Isospin mass differences of singly heavy baryons

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We study the isospin mass differences of singly heavy baryons, based on a pion mean-field approach. We consider both the electromagnetic interactions and the hadronic contributions that arise from the mass difference of the up and down quarks. The relevant parameters have been already fixed by the baryon octet. In addition, we introduce the strong hyperfine interactions between the light quarks inside a chiral soliton and the Coulomb interactions between the chiral soliton and a heavy quark. The numerical results are in good agreement with the experimental data. In particular, the results for the neutral mass relations, which contain only the electromagnetic contributions, are in remarkable agreement with the data, which implies that the pion mean field approach provides a good description of the singly heavy baryons.

Keywords: heavy baryons, isospin mass differences, pion mean fields
I. INTRODUCTION

Since W. Heisenberg proposed isospin symmetry to deal with the proton and the neutron on an equal footing \[1\], isospin symmetry has played an essential role in classifying the identities of newly found hadrons \[2,3\]. The breaking of isospin symmetry is attributed to two difference sources: one comes from the electromagnetic (EM) interaction \[4\], which however caused several conflicts with experiments such as the mass difference between the proton and the neutron \[6,9\]. After the advent of quantum chromodynamics, the current-quark mass difference of the up and down quarks \[10,11\] was known to the other source of isospin symmetry breaking. To explain the isospin mass differences of singly heavy baryons, one has to consider these two contributions in a consistent manner. The masses of singly heavy baryons have been extensively investigated within various different theoretical approaches well over decades \[12,37\]. While most of these works were based on variants of the constituent quark models with phenomenological inter-quark interactions, the MIT Bag model \[24\] and chiral perturbation theory \[35\] were also used to compute the isospin mass differences of singly heavy baryons. As shown in a seminal paper \[38\], the masses of hadrons have been described based on various different strong and EM quark-quark interactions. This idea was also applied to the description of the isospin mass differences of singly heavy baryons. The quark-quark interactions in describing theses differences consist of the chromomagnetic, EM hyperfine, and Coulomb interactions. Thus, it is inevitable to introduce several parameters to be fixed. However, it is of great importance to reduce uncertainties arising from these parameters such that the isospin mass differences of singly heavy baryons should be predicted consistently both in the charmed and beauty sectors.

Recently, a pion mean-field approach has been developed to explain various properties of singly heavy baryons \[39-47\]. This approach has a great virtue that the light and heavy baryons are investigated within the same framework without introducing almost no free parameter. For example, it was shown in Ref. \[39\] that all the parameters can be taken from those already determined in the light baryon sector. The only parameter that was introduced additionally was the heavy quark hyperfine interactions to remove the spin degeneracy in the baryon sextet. The principal idea of the pion mean-field approach was first proposed by Witten \[48,49\]. In this approach, a singly heavy baryon can be viewed as a state consisting of \(N_c - 1\) valence quarks bound by the pion mean fields, which are created in the presence of the \(N_c - 1\) valence quarks self-consistently, while a heavy quark inside a singly heavy baryon is considered to be a mere static color source in the limit of the infinitely heavy quark mass. The SU(3) representations of singly heavy baryons, i.e., the baryon antitriplet and sextet naturally appear in the pion mean-field approach.

In the present work, we want to scrutinize the isospin mass differences of singly heavy baryons, based on the pion mean-field approach. As mentioned previously, the light and heavy baryons are treated on an equal footing within this approach. This means that we can use the fixed parameters for the EM corrections to the baryon octet, which was already done in Ref. \[50\]. These parameters provide directly the EM corrections to the isospin mass differences of singly heavy baryons, which come from two light quarks inside a singly heavy baryon. Moreover, the parameters, which are responsible for contributions from the mass difference of the up and down quarks, were also determined already by describing the isospin mass differences of the baryon octet and decuplet. Therefore, in the present work, we only need to introduce two physical parameters, which arise from the strong hyperfine interactions between the two light quarks inside a chiral soliton, and the EM Coulomb interactions between the soliton and a heavy quark for both the charmed and beauty baryons. We will select two experimental data on the isospin mass differences of the charmed baryons to fix these two parameters. With all these fixed parameters, we will show that we describe very well the remaining isospin mass differences of the charmed baryons. Then the isospin mass differences of the beauty baryons are well predicted within the present framework.

The present work is organized as follows: In Section I I we briefly show how to compute the isospin mass differences of singly heavy baryons from the pion mean-field approach. In addition, we derive various mass sum rules such as the Coleman-Glashow-like and Guadagnini-like ones. In Section I I I we present the numerical results for the isospin mass differences of both the charmed and beauty baryons. We discuss in detail each contribution to them, comparing the corresponding result with the existing experimental data. We finally compare the present results with those from other works. The last Section is devoted to the summary and conclusion of the present work.

II. PION MEAN FIELDS AND ISOSPIN SYMMETRY BREAKING

As mentioned previously, the effects of isospin symmetry breaking are attributed to two different sources: the EM interaction and the difference of the up and down quark masses. Thus, we will first consider the corrections to the masses of the singly heavy baryons from the EM self-energies. The EM corrections to SU(3) baryon masses were already discussed in Ref. \[50\]. As will be shown in this work, the present pion mean-field approach has a great virtue in dealing with the light and heavy baryons on an equal footing also in the case of the isospin symmetry breaking due
to the EM interactions. In general, the EM contribution to a baryon mass is expressed as \[ \mathcal{M}^\text{EM}_B = \frac{1}{2} \int d^3x \, d^3y \, \langle B | J^\mu(x) J^\nu(y) | B \rangle \, D_{\gamma}(x,y) = \langle B | \mathcal{O}^\text{EM} | B \rangle, \] (1)

where \( |B \rangle \) denotes the baryon state and \( J^\mu \) the EM current defined as \[ J^\mu(x) = e \bar{\psi}(x) \gamma_\mu \tilde{\Psi}(x), \] (2)

with the electric charge \( e \) and the quark charge operator \[ \tilde{\Psi} = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix} = \frac{1}{2} \left( \lambda_3 + \frac{1}{\sqrt{3}} \lambda_8 \right). \] (3)

Equation (3) is the well-known Gell-Mann-Nishijima relation. \( \lambda_3 \) and \( \lambda_8 \) denote the Gell-mann matrices in SU(3). The \( D_{\gamma} \) represents the static photon propagator. This is absorbed in parameters that can be fitted to the experimental data. Since the EM current is considered as an flavor octet operator, we write the most general form of the \( \mathcal{O}^\text{EM} \) as a collective operator \[ \mathcal{O}^\text{EM} = \alpha_1 \sum_{i=1}^{3} D_{Q_i}^{(8)} D_{Q_i}^{(8)} + \alpha_2 \sum_{p=4}^{7} D_{Q_p}^{(8)} D_{Q_p}^{(8)} + \alpha_3 D_{Q_8}^{(8)} D_{Q_8}^{(8)}, \] (4)

where \( D_{Q_a}^{(8)} \) denotes the combinations of the SU(3) Wigner \( D \) functions, which are defined by \[ D_{Q_a}^{(8)}(A) = \frac{1}{2} \left( D_{3a}^{(8)}(A) + \frac{1}{\sqrt{3}} D_{8a}^{(8)}(A) \right). \] (5)

Here, \( A \) designates the rotation in flavor SU(3) space. The parameters \( \alpha_i \) encode specific dynamics of chiral solitonic models and contain the photon static propagator. They were already fixed by the empirical data on the EM mass differences of the baryon octet [50]. The product of two octet operators can be expanded as the SU(3) Clebsch-Gordan (CG) series: \( 1 \oplus 8_s \oplus 8_s \oplus 10 \oplus \overline{10} \oplus 27 \). Note, however, that because of Bose symmetry we are left only with the singlet, the octet, and the eikosiheptaplet (27), which are all symmetric. Thus, \( \mathcal{O}^\text{EM} \) contains only the CG series \( 1 \oplus 8_s \oplus 27 \), written as \[ \mathcal{O}^\text{EM} = c^{(27)} \left( \sqrt{5} D_{\Sigma_3}^{(27)} + \sqrt{3} D_{\Lambda_2}^{(27)} + D_{\Sigma_3}^{(27)} \right) + c^{(8)} \left( \sqrt{3} D_{\Sigma_1}^{(8)} + D_{\Lambda_1}^{(8)} \right) + c^{(1)} D_{\Lambda_1}^{(1)}. \] (6)

The explicit definitions of the Wigner \( D \) functions \( D_{B,B}^{(R)}(A) \) can be found in Ref. [52]. The new set of parameters \( c^{(R)} \) can be expressed in terms of \( \alpha_i \). The superscript \( (R) \) stands for the corresponding irreducible representation. The last term in Eq. (6) does not contribute to the mass splittings due to isospin symmetry breaking, since the contributions with that are cancelled out. The EM masses of singly heavy baryons can be obtained by sandwiching the collective wavefunction to the heavy quark spinor such that the heavy baryon state becomes a color singlet. Thus, the wavefunction for a singly heavy baryon should be written as \[ \Psi^{(R)}_{B_h}(A) = \sum_{J'_s, J''_s} C^{J'_s J''_s}_{J_s J_s} \chi_{J_s} \psi^{(R; Y, T, T_3 = 3)}_{Y, J_s} \psi_{(R; Y, T, T_3)}(A), \] (8)

where \( \chi_{J_s} \) denote the Pauli spinors and \( C^{J'_s J''_s}_{J_s J_s} \) the CG coefficients. The subscript \( h \) denotes generically a heavy baryon with both charmed and beauty quarks.
Having computed the matrix elements of the collective EM operator $\mathcal{O}^{EM}$ between the collective wavefunctions for singly heavy baryons given in Eq. (8), we obtain the EM mass corrections

$$M^\text{EM}_{A_{h\text{sol}}} = \frac{1}{4} c^{(8)} + c^{(1)},$$

$$M^\text{EM}_{\Sigma_{h\text{sol}}} = \frac{3}{4} \left( T_3 - \frac{1}{6} \right) c^{(8)} + c^{(1)},$$

$$M^\text{EM}_{\Xi_{h\text{sol}}} = \frac{3}{10} \left( T_3 + \frac{1}{3} \right) c^{(8)} + \frac{1}{9} \left( T_3^2 + \frac{1}{5} T_3 - \frac{3}{5} \right) c^{(27)} + c^{(1)},$$

$$M^\text{EM}_{\Omega_{h\text{sol}}} = -\frac{1}{5} \left( c^{(8)} - \frac{1}{9} c^{(27)} \right) + c^{(1)},$$

(9)

to the baryon antitriplet, and

$$M^\text{EM}_{\Sigma^{(\gamma)}_{h\text{sol}}} = \frac{3}{10} \left( T_3 - \frac{1}{6} \right) c^{(8)} - \frac{2}{45} \left( T_3^2 + 2T_3 + \frac{1}{4} \right) c^{(27)} + c^{(1)},$$

$$M^\text{EM}_{\Omega^{(\gamma)}_{h\text{sol}}} = -\frac{1}{5} \left( c^{(8)} - \frac{1}{9} c^{(27)} \right) + c^{(1)},$$

(10)
to the baryon sextet. Note that the EM mass corrections we consider here are only taken from the chiral soliton that consists of the light quarks inside the singly heavy baryons. It is of great interest to see that the eikosiheptaplet parts of $\mathcal{O}^{EM}$ do not contribute to the baryon antitriplet. The EM corrections have the same effects on the baryon sextet with both spin $1/2$ and $3/2$. The contributions from the flavor singlet with $c^{(1)}$ are absorbed in the classical mass of the chiral soliton.

We now turn to the mass corrections from the isospin symmetry breaking by the mass difference between up and down quarks. We call them the hadronic (H) corrections to the masses of the singly heavy baryons. As discussed in Ref. [52], the collective Hamiltonian with isospin symmetry breaking is expressed as

$$H^\text{iso}_{sb} = (m_d - m_u) \left( \frac{\sqrt{3}}{2} \alpha D^{(8)}_{38} (R) + \beta T_3 + \frac{1}{2} \gamma \sum_{i=1}^{3} D^{(8)}_{3i} (R) \hat{J}_i \right),$$

(11)

where $\alpha$, $\beta$, and $\gamma$ can be fixed by using the experimental data on the masses of the baryon octet. When it comes to the mass spectra of the singly heavy baryons, the presence of $N_c - 1$ valence quarks instead of $N_c$ valence quarks makes the pion mean fields undergo changes, as mentioned previously in Introduction. While parameters $\beta$ and $\gamma$ are kept intact in the course of changing the mean fields, $\alpha$ should be modified. Therefore, as we did for the dynamical parameters for SU(3) symmetry breaking in Ref. [39], we have to replace the $N_c$ factor by $N_c - 1$, which reflects the presence of the $N_c - 1$ valence quarks. Thus, we use $\bar{\alpha}$ to distinguish from the original parameter $\alpha$. Using the results from Ref. [52], we can immediately obtain the numerical values of $\bar{\alpha}$, $\beta$, and $\gamma$ as

$$(m_d - m_u) \bar{\alpha} = (-2.93 \pm 0.002) \text{ MeV},$$

$$(m_d - m_u) \beta = (-2.41 \pm 0.001) \text{ MeV},$$

$$(m_d - m_u) \gamma = (-1.74 \pm 0.006) \text{ MeV}.$$

(12)

We can redefine the parameters for each contribution to the baryon antitriplet and sextet, i.e. $\bar{\delta}_5^\text{iso}$ and $\bar{\delta}_6^\text{iso}$, which are explicitly written as

$$\bar{\delta}_5^\text{iso} = \frac{9}{16} (m_d - m_u) \left( \bar{\alpha} + \frac{16}{9} \beta \right),$$

$$\bar{\delta}_6^\text{iso} = \frac{9}{40} (m_d - m_u) \left( \bar{\alpha} + \frac{40}{9} \beta - \frac{4}{3} \gamma \right).$$

(13)

Note that $\gamma$ contributes only to the baryon sextet. As displayed in Eq. (11), the last term with $\gamma$ is related to the spin of a singly heavy baryon. Since the light quarks in the baryon antitriplet compose the spin $J_{sol} = 0$ state, i.e. a soliton with $J_{sol} = 0$, the baryon antitriplet does not get any contribution from $\gamma$.

The baryon sextet with spin $J = 1/2$ or $J = 3/2$ is constructed by the soliton with $J_{sol} = 1$ and a heavy quark with $J_h = 1/2$ ($J = J_{sol} + J_h$), whereas the baryon antitriplet with spin $1/2$ is formed by combining the soliton with $J_{sol} = 0$ and a heavy quark with $J_h = 1/2$. This implies that the contributions of the strong hyperfine interactions should come into play due to the spin arrangement of the light quarks. In Ref. [39], we did not need to consider the strong hyperfine splittings between the light quarks, since Ref. [39] investigated the mass splittings of the singly...
heavy baryons only within each flavor SU(3) representation. On the other hand, it is essential to take into account the strong hyperfine interactions between the light quarks in the present case, because the configuration of the light-quark spin will definitely affect the mass of each baryon in the baryon antitriplet and sextet, as shown in various quark models [21, 25, 37]. Thus, we introduce explicitly the strong hyperfine interaction for the light quarks inside a heavy baryon

\[ H_{\text{hf}} = \delta_{\text{hf}} S_1 \cdot S_2, \]

where \( S_1 \) and \( S_2 \) denote the spin operators for the light quarks inside a soliton, which yield the soliton spin \( J_{\text{sol}} = S_1 + S_2 \). Parameter \( \delta_{\text{hf}} \) contains the masses of the up and down quarks, and the strength of the strong hyperfine interaction. However, we will simply fit it to the experimental data, which will be shown soon.

The EM interactions between the soliton and a heavy quark should be involved in the isospin symmetry breaking of corresponding isospin mass differences, whereas the fourth and fifth ones list the contributions of the up and down quark mass differences, respectively. The last column lists the corrections from the Coulomb interaction of which the form is taken to be

\[ H_{\text{Coul}} = \alpha_{\text{sol-h}} \hat{Q}_{\text{sol}} \hat{Q}_h, \]

where \( \hat{Q}_{\text{sol}} \) and \( \hat{Q}_h \) represent the charge operators acting on the soliton and a heavy quark, respectively. \( \alpha_{\text{sol-h}} \) consists of the expectation value of the inverse distance and the fine structure constant. However, we will determine its value by adjusting it to the experimental data.

The parameters \( \delta_{\text{hf}} \) and \( \alpha_{\text{sol-h}} \) can be fixed by using the experimental data on the isospin mass differences [55]:

\[
\begin{align*}
M_{\Xi^+} - M_{\Xi^0} &= (-2.98 \pm 0.22) \text{ MeV}, \\
M_{\Sigma^+} - M_{\Sigma^0} &= (-0.9 \pm 0.4) \text{ MeV}.
\end{align*}
\]

Thus, they are determined to be

\[
\begin{align*}
\delta_{\text{hf}} &= (0.40 \pm 0.06) \text{ MeV}, \\
\alpha_{\text{sol-h}} &= (2.76 \pm 0.28) \text{ MeV}.
\end{align*}
\]

We want to emphasize that except for these two parameters we have no free parameters at all, since all other parameters were fixed in the light baryon sector. In Table I, we compile the formulae for each contribution to the isospin mass differences.

### Table I: Compilation of the formulae for each contribution to the isospin mass differences in the baryon antitriplet and sextet

| \( \mathcal{R},J \) | \( B_h \) | \( \Delta M_{\text{sol}} \Delta M_{\text{sb}} \Delta M_{\text{iso}} \Delta M_{\text{hf}} \Delta M_{\text{Coul}} \) |
|---|---|---|---|---|
| \( \mathcal{F}_{1/2} \) | \( \Xi^+ - \Xi^0 \) | \( \frac{3}{4} c(8) \) | \( \delta_{\text{iso}} \) | \(-3\delta_{\text{hf}} \) | \( 2\alpha_{\text{iso}} \) |
| \( \Xi^0 - \Xi^0 \) | \( \frac{3}{10} \) | \( \frac{4}{9} c(27) \) | \( \delta_{\text{iso}} \) | \( \delta_{\text{hf}} \) | \( \alpha_{\text{iso}} \) |
| \( \Sigma^+ - \Sigma^+ \) | \( \frac{3}{10} \) | \( c(8) \) | \( \delta_{\text{iso}} \) | \( \delta_{\text{hf}} \) | \( \alpha_{\text{iso}} \) |
| \( \Sigma^0 - \Sigma^0 \) | \( \frac{3}{10} \) | \( -c(8) \) | \( c(27) \) | \( \delta_{\text{iso}} \) | \( \delta_{\text{hf}} \) | \( \alpha_{\text{iso}} \) |
| \( \Xi^+ - \Xi^0 \) | \( \frac{3}{10} \) | \( c(8) \) | \( -c(27) \) | \( \delta_{\text{iso}} \) | \( \delta_{\text{hf}} \) | \( \alpha_{\text{iso}} \) |
| \( \Xi^0 - \Xi^0 \) | \( \frac{3}{10} \) | \( c(8) \) | \( c(27) \) | \( \delta_{\text{iso}} \) | \( \delta_{\text{hf}} \) | \( \alpha_{\text{iso}} \) |
| \( \Sigma^+ - \Sigma^0 \) | \( \frac{2}{c(27)} \) | \( c(8) \) | \( \delta_{\text{iso}} \) | \( \delta_{\text{hf}} \) | \( \alpha_{\text{iso}} \) |
| \( \Sigma^0 - \Sigma^0 \) | \( \frac{2}{c(27)} \) | \( c(8) \) | \( c(27) \) | \( \delta_{\text{iso}} \) | \( \delta_{\text{hf}} \) | \( \alpha_{\text{iso}} \) |
differences in the baryon antitriplet and sextet representations.

It is of great interest to examine the mass sum rules or the relations between the isospin mass differences of the singly heavy baryons. The present work provide various relations among isospin multiplets. When $\Delta T_3 = 1$, we find the following relations

\[
M_{\Sigma^{++}} - M_{\Sigma^0} = M_{\Sigma^{'++}} - M_{\Sigma'{}^0}, \quad (18)
\]

\[
M_{\Sigma^+_b} - M_{\Xi^0_b} = M_{\Xi^{++}_b} - M_{\Xi^{+0}_b}, \quad (19)
\]

In Eq. (18), $M_{\Sigma^{++}} - M_{\Sigma^+}$ and $M_{\Sigma^{'++}} - M_{\Sigma'{}^+}$ belong to the baryon sextet with spin 1/2 and 3/2, respectively. In Eq. (19), we gives the same relation for the beauty sextet. We can obtain another relation for the isospin mass differences of the charmed baryons in the sextet as follows:

\[
\begin{align*}
M_{\Sigma^{++}_c} - M_{\Sigma^0_c} &= M_{\Sigma^{++}_c} - M_{\Sigma^0_c}, \\
(-0.9 \pm 0.4) \text{ MeV} &= (-0.8 \pm 0.6) \text{ MeV} \quad (\text{20})
\end{align*}
\]

Having inserted the experimental values into each term in Eq. (20), we find that the relation is in good agreement with the data. We also observe that for $\Delta T_3 = 1$ the $\Sigma_b$ and $\Xi_b$ multiplets in the sextet satisfy the following relation:

\[
M_{\Sigma^0_b} - M_{\Sigma^+_b} = M_{\Xi^0_b} - M_{\Xi^+_b} = M_{\Xi^0_b} - M_{\Xi^+_b}, \quad (21)
\]

which are simply the same as that in Eq. (20) except for the heavy-quark flavor. The relations given in Eqs. (20) and (21) remind us the Coleman-Glashow mass formula [6] that relates the isospin mass differences of the baryon octet. As for the $\Delta T_3 = 2$, we obtain the relations for the baryon sextet with both spin 1/2 and 3/2:

\[
\begin{align*}
M_{\Sigma^{++}_b} - M_{\Sigma^0_b} &= M_{\Sigma^{'++}_b} - M_{\Sigma'{}^0_b}, \\
(0.220 \pm 0.013) \text{ MeV} &= (0.01 \pm 0.15) \text{ MeV} \quad (\text{22})
\end{align*}
\]

\[
\begin{align*}
M_{\Sigma^+_b} - M_{\Xi^0_b} &= M_{\Sigma^{'++}_b} - M_{\Xi'{}^0_b}, \\
(-5.06 \pm 0.18) \text{ MeV} &= (-4.37 \pm 0.33) \text{ MeV} \quad (\text{23})
\end{align*}
\]

We put the experimental values to check the relations. As shown in the underbraces, it seems that they deviate slightly from the experimental data.

There is also a neutral mass sum rule [12], in which each sum of the heavy baryon masses have the neutral charge:

\[
(M_{\Sigma^{++}} + 2M_{\Sigma^0} - 2M_{\Sigma^+_c}) = (M_{\Sigma^{++}} + 2M_{\Sigma^0} - 2M_{\Sigma^+_c})
\]

\[
= (M_{\Sigma^+_b} + 2M_{\Xi^0_b} - 2M_{\Xi^+_b}) = (M_{\Sigma^+_b} + 2M_{\Xi^0_b} - 2M_{\Xi^+_b})
\]

\[
= \frac{1}{2}(M_{\Sigma^{++}} + 2M_{\Sigma^0} - 2M_{\Sigma^+_c}) \approx \frac{3}{2}(M_{\Sigma^{++}} + 2M_{\Xi^0} - 2M_{\Xi^+_b}). \quad (24)
\]

The neutral mass sum rule in Eq. (24) is usually given for the charmed baryons. However, we can extend this sum rule by including the beauty baryons as shown in Eq. (24). The neutral mass sum rule has a unique feature. All the hadronic contributions are cancelled each other, so that only the EM corrections remain [11]. Thus, Eq. (24) can be considered as the EM mass relation. As will be explicitly shown later, the neutral mass sum rule in Eq. (24) is in remarkable agreement with the experimental data. It implies that the present treatment for the EM contributions, which was determined already in the light baryon sector, describes universally well both the EM isospin mass differences of light and heavy baryons.

The Guadagnini mass formula [57] relates the mass difference in the baryon decuplet to that in the octet. Similarly, we find that the isospin mass differences in the baryon antitriplet can be related to those in the baryon sextet as
follows:

$$
\left( M_{\Xi^+} - M_{\Xi^0} \right) - \left( M_{\Xi_b^0} - M_{\Xi_b^-} \right) = \frac{1}{2} \left[ \left( M_{\Sigma_c^{(*)}++} - M_{\Sigma_c^{(*)}0} \right) - \left( M_{\Sigma_b^{(*)}+} - M_{\Sigma_b^{(*)}0} \right) \right] 
(2.92 \pm 0.64) \text{ MeV}
$$

$$
= \frac{1}{2} \left[ \left( M_{\Sigma_c^{(*)}+} - M_{\Sigma_c^{(*)}0} \right) - \left( M_{\Sigma_b^{(*)}+} - M_{\Sigma_b^{(*)}0} \right) \right] 
(2.64 \pm 0.09) \text{ MeV}
$$

$$
= \frac{1}{2} \left[ \left( M_{\Sigma_c^{(*)}++} - M_{\Sigma_c^{(*)}0} \right) - \left( M_{\Sigma_b^{(*)}+} - M_{\Sigma_b^{(*)}0} \right) \right] 
(2.19 \pm 0.18) \text{ MeV}
$$

$$
= \frac{1}{2} \left[ \left( M_{\Sigma_c^{(*)}+} - M_{\Sigma_c^{(*)}0} \right) - \left( M_{\Sigma_b^{(*)}+} - M_{\Sigma_b^{(*)}0} \right) \right] 
(2.29 \pm 0.17) \text{ MeV}
$$

$$
= \frac{1}{2} \left[ \left( M_{\Sigma_c^{(*)}++} - M_{\Sigma_c^{(*)}0} \right) - \left( M_{\Sigma_b^{(*)}+} - M_{\Sigma_b^{(*)}0} \right) \right] 
(2.53 \pm 0.12) \text{ MeV}
$$

$$
= \left( M_{\Xi^+} - M_{\Xi^0} \right) - \left( M_{\Xi_b^0} - M_{\Xi_b^-} \right), \quad (25)
$$

which are well satisfied. We can also obtain two more mass relations among the baryon sextet:

$$
\left( M_{\Sigma_c^{(*)}++} - M_{\Sigma_c^{(*)}+} \right) + \left( M_{\Xi^0} - M_{\Xi^0} \right) = \frac{1}{2} \left[ \left( M_{\Sigma_c^{(*)}++} - M_{\Sigma_c^{(*)}0} \right) + \left( M_{\Sigma_b^{(*)}+} - M_{\Sigma_b^{(*)}0} \right) \right], \quad (26)
$$

$$
\left( M_{\Sigma_c^{(*)}+} - M_{\Sigma_c^{(*)}0} \right) = \left( M_{\Sigma_b^{(*)}+} - M_{\Sigma_b^{(*)}0} \right) + \left( M_{\Xi^0} - M_{\Xi^0} \right). \quad (27)
$$

### III. RESULTS AND DISCUSSION

We are now in a position to discuss the results from the present work. In Table II we list the results of each contribution to the isospin mass differences of the charmed baryon antitriplet and sextet. We want to emphasize again that except for the contributions from the strong hyperfine interactions, $\Delta M_{hf}$, and the Coulomb interactions, $\Delta M_{Coul}^{\text{sol-h}}$, all the other terms were determined by using the parameters fixed already in the light baryon sectors. Comparing the values of $\Delta M_{sol}^{\text{EM}}$ with the experimental data that are listed in the eighth and last columns, we interestingly observe that the EM corrections from the light quarks describe almost quantitatively the experimental data apart from the baryon antitriplet. It indicates that the hadronic contributions from the mass difference of the up and down quarks, $\Delta M_{sb}^{\text{iso}}$, are almost cancelled out by both the strong hyperfine interactions, $\Delta M_{hf}$, and the Coulomb interactions between the soliton and a heavy quark, $\Delta M_{Coul}^{\text{sol-h}}$. However, the small amounts that are left after the cancellations are rather important to describe the experimental data quantitatively. On the other hand, the effects of the hadronic part that includes $m_d - m_u$ are the most dominant ones. The total results are generally in good agreement with the experimental data. As explained previously, the neutral mass differences, $M_{\Sigma_c^{(*)}++} + M_{\Sigma_c^{(*)}+} - 2M_{\Sigma_c^{(*)}0}$ and $M_{\Sigma_b^{(*)}+} + M_{\Sigma_b^{(*)}0} - 2M_{\Sigma_b^{(*)}+}$, only contain the EM contributions. We emphasize that all the parameters have been fixed already by reproducing the EM mass differences for the baryon octet. Moreover, the present results are in remarkably good agreement with the experimental data, as shown in Table II. This implies that the present pion mean-field approach indeed explains consistently both the SU(3) light baryons and the singly heavy baryons.

Table III lists the results of each contribution to the isospin mass differences of the beauty baryon antitriplet and sextet. Since we have already fixed the parameters for the strong hyperfine and Coulomb interactions, the results listed in Table III are the predictions of the present work. Note that there are only three experimental data. In addition we can extract one more data by using the experimental values of the $\Xi^*_b^{0}$ and $\Xi^*_b^{-}$ masses. The present results describe the experimental data very well. For example, the value of the isospin mass difference of the beauty baryon antitriplet is in very good agreement with the corresponding data by approximately 3%. This implies that heavy-quark flavor symmetry is well satisfied. Note that in the present work, we do not have any contribution from heavy quark mass corrections such as chromoelectric or chromomagnetic interactions that are proportional to $1/m_b$. 
TABLE II. Isospin mass differences of the charmed baryon antitriplet and sextet in units of MeV. The first column shows the SU(3) representations of the singly heavy baryons. The second one indicates the isospin mass difference in a given representation. The third one lists the results of the electromagnetic corrections of the light quarks to the mass of a singly heavy baryon, $\Delta M_{EM}$. The fourth one gives the corrections from the mass difference of the up and down quarks, $\Delta M_{iso}$. The fifth one lists the corrections from the strong hyperfine interactions between the light quarks, $\Delta M_{hf}$. The sixth one presents the Coulomb interactions between the soliton and a heavy quark, $\Delta M_{sol}$. The seventh one lists the total results of the isospin mass differences of the charmed baryon antitriplet and sextet. The eighth one lists the corresponding experimental data on the isospin mass differences taken from the Particle Data Group (PDG) [55]. In the last column, we list the derived values of the isospin mass differences, using the experimental data on the masses of the corresponding singly heavy baryons.

| $\mathcal{R}_J$ | $B_b$ | $\Delta M_{EM}$ | $\Delta M_{iso}$ | $\Delta M_{hf}$ | $\Delta M_{sol-h}$ | $\Delta M_{total}$ | PDG [55] | PDG$^\dagger$ |
|----------------|-------|-----------------|-----------------|-----------------|-------------------|-----------------|----------|----------|
| $3_{1/2}$      | $\Xi_b^+ - \Xi_b^0$ | -0.11 ± 0.17    | 3.51 ± 0.18     | -0.92 ± 0.09    | -5.74 ± 0.27      | -5.9 ± 0.6      | -         | -         |
|                | $\Sigma_b^+ - \Sigma_b^0$ | 1.10 ± 0.33    | -2.33 ± 0.40    | 0.92 ± 0.09     | -1.74 ± 0.34      | -                | -         | -         |
| $6_{1/2}$      | $\Xi_b^0 - \Xi_b^-$ | -0.81 ± 0.22   | -2.33 ± 0.40    | -0.92 ± 0.09    | -3.66 ± 0.25      | -                | -         | -         |
|                | $\Xi_b^+$ - $\Xi_b^0$ | 0.81 ± 0.22    | -2.33 ± 0.40    | -0.92 ± 0.09    | -3.66 ± 0.25      | -                | -         | -         |
|                | $\Sigma_b^+ - \Sigma_b^0$ | 0.29 ± 0.17    | -4.66 ± 0.80    | 1.84 ± 0.19     | -5.40 ± 0.28      | -5.06 ± 0.18    | -         | -         |
|                | $\Sigma_b^+ + \Sigma_b^- - 2\Sigma_b^0$ | 1.92 ± 0.53    | 0               | 0               | 1.92 ± 0.53       | -1.89 ± 0.64    | -         | -         |

TABLE III. Isospin mass differences of the beauty baryon antitriplet and sextet in units of MeV. The notations are the same as in Table II.

| $\mathcal{R}_J$ | $B_b$ | $\Delta M_{EM}$ | $\Delta M_{iso}$ | $\Delta M_{hf}$ | $\Delta M_{sol-h}$ | $\Delta M_{total}$ | PDG [55] | PDG$^\dagger$ |
|----------------|-------|-----------------|-----------------|-----------------|-------------------|-----------------|----------|----------|
| $3_{1/2}$      | $\Xi_b^+ - \Xi_b^0$ | -0.11 ± 0.17    | 3.51 ± 0.18     | -0.92 ± 0.09    | -5.74 ± 0.27      | -5.9 ± 0.6      | -         | -         |
|                | $\Sigma_b^+ - \Sigma_b^0$ | 1.10 ± 0.33    | -2.33 ± 0.40    | 0.92 ± 0.09     | -1.74 ± 0.34      | -                | -         | -         |
| $6_{1/2}$      | $\Xi_b^0 - \Xi_b^-$ | -0.81 ± 0.22   | -2.33 ± 0.40    | -0.92 ± 0.09    | -3.66 ± 0.25      | -                | -         | -         |
|                | $\Xi_b^+$ - $\Xi_b^0$ | 0.81 ± 0.22    | -2.33 ± 0.40    | -0.92 ± 0.09    | -3.66 ± 0.25      | -                | -         | -         |
|                | $\Sigma_b^+ - \Sigma_b^0$ | 0.29 ± 0.17    | -4.66 ± 0.80    | 1.84 ± 0.19     | -5.40 ± 0.28      | -5.06 ± 0.18    | -         | -         |
|                | $\Sigma_b^+ + \Sigma_b^- - 2\Sigma_b^0$ | 1.92 ± 0.53    | 0               | 0               | 1.92 ± 0.53       | -1.89 ± 0.64    | -         | -         |

To compare the present results with the experimental data more visibly, we illustrate in Fig. [4] the results for the isospin mass difference of both the charmed and beauty baryons, which we have already discussed in Tables II and III.
FIG. 1. Comparison of the present results with the corresponding experimental data. The $x$-axis denotes the values of the isospin mass differences between singly heavy baryons in units of MeV whereas the $y$-axis designates the corresponding mass difference of the isospin multiplet. The filled circles stand for the experimental data taken from the PDG [55], the filled triangles represent the data obtained by using the experimental values of the masses of the corresponding heavy baryons [55], and the open circles designate the data taken to be as input. The shaded rectangles represent the present results.
TABLE IV. Comparison of the present results for the isospin mass differences of the charmed baryon antitriplet and sextet with those from a quark model (QM), a relativized quark model (RQM), the semirelativistic quark model (SRQM), a constituent quark model (CQM), a potential model (PM), the chiral bag model (χBM), and the MIT bag model in units of MeV. The numerical result of the mass difference for Ξ_{−0} is given as (−0.2 ± 0.6) MeV.

| \( \mathcal{R}_J \) | \( B_c \) | PDG | PDG | This work | QM | RM | SRQM | CQM | PM | χBM | MIT |
|---|---|---|---|---|---|---|---|---|---|---|---|
| \( 3_{1/2} \) | \( \Xi_c^+ - \Xi_c^0 \) | 2.98 ± 0.22 | 3.00 ± 0.43 | input | 3.20 | 2.6 | 1.2 | 1.56 | 3.5 | 0.82 |
| \( 6_{1/2} \) | \( \Sigma_c^+ - \Sigma_c^0 \) | 1.07 ± 0.42 | 1.02 ± 0.38 | 1.05 | 1.2 | 1.56 | 3.5 | 0.82 |
| \( 6_{3/2} \) | \( \Xi_c^+ - \Xi_c^0 \) | 0.91 ± 2.31 | 1.02 ± 0.38 | 0.81 | 1.7 | 0.9 | 3.3 | 0.63 |
| \( \Sigma_c^+ - \Sigma_c^0 \) | 1.92 ± 0.82 | 1.92 ± 0.53 | 1.92 ± 0.53 | 1.92 ± 0.53 |

TABLE V. Comparison of the present results for the isospin mass differences of the beauty baryon antitriplet and sextet with those from the quark model (QM), heavy-quark effective theory (HQET), chiral perturbation theory (χPT), a relativized quark model (RQM), the semirelativistic quark model (SRQM), a potential model (PM), and the chiral bag model (χBM) in units of MeV.

| \( \mathcal{R}_J \) | \( B_b \) | PDG | PDG | This work | QM | HQET | χPT | RQM | SRQM | PM | χBM |
|---|---|---|---|---|---|---|---|---|---|---|---|
| \( 3_{1/2} \) | \( \Xi_b^0 - \Xi_b^+ \) | 5.9 ± 0.6 | 5.10 ± 1.03 | 5.74 ± 0.27 | 6.16 | 6.9 ± 1.1 | 5.39 | 5.39 |
| \( \Sigma_b^+ - \Sigma_b^0 \) | 1.74 ± 0.34 | 2.17 | - | - | - | - | - | - | - | - | -
| \( \Xi_b^0 - \Xi_b^+ \) | 3.66 ± 0.25 | 3.95 | -4.7 ± 1.0 | -4.9 ± 1.9 | -3.7 | -2.74 | -2.51 | -2.51 | -2.51 | -2.51 | -2.51 |
| \( \Sigma_b^+ - \Sigma_b^0 \) | 5.06 ± 0.18 | 5.08 ± 0.37 | 5.40 ± 0.28 | -6.07 | -5.6 | -3.70 | -3.57 | -7.1 | -7.1 | -7.1 | -7.1 |
| \( \Sigma_b^+ + \Sigma_b^- - 2\Xi_b^0 \) | 1.92 ± 0.53 | 1.92 ± 0.53 | 1.92 ± 0.53 | 1.92 ± 0.53 | 1.92 ± 0.53 |

| \( \Sigma_b^+ - \Sigma_b^0 \) | 1.74 ± 0.34 | 2.02 | - | - | - | - | - | - | - | - | -
| \( \Xi_b^0 - \Xi_b^+ \) | 3.66 ± 0.25 | 3.80 | -3.6 | -3.21 | -3.21 | -3.21 | -3.21 | -3.21 | -3.21 | -3.21 | -3.21 |
| \( \Xi_b^0 - \Xi_b^+ \) | 3.03 ± 0.91 | 3.66 ± 0.25 | -4.01 | -3.08 | -3.08 | -3.08 | -3.08 | -3.08 | -3.08 | -3.08 | -3.08 |
| \( \Sigma_b^+ - \Sigma_b^- \) | 4.37 ± 0.33 | 4.42 ± 0.40 | 5.40 ± 0.28 | -5.85 | -5.4 | -4.30 | -6.5 | -6.5 | -6.5 | -6.5 | -6.5 |
| \( \Sigma_b^+ + \Sigma_b^- - 2\Xi_b^0 \) | 1.92 ± 0.53 | 1.92 ± 0.53 | 1.92 ± 0.53 | 1.92 ± 0.53 | 1.92 ± 0.53 |
IV. SUMMARY AND OUTLOOK

In the present work, we investigated the isospin mass differences of the singly heavy baryons within the framework of a pion mean-field approach. Since there are two different sources that break the isospin symmetry, i.e., the electromagnetic interactions and the mass difference of the up and down quarks, we need to treat them within the same framework. We started with computing the electromagnetic contributions to the masses of the baryon antitriplet and sextet. Because of Bose symmetry of photons, we have only symmetric contributions such as the SU(3) singlet, octet, and the ekiosheptaplet (27). This means that we have three independent parameters, of which the singlet terms can be absorbed in the classical soliton mass. The remaining two parameters can be fixed by using the empirical data on the electromagnetic mass differences of the baryon octet. The hadronic contributions from the mass difference of the up and down quarks can be taken into account perturbatively. These contributions were also fixed by reproducing the masses of the baryon octet. Thus, we have no free parameters related to the light-quark sector. In doing so, we have one free parameter to be fixed by using the experimental data on the isospin mass differences of the charmed baryons. While a heavy quark inside a singly heavy baryon is regarded as a mere static color source, we still need to consider the electromagnetic Coulomb interactions between the soliton and a heavy quark. It brings an additional free parameter, which will be also fixed by using the experimental data as done with that of the strong hyperfine interactions. Once we fixed these two free parameters, we were able to produce all possible isospin mass differences of both the charmed and beauty baryons. We derived sum rules among the isospin mass differences of the baryon antitriplet to those of the baryon sextet. These mass relations resemble the Guadagnini mass formula. These sum rules are in good agreement with the experimental data. In addition, we also obtained the mass relations among the members of the baryon sextet.

With the two parameters fixed by using two of the experimental data, we proceeded to computing the isospin mass differences of the charmed baryons. We observed that the electromagnetic contributions from the light quarks described already the experimental data on the isospin mass differences of the baryon sextet almost quantitatively. Having included all the contributions, we showed that the present results are in very good agreement with the experimental data. Interestingly, the neutral mass relations for the charmed baryons $\Sigma_c^c$, which are written as $M_{\Sigma^c}^c + 4M_{\Sigma^0}^c - 2M_{\Sigma^+}^c + 2M_{\Sigma^*}^c + 2M_{\Sigma^*}^c$, contain only the electromagnetic interactions of the light quarks. This indicates that the neutral mass relations provide a stringent test on the present framework. Indeed, the results of the neutral sum rules are in remarkable agreement with the experimental data. This implies that the present pion mean-field approach explains consistently both the isospin mass differences of the light baryons and singly heavy baryons. Finally, we compared the present results with those from various different works.

ACKNOWLEDGMENTS

The present work was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2019R1A2C1010443 (Gh.-S. Y.), 2018R1A2B2001752, and 2018R1A5A1025563 (H.-Ch.K.)).

Appendix A: Relations of the electromagnetic mass differences

For completeness, we present various relations of the EM mass differences as

$$
\Delta M \left[ \Sigma^c_{++} - \Sigma^c_{00} \right]_{\text{sol}} = \Delta M \left[ \Sigma^c_{++} - \Sigma^c_{00} \right]_{\text{sol}} + \Delta M \left[ \Sigma^c_{++} - \Sigma^c_{00} \right]_{\text{sol}}
$$

$$
= \Delta M \left[ \Sigma^c_{++} - \Sigma^c_{00} \right]_{\text{sol}} + \Delta M \left[ \Sigma^c_{++} - \Sigma^c_{00} \right]_{\text{sol}} + \Delta M \left[ \Sigma^c_{++} - \Sigma^c_{00} \right]_{\text{sol}}
$$

$$
\Delta M \left[ \Sigma^c_{++} - \Sigma^c_{00} \right]_{\text{sol}} = \Delta M \left[ \Sigma^c_{++} - \Sigma^c_{00} \right]_{\text{sol}} + \Delta M \left[ \Sigma^c_{++} - \Sigma^c_{00} \right]_{\text{sol}}
$$

$$
\Delta M \left[ \Sigma^c_{++} - \Sigma^c_{00} \right]_{\text{sol}} = \Delta M \left[ \Sigma^c_{++} - \Sigma^c_{00} \right]_{\text{sol}} + \Delta M \left[ \Sigma^c_{++} - \Sigma^c_{00} \right]_{\text{sol}}
$$

$$
\Delta M \left[ \Sigma^c_{++} - \Sigma^c_{00} \right]_{\text{sol}} = \Delta M \left[ \Sigma^c_{++} - \Sigma^c_{00} \right]_{\text{sol}} + \frac{1}{2} \Delta M \left[ \Sigma^c_{++} - \Sigma^c_{00} \right]_{\text{sol}}
$$

$$
= \Delta M \left[ \Sigma^c_{++} - \Sigma^c_{00} \right]_{\text{sol}} + \Delta M \left[ \Sigma^c_{++} - \Sigma^c_{00} \right]_{\text{sol}} + \frac{1}{2} \Delta M \left[ \Sigma^c_{++} - \Sigma^c_{00} \right]_{\text{sol}}
$$

(A1)
Note that these relations show the heavy-quark flavor symmetry in the limit of $m_Q \rightarrow \infty$.

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