An Improved FPT Algorithm for the Flip Distance Problem∗†

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Abstract

Given a set \( P \) of points in the Euclidean plane and two triangulations of \( P \), the flip distance between these two triangulations is the minimum number of flips required to transform one triangulation into the other. Parameterized Flip Distance problem is to decide if the flip distance between two given triangulations is equal to a given integer \( k \). The previous best FPT algorithm runs in time \( O^*(k \cdot c^k) \) (\( c \leq 2 \times 14^{11} \)), where each step has fourteen possible choices, and the length of the action sequence is bounded by \( 11k \). By applying the backtracking strategy and analyzing the underlying property of the flip sequence, each step of our algorithm has only five possible choices. Based on an auxiliary graph \( G \), we prove that the length of the action sequence for our algorithm is bounded by \( 2 |G| \). As a result, we present an FPT algorithm running in time \( O^*(k \cdot 32^k) \).

1 Introduction

Given a set \( P \) of \( n \) points in the Euclidean plane, a triangulation of \( P \) is a maximal planar subdivision whose vertex set is \( P \). A flip operation to one diagonal \( e \) of a convex quadrilateral in a triangulation is to remove \( e \) and insert the other diagonal into this quadrilateral. Note that if the quadrilateral associated with \( e \) is not convex, the flip operation is not allowed. The flip distance between two triangulations is the minimum number of flips required to transform one triangulation into the other.

Triangulations play an important role in computational geometry, which are applied in areas such as computer-aided geometric design and numerical analysis [7, 8, 16].

Given a point set \( P \) in the Euclidean plane, we can construct a graph \( G_T(P) \) in which every triangulation of \( P \) is represented by a vertex, and two vertices are adjacent if their corresponding triangulations can be transformed into each other through one flip operation. \( G_T(P) \) is called the triangulations graph of \( P \). Properties of the triangulations graph are studied in the literature. Aichholzer et al. [1] showed that the lower bound of the number of vertices of \( G_T(P) \) is \( \Omega(2^{0.33n}) \). Lawson and Charles [12] showed that the diameter of \( G_T(P) \) is \( O(n^2) \). Hurtado et al. [9] proved that the bound is tight. Since \( G_T(P) \) is connected [12], any two triangulations of \( P \) can be transformed into each other through a certain number of flips.

Flip Distance problem consists in computing the flip distance between two triangulations of \( P \), which was proved to be NP-complete by Lubiw and Pathak [13]. Pilz showed that the Flip Distance is APX-hard [15]. Aichholzer et al. [2] proved that Parameterized Flip Distance is NP-complete on triangulations of simple polygons. However, the complexity of Flip Distance on triangulations of convex polygons has been open for many years, which is equivalent to the problem of computing the rotation distance between two rooted binary trees [17].

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PARAMETERIZED FLIP DISTANCE problem is: given two triangulations of a set of points in the plane and an integer \( k \), deciding if the flip distance between these two triangulations is equal to \( k \). For PARAMETERIZED FLIP DISTANCE on triangulations of a convex polygon, Lucas \[14\] gave a kernel of size \( 2k \) and an \( O^*(k^6) \)-time algorithm. Kanj and Xia \[11\] studied PARAMETERIZED FLIP DISTANCE on triangulations of a set of points in the plane, and presented an \( O^*(k \cdot c^k) \)-time algorithm \((c \leq 2 \cdot 14^{11})\), which applies to triangulations of general polygonal regions (even with holes or points inside it).

In this paper, we exploit PARAMETERIZED FLIP DISTANCE further. At first, we give a nondeterministic construction process to illustrate our idea. The nondeterministic construction process contains only two types of actions, which are the moving action as well as the flipping and backing action. Given two triangulations and a parameter \( k \), we prove that either there exists a sequence of actions of length at most \( 2k \), following which we can transform one triangulation into the other, or we can conclude that no valid sequence of length \( k \) exists. Thus we get an improved \( O^*(k \cdot 32^k) \)-time FPT algorithm, which also applies to triangulations of general polygonal regions (even with holes or points inside it).

2 Preliminaries

In a triangulation \( T \), a flip operation \( f \) to an edge \( e \) that is the diagonal of a convex quadrilateral \( Q \) is to delete \( e \) and insert the other diagonal \( e' \) into \( Q \). We define \( e \) as the underlying edge of \( f \), denoted by \( \varepsilon(f) \), and \( e' \) as the resulting edge of \( f \), denoted by \( \varphi(f) \). (For consistency and clarity, we continue to use some symbols and definitions from \[11\]). Note that if \( e \) is not a diagonal of any convex quadrilateral in the triangulation, flipping \( e \) is not allowed. Suppose that we perform a flip operation \( f \) on a triangulation \( T_1 \) and get a new triangulation \( T_2 \). We say \( f \) transforms \( T_1 \) into \( T_2 \). \( T_1 \) is called an underlying triangulation of \( f \), and \( T_2 \) is called a resulting triangulation of \( f \). Given a set \( P \) of \( n \) points in the Euclidean plane, let \( T_{start} \) and \( T_{end} \) be two triangulations of \( P \), in which \( T_{start} \) is the initial triangulation and \( T_{end} \) is the objective triangulation. Let \( F = \langle f_1, f_2, \ldots, f_r \rangle \) be a sequence of flips, and \( \langle T_0, T_1, \ldots, T_r \rangle \) be a sequence of triangulations of \( P \) in which \( T_0 = T_{start} \) and \( T_r = T_{end} \). If \( T_{i-1} \) is an underlying triangulation of \( f_i \), and \( T_i \) is a resulting triangulation of \( f_i \) for each \( i = 1, 2, \ldots, r \), we say \( F \) transforms \( T_{start} \) into \( T_{end} \), or \( F \) is a valid sequence, denoted by \( T_{start} \xrightarrow{F} T_{end} \). The flip distance between \( T_{start} \) and \( T_{end} \) is the length of a shortest valid flip sequence.

Now we give the formal definition of PARAMETERIZED FLIP DISTANCE problem.

**PARAMETERIZED FLIP DISTANCE**

**Input:** Two triangulations \( T_{start} \) and \( T_{end} \) of \( P \) and an integer \( k \).

**Question:** Decide if the flip distance between \( T_{start} \) and \( T_{end} \) is equal to \( k \).

The triangulation on which we are performing a flip operation is called the current triangulation. An edge \( e \) which belongs to the current triangulation but does not belong to \( T_{end} \) is called a necessary edge in the current triangulation. It is easy to see that for any necessary edge \( e \), there must exist a flip operation \( f \) in a valid sequence such that \( e = \varepsilon(f) \). Otherwise, we cannot get the objective triangulation \( T_{end} \).

For a directed graph \( D \), a maximal connected component of its underlying graph is called a weakly connected component of \( D \). We define the size of an undirected tree as the number of its vertices. A node in \( D \) is called a source node if the indegree of this node is 0.

A parameterized problem is a decision problem for which every instance is of the form \((x,k)\), where \( x \) is the input instance and \( k \in \mathbb{N} \) is the parameter. A parameterized problem is fixed-parameter tractable (FPT) if it can be solved by an algorithm (FPT algorithm) in \( O(f(k)|x|^{O(1)}) \) time, where \( f(k) \) is a computable function of \( k \). For a further introduction to parameterized algorithms, readers could refer to \[3\,\[5\].

3 The Improved Algorithm for Parameterized Flip Distance

Given \( T_{start} \) and \( T_{end} \), let \( F = \langle f_1, f_2, \ldots, f_r \rangle \) be a valid sequence, that is, \( T_{start} \xrightarrow{F} T_{end} \). Definition \[1\] defines the adjacency of two flips in \( F \).
Definition 1. Let $f_i$ and $f_j$ be two flips in $F$ ($1 \leq i < j \leq r$). We define that flip $f_j$ is adjacent to flip $f_i$, denoted by $f_i \rightarrow f_j$, if the following two conditions are satisfied:

(1) either $\varphi(f_i) = \varepsilon(f_j)$, or $\varphi(f_i)$ and $\varepsilon(f_j)$ share a triangle in triangulation $T_{j-1}$;

(2) $\varphi(f_i)$ is not flipped between $f_i$ and $f_j$, that is, there does not exist a flip $f_p$ in $F$, where $i < p < j$, such that $\varphi(f_p) = \varepsilon(f_p)$.

By Definition 1 we can construct a directed acyclic graph (DAG), denoted by $D_F$. Every node in $D_F$ represents a flip operation of $F$, and there is an arc from $f_i$ to $f_j$ if $f_j$ is adjacent to $f_i$. For convenience, we label the nodes in $D_F$ using labels of the corresponding flip operations. In other words, we can see a node in $D_F$ as a flip operation and vice versa.

The intuition of Definition 1 is that if there is an arc from $f_i$ to $f_j$, then $f_j$ cannot be flipped before $f_i$ because the quadrilateral corresponding to the flip $f_j$ is formed after $f_i$ or the underlying edge of $f_j$, namely $\varepsilon(f_j)$ is the resulting edge of $f_i$, namely $\varphi(f_i)$. The following lemma gives a stronger statement: any topological sorting of $D_F$ is a valid sequence.

Lemma 1. Let $T_0$ and $T_r$ be two triangulations and $F = \langle f_1, f_2, ..., f_r \rangle$ be a sequence of flips such that $T_0 F T_r$. Let $\pi(F)$ be a permutation of the flips in $F$ such that $\pi(F)$ is a topological sorting of $D_F$. Then $\pi(F)$ is a valid sequence of flips such that $T_0 \pi(F) T_r$.

Lemma 1 ensures that if we repeatedly remove a source node from $D_F$ and flip the underlying edge of this node until $D_F$ becomes empty, we can get a valid sequence and the objective triangulation $T_{end}$. On the basis of Lemma 1 the essential task of our algorithm is to find an edge which is the underlying edge of a source node. Thus we introduce the definition of a walk, which describes the “track” to find such an edge.

Definition 2. A walk in a triangulation $T$ (starting from an edge $e \in T$) is a sequence of edges of $T$ beginning with $e$ in which any two consecutive edges share a triangle in $T$.

According to Lemma 1 if there is a valid sequence $F$ for the input instance, any topological sorting of $D_F$ is also a valid sequence for the given instance. The difficulty is that $F$ is unknown. In order to find the topological sorting of $D_F$, the algorithm of Kanj and Xia takes a nondeterministic walk to find an edge $e$ which is the underlying edge of a source node, flips this edge (removing the corresponding node from $D_F$), nondeterministically walks to an edge which shares a triangle with $e$ and recursively searches for an edge corresponding to a source node. Their algorithm deals with weakly connected components of $D_F$ one after another (refer to Corollary 4 in [11]), that is, the algorithm tries to find a solution $F$ in which all flips belonging to the same weakly connected component of $D_F$ appear consecutively. In order to keep this procedure within the current weakly connected component, the algorithm uses a stack to preserve the nodes (defined as connecting point in [11]) whose removal separates the current weakly connected component into small weakly connected components. When removing all nodes of a small component, their algorithm jumps to the connecting point at the top of the stack in order to find another small component.

We observe that it is not necessary to remove all nodes of a weakly connected component before dealing with other weakly connected components, that is, our algorithm may find a solution $F$ in which the nodes belonging to the same weakly connected components appear dispersedly. Thus our algorithm leaves out the stack which is used to preserve connecting points. We show that it suffices to use two types of actions (Section 3.2) instead of the five types in [11]. Moreover, every time our algorithm finds a source node, it removes the node, flips the underlying edge and backtracks to the previous edge in the walk instead of searching for the next node, thus reducing the number of choices for the actions. In a word, our algorithm traverses $D_F$ in a reverse way. However, two adjacent edges in a walk may not correspond to two adjacent vertices in $D_F$. In order to emphasize this fact and make it more convenient for the proof, we construct an auxiliary graph $G$ and prove that $G$ is a forest. Actually $G$ can be seen as the track of the reverse traversal of $D_F$. However, $G$ is not the underlying graph of $D_F$ (Fig. 3). Since there is a bijection between nondeterministic actions and nodes as well as edges of $G$, we prove that there exists a sequence of actions of length at most $2|D_F|$, which is smaller than $11|D_F|$ in [11]. In addition, we make some optimization on the strategy of finding the objective sequence. As a result, we improve the running time of the algorithm from $O^*(k \cdot c^k)$ where $c \leq 2 \cdot 1411$ to $O^*(k \cdot 32^k)$.
3.1 Nondeterministic construction process

Now we give a description of our nondeterministic construction process \textsc{NDTRV} (see Fig. 1). The construction is nondeterministic, that is, we suppose it always guesses the optimal choice correctly when running. The actual deterministic algorithm enumerates all possible choices to simulate the nondeterministic actions (see Fig. 3). Readers could refer to [4] as an example of nondeterministic algorithm. We present this construction process in order to depict the idea behind our deterministic algorithm clearly and vividly.

Let \( T_{\text{start}} \) be the initial triangulation, and \( T_{\text{end}} \) be the objective triangulation. Suppose that \( F \) is a shortest valid sequence, that is, a shortest length among all valid sequences. Let \( D_F \) be the DAG constructed after \( F \) according to Definition 1. \textsc{NDTRV} traverses \( D_F \) reversely, removes the vertices of \( D_F \) in a topologically-sorted order and transforms \( T_{\text{start}} \) into \( T_{\text{end}} \). Although \( D_F \) is unknown, for further analysis, we assume that \textsc{NDTRV} can remove and copy nodes in \( D_F \) so that it can construct an auxiliary undirected graph \( G \) and a list \( L \) during the traversal. In later analysis we show that \( G \) is a forest. Moreover, there is a bijection between flipping actions of \textsc{NDTRV} and nodes of \( G \) while there is a bijection between moving actions of \textsc{NDTRV} and edges of \( G \). Obviously \( G \) and \( L \) are unknown as well. We just show that if a shortest valid sequence \( F \) exists, then \( D_F \) exists. So do \( G \) and \( L \). We can see \( D_F \) and \( G \) as conceptual or dummy graphs. We construct \( G \) instead of analysing a subgraph of \( D_F \) because one moving action (see Section 3.2) of \textsc{NDTRV} may correspond to one or more edges in \( D_F \) (see Fig. 3), while there is a one-to-one correspondence between moving actions and edges in \( G \).

At the beginning of an iteration, \textsc{NDTRV} picks a necessary edge \( e = \epsilon(f_h) \) arbitrarily and nondeterministically guesses a walk \( W \) to find the underlying edge of a source node \( f_s \). Lemma 2 shows that there exists such a walk \( W \) whose length is bounded by the length of a directed path \( B \) from \( f_s \) to \( f_h \), and every edge \( e' \) in \( W \) is the underlying edge of some flip \( f' \) on \( B \). \textsc{NDTRV} uses \( L \) to preserve a sequence of nodes \( \Gamma = (f_s = v_1, \ldots, f_h = v_l) \) on \( B \), whose underlying edges are in \( W \). Simultaneously \textsc{NDTRV} constructs a path \( S \) by copying all nodes in \( \Gamma \) as well as adding an undirected edge between the copy of \( v_i \) and \( v_{i+1} \) for \( i = 1, \ldots, l \). \( S \) is defined as a searching path. The node \( f_h \) is called a starting node. If a starting node is precisely a source node in \( D_F \), the searching path consists only of the copy of this starting node. When finding \( \epsilon(f_s) \), \textsc{NDTRV} removes \( f_s \) from \( D_F \), flips \( \epsilon(f_s) \) and moves back(backtracks) to the previous edge \( \epsilon(v_2) \) of \( \epsilon(f_s) \) in \( W \). If \( v_2 \) becomes a source node of \( D_F \), \textsc{NDTRV} removes \( v_2 \) from \( D_F \), flips \( \epsilon(v_2) \) and moves back to the previous edge \( \epsilon(v_3) \). \textsc{NDTRV} repeats the above operations until finding a node \( v_i \) in \( \Gamma \) which is not a source node in \( D_F \). Then \textsc{NDTRV} uses \( v_i \) as a new starting node, and recursively guesses a walk nondeterministically from \( \epsilon(v_i) \) to find another edge which is the underlying edge of a source node as above. \textsc{NDTRV} performs these operations until the initial starting node \( f_h \) becomes a source node in \( D_F \). Finally \textsc{NDTRV} removes \( f_h \) and flips \( \epsilon(f_h) \), terminating this iteration. If \( T_{\text{current}} \) is not equal to \( T_{\text{end}} \),
Lemma 2. Suppose that a sequence of flips algorithm. We give the formal presentation of NDTRV in Fig. 2 and an example in Fig. 1 and Fig. 3.

3.2 Actions of the construction

Our construction process contains two types of actions operating on triangulations. The edge which the algorithm is operating on is called the current edge. The current triangulation is denoted by $T_{current}$.

(i) Move to one edge that shares a triangle with the current edge in $T_{current}$. We formalize it as $(move, e_1 \mapsto e_2)$, where $e_1$ is the current edge and $e_2$ shares a triangle with $e_1$.

(ii) Flip the current edge and move back to the previous edge of the current edge in $W$. We formalize it as $(f, e_4 \mapsto e_3)$, where $f$ is the flip performed on the current edge, $e_4$ equals $\varphi(f)$ and $e_3$ is the previous edge of $\varepsilon(f)$ in the current walk $W$.

Since there are four edges that share a triangle with the current edge, there are at most four directions for an action of type (i). However, there is only one choice for an action of type (ii).

3.3 The sequence of actions

The following theorem is the main theorem for the deterministic algorithm FLIPDT, which bounds the length of the sequence of actions by $2|V(D_F)|$.

Theorem 1. There exists a sequence of actions of length at most $2|V(D_F)|$ following which we can perform a sequence of flips $F'$ of length $|V(D_F)|$, starting from a necessary edge in $T_{start}$, such that $F'$ is a topological sorting of $D_F$.

In order to prove theorem 1, we need to introduce some lemmas. We will give the proof for Theorem 1 at the end of Section 3.3.

Lemma 2. Suppose that a sequence of flips $F^-$ is performed such that every time we flip an edge, we delete the corresponding source node in the DAG resulting from preceding deleting operations. Let $f_h$ be a node in the remaining DAG such that $\varepsilon(f_h)$ is an edge in the triangulation $T$ resulting from performing the sequence of flips $F^-$. There is a source node $f_s$ in the remaining DAG satisfying:

1. There is a walk $W$ in $T$ from $\varepsilon(f_h)$ to $\varepsilon(f_s)$.
2. There is a directed path $B$ from $f_s$ to $f_h$ in the remaining DAG that we refer to as the backbone of the DAG.
3. The length of $W$ is at most that of $B$.
4. Any edge in $W$ is the underlying edge of a flip in $B$, that is, $W = (\varepsilon(v_1), ..., \varepsilon(v_t))$, where $v_1 = f_s, ..., v_t = f_h$ are nodes in $B$ and there is a directed path $B_i$ from $v_i$ to $v_{i+1}$ for $i = 1, ..., t - 1$ such that $B_i \subset B$.

Lemma 3. NDTRV transforms $T_{start}$ into $T_{end}$ with the minimum number of flips and stops in polynomial time if it correctly guesses every moving and flipping action.

Proof. Suppose that $F$ is a shortest valid sequence. According to Lemma 2, every edge flipped in NDTRV is the underlying edge of a source node in the remaining graph of $D_F$, and every node removed from the remaining graph of $D_F$ in NDTRV is a source node. If $T_{current}$ is equal to $T_{end}$ but $D_F$ is not empty, then there exists a valid sequence $F'$ which is shorter than $F$, contradicting that $F$ is a shortest valid sequence. Thus NDTRV traverses $D_F$, removes all nodes of $D_F$ in a topologically-sorted order and transforms $T_{start}$ into $T_{end}$ with the minimum number of flips by Lemma 1. Since the diameter of a transformations graph $G_T(P)$ is $O(n^2)$ [12], NDTRV stops in polynomial time. □
NDTRV(T_{start}, T_{end}; D_F)

Input: the initial triangulation T_{start} and objective triangulation T_{end}.

/*Assume that F is a shortest sequence, D_F is the corresponding DAG by Definition 4*/
/*G is an auxiliary undirected graph*/
/*L is a list keeping track of searching paths for backtracking*/
/*Q is a list preserving the sequence of nondeterministic actions*/
/*T_{current} is the current triangulation*/

a. Let V(G) and E(G) be empty sets, L and Q be empty lists;
b. T_{current} = T_{start};
c. While T_{current} ≠ T_{end} do
c.1. Pick a necessary edge e = ε(f_h) in T_{current} arbitrarily;
c.2. Add a copy of f_h to G;
c.3. Append f_h to L;
c.4. TrackTree(T_{current}, e, D_F, G, L, Q);

c. TrackTree(T_{current}, ε(f_h), D_F, G, L, Q)
   1. Nondeterministically guess a walk in T_{current} from ε(f_h) to find ε(f_s) according to Lemma 2, let Γ = ⟨f_s = v_1, ..., f_h = v_ℓ⟩, where f_s is a source node in D_F, be a sequence of nodes on the backbone B whose underlying edges are in the walk W such that ε(v_i) and ε(v_{i+1}) are consecutive in W for i = 1, ..., ℓ - 1;
   2. Add a copy of v_1,...,v_{ℓ-1} to G respectively;
   3. Connect the copies of v_1,...,v_{ℓ-1} in G into a path;
   4. Append v_{ℓ-1},...,v_1 to L;
   5. Append (move, ε(v_1) → ε(v_{ℓ-1})),...(move, ε(v_2) → ε(v_1)) to Q; /*record actions*/
   6. Remove f_s = v_1 from L;
   7. Remove f_s = v_1 from D_F;
   8. Flip ε(f_s) in T_{current} and move back to ε(v_2);
   9. Append (f_s, ϕ(v_1) → ε(v_2)) to Q; /*record actions*/
10. For i = 2 to ℓ do
10.1 Nondeterministically guess if v_i is a source node in D_F;
10.2 If v_i is a source node of D_F then /*flip and move back*/
10.2.1 Remove v_i from L;
10.2.2 Remove v_i from D_F;
10.2.3 Flip ε(v_i) in T_{current} and move back to ε(v_{i+1});
10.2.4 Append (v_i, ϕ(v_i) → ε(v_{i+1})) to Q;
10.3 Else
10.3.1 TrackTree(T_{current}, ε(v_i), D_F, G, L, Q);

Figure 2: Nondeterministic construction NDTRV
Lemma 4. The auxiliary graph $G$ constructed during NDTRV is a forest and $|V(G)| = |V(D_F)|$. Moreover, $G$ consists of a set of vertex-disjoint trees called track trees. Each track tree is created during an iteration of NDTRV.

Proof. Since NDTRV makes a topological sorting during execution and the copy of a vertex of $D_F$ is added to $G$ only when it is removed from $D_F$, it follows that $|V(G)| = |V(D_F)|$. Suppose that there is a cycle in $G$. According to Lemma 2 and NDTRV, we can find a directed cycle in $D_F$, contradicting that $D_F$ is a directed acyclic graph. Thus $G$ is a forest. From the execution of NDTRV, we get that it creates a connected subgraph in $G$ during every iteration and the subgraphs created during each iteration are vertex-disjoint. Thus the subgraph created during each iteration is a tree. This concludes the proof.

We give the proof of Theorem 1 below.

Proof. (Theorem 1) During the procedure of NDTRV (Fig. 2), it constructs a list $Q$ consisting of actions of type (i) and (ii). We claim that $Q$ is exactly the sequence satisfying the requirement of this theorem. NDTRV appends an action of type (ii) to $Q$ if and only if it adds a vertex to $G$. Meanwhile, NDTRV appends an action of type (i) to $Q$ if and only if it adds an edge to $G$. It follows that there is a one-to-one correspondence between actions of type (i) in $Q$ and $E(G)$ and there is a one-to-one correspondence between actions of type (ii) in $Q$ and $V(G)$. According to Lemma 4, $G$ is a forest. As a result, $|E(G)| \leq |V(G)|$, and the length of $Q$ is bounded by $|E(G)| + |V(G)| \leq 2|V(G)| = 2|V(D_F)|$.

3.4 The deterministic algorithm

Now we are ready to give the deterministic algorithm FLIPDT for Parameterized Flip Distance. The specific algorithm is presented in Fig. 4. As mentioned above, we assume that NDTRV is always able to guess the optimal choice correctly. In fact, FLIPDT achieves this by trying all possible sequences of actions and partitions of $k$. At the top level, FLIPDT branches into all partitions of $k$, namely $(k_1, ..., k_t)$ satisfying $k_1 + \ldots + k_t = k$ and $k_1, ..., k_t \geq 1$, in which $k_i$ ($i = 1, ..., t$) equals the size of the track tree $A_i$ constructed during the $i$-th iteration.

Suppose that FLIPDT is under some partition $(k_1, ..., k_t)$. Let $T_i^{\text{iteration}} = T_{\text{start}}$. FLIPDT permutes all necessary edges in $T_{\text{start}}$ in the lexicographical order, and the ordering is denoted by $O_{\text{lex}}$. Here we number the given points of $P$ in the Euclidean plane from 1 to $n$ arbitrarily and label one edge by a tuple consisting of two numbers of its endpoints. Thus we can order the edges lexicographically. FLIPDT performs $t$ iterations. At the beginning of the $i$-th iteration, $i = 1, ..., t$, we denote the current triangulation...
by $T_{iteration}^{i−1}$. For $i = 1, ..., t$, $T_{iteration}^i$ is also the triangulation resulting from the execution of the first $i$ iterations. At the beginning of the $i$-th iteration ($i = 1, ..., t$), FLIPDT repeatedly picks the next edge in $O_{lex}$ until finding a necessary edge $e$ belonging to $T_{iteration}^{i−1}$. Note that one edge in $O_{lex}$ may not be a necessary edge anymore with respect to $T_{iteration}^{i−1}$. Moreover, if FLIPDT reaches the end of $O_{lex}$ but does not find a necessary edge belonging to $T_{iteration}^{i−1}$, it needs to update $O_{lex}$ by clearing $O_{lex}$ and permuting all necessary edges in $T_{iteration}^{i−1}$ lexicographically, and choose the first edge in the updated ordering $O_{lex}$. Then FLIPDT branches into every possible sequence of actions $seq_i$ of length $2k_i − 1$.

Under each enumeration of $seq_i$, FLIPDT branches into every possible sequence of actions $seq_{i+1}$ of length $2k_{i+1} − 1$. FLIPDT proceeds as above. When FLIPDT finishes the last iteration, it judges if the resulting triangulation $T_{iteration}$ is equal to $T_{end}$. If they are equal, the input instance is a yes-instance. Otherwise, FLIPDT rejects this branch and proceeds.

Now we analyse how to enumerate all possible sequences of length $2k_i − 1$. According to Lemma 1 and Theorem 1 in every iteration NDTRV constructs a track tree in which a node corresponds to an action of type (ii) while an edge corresponds to an action of type (i). It follows that the number of actions of type (ii) is $k_i$, and the number of actions of type (i) is $k_i − 1$. According to NDTRV, the last action $\gamma$ in $seq_i$ must be of type (ii), and in any prefix of $seq_i − \gamma$ the number of actions of type (i) must not be less than that of type (ii). Thus FLIPDT only needs to enumerate all sequences of length $2k_i − 1$ satisfying the above constraints.

**FLIPDT($T_{start}, T_{end}, k$)**

Input: two triangulations $T_{start}$ and $T_{end}$ of a point set $P$ in the Euclidean plane and an integer $k$.

Output: return YES if there exists a sequence of flips of length $k$ that transforms $T_{start}$ into $T_{end}$; otherwise return NO.

1. For each partition $(k_1, ..., k_t)$ of $k$ satisfying $k_1 + k_2 + ... + k_t = k$ and $k_1, ..., k_t \geq 1$ do
2. 1. Order all necessary edges in $T_{start}$ lexicographically and denote this ordering by $O_{lex}$;
2. 1.2 FDSearch($T_{start}, 1, (k_1, ..., k_t)$); /*iteration 1 distributed with $k_1$*/
2. Return NO;

**FDSearch($T, i, (k_1, ..., k_t)$)** /*the concrete branching procedure*/

Input: a triangulation $T$, an integer $i$ denoting that the algorithm is at the $i$-th iteration and a partition $(k_1, ..., k_t)$ of $k$.

Output: return YES if the instance is accepted.

1. Repeatedly pick the next edge in $O_{lex}$ until finding a necessary edge $e$ with respect to $T$ and $T_{end}$;
2. If it reaches the end of $O_{lex}$, but finds no necessary edge in $T$ then
2. 1. Update $O_{lex}$ by permuting all necessary edges in $T_{iteration}^{i−1}$ in lexicographical order, and pick the first edge $e$ in $O_{lex}$;
3. For each possible sequence of actions $seq_i$ of length $2k_i − 1$ do
3. 1. $T' = \text{Transform}(T, seq_i, e)$;
3. 2. If $i < t$ then /*continue to the next iteration distributed with $k_{i+1}$*/
3. 2.1 FDSearch($T', i+1, (k_1, ..., k_t)$);
3. 3. Else if $i = t$ and $T' = T_{end}$ then /*compare $T'$ with $T_{end}$*/
3. 3.1 Return YES;

**Transform($T, s, e$)** /*subprocess for transforming triangulations*/

Input: a triangulation $T$, a sequence of actions $s$ and a starting edges $e$.

Output: a new triangulation $T'$.

1. Perform a sequence of actions $s$ starting from $e$ in $T$, getting a new triangulation $T'$;
2. Return $T'$;

Figure 4: The deterministic algorithm for PARAMETERIZED FLIP DISTANCE
The following theorem proves the correctness of the algorithm FLIPDT.

**Theorem 2.** Let \((T_{\text{start}}, T_{\text{end}}, k)\) be an input instance. FLIPDT is correct and runs in time \(O^*(k \cdot 32^k)\).

**Proof.** Suppose that \((T_{\text{start}}, T_{\text{end}}, k)\) is a yes-instance. There must exist a sequence of flips \(F\) of length \(k\) such that \(T_{\text{start}} \overset{F}{\rightarrow} T_{\text{end}}\). Thus \(D_F\) exists according to Definition 7. By NDTRV and Lemma 4, there exists an undirected graph \(G\) consisting of a set of vertex-disjoint track trees \(A_1, \ldots, A_t\). Moreover, Theorem 3 shows that there exists a sequence of actions \(Q\) following which we can perform all flips of \(D_F\) in a topologically-sorted order. Due to NDTRV, \(Q\) consists of several subsequences \(seq_1, \ldots, seq_t\), in which \(seq_i\) is constructed in the \(i\)-th iteration and corresponds to the track tree \(A_i\) for \(i = 1, \ldots, t\). The length of \(seq_i\) is \(2\lambda_i - 1\). Since FLIPDT tries every distribution in \(\{1, \ldots, k\}\) for the first iteration and \(1 \leq \lambda_i \leq k\), there is a correct guess of the distribution equal to \(\lambda_i\) for this iteration.

We claim that FLIPDT is able to perform a sequence \(\Sigma\) of actions which correctly guesses every subsequence \(seq_1, \ldots, seq_t\) of the objective sequence \(Q\), that is, \(\Sigma\) is a concatenation of \(seq_1, \ldots, seq_t\). Suppose that FLIPDT has completed \(i\) iterations. We prove this claim by induction on \(i\). At the first iteration, FLIPDT starts by picking the first necessary edge \(e_1\) in list \(O_{\text{lex}}\). In the first iteration of constructing \(Q\), NDTRV starts by picking an arbitrary necessary edge. Without loss of generality, it chooses \(e_1\) and construct \(seq_1\) starting from \(e_1\). The length of \(seq_1\) is \(2\lambda_1 - 1\). Since FLIPDT tries every distribution in \(\{1, \ldots, k\}\) for the first iteration and \(1 \leq \lambda_1 \leq k\), there is a correct guess of the distribution equal to \(\lambda_1\) for this iteration.

Suppose that the claim is true for any first \(i\) iterations (\(1 \leq i < t\)). That is, under some guess for the partition of \(k\), \(\lambda_1, \ldots, \lambda_t\) are distributed to the first \(i\) iterations respectively. Moreover, FLIPDT has completed \(i\) iterations and performed a sequence of actions \(seq_{\text{concat}}, i\), which is equal to the concatenation of \(seq_1, \ldots, seq_i\), resulting in a triangulation \(T_i\). Based on \(T_i\) and \(seq_{\text{concat}}, i\), FLIPDT is ready to perform the \((i + 1)\)-th iteration. Suppose that FLIPDT picks \(e_{i+1}\) from \(O_{\text{lex}}\). Let us see the construction of \(Q\) in NDTRV. Suppose NDTRV has constructed the first \(i\) track trees \(A_1, \ldots, A_i\), and it is ready to begin a new iteration by arbitrarily picking a necessary edge in the current triangulation. Since FLIPDT correctly guessed and performed the first \(i\) subsequences of \(Q\), \(T_i\) is exactly equal to the current triangulation in NDTRV. Thus \(e_{i+1}\) is a candidate edge belonging to the set of all selectable necessary edges for NDTRV in this iteration. Without loss of generality, it chooses \(e_{i+1}\) and constructs \(seq_{i+1}\) of length \(2\lambda_{i+1} - 1\) starting from \(e_{i+1}\). Since the sizes of \(A_1, \ldots, A_i\) are \(\lambda_1, \ldots, \lambda_i\) respectively, we get that \(1 \leq \lambda_{i+1} \leq k - (\lambda_1 + \ldots + \lambda_i)\). We argue that FLIPDT is able to perform a sequence that is equal to the concatenation of \(seq_1, \ldots, seq_i, seq_{i+1}\). Since the edges in \(O_{\text{lex}}\) are ordered lexicographically and FLIPDT chooses necessary edges in a fixed manner, FLIPDT is sure to choose \(e_{i+1}\) to begin the \((i + 1)\)-th iteration for every guessed sequence in which the first \(i\) subsequences are equal to \(seq_1, \ldots, seq_i\) respectively. Thus FLIPDT actually tries every distribution in \(\{1, \ldots, k - (\lambda_1 + \ldots + \lambda_i)\}\) for the \((i + 1)\)-th iteration starting from \(e_{i+1}\) based on \(T_i\) and \(seq_{\text{concat}}, i\). It follows that there is a correct guess of distribution for the \((i + 1)\)-th iteration which is equal to \(\lambda_{i+1}\).

Under this correct guess of distribution, FLIPDT tries all possible sequences of length \(2\lambda_{i+1} - 1\) starting from \(e_{i+1}\) on \(T_i\) based on \(seq_{\text{concat}}, i\), ensuring that one of them is equal to \(seq_{i+1}\). It follows that the claim is true for the first \(i + 1\) iterations. This completes the inductive proof for the claim.

If \((T_{\text{start}}, T_{\text{end}}, k)\) is a yes-instance, the action sequence \(Q\) of length at most \(2k\) exists and the deterministic algorithm can find such a sequence. Otherwise, there is no valid sequence \(F\) of length \(k\). Thus there is no such action sequence \(Q\). As a result, FLIPDT returns NO. It is proved that FLIPDT decides the given instance \((T_{\text{start}}, T_{\text{end}}, k)\) correctly.

Finding and ordering all necessary edges in \(T_{\text{start}}\) takes \(O(n + k \log k)\) time, and FLIPDT may update the ordering \(O_{\text{lex}}\) at the beginning of each iteration. The number of partitions of \(k\) is known as the composition number of \(k\), which is \(2^{k-1}\). Under each partition \((k_1, \ldots, k_t)\) of \(k\) and for each \(k_i, i = 1, \ldots, t\), we enumerate all possible subsequences of actions in which there are \(k_i\) actions of type (ii) and \(k_i - 1\) actions of type (i).
It follows that the number of all possible subsequences is bounded by \((\binom{2(k_i-1)}{k_i-1}) \times 4^{k_i-1} = O^*(16^{k_i})\) since there are four choices for action (i) and one choice for action (ii). Here we use Stirling’s approximation \(n! \approx \sqrt{2\pi n} (n/e)^n\) and get that \((\binom{2(k_i-1)}{k_i-1}) = O^*(4^{k_i})\). It follows that there are \(O^*(16^{k_1}) \times O^*(16^{k_2}) \times \ldots \times O^*(16^{k_t}) = O^*(16^{k})\) cases under each partition. Since for each case we can perform the sequence of actions in \(O(k)\) time, and the resulting triangulation can be compared to \(T_{end}\) in \(O(k)\) time, the running time of the whole algorithm is bounded by \(O^*(k \cdot 2^{k-1} \cdot (n + k \log k) + k \cdot 2^{k-1} \cdot 16^k) = O^*(k \cdot 32^k)\).

According to the definition of PARAMETERIZED FLIP DISTANCE, we need to check if we can find a shorter valid sequence for the given triangulations \(T_{start}\) and \(T_{end}\). This is achieved by calling \textsc{flipdt} on each instance \((T_{start}, T_{end}, k')\) for \(k' = 0, \ldots, k\). The running time is bounded by \(\sum_{k'=0}^{k} O^*(k' \cdot 32^{k'}) = O^*(k \cdot 32^k)\).

4 Conclusion

In this paper we presented an FPT algorithm running in time \(O^*(k \cdot 32^k)\) for PARAMETERIZED FLIP DISTANCE, improving the previous \(O^*(k \cdot c^k)\)-time \((c \leq 2 \times 14^{11})\) FPT algorithm by Kanj and Xia \cite{kanj2015}. An important related problem is computing the flip distance between triangulations of a convex polygon, whose traditional complexity is still unknown. Although our algorithm can be applied to the case of convex polygon, it seems that an \(O(c^k)\) algorithm with smaller \(c\) for this case probably exists due to its more restrictive geometric property. In addition, whether there exists a polynomial kernel for PARAMETERIZED FLIP DISTANCE is also an interesting problem.

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