CONSERVATION LAWS FOR AN EQUATION MODELING
ROOTS OF POLYNOMIALS UNDER DIFFERENTIATION

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Abstract. Let \( p_n \) be a polynomial of degree \( n \) having \( n \) distinct, real roots distributed according to a nice probability distribution \( u(0,x)dx \) on \( \mathbb{R} \). One natural problem is to understand the density \( u(t,x) \) of the roots of the \((t\cdot n)\)-th derivative of \( p_n \) where \( 0 < t < 1 \) as \( n \to \infty \). The author suggested that these densities might satisfy the partial differential equation of transport type

\[
\frac{\partial u}{\partial t} + \frac{1}{\pi} \frac{\partial}{\partial x} \arctan \left( \frac{H u}{u} \right) = 0 \quad \text{on } \{ x : u(x) > 0 \},
\]

where \( H \) is the Hilbert transform. Conditional on this being the case, we derive an infinite number of conversation laws of which the first three are

\[
\int_{\mathbb{R}} u(t,x) \, dx = 1 - t, \quad \int_{\mathbb{R}} u(t,x)x \, dx = (1 - t) \int_{\mathbb{R}} u(0,x)x \, dx,
\]

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} u(t,x)(x - y)^2u(t,y) \, dxdy = (1 - t)^3 \int_{\mathbb{R}} \int_{\mathbb{R}} u(0,x)(x - y)^2u(0,y) \, dxdy.
\]

In particular, we present two closed-form solutions for which all infinitely many conservation laws are valid. This raises a number of interesting questions.

1. Introduction

1.1. Introduction. Let \( p_n : \mathbb{R} \to \mathbb{R} \) be a polynomial of degree \( n \) having \( n \) distinct real roots. Rolle’s theorem implies that the derivative \( p_n' \) has exactly \( n - 1 \) real roots. Moreover, between any two roots of \( p_n \) there is exactly one root of \( p_n' \).

These objects are classical and much is known about them. Indeed, the study of the distribution of roots of \( p_n' \) depending on \( p_n \) is an active field [4, 5, 6, 8].

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Figure 1. The three Hermite polynomials \( H_1, H_2, H_3 \) (rescaled): between any two roots of \( H_k \) there is a root of \( H_{k-1} \).
A basic result is commonly attributed to Riesz \cite{11,30}: denoting the smallest gap of a polynomial $p_n$ having $n$ real roots \{x_1,\ldots,x_n\} by

$$G(p_n) = \min_{i \neq j} |x_i - x_j|,$$

we have $G(p'_n) \geq G(p_n)$: the minimum gap grows under differentiation. A simple proof is given by Farmer & Rhoades \cite{11}. Moreover, the Gauss-Lucas theorem \cite{12,19} states that the convex hull of the roots of $p'_n$ is contained in the convex hull of the roots of $p_n$: the roots of $p'_n$ are trapped by the roots of $p_n$. Put differently, the roots are localized in space and the minimal gaps between roots do not shrink under differentiation, they spread out. This motivates a simple question: do the roots of such a polynomial become ‘more regular’ under iterated differentiation? This could be made precise in many different ways and many of these different formulations would be interesting. Polya \cite{26} has asked similar problems for transcendental functions, we refer to Farmer & Rhoades \cite{11}. We note this informally as

**Conjecture A** (Polya \cite{26}, Farmer & Rhoades \cite{11}). The roots of a polynomial $p_n$ having $n$ distinct roots become more regular under iterated differentiation.

The formulation of Conjecture A is intentionally vague, to the best of our knowledge no precise formulation exists. One possible weak formulation is, for example, as follows (with scales chosen more or less at random): assuming some mild conditions on the roots of $p_n$ (say, their minimal distance is $\sim n^{-1}$), for any $1/6 < t < 1/3$, the $n/3$−th largest root of the $t \cdot n$−th derivative of $p_n$ and its $\sim \log n$ nearest neighbors form an arithmetic progression (up to a small error of size $o(n^{-1})$). To the best of our knowledge, there are no rigorous results in this direction. Farmer & Rhoades \cite{11} discuss the case of entire functions and obtain results in this setting. Having already mentioned the result of Riesz \cite{30}, we also mention a little known result of Sz-Nagy \cite{33} (see also Rahman & Schmeisser \cite{27}): Sz-Nagy \cite{33} proves that the average distance between consecutive pairs of roots increases under differentiation whereas the square mean and the variance decrease. One of these observations is equivalent to the third conservation law in the infinite family that we describe.

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**Figure 2.** Two closed-form solutions of the PDE for different times $0 < t < 1$: the semicircle solution and the Marchenko-Pastur solution. The solutions shrink and vanish at time $= 1$.

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1.2. A Partial Differential Equation. Recently, the author \cite{29} studied the following question: if the roots of $p_n(x)$ are distributed according to some nice smooth $C^\infty_c(\mathbb{R})$ function $u(0,x)$, what can be said about the distribution of the roots of the $(t \cdot n)$−th derivative of $p_n$, where $0 < t < 1$? If we denote their
distribution by \( u(t, x) \), how is \( u(t, x) \) connected to the original distribution \( u(0, x) \)? The author [29] suggested that if \( \{ x \in \mathbb{R} : u(0, x) > 0 \} \) is a finite interval, then this process may be governed by the nonlinear partial differential equation

\[
\frac{\partial u}{\partial t} + \frac{1}{\pi} \frac{\partial}{\partial x} \arctan \left( \frac{H u}{u} \right) = 0 \quad \text{on} \quad \{ x : u(x) > 0 \},
\]

where

\[
H f(x) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x - y} dy
\]

is the Hilbert transform.

The partial differential equation gives the correct prediction for Hermite polynomials, Laguerre polynomials and a class of orthogonal polynomials, however, a rigorous derivation is still outstanding. The derivation of the partial differential equation implicitly assumes a smoothing phenomenon at the level of the roots, one would assume that this is reflected in the behavior of the partial differential equation itself: that it has smoothing properties. This we state as

**Conjecture B (29).** The partial differential equation

\[
\frac{\partial u}{\partial t} + \frac{1}{\pi} \frac{\partial}{\partial x} \arctan \left( \frac{H u}{u} \right) = 0 \quad \text{on} \quad \{ x : u(x) > 0 \},
\]

has smoothing properties.

This is intentionally stated in a vague sense and could again be made precise in various ways. Granero-Belinchón [14] has studied a similar partial differential equation on \( \mathbb{S}^1 \). Conjecture B should be more accessible than Conjecture A. Conjecture B, or rather the underlying smoothing mechanisms, could also point towards the kind of quantitative results one would hope to obtain for Conjecture A.

### 1.3. The Complex Case.

The same question is meaningful for general polynomials \( p_n : \mathbb{C} \to \mathbb{C} \) having a distribution of roots being given by a smooth probability distribution \( u(0, z) : \mathbb{C} \to \mathbb{R}_{\geq 0} \) in the complex plane. In case the limiting measure \( u(0, z) \) is radial, the problem was studied by O’Rourke and the author [23] who (non-rigorously) derived a nonlocal transport equation

\[
\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial x} \left( \frac{1}{x} \int_0^x \psi(s) ds \right)^{-1} \psi(x)
\]

for the evolution of the radial profile. We emphasize that the conservation laws we derive in this paper should also apply to any evolution equation derived for the general complex case \( p_n : \mathbb{C} \to \mathbb{C} \) (since our derivation is algebraic in nature and does not distinguish between the real and complex numbers); one would expect them to be less interesting for radial data which is why we are not pursuing this question further at this point. We emphasize that a rigorous understanding of the complex case (both a rigorous derivation of a partial differential equations as well as a study of its properties) remains open and seems to be an interesting problem.

### 2. Main Results

#### 2.1. A Word of Warning.

The original derivation of the partial differential equation is only valid for data for which \( \{ x \in \mathbb{R} : u(0, x) > 0 \} \) is a single compact interval. More precisely, if \( \{ x \in \mathbb{R} : u(0, x) > 0 \} \) is, say, the union of two disjoint intervals, then the currently existing analysis would yield a prediction for the flow of roots within the two intervals but it is not at all clear what would happen in
the (initially empty) interval between the two intervals on which the distribution is supported. Roots of the iterative derivatives are going to move there but it is not currently clear how this happens. In particular, since our approach predicts conservation laws as well as Hilbert transform identities, these statements are a priori only to be expected for functions for which \( \{ x \in \mathbb{R} : u(0, x) > 0 \} \) is an interval. If the more general case was understood, it would imply conservation laws and Hilbert transform identities by the same mechanism that we describe here.

\[ \text{Figure 3. A case not covered by the current analysis.} \]

2.2. Infinitely Many Conservation Laws. This section presents the main contribution of the paper: if the equation does indeed describe the roots of polynomials under differentiation, then it has infinitely many conservation laws.

**Theorem 1 (Conservation Laws.).** Conditional on the equation

\[
\frac{\partial u}{\partial t} + \frac{1}{\pi} \frac{\partial}{\partial x} \arctan \left( \frac{Hu}{u} \right) = 0 \quad \text{on} \quad \{ x : u(x) > 0 \}
\]

describing the roots of polynomials under differentiation, there are infinitely many conservation laws for which we can give a closed form expression.

Any smooth solution has to vanish at time \( t = 1 \), so we understand ‘conservation law’ in a flexible sense, i.e. an algebraic identity involving the solution at time \( 0 < t < 1 \) and the solution at time 0. The first three laws, which are somewhat easier to write down than the subsequent ones, are

\[
\int_{\mathbb{R}} u(t, x) \, dx = 1 - t
\]

\[
\int_{\mathbb{R}} u(t, x) x \, dx = (1 - t) \int_{\mathbb{R}} u(0, x) x \, dx
\]

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} u(t, x)(x - y)^2 u(t, y) \, dxdy = (1 - t)^3 \int_{\mathbb{R}} \int_{\mathbb{R}} u(0, x)(x - y)^2 u(0, y) \, dxdy
\]

We will derive the conservation laws in §3. We also emphasize that Theorem 1 is not purely conditional: as derived in [29], there are two explicit closed-form solutions that indeed accurately describe the asymptotic behavior of roots: the asymptotic distributions of Hermite polynomials and associated Laguerre polynomials satisfy the partial differential equation. Theorem 1 therefore applies to these two closed-form solutions for which we have thus found an infinite number of conservation laws. Naturally, this raises the question of whether (1) the partial differential equation only correctly predicts the roots of polynomials under differentiation for some very particular classes of polynomials or (2) whether the partial differential equation might have other properties reminiscent of completely integrable systems as well.
2.3. Hilbert Transform Identities. The classical way to prove a conservation law is to differentiate in time, use the partial differential equation to replace the $\partial_t u$ term and simplify the arising terms (often by integration by parts). This is how we showed
\[
\int_{\mathbb{R}} u(t, x) \, dx = 1 - t
\]
in [29]. However, once we proceed to the next conservation law
\[
\int_{\mathbb{R}} u(t, x) x \, dx = (1 - t) \int_{\mathbb{R}} u(0, x) x \, dx
\]
this becomes slightly more nontrivial. Differentiating in time for $t = 0$, we obtain
\[
\int_{\{u > 0\}} u(0, x) x \, dx = -\frac{\partial}{\partial t} \bigg|_{t=0} \int_{\{u > 0\}} u(t, x) x \, dx
\]
\[
= \frac{1}{\pi} \int_{\{u > 0\}} x \frac{\partial}{\partial x} \arctan \left( \frac{H u}{u} \right) \, dx
\]
resulting in the identity, for probability distributions $u(0, x)$ for which the support $\{x \in \mathbb{R} : u(0, x) > 0\}$ is a finite interval,
\[
\frac{1}{\pi} \int_{\{u > 0\}} x \frac{\partial}{\partial x} \arctan \left( \frac{H u}{u} \right) \, dx = \int_{\{u > 0\}} u(0, x) x \, dx.
\]
We can remove the scaling condition and obtain a general identity for functions $u : \mathbb{R} \to \mathbb{R}$ for which $\{x : u(x) > 0\}$ is an interval
\[
\left( \int_{\{u > 0\}} x \frac{\partial}{\partial x} \arctan \left( \frac{H u}{u} \right) \, dx \right) \left( \int_{\mathbb{R}} u(x) \, dx \right) = \pi \int_{\{u > 0\}} x \cdot u(x) \, dx.
\]
We are not aware of any statement of such a flavor being stated anywhere. Similarly, plugging in the third conservation law suggests, after some minor computation, that, again for smooth, compactly supported, probability densities $u(x)$ for which $\{x \in \mathbb{R} : u(x) > 0\}$ is a finite interval,
\[
\int_{\{u > 0\}} x^2 \frac{\partial}{\partial x} \arctan \left( \frac{H u}{u} \right) \, dx = 2\pi \int_{\{u > 0\}} x^2 \cdot u(x) \, dx - \pi \left( \int_{\{u > 0\}} x \cdot u(x) \, dx \right)^2
\]
We remark that if any of these identities were to fail on an explicit function $u(x)$ (satisfying all the conditions), then this would imply that the partial differential equation does not accurately model the roots of polynomials distributed according to $u(x)$. Conversely, establishing these identities would prove that the conservation laws are valid along solutions moving in the class of functions for which the identities have been established.

2.4. A Monotone Quantity for the Size of the Support. This section quickly discusses a consequence of a little-known result of Sz-Nagy [33]. Conditional on our partial differential equation describing the evolution of the distribution accurately, it will turn into a monotone quantity. We first summarize the original argument of Sz-Nagy. Let $p_n : \mathbb{R} \to \mathbb{R}$ denote a polynomial of degree $n$ with $n$ roots on the real line $x_1 < x_2 < \cdots < x_n$. Sz-Nagy introduces the average distance between consecutive roots
\[
\text{av}(p_n) = \frac{x_n - x_1}{n - 1}.
\]
Theorem (Sz-Nagy [33]). If \( p_n : \mathbb{R} \to \mathbb{R} \) is a polynomial of degree at least 3 having \( n \) distinct real roots, then
\[
\text{av}(p_n') \geq \text{av}(p_n).
\]

The proof is a very simple consequence of a result proved by Sz-Nagy [32] three decades earlier in 1918: let us define the span of such a polynomial by
\[
\text{span}(p_n) = x_n - x_1 = (n - 1)\text{av}(p_n).
\]

The earlier result of Sz-Nagy [32] shows that the span cannot shrink too quickly under differentiation and
\[
\frac{\text{span}(p_n^{(k)})}{\text{span}(p_n)} \geq \sqrt{\frac{(n - k)(n - k - 1)}{n(n - 1)}}
\]
with equality if and only if (up to symmetries)
\[
p_n(x) = (x^2 - c) x^{n-2}
\]
for suitable \( c \).

This, in turn, follows rather easily from Gauss’ electrostatic interpretation (a proof can be found in the book of Rahman & Schmeisser [27]) and implies the original claim. From this we deduce the following monotone quantity.

**Theorem 2.** Conditional on the partial differential equation describing the evolution of roots of polynomials under differentiation, the quantity
\[
\frac{|\{x \in \mathbb{R} : u(t, x) > 0\}|}{\sqrt{1 - t}}
\]
is non-decreasing in time.

This statement implies that the support cannot shrink faster than linearly. We observe that this extremal example in the Theorem of Sz-Nagy is quite degenerate, its roots have a singular limiting distribution. This suggests that stronger monotone quantities might exist. We recall that \( u(t, x) = \sqrt{1 - t - x^2} \) is a particular solution of the partial differential equation (corresponding to Hermite polynomials) and
\[
\frac{|\{x \in \mathbb{R} : u(t, x) > 0\}|}{\sqrt{1 - t}} = 1.
\]

Interestingly, a similar relation holds for the other closed-form solution as well. We recall that the Marchenko-Pastur solution has the following form: for \( c > 0 \), we introduce the function
\[
v(c, x) = \frac{\sqrt{(x^2 - x)(x - x^-)}}{2\pi x} \chi(x^-, x^+) \quad \text{where} \quad x_{\pm} = (\sqrt{c}+1 \pm 1)^2.
\]
The corresponding Marchenko-Pastur solution of the equation is then given by
\[
u_c(t, x) = v\left(\frac{c + t}{1 - t}, \frac{x}{1 - t}\right).
\]

We observe that
\[
\{u_c(t, x) > 0\} = \left(\left(\frac{c + t}{\sqrt{1 - t} + 1 - 1}\right)^2, \left(\frac{c + t}{\sqrt{1 - t} + 1 + 1}\right)^2\right)
\]
and thus
\[
\frac{|\{u_c(t, x) > 0\}|}{\sqrt{1 - t}} = 4\sqrt{1 + c}
\]
which is constant along the flow. One would assume, also in light of the conservation laws discussed above, that the actual scaling should be on the order of \( \sqrt{1 - t} \). While
we do not know whether monotonicity at that scale is generally true, we can show that it is the right order of magnitude.

**Theorem 3.** Conditional on the partial differential equation describing the evolution of roots of polynomials under differentiation, for each sufficiently smooth initial distribution for which \( \{ x : u(t,x) > 0 \} \) is an interval, we have, for \( 0 \leq t < 1 \),

\[
\frac{|\{ u(t,x) > 0 \}|}{\sqrt{1-t}} \geq \left( \int_{\mathbb{R}} \int_{\mathbb{R}} u(0,x)(x-y)^2 u(0,y) \, dx \, dy \right)^{1/2} > 0.
\]

**Proof.** We use the first conservation law

\[
\int_{\mathbb{R}} u(t,x) \, dx = (1-t) \int_{\mathbb{R}} u(0,x) \, dx
\]
to rewrite the third one as

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{u(t,x)}{1-t} (x-y)^2 \frac{u(t,y)}{1-t} \, dx \, dy = (1-t)c(u(0,x)),
\]

where \( c(u(0,x)) \) is the value of the left-hand side for \( t = 0 \). We note that \( u(t,x)/(1-t) \) can be interpreted as a probability distribution. The integral is invariant under translations, so we can assume that the mean value of the probability distribution is 0. (The first conservation law would imply that this is preserved under the flow but this is not important here). Then the integral simplifies to

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{u(t,x)}{1-t} (x-y)^2 \frac{u(t,y)}{1-t} \, dx \, dy = 2 \int_{\mathbb{R}} \frac{u(t,x)}{1-t} x^2 \, dx.
\]

This quantity is twice the variance of the distribution given by \( u(t,x)/(1-t)dx \). An elementary inequality of Bhatia & Davis \[2\] implies that if \( u(t,x)/(1-t) \) is a probability distribution with mean 0 and \( \{ x : u(t,x) > 0 \} = (m,M) \) (where \( m < 0 < M \)), then

\[
2 \int_{\mathbb{R}} u(t,x)x^2dx \leq 2M(-m).
\]
The Cauchy-Schwarz inequality implies

\[
2 \int_{\mathbb{R}} u(t,x)x^2dx \leq 2M(-m) \leq m^2 + M^2.
\]

Altogether, we have

\[
(1-t)c(u(0,x)) \leq m^2 + M^2.
\]

Altogether

\[
|\{ u(t,x) > 0 \}| = |m| + M \geq \sqrt{m^2 + M^2} \geq \sqrt{1-t} \sqrt{c(u(0,x))}.
\]

\[ \square \]

### 3. Proof of Theorem 1

**Proof.** We explain how to derive these infinitely many conservation laws. We first explain the rough idea and then explicitly derive the first three conservation laws. After that, we explain the general principle in greater detail. The main idea, in short, is as follows. Let

\[
p_n(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = \prod_{k=1}^{n} (x - x_k).
\]
Multiplication shows, for every $1 \leq k \leq n$, the Vieta formula

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \left( \prod_{\ell=1}^{k} x_{i_\ell} \right) = (-1)^k a_{n-k}.$$ 

Let us now consider the $\ell$–th derivative of the polynomial

$$p^{(\ell)}_n(x) = \frac{n!}{(n-\ell)!} x^{n-\ell} + \cdots = \frac{n!}{(n-\ell)!} \prod_{i=1}^{n-\ell} (x - y_i).$$

Let us fix an integer $k$. The Vieta formula relates the size of all $k$–products of the roots of a polynomial of degree $n$ to the $a_{n-k}$–th coefficient. However, this coefficient $a_{n-k}$ is preserved under differentiation (except for multiplication with explicit factors). This allows us to obtain algebraic expression of the roots and the same expression for the roots of an arbitrary derivative. Algebraic manipulation allows to rewrite these algebraic terms into expressions that converge to integral expressions (if the roots of a polynomial indeed approximate a certain density in the limit). These, in turn, have to be laws that are also obeyed by the partial differential equation if the equation does indeed model roots under differentiation. Special cases are rather easily illustrated.

3.0.1. The case $k = 0$. The case ‘$k = 0$’ can be interpreted as saying that a polynomial of degree $n$ has $n$ roots. Under differentiation, we lose a root at each step. Taking an appropriate limit, we see that we would expect

$$\int_{\mathbb{R}} u(t, x) \, dx = 1 - t$$

which was already shown in [29].

3.0.2. The case $k = 1$. The case $k = 1$ turns into the statement

$$\sum_{i=1}^{n} x_i = -a_{n-1}.$$ 

Let us now consider the $\ell$–th derivative of the polynomial $p_n(x)$

$$p^{(\ell)}_n(x) = \frac{n!}{(n-\ell)!} x^{n-\ell} + a_{n-1} \frac{(n-1)!}{(n-\ell-1)!} x^{n-\ell-1} + \cdots = \frac{n!}{(n-\ell)!} \prod_{i=1}^{n-\ell} (x - y_i).$$

Normalizing the polynomial to be monic (which has no effect on the roots) and applying again Vieta’s formula shows that

$$\sum_{i=1}^{n-\ell} y_i = -a_{n-1} \frac{n-\ell}{n}.$$ 

Letting both $n$ and $\ell$ tend to infinity in such a way that $\ell/n \rightarrow t$, we obtain the identity

$$\int_{\mathbb{R}} u(t, x) x \, dx = (1 - t) \int_{\mathbb{R}} u(0, x) x \, dx.$$ 

In particular, if $u(0, x)$ has mean value 0, then this is preserved by the evolution.
The case $k = 2$. The case $k = 2$ presents one with greater algebraic flexibility since the previous laws can be combined in different ways; nonetheless, there is only one conservation law. The Vieta identity for $k = 2$ can be written as

$$\left( \sum_{i=1}^{n} x_i \right)^2 - \sum_{i=1}^{n} x_i^2 = 2a_{n-2}.$$ 

Using the conservation law from $k = 1$, we can assume w.l.o.g. that the mean value of the roots is 0 and this is then also preserved under the evolution. Differentiating and arguing as above, we obtain

$$\left( \sum_{i=1}^{n-\ell} y_i \right)^2 - \sum_{i=1}^{n-\ell} y_i^2 = \frac{(n-\ell)(n-\ell-1)}{n(n-1)} \left[ \left( \sum_{i=1}^{n} x_i \right)^2 - \sum_{i=1}^{n} x_i^2 \right].$$

Taking again appropriate limits and using the predicted conservation law for $k = 1$, we can simplify the statement to

$$\int_{\mathbb{R}} u(0, x) dx = 0 \implies \int_{\mathbb{R}} u(t, x)^2 dx = (1-t)^2 \int_{\mathbb{R}} u(0, x)^2 dx.$$

A different formulation is in terms of variance and can be found in the book of Rahman & Schmeisser [27, Lemma 6.1.5]. Using $y_i$ to denote the roots of $\ell$–th derivative, we have

$$\frac{1}{n^2(n-1)} \sum_{1 \leq i < j \leq n} (x_i - x_j)^2 = \frac{1}{(n-\ell)^2(n-\ell-1)} \sum_{1 \leq i < j \leq n-\ell} (y_i - y_j)^2.$$ 

Taking appropriate limits, this predicts the conservation law for $k = 2$ in the form

$$\int_{\mathbb{R}} \int_{\mathbb{R}} u(t, x)(x-y)^2 u(t, y) \, dx \, dy = (1-t)^3 \int_{\mathbb{R}} \int_{\mathbb{R}} u(0, x)(x-y)^2 u(0, y) \, dx \, dy$$

We emphasize that this second conservation law, coupled with the first two, already gives some insight into what the dynamic looks like. The density increases linearly according to $1-t$ (the $k = 0$ conservation law). This, coupled with the conservation law for $k = 1$, implies that the mean value is preserved. The conservation law for $k = 2$ coupled with the conservation law for $k = 0$ implies

$$\frac{\int_{\mathbb{R}} \int_{\mathbb{R}} u(t, x)(x-y)^2 u(t, y) \, dx \, dy}{\int_{\mathbb{R}} \int_{\mathbb{R}} u(t, x) u(t, y) \, dx \, dy} = \text{const}(u_0) \cdot (1-t),$$

where the constant $\text{const}(u_0)$ depends on the explicit solution. This means that the typical scale of mass distribution as $t \to 1$ is $(x-y)^2 \sim 1-t$ which shows shrinking at scale $\sqrt{1-t}$ (much like the semicircle solution).
3.0.4. The case $k = 3$. This is the last case that we derive ‘by hand’. Let us assume the distinct roots are given by $x_1, \ldots, x_n$. An algebraic expansion shows that

\[
\left( \sum_{i=1}^{n} x_i \right)^3 = \sum_{i=1}^{n} x_i^3 + 3 \sum_{i \neq j} x_i^2 x_j + \sum_{i \neq j \neq k} x_i x_j x_k
\]

\[
= -2 \sum_{i=1}^{n} x_i^3 + 3 \sum_{i,j} x_i^2 x_j + \sum_{i \neq j \neq k} x_i x_j x_k
\]

\[
= -2 \sum_{i=1}^{n} x_i^3 + 3 \left( \sum_{i=1}^{n} x_i \right) \left( \sum_{i=1}^{n} x_i^2 \right) + \sum_{i \neq j \neq k} x_i x_j x_k.
\]

We recall, from the Vieta identities, that

\[
\sum_{i=1}^{n} x_i = -a_{n-1} \quad \text{and} \quad \sum_{i=1}^{n} x_i^2 = a_{n-1}^2 - 2a_{n-2}.
\]

Moreover, as the new ingredient,

\[
\sum_{i \neq j \neq k} x_i x_j x_k = -6a_{n-3}.
\]

Collecting all these estimates results in

\[
\sum_{i=1}^{n} x_i^3 = -a_{n-1}^3 + 3a_{n-1}a_{n-2} - 3a_{n-3}.
\]

Rewriting everything in terms of power sums, we obtain

\[
3a_{n-3} = - \left( \sum_{i=1}^{n} x_i \right)^3 + 3 \left( - \sum_{i=1}^{n} x_i \right) \frac{1}{2} \left( \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i^2 \right) - \sum_{i=1}^{n} x_i^3.
\]

We know that $a_{n-3}$ behaves under iterated differentiation and normalization to a monic polynomial like $(1 - t)^3$. This then results in the same decay behavior for the functional

\[
J(f(x)) = \frac{3}{2} \left( \int_{\mathbb{R}} f(x) x dx \right) \left( \left( \int_{\mathbb{R}} f(x)x dx \right)^2 - \int_{\mathbb{R}} f(x)x^2 dx \right)
\]

\[
+ \left( \int_{\mathbb{R}} f(x)x dx \right)^3 + \left( \int_{\mathbb{R}} f(x)x^2 dx \right)
\]

which then satisfies

\[
J(u(t, x)) = (1 - t)^3 \cdot J(u(0, x)).
\]

We note that, as is frequently the case, the higher-order conservation laws do not seem to have as straightforward an interpretation as the first few.

3.0.5. The general case. We are now ready to discuss the general case. We use the notation $e_k(x_1, \ldots, x_n)$ to denote the $k$-th elementary symmetric polynomial on $n$
variables, i.e.

\[ e_0(x_1, \ldots, x_n) = 1 \]
\[ e_1(x_1, \ldots, x_n) = x_1 + \cdots + x_n \]
\[ e_2(x_1, \ldots, x_n) = \sum_{i<j} x_i x_j \]

and so on for \( k \leq n \). For \( k > n \), we set \( e_k(x_1, \ldots, x_n) = 0 \). We define the \( k \)-th power sum as

\[ p_k(x_1, \ldots, x_n) = x_1^k + x_2^k + \cdots + x_n^k. \]

The elementary symmetric polynomials arise naturally from

\[ \prod_{i=1}^{n} (x - x_i) = \sum_{k=0}^{n} (-1)^k e_k(x_1, \ldots, x_n) x^{n-k}. \]

If we differentiate such a polynomial \( \ell \)-times and obtain the roots \( y_1, \ldots, y_{n-\ell} \), then

\[ \frac{d}{dx} \sum_{k=0}^{n} (-1)^k e_k(x_1, \ldots, x_n) x^{n-k} = \sum_{k=0}^{n-\ell} (-1)^k e_k(x_1, \ldots, x_n) \frac{(n-k)!}{(n-k-\ell)!} x^{n-k-\ell} \]

which, normalized to be a monic polynomial, then has the form

\[ \prod_{i=1}^{n-\ell} (x - y_i) = \frac{(n-\ell)!}{n!} \sum_{k=0}^{n-\ell} (-1)^k e_k(x_1, \ldots, x_n) \frac{(n-k)!}{(n-k-\ell)!} x^{n-k-\ell}. \]

If \( k \in \mathbb{N} \) is fixed and \( n, \ell \) tend to infinity in such a way that \( \ell/n \to t \) for some \( 0 < t < 1 \), then

\[ \frac{(n-\ell)!}{n!} \frac{(n-k)!}{(n-k-\ell)!} \sim \frac{(n-\ell)^k}{n^k} \to (1-t)^k. \]

At the same time, by definition, we have

\[ \prod_{i=1}^{n-\ell} (x - y_i) = \sum_{k=0}^{n-\ell} (-1)^k e_k(y_1, \ldots, y_{n-\ell}) x^{n-\ell-k}. \]

This shows that, as \( n \) and \( \ell \) get large in such a way that \( \ell/n \to t \) converges,

\[ \frac{e_k(y_1, \ldots, y_{n-\ell})}{e_k(x_1, \ldots, x_n)} \to (1-t)^k. \]

Our final ingredient are the Newton’s identities: we can expand the elementary symmetric polynomials in terms of power sums via

\[ ke_k(x_1, \ldots, x_n) = \sum_{i=1}^{k} (-1)^{i-1} e_{k-i}(x_1, \ldots, x_n) p_i(x_1, \ldots, x_n). \]

We observe that this representation contains also other elementary symmetric polynomials but of lower degree; in particular, an iterative application of the formula
yields representation formulas of $e_k$ purely in terms of $p_i$ for $i \leq k$. For example, we have
\begin{align*}
e_1 &= p_1 \\
2e_2 &= p_1^2 - p_2 \\
3e_3 &= \frac{p_1^3}{2} - \frac{3}{2}p_1p_2 + p_3 \\
4e_4 &= \frac{p_1^4}{6} - p_1^2p_2 + \frac{4p_1p_3}{3} + \frac{p_2^2}{2} - p_4.
\end{align*}
However, the power sums have a clear limiting behavior since
\[p_k(x_1, \ldots, x_n) \to n \int_R u(0, x) x^k \, dx\]
as $n \to \infty$ whenever the roots are indeed distributed according to a nice distribution $u(0, x)$. The conservation law for $e_k$, expressible in terms of quantities that are meaningful, decays like $(1 - t)^k$ along solutions of the flow. \qed

4. More on Hilbert Transform Identities

4.1. Some further evidence. We study the conjectured identity
\[\left( \int_{\{u>0\}} x \frac{\partial}{\partial x} \arctan \left( \frac{Hu}{u} \right) \, dx \right) \left( \int_R u(x) \, dx \right) = \pi \int_{\{u>0\}} x \cdot u(x) \, dx\]
and linearize it around the explicit function $u(x) = (2/\pi)\sqrt{1 - x^2}$. More precisely, we assume the identities are valid and consider the function
\[u(x) = \frac{2}{\pi} \sqrt{1 - x^2} + \varepsilon f(x)\]
where $f \in C^\infty_c(-1, 1)$ and we consider the limit $\varepsilon \to 0$. We use that
\[H \frac{2}{\pi} \sqrt{1 - x^2} = \frac{2x}{\pi} \quad \text{for} \quad -1 < x < 1\]
and Taylor series expansion to compute that, in the limit $\varepsilon \to 0$, the arising expression is
\[\int_{-1}^1 x \frac{\partial}{\partial x} \left( \sqrt{1 - x^2} (Hf)(x) \right) \, dx = \int_{-1}^1 xf(x) \, dx.
\]
As it turns out, the arising linearized relation can be explicitly proven.

**Proposition.** Let $f \in C^\infty_c(-1, 1)$. Then
\[\int_{-1}^1 x \frac{\partial}{\partial x} \left( \sqrt{1 - x^2} (Hf)(x) \right) \, dx = \int_{-1}^1 xf(x) \, dx.
\]

**Proof.** We introduce the Chebyshev polynomials $T_k$
\[T_0(x) = 1, \quad T_1(x) = x \quad \text{and} \quad T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x),\]
as well as Chebyshev polynomials of the second kind $U_k$ given by
\[U_0(x) = 1, \quad U_1(x) = 2x \quad \text{and} \quad U_{k+1}(x) = 2xU_k(x) - U_{k-1}(x).
\]
These sequences of polynomials satisfy for $n, m \geq 1$,
\[\frac{2}{\pi} \int_{-1}^1 T_n(x)T_m(x) \frac{dx}{\sqrt{1 - x^2}} = \delta_{nm}.
\]
and
\[ \frac{2}{\pi} \int_{-1}^{1} U_n(x) U_m(x) \sqrt{1 - x^2} dx = \delta_{nm}. \]

A crucial identity is
\[ \frac{1}{\pi} \int_{-1}^{1} \frac{a_k T_k(y)}{(x - y) \sqrt{1 - y^2}} dy = a_k U_{k-1}(x). \]

We now introduce
\[ g(x) = f(x) \sqrt{1 - x^2} \]
and expand \( g \) into Chebychev polynomials via
\[ g(x) = \sum_{k=0}^{\infty} a_k T_k(x). \]

Smoothness of \( f \) implies decay of the coefficients allowing us to write
\[ Hf = H \frac{g}{\sqrt{1 - x^2}} = \sum_{k=0}^{\infty} \frac{1}{\pi} \int_{-1}^{1} \frac{a_k T_k(y)}{(x - y) \sqrt{1 - y^2}} dy = \sum_{k=1}^{\infty} a_k U_{k-1}(x). \]

Thus
\[ \frac{\partial}{\partial x} \sqrt{1 - x^2} (Hf)(x) = \frac{\partial}{\partial x} \sqrt{1 - x^2} \sum_{k=1}^{\infty} a_k U_{k-1}(x) \]
\[ = -\frac{x}{\sqrt{1 - x^2}} \sum_{k=1}^{\infty} a_k U_{k-1}(x) + \sqrt{1 - x^2} \sum_{k=1}^{\infty} a_k \frac{\partial}{\partial x} U_{k-1}(x). \]

We use the identity
\[ \frac{\partial}{\partial x} U_{k-1}(x) = \frac{k T_k(x) - x U_{k-1}(x)}{x^2 - 1} \]
to simplify
\[ \frac{\partial}{\partial x} \sqrt{1 - x^2} (Hf)(x) = \frac{1}{\sqrt{1 - x^2}} \sum_{k=1}^{\infty} a_k k T_k(x). \]

This allows us to compute the contribution of the first term of the first integral since
\[ \int_{-1}^{1} \frac{\partial}{\partial x} \sqrt{1 - x^2} (Hf)(x) dx = \int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2}} \left( \sum_{k=1}^{\infty} a_k k T_k(x) \right) \]
\[ = \int_{-1}^{1} T_1(x) \left( \sum_{k=1}^{\infty} a_k k T_k(x) \right) \frac{dx}{\sqrt{1 - x^2}} \]
\[ = \sum_{k=1}^{\infty} a_k \int_{-1}^{1} T_1(x) (k T_k(x)) \frac{dx}{\sqrt{1 - x^2}} \]
\[ = a_1 \pi \frac{\pi}{2}. \]
It remains to compute the integral on the other side. We obtain
\[
\int_{-1}^{1} xf(x) dx = \sum_{k=0}^{\infty} a_k \int_{-1}^{1} xT_k(x) \frac{dx}{\sqrt{1-x^2}} = \sum_{k=0}^{\infty} \frac{a_k}{\sqrt{1-x^2}} = \frac{a_1 \pi}{2}.
\]

4.2. Unbounded Support. An interesting question is how many of these arguments would also hold in the case of unbounded support. The derivation of the partial differential equation is, a priori, also meaningful for functions with unbounded support, however, it becomes more difficult to interpret what this could mean in terms of roots of polynomials. However, when the function \( u(0, x) \) has sufficient decay, one would certainly expect things to remain meaningful in a certain sense. We illustrate this with the density
\[
u(0, x) = \frac{e^{-(x-1)^2}}{\sqrt{\pi}}.
\]
The translation by 1 is to avoid trivial symmetries further below. The Hilbert transform of \( u(0, x) \) has an explicit representation in terms of the Dawson function
\[Hu(0, x) = \frac{2}{\pi} \text{DawsonF}(x-1) .\]
We can explicitly compute
\[
\frac{\partial}{\partial x} \arctan \left( \frac{Hu}{u} \right) = \frac{2}{\sqrt{\pi}} \frac{e^{(x-1)^2}}{1 + \text{erfi}^2(1-x)}
\]
which is sufficient to show that
\[
\left( \int_{\{u>0\}} x \frac{\partial}{\partial x} \arctan \left( \frac{Hu}{u} \right) dx \right) \left( \int u(x) dx \right) = \pi \int_{\{u>0\}} x \cdot u(x) dx.
\]
In particular, there exists a nonempty set of functions with unbounded support for which some of our arguments apply.

4.3. Another Interpretation. The purpose of this section is to connect the partial differential equation to complex analysis methods in signal processing and the notion of instantaneous frequency. Let us assume \( u : [-1, 1] \rightarrow \mathbb{R} \) is a smooth function that is positive in \((-1, 1)\) and vanishes at the boundary. One thing that would be nice to have is a local measure of how much the function oscillates locally at a point. For example, if \( u(x) = \sin(ax) \) for some \( a > 0 \), then this instantaneous frequency should be \( a \). There is no canonical way to meaningfully defined this notion because of the uncertainty principle (nicely explained in the book by Gröchenig [15]). Nonetheless, given the importance of the question in signal processing (which dates back to the 1920s), various methods have been proposed. One basic notion in that context is the analytic signal: the function
\[f(x) = u(x) + i(Hu)(x)\]
is a complex-valued function on $\mathbb{R}$ that admits a holomorphic extension to the upper-half plane. We can then write this expression in polar coordinates

$$f(x) = r(x)e^{i\phi(x)} \quad \text{and} \quad \frac{d}{dx}\phi(x)$$

is then a possible definition for the instantaneous frequency. We refer to the book of Cohen [7] for more details. However, this can also be expressed as

$$\frac{d}{dx}\phi(x) = \frac{d}{dx}\arctan\left(\frac{H u}{u}\right)$$

which is exactly the driving term in the partial differential equation. Put differently, the speed of transport is determined by the instantaneous frequency. We believe this to be more than just an algebraic coincidence and fundamental for any deeper understanding of the partial differential equation. It presumably also suggests a way of establishing the identities for the Hilbert transform though it is not clear to us at this point how one would establish all of them in a unified manner.

5. A Connection to Random Matrices?

5.1. Similarities. Various objects that appear in this paper also appear in the context of random matrices. In particular, both the semicircle distribution

$$\frac{2}{\pi}\sqrt{1 - x^2}$$

as well as the Marchenko-Pastur distribution

$$\frac{\sqrt{(x_+ - x)(x - x_-)}}{2\pi x} \chi(x_-..x_+) \quad \text{where} \quad x_\pm = (\sqrt{c} + 1 \pm 1)^2$$

arise naturally as limiting statistics of random matrices (Gaussian random matrices and Wishart random matrices, respectively). Both in random matrix theory and our approach, the Hilbert transform (or Cauchy transform, respectively) play a crucial role. Another common occurrence is the appearance of

the $k$-th moment $\int_{\mathbb{R}} x^k d\mu$,

which is both a crucial ingredient in random matrix theory but also the elementary quantities in which our conservation laws are phrased. The following fact was pointed out to us by Christopher Xue [37]: if $A \in \mathbb{R}^{n \times n}$ is a symmetric matrix and $p_A(x)$ is its characteristic polynomial, then

$$p_A'(x) = \sum_{i=1}^{n} p_{A_i}(x),$$

where $p_{A_i}$ is the characteristic polynomial of the $i$-th minor of $A$ (obtained from $A$ by deleting the $i$-th row and column). Moreover, by Cauchy’s interlacing theorem, the eigenvalues of each minor are interlacing. This strongly suggests a connection between our partial differential equation and the eigenvalue distribution of submatrices of random matrices.
5.2. A Conjecture. One specific interpretation could be the following: if we take a random minor $A_i$ (by picking $i \in \{1, \ldots, n\}$ uniformly at random), then a first guess would be that the expected eigenvalue of $A_i$ between two consecutive eigenvalues of $A$ should behave approximately like in the case of differentiation. In particular, if we pick $k$ successive minors at random where $k \sim tn$ is large compared to $n$, then there might be an emerging smoothing effect of the same type. The simple example of a diagonal matrix shows that any type of limiting statement cannot be true for fixed matrices. We arrive at the following question.

**Question.** Let $\lambda_1, \ldots, \lambda_n$ be $n$ distinct numbers drawn from a sufficiently smooth distribution $u(0, x)$. Fix a randomly chosen orthonormal basis $\{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ and consider the matrix

$$A = \sum_{i=1}^{n} \lambda_i v_i \otimes v_i.$$ 

Let $J \subset \{1, 2, \ldots, n\}$ be a subset of size $(1-t)\cdot n$ where $0 < t < 1$ and let $B$ be matrix induced by restricting $A$ onto rows and columns indexed by $J$. Are the eigenvalues of $B$ predicted by the PDE started at $u(0, x)$ at time $t$?

We carried out the following explicit experiment: we fixed $n = 200$ and created a random orthonormal basis $\{v_1, \ldots, v_{200}\}$. We then considered a specific matrix

$$A = \sum_{k=1}^{200} \frac{k}{200} v_i \otimes v_i$$

and then considered 1000 random principal submatrices of size 100. We sort their eigenvalues by size and then average the $\ell$–th eigenvalue for $1 \leq \ell \leq 100$.

![Figure 4](image)

**Figure 4.** The eigenvalue distribution arising from random principal submatrices (left), distribution of the roots $q^{(100)}(x)$ (right).

We compare this to the roots of the 100–th derivative of

$$q(x) = \prod_{k=1}^{200} \left(x - \frac{k}{200}\right).$$

The prediction would be that the eigenvalues following from a random submatrix follow exactly the distribution of roots of the derivative of $q$. Both distributions are shown in Figure 4 and seem to confirm this suspicion. The question can be interpreted in a variety of ways. Possibly the most accessible is to interpret it as a statement for the arising distribution *averaged* over all choices of orthonormal basis. However, we point out that the numerical experiments reported above were
carried out on a single explicit matrix generated by a single explicit orthonormal basis in $\mathbb{R}^n$. A stronger formulation of the problem would thus be the existence of a concentration phenomenon: is it true that among all bases in $\mathbb{R}^n$ all but an exponentially small (in $n$) set has the property that the eigenvalue distribution of a random principal matrix is approximately described by the solution of the partial differential equation? A simpler version, circumventing the connection between the PDE and the roots of a polynomial, would be to phrase it in terms of roots of the characteristic polynomial: is there a smoothing effect acting on iterated random choice of principal minors that has an effective limit given by differentiation of the characteristic polynomial?

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