GRAPHS WITH MULTIPLE SHEETED PLURIPOLAR HULLS

EVGENY POLETSKY AND JAN WIEGIERINCK

Abstract. In this paper we study the pluripolar hulls of analytic sets. In particular, we show that hulls of graphs of analytic functions can be multiple sheeted and sheets can be separated by a set of zero dimension.

1. Introduction

One of the oldest interesting topics in complex analysis is the problem of analytic extensions: find the maximal analytic object containing a given one. For example, if $f$ is an analytic function we are looking for its analytic continuation and if $A$ is an irreducible analytic set we try to find another one of the same dimension containing $A$.

The counterpart of analytic extension in pluripotential theory is the so-called pluripolar hull. There are two types of pluripolar hulls of a set $A$ in a domain $D \subset \mathbb{C}^n$. Let $\text{PSH}(D)$ be the set of all plurisubharmonic functions on $D$ and $\text{PSH}_0(D)$ the set of all negative functions from $\text{PSH}(D)$. Define

$$A^*_D = \{z \in D : \forall h \in \text{PSH}(D) \ h|_A = -\infty \Rightarrow h(z) = -\infty\}$$

and

$$A^*_D = \{z \in D : \forall h \in \text{PSH}_0(D) \ h|_A = -\infty \Rightarrow h(z) = -\infty\}.$$ 

For example, if $A$ is an analytic set in pseudoconvex domain $D$, then every point of $A$ has a neighborhood $V$ where $A \cap V = \{h_1 = \cdots = h_k = 0\}$ and the functions $h_k$ are holomorphic on this neighborhood. Hence $A \cap V = \{\log \max\{|h_1|, \ldots, |h_k|\} = -\infty\}$. In fact, $A^*_D = A$, because by [1, Cor. 1] there even exists $v \in \text{PSH}(D)$ such that $A = \{v = -\infty\}$.

If such a $v \in \text{PSH}(D)$ exists for $A$, we call $A$ pluricomplete in $D$. In general, an analytic extension of $A$ is contained in $A^*_D$.

In the case when $A = \Gamma_f$ is the graph of an analytic function $f$ it was boldly conjectured in [8] that the closure of the analytic extension.

2000 Mathematics Subject Classification. Primary: 32U15; secondary: 32D15.

Key words and phrases. Pluripotential theory, pluripolar hulls.

The first author was partially supported by an NSF Grant.
of \( A \) coincides with \( A_D^* \). However, A. Edigarian and the second author found in [2] an analytic function \( f \) on the unit disk that does not extend analytically while the pluripolar hull of its graph is a graph of a function defined on almost the whole plane, cf. [10]. This example raised the question: what are pluripolar hulls or, better to say, extensions of analytically non-extendible analytic sets?

The pluripolar hull of the graph \( \Gamma_f \) of a holomorphic function \( f(z) \) on a domain \( D \) may well be multi-sheeted over \( D \). The principal value of \( \sqrt{z} \) on \( \{ \Re z > 0 \} \) provides the easiest example. Only recently Zwonek [11], and, independently, Edlund & Jöricke [5] gave examples of holomorphic functions \( f \) on their domain of existence \( D \) with the property that the pluripolar hull \( (\Gamma_f)^* \subset C \) is multi-sheeted over (parts of) \( D \).

In these examples sheets can be separated only by a cut whose projection on \( z \)-plane has the dimension 1. As we show in Section 2 this is an intrinsic property of analytic extensions.

In the present note we show that there is a Cantor type set \( K \) and a holomorphic function \( f(z) \) on \( D = \mathbb{C} \setminus K \) such that \( (\Gamma_f)^* \) is 2-sheeted over \( D \). So for pluripolar extensions sheets can be separated by a 0-dimensional cut. As a by-product we obtain an example of a uniformly convergent sequence of holomorphic functions such that their pluripolar hulls do not converge to the pluripolar hull of the limit.

The set \( K \) should be sufficiently fat. Edigarian and the second author showed that if \( D = \mathbb{C} \setminus K \), with \( K \) a polar compact set in \( \mathbb{C} \), and if \( f \) is not extendible over \( K \), then \( (\Gamma_f)^* \cap D \times \mathbb{C} = \Gamma_f \); see [3], and [4] for the fact that also over \( K \) the hull is at most single sheeted.

2. Pluripolar Extensions

Let \( E \) be a closed set in a pseudoconvex domain \( D \subset \mathbb{C}^n \). If \( A \subset D \setminus E \) then, in general, \( A_{D \setminus E}^* \) is a proper subset of \( A_D^* \setminus E \). However, as the following statement shows, these sets coincide when \( E \) is pluripolar.

**Proposition 2.1.** If \( E \) is a closed pluripolar set in a pseudoconvex domain \( D \subset \mathbb{C}^n \) and \( A \subset D \setminus E \) then, \( A_D^* \setminus E = A_{D \setminus E}^* \).

**Proof.** Let \( \{D_j\} \) be an increasing sequence of relatively compact subdomains with \( \bigcup_j D_j = D \). By [9, Thm. 2.4] \( A_D^* = \bigcup_j (A \cap D_j)^* \). If \( u \in \text{PSH}_0(D_j \setminus E) \), then \( u \) extends as a negative plurisubharmonic function to \( D_j \) (see [6, Thm. 2.9.22]). Therefore, \( (A \cap D_j)^* \setminus E = (A \cap D_j)^* \setminus D_j \). Since \( (A \cap D_j)^* \setminus E \subset A_{D \setminus E}^* \), we see that \( A_D^* \setminus E \subset A_{D \setminus E}^* \) and, consequently, \( A_D^* \setminus E = A_{D \setminus E}^* \). \( \square \)
The proposition below describes the situation when $E$ is a closed set in a pseudoconvex domain $D \subset \mathbb{C}^n$, $A \subset D \setminus E$ is an analytic set and $A_D^*$ is also analytic.

**Proposition 2.2.** Suppose that $E$ is a closed set in a pseudoconvex domain $D \subset \mathbb{C}^n$ and $A \subset D \setminus E$ is an analytic set. If the set $A_D^*$ is analytic then every irreducible component of $A_D^*$ contains a component of $A$ of the same dimension.

**Proof.** Let $X$ be an irreducible component of $A_D^*$. We represent $A_D^*$ as $X \cup Y$, where $Y$ is another analytic set in $D$ and $\dim X \cap Y < \dim X$. As we indicated in Section 1 the set $Y$ is pluricomplete and $Y_D^* = Y$. It is easy to check that if sets $F, G \subset D$, then $(F \cup G)_D^* = F_D^* \cup G_D^*$. So if $B = A \cap (X \setminus Y)$ then $X \setminus Y \subset B_D^*$.

Suppose that $\dim X > \dim B$ and let $R$ be the set of regular points of $X$. The set of singular points of $X$ is analytic and, consequently, pluricomplete. By the argument above $X \setminus Y$ belongs to the pluripolar hull of the set $B' = B \cap R$.

We may assume that $0 \in R$ and let $T$ be the tangent plane to $R$ at 0. If $p$ is a projection of $\mathbb{C}^n$ on $T$, then the set $p(B')$ is pluripolar in $T$ and, consequently, there is a plurisubharmonic function $u$ on $T$ equal to $-\infty$ on $p(B')$. The set $p(R)$ has a non-empty interior in $T$ and, therefore, there is a point $z_0 \in R$ such that $u(p(z_0)) \neq -\infty$. Then the function $v = u \circ p$ is plurisubharmonic on $\mathbb{C}^n$, equal to $-\infty$ on $B'$ and $v(z_0) > -\infty$. Thus $R$ does not belong to the pluripolar hull of the set $B'$. This contradiction proves the proposition. \[\square\]

Suppose that $A$ is a pluricomplete analytic set of pure dimension $m$ in $D \setminus E$. If $A_D^*$ is an analytic set in $D$ and $A$ is a proper subset of $A_D^* \setminus E$, then the set $E \cap A_D^*$ cuts $A_D^*$ into several pieces and, therefore, its topological dimension must be at least $2m - 1$.

For example, let $D = \{ (z, w) \in \mathbb{C}^2 \}$ and $E = \{ 3z = 0, \Re z \geq 0 \}$. Take a branch $w = f(z)$ of the function $w = \sqrt{z}$ over $\mathbb{C} \setminus E$ and let $A = \{ (z, f(z)) : z \in \mathbb{C} \}$. The pluripolar hull of $A$ in $\mathbb{C}^2 \setminus E$ is $A$ because the function $\log|f(z) - z|$ is equal to $-\infty$ exactly on $A$. But $A_D^* = \{ (z, w) : z = w^2 \}$. In this example $A_D^*$ is an analytic set and the set $A_D^* \cap E$ is the real curve $\{ (x^2, x) : x \in \mathbb{R} \}$ which projects 2 to 1 except at 0 and its projection has dimension 1.

As the following statement shows this is the minimal possible dimension.

**Proposition 2.3.** Let $E$ be a closed set in a pseudoconvex domain $D \subset \mathbb{C}^n$ and let $A$ be an irreducible analytic set of dimension $m$ in $D \setminus E$ such that $A_D^*$ is also analytic. If $p$ is a projection of $\mathbb{C}^n$ onto
$\mathbb{C}^m \subset \mathbb{C}^n$ such that $p(A)$ has a non-empty interior in $\mathbb{C}^m$ and the topological dimension of $p(E)$ is less than $2m - 1$, then $A^*_D \setminus E = \emptyset$.

**Proof.** By Proposition 2.2 every irreducible component of $A^*_D$ contains a component of $A$ of the same dimension. Thus $A^*_D$ is also irreducible and has dimension $m$. We denote by $X$ the set of regular points of $A^*_D$ and let $X'$ be the subset of $X$ where the restriction of the projection $p$ to $X$ has maximal rank $m$. Since $p(A)$ has non-empty interior in $\mathbb{C}^m$, $x'$ is non-empty and relatively open in $X$. The set $X \setminus X'$ is analytic and, therefore, has dimension at most $m - 1$. Hence the set $X'$ is connected and the set $A \cap X'$ is not empty.

Suppose that the topological dimension of $p(E)$ is smaller than $2m - 1$. Choose points $z_0 \in A \cap X'$ and $z_1 \in X'$ such that $p(z_0)$ and $p(z_1)$ do not belong to $p(E)$. We can connect these points by a real analytic curve $\gamma$ in $X'$. The upper bound on the dimension of $p(E)$ implies that we can slightly shift $\gamma$ so that $p(\gamma)$ does not meet $p(E)$. Since $E$ is closed there is a relatively open neighborhood of $\gamma$ in $X'$ which does not contain points of $E$. But $A$ is a relatively open analytic subset of $X' \setminus E$. Hence $\gamma \subset A$ and $z_1 \in A$.

If $p(z_1) \in p(E)$ but $z_1 \not\in E$, then we can take a neighborhood of $z_1$ in $X'$ where points $z$ with $p(z) \in E$ form a set with empty interior. Since other points are in $A$, the whole neighborhood is there also. Thus $X' \setminus E \subset A$. Since $A$ is closed in $A^*_D \setminus E$, $A^*_D \setminus E = A$. \hfill $\square$

When $A$ is an analytic set in $D \setminus E$ we denote by $A_E$ the intersection of the closure of $A$ in $D$ with $E$. If $A^*_D \setminus E \neq A$ we will say that $A$ has a non-trivial pluripolar extension through $E$ in $D$.

The following theorem lists some limitations on the set $A_E$ when a non-trivial pluripolar extension takes place. Following [7] we call a set $G$ in a domain $Y \subset \mathbb{C}^p$ locally removable if $G$ is closed and for every open set $V$ in $Y$ every bounded holomorphic function $f$ on $V \setminus G$ extends holomorphically to $V$.

**Theorem 2.4.** Suppose that $D$ is a pseudoconvex domain in $\mathbb{C}^n$, $E$ is a closed set in $D$ and $A$ is an analytic set of pure dimension $m$ in $D \setminus E$ with a non-trivial pluripolar extension through $E$ in $D$. Then the $(2m - 1)$-Hausdorff measure of the set $A_E$ is not equal to zero and if, additionally, $p : \mathbb{C}^n \to \mathbb{C}^m$ is a projection such that the restriction $p|_{A_E}$ is proper and $A \cap p^{-1}(z)$ is empty for all $z \in p(A_E)$, then $p(A_E)$ is not locally removable in $\mathbb{C}^m$.

**Proof.** The set $A$ is analytic in $D \setminus A_E$. If the $(2m - 1)$-Hausdorff measure of $E$ is zero, then by Shiffman’s theorem (see [7], 4.4) the closure $\bar{A}$ of $A$ in $D$ is an analytic set in $D$. Since the domain $D$
is pseudoconvex there is a holomorphic function \( f \) on \( D \) such that \( \bar{A} = \{ f = 0 \} \). Thus \( A_D = \bar{A} \) and this contradicts the assumption that the extension is non-trivial.

In the second case if \( p(A_E) \) is locally removable in \( \mathbb{C}^m \), then by the proposition in \( [18.1] \) the closure \( \bar{A} \) of \( A \) is an analytic set in \( D \) as before and the same argument leads to a contradiction.

In our main example \( n = 2, m = 2 \) and \( m = 1 \). In this case Theorem 2.4 can be reformulated as follows:

**Corollary 2.5.** If in the assumptions of Theorem 2.4 \( n = 2 \) and \( m = 1 \), then the first Hausdorff measure of \( A_E \) is not zero and, under additional assumptions, the first Hausdorff measure of \( p(A_E) \) are not zeros.

### 3. A holomorphic function on the complement of a Cantor type set with 2-sheeted hull

**Definition 3.1.** A Cantor type set \( K \) will be a compact perfect subset of \( \mathbb{R} \) with empty interior.

It is a well known fact from elementary point set topology that such a \( K \) is homeomorphic with Cantor’s middle third set. It is of the form \([a_0, b_0] \setminus \bigcup_{j=1}^{\infty} I_j\) where \( I_j \) are open intervals in \([a_0, b_0]\), \( I_j \cap I_k = \emptyset \) if \( j \neq k \) and \( \bigcup_{j=1}^{\infty} I_j \) is dense in \([a, b]\). We can assume that the length of \( I_j = (a_j, b_j) \) decreases with \( j \).

It is useful to enumerate the set \( \{a_j, b_j, j = 0, \ldots, n\} \) as \( \{\alpha_{jn}, \beta_{jn}\} \) so that \( \alpha_{0n} = a_0, \alpha_{jn} < \beta_{jn} < \alpha_{j+1,n} \) and \( \beta_{nn} = b_0 \). Note that \([a_0, b_0] \setminus \bigcup_{j=0}^{n} I_j = \bigcup_{j=0}^{n}[\alpha_{jn}, \beta_{jn}] \) and that \( I_m \cap [\alpha_{jn}, \beta_{jn}] \neq \emptyset \) implies that \( I_m \subseteq [\alpha_{jn}, \beta_{jn}] \neq \emptyset \).

Let

\[
g_n(z) = \frac{z - a_0}{z - b_0} \frac{z - b_1}{z - a_1} \cdots \frac{z - b_n}{z - a_n}.
\]

Then

\[
g_n(z) = \frac{z - \alpha_{0n}}{z - \beta_{0n}} \frac{z - \alpha_{1n}}{z - \beta_{1n}} \cdots \frac{z - \alpha_{nn}}{z - \beta_{nn}}.
\]

Each fraction \( \frac{z - \alpha_{jn}}{z - \beta_{jn}}, j = 0, 1, \ldots, n, \) has a holomorphic branch \( \sqrt{\frac{z - \alpha_{jn}}{z - \beta_{jn}}} \) of its square root outside \([\alpha_{jn}, \beta_{jn}]\) that equals 1 at infinity. Let

\[
f_n(z) = \sqrt{\frac{z - \alpha_{0n}}{z - \beta_{0n}}} \sqrt{\frac{z - \alpha_{1n}}{z - \beta_{1n}}} \cdots \sqrt{\frac{z - \alpha_{nn}}{z - \beta_{nn}}} = \sqrt{g_n(z)}.
\]

Then \( f_n(\infty) = 1, f_n^2 = g_n \) and \( f_n \) is holomorphic on \( G_n = (\mathbb{C} \setminus [a_0, b_0]) \cup_{j=1}^{n} I_j \).
The maximal analytic extension of $f_n$ is a branched two sheeted cover $X_n = \{(z, w) : w^2 = g_n(z)\}$ of $\mathbb{C}$ that branches over $\{a_j, b_j : j = 0, 1, \ldots, n\}$. The pluripolar hull $(\Gamma_{f_n})^*$ equals $X_n$.

**Lemma 3.2.** Keeping the notation as above, the sequence $\{g_n\}$ converges normally to an analytic function $g$ on $\mathbb{C} \setminus K$. Moreover, the function $g$ extends analytically over a point $x \in K$ if and only if for some $\alpha < x < \beta$ the length of the set $K \cap (\alpha, \beta)$ is zero.

**Proof.** Let $L$ be a compact set in $\mathbb{C} \setminus K$. Let us show that

$$
(3.2) \quad \frac{z - a_0}{z - b_0} \prod_{j=1}^{\infty} \frac{z - b_j}{z - a_j} = \lim_{n \to \infty} g_n(z)
$$

is uniformly convergent on $L$. There exists $n_0$ such that $L \subset G_n$ for $n > n_0$ and moreover, for some $\delta > 0$

$$
L \subset \{z : |z - a_j| > \delta, j = 0, 1, \ldots\}.
$$

Hence, for $z \in L$,

$$
\left| \frac{z - b_j}{z - a_j} - 1 \right| = \left| \frac{b_j - a_j}{z - a_j} \right| \leq \frac{b_j - a_j}{\delta}.
$$

Since $\sum (b_j - a_j)$ is finite, the product in (3.2) converges uniformly on $L$ to a function $g$ that is holomorphic on $\mathbb{C} \setminus K$.

Suppose that the function $g$ extends analytically over a point $x \in K$ so that $g$ is analytic on $(\mathbb{C} \setminus K) \cup (\alpha, \beta)$. We may assume that $\alpha \in I_k$, $b_k \leq x$, and $\beta \in I_m$, $a_m \geq x$ and

$$
g_{1n}(z) = \prod \frac{z - \alpha_{jn}}{z - \beta_{jn}},
$$

where the product runs over all $j$ such that either $\beta_{jn} < \alpha$ or $\alpha_{jn} > \beta$. Let

$$
g_{2n}(z) = \prod \frac{z - \alpha_{jn}}{z - \beta_{jn}},
$$

where the product runs over all $j$ such that $\alpha < \alpha_{jn}$ and $\beta_{jn} < \beta$. Then $g_n = g_{1n}g_{2n}$ and by the argument above the sequences $\{g_{1n}\}$ and $\{g_{2n}\}$ converge uniformly on compacta on $\mathbb{C} \setminus (K \setminus (\alpha, \beta))$ and $\mathbb{C} \setminus (K \cap (\alpha, \beta))$ respectively. We denote their respective limits by $g_1$ and $g_2$.

The derivative $g_{2n}(\infty) = \sum (\beta_{jn} - \alpha_{jn}) = l_n$, where the sum runs over all $j$ such that $\alpha < \alpha_{jn}$ and $\beta_{jn} < \beta$. Thus $g_{2n}(\infty)$ is equal to the length $l_n$ of the intervals $(\alpha_{jn}, \beta_{jn})$ lying in $(\alpha, \beta)$ and $g_2(\infty)$ is the length of the set $K \cap (\alpha, \beta)$. If this length is positive, then the function $g_2$ is not constant and, therefore, does not extend to $K \cap (\alpha, \beta)$. 

If this length is 0 then for $z \in \mathbb{C}$ such that $|z - y| \geq 1$ for all $y \in (\alpha, \beta)$ we have

$$|1 - g_{2n}(z)| = \left| 1 - \prod \left( 1 + \frac{\beta jn - \alpha jn}{z - \beta jn} \right) \right| \leq e^{\lambda n} - 1.$$ 

Hence the sequence $\{g_{2n}\}$ converges to 1 near $\infty$, $g_2 \equiv 1$ and $g$ extends analytically over $(\alpha, \beta)$. □

Lemma 3.3. If $f = f_K$ and the length of $K$ is positive, then the union $\bar{\Gamma}_f \cup \bar{\Gamma}_{-f}$ of the closures of the graphs of $f$ and $-f$ is not an analytic set.

Proof. If $A = \bar{\Gamma}_f \cup \bar{\Gamma}_{-f}$ is an analytic set, then there is a holomorphic function $h = h(z, w)$ on $\mathbb{C}^2$ such that $h \equiv 0$ on $A$. We have $A = \Gamma_f \cup \Gamma_{-f} \cup E$, where $E \subset K \times \mathbb{C}$.

Let us show that for every $z_0 \in K$ the analytic set $E_{z_0} = \{w : (z_0, w) \in E\} = A \cap \{z_0\} \times \mathbb{C}$ consists of at most two points. If it contains three points, then at least two of them belong to, say, $\bar{\Gamma}_f$. Thus there are sequences $\{z_j\}$ and $\{z'_j\}$ converging to $z_0$ such that the sequences $\{f(z_j)\}$ and $\{f(z'_j)\}$ have distinct limits. Connecting each $z_j$ and $z'_j$ by small curves in $\mathbb{C} \setminus K$ and looking at their limits we see that the cluster set of $f$ at $z_0$ contains a continuum. Hence, $E_{z_0} = \mathbb{C}$ and $h(z_0, w) \equiv 0$.

From the Taylor expansion of $h$ we immediately derive that $h(z, w) = (z - z_0)^n h_1(z_0, w)$, where $h_1$ is holomorphic on $\mathbb{C}^2$ and $h_1(z_0, w) \neq 0$. But for every point $w \in \mathbb{C}$ there is a sequence of $z_j$ converging to $z_0$ such that, say, $f(z_j)$ converges to $w$. Since $h(z_j, f(z_j)) = 0$ we see that $h_1(z_0, w) = 0$. This contradiction shows that $E_{z_0}$ has at most two points and the intersection of $\bar{\Gamma}_f$ or $\bar{\Gamma}_{-f}$ with $E$ consists of at most one point.

It follows that $f$ extends continuously to $K$. Since $K$ lies on the real line $f$ extends holomorphically to $\mathbb{C}$ but this impossible by Lemma 3.2. □

Example 3.4. If the set $K$ has Lebesgue-measure 0, then

$$\lim_{n \to \infty} g_n(z) = 1,$$

uniformly on any compact set $L$ not meeting $K$. It follows that $f \equiv 1$ and $(\Gamma_f)^{\times 2} = \{(z, 1)\}$. But the Hausdorff limit of the sets $X_n$ over $D$ equals $\{(z, w) : w = 1 \text{ or } -1\}$.

We will need the next lemma whose proof is similar to the proof of Theorem 2.1 in [3].
Lemma 3.5. Let \( f \) be a holomorphic function on a domain \( V \subset \mathbb{C}^n \) containing a closed ball \( B \) and let \( \{r_n\}_n \) be a sequence of rational functions of degree \( n \) with poles outside \( V \) and such that the sup-norm \( \|f - r_n\|_B^{1/n} \to 0 \) as \( n \to \infty \). Then there is a plurisubharmonic function \( v \) on \( \mathbb{C}^{n+1} \) such that \( \{v = -\infty\} \cap (V \times \mathbb{C}) = \Gamma_f \). Thus, \( (\Gamma_f)^{\ast} \cap (V \times \mathbb{C}) = \Gamma_f \).

Proof. The functions \( r_n \) are ratios of polynomials \( p_n \) and \( q_n \) of degree \( n \). We may assume that \( B \) is the closed unit ball centered at the origin and \( \|q_n\|_B = 1 \). Then \( |q_n(z)| \leq \max\{1, |z|^n\} \), \( \|p_n\|_B \) does not exceed some constant \( C \) and \( |p_n(z)| \leq C \max\{1, |z|^n\} \).

Consider the plurisubharmonic functions

\[
u_n(z, w) = \frac{1}{n} \log |q_n(z)w - p_n(z)|\]

on \( \mathbb{C}^{n+1} \). From the estimates on \( p_n \) and \( q_n \) there is a constant \( C_1 \) such that \( \nu_n(z, w) \leq 2 \log |z| + \log |w| + C_1 \) when \( |z| \geq 1 \) and \( \nu_n(z, w) \leq \log |w| + C_1 \) when \( |z| \leq 1 \).

We take \( z_n \in B \) such that \( q_n(z_n) = a_n, \ |a_n| = 1 \) and let \( w_n = (p_n(z_n) + 1)/a_n \). Then \( |w_n| \leq C + 1 \) and \( \nu_n(z_n, w_n) = 0 \). If \( B_n \) is a ball in \( \mathbb{C}^{n+1} \) centered at \( (z_n, w_n) \) and of radius \( r_n = C + 5 \), then \( B_n \) contains the unit ball \( B' \) centered at the origin and

\[
\int_{B_n} \nu_n dV \geq c \nu_n(z_n, w_n) = 0.
\]

It is immediate from the upper estimates on \( \nu_n \) that there is a constant \( C_2 \) such that

\[
\int_{B'} \nu_n dV \geq C_2,
\]

By our assumption there is a sequence \( \{d_n\} \) converging to \( \infty \) such that

\[
u_n(z, f_n(z)) = \frac{1}{n} \log |q_n(z)| + \frac{1}{n} \log \left| f(z) - \frac{p_n(z)}{q_n(z)} \right| \leq -d_n
\]

when \( |z| \leq 1 \).

Let us take a sequence \( \{c_n\} \) of positive reals such that \( \sum c_n = 1 \) while \( \sum c_n d_n = \infty \). Let

\[
v(z, w) = \sum_{n=1}^{\infty} c_n \max\{u_n(z, w), d_n\}.
\]
Since
\[ \int_B v \, dV \geq C_2, \]
v \neq -\infty and, therefore, is a plurisubharmonic function on \( \mathbb{C}^{n+1} \).
Clearly, \( v(z, f(z)) = -\infty \) when \( |z| \leq 1 \). Therefore, \( v = -\infty \) on \( \Gamma_f \).

The zeros of the polynomials \( q_n \) are in \( \mathbb{C}^n \setminus V \). Hence the functions \( h_n(z) = \frac{1}{n} \log |q_n(z)| \) are harmonic on \( V \) and uniformly bounded above on compacta. So if \( z \in V \) and \( \liminf h_n(z) = -\infty \), then there is a subsequence \( \{n_k\} \) such that \( \lim h_{n_k}(z) = -\infty \). Therefore, functions \( h_{n_k} \) converge to \(-\infty\) uniformly on compacta in \( V \). But \( h_{n_k}(z_{n_k}) = 0 \) and this contradiction tells us that \( \liminf h_n(z) > -\infty \). So if \( w \neq f(z) \), then
\[
v(z, w) \geq \sum_{n=1}^{\infty} c_n h_n(z) + \sum_{n=1}^{\infty} \frac{c_n}{n} \log \left| w - \frac{p_n(z)}{q_n(z)} \right| > -\infty.
\]
Hence \( \{v = -\infty\} \cap (V \times \mathbb{C}) = \Gamma_f \) and \( (\Gamma_f)^{\ast}_{\mathbb{C}^{n+1}} \cap (V \times \mathbb{C}) = \Gamma_f \). \( \square \)

Now we can present our main example.

**Theorem 3.6.** There exists a Cantor type set \( K \) obtained by deleting intervals \( I_i = (a_i, b_i) \) from \([-1, 1]\), such that the function \( f = f_K \) given by Lemma 3.3 has the following property:
\[
(\Gamma_f)^{\ast}_{\mathbb{C}^2} \cap (\mathbb{C} \setminus K) \times \mathbb{C} = \Gamma_f \cup \Gamma_{-f}.
\]

**Proof.** We will construct \( K \) by deleting a sequence of open intervals \((a_i, b_i)\) from the interval \([-1, 1]\). For convenience, set \( a_0 = 1 \), \( b_0 = -1 \).
In order to choose the intervals appropriately, we have to construct certain subdomains \( D_n \) in the open unit disk \( \mathbb{D} \) in the process. The domains \( D_n \) will contain the set \( \mathbb{D} \cap \{|\Im z| > 1/2\} \). Thus the closed discs \( S = \{ |z + 3i/4| \leq 1/8 \} \) and \( X = \{ |z - 3i/4| \leq 1/8 \} \) will be contained in \( D_n \).

For a compact set \( F \) in a domain \( D \subset \mathbb{C} \) let
\[
\omega(z, F, D) = -\sup \{ h(z) : h \in \text{PSH}_0(D), \limsup_{w \to K} h(w) \leq -1 \}
\]
be the harmonic measure of \( F \) in \( D \). Set \( D_0 = \mathbb{D} \) and observe that \( \omega(z, S, D_0) > c_0 \) for some positive \( c_0 \).

Let \( \{c_n\} \) be a sequence of positive real numbers converging to \( \infty \). Suppose that the intervals \( I_1, \ldots, I_n \) have been chosen. We take as \( a_{n+1} \)
the midpoint of the largest interval in their complement. Next take \( b_{n+1} > a_{n+1} \) so small that the interval \([a_{n+1}, b_{n+1}]\) does not intersect the intervals \( I_1, \ldots, I_n \). \( d_{n+1} = b_{n+1} - a_{n+1} < 4^{-(n+1)c_{n+1}} \) and, moreover,
\[
\omega(z, S, D_{n+1}) > c_0, \quad z \in X.
\]
Here we define \( D_{n+1} = D_n \setminus \mathbb{D}(a_{n+1}, (b_{n+1} - a_{n+1})2^{n+1}) \), where \( \mathbb{D}(a, r) \) is the open disk centered at \( a \) and of radius \( r \).

Observe that for \( j \leq n \)
\[
\left| \frac{z - b_j}{z - a_j} - 1 \right| = \left| \frac{d_j}{z - a_j} \right| < 1/2^j
\]
on \( D_n \). It follows that
\[
\prod_{j=1}^{n} \frac{z - b_j}{z - a_j}
\]
is bounded independently of \( n \) on \( D_n \).

Let \( z_0 \in X \). We will show that \((z_0, -f(z_0)) \in (\Gamma_f)_{\mathbb{C}^2}^* \). Then \( \Gamma_{-f} \) is also in the hull and we are done. Consider the function \( g_n \) defined on \( \mathbb{D} \setminus (\bigcup_{j=1}^n I_j) \) by
\[
g_n(z) = \begin{cases} 
  f_n(z) & \text{if } \Im z < 0; \\
  -f_n(z) & \text{if } \Im z > 0; \\
  \lim_{y \to 0} f(x + iy) & \text{if } x \in [-1, 1] \setminus \bigcup I_j.
\end{cases}
\]

The function \( g_n \) is holomorphic. Let \( c_n = g_n(z_0) + f_K(z_0) \). Then \( c_n \to 0 \) as \( n \to \infty \). The functions \( h_n = g_n - c_n \) tend to \( f_K \) uniformly on compact sets in \( \mathbb{D} \cap \{ \Im z < 0 \} \) and \( h_n(z_0) = -f_K(z_0) \).

Now let \( u \) be a plurisubharmonic function on \( \mathbb{C}^2 \) that equals \(-\infty \) on \( \Gamma_f \). The function \( u(z, h_n(z)) \) is subharmonic on the domain \( D_n \) and because \( h_n(z) \) is bounded independently of \( n \) on \( D_n \), \( u(z, h_n(z)) \) is bounded by a constant \( M \) independently of \( n \).

Next we apply the two constant theorem and find
\[
(3.3) \quad u(z_0, -f_K(z_0)) \leq M(1 - \omega(z_0, S, D_n) + \max_{z \in S} u(z, h_n(z))\omega(z_0, S, D_n) \to -\infty, \quad \text{if } n \to \infty.
\]

Hence, \((\Gamma_f)_{\mathbb{C}^2}^* \supset \Gamma_f \cup \Gamma_{-f} \).

To get the equality we will show that the sup-norm
\[
\|g - g_n\|_{L^\infty}^{1/n} \to 0, \quad n \to \infty
\]
on compacta \( L \) outside \( K \). For this we write
\[
|g - g_n| = |g_n| \left| \prod_{k=n+1}^\infty \frac{z - a_k}{z - b_k} - 1 \right|.
\]
The first factor is bounded by a constant \( C \) depending on \( L \). To estimate the second factor we let \( \delta \) be the distance from \( L \) to \( K \) and write
the factor as
\[
\left| \prod_{k=n+1}^{\infty} \left( 1 + \frac{d_k}{z - b_k} \right) - 1 \right| \leq \prod_{k=n+1}^{\infty} \left( 1 + \frac{d_k}{\delta} \right) - 1 \leq \exp \left( \sum_{k=n+1}^{\infty} \frac{d_k}{\delta} \right) - 1.
\]

Since \(d_k < 4^{-kc_k}\) we see that
\[
\left| \prod_{k=n+1}^{\infty} \frac{z - a_k}{z - b_k} - 1 \right| \leq \exp \left( \frac{4^{-nc_n}}{(1 - 4^{-c_n})\delta} \right) - 1.
\]

Hence
\[
\| g - g_n \|_{L^1}^{1/n} \leq 2 \left( \frac{C_n}{\delta} \right)^{1/n} 4^{-c_n}
\]
when \(n\) is sufficiently large and
\[
\| g - g_n \|_{L^1}^{1/n} \to 0, \quad n \to \infty.
\]

By Lemma 3.5 the pluripolar hull \(\Gamma_g^* = \Gamma_g^* \cap (K \times \mathbb{C})\). Thus for any points \((z_0, w_0), z_0 \in \mathbb{C} \setminus K, w_0 \neq g(z_0)\), there is a function \(u \in \text{PSH}(\mathbb{C}^2)\) such that \(u|_\Gamma_g^* = -\infty\) and \(u(z_0, w_0) \neq -\infty\). Then the function \(v(z, w) = u(z, w^2)\) is equal to \(-\infty\) on \(\Gamma_f \cup \Gamma_{-f}\) and \(v(z_0, \pm \sqrt{w_0}) \neq -\infty\). Hence \((\Gamma_f)^{\times \mathbb{C}^2} \cap (\mathbb{C} \setminus K) \times \mathbb{C} = \Gamma_f \cup \Gamma_{-f}\).

\[\square\]

References

1. M. Colțoiu, Complete locally pluripolar sets, J. Reine Angew. Math., 412(1990), 108–112.
2. A. Edigarian & J. Wiegerinck, Graphs that are not complete pluripolar, Proc. Amer. Math. Soc. 131 (2003), 2459-2465.
3. A. Edigarian & J. Wiegerinck, The pluripolar hull of the graph of a holomorphic function with polar singularities, Indiana Univ. Math. J., 52 no 6 (2003) 1663-1680.
4. A. Edigarian & J. Wiegerinck, Determination of the pluripolar hull of graphs of certain holomorphic functions, Ann. Inst. Fourier, 54 (2004), 2085–2104.
5. T. Edlund & B. Jörickie The pluripolar hull of a graph and fine analytic continuation preprint, MathArXiv: math CV/0405025.
6. M. Klimek, Pluripotential Theory, London Math. Soc. Monographs, 6, Clarendon Press, 1991.
7. E. M. Chirka, Complex Analytic sets, Nauka, 1985
8. N. Levenberg, G. Martin & E.A. Poletsky, Analytic disks and pluripolar sets, Indiana Univ. Math. J., 41 (1992), 515–532.
9. N. Levenberg & E.A. Poletsky, Pluripolar hulls, Michigan Math. J., 46 (1999), 151–162.
10. J. Siciak, Pluripolar sets and pseudocontinuation, Complex analysis and Dynamical systems II (Nahariya 2003), AMS, Contemp. Math., (to appear).
11. W. Zwonek, A note on pluripolar hulls of Blaschke products Preprint
Department of Mathematics, 215 Carnegie Hall, Syracuse University, Syracuse, NY 13244
E-mail address: eapolets@syr.edu

Korteweg–de Vries Institute for Mathematics, University of Amsterdam, Plantage Muidergracht 24, 1018 TV, Amsterdam, The Netherlands
E-mail address: janwieg@science.uva.nl