EMBEDDINGS OF SURFACES INTO 3-SPACE
AND QUADRUPLE POINTS OF REGULAR HOMOTOPIES

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Abstract. Let $F$ be a closed orientable surface. We give an explicit formula for the number mod 2 of quadruple points occurring in any generic regular homotopy between any two regularly homotopic embeddings $e, e': F \to \mathbb{R}^3$. The formula is in terms of homological data extracted from the two embeddings.

1. Introduction

For $F$ a closed surface and $i, i': F \to \mathbb{R}^3$ two regularly homotopic generic immersions, we are interested in the number mod 2 of quadruple points occurring in generic regular homotopies between $i$ and $i'$. It has been shown in [N1] that this number is the same for all such regular homotopies, and so it is a function of $i$ and $i'$ which we denote $Q(i, i') \in \mathbb{Z}/2$. For $F$ orientable and $e, e': F \to \mathbb{R}^3$ two regularly homotopic embeddings, we give an explicit formula for $Q(e, e')$ which depends on the following data: If $e: F \to \mathbb{R}^3$ is an embedding then $e(F)$ splits $\mathbb{R}^3$ into two pieces, one compact which will be denoted $M^0(e)$ and the other non-compact which will be denoted $M^1(e)$. By restriction of range $e$ induces maps $e^k: F \to M^k(e)$ ($k = 0, 1$) and let $A^k(e) \subseteq H_1(F, \mathbb{Z}/2)$ be the kernel of the map induced by $e^k$ on $H_1(\cdot, \mathbb{Z}/2)$. Let $o(e)$ be the orientation on $F$ which is induced from $M^0(e)$ to $\partial M^0(e) = e(F)$ and then via $e$ to $F$. Our formula for $Q(e, e')$ will be in terms of the two triplets $A^0(e), A^1(e), o(e)$ and $A^0(e'), A^1(e'), o(e')$. Our formula will be also easily extended to finite unions of closed orientable surfaces.

For two special cases a formula for $Q(e, e')$ (for $e, e'$ embeddings) has already been known: The case where $F$ is a sphere has appeared in [MB] and [N1], and the case where $F$ is a torus has appeared in [N1]. The starting point for our work will be [N2] where an explicit formula has been given for $Q(i, i \circ h)$, where $i: F \to \mathbb{R}^3$ is any generic immersion and $h: F \to F$ is any diffeomorphism such that $i$ and $i \circ h$ are regularly homotopic.

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2. Totally Singular Decompositions

Let $V$ be a finite dimensional vector space over $\mathbb{Z}/2$. A function $g : V \to \mathbb{Z}/2$ is called a quadratic form if $g$ satisfies: $g(x + y) = g(x) + g(y) + B(x, y)$ for all $x, y \in V$, where $B(x, y)$ is a bilinear form. The following properties follow: (a) $g(0) = 0$. (b) $B(x, x) = 0$ for all $x \in V$. (c) $B(x, y) = B(y, x)$ for all $x, y \in V$. $g$ is called non-degenerate if $B$ is non-degenerate, i.e. for any $0 \neq x \in V$ there is $y \in V$ with $B(x, y) \neq 0$. For an exposition of quadratic forms see [C].

In what follows we always assume that our vector space $V$ is equipped with a non-degenerate quadratic form $g$. It then follows that $\dim V$ is even. A subspace $A \subseteq V$ such that $g|_A \equiv 0$ is called a totally singular subspace. A pair $(A, B)$ of subspaces of $V$ will be called a totally singular decomposition (abbreviated TSD) of $V$ if $V = A \oplus B$ and both $A$ and $B$ are totally singular. It then follows that $\dim A = \dim B$. (We remark that TSDs do not always exist. They will however always exist for the quadratic forms which will arise in our geometric considerations, as seen in Lemma 3.3 below.) A linear map $T : V \to V$ is called orthogonal if $g(T(x)) = g(x)$ for all $x \in V$. It then follows that $B(T(x), T(y)) = B(x, y)$ for all $x, y \in V$ and that $T$ is invertible. The group of all orthogonal maps of $V$ with respect to $g$ will be denoted $O(V, g)$.

The proof of the following lemma appears in [C]:

**Lemma 2.1.** Let $\dim V = 2n$.

1. If $A \subseteq V$ is a totally singular subspace of dimension $n$ then there exists a $B \subseteq V$ such that $(A, B)$ is a TSD of $V$.

2. If $(A, B)$ is a TSD of $V$ and $a_1, \ldots, a_n$ is a given basis for $A$ then there is a basis $b_1, \ldots, b_n$ for $B$ such that $B(a_i, b_j) = \delta_{ij}$.

**Definition 2.2.** If $(A, B)$ is a TSD of $V$ then a basis $a_1, \ldots, a_n, b_1, \ldots, b_n$ of $V$ will be called $(A, B)$-good if $a_i \in A$, $b_i \in B$ and $B(a_i, b_j) = \delta_{ij}$.

The following two lemmas follow directly from the definition of quadratic form:

**Lemma 2.3.** Let $(A, B)$ be a TSD of $V$ and $a_1, \ldots, a_n, b_1, \ldots, b_n$ an $(A, B)$-good basis for $V$. If $v = \sum x_i a_i + \sum y_i b_i$ and $v' = \sum x'_i a_i + \sum y'_i b_i$ then $g(v) = \sum x_i y_i$ and $B(v, v') = \sum x_i y'_i + \sum y_i x'_i$. 

Lemma 2.4. Let \((A, B)\) and \((A', B')\) be two TSDs of \(V\). Let \(a_1, \ldots, a_n, b_1, \ldots, b_n\) be an \((A, B)\)-good basis for \(V\) and \(a'_1, \ldots, a'_n, b'_1, \ldots, b'_n\) an \((A', B')\)-good basis for \(V\). If \(T : V \to V\) is the linear map defined by \(a_i \mapsto a'_i\), \(b_i \mapsto b'_i\) then \(T \in O(V, g)\).

For \(T \in O(V, g)\) we define \(\psi(T) \in \mathbb{Z}/2\) by:

\[
\psi(T) = \text{rank}(T - Id) \mod 2.
\]

It has been shown in [N2] that \(\psi : O(V, g) \to \mathbb{Z}/2\) is a (non-trivial) homomorphism.

Lemma 2.5. If \((A, B)\) is a TSD of \(V\) and \(T \in O(V, g)\) satisfies \(T(A) = A\) and \(T(B) = B\) then \(\psi(T) = 0\).

Proof. By Lemma 2.1 there exists an \((A, B)\)-good basis \(a_1, \ldots, a_n, b_1, \ldots, b_n\) for \(V\). Using Lemma 2.3 it is easy to verify that the matrix of \(T\) with respect to such a basis has the form:

\[
\begin{pmatrix}
S' & 0 \\
0 & S^{-1}
\end{pmatrix}
\]

where \(S \in GL_n(\mathbb{Z}/2)\). It follows that \(\psi(T) = 0\).

Given two TSDs \((A, B), (A', B')\) of \(V\) then by Lemmas 2.1 and 2.4 there exists a \(T \in O(V, g)\) such that \(T(A) = A'\) and \(T(B) = B'\). It follows from Lemma 2.5 that if \(T_1, T_2\) are two such \(T\)s then \(\psi(T_1) = \psi(T_2)\). And so the following is well defined:

Definition 2.6. For a pair \((A, B), (A', B')\) of TSDs of \(V\) let \(\hat{\psi}(A, B; A', B') \in \mathbb{Z}/2\) be defined by \(\hat{\psi}(A, B; A', B') = \psi(T)\) for some (thus all) \(T \in O(V, g)\) with \(T(A) = A'\) and \(T(B) = B'\).

Definition 2.7. For two TSDs \((A, B), (A', B')\) of \(V\), we will write \((A, B) \sim (A', B')\) if \(\hat{\psi}(A, B; A', B') = 0\).

Since \(\psi\) is a homomorphism, \(\hat{\psi}(A, B; A'', B'') = \hat{\psi}(A, B; A', B') + \hat{\psi}(A', B'; A'', B'')\) for any three TSDs \((A, B), (A', B'), (A'', B'')\). It follows that \(\sim\) is an equivalence relation with precisely two equivalence classes and that \(\hat{\psi}(A, B; A'', B'') = \hat{\psi}(A', B'; A'', B'')\) whenever \((A, B) \sim (A', B')\).

Lemma 2.8. Let \(\dim V = 2n\) and let \(A \subseteq V\) be a totally singular subspace of dimension \(n\). If \(T \in O(V, g)\) satisfies \(T(x) = x\) for every \(x \in A\) then \(\psi(T) = 0\).

Proof. By Lemma 2.1 there is a \(B \subseteq V\) such that \((A, B)\) is a TSD of \(V\) and an \((A, B)\)-good basis \(a_1, \ldots, a_n, b_1, \ldots, b_n\) for \(V\). Using Lemma 2.3 it is easy to verify that the matrix of \(T\)
with respect to such a basis has the form: \[
\begin{pmatrix}
I & S \\
0 & I
\end{pmatrix}
\]
where \(I\) is the \(n \times n\) identity matrix and \(S \in M_n(\mathbb{Z}/2)\) is an alternating matrix, i.e. if \(S = \{s_{ij}\}\) then \(s_{ii} = 0\) and \(s_{ij} = s_{ji}\). Since alternating matrices have even rank, it follows that \(\psi(T) = 0\).

\[\square\]

**Corollary 2.9.** Let \((A, B)\) and \((A', B')\) be two TSDs of \(V\). If \(A = A'\) or \(B = B'\) then \((A, B) \sim (A', B')\).

**Proof.** Say \(A = A'\). By Lemmas 2.1 and 2.4 there exists a \(T \in O(V, g)\) with \(T(x) = x\) for all \(x \in A = A'\) and \(T(B) = B'\). The conclusion follows from Lemma 2.8.

Let \(V_0, V_1 \subseteq V\) be two subspaces of \(V\). We will write \(V_0 \perp V_1\) if \(B(x, y) = 0\) for every \(x \in V_0, y \in V_1\). The following is clear:

**Lemma 2.10.** Let \(V_0, V_1 \subseteq V\) satisfy \(V = V_0 \oplus V_1\) and \(V_0 \perp V_1\).

1. If for \(l = 0, 1\), \((A_l, B_l)\) is a TSD of \(V_l\) (with respect to \(g|_{V_l}\) which is indeed non-degenerate) then \((A_0 + A_1, B_0 + B_1)\) is a TSD of \(V\).
2. If \((A_l', B_l')\) is another TSD of \(V_l\) and \((A_l, B_l) \sim (A_l', B_l')\) \((l = 0, 1)\) then \((A_0 + A_1, B_0 + B_1) \sim (A_0' + A_1', B_0' + B_1')\).

### 3. Statement of Main Result

A surface is by definition assumed connected. A finite union of surfaces will be called a system of surfaces. Let \(S\) be a system of closed surfaces and \(H_t : S \to \mathbb{R}^3\) a generic regular homotopy. We denote by \(q(H_t) \in \mathbb{Z}/2\) the number mod 2 of quadruple points occurring in \(H_t\). The following has been shown in [N1]:

**Theorem 3.1.** Let \(S\) be a system of closed surfaces (not necessarily orientable.) If \(H_t, G_t : S \to \mathbb{R}^3\) are two generic regular homotopies between the same two generic immersions, then \(q(H_t) = q(G_t)\).

**Definition 3.2.** Let \(S\) be a system of closed surfaces and \(i, i' : S \to \mathbb{R}^3\) two regularly homotopic generic immersions. We define \(Q(i, i') \in \mathbb{Z}/2\) by \(Q(i, i') = q(H_t)\), where \(H_t\) is any generic regular homotopy between \(i\) and \(i'\). This is well defined by Theorem 3.1.

Let \(F\) from now on denote a closed orientable surface. A simple closed curve in \(F\) will be called a circle. If \(c\) is a circle in \(F\), the homology class of \(c\) in \(H_1(F, \mathbb{Z}/2)\) will be denoted
by \( [c] \). Any immersion \( i : F \to \mathbb{R}^3 \) induces a quadratic form \( g^i : H_1(F, \mathbb{Z}/2) \to \mathbb{Z}/2 \) whose associated bilinear form \( B(x, y) \) is the algebraic intersection form \( x \cdot y \) of \( H_1(F, \mathbb{Z}/2) \), as follows: For \( x \in H_1(F, \mathbb{Z}/2) \) let \( A \subseteq F \) be an annulus bounded by circles \( c, c' \) with \( [c] = x \), let \( j : A \to \mathbb{R}^3 \) be an embedding which is regularly homotopic to \( i|_A \) and define \( g^i(x) \) to be the \( \mathbb{Z}/2 \) linking number between \( j(c) \) and \( j(c') \) in \( \mathbb{R}^3 \). One needs to verify that \( g^i(x) \) is independent of the choices being made and that \( g^i(x + y) = g^i(x) + g^i(y) + x \cdot y \). This has been done in \([P]\). Also, \( i, i' : F \to \mathbb{R}^3 \) are regularly homotopic iff \( g^i = g^{i'} \).

If \( e : F \to \mathbb{R}^3 \) is an embedding then \( e(F) \) splits \( \mathbb{R}^3 \) into two pieces one compact and one non-compact. We denote the compact piece by \( M^0(e) \) and the non-compact piece by \( M^1(e) \). By restriction of range, \( e \) induces maps \( e^k : F \to M^k(e), k = 0, 1 \). Let \( e^k : H_1(F, \mathbb{Z}/2) \to H_1(M^k(e), \mathbb{Z}/2) \) be the maps induced on homology and finally let \( A^k(e) = \ker e^k, k = 0, 1 \).

**Lemma 3.3.** Let \( e : F \to \mathbb{R}^3 \) be an embedding, then \((A^0(e), A^1(e))\) is a TSD of \( H_1(F, \mathbb{Z}/2) \) with respect to the quadratic form \( g^e \).

**Proof.** We first show that each \( A^k(e) \) is totally singular: For \( x \in A^k(e) \) let \( A, c, c' \) be as in the definition of \( g^e(x) \) and simply take \( j = e|_A \). Since \( e^k_*(x) = 0 \), \( e(c) \) bounds a properly embedded (perhaps non-orientable) surface \( S \) in \( M^k(e) \). Since \( e(c') \) is disjoint from \( S \), the \( \mathbb{Z}/2 \) linking number between \( e(c) \) and \( e(c') \) in \( \mathbb{R}^3 \) is 0, and so \( g^e(x) = 0 \). Now, the fact that \( H_1(F, \mathbb{Z}/2) = A^0(e) \oplus A^1(e) \) is a consequence of the \( \mathbb{Z}/2 \) Mayer-Vietoris sequence for \( \mathbb{R}^3 = M^0(e) \cup M^1(e) \) where \( F \) is identified with \( M^0(e) \cap M^1(e) \) via \( e \). \( \square \)

If \( e, e' : F \to \mathbb{R}^3 \) are two regularly homotopic embeddings then \( g^e = g^{e'} \) so \((A^0(e), A^1(e))\) and \((A^0(e'), A^1(e'))\) are TSDs of \( H_1(F, \mathbb{Z}/2) \) with respect to the same quadratic form and so \( \hat{\psi}(A^0(e), A^1(e); A^0(e'), A^1(e')) \) is defined. We spell out the actual computation involved in \( \hat{\psi}(A^0(e), A^1(e); A^0(e'), A^1(e')) \):

1. Find a basis \( a_1, \ldots, a_n, b_1, \ldots, b_n \) for \( H_1(F, \mathbb{Z}/2) \) such that \( e^0_*(a_i) = 0, e^1_*(b_i) = 0 \) and \( a_i \cdot b_j = \delta_{ij} \).
2. Find a similar basis \( a'_1, \ldots, a'_n, b'_1, \ldots, b'_n \) using \( e' \) in place of \( e \).
3. Let \( m \) be the dimension of the subspace of \( H_1(F, \mathbb{Z}/2) \) spanned by:
   \[
   a'_1 - a_1 , \ldots, a'_n - a_n, b'_1 - b_1, \ldots, b'_n - b_n.
   \]
4. \( \hat{\psi}(A^0(e), A^1(e); A^0(e'), A^1(e')) = m \mod 2 \), (an element in \( \mathbb{Z}/2 \).)
Definition 3.4. If $e : F \to \mathbb{R}^3$ is an embedding then we define $o(e)$ to be the orientation on $F$ which is induced from $M^0(e)$ to $\partial M^0(e) = e(F)$ and then via $e$ to $F$ (and where the orientation on $M^0(e)$ is the restriction of the orientation of $\mathbb{R}^3$.) If $e, e' : F \to \mathbb{R}^3$ are two embeddings then we define $\widehat{e}(e, e') \in \mathbb{Z}/2$ to be $0$ if $o(e) = o(e')$ and $1$ if $o(e) \neq o(e')$.

Our purpose in this work is to show:

Theorem 3.5. Let $n$ be the genus of $F$. If $e, e' : F \to \mathbb{R}^3$ are two regularly homotopic embeddings then:

$$Q(e, e') = \widehat{\psi}(A^0(e), A^1(e); A^0(e'), A^1(e')) + (n + 1)\widehat{e}(e, e').$$

Our starting point is the following theorem which has been proved in [N2]:

Theorem 3.6. For any generic immersion $i : F \to \mathbb{R}^3$ and any diffeomorphism $h : F \to F$ such that $i$ and $i \circ h$ are regularly homotopic,

$$Q(i, i \circ h) = \psi(h_*) + (n + 1)\epsilon(h),$$

where $h_*$ is the map induced by $h$ on $H_1(F, \mathbb{Z}/2)$, $n$ is the genus of $F$ and $\epsilon(h) \in \mathbb{Z}/2$ is $0$ or $1$ according to whether $h$ is orientation preserving or reversing, respectively.

4. Equivalent Embeddings and $k$-Extendible Regular Homotopies

Let $e : F \to \mathbb{R}^3$ be an embedding, let $P \subseteq \mathbb{R}^3$ be a plane and assume $e(F)$ intersects $P$ transversally in a unique circle. Let $c = e^{-1}(P)$ then $c$ is a separating circle in $F$. Let $A$ be a regular neighborhood of $c$ in $F$ and let $F_0, F_1$ be the connected components of $F - \text{int}A$. (A lower index will always be related to the splitting of $\mathbb{R}^3$ via a plane, the assignment of $0$ and $1$ to the two sides being arbitrary. An upper index on the other hand is related to the splitting of $\mathbb{R}^3$ via the image of a closed surface, assigning $0$ to the compact side and $1$ to the non-compact side.) Let $\bar{F}_l$ ($l = 0, 1$) be the closed surface obtained by gluing a disc $D_l$ to $F_l$. Let $e_l : \bar{F}_l \to \mathbb{R}^3$ be the embedding such that $e_l|_{F_l} = e|_{F_l}$ and $e_l(D_l)$ is parallel to $P$. Let $i_{F_l} : F_l \to F$ and $i_{\bar{F}_l} : F_l \to \bar{F}_l$ denote the inclusion maps. The induced map $i_{\bar{F}_l} : H_1(F_l, \mathbb{Z}/2) \to H_1(\bar{F}_l, \mathbb{Z}/2)$ is an isomorphism and let $h_l : H_1(\bar{F}_l, \mathbb{Z}/2) \to H_1(F, \mathbb{Z}/2)$ be the map $h_l = i_{F_l} \circ (i_{\bar{F}_l})^{-1}$.

Lemma 4.1. Under the above assumptions and definitions: $A^k(e) = h_0(A^k(e_0)) + h_1(A^k(e_1))$, $k = 0, 1$. 

Proof. This follows from the fact that the inclusions $F_0 \cup F_1 \rightarrow \bar{F}_0 \cup \bar{F}_1$, $F_0 \cup F_1 \rightarrow F$, $M^0(e_0) \cup M^0(e_1) \rightarrow M^0(e)$ and $M^1(e) \rightarrow \mathbb{R}^3 - (M^0(e_0) \cup M^0(e_1))$ all induce isomorphisms on $H_1(\cdot, \mathbb{Z}/2)$ and the splitting of each of the above spaces via $P$ induces a direct sum decomposition. We only check that the inclusion $M^1(e) \rightarrow \mathbb{R}^3 - (M^0(e_0) \cup M^0(e_1))$ induces isomorphism on $H_1(\cdot, \mathbb{Z}/2)$. Indeed $\mathbb{R}^3 - (M^0(e_0) \cup M^0(e_1))$ is obtained from $M^1(e)$ by gluing a 2-handle along $e(A)$, and the inclusion of $e(A)$ in $M^1(e)$ is null-homotopic. \hfill \qed

Definition 4.2. Two embeddings $e, f : F \rightarrow \mathbb{R}^3$ will be called equivalent if:

1. There is a regular homotopy between $e$ and $f$ with no quadruple points.
2. $(A^0(e), A^1(e)) \sim (A^0(f), A^1(f))$.
3. $o(e) = o(f)$

Definition 4.3. An embedding $e : F \rightarrow \mathbb{R}^3$ will be called standard if its image $e(F)$ is a surface in $\mathbb{R}^3$ as in Figure 1.

![Figure 1. Image of a standard embedding.](image)

In Proposition 4.8 below we will show that any embedding $e : F \rightarrow \mathbb{R}^3$ is equivalent to a standard embedding. The following lemma will be used in the induction step:

Lemma 4.4. Let $e : F \rightarrow \mathbb{R}^3$ be an embedding. Assume $e(F)$ intersects a plane $P \subseteq \mathbb{R}^3$ transversally in one circle and let $c, A, F_l, \bar{F}_l, D_l, e_l$ be as above. If $e_l : \bar{F}_l \rightarrow \mathbb{R}^3$ $(l = 0, 1)$ are both equivalent to standard embeddings, then $e$ is equivalent to a standard embedding.

Proof. Changing $e$ by isotopy, we may assume $e(A)$ is a very thin tube. $e_l : \bar{F}_l \rightarrow \mathbb{R}^3$ is equivalent to a standard embedding $f_l$ via a regular homotopy $(H_l)_t : F_l \rightarrow \mathbb{R}^3$ having no quadruple points. We may further assume that each $(H_l)_t$ moves $\bar{F}_l$ only within the corresponding half-space defined by $P$, that each $f_l(D_l)$ is situated at the point of $f_l(\bar{F}_l)$ which is closest to $P$ and that these two points are opposite each other with respect to $P$. We now perform both $(H_l)_t$, letting the thin tube $A$ be carried along. If we make sure the thin tube $A$ does not pass triple points occurring in $F_1$ and $F_2$ then the regular
homotopy $H_t$ induced on $F$ in this way will also have no quadruple points. Since $e(A)$ has approached $e_i(\bar{F}_i)$ from $M^1(e_i)$ and since $o_{e_i} = o_{f_i}$, we also have at the end of $H_t$ that $A$ approaches $f_i(\bar{F}_i)$ from $M^1(f_i)$. And so we may continue moving the tube $A$ until it is all situated in the region between $f_0(\bar{F}_0)$ and $f_1(\bar{F}_1)$, then canceling all knotting by having the thin tube pass itself (this involves only double lines) until $A$ is embedded as a straight tube connecting $f_0(\bar{F}_0)$ to $f_1(\bar{F}_1)$ and so the final map $f : F \to \mathbb{R}^3$ thus obtained is indeed a standard embedding. By assumption $(A^0(e_i), A^1(e_i)) \sim (A^0(f_i), A^1(f_i))$, $l = 0, 1$ which implies that $(h_l(A^0(e_i)), h_l(A^1(e_i))) \sim (h_l(A^0(f_i)), h_l(A^1(f_i)))$, $l = 0, 1$ as TSDs of $V_i = h_l(H_1(\bar{F}_i, \mathbb{Z}/2)) \subseteq H_1(F, \mathbb{Z}/2)$. (Note that $h_l$ preserves the corresponding quadratic forms.) But $H_1(F, \mathbb{Z}/2) = V_0 \oplus V_1$ and $V_0 \perp V_1$ and so by Lemma 2.10 and Lemma 1.1 $(A^0(e), A^1(e)) \sim (A^0(f), A^1(f))$. Finally, from $o_{e_i} = o_{f_i}$ it follows that $o(e) = o(f)$. \hfill \Box

**Definition 4.5.** Let $e, f : F \to \mathbb{R}^3$ be two embeddings. A regular homotopy $H_t : F \to \mathbb{R}^3$ ($a \leq t \leq b$) with $H_a = e$, $H_b = f$ will be called $k$-extendible (where $k$ is either 0 or 1) if there exists a regular homotopy $G_t : M^k(e) \to \mathbb{R}^3$ ($a \leq t \leq b$) satisfying:

1. $G_a$ is the inclusion map of $M^k(e)$ in $\mathbb{R}^3$.
2. $H_t = G_t \circ e^k$. (Recall that $e^k : F \to M^k(e)$ is simply $e$ with range restricted to $M^k(e)$.)
3. $G_b$ is an embedding with $G_b(M^k(e)) = M^k(f)$.

**Lemma 4.6.** If for a given $k$ there is a $k$-extendible regular homotopy between the embeddings $e$ and $f$ then $A^k(e) = A^k(f)$.

*Proof.* $f = H_b = G_b \circ e^k$ and so $f^k = G_b^k \circ e^k$ where $G_b^k : M^k(e) \to M^k(f)$ is the map $G_b$ with range restricted to $M^k(f)$. Since $G_b^k$ is a diffeomorphism it follows that $\ker f^k_* = \ker e^k_*$. \hfill \Box

**Corollary 4.7.** If there is a $k$-extendible regular homotopy between the embeddings $e$ and $f$ for either $k = 0$ or $k = 1$ then:

1. $(A^0(e), A^1(e)) \sim (A^0(f), A^1(f))$.
2. $o(e) = o(f)$.

*Proof.* 1 follows from Lemma 1.6 and Corollary 2.3. Since $G_a$ is the inclusion and $G_t$ is a regular homotopy it follows that $G_b$ is orientation preserving. This implies 2. \hfill \Box

**Proposition 4.8.** Every embedding $e : F \to \mathbb{R}^3$ is equivalent to a standard embedding.
Proof. The proof is by induction on the genus of $F$. If $F = S^2$ then any $e$ is isotopic to a standard embedding and isotopic embeddings are equivalent. So assume $F$ is of positive genus and so there is a compressing disc $D$ for $e(F)$ in $\mathbb{R}^3$ (i.e. $D \cap e(F) = \partial D$ and $\partial D$ does not bound a disc in $e(F)$.) Let $c = e^{-1}(\partial D) \subseteq F$ and let $A$ be a regular neighborhood of $c$ in $F$. Isotoping $A$ along $D$ as before we may assume $A$ is embedded as a thin tube. There are four cases to be considered according to whether $D$ is contained in $M^0(e)$ or $M^1(e)$ and whether $\partial D$ separates or does not separate $e(F)$.

Case 1: $D \subseteq M^0(e)$ and $\partial D$ separates $e(F)$. It then follows that $D$ separates $M^0(e)$. If $F_0, F_1$ denote the two components of $F - \text{int}A$ and $e_l : \bar{F}_l \to \mathbb{R}^3$ are defined as before then it follows from the assumptions of this case that $M^0(e_0)$ and $M^0(e_1)$ are disjoint and the tube $e(A)$ approaches each $e_l(\bar{F}_l)$ from its non-compact side, i.e. from $M^1(e_l)$. Move each foot of the tube $e(A)$ (see Figure 2) along the corresponding surface $e_l(\bar{F}_l)$ until they are each situated at the point $p_l$ of $e_l(\bar{F}_l)$ having maximal $z$-coordinate. In particular it follows that now $e(A)$ approaches each $e_l(\bar{F}_l)$ from above. We now uniformly shrink each $e(F_l)$ towards the point $p_l$ until it is contained in a tiny ball $B_l$ attached from below to the corresponding foot of $e(A)$, arriving at a new embedding $e' : F \to \mathbb{R}^3$. This regular homotopy is clearly 0-extendible, and since no self intersection may occur within each of $F_0$, $F_1$ and $A$, this regular homotopy has no quadruple points. And so by Corollary 4.7, $e'$ is equivalent to $e$. We now continue by isotopy, deforming the thin tube $e'(A)$ until it is a straight tube, and rigidly carrying $B_0$ and $B_1$ along. We finally arrive at an embedding $e''$ for which there is a plane $P$ intersecting $e''(F)$ as in Lemma 4.4 with our $F_0$ and $F_1$ on the two sides of $P$. Since the genus of both $\bar{F}_0$ and $\bar{F}_1$ is smaller than that of $F$ then by our induction hypothesis and Lemma 4.4, $e''$ is equivalent to a standard embedding.

Figure 2. Moving the foot of a tube.

Case 2: $D \subseteq M^1(e)$ and $\partial D$ separates $e(F)$. This time either $M^0(e_0) \subseteq M^0(e_1)$ or $M^0(e_1) \subseteq M^0(e_0)$ and assume the former holds. In this case $e(A)$ approaches only $e_0(\bar{F}_0)$ from its non-compact side and so we push the tube and perform the uniform shrinking as above only with $F_0$. This is a 1-extendible regular homotopy since we are shrinking $M^0(e_0)$
which is part of $M^1(e)$. Now, if $B$ is the tiny ball into which we have shrunken $e(F_0)$ then
$\partial B$ supplies separating compressing discs on both sides of $e(F)$ and so we are done by Case 1.

Case 3: $D \subseteq M^0(e)$ and $\partial D$ does not separate $e(F)$. If $F' = F - \text{int} A$ and $e' : \bar{F}' \to \mathbb{R}^3$
is induced as above (where $\bar{F}'$ is the surface obtained from $F'$ by gluing two discs to it) then both feet of the tube $e(A)$
approach $e'(\bar{F}')$ from its non-compact side. Push the feet of $e(A)$ until they are both situated near the same point $p$ in $e'(\bar{F}')$
having maximal $z$ coordinate. Let $P$ be a horizontal plane passing slightly below $p$ (so that in a neighborhood of $p$ it intersects $F$ in only one circle.) We may pull the tube $e(A)$ until it is all above $P$. We then let it pass through itself until it is unknotted. This is a 0-extendible regular homotopy with no quadruple points, at the end of which we have an embedding intersecting $P$ as in Lemma 4.4 with an embedding of a torus above the plane $P$, this embedding being already standard and an embedding of a subsurface $F''$ of $F$ below the plane $P$, $F''$ being of smaller genus than that of $F$. Again we are done by induction and Lemma 4.4.

Case 4: $D \subseteq M^1(e)$ and $\partial D$ does not separate $e(F)$. We may proceed as in Case 3 (this time via a 1-extendible regular homotopy) to obtain a standard embedding of a torus connected with a tube to $e'(\bar{F}')$ but this time the torus is contained in $M^0(e')$ and the tube connects to $e'(\bar{F}')$ from its compact side. But once we have such an embedding then the little standardly embedded torus has non-separating compressing discs on both sides and so we are done by Case 3.

\[ \square \]

Lemma 4.9. If $e : F \to \mathbb{R}^3$ is an embedding and $h : F \to F$ is a diffeomorphism such that $e$ and $e \circ h$ are regularly homotopic, then $\hat{\psi}(A^0(e), A^1(e); A^0(e \circ h), A^1(e \circ h)) = \psi(h_\ast)$ and $\hat{\gamma}(e, e \circ h) = \epsilon(h)$. (Recall that $h_\ast$ is the map induced by $h$ on $H_1(F, \mathbb{Z}/2)$ and $\epsilon(h) \in \mathbb{Z}/2$ is 0 or 1 according to whether $h$ is orientation preserving or reversing.)

Proof. $x \in \ker(e \circ h)^k_\ast$ iff $h_\ast(x) \in \ker e^k_\ast$ and so $A^k(e \circ h) = h_\ast^{-1}(A^k(e))$, $k = 0, 1$. By definition then $\hat{\psi}(A^0(e), A^1(e); A^0(e \circ h), A^1(e \circ h)) = \psi(h_\ast^{-1}) = \psi(h_\ast)$. (Note that if $e$ and $e \circ h$ are regularly homotopic then indeed $h_\ast^{-1} \in O(H_1(F, \mathbb{Z}/2), g^e)$.) $\hat{\gamma}(e, e \circ h) = \epsilon(h)$ is clear. \[ \square \]

We are now ready to prove Theorem 3.5. For two regularly homotopic embeddings $e, e' : F \to \mathbb{R}^3$ let $\hat{\Psi}(e, e') = \hat{\psi}(A^0(e), A^1(e); A^0(e'), A^1(e')) + (n + 1)\hat{\gamma}(e, e')$. We need to show $Q(e, e') = \hat{\Psi}(e, e')$. If $e'' : F \to \mathbb{R}^3$ is also in the same regular homotopy class then $Q(e, e'') = \hat{\Psi}(e, e'')$.
$Q(e,e') + Q(e',e'') = \tilde{\Psi}(e,e') + \tilde{\Psi}(e',e'')$. And so if $e'$ is equivalent to $e''$ and $Q(e,e'') = \tilde{\Psi}(e,e'')$ then also $Q(e,e') = \tilde{\Psi}(e,e')$. And so we may replace $e$ with an equivalent standard embedding $f$ (Proposition 4.8) and similarly replace $e'$ with an equivalent standard embedding $f'$. Now $f$ and $f'$ have isotopic images and so after isotopy we may assume $f(F) = f'(F)$ and so $f' = f \circ h$ for some diffeomorphism $h: F \to F$. By Lemma 4.9 and Theorem 3.6 the proof of Theorem 3.5 is complete.

We conclude with a remark on systems of surfaces. If $S = F_1 \cup \cdots \cup F_r$ is a system of closed orientable surfaces, and $e : S \to \mathbb{R}^3$ is an embedding, then we can rigidly move $e(F_i)$ one by one, until they are all contained in large disjoint balls. When it is the turn of $F_i$ to be rigidly moved, then the union of all other components is embedded and so only double lines occur. If $e' : S \to \mathbb{R}^3$ is another embedding then we can similarly move $e'(F_i)$ into the corresponding balls. It follows that $Q(e,e') = \sum_{i=1}^r Q(e|_{F_i}, e'|_{F_i})$ and so we obtain a formula for systems of surfaces, namely: $Q(e,e') = \sum_{i=1}^r \tilde{\Psi}(e|_{F_i}, e'|_{F_i})$.

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