Low Temperature Expansions for Potts Models

by

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On simple cubic lattices, we compute low temperature series expansions for the energy, magnetization and susceptibility of the three-state Potts model in $D = 2$ and $D = 3$ to 45 and 39 excited bonds respectively, and the eight-state Potts model in $D = 2$ to 25 excited bonds. We use a recursive procedure which enumerates states explicitly. We analyze the series using Dlog Pade analysis and inhomogeneous differential approximants.

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A subset of the present authors recently described a method [1,2] similar to the finite lattice method [3] for generating low temperature series for discrete models. This method is based on a recursive computer enumeration of configurations and has resulted in series expansions for the $D = 3$ Ising model that extend available series by several terms [2-4].

In this paper, we present results from a similar analysis for the low temperature expansions of Potts models in two and three dimensions on a simple cubic lattice. We will not describe the method used in much detail. It has already been outlined in Ref [2] and will be described in detail in a separate paper.

The energy for the $q$-state Potts model is defined to be

$$E = \sum_{\langle ij \rangle} \left[ 1 - \delta_{\sigma_i, \sigma_j} \right]$$

(1)

where $\sigma_i$ is a site-defined field that takes $q$ possible values. The sum is taken over all nearest neighbor pairs of spins with $\delta$ being the Kronecker symbol.

The partition function is the sum of the Boltzmann weights over all configurations

$$Z = \sum_{\{\sigma\}} e^{-\beta E}$$

(2)

Sorting configurations by energy, we rewrite this as a sum over $E$. Defining $P(E)$ to be the number of states with a given energy $E$, we have

$$Z = \sum_{E=0}^{dN} P(E) u^E$$

(3)

where $d$ is the number of dimensions, $N$ is the number of sites and $u = e^{-\beta}$.

We compute the coefficients $P(E)$ exactly on small systems by recursively assembling the system one site at a time. The method enables us to build up a lattice with arbitrary length in one direction. Intermediate stages require an explicit enumeration of exposed slices transverse to this direction. This effectively reduces the computational complexity to that of a system of one less dimension.

The starting point is a list of all states and corresponding energies for a single transverse layer of the lattice. In $D = 2$, the transverse layer is a line of
spins, in \(D = 3\), it is a plane of spins. All the spins outside this transverse layer are frozen to the same value; that is, the boundary conditions in the longitudinal direction are cold. Spins are then sequentially freed to build up the lattice in the longitudinal direction. We store the number of states with a given energy, \(E\), and exposed top layer in an array \(p_0(E, I)\), where the integer \(I\) is an index which specifies the exact configuration of the exposed transverse layer using bit-coding. When a new spin is added, we obtain the new counts \(p'_0(E, I)\) as a sum over the old counts

\[
p'_0(E, I) = \sum_{I'} p_0(E - \Delta(I, I'), I').
\] (4)

Here \(I'\) can differ from \(I\) only in the bits representing the newly covered spins, and \(\Delta(I, I')\) is the change in energy due to any newly changed bonds. For the present analysis we add the spins one at a time. Thus, the sum in the above equation is only over \(q\) terms, representing the \(q\) possible values of the newly covered spin. After the lattice is grown, a sum over the top layers gives the resulting \(P(E) = \sum_I p_0(E, I)\). We always continue this recursion sufficiently to avoid finite size errors in the longitudinal direction.

As the temperature goes to zero, so does the variable \(u\). Thus, what we have in Eq. (3) is the low temperature expansion for \(Z\). From it, we compute the series for the average energy, \(\langle E \rangle = (u \frac{\partial}{\partial u}) \log(Z)\). Subtracting this expectation value before adding the last spin from its value after adding the last spin, we obtain the average energy per new site. This also eliminates the effect of the fixed end boundaries. Writing,

\[
\langle E/N \rangle = \sum_j e_j u^j
\] (5)

the low temperature expansion amounts to listing the coefficients \(e_j\).

The recursive technique can be extended to enable calculation of quantities such as the magnetization and susceptibility. We define a magnetization in the Potts model by

\[
\langle M \rangle = \sum_i \langle \delta_{\sigma_i,0} \rangle = N \sum_j m_j u^j
\] (6)
assigning to each unexcited spin the value one, and to each excited spin the value zero. The calculation of susceptibility is carried out using the fluctuation-dissipation theorem and we define the low temperature series coefficients $\chi_j$ as follows:

$$N\chi = \langle M^2 \rangle - \langle M \rangle^2 = N \sum_j \chi_j u^j \quad (7)$$

Let $p(E, M, I)$ to be the number of states with given energy, magnetization and exposed top layer $I$. To compute any moment of the magnetization, it would be sufficient to compute $p(E, M, I)$. However, one can avoid computing this memory expensive quantity. Let us demonstrate this for the case of the magnetization.

First, note that $p_0(E, I) = \sum_M p(E, M, I)$ is the count we had before. To compute the magnetization, we need one more count: $p_1(E, I) = \sum_M M p(E, M, I)$. This is because the expectation value of magnetization can be written as,

$$\langle M \rangle = \frac{\sum_E P_1(E) u^E}{Z}, \quad (8)$$

with $P_1(E) = \sum_I p_1(E, I)$ and $Z = \sum_{E,I} p_0(E,I) u^E$. The counting scheme for $p_1(E, I)$ is easy to derive. In analogy with Eq. (4) one can write,

$$p_1'(E, I) = \sum_M M p'_1(E, M, I) = \sum_{M,I'} M p(E - \Delta_e, M - \Delta_m, I') =$$

$$= \sum_{M,I'} (M - \Delta_m + \Delta_m) p(E - \Delta_e, M - \Delta_m, I') =$$

$$= \sum_{I'} [p_1(E - \Delta_e, I') + \Delta_m p_0(E - \Delta_e, I')] \quad (9)$$

Here $\Delta_e \equiv \Delta_e(I, I')$ and $\Delta_m \equiv \Delta_m(I, I')$ denote the change in energy and magnetization when adding the new spin. Thus, computation of the magnetization series requires just the introduction of one additional count, (which only doubles the memory requirement) and we can calculate the magnetization series to essentially the same order as the energy series.

For the susceptibility series, we need to compute $\langle M^2 \rangle$. This requires a count $p_2(E, I) = \sum_M M^2 p(E, M, I)$. It is easy to see that $p_2$ obeys the recursion
relation,

\[ p'_2(E, I) = \sum_{I'} \left[ p_2(E - \Delta_e, I') + 2\Delta_m p_1(E - \Delta_e, I') + \Delta^2_m p_0(E - \Delta_e, I') \right] \] (10)

As discussed in Ref [2], we work on generalized helical lattices and label our lattice points by their ordinal number on a helix. In three dimensions, the nearest neighbors on the lattice in the \( x \), \( y \) and \( z \) directions are separated by \( h_x \), \( h_y \) and \( h_z \) steps along the helix respectively. We assume that the \( h \)'s are ordered so that \( h_x < h_y < h_z \). Then, our numerical method requires us to keep track of at most \( q^{h_z} \) states and so we try to make \( h_z \) as small as possible. Let \( n \) be the effective lattice size, defined as the length of the shortest closed path on the helical lattice. For a given set of \( h \) values, if we compute the set of nonzero vectors \( S = \{ n_x, n_y, n_z; n_x h_x + n_y h_y + n_z h_z = 0 \} \) then \( n = \text{Min}_S(|n_x| + |n_y| + |n_z|) \). The series expansion will be correct up to the order \( u^{(4n-1)} \). Higher orders are corrupted by contributions from graphs that wrap around the lattice. However, as described in Ref [2], we can combine results from different helical lattices to cancel these finite size effects to some order in the series. In two dimensions, there is not enough complexity for this cancellation mechanism to work. Instead, one observes that keeping \( h_y \) spins in the top layer, the optimal choice of the lattice is \( h_x = h_y - 1 \). This gives the series correct to order \( 4h_y - 3 \).

Our series are listed in Tables I-III. The series for \( D = 2 \) and \( D = 3 \) Potts Models were computed on a CM-200/CM-2 Connection Machine using CM-Fortran and C* programs. The \( D = 2 \), 8-states model series were computed on a CRAY-2 using a C code and checked on a CM-2 using CM-Fortran code. To get 3d series up to 39 excited bonds, we used lattices of effective size up to 10. This required the top layer to have at most 15 spins. In Table IV we show the lattices and combination factors used.

Note that our definition of \( M \) in Eq. 6 is such that in the completely disordered state it has the value \( N/q \). The proper order parameter for Potts models is the so called reduced magnetization \( M_R \) which is related to \( M \) by the formula \( M_R = [qM - N]/(q - 1) \). The reduced magnetization takes the values \( N \) and
0 in the completely ordered and disordered states respectively. The results we
give below from our analysis of series are for the reduced magnetization and the
corresponding susceptibility.

In addition to the usual dlog Pade (DlP) method [5,6], we will use the method
of inhomogeneous differential approximants (IDA) introduced by Fisher and Au-
Yang [7] (see also [8]). These are useful in handling singularities of the form,

\[ F(u) = A(u)(1 - u/u_c)^\zeta + B(u) \]  

where \( A \) and \( B \) are analytic in \( u \).

Given a series expansion for \( F(u) \) to \( N \)-th order, \( F_N(u) = 1 + \sum_{i=1}^{N} f_i u^i \),
(we will use the simplification that one can always normalize the series so that
the constant term is unity), one computes coefficients for polynomials \( Q_L(u) = \sum_{i=0}^{L} q_i u^i \), \( R_M(u) = 1 + \sum_{i=1}^{M} r_i u^i \) and \( S_J(u) = \sum_{i=0}^{J} s_i u^i \), which satisfy,

\[ F_N Q_L + S_J = F'_N R_M \]  

to order \( N \), with \( L + M + J = N - 2 \). Note that for \( S_J = 0 \) one gets the usual
Dlog Pade ratio from \( Q_L/R_M \). It is easy to see that potential critical points
\( u_c \) are the zeros of \( R_M \) and for each of these, the exponent \( \zeta \) is estimated as
\( \zeta = -Q_L(u_c)/R'_M(u_c) \).

Consider first the \( D = 2 \) Potts models. Here, we know from self-duality that
the critical point is at \( u_c = 1/(\sqrt{q} + 1) \). For \( q \leq 4 \) the transition is continuous and
the critical exponents are known exactly (see [9] and references therein). Models
with \( q > 4 \) undergo a first order phase transition. Having results from both of
the above categories available, our \( D = 2 \) series offer themselves as a good testing
ground for series analysis methods.

Given the low temperature series, does one has enough information to de-
termine the nature of the transition, assuming that the critical temperature is
exactly known. In \( D = 2 \), because of self duality, this is easy if the series at hand
has a sufficient number of terms. To illustrate this, we plot in Fig. 1a the energy
as a function of \( u \) from the low temperature series and its dual high temperature
series for $q = 3$ and $q = 8$. In Fig. 1b and Fig. 1c, we plot the latent heat $L(n)$ derived using duality at the known critical point as a function of the number of terms $n$ in the series. The fits of $L$ to a power law in $1/n$ (Fig. 1b and 1c) convincingly demonstrate that the $q = 3$ model has a second order transition while the $q = 8$ model has a first order transition with the latent heat equal to $1/2$ to 2 parts in a thousand.

In general however, self duality is not available as a symmetry. In this case, one must rely on DIP and IDA analysis on the low temperature series to determine the critical properties. Our arguments below are similar in spirit to the discussion presented by Enting and Guttmann [10].

If the system undergoes a second order phase transition, one expects in general that the order parameter $M_R/N$ vanishes at the critical point, approaching it with infinite slope. Estimates of the critical temperature (poles) from DIP-s should then cluster well around the exact value and estimates of the critical exponent $\beta$ (residues) should also be quite accurate. On the other hand, at a first order transition, the magnetization is finite and nonzero and its slope can be either finite or infinite. In this case one would expect the approximants to continue the curve beyond the critical temperature along the so called pseudo-spinodal line [11]. This line intersects the temperature axis at the point $u_S$ with corresponding exponent $\beta_S$. Applying DIP-s in this case should then result in a systematic overestimation of the critical temperature because it is $u_S$ that the Pade is trying to fit.

In case of a first order transition with a divergent slope of the magnetization as the transition is approached, DIP-s still tend to overestimate the transition point because the finite value of the magnetization is not modelled in the DIP-s (more detailed reasons can be found in [12]). However, for this case, the IDA-s should treat the situation better because they can account for a finite $<M/N>$ at the critical point. Thus, comparing the results of the two types of approximants one might be able to determine the order of the transition.

Applying DIP-s to the 45-term magnetization series of the 3-state model in
$D = 2$ leads to a slight systematic underestimation of the critical point. Taking into account seven most central approximants we got $u_c = 0.36595 \pm 0.00003$ which is to be compared with the exact value $u_c = 0.36602$. The error here corresponds to the scattering of values from the different DlP-s. In the light of the above discussion, this suggests that the transition is continuous. We estimate the critical exponent $\beta = 0.1084 \pm 0.0002$ by evaluating it at the known critical point for this model. The error bar is of course meaningless as it comes only from the error on the extrapolation and ignores the systematic effects of the finiteness of the series. The value obtained is about 2.5% below the exact result $\beta = 1/9$.

In the 8-state model in $D = 2$ on the other hand, DlP-s show a critical point at $u_c = 0.2628 \pm 0.0003$ which is substantially beyond the true value $u_c = 0.2612$. This suggests a first order phase transition. In Fig. 2 we plot $u_c$ versus $\beta$ for small values of $J$. The points for different $J$ lie fairly well on a line with an obvious tendency to overestimate the critical point again. This again establishes the first order nature of the transition. The corresponding pseudo-exponent estimated from DlP-s has the value $\beta_S = 0.059 \pm 0.005$.

Similar ideas can be applied to the energy and specific heat series. At a first order phase transition there is a finite latent heat but the energy curve can have either finite or infinite slope (specific heat) as that point is approached. DlP analysis of the $q = 3$ specific heat series in $D = 2$ shows a slight overestimate of the critical point, namely $u_c = 0.36626 \pm 0.00001$. IDA-s on the other hand lead to a small underestimate (see Fig. 3) giving an overall consistency with the second order phase transition present. DlP-s average for critical exponent $\alpha = 0.412 \pm 0.001$ is rather poor when compared to the exact value $\alpha = 1/3$. This is probably due to the strong confluent singularity present in this case [13]. The results of the IDA analysis is shown in Fig. 3 where we plot $u_c$ versus $\alpha$ for various $J$ values from 0 to 20 with $L$ and $M$ chosen to be equal or differing by at most one (see Eq. 12). Notice that if we fit the data to a straight line and compute the value of $\alpha$ at the exactly known critical point (vertical line in Fig. 3), we get a result which differs from the exact value by about 1%.
In the $q = 8$ model the results from the specific heat series and magnetization series are very consistent with each other. There is an overestimate of the critical point by DlP-s ($u_S = 0.2620 \pm 0.0001$) as well as by IDA-s. The averaged pseudo-exponent from DlP-s is $\alpha_S = 0.592 \pm 0.004$.

Finally, an analysis of the susceptibility series for the $q = 3$ model using the Dlog Pade and IDA analysis gave $\gamma = 1.47 \pm 0.02$ by extrapolating to the known critical point as was done above for $\alpha$ and $\beta$. This is to be compared with the exact result $\gamma = 13/9 = 1.444\ldots$. For the $q = 8$ model, we estimate $u_S = 0.2629 \pm 0.0009$, $\gamma_S = 1.16 \pm 0.07$.

Let us now turn to the series for the $q = 3$ Potts model in $D = 3$ given in Table III. Theoretically, this is the most interesting case of those considered in this paper, because of its connection to the SU(3) lattice gauge theory in $D = 4$ [14] and because of the lack of any exact results. There was a good deal of confusion about the nature of the transition in the past but by now the first order nature of this transition seems to be well established [15]. Although the transition temperature is not known exactly, there are very accurate Monte Carlo estimates for it. For the purpose of our analysis we will assume that the value $u_c = 0.57659(1)$ estimated in Ref. [15] is the exact result. We will do so because we found that neither the DIP nor the IDA analyses can yield a more accurate value.

Consider first the magnetization series. In Fig. 4 we show the results from central Dlog Pades. The data clusters well around the value $u_c = 0.5785 \pm 0.0003$, quite far from 0.57659. IDA-s show the same tendency as can be seen in Fig. 5. Here, the results from small $J$ fall very nicely on a straight line beyond the critical point which is marked by a cross. These results support the conclusion that this model has a first order phase transition in agreement with [10] and Monte Carlo data [15]. The critical pseudo-exponent from DIP-s has the value $\beta_S = 0.204 \pm 0.002$ which agrees very accurately with results of Miyashita et al. [16] who analyzed a shorter series, and also with numerical simulations [17].

Next consider the specific heat series. Here one gets stable results from many
central Dlog Pades. Also, the IDA-s are quite stable for small $J$. Fig. 6 shows the results of these analyses. The circles correspond to the DlP-s and the other symbols are the results from the IDA-s for $J \leq 4$. There is no clear evidence for systematic overestimation of the critical point by neither DlP-s nor IDA-s suggesting that the transition is weakly first order in this variable. The straight lines in Fig. 6 are least square fits to IDA-s and DlP-s. Since the latent heat is small, one would expect that these should intersect at the critical point where they are both dominated by the singularity. Away from the critical point, the Dlog Pade and the IDA-s treat the non-leading corrections differently and so the results from them could be different. Indeed the lines in Fig. 6 intersect at $u_c = 0.5766(2), \alpha = 0.421(2)$. We have estimated the error on these parameters from the errors in the fitted parameters for the straight lines.

Finally, we analyzed the $q = 3$ susceptibility series in three dimensions. Here the combined data for DlP and IDA fall nicely on a line. We estimate $\gamma = 1.085 \pm 0.005$ by evaluating the fitted line at $u_c = 0.57659$.

Recently, Vohwinkel [18] has extended the shadow lattice method and shown how one can obtain extremely high order low temperature expansions. His series for the magnetization has several more terms than ours and although he does not generate series for the other quantities we measure in the present paper, we presume he can do so. A challenge now is to see if the ideas of Ref. [18] can be incorporated into our method.

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Figure Captions

Figure 1a. The average energy from the series expansions in $D = 2$ for $q = 3$ and $q = 8$. Duality was used to get the series in the high temperature phase from the series in the low temperature phase. The exactly known transition points are shown as vertical lines.

Figure 1b. The Latent heat $L(n)$ as a function of $n$ for $q = 3$ in $D = 2$. The solid line is a fit to a power law and demonstrates that for $n = \infty$, the latent heat vanishes.

Figure 1c. The Latent heat $L(n)$ as a function of $n$ for $q = 8$ in $D = 2$. The solid line is a fit to a power law plus a constant and demonstrates that for $n = \infty$, the latent heat is about $1/2$.

Figure 2. $u_c$ versus the exponent $\beta$ from the magnetization series for the $q = 8$ model in $D = 2$. The exact value of $u_c$ is the vertical line.

Figure 3. $u_c$ versus the exponent $\alpha$ from the series for the specific heat for $q = 3$ in $D = 2$ from IDA analysis. The vertical line is the exact value of $u_c$.

Figure 4. $u_c$ versus the exponent $\beta$ from the magnetization series for the $q = 3$ model in $D = 3$ using DlP-s.

Figure 5. $u_c$ versus the exponent $\beta$ from the magnetization series for the $q = 3$ model in $D = 3$ using IDA-s with small $J$ values. The ‘exact’ value of $u_c$ is marked with a plus and is a Monte Carlo result from Ref. 15.

Figure 6. $u_c$ versus the exponent $\alpha$ from the series for the specific heat for $q = 3$ in $D = 3$ from DlP and IDA analysis. The transition point is accurately determined by the crossing of the lines for DlP-s and IDA-s.
**Table I:** The low temperature expansion coefficients $e_i$, $m_i$ and $\chi_i$ for the energy, magnetization and susceptibility series for the $q = 3$ Potts model in $D = 2$ on a simple cubic lattice.

| i  | $e_i$  | $m_i$  | $c_i$  |
|----|--------|--------|--------|
| 0  | 0      | 1      | 0      |
| 1  | 0      | 0      | 0      |
| 2  | 0      | 0      | 0      |
| 3  | 0      | 0      | 0      |
| 4  | 8      | -2     | 2      |
| 5  | 0      | 0      | 0      |
| 6  | 24     | -8     | 16     |
| 7  | 28     | -8     | 16     |
| 8  | 32     | -24    | 100    |
| 9  | 216    | -72    | 216    |
| 10 | 160    | -140   | 844    |
| 11 | 660    | -320   | 1,552  |
| 12 | 2,072  | -1,164 | 7,844  |
| 13 | 1,664  | -1,560 | 12,112 |
| 14 | 11,760 | -7,044 | 60,268 |
| 15 | 17,700 | -13,000| 118,944|
| 16 | 41,088 | -35,984| 424,072|
| 17 | 156,468| -101,736| 1,081,392|
| 18 | 207,240| -219,616| 3,201,728|
| 19 | 849,300| -647,536| 8,670,688|
| 20 | 1,817,048| -1,602,194| 25,713,154|
| 21 | 4,021,780| -3,970,384| 67,206,560|
| 22 | 13,178,264| -11,239,056| 203,077,760|
| 23 | 25,754,296| -26,891,584| 532,881,432|
| 24 | 75,653,408| -73,534,214| 1,558,159,918|
| 25 | 193,458,400| -191,374,464| 4,250,639,632|
| 26 | 440,725,376| -486,815,472| 11,956,293,152|
| 27 | 1,296,485,460| -1,323,802,480| 33,296,697,848|
| 28 | 3,009,317,200| -3,380,001,144| 92,820,406,096|
| 29 | 7,977,739,920| -8,964,296,480| 257,249,275,776|
| 30 | 21,217,637,824| -23,766,809,488| 721,023,458,656|
| 31 | 51,359,965,976| -61,628,612,552| 1,986,080,278,600|
| 32 | 140,886,970,816| -165,024,619,666| 5,561,045,323,298|
| 33 | 354,038,121,756| -432,231,505,864| 15,359,165,767,512|
| 34 | 916,153,258,448| -1,142,608,252,368| 42,717,426,328,784|
| 35 | 2,439,917,893,708| -3,039,729,276,192| 118,457,421,095,792|
| 36 | 6,161,990,034,800| -7,994,207,679,356| 328,170,466,563,836|
| 37 | 16,397,314,674,708| -21,295,402,476,752| 909,829,346,983,664|
| 38 | 42,540,620,667,584| -56,399,959,949,412| 2,520,622,606,225,868|
| 39 | 110,314,458,936,968| -149,510,058,508,096| 6,973,368,153,491,880|
| 40 | 292,427,669,006,272| -398,341,255,729,746| 19,322,697,243,220,158|
| 41 | 756,553,239,055,504| -1,056,154,269,407,136| 53,409,977,638,363,032|
| 42 | 1,994,873,374,110,312| -2,813,530,068,950,904|
| 43 | 5,238,354,130,103,568| -7,489,714,245,193,504|
| 44 | 13,686,401,970,717,088| -19,928,407,714,223,232|
| 45 | 36,195,015,152,016,276| -53,175,417,534,052,136|
Table II: The low temperature expansion coefficients $e_i$, $m_i$ and $\chi_i$ for the energy, magnetization and susceptibility series for the $q = 8$ Potts model in $D = 2$ on a simple cubic lattice.

| $i$ | $e_i$       | $m_i$       | $c_i$       |
|-----|-------------|-------------|-------------|
| 0   | 0           | 1           | 0           |
| 1   | 0           | 0           | 0           |
| 2   | 0           | 0           | 0           |
| 3   | 0           | 0           | 0           |
| 4   | 28          | -7          | 7           |
| 5   | 0           | 0           | 0           |
| 6   | 84          | -28         | 56          |
| 7   | 588         | -168        | 336         |
| 8   | -588        | 91          | 0           |
| 9   | 4,536       | -1,512      | 4,536       |
| 10  | 11,760      | -4,060      | 14,504      |
| 11  | -13,860     | 0           | 15,792      |
| 12  | 205,072     | -68,859     | 288,169     |
| 13  | 144,144     | -84,840     | 556,752     |
| 14  | 271,460     | -256,424    | 2,062,088   |
| 15  | 7,553,700   | -2,678,760  | 15,132,264  |
| 16  | -713,692    | -2,049,229  | 25,582,802  |
| 17  | 45,219,048  | -21,023,016 | 165,495,792 |
| 18  | 232,853,880 | -93,466,856 | 720,185,368 |
| 19  | -14,850,780 | -107,162,496| 1,588,846,728|
| 20  | 2,822,644,748| -1,187,630,969| 10,588,862,669|
| 21  | 6,212,314,080| -3,159,741,984| 33,856,668,720|
| 22  | 8,166,041,884| -7,756,117,236| 108,773,186,200|
| 23  | 131,708,763,816| -56,277,329,304| 596,266,427,232|
| 24  | 167,481,870,528| -118,516,443,339| 1,709,093,729,238|
| 25  | 846,878,642,400| -506,752,816,584| 7,126,592,218,032|
Table III: The low temperature expansion coefficients $e_i$, $m_i$ and $\chi_i$ for the energy, magnetization and susceptibility series for the $q = 3$ Potts model in $D = 3$ on a simple cubic lattice.

| $i$ | $e_i$  | $m_i$  | $c_i$  |
|-----|--------|--------|--------|
| 0   | 0      | 1      | 0      |
| 1   | 0      | 0      | 0      |
| 2   | 0      | 0      | 0      |
| 3   | 0      | 0      | 0      |
| 4   | 0      | 0      | 0      |
| 5   | 0      | 0      | 0      |
| 6   | 12     | -2     | 2      |
| 7   | 0      | 0      | 0      |
| 8   | 0      | 0      | 0      |
| 9   | 0      | 0      | 0      |
| 10  | 60     | -12    | 24     |
| 11  | 66     | -12    | 24     |
| 12  | -168   | 28     | -56    |
| 13  | 0      | 0      | 0      |
| 14  | 420    | -90    | 270    |
| 15  | 900    | -180   | 540    |
| 16  | -1,728 | 318    | -930   |
| 17  | -2,448 | 432    | -1,296 |
| 18  | 6,708  | -1,320 | 4,768  |
| 19  | 9,462  | -1,992 | 7,968  |
| 20  | -14,280| 2,760  | -10,560|
| 21  | -49,686| 9,368  | -36,992|
| 22  | 71,940 | -14,460| 64,812 |
| 23  | 177,192| -35,280| 163,440|
| 24  | -194,544| 36,680   | -16,5464|
| 25  | -684,300| 134,568 | -659,088|
| 26  | 515,892 | -108,516| 600,024|
| 27  | 3,087,234| -609,692| 3,278,256|
| 28  | -1,927,296| 370,500   | -1,980,408|
| 29  | -10,943,904| 2,153,016| -12,285,816|
| 30  | 3,863,712 | -792,218 | 5,005,014|
| 31  | 44,383,506| -8,867,580| 55,200,864|
| 32  | -4,406,976| 935,124   | -6,062,712|
| 33  | -177,069,948| 34,889,512 | -227,203,096|
| 34  | -1,133,220| 63,834    | 1,954,650|
| 35  | 652,560,090| -130,265,472| 914,339,736|
| 36  | 199,263,288| -39,322,372 | -|
| 37  | -2,553,456,210| 507,892,056 | -|
| 38  | -1,235,636,652| 239,776,590 | -|
| 39  | 9,742,992,324 | -1,940,344,524 | -|
**Table IV**: The lattices parameters and combination factors that give the series accurate to 39 excited bonds in $D = 3$.

| $h_x$ | $h_y$ | $h_z$ | Coefficient |
|-------|-------|-------|-------------|
| 9     | 14    | 15    | 2           |
| 11    | 12    | 15    | -1          |
| 9     | 11    | 15    | -2          |
| 10    | 13    | 14    | 1           |
| 11    | 12    | 14    | 5           |
| 9     | 11    | 14    | -1          |
| 7     | 12    | 13    | 1           |
| 10    | 11    | 13    | -3          |
| 8     | 10    | 13    | 1           |
| 5     | 11    | 12    | 3           |
| 7     | 10    | 12    | -5          |