BIRATIONAL MAPS WITH COHEN-MACaulAY GRAPHS

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Abstract. A rational map whose source and image are projectively embedded varieties has a Cohen–Macaulay graph if the Rees algebra of one (hence any) of its base ideals is a Cohen–Macaulay ring. If the map is birational onto the image one considers how this property forces an upper bound on the degree of a representative of the map. In the plane case a complete description is given of the Cremona maps with Cohen–Macaulay graph, while in arbitrary dimension $n$ it is shown that a Cremona map with Cohen–Macaulay graph has degree at most $n^2$.

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Introduction

The overall goal of this work is to establish upper bounds for the degrees of representatives of a birational map. The guiding idea is to obtain such bounds that are

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simply expressed in terms of known numerical invariants, such as dimension, relation type, regularity and so forth.

A more specific goal of this paper is to study how the Cohen–Macaulay property of the graph of the map has any impact on these bounds. Here we think of the graph of the map in terms of the Rees algebra $\mathcal{R}_A(I)$ of a base ideal $I \subset A$ of the map, where $A$ denotes the homogeneous coordinate ring of the source projective variety. The algebra $\mathcal{R}_A(I)$ is sufficiently stable to warrant a bona fide replacement of the graph - this and other aspects of the theory will be explained in the later sections. To avoid any confusion, we clarify that assuming that $\mathcal{R}_A(I)$ is a Cohen–Macaulay ring is not the same as assuming that the bi-projective structure of the graph is a Cohen–Macaulay variety since the latter property is a local property at geometric points off the bi-irrelevant ideal.

We now describe the contents of the sections.

The first section is a recap of the main terminology and tools, with a pointer to the aspects that will play a central role in the statements and proofs of subsequent sections. As such, Section 1.2 brings up how one deals appropriately with the notion of a base ideal and its associated algebraic constructions.

The second section contains the main results. These are bounds for the degrees of the forms defining a rational map in case the map is birational and/or the associated graph is Cohen–Macaulay.

The first of these results gives a full description of the plane Cremona maps having a saturated base ideal $I \subset A$ such that $\mathcal{R}_A(I)$ is Cohen–Macaulay. The proof is mostly a sharp elementary calculation culminating with the upfront use of the main criterion of [DHS]. Later in the section we give a homological argument showing that in the plane case the assumption that $\mathcal{R}_A(I)$ is Cohen–Macaulay actually implies that $I$ be saturated.

The second result states two bounds for a birational map onto the image, the second of these holding when the graph is in addition Cohen–Macaulay. Both bounds seem to be new in the literature.

We then state a crucial lemma which encapsulates a couple of estimates around numerical homological invariants of an ideal of a local ring. This lemma is applied to derive some stronger bounds in the case of a Cremona map in arbitrary dimension whose graph is Cohen–Macaulay. In addition one proves that a plane Cremona map having Cohen–Macaulay graph has a saturated base ideal. Again such results seem to be new.

The third section contains a few ramblings on the subject. We first focus on bounding the defining degrees of a rational map having a base ideal $I$ of grade at least 2 – a hypothesis saying that the map has essentially a unique base ideal. The statement gives an upper bound for the degree of the generators of $I$ by invoking its syzygies.
Next one proceeds to a comparison between some of the bounds and to some questions on the asymptotic behavior of the regularity of the powers of the base ideal of a birational map.

The section is wrapped up with a weak converse to a result of [SUV] concerning an upper bound for the \( a \)-invariant. The result implies that if \( \mathbb{P}_k^{d-1} \rightarrow Y \subseteq \mathbb{P}_k^m \) is a rational representation of \( \mathbb{P}_k^{d-1} \) whose base ideal \( I \subset A = k[x_0, \ldots, x_{d-1}] \) has codimension 2 and is not generically a complete intersection then \( G = G_A(I) \) is not Gorenstein.

1. Setup and terminology

1.1. Homological recap. Let \((A, \mathfrak{m})\) denote a Noetherian local ring of dimension \( d \) and let \( I \) stand for a proper ideal of \( A \). Set \( \mathcal{R}_A(I) := A[t] \subset A[t] \), the Rees algebra of \( I \) over \( A \) and \( \mathcal{G}_I(A) := A/I \otimes_A \mathcal{R}_A(I) \), the associated graded algebra. Let \( S = A[y_1, \ldots, y_r] \) denote a standard graded \(*\)local polynomial ring over \( A \) and let \( \mathcal{J} \) stand for a presentation ideal of \( \mathcal{R}_A(I) \) over \( S \). Clearly, then \( \mathcal{R}_A(I) \cong S/\mathcal{J} \) is standard graded \(*\)local, with homogeneous maximal ideal \( \mathfrak{N} := (\mathfrak{m}, (\mathcal{R}_A(I))_+) \). The ideal \( \mathcal{J} \) is a homogeneous ideal of \( S \). The maximum degree of the elements of a minimal generating set of \( \mathcal{J} \) is called the relation type of \( I \). The relation type depends only on the ideal \( I \) and is independent of its generators. Similarly, residue algebra \( \mathcal{G}_I(A) \) is a \(*\)local ring with \(*\)maximal ideal \( \mathfrak{N} := (\mathfrak{m}, \mathcal{G}_I(A)_+) \).

We will be interested in the case where \( \mathcal{R}_A(I) \) is a Cohen–Macaulay ring. For this, recall more generally that if \( S \) is a finitely generated \( \mathbb{N} \)-graded algebra over a local ring \((A, \mathfrak{m})\) and its maximal ideal or a standard graded algebra over a field and its irrelevant ideal, with \( S_0 = A \), then \( S \) is a Cohen–Macaulay ring if and only if the local ring 
\[
T := S_{(\mathfrak{m}, S_+)} \text{ is Cohen–Macaulay (\cite{Hochster} or \cite{BrunsHuneke} Exercise 2.1.17)}.
\]

An ideal \( J \subset I \) is called a reduction of \( I \) if \( JJ^n = I^{n+1} \) for some integer \( n \). The reduction \( J \) is said to be a \textit{minimal reduction} if it does not contain properly another reduction of \( I \). The \textit{reduction exponent} of \( I \) with respect to the minimal reduction \( J \), denoted by \( r_J(I) \), is the least integer \( n \geq 0 \) such that \( JJ^n = I^{n+1} \). The (absolute) reduction exponent of \( I \) is \( min\{r_J(I) : J \text{ is a minimal reduction of } I\} \). The \textit{analytic spread} of \( I \) is \( \ell(I) := \dim \mathcal{R}_A(I)/\mathfrak{m}\mathcal{R}_A(I) \). If the residue field \( A/\mathfrak{m} \) is infinite then the analytic spread coincides with the minimum number of the generators of a minimal reduction of \( I \).

Let \( R = \bigoplus_{n \geq 0} R_n \) be a positively graded \(*\)local Noetherian ring of dimension \( d \) with graded maximal ideal \( \mathfrak{n} \). For a graded \( R \)-module \( M \), \( \text{indeg}(M) := \inf\{i : M_i \neq 0\} \) and \( \text{end}(M) := \sup\{i : M_i \neq 0\} \).

If \( M \) is a finitely generated graded \( R \)-module, the Castelnuovo-Mumford regularity of \( M \) is defined as 
\[
\text{reg}(M) := \max\{\text{end}(H^i_{R_+}(M)) + i\}.
\]
One can also introduce the regularity with respect to the maximal ideal \( \mathcal{M} \), namely,
\[
\text{reg}_{\mathcal{M}}(M) := \max\{\text{end}(H^i_{\mathcal{M}}(M)) + i\}.
\]
Likewise, set \( a^*_d(M) := \text{end}(H^d_{\mathcal{M}}(G_I(A))) \), an invariant of relevance in this work.

In particular, if \( \dim A = d \) then \( a^*_d(G_I(A)) = \text{end}(H^d_{\mathcal{M}}(G_I(A))) \), an invariant of relevance in this work.

Finally, if \( A \) is standard graded and \( I \subset A \) is an ideal generated by forms of the same degree, then one can also consider \((x)\)-regularity by defining it to be \( \text{reg}_x(R_A(I)) := \max\{\text{end}(H^i_x(R_A(I))) + i\} \), where the grading has been reseted to \( \deg(x) = 1 \) and \( \deg(y) = 0 \). This notion will be frequently used throughout.

The above four notions – Cohen–Macaulayness, reduction number, regularity and \( a^- \)-invariant – are inextricably tied together and there is quite a large publication list on the subject by several authors going back at least to the early nineties. Our purpose is to draw on parts of this knowledge in order to prove one of the main theorems in the paper.

1.2. Elements of birational maps. Let \( \tilde{\mathcal{F}} : X \dashrightarrow Y \) denote a birational map of (reduced and irreducible) projective varieties over an algebraically closed field \( k \). For simplicity, assume that \( X \subset \mathbb{P}^n_k \) and \( Y \subset \mathbb{P}^m_k \) are non-degenerate embeddings. Let \( A := S(X) \) and \( B := S(Y) \) stand for the corresponding homogeneous coordinate rings. Set \( d = \dim A = \dim B \).

Now, quite generally, a rational map \( \mathcal{F} : X \dashrightarrow \mathbb{P}^m_k \) is defined by forms \( f_0, \ldots, f_m \in A \) of the same degree – the tuple \((f_0, \ldots, f_m)\) is called a representative of the map and the \( f_j \)'s are its coordinates. If no confusion arises, we call the common degree of the \( f_j \)'s the degree of the representative. Set \( I := (f_0, \ldots, f_m) \subset A \) for the ideal generated by the coordinates of the representative.

It is a fact that the set of representatives of \( \mathcal{F} \) correspond bijectively to the homogeneous vectors in the rank one graded \( A \)-module \( \text{Hom}(I, A) \). In particular, \( \mathcal{F} \) is uniquely represented up to proportionality if and only if \( I \) has grade at least two (see [S, Proposition 1.1 and Definition 1.2]). We emphasize that two representatives of \( \mathcal{F} \) share the same module of syzygies, generate isomorphic \( k \)-subalgebras of \( A \) and the corresponding base ideals of \( A \) have naturally isomorphic Rees algebras (see [S] and [DHS] for the details).

Choosing a (finite) minimal set of homogeneous generators of \( \text{Hom}(I, A) \) gives a well defined finite set of degrees of representatives of the map \( \mathcal{F} \).

For any given representative \((f_0, \ldots, f_m)\) of the rational map, one calls base ideal the corresponding ideal \( I = (f_0, \ldots, f_m) \subset A \). The base locus, in the sense of the exact subscheme where the map is not defined is given by another ideal containing every such \( I \) (see [S, Proposition 1.1 (2)])..

An effective algebraic criterion for the birationality of \( \mathcal{F} \) onto \( Y \) was established in [DHS, Theorem 2.18]. The appropriate tool to deal with this question is the graph of \( \mathcal{F} \).
which, algebraically, is thought of as the Rees algebra of a given base ideal \( (f) \subset A(X) \) – this idea goes back at least to [RuS].

2. Upper bounds when the graph is Cohen–Macaulay

2.1. Plane Cremona case. Although our main concern is the use of homological tools to understand the impact of the Cohen–Macaulay property, in a special case one can get around with more elementary methods – this is the purpose of this short part.

Namely, we have the following result:

**Proposition 2.1.** Let \( I \subset A := k[x, y, z] \) denote the base ideal of a plane Cremona map of degree \( \delta \geq 1 \). The following two conditions are equivalent:

(a) \( I \) is saturated and the Rees algebra \( R := R_A(I) \) of \( I \) is Cohen–Macaulay.

(b) \( \delta \leq 3 \) or else \( \delta = 4 \) and the map is not a de Jonquières map.

**Proof.** (a) \( \Rightarrow \) (b) First, since \( R \) has codimension 2, the Cohen–Macaulayness condition implies that the defining ideal \( J \) of \( R \) on the polynomial ring \( A[t, u, v] \) (\( t, u, v \) variables for the target projective space \( \mathbb{P}^2 \)) is a codimension two perfect ideal; as such it is generated by the maximal minors of an \((m + 1) \times m\) matrix \( \Psi \) with bigraded entries.

Among the minimal generators of \( J \) are included the ones of bidegree \((r, 1)\) coming from a complete set of minimal syzygies of \( I \). Since we are assuming that \( I \) is saturated, then \( I \) is generated by the 2-minors of a rank two \( 3 \times 2 \) matrix \( \phi \). Letting \( r, r' \) denote the (standard) degrees of the two columns of \( \phi \) (hence, \( d = r + r' \)), the ideal \( J \) has two minimal generators of bidegrees \((r, 1)\) and \((r', 1)\). As these are maximal minors \( \Delta \) and \( \Delta' \) of \( \Psi \), we may assume that they are, respectively, the minor omitting the last row and the minor omitting the first row. By suitably permuting columns one may assume that \( \Psi \) has the form

\[
\Psi = \begin{pmatrix}
  a_1 & a_2 & \cdots & b_1 \\
  a_3 & \cdots & & \\
  b_2 & \cdots & & \\
  \vdots & \ddots & \ddots & \\
  b_m & \cdots & & \nonumber
\end{pmatrix}
\]

where \( \Delta = a_1a_2\cdots a_m + \cdots \) and, similarly, \( \Delta' = b_1b_2\cdots b_m + \cdots \) (note the random positioning of the \( b_i \)'s as one may not be able to similarly dispose them along a subdiagonal without disrupting the diagonal arrangement of the \( a_i \)'s, but this will turn out to be irrelevant).

Setting \( \text{bideg}(a_i) := (r_i, s_i), 1 \leq i \leq m \), and similarly, \( \text{bideg}(b_i) := (r'_i, s'_i), 1 \leq i \leq m \), a minute thought, keeping in mind that \( \Delta \) (respectively, \( \Delta' \)) has bidegree \((r, 1)\)
sufficiently well-known: the ideal is generated by the 2-minors of a rank two 2}
\leq
2 \iff \text{2 and is necessarily non de Jonqui`eres.}
\text{two minimal generators of bidegree (1, s)}', 
\text{but this not possible by [DHS, Proposition 3.4]. Thus, we complete intersection and, hence of linear type since it is an almost complete intersection; }
\text{hence we must have } r \leq r' = r_1 - r'_m.
\text{Now, any other minimal generator of } J \text{ being still a maximal minor thereof has to include the first and last rows of } \Psi. \text{ This leaves no choice for the bidegrees of the remaining minimal generators of } J \text{ except to have the form } (t_j, 2), \text{ for some } t_j \geq 2, \text{ } 2 \leq j \leq m. \text{ Moreover, necessarily, } t_j = r_1 + r_2 + \cdots + r_j + \cdots + r_m + r'_m.
\text{On the other hand, since } I \text{ is the base ideal of a Cremona map, then } J \text{ admits at least two minimal generators of bidegree (1, s), for some } s \geq 1 \text{ (see [DHS, Theorem 2.18]).}
\text{Say, } r' \leq r. \text{ If } r' = 1, \text{ and since one is assuming that } I \text{ is saturated, then one can show that the map is a de Jonqui`eres map – a full proof is given in [RaS, Proposition 3.4 ]), but the main line of argument is as follows: if cod}(I_1(\phi)) = 3 \text{ then } I \text{ is a generically a complete intersection and, hence of linear type since it is an almost complete intersection; but this not possible by [DHS, Proposition 3.4]. Thus, } I_1(\phi) \text{ has codimension } \leq 2 \text{ and up to a change of variables one may assume that the first column of } \phi \text{ has coordinates } x, y, 0. \text{ Since the map is assumed to a Cremona map, the coordinates of the second column can be taken to be } z\text{-monoids (see, e.g., [Pan, Corollaire 2.3]).}
\text{Now, by the “only if” part of [HIS Theorem 2.7 (iii)], one has } d \leq 3.
\text{Thus, assume that } r' \geq 2. \text{ Then there are two distinct minimal generators of bidegree (1, 2). Letting } \Delta_1 \text{ be one of them, we must have } 1 = t_j = r_1 + r_2 + \cdots + r_j + \cdots + r_m + r'_m, \text{ for for some } 2 \leq j \leq m. \text{ It follows that } r = (r_1 + r_2 + \cdots + r_j + \cdots + r_m) + r_j \leq r_j + 1. \text{ Since the row where the entry } a_j \text{ appears is omitted by the minor } \Delta_1, \text{ it must appear in the other minor } \Delta_2 \text{ of bidegree (1, 2). By a similar token, } 1 = t_{j'}, \text{ with } j' \neq j, \text{ and hence we must have } r_j \leq 1. \text{ We conclude that } r \leq r_j + 1 \leq 2. \text{ Since we are assuming that } 2 \leq r' \leq r, \text{ it follows finally that } r = r' = 2, \text{ in which case the map has degree 4 and is necessarily non de Jonqui`eres.}
\text{(b) } \Rightarrow \text{ (a) This implication is mostly an educated exercise.}
\text{Assume first that } \delta = 2, \text{ i.e., the map is a quadratic Cremona map. This case is sufficiently well-known: the ideal is generated by the 2-minors of a rank two } 2 \times 3 \text{ matrix}.
with linear entries. Therefore, \( A/I \) is a Cohen–Macaulay ring – equivalently here \( I \) is saturated – and, moreover, \( I \) is an ideal of linear type, i.e., \( \mathcal{R} \) coincides with the symmetric algebra of \( I \), the latter being a complete intersection.

If \( \delta = 3 \) the map is a de Jonquières map as stems easily from the equations of condition, hence the result follows, e.g., from \([\text{HS}, \text{Corollary 2.5}]\) for the saturation and from the “if” part of \([\text{HS}, \text{Theorem 2.7 (iii)}]\), for the Cohen–Macaulayness.

Finally, assume that \( \delta = 4 \) and that the map is not a de Jonquières map. It is classically known – or easy to verify – that there is only one additional proper homaloidal type for this degree, namely, \((4; 2^3, 1^3)\). Still there is the question here of the base points, since if they are not general it may happen that such a Cremona map or its inverse, though of the same homaloidal type, may have infinitely near base points. Alas, it is not clear a priori whether these are obstacles for showing the stated result. Fortunately, somehow mysteriously, the algebra gets around it in all cases! For the proof we refer to the recent \([\text{CRS}, \text{Proposition A.1 and Remark A.3}]\), where all is needed is that we may assume that three of the base points be proper and non-aligned. \( \square \)

**Remark 2.2.** If \( I \) is not saturated then to proceed elementarily as in the above argument, would require knowing the number and degrees of a set of minimal syzygies of \( I \). There is a strong suspicion about how these ought to be, while there is at least one large class matching this prescription (\([\text{RS}, \text{Corollary 4.15}]\)). Note that, for an arbitrary ideal \( I \) the hypothesis that \( \mathcal{R} \) is Cohen–Macaulay does not imply that \( I \) is saturated. For example, this is the case of an almost complete intersection \( I \subset R \) in a Cohen–Macaulay ring, such that \( I \) is a non-saturated generically complete intersection. It is known – see, e.g., \([\text{HSV}]\) –, that \( I \) is of linear type (in particular, has maximal analytic spread) and its symmetric algebra is Cohen–Macaulay. (Perhaps the simplest explicit example has \( R = k[x, y, z] \) and \( I \) the ideal generated by the partial derivatives of the binomial \( y^2z - x^3 \).)

Note, however, that the base ideal of a plane Cremona map, although being an almost complete intersection of maximal analytic spread, is very rarely generically a complete intersection (\([\text{DHS}, \text{Corollary 3.6}]\)). And in fact, we will prove in Proposition 2.6 that, for the base ideal of a plane Cremona map, the hypothesis that \( \mathcal{R} \) is Cohen–Macaulay does imply that \( I \) is saturated. This will allow us to remove the extra assumption in item (a) of the above proposition.

### 2.2. General case.

From previous parts, given a rational map \( \mathfrak{F} : X \dashrightarrow \mathbb{P}^m \), the Rees algebra \( \mathcal{R}_A(I) \) of one of its base ideals \( I \subset A \) depends only on the map and not on this particular base ideal. Since the biprojective variety defined by \( \mathcal{R}_A(I) \) gives a version of the graph of \( \mathfrak{F} \), if no confusion arises any property of the Rees algebra will be said to be a property of the latter.
Recall that $\delta(F)$ stands for the least possible degree of the base ideals of the map. For the next proposition we introduce a new numerical invariant of a rational map $F : X \subset \mathbb{P}^n_k \rightarrow \mathbb{P}^m_k$. It will be a consequence of the following lemma.

**Lemma 2.3.** Let $I, J \subset A$ denote two base ideals of a rational map $F : X \subset \mathbb{P}^n_k \rightarrow \mathbb{P}^m_k$, where $A$ stands for the homogeneous coordinate ring of $X$. Let $\delta_I, \delta_J$ denote the respective degrees of the generators. Then

$$\text{reg}(I^r) - r\delta_I = \text{reg}(J^r) - r\delta_J,$$

for every $r \geq 1$.

**Proof.** As observed in Section 1.2, $I$ and $J$ have the same syzygies. This is because being base ideals of the same rational map they come from equivalent representatives in the sense of [DHS, Section 2.2] – that is, two sets of homogeneous coordinates of the same point of $\mathbb{P}^n(A)$, where $K(R)$ is the total ring of fractions of $A$. It follows that, for any $r \geq 1$, the powers $I^r$ and $J^r$ are base ideals of another rational map, and hence by the same token they too have the same syzygies. Writing $A$ as a residue of the polynomial ring $R = k[x_0, \ldots, x_n]$, let

$$\cdots \rightarrow \oplus_i R(-r \delta_I - i) \rightarrow R(-r \delta_I)^{N_r} \rightarrow I^r \rightarrow 0 \quad (3)$$

denote the minimal free resolution of $I^r$ over $R$, for suitable $N_r$. Then the above discussion implies that the minimal free resolution of $J^r$ over $R$ has the form

$$\cdots \rightarrow \oplus_i R(-r \delta_J - i) \rightarrow R(-r \delta_J)^{N_r} \rightarrow J^r \rightarrow 0$$

where the maps and the standard degrees $i$’s are the same as those of (3).

The stated equality follows immediately from this. \qed

Clearly, $\text{reg}(I^r) \geq r\delta_I$ for every $r \geq 1$. Thus, one gets a numerical function $f : \mathbb{N} \rightarrow \mathbb{N}$ which depends solely on the map $F$, defined by $f(r) = \text{reg}(I^r) - r\delta_I$, for any choice of a base ideal $I$. This function attains a maximum which, according to Chardin ([Ch, Theorem 3.5]), is the $x$-regularity $\text{reg}_x(R_I(A))$ introduced in Section 1.1, thus retrieving another numerical invariant of $F$ (since as remarked in Section 1.2 the Rees algebra is independent of the chosen base ideal).

In the case where $F$ is birational onto its image, one can state a bound for the minimum degree $\delta(F)$ of a representative:

**Theorem 2.4.** Let $F : X \subset \mathbb{P}^n_k \rightarrow Y \subset \mathbb{P}^m_k$ denote a birational map of reduced and irreducible nondegenerate projective varieties over an algebraically closed field $k$ and let $I$ stand for a base ideal of $F$ generated in degree $\delta$. Then

$$\delta(F) \leq m \sup_{r \geq 1}\{\text{reg}(I^r) - r\delta + 1\}, \quad (4)$$
If moreover the graph of $\mathfrak{F}$ is Cohen-Macaulay then

$$\delta(\mathfrak{F}) \leq m \sup_{r \geq 1} \{ \text{reg}(I^r) - r\delta + 1 \} \leq m d$$  \hspace{1cm} (5)$$

where $d = \dim(X) + 1$.

Proof. Let $A$ and $B$ stand, respectively, for the homogeneous coordinate rings of $X$ and $Y$. Let $I' \subset B$ denote the ideal generated by the forms of a representative of the inverse map to $\mathfrak{F}$. Consider a presentation $\mathcal{R}_B(I') \simeq B[x_0, \ldots, x_n]/\mathcal{J}_{I'}$, where $B$ is concentrated in degree zero. According to [DHS, Theorem 2.18 Supplement (ii)], the map $\mathfrak{F}$, considered as the inverse map to its own inverse, has a representative whose coordinates are suitable $m \times m$ minors of the Jacobian matrix Jacoby($((\mathcal{J}_{I'})(x,1))$ and, clearly, the degree of these coordinates is an upper bound for the minimum degree $\delta(\mathfrak{F})$ of a representative. Therefore, to bound $\delta(\mathfrak{F})$ one is led to bounding from above the $x$-degree of the minimal generators of $\mathcal{J}_{I'}$, i.e. the relation type of the ideal $I' \subset B$.

Now, quite generally, the relation type of the ideal $I' \subset B$ is bounded by $\text{reg}(\mathcal{R}_B(I')) + 1$, where the regularity is computed with respect to the the irrelevant ideal $\mathcal{R}_B(I')_+$. We claim that $\text{reg}_x(\mathcal{R}_A(I)) = \text{reg}(\mathcal{R}_B(I'))$.

For this, we note that birationality implies an isomorphism $\mathcal{R}_B(I') \simeq \mathcal{R}_A(I)$ of bigraded $k$-algebras ([S, Proposition 2.1]) naturally based on the standard bigraded structure of the polynomial ring $R[y] = k[x,y]$. If one changes this grading by setting $\deg(y) = 0$ and $\deg(x) = 1$, it obtains an isomorphism of standard graded $k$-algebras. It follows that $H^i_{(x)}(\mathcal{R}_B(I')) \simeq H^i_{(x)}(\mathcal{R}_A(I))$ for every $i$. Since $\text{reg}(\mathcal{R}_B(I')) + 1$ is computed with respect to the the irrelevant ideal $\mathcal{R}_B(I')_+$, it is defined in terms of local cohomology based on the ideal $B[x]_+ = (x)$. Then the above isomorphisms of local cohomology modules imply that $\text{reg}_x(\mathcal{R}_A(I)) = \text{reg}(\mathcal{R}_B(I'))$.

As a consequence, $\delta(\mathfrak{F}) \leq m(\text{reg}_x(\mathcal{R}_A(I)) + 1)$. Applying Chardin’s equality mentioned earlier, we obtain the first inequality.

If, in addition, the graph of $\mathfrak{F}$ is Cohen-Macaulay, then the isomorphism $\mathcal{R}_B(I') \simeq \mathcal{R}_A(I)$ of bigraded structures implies that $\mathcal{R}_B(I')$ is Cohen-Macaulay. By [AHT, Proposition 6.2], one has $\text{reg}(\mathcal{R}_B(I')) \leq \ell(I') - 1$. Since $\ell(I') = \dim(X) + 1$ we are through for the second bound. 

For the subsequent result we will use the following lemma – essentially a collection of known results in a form that suits our purpose in this part.
Lemma 2.5. Let \((A, \mathfrak{m})\) be a Cohen–Macaulay local ring of dimension \(d\) and let \(I\) be an ideal of \(A\) with analytic spread \(\ell\) and reduction number \(r(I)\). If \(\mathcal{R}_A(I)\) is Cohen–Macaulay, one has

\[
\text{reg}(\mathcal{R}_A(I)) \leq \begin{cases} 
    r(I) & \text{if } \ell = \text{ht}(I) \\
    \max\{r(I), \ell - 2\} & \text{otherwise.}
\end{cases}
\]

Proof. Set \(\mathcal{R} := \mathcal{R}_A(I)\) and \(\mathcal{G} := G_I(A)\). Since both \(A\) and \(\mathcal{R}\) are Cohen-Macaulay then \(\mathcal{G}\) is Cohen-Macaulay and moreover \(a_d^*(\mathcal{G}) \leq -1\) [TT Theorem 1.1]. According to [HIHK Corollary 4.2], one has

\[
a_d^*(\mathcal{G}) = \max\{r(I) - \ell(I), a(\mathcal{G}_p) \mid p \in V(I) \setminus \mathfrak{m}\}
\]

where \(a(\mathcal{G}_p)\) is the \(a\)-invariant of \(\mathcal{G}_p\) over \(A_p\). A recursion then yields

\[
a_d^*(\mathcal{G}) = \max\{r(I_p) - \ell(I_p) \mid p \in V(I)\}. \tag{6}
\]

On the other hand, by [AHT Corollary 4.11], \(\text{reg}(\mathcal{G}) \leq \max\{r_{\ell-1}(I), r(I)\}\), where

\[
r_{\ell-1}(I) = \begin{cases} 
    -1, & \text{if } \ell = \text{ht}(I) \\
    \max\{r(I_p) - \ell(I_p) : \text{ht}(p) \leq \ell - 1 \text{ and } \ell(I_p) = \text{ht}(p)\} + \ell - 1, & \text{otherwise.}
\end{cases}
\]

Since \(a_d^*(\mathcal{G}) \leq -1\), (6) implies that \(r(I_p) - \ell(I_p) \leq -1\) for all \(p \in V(I)\). Therefore \(r_{\ell-1}(I) \leq \ell - 2\) if \(\ell \neq \text{ht}(I)\). The assertion now follows because \(\text{reg}(\mathcal{R}) = \text{reg}(\mathcal{G})\) by [TT Corollary 3.3].

Theorem 2.6. \((k \text{ infinite})\) Let \(\mathfrak{F} : \mathbb{P}_k^n \rightarrow \mathbb{P}_k^n\) be a Cremona map having Cohen–Macaulay graph. Then the respective base ideals of \(\mathfrak{F}\) and its inverse are generated in degree \(\leq n^2\).

Furthermore, if \(I \subset A = k[x_0, \ldots, x_n]\) stands for the base ideal of \(\mathfrak{F}\), then the saturation of \(I\) is the ideal \((I : A A^n_{+}^{n-2})\). In particular, the base ideal of a plane Cremona map with Cohen–Macaulay graph is saturated.

Proof. Set \(I \subset A\) for the base ideal of the map. Since the generators of \(I\) are analytically independent and \(k\) is infinite, the reduction number \(r(I) = 0\). From Lemma 2.5 one has \(\text{reg}(\mathcal{R}_A(I)) \leq n - 1\), hence the relation type of \(I\) is \(\leq n\). Therefore the same reasoning as in the proof of Theorem 2.4 shows that \(\delta(\mathfrak{F})\) and \(\delta(\mathfrak{F}^{-1})\) are at most \(n^2\).

As earlier, twisting around by considering the graph of the inverse map, we derive the inequality \(\text{reg}_x(\mathcal{R}_A(I)) \leq n - 1\). It then follows from Chardin’s equality that \(\text{reg}(I) - \delta \leq n - 1\) where \(\delta\) is the degree of the generators of \(I\). Therefore \(\text{reg}(A/I) \leq \delta + n - 2\). In particular \(\text{end}(H^0_\infty(A/I)) = \text{end}(I^{\text{sat}}/I) \leq \delta + n - 2\). On the other hand, by [PR Proposition 1.2 (i)] (also [D Proposition 1.3.6]) one always has \(\text{indeg}(I^{\text{sat}}/I) \geq \delta + 1\) whenever \(I\) is the base ideal of a Cremona map. Therefore \((A_+)^{n-2}I^{\text{sat}} = 0\) which proves the assertion. \(\square\)
The bounds just obtained are expressed solely in terms of the ambient dimension, whereas the Cohen-Macaulayness of the graph is a property of the map. A tantalizing question is:

**Question 2.7.** Let $\mathfrak{F} : \mathbb{P}^n \to \mathbb{P}^n$ be an arbitrary Cremona map. Can one bound the degree of the generators of the base ideal of $\mathfrak{F}^{-1}$ in terms of the degree of the generators of the base ideal of $\mathfrak{F}$?

### 3. Upper bounds for the degree of a linear system

#### 3.1. A naïve upper bound.

In this short section we give an upper bound for the degree of a linear system defining a rational map, under the assumption that it generates an ideal of grade at least 2— an assumption meaning that the map has essentially a unique base ideal.

**Proposition 3.1.** Let $\mathfrak{F} : X \subseteq \mathbb{P}^n_k \to \mathbb{P}^m_k$ denote a rational map. Suppose that $\mathfrak{F}$ admits a base ideal $I \subset A$ of grade at least 2. If $\delta$ denote the degree of the forms generating $I$, one has

$$\delta \leq \frac{m}{m+1} e,$$

where $e$ is the largest twist in the graded presentation of $I$ over $A$.

The proposition is an immediate consequence of the following general lemma which seems to be folklore.

**Lemma 3.2.** Let $R$ be a commutative Noetherian ring and $I \subset R$ be an ideal of grade at least two. Fixing a set $\{f_0, \ldots, f_m\}$ of generators of $I$, let $M$ denote a corresponding presentation matrix. Then there exists an $(m+1) \times m$ submatrix of $M$ of rank $m$, and a non-zero element $h \in R$ such that $hf_i = \Delta_i$, where $\Delta_i$ denotes the $i$-th minor of this submatrix.

In particular, if $R$ is standard graded and if $I$ is homogenous and generated in a single degree $\delta$ and related in degree at most $b_1(I)$ then $b_1(I) \geq \frac{m+1}{m} \delta$.

**Proof.** Consider the free presentation $R^p \xrightarrow{\partial} R^{m+1} \xrightarrow{f} R \to R/I \to 0$, where $f$ denotes the vector $(f_0 \cdots f_m)$. Since $\text{grade}(I) \geq 2$, dualizing into $R$ yields the exact sequence $0 \to R^* \xrightarrow{f^*} R^{m+1} \xrightarrow{\partial^*} R^p$ and moreover $\text{Im}(M)$ has rank $m$. Let $N$ be an $m \times (m+1)$ submatrix of $M^t$ of rank $m$ and denote the (ordered) $m \times m$ minors of $N$ by $\Delta = \Delta_0, \ldots, \Delta_m$. For an arbitrary row of $M^t$ say $(r_0 \cdots r_m)$, $\sum_{j=0}^m r_{ji}\Delta_j$ is the determinant of an $(m+1) \times (m+1)$ matrix which is either a matrix with two equal rows or it is a $(m+1) \times (m+1)$ submatrix of $M^t$. In any case the associated determinant is zero. Therefore in the following diagram the compositions of any two compatible maps are
Since \( \ker(M^t) = \text{Im} f^t \), one has \( \Delta = (\Delta_0 \cdots \Delta_n) \in \text{Im}(f^t) \); which proves the first assertion. To see the degree assertion in the equihomogeneous case, note that in the graded case \( h \) is homogeneous, hence the equality \( hf_i = \Delta_i \) implies that \( \delta \leq \deg(\Delta_i) \leq m(b_1(I) - \delta) \). \( \square \)

3.2. **Confronting bounds in the birational case.** Since the bound in Proposition 3.1 is quite natural, it makes sense confronting it with the birational bound obtained in Theorem 2.4. For this we assume that the map \( \mathfrak{F} \) is birational, as in Theorem 2.4, and \( I \subset A \) is a base ideal of grade \( \geq 2 \), as in Proposition 3.1. In this situation, the representative of \( \mathfrak{F} \) is uniquely defined up to scalars (see [S, Proposition 1.1]), hence \( \delta(\mathfrak{F}) = \delta \). Let us suppose in addition that the function \( f \) attains its maximum at \( r = 1 \) – a hypothesis we will digress on in a minute.

Then the second bound gives \( \delta \leq m(\text{reg}(I) - \delta + 1) \), i.e., \( \delta \leq \frac{m}{m+1}(\text{reg}(I) + 1) \). Since \( \text{reg}(I) \geq e - 1 \), we see that the first bound is better.

On the other hand, the second bound is more comprehensive and besides it has the advantage of providing a lower bound for \( f \) in terms of an invariant of the map \( \mathfrak{F} \) and not of a particular base ideal \( I \).

A question arises naturally in the birational case, as to what values of \( r \) give the maximum of \( f \) defined by \( f(r) = \text{reg}(I^r) - r\delta_I \), for any choice of a base ideal \( I \). It is conceivable that, for arbitrary homogeneous ideals generated in a fixed degree, the behavior is quite erratic. However, for a base ideal \( I \) of a birational map, one knows, for example, that far out powers of \( I \) will have quite a bit of linear syzygies, while the regularity is quite sensitive to those.

And, in fact, the maximum may fail to be attained at \( r = 1 \), as is shown by the following example of Terai (see also [St]):

**Example 3.3.** Let

\[
I = (f) := (abc, abf, ace, ade, adf, bcd, bde, bef, cdf, cef) \subset A := \mathbb{Q}[a, b, c, d, e, f].
\]

The main highlights of this example are:

1. \( I \) has linear resolution – in particular, \( \text{reg}(I) = 3 \).
2. \( I^2 \) has codepth zero and the last syzygies have standard degree 2, hence \( \text{reg}(I^2) \geq 12 - 5 = 7 \) (in fact, one has \( = 7 \)).
3. \( I \) defines a birational map onto the image.
The first two features are directly checked in a computer program. To see (3), it suffices to know that \( I \) is linearly presented (or even still that the linear part of the syzygies has maximal rank, i.e., 9) and that \( \dim[f] = 6 \) (maximum). These two properties imply birationality by [DHS, Theorem 3.2]. For the dimension, since we are in characteristic zero, it suffices to check that the Jacobian matrix of \( f \) has rank 6.

The above question gets streamlined in the particular case of Cremona maps. Here it might be tempting to guess that the maximum is attained at \( r = 1 \), but at the moment one lacks sufficient preliminary evidence.

### 3.3. Cohen–Macaulay graphs and the \( a \)-invariant

As a final piece, we comment on the \( a \)-invariant vis-à-vis the Cohen–Macaulay property. We have seen the relevance of the inequality \( a^*_d(\mathcal{G}) \leq -1 \) in the proof of Lemma 2.5. Bounding the \( a \)-invariant of \( \mathcal{G} \) below \(-1\) may require strong additional assumptions. In [SUV, Theorem 2.4], e.g., it is proved that, for an ideal \( I \) of codimension \( g \) such that \( \mathcal{G} := \mathcal{G}_A(I) \) is Cohen-Macaulay and \( I \) satisfies the condition \((F_1)\), one has \( a^*_d(\mathcal{G}) \leq -g \). The latter condition means that \( \mu(I_p) \leq \text{ht}(p) \) for any prime \( p \supseteq I \), thus imposing a severe restriction on the nature of \( I \). In particular, it implies that \( I \) is generically a complete intersection which, as remarked earlier, is a rare event among base ideals of Cremona maps ([DHS, Corollary 3.6]). Curiously, we can file the following weak converse to the above result.

**Proposition 3.4.** Let \((A, \mathfrak{m})\) denote a Cohen–Macaulay local ring of dimension \( d \) and with infinite residue field. Let \( I \subset A \) stand for an equidimensional ideal of codimension \( g \) such that \( \mathcal{G} := \mathcal{G}_A(I) \) is Cohen-Macaulay. If \( a^*_d(\mathcal{G}) \leq -g \) then \( I \) is generically complete intersection.

**Proof.** [HHK, 4.2] entails the equality \( a^*_d(\mathcal{G}) = \max\{r(I_p) - \ell(I_p) \mid p \in V(I)\} \). Let \( p \in V(I) \) have codimension \( g \). Since \( a^*_d(\mathcal{G}) \leq -g \), one has \( r(I_p) - \ell(I_p) \leq -g \), and since \( I \) is assumed to be equidimensional, then \( \ell(I_p) = \text{ht}(p) = g \). Thus, \( r(I_p) \leq 0 \) which means that \( I_p \) has no proper reduction. On the other hand, since the the residue field is infinite, \( I_p \) possesses a minimal reduction with \( \ell(I_p) \) generators. Of course, \( \ell(I_p) \leq \mu(I_p) \). Necessarily then \( \ell(I_p) = \mu(I_p) \), that is \( \mu(I_p) = g \) which proves the assertion.

In particular, if \( \mathbb{P}^{d-1}_k \rightarrow Y \subseteq \mathbb{P}^m_k \) is a rational representation of \( \mathbb{P}^{d-1}_k \) whose base ideal \( I \subset A = k[x_0, \ldots, x_{d-1}] \) has codimension 2 and is not generically a complete intersection then \( \mathcal{G} = \mathcal{G}_A(I) \) is not Gorenstein, as otherwise it would imply the inequality \( a^*_d(\mathcal{G}) \leq -2 \) by [HHK, Theorem 5.2 and Remark 5.3].

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