Irreducibility criterion for a finite-dimensional highest weight representation of the $sl_2$ loop algebra and the dimensions of reducible representations

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Abstract. We present a necessary and sufficient condition for a finite-dimensional highest weight representation of the $sl_2$ loop algebra to be irreducible. In particular, for a highest weight representation with degenerate parameters of the highest weight, we can explicitly determine whether it is irreducible or not. We also present an algorithm for constructing finite-dimensional highest weight representations with a given highest weight. We give a conjecture that all of the highest weight representations with the same highest weight can be constructed by the algorithm. For some examples we show the conjecture explicitly. The result should be useful in analysing the spectra of integrable lattice models related to roots of unity representations of quantum groups, in particular, the spectral degeneracy of the XXZ spin chain at roots of unity associated with the $sl_2$ loop algebra.

Keywords: algebraic structures of integrable models, solvable lattice models, symmetries of integrable models

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1. Introduction

Symmetry operators play a central role in the spectra of quantum systems. Recently, it was shown that the XXZ spin chain at roots of unity commutes with the $sl_2$ loop algebra [1], and there exist large spectral degeneracies associated with the symmetry [2]–[7]. The Hamiltonian of the XXZ spin chain under the periodic boundary conditions is given by

$$H_{XXZ} = \frac{1}{2} \sum_{j=1}^{L} (\sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y + \Delta \sigma_j^Z \sigma_{j+1}^Z).$$

(1)
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Here the XXZ anisotropic coupling $\Delta$ is related to the $q$ parameter by $\Delta = (q + q^{-1})/2$. The symmetry of the $sl_2$ loop algebra appears when $q$ is a root of unity. The symmetry also appears in the spectrum of the transfer matrix of the six-vertex model at roots of unity [1]. We note that the XXZ Hamiltonian is derived from the logarithmic derivative of the transfer matrix of the six-vertex model.

Let us introduce the $sl_2$ loop algebra, $U(L(sl_2))$ [8]–[10]. The generators $x_k^\pm$ and $h_k$ for $k \in \mathbb{Z}$ satisfy

$$[h_j, x_k^\pm] = \pm 2 x_{j+k}^\pm, \quad [x_j^+, x_k^-] = h_{j+k},$$
$$[h_j, h_k] = 0, \quad [x_j^+, x_k^+] = 0, \quad \text{for} \ j, k \in \mathbb{Z}. \tag{2}$$

In a representation of $U(L(sl_2))$, we call a vector $\Omega$ highest weight if $\Omega$ is annihilated by generators $x_k^+$ for all integers $k$ and such that $\Omega$ is a simultaneous eigenvector of every generator $h_k$ ($k \in \mathbb{Z}$) [9]–[11]:

$$x_k^+ \Omega = 0, \quad \text{for} \ k \in \mathbb{Z}, \tag{3}$$

$$h_k \Omega = d_k \Omega, \quad \text{for} \ k \in \mathbb{Z}. \tag{4}$$

We call the set of eigenvalues $d_k$ the highest weight of $\Omega$. The representation generated by a highest weight vector $\Omega$ is called the highest weight representation of $\Omega$. In this paper we assume that $\Omega$ generates a finite-dimensional representation.

It is easy to show that the weight $d_0$ is given by a non-negative integer. We denote it by $r$. We shall show that $\Omega$ is a simultaneous eigenvector of operators $(x_0^+)^n(x_1^-)^n$:

$$(x_0^+)^n(x_1^-)^n \Omega = (n!)^2 \lambda_n \Omega, \quad \text{for} \ n = 1, 2, \ldots, r. \tag{5}$$

In terms of $\lambda_n$’s, we define a polynomial $P_\lambda(u)$ as follows:

$$P_\lambda(u) = \sum_{k=0}^r (-1)^k \lambda_k u^k. \tag{6}$$

Here $\lambda$ denotes a sequence of $\lambda_n$’s, i.e. $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$. We call $P_\lambda(u)$ the highest weight polynomial for the highest weight $d_k$. The highest weight polynomial generalizes the Drinfeld polynomial. In fact, every finite-dimensional highest weight representation has its highest weight polynomial. If the representation is irreducible, the highest weight polynomial $P_\lambda(u)$ is nothing but the Drinfeld polynomial [10]. Let us factorize polynomial $P_\lambda(u)$ as follows:

$$P_\lambda(u) = (1 - \hat{a}_1 u) \cdots (1 - \hat{a}_r u). \tag{7}$$

We call parameters $\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_r$, the highest weight parameters of $\Omega$. In terms of highest weight parameters $\hat{a}_j$, the highest weight $d_k$ of $\Omega$ is expressed as

$$d_k = \sum_{j=1}^r \hat{a}_j^k \quad \text{for} \ k \in \mathbb{Z}, \tag{8}$$

and the eigenvalues $\lambda_k$ are expressed as

$$\lambda_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq r} \hat{a}_{i_1} \hat{a}_{i_2} \cdots \hat{a}_{i_k}. \tag{9}$$

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Let us discuss how to evaluate degenerate multiplicities in the spectrum of some model that has the sl₂ loop algebra symmetry. They are given by the dimensions of some finite-dimensional representations of the sl₂ loop algebra. Here we have a conjecture that every finite-dimensional representation is decomposed into a collection of finite-dimensional highest weight representations. Thus, the degenerate multiplicities should be evaluated essentially in terms of the dimensions of corresponding finite-dimensional highest weight representations. The dimensions of irreducible representations of $U(L(sl₂))$ and those of $U_q(L(sl₂))$ are known [9]–[11]. Furthermore, it was shown by Chari and Pressley [12] that corresponding to each irreducible finite-dimensional representation with highest weight parameters $\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_r$, there exists a unique finite-dimensional highest weight module $W$ with the highest weight parameters $\hat{a}_j$ for $j = 1, 2, \ldots, r$, such that any finite-dimensional highest weight module $V$ with the same highest weight parameters $\hat{a}_j$ ($j = 1, 2, \ldots, r$), is a quotient of $W$. We call $W$ the Weyl module of the highest weight parameters $\hat{a}_j$. It has also been shown [12] that a Weyl module is irreducible if and only if the highest weight parameters $\hat{a}_j$ are distinct. However, if some of the highest weight parameters are degenerate, it is not trivial whether the representation generated by $\Omega$ is irreducible or not.

Here, we prove a necessary and sufficient condition for a finite-dimensional highest weight representation to be irreducible. Suppose that a highest weight vector $\Omega$ for $j = 1, 2, \ldots, r$, generates a finite-dimensional representation $U\Omega$. We also assume that the highest weight parameters $\hat{a}_j$ for $j = 1, 2, \ldots, r$ are given by a set of distinct parameters $a_j$ with multiplicities $m_j$ for $1 \leq j \leq s$. Then, we shall show that $U\Omega$ is irreducible if and only if the following condition holds:

$$
\sum_{j=0}^{s} (-1)^{s-j} \mu_{s-j} x_j \Omega = 0,
$$

(10)

where coefficients $\mu_k$ ($k = 1, 2, \ldots, s$) are given by

$$
\mu_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq s} a_{i_1} a_{i_2} \cdots a_{i_k}.
$$

(11)

If $U\Omega$ is irreducible, the dimensionality is given by

$$
\dim U\Omega = \prod_{j=1}^{s} (m_j + 1).
$$

(12)

Here we note that if the highest weight parameters $\hat{a}_j$ are distinct (i.e. $s = r$), condition (10) holds trivially. Furthermore, we introduce an algorithm by which we can construct practically all finite-dimensional highest weight modules with a given set of highest weight parameters $\hat{a}_j$. It is a conjecture that all such representations are constructed by the algorithm. Here we note that the algorithm is not complete in the sense that we use some conjectured relations among products of generators acting on $\Omega$, by which we exclude some redundant quotients of submodules. For some simple cases, however, we explicitly calculate the dimensions of all possible reducible highest weight representations that have the same given set of highest weight parameters $\hat{a}_j$.

As an illustration, let us consider the case of $r = 3$ where two of the three highest weight parameters are degenerate, i.e. $(\hat{a}_1, \hat{a}_2, \hat{a}_3) = (a_1, a_1, a_2)$. We have

$$
P_\lambda(u) = (1 - a_1 u)^2 (1 - a_2 u) = 1 - (2a_1 + a_2)u + (a_1^2 + 2a_1 a_2)u^2 - a_1^2 a_2 u^3.
$$

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For any highest weight vector $\Omega$ with the same highest weight we have
\[(x_3^2 - (2a_1 + a_2)x_2^2 + (a_1^2 + 2a_1a_2)x_1^2 - a_1^2a_2x_0^2) \Omega = 0.\]
However, if $\Omega$ satisfies the following relation:
\[(x_2^2 - (a_1 + a_2)x_1^2 + a_1a_2x_0^2)\Omega = 0,\]
then it generates an irreducible representation.

The highest weight polynomial should be useful for physical applications. Let us recall that every finite-dimensional highest weight representation has a unique highest weight polynomial $P_{\lambda}(u)$, while it has the Drinfeld polynomial only if it is irreducible. We also recall that the degenerate eigenspaces of some physical system that has the $sl_2$ loop algebra symmetry should be given by collections of finite-dimensional highest weight representations. However, it is not certain whether they are irreducible or not. We therefore introduce the highest weight polynomial, by which we can investigate highest weight representations that are not necessarily irreducible.

The general criterion of irreducibility should be useful for studying the spectra of some integrable models associated with roots of unity representations of the quantum groups, in particular, the spectral degeneracy of the XXZ spin chain and the six-vertex model at roots of unity [5]. Recall that in order to derive the degenerate multiplicities of the $sl_2$ loop algebra symmetry rigorously, one has to calculate the dimensions of highest weight representations generated by the corresponding Bethe vectors [4, 5]. However, it is not trivial whether they are irreducible or not. In fact, there exists such a Bethe vector that is highest weight and generates a reducible representation [5]. Furthermore, it has not been discussed how to derive the dimensions of reducible highest weight representations in the general case. Thus, through the irreducibility criterion and the algorithm for constructing practically all reducible representations with the same highest weight, we can evaluate the degenerate multiplicities systematically. It should be stated that the degenerate eigenvectors of the $sl_2$ loop algebra can also be discussed in terms of Bethe vectors through some limiting procedure [13]. However, such a limiting procedure is not always straightforward [2, 3], and it does not give a systematic method. We also note that the irreducibility criterion (10) has been announced in [5]–[7] without an explicit proof for the general case.

There are several important viewpoints associated with the $sl_2$ loop algebra symmetry of the six-vertex model at roots of unity. (i) Roots of unity representations of the quantum groups have many subtle and interesting properties [14]–[17]. They have several connections to the six-vertex model and the XXZ spin chain at roots of unity [18, 19]. Roots of unity representations are also related to the chiral Potts model [20, 21]. (ii) It has been shown for the $sl_2$ loop algebra that every irreducible representation is given by a tensor product of evaluation modules with distinct evaluation parameters [9]. However, it has not been discussed how to determine whether a given highest weight representation is irreducible or not. For $U_q(L(sl_2))$, the irreducibility criterion for tensor products of evaluation modules has been shown in [10]. For Yangians, an irreducibility criterion for tensor products of Yangian evaluation modules has been derived in [22]. (iii) The Bethe ansatz equations of integrable lattice models associated with Yangians and reflection algebras are discussed [23]. Here, the Drinfeld polynomials play an important role. (iv) Quantum groups at roots of unity have cyclic representations which have no highest weight vector [14]. It has been conjectured that some highest weight representation of the $sl_2$
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The $sl_2$ loop algebra symmetry of the XXZ spin chain at roots of unity should also be interesting in the spectral analysis of quantum Hamiltonians. In the spectral flow of the XXZ spin chain with respect to $\Delta$, level crossings with exponentially large degenerate multiplicities appear at some discrete values of $\Delta$. According to a theorem by von Neumann and Wigner, it is far more likely to have spectral degeneracies if one or more symmetries exist than if no symmetries exist [25]. In association with it, the non-crossing rule is stated as follows: in the spectrum of a Hamiltonian that depends on a real parameter, levels of the same symmetry never cross each other as the parameter varies, where all levels are classified by symmetry quantum numbers. Here we call an operator commuting with the Hamiltonian a symmetry operator if it is independent of the model parameter [26]. However, the spectrum of a Hamiltonian can have degeneracies. Novel counterexamples to the non-crossing rule were discussed in the spectrum of the one-dimensional Hubbard Hamiltonian [26].

This paper is organized as follows. In section 2, we introduce generators of the $sl_2$ loop algebra with parameters. In section 3, we discuss sectors of highest weight representations. We show that every finite-dimensional highest weight representation has non-zero and finite highest weight parameters. In section 4, we define highest weight parameters and highest weight polynomials, explicitly. In section 5, we prove the irreducibility criterion. In section 6, we formulate an algorithm by which we can construct practically all reducible or irreducible highest weight representations that have the same highest weight. For two simple cases, we derive dimensions of all reducible highest weight modules that have the same highest weight, explicitly.

Throughout the paper we assume that $\Omega$ is a non-zero highest weight vector with highest weight $d_k$ in a finite-dimensional representation of $U(L(sl_2))$. Thus, it generates a finite-dimensional representation, which we denote by $U\Omega$. We denote the highest weight $d_0$ by $r$, i.e. $h_0 \Omega = r \Omega$. As shown in section 3, $r$ is given by a non-negative integer, and it is equal to the number of highest weight parameters $\hat{a}_j$ of $\Omega$.

2. Loop algebra generators with parameters

Let $\alpha$ denote a finite sequence of complex parameters such as $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. We define generators with $n$ parameters, $x^\pm_m(\alpha)$ and $h_m(\alpha)$, as follows [6, 7]:

\[
\begin{align*}
x^\pm_m(\alpha) &= \sum_{k=0}^{n} (-1)^k x^\pm_{m-k} \sum_{\{i_1, \ldots, i_k \} \subset \{1, \ldots, n\}} \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k}, \\
h_m(\alpha) &= \sum_{k=0}^{n} (-1)^k h_{m-k} \sum_{\{i_1, \ldots, i_k \} \subset \{1, \ldots, n\}} \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k}.
\end{align*}
\]

Let $\alpha$ and $\beta$ be arbitrary sequences of $n$ and $p$ parameters, respectively. In terms of generators with parameters we express the defining relations of the $sl_2$ loop algebra as

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introduce the following symbol: \( \alpha \)

Let us introduce a symbol \( \alpha \beta \) denotes the composite sequence of \( \alpha \) and \( \beta \):

\[
\alpha \beta = (\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_p).
\]

Using relations (14), we can show the following relations for \( t \in \mathbb{Z}_{\geq 0} \):

\[
[x^+_\ell(\alpha), x^-_m(\beta)] = h_{\ell+m}(\alpha \beta), \quad [h_\ell(\alpha), x^+_m(\beta)] = \pm 2x^+_\ell(\alpha \beta).
\]

Here the symbol \( \alpha \beta \) denotes the composite sequence of \( \alpha \) and \( \beta \):

\[
\alpha \beta = (\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_p).
\]

Using relations (14), we can show the following relations for \( t \in \mathbb{Z}_{\geq 0} \):

\[
[(x^+_m(\alpha))^{(t)}, x^-_\ell(\beta)] = (x^+_m(\alpha))^{(t-1)}h_{\ell+m}(\alpha \beta) + x^+_\ell+2m(\alpha \beta)(x^+_m(\alpha))^{(t-2)},
\]

\[
[x^+_\ell(\alpha), (x^-_m(\beta))^{(t)}] = (x^-_m(\beta))^{(t-1)}h_{\ell+m}(\alpha \beta) - x^-_{\ell+2m}(\alpha \beta)(x^-_m(\beta))^{(t-2)},
\]

\[
[h_\ell(\alpha), (x^-_m(\beta))^{(t)}] = \pm 2(x^+_m(\beta))^{(t-1)}x^+_\ell+m(\alpha \beta).
\]

Here the symbol \((X)^{(n)}\) denotes the \( n \)th power of operator \( X \) divided by the \( n \) factorial, i.e. \((X)^{(n)} = X/n!\).

For a given sequence of \( m \) parameters, \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \), let us denote by \( \alpha_j \) the sequence of parameters of \( \alpha \) other than \( \alpha_j \):

\[
\alpha_j = (\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_m).
\]

Here we assume that parameters \( \alpha_j \) take distinct values for \( j = 1, 2, \ldots, m \). We now introduce the following symbol:

\[
\rho^+_j(\alpha; m) = x^+_{m-1}(\alpha_j) \quad \text{for} \quad j = 1, 2, \ldots, m.
\]

The generators \( x^+_j \) for \( j = 0, 1, \ldots, m-1 \), are expressed as linear combinations of \( \rho^+_j(\alpha; m) \). Let us introduce a symbol \( \alpha_{kj} \) by \( \alpha_{kj} = \alpha_k - \alpha_j \). It is easy to show the following:

\[
(-1)^{n-1}\sum_{j=1}^{n}\frac{\rho^+_j(\alpha; m)}{\prod_{k=1; k\neq j}^{n}\alpha_{kj}} = x^+_{m-n}(\alpha_{n+1}, \ldots, \alpha_m) \quad (1 \leq n \leq m).
\]

It can be shown by making use of equation (19) that \( x^+_k \) \((0 \leq k \leq m - 1)\) are expressed in terms of linear combinations of \( \rho^+_j(\alpha; m) \) with \( 1 \leq j \leq m \).

3. Sectors of a highest weight representation

Let us briefly introduce elementary representation theory of \( sl_2 \) as follows.

**Lemma 1.** Let \( e, f \) and \( h \) be standard generators of \( sl_2 \). If \( u \) is a non-zero vector in a finite-dimensional representation of \( sl_2 \) such that \( eu = 0 \) and \( hu = ru \), then we have the following: (i) \( r \) is a non-negative integer; (ii) \( U(sl_2)u \) is an irreducible \((r+1)\)-dimensional representation; (iii) \( f^ru \neq 0 \) and \( f^{r+1}u = 0 \).

We now consider the \( sl_2 \)-subalgebra \( U_k \) generated by \( x^+_k, x^-_k \) and \( h_0 \) for an integer \( k \). Here we recall that \( \Omega \) is a non-zero highest weight vector with highest weight \( d_k \) in a finite-dimensional representation of \( U(L(sl_2)) \). Applying the Poincaré–Birkhoff–Witt theorem [27] to \( U(L(sl_2)) \), we can show that in \( U\Omega \) eigenvalues of \( h_0 \) are given by integers. We call them weights. It also follows that \( U\Omega \) is given by the direct sum of subspaces of weights, i.e. sectors with respect to eigenvalues of \( h_0 \). Applying lemma 1 to \( U_k \) we have the following.

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Corollary 2. The highest weight $d_0$ is given by a non-negative integer, $r$, and we have $(x_k^-)^r \Omega \neq 0$ and $(x_k^-)^{r+1} \Omega = 0$ for $k \in \mathbb{Z}$.

Proposition 3. The subspace of weight $-r$ of $U \Omega$ is one-dimensional.

We shall discuss an explicit proof of proposition 3 in appendix A.

Lemma 4. $\Omega$ is a simultaneous eigenvector of $(x_0^+)^{(n)}(x_1^-)^{(n)}$:

$$(x_0^+)^{(j)}(x_1^-)^{(j)} \Omega = \lambda_j \Omega, \quad \text{for } j = 1, 2, \ldots, r. \quad (20)$$

Here $\lambda_j$ are eigenvalues.

Proof. Applying the Poincaré–Birkhoff–Witt theorem [27] to $U(L(sl_2))$, we derive that the subspace of weight $r$ in $U \Omega$ is one-dimensional. Here we note that $(x_0^+)^{(j)}(x_1^-)^{(j)} \Omega$ is in the subspace of weight $r$ in $U \Omega$. Therefore, $(x_0^+)^{(j)}(x_1^-)^{(j)} \Omega$ is proportional to the basis vector $\Omega$, and hence we have (20). $\square$

Proposition 5. Eigenvalue $\lambda_r$ is non-zero. Here we recall $(x_0^+)^{(r)}(x_1^-)^{(r)} \Omega = \lambda_r \Omega$.

Proof. Recall corollary 2 that $(x_1^-)^r \Omega \neq 0$ and $(x_0^-)^r \Omega \neq 0$. It follows from proposition 3 that they are linearly dependent, i.e. we have $(x_1^-)^r \Omega = A_1(x_0^-)^r \Omega$ with a non-zero constant $A_1$. The eigenvalue $\lambda_r$ is given by $A_1$ as follows:

$$\lambda_r (r!)^2 \Omega = (x_0^+)^{(r)}(x_1^-)^{(r)} \Omega = A_1(x_0^-)^{(r)}(x_0^-)^{(r)} \Omega = A_1 (r!)^2 \Omega. \quad (21)$$

We thus obtain $\lambda_r = A_1$ and $\lambda_r \neq 0$. $\square$

4. Highest weight polynomials

4.1. Parameters expressing the highest weight

We now introduce parameters expressing the highest weight of $\Omega$. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ denote the sequence of eigenvalues $\lambda_k$ which are defined in equation (20). We define a polynomial $P_\lambda(u)$ by the following relation [28]:

$$P_\lambda(u) = \sum_{k=0}^{r} \lambda_k (-u)^k. \quad (22)$$

We call it the highest weight polynomial of $\Omega$.

Let us factorize polynomial $P_\lambda(u)$ as follows

$$P_\lambda(u) = \prod_{k=1}^{s} (1 - a_k u)^{m_k}, \quad (23)$$

where $a_1, a_2, \ldots, a_s$ are distinct, and their multiplicities are given by $m_1, m_2, \ldots, m_s$, respectively. We denote by $a$ the sequence of $s$ parameters $a_j$:

$$a = (a_1, a_2, \ldots, a_s). \quad (24)$$

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Here we note that \( r \) is equal to the sum of multiplicities \( m_j: r = m_1 + \cdots + m_s \). We define parameters \( \hat{a}_i \) for \( i = 1, 2, \ldots, r \), as follows.

\[
\hat{a}_i = a_k \quad \text{if } m_1 + m_2 + \cdots + m_{k-1} < i \leq m_1 + \cdots + m_{k-1} + m_k. \tag{25}
\]

Then, the set \( \{\hat{a}_j | j = 1, 2, \ldots, r\} \) corresponds to the set of parameters \( a_j \) with multiplicities \( m_j \) for \( j = 1, 2, \ldots, s \). We denote by \( \hat{a} \) the sequence of \( r \) parameters \( \hat{a}_i \):

\[
\hat{a} = (\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_r). \tag{26}
\]

We call parameters \( \hat{a}_i \) the highest weight parameters of \( \Omega \). It follows from the definition of highest weight polynomial \( P_\lambda(u) \) given by (22) and that of highest weight parameters (23) that we have

\[
\lambda_n = \sum_{1 \leq j_1 < \cdots < j_n \leq r} \hat{a}_{j_1} \cdots \hat{a}_{j_n}. \tag{27}
\]

**Proposition 6.** The roots of highest weight polynomial \( P_\lambda(u) \) are non-zero, and the degree is given by \( r \).

**Proof.** Recall proposition 5 that \( \lambda_r \neq 0 \). We note that \( \lambda_r = \prod_{j=1}^r \hat{a}_j = \prod_{j=1}^s a_j^{m_j} \). Therefore, \( a_j \) are non-zero for \( j = 1, 2, \ldots, s \).

\[\Box\]

### 4.2. Recursive lemmas

Let \( a \) be an arbitrary complex number. We denote by \( (a)^n \) the sequence of parameter \( a \) with multiplicity \( n \), i.e. \( (a)^n = (a, a, \ldots, a) \). For \( n = 1 \) we write \( x_n^\pm((a)^1) = x_n^\pm - ax_{n-1}^\pm \) simply as \( x_n^\pm(a) \). We also introduce the following:

\[
\lambda_n(a) = \sum_{1 \leq j_1 < \cdots < j_n \leq r} (\hat{a}_{j_1} - a) \cdots (\hat{a}_{j_n} - a). \tag{28}
\]

**Lemma 7.** For a given integer \( \ell \), we have

\[
(x_\ell^+(a))^{(n)}(x_{\ell-\ell}^{-}(a))^{(n+1)} = x_{\ell-\ell}^{-}(a)(x_\ell^+(a))^{(n)}(x_{\ell-\ell}^{-}(a))^{(n)} + \frac{1}{2}[h_1(a), (x_\ell^+(a))^{(n-1)}(x_{\ell-\ell}^{-}(a))^{(n)}] - (x_\ell^+(a))^{(n-1)}(x_{\ell-\ell}^{-}(a))^{(n+1)}x_\ell^+, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{29}
\]

**Proof.** We first show the following relations by induction on \( n \):

\[
[h_1(a), (x_\ell^+(a))^{(n)}] = 2x_{\ell+1}^-(a)(x_\ell^+(a))^{(n-1)},
\]

\[
[(x_\ell^+(a))^{(n)}, x_{\ell-\ell}^-(a)] = (x_\ell^+(a))^{(n-1)}h_1(a) + x_{\ell+1}^+(a)(x_\ell^+(a))^{(n-2)},
\]

\[
[h_1(a), (x_{\ell-\ell}^{-}(a))^{(n)}] = -(2)x_{\ell-1}^-(a)(x_{\ell-\ell}^-(a))^{(n-1)},
\]

\[
[x_\ell^+, (x_{\ell-\ell}^{-}(a))^{(n)}] = (x_{\ell-\ell}^{-}(a))^{(n-1)}h_1(a) - x_{\ell-1}^-(a)(x_{\ell-\ell}^{-}(a))^{(n-2)}. \tag{30}
\]

Making use of relations (30) we can show relation (29). Some details will be shown in appendix B. \[\Box\]

In the case of \( a = 0 \) and \( \ell = 0 \) the relation (29) has been shown for the case of \( \mathcal{U}_q(sl(2)) \) \[10\].

Let \( U \) denote the \( sl_2 \) loop algebra \( U(L(sl_2)) \). For a given integer \( \ell \), let \( U(\mathcal{B}_\ell) \) be the subalgebra of \( U(L(sl_2)) \) generated by \( h_k, x_{\ell+k}^+ \) and \( x_{\ell+1+k}^- \) for \( k \in \mathbb{Z}_{\geq 0} \). We denote by \( \mathcal{B}_\ell^+ \) such a subalgebra of \( U(\mathcal{B}_\ell) \) that is generated by \( x_{\ell+k}^+ \) for \( k \in \mathbb{Z}_{\geq 0} \).

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Lemma 8. For a given integer \( \ell \) we have the following recursive relations for \( n \in Z \):

\[
\begin{align*}
(A_n): & \quad (x_\ell^+)^{(n-1)}(x_{-\ell}^-)(a)^{(n)} = \sum_{k=1}^{n} (-1)^{k-1} x_{-\ell}^-((a)^{(k)})(x_\ell^+)^{(n-k)}(x_{-\ell}^-)(a)^{(n-k)} \mod U(B_\ell)B_\ell^+, \\
(B_n): & \quad (x_\ell^+)^{(n)}(x_{-\ell}^-)(a)^{(n)} = \frac{1}{n} \sum_{k=1}^{n} (-1)^{k-1} h_k((a)^{(k)})(x_\ell^+)^{(n-k)}(x_{-\ell}^-)(a)^{(n-k)} \mod U(B_\ell)B_\ell^+, \\
(C_n): & \quad [h_j(a),(x_\ell^+)^{(m)}(x_{-\ell}^-)(a)^{(m)}] = 0 \mod U(B_\ell)B_\ell^+ \quad \text{for } m \leq n \text{ and } j \in Z.
\end{align*}
\]

Proof. We now show relations \((A_n), (B_n), \) and \((C_n)\), inductively on \( n \) as follows. We first show \((A_1), (A_2), (B_1), \) and \((C_1), \) directly. Then, relation \((A_n)\) is derived from \((A_{n-1})\) and \((C_{n-2})\). Here we make use of formula (29). Relation \((B_n)\) is derived from \((A_n), (C_{n-1})\) and \((B_m)\) for \( m \leq n - 1 \). We multiply both sides of \((A_n)\) by \( x_\ell^+ \) from the left. We show \((x_\ell^+)^{(n)}(x_{-\ell}^-)(a)^{(m)} \in U(B_\ell)B_\ell^+ \) for \( m \leq n - 1 \) by induction on \( m \). Here we make use of \((B_m)\) for \( m \leq n - 1 \) and \( C_{n-1} \). Finally, \((C_n)\) is derived from \((B_{n-1})\) and \((C_{n-1})\). Thus, the cycle of induction process, \((A_n), (B_n), \) and \((C_n), \) is closed. \( \square \)

Applying relations \((B_n)\) of lemma 8, we shall prove an irreducibility criterion in section 5.

4.3. Reduction relations

Let us define the elementary symmetric polynomials \( p_m \) in \( x_1, \ldots, x_r \), as follows:

\[
p_m = \sum_{1 \leq i_1 < \cdots < i_m \leq r} x_{i_1} \cdots x_{i_m}. \tag{31}
\]

We introduce symmetric polynomials \( s_k \) by \( s_k = \sum_{j=1}^{r} x_j^k \). Then, Newton’s formulae are given as follows:

\[
p_n = \frac{1}{n} \sum_{k=1}^{n} (-1)^{k-1} s_k p_{n-k} \quad \text{for } n \leq r, \tag{32}
\]

\[
s_{r+1+j} = \sum_{k=1}^{r} (-1)^{r-k} s_{k+j} p_{r+1-k} \quad \text{for } j \in Z. \tag{33}
\]

We now show systematically that highest weights \( d_k \) are expressed as the symmetric polynomials \( s_k \) of highest weight parameters \( \hat{a}_j \). Substituting \( \ell = 0 \) and \( a = 0 \) in \((B_n)\) of lemma 8 and making use of (4), we have the following.

Corollary 9. \[
\lambda_n = \frac{1}{n} \sum_{k=1}^{n} (-1)^{k-1} d_k \lambda_{n-k}, \quad \text{for } n = 1, 2, \ldots, r. \tag{34}
\]

Lemma 10. For any integer \( \ell \) we have

\[
(x_\ell^+)^{(n)}(x_{-\ell}^-)^{(n)} \Omega = \lambda_n \Omega, \quad \text{for } n = 1, 2, \ldots, r. \tag{35}
\]

\[\text{doi:10.1088/1742-5468/2007/05/P05007}\]
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Proof. Making use of (B$_n$) with $a = 0$, we show (35) by induction on $n$. Here we also make use of (34) and (4).

Proposition 11 (Reduction relations).

\[ x_{r+1-\ell}\Omega = \sum_{k=1}^{r} (-1)^{r-k} \lambda_{r+1-k} x_{k-\ell}\Omega, \quad \text{for} \ \ell \in \mathbb{Z}, \quad (36) \]

\[ d_{r+1-\ell} = \sum_{k=1}^{r} (-1)^{r-k} \lambda_{r+1-k} d_{k-\ell}, \quad \text{for} \ \ell \in \mathbb{Z}. \quad (37) \]

Proof. We derive reduction relations (36) from (A$_{r+1}$) of lemma 8 with $a = 0$ and lemma 10. Applying $x_0^+$ to (36) from the left, we derive relations (37).

Proposition 12. The highest weights $d_n$ are given by the following symmetric polynomials of highest weight parameters $\hat{a}_j$:

\[ d_n = \sum_{j=1}^{r} \hat{a}_j^n, \quad \text{for} \ n \in \mathbb{Z}. \quad (38) \]

Proof. Making use of (34) and Newton’s formula (32), we show (38) by induction on $n$ for $n = 1, 2, \ldots, r$. Then, we generalize (38) to the case of arbitrary integers $n$ through Newton’s formula (33).

Proposition 13. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a sequence of arbitrary complex parameters. For a given integer $m$ we have

\[ h_m(\alpha)\Omega = d_m(\alpha)\Omega, \quad (39) \]

where $d_m(\alpha)$ is given by

\[ d_m(\alpha) = \sum_{j=1}^{r} \hat{a}_j^{m-n} \prod_{i=1}^{n}(\hat{a}_j - \alpha_i). \quad (40) \]

In particular, if the set $\alpha$ contains $a_1, a_2, \ldots, a_s$, we have

\[ h_m(\alpha)\Omega = 0. \quad (41) \]

Proof. Through (37) and (33), we show (38) for any integer $n$. Substituting relations (38) for $n \in \mathbb{Z}$ into (13), we obtain (39).

Reduction relations (36) are expressed as follows:

\[ x_{r+1-\ell}(\hat{a})\Omega = 0 \quad \text{for} \ \ell \in \mathbb{Z}. \quad (42) \]

Reduction relations (42) are fundamental when we construct a reducible or irreducible highest weight representation. Here we note that relations (36) and (42) play a similar role as the characteristic equations of matrices.
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Lemma 14. Let $\ell$ be an integer. If $x_\ell^-(A)\Omega = 0$ for a sequence of parameters $A$, then for any sequence of parameters $B$ that contains $A$ as a set, we have

$$x_m^-(B)\Omega = 0 \quad \text{for } m \in \mathbb{Z}. \quad (43)$$

Proof. Let us denote by $C$ a sequence of elements of the complementary set $B \setminus A$. When $A \subset B$, we can permute the elements of sequence $B$ so that it is given by $CA$, a composite sequence of $A$ and $C$. Here we note that $x_m^-(B) = x_m^-(CA)$. We have

$$(-2)x_m^-(CA)\Omega = [h_{m-\ell}(C), x_\ell^-(A)]\Omega = h_{m-\ell}(C)x_\ell^-(A)\Omega - d_{m-\ell}(C)x_\ell^-(A)\Omega = 0.$$  

Here we have made use of (40). Thus, we obtain $x_m^-(B)\Omega = 0$. $\square$

In appendix C, we show reduction relations for $a \neq 0$, which generalize (36).

5. Derivation of irreducibility criterion

For any given highest weight vector $\Omega$, we have reduction relation (42) with respect to the highest weight parameters $\hat{a}_j$ for $j = 1, 2, \ldots, r$. It is of an $r$th order. However, it is not always the case that the following $s$th order relation holds:

$$x_s^-(a)\Omega = 0. \quad (44)$$

Here we recall that $s \leq r$.

We shall show that $U\Omega$ is irreducible if the $s$th order relation (44) holds. Here we recall that the highest weight parameters $\hat{a}_k$ are given by distinct parameters $a_j$ with multiplicities $m_j$ for $j = 1, 2, \ldots, s$ where $s \leq r$ and $a = (a_1, a_2, \ldots, a_s)$. Here we also note that condition (44) is similar to the criterion for a matrix to have no Jordan blocks.

It is easy to show that the $s$th order relation (44) is necessary for $U\Omega$ to be irreducible. Let us show that $x_n^+x_s^-(a)\Omega = 0$ for all $n \in \mathbb{Z}$. Here, from proposition 13 we have

$$x_n^+x_s^-(a)\Omega = h_{n+s}(a)\Omega = \sum_{j=1}^{r} \hat{a}_j^n \prod_{i=1}^{s} (\hat{a}_j - a_i)\Omega = 0.$$  

Therefore, if $x_s^-(a)\Omega \neq 0$, then $Ux_s^-(a)\Omega$ is a proper submodule of $U\Omega$, and hence $U\Omega$ is reducible. It thus follows that $x_s^-(a)\Omega = 0$ if $U\Omega$ is irreducible.

Lemma 15. Let $\alpha$ and $\beta$ be sequences of complex parameters such as $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$. If $x_\ell^-(\alpha)\Omega = 0$, we have

$$x_{\ell+n-1}(\alpha_j, \beta)\Omega = \prod_{k=1}^{n} (\alpha_j - \beta_k)x_{\ell-1}^-(\alpha_j)\Omega, \quad \text{for } j = 1, 2, \ldots, s. \quad (45)$$

Proof. We show it by induction on $n$. Here we note that $n$ denotes the number of parameters $\beta_k$ in equation (45). For $n = 1$ we note

$$x_\ell^-(\alpha_j, \beta_1) - x_\ell^-(\alpha) = (\alpha_j - \beta_1)x_{\ell-1}^-(\alpha_j).$$
Thus, we have $x^- \alpha_j(\beta_1) = (\alpha_j - \beta_1)x^- \alpha_j(\beta_1)\Omega$ if $x^- \alpha(\beta)\Omega = 0$. Let us assume (45) in the case of $n - 1$. Here we recall $\beta_n = (\beta_1, \ldots, \beta_{n-1})$. We note the following:

$$x^-_{\ell+n-1}(\alpha_j) - x^-_{\ell+n-1}(\alpha_j\beta_n) = (\alpha_j - \beta_n)x^-_{\ell+n-2}(\alpha_j\beta_n).$$

(46)

It follows from lemma 14 that $x^-_{\ell+n-1}(\alpha_j\beta_n) = 0$ if $x^- \alpha(\beta)\Omega = 0$. Thus, applying each side of equation (46) to $\Omega$, we have (45) in the case of $n$.

\[\square\]

**Lemma 16.** If $x^- \alpha(\beta)\Omega = 0$, we have the following relations for $j = 1, 2, \ldots, s$:

$$x^n_+(\rho_j^- (\alpha; s))^{(m_j+1)}\Omega = 0, \quad \text{for } n \in \mathbb{Z}.\quad (47)$$

**Proof.** Using equations (16) we have

$$x^n_+(\rho_j^- (\alpha; s))^{(m_j+1)}\Omega = (\rho_j^- (\alpha; s))^{(m_j)}h_{s-1}(\alpha_j)\Omega - (\rho_j^- (\alpha; s))^{(m_j-1)}x^-_{2s-2}(\alpha_j)\Omega.\quad (48)$$

In terms of $a_{kj} = a_k - a_j$, we have

$$h_{s-1}(\alpha_j)\Omega = m_j \left( \prod_{k=1; k \neq j}^s a_{jk} \right)\Omega.\quad (49)$$

Making use of lemma 15 with $\alpha$ given by $\alpha$ we have

$$x^-_{2s-2}(\alpha_j)\Omega = \left( \prod_{k=1; k \neq j}^s a_{jk} \right)x^-_{s-1}(\alpha_j)\Omega = \left( \prod_{k=1; k \neq j}^s a_{jk} \right)\rho_j^- (\alpha; s)\Omega.\quad (50)$$

Putting (49) and (50) into (48), we obtain equation (47) for $n = 0$. For any given integer $n$, we have

$$x^n_+(\alpha_j)(\rho_j^- (\alpha; s))^{(m_j+1)}\Omega = (\rho_j^- (\alpha; s))^{(m_j)}h_{n+s-1}(\alpha_j)\Omega - (\rho_j^- (\alpha; s))^{(m_j-1)}x^-_{n+2s-2}(\alpha_j)\Omega = 0.\quad (51)$$

Here we note that $h_{n+s-1}(\alpha_j)\Omega = 0$, and it follows from lemma 14 that we have $x^-_{n+2s-2}(\alpha_j)\Omega = 0$. We now derive from equation (51) the following recursive relation with respect to $n$:

$$x^n_+(\rho_j^- (\alpha; s))^{(m_j+1)}\Omega = a_j x^n_+(\rho_j^- (\alpha; s))^{(m_j+1)}\Omega.\quad (52)$$

Making use of relation (52), we derive (47) for all $n$ from that of $n = 0$.

\[\square\]

**Lemma 17.** If $x^- \alpha(\beta)\Omega = 0$, we have

$$[x^n_0(\alpha_j)(\rho_j^- (\alpha; s))^{(m_j+1)}\Omega = 0, \quad \text{for } n \in \mathbb{Z}.$$

(53)

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Proof. We show it by induction on $n$. First, we have for any positive integer $k$
\begin{equation}
\label{eq:54}
[h_k((a_j)^k), \rho_j^{-}(a; s)]^{(m_j+1)} = (-2)(\rho_j^{-}(a; s))^{(m_j)}x_{s+k-1}^{-}(a(a_j)^{k-1})\Omega = 0.
\end{equation}
Secondly, we derive relation (53) in the case of $n = 1$ as follows:
\begin{align*}
[(x_0^+)(x_1^-)(a_j)), (\rho_j^{-}(a; s))]^{(m_j+1)}\Omega \\
= [h_1(a_j), (\rho_j^{-}(a; s))]^{(m_j+1)}\Omega + x_1^- (a_j)x_0^+(\rho_j^{-}(a; s))^{(m_j+1)}\Omega \\
= 0.
\end{align*}
Thirdly, assuming relations (53) for the cases of $n < p$, we show the case of $n = p$ as follows: using $(B_p)$ of lemma 8 with $a = a_j$ and $\ell = 0$, for some element $x^+$ of $U(\mathcal{B}_0^+\mathcal{B}_0^*)$ we have
\begin{align}
(x_0^+)^{(p)}(x_1^-)(a_j))^{(p)} &= -1^{p-1}\frac{1}{p}h_p((a_j)^p) \\
&+ 1\sum_{k=1}^{p-1}(-1)^{k-1}h_k((a_j)^k)(x_0^+)^{(p-k)}(x_1^-)(a_j))^{(p-k)} + x^+.
\end{align}
Substituting (55) into the commutator, we have
\begin{align}
[(x_0^+)^{(p)}(x_1^-)(a_j))^{(p)}, (\rho_j^{-}(a; s))]^{(m_j+1)}\Omega \\
&= -1^{p-1}\frac{1}{p}[h_p((a_j)^p), (\rho_j^{-}(a; s))]^{(m_j+1)}\Omega + [x^+, (\rho_j^{-}(a; s))]^{(m_j+1)}\Omega \\
&+ 1\sum_{k=1}^{p-1}(-1)^{k-1}[h_k((a_j)^k)(x_0^+)^{(p-k)}(x_1^-)(a_j))^{(p-k)}, (\rho_j^{-}(a; s))]^{(m_j+1)}\Omega \\
&= 0.
\end{align}
Here we have used (54) and (47). We thus obtain (53). \hfill \Box

Proposition 18. If $x_s^-(a)\Omega = 0$, we have
\begin{equation}
\label{eq:57}
(\rho_j^{-}(a; s))^{m_j+1}\Omega = 0.
\end{equation}
Proof. We recall $h_0\Omega = r\Omega$. We have
\begin{equation}
\label{eq:58}
(x_1^-)(a_j))^{r-m_j}(\rho_j^{-}(a; s))^{m_j+1}\Omega = 0.
\end{equation}
Here we note $(r - m_j) + (m_j + 1) > r$, and there is no non-zero element in the sector of $h_0 = -r - 2$ in $U\Omega$. We now apply $(x_0^+)^{(r-m_j)}(x_1^-)(a_j))^{(r-m_j)}$ to $(\rho_j^{-}(a; s))^{(m_j+1)}\Omega$. The product therefore vanishes:
\begin{equation}
\label{eq:59}
(x_0^+)^{(r-m_j)}(x_1^-)(a_j))^{(r-m_j)} \times (\rho_j^{-}(a; s))^{(m_j+1)}\Omega = 0.
\end{equation}
It follows from commutation relation (53) that the left-hand side of (59) is given by
\begin{align}
(\rho_j^{-}(a; s))^{(m_j+1)} \times (x_0^+)^{(r-m_j)}(x_1^-)(a_j))^{(r-m_j)}\Omega \\
&= \sum_{1 \leq k_1 < \cdots < k_n \leq r} (\hat{a}_{k_1} - a_j) \cdots (\hat{a}_{k_n} - a_j) \times (\rho_j^{-}(a; s))^{(m_j+1)}\Omega \\
&= \left( \prod_{k=1; k \neq j}^{s} a_{k,j}^{m_j} \right) \times (\rho_j^{-}(a; s))^{(m_j+1)}\Omega,
\end{align}
\text{doi:10.1088/1742-5468/2007/05/P05007}
where \( n \) denotes \( r - m_j \) (see also lemma C.1). Here, the number of such parameters \( \hat{a}_k \) that are not equal to \( a_j \) is given by \( n = r - m_j \), and hence we have the last line of (60). In fact, we have only one set of integers \( k_1 < \cdots < k_n \) that make the following product non-zero: \((\hat{a}_{k_1} - a_j) \cdots (\hat{a}_{k_n} - a_j)\) for \( n = r - m_j \). Since the product is non-zero, i.e. \( \prod_{k=1;k \neq j}^n a_{jk}^{m_k} \neq 0 \), we obtain (57).

Let us define the binomial coefficients for integers \( n \) and \( k \) with \( n \geq k \geq 0 \) as follows:

\[
\binom{n}{k} = \frac{n!}{(n-k)!k!}.
\]

**Lemma 19.** Suppose that we have \( x_s^-(a)\Omega = 0 \). Let \( n \) be a non-negative integer. We take such non-negative integers \( \ell_j \) and \( k_j \) for \( j = 1, 2, \ldots, s \) that satisfy \( \ell_1 + \cdots + \ell_s = k_1 + \cdots + k_s = n \). Then, we have

\[
\prod_{j=1}^s (\rho_j^+(a; s))^{(\ell_j)} \times \prod_{j=1}^s (\rho_j^-(a; s))^{(k_j)} \Omega = \prod_{j=1}^s \left( \delta_{\ell_j,k_j} \frac{m_j}{k_j} \prod_{t=1; t \neq j}^s a_{jt}^{2k_j} \right) \Omega.
\]

Here \( \delta_{j,k} \) denotes the Kronecker delta.

**Proof.** First, it is easy to show the following:

\[
\rho_i^+(a; s) \times (\rho_j^-(a; s))^{(k_j)} \Omega = \delta_{i,j}(m_j - k_j + 1) \prod_{t=1; t \neq j}^s a_{jt}^2 (\rho_j^-(a; s))^{(k_j-1)} \Omega.
\]

Relation (63) is derived through a similar method to that for equation (47). By induction on \( k_j \) and making use of (63), we can show the following:

\[
(\rho_j^+(a; s))^{(k_j)} \times (\rho_j^-(a; s))^{(k_j)} \Omega = \left( \frac{m_j}{k_j} \prod_{t=1; t \neq j}^s a_{jt}^{2k_j} \right) \Omega.
\]

We thus obtain (62).

**Proposition 20.** If \( x_s^-(a)\Omega = 0 \), the set of vectors \( \prod_{j=1}^s (\rho_j^-(a; s))^{(k_j)} \Omega \) where \( 0 \leq k_j \leq m_j \) for \( j = 1, 2, \ldots, s \), gives a basis of \( U\Omega \).

**Proof.** It is derived from lemma 19 that vectors \( \prod_{j=1}^s (\rho_j^-(a; s))^{(k_j)} \Omega \) are non-zero, if we have \( 0 \leq k_j \leq m_j \) for \( j = 1, 2, \ldots, s \). It follows from proposition 18, the definition of \( \rho_j^-(a; s) \) and the condition: \( x_s^-(a)\Omega = 0 \) that every vector \( v_n \) in the sector of \( h_0 = r - 2n \) is expressed as a linear combination of vectors \( (\rho_j^-(a; s))^{(k_1)} \cdots (\rho_j^-(a; s))^{(k_s)} \Omega \) over sets of such integers \( k_1, \ldots, k_s \in \mathbb{Z}_{\geq 0} \) that satisfy \( k_1 + \cdots + k_s = n \) where \( 0 \leq k_j \leq m_j \) for \( j = 1, 2, \ldots, s \). We thus have

\[
v_n = \sum_{k_1=0}^{m_1} \sum_{k_2=0}^{m_2} \cdots \sum_{k_s=0}^{m_s} \delta_{k_1+\cdots+k_s,n} C_{k_1,\ldots,k_s} \prod_{j=1}^s (\rho_j^-(a; s))^{(k_j)} \Omega.
\]

In fact, we have from lemma 19

\[
\prod_{j=1}^s (\rho_j^+(a; s))^{(\ell_j)} \times v_n = C_{\ell_1,\ldots,\ell_s} \times \prod_{j=1}^s \left( \frac{m_j}{\ell_j} \prod_{t=1; t \neq j}^s a_{jt}^{2\ell_j} \right) \Omega.
\]

It thus follows that if \( v_n = 0 \), all the coefficients \( C_{k_1,\ldots,k_s} \) are zero.
Irreducibility criterion for a finite-dimensional highest weight representation of the $sl_2$ loop algebra

**Corollary 21.** If $x^{-}_s(a)\Omega = 0$, the dimension of $U\Omega$ is given by

$$
\dim U\Omega = \prod_{j=1}^s (m_j + 1).
$$

**Theorem 22.** Let $\Omega$ be a highest weight vector with highest weight parameters $\hat{a}_k$ for $k = 1, 2, \ldots, r$. Here, $\hat{a}_k$ are given by distinct parameters $a_j$ with multiplicities $m_j$ for $j = 1, 2, \ldots, s$, which are expressed as $a = (a_1, a_2, \ldots, a_s)$. We assume that $\Omega$ generates a finite-dimensional representation, and we denote it by $U\Omega$. Then, $U\Omega$ is irreducible if and only if $x^{-}_s(a)\Omega = 0$.

**Proof.** We now show that if $x^{-}_s(a)\Omega = 0$, every non-zero vector of $U\Omega$ has such an element of the loop algebra that maps it to $\Omega$. Suppose that there is a non-zero vector $v_n$ in the sector $h_0 = r - 2n$ of $U\Omega$ that has no such element. Then, we have

$$
\left( \sum_{k_1, \ldots, k_n} C_{k_1, \ldots, k_n} x_{k_1}^+ \cdots x_{k_n}^+ \right) v_n = 0
$$

for all linear combinations of monomial elements $x_{k_1}^+ \cdots x_{k_n}^+$. Let us express $v_n$ in terms of basis vectors $(\rho_{1}^{-}(a; s))^{(k_1)} \cdots (\rho_{s}^{-}(a; s))^{(k_s)} \Omega$ with coefficients $C_{k_1, \ldots, k_s}$ where $k_1, \ldots, k_s \in \mathbb{Z}_{\geq 0}$ satisfy $k_1 + \cdots + k_s = n$, as shown in (65). We multiply $v_n$ with $(\rho_{1}^{+}(a; s))^{(j_1)} \cdots (\rho_{s}^{+}(a; s))^{(j_s)}$ for a set of non-negative integers $j_1, \ldots, j_s$ satisfying $j_1 + \cdots + j_s = n$. Then, it follows from equation (62) that the coefficient $C_{j_1, \ldots, j_s}$ vanishes. We thus have shown that all the coefficients $C_{j_1, \ldots, j_s}$ vanish. However, this contradicts the assumption that $v_n$ is non-zero. It therefore follows that $v_n$ has such an element that maps it to $\Omega$. \hfill \Box

**6. Reducible highest weight representations**

Let us recall that $\Omega$ denotes a non-zero highest weight vector with highest weight $d_k$ where $d_0 = r$ and $U\Omega$ is finite dimensional. In section 6 we shall formulate a method for constructing practically all finite-dimensional representations generated by such highest weight vectors that have the same highest weight parameters $\hat{a} = (\hat{a}_1, \ldots, \hat{a}_s)$. Throughout section 6, we assume that the highest weight parameters $\hat{a}_1, \ldots, \hat{a}_s$ are given by distinct parameters $a_j$ with multiplicities $m_j$ for $j = 1, 2, \ldots, s$. We denote them by $a = (a_1, a_2, \ldots, a_s)$. Here we also recall rule (25).

**6.1. Summary of the irreducible case**

As shown in (42), we always have

$$
x_{s}^{-}(a)\Omega = 0.
$$

However, if we have the following relation:

$$
x_{s}^{-}(a)\Omega = 0,
$$

then it follows from theorem 22 that the highest weight representation $U\Omega$ of $\Omega$ is irreducible, and the dimensionality of $U\Omega$ is given by

$$
\dim U\Omega = \prod_{j=1}^s (m_j + 1).
$$

Here we recall the symbol: $a = (a_1, a_2, \ldots, a_s)$.  

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If \( x^{-}(\alpha)\Omega \neq 0 \), then \( U\Omega \) is reducible. In order to determine the dimensions of reducible highest weight representations in \( U\Omega \), we shall discuss further conditions of \( \Omega \) in the next subsection.

### 6.2. Highest weight representations with the same given highest weight

#### 6.2.1. Some notation

Let \( i_{1}, \ldots, i_{m} \) be a set of integers satisfying \( 1 \leq i_{1} < \cdots < i_{m} \leq r \). We consider a subsequence of \( \hat{a} \) with respect to \( i_{1}, \ldots, i_{m} \) and denote it by \( A \), i.e. \( A = (\hat{a}_{i_{1}}, \ldots, \hat{a}_{i_{m}}) \). We denote by \( \hat{a} \setminus A \) such a subsequence of \( \hat{a} \) that is obtained by removing \( \hat{a}_{i_{1}}, \ldots, \hat{a}_{i_{m}} \) from sequence \( \hat{a} \):

\[
\hat{a} \setminus A = (\hat{a}_{1}, \ldots, \hat{a}_{i_{1} - 1}, \hat{a}_{i_{1} + 1}, \ldots, \hat{a}_{i_{2} - 1}, \hat{a}_{i_{2} + 1}, \ldots, \hat{a}_{r}).
\]

(69)

We also express it as \( \hat{a}_{A} \), briefly. We now define operators \( w_{A}(\hat{a}) \) by

\[
w_{A}(\hat{a}) = x^{-}_{r - m}(\hat{a}_{A}).
\]

(70)

Let us consider subsequence \( A = ((a_{1})^{k_{1}}(a_{2})^{k_{2}} \cdots (a_{s})^{k_{s}}) \), in which there are \( k_{j} \) copies of parameters \( a_{j} \) for \( j = 1, 2, \ldots, s \). We denote it simply as \( 1^{k_{1}}2^{k_{2}}\cdots s^{k_{s}} \). If \( k_{i} \neq 0 \) only for \( i = j \) and \( k_{j} = k \), we express it as \( j^{k} \), i.e. we have

\[
j^{k} = (a_{j})^{k}.
\]

(71)

Let us now consider the following operators: \( \rho_{j}^{-}(\alpha; s) \) for \( j = 1, 2, \ldots, s \), and \( w_{j,s}(\hat{a}) \) for \( k = 1, 2, \ldots, m_{j} - 1 \) and \( j = 1, 2, \ldots, s \). Here we note that there are \( r \) operators in total, since we have \( s + (m_{1} - 1) + \cdots + (m_{s} - 1) = m_{1} + \cdots + m_{s} = r \).

**Lemma 23.** Let \( n \) be an integer satisfying \( 0 \leq n \leq r \). Every vector \( v_{n} \) in the sector of \( h_{0} = r - 2n \) of \( U\Omega \) is expressed as a linear combination of monomial vectors consisting of products of \( \rho_{j}^{-}(\alpha; s) \) and \( w_{j,s}(\hat{a}) \) acting on \( \Omega \) for \( j = 1, 2, \ldots, s \), and \( k = 1, 2, \ldots, m_{j} - 1 \).

**Proof.** By the Poincaré–Birkhoff–Witt theorem [27] applied to \( U(L(sl_{2})) \), every vector \( v_{n} \) in the subspace of weight \( r - 2n \) of \( U\Omega \) is expressed as a linear combination of monomial vectors \( \prod_{i=1}^{n} x^{-}_{j_{i}}\Omega \) for some sets of integers \( j_{1} \leq \cdots \leq j_{n} \). It follows from the \( r \)th order reduction relations (42) (i.e. (36)) that every factor \( x^{-}_{j_{t}} \) of the monomial vectors \( x^{-}_{j_{1}}\cdots x^{-}_{j_{n}}\Omega \) is expressed as a linear combination of \( \rho_{j_{t}}^{-}(\alpha; s) \) and \( w_{j,s}(\hat{a}) \). We therefore obtain lemma 23. \( \square \)

Let \( \Sigma \) be a sequence of such subsequences of \( \hat{a} \) that are of the form \( (a_{j})^{k} \) where integer \( k \) satisfies \( 1 \leq k \leq m_{j} - 1 \). For notational convenience, we also regard \( \Sigma \) as a set. If a sequence \( A \) is a component of \( \Sigma \), we express it as follows: \( A \in \Sigma \). We now take the product of \( w_{A}(\hat{a}) \) over \( A \in \Sigma \), and apply it to \( \Omega \). We denote it by \( \omega_{\Sigma} \) as follows:

\[
\omega_{\Sigma} = \left( \prod_{A \in \Sigma} w_{A}(\hat{a}) \right) \Omega.
\]

(72)

Let \( \ell_{j} \) be non-negative integers for \( j = 1, 2, \ldots, s \). We take sequences of \( \ell_{j} \) integers \( k_{j}(\ell_{j}) = (k_{j}(1), k_{j}(2), \ldots, k_{j}(\ell_{j})) \) which satisfy \( 1 \leq k_{j}(1) \leq \cdots \leq k_{j}(\ell_{j}) < m_{j} \) for \( j = 1, 2, \ldots, s \). Here we assume that if \( \ell_{j} = 0 \), \( k_{j}(\ell_{j}) \) is given by an empty set. We define \( \Sigma(k_{j}(\ell_{j})) \) by

\[
\Sigma(k_{j}(\ell_{j})) = (j^{k_{j}(1)}, j^{k_{j}(2)}, \ldots, j^{k_{j}(\ell_{j}))}.
\]

(73)
We introduce the following symbol: \( \omega_{\Sigma(k_j(\ell_j))} = w_{j,\ell_j(1)}(\hat{a}) \cdots w_{j,\ell_j(n)}(\hat{a})\Omega \). We denote by \( k(\ell) \) a set of sequences as follows:

\[
k(\ell) = (k_1(\ell_1), k_2(\ell_2), \ldots, k_s(\ell_s)).
\]

Here \( \ell = (\ell_1, \ell_2, \ldots, \ell_s) \). Furthermore, we denote by \( \Sigma(k(\ell)) \) the following sequence of \( \Sigma(k_j(\ell_j))s \):

\[
\Sigma(k(\ell)) = (\Sigma(k_1(\ell_1)), \ldots, \Sigma(k_s(\ell_s))).
\]

We now define \( \omega_{\Sigma(k(\ell))} \) by the following vector:

\[
\omega_{\Sigma(k(\ell))} = \prod_{j=1}^{s} (w_{j,\ell_j(1)}(\hat{a}) \cdots w_{j,\ell_j(n)}(\hat{a}))\Omega.
\]

For some examples, we shall show that the highest weight vectors of irreducible quotients of submodules in \( U\Omega \) are given by vectors \( \omega_{\Sigma(k(\ell))} \). Here we have the following.

**Conjecture 24.** Every irreducible quotient of submodules in \( U\Omega \) has a highest weight vector of the form \( \omega_{\Sigma(k(\ell))} \).

### 6.2.2. Useful lemmas and propositions

**Definition 25.** Let \( V \) be a submodule of \( U\Omega \). We say that \( \omega \in U\Omega \) is the highest weight modulo \( V \) if \( V \subset U\omega \) and we have the following conditions:

\[
x_n^+ \omega = 0 \quad \text{mod } V \quad \text{for } n \in \mathbb{Z},
\]

\[
h_n \omega = \tilde{d}_n \omega \quad \text{mod } V \quad \text{for } n \in \mathbb{Z}.
\]

Here eigenvalues \( \tilde{d}_n \) are given by some complex numbers.

Here we remark that if \( \omega \) is the highest weight modulo \( V \) for a submodule \( V \) of \( U\Omega \) and we denote \( U_\omega \) by \( W \), then the dimension of \( W \) is given by

\[
dim W = \dim W/V + \dim V.
\]

Let us extend the definition of the binomial coefficient (61) into the case of negative integers \( n \) as follows:

\[
\binom{n}{k} = \frac{n(n-1) \cdots (n-k+1)}{k!} \quad \text{for } n \in \mathbb{Z}.
\]

We also denote it by \( nC_k \). Applying \( h_n \) to \( w_{j,t}(\hat{a})\Omega \), we have linear combinations of vectors \( w_{j,t-1}(\hat{a})\Omega \) for \( t = 1, 2, \ldots, k-1 \).

**Lemma 26.** For integers \( k \) and \( k' \) satisfying \( 1 \leq k, k' \leq m_j - 1 \), we have the following:

\[
x_n^+ w_{j,t}(\hat{a})\Omega = 0 \quad \text{for } n \in \mathbb{Z},
\]

\[
x_n^+ w_{j,t}(\hat{a}) w_{j,t'}(\hat{a})\Omega = 0 \quad \text{for } n \in \mathbb{Z} \quad \text{if } k + k' \leq m_j,
\]

\[
[h_{n}, w_{j,t}(\hat{a})]\Omega = (-2) \sum_{t=0; t \leq k}^{n} \binom{n}{t} a_j^{n-t} w_{j,t-1}(\hat{a})\Omega,
\]

\[
[h_{-n}, w_{j,t}(\hat{a})]\Omega = (-2) \sum_{t=0; t \leq k}^{n} \binom{-n}{t} a_j^{-n-t} w_{j,t-1}(\hat{a})\Omega, \quad \text{for } n \in \mathbb{Z}_{>0}.
\]
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**Proof.** It is straightforward to show (80). Let us now show (81). If $m_j \geq k + k'$, we have
\[
x_n^+ w_{j,k}(\hat{a}) \Omega = h_{\tau-k+n}(\hat{a} \setminus j^k) w_{j,k}(\hat{a}) \Omega = (-2)x_{2r-k+n}(\hat{a} \setminus j^m) j^{m_{\tau-k-k'}} \Omega = 0.
\]
We can show (82) by induction on $n$. We shall discuss it in appendix D. \hfill \Box

As a corollary of lemma 26 we have the following.

**Corollary 27.** Vectors $w_{j,k}(\hat{a}) \Omega$ are the highest weight modulo $U w_{j,k-1}(\hat{a}) \Omega$ ($1 \leq k \leq m_j - 1$) for $j = 1, 2, \ldots, s$:
\[
x_n^+ w_{j,k}(\hat{a}) \Omega = 0 \quad \text{for } n \in \mathbb{Z},
\]
\[
h_n w_{j,k}(\hat{a}) \Omega = (d_n - 2a_n^k) w_{j,k}(\hat{a}) \Omega \mod U w_{j,k-1}(\hat{a}) \Omega \quad \text{for } n \in \mathbb{Z}.
\]
Let $\{k(\ell) = (k(1), k(2), \ldots, k(\ell))\}$ be a sequence of integers satisfying $1 \leq k(1) \leq k(2) \leq \cdots \leq k(\ell) \leq m_j - 1$. If $k(i) + k(t) \leq m_j$ for all pairs of integers $i$ and $t$ satisfying $1 \leq i < t \leq \ell$, we have
\[
x_n^+ \prod_{i=1}^{\ell} w_{j,k(i)}(\hat{a}) \Omega = 0 \quad \text{for } n \in \mathbb{Z}.
\] (83)

From lemma 26 and corollary 27 we have the following lemma.

**Lemma 28.** For a set of subsequences $\Sigma = \{j_{1,1}^{k_1}, \ldots, j_{s,1}^{k_s}\}$ where $1 \leq j_1 \leq \cdots \leq j_{s} \leq s$ and $1 \leq k_t \leq m_{j_t} - 1$ for $t = 1, 2, \ldots, p$, we denote by $\nu_{\Sigma}(j)$ the number of such integers in the set $\{j_1, j_2, \ldots, j_{s}\}$ that are equal to $j$. Let us denote by $\omega'$ the following vector:
\[
\omega' = w_{j_{1,1}}(\hat{a}) \cdots w_{j_{s,1}}(\hat{a}) \Omega. \quad (84)
\]
If $\omega'$ is the highest weight modulo $V$ where $V$ is a submodule of $U \Omega$, then the highest weight parameters $\hat{a}'_j$ of $\omega'$ are given by parameters $a_j$ with multiplicities $m'_j$ where we have
\[
m'_j = m_j - 2 \nu_{\Sigma}(j) \quad \text{for } j = 1, 2, \ldots, s, \quad (85)
\]
and eigenvalues $d'_n$ of $h_n$ are given by
\[
d'_n = d_n - 2 \sum_{i=1}^{p} a_i^n, \quad (86)
\]
where $d'_n$ have been defined by the following:
\[
h_n \omega' = d'_n \omega' \mod V. \quad (87)
\]
Here we recall $h_n \Omega = d_n \Omega$.

**Lemma 29.** Let $V$ be a submodule of $U \Omega$. If vector $\omega'$ is the highest weight modulo $V$ and it has highest weight parameters $\hat{a}'_j$ which are given by distinct parameters $a_j$ with multiplicities $m'_j$, then we have
\[
x_{r+1}^{-} \omega' = \sum_{j=1}^{r'} (-1)^{r'+j} \lambda_{r+1-j}^{-} x_j^{-} \omega' \mod V \quad \text{for } \ell \in \mathbb{Z}. \quad (88)
\]
Here $r'$ is given by $h_0 \omega' = r' \omega'$ and $\lambda'_n$ are defined by
\[
\lambda_n' = \sum_{0 \leq k_1 < \cdots < k_n \leq r'} \hat{a}_{k_1} \hat{a}_{k_2} \cdots \hat{a}_{k_n}. \quad (89)
\] doi:10.1088/1742-5468/2007/05/P05007
Similarly to (47), reduction relation (88) leads to the following.

**Lemma 30.** Let $V$ be a submodule of $U\Omega$. If vector $\omega'$ is the highest weight modulo $V$ and it has such highest weight parameters that are given by distinct parameters $a_j$ with multiplicities $m_j'$ for $j = 1, 2, \ldots, s$, and furthermore if $x_+(\alpha)\Omega = 0$, then we have

$$ (\rho_j^*(\alpha; s))^{(m_j'+1)}\omega' = 0 \mod V. $$

For given submodules of $U\Omega$, $V_1, V_2, \ldots, V_p$, we denote by $V_1 + \cdots + V_p$ or $\sum_{j=1}^p V_j$ the module generated by $V_1 \cup V_2 \cup \cdots \cup V_p$. Suppose that $U\omega_{\Sigma}$ is the highest weight modulo submodule $\sum_{j=1}^p U\omega_{\Sigma_j}$ for some $\Sigma_j$s and also that the quotient $U\omega_{\Sigma}/\sum_j U\omega_{\Sigma_j}$ is irreducible. Then, we can determine the dimension of the quotient, making use of theorem 22 or lemmas 28, 29 and 30.

**Proposition 31.** Let $j$ be an integer satisfying $1 \leq j \leq s$, and $(k_1, k_2, \ldots, k_\ell)$ be a sequence of integers satisfying $1 \leq k_1 \leq \cdots \leq k_\ell < m_j$. We assume that $\omega' = w_{j^k_1}(\tilde{\alpha}) \cdots w_{j^k_{\ell-1}}(\tilde{\alpha})\Omega$ is the highest weight modulo $V$. If $k_\ell \leq 2\ell - 2$, we have

$$ w_{j^k_\ell}(\tilde{\alpha})\omega' = 0 \mod V. $$

**Proof.** Vector $\omega'$ has weight $r' = r - 2\ell + 2$. From lemma 29, $\omega'$ has the $r'$th order reduction relation as follows: $w_{j^{2\ell-2}}(\tilde{\alpha})\omega' = 0 \mod V$. It thus follows from lemma 14 that $w_{j^k_\ell}(\tilde{\alpha})\omega' = 0 \mod V$. \hfill \Box

**Proposition 32.** Let $j$ be an integer satisfying $1 \leq j \leq s$. For an integer $\ell$ satisfying $1 \leq \ell < (m_j + 1)/2$, we define $k_j^{(0)}(\ell)$ by $k_j^{(0)}(\ell) = 2t - 1$ for $t = 1, 2, \ldots, \ell$:

$$ k_j^{(0)}(\ell) = (1, 3, \ldots, 2\ell - 1). $$

Then, $\omega_{\Sigma(k_j^{(0)}(\ell))}$ is highest weight, i.e. we have the following:

$$ x_+^{2n}\omega_{\Sigma(k_j^{(0)}(\ell))} = 0 \quad \text{for } n \in \mathbb{Z}, $$

$$ h_n\omega_{\Sigma(k_j^{(0)}(\ell))} = (d_n - 2\ell a_j^n)\omega_{\Sigma(k_j^{(0)}(\ell))} \quad \text{for } n \in \mathbb{Z}. $$

Here we recall $\omega_{\Sigma(k_j^{(0)}(\ell))} = \prod_{k=1}^\ell w_{j^{2\ell-1}}(\tilde{\alpha})\Omega$.

**Proof.** From (94) and induction on $\ell$ we have (93) for all $\ell$. We now show (94) by induction on $\ell$. For $\ell = 1$, we have $\omega_{\Sigma(k_j^{(0)}(1))} = w_j(\tilde{\alpha})\Omega$. It is readily derived from lemma 26 that $w_j(\tilde{\alpha})\Omega$ is highest weight. Let us assume (94) for $\ell$. We have

$$ h_n\omega_{\Sigma(k_j^{(0)}(\ell+1))} = h_n w_{j^{2\ell+1}}(\tilde{\alpha}) \omega_{\Sigma(k_j^{(0)}(\ell))} $$

$$ = w_{j^{2\ell+1}}(\tilde{\alpha}) h_n \omega_{\Sigma(k_j^{(0)}(\ell))} + [h_n, w_{j^{2\ell+1}}(\tilde{\alpha})] \omega_{\Sigma(k_j^{(0)}(\ell))} $$

$$ = (d_n - 2\ell a_j^n) w_{j^{2\ell+1}}(\tilde{\alpha}) \omega_{\Sigma(k_j^{(0)}(\ell))} + (-2)x_{r-2\ell-1+n}(\tilde{\alpha} \setminus j^{2\ell+1}) \omega_{\Sigma(k_j^{(0)}(\ell))}. $$

Here we note that $r' = r - 2\ell$ for $\omega_{\Sigma(k_j^{(0)}(\ell))}$. Therefore, making use of lemma 15 we have

$$ x_{r-2\ell-1+n}(\tilde{\alpha} \setminus j^{2\ell+1}) \omega_{\Sigma(k_j^{(0)}(\ell))} = a_j^n x_{r-2\ell-1}(\tilde{\alpha} \setminus j^{2\ell+1}) \omega_{\Sigma(k_j^{(0)}(\ell))}. $$

Thus, we obtain (94) for $\ell + 1$. \hfill \Box

Let us denote by $\ell_j^{\max}$ the largest integer $\ell_j$ satisfying $\ell_j < (m_j + 1)/2$, for $j = 1, 2, \ldots, s$. We also denote $\ell_j^{\max}$ by $\ell_j^{(0)}$. 

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**Proposition 33.** Let us define vector $\omega^{\text{max}}$ by

$$ \omega^{\text{max}} = \prod_{j=1}^{s} \left( \prod_{k=1}^{\ell_j^{\text{max}}} w_{j,k-1}(\hat{a}) \right) \Omega. \quad (95) $$

Then, it is highest weight. The representation generated by $\omega^{\text{max}}$ is irreducible and has the following dimension:

$$ \dim(U\omega^{\text{max}}) = \prod_{k=1}^{s} (m_k + 1 - 2\ell_k^{\text{max}}). \quad (96) $$

**Proof.** It follows from proposition 32 and lemma 26 that $\omega^{\text{max}}$ is highest weight. Making use of lemma 28, we evaluate the multiplicities of parameters $a_j$ describing the highest weight parameters of $\omega^{\text{max}}$, and we have $m_j' = m_j - 2\ell_j^{\text{max}}$ for each $j$. It follows from theorem 22 and lemma 29 that the representation generated by $\omega^{\text{max}}$ is irreducible. $\square$

We remark that $\omega^{\text{max}}$ is also expressed as follows:

$$ \omega^{\text{max}} = \omega_{\Sigma(\ell^{(0)})}. \quad (97) $$

Here $k^{(0)}(\ell^{(0)})$ denotes the set $\{k_1^{(0)}(\ell_1^{(0)}), k_2^{(0)}(\ell_2^{(0)}), \ldots, k_s^{(0)}(\ell_s^{(0)})\}$.

6.2.3. Algorithm for constructing reducible highest weight representations. Let us now formulate an algorithm for constructing irreducible quotients of submodules of $U\Omega$. We first construct a network of submodules $U\omega_{\Sigma(k(\ell))}$. The network consists of vertices and edges, where each of the vertices corresponds to a submodule $U\omega_{\Sigma(k(\ell))}$ and each of the edges has an arrow which goes from a parental submodule to a daughter submodule.

For a given vector $\omega_{\Sigma(k(\ell))}$, which we call a parental vector, we construct a daughter vector $\omega_{\Sigma(k'(\ell'))}$ by the procedures from (i) to (v) in the following:

1. We select an integer $j$ satisfying $1 \leq j \leq s$.
2. If $\ell_j = \ell_j^{(0)}$ and $k_j(\ell_j) + 1 < m_j$, then we set $k_j'(\ell_j) = k_j(\ell_j) + 1$ and define descendant $\omega_{\Sigma(k'(\ell'))}$ by

$$ \omega_{\Sigma(k'(\ell'))} = w_{j,k_j}(\hat{a}) \cdots w_{j,k_j(\ell_j)+1}(\hat{a}) \prod_{t=1; t \neq j}^{s} (w_{t,k_t}(\hat{a}) \cdots w_{t,k_t(\ell_t)}(\hat{a})) \Omega. $$

3. If $\ell_j = \ell_j^{(0)}$ and $k_j(\ell_j) + 1 = m_j$, then we set $\ell_j' = \ell_j - 1$ and we define descendant $\omega_{\Sigma(k'(\ell'))}$ by

$$ \omega_{\Sigma(k'(\ell'))} = w_{j,k_j}(\hat{a}) \cdots w_{j,k_j(\ell_j-1)}(\hat{a}) \prod_{t=1; t \neq j}^{s} (w_{t,k_t}(\hat{a}) \cdots w_{t,k_t(\ell_t)}(\hat{a})) \Omega. $$

4. If $\ell_j < \ell_j^{(0)}$ and $k_j(\ell_j) < m_j - 1$, then we set $k_j'(\ell_j) = k_j(\ell_j) + 1$. Furthermore, we set $k_j'(\ell_j + i) = \max\{k_j(\ell_j) + 1, 2(\ell_j + i) - 1\}$ for $i = 1, 2, \ldots, \ell_j^{(0)} - \ell_j$, and we set $\ell_j' = \ell_j^{(0)}$. We define descendant $\omega_{\Sigma(k'(\ell'))}$ by

$$ \omega_{\Sigma(k'(\ell'))} = w_{j,k_j(\ell_j)}(\hat{a}) \cdots w_{j,k_j(\ell_j-1)}(\hat{a}) \prod_{t=1; t \neq j}^{s} (w_{t,k_t}(\hat{a}) \cdots w_{t,k_t(\ell_t)}(\hat{a})) \Omega. $$
We have four-dimensional irreducible quotients: 

\[ U_{w} \]

eight-dimensional irreducible quotients:

\[ ((1^k, 0^m)) = (\Omega, (1^k, 0^m)) \]

Here we have specified only such elements of \( k'(\ell') \) that are changed from the parental one \( k(\ell) \).

First, we put vector \( \omega^{\text{max}} \) at the starting point of the network. That is, we set \( \ell_j = \ell^0_j \) and put \( k_j(\ell_j) = k_j^0(\ell_j^0) \) for \( j = 1, 2, \ldots, s \), as initial conditions. Then, we apply the procedures of (i), (ii), \ldots, and (v) to the parental vector \( \omega^{\Sigma}(k(\ell)) \), then we derive daughter vectors \( \omega^{\Sigma}(k(\ell')) \) for each \( j \) (\( 1 \leq j \leq s \)). Then, we choose one of \( k'(\ell') \), and we set \( k(\ell) = k'(\ell') \). We repeat the procedure again. Finally, we arrive at the end point of the network, where \( k(\ell) \) is given by an empty set, \( \emptyset \).

Applying lemma 28 to the derived network of submodules, we can calculate practically all of the dimensions of reducible highest weight representations with the same given highest weight. Suppose that a submodule \( V \) in the network has parental submodules \( V_1, \ldots, V_p \). We take the quotient of \( V \) with respect to the sum over all the parental submodules; we have \( V/(V_1 + \cdots + V_p) \). Here we remark that if all of the parental submodules are irreducible, the sum \( V_1 + \cdots + V_p \) is given by the direct sum.

If the quotient \( V/(V_1 + \cdots + V_p) \) does not vanish, then it is irreducible. Evaluating the multiplicities \( m'_{j} \) through lemma 28, we derive the dimension of the irreducible quotient by corollary 21. We then cut the network into two parts such that one has the starting point, while another has the end point, respectively. Then, the subnetwork that has the end point corresponds to a reducible (or irreducible) highest weight representation. We obtain the dimension of the representation taking the sum of all of the dimensions of the irreducible quotients in the remaining part of the network.

As an illustration, let us consider the case of \( r = 6 \) with \( (m_1, m_2) = (3, 3) \). Here we have \( \ell_1^{\text{max}} = \ell_2^{\text{max}} = 1 \). We put \( \ell^0_1 = \ell^0_2 = 1 \) and start with \( k^0(\ell^0) = ((1^1), (2^1)) \) where we have \( k^0_1 = (1^1) \) and \( k^0_2 = (2^1) \). The highest weight vector \( \omega^{\text{max}} \) is given by \( w_{11}w_{21}\Omega \), and it generates a four-dimensional irreducible module. Here we recall proposition 33. Through the procedures of (i)–(v), we now derive all of the daughter vectors. For a given \( k(\ell) \), we show all \( k'(\ell') \) after the symbol ‘\( \rightarrow \)’, as follows:

\[
\begin{align*}
((1^1), (2^1)) & \rightarrow ((1^2), (2^1)), ((1^1), (2^2)); \\
((1^2), (2^1)) & \rightarrow ((\emptyset), (2^1)), ((1^2), (2^2)); \\
((1^1), (2^2)) & \rightarrow ((1^2), (2^2)), ((1^1), (\emptyset)); \\
((1^2), (2^2)) & \rightarrow ((\emptyset), (2^2)), ((1^2), (\emptyset)); \\
((1^1), (\emptyset)) & \rightarrow ((1^2), (\emptyset)); \\
((\emptyset), (2^1)) & \rightarrow ((\emptyset), (2^2)); \\
((1^2), (\emptyset)) & \rightarrow ((\emptyset), (\emptyset)); \\
((\emptyset), (\emptyset)) & \rightarrow ((\emptyset), (\emptyset)).
\end{align*}
\]

We have four-dimensional irreducible quotients:

\[
U_{w_1}w_{21}\Omega/U_{w_1}w_{21}\Omega, \quad U_{w_1}w_{22}\Omega/U_{w_1}w_{21}\Omega,
\]

\[
U_{w_1}w_{22}\Omega/(U_{w_1}w_{22}\Omega + U_{w_1}w_{22}\Omega),
\]

eight-dimensional irreducible quotients:

\[
U_{w_2}\Omega/U_{w_1}w_{22}\Omega, \quad U_{w_1}\Omega/U_{w_1}w_{22}\Omega, \quad U_{w_1}\Omega/(U_{w_1}\Omega + U_{w_1}w_{22}\Omega),
\]

\[
U_{w_2}\Omega/(U_{w_2}\Omega + U_{w_1}w_{22}\Omega),
\]

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and a 16-dimensional irreducible quotient:

$$U\Omega/(Uw_1\Omega + Uw_2\Omega).$$

Here we recall the following: $$w_{11} = x_5^-((a_1)^2(a_2)^3), w_{12} = x_4^-((a_1)^4(a_2)^3), w_{21} = x_5^-((a_1)^3(a_2)^2),$$ and so on. In total, we have 64 dimensions as follows:

$$4 + (4 + 4 + 4) + (8 + 8 + 8 + 8) + 16 = 64.$$ 

Here we note that it is given by $$2^6.$$ Let us now construct an example of a reducible highest weight module. For instance, suppose that $$w_1\Omega = 0, w_1 w_2 \Omega = 0$$ and $$w_2 \Omega = 0.$$ We then take the sum of two irreducible quotients for $$\mathbf{k}(n) = ((\emptyset), (\emptyset))$$ and $$((1^2), (\emptyset))$$ as follows:

$$U\Omega/(Uw_1\Omega + Uw_2\Omega) \oplus Uw_1\Omega/(Uw_1\Omega + Uw_2 w_2\Omega).$$

The reducible module has $$16 + 8 = 24$$ dimensions. We thus obtain a 24-dimensional reducible highest weight representation.

In summary, the algorithm consists of the following. We first derive sequences of irreducible quotients of highest weight submodules. Here they form a network of irreducible quotients. We terminate some sequences in the network at some points, and make it into two subnetworks. We then take the sum of the irreducible quotients in the subnetwork which has the end point of the network. We thus obtain a reducible highest weight submodule.

6.2.4. Conjectured relations. We remark that in the procedures from (ii) to (v), the quotient of a daughter submodule $$U\omega_{\mathbf{k}(\mathbf{w}')}$$ modulo the sum of all the parental submodules may have zero dimension. Up to $$m_j = 5,$$ we have confirmed that such vanishing cases are determined by using the following conjecture.

**Conjecture 34.** For $$0 \leq n \leq m_j$$ we have

$$\sum_{k=1}^{n} k w_{j+1}(\hat{a}) w_{j+1-k}(\hat{a}) \Omega = 0. \quad (98)$$

Up to the case of $$n = 3,$$ we have shown relation (98), applying relation (A_m) of lemma 8 with $$m = r - n$$ to vector $$w_{j}\Omega.$$

As an illustration, we consider the case of $$r = 6$$ and $$(m_1, m_2) = (5, 1).$$ Here we have $$\ell_1^{\text{max}} = 2$$ and $$\ell_2^{\text{max}} = 0.$$ We put $$\ell_1(0) = 2$$ and $$\ell_2(0) = 0.$$ Starting with $$\mathbf{k}(0) = (1^1, 1^3)$$ we have the following sequence of descendants: $$(1^1, 1^3) \rightarrow (1^1, 1^4) \rightarrow (1^1) \rightarrow (1^2, 1^3) \rightarrow (1^2, 1^4) \rightarrow (1^2) \rightarrow (1^3, 1^3) \rightarrow (1^3, 1^4) \rightarrow (1^3) \rightarrow (1^4, 1^4) \rightarrow (1^4) \rightarrow (\emptyset).$$ It follows from relations (98) that quotients corresponding to $$(1^2, 1^3)$$ and $$(1^3, 1^3)$$ have zero dimension. Irreducible quotients for $$(1^1, 1^3), (1^1, 1^4), (1^2, 1^4), (1^3, 1^4),$$ and $$(1^4, 1^4)$$ have four dimensions, while irreducible quotients for $$(1^1), (1^2), (1^3),$$ and $$(1^4)$$ have eight dimensions. The dimension of the irreducible quotient corresponding to $$(\emptyset)$$ is given by $$((5+1)(1+1)) = 12.$$ In total, we have $$2^6 = 64$$ dimensions as follows:

$$(4 + 4 + 4 + 4 + 4) + (8 + 8 + 8 + 8) + 12 = 64.$$ 

For $$m_j > 5$$ or 6, we may have some relations consisting of products of three or more $$w_{j}\mathbf{a}(\hat{a})$$'s which generalize relations (98), and they should be useful for constructing reducible highest weight representations.
6.3. Examples of reducible representations

We have a conjecture that the algorithm of section 6.2.3 leads to all of the reducible highest weight representations with the same given highest weight. For some explicit examples, we shall now construct all reducible representations with a given highest weight by the algorithm.

Hereafter, we write $\rho_j(a; s)$ and $w_j(\hat{a})$ simply as $\rho_j$ and $w_j$, respectively.

6.3.1. The case of $r = 3$. Let us consider the case of $r = 3$ with $m_1 = 2$ and $m_2 = 1$. The highest weight parameters of $\Omega$ are given by $\hat{a} = (a_1, a_1, a_2)$. It follows that the highest weight representation $U \Omega$ has four sectors of $h_0 = 3, 1, -1, \text{and} 3$, respectively. Here we recall some symbols.

$$ \rho_1 = x_1(a_2), \quad \rho_2 = x_1(a_1), \quad w_1 = x_2(a_1, a_2). $$

Here we recall that $x_2^-(a_1, a_2)$ denotes $x_2^-(B)$ with $B = (a_1, a_2)$, and $w_1$ abbreviates $w_1$. It is easy to show from reduction relations (42), i.e. $x_n^-(\hat{a})\Omega = 0 \ (n \in \mathbb{Z})$, that $x_n^-\Omega$ is expressed in terms of $\rho_1\Omega$, $\rho_2\Omega$, and $w_1\Omega$ as follows:

$$ x_n^-\Omega = \frac{a_1^n}{a_{12}}\rho_1\Omega + \frac{a_2^n}{a_{21}}\rho_2\Omega + \left(\frac{na_1^{n-1}}{a_{12}} - \frac{a_1^n - a_2^n}{a_{12}}\right)w_1\Omega \quad \text{for} \ n \in \mathbb{Z}. \quad (100) $$

It is straightforward to show

$$ x_n^+w_1\Omega = 0, \quad \text{for} \ n \in \mathbb{Z}. \quad (101) $$

It thus follows that $U\Omega$ is reducible and indecomposable if $w_1\Omega \neq 0$. From lemmas 29 and 30 we have

$$ w_1^2\Omega = 0. \quad (102) $$

The following vectors do not vanish, if and only if $\Omega$ does not vanish (i.e. $\Omega \neq 0$):

$$ \rho_2\Omega \neq 0; \quad \rho_1\rho_2\Omega \neq 0; \quad w_1^2\Omega \neq 0. \quad (103) $$

In fact, $\rho_2^2\rho_2\Omega = 0$, applying $\rho_1^+(\hat{a}; 2)$ and $\rho_2^+(\hat{a}; 2)$ to it, we derive that $\Omega = 0$. From the viewpoint of lemma 29, the highest weight vector $w_1\Omega$ has only one highest weight parameter $a_2$, i.e. $m'_1 = 0$ and $m'_2 = 1$, and hence we have the following reduction relation:

$$ \rho_1w_1\Omega = 0. \quad (104) $$

Furthermore, we have $\rho_2w_1\Omega \neq 0$ if $w_1\Omega \neq 0$. In the four sectors of $U\Omega$ the basis vectors are given as follows:

$$ \Omega, \quad \rho_1\Omega, \quad \rho_2\Omega, \quad w_1\Omega, \quad \text{for} \ h_0 = 3; $$

$$ \rho_1^2\Omega, \quad \rho_1\rho_2\Omega, \quad w_1\Omega, \quad \text{for} \ h_0 = -1; $$

$$ \rho_1^2\rho_2\Omega, \quad \text{for} \ h_0 = -3. $$

Consequently, we have the following result.

**Proposition 35.** The highest weight representation with three highest weight parameters: $(\hat{a}) = (a_1, a_1, a_2)$, is reducible, indecomposable and of $2^3$ dimensions, if and only if $w_1\Omega \neq 0$. It is irreducible and of 6 dimensions, if and only if $w_1\Omega = 0$. 

The highest weight representation with three highest weight parameters: $(\hat{a}) = (a_1, a_1, a_2)$, is reducible, indecomposable and of $2^3$ dimensions, if and only if $w_1\Omega \neq 0$. It is irreducible and of 6 dimensions, if and only if $w_1\Omega = 0$. 

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6.3.2. The case of four highest weight parameters. Let us consider the case of \( r = 4 \) with \( m_1 = 2 \) and \( m_2 = 2 \). Here we note \((\hat{a}) = (a_1, a_1, a_2, a_2)\). The highest weight representation \( U\Omega \) has five sectors of \( h_0 = 4, 2, 0, -2, \) and \(-4\). With the reduction relations \( x_n^- (\hat{a}) \Omega = 0 \) \((n \in \mathbb{Z})\) as given in equation (42), we consider the following four operators:

\[
\rho_1 = x_1^- (a_2), \quad \rho_2 = x_1^- (a_1), \quad w_1 = x_3^- (a_1, a_2, a_2), \quad w_2 = x_3^- (a_1, a_1, a_2).
\]

(105)

Vectors \( x_n^- \Omega \) for \( n \in \mathbb{Z} \) are expressed in terms of \( \rho_1 \Omega, \rho_2 \Omega, w_1 \Omega \) and \( w_2 \Omega \) as follows:

\[
x_n^- \Omega = \frac{a_n^1}{a_{12}} \rho_1 \Omega + \frac{a_n^2}{a_{21}} \rho_2 \Omega + \frac{1}{a_{12}^2} \left( na_{11}^{-1} - \frac{a_n^1 - a_n^2}{a_{12}} \right) w_1 \Omega + \frac{1}{a_{12}^2} \left( na_{22}^{-1} - \frac{a_n^1 - a_n^2}{a_{12}} \right) w_2 \Omega.
\]

(106)

Making use of proposition 31 (or directly from lemma 29) we have

\[
w_1^2 \Omega = w_2^2 \Omega = 0.
\]

(107)

Making use of lemma 29, we have

\[
w_1 w_2 \Omega = a_{12}^2 \rho_1 w_1 \Omega = a_{12}^2 \rho_2 w_2 \Omega.
\]

(108)

It is also straightforward to show the following:

\[
x_n^+ w_1 \Omega = x_n^+ w_2 \Omega = 0, \quad \text{for } n \in \mathbb{Z},
\]

\[
x_n^+ w_1 w_2 \Omega = 0 \quad (x_n^+ \rho_1 w_1 \Omega = x_n^+ \rho_2 w_2 \Omega = 0), \quad \text{for } n \in \mathbb{Z}.
\]

(109)

It thus follows that \( U\Omega \) is reducible if \( w_1 \Omega \neq 0, w_2 \Omega \neq 0 \) or \( w_1 w_2 \Omega \neq 0 \). From lemma 30 and proposition 31 we also have the following:

\[
\rho_1^3 \Omega \neq 0, \quad \rho_2^3 \Omega = 0; \quad \rho_2^3 \Omega \neq 0, \quad \rho_2^3 \Omega = 0; \quad w_2^3 \Omega = 0.
\]

(110)

The basis vectors of \( U\Omega \) are given by

\[
\Omega, \quad \rho_1 \Omega, \quad \rho_2 \Omega, \quad w_1 \Omega, \quad w_2 \Omega, \quad w_1 w_2 \Omega, \quad w_1 w_2 \Omega, \quad w_1 w_2 \Omega, \quad w_1 w_2 \Omega, \quad w_1 w_2 \Omega.
\]

As an illustration, let us apply the algorithm of section 6.2.3 for constructing reducible highest weight representations. Here we recall that \( m_1 = 2 \) and \( m_2 = 2 \). Here we have \( \ell_1 < (m_1 + 1)/2 = 3/2 \), and we obtain \( \ell_1 \text{max} = 1 \). Similarly, we have \( \ell_2 \text{max} = 1 \). Thus, we first consider \( \omega \text{max} = Uw_1, w_2 \Omega \).

1. \( Uw_1 w_2 \Omega \) has \( r' = 4 - 2 \times 2 = 0 \) and it has one dimension;
2. \( Uw_1 \Omega / Uw_1 w_2 \Omega \) has \( r' = 4 - 1 \times 2 = 2 \) where \((\hat{a}'_1, \hat{a}'_2) = (a_1, a_1)\), and it has three dimensions since \((2 + 1) = 3\);
3. \( Uw_2 \Omega / Uw_1 w_2 \Omega \) has \( r' = 4 - 1 \times 2 = 2 \) where \((\hat{a}'_1, \hat{a}'_2) = (a_2, a_2)\), and it has three dimensions since \((2 + 1) = 3\);
4. \( U\Omega \) modulus \( Uw_1 \Omega \) and \( Uw_2 \Omega \) has \( r = 4 \) where \((\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4) = (a_1, a_1, a_2, a_2)\), and it has \((2 + 1) \times (2 + 1) = 9\), i.e. nine dimensions.

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We note that \( w_1 - w_2 = a_1 \bar{a}_2 (a_1, a_2) \). Thus, if \( w_1 \Omega = 0 \) and \( w_2 \Omega = 0 \), we have \( \bar{a}_2 (a_1, a_2) \Omega = 0 \). That is, \( \Omega \) generates an irreducible representation.

In summary, for all possible dimensions of reducible and irreducible highest weight representations with highest weight parameters \((a_1, a_2, a_2)\), we have the following result.

**Proposition 36.** If the highest weight representation with highest weight parameters \( \hat{a} = (a_1, a_1, a_2, a_2) \) is reducible, there are the following four cases: (i) \( w_1 \Omega \neq 0, w_2 \Omega \neq 0, \) and \( w_1 w_2 \Omega \neq 0; \) (ii) \( w_1 \Omega = 0, w_2 \Omega \neq 0, \) and \( w_1 w_2 \Omega = 0; \) (iii) \( w_1 \Omega \neq 0, w_2 \Omega = 0, \) and \( w_1 w_2 \Omega = 0; \) (iv) \( w_1 \Omega = 0, w_2 \Omega = 0, \) and \( w_1 w_2 \Omega = 0. \) It has dimensions 16, 15, 12, and 12, for cases (i), (ii), (iii) and (iv), respectively. It is irreducible if and only if \( w_1 \Omega = 0 \) and \( w_2 \Omega = 0 \). If it is irreducible, it has nine dimensions.

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**Appendix A. Proof of proposition 3**

Applying the Poincaré–Birkhoff–Witt theorem [27] we can show that every vector \( v \) in the subspace of weight \(-r\) of \( U \Omega \) is written as follows

\[
v = \sum_{k_1 \leq \ldots \leq k_r} C_{k_1, \ldots, k_r} x_{k_1}^{-} \cdots x_{k_r}^{-} \Omega.
\]

(A.1)

Here the coefficients \( C_{k_1, \ldots, k_r} \) are some complex numbers. Then, we obtain proposition 3 from the following lemma.

**Lemma A.1.** Let \( n \) be a non-negative integer and \( n \leq r \). For any given set of integers \( k_1, \ldots, k_n \), we have

\[
(x_0^{-})^{-n} x_{k_1}^{-} \cdots x_{k_n}^{-} \Omega = A_{k_1, \ldots, k_n} (x_0^{-})^r \Omega.
\]

(A.2)

Here \( A_{k_1, \ldots, k_n} \) is a complex number.

**Proof.** We show it by induction on \( n \). The case of \( n = 0 \) is trivial. Suppose that relations (A.2) hold for the cases \( n - 1 \) and \( n \). We show the case of \( n + 1 \) as follows. We have from (A.2) in the case of \( n \) the following:

\[
x_m^+(x_0^{-})^{-n+1} \prod_{j=1}^{n} x_{k_j}^{-} \Omega = x_m^+ x_0^{-} \cdot A_{k_1, \ldots, k_n} (x_0^{-})^r \Omega
\]

\[
= A_{k_1, \ldots, k_n} x_m^+ (x_0^{-})^{r+1} \Omega = 0.
\]

Calculating the commutation relation: \( [x_m^+, (x_0^{-})^{-1-n}] \prod_{j=1}^{n} x_{k_j}^{-} \), we have

\[
(x_0^{-})^{-n+1} x_m^+ \prod_{j=1}^{n} x_{k_j}^{-} \Omega = d_{m+k_i} (x_0^{-})^{-n} \prod_{j=1}^{n} x_{k_j}^{-} \Omega + (-2) \sum_{i=1}^{n} (x_0^{-})^{-n} x_{k_i+1} x_{k_{j \neq i}}^{-} \prod_{j=1; j \neq i}^{n} x_{k_j}^{-} \Omega
\]

\[
+ \sum_{i=1}^{n} d_{m+1+k_i} (x_0^{-})^{-n+1} \prod_{j=1; j \neq i}^{n} x_{k_j}^{-} \Omega
\]

\[
+ (-2) \sum_{1 \leq i_1 < i_2 \leq n} (x_0^{-})^{-n} x_{m+1+k_{i_1}+k_{i_2}} x_{k_{j \neq i_1, j \neq i_2}}^{-} \prod_{j=1; j \neq i_1, i_2}^{n} x_{k_j}^{-} \Omega.
\]

(A.3)

Denoting \( m \) by \( k_{n+1} \), we thus obtain relation (A.2) for the case of \( n + 1 \).
Appendix B. Recursive relations for Drinfeld generators

We now show lemma 7. We recall that \((X)^{(n)}\) denotes \((X)^{(n)} = X^n/n!\).

Lemma B.1. The following recursive formula with respect to \(n\) holds for products of operators \((x^+_{\ell})^{(n-1)}(x^-_{1-\ell}(a))^{(n)}\):

\[
(x^+_{\ell})^{(n)}(x^-_{1-\ell}(a))^{(n+1)} = x^-_{1-\ell}(a)(x^+_{\ell})^{(n)}(x^-_{1-\ell}(a))^{(n)} + \frac{1}{2}[h_1, (x^+_{\ell})^{(n-1)}(x^-_{1-\ell}(a))^{(n)}]
- (x^+_{\ell})^{(n-1)}(x^-_{1-\ell}(a))^{(n+1)} x^+_{\ell}, \quad \text{for } \ell \in \mathbb{Z}.
\]  

(B.1)

Proof. Applying relations (30) we show the following:

\[
(n+1)(x^+_{\ell})^{(n)}(x^-_{1-\ell}(a))^{(n+1)} = (x^+_{\ell})^{(n)} x^-_{1-\ell}(a)(x^-_{1-\ell}(a))^{(n)}
- x^-_{1-\ell}(a)(x^+_{\ell})^{(n)}(x^-_{1-\ell}(a))^{(n)} + [(x^+_{\ell})^{(n)}, x^-_{1-\ell}(a)](x^-_{1-\ell}(a))^{(n)}
+ (x^+_{\ell})^{(n-1)} h_1(a)(x^-_{1-\ell}(a))^{(n)} + x^+_{\ell+1}(a)(x^-_{1-\ell}(a))^{(n-2)}(x^-_{1-\ell}(a))^{(n)}
= x^-_{1-\ell}(a)(x^+_{\ell})^{(n)}(x^-_{1-\ell}(a))^{(n)}
+ \left\{ (x^+_{\ell})^{(n-1)}(x^-_{1-\ell}(a))^{(n)} h_1(a) + (x^+_{\ell})^{(n-1)}[h_1(a), (x^-_{1-\ell}(a))^{(n)}] \right\}
+ \frac{1}{2}[h_1(a), (x^+_{\ell})^{(n-1)}](x^-_{1-\ell}(a))^{(n)}.
\]  

(B.2)

Here, substituting the product \((x^-_{1-\ell}(a))^{(n)} h_1(a)\) by

\[
[x^+_{\ell}, (x^-_{1-\ell}(a))^{(n+1)}] + 2 x^-_{2-\ell}(a)^2(x^-_{1-\ell}(a))^{(n-1)}
= x^+_{\ell}(x^-_{1-\ell}(a))^{(n+1)} + 2 x^-_{2-\ell}(a)^2(x^-_{1-\ell}(a))^{(n-1)} - (x^-_{1-\ell}(a))^{(n+1)} x^+_{\ell},
\]

we show that the second term \((x^+_{\ell})^{(n-1)}(x^-_{1-\ell}(a))^{(n)} h_1(a)\) in the last lines of (B.2) is equal to the following:

\[
n(x^+_{\ell})^{(n)}(x^-_{1-\ell}(a))^{(n+1)} + (x^+_{\ell})^{(n-1)} x^-_{2-\ell}(a)^2(x^-_{1-\ell}(a))^{(n-1)} - (x^+_{\ell})^{(n-1)}(x^-_{1-\ell}(a))^{(n+1)} x^+_{\ell}.
\]

Thus, we have

\[
\{(n+1) - n\}(x^+_{\ell})^{(n)}(x^-_{1-\ell}(a))^{(n+1)}
= x^-_{1-\ell}(a)(x^+_{\ell})^{(n)}(x^-_{1-\ell}(a))^{(n)} + (x^+_{\ell})^{(n-1)} x^-_{2-\ell}(a)^2(x^-_{1-\ell}(a))^{(n-1)}
+ (x^+_{\ell})^{(n-1)}[h_1(a), (x^-_{1-\ell}(a))^{(n)}]
+ \frac{1}{2}[h_1(a), (x^+_{\ell})^{(n-1)}](x^-_{1-\ell}(a))^{(n)} - (x^+_{\ell})^{(n-1)}(x^-_{1-\ell}(a))^{(n+1)} x^+_{\ell}.
\]

(B.3)

Putting \(x^-_{2-\ell}(a)^2(x^-_{1-\ell}(a))^{(n-1)} = -(1/2)[h_1(a), (x^-_{1-\ell}(a))^{(n)}]\) into (B.3) we have relation (B.1). \(\square\)
Appendix C. Reduction relations for \( a \neq 0 \)

Substituting \( \ell = 0 \) in \((B_n)\) and making use of (39), we have the following.

**Corollary C.1.**

\[
\lambda_n(a) = \frac{1}{n} \sum_{k=1}^{n} (-1)^{k-1} d_k((a)^k) \lambda_{n-k}(a), \quad \text{for } n = 1, 2, \ldots, r. \tag{C.1}
\]

**Lemma C.1.** For any integer \( \ell \) we have

\[
(x^\ell)(k)(x_{1-\ell}^-((a))^{(k)}(a)\Omega = \lambda_k(a)\Omega, \quad \text{for } k = 1, 2, \ldots, r. \tag{C.2}
\]

**Proof.** We show (C.2) by induction on \( k \). The \( k = 1 \) case is shown directly. Assuming (C.2) for \( k \leq n - 1 \), we derive the \( k = n \) case by \((B_n)\) of lemma 8, (C.1) and (39).

**Proposition C.1.** We have

\[
x_{r+1-\ell}^-(a)\Omega = \sum_{j=1}^{r} (-1)^{r-j} \lambda_{r+1-j}((a)j)\Omega, \quad \text{for } \ell \in \mathbb{Z}. \tag{C.3}
\]

Here we recall that \( \lambda_j(a) \) is defined by (28) with the highest weight parameters.

**Proof.** We derive reduction relation (C.3) from \((A_{r+1})\) of lemmas 8 and C.1.

Appendix D. Proof of equation (82) of lemma 26

In order to derive the first relation of (82), we first show the following:

\[
[h_n, w_{jk}(\hat{a})] = (-2) \sum_{t=0}^{n} \binom{n}{t} a_j^{n-t} w_{jk-l}(\hat{a}) \quad \text{for } n > 0. \tag{D.1}
\]

We show it by induction on \( n \). For \( n = 1 \) we have

\[
[h_1, w_{jk}(\hat{a})]/(-2) = x_{r-k+1}^-((\hat{a}) \setminus j^k)
= x_{r-k+1}^-((\hat{a}) \setminus j^{k-1}) + a_j x_{r-k}^-((\hat{a}) \setminus j^k)
= w_{j-1}((\hat{a}) + a_j w_{jk}(\hat{a}).
\]

Let us assume (D.1) in the case of \( n \). In order to derive (D.1) for the case of \( n + 1 \), we first make use of the following:

\[
[h_{n+1}, w_{jk}(\hat{a})]/(-2) = x_{r-k+n+1}^-((\hat{a}) \setminus j^k)
= x_{r-(k-1)+n}^-((\hat{a}) \setminus j^{k-1}) + a_j x_{r-k+n}^-((\hat{a}) \setminus j^k),
= [h_n, w_{jk-1}(\hat{a})] + a_j [h_n, w_{jk}(\hat{a})]. \tag{D.2}
\]

Substituting relation (D.1) for \( n \) into (D.2) and making use of the recursive relation: \( n+1 C_t = n C_t + n C_{t-1} \), we obtain relation (D.1) for the case of \( n + 1 \).
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We now discuss the second relation of (82). We show it by induction on $n$ and $k$. We first derive it for the case of $n = 1$ and for arbitrary $k$ with $k \geq 1$. Through induction on $k$, it is easy to show the following:

$$[h_{-1}, w_{j+k}(\hat{a})]\Omega = (-2) \sum_{t=0}^{n-1} \binom{-1}{t} a_j^{-1-t} w_{j+k-t}(\hat{a})\Omega.$$  \hfill (D.3)

Assuming the case of $n$ and $k$, we now show the case of $n+1$ and $k$. We first note

$$x_{r-m-n-1}^{-} (\hat{a}/j^k) = a_1^{-1} x_{r-m-n}^{-} (\hat{a}/j^k) - a_1^{-1} x_{r-m-n}^{-} (\hat{a}/j^{k-1})$$ \hfill (D.4)

and make use of the following relation:

$$\binom{-n-1}{k} = \binom{-n}{t} - \sum_{\ell=1}^{k} \binom{-1}{\ell-1} \binom{-n}{k-\ell},$$ \hfill (D.5)

we obtain the second relation of (82) for the case of $n+1$ and $k$.

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