Singular Polynomials for the Rational Cherednik Algebra for $G(r, 1, 2)$

Armin Gusenbauer

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Abstract

We study the rational Cherednik algebra attached to the complex reflection group $G(r, 1, 2)$. Each irreducible representation $S^\lambda$ of $G(r, 1, 2)$ corresponds to a standard module $\Delta(\lambda)$ for the rational Cherednik algebra. We give necessary and sufficient conditions for the existence of morphism between two of these modules and explicit formulas for them when they exist.

Keywords Cherednik algebras · Dunkl operators · Singular polynomials · Highest weight category

Mathematics Subject Classification (2010) 20C08 (16G99 20F55)

1 Introduction

The rational Cherednik algebra $\mathbb{H}$ is an algebra attached to a complex reflection group $W$, depending on a set of parameters indexed by the conjugacy classes of reflection in $W$. The algebra $\mathbb{H}$ possesses a triangular decomposition ([1] and [5]) allowing the construction of induced modules called standard modules, and the Serre subcategory of $\mathbb{H}$-mod generated by these, category $\mathcal{O}$, has been the object of intense study during the last fifteen years. Part of the structure of the category $\mathcal{O}$ is encoded by the homomorphisms between standard modules, and the classification and construction of these homomorphisms seems to be a difficult problem.

The first work on this problem is due to Dunkl [2, 3], who solved it for $W = S_n$ the symmetric group and codomain the standard module parabolically induced from the trivial representation. Subsequently Griffeth [6] solved it for $W = G(r, 1, n)$, but with a certain genericity condition in the parameters. We will specialize to $W = G(r, 1, 2)$ and solve the problem without any restriction on the parameters.

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Armin Gusenbauer
armingk@inst-mat.ualca.cl

1 Universidad de Talca, Talca, Chile
The parameters space for $W = G(r, 1, 2)$ is $r$-dimensional with coordinates $c_0, d_0, d_1, \ldots, d_{r-1}$ subject to the requirement

$$d_0 + d_1 + d_2 + \ldots + d_{r-1} = 0.$$ 

The irreducible representations of $G(r, 1, n)$ are indexed by $r$-partitions of $n$. So for $n = 2$ there are three kinds of irreducible representations $\{\lambda_i, \lambda^j, \lambda_{i,j} | 0 \leq i \neq j \leq r - 1\}$. Our main theorem gives necessary and sufficient conditions for the existence of morphisms between the corresponding standard modules (see Theorem 4.2).

For the necessary conditions we start by using Theorem 5.1 of [8]. For the sufficient conditions we construct the morphisms explicitly. This amounts to finding elements of the codomain that are annihilated by the Dunkl operators. In other words, we are looking for a generalized version of singular polynomials.

For $G(r, 1, 2)$ the dimension of the homomorphism space between two standard modules is always at most two. The next theorem gives sufficient conditions for the dimension to be equal to 2 (we suspect that this the only way this can happen).

**Theorem 1.1** If we have the conditions

- $d_i - d_k + c_0 r = i - k + m_1 r > 0$
- $d_i - d_k - c_0 r = i - k + m_2 r > 0$
- $d_j - d_i + c_0 r = j - i + m_3 r > 0$
- $d_j - d_i - c_0 r = j - i + m_4 r > 0$

where $m_i$ is an integer for $i = 1, 2, 3, 4$, then we have

$$\text{Dim} (\text{Hom}(\Delta(\lambda_{i,k}), \Delta(\lambda_{i,j}))) = 2.$$ 

As we say before in order to prove our results we combine the necessary conditions from [8] with explicit computations involving Dunkl operators acting on vector-valued polynomial functions. A standard technique for answering the questions we pose here is to apply the KZ-functor and use known results about Hecke algebras. The obstruction in our case is that we do not have good control over the KZ images of the standard modules (except for parameters in a certain cone).

One might hope that our results would compute the simple modules in category $\mathcal{O}$. However, it is quite rare that the radical of the standard module is generated by the singular polynomials it contains. For instance if the radical of every standard module is generated by singular polynomials then every simple object in category $\mathcal{O}$ has a BGG resolution by standard modules (Theorem 1.1 of [9]).

Category $\mathcal{O}$ is a highest weight category with BGG reciprocity so by Lemma 4.5 of [8] it is equipped with a canonical coarsest order. In the example at the end of the paper we observe that this poset is graded and self-dual. This raises the question if this is always so and if there is a structural reason for this phenomenon.

### 2 Notation and Preliminaries

Let $n$ and $r$ be positive integers. A partition of $n$ is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ of positive integers such that $|\lambda| := \sum_{i=1}^{l} \lambda_i = n$ and $\lambda_i \geq \lambda_{i+1}$. The Young diagram of $\lambda$ is the set

$$[\lambda] = \{(i, j) \in \mathbb{N} \times \mathbb{N} | 1 \leq j \leq \lambda_i \text{ and } i \geq 1\}$$

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It is useful to think about the elements of $[\lambda]$ as an array of boxes in the plane, with the indices following matrix conventions. This is, the box with label $(i, j)$ belongs to the $i$-th row and the $j$-th column. An $r$-partition of $n$ is a sequence $\lambda = (\lambda^{(0)}, \ldots, \lambda^{(r-1)})$ of partitions such that $|\lambda| := \sum_{i=0}^{r-1} |\lambda^{(i)}| = n$. By the Young diagram of an $r$-partition $\lambda$ of $n$ we mean the set

$$[\lambda] = \{(i, j, k) \in \mathbb{N} \times \mathbb{N} \times \{0, 1, ..., r-1\} | 1 \leq j \leq (\lambda^{(k)})_i \text{ and } i \geq 1\}.$$ 

We can see $[\lambda]$ as an $r$-tuple of usual Young diagrams, which we called the components of $[\lambda]$. Let $\lambda$ be an $r$-partition. A $\lambda$-tableau is a bijection $T : [\lambda] \rightarrow \{1, 2, ..., n\}$.

We say that $\lambda$-tableaux have a unique element. On the other hand, the set $SYT(\lambda)$ has two elements, which are depicted below.

$$T_1 = \left(\emptyset, \ldots, \begin{array}{|c|}
\hline
1 \\
\hline
\end{array}, \ldots, \emptyset \right) \quad \text{and} \quad T_2 = \left(\emptyset, \ldots, \begin{array}{|c|}
\hline
2 \\
\hline
\end{array}, \ldots, \emptyset \right). \quad (2.1)$$

### 2.1 The Rational Cherednik Algebra for $G(r, 1, 2)$

Let $W = G(r, 1, 2)$ be the group of $2 \times 2$ monomial matrices whose entries are $r$-roots of unity. Set $\zeta = e^{\frac{2\pi i}{r}}$. We define

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \zeta_1 = \begin{pmatrix} \zeta & 0 \\ 0 & \bar{\zeta} \end{pmatrix} \quad \text{and} \quad \zeta_2 = \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}.$$ 

We recall that $W$ is generated by $S$, $\zeta_1$ and $\zeta_2$. Let $y_1 = (1, 0)$, $y_2 = (0, 1)$, $x_1 = (1, 0)'$ and $x_2 = (0, 1)'$. The group $W$ acts on $x_1$, $x_2$, $y_1$ and $y_2$ according to the following rule

$$\zeta_i \cdot x_i = \zeta^{-1} \cdot x_i \quad \zeta_i \cdot x_j = x_j \quad S \cdot x_i = x_j$$

$$\zeta_i \cdot y_i = \zeta^{-1} \cdot y_i \quad \zeta_i \cdot y_j = y_j \quad S \cdot y_i = y_j$$

for $i, j \in \{1, 2\}$ and $i \neq j$.

Henceforth, we fix parameters $c_0, d_0, d_1, \ldots, d_{r-1} \in \mathbb{C}$ such that $d_0 + d_1 + \ldots + d_{r-1} = 0$. For $i \in \mathbb{Z}$, we define $d_i$ by the rule $d_i = d_j$ if $i = j \mod r$.

### Definition 2.2

The rational Cherednik algebra $\mathcal{H} = \mathcal{H}(c_0, d_0, \ldots, d_{r-1})$ for $W = G(r, 1, 2)$ is the $\mathbb{C}$-algebra generated by $\mathbb{C}[x_1, x_2]$, $\mathbb{C}[y_1, y_2]$ and $\{w \mid w \in W\}$ subject to the relations

$$\bar{w}v = \bar{v}w, \quad \bar{w}x = (wx)\bar{w}, \quad \bar{w}y = (wy)\bar{w} \quad (2.2)$$
for \( w, v \in W, x \in \mathbb{C}[x_1, x_2], \) and \( y \in \mathbb{C}[y_1, y_2], \)
\[
y_i x_j = x_j y_i + c_0 \sum_{l=0}^{r-1} \xi^{-l} \xi_i^l S \xi_i^{-l}
\]
for \( 1 \leq i \neq j \leq 2, \)
\[
y_i x_i = x_i y_i + \kappa - \sum_{l=1}^{r-1} (d_j - d_{j-1}) e_{ij} - c_0 \sum_{j \neq i} \sum_{l=0}^{r-1} \xi_i^l S \xi_i^{-l}
\]
for \( 1 \leq i \leq 2, \) where \( e_{ij} \in \mathbb{C}W \) is the idempotent
\[
e_{ij} = \frac{1}{r} \sum_{l=0}^{r-1} \xi^{-lj} \xi_i^l.
\]

The PBW theorem (see for example \cite{7}) for \( \mathbb{H} \) asserts that
\[
\mathbb{H} \simeq \mathbb{C}[x_1, x_2] \otimes \mathbb{C}W \otimes \mathbb{C}[y_1, y_2],
\]
as \( \mathbb{C} \)-vector spaces. The following proposition is a particular case of Proposition 4.1 of \cite{6} for \( n = 2. \)

**Proposition 2.3** In \( \mathbb{H} \) the following relations hold

\[
y_1 x_1^n x_2^m = x_1^n x_2^m y_1 + x_1^{n-1} x_2^m \left( n - \sum_{j=0}^{r-1} \frac{d_j}{r} \sum_{l=0}^{r-1} \xi^{-lj} \left( (1 - \xi^{-ln}) \left( \begin{smallmatrix} \xi^l & 0 \\ 0 & 1 \end{smallmatrix} \right) \right) \right)
\]
\[
- c_0 \sum_{l=0}^{r-1} \frac{x_1^n x_2^m - \left( 0 \right. \xi^l \left. \begin{smallmatrix} \xi^{-l} & 0 \\ 0 & 1 \end{smallmatrix} \right)}{x_1 - \xi^l x_2} \left( \begin{smallmatrix} 0 \\ \xi^l \end{smallmatrix} \right) .
\]
\[
y_2 x_1^n x_2^m = x_1^n x_2^m y_2 + x_1^{n-1} x_2^m \left( m - \sum_{j=0}^{r-1} \frac{d_j}{r} \sum_{l=0}^{r-1} \xi^{-lj} \left( (1 - \xi^{-lm}) \left( 0 \begin{smallmatrix} 1 \\ 0 \xi^l \end{smallmatrix} \right) \right) \right)
\]
\[
- c_0 \sum_{l=0}^{r-1} \frac{x_1^n x_2^m - \left( 0 \right. \xi^{-l} \left. \begin{smallmatrix} \xi^l & 0 \\ 0 & 1 \end{smallmatrix} \right)}{x_2 - \xi^l x_1} \left( \begin{smallmatrix} 0 \\ \xi^{-l} \end{smallmatrix} \right) .
\]

### 2.2 Standard Modules for the Rational Cherednik Algebra

The irreducible representations of \( W \) are parametrized by the \( r \)-partitions of 2. Given an \( r \)-partition \( \lambda \) of 2 we denote by \( S^\lambda \) the corresponding irreducible representation. It is well-known that \( S^\lambda \) has a basis \( \{ v_T \} \) indexed by the elements of \( SYT(\lambda) \) (see Theorem 3.1 of \cite{6}). Define the standard module \( \Delta(\lambda) \) to be the induced module

\[
\Delta(\lambda) = \text{Ind}^{\mathbb{H}}_{\mathbb{C}W \otimes \mathbb{C}[y_1, y_2]} S^\lambda
\]
and we define the \( \mathbb{C}[y_1, y_2] \) action on \( S^\lambda \) by

\[
y_i \cdot v = 0 \quad \text{for} \quad 1 \leq i \leq 2 \quad \text{and} \quad v \in S^\lambda.
\]

By the PBW theorem for \( \mathbb{H} \) we have an isomorphism of \( \mathbb{C} \)-vector spaces

\[
\Delta(\lambda) \simeq \mathbb{C}[x_1, x_2] \otimes \mathbb{C} S^\lambda.
\]
We are going to describe the action of $H$ on the standard modules. As we already pointed out, there are three types of $r$-partitions of 2: $\lambda_i$, $\lambda_i^t$ and $\lambda_{i,j}$. The representations $S^{\lambda_i}$ and $S^{\lambda_i^t}$ are one-dimensional with basis $\{v_T\}$. The representation $S^{\lambda_{i,j}}$ is two-dimensional with basis $\{v_{T_1}, v_{T_2}\}$, where $T_1$ and $T_2$ are as in Eq. 2.1. The action of $W$ on $S^{\lambda}$ is described in the table below.

| $\lambda_i$ | $\lambda_i^t$ |
|-------------|---------------|
| $\zeta_2 \cdot v_T = \zeta^t v_T$ | $\zeta_2 \cdot v_T = \zeta^t v_T$ |
| $\zeta_1 \cdot v_T = \zeta^t v_T$ | $\zeta_1 \cdot v_T = \zeta^t v_T$ |
| $S \cdot v_T = v_T$ | $S \cdot v_T = -v_T$ |

(2.5)

| $\lambda_{i,j}$ |
|------------------|
| $\zeta_2 \cdot v_{T_1} = \zeta^t v_{T_1}$ |
| $\zeta_1 \cdot v_{T_1} = \zeta^t v_{T_1}$ |
| $S \cdot v_{T_1} = v_{T_2}$ |

2.3 The Action on $\Delta(\lambda)$

The elements of $\Delta(\lambda)$ are sums of elements of the form $x_1^n x_2^m \otimes v_T$ for $n$ and $m$ positive integers. We focus on how $H$ acts in the elements of this form. The elements of $\mathbb{C}[x_1, x_2]$ act by multiplication and the group elements act in the obvious way. Therefore, we only need to concentrate on the action of $y_1$ and $y_2$ on an element of the form $x_1^n x_2^m \otimes v_T$. In what follows $[A]$ will denote the floor function of $A$.

**Proposition 2.4** For $\lambda = \lambda_i$ the action of $y_1$ and $y_2$ on an element of the form $x_1^n x_2^m \otimes v_T$ is given by

(1) $y_1 \cdot x_1^n x_2^m \otimes v_T = \begin{cases} (n-d_i + d_i-n-c_0 r)x_1^n x_2^m - c_0 r \sum_{k=1}^{[n-m]} x_1^{n-kr} x_2^{m+kr} \otimes v_T, & \text{if } n > m; \\
(n-d_i + d_i-n)x_1^n x_2^m + c_0 r \sum_{k=1}^{[n-m]} x_1^{n+kr} x_2^{m-kr} \otimes v_T, & \text{if } n \leq m. \end{cases}$

(2.6)

(2) $y_2 \cdot x_1^n x_2^m \otimes v_T = \begin{cases} (m-d_i + d_i-m)x_1^n x_2^m - c_0 r \sum_{k=1}^{[m-n]} x_1^{n-kr} x_2^{m+kr} \otimes v_T, & \text{if } n \geq m; \\
(m-d_i + d_i-m-c_0 r)x_1^n x_2^m - c_0 r \sum_{k=1}^{[m-n-1]} x_1^{n+kr} x_2^{m-kr} \otimes v_T, & \text{if } n < m. \end{cases}$

(2.7)

**Proof** We only prove the formula for $y_1$ since the formula for $y_2$ is obtained in the same way. We know that $y_1 \cdot (x_1^n x_2^m \otimes v_T) = y_1 x_1^n x_2^m \otimes v_T$. In order to obtain the desired formula we use the commutation rules given in Proposition 2.3.
The leftmost term in the expression for \( y_1 x_1^n x_2^m \) in Proposition 2.3 has a \( y_1 \) on its right. Therefore, it acts by zero on \( S^0 \) and we can disregard such a term. Henceforth, we omit the \( \otimes \gamma_T \) part at the end of each equality. We have

\[
y_1 x_1^n x_2^m = x_1^{n-1} x_2^m \left( n - \sum_{j=0}^{r-1} \frac{d_j}{r} \sum_{l=0}^{r-1} \zeta^{-lj} (1 - \zeta^{-ln}) \zeta^{il} \right) - c_0 \sum_{l=0}^{r-1} x_1^{n-m} \left( \frac{0}{\zeta - l} \right) \cdot x_1 x_2^m \\
= x_1^{n-1} x_2^m \left( n - \sum_{j=0}^{r-1} \frac{d_j}{r} \sum_{l=0}^{r-1} (\zeta^{l(i-j)} - \zeta^{l(i-j-n)}) \right) - c_0 \sum_{l=0}^{r-1} x_1^{n-m} \left( \frac{\zeta^{l(n-m)}}{\zeta - l} \right) \cdot x_1 x_2^m \\
= x_1^{n-1} x_2^m (n - d_i + d_{i-n}) - c_0 \sum_{l=0}^{r-1} x_1 x_2^m \left( \frac{\zeta^{l(n-m)} x_1 x_2^m}{x_1 - \zeta^l x_2} \right). \tag{2.8}
\]

Some remarks are in order. The first equality is obtained by combining Eq. 2.5 and Proposition 2.3. We notice that the matrices occurring on the right of the expression in Proposition 2.3 are replaced by certain scalars by using the formulas in Eq. 2.5. The sum on the right of the second equality is obtained in the same way, while the term inside the parenthesis is obtained from the previous one by expanding the powers of \( \zeta \). Finally, the last equation is easily obtained once we note that

\[
\sum_{l=0}^{r-1} \zeta^{kl} = \begin{cases} r, & \text{if } k \equiv 0 \mod r; \\ 0, & \text{otherwise}. \end{cases} \tag{2.9}
\]

We need to split the argument into two cases according to whether \( n > m \) or \( n \leq m \). We begin by treating the case \( n > m \). Let us consider the sum appearing in the expression of \( y_1 x_1^n x_2^m \) obtained in Eq. 2.8. Since \( n > m \) we can factor this term by \( x_1^m x_2^m \) in order to obtain

\[
\sum_{l=0}^{r-1} x_1 x_2^m - \zeta^{l(n-m)} x_1 x_2^m = \sum_{l=0}^{r-1} x_1 x_2^m (x_1 x_2^m - (\zeta^l x_2)^{n-m}). \tag{2.10}
\]

By using the identity

\[
(a^s - b^s) = (a - b) \sum_{k=0}^{s-1} a^{s-1-k} b^k \tag{2.11}
\]

for \( a = x_1, b = \zeta^l x_2 \) and \( s = n - m \) we obtain

\[
\sum_{l=0}^{r-1} x_1 x_2^m - \zeta^{l(n-m)} x_1 x_2^m = \sum_{l=0}^{r-1} (\zeta^{lk} x_1 x_2^m - \zeta^{l(n-m)} x_1 x_2^m) = \sum_{k=0}^{n-m-1} \sum_{l=0}^{r-1} \zeta^{lk} x_1^{n-1-k} x_2^m + k. \tag{2.12}
\]

By Eq. 2.9 \( \sum_{l=0}^{r-1} \zeta^{lk} \) vanishes unless \( k \equiv 0 \mod r \). Then, we have

\[
\sum_{l=0}^{r-1} x_1 x_2^m (x_1 x_2^m - (\zeta^l x_2)^{n-m}) = \sum_{l=0}^{r-1} x_1 x_2^m \left( \frac{n-m-1}{r} \right) \sum_{t=0}^{r-1} x_1^{n-1-tr} x_2^{m+tr}. \tag{2.13}
\]
If we now isolate the term corresponding to \( t = 0 \) in the sum on the right-hand side of Eq. 2.13 we obtain

\[
\sum_{l=0}^{r-1} \frac{x_1^m x_2^m (x_1^{n-m} - (\xi^l x_2)^{n-m})}{x_1 - \xi^l x_2} = r x_1^{n-1} x_2^m + r \sum_{l=1}^{[\frac{m-1}{r}]} x_1^{n-tr-1} x_2^{m+tr}.
\]  

Finally, a combination of Eqs. 2.8 and 2.14 yields

\[
y_1 x_1^m x_2^m = x_1^{n-1} x_2^m (n - d_i + d_{i-n} - c_0 r) - c_0 r \sum_{k=1}^{[\frac{m-n}{r}]} x_1^{n-kr-1} x_2^{m+kr}, 
\]  

which is what we wanted to show. The case \( n \leq m \) is treated similarly, but in this case we factor by \( x_1^m x_2^n \) rather than \( x_1^m x_2^m \). For the sake of brevity we omit the proof of this case.

**Proposition 2.5** For \( \lambda = \lambda^i \) the action of \( y_1 \) and \( y_2 \) on an element of the form \( x_1^n x_2^m \otimes v_T \) is given by

\[
(1) \quad y_1 \cdot x_1^n x_2^m \otimes v_T = \begin{cases} 
(n - d_i + d_{i-n} - c_0 r)x_1^{n-1} x_2^m + c_0 r \sum_{k=1}^{[\frac{m-n}{r}]} x_1^{n-kr-1} x_2^{m+kr} \otimes v_T, & \text{if } n > m; \\
(n - d_i + d_{i-n})x_1^{n-1} x_2^m - c_0 r \sum_{k=1}^{[\frac{m-n}{r}]} x_1^{n+kr-1} x_2^{m-kr} \otimes v_T, & \text{if } n \leq m.
\end{cases}
\]

\[
(2) \quad y_2 \cdot x_1^n x_2^m \otimes v_T = \begin{cases} 
(m - d_i + d_{i-m})x_1^n x_2^{m-1} - c_0 r \sum_{k=1}^{[\frac{m-n}{r}]} x_1^{n-kr} x_2^{m+kr-1} \otimes v_T, & \text{if } n \geq m; \\
(m - d_i + d_{i-m} + c_0 r)x_1^n x_2^{m-1} + c_0 r \sum_{k=1}^{[\frac{m-n-1}{r}]} x_1^{n+kr} x_2^{m-kr-1} \otimes v_T, & \text{if } n < m.
\end{cases}
\]

**Proof** The same argument given in the proof of Proposition 2.4 applies to this case.

In \( \lambda = \lambda_{i,j} \) we have two generators of \( S^\lambda \), called \( v_{T_1} \) and \( v_{T_2} \).

**Proposition 2.6** When \( \lambda = \lambda_{i,j} \) the action of \( y_1 \) and \( y_2 \) in a generic \( x_1^n x_2^m \otimes v_{T_1} \) or a generic \( x_1^n x_2^m \otimes v_{T_2} \) is given by:
1) \( y_1 \cdot x_1^m x_2^n \otimes v_{T1} = \)
\[
\begin{cases}
(n - d_j + d_j - n) x_1^{n-1} x_2^m \otimes v_{T1} + r c_0 \sum_{k=0}^{\left[\frac{n-m-i+j}{r}\right]} x_1^{n+kr+j-i-1} x_2^{m-rk+i-j-1} \otimes v_{T2}, \text{ if } n < m; \\
(n - d_j + d_j - n) x_1^{n-1} x_2^m \otimes v_{T1} - r c_0 \sum_{k=1}^{\left[\frac{n-m-i+j}{r}\right]} x_1^{n-kr+j-i-1} x_2^{m+rk+i+j} \otimes v_{T2}, \text{ if } n > m;
\end{cases}
\]

\[ (n - d_j + d_j - n) x_1^{n-1} x_2^n \otimes v_{T2}, \quad \text{if } n = m. \quad (2.19) \]

2) \( y_1 \cdot x_1^m x_2^n \otimes v_{T2} = \)
\[
\begin{cases}
(n - d_j + d_j - n) x_1^{n-1} x_2^m \otimes v_{T2} + r c_0 \sum_{k=0}^{\left[\frac{n-m+i-j}{r}\right]} x_1^{n+kr+j+i-1} x_2^{m-rk-i-j} \otimes v_{T2}, \text{ if } n > m; \\
(n - d_j + d_j - n) x_1^{n-1} x_2^m \otimes v_{T2} + r c_0 \sum_{k=1}^{\left[\frac{n-m+i-j}{r}\right]} x_1^{n+kr+i-1} x_2^{m-rk-j-1} \otimes v_{T2}, \text{ if } n < m;
\end{cases}
\]

\[ (n - d_j + d_j - n) x_1^{n-1} x_2^n \otimes v_{T2}, \quad \text{if } n = m. \quad (2.20) \]

3) \( y_2 \cdot x_1^m x_2^n \otimes v_{T1} = \)
\[
\begin{cases}
(m - d_j + d_j - m) x_1^{n-1} x_2^{m-1} \otimes v_{T1} + r c_0 \sum_{k=0}^{\left[\frac{n-m-i+j}{r}\right]} x_1^{n-kr+i-j} x_2^{m-rk+i-j-1} \otimes v_{T2}, \text{ if } n > m; \\
(m - d_j + d_j - m) x_1^{n-1} x_2^{m-1} \otimes v_{T1} - r c_0 \sum_{k=1}^{\left[\frac{n-m+i-j}{r}\right]} x_1^{n+kr-i+j} x_2^{m+rk-i-j-1} \otimes v_{T2}, \text{ if } n < m;
\end{cases}
\]

\[ (n - d_j + d_j - n) x_1^{n-1} x_2^n \otimes v_{T1}, \quad \text{if } n = m. \quad (2.21) \]

4) \( y_2 \cdot x_1^m x_2^n \otimes v_{T2} = \)
\[
\begin{cases}
(m - d_i + d_i - m) x_1^{n-1} x_2^{m-1} \otimes v_{T2} + r c_0 \sum_{k=0}^{\left[\frac{n-m+i-j}{r}\right]} x_1^{n-kr+i-j} x_2^{m-rk+i-j-1} \otimes v_{T1}, \text{ if } n > m; \\
(m - d_i + d_i - m) x_1^{n-1} x_2^{m-1} \otimes v_{T2} - r c_0 \sum_{k=1}^{\left[\frac{n-m+i-j}{r}\right]} x_1^{n+kr+i-j} x_2^{m+rk-i-j-1} \otimes v_{T1}, \text{ if } n < m;
\end{cases}
\]

\[ (n - d_i + d_i - m) x_1^{n-1} x_2^n \otimes v_{T2}, \quad \text{if } n = m. \quad (2.22) \]

\[ (2.21) \]

\[ (2.20) \]

\[ (2.19) \]

\[ (2.18) \]
Proof  We only prove the formula for \( y_1 \cdot x_1^n x_2^m \otimes v_T \), since the others formulas are obtained in the same way. In order to obtain the desired formula we use the commutation rules given in Proposition 2.3. As we mentioned in the proof of Proposition 2.4 the leftmost term in the expression for \( y_1 x_1^n x_2^m \) in Proposition 2.3 has a \( \lambda \) and we can disregard such a term. A combination of Proposition 2.3, Eqs. 2.5 and 2.9 yields

\[
y_1 \cdot x_1^n x_2^m \otimes v_T = x_1^{n-1} x_2^m \left( n - \sum_{s=0}^{r-1} \frac{d_s}{r} \sum_{l=0}^{r-1} \zeta^{-ls} (1 - \zeta^{-ln}) \zeta^{il} \right) \otimes v_T - c_0 \sum_{l=0}^{r-1} \frac{x_1^n x_2^m - \zeta^{(n-m)l} x_1^m x_2^n}{x_1 - \zeta^l x_2} \zeta^{(j-i)l} \otimes v_T_2
\]

\[
= x_1^{n-1} x_2^m \left( n - \sum_{s=0}^{r-1} \frac{d_s}{r} \sum_{l=0}^{r-1} \zeta^{(i-s)l} - \zeta^{(i-s-n)l} \right) \otimes v_T - c_0 \sum_{l=0}^{r-1} \frac{x_1^n x_2^m - \zeta^{(n-m)l} x_1^m x_2^n}{x_1 - \zeta^l x_2} \zeta^{(j-i)l} \otimes v_T_2
\]

\[
= (n - d_i + d_i - n) x_1^{n-1} x_2^m \otimes v_T - c_0 \sum_{l=0}^{r-1} \frac{x_1^n x_2^m - \zeta^{(n-m)l} x_1^m x_2^n}{x_1 - \zeta^l x_2} \zeta^{(j-i)l} \otimes v_T_2
\]

We need to split the argument in three cases according to whether \( n > m, n < m \) and \( n = m \). In the three cases the argument is the same as the one used in the proof of Proposition 2.4, but in this case we only work with the expression on the right. This is, we first factor by some monomial in \( x_1 \) and \( x_2 \), then we use Eq. 2.11 in order to eliminate the denominator, and finally, we use Eq. 2.9 to obtain the desired expression.

(a) \( n > m \)

\[
y_1 \cdot x_1^n x_2^m \otimes v_T = (n - d_i + d_i - n) x_1^{n-1} x_2^m \otimes v_T - c_0 \sum_{l=0}^{r-1} \frac{x_1^n x_2^m - \zeta^{(n-m)l} x_1^m x_2^n}{x_1 - \zeta^l x_2} \zeta^{(j-i)l} \otimes v_T_2
\]

\[
= (n - d_i + d_i - n) x_1^{n-1} x_2^m \otimes v_T - c_0 \sum_{l=0}^{r-1} \frac{x_1^n x_2^m}{x_1 - \zeta^l x_2} (x_1^{n-m} - (\zeta^l x_2)^{n-m}) \zeta^{(j-i)l} \otimes v_T_2
\]

\[
= (n - d_i + d_i - n) x_1^{n-1} x_2^m \otimes v_T - c_0 \sum_{l=0}^{r-1} x_1^n x_2^m \sum_{k=0}^{n-m-1} x_1^{n-m-1-k} \zeta^k \zeta^{(j-i)l} \otimes v_T_2
\]

\[
= (n - d_i + d_i - n) x_1^{n-1} x_2^m \otimes v_T - c_0 \sum_{l=0}^{r-1} x_1^n x_2^m \sum_{k=0}^{n-m-1} x_1^{n-m-1-k} x_2^{m+k} \zeta^{(k+j-i)l} \otimes v_T_2
\]

\[
= (n - d_i + d_i - n) x_1^{n-1} x_2^m \otimes v_T - c_0 \sum_{k=0}^{n-m-1} x_1^{n-1-k} x_2^{m+k} \sum_{l=0}^{r-1} \zeta^{(k+j-i)l} \otimes v_T_2
\]

\[
= (n - d_i + d_i - n) x_1^{n-1} x_2^m \otimes v_T - r c_0 \sum_{k=1}^{n-m-1+j-i} x_1^{n-1-kr-i+j} x_2^{m+kr+i-j} \otimes v_T_2
\]
(b) \( (n < m) \)
\[
(y_1 \cdot x_1^n x_2^m) \otimes vT_1
\]
\[
= (n - d_i + d_{i-n}) x_1^{n-1} x_2^m \otimes vT_1 - c_0 \sum_{l=0}^{r-1} x_1^n x_2^m - \zeta^{(n-m)l} x_1^m x_2^n \zeta^{(j-i)l} \otimes vT_2
\]
\[
= (n - d_i + d_{i-n}) x_1^{n-1} x_2^m \otimes vT_1 - c_0 \sum_{l=0}^{r-1} x_1^n x_2^m (x_2^{m-n} - (\zeta^{-l} x_1^{m-n})) \zeta^{(j-i)l} \otimes vT_2
\]
\[
= (n - d_i + d_{i-n}) x_1^{n-1} x_2^m \otimes vT_1 + c_0 \sum_{l=0}^{r-1} x_1^n x_2^m \zeta^{-l} x_1^k x_2^m - (\zeta^{(j-i)l} \otimes vT_2)
\]
\[
= (n - d_i + d_{i-n}) x_1^{n-1} x_2^m \otimes vT_1 + c_0 \sum_{l=0}^{r-1} x_1^n x_2^m \zeta^{-l} x_1^k x_2^m - (\zeta^{(j-i)l} \otimes vT_2)
\]
\[
= (n - d_i + d_{i-n}) x_1^{n-1} x_2^m \otimes vT_1 + c_0 \sum_{l=0}^{r-1} x_1^n x_2^m \zeta^{-l} x_1^k x_2^m - (\zeta^{(j-i)l} \otimes vT_2)
\]
\[
= (n - d_i + d_{i-n}) x_1^{n-1} x_2^m \otimes vT_1 + r c_0 \sum_{l=0}^{r-1} x_1^n x_2^m \zeta^{(j-i)l} \otimes vT_2
\]

(c) \( (n = m) \)
\[
(y_1 \cdot x_1^n x_2^m) \otimes vT_1
\]
\[
= (n - d_i + d_{i-n}) x_1^{n-1} x_2^m \otimes vT_1 - c_0 \sum_{l=0}^{r-1} x_1^n x_2^m - \zeta^{(n-m)l} x_1^m x_2^n \zeta^{(j-i)l} \otimes vT_2
\]
\[
= (n - d_i + d_{i-n}) x_1^{n-1} x_2^m \otimes vT_1 - c_0 \sum_{l=0}^{r-1} x_1^n x_2^m - \zeta^{(n-m)l} x_1^m x_2^n \zeta^{(j-i)l} \otimes vT_2
\]

So far we have proved the formula for \( y_1 \cdot x_1^n x_2^m \otimes vT_1 \). The result for \( y_1 \cdot x_1^n x_2^m \otimes vT_2 \) is obtained in the same way by replacing \( i \) by \( j \) and vice versa and \( vT_1 \) by \( vT_2 \) and vice versa. In the same vein, the proof for the formula for \( y_2 \cdot x_1^n x_2^m \otimes vT_1 \) is obtained using the same proof as for \( y_1 \cdot x_1^n x_2^m \otimes vT_1 \) by replacing \( x_1 \) by \( x_2 \) and vice versa and \( i \) by \( j \) and vice versa. It is worth mentioning that the above interchanging in the variables also produces an interchanging in the cases, this is, the case \( n < m \) is now the case \( n > m \) and vice versa. Finally, for \( y_2 \cdot x_1^n x_2^m \otimes vT_2 \) the proof follows by interchanging \( i, x_1 \), and \( n \) with \( j, x_2 \) and \( m \), respectively. \( \square \)

## 3 Singular Polynomials

We recall that our goal is to determine all the morphisms between two standard modules. In this section we describe explicitly a family of polynomials which will allow us to construct such morphisms.

**Definition 3.1** A singular polynomial is an element \( m \in \Delta(\lambda) \) such that \( y_1 \cdot m = 0 \) and \( y_2 \cdot m = 0 \).

It is a consequence of the definition of the standard module \( \Delta(\lambda) \) that for any \( \mathfrak{H} \)-module \( M \) the map

\[
\text{Hom}_\mathfrak{H}(\Delta(\lambda), M) \xrightarrow{\sim} \text{Hom}_{\mathcal{W}}(S^\lambda, \text{Sing}(M))
\]
defined by
\[ \phi \mapsto \phi|_\lambda \]
is a bijection, where \( \text{Sing}(M) = \{ m \in M | y \cdot m = 0 \ \forall y \in \mathbb{C}^2 \} \). Motivated by the above isomorphism we undertake the task of determine several elements in \( \text{Sing}(\Delta(\lambda)) \), for the different values of \( \lambda \). The polynomials we are going to define have already appeared in the literature, see for example [6]. However, they are not described explicitly. We provide a concrete definition for these polynomials which will be the key for the construction of the morphisms in the forthcoming sections.

**Proposition 3.2** The following polynomials are singular polynomials in \( \Delta(\lambda_i) \):

(a) \( (x_1^r - x_2^r)^k \otimes v_T, \) if \( c_0 = \frac{k}{2} \) and \( k \) is an odd positive integer.

(b) \( x_1^t x_2^n \otimes v_T, \) if \( n - d_i + d_{i-n} = 0. \)

(c) \( p(x_1, x_2) = \left( x_1^n + \sum_{t=1}^{\left[ \frac{k-1}{2} \right]} \alpha_t \beta_{t-1} x_1^{n-t} x_2^{t_r} + \sum_{t=0}^{\left[ \frac{k}{2} \right]} \alpha_t \beta_{t} x_1^{n-(k-t)r} x_2^{(k-t)r} \right) \otimes v_T, \)
if \( n - d_i + d_{i-n} - c_0r = 0, \) \( n \not\equiv 0 \mod r \) where \( k = \left[ \frac{n}{r} \right], \ \alpha_t = \binom{k}{t} \) and 
\[ \beta_t = \frac{c_0(c_0 - 1) \ldots (c_0 - t)}{(c_0 - k)(c_0 - (k - 1)) \ldots (c_0 - (k - t))}. \]

**Proof** We begin by noticing that if \( p(x_1, x_2) \) is a singular polynomial then \( cp(x_1, x_2) \) is also singular, for all \( c \in \mathbb{C} \). Since the coefficients occurring in our polynomials are defined by fractions, it is possible that some of them are undefined for some choices of the parameters. If we are in this case, rather than consider \( p(x_1, x_2) \) we consider an scalar multiple of it. More precisely if \( c_0 = m, \) then we redefine \( p(x_1, x_2) \) as \( (c_0 - m)p(x_1, x_2) \).

The fact that the polynomial in (a) is singular follows by [4, Proposition 5.2]. On the other hand, by Proposition 2.4 we obtain
\[ y_1 \cdot x_1^n x_2^n \otimes v_T = (n - d_i + d_{i-n}) x_1^{n-1} x_2^{n} \otimes v_T = 0, \]
\[ y_2 \cdot x_1^n x_2^n \otimes v_T = (n - d_i + d_{i-n}) x_1^{n} x_2^{n-1} \otimes v_T = 0. \]
It follows by the condition \( n - d_i + d_{i-n} = 0 \) that the polynomial in (b) is singular.

We are now going to prove that the polynomial in (c) is singular. A precise inspection of Eq. 2.6 and the definition of \( p(x_1, x_2) \) reveals that
\[ y_1 \cdot p(x_1, x_2) = a_0 x_1^{n-1} + a_1 x_1^{n-r-1} x_2^r + \ldots + a_k x_1^{n-kr-1} x_2^{kr}. \] (3.1)
Therefore, in order to prove that \( p(x_1, x_2) \) is singular we need to show that \( a_l = 0, \) for all \( 0 \leq l \leq k. \)

Given a polynomial \( q(x_1, x_2) = \sum_{0 \leq i,j} a_{i,j} x_1^i x_2^j \in \mathbb{C}[x_1, x_2] \) and two non-negative integers \( s \) and \( t \) we define \( [q(x_1, x_2)]_{s,t} := a_{s,t}. \) By the first part of Eq. 2.6 we have
\[ \left[ y_1 \cdot x_1^n x_2^n \right]_{n-1,0} = n - d_i + d_{i-n} - c_0r \]
\[ \left[ y_1 \cdot x_1^{n-lr} x_2^{lr} \right]_{n-1,0} = 0 \] for \( 1 \leq l \leq k. \)
Therefore, we conclude that \( a_0 = [y_1 \cdot p(x_1, x_2)]_{n-1,0} = n - d_i + d_{i-n} - c_0r = 0. \)
In order to show that \( a_l = 0 \) for \( 1 \leq l \leq k, \) we first need the following result.
Claim 3.3 The values of $a_l$ are given by

$$a_l = \begin{cases} -l r \alpha_l \beta_{l-1} - c_0 r - \sum_{j=1}^{l-1} c_0 r \alpha_j \beta_{j-1} + \sum_{j=k-l+1}^{k} c_0 r \alpha_{k-j} \beta_{k-j}, & \text{if } 1 \leq l \leq \frac{k}{2}; \\ -c_0 r - \sum_{j=1}^{k-l} c_0 r \alpha_j \beta_{j-1} + \sum_{j=l+1}^{k} c_0 r \alpha_{k-j} \beta_{k-j} + (c_0 - l) r \alpha_{k-l} \beta_{k-l}, & \text{if } \frac{k}{2} < l \leq k. \end{cases} \tag{3.2}$$

Proof We first assume that $1 \leq l \leq \frac{k}{2}$. We consider the action of $y_1$ on monomials of the form $x_1^{n-j} x_2^{j r}$. First suppose that $j \leq \frac{k}{2}$. In this case we have that $n - j r > j r$. Then, the first formula in Eq. 2.6 yields

$$y_1 \cdot x_1^{n-j} x_2^{j r} = (n - j r - d_1 + d_{l-n+j} - c_0 r) x_1^{n-j-1} x_2^{j r} - c_0 r \sum_{\kappa=1}^{n-2j} x_1^{n-jr-\kappa} x_2^{jr+\kappa}. \tag{3.3}$$

It follows that if $[y_1 \cdot x_1^{n-j} x_2^{j r}]_{n-lr-1,lr} \neq 0$ then $j = l$ or $\kappa = l - j$ for some $1 \leq \kappa \leq \left[ \frac{n-2j-1}{r} \right]$. If $j = l$ then by using the identity $d_{l-n+lr} = d_{l-n}$ and the condition $n-d_1 + d_{l-n} - c_0 r = 0$ we have that

$$[y_1 \cdot x_1^{n-lr} x_2^{j r}]_{n-lr-1,lr} = (n - lr - d_1 + d_{l-n+lr} - c_0 r) = (n - lr - d_1 + d_{l-n} - c_0 r) = -lr.$$

We conclude that

$$[y_1 \cdot \alpha_l \beta_{l-1} x_1^{n-lr} x_2^{j r}]_{n-lr-1,lr} = -l r \alpha_l \beta_{l-1}. \tag{3.4}$$

Suppose now that $\kappa = l - j$. We notice that $\left[ \frac{n-2j-1}{r} \right] = k - 2j$. The condition $1 \leq \kappa \leq k - 2j$ is equivalent to $0 \leq j \leq l - 1$. For these values of $j$ we have

$$[y_1 \cdot x_1^n]_{n-lr-1,lr} = -c_0 r$$

and

$$[y_1 \cdot \alpha_j \beta_{j-1} x_1^{n-jr} x_2^{j r}]_{n-lr-1,lr} = -c_0 r \alpha_j \beta_{j-1} \text{ for } 1 \leq j \leq l - 1. \tag{3.5}$$

We now assume that $j \geq \frac{k}{2}$. In this case we have $n - j r < j r$. Then, the second formula in Eq. 2.6 yields

$$y_1 \cdot x_1^{n-j} x_2^{j r} = (n - j r - d_1 + d_{l-n+j} - c_0 r) x_1^{n-jr-1} x_2^{j r} + c_0 r \sum_{\kappa=1}^{2j} x_1^{n-rj-2} x_2^{j r-2}. \tag{3.6}$$

It follows that if $[y_1 \cdot x_1^{n-j} x_2^{j r}]_{n-lr-1,lr} \neq 0$ then $\kappa = j - l$ for $1 \leq \kappa \leq \left[ \frac{2j-r}{n} \right]$. We notice that $\left[ \frac{2j-r}{n} \right] = 2j - k - 1$. The condition $1 \leq \kappa \leq 2j - k - 1$ is equivalent to $k - l + 1 \leq j \leq k$. For these values of $j$ we have

$$[y_1 \cdot \alpha_k \beta_{k-j} x_1^{n-jr} x_2^{j r}]_{n-lr-1,lr} = c_0 r \alpha_k \beta_{k-j} \text{ for } k - l + 1 \leq j \leq k. \tag{3.7}$$

By combining Eqs. 3.4, 3.5 and 3.7 we obtain the desired formula for $a_l$, for $1 \leq l \leq \frac{k}{2}$. The case $\frac{k}{2} < l \leq k$ is treated similarly and for this reason we omit the details. \hfill \square
Let us to return to the proof of the Proposition. We recall that our goal is to demonstrate that $a_l = 0$ for all $1 \leq l \leq k$. Suppose that $l \leq \frac{k}{2}$. By Claim 3.3 we have

$$a_{k-1+1} = -c_0 r - \sum_{j=1}^{l-1} c_0 r \alpha_j \beta_{j-1} + \sum_{j=k-l+2}^{k} c_0 r \alpha_{k-j} \beta_{k-j} + (c_0 - k + l - 1) r \alpha_{l-1} \beta_{l-1}.$$  

We notice that 

$$\sum_{j=k-l+2}^{k} c_0 r \alpha_{k-j} \beta_{k-j} = \sum_{j=1}^{l-1} c_0 r \alpha_{j-1} \beta_{j-1}.$$  

Then we conclude that

$$a_{k-1+1} = -c_0 r \left(1 + \sum_{j=1}^{l-1} (\alpha_j - \alpha_{j-1}) \beta_{j-1}\right) + (c_0 - k + l - 1) r \alpha_{l-1} \beta_{l-1}. \quad (3.8)$$

We prove that $a_{k-1+1} = 0$ for all $1 \leq l \leq \frac{k}{2}$. We proceed by induction on $l$. For $l = 1$ we have

$$a_k = -c_0 r + (c_0 - k) r \alpha_0 \beta_0 = -c_0 r + (c_0 - k) \frac{c_0}{c_0 - k} = 0.$$  

We now assume that $a_{k-l+1} = 0$. We notice $\alpha_l = \frac{k - l + 1}{l} \alpha_{l-1}$ and $\beta_l = \frac{c_0 - l}{c_0 - k + l} \beta_{l-1}$. Then, Claim 3.3 yields

$$a_{k-(l+1)+1} = a_{k-l}$$

$$= -c_0 r \left(1 + \sum_{j=1}^{l-1} (\alpha_j - \alpha_{j-1}) \beta_{j-1}\right) + (c_0 - (k-l)) r \alpha_l \beta_l$$

$$= -c_0 r \left(1 + \sum_{j=1}^{l-1} (\alpha_j - \alpha_{j-1}) \beta_{j-1} + (\alpha_l - \alpha_{l-1}) \beta_{l-1}\right) + (c_0 - (k-l)) r \alpha_l \beta_l$$

$$= -c_0 r \left(1 + \sum_{j=1}^{l-1} (\alpha_j - \alpha_{j-1}) \beta_{j-1}\right) + (c_0 - (k-l+1)) r \alpha_{l-1} \beta_{l-1}$$

$$- (c_0 - (k-l+1)) r \alpha_{l-1} \beta_{l-1} - c_0 r (\alpha_l - \alpha_{l-1}) \beta_{l-1} + (c_0 - (k-l)) r \alpha_l \beta_l$$

$$= a_{k-l+1} - c_0 r \alpha_{l-1} \beta_{l-1} + (k-l+1) r \alpha_{l-1} \beta_{l-1} - c_0 r \left(\frac{k - l + 1}{l} \alpha_{l-1} - \alpha_{l-1}\right) \beta_{l-1}$$

$$+ (c_0 - k + l) \frac{k - l + 1}{l} r \alpha_{l-1} \frac{c_0 - l}{c_0 - k + l} \beta_{l-1}$$

$$= \left(-c_0 + k - l + 1 - c_0 \frac{k - l + 1}{l} + c_0 + \frac{(k - l + 1)(c_0 - l)}{l}\right) r \alpha_{l-1} \beta_{l-1}$$

$$= \frac{(k - l + 1)(l) - c_0(k - l + 1) + (k - l + 1)(c_0 - l)}{l} r \alpha_{l-1} \beta_{l-1}$$

$$= 0.$$
This completes the proof of \( a_{k-l+1} = 0 \) for \( 1 \leq l \leq \frac{k}{2} \). We now prove that \( a_l = a_{k-l+1} \) for \( 1 \leq l \leq \frac{k}{2} \). By Claim 3.3 we note that

\[
a_l = -l r \alpha_i \beta_{i-l} - c_0 r - \sum_{j=1}^{l-1} c_0 r \alpha_j \beta_{j-l} + \sum_{j=k-l+1}^{k} c_0 r \alpha_{k-j} \beta_{k-j} \\
a_{k-l+1} = -c_0 r - \sum_{j=1}^{l-1} c_0 r \alpha_j \beta_{j-l} + \sum_{j=k-l+2}^{k} c_0 r \alpha_{k-j} \beta_{k-j} + (c_0 - k + l - 1)r \alpha_{l-1} \beta_{l-1}.
\]

We conclude that

\[
a_l - a_{k-l+1} = -l r \alpha_i \beta_{i-l} - c_0 r \alpha_{l-1} \beta_{l-1} - (c_0 - k + l - 1)r \alpha_{l-1} \beta_{l-1}. \tag{3.9}
\]

The right-hand side of Eq. 3.9 vanishes since \( l \alpha_l = (k-l+1) \alpha_{l-1} \). Therefore, \( a_l = a_{k-l+1} \) for \( 1 \leq l \leq \frac{k}{2} \). The above shows that \( a_l = 0 \) for all \( 1 \leq l \leq k \). Hence \( \gamma_l \cdot p(x_1, x_2) = 0 \).

The fact that \( y_2 \cdot p(x_1, x_2) = 0 \) follows by using the same argument. The only difference in this case is that we use Eq. 2.7 rather than Eq. 2.6. This completes the proof of the singularity of the polynomial \( p(x_1, x_2) \).

**Proposition 3.4** The following polynomials are singular polynomials in \( \Delta(\lambda^i) \):

(a) \( (x_1^r - x_2^r)^k \otimes v_t \) when \( c_0 = -\frac{k}{2} \) for positive odd \( k \).

(b) \( x_1^n x_2^n \otimes v_t \) when \( n - d_i + d_{i-n} = 0 \)

(c) \( p(x_1, x_2) = \left( x_1^n + \sum_{t=1}^{\left\lceil \frac{k}{2} \right\rceil} \alpha_i \beta_{i-t-1} x_1^{n-tr} x_2^{tr} + \sum_{t=0}^{\left\lceil \frac{k}{2} \right\rceil} \alpha_i \beta_1 x_1^{n-(k-t)r} x_2^{(k-t)r} \right) \otimes v_T, \)

if \( n - d_i + d_{i-n} + c_0 r = 0, n \neq 0 \mod r \) and defining \( k = \left\lceil \frac{n}{r} \right\rceil, \alpha_l = \left( \begin{array}{c} k \\ l \end{array} \right) \) and \( \beta_l = \frac{c_0(c_0 + 1) \ldots (c_0 + t)}{(c_0 + k)(c_0 + (k-1)) \ldots (c_0 + (k-t))} \).

**Proof** The proof in this case is as in the \( \lambda_i \) case. We only need to change \( c_0 \) into \(-c_0\). \( \square \)

**Proposition 3.5** Let \( 0 \leq i < j \leq r - 1 \). The following polynomials are singular in \( \Delta(\lambda_{i,j}) \).

(1) (a) \( p(x_1, x_2) = \left( x_1^n + \sum_{t=1}^{k-1} b_t x_1^{n-tr} x_2^{tr} \right) \otimes v_T + \sum_{t=1}^{k} a_t x_1^{n-tr+j-i} x_2^{tr-j+i} \otimes v_T, \)

if \( n - d_i + d_{i-n} = 0, n + j - i \equiv 0 \mod r \) and \( s_t = j - i - d_j + d_{j-t} - tr \), where \( k = \left\lceil \frac{n + j - i}{r} \right\rceil \).

(b) \( p(x_1, x_2) = \left( x_2^n + \sum_{t=1}^{k-1} b_t x_1^{tr} x_2^{n-tr} \right) \otimes v_T + \sum_{t=0}^{k-1} a_{t+1} x_1^{n-tr+j-i} x_2^{n-tr-j+i} \otimes v_T, \)

if \( n - d_j + d_{j-n} = 0 \) and \( s_t = i - j - d_i + d_{j-(t-1)r} \), where \( k = \left\lceil \frac{n + i - j}{r} \right\rceil + 1. \)

In both cases the coefficients \( a_t \) and \( b_t \) are defined by the following recurrence:

\[
s_1 a_1 = c_0 r \tag{3.10}
\]
(2) \( p(x_1, x_2) = (x_1^n \otimes v_{T_1} - x_2^n \otimes v_{T_2}) + \sum_{l=1}^{k} a_l \left( x_1^{n-r_l} x_2^{r_l} \otimes v_{T_1} - x_1^{r_l} x_2^{n-r_l} \otimes v_{T_2} \right) \),

if \( n = i - j + (k + 1)r \) and \( n = d_i - d_j + r c_0 \).

(b) \( p(x_1, x_2) = (x_1^n \otimes v_{T_2} - x_2^n \otimes v_{T_1}) + \sum_{l=1}^{k} a_l \left( x_1^{n-r_l} x_2^{r_l} \otimes v_{T_2} - x_1^{r_l} x_2^{n-r_l} \otimes v_{T_1} \right) \),

if \( n = j - i + kr \) and \( n = d_j - d_i + r c_0 \).

In both cases the coefficients are given by:

\[
\begin{align*}
    a_l &= \frac{1}{l!} \frac{c_0 (c_0 - 1) \cdots (c_0 - (l - 1)) k (k - 1) \cdots (k - (l - 1))}{(c_0 - k) (c_0 - (k - 1)) \cdots (c_0 - (k - (l - 1)))} \quad \text{for} \quad 1 \leq l \leq \left[ \frac{k}{2} \right] \\
    a_{k-l} &= \frac{1}{l!} \frac{c_0 (c_0 - 1) \cdots (c_0 - (l - 2)) k (k - 1) \cdots (k - (l - 1))}{(c_0 - k) (c_0 - (k - 1)) \cdots (c_0 - (k - (l - 1)))} \quad \text{for} \quad 1 \leq l \leq \left[ \frac{k}{2} \right] \\
    a_k &= \frac{c_0}{c_0 - k}.
\end{align*}
\]

(3) \( p(x_1, x_2) = (x_1^n \otimes v_{T_1} + x_2^n \otimes v_{T_2}) + \sum_{l=1}^{k} a_l \left( x_1^{n-r_l} x_2^{r_l} \otimes v_{T_1} + x_1^{r_l} x_2^{n-r_l} \otimes v_{T_2} \right) \),

if \( n = i - j + (k + 1)r \) and \( n = d_i - d_j - r c_0 \).

(b) \( p(x_1, x_2) = (x_1^n \otimes v_{T_2} + x_2^n \otimes v_{T_1}) + \sum_{l=1}^{k} a_l \left( x_1^{n-r_l} x_2^{r_l} \otimes v_{T_2} + x_1^{r_l} x_2^{n-r_l} \otimes v_{T_1} \right) \),

if \( n = j - i + kr \) and \( n = d_j - d_i - r c_0 \).

In both cases the coefficients are given by:

\[
\begin{align*}
    a_l &= \frac{1}{l!} \frac{c_0 (c_0 + 1) \cdots (c_0 + (l - 1)) k (k - 1) \cdots (k - (l - 1))}{(c_0 + k) (c_0 + (k - 1)) \cdots (c_0 + (k - (l - 1)))} \quad \text{for} \quad 1 \leq l \leq \left[ \frac{k}{2} \right] \\
    a_{k-l} &= \frac{1}{l!} \frac{c_0 (c_0 + 1) \cdots (c_0 + (l - 2)) k (k - 1) \cdots (k - (l - 1))}{(c_0 + k) (c_0 + (k - 1)) \cdots (c_0 + (k - (l - 1)))} \quad \text{for} \quad 1 \leq l \leq \left[ \frac{k}{2} \right] \\
    a_k &= \frac{c_0}{c_0 + k}.
\end{align*}
\]

For the polynomials in (2)(a), (2)(b), (3)(a) and (3)(b), if \( k \) is an even integer we have two different ways to compute \( a_{\frac{k}{2}} \), and both of these give us distinct values. In this case we use the definition of \( a_1 \) rather than \( a_{k-1} \).

**Proof** We begin by noticing that if \( p(x_1, x_2) \) is a singular polynomial then \( cp(x_1, x_2) \) is also singular, for all \( c \in \mathbb{C} \). Since the coefficients occurring in our polynomials are defined by
fractions, it is possible that some of them are undefined for some choices of the parameters. If we are in this case, rather than consider $p(x_1, x_2)$ we consider an scalar multiple of it. More precisely if $c_0 = m, c_0 = -m$ or $s_i = 0$, then we redefine $p(x_1, x_2)$ as $(c_0 - m)p(x_1, x_2), (c_0 + m)p(x_1, x_2)$ or $s_ip(x_1, x_2)$, respectively.

Given a polynomial $q(x_1, x_2) = \sum_{0 \leq i, j} a_i,j x_1^i x_2^j \otimes v_{ij} + b_i,j x_1^i x_2^j \otimes v_{ij} \in \Delta(\lambda_i,j)$ and two non-negative integers $s$ and $t$ we define $[q(x_1, x_2)]_{s,t}^1 := a_{s,t}$ and $[q(x_1, x_2)]_{s,t}^2 := b_{s,t}$.

1. (a) A precise inspection of Proposition 2.6 and the definition of $m)p(x_2$ reveals that

$$y_1 \cdot p(x_1, x_2) = (\alpha_0 x_1^{n-1} + \alpha_1 x_1^{n-r-1} x_2^r + \ldots + \alpha_k x_1^{n-(k-1)r-1} x_2^{(k-1)r}) \otimes v_{T_1} + (\beta_1 x_1^{n+j-l-r} x_2^{l-j+r} + \ldots + \beta_k x_1^{n+j-i-kr-1} x_2^{l-j+kr}) \otimes v_{T_2}$$

Therefore, in order to prove that $p(x_1, x_2)$ is singular we need to show that $\alpha_l$ and $\beta_l$ are both zero. In order to show that $\alpha_l = 0$ for $0 \leq l \leq k - 1$, we need the following result.

**Claim 3.6** The coefficients $\alpha_l$ are given by

$$\alpha_l = \begin{cases} 
-lb_l r - c_0 r \sum_{l=1}^k a_l + c_0 r \sum_{l=k-l+1}^k a_l, & \text{if } 0 \leq l \leq \left[ \frac{k}{2} \right]; \\
-lb_l r - c_0 r \sum_{l=0}^{k-l} a_l + c_0 r \sum_{l=k-l+1}^k a_l, & \text{if } \left[ \frac{k}{2} \right] < l \leq k - 1. 
\end{cases}$$

**Proof** Using Eq. 2.18 we get

$$y_1 \cdot x_1^{n-tr} x_2^r \otimes v_{T_1} = (n-tr - d_i + d_i - n + tr)x_1^{n-tr-1} x_2^r \otimes v_{T_1} + X \otimes v_{T_2},$$

where $X$ is some polynomial in $\mathbb{C}[x_1, x_2]$. It follows that if $[y_1 \cdot x_1^{n-tr} x_2^r \otimes v_{T_1}]_{n-tr-1,lr}^1 \neq 0$ then $t = l$. If $t = l$ then, by using the identity $d_i - n + lr = d_i - n$ and the condition $n - d_i + d_i - n = 0$, we have that

$$\left[ y_1 \cdot \left( x_1^n + \sum_{t=1}^{k-1} b_t x_1^{n-tr} x_2^r \right) \otimes v_{T_1} \right]_{n-tr-1,lr}^1 = b_l(n - lr - d_i + d_i - n + lr) = -b_l r.$$  

(3.18)

Now we want to compute $[y_1 \cdot x_1^{n-tr+j-i} x_2^{tr-j+i} \otimes v_{T_2}]_{n-tr-j+i}^1$. First suppose that $t \leq \left[ \frac{k}{2} \right]$. This condition implies that $2tr \leq kr$. Furthermore, the definition of $k$ implies $kr < n + j - i$ so we conclude that $2tr < n + j - i$. The above is equivalent to $2tr - n - j + i < 0$.

Now suppose that $n - tr + j - i \leq tr - j + i$. In this case we must use the second part of Eq. 2.19. The floor function over the sum in Eq. 2.19 is

$$\frac{tr - j + i - n + tr - j + i + j - i}{r}.$$  

(3.19)
The numerator in Eq. 3.19 reduces to \(2t^r - n - j + i\), which is negative. Therefore we conclude that there are no \(\otimes v_T\) terms in this case. By this observation we can assume that \(n - tr + j - i > tr + j + i\). The first part of Eq. 2.19 yields

\[
y_1 \cdot x_1^{n-tr+j-i} x_2^{tr-j+i} \otimes v_T = X \otimes v_T - c_0 r \sum_{\kappa=0}^{[n-2tr+j-i-1]} \frac{1}{x_1^{n-tr-\kappa-1} x_2^{tr+\kappa} \otimes v_T},
\]

where \(X\) is some polynomial in \(\mathbb{C}[x_1, x_2]\). It follows that if \([y_1 \cdot x_1^{n-tr+j-i} x_2^{tr-j+i} \otimes v_T]_{n-tr-1, lr} \neq 0\) then \(\kappa = l - t\) for some \(0 \leq \kappa \leq \left\lfloor \frac{n-2tr+j-i-1}{r} \right\rfloor\). The condition \(0 \leq \kappa\) is equivalent to \(t \leq l\). On the other hand, a careful analysis reveals that \(\kappa \leq \left\lfloor \frac{n-2tr+j-i-1}{r} \right\rfloor\) is equivalent to \(t \leq k - l\). We conclude that \(\kappa\) belongs to the required range if and only if \(1 \leq t \leq \mu\), where \(\mu = \min(l, k - l)\).

If \(l \leq \left\lfloor \frac{k}{2} \right\rfloor\) we have \(\mu = l\) and \([y_1 \cdot x_1^{n-tr} x_2^{tr} \otimes v_T]_{n-tr-1, lr} = -c_0 r\) for each \(1 \leq t \leq l\). The above implies that

\[
\left[ y_1 \cdot \sum_{t=1}^{\left\lfloor \frac{k}{2} \right\rfloor} a_t x_1^{n-tr+j-i} x_2^{tr-j+i} \otimes v_T \right]_{n-tr-1, lr}^1 = -c_0 r \sum_{t=1}^{l} a_t. \tag{3.21}
\]

If \(l > \left\lfloor \frac{k}{2} \right\rfloor\) we have that \(\mu = k - l\) and \([y_1 \cdot x_1^{n-tr} x_2^{tr} \otimes v_T]_{n-tr-1, lr} = -c_0 r\) for each \(1 \leq t \leq k - l\). The above implies that

\[
\left[ y_1 \cdot \sum_{t=1}^{\left\lfloor \frac{k}{2} \right\rfloor} a_t x_1^{n-tr+j-i} x_2^{tr-j+i} \otimes v_T \right]_{n-tr-1, lr}^1 = -c_0 r \sum_{t=1}^{k-l} a_t. \tag{3.22}
\]

Let us now assume that \(t > \left\lfloor \frac{k}{2} \right\rfloor\). By arguing as in the previous case and considering the second part of Eq. 2.19 we obtain that

\[
\left[ y_1 \cdot \sum_{t=\left\lceil \frac{k}{2} \right\rceil+1}^{k} a_t x_1^{n-tr+j-i} x_2^{tr-j+i} \otimes v_T \right]_{n-tr-1, lr}^1 = \begin{cases} 
  c_0 r \sum_{t=k-l+1}^{k} a_t, & \text{if } 0 \leq l \leq \left\lceil \frac{k}{2} \right\rceil; \\
  c_0 r \sum_{t=l+1}^{k} a_t, & \text{if } \left\lfloor \frac{k}{2} \right\rfloor < l \leq k - 1.
\end{cases} \tag{3.23}
\]

This completes the proof of the Claim. \(\square\)

In order to show that \(\beta_l = 0\) for \(1 \leq l \leq k\), we need the following result.
Claim 3.7 The coefficients $\beta_l$ are given by

$$\beta_l = \begin{cases} 
  s_j a_l - c_0 r - c_0 r \sum_{t=1}^{l-1} b_t + c_0 r \sum_{t=k-l+1}^{k-1} b_t, & \text{if } 1 \leq l \leq \left[ \frac{k}{2} \right]; \\
  s_j a_l - c_0 r - c_0 r \sum_{t=1}^{k-l} b_t + c_0 r \sum_{t=l}^{k-1} b_t, & \text{if } \left[ \frac{k}{2} \right] < l \leq k.
\end{cases} \quad (3.24)$$

Proof The proof of this claim is treated similarly and for this we omit details.

Let us to return to the proof of the Proposition. We recall that our goal is to demonstrate that $\alpha_l$ and $\beta_l$ are both zero.

For this we first observe that $\beta_1 = s_1 a_1 - c_0 r = 0$ because of Eq. 3.10. Now if $1 \leq l \leq \left[ \frac{k}{2} \right]$, the definition of $\beta_l$ and Eq. 3.11 yields

$$\beta_l - \beta_{k-l+1} = s_l a_l - s_{k-l+1} a_{k-l+1} = 0. \quad (3.25)$$

We have that $\beta_l = \beta_{k-l+1}$ for $1 \leq l \leq \left[ \frac{k}{2} \right]$. This implies that we only need to prove that $\beta_l = 0$ for $1 \leq l \leq \left[ \frac{k}{2} \right]$. To see this we notice that

$$\beta_l = s_l a_l - c_0 r - c_0 r \sum_{t=1}^{l-1} b_t + c_0 r \sum_{t=k-l+1}^{k-1} b_t = s_l a_l - c_0 r - c_0 r \sum_{t=1}^{l-1} b_t + c_0 r \sum_{t=1}^{l-1} b_{k-t}. \quad (3.26)$$

So far we have just rewritten the last sum of the definition of $\beta_l$. By Eq. 3.12 we obtain

$$\beta_l = s_l a_l - c_0 r - c_0 r \sum_{t=1}^{l-1} \left( \frac{k-t}{t} - 1 \right) b_{k-t} = s_l a_l - c_0 r - c_0 r \sum_{t=1}^{l-1} \left( \frac{k-2t}{t} \right) b_{k-t}. \quad (3.27)$$

Finally, Eq. 3.13 yields

$$\beta_l = s_l \left( \frac{c_0 r}{s_l} \left( 1 + \sum_{t=1}^{l-1} \frac{k-2t}{t} b_{k-t} \right) \right) - c_0 r - c_0 r \sum_{t=1}^{l-1} \left( \frac{k-2t}{t} \right) b_{k-t} = 0. \quad (3.28)$$

Therefore we have proved that $\beta_l = 0$ for $1 \leq l \leq k$.

We now focus on to show that $\alpha_l = 0$ for $0 \leq l \leq k - 1$. By definition $\alpha_0 = 0$. Now if $1 \leq l \leq \left[ \frac{k}{2} \right]$, the definition of $\alpha_l$ and Eq. 3.12 yields

$$\alpha_l - \alpha_{k-l} = -l b_l r + (k - l) b_{k-l} = 0. \quad (3.29)$$

We have that $\alpha_l = \alpha_{k-l}$ for $1 \leq l \leq \left[ \frac{k}{2} \right]$. This implies that we only need to prove that $\alpha_l = 0$ for $1 \leq l \leq \left[ \frac{k}{2} \right]$. To see this we notice that

$$\alpha_l = -l b_l r - c_0 r \sum_{t=1}^{l} a_t + c_0 r \sum_{t=k-l+1}^{k} a_t = -l b_l r - c_0 r \sum_{t=0}^{l-1} a_{t+1} + c_0 r \sum_{t=0}^{l-1} a_{k-t}. \quad (3.30)$$
So far we have just rewritten the last sum of the definition of $\alpha_l$. By Eq. 3.11 we obtain
\[ \alpha_l = -lbfr - c_0r \sum_{t=0}^{l-1} \left( 1 - \frac{s_{t+1}}{s_{k-t}} \right) a_{t+1} = -lbfr + c_0r \sum_{t=0}^{l-1} \left( \frac{(k-2t-1)r}{s_{k-t}} \right) a_{t+1}. \]

Finally, Eq. 3.14 yields
\[ \alpha_l = -lr \left( \frac{c_0}{r} \sum_{t=0}^{l-1} \left( \frac{(k-2t-1)r}{s_{k-t}} \right) a_{t+1} \right) + c_0r \sum_{t=0}^{l-1} \left( \frac{(k-2t-1)r}{s_{k-t}} \right) a_{t+1} = 0. \]

Therefore we have proved that $\alpha_l = 0$ for $0 \leq l \leq k$. Summing up we have proved that the polynomial $p(x_1, x_2)$ is singular.

(b) The arguments given in the previous case works for this case as well.

(2) (a) A precise inspection of Proposition 2.6 and the definition of $p(x_1, x_2)$ reveals that
\[
y_1 \cdot p(x_1, x_2) = (\alpha_0 x_1^{n-1} + \alpha_1 x_1^{n-r-1} x_2 + \ldots + \alpha_k x_1^{n-r+1} x_2^k) \otimes v_T_1 + (\beta_1 x_1^{n-r} + \ldots + \beta_k x_1^{n-r+1} x_2^k) \otimes v_T_2 \]

Therefore, in order to prove that $p(x_1, x_2)$ is singular we need to show that $\alpha_l$ and $\beta_l$ are both zero. In order to show that $\alpha_l = 0$ for $0 \leq l \leq k - 1$, we need the following result.

**Claim 3.8** The coefficient $\alpha_l$ are given by
\[
\alpha_l = \left\{ \begin{array}{ll}
& c_0r - cr = 0 \quad \text{if} \quad l = 0; \\
& a_1(c_0r - lr) - c_0r - cr \sum_{t=1}^{l} a_t + c_0r \sum_{t=k-l+1}^{k} a_t, \quad \text{if} \quad 1 \leq l \leq \left\lfloor \frac{k}{2} \right\rfloor; \\
& a_1(c_0r - lr) - c_0r - cr \sum_{t=1}^{k-l} a_t + c_0r \sum_{t=k-l+1}^{k} a_t, \quad \text{if} \quad \left\lfloor \frac{k}{2} \right\rfloor < l \leq k.
\end{array} \right.
\]

**Proof** Using Eq. 2.18 we get
\[
y_1 \cdot x_1^{n-lr} x_2^r \otimes v_T_1 = (n - lr - d_l + d_{l-n+lr}) x_1^{n-lr-1} x_2^r \otimes v_T_1 + X \otimes v_T_2,
\]
where $X$ is some polynomial in $\mathbb{C}[x_1, x_2]$. It follows that if $[y_1 \cdot x_1^{n-lr} x_2^r \otimes v_T_1]_{n-lr-1, lr} \neq 0$ then $t = l$. If $t = l$ then, by using $d_i = d_{i-(i-j+(k+1)r)+lr} = d_j$ and the condition $n = d_i - d_j + c_0r$, we have that
\[
[y_1 \cdot x_1^{n-lr} x_2^r \otimes v_T_1]_{n-lr-1, lr} = c_0r - lr.
\]

This implies that
\[
[y_1 \cdot x_1^n \otimes v_T_1]_{n-1, 0} = c_0r
\]
and
\[
[y_1 \cdot \left( \sum_{t=1}^{k} a_t x_1^{n-lr} x_2^r \otimes v_T_1 \right)]_{n-lr-1, lr} = a_l(c_0r - lr) \quad \text{for} \quad 1 \leq l \leq k.
\]
Now we want to compute \( [y_1 \cdot x_1^{tr} x_2^{n-2tr} \otimes v_{T_2}]_{n-lr-1,lr} \). For this we use Eq. 2.19. First suppose that \( t \leq \left[ \frac{k}{2} \right] \). This condition implies that \( tr < n - tr \). The second part of Eq. 2.19 yields
\[
y_1 \cdot x_1^{tr} x_2^{n-2tr} \otimes v_{T_2} = X \otimes v_{T_2} + c_0 r \sum_{k=1}^{k+1-2t} x_1^{tr+kr-j+i-1} x_2^{n-2tr+kr+j-i} \otimes v_{T_1},
\]
where \( X \) is some polynomial in \( \mathbb{C}[x_1, x_2] \). A consequence if the last equation is that
\[
[y_1 \cdot (x_2^n \otimes v_{T_2})]_{n-lr-1,lr} = -c_0 r. \quad \text{for} \ 0 \leq l \leq k.
\]
We have that \( tr + kr - j + i - 1 = n + (t + k - 1)r - 1 \). It follows that if \( [y_1 \cdot x_1^{tr} x_2^{n-2tr} \otimes v_{T_2}]_{n-lr-1,lr} \neq 0 \) then \( \kappa = k + 1 - tr - l \) for some \( 1 \leq \kappa \leq k + 1 - 2t \). If \( l \leq \left[ \frac{k}{2} \right] \) we conclude that \( \kappa \) belongs to the required range if and only if \( 1 \leq t \leq k \). The above implies that
\[
\left[ y_1 \cdot \left( -\sum_{t=1}^{\left[ \frac{k}{2} \right]} a_t x_1^{tr} x_2^{n-2tr} \otimes v_{T_2} \right) \right]_{n-lr-1,lr} = \sum_{t=1}^{l} -c_0 r a_t. \quad (3.41)
\]
If \( l > \left[ \frac{k}{2} \right] \) we conclude that \( \kappa \) belongs to the required range if and only if \( 1 \leq t \leq k - l \). The above implies that
\[
\left[ y_1 \cdot \left( -\sum_{t=1}^{\left[ \frac{k}{2} \right]} a_t x_1^{tr} x_2^{n-2tr} \otimes v_{T_2} \right) \right]_{n-lr-1,lr} = \sum_{t=1}^{k-l} -c_0 r a_t. \quad (3.42)
\]
Let us now assume that \( t > \left[ \frac{k}{2} \right] \). By arguing as in the previous case and considering the first part Eq. 2.19 we obtain if
\[
\left[ y_1 \cdot \sum_{t=\left[ \frac{k}{2} \right]+1}^{k} a_t x_1^{tr} x_2^{n-2tr} \otimes v_{T_2} \right]_{n-lr-1,lr} = \begin{cases} c_0 r \sum_{t=k-l+1}^{k} a_t, & \text{if } 0 \leq l \leq \left[ \frac{k}{2} \right]; \\ c_0 r \sum_{t=l+1}^{k} a_t, & \text{if } \left[ \frac{k}{2} \right] < l \leq k. \end{cases} \quad (3.43)
\]
This completes the proof of the Claim.

\( \square \)

In order to show that \( \beta_l = 0 \) for \( 1 \leq l \leq k \), we need the following result.

**Claim 3.9** The Coefficient \( \beta_l \) are given by
\[
\beta_l = \begin{cases} -a_l r - c_0 r \sum_{t=1}^{k-l} a_t + c_0 r \sum_{t=k-l+1}^{k} a_t, & \text{if } 1 \leq l \leq \left[ \frac{k}{2} \right]; \\ -a_l r - c_0 r \sum_{t=1}^{k-l} a_t + c_0 r \sum_{t=l}^{k} a_t, & \text{if } \left[ \frac{k}{2} \right] < l \leq k. \end{cases} \quad (3.44)
\]
Proof The proof of this claim is treated similarly and for this we omit details.

Let us to return to the proof of the Proposition. We recall that our goal is to demonstrate that $a_l$ and $\beta_l$ are both zero. For this we first observe that $a_0 = 0$ by Claim 3.8. It is easy to see that $a_1 = \beta_1$ for $1 \leq l \leq k$. Therefore, we are done if we are able to prove that $\beta_1 = 0$.

Arguing as before we can see that $\beta_l = \beta_{k-l+1}$. Summing up we only have to show that $\beta_l = 0$ for $1 \leq l \leq \left[ \frac{k}{2} \right]$. We proceed by induction on $l$. If $l = 1$ we have

$$
\beta_1 = -a_1 r - c_0 r + c_0 r c_0 = -\frac{c_0 k}{c_0 - k} r - c_0 r + c_0 r \frac{c_0}{c_0 - k} = 0.
$$

Let $m > 1$ and suppose that

$$
\beta_m = -a_m mr - c_0 r - c_0 r \sum_{t=1}^{m-1} a_t + c_0 r \sum_{t=m+1}^k a_t = 0.
$$

On the other hand, by using the definition of $a_l$ we can see that

$$
a_l = a_{k-l+1}(k-l+1) \quad \text{and} \quad a_l \frac{(c_0 - l)(k-l)}{(l+1)(c_0 - (k-l))} = a_{l+1}
$$

By combining Eqs. 3.46 and 3.47 we have

$$
\beta_{m+1} = -a_{m+1} (m+1) r - c_0 r - c_0 r \sum_{t=1}^{m-1} a_t + c_0 r \sum_{t=m+1}^k a_t
$$

$$
= -a_{m+1} (m+1) r - c_0 r a_m + c_0 r a_{k-m} + a_m mr - a_m mr - c_0 r
$$

$$
= -a_{m+1} (m+1) r + a_m (m - c_0) r + c_0 r a_{k-m} + \beta_m
$$

$$
= -a_{m+1} (m+1) r + a_{m+1} \frac{(m+1)(c_0 -(k-m))}{(c_0-m)(k-m)} (m-c_0) r + c_0 r a_{m+1} \frac{m+1}{k-m}
$$

$$
= a_{m+1} r \left( \frac{c_0(m+1)}{k-m} + \frac{(m+1)(c_0 -(k-m))}{(c_0-m)(k-m)} (m-c_0) - (m+1) \right)
$$

$$
= 0.
$$

Therefore $\beta_l = 0$ for $1 \leq l \leq k$ and this completes the proof in this case.

(b) The proof of this case follows in a very similar way to the last one.

(3) The proof of this case follows in a very similar way to case (2). 

---

Example 3.10 We illustrate how to compute the polynomial $p(x_1, x_2)$ considered in the case (1)(a) of Proposition 3.5. Suppose that we have the following data:

- $r = 4$
- $d_0 = 13$
- $d_1 = -13$
- $d_2 = 0$
- $d_3 = 0$
- $c_0 = -3$

If we consider $n = 13$ for $\lambda_{0,1}$ then we have that

$$
n - d_i + d_{i-n} = 13 - d_0 + d_{0-13} = 13 - 13 - 0 = 0
$$
and
\[ k = \left\lceil \frac{n + j - i}{r} \right\rceil = \left\lceil \frac{13 + 1 - 0}{4} \right\rceil \]
thus \( k = 3 \). The polynomial annihilated is:
\[ p(x_1, x_2) = \left( x_1^{13} + b_1 x_1^9 x_2^4 + b_2 x_1^5 x_2^8 \right) \otimes \nu T_1 + \left( a_1 x_1^{10} x_2^3 + a_2 x_1^6 x_2^7 + a_3 x_1^2 x_2^{11} \right) \otimes \nu T_2. \]
We need to compute the coefficients. In this case
\[ s_1 = 23, \ s_2 = 19, \ s_3 = 15. \]
It follows that Eq. 3.10 implies
\[ a_1 = -\frac{12}{23}. \]
Using Eq. 3.11 we have \( s_1 a_1 = s_3 a_3 \). This implies that
\[ a_3 = -\frac{4}{5}. \]
If we compute \( b_1 \) using Eq. 3.14 we have \( b_1 = c_0 \left( \frac{2 \epsilon}{\delta^0} \right) a_1 \). This implies that
\[ b_1 = \frac{96}{115}. \]
Using Eq. 3.12 we have that \( b_1 = 2b_2 \). This implies
\[ b_2 = \frac{48}{115}. \]
We finish computing \( a_2 \).
\[ a_2 = \frac{c_0 r}{s_2} (b_2 + 1) = -\frac{1956}{2185}. \]
Finally the polynomial is:
\[ p(x_1, x_2) = \left( x_1^{13} + \frac{96}{115} x_1^9 x_2^4 + \frac{48}{115} x_1^5 x_2^8 \right) \otimes \nu T_1 - \left( \frac{12}{23} x_1^{10} x_2^3 + \frac{1956}{2185} x_1^6 x_2^7 + \frac{4}{5} x_1^2 x_2^{11} \right) \otimes \nu T_2. \]

4 Main Theorems

If we have an \( r \)-partition \( \lambda = (\lambda^{(0)}, \lambda^{(1)}, ..., \lambda^{(r-1)}) \), define the content of a box \( b \in \lambda^{(i)} \) to be the integer \( ct(b) := j - k \), if \( b \) is in the \( k \) row and in the \( j \) column from \( \lambda^{(i)} \). If \( T \) is a standard Young tableau associated to \( \lambda \), let be \( T(i) \) for the box \( b \) of \( \lambda \), in which \( i \) appears.
And define the function \( \beta \) over the set of all boxes of \( \lambda \) as follows:
\[ \beta(b) = i \text{ if } b \in \lambda^{(i)}. \]
We also define the charged content \( c(b) \) of a box \( b \) of \( \lambda \) by the equation
\[ c(b) = ct(b)r c_0 + d_\beta(b). \]
Now we are able to enunciate Theorem 5.1 of [8].

**Theorem 4.1** If there is a non-zero morphism \( \Delta(\lambda) \rightarrow \Delta(\mu) \), then there are \( T \in SYT(\lambda) \) and \( U \in SYT(\mu) \) such that for all \( i \) we have
\[ c(U(i)) - c(T(i)) \in \mathbb{Z}_{\geq 0} \text{ and } c(U(i)) - c(T(i)) = \beta(U(i)) - \beta(T(i)) \mod r. \]
We use this theorem to prove the necessary conditions for the existence of morphisms between standard modules and for the sufficient conditions we use the singular polynomials described in Propositions 3.2, 3.4 and 3.5.

**Theorem 4.2** The necessary and sufficient conditions for the existence of a morphism between standard modules for $G(r, 1, 2)$ are shown in the followings tables:

| $\Delta(\lambda_i)$ | $\Delta(\lambda_j)$ | $\Delta(\lambda')$ | $\Delta(\lambda^j)$ | $\Delta(\lambda_{i,j})$ | $\Delta(\lambda_{i,k})$ |
|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| $\Delta(\lambda_i)$ | $d_j - d_i$ | $c_0 = -k_2$ | $d_j - d_i$ | $d_j - d_i - c_0 r$ | $d_j - d_i$ |
| $\Delta(\lambda^j)$ | $c_0 = -k_2$ | $d_j - d_i$ | $d_j - d_i + c_0 r$ | $d_k - d_j - c_0 r$ |
| $\Delta(\lambda_{i,j})$ | $d_j - d_i - c_0 r$ | $d_j - d_i$ | $d_j - d_i + c_0 r$ | $d_k - d_j + c_0 r$ |
| $\Delta(\lambda_{i,k})$ | $d_j - d_i$ | $d_k - d_j$ | $d_k - d_j$ |

Columns represent the domain, rows represent the codomain and the entries represent conditions on the parameters. If more than one condition appears, this means that both must hold. The dots mean that there is no condition. The condition $d_j - d_i$ means that $d_j - d_i \in \mathbb{Z}_{\geq 0}$ and $d_j - d_i = i - j \mod r$. The condition $d_j - d_i \pm c_0 r$ means $d_j - d_i \pm c_0 r \in \mathbb{Z}_{\geq 0}$ and $d_j - d_i \pm c_0 r = i - j \mod r$. Finally the conditions $c_0 = \pm k_2$ means that $c_0 = \pm k_2$ for $k$ a positive odd integer.

**Proof** For the necessary conditions we use Theorem 4.1 attached to our $r$-partitions. This theorem gives us almost all the conditions. Nevertheless in the cases of $\lambda_i \rightarrow \lambda_j$ and $\lambda^j \rightarrow \lambda_i$ the theorem shows in the first case that $c_0 = -k_2$ and in the second case that $c_0 = k_2$, without the condition that $k$ is odd. To get this condition we apply Theorem 1.2 of [8] with $G_S = G(1, 1, 2)$ to obtain a nonzero morphisms $\Delta c_0(sign) \rightarrow \Delta c_0(triv)$ for the rational Cherednik algebra for $G(1, 1, 2)$. This implies that $c_0 = k_2$ with the condition that $k$ is odd.

To prove that these conditions are sufficient we construct explicit homomorphisms using the singular polynomials described in Propositions 3.2, 3.4 and 3.5. We start with the cases that only have one condition.

1) $\Delta(\lambda_i) \rightarrow \Delta(\lambda_{i,j})$.
   In this case the condition is $d_j - d_i$. If we use $n = d_j - d_i$, we have the condition (b) of Proposition 3.2. In this case the morphism is given by sending $1 \otimes v_T \rightarrow x_1^a x_2^b \otimes v_T$.

2) $\Delta(\lambda_i) \rightarrow \Delta(\lambda^j)$.
   In this case the condition is $c_0 = -k_2$. We have the condition (a) of Proposition 3.4. In this case the morphism is given by sending $1 \otimes v_T \rightarrow (x_1^a - x_2^b)^k \otimes v_T$.

3) $\Delta(\lambda_j) \rightarrow \Delta(\lambda_{i,j})$.
   In this case the condition is $d_j - d_i - c_0 r$. We have two options:
(a) \( i < j \). If we use \( n = d_j - d_i - c_0 r \) we have the condition (3)(b) of Proposition 3.5. In this case the morphism is given by sending \( 1 \otimes v_T \rightarrow p(x_1, x_2) \) where \( p(x_1, x_2) \) is the singular polynomial of case (3)(b) of Proposition 3.5.

(b) \( i > j \). If we use \( n = d_j - d_i - c_0 r \), we have the condition (3)(a) of Proposition 3.5. In this case the morphism is given by sending \( 1 \otimes v_T \rightarrow p(x_1, x_2) \) where \( p(x_1, x_2) \) is the singular polynomial of case (3)(a) of Proposition 3.5.

4) \( \Delta(\lambda^i) \rightarrow \Delta(\lambda^i_1) \).
   In this case the condition is \( c_0 = \frac{k}{2} \). We have the condition (a) of Proposition 3.2. In this case the morphism is given by sending \( 1 \otimes v_T \rightarrow (x^T_1 - x^T_2)^k \otimes v_T \).

5) \( \Delta(\lambda^i) \rightarrow \Delta(\lambda^i_1) \).
   In this case the condition is \( d_j - d_i \). If we use \( n = d_j - d_i \), we have the condition (b) of Proposition 3.4. In this case the morphism is given by sending \( 1 \otimes v_T \rightarrow x^n_1 x^n_2 \otimes v_T \).

6) \( \Delta(\lambda^i) \rightarrow \Delta(\lambda^i_1) \).
   In this case the condition is \( d_j - d_i + c_0 r \). We have two options:
   (a) \( i < j \). If we use \( n = d_j - d_i + c_0 r \), we have the condition (2)(b) of Proposition 3.5. In this case the morphism is given by sending \( 1 \otimes v_T \rightarrow p(x_1, x_2) \) where \( p(x_1, x_2) \) is the singular polynomial of case (2)(b) of Proposition 3.5.
   (b) \( i > j \). If we use \( n = d_j - d_i - c_0 r \), we have the condition (2)(a) of Proposition 3.5. In this case the morphism is given by sending \( 1 \otimes v_T \rightarrow p(x_1, x_2) \) where \( p(x_1, x_2) \) is the singular polynomial of case (2)(a) of Proposition 3.5.

7) \( \Delta(\lambda^i_{1,j}) \rightarrow \Delta(\lambda^i_1) \).
   In this case the condition is \( d_i - d_j + c_0 r \). If we use \( n = d_i - d_j + c_0 r \) we are in case (c) of Proposition 3.2. We have two options:
   (a) \( i < j \). The morphism is given by sending \( 1 \otimes v_{T_2} \rightarrow p(x_1, x_2) \otimes v_T \) where \( p(x_1, x_2) \) is the singular polynomial of case (c) of Proposition 3.2.
   (b) \( i > j \). The morphism is given by sending \( 1 \otimes v_{T_1} \rightarrow p(x_1, x_2) \otimes v_T \) where \( p(x_1, x_2) \) is the singular polynomial of case (c) of Proposition 3.2.

8) \( \Delta(\lambda^i_{1,j}) \rightarrow \Delta(\lambda^i_1) \).
   In this case the condition is \( d_i - d_j - c_0 r \). If we use \( n = d_i - d_j - c_0 r \) we are in case (c) of Proposition 3.4. We have two options:
   (a) \( i < j \). The morphism is given by sending \( 1 \otimes v_{T_2} \rightarrow p(x_1, x_2) \otimes v_T \) where \( p(x_1, x_2) \) is the singular polynomial of case (c) of Proposition 3.4.
   (b) \( i > j \). The morphism is given by sending \( 1 \otimes v_{T_1} \rightarrow p(x_1, x_2) \otimes v_T \) where \( p(x_1, x_2) \) is the singular polynomial of case (c) of Proposition 3.4.

9) \( \Delta(\lambda^i_{1,j}) \rightarrow \Delta(\lambda^i_{1,k}) \).
   In this case the condition is \( d_k - d_j \). We have four options:
   (a) \( i < j \) and \( i < k \). If we use \( n = d_k - d_j \), we are in case (1)(b) of Proposition 3.5. In this case the morphism is given by sending \( 1 \otimes v_{T_1} \rightarrow p(x_1, x_2) \) where \( p(x_1, x_2) \) is the singular polynomial of case (1)(b) of Proposition 3.5.
   (b) \( i < j \) and \( i > k \). If we use \( n = d_k - d_j \), we are in case (1)(a) of Proposition 3.5. In this case the morphism is given by sending \( 1 \otimes v_{T_2} \rightarrow p(x_1, x_2) \) where \( p(x_1, x_2) \) is the singular polynomial of case (1)(a) of Proposition 3.5.
We continue with the cases that have two conditions. There are seven cases with two conditions:

(a) $\Delta(\lambda_i) \to \Delta(\lambda_j)$ or $\Delta(\lambda_j) \to \Delta(\lambda_i)$.

For $\Delta(\lambda_i) \to \Delta(\lambda_j)$ we have the conditions $d_j - d_i$ and $c_0 = -\frac{k}{2}$. The condition $c_0 = -\frac{k}{2}$ allows the construction of the morphism $\Delta(\lambda_i) \to \Delta(\lambda_j)$. The condition $d_j - d_i$ allows the construction of the morphism $\Delta(\lambda_j) \to \Delta(\lambda_i)$. The composition of these two morphisms is a morphism from $\Delta(\lambda_i)$ to $\Delta(\lambda_j)$. This is a non-zero composition, because it is of the form $1 \otimes v_T \sim p q \otimes v_T$ where $p$ and $q$ are non-zero polynomials.

For $\Delta(\lambda_j) \to \Delta(\lambda_i)$ we use the same arguments as before attached to this case.

(b) $\Delta(\lambda_i) \to \Delta(\lambda_{j,k})$ or $\Delta(\lambda_j) \to \Delta(\lambda_{i,k})$.

For $\Delta(\lambda_i) \to \Delta(\lambda_{j,k})$ we have the conditions $d_j - d_i$ and $d_k - d_j - c_0 r$. The condition $d_j - d_i$ allows the construction of the morphism $\Delta(\lambda_i) \to \Delta(\lambda_{j,k})$. The condition $d_k - d_j - c_0 r$ allows the construction of the morphism $\Delta(\lambda_{j,k}) \to \Delta(\lambda_i)$. The composition of these two morphisms is a morphism from $\Delta(\lambda_i)$ to $\Delta(\lambda_{j,k})$. This is a non-zero composition, because it is of the form $1 \otimes v_T \sim p q \otimes v_T + p r \otimes v_T$ where $p, q, r$ are non-zero polynomials. For $\Delta(\lambda_j) \to \Delta(\lambda_{i,k})$ we use the same arguments as before attached to this case.

(c) $\Delta(\lambda_{i,j}) \to \Delta(\lambda_k)$ or $\Delta(\lambda_{i,j}) \to \Delta(\lambda_{k,j})$.

For $\Delta(\lambda_{i,j}) \to \Delta(\lambda_k)$ we have the conditions $d_k - d_i$ and $d_k - d_i + c_0 r$. The condition $d_k - d_i$ allows the construction of the morphism $\Delta(\lambda_{i,j}) \to \Delta(\lambda_k)$. The condition $d_k - d_i + c_0 r$ allows the construction of the morphism $\Delta(\lambda_{j,k}) \to \Delta(\lambda_k)$. The composition of these two morphisms is a morphism from $\Delta(\lambda_{i,j})$ to $\Delta(\lambda_k)$. This composition is of the form $1 \otimes v_{T_1} \sim (p r + q r') \otimes v_T$ where $p, q, r$ are non-zero polynomials and $r'$ is just interchanging $x_1$ and $x_2$ in $r$. Looking at the coefficients of the polynomials involved we can see that $(p r + q r')$ is a non-zero polynomial. For $\Delta(\lambda_{i,j}) \to \Delta(\lambda_k)$ we use the same arguments as before attached to this case.

(d) $\Delta(\lambda_{i,j}) \to \Delta(\lambda_{k,s})$.

For this case we have the conditions $d_k - d_i$ and $d_s - d_j$ (or $d_s - d_i$ and $d_k - d_j$). The condition $d_k - d_i$ allows the construction of the morphism $\Delta(\lambda_{i,j}) \to \Delta(\lambda_{k,s})$. The condition $d_s - d_j$ allows the construction of the morphism $\Delta(\lambda_{k,s}) \to \Delta(\lambda_{k,j})$. This composition is of the form $1 \otimes v_{T_1} \sim (p r + q r') \otimes v_{T_1} + (p s + q s') \otimes v_{T_2}$ where $p, q, r, s$ are non-zero polynomials and to obtain $r'$ and $s'$ we just interchange $x_1$ and $x_2$ in $r$ and $s$. Looking at the coefficients of the polynomials involved we can see that $(p r + q r')$ or $(p s + q s')$ is a non-zero polynomial. For the condition $d_s - d_i$ and $d_k - d_j$ we can do the same.

\[\Box\]

### 5 Dimension

In this section we establish sufficient conditions so that the dimension of the homomorphisms space between two standard modules is two. We suspect that these sufficient
conditions are also necessary conditions for having a two-dimensional space of morphisms of any standard module.

**Theorem 5.1** If we have the following conditions

- \( d_i - d_k + c_0r \in \mathbb{Z}_{\geq 0} \) and \( d_i - d_k + c_0r \equiv i - k \mod r \).
- \( d_i - d_k - c_0r \in \mathbb{Z}_{\geq 0} \) and \( d_i - d_k - c_0r \equiv i - k \mod r \).
- \( d_j - d_l + c_0r \in \mathbb{Z}_{\geq 0} \) and \( d_j - d_l + c_0r \equiv j - l \mod r \).
- \( d_j - d_l - c_0r \in \mathbb{Z}_{\geq 0} \) and \( d_j - d_l - c_0r \equiv j - l \mod r \).

where \( c_0 \) is a non-zero integer, then we have that

\[
\text{Dim}(\text{Hom}_{\mathbb{Z}}(\Delta(\lambda_{i,k}), \Delta(\lambda_{i,j}))) = 2.
\]

**Proof** We have that this fourth condition allows the construction of morphisms between some standard modules. In particular we have that

(a) \( d_i - d_k + c_0r \) allows a morphisms \( \Delta(\lambda_{i,k}) \to \Delta(\lambda_i) \).

(b) \( d_i - d_k - c_0r \) allows a morphisms \( \Delta(\lambda_{i,k}) \to \Delta(\lambda_i) \).

(c) \( d_j - d_l + c_0r \) allows a morphisms \( \Delta(\lambda_i) \to \Delta(\lambda_{i,j}) \).

(d) \( d_j - d_l - c_0r \) allows a morphisms \( \Delta(\lambda_i) \to \Delta(\lambda_{i,j}) \).

We can see that we have two ways to go from \( \Delta(\lambda_{i,k}) \) to \( \Delta(\lambda_{i,j}) \). We are proving that these two ways are linearly independent. Let

\[
\begin{align*}
   n_1 &= d_i - d_k + c_0r \\
   m_1 &= d_i - d_k - c_0r \\
   n_2 &= d_j - d_l + c_0r \\
   m_2 &= d_j - d_l - c_0r.
\end{align*}
\]

This integers are the degrees of the corresponding singular polynomials of each morphisms. The fact that the composition of morphisms correspond to multiplying the singular polynomials implies that \( n_1 + m_1 = n_2 + m_2 \). This last equality implies that \( n_1 = n_2 \) and \( m_1 = m_2 \). If we call \( p(x_1, x_2) \otimes v_T \) to the corresponding polynomial for \( n_1 \) then we have that \( p(x_1, x_2) \otimes v_{T_1} + q'(x_1, x_2) \otimes v_{T_2} \) is the polynomial corresponding to \( n_2 \), where \( q'(x_1, x_2) \) correspond to the polynomial obtained if we interchange \( x_1 \) by \( x_2 \) and vice versa in \( p(x_1, x_2) \). Using the same arguments we have that if \( q(x_1, x_2) \otimes v_T \) is the polynomial corresponding to \( m_1 \), then \( q(x_1, x_2) \otimes v_{T_1} + q'(x_1, x_2) \otimes v_{T_2} \) is the polynomial corresponding to \( m_2 \). Now the corresponding compositions are:

- \( p(x_1, x_2)q(x_1, x_2) \otimes v_{T_1} + q'(x_1, x_2) \otimes v_{T_2} = p(x_1, x_2)q(x_1, x_2) \otimes v_{T_1} + p(x_1, x_2)q'(x_1, x_2) \otimes v_{T_2} \)

- \( q(x_1, x_2)p(x_1, x_2) \otimes v_{T_1} - p'(x_1, x_2) \otimes v_{T_2} = q(x_1, x_2)p(x_1, x_2) \otimes v_{T_1} - q(x_1, x_2)p'(x_1, x_2) \otimes v_{T_2} \).

Looking at the coefficients of the polynomials involved we can establish that \( p(x_1, x_2)q'(x_1, x_2) \neq -q(x_1, x_2)p'(x_1, x_2) \), and this implies that this two morphisms are linearly independent. In conclusion we have two linearly independent ways to go from \( \Delta(\lambda_{i,k}) \) to \( \Delta(\lambda_{i,j}) \). This implies that the dimension of the space of homomorphism is 2.

\[ \square \]

**6 Example**

In this section we give an explicit example.
**Example 6.1** For this example we work with $r = 3$. Suppose first that $10 - d_0 + d_2 = 0$. This is a condition of the form $d_0 - d_2$ and allows the construction of some morphisms. Next we add the condition $5 - d_0 + d_1 = 0$. This is a condition from the form $d_0 - d_1$. With these two conditions we can form a new one by subtracting the second condition from the first one. This new condition is $5 - d_1 + d_2 = 0$ and is from the form $d_1 - d_2$. This way it is possible to construct more morphisms. Finally, if we add the condition $c_0 = 1$, then we have 6 new conditions:

$$13 - d_0 + d_2 - c_0r = 0 \quad (d_0 - d_2 + c_0r)$$
$$7 - d_0 + d_2 + c_0r = 0 \quad (d_0 - d_2 - c_0r)$$
$$8 - d_0 + d_1 - c_0r = 0 \quad (d_0 - d_1 + c_0r)$$
$$2 - d_0 + d_1 + c_0r = 0 \quad (d_0 - d_1 - c_0r)$$
$$8 - d_1 + d_2 - c_0r = 0 \quad (d_1 - d_2 + c_0r)$$
$$2 - d_1 + d_2 + c_0r = 0 \quad (d_1 - d_2 - c_0r)$$

and this allows the construction of 12 new morphisms. Table 1 shows all the morphisms constructed by the corresponding conditions.

In Table 1 we have enlisted all the morphisms. We obtain the diagram in Fig. 1.

Now we describe each of the 21 morphisms using the singular polynomials (Table 2).

In the diagram in Figure 1 there are many morphisms that can be obtained as a composition of two other morphisms. For example, morphism 2 can be obtained by composing morphisms 6 and 7. If we eliminate all the redundant morphisms, then we get the diagram in Fig. 2. In this diagram we have three morphisms from $\Delta(\lambda_{1,2})$ to $\Delta(\lambda_{0,1})$. Any two of

| Conditions | Description | Number |
|------------|-------------|--------|
| $d_0 - d_2$ | $\Delta(\lambda_{2}) \rightarrow \Delta(\lambda_{0})$ | 1 |
| $\Delta(\lambda_{2}) \rightarrow \Delta(\lambda_{0})$ | 2 |
| $\Delta(\lambda_{1,2}) \rightarrow \Delta(\lambda_{0,1})$ | 3 |
| $d_0 - d_1$ | $\Delta(\lambda_{2}) \rightarrow \Delta(\lambda_{1})$ | 4 |
| $\Delta(\lambda_{1}) \rightarrow \Delta(\lambda_{0})$ | 5 |
| $\Delta(\lambda_{2}) \rightarrow \Delta(\lambda_{1})$ | 6 |
| $\Delta(\lambda_{1}) \rightarrow \Delta(\lambda_{0})$ | 7 |
| $\Delta(\lambda_{1,2}) \rightarrow \Delta(\lambda_{0,2})$ | 8 |
| $\Delta(\lambda_{0,2}) \rightarrow \Delta(\lambda_{0,1})$ | 9 |
| $c_0 = 1$ | $\Delta(\lambda_{0,1}) \rightarrow \Delta(\lambda_{0})$ | 10 |
| $d_0 - d_2 + c_0r$ | $\Delta(\lambda_{2}) \rightarrow \Delta(\lambda_{1,2})$ | 11 |
| $\Delta(\lambda_{0,1}) \rightarrow \Delta(\lambda_{0})$ | 12 |
| $\Delta(\lambda_{2}) \rightarrow \Delta(\lambda_{1,2})$ | 13 |
| $\Delta(\lambda_{1,2}) \rightarrow \Delta(\lambda_{0,2})$ | 14 |
| $d_1 - d_2 + c_0r$ | $\Delta(\lambda_{1,2}) \rightarrow \Delta(\lambda_{1})$ | 15 |
| $\Delta(\lambda_{1,2}) \rightarrow \Delta(\lambda_{0,2})$ | 16 |
| $\Delta(\lambda_{1}) \rightarrow \Delta(\lambda_{0,1})$ | 17 |
| $\Delta(\lambda_{0,2}) \rightarrow \Delta(\lambda_{0})$ | 18 |
| $\Delta(\lambda_{0,2}) \rightarrow \Delta(\lambda_{0})$ | 19 |
| $\Delta(\lambda_{0,2}) \rightarrow \Delta(\lambda_{0})$ | 20 |
| $\Delta(\lambda_{2}) \rightarrow \Delta(\lambda_{0,2})$ | 21 |
Fig. 1 Complete diagram of morphisms

Table 2 Singular polynomials

|   |   |
|---|---|
| 1 | $x_1^{10} x_2^{10}$ |
| 2 | $x_1^{10} x_2^{10}$ |
| 3 | $x_1^5 x_2^5$ $\otimes vT_1$ |
| 4 | $x_1^5 x_2^5$ |
| 5 | $x_1^5 x_2^5$ |
| 6 | $x_1^5 x_2^5$ |
| 7 | $x_1^5 x_2^5$ |
| 8 | $(x_1^5 + \frac{1}{5} x_1^7 x_2^3) \otimes vT_1 + (\frac{1}{5} x_1^4 x_2^4 + \frac{1}{3} x_1 x_2^2) \otimes vT_2$ |
| 9 | $(x_1^5 + \frac{1}{5} x_1^7 x_2^3) \otimes vT_1 - (\frac{1}{5} x_1^4 x_2^4 + \frac{1}{3} x_1 x_2^2) \otimes vT_2$ |
| 10 | $x_1^8 - x_1^7 x_2^3 - 2 x_1^5 x_2^5$ |
| 11 | $x_1^2 \otimes vT_1 + x_2^2 \otimes vT_2$ |
| 12 | $x_1^2$ |
| 13 | $(x_1^8 - 3 x_1^7 x_2^3 - x_1^5 x_2^5) \otimes vT_1 - (x_1^8 - 3 x_1^7 x_2^3 - x_1^5 x_2^5) \otimes vT_2$ |
| 14 | $(x_1^7 + \frac{2}{5} x_1^4 x_2^4 + \frac{1}{3} x_1 x_2^2) \otimes vT_1 - (x_1^7 + \frac{2}{5} x_1^4 x_2^4 + \frac{1}{3} x_1 x_2^2) \otimes vT_2$ |
| 15 | $x_1^8 - x_1^7 x_2^3 - 2 x_1^5 x_2^5$ |
| 16 | $x_1^2$ |
| 17 | $(x_1^8 - 2 x_1^7 x_2^3 - x_1^5 x_2^5) \otimes vT_1 - (x_1^8 - 2 x_1^7 x_2^3 - x_1^5 x_2^5) \otimes vT_2$ |
| 18 | $x_1^2 \otimes vT_1 + x_2^2 \otimes vT_2$ |
| 19 | $x_1^{13} - \frac{1}{5} x_1 x_2^2 - \frac{4}{5} x_1^{10} x_2^3$ |
| 20 | $x_1^7 + \frac{1}{5} x_1 x_2^2 - \frac{2}{5} x_1^4 x_2^3$ |
| 21 | $(x_1^{13} - \frac{1}{5} x_1 x_2^2 - \frac{4}{5} x_1^{10} x_2^3) \otimes vT_1 - (x_1^{13} - \frac{1}{5} x_1 x_2^2 - \frac{4}{5} x_1^{10} x_2^3) \otimes vT_2$ |
these morphisms form a basis for $\text{Hom}_{\mathbb{H}}(\Delta(\lambda_{1,2}), \Delta(\lambda_{0,1}))$, so that the dimension of this space is two as predicted by Theorem 5.1.

We conclude this section by providing several examples illustrating our construction. For all these examples we keep the condition $r = 3$ (Figs. 3, 4, 5, 6, 7 and 8).
Fig. 4  $d_0 = -\frac{3}{2}, d_1 = -2, d_2 = \frac{1}{2}, c_0 = \frac{1}{2}$. The only condition satisfied is $d_0 - d_2$

Fig. 5  $d_0 = \frac{1}{2}, d_1 = -1, d_2 = \frac{1}{2}, c_0 = \frac{1}{2}$. The only condition satisfied is $d_0 - d_2 + c_0 r$

Fig. 6  $d_0 = \frac{3}{2}, d_1 = -2, d_2 = \frac{1}{2}, c_0 = \frac{1}{2}$. The conditions are $d_0 - d_2$ and $c_0 = \frac{k}{2}$
Fig. 7  \( d_0 = \frac{5}{3}, d_1 = -\frac{7}{3}, d_2 = \frac{2}{3}, c_0 = \frac{1}{3} \). The conditions are \( d_0 - d_2, d_0 - d_1 + c_0r \) and \( d_2 - d_1 + c_0r \)

Fig. 8  \( d_0 = \frac{3}{4}, d_1 = 1, d_2 = -\frac{7}{4}, c_0 = \frac{1}{2} \). The conditions are \( d_0 - d_2 + c_0r, d_0 - d_2 - c_0r \) and \( c_0 = \frac{k}{2} \)

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