Description of diffraction grating experiments for photons and electrons in Feynman’s space-time formulation of quantum mechanics: The quantum origins of classical wave theories of light and massive particles

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Abstract

The five laws of relativistic quantum mechanics, according to Feynman’s path integral formulation, are concisely stated and applied to experiments. Reflection diffraction grating experiments for both photons and electrons are analysed, in particular the Davisson-Germer experiment in which the wave-like property of electrons was first established. It is shown how classical, purely spatial, effective wave theories for both photons and electrons are predicted by the path integral formulation of quantum mechanics. The standard Copenhagen interpretation of wave mechanics is critically discussed in the light of the described experimental applications of the path integral formulation.

PACS 03.30.+p
1 Introduction

Feynman’s path integral formulation of quantum mechanics [1, 2, 3] based on cited earlier work of Heisenberg [4] and Dirac [5] has two distinct parts. The first is a set of rules (the laws I–V of Section 2 below) concerning the construction and interpretation of probability amplitudes for space-time experiments in quantum mechanics[2]. These rules are valid in both the non-relativistic and relativistic theories. The second part is the detailed mathematical development of the non-relativistic limit [3]. The most detailed working-out of Feynman’s space-time concepts for quantum mechanics is to be found, not in the research literature or text books, but in the popular book ‘QED the strange story of light and matter’ [6] published shortly before his death. The photons of which light consists are, of course, unlike the particles considered in Refs. [1, 2, 3], ultra-relativistic. In this book, many experiments on propagation, reflection, refraction, diffraction and interference of light, which are conventionally described by the classical wave theory, are all treated as applications of Feynman’s path integral formulation of quantum mechanics. Although a complete physical description of the experiments was given (neglecting only polarisation effects) no equations were employed. Many of the experiments described in this book have been worked out in full mathematical detail in a previous paper [7] by the present author. The present paper also describes physical optics experiments for both photons and massive particles (electrons) in a similar manner to Ref. [7], but has more avowedly pedagogical goals. In particular, the path integral analysis of the experiments, is confronted, in a critical manner, with typical concepts of the ‘Copenhagen Interpretation’ that are found in quantum mechanical text books.

Before presenting a space-time analysis\(^1\) of the original Davisson-Germer experiment [8] in which ‘matter waves’ were discovered, a reflection diffraction grating experiment with a similar geometry, but using photons produced in the decays of excited atoms, is analysed. Although the two experiments are explained similarly by classical, spatial, wave theories with similarly defined phenomenological de Broglie wavelengths, the underlying space-time physics is seen to be entirely different in the two cases. This distinction was previously pointed out in Ref. [7] for the case of Young double slit experiments using photons or electrons.

The plan of this paper is as follows: In the following section Feynman’s rules for constructing and interpreting probability amplitudes [1, 2, 3, 9, 7] are reviewed. In Section 3 the space-time propagator for a free relativistic particle is derived. The following three sections present space-time analyses of different experiments: in Section 4, one in which a photon, produced in the decay of an excited atom, is detected, in Section 5, an experiment with a similarly produced photon scattered from a reflection diffraction grating, and in Section 6 a similar experiment using an electron beam —the Davisson-Germer experiment. In Section 7 it is shown, following, Ref. [7], how the path integral formulation of quantum mechanics leads to similar, purely spatial, classical wave theories for both photons and massive particles, in spite of completely different underlying space-time processes in the two cases. The final section contains a critical discussion, in the light of the experiments presented, of some basic concepts within the standard ‘Copenhagen Interpretation’ [10, 11] of wave-mechanics, such as wave packets and uncertainty relations.

\(^1\)As in Refs. [6, 7] polarisation effects are neglected throughout the present paper.
Finally the interpretation of the famous ‘Schrödinger’s cat’ experiment [12] within the path integral formulation is considered.

2 Feynman’s conceptual formulation of quantum mechanics

Feynman’s path integral formulation of quantum mechanics is the development of earlier work by Heisenberg and Dirac in which the axiomatic basis of the formulation was already specified. Heisenberg’s work [4], concerned the manner in which probability amplitudes are to be constructed and interpreted. Dirac’s seminal paper ‘The Lagrangian in Quantum Mechanics’ [5] provided the dynamical foundation of the theory by specifying the connection between a quantum mechanical path amplitude and the Lagrangian function of the corresponding classical system. This work of Heisenberg and Dirac is valid in both the non-relativistic and relativistic limits of the theory. Feynman chose, in his original path integral paper [2], to consider only the non-relativistic case for detailed mathematical treatment. However, Section 2 of this paper, ‘The superposition of probability amplitudes’, reviewing the earlier work of Heisenberg [4], is applicable also to the relativistic theory. The construction of probability amplitudes for a space-time quantum mechanical experiment, and how such an experiment differs from the corresponding classical experiment, which, according to Heisenberg, ‘when stated in a sufficiently general form’ is ‘the center of the whole quantum theory’ [4], will now be described.

Suppose that some quantum mechanical system is prepared in the state \( |i\rangle \) and measured to be in the state \( |f\rangle \), having, at some intermediate time, passed through the state \( |k\rangle \)\(^2\). The probability amplitude to measure the state \( f \) given the prepared state \( i \) is:

\[
A_{fi} = \langle f |k| i \rangle = \langle f |k\rangle \langle k |i \rangle \equiv A_{fk} A_{ki}
\] (2.1)

where the Dirac Bra-Ket notation [13] for states and transition amplitudes is employed. In this formula the amplitudes \( A \) are complex numbers determined by the underlying physics of the process considered. They are typically either space-time propagators giving the amplitude to find a particle at some position when it is at a known position at some earlier time, or the amplitude for a particle scattering or production process.

According to the Born rule [14] the probability to observe the state \( f \) given the states \( i \) and \( k \) is:

\[
P_{fi} = |\langle f |k| i \rangle|^2 = |\langle f |k\rangle|^2 |\langle k |i \rangle|^2 = P_{fk} P_{ki},
\] (2.2)

This formula expresses the law of conditional probability: the probability of \( f \) given \( k \) and \( i \) is the probability of \( k \) given \( i \) times the probability of \( f \) given \( k \). If \( k \) takes, with equal probabilities, several different, but unknown, values, the overall probability \( P^{\text{Cl}}_{fi} \) (where the suffix Cl stands for ‘classical’) of \( f \), given \( i \), is:

\[
P^{\text{Cl}}_{fi} = \sum_k P^k_{fi} = \sum_k P_{fk} P_{ki} = \sum_k |A_{fk}|^2 |A_{ki}|^2.
\] (2.3)

\(^2\)For concretness, following Feynman [2], the case of a particle initially at some spatial position in the state \( |x_i\rangle \) and detected in another \( |x_f\rangle \) having passed through an intermediate state \( |x_k\rangle \) may be considered, though the formula (2.1) is of quite general validity.
i.e. the overall probability is the sum of the conditional probabilities. In quantum mechanics, the formula (2.3) for the case that the intermediate state $|k\rangle$ is one of a set specified by the label $k$, but the value of $k$ is unknown, is replaced by:

$$P_{fi} = |A_{fi}|^2 = |\sum_k A_{f_i}^k|^2 = |\sum_k A_{fk}A_{ki}|^2$$

(2.4)

where

$$A_{fi} \equiv \sum_k A_{f_i}^k = \sum_k A_{fk}A_{ki}$$

(2.5)

expresses the principle of superposition of the path amplitudes $A_{f_i}^k$.

The difference between (2.3) and (2.4) and the non-intuitive nature of the latter equation constitute what Feynman called (as exemplified in the Young double slit experiment) the only mystery of interpretation of quantum mechanics [15]. It is the same as what Heisenberg earlier called ‘the center of the whole quantum theory’ [4].

Although it is not possible to ‘explain’ the fundamental formula (2.4) in terms of our understanding of space, time and causality in the world of everyday experience, it is at least possible to state how the classical formula (2.3) differs from the quantum mechanical one (2.4) that describes the real world. Both (2.3) and (2.4) make predictions for probabilities concerning a statistical ensemble of experiments in which $k$ is allowed to vary for fixed values of $i$ and $f$. Both predictions are ‘probabilistic’. It is not the case that the classical formula is deterministic and the quantum one probabilistic. However, the classical formula is probabilistic because the value of $k$, although existing, (i.e. corresponding to a physically existing trajectory of a particle or a definite time-ordered sequence of space-time events) is not known, whereas for the quantum formula the value of $k$ is not simply unknown, but in principle unknowable, without destroying the assumed experimental conditions. The probabilistic nature of the quantum formula is therefore not a consequence of simple ignorance of the values of some actually-existing physical parameter. The logical basis of (2.3) is that, in the different experiments of the ensemble, each one corresponds to a unique value of $k$. Say $k = k'$ for one experiment and $k = k''$ for another, corresponding, according to (2.2) to the conditional probabilities $P_{fi}^{k'}$ and $P_{fi}^{k''}$, which are added in as contributions to the overall probability in (2.3). That is, each value of $k$ corresponds to a distinct causal chain in space-time which may be called a ‘classical history’. On the other hand, in (2.4), different values of $k$ contribute to every measurement of $f$, as if, loosely speaking, the quantum system is simultaneously occupying all possible intermediate states allowed by the experimental configuration. However, the conventional notions of space and time must still be applied to calculate correctly the path amplitudes $A_{f_i}$, i.e. particles of known identity and kinematical properties are assumed to be produced, destroyed or scattered and to propagate in space-time according to the classical laws of space-time geometry and kinematics3. Particle concepts are therefore essential to calculate the path amplitudes and any ‘wave’ concept is irrelevant. This will become clear in the specific space-time experiments to be discussed in the following sections. When the path amplitudes $A_{f_i}^k$ are combined, according to the superposition law (2.5), to give the quantum probability amplitude $A_{fi}$, the modulus squared of which gives the probability

3‘classical’ is used here in the sense of ‘non-quantum’. The kinematical formulas are, in general, those of relativistic, not classical, mechanics.
to measure the state $|f\rangle$ when the state $|i\rangle$ is prepared, but the experimental configuration allows all members of the set of intermediate states $|k\rangle$, there is a breakdown of the correspondence between different values of $k$ and different classical histories which occurs in (2.3). Quantum mechanical superposition requires that amplitudes (not probabilities) for different classical histories are added to form the overall probability amplitude for the experiment, like the simultaneous overlapping of different melodies (each one musically coherent) in musical counterpoint. It is as though the unique causal chain of classical physics is replaced, in constructing the probability amplitude, by an, in general, infinite number of such chains in one-to-one correspondence with the path amplitudes when a single measurement of the final state $|f\rangle$ is performed. This is reminiscent of Everett’s ‘Many Worlds’ interpretation [16] except that the correspondence is between the one unique actual world described by quantum mechanics and all the different distinct classical histories that must be considered in the calculation of the overall probability amplitude for an experiment. The essentially non-classical (and counter intuitive) aspect of the situation is that a correspondence exists between a single observed quantum system (for example the photon or electron discussed in the following sections of the present paper) and all path amplitudes consistent with the experimental configuration.

The fundamental formula (2.4) generalises, by iteration, to give [7]:

$$P_{FI} = \sum_{m} \sum_{l} \left| \sum_{k_n} \sum_{k_2} \sum_{k_1} \langle f_m | k_n, \ldots, k_2, k_1 | i_l \rangle \right|^2. \quad (2.6)$$

The quantity $P_{FI}$ is the probability to observe any one of the set of final states $F$: $|f_m\rangle$, $m = 1, 2, 3, \ldots$ when any one of the set of initial states $I$: $|i_l\rangle$, $l = 1, 2, 3, \ldots$ is prepared and where $|k_1\rangle, |k_2\rangle, \ldots |k_n\rangle$ are intermediate states that are unobserved, but which must be specified in order to calculate the path amplitude:

$$\langle f | k_n, \ldots, k_2, k_1 | i \rangle \equiv A_{fi}^{k_n, \ldots, k_2, k_1} = \langle f | k_n \rangle \langle k_n | k_{n-1} \rangle \ldots \langle k_2 | k_1 \rangle \langle k_1 | i \rangle. \quad (2.7)$$

Three fundamental quantum mechanical laws [2, 3, 9, 7] are incorporated in (2.6):

(I) The Born probability interpretation of the amplitudes:

$$P_{fi} = |A_{fi}|^2. \quad (2.8)$$

(II) Sequential factorisation of temporally-ordered amplitudes:

$$A_{fi}^k = A_{fk} A_{ki}. \quad (2.9)$$

(III) Superposition of the path amplitudes $A_{fi}^k$:

$$A_{fi} = \sum_k A_{fi}^k = \sum_k A_{fk} A_{ki}. \quad (2.10)$$

Notice that matrix multiplication, introduced into quantum mechanics by Heisenberg [17] and refined by the work of Born and Jordan [18] and by Dirac in his ‘transformation theory’ [19], is axiomatically embodied in the laws II and III.
A fourth important quantum mechanical law is applicable when the initial and final states are tensor products such as $|i(1)\rangle \otimes |i(2)\rangle$, $|f(1)\rangle \otimes |f(2)\rangle$. Such states occur in experiments where two or more particles are detected in coincidence in the final state. Such experiments are conventionally described as having an 'entangled wavefunction' [12]. In the case that there are no intermediate states in common in the path amplitudes $\langle f(1)|k_n^{(1)},...k_2^{(1)},k_1^{(1)}|i(1)\rangle$, $\langle f(2)|k_n^{(2)},...k_2^{(2)},k_1^{(2)}|i(2)\rangle$, the probability amplitude is given by the law:

(IV) Composite factorisation:

$$A_{f(1)f(2)i(1)i(2)} = A_{f(1)i(1)}^{k_n^{(1)},...k_2^{(1)},k_1^{(1)}} A_{f(2)i(2)}^{k_n^{(2)},...k_2^{(2)},k_1^{(2)}}.$$  \hspace{1cm} (2.11)

Since there is no discussion, in the present paper, of experiments with entangled wavefunctions, no applications of (2.11) are considered. A brief discussion of composite factorisation in the annihilation of para-positronium: $e^+e^-(1S_0) \rightarrow \gamma\gamma$ may be found in Ref. [7].

The laws I-IV above describe simply how probability amplitudes are to be constructed and interpreted. They have no dynamical content. Feynman [2], restricting his discussion to the trajectories of particles in space-time, introduced dynamics into the problem, following Dirac [5], in the second of the two postulates:

A If an ideal measurement is performed to determine whether a particle has a path lying in a region of spacetime, the probability that the result will be affirmative is the absolute square of a sum of complex contributions, one from each path in the region.

B The paths contribute equally in magnitude but the phase of their contribution is the classical action (in units of $\hbar$) i.e. the time integral of the Lagrangian taken along the path.

These postulates give the following law$^4$ specifying the probability amplitude, $A_{BA}$, that a particle, initially, at time $t_A$, at position $\vec{x}_A$, will be found at later time $t_B$, at position $\vec{x}_B$, having followed the spacetime paths: $[\vec{x}^j(t)]$, $j = 1, 2, 3,...$:

(V) The Feynman path integral:

$$A_{BA} = \sum_j A_{jBA}^{(j)} = \sum_j N^{(j)} \exp \left\{ i \frac{S_{BA}([\vec{x}^{(j)}(t)])}{\hbar} \right\}$$ \hspace{1cm} (2.12)

where

$$S_{BA}([\vec{x}^{(j)}(t)]) = \int_{t_A}^{t_B} L([\vec{x}^{(j)}(t)], t) dt.$$ \hspace{1cm} (2.13)

$^4$Note that this law actually incorporates those of sequential factorisation, II, and superposition, III.
In (2.12) \( N^{(j)} \) is a (possibly space-time dependent) normalisation factor. A particular path \( \vec{x}(t)^{(j)} \) is specified by an array of space-time coordinates. Considering one spatial dimension:

\[
[x(t)^{(j)}] : \quad x_A^{(j)}, t_A^{(j)}; \quad x_1^{(j)}, t_1^{(j)}; \quad x_2^{(j)}, t_2^{(j)}; \ldots \quad x_n^{(j)}, t_n^{(j)}; \ldots \quad x_B^{(j)}, t_B^{(j)}
\]

\( t_A^{(j)} < t_1^{(j)} < t_2^{(j)} \ldots < t_n^{(j)} \ldots < t_B^{(j)} \).

Notice that the velocity argument, \( \dot{x}(t) \equiv dx(t)/dt \), of the Lagrangian is implicit in the specification of the path \( x(t)^{(j)} \):

\[
\dot{x}^{(j)}(t_n^{(j)}) = \text{Lim}(t_{n+1}^{(j)} \to t_{n-1}^{(j)}) \frac{x_{n+1}^{(j)} - x_{n-1}^{(j)}}{t_{n+1}^{(j)} - t_{n-1}^{(j)}}.
\]

This completes the presentation of the laws of Feynman’s formulation of quantum mechanics. All the laws are equally valid in both the non-relativistic and relativistic limits. Indeed a prime motivation for Dirac’s introduction, in (2.12), of the Lagrangian was to enable the construction of a relativistic theory, which is not possible in a Hamiltonian-based formulation due to the lack of symmetry between temporal and spatial coordinates.

3 The relativistic space-time propagator of a free particle

Writing the exponential factor in Eq. (2.12) as \( \exp(i\phi) \) the phase, \( \phi \), is given for a particle of Newtonian mass \( m \), moving with velocity \( \vec{v} \) in free space, by the relations:

\[
\phi = \int_{t_A}^{t_B} \frac{L}{\hbar} dt = -\frac{mc^2}{\hbar} \left( \sqrt{1 - \beta^2} \right) t \equiv -\frac{mc^2}{\hbar} t = -\frac{mc^2}{\hbar} \gamma t = -\frac{P \cdot R}{\hbar} = \frac{-Et + \vec{p} \cdot \vec{r}}{\hbar}.
\]

where \( \beta \equiv v/c, t \equiv t_B - t_A \), \( \tau \) is a proper time interval for the particle, \( \vec{r} \equiv \vec{x}_B - \vec{x}_A \) and \( R \) and \( P \) are space-time and energy-momentum four-vectors of the particle:

\[
R = (R^0, \vec{R}) \equiv (ct, \vec{r}),
\]

\[
P = (P^0, \vec{P}) \equiv (\gamma mc, \gamma m\vec{v}) \equiv \left( \frac{E}{c}, \vec{p} \right),
\]

\[
P \cdot R \equiv P^0 R^0 - \vec{R} \cdot \vec{P}
\]

where \( E \) and \( \vec{p} \) are the relativistic energy and momentum of the particle. The expression (3.1) incorporates the well-known [20, 21] non-covariant relativistic Lagrangian: \( L = -mc^2 \sqrt{1 - \beta^2} \) for a free particle as well as the relativistic time-dilation relation \( t = \gamma \tau \). Denoting, following Feynman, the amplitude to find a particle, originally at \( \vec{x}_A \) at time \( t_A \), at position \( \vec{x}_B \) at time \( t_B \), the ‘kernel’, or ‘Green’s function’ or ‘space-time propagator’, by \( K(\vec{x}_B, t_B; \vec{x}_A, t_A) \), it follows from (2.12) and (3.1) that:

\[
K(\vec{x}_B, t_B; \vec{x}_A, t_A) = K(\vec{r}, t) = N \exp \left\{ -i \frac{(Et - \vec{p} \cdot \vec{r})}{\hbar} \right\}.
\]
The functional dependence of the kernel on $\vec{r}$ is a consequence of translational invariance. Because the vectors $\vec{r}$ and $\vec{p}$ are parallel, the phase in (3.1) can be written as

$$\hbar \phi = \vec{p} \cdot \vec{r} - Et = pr - Et = p(r - v_\phi t)$$  \hspace{1cm} (3.6)$$

where a hypothetical, superluminal, 'phase velocity':

$$v_\phi \equiv \frac{E}{p} = \frac{c^2}{v} \geq c$$  \hspace{1cm} (3.7)$$

has been defined by mathematical substitution. The kernel can then be written in a 'wave-like' manner as:

$$K = N \exp \{ i \phi \} = N f(r - v_\phi t)$$  \hspace{1cm} (3.8)$$

where the last member of (3.8) shows the functional dependence of $K$ on $r$ and $t$. Because of this dependence, the kernel satisfies a classical wave equation with phase velocity $v_\phi$.

The kernel will therefore respect the spherical symmetry of free space if it satisfies the differential equation:

$$\nabla^2 K = \frac{\partial^2 K}{\partial r^2} + \frac{2}{r} \frac{\partial K}{\partial r} = \frac{1}{v_\phi^2} \frac{\partial^2 K}{\partial t^2}$$  \hspace{1cm} (3.9)$$

or

$$\frac{\partial^2 (rK)}{\partial r^2} = \frac{1}{v_\phi^2} \frac{\partial^2 (rK)}{\partial t^2}$$  \hspace{1cm} (3.10)$$

which has the general solution [22]:

$$K = \frac{1}{r} \left[ f(r - v_\phi t) + g(r + v_\phi t) \right].$$  \hspace{1cm} (3.11)$$

Comparing (3.8) and (3.11) it is seen that

$$N = \frac{1}{r}$$  \hspace{1cm} (3.12)$$

so that $K$ has the space-time dependence of a harmonic 'spherical wave'.

An alternative derivation of (3.12) is provided by adapting, to the relativistic case, Feynman’s original calculation of normalisation constants in the non-relativistic path integral. The kernel $K$ determines the space-time evolution of a wavefunction $\psi(\vec{x}, t)$ according to the integral equation:

$$\psi(\vec{x}_B, t_B) = \int \int \int K(\vec{x}_B, t_B; \vec{x}_A, t_A) \psi(\vec{x}_A, t_A) d^3 x_A.$$  \hspace{1cm} (3.13)$$

Setting $t_B = t$, $t_A = t - \epsilon$, where $\epsilon$ is a small fixed increment of time, and $\vec{x}_B - \vec{x}_A = \vec{\eta}$, as well as choosing the origin of spatial coordinates so that $\vec{x}_B = 0$, gives:

$$\psi(0, 0, 0, t) = \int \int \int K(0, t; -\vec{\eta}, t - \epsilon) \psi(-\vec{\eta}, t - \epsilon) d^3 \eta.$$  \hspace{1cm} (3.14)$$

Substituting for the kernel from Eqs. (3.5) and (3.1) and choosing the $\eta_1$ axis parallel to the momentum of the particle gives:

$$\psi(0, 0, 0, t) = \int \int \int N \exp \left\{ \frac{i \epsilon}{\hbar} L \left( \frac{|\vec{\eta}|}{\epsilon} \right) \right\} \psi(-\vec{\eta}, t - \epsilon) \delta(\eta_2) \delta(\eta_3) d^3 \eta.$$  \hspace{1cm} (3.15)$$

\text{---}

\text{See Ref. [3] Eq. (3.42) p. 57.}
Integrating over $\eta_2$ and $\eta_3$, noting that $\eta_1 = \eta = |\vec{\eta}|$ and substituting $L$ as given in (3.1):

$$
\psi(0, 0, 0, t) = \int N \exp\left\{ -i\frac{e}{\hbar} mc^2 \sqrt{1 - \left(\frac{\eta}{c\epsilon}\right)^2} \right\} \psi(-\eta, 0, 0, t - \epsilon) d\eta.
$$

(3.16)

Making Taylor expansions of $\psi(-\eta, 0, 0, t - \epsilon)$ in the small quantities $\eta$ and $\epsilon$ and retaining only the zeroth order term gives

$$
\psi(0, 0, 0, t) = \int N \exp\left\{ -i\frac{e}{\hbar} mc^2 \sqrt{1 - \left(\frac{\eta}{c\epsilon}\right)^2} \right\} \psi(0, 0, 0, t) d\eta.
$$

(3.17)

Consistency of the zeroth order terms in $\epsilon$ on both sides of this equation requires

$$
1 = N \int d\eta = N\eta
$$

(3.18)

or

$$
N = \frac{1}{\eta}
$$

(3.19)
in agreement with (3.12) above.

Combining (3.5) with (3.12) or (3.19) then gives, for the relativistic space-time propagator of a free particle of mass $m$ and relativistic momentum and energy $\vec{p}$ and $E$, the expression:

$$
K(\vec{x}_B, t_B; \vec{x}_A, t_A) = \frac{1}{|\vec{x}_B - \vec{x}_A|} \exp\left\{ -i \frac{mc^2}{\hbar} (\tau_B - \tau_A) \right\}
$$

$$
= \frac{1}{|\vec{x}_B - \vec{x}_A|} \exp\left\{ -i \frac{E(t_B - t_A) - \vec{p} \cdot (\vec{x}_B - \vec{x}_A)}{\hbar} \right\}.
$$

(3.20)

The phase of the complex exponential in this equation is in agreement with that of the Fourier transform of the Lorentz invariant momentum-space propagator, in the limit of large time-like space-time intervals, as given in Feynman’s first QED paper [23]. The $1/r$ factor in the propagator has been given by Feynman in an introductory discussion of probability amplitudes\textsuperscript{6} as well as the popular book ‘QED’ [6] \textsuperscript{7}.

### 4 The path amplitude for a photon produced in the decay of an excited atom

As a simple illustration of the laws I and II of Eqs. (2.8) and (2.9), and the application of the formula (3.20) for the space-time propagator of a free particle, consider the probability amplitude to detect a photon that originates in the decay of an excited atom at rest. If the excited state, $i$, \textsuperscript{9}, with relativistic energy $E_i$, is produced at time $t_0$ and decays at

\textsuperscript{6}See Ref. [15], Vol III ‘Quantum Mechanics’, Ch. 3 Eq. (3.7).

\textsuperscript{7}‘The length of the arrow’ (i.e. the modulus of the complex probability amplitude) ‘is inversely proportional to the distance the light goes.’\textsuperscript{8} (Feynman’s italics)

\textsuperscript{9}Whether the symbol $i$ denotes a state label or $\sqrt{-1}$ in an equation is clear from its context.
the later time $t_\gamma$, into the stable ground state of energy $E_f$, the probability amplitude to create the photon may be written as $\langle f,t_\gamma|i,t_\gamma \rangle$. Neglecting the recoil of the daughter atom, the time evolution of the initial and final states is given by (3.20) as:

$$|i,t_\gamma\rangle = \exp \left\{ -i \left( \frac{E_i - i \Gamma_i/2}{\hbar} \right) (t_\gamma - t_0) \right\} |i,t_0\rangle$$

(4.1)

$$|f,t_\gamma\rangle = \exp \left\{ -i \frac{E_f}{\hbar} (t_\gamma - t_0) \right\} |f,t_0\rangle$$

(4.2)

so that:

$$\langle f,t_\gamma|i,t_\gamma \rangle = \langle f,t_0|i,t_0 \rangle \exp \left\{ -i \frac{\bar{\hbar}}{2} (E_i - E_f - i \frac{\Gamma_i}{2}) (t_\gamma - t_0) \right\}$$

(4.3)

where $\Gamma_i$ is the natural width of the state $i$, related to the mean lifetime $\tau_i$ of the state by the energy-time uncertainty relation:

$$\Gamma_i = \frac{\hbar}{\tau_i}.$$ 

(4.4)

The factor $\exp[-\Gamma_i/2] (t_\gamma - t_0)$ in (4.3) takes into account the exponential decay law of the excited atom. If the photon is detected at time $t_D$ at distance $r$ from the source atom, the photon path amplitude is, from (3.20):

$$\langle \gamma,t_D|\gamma,t_\gamma \rangle = \frac{1}{r} \exp \left\{ -i \left( \frac{E_\gamma - i \Gamma_i/2}{\bar{\hbar}} \right) (t_D - t_\gamma) \right\}$$

(4.5)

where

$$E_\gamma = E_i - E_f.$$ 

(4.6)

Since

$$\frac{E_\gamma}{p_\gamma} = c = \frac{r}{t_D - t_\gamma}$$ 

(4.7)

the phase in (4.5) vanishes and

$$\langle \gamma,t_D|\gamma,t_\gamma \rangle = \frac{1}{r}. $$

(4.8)

As remarked by Feynman [6]: ‘Once a photon has been emitted there is no further turning of the arrow as the photon goes from one point to another in space-time.’ The direction of the ‘arrow’ is the phase of the path amplitude. The full probability amplitude, incorporating photon production, and propagation, analogous to the ‘wavefunction’ of conventional quantum mechanics, is:

$$\psi_\gamma(\vec{r},t_D,t_\gamma,t_0) = A_0 \frac{1}{r} \exp \left\{ -i \left[ E_\gamma - i \frac{\Gamma_i}{2} \right] (t_\gamma - t_0) \right\} \langle f,t_0|i,t_0 \rangle \delta[r - c(t_D - t_\gamma)]$$

(4.9)

where $A_0$ is a normalisation constant and the $\delta$-function imposes the space-time geometrical constraint relating $r$, $t_D$ and $t_\gamma$. The constant $A_0$ is chosen in such a way that $\psi_\gamma$ has the usual Born probabilistic interpretation (Law I):

$$P_\gamma = \int |\psi_\gamma|^2 dV dt_D = \int |\psi_\gamma|^2 d\Omega r^2 dr dt_D = 1$$

(4.10)

$^{10}$See Ref. [13], Section 28 Eq. (20).
where $dV$ is a spatial volume element and $d\Omega$ is an element of solid angle containing the photon path. That is, the probability is unity that the photon, once created, can be detected at some position at some later time. Notice that the probability amplitude $\psi_{\gamma}$ is a function, not only of the spatial position $\vec{r}$, but also of the times $t_D$, $t_\gamma$ and $t_0$ as well as $E_\gamma$ and $\Gamma_i$. However, $P_\gamma$ can only be normalised, as in Eq. (4.10), by a suitable choice of $A_0$ providing that the the modulus of $\psi_{\gamma}$ has the spatial dependence $1/r$.

The above example also demonstrates the importance of taking into account all relevant physical parameters when analysing a realistic space-time experiment. In the example just analysed, the photon travels in a classical manner. The only place where ‘Heisenberg uncertainty’ plays a role is in the energy/time uncertainty relation (4.4). With a typical value of the lifetime of the initial state atom of $\tau_i = 10^{-8}$ s the momentum/space uncertainty relation: $\Delta x_\gamma = \hbar/\Delta p_\gamma$ where $\Delta p_\gamma = \Gamma_i/c = \hbar/(c\tau_i)$ gives $\Delta x_\gamma = 3$ meters. A ‘wavepacket’ of this width clearly has no relevance in the simple experiment just described. Using a photon detector with a spatial resolution of $10\mu m$, easily obtained using modern solid-state technology, the position of the photon, at the instant of detection, can be determined, given the prior knowledge of the momentum of the photon, with a precision 33000 times better than ‘allowed’ by the momentum/space uncertainty relation! A correct application of this relation would be to the momentum and spatial position of an electron in the bound state of an atom. Their distributions are related by a Fourier transform which shows that the widths of the distributions are related as in the momentum/space uncertainty relation. In general, no physical significance can be attached to the ‘plane wave’ represented by the phase factor in Eq. (3.1) or (3.6), in the absence of the knowledge of other physical parameters necessary to specify any realistic space-time experiment where particles travel in free space. In particular, a free particle with a known small momentum uncertainty, due to its production process, does not have to have a very large uncertainty in position, as prescribed by the corresponding uncertainty relation.

5 Reflection diffraction grating for photons

The experiment discussed in the previous section required only the application of the laws I (the the Born probability interpretation) and II (sequential factorisation) as well as the formula (3.20) for the space-time propagator of a free particle. The experiments to be discussed in the present and following section also bring into play the superposition law III. The geometry of the experiment, which is chosen to be the same as that, using electrons, of Davisson and Germer [8], to be discussed in the following section, is shown in Fig. 1. As in the previous section, the probability amplitude for an experiment where a single photon, produced by spontaneous decay of a single excited atom, is considered.

---

11 This connection between Fourier transforms and uncertainty relations was noted at an early date by Bohr [10].

12 The experiment can be readily performed, in a qualitative manner, by the reader by holding up a compact disc so that light from a localised source falls on it. The bright bands, changing rapidly in
Figure 1: Geometry of a reflection diffraction grating. A single photon, $\gamma$, is produced in the decay of an excited atom at S, reflected from the grating and observed at O. See text for discussion.
This atom, constituting the source, $S$, is a distance $r_s$ from a ruled reflection grating of total width $W$ ($W \ll r_s$) and strip separation $d$. The width of the individual strips is $b$. The $x$ and $y$ axes of a rectangular Cartesian coordinate system lie in the plane of the grating with the $y$-axis parallel to the strips. The source is centered on the grating and lies on the $z$-axis. Since $r_s \gg W$ all initial photon paths can be considered to be parallel to the $z$-axis. The plane of the observer $O$ and the source $S$ includes the $z$-axis, and is perpendicular to the $y$-axis, so that consideration is limited to paths lying in the $x$-$z$ plane. The $N$ reflecting strips are labelled $0, 1, 2, ..., n$ so that $N = n + 1$, the strip 0 being distant $r_O$ ($r_O \gg W$) from the observer. The $x$ coordinate of the centre of the $k$th strip is: $x_k = kd$.

The excited atom is created at time $t_0$ and subsequently decays at $t_\gamma$ — the creation time of the photon. In Fig. 1a is shown the path amplitude of a photon reflected from strip 3 — it is shown at the instant after reflection from the strip. The photon is later observed at time $t_D$. The initial state of the experiment is that of the excited atom at its instant of creation, $t_0$. The final state corresponds to observation (and destruction) of the photon at time $t_D$. The times $t_0$ and $t_D$ thus label the initial and final states respectively, which are the same for all paths considered, like the times $t_A$ and $t_B$ at the limits of the path amplitudes discussed in Sections 2 and 3, or the initial and final state labels $i$ and $f$ in the superposition law (2.10).

Suitably adapting the formula (4.9), noting the geometry of Fig. 1a, the probability amplitude for a photon diffracted from the $k$th strip is:

$$
|t_D \rangle |t_\gamma \rangle |t_0 \rangle = A_0 A_{\text{det}}^\gamma \frac{1}{r_O + x_k \sin \theta} A_{\text{Diff}}(\theta) \frac{1}{r_s} e^{\frac{(t_\gamma - t_0)}{2\tau_i}} e^{-\frac{i}{\hbar}E_\gamma(t_\gamma - t_0)} |f, t_0 \rangle |i, t_0 \rangle
$$

(5.1)

where $A_{\text{det}}^\gamma$ is the amplitude of the photon detection process which includes acceptance factors for solid angles subtended at the grating by the source and by the observer at the grating. The energy-time uncertainty relation (4.4) is used to replace $\Gamma$, in (4.9) by $h/\tau_i$. $A_{\text{Diff}}(\theta)$ is the amplitude for diffraction through the angle $\theta$ from any single strip. Assuming $x_k \sin \theta \ll r_O$ and using the space-time geometrical constraint (see Fig. 1a):

$$
t_\gamma = t_D - r_s + r_O + kd \sin \theta, \quad (5.2)
$$

the decay time $t_\gamma$ can be eliminated from (5.1) in favour of the strip label $k$ to give:

$$
A_{FI}^k \equiv |t_D \rangle |t_\gamma \rangle |t_0 \rangle = A_0 A_{\text{det}}^\gamma \frac{1}{r_O} A_{\text{Diff}}(\theta) \frac{1}{r_s} \exp[i\phi(t_D, t_0)] \exp(ik\alpha) |f, t_0 \rangle |i, t_0 \rangle

= \tilde{A}_0 A_{\text{Diff}}(\theta) e^{ik\alpha} \tilde{A}_{0D} e^{ik\alpha}
$$

(5.3)

where

$$
\phi(t_D, t_0) \equiv -\left(\frac{E_\gamma}{\hbar} + \frac{1}{2i\tau_i}\right) \left[t_D - \left(r_s + r_O\right) \frac{1}{c} - t_0\right]
$$

(5.4)

and

$$
\tilde{A}_0 \equiv \frac{A_0 A_{\text{det}}^\gamma |f, t_0 \rangle |i, t_0 \rangle}{r_Or_S} \exp[i\phi(t_D, t_0)],
$$

(5.5)

colour, which are seen as the plane of the disc is rotated about a diameter, is the diffraction pattern; the observer’s eye serves as the photon detector.
Applying the Born interpretation, law I, of Eq. (2.8) the final probability, $dP$ gives:

\[ D\text{-lines}) \text{ and a natural atomic lifetime } \tau \]

\[ E \]

Considering a typical visual photon energy, $E$ of 2 eV (e.g. the yellow light of the sodium D-lines) and a natural atomic lifetime $\tau$ of 10 ns it is found that

\[ \frac{q}{r} = \frac{\hbar}{\tau E} = 1.64 \times 10^{-8}. \]

Note that, in virtue of the energy-time uncertainty relation (4.4), the condition $q/r \ll 1$ implies that $\Gamma \ll E$. Neglecting the imaginary parts of $\alpha$ and $\beta$, since $q \ll r$, and integrating the detection time $t_D$ from its lower limit of $t_0 + (r_S + r_O)/c$ to infinity, (5.10) gives:

\[ P_{FI} = 4\tau_i |\tilde{A}_0 a|^2 \sin^2 \frac{rb_0}{2} \sin^2 \frac{Nrd}{2} \]

\[ \tau_i |\tilde{A}_0 a|^2 \left[ \sin \left( \frac{E \omega \sin \theta}{2\hbar c} \right) \right]^2 \left[ \sin \left( \frac{NEc \omega \sin \theta}{2\hbar c} \right) \right]^2. \]
The $\theta$ dependence of this formula, familiar from the classical wave theory of light [24], has here been derived from the path integral formulation of quantum mechanics, without introducing any ‘wave’ concept whatever.

Neglecting the imaginary part of $\alpha$ and setting $\alpha = 2\pi l$, where $l$ is an integer, (5.7) gives:

$$A_{FI} = \tilde{A}_{0D}[1 + e^{i2\pi l} + e^{i4\pi l} + ... + e^{i2n\pi l}]$$

$$= \tilde{A}_{0D}[1 + 1 + 1 + ... + 1] = N\tilde{A}_0$$  (5.12)

so that all path amplitudes are in phase and add constructively. According to (5.6) this occurs for all angles $\theta_{\text{const}}^l$ such that

$$\sin \theta_{\text{const}}^l = \frac{2\pi lh\bar{c}}{E_\gamma d}$$  \hspace{1cm} l = 1, 2, 3, ...  (5.13)

For $\alpha = (2l + 1)\pi$, $l = 0, 1, 2, ...$ (5.7) gives

$$A_{FI} = \tilde{A}_{0D}[1 + e^{i(2l+1)\pi} + e^{i2(2l+1)\pi} + ... + e^{i(n(2l+1)\pi]}$$

$$= \tilde{A}_{0D}[1 - 1 + 1 - ...]$$

$$= 0 \quad (n \text{ odd})$$

$$= \tilde{A}_{0D} \quad (n \text{ even})$$  (5.15)

corresponding to destructive interference which is complete for $n$ odd (i.e. for an even number of strips) and partial for $n$ even. This occurs for all angles $\theta_{\text{dest}}^l$ such that

$$\sin \theta_{\text{dest}}^l = \frac{(2l + 1)\pi h\bar{c}}{E_\gamma d}$$  \hspace{1cm} l = 0, 1, 2, ...  (5.16)

If the first interference maximum occurs for $\theta = 5^\circ$ for a photon energy of 2 eV then (5.12) gives

$$d = \frac{hc}{E_\gamma \sin 5^\circ} = 7.1 \mu\text{m.}$$  (5.17)

For a grating with $b = d$ and 1000 strips (i.e. with a total width of 14.2 mm)

$$2Nqd = \frac{Nd \sin \theta}{ct_i} < \frac{Nd}{ct_i} = 2.4 \times 10^{-3}.$$  (5.18)

The approximation: $\exp(2Nqd) \simeq 1$ used to obtain (5.11) from (5.10) is therefore good to within a few parts in a thousand for the above choice of diffraction grating parameters and source atom. It also follows in this case that damping of the interference pattern by the $1/\cosh Nqd$ factor multiplying the $\cos Nrd$ term in (5.10) is small.

As required by the fundamental law of superposition (2.10), all paths have the same initial and final states. In all paths the source atom is produced at the same time $t_0$ and the photon is detected at the same time $t_D$. The phases of the different path amplitudes are different because the decay time $t_\gamma$ of the excited atom is different in the different paths, so that each path corresponds to a different classical history. This important point was clearly stated in Feynman’s last work of popular physics [6] published a quarter of a century ago now. However, this realisation of the crucial importance of time for a correct
understanding of quantum mechanical superposition has still not yet penetrated into the relevant research literature\textsuperscript{13}, much less into textbooks or the pedagogical literature.

6 Reflection diffraction grating for electrons: the Davisson-Germer experiment

The Davisson-Germer experiment \cite{8}, performed in 1927, in which the wave-like aspect of massive particles was first demonstrated, had the same geometrical configuration as the light diffraction experiment shown in Fig. 1. Electrons produced by thermionic emission from a heated tungsten filament were accelerated in an electric field. The electron beam thus produced struck at normal incidence the [111] face of a face-centered-cubic crystal of nickel atoms with lattice spacing 3.51 Å. The electrons were scattered through an angle $\pi - \theta$ by rows of nickel atoms, with a spacing $d = 2.15$ Å, that correspond to the reflecting strips in the photon experiment described above. The current of scattered electrons was measured by a Faraday box collector over the adjustable angular range $20^\circ < \theta < 90^\circ$. The paths of the incident and scattered beams were of lengths $r_S = 27$mm and $r_O = 23$mm, respectively. A clear and simple description of the experiment and its results can be found in Ref. \cite{27}.

In this and similar experiments, the electrons are produced not by a well-defined quantum-mechanical process as in the photon diffraction experiment described above but by the stochastic process of thermionic emission. There is therefore no known phase relation for the amplitudes of electron emission events at different times, and the transit time of an electron is determined not, as for a photon, uniquely by the length of its path but also by its velocity. Indeed it will be seen that the spread in electron velocities is crucial for the observed quantum interference effects. The electrons are emitted from the filament with an absolute temperature, $T$, around 2500° K, with a Maxwell-Boltzmann velocity distribution \cite{28, 29} corresponding to this temperature. The RMS velocity at emission is then

\[
\bar{v}_{\text{emit}} = \sqrt{\frac{3kT}{m_e}} = 1.12 \times 10^{-3}c
\]  

where $k$ is the Boltzmann constant and $m_e$ the mass of the electron. The corresponding kinetic energy is 0.32 eV. For comparison, the electron, after acceleration to a typical kinetic energy of 50 eV, has a velocity of $1.4 \times 10^{-2}c$, so that the spread of electron velocities is about 6%. As will be shown, this is many orders of magnitude greater than the spread in electron velocities that comes into play in the quantum interference effects.

Unlike the photon experiment, where the photon propagator gives a vanishing contribution to the phase of the path amplitudes, this phase for the electron experiment originates entirely from the electron propagator. Writing the propagator phase, $\phi$, as in

\textsuperscript{13}One isolated example occurred in an analysis of neutrino oscillations \cite{25} where it was stated that: ‘Since the mass eigenstates propagate with different velocities (for fixed energy) the desired interference is between neutrinos emitted at slightly different times.’ Actually \cite{26} the neutrinos do not have ‘fixed energy’ but certainly have different velocities, so that the statement is a correct one within the path integral formulation.
In a similar manner to (5.3) the path amplitude when the electron scatters from the kth row of nickel atoms is:

\[
A_{FI}^k = A_0 \frac{A_0 A_{det} A_{Scat}(\theta) A_P^e}{r_{SR O}} \int f(p_k) \exp[i\phi_k] dp_k
\]

where \( A_P^e \) is the production amplitude of the electron,

\[
\phi_k \equiv -\frac{(mc)^2}{\hbar p_k}s_k = -\frac{(mc)^2}{\hbar p_k}(r_S + r_O + k d \sin \theta)
\]

and \( f(p_k) \) is the amplitude for production of an electron that has momentum \( p_k \) at the scattering event. It is distributed around the average momentum \( \langle p \rangle \) according to a Gaussian with a width determined by the Maxwell-Boltzmann distribution corresponding to the thermionic emission process:

\[
f(p) = \frac{1}{\sqrt{\pi} \sigma_p} \exp \left[-\frac{(p - \langle p \rangle)^2}{2 \sigma_p^2}\right]
\]

where \( \sigma_p = \sqrt{2mc kT}, \; T = 2500^\circ \text{K}. \) (6.6)

It will be found convenient to discuss first the probability amplitude given by superposition of the path amplitudes from two adjacent rows of nickel atoms labelled \( k \) and \( k + 1 \):

\[
A_{FI}^{(2)} = A_{FI}^k + A_{FI}^{k+1} = \tilde{A}_0^e(\theta) \left[ \int f(p_k) \exp[i\phi_k] dp_k + \int f(p_{k+1}) \exp[i\phi_{k+1}] dp_{k+1} \right].
\] (6.7)

The corresponding probability, given by the Born interpretation (2.8) is

\[
P_{FI}^{(2)} = |A_{FI}^k + A_{FI}^{k+1}|^2 = |A_{FI}^k|^2 + |A_{FI}^{k+1}|^2 + 2Re \left[ (A_{FI}^k)^* A_{FI}^{k+1} \right]
\] (6.8)

where

\[
|A_{FI}^k|^2 = |A_{FI}^{k+1}|^2 = |\tilde{A}_0^e|^2 \int \int \exp[i(\phi_k(p) - \phi_k(p'))] f(p) f(p') \delta(p - p') dp dp'
\]

\[
= |\tilde{A}_0^e(\theta)|^2
\] (6.9)

and

\[
2Re \left[ (A_{FI}^k)^* A_{FI}^{k+1} \right] = 2|\tilde{A}_0^e(\theta)|^2 Re \left[ \int \int \exp[i(\phi_{k+1}(p_{k+1}) - \phi_k(p_k))] f(p_{k+1}) f(p_k) \delta(p_{k+1} - p_k - \Delta p) dp_{k+1} dp_k \right]
\]

\[
\equiv 2|\tilde{A}_0^e(\theta)|^2 Re(I^{(2)})
\] (6.10)
where $\Delta p \equiv p_{k+1} - p_k$.

From (6.2),
\[
\Delta \phi \equiv \phi_{k+1}(p_{k+1}) - \phi_k(p_k) = \frac{(mc)^2}{\hbar} \left( \frac{s_k}{p_k} - \frac{s_{k+1}}{p_{k+1}} \right).
\]  

(6.11)

The relativistic formula for $\Delta \phi$, correct to first order in $\Delta s/\bar{s}$ where $\Delta s \equiv s_{k+1} - s_k$ and $\bar{s} \equiv (s_{k+1} + s_k)/2$ is derived in the Appendix:
\[
\Delta \phi = \frac{(mc)^2}{\hbar} \left[ -\frac{\Delta s}{\bar{p}} + \frac{\bar{s}}{\bar{p}} \left( 1 + \frac{\bar{p}^2}{(mc)^2} \right) \left( \frac{\mathcal{R}_{\text{EV}}}{\mathcal{R}} - 1 \right) \right]
\]  

(6.12)

where $\bar{p} \equiv (p_{k+1} + p_k)/2$, $\mathcal{R} \equiv t_{k+1}/t_k$ and $\mathcal{R}_{\text{EV}} = s_{k+1}/s_k$. Of particular interest are the cases $\mathcal{R} = 1$, corresponding to equal production times of the electron in the two paths and $\mathcal{R} = \mathcal{R}_{\text{EV}}$ which holds for equal velocities (hence the label 'EV') in the two paths.

For $1 < \mathcal{R} < \mathcal{R}_{\text{EV}}$ and $\mathcal{R} > \mathcal{R}_{\text{EV}}$ both production times and velocities are different in the two paths. It is shown in the Appendix that (6.12) yields the following predictions for $\Delta \phi$ to first order in $\Delta s/\bar{s}$:
\[
\Delta \phi = \frac{\bar{p}\Delta s}{\hbar} = \frac{\bar{p}}{\hbar} \frac{d \sin \theta}{\theta} \quad \text{(equal production times)},
\]
\[
\Delta \phi = -\frac{(mc)^2 \Delta s}{\hbar \bar{p}} = -\frac{(mc)^2}{\hbar} \frac{d \sin \theta}{\theta} \quad \text{(equal velocities)}.
\]  

(6.13) (6.14)

Considering first the equal production time case and integrating first over $p_k$ gives, for the integral $I^{(2)}$ in (6.10):
\[
I^{(2)} = \frac{1}{\sqrt{\pi} \sigma_p} \int e^{\frac{(p_{k+1} - \Delta p/2)}{\sigma_p^2}} \Delta s e^{\frac{(p_{k+1} - \bar{p})^2}{2\bar{p}^2}} e^{\frac{(p_{k+1} - \Delta p - \bar{p})^2}{2\bar{p}^2}} dp_{k+1}.
\]  

(6.15)

Performing the integrals over Gaussians in (6.15) by ‘completing the square’
\footnote{With the change of variable $p_{k+1} - \langle p \rangle = \bar{p}$ the integration limits of $\bar{p}$ are $-\langle p \rangle$ to infinity. Since $\sigma_p \ll \langle p \rangle$ the lower limit is set to $-\infty$ in performing the integrals in (6.9) and (6.15).} the result for $I^{(2)}$ in (6.10) gives:
\[
2\text{Re} \left[ \left( A_{FI}^k \right)^* A_{FI}^{k+1} \right] = |\bar{A}_0(\theta)|^2 e^{-\frac{(\Delta \phi)^2}{\sigma_\phi^2}} e^{-\frac{(\sigma_{\bar{p}}^2 \Delta \phi)^2}{2\sigma_\phi^2}} \cos \left[ \frac{\langle p \rangle}{\bar{p}} \Delta s \right].
\]

(6.16)

The integrals over the momentum distributions have the effect of replacing the quantity:
\[
p_{k+1} - \frac{\Delta p}{2} = \frac{p_{k+1} + p_k}{2} = \bar{p}
\]  

(6.17)

which is the mean momentum of the particle in the two interfering paths, by the quantity $\langle p \rangle$. With equal production times:
\[
t_{k+1} = t_k = \frac{s_{k+1}}{v_{k+1}} = \frac{s_k}{v_k}.
\]

(6.18)

Since $\beta \simeq 10^{-2} \ll 1$ in the Davisson-Germer experiment the non-relativistic momentum formula, $p = mv$, holds so that, from (6.18), to first order in $\Delta s/\bar{s}$,
\[
\frac{\Delta s}{\bar{s}} = \frac{\Delta v}{v} = \frac{\Delta p}{\bar{p}}.
\]

(6.19)
Also, with \( T = 2500^\circ \) and \( \langle p \rangle = 7.2 \times 10^{-3} \) MeV/c:

\[
\frac{\sigma_p}{\langle p \rangle} = \frac{\sqrt{2m_e kT}}{\langle p \rangle} = 6 \times 10^{-2}.
\]  
(6.20)

It follows from (6.19) and (6.20) that

\[
\frac{\Delta p}{\sigma_p} \sim \frac{\langle p \rangle \Delta s}{\sigma_p / \bar{s}} = 16.7 \Delta s / \bar{s}.
\]  
(6.21)

Because \( \Delta s = d \sin\theta < d = 2.15 \) Å, the upper limit on \( \Delta p/(2\sigma_p) \) is

\[
\frac{\Delta p}{2\sigma_p} = 8.35 \frac{\Delta s}{\bar{s}} < \frac{8.35d}{r_S + r_O} = 4.2 \times 10^{-8}.
\]  
(6.22)

Since (6.20) gives \( \sigma_p = 4.3 \times 10^{-4} \) MeV/c then

\[
\frac{\sigma_p \Delta s}{2\hbar} = \frac{\sigma_p d \sin\theta}{2\hbar} < \frac{\sigma_p d}{2\hbar} = 8.9 \times 10^{-9}.
\]  
(6.23)

Finally (6.20) and (6.22) give

\[
\frac{\Delta p}{\bar{p}} \sim \frac{\Delta p}{\langle p \rangle} < 5.0 \times 10^{-9}.
\]  
(6.24)

The momentum difference \( \Delta p \) that is the physical basis of the interference of the path amplitudes of electrons scattering from adjacent rows of atoms is therefore seven orders of magnitude smaller than the momentum spread of the beam in the experiment.

The left sides of (6.22) and (6.23) appear squared as the arguments of the negative exponential factors in (6.16). The interference damping given by these factors it therefore completely negligible. Setting therefore the damping factors to unity, (6.8), (6.10) and (6.16) may be combined to give:

\[
P_{FI}^{(2)} = |A_{FI}^k + A_{FI}^{k+1}|^2 = |\tilde{A}_0^k(\theta)|^2 \left[ 1 + \cos \left( \frac{\langle p \rangle d \sin\theta}{\hbar} \right) \right] \text{ (equal production times).}
\]  
(6.25)

Notice the important point that, although the damping produced by the factor \( \exp[-(\Delta p/2\sigma_p)^2] \) in (6.16) is completely negligible, this factor, and hence the interference term, vanishes for vanishing \( \sigma_p \) for any finite value of \( \Delta p \). The interference effect therefore requires different momenta in the two paths for equal production times in the paths. The factor \( \exp[-(\sigma_p \Delta s/2\hbar)^2] \) in (6.16) shows that the interference effect is also destroyed for large values of the path difference \( \Delta s \gg 2\hbar/\sigma_p \), i.e. for values such that \( \Delta s \) is much larger than the width of a hypothetical spatial electron ‘wave packet’ given by the Fourier transform of the amplitude \( f(p) \) in Eq. (6.5).

For the equal velocity case, the \( \delta \)-function \( \delta(p_{k+1} - p_k - \Delta p) \) in (6.10) is replaced by \( \delta(p_{k+1} - p_k) \) so that the integrals over electron momenta are similar to that in (6.9) and there are no damping factors containing \( \sigma_p \). Also, since the effect of integration over the electron momentum distributions is to replace \( \bar{p} \) in the interference term by \( \langle p \rangle \) (see Eqs. (6.15), (6.16) and (6.17)) the probability distribution for equal velocities is:

\[
P_{FI}^{(2)} = |\tilde{A}_0^e(\theta)|^2 \left[ 1 + \cos \left( \frac{(m_e c)^2 d \sin\theta}{\hbar \langle p \rangle} \right) \right] \text{ (equal velocities).}
\]  
(6.26)
With the definition:

\[ \alpha_e \equiv \frac{\langle p \rangle d \sin \theta}{\hbar} \]  \hspace{1cm} (6.27)

then, since

\[ \sin^2 2 \left( \frac{\alpha_e}{2} \right) = \frac{4 \sin^2 \alpha_e \cos^2 \alpha_e}{\sin^2 \alpha_e / 2} = 4 \cos^2 \frac{\alpha_e}{2} = 2(1 + \cos \alpha_e) \].  \hspace{1cm} (6.28)

the equal production time formula (6.25) has the same \( \theta \) dependence as given by the replacements: \( N = 2 \), \( E_\gamma / c = p_\gamma \rightarrow \langle p \rangle \) in the last factor of the equal velocity, different production time, formula (5.15) for the photon experiment. The quantity \( P_{FI}^{(N)} \) for \( N > 2 \) is therefore also expected to be proportional to \( \sin^2(N\alpha_e/2) / \sin^2(\alpha_e/2) \). Since the \( \sigma_p \) dependent damping factors may be set to unity, this result may be derived by introducing momentum-averaged phases for the paths according to the equations:

\[ \langle \phi_{k}^{\text{EPT}} \rangle = \frac{\langle p \rangle s_k}{\hbar} = k\alpha_e^{\text{EPT}} + \phi_0^{\text{EPT}} \] (equal production times),  \hspace{1cm} (6.29)

\[ \langle \phi_{k}^{\text{EV}} \rangle = -\frac{(m_e c)^2 d \sin \theta}{\hbar \langle p \rangle} s_k = k\alpha_e^{\text{EV}} + \phi_0^{\text{EV}} \] (equal velocities)  \hspace{1cm} (6.30)

where

\[ \alpha_e^{\text{EPT}} \equiv \frac{\langle p \rangle d \sin \theta}{\hbar}, \quad \alpha_e^{\text{EV}} \equiv -\frac{(m_e c)^2 d \sin \theta}{\hbar \langle p \rangle}, \]  \hspace{1cm} (6.31)

\[ \phi_0^{\text{EPT}} \equiv \frac{\langle p \rangle (r_S + r_O)}{\hbar}, \quad \phi_0^{\text{EV}} \equiv -\frac{(m_e c)^2 d \sin \theta}{\hbar \langle p \rangle} (r_S + r_O). \]  \hspace{1cm} (6.32)

Eq. (6.7) then generalises to:

\[ A_{FI}^{(N)}(\theta) = \tilde{A}_0^e(\theta) \left[ e^{i\phi_0} + e^{i\phi_1} + \ldots + e^{i(\phi_{N-1})} \right] \]

\[ = \tilde{A}_0^e(\theta) \left[ 1 + e^{i\alpha_e} + e^{i2\alpha_e} + \ldots + e^{i(N-1)\alpha_e} \right] e^{i\phi_0} \]

\[ = \tilde{A}_0^e(\theta) e^{i\phi_0} \frac{1 - e^{iN\alpha_e}}{1 - e^{i\alpha_e}} \] \hspace{1cm} (6.33)

so that the formula for the angular distribution of an electron diffracted from \( N \) rows of nickel atoms in the Davisson-Germer experiment is:

\[ P_{FI}^{(N)}(\theta) = |A_{FI}^{(N)}|^2 = |\tilde{A}_0^e(\theta)|^2 \frac{\sin^2 N \left( \frac{\alpha_e}{2} \right)}{\sin^2 \alpha_e / 2} \] \hspace{1cm} (6.34)

where \( \alpha_e = \alpha_e^{\text{EPT}} \) or \( \alpha_e^{\text{EV}} \). The first diffraction maximum occurs when \( |\alpha_e| = 2\pi \) giving the predictions:

\[ \sin \theta_{\text{max,1}}^{\text{EPT}} = \frac{2\pi \hbar}{\langle p \rangle d} \] (equal production times), \hspace{1cm} (6.35)

\[ \sin \theta_{\text{max,1}}^{\text{EV}} = \frac{2\pi \hbar \langle p \rangle}{(m_e c)^2 d} \] (equal velocities). \hspace{1cm} (6.36)

Substituting \( \langle p \rangle = 7.43 \times 10^{-3} \text{MeV}/c \) corresponding to \( T_e = 54 \text{ eV} \) and \( d = 2.15 \text{Å} \) in these formulas gives \( \theta_{\text{max,1}}^{\text{EPT}} = 51^\circ \) in good agreement with the observation of the Davisson-Germer experiment [8, 27] whereas \( \theta_{\text{max,1}}^{\text{EV}} = 0.0094^\circ \). Only the equal production time
hypothesis is therefore consistent with the experiment. A consequence is that the prediction does not depend on any phase, with a possibly stochastic time dependence, of the electron production amplitude \( A_p \), which is not the case for equal velocities and different production times.

The above analysis neglects the angular dependence of the scattering amplitude \( A_{\text{Scat}}(\theta) \) of an electron on a single nickel atom, which can only be obtained by performing the appropriate quantum-mechanical calculation. For the non-relativistic electrons in the Davisson-Germer experiment the corresponding angular distribution is expected to be isotropic, or in any case, much less rapidly-varying than the single-strip diffraction factor for photons in (6.34). The approximation of assuming an angle-independent scattering amplitude in calculating the shape of the overall diffraction pattern is therefore expected to be a good one.

### 7 Classical wave theories of photons and massive particles

Using Eq. (5.2) the photon path amplitude of Eq. (5.3) may be split into constant, time-dependent and spatially-dependent factors as:

\[
A_{FI}^k = \frac{A_0 A_0^\gamma A_{\text{Diff}}(\theta) \langle f, t_0 | i, t_0 \rangle}{r_k r_S} \\
\times \exp \left[ -\frac{i}{\hbar} \left( E_\gamma - i \frac{\Gamma_i}{2} \right) (t_D - t_0) \right] \\
\times \exp \left[ \frac{i}{\hbar} \left( E_\gamma - i \frac{\Gamma_i}{2} \right) (r_k + r_S) \right]
\]

(7.1)

where

\[
r_k \equiv r_O + kd \sin \theta.
\]

(7.2)

The \( 1/r_k \) dependence of (7.1), unlike the \( 1/r_O \) dependence of (5.3) is exact to first order in \( \Delta s/s \). Consider now observation of the photon at different positions for a fixed source position. Then \( r_S, t_D \) and \( t_0 \) are constants for different paths labelled by \( k \) so that \( A_{FI}^k \) is a function only of \( r_k \):

\[
A_{FI}^k \rightarrow U_\gamma(r_k) = \frac{(\bar{A}_O^S)^\gamma}{r_k} \exp \left[ \frac{i}{\hbar c} \left( E_\gamma - i \frac{\Gamma_i}{2} \right) r_k \right]
\]

(7.3)

where the label S in the constant amplitude \( (\bar{A}_O^S)^\gamma \) stands for ‘source’. If \( \Gamma_i/(2\hbar c) r_k \ll 1 \), so that damping effects due to the finite lifetime of the source atom are negligible, (7.3) simplifies to:

\[
U_\gamma(r_k) = \frac{(\bar{A}_O^S)^\gamma}{r_k} \exp \left[ \frac{iE_\gamma r_k}{\hbar c} \right] = \frac{(\bar{A}_O^S)^\gamma}{r_k} \exp \left[ \frac{i2\pi\gamma r_k}{\lambda_\gamma} \right]
\]

(7.4)

where

\[
\frac{E_\gamma}{\hbar c} \equiv \frac{2\pi}{\lambda_\gamma} \equiv \kappa_\gamma.
\]

(7.5)
The quantity $\lambda_\gamma = \frac{hc}{E_\gamma} = \frac{h}{p_\gamma}$ is the ‘de Broglie wavelength’ of the photon. However the discussion in Section 5 above shows that the phase in (7.4) originates entirely from the time dependence of the decay amplitude of the unstable atom (Eq. (4.3)) and so is in no sense an attribute of the photon itself. Eq. (7.4) shows that the photon path amplitude is equivalent to the spatial part of a classical wave with phase velocity $c$ and wavelength $\lambda_\gamma$. The relation $c = \lambda_\gamma \nu$ follows from (7.5) and the Planck-Einstein relation for photons: $E_\gamma = h\nu$. All predictions of interference and diffraction effects for light in the case that the lifetime of the excited atom can be considered to be infinitely long can therefore be obtained from (7.4) in which only spatially dependent waves are considered. This is the 19th Century ‘classical wave theory of light’. The theory is ‘classical’ because Planck’s constant does not appear in the equations, its effect being hidden in the phenomenological photon wavelength parameter. The temporal physical origin of the phase, evident in Eq. (4.3), is transformed away when $t_\gamma$ is eliminated from (5.3) using (5.2) to yield (7.1).

Considering, for electrons, the equal production time case, as experimentally verified in the Davisson-Germer experiment, the path amplitude formula analogous to (7.3) is:

$$A^k_{FI} = \frac{A_0 A^s_{det} A^s_{scat}(\theta) A^c_p}{r_k r_S} \exp \left[ \frac{i \langle p \rangle}{\hbar} (r_k + r_S) \right] \to \quad U_e(r_k) \equiv \frac{\langle \tilde{A}^s_S \rangle_e}{r_k} \exp \left[ \frac{i \langle p \rangle r_k}{\hbar} \right] = \frac{\langle \tilde{A}^s_S \rangle_e}{r_k} \exp \left[ \frac{i 2\pi r_k}{\lambda_e} \right] \quad (7.6)$$

where the ‘de Broglie wavelength’ of the electron: $\lambda_e \equiv \frac{h}{\langle p \rangle} \equiv \frac{2\pi}{\kappa_e}$, although defined in a similar way, as $h/p$, is, unlike that of the photon, an attribute of the electron originating in its space-time propagator. There is no time dependence in the path amplitude (7.6) as a consequence of the equal time condition for different paths (i.e. for different values of $k$). The formula (7.6) gives a classical theory of ‘matter waves’ strictly analogous to the classical wave theory of light.

The spherical spatial waves of (7.4) or (7.6) are solutions of the Helmholtz equation:

$$\nabla^2 U + \kappa^2 U = 0 \quad (7.7)$$

which serves as the basis of classical wave theories of both light and material particles, even though the underlying space-time physics is quite different in the two cases. As a consequence of (7.7), diffraction is described by Kirchoff’s equation [30], and Huygen’s construction can be used to perform a purely spatial analysis of wavefronts in conjunction with the phenomenological wavelength parameters. Planck’s constant is thereby banished from all equations and the purely quantum mechanical nature of the fundamental underlying physics is hidden. Indeed the simple classical wave formalism of Eq. (7.4) gives many quantitatively correct results in physical optics and the classical wave formula Eq. (7.6) was used in the original interpretation of the Davisson-Germer experiment, to correctly predict the observed diffraction effects. Historically, of course, the discovery of the phenomenological wave theory of light, to explain the interference experiments of Fresnel and Young, predated by a century that of quantum mechanics and lead to the false ontological identification of light with classical waves, i.e. as a disturbance of some material medium. This misidentification was reinforced by the advent of Maxwell’s electromagnetic wave theory of light and the associated models of a material luminiferous aether to support the putative wave motion. After a century of experimental particle physics
it is now known, beyond reasonable doubt, that both photons and electrons are indeed particles in the ontological sense. The temporal aspects of their motion in space-time, inexplicable by classical wave theory, may be clearly demonstrated by considering suitably chosen experiments. An example of this is the damping effect of the \( \frac{1}{\cosh} \) factors in Eq. (5.10) reflecting the finite lifetime of the source atom. Some other examples may be found in Ref. [7].

8 Conclusions for the physical interpretation of quantum mechanics

Some remarks are now made concerning the ‘interpretation of quantum mechanics’ in the context of Feynman’s space path integral formulation and in the light of the space-time analyses of the experiments presented in Sections 4, 5 and 6 above. In typical textbook presentations of quantum mechanics the primary physical concept is not, as in Feynman’s formulation, the probability amplitude \( A_{fi} \) but the wavefunction \( \psi(\vec{x}_1, \vec{x}_2, ..., \vec{x}_N; t_1, t_2, ..., t_N) \) of a quantum mechanical system composed of \( N \) particles. For a single particle, the wavefunction is assumed to be a function of only of the space-time coordinates of this particle and so constitutes a ‘field’. In the non-relativistic theory the properties of the wavefunction are assumed to be those of a solution of the Schrödinger equation. In contrast, as demonstrated by the calculations presented in Sections 4, 5 and 6 above, the probability amplitude for a quantum experiment depends also on the space-time coordinates of the source of particles considered as well as those of the particle (or particles) which are detected in the experiment.

In the ‘Copenhagen Interpretation’ of quantum mechanics [11, 10], which is the one adopted by essentially all text books on the subject, as well as in the pedagogical literature and many popular accounts, there is supposed to be an ontological dualism between ‘waves’ and ‘particles’, termed ‘wave-particle duality’. It is commonly stated that an electron ‘sometimes behaves like a wave’ and ‘sometimes behaves like a particle’. This begs the obvious question: What \( \text{is} \) an electron, is it a particle or is it a wave? Every particle physicist who has ever built or worked on an actual experiment knows, beyond any doubt, that, operationally speaking, the electron (and the photon) \( \text{are} \) particles in the same sense that this word was understood by Newton. This is the essential concept required to design and understand what are correctly called ‘particle physics’ experiments. Even experiments such as those described in Sections 5 and 6 above where the ‘wave-like’ properties of particles are demonstrated are analysed, in Feynman’s formulation of quantum mechanics, entirely in terms of ‘what happens’ to entities localised in time and space and moving classically — particles.

The formulas of the path integral formulation contain only space-time coordinates and kinematical quantities (energies, momenta and masses) of \( \text{particles} \) so the question of ‘wave-particle duality’ cannot even arise in ‘interpretations’ of this formulation. So where do the ‘waves’ come from in the traditional Schrödinger formulation? By formal mathematical substitution it is possible to eliminate Planck’s constant, \( \hbar \), completely from all equations, in favour of the de Broglie wave length, \( \lambda \), by using the relation \( \hbar = p\lambda \). This leads to the ‘wave-like’ space-time functional dependence of the free particle propagator as
in Eq. (3.8) with a superluminal ‘phase velocity’: \( v_{\phi} = c^2/v \). By introducing a ‘refractive index’ of free space: \( n \equiv v/v_{\phi} \) and using Lord Rayleigh’s group velocity formula: \( 1/u = (1/c)d(n\nu)/d\nu \) de Broglie showed [31, 32, 33] that \( u = v \) i.e. that the particle velocity is equal to the group velocity of a packet of ‘phase waves’ with average velocity \( v_{\phi} \). The same conclusion may be drawn by defining the group velocity as \( u \equiv dE/dP \) and using the relativistic formula relating energy, momentum and mass together with the Planck-Einstein relation \( E = h\nu \) [33]. A consequence of this purely mathematical manipulation is that many textbooks state, in an introductory chapter, that a ‘particle’ in quantum mechanics is a packet of phase waves with widths in momentum space \( \Delta p \) and physical space \( \Delta x \) that satisfy the Heisenberg Uncertainty relation \( \Delta p \Delta x = \hbar \). In the analysis of the photon experiments described in Sections 4 and 5 above no such ‘wave packet’ occurs. This identification of particles with wave packets is part of Bohr’s original specification of the ‘Copenhagen Interpretation’ [10]:

The circumstance that the (phase velocity) is in general greater than the velocity of light emphasises the symbolic character of these considerations. At the same time the possibility of indentifying the velocity of the particle with the group velocity indicates the field of applicability of space-time pictures in the quantum theory.

The photon propagates in space time, within each path, not as a ‘wave packet’ but as a classical particle with fixed momentum and a constant velocity\(^{15}\) that is the same in all paths. In the Davisson-Germer experiment, momentum wave packets do occur in the electron path amplitudes in virtue of the thermionic emission process that liberates the electrons. The electrons then propagate, with a constant, but path-dependent, velocity in each path. The corresponding spatial wave packet, derived from Eq. (6.5) by Fourier transformation will have, as first pointed out by Bohr [10], a width \( \sigma_x \) that respects the uncertainty-like relation \( \sigma_p \sigma_x = \hbar \), but this hypothetical spatial wave packet, unlike the Maxwellian momentum distribution, plays no role in the space-time analysis of the Davisson-Germer experiment. The only relevant application of a Heisenberg Uncertainty relation in the experiments described above is the use of energy-time relation (4.4), which connects the decay width of the excited state to its mean lifetime. As explained at the end of Section 5, there is no corresponding photon wave packet, and the position and momentum of the photon can both be known with a precision much greater than allowed by a putative momentum-space uncertainty relation with \( \Delta p_{\gamma} = \Gamma_i/c \).

To make clear the different ontological nature of particles and the ‘waves’ of wave mechanics or the probability amplitudes of Feynman’s formulation, it is instructive to consider first a typical system described by classical mechanics. It consists of a part that exists in the real world (e.g. the Earth-Moon system) and is described by an abstract mathematical entity (e.g. a Lagrangian) from which its space-time behaviour (its motion) can be derived. For a quantum mechanical system, say a particle moving in a region of known potential energy, there is also a part that exists in the real world (the particle and the source of the potential) as well as the probability amplitude, which is a useful mathematical abstraction, from which statistical information on the space-time evolution of the system can be derived. It turns out that, in Feynman’s formulation, the essential ingredient of the probability amplitude is, according to the law V, the same as for the

\(^{15}\)This is only the case in ‘flat space’ where all gravitational effects are negligible.
classical example, the (classical) Lagrangian function of the system. In fact as pointed out by Dirac [5], the classical motion is recovered from the probability amplitude in the limit $h \ll A_{\text{min}}$ where $A_{\text{min}}$ is the minimum size of any quantity with dimensions of action that enters into the physical description of the system under consideration. The ontological confusion of statements like ‘an electron sometimes behaves like a particle, and sometimes behaves like a Lagrangian’ or ‘an electron is sometimes a particle and sometimes a Lagrangian’ is quite evident. In fact an electron is always a particle and its motion in space-time is always described by an appropriate (classical) Lagrangian in both classical and quantum mechanics in the space-time formulation due to Dirac and Feynman. Quite simply, in this case, there is no ontological duality.

The failure of the matrix mechanics and wave mechanics formulations to incorporate space-time concepts in a transparent manner, as well as the ontological confusion between ‘particles’ and ‘waves’ which has persisted until the time of this writing is primarily due to two circumstances: (1) Because of the quantitative success of the 19th century classical wave theory of light as developed by Young, Fresnel and others, as well as Maxwell’s identification of light with electromagnetic waves, the concept of ‘light waves’ was already firmly woven into the fabric of physics at the time of the advent of quantum mechanics. (2) The problem which was addressed by the pioneers of quantum mechanics — the theory of atomic structure and atomic radiative transitions — was one one in which space-time concepts play no essential role. As correctly remarked by Bohr [10]:

For example, the experiments regarding the excitation of spectra by electronic impacts and by radiation are adequately accounted for on the assumption of discrete stationary states and individual transition processes. This is primarily due to the circumstance that in these questions no closer description of the space-time behaviour of the processes is required.

Indeed, the wavefunction concept and the Schrödinger equation are essential for the quantum description of atomic states and atomic transitions (which may be characterised as ‘quantum statics’) but they are, unlike the probability amplitude of Feynman’s formulation, ill-adapted to the description of dynamical experiments where space-time ideas are primordial, such as those described in Sections 4, 5 and 6 above or [7].

Another subject to be found in typical quantum mechanical text books is ‘measurement theory’ based on projection operators in an abstract Hilbert space, following the methodology introduced by von Neumann in the early 1930’s [34]. ‘Measurement’ is defined as the projection of an eigenfunction out of some general Hilbert space state vector. For example16:

...we see that a measurement always causes a system to jump into an eigenstate of the dynamical variable that that is being measured, the eigenvalue of this eigenstate being equal to the result of the measurement.

In the case of the experiments described in Ref. [7] and in Sections 4 and 5 above ‘measurement’ (i.e. detection of the photon) does not produce a photon in ‘an eigenstate of position’ but simply destroys it. The electrons which are detected in the Faraday box

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16See Ref. [13], Sect. 10, p. 36.
of the Davisson-Germer experiment are also not left in ‘position eigenstates’ after detection. To state the situation bluntly, the basic concepts of textbook quantum mechanical ‘measurement theory’, although mathematically elegant, have little direct applicability to most real-world experiments. Rigorous conclusions are obtained within a limited, well-defined, but abstract model. However this model is, in general, too simplistic to address the complexity of actual, real-world experiments like those considered in Sections 5 and 6 above.

As a last application of the path integral formalism, consider the well-known ‘Schrödinger’s cat’ thought experiment of wave mechanics [12]:

A cat is penned up in a steel chamber along with the following diabolical device (which must be secured against direct influence by the cat): in a Geiger counter there is a tiny bit of radioactive substance so small that perhaps in the course of one hour one of the atoms decays, but also, with equal probability, perhaps none; if it happens, the counter tube discharges and through a relay releases a small hammer which shatters a small flask of hydrocyanic acid. If one has left this entire system to itself for an hour one would say that the cat still lives if meanwhile no atom has decayed. The first atomic decay would have poisoned it. The \( \psi \) function of the entire system would express this by having in it the living and dead cat (pardon the expression) mixed, or smeared out in equal parts. (italics in the original)

The conditions of the experiment are met by considering a single radioactive atom of mean lifetime \( \tau = 1.44 \text{ hr} \). What is relevant here is not ‘The \( \psi \) function of the entire system’ but the time-dependent transition amplitude: \( \langle f, t | i, t \rangle \) of the decay process where \( i \) and \( f \) are the initial and final states of the unstable radioactive nucleus. Using (4.9) and (2.8) the probability that the cat is still alive after a time interval \( t \) is:

\[
P(\text{cat alive}) = \frac{\left| \langle f, t - t_{\text{Del}} | i, t - t_{\text{Del}} \rangle \right|^2}{\left| \langle f, 0 | i, 0 \rangle \right|^2} = \exp \left\{ -\frac{(t - t_{\text{Del}})}{\tau} \right\} \tag{8.1}
\]

where \( t_{\text{Del}} \) is the delay time, determined by a chain of events related by classical causality, after the decay of the radioactive atom, before the cat actually dies. The cat is dead at, or after, the time interval \( t \) if the atom decays within the time interval \( t - t_{\text{Del}} \) so that:

\[
P(\text{cat dead}) = 1 - P(\text{cat alive}) = 1 - \exp \left\{ -\frac{(t - t_{\text{Del}})}{\tau} \right\} \tag{8.2}
\]

since the atom either decays, or does not decay, within the time interval \( t - t_{\text{Del}} \). The probabilities in (8.1) and (8.2) are independent of whether the chamber is opened or remains shut after the experiment, i.e. whether the cat is observable or not. At each instant during the experiment the cat is either alive or dead so its wavefunction is either \( \psi(\text{cat alive}) \) or \( \psi(\text{cat dead}) \). Schrödinger’s putative wavefunction, after one hour, (neglecting the delay \( t_{\text{Del}} \)) of \( \psi = [\psi(\text{cat alive}) + \psi(\text{cat dead})]/\sqrt{2} \) therefore has no relevance to the analysis of the experiment. Note that the quantum mechanical superposition evident in Schrödinger’s wavefunction plays no role in the derivation of the prediction (8.2) in Feynman’s formulation, much less ‘entangled’ wavefunctions, that are appropriate only to quantum experiments with dual final states.
Consider paths A or B followed by a free particle of mass \( m \) with velocities \( v_A \) or \( v_B \). If the corresponding times of transit are \( t_A, t_B \) then the lengths of the paths are: \( s_A = v_A t_A, \quad s_B = v_B t_B \). With the definitions:

\[
\mathcal{R} \equiv \frac{t_A}{t_B}, \quad \mathcal{D} \equiv t_A - t_B
\]

(A.1)

it follows that

\[
v_A = \frac{(\mathcal{R} - 1)s_A}{\mathcal{D}}, \quad v_B = \frac{(\mathcal{R} - 1)s_B}{\mathcal{D}}.
\]

(A.2)

If \( t_A \geq t_B \) then the physically allowed regions of \( \mathcal{R} \) and \( \mathcal{D} \) are:

\[
1 \leq \mathcal{R} < \infty, \quad 0 \leq \mathcal{D} < \infty.
\]

(A.3)

The conditions \( \mathcal{R} = 1 \) and \( \mathcal{D} = 0 \) correspond to equal transit times. Also of particular interest is the condition:

\[
\mathcal{R} = \mathcal{R}_{EV} \equiv \frac{s_A}{s_B}
\]

(A.4)

which can be seen, from (A.2), to correspond to equal velocities \( v_A = v_B \). Defining

\[
\Delta v \equiv v_A - v_B, \quad \bar{v} \equiv \frac{v_A + v_B}{2}
\]

(A.5)

and combining (A.2), (A.4) and (A.5) gives

\[
\Delta v = \frac{2\bar{v}(\mathcal{R}_{EV} - 1)}{(\mathcal{R}_{EV} + \mathcal{R})} = \frac{\bar{v}(\mathcal{R}_{EV} - 1)}{\mathcal{R}} + O \left[ \left( \frac{\Delta s}{\bar{s}} \right)^2 \right]
\]

(A.6)

where

\[
\Delta s \equiv s_A - s_B, \quad \bar{s} \equiv \frac{s_A + s_B}{2},
\]

(A.7)

It follows from the relativistic relations:

\[
v = \frac{pc^2}{E} = \frac{pc^2}{\sqrt{m^2c^4 + p^2c^2}}
\]

(A.8)

that

\[
\Delta p = \frac{E^3c^2}{m^2c^6} \Delta v.
\]

(A.9)

Retaining only terms of order \( \Delta s/\bar{s} \) in the phase difference formula (6.11) gives

\[
\Delta \phi \equiv \phi_A - \phi_B = \frac{(mc)^2}{\hbar} \left[ \frac{s_B - s_A}{p_B - p_A} \right]
\]

\[
= \frac{(mc)^2}{\hbar} \left[ \frac{\Delta p}{p^2} - \frac{\Delta s}{p} \right] + O \left[ \left( \frac{\Delta s}{\bar{s}} \right)^2 \right]
\]

\[
= \frac{(mc)^2}{\hbar} \left[ -\frac{\Delta s}{\bar{p}} + \frac{\bar{s}}{\bar{p}} \left( 1 + \frac{p^2}{(mc)^2} \right) \frac{\mathcal{R}_{EV} - \mathcal{R}}{\mathcal{R}} \right] + O \left[ \left( \frac{\Delta s}{\bar{s}} \right)^2 \right]
\]

(A.10)

where, in the last line, (A.6), (A.8) and (A.9) have been used.
Setting $\mathcal{R} = \mathcal{R}_{EV}$ (equal velocities, and so different production times in the paths A and B) gives

$$\Delta \phi = -\frac{(mc)^2}{\hbar} \frac{\Delta s}{\bar{p}} \quad \text{(equal velocities).} \quad (A.11)$$

Setting $\mathcal{R} = 1$ so that $t_A = t_B$ and the paths have the same initial time, but are executed with different velocities, gives

$$\Delta \phi = \frac{(mc)^2}{\hbar} \left[ -\frac{\Delta s}{\bar{p}} + \bar{s} \left( \frac{1}{\bar{p}} + \frac{\bar{p}}{(mc)^2} \right) \frac{\Delta s}{s_B} \right]$$

$$= \frac{(mc)^2}{\hbar} \left[ -\frac{\Delta s}{\bar{p}} + \frac{\Delta s}{\bar{p}} + \frac{\bar{p} \Delta s}{(mc)^2} \right] + O \left[ \left( \frac{\Delta s}{\bar{s}} \right)^2 \right]$$

$$= \frac{\bar{p} \Delta s}{\hbar} \quad \text{(equal times)} \quad (A.12)$$

where, in the last line, only terms of order $\Delta s/\bar{s}$ have been retained. Equations (A.11) and (A.12) are Eqs. (6.14) and (6.13), respectively, of the main text.
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