UNIQUENESS OF GENERATING HAMILTONIANS FOR CONTINUOUS HAMILTONIAN FLOWS

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Abstract. We prove that a continuous Hamiltonian flow as defined by Oh and Müller [OM], has a unique $L^{(1,\infty)}$ generating Hamiltonian. This answers a question raised by Oh and Müller in [OM], and improves a result of Viterbo [V].

1. Introduction

Let $(M, \omega)$ be a symplectic manifold, which is closed and connected. Throughout the paper we assume that all Hamiltonians are normalized in the following way: given a time dependent Hamiltonian $H : [0, 1] \times M \to \mathbb{R}$ we require that $\int_M H(t, x)\omega^n = 0, \forall t \in [0, 1]$. For a given open subset $U \subset M$, we denote by $\text{Ham}_U(M, \omega)$ the set of all time-1 maps of smooth Hamiltonian flows that coincide with the identity flow on $M \setminus U$.

We denote by $\text{PHam}(M, \omega)$ the space of smooth Hamiltonian flows. Clearly, given $\Phi^t \in \text{PHam}(M, \omega)$, there exists a unique normalized Hamiltonian $H$, that generates the flow $\Phi^t$. The main purpose of this paper is to prove the above uniqueness result for Hamiltonian generators of topological Hamiltonian paths, as defined in [OM]. This “uniqueness of generating Hamiltonians” turns out to be essential to extending various constructions on spaces $\text{Ham}(M, \omega)$ and $\text{PHam}(M, \omega)$, to the case of continuous Hamiltonian flows [Oh-1].

The study of continuous symplectic geometry began with the celebrated Gromov-Eliashberg rigidity theorem, which states that the group $\text{Symp}(M, \omega)$ of symplectomorphisms of $(M, \omega)$ is $C^0$ closed in

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the group of diffeomorphisms of $M$. This theorem motivates the following definition of symplectic homeomorphisms. The group of symplectic homeomorphisms $\text{Sympeo}(M, \omega)$ is defined as the $C^0$ closure of $\text{Symp}(M, \omega)$ in the group of homeomorphisms of $M$. Extending the notion of Hamiltonian flows turns out to be more complicated.

In [OM], Oh and Müller introduce the notions of topological Hamiltonian paths (or continuous Hamiltonian flows), and Hamiltonian homeomorphisms. By definition, a continuous path of homeomorphisms $\Phi^t : M \to \mathbb{R}$ is called a continuous Hamiltonian flow, generated by $H \in L^{(1, \infty)}([0, 1] \times M)$, if there exists a sequence of smooth Hamiltonian flows, $\Phi^t_{H_i}$, with generating Hamiltonians $H_i \in C^\infty([0, 1] \times M)$, such that

$$\Phi^t = (C^0) \lim_{i \to \infty} \Phi^t_{H_i},$$

$$H = (L^{(1, \infty)}) \lim_{i \to \infty} H_i,$$

that is, the first convergence is in the uniform topology, and the second convergence is in the $L^{(1, \infty)}$ topology. The $L^{(1, \infty)}$ norm, also known as the Hofer [HZ] norm of a Hamiltonian is defined as

$$\|K\|^{(1, \infty)} = \int_0^1 \max_x (\max_t K(t, x) - \min_t K(t, x)) \, dt.$$

We denote by $\text{PHameo}(M, \omega)$ the space of all pairs $(\Phi^t, H)$ of a continuous Hamiltonian flow $\Phi^t$ and a $L^{(1, \infty)}$ Hamiltonian $H$, that generates $\Phi^t$.

The space $\text{Hameo}(M, \omega)$ of Hamiltonian homeomorphisms is defined to be the set of all time-1 maps of continuous Hamiltonian flows.

**Question 1.** Does a continuous Hamiltonian flow $\Phi^t$ have a unique generating Hamiltonian? In other words, assume we have two (smooth) sequences $(\Phi^t_{H_i}, H_i), (\Phi^t_{K_i}, K_i) \in \text{PHam}(M, \omega)$ satisfying

$$(C^0) \lim \Phi^t_{H_i} = (C^0) \lim \Phi^t_{K_i} = \Phi^t,$$

$$(L^{(1, \infty)}) \lim H_i = H,$$

$$(L^{(1, \infty)}) \lim K_i = K.$$

Does this imply $K = H$, as $L^{(1, \infty)}$ functions?

This question was raised by Oh and Müller [OM]. The goal of this paper is to give an affirmative answer to the above question.

Going back to the case of smooth Hamiltonian flows, for given $\Phi^t_{H}, \Phi^t_K \in \text{PHam}(M, \omega)$, generated by smooth Hamiltonians $H, K$, we have the
following well known formulae for the Hamiltonian functions of a composition of flows and an inverse of a flow:

\[ \Phi_t^H \circ \Phi_t^K = \Phi_t^G, \text{ where } G = H \# K(t, x) := H(t, x) + K(t, (\Phi_t^H)^{-1}(x)). \]

\[ (\Phi_t^H)^{-1} = \Phi_t^{-H}, \text{ where } \overline{H}(t, x) := -H(t, \Phi_t^H(x)). \]

It was shown by Oh and Müller [OM] that these operations admit a natural generalization to the space \( \text{PHameo}(M, \omega) \). It follows that given two pairs \((\Phi^t, H), (\Phi^t, K) \in \text{PHameo}(M, \omega)\) with common continuous Hamiltonian flow, we get the identity flow \( \text{Id}^t = (\Phi^t)^{-1} \circ \Phi^t \) generated by the Hamiltonian

\[ \overline{H} \# K(t, x) = -H(t, \Phi^t(x)) + K(t, \Phi^t(x)). \]

Hence, question 1 simplifies to:

**Question 2.** Assume we have a sequence of smooth Hamiltonian paths \((\Phi^t_H, H_i) \in \text{PHam}(M, \omega)\) satisfying \((C^0) \lim \Phi^t_H = \text{Id}^t\) and \((L^{(1,\infty)}) \lim H_i = H\). Does this imply \( H = 0 \), as an \( L^{(1,\infty)} \) function?

In [V], Viterbo gives an affirmative answer to the above question assuming \((C^0) \lim H_i = H\). Note that \((C^0) \lim H_i = H\) implies

\[(L^{(1,\infty)}) \lim H_i = H.\]

The methods employed in this paper are very different than those used in [V].

Section 2 contains the statement of our main result and a formulation of a sequence of lemmata, that are used in its proof. In Section 3 we present the proof of the main result. Section 4 studies the local uniqueness for \( L^{(1,\infty)} \) Hamiltonians and for continuous Hamiltonian flows. Here we state and prove the generalization of Theorem 1.3 from [Oh-2], to the \( L^{(1,\infty)} \) case. We derive two consequences of this local uniqueness result. First, on any closed symplectic manifold we construct an example of a continuous function, that fails to be a generator of any continuous Hamiltonian flow. Second, we give an example of a continuous flow on any closed 2-dimensional surface, which is the \( C^0 \) limit of smooth Hamiltonian flows, but is not a continuous Hamiltonian flow.

**Remark 3.** All the results in the present paper can be directly generalized to the case of an open symplectic manifold \((M, \omega)\), where in this case we consider the space of compactly supported continuous Hamiltonian flows, or equivalently, the space of compactly supported topological Hamiltonian paths (see [OM] for the definition).
2. Main result

In this section we present our main result.

Here's our answer to Question 2:

**Theorem 4.** Denote by $\text{Id}^t : M \to M$ the identity flow. If we have $H \in L^{1,\infty}([0, 1] \times M)$, such that $(\text{Id}^t, H) \in \text{PHameo}(M, \omega)$, then we have $H = 0$ in $L^{1,\infty}([0, 1] \times M)$.

We will use the following definition in our proof.

**Definition 5.** (Null Hamiltonians) Define $\mathcal{H}_0 = \{ H \in L^{1,\infty}([0, 1] \times M) | (\text{Id}^t, H) \in \text{PHameo}(M, \omega) \}$, this is the space of null Hamiltonians.

$\mathcal{H}_0^{st} = \{ H \in \mathcal{H}_0 | H \text{ is time independent} \}$.

Since the space $\mathcal{H}_0^{st}$ consists of time-independent null Hamiltonians, we identify it with a subspace of $C(M)$.

We divide the proof of Theorem 4 into a sequence of lemmata. Lemma 6 is the smooth case of Theorem 4. It has been proven in the past, see e.g. [OM] or [HZ].

**Lemma 6.** If $H \in \mathcal{H}_0 \cap C^\infty([0, 1] \times M)$, then for all $t \in [0, 1]$ we have $H(t, x) \equiv 0$.

**Lemma 7.** The spaces $\mathcal{H}_0, \mathcal{H}_0^{st}$ have the following properties:

1. $\mathcal{H}_0$ is closed under the sum operation and the minus operation.
   In other words, if $H, K \in \mathcal{H}_0$, then $-H, H + K \in \mathcal{H}_0$.
2. $\mathcal{H}_0$ is closed in $L^{1,\infty}$ topology. $\mathcal{H}_0^{st}$ is closed in $C^0$-topology.
3. If $H \in \mathcal{H}_0$, then for any smooth increasing function $\alpha : [0, 1] \to [0, 1]$ the Hamiltonian $K(t, x) = \alpha'(t)H(\alpha(t), x)$ belongs to $\mathcal{H}_0$ as well.
4. $\mathcal{H}_0^{st}$ is a vector space over $\mathbb{R}$.
5. If $H \in \mathcal{H}_0^{st}$, then for any $\Phi \in \text{Symp}(M, \omega)$ we have $\Phi^*H = H \circ \Phi \in \mathcal{H}_0^{st}$.

**Lemma 8.** (Lebesgue’s differentiation theorem) Let $H_i \in C([0, 1] \times M)$ be a sequence converging in the $L^{1,\infty}$ topology to a function $H \in L^{1,\infty}([0, 1] \times M)$. Then $H$ can be represented by a function $H : [0, 1] \times M \rightarrow \mathbb{R}$ (we use the same notation for the function as well), such that for any $t \in [0, 1]$ we have $H(t, \cdot) \in C(M)$. Moreover, almost everywhere in $t \in [0, 1)$ we have

$$\lim_{h \to 0^+} \frac{1}{h} \int_{t}^{t+h} \|H(s, \cdot) - H(t, \cdot)\|_\infty ds = 0.$$
Lemma 9. Let \( H \in \mathcal{H}_0 \), and denote by the same notation \( H \) its functional representative, as in Lemma 8. Then for almost any \( t \in [0,1] \), the time-independent Hamiltonian \( h(x) = H(t,x) \) lies inside \( \mathcal{H}_0^{st} \).

Lemma 10. If \( H \in \mathcal{H}_0^{st} \), then \( H \equiv 0 \).

3. PROOFS

Proof of Lemma 9. Assume for a contradiction, that \( H \) is not constantly zero. Let \( \Phi_H^t \) denote the flow of \( H \). Since \( H \) is not constantly zero we conclude that \( \Phi_H^t \) is not the identity map, for some \( T \in [0,1] \).

Since \((\text{Id}^t, H) \in \text{PHameo}(M,\omega)\), there exists a smooth sequence \((\Phi_{H_i}^t, H_i) \in \text{PHam}(M,\omega)\) which converges to \((\text{Id}^t, H)\). This implies that \((\Phi_{H_i}^T)^{-1} \circ \Phi_H^T C^0 \) converges to \( \Phi_H^T \). Pick a point \( x \in M \) such that \( \Phi_H^T(x) \neq x \). There exists a small open neighborhood, \( U \) of \( x \), which is displaced by \((\Phi_{H_i}^T)^{-1} \circ \Phi_H^T \), for \( i \) large enough. The energy-capacity inequality \([H] \), implies that the Hofer norm of \((\Phi_{H_i}^T)^{-1} \circ \Phi_H^T \) is bounded below by a positive constant, \( e(U) \). But this norm is bounded from above by \( \| \Phi_{H_i}^T \# H \|_{(1,\infty)} = \| -H_i(t, \Phi_{H_i}^t(x)) + H(t, \Phi_{H_i}^t(x)) \|_{(1,\infty)} = \| -H_i + H \|_{(1,\infty)} \), which contradicts the \( L^{(1,\infty)} \) convergence of \( H_i \) to \( H \).

Proof of Lemma 7.

(1): If \((\lambda', H), (\mu', K) \in \text{PHameo}(M,\omega)\), then the composition of the pairs, \((\lambda' \circ \mu', H \# K)\), and the inverse flow \(( (\lambda')^{-1}, \mathcal{H} ) \) are also in \( \text{PHameo}(M,\omega) \). Since \( \lambda' = \mu' = \text{Id}^t \), we have \( H \# K = H + K, \mathcal{H} = -H \).

(2): This is clear from the definition of \( \mathcal{H}_0 \) and of \( \mathcal{H}_0^{st} \).

(3): If \( \Phi_G^t \) is a smooth Hamiltonian flow generated by \( G \), then its reparameterized flow \( \Phi_G^{\alpha(t)} \) is generated by \( L(t,x) = \alpha'(t)G(\alpha(t),x) \). If we assume that \( H \in \mathcal{H}_0 \), then there exists a sequence \( H_i(t,x) \) of smooth Hamiltonians, such that we have \( (C^0) \lim \Phi_{H_i}^t = \text{Id}^t \), and \( (L^{(1,\infty)}) \lim H_i = H \). Then the reparameterized flows \( \Phi_{H_i}^{\alpha(t)} \) are generated by \( K_i(t,x) = \alpha'(t)H_i(\alpha(t),x) \). It is clear, that \( (C^0) \lim \Phi_{K_i}^t = (C^0) \lim \Phi_{H_i}^{\alpha(t)} = \text{Id}^t \), and also \( (L^{(1,\infty)}) \lim K_i(t,x) = (L^{(1,\infty)}) \lim \alpha'(t)H_i(\alpha(t),x) = \alpha'(t)H(\alpha(t),x) \).

Therefore \( K(t,x) = \alpha'(t)H(\alpha(t),x) \in \mathcal{H}_0 \).

(4): This follows from the previous results. Suppose \( H \in \mathcal{H}_0^{st} \) with the Hamiltonian flow \( \Phi_H^t \). For any \( 0 < a < 1 \), apply (3) with \( \alpha(t) = at \)
to obtain that \( aH \in \mathcal{H}_0 \) and hence \( aH \in \mathcal{H}^*_0 \). Then, the case of general \( a \in \mathbb{R} \) follows from (1).

(5): In the smooth case, if \( H \) generates the Hamiltonian flow \( \Phi^t \), then \( \Psi^*H \) generates the Hamiltonian flow \( \Psi^{-1}\Phi^t\Psi \). This property extends to topological Hamiltonian flows \( [OM] \), and hence the result follows.

\[
\square
\]

**Proof of Lemma 8.** We think of each \( H_i \) as a continuous path \( t \mapsto H_i(t, \cdot) \) in the Banach space \( L^\infty(M) \). The images of these paths are contained in the subspace \( C(M) \subset L^\infty(M) \). The sequence of paths \( H_i \) converges to \( H \) in the \( L^1 \) norm, or equivalently the sequence of functions \( \|H_i(t, \cdot) - H(t, \cdot)\|_\infty : [0, 1] \mapsto \mathbb{R} \), \( L^1 \) converges to zero. It is well known that every \( L^1 \) converging sequence of functions has a subsequence which converges almost everywhere, see [F]. So we may assume that, for almost every \( t \in [0, 1] \) the sequence \( \|H_i(t, \cdot) - H(t, \cdot)\|_\infty \) converges to 0. This implies that \( H(t, \cdot) \in C(M) \) for almost every \( t \in [0, 1] \).

By changing the function \( H(t, \cdot) \) to coincide with a continuous function for other values of \( t \), we obtain \( H(t, \cdot) \in C(M) \) for all \( t \in [0, 1] \).

The second part of the theorem is a reformulation of Lebesgue’s differentiation theorem for \( L^1 \) maps from \([0,1]\) to the Banach space \( C(M) \). The functions \( H_i \) are continuous and hence they satisfy

\[
\lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \|H_i(s, \cdot) - H_i(t, \cdot)\|_\infty ds = 0
\]

for all \( t \in [0, 1] \).

Denote \( F_i = H - H_i \). Then for \( t \in [0, 1] \) we have

\[
\limsup_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \|H(s, \cdot) - H(t, \cdot)\|_\infty ds
\]

\[
\leq \left( \limsup_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \|F_i(s, \cdot) - F_i(t, \cdot)\|_\infty ds \right) + \left( \limsup_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \|H_i(s, \cdot) - H_i(t, \cdot)\|_\infty ds \right)
\]

\[
= \limsup_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \|F_i(s, \cdot) - F_i(t, \cdot)\|_\infty ds
\]

\[
\leq \limsup_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \|F_i(s, \cdot)\|_\infty + \|F_i(t, \cdot)\|_\infty ds
\]

\[
= \|F_i(t, \cdot)\|_\infty + \limsup_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \|F_i(s, \cdot)\|_\infty ds.
\]
Denote \( f_i(t) := \|F_i(t, \cdot)\|_\infty \), we have \( f_i \in L^1([0, 1]) \). By the standard Lebesgue differentiation theorem, for any \( i \), we have

\[
\lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} f_i(s) ds = f_i(t),
\]

or

\[
\lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \|F_i(s, \cdot)\|_\infty ds = \|F_i(t, \cdot)\|_\infty
\]

for almost every \( t \in [0, 1) \). Therefore for any \( i \) we have

\[
\lim \sup_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \|H(s, \cdot) - H(t, \cdot)\|_\infty ds \leq 2 f_i(t) = 2 \|F_i(t, \cdot)\|_\infty
\]

for almost every \( t \in [0, 1) \).

The sequence of functions, \( f_i(t) \), \( L^1 \) converges to zero. Every \( L^1 \) converging sequence has a subsequence that converges almost everywhere. Hence, by passing to a subsequence we may assume \( f_i(t) \) converges to zero for almost every \( t \in [0, 1) \).

\[ \square \]

**Proof of Lemma 9.** Because of Lemma 8, \( H \) can be represented by a function \( H : [0, 1] \times M \to \mathbb{R} \), such that for any \( t \in [0, 1) \), the function \( H(t, \cdot) \in C(M) \) is continuous, and moreover for almost any \( t \in [0, 1) \) we have

\[
\lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \|H(s, \cdot) - H(t, \cdot)\|_\infty ds = 0.
\]

Consider such a value of \( t \in [0, 1) \). Take \( N \in \mathbb{N} \) large enough. Applying Lemma (3) for \( \alpha(s) = t + \frac{\phi}{N} \), we obtain a Hamiltonian \( G_N(s, x) = \frac{1}{N} H(t + \frac{\phi}{N}, x) \in \mathcal{H}_0 \). Applying Lemma (1), we get \( H_N(s, x) = NG_N(s, x) = H(t + \frac{\phi}{N}, x) \in \mathcal{H}_0 \). Denote \( h(x) = H(t, x) \) for \( x \in M \). We have

\[
\int_0^1 \|H_N(s, \cdot) - h(\cdot)\|_\infty ds = N \int_t^{t+\frac{\phi}{N}} \|H(\tau, \cdot) - H(t, \cdot)\|_\infty d\tau \to N \to \infty 0,
\]

where we made the substitution \( \tau = t + \frac{\phi}{N} \). Therefore, because of Lemma (2), we have \( h \in \mathcal{H}_0 \), and being time-independent, \( h \in \mathcal{H}^t_0 \).

\[ \square \]

**Proof of Lemma 10.** Let \( H \in \mathcal{H}^t_0 \), and assume by contradiction that \( H \) is a non-zero function. Let us show, that then there exists a non-zero function \( h(x) \in \mathcal{H}^t_0 \cap C^\infty(M) \). First, there exists a point in \( M \) such that \( H \) is not constant in any neighborhood of it (otherwise \( H \) is
Take such a point $x_0$, and consider an open neighborhood $x_0 \in U$, such that $U \subset M$ is moreover a Darboux chart. Take $y_0 \in U$, such that $H(x_0) \neq H(y_0)$. There exists $\Phi \in \text{Ham}_U(M, \omega)$, such that $\Phi(x_0) = y_0$. Define $K = H \circ \Phi - H$. Then $K \in H^s_0$, because of Lemma 4, (4), (5). Moreover $K$ is a non-zero function, and $\text{supp}(K) \subset U$. Consider the $C^0$- closure $\mathcal{L}$ of the linear span of all functions of the form $\Phi^*K$, where $\Phi \in \text{Ham}_U(M, \omega)$. In view of Lemma 1 (2), (4), (5), we have $\mathcal{L} \subset H^s_0$. Let us show, that $\mathcal{L}$ contains a non-constant smooth function. Since $U$ is a Darboux neighborhood, and the latter statement has a local nature, we can further assume, that $U \subset (\mathbb{R}^{2n}, \omega_{std})$, and moreover we have $K : U \to \mathbb{R}$ with $K \neq 0$, and moreover $K = 0$ near $\partial U$. Extend $K$ as a function $K : \mathbb{R}^{2n} \to \mathbb{R}$ by 0 outside $U$. In this new situation, where we replaced the manifold $M$ by $\mathbb{R}^{2n}$, we keep the notation $\mathcal{L}$ for the $C^0$- closure of the linear span of all functions of the form $\Phi^*K$, where $\Phi \in \text{Ham}_U(\mathbb{R}^{2n}, \omega_{std})$. For $v \in \mathbb{R}^{2n}$, we denote $K_v(x) = K(x - v)$. Let us show, that when the norm $\|v\|$ is small enough, we have $K_v \in \mathcal{L}$. Take a neighborhood $W$ of $\text{supp}(K)$, such that $\overline{W} \subset U$. Pick a function $\phi : \mathbb{R}^{2n} \to \mathbb{R}$, such that $\text{supp}(\phi) \subset U$ and moreover $\phi \equiv 1$ on $W$. For any $v \in \mathbb{R}^{2n}$ define a Hamiltonian $G_v : \mathbb{R}^{2n} \to \mathbb{R}$ as $G_v(x) = \omega_{std}(v, x)\phi(x)$ for $x \in \mathbb{R}^{2n}$. Then, for small $\|v\|$, the time one map of its Hamiltonian flow coincides on $\text{supp}(K)$ with the translation $x \mapsto x + v$. Therefore we will have $K_v = (\Phi_{G_v}^{-1})^*K$, and hence $K_v \in \mathcal{L}$. Here we denote by $\Phi_{G_v}$ the Hamiltonian flow of $G_v$, for $t \in \mathbb{R}$.

Therefore we have shown, that $K_v \in \mathcal{L}$ for small $\|v\|$. As a conclusion, we have that for a smooth function $\chi$ with support lying in a sufficiently small neighborhood of 0, we have that the convolution $K \ast \chi$ lies in $\mathcal{L}$ as well. We see this from the fact, that $K \ast \chi$ is a $C^0$ limit of a sequence of finite sums $\sum_{k=1}^{m} c_k K_{\chi_k}$, coming from the approximation of the Riemann integral by Riemann sums. But of course, the function $K \ast \chi$ is smooth, provided that the function $\chi$ is smooth. Moreover, $K$ is a non-zero function on $U$. Choose a sequence $\chi_k$ of smooth mollifiers approximating the $\delta_0$ - function, having sufficiently small supports. Then we have $K \ast \chi_k \to K$ in the $C^0$ topology, and hence the function $K \ast \chi_k$ is a non-zero function too when $k$ is large. This shows, that $\mathcal{L}$ contains a non-zero smooth function. Therefore we conclude, that the space $H^s_0 \cap C^\infty(M)$ contains a non-zero smooth function, what contradicts Lemma 6.

□
**Proof of Theorem 4.** Assume that $H \in L^{(1, \infty)}([0, 1] \times M)$, such that $(\text{Id}^t, H) \in \text{PHameo}(M, \omega)$. Then $H \in \mathcal{H}_0$, and Lemma 9 implies that for almost any $t \in [0, 1]$, the time-independent Hamiltonian $h(x) = H(x, t)$ lies inside $\mathcal{H}^n_0$. Then, for such values of $t$, the function $H(\cdot, t)$ is zero, by Lemma 10. Therefore $H = 0$ in $L^{(1, \infty)}([0, 1] \times M)$.

**4. Local uniqueness**

In this section we present a generalization of Theorem 1.3 from [Oh-2], to the $L^{(1, \infty)}$ case. As an application we give an example of a continuous function which fails to be a generator of any continuous Hamiltonian flow. As another application, we give an example of a continuous flow, which is a $C^0$ limit of smooth Hamiltonian flows, but is not a continuous Hamiltonian flow.

**4.1. Local uniqueness for $L^{(1, \infty)}$ Hamiltonians.** The uniqueness result from Theorem 4 admits a generalization, which is a local analogue of it. The following result holds.

**Theorem 11.** Suppose that we have $H \in L^{(1, \infty)}([0, 1] \times M)$ and a continuous flow $\Phi^t$ on $M$, such that $(\Phi^t, H) \in \text{PHameo}(M, \omega)$. Assume in addition, that the flow $\Phi^t$ equals to the identity flow on some open subset $U \subset M$, i.e. for any $x \in U$ and $t \in [0, 1]$ we have $\Phi^t(x) = x$. Then for almost all $t \in [0, 1]$, the restriction $H(t, \cdot)|_U$ is a constant function.

**Proof of Theorem 11.** Let $\Psi \in \text{Ham}_U(M, \omega)$. Then we have $\Psi^{-1} \circ \Phi^t \circ \Psi = \Phi^t$ for any $t \in [0, 1]$. On the other hand, the Hamiltonian function of the flow $\Psi^{-1} \circ \Phi^t \circ \Psi$ equals to $\Psi^* H$, while the Hamiltonian function of the flow $\Phi^t$ equals $H$. We can apply the uniqueness result for the Hamiltonian function, corresponding to a continuous Hamiltonian flow, which follows from Theorem 4. We conclude that $H(t, \Psi(x)) = H(t, x)$ in $L^{(1, \infty)}([0, 1] \times M)$, for any $\Psi \in \text{Ham}_U(M, \omega)$. Let us derive the result of the theorem from this. Choose a dense countable subset of $U$, $X = \{x_0, x_1, x_2, \ldots \} \subset U$. For every $i \in \mathbb{N}$ pick some $\Psi_i \in \text{Ham}_U(M, \omega)$ satisfying $\Psi_i(x_0) = x_i$. Then for each $i \in \mathbb{N}$ there exists a zero-measurable set $S_i \subset [0, 1]$, such that $H(t, \Psi_i(x)) = H(t, x)$ for any $t \notin S_i$ and $x \in M$. In particular $H(t, x_i) = H(t, x_0)$ for any $t \notin S_i$. Denote $S = \bigcup_{i=1}^{\infty} S_i$. Then $S \subset [0, 1]$ is of measure 0, and moreover we have $H(t, x_i) = H(t, x_0)$ for any $t \notin S$. Fix arbitrary $t \in S$. The function $H(t, \cdot)$ is continuous on $M$, and we have $H(t, x) = H(t, x_0)$ for any $x \in X$, while $X \subset U$ is a dense subset. We conclude that $H(t, \cdot)|_U = \text{const}$ for any $t \in S$. □
4.2. Local uniqueness for continuous Hamiltonian flows.

Theorem 12.

(1) Let \( H \in L^{(1, \infty)}([0, 1] \times M) \) be a Hamiltonian function, that generates a continuous Hamiltonian flow \( \Phi^t_H \). Assume that for almost all \( t \in [0, 1] \), the restriction \( H(t, \cdot)|_U \) is a constant function, say \( c(t) \). Then \( \Phi^t_H(x) = x \) for any \( x \in U \), \( t \in [0, 1] \).

(2) Let \( H, K \in L^{(1, \infty)}([0, 1] \times M) \) be two Hamiltonian functions, that generate continuous Hamiltonian flows \( \Phi^t_H, \Phi^t_K \). Assume that for any \( t \in [0, 1] \) we have \( H(t, x) = K(t, x) \forall x \in \Phi^t_H(U) \). Then we have \( \Phi^t_H(x) = \Phi^t_K(x) \) for any \( x \in U \), \( t \in [0, 1] \).

The proof of Theorem 12 (1) is similar to the proof of Theorem 3.1 from [Oh-2].

Proof of Theorem 12

(1) We know that there exists a sequence of smooth Hamiltonians \( H_i, L^{(1, \infty)} \) converging to \( H \) whose flows \( \Phi^t_{H_i} \in C^0 \) converge to \( \Phi^t_H \). For a given point \( x \in U \), pick a neighborhood of it \( V \) which is compactly contained in \( U \), and take a smooth cut off function \( \beta \) such that support of \( \beta \) is contained in \( U \) and \( \beta = 1 \) on \( V \). For any \( t \in [0, 1] \), denote

\[
d(t) = \frac{\int_M \beta(x)(H_i(t, x) - c(t))\omega^n}{\int_M \omega^n},
\]

and then define new smooth normalized Hamiltonians

\[
G_i(t, x) = \beta(x)(H_i(t, x) - c(t)) - d(t).
\]

Then \( G_i(t, x) = H_i(t, x) - c(t) - d(t) \) on \( V \) and \( G_i = 0 \) outside \( U \). It is easy to see that \( (L^{(1, \infty)}) \lim G_i = 0 \). Assume for a contradiction, that for some \( t \in [0, 1] \) we have \( \Phi^t_{H_i}(x) \neq x \). Then we can find some \( 0 < T \leq 1 \) such that \( \Phi^T_{H_i}(x) \neq x \) and moreover \( \Phi^t_{H_i}(x) \in V \) for all \( t \in [0, T] \). Therefore, since \( (C^0) \lim \Phi^t_{H_i} = \Phi^t_H \), there exists a small enough open neighborhood \( W \) of \( x \), \( x \in W \subset V \), such that \( \Phi^T_{H_i}(W) \cap W = \emptyset \) and moreover \( \Phi^t_{H_i}(W) \subset V \) for all \( t \in [0, T] \), for sufficiently large \( i \). Because \( G_i(t, x) = H_i(t, x) - c(t) - d(t) \) on all of \( V \), we have \( \Phi^T_{G_i}(W) \cap W = \emptyset \) as well, for \( i \) large enough. Then the energy-capacity inequality implies that \( \|G_i\|_{(1, \infty)} \) is bounded below by the displacement energy of \( W \), which is known to be positive. But we know that \( G_i \in L^{(1, \infty)} \)-converges to \( 0 \), and this is a contradiction.

(2) Consider the flow \( (\Phi^t_H)^{-1} \circ \Phi^t_K \). This flow is generated by the Hamiltonian \( \overline{H} = \#K(t, x) = -H(t, \Phi^t_H(x)) + K(t, \Phi^t_H(x)) \). By our assumption, this Hamiltonian is zero on \( U \). Therefore, by (1) we obtain
\[ (\Phi^t_H)^{-1} \circ \Phi^t_K(x) = x, \] and hence, \[ \Phi^t_H(x) = \Phi^t_K(x) \] for any \( x \in U, t \in [0,1] \).

4.3. Example of a non-generator. We will now construct an example of a continuous function which does not generate a continuous Hamiltonian flow. Let \((M,\omega)\) be a closed symplectic manifold. Consider some Darboux chart \(W \subset M\), endowed with symplectic coordinates \((x_1, y_1, \ldots, x_n, y_n)\), and assume for simplicity that

\[ 0 = (0,0,\ldots,0,0) \in W. \]

Take any continuous function \(K : M \to \mathbb{R}\), such that for every point

\[ (x_1, y_1, \ldots, x_n, y_n) \in W, \]

sufficiently close to \(0 \in W\), we have \(K(x_1, y_1, \ldots, x_n, y_n) = |x_1|\). Let us show that such a function does not generate a continuous Hamiltonian flow. Assume for a contradiction, that \(K\) generates a continuous Hamiltonian flow \(\Phi^t_K\) on \(M\). There exists \(\epsilon > 0\), such that we have

\[ (x_1, y_1, \ldots, x_n, y_n) \in W \]

and

\[ K(x_1, y_1, \ldots, x_n, y_n) = |x_1|, \]

provided \(|x_i|, |y_i| \leq \epsilon, i = 1,2,\ldots,n\). Consider any smooth function \(\phi : M \to \mathbb{R}\) supported in \(W\), such that \(\phi(x) = 1\) for \(x = (x_1, y_1, \ldots, x_n, y_n) \in W\) with \(|x_i|, |y_i| \leq \epsilon, i = 1,2,\ldots,n\).

Define \(H_1 : M \to \mathbb{R}\) as \(H_1(x) = x_1 \phi(x)\), for \(x = (x_1, y_1, \ldots, x_n, y_n) \in W\), and as \(H_1(x) = 0\) for \(x \in M \setminus W\). Define \(U_1 \subset W\) to be the set of all \((x_1, y_1, \ldots, x_n, y_n) \in W\), such that \(0 < x_1 < \epsilon, |y_1| < \frac{\epsilon}{2}\), and \(|x_i|, |y_i| < \epsilon\) for \(i = 2,3,\ldots,n\). Apply Theorem 12 (2), to \(H_1, K, U_1\) in the time interval \([0,\frac{\epsilon}{2}]\) (of course, the time interval \([0,1]\) in Theorem 12 can be replaced by any other time interval). We conclude that

\[ \Phi^t_K(x_1, y_1, \ldots, x_n, y_n) = \Phi^t_{H_1}(x_1, y_1, \ldots, x_n, y_n) = (x_1, y_1 - t, \ldots, x_n, y_n) \]

for any \(0 \leq t \leq \frac{\epsilon}{2}\), for any \((x_1, y_1, \ldots, x_n, y_n) \in W\), provided \(0 < x_1 < \epsilon, |y_1| < \frac{\epsilon}{2}, |x_i|, |y_i| < \epsilon\) for \(i = 2,3,\ldots,n\).

Now define \(H_2 : M \to \mathbb{R}\) as \(H_2(x) = -H_1(x)\), and let \(U_2 \subset W\) be the set of all \((x_1, y_1, \ldots, x_n, y_n) \in W\), such that \(-\epsilon < x_1 < 0, |y_1| < \frac{\epsilon}{2}, |y_1| < \epsilon\) for \(i = 2,3,\ldots,n\). Applying Theorem 12 (2), to \(H_2, K, U_2\) in the time interval \([0,\frac{\epsilon}{2}]\), in a similar way we obtain

\[ \Phi^t_K(x_1, y_1, \ldots, x_n, y_n) = \Phi^t_{H_2}(x_1, y_1, \ldots, x_n, y_n) = (x_1, y_1 + t, \ldots, x_n, y_n) \]

for any \(0 \leq t \leq \frac{\epsilon}{2}\), for any \((x_1, y_1, \ldots, x_n, y_n) \in W\), provided \(-\epsilon < x_1 < 0, |y_1| < \frac{\epsilon}{2}, |x_i|, |y_i| < \epsilon\) for \(i = 2,3,\ldots,n\).
Clearly, such flow $\Phi^K_t$ is not continuous, and we come to a contradiction.

4.4. Example of a non-flow. In this section we construct a continuous, area-preserving flow, i.e. a path in the group $Homeo(M, \Omega)$, which is a $C^0$ limit of smooth Hamiltonian flows but not a continuous Hamiltonian flow. This flow fails to be a continuous Hamiltonian flow, because there exist no $H \in L^{(1, \infty)}(M)$ generating the flow.

The following example is similar to the one considered by Oh and Muller [OM]. Let $(S^2, \omega)$ be a 2-dimensional closed surface with area form $\omega$. Consider an embedding of a small disc $i : (D^2(a), \omega_{std}) \hookrightarrow (S^2, \omega)$, denote $V = i(D^2(a))$. Take polar coordinates $(r, \theta)$ on $V$, that come from $D^2(a)$, and we have $\omega = rdr \wedge d\theta$ on $V$. For a smooth function $h : (0, a) \to \mathbb{R}$, which is zero near $a$, define a Hamiltonian $H : S \setminus \{i(0)\} \to S \setminus \{i(0)\}$ as $H(r, \theta) = h(r)$ for $(r, \theta) \in V \setminus \{i(0)\}$, and $H(x) = 0$ for $x \in S \setminus V$.

Then $H$ has a well defined smooth Hamiltonian flow $\Psi^t : S \setminus \{i(0)\} \to S \setminus \{i(0)\}$, and we extend $\Psi^t$ to a continuous flow on $S$, by setting $\Psi^t(i(0)) = i(0)$. Note that we have $\Psi^t(r, \theta) = (r, \theta + th'(r))$ for $(r, \theta) \in V \setminus \{i(0)\}$. Moreover, in the case when $h(r) = 0$ for small $r$, the flow $\Psi^t$ is Hamiltonian, where the (un-normalized) Hamiltonian function equals $H$ on $S \setminus \{i(0)\}$, and equals 0 at $i(0)$. We say that $\Psi^t$ is the rotation associated to $h$.

Now, consider a smooth function $f : (0, a) \to \mathbb{R}$, such that $f(r) = \frac{2\pi}{r}$ for $r \in (0, \frac{a}{3})$, and also $f(r) = 0$ for $r \in (\frac{2a}{3}, a)$. Let $\Psi^t : S \to S$ be the rotation, associated to $f$. Then the flow $\Psi^t$ is a $C^0$-limit of smooth Hamiltonian flows. Indeed, take a sequence of smooth functions $f_n : (0, a) \to \mathbb{R}$, such that $f_n(r) = 0$ for $r \in (0, \frac{1}{n})$, $f_n(r) = f(r)$ for $r \in (\frac{2}{n}, a)$, and for each $n$ define $\Psi^t_n$ to be the rotation associated to $f_n$. Then $\Psi^t_n$ is the needed sequence of smooth Hamiltonian flows.

Assume by contradiction, that the continuous flow $\Psi^t$ is in fact a continuous Hamiltonian flow. Then denote by $H(t, x)$ its Hamiltonian function. Take $f_n$, $\Psi^t_n$ as above, and denote by $H_n(x)$ the Hamiltonian function of $\Psi^t_n$. We obtain, that the flow $(\Psi^t_n)^{-1} \circ \Psi^t$ is generated by $K_n(t, x) = -H_n(\Psi^t_n(x)) + H(t, \Psi^t_n(x))$. Moreover, we have $(\Psi^t_n)^{-1} \circ \Psi^t = Id^t$ on $V_n := \{(r, \theta) \in V \mid r > \frac{2}{n}\}$. Then, from Theorem 11 we have $K_n(t, x) = -H_n(\Psi^t_n(x)) + H(t, \Psi^t_n(x)) = c_n(t)$ for almost all $t$, for $x \in V_n$. Since $\Psi^t_n(V_n) = V_n$, we get $H(t, x) = H_n(x) + c_n(t)$ for almost all $t$, for $x \in V_n$. This immediately implies that for almost any fixed $t$, the function $H(t, \cdot)$ is unbounded, and we come to contradiction.
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