On generalized Flett’s mean value theorem

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Abstract. We present a new proof of generalized Flett’s mean value theorem due to Pawlikowska (from 1999) using only the original Flett’s mean value theorem. Also, a Trahan-type condition is established in general case.

Key words and phrases. Flett’s mean value theorem, real function, differentiability, Taylor polynomial

1 Introduction

Mean value theorems play an essential role in analysis. The simplest form of the mean value theorem due to Rolle is well-known.

Theorem 1.1 (Rolle’s mean value theorem) If \( f : (a, b) \to \mathbb{R} \) is continuous on \( (a, b) \), differentiable on \( (a, b) \) and \( f(a) = f(b) \), then there exists a number \( \eta \in (a, b) \) such that \( f'(\eta) = 0 \).

A geometric interpretation of Theorem 1.1 states that if the curve \( y = f(x) \) has a tangent at each point in \( (a, b) \) and \( f(a) = f(b) \), then there exists a point \( \eta \in (a, b) \) such that the tangent at \( (\eta, f(\eta)) \) is parallel to the x-axis. One may ask a natural question: What if we remove the boundary condition \( f(a) = f(b) \)? The answer is well-known as the Lagrange’s mean value theorem. For the sake of brevity put

\[
\frac{b}{a}K(f^{(n)}, g^{(n)}) = \frac{f^{(n)}(b) - f^{(n)}(a)}{g^{(n)}(b) - g^{(n)}(a)}, \quad n \in \mathbb{N} \cup \{0\},
\]

for functions \( f, g \) defined on \( (a, b) \) (for which the expression has a sense). If \( g^{(n)}(b) - g^{(n)}(a) = b - a \), we simply write \( \frac{b}{a}K(f^{(n)}) \).

Theorem 1.2 (Lagrange’s mean value theorem) If \( f : (a, b) \to \mathbb{R} \) is continuous on \( (a, b) \) and differentiable on \( (a, b) \), then there exists a number \( \eta \in (a, b) \) such that \( f'(\eta) = \frac{b}{a}K(f) \).

Clearly, Theorem 1.2 reduces to Theorem 1.1 if \( f(a) = f(b) \). Geometrically, Theorem 1.2 states that given a line \( \ell \) joining two points on the graph of a differentiable function \( f \), namely \( (a, f(a)) \) and \( (b, f(b)) \), then there exists a point \( \eta \in (a, b) \) such that the tangent at \( (\eta, f(\eta)) \) is parallel to the given line \( \ell \).

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In connection with Theorem 1.1 the following question may arise: Are there changes if in Theorem 1.1 the hypothesis $f(a) = f(b)$ refers to higher-order derivatives? T. M. Flett, see [3], first proved in 1958 the following answer to this question for $n = 1$ which gives a variant of Lagrange’s mean value theorem with Rolle-type condition.

**Theorem 1.3 (Flett’s mean value theorem)** If $f : \langle a, b \rangle \rightarrow \mathbb{R}$ is a differentiable function on $\langle a, b \rangle$ and $f'(a) = f'(b)$, then there exists a number $\eta \in (a, b)$ such that

$$f'(\eta) = \frac{a}{2} K(f).$$  

Flett’s original proof, see [3], uses Theorem 1.1. A slightly different proof which uses Fermat’s theorem instead of Rolle’s can be found in [10]. There is a nice geometric interpretation of Theorem 1.3 if the curve $y = f(x)$ has a tangent at each point in $\langle a, b \rangle$ and if the tangents at $(a, f(a))$ and $(b, f(b))$ are parallel, then there exists a point $\eta \in (a, b)$ such that the tangent at $(\eta, f(\eta))$ passes through the point $(a, f(a))$, see Figure 1.

Similarly as in the case of Rolle’s theorem we may ask about possibility to remove the boundary assumption $f'(a) = f'(b)$ in Theorem 1.3. As far as we know the first result of that kind is given in book [11].

**Theorem 1.4 (Riedel-Sahoo)** If $f : \langle a, b \rangle \rightarrow \mathbb{R}$ is a differentiable function on $\langle a, b \rangle$, then there exists a number $\eta \in (a, b)$ such that

$$f'(\eta) = \frac{a}{2} K(f) + \frac{b}{a} K(f') \cdot \frac{\eta - a}{2}. $$

We point out that there are also other sufficient conditions guaranteeing the existence of a point $\eta \in (a, b)$ satisfying (1). First such a condition was published in Trahan’s work [13]. An interesting idea is presented in paper [12] where the discrete and integral arithmetic mean is used. We suppose that this idea may be further generalized for the case of means studied e.g. in [5, 6, 7].

In recent years there has been renewed interest in Flett’s mean value theorem. Among the many other extensions and generalizations of Theorem 1.3, see

![Figure 1: Geometric interpretation of Flett’s mean value theorem](image)
e.g. [1], [2], [3], [9], we focus on that of Iwona Pawlikowska [8] solving the question of Zsolt Pales raised at the 35-th International Symposium on Functional Equations held in Graz in 1997.

**Theorem 1.5 (Pawlikowska)** Let $f$ be $n$-times differentiable on $(a, b)$ and $f^{(n)}(a) = f^{(n)}(b)$. Then there exists $\eta \in (a, b)$ such that

$$f(\eta) - f(a) = \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i!} (\eta - a)^i f^{(i)}(\eta). \quad (2)$$

Observe that the Pawlikowska’s theorem has a close relationship with the $n$-th Taylor polynomial of $f$. Indeed, for

$$T_n(f, x_0)(x) = f(x_0) + \frac{f'(x_0)}{1!} (x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

the Pawlikowska’s theorem has the following very easy form $f(a) = T_n(f, \eta)(a)$.

Pawlikowska’s proof follows up the original idea of Flett, see [3], considering the auxiliary function

$$G_f(x) = \left\{ \begin{array}{ll}
g^{(n-1)}(x), & x \in (a, b) \\
\frac{1}{n} f(n)(a), & x = a \end{array} \right.$$ 

where $g(x) = \frac{x}{a} K(f)$ for $x \in (a, b)$ and using Theorem 1.1. In what follows we provide a different proof of Theorem 1.5 which uses only iterations of an appropriate auxiliary function and Theorem 1.3. In Section 3 we give a general version of Trahan condition, cf. [13] under which Pawlikowska’s theorem holds.

## 2 New proof of Pawlikowska’s theorem

The key tool in our proof consists in using the auxiliary function

$$\varphi_k(x) = x f^{(n-k+1)}(a) + \sum_{i=0}^{k} \frac{(-1)^{i+1}}{i!} (k-i)(x-a)^i f^{(n-k+i)}(x), \quad k = 1, 2, \ldots, n.$$ 

Running through all indices $k = 1, 2, \ldots, n$ we show that its derivative fulfills assumptions of Flett’s mean value theorem and it implies the validity of Flett’s mean value theorem for $l$-th derivative of $f$, where $l = n-1, n-2, \ldots, 1$.

Indeed, for $k = 1$ we have $\varphi_1(x) = -f^{(n-1)}(x) + x f^{(n)}(a)$ and $\varphi_1'(x) = -f^{(n)}(x) + f^{(n)}(a)$. Clearly, $\varphi_1'(a) = 0 = \varphi_1'(b)$, so applying the Flett’s mean value theorem for $\varphi_1$ on $(a, b)$ there exists $u_1 \in (a, b)$ such that $\varphi_1'(u_1)(u_1-a) = \varphi_1(u_1) - \varphi_1(a)$, i.e.

$$f^{(n-1)}(u_1) - f^{(n-1)}(a) = (u_1-a) f^{(n)}(u_1). \quad (3)$$

Then for $\varphi_2(x) = -2 f^{(n-2)}(x) + (x-a) f^{(n-1)}(x) + x f^{(n-1)}(a)$ we get

$$\varphi_2'(x) = -f^{(n-1)}(x) + (x-a) f^{(n)}(x) + f^{(n-1)}(a)$$

and $\varphi_2'(a) = 0 = \varphi_2'(u_1)$ by (3). So, by Flett’s mean value theorem for $\varphi_2$ on $(a, u_1)$ there exists $u_2 \in (a, u_1) \subset (a, b)$ such that $\varphi_2'(u_2)(u_2-a) = \varphi_2(u_2) - \varphi_2(a)$, which is equivalent to

$$f^{(n-2)}(u_2) - f^{(n-2)}(a) = (u_2-a) f^{(n-1)}(u_2) - \frac{1}{2} (u_2-a)^2 f^{(n)}(u_2).$$
Continuing this way after $n - 1$ steps, $n \geq 2$, there exists $u_{n-1} \in (a, b)$ such that
\[
f'(u_{n-1}) - f'(a) = \sum_{i=1}^{n-1} \frac{(-1)^{i+1}}{i!} (u_{n-1} - a)^i f^{(i)}(u_{n-1}). \]  
(4)

Considering the function $\varphi_n$ we get
\[
\varphi'_n(x) = -f'(x) + f'(a) + \sum_{i=1}^{n-1} \frac{(-1)^{i+1}}{i!} (x - a)^i f^{(i)}(x)
= f'(a) + \sum_{i=0}^{n-1} \frac{(-1)^{i+1}}{i!} (x - a)^i f^{(i+1)}(x).
\]

Clearly, $\varphi'_n(a) = 0 = \varphi'_n(u_{n-1})$ by (1). Then by Flett’s mean value theorem for $\varphi_n$ on $\langle a, u_{n-1} \rangle$ there exists $\eta \in (a, u_{n-1}) \subset (a, b)$ such that
\[
\varphi'_n(\eta)(\eta - a) = \varphi_n(\eta) - \varphi_n(a).
\]  
(5)

Since
\[
\varphi_n(\eta)(\eta - a) = (\eta - a)f'(a) + \sum_{i=1}^{n} \frac{(-1)^{i}}{(i-1)!} (\eta - a)^i f^{(i)}(\eta)
\]
and
\[
\varphi_n(\eta) - \varphi_n(a) = (\eta - a)f'(a) - n(f(\eta) - f(a)) + \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i!} (n - i)(\eta - a)^i f^{(i)}(\eta),
\]
the equality (5) yields
\[
-n(f(\eta) - f(a)) = \sum_{i=1}^{n} \frac{(-1)^{i}}{(i-1)!} (\eta - a)^i f^{(i)}(\eta) \left( 1 + \frac{n - i}{i} \right)
= n \sum_{i=1}^{n} \frac{(-1)^{i}}{i!} (\eta - a)^i f^{(i)}(\eta),
\]
which corresponds to (2).

It is also possible to state the result which no longer requires any endpoint conditions. If we consider the auxiliary function
\[
\psi_k(x) = \varphi_k(x) + \frac{(-1)^{k+1}}{(k+1)!} (x - a)^{k+1} \cdot b K \left( f^{(k)} \right),
\]
then the analogous way as in the proof of Theorem 1.5 yields the following result also given in [8] including Riedel-Sahoo’s Theorem 1.4 as a special case ($n = 1$).

**Theorem 2.1** If $f : (a, b) \to \mathbb{R}$ is $n$-times differentiable on $(a, b)$, then there exists $\eta \in (a, b)$ such that
\[
f(a) = T_n(f, \eta)(a) + \frac{(a - \eta)^{n+1}}{(n+1)!} \cdot b K \left( f^{(n)} \right).
\]
Note that the case \( n = 1 \) is used to extend Flett’s mean value theorem for holomorphic functions, see [1]. An easy generalization of Pawlikowska’s theorem involving two functions is the following one.

**Theorem 2.2** Let \( f, g \) be \( n \)-times differentiable on \((a, b)\) and \( g^{(n)}(a) \neq g^{(n)}(b) \). Then there exists \( \eta \in (a, b) \) such that

\[
f(a) - T_n(f, \eta)(a) = \frac{b}{a}K\left(f^{(n)}, g^{(n)}\right) \cdot [g(a) - T_n(g, \eta)(a)].
\]

This may be verified applying the Pawlikowska’s theorem to the auxiliary function

\[
h(x) = f(x) - \frac{b}{a}K\left(f^{(n)}, g^{(n)}\right) \cdot g(x), \quad x \in (a, b).
\]

A different proof will be presented in the following section.

### 3 A Trahan-type condition

In [13] Trahan gave a sufficient condition for the existence of a point \( \eta \in (a, b) \) satisfying the assumptions of differentiability of \( f \) on \((a, b)\) and inequality

\[
(f'(b) - \frac{b}{a}K(f)) \cdot (f'(a) - \frac{b}{a}K(f)) \geq 0.
\]

Modifying the Trahan’s original proof using the Pawlikowska’s auxiliary function \( G_f \) we are able to state the following condition for validity [2].

**Theorem 3.1** Let \( f \) be \( n \)-times differentiable on \((a, b)\) and

\[
\left(\frac{f^{(n)}(a)(a - b)^n}{n!} + M_f\right) \left(\frac{f^{(n)}(b)(a - b)^n}{n!} + M_f\right) \geq 0,
\]

where \( M_f = T_{n-1}(f, b)(a) - f(a) \). Then there exists \( \eta \in (a, b) \) satisfying [3].

**Proof.** Since \( G_f \) is continuous on \((a, b)\) and differentiable on \((a, b)\) with

\[
G'_f(x) = g^{(n)}(x) = \frac{(-1)^n n!}{(x - a)^{n+1}} \left(f(x) - f(a) + \sum_{i=1}^{n} \frac{(-1)^i}{i!} (x - a)^i f^{(i)}(x)\right),
\]

for \( x \in (a, b) \), see [5], then

\[
(G_f(b) - G_f(a))G'_f(b) = \left(g^{(n-1)}(b) - \frac{1}{n} f^{(n)}(a)\right) g^{(n)}(b)
\]

\[
= - \frac{n! (n-1)!}{(b - a)^{2n+1}} \left(\frac{f^{(n)}(a)(a - b)^n}{n!} + T_{n-1}(f, b)(a) - f(a)\right)
\]

\[
\cdot \left(\frac{f^{(n)}(b)(a - b)^n}{n!} + T_{n-1}(f, b)(a) - f(a)\right) \leq 0.
\]

According to Lemma 1 in [13] there exists \( \eta \in (a, b) \) such that \( G'_f(\eta) = 0 \) which corresponds to [2].

Now we provide an alternative proof of Theorem 2.2 which does not use original Pawlikowska’s theorem.
Proof of Theorem 2.2. For $x \in (a, b)$ put $\varphi(x) = \frac{b}{a} K(f)$ and $\psi(x) = \frac{b}{a} K(g)$.

Define the auxiliary function $F$ as follows

$$F(x) = \begin{cases} 
\varphi^{(n-1)}(x) - bK\left(f^{(n)}, g^{(n)}\right) \cdot \psi^{(n-1)}(x), & x \in (a, b) \\
\frac{1}{n!} \left[f^{(n)}(a) - bK\left(f^{(n)}, g^{(n)}\right) \cdot g^{(n)}(a)\right], & x = a.
\end{cases}$$

Clearly, $F$ is continuous on $(a, b)$, differentiable on $(a, b)$, and for $x \in (a, b)$ there holds

$$F'(x) = \varphi^{(n)}(x) - bK\left(f^{(n)}, g^{(n)}\right) \cdot \psi^{(n)}(x)$$

$$= \frac{(-1)^n n!}{(x-a)^{n+1}} \left[T_n(f, x)(a) - f(a) - bK\left(f^{(n)}, g^{(n)}\right) \cdot \left(T_n(g, x)(a) - g(a)\right)\right].$$

Then

$$F'(b)[F(b) - F(a)] = -\frac{n!(n-1)!}{(b-a)^{2n+1}} \left(F(b) - F(a)\right)^2 \leq 0,$$

and by Lemma 1 in [13] there exists $\eta \in (a, b)$ such that $F'(\eta) = 0$, i.e.,

$$f(a) - T_n(f, \eta)(a) = bK\left(f^{(n)}, g^{(n)}\right) \cdot \left(g(a) - T_n(g, \eta)(a)\right).$$

$\square$

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