Stability for weighted composition $C_0$-semigroups on Lebesgue and Sobolev spaces

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Abstract Stability of weighted composition strongly continuous semigroups acting on Lebesgue and Sobolev spaces is studied, without the use of spectral conditions on the generator of the semigroup. Applications to the generalized von Foerster–Lasota semigroup and a comparison with hypercyclicity conditions are presented.

Keywords Stable semigroups · Weighted Lebesgue spaces · von Foerster-Lasota equation

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $\varphi : [0, \infty[ \times \Omega \to \Omega$ be a semiflow, i.e. a continuous function such that $\varphi(0, \cdot) = id_\Omega$, $\varphi(t, \cdot) \circ \varphi(s, \cdot) = \varphi(t+s, \cdot)$ for all $t, s \geq 0$ and such that $\varphi(t, \cdot)$ is injective for all $t \geq 0$. For example, $\varphi$ could be the solution of the initial value problem

$$x' = F(x), \quad x(0) = x_0$$

(1.1)
on $\Omega$, where $F : \Omega \to \mathbb{R}$ is a locally Lipschitz continuous vector field over $\Omega$ and $x_0 \in \Omega$. 

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Let $h : \Omega \to \mathbb{R}$ be a continuous function and define for every $t \geq 0$

$$h_t : \Omega \to ]0, \infty[, \, x \mapsto \exp \left( \int_0^t h(\varphi(s, x)) \, ds \right).$$

Denoting by $\lambda$ the $N$-dimensional Lebesgue measure, consider a $\lambda$-measurable and locally integrable function $\rho : \Omega \to ]0, \infty[$ and, for every $1 \leq p < \infty$, the space

$$L^p_\rho(\Omega) = \{ u : \Omega \to \mathbb{R} \mid u \text{ $\lambda$-measurable and } \| u \|_p < \infty \},$$

where $\| u \|_p = \left( \int_\Omega |u(t)|^p \rho(t) \, dt \right)^{\frac{1}{p}}$. If $p = \infty$, as usual, we consider the space

$$L^\infty(\Omega) := \{ u : \Omega \to \mathbb{R} \mid u \text{ $\lambda$-measurable and } \exists C > 0 \text{ s.t. } |u(x)| \leq C \text{ $\lambda$-a.e. in } \Omega \},$$

with the norm

$$\| u \|_\infty = \inf\{ C > 0 \mid |u(x)| \leq C \text{ $\lambda$-a.e. in } \Omega \}.$$

Finally set

$$(T(t)f)(x) := h_t(x) f(\varphi(t, x)), \quad f \in L^p_\rho(\Omega), \, x \in \Omega, \, t \geq 0. \quad (1.2)$$

If $\varphi$ is continuously differentiable, it has been proved in [13, Proposition 4.12, Theorem 4.7] that $T = (T(t))_{t \geq 0}$ is a $C_0$-semigroup in $L^p_\rho(\Omega)$ if and only if there exist $M \geq 1$ and $\omega \in \mathcal{S}$ such that, for every $t \geq 0$, we have

$$h^p_t \rho \leq M e^{\omega t} (\rho(\varphi(t, \cdot)) |\det D\varphi(t, \cdot)| \quad \lambda\text{-a.e. on } \Omega$$

where $D\varphi(t, \cdot)$ is the Jacobian of $\varphi(t, \cdot)$, or equivalently

$$h^p_t (\rho(\varphi(-t, \cdot))) \rho(\varphi(-t, \cdot)) |\det D\varphi(-t, \cdot)| \leq M e^{\omega t} \quad \lambda\text{-a.e. on } \Omega, \quad (1.3)$$

where $\varphi(-t, \cdot)$ is the inverse function of $\varphi(t, \cdot)$ and $D\varphi(-t, \cdot)$ is its Jacobian. In this case $\rho$ is said to be a $p$-admissible weight and for every $t \geq 0$ we have

$$\| (T(t)) \| = \left\| \frac{\rho_t}{\rho} \right\|_\infty,$$

where

$$\rho_{t, p} = \chi_{\varphi(t, \Omega)} h_t (\varphi(-t, \cdot))^p \rho(\varphi(-t, \cdot)) |\det D\varphi(-t, \cdot)|.$$

In the following, we will consider a continuous $p$-admissible function. In this case, clearly (1.3) holds true for every $x \in \Omega$.

Our aim is to characterize the stability of the $C_0$-semigroup $T$ on $L^p_\rho(\Omega)$, namely when

$$\forall f \in L^p_\rho(\Omega) \quad \lim_{t \to \infty} \| T(t)f \|_p = 0.$$

The classical approach to stability is to analyze the spectrum of the generator of the semigroup (see e.g. [10]). We will instead characterize stability in terms of the functions $\rho_{t, p}$.

After studying general conditions in the multidimensional case, we will focus on the one-dimensional case, obtaining explicit characterizations when the semiflow is associated to a function $F$ as in (1.1). Similar characterizations for weighted composition semigroups acting on Sobolev spaces $W^{1, p}(a, b)$ are considered. In Sect. 4 applications to the semigroups associated with the (generalized) linear von Foerster–Lasota equations are investigated. Finally
Sect. 5 is devoted to the comparison of stability conditions for the semigroups defined as (1.2) and the hypercyclicity conditions that were obtained in [3,13,14].

2 The multidimensional case

For sake of completeness, we state an essentially known lemma about convergence with respect to the $w^s$-topology in $L^\infty(\Omega, \mu)$ (see e.g. [5, Ex.4.22]).

**Lemma 2.1** Let $\Omega$ be a locally compact Hausdorff space, $\mu$ a regular Borel $\sigma$-finite measure on $\Omega$, and $\psi : [0, \infty[ \to L^\infty(\Omega, \mu)$ a locally bounded function. Then $w^s\lim_{t \to \infty} \psi(t) = 0$ in $L^\infty(\Omega, \mu)$ if and only if

1. $\sup_{t \geq 0} ||\psi(t)||_\infty < \infty$;
2. for every compact set $K \subseteq \Omega$, it holds

$$\lim_{t \to \infty} \int_K \psi(t)d\mu = 0.$$

**Theorem 2.2** Let $\Omega$ be an open subset of $\mathbb{R}^N$, $\varphi$ a continuously differentiable semiflow on $\Omega$, $h \in C(\Omega)$, $\rho$ a continuous $p$-admissible function, and let $T$ be the semigroup on $L^p_\rho(\Omega)$ defined by (1.2). Then the following conditions are equivalent:

1. $T$ is stable on $L^p_\rho(\Omega)$,
2. $T$ is bounded.
3. for every compact set $K \subseteq \Omega$,

$$\lim_{t \to \infty} \int_K \rho_t, p(x)dx = 0.$$

**Proof** Observe that, by (1.4), $T$ is bounded if and only if $\sup_{t \geq 0} \left\| \frac{\partial_t}{\rho} \right\|_\infty < \infty$.

Then $T$ is stable on $L^p_\rho(\Omega)$ if and only if

$$\lim_{t \to \infty} \int_\Omega |h_t(x)f(\varphi(t, x))|^p \rho(x)dx = 0, \quad \forall f \in L^p_\rho(\Omega),$$

or, equivalently, with a change of variable (see also the argument in the proof of [13, Proposition 3.11]):

$$\lim_{t \to \infty} \int_\Omega |f(y)|^p \rho_t, p(y)dy = 0, \quad \forall f \in L^p_\rho(\Omega),$$

that is

$$\lim_{t \to \infty} \int_\Omega |g(y)| \rho_t, p(y)dy = 0, \quad \forall g \in L^1_\rho(\Omega).$$

By applying the previous equality to $g^+ = g \vee 0$ and $g^- = (-g) \vee 0$, we get that it is equivalent to

$$\lim_{t \to \infty} \int_\Omega g(x) \frac{\rho_t, p(x)}{\rho(x)} \rho(x)dx = 0, \quad \forall g \in L^1_\rho(\Omega).$$

(2.5)
Setting $\mu = \rho \cdot \lambda$, it holds that $\mu$ is a regular $\sigma$-finite Borel measure. Moreover, by observing that $\rho > 0$, we get that the dual of $L^p_\rho(\Omega)$ is

$$\{u : \Omega \to \mathbb{R} \mid u \text{ } \mu\text{-measurable and } \exists C > 0 \text{ s.t. } |u(x)| \leq C \mu\text{-a.e. in } \Omega\} = L^\infty(\Omega).$$

So, (2.5) is equivalent to say that $\lim_{t \to \infty} \frac{\rho (t)}{\rho (1)} \to 0$ with respect to the $w^*$-topology on $L^\infty(\Omega)$ induced by $L^1_\rho(\Omega)$. Since the function $t \mapsto \rho_{t,p}/\rho \in L^\infty(\Omega)$ is locally bounded by (1.3), the last assertion is equivalent to conditions (1) and (2) of Lemma 2.1.

**Remark 2.3** Observe that, by the continuity of $\rho$, $T$ is bounded if and only if

$$\exists C > 0 \forall x \in \Omega \forall t \geq 0 \quad \rho_{t,p}(x) \leq C \cdot \rho(x). \quad (2.6)$$

**Remark 2.4** We could also consider the complex valued function space $L^p_\rho(\Omega, \mathbb{C})$, $h \in C(\Omega, \mathbb{C})$ and define for every $f \in L^p_\rho(\Omega, \mathbb{C})$ and every $t \geq 0$

$$\left( T^C_h(t)f \right)(x) = \exp \left( \int_0^t h(\varphi(s, x)) \right) f(\varphi(x, t)).$$

By the quoted results in [13], if $\rho$ is a $p$-admissible weight, then $T^C_h = (T^C_h(t))_{t \geq 0}$ is a strongly continuous semigroup on $L^p_\rho(\Omega, \mathbb{C})$. Consider, for every $f \in L^p_\rho(\Omega, \mathbb{R})$ and $t \geq 0$

$$\left( T^R_{Reh}(t)f \right)(x) = \exp \left( \int_0^t h(\varphi(s, x)) \right) f(\varphi(x, t)).$$

It holds that $T^C_h$ is stable on $L^p_\rho(\Omega, \mathbb{C})$ if and only if $T^R_{Reh}$ is stable on $L^p_\rho(\Omega, \mathbb{R})$. Indeed, if $f \in L^p_\rho(\Omega, \mathbb{R})$, then

$$\left| \left( T^C_h(t)f \right)(x) \right| = \left| \left( T^R_{Reh}(t)f \right)(x) \right|$$

and we get immediately that if $T^C_h$ is stable then $T^R_{Reh}$ is stable. Conversely, if $f \in L^p_\rho(\Omega, \mathbb{C})$, then $|f| \in L^p_\rho(\Omega, \mathbb{R})$ and

$$\left| \left( T^C_h(t)f \right)(x) \right| = \left| \left( T^R_{Reh}(t)|f| \right)(x) \right|$$

and again we get that if $T^R_{Reh}$ is stable, then $T^C_h$ is stable too.

Taking into account the previous considerations, we will only consider the real valued function space $L^p_\rho(\Omega, \mathbb{R})$.

### 3 One-dimensional case on Lebesgue spaces

In the following sections we assume that:

- (H1) $\Omega \subseteq \mathbb{R}$ open, $h \in C(\Omega)$;
- (H2) $F \in C^1(\Omega)$ and $\varphi$ is the semiflow associated with $F$; i.e. for every $x_0 \in \Omega$, $\varphi(\cdot, x_0) : J(x_0) \to \mathbb{R}$ is the unique solution of the initial value problem

$$\dot{x} = F(x), \quad x(0) = x_0$$

where the $J(x_0) \subseteq \mathbb{R}$ is the maximal domain of $\varphi(\cdot, x_0)$. It is known that $J(x_0)$ is an open interval such that $0 \in J(x_0)$;

- (H3) $[0, \infty] \subseteq J(x_0)$ for every $x_0 \in \Omega$. 

(H4) $\rho : \Omega \to ]0, \infty[ $ is a continuous $p$-admissible function.

We refer to the monograph of Amann [1] for further results on the topic of flows. Let us define the following subsets of $\Omega$:

$$
\Omega_0 := \{ x \in \Omega \mid F(x) = 0 \}, \quad \Omega_1 = \Omega \setminus \Omega_0.
$$

To emphasize the dependence on $F$ and $h$, we will denote the semigroup defined in (1.2) by $T_{F,h} = (T_{F,h}(t))_{t \geq 0}$.

We recall the following results; the proof of the first part can be found in [14, Lemma 7], while the second part can be found in [3, Corollary 12].

**Lemma 3.1** Assume that $\Omega \subseteq \mathbb{R}$, $F, h, \rho$ satisfy (H1)–(H4).

1. Let $[a, b] \subseteq \Omega_1$ and set $\alpha := a, \beta := b$ if $F_{|[a,b]} > 0$, respectively $\alpha := b, \beta := a$ if $F_{|[a,b]} < 0$. There is a constant $C > 0$ such that

$$
\forall x \in [a, b] : \quad \frac{1}{C} \leq \rho(x) \leq C
$$

and

$$
\forall t \geq 0, x \in [a, b] : \quad \frac{1}{C} \rho(t,x) \leq \rho(t,x) \leq C \rho(t,x).
$$

2. For all $t \geq 0$ and $x \in \Omega$,

$$
\rho_{t,p}(x) = \chi_{\varphi(t,\Omega)}(x) \exp \left( p \int_{-t}^{0} \left[ h(\varphi(s,x)) - \frac{1}{p} F'(\varphi(s,x)) \right] ds \right) \rho(\varphi(-t,x))
$$

$$
= \begin{cases} 
\exp \left( pt[h(x) - \frac{1}{p} F'(x)] \right) \rho(x), & x \in \Omega_0, \\
\chi_{\varphi(t,\Omega)}(x) \exp \left( p \int_{\varphi(-t,x)}^{x} \frac{h(y) - \frac{1}{p} F'(y)}{F(y)} dy \right) \rho(\varphi(-t,x)), & x \in \Omega_1
\end{cases}
$$

**Remark 3.2** Observe that, if $\lambda(\Omega_0) > 0$, then

$$
\rho_{t,p}(x) = e^{pt h(x)} \rho(x), \quad \lambda - \text{a.e.} \ x \in \Omega_0.
$$

Indeed, by [11, Lemma 7.7], $F' = 0$ in $\Omega_0$.

A first consequence of Theorem 2.2 is a characterization of stability for the $C_0$-semigroup $T_{F,h}$ on $L^p(\rho)$.

**Theorem 3.3** Assume that $\Omega \subseteq \mathbb{R}$, $F, h, \rho$ satisfy (H1)–(H4). Then the following conditions are equivalent:

1. $T_{F,h}$ is stable on $L^p(\rho)$,
2. It holds:
   1. $T_{F,h}$ is bounded;
   2. $\lim_{t \to \infty} \rho_{t,p}(x) = 0$ for all $x \in \Omega_1$
   3. if $\lambda(\Omega_0) > 0$, $h(x) < 0$ $\lambda$-a.e. in $\Omega_0$. 

Proof First observe that if we define

$$X_i = \{ f \in L_p^p(\Omega) \mid f = 0 \text{ a.e. in } \Omega_i \}, \quad i = 0, 1,$$

clearly we can identify $X_i$ with $L_p^p(\Omega_i)$ and

$$L_p^p(\Omega) = X_0 \oplus X_1 = L_p^p(\Omega_0) \oplus L_p^p(\Omega_1).$$

If $\lambda(\Omega_0) = 0$, then $X_0$ reduces to $\{0\}$ and $L_p^p(\Omega)$ can be identified with $L_p^p(\Omega_1)$.

For every $x_0 \in \Omega_0$ we have that $\varphi(t, x_0) = x_0$ for every $t \geq 0$. By the uniqueness of the solutions of the initial value problems

$$\dot{x} = F(x), \quad x(0) = x_0 (x_0 \in \Omega),$$

$\varphi(t, \Omega_0) \subseteq \Omega_0$ and $\varphi(t, \Omega_1) \subseteq \Omega_1$. This implies that $L_p^p(\Omega_i)$ is invariant under $T_{F,h}$ for $i = 0, 1$. Thus we can define $T_{F,h}^i = T_{F,h|L_p^p(\Omega_i)}$ and we can write

$$T_{F,h} = T_{F,h}^0 \oplus T_{F,h}^1.$$

Clearly $T_{F,h}$ is stable on $L_p^p(\Omega)$ if and only if $T_{F,h}^i, i = 0, 1$, are stable on $L_p^p(\Omega_i)$.

Observe moreover that $(T_{F,h}^i(f))(x) = \exp(t h(x)) f(x)$ for every $f \in L_p^p(\Omega_0)$ and $x \in \Omega_0$.

(i) $\Rightarrow$ (ii) Let $x \in \Omega_1$. If $F(x) > 0$ there exists $r > 0$ such that $[x, x + r] \subseteq \Omega_1$ with $F(s) > 0$ for $s \in [x, x + r]$. By Lemma 3.1, there exists $C > 0$ such that

$$\rho_t(p) \leq C \rho_t(p) \quad s \in [x, x + r],$$

hence

$$\rho_t(p)(x) = \frac{1}{r} \int_x^{x+r} \rho_t(p)(x) ds \leq C \int_x^{x+r} \rho_t(p)(s) ds,$$

and therefore, by the assumptions and Theorem 2.2, $\lim_{t \to \infty} \rho_t(p)(x) = 0$. If $F(x) < 0$, we consider an interval $[x - r, x]$ with $F(s) < 0$ for $s \in [x - r, x]$ and we get the assertion again by Lemma 3.1 arguing as in the case $F(x) > 0$.

By the stability of $T_{F,h}^0$, we get that for every $f \in L_p^p(\Omega_0)$

$$\lim_{t \to \infty} \int_{\Omega_0} e^{p t h(x)} |f(x)|^p \rho(x) dx = 0$$

hence either $\lambda(\Omega_0) = 0$ or, if $\lambda(\Omega_0) > 0$, we have $h(x) < 0$ $\lambda$-a.e. in $\Omega_0$.

(ii) $\Leftarrow$ (i) By (iia) and (2.6), there exists $C > 0$ such that $\rho_t(p) \leq C \rho$ in $\Omega$ for every $t \geq 0$. Then, being $\rho$ integrable on compact subsets of $\Omega$, we can apply the dominated convergence theorem to get that for any compact set $K \subseteq \Omega$:

$$\lim_{t \to \infty} \int_{K} \rho_t(p)(s) ds = 0.$$

$\square$

Example 3.4 (Left translation semigroup) Let $\Omega = \mathbb{R}$, $F = 1$, $h = 0$, $\rho : \mathbb{R} \to [0, \infty[$ a continuous function. It is easily seen that $\rho$ is p-admissible for $F$ and $h$ for some $p \in [1, \infty)$ if the same holds for every $p \in [1, \infty)$ and the $C_0$-semigroup $T_{F,h}$ on $L_p^p(\mathbb{R})$ is defined by $(T_{F,h}(t)f)(x) = f(x + t)$. Moreover, we have $\Omega_0 = \emptyset$ and $\rho_t(p)(x) = \rho(x - t)$.
By Theorem 3.3 and Remark 2.3, $T_{F,h}$ is stable on $L^p_\rho(\mathbb{R})$ if and only if there exists a real finite constant $C \geq 0$ such that
\[
\rho(x - t) \leq C\rho(x), \quad \forall \, t \geq 0, \, \forall \, x \in \mathbb{R}
\]
and $\lim_{x \to -\infty} \rho(x) = 0$. This condition is independent of $p$.

**Example 3.5** Let $\Omega = \mathbb{R}$, $F(x) := 1 - x$, $h(x) = 0$. In this case $h_\ell(x) = 1$, $\varphi(t, x) = 1 + (x - 1)e^{-t}$, $\partial_2 \varphi(t, x) = e^{-t}$, and $T_{F,h}$ is given by $(T_{F,h}(t)f)(x) = f(1 + (x - 1)e^{-t})$. Furthermore, $\omega_0 = \{1\}$, thus $\lambda(\omega_0) = 0$, and
\[
\forall \, t \geq 0 \, \forall \, x \in \Omega_1 = \mathbb{R}\setminus\{-1\} \quad \rho_{\ell, p}(x) = \rho(1 + (x - 1)e^t)e^t,
\]
since $\varphi(t, \mathbb{R}) = \mathbb{R}$. By Theorem 3.3, $T_{F,h}$ is stable on $L^p_\rho(\mathbb{R})$ if and only if there exists a real finite constant $C \geq 0$ such that
\[
\rho(1 + (x - 1)e^t)e^t \leq C\rho(x), \quad \forall \, t \geq 0, \, \forall \, x \in \mathbb{R}\setminus\{-1\}
\]
and $\lim_{|r| \to \infty} \rho(r)r = 0$.

We can simplify the assumptions of Theorem 3.3, if we have more information about the flow $\varphi$.

**Corollary 3.6** Assume that $\Omega \subseteq \mathbb{R}$, $F$, $h$, $\rho$ satisfy (H1)–(H4) and that $\forall \, x \in \Omega_1 \exists \, \ell > 0 \quad x \not\in \varphi(\ell, \Omega)$.

where $\varphi$ is the semiflow associated with $F$. Then the following conditions are equivalent:

(i) $T_{F,h}$ is stable on $L^p_\rho(\Omega)$,

(ii) It holds:

\( iia \) $T_{F,h}$ is bounded,

\( iib \) if $\lambda(\omega_0) > 0$, $h(x) < 0$ $\lambda$-a.e. in $\omega_0$.

**Proof** Simply observe that the assumption implies that $x \not\in \varphi(t, \Omega)$ for every $t > \ell$ and for every $x \in \omega_1$, and therefore $\lim_{t \to \infty} \rho_{t,p}(x) = 0$.

We will apply this corollary to the von Foerster–Lasota semigroup in Sect. 4.

If $\rho = 1$, a straightforward consequence of Lemma 3.1(2) gives the following characterization.

**Theorem 3.7** Assume that $\Omega \subseteq \mathbb{R}$, $F$, $h$ and $\rho = 1$ satisfy (H1)–(H4). Assume that $F(x) \neq 0$ for every $x \in \Omega$.

(1) If $\varphi(t, \Omega) = \Omega$ for every $t > 0$, then the following conditions are equivalent:

(i) $T_{F,h}$ is stable on $L^p(\Omega)$;

(ii) It holds:

\( iia \) there exists $C \in \mathbb{R}$ such that for all $t \geq 0$ and for all $y \in \Omega$

\[
\int_{\gamma}^{\varphi(t,y)} \frac{h(s) - \frac{1}{p} F'(s)}{F(s)} ds \leq C,
\]
or, equivalently, $T_{F,h}$ is bounded;
(ii) for every \( y \in \Omega \)
\[
\lim_{t \to \infty} \int_{y}^{\psi(t,y)} \frac{h(s) - \frac{1}{p} F'(s)}{F(s)} \, ds = -\infty.
\]

(2) If
\[
\forall x \in \Omega_1 \exists \tilde{t} > 0 \quad x \notin \varphi(\tilde{t}, \Omega),
\]
then the following conditions are equivalent:

(i) \( T_{F, h} \) is stable on \( L^p(\Omega) \);

(ii) there exists \( C \in \mathbb{R} \) such that for all \( t \geq 0 \) and for all \( y \in \Omega \)
\[
\int_{y}^{\psi(t,y)} \frac{h(s) - \frac{1}{p} F'(s)}{F(s)} \, ds \leq C
\]

or, equivalently, \( T_{F, h} \) is bounded.

4 Stability on Sobolev spaces

Throughout this section, let \( I = ]a, b[ \) be a bounded open interval of \( \mathbb{R} \). For \( 1 \leq p \leq \infty \) we set as usual
\[
W^{1, p}(I) = \{ u \in L^p(I); u' \in L^p(I) \},
\]
where \( u' \) denotes the distributional derivative of \( u \). Endowed with the norm
\[
\| u \|_{1, p} = \| u \|_p + \| u' \|_p,
\]
\( W^{1, p}(I) \) is a Banach space. It holds that \( W^{1, p}(I) \subseteq C([a, b]) \) and that for any \( x \in [a, b] \) the point evaluation \( \delta_x \) in \( x \) is a continuous linear form on \( W^{1, p}(I) \). We are interested also in the following closed subspace of \( W^{1, p}(I) \),
\[
W^{1, p}_s(I) := \ker \delta_a.
\]

Being \( I \) a bounded interval, we have the topological direct sum
\[
W^{1, p}(I) = W^{1, p}_s(I) \oplus \text{span } \{ 1 \},
\]
where \( 1 \) denotes the constant function with value 1.

Let \( F : [a, b] \to \mathbb{R} \) a \( C^1 \)-function satisfying (H2)–(H3) with \( F(a) = 0, h \in C([a, b]) \) and consider the restriction \( S_{F, h} \) of \( T_{F, h} \) to \( W^{1, p}(I) \), for every \( 1 \leq p < \infty \). Then \( S_{F, h} \) is a \( C_0 \)-semigroup on \( W^{1, p}(I) \) and \( W^{1, p}_s(I) \) is invariant for \( S_{F, h} \) (see [3, Proposition 23]).

We recall that two \( C_0 \)-semigroups \( \overline{T} \) and \( S \), on Banach spaces \( Y \) and \( X \) respectively, are said to be similar if there exists an isomorphism \( \phi \) from \( Y \) onto \( X \) such that \( \phi \circ T(t) = S(t) \circ \phi \), for every \( t \geq 0 \) (see e.g. [10, Chapter II. 5.10]). It is immediate that stability is invariant under similarity.

**Theorem 4.1** Let \( I = ]a, b[ \) be a bounded interval, \( 1 \leq p < \infty \), \( F \in C^1([a, b]) \) satisfying (H2), (H3) with \( F(a) = 0 \) and \( F(x) \neq 0 \) in \( ]a, b[ \) and \( h \in W^{1, \infty}(I) \). Assume that the function \( [a, b[ \to \mathbb{R}, y \mapsto \frac{h(y) - h(a)}{F(y)} \) belongs to \( L^\infty(I) \). Then:

(1) If \( \psi(t, I) = 1 \) for every \( t > 0 \), then the following conditions are equivalent:
Moreover, in order to simplify the notation, throughout this section we will denote

5.1 The linear von Foerster–Lasota equation

\[ S \]  

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(i) \( S_{F,h} \) is stable on \( W^{1,p}_* (I) \);  
(ii) It holds:  

(iia) there exists \( C \in \mathbb{R} \) such that for all \( t \geq 0 \) and for all \( y \in I \)

\[
\int_y^{\varphi(t,y)} h(a) - \left( \frac{1}{p} - 1 \right) F'(s) ds \leq C
\]

(iiib) for every \( y \in I \)

\[
\lim_{t \to \infty} \int_y^{\varphi(t,y)} h(a) - \left( \frac{1}{p} - 1 \right) F'(s) ds = -\infty.
\]

(2) If

\[
\forall x \in I \exists \tilde{t} > 0 \quad x \notin \varphi(\tilde{t}, I)
\]

then the following conditions are equivalent:

(i) \( S_{F,h} \) is stable on \( W^{1,p}_* (I) \);  
(ii) there exists \( C \in \mathbb{R} \) such that for all \( t \geq 0 \) and for all \( y \in I \)

\[
\int_y^{\varphi(t,y)} h(a) - \left( \frac{1}{p} - 1 \right) F'(s) ds \leq C.
\]

Moreover, \( S_{F,h} \) is stable on \( W^{1,p}_* (I) \) if and only if it is stable on \( W^{1,p}_* (I) \) and \( h(a) < 0 \).

**Proof** By the discussion in [3, Section 3], the \( C_0 \)-semigroup \( S_{F,h} \) on \( W^{1,p}_* (I) \) is similar to \( T_{F,F' + h(a)} \) on \( L^p(I) \). Thus, \( S_{F,h} \) is stable on \( W^{1,p}_* (I) \) if and only if \( T_{F,F' + h(a)} \) is stable on \( L^p(I) \).

A short calculation shows that \( T_{F,F' + h(a)} \) is stable on \( L^p(I) \) if and only if condition (ii) holds. In fact, for \( T_{F,F' + h(a)} \) observe that \( h_1(x) = e^{t h(a)} \partial_2 \varphi(t, x) \) and \( \rho(x) = 1 \) is \( p \)-admissible with respect to \( F \) and \( F' + h(a) \) taking, for example, \( M = 1 \) and \( \omega = ph(a) + (p - 1)\|F'\|_\infty \). Depending on the behavior of \( \varphi(t, \Omega) \) for \( t \geq 0 \), the proof on \( W^{1,p}_* (I) \) follows if we replace \( h \) by \( F' + h(a) \) in the conditions of Theorem 3.7.

Finally, observe that

\[
S_{F,h(a)} = S_{F,h(a)}|_{W^{1,p}_* (I)} \oplus S_{F,h(a)}|_{\text{span}\{1\}}
\]

Thus \( S_{F,h(a)} \) is stable if and only if \( S_{F,h(a)} \) is stable on \( W^{1,p}_* (I) \) and on span \( \{1\} \), i.e., \( h(a) < 0 \). We get the assertion since, by the discussion in [3, Section 3], the semigroups \( S_{F,h(a)} \) and \( S_{F,h} \) are similar. \( \square \)

5 The von Foerster–Lasota equation

5.1 The linear von Foerster–Lasota equation

In order to simplify the notation, throughout this section we will denote \( L^p(0, 1) \) by \( L^p(0, 1) \) and \( W^{1,p}(0, 1) \) by \( W^{1,p}(0, 1) \).

Let \( h \in C([0, 1]) \). Consider the linear von Foerster–Lasota equation

\[
\frac{\partial}{\partial t} u(t, x) + x \frac{\partial}{\partial x} u(t, x) = h(x) u(t, x), \quad t \geq 0, \ 0 < x < 1
\]

(5.7)
with the initial condition
\[ u(0, x) = v(x), \quad 0 < x < 1, \]
where \( v \) is a given function. This equation is a particular case of the equation
\[ \frac{\partial u}{\partial t}(t, x) + c(x) \frac{\partial u}{\partial x}(t, x) = f(x, u(t, x)) \quad t \geq 0, \quad x \in [0, 1] \]
that was introduced in [18] to describe the reproduction of a population of red blood cells, mainly in connection with studies about anemia. Defining
\[ (T_h(t)v)(x) = \exp \left( \int_{-t}^{0} h(xe^{-s})ds \right) v(xe^{-t}), \quad t \geq 0, \quad x \in [0, 1], \quad (5.8) \]
the family \( T_h = (T_h(t))_{t \geq 0} \) is a \( C_0 \)-semigroup in \( L^p(0, 1) \) with \( 1 \leq p < \infty \), and \( u(t, x) = (T_h(t)v)(x), t \geq 0, x \in [0, 1] \) is the solution of the Eq. (5.7) with initial value \( v \in L^p(0, 1) \).

Clearly \( T_h = T_{F,h} \) with \( F(x) = -x \). In analogy with the previous section, we denote by \( S_h \) the restrictions of \( T_h \) to \( W^{1,p}(0, 1) \).

After the paper [17], the asymptotic behaviour of \( T_h \) has been studied in different function spaces by several authors (see e.g. [6,7,19] and the references quoted therein). We recover their results about stability by applying the discussion of Sects. 2 and 3.

**Theorem 5.1** (1) Assume that for \( h \in C([0, 1]) \) the function
\[ ]0, 1[\rightarrow \mathbb{R}, \quad x \mapsto \frac{h(x) - h(0)}{x} \]
belongs to \( L^1(0, 1) \). Then the following properties of the associated von Foerster–Lasota semigroup \( T_h \) on \( L^p(0, 1) \) are equivalent.

(i) \( T_h \) is stable on \( L^p(0, 1) \).

(ii) \( h(0) \leq -\frac{1}{p} \).

(2) Assume that for \( h \in W^{1,\infty}(0, 1) \) the function
\[ ]0, 1[\rightarrow \mathbb{R}, \quad x \mapsto \frac{h(x) - h(0)}{x} \]
belongs to \( L^\infty(0, 1) \). Then, for the von Foerster–Lasota semigroup \( S_h \), t.f.a.e.:

(i) \( S_{F,h} \) is stable on \( W^{1,p}_*(0, 1) \);

(ii) \( h(0) \leq 1 - \frac{1}{p} \).

Moreover, \( S_h \) is stable on \( W^{1,p}(0, 1) \) if and only if \( h(0) < 0 \).

**Proof** First observe that, for \( F(x) = -x \), we have \( \varphi(t, x) = xe^{-t} \) thus, for every \( x \in (0, 1) \) we get that \( x \notin \varphi(t, \Omega) \) if \( t > -\log x \).

Proof of part (1).

For every \( x \in [0, 1] \) and for all \( t \geq 0 \),
\[ \int_{x}^{xe^{-t}} \frac{h(y) + \frac{1}{p}}{-y} dy = \int_{xe^{-t}}^{x} \frac{h(y) - h(0) + h(0) + \frac{1}{p}}{y} dy. \]

Since
\[ ]0, 1[\rightarrow \mathbb{R}, \quad x \mapsto \frac{h(x) - h(0)}{x} \]
belongs to $L^1(0, 1)$, we obtain for some constant $K \geq 0$ that
\[
\int_{xe^{-t}}^{x} \frac{h(y) - h(0)+h(0)+\frac{1}{p}dy}{y} \leq K + \int_{xe^{-t}}^{x} \frac{h(0)+\frac{1}{p}dy}{y}.
\]

By Theorem 3.7(2), $T_h$ is stable if and only if there exists $C \in \mathbb{R}$ such that for $\lambda$-a.e. $x \in I$ and for all $t \geq 0$
\[
\int_{xe^{-t}}^{x} \frac{h(0)+\frac{1}{p}dy}{y} \leq C.
\]

Observe that
\[
\int_{xe^{-t}}^{x} \frac{h(0)+\frac{1}{p}dy}{y} = \left(h(0)+\frac{1}{p}\right)t.
\]

Then $T_h$ is stable if and only if $h(0) \leq -\frac{1}{p}$.

Proof of part (2).

It follows with straightforward calculations from Theorem 4.1(2). \(\square\)

### 5.2 Generalized von Foerster–Lasota equation

Let $h \in C([0, 1])$, $r > 1$ and consider the first order partial differential equation
\[
\frac{\partial}{\partial t} u(t, x) + x^r \frac{\partial}{\partial x} u(t, x) = h(x) u(t, x), \quad t \geq 0, \quad 0 < x < 1
\]
with the initial condition
\[
u(0, x) = v(x), \quad 0 < x < 1,
\]
where $v$ is a given function. Denote by $T_{r,h}$ the $C_0$-semigroup on $L^p(0, 1)$ associated with $F(x) = -x^r$ and $h$ and by $S_{r,h}$ its restriction to $W^{1,p}(0, 1)$.

**Theorem 5.2** (1) Let $h \in C([0, 1])$ and assume that the function
\[
]0, 1[ \rightarrow \mathbb{R}, \quad x \mapsto \frac{h(x) - x^{r-1}h(0)}{x^r}
\]
belongs to $L^1(0, 1)$. Then the following properties of $T_{r,h}$ on $L^p(0, 1)$ are equivalent.

(i) $T_{r,h}$ is stable.
(ii) $h(0) \leq -\frac{r}{p}$.

(2) Let $h \in W^{1,\infty}(0, 1)$ and assume that the function
\[
]0, 1[ \rightarrow \mathbb{R}, \quad x \mapsto \frac{h(x) - h(0)}{x^r}
\]
belongs to $L^\infty(0, 1)$. Then the following are equivalent.

(i) $S_{r,h}$ is stable on $W^{1,p}_*(0, 1)$.
(ii) $h(0) \leq 0$.

Moreover, $S_h$ is stable on $W^{1,p}(0, 1)$ if and only if $h(0) < 0$. 

Proof. Recall that $F(x) = -x^r$ thus, $\varphi(t, x) = ((r - 1)t + x^{1-r})^{\frac{1}{r-1}}$ and

$$\forall \ t \geq 0, \ x \in [0, 1[ \quad \partial_2 \varphi(t, x) = x^{-r}((r - 1)t + x^{1-r})^{\frac{1}{r-1}}$$

For all $x \in ]0, 1[$, choosing $\overline{t} > \frac{x^{1-r} - 1}{r-1}$ we have $x \not\in \varphi(\overline{t}, ]0, 1[)$. By applying Theorem 3.7(2) and Theorem 4.1(2) we get the proofs of (1) and (2) respectively. In fact, for the proof of part (1) since the function

$$]0, 1[ \rightarrow \mathbb{R}, \quad x \mapsto \frac{h(x) - x^{r-1}h(0)}{x^r}$$

belongs to $L^1(0, 1)$, if we denote by $K$ its norm, we obtain for $y \in ]0, 1[$ and $t \geq 0$

$$\int_y^{\varphi(t, y)} \frac{h(s) - \frac{1}{p} F'(s)}{F(s)} \, ds = \int_y^{\varphi(t, y)} \frac{h(s) + \frac{r}{p} s^{r-1}}{s^r} \, ds$$

$$= \int_y^{\varphi(t, y)} \frac{h(s) + s^{r-1}h(0)}{s^r} \, ds + \int_y^{\varphi(t, y)} \frac{h(0) + \frac{r}{p}}{s} \, ds$$

$$\leq K + \int_y^{\varphi(t, y)} \frac{h(0) + \frac{r}{p}}{s} \, ds$$

$$= K + \left( h(0) + \frac{r}{p} \right) \log \left( \frac{y}{\varphi(t, y)} \right)$$

$$= K + \frac{1}{r-1} \left( h(0) + \frac{r}{p} \right) \log \left( 1 + (r-1)ty^{r-1} \right)$$

Then $T_{r, h}$ is stable on $L^p(0, 1)$ if and only if $\left( h(0) + \frac{r}{p} \right) \leq 0$, by Theorem 3.7(2).

Concerning part (2), by using Theorem 4.1(2) and by observing that, since $p \geq 1$, $r > 1$, and $\frac{y}{\varphi(t, y)} = (1 + (r-1)ty^{r-1})^{\frac{1}{r-1}}$, we have for some $C > 0$

$$\left( \frac{1}{p} - 1 \right) r \log \left( \frac{y}{\varphi(t, y)} \right) \leq 0,$$

we get that

$$\int_y^{\varphi(t, y)} \frac{h(0) - \left( \frac{1}{p} - 1 \right) F'(s)}{F(s)} \, ds \leq - \frac{h(0)}{r-1} \left( y^{1-r} - \varphi(t, y)^{1-r} \right) + C$$

$$= h(0)t + C.$$

Finally, $S_{r, h}$ is stable on $W^{1,p}_1(0, 1)$ if and only if $h(0) \leq 0$, by Theorem 4.1(2). Of course, $S_{r, h}$ is stable on $W^{1,p}_1(0, 1)$ if and only if $h(0) < 0$ by the same Theorem. \hfill $\Box$

### 6 Stability and hypercyclicity

We would like now to compare the previous stability conditions with hypercyclicity conditions. We recall that a $C_0$-semigroup $T = (T(t))_{t \geq 0}$ on a separable Banach space $X$, is said to be hypercyclic if there exists $x \in X$, called hypercyclic vector, such that its orbit $\{T(t)x \mid t \geq 0\}$ is dense in $X$. We refer to the monographs [4,12] and to the review paper.
The hypercyclic behaviour of the weighted composition semigroups $T_{F,h}$ has been characterized by Kalmes in [13, Theorem 9] (see also [15,16] for further hypercyclicity related conditions on the same semigroups):

**Theorem 6.1** Assume that $\Omega \subseteq \mathbb{R}$, $F$, $h$, $\rho$ satisfy (H1)–(H4). For the $C_0$-semigroup $T_{F,h}$ on $L^p_\rho(\Omega)$ the following are equivalent.

i) $T_{F,h}$ is hypercyclic.

ii) $\lambda(\Omega_0) = 0$ and for every $m \in \mathbb{N}$ for which there are $m$ different connected components $C_1, \ldots, C_m$ of $\Omega_1$, for $\lambda^m$-almost all choices of $(x_1, \ldots, x_m) \in \prod_{j=1}^m C_j$ there is a sequence of positive numbers $(t_n)_{n \in \mathbb{N}}$ tending to infinity such that

$$\forall 1 \leq j \leq m : \lim_{n \to \infty} \rho_{t_n,p}(x_j) = \lim_{n \to \infty} \rho_{t_n-p}(x_j) = 0,$$

where $\rho_{t-p} : \Omega \to [0, \infty)$, $\rho_{t-p}(x) := h_t(x) - \rho(\varphi(t, x)) \partial_\varphi(t, x)$.

Clearly if a semigroup is hypercyclic, then it cannot be stable and, in general, a semigroup can be not stable and not hypercyclic. Indeed, if we consider a semigroup $T_{F,h}$ such that $\lambda(\Omega_0) > 0$ and $h(x) > 0$ on $\Omega_0$ then we get a semigroup which is not stable by Theorem 3.3 and not hypercyclic by Theorem 6.1. Nevertheless, there are cases in which stability and not hypercyclicity are equivalent. This happens, under suitable assumptions, e.g. for the von Foerster–Lasota semigroup, as it has been proved by Dawidowicz and Poskrobko in [9].

This result will be covered by the next theorem, in the particular case $F(x) = -x$.

**Theorem 6.2** Consider a bounded interval $]a, b[ \subseteq \mathbb{R}$, $F \in C^1([a, b])$ satisfying (H2)–(H3). Assume $F$ decreasing, $F(x) < 0$ for each $x \in ]a, b]$, $F(a) = 0$ and such that

$$\forall x \in ]a, b[ \exists t > 0 \quad x \notin \varphi(t, ]a, b[).$$

Let $h = -\lambda F'$ for some $\lambda \in \mathbb{R}$.

Then, for the $C_0$-semigroup $T_{F,h}$ on $L^p(a, b)$ the following are equivalent.

(i) $T_{F,h}$ is stable,

(ii) $T_{F,h}$ is not hypercyclic,

(iii) $\lambda \leq -\frac{1}{p}$.

**Proof** Observe that $\rho = 1$ is $p$-admissible for $F$ and $h$ by [3, Lemma 19]. If $\lambda = -\frac{1}{p}$, then clearly the semigroup is stable and not hypercyclic. Then assume that $\lambda \neq -\frac{1}{p}$.

First observe that for every $y, z \in ]a, b[$

$$\int_y^z \frac{h(s) - \frac{1}{p} F'(s)}{F(s)} ds = -\left(\lambda + \frac{1}{p}\right) \log \frac{F(z)}{F(y)}.$$

In particular, it follows that for every $z \in ]a, b[$

$$\exists \lim_{s \to a} \int_y^z \frac{h(s) - \frac{1}{p} F'(s)}{F(s)} ds = -\left(\lambda + \frac{1}{p}\right) (+\infty)$$

thus $T_{F,h}$ is hypercyclic if and only if $\lambda > -\frac{1}{p}$.

On the other hand, $T_{F,h}$ is stable if and only if there exists $C > 0$ such that

$$\forall x \in ]a, b[ \forall t > 0 : \int_x^{\varphi(t,x)} \frac{h(s) - \frac{1}{p} F'(s)}{F(s)} ds \leq C,$$
that is
\[ \forall x \in ]a, b[ \ \forall t > 0 : \ - \left( \lambda + \frac{1}{p} \right) \log \frac{F(\varphi(t, x))}{F(x)} \leq C. \]

Being \( F < 0 \) in \( ]a, b[ \), we have that \( \varphi(\cdot, x) \) is strictly decreasing, hence \( \varphi(t, x) < x \) for every \( t > 0 \). Then, since \( F \) is decreasing,
\[ \forall x \in ]a, b[ \ \forall t > 0 \quad \frac{F(\varphi(t, x))}{F(x)} \leq 1. \]
Thus, if \( \lambda < -\frac{1}{p} \), then
\[ \forall x \in ]a, b[ \ \forall t > 0 \quad - \left( \lambda + \frac{1}{p} \right) \log \frac{F(\varphi(t, x))}{F(x)} \leq 0, \]
and so \( T_{F,h} \) is stable. \( \square \)

**Remark 6.3** Of course it is possible to characterize stability for \( T_{F,h} \) if \( F \) is strictly positive, taking into account that \( \varphi \) would be increasing. In this case, we only need to change condition (iii) by \( \lambda \geq -\frac{1}{p} \).

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