Practical Distributed Control for VTOL UAVs to Pass a Virtual Tube
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Abstract—Unmanned Aerial Vehicles (UAVs) are now becoming increasingly accessible to amateur and commercial users alike. An air traffic management (ATM) system is needed to help ensure that this newest entrant into the skies does not collide with others. In an ATM, airspace can be composed of airways, intersections and nodes. In this paper, for simplicity, distributed coordinating the motions of Vertical TakeOff and Landing (VTOL) UAVs to pass an airway is focused. This is formulated as a virtual tube passing problem, which includes passing a virtual tube, inter-agent collision avoidance and keeping within the virtual tube. Lyapunov-like functions are designed elaborately, and formal analysis based on invariant set theorem is made to show that all UAVs can pass the virtual tube without getting trapped, avoid collision and keep within the virtual tube. What is more, by the proposed distributed control, a VTOL UAV can keep away from another VTOL UAV or return back to the virtual tube as soon as possible, once it enters into the safety area of another or has a collision with the virtual tube during it is passing the virtual tube. Simulations and experiments are carried out to show the effectiveness of the proposed method and the comparison with other methods.

Index Terms—Distributed control, swarm, UAVs, air traffic, virtual tube.

I. INTRODUCTION
Airspace is utilized today by far lesser aircraft than it can accommodate, especially low altitude airspace. There are more and more applications for UAVs in low altitude airspace, ranging from the on-demand package delivery to traffic and wildlife surveillance, inspection of infrastructure, search and rescue, agriculture, and cinematography. Moreover, since UAVs are usually small owing to portability requirements, it is often necessary to deploy a team of UAVs to accomplish certain missions. All these applications share a common need for both navigation and airspace management. One good starting point is NASA’s Unmanned Aerial System Traffic Management (UTM) project, which organized a symposium to begin preparations of a solution for low altitude traffic management to be proposed to the Federal Aviation Administration. What is more, air traffic for UAVs is attracted more and more research [1], [2]. Traditionally, the main role of air traffic management (ATM) is to keep a prescribed separation among all aircraft by using centralized control. However, it is infeasible for increasing UAVs because the traditional control method lacks scalability. In order to address such a problem, free flight is a developing air traffic control method that uses no centralized control. Instead, parts of airspace are reserved dynamically and automatically in a distributed way using computer communication to ensure the required separation among aircraft. This new system may be implemented into the U.S. air traffic control system in the next decade. Airspace may be allocated temporarily by an ATM for a special task within a given time interval. In this airspace, these aircraft have to be managed so that they can complete their tasks meanwhile avoiding collision. In [1], the airspace is structured similarly to the road network as shown in Figure 1(a). Aircraft are only allowed inside the following three: airways playing a similar role to roads or virtual tubes, intersections formed by at least two airways, and nodes which are the points of interest reachable through an alternating sequence of airways and intersections.

Fig. 1. Practical application scenarios of virtual tube passing problem.

In this paper, for simplicity, coordinating the motions of VTOL UAVs to pass an airway is considered, which can be taken as a virtual tube or corridor in the air. Concretely, the main problem is to coordinate the motions of VTOL UAVs include passing a virtual tube, inter-agent conflict (coming within the minimum allowed distance between each other, not to be confused with a collision) avoidance and keeping within the virtual tube, which is called the virtual tube passing problem here, which is very common in practice. For example, virtual tubes can be paths connecting two places, designed to bypass areas having the dense population or to be covered by wireless mobile networks (4G or 5G), virtual tubes can also be gates, corridors or windows, because they can be viewed as virtual virtual tubes, as shown in Figure 1(b). Such problems of coordination of multiple agents have been addressed partly using different approaches, various stability criteria and numerous control techniques [3], [4], [6], [7], [8], [9]. A commonly-used method, namely the dynamic region-following formation control, is to organize multiple agents as a group
inside a region and then move the group to pass virtual tubes, where the size of the region can vary according to the virtual tubes [10, 11, 12, 13]. However, the formation control is not very suited for the air traffic control problem considered. First, each UAV has its own task, while the formation control means that one has to wait for other ones. Second, higher-level coordination should be made to decide which ones should be in one group. What is more, the number of UAVs in airspace is varying dynamically, which increases the design difficulty of the higher-level coordination. Another way is to plan the trajectories for UAVs [14, 15, 16]. However, planning often depends on global information and may have to be updated due to uncertainties in practice, which brings more complex calculations.

According to the consideration above, we propose distributed control for VTOL UAV swarm, each one having the same control protocol. Distributed control will not use the global information so that the computation only depends on the number of local UAVs [17, 18, 19, 20]. This framework is applicable to dense air traffic. By the proposed protocol, every UAV can pass a virtual tube freely not in formation, meanwhile avoiding conflict with each other and keeping within the virtual tube once it enters into the virtual tube. During the process, some UAVs with high speed will overtake slow ones. The idea used is similar to artificial potential field methods because of its ease-of-use, where designed barrier functions [21] are taken as artificial potential functions. The distributed control laws use the negative gradient of mixing of attractive and repulsive potential functions to produce vector fields that ensure the passing and conflict avoidance, respectively. However, it is not easy to use such an idea, with two reasons in the following.

- The guidance strategy for each UAV has to design. An easy method is to set a chain of waypoints for each UAV. However, UAVs may get trapped when using this method. Namely, they have not arrived at their corresponding waypoints, but velocities are zero. Consequently, in order to avoid trap, a higher-level decision should be made to set these waypoints. As indicated by [22], the complexity of the calculation of undesired equilibria remains an open problem. An example is proposed in [16], modelling the virtual tube passing problem as an objective optimization problem, which can get the optimal path to minimize the length, time or energy by designing suitable optimization-based algorithm and objective functions. This algorithm works well for offline path-planning. However, if the obstacles to avoid are dynamic, the optimization-based algorithm will consume a lot of time to update global information and the corresponding constraints during the online path-planning process, which is not suitable for dense air traffic because of lacking real-time of control. Compared to the optimal solution of targets, safety, real-time and reliability are more necessary in practice.

- Besides this problem, the second problem is also encountered in practice especially for UAVs outdoor. The conflict of two agents is often defined in control strategies that their distance is less than a safety distance. The area is called safety area of an agent if the distance to the agent is less than the safety distance. However, a conflict will happen in practice even if conflict avoidance is proved formally because some assumptions will be violated in practice. For example, a UAV may enter into the safety area of another due to an unpredictable communication delay. On the other hand, most likely, two UAVs may not have a real collision in physics because the safety distance is often set large by considering various uncertainties, such as estimate error, communication delay, and control delay. This is a big difference from some indoor robots with a highly accurate position estimation and control. In most literature, if their distance is less than a safety distance, then their control schemes either do not work or even push the agent towards the center of the safety area rather than leaving the safety area. For example, some studies have used the following barrier function terms for collision avoidance, such as $1/\left(\|p_i - p_j\|^2 - R\right)$ [23, p. 323] or $\ln(\|p_i - p_j\| - R)$ [24], where $p_i$, $p_j$ are two UAVs’ positions, and $R > 0$ is the separation distance. The principle is to design a controller to make the barrier function terms bounded so that $\|p_i - p_j\|^2 > R$ if $\|p_i(0) - p_j(0)\|^2 > R$. Otherwise, $\|p_i - p_j\|^2 = R$ will make the barrier function term unbounded. The separation distance for robots indoor is often the sum of the two robots’ physical radius, namely $\|p_i - p_j\|^2 < R$ will not happen in practice. But, the separation distance is set largely for UAVs compared with their sizes. Due to some uncertainties such as communication delay, $\|p_i - p_j\|^2 < R$ will happen in the air. As a consequence, the control corresponding to the barrier function terms mentioned above will make $\|p_i - p_j\|^2 \to 0$ if $1/\left(\|p_i - p_j\|^2 - R\right)$ is used (the two UAVs are pushed together by the design controller) or appear numerical computation error if $\ln(\|p_i - p_j\| - R)$ is used.

Motivated by these problems, practical distributed control is proposed here to solve the virtual tube passing problem. Such a problem can be classified into a basic virtual tube passing problem and a general virtual tube passing problem. The former only considers that all UAVs are within the virtual tube at the beginning, while the latter allows UAVs at everywhere in the beginning. For the basic virtual tube passing problem, in light of artificial potential field methods, one Lyapunov function and two barrier functions are designed elaborately for approaching the finishing line, avoiding conflict with other UAVs, and keeping within the virtual tube, respectively. The distributed controller design is based on the combination of the three Lyapunov-like functions. A formal proof is given to show that all UAVs can pass the virtual tube without trapping, avoid conflict and keep within the virtual tube. What is more, by the proposed control, a UAV can keep away from another UAV or return back to the virtual tube as soon as possible, once it enters into the safety area of another UAV or has a conflict with the virtual tube during it is passing the virtual tube. For the general virtual tube passing problem, several virtual tube type areas are defined to cover the whole airspace. As a consequence, the general virtual tube passing problem is decomposed into several basic virtual tube passing problems.
As a result, for UAVs in different areas, they have different controllers according to the design of the basic virtual tube passing problem. By switching these controllers, the general virtual tube passing problem can be solved.

The practicability of the proposed distributed control lies on the following six features:

- **No ID required.** Unlike the formation control, neighboring UAVs’ IDs of a UAV are not required by the proposed distributed control. Some active detection devices such as cameras or radars may only detect neighboring UAVs’ position and velocity but no IDs, because these UAVs may look similarly. This implies that the proposed distributed control can work autonomously without communication.

- **Practical model used.** A double integral model with the given velocity command as input is proposed for UAVs. This model is simple and easy to obtain. What is more, distributed control is developed for various tasks based on commercial semi-autonomous autopilots.

- **Control saturation.** The maximum velocity command in the proposed distributed controller is confined according to the requirement of semi-autonomous autopilots.

- **Conflict-free.** A formal proof about conflict avoidance and keeping within the virtual tube is given. Even if a UAV enters into the safety area of another UAV or has a conflict with the virtual tube, it can keep away from the UAV or can return back to the virtual tube as soon as possible.

- **Convergence.** A formal proof is given to show that all UAVs pass the finishing line without getting trapped.

- **Low time complexity.** The proposed control protocol is simple and can be computed at high speed, which is more suitable for dense air traffic than optimization-based algorithms. The calculation time of finding feasible solutions for different strategies will be compared in simulation.

The paper is organised as follows. In Section II, a UAV control model is proposed, which contains the filtered position model and three types of areas, to formulate the virtual tube passing problem composed of a basic one and a general one. Some functions for different purposes are introduced in Section III for controller design. In Section IV, a controller is designed based on Lyapunov-like functions to solve the basic virtual tube passing problem, where the stability analysis is made. In Section V, a controller is designed based on Lyapunov-like functions to solve the general virtual tube passing problem, by decomposing this problem into several basic virtual tube passing problems. The effectiveness of the proposed method is demonstrated by simulation and flight experiments in Section VI. The conclusions are given in Section VII. Some details of mathematical proof process are given in Section VIII as appendix.

II. PROBLEM FORMULATION

In this section, a UAV control model is introduced first, including three types of areas, namely safety area, avoidance area, and detection area, used for control. Then, the virtual tube passing problem is formulated into a basic one and a general one depending on the initial places of UAVs.

A. UAV Control Model

1) Position Model: There are $M$ VTOL UAVs in local airspace at the same altitude satisfying the following model

$$\dot{p}_i = v_i$$
$$\dot{v}_i = -l_i (v_i - v_{c,i})$$

where $l_i > 0$, $p_i \in \mathbb{R}^2$ and $v_i \in \mathbb{R}^2$ are the position and velocity of the $i$th VTOL UAV, $v_{c,i} \in \mathbb{R}^2$ is the velocity command of the $i$th UAV, $i = 1, 2, \cdots, M$. The control gain $l_i$ depends on the $i$th UAV and the semi-autonomous autopilot used, which can be obtained through flight experiments. From the model (1), $\lim_{t \to \infty} \|v_i(t) - v_{c,i}\| = 0$ if $v_{c,i}$ is constant. Here, the velocity command $v_{c,i}$ for the $i$th VTOL UAV, is subject to a saturation defined as where $v_{m,i} > 0$ is the maximum speed of the $i$th VTOL UAVs, $i = 1, 2, \cdots, M$, $v = [v_1 \ v_2]^T \in \mathbb{R}^2$. The saturation function sat$(v, v_{m,i})$ and the vector $v$ are parallel all the time so it can keep the flying direction the same if $\|v\| > v_{m,i}$ [23]. The saturation function can be rewritten as

$$\text{sat}(v, v_{m,i}) = \kappa_{v_{m,i}}(v) v$$

where

$$\kappa_{v_{m,i}}(v) = \begin{cases} 1, & \|v\| \leq v_{m,i} \\ \|v\|, & \|v\| > v_{m,i} \end{cases}$$

It is obvious that $0 < \kappa_{v_{m,i}}(v) \leq 1$. Sometimes, $\kappa_{v_{m,i}}(v)$ will be written as $\kappa_v$ for short. According to this, if and only if $v = 0$, then

$$v^T \text{sat}(v, v_{m,i}) = 0.$$

Remark 1. It is well-known that a typical multicopter is a physical system with underactuated dynamics [23 pp.126-130]. But, many organizations or companies have designed some open source semi-autonomous autopilots or offered semi-autonomous autopilots with software development kits. The semi-autonomous autopilots can be used for velocity control of VTOL UAVs. For example, A3 autopilots released by DJI allow the range of the horizontal velocity command from $-10\text{m/s} \sim 10\text{m/s}$ [23]. With such an autopilot, the velocity of a VTOL UAV can track a given velocity command in a reasonable time. Not only can this avoid the trouble of modifying the low-level source code of autopilots, but also it can utilize commercial autopilots to complete various tasks. So, the dynamics (1) is practical especially for higher-level control.

2) Filtered Position Model: In this section, the motion of each VTOL UAV is transformed into a single integrator form to simplify the controller design and analysis. As shown in Figure 2, although the position distances of the three cases are the same, namely a marginal avoidance distance, the case in Figure 2(b) needs to carry out avoidance urgently by considering the velocity. However, the case in Figure 2(a) in fact does not need to be considered to perform collision avoidance. With such an intuition, a filtered position is defined as follows:

$$\xi_i = p_i + \frac{1}{l_i} v_i.$$
Then
\[ \dot{\xi}_i = \dot{p}_i + \frac{1}{l_i} \dot{v}_i = v_i - \frac{1}{l_i} l_i (v_i - v_{c,i}) = v_{c,i} \]

where \( i = 1, 2, \cdots, M \). Let
\[ r_v = \max_i \frac{v_{m,i}}{l_i} . \]

In the following, a relationship between the position error and the filtered position error is shown.

**Proposition 1.** Given any \( r > 0 \), for the \( i \)th and \( j \)th VTOL UAVs, if \( \|v_i(0)\| \leq v_{m,i} \) and the filtered position error satisfies \( \|\xi_i(t) \| - \|\xi_j(t)\| \geq r + 2r_v \), then
\[ \|p_i(t) - p_j(t)\| \geq r \]

\( t \geq 0 \), where \( i, j = 1, 2, \cdots, M, i \neq j, r > 0. \)

*Proof.* See Appendix. □

**B. Three Types of Areas around a UAV**

In light of [17], three types of areas used for control, namely safety area, avoidance area, and detection area, are defined. Unlike [17], these areas are suit for UAVs and the velocity is further introduced.

1) **Safety Area:** In order to avoid a conflict, as shown in Figure 5 the safety radius \( r_s \) of a UAV is defined as
\[ S_i = \{ x \in \mathbb{R}^2 \mid \|x - \xi_i\| \leq r_s \} \]

where \( r_s > 0 \) is the safety radius, \( i = 1, 2, \cdots, M \). It should be noted that we consider the velocity of the \( i \)th UAV in the definition of \( S_i \). For all UAVs, no confliction with each other implies
\[ S_i \cap S_j = \emptyset \]

namely
\[ \|\xi_j - \xi_i\| > 2r_s . \]

**Proposition 1** implies that two VTOL UAVs will be separated largely enough if (8) is satisfied with a safety radius \( r_s \) large enough.

2) **Avoidance Area:** Besides the safety area, there exists an avoidance area used for starting avoidance control. If another UAV is out of the avoidance area of the \( i \)th UAV, then the object will not need to be avoided. For the \( i \)th UAV, the avoidance area for other UAVs is defined as
\[ A_i = \{ x \in \mathbb{R}^2 \mid \|x - \xi_i\| \leq r_a \} \]

where \( r_a > 0 \) is the avoidance radius, \( i = 1, 2, \cdots, M \). It should be noted that we consider the velocity of the \( i \)th UAV in the definition of \( A_i \). If
\[ A_i \cap S_j \neq \emptyset , \]

namely
\[ \|\xi_i - \xi_j\| \leq r_a + r_s \]

then the \( j \)th UAV should be avoided by the \( i \)th UAV. Since
\[ A_i \cap S_j \neq \emptyset \Leftrightarrow A_j \cap S_i \neq \emptyset \]

according to the definition of \( A_i \), the \( i \)th UAV should be avoided by the \( j \)th UAV at the same time. When the \( j \)th UAV just enters into the avoidance area of the \( i \)th UAV, it is required that they have not conflicted at the beginning. Therefore, we require
\[ r_a > r_s . \]

3) **Detection Area:** By cameras, radars, or Vehicle to Vehicle (V2V) communication, the UAVs can receive the positions and velocities of their neighboring UAVs. The detection area only depends on the detection range of the used devices, which is only related to its position. For the \( i \)th UAV, this area is defined as
\[ D_i = \{ x \in \mathbb{R}^2 \mid \|x - p_i\| \leq r_d \} \]

where \( r_d > 0 \) is the detection radius, \( i = 1, 2, \cdots, M \). When another UAV is within this area, it can be detected.

**Proposition 2.** Suppose \( r_d > r_s + r_a + 2r_v, i = 1, 2, \cdots, M \). Then for any \( i \neq j \), if \( A_i \cap S_j \neq \emptyset \), then \( p_j \in D_i, i, j = 1, 2, \cdots, M \).

*Proof.* It is similar to proof of Proposition 1. □

To simplify the following problems, we have the following assumption for all VTOL UAVs.

**Assumption 1.** The radius of the detection area satisfies \( r_d > r_s + r_a + 2r_v \).
According to Assumption 1, for the $i$th UAV, any other UAV entering into its avoidance area can be detected by the $i$th UAV and will not conflict with the $i$th UAV initially, $i = 1, 2, \cdots, M$.

### C. virtual tube Passing Problem Formulation

In a horizontal plane, as shown in Figure 4 a virtual tube (analogous to an *airway* or a *highway* on the ground) here is a horizontal long band with the width $2r_i$ and centerline starting from $p_{i,1} \in \mathbb{R}^2$ to $p_{i,2} \in \mathbb{R}^2$, where $r_i > L_{ra}$, where $L \in \mathbb{Z}^+$ is the lane number in the virtual tube allowed for UAVs.

Define

$$A_{i,12} (p_{i,1}, p_{i,2}) \triangleq I_2 - \frac{(p_{i,1} - p_{i,2})(p_{i,1} - p_{i,2})^T}{\|p_{i,1} - p_{i,2}\|^2}$$

$$A_{i,23} (p_{i,2}, p_{i,3}) \triangleq I_2 - \frac{(p_{i,2} - p_{i,3})(p_{i,2} - p_{i,3})^T}{\|p_{i,2} - p_{i,3}\|^2}.$$\hspace{1cm} (11)

Here, matrix $A_{i,12} = A_{i,12}^T$, $A_{i,23} = A_{i,23}^T \in \mathbb{R}^{2 \times 2}$ are positive semi-definite matrices. According to the projection operator [26, p. 480], the value $\|A_{i,12} (p - p_{i,1})\|$ is the distance from $p \in \mathbb{R}^2$ to the straight line $p_{i,1}p_{i,2}$ as shown in Figure 5. Particularly, the equation $\|A_{i,12} (p - p_{i,1})\| = 0$ implies that $p$ is on the straight-line $p_{i,1}p_{i,2}$. Similarly, the value $\|A_{i,23} (p - p_{i,2})\|$ is the distance from $p$ to the finishing line $p_{i,2}p_{i,3}$.

Define position errors as

$$\tilde{p}_{i,j} \triangleq A_{i,23} (p_i - p_{i,2})$$

$$\tilde{p}_{m,i,j} \triangleq p_i - p_j$$

$$\tilde{p}_{i,j} \triangleq A_{i,12} (p_i - p_{i,2})$$

and the filtered position errors as

$$\tilde{x}_{i,j} \triangleq A_{i,23} (\xi_i - p_{i,2})$$

$$\tilde{x}_{m,i,j} \triangleq \xi_i - \xi_j$$

$$\tilde{x}_{i,j} \triangleq A_{i,12} (\xi_i - p_{i,2})$$

where $i, j = 1, 2, \cdots, M$. With the definitions above, according to (11), the derivatives of the filtered errors are

$$\dot{\tilde{x}}_{i,j} = A_{i,23} v_{c,i,j} \hspace{1cm} (12)$$

$$\dot{\tilde{x}}_{m,i,j} = v_{c,i} - v_{c,j} \hspace{1cm} (13)$$

$$\dot{\tilde{x}}_{i,j} = A_{i,12} v_{c,i} \hspace{1cm} (14)$$

where $i, j = 1, 2, \cdots, M$.

Fig. 5. Diagram of the projective operator.

With the description above, the following assumptions are proposed.

**Assumption 2.** As shown in Figure 4 the initial condition $p_i (0), \xi_i (0), i = 1, 2, \cdots, M$ are all within the virtual tube or its extension, namely

$$\left(\frac{p_i - p_{i,1}}{\|p_i - p_{i,1}\|}\right)^T (p_i (0) - p_{i,2}) < 0$$

$$\left(\frac{p_i - p_{i,1}}{\|p_i - p_{i,1}\|}\right)^T (\xi_i (0) - p_{i,2}) < 0$$

where $p_{i,2}p_{i,3}$ is perpendicular to $p_{i,1}p_{i,2}$ with $\|p_{i,2} - p_{i,3}\| = r_i$.

**Assumption 2’.** As shown in Figure 4 the initial condition $\xi_i (0)$ are not all within the virtual tube or its extension, but locate the left of the finishing line $p_{i,2}p_{i,3}$.

**Assumption 3.** The UAVs’ initial conditions satisfy

$$\|\xi_i (0) - \xi_j (0)\| > 2r_i, i \neq j$$

and $\|v_i (0)\| \leq v_m$, where $i, j = 1, 2, \cdots, M$.

**Assumption 4.** Once a UAV arrives near the finishing line $p_{i,2}p_{i,3}$, then it will quit the virtual tube not to affect the UAVs behind. Mathematically, given $\epsilon_0 \in \mathbb{R}^{+}$, a UAV arrives near the finishing line $p_{i,2}p_{i,3}$ if

$$(p_{i,2} - p_{i,1})^T A_{i,23} (p_i - p_{i,2}) \geq -\epsilon_0.$$\hspace{1cm} (15)

**Neighboring Set.** Let the set $N_{m,i}$ be the collection of all mark numbers of other VTOL UAVs whose safety areas enter into the avoidance area of the $i$th UAV, namely

$$N_{m,i} = \{ j | S_j \cap A_i \neq \emptyset, j = 1, \cdots, M, i \neq j \} .$$

For example, if the safety areas of the 1th, 2th VTOL UAVs enter into the avoidance area of the 3th UAV, then $N_{m,3} = \{1, 2\}$. Based on Assumptions and definition above, two types of virtual tube passing problems are stated in the following.

- **Basic virtual tube passing problem.** Under Assumptions 1-4, design the velocity input $v_{c,i}$ for the $i$th UAV with local information from $N_{m,i}$ to guide it to fly to pass the virtual tube until it arrives near the finishing line $p_{i,2}p_{i,3}$, meanwhile avoiding colliding other UAVs ($\|\tilde{x}_{m,i,j}\| > 2r_i$) and keeping within the virtual tube ($\|\tilde{x}_{i,j}\| < r_i - r_s$) while passing it, $i = 1, 2, \cdots, M$.

- **General virtual tube passing problem.** Under Assumptions 1,2',3,4, design the velocity input $v_{c,i}$ for the $i$th UAV.
UAV with local information from $N_{m,i}$ to guide it to fly to pass the virtual tube until it arrives near the finishing line $p_{i,2}p_{i,3}$, meanwhile avoiding conflict with other UAVs ($\|\tilde{\xi}_{m,ij}\| > 2r_i$) and keeping within the virtual tube when passing it ($\|\tilde{\xi}_{i}\| < r_i - r_v$, $i = 1, 2, \ldots, M$).

**Remark 1.** As shown in Figure 6 if $(p_{i,3} - p_{i,1})^T(p_i - p_{i,1}) > 0$, then $p_i$ locates the right side of the finishing line $p_{i,2}p_{i,3}$; if $(p_{i,3} - p_{i,1})^T(p_i - p_{i,2}) < 0$, then $p_i$ locates the left side of the finishing line $p_{i,2}p_{i,3}$.

![Fig. 6. Position relative to the finishing line.](image)

**Remark 2.** For Assumption 2, all UAVs are within the virtual tube (like Place 0 in Figure 4) or its extension (like Place 1 in Figure 4). For Assumption 2', UAVs are not all within the virtual tube or its extension. This implies that UAVs may locate everywhere. For example, UAVs may locate the places, like Place 0, ... Place 5 shown in Figure 4.

### III. Preliminaries

#### A. Line Integral Lyapunov Function

In the following, we will design a new type of Lyapunov functions, called Line Integral Lyapunov Function. This type of Lyapunov functions is inspired by its scalar form [27, p.74]. If $xf(x) > 0$ for $x \neq 0$, then $V^i_h(y) = \int_0^y f(x)dx > 0$ when $y \neq 0$. The derivative is $V^i_h(y) = f(y)\dot{y}$. A line integral Lyapunov function for vectors is defined as

$$V^i_h(y) = \int_{C_y} \kappa_a(x)Tdx$$

(16)

where $a > 0$, $x \in \mathbb{R}^n$, $C_y$ is a line from 0 to $y \in \mathbb{R}^n$. In the following lemma, we will show its properties.

**Lemma 1.** Suppose that the line integral Lyapunov function $V^i_h(y)$ is defined as (16). Then (i) if $V^i_h(y) > 0$ if $\|y\| \neq 0$; (ii) if $\|y\| \to \infty$, then $V^i_h(y) \to \infty$; (iii) if $V^i_h(y)$ is bounded, then $\|y\|$ is bounded.

**Proof.** Since

$$\text{sat}(x, a) = \kappa_a(x)x$$

the function (16) can be written as

$$V^i_h(y) = \int_{C_y} \kappa_a(x)x^Tdx$$

(17)

where

$$\kappa_a(x) = \begin{cases} 1, & \|x\| \leq a \\ \frac{a}{\|x\|}, & \|x\| > a \end{cases}$$

Let $z = \|x\|$. Then the function (17) becomes

$$V^i_h(y) = \int_{C_y} \frac{\kappa_a(x)d}{2z^2}$$

$$= \int_0^{\|y\|} \kappa_a(x)zd\zeta.$$

- If $\|y\| \leq a$, then $\kappa_a(x) = 1$. Consequently,
  $$V^i_h(y) = \frac{1}{2}\|y\|^2.$$  
  (18)

- If $\|y\| > a$, then
  $$\int_0^{\|y\|} \kappa_a(x)zd\zeta = \int_0^a zd\zeta + \int_a^{\|y\|} \frac{a}{\|x\|}zd\zeta.$$  
  Since $z = \|x\|$, we have
  $$V^i_h(y) \geq \frac{1}{2}a^2 + a(\|y\| - a).$$  
  (19)

Therefore, from the form of (18) and (19), we have (i) $V^i_h(y) > 0$ if $\|y\| \neq 0$. (ii) if $\|y\| \to \infty$, then $V^i_h(y) \to \infty$; (iii) if $V^i_h(y)$ is bounded, then $\|y\|$ is bounded. □

#### B. Two Smooth Functions

Two smooth functions are defined for the following Lyapunov-like function design. As shown in Figure 7 (upper plot), define a second-order differentiable ‘bump’ function as [24]

$$\sigma(x, d_1, d_2) = \begin{cases} 1 & \text{if } x \leq d_1 \\ A x^3 + B x^2 + C x + D & \text{if } d_1 \leq x \leq d_2 \\ 0 & \text{if } d_2 \leq x \end{cases}$$

(20)

with $A = -2/(d_1 - d_2)^3$, $B = 3(d_1 + d_2)/(d_1 - d_2)^3$, $C = -6d_1 d_2/(d_1 - d_2)^3$ and $D = d_2^2(3d_1 - d_2)/(d_1 - d_2)^3$. The derivative of $\sigma(x, d_1, d_2)$ with respect to $x$ is

$$\frac{\partial \sigma(x, d_1, d_2)}{\partial x} = \begin{cases} 0 & \text{if } x \leq d_1 \\ 3Ax^2 + 2Bx + C & \text{if } d_1 \leq x \leq d_2 \\ 0 & \text{if } d_2 \leq x \end{cases}$$

Define another smooth function as shown in Figure 7 (lower plot) to approximate a saturation function

$$\bar{s}(x) = \min(x, 1), x \geq 0$$

that

$$s(x, \epsilon_2) = \begin{cases} (1 - \epsilon_2) + \sqrt{\epsilon_2^2 - (x - x_2)^2} & 0 \leq x \leq x_1 \\ 1 & x_1 \leq x \leq x_2 \\ 0 & x_2 \leq x \end{cases}$$

(21)

with $x_2 = 1 + \frac{1}{\tan 67.5^\circ} \epsilon_2$ and $x_1 = x_2 - \sin 45^\circ \epsilon_2$. Since it is required $x_1 \geq 0$, one has $\epsilon_2 \leq \frac{\tan 67.5^\circ}{\tan 67.5^\circ + \sin 45^\circ - 1}$. For any $\epsilon_2 \in [0, \frac{\tan 67.5^\circ}{\tan 67.5^\circ + \sin 45^\circ - 1}]$, it is easy to see

$$s(x, \epsilon_2) \leq \bar{s}(x)$$

(22)

and

$$\lim_{\epsilon_2 \to 0, x \geq 0} |\bar{s}(x) - s(x, \epsilon_2)| = 0.$$  
(23)
where \( k_2 \) is an adjustable parameter, \( i = 1, 2, \ldots, M \). From the definition, \( V_{i,i} \geq 0 \). According to Thomas’ Calculus [28, p. 911], one has
\[
V_{i,i} = \int_0^t \text{sat} \left( k_1 \xi_{li} (\tau), v_{m,i} \right)^T \dot{\xi}_{li} (\tau) d\tau. \quad (25)
\]
The objective of the designed velocity command is to make \( V_{i,i} \) be zero. This implies that \( \| \xi_{li} \| \) goes down to zero according to the property (24), namely the \( i \)-th UAV approaches the finishing line \( \mathbf{p}_{n,2} \mathbf{p}_{n,3} \).

Fig. 7. Two smooth functions. For a smooth saturation function, \( \theta_i = 67.5^\circ \).

The derivative of \( s(x, \epsilon_s) \) with respect to \( x \) is
\[
\frac{\partial s(x, \epsilon_s)}{\partial x} = \begin{cases} 
\frac{1}{x_2 - x} & 0 \leq x \leq x_1 \\
\frac{1}{\sqrt{r^2 - (x - x_2)^2}} & x_1 \leq x \leq x_2 \\
0 & x \geq x_2
\end{cases}.
\]

For any \( \epsilon_s > 0 \), we have \( \sup_{x \geq 0} |\partial s(x, \epsilon_s) / \partial x| \leq 1 \).

IV. BASIC VIRTUAL TUBE PASSING PROBLEM

In this section, three Lyapunov-like functions for approaching the finishing line, avoiding conflict, and keeping within the virtual tube are established. Based on them, a controller to solve the basic virtual tube passing problem is derived and then a formal analysis is made.

A. Lyapunov-Like Function Design and Analysis

For the basic virtual tube passing problem, three subproblems are required to solve, namely approaching the finishing line \( \mathbf{p}_{n,2} \mathbf{p}_{n,3} \), avoiding conflict with other UAVs, and keeping within the virtual tube. Correspondingly, three Lyapunov-like functions are proposed.

1) Integral Lyapunov Function for Approaching Finishing Line: Define a smooth curve \( C_{\xi_{li},s} \) from 0 to \( \xi_{li} \). Then, the line integral of sat \( (x, v_{m,i}) \) along \( C_{\xi_{li},s} \) is
\[
V_{i,i} = \int_{C_{\xi_{li},s}} \text{sat} \left( k_1 x, v_{m,i} \right)^T dx \quad (24)
\]
The objective of the designed velocity command is to make 

\[ V_{m,i} \approx \frac{k_2}{\epsilon_m \left\| \tilde{\xi}_{m,i} \right\|} \geq \frac{k_2}{2 \epsilon_m r_s}. \]  

(27)

The objective of the designed velocity command is to make 

\[ V_{m,i} \approx \frac{k_2}{\epsilon_m \left\| \tilde{\xi}_{m,i} \right\|} \geq \frac{k_2}{2 \epsilon_m r_s}. \]  

(27)

The objective of the designed velocity command is to make 

\[ V_{m,i} \approx \frac{k_2}{\epsilon_m \left\| \tilde{\xi}_{m,i} \right\|} \geq \frac{k_2}{2 \epsilon_m r_s}. \]  

(27)

\[ V_{m,i} = \frac{k_3 \sigma_i (r_i - \left\| \tilde{\xi}_{i,t} \right\|)}{(r_i - r_s) - \left\| \tilde{\xi}_{i,t} \right\| s \left( \frac{r_i - r_s}{\left\| \tilde{\xi}_{i,t} \right\| + \epsilon_i} \right)} \]

where \( \sigma_i (x) \triangleq \sigma (r_i, r_s, r_a) \). When \( r_i = 50, r_s = 10, r_a = 20, \), \( \epsilon_i = 10^{-6}, k_3 = 1 \), the function \( V_{i,t} (x) \) is shown in Figure 8 (lower plot), where \( V_{i,t} (x) = 0 \) as \( x \leq r_i - r_a = 30 \) and \( V_{i,t} (x) \) is increased sharply as \( x \to 40 \) from \( x = 30 \). The function \( V_{i,t} \) has the following properties:

(i) \( \partial V_{i,t} / \partial \left\| \tilde{\xi}_{i,t} \right\| \geq 0 \) as \( V_{i,t} \) is a nondecreasing function with respect to \( \left\| \tilde{\xi}_{i,t} \right\| \);

(ii) if \( r_i - \left\| \tilde{\xi}_{i,t} \right\| \geq r_s \), namely the edges of the virtual tube are out of the avoidance area of the \( i \)th UAV, then \( \sigma_i (r_i - \left\| \tilde{\xi}_{i,t} \right\|) = 0 \); consequently, \( V_{i,t} = 0 \) and \( \partial V_{i,t} / \partial \left\| \tilde{\xi}_{i,t} \right\| = 0 \);

(iii) if \( r_i - \left\| \tilde{\xi}_{i,t} \right\| < r_s \), namely one edge of the virtual tube has entered into the safety area of the \( i \)th UAV, then

\[ \sigma_i (r_i - \left\| \tilde{\xi}_{i,t} \right\|) = 1 \]

and there exists a sufficiently small \( \epsilon_s > 0 \) such that

\[ s \left( \frac{r_i - r_s}{\left\| \tilde{\xi}_{i,t} \right\| + \epsilon_i} \right) \geq \frac{r_i - r_s}{\left\| \tilde{\xi}_{i,t} \right\| + \epsilon_i} < 1. \]

As a result,

\[ V_{i,t} \approx \frac{k_3}{\epsilon_i (r_i - r_s)} \]

which will be very large if \( \epsilon_i \) is very small.

The objective of the designed velocity command is to make \( V_{i,t} \) be zero. This implies \( r_i - \left\| \tilde{\xi}_{i,t} \right\| \geq r_s \) according to property (ii), namely the \( i \)th UAV will keep within the virtual tube.

\[ V_{c,i} = -sat \left( A_{i23} \text{sat} \left( k_1 \tilde{\xi}_{i,t} + v_{m,i} \right) + \sum_{j \in A_{m,i}} -b_{ij} \tilde{\xi}_{m,i} \right) \]

Line Approaching \[ \text{UAV Avoidance} \]

Tunnel Keeping \[ \text{UAV Avoidance} \]

(29)

\[ b_{ij} = -\frac{\partial V_{m,i}}{\partial \left\| \tilde{\xi}_{m,i} \right\|} \frac{1}{\left\| \tilde{\xi}_{m,i} \right\|} \]

(30)

\[ c_i = \frac{\partial V_{i,t}}{\partial \left\| \tilde{\xi}_{i,t} \right\|} \frac{1}{\left\| \tilde{\xi}_{i,t} \right\|} \]

(31)

This is a distributed control form. Unlike the formation control, neighboring UAVs’ IDs of a UAV are not required. By active detection devices such as cameras or radars may only detect neighboring UAVs’ position and velocity but no IDs, because these UAVs may look alike. This implies that the proposed distributed control can work autonomously without communication.

Remark 3. It is noticed that the velocity command (29) is saturated, whose norm will not exceed \( v_{m,i} \). If the case such as \( \left\| \tilde{\xi}_{m,i} \right\| < 2 r_s \) happens in practice due to unpredictable uncertainties out of the assumptions we make, this may not imply that the \( i \)th UAV has collided the \( j \)th UAV physically. In this case, the velocity command (29) degenerates to be

\[ v_{c,i} = -sat \left( A_{i23} \text{sat} \left( k_1 \tilde{\xi}_{i,t} + v_{m,i} \right) - \sum_{j=1,j \neq i}^{M} b_{ij} \tilde{\xi}_{m,i} \right) + c_i A_{i12} \tilde{\xi}_{i,t} + b_{ij} \tilde{\xi}_{m,i}, v_{m,i} \]

with \( b_{ij} \approx \frac{k^2}{\epsilon_m \left\| \tilde{\xi}_{m,i} \right\|} \).

Since \( \epsilon_m \) is chosen to be sufficiently small, the term \( b_{ij} \tilde{\xi}_{m,i} \) will dominate so that the velocity command \( v_{c,i} \) becomes

\[ v_{c,i} \approx sat \left( \frac{k_2}{\epsilon_m \left\| \tilde{\xi}_{m,i} \right\|} \frac{1}{\left\| \tilde{\xi}_{m,i} \right\|} v_{m,i} \right). \]

This implies that, by recalling (13), \( \left\| \tilde{\xi}_{m,i} \right\| \) will be increased very fast so that the \( i \)th UAV can keep away from the \( j \)th UAV immediately.

Remark 4. In practice, the case such as \( r_s - \left\| \tilde{\xi}_{h,i} \right\| < r_s \) may still happen in practice due to unpredictable uncertainties out of the assumptions we make. In this case, since \( \epsilon_i \) is chosen to be sufficiently small, the velocity command \( v_{c,i} \) becomes

\[ v_{c,i} \approx sat \left( -\frac{k_3}{\epsilon_i (r_i - r_s)} A_{i12} \tilde{\xi}_{i,t} + v_{m,i} \right). \]

\[ 1 \text{is chosen to the property (i) of } V_{m,i} \text{; } c_i \geq 0 \text{ according to the property (i) of } V_{i,t}. \]

\[ 2 \text{Furthermore, we assume that the } i \text{th UAV does not conflict with others except for the} j \text{th UAV, or not very close to the edges of the virtual tube.} \]
This implies that, by recalling (14), \( \| \mathbf{A}_{t,23} \mathbf{\hat{e}}_{i,t} \| \) will be decreased so that the \( i \)th UAV can return back to the virtual tube immediately.

\[ \mathbf{V} = \sum_{i=1}^{M} \left( \mathbf{V}_{i,i} + \frac{1}{2} \sum_{j=1,j \neq i}^{M} \mathbf{V}_{m,ij} + \mathbf{V}_{i,i} \right). \]

The derivative of \( \mathbf{V} \) along the solution to (12), (13), (14) is

\[ \dot{\mathbf{V}} = \sum_{i=1}^{M} \left( \mathbf{s}(k_1 \mathbf{\hat{e}}_{i,t}, v_{m,i}) + \mathbf{A}_{t,23} \mathbf{V}_{c,i} \right) \]

\[ - \frac{1}{2} \sum_{j=1,j \neq i}^{M} b_{ij} \mathbf{\hat{e}}_{m,ij} (\mathbf{V}_{c,i} - \mathbf{v}_{c,j}) + c_i \mathbf{A}_{t,12} \mathbf{V}_{c,i} \]

\[ = \sum_{i=1}^{M} \mathbf{A}_{t,23} \mathbf{s}(k_1 \mathbf{\hat{e}}_{i,t}, v_{m,i}) - \sum_{j=1,j \neq i}^{M} b_{ij} \mathbf{\hat{e}}_{m,ij} \]

\[ + c_i \mathbf{A}_{t,12} \mathbf{\hat{e}}_{i,t} \mathbf{v}_{c,i} \text{T} \]

Since

\[ \sum_{j=1,j \neq i}^{M} b_{ij} \mathbf{\hat{e}}_{m,ij} = \sum_{j \in M, i} b_{ij} \mathbf{\hat{e}}_{m,ij} \]

by using the velocity input (28), \( \dot{\mathbf{V}} \) becomes

\[ \dot{\mathbf{V}} \leq 0. \]

Before introducing the main result, two lemmas are needed. **Lemma 2.** Under Assumptions 1-4, suppose that the velocity command is designed as (28). Then there exist sufficiently small \( \epsilon_m, \epsilon_c > 0 \) in \( b_{ij} \) and \( \epsilon_i > 0 \) in \( c_i \) such that \( \| \mathbf{\hat{e}}_{m,ij}(t) \| > 2r_s, \| \mathbf{\hat{e}}_{i,t} \| < r_t - r_s, t \in [0, \infty) \) for all \( \mathbf{p}_i(0), i = 1, \cdots, M. \)

**Proof.** See Appendix. \( \square \)

With Lemmas 1-2 in hand, we can state the main result. **Theorem 1.** Under Assumptions 1-4, suppose (i) the velocity command in the distributed form is designed as in (28), (ii) given \( \epsilon_0 \in \mathbb{R}_+ \), if (15) is satisfied, then \( b_{ij} \equiv 0 \) and \( c_i \equiv 0 \) (this implies that the \( i \)th UAV is removed from the virtual tube mathematically). Then, for given \( \epsilon_0 \in \mathbb{R}_+ \), there exist sufficiently small \( \epsilon_m, \epsilon_c > 0 \) in \( b_{ij}, \epsilon_i \in \mathbb{R}_+ \) in \( c_i \) and \( t_1 \in \mathbb{R}_+ \) such that all UAVs can satisfy (15) as \( t \geq t_1 \), meanwhile

\[ \| \mathbf{\hat{e}}_{m,ij}(t) \| > 2r_s, \| \mathbf{\hat{e}}_{i,t} \| < r_t - r_s, t \in [0, \infty) \) for all \( \mathbf{p}_i(0), i = 1, \cdots, M. \)

**Proof.** According to Lemma 2, these VTOL UAVs are able to avoid conflict with each other and keep within the virtual tube, namely \( \| \mathbf{\hat{e}}_{m,ij}(t) \| > 2r_s, \| \mathbf{\hat{e}}_{i,t} \| < r_t, i \neq j, i, j = 1, 2, \cdots, M. \) In the following, the reason why the \( i \)th UAV is able to approach the finishing line \( \mathbf{p}_{m,1} \) is given.

The function \( V \) is not a Lyapunov function. The invariant set theorem [27] p. 69] is used to do the analysis.

- First, we will study the property of function \( V \). Let \( \Omega = \{ \xi_1, \cdots, \xi_M | V(\xi_1, \cdots, \xi_M) \leq l \}, l > 0. \) According to
V. CONTROLLER DESIGN FOR GENERAL VIRTUAL TUBE PASSING PROBLEM

So far, we have solved the basic virtual tube passing problem. Then, we are going to solve the general virtual tube passing problem. First, we define different areas for the whole airspace. Then, the general virtual tube passing problem is decomposed into several basic virtual tube passing problems. As a result, for UAVs in different areas, they have different controllers, like (28). Combining them together, the final controller is obtained.

A. Area Definition

As shown in Figure 9(a), the whole airspace is divided into six areas, namely Left Standby Area, Left Ready Area, Right Standby Area, Right Ready Area, virtual tube, and virtual tube Extension. Moreover, the Earth-fixed coordinate frame is built. For simplicity, let \( \mathbf{p}_{k,1} = 0 \) with \( x \)-axis pointing to \( \mathbf{p}_{k,2} \) and \( y \)-axis pointing to its left side.

- **Left Standby Area** and **Right Standby Area** are the areas on the outside of virtual tube and the right side of **Starting Line** \( \mathbf{p}_{k,1}\mathbf{p}_{k,2} \), where \( \mathbf{p}_{k,1} = [0 \ r_1]^T \). Concretely, if

  \[
  \xi_i (1) > 0, \quad \xi_i (2) > r_1
  \]

  then \( \xi_i \) is in **Left Standby Area**. If

  \[
  \xi_i (1) > 0, \quad \xi_i (2) < -r_1
  \]

  then \( \xi_i \) is in **Right Standby Area**.

- **Left Ready Area** and **Right Ready Area** are the areas on the outside of virtual tube Extension and the left side of **Starting Line** \( \mathbf{p}_{k,1}\mathbf{p}_{k,4} \). Concretely, if

  \[
  \xi_i (1) \leq 0, \quad \xi_i (2) > r_1
  \]

  then \( \xi_i \) is in **Left Standby Area**. If

  \[
  \xi_i (1) \leq 0, \quad \xi_i (2) < -r_1
  \]

  then \( \xi_i \) is in **Right Standby Area**.

- **Virtual tube** and **virtual tube Extension** are a band. Concretely, if

  \[
  \begin{align*}
  \xi_i (1) &\leq 0 \land \xi_i (2) > \| \mathbf{p}_{k,1} - \mathbf{p}_{k,2} \| \\
  -r_1 &\leq \xi_i (2) \leq r_1
  \end{align*}
  \]

  then \( \xi_i \) is in **virtual tube**.

B. virtual tube Passing Scheme and Requirements

As Assumption 2' points, at the beginning, UAVs may locate in the six areas. A flight sequence is given shown in Figure 9(b). The requirement is as follows.

- From **Left/Right Standby Area** to **Left/Right Ready Area**. UAVs are required to fly into **Left/Right Ready Area**, meanwhile avoiding conflict with other UAVs and keeping away from virtual tube and virtual tube Extension.

- From **Left/Right Ready Area** to **virtual tube Extension**. UAVs are required to fly into **virtual tube Extension**, meanwhile avoiding conflict with other UAVs and keeping away from virtual tube.

- From **virtual tube and virtual tube Extension** to **Finishing Line** \( \mathbf{p}_{k,2}\mathbf{p}_{k,3} \). UAVs are required to pass the virtual tube until it arrives near the finishing line \( \mathbf{p}_{k,2}\mathbf{p}_{k,3} \), meanwhile avoiding conflict other UAVs and keeping within the virtual tube and its extension.

C. Controller Design

As Assumption 2' points, at the beginning, UAVs may locate in the six areas. A flight sequence is given shown in Figure 9(b). The requirement is as follows.

1) **From Standby Area to Ready Area**: As shown in Figure 10 a virtual virtual tube, named **LS2R virtual tube**, is designed.
Theorem 1 is designed as colliding other UA Vs and keeping within \( p_2 \) and virtual tube \( \text{namely keeping away from } p_2 \). Moreover, all UAVs in \( \text{Left Standby Area} \) approach the finishing line \( p_{st,2}p_{st,3} \). Here

\[
\begin{align*}
P_{st,1} &= \begin{bmatrix} p_{t,2} (1) \end{bmatrix}, P_{st,2} = \begin{bmatrix} -r_b r_b \end{bmatrix}, P_{st,3} = \begin{bmatrix} -r_b r_b \end{bmatrix},
\end{align*}
\]

where \( r_b > 0 \), \( r_b = r_a \) for example. The intersection of \( \text{LS2R virtual tube} \) and \( \text{Left Ready Area} \) is a buffer with length \( r_b \), which can make a UA V fly into \( \text{Left Ready Area} \) not only approaching it. According to (15), the controller is designed as \( v_{c,i} = v_{st,i} \), where

\[
\begin{align*}
v_{st,i} &= \text{sat}\left( v_{i,i}, (k_1, p_{st,2}, p_{st,3}) + v_{m,i} (k_2) \right) + v_{t,i} (k_3, r_s, p_{st,1}, p_{st,2}, v_{m,i})
\end{align*}
\]

(38)

where

\[
\begin{align*}
v_{i,i} (k_1, p_{st,2}, p_{st,3}) &= -A_{t,23} (p_{st,2}, p_{st,3}) \text{sat}\left( k_1 \xi_{i,i}, v_{m,i} \right)
\end{align*}
\]

(39)

\[
\begin{align*}
v_{m,i} (k_2) &= \sum_{j \in N_{m,i}} b_{ij} (k_2) \xi_{m,ij}
\end{align*}
\]

(40)

\[
\begin{align*}
v_{t,i} (k_3, r_s, p_{st,1}, p_{st,2}) &= -c_i (k_3, r_t) A_{l,12} (p_{st,1}, p_{st,2}) \xi_{t,i}
\end{align*}
\]

(41)

Furthermore, according to Theorem 1, UAVs in \( \text{Left Standby Area} \) will fly into \( \text{Left Ready Area} \), meaning while avoiding colliding other UAVs and keeping within \( \text{LS2R virtual tube} \), namely keeping away from \( \text{virtual tube} \) and \( \text{virtual tube Extension} \).

Similarly, the controller for from \( \text{Right Standby Area} \) to \( \text{Right Ready Area} \) is designed as \( v_{c,i} = v_{st,i} \), where

\[
\begin{align*}
v'_{st,i} &= \text{sat}\left( v_{i,i}, (k_1, p'_{st,2}, p'_{st,3}) + v_{m,i} (k_2) \right) + v_{t,i} (k_3, r_s, p'_{st,1}, p'_{st,2}, v_{m,i})
\end{align*}
\]

(42)

with

\[
\begin{align*}
p'_{st,1} &= \begin{bmatrix} p_{t,2} (1) \end{bmatrix}, p'_{st,2} = \begin{bmatrix} -r_b \end{bmatrix}, p'_{st,3} = \begin{bmatrix} -r_b \end{bmatrix},
\end{align*}
\]

2) From \( \text{Ready Area} \) to \( \text{virtual tube Extension} \): As shown in Figure 11 a virtual virtual tube, named LR2T virtual tube, is designed with the width \( 2r_t \) and centerline starting from \( p_{nt,1} \in \mathbb{R}^2 \) to \( p_{nt,2} \in \mathbb{R}^2 \), where \( r_t > 0 \) is sufficiently large that takes all UAVs in \( \text{Left Ready Area} \) in the virtual virtual tube. Moreover, all UAVs in \( \text{Left Ready Area} \) approach the finishing line \( p_{nt,2}p_{nt,3} \). Here

\[
\begin{align*}
P_{nt,1} &= \begin{bmatrix} -r_t \end{bmatrix}, P_{nt,2} = \begin{bmatrix} -r_t \end{bmatrix}, P_{nt,3} = \begin{bmatrix} 0 \end{bmatrix},
\end{align*}
\]

The intersection of \( \text{LR2T virtual tube} \) and \( \text{virtual tube Extension} \) is a buffer with length \( r_b \), which can make a UA V fly into the virtual tube Extension not only approaching it. According to (15), the controller is designed as \( v_{c,i} = v_{nt,i} \), where

\[
\begin{align*}
v_{nt,i} &= \text{sat}\left( v_{i,i}, (k_1, p_{nt,2}, p_{nt,3}) + v_{m,i} (k_2) \right) + v_{t,i} (k_3, r_t, p_{nt,1}, p_{nt,2}, v_{m,i})
\end{align*}
\]

(43)

with

\[
\begin{align*}
p'_{nt,1} &= \begin{bmatrix} -r_t \end{bmatrix}, p'_{nt,2} = \begin{bmatrix} -r_t \end{bmatrix}, p'_{nt,3} = \begin{bmatrix} 0 \end{bmatrix},
\end{align*}
\]

3) Final Controller: With the design above, the final controller is designed as

\[
\begin{align*}
v_{nt,i} &= v_{nt,i} \quad \text{if } \xi_i \text{ in } \text{Tunnel and Tunnel Extension} \\
v_{st,i} &= v_{st,i} \quad \text{if } \xi_i \text{ in } \text{Left Standby Area} \\
v'_{nt,i} &= v'_{nt,i} \quad \text{if } \xi_i \text{ in } \text{Right Standby Area} \\
v'_{st,i} &= v'_{st,i} \quad \text{if } \xi_i \text{ in } \text{Right Ready Area}
\end{align*}
\]

(45)

where \( i = 1, 2, \cdots, M \). Then, for given \( \epsilon_0 \in \mathbb{R}_+ \), there exist sufficiently small \( \epsilon_m, r_5 \in \mathbb{R}_+ \) in \( b_{ij}, c_i \in \mathbb{R}_+ \) and \( r_1 \in \mathbb{R}_+ \) such that all UAVs can satisfy (15) as \( t \geq t_1 \), meanwhile \( \| \xi_{m,ij} \| > 2r_s \) and \( \| \xi_{i,j} \| < r_l - r_s \) when passing the virtual tube, \( t \in [0, \infty) \) for all \( p_{nt,0} \), \( i = 1, 2, \cdots, M \).

VI. Simulation and Experiment

Simulations and experiments are given in the following to show the effectiveness of the proposed method, where a video about simulations and experiments is available on https://youtu.be/LTN9jWIBdY or http://weibo.ws/nJiiXJ.
A. Simulation

1) Simulation with 40 UAVs: We consider a scenario of $M = 40$ UAVs with $r_s = 20$m, $r_a = 30$m, $r_d = 80$m, $l_i = 5$ and $v_{m,i} = 5 + i \cdot 4$m/s, $i = 1, 2, \ldots, M$. The virtual tube is a long horizontal band with the width $2r_t = 300$m and centerline through $p_{1,1} = [0 \ 0]^T$m and $p_{1,2} = [500 \ 0]^T$m. For the $i$th instance, the constraints are $\epsilon_m = \epsilon_i = 10^{-6}$, $k_1 = k_2 = k_3 = 1$. With these conditions and parameters, the controller is performed for the initial and online path-planning problem. We design a scenario that contains 10 UAVs with $r_s = 5$m, $r_a = 7.5$m, $r_d = 20$m, and a virtual tube with the width $2r_t = 200$m, centerline through $p_{1,1} = [0 \ 0]^T$m and $p_{1,2} = [100 \ 0]^T$m. The initial position of the 1st UAV $p_1(0) = p_{1,1} = [0 \ 0]^T$, while the other UAVs are distributed inside the virtual tube randomly with the same velocity $v_i = [1 \ 0]^T$m/s, $i = 2, \ldots, M$. For two different algorithms, we design 10 sets of random initial positions for the other UAVs, run the simulation on the same computer and record the average calculation time when the 1st UAV passes the finishing line of the virtual tube. Figure 14 shows the performance of the density on calculation speed by the changing the safety radius and the number of UAVs separately. As shown in Figure 14 for the same airspace, if the number of UAVs increases or the safety radius of UAVs gets larger, the calculation speed of optimization-based algorithm will decrease rapidly because the probability of constraint being triggered is increasing, which brings more complex calculations. On the contrary, the proposed controller can better deal with such situation, in other words, be suitable for dense and complex environment.

Fig. 12. UAVs’ positions at different time.

Fig. 13. Minimum distance among all UAVs and minimum distance from the virtual tube edge to all UAVs.

Fig. 14. The calculation speed of different algorithms.
B. Flight Experiments

As shown in Fig. 16, a set of experiments is carried out in a laboratory room with an OptiTrack motion capture system installed, which provides the positions and orientations of UAVs for distributed control. The laptop is connected to the Tello UAVs and OptiTrack by a local network, running the proposed controller and a real-time position plotting module.

In the first experiment, there are $M = 6$ UAVs to pass a virtual tube (indicated by a purple rectangle) as shown in Fig. 15. The virtual tube is a long horizontal band with the width $2r_t = 1.7\text{m}$ and centerline through $p_{t,1} = [1.4 0]_T\text{m}$ and $p_{t,2} = [-1.4 0]_T\text{m}$. Assuming that $r_s = 0.16\text{m}$, $r_a = 0.4\text{m}$, $v_{m,1} = v_{m,2} = 0.1\text{m/s}$ and $v_{m,i} = 0.2\text{m/s}$, $i = 3, 4, \cdots , M$. The parameters of controller (45) are $\epsilon = r_s = 10^{-6}$, $k_1 = k_2 = k_3 = 1$ and $r_b = r_a$, $r_{st} = r_{tt} = 10000\text{m}$. As shown in Fig 15(a), the initial positions of all UAVs (indicated by the dotted circle) are at everywhere with initial velocities being zero. We track UAVs from three perspectives: the main view, local view and the Graphical User Interface(GUI) view. As shown in Fig. 15(b), after take-off they all enter into the virtual tube according to the sequence given in Figure 9(b). By comparing with Fig. 15(c), we can get the result that the slower UAVs are gradually overtaken by the faster ones. Finally, as shown in Fig. 15(c), all UAVs complete their routes at 213.88 second, and they always keep a safe distance from others. This means each UAV passes the virtual tube and avoids each other successfully.

In the experiment, three virtual tubes are added into the flight progress as shown by $t = 0s$ in Figure 17 UAVs need to pass the three virtual tubes in order of clockwise rotation (indicated by the purple arrow), during which the faster ones will overtake the slow ones. As shown in Figure 17 (lower plot), the faster UAVs pass the three virtual tube at 195.54 second, while the slower ones are still on the way. By comparing with Figure 17 (upper plot), we can observe that slow UAVs are overtaken by faster ones.

VII. CONCLUSIONS

The virtual tube passing problem, which includes passing a virtual tube, inter-agent conflict avoidance and keeping within the virtual tube, is studied in this paper. Based on the velocity control model of UAVs with control saturation, practical distributed control is proposed for multiple UAVs to pass a virtual tube. Every UAV has the same and simple control protocol. Lyapunov-like functions are designed elaborately, and formal analysis and proofs are made to show that the virtual tube passing problem can be solved, namely passing the virtual tube without getting trapped, avoiding conflict and keeping within the virtual tube. Besides the functional requirement, the safety requirement is also satisfied. By the proposed distributed control, a UAV can keep away from another one or return back to the virtual tube as soon as possible, once it enters into the safety area of another UAV or has a conflict with the virtual tube accidentally during it is passing the virtual tube. This is very necessary to guarantee the
flight safety. Simulations and experiments are given to show the advantages of the proposed method over other algorithms in terms of calculation speed of finding feasible solutions, and the effectiveness of the proposed method from the functional requirement and the safety requirement.

VIII. APPENDIX

A. Proof of Proposition 1

Proof. According to (1), we have

\[ v_i^T \dot{v}_i = -l_i \left( v_i^T v_i - v_i^T v_{c,i} \right) \]

Then

\[ \frac{d\|v_i\|}{dt} = -l_i \|v_i\| + \frac{1}{\|v_i\|} v_i^T v_{c,i} \]

whose solution is

\[ \|v_i(t)\| = e^{-l_i t} \|v_i(0)\| + \int_0^t e^{-l_i (t-\tau)} \frac{1}{\|v_i(\tau)\|} v_i^T v_{c,i}(\tau) \, d\tau \]

\[ \leq e^{-l_i t} \|v_i(0)\| + \int_0^t e^{-l_i (t-\tau)} v_{m,i} \, d\tau. \]

If \( \|v_i(0)\| \leq v_{m,i} \), then

\[ \|v_i(t)\| \leq e^{l_i t} v_{m,i}. \]

With the result, one has

\[ \|\xi_i(t) - \xi_j(t)\| \leq \|p_i(t) - p_j(t)\| + \left\| \frac{1}{l_i} v_i \right\| + \left\| \frac{1}{l_j} v_j \right\| \]

\[ \leq \|p_i(t) - p_j(t)\| + 2 r_v. \]

Since \( \|\xi_i(t) - \xi_j(t)\| \geq r + 2 r_v \), one further has

\[ \|p_i(t) - p_j(t)\| + r_v \geq r + r_v. \]

Then \( \|p_i(t) - p_j(t)\| > r. \)

B. Proof of Lemma 2

The reason why these VTOL UAVs are able to avoid conflict with each other, which will be proved by contradiction. Without loss of generality, assume that \( \|\xi_{m,ij}(t_0)\| = 2 r_s \) occurs at \( t_0 > 0 \) first, i.e., a conflict between the \( i \)th UAV and the \( j \)th UAV happening. Then, \( \|\xi_{m,ij}(t_0)\| > 2 r_s \) for \( j \neq j \). Consequently, \( V_{m,ij}(t_0) \geq 0 \) if \( j \neq j \). Since \( V(0) > 0 \) and \( V(t) \leq 0 \), the function \( V \) satisfies \( V(t_0) \leq V(0) \), \( t \in [0, \infty) \). By the definition of \( V \), we have

\[ V_{m,ij}(t_0) \leq V(0). \]

According to (23), given any \( \epsilon_{rs} > 0 \), there exists a \( \epsilon_s > 0 \), such that

\[ s(1, \epsilon_s) = 1 - \epsilon_{rs}. \]

Then, at time \( t_0 \), the denominator of \( V_{m,ij} \) defined in (26) is

\[ \frac{(1 + \epsilon_m)\|\xi_{m,ij}(t_0)\|}{2 r_s} \]

\[ = 2 r_s (1 + \epsilon_m) - 2 r_s (1 - \epsilon_{rs}) \]

\[ = 2 r_s (\epsilon_m + \epsilon_{rs}) \]

where \( \epsilon_{rs} > 0 \) can be sufficiently small if \( \epsilon_s \) is sufficiently small according to (23). According to the definition in (26), we have

\[ \frac{1}{2 r_s (\epsilon_m + \epsilon_{rs})} = \frac{V_{m,ij}(t_0)}{k_2} \leq \frac{V(0)}{k_2} \]

where \( \sigma_m \left( \|\xi_{m,ij}\| \right) = 1 \) is used. Consequently, \( V(0) \) is unbounded as \( \epsilon_m \to 0 \) and \( \epsilon_{rs} \to 0 \). On the other hand, for any \( j \), we have \( \|\xi_{m,ij}(0)\| > 2 r_s \) by Assumption 3. Let

\[ \|\xi_{m,ij}(0)\| = 2 r_s + \epsilon_{m,ij}, \epsilon_{m,ij} > 0. \]

Then, at time \( t = 0 \), the denominator of \( V_{m,ij} \) defined in (26) is

\[ (1 + \epsilon_m)\|\xi_{m,ij}(0)\| - 2 r_s s \left( \frac{\|\xi_{m,ij}(0)\|}{2 r_s} , \epsilon_s \right) \]

\[ \geq (1 + \epsilon_m) (2 r_s + \epsilon_{m,ij}) - 2 r_s s \left( \frac{\|\xi_{m,ij}(0)\|}{2 r_s} \right) \]

\[ = 2 r_s \epsilon_m + (1 + \epsilon_m) \epsilon_{m,ij}. \]

Then

\[ V_{m,ij}(0) \leq \frac{k_2}{2 r_s \epsilon_m + (1 + \epsilon_m) \epsilon_{m,ij}}. \]

Consequently, \( V_{m,ij}(0) \) is still bounded as \( \epsilon_m \to 0 \) no matter what \( \epsilon_{rs} \) is. According to the definition of \( V(0) \), \( V(0) \) is still bounded as \( \epsilon_m \to 0 \) and \( \epsilon_{rs} \to 0 \). This is a contradiction. Thus

\[ \|\xi_{m,ij}(t)\| > 2 r_s, i \neq j \]

for \( i, j = 1, 2, \cdots, N, t \in [0, \infty) \). Therefore, the UAV can avoid another UAV by the velocity command (28).

The reason why a UAV can stay within the virtual tube is similar to the above proof. It can be proved by contradiction as well. Without loss of generality, assume that \( \|\xi_{s,i}(t_0)\| = r_i - r_s \) occurs at \( t_0 > 0 \), i.e., a conflict happening first, while \( \|\xi_{s,i}(t_0)\| > r_i - r_s \) for \( i = 1, \cdots, M \). Similar to the above proof, one can also get a contradiction.

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