Achieving SDP Tightness Through SOCP Relaxation with Cycle-Based SDP Feasibility Constraints for AC OPF

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Abstract—In this paper, we show that the standard semidefinite programming (SDP) relaxation of altering current optimal power flow (AC OPF) can be equivalently reformulated as second-order cone programming (SOCP) relaxation with maximal clique- and cycle-based SDP feasibility constraints. The formulation is based on the positive semi-definite (PSD) matrix completion theorem, which states that if all sub matrices corresponding to maximal cliques in a chordal graph are PSD, then the partial matrix related to the chordal graph can be completed as a full PSD matrix. Existing methods in [1] first construct a chordal graph through Cholesky factorization. In this paper, we identify maximal cliques and minimal chordless cycles first. Enforcing the submatrices related to the maximal cliques and cycles PSD will guarantee a PSD full matrix. Further, we conduct chordal relaxation for the minimal chordless cycles by adding virtual lines and decompose each chordless cycle to 3-node cycles. Thus, the entire graph consists of trees, maximal cliques, and 3-node cycles. The submatrices related to the maximal cliques and 3-node cycles are enforced to be PSD to achieve a full PSD matrix. As majority power grids having the size of maximal cliques limited to 4-node, this graph decomposition method results in a low-rank full PSD matrix. The proposed method significantly reduces computing time for SDP relaxation of AC OPF and can handle power systems with thousands of buses.

Index Terms—SOCP; SDP; AC OPF; Chordal Relaxation; Sparsity Technique

I. INTRODUCTION

SDP relaxation of AC OPF has shown to be a very strong convex relaxation to the original non-convex formulation [2], [3]. The disadvantage of SDP relaxation is its expensive computational cost. For that reason, power grid sparsity was exploited in the literature. Positive semidefinite (PSD) matrix completion theorem has been used for sparsity exploitation for SDP relaxation in [1]. Detailed implementation procedure was presented in [4]. This sparsity technique has been also implemented for moment-based relaxation for OPF problems [5] to handle power grids up to 300 buses. Based on the PSD matrix completion theorem [6], if every submatrix related to every maximal clique in a chordal graph is PSD, then the partial symmetric matrix \( X_{ch} \) corresponding to the chordal graph can be completed as a full PSD matrix \( X \succeq 0 \). In addition, [3], [7] show that SDP relaxation based on a full matrix is equivalent to SDP relaxation based on a partial matrix related to the chordal extension of the power grid. The definition of the partial matrix in [3], [7] is different from that in [1]. In [1], the dimension of the partial matrix is the same as the original matrix. While in [3], [7], the matrix related to the chordal extension is much larger. Indeed, that matrix is related to the clique tree. Instead of finding maximal cliques and further a clique tree through Cholesky factorization tree decomposition, tree decomposition based on graph can also be used to find a clique tree. Tree width decomposition-based SDP relaxation is presented in [8] and has been implemented in a software package for SDP relaxations of OPF [9].

In this paper, we explore an alternative computationally friendly method that can achieve the strength of SDP relaxation. Specifically, instead of directly dealing with SDP relaxations, we consider SOCP relaxation first. Note that compared to SDP relaxation, SOCP relaxation is computationally more efficient. Nevertheless, the feasible region of SOCP relaxation is less tighter. Strengthening SOCP relaxation has been studied in [10], [11] by implementing cutting plane algorithms, i.e., iteratively adding valid inequalities (aka. cuts), including SDP based ones.

Instead of iteratively solving and strengthening the SOCP relaxation by cutting planes, we propose to directly add maximal clique-based and cycle-based SDP feasibility constraints in the SOCP relaxation. Those added constraints enforce the submatrices related to maximal cliques and cycles to be PSD. According to PSD matrix completion theorem [6], a matrix is PSD if and only if every submatrix related to its maximal cliques is PSD. A chordless cycle with \( n \geq 4 \) nodes can be converted to a clique by adding \( \frac{n(n-3)}{2} \) edges. In a power grid, the added edges can be viewed as virtual lines: Two buses without a direct connection are connected through a line with an impedance approaching infinity. Hence, with the submatrices related to cycles being PSD, the full matrix corresponding to the entire grid is PSD. The mathematical programming problem formulated is indeed SDP relaxation of AC OPF.

Further, we conduct chordal relaxation based on a power grid’s topology to obtain a chordal graph. A power grid graph consists of trees, maximal cliques and minimal chordless cycles. A chordless cycle can be decomposed into 3-node cycles or cliques by adding \( (n-3) \) virtual edges. If the submatrices related to all 3-node cycles are PSD, then the matrix related to the entire chordal graph or the grid is PSD. In the second formulation, the cycle-based SDP feasibility constraints are replaced by 3-node cycle-based SDP constraints. Compared to the maximal clique identification method through Cholesky
factorization in [1] and the tree decomposition method in [9], the 3-node cycle decomposition method yields submatrices with much smaller sizes. For a 2383-bus system, the maximum size of a bag in a clique tree is more than 20 nodes [9], while the maximum size of maximal cliques using the method in [1] is more than 200 nodes from the experiments conducted in this paper.

We highlight that the sparsity technique proposed in this paper yielding 3-node cycles is based on an understanding of the power network physics: Two buses with no direct connection can be viewed as with a direct connection, however the impedance of the connecting line is infinity. This fact enables the addition of virtual lines to facilitate graph decomposition.

We conducted case studies for systems with various sizes from 3 buses to 2383 buses. For the systems with size less than or equal to 300 buses, we compared gaps from the above two formulations as well as the standard SDP relaxation for systems. Results show that the proposed formulations produce results same as those of SDP relaxation of AC OPF. The two large-scale power grids (1354-bus and 2383-bus) are used as examples to test the second formulation for computing speed.

The rest of the paper is organized as follows. Section II presents SOCP and SDP relaxations of AC OPF. Section III presents the maximal clique- and cycle-based SDP formulations. Numerical results are presented in Section IV. Section V concludes the paper.

II. SOCP AND SDP RELAXATIONS OF AC OPF

A. Nonconvex AC OPF Formulation

The decision variables of AC OPF are voltage magnitudes $V$, phase angles $\theta$ of buses (the bus set is notated as $\mathcal{B}$), and generators’ real and reactive power outputs, notated as $P_i^g$, $Q_i^g$, where $i \in \mathcal{G}$, and $\mathcal{G} \subseteq \mathcal{B}$ is the subset of the buses. The set of the branches is notates as $\mathcal{L}$. The mathematical programming problem of AC OPF is presented in (1).

$$\begin{align*}
\min & \quad f_i(P_i^g) \\
\text{s.t.} & \quad P_i^g - P_i^d = P_i(V_i, \theta_i) = 0, \quad i \in \mathcal{B} \\
& \quad Q_i^g - Q_i^d = Q_i(V_i, \theta_i) = 0, \quad i \in \mathcal{B} \\
& \quad |S_{ij}(V_i, \theta_i)| - S_{ij}^{max} \leq 0, \quad (i, j) \in \mathcal{L} \\
& \quad V_i^{min} \leq V_i \leq V_i^{max}, \quad i \in \mathcal{B} \\
& \quad P_i^{g, min} \leq P_i^g \leq P_i^{g, max}, \quad i \in \mathcal{G} \\
& \quad Q_i^{g, min} \leq Q_i^g \leq Q_i^{g, max}, \quad i \in \mathcal{G}
\end{align*}$$

where $f(.)$ is the cost function, $P_i^d$, $Q_i^d$ are the real and reactive power load consumption at Bus $i$, $P_i(V_i, \theta_i)$ and $Q_i(V_i, \theta_i)$ are Bus $i$’s power injection expressions in terms of bus voltage magnitudes and phase angles, and $S_{ij}(V_i, \theta_i)$ is the complex power flow from Bus $i$ to Bus $j$ on the branch connecting the two buses. The dimension of the decision variable vector is $2|\mathcal{G}| + 2|\mathcal{B}|$.

The aforementioned AC OPF formulation is a nonconvex optimization problem. This can be shown by the power injection equality constraints. Given the system admittance matrix $Y = G + jB$, the power injection at every node can be expressed by $V$ and $\theta$.

$$\begin{align*}
P_i^g - P_i^d &= \sum_{j \in \mathcal{B}} V_i V_j (G_{ij} \cos(\theta_i - \theta_j) + B_{ij} \sin(\theta_i - \theta_j)) \\
Q_i^g - Q_i^d &= \sum_{j \in \mathcal{B}} V_i V_j (G_{ij} \sin(\theta_i - \theta_j) - B_{ij} \cos(\theta_i - \theta_j))
\end{align*}$$

where superscript $g$ notates generator’s output and $d$ notates load consumption. $\delta_i$ is the set of the buses that are directly connected to Bus $i$ through branches.

Note that the equality constraints of power injections are nonlinear in terms of $V$ and $\theta$. This yield the AC OPF problem nonconvex. Relaxations have been developed in the literature to have a convex feasible region. These methods deal with new sets of decision variables to replace $V$ and $\theta$.

B. SOCP and SDP relaxations

In SOCP relaxation [12], a new set of variables $c_{ij}$ and $s_{ij}$ is used to replace the voltage phasors $V_i \angle \theta_i, i \in \mathcal{B}$.

$$\begin{align*}
c_{ii} &= V_i^2, \quad i \in \mathcal{B} \\
c_{ij} &= V_i V_j \cos(\theta_i - \theta_j), \quad (i, j) \in \mathcal{L} \\
s_{ij} &= -V_i V_j \sin(\theta_i - \theta_j), \quad (i, j) \in \mathcal{L}
\end{align*}$$

where $c_{ij} = c_{ji}$ and $s_{ij} = -s_{ji}$.

It is easy to find the following relationship:

$$c_{ij}^2 + s_{ij}^2 = V_i^2 V_j^2 = c_{ii} c_{jj}. \quad (3)$$

There will be $|\mathcal{L}|$ number $c_{ij}$ and $s_{ij}$ to be defined as decision variables. If there is no direct connection between Bus $i$ and Bus $j$, the power injection equations will not contain $c_{ij}$ nor $s_{ij}$. The decision variables $V$ and $\theta$ are now replaced by $c_{ii}, i \in \mathcal{B}$, and $c_{ij}, s_{ij}, (i, j) \in \mathcal{L}$. The dimension of the new set of the variables is $2|\mathcal{G}| + |\mathcal{B}| + 2|\mathcal{L}|$.

With the new set of variables, power injection equations (2) are now linear. The line flow limit constraints in (1d) are second-order cone constraints. The voltage limits (1e) can be replaced by:

$$c_{ii}^{min} \leq c_{ii} \leq c_{ii}^{max}, \quad i \in \mathcal{B} \quad (4)$$

where $c_{ii}^{min} = (V_i^{min})^2$ and $c_{ii}^{max} = (V_i^{max})^2$.

The only constraint that makes the program nonconvex is (3). This constraint can be relaxed as a second-order cone:

$$c_{ij}^2 + s_{ij}^2 \leq c_{ii} c_{jj}, \quad (i, j) \in \mathcal{L} \quad (5)$$

This relaxation is first proposed in [12] for AC OPF and named as SOCP relaxation.

It has been shown that in spanning tree power networks, under a mild condition (e.g., voltage upper bounds not binding), SOCP relaxations are exact [13].

In SDP relaxation, rectangular expressions are used to represent the voltage phasors.

$$V_i = V_i \angle \theta_i = \underbrace{V_i \cos \theta_i + j V_i \sin \theta_i}_{e_i}$$

where $e_i$ and $f_i$ are the active and reactive power components of Bus $i$.
A matrix $W$ is defined as follows

$$W = \begin{bmatrix} e^T & f^T \end{bmatrix} = \begin{bmatrix} e & e^T \\ f & f^T \end{bmatrix} \quad (7)$$

where $e = (e_1, e_2, \cdots, e_n)^T$, $f = (f_1, f_2, \cdots, f_n)^T$, $n = |\mathcal{B}|$.

Alternatively, we may define a complex matrix $X = \overline{V} V^H$, where $X_{ij} = \overline{V_i V_j}$ and superscript $H$ means complex conjugate transpose.

It is obvious to find the following characteristics:

$$W = W^T, \quad W \succeq 0, \text{ and } \text{rank}(W) = 1 \quad (8)$$

$$X = X^H, \quad X \succeq 0, \text{ and } \text{rank}(X) = 1 \quad (9)$$

where superscript $T$ denotes transpose and superscript $H$ denotes Hermitian transpose. $W \succeq 0$ means that this matrix is positive semi-definite (PSD).

The power injection constraints are linear with the elements of $W$. With the rank 1 constraint relaxed, the problem is a convex problem: SDP relaxation of AC OPF.

For tree networks, SOCP relaxation and SDP relaxation are equivalent [3]. For meshed network, then the SOCP constraint (5) enforces only $2 \times 2$ submatrices PSD of a full matrix $X$. For cliques or cycles with more than 2 nodes, SOCP relaxation does not guarantee the related submatrices PSD.

In SDP relaxation of OPF, $W$ or $X$ is treated as a decision variable and $W$ or $X$ should be PSD. In SOCP relaxation of OPF, $c_{ij}$, $c_{ij}$ and $s_{ij}$ are the decision variables. The two set of the decision variables have the following relationship.

Define

$$i' = i + |\mathcal{B}|, \quad j' = j + |\mathcal{B}|, \quad (10)$$

where $|.|$ notates the cardinality of a set.

$$c_{ij} = e_i e_j + f_i f_j = W_{ij} + W_{i'j'}, \quad (11)$$

$$s_{ij} = e_i f_j - e_j f_i = W_{ij} - W_{ij'}, \quad$$

$$c_{ii} = e_i^2 + f_i^2 = W_{ii} + W_{i'i'}.$$ 

For every $c_{ij}$, $s_{ij}$ and $c_{ii}$, we can notate them as $z_l$ ($l = 1, \cdots, (|\mathcal{B}| + 2)|\mathcal{C}|)$ and express them to be the Frobenius product related to the (PSD) matrix $W$:

$$z_l = A_l \bullet W,$$

where $\bullet$ denotes Frobenius product.

### III. MAXIMAL CLIQUE AND CYCLE-BASED SDP/SOCP FORMULATIONS

Instead of dealing with a $W$ for the entire grid, for each maximal clique and each cycle in the cycle basis of the network, we may impose the relationship between the two sets of the decision variables. For each cycle, we then impose the PSD constraint for the corresponding submatrix $\overline{W}$. This approach can save computing cost due to the reduction of the size of the PSD matrices.

#### A. Maximal Cliques Identification

Given a graph’s boolean adjacency matrix, all maximal cliques can be identified using Bron-Kerbosch algorithm [14]. In this project, a MATLAB toolbox [15] based on Bron-Kerbosch algorithm has been used to identify maximal cliques. Table I presents the size of the largest maximal cliques in test cases. We may observe that all grids have 3-node cliques or cycles as the largest maximal cliques, except IEEE 118-bus system. This system has a maximal clique of size 4.

| Test case | Size | Test case | Size |
|-----------|------|-----------|------|
| case1_lmbd | 3 | case4_gs | 2 |
| case5_pim | 3 | case14_ieee | 3 |
| case30_ieee | 3 | case57_ieee | 3 |
| case118_ieee | 4 | case300_ieee | 3 |
| case1354_pegase | 3 | case2383wp | 3 |

After identifying the maximal cliques in a power grid, the next step is to identify minimal chordless cycles. The identification is achieved in two steps. First, cycle basis identification is carried out to identify a set of cycles in a cycle basis. These cycles are not guaranteed to be minimal chordless cycles. To construct a chordal graph, minimal chordless cycles should be identified. Thus, the second step is to compare the cycles with each other and maximal cliques and construct minimal chordless cycles.

#### B. Cycle Basis Identification

Cycle basis identification algorithm in [16] is used to identify the cycle basis. A related MATLAB toolbox is also available [17]. The five-bus system shown in Fig. 1 has two cycles $\{1, 4, 5\}$ and $\{1, 2, 3, 4\}$ in its cycle basis.

Fig. 1. Five-bus test case with two cycles. Cycle 1: nodes 1, 2, 3, 4, branches 1, 2, 4, 5; Cycle 2: nodes 1, 4, 5, branches 2, 3, 6.

The cycle basis identification algorithm first identifies the system’s minimum spanning tree and then examines the rest of the edges one by one. For example, the spanning tree starts from Bus 1 has three branches: $1 \rightarrow 4$, $1 \rightarrow 2 \rightarrow 3$, and $1 \rightarrow 5$.

Fig. 2. Spanning tree of the five-bus test system.

The edges not included in the spanning trees are 3 – 4 and 4 – 5. This indicates the system has two cycles in its cycle basis. With the first edge $(3, 4)$ added to the spanning tree, the
first cycle is identified as \{1, 2, 3, 4\}. With the second edge \(1 \rightarrow 4\) added to the spanning tree, the second cycle is identified as \{1, 4, 5\}.

In the following IEEE 14-bus example (shown in Fig. 3), a spanning tree is be built (shown in Fig. 4).

![Fig. 3. Topology of IEEE 14-bus system. Shaded regions indicate five maximal cliques.](image1)

The spanning tree will lead to a set of cycles in a cycle basis as shown in Fig. 5. We can see that the set of cycles has the node sizes as 3, 3, 3, 3, 4, 8, 8.

![Fig. 4. The spanning tree of IEEE 14-bus system.](image2)

![Fig. 5. Cycles obtained using the first spanning tree.](image3)

C. Minimal Chordless Cycle Identification

Minimal chordless cycles are desired for chordal graph construction purpose. Fig. 6 presents an example graph to illustrate the chordal graph construction and why minimal chordless cycles are desired. Fig. 6(a) presents the original topology of a graph. The definition of a chordal cycle is that all cycles of four or more vertices have a chord. This original graph is not a chordal graph since there is no chord for Cycle \{2, 3, 4, 5\}.

As cycle basis identification algorithm does not guarantee minimal chordless cycles, we may end up with two cycles identified for Fig. 6(a): \{1, 2, 5\} and \{1, 2, 3, 4, 5\}. Suppose that for the second cycle identified, two lines: \(1 \rightarrow 4\) and \(1 \rightarrow 5\) are added. The resulting graph shown in Fig. 6(b) is not a chordal graph since there is no chord in Cycle \{2, 3, 4, 5\}.

![Fig. 6. Chordal graph construction explanation.](image4)

On the other hand, if we are able to identify the two minimal chordless cycles as \{1, 2, 5\} and \{2, 3, 4, 5\}, we may add a chord in the 4-node cycle (line 2 \rightarrow 4 or line 3 \rightarrow 5). The resulting graphs shown in Fig. 6(c)(d) are two chordal graphs. For chordal graphs, we may apply PSD matrix completion theorem. For this system, as long as the submatrices related to three 3-node cycles are PSD, the full matrix is PSD.

Therefore, minimal chordless cycle identification is needed. We will use the IEEE 14-bus system to explain the minimal chordless cycle generation procedure developed for this research. First, five maximal cliques of this system can be identified using maximal clique identification toolbox [15]. The cliques are all 3-node cycles and have been marked as the shaded boxes in Fig. 3. Note that the cycle basis includes only four maximal cliques. Cycle \{2, 4, 5\} is not included. In the cycle basis, the 4-node cycle \{1, 2, 4, 5\}, the two 8-node cycles \{1, 2, 4, 9, 10, 11, 6, 5\}, and \{1, 2, 4, 9, 14, 13, 6, 5\} are not minimal chordless cycles.

For example, the last cycle (the 7th) \{1, 2, 4, 9, 14, 13, 6, 5\} can be reduced to a smaller cycle by comparing this cycle with the 5th cycle \{1, 2, 4, 5\}, getting rid of overlapping edges and adding back an edge (4 \rightarrow 5). The resulting cycle is \{4, 5, 6, 13, 14, 9\}.

![Fig. 7. Cycles in IEEE 14-bus system.](image5)

After the reduction step, the seven cycles are shown in Fig. 7. Note the two cycles with the largest sizes are shown with dotted lines connected. These lines are virtual lines that will be discussed in Subsection II.F.
D. Positive Semidefinite Matrix Completion

The PSD matrix completion theorem [6] states that if and only if the submatrices corresponding to all maximal cliques of a chordal graph are PSD, then the partial symmetric matrix $X$ corresponding to the chordal graph with some elements unknown can be completed to a full PSD matrix $X$.

The graph related to a power grid can viewed as the combination of spanning trees, maximal cliques, and chordless cycles. For spanning trees, the maximal cliques are two nodes (e.g. $i$ and $j$) connected by one line. Therefore, the related complex submatrix is $2 \times 2$. If $X$ is PSD, then the submatrix should be PSD. The determinant of the submatrix should be greater than zero to guarantee PSD.

\[
\begin{bmatrix}
X_{ii} & X_{ij} \\
X_{ji} & X_{jj}
\end{bmatrix} = X_{ii}X_{jj} - X_{ij}X_{ji} \geq 0
\] (12)

Eq. (12) can be written as

\[
V_i^2 V_j^2 - V_i V_j V_i V_j^* \geq 0
\implies c_{ij}^2 + i_{ij}^2 \leq c_{ij} I_{ij}
\]

Thus, the second-order cone constraint (5) imposed for every line guarantees the PSD condition for all maximal cliques with 2 nodes.

For chordless cycles, they can be first relaxed to be chordal graphs by adding virtual edges. In power grid, this is similar to claim that any two nodes without direct line connection can be viewed as connected through a line with infinite impedance.

We may further consider that each cycle with $n$ nodes can be viewed as a maximal clique with $n(n-3)/2$ virtual lines added (see Fig. 8). For a cycle of 4 nodes with ring connection as $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4 \leftrightarrow 1$, we may add two virtual lines $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$. For a cycle of 5 nodes with ring connection as $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4 \leftrightarrow 5 \leftrightarrow 1$, we may add 5 virtual lines $1 \leftrightarrow 3$, $1 \leftrightarrow 4$, $2 \leftrightarrow 4$, $2 \leftrightarrow 5$, and $3 \leftrightarrow 5$.

Fig. 8. Cycles become cliques with virtual lines. 4-node cycle: 2 virtual lines. 5-node cycle: 5 virtual lines.

If every submatrix corresponding each cycle are PSD, then the entire grid’s matrix $W$ or $X$ is PSD.

For the 5-bus system presented in Fig. 1, the two submatrices $X([1, 4, 5], [1, 4, 5])$ and $X([1 : 4], [1 : 4])$ should be PSD to have a full PSD matrix $X$.

E. Formulation 1

Based on the above argument, we propose the first maximal clique- and cycle-based formulation.

The decision variables include those in SOCP relaxation $(c_{ii}, c_{ij}, s_{ij})$ and submatrices $\tilde{W}$ related to each cycle (with size greater or equal to 4) in a cycle basis and the maximal cliques (with size greater or equal to 3)

\[
\min \ f(P_g)
\]

\[
s.t. \quad P_i^g - P_i = \sum_{j \in S} (G_{ij}c_{ij} - B_{ij}s_{ij}), \quad i \in B
\]

\[
Q_i^g - Q_i = \sum_{j \in S} (-G_{ij}s_{ij} - B_{ij}c_{ij}), \quad i \in B
\]

\[
P_{ij} = g_{ij}(c_{ii} - c_{ij}) - b_{ij}s_{ij}, \quad (i, j) \in \mathcal{L}
\]

\[
Q_{ij} = -b_{ij}(c_{ii} - c_{ij}) - g_{ij}s_{ij}, \quad (i, j) \in \mathcal{L}
\]

\[
|P_{ij}^2 + Q_{ij}^2| \leq S_{\max}^2, \quad (i, j) \in \mathcal{L}
\]

\[
(V_i^\text{max})^2 \leq c_{ii} \leq (V_i^\text{max})^2, \quad i \in \mathcal{B}
\]

\[
c_{ij}^2 + s_{ij}^2 \leq c_{ij} c_{ij}, \quad (i, j) \in \mathcal{L}
\]

\[
P_g^\text{min} \leq P_g \leq P_g^\text{max}, \quad Q_g^\text{min} \leq Q_g \leq Q_g^\text{max}
\]

for each maximal clique $MC$,

\[
z_l = A_l \cdot \tilde{W}_i, \quad l = 1, \ldots, |MC|/2
\]

\[
\tilde{W}_i \geq 0, \quad i \in S_{MC}
\]

for each cycle identified $C$,

\[
z_l = A_l \cdot \tilde{W}_i, \quad l = 1, \ldots, 3|C|
\]

\[
\tilde{W}_i \geq 0, \quad i \in S_C
\]

where $S_{MC}$ notates the set of maximal cliques with sizes greater than or equal to 3, $|MC|$ is the size of a maximal clique, $V_i^\text{max}$ is the combination of every two nodes, $S_C$ notates the set of cycles with sizes greater than 3, $g_{ij} = \text{real}(y_{ij})$ and $b_{ij} = \text{imag}(y_{ij})$ is a branch (between Bus $i$ and Bus $j$)’s admittance, (13b) and (13c) represent the line power injection constraints at each bus. (13f) is the line flow limit constraint. (13g) is the voltage limit constraint. (13h) is the second-order cone relaxation inequality. (13i) are the generator power limit constraints, and (13l) and (13m) are the SDP feasibility constraints for each cycle.

For the 5-bus example in Fig. 1, a full PSD matrix $W$ is a symmetric $10 \times 10$ matrix with $10 \times 11 = 55$ independent matrix elements. With two cycles identified, the two submatrices having the sizes as $6 \times 6$ and $8 \times 8$ with Branch 1–4 shared by both cycles. We may define a sparse matrix $W'$ with the two submatrices related to Cycle $\{1, 2, 3, 4\}$ and Cycle $\{1, 4, 5\}$. Consider its symmetric feature, the number of independent matrix elements is 47. $W'$ will be used in the optimization programming problem and coding instead of $W$.

Fig. 9 present the structure of the sparse matrix $W'$. $W'$ has $2 \times 2$ blocks, each block with the same sparsity structure. For large-scale power grids, we expect the exploiting of sparsity can reduce computing time related to SDP relaxation.

F. Formulation 2

Computing experiments conducted for Formulation 1 with systems with thousands of buses show that computing time is very long. An examination of an example 2283-bus system shows that the size of cycles ranges from 3 nodes to 45 nodes.
Fig. 9. The structure of the sparse matrix $W'$ in Formulation 1.

Fig. 10 shows the histogram of the cycle sizes. It can be seen that many cycles have more than 10 nodes. One cycle has a size of 45 nodes while 36 cycles are 10-node cycles.

Fig. 10. Histogram of the sizes of cycles in a 2383-bus power grid.

As a comparison, the method in [1] is used to identify maximal cliques. The admittance matrix of the 2383-bus system is first obtained. This matrix is permuted based on minimum-degree ordering heuristic. The Cholesky factorization of the absolute value of the imaginary part of the permuted admittance matrix is computed. The resulting matrix after Cholesky factorization defines the sparsity pattern of the chordal relaxation of the original network. An adjacency matrix for the chordal graph is then defined and maximal cliques are identified based on this adjacency matrix using a MATLAB toolbox developed based on Bron-Kerbosch algorithm [15].

It can be seen from the histogram in Fig. 11 that although Cholesky factorization can lead to many maximal cliques with small sizes, it can also lead to a number of maximal cliques with large sizes (up to 210 nodes).

Fig. 11. Histogram of the sizes of maximal cliques in a 2383-bus power grid.

In Formulation 2, a further graph decomposition strategy is employed to have virtual lines added and have any minimal chordless cycles with size $\geq 4$ to be decomposed into cycles with 3 nodes. The number of virtual lines added is $n-3$ where $n$ is the number of nodes.

For the IEEE 14-bus system, for the two cycles with node size greater than 3, virtual lines are first added as shown in Fig. 7. Total 13 3-node cycles can be identified. The submatrices related to these 13 cycles will be enforced to be PSD to guarantee a full PSD matrix.

The proposed graph decomposition indeed offers a sparse matrix technique. Instead of seeking a full matrix $W$ or $X$, we seek a sparse matrix $W'$ or $X'$. The submatrices related to the maximal cliques (including 3-node cycles) of $W$ and $W'$ are exactly the same. For large-scale systems, the 3-node graph decomposition strategy will yield to a more sparse matrix $W'$ compared to the cycle-based decomposition strategy. In turn, computational efficiency will be significantly improved.

What’s more, the full matrix $X$ or $W$ will have a low rank. The rank of $X$ equals to the maximum rank of submatrices related to maximal cliques in the chordal graph [18]. For the 3-node cycle-based decomposition, the rank of the full matrix is the maximum of the ranks of the submatrices related to the maximal cliques and 3-node cycles. If the largest size of maximal cliques is 4, then the rank is less than or equal to 4.

The mathematical programming problem related to Formulation 2 is shown in (14).

\[
\begin{align}
\text{min} & \quad f(P_g) \\
\text{s.t.} & \quad P^g_i - P^d_i = \sum_{j \in \delta_i} (G_{ij}c_{ij} - B_{ij}s_{ij}), \quad i \in B \\
& \quad Q^g_i - Q^d_i = \sum_{j \in \delta_i} (-G_{ij}s_{ij} - B_{ij}c_{ij}), \quad i \in B \\
& \quad P_{ij} = g_{ij}(c_{ii} - c_{ij}) - b_{ij}s_{ij}, \quad (i, j) \in \mathcal{L} \\
& \quad Q_{ij} = -b_{ij}(c_{ii} - c_{ij}) - g_{ij}s_{ij}, \quad (i, j) \in \mathcal{L} \\
& \quad |P^2_{ij} + Q^2_{ij} - S_{ij}^\max| \leq 0, \quad (i, j) \in \mathcal{L}
\end{align}
\]
(V_{\text{min}})^2 \leq c_{ii} \leq (V_{\text{max}})^2, \quad i \in B \quad (14g)
\|c_{ij}^2 + s_{ij}^2 - c_{ii}c_{jj}\|, \quad (i,j) \in L \quad (14h)
\sum_{i,j} c_{ij}^2 s_{ij} \leq \sum_i c_{ii} c_{jj}, \quad (i,j) \in L \quad (14i)

for each maximal clique $MC$, $z_l = A_l \cdot \tilde{W}_l, \quad l = 1, \cdots, C^2_{|MC|} \quad (14j)
\tilde{W}_l \succeq 0, \quad i \in S_{MC} \quad (14k)$

Here $S_{MC}$ includes all maximal cliques with size greater than or equal to 3 related to the chordal graph constructed. The 3-node cycles have all been included. Every $\tilde{W}_l$ is a submatrix of the sparse matrix $W'$.

IV. NUMERICAL EXAMPLES

Ten test cases (eight cases with bus number less than or equal to 300 and two cases with 1354 buses and 2383 buses) from the NICTA test archive [19] were tested using the proposed formulations. The ten cases were solved by MATPOWER to obtain feasible solutions as upper bounds. In addition, the first eight cases were solved by the formulation considering a full PSD matrix (notated as SDP in Table II), Formulation 1 considering PSD submatrices related to cliques, and Formulation 2 considering PSD submatrices related to maximal cliques and 3-node cycles. Programs were coded in MATLAB 2013b in CVX environment [20] with Moscak 7.1 [21] as the solver. The programs were run in a PC with 3.40GHz CPU and 16.0 GB RAM. The last two cases can only be solved by Formulation 2 and hence are presented in a separate table.

**TABLE II**

| Test Case | UBr($\$/h) | Gap (%) | $t$(sec) | MCs | Cycles |
|-----------|------------|---------|----------|-----|--------|
| case14_ieee  | 5812.64    | 1.26    | 0.49     | 0   | 1      |
| Form 1     | 1.26       | 0.49    | 0        | 1   |        |
| Form 2     | 1.26       | 0.49    | 0        | 1   |        |
| case4_gs   | 156.4      | 0.00    | 0.56     | -   | 1      |
| Form 1     | 0.00       | 0.56    | -        | 1   |        |
| Form 2     | 0.00       | 0.68    | 0        | 1   |        |
| case1354_pegase   | 17551.9   | 8.27    | 0.68     | -   | 2      |
| Form 1     | 8.27       | 0.68    | -        | 2   |        |
| Form 2     | 8.27       | 0.66    | 0        | 3   |        |
| case30_ieee  | 244.1      | 0.00    | 1.32     | -   | 15     |
| Form 1     | 0.00       | 1.32    | -        | 7   |        |
| Form 2     | 0.00       | 1.20    | 0        | 15  |        |
| case57_ieee  | 205.0      | 0.0     | 2.83     | -   | 12     |
| Form 1     | 0.0        | 2.83    | 0        | 12  |        |
| Form 2     | 0.0        | 2.85    | 0        | 29  |        |
| case118_ieee | 1143.2    | 0.0     | 10.5     | -   | 22     |
| Form 1     | 0.0        | 6.1     | 0        | 22  |        |
| Form 2     | 0.0        | 6.5     | 0        | 97  |        |
| case300_ieee  | 3715.7     | 0.88    | 28.20    | -   | 175    |
| Form 1     | 0.88       | 28.20   | 1        | 62  |        |
| Form 2     | 0.88       | 13.45   | 1        | 175 |        |
| case1354_pegase   | 16891.3    | 0.07    | 169.03   | -   | 364    |
| Form 1     | 0.07       | 77.72   | 0        | 119 |        |
| Form 2     | 0.07       | 31.69   | 0        | 364 |        |

Note MCs are maximal cliques with size $\geq 4$.

For Formulation 2, the cycles are all 3-node cycles.

Table III presents the results of the 1354-bus system and the 2383-bus system.

From both tables, it can be seen that Formulation 2 can significantly reduce the computing time when the size of the system increases. Especially for systems with more than a thousand buses, Formulation 2 is the only approach to give solutions. For the first 6 cases, all formulations give the same results as the SDP formulation.

Figs. 13 and 14 present the structure of the sparse matrix $W'$ defined in Formulation 2 for the 1354-bus case and the 2383-bus case. The $W'$ matrix has $2 \times 2$ blocks with the same sparsity structure. Hence only the first block is presented. It can be seen that with large-scale systems, exploiting sparsity can significantly reduce the number of matrix elements and hence save computing cost.

**TABLE III**

| Test Case | $t$(sec) | MCs | Cycles |
|-----------|----------|-----|--------|
| case1354_pegase   | 74069    | 0.0 | 357    |
| case2383wp        | 1688500  | 1.18| 504    |
| case2383wp        | 1580.3   | 1379|

**Fig. 13.** The structure of the sparse matrix $W'$'s first block in Formulation 2 of the 1354-bus case.

V. CONCLUSION

This paper presents two SOCP formulations with maximal clique-based and cycle-based SDP constraints for AC OPF problems. Both of them need cycle basis identification. In the first formulation, the cycles are viewed as maximal cliques with additional virtual lines. Enforcing the submatrices related to those cliques as PSD guarantees the full matrix PSD. In the second formulation, we further decompose the minimal chordless cycles in a cycle basic into 3-node cycles. We then enforce PSD constraints for all submatrices related to the 3-node cycles and other maximal cliques. Both formulations will lead to SDP relaxation of AC OPF while Formulation 2 is more computationally efficient.

REFERENCES

[1] R. A. Jabr, “Exploiting sparsity in sdp relaxations of the opf problem,” IEEE Transactions on Power Systems, vol. 27, no. 2, pp. 1138–1139, 2012.
Fig. 14. The structure of the sparse matrix $W'$'s first block in Formulation 2 of the 2383-bus case.