Bilattice Logic Properly Displayed*

Giuseppe Greco¹, Fei Liang¹,², Alessandra Palmigiano¹,³, and Umberto Rivieccio⁴

¹Delft University of Technology, the Netherlands
²Institute of Logic and Cognition, Sun Yat-sen University, China
³University of Johannesburg, South Africa
⁴Federal University of Rio Grande do Norte, Brazil

Abstract

We introduce a proper multi-type display calculus for bilattice logic (with conflation) for which we prove soundness, completeness, conservativity, standard subformula property and cut-elimination. Our proposal builds on the product representation of bilattices and applies the guidelines of the multi-type methodology in the design of display calculi.

Keywords: Non-classical logics, bilattice logic, many-valued logics, substructural logics, algebraic proof theory, sequent calculi, cut elimination, display calculi, multi-type calculi.

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1 Introduction

Bilattices are algebraic structures introduced in [20] in the context of a multivalued approach to deductive reasoning, and have subsequently found applications in a variety of areas in computer science and artificial intelligence. The basic intuition behind the bilattice formalism, which can be traced back to the work of Dunn and Belnap [12, 4, 5], is to carry out reasoning within a space of truth-values that results from expanding the classical set \{f, t\} with a value \(\perp\), representing lack of information, and a value \(\top\), representing over-defined or contradictory information.

During the last two decades, the theory of bilattices has been investigated in depth from a logical and algebraic point of view: complete (Hilbert- and Gentzen-style) presentations of bilattice-based logics were introduced in [1, 2], followed by [8] which focuses on the implication-free reduct of the logic. The calculi introduced in these papers have many common aspects with those considered e.g. in [13] for the Belnap-Dunn logic, of which bilattice logics are conservative expansions.

Negation plays a very special role, and it is in fact due to the negation connective that bilattice logics are not self-extensional [33] (or, as other authors say, congruential), i.e. the inter-derivability relation of the logic is not a congruence of the formula algebra. This means that there are formulas such that \(\varphi \dashv \vdash \psi\) and yet \(\neg \varphi \nvDash \neg \psi\) (which did not happen in the Belnap-Dunn logic that is indeed self-extensional). In the Gentzen-style calculus for bilattice logic GBL introduced in [1 Section 3.2], each binary connective is introduced via four different logical rules, two of which are standard, and introduce it as main connective on the left and on the right of the turnstyle, and two non-standard rules, which introduce the same connective under the scope of a negation. From a proof-theoretic perspective, this solution presents the disadvantage that the resulting calculus is

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not fully modular, does not support a proof-theoretic semantics, and does not enjoy the standard subformula property.

In this paper we introduce a proper multi-type display calculus for bilattice logic that circumvents all the above-mentioned disadvantages. The design of our calculus follows the principles of the multi-type methodology introduced in [21, 16, 14, 15] with the aim of displaying dynamic epistemic logic and propositional dynamic logic and subsequently applied to several other logics (e.g. linear logic with exponentials [25], inquisitive logic [17], semi-De Morgan logic [22], lattice logic [24]) which are not properly displayable in their single-type presentation, which also inspired the design of novel logics [6]. Our multi-type syntactic presentation of bilattice logic is based on the algebraic insight provided by the product representation theorems (see e.g. [7]) and possesses all the desirable properties of proper display calculi. In particular, our calculus enjoys the standard subformula property, supports a proof-theoretic semantics and is fully modular.

Structure of the paper In Section 2 we recall basic definitions and results about bilattices and bilattice logics. Section 3 presents an algebraic analysis of bilattices as heterogeneous structures which provides a basis for our multi-type approach to their proof theory. Our display calculus is introduced in Section 4 where we also prove soundness, completeness, conservativity, subformula property and cut-elimination. In Section 6 we outline some directions for future work.

2 Preliminaries on bilattices

The following definitions and results can be found e.g. in [1,8].

Definition 2.1. A bilattice is a structure \( \mathbb{B} = (B, \leq_t, \leq_k, \neg) \) such that \( B \) is a non-empty set, \( (B, \leq_t) \), \( (B, \leq_k) \) are lattices, and \( \neg \) is a unary operation on \( B \) having the following properties:

- if \( a \leq_t b \), then \( \neg b \leq_t \neg a \),
- if \( a \leq_k b \), then \( \neg a \leq_k \neg b \),
- \( \neg \neg a = a \).

We use \( \land, \lor \) for the lattice operations which correspond to \( \leq_t \) and \( \otimes, \oplus \) for those that correspond to \( \leq_k \). If present, the lattice bounds of \( \leq_t \) are denoted by \( f \) and \( t \) (minimum and maximum, respectively) and those of \( \leq_k \) by \( \bot \) and \( \top \). The smallest non-trivial bilattice is the four-element one (called Four) with universe \( \{f, t, \bot, \top\} \).

Fact 2.2. The following equations (De Morgan laws for negation) hold in any bilattice:

\[
\neg(x \land y) = \neg x \lor \neg y, \quad \neg(x \lor y) = \neg x \land \neg y,
\neg(x \otimes y) = \neg x \oplus \neg y, \quad \neg(x \oplus y) = \neg x \otimes \neg y.
\]

Moreover, if the bilattice is bounded, then

\[
\neg t = f, \quad \neg f = t, \quad \neg \top = \bot, \quad \neg \bot = \top.
\]

Definition 2.3. A bilattice is called distributive when all possible distributive laws concerning the four lattice operations, i.e., all identities of the following form, hold:

\[
x \circ (y \bullet z) \approx (x \circ y) \bullet (x \circ z) \quad \text{for all } \circ, \bullet \in \{\land, \lor, \otimes, \oplus\}
\]

\(1\)The notion of proper display calculus has been introduced in [32]. Properly displayable logics, i.e. those which can be captured by some proper display calculus, have been characterized in a purely proof-theoretic way in [10]. In [23], an alternative characterization of properly displayable logics was introduced which builds on the algebraic theory of unified correspondence [11].
If a distributive bilattice is bounded, then

\[ t \otimes f = \perp, \ t \oplus f = \top, \ \top \land \perp = f, \ \top \lor \perp = t. \]

In the following, we use \( B \) to denote the class of bounded distributive bilattices.

**Theorem 2.4** (Representation of distributive bilattices). *Let \( \mathbb{L} \) be a bounded distributive lattice with join \( \sqcup \) and meet \( \sqcap \). Then the algebra \( \mathbb{L} \times \mathbb{L} \) having as universe the direct product \( L \times L \) is a distributive bilattice with the following operations:

\[
\begin{align*}
\langle a_1, a_2 \rangle \land \langle b_1, b_2 \rangle & := \langle a_1 \sqcap b_1, a_2 \sqcup b_2 \rangle \\
\langle a_1, a_2 \rangle \lor \langle b_1, b_2 \rangle & := \langle a_1 \sqcup b_1, a_2 \sqcap b_2 \rangle \\
\langle a_1, a_2 \rangle \otimes \langle b_1, b_2 \rangle & := \langle a_1 \sqcap b_1, a_2 \sqcap b_2 \rangle \\
\langle a_1, a_2 \rangle \oplus \langle b_1, b_2 \rangle & := \langle a_1 \sqcup b_1, a_2 \sqcup b_2 \rangle \\
\neg \langle a_1, a_2 \rangle & := \langle a_2, a_1 \rangle \\
f & := \langle 0, 1 \rangle \\
t & := \langle 1, 0 \rangle \\
\perp & := \langle 0, 0 \rangle \\
\top & := \langle 1, 1 \rangle
\end{align*}
\]

**Theorem 2.5.** *Every distributive bilattice is isomorphic to \( \mathbb{L} \times \mathbb{L} \) for some distributive lattice \( \mathbb{L} \).*

**Definition 2.6.** *A structure \( \mathbb{B} = (B, \leq_t, \leq_k, \neg, -) \) is a bilattice with conflation if the reduct \( (B, \leq_t, \leq_k, \neg) \) is a bilattice and the conflation \(- : B \to B\) is an operation satisfying:

- if \( a \leq_t b \), then \(-a \leq_t -b;\)
- if \( a \leq_k b \), then \(-b \leq_k -a;\)
- \( -(-a) = a.\)

We say that \( \mathbb{B} \) is commutative if it also satisfies the equation: \( \neg -a = -\neg a.\)

**Fact 2.7.** *The following equations (De Morgan laws for conflation) hold in any bilattice with conflation:

\[
\begin{align*}
-(x \land y) &= \neg x \land \neg y \\
-(x \lor y) &= \neg x \lor \neg y \\
-(x \otimes y) &= \neg x \oplus \neg y \\
-(x \oplus y) &= \neg x \otimes \neg y
\end{align*}
\]

Moreover, if the bilattice is bounded, then

\( \neg \top = \bot, \neg \bot = \top, \neg \bot \oplus \neg \bot = \bot, \neg \bot \otimes \neg \bot = \bot.\)

We denote by \( \text{CB} \) the class of bounded commutative distributive bilattices with conflation.

**Theorem 2.8.** *Let \( \mathbb{D} = (D, \sqcap, \sqcup, \sim, 0, 1) \) be a De Morgan algebra, then \( \mathbb{D} \times \mathbb{D} \) is a bounded commutative distributive bilattice with conflation where:

- \( \sim(0, 1) \sqcap \sim(1, 0) \sqcup \sim(1, 1) \);
- \( \neg(a, b) = (\sim b, \sim a);\)

**Theorem 2.9.** *Every bounded commutative distributive bilattice with conflation is isomorphic to \( \mathbb{D} \times \mathbb{D} \) for some De Morgan algebra \( \mathbb{D} \).*
A calculus for bilattice logic

The language of bilattice logic \( \mathcal{L} \) over a denumerable set \( \text{AtProp} = \{p, q, r, \ldots\} \) of atomic propositions is generated as follows:

\[
A ::= p \mid t \mid \mathbf{f} \mid \top \mid \bot \mid \neg A \mid A \land A \mid A \lor A \mid A \otimes A \mid A \oplus A,
\]

the language of bilattice logic with conflation also includes the conflation formula \( \neg A \).

The calculus for bilattice logic \( \text{BL} \) consists of the following axioms:

\[
A \vdash A, \quad \neg \neg A \vdash A,
\]

\[
\mathbf{f} \vdash A, \quad A \vdash t, \quad \bot \vdash A, \quad A \vdash \top,
\]

\[
A \vdash \neg \mathbf{f}, \quad \neg \top \vdash A, \quad \neg \bot \vdash A, \quad A \vdash \neg \neg A,
\]

\[
A \land B \vdash A, \quad A \land B \vdash B, \quad A \vdash A \lor B, \quad B \vdash A \lor B,
\]

\[
A \otimes B \vdash A, \quad A \otimes B \vdash B, \quad A \vdash A \oplus B, \quad B \vdash A \oplus B,
\]

\[
A \land (B \lor C) \vdash (A \land B) \lor (A \land C),
\]

\[
A \otimes (B \oplus C) \vdash (A \otimes B) \lor (A \oplus C),
\]

\[
\neg (A \land B) \vdash \neg A \lor \neg B, \quad \neg (A \lor B) \vdash \neg A \land \neg B,
\]

\[
\neg (A \otimes B) \vdash \neg A \otimes \neg B, \quad \neg (A \oplus B) \vdash \neg A \oplus \neg B.
\]

and the following rules:

\[
\begin{array}{ccc}
A \vdash B & \quad B \vdash C & \quad A \vdash C \\
\hline
A \vdash C
\end{array}
\]

\[
\begin{array}{ccc}
A \vdash B & \quad A \vdash C & \quad A \vdash B \land C \\
\hline
A \vdash B \land C
\end{array}
\]

\[
\begin{array}{ccc}
A \vdash B & \quad C \vdash B & \quad A \vdash C \\
\hline
A \vdash C \lor B
\end{array}
\]

\[
\begin{array}{ccc}
A \vdash B & \quad A \vdash C & \quad A \vdash B \oplus C \\
\hline
A \vdash B \oplus C
\end{array}
\]

\[
\begin{array}{ccc}
A \vdash B & \quad C \vdash B & \quad A \vdash C \\
\hline
A \vdash C \oplus B
\end{array}
\]

The calculus for bilattice logic with conflation \( \text{CBL} \) consists of the axioms and rules of \( \text{BL} \) plus the following axioms:

\[
\neg \neg A \vdash A, \quad \neg A \vdash \neg \neg A,
\]

\[
\neg \mathbf{f} \vdash A, \quad A \vdash \bot, \quad \neg \top \vdash A, \quad A \vdash \bot,
\]

\[
-(A \land B) \vdash -A \land -B, \quad -(A \lor B) \vdash -A \lor -B,
\]

\[
-(A \otimes B) \vdash -A \otimes -B, \quad -(A \oplus B) \vdash -A \oplus -B.
\]

The algebraic semantics of \( \text{BL} \) (resp. \( \text{CBL} \)) is given by \( \mathcal{B} \) (resp. \( \mathcal{CB} \)). We use \( A \models_{\mathcal{B}} C \) (resp. \( A \models_{\mathcal{CB}} C \)) to mean: for any \( \mathcal{B} \in \mathcal{B} \) (resp. \( \mathcal{B} \in \mathcal{CB} \)), if \( A^\mathcal{B} \in F_\mathcal{B} \) then \( C^\mathcal{B} \in F_\mathcal{B} \). Here \( A^\mathcal{B}, C^\mathcal{B} \) mean the interpretations of \( A \) and \( C \) in \( \mathcal{B} \), respectively; and \( F_\mathcal{B} = \{a \in B : t \leq_a a\} \) is the set of designated elements of \( \mathcal{B} \) (using the terminology of [11] Definition 2.13, \( F_\mathcal{B} \) is the least bifilter of \( \mathcal{B} \)).

Soundness of \( \text{BL} \) (resp. \( \text{CBL} \)) is straightforward. In order to show completeness, we can prove that every axiom and rule of Arieli and Avron’s \( \text{GBL} \) (resp. \( \text{GBS} \), cf. [11]) is derivable in \( \text{BL} \) (resp. \( \text{CBL} \)). Then the completeness of \( \text{BL} \) (resp. \( \text{CBL} \)) follows from the completeness of \( \text{GBL} \) (resp. \( \text{GBS} \), [11] Theorem 3.7]).

**Theorem 2.10** (Completeness). \( A \vdash_{\text{BL}} C \) iff \( A \models_{\mathcal{B}} C \) (resp. \( A \vdash_{\text{CBL}} C \) iff \( A \models_{\mathcal{CB}} C \)).

\(^2\)In order to do this, we view a sequent \( \Gamma \Rightarrow \Delta \) of \( \text{GBL} \) (\( \text{GBS} \)) as the equivalent sequent \( \land \Gamma \Rightarrow \lor \Delta \).
3 Multi-type algebraic presentation

In the present section we introduce the algebraic environment which justifies semantically the multi-type approach to bilattice logic presented in Section 4. The main insight is that (bounded) bilattices (with conflation) can be equivalently presented as heterogeneous structures, i.e. tuples consisting of two (bounded) distributive lattices (De Morgan algebras) together with two maps between them.

Multi-type semantic environment

For a bilattice \( \mathcal{B} \), let \( \text{Reg}(\mathcal{B}) = \{a \in \mathcal{B} : a = \neg a\} \) be the set of regular elements \([7]\). It is easy to show that \( \text{Reg}(\mathcal{B}) \) is closed under \( \otimes \) and \( \oplus \), hence \( (\text{Reg}(\mathcal{B}), \otimes, \oplus) \) is a sublattice of \( (\mathcal{B}, \otimes, \oplus) \). For every \( a \in \mathcal{B} \), we let
\[
\text{reg}(a) := (a \vee (a \otimes \neg a)) \oplus (a \vee (a \otimes \neg a))
\]
be the regular element associated with \( a \). It follows from the representation result of \([7, \text{Theorem 3.2}]\) that
\[
\mathcal{B} \cong (\text{Reg}(\mathcal{B}), \otimes, \oplus) \odot (\text{Reg}(\mathcal{B}), \otimes, \oplus)
\]
where the isomorphism \( \pi : B \rightarrow \text{Reg}(\mathcal{B}) \times \text{Reg}(\mathcal{B}) \) is defined, for all \( a \in \mathcal{B} \), as
\[
\pi(a) = (\text{reg}(a), \text{reg}(\neg a)).
\]
The inverse map \( f : \text{Reg}(\mathcal{B}) \times \text{Reg}(\mathcal{B}) \rightarrow \mathcal{B} \) is defined, for all \( (a, b) \in \text{Reg}(\mathcal{B}) \times \text{Reg}(\mathcal{B}) \), as
\[
f((a, b)) := (a \otimes (a \vee b)) \oplus (b \otimes (a \wedge b)).
\]

Heterogeneous Bilattices

**Definition 3.1.** A distributive lattice \( \mathbb{A} \) is perfect (cf. \([18]\)) if it is complete, completely distributive and completely join-generated by the set \( J^\infty(\mathbb{A}) \) of its completely join-irreducible elements (as well as completely meet-generated by the set \( M^\infty(\mathbb{A}) \) of its completely meet-irreducible elements).

A lattice isomomorphism \( h : \mathbb{L} \rightarrow \mathbb{L}' \) is complete if it satisfies the following properties for each \( X \subseteq \mathbb{L} \):
\[
h(\lor X) = \lor h(X) \quad h(\land X) = \land h(X),
\]

**Definition 3.2.** A heterogeneous bilattice \((HBL)\) is a tuple \((\mathbb{H} = (\mathbb{L}_1, \mathbb{L}_2, n, p))\) satisfying the following conditions:

(H1) \( \mathbb{L}_1, \mathbb{L}_2 \) are bounded distributive lattices.

(H2) \( n : \mathbb{L}_1 \rightarrow \mathbb{L}_2 \) and \( p : \mathbb{L}_2 \rightarrow \mathbb{L}_1 \) are mutually inverse lattice isomorphisms.

An HBL is perfect if:

(H3) both \( \mathbb{L}_1 \) and \( \mathbb{L}_2 \) are perfect lattices;

(H4) \( p, n \) are complete lattice isomorphisms.

By (H2) we have that \( np = \text{Id}_{\mathbb{L}_1} \) and \( pn = \text{Id}_{\mathbb{L}_2} \). The definition of the heterogeneous bilattice with conflation \((HCBL)\) is analogous, except that we replace (H1) with (H1’): \( \mathbb{L}_1 \) and \( \mathbb{L}_2 \) are De Morgan algebras.

The following lemma is an easy consequence of the results in \([19, \text{Theorems 2.3 and 3.2}]\).

**Lemma 3.3.** If \((\mathbb{L}_1, \mathbb{L}_2, n, p)\) is an HBL \((HCBL)\), then \((\mathbb{L}_1^\delta, \mathbb{D}_1^\delta, n_1^\delta, p_1^\delta)\) is a perfect HBL \((HCBL)\).
Equivalence of the two presentations

The following result is an immediate consequence of Definition 3.2.

Proposition 3.4. For any bounded distributive bilattice $B$, the tuple $B^+ = (L_1 = \text{Reg}(B), L_2 = \text{Reg}(B), p = \text{Id}_{\text{Reg}(B)}, n = \text{Id}_{\text{Reg}(B)})$ is an HBL, where $\cap_1 = \cap_2 = \emptyset; \cup_1 = \cup_2 = \emptyset; l_1 = l_2 = T$ and $0_1 = 0_2 = \bot$.

For any CB $B$, $B^+ = (L_1 = (\text{Reg}(B), \neg), L_2 = (\text{Reg}(B), \neg), p = \text{Id}_{\text{Reg}(B)}, n = \text{Id}_{\text{Reg}(B)})$ is an HCBL, where $\sim_2 = \sim_1 = \sim$.

Proposition 3.5. If $(\mathbb{L}_1, \mathbb{L}_2, n, p)$ is an HBL (resp. HCBL), then $L_1 \times L_2$ can be endowed with the following structure:

$$
\langle a_1, a_2 \rangle \odot \langle b_1, b_2 \rangle := \langle a_1 \cap_1 b_1, a_2 \cap_2 b_2 \rangle
$$

$$
\langle a_1, a_2 \rangle \oplus \langle b_1, b_2 \rangle := \langle a_1 \cup_1 b_1, a_2 \cup_2 b_2 \rangle
$$

$$
\langle a_1, a_2 \rangle \land \langle b_1, b_2 \rangle := \langle a_1 \cap_1 b_1, a_2 \cap_2 b_2 \rangle
$$

$$
\neg \langle a_1, a_2 \rangle := \langle p(a_2), n(a_1) \rangle
$$

$$
\langle a_1, a_2 \rangle := \langle p(\sim_2 a_2), n(\sim_1 a_1) \rangle
$$

$$
f := (0, 1)
$$

$$
t := (1, 0)
$$

$$
\bot := (0, 0)
$$

$$
\top := (1, 1)
$$

Proof. Firstly, we show that $(\mathbb{L}_1 \times \mathbb{L}_2, \odot, \oplus)$ and $(\mathbb{L}_1 \times \mathbb{L}_2, \land, \lor)$ are bounded distributive lattices. It is obvious that they are both bounded lattices. We only need to show that the distributivity law holds. We have:

$$
\langle a_1, a_2 \rangle \odot (\langle b_1, b_2 \rangle \oplus (c_1, c_2)) =
$$

$$
= \langle a_1 \cap_1 (b_1 \cup_1 c_1), a_2 \cap_2 (b_2 \cup_2 c_2) \rangle
$$

(Def. of $\odot$)

$$
= \langle a_1 \cap_1 (b_1 \cup_1 c_1), a_2 \cap_2 (b_2 \cup_2 c_2) \rangle
$$

(Def. of $\cap$)

$$
= \langle a_1 \cap_1 (b_1 \cup_1 c_1), (a_2 \cap_2 b_2) \cup_2 (a_2 \cap_2 c_2) \rangle
$$

(Distributivity of $\cap_1$ and $\cup_2$)

$$
= \langle a_1 \cap_1 (b_1 \cup_1 c_1), (a_2 \cap_2 b_2) \rangle \oplus \langle a_1 \cap_1 c_1, (a_2 \cap_2 c_2) \rangle
$$

(Def. of $\oplus$)

$$
= \langle a_1, a_2 \rangle \odot (b_1, b_2) \oplus \langle a_1, a_2 \rangle \odot (c_1, c_2)
$$

(Def. of $\odot$)

As to $(\mathbb{L}_1 \times \mathbb{L}_2, \land, \lor)$, the argument is analogous.

Now we show that the properties of $\neg$ are also met. Assume that $\langle a_1, a_2 \rangle \leq_1 \langle b_1, b_2 \rangle$, equivalently, $a_1 \leq_1 b_1$ and $b_2 \leq_2 a_2$. By the definition of $\neg$, we have $\neg \langle a_1, a_2 \rangle = \langle p(a_2), n(\neg_1 a_1) \rangle$ and $\neg \langle b_1, b_2 \rangle = \langle p(b_2), n(\neg_1 b_1) \rangle$. Hence $p(b_2) \leq_1 p(a_2)$ and $n_1 a_2 \leq_2 n_1 b_1$ by (H2). Thus $\langle b_1, b_2 \rangle \leq_1 \neg_1 \langle a_1, a_2 \rangle$. A similar reasoning shows that the corresponding property involving $\neg$ and $\leq_2$ also holds. The following argument shows that $\neg$ is involutive.
\[\neg\langle a_1, a_2 \rangle =\]
\[\equiv \neg(p(a_2), nA_1) \quad \text{Def. of } \neg\]
\[\equiv \langle pnA_1, np(a_2) \rangle \quad \text{Def. of } \neg\]
\[\equiv \langle a_1, a_2 \rangle \quad np = \text{Id}_{L_1}, \text{ and } pn = \text{Id}_{L_2}\]

As to conflation, assume \(\langle a_1, a_2 \rangle \leq_1 \langle b_1, b_2 \rangle\), equivalently, \(a_1 \leq_1 b_1 \) and \(b_2 \leq_2 a_2\). By the defi-

\[\neg\langle a_1, a_2 \rangle =\]
\[\equiv -\langle p(\neg a_2), n(\sim a_1) \rangle \quad \text{(Def. of } \neg)\]
\[\equiv \langle p(\neg 2n(\sim a_1)), n(\sim_1 p(\sim a_2)) \rangle \quad \text{(Def. of } \neg)\]
\[\equiv \langle p(\sim 2n(a_1)), n(\sim_1 \sim p(a_2)) \rangle \quad \text{(H2)}\]
\[\equiv \langle pn(a_1), np(a_2) \rangle \quad \text{(H1)}\]
\[\equiv \langle a_1, a_2 \rangle \quad \text{(np = } \text{Id}_{L_1}, \text{ and } \text{pn = } \text{Id}_{L_2}).\]

\[\neg\langle a_1, a_2 \rangle =\]
\[\equiv -\langle p(a_2), n(a_1) \rangle \quad \text{(Def. of } \neg)\]
\[\equiv \langle p(\sim_2 n(a_1)), n(\sim_1 p(a_2)) \rangle \quad \text{(Def. of } \neg)\]
\[\equiv \langle \neg_1 p(a_2), \sim_2 n(a_1) \rangle \quad \text{(Def. of } \neg)\]
\[\equiv \langle \neg(\sim_2 a_2), n(\sim_1 a_2) \rangle \quad \text{(H2)}\]
\[\equiv -\langle a_1, a_2 \rangle \quad \text{(Def. of } \neg).\]

**Definition 3.6.** For any HBL \(\mathbb{H} = (L_1, L_2, n, p)\), we denote by \(\mathbb{H}_+ = (B, \wedge, \vee, \otimes, \oplus, \neg)\) the product algebra where the four lattice operations are defined as in \(\mathbb{L}_1 \odot \mathbb{L}_2\) (Theorem 2.7) and the negation is given by \(\neg\langle a_1, a_2 \rangle := \langle p(a_2), nA_1 \rangle\) for all \(\langle a_1, a_2 \rangle \in B\). If \(L_1\) and \(L_2\) are isomorphic De Morgan algebras, then we define \(\mathbb{H}_+ = (B, \wedge, \vee, \otimes, \oplus, \neg, \sim)\) as before, with the conflation given by \(\sim\langle a_1, a_2 \rangle := \langle p(\sim_2 a_2), n(\sim_1 a_1) \rangle\) for all \(\langle a_1, a_2 \rangle \in B\).

**Proposition 3.7.** For any \(\mathbb{B} \in B\) (resp. \(\mathbb{B} \in CB\)) and any HBL (resp. HCBL) \(\mathbb{H}\), we have
\[
\mathbb{B} \cong (\mathbb{B}^+) \quad \text{and} \quad \mathbb{H} \cong (\mathbb{H}_+)^+.
\]

**Proof.** Immediately follows from Propositions 3.4 and 3.5. \(\square\)

## 4 Multi-type proper display calculus

In this section we introduce the proper display calculus D.BL (D.CBL) for bilattice logic (with conflation).

**Language**

The language \(L_{MF}\) of D.BL is given by the union of the sets \(L_1\) and \(L_2\) defined as follows. \(L_1\) is given by simultaneous induction over the set \(\text{AtProp}_1 = \{p_1, q_1, r_1, \ldots\}\) of \(L_1\)-type atomic propositions as follows:

\[
A_1 := p_1 | 1_1 | 0_1 | pA_2 | A_1 \cap_1 A_1 | A_1 \cup_1 A_1
\]
\[
X_1 := A_1 | 1_1 | 0_1 | PX_2 | X_1 \hat{\cap}_1 X_1 | X_1 \hat{\cup}_1 X_1
\]

\(L_2\) is given by simultaneous induction over the set \(\text{AtProp}_2 = \{p_2, q_2, r_2, \ldots\}\) of \(L_2\)-type atomic propositions as follows:
\[
A_2 ::= p_2 | l_2 | 0_2 | nA_1 | A_2 \cap_l A_2 | A_2 \cup_l A_2 \\
X_2 ::= A_2 | l_2 | 0_2 | NX_1 | X_1 \cap_l X_1 | X_1 \cup_l X_1
\]

The language of D.CBL can be obtained by adding structural operators \(*_1\) and \(*_2\) and their corresponding connectives \(\sim_1\), \(\sim_2\) to \(\mathcal{L}_1\) and \(\mathcal{L}_2\) respectively.

**Rules**

For \(i \in \{1, 2\},\)

- Pure \(\mathcal{L}_i\)-type display rules
  \[
  \begin{array}{c}
  \text{res} \quad X_i \hat{\cap}_i Y_i \vdash Z_i \\
  \hline
  \text{res} \\
  \end{array}
  \quad\quad
  \begin{array}{c}
  \hline
  \text{res} \\
  \end{array}
  \]

- Multi-type display rules
  \[
  \begin{array}{c}
  \text{adj} \quad PX_2 \vdash Y_1 \\
  \hline
  \text{adj} \\
  \end{array}
  \quad\quad
  \begin{array}{c}
  \hline
  \text{adj} \\
  \end{array}
  \]

- Pure \(\mathcal{L}_i\)-type identity and cut rules
  \[
  \begin{array}{c}
  \text{Id}_i \quad p_i \vdash p_i \\
  \hline
  X_i \vdash Y_i \\
  \text{Cut} \\
  \end{array}
  \]

- Pure \(\mathcal{L}_i\)-type structural rules
  \[
  \begin{array}{c}
  i \quad X_i \hcap_i 1_i \vdash Y_i \\
  \hline
  X_i \vdash Y_i \\
  \end{array}
  \quad\quad
  \begin{array}{c}
  E \quad X_i \hcap_i Y_i \vdash Z_i \\
  \hline
  Y_i \hcap_i X_i \vdash Z_i \\
  \end{array}
  \quad\quad
  \begin{array}{c}
  A \quad (X_i \hcap_i Y_i) \hcap_i Z_i \vdash W_i \\
  \hline
  X_i \hcap_i (Y_i \hcap_i Z_i) \vdash W_i \\
  \end{array}
  \quad\quad
  \begin{array}{c}
  W \quad X_i \vdash Z_i \\
  \hline
  X_i \hcap_i Y_i \vdash Z_i \\
  \end{array}
  \quad\quad
  \begin{array}{c}
  C \quad X_i \hcap_i X_i \vdash Z_i \\
  \hline
  X_i \vdash Y_i \hcup_i Z_i \\
  \end{array}
  \]

- Pure \(\mathcal{L}_i\) type operational rules
  \[
  \begin{array}{c}
  i \quad \hat{1}_i \vdash X_i \\
  \hline
  \hat{1}_i \vdash 1_i \\
  \end{array}
  \quad\quad
  \begin{array}{c}
  0_i \vdash \hat{0}_i \\
  \hline
  X_i \vdash \hat{0}_i \\
  \end{array}
  \quad\quad
  \begin{array}{c}
  \cap_i \quad A_i \hcap_i B_i \vdash X_i \\
  \hline
  A_i \hcap_i B_i \vdash X_i \\
  \end{array}
  \quad\quad
  \begin{array}{c}
  \cup_i \quad A_i \vdash X_i \\
  \hline
  A_i \hcup_i B_i \vdash X_i \\
  \end{array}
  \]

- Multi-type structural rules
\[
\begin{array}{c}
N X_1 \vdash Y_1 \\
N X_1 \vdash N Y_1
\end{array}
\quad \begin{array}{c}
X_2 \vdash Y_2 \\
PA_2 \vdash PY_2
\end{array}
\]

\[
\begin{array}{c}
p_0 \vdash X_1 \\
p_0 \vdash X_1
\end{array}
\quad \begin{array}{c}
0_1 \vdash X_1 \\
0_1 \vdash X_1
\end{array}
\quad \begin{array}{c}
X_1 \vdash I_1 \\
X_1 \vdash PI_2
\end{array}
\]

- Multi-type operational rules

\[
\begin{array}{c}
\begin{array}{c}
A_1 + X_2 \\
A_1 + X_2
\end{array}
\quad \begin{array}{c}
X_2 \vdash A_1 \\
X_2 \vdash A_1
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
A_2 + X_1 \\
A_2 + X_1
\end{array}
\quad \begin{array}{c}
X_1 \vdash A_2 \\
X_1 \vdash A_2
\end{array}
\end{array}
\end{array}
\]

The multi-type display calculus D.CBL also includes the following rules:

- Pure \( L_i \) display structural rules:

\[
\begin{array}{c}
\begin{array}{c}
\vdash X_i \\
\vdash X_i
\end{array}
\quad \begin{array}{c}
\vdash Y_i \\
\vdash Y_i
\end{array}
\end{array}
\quad \begin{array}{c}
\vdash *X_i \\
\vdash *Y_i
\end{array}
\quad \begin{array}{c}
\vdash *X_i \\
\vdash *Y_i
\end{array}
\quad \begin{array}{c}
\vdash X_i \\
\vdash Y_i
\end{array}
\]

- Pure \( L_i \) structural rules:

\[
\begin{array}{c}
\begin{array}{c}
X_i \\
Y_i
\end{array}
\end{array}
\quad \begin{array}{c}
\vdash X_i \\
\vdash Y_i
\end{array}
\quad \begin{array}{c}
\vdash *X_i \\
\vdash *Y_i
\end{array}
\]

- Multi-type structural rules:

\[
\begin{array}{c}
\begin{array}{c}
X_1 \vdash Y_2 \\
X_2 \vdash Y_1
\end{array}
\quad \begin{array}{c}
\vdash *X_1 \\
\vdash *Y_1
\end{array}
\quad \begin{array}{c}
\vdash *X_2 \\
\vdash *Y_2
\end{array}
\end{array}
\quad \begin{array}{c}
\vdash N X_1 \vdash Y_2 \\
N X_1 \vdash Y_1
\end{array}
\quad \begin{array}{c}
\vdash N X_2 \vdash Y_1 \\
X_2 \vdash N Y_1
\end{array}
\quad \begin{array}{c}
\vdash *N X_1 \vdash Y_2 \\
\vdash *N X_2 \vdash Y_1
\end{array}
\quad \begin{array}{c}
\vdash *N X_1 \vdash Y_2 \\
\vdash *N X_2 \vdash Y_1
\end{array}
\]

- Pure \( L_i \) operational rules:

\[
\begin{array}{c}
\begin{array}{c}
X_i \vdash Y_i \\
Y_i \vdash X_i
\end{array}
\end{array}
\quad \begin{array}{c}
\vdash X_i \\
\vdash Y_i
\end{array}
\quad \begin{array}{c}
\vdash *X_i \\
\vdash *Y_i
\end{array}
\quad \begin{array}{c}
\vdash X_i \\
\vdash Y_i
\end{array}
\]

An essential feature of our calculus is that the logical rules are standard introduction rules of display calculi. This is key for achieving a canonical proof of cut elimination. The special behaviour of negation is captured by a suitable translation in a multi-type environment, which makes it possible to circumvent the technical difficulties created by the non-standard introduction rules of \([\text{[]]}\).

5 Properties

Soundness

We outline the verification of soundness of the rules of D.BL (resp. D.CBL) w.r.t. the semantics of perfect HBL (resp. HCBL). The first step consists in interpreting structural symbols as logical symbols according to their (precedent or succedent) position. This makes it possible to interpret sequents as inequalities, and rules as quasi-inequalities. The verification of soundness of the rules of D.BL (resp. D.CBL) then consists in checking the validity of their corresponding quasi-inequalities in perfect HBL (resp. HCBL). For example, the rules on the left-hand side below are interpreted as the quasi-inequalities on the right-hand side:

\[
\begin{array}{c}
PX_2 \vdash Y_1 \\
X_2 \vdash N Y_1
\end{array}
\quad \implies \forall a_1 \forall a_2 [p(a_2) \leq_1 b_1 \Leftrightarrow a_2 \leq_2 n(b_1)]
\]

\[
\begin{array}{c}
X_i \vdash Y_i \\
*Y_i \vdash *X_i
\end{array}
\quad \implies \forall a_i \forall b_i [a_i \leq_1 b_i \Leftrightarrow \sim b_i \leq_1 \sim a_i]
\]
The verification of soundness of pure-type rules and of the introduction rules following this procedure is routine, and is omitted. The validity of the quasi-inequalities corresponding to multi-type structural rules follows straightforwardly from the observation that the quasi-inequality corresponding to each rule is obtained by running the algorithm ALBA [23, Section 3.4] on one of the defining inequalities of HBL (resp. HCBL). For instance, the soundness of the first rule above is due to \( p \) and \( n \) being isomorphisms (by (H2) in Definition 3.2).

**Completeness**

In order to prove completeness, we shall introduce translations from sequents in the language of BL (resp. CBL) into sequents in the language of D.BL (resp. D.CBL).

Let \( t_1(\cdot), t_2(\cdot) : \mathcal{L} \to \mathcal{L}_{MT} \) be maps between the language \( \mathcal{L} \) of BL and \( \mathcal{L}_{MT} \) of D.BL inductively defined as follows:

\[
\begin{align*}
  t_1(p) & := p_1 & t_2(p) & := p_2 \\
  t_1(\mathbf{t}) & := 1_1 & t_2(\mathbf{t}) & := 0_2 \\
  t_1(\mathbf{E}) & := 0_1 & t_2(\mathbf{E}) & := 1_2 \\
  t_1(\mathbf{T}) & := 1_1 & t_2(\mathbf{T}) & := 1_2 \\
  t_1(\bot) & := 0_1 & t_2(\bot) & := 0_2 \\
  t_1(A \land B) & := t_1(A) \cap t_1(B) & t_2(A \land B) & := t_2(A) \cup t_2(B) \\
  t_1(A \lor B) & := t_1(A) \cup t_1(B) & t_2(A \lor B) & := t_2(A) \cap t_2(B) \\
  t_1(A \otimes B) & := t_1(A) \cap t_1(B) & t_2(A \otimes B) & := t_2(A) \cup t_2(B) \\
  t_1(A \oplus B) & := t_1(A) \cup t_1(B) & t_2(A \oplus B) & := t_2(A) \lor t_2(B) \\
  t_1(\neg A) & := p t_2(A) & t_2(\neg A) & := n t_1(A)
\end{align*}
\]

A sequent \( A \vdash B \) of BL is translated as \( t_1(A) \vdash t_1(B) \) of D.BL. For CBL we also need the following translation for the conflation connective:

\[
t_1(\neg A) := p \neg t_2(A) \quad t_2(\neg A) := n \neg t_1(A)
\]

The following proposition is immediate.

**Proposition 5.1.** For every formula \( A \) of BL (resp. CBL), the sequents \( t_1(A) \vdash t_1(A) \) and \( t_2(A) \vdash t_2(A) \) are derivable in D.BL (resp. D.CBL).

**Proof.** By induction on the complexity of the formula \( A \). If \( A \) is an atomic formula, the translation of \( t_i(A) \vdash t_i(A) \) with \( i \in \{1, 2\} \) is \( A_i \vdash A_i \), hence it is derivable using (Id) in \( L_1 \) and \( L_2 \), respectively. If \( A = A_1 \otimes A_2 \), then \( t_i(A_1 \otimes A_2) = t_i(A_1) \cap t_i(A_2) \) and if \( A = A_1 \oplus A_2 \), then \( t_i(A_1 \oplus A_2) = t_i(A_1) \lor t_i(A_2) \).

By induction hypothesis, \( t_i(A_1) \vdash t_i(A_1) \). So, it is enough to show that:

\[
\begin{align*}
  t_i(A_1) \vdash t_i(A_1) & \quad t_i(A_2) \vdash t_i(A_2) \\
  t_i(A_1) \land t_i(A_2) & \vdash t_i(A_1) & t_i(A_2) & \vdash t_i(A_2) \\
  t_i(A_1) \lor t_i(A_2) & \vdash t_i(A_1) & t_i(A_2) & \vdash t_i(A_2) \\
  t_i(A_1) \otimes t_i(A_2) & \vdash t_i(A_1) & t_i(A_2) & \vdash t_i(A_2) \\
  t_i(A_1) \oplus t_i(A_2) & \vdash t_i(A_1) & t_i(A_2) & \vdash t_i(A_2)
\end{align*}
\]

\[\text{E} \quad \text{C} \quad \text{W} \quad \text{E} \quad \text{C}\]

\(3\)As discussed in [23], the soundness of the rewriting rules of ALBA only depends on the order-theoretic properties of the interpretation of the logical connectives and their adjoints and residuals. The fact that some of these maps are not internal operations but have different domains and codomains does not make any substantial difference.
The arguments for \( A = A_1 \wedge A_2 \) and \( A = A_1 \lor A_2 \) are similar and they are omitted.

If \( A = \neg B \), then \( t_1(\neg B) = \text{pr}_2(B) \) and \( t_2(\neg B) = \text{ntr}_1(B) \). By induction hypothesis \( t_i(A) \vdash t_i(A) \).

Hence it is enough to show that:

\[
\frac{t_2(B) \vdash t_2(B)\quad \text{Pr}_2(B) \vdash \text{Pr}_2(B)}{\text{Pr}_2(B) \vdash \text{Pr}_2(B)}
\]

\[
\frac{\text{Pr}_2(B) \vdash \text{pr}_2(B)}{\text{Pr}_2(B) \vdash \text{pr}_2(B)}
\]

\[
\frac{t_1(B) \vdash t_1(B)\quad \text{Ntr}_1(B) \vdash \text{Ntr}_1(B)}{\text{Ntr}_1(B) \vdash \text{Ntr}_1(B)}
\]

\[
\frac{\text{Ntr}_1(B) \vdash \text{ntr}_1(B)}{\text{ntr}_1(B) \vdash \text{ntr}_1(B)}
\]

If \( A = -B \), then \( t_1(-B) = p \sim t_2(B) \) and \( t_2(-B) = n \sim t_1(B) \). By induction hypothesis \( t_i(B) \vdash t_i(B) \).

Hence it is enough to show that:

\[
\frac{t_2(B) \vdash t_2(B)\quad \text{Pr}_2(B) \vdash \text{Pr}_2(B)}{\text{Pr}_2(B) \vdash \text{Pr}_2(B)}
\]

\[
\frac{\text{Pr}_2(B) \vdash \text{pr}_2(B)}{\text{Pr}_2(B) \vdash \text{pr}_2(B)}
\]

\[
\frac{t_1(B) \vdash t_1(B)\quad \text{Ntr}_1(B) \vdash \text{Ntr}_1(B)}{\text{Ntr}_1(B) \vdash \text{Ntr}_1(B)}
\]

\[
\frac{\text{Ntr}_1(B) \vdash \text{ntr}_1(B)}{\text{ntr}_1(B) \vdash \text{ntr}_1(B)}
\]

\[\square\]

**Proposition 5.2.** For all formulas \( A, B \) of BL (resp. CBL), if \( A \vdash B \) is derivable in BL (resp. CBL), then \( t_1(A) \vdash t_1(B) \) is derivable in D.BL (resp. D.CBL).

**Proof.** In what follows we show that the translations of the axioms and rules of BL (resp C.BL) are derivable in D.BL (resp. D.CBL). Since BL (resp C.BL) is complete w.r.t. the class of bilattice algebras (by Theorem 2.10), and hence w.r.t their associated heterogeneous algebras (by Propositions 3.4 and 3.5), this is enough to show the completeness of D.BL (resp. D.CBL). For the sake of readability, although each formula \( A \) in precedent (resp. succeedent) position should be written as \( t_1(A) \), we suppress it in the derivation trees of the axioms.

The Identity axiom \( A \vdash A \) is proved in Proposition 5.1.

The derivations of the binary rules are standard and we omit them.

The translations of the axioms \( \text{\&} \vdash A, A \vdash \top, A \vdash \bot \) are \( 0_1 \vdash A_1, A_1 \vdash 1_1, 0_1 \vdash A_1 \), and \( A_1 \vdash 1_1 \), respectively. The derivations are straightforward and they are omitted, in particular they make use of the introduction rules of \( 1_1 \) and \( 0_1 \), Weakening (W) and the structural rules for the neutral element \( (1_1 \text{ and } 0_1) \).

The translations of the axioms \( A \vdash \neg \text{\&} \vdash A, \neg \vdash A \vdash \neg \top \) are \( A_1 \vdash \text{p}1_2, \text{p}0_2 \vdash A_1, \text{p}0_2 \vdash A_1, \text{p}1_2 \vdash A_1, \) and \( A_1 \vdash \text{p}1_2 \), respectively. The derivations are straightforward and they are omitted, in particular they make use of the introduction rules of \( 1_2 \) and \( 0_2 \), the structural rules \( \text{P}0_2 \) and \( \text{P}1_2 \).

\[
\neg A \vdash A \quad \Rightarrow \quad \text{pn}A_1 \vdash A_1 \quad \text{and} \quad A_1 \vdash \text{pn}A_1
\]

\[
\frac{A_1 \vdash A_1}{\text{NA}_1 \vdash \text{NA}_1}
\]

\[
\frac{\text{PN}A_1 \vdash A_1}{\text{pn}A_1 \vdash A_1}
\]

\[
\frac{A_1 \vdash A_1}{\text{N}A_1 \vdash \text{N}A_1}
\]

\[
\frac{A_1 \vdash \text{pn}A_1}{A_1 \vdash \text{pn}A_1}
\]

\[
\neg A \vdash A \quad \Rightarrow \quad p \sim n \sim A_1 \vdash A_1
\]
\[-(A \land B) \rightarrow \neg A \land \neg B \quad \rightsquigarrow \quad p \sim_2 (A_2 \cup_2 B_2) \leftrightarrow p \sim_2 A_2 \cap_1 p \sim_2 B_2\]
To argue that the calculus introduced in Section 4 is conservative w.r.t. BL (resp. CBL), we need to show that for all \( \phi, \psi \in \mathcal{L} \) such that \( \mathcal{B} \vdash \phi \) and \( \mathcal{B} \vdash \psi \), it follows that \( \mathcal{B} \vdash \phi \wedge \psi \) and \( \mathcal{B} \vdash \phi \lor \psi \). This can be done by constructing appropriate proofs in the calculus. The details of this proof are technical and involve the use of the calculus rules for \( \mathcal{L} \) and the corresponding rules for \( \mathcal{B} \).
\[ \models_{\text{HBL}} \text{ (resp. } \models_{\text{HCBL}}) \] the semantic consequence relation arising from (perfect) HBL (resp. HCBL). We need to show that, for all formulas \( A \) and \( B \) of the original language of BL (resp. CBL), if \( t_1(A) \vdash t_1(B) \) is a D.BL-derivable (resp. D.CBL-derivable) sequent, then \( A \vdash_{\text{BL}} B \) (resp. \( A \vdash_{\text{CBL}} B \)). This can be proved using the following facts: (a) the rules of D.BL (resp. D.CBL) are sound w.r.t. perfect HBL-algebras (resp. HCBL-algebras); (b) BL (resp. CBL) is complete w.r.t. B (resp. CB); and (c) B (resp. CB) are equivalently presented as (perfect) HBL-algebras (resp. cf. HCBL-algebras, Section 3), so that the semantic consequence relations arising from each type of structures preserve and reflect the translation (cf. Propositions 5.1 and 5.2). Let then \( A, B \) be formulas of the original BL (resp. CBL)-language. If \( t_1(A) \vdash t_1(B) \) is a D.BL (resp. D.CBL)-derivable sequent, then, by (a), \( t_1(A) \models_{\text{HBL}} t_1(B) \) (resp. \( t_1(A) \models_{\text{HCBL}} t_1(B) \)). By (c) and Proposition 5.3, this implies that \( A \models B \) (resp. \( A \models_{\text{CB}} B \)). By (b), this implies that \( A \vdash_{\text{BL}} B \) (resp. \( A \vdash_{\text{CBL}} B \)), as required.

Subformula property and cut elimination

Let us briefly sketch the proof of cut elimination and subformula property for D.BL (resp. D.CBL). As discussed earlier on, proper display calculi have been designed so that the cut elimination and subformula property can be inferred from a meta-theorem, following the strategy introduced by Belnap for display calculi [4]. The meta-theorem to which we will appeal for D.BL (resp. D.CBL) was proved in [15].

All conditions in [15] Theorem 4.1 except \( C'_8 \) are readily seen to be satisfied by inspection of the rules. Condition \( C'_8 \) requires to check that reduction steps are available for every application of the cut rule in which both cut-formulas are principal, which either remove the original cut altogether or replace it by one or more cuts on formulas of strictly lower complexity. In what follows, we only show \( C'_8 \) for the unary connectives \( \sim \) and \( n \) (the proof for \( p \) is analogous). The cases of lattice connectives are standard and they are omitted.

L-type connectives

\[
\begin{align*}
\frac{\vdash \pi_1 \\ X_i \vdash \sim \iota \lambda_i}{\vdash \pi_1} & \quad \frac{\vdash \pi_2 \\ \gamma_i \lambda_i \vdash \lambda_i \iota_i}{\vdash \pi_2} \\
\frac{\vdash \pi_1 \\ \gamma_i \lambda_i \vdash \lambda_i \iota_i}{\vdash \pi_1} & \quad \frac{\vdash \pi_2 \\ X_i \vdash \sim \iota \lambda_i \iota_i}{\vdash \pi_2}
\end{align*}
\]

\[
\begin{align*}
\frac{\vdash \pi_1}{\vdash \pi_1} & \quad \frac{\vdash \pi_2}{\vdash \pi_2} \\
\frac{X_i \vdash \gamma_i \lambda_i \iota_i}{X_i \vdash \gamma_i \lambda_i \iota_i} & \quad \frac{X_i \vdash \gamma_i \lambda_i \iota_i}{X_i \vdash \gamma_i \lambda_i \iota_i}
\end{align*}
\]

Multi-type connectives

\[
\begin{align*}
\frac{\vdash \pi_1}{\vdash \pi_1} & \quad \frac{\vdash \pi_2}{\vdash \pi_2} \\
\frac{X_2 \vdash \lambda a_1}{X_2 \vdash \lambda a_1} & \quad \frac{X_2 \vdash \lambda a_1}{X_2 \vdash \lambda a_1}
\end{align*}
\]

\[
\begin{align*}
\frac{\vdash \pi_1}{\vdash \pi_1} & \quad \frac{\vdash \pi_2}{\vdash \pi_2} \\
\frac{X_2 \vdash \lambda a_1}{X_2 \vdash \lambda a_1} & \quad \frac{X_2 \vdash \lambda a_1}{X_2 \vdash \lambda a_1}
\end{align*}
\]

6 Conclusions and future work

The modular character of proper multi-type display calculi makes it possible to easily extend our formalism in order to axiomatize axiomatic extensions (e.g. the logic of classical bilattices with conflation [11 Definition 2.11]) as well as language expansions of the basic bilattice logics treated in the present paper. Expansions of bilattice logic have been extensively studied in the literature as early as in [11], which introduces an implication enjoying the deduction-detachment theorem (see also [9]). More recently, modal operators have been added to bilattice logics, motivated by potential applications to computer science and in particular verification of programs [27, 30]; as
well as dynamic modalities, motivated by applications in the area of dynamic epistemic logic \cite{28,29}.

Yet more recently, bilattices with a negation not necessarily satisfying the involution law \((\neg\neg a = a)\) have been introduced with motivations of domain theory and topological duality (see \cite{26}), and the study of the corresponding logics has been started \cite{31}. These logics are weaker than the one considered in the present paper, and so adapting our display calculus formalism to them might prove a more challenging task (in particular, the translations introduced in Section 5 may need to be redefined, as they rely on the maps \(p\) and \(n\) being lattice isomorphisms, which is no longer true in the non-involutive case).

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