A simple formula for the thermal pair annihilation line emissivity

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Abstract. We introduce a simple and convenient fitting formula for the thermal annihilation line from pair plasmas in cosmic sources. The fitting formula is accurate to 0.04% and is valid at all photon energies and temperatures of interest. The commonly used Gaussian line profile is not a good approximation for broader lines.

Key words: gamma rays: theory – line: profiles – radiative transfer mechanisms: thermal

1. Introduction

For two reasons a need has developed to be able to compute the annihilation line emissivity in a thermal plasma both exactly and rapidly. First, relatively broad and resolved annihilation lines have been observed in the galactic black hole candidates Nova Muscae (Gilfanov et al. 1993), and 1E 1740.7-2942 (Bouchet et al. 1991) as well as from active galactic nuclei, galactic black hole candidates, and some exceptions (e.g. Macioci/sequi-Niedzwiecki & Zdziarski 1994) a Gaussian line profile is normally used when fitting the observations. A Gaussian profile does not have the correct shape when the temperature becomes sufficiently large. Future gamma ray missions such as INTEGRAL will be able to determine the annihilation line shape to such accuracy that the exact annihilation line shape must be used when fitting the line. Second, the theoretical modelling of radiative transfer in the hot thermal plasmas thought to be responsible for the X-ray and γ-ray emission from e.g. active galactic nuclei, galactic black hole candidates, and gamma-ray bursts requires a fast way to evaluate the thermal annihilation line emissivity.

The thermal annihilation line emissivity is exactly determined by a single, one-parameter integral first derived by Svensson (1983, see also Dermer 1984). Svensson (1983) also gave some less convenient but very accurate polynomial fits to that integral. Our aim here is to provide a single, simple, but accurate fitting formula valid at all temperatures and photon energies of interest. This fitting formula can then be used when modelling observations. It also allows rapid evaluations of the annihilation line emissivity. Svensson (1983) determined by a single, one-parameter integral first derived by Svensson (1983, see also Dermer 1984). Svensson (1983) also gave some less convenient but very accurate polynomial fits to that integral. Our aim here is to provide a simple formula for the thermal pair annihilation line emissivity in radiative transfer calculations.

The structure of the paper is as follows. In § 2 we summarize previous theoretical results regarding the thermal annihilation line emissivity and in § 3 we introduce a very accurate fitting formula for the line shape. Finally, in § 4 we summarize our results.

2. Theory of the thermal pair annihilation line

Consider a plasma consisting of electrons and positrons of densities $n_-$ and $n_+$, respectively. The electrons and positrons are assumed to have Maxwell-Boltzmann energy distributions at the same temperature, $T$. Svensson (1983) showed using simple detailed balance arguments that the emission rate, $(dn/dt)(x)dx$ cm$^{-3}$ s$^{-1}$, of annihilation photons of dimensionless energy, $x = h\nu/m_e c^2$, in the energy interval $dx$ in such a plasma is given by an integral over the cross section for the reverse process, photon-photon pair production:

$$dn/dt(x, \theta)dx = n_+n_-cdx \times$$

$$\frac{2}{\theta [K_2(1/\theta)]^2} e^{-x/\theta} \int_{1}^{\infty} dss\sigma_{\gamma\gamma}(s) e^{-s/x\theta}. \quad (1)$$

Here, $K_2(1/\theta)$ is the modified Bessel function of second kind of order 2, the dimensionless temperature $\theta$ is defined as $\theta \equiv kT/m_e c^2$, and $\sigma_{\gamma\gamma}(s)$ is the cross section for photon-photon pair production, $\gamma\gamma \rightarrow e^+e^-$ (Jauch & Rohrlich 1976),

$$\sigma_{\gamma\gamma}(s) = \sigma_T \frac{3}{8s} \left[ \left( 2 + \frac{2}{s} - \frac{1}{s^2} \right) \ln \left( \sqrt{s} + \sqrt{s-1} \right) - \left( 1 + \frac{1}{s} \right) \left( 1 - \frac{1}{s} \right)^{1/2} \right], \quad (2)$$

where $s \equiv x_{cm}^2$ with $x_{cm}$ being the photon energy in the center-of-momentum frame. Near pair production thresh-
old, \( s - 1 \ll 1 \), we have \( \sigma_\gamma(s) = \sigma_T(3/8)(s - 1)^{1/2} \),
while far above the threshold, \( s \gg 1 \), we have \( \sigma_\gamma(s) = \sigma_T(3/8s)(\ln 4s - 1) \).

Dermer (1984) developed a general theory for production spectra from binary collisions in thermal plasmas by
reducing a six-dimensional integral over the differential
cross section to a double integral. Dermer applied this result to the pair annihilation process, in which case the
double integral reduces to a single integral over the pair
annihilation cross section:

\[
\frac{dn}{dt}(x, \theta)dx = n_+ n_- c dx \times \frac{2}{\theta [K_2(1/\theta)]^2} \times e^{-x/\theta} \int_1^\infty ds 2(s-1)\sigma_{ann}(s)e^{-s/x\theta}.
\]

(3)

Here, \( \sigma_{ann}(s) \) is the pair annihilation cross section, and
\( s \equiv \gamma_{cm}^2 \), with \( \gamma_{cm} \) being the particle Lorentz factors in the
center-of-momentum frame. We have rewritten Dermer’s integral in terms of \( s \) instead of his chosen integration
variable, \( \gamma_\parallel = 2s + 1 \), i.e. the Lorentz factor of one of the
particles in the rest frame of the other. The cross sections
for the annihilation process, \( \sigma_{ann} \), and its inverse, \( \sigma_\gamma \), are
related to each other due to microscopic detailed balance:

\[
\sigma_{ann}(s) = \frac{s}{2(s-1)} \sigma_\gamma(s); \quad \gamma_{cm} = \gamma_{cm}. \tag{4}
\]

Using this relation we find that the two expressions (1) and (3) for the annihilation emissivity, obtained using two
different methods, are identical.

We now need too evaluate the Bessel function factor
as a function of \( \theta \) and the integral as a function of the
parameter \( x/\theta \). To an accuracy of 0.06\% the Bessel function
factor can be approximated with (Svensson 1983)

\[
[K_2(1/\theta)]^2 = 4\theta^2 e^{-3/\theta} \times [1 + 2.0049\theta^{-1} + 1.4774\theta^{-2} + \pi(2\theta)^{-3}], \tag{5}
\]

which has the correct analytic behavior for \( \theta \ll 1 \) and \( \theta \gg 1 \). Using the two asymptotic forms for the cross section
at \( s \ll 1 \) and \( s \gg 1 \) allows the integral to be solved analytically in the two limits \( x/\theta \ll 1 \) and \( x/\theta \gg 1 \) giving

\[
\int_1^\infty ds s\sigma_\gamma(s) e^{-s/x\theta} = \begin{cases} \sigma_T \frac{3}{8} \pi^{3/2}(x\theta)^{3/2} e^{-1/x\theta}; & x\theta \ll 1, \\ \sigma_T \frac{3}{8} \pi \theta (\ln 4\pi x\theta - 1); & x\theta \gg 1, \end{cases} \tag{6}
\]

where \( \gamma_\parallel = \exp(-\gamma_{cm}) = 0.56146... \), \( \gamma_{cm} \) being Euler’s constant.
Using the asymptotic analytic forms of both the
Bessel function factor and the integral gives the two following useful asymptotic expressions for the annihilation
emissivity

\[
\frac{dn}{dt}(x, \theta)dx = n_+ n_- c \sigma_T dx \times \begin{cases} \frac{3}{4\pi \theta^{3/2}} x^{3/2} e^{-(x-1)^2/x\theta}; & x \ll \frac{1}{\theta}, \; \theta \ll 1, \\ \frac{3}{16\pi} (\ln 4\pi x\theta - 1) xe^{-x/\theta}; & x \gg \frac{1}{\theta}, \; \theta \gg 1. \end{cases} \tag{7}
\]

At relativistic temperatures, \( \theta \gg 1 \), the kinetic energy of
the annihilating pair dominates over the rest mass energy,
and the annihilation spectrum mimics a Maxwellian with
a peak at \( x \approx \theta \). At nonrelativistic temperatures, \( \theta \ll 1 \), the
rest mass energy dominates and the annihilation line
peaks at \( x \approx 1 + 3\theta/4 \) with the slight blue shift and broadening
being due to the kinetic energy of the annihilating
particles (Ramaty & Mészáros 1981). The nonrelativistic
expression is valid throughout the whole low energy wing
of the line, and far into the high energy wing up to a photon
energy of \( 1/\theta \). If one only considers the line core near
\( x \sim 1 \), or, more precisely \( x - 1 \ll 1 \), one gets

\[
\left. \left( \frac{dn}{dt} \right) \right|_G (x, \theta) dx = n_+ n_- c \sigma_T dx \times \frac{3}{4\pi \theta^{3/2}} x^{3/2} e^{-(x-1)^2/x\theta};
\]

\[
x - 1 \ll 1, \quad \theta \ll 1, \tag{8}
\]

which shows that the line core has a Gaussian shape for
\( \theta \ll 1 \). It is this nonrelativistic line core approximation
that is normally used when fitting observed annihilation
features. Then, however, an exponent of \( (x - x_c)^2/\theta \) is
used with two fitting parameters, the line center energy,
\( x_c \), and the temperature, \( \theta \).

It is appropriate at this point to briefly mention other
work on the annihilation line profile. Aharonian, Atoyan,
Sunyaev (1983) also derived an single integral expression
for the thermal annihilation line emissivity, but this
differs from the self consistent and correct expressions (1)
and (3) obtained by Svensson (1983) and Dermer (1984).
While analytical (or numerical) integration of the correct
expression over photon energy gives the well known
analytical expression for the annihilation rate already
obtained by Weaver (1976), a similar integration over Aharonian et al.’s expression does not. The reason for the
discrepancy is the second term \( I_2 \) in the integrand of their
equation (A.4) originating from the term \( \propto \cos^2 \chi' \) in their
equation (33). Using \( I_1 \) only, givess the correct analytical
result.

The multi-dimensional integral for the emissivity that
Dermer started out with in his derivation has been evaluated
numerically using Monte Carlo methods by Zdziarski
(1980), Ramaty & Mészáros (1981), and Yahel (1982). The
results of the first two papers agree with the correct analytical
results, while the calculations by Yahel give quite a
different line shape. The reason for the discrepancy is due
to Yahel confusing the angle between the annihilation photon
and the particle momentum with the angle between
the direction of transformation to the lab frame and the
particle momentum, all in the center-of-momentum frame.
The existence of simple analytical expressions for the thermal annihilation emissivity has now eliminated the need for the time-consuming Monte Carlo calculations appearing in these papers.

### 3. An excellent fitting formula

We now attempt to provide fitting formulas for the integral in equation (1) valid for all values of $x\theta$. As the annihilation line is swamped by bremsstrahlung emission for $\theta$ larger than about $3$ (Svensson 1982, Maciolek-Niedźwiecki et al. 1995), a useful formula for fitting observable lines would need to be valid for, say, $x\theta < 20$. We therefore write the integral as

$$
\int_1^\infty dss\sigma_{\gamma\gamma}(s)e^{-s/x\theta} = \sigma_T \frac{3}{16}\pi^{1/2} (x\theta)^{3/2} e^{-1/x\theta} C(x\theta); \quad x\theta < 20, \quad (9)
$$

where $C(y)$ is the correction factor to the asymptotic expression (6) for $x\theta \ll 1$. The annihilation emissivity, equation (1) then becomes

$$
dn\frac{dt}{dt}(x, \theta)dx = n_+n_-c\sigma_Tdx \frac{3}{8}(\pi\theta)^{1/2}x^{3/2} \times
$$

$$
\exp \left( -\frac{x + x^{-1}}{\theta} \right) \frac{C(x\theta)}{[K_2(1/\theta)]^2}; \quad \text{any } \theta, \ x\theta < 20. \quad (10)
$$

The Bessel function factor is given by the approximative equation (5) while $C(y)$ was computed to an accuracy of $10^{-5}$ using integration routines from Numerical Recipes (Press et al. 1986). The resulting $C(y)$ is shown in the top panel in Figure 1. In order to find accurate fitting formulas for $C(y)$, we use two different Padé approximations with increasing accuracy. Fitting the numerical results of Figure 1 with the Padé approximation

$$
C(y) = \frac{1 + a_1 y + a_2 y^2}{1 + b_1 y + b_2 y^2}; \quad y < 20, \quad (11)
$$

gave the following coefficients: $a_1 = 7.308$, $a_2 = 0.4946$, $b_1 = 5.002$, and $b_2 = 0.8075$. The error as a function of $y \equiv x\theta$ is shown in the middle panel in Figure 1. The maximum error is about 1.4% for $x\theta < 20$ and 0.5% for $x\theta < 10$. Similarly, using the expression

$$
C(y) = \frac{1 + c_1 y + c_2 y^2 + c_3 y^3}{1 + d_1 y + d_2 y^2 + d_3 y^3}; \quad y < 20, \quad (12)
$$

we find the coefficients $c_1 = 6.8515487$, $c_2 = 1.4251694$, $c_3 = 0.017790149$, $d_1 = 4.63115589$, $d_2 = 1.5253007$, and $d_3 = 0.04522338$. The error of this fit is shown in the lower panel of Figure 1, where the maximum error is seen to be 0.04 per cent.

In rare circumstances in numerical simulations one may need to know the emissivity for $x\theta > 20$. Applying a correction factor, $C_R(y)$, to the asymptotic limit of the integral, equation (6) for $x\theta \gg 1$, we have

$$
\int_1^\infty dss\sigma_{\gamma\gamma}(s)e^{-s/x\theta} = \sigma_T \frac{3}{8}x\theta(\ln 4\eta x\theta - 1)C_R(x\theta); \quad x\theta > 20. \quad (13)
$$

Then the annihilation emissivity becomes

$$
dn\frac{dt}{dt}(x, \theta)dx = n_+n_-c\sigma_Tdx \frac{3}{4}x(\ln 4\eta x\theta - 1) \times
$$

$$
e^{-x/\theta} \frac{C_R(x\theta)}{[K_2(1/\theta)]^2}; \quad \text{any } \theta, \ x\theta > 20. \quad (14)
$$

Fitting the results from numerically evaluating the integral gives

$$
C_R(y) = 1 + 2.712y^{-1} - 55.60y^{-2} + 1039.8y^{-3} - 7800y^{-4}; \quad \text{any } \theta, \ y > 20 \quad (15)
$$

to an accuracy of 0.1 per cent.

Our finale recipe for computing the annihilation emissivity for $x\theta < 20$ is then to use equation (10) together with the fit for $K_2(1/\theta)$, equation (5), and one of the fits for $C(y)$, equations (11) or (12). The total error is at most 0.1 per cent for the latter case. For those rare cases when one is interested in the annihilation emissivity for $x\theta > 20$, one should use equation (14) together with the fits, equations (5) and (15). The total error is at most 0.15 per cent. In Figure 1 we show the dimensionless annihilation line shapes, $dn/dt(x, \theta)/n_+n_-c\sigma_T$, using the recipe given above for different dimensionless temperatures, $\theta$. The line
core is approximately Gaussian for temperatures less than about $10^8$ K, but becomes strongly asymmetric for larger temperatures.

4. Summary and Discussion

We have derived very convenient formula to use when fitting observations of thermal annihilation lines or for the rapid evaluation of the annihilation line emissivity in radiative transfer calculations.

The main result is equation (10) to be used together with equations (5), and (13) or (12) when computing the annihilation emissivity for $x\theta < 20$. Using equation (13) gives an error of less than 0.1 per cent. The error can be reduced to less than 0.04 per cent by evaluating $K_2(1/\theta)$ more accurately using, e.g., routines in Numerical Recipes (Press et al. 1986). For $x\theta > 20$, one should use equations (14) together with equations (5) and (15). The total error is at most 0.15 per cent.

There are a few limitations to the expressions we give. They assume a Maxwell-Boltzmann distribution for the annihilating particles. The mechanism normally thought to be responsible for maintaining such a thermal distribution is energy exchange through Coulomb scattering. This fails for temperatures $kT$ of order $m_e c^2$, i.e. $\theta > 1$ as discussed by e.g. Ghisellini, Haardt & Fabian (1993). Other thermalizing mechanisms may, however, be operating. Ghisellini, Guilbert, & Svensson (1988) and Ghisellini & Svensson (1990) show that synchrotron self-absorption may act as an extremely efficient thermalizing mechanism.

The expressions given are in the Born-approximation. Below temperatures of about $10^8$ K, Coulomb corrections must be included. At such low temperatures, the annihilation takes place through positronium formation and one must also include the positronium continuum due to three-photon annihilation (Ore & Powell 1949).

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