad \square
\]

5. **Contact Structures**

In this section we comment on the interaction between the Poisson and contact structures in near-symplectic manifolds. In particular, we focus on the Poisson bivector \( \pi_U \) of Proposition 3.1 and a contact structure on the tubular neighbourhood of \( Z_\omega \). In dimension 4, it has been shown that there is an overtwisted contact structure on the boundary of \( U_2 \).

**Theorem 5.1.** [7, 12] Let \((M, \omega)\) be a near-symplectic 4-manifold. There is an overtwisted contact structure \( \xi = \ker(\alpha) \) on \( \partial U_2 \cong S^1 \times S^2 \) such that \( d\alpha = i^* \omega \), where \( i : S^1 \times S^2 \to S^1 \times D^3 \).

Recall that a contact structure on an \((2n-1)\)-dimensional manifold \( N \) is a maximally non-integrable hyperplane distribution \( \xi \subset TN \) determined by the kernel of a globally defined 1-form \( \alpha \) satisfying \( \alpha \wedge d\alpha^{n-1} \neq 0 \). Contact structures on 3-manifolds \( N^3 \) are classified as tight or overtwisted. A contact structure is called overtwisted if \((N^3, \xi)\) contains an embedding of a disk \( D^2 \hookrightarrow N^3 \) such that for its characteristic foliation \( \Delta = T_pD \cap \xi_p \): (i) the boundary \( \partial D \) is a closed leaf, and (ii) there is a unique elliptic singular point in the interior. If there is no such a disk, then the contact structure is said to be tight.

With respect to the local model of \( \omega \) on \( U_2 = S^1 \times D^3 \) as in (5) with \( \omega_{S^2} = 0 \) [12, Sec. 5], the defining contact form for \( \xi = \ker(\alpha) \) is
\[
\alpha = \frac{1}{2} (x_1^2 - 2x_2^2 + x_3^2) \, d\theta + x_2(x_1 dx_3 - x_3 dx_1).
\]
Now we look at the action of $\pi^t$ on this contact form. Consider the Poisson bivector $\pi_U = \eta + \ast \eta$ on $U_2$ as in (7) and (8). Then

$$\pi^t(\alpha) = -\frac{1}{2} \left(x_1^2 - 2x_2^2 + x_3^2\right) \pi^t(d\theta) + x_2(x_1^2 + x_3^2) \partial_2.$$ 

After a change of coordinates $\theta = \theta, x_1 = r \cos(\phi), x_2 = z, x_3 = r \sin(\phi)$, the previous expression becomes

$$\pi^t(\alpha) = \left(\frac{1}{2}r^2 - z^2\right) V_{\text{Ham}_1} + (r^2 \cdot z) \partial_2.$$ 

This vector field is clearly zero on $Z_\omega = S^1 \times \{0\}$. As it moves to the boundary $S^1 \times S^2$ the action of $\pi^t$ on the contact form is a combination of the Hamiltonian vector field $V_{\text{Ham}_1} = \pi^t(d\theta)$ and the Poisson vector field $\partial_2$.

In [12] the author also provides the Reeb vector field of the contact structure, i.e. the unique vector field $Y$ such that $Y \in \ker(d\alpha)$ and $\alpha(Y) = 1$. Up to a multiple the Reeb vector field is given by

$$Y = \frac{1}{f} \left(x_1^2 - 2x_2^2 + x_3^2\right) \partial_\theta + 3x_2(-x_3 \partial_1 + x_1 \partial_3)$$

where $f = -\frac{1}{2} \left[(x_1^2 + x_3^2)(x_1^2 - 2x_2^2 + x_3^2) + 4x_3^2\right]$. In terms of the Poisson structure $\pi_U$, the Reeb vector field can be expressed using the modular vector field and a Hamiltonian vector field

(48) $$Y = \frac{1}{f} \left(x_1^2 - 2x_2^2 + x_3^2\right) V_{\text{mod}} + (3x_2) V_{\text{Ham}_2},$$

where $V_{\text{mod}} = \partial_\theta$ and $V_{\text{Ham}_2} = (\pi^t(dx_2)) = x_3 \partial_1 - x_1 \partial_3$.

Denote by $\text{pt}_N := \{(0, 1, 0)\}, \text{pt}_S := \{(0, -1, 0)\}$ the north and south poles of $S^2$. The closed orbits of the Reeb vector field are

(49) $$S^1 \times \{\text{pt}_N\} \ , \ S^1 \times \{\text{pt}_S\} \ , \ S^1 \times \{(x_1, 0, x_3)\}$$

with $x_1^2 + x_3^2 = 1$ and $x_1, x_3$ fixed. Hence, at the closed orbits, $Y$ is a constant multiple of the modular vector field and of a Hamiltonian vector field. Thus, by Proposition 4.7 one can summarize the previous observations in the following corollary.

**Corollary 5.2.**

*Let $(M, \omega)$ be a near-symplectic 4–manifold. Along closed orbits, the Reeb vector field of the contact structure $(\partial U_2, \xi)$ of Theorem 5.1 is in the Poisson cohomology class of $[\partial_\theta] \in H^1_{\mathbb{Z}}$.***

In higher dimensions, the situation is unknown. On one hand, it is not clear if there is a contact structure in some submanifold of a near-symplectic manifold. On the other, the Poisson cohomology for $(M, \pi)$ would require other techniques for its computation.

**References**

[1] I. Androulidakis, M. Zambon, *Almost regular Poisson structures and their holonomy groupoids*, Sel. Math. New Ser. (2017) 23:2291.

[2] D. Auroux, S. K. Donaldson, L. Katzarkov, *Singular Lefschetz pencils*, Geom. Topol., Vol. 9 (2005) 1043–1114.

[3] P. Batakidis, N. Papalexiou, *$W^*$- algebras and Duflo Isomorphism*, J. Algebra Appl., 11 (2018) 1850041.
[4] H. Bursztyn, H. Lima, E. Meinrenken, Splitting theorems for Poisson and related structures. J. Reine Angew. Math., (2017). Published online 2017-03-18, doi:10.1515/crelle-2017-0014.
[5] E. Calabi, An intrinsic characterization of harmonic one-forms, Global Analysis, Papers in Honor of K. Kodaira (1969), 101-107.
[6] L. Garcia-Naranjo, P. Suárez-Serrato, R. Vera, Poisson structures on smooth 4-manifolds, Lett. Math. Phys. Vol. 105, no. 11 (2015), 1533–1550.
[7] D. T. Gay, R. Kirby, Constructing symplectic forms on 4-manifolds which vanish on circles, Geom. Topol. 8 (2004) 743–777.
[8] V. Ginzburg, Momentum mappings and Poisson cohomology, Int. J. Math., 07, 329 (1996). DOI: http://dx.doi.org/10.1142/S0129167X96000207.
[9] V. L. Ginzburg, A. Weinstein, Lie-Poisson structure on some Poisson-Lie groups, J. American Math. Soc., 5 (1992), 445–453.
[10] M. Gualtieri, S. Li, Symplectic groupoids of log-symplectic manifolds.
[11] M. Pflaum, H. Posthuma, X. Tang, Quantization of Whitney functions.
[12] K. Honda, Local properties of self-dual harmonic 2-forms on a 4-manifold, J. reine angew. Math. 577 (2004), 105–116.
[13] C. Laurent-Gengoux, A. Pichereau, P. Vanhaecke, Poisson structures, Grundlehren der Mathematischen Wissenschaften, 347. Springer, Heidelberg, 2013.
[14] C. LeBrun, Yamabe constants and the perturbed Seiberg-Witten equations, Comm. Anal. Geom. 5 (1997), 535–553.
[15] Y. Lekili, Wrinkled fibrations on near-symplectic manifolds, Geom. Topol., 13 (2009), 277–318.
[16] A. Lichnerowicz, Les variétés de Poisson et leurs algèbres de Lie associées, J. Differential Geom. 12 (2), (1977) 253-300.
[17] J.-H. Lu, Multiplicative and affine Poisson structures on Lie groups, Thesis, U.C. Berkeley, 1990.
[18] K. Luttinger, C. Simpson, A Normal Form for the Birth/Flight of Closed Self-dual 2-Form Degeneracies, ETH preprint (1996).
[19] I. Markcut, B. Osorno Torres, Deformations of log-symplectic structures, Journal of the London Mathematical Society, 90, (2014) 197–212.
[20] I. Markcut, B. Osorno Torres, On cohomological obstructions for the existence of log-symplectic structures, Journal of Symplectic Geometry, 12, (2014) 863–866.
[21] T. Perutz, Zero-sets of near-symplectic forms, J. Symplectic Geom., Vol.4, no.3, (2007) 237–257.
[22] M. Pflaum, H. Posthuma, X. Tang, Quantization of Whitney functions, Trav. Math., 20, (2012), 153–165.
[23] O. Radko, A classification of topologically stable Poisson structures on a compact oriented surface, J. Symplectic Geom., Vol. 1, no. 3 (2002) 523–542.
[24] D. Roytenberg, Poisson cohomology of $SU(2)$-covariant “necklace” Poisson structures on $S^2$, J. Nonlinear Math. Phys. 9 (2002), no. 3, 347–356.
[25] C. Taubes, The structure of pseudo-holomorphic subvarieties for a degenerate almost complex structure and symplectic form on $S^1 \times H^3$, Geom. Topol., Vol. 2 (1998), 221–332.
[26] C. Taubes, The geometry of the Seiberg-Witten invariants, Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998). Doc. Math. 1998, Extra Vol. II, 493-504.
[27] C. Taubes, Seiberg-Witten invariants and pseudo-holomorphic subvarieties for self-dual, harmonic 2-forms, Geom. Topol., Vol. 3 (1999) 167–210.
[28] C. Taubes, A proof of a Theorem of Luttinger and Simpson about the Number of Vanishing Circles of a Near-symplectic Form on a 4-dimensional Manifold, Math. Res. Lett. 13 (2006), no. 4, 557-570.
[29] I. Vaisman, Geometric Quantization on Presymplectic Manifolds, Monatshefte fur Math., 96 (1983), 293–310.
[30] R. Vera, Near-symplectic 2n-manifolds, Alg. Geom. Topol. 16 no.3 (2016), 1403–1426.

DEPARTMENT OF MATHEMATICS & STATISTICS, UNIVERSITY OF CYPRUS, NICOSIA, 1678, CYPRUS
E-mail address: batakidis@gmail.com

INSTITUTE OF MATHEMATICS - UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO, CIRCUITO EXTERIOR, CIUDAD UNIVERSITARIA, COYOACÁN, 04510, MEXICO CITY, MEXICO
E-mail address: vera@im.unam.mx, rvera.math@gmail.com