Intrinsic Dynamics of Symplectic Manifolds: Membrane Representation and Phase Product

Mikhail Karasev*

Moscow Institute of Electronics and Mathematics
Moscow 109028, Russia
karasev@miem.edu.ru

Abstract

On general symplectic manifolds we describe a correspondence between symplectic transformations and their phase functions. On the quantum level, this is a correspondence between unitary operators and phase functions of the WKB-approximation. We represent generic functions via symplectic area of membranes and consider related geometric properties of the noncommutative phase product. An interpretation of the phase product in terms of symplectic groupoids and the groupoid extension of Lagrangian submanifolds are described. The membrane representations of corresponding Lagrangian phase functions are obtained. This paper uses the intrinsic dynamic approach based on the notion of Ether Hamiltonian which is a generalization of the notion of symplectic connection. We demonstrate that this approach works for torsion case as well.

1 Introduction

The intrinsic dynamics of symplectic manifolds is generated by symplectic connections, but this dynamics is not equivalent to the kinematics of parallel

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translations of vectors or tensors by means of the connection. The intrinsic dynamics is based on a certain object (called the Ether Hamiltonian [1]) which can be considered as a connection in the function bundle over the manifold. This object allows one to translate functions, rather than vectors, and thus to exploit certain hidden geometry which we never see in the habitual analysis produced by the infinitesimal geometrical tools.

In the given paper, we apply these ideas to investigate the following question: how to associate functions on a phase space with symplectic transformations of this space and what is the product of functions corresponding to the composition of transformations? On the Euclidean phase spaces with the canonical structure \( dp \wedge dq \) this is a routine question about phase functions (or generating functions) of symplectic transformations [2, 3]. The choice of a phase function depends on the choice of polarization on the double phase space.

In the general case, the given Ether Hamiltonian on the symplectic manifold determines a natural choice of polarization, and thus one obtains a correspondence between functions and symplectic transformations.

Moreover, each function close enough to constant can be represented as the symplectic area of a membrane whose boundary is organized by means of Ether geodesics and the related symplectic transformation. Composition of transformations makes up the composition of membranes. Via the Stokes theorem, the area of this composition represents a noncommutative product of functions.

The corresponding composition of Lagrangian submanifolds is generated by the groupoid multiplication operation. This operation can also be used for transformations and extensions of Lagrangian submanifolds, and for various types of membrane representations of generating functions related to these submanifolds.

In the last section we consider a generalization of the results to the torsion case.

## 2 Ether Hamiltonian

Let \( \mathfrak{X} \) be a manifold with symplectic form \( \omega \) and symplectic torsion free connection \( \Gamma \) (that is, \( \omega \) is covariantly constant in the sense of \( \Gamma \)).

The connection generates the kinematic geometry on \( \mathfrak{X} \): parallel translations of vectors, geodesics, exponential mappings, etc. There are no forces
in this geometry, no changing of momenta, and no opportunities to translate functions or curves and surfaces. That is why one needs to integrate the pair \((\omega, \Gamma)\) up to a more substantial object which we call the Ether Hamiltonian. It generates a dynamic geometry on \(\mathfrak{X}\). We recall some notions from [1].

The Ether Hamiltonian is a 1-form of \(\mathfrak{X}\) with values in \(C^\infty(\mathfrak{X})\) satisfying the zero curvature equation and some boundary and skew-symmetry conditions. We use the following notation for this Hamiltonian:

\[
H_x(z) = \sum_{j=1}^{2n} H_x(z) dx^j, \quad 2n = \dim \mathfrak{X}, \ x, z \in \mathfrak{X}.
\]

The zero curvature equation is

\[
(2.1) \quad \partial H + \frac{1}{2} \{H \wedge H\} = 0.
\]

Here the Poisson brackets \(\{\cdot, \cdot\}\) are taken with respect to the symplectic form \(\omega = \omega(z)\), and \(\partial = \partial_z\) denotes the differential of a form at the point \(x\).

The boundary conditions are

\[
(2.2) \quad H_x(z) \bigg|_{x=z} = 0, \quad DH_x(z) \bigg|_{x=z} = 2\omega(z), \quad D^2H_x(z) \bigg|_{x=z} = 2\omega(z)\Gamma(z).
\]

Here \(D = D_z\) is the derivative with respect to \(z\). Note that everywhere we use identical notations both for a differential 2-form itself and for the matrix of coefficients of this form in local coordinates. In (2.2), of course, \(\omega(z)\) stays for the matrix of coefficients of the form \(\omega\) at the point \(z\), and \(\Gamma(z)\) is the matrix of Christoffel symbols of the connection \(\Gamma\) at the point \(z\).

Let \(z \rightarrow s_x(z)\) be a trajectory of the Ether Hamiltonian (that is, the “time” derivative \(\partial s_x(z)\) coincides with the Hamiltonian vector field of \(H_x\) at the point \(s_x(z)\)), and the “initial” data are \(s_x(z) \bigg|_{x=z} = z\), see (3.4) below.

The skew-symmetry condition is

\[
(2.3) \quad H_x(s_x(z)) = -H_x(z).
\]

For each \(x \in \mathfrak{X}\) and \(v \in T_x\mathfrak{X}\), by \(\text{Exp}_x(vt)\) we denote the Hamilton trajectory on \(\mathfrak{X}\) which corresponds to the Hamilton function \(\frac{1}{2}vH_x\) and starts at \(x\) when \(t = 0\).

**Theorem 2.1.** (i) In a neighborhood of the diagonal \(z = x\), there exists a solution of Eq. (2.1) satisfying conditions (2.2), (2.3).
(ii) Mappings $s_x$ are symplectic transformations of $\mathcal{X}$. The point $x$ is an isolated fixed point of $s_x$:

$$s_x(x) = x.$$  

The mapping $s_x$ is an involution:

$$s_x^2 = \text{id}.$$  

Moreover, the family $\{s_x\}$ is related to the connection $\Gamma$ by the formula

$$\Gamma(z) = -\frac{1}{2}D^2s_x(z) \bigg|_{x=z}.$$  

(iii) If $\{s_x(z) \mid x \in \mathcal{X}\}$ is a smooth family of symplectic transformations of $\mathcal{X}$ satisfying (2.4), (2.5), then formula (2.6) determines a symplectic connection on $\mathcal{X}$, and formula

$$\mathcal{H}_x(z) = \int_z^x \langle \partial s_x(s_x(z)), \omega(z) \rangle \, dz$$  

determines the Ether Hamiltonian (the solution of (2.1)–(2.3)).

(iv) The mappings $\text{Exp}_x$ are related to $\mathcal{H}_x$ and to the family $\{s_x\}$ as follows:

$$\mathcal{H}_x(\text{Exp}_x(v)) = -\mathcal{H}_x(\text{Exp}_x(-v)), \quad s_x(\text{Exp}_x(v)) = \text{Exp}_x(-v).$$  

We call the mapping $s_x$ a reflection, and we call $\text{Exp}_x$ an Ether exponential mapping. The curve $\{\text{Exp}_x(vt) \mid -\varepsilon < t < \varepsilon\}$ is called the Ether geodesics through the mid-point $x$.

The composition of two reflections $g_{x,y} = s_x \circ s_y$ we call the Ether translation. This map coincides with the shift along trajectories of the Ether dynamical system with Hamiltonian $\mathcal{H}_x$ when the “time” varies from $y$ to $x$ (see in [II]).

In general, the reflections $s_x$ do not preserve the connection $\Gamma$, the mappings $\text{Exp}_x$ do not coincide with the exponential mappings $\exp_x$ generated by $\Gamma$, and so the Ether geodesics do not coincide with the $\Gamma$-geodesics.
3 Phase functions

In the double $\mathfrak{X} \times \mathfrak{X}$ with the symplectic structure $\omega \ominus \omega$ we have the Lagrangian fibration $\mathcal{S} = \{ \mathcal{S}_x \mid x \in \mathfrak{X} \}$ whose fibers are graphs of reflections: $\mathcal{S}_x = \text{Graph}(s_x)$.

If $\gamma$ is a transformation of $\mathfrak{X}$, then its graph $\text{Graph}(\gamma)$ intersects the fiber $\mathcal{S}_x$ at a point $(\gamma(\tilde{x}), \tilde{x})$, where $\tilde{x}$ is a fixed point of the mapping $s_x \circ \gamma$. If $\gamma$ is close enough to the identity mapping, then the fixed point $\tilde{x}$ is close to $x$ and unique. We will also denote $\tilde{x}$ by $\tilde{x}^\gamma$ to indicate what mapping $\gamma$ generates this fixed point.

The correspondence $x \to \tilde{x}$ is a local diffeomorphism. The inverse mapping $\tilde{x}^\gamma : \tilde{x} \to x$ we call the mid-transformation related to $\gamma$.

**Theorem 3.1.** (i) The transformation $\gamma$ is reconstructed via reflections and the mid-transformation $\tilde{\gamma}$ by the formula

$$\gamma(z) = s_{\tilde{\gamma}(z)}(z).$$

(ii) Let $\gamma$ be symplectic, and let $\tilde{x} = \tilde{x}^\gamma$ be the fixed point of $s_x \circ \gamma$. Then the 1-form

$$-\mathcal{H}_x(\tilde{x}) \equiv \mathcal{H}_x(\gamma(\tilde{x}))$$

is closed. In the simply connected case there is a function $\Phi^\gamma$ such that

$$d\Phi^\gamma(x) + \mathcal{H}_x(\tilde{x}) = 0.$$  

(iii) Stationary points of the function $\Phi^\gamma$ are fixed points of the symplectic transformation $\gamma$. The differential of $\gamma$ and the matrix of the second derivatives of $\Phi^\gamma$ at the fixed (stationary) point are related to each other by the formula

$$d\gamma = \frac{I + \frac{1}{2} \Psi \cdot D^2 \Phi^\gamma}{I - \frac{1}{2} \Psi \cdot D^2 \Phi^\gamma}, \quad \text{or} \quad D^2 \Phi^\gamma = 2\omega \cdot \frac{d\gamma - I}{d\gamma + I}$$

[at the fixed point].

*Here $\Psi = \omega^{-1}$ is the Poisson tensor on $\mathfrak{X}$.*

**Proof.** Formula (3.1) is just a consequence of the definition of $\tilde{x}$ and $\tilde{\gamma}$. Assertion (ii) is another statement of the fact that $\gamma$ is symplectic (the submanifold $\text{Graph}(\gamma)$ is Lagrangian in $\mathfrak{X} \times \mathfrak{X}$). Assertion (iii) follows from (3.2) and the boundary conditions (2.2).  

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We call $\Phi^\gamma$ the *phase function* corresponding to the transformation $\gamma$. This name is due to the fact that $\Phi^\gamma$ is the phase of the WKB-approximation of the quantum operator related to $\gamma$.

Note that the phase function is uniquely defined if some additional Cauchy data are fixed, say, zero data at some point $y \in \mathfrak{X}$. We denote such a normalized phase function corresponding to $\gamma$ by $\Phi^\gamma_y$. Thus, by definition, the function $\Phi^\gamma_y$ obeys (3.2) and $\Phi^\gamma_y(y) = 0$.

Hereafter, we assume that the form $\omega$ is exact. We need this condition to be sure that integrals of the symplectic form along membranes, which we consider below, do not depend on the choice of membrane surface. Actually, in applications of this phase analysis in quantum theory all the integrals are staying in the exponent and we can replace the exactness of $\omega$ by the quantization condition on the cohomology classes.

**Theorem 3.2.** (i) The normalized phase function of the symplectic transformation $\gamma$ is given by the formula

\begin{equation}
\Phi^\gamma_y(x) = \int_{\Sigma^\gamma(x,y)} \omega.
\end{equation}

Here $\Sigma^\gamma(x,y)$ is a membrane in $\mathfrak{X}$ whose boundary is composed by four pieces: an arbitrary curve $c$ connecting $\tilde{x}$ with $\tilde{y}$, the Ether geodesic from $\tilde{y}$ to $\gamma(\tilde{y})$ through the mid-point $y$, the curve $\gamma(c)$ (with the opposite orientation) connecting $\gamma(\tilde{y})$ with $\gamma(\tilde{x})$, and the Ether geodesic from $\gamma(\tilde{x})$ to $\tilde{x}$ through the mid-point $x$.

(ii) The following cocyclic properties hold:

\[
\Phi^\gamma_x(y) + \Phi^\gamma_y(x) = 0, \quad \Phi^\gamma_y(x) + \Phi^\gamma_w(y) + \Phi^\gamma_x(w) = 0,
\]

\[
\Phi^\gamma_y^{-1}(x) + \Phi^\gamma_y(x) = 0, \quad \forall x, y, w \in \mathfrak{X}.
\]

**Proof.** Assertion (i) can be proved in the same way as Lemma 7.1 (ii) in [1]; see also the proof of Theorem 8.1 below. Assertion (ii) is a direct consequence of (3.3) and the Stokes theorem. \qed

In the conclusion of this section, let us make some remarks about the case of symmetric symplectic spaces [4].

First, note that in the construction of Theorem 3.1 the main role is played by the Lagrangian fibration generated by graphs of reflections. The Lagrangiancy is equivalent to symplecticity of reflections. In our approach the
symplecticity is an automatic consequence of the definition of reflections $s = s_x$ by means of the Hamilton type dynamic equation

$$\frac{\partial}{\partial x}s = D\mathcal{H}_x(s)\Psi(s), \quad s \bigg|_{x=z} = z. \tag{3.4}$$

The Ether Hamiltonian $\mathcal{H}$ is the object which generates this dynamics.

So, the symplecticity of $s_x$ is the priority. At the same time, we lose (in general) the usual relationship of reflections with the exponential mappings. Our reflections are not geodesic reflections:

$$s_x \neq s_0^x, \quad \text{where} \quad s_0^x(z) \overset{\text{def}}{=} \exp_x \left( - \exp_x^{-1}(z) \right).$$

Instead of this, we have

$$s_x(z) = \text{Exp}_x \left( - \text{Exp}_x^{-1}(z) \right),$$

where $\text{Exp}_x$ is the Ether exponential mapping. That is why we use in Theorem 3.2 and everywhere below the Ether geodesics, but not the usual $\Gamma$-geodesics.

In the classical theory of symmetric spaces, initiated by E. Cartan \cite{5}, the geodesics and the geodesic reflections $s_0^x$ play an exclusive role. In the symplectic situation \cite{4}, the geodesic reflections in order to be symplectic mappings must satisfy the Loos condition \cite{6} $s_0^x s_0^y s_0^x = s_0^y s_0^y s_0^x$ or the Cartan condition $\nabla R = 0$. One cannot avoid these conditions if only geodesic reflections are considered.

Under these conditions, i.e., in the framework of symmetric symplectic spaces, the correspondence between symplectic transformations and phase functions was studied in \cite{7}. The construction in \cite{7} is based on the use of exponential mappings $\exp_x$, which are the usual “kinematic” tools in this framework. But one can clearly see that, besides the local character, the presence of exponential mappings in all formulas actually implies loosing some important geometric structures like, for instance, the dynamic equation (3.4) or the zero curvature equation (2.1). The information carried by the Ether Hamiltonian allows one to avoid these kinematic difficulties and makes the dynamic view (3.4) to be the basic point. It seems that Eq. (3.4), even for $s = s^0$ in the symmetric case, was not used in the literature.
4 Dynamic phase functions

The most important examples of symplectic transformations are translations along trajectories of Hamiltonian systems. We denote such a translation by \( \gamma_H^t \), where \( H \) is the Hamilton function and \( t \) is the time variable. If \( t \) is close to zero, then the corresponding quantum flow in the WKB-approximation is described by a dynamic phase function \( \Phi_t \), namely,

\[
\exp \left\{ -\frac{it}{\hbar} \hat{H} \right\} = \hat{G}_t, \quad G_t = \exp \left\{ \frac{i}{\hbar} \Phi_t \right\} \varphi_t + O(\hbar)
\]

(in a semiclassically-simple domain); see details in [1].

The following formula for \( \Phi_t \) was derived in [1] via the membrane area:

\[
(4.1) \quad \Phi_t(x) = \int_{\Sigma_t(x)} \omega - tH(\tilde{x}).
\]

Here \( \Sigma_t(x) \) is a dynamic segment bounded by the Hamiltonian trajectory (whose time-length is \( t \)) and by the Ether geodesics connecting the ends of the trajectory and passing through the mid-point \( x \). The value of the Hamilton function \( H \) in (4.1) is taken on the trajectory-side of \( \Sigma_t(x) \).

Formula (4.1) is a generalization of the mid-point formulas found by Berry and Marinov in Euclidean phase spaces [8, 9], see also [7] for symmetric spaces. A certain modification using membranes with “wings” was suggested in [10] for magnetic phase spaces.

**Theorem 4.1.** The function (4.1) is the phase function corresponding to the Hamiltonian translation \( \gamma_H^t \). The normalized phase function of \( \gamma_H^t \) given by (3.3) is related to (4.1) via the identity:

\[
(4.2) \quad \Phi_{\gamma_H^t}(x) = \Phi_t(x) - \Phi_t(y).
\]

**Proof.** The first statement was proved in [1]. Formula (4.2) is a consequence of the Stokes theorem and the following version of the Poincare-Cartan “integral invariant” formula

\[
(4.3) \quad t(H(z) - H(w)) = \int_{\Sigma_{z,w}} \omega.
\]

Here the boundary of \( \Sigma_{z,w} \) consists of two pieces of the Hamiltonian trajectories passing through \( z \) and \( w \), of a path \( c \) connecting \( w \) with \( z \), and of the path \( \gamma_H^t(c) \). \qed
5 Membrane representation

The name “phase function” which we use for solutions of Eq. (3.2) is not the only natural candidate. At the same time one can use for $\Phi^\gamma$ the name: generating function of $\gamma$. This is due to the following statement.

**Theorem 5.1.** Let a real smooth function $\Phi$ be close enough to a constant, so that the equation

$$d\Phi(x) + \mathcal{H}_x(z) = 0,$$

has a smooth solution $x = \tilde{\gamma}(z)$ close enough to the identical $x = z$. Then the transformation $\gamma$ defined by (3.1) is symplectic, and $\tilde{\gamma}$ is its mid-transformation.

Thus the function $\Phi$ generates a symplectic transformation of $\mathcal{X}$.

**Corollary 5.2.** Any function $\Phi$ satisfying the conditions of Theorem 5.1 can be represented via symplectic area as follows:

$$\Phi(x) = \int_{\Sigma^\gamma(x,y)} \omega + \Phi(y), \quad \forall x \in \mathcal{X}.$$

Here $\gamma$ is the symplectic transformation generated by $\Phi$ as in Theorem 5.1, and $\Sigma^\gamma(x,y)$ is the membrane in $\mathcal{X}$ defined in Theorem 3.2, (i). The point $y \in \mathcal{X}$ is fixed.

We call formula (5.2) the membrane representation of the function $\Phi$.

6 Geometric phase product

The product of quantum operators corresponds to a noncommutative product of functions over $\mathcal{X}$. The integral kernel of the product operation in semiclassical approximation can be described by the phase function

$$\Phi_{y,z}(x) = \int_{\Delta(x,y,z)} \omega.$$

Here the membrane $\Delta(x, y, z)$ in $\mathcal{X}$ is composed by three Ether geodesics passing through mid-points $z, y, \text{ and } x$ close enough to each other (see [Diagram]).
On the level of phase functions the noncommutative product is given by

\[
(\Phi'' \circ \Phi')(x) \overset{\text{def}}{=} \left[ \Phi''(x'') + \Phi'(x') + \Phi_{x'',x'}(x) \right]_{x' = X'(x), x'' = X''(x)},
\]

where \(X'(x)\) and \(X''(x)\) are stationary points of the right-hand side of (6.2) with respect to \(x'\) and \(x''\).

In the case of Euclidean space, the phase product (6.2) was considered in many papers dealing with the Fourier integral operators and the Maslov canonical operator, for instance, in [3, 12, 13, 14], and in the case of symmetric symplectic manifolds, the detailed study was done in [7].

We call (6.2) a phase product over \(X\).

**Theorem 6.1.** (i) The phase product is associative and has the unity element \(\Phi = 0\).

(ii) The phase product (6.2) of phase functions of symplectic transformations (close enough to the identity) is the phase function of a composition of these transformations.

(iii) The family of functions (4.1) forms a one-parameter local group with respect to the phase product (6.2).

(iv) The triangle area (6.1) is the phase (or generating) function corresponding to the Ether translation \(g_{y,z} = s_y \circ s_z\). For this transformation, the membrane formula (3.3) is equivalent to (6.1):

\[
\Phi_{y,z} = \Phi_{y,z}.
\]

Now let us take two functions, consider the corresponding symplectic transformations, and make up the phase product of their phase functions.

**Theorem 6.2.** Let functions \(\Phi'\) and \(\Phi''\) satisfy the conditions of Theorem 5.1. Let \(\gamma'\) and \(\gamma''\) be the corresponding symplectic transformations, and let \(\gamma = \gamma'' \circ \gamma'\) be their composition. For any \(x \in X\) consider the fixed point \(\bar{x}\) of the mapping \(s_x \circ \gamma\). Let \(x'\) be the mid-point of the Ether geodesics between \(\bar{x}\) and \(\gamma'(\bar{x})\), and let \(x''\) be the mid-point of the Ether geodesics between \(\gamma'(\bar{x})\) and \(\gamma(\bar{x})\). Then the phase product of functions \(\Phi'\) and \(\Phi''\) is given by

\[
(\Phi'' \circ \Phi')(x) = \Phi''(x'') + \Phi'(x') + \int_{\Delta(x'',x',x)} \omega.
\]

The differential of this phase product is given by the Ether Hamiltonian:

\[
d(\Phi'' \circ \Phi')(x) = \mathcal{H}_x(\gamma(\bar{x})).
\]
Proof. By (3.2), we know that \( d\Phi''(x'') = -\mathcal{H}_{x''}(\tilde{x}'') \), where \( \tilde{x}'' \) is the fixed point of \( s_{x''} \circ \gamma'' \). From [1], formula (7.9), we have \( \partial_{x''}\Phi_{x'',x'} = \mathcal{H}_{x''}(a) \), where \( a \) is the vertex of \( \Delta(x'',x',x) \) common for sides with mid-points \( x' \) and \( x'' \). So, the stationary phase condition in (6.2) implies \( a = \tilde{x}'' \). In the same way one can prove that \( \tilde{x} = \tilde{x}' \) is the vertex of \( \Delta(x'',x',x) \) common for sides with mid-points \( x \) and \( x' \). Thus the triple \( (X'',X',x) \) in (6.2) coincides with the triple \( (x'',x',x) \) in Theorem 6.2, and then (6.3) follows from (6.2) and (6.1). The differential is given by \( d(\Phi'' \circ \Phi') = d\Phi x''_{x',x'}(x) + \int_{\Delta(x'',x',x)} \omega \). Since \( \gamma'(\tilde{x}) \) is the vertex of \( \Delta(x'',x',x) \) common for sides with mid-points \( x \) and \( x'' \).

Corollary 6.3. Let a function \( \Phi^0 \) be close enough to constant (as in Theorem 5.1), and \( \Phi^t \) be the dynamic phase function (4.1). Then the phase product \( \Phi(x,t) = (\Phi^t \circ \Phi^0)(x) \) is given by

\[
\Phi(x,t) = \Phi^0(x') + \Phi^t(x'') + \int_{\Delta(x'',x',x)} \omega.
\]

Here \( x' \) is defined by \( d\Phi^0(x') = \mathcal{H}_{x'}(a) \), where \( a \) is the vertex of the map \( s_{x''} \circ s_x \circ \gamma_H^\tau \), and \( x'' \) is the mid-point of the Ether geodesic from \( a \) to \( \gamma_H^\tau(a) \).

One can call formulas like (6.3) and (6.5) the geometric representation of the phase product.

In particular, consider the family of dynamic phase functions \( \Phi^t \) (4.1). On the membrane level we have

\[
\Sigma^{\tau+t}(x) = \Sigma^{\tau}(X^{\tau}) \cup \Sigma^{t}(X^{t}) \cup \Delta(X^{\tau},X^{t},x).
\]

Here \( X^{t} \) is the mid-point of the Ether geodesics between \( \tilde{x} \) and \( \gamma^{t}(\tilde{x}) \), \( X^{\tau} \) is the mid-point between \( \gamma^{t}(\tilde{x}) \) and \( \gamma^{\tau+t}(\tilde{x}) \), and we denote by \( \tilde{x} = \tilde{x}^{\tau+t} \) the fixed point of \( s_x \circ \gamma^{\tau+t} \).

Thus by formula (4.1) we have

\[
\Phi^{\tau+t}(x) = \int_{\Sigma^{\tau+t}(x)} \omega - (\tau + t)H(\tilde{x})
\]

see (6.6)

\[
(\int_{\Sigma^{\tau}(X^{\tau})} \omega - \tau H) + \left( \int_{\Sigma^{t}(X^{t})} \omega - t H \right) + \int_{\Delta(X^{\tau},X^{t},x)} \omega
\]

see (6.3)

\[
(\Phi^{\tau} \circ \Phi^{t})(x).
\]
Formula (6.6) and the last calculation geometrically represent the statement of Theorem 6.1, (iii): the group property of the family of phase functions $\Phi^t$ with respect to the phase product on general symplectic manifolds.

In Euclidean spaces this calculation was first demonstrated in [9].

Formula (6.5) can also be easily interpreted in this way. Actually, there is a natural extension of this representation for phase functions of general symplectic transformations.

Let $(x'', x', x)$ be a triple of points related to symplectic transformations $\gamma'$ and $\gamma''$ as in Theorem 6.2, and let $(y'', y', y)$ be another such triple. Then, on the membrane level, we have

$$
\Sigma^{\gamma'' \circ \gamma'}(x,y) \cup \Delta(y'', y', y) = \Sigma^{\gamma''}(x'', y'') \cup \Sigma^{\gamma'}(x', y') \cup \Delta(x'', x', x).
$$

By applying the Stokes theorem and formula (6.3), one obtains:

$$
\Phi^{\gamma''}(y'' \circ \gamma') = \Phi^{\gamma''} + \Phi_{y'', y'}(y).
$$

Here, on the left, we have the phase product of normalized generating functions of two symplectic transformations, and, on the right, we have a generating function of the composition of these transformations.

The normalized generating functions are given by the membrane area via (3.3). Thus formulas (6.7), (6.8) represent the statement of Theorem 6.1, (ii) geometrically via symplectic areas of membranes on general symplectic manifolds.

7 Groupoid interpretation of the phase product

The boundary condition (2.2) guarantees that the matrix $DH_x(z)$ is not degenerate at the diagonal $\{x = z\} \subset X \times X$. Denote by $X^\# \subset X \times X$ a connected reflective neighborhood of the diagonal where $\det D\mathcal{H}_x(z) \neq 0$. Also denote by $\mathcal{E}$ the corresponding neighborhood of the zero section in $T^*X$:

$$
\mathcal{E} = \{(x, p) \in T^*X \mid p = \mathcal{H}(z), (x, z) \in X^\#\}.
$$

Then there is a fibration of $\mathcal{E}$ over $X$,

$$
\ell : \mathcal{E} \rightarrow X, \quad \ell(x, p) \overset{\text{def}}{=} z \quad \text{if} \quad p = \mathcal{H}_x(z),
$$

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and a dual fibration

\[(7.1\ a)\quad r : \mathcal{E} \to \mathfrak{X}, \quad r(x, p) \overset{\text{def}}{=} \ell(x, -p).\]

In view of (2.3), we have

\[(7.2)\quad \ell(x, p) = s_x(r(x, p)).\]

The zero curvature equation (2.1) implies that $\ell$ is a Poissonian mapping and $r$ is an anti-Poissonian mapping commuting with $\ell$, that is,

\[(7.3)\quad \{\ell^j, \ell^k\} = \Psi^{jk}(\ell), \quad \{r^j, r^k\} = \Psi^{jk}(\ell), \quad \{\ell^j, r^k\} = 0,
\]

where the brackets $\{\cdot, \cdot\}$ correspond to the standard symplectic form $dp \wedge dx$ on $\mathcal{E}$. System (7.3) is known as the Lie–Engel system.

Thus we conclude that in the sense of [15, 16] the space $\mathcal{E}$ is a phase space over $\mathfrak{X}$ equipped with the Poisson bifibration (7.1), (7.1 a), and (7.3). Actually, such bifibrations (in the case of Poisson brackets of constant rank) were first considered by S. Lie; see references for the general Poisson case in [15] and for more details in [16]–[18].

The boundary conditions (2.2) imply

\[(7.4)\quad \ell(x, p) \Big|_{p=0} = x, \quad \frac{\partial \ell(x, p)}{\partial p} \Big|_{p=0} = \frac{1}{2} \Psi(x), \quad \frac{\partial^2 \ell(x, p)}{\partial p \partial p} \Big|_{p=0} = \frac{1}{4} \Psi(x) \Gamma(x) \Psi(x).
\]

Eqs. (7.2) and (7.4) relate the phase space structure on $\mathcal{E}$ with the reflective structure on $\mathfrak{X}$, in particular, with the symplectic connection $\Gamma$ on $\mathfrak{X}$ (see [1]).

Each function $H$ on $\mathfrak{X}$ is lifted up to the function $\ell^* H$ on $\mathcal{E} \subset T^* \mathfrak{X}$. The last one can be considered as a Hamilton function for the Hamilton–Jacobi equation over $\mathfrak{X}$ (see in [15] for the general Poisson case):

\[(7.5)\quad \frac{\partial \Phi}{\partial t} + H \left( \ell \left( x, \frac{\partial \Phi}{\partial x} \right) \right) = 0.
\]

**Theorem 7.1.** The dynamic phase function $\Phi^t$ (4.1) is the solution of the Hamilton–Jacobi equation (7.5) with zero Cauchy data. The phase product function $\Phi$ (6.5) is the solution of (7.5) with the Cauchy data $\Phi \big|_{t=0} = \Phi^0$.  

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Proof. The first statement was proved in [1]. From formula (6.3) we see that

\[ \frac{\partial}{\partial t} \Phi(x, t) = \frac{\partial \Phi^t}{\partial t}(x') = -H(\ell(x', d\Phi^t(x'))). \]

As in the proof of Theorem 6.2, we know that \( d\Phi^t(x') = \mathcal{H}_x(\gamma^t(a)) \) and thus, \( \ell(x', d\Phi^t(x')) = \gamma^t_H(a) = \gamma^t_H(\gamma^0(\tilde{x})) \), where \( \gamma^0 \) is the symplectic transformation generated by the function \( \Phi^0 \).

From (6.4) we have

\[ d_x \Phi(x, t) = d(\Phi^t \circ \Phi^0)(x) = \mathcal{H}_x(\gamma^t_H(\gamma^0(\tilde{x}))) \]

and thus \( \ell(x, d_x \Phi(x, t)) = \gamma^t_H(\gamma^0(\tilde{x})) \).

So we see that \( \ell(x', d\Phi^t(x')) = \ell(x, d_x \Phi(x, t)) \). Then Eq. (7.5) follows from (7.6).

Let us now consider the phase product (6.3) from the groupoid point of view.

The phase space \( \mathcal{E} \subset T^*\mathfrak{X} \) can be represented in a neighborhood of the diagonal in \( \mathfrak{X} \times \mathfrak{X} \) by means of the Poisson bifibration

\[ \ell \times r : \mathcal{E} \to \mathfrak{X} \times \mathfrak{X}^{(-)}, \quad (\ell \times r)(x, p) \overset{\text{def}}{=} (\ell(x, p), r(x, p)), \]

\[ (\ell \times r)^* dp \land dx = \omega(\ell) - \omega(r). \]

In the direct product \( \mathfrak{X} \times \mathfrak{X} \), there is a natural groupoid multiplication

\[ (x, y) \otimes (y, z) = (x, z). \]

Transporting this multiplication back to \( \mathcal{E} \) by means of \( (l \times r)^{-1} \), we obtain the groupoid structure

\[ (x, p) \otimes (y, \xi) \overset{\text{def}}{=} (z, \eta) \]

just by solving the system of equations

\[ \ell(z, \eta) = \ell(x, p), \quad r(z, \eta) = r(y, \xi), \]

\[ r(x, p) = \ell(y, \xi). \]

Here \( p \in T^*_x \mathfrak{X}, \xi \in T^*_y \mathfrak{X}, \eta \in T^*_z \mathfrak{X} \). In the symplectic case that we consider, the multiplication (7.8) coincides with that described in [17] for the case of general Poisson manifolds.
With respect to the multiplication (7.8), the mappings \( \ell \) and \( r \) are just the left and right (or the source and target) groupoid mappings:

\[
\ell(m) = m \odot m^{-1}, \quad r(m) = m^{-1} \odot m, \quad m \in E.
\]

There is a consistency condition between the groupoid multiplication (7.8) and the symplectic structure in \( E \); namely, the mapping \( \odot \) preserves the symplectic structure or the graph of \( \odot \) in a Lagrangian submanifold in \( E \times E \times E^{(-)} \), see details in [17, 18, 20].

For any two subsets \( \Lambda', \Lambda'' \subset E \), one can define their groupoid product:

\[
\Lambda'' \odot \Lambda' \overset{\text{def}}{=} \{ m'' \odot m' \mid m'' \in \Lambda'', m' \in \Lambda' \}.
\]

**Lemma 7.2.** (i) Let \( \Lambda' \) and \( \Lambda'' \) be submanifolds, the left mapping \( \ell \) (7.9) restricted to \( \Lambda' \) be a diffeomorphism, and the right mapping \( r \) (7.9) restricted to \( \Lambda'' \) be a diffeomorphism. Then the subset (7.10) is a submanifold.

(ii) If \( \Lambda'' \) and \( \Lambda' \) are Lagrangian, then \( \Lambda'' \odot \Lambda' \) is Lagrangian at every point where it is a submanifold.

Now, to each function \( \Phi \) on \( X \) one can assign a Lagrangian submanifold in \( T^*X \):

\[
\Lambda^{\Phi} = \{(x, p) \mid p = d\Phi(x)\}.
\]

If the function \( \Phi \) is close to constant, then the submanifold \( \Lambda^{\Phi} \) belongs to the groupoid \( E \subset T^*X \). We call \( \Phi \) the generating function of \( \Lambda^{\Phi} \).

Let us take two functions \( \Phi' \) and \( \Phi'' \) that satisfy conditions of Theorem 5.1 and consider their phase product \( \Phi' \circ \Phi'' \) as in Theorem 6.2. The following statement is just a reformulation of the construction described in Theorem 6.2.

**Theorem 7.3.** The phase product (6.2) of functions over \( X \) corresponds to the groupoid product (7.10) of Lagrangian submanifolds in \( E \subset T^*X \), that is,

\[
\Lambda^{\Phi''} \odot \Lambda^{\Phi'} = \Lambda^{\Phi'' \circ \Phi'},
\]

\[
\Lambda^{\Phi} \odot \Lambda^{-\Phi} \subset \Lambda^{0} \equiv X,
\]

\[
\Lambda^{\Phi} \odot \Lambda^{0} = \Lambda^{0} \odot \Lambda^{\Phi} = \Lambda^{\Phi}.
\]

Here the submanifold \( \Lambda^{0} = X \) (the zero section in \( T^*X \)) is assigned to the zero function \( \Phi = 0 \), and so the base manifold \( X \) plays the role of the unity element for the product (7.10).
8 Chord submanifolds

Actually, Theorem 7.3 is a consequence of the general observation [17] that on the quantum level the groupoid product \(\odot\) corresponds to a noncommutative algebra of operators (quantum observables). This algebra admits very tricky constructions. Thus we can expect that the groupoid product might be used to create some interesting extensions of objects of symplectic geometry.

The simplest construction of such a type is inspired by the known quantum Weyl–Wigner isomorphism between integral kernels and symbols of operators. In the symplectic geometry this isomorphism is represented by the left-right groupoid mapping \(\ell \times r : \mathcal{E} \to \mathfrak{X} \times \mathfrak{X}^{(-)}\).

To each Lagrangian submanifold \(M \subset \mathfrak{X} \times \mathfrak{X}^{(-)}\) (on which the symplectic form \(\omega \odot \omega\) is annulled), one can assign a Lagrangian submanifold in \(\mathcal{E}\):

\[
\Lambda_M \overset{\text{def}}{=} \{ m \in \mathcal{E} \mid (m \odot m^{-1}, m^{-1} \odot m) \in M \} = (\ell \times r)^{-1}(M).
\]

In particular, if \(M = \text{Graph}(\gamma)\), where \(\gamma\) is a symplectic transformation of \(\mathfrak{X}\), then

\[
\Lambda_{\text{Graph}(\gamma)} = \{ m \in \mathcal{E} \mid m \odot m^{-1} = \gamma(m^{-1} \odot m) \}.
\]

Assuming that \(\gamma\) is close enough to the identical mapping, we can represent this Lagrangian submanifold in the form (7.10), that is,

\[
\Lambda_{\text{Graph}(\gamma)} = \Lambda_{\Phi^\gamma},
\]

where \(\Phi^\gamma\) is the generating function corresponding to \(\gamma\) via Theorem 5.1.

Another important case is \(M = \lambda \times \lambda\), where \(\lambda\) is a Lagrangian submanifold in \(\mathfrak{X}\). In this case

\[
(8.1) \quad \Lambda_{\lambda \times \lambda} = \{ m \in \mathcal{E} \mid m \odot m^{-1} \in \lambda, m^{-1} \odot m \in \lambda \}.
\]

For each \(m = (x, p) \in \Lambda_{\lambda \times \lambda}\) we have two points \(b \overset{\text{def}}{=} r(x, p)\) and \(a \overset{\text{def}}{=} \ell(x, p)\) belonging to \(\lambda\). One can identify \(m\) with the Ether geodesic connecting \(b\) with \(a\) and passing through the mid-point \(x\). Such a geodesic can be called a chord of the submanifold \(\lambda\). So \(\Lambda_{\lambda \times \lambda}\) can be considered as a set of all chords. We call \(\Lambda_{\lambda \times \lambda}\) a chord submanifold.

Note that here we have in mind the oriented chords, but one can consider the nonoriented chords as well.
Theorem 8.1. Let \( \tilde{X}_\lambda \subset X \) be a connected domain such that for any \( x \in \tilde{X}_\lambda \) there is a unique nonoriented chord of \( \lambda \) passing through the mid-point \( x \).

Then over the domain \( \tilde{X}_\lambda \) the chord Lagrangian submanifold \( \Lambda_{\lambda \times \lambda} \) (8.1) can be represented in the form analogous to (7.10):

\[
\Lambda_{\lambda \times \lambda} = \{ (x, p) \mid p = \pm d\Phi_\lambda(x) \},
\]

where the function \( \Phi_\lambda \) is given by the integral

\[
(8.2) \quad \Phi_\lambda(x) = \int_{\Sigma_\lambda(x)} \omega.
\]

Here \( \Sigma_\lambda(x) \) is a membrane in \( \tilde{X} \) composed of the chord and the end-points of this chord. The differential of \( \Phi_\lambda \) is given by

\[
(8.3) \quad d\Phi_\lambda(x) = \mathcal{H}_x(a),
\]

where \( a \) is the end-point of the chord.

Proof. We need to prove that the end-points \( a, b \in \lambda \) of the Ether geodesics passing through the mid-point \( x \in \tilde{X}_\lambda \) are related to the function (8.2) by means of the equations

\[
(8.4) \quad a = \ell(x, d\Phi_\lambda(x)), \quad b = r(x, d\Phi_\lambda(x)).
\]

Let us fix a certain point \( x_0 \in \tilde{X}_\lambda \) close enough to \( x \) and denote by \( a_0, b_0 \in \lambda \) the end-points of the corresponding Ether geodesic through the mid-point \( x_0 \). The function \( \Phi_\lambda(x) \) (8.2) can be represented as

\[
(8.5) \quad \Phi_\lambda(x) = \Phi_\lambda(x_0) + \int_{\Sigma_\lambda(x, x_0)} \omega.
\]

Here \( \Sigma_\lambda(x, x_0) \) is a slice between two fixed Ether geodesics passing through the mid-points \( x \) and \( x_0 \). The part of the boundary of \( \Sigma_\lambda(x, x_0) \) belonging to \( \lambda \) consists of two paths \( A = \{ A(t) \mid t \in [0, 1] \} \) and \( B = \{ B(t) \mid t \in [0, 1] \} \), so that \( A(0) = a_0, A(1) = a \), and \( B(0) = b_0, B(1) = b \). The points \( A(t) \) and \( B(t) \) are just the left and right ends of the Ether geodesics fibrating the slice \( \Sigma_\lambda(x, x_0) \). Let \( X = \{ X(t) \mid t \in [0, 1] \} \) be the corresponding curve of mid-points of those Ether geodesics, so that \( X(0) = x_0, X(1) = x \).
Then the curve $X$ separates the slice $\Sigma_\lambda(x, x_0)$ in two segments (the left and the right):

\begin{equation}
\Sigma_\lambda(x, x_0) = \Sigma_{\lambda}^{\text{left}} \cup \Sigma_{\lambda}^{\text{right}}.
\end{equation}

The left segment $\Sigma_{\lambda}^{\text{left}}$ is bounded by the left ends $A(t)$ of the Ether geodesics and the right segment $\Sigma_{\lambda}^{\text{right}}$ is bounded by the right ends $B(t)$.

Note that both segments are images of one and the same “vertical” membrane $\sigma \subset T^*\tilde{X}_\lambda$, that is

\begin{equation}
\Sigma_{\lambda}^{\text{left}} = \ell(\sigma), \quad \Sigma_{\lambda}^{\text{right}} = r(\sigma^{(-)}),
\end{equation}

where the sign minus marks the inversion of the orientation. The boundary of the membrane $\sigma$ is composed of four pieces. The part of the boundary belonging to $\tilde{X}_\lambda$ is the curve $X$. The vertical part of the boundary of $\sigma$ consists of two paths in the vertical fibers $T_{x_0}^*\tilde{X}_\lambda$ and $T_x^*\tilde{X}_\lambda$. Each vertical path is projected by the mappings $\ell$ and $r$ onto the left and right parts of the Ether geodesics through $x_0$ and $x$. The last piece of the boundary of $\sigma$ is a curve $m = \{(X(t), P(t)) \mid t \in [0, 1]\} \subset \Lambda_{\lambda \times \lambda}$.

The points of this curve are mapped by the left and right mappings to the left and right pieces of the boundary of $\Sigma_\lambda(x, x_0)$, that is,

\begin{equation}
\ell(X(t), P(t)) = A(t), \quad r(X(t), P(t)) = B(t).
\end{equation}

In view of (8.6) and (8.7), we have

\begin{equation}
\int_{\Sigma_\lambda(x, x_0)} \omega = \int_{\Sigma_{\lambda}^{\text{left}}} \omega + \int_{\Sigma_{\lambda}^{\text{right}}} \omega = \int_{\ell(\sigma)} \omega - \int_{r(\sigma)} \omega = \int_{\sigma} (\ell \times r)^* (\omega \otimes \omega)
\end{equation}

\begin{equation}
= \int_{m} dp \wedge dx = \int_{m} pdx = \int_0^1 P(t) dX(t).
\end{equation}

Using formula (8.5), we conclude that

\begin{equation}
d\Phi_\lambda(X(t)) = P(t) = \mathcal{H}_{X(t)}(A(t)).
\end{equation}

Now taking into account (8.8) and setting $t = 1$, we obtain (8.4) and (8.3). \qed

Note that one may use a description of the Lagrangian submanifold $\lambda \subset \mathcal{X}$ as the joint energy level of Hamiltonians in involution (on $\lambda$), namely,

$$\lambda = \{x \in \mathcal{X} \mid H_1(x) = E_1, \ldots, H_n(x) = E_n\}.$$
where $E_j$ are constants. Then the chord function $\Phi_\lambda$ (8.2) can be considered as a solution of the system of Hamilton–Jacobi equations

\begin{equation}
H_j(\ell(x, d\Phi_\lambda(x))) = H_j(r(x, d\Phi_\lambda(x))) = E_j, \quad j = 1, 2, \ldots, n.
\end{equation}

Indeed, this fact follows from (8.4) if one notes that $H_j(a) = H_j(b) = E_j$ since $a, b \in \lambda$.

Of course, the function $-\Phi_\lambda$ also satisfies system (8.11).

Actually, the sequence of statements (8.5), (8.9), (8.10) can be revised to obtain the following result.

**Corollary 8.2.** In the domain $\tilde{X}_\lambda$ the chord function $\Phi_\lambda$ (8.2) is a unique (up to the sign) solution of the system of Hamilton–Jacobi equations (8.11) obeying the boundary condition $\Phi_\lambda|_{\lambda} = 0$.

In the Euclidean 2-dimensional case $X = \mathbb{R}^2$, the membrane formula (8.2) for the WKB-phase of the Wigner function was established by M. Berry in the framework of the semiclassical approximation theory. In this case, the Ether geodesics in the definition of the membrane $\Sigma_\lambda(x)$ in (8.2) is just a straight chord connecting a pair of points of the curve $\lambda \subset \mathbb{R}^2$.

**Remark 8.3.** Formula (8.2) is easily generalized to the case of fibred co-isotropic submanifolds. Namely, let $\lambda \subset X$ be coisotropic and its isotropic foliation be a fibration (see [21]). Denote by $\lambda^\#$ the Whitney sum of two copies of $\lambda$ with respect to this isotropic fibration. Then $\lambda^\#$ is a Lagrangian submanifold in $X \times X^\#$ and we can assign to it the Lagrangian submanifold

$$\Lambda_{\lambda^\#} \overset{\text{def}}{=} \{ m \in E \mid m \otimes m^{-1} \text{ and } m^{-1} \otimes m \text{ belong to one and the same isotropic fiber in } \lambda \}.$$ 

Then in a certain domain $\tilde{X}_\lambda$, one can represent this submanifold as

$$\Lambda_{\lambda^\#} \overset{\text{def}}{=} \{(x, p) \mid p = \pm d\Phi_\lambda(x)\}.$$ 

Formula (8.2) works in this fibred isotropic case as well, but in the construction of the membrane $\Sigma_\lambda(x)$ one has to consider only those paths on $\lambda$ whose ends belong to one and the same fiber.

We conclude this section with an application of the groupoid product construction (7.10) and the phase product construction (6.3) to the case of chord submanifolds.
Theorem 8.4. (i) Let $\lambda \subset \mathfrak{X}$ be a Lagrangian submanifold, and let $\gamma : \mathfrak{X} \to \mathfrak{X}$ be a symplectic mapping. Then

$$
\Lambda_{\text{Graph}(\gamma)} \odot \Lambda_{\lambda \times \lambda} = \Lambda_{\gamma(\lambda) \times \lambda}.
$$

Let $\tilde{\mathfrak{X}}_\gamma \subset \mathfrak{X}$ be a connected domain such that for any $x \in \tilde{\mathfrak{X}}_\gamma$ there is a unique Ether geodesic through the mid-point $x$ connecting $\lambda$ with $\gamma(\lambda)$. Let $y \in \tilde{\mathfrak{X}}_\gamma$ be a point such that $\tilde{y}^\gamma \in \lambda$ (see Sec. 3). Then the phase product of the generating function of $\gamma$ and the chord function (8.2) is given by

$$
(8.12) \quad (\Phi^\gamma_y \circ \Phi^\lambda)(x) = \int_{\Sigma^\lambda_{\gamma}(x,y)} \omega.
$$

Here the boundary of the membrane $\Sigma^\lambda_{\gamma}(x,y)$ consists of two Ether geodesics through the mid-points $x$ and $y$ connecting $\lambda$ with $\gamma(\lambda)$ and of two paths on $\lambda$ and on $\gamma(\lambda)$ connecting the end-points of those Ether geodesics.

(ii) Let $\Phi^t$ be the dynamic phase function (4.1) corresponding to a Hamilton flow $\gamma^t_H$. Then over the domain $\tilde{\mathfrak{X}}^t_{\gamma_H} \subset \mathfrak{X}$ the phase product of $\Phi^t$ with the chord function (8.2) is given by

$$
(8.13) \quad (\Phi^t \circ \Phi^\lambda)(x) = \int_{\Sigma^t_{\lambda}} \omega - Ht.
$$

Here the boundary of the membrane $\Sigma^t_{\lambda}(x,y)$ is composed by the Ether geodesic through the mid-point $x$ connecting $\lambda$ with $\gamma^t_H(\lambda)$, by a Hamiltonian trajectory (whose time length is $t$) coming from $\lambda$ to $\gamma^t_H(\lambda)$, and by a path on $\lambda$ connecting the origin of this trajectory with the end-points of those Ether geodesics. The function (8.13) is the solution of the Hamilton–Jacobi equation (7.5) with the initial data $\Phi^t|_{t=0} = \Phi^\lambda$.

9 Groupoid extension of Lagrangian submanifolds

The groupoid multiplication provides an extension of submanifolds. For any $\Lambda \subset \mathcal{E}$ we defined the extension $\Lambda^\# \subset \mathcal{E} \times \mathcal{E}$ as follows

$$
(9.1) \quad \Lambda^\# \overset{\text{def}}{=} \{(m'', m') \mid m'' \otimes m' \in \Lambda\}.
$$

Thus $\Lambda^\#$ consists of those pairs of multiplicable elements of the groupoid $\mathcal{E}$ whose product belongs to $\Lambda$. 20
Lemma 9.1. (i) If $\Lambda$ is Lagrangian, then $\Lambda^\#$ is Lagrangian (at all points where it is a submanifold).

(ii) If $\Phi$ is the generating function of $\Lambda \subset \mathcal{E}$ in the sense of (7.11), then the generating function of $\Lambda^\# \subset \mathcal{E} \times \mathcal{E}$ is given by

\begin{equation}
\Phi^\#(x, y) = \left[ \Phi(z) + \Phi_y(x, z) \right] \bigg|_{z = Z(x, y)},
\end{equation}

where $Z(x, y)$ is the stationary point of the right-hand side of (9.2) with respect to $z$.

(iii) Let $\Lambda = \Lambda_M$, where $M$ is a Lagrangian submanifold in $\mathfrak{X} \times \mathfrak{X}$. For each pair $(x, y) \in \mathfrak{X} \times \mathfrak{X}$ close to the diagonal, let us denote by $(a, b)$ the point of intersection of $M$ with the graph of the Ether translation $s_x \circ s_y$. Then the stationary point $Z(x, y)$ in (9.2) coincides with the mid-point of the Ether geodesic connecting $a$ and $b$. The generating function $\Phi^\#_M$ of $\Lambda_M^\#$ obeys the equations

\begin{equation}
\partial_x \Phi^\#_M(x, y) = \mathcal{H}_x(b), \quad \partial_y \Phi^\#_M(x, y) = -\mathcal{H}_y(a).
\end{equation}

(iv) If $M = \lambda \times \lambda$, where $\lambda$ is a Lagrangian submanifold in $\mathfrak{X}$, then

\begin{equation}
\Phi^\#_{\lambda \times \lambda}(x, y) = \int_{\Sigma_{\lambda}(Z(x, y))} \omega + \int_{\Delta(Z(x, y), y, x)} \omega,
\end{equation}

where the membrane $\Sigma_{\lambda}$ is defined in Theorem 8.1, and $\Delta$ is the triangle from (6.1).

(v) Let $M = \text{Graph}(\gamma'_H)$, where $\gamma'_H$ is a Hamiltonian flow in $\mathfrak{X}$. Then the generating function $\Phi^\#_{\text{Graph}(\gamma'_H)}$ is given by a formula similar to (9.4) with the first summand replaced by the dynamic phase function $\Phi^t(Z(x, y))$ given by (4.1).

(vi) Let $y = Y(x)$ be a point such that the pair $(s_x(y), y)$ belongs to $(\ell \times r)(\Lambda)$. Then $y = Y(x)$ is the stationary point of $\Phi^\#(x, y)$ with respect to $y$, and

\begin{equation}
\Phi^\#(x, y) \bigg|_{y = Y(x)} = \Phi(x).
\end{equation}

Now let us note that submanifolds in $\mathcal{E} \times \mathcal{E}$ can be considered as “operators” acting in the space of submanifolds in $\mathcal{E}$. Namely, if $A \subset \mathcal{E} \times \mathcal{E}$ and $L \subset \mathcal{E}$, then

\begin{equation}
A(L) \overset{\text{def}}{=} \{ m \in \mathcal{E} \mid \exists \tilde{m} \in \mathcal{E} : (m, \tilde{m}) \in A, \tilde{m}^{-1} \in L \}.
\end{equation}
The role of the unity operator is played by the submanifold
\[ \mathcal{X}^\# \overset{\text{def}}{=} \{ (m, m^{-1}) \mid m \in \mathcal{E} \} . \]
Indeed, \( \mathcal{X}^\#(L) = L \) for any \( L \subset \mathcal{E} \).

The composition of two “operators” \( A, B \subset \mathcal{E} \times \mathcal{E} \) is defined as
\[ A \otimes B \overset{\text{def}}{=} \{ (m'', m') \mid \exists m : (m'', m) \in A, (m^{-1}, m') \in B \} . \]

Obviously, this is an associative product, and the element (9.7) is the unity element with respect to this product:
\[ A \otimes \mathcal{X}^\# = \mathcal{X}^\# \otimes A = A . \]

There is a natural inverse
\[ A^{-1} \overset{\text{def}}{=} \{ (m, \tilde{m}) \mid (\tilde{m}^{-1}, m^{-1}) \in A \} . \]
It is easy to check that
\[ A \otimes A^{-1} \subset \mathcal{X}^\# \quad \text{(or} \quad A^{-1} \otimes A \subset \mathcal{X}^\#) \]
if \( A \) is one-to-one projected to the right (or the left) multiplier in \( \mathcal{E} \times \mathcal{E} \).

The product (9.8) is consistent with the action (9.6), that is,
\[ A(B(L)) = (A \otimes B)(L) . \]

The permutation operation \( A \to A' \),
\[ A' \overset{\text{def}}{=} \{ (m, \tilde{m}) \mid (\tilde{m}, m) \in A \} , \]
is also consistent with the product (9.8):
\[ (A \otimes B)' = B' \otimes A' . \]

For each \( \Lambda \subset \mathcal{E} \), we denote \( \Lambda^\kappa \overset{\text{def}}{=} (\Lambda^\#)' \).

**Theorem 9.2.** The following properties hold:
\[ \Lambda_1^\# (\Lambda_2) = \Lambda_1 \otimes \Lambda_2 , \]
\[ \Lambda_2^\kappa (\Lambda_2) = \Lambda_2 \otimes \Lambda_1 , \]
\[ \Lambda_1^\# \otimes \Lambda_2^\# = (\Lambda_1 \otimes \Lambda_2)^\# , \]
\[ \Lambda_1^\kappa \otimes \Lambda_2^\kappa = (\Lambda_2 \otimes \Lambda_1)^\kappa , \]
\[ \Lambda_1^\# \otimes \Lambda_2^\kappa = \Lambda_2^\kappa \otimes \Lambda_1^\# . \]

These properties repeat, on the level of Lagrangian submanifolds, the properties the left and right quantum left mappings related to the \( \ast \)-product operation over \( \mathfrak{X} \) (see [11, 18]).
10 Generalization to the torsion case

The symplectic groupoid approach described in Secs. 7–9 is mostly based on the Poisson bifibration (7.1)–(7.3). In definition (7.1), we used the Ether Hamiltonian $\mathcal{H}$ which obeys the zero curvature condition (2.1) and the additional boundary and skew-symmetry conditions (2.2), (2.3). These additional conditions actually are not necessary. One can generalize them, but still keep the appropriate properties of the basic dynamic equation (3.4) for the family of symplectic mappings $\{s_x\}$.

One can keep the fixed point property (2.4), but refuse the involution property (2.5). In this case, the family $\{s_x\}$ will still generate a symplectic connection on $\mathfrak{X}$, but in a more complicated way than via the simplest formula (2.6), and this connection will no longer be torsion free.

The goal in this section is to consider this torsion case and to show at which places the differences from the torsion free case do appear.

The basic zero curvature equation (2.1) for $\mathcal{H}$ is kept unchanged, as well as the first boundary condition (2.2), that is

$$\mathcal{H}|_{\text{diag}} = 0.$$ 

Under these general conditions we call $\mathcal{H}$ an internal Hamiltonian over $\mathfrak{X}$. The family of symplectic mappings $\{s_x\}$ is defined by (3.4), so that condition (2.4) still holds. We call $s_x$ inversions.

The connection on $\mathfrak{X}$ generated by the family $\{s_x\}$ is given by

$$\Gamma^l_{jk}(x) = -\frac{\partial^2 s^l_x(z)}{\partial z^m \partial x^r} \left[ \frac{\partial s_x(z)}{\partial z} \right]^{-1}_m \left[ \frac{\partial s_x(z)}{\partial x} \right]^{-1}_r \bigg|_{z=x}.$$

**Lemma 10.1.** The connection $\Gamma$ (10.1) is symplectic.

In general, this connection is not torsion free.

The second and the third boundary conditions in (2.2), as well as the skew-symmetry condition (2.3), do not hold for the internal Hamiltonian $\mathcal{H}$, in general. But still the following boundary condition holds:

$$\nabla_l \nabla_m \mathcal{H}_k \bigg|_{\text{diag}} + \nabla_s \mathcal{H}_k \bigg|_{\text{diag}} \Psi^{sr} T^l_{ri} \omega_{jm} = 0,$$

where $\nabla$ is taken with respect to the connection $\Gamma$ (10.1), and $T$ is the torsion tensor of $\Gamma$. 

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Now, the left mapping $\ell$ of the bifibration of the phase space over $\mathfrak{X}$ is defined by the same formula (7.1), and the right mapping $r$ is defined by (7.2), i.e., $r(x,p) = s_x^{-1}(\ell(x,p))$, where $s_x$ is the inversion mapping.

**Lemma 10.2.** The components of the mappings $\ell, r$ defined above obey the Lie–Engel system (7.3).

The family of inversions $\{s_x\}$ still defines a Lagrangian fibration of a neighborhood of the diagonal in $\mathfrak{X} \times \mathfrak{X}$. The fibers are graphs of the mappings $s_x$.

If $\gamma$ is a transformation of $\mathfrak{X}$, the Graph($\gamma$) intersects the fiber of $s_x$ at a point $(\gamma(\bar{x}), \bar{x})$, where $\bar{x}$ is a fixed point of the mapping $s_x^{-1} \circ \gamma$ (compare with Sec. 3).

**Lemma 10.3.** Theorem 3.1, (i) still holds. If the mapping is symplectic, then the form $\mathcal{H}_x(\gamma(\bar{x}))$ is closed, and we can define a function $\Phi^\gamma$ by the formula

$$D\Phi^\gamma(x) = \mathcal{H}_x(\gamma(\bar{x}))$$

(compare with (3.2)).

Let us now define internal exponential mappings $\text{Exp}_x$ in the same way as in Sec. 2.

Consider the curve $\sigma_x = \sigma_x^+ \cup \sigma_x^-$ composed of two pieces: $\sigma_x^+ = \{\text{Exp}_x(vt) \mid 0 \leq t \leq 1\}$ and $\sigma_x^- = \{s_x^{-1}(\text{Exp}_x(vt))\}$. Denote $z = \text{Exp}_x(v), y = s_x^{-1}(z)$. We call $\sigma_x$ the internal geodesic form $y$ to $z$ through the center $x$.

Since $s_x(\sigma_x^+) = \sigma_x^-$, such an internal geodesic $\sigma_x$ is an inversive curve with the center point $x$.

**Lemma 10.4.** (i) The membrane formula (3.3) for the phase function of the symplectic transformation $\gamma$ still holds if in the construction of the membrane $\Sigma^\gamma(x,y)$ the Ether geodesics are replaced by the internal geodesics and the term “mid-point” is replaced by the term “center-point.”

(ii) The membrane formulas (4.1), (6.1), the Hamilton–Jacobi equation (7.5), the identity (4.2), as well as the composition formulas (6.3), (6.8) and all other formulas of Secs. 7–9, hold with the same replacement agreement as in case (i).
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