BORDERLINE WEAK–TYPE ESTIMATES FOR SPARSE BILINEAR FORMS INVOLVING $A_\infty$ MAXIMAL FUNCTIONS

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Abstract. For any operator $T$ whose bilinear form can be dominated by a sparse bilinear form, we prove that $T$ is bounded as a map from $L^1(\tilde{M}w)$ into weak-$L^1(w)$. Our main innovation is that $\tilde{M}$ is a maximal function defined by directly using the local $A_\infty$ characteristic of the weight (rather than Orlicz norms). Prior results are due to Coifman&Fefferman, Pérez, Hytönen&Pérez, and Domingo-Salazar&Lacey&Rey.

1. Introduction

We study weighted endpoint estimates for those operators whose bilinear form has a sparse domination. Our estimates are in the spirit of Fefferman–Stein \cite{9}; in particular, for an operator $T$, a function $f$ and a non–negative weight $w$ (by non–negative weight we mean a non–negative locally integrable function) we will prove:

$$\lambda w(\{Tf > \lambda\}) \leq \int_{\mathbb{R}^d} |f(y)| M_\varepsilon w(y) dy,$$

where $M_\varepsilon$ is a certain maximal function that is pointwise larger than the Hardy–Littlewood maximal function. We take an “entropy bump” point of view – which is our main innovation – and define $M_\varepsilon$ in terms of these entropy bumps. This innovation makes explicit use of the important $A_\infty$ characteristic and using this framework we can we can prove results of, for example, Hytönen-Pérez \cite{12}.

We now prepare to state our main results. Recall that if $D$ is a dyadic lattice, a subset $S$ of $D$ is said to be $t$–sparse (or simply “sparse”) ($0 < t < 1$) if every $Q \in S$ there is a subset $E_Q \subset Q$ such that $|E_Q| > t |Q|$ and the sets $\{E_Q : Q \in S\}$ are pairwise disjoint.

When we say that $T$ has a bilinear sparse domination we mean that for all bounded and compactly supported functions $f_1, f_2$ there are $K < \infty$ sparse sets such that there holds:

$$|(\langle f_1, f_2 \rangle| \leq \sum_{k=1}^K \sum_{Q \in \delta_k} |Q| \langle f_1 \rangle_Q \langle f_2 \rangle_Q,$$

where $\langle f \rangle_Q := \frac{1}{|Q|} \int_Q |f|$ (note the presence of the absolute value inside the integral). The exact value $K$ isn’t important for what we are doing, and is different for various operators; it should be pointed out that $K$ depends on $T$ but not $f_1$ nor $f_2$.

2010 Mathematics Subject Classification. Primary:42B20 Secondary: 42B25.

Key words and phrases. weak–type estimate, maximal function.
The class of operators that have such a sparse domination is vast. Any Calderón–Zygmund operator has such a domination, for example ([14]). See also [1, 3, 4] and the references therein, for example. Thus, the theorem covers many operators from harmonic analysis.

Let \( \varepsilon : [1, \infty] \to [1, \infty] \) be an increasing function with \( K_\varepsilon := \sum_{k=-1}^{\infty} \varepsilon(2^{2k})^{-1} < \infty \).

The example you should keep in mind is essentially \( \varepsilon(t) = (\log \log t)(\log \log \log t)^{1+\varepsilon} \).

For a cube \( Q \) and a weight \( w(x) \geq 0 \) define:

\[
\rho_w(Q) := \frac{1}{w(Q)} \int_Q M(\mathbb{I}_Q w)(x) \, dx,
\]

where \( M \) is the usual Hardy–Littlewood maximal function and \( w(Q) := \int_Q w(y) \, dy \). For a collection \( S \) of cubes (e.g. a dyadic lattice or a sparse subset of a dyadic lattice) define the following maximal function:

\[
M_\varepsilon w := \sup_{Q \in S} \mathbb{I}_Q(w)Q \log \rho_w(Q) \varepsilon(\rho_w(Q)).
\]

These are our main theorems: The first is a result of Hytönen–Pérez [11, Corollary 1.4]:

**Theorem 1.1.** Let \( T \) be an operator that has a sparse bilinear domination as in (1.2) and let \( w \) be an \( A_1 \) weight. Then we have the following quantitative estimate:

\[
\|T : L^1(w) \to L^{1,\infty}(w)\| \lesssim [w]_{A_1} \log (e + [w]_{A_\infty}),
\]

where

\[
[w]_{A_1} := \sup_{Q \text{ a cube}} \frac{M(\mathbb{I}_Q w)(x)}{w(x)} \quad \text{and} \quad [w]_{A_\infty} := \sup_{Q \text{ a cube}} \frac{1}{w(Q)} \int_Q M(\mathbb{I}_Q w)(x) \, dx.
\]

The main theorem (and the new one) is:

**Theorem 1.2.** Let \( T \) be an operator that has a sparse bilinear domination as in (1.2) and let \( \varepsilon \) be a function as above. Then for any weight \( w(x) \geq 0 \) we have:

\[
\|T : L^1(M_\varepsilon w) \to L^{1,\infty}(w)\| \lesssim K_\varepsilon.
\]

The paper is organized as follows. In the next section, we discuss the main result and give some motivation. Following that, in Section 3 we give some background information and preliminary information and then in Section 4 we prove Theorems 1.2 and 1.1.

**Acknowledgment.** I am grateful to a very thorough anonymous referee who not only caught many typos, mistakes, etc, but also gave constructive suggestions to make the exposition better. Of course, any missteps are my fault.
2. Discussion of Main Results and Previous Results

For the remainder of the paper, the function \( \log t = \log t \) is the function that satisfies \( 2^{\log t} = 2 + t \). That is, the \( \log \) we’re using here is really \( \log t = \log_2(2 + t) \).

One would like to replace \( M_\varepsilon \) in (1.1) with the smaller Hardy–Littlewood maximal function \( M \). However, this is not possible; see for example [10, 18, 19]. It is of interest then to determine the smallest maximal function for which (1.1) holds.

Observe that one way to write \( \langle w \rangle_Q \) is \( \|w\|_{L^1(\frac{dt}{|Q|})} \). Thus to make \( M \) slightly larger, we can choose a norm that is slightly larger than the normalized \( L^1 \) norm. A common approach has been to use Orlicz norms. That is, given a positive non–decreasing function \( \Phi \) define:

\[
\|w\|_{Q,\Phi} := \inf(\lambda > 0 : \frac{1}{|Q|} \int_Q \Phi(\frac{w(x)}{\lambda}) dx \leq 1).
\]

When \( \Phi(t) = t^r \), then \( \|w\|_{Q,\Phi} = \|w\|_{L^r(\frac{dx}{|Q|})} \), which is bigger than normalized \( L^1 \) norm for \( 1 < r \). Maximal functions created from these “power bumps” were studied in [2]. In 1994, Pérez shows that for singular integral operators (in fact maximal truncations), (1.1) holds when \( \tilde{M} \) is the maximal function based on \( \Phi(t) = (t log t)^{1+\varepsilon} \) and this was result was recently quantified (in terms of \( \varepsilon \)) by Hytönen–Pérez [12, 17]. The best known result so far is due to Domingo-Salazar, Lacey, and Rey [8] where \( \Phi(t) = (\log \log t)(\log \log \log t)^{1+\varepsilon} \).

See the papers listed in the references for more detailed information about these maximal functions and Orlicz norms.

In this paper, we take a slightly different approach and use the so–called “entropy bumps” introduced by Treil–Volberg [20]. More precisely we consider an increasing function \( \varepsilon : [1, \infty) \rightarrow [1, \infty) \) that is just barely summable in the sense that \( K_\varepsilon := \sum_{k \geq 1} \varepsilon(2^k)^{-1} < \infty \). For a cube \( Q \) we define:

\[
\|w\|_{Q,\rho_\varepsilon(Q)} := \langle w \rangle_Q \rho_w(Q) \varepsilon(\rho_w(Q)).
\]

In [20] it is shown that \( t \mapsto \Phi(t)/t \log t \) is increasing for sufficiently large \( t \) and \( \int_0^\infty \frac{dt}{\Phi(t)} < \infty \), then there is a function \( \varepsilon \) as above with \( \|w\|_{Q,\rho_\varepsilon(Q)} \leq \|w\|_{Q,\Phi} \).

We will use entropy norms that are smaller than the entropy norms defined above. In particular we will use the following:

\[
\|w\|_{Q,\log \rho(Q)} = \langle w \rangle_Q \log(\rho_w(Q)) \varepsilon(\rho_w(Q)).
\]

The result mentioned above in [20] does not imply that these entropy bumps are smaller than the ones in [8], and, in fact, the Orlicz and entropy maximal functions are probably not directly comparable. One advantage of the entropy bumps is that they explicitly use \( A_\infty \) data on the weight \( w \) and it has become clear that this data is important. An advantage of the Orlicz norm is of a qualitative nature: the results in [8] require just a bit more than local \( L \log \log L \) integrability of the weight, while the entropy bumps require local \( L \log L \) integrability (since \( \rho_w(Q) \approx \|w\|_{L \log L(Q)} \)).
The proof(s) in this paper are modifications of the proofs in [7, 8] to the present setting.

3. Background Information and Preliminaries

In the proof of the theorems, we will need the collections to satisfy the following stronger condition: for every \( Q \in S \) there holds:

\[
\left| \bigcup_{Q' \in S, Q' \subsetneq Q} Q' \right| \leq \frac{1}{4} |Q|.
\]  

(3.1)

The following lemma says that every sparse collection is a finite union of sparse collections that satisfy this stronger condition.

**Lemma 3.1.** Let \( S \) be \( 2^{-l} \)-sparse for some \( l \in \mathbb{N} \) that contains a "top cube" (i.e. a cube that contains all other cubes in \( S \)). Then \( S = \bigcup_{i=1}^{2^{l+2}} S^i \) where each \( S^i \) satisfies the stronger condition (3.1).

**Proof.** The sparse condition implies the "Carleson" condition (recall that if \( Q' \in S \) then \( |E_{Q'}| > 2^{-l} |Q'| \)): for every \( Q \in S \) we have:

\[
\sum_{Q' \in S, Q' \subsetneq Q} |Q'| \leq 2^{l} \sum_{Q' \in S, Q' \subsetneq Q} |E_{Q'}| \leq 2^{l} |Q|.
\]

For a \( Q \in S \) define \( S_1(Q) \) be those cubes in \( S \) that are contained in \( Q \) and are maximal in \( Q \) (i.e. \( Q' \in S_1(Q) \) if \( Q' \in S \), \( Q' \subset Q \) and there is no cube \( Q'' \in S \) with \( Q' \subseteq Q'' \subseteq Q \)). Inductively, define \( S_{k+1}(Q) \) to be those cubes that are in \( S \), are contained in some \( R \in S_k(Q) \) and are maximal in that cube (i.e. \( Q' \in S_{k+1}(Q) \) if \( Q' \in S \), \( Q' \subset R \) for some \( R \in S_k(Q) \) and there is no \( Q'' \in S \) with \( Q' \subseteq Q'' \subseteq R \)). Informally, \( S_k(Q) \) are the cubes in \( S \) that are \( k \)-generations down in \( S \) from \( Q \) (or the \( k \)-children of \( Q \) in \( S \)).

We claim that \( \left| \bigcup_{Q' \in S_{2^{l+2}}(Q)} Q' \right| \leq \frac{1}{4} |Q| \). Indeed, suppose not; then we would have:

\[
\sum_{Q' \in S, Q' \subsetneq Q} |Q| > \sum_{k=1}^{2^{l+2}} \sum_{Q' \in S_k(Q)} |Q'| > \sum_{k=1}^{2^{l+2}} \frac{1}{4} |Q| = \frac{2^{l+2}}{4} |Q| = 2^l |Q|,
\]

which violates the Carleson condition. It is now easy to see how to separate \( S \) into \( 2^{l+2} \) sub-collections: let \( Q_0 \) be the top cube in \( S \). For \( k = 1, \ldots, 2^{l+2} \) let

\[
S_k = \bigcup_{n \geq 0} \bigcup_{Q \in S_{2^{l+2}n+k}(Q_0)} Q.
\]

Thus, for each \( Q \in S_k \), the cubes one generation down in \( S_k \) are \( 2^{l+2} \) generations down in \( S \) and so we have the stronger sparse condition:

\[
\left| \bigcup_{Q' \in S_k, Q' \subsetneq Q} Q' \right| \leq \frac{1}{4} |Q|,
\]

as desired. \( \square \)
We now have a variant of the classic Fefferman–Stein Inequality (see also [6]). (Observe that the lemma and its proof are classical and well–known; however, since this maximal function only takes a sup over a subset of $\mathcal{S}$, we can’t quote earlier results). Let $\mathcal{S}$ be a subset of some dyadic lattice and define $M_{\mathcal{S}}f := \sup_{Q \in \mathcal{S}} \Pi_Q(f)_Q$.

**Lemma 3.2.** For every $f$ and $\lambda > 0$ we have:

$$
\lambda w(\{M_{\mathcal{S}}f > \lambda\}) \leq \int_{\mathbb{R}^d} |f(y)||M_{\mathcal{S}}w(y)| dy.
$$

**Proof.** For $\lambda > 0$, let $\Omega_\lambda$ be the maximal cubes in $\mathcal{S}$ with $(f)_Q > \lambda$. Then using the fact that the cubes in $\Omega_\lambda$ are pairwise disjoint, there holds

$$
\lambda w(\{M_{\mathcal{S}}f > \lambda\}) \leq \sum_{Q \in \Omega_\lambda} (f)_Q w(Q)
$$

$$
= \sum_{Q \in \Omega_\lambda} \int_Q |f(y)| |M_{\mathcal{S}}w(y)| \frac{w(Q)}{|Q|} dy
$$

$$
\leq \int_{\mathbb{R}^d} |f(y)| M_{\mathcal{S}}w(y) dy,
$$

as desired. $\square$

There are many ways to define the $[w]_{A_{\infty}}$ characteristic of a weight. The one we use – and the one that seems most useful and popular – is the one of Wilson [21]; see also [12] for more information. For a dyadic lattice $\mathcal{D}$, let $M_{\mathcal{D}}$ be the dyadic maximal function. For $Q \in \mathcal{D}$ define:

$$
\rho_w(Q) := \frac{1}{w(Q)} \int_Q M_{\mathcal{D}}(w \Pi_Q)(x) dx
$$

The following is [11, Lemma 6.6]:

**Lemma 3.3.** For a cube $Q \in \mathcal{D}$ and a subset $E \subset Q$ we have

$$
w(E) \leq w(Q) \frac{\rho_w(Q)}{\log \frac{|E|}{|Q|}}.
$$

4. **Proofs of Theorems 1.1 and 1.2**

Recall that we are dealing with operators $T$ whose bilinear form has a sparse domination. That is, there are $\mathcal{K} < \infty$ sparse sets such that

$$
|\langle Tf_1, f_2 \rangle| \leq \sum_{k=1}^{\mathcal{K}} \sum_{Q \in S_k} |Q| \langle |f_1|_Q \rangle \langle |f_2|_Q \rangle.
$$

The choice of sparse sets depend on $f_1$, $f_2$ and, $T$, but $\mathcal{K}$ and the implied constant are independent of $f_1$ and $f_2$, but depend on $T$ and the geometry of the space (i.e. $\mathbb{R}^d$).
The rest of the section is used to prove the following proposition, but before we prove the proposition, we show how Theorems 1.1 and 1.2 are corollaries of the proposition. (It might seem strange that there is a $2^{2^r}$ and $2^r$ in the proposition below, instead of $r$ and $\log r$, but when this proposition is applied, it is applied with $2^{2^r}$ and $2^r$).

**Proposition 4.1.** Let $w$ be a weight (i.e. non-negative locally integrable function). Then for any sparse collection $S$ with $\sup_{Q \in S} \rho_w(Q) \leq 2^{2^r}$ for some $r \in \mathbb{N}$ and for all non-negative locally integrable functions $f$ there holds

$$\sup_{G \subset \mathbb{R}^d} \inf_{0<w(G)<\infty} \sum_{Q \in S} |Q| \langle f \rangle_Q \langle w \mathbb{1}_{G'} \rangle_Q \lesssim 2^r \int_{\mathbb{R}^d} |f(y)| M_S w(y) \, dy.$$  

where $M_S$ is the maximal function restricted to cubes in $S$:

$$M_S f(x) := \sup_{Q \in S} \langle |f| \rangle_Q \mathbb{1}_Q(x).$$

4.1. **Deducing the Main Theorems.** First, recall that the weighted weak $L^1$ norm can be computed as follows:

$$\|g\|_{L^{1,\infty}(\mu)} = \sup_{G \subset \mathbb{R}^d} \inf_{0<\mu(G)<\infty} |\langle g, \mu \mathbb{1}_G \rangle|.$$

Thus, for a weight $w$ and an operator $T$:

$$\|Tf\|_{L^{1,\infty}(w)} = \sup_{G \subset \mathbb{R}^d} \inf_{0<w(G)<\infty} |\langle Tf, w \mathbb{1}_G \rangle|.$$  

Therefore if $T$ has a bilinear domination as defined in Section 1:

$$|\langle Tf, f \rangle| \leq \sum_{k=1}^K \sum_{Q \in S_k} |Q| \langle f_1 \rangle_Q \langle f_2 \rangle_Q,$$

then via standard reductions, and (4.1), it is enough to estimate:

$$\sup_{G \subset \mathbb{R}^d} \inf_{0<w(G)<\infty} \sum_{Q \in S} |Q| \langle f \rangle_Q \langle w \mathbb{1}_{G'} \rangle_Q,$$

uniformly over all sparse sets $S$ and non-negative, compactly supported functions $f$. Furthermore, by (for example) the monotone convergence theorem, we may assume that there is a finite number of cubes in $S$ (to avoid any convergence issues in the estimates below) and we may assume there is a "top cube", that is a cube in $S$ that contains all other cubes in $S$ (so we can apply Lemma 3.1).

**Proof of Theorem 1.1.** To deduce Theorem 1.1 from Proposition 4.1 observe that if $w \in A_\infty$ and $2^{2^{r-1}} \leq [w]_{A_\infty} \leq 2^r$, then we may take any sparse collection to be the collection
in Proposition 4.1. Since \(2^r \simeq \log[w]_{A_\infty}\) and \(M_w(y) \leq [w]_{A_1} w(y)\), Proposition 4.1 and the above reductions assert that:

\[
\|T\|_{L^{1,\infty}(w)} \leq 2^r \int_{\mathbb{R}^d} |f(y)| M_S w(y) \, dy \leq [w]_{A_1} \log[w]_{A_\infty} \int_{\mathbb{R}^d} |f(y)| w(y) \, dy,
\]

which is Theorem 1.1. \(\square\)

Now we deduce Theorem 1.2 from Proposition 4.1.

**Proof of Theorem 1.2.** We estimate:

\[
\sup_{G \subset \mathbb{R}^d} \inf_{0 < w(G) < \infty} \sum_{Q \in S} |Q| \langle f \rangle_Q \langle w \mathbb{1}_G \rangle_Q \leq 2^r \int_{\mathbb{R}^d} |f(y)| M_{S_r} w(y) \, dy.
\]

We now have a critical observation. The maximal function \(M_{S_r}\) only considers those cubes in \(S_r\). This means that for those cubes, \(\log \rho_w(Q) \simeq 2^r\) and similarly, using the fact that \(\varepsilon\) is increasing, \(\varepsilon(2^{2^r - 1}) < \varepsilon(\rho_w(Q))\). Therefore we may estimate:

\[
2^r M_{S_r} w(y) = \frac{1}{\varepsilon(2^{2^r - 1})} \sup_{Q \in S_r} \langle w \rangle_Q 2^r \varepsilon(2^{2^r - 1}) \mathbb{1}_Q(y)
\]

\[
\simeq \frac{1}{\varepsilon(2^{2^r - 1})} \sup_{Q \in S_r} \langle w \rangle_Q \log(\rho_w(Q)) \varepsilon(2^{2^r - 1}) \mathbb{1}_Q(y)
\]

\[
\leq \frac{1}{\varepsilon(2^{2^r - 1})} \sup_{Q \in S_r} \log(\rho_w(Q)) \varepsilon(\rho_w(Q)) \mathbb{1}_Q(y)
\]

\[
\leq \frac{1}{\varepsilon(2^{2^r - 1})} M_w(y).
\]

Thus, the estimate in (4.2) can be continued as

\[
\sup_{G \subset \mathbb{R}^d} \inf_{0 < w(G) < \infty} \sum_{Q \in S_r} |Q| \langle f \rangle_Q \langle w \mathbb{1}_G \rangle_Q \leq 2^r \int_{\mathbb{R}^d} |f(y)| M_{S_r} w(y) \, dy
\]

\[
\leq \frac{1}{\varepsilon(2^{2^r - 1})} \int_{\mathbb{R}^d} |f(y)| M_w(y) \, dy.
\]

Using the summability condition on \(\varepsilon\), this estimate can be summed in \(r\) to the desired estimate in Theorem 1.2. \(\square\)
4.2. **Proving Proposition 4.1.** We now turn to proving Proposition 4.1. By homogeneity, it suffices to prove

\[
\sup_{f \geq 0 \|f\|_{L^1(Sw)} = 1} \sup_{G \subset \mathbb{R}^d} \inf_{0 < w(G) < \infty, \|w(G)\|_{L^1} \leq 2w(G')} \sum_{Q : f \in G(G')} |Q| \langle f \rangle_Q \langle w \| G' \rangle_Q \leq 2^r,
\]

where this estimate is uniform over all sparse collections with \( \sup_{Q \in S} w(Q) \leq 2^2 \).

Fix a set \( G \) with \( 0 < w(G) < \infty \) and a compactly supported non-negative function \( f \) with \( \|f\|_{L^1(Sw)} = 1 \). Since \( f \) is bounded and compactly supported, we may assume that \( f \) is supported on a cube \( Q_0 \in S \) and that \( \langle f \rangle_{Q_0} < 4w(G)^{-1} \). Furthermore, we may assume that \( Q_0 \) contains all cubes in \( S \).

Let \( H \) be the maximal cubes in \( S \) with \( \langle f \rangle_Q > 4w(G)^{-1} \) and set \( H = \bigcup_{Q \in H} Q \). By the Fefferman–Stein Inequality (Lemma 3.2) we have \( w(H) \leq \frac{1}{4} w(G) \). Indeed, taking \( \lambda = 4w(G)^{-1} \) and noting that \( \|f\|_{L^1(Sw)} = 1 \) there holds

\[
w(H) = w(\{Msf > \lambda\}) \leq \frac{1}{\lambda} \int_{\mathbb{R}^d} |f| Msw \leq \frac{1}{\lambda} \int_{\mathbb{R}^d} |f| Msw = \frac{w(G)}{4}.
\]

Set \( G' = G \cap H^c \) and note that:

\[
w(G) = w(G \cap H) + w(G \cap H^c) \leq w(H) + w(G') \leq \frac{1}{4} w(G) + w(G'),
\]

and so \( w(G) \leq 2w(G') \).

We may now assume that the cubes in \( S \) satisfy \( \langle f \rangle_Q \leq 4w(G)^{-1} \). If not then \( Q \) is either a cube in \( H \) or is contained in a cube in \( H \); either way, \( Q \) is contained in \( H \). But \( \|G'\|_Q \) is zero on \( H \) and so \( \langle \|G'\rangle_Q \rangle = 0 \). Thus, we assume that \( \langle f \rangle_Q \leq 4w(G)^{-1} \).

For \( k \geq 1 \), let

\[
S_k := \{Q \in S : 4^{-k} w(G)^{-1} < \langle f \rangle_Q \leq 2 \cdot 4^{-k} w(G)^{-1}\}.
\]

For fixed \( k \), let \( S_k^j \) be the maximal cubes in \( S_k \) and for \( j \geq 1 \) let \( S_k^j \) be the maximal cubes in \( S_k \setminus \bigcup_{l=0}^{j-1} S_k^l \). For \( Q \in S_k^j \) let \( E_Q = Q \cup \bigcup_{Q' \in S_k^{j+1}} Q \). Observe that the sets \( \{E_Q : Q \in S_k\} \) are pairwise disjoint. The sparsity condition (3.1) implies:

**Lemma 4.2.** \( \int_Q |f(y)| \, dy \simeq \int_{E_Q} |f(y)| \, dy \).

**Proof of Lemma 4.2.** If \( S_k^j(Q) \) are those cubes in \( S_k^{j+1} \) that are contained in \( Q \), there holds:

\[
\int_Q |f(y)| \, dy = \int_{E_Q} |f(y)| \, dy + \sum_{Q' \in S_k^j(Q)} |Q'| \langle f \rangle_{Q'} \leq \int_{E_Q} |f(y)| \, dy + \sum_{Q' \in S_k^j(Q)} |Q'| 2 \cdot 4^{-k} w(G)^{-1}.
\]
Using the definition of $S_k$ we see that $4^{-k}w(G)^{-1} < \langle |f| \rangle_Q$. This combined with the condition (3.1) allows us to continue the estimate above as:

$$\int_Q |f(y)| \, dy = \int_{E_Q} |f(y)| \, dy + \sum_{Q' \in S^1_k(Q)} |Q'| 2 \cdot 4^{-k}w(G)^{-1} \leq \int_{E_Q} |f(y)| \, dy + \frac{1}{2} \int_Q |f(y)| \, dy,$$

from which we conclude $\int_Q |f(y)| \, dy \simeq \int_{E_Q} |f(y)| \, dy$ as desired. □

We break the $k$ sum into two pieces

$$\sum_{k=-1}^{10 \cdot 2^r} \sum_{Q \in S^1_k} |Q| \langle f \rangle_Q \langle w11_{G'} \rangle_Q + \sum_{Q \in S^1_k} |Q| \langle f \rangle_Q \langle w11_{G'} \rangle_Q$$

and show each piece is controlled by $2^r$ and 1, respectively. We handle each piece in the following two subsections.

### 4.3. Main Portion ($k \leq 10 \cdot 2^r$)

Observe:

$$|Q| \langle w11_{G'} \rangle_Q = w(G' \cap Q).$$

The goal of this subsection is to prove:

**Lemma 4.3.**

$$\sum_{k=-1}^{10 \cdot 2^r} \sum_{Q \in S^1_k} \langle f \rangle_Q w(G' \cap Q) \leq 2^r.$$

**Proof of Lemma 4.3.** This follows from the following estimates.

First, using Lemma 4.2 and the pairwise disjointness of the sets $\{E_Q\}$, for fixed $k$ there holds:

$$\sum_{Q \in S_k} \langle f \rangle_Q w(G' \cap Q) \simeq \sum_{Q \in S_k} \int_{E_Q} |f(y)| \, dy \langle w \rangle_Q \leq \int_{\mathbb{R}^d} |f(y)| \langle M_{S_k} w \rangle dy,$$

where $M_{S_k}$ is the maximal function where the supremum is taken over cubes in $S_k$. Using the fact that this estimate holds uniformly in $k$, we can make the following coarse estimate:

$$\sum_{k=-1}^{10 \cdot 2^r} \sum_{Q \in S_k} |Q| \langle f \rangle_Q w(G' \cap Q) \leq 10 \cdot 2^r \int_{\mathbb{R}^d} |f(y)| \langle M_{S} w \rangle dy,$$

and using the fact that $\|f\|_{L^1(M^{1/2}w)} = 1$ this completes the proof of Lemma 4.3. □
4.4. The Tail \((k \geq 10 \cdot 2^t)\). The goal of this subsection is to prove

**Lemma 4.4.**

\[
\sum_{k=10 \cdot 2^t}^{\infty} \sum_{Q \in S_{r,k}} \langle f \rangle_Q w(G' \cap Q) \leq 1.
\]

For a cube \(Q\) in \(S_k^i\), let \(S_k^{i+1}(Q)\) be the cubes in \(S_k^{i+1}\) contained in \(Q\) and define \(Q_t := \bigcup_{Q' \in S_k^{i+1}(Q)} Q'\) where \(t = 2^k\). The sparse condition implies that \(|Q_t| \leq 4^{-t} |Q|\) and Lemma 3.3 implies that \(w(Q_t) \leq 2^{2r} 2^{-k} w(Q)\). Note that we may write:

\[
Q = Q_t \cup \bigcup_{l=0}^{l-1} \bigcup_{Q' \in S_k^{i+1}(Q)} E_Q.
\]

Concerning the \(Q_t\) portion, for \(Q\) in \(S_k\) and Lemma 4.2:

\[
\langle f \rangle_Q w(G' \cap Q_t) \leq \frac{2^{2r}}{2^k} \langle f \rangle_Q w(Q) = \frac{2^{2r}}{2^k} \int_Q |f(y)| \, dy \langle w \rangle_Q \approx \frac{2^{2r}}{2^k} \int_{E_Q} |f(y)| \, dy \langle w \rangle_Q.
\]

(The "\(\leq\)" is Lemma 3.3 and the "\(\approx\)" is Lemma 4.2). Thus for fixed \(k\) we have – using the pairwise disjointness of the sets \(\{E_Q : Q \in S_k\}\):

\[
\sum_{Q \in S_k} \langle f \rangle_Q w(G' \cap Q_t) \leq \frac{2^{2r}}{2^k} \int_{\mathbb{R}^d} |f(y)| Mw(y) \leq \frac{2^{2r}}{2^k}.
\]

This can be summed in \(k \geq 10 \cdot 2^t\) to the desired estimate.

We must now handle the portion involving \(Q \setminus Q_t\). Note that for fixed \(l\) and \(k\), the sets \(\{E_Q' : Q' \in S_k^{i+1}(Q)\) and \(Q \in S_k^i, j \geq 0\) are pairwise disjoint. Thus for fixed \(k\) we have:

\[
\sum_{j \geq 0} \sum_{Q \in S_k^i} \sum_{l=0}^{l-1} \sum_{Q' \in S_k^{i+1}, Q' \subset Q} \langle f \rangle_Q w(G' \cap E_Q') \leq 4^{-k} w(G)^{-1} t \sum_{j \geq 0} \sum_{Q \in S_k^i} w(G' \cap E_Q),
\]

where the sets \(E_Q\) are pairwise disjoint according to the observation above. Therefore this term is bounded by \(4^{-k} t w(G)^{-1} w(G') \leq 2^{-k}\). This can be summed in \(k \geq 10 \cdot 2^t\) to the desired estimate.

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