Existence and stability of global large strong solutions for the Hall-MHD system

Maicon J. Benvenutti* & Lucas C. F. Ferreira†

Abstract

We consider the 3D incompressible Hall-MHD system and prove a stability theorem for global large solutions under a suitable integrable hypothesis in which one of the parcels is linked to the Hall-term. After, a special large solution is constructed by assuming the condition of curl-free magnetic fields. Combining this with the stability result, we provide a class of three-dimensional global large solutions with approximately symmetric initial velocity and approximately irrotational initial magnetic field. Moreover, we prove the local in time well-posedness of $H^2$-strong solutions which improves previous regularity conditions on initial data.

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1 Introduction

This paper is concerned with the 3D incompressible Hall-MHD system

\[
\begin{align*}
\partial_t u + [u, \nabla] u + \nabla p - (\nabla \times b) \times b &= \mu \Delta u \quad \text{in} \quad (x, t) \in \mathbb{R}^3 \times [0, \infty); \\
\partial_t b - \nabla \times (u \times b) + \nabla \times ((\nabla \times b) \times b) &= \gamma \Delta b \quad \text{in} \quad (x, t) \in \mathbb{R}^3 \times [0, \infty); \\
\text{div} u &= 0 \quad \text{in} \quad (x, t) \in \mathbb{R}^3 \times [0, \infty),
\end{align*}
\]

where $u = (u_1(x, t), u_2(x, t), u_3(x, t))$ is the velocity field, $p = p(x, t)$ is the scalar pressure field, $b = (b_1(x, t), b_2(x, t), b_3(x, t))$ is the magnetic field induced by the charged fluid, $\mu > 0$ and $\gamma > 0$ are respectively the viscosity and resistivity coefficients, $[u, \nabla] = \sum_{i=1}^{3} u_i \partial_{x_i}$ and the symbol $\times$ stands for the usual three-dimensional cross-product. The density of the fluid is assumed to be one by normalization.

The system (1.1) has been studied in the physics literature for decades (see e.g. [2], [27] and their references) and has application in a number of physical fields such as geo-dynamo [31], neutron stars [37]

*Universidade Estadual de Campinas, IMECC-Departamento de Matemática, CEP 13083-859, Campinas-SP, Brazil. Email: mbenvenuiti@hotmail.com. MJB was supported by FAPESP, Brazil.
†Universidade Estadual de Campinas, IMECC-Departamento de Matemática, CEP 13083-859, Campinas-SP, Brazil. Email: lcff@ime.unicamp.br. LCFF was supported by FAPESP and CNPQ, Brazil. (corresponding author)
and magnetic reconnection in plasmas [20]. The reader is referred to [2] (see also [6]) for a deduction of (1.1) from two-fluids model, as well as from kinetic model, considering a generalized Ohm law. In comparison with the usual incompressible MHD system (see [36]), we have the new term $\nabla \times ((\nabla \times b) \times b)$ which is due to Hall effect and prevents straightforward adaptations from arguments used in the mathematical analysis of Navier-Stokes and related models.

Unlike MHD system that has an extensive variety of studies in classical subjects such as existence of solutions, regularity criteria and stability (see e.g. [14], [16], [19], [36], [39], [40] and references therein), the influence of the Hall term has been little explored on these topics. Indeed, Hall-MHD has appeared in the literature only recently in the mathematical literature and there are relatively a few works with this type of approach which are reviewed in what follows. In [2], by using Galerkin’s method, global in time existence of weak solutions is proved in the periodic setting $L^2([0,1]^3)$ for the resistive ($\gamma > 0$) and viscous case ($\mu > 0$). The uniqueness of weak solutions is still an open problem. Considering $\mu \geq 0$ and $\gamma > 0$, the authors of [8] obtained, via energy method, local-in-time well-posedness of strong solutions in $H^m(\mathbb{R}^3)$ with $m > \frac{7}{2}$ as well as global well-posedness under small conditions. They also showed blow-up criteria of first type for strong solutions and a Liouville theorem for smooth stationary solutions. The main point in [8] was the control of the Hall term via diffusion induced by the resistivity (see more details in the next paragraph). In [9] some blow-up criteria are studied and it is obtained a global well-posedness result for small initial data in terms of Besov norm which can be considered optimal in a suitable way that takes into account the scaling property for the system with null velocity. A subclass of global strong axisymmetric solutions was obtained in [17]. By employing Fourier splitting method, time-decay of Sobolev norms is showed in [10] for a class of weak solutions. A version of (1.1) with magnetic fractional diffusion $(-\Delta)\alpha$ was considered in [12], where it was proved local well-posedness in Sobolev spaces for any $\alpha > \frac{3}{2}$ by using the smoothing effects of the dissipation and local bounds for the Sobolev norms through a multi-stage process. Regularity criteria for the density-dependent case is studied in [18]. In [11] it is shown that the non-resistive system ($\gamma = 0$) is not globally well-posed in any Sobolev space $H^m(\mathbb{R}^3)$ with $m > \frac{7}{2}$ in the sense that either it is locally ill-posed or it is locally well-posed but there exists an axisymmetric solution that loses the initial regularity in finite time.

Due to the spatial derivative of high-order in a nonlinear term, the Hall-MHD leads us to deal with higher regularity in the energy estimates (see e.g. (4.16)-(4.18) and (4.24)-(4.30) in Section 4), which introduces further difficulties in handling the system. For comparison, let us recall briefly about the issue of well-posedness for the incompressible Navier-Stokes and Euler equations: the local well-posedness with large data and the global one with small initial data in $H^1(\mathbb{R}^3)$ for the Navier-Stokes equations is obtained via an energy inequality where the nonlinearity, which is of first order ($[u, \nabla]u$), is estimated by using Gagliardo-Nirenberg inequality and the diffusion controls the generated second-order derivative (see [38]). In the inviscid case, the local well-posedness is obtained only in $H^m(\mathbb{R}^3)$ with $m > \frac{5}{2}$ where the key inclusion $H^m(\mathbb{R}^3) \hookrightarrow W^{1,\infty}(\mathbb{R}^3)$ holds true (see [30]). In [8], Chae et al. mixed these two approaches to prove a local existence theorem in $H^m(\mathbb{R}^3)$, for $\gamma > 0$ and $m > \frac{5}{2}$. Another structural difference is that the second-order derivatives in the Hall term seem to obstruct the parabolic regularization effect exhibited for instance by MHD and Navier-Stokes systems.

Our first result improves (in the viscosity case) the one of [8] by proving local in time well-posedness of (1.1) in $H^2(\mathbb{R}^3)$, and global well-posedness for small $H^2$-initial data (see Theorem 3.1). Here we use accurate energy estimates and also the particular structure of the Hall term.

Global existence of strong solutions of (1.1) for large initial data is still an open challenging problem. With respect to this matter, as far as we know, there are just the above mentioned class of $2\frac{1}{2}$ dimensional solutions of the form $(b, u) = (b(r, z)e^\beta, u(r, z)e^r + u(r, z)e^z)$ as proved in [17]. We observe that the
two-dimensional symmetry is not tractable due to the fact that in this situation the Hall term has just the third component nonzero. Despite the helical symmetry is conserved for the system, it is an open question to prove that they are global in time. Let us again make a comparison with the Navier-Stokes and other classical systems. For Navier-Stokes equations, there are global strong solutions in the two-dimensional case (see e.g. [38]), under the condition of axial symmetry without swirl [24], and in the presence of helical symmetry [29]. In [34], Ponce et al. proved that global solutions with a suitable property are stable in the sense that solutions close to them are global as well. Fortunately, symmetric and two-dimensional solutions satisfy the hypothesis required and this gives a class of global large solutions which are genuinely three-dimensional (although approximately symmetric or two-dimensional). Related results can be found in [4], [5], [13], [21], [22], [23], [32] and [35]. There are similar theorems for inhomogeneous Navier-Stokes equations [17], Boussinesq system [26, 28] and MHD system [25].

In this paper we extend the stability result of [34] to the system (1.1). Again, the main difficult is the Hall term that requires precise estimates to deal with the higher derivatives in the nonlinear term (see Theorem 3.2). Considering curl-free magnetic fields, we decouple the first equation in (1.1) and obtain a special large solution by solving first the velocity equation (via results of [34]) and afterwards the equation for $b$ (see Theorem 3.3). Using this solution in Theorem 3.2 we provide a first class of global large strong solutions $(v, h)$ for (1.1) genuinely three-dimensional where the initial velocity $v_0$ is approximately symmetry or two-dimensional and the initial magnetic field $h_0$ is approximately irrotational (see Theorem 3.4).

This paper is organized in the following way: in Section 2 we give some definitions, recall some basic inequalities and vector identities, and discuss the formulation of the problem as well as the notions of weak and strong solutions. Section 3 is devoted to state our results. In Section 4 we obtain key estimates to deal with the system. Finally, the results are proved in Section 5.

## 2 Preliminary

### 2.1 Basic definitions and inequalities

Let us start with some basic definitions in order to formulate the problem. We denote by $H^m(\mathbb{R}^3)$ and $W^{m,p}(\mathbb{R}^3)$ the usual three-dimensional vector Sobolev spaces (see [38]). The subscript $\sigma$ in $H^m(\mathbb{R}^3)$ or in $L^p$ means that the vector fields are divergence-free. The same is true for $p$ with relation to curl-free vector fields (we will remember it later). The classical Helmholtz orthogonal projection $\mathbb{P}$ onto the space of the solenoidal functions is denoted by

$$\mathbb{P} : L^2(\mathbb{R}^3) \mapsto L^2_\sigma(\mathbb{R}^3).$$

We recall the following particular cases of the Gagliardo-Nirenberg inequality in $\mathbb{R}^3$ (see [38])

$$
\begin{align*}
\|f\|_{L^6(\mathbb{R}^3)} & \leq C\|\nabla f\|_{L^2(\mathbb{R}^3)}, & \forall f \in H^1(\mathbb{R}^3), \\
\|f\|_{L^3(\mathbb{R}^3)} & \leq C\|f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|\nabla f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}, & \forall f \in H^1(\mathbb{R}^3), \\
\|f\|_{L^\infty(\mathbb{R}^3)} & \leq C\|\nabla f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|\nabla^2 f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}, & \forall f \in H^2(\mathbb{R}^3).
\end{align*}
$$

(2.1)
Also, we will use some equivalent seminorms and norms that can be obtained easily by Fourier transform. We have that

\[
\begin{align*}
\|\nabla^2 f\|_{L^2(\mathbb{R}^3)} &\approx \|\Delta f\|_{L^2(\mathbb{R}^3)} = \|\nabla \times \nabla \times f\|_{L^2(\mathbb{R}^3)} + \|\nabla \div f\|_{L^2(\mathbb{R}^3)} \quad \text{in } H^2(\mathbb{R}^3), \\
\|\nabla^3 f\|_{L^2(\mathbb{R}^3)} &\approx \|\nabla \times \Delta f\|_{L^2(\mathbb{R}^3)} + \|\nabla \times f\|_{L^2(\mathbb{R}^3)} \quad \text{in } H^3(\mathbb{R}^3), \\
\|\nabla^3 f\|_{L^2(\mathbb{R}^3)} &\approx \|\nabla \times f\|_{L^2(\mathbb{R}^3)} \quad \text{in } H^3(\mathbb{R}^3), \\
\|\nabla^3 f\|_{L^2(\mathbb{R}^3)} &\approx \|\nabla \div f\|_{L^2(\mathbb{R}^3)} \quad \text{in } H^3(\mathbb{R}^3),
\end{align*}
\]

and

\[
\begin{align*}
\|f\|_{H^2(\mathbb{R}^3)} &\approx \|f\|_{L^2(\mathbb{R}^3)} + \|\nabla f\|_{L^2(\mathbb{R}^3)} + \|\Delta f\|_{L^2(\mathbb{R}^3)}, \\
\|f\|_{H^3(\mathbb{R}^3)} &\approx \|f\|_{L^2(\mathbb{R}^3)} + \|\nabla f\|_{L^2(\mathbb{R}^3)} + \|\Delta f\|_{L^2(\mathbb{R}^3)} + \|\nabla \div f\|_{L^2(\mathbb{R}^3)} + \|\nabla \times f\|_{L^2(\mathbb{R}^3)}, \\
\|f\|_{H^3(\mathbb{R}^3)} &\approx \|f\|_{L^2(\mathbb{R}^3)} + \|\nabla f\|_{L^2(\mathbb{R}^3)} + \|\Delta f\|_{L^2(\mathbb{R}^3)} + \|\nabla \times f\|_{L^2(\mathbb{R}^3)} + \|\nabla \div f\|_{L^2(\mathbb{R}^3)}, \\
\|f\|_{H^3(\mathbb{R}^3)} &\approx \|f\|_{L^2(\mathbb{R}^3)} + \|\nabla f\|_{L^2(\mathbb{R}^3)} + \|\Delta f\|_{L^2(\mathbb{R}^3)} + \|\nabla \div f\|_{L^2(\mathbb{R}^3)}.
\end{align*}
\]

\section{Vector identities}

Here we recall some vector equalities which will be useful in order to deal with the Hall term. We have (see \cite{15})

\[
\begin{align*}
\Delta A &= \nabla \div A - \nabla \times \nabla \times A, \\
(\nabla \times A) \times B &= [A.\nabla]B + [B.\nabla]A + A \times (\nabla \times B) - \nabla (A.B), \\
\nabla \times (A \times B) &= A(\div B) - B(\div A) + [B.\nabla]A - [A.\nabla]B,
\end{align*}
\]

from which we obtain

\[
\nabla \times ((\nabla \times A) \times B) - (\nabla \times \nabla \times A) \times B = (\nabla \times A)(\div B) - 2[(\nabla \times A) \cdot \nabla]B - (\nabla \times A) \times (\nabla \times B) + \nabla ((\nabla \times A) \cdot B) \tag{2.4}
\]

and

\[
\nabla \times ((\nabla \times \nabla \times A) \times B) - (\nabla \times \nabla \times \nabla \times A) \times B = (\nabla \times \nabla \times A)(\div B) - 2[(\nabla \times \nabla \times A) \cdot \nabla]B - (\nabla \times \nabla \times A) \times (\nabla \times B) + \nabla ((\nabla \times \nabla \times A) \cdot B). \tag{2.5}
\]

\section{Weak and strong solutions}
Consider the operators

\[ A_1 : H^1_\sigma(\mathbb{R}^3) \rightarrow H^{-1}_\sigma(\mathbb{R}^3) \text{ defined by } \langle A_1[u], v \rangle = \mu \int_{\mathbb{R}^2} \nabla u . \nabla v \, dx; \]

\[ A_2 : H^1_\sigma(\mathbb{R}^3) \times H^1_\sigma(\mathbb{R}^3) \rightarrow H^{-1}_\sigma(\mathbb{R}^3) \text{ defined by } \langle A_2[u, h], v \rangle = \int_{\mathbb{R}^2} ([u. \nabla] h).v \, dx; \]

\[ A_3 : H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \rightarrow H^{-1}_\sigma(\mathbb{R}^3) \text{ defined by } \langle A_3[b, h], v \rangle = -\int_{\mathbb{R}^2} ((\nabla \times b) \times h).v \, dx; \]

\[ B_1 : H^1(\mathbb{R}^3) \rightarrow H^{-2}(\mathbb{R}^3) \text{ defined by } \langle B_1[b], w \rangle = \gamma \int_{\mathbb{R}^2} \nabla b . \nabla w \, dx; \]

\[ B_2 : H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \rightarrow H^{-2}(\mathbb{R}^3) \text{ defined by } \langle B_2[u, b], w \rangle = -\int_{\mathbb{R}^2} ((\nabla \times (u \times b)).w \, dx; \]

\[ B_3 : H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \rightarrow H^{-2}(\mathbb{R}^3) \text{ defined by } \langle B_3[b, h], w \rangle = \int_{\mathbb{R}^2} ((\nabla \times b) \times h).(\nabla \times w) \, dx. \]

It is straightforward to prove by Gagliardo-Nirenberg type inequalities that these operators are well-defined and continuous. We consider the usual weak formulation for (1.1)

\[
\begin{align*}
\frac{d}{dt}u + A_1[u] + A_2[u, u] + A_3[b, b] & = 0 \text{ in } L^1((0, T), H^{-1}_\sigma(\mathbb{R}^3)); \\
\frac{d}{dt}b + B_1[b] + B_2[u, b] + B_3[b, b] & = 0 \text{ in } L^1((0, T), H^{-2}(\mathbb{R}^3)); \\
\mathcal{P}[u] & = u.
\end{align*}
\]

For \((u_0, b_0) \in L^2_\sigma(\mathbb{R}^3) \times L^2(\mathbb{R}^3)\) and \(0 < T < \infty\), we say that \((u, b)\) is a weak solution in \((0, T)\) for (1.1) with initial data \((u_0, b_0)\) if

\[
\begin{align*}
\quad u & \in L^2((0, T); H^1_\sigma(\mathbb{R}^3)) \cap L^\infty((0, T); L^2_\sigma(\mathbb{R}^3)), \\
\quad b & \in L^2((0, T); H^1(\mathbb{R}^3)) \cap L^\infty((0, T); L^2(\mathbb{R}^3)), \\
\quad \frac{d}{dt}u & \in L^1((0, T), H^{-1}_\sigma(\mathbb{R}^3)), \\
\quad \frac{d}{dt}b & \in L^1((0, T), H^{-2}(\mathbb{R}^3))
\end{align*}
\]

and \((u, b)\) satisfies (2.6). In the case \((0, \infty)\) (global solutions), we assume that \((u, b)\) satisfies (2.6) and (2.7) for all \(0 < T < \infty\).

**Remark 2.1.** If \((u, b)\) is a weak solution, then \((u, b) \in C_w([0, T], L^2(\mathbb{R}^3))\) and the initial data condition is satisfied in an appropriate sense of weak limit (see [38]).

Inspired on the classical mathematical literature, it is natural to consider class of solutions in spaces where energy estimates provide, at least, local well-posedness. For Navier-Stokes equations (and also MHD), the space \(L^2((0, T); H^2) \cap L^\infty((0, T); H^1)\) is commonly used together with \(H^1\) initial data. These solutions are strong in the sense that they have \(H^1\)-continuous orbits (i.e., belong to \(C([0, T], H^1)\)) and satisfy their respective systems in \(L^2\) for almost everywhere \(t \in (0, T)\).

Due the second-order derivative in the non-linear part of (1.1), the above space is not appropriated to perform suitable energy estimates. However, using the special structure of Hall term \(\nabla \times ((\nabla \times b) \times b)\), we will prove the local well-posedness in \(L^2((0, T); H^2_\sigma \times H^3) \cap L^\infty((0, T); H^2_\sigma \times H^2)\) for \(H^2\) initial data and these solutions have \(H^2\)-continuous orbits and satisfy the system in \(L^2\), for almost everywhere \(t \in (0, T)\). So, we establish the following definition.
Definition 2.2. (Strong solution) Let \((u_0, b_0) \in H^2_{\sigma}(\mathbb{R}^3) \times H^2(\mathbb{R}^3)\). For \(0 < T < \infty\), we say that \((u, b)\) is a strong solution in \((0, T)\) for (1.1) with initial data \((u_0, b_0)\) if \((u, b)\) verifies (2.6) and belongs to class \(L^2((0, T); H^2_{\sigma}(\mathbb{R}^3)) \cap L^\infty((0, T); H^2(\mathbb{R}^3))\).

\begin{align*}
\left\{ \begin{array}{l}
u \in L^2((0, T); H^2_{\sigma}(\mathbb{R}^3)) \cap L^\infty((0, T); H^2(\mathbb{R}^3)), \\
\quad b \in L^2((0, T); H^3(\mathbb{R}^3)) \cap L^\infty((0, T); H^2(\mathbb{R}^3)).
\end{array} \right.
\end{align*}

In the case of \((0, \infty)\) (global solutions), we assume that \((u, b)\) satisfies (2.6) and (2.8) for all \(0 < T < \infty\).

Remark 2.3. Indeed, we are going to prove uniqueness of weak solutions in a class larger than (2.8), namely \(L^4((0, T), H^1_{\sigma}(\mathbb{R}^3)) \times L^4((0, T), H^2(\mathbb{R}^3))\) (see Theorem 3.1).

Remark 2.4. It is straightforward to check that a strong solution of (1.1) satisfies

\[\frac{d}{dt} \Delta u \in L^2((0, T); H^{-1}(\mathbb{R}^3)) \quad \text{and} \quad \frac{d}{dt} \Delta b \in L^2((0, T); H^{-1}(\mathbb{R}^3)).\]

Therefore

\[u, b \in C([0, T), H^2(\mathbb{R}^3)),\]

\[\langle \frac{d}{dt} \Delta u, \Delta u \rangle = \frac{1}{2} \frac{d}{dt} \|\Delta u\|_{L^2(\mathbb{R}^3)}^2 \quad \text{and} \quad \langle \frac{d}{dt} \Delta b, \Delta b \rangle = \frac{1}{2} \frac{d}{dt} \|\Delta b\|_{L^2(\mathbb{R}^3)}^2\] (see [38] for further details). The same is obviously true for spatial derivatives of lower order.

3 Results

In this section we state our results. We start with a result which improves the initial data regularity condition in [38] for local in time well-posedness.

3.1 Local in time well-posedness in \(H^2\)

Theorem 3.1. Let \((u_0, b_0) \in H^2_{\sigma}(\mathbb{R}^3) \times H^2(\mathbb{R}^3)\). Then, there exist \(T = T(\|u_0\|_{H^2(\mathbb{R}^3)}, \|B_0\|_{H^2(\mathbb{R}^3)}) > 0\) and a strong solution \((u, b)\) of (1.1) in \((0, T)\) with initial data \((u_0, b_0)\). This solution is the unique weak solution in \(L^4((0, T), H^1_{\sigma}(\mathbb{R}^3)) \times L^4((0, T), H^2(\mathbb{R}^3))\). Furthermore, if \(\|u_0\|_{H^2(\mathbb{R}^3)}^2 + \|b_0\|_{H^2(\mathbb{R}^3)}^2\) is small enough, then the solution is global in time. Finally, if \(T < \infty\) is the maximal existence time, then

\[\int_0^T \left(\|\nabla u\|_{L^2(\mathbb{R}^3)}^4 + \|\nabla b\|_{L^2(\mathbb{R}^3)}^4 + \|\Delta b\|_{L^2(\mathbb{R}^3)}^4\right) dt = \infty.\]

3.2 Global stability of large solutions

In the next theorem we obtain stability of large global strong solutions whose corresponding integral in (3.1) is finite for \(T = \infty\). Notice that this condition is natural because we are dealing with global solutions.
There exists \( \delta > 0 \) such that if \( (v_0, h_0) \in H^2_p(\mathbb{R}^3) \times H^2(\mathbb{R}^3) \) and
\[
\|u_0 - v_0\|^2_{H^2_p(\mathbb{R}^3)} + \|b_0 - h_0\|^2_{H^2(\mathbb{R}^3)} < \delta,
\]
then the strong solution \((v, h)\) with initial data \((v_0, h_0)\) is global in time. Furthermore, there exists \( M = M(\delta) \) with \( M(\delta) \xrightarrow{\delta \to 0} 0 \) such that
\[
\sup_{t \geq 0} \left( \|u(t) - v(t)\|^2_{H^2_p(\mathbb{R}^3)} + \|b(t) - h(t)\|^2_{H^2(\mathbb{R}^3)} \right) \leq M(\delta).
\]

### 3.3 Irrotational magnetic field and global strong solutions

The aim of this subsection is to provide a class of global large strong solutions for \((1.1)\). For that matter, we first obtain existence of a special strong solution \((u, b)\) corresponding to the data \((u_0, b_0) \in H^2_p(\mathbb{R}^3) \times H^2(\mathbb{R}^3)\) and satisfying \((3.2)\). Afterwards, this solution is used in Theorem 3.2 in order to obtain a class of global large strong solutions \((v, h)\) with data \((v_0, h_0)\) verifying \((3.5)\).

Let us suppose that the magnetic field is curl-free, this is, \(b\) belongs to
\[
H^2_p(\mathbb{R}^3) = \{b \in H^2(\mathbb{R}^3); \nabla \times b = 0\}. \tag{3.4}
\]

The system \((1.1)\) can be split into two ones. The first is the classical incompressible Navier-Stokes equations
\[
\begin{align*}
\partial_t u + [u, \nabla]u + \nabla p &= \mu \Delta u \quad \text{in} \quad (x, t) \in \mathbb{R}^3 \times [0, \infty); \\
\text{div } u &= 0 \quad \text{in} \quad (x, t) \in \mathbb{R}^3 \times [0, \infty),
\end{align*}
\tag{3.5}
\]
and the second corresponds to the dynamics of the irrotational magnetic field
\[
\begin{align*}
\partial_t b - \nabla \times (u \times b) &= \gamma \Delta b \quad \text{in} \quad (x, t) \in \mathbb{R}^3 \times [0, \infty); \\
\nabla \times b &= 0 \quad \text{in} \quad (x, t) \in \mathbb{R}^3 \times [0, \infty). \tag{3.6}
\end{align*}
\]

As pointed out in Introduction, the paper [34] provides global large solutions for \((3.5)\) that satisfy
\[
\int_0^\infty \|\nabla u(s)\|^2_{L^2(\mathbb{R}^3)} ds < \infty. \tag{3.7}
\]

So, we have the following result:

**Theorem 3.3.** Let \(u\) be a global solution of \((3.5)\) with initial data \(u_0 \in H^2_p(\mathbb{R}^3)\) and satisfying \((3.7)\) and
\[
u \in L^2((0, T); H^3_p(\mathbb{R}^3)) \cap L^\infty((0, T); H^2_p(\mathbb{R}^3)), \quad \text{for all } T > 0.
\]

If \(b_0 \in H^2_p(\mathbb{R}^3)\) then there exists a global solution \(b\) of \((3.6)\) with initial data \(b_0\) such that
\[
\begin{align*}
b &\in L^2((0, T); H^3_p(\mathbb{R}^3)) \cap L^\infty((0, T); H^2_p(\mathbb{R}^3)), \quad \text{for all } T > 0.
\end{align*}
\]

Furthermore, \((u, b)\) is a global strong solution of \((1.1)\) and verifies \((3.2)\).
Theorem 3.4. Let \((u, b)\) be as in Theorem 3.3 Then, there exists \(\delta > 0\) such that if \((v_0, h_0) \in H^2_0(\mathbb{R}^3) \times H^2(\mathbb{R}^3)\) and
\[
\|u_0 - v_0\|_{H^2(\mathbb{R}^3)}^2 + \|b_0 - h_0\|_{H^2(\mathbb{R}^3)}^2 < \delta,
\]
then the local in time strong solution \((v, h)\) of (1.1) with initial data \((v_0, h_0)\) is global.

4 Key estimates

We start with two lemmas which contain energy estimates that will be used to prove the results stated in Section 3. For the sake of presentation, the proof of Lemma 4.1 is postponed for Subsection 4.1.

Lemma 4.1. Let \((u, b)\) be a strong solution of (1.1) in \((0, T)\) according to Definition 2.2. We have that
\[
\frac{1}{2} \frac{d}{dt} \left( \|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 \right) + \mu \|\nabla u(t)\|_{L^2}^2 + \gamma \|\nabla b(t)\|_{L^2}^2 = 0, \quad \forall 0 \leq t < T. \tag{4.1}
\]
Furthermore, there are constants \(C_0 = C_0(\mu, \gamma) > 0\) and \(C_1 = C_1(\mu, \gamma) > 0\) such that
\[
\frac{1}{2} \frac{d}{dt} \left( \|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2 \right) + \frac{\mu}{2} \|\Delta u(t)\|_{L^2}^2 + \frac{\gamma}{2} \|\Delta b(t)\|_{L^2}^2 \\
\leq C_0 \left( \|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2 \right)^3 + \|\Delta b(t)\|_{L^2}^2 \|\nabla b(t)\|_{L^2}^2, \quad \forall 0 \leq t < T, \tag{4.2}
\]
and
\[
\frac{1}{2} \frac{d}{dt} \left( \|\Delta u(t)\|_{L^2}^2 + \|\Delta b(t)\|_{L^2}^2 \right) + \frac{\mu}{4} \|\nabla \times \Delta u(t)\|_{L^2}^2 + \frac{\gamma}{4} \left( \|\nabla \times \Delta b(t)\|_{L^2}^2 + \|\text{div} \Delta b(t)\|_{L^2}^2 \right) \\
\leq \frac{\gamma}{4} \|\Delta b(t)\|_{L^2}^2 + C_1 \left( \|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2 \right)^3 + \|\Delta u(t)\|_{L^2}^2 \left( \|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2 \right), \quad \forall 0 \leq t < T. \tag{4.3}
\]

Remark 4.2. Using (4.1)–(4.3) together with the equivalent norms (2.2)–(2.3), we have that the above solution satisfies
\[
\frac{1}{2} \frac{d}{dt} \left( \|u(t)\|_{H^2(\mathbb{R}^3)}^2 + \|b(t)\|_{H^2(\mathbb{R}^3)}^2 \right) + \frac{\mu}{4} \left( \|\nabla u(t)\|_{L^2(\mathbb{R}^3)}^2 + \|\Delta u(t)\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \times \Delta u(t)\|_{L^2(\mathbb{R}^3)}^2 \right) \\
+ \frac{\gamma}{4} \left( \|\nabla b(t)\|_{L^2(\mathbb{R}^3)}^2 + \|\Delta b(t)\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \times \Delta b(t)\|_{L^2(\mathbb{R}^3)}^2 + \|\text{div} \Delta b(t)\|_{L^2(\mathbb{R}^3)}^2 \right) \\
\leq C \left( \|u(t)\|_{H^2(\mathbb{R}^3)}^2 + \|b(t)\|_{H^2(\mathbb{R}^3)}^2 \right) \left( \|\nabla u(t)\|_{L^2(\mathbb{R}^3)}^4 + \|\nabla b(t)\|_{L^2(\mathbb{R}^3)}^4 + \|\Delta b(t)\|_{L^2(\mathbb{R}^3)}^4 \right). \tag{4.4}
\]

The subject of the next lemma is to show that solutions as in Lemma 4.1 under the condition (3.2) satisfy a stronger estimate that gives some control on the Hall-term.
Lemma 4.3. Let \((u, b)\) be a global strong solution that satisfies \((3.2)\). Then
\[
\int_0^\infty \|\nabla u(s)\|_{L^2}^2 + \|\nabla b(s)\|_{L^2}^2 + \|\Delta u(s)\|_{L^2}^2 + \|\Delta b(s)\|_{L^2}^2 + \|\nabla \times \Delta b(s)\|_{L^2}^2 + \|\text{div} \Delta b(s)\|_{L^2}^2 \, ds < \infty.
\]

Proof. Let \(f(t) = \|u(t)\|_{H^2(\mathbb{R}^3)}^2 + \|b(t)\|_{H^2(\mathbb{R}^3)}^2\). So, by \((4.4)\), we have
\[
\frac{1}{2} \frac{d}{dt} f(t) \leq C f(t) \left( \|\nabla u(s)\|_{L^2(\mathbb{R}^3)}^4 + \|\nabla b(s)\|_{L^2(\mathbb{R}^3)}^4 + \|\Delta b(s)\|_{L^2(\mathbb{R}^3)}^4 \right).
\]
If we define
\[
M = \int_0^\infty \left( \|\nabla u(s)\|_{L^2(\mathbb{R}^3)}^4 + \|\nabla b(s)\|_{L^2(\mathbb{R}^3)}^4 + \|\Delta b(s)\|_{L^2(\mathbb{R}^3)}^4 \right) \, ds < \infty
\]
and use Gronwall inequality in \((4.6)\), we obtain
\[
\|u(t)\|_{H^2(\mathbb{R}^3)}^2 + \|b(t)\|_{H^2(\mathbb{R}^3)}^2 \leq e^{2CM} \left( \|u(0)\|_{H^2(\mathbb{R}^3)}^2 + \|b(0)\|_{H^2(\mathbb{R}^3)}^2 \right).
\]
On the other side, integrating \((4.4)\), we also get
\[
\frac{\gamma}{4} \int_0^t \left( \|\nabla \times \Delta b(s)\|_{L^2(\mathbb{R}^3)}^2 + \|\text{div} \Delta b(s)\|_{L^2(\mathbb{R}^3)}^2 \right) \, ds + \frac{\mu}{4} \int_0^t \|\Delta u(s)\|_{L^2(\mathbb{R}^3)}^2 \, ds
\leq CM \sup_{s > 0} \left\{ \|u(s)\|_{H^2(\mathbb{R}^3)}^2 + \|b(s)\|_{H^2(\mathbb{R}^3)}^2 \right\}.
\]
The condition \((3.2)\) and inequalities \((4.7)-(4.8)\) give \((4.5)\).

In order to obtain the stability result, we need to estimate the difference between two strong solutions of \((1.1)\). For this, we have two lemmas whose proofs are relatively long and so we also postpone them for later (see subsections \((4.2)\) and \((4.3)\)).

Lemma 4.4. Let \((u, b)\) and \((v, h)\) be strong solutions of \((1.1)\). If \(U = v - u\) and \(B = h - b\) then
\[
\begin{align*}
\partial_t U - \mu \Delta U &= -P[\nabla U, U] - P[\nabla U, u] - P[\nabla U, U] + P[\nabla \times B \times B] \\
&+ P[\nabla \times B \times b] + P[\nabla \times b \times B]; \\
\partial_t B - \gamma \Delta B &= \nabla \times (U \times B) + \nabla \times (U \times b) + \nabla \times (u \times B) - \nabla \times ((\nabla \times B) \times B) \\
&- \nabla \times ((\nabla \times B) \times b) - \nabla \times ((\nabla \times b) \times B); \\
\text{div} U &= 0.
\end{align*}
\]
Furthermore, there are constants \(C_2 = C_2(\mu, \gamma) > 0\) and \(C_3 = C_3(\mu, \gamma) > 0\) such that
\[
\frac{1}{2} \frac{d}{dt} \left( \|U(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2 \right) + \frac{\mu}{2} \|\nabla U(t)\|_{L^2}^2 + \frac{\gamma}{2} \|\nabla B(t)\|_{L^2}^2 \\
\leq C_2 \left( \|\nabla u(t)\|_{L^2}^4 + \|\nabla b(t)\|_{L^2}^4 + \|\Delta b(t)\|_{L^2}^4 \right) \left( \|U(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2 \right), \forall 0 \leq t < T,
\]
and
\[
\frac{1}{2} \frac{d}{dt} \left( \|U(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2 \right) + \mu \|\nabla U(t)\|_{L^2}^2 + \frac{\mu}{2} \|\Delta U(t)\|_{L^2}^2 + \frac{\gamma}{2} \|\Delta B(t)\|_{L^2}^2 \\
\leq C_3 \left( \|\nabla U(t)\|_{L^2}^2 + \|
abla B(t)\|_{L^2}^2 + \|\Delta B(t)\|_{L^2}^2 \right) \left( \|\nabla u(t)\|_{L^2}^2 + \|\nabla b(t)\|_{L^2}^2 + \|\Delta b(t)\|_{L^2}^2 \right) \\
+ C_3 \left( \|\nabla U(t)\|_{L^2}^2 + \|
abla B(t)\|_{L^2}^2 + \|\Delta B(t)\|_{L^2}^2 \right)^3 , \quad \forall 0 \leq t < T.
\]
(4.11)

Remark 4.5. The inequality (4.10) holds true for two weak solutions \((u, b)\) and \((v, h)\) belonging to \(L^4((0, T), H^1_0(\mathbb{R}^3)) \times L^4((0, T), H^2(\mathbb{R}^3))\).

Lemma 4.6. Let \((u, b), (v, h)\) and \((U, B)\) as in Lemma 4.4. There is a constant \(C_4 = C_4(\mu, \gamma) > 0\) such that
\[
\frac{1}{2} \frac{d}{dt} \left( \|\Delta U\|_{L^2}^2 + \|\Delta B\|_{L^2}^2 \right) + \frac{\mu}{4} \|\nabla \times \Delta U\|_{L^2}^2 + \frac{\gamma}{4} \left( \|\nabla \times \Delta B\|_{L^2}^2 + \|\text{div} \Delta B\|_{L^2}^2 \right) \\
\leq C_4 \left( \|\nabla U\|_{L^2}^2 + \|
abla B\|_{L^2}^2 + \|\Delta U\|_{L^2}^2 + \|\Delta B\|_{L^2}^2 \right) \left( \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2 \right) \\
+ \frac{\mu}{4} \|\Delta U\|_{L^2}^2 + \frac{\gamma}{4} \|\Delta B\|_{L^2}^2 + C_4 \left( \|\nabla B\|_{L^2}^2 + \|\Delta B\|_{L^2}^2 \right) \left( \|\nabla \times \Delta b\|_{L^2}^2 + \|\text{div} \Delta b\|_{L^2}^2 \right) \\
+ C_4 \left( \|\nabla U\|_{L^2}^2 + \|
abla B\|_{L^2}^2 + \|\Delta U\|_{L^2}^2 + \|\Delta B\|_{L^2}^2 \right)^3 , \quad \forall 0 \leq t < T.
\]
(4.12)

Remark 4.7. Let \(L(t) = \|\nabla u\|_{L^2}^2 + \|
abla b\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2 + \|
abla \times \Delta b\|_{L^2}^2 + \|\text{div} \Delta b\|_{L^2}^2\).

It follows from (4.10)-(4.12) that
\[
\frac{d}{dt} \left( \|U\|_{H^2}^2 + \|B\|_{H^2}^2 \right) + \frac{\mu}{4} \left( \|\nabla U\|_{L^2(\mathbb{R}^3)}^2 + \|\Delta U\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \times U\|_{L^2(\mathbb{R}^3)}^2 \right) \\
+ \frac{\gamma}{4} \left( \|\nabla B\|_{L^2(\mathbb{R}^3)}^2 + \|\Delta B\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \times B\|_{L^2(\mathbb{R}^3)}^2 + \|\text{div} \Delta B\|_{L^2(\mathbb{R}^3)}^2 \right) \\
\leq C \left( \|U\|_{H^2}^2 + \|B\|_{H^2}^2 \right)^3 + C \left( \|U\|_{H^2}^2 + \|B\|_{H^2}^2 \right) L(t).
\]

4.1 Proof of Lemma 4.1

We multiply the first and second equations in (2.6) by \(u\) and \(b\), respectively, and afterwards we integrate to obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \|u\|_{L^2}^2 + \|b\|_{L^2}^2 \right) + \mu \|\nabla u\|_{L^2}^2 + \gamma \|\nabla b\|_{L^2}^2 = - \left( \frac{l_1}{2}, \int_{\mathbb{R}^3} (u, \nabla u) \right)_{L^2} \\
+ \left( \frac{l_2}{2}, \int_{\mathbb{R}^3} (\nabla \times b, u) \right)_{L^2} + \left( \frac{l_3}{2}, \int_{\mathbb{R}^3} (\nabla \times (u \times b), b) \right)_{L^2} \\
- \left( \frac{l_4}{2}, \int_{\mathbb{R}^3} ((\nabla \times b) \times b, b) \right)_{L^2}.
\]
(4.13)
Performing an integration by parts and using the incompressible condition to \( u \), we get

\[ I_1 = ([u.]u, u)_{L^2} = -(u.\nabla)u, u)_{L^2} = 0, \]

\[ I_2 = ((\nabla \times b) \times b, u)_{L^2} = ([b.]b - \frac{1}{2}\nabla|b|^2, u)_{L^2} = ([b.]b, u)_{L^2}, \]

\[ I_3 = (\nabla \times (u \times b), b)_{L^2} = (u(div b) + [b.]u - [u.]b, b)_{L^2} = -(u(div b) + [b.]u, b)_{L^2} \]
\[ = -(|b.|b, u)_{L^2} = -I_2, \]

\[ I_4 = ((\nabla \times b) \times b, \nabla \times b)_{L^2} = 0. \]

Inserting the above equalities in (4.13), we obtain (4.1).

Now, we multiply the first equation of (2.6) by \(-\Delta u\) and the second by \(-\Delta b\) in order to obtain

\[
\frac{1}{2} \frac{d}{dt} (\|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \mu \|\Delta u\|_{L^2}^2 + \gamma \|\Delta b\|_{L^2}^2 = \underbrace{(P([u.]u, \Delta u)]_{L^2})}_{J_1} - \underbrace{(P(\nabla \times b, \Delta u)]_{L^2})}_{J_2} - \underbrace{(\nabla \times (u \times b), \Delta b)]_{L^2})}_{J_3} + \underbrace{(\nabla \times ((\nabla \times b) \times b), \Delta b)]_{L^2})}_{J_4}.
\]

(4.14)

The terms \( J_i \) on the right side of (4.14) can be estimated by using Gagliardo-Nirenberg inequalities and Young inequality with \( \epsilon \). Hereafter \( C \) will be positive constants that can change in each line and they may depend of \( \mu, \gamma \) and \( \epsilon \). Precisely, we have

\[ |J_1| = |([u.]u, \Delta u)_{L^2}| \leq C\|\Delta u\|_{L^2}\|\nabla u\|_{L^3}\|u\|_{L^6} \]
\[ \leq C\|\Delta u\|_{L^2}\left(\|\nabla u\|_{L^2}^\frac{3}{2}\|u\|_{L^6}\right) \|\nabla u\|_{L^2} \leq \epsilon\|\Delta u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^6, \]

\[ |J_2| = |((\nabla \times b) \times b, \Delta u)_{L^2}| \leq C\|\Delta u\|_{L^2}\|\nabla b\|_{L^2}\|b\|_{L^6} \]
\[ \leq C\|\Delta u\|_{L^2}\left(\|\nabla b\|_{L^2}^\frac{3}{2}\|\Delta b\|_{L^2}^\frac{1}{2}\right) \|\nabla b\|_{L^2} \leq \epsilon\|\Delta u\|_{L^2}^2 + \epsilon\|\Delta b\|_{L^2}^2 + C\|\nabla b\|_{L^2}^6, \]

\[ |J_3| = |(\nabla \times (u \times b), \Delta b)_{L^2}| \leq C\|\Delta b\|_{L^2}\left(\|\nabla b\|_{L^2}^\frac{3}{2}\|\Delta b\|_{L^2}^\frac{1}{2}\right) \|\nabla u\|_{L^2} \leq \epsilon\|\Delta u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4\|\nabla b\|_{L^2}^2, \]

\[ |J_4| = |(\nabla \times ((\nabla \times b) \times b), \Delta b)_{L^2}| \leq C\|\Delta b\|_{L^2}\left(\|\nabla b\|_{L^2}^\frac{3}{2}\|\Delta b\|_{L^2}^\frac{1}{2}\right) \|\Delta b\|_{L^2} \leq \epsilon\|\Delta b\|_{L^2}^2 + C\|\Delta b\|_{L^2}^2\|\nabla b\|_{L^2}^2. \]

The inequality (4.2) follows by inserting the above estimates in (4.14) and choosing \( \epsilon > 0 \) small enough.
Finally, we deal with (4.3). After applying $\nabla \times$ in (2.6), we multiply the first equation by $-\nabla \times \Delta u$ and the second by $-\nabla \times \Delta b$. Also we apply $\text{div}$ in the second equation of (2.6) and then multiply it by $-\text{div} \Delta b$. With these manipulations, we get the following equality

$$
\frac{1}{2} \frac{d}{dt} (\|\Delta u\|_{L^2}^2 + \|\Delta b\|_{L^2}^2) + \mu \|\nabla \times \Delta u\|_{L^2}^2 + \gamma (\|\nabla \times \Delta b\|_{L^2}^2 + \|\text{div} \Delta b\|_{L^2}^2)
$$

$$
= (\nabla \times (u, \nabla u), \nabla \times \Delta u)_{L^2} - (\nabla \times ((\nabla \times b) \cdot b), \nabla \times \Delta u)_{L^2}
$$

$$
- (\nabla \times \Delta \times (u \times b), \nabla \times \Delta b)_{L^2} + (\nabla \times \Delta \times ((\nabla \times b) \cdot b), \nabla \times \Delta b)_{L^2}.
$$

(4.15)

For the first three terms on the right side of (4.15), we have

$$
|K_1| \leq C \|\nabla \times \Delta u\|_{L^2} (\|u\|_{L^6} \|\Delta u\|_{L^3} + \|\nabla u\|_{L^\infty} \|\nabla u\|_{L^2})
$$

$$
\leq C \|\nabla \times \Delta u\|_{L^2} \left(\|\nabla \times \Delta u\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2}^{\frac{3}{2}}\right) \|\nabla u\|_{L^2}
$$

$$
\leq \epsilon \|\nabla \times \Delta u\|_{L^2}^2 + C \|\Delta u\|_{L^2}^2 \|\nabla u\|_{L^2}^4,
$$

and

$$
|K_2| \leq C \|\nabla \times \Delta u\|_{L^2} (\|b\|_{L^\infty} \|\Delta b\|_{L^2} + \|\nabla b\|_{L^3} \|\nabla b\|_{L^6})
$$

$$
\leq C \|\nabla \times \Delta u\|_{L^2} \left(\|\nabla b\|_{L^2}^{\frac{1}{2}} \|\Delta b\|_{L^2}^{\frac{3}{2}}\right) \|\Delta b\|_{L^2}
$$

$$
\leq \epsilon \|\nabla \times \Delta u\|_{L^2}^2 + \epsilon \|\Delta b\|_{L^2}^2 + C \|\Delta b\|_{L^2}^2 \|\nabla b\|_{L^2}^2,
$$

and

$$
|K_3| \leq C \|\nabla \times \Delta b\|_{L^2} (\|u\|_{L^6} \|\Delta b\|_{L^3} + \|\nabla u\|_{L^2} \|\nabla b\|_{L^\infty})
$$

$$
\quad + C \|\nabla \times \Delta b\|_{L^2} (\|b\|_{L^6} \|\Delta u\|_{L^3})
$$

$$
\leq C \|\nabla \times \Delta b\|_{L^2} \left(\|\Delta b\|_{L^2}^{\frac{1}{2}} \left(\|\nabla \times \Delta b\|_{L^2}^{\frac{1}{2}} + \|\text{div} \Delta b\|_{L^2}^{\frac{1}{2}}\right)\right) \|\nabla u\|_{L^2}
$$

$$
\quad + C \|\nabla \times \Delta b\|_{L^2} \left(\|\nabla \times \Delta u\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2}^{\frac{1}{2}}\right) \|\nabla b\|_{L^2}
$$

$$
\leq \epsilon \|\nabla \times \Delta u\|_{L^2}^2 + \epsilon \|\nabla \times \Delta b\|_{L^2}^2 + \epsilon \|\text{div} \Delta b\|_{L^2}^2 + C \|\Delta b\|_{L^2}^2 \|\nabla u\|_{L^2}^4 + C \|\Delta u\|_{L^2}^2 \|\nabla b\|_{L^2}^4.
$$

For the Hall-term, using the vector identity $(A \times B) \cdot A = 0$, we get

$$
K_4 = (\nabla \times \nabla \times ((\nabla \times b) \cdot b), -\nabla \times \nabla \times \nabla \times b)_{L^2}
$$

$$
= \left(\nabla \times \nabla \times ((\nabla \times b) \cdot b) - \nabla \times ((\nabla \times \nabla \times b) \cdot b), -\nabla \times \nabla \times \nabla \times b\right)_{L^2}
$$

$$
+ \left(\nabla \times ((\nabla \times \nabla \times b) \cdot b) - (\nabla \times \nabla \times \nabla \times b) \cdot b, -\nabla \times \nabla \times \nabla \times b\right)_{L^2}.
$$

(4.16)
Now, by identities (2.4)-(2.5), we obtain
\[ |K_5| = |(\nabla \times (\nabla \times (\nabla \times b) - (\nabla \times \nabla \times b) \times \nabla \times b))_L^2| \]
\[ = |(\nabla \times (\nabla \times b)(\div b) - 2(\nabla \times b, \nabla b), \nabla \times \Delta b)_L^2| \]
\[ \leq C\| \nabla \times \Delta b\|_L^2 \|\nabla b\|_L^6 \|\Delta b\|_L^3 \]
\[ \leq C\| \nabla \times \Delta b\|_L^2 \|\Delta b\|_L^2 \left(\|\nabla \times \Delta b\|_L^2 + \|\div \Delta b\|_L^2 \right) \]
\[ \leq \epsilon \| \nabla \times \Delta b\|^2_2 + \epsilon \| \div \Delta b\|^2_2 + C\| \Delta b\|^6_2 \]  
(4.17)

and
\[ |K_6| = |(\nabla \times \nabla \times b)(\div b) - 2(\nabla \times \nabla \times b), \nabla b - (\nabla \times \nabla \times b) \times (\nabla \times b), \nabla \times \Delta b)_L^2| \]
\[ \leq C\| \nabla \times \Delta b\|_L^2 \|\Delta b\|_L^3 \|\nabla b\|_L^6 \]
\[ \leq C\| \nabla \times \Delta b\|_L^2 \left(\|\nabla \times \Delta b\|_L^2 + \|\div \Delta b\|_L^2 \right) \| \Delta b\|_L^2 \]
\[ \leq \epsilon \| \nabla \times \Delta b\|^2_2 + \epsilon \| \div \Delta b\|^2_2 + C\| \Delta b\|^6_2. \]  
(4.18)

We obtain (4.3) from (4.15), the above estimates for $|K_1|$, and choosing a suitable $\epsilon > 0$ small enough.

### 4.2 Proof of Lemma 4.4

The proof of (4.9) is a straightforward calculation. Now we multiply the first equation of (4.9) by $U$ and the second equation by $B$ to obtain
\[ \frac{1}{2} \frac{d}{dt} (\|U\|^2_{L^2} + \|B\|^2_{L^2}) + \mu\|\nabla U\|^2_{L^2} + \gamma\|\nabla B\|^2_{L^2} = L_1 L_2 \]
\[ - (P[[U, \nabla]U], U)_{L^2} - (P[[U, \nabla]U], U)_{L^2} \]
\[ + (P[[U, \nabla]U], U)_{L^2} + (P[[\nabla \times B] \times B, U)_{L^2} \]
\[ + (\nabla \times (u \times B), B)_{L^2} - (\nabla \times ((\nabla \times B) \times B), B)_{L^2} \]
\[ - (\nabla \times ((\nabla \times b) \times B), B)_{L^2}. \]  
(4.19)

We have that $L_1 = L_3 = L_{10} = L_{11} = 0$ and $L_4 = -L_7$. Then, we need to estimate the remainder terms. Proceeding as in the proof of Lemma 4.1, we obtain
\[ |L_2| \leq \epsilon \|\nabla U\|^2_{L^2} + C\|\nabla u\|^2_{L^2} \|U\|^2_{L^2}, \]
\[ |L_5| \leq \epsilon \|\nabla B\|^2_{L^2} + C\|\nabla b\|^2_{L^2} \|B\|^2_{L^2}, \]
\[ |L_6| \leq \epsilon \|\nabla B\|^2_{L^2} + \epsilon \|\nabla U\|^2_{L^2} + C\|\nabla b\|^4_{L^2} \|B\|^2_{L^2}, \]
\[ |L_8| \leq \epsilon \|\nabla B\|^2_{L^2} + \epsilon \|\nabla U\|^2_{L^2} + C\|\nabla b\|^4_{L^2} \|B\|^2_{L^2}, \]
\[ |L_9| \leq \epsilon \|\nabla B\|^2_{L^2} + C\|\nabla u\|^2_{L^2} \|B\|^2_{L^2}, \]
\[ |L_{12}| \leq \epsilon \|\nabla B\|^2_{L^2} + C\|\Delta b\|^4_{L^2} \|B\|^2_{L^2}. \]
Now we obtain (4.10) after inserting the above inequalities in (4.19) and taking $\epsilon > 0$ small enough.

Next we prove (4.11). Multiplying the first equation of (4.9) by $-\Delta U$ and the second by $-\Delta B$, we get

\[
\frac{1}{2} \frac{d}{dt} \left( ||\nabla U||^2_{L^2} + ||\nabla B||^2_{L^2} \right) + \mu ||\Delta U||^2_{L^2} + \gamma ||\Delta B||^2_{L^2} = (P[U, \nabla U], \Delta U)_{L^2} + (P[U, \nabla u], \Delta U)_{L^2} \\
+ (P[u, \nabla U], \Delta U)_{L^2} - (P[(\nabla \times B) \times B], \Delta U)_{L^2} - (P[(\nabla \times B) \times b], \Delta U)_{L^2} \\
- (P[(\nabla \times b) \times B], \Delta U)_{L^2} - (\nabla \times (U \times B), \Delta B)_{L^2} - (\nabla \times (U \times b), \Delta B)_{L^2} \\
- (\nabla \times ((\nabla \times b) \times b), \Delta B)_{L^2} + (\nabla \times ((\nabla \times b) \times B), \Delta B)_{L^2}.
\] (4.20)

We have the following estimates:

\[
|M_1| \leq \epsilon ||\Delta U||^2_{L^2} + C||\nabla U||^6_{L^2}, \\
|M_2| \leq \epsilon ||\Delta U||^2_{L^2} + C||\nabla u||^2_{L^2}||\nabla U||^2_{L^2}, \\
|M_3| \leq \epsilon ||\Delta U||^2_{L^2} + C||\nabla u||^4_{L^2}||\nabla U||^2_{L^2}, \\
|M_4| \leq \epsilon ||\Delta U||^2_{L^2} + \epsilon ||\Delta B||^2_{L^2} + C||\nabla B||^6_{L^2}, \\
|M_5| \leq \epsilon ||\Delta U||^2_{L^2} + \epsilon ||\Delta B||^2_{L^2} + C||\nabla b||^4_{L^2}||\nabla B||^2_{L^2}, \\
|M_6| \leq \epsilon ||\Delta U||^2_{L^2} + \epsilon ||\Delta B||^2_{L^2} + C||\nabla b||^4_{L^2}||\nabla B||^2_{L^2}, \\
|M_7| \leq \epsilon ||\Delta U||^2_{L^2} + \epsilon ||\Delta B||^2_{L^2} + C||\nabla B||^4_{L^2}||\nabla U||^2_{L^2}, \\
|M_8| \leq \epsilon ||\Delta U||^2_{L^2} + \epsilon ||\Delta B||^2_{L^2} + C||\nabla b||^4_{L^2}||\nabla U||^2_{L^2}, \\
|M_9| \leq \epsilon ||\Delta B||^2_{L^2} + C||\nabla u||^2_{L^2}||\nabla B||^2_{L^2}, \\
|M_{10}| \leq \epsilon ||\Delta B||^2_{L^2} + C||\Delta B||^4_{L^2}||\nabla B||^2_{L^2};
\]

and

\[
|M_{11}| \leq \epsilon ||\Delta B||^2_{L^2} + C||\Delta b||^4_{L^2}||\nabla B||^2_{L^2} + C||\Delta B||^2_{L^2}||\nabla b||^4_{L^2} + C||\Delta B||^2_{L^2}||\Delta b||^4_{L^2}, \\
|M_{12}| \leq \epsilon ||\Delta B||^2_{L^2} + C||\Delta b||^4_{L^2}||\nabla B||^2_{L^2}.
\]

Again, taking $\epsilon > 0$ small enough, the above estimates together with (4.20) give (4.11).

\[\blacksquare\]

### 4.3 Proof of Lemma 4.6

First we apply $\nabla \times$ in (4.9) and then multiply the first equation by $-\nabla \times \Delta U$ and the second equation by $-\nabla \times \Delta B$. Also, we apply $div$ in the second equation of (4.9) and we multiply it by $-div \Delta B$. 
After this manipulations, we obtain

\[
\frac{1}{2} \frac{d}{dt} \left( \| \Delta U \|_{L^2}^2 \right) + \| \Delta B \|_{L^2}^2 + \mu \| \nabla \times \Delta U \|_{L^2}^2 + \gamma \left( \| \nabla \times \Delta B \|_{L^2}^2 + \| \text{div} \Delta B \|_{L^2}^2 \right) \\
= \left( \nabla \times ([U, \nabla] U), \nabla \times \Delta U \right)_{L^2} \color{red}{\nabla \times \left( U \nabla \Delta U \right)}_{L^2} + \left( \nabla \times ([U, \nabla] u), \nabla \times \Delta U \right)_{L^2} + \left( \nabla \times [u, \nabla] U, \nabla \times \Delta U \right)_{L^2} - \left( \nabla \times ([\nabla \times B] \times B), \nabla \times \Delta U \right)_{L^2} \\
- \left( \nabla \times ([\nabla \times B] \times b), \nabla \times \Delta U \right)_{L^2} - \left( \nabla \times ([\nabla \times B] \times B), \nabla \times \Delta U \right)_{L^2} - \left( \nabla \times \nabla \times (U \times B), \nabla \times \Delta B \right)_{L^2} - \left( \nabla \times \nabla \times (U \times b), \nabla \times \Delta B \right)_{L^2} \\
- \left( \nabla \times \nabla \times (u \times B), \nabla \times \Delta B \right)_{L^2} + \left( \nabla \times \nabla \times (u \times B), \nabla \times \Delta B \right)_{L^2} + \left( \nabla \times \nabla \times (u \times B), \nabla \times \Delta B \right)_{L^2} \\
+ \left( \nabla \times \nabla \times ((\nabla \times B) \times B), \nabla \times \Delta U \right)_{L^2} + \left( \nabla \times \nabla \times ((\nabla \times B) \times B), \nabla \times \Delta B \right)_{L^2}. \]

(4.21)

As before, we can estimate

\[
|N_1| \leq \epsilon \| \nabla \times \Delta U \|_{L^2}^2 + \epsilon \| \Delta U \|_{L^2}^2 + C \| \Delta U \|_{L^2}^2 \| \nabla U \|_{L^2}^2, \quad (4.22) \\
|N_2| \leq \epsilon \| \nabla \times \Delta U \|_{L^2}^2 + \epsilon \| \Delta U \|_{L^2}^2 + C \| \Delta u \|_{L^2}^2 \| \nabla U \|_{L^2}^2, \\
|N_3| \leq \epsilon \| \nabla \times \Delta U \|_{L^2}^2 + C \| \nabla u \|_{L^2}^2 \| \Delta U \|_{L^2}^2, \\
|N_4| \leq \epsilon \| \nabla \times \Delta U \|_{L^2}^2 + \epsilon \| \Delta B \|_{L^2}^2 + C \| \nabla B \|_{L^2}^2 \| \Delta B \|_{L^2}^2, \\
|N_5| \leq \epsilon \| \nabla \times \Delta U \|_{L^2}^2 + \epsilon \| \nabla \times \Delta B \|_{L^2}^2 + \epsilon \| \text{div} \Delta B \|_{L^2}^2 + C \| \nabla b \|_{L^2}^2 \| \Delta B \|_{L^2}^2, \]

and

\[
|N_6| \leq \epsilon \| \nabla \times \Delta U \|_{L^2}^2 + \epsilon \| \Delta B \|_{L^2}^2 + C \| \Delta b \|_{L^2} \| \nabla B \|_{L^2}, \quad (4.23) \\
|N_7| \leq \epsilon \| \Delta U \|_{L^2}^2 + \epsilon \| \Delta B \|_{L^2}^2 + \epsilon \| \nabla \times \Delta U \|_{L^2}^2 + C \| \Delta B \|_{L^2} \| \nabla U \|_{L^2} \| \nabla \nabla \times \Delta U \|_{L^2}^2 + C \| \Delta U \|_{L^2} \| \nabla B \|_{L^2}^2, \\
|N_8| \leq \epsilon \| \nabla \times \Delta U \|_{L^2}^2 + \epsilon \| \Delta B \|_{L^2}^2 + \epsilon \| \nabla \times \Delta U \|_{L^2}^2 + C \| \nabla b \|_{L^2} \| \Delta U \|_{L^2} \| \nabla \times \Delta U \|_{L^2}^2 + C \| \nabla U \|_{L^2} \| \Delta b \|_{L^2}, \\
|N_9| \leq \epsilon \| \nabla \times \Delta B \|_{L^2}^2 + \epsilon \| \text{div} \Delta B \|_{L^2}^2 + \epsilon \| \Delta B \|_{L^2}^2 + C \| \nabla u \|_{L^2} \| \Delta B \|_{L^2} \| \nabla \nabla \times \Delta B \|_{L^2}^2 + C \| \nabla B \|_{L^2} \| \Delta u \|_{L^2}^2. \\
\]

For the Hall-term, using the vector identity \((A \times B) \cdot A = 0\), we get

\[
N_{10} = (\nabla \times \nabla \times ((\nabla \times B) \times B), -\nabla \times \nabla \times \nabla \times B)_{L^2} \\
= \left( \nabla \times \nabla \times ((\nabla \times B) \times B) - \nabla \times ((\nabla \times \nabla \times B) \times B) \right)_{L^2} \\
+ \left( \nabla \times ((\nabla \times \nabla \times B) \times B) - (\nabla \times \nabla \times B) \times B, -\nabla \times \nabla \times \nabla \times B \right)_{L^2}. \quad (4.24)
\]
and

\[ N_{11} = (\nabla \times \nabla \times ((\nabla \times B) \times b), -\nabla \times \nabla \times \nabla \times B)_{L^2} \]

\[ = (\nabla \times \nabla \times ((\nabla \times B) \times b) - \nabla \times ((\nabla \times \nabla \times B) \times b), -\nabla \times \nabla \times \nabla \times B)_{L^2} \]

\[ + (\nabla \times ((\nabla \times \nabla \times B) \times b) - (\nabla \times \nabla \times \nabla \times B) \times b, -\nabla \times \nabla \times \nabla \times B)_{L^2}. \quad (4.25) \]

Now, by using identities (2.4)-(2.5), we obtain

\[ |N_{10,a}| = |(\nabla \times \{ \nabla \times ((\nabla \times B) \times B) - (\nabla \times \nabla \times B) \times B \}, \nabla \times \Delta B)_{L^2}| \]

\[ = |(\nabla \times \{ (\nabla \times B) (\text{div} B) - 2[(\nabla \times B) \cdot \nabla] B - (\nabla \times \nabla \times B) \times (\nabla \times B), \nabla \times \Delta B)_{L^2}| \]

\[ \leq C \| \Delta B \|_{L^2} \| \nabla B \|_{L^\infty} \| \nabla \times \Delta B \|_{L^2} \]

\[ \leq C \| \Delta B \|_{L^2}^2 \left( \|\nabla \times \Delta B\|_{L^2}^2 + \|\text{div} \Delta B\|_{L^2}^2 \right) \| \nabla \times \Delta B \|_{L^2} \]

\[ \leq \epsilon \| \nabla \times \Delta B \|_{L^2}^2 + \epsilon \|\text{div} \Delta B\|_{L^2}^2 + C \| \Delta B \|_{L^2}^6, \quad (4.26) \]

\[ |N_{10,b}| = |((\nabla \times \nabla \times B) (\text{div} B) - 2[(\nabla \times \nabla \times B) \cdot \nabla] B - (\nabla \times \nabla \times B) \times (\nabla \times B), \nabla \times \Delta B)_{L^2}| \]

\[ \leq C \| \Delta B \|_{L^2} \| \nabla B \|_{L^\infty} \| \nabla \times \Delta B \|_{L^2} \]

\[ \leq C \| \Delta B \|_{L^2}^2 \left( \|\nabla \times \Delta B\|_{L^2}^2 + \|\text{div} \Delta B\|_{L^2}^2 \right) \| \nabla \times \Delta B \|_{L^2} \]

\[ \leq \epsilon \| \nabla \times \Delta B \|_{L^2}^2 + \epsilon \|\text{div} \Delta B\|_{L^2}^2 + C \| \Delta B \|_{L^2}^6, \quad (4.27) \]

\[ |N_{11,a}| = |(\nabla \times \{ (\nabla \times B) (\text{div} B) - 2[(\nabla \times B) \cdot \nabla] B - (\nabla \times \nabla \times B) \times (\nabla \times b), \nabla \times \Delta B)_{L^2}| \]

\[ \leq C \| \Delta B \|_{L^2} \| \nabla \times B \|_{L^\infty} \| \nabla \times \Delta B \|_{L^2} \]

\[ \leq C \| \Delta B \|_{L^2}^2 \left( \|\nabla \times \Delta B\|_{L^2}^2 + \|\text{div} \Delta B\|_{L^2}^2 \right) \| \nabla \times \Delta B \|_{L^2} \]

\[ \leq \epsilon \| \nabla \times \Delta B \|_{L^2}^2 + \epsilon \|\text{div} \Delta B\|_{L^2}^2 + C \| \Delta B \|_{L^2}^4, \quad (4.28) \]

\[ |N_{11,b}| = |((\nabla \times \nabla \times B) (\text{div} B) - 2[(\nabla \times \nabla \times B) \cdot \nabla] B - (\nabla \times \nabla \times B) \times (\nabla \times b), \nabla \times \Delta B)_{L^2}| \]

\[ \leq C \| \Delta B \|_{L^2} \| \nabla \times B \|_{L^\infty} \| \nabla \times \Delta B \|_{L^2} \]

\[ \leq C \| \Delta B \|_{L^2}^2 \left( \|\nabla \times \Delta B\|_{L^2}^2 + \|\text{div} \Delta B\|_{L^2}^2 \right) \| \Delta b \|_{L^2} \| \nabla \times \Delta B \|_{L^2} \]

\[ \leq \epsilon \| \nabla \times \Delta B \|_{L^2}^2 + \epsilon \|\text{div} \Delta B\|_{L^2}^2 + C \| \Delta B \|_{L^2}^2 \| \Delta b \|_{L^2}^4, \quad (4.29) \]

and

\[ |N_{12}| = |(\nabla \times \nabla \times ((\nabla \times B) \times B), \nabla \times \Delta B)_{L^2}| \]

\[ \leq C \| \Delta B \|_{L^2} \| \nabla \times B \|_{L^\infty} \| \nabla \times \Delta B \|_{L^2} \]

\[ + \| \Delta B \|_{L^2} \| \nabla \times B \|_{L^\infty} \| \nabla \times \Delta B \|_{L^2} \]

\[ \leq C \| \Delta B \|_{L^2}^2 \left( \|\nabla \times \Delta B\|_{L^2}^2 + \|\text{div} \Delta B\|_{L^2}^2 \right) \| \Delta b \|_{L^2} \| \nabla \times \Delta B \|_{L^2} \]

\[ \leq \epsilon \| \nabla \times \Delta B \|_{L^2}^2 + \epsilon \|\text{div} \Delta B\|_{L^2}^2 + C \| \Delta B \|_{L^2}^2 \| \Delta b \|_{L^2}^4 \]

\[ + C \| \nabla B \|_{L^2}^2 + \| \Delta B \|_{L^2}^2 \). \quad (4.30) \]
We conclude the proof of (4.12) by considering the estimates (4.22)-(4.30) in (4.21) and taking a suitable \( \epsilon > 0 \).

\[ \]
Then, using the boundedness of \( (f_n(0))_{n \in \mathbb{N}} \) by \( f_0 = \| u_0 \|_{L^2(\mathbb{R}^3)}^2 + \| b_0 \|_{H^2(\mathbb{R}^3)}^2 \) and fixing \( 0 < T^* < \frac{1}{2Cf_0} \), we obtain that
\[
(u_n)_{n \in \mathbb{N}} \text{ and } (b_n)_{n \in \mathbb{N}} \text{ are bounded in } L^\infty((0, T^*), H^2(\mathbb{R}^3)).
\] (5.3)

Using again (5.2) and equations in (5.1), we can prove that
\[
(J_n u_n)_{n \in \mathbb{N}} \text{ and } (J_n b_n)_{n \in \mathbb{N}} \text{ are bounded in } L^2((0, T^*), H^3(\mathbb{R}^3)),
\] (5.4)
\[
(\partial_t u_n)_{n \in \mathbb{N}} \text{ and } (\partial_t b_n)_{n \in \mathbb{N}} \text{ are bounded in } L^\infty((0, T^*), L^2(\mathbb{R}^3)).
\] (5.5)

By (5.3)-(5.5), there exist a sub-sequence of \( (u_n, b_n)_{n \in \mathbb{N}} \) (still indexed by \( n \)) and functions \( u, b \in L^\infty((0, T^*), H^2(\mathbb{R}^3)) \cap L^2((0, T^*), H^3(\mathbb{R}^3)) \) such that (see \[38\])
\[
(u_n, b_n) \xrightarrow{n \to \infty} (u, b) \text{ weak-}^* \text{ in } L^\infty((0, T^*), H^2(\mathbb{R}^3) \times H^2(\mathbb{R}^3)),
\]
\[
(J_n u_n, J_n b_n) \xrightarrow{n \to \infty} (u, b) \text{ weak in } L^2((0, T^*), H^3(\mathbb{R}^3) \times H^3(\mathbb{R}^3)),
\]
\[
(u_n, b_n) \xrightarrow{n \to \infty} (u, b) \text{ strong in } L^2((0, T^*), L^2_{\text{loc}}(\mathbb{R}^3) \times L^2_{\text{loc}}(\mathbb{R}^3)),
\]
\[
(\partial_t u_n, \partial_t b_n) \xrightarrow{n \to \infty} (u, b) \text{ weak-}^* \text{ in } L^\infty((0, T^*), L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)).
\]

With these properties, we can apply the weak limit in (5.1) and prove that \( (u, b) \) is a local strong solution of (1.1) with initial data \( (u_0, b_0) \).

Let us to prove the uniqueness. Suppose that \( (u, b) \) and \( (v, h) \) are two weak solutions of (1.1) with the same initial data. Let \( U = v - u \) and \( B = h - b \). So, by inequality (4.10) (see Remark 4.1), we have that
\[
\frac{1}{2} \frac{d}{dt} (\| U(t) \|_{L^2}^2 + \| B(t) \|_{L^2}^2) \leq C_2 \left( \| \nabla u(t) \|_{L^2}^2 + \| \nabla b(t) \|_{L^2}^2 + \| \Delta b(t) \|_{L^2}^2 \right) \left( \| U(t) \|_{L^2}^2 + \| B(t) \|_{L^2}^2 \right).
\] (5.6)

Now the uniqueness of solutions in \( L^4((0, T^*), H^{\frac{3}{2}}(\mathbb{R}^3) \times H^2(\mathbb{R}^3)) \) follows from (5.6) and Gronwall inequality.

In what follows, we prove the blow-up criterion (3.1). If fact, suppose that \( (u, b) \) satisfies
\[
M := \int_0^T \left( \| \nabla u(s) \|_{L^2(\mathbb{R}^3)}^4 + \| \nabla b(s) \|_{L^2(\mathbb{R}^3)}^4 + \| \Delta b(s) \|_{L^2(\mathbb{R}^3)}^4 \right) \, ds < \infty.
\]

Proceeding in an analogous way to the proof of Lemma 4.3 we obtain
\[
\| u(t) \|_{H^2(\mathbb{R}^3)}^2 + \| b(t) \|_{H^2(\mathbb{R}^3)}^2 \leq e^{2CM} \left( \| u(0) \|_{H^2(\mathbb{R}^3)}^2 + \| b(0) \|_{H^2(\mathbb{R}^3)}^2 \right).
\]

So, by using the usual blow-up criterion of time-continuous \( H^2 \)-solutions, we have that the solution can be extended beyond \( T \) (see [9] and [38]).

Finally, let us prove global solutions for small initial data. By inequality (4.4) and equivalences (2.2)-(2.3), there exist \( C_5 > 0 \) and \( C_6 > 0 \) such that
\[
\frac{d}{dt} (\| b(t) \|_{H^2}^2 + \| u(t) \|_{H^2}^2) + C_5 \left( \| u(t) \|_{H^2}^2 + \| b(t) \|_{H^2}^2 \right) \leq C_6 \left( \| b(t) \|_{H^2}^2 + \| u(t) \|_{H^2}^2 \right)^3 + C_5 \left( \| u(t) \|_{L^2}^2 + \| b(t) \|_{L^2}^2 \right).
\] (5.7)

Suppose that the initial data is small enough to satisfy
\[
\| u(0) \|_{H^2}^2 + \| b(0) \|_{H^2}^2 \leq \frac{1}{12} \sqrt{\frac{C_5}{C_6}}.
\]
Let \( T^* > 0 \) be the supremum over all finite \( \tilde{T} > 0 \) such that
\[
\|u(t)\|_{H^2}^2 + \|b(t)\|_{H^2}^2 \leq \sqrt{\frac{C_5}{2C_6}}, \quad \forall 0 \leq t \leq \tilde{T}.
\]
By contradiction, let us assume that \( 0 < T^* < \infty \). By (5.7) we get
\[
\frac{d}{dt} (\|b(t)\|_{H^2}^2 + \|u(t)\|_{H^2}^2) + \frac{C_5}{2} (\|u(t)\|_{H^2}^2 + \|b(t)\|_{H^2}^2) \leq C_5 (\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2), \quad \forall 0 \leq t \leq \tilde{T},
\]
for all \( 0 < \tilde{T} < T^* \). Then, Gronwall type inequality and the time-uniform boundedness \( \|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 \leq \|u(0)\|_{L^2}^2 + \|b(0)\|_{L^2}^2 \) give
\[
\|b(t)\|_{H^2}^2 + \|u(t)\|_{H^2}^2 \leq e^{-\frac{C_5}{4}t} \left( \|u(0)\|_{H^2}^2 + \|b(0)\|_{H^2}^2 + \int_0^t e^{-\frac{C_5}{4}s} \left( \|u(s)\|_{L^2}^2 + \|b(s)\|_{L^2}^2 \right) ds \right)
\]
\[
\leq 3 (\|u(0)\|_{H^2}^2 + \|b(0)\|_{H^2}^2) \leq \frac{1}{4} \sqrt{\frac{C_5}{C_6}}, \quad \forall 0 \leq t \leq \tilde{T},
\]
for all \( 0 < \tilde{T} < T^* \). In view of the time-continuity of \((u, b)\) (see Remark 2.4) and \( \frac{1}{4} \sqrt{\frac{C_5}{C_6}} < \sqrt{\frac{C_5}{2C_6}} \), the estimate (5.8) contradicts the maximality of \( T^* \). So, \( T^* = \infty \) and the solution is global in time.

\[\Box\]

### 5.2 Proof of Theorem 3.2

Let \( U = v - u, B = h - b \) and
\[
L(t) = \|\nabla u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 + \|\Delta u\|_{L^2}^4 + \|\Delta b\|_{L^2}^4 + \|\nabla \times \Delta b\|_{L^2}^2 + \|\text{div} \Delta b\|_{L^2}^2.
\]
By Remark 4.7, there are \( C_{13} > 0, C_{14} > 0 \) and \( C_{15} > 0 \) such that
\[
\frac{d}{dt} (\|U\|_{H^2}^2 + \|B\|_{H^2}^2) + C_{13} (\|U\|_{H^2}^2 + \|B\|_{H^2}^2) \leq C_{14} (\|U\|_{H^2}^2 + \|B\|_{H^2}^2)^3 + C_{15} (\|U\|_{H^2}^2 + \|B\|_{H^2}^2) L(t) + C_{13} (\|U\|_{L^2}^2 + \|B\|_{L^2}^2).
\]
(5.9)

On the other side, using Lemma 4.5, we have that if (5.2) holds, then
\[
\int_0^\infty L(s)ds < \infty.
\]
Suppose that the initial data \((v(0), h(0))\) is close to \((u(0), b(0))\) to satisfy (here \( C_2 > 0 \) is given in (4.10))
\[
\|U(0)\|_{H^2}^2 + \|B(0)\|_{H^2}^2 \leq \frac{1}{12} \sqrt{\frac{C_{13}}{C_{14}}} \frac{1}{e^{(C_{15} + 2C_2) \int_0^\infty L(s)ds}}.
\]
(5.10)

Let \( T^* > 0 \) be the supremum over all finite \( \tilde{T} > 0 \) such that
\[
\|U(t)\|_{H^2}^2 + \|B(t)\|_{H^2}^2 \leq \sqrt{\frac{C_{13}}{2C_{14}}}, \quad \forall 0 \leq t \leq \tilde{T}.
\]
Assume by contradiction that $T^* < \infty$. So, for all $0 < \tilde{T} < T^*$, we get
\[
\frac{d}{dt} \left( \|U\|^2_{H^2} + \|B\|^2_{H^2} \right) + \frac{C_{13}}{2} \left( \|U\|^2_{H^2} + \|B\|^2_{H^2} \right) \leq C_{15} \left( \|U\|^2_{H^2} + \|B\|^2_{H^2} \right) L(t) \\
+ C_{13} \left( \|U\|^2_{L^2} + \|B\|^2_{L^2} \right) \forall 0 \leq t \leq \tilde{T}.
\]
By Gronwall type inequality, for all $0 < \tilde{T} < T^*$, we obtain
\[
\|U(t)\|^2_{H^2} + \|B(t)\|^2_{H^2} \leq e^{\frac{C_{13}}{2} t + C_{15} \int_0^t L(s) ds} \left( \|U(0)\|^2_{H^2} + \|B(0)\|^2_{H^2} \right) \\
+ e^{\frac{C_{13}}{2} t + C_{15} \int_0^t L(s) ds} C_{13} \int_0^t e^{\frac{C_{13}}{2} s} \left( \|U(s)\|^2_{L^2} + \|B(s)\|^2_{L^2} \right) ds \\
\leq \frac{1}{12} \sqrt{\frac{C_{13}}{C_{14}}} + 2e^{C_{15} \int_0^\infty L(s) ds} \sup_{s > 0} \left\{ \|U(s)\|^2_{L^2} + \|B(s)\|^2_{L^2} \right\}, \forall 0 \leq t \leq \tilde{T}.
\]
Finally, it follows from (4.10) and Gronwall inequality that
\[
\sup_{s > 0} \left\{ \|U(s)\|^2_{L^2} + \|B(s)\|^2_{L^2} \right\} \leq e^{2C_2 \int_0^\infty L(s) ds} \left( \|U(0)\|^2_{L^2} + \|B(0)\|^2_{L^2} \right), \forall 0 \leq t \leq \tilde{T}.
\]
So
\[
\|U(t)\|^2_{H^2} + \|B(t)\|^2_{H^2} \leq \frac{1}{4} \sqrt{\frac{C_{13}}{C_{14}}}, \forall 0 \leq t \leq \tilde{T}, \tag{5.11}
\]
for all $0 < \tilde{T} < T^*$. The estimate (5.11) contradicts the maximality of $T^*$ because $\frac{1}{4} \sqrt{\frac{C_{13}}{C_{14}}} < \sqrt{\frac{C_{14}}{2C_{15}}} \int_0^\infty L(s) ds$ and $(U, B)$ is time-continuous (see Remark 2.4). It follows that $T^* = \infty$. As $(u, b)$ is a global solution in $H^2(\mathbb{R}^3)$, the above inequality implies that $(v, h)$ is as well. Furthermore, by repeating steps between (5.10)-(5.11), one can check that for initial data less than $\delta$, where $0 < \delta < \frac{1}{12} \sqrt{\frac{C_{13}}{C_{14}}} e^{(C_{15}+2C_2) \int_0^\infty L(s) ds}$, we can take $M(\delta) = 3\delta e^{C_{15}+2C_2} \int_0^\infty L(s) ds$, for $M(\delta)$ as in the statement of the theorem. This concludes the proof.

\section{5.3 Proof of Theorem 3.3}

The proof of the local in time existence of $b$ is standard and it follows the previous ideas (see the proof of Theorem 3.1). Recall that $u$ is a global strong solution and verifies (3.7). Let us now prove that $(u, b)$ is global and satisfies (3.2). In fact, these facts can be obtained just following the above steps and observing that
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|b\|^2_{L^2} + \gamma \|\nabla b\|^2_{L^2} & = 0, \\
\frac{1}{2} \frac{d}{dt} \|\nabla b\|^2_{L^2} + \frac{\gamma}{2} \|\Delta b\|^2_{L^2} & \leq C \|\nabla u\|^4_{L^2} \|
abla b\|^2_{L^2}, \\
\frac{1}{2} \frac{d}{dt} \|\Delta b\|^2_{L^2} + \gamma \|\text{div} \Delta b\|^2_{L^2} & = 0.
\end{align*}
\]
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