TOPOLOGICAL ENTROPY AND HAUSDORFF DIMENSION OF
IRREGULAR SETS FOR NON-HYPERBOLIC DYNAMICAL SYSTEMS

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Abstract. We systematically investigate examples of non-hyperbolic dynamical systems having irregular sets of full topological entropy and full Hausdorff dimension. The examples include some partially hyperbolic systems and geometric Lorenz flows. We also pose interesting questions for future research.

1. Introduction

Let us consider a continuous map $f$ of a totally bounded metric space $X$. We define the irregular set $I(f)$ of $f$ as the set of points $x \in X$ such that there exists a continuous function $\phi : X \to \mathbb{R}$ for which the time average

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x))$$

does not exist. It turns out that $I(f)$ has zero measure for any invariant measure by Birkhoff’s ergodic theorem. Although this set is negligible from the point of view of Ergodic Theory, it is still possible to be topologically large or to have large topological entropy and Hausdorff dimension. In fact, dynamical systems with irregular sets of full topological entropy and full Hausdorff dimension have been intensively studied in the literature [PP84, BS00, FFW01, EKS05, Tho10, Tia17, MY17a, DOT18, BLV18, FKKOT21]. However, most of these works only deal with continuous/smooth maps having inherent properties from (non)uniform hyperbolicity, such as the specification-like property or the shadowing property. In this paper, we establish full topological entropy and full Hausdorff dimension of the irregular set for abundant examples of dynamics without such properties, but well approximated by some hyperbolic structures. To our best knowledge, this is the first systematic investigation of such phenomena beyond the specification-like or shadowing property.

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In Section 2, we study the topological entropy of irregular sets. In lower dimensions, we show that $I(f)$ has full topological entropy when $f$ is a continuous map or a piecewise monotonic map of a compact interval (Theorem 2.4) and when $f$ is a $C^{1+\alpha}$-diffeomorphism of a compact surface (Corollary 2.5). We obtain the same result for some interesting higher dimensional systems. These include partially hyperbolic diffeomorphisms having a one-dimensional center direction and minimal strong foliations, partially hyperbolic diffeomorphisms on three dimensional nilmanifolds, perturbations of time-one map of Anosov flows, generic symplectic diffeomorphisms with no dominated splitting and diffeomorphisms isotopic to Anosov (see §2.4).

In Section 3, we study the Hausdorff dimension of irregular sets. We review the known results in lower dimensions on surfaces and in parametric families, concluding that $I(f)$ has full Hausdorff dimension in these examples. In Section 3.5 we look at skew-products over the symbolic shift with interval fibers. These examples are related to partially hyperbolic sets, such as blenders and porcupines. Afterward, we study mostly contracting partially hyperbolic systems (§3.6) and systems having no dominated splitting (§3.7), where we conclude full Hausdorff dimension of the irregular set in the generic context.

In Section 4, we prove that the irregular set of geometric Lorenz flows $C^1$-robustly has full topological entropy and full Hausdorff dimension (Theorem 4.1). The longer and more technical proofs are collected in Section 5 in order not to obscure the main ideas of this paper.

## 2. Systems with the irregular set having full topological entropy

Since the irregular set $I(f)$ is not necessarily compact, we need to adapt the notion of topological entropy to this setting. In what follows, as in the previous literature [BS00, Tho10], we consider the topological entropy $h_{\text{top}}(A, f)$ in the sense of Pesin and Pitskel’ [PP84]. This entropy is defined for a continuous map $f$ of a totally bounded metric space $X$ and an invariant subset $A$ as a Carathéodory characteristic of dimension type and we refer to [Pes97] for the precise definition. Observe that this notion allows us to deal also with a discontinuous map $f : X \to X$ by restricting $f$ to $Y = X \setminus \{f^{-n}(Z) : n \geq 0\}$ where $Z$ is the set of discontinuity points. The variational principle also holds for this topological entropy and, in the case that $f$ is continuous, $h_{\text{top}}(A, f)$ coincides with the Bowen-Hausdorff topological entropy of a (not necessarily compact) set $A$ introduced by Bowen in [Bow73]. In particular, when $A$ is a compact subset of $X$, $h_{\text{top}}(A, f)$ coincides with the usual topological entropy (i.e. the Bowen-Dinaburg topological entropy [Din70, Bow71]). We simply denote $h_{\text{top}}(X, f)$ by $h_{\text{top}}(f)$.
2.1. **Approximation of entropy by horseshoes.** We start by the following basic observation, which involves invariant compact sets with the specification property (see definition in [KLO16, Def. 3]).

**Proposition 2.1.** Let \( f \) be a continuous map of a totally bounded metric space \( X \). Then,
\[
h_{\text{top}}(I(f), f) \geq \sup \ h_{\text{top}}(\Lambda, f)
\]
where the supremum is with respect to all compact subsets \( \Lambda \) of \( X \), which are invariant and satisfy the specification property for some iterate of \( f \).

**Proof.** Assume that \( \Lambda \) is a compact \( f^k \)-invariant subset of \( X \) with the specification property for some \( k \geq 1 \). It is easy to see that \( I(f^k) \subset I(f) \). Also, observe that in general, one has that \( I(f^k) \cap \Lambda \subset I(f^k|_{\Lambda}) \). On the left-hand of the inclusion, we consider the time average of real-valued continuous observables \( \phi \) on \( X \), while on the right-hand, observables are only defined on \( \Lambda \). However, since \( \Lambda \) is a closed set of \( X \), by Tietze extension theorem we can extend continuously the observables to the whole space \( X \) and get the other inclusion. Thus, we have that \( I(f^k|_{\Lambda}) = I(f^k) \cap \Lambda \).

Hence, according to [EKS05, Corollary 3.11] it follows that
\[
h_{\text{top}}(\Lambda, f^k) = h_{\text{top}}(I(f^k|_{\Lambda}), f^k) = h_{\text{top}}(I(f^k) \cap \Lambda, f^k) \leq h_{\text{top}}(I(f) \cap \Lambda, f^k).
\]
(1)

On the other hand, from [Bow73, Proposition 2(d)], we get
\[
h_{\text{top}}(I(f) \cap \Lambda, f^k) = k \cdot h_{\text{top}}(I(f) \cap \Lambda, f) \quad \text{and} \quad h_{\text{top}}(\Lambda, f^k) = k \cdot h_{\text{top}}(\Lambda, f).
\]
(2)

Putting together (1) and (2) we obtain that
\[
h_{\text{top}}(\Lambda, f) \leq h_{\text{top}}(I(f) \cap \Lambda, f) \leq h_{\text{top}}(I(f), f)
\]
which completes the proof. \( \square \)

We shall apply the above basic fact to a diffeomorphism \( f \) of a compact manifold \( M \) as follows. Recall that an ergodic \( f \)-invariant probability measure \( \mu \) is said to be hyperbolic if all of its Lyapunov exponents are nonzero. On the other hand, an elementary horseshoe of \( f \) is a Cantor set \( \Lambda \) which is a topologically mixing locally maximal invariant hyperbolic set for some iterate of \( f \). Let \( f \) be a \( C^{1+\alpha} \)-diffeomorphism (i.e. a \( C^1 \)-diffeomorphism with \( \alpha \)-Hölder derivative) and consider a hyperbolic ergodic \( f \)-invariant probability measure \( \mu \). Then, according to [Ge16, Theorem 1] and [KH99], there is a sequence of elementary horseshoes \((\Lambda_n)_{n \geq 1}\) of \( f \) such that \( h_{\text{top}}(\Lambda_n, f) \rightarrow h_\mu(f) \)

\[1\]This can be seen by using the following decomposition which relates the finite Birkhoff sum of \( f \) with that of \( f^k \) and then analyzing the limits inferior and superior,
\[
\frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x)) = \frac{\ell_n}{n} \cdot \frac{1}{\ell_n} \sum_{j=0}^{\ell_n-1} \phi(f^j(x)) + \frac{1}{n} \sum_{r=1}^{k-1} \sum_{j=0}^{\ell_n-1} \phi(f^{k+r}(x)) + \frac{1}{n} \sum_{r=0}^{r_n} \phi(f^{\ell_n+k+r}(x))
\]
where \( 0 \leq r_n < k \) and \( \ell_nk + r_n = n - 1 \).
as $n \to \infty$ where $h_\mu(f)$ denotes the (Kolmogorov-Sinai) metric entropy of $\mu$. Since $\Lambda_n$ is an elementary horseshoe (and in particular expansive) for some iterate $f^k$ of $f$, then the specification property holds on $\Lambda_n$ (for $f^k$) [DGS76]. Hence, from Proposition 2.1 it follows that

$$h_{\text{top}}(I(f), f) \geq h_\mu(f).$$  

(3)

In light of the above observation, we introduce the hyperbolic entropy $H(f)$ of $f$ as

$$H(f) = \sup \{h_\mu(f) : \mu \text{ is a hyperbolic ergodic } f\text{-invariant probability measure}\}$$

and get the following.

**Corollary 2.2.** Let $f$ be a $C^{1+\alpha}$-diffeomorphism of a compact manifold $M$. Then, the topological entropy of the irregular set of $f$ is bounded from below by the hyperbolic entropy, that is,

$$h_{\text{top}}(I(f), f) \geq H(f).$$

We observe that the above result already appeared in [DT, Corollary B], nevertheless we decided to include the previous proof for the completeness of the article.

**Remark 2.3.** Corollary 2.2 can be similarly stated for $C^1$-diffeomorphisms having a dominated splitting by using a version of the theorem of Katok on approximation by horseshoes obtained in [Gel16] (see also [ST]). To do this, it requires assuming that the hyperbolic measures have an Oseledets splitting coinciding with the dominated splitting in the hypothesis.

Thus, one approach to obtain full topological entropy of the irregular set $I(f)$ is to find hyperbolic measures whose metric entropy approaches the topological entropy of the whole space $X$. Similar estimates can also be obtained for the topological pressure of irregular sets using the arguments and the results in [Tho10] and [SS17].

2.2. Examples in dimension one. Recently, the second author of this paper obtained in [NY21] the following result. The irregular set of a piecewise monotonic map of a compact interval has full topological entropy if $f$ is transitive and the set of periodic measures of $f$ is dense in the set of ergodic measures of $f$. Recall that a map $f : I \to I$ is said to be a *piecewise monotonic map* of a compact interval $I$ if there exists a partition of $I$ into finitely many pairwise disjoints intervals $I_1, \ldots, I_k$ such that $f|_{I_j}$ is strictly monotone and continuous for $j = 1, \ldots, k$. Here, we extend this result to any continuous map and any piecewise monotonic map of a compact interval.

**Theorem 2.4.** Let $f$ be either a continuous or piecewise monotonic map of a compact interval. Then, the irregular set has full topological entropy, that is, $h_{\text{top}}(I(f), f) = h_{\text{top}}(f)$. 

The proof of Theorem 2.4 will be given in Section 5.1. In the proof, we will use the previously mentioned idea of approximation of the topological entropy of $f$ by horseshoes, which is now achieved by Misiurewicz’s theorem. We remark that these horseshoes are of $f$ and not of a lift of $f$ (that is, the Hofbauer’s Markov diagram of $f$) as in the approach of [NY21].

2.3. Examples in dimension two. By Ruelle’s inequality in dimension two, any ergodic invariant probability measure of a $C^1$-diffeomorphism with positive metric entropy is hyperbolic. Thus, by the variational principle the topological entropy of any surface $C^1$-diffeomorphism $f$ coincides with the hyperbolic entropy of $f$. Hence, Corollary 2.2 implies that the irregular set of $f$ has full topological entropy if $f$ is a $C^{1+\alpha}$-diffeomorphism.

Moreover, one can restrict the attention to any homoclinic class $H(\mathcal{O})$ with positive topological entropy. Recall that a homoclinic class of a hyperbolic periodic orbit $\mathcal{O}$ is the closure of the transverse intersections between the stable and unstable manifold of $\mathcal{O}$. Any homoclinic class is a compact transitive invariant set. As a consequence again of the Ruelle inequality and the variational principle for the restriction of $f$ to $H(\mathcal{O})$ we obtain that $h_{\text{top}}(H(\mathcal{O}), f)$ can be approximated by the metric entropy $h_{\mu}(f)$ of a hyperbolic invariant measure $\mu$ supported on $H(\mathcal{O})$. Thus, from Corollary 2.2 and the fact that $I(f|_{H(\mathcal{O})}) = I(f) \cap H(\mathcal{O})$ it follows that $I(f)$ has full topological entropy in any homoclinic class $H(\mathcal{O})$, that is,

$$h_{\text{top}}(I(f) \cap H(\mathcal{O}), f) = h_{\text{top}}(H(\mathcal{O}), f).$$

To summarize the above arguments, we get the following:

**Corollary 2.5.** Let $f$ be a $C^{1+\alpha}$-diffeomorphism of a compact surface. Then the irregular set $I(f)$ has full topological entropy. Moreover, $I(f)$ has full topological entropy in any homoclinic class $H(\mathcal{O})$.

We were recently informed that the above result was also obtained by D. Sanhueza in his Ph.D. thesis [San20]. See also [DT, Corollary C].

2.4. Examples in higher dimensions. In what follows, we describe some interesting examples in dimension greater than two whose irregular sets have full topological entropy.

2.4.1. Non-hyperbolic homoclinic classes. First, we focus on the case of non-hyperbolic homoclinic classes with index-variation, a question that was asked in [BKN’20, Question 1]. Let $M$ be a compact manifold with $\dim(M) \geq 3$ and denote by $\text{Diff}^1(M)$ the set of $C^1$-diffeomorphisms of $M$. Following the recent works [DGS20, YZ20]
we consider the open set $\mathcal{U} \subset \text{Diff}^1(M)$ consisting of partially hyperbolic systems\footnote{A diffeomorphism $f$ is called partially hyperbolic if there exists a decomposition $TM = E^s \oplus E^c \oplus E^u$ of the tangent bundle into three continuous invariant sub-bundles such that (i) $Df|E^s$ is uniformly contracting, $Df|E^u$ is uniformly expanding and (ii) for any unit vectors $v^s \in E^s$, $v^c \in E^c$, $v^u \in E^u$ and any $x \in M$, we have that $\|Df(x)v^s\| \cdot \|Df(x)v^c\|^{-1} < 1/2$, and $\|Df(x)v^c\| \cdot \|Df(x)v^u\|^{-1} < 1/2$.} with one-dimensional central bundle, having minimal strong stable and unstable foliations and a pair of hyperbolic periodic points $P$ and $Q$ with different indices (dimension of the stable bundle). In particular, these systems are $C^1$-robustly transitive and non-hyperbolic. Moreover, there exists an open and dense subset $\mathcal{V}$ of $\mathcal{U}$ such that for any $f \in \mathcal{V}$, it holds that $H(P) = H(Q) = M$. This kind of diffeomorphisms is quite abundant among partially hyperbolic skew-products in dimension 3 having circles as the central fibers. The authors in [DGS20, YZ20] proved that for all $f \in \mathcal{V}$ the hyperbolic entropy of $f$ is equal to the topological entropy. Consequently, it follows from Corollary 2.2, Remark 2.3 that the irregular set $I(f)$ has full topological entropy for any $f \in \mathcal{V}$.

2.4.2. Volume-preserving or symplectic diffeomorphisms with no dominated splitting. Let $\omega$ be a volume or a symplectic form of a compact manifold $M$. We denote by $\mathcal{E}_\omega(M)$ the interior of the set of $C^1$ diffeomorphisms which preserve $\omega$ and having no dominated splitting\footnote{A splitting $T_x M = E(x) \oplus F(x)$ is called dominated if it is invariant under the derivative, varies continuously with $x$, and there exists $C > 0$, $\lambda < 1$ such that for every $x \in M$ and every pair of unit vectors $u \in E(x)$, $v \in F(x)$, one has $\|Df^n(x)u\| \cdot \|Df^n(x)v\|^{-1} < C\lambda^n$, for every $n \geq 1$ (cf. [BDV06]).}. Buzzi, Crovisier and Fisher in [BCF18] proved that the topological entropy of a generic $f \in \mathcal{E}_\omega(M)$ is the supremum of the topological entropy of horseshoes of $f$. Although a priori these horseshoes $\Lambda$ need not be elementary, one can use the Spectral Decomposition Theorem [KH99] and obtain an elementary subhorseshoe $\Lambda'$ having the entropy of $\Lambda$. Thus, we can always approximate $h_{\text{top}}(f)$ by the topological entropy of elementary horseshoes of $f$. Therefore, Proposition 2.1 concludes that the irregular set of generic diffeomorphisms $f \in \mathcal{E}_\omega(M)$ has full topological entropy.

2.4.3. Diffeomorphisms of tori isotopic to Anosov. The well-known DA (derived from Anosov)-example by Mañé ([BDV06]) of a partially hyperbolic diffeomorphism $f$ on the torus $\mathbb{T}^3$ has a unique measure of maximal entropy. This measure is in fact a hyperbolic measure. This example was extended by Ures [Ure12] to all absolutely partially hyperbolic diffeomorphisms of $\mathbb{T}^3$ which are isotopic to a linear Anosov diffeomorphism. In higher dimensions, Fisher, Potrie and Sambarino [FPS14] proved that every partially hyperbolic diffeomorphism of $\mathbb{T}^d$ which is isotopic to a linear Anosov diffeomorphism along a path of partially hyperbolic diffeomorphisms with a one-dimensional center bundle has a unique measure of maximal entropy. In [Rol16], the last author of this paper proved that this measure is in fact a hyperbolic
measure. Thus, it follows from Corollary 2.2 that the irregular set of such systems has full topological entropy. Furthermore, for this type of diffeomorphisms of \( T^d \), but now assuming a compact two-dimensional center direction and \( C^2 \)-regularity of the diffeomorphism, in [\text{A20}] the author proved the hyperbolicity of any maximal entropy measure. Hence, again by Corollary 2.2, we may conclude full entropy of the irregular set.

2.4.4. Trivial factors over Anosov: Diffeomorphisms of 3-dimensional nilmanifolds. Let us say that a diffeomorphism \( f \) of a compact manifold \( M \) trivially factors over Anosov if there exists an Anosov map \( \Lambda : T^d \to T^d \) on the torus \( T^d \), and a continuous surjective map \( \pi : M \to T^d \) such that \( \pi \circ f = \Lambda \circ \pi \) and \( h_{\text{top}}(\pi^{-1}(y), f) = 0 \) for all \( y \in T^d \).

**Proposition 2.6.** If \( f \) is a \( C^1 \)-diffeomorphism that trivially factors over an Anosov \( \Lambda \) then \( h_{\text{top}}(I(f), f) = h_{\text{top}}(f) = h_{\text{top}}(\Lambda) \).

**Proof.** According to Bowen [Bow71, Thm. 17] and since the topological entropy of \( f \) restricted to the fibers \( \pi^{-1}(y) \) is zero it holds that

\[
h_{\text{top}}(f) \leq h_{\text{top}}(\Lambda) + \sup_{y \in T^d} h_{\text{top}}(\pi^{-1}(A), f) = h_{\text{top}}(\Lambda).
\]

Moreover, since the irregular set \( I(\Lambda) \) of the Anosov map \( \Lambda \) has full topological entropy (actually this is true for any Axiom A from Proposition 2.1), we obtain that

\[
h_{\text{top}}(\Lambda) = h_{\text{top}}(I(\Lambda), A) \leq h_{\text{top}}(\pi^{-1}(I(\Lambda)), f) \leq h_{\text{top}}(I(f), f).
\]

The last inequality above follows from the inclusion \( \pi^{-1}(I(\Lambda)) \subset I(f) \), which is a straightforward consequence of Lemma 5.1. The first inequality follows from [PP84, Prop. 2]. Putting together the above inequalities we conclude the proof. \( \square \)

We will now apply the above proposition to partially hyperbolic diffeomorphisms on a 3-dimensional nilmanifold \( M \neq T^3 \). Recall that 3-dimensional nilmanifolds are circle bundles over the torus \( T^2 \). The results of [Ham13, HP14] imply that in this case there exists a center foliation \( \mathcal{F}^c \) for \( f \) (dynamical coherence) such that each leaf is a circle tangent to \( E^c \). The foliation \( \mathcal{F}^c \) forms a circle bundle and let us consider the quotient space \( M/\mathcal{F}^c \), which is actually the 2-torus \( T^2 \). Then, the map induced on the quotient space will be a transitive Anosov homeomorphism which is conjugate to a linear Anosov diffeomorphism which we denote by \( \Lambda \). In particular, we get that \( f \) is semi-conjugate to \( \Lambda \) by means of the canonical quotient map \( \pi \). Since the center leaves are 1-dimensional circles restricted to which \( f \) is a homeomorphism, the fibers \( \pi^{-1}(y) \) have zero topological entropy. Thus, \( f \) trivially factors over Anosov. Consequently, from Proposition 2.6 we have that any partially hyperbolic diffeomorphism on a 3-dimensional nilmanifold (other than the torus \( T^3 \)) has full topological entropy of the irregular set.
In dimensions larger than three, the same argument can be applied to partially hyperbolic $C^1$-diffeomorphisms which are dynamically coherent and have a 1-dimensional center direction so that the center foliation $\mathcal{F}^c$ forms a circle bundle. Such diffeomorphisms trivially factor over Anosov (see for example [UVYY20, Prop. 7.1]). Consequently, the irregular set has full topological entropy.

Finally, we can apply the above proposition to the recent results on partially hyperbolic DA diffeomorphisms with center bundle of arbitrary dimension in [CLPV21, Thm. A. See also Sec. 2.3 & Sec. 4] and obtain full topological entropy of the irregular set for this class of systems.

2.4.5. *Perturbations of time-one map of Anosov flows.* A recent theorem of [BFT, Theorem 1.1] describes the following dichotomy for measures of maximal entropy with respect to perturbations of time-one map of Anosov flows. Let $\phi^t$ be a transitive Anosov flow on a compact manifold $M$. Then, there is an open set $U$ in $\text{Diff}^1(M)$ containing the time-one map $\phi^1$ in its closure, such that for any $f \in U \cap \text{Diff}^2(M)$ it holds that: either (i) there are exactly two hyperbolic measures of maximal entropy or (ii) all measures of maximal entropy have zero central Lyapunov exponents. It is believed that the first case is dense and $C^2$-open in $U$ and there are preliminary announcements of these results by Crovisier and Poletti [BFT, Remark 1.2]. In particular, if $f$ satisfies case (i) then, as before, by using Corollary 2.2 we conclude that the irregular set $I(f)$ has full topological entropy.

3. Systems with the irregular set of full Hausdorff dimension

We denote by $\dim_H A$ the Hausdorff dimension of a set $A$ (see e.g. [Pes97] for its definition). In what follows, we will study the Hausdorff dimension of $I(f)$.

3.1. **Approximation of Hausdorff dimension by $u$-conformal horseshoes.** Let $\Lambda$ be a locally maximal hyperbolic set of a $C^1$-diffeomorphism $f$ on a compact manifold. Recall that $\Lambda$ is said to be basic set if additionally it is a transitive set (i.e., if it has a dense orbit). The set $\Lambda$ is said to be $u$-conformal if there exists a continuous function $a^u: \Lambda \to \mathbb{R}$ such that $Df(x)|_{E^u(x)} = a^u(x) \cdot \text{Isom}_x$ for every $x \in \Lambda$ where $\text{Isom}_x$ denotes an isometry. Similarly, $s$-conformal hyperbolic sets are defined and we say that $\Lambda$ is conformal if it is both $s$-conformal and $u$-conformal. For instance, on surfaces any hyperbolic set is conformal. According to [Pes97, Theorem 22.1] and [Bar11, Theorem 6.2.8], if $\Lambda$ is a $u$-conformal (resp. $s$-conformal) basic set

$$d^u(\Lambda) \overset{\text{def}}{=} \dim_H W^{u}_{\text{loc}}(\Lambda) \cap \Lambda \quad \text{(resp. } d^s(\Lambda) \overset{\text{def}}{=} \dim_H W^{s}_{\text{loc}}(\Lambda) \cap \Lambda)$$

\footnote{Observe that due to the spectral decomposition theorem, it is enough to assume that the basic set is only transitive and not necessarily mixing, see [Pes97, p. 228] and [Bar11, pg.86 and 123].}
does not depend on \(x \in \Lambda\). Moreover, if \(\Lambda\) is conformal then it holds that
\[
\dim_H \Lambda = d^u(\Lambda) + d^s(\Lambda).
\]

This result was obtained first, independently, by McCluskey and Manning in [MM83] and Palis and Viana in [PV88] for surface diffeomorphisms. Barreira and Schmeling in [BS00, Theorem 4.2] use this result to prove that if \(\Lambda\) is a conformal elementary horseshoe of a \(C^{1+\alpha}\) diffeomorphism \(f\) then,
\[
\dim_H I(f) \geq \dim_H I(f) \cap \Lambda = \dim_H \Lambda.
\]

Following essentially similar arguments as in [BS00] to get the above inequality, we can get the following improvement. First, recall that by an \textit{elementary (hyperbolic) set} of \(f\) is understood as a topologically mixing locally maximal hyperbolic set of some iterate \(f^k\) of \(f\).

**Proposition 3.1.** Let \(f\) be a \(C^{1+\alpha}\)-diffeomorphism of a compact manifold. Assume that there exists a \(u\)-conformal elementary hyperbolic set \(\Lambda\) of \(s\)-index \(d_s(\Lambda)\) (i.e. \(d_s(\Lambda) = \dim E^s\)). Then
\[
\dim_H I(f) \geq d_s(\Lambda) + d^u(\Lambda).
\]

**Proof.** Let \(k\) be the period of the elementary set \(\Lambda\) (i.e., the smallest positive integer such that \(f^k(\Lambda) = \Lambda\)). Since \(I(f^k) \subset I(f)\), we may assume for simplicity that \(k = 1\). Let \(p\) be a fixed point in \(\Lambda\). We will need the following fact, which we explain below:

\[
\dim H W^u(p) \cap I(f|_\Lambda) = \dim H W^u(p) \cap \Lambda. \tag{4}
\]

Indeed, the assumption of \(u\)-conformality implies that the dynamics of \(f\) restricted to \(W^u(p) \cap \Lambda\) can be seen as a conformal expanding map. Applying [BS00, Thm. 7.5], (to \(X = W^u(p) \cap \Lambda, m = 1, \varrho = 1_X\) indicator of \(X\) and \(\phi_1\) to be any \(\text{Hölder} \) continuous function on \(W^u(p) \cap \Lambda\) which is not cohomologous to a constant function), we get
\[
\dim H W^u(p) \cap I(f|_\Lambda) = \dim H I(f|_{W^u(p) \cap \Lambda}) \geq \dim H (\phi_1, f|_{W^u(p) \cap \Lambda}) = \dim H W^u(p) \cap \Lambda
\]

where \(I(\phi_1, f|_{W^u(p) \cap \Lambda})\) is \(\phi_1\)-irregular set (see definition in Remark 3.4). The converse inequality is obviously true and therefore we can conclude (4).

Now there exists a point \(x \in W^u(p)\) such that
\[
\dim H W^u_{loc}(x) \cap I(f|_\Lambda) = \dim H W^u(p) \cap I(f|_\Lambda). \tag{5}
\]

Fix such \(x\) and let \(V\) be a small neighborhood of \(x\). Let us observe that if \(y \in I(f)\), then \(W^s(y) \subset I(f)\). Define the set \(F\) given by the set \(I(f|_\Lambda) \cap V\) saturated by the stable leaves
\[
F = \{z \in V : z \in W^s(y) \text{ and } y \in I(f) \cap \Lambda = I(f|_\Lambda)\}.
\]

Then \(F \subset I(f)\) and as a consequence \(\dim H I(f) \geq \dim H F\). Hence, it suffices to calculate the Hausdorff dimension of \(F\).
Consider the $s$-holonomy map $h^s : V \to W^u_{\text{loc}}(x)$ defined using the stable foliation and the transversal section $W^u_{\text{loc}}(x)$ (here "loc" means the local intersection with $V$). Let $\phi(z) = (\pi(z), h^s(z))$ for $z \in V$, where $\pi : V \to W^s_{\text{loc}}(x)$ is the orthogonal projection on $W^s_{\text{loc}}(x)$. Observe that $\phi(F) = W^s_{\text{loc}}(x) \times (W^u_{\text{loc}}(x) \cap I(f|_\Lambda))$, and so $F$ can be identified with a product given by $W^s_{\text{loc}}(x) \times (W^u_{\text{loc}}(x) \cap I(f|_\Lambda))$. In general the $s$-holonomy is only Hölder continuous with exponent $0 < \alpha \leq 1$. Then, a priori $\phi$ is just an $\alpha$-Hölder continuous map and thus

$$\dim_H \phi(F) \leq \frac{1}{\alpha} \cdot \dim_H F.$$  

However, since $\Lambda$ is $u$-conformal then the Hölder exponent $\alpha$ can be taken arbitrarily close to 1 (cf. [PV88], [SSS92] and [BCF18, proof of Proposition 4.6]). This implies that $\dim_H \phi(F) \leq \dim_H F$. Thus, one gets that

$$\dim_H I(f) \geq \dim_H F \geq \dim_H W^s_{\text{loc}}(x) + \dim_H W^u_{\text{loc}}(x) \cap I(f|_\Lambda).$$

Using (4) and (5),

$$\dim_H I(f) \geq \dim_H W^s_{\text{loc}}(x) + \dim_H W^u(p) \cap \Lambda.$$  

On the other hand, $\dim_H W^u(p) \cap \Lambda = \dim_H W^u_{\text{loc}}(x) \cap \Lambda = d^u(\Lambda)$, see [Pes97, Thm.22.1]. Thus, taking into account that $\dim_H W^s_{\text{loc}}(x) = d_s(\Lambda)$, we obtain that $\dim_H I(f) \geq d_s(\Lambda) + d^u(\Lambda)$ concluding the proof of the proposition.

**Remark 3.2.** Let $\Lambda$ be a $u$-conformal basic set of $f$ a $C^{1+\epsilon}$-diffeomorphism. It follows from the Spectral Decomposition Theorem [KH99] that $\Lambda$ is a finite union of elementary sets $\Lambda_1, \ldots, \Lambda_k$ which are cyclically permuted by $f$. Thus, $d_s(\Lambda) = d_s(\Lambda_i)$ and $d^u(\Lambda) = d^u(\Lambda_i)$ for all $i = 1, \ldots, k$. Hence, applying Proposition 3.1, we get that

$$\dim_H I(f) \geq \max_{1 \leq i \leq k} (d_s(\Lambda_i) + d^u(\Lambda_i)) \geq d_s(\Lambda) + d^u(\Lambda).$$

Proposition 3.1 with Remark 3.2 can be applied to diffeomorphisms having a $u$-conformal uniformly hyperbolic attractor $\Lambda$, which is a basic set. Since for every $y \in \Lambda$ the unstable manifold $W^u(y)$ is contained in $\Lambda$, then $d^u(\Lambda)$ is the $u$-index $d_u(\Lambda)$ of $\Lambda$ (i.e., $\dim E^u_{\Lambda}$). Thus, we obtain the following corollary.

**Corollary 3.3.** Let $f$ be a $C^{1+\epsilon}$-diffeomorphism of a manifold $M$, having a non-trivial $u$-conformal uniformly hyperbolic attractor $\Lambda$. Then, $\dim_H I(f) = \dim M$ (full Hausdorff dimension of irregular set).

Hence, one approach to obtain full Hausdorff dimension of the irregular set is to find a sequence of $u$-conformal horseshoes $(\Lambda_n)_{n \geq 1}$ of $f$ such that the $s$-index of $\Lambda_n$ together with the unstable Hausdorff dimension of $\Lambda_n$ approach the dimension of the whole space.
Remark 3.4. Given a continuous map $\phi : X \to \mathbb{R}$, we denote by $I(\phi, f)$ the subset of irregular points of $f$ for which the time average with respect to the potential $\phi$ does not converge. This set is usually called the $\phi$-irregular set. Feng, Lau and Wu [FLW02] showed that if $f$ is of class $C^{1+\alpha}$ and $\Lambda$ is a topologically mixing repeller of $f$, then for each continuous function $\phi$ such that $I(\phi, f|_\Lambda) \neq \emptyset$,

$$\dim_H I(\phi, f|_\Lambda) = \dim_H \Lambda$$

see [FLW02, Theorem 1.2]. That is, the $\phi$-irregular set has full Hausdorff dimension. The statement also holds if $f$ is a topologically mixing subshift of finite type [FFW01, Sec. 4]. This can be compared with [BS00, Theorem 4.2 (1) and 7.1] where it was shown only for Hölder potentials $\phi$. Therefore, the proof of Proposition 3.1 works to prove that

$$I(\phi, f|_\Lambda) \neq \emptyset \implies \dim_H I(\phi, f|_\Lambda) \geq d_s(\Lambda) + d_u(\Lambda)$$

for each $C^{1+\alpha}$-diffeomorphism $f$ of a compact manifold with a $\mu$-conformal elementary hyperbolic set $\Lambda$ of $s$-index $d_s(\Lambda)$. Consequently, as in Corollary 3.3, we get full Hausdorff dimension of any non-empty $\phi$-irregular sets of a non-trivial $\mu$-conformal uniformly hyperbolic attractor of a $C^{1+\alpha}$-diffeomorphism.

3.2. One dimensional examples. For one-dimensional uniform expanding dynamics and, in general, for conformal repellers in higher dimension, it follows from [BS00, FFW01, FLW02] that the irregular set has full Hausdorff dimension. This result was extended in [Chu10] for topologically exact $C^2$-interval maps admitting an absolute continuous invariant probability measure (see Proposition 8 and the remark before section 4 of [Chu10]). Another extension was made in [MY17b] for a class of non-uniformly expanding one-dimensional dynamics which includes Manneville-Pomeau maps, that is, the map $T : [0, 1] \to [0, 1]$ defined by $T(x) = x + x^{1+\alpha} \mod 1$, where $0 < \alpha < 1$.

3.3. Surface diffeomorphisms. For $C^{1+\alpha}$ surface diffeomorphisms any hyperbolic set is $\mu$-conformal and in particular Proposition 3.1 and Corollary 3.3 can be applied. Homoclinic tangencies are the obstruction to hyperbolicity for surface diffeomorphisms. In [DN05, Theorem 1.6], Downarowicz and Newhouse constructed a residual subset $\mathcal{R}$ of $\text{Diff}^r(M)$ ($r \geq 2$) for a compact surface $M$, such that if $f \in \mathcal{R}$ and $f$ has a homoclinic tangency, then $f$ has a sequence of elementary horseshoes $(\Lambda_n)_{n \geq 1}$ with $\dim_H \Lambda_n \to 2$ as $n \to \infty$. Therefore, combining this with Corollary 2.5 and Proposition 3.1, we obtain full entropy and Hausdorff dimension of irregular set in this setting.

Corollary 3.5. There exists a residual subset $\mathcal{R}$ of $\text{Diff}^r(M)$ for a compact surface $M$ with $r \geq 2$ such that if $f \in \mathcal{R}$ and $f$ has a homoclinic tangency, then $\dim_H I(f) = 2$. In particular, the irregular set of $f$ has full topological entropy and full Hausdorff dimension.
3.4. Parametric families in low dimensions.

3.4.1. Quadratic family. As previously mentioned, it follows from [Chu10] that if $f$ is a topologically exact $C^2$-interval map admitting an absolute continuous invariant probability measure, then $\dim_H I(f) = 1$. On the other hand, it is well known that these conditions are satisfied for the quadratic map $f_a(x) = x^2 + a$ for parameters $a$ in a Lebesgue positive measure set $E$ close to $a = 2$ (cf. [Jak81]). Thus, the irregular set of the quadratic map $f_a$ has full Hausdorff dimension for all $a \in E$.

3.4.2. The Standard map. The standard family is defined on the torus by $f_k(x, y) = (-y + 2x + k \sin(2\pi x), x) \mod \mathbb{Z}^2$. Gorodetski [Gor12] proved that for each $k$ in a residual set of $(k_0, \infty)$ with a large $k_0$, there exists a sequence of elementary horseshoes $(\Lambda^k_n)_{n \geq 1}$ of $f_k$ satisfying $\dim_H \Lambda^k_n \to 2$ as $n \to \infty$. Thus, again by Proposition 3.1 we conclude that $\dim_H I(f_k) = 2$ for all $k$ in the residual set (full Hausdorff dimension). We note that the Gorodetski’s approximation theorem by horseshoes holds for every area-preserving one-parameter family $(g_t)$ generically unfolding a quadratic homoclinic tangency at $g_0$.

3.5. Examples in skew-products over a symbolic shift.

3.5.1. Hyperbolic graphs. Consider a skew product $F$ of the form $F(x, y) = (f(x), g_x(y))$ defined in $\Lambda \times \mathbb{R}^n$, where $\Lambda$ is an elementary horseshoe of a two-dimensional $C^2$-diffeomorphism $f$. As mentioned before, from [BS00] we have that $\dim_H (I(f) \cap \Lambda) = \dim_H \Lambda$. Let us further observe the following. Given $x_0 \in I(f)$, there exists a continuous function $\phi_0 : \Lambda \to \mathbb{R}$ for which the Birkhoff time average
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi_0(f^j(x_0))
\]
does not exist. Considering $\tilde{\phi}_0(x, y) = \phi_0(x)$ and because of the skew-product structure of $F$, the above implies that for every $y \in \mathbb{R}^n$, the Birkhoff time average with respect to the point $(x_0, y)$ also does not converge. Therefore, we obtain in this case full Hausdorff dimension of the irregular set of $F$:

**Lemma 3.6.** $I(f) \times \mathbb{R}^n \subset I(F)$ and $\dim_H I(F) = \dim_H \Lambda + n$ (full Hausdorff dimension).

Thus, in this setting, it makes more sense to ask the following. Is it true that if $I(\phi, F)$ is non-empty, then it has full Hausdorff dimension? Recall that given a continuous function $\phi : \Lambda \times \mathbb{R}^n \to \mathbb{R}$, the $\phi$-irregular set $I(\phi, F)$ is defined as the set of points $(x, y) \in \Lambda \times \mathbb{R}^n$ for which $\lim_{n \to \infty} n^{-1} \sum_{j=0}^{n-1} \phi(F^j(x, y))$ does not exist.
Assume from now that the fiber map $g_x$ is a contraction for every $x \in \Lambda$. Then, the maximal invariant set of $F$ is unique and is the graph $\{(x, \Phi(x)) \mid x \in \Lambda\}$ of an invariant map $\Phi : \Lambda \rightarrow \mathbb{R}^n$, that is $g_x(\Phi(x)) = \Phi(f(x))$ for every $x \in \Lambda$. This class includes an important example of horseshoes called blenders (cf. [BDV06]). We emphasize that compared with the one-dimensional or two-dimensional cases, in higher dimensions, there is no explicit way to calculate the Hausdorff dimension of the maximal invariant set. Let us mention the work [DGGJ19], where Díaz et al. calculated the box dimension of the maximal invariant set in dimension 3 of a variety of examples of skew-products with contracting/expanding fibers, including blenders. Nevertheless, in the special case when the fiber maps are contractions, we can give a complete answer to the above question about the Hausdorff dimension of $\phi$-irregular sets:

**Proposition 3.7.** Assume that $g_x$ is a contraction for all $x \in \Lambda$. Then, for any continuous function $\phi : \Lambda \times \mathbb{R}^n \rightarrow \mathbb{R}$, either $I(\phi, F) = \emptyset$ or $\dim_H I(\phi, F) = \dim_H \Lambda + n$.

**Proof.** Denote by $\Phi : \Lambda \rightarrow \mathbb{R}^n$ the invariant map of $F$ and by $\Gamma$ the graph of $\Phi$. Let $\phi$ be a continuous function on $\Lambda \times \mathbb{R}^n$. Then it induces a continuous function $\bar{\phi}$ in $\Lambda$, defined by $\bar{\phi}(x) = \phi(x, \Phi(x))$. We claim that

$$I(\bar{\phi}, f|_{\Lambda}) \times \mathbb{R}^n \subset I(\phi, F).$$

Indeed, if $(x, y) \in I(\bar{\phi}, f|_{\Lambda}) \times \mathbb{R}^n$ then $(x, \Phi(x)) \in I(\phi, F)$. Hence, since $\{x\} \times \mathbb{R}^n = \Lambda^e(x, \Phi(x)) \subset I(\phi, F)$ we get that $(x, y) \in I(\phi, F)$. That is, (6) holds.

Suppose that $I(\phi, F)$ is non-empty and take $(x, y) \in I(\phi, F)$. Observe that $(x, y) \in \Lambda^e(x, \Phi(x))$. Then $(x, \Phi(x)) \in I(\phi, F|_{\Gamma})$ and so $I(\phi, F|_{\Gamma})$ is non-empty. Thus $I(\bar{\phi}, f|_{\Lambda})$ is also non-empty. Since $\Lambda$ is a two-dimensional horseshoe, it follows (see Remark 3.4) that $I(\bar{\phi}, f|_{\Lambda})$ has full Hausdorff dimension. Combining this with (6) we obtain that

$$\dim_H I(\phi, F) = \dim_H \Lambda + n$$

completing the proof.

3.5.2. Porcupines and bony graphs. Porcupines were defined in [GD12] and form an important class of partially hyperbolic sets, mixing contracting and expanding behaviour. Below we describe the setting borrowing the notation from [DG12]. Let $\Sigma_2 = [0, 1]^2$ and consider a skew-product map $F$ of $\Sigma_2 \times [0, 1]$ defined by

$$F(\xi, x) = (\sigma(\xi), f_{\xi_0}(x))$$

where $\sigma : \Sigma_2 \rightarrow \Sigma_2$ denotes the shift map and $\xi_0$ is the zeroth coordinate of $\xi = (\xi_i)_{i \in \mathbb{Z}}$. We actually study a one-parameter family $(F_t)$ given by $F_t(\xi, x) = (\sigma(\xi), f_{\xi_0, t}(x))$, where $f_{0, t} = f_0$ is an increasing concave $C^2$-map independent of $t$ with two fixed hyperbolic points $f_0(0) = 0$, $f_0(1) = 1$ and $f_{1, t}$ is the explicit affine map $f_{1, t}(x) = t(1 - x)$. 
Denote by $\Lambda_t$ the maximal invariant set of $F_t$. The set $\Lambda_t$ is semi-conjugate to the shift map in $\Sigma_2$. Namely, if $\Pi_t$ denotes the projection of $\Sigma_2 \times [0, 1]$ onto the first coordinate, then $\Pi_t \circ F_t = \sigma \circ \Pi_t$. For each $\xi \in \Sigma_2$ we consider the set

$$\Pi_t^{-1}(\xi) \triangleq \Pi_t^{-1}(\xi) \cap \Lambda_t = \{\xi\} \times I_{\xi}$$

called a spine of $\Lambda_t$. Here $I_{\xi}$ is an interval of $[0, 1]$. The spine $I_{\xi}$ is said to be non-trivial if it is not a singleton and trivial otherwise. In this way, we split the set $\Sigma_2$ into two disjoint invariant sets $\Sigma^{non}_{2,t}$ and $\Sigma^{trv}_{2,t}$ consisting of sequences with non-trivial and trivial spines, respectively. Moreover, $\Lambda_t$ is a porcupine-like horseshoe (shortly, a porcupine), that is, a topologically transitive set of $F_t$ such that the sets $\Sigma^{non}_{2,t}$ and $\Sigma^{trv}_{2,t}$ are both dense and uncountable in $\Sigma_2$. We observe that the topological entropy of $F_t$ is always $\log(2)$ because $F_t$ is semi-conjugate to the full shift $\Sigma_2$ and the fiber maps do not have critical points.

In the set $\Sigma_2$ consider the distance defined by

$$d(\xi, \zeta) = 2^{1/2}2^{-|\eta|} \quad \text{where } |\eta| \text{ is the smallest value with } \xi_n \neq \zeta_n.$$ 

With this distance the Hausdorff dimension of $\Sigma_2$ is $2$. Denote by $b_{1/2}$ the $(1/2, 1/2)$-Bernoulli measure on $\Sigma_2$, which in this case coincides with the $2$-dimensional Hausdorff outer measure $m_2$. It is shown in [DG12, Theorems 1 and 2] that, for every $t \in (0, 1)$,

$$\dim_H(\Sigma^{non}_{2,t}) < 2 \quad \text{and} \quad b_{1/2}(\Sigma^{trv}_{2,t}) = m_2(\Sigma^{trv}_{2,t}) = 1.$$

Consider the following set

$$G_t = \{ (\xi, g(\xi)) \in \Lambda_t : \xi \in \Sigma^{trv}_{2,t} \}$$

of a $b_{1/2}$-almost everywhere defined function $g$ from the base of the skew product to the fiber. In particular, one gets that $\Lambda_t$ is a union of the almost-everywhere defined graph $G_t$ with some non-trivial spines (called bones), and this is termed a bony graph.

**Proposition 3.8.** For every continuous $\phi$ on $\Sigma_2 \times [0, 1]$ and $t \in (0, 1)$, either $I(\phi, F_t) \cap G_t = \emptyset$ or $\dim_H(I(\phi, F_t)) \geq 2$. In the latter case $I(\phi, F_t)$ also has full entropy.

**Proof.** Consider the continuous function $\tilde{\phi}_t = \phi \circ \Pi_t^{-1}$, which is a priori well-defined only in $\Sigma^{trv}_{2,t}$ but can be extended to a continuous function on $\Sigma_2$. If $I(\phi, F_t) \cap G_t \neq \emptyset$, then $I(\tilde{\phi}_t, \sigma) \neq \emptyset$ and so it follows (Remark 3.4) that $I(\tilde{\phi}_t, \sigma)$ has full Hausdorff dimension. From [DG12, Theorem 1], we have that $\dim_H(\Sigma^{non}_{2,t}) < 2$. As $\Sigma^{non}_{2,t} \cup \Sigma^{trv}_{2,t} = \Sigma_2$, therefore, $I(\tilde{\phi}_t, \sigma) \cap \Sigma^{trv}_{2,t}$ also has full Hausdorff dimension. In the case of the symbolic shift $\Sigma_2$, full Hausdorff dimension also implies full entropy of the set $I(\tilde{\phi}_t, \sigma) \cap \Sigma^{trv}_{2,t}$, which is $\log(2)$ and coincides with the entropy of $F_t$. 

Consider the graph over $I(\tilde{\phi}_t, \sigma) \cap \Sigma^\nu_{2,J}$,

$\tilde{G}_t = \{ (\xi, x) \in \Lambda_t : \xi \in I(\tilde{\phi}_t, \sigma) \cap \Sigma^\nu_{2,J} \}$.

Then $\dim H(\tilde{G}_t) \geq \dim H(I(\tilde{\phi}_t, \sigma) \cap \Sigma^\nu_{2,J}) \geq 2$ and $h_{\text{top}}(\tilde{G}_t) = \log(2)$. Since by construction $\tilde{G}_t \subset I(\phi, F_t)$, we reach the conclusion required for $I(\phi, F_t)$.  

**Question 3.9.** Using the fact that there is nonuniform contraction along the trivial fibers is it possible to show that actually $\dim H(I(\phi, F_t)) = 3$ (full Hausdorff dimension) for $F_t$ in the above proposition? Do similar statements hold for the skew-products over the shift with circle fibers studied in [GDR17], the bony attractors in the results of Kleptsyn and Volk [KV14], and the higher dimensional bony-attractors of Kudryashov [Kud10]? 

### 3.6. mostly contracting systems.

Consider a $C^{1+\alpha}$ partially hyperbolic diffeomorphism $f$ on a compact manifold $M$. Recall this means that the tangent bundle $TM$ splits in three $Df$-invariant subbundles $E^s \oplus E^c \oplus E^u$, where $E^s$ is uniformly contracting, $E^u$ is uniformly expanding and $E^c$ called the central bundle is in-between. The extremal subbundles, $E^s$ and $E^u$, can be uniquely integrated obtaining the so-called strong stable and unstable foliations of $M$. A Gibbs $u$-state is an invariant probability measure whose conditional probabilities along strong unstable leaves are absolutely continuous with respect to the Lebesgue measure on the leaves (see [BDV06] for an introduction). Gibbs $u$-states are important in the study of physical measures and in particular, every ergodic Gibbs $u$-state with negative center Lyapunov exponents is a physical measure. It said that $f$ has mostly contracting center if all of its Gibbs $u$-states have only negative center Lyapunov exponents. These systems were first studied in [BV00] where the existence and finiteness of physical measures were shown. The key property for us is that the Pesin formula is satisfied with respect to an ergodic Gibbs $u$-state $\mu$: the metric entropy $h_{\mu}(f)$ is equal to the sum of positive Lyapunov exponents (counting with multiplicity).

Let $\text{Diff}^{1+}(M)$ denote the set of all diffeomorphisms which are $C^{1+\alpha}$ for some $\alpha$. This set can be endowed with the $C^1$-topology, and systems with mostly contracting center form an open set in $\text{Diff}^{1+}(M)$ with this topology [Yan]. One can similarly define systems having a mostly expanding center and analogous properties hold for them, including the Pesin formula ([ABV00, Yan, RHUY]).

**Theorem 3.10.** Let $f$ be $C^{1+\alpha}$ partially-hyperbolic diffeomorphism with mostly contracting center and assume $\dim(E^u) = 1$. Then there exists an open neighborhood $\mathcal{U}$ of $f$ in $\text{Diff}^{1+}(M)$ and a residual set $\mathcal{R} \subset \mathcal{U}$ such that for $g \in \mathcal{R}$, the irregular set has full Hausdorff dimension, that is $\dim_H I(g) = \dim M$. 

The proof of the above theorem is given in Section 5.2. Here we will describe the key steps, which are as follows.

i) Existence of a dense set of maps in $U$ having a hyperbolic measure satisfying the Pesin Formula.

ii) Approximating the measure in entropy by a horseshoe and afterwards perturbing the horseshoe in $C^1$-topology to the so-called standard affine horseshoe.

iii) Projecting the horseshoe in the unstable direction, we can obtain an attractor of an affine iterated function system on the line. This permits to calculate the Hausdorff dimension of this set, which is going to be relatively big because of the Pesin formula.

iv) Calculating the dimension of the irregular set of the horseshoe.

3.6.1. Remarks on possible extensions of Theorem 3.10. We are interested in extending Theorem 3.10 to diffeomorphisms with mostly expanding center or conservative systems in dimension 3. The first three of the above steps actually still hold in these cases. Except that now we would be dealing with an attractor of an affine iterated function system in the plane as the exponent in the center direction will be also positive. Under some algebraic assumptions, it is possible to conclude that the Hausdorff dimension of the attractor is equal to the affinity dimension for a system of affine contractions in the plane with a diagonal linear part ([MS19, Proposition 7.2]). Then, one can calculate the affinity dimension using, for example, the formulas in [Mor19]. The main problem is that it is unknown how to calculate the Hausdorff dimension of the irregular set for these iterated systems. Since they are non-conformal, we cannot apply directly [BS00], as was done in Proposition 3.1. This leads to the following question.

Question 3.11. Consider a planar contractive, affine, diagonal iterated function system as in [MS19](or [PW94]) for which the Hausdorff dimension of the attractor is known explicitly. Show that the irregular set has full Hausdorff dimension.

3.7. $C^1$-generic diffeomorphisms with no dominated splitting. Motivated by the Newhouse’s theorem on approximation by horseshoes with large Hausdorff dimension for surface diffeomorphisms with homoclinic tangencies [New78], Buzzi, Crovisier and Fisher have recently obtained a similar result in higher dimension in [BCF18, Theorem 6]. Using this extension and Proposition 3.1 we can obtain the next result on the full dimension of the irregular set. To state the result precisely, we need some notation. Each periodic point $P$ of a diffeomorphisms $f$ of a compact manifold of dimension $d$ admits Lyapunov exponents $\lambda_1(f, P) \geq \cdots \geq \lambda_d(f, P)$, listed
with multiplicity. We set $\lambda_i^+ = \max(\lambda_i, 0)$ and $\lambda_i^- = \max(-\lambda_i, 0)$. Also we denote

$$\Delta^+(f, P) = \sum_{i=1}^d \lambda_i^+ \quad \text{and} \quad \Delta^-(f, P) = \sum_{i=1}^d \lambda_i^-.$$ 

**Theorem 3.12.** For a $C^1$-generic diffeomorphism $f$ of a compact manifold $M$, if $f$ has a periodic point $P$ such that the homoclinic class $H(P)$ of $P$ has no dominated splitting and $\Delta^-(f, P) \geq \Delta^+(f, P)$, then $\dim_H I(f) = \dim M$.

See Section 5.3 for the proof.

### 4. Geometric Lorenz flows

In this section, we show that every geometric Lorenz flow has the irregular set of full topological entropy and full Hausdorff dimension.

The irregular set $I(f)$ of a continuous flow $f = \{f_t\}_{t \in \mathbb{R}}$ on a compact metric space $X$ is defined as the set of points $x \in X$ such that there exists a continuous function $\phi : X \to \mathbb{R}$ for which the time average

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \phi \circ f^t(x) \, dt$$

(7)

does not exist. We will also use the set $I(\phi, f)$ defined as the set of points $x \in X$ such that (7) does not exist. Recall that the Abramov-Bowen formula ([Bow71, Proposition 21]; see also [Abr59] as its origin for measurable flows) establishes that

$$h_{\text{top}}(f^t) = \frac{1}{|t|} h_{\text{top}}(f^t) \quad \text{for all } t \neq 0.$$

So, the topological entropy $h_{\text{top}}(f)$ of $f$ is defined as the topological entropy of the time one map $f^1$, i.e. $h_{\text{top}}(f) = h_{\text{top}}(f^1)$. We also refer to [BR75, p. 185] for an equivalent definition of the topological entropy of $f$ given as a straightforward analogue of the usual (i.e. Bowen-Dinaburg’s) topological entropy for a continuous map on a compact metric space.

In order to define the topological entropy $h_{\text{top}}(A, f)$ of $f$ on a (not necessarily compact) invariant set $A$ of a totally bounded space $X$, we have two alternatives in literature. The first alternative is to extend the definition of the Pesin-Pitskel’ topological entropy for maps [PP84] to flows, which was done by Thompson [Tho10, Section 4] and Shen-Zhao [SZ12]. For this definition of the topological entropy for flows, the Abramov-Bowen type formula is also proven, i.e.

$$h_{\text{top}}(A, f) = \frac{1}{|t|} h_{\text{top}}(A, f^t) \quad \text{for all } A \subset X \text{ and } t \neq 0.$$
The other alternative is, as Pacifico and Sanhueza [PS19] recently did, to extend Bowen-Hausdorff version of the topological entropy for maps [Bow73] to flows. They also showed the Abramov-Bowen type formula for this topological entropy. When $A$ is compact, as we notified in Section 2 all the previously-mentioned versions of the topological entropy of a continuous map on $A$ coincide, and so is a continuous flow on $A$.

To state our main result clearly, we employ the construction of geometric Lorenz flows of [MPRI20]. Given a $C^1$ vector field $V$ of $\mathbb{R}^3$, we denote by $f_V = \{f_V^t\}_{t \in \mathbb{R}}$ the continuous flow generated by $V$. We recall that in the setting of [MPRI20, Section 2], a geometric Lorenz flow $f_V$ generated by a $C^1$ vector field $V$ satisfies that

$$V(x, y, z) = (\lambda_1 x, \lambda_2 y, \lambda_3 z) \text{ on } [-1, 1]^3 \text{ with } 0 < -\lambda_3 < \lambda_1 < -\lambda_2.$$ 

Moreover, the Poincaré map $P : S_0 \to S$ of $f_V$ with

$$S = [-1, 1]^2 \times \{1\} \text{ and } S_0 = \{(x, y, 1) \in S \mid x \neq 0\}$$

is a skew-product map of the form

$$P(x, y, 1) = (F(x), G(x, y), 1) \text{ for } x \in [-1, 1] \setminus \{0\} \text{ and } y \in [-1, 1]$$

where $F : [-1, 1] \setminus \{0\} \to [-1, 1]$ is a $C^2$ piecewise expanding map and $y \mapsto G(x, y)$ is a contraction map for every $x \in [-1, 1] \setminus \{0\}$. Let $K$ be the closure of $\{f_V^t(x) \mid x \in S_0, t \geq 0\}$ and

$$\Lambda_V := \bigcap_{t \geq 0} f_V^t(K).$$

Then, one can take an open set $D$ including $\Lambda_V$ such that for any $C^1$-close vector field $W$ to $V$, the set $\Lambda_W := \bigcap_{t \geq 0} f_W^t(D)$ is transitive and singular-hyperbolic. That is, $f_W$ is partially hyperbolic with volume expanding center-unstable subbundle on $\Lambda_W$ and all singularities of $W$ in $\Lambda_W$ are hyperbolic. The set $\Lambda_W$ is called the geometric Lorenz attractor of $f_W$ while $D$ is said to be a trapping region. Finally, we denote by $\mathcal{X}(D)$ the space of $C^1$ vector fields on $D$.

**Theorem 4.1.** Let $V$ be a $C^1$ vector field which defines a geometric Lorenz flow $f_V$ with a trapping region $D$. Then, there is a neighborhood $\mathcal{N} \subset \mathcal{X}(D)$ of $V$ such that for any $W \in \mathcal{N}$,

(i) $h_{\text{top}}(I(f_W), f_W) = h_{\text{top}}(f_W)$ (full topological entropy).

Furthermore, if $-\frac{1}{\lambda_1} > -\frac{1}{\lambda_1} + 2$, then for any $W \in \mathcal{N}$,

(ii) $\dim_H I(f_W) = 3$ (full Hausdorff dimension).

In [KLS16] (see also [Yan20]), it was proven that the irregular set of any geometric Lorenz flow is residual in the trapping region. More precisely, they constructed a residual set $\mathcal{R}$ consisting of points whose partial time averages for some continuous
function oscillate between the Dirac measures of the singularity and a periodic orbit. However, since both of the measures have zero topological entropy and zero Hausdorff dimension, it seems that the topological entropy and Hausdorff dimension of \( R \) is zero. (cf. [ZC13])

**Remark 4.2.** The condition
\[
-\frac{\lambda_2}{\lambda_1} > -\frac{\lambda_3}{\lambda_1} + 2
\]

in Theorem 4.1 is only used to ensure the regularity of the quotient map along the strong stable foliation of the Poincaré map \( P_W \) of \( f_W \). This regularity is automatically satisfied when \( W = V \) by construction (we will explain it more precisely in §4.2). Thus, one gets that \( \dim_H I(f_V) = 3 \) without the extra condition.

As in the map case, the key ingredient is an approximation theorem by horseshoes, so before starting the proof of Theorem 4.1, we recall the definition of horseshoes for continuous flows. Given a homeomorphism \( T : X \to X \) on a compact metric space \( X \) and a continuous function \( \rho : X \to \mathbb{R}_+ := \{z > 0\} \), the suspension flow \( T_\rho = \{(T^t)_\rho\}_{t \in \mathbb{R}} \) of \( T \) over \( \rho \) is defined as
\[
T_\rho^t : X_\rho \to X_\rho, \quad T_\rho^t(x, z) = (x, z + t)
\]
where \( X_\rho \) is the \( \rho \)-suspension space given by
\[
X_\rho = \{(x, z) \mid x \in X, \ 0 \leq z \leq \rho(x)\} / \sim
\]
with \((x, \rho(x)) \sim (T(x), 0)\). For a continuous flow \( f = \{f^t\}_{t \in \mathbb{R}} \), a compact \( f \)-invariant set \( \Gamma \) is called a horseshoe of \( f \), if there exists a suspension flow \( \sigma_\rho = \{\sigma^t_\rho\}_{t \in \mathbb{R}} \) of the left-shift operation \( \sigma : \Sigma \to \Sigma \) of a two-sided topologically mixing subshift of finite type with a finite alphabet over a continuous function \( \rho \), and a homeomorphism \( \pi : \Sigma_\rho \to \Gamma \) such that
\[
f^t \circ \pi = \pi \circ \sigma^t_\rho \quad \text{for every} \ t \in \mathbb{R}.
\]
We will use the following preliminary lemma, whose proof will be given in §5.4.

**Lemma 4.3.** Let \( \Sigma \subset \{1, 2, \ldots, N\}^\mathbb{Z} \) be a topologically mixing subshift of finite type with \( N \geq 2 \) and \( \sigma : \Sigma \to \Sigma \) the left-shift operation. Let \( \rho : \Sigma \to \mathbb{R}_+ \) be a continuous function. Then, there exists a continuous function \( \phi : \Sigma_\rho \to \mathbb{R} \) such that \( I(\phi, \sigma_\rho) \neq \emptyset \). Moreover,

(i) if \( \Gamma \) is a horseshoe of a continuous flow \( f \) then \( I(f|_\Lambda) \neq \emptyset \);

(ii) if \( \sigma : \Sigma \to \Sigma \) is topologically conjugate by a homeomorphism \( \Pi : \Lambda \to \Sigma \) to the restriction of a \( C^{1+\alpha} \)-diffeomorphism \( T \) on a \( u \)-conformal elementary horseshoe \( \Lambda \),
\[
\dim_H I(\hat{\phi}, (T|_\Lambda)_\rho) \geq 1 + d_s(\Lambda) + d_u(\Lambda),
\]
where \( \rho = \rho \circ \Pi \) and \( \hat{\phi} = \phi \circ \hat{\Pi} \) with \( \hat{\Pi} : \Lambda_\rho \to \Sigma_\rho \) given by \( \hat{\Pi}(x, t) = (\Pi(x), t) \).
4.1. **Topological entropy.** We first show that the irregular set of any geometric Lorenz flow has full topological entropy.

**Remark 4.4.** Pacifico and Sanhueza recently showed that for any continuous flow \( f \) on a compact metric space with (almost) specification property, the irregular set \( I(f) \) of \( f \) has full topological entropy ([PS19, Theorem 5.8]). However, it is known that any geometric Lorenz flow does not satisfy the specification property ([SVY15]). Furthermore, Thompson showed in [Tho10] that for any suspension flow \( f \) of a homeomorphism with the specification property, the irregular set \( I(f) \) has full topological entropy (\( f \) itself is not required to satisfy the specification property). However, again, it is unclear that the Poincaré map of a geometric Lorenz flow satisfies the specification property in general.

**Proof of Proposition 4.1 (i).** Let \( V \) be a \( C^1 \) vector field which defines a geometric Lorenz attractor \( \Lambda \). We first show that the irregular set of \( f_V \) has full topological entropy. If the topological entropy of \( f_V \) is zero, then we have nothing to prove. So, we assume that \( h_{\text{top}}(f_V) > 0 \). Recall that for any \( C^1 \)-close vector field \( W \) to \( V \), \( \Lambda_W \) is a singular-hyperbolic attractor with the trapping region \( D \). In particular, all singularities of \( W \) in \( D \) are hyperbolic. This implies that \( f_V \) satisfies the star property, i.e. there is a neighborhood \( N \subset X_1(D) \) of \( V \) such that for any \( W \in N \), all singularities and all periodic orbits of \( f_W \) are hyperbolic. Hence, we can apply the entropy approximation theorem of star flows by horseshoes ([LSWW20, Proposition 1.1]) to \( f_W \) for any \( W \in N \):

\[
\text{for every } \epsilon > 0, \text{ there exists a horseshoe } \Gamma_\epsilon \text{ of } f_W \text{ such that }
\]

\[
h_{\text{top}}(\Gamma_\epsilon, f_W) > h_{\text{top}}(\Lambda_W, f_W) - \epsilon = h_{\text{top}}(f_W) - \epsilon. \tag{9}
\]

Let \( \sigma_{\rho_\epsilon} \) be the suspension flow of the left-shift operator \( \sigma \) on the topologically mixing shift of finite type \( \Sigma_\epsilon \) over a continuous function \( \rho_\epsilon \) which is conjugate to \( f_W \) on \( \Gamma_\epsilon \). Then, since \( I(\sigma_{\rho_\epsilon}) \neq \emptyset \) by Lemma 4.3 and a subshift of finite type is topologically mixing if and only if it has the periodic specification property (cf. [KLO16]), it follows from the previously-mentioned Thompson’s theorem ([Tho10, Theorem 5.1 and Lemma 5.4]) that

\[
h_{\text{top}}(I(\sigma_{\rho_\epsilon}), \sigma_{\rho_\epsilon}) = h_{\text{top}}(\sigma_{\rho_\epsilon}) = h_{\text{top}}(\Gamma_\epsilon, f_W).
\]

On the other hand,

\[
h_{\text{top}}(I(\sigma_{\rho_\epsilon}), \sigma_{\rho_\epsilon}) \leq h_{\text{top}}(I(f_W) \cap \Gamma_\epsilon, f_W) \leq h_{\text{top}}(I(f_W), f_W) \tag{10}
\]

(the first inequality will be proven in a slightly more general form, see Remark 5.2). Combining these estimates with (9), we have

\[
h_{\text{top}}(f_W) - \epsilon < h_{\text{top}}(I(f_W), f_W).
\]
Since $\epsilon$ is arbitrary, we conclude that the irregular set of $f_W$ has full topological entropy. \hfill $\square$

4.2. Hausdorff dimension. Next, we will show that the irregular set of any geometric Lorenz flow has full Hausdorff dimension.

The proof of Proposition 4.1 (ii). Recall (8) for the Poincaré map $P$ of $f_V$. We identify $P$ with the two-dimensional map $(x, y) \mapsto (F(x), G(x, y))$. By [MPRI20, Thm. 1], there is an increasing sequence of regular Cantor sets $(C_k)_{k \in \mathbb{N}}$ for $F$ such that $\dim_H C_k \to 1$ as $k \to \infty$. Fix $k \geq 1$. We consider

$$\Lambda_k = \{(x, y) \in [-1, 1]^2 \mid (x, y, 1) \in \Lambda_V, x \in C_k\}.$$ 

Since $C_k$ is a regular Cantor set, $\Lambda_k$ is an elementary horseshoe of $P$. Hence, the restriction $P|_{\Lambda_k}$ of $P$ on $\Lambda_k$ is topologically conjugate to the shift operation $\sigma : \Sigma_k \to \Sigma_k$ on a topologically mixing subshift of finite type $\Sigma_k$ with a finite alphabet. That is, there is a homeomorphism $\Pi : \Lambda_k \to \Sigma_k$ such that $\sigma \circ \Pi = \Pi \circ P$. Let $\hat{\rho}(x, y)$ be the first return time of $(x, y, 1)$ for $(x, y) \in \Lambda_k$ to the Poincaré section $S$ by $f_V$. Since $C_k$ is far from the singularity $0$, it is straightforward to see that $\hat{\rho} : \Lambda_k \to \mathbb{R}_+$ is uniformly bounded and continuous. Set $\rho = \hat{\rho} \circ \Pi^{-1}$. From Lemma 4.3, we get that there exists a continuous function $\hat{\phi} : (\Sigma_k)_\rho \to \mathbb{R}$ such that

$$\dim_H I(\hat{\phi}, (P|_{\Lambda_k})_\rho) \geq 1 + d_s(\Lambda_k) + d^u(\Lambda_k),$$

where $\hat{\phi} = \phi \circ \hat{\Pi}$ with $\hat{\Pi} : (\Lambda_k)_\rho \to (\Sigma_k)_\rho$ given by $\hat{\Pi}(x, y, t) = (\Pi(x, y), t)$. Moreover,

$$d_s(\Lambda_k) = 1 \quad \text{and} \quad d^u(\Lambda_k) = \dim_H (W^u_{loc}(x) \cap \Lambda_k) = \dim_H C_k \quad \text{for each } x \in \Lambda_k.$$

Consider now

$$\Gamma_k = \{f_V^t(x, y, 1) \mid (x, y) \in \Lambda_k, 0 \leq t < \hat{\rho}(x, y)\}.$$

Notice that $f|_{\Gamma_k}$ is topologically conjugate to the suspension flow $(P|_{\Lambda_k})_\rho$ of $P|_{\Lambda_k}$ over $\rho$ by the map $\pi : (\Lambda_k)_\rho \to \Gamma_k$, where $\pi(x, y, t) = f_V^t(x, y, 1)$. Observe that from the smooth dependence of the flow with respect to the initial conditions, since $V$ is a $C^1$ vector field, $\pi$ is also of class $C^1$ (cf. [DK00, Appendix B]). In particular, since

$$I(\hat{\phi} \circ \pi, f_V|_{\Gamma_k}) = \pi \left( I(\hat{\phi}, (P|_{\Lambda_k})_\rho) \right),$$

it holds that $\dim_H I(\hat{\phi} \circ \pi, f_V|_{\Gamma_k}) = \dim_H I(\hat{\phi}, (P|_{\Lambda_k})_\rho)$. So, we get

$$\dim_H I(f_V) \geq \dim_H I(f_V|_{\Gamma_k}) \geq \dim_H I(\hat{\phi} \circ \pi, f_V|_{\Gamma_k}) \geq 2 + \dim_H C_k \to 3$$

as $k \to \infty$. That is, the irregular set of $f_V$ has full Hausdorff dimension.

Now, we argue that the same is true for any vector field $W C^1$-close to $V$ under the assumption $-\frac{\lambda_2}{\lambda_1} > -\frac{\lambda_3}{\lambda_1} + 2$. Recall that there exists a $C^1$-neighborhood $\mathcal{N} \subset \mathcal{X}^1(D)$ of
such that for each $W \in N$, the maximal invariant set $\Lambda_W = \bigcap_{t \geq 0} f^t_W(D)$ is singular-hyperbolic. Moreover, it is known that the associated Poincaré map $P_W$ preserves the strong stable foliation $\mathcal{F}_W$ with $C^1$ leaves and the holonomies along the leaves are of class $C^1$ (cf. [MPRI20, Section 2.2]). On the one hand, it follows from Proposition 1 and Corollary D of [MPRI20] that, by taking $N$ small if necessary, for each $W \in N$, the quotient map $F_W : S_0/\mathcal{F}_W \to S/\mathcal{F}_W$ associated with the Poincaré return map $P_W : S_0 \to S$ has a sequence of Cantor sets $(C_W^k)_{k \in \mathbb{N}}$ with $\dim_H(C_W^k) \to 1$, provided that $F_W$ is a $C^2$ map. On the other hand, if it satisfies that $-\frac{\lambda_2}{\lambda_1} > -\frac{\lambda_3}{\lambda_1} + r$ with some $r \geq 2$, then $F_W$ is a $C^r$ smooth foliation and the quotient map is also of class $C^r$ (see Section 2.2 of [MPRI20] for details). Therefore, under our hypothesis, the quotient map $F_W$ is a $C^2$ map and thus there is a sequence of Cantor sets $(C_W^k)_{k \in \mathbb{N}}$ for each $W \in N$. Since the Poincaré map $P_W$ preserves the strong stable foliation $\mathcal{F}_W$, we can get the conclusion by repeating the argument for the proof of full Hausdorff dimension of the irregular set of $f_V$. □

Finally, we ask two questions about the generalization of Theorem 4.1.

**Question 4.5.** Does there exist an open and dense set $U$ of the space of three-dimensional smooth vector fields defining a singular-hyperbolic attractor such that the irregular set of $f_V$ has full topological entropy and full Hausdorff dimension for every $V \in U$?

**Question 4.6.** If $f_V$ is the geometric Lorenz flow, then does the completely irregular set $CI(f)$ (cf. [Tia17]) have full topological entropy and full Hausdorff dimension? Moreover, does this hold for $f_W$ for any vector field $W$ $C^1$-close to $V$?

5. The proofs

5.1. Proof of Theorem 2.4. We start from the following preliminary lemma.

**Lemma 5.1.** Let $T : X' \to X'$ and $S : Y \to Y$ be continuous maps on metric spaces $X'$ and $Y$. Assume that there exist a closed invariant subset $X \subset X'$, invariant subsets $X_0 \subset X$ and $Y_0 \subset Y$, and a continuous map $\varphi : X \to Y$ such that $\varphi : X_0 \to Y_0$ is a surjection and $\varphi \circ T = S \circ \varphi$ on $X_0$ (i.e. $T : X_0 \to X_0$ and $S : Y_0 \to Y_0$ are topologically semi-conjugate by $\varphi$). Then, it holds that $\varphi^{-1}(Y_0 \cap I(S)) \subset X_0 \cap I(T)$.

**Proof.** Let $x \in \varphi^{-1}(Y_0 \cap I(S))$. Then $x \in \varphi^{-1}(Y_0) = X_0$ and $\varphi(x) \in Y_0 \cap I(S)$. Thus, there exists a continuous map $\phi : Y \to \mathbb{R}$ such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(S^j(\varphi(x)))$$
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Let \( \psi : X \to \mathbb{R} \) be a continuous function given by \( \psi = \phi \circ \varphi \), and extend it to a continuous function on \( X' \) by the Tietze extension theorem. Then,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(T^j(x)) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ \psi(T^j(x)) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(S^j(\varphi(x)))
\]

also does not exist. Hence, \( x \in X_0 \cap I(T) \).

Remark 5.2. By a similar argument, we can show the flow version of Lemma 5.1, that is, Lemma 5.1 with continuous flows \( T = \{ T^t \}_{t \in \mathbb{R}} \) and \( S = \{ S^t \}_{t \in \mathbb{R}} \) in place of continuous maps \( T \) and \( S \). Note that (10) immediately follows by applying it to \( X' = X = D, X_0 = \Gamma_\varepsilon, T = f_V, Y = Y_0 = \Sigma_\rho, S = \sigma_\rho \) and \( \varphi = \pi^{-1} \).

Now we prove Theorem 2.4. If the topological entropy of \( f \) is zero, then we have nothing to prove. Thus, we can assume that \( h_{\text{top}}(f) > 0 \). According to Misiurewicz’s theorem (see [Rue17, Theorem 4.7]) for every \( 0 < \lambda < h_{\text{top}}(f) \), there exist intervals \( J_1, \ldots, J_p \) and an integer \( k \) such that \( (J_1, \ldots, J_p) \) is a strict \( p \)-horseshoe of \( f^k \) and

\[
\frac{\log p}{k} \geq \lambda.
\]

A strict \( p \)-horseshoe of \( f \) is a collection of \( p \) pairwise disjoints intervals \( (J_1, \ldots, J_p) \) such that \( J_1 \cup \cdots \cup J_p \subset f(J_i) \) for all \( i = 1, \ldots, p \). Moreover, according to Theorem 5.15 and the remarks following Theorem 5.8 of [Rue17], there exist an \( f^k \)-invariant Cantor set \( X \subset J_1 \cup \cdots \cup J_p \) and a continuous map \( \varphi : X \to \Sigma \) with \( \Sigma = \{1, \ldots, p\}^\mathbb{N} \) such that

i) \( \varphi \) is a semi-conjugacy between \( f^k|_X \) and the shift map \( \sigma : \Sigma \to \Sigma \);

ii) there exists an \( f^k \)-invariant countable set \( E \subset X \) such that \( \varphi \) is one-to-one on \( X \setminus E \) and two-to-one on \( E \).

In particular, since \( \sigma : \Sigma \to \Sigma \) is a factor of \( f^k|_X \), we have

\[
k \cdot h_{\text{top}}(X, f) = h_{\text{top}}(X, f^k) \geq h_{\text{top}}(\sigma) = \log p
\]

and consequently, we get that

\[
h_{\text{top}}(f) \geq h_{\text{top}}(X, f) \geq \frac{\log p}{k} \geq \lambda.
\]

That is, the topological entropy of \( f \) is approximated by the topological entropy of \( X \) (a horseshoe). However, we do not know if \( X \) has specification property for \( f^k \) (notice that \( f^k|_X \) is an extension of the full shift but the specification property is preserved a priori only by factors). Thus, we cannot directly apply the argument in Subsection 2.1 of approximation by horseshoes with specification. However, since the semi-conjugacy is, in fact, almost a conjugacy (that is, \( \varphi \) is a one-to-one map except a countable set), we can still approximate the topological entropy of \( f \) by the topological entropy of the irregular set contained in the horseshoe as follows:
Lemma 5.3. $h_{\text{top}}(X \cap I(f), f) \geq \frac{\log p}{\lambda}$.

Proof. Let $X_0 = X \setminus E$ and $\Sigma_0 = \Sigma \setminus A$, where $A = \varphi(E)$. Notice that $A$ is also a countable set and that $I(\sigma) \setminus A = \Sigma_0 \cap I(\sigma)$ and $X_0 \cap I(f) = X \cap I(f) \setminus E$. Observe that since the topological entropy of the countable union of sets equals to the supremum of the topological entropy of each set (see [Bow73, Proposition 2(c)]), the topological entropy of a countable set is zero. We also have that $\varphi : X_0 \to \Sigma_0$ is a continuous bijection. By virtue of (5.9) in [Rue17, Theorem 5.15], it is not difficult to see that $\varphi : X_0 \to \Sigma_0$ is a homeomorphism. Thus, $f^k : X_0 \to X_0$ and $\sigma : \Sigma_0 \to \Sigma_0$ are conjugate. Since topological entropy is invariant under conjugacy (see [Bow73, Proposition 2(a)]), we get that $h_{\text{top}}(Z, \sigma) = h_{\text{top}}(\varphi^{-1}(Z), f^k)$ for all $Z \subset \Sigma_0$. Hence, according to [FFW01, Theorem B] (see also [BS00]) we get that

$$\log p = h_{\text{top}}(\sigma) = h_{\text{top}}(I(\sigma), \sigma)$$

$$= h_{\text{top}}(I(\sigma) \setminus A, \sigma) = h_{\text{top}}(\Sigma_0 \cap I(\sigma), \sigma) = h_{\text{top}}(\varphi^{-1}(\Sigma_0 \cap I(\sigma)), f^k)$$

$$\leq h_{\text{top}}(X_0 \cap I(f), f^k) = h_{\text{top}}(X \cap I(f) \setminus E, f^k)$$

$$= h_{\text{top}}(X \cap I(f), f^k) = k \cdot h_{\text{top}}(X \cap I(f), f).$$

The inequality above follows from the inclusion $\varphi^{-1}(\Sigma_0 \cap I(\sigma)) \subset X_0 \cap I(f)$, which is a straightforward consequence of Lemma 5.1. \hfill \Box

Finally, Lemma 5.3 and (11) imply that

$$h_{\text{top}}(f) \geq h_{\text{top}}(I(f), f) \geq h_{\text{top}}(X \cap I(f), f) \geq \lambda.$$

Since $\lambda < h_{\text{top}}(f)$ is arbitrary, this immediately completes the proof of Theorem 2.4.

5.2. Proof of Theorem 3.10. Let $u$ be the $C^1$-neighborhood of $f$ given in [Yan] so that any $C^{1+}$ diffeomorphism in $u$ has mostly contracting center.

Lemma 5.4. Given $\epsilon > 0$ there exists a dense set $D_{\epsilon} \subset u$ such that for $g \in D_{\epsilon}$ there is an elementary horseshoe $\Lambda_\epsilon$ with $d\mu(\Lambda_\epsilon) > 1 - \epsilon$.

Proof. Since $C^r$-maps are dense in $u$, any element of $u$ can be $C^1$-approximated by $\tilde{f}$ which is $C^2$ and so has mostly contracting center. Consider $\mu$ an ergodic Gibbs u-state for $\tilde{f}$. Since $\dim E^u = 1$ and $\tilde{f}$ is mostly contracting, $\mu$ can only have one positive Lyapunov exponent, which we denote by $\lambda^u$. As $\mu$ is a Gibbs u-state, then the Pesin formula is satisfied and $h_\mu(\tilde{f}) = \lambda^u$.

Because $\mu$ is a hyperbolic measure, by the theorem of Katok [KH99] we can approximate in entropy the measure by elementary horseshoes. One of the results of the recent work of [ACW21] states that this horseshoe can be linearized in $C^1$-topology. More specifically, applying [ACW21, Theorem B'], we can $C^1$-approximate
\( f \) by a \( C^2 \) map \( g \), which has in certain coordinates an affine linear horseshoe \( \Lambda_g \). That is, in a neighborhood of each point in \( \Lambda_g \) there exists a change of coordinates so that the map \( g \) coincides with an affine linear transformation. The constant linear part of the transformation is given by a diagonal matrix \( A \) whose entries are independent of the point in \( \Lambda_g \). Moreover, the logarithms of the eigenvalues of \( A \) are arbitrary close to the exponents of \( \mu \) and the topological entropy of \( \Lambda_g \) is close to the metric entropy \( h_\mu(g) \). Since \( \mu \) has only one positive Lyapunov exponent, thus \( A \) has a unique unstable eigenvalue which we denote by \( \lambda^u_A \).

Actually, we will take the horseshoe to be the so-called standard affine horseshoe, \( \Lambda'_g \subset \Lambda_g \) (see [ACW21, Definition 7.4 and Proposition 7.8]). In this definition, the horseshoe satisfies some extra properties based on the classical model. We observe that the standard affine horseshoe of [ACW21, Proposition 7.8] is with respect to some iterate \( g^N \) of the map \( g \) satisfying \( \frac{1}{N} h_{\text{top}}(\Lambda'_g, g^N) \geq h_{\text{top}}(\Lambda'_g, g) - \epsilon \) for some arbitrary small \( \epsilon \). Observe that the Lyapunov exponents of \( \Lambda'_g \) with respect to \( g^N \) then will also be multiplied by \( N \). In the arguments below, without loss of generality, we may assume that \( \Lambda_g \) is the standard affine horseshoe for \( g \). The horseshoe \( \Lambda_g \) is defined by affine diagonal maps, and projecting these maps to the line spanned by the unstable direction, one obtains an iterated function system on the line given by affine maps of the form: \( T_i(x) = A^u x + c_i \). By taking the inverses \( T_i^{-1} \) we may assume we are working with affine contractions in the line having an invariant attractor \( \Lambda^u_g \).

With respect to the horseshoe \( \Lambda'_g \), it is a standard affine horseshoe, and in this case the entropy \( h_{\text{top}}(\Lambda'_g, g) \) is related to the number of expanding “legs” of the horseshoe, which in turn is equal to the number of the affine maps in the iterated system. In particular, the entropy \( h_{\text{top}}(\Lambda'_g, g) \) is equal to \( \log(k) \), where \( k \) is the number of maps of the iterated system \( \{T_i\} \).

To estimate the Hausdorff dimension of the invariant set \( \Lambda^u_g \) we will use the classical result of Falconer [Fall14] for affine contractions on the line satisfying the open set condition. The open set condition has to do with disjointness of the images of \( T_i \) and since \( \Lambda_g \) is a standard affine horseshoe, the iterated system \( \{T_i\}^{-1} \) will satisfy this property ([ACW21, Definition 7.4]). To calculate \( \dim_H(\Lambda^u_g) \), one has to resolve for \( s \) the equality \( k \cdot (A^u)^{-s} = 1 \). Then we obtain the classical formula

\[
\dim_H(\Lambda^u_g) = \log k \cdot \log(A^u)^{-1} = h_{\text{top}}(\Lambda_g, g) \cdot (\lambda^u_A)^{-1}.
\]

By construction, \( h_{\text{top}}(\Lambda_g) \) is arbitrary close to \( h_\mu(\tilde{f}) \) and \( \lambda^u_A \) is close to the unique unstable exponent \( \lambda^u_\mu \) of \( \mu \). Since \( h_\mu(\tilde{f}) = \lambda^u_\mu \), then \( h_{\text{top}}(\Lambda_g, g) \) is arbitrarily close to \( \lambda^u_A \) and consequently \( \dim_H(\Lambda^u_g) \) is arbitrary close to 1. Moreover, observing that \( \dim_H(\Lambda^u_g) = d^+(\Lambda_g) \) we conclude the proof of the lemma. \( \square \)
Let $g$ be as in the previous lemma. Consider $\tilde{g}$ that is $C^1$-close to $g$ and its respective horseshoe $\Lambda_{\tilde{g}}$, which is the continuation of $\Lambda_g$. There exists a Hölder homeomorphism $\phi$ which conjugates $\Lambda_g$ with $\Lambda_{\tilde{g}}$. Moreover since the unstable dimension is one, for every $x \in \Lambda_g$, the Hölder constant of $\phi|_{\Lambda_g \cap W^u(g)}$ is arbitrary close to 1 (see [KH99, ch.19]). In particular, $d^u(\Lambda_{\tilde{g}})$ varies continuously with respect to the continuation of the horseshoe $\Lambda_{\tilde{g}}$ in a $C^1$-neighborhood of $g$. Thus, we actually have the following.

**Lemma 5.5.** Given $\epsilon > 0$ there exists an open and dense set $R_\epsilon \subset U$ such that for $g \in R_\epsilon$ there exists an elementary horseshoe $\Lambda_g$ whose $u$-index (dimension of the unstable bundle) is 1 and with $d^u(\Lambda_g) > 1 - \epsilon$.

Then taking $\epsilon_n = 1/n$ and $R = \bigcap R_{\epsilon_n}$, we obtain a residual set $R$ in $U$ so that for $g \in R$

$$\sup \{ d^u(\Lambda_g) : \Lambda_g \text{ is an elementary horseshoe with } u\text{-index 1} \} = 1.$$  

Consider now a map $g \in R$ and any elementary horseshoe $\Lambda_g$ with $u$-index 1. In particular, $d_u(\Lambda_g) = \dim(M) - 1$. Applying Proposition 3.1 we obtain that

$$\dim_H I(g) \geq \sup \{ d_s(\Lambda_g) + d^u(\Lambda_g) : \Lambda_g \text{ is an elementary horseshoe with } u\text{-index 1} \} = \dim(M).$$

This completes the proof of the theorem.

### 5.3. Proof of Theorem 3.12

We will need to explain some of the results coming from [BCF18]. Using the non-existence assumption of dominated splitting and the hypothesis that $\Delta^+(f, P) \geq \Delta^+(f, P)$, we can $C^1$-approximate $f$ by a $C^2$-diffeomorphism $g$ having a $u$-conformal affine linear horseshoe $K_g$ with unstable index $d_u = \dim E^u_p$ and stable index $d_s = \dim E^s_p$ (see Theorem 4.1 and Proposition 4.6 of [BCF18]). Moreover, as explained in the proof of [BCF18, Proposition 4.6] this horseshoe has unstable dimension $d^u(K_g) \geq d_u$. Relating this with Proposition 3.1 we get that $\dim_H I(g) \geq d_s + d_u = \dim M$.

Now consider a function $h$ that is $C^1$-close to $g$ and its respective horseshoe $K_h$, which is the continuation of $K_g$. Then there exists a (unique) homeomorphism $\phi$, close to the identity, which conjugates $K_g$ with $K_h$. The functions $\phi$ and $\phi^{-1}$ are actually Hölder continuous with the Hölder constant arbitrarily close to 1 as $h$ gets closer to $g$ (see again the proof of Proposition 4.6 of [BCF18]). Using this conjugation with respect to the unstable irregular set of $K_g$, and applying similar reasoning as in Proposition 3.1 we obtain the following. Given $\epsilon > 0$ there exists a neighborhood $U_\epsilon$ of $g$ so that each $h \in U_\epsilon$ satisfies that $\dim_H I(h) > \dim M - \epsilon$. 


To conclude the $C^1$-genericity in the statement of the theorem, one can use known $C^1$-generic properties together with standard Baire arguments. This is described in [BCF18], particularly in the proof of Theorem 5.

5.4. **Proof of Lemma 4.3.** We first recall results of [BS00] and [Tho10] for preparation. Let $f$ be a continuous map on a compact invariant set $\Lambda$ of a smooth manifold and $\rho : \Lambda \to \mathbb{R}_+$ a continuous function. Given a continuous function $\phi : \Lambda \to \mathbb{R}_+$, we define a $\rho$-weighted $\phi$-irregular set $I_\rho(\phi, f)$ by

$$I_\rho(\phi, f) = \left\{ x \in \Lambda : \lim_{n \to \infty} \frac{S_n \phi(x)}{S_n \rho(x)} \text{ does not exists} \right\},$$  \hfill (12)

where

$$S_n \psi(x) = \frac{1}{n} \sum_{j=0}^{n-1} \psi \circ f^j(x)$$

for each real-valued continuous function $\psi$. Feng, Lau and Wu showed that if $f$ is of class $C^{1+\alpha}$ and $\Lambda$ is a repeller of $f$, then for each continuous function $\phi$ such that $I_\rho(\phi, f|_\Lambda) \neq \emptyset$,

$$\dim_H I_\rho(\phi, f|_\Lambda) = \dim_H \Lambda,$$

see [FLW02, Theorem 1.2] (compare it with [BS00, Theorem 4.2 (1) and 7.1]). Therefore, the proof of Proposition 3.1 works to prove that

$$I_\rho(\phi, f|_\Lambda) \neq \emptyset \quad \Rightarrow \quad \dim_H I_\rho(\phi, f|_\Lambda) \geq d_s(\Lambda) + d^u(\Lambda)$$  \hfill (13)

for each $C^{1+\alpha}$-diffeomorphism $f$ of a compact manifold with a $u$-conformal elementary horseshoe $\Lambda$ of $s$-index $d_s(\Lambda)$.

Moreover, it follows from [Tho10, Lemma 5.4] that for any continuous function $\phi : X_\rho \to \mathbb{R}$, if we define a continuous function $\iota(\phi) : X \to \mathbb{R}$ by

$$\iota(\phi)(x) = \int_0^{\rho(x)} \phi(x, z) \, dz \quad \text{for } x \in X,$$

then we have

$$I(\phi, T_\rho) = \left\{ (x, z) : x \in I_\rho(\iota(\phi), T), \ 0 \leq z < \rho(x) \right\}. \hfill (14)$$

In particular, $I(\phi, T_\rho) \neq \emptyset$ if and only if $I_\rho(\iota(\phi), T) \neq \emptyset$.

**Proof of Lemma 4.3.** We first prove the first assertion in Lemma 4.3 (i.e. the existence of a continuous function $\phi$ for which $I(\phi, \sigma_\rho) \neq \emptyset$). Indeed, we will construct a continuous map $\phi : \Sigma_\rho \to \mathbb{R}$ such that $I_\rho(\iota(\phi), \sigma) \neq \emptyset$. Due to (14), this immediately
implies that $I(\phi, \sigma_p) \neq \emptyset$. We follow the argument in [Tak08, Section 4]. For clarity, we let $\Sigma$ endowed with a standard metric $d_\Sigma$ given by

$$d_\Sigma(x, y) = \sum_{m \in \mathbb{Z}} \frac{|x_m - y_m|}{\beta^{|m|}}$$

for $x = (x_m)_{m \in \mathbb{Z}}$ and $y = (y_m)_{m \in \mathbb{Z}}$, where $\beta > 1$.

Set $\mathcal{L}(\Sigma) = \{x \in \bigcup_{n \geq 1} [1, 2, \ldots, N]^n \mid C(x) \neq \emptyset\}$, where $C(x) = \{y \in \Sigma \mid (y_1, \ldots, y_n) = x\}$ for $x \in \{1, 2, \ldots, N\}^n$. Since $\Sigma$ is a topologically mixing subshift of finite type with $N \geq 2$, there are two different periodic points $p^0$ and $p^1$, and $\sigma : \Sigma \to \Sigma$ satisfies the specification property, i.e. there is an integer $L > 0$ such that for any $x, y \in \mathcal{L}(\Sigma)$, one can find $z \in \mathcal{L}(\Sigma)$ with $|z| = L$ such that $zxy \in \mathcal{L}(\Sigma)$, where $|z|$ is the length of the word $z$ and $zxy$ is the concatenation of $x$, $z$ and $y$. Let $L_j$ be the period of $p^j$ ($j = 0, 1$). Since $\Sigma$ is compact and $\rho : \Sigma \to \mathbb{R}$ is continuous, there is a constant $C > 1$ such that

$$C^{-1} \leq \rho(x) \leq C \quad \text{for every } x \in \Sigma. \quad (15)$$

Take a strictly increasing sequence of positive integers $(n_j)_{j \in \mathbb{N}}$ such that

$$\liminf_{m \to \infty} \frac{n_{2m-1}L_1}{N_{2m-1}} > \frac{2}{3}, \quad \liminf_{m \to \infty} \frac{n_{2m}L_0}{N_{2m}} > 1 - \frac{1}{3C^2}, \quad N_m = mL + \sum_{k=1}^{m} n_kL(k \mod 2). \quad (16)$$

Let $A_j$ be a small neighborhood of the orbit of $p^j$ ($j = 0, 1$) such that the closures of $A_1$ and $A_2$ are disjoint. We then let $\hat{x} = (\hat{x}_m)_{m \in \mathbb{Z}}$ be a point such that

$$[\hat{x}_0] = z_1[p^1]_{L_1}^{n_{L_1}-1}z_2[p^0]_{L_0}^{n_{L_0}-1}z_3[p^1]_{L_1}^{n_{L_1}-1}z_4[p^0]_{L_0}^{n_{L_0}-1}z_5[p^1]_{L_1}^{n_{L_1}-1}z_6[p^0]_{L_0}^{n_{L_0}-1} \cdots,$n_{L_1}^{n_{L_1}-1}\cdots,$

with some words $z_j \in \mathcal{L}(\Sigma)$ of length $L$, where $[x]_n = (x_n x_{n+1} \cdots x_m)$ for $x = (x_m)_{m \in \mathbb{Z}}$. Notice that we have no requirement on $\hat{x}_m$ with $m \leq -1$. Fix $\epsilon > 0$ such that the $\epsilon$-neighborhood of $p^j$ is included in $A_j$ for each $j = 0, 1$. Let $m_0$ be a positive integer such that

$$\sum_{|m| > m_0} \frac{1}{\beta^{|m|}} < \epsilon.$$

In other words,

$$x_m = y_m \quad \text{for all } |m| \leq m_0 \quad \Rightarrow \quad d_\Sigma(x, y) < \epsilon.$$

Since $(n_j)_{j \in \mathbb{N}}$ is increasing, one can find $j_1$ such that $n_j \min\{L_1, L_2\} \geq 2m_0 + 1$ for all $j \geq j_1$. For each $j \geq j_1$, consider $\hat{x}^j = \sigma^{|n_j|+L+m_0}(\hat{x})$. In the case that $j$ is odd, it satisfies

$$[\hat{x}^j]_{n_jL_1-m_0}^{n_jL_1} = [p^1]_{L_1}^{n_jL_1-1}.$$

Similar expression with $L_0$, $p^0$ instead of $L_1$, $p^1$ holds if $j$ is even. Thus,

$$d_\Sigma\left(\sigma^{N_j+L+\ell}(\hat{x}), \sigma^{\ell}(p^{(j \mod 2)})\right) < \epsilon \quad \text{for all } m_0 \leq \ell \leq n_jL(j \mod 2) - m_0 - 1. \quad (17)$$
Take a continuous function \( \psi : \Sigma \to \mathbb{R} \) such that
\[
0 \leq \psi(x) \leq 1 \quad \text{for each } x \in \Sigma, \quad \psi(x) = j \quad \text{for each } x \in A_j
\] 
with \( j = 0, 1 \), and define a function \( \phi : \Sigma_0 \to \mathbb{R} \) by
\[
\phi(x, z) = \psi(x) + \frac{z}{\rho(x)}(\psi \circ \sigma(x) - \psi(x)).
\]
Then, it is straightforward to see that \( \phi \) is continuous and \( \psi = \iota(\phi) \). Since the \( \epsilon \)-neighborhood of \( p^j \) is included in \( A_j \), by (17)
\[
\psi \circ \sigma^{N_j+1+L+\ell}(\hat{x}) = \begin{cases} 
1 & \text{if } j \text{ is odd} \\
0 & \text{if } j \text{ is even}
\end{cases}
\quad \text{for all } m_0 \leq \ell \leq n_j L_{(j \mod 2)} - m_0 - 1. 
\] 

On the other hand, by (18), it also holds that \( \psi \circ \sigma^\ell(\hat{x}) \geq 0 \) for all \( \ell \geq 0 \). Therefore, taking into account (15) and (19), for each odd \( j \) we have
\[
\frac{S_{N_j} \psi(\hat{x})}{S_{N_j} \rho(\hat{x})} \geq \frac{1}{CN_j} \sum_{n=0}^{N_j-1} \psi \circ \sigma^n(\hat{x}) + \frac{1}{CN_j} \sum_{\ell=m_0}^{n_j-m_0-1} \psi \circ \sigma^{N_j+1+L+\ell}(\hat{x})
\]
\[
= \frac{(n_j L_1 - m_0 - 1) - m_0 + 1}{CN_j} = \frac{n_j L_1}{CN_j} - \frac{2m_0}{CN_j}.
\]

Similarly, for even \( j \), with a discrete interval \( I_j := \{ n \in \mathbb{Z} \mid 0 \leq n \leq N_j - 1 \} \setminus \{ N_j - 1 + L + \ell \mid m_0 \leq \ell \leq n_j L_0 - m_0 - 1 \},
\[
\frac{S_{N_j} \psi(\hat{x})}{S_{N_j} \rho(\hat{x})} \leq \frac{1}{C-1 N_j} \sum_{n=0}^{N_j-1} \psi \circ \sigma^n(\hat{x}) = \frac{1}{C-1 N_j} \sum_{n \in I_j} \psi \circ \sigma^n(\hat{x})
\]
\[
\leq \frac{\#I_j}{C-1 N_j} = \frac{N_j - [(n_j L_0 - m_0 - 1) - m_0 + 1]}{C-1 N_j} = C \left( 1 - \frac{n_j L_0}{N_j} \right) + \frac{2Cm_0}{N_j}.
\]

Therefore, by (16) we get that
\[
\liminf_{m \to \infty} \frac{S_{N_{2m-1}} \psi(\hat{x})}{S_{N_{2m-1}} \rho(\hat{x})} > \frac{2}{3C} = \limsup_{m \to \infty} \frac{S_{N_{2m}} \psi(\hat{x})}{S_{N_{2m}} \rho(\hat{x})}.
\]

In particular, \( \hat{x} \in I_{\rho}(\psi, \sigma) = I_{\rho}(\iota(\phi), \sigma) \). This completes the proof of the first assertion in Lemma 4.3.

We next show the items (i) and (ii) of Lemma 4.3. Let \( T \) be a \( C^{1+\alpha} \)-diffeomorphism \( T \) having a \( u \)-conformal elementary horseshoe \( \Lambda \) for which \( |T|_{\Lambda} \) is topologically conjugate to \( \sigma \) by a homeomorphism \( \Pi : \Lambda \to \Sigma \). Then, it is straightforward to see that
\[
I(\hat{\phi}, (T|_{\Lambda})_{\hat{\rho}}) = \Pi^{-1} \left( I(\phi, \sigma_{\hat{\rho}}) \right)
\]
with \( \hat{\phi} \) (induced by the function \( \phi \) constructed in the above argument) and \( \hat{\rho} \) given in the statement of Lemma 4.3. This immediately implies that \( I(\hat{\phi}, (T|_{\Lambda})_{\hat{\rho}}) \neq \emptyset \),
particularly the item (i), because of the previous result $I(\phi, \sigma_p) \neq \emptyset$. Therefore, it follows from (14) that

$$\dim_H I(\hat{\phi}, (T_\Lambda)_\rho) = 1 + \dim_H I_p(\iota(\hat{\phi}), T_\Lambda).$$

Combining this with (13) and (14), we get that

$$\dim_H I(\hat{\phi}, (T_\Lambda)_\rho) \geq 1 + d_\rho(\Lambda) + d^u(\Lambda)$$

which completes the proof. □

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