Asymptotics for Kotz Type III Elliptical Distributions

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Abstract: Let $X$ be a Kotz Type III elliptical random vector in $\mathbb{R}^k$, $k \geq 2$, and let $t_n, n \geq 1$ be positive constants such that $\lim_{n \to \infty} t_n = \infty$. In this article we obtain an asymptotic expansion of the tail probability $P\{X > t_n a\}, a \in \mathbb{R}^k$. As an application we derive an approximation for the conditional excess distribution. Furthermore, we discuss the asymptotic dependence of Kotz Type III triangular arrays and provide some details on the estimation of conditional excess distributions and survivor function of Kotz Type III distributions.

Key words and phrases: Exact tail asymptotics; Kotz Type III elliptical distribution; Gumbel max-domain of attraction; maxima of triangular arrays; estimation of joint survivor probability; estimation of conditional excess distribution; quadratic programming.

1 Introduction

Consider a Kotz Type III elliptical random vector $X$ in $\mathbb{R}^k$, $k \geq 2$, with the stochastic representation

$$X \overset{d}{=} A^\top RU,$$

(1.1)

where $A \in \mathbb{R}^{k \times k}$ is a non-singular matrix; the associated random radius $R > 0$ has the tail asymptotic behaviour

$$P\{R > u\} = (1 + o(1))pu^N \exp(-qu^d), \quad \text{with } \delta > 0, p > 0, q > 0, N \in \mathbb{R}, \quad u \to \infty.$$  

(1.2)

Furthermore, $R$ is independent of the random vector $U$ which is uniformly distributed over the unit $k$-sphere of $\mathbb{R}^k$. Here $\overset{d}{=}$ denotes the equality of distribution functions and $^\top$ is the transpose sign.

Prominent examples of the Kotz Type III elliptical random vectors are the Gaussian ones where $R^2$ is chi-squared distributed with $k$ degrees of freedom (see e.g., Kotz et al. (2000)) and the broader class of Kotz Type I elliptical random vectors with $R^2$ a Gamma distributed random variable. See Kotz (1975) and Nadarajah (2003) for the main properties of Kotz Type I elliptical random vectors.

Since $X$ with stochastic representation (1.1) is an elliptical random vector its basic distributional properties are well-known; see e.g., Cambanis et al. (1981), Fang et al. (1990), or Kotz et al. (2000).

The main goal of this paper is to investigate some asymptotical properties of the Kotz Type III elliptical random vectors. In the recent papers Hashorva (2006a, 2007a,b) it is shown that such random vectors have an asymptotic behaviour similar to that of the Gaussian random vectors. This fact is at first sight surprising since we do not specify the distribution of $R$, assuming only the asymptotic relation in (1.2). The main reason for this similarity is the fact that the associated random radius $R$ has distribution function in the Gumbel max-domain of attraction.

Our primary interest in this paper is the tail asymptotic behaviour of $X$. Explicitly, if $t_n a, n \geq 1, a \in \mathbb{R}^k$ are given thresholds in $\mathbb{R}^k, k \geq 2$ such that $\lim_{n \to \infty} t_n = \infty$ and $a$ has at least one positive component, then $\lim_{n \to \infty} P\{X > t_n a\} = 0$. Of interest is therefore to determine the speed of the convergence to $0$ of the tail probability $P\{X > t_n a\}$. The bivariate setup is discussed in Hashorva (2007a). We extend that result to the multivariate setup by utilising some general results for elliptical distributions obtained in Hashorva (2007b).

Two applications of the tail asymptotic expansion which we present here are:

a) an approximation of the conditional excess distribution function, b) the asymptotic dependence of Kotz Type III elliptical triangular arrays.

The Kotz Type distributions are encountered in various statistical applications (see Nadarajah (2003)). In the light of our new asymptotic results it is possible to estimate the survivor function and the conditional excess distribution of a Kotz Type III random vector.

Organisation of the paper: In the next section we present some preliminary results. The exact tail asymptotics is discussed in Section 3 which is then followed by two short sections devoted to approximation of the conditional excess distribution and the asymptotic independence of the Kotz Type III elliptical triangular arrays. In Section 6 we discuss briefly estimation of the survivor function and the conditional excess distribution. Proofs of all the results are relegated to Section 7.
2 Preliminaries

We shall introduce some standard notation. Let \( \mathbf{x} = (x_1, \ldots, x_k) \top \in \mathbb{R}^k \) be a vector in \( \mathbb{R}^k \), \( k \geq 2 \), and let in the following \( I \) be a non-empty index sets of \( \{1, \ldots, k\} \). Denote by \(|I|\) the number of elements of \( I \) and set \( J := \{1, \ldots, k\} \setminus I \). We define the subvector of \( \mathbf{x} \) with respect to \( I \) by \( \mathbf{x}_I := (x_i, i \in I) \top \in \mathbb{R}^k \). If \( A \in \mathbb{R}^{k \times k} \) is a given matrix, then the submatrix \( A_{IJ} \) of \( A \) is obtained by deleting both the rows and the columns of \( A \) with indices in \( J \) and in \( I \), respectively. \( A_{II}, A_{JJ}, A_{IJ} \) are similarly defined. We set \( \Sigma := A^\top A \) where \( A \) is assumed to have a positive determinant \(|A|\) implying that the inverse matrix \( \Sigma^{-1} \) of \( \Sigma \) exists. For notational simplicity we shall write \( \mathbf{x}_J^\top, \Sigma_J \) instead of \( \left( \mathbf{x}_J \right)^\top, \left( \Sigma_{JJ} \right)^{-1} \), respectively. Given \( \mathbf{a}, \mathbf{x}, \mathbf{y} \in \mathbb{R}^k \) we shall define

\[
\mathbf{x} > \mathbf{y}, \text{ if } x_i > y_i, \quad \forall i = 1, \ldots, k, \\
\mathbf{x} \geq \mathbf{y}, \text{ if } x_i \geq y_i, \quad \forall i = 1, \ldots, k, \\
\mathbf{x} + \mathbf{y} := (x_1 + y_1, \ldots, x_k + y_k)^\top, \\
\mathbf{c} \mathbf{x} := (c x_1, \ldots, c x_k)^\top, \quad \mathbf{c} \in \mathbb{R}, \\
\mathbf{0} := (0, \ldots, 0)^\top \in \mathbb{R}^k, \quad 1 := (1, \ldots, 1)^\top \in \mathbb{R}^k, \quad \text{and } \|\mathbf{x}\|^2 := x_1^2 + \cdots + x_k^2.
\]

Without loss of generality we shall assume that \( \Sigma \) is a correlation matrix, i.e., all entries of the main diagonal of \( \Sigma \) are equal to 1. If \( \mathbf{X} \) is an elliptical random vector in \( \mathbb{R}^k, k \geq 2 \) with stochastic representation \((\mathbf{U}, \mathbf{M})\) in view of Lemma 12.1.2 in Berman (1992) we have

\[
X_i \overset{d}{=} \mathbf{R}U_1, \quad 1 \leq i \leq k, \quad (2.3)
\]

where \( X_i \) is the \( i \)-th component of \( \mathbf{X} \) and \( U_1 \) is the first component of \( \mathbf{U} \). If further the associated random radius \( R \) has the tail asymptotics \((2.2)\), then for any \( x \in \mathbb{R} \)

\[
\frac{P\{R > u + x/(q\delta u^{\delta - 1})\}}{P\{R > u\}} \to \exp(-x), \quad u \to \infty, \quad (2.4)
\]

hence \( R \) has distribution function \( F \) in the max-domain of attraction of the Gumbel distribution \( \Lambda(x) = \exp(-\exp(-x)), x \in \mathbb{R} \). From the extreme value theory a distribution function \( F \) with upper endpoint \( \infty \) belongs to the max-domain of attraction of \( \Lambda \) if

\[
\lim_{u \to -\infty} \frac{1 - F(u + x/w(u))}{1 - F(u)} = \exp(-x), \quad \forall x \in \mathbb{R}, \quad (2.5)
\]

where \( w(\cdot) \) is a positive scaling function (see e.g., Resnick (1987), Reiss (1989), Embrechts et al. (1997), Falk et al. (2004), Kotz and Nadarajah (2005) or de Haan and Ferreira (2006)). If the associated random radius \( R \) has tail asymptotics given by \((2.2)\), then \((2.4)\) implies that the distribution function of \( R \) is in the Gumbel max-domain of attraction with the scaling function \( w(\cdot) \) defined by

\[
w(u) = (1 + o(1))q\delta u^{\delta - 1}, \quad u \to \infty. \quad (2.6)
\]

When the associated random radius \( R \) possesses the chi-squared distribution with \( k \) degrees of freedom we obtain

\[
P\{R > u\} = \frac{(1 + o(1))\exp(-u^2/2)u^{k-2}}{2^{k/2-1}\Gamma(k/2)}, \quad u \to \infty,
\]

where \( \Gamma(\cdot) \) is the Gamma function. Hence in this case \((1.2)\) holds with

\[
p := \frac{1}{2^{k/2-1}\Gamma(k/2)}, \quad q := 1/2, \quad \delta := 2, \quad \text{and } N := k - 2 \quad (2.7)
\]

implying that standard Gaussian random vectors belong to the class of the Kotz Type III elliptical random vectors. Tail asymptotics of the Gaussian random vectors is discussed Dai and Mukherjea (2001), Hashorva and Hüsler (2003), and Hashorva (2003, 2005) among several other papers.

The solution of the following quadratic programming problem:

\[
Q(\Sigma, \mathbf{a}) : \text{minimise } \|\mathbf{x}\|^2 \text{ under the linear constraint } \mathbf{x} \geq \mathbf{a}, \quad (2.8)
\]

with \( \mathbf{a} \in \mathbb{R}^k \setminus (-\infty, 0]^k \) is the main ingredient for determining the tail asymptotics under consideration.
Proposition 2.1. Let $\Sigma \in \mathbb{R}^{k \times k}$, $k \geq 2$ be a positive definite correlation matrix and let $a \in \mathbb{R}^k \setminus (-\infty,0]^k$ be a given vector. Then the quadratic programming problem $Q(\Sigma, a)$ has a unique solution $\tilde{a}$ defined by a unique non-empty index set $I \subset \{1,\ldots,k\}$ such that
\[ \min_{x \geq a} \|x\|^2 = \min_{x \geq a} x^\top \Sigma^{-1} x = \| \tilde{a} \|^2 = \| a_l \|^2 = a_l^\top \Sigma^{-1} a_l > 0 \quad (2.9) \]
and in the case when $|I| < k$ we have for $J := \{1,\ldots,d\} \setminus I$
\[ \tilde{a}_J = -((\Sigma^{-1})_{JJ})^{-1}(\Sigma^{-1})_{Jl}a_l = \Sigma_{JJ}^{-1} a_l \geq a_J. \quad (2.11) \]
Furthermore, for any $x \in \mathbb{R}^k$
\[ x^\top \Sigma^{-1} \tilde{a} = x_l^\top \Sigma^{-1} a_l = x_l^\top \Sigma_{ll}^{-1} a_l = \sum_{i \in I} x_i e_i^\top \Sigma^{-1} a_l, \quad (2.12) \]
with $e_i$ being the $i$-th unit vector in $\mathbb{R}^d$ and $e_i^\top \Sigma_{ll}^{-1} a_l > 0, i \in I$. If $a = c1, c \in (0,\infty)$, we have $2 \leq |I| \leq k$
where $|I|$ denotes the number of elements of $I$.

Below we will refer to the index set $I$ as the minimal index set. Note in passing that
\[ \| c x \| = c \| x \|, \quad \forall c > 0, \quad x \in \mathbb{R}^k, \]
hence the unique solution of $Q(\Sigma, ta), t > 0, a \in \mathbb{R}^k$ coincides with the unique solution of $Q(\Sigma, a)$ multiplied by $t$.

Next, we provide a general result for elliptical random vectors which follows from Theorem 3.1 and Theorem 3.4 in Hashorva (2007b). If $t_n$ is a given vector in $\mathbb{R}^d$ then we write for notational simplicity $t_{n,K}$ instead $(t_n)_K$ with $K \subset \{1,\ldots,k\}$.

Theorem 2.2. Let $X \overset{d}{=} A^\top RU$ be an elliptical random vector in $\mathbb{R}^k$, $k \geq 2$ where $R > 0$ has distribution function $F$ with an infinite upper endpoint being independent of $U$ that is uniformly distributed on the unit sphere of $\mathbb{R}^k$ and $A$ is a non-singular $k$-dimensional square matrix such that $\Sigma := A^\top A$ is a correlation matrix. Assume that the distribution function $F$ is in the Gumbel max-domain of attraction with the positive scaling function $w$ and let $t_{n}, n \geq 1$ be positive constants converging to $\infty$. Denote by $I$ the minimal index set of the quadratic programming problem $Q(\Sigma, a), a \in (\mathbb{R}^k \setminus (-\infty,0]^k$. Assume, for simplicity, that $\|a_l\| = 1$. If $t_n, n \geq 1$ are given vectors in $\mathbb{R}^k$ such that
\[ \lim_{n \to \infty} w(t_n)(t_{n,J} - t_{n}a_l) = q_J \in \mathbb{R}^m, \quad (2.13) \]
where $m := |I|$ and furthermore if $m < k$ for $J := \{1,\ldots,k\} \setminus I$
\[ \lim_{n \to \infty} \sqrt{t_n w(t_n)(t_{n,J} - \Sigma_{JJ}^{-1} a_l)} = q_J \in [-\infty,\infty)^{k-m} \quad (2.14) \]
holds, then we have
\[ P\{X > t_n\} = (1 + o(1)) \exp(-q_J^\top \Sigma_{JJ}^{-1} a_l) \frac{\Gamma(k/2)2^{k/2-1}}{(2\pi)^{l/2}|\Sigma_{JJ}|^{1/2}} \prod_{i \in I} a_l \Sigma_{II}^{-1} e_i \times \sqrt{t_n w(t_n)}^{1-(k+m)/2} (1 - F(t_n)), \quad n \to \infty, \quad (2.15) \]
where $Z$ is a standard Gaussian random vector in $\mathbb{R}^k$ with covariance matrix $\Sigma$, $P\{Z_J > q_J|Z_I = 0_I\} = 1$ if $m = k$, and $e_i$ is the $i$-th unit vector in $\mathbb{R}^m$.

3 Exact Tail Asymptotics

As shown in Theorem 2.2, the exact asymptotic behaviour of elliptical random vectors can be derived provided that the distribution function of the associated random radius $R$ is in the Gumbel max-domain of attraction. As mentioned previously the associated random radius of Kotz Type III elliptical random vectors is in the Gumbel max-domain of attraction. This fact and the above theorem lead us to the following result:
Assume for simplicity that the parameters \( q, \delta \) are specified by

\[ Z \] where

\[ \{ X_i \} \in -IR \]

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Theorem 3.1. Let \( X \) be a Kotz Type III elliptical random vector in \( IR^k \), \( k \geq 2 \) where the associated random variable \( R \) has the tail asymptotic given in \( \{ 1 \} \), and \( A \in IR^{k×k} \) be a non-singular matrix with \( \Sigma := A^T A \) a correlation matrix. Let \( t_n, n \geq 1 \) be a positive sequence and let \( a \in IR^k \setminus (-\infty, 0]^k \) be a given vector. Denote by \( I \) the minimal index set related to the quadratic programming problem \( Q(\Sigma, a) \) and define \( v_n \in IR^k, n \geq 1 \) by

\[ (v_n)_I := q\delta(t_n\|a_I\|)\delta^{-1}1_I, \] and for \( |I| < k \) set \( (v_n)_J := \sqrt{q\delta(t_n\|a_I\|)}\delta^{1/2-1}1_J. \]

If \( I \) has \( m \) elements and \( \lim_{n \to \infty} t_n = \infty \), then for any \( x \in IR^k \) we have

\[
P\{ X > t_n a + x/v_n \} = (1 + o(1))p(q\delta)^{(1-\frac{k}{2}(1-k)/2)\|a_I\|^{k+N+\delta(1-k)/2}}/\Gamma(k/2)2^{k/2-1}/(2\pi)^{m/2}\Sigma_{II}^{1/2}1_{I_n}a_j\Sigma_{II}^{-1}e_i \times P\{ Z_J > (a_J - \Sigma_{II}^{-1}a_I) + x_j|Z_J = 0_I \}t_n^{1/2+\delta(1-k)/2}\exp(-q\|a_I\|\delta a_I) = n \to \infty, \quad (3.16)\]

where \( Z \) is a standard Gaussian random variable in \( IR^k \) with covariance matrix \( \Sigma \).

Set \( P\{ Z_J > (a_J - \Sigma_{II}^{-1}a_I) + x_j|Z_J = 0_I \} \) to 1 if \( J \) is empty and substitute above \( + \cdot 0 \) by 0.

Remark 3.2. a) In the above theorem the index sets \( I, J \) do not depend on the choice of \( x \in IR^k \).

b) If \( S \) is a Kotz Type III random vector with covariance matrix \( I \) which is the identity matrix we have by the amalgamation property (see e.g., Fang et al. (1990)) of the spherical random vector that

\[ S \triangleq (I_1|S_1|, \ldots, I_k|S_k|), \]

where \( I_1, \ldots, I_k \) are independent random variables taking values \(-1, 1\) with probability \( 1/2 \). The above theorem can be easily extended to the case of weighted Kotz Type III elliptical random vector defined via the stochastic representation

\[ X \triangleq A(I_1^*|S_1|, \ldots, I_k^*|S_k|) \]

where \( A \) is a \( k \)-dimensional real square matrix and \( I_i^*, i \leq k \) are independent random variables taking two values with \(-1, 1\), with \( P\{ I_i^* = 1 \} \in (0, 1], i \leq k \).

We present next an illustrating example.

Example 1. Let \( X \in IR^k, k \geq 2 \) be a Kotz Type III random vector with underlying covariance matrix \( \Sigma \) specified by

\[ \Sigma = (1 - \rho)I + \rho 11^T, \quad \rho \in (-1/(k-1), 1), \quad (3.17) \]

where \( I \in IR^{k×k} \) is the identity matrix. We are interested on the asymptotic behaviour of \( P\{ X > t_n1 \} \) where \( t_n, n \geq 1 \) are positive constants tending to infinity as \( n \to \infty \). Since necessarily \( \rho > -1/(k-1) \) we have

\[ \Sigma^{-1}1 = (1 - \rho)I + \rho(11^T)^{-1}1 = \frac{1}{1 + (k-1)\rho} > 0, \]

hence the quadratic programming problem \( Q(\Sigma, a) \) has the unique solution \( t_n1 \) with minimal index set \( I = \{1, \ldots, k\} \). Simple calculations yield

\[ \|1\|^2 = 1^T\Sigma^{-1}1 = \frac{k}{1 + (k-1)\rho} =: C_\rho^2 > 0, \]

and

\[ \|\Sigma\| = (1 - \rho)^{(k-1)}(1 + (k-1)\rho) > 0. \]

Assume for simplicity that the parameters \( q, \delta \) defining the tail asymptotic of the associated random radius of \( X \) satisfy \( q\delta = 1 \). Applying Theorem \ref{theo:3.1} we obtain for any \( x \in IR \)

\[
P\{ X > t_n1 + (C_\rho t_n)^{-1}x \} = (1 + o(1))pC_\rho^{(k+N+\delta(1-k)/2)}\Gamma(k/2)2^{k/2-1}/(2\pi)^{k/2}(1 - \rho)^{(k-1)/2} \]


easy to find norming constants h

As demonstrated in the main result of the previous section the asymptotic expansion of the tail probability of

where a

then the vector Σ

In view of Proposition 2.1 if for a given non-empty index set I, J are partitions of \{1, ..., k\}, it is of some interest to investigate the asymptotic behaviour of the distribution function of X_{J} given the partial information X_{I} > ta_{I}, with t tending to infinity. Thomas and Reiss (2007) provide a detailed treatment of statistical applications related to conditional distributions. In the context of extreme value estimation of conditional distribution relies on asymptotic results derived for large thresholds; in our context it means that further investigation of the quantities of interest requires that t tends to infinity. Explicitly, we discuss next the asymptotic properties (n → ∞) of two random sequences V_{n,a}, V_{n:t,a}, n \geq 1, a \in \mathbb{R}^k defined via the stochastic representation

V_{n,a}^d = X - ta_{a} | X > t_{a}a_{a}, \quad \text{and} \quad V_{n:t,a}^d = X_{J} | X_{I} > t_{a}a_{I}, \quad t_{n} \in \mathbb{R}.

The asymptotic behaviour of both random sequences under consideration are closely related to the tail asymptotics of the Kotz Type III elliptical random vectors. We give next our first results.

**Theorem 4.1.** Under the assumptions and the notation of Theorem 3.1 we have the convergence in distribution

\[ v_{n} V_{n,a} \overset{d}{\rightarrow} W_{I}, \quad n \rightarrow \infty, \quad (4.18) \]

where W_{I} has independent components with survivor function exp(-\sigma_{I}^{-1}e_{i}/\|a_{I}\|), i \in I, s > 0. Furthermore if \(|I| < k\), then W_{I} is independent of W_{J} which has partial survivor function

\[ P(W_{L} > x_{L}) = \frac{P(Z_{J} > \infty(a_{J} - \Sigma_{J}^{-1}a_{J}) + x_{J}^{*}|Z_{I} = 0_{I})}{P(Z_{J} > \infty(a_{J} - \Sigma_{J}^{-1}a_{J})|Z_{I} = 0_{I})}, \quad \forall x \in [0, \infty)^{k}, \quad L \subset J \quad (4.19) \]

where x^{*} ∈ \mathbb{R}^{|J|} with x_{L}^{*} := x_{L} and if \(|J| > |L|\) set x_{J\setminus L}^{*} := 0_{J\setminus L}.

As demonstrated in the main result of the previous section the asymptotic expansion of the tail probability of interest is determined via the unique index set I of the related quadratic programming problem. It is therefore easy to find norming constants h_{n}, b_{n}, n \geq 1 so that h_{n}(X - b_{n})_{J} given X_{I} > t_{n}a_{I} converges in distribution if I is the unique index set determining the solution of the quadratic programming problem Q(Σ, a). We arrive thus at the following result:

**Theorem 4.2.** Under the assumptions and the notation of Theorem 3.1 if further the index set J is non-empty, then we have the convergence in distribution

\[ h_{n}(V_{n:t,a} - t_{n}\Sigma_{J}^{-1}a_{I}) \overset{d}{\rightarrow} Z_{J}|Z_{I} = 0_{I}, \quad n \rightarrow \infty, \quad (4.20) \]

with h_{n} := \sqrt{q_{0}(t_{n}|a_{I}|)^{5/2-1}}, n \geq 1.

In view of Proposition 2.1 if for a given non-empty index set I ⊂ \{1, ..., k\} the vector a_{I} ∈ \mathbb{R}^{|I|} is such that

\[ \Sigma_{J}^{-1}a_{J} > 0_{I}, \]

then the vector a^{*} ∈ \mathbb{R}^k with components a_{J}^{*} := a_{J}, a_{J}^{*} := \Sigma_{J}^{-1}a_{J} is the solution of the quadratic programming problem Q(Σ, a^{*}). Consequently the above corollary can be formulated for every vector a and a non-empty index set I such that the vector \Sigma_{J}^{-1}a_{J} has positive components.

Instead of conditioning on the event X_{I} > t_{n}a_{I} which is initially dealt with in Berman (1982, 1983), since X_{I} possesses an absolute continuous distribution function when |I| < k, we may consider conditioning on X_{I} = t_{n}a_{I}. This was first suggested in Hashorva (2006a). We reformulate Theorem 3.2 therein for our specific setup. Define therefore a sequence of random vectors V_{n:t,a}, n \geq 1 in the same probability space such that

\[ V_{n:t,a} \overset{d}{\rightarrow} X_{J}|X_{I} = t_{n}a_{I}, n \geq 1. \]
Theorem 4.3. Let $X, \Sigma, t_n, n \geq 1$ be as in Theorem 4.1. If $I, J$ is a partition of $\{1, \ldots, k\}$ and $a \in \mathbb{R}^k$ is such that $\|a_I\| > 0$, then we have the convergence in distribution

$$h_n \left(V^*_{n,1} - t_n \Sigma_{IJ} \Sigma^{-1}_{IJ} a_I \right) X_I = t_n a_J \quad \xrightarrow{d} \quad Z_J | Z_I = 0, \quad n \to \infty,$$

with $h_n := \sqrt{n} \|a_I\|^{\delta/2} - 1, n \geq 1$ and $Z$ a standard Gaussian random vector in $\mathbb{R}^k$ with covariance matrix $\Sigma$.

Remark 4.4. a) The random vector $Z_J | Z_I = 0$ is a Gaussian random vector in $\mathbb{R}^{|J|}$ with mean zero and positive definite covariance matrix $\Sigma_{IJ} - \Sigma_{II} \Sigma^{-1}_{IJ} \Sigma_{JJ}$.

b) It is remarkable that in both (4.20) and (4.21) the same limiting random vector appears. In Theorem 4.3 the index sets $I, J$ are not related to the quadratic programming problem $Q(\Sigma, a)$ which is the case in Theorem 4.2.

Example 2. Let $X, C_\rho$ be as in Example 1 (recall we set $\rho = 1$). Since the index set $J$ is empty we obtain applying Theorem 4.1

$$(t_n C_\rho)^{\delta-1} V_{n,1} \quad \xrightarrow{d} \quad W, \quad n \to \infty,$$

where $W$ has independent unit Exponential components.

In the Gaussian case $\delta = 2$ hence we have the convergence in distribution

$$t_n C_\rho V_{n,1} \quad \xrightarrow{d} \quad W, \quad n \to \infty.$$  

5 Maxima of Triangular Arrays of Kotz Type III Random Vectors

As the Gaussian distribution, the Kotz Type III multivariate distribution possess some interesting asymptotic properties with respect to the asymptotic dependence and asymptotic behaviour of sample extremes. In order to present those properties, we deal next with a random sequence of Kotz Type III vectors.

Let therefore $X, X_n, n \geq 1$ be independent random vectors in $\mathbb{R}^k, k \geq 2$ with common distribution $G$ such that $X$ has stochastic representation (1.1). In view of (2.3) the marginal distributions of $G$ are identical and furthermore Theorem 12.3.1 in Berman (1992) implies that each marginal distribution is in the Gumbel max-domain of attraction with the scaling function $w(u) = q \delta u^{\delta-1}, u > 0$. For any two components $X_i, X_j, i \neq j, i, j \leq k$ we have applying Theorem 3.3

$$\lim_{u \to \infty} \frac{P\{X_i > u, X_j > u\}}{P\{X_i > u\}} = 0.$$ 

The above asymptotics implies that the sample maxima has independent components (see e.g., Reiss (1989)). Asymptotic independence means that the componentwise sample maxima $M_n, n \geq 1$ converges to a random vector with independent unit Gumbel components. Explicitly, we have

$$\left( (M_{n1} - b_n)/a_n, \ldots, (M_{nk} - b_n)/a_n \right) \quad \xrightarrow{d} \quad (M_1, \ldots, M_k), \quad n \to \infty,$$

where

$$a_n := b_n^{\delta-1}/(q \delta), \quad b_n := G_1^{-1}(1 - 1/n), \quad n > 1,$$

and $G_1^{-1}$ is the inverse of the marginal distribution function $G_1$ of $G$. Hüsler and Reiss (1989) have shown that a triangular array of Gaussian random vectors can be constructed such that the limiting distribution function of the sample maxima is a random vector in $\mathbb{R}^k$ with dependent components and max-stable multivariate distribution which possesses unit Gumbel marginal distribution.

In view of Hashorva (2006b) the same asymptotic results hold in the more general case of the Kotz Type III distribution. Explicitly, let us consider the Kotz Type III multivariate elliptical triangular array with stochastic representation

$$(X_n^{(j)} = X_n^{(j)} \frac{d}{d} A_n^\top R U, \quad 1 \leq j \leq n, \quad n \geq 1, \quad (5.25)$$

where $R > 0$ has tail asymptotic behaviour as in (1.2), independent of $U$ which is uniformly distributed on the unit sphere, and $A_n, n \geq 1$ is a sequence of $k$-dimensional non-singular square matrix. If $\Sigma_n := A_n^\top A_n, n \geq 1$ has all main diagonal entries equal 1, then the convergence in distribution in (5.25) holds provided that

$$\lim_{n \to \infty} (11^\top - \Sigma_n) b_n/2a_n = C \in (0, \infty)^{k \times k}, \quad (5.26)$$
with $a_n, b_n, n \geq 1$ given by
\[ a_n := (q^{-1} \ln n)^{1/\delta - 1} / (q \delta), \quad b_n := (q^{-1} \ln n)^{1/\delta} + a_n \left[ N \ln(q^{-1} \ln n) / \delta + \ln p \right], \quad n > 1. \]
If $q \delta = 1, \delta = 2$, then we have
\[ a_n := (2 \ln n)^{-1/2}, \quad b_n := (2 \ln n)^{1/2} + a_n \left[ N \ln(2 \ln n) / 2 + \ln p \right], \quad n > 1. \]
Condition (5.26) in this case agrees with the one imposed in Hüsler and Reiss (1989) for the Gaussian setup.

The bivariate distribution $\mathcal{G}$ of $(M_1, M_2)$ is given by
\[ G_{\gamma}(x, y) = \exp \left( -\Phi \left( \gamma + \frac{x-y}{2\gamma} \right) \exp(-y) - \Phi \left( \gamma + \frac{y-x}{2\gamma} \right) \exp(-x) \right), \quad x, y \in \mathbb{R}, \]
where $\Phi$ is the standard Gaussian distribution function on $\mathbb{R}$ and $\gamma^2 := \lim_{n \to \infty} (1 - \sigma_{12,n}) \ln n \in (0, \infty)$.

6 Estimation of Joint Survivor and Conditional Excess Distribution

Let $X, X_1, \ldots, X_n$ be independent random vectors in $\mathbb{R}^k, k \geq 2$ with common distribution function $G$ such that $X$ is a Kotz Type III random vector with stochastic representation (1.1) and matrix $A$ such that $\Sigma := A^\top A$ is a positive definite correlation matrix. In view of Lemma 12.1.2 in Berman (1992) the marginal distributions $G_i, i \leq k$ of $G$ are equal. Furthermore, by Theorem 12.3.1 in Berman (1992) we obtain
\[ 1 - G_1(t) = (1 + o(1)) \frac{1}{2 \Gamma(1/2)} (t w(t))^{(k-1)/2} 2^{(k-1)/2} P\{R > t\}, \quad t \to \infty. \quad (6.27) \]

In various statistical applications given the finite sample $X_1, \ldots, X_n, n > 1$, estimation of the joint survivor probability and the conditional excess $\psi_t, \psi_{t,x}^*$
\[ \psi_t := P\{X > t1\}, \quad \psi_{t,x}^* := P\{X - t1 > x|X > t1\}, \quad x \in \mathbb{R}^k \]
for $t$ large enough is of certain interest. Let in the following $\hat{\psi}_{n,t}, \hat{\psi}_{n,t,x}^*$ denote two estimators of these quantities which we specify below. In view of our asymptotic results in order to estimate $\psi_t$ we need to estimate $p, N$ and $q, \delta, \Sigma^{-1}$, whereas for estimating $\psi_{t,x}^*$ we need to estimate only $p, \delta$ and $\Sigma^{-1}$. We note in passing that if $X$ is a Gaussian random vector, then $N = k - 2$ and $\delta = 2$. We assume for simplicity below that in our setup of Kotz Type III elliptical random vectors the constants $p, N$ are known.

Estimation of $\Sigma$ is dealt with in several recent papers, see for instance see e.g., Schmidt and Schmidt (2006), Schmidt and Schmieder (2007), or Sarr and Gupta (2008). Let $\Sigma_n^{-1}$ denote an estimator of $\Sigma^{-1}$. We note that estimation of the precision matrix $\Sigma^{-1}$ is important, since we implicitly determine (estimate) the unique index set $I$ related to the quadratic programming problem $Q(\Sigma, 1)$. Under a more restrictive assumption on $R$, for instance $R^a$ is Gamma distributed with positive parameters $a, b$, then for estimating the dispersion matrix $\Sigma^{-1}$ we can utilise the recent results of Sarr and Gupta (2008).

Estimation of $q$ and $\delta$ is closely related to the estimation of the scaling function $w(u) = q \delta u^{\delta - 1}, u > 0$. As in Hashorva (2007c) we can estimate $q, \delta$ borrowing the idea of Abdous et al. (2008). Next, write $Y_{1:n} \leq \cdots \leq Y_{n:n}$ for the associated order statistics of $X_{i,1}, i \leq n$ and define the following Gardes-Girard estimator of $\delta$ by
\[ \hat{\delta}_n := \frac{1}{T_n} \sum_{i=1}^n \left( \log Y_{n-i+1:n} - \log Y_{n-k_n+1:n} \right), \quad j = 1, 2, \]
with $1 \leq k_n \leq n, T_n > 0, n \geq 1$ given constants satisfying
\[ \lim_{n \to \infty} k_n = \infty, \quad \lim_{n \to \infty} \frac{k_n}{n} = 0, \quad \lim_{n \to \infty} \log(T_n/k_n) = 1, \quad \lim_{n \to \infty} \sqrt{k_n} b(\log(n/k_n)) \to \lambda \in \mathbb{R}, \]
where $b$ is some regularly varying function with index $-1$ related to the asymptotics of $\ln(1 - G_1(t)), t \to \infty$ (see Gardes and Girard (2006)). The scaling coefficient $q$ can be estimated by (see Abdous et al. (2008))
\[ \hat{q}_n := \frac{1}{k_n} \sum_{i=1}^{k_n} \log(Y_{n-i+1:n}^{1/k_n})^j, \quad j = 1, 2, n > 1. \quad (6.28) \]

The estimators of $\hat{\psi}_{n,t}, \hat{\psi}_{n,t,x}^*$ can now be defined by plugging in $\hat{q}_n, \hat{\delta}_n, \hat{\Sigma}_n^{-1}$ in (3.16) and (3.19), respectively. Based on the known asymptotic properties of these estimators it is possible to construct further confidence intervals for both the survivor and the conditional excess function.
7 Proofs

Proof of Proposition 2.1 The claim follows from Proposition 2.1 in Hashorva and Hüsler (2003) and Proposition 2.1 in Hashorva (2005).

Proof of Theorem 3.1 In view of Proposition 2.1 we have $a_i^T \Sigma^{-1}_{ij} e_i > 0$ holds for all $i \in I$. Assume for simplicity that the index set $I$ has less than $k$ elements. Then we have further $a_j \leq \Sigma_{ij} \Sigma^{-1}_{jj} a_j, \|a_j\| > 0$. (2.6) implies that the associated random radius $R$ has distribution function $F$ in the Gumbel max-domain of attraction with the scaling function $w(u) = q u^{d-1}, u > 0$. Set

$$h_n := t_n \|a_i\| > 0, \quad t_n := t \alpha + x/v_n, n \geq 1,$$

where $(v_n)_I = w(h_n) 1_I$ and $(v_n)_J = \sqrt{w(h_n)/h_n} 1_J$. We have

$$\lim_{n \to \infty} w(h_n)(t_n - t \alpha)_I = x_I$$

and

$$\lim_{n \to \infty} \left( \frac{q \delta h_n^{d-1}}{h_n} \right)^{1/2} t_n \left( a_J - \Sigma_{ij} \Sigma^{-1}_{jj} a_I \right) = \infty (a_J - \Sigma_{ij} \Sigma^{-1}_{jj} a_I) \in [-\infty, 0]^{|J|},$$

(7.29)

where we interpret $\infty \cdot 0$ as 0. The assumptions of Theorem 3.2 are thus fulfilled, hence we may further write

$$P \{ X > t \alpha \} = (1 + o(1)) \exp(-x_I^T \Sigma^{-1}_{ij} a_I/\|a_I\|) \|a_I\|^{1/|I|} \times \Gamma(k/2) 2^{k/2-1} P \{ Z > \infty (a_J - \Sigma_{ij} \Sigma^{-1}_{jj} a_I) + x_J/|Z_I| = 0 \} \times (q \delta h_n^{d-1})^{1/|J|} \|a_J\|^{1/|J|} \times P \{ R > h_n \}$$

$$= (1 + o(1)) \exp(-x_I^T \Sigma^{-1}_{ij} a_I/\|a_I\|) \|a_I\|^{1/|I|} \times \Gamma(k/2) 2^{k/2-1} P \{ Z > \infty (a_J - \Sigma_{ij} \Sigma^{-1}_{jj} a_I) + x_J/|Z_I| = 0 \} \times (q \delta h_n^{d-1})^{1/|J|} \|a_J\|^{1/|J|} \times P \{ R > h_n \}$$

$$= (1 + o(1)) \exp(-x_I^T \Sigma^{-1}_{ij} a_I/\|a_I\|) \|a_I\|^{1/|I|} \times \Gamma(k/2) 2^{k/2-1} P \{ Z > \infty (a_J - \Sigma_{ij} \Sigma^{-1}_{jj} a_I) + x_J/|Z_I| = 0 \} \times (q \delta h_n^{d-1})^{1/|J|} \|a_J\|^{1/|J|} \times P \{ R > h_n \}$$

hence the proof follows.

Proof of Theorem 4.1 Let $L$ be a non-empty index set of $\{1, \ldots, k\}$ and set $M := \{1, \ldots, k\} \setminus L$. For any $x \in \mathbb{R}^k$ we may write

$$P \{ X > t \alpha \} P \{ (X - t \alpha)_L > (x/v_n)_L \mid X > t \alpha \} = P \{ X_L > t \alpha_L + (x/v_n)_L, X_M > t \alpha_M \}$$

$$= P \{ X > t \alpha + x^*/v_n \},$$

where $x^*$ is a vector in $\mathbb{R}^k$ with $x^*_L := x_L, x^*_M := 0_M$. Assume for simplicity that $|J| < k$ (thus $J$ is non-empty). Utilising Theorem 3.1 for any $x$ such that $x_L \in [0, \infty)^{|L|}$ we have

$$\lim_{n \to \infty} P \{ (X - t \alpha)_L > (x/v_n)_L \mid X > t \alpha \}$$

$$= \lim_{n \to \infty} \frac{P \{ X > t \alpha + x^*/v_n \}}{P \{ X > t \alpha \}}$$

$$= \exp(-a_I^T \Sigma^{-1}_{ij}(x^*_j)) \frac{P \{ Z_I > \infty (a_J - \Sigma_{ij} \Sigma^{-1}_{jj} a_I) + x^*_J/|Z_I| = 0 \}}{P \{ Z_I > \infty (a_J - \Sigma_{ij} \Sigma^{-1}_{jj} a_I) | Z_I = 0 \}},$$

as $n \to \infty$, hence the proof follows.

Proof of Theorem 4.2 For any non-empty index set $L \subset J$ set

$$K := L \cup I, \quad w(u) := q u^{d-1}, u > 0, \quad h_n := \frac{(w(t_n \|a_I\|))^{1/2}}{t_n \|a_I\|} > 0, \quad n \geq 1.$$
We may write
\[
P\{X_I > t_n a_I\} P\{h_n(X_I - t_n a_I)_L > y_L \mid X_I > t_n a_I\}
= P\{X_L - t_n (\Sigma J \Sigma_I^{-1} a_I)_L > h_n^{-1} y_L, X_I > t_n a_I\}
= P\{X_K > t_n a_K + h_n^{-1} y_K^*\}, \quad y \in \mathbb{R},
\]
where \(a^*, y^*\) are vectors in \(\mathbb{R}^{|L| + |I|}\) with
\[
a^*_L := (\Sigma J \Sigma_I^{-1} a_I)_L, \quad a^*_I := a_I, \quad \text{and} \quad y^*_L := y_L, \quad y^*_I := 0.
\]
By Proposition 2.1 \(\Sigma J \Sigma_I^{-1} a_I \geq a_I\), implying \(a^*_I \geq a_L\). Further since \(\Sigma_I^{-1} a_I\) has all components positive we have that \(a^*_I\) is the unique solution of the quadratic programming problem \(Q(B^{-1}, a^*_I)\), where \(B := \Sigma_{K,K}\).

Applying Theorem 3.1 we obtain as \(n \to \infty\)
\[
P\{h_n(X_I - t_n a_I)_L > y_L \mid X_I > t_n a_I\} = P\{Z_L > \infty 0_L + y^*_L \mid Z_I = 0_I\}
= P\{Z_L > y^*_L \mid Z_I = 0_I\},
\]
hence the proof follows.

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