Distribution of a Second-Class Particle’s Position in the Two-Species ASEP with a Special Initial Configuration

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Abstract
In this paper, we consider the two-species asymmetric simple exclusion process (ASEP) consisting of \( N - 1 \) first-class particles and one second-class particle. We assume that all particles are located at arbitrary positions but the second-class particle is the rightmost particle at time \( t = 0 \). We find the exact formula of the distribution of the second-class particle’s position at time \( t \) by directly using the transition probabilities of the two-species ASEP, which is a different approach from the coupling method Tracy and Widom used in [J Phys A 42:425002, 2009].

Keywords ASEP · TASEP · Multi-species · Second-class particle

1 Introduction

The multi-species asymmetric simple exclusion process is a generalization of the asymmetric simple exclusion process (ASEP) in the sense that each particle may belong to a different species. In principle, particles follow the rules of the ASEP, that is, a particle tries to jump to the right with probability \( p \) or to the left with probability \( q = 1 - p \) after exponential waiting time with parameter 1. But a particle belonging to a higher species is given priority when it tries to jump to a site already occupied by a lower species. To be more specific, let us denote a species by an integer \( l \). If a particle belonging to a species \( l \) tries to jump to a site already occupied by a particle belonging to \( l' < l \), then the particle belonging to \( l' \) can jump by interchanging its position with the position of the particle belonging to \( l \). However, if it tries to jump to a site occupied by a particle \( l'' \geq l \), the jump is not allowed. In the two-species ASEP, it is common that a lower species particle is called a second-class particle and a higher species particle is called a first-class particle.

The multi-species ASEP is mainly being considered on the one-dimensional finite integer lattice or infinite lattice \( \mathbb{Z} \). Some featured results about the multi-species ASEP on a finite integer lattice are found in [1, 5, 6, 8, 19] where the steady state of the process is one of...
the main interests. In the multi-species ASEP on \( \mathbb{Z} \), the exact formulas of the transition probability and some probability distributions were studied in [4, 10, 11, 13–16, 21, 22]. Also, the two-species with a special initial configuration, called the step initial condition, has drawn an attention in studying a partial differential equation as the hydrodynamic limit of the one-dimensional ASEP [7, 17], the competition interface in the last-passage percolation [9] and some asymptotics in the realm of the KPZ statistics [18]. Moreover, the multi-species ASEP was recently studied in more general context, the coloured stochastic vertex model [2, 3].

In this paper, we consider the two-species ASEP. The main purpose of this paper is to provide the exact formula of the probability distribution of the second-class particle’s position at time \( t \) for an \( N \)-particle system with \((N - 1)\) first-class particles and one second-class particle. The initial configuration of our interest is that the second-class particle is the rightmost particle and the positions of the particles are arbitrarily fixed. In fact, Tracy and Widom have already found the probability distribution of the second-class particle’s position at time \( t \) for an infinite system with all positive sites occupied by first-class particles and the origin occupied by a second-class particle and all negative sites unoccupied at \( t = 0 \) [21]. The method Tracy and Widom used is the coupling method for two (single-species) ASEP with a single discrepancy in their initial configurations where the discrepancy can be viewed as the second-class particle. But the coupling method in [21, Section 2] is for a system with only one second-class particle, and it is not clear if the coupling method in Tracy and Widom’s work can be extended to a system with multiple second-class particles. In this paper, we provide a different method for the probability distribution of the second-class particle’s position. Our approach is to use the transition probabilities of the two-species ASEP. Since the transition probabilities of the two-species ASEP are fully known [15], it should be possible to find the probability distribution of a second-class particle’s position even when there are multiple second-class particles initially. The method using the transition probabilities of the two-species ASEP is straightforward in the sense that the required probability is obtained by summing the transition probabilities over all possible configurations but a huge computation is needed for a large system or the proof for a general \( N \)-particle system. One contribution of this paper is to provide a method for the huge computation although we focus on the system with one second-class particle. It remains to be seen how much useful this method is for multiple second-class particle cases, but in the authors’ another ongoing work, we could successfully obtain the exact formula of the probability distribution of the leftmost second-class particle’s position in a small system (with one first-class particle and three second-class particles) by using some techniques in this paper. Also, since we assume that the positions of the particles are arbitrarily fixed initially (but the second-class particle is the rightmost particle) in this paper, we expect that our formula can be used for a different special initial condition such as flat initial condition.

To overview our approach and state the main result, let us denote a state of the process by \((X, v)\) where \( X = (x_1, \ldots, x_N) \) with \( x_1 < \cdots < x_N \) represents the positions of \( N \) particles, and \( v \) is a permutation of the multi-set \([1, 2, \ldots, 2]\) with cardinality \( N \), representing the order of particles. (A permutation of a multi-set is an ordered arrangement of all elements of the multi-set. Thus, the multi-set \([1, 2, \ldots, 2]\) with cardinality \( N \) has \( N \) permutations.) Here, “1” represents a particle belonging to species 1 (a second-class particle) and “2” represents a particle belonging to species 2 (a first-class particle). Let us denote by \( v(n) \) the \( n \)th number in the permutation \( v \) and denote by \( v_n \) the permutation \( v \) with \( v(n) = 1 \). The probability that the process is at state \((X, \pi)\) at time \( t \) given that the initial configuration is \((Y, v)\) is denoted by \( P_{(Y,v)}(X, \pi; t) \). For simplicity of notations, we write \( P_{(Y,v_N)}(X, v_n; t) = P_Y(X, v_n; t) \) because, in this paper, we will consider only \( v_N \) for the initial order of particles. One of
the authors of this paper found the transition probabilities of the $N$-particle multi-species ASEP with arbitrary combination of species [15]. In particular, the transition probabilities of interest in this paper are given by

$$P_Y(X, v_n; t) = \sum_{\sigma \in S_N} \int \cdots \int \left[ A_\sigma \right]_{v_n, v_N} \prod_{i=1}^N \left( \frac{x_i^{v_\sigma(i)} - y_\sigma(i) - 1}{p} e^{(\frac{p}{q} + q) - 1} \right) d\xi_1 \cdots d\xi_N$$  \hspace{1cm} (1)

where $c$ is a counterclockwise circle centered at the origin with sufficiently small radius $r$ and $[A_\sigma]_{v_n, v_N}$ is in the form of

$$\prod_{(\beta, \alpha)} R_{\beta \alpha}$$

where the product is taken over all inversions $(\beta, \alpha)$ in $\sigma$ and $R_{\beta \alpha}$ is one of

$$S_{\beta \alpha} = -\frac{p + q \xi_\alpha \xi_\beta - \xi_\beta}{p + q \xi_\alpha \xi_\beta - \xi_\alpha}, \quad pT_{\beta \alpha} = \frac{p(\xi_\beta - \xi_\alpha)}{p + q \xi_\alpha \xi_\beta - \xi_\alpha}, \quad Q_{\beta \alpha} = \frac{(p - q \xi_\beta)(\xi_\alpha - 1)}{p + q \xi_\alpha \xi_\beta - \xi_\alpha}$$  \hspace{1cm} (2)

or zero [15, 16]. How to choose $R_{\beta \alpha}$ from $S_{\beta \alpha}$, $pT_{\beta \alpha}$, $Q_{\beta \alpha}$, $0$ for each inversion $(\beta, \alpha)$ depends on $\sigma$ and $v_n$. See Appendix A for the details on $[A_\sigma]_{v_n, v_N}$ or directly see Theorem 1.2 (a), Theorem 1.4, Theorem 1.7, and Proposition 1.8 in [16] for the explicit formulas of $[A_\sigma]_{v_n, v_N}$. Here, $\sum_{\sigma \in S_N}$ implies the sum over all permutations of the symmetric group $S_N$ and $\int$ implies $(1/2\pi i) \int$ throughout this paper. We recall that, in the case of the single-species ASEP, the formula of the transition probabilities,

$$P_Y(X; t) = \sum_{\sigma \in S_N} \int \cdots \int A_\sigma \prod_{i=1}^N \left( \frac{x_i^{v_\sigma(i)} - y_\sigma(i) - 1}{p} e^{(\frac{p}{q} + q) - 1} \right) d\xi_1 \cdots d\xi_N$$

where

$$A_\sigma = \prod_{(\beta, \alpha)} S_{\beta \alpha}$$  \hspace{1cm} (3)

was obtained by Tracy and Widom [20].

### 1.1 Main Results

Let us denote by $\mathbb{P}$ the probability measure of the process with the initial state $(Y, v_N)$ and let $\eta(t)$ be the random variable of the position of the second-class particle at time $t$. Then, it is obvious that

$$\mathbb{P}(\eta(t) = x) = \sum_{X \text{ with } x_N = x} P_Y(X, v_N; t) + \cdots + \sum_{X \text{ with } x_2 = x} P_Y(X, v_2; t) + \sum_{X \text{ with } x_1 = x} P_Y(X, v_1; t).$$  \hspace{1cm} (4)

It is not hard to compute (4) for a small system such as $N = 2$, but even for $N = 3, 4$, it requires a tedious and long computation together with contour deformations. Moreover, it is not easy to even conjecture the formula for general $N$ from the results for $N = 2, 3, 4$. One contribution of this paper is to provide a novel idea to evaluate the sum (4) to enable us to prove the formula for general $N$. We introduce some notations that will be used throughout the paper.
1.1.1 Notations

We write $S_U$ for the set of all permutations on a finite set $U$. If $U = \{1, \ldots, n\}$, we simply write $S_U = S_n$. For a set $U = \{u_1, \ldots, u_n\}$ of positive integers with $u_1 < \cdots < u_n$, we write $\xi_U = (\xi_{u_1}, \ldots, \xi_{u_n}) \in \mathbb{C}^n$ and $Y_U = (y_{u_1}, \ldots, y_{u_n})$, an ordered $n$-tuple of integers with $y_{u_1} < \cdots < y_{u_n}$. If $U = \{1, \ldots, N\}$ where $N$ is the total number of particles in the system, then we simply write $\xi_U = \xi$ and $Y_U = Y$. For a nonempty finite set of positive integers $U$, define $g_U : U \to \mathbb{Z}$ by $g_U(u) = i$ where $i$ implies that $u$ is the $i$th smallest number in $U$. For a nonempty subset $S$ of $U$, we define

$$\Sigma_U(S) := \sum_{s \in S} g_U(s)$$

and $\Sigma_U(\emptyset) := 0$. For example, if $U = \{2, 4, 7\}$, then $g_U(2) = 1, g_U(4) = 2, g_U(7) = 3$, and for $S = \{2, 7\} \subset U$,

$$\Sigma_U(S) := g_U(2) + g_U(7) = 4.$$

If $U = \{1, \ldots, n\}$ for a given positive integer $n$, then we write $\Sigma_U(S) = \Sigma(S)$ for simplicity, which implies the sum of all elements in $S$. For $U = \{u_1, \ldots, u_n\}$, let

$$J(\xi_U) = J(\xi_{u_1}, \ldots, \xi_{u_n}) = \frac{1}{\prod_{u \in U} (\xi_u - 1)}, \quad I(\xi_U) = (\xi_{u_1} \cdots \xi_{u_n} - 1)J(\xi_U)$$

and

$$W_{t,x,Y_U}(\xi_U) = \prod_{i=1}^n \left( \frac{x - y_{u_i} - 1}{x + y_{u_i} - 1} \right)^{\beta_0}(\frac{e^{\rho x}q^{\xi_{u_i} - 1}}{e^{\rho x}q^{\xi_{u_i} - 1}})^{\beta_0}.$$

We define

$$\prod_{i=m}^n a_i := \begin{cases} a_m a_{m+1} \cdots a_n & \text{if } n \geq m, \\ 1 & \text{if } n = m - 1. \end{cases}$$

1.1.2 Statements of Results

**Theorem 1.1** If the initial configuration is $(Y, 2 \cdots 21)$, then the probability distribution of the second-class particle’s position at time $t$, denoted by $\eta(t)$, is

$$\mathbb{P}(\eta(t) = x) = \sum_{S \neq \emptyset \subset \{1, \ldots, N\}} c_S \int_c \cdots \int_c \left( \prod_{\alpha < \beta} T_{\beta \alpha} \right) I(\xi_S)W_{t,x,Y_S}(\xi_S)d\xi_S$$

where

$$c_S = \begin{cases} \left( \prod_{i=1}^{[S]-1} (q^j - p^j) \right) \left( \frac{q}{p} \right)^{\Sigma(S') - [S']([S'] + 1)/2} & \text{if } N \in S, \\ \frac{1}{p^{[S]}} \left( \prod_{i=1}^{[S]} (q^j - p^j) \right) \left( \frac{q}{p} \right)^{\Sigma(S') - [S']([S'] - 1)/2 - N} & \text{if } N \notin S \end{cases}$$

where $S' = \{1, \ldots, N\} \setminus S$ and $T_{\beta \alpha}$ is given in (2).

The formula (11) in [21] by Tracy and Widom for an infinite system with step initial condition was given by the contour integrals with large contours. On the other hand, our
formula (5) which is for a finite system is given by the contour integrals with small contours. We did not try to rediscover Tracy and Widom’s formula from our result but we believe that it is possible. Also, to the best of authors’ knowledge, the probability distribution of the second-class particle for flat initial condition has not been obtained. It would be interesting to see if our formula can be used to find a formula for flat initial condition (see [12] for some exact formulas in the single-species ASEP with flat initial condition).

A simple but interesting result can be obtained in the limit $p \to \frac{1}{2}$.

**Corollary 1.2** In the two-species symmetric exclusion process

$$
\mathbb{P}(\eta(t) = x) = \int_c^c \xi^{x-y_N-1} e^{\left(\frac{1}{2} + \frac{\xi}{z} - 1\right)t} d\xi.
$$

(7)

Note that $y_N$ in (7) is the initial position of the second-class particle. It is interesting that the formula (7) is the same as the probability distribution of the continuous-time symmetric random walk on $\mathbb{Z}$ when the random walk’s initial position is $y_N$. Hence, roughly speaking, our result implies that the second-class particle behaves like the symmetric random walk although it is affected by the first-class particles.

### 1.2 Organization of the Paper

In Sect. 2, we provide some lemmas for the proof of Theorem 1.1. In Sect. 3, we find the exact formula of the probability distribution for a small system with $N = 3$. Section 3 will serve as a warmup for proving the general formula for $N$-particle system. The main idea of the proof for the general formula in Sect. 4 is based on the techniques in Sect. 3. Finally, we prove our main result, Theorem 1.1, in Sect. 4.

## 2 Lemmas

In this section, we provide some results which will be used for the proof of our main theorem. First, Lemma 2.1 is a part of the proof of Lemma 3.1 in [20] and is used to prove Lemma 2.2.

**Lemma 2.1** [20] Suppose that $f(\xi_1, \ldots, \xi_n)$ is analytic for all $\xi_i \neq 0$ and that for $i > k$, \[ f(\xi_1, \ldots, \xi_n) \bigg|_{\xi_i \to (\xi_k - p)/q\xi_k} = O(\xi_k), \] as $\xi_k \to 0$, uniformly when all $\xi_j$ with $j \neq k$ are bounded and bounded away from zero. Then,

$$
\int_c \cdots \int_c \left( \prod_{i < j} \frac{\xi_j - \xi_i}{p + q\xi_i\xi_j - \xi_i \prod_i(1 - \xi_i)} \right) \bigg|_{\xi_n \to (\xi_k - p)/q\xi_k} d\xi_1 \cdots d\xi_k \cdots d\xi_{n-1} = 0.
$$

If, in particular,

$$
f(\xi_1, \ldots, \xi_n) = \frac{1}{(\xi_1, \ldots, \xi_n)^z} \times W_{t,x,y_{1,\ldots,n}}(\xi_1, \ldots, \xi_n), \ z = 1, 2, \ldots,
$$

then $f(\xi_1, \ldots, \xi_n)$ satisfies the hypothesis of Lemma 2.1. (See Proof of Theorem 3.2 in [20]).

Now, we provide Lemma 2.2 which will be used to prove Lemma 2.3.
Lemma 2.2 Let $U = \{u_1, \ldots, u_n\} \subset \{1, 2, \ldots\}$ for $n \geq 2$, $T_{\beta\alpha}$ be given in (2), $A_{\sigma}$ with a permutation $\sigma$ on $U$ be given by (3), and

$$c_S = q^{n(n-1)/2} \frac{q^{\Sigma_U(S')-n|S'|}}{p^{\Sigma_U(S')-|S'|(|S'|+1)/2}}. \quad (8)$$

If

$$\sum_{z_1, \ldots, z_{n-1}=1}^{\infty} \int \cdots \int \sum_{\sigma \in S_U} A_{\sigma} \prod_{i=1}^{n-1} (\xi_{\sigma(i)} - z_i W_{t,x,y,U}(\xi_U)) d\xi_U$$

$$= \sum_{S \subset U} c_S \int \cdots \int \left( \prod_{\alpha < \beta, \alpha, \beta \in S} T_{\beta\alpha} \right) I(\xi_S) W_{t,x,y,S}(\xi_S) d\xi_S \quad (9)$$
is true for $n$, then

$$\sum_{z_1, \ldots, z_n=1}^{\infty} \int \cdots \int \sum_{\sigma \in S_n} A_{\sigma} \prod_{i=1}^{n} (\xi_{\sigma(1)} - \xi_{\sigma(i)})^{-z_i} W_{t,x,y,U}(\xi_U) d\xi_U$$

$$= 1 + \sum_{S \subset U} c_S \int \cdots \int \left( \prod_{\alpha < \beta, \alpha, \beta \in S} T_{\beta\alpha} \right) J(\xi_S) W_{t,x,y,S}(\xi_S) d\xi_S \quad (10)$$
is true for $n$. ($\sum_{S \subset U}$ implies the sum over all nonempty subsets $S \subset U$.)

Proof We will prove for the case of $U = \{1, \ldots, n\}$ without loss of generality. Let us write the left-hand side of (10) as

$$\sum_{z_n=1}^{\infty} \left( \sum_{z_1, \ldots, z_{n-1}=1}^{\infty} \int \cdots \int \sum_{\sigma \in S_n} A_{\sigma} \prod_{i=1}^{n-1} (\xi_{\sigma(1)} - \xi_{\sigma(i)})^{-z_i} W_{t,x,y,z_n,Y,U}(\xi_U) d\xi_U \right). \quad (11)$$

Since we assume that (9) is true, (11) becomes

$$\sum_{z_n=1}^{\infty} \sum_{S \subset \{1, \ldots, n\}, S \neq \emptyset} c_S \int \cdots \int \left( \prod_{\alpha < \beta, \alpha, \beta \in S} T_{\beta\alpha} \right) I(\xi_S) W_{t,x,y,z_n,Y,S}(\xi_S) d\xi_S. \quad (12)$$

Note that any nonempty subset of $\{1, \ldots, n\}$ is a nonempty subset $S'$ of $\{1, \ldots, n-1\}$ or $S' \cup \{n\}$ or $\{n\}$. Let us consider a pair of integral terms for a set $\emptyset \neq S' = \{s_1, \ldots, s_r\} \subset \{1, \ldots, n-1\}$ and $S = S' \cup \{n\}$ in (12):

$$\sum_{z_n=1}^{\infty} c_S \int \cdots \int \left( \prod_{\alpha < \beta, \alpha, \beta \in S} T_{\beta\alpha} \right) I(\xi_S) W_{t,x,y,S}(\xi_S,\xi_S,\xi_n) d\xi_S \quad (13)$$

$$\sum_{z_n=1}^{\infty} c_S' \int \cdots \int \left( \prod_{\alpha < \beta, \alpha, \beta \in S'} T_{\beta\alpha} \right) I(\xi_{S'}) W_{t,x,y,S'}(\xi_{S'},\xi_n) d\xi_{S'}. \quad (14)$$

In (13), if we enlarge the contour $c$ for the variable $\xi_n$ to a sufficiently large circle $C$ for the convergence of the sum over $z_n$, then we encounter a pole at $\xi_n = 1$ that comes from
\[ I(\xi_S) \text{ and } \xi_n = (\xi_n - \rho)/q \xi_n \text{ that come from } T_{n\alpha}. \] By Lemma 2.1, the residue at \( \xi_n = (\xi_n - \rho)/q \xi_n \) is zero when integrated over \( \xi_n \). Hence, (13) equals

\[
\sum_{z_n=1}^{\infty} c_S \int_C \cdots \int_C \left( \prod_{\alpha<\beta, \alpha, \beta \in S} T_{\beta \alpha} \right) I(\xi_S) W_{t,x,Y_S}(\xi_S)(\xi_{s_1}, \ldots, \xi_{s_n}, \xi_n)^{-z_n} d\xi_S
\]

\[
- \sum_{z_n=1}^{\infty} c_{S'} \int_C \cdots \int_C \left( \prod_{\alpha<\beta, \alpha, \beta \in S'} T_{\beta \alpha} \right) I(\xi_{S'}) W_{t,x,Y_{S'}}(\xi_{S'})(\xi_{s_1}, \ldots, \xi_{s_n}, \xi_n)^{-z_n} d\xi_{S'}
\]

(15)

where \( c_{S'} \) is due to the facts that

\[
c_S = p^{|S'|} \times c_{S'}, \quad \frac{1}{p^{|S'|}} = \left( \prod_{\alpha \in S'} T_{\alpha \alpha} \right) \bigg|_{z_n=1}.
\]

We note that the second term in (15) can cancel with (14). Evaluating the first sum in (15) and then shrinking \( C \) back to \( c \), we obtain the sum of (13) and (14) equal to

\[
c_S \int_C \cdots \int_C \left( \prod_{\alpha<\beta, \alpha, \beta \in S} T_{\beta \alpha} \right) J(\xi_S) W_{t,x,Y_S}(\xi_S) d\xi_S + c_{S'} \int_C \cdots \int_C \left( \prod_{\alpha<\beta, \alpha, \beta \in S'} T_{\beta \alpha} \right) J(\xi_{S'}) W_{t,x,Y_{S'}}(\xi_{S'}) d\xi_{S'}.
\]

Finally, the sum of the integral for \( S = \{n\} \) over \( z_n \) in (12) is

\[
\sum_{z_n=1}^{\infty} c_{|n|} \int_C \xi_n^{-z_n} W_{t,x,y_n}(\xi_n) d\xi_n = 1 + \int_C J(\xi_n) W_{t,x,y_n}(\xi_n) d\xi_n
\]

because \( W_{t,x,y_n}(1) = 1 \) and \( c_{|n|} = 1 \). Hence, (12) equals the right-hand side of (10).

**Remark 2.1** Recall that \( W_{t,x,Y_U}(\xi_U) \) is analytic for each variable \( \xi_i \), except at the origin. If \( W_{t,x,Y_U}(\xi_U) \) and \( W_{t,x,Y_S}(\xi_S) \) are replaced with \( f(\xi_U) \) and \( f(\xi_S) |_{\xi_i=1, \xi_S} \), respectively, for any analytic function \( f(\xi_U) \) for each variable \( \xi_i \), inside a sufficiently large circle centered at the origin except at the origin, then the constant 1 on the right-hand side of (10) is replaced with \( f(1, \ldots, 1) \). The proofs are almost the same.

For nonnegative integer \( n \), we define

\[
[n] = \frac{p^n - q^n}{p - q}
\]

and

\[
[n]! = \begin{cases} [n][n-1] \cdots [1] & \text{if } n \geq 1, \\ 1 & \text{if } n = 0, \end{cases}
\]

\[
\left[ \begin{array}{c} n \\ m \end{array} \right] = \begin{cases} [n]! \\ [n-m]![m]! \end{cases} & \text{if } m \leq n,
\]

\[ 0 & \text{if } m > n. \]

(17)

Let \( U = \{u_1, \ldots, u_n\} \subset \{1, 2, \ldots\} \) and \( S \) be a nonempty subset of \( U \).

**Lemma 2.3** For \( U = \{u_1, \ldots, u_n\} \subset \{1, 2, \ldots\} \), (9) is true for all \( n \geq 2 \).
Proof We will prove by mathematical induction for the case of $U = \{1, \ldots, n\}$ without loss of generality. When $n = 2$, a simple manipulation of the contour (that is, enlarging and then shrinking back after summation) shows that

$$
\sum_{z=1}^{\infty} \int C \int C \xi_1^{-z} W_{t, x, y}(\xi_1, \xi_2) d\xi_1 d\xi_2 \\
= \int C \int C \frac{1}{\xi_1 - 1} W_{t, x, y}(\xi_1, \xi_2) d\xi_1 d\xi_2 + \int C W_{t, x, y}(\xi_2) d\xi_2
$$

(18)

and

$$
\sum_{z=1}^{\infty} \int C \int C S_2 \xi_2^{-z} W_{t, x, y}(\xi_1, \xi_2) d\xi_1 d\xi_2 \\
= \int C \int C \frac{1}{\xi_2 - 1} S_2 W_{t, x, y}(\xi_1, \xi_2) d\xi_1 d\xi_2 + \frac{q}{p} \int C W_{t, x, y}(\xi_1) d\xi_1,
$$

(19)

where the single variable integrals in (18) and (19) are the residues at $\xi_1 = 1$ and $\xi_2 = 1$, respectively, and the sum of (18) and (19) shows that (9) holds for $n = 2$.

Suppose that (9) holds for $n - 1$. Let us decompose $\sum_{\sigma \in S_n}$ in the left-hand side of (9) as follows:

$$
\sum_{\sigma \in S_n} ( \cdots ) = \sum_{l=1}^{n} \sum_{\sigma \in S_n \text{ with } \sigma(n) = l} ( \cdots ).
$$

For $l \in \{1, \ldots, n\}$ and $\sigma \in S_n$ with $\sigma(n) = l$, let $\sigma' = \sigma(1) \cdots \sigma(n - 1) \in S_{\{1, \ldots, n\}\{l\}}$. Then,

$$
A_{\sigma} = A_{\sigma'} \times \prod_{l < \beta} S_{\beta l}
$$

and the left-hand side of (9) becomes

$$
\sum_{l=1}^{n} \int C \left( \sum_{z_1, \ldots, z_{n-1} = 1}^{\infty} \int C \cdots \int C \sum_{\sigma' \in S_{\{1, \ldots, n\}\{l\}}} A_{\sigma'} \prod_{i=1}^{n-1} (\xi_{\sigma(1)} \cdots \xi_{\sigma(i)})^{-z_i} \right. \\
\left. \times \left( \prod_{l < \beta} S_{\beta l} \right) W_{t, x, y}(\xi) d\xi_{\sigma(1)} \cdots d\xi_{\sigma(n - 1)} \right) d\xi_l.
$$

(20)

Note that (⋆) is analytic for each variable $\xi_i \neq \xi_j$ inside a sufficiently large circle centered at the origin except at the origin. Hence, by Remark 2.1 which supplements Lemma 2.2, (20) becomes

$$
\sum_{l=1}^{n} \int C \left( \prod_{l < \beta} S_{\beta l} \right) W_{t, x, y}(\xi) \left|_{\xi_i = 1, i \neq l} + \sum_{S' \subset \{1, \ldots, n\}\{l\}, S' \neq \emptyset} c_{S'} \int C \cdots \int C \left( \prod_{\alpha < \beta, \alpha \in S'} T_{\beta \alpha} \right) \times J(\xi_{S'}) \left( \prod_{l < \beta} S_{\beta l} \right) W_{t, x, y}(\xi) \right|_{\xi_i = 1, i \neq S' \cup \{l\}} d\xi_l
$$

(21)

where

$$
c_{S'} = q^{(n-1)(n-2)/2} \times \frac{q^{\sum_{A \setminus S'} -(n-1)\left|A \setminus S'\right|}}{p^{\sum_{A \setminus S'} - \left|A \setminus S'\right| \left|\left(A \setminus S'\right) \setminus \{l\}\right|}}
$$
with \( A = \{1, \ldots, n\} \setminus \{l\} \). If \( l > 1 \) is the number of elements in \( A \setminus S' \) larger than \( l \),

\[
\left( \prod_{l < \beta} S_{\beta l} \right) W_{l,x,Y}(\xi) \bigg|_{\xi_i = 1 \text{ if } i \not\in S' \cup \{l\}} = \left( \frac{q}{\rho} \right)^{<l>} \left( \prod_{l < \beta \in S'} S_{\beta l} \right) W_{l,x,Y_{S' \cup \{l\}}}(\xi_{S' \cup \{l\}})
\]

because \( (S_{\beta l})|_{\xi_i = 1} = \frac{q}{\rho} \). Also, noting that \( \sum_A(A \setminus S') = \sum(A \setminus S') - \langle l \rangle > 0 \), we obtain that

\[
c_{S'} \times (q / \rho)^{<l>} = q^{(n-1)(n-2)/2} \frac{\rho \sum_{\sum_A(A \setminus S') - n} \sum_{\sum_A(A \setminus S')} \rangle}{\rho \sum_{\sum_A(S' \cup \{l\}) + \langle l \rangle} = \tilde{c}_{S',l}.
\]

Now, we compute

\[
\sum_{l=1}^n \sum_{S' \subset \{1, \ldots, n\} \setminus \{l\}, S' \neq \emptyset} c_{S',l} \int_c \cdots \int_c \left( \prod_{\alpha < \beta, \alpha \in S'} T_{\beta \alpha} \right) \left( \prod_{l < \beta \in S'} S_{\beta l} \right) J(\xi_{S'}) W_{l,x,Y_{S' \cup \{l\}}}(\xi_{S' \cup \{l\}}) d\xi_{S' \cup \{l\}}. \tag{22}
\]

Note that if we let \( S = S' \cup \{l\} \), then \( \Sigma(A \setminus S') = \Sigma(S') \) and \( |A \setminus S'| = |S'| \). This implies that if a pair \((S', l)\) with \( S' \subset \{1, \ldots, n\} \setminus \{l\}\) and a pair \((S'', l')\) with \( S'' \subset \{1, \ldots, n\} \setminus \{l'\}\) satisfy \( S' \cup \{l\} = S'' \cup \{l'\} \), then \( \tilde{c}_{S',l} = \tilde{c}_{S'',l'} \). Hence, the sum (22) is equivalent to

\[
\sum_{|S| \geq 2} \sum_{(S', l)} c_{S} \int_c \cdots \int_c \left( \prod_{\alpha < \beta, \alpha \in S'} T_{\beta \alpha} \right) \left( \prod_{l < \beta \in S'} S_{\beta l} \right) J(\xi_{S'}) W_{l,x,Y}(\xi_{S}) d\xi_{S}
\]

where \( \sum_{(S', l)} \) implies the sum over partitions of \( S = S' \cup \{l\} \) and

\[
c_{S} = q^{(n-1)(n-2)/2} \frac{\rho \sum_{\Sigma(S') - n} \sum_{\Sigma(S')} \rangle}{\rho \sum_{\Sigma(S') + |S'| + 1/2} = \tilde{c}_{S}. \tag{23}
\]

Replacing \( N \) by \(|S|\) and \( m \) by \(|S| - 1\) in the identity (1.9) in [20], which is

\[
\sum_{\|A\|=m, \sum_{\|A\|=1}} \prod_{\|A\|=m, \sum_{\|A\|=1}} p + q \xi_i \xi_j - \xi_i \cdot (1 - \prod_{j \in A^c} \xi_j) = q^m \left( \frac{N-1}{m} \right) \left( 1 - \prod_{j=1}^N \xi_j \right),
\]

and recalling the form of \( T_{\beta \alpha} \) in (2), we obtain

\[
\sum_{(S', l)} \left( \prod_{\alpha < \beta, \alpha \in S'} T_{\beta \alpha} \right) \left( \prod_{l < \beta \in S'} S_{\beta l} \right) J(\xi_{S'}) = q^{|S| - 1} \left( \prod_{\alpha < \beta, \alpha \in S'} T_{\beta \alpha} \right) I(\xi_{S})
\]

and a simple calculation shows that

\[
c_{S} \times q^{|S| - 1} = q^{(n-1)/2} \frac{\rho \sum_{\Sigma(S') - n} \sum_{\Sigma(S')} \rangle}{\rho \sum_{\Sigma(S') + |S'| + 1/2} = c_{S}. \tag{23}
\]

Hence, we showed that (22) equals

\[
\sum_{|S| \geq 2} c_{S} \int_c \cdots \int_c \left( \prod_{\alpha < \beta, \alpha \in S'} T_{\beta \alpha} \right) I(\xi_{S}) W_{l,x,Y_{S}}(\xi_{S}) d\xi_{S}.
\]

Finally,

\[
\sum_{l=1}^n \int_c \left( \prod_{l < \beta} S_{\beta l} \right) W_{l,x,Y}(\xi) \bigg|_{\xi_i = 1, i \neq l} d\xi_{1, i \neq l} = \sum_{l=1}^n \left( \frac{q}{\rho} \right)^{n-l} \int_c W_{l,x,Y}(\xi_{l}) d\xi_{l},
\]
Lemma 2.5 For any $U = \{u_1, \ldots, u_n\} \subset \{1, 2, \ldots\}$, (10) is true for all $n \geq 2$.

Proof It is clear that Lemmas 2.2 and 2.3 imply Lemma 2.4.

Lemma 2.5 Let $S \subset \{1, \ldots, n\}$ with $|S| = m$. Then,

$$\sum_{S \text{ with } |S|=m} \left( \prod_{\alpha<\beta, \alpha, \beta \in S^c} T_{\beta \alpha} \right) \left( \prod_{\alpha<\beta, \alpha, \beta \in S} T_{\beta \alpha} \right) \left( \prod_{\alpha<\beta, \alpha, \beta \in S^c} S_{\beta \alpha} \right) = \left[ \begin{array}{c} n \\ m \end{array} \right] \left( \prod_{\alpha<\beta, \alpha, \beta \in \{1,\ldots,n\}} T_{\beta \alpha} \right).$$

Proof Let us recall the forms of $S_{\beta \alpha}$ and $T_{\beta \alpha}$ in (2). Dividing both sides of (24) by

$$\left( \prod_{\alpha<\beta, \alpha, \beta \in \{1,\ldots,n\}} T_{\beta \alpha} \right),$$

we obtain

$$\sum_{S \text{ with } |S|=m} \left( \prod_{\alpha<\beta, \alpha, \beta \in S^c} S_{\beta \alpha} \right) \left( \prod_{\alpha<\beta, \alpha, \beta \in S} T_{\beta \alpha} \right) \left( \prod_{\alpha>\beta, \alpha, \beta \in S} T_{\alpha \beta} \right) = \sum_{S \text{ with } |S|=m} \prod_{\alpha<\beta, \alpha, \beta \in S^c} \left( \frac{p + q\xi_{\alpha} \xi_{\beta} - \xi_{\beta}}{\xi_{\alpha} - \xi_{\beta}} \right) \prod_{\alpha>\beta, \alpha, \beta \in S} \left( p + q\xi_{\alpha} \xi_{\beta} - \xi_{\beta} \right) = \sum_{S \text{ with } |S|=m} \prod_{\alpha \in S, \beta \in S^c} \left( p + q\xi_{\alpha} \xi_{\beta} - \xi_{\beta} \right) \left( \xi_{\alpha} - \xi_{\beta} \right) = \left[ \begin{array}{c} n \\ n-m \end{array} \right] \left[ \begin{array}{c} n \\ m \end{array} \right]$$

where the third equality follows from the identity (6.3) in [20], which is

$$\sum_{A \subset \{1, \ldots, N\}, |A|=m} \prod_{i \in A, j \in A^c} \frac{p + q\xi_i \xi_j - \xi_i}{\xi_j - \xi_i} = \left[ \begin{array}{c} N \\ m \end{array} \right].$$

Proof Let us recall the Cauchy binomial theorem,

$$\prod_{k=1}^{n} (1 + y\tau^k) = \sum_{k=0}^{n} y^k \tau^{k(k+1)/2} \left[ \begin{array}{c} n \\ k \end{array} \right] \tau^k.$$  

where $\left[ \begin{array}{c} n \\ k \end{array} \right] \tau$ implies the usual q-Binomial Coefficient. Setting $y = -1$ and $\tau = q/p$ in (26), we obtain the following identity.

Lemma 2.6 For each $l = 1, 2, \ldots$,

$$\prod_{i=1}^{l-1} (q^i - p^i) = \sum_{k=0}^{l-1} (-1)^k \left[ \begin{array}{c} l-1 \\ k \end{array} \right] p^{k(k+1)/2} q^{(l-k)(l-k-1)/2}.$$  

\[ \square \] Springer
In this section, we find $P(\eta(t) = x)$ for $N = 3$. The methods used in the section are further generalized in the next section for general $N$-particle systems.

### 3.1 Decomposition of the Transition Probability

For $N = 3$, it is straightforward that

$$P(\eta(t) = x) = \sum_{X \text{ with } x_3 = x} P_Y(X, v_3; t) + \sum_{X \text{ with } x_2 = x} P_Y(X, v_2; t) + \sum_{X \text{ with } x_1 = x} P_Y(X, v_1; t),$$

(28)

and

$$P_Y(X, v_n; t) = \sum_{\sigma \in S_3} \int_3 \int_3 \int_3 [A_{\sigma}]_{v_n, 221} \prod_{i=1}^{3} \left( \xi_{\sigma(i)}^{x_i - y_{\sigma(i)}} - 1 \right) e^{(\xi_{\sigma(i)}^{x_i} + q^{\xi_{\sigma(i)} - 1})t} d\xi_1 d\xi_2 d\xi_3$$

(29)

where the matrix elements $[A_{\sigma}]_{v_n, 221}$ are given as in Table 1, which can be obtained by the formulas of $[A_{\sigma}]_{v_n,v_n}$ in [16].

Using that

$$Q_{\beta\alpha} = S_{\beta\alpha} - pT_{\beta\alpha},$$

(30)

we write $[A_{\sigma}]_{v_n, 221}$ as

$$[A_{\sigma}]_{v_n, 221} = [A_{\sigma}]^+_{v_n, 221} + [A_{\sigma}]^-_{v_n, 221}$$

where $[A_{\sigma}]^+_{v_n, 221}$ and $[A_{\sigma}]^-_{v_n, 221}$ are given in Table 2.
Following the convention above, we decompose the transition probabilities $P_Y(X, v_\eta; t)$ as follows:

$$P_Y(X, v_\eta; t) = P_Y^+(X, v_\eta; t) + P_Y^-(X, v_\eta; t)$$

where $P_Y^+(X, v_\eta; t)$ and $P_Y^-(X, v_\eta; t)$ are the components of $P_Y(X, v_\eta; t)$ obtained by replacing $[A_\sigma]_{v_\eta, 221}$ by $[A_\sigma]_{v_\eta, 221}$ and $[A_\sigma]_{v_\eta, 221}$, respectively, in (29). With these settings, we will compute the sum (28) in the following manner:

$$\mathbb{P}(\eta(t) = x) = \sum_{X \text{ with } x_3 = x} P_Y^+(X, v_3; t) + \left( \sum_{X \text{ with } x_3 = x} P_Y^-(X, v_3; t) + \sum_{X \text{ with } x_2 = x} P_Y^+(X, v_2; t) \right)$$

$$+ \left( \sum_{X \text{ with } x_1 = x} P_Y^-(X, v_2; t) + \sum_{X \text{ with } x_1 = x} P_Y(X, v_1; t) \right).$$

(31)

### 3.2 Computation of (a) in (31)

We observe that $[A_\sigma]_{221}^+$ is the same as $A_\sigma$ in (3) for the single-species ASEP. Hence, the sum (a) in (31) is the same as the probability distribution of the rightmost particle’s position in the single-species ASEP. Using Theorem 5.1 with $N = m = 3$ in [20], we obtain (a) equal to

$$q^3 \int_c \int_c \int_c T_{32}T_{31}t_{21}I(\xi_1, \xi_2, \xi_3)W_{r,x,y_1,y_2,y_3}(\xi_1, \xi_2, \xi_3)d\xi_1d\xi_2d\xi_3$$

$$+ \frac{q^3}{p^2} \int_c \int_c \int_c T_{21}I(\xi_1, \xi_2)W_{r,x,y_1,y_2}(\xi_1, \xi_2)d\xi_1d\xi_2$$

$$+ \frac{q^2}{p} \int_c \int_c \int_c T_{31}I(\xi_1, \xi_3)W_{r,x,y_1,y_3}(\xi_1, \xi_3)d\xi_1d\xi_3$$

$$+ \frac{q}{p^2} \int_c \int_c \int_c T_{32}I(\xi_2, \xi_3)W_{r,x,y_2,y_3}(\xi_2, \xi_3)d\xi_2d\xi_3$$

$$+ \frac{q^2}{p^2} \int_c \int_c W_{t,x,y_1}(\xi_1)d\xi_1 + \frac{q}{p} \int_c \int_c \int_c W_{t,x,y_2}(\xi_2)d\xi_2 + \int_c \int_c W_{t,x,y_3}(\xi_3)d\xi_3. \quad (32)$$

### 3.3 Computation of (b) in (31)

Recall that $P_Y^-(X, v_3; t)$ and $P_Y^+(X, v_2; t)$ are in the form of

$$\sum_{\sigma \in \mathcal{S}_3} \int_c \int_c \int_c (\cdots).$$

We decompose $\sum_{\sigma \in \mathcal{S}_3}$ as in

$$\sum_{\sigma \in \mathcal{S}_3} (\cdots) = \sum_{l=1}^{3} \sum_{\sigma \text{ with } \sigma(3)=l} (\cdots)$$
and evaluate the sum (b) in the following manner:

\[
(b) = \sum_{l=1}^{3} \left( \sum_{\sigma \text{ with } X \text{ with } \sigma(3)=x} \int \int \int [A_\sigma]_{211,221}^{l} \prod_{i=1}^{3} \left( \xi_{\sigma(i)} - \gamma_{\sigma(i)} - 1 \right) e^{\left( q \xi_{\sigma(i)} - 1 \right) t} d\xi_1 d\xi_2 d\xi_3 \right) \\
+ \sum_{\sigma \text{ with } X \text{ with } \sigma(3)=l} \int \int \int [A_\sigma]_{212,221}^{l} \prod_{i=1}^{3} \left( \xi_{\sigma(i)} - \gamma_{\sigma(i)} - 1 \right) e^{\left( q \xi_{\sigma(i)} - 1 \right) t} d\xi_1 d\xi_2 d\xi_3 \right).
\]

Note that when \( l = 3 \), both \((*)\) and \((***)\) are zero because \([A_\sigma]_{211,221} = [A_\sigma]_{212,221} = 0\) in this case (see Table 2). When \( l = 2 \), \((*)\) can be written

\[
- \sum_{z_1, z_2=1}^{\infty} \int \int \int \left( (\xi_1 \xi_3)^{-z_1} (\xi_1)^{-z_2} + S_{31} (\xi_3 \xi_1)^{-z_1} (\xi_3)^{-z_2} \right) p T_{32} \\
\times W_{t,x,y_1,y_2,y_3} (\xi_1, \xi_2, \xi_3) d\xi_1 d\xi_2 d\xi_3.
\]

(33)

If we apply Lemma 2.4 with \( U = \{1, 3\} \) and Remark 2.1 to the sum of double integral with respect to \( \xi_1, \xi_3 \) in (33), then (33) equals

\[
-q p \sum_{z_1, z_2=1}^{\infty} \int \int \int T_{31} T_{32} J(\xi_1, \xi_3) W_{t,x,y_1,y_2,y_3}(\xi_1, \xi_2, \xi_3) d\xi_1 d\xi_2 d\xi_3 \\
- \frac{q}{p} \sum_{z_1, z_2=1}^{\infty} \int \int J(\xi_1) W_{t,x,y_1,y_2}(\xi_1, \xi_2) d\xi_1 d\xi_2 - p \sum_{z_1, z_2=1}^{\infty} \int \int T_{32} J(\xi_3) W_{t,x,y_2,y_3}(\xi_2, \xi_3) d\xi_2 d\xi_3 \\
- \int \int W_{t,x,y_2}(\xi_2) d\xi_2.
\]

(34)

When \( l = 2 \), \((***)\) equals

\[
\sum_{\sigma \text{ with } \sigma(3)=2}^{\infty} \sum_{z_1, z_2=1}^{\infty} \int \int \int [A_\sigma]_{212,221}^{z_1} \xi_{\sigma(1)}^{-z_2} \xi_{\sigma(2)}^{-z_2} W_{t,x,y_1,y_2,y_3}(\xi_1, \xi_2, \xi_3) d\xi_1 d\xi_2 d\xi_3 \\
= \sum_{\sigma \text{ with } \sigma(3)=2}^{\infty} \sum_{z_1, z_2=1}^{\infty} \int \int \int [A_\sigma]_{212,221}^{z_1} \xi_{\sigma(1)}^{-z_2} \xi_{\sigma(2)}^{-z_2} \frac{\xi_2}{1 - \xi_2} W_{t,x,y_1,y_2,y_3}(\xi_1, \xi_2, \xi_3) d\xi_1 d\xi_2 d\xi_3 \\
= \sum_{z_1=1}^{\infty} \int \int \int \left( (\xi_1)^{-z_2} + S_{31} (\xi_3)^{-z_2} \right) p T_{32} W_{t,x,y_1,y_2,y_3}(\xi_1, \xi_2, \xi_3) d\xi_1 d\xi_2 d\xi_3 \frac{\xi_2}{1 - \xi_2} d\xi_2.
\]

(35)

If we apply Lemma 2.3 with \( U = \{1, 3\} \) and Remark 2.1 to the sum of the double integral with respect to \( \xi_1, \xi_3 \) in (35), then (35) equals

\[
q p \sum_{z_1, z_2=1}^{\infty} \int \int \int T_{31} T_{32} I(\xi_1, \xi_3) \frac{\xi_2}{1 - \xi_2} W_{t,x,y_1,y_2,y_3}(\xi_1, \xi_2, \xi_3) d\xi_1 d\xi_2 d\xi_3 \\
+ \frac{q}{p} \sum_{z_1, z_2=1}^{\infty} \int \int \frac{\xi_2}{1 - \xi_2} W_{t,x,y_1,y_2}(\xi_1, \xi_2) d\xi_1 d\xi_2 + p \sum_{z_1, z_2=1}^{\infty} \int \int T_{32} \frac{\xi_2}{1 - \xi_2} W_{t,x,y_2,y_3}(\xi_2, \xi_3) d\xi_2 d\xi_3.
\]

(36)
Summing (34) and (36), we obtain
\[-qp \int_c \int_c \int_c T_{31} T_{32} I(\xi_1, \xi_2, \xi_3) W_{t,x,y_1,y_2,y_3} (\xi_1, \xi_2, \xi_3) d\xi_1 d\xi_2 d\xi_3\]
\[-q \int_c \int_c \int_c I(\xi_1, \xi_2) W_{t,x,y_1,y_2} (\xi_1, \xi_2) d\xi_1 d\xi_2\]
\[-p \int_c \int_c \int_c T_{32} I(\xi_2, \xi_3) W_{t,x,y_2,y_3} (\xi_2, \xi_3) d\xi_2 d\xi_3 - \int_c W_{t,x,y_2} (\xi_2) d\xi_2.\quad (37)\]

Similarly, when \(\sigma(3) = 1\), we obtain
\[(*) + (**) = -qp \int_c \int_c \int_c T_{31} T_{32} S_{21} I(\xi_1, \xi_2, \xi_3) W_{t,x,y_1,y_2,y_3} (\xi_1, \xi_2, \xi_3) d\xi_1 d\xi_2 d\xi_3\]
\[-q \int_c \int_c \int_c S_{21} I(\xi_1, \xi_2) W_{t,x,y_1,y_2} (\xi_1, \xi_2) d\xi_1 d\xi_2\]
\[-q \int_c \int_c \int_c T_{31} I(\xi_1, \xi_3) W_{t,x,y_1,y_3} (\xi_1, \xi_3) d\xi_1 d\xi_3 - q \int_c W_{t,x,y_1} (\xi_1) d\xi_1.\quad (38)\]

Finally, using the fact that
\[1 + S_{\beta\alpha} = T_{\beta\alpha},\]
we find that the sum of (37) and (38), that is, (b) equals
\[-qp \int_c \int_c \int_c T_{31} T_{32} T_{21} I(\xi_1, \xi_2, \xi_3) W_{t,x,y_1,y_2,y_3} (\xi_1, \xi_2, \xi_3) d\xi_1 d\xi_2 d\xi_3\]
\[-q \int_c \int_c \int_c T_{21} I(\xi_1, \xi_2) W_{t,x,y_1,y_2} (\xi_1, \xi_2) d\xi_1 d\xi_2 - q \int_c \int_c \int_c T_{31} I(\xi_1, \xi_3) W_{t,x,y_1,y_3} (\xi_1, \xi_3) d\xi_1 d\xi_3\]
\[-p \int_c \int_c \int_c T_{32} I(\xi_2, \xi_3) W_{t,x,y_2,y_3} (\xi_2, \xi_3) d\xi_2 d\xi_3\]
\[-q \int_c \int_c I(\xi_1) W_{t,x,y_1} (\xi_1) d\xi_1 - \int_c I(\xi_2) W_{t,x,y_2} (\xi_2) d\xi_2.\quad (39)\]

### 3.4 Computation of (c) in (31)

It can be easily obtained that the first sum in (c) equals
\[p^3 \int_c \int_c \int_c T_{21} T_{31} T_{32} \frac{\xi_1 \xi_2 - 1}{(\xi_1 - 1)(\xi_2 - 1)(\xi_3 - 1)} W_{t,x,y_1,y_2,y_3} (\xi_1, \xi_2, \xi_3) d\xi_1 d\xi_2 d\xi_3 + p \int_c \int_c \int_c T_{21} I(\xi_1, \xi_2) W_{t,x,y_1,y_2} (\xi_1, \xi_2) d\xi_1 d\xi_2\]

and the second sum in (c) equals
\[p^3 \int_c \int_c \int_c T_{21} T_{31} T_{32} \frac{\xi_1 \xi_2}{(1 - \xi_1)(1 - \xi_2)} W_{t,x,y_1,y_2,y_3} (\xi_1, \xi_2, \xi_3) d\xi_1 d\xi_2 d\xi_3.\]

Summing these two results, we can see that (c) is equal to
\[p^3 \int_c \int_c \int_c T_{21} T_{31} T_{32} I(\xi_1, \xi_2, \xi_3) W_{t,x,y_1,y_2,y_3} (\xi_1, \xi_2, \xi_3) d\xi_1 d\xi_2 d\xi_3 + p \int_c \int_c \int_c T_{21} I(\xi_1, \xi_2) W_{t,x,y_1,y_2} (\xi_1, \xi_2) d\xi_1 d\xi_2.\quad (40)\]
3.5 \((a) + (b) + (c)\) in (31)

We observe that the integrals of the same variables in (32), (39) and (40) have the same integrand, so their sum will be the integral multiplied by the sum of coefficients of those integrals. Moreover, the sum of these coefficients can be factorized. The final form of \((a) + (b) + (c)\) is as follows:

\[
\begin{align*}
\mathbb{P}(\eta(t) = x) &= (q - p)^2 \int \int \int I(\xi_1, \xi_2, \xi_3) W_{t, x, y_1, y_2, y_3}(\xi_1, \xi_2, \xi_3) d\xi_1 d\xi_2 d\xi_3 \\
+ &\frac{(q - p)^2}{p^2} \int \int \int I(\xi_1, \xi_2) W_{t, x, y_1, y_2}(\xi_1, \xi_2) d\xi_1 d\xi_2 \\
+ &\frac{q}{p} (q - p) \int \int \int I(\xi_1, \xi_3) W_{t, x, y_1, y_3}(\xi_1, \xi_3) d\xi_1 d\xi_3 \\
+ &\frac{q}{p^2} (q - p) \int \int \int I(\xi_2, \xi_3) W_{t, x, y_2, y_3}(\xi_2, \xi_3) d\xi_2 d\xi_3 \\
+ &\frac{q}{p} (q - p) \int \int \int I(\xi_1) W_{t, x, y_1}(\xi_1) d\xi_1 \\
+ &\frac{(q - p)}{p} \int \int \int I(\xi_2) W_{t, x, y_2}(\xi_2) d\xi_2 + \int \int I(\xi_3) W_{t, x, y_3}(\xi_3) d\xi_3.
\end{align*}
\]  

(41)

4 Proof of Theorem 1.1

In this section, we extend the idea used in Sect. 3 to prove Theorem 1.1. For \([A_\sigma]_{v_\alpha, v_N}\) containing \(Q_{\beta\alpha}\), let denote such \([A_\sigma]_{v_\alpha, v_N}\) by \([A_\sigma]^+_{v_\alpha, v_N}\) if \(Q_{\beta\alpha}\) is replaced by \(S_{\beta\alpha}\), and by \([A_\sigma]^-_{v_\alpha, v_N}\) if \(Q_{\beta\alpha}\) is replaced by \(- p T_{\beta\alpha}\). For \([A_\sigma]_{v_\alpha, v_N}\) containing no \(Q_{\beta\alpha}\), we set \([A_\sigma]_{v_\alpha, v_N} = [A_\sigma]^+_{v_\alpha, v_N}\) and \([A_\sigma]^-_{v_\alpha, v_N} = 0\). Then, we have an expression

\[
[A_\sigma]_{v_\alpha, v_N} = [A_\sigma]^+_{v_\alpha, v_N} + [A_\sigma]^-_{v_\alpha, v_N}
\]

because \(Q_{\beta\alpha} = S_{\beta\alpha} - p T_{\beta\alpha}\). See Tables 1 and 2 for \(N = 3\), for example. Following the convention above, we decompose the transition probabilities \(P_Y(X, v_\alpha; t)\) as follows:

\[
P_Y(X, v_\alpha; t) = P_Y^+(X, v_\alpha; t) + P_Y^-(X, v_\alpha; t)
\]

where \(P_Y^+(X, v_\alpha; t)\) and \(P_Y^-(X, v_\alpha; t)\) are the components of \(P_Y(X, v_\alpha; t)\) obtained by replacing \([A_\sigma]_{v_\alpha, v_N}\) by \([A_\sigma]^+_{v_\alpha, v_N}\) and \([A_\sigma]^-_{v_\alpha, v_N}\), respectively, in (1). Hence, (4) is written as

\[
\begin{align*}
\mathbb{P}(\eta(t) = x) &= \sum_{X \text{ with } x_N = x} P_Y^+(X, v_N; t) + \left( \sum_{X \text{ with } x_N = x} P_Y^-(X, v_N; t) + \sum_{X \text{ with } x_{N-1} = x} P_Y^+(X, v_{N-1}; t) \right) \\
+ & \cdots + \left( \sum_{X \text{ with } x_2 = x} P_Y^-(X, v_2; t) + \sum_{X \text{ with } x_1 = x} P_Y(X, v_1; t) \right).
\end{align*}
\]  

(42)

We will find the general formula of

\[
\left( \sum_{X \text{ with } x_{n+1} = x} P_Y^-(X, v_{n+1}; t) + \sum_{X \text{ with } x_n = x} P_Y^+(X, v_n; t) \right)
\]  

(43)
for \( n = 1, \ldots, N \) with conventions \( P^-(Y, v_{N+1}; t) = 0 \) and \( P^+(Y, v_1; t) = P^+(Y, v_1; t) \), and then sum the result over \( n = 1, \ldots, N \).

**Proposition 4.1** For \( n = 1, \ldots, N \),

\[
\sum_{X \text{ with } \lambda_{n+1} = x} P^-(Y, v_{n+1}; t) + \sum_{X \text{ with } \lambda_n = x} P^+(Y, v_n; t) = \sum_{\theta \neq S \subseteq \{1, \ldots, N\}, |S| \geq N-n} \tilde{c}_S \int_{\mathcal{F}} \cdots \int_{\mathcal{F}} \left( \prod_{\alpha < \beta, \alpha, \beta \in S} T_{\beta \alpha} \right) I(\xi_S) W_{t, X, Y_S}(\xi_S) d\xi_S
\]

(44)

where

\[
\tilde{c}_S = \begin{cases} 
(1)^{N-n} q^{n(n-1)/2} p^{(N-n)(N-n+1)/2} \frac{q^{-\sum(S^c) - n|S^c|}}{p^{\sum(S^c)|S^c|+(N-n+1)/2} [\frac{|S|-1}{N-n}]} & \text{if } N \in S, \\
(1)^{N-n} q^{n(n-1)/2} p^{(N-n)(N-n+1)/2} \frac{q^{-\sum(S^c) - n|S^c|-(N-n)}}{p^{\sum(S^c)|S^c|+(N-n-1)/2} [\frac{|S|}{N-n}]} & \text{if } N \notin S.
\end{cases}
\]

**Proof** First, we consider the case of \( n = N \). Since \([A_{\sigma}]^+_{v_n, v_N} = 0 \) if \( \sigma^{-1}(N) > n \), so the components of the transition probabilities are written

\[
P^\pm(Y, v_n; t) = \sum_{\sigma \text{ with } \sigma^{-1}(N) \leq n} \int_{\mathcal{F}} \cdots \int_{\mathcal{F}} [A_{\sigma}]^\pm_{v_n, v_N} \prod_{i=1}^{N} \left( \xi_{\sigma(t)}^{x_i - y_{\sigma(i)}} - 1 e^{(q \xi_i + q \xi_i^{-1})t} \right) d\xi.
\]

(45)

The sum in (45) is written

\[
\sum_{\text{all } A \subseteq \{1, \ldots, N\} \text{ with } |A| = n, N \in A} \sum_{\sigma \text{ with } \sigma(1), \ldots, \sigma(n) \in A} \sum_{\sigma \text{ with } \sigma(n+1), \ldots, \sigma(N) \in A^c} \sum_{\sigma(1), \ldots, \sigma(n) \text{ fixed}} \int_{\mathcal{F}} \cdots \int_{\mathcal{F}} [A_{\sigma}]^+_{v_n, v_N} \left( \xi_{\sigma(i)}^{x_i - y_{\sigma(i)}} - 1 e^{(q \xi_i + q \xi_i^{-1})t} \right) d\xi.
\]

(46)

and using this, we evaluate the sum on the left-hand side of (44) as in the following manner:

(i) **Step 1.** In this step, we compute the second sum on the left-hand side of (44). The second sum is written

\[
\sum_{\text{all } A \subseteq \{1, \ldots, N\} \text{ with } |A| = n, N \in A} \left( \sum_{X \text{ with } \lambda_n = x} \sum_{\sigma \text{ with } \sigma(1), \ldots, \sigma(n) \in A} \sum_{\sigma \text{ with } \sigma(n+1), \ldots, \sigma(N) \in A^c} \sum_{\sigma(1), \ldots, \sigma(n) \text{ fixed}} \int_{\mathcal{F}} \cdots \int_{\mathcal{F}} [A_{\sigma}]^+_{v_n, v_N} \left( \xi_{\sigma(i)}^{x_i - y_{\sigma(i)}} - 1 e^{(q \xi_i + q \xi_i^{-1})t} \right) d\xi \right).
\]

(\#)

We compute (#) for a fixed \( A \subseteq \{1, \ldots, N\} \) with \( |A| = n \) and \( N \in A \). Set \( x_n = x \) and

\[
x_{n-1} = x - z_{n-1}, x_{n-2} = x - z_{n-1} - z_{n-2}, \ldots, x_1 = x - z_{n-1} - \cdots - z_1, \\
x_{n+1} = x + v_1, x_{n+2} = x + v_1 + v_2, \ldots, x_N = x + v_1 + \cdots + v_{N-n}
\]
where \( z_i \) and \( v_i \) are positive integers. Then, (\#) is written

\[
\sum_{z_1, \ldots, z_{n-1}} \sum_{v_1, \ldots, v_{N-n}} \sum_{\sigma \text{ with } \sigma(1), \ldots, \sigma(n) \in A} \sum_{\sigma \text{ with } \sigma(1), \ldots, \sigma(n) \in A^c} \int_c \cdots \int_c [A_\sigma]_{v_1, v_N}^{n-1} \prod_{i=1}^{n-1} (\xi_\sigma(1) \cdots \xi_\sigma(i))^{-z_i} \prod_{i=1}^{N-n} (\xi_\sigma(n+i) \cdots \xi_\sigma(N))^v_i W_{t, x, y}(\xi) d\xi.
\]

(47)

By Theorem 1.7, Proposition 1.8 in [16] and (30), if \( \sigma(1), \ldots, \sigma(n) \in A \) and \( N \in A \),

\[
[A_\sigma]_{v_1, v_N} = \left( \prod_{\alpha \in A^c} p_{T_{N\alpha}} \right) \times \left( \prod_{\alpha < \beta \neq N} S_{\beta\alpha} \right)
\times \left( \prod_{\text{all inversions } (\beta, \alpha) \text{ in } \sigma(1) \cdots \sigma(n)} S_{\beta\alpha} \right)
\times \left( \prod_{\text{all inversions } (\beta, \alpha) \text{ in } \sigma(n+1) \cdots \sigma(N)} S_{\beta\alpha} \right).
\]

(48)

Let us sum the integral in (47) over \( v_1, \ldots, v_{N-n} \) and then over all permutations \( \sigma \) with \( \sigma(n+1), \ldots, \sigma(N) \in A^c \) and \( \sigma(1), \ldots, \sigma(n) \) fixed. Then, (47) becomes

\[
\sum_{z_1, \ldots, z_{n-1}} \sum_{\sigma \text{ with } \sigma(1), \ldots, \sigma(n) \in A} \int_c \cdots \int_c f(\xi) \left( \prod_{\text{all inversions } (\beta, \alpha) \text{ in } \sigma(1) \cdots \sigma(n)} S_{\beta\alpha} \right) \prod_{i=1}^{n-1} (\xi_\sigma(1) \cdots \xi_\sigma(i))^{-z_i} d\xi
\]

(49)

where

\[
f(\xi) = p^{(N-n)(N-n-1)/2} \left( \prod_{\alpha \in A^c} p_{T_{N\alpha}} \right) \left( \prod_{\alpha < \beta \neq N} S_{\beta\alpha} \right)
\times \left( \prod_{\alpha < \beta} T_{\beta\alpha} \right) \prod_{\alpha \in A^c} \xi_{\alpha} \prod_{\alpha \in A^c} (1 - \xi_{\alpha}) W_{t, x, y}(\xi).
\]

by using the identity (1.6) in [20]. Here, we note that \( f(\xi) \) can be made analytic inside a sufficiently large circle centered at the origin except at the origin for variable \( \xi_i \) where \( i \in A \) for the use of Lemma 2.3 (see Remark 2.1). Then, by Lemma 2.3 and Remark 2.1, (49) becomes

\[
\sum_{\emptyset \neq S' \subset A} \tilde{c}_{S'} \int_c \cdots \int_c \left( \prod_{\alpha < \beta, \alpha, \beta \in S'} T_{\beta\alpha} \right) I(\xi_{S'}) f(\xi) \bigg|_{\xi_i = 1, i \in A \setminus S'} d\xi_{S'} d\xi_{A^c}
\]

(50)

where

\[
\tilde{c}_{S'} = q^{(n-1)/2} \times \frac{q^{\sum_{A \setminus S'} - n|A \setminus S'|}}{p^{\sum_{A \setminus S'} - |A \setminus S'|[|A \setminus S'|+1]/2}}.
\]

(51)
Now, we evaluate \( f(\xi) \mid_{\xi_i = 1, i \in A \setminus S'} \). Let \( |a| > \) be the number of elements in \( A \setminus S' \) larger than \( a \in A^c \). Then, noting that \( pT_{\beta\alpha} \mid_{\xi_\beta = 1} = 1 \) and \( S_{\beta\alpha} \mid_{\xi_\beta = 1} = q/p \), we obtain

\[
f(\xi) \mid_{\xi_i = 1, i \in A \setminus S'} = p^{(N-n)(N-n-1)/2} \left( \prod_{\alpha < \beta} T_{\beta\alpha} \right) \frac{\prod_{\alpha \in A^c} \xi_\alpha}{\prod_{\alpha \in A^c} (1 - \xi_\alpha)} W_{L,x,Y_{S' \cup A'}}(\xi_{S' \cup A'})
\]

\[
\times \begin{cases} 
\left( \prod_{\alpha \in A^c} pT_{\alpha \alpha} \right) \left( \prod_{\alpha < \beta \neq N} S_{\beta\alpha} \right) \left( q/p \right) \sum_{a \in A^c < a'} & \text{if } N \in S', \\
\left( \prod_{\alpha < \beta \neq N} S_{\beta\alpha} \right) \left( q/p \right) \sum_{a \in A^c < a'} & \text{if } N \notin S'.
\end{cases}
\]

Hence, noting that

\[
\Sigma_A(A \setminus S') = \Sigma(A \setminus S') - \sum_{a \in A^c < a'},
\]

we obtain

\[
\# = \sum_{S' \subset A, \ N \in S'} q^{n(n-1)/2} p^{(N-n)(N-n-1)/2} \frac{q^{\sum(A \setminus S') - n|A \setminus S'|}}{p^{\sum(A \setminus S') - |A \setminus S'|(|A \setminus S'|+1)/2}}
\]

\[
\times \prod_{c} \int_{c} \left( \prod_{\alpha < \beta} T_{\beta\alpha} \right) \left( \prod_{\alpha < \beta} T_{\beta\alpha} \right) \left( \prod_{\alpha < \beta} S_{\beta\alpha} \right) \left( \prod_{\alpha < \beta} T_{\alpha \alpha} \right)
\]

\[
\times I(\xi_{S'}) \prod_{\alpha \in A^c} \xi_\alpha \frac{W_{L,x,Y_{S' \cup A'}}(\xi_{S' \cup A'})}{1 - \xi_\alpha} d\xi_{S' \cup A'}
\]

\[
+ \sum_{\emptyset \neq S' \subset A, \ N \notin S'} q^{n(n-1)/2} p^{(N-n)(N-n-1)/2} \frac{q^{\sum(A \setminus S') - n|A \setminus S'| - |A^c|}}{p^{\sum(A \setminus S') - |A \setminus S'|(|A \setminus S'|+1)/2 - |A^c|}}
\]

\[
\times \prod_{c} \int_{c} \left( \prod_{\alpha < \beta} T_{\beta\alpha} \right) \left( \prod_{\alpha < \beta} T_{\beta\alpha} \right) \left( \prod_{\alpha < \beta} S_{\beta\alpha} \right)
\]

\[
\times I(\xi_{S'}) \prod_{\alpha \in A^c} \xi_\alpha \frac{W_{L,x,Y_{S' \cup A'}}(\xi_{S' \cup A'})}{1 - \xi_\alpha} d\xi_{S' \cup A'}.
\]

(ii) **Step 2.** In this step, we compute the first sum on the left-hand side of (44). The first sum is written

\[
\sum_{A \subset \{1, \ldots, N\} \atop |A| = n, \ N \in A} \left( \sum_{X \text{ with } x_{n+1} = x} \sum_{\sigma \text{ with } \sigma(1), \ldots, \sigma(n) \in A} \sum_{\sigma(n+1), \ldots, \sigma(N) \in A^c, \ \sigma(1), \ldots, \sigma(n) \text{ fixed}} \int_{c} \int_{c} \int_{c} \left[ A_{\sigma} \right]_{\gamma_{n+1}, \gamma_{\infty}} \left( \xi_{\sigma(i)} - \gamma_{\sigma(i) - 1} e^{\left( \frac{p}{n} + q \xi_{i-1} \right) r} \right) d\xi \right).
\]
In a similar way to Step 1, we write (###) as

$$
\sum_{v_1,\ldots,v_{N-n-1}} \sum_{\sigma \text{ with } \sigma(1) = \sigma(n) = 1, \sigma(n+1),\ldots,\sigma(N) \in A^c, \sigma(1) = \sigma(n) = 1, \sigma(n+1),\ldots,\sigma(N) \text{ fixed}} \int_{c} \cdots \int_{c} [A_{\sigma}]_{v_1,\ldots,v_{N-n-1}}^{n} \frac{1}{(\xi_{\sigma(1)} \cdots \xi_{\sigma(i)})^{-z_i}} \prod_{i=1}^{N-n-1} \left( \frac{1}{(\xi_{\sigma(n+i)} \cdots \xi_{\sigma(N)})} \right)^{\nu_i} W_{t,x,y}(\xi) d\xi.
$$

(54)

Note that $[A_{\sigma}]_{v_1,\ldots,v_{N-n-1}} = -[A_{\sigma}]_{v_1,\ldots,v_{N-n-1}}$ by Theorem 1.7, Proposition 1.8 in [16] and (30). In (54), we first sum over $v_1,\ldots,v_{N-n-1}$ and then sum over all permutations $\sigma$ with $\sigma(n+1),\ldots,\sigma(N) \in A^c$ and $\sigma(1),\ldots,\sigma(n) \text{ fixed}$. Then, (54) becomes

$$
\sum_{v_1,\ldots,v_{N-n-1}} \sum_{\sigma \text{ with } \sigma(1) = \sigma(n) = 1, \sigma(n+1),\ldots,\sigma(N) \in A^c, \sigma(n+1),\ldots,\sigma(n) \text{ fixed}} \int_{c} \cdots \int_{c} f(\xi) \left( \prod_{i=1}^{N-n-1} S_{\beta\alpha} \right) \prod_{i=1}^{n} \left( \frac{1}{(\xi_{\sigma(1)} \cdots \xi_{\sigma(i)})} \right)^{z_i} d\xi
$$

(55)

where

$$
f(\xi) = p^{(N-n)(N-n-1)/2} \left( \prod_{\alpha \in A^c} p_{T_{\alpha}} \right) \left( \prod_{\alpha < \beta \neq N} S_{\beta\alpha} \right) \times \left( \prod_{\alpha < \beta} T_{\beta\alpha} \right) (-1)^{A^c} I(\xi_{A^c}) W_{t,x,y}(\xi)
$$

(56)

by using the identity (1.6) in [20]. Here, we note that (56) can be made analytic inside a sufficiently large circle centered at the origin except at the origin for variable $\xi_i$ where $i \in A$ for the use of Lemma 2.4 (see Remark 2.1). Then, by Lemma 2.4 and Remark 2.1, (55) becomes

$$
\int_{c} \cdots \int_{c} f(\xi) \bigg|_{\xi_i = 1, i \in A} d\xi_{A^c} + \sum_{\emptyset \neq S' \subset A} \tilde{c}_{S'} \int_{c} \cdots \int_{c} \left( \prod_{\alpha < \beta} T_{\beta\alpha} \right) I(\xi_{S'}) f(\xi) \bigg|_{\xi_i = 1, i \in A \setminus S'} d\xi_{S'} d\xi_{A^c}
$$

(57)

where $\tilde{c}_{S'}$ is given by (51). Now, if we evaluate $f(\xi) \big|_{\xi_i = 1, i \in A}$ and $f(\xi) \big|_{\xi_i = 1, i \in A \setminus S'}$ in (57) by the same method as in Step 1, then we can show

(###) = $\sum_{\emptyset \neq S' \subset A, N \in S'} q^{n(n-1)/2} p^{(N-n)(N-n+1)/2} \frac{q^{\sum (A \setminus S') \cdot n (A \setminus S')}}{p^{\sum (A \setminus S') \cdot |(A \setminus S')\setminus (A \setminus S')| + 1/2}}

$$
\times \int_{c} \cdots \int_{c} \left( \prod_{\alpha < \beta} T_{\beta\alpha} \right) \left( \prod_{\alpha < \beta} T_{\beta\alpha} \right) \left( \prod_{\alpha < \beta \neq N} S_{\beta\alpha} \right) \left( \prod_{\alpha \in A^c} T_{\alpha} \right)

\times \int_{c} \cdots \int_{c} \left( \prod_{\alpha < \beta} T_{\beta\alpha} \right) \left( \prod_{\alpha < \beta \neq N} S_{\beta\alpha} \right) \left( \prod_{\alpha \in A^c} T_{\alpha} \right)

\times J(\xi_{S'}) I(\xi_{A^c}) (-1)^{|A^c|} W_{t,x,y_{S'\cup A^c}}(\xi_{S'\cup A^c}) d\xi_{S'\cup A^c}
$$

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Step 3. In this step, we sum \((61)\) and \((62)\). Recall that all \(S'\) in the sum (\#) and \(S'\) in the sum (\#\#) can be the empty set. Using that when \(S' \neq \emptyset\),

\[
I(\xi_{S'}) \prod_{a \in A^c} \frac{\xi_a}{(1 - \xi_a)} + J(\xi_{S'}) I(\xi_{A^c})(-1)^{|A^c|} = (-1)^{|A^c|} I(\xi_{S' \cup A^c}),
\]

we obtain the sum of (\#) and (\#\#) in the form of

\[
\sum_{S' \subset A} c_{A, S'} \int_{\mathcal{C}} \cdots \int_{\mathcal{C}} K(A^c, S') I(\xi_{S' \cup A^c}) W_{t, x, Y_{S' \cup A^c}}(\xi_{S' \cup A^c}) d\xi_{S' \cup A^c}
\]

(59)

where

\[
c_{A, S'} = \begin{cases} 
(-1)^{N-n} q^{n(n+1)/2} p^{N-n}(N-n+1)/2 & \text{if } N \in S', \\
(-1)^{N-n} q^{n(n+1)/2} p^{N-n}(N-n+1)/2 & \text{if } N \notin S', 
\end{cases}
\]

(60)

and

\[
K(A^c, S') = \begin{cases} 
\left( \prod_{a < \beta} T_{\beta a} \right) \left( \prod_{a < \beta} T_{\beta a} \right) \left( \prod_{a < \beta} T_{\beta a} \right) & \text{if } N \in S', \\
\left( \prod_{a < \beta} T_{\beta a} \right) \left( \prod_{a < \beta} T_{\beta a} \right) \left( \prod_{a < \beta} T_{\beta a} \right) & \text{if } N \notin S'. 
\end{cases}
\]

(iv) Step 4. Finally, we compute

\[
\sum_{|A| = n, N \in A} \sum_{S' \subset A} c_{A, S'} \int_{\mathcal{C}} \cdots \int_{\mathcal{C}} K(A^c, S') I(\xi_{S' \cup A^c}) W_{t, x, Y_{S' \cup A^c}}(\xi_{S' \cup A^c}) d\xi_{S' \cup A^c}.
\]

(61)

Note that if we let \(S = S' \cup A^c\), then \(\Sigma(A \setminus S') = \Sigma(S^c)\) and \(|A \setminus S'| = |S^c|\). This implies that if a pair \((A, S')\) such that \(N \in A \subset \{1, \ldots, N\}\) with \(|A| = n\) and \(S' \subset A\), and another pair \((A', S'')\) such that \(N \in A' \subset \{1, \ldots, N\}\) with \(|A'| = n\) and \(S'' \subset A'\) satisfy \(S' \cup A^c = S'' \cup A^c\), then \(c_{A, S'} = c_{A', S''}\). Hence, the sum (61) is equivalent to

\[
\sum_{S \text{ with } |S'| = N-n} \sum_{(S_1, S_2)} \hat{c}_S \int_{\mathcal{C}} \cdots \int_{\mathcal{C}} K(S_2, S_1) I(\xi_S) W_{t, x, Y_S}(\xi_S) d\xi_S
\]

\[
+ \sum_{S \text{ with } |S'| = N-n, N \notin S} \hat{c}_S \int_{\mathcal{C}} \cdots \int_{\mathcal{C}} K(S, \emptyset) I(\xi_S) W_{t, x, Y_S}(\xi_S) d\xi_S.
\]

(62)
where

\[
\hat{c}_{S} = \begin{cases} 
(1)^{N-n} q^{n(n-1)/2} p^{(N-n)(N-n+1)/2} \sum_{(S', S'')} q^{(S')-n(S'')} & \text{if } N \in S, \\
(1)^{N-n} q^{n(n-1)/2} p^{(N-n)(N-n-1)/2} \sum_{(S', S'')} q^{(S')-n(S'')} & \text{if } N \notin S
\end{cases}
\]

and \(\sum_{(S_1, S_2)}\) is the sum over all partitions of \(S = S_1 \cup S_2\) such that \(|S_2| = N - n\) and \(N \notin S_2\). By using Lemma 2.5,

\[
\sum_{(S_1, S_2)} K(S_2, S_1) = \begin{cases} 
\left[\binom{|S|-1}{N-n}\right] \prod_{\alpha < \beta, \alpha, \beta \in S} T_{\beta \alpha} & \text{if } N \in S, \\
\left[\binom{|S|}{N-n}\right] \prod_{\alpha < \beta, \alpha, \beta \in S} T_{\beta \alpha} & \text{if } N \notin S
\end{cases}
\]

and we note that

\[
K(S, \emptyset) = \prod_{\alpha < \beta, \alpha, \beta \in S} T_{\beta \alpha}.
\]

Hence, (62) implies (44).

\(\square\)

(Continuing the proof of Theorem 1.1) Now, finally, we sum (44) over \(n = 1, \ldots, N\). Note that by the convention (17),

\[
\sum_{n=1}^{N} \hat{c}_{S} = \begin{cases} 
\sum_{n=N-|S|+1}^{N} \hat{c}_{S} & \text{if } N \in S, \\
\sum_{n=N-|S|+1}^{N} \hat{c}_{S} & \text{if } N \notin S
\end{cases}
\]

If \(N \in S\), then

\[
\sum_{n=N-|S|+1}^{N} \hat{c}_{S} = \left(\prod_{i=1}^{|S|-1} (q^i - p^i)\right) \left(\frac{q}{p}\right)^{\Sigma(S')-|S'|(|S'|+1)/2}
\]

by Lemma 2.6 where \(l - 1\) is replaced by \(|S| - 1\) and \(k\) is replaced by \(N - n\). For the case of \(N \notin S\), observe that

\[
\left(\frac{q}{p}\right)^{\Sigma(S')-|S'|(|S'|+1)/2} = \left(\frac{q}{p}\right)^{|S|} \left(\frac{q}{p}\right)^{\Sigma(S')-|S'|(|S'|-1)/2-N}
\]

and then apply Lemma 2.6 where \(l - 1\) is replaced by \(|S|\) and \(k\) is replaced by \(N - n\). Then, we obtain

\[
\sum_{n=N-|S|}^{N} \hat{c}_{S} = \frac{1}{p^{|S|}} \left(\prod_{i=1}^{|S|} (q^i - p^i)\right) \left(\frac{q}{p}\right)^{\Sigma(S')-|S'|(|S'|-1)/2-N}.
\]
Hence,
\[
\begin{align*}
    c_S &= \begin{cases} 
        \left( \prod_{i=1}^{\lvert S \rvert-1} (q^i - p^i) \right) \left( \frac{q}{p} \right)^{\Sigma(\lvert S^c \rvert) - \lvert S^c \rvert (\lvert S^c \rvert + 1)/2} & \text{if } N \in S, \\
        \frac{1}{p^{\lvert S \rvert}} \left( \prod_{i=1}^{\lvert S \rvert} (q^i - p^i) \right) \left( \frac{q}{p} \right)^{\Sigma(\lvert S^c \rvert) - \lvert S^c \rvert (\lvert S^c \rvert - 1)/2 - N} & \text{if } N \not\in S.
    \end{cases}
\end{align*}
\]

This completes the proof of Theorem 1.1. \hfill \Box

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**Declarations**

**Conflict of interest**  The authors declare that they have no conflict of interest.

**Appendix A: Formulas of \([A_\sigma]_{\nu_1, \nu_2}\) in (1)**

In this appendix, we briefly summarize how to find the formulas of \([A_\sigma]_{\nu_1, \nu_2}\) in (1), which were developed in [15, 16]. For a permutation \(\sigma\) in the symmetric group \(S_N\), \(A_\sigma\) is an \(N^N \times N^N\) matrix which is obtained as follows. Let \(T_i\) be the simple transposition which interchanges the \(i^{th}\) element and the \((i + 1)^{st}\) element and leaves everything else fixed. Then, any permutation \(\sigma \in S_N\) can be written as a product of simple transpositions, that is,
\[
\sigma = T_{i_1} \cdots T_{i_j}
\]  \hspace{1cm} (63)

for some \(i_1, \ldots, i_j \in \{1, \ldots, N - 1\}\). Of course, the form of (63) is not unique. For example, a permutation \((321) \in S_3\) is written as
\[
(321) = T_1T_2T_1 = T_2T_1T_2.
\]

If \(T_i\) interchanges \(\alpha\) at the \(i^{th}\) slot and \(\beta\) at the \((i + 1)^{st}\) slot in a permutation, that is,
\[
T_i(\cdots \alpha \beta \cdots) = (\cdots \beta \alpha \cdots),
\]
we denote this \(T_i\) by \(T_i(\beta, \alpha)\) to show explicitly which numbers are interchanged. For example,
\[
(321) = T_1(3, 2)T_2(3, 1)T_1(2, 1).
\]

Hence, we may have an expression
\[
\sigma = T_{i_1}(\beta_1, \alpha_1) \cdots T_{i_j}(\beta_j, \alpha_j)
\]  \hspace{1cm} (64)

for any permutation \(\sigma\). Given \(T_i(\beta, \alpha)\), we define \(N^N \times N^N\) matrix \(T_i(\beta, \alpha)\) by
\[
T_i(\beta, \alpha) = \underbrace{I_N \otimes \cdots \otimes I_N}_{(i-1) \text{ times}} \otimes R_{\beta\alpha} \otimes \underbrace{I_N \otimes \cdots \otimes I_N}_{(N-i-1) \text{ times}}
\]
where \( I_N \) is the \( N \times N \) identity matrix and \( R_{\beta\alpha} \) is an \( N^2 \times N^2 \) matrix, where \( R_{\beta\alpha} \) is defined as follows. Let us label all columns and rows of \( N^2 \times N^2 \) matrices by \( i j \) where \( i, j = 1, \ldots, N \) in the lexicographical order. The \((ij, kl)\) entry of the matrix \( R_{\beta\alpha} \) is defined to be

\[
[R_{\beta\alpha}]_{ij,kl} = \begin{cases} 
S_{\beta\alpha} & \text{if } ij = kl \text{ with } i = j; \\
P_{\beta\alpha} & \text{if } ij = kl \text{ with } i < j; \\
Q_{\beta\alpha} & \text{if } ij = kl \text{ with } i > j; \\
pT_{\beta\alpha} & \text{if } ij = lk \text{ with } i < j; \\
qT_{\beta\alpha} & \text{if } ij = lk \text{ with } i > j; \\
0 & \text{for all other cases}
\end{cases}
\]

where

\[
S_{\beta\alpha} = \frac{p + q \xi_\alpha \xi_\beta - \xi_\beta}{p + q \xi_\alpha \xi_\beta - \xi_\alpha}, \quad P_{\beta\alpha} = \frac{(p - q \xi_\alpha)(\xi_\beta - 1)}{p + q \xi_\alpha \xi_\beta - \xi_\alpha},
\]

\[
T_{\beta\alpha} = \frac{\xi_\beta - \xi_\alpha}{p + q \xi_\alpha \xi_\beta - \xi_\alpha}, \quad Q_{\beta\alpha} = \frac{(p - q \xi_\beta)(\xi_\alpha - 1)}{p + q \xi_\alpha \xi_\beta - \xi_\alpha}.
\]

Now, for a given expression (64), define

\[
A_{\sigma} := T_{i_1}(\beta_1, \alpha_1) \cdots T_{i_N}(\beta_N, \alpha_N).
\]

It is a known fact that \( A_{\sigma} \) is well-defined in the sense that (65) represents the same matrix for any expression (64). We have just constructed an \( N \times N \) matrix \( A_{\sigma} \) for a given permutation \( \sigma \). Now, let us label all columns and rows of \( A_{\sigma} \) by \( v = i_1 \cdots i_N \) where \( i_j \in \{1, \ldots, N\}, j = 1, \ldots, N \) in the lexicographical order. In this paper, we denote 2 \( \cdots \) 212 \( \cdots \) 2 by \( v_N \) if 1 is the \( n^{th} \) lefmost number.

Since \( A_{\sigma} \) is a product of matrices, in general, the matrix elements \([A_{\sigma}]_{\pi,\nu}\) are possibly written as a sum of the products of the matrix elements of \( T_{i_j}(\beta_j, \alpha_j) \), \( \ldots \), \( T_{i_1}(\beta_1, \alpha_1) \) in (65). However, it was shown in [16] that, for some special cases of initial order of particles \( v \), \([A_{\sigma}]_{\pi,\nu}\) are expressed as a product of the matrix elements of \( T_{i_j}(\beta_j, \alpha_j) \), \( \ldots \), \( T_{i_1}(\beta_1, \alpha_1) \), not a sum of the products. In particular, for the initial order of particles \( v_N \), that is, \( 2 \cdots 21 \), the formulas \([A_{\sigma}]_{\pi,\nu_N}\) are given in Theorem 1.2 (a), Theorem 1.4, Theorem 1.6, Theorem 1.7, and Proposition 1.8 in [16].

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