PURITY, ALGEBRAIC COMPACTNESS, DIRECT SUM DECOMPOSITIONS, AND REPRESENTATION TYPE

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INTRODUCTION

The idea of purity pervades human culture, from religion through sexual and moral codes, cleansing rituals and dietary rules, to stratifications of societies into castes. It does so to an extent which is hardly backed up by the following definition found in the Oxford English Dictionary: freedom from admixture of any foreign substance or matter. In fact, in a process of blending and fusing of ideas and ideals (which stands in stark contrast to the dictionary meaning of the term itself), the concept has acquired dozens of additional connotations. They have proved notoriously hard to pin down, to judge by the incongruence of the emotional and poetic reactions they have elicited. Here is a small sample:

Purity is obscurity. Ogden Nash.
One cannot be precise and still be pure. Marc Chagall.
Unto the pure all things are pure. New Testament.
To the pure all things are impure. Mark Twain.
To the pure all things are indecent. Oscar Wilde.

Blessed are the pure in heart for they have so much more to talk about. Edith Wharton.
Be thou as chaste as ice, as pure as snow, thou shalt not escape calumny. Get thee to a nunnery, go. W. Shakespeare.

Necessary, for ever necessary, to burn out false shames and smelt the heaviest ore of the body into purity. D. H. Lawrence.

Mathematics possesses not only truth, but supreme beauty – a beauty cold and austere, like that of a sculpture, without appeal to any part of our weaker nature, sublimely pure, and capable of a stern perfection such as only the greatest art can show. Bertrand Russell.

Purity strikes me as the most mysterious of the virtues and the more I think about it the less I know about it. Flannery O’Connor.

Even in the ‘sublimely pure’ subject of mathematics, several notions of purity were competing with each other a few decades ago. However, while the state of affairs is still
rather muddled in contemporary society at large, at least mathematicians have come to an agreement in the meantime. Section 1 traces the development of the concept within algebra in rough strokes. Due to the large number of contributions to the subject, we can only highlight a selection.

The survey to follow should be read in conjunction with those of M. Prest and G. Zwara ([59] and [84]). The first emphasizes the model-theoretic aspect of the topic, while the second describes the impact of generic modules on representation theory. The generic modules form a distinguished subclass of the class of pure injective (= algebraically compact) modules, the importance of which became apparent through the seminal work of Crawley-Boevey [16]. Since we wanted to minimize the overlap with an overview article on endofinite modules by the latter author, the article [17] should be consulted as another supplement to our survey. For the sake of orientation, we offer the following diagram displaying a hierarchy of noteworthy classes of algebraically compact modules.

In the present overview, we mainly address the outer layers of the stack. We conclude with a discussion of product-complete modules, the study of which provides a natural bridge to the class of endofinite modules. The interest of the latter class also becomes clear as one approaches it from alternate angles, e.g., via the duality theory developed in [17]. Zwara’s survey picks up the story at this point and zeroes in on the generic modules, i.e., the non-finitely generated indecomposable endofinite modules. The extensive spread of applications of these concepts is further illustrated by the contributions of D. Benson and H. Krause to this volume ([10] and [50]): Benson encounters them in his study of phantom maps between modules over group algebras, while Krause uses them towards new characterizations of tame representation type.

What follows below is almost exclusively expository. However, a few results rounding off the picture (e.g., the characterization of purity in terms of matrix groups in Proposition 4) appear to be new, as are several of our arguments. Moreover, in some instances, we streamline results which can essentially be found in the literature by taking the present state of the art into account.

The red thread that will lead us through this discussion is provided in Section 2, in the form of a number of global decomposition problems going back to work of Koethe (1935)
and Cohen-Kaplansky (1951). Long before the impact of the subject on the representation theory of tame finite dimensional algebras surfaced, these problems had put a spotlight on (\(\Sigma\)-)algebraic compactness and linked it to finite representation type. Section 3 contains the most important characterizations of the algebraically compact and the \(\Sigma\)-algebraically compact modules, as well as the functorial underpinnings on which they are based. Section 4 takes us back to the decomposition problems stated at the outset, and Section 5 evaluates the outcome in representation-theoretic terms. The topic of product-completeness, addressed in Section 6, supplements both the decomposition theory of direct powers of modules as described in Theorem 10 of Section 4, and the impact of ‘globally nice’ decompositions of products on the representation type of the underlying ring discussed in Section 5. The very short final section should be seen as an appendix, meant to trigger further research in a direction that has been somewhat neglected in the recent past.

1. **Purity and algebraic compactness – definitions and a brief history**

It was already in the first half of this century that the concept of a pure subgroup of an abelian group proved pivotal in accessing the structure, first of \(p\)-groups, then also of torsionfree and mixed abelian groups, as well as of modules over PID’s. Given a PID \(R\), a submodule \(A\) of an \(R\)-module \(B\) is called pure if, for all \(r \in R\), the intersection \(A \cap rB\) equals \(rA\). As is well-known, the submodule \(T(B)\) consisting of the torsion elements of \(B\) is always pure, and in exploring the \(p\)-primary components of \(T(B)\), major headway is gained by studying a tell-tale class of pure submodules of the simplest possible structure, namely the ‘basic’ ones. They are determined up to isomorphism by the following requirements: Given a prime \(p \in R\), a submodule \(B_0\) of a \(p\)-primary \(R\)-module \(B\) is called basic if it is a direct sum of cyclic groups which is pure in \(B\) and has the additional property of making \(B/B_0\) divisible. In fact, a natural extension of this concept to arbitrary modules over discrete valuation domains provides us with one of the reference points in the classification of the algebraically compact abelian groups sketched below.

In the 1950’s, it became apparent that suitable variations of the original notion of purity should yield important generalizations of split embeddings in far more general contexts, and several extensions of the concept, naturally all somewhat akin in spirit, appeared in the literature. Finally, in 1959, Cohn’s definition \([14]\) won the ‘contest’. Given left modules \(A\) and \(B\) over an arbitrary associative ring \(R\), Cohn called a monomorphism \(f : A \to B\) pure in case tensoring with any right \(R\)-module \(X\) preserves injectivity in the induced map \(id \otimes f : X \otimes_R A \to X \otimes_R B\); by extension, a short exact sequence \(0 \to A \to B \to C \to 0\) is labeled pure exact in case the monomorphism from \(A\) to \(B\) is pure. When, in 1960, Maranda \([54]\) undertook a study of the modules \(M\) for which the contravariant Hom-functor \(\text{Hom}_R(-, M)\) takes maps from certain restricted classes of monomorphisms in \(R\)-Mod to epimorphisms in the category of abelian groups, the class which quickly became the most popular in this game was that of pure monomorphisms. Accordingly, modules \(M\) such that \(\text{Hom}_R(-, M)\) preserves exactness in pure exact sequences were labelled pure injective.

One of the fundamental demands on a good concept of purity is this: It should yield convenient criteria for recognizing direct summands – the philosophy being that purity is
a first step toward splitting. One hopes for readily verifiable conditions which take a pure inclusion the rest of the way to a split one. Here are two sample statements of this flavor for Cohn’s purity; we will re-encounter them in Section 3.

**Observations 0.**

- If $R$ is a Dedekind domain and $M$ a pure submodule of an $R$-module $N$ such that $\text{Ann}_{R}(M) \neq 0$, then $M$ is a direct summand of $N$.

- If $R$ is a left perfect ring and $Q$ a pure submodule of a projective left $R$-module $P$, then $Q$ is a direct summand of $P$.

**Proof.** The first statement will arise as an immediate consequence of Theorem 6 in Section 3. To establish the second we will show that, given a pure exact sequence

$$0 \to Q \to P \to P/Q \to 0$$

of left modules over a ring $R$, such that $P$ is flat, the quotient $P/Q$ is flat as well. This will yield our claim since a perfect base ring will make flat modules projective.

To check flatness of $P/Q$, let $0 \to A \to B \to C \to 0$ be any short exact sequence of right $R$-modules, and consider the following diagram which has exact rows and columns due to our setup.

Applying the Snake Lemma and using the fact that the map $\psi$ is injective by hypothesis, we deduce that $\text{Ker}(\phi) = 0$ as required. □

The theory of purity and pure injectivity could also be baptized the theory of systems of linear equations for modules. A typical such system has the following format: Starting with a left $R$-module $M$, an element $(m_i)_{i \in I} \in M^I$, and a row-finite matrix $(r_{ij})_{i \in I, j \in J}$ of elements of $R$, one considers the system

$$(\dagger) \quad \sum_{j \in J} r_{ij} X_j = m_i \quad (i \in I),$$
and calls each element in $M^J$ which satisfies all equations of (†) a solution in $M$. A first indication of a connection between such systems and the theory of purity and pure injectivity was exhibited by Fieldhouse (see [22]), who observed that purity of a submodule $A$ of an $R$-module $B$ is equivalent to the following equational condition: Every finite system of linear equations with right-hand sides in $A$, which is solvable in $B$, has a solution in $A$. This result makes one anticipate a bridge between linear systems and pure injectivity as well. Such a bridge does in fact exist, but relies on systems that need not be finite. Here it is:

**Theorem 1.** (Warfield, 1969 [72]) For $M \in R\text{-Mod}$, the following statements are equivalent:

1. $M$ is pure injective.
2. Any system of the form (†) which is finitely solvable in $M$ (i.e., has the property that, for any finite subset $I' \subseteq I$, the finite subsystem of equations labeled by $i \in I'$ is solvable in $M$) has a global solution in $M$.
3. $M$ is a direct summand of a compact Hausdorff $R$-module $N$ (the latter is to mean that $N$ is a compact Hausdorff abelian group such that all multiplications by elements of $R$ are continuous).

In this instance, Warfield acted primarily as a coordinator of results and ideas from various parts of the literature, most of the implications having been previously known in a variety of specialized contexts, some in full generality. It is quite enlightening to follow the line of successive modifications of the pertinent concepts. From work of Kaplansky [45], Loś [53], and Balcerzyk [9] done in the fifties, through papers of Mycielski [55], Butler-Horrocks [11], Kiepiński [46], Weglorz [75], Fuchs [23], and Stenström [71] scattered over the sixties, it incrementally leads to the present theory.

Warfield’s coordination was very much called for, as the obvious interest of the topological condition (3) of Theorem 1 had triggered various exploratory trips in its own right. In 1954, Kaplansky published a characterization of those abelian groups which arise as algebraic direct summands of compact Hausdorff groups, referring to them as ‘algebraically compact’ [45]. Originally, he had set out to describe the compact abelian groups, but realized that, from an algebraic viewpoint, the former class allowed for a cleaner description and appeared more natural. In particular, he provided a complete classification of the algebraically compact abelian groups: They are precisely the direct sums of divisible groups and groups which are Hausdorff and complete in their $\mathbb{Z}$-adic topologies. The latter can be written uniquely as direct products of factors $A_p$ which are complete and Hausdorff in their $p$-adic topologies, respectively, with $p$ tracing the primes. Each $A_p$, in turn, can be canonically viewed as a module over the ring of $p$-adic integers and pinned down up to isomorphism in terms of its basic submodule, a direct sum of cyclic $p$-groups and copies of the $p$-adic integers, thus leading to a convenient full set of invariants. Subsequently, equivalent descriptions of the class of groups exhibited by Kaplansky were given by Mycielski, Loś, Weglorz, and Fuchs (in the setting of noetherian rings). Most notable were the descriptions in terms of ‘equational compactness’ conditions. The first general concept of algebraic compactness was introduced by Mycielski in 1964, for arbitrary algebraic systems.
in fact. Restricted to modules, it just amounts to condition (2) of Theorem 1. For the sake of emphasis, we repeat:

**Definition.** Given an associative ring $R$ with identity, a left $R$-module $M$ is called *algebraically compact* if any system

$$\sum_{j \in J} r_{ij} X_j = m_i \quad (i \in I),$$

based on a row-finite matrix $(r_{ij})$ with entries in $R$ and $m_i \in M$, which is finitely solvable in $M$, has a global solution in $M$.

Moreover, $M$ is said to be \(\Sigma\)-*algebraically compact* in case all direct sums of copies of $M$ are algebraically compact.

So, in particular, Theorem 1 tells us that the pure injective and the algebraically compact modules coincide. We will give the latter terminology preference in the sequel.

**Remark.** It is of course natural to also wonder about characterizations of pure projective modules, i.e., of those modules which are projective relative to pure exact sequences. Such characterizations are much more readily obtained than useful descriptions of their pure injective counterparts. According to Warfield [72] and Fieldhouse [22], the pure projective modules are precisely the direct summands of direct sums of finitely presented modules.

With little effort, one obtains the following first list of examples of algebraically compact (alias pure injective) modules that goes beyond the most obvious ones, the injectives; a second installment of examples can be found in part B of Section 3.

1. [72] Any module $M$ can be purely embedded into a pure injective module, namely its ‘Bohr compactification’, as follows:

$$M \to \text{Hom}_\mathbb{Z}(\text{Hom}_\mathbb{Z}(M, \mathbb{R}/\mathbb{Z}), \mathbb{R}/\mathbb{Z}),$$

the assignment being evaluation. If one equips \(\text{Hom}_\mathbb{Z}(M, \mathbb{R}/\mathbb{Z})\) with the discrete topology and the ‘double dual’ of $M$ with the compact-open topology, one thus arrives at a compact Hausdorff $R$-module. As a consequence, one can construct pure injective resolutions of a module $M$ and measure the deviation of $M$ from pure injectivity by means of its ‘pure injective dimension’.

2. [72] If $R$ is a commutative local noetherian domain with maximal ideal $\mathfrak{m}$, complete in its $\mathfrak{m}$-adic topology, then $R$ is algebraically compact (as an $R$-module). In particular, this is true for the ring of $p$-adic integers. Note, however, that the ring of $p$-adic integers fails to be \(\Sigma\)-algebraically compact (both as an abelian group and as a module over itself).

3. [24] If $R$ is a countable ring, $A$ any left $R$-module, and $\mathcal{F}$ a non-principal ultrafilter on $\mathbb{N}$, then the ultrapower $A^{\mathbb{N}}/\mathcal{F}$ is algebraically compact.

4. Every artinian module over a commutative ring is \(\Sigma\)-algebraically compact (see the examples following Theorem 6 below).

5. If $R$ is a commutative artinian principal ideal ring, then all $R$-modules are direct sums of cyclic submodules. The latter being finite in number, up to isomorphism, Example 4 shows all objects of $R$-$\text{Mod}$ to be algebraically compact in that case.
We will sketch a proof for the fifth remark (apparently folklore), since it is part of the red thread that will lead us through this survey. By the Chinese Remainder Theorem, $R$ is a finite direct product of local rings, and hence it is harmless to assume that $R$ is a local artinian principal ideal ring with maximal ideal $m$ say. Now all ideals of $R$ are powers of $m$, and each homomorphism from a power $m^n$ to $R$ sends $m^n$ back to $m^n$ and can therefore be extended to $R$ — just use the fact that $m$ is principal. This shows that $R$ is self-injective; in fact, $R$ being an artinian ring, $R$ is $\Sigma$-injective as an $R$-module. Now let $M$ be any nonzero module. We provide a decomposition of the desired ilk by induction on the least natural number $N$ such that $m^{N+1}M = 0$. Clearly $M$ is a module over the ring $R/m^{N+1}$ which, as we just saw, is $\Sigma$-injective over itself. Let $(x_i)_{i \in I}$ be a maximal family of elements of $M$ such that the sum of the $Rx_i$ is direct, with each of the $Rx_i$ isomorphic to $R$. Due to injectivity, the sum $\sum_{i \in I} Rx_i$ is then a direct summand of $M$, say $M = \sum_{i \in I} Rx_i \oplus M'$, and due to the maximal choice of our family, $M'$ does not contain a copy of $R/m^{N+1}$, i.e., $M'$ is annihilated by $m^N$. Our claim follows by induction.

Point 5 shows, in particular, that all artinian principal ideal rings have finite representation type in the sense to follow. In fact, it has long been known that, among the commutative artinian rings, the principal ideal rings are precisely the ones having this property.

**Definition.** A ring $R$ is said to have finite representation type if it is left artinian and if, up to isomorphism, there are only finitely many indecomposable finitely generated left $R$-modules.

Finite representation type is actually left-right symmetric, as was shown by Eisenbud and Griffith in [18].

### 2. A Global Decomposition Problem

For better focus, we interject two problems which, on the face of it, are only loosely connected with our main theme. The connection turns out to be much closer than anticipated at first sight. In fact, these problems have motivated a major portion of the subsequent work on algebraic compactness.

**Global Problems.** (Koethe [48], Cohen-Kaplansky [13]) For which rings $R$ is every right $R$-module

(a) a direct sum of finitely generated modules?

(b) a direct sum of indecomposable modules?

Work on the commutative case was initiated by Koethe in 1935, continued by Cohen-Kaplansky in 1951, and completed by Griffith in 1970 for part (a), and by Warfield in 1972 for part (b).

**Theorem 2.** ([48, 13, 28, 74]) For any commutative ring $R$, conditions (a) and (b) above are equivalent and satisfied if and only if $R$ is an artinian principal ideal ring.

As we noted at the end of Section 1, among the commutative rings, the artinian principal ideal rings are precisely the ones having finite representation type. Moreover, we observed
that these rings enjoy the property that all their modules are algebraically compact. As we will see in Theorem 13 of Section 4, this property in turn characterizes the artinian principal ideal rings, which rounds off the ‘commutative solution’ to our problems. The reasons for the most interesting implications, namely that either of the two conditions (a), (b) forces \( R \) to be an artinian principal ideal ring, can be roughly summarized as follows:

In general, large direct products of modules exhibit a high resistance to infinite direct sum decompositions; more precisely, in most cases, infinite direct sum decompositions of direct products can be traced back to infinite decompositions of finite sub-products.

In the noncommutative situation, the problems become far more challenging. In his 1972 paper, Warfield stated: “For non-commutative rings, the questions raised in this paper seem to be much more difficult. All that seems to be known is that any ring satisfying [the above conditions (a), (b)] is necessarily [left] artinian.” At present, there is still at least one link missing to a truly satisfactory resolution of these problems. The main key to what is known is an equivalent characterization of \( \Sigma\)-algebraic compactness, which will be presented in the next section.

3. Characterizations of \((\Sigma\)-)algebraically compact modules

We begin with a subsection dedicated to the main technical resource of the subject, introduced independently by Gruson-Jensen [30] and W. Zimmermann [80]. It is the more general functorial framework of [80] and [37] which we will describe below, since the extra generality adds to the transparency of the arguments.

A. Product-compatible functors and matrix functors.

Definition. (1) A \( p\)-functor on \( R\)-Mod is a subfunctor \( P \) of the forgetful functor \( R\)-Mod \( \rightarrow \) \( \text{Ab} \) which commutes with direct products, i.e., \( P \) assigns to each left \( R\)-module \( M \) a subgroup \( PM \) such that \( f(PM) \subseteq PN \) for any homomorphism \( f : M \rightarrow N \), and \( P(\prod_{i \in I} M_i) = \prod_{i \in I}(PM_i) \) for any direct product \( \prod_{i \in I} M_i \) in \( R\)-Mod.

Note that any \( p\)-functor automatically commutes with direct sums (this being actually true for any subfunctor of the forgetful functor).

(2) A pointed matrix over \( R \) is a row-finite matrix \( A = (a_{ij})_{i \in I, j \in J} \) of elements from \( R \), paired with a column index \( \alpha \in J \). Given a pointed matrix \( (A, \alpha) \), we call the following \( p\)-functor \( [A, \alpha] \) on \( R\)-Mod a matrix-functor: For any \( R\)-module \( M \), the subgroup \( [A, \alpha]M \) is defined to be the \( \alpha \)-th projection of the solution set in \( M \) of the homogeneous system

\[
\sum_{j \in J} a_{ij}X_j = 0 \quad \text{for all} \quad i \in I;
\]

in other words,

\[
[A, \alpha]M = \{m \in M \mid \exists \text{ a solution } (m_j) \in M^J \text{ of the above system with } m_\alpha = m\}.
\]

Further, we call \( [A, \alpha] \) a finite matrix functor in case the matrix \( A \) is finite.

Given a (finite) matrix functor \( [A, \alpha] \) on \( R\)-Mod and a left \( R\)-module \( M \), we will call the subgroup \( [A, \alpha]M \) a (finite) matrix subgroup of \( M \). The finite matrix subgroups were labeled “sousgroupes de définition fini” by Gruson and Jensen, and “\( pp\)-definable subgroups” by the model theorists, Prest, Herzog, Rothmaler, and others.
Observation 3. Basic properties of matrix functors. The first two give alternate descriptions of matrix functors.

1. \([A, \alpha] = \text{Hom}_R(Z, -)(z)\) for a suitable left \(R\)-module \(Z\) and \(z \in Z\); conversely, every functor of the form \(\text{Hom}_R(Z, -)(z)\) is a matrix functor for a suitable matrix \(A\). The finite matrix functors are precisely the functors \(\text{Hom}_R(Z, -)(z)\) with finitely presented \(Z\).

2. The finite matrix subgroups of \(M\) moreover coincide with the kernels of the \(Z\)-linear maps \(M \to Z \otimes_R M, m \mapsto z \otimes m\), where \(Z\) is a finitely presented \(R\)-module and \(z \in Z\).

3. The class of matrix functors is closed under arbitrary intersections and finite sums. The finite matrix functors are closed under finite intersections and finite sums.

4. If \(M\) is an \(R\)-\(S\)-bimodule, then every matrix subgroup of \(R\)\(M\) is an \(S\)-submodule of \(M\).

In the context of part (4), the most important candidates for \(S\) will be the opposite of the endomorphism ring of \(M\), as well as subrings of the center of \(R\). How easily matrix functors can be manipulated is evidenced by the easy proofs of the above observations; we include one sample argument to make our point.

Proof of part of (3). We will show that finite sums of matrix functors are again of that ilk. So let \((A = (a_{ij})_{i \in I, j \in J}, \alpha)\) and \((B = (b_{kl})_{k \in K, l \in L}, \beta)\) be two pointed matrices with entries in \(R\). It is clearly harmless to assume that the occurring index sets are all disjoint. Let \((C = (c_{uv})_{u \in U, v \in V}, \gamma)\) be defined as follows: Assuming that \(\gamma\) belongs to none of the sets \(I, J, K, L\), we set \(U = \{\gamma\} \cup I \cup K\) and \(V = \{\gamma\} \cup J \cup L\), and define \(c_{uv}\) via \(c_{\gamma\gamma} = 1\), \(c_{\gamma\alpha} = c_{\gamma\beta} = -1\), \(c_{ij} = a_{ij}\) whenever \((i, j) \in I \times J\), \(c_{kl} = b_{kl}\) whenever \((k, l) \in K \times L\), and \(c_{uv} = 0\) in all other cases. It is straightforward to check that, for any left \(R\)-module \(M\), we have \([A, \alpha]M + [B, \beta]M = [C, \gamma]M\). Moreover, we remark that \(C\) is finite if \(A\) and \(B\) are. \(\square\)

Important instances of matrix functors.

- Whenever \(a\) is a finitely generated right ideal of \(R\), the assignment \(M \mapsto aM\) defines a finite matrix functor on \(R\)-Mod.
- For every subset \(T\) of \(R\), the assignment \(M \mapsto \text{Ann}_M(T)\) is a matrix functor; it is finite in case \(T\) is.

More generally: Whenever \(a\) is a finitely generated right ideal and \(T\) a subset of \(R\), the conductor \((aM : T)\) is a matrix subgroup of \(M\).

- All finitely generated \(\text{End}_R(M)\)-submodules of \(M\) are matrix subgroups.

Pinning down matrices for the first three types of examples is straightforward. To verify the last: By Observation 3(1), each cyclic \(\text{End}_R(M)\)-submodule of \(M\) is a matrix subgroup; now use Observation 3(3) to move to finite sums.

As a first application of matrix functors to our present objects of interest, we will give a characterization of purity in terms of finite matrix subgroups. Ironically, this description provides a perfect formal parallel to that of purity over PID’s, while all of the initial attempts to generalize the concept of purity, which were based on formal analogy (with requirements such as \(aN \cap M = aM\) for all cyclic or finitely generated right ideals of
the base ring), were eventually discarded as not quite strong enough to best serve their purpose.

**Proposition 4.** A submodule \( M \) of a left \( R \)-module \( N \) is pure if and only if \([A, \alpha]N \cap M = [A, \alpha]M\) for all finite matrix functors \([A, \alpha]\) on \( R\text{-Mod}\).

**Proof.** The proof for pure left exactness of finite matrix functors is straightforward (cf. [79]).

Now assume that \( M \subseteq N \) satisfies the above intersection property. To verify purity of the inclusion, let

\[ \sum_{j=1}^{s} a_{ij}X_j = m_i \quad (1 \leq i \leq r) \quad (\dagger) \]

be a finite linear system with \( m_i \in M \), which is solvable in \( N \). We prove its solvability in \( M \) by induction on \( r \). For \( r = 1 \), let \( A = (-1, a_{11}, \ldots, a_{1s}) \) be a single row with the first column labeled \( \alpha \); then \( m_1 \in [A, \alpha]N \cap M = [A, \alpha]M \) by construction and hypothesis, which provides a solution of (\dagger) in \( M \). Next suppose that \( r \geq 2 \), pick a solution \( (v_j) \) of (\dagger) in \( N \), and use the induction hypothesis to procure a solution \( (u_j) \) in \( M \) of the first \( r - 1 \) equations. Define \( m'_r := \sum_{j=1}^{s} a_{rj}(v_j - u_j) = m_r - \sum_{j=1}^{s} a_{rj}u_j \in M \), and observe that \( \sum_{j=1}^{s} a_{ij}(v_j - u_j) = 0 \) for \( i = 1, \ldots, r - 1 \). We infer that \( m'_r \) belongs to \([B, \beta]N \cap M\), where \( B \) is the matrix

\[
\begin{pmatrix}
0 & a_{11} & a_{12} & \cdots & a_{1s} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & a_{r-1,1} & a_{r-1,2} & \cdots & a_{r-1,s} \\
-1 & a_{r1} & a_{r2} & \cdots & a_{rs}
\end{pmatrix},
\]

and \( \beta \) labels the first column of \( B \). By hypothesis, we thus have \( m'_r \in [B, \beta]M \), which provides us with a family \((u'_j)\) in \( M \) satisfying \( \sum_{j=1}^{s} a_{ij}u'_j = 0 \) for \( 1 \leq i \leq r - 1 \) and \( \sum_{j=1}^{s} a_{rj}u'_j = m'_r \). This yields a solution \((u_j + u'_j)\) of (\dagger) in \( M \) as required. \( \square \)

Moreover, numerous properties of a ring \( R \) linked to the behavior of direct products in \( R\text{-Mod}\) can be conveniently recast in terms of matrix functors. We give two examples taken from [79, 80]:

- A left \( R \)-module \( M \) is flat if and only if \([A, \alpha]M = [A, \alpha]R \cdot M\) for arbitrary finite matrix functors \([A, \alpha]\). One deduces that all direct sums of copies of a flat module \( M \) are flat if and only if, for each finite matrix functor \([A, \alpha]\), there exists a finitely generated right ideal \( a \subseteq [A, \alpha]R \) with \([A, \alpha]M = aM\).

- Thus: flatness is inherited by arbitrary direct products of flat left \( R \)-modules (i.e., \( R \) is right coherent) if and only if all finite matrix subgroups of \( R \cdot R \) are finitely generated right ideals.

**B. \( p \)-functors and algebraic compactness.**

The following equivalent descriptions of algebraically compact and \( \Sigma \)-algebraically compact modules will prove extremely useful in the sequel. In each of the two theorems, the
equivalence of (1) and (3) was independently established by Gruson-Jensen and Zimmermann ([31, 80]); condition (2) – a significant strengthening of the necessary side, in light of the third of the above instances of matrix subgroups – was added by the latter.

**Theorem 5.** The following statements are equivalent for any left $R$-module $M$:

1. $M$ is algebraically compact.
2. Every family of residue classes $(m_l + P_l M)_{l \in L}$ with $m_l \in M$ and $p$-functors $P_l$, which has the finite intersection property, has non-empty intersection.
3. Same as (2) with $P_l$ replaced by finite matrix functors.

In the special situation of a Prüfer domain $R$ (i.e., a commutative semihereditary integral domain), a specialized version of this result had already been obtained by Warfield [72]: In that case, the matrix subgroups of the form $(a M : T)$, where $a$ is a finitely generated ideal and $T$ a finite subset of $R$, are representative.

Since Zimmermann’s elementary proof of this theorem [80, Satz 2.1] does not exist in English translation, we include a sketch below. We begin with a remark which will come in handy in other contexts as well. Namely, locally – i.e., on the closure of a given module $M$ under direct sums and products in $R$-$Mod$ – any p-functor acts like a matrix functor. Indeed, given a p-functor $P$ on $R$-$Mod$, we write $PM = \{m_i \mid i \in I\}$, let $M_i$ be a copy of $M$ for each $i$, and set $\overline{m} = (m_i)_{i \in I} \in \prod_{i \in I} M_i$. Then $PM = \text{Hom}_R(\prod_{i \in I} M_i, M)(\overline{m})$, and Observation 3(1) yields our claim.

**Proof of Theorem 5.** ‘(1) $\implies$ (2)’: Assuming (1), we start with a family of residue classes $(m_l + P_l M)_{l \in L}$ having the finite intersection property. By the preceding remark, we may assume that the $P_l$ are matrix functors, say $P_l = [A_l, \alpha_l]$ with $A_l = (a_{ij})_{i \in I_l, j \in J_l}$. We construct a system of equations which is finitely solvable in $M$, and a global solution to which will provide us with an element in the intersection of our family. It is clearly harmless to assume that all of the index sets $I_l$ and $J_l$ are pairwise disjoint. Choose an element $\alpha$ contained in none of these. We set $I = \bigcup_{l \in L} I_l$, $J = \{\alpha\} \cup \bigcup_{l \in L} J_l \setminus \{\alpha_l\}$, and define a row-finite matrix $A = (a_{ij})_{i \in I, j \in J}$ as follows: $a_{i\alpha} = a_{i\alpha_l}$ if $i \in I_l$; $a_{ij} = a_{ij}^l$ if $i \in I_l$ and $j \in J_l \setminus \{\alpha_l\}$; finally we set $a_{ij} = 0$ in all other cases. It is immediate that $[A, \alpha]M = \bigcap_{l \in L} [A_l, \alpha_l] M$. The matrix $A$ will serve as coefficient matrix of our system. As for its right-hand side, we define $m \in M^I$ by stringing up the elements $m_i = a_{i\alpha} m_l$ for $i \in I_l$ in the only plausible fashion. It is then straightforward to check that the system $\sum_{j \in J} a_{ij} X_j = m_i$ for $i \in I$ is finitely solvable by construction. Hence it has a global solution, say $(z_i)_{i \in I}$, and one readily verifies that $z_\alpha$ belongs to the intersection of the family of residue classes with which we started out.

‘(3) $\implies$ (1)’: This time, we begin with a system

\[(\dag) \quad \sum_{j \in J} a_{ij} X_j = m_i \quad \text{for} \quad i \in I,
\]

which is finitely solvable in $M$. The crucial step of our argument is the following easy consequence of (3) and finite solvability of (\dag): Namely, for each index $\alpha \in J$, there exists
an element \( y_\alpha \in M \) such the the system

\[
\sum_{j \in J \setminus \{\alpha\}} a_{ij}X_j = m_i - a_{i\alpha}y_\alpha
\]

is again finitely solvable, as follows. Given any finite subset \( I' \subseteq I \), let \( y_\alpha(I') \) be the \( \alpha \)-th component of a solution of the finite system \( \sum_{j \in J} a_{ij}X_j = m_i \) for \( i \in I' \), and let \( A(I') \) be the matrix consisting of the rows of \( A \) labelled by \( I' \). Then the family \( (y_\alpha(I') + [A(I'), \alpha]M) \), where \( I' \) runs through the finite subsets of \( I \), has the finite intersection property. Therefore its intersection is nonempty by (3), and any element \( y_\alpha \) in this intersection satisfies our requirement.

Next we consider the set \( K \) of all pairs \( (K, y) \), where \( K \) is a subset of \( J \) and \( y = (y_k) \in M^K \) is such that the system

\[
\sum_{j \in J \setminus K} a_{ij}X_j = m_i - \sum_{k \in K} a_{ik}y_k \quad \text{for} \quad i \in I
\]

is in turn finitely solvable in \( M \). We equip this set of pairs with the standard order, namely \( (K, y) \leq (K', y') \) if \( K \subseteq K' \) and \( y_k = y'_k \) whenever \( k \in K \). Clearly, \( K \neq \emptyset \). One checks that the set \( K \) is inductively ordered, and denotes by \( (K_0, z) \) a maximal element of \( K \). Our initial statement, applied to the finitely solvable system \( \sum_{j \in J \setminus K_0} a_{ij}X_j = m_i - \sum_{k \in K_0} a_{ik}z_k \), now yields \( J = K_0 \), which makes \( z \) a global solution of (1). \( \square \)

**Theorem 6.** The following statements are equivalent for any left \( R \)-module \( M \):

1. \( M \) is \( \Sigma \)-algebraically compact.
2. Every countable descending chain \( P_1 \supseteq P_2 \supseteq P_3 \supseteq \cdots \) of \( p \)-functors becomes stationary on \( M \).
3. Same as (2) with \( P_1 \) replaced by finite matrix functors.

We point out that conditions (2) and (3) could just as well have been phrased as follows: ‘\( M \) has the descending chain condition for \( p \)-functorial subgroups’ (or, equivalently, ‘\( M \) has the descending chain condition for finite matrix subgroups’), since the classes of \( p \)-functors and finite matrix functors are both closed under finite intersections. Indeed, given a descending chain \( P_1M \supseteq P_2M \supseteq \cdots \) of \( p \)-functorial subgroups of \( M \), we obtain \( P_iM = Q_iM \) for the descending chain \( Q_i = P_1 \cap \cdots \cap P_i \) of \( p \)-functors.

**Proof of Theorem 6.** In view of Theorem 5, the implication ‘(3) \( \implies \) (1)’ is just an analogue of the much older result that artinian modules are linearly compact [77]; it is left as an exercise. In the arguments given by Zimmermann and Gruson-Jensen, the novel implication ‘(1) \( \implies \) (2)’ is obtained via a detour through direct products, which we sketch because it pinpoints the importance of product-compatibility of the functors we are considering: Clearly, \( \Sigma \)-algebraic compactness of \( M \) forces the natural (pure) embedding of the direct sum \( M^{(\mathbb{N})} \) in the direct product \( M^{\mathbb{N}} \) to split, say \( \prod_{n \in \mathbb{N}} M_n = C \oplus \bigoplus_{n \in \mathbb{N}} M_n \), where each \( M_n \) is a copy of \( M \). Let \( \pi : \prod_{n \in \mathbb{N}} M_n \to C \) be the corresponding projection, and \( \pi_n : \prod_{i \in \mathbb{N}} M_i \to M_n \) the canonical maps. Moreover, assume we have a descending
chain $P_1 \supseteq P_2 \supseteq P_3 \supseteq \ldots$ of $p$-functors with $m_n \in P_n M_n \setminus P_{n+1} M_n$ for $n \in \mathbb{N}$. Set $x = (m_n)$, $y_n = (m_1, \ldots, m_n, 0, \ldots)$, $x_n = x - y_n$, and decompose $x$ and $x_n$ in the form $x = s + c$ and $x_n = s_n + c_n$ with $s, s_n \in \bigoplus_{n \in \mathbb{N}} M_n$ and $c, c_n \in C$; then, clearly $s = y_n + s_n$. Since $x_n$ belongs to $\prod_{n \in \mathbb{N}} P_{n+1} M_n = P_{n+1}(\prod_{n \in \mathbb{N}} M_n)$ by construction, $s_n$ belongs to $P_{n+1}(\bigoplus_{n \in \mathbb{N}} M_n)$. So if we choose an index $N \in \mathbb{N}$ with $\pi_N(s) = 0$, we infer that $m_N = \pi_N(y_N) = \pi_N(s - s_N) = -\pi_N(s_N)$ lies in $P_{N+1} M_N$, which contradicts our choice of $m_N$. □

In Section 4, these concepts and results will turn out tailored to measure for tackling the problems advertized in Section 2. However, the two preceding theorems lend themselves to a large number of further applications. We present a small selection, starting with a few additional examples; most of them are obvious in light of the above theory, the others we tag with references.

**Further examples of (Σ-)algebraically compact modules.**

- Whenever $M$ is an $R$-$S$-bimodule which is artinian over $S$, $M$ is Σ-algebraically compact as an $R$-module. This is, for instance, true if $M$ is an $R$-module which is artinian over its endomorphism ring or over the center of $R$. In particular, for any Artin algebra $R$, the category $R$-mod consists entirely of Σ-algebraically compact modules.

- Suppose that $M$ is a (Σ-)algebraically compact left $R$-module, $P$ a $p$-functor on $R$-Mod and $S$ a subring of $R$ such that $PM$ is an $S$-submodule of $M$. Then both $PM$ and $M/PM$ are (Σ-)algebraically compact over $S$. In particular: If $a$ is an ideal of $R$ which is finitely generated on the right, then $aM$ and $M/aM$ are (Σ-)algebraically compact (see [37]).

- If $M$ is a module over a Dedekind domain $R$, then $M$ is Σ-algebraically compact if and only if $M = M_1 \oplus M_2$ where $M_1$ is divisible and $M_2$ has nonzero $R$-annihilator.

- If $M$ is finitely generated over a commutative noetherian ring, then $M$ is (Σ)-algebraically compact precisely when $M$ is artinian. (For the nontrivial implication, see [80].)

- Suppose $S$ is a left Σ-algebraically compact ring (e.g. a field), $(X_i)_{i \in I}$ a family of independent indeterminates over $S$, and $k$ any positive integer. Then the truncated polynomial ring

$$ R = S[X_i \mid i \in I]/(X_i \mid i \in I)^k $$

is in turn Σ-algebraically compact as a left module over itself (see [37]).

- If $S$ is left algebraically compact, then so is every power series ring $R = S[[X_i \mid i \in I]]$ ([loc. cit.]). Note, however, that $R$ fails to be Σ-algebraically compact in case $I \neq \emptyset$.

- If $S$ is any ring and $G$ a group, then the group ring $SG$ is left algebraically compact if and only if $S$ has this property and $G$ is finite (see [81]).

- Suppose $M$ is an $R$-$S$-bimodule. If $A$ is an algebraically compact left $R$-module, then the left $S$-module $\text{Hom}_R(M, A)$ is algebraically compact as well – see below. (This ‘compactification by passage to a suitable dual’ is akin to the Pontrjagin dual.) In particular, the endomorphism ring $S$ of any algebraically compact left module is algebraically compact on the right, the change of side being due to the convention that $S$ acts on the left of $M$.

Since the last of the above remarks, concerning the passage of algebraic compactness from $R A$ to $S \text{Hom}_R(M, A)$, will be relevant in the sequel, we include a justification based
on the adjointness of Hom and tensor product: Start with a pure monomorphism $U \to V$ of left $S$-modules, and observe that the pure injectivity of $A$ forces the upper row (and hence also the lower one) of the following commutative diagram to be an epimorphism:

\[
\begin{array}{ccc}
\text{Hom}_R(M \otimes_S V, A) & \longrightarrow & \text{Hom}_R(M \otimes_S U, A) \\
\downarrow \cong & & \downarrow \cong \\
\text{Hom}_S(V, \text{Hom}_R(M, A)) & \longrightarrow & \text{Hom}_S(U, \text{Hom}_R(M, A))
\end{array}
\]

The following equivalence is due to Faith [21].

**Corollary 7.** For any injective left $R$-module $M$, the following statements are equivalent:

1. $M$ is $\Sigma$-injective (meaning that every direct sum of copies of $M$ is injective).
2. $R$ has the ascending chain condition for $M$-annihilators, i.e., for left ideals of the form $\text{Ann}_R(U)$, where $U$ is a subset of $M$.

**Proof.** Note that condition (1) is tantamount to $\Sigma$-algebraic compactness of $M$ under our hypotheses, since $M^I$ is pure in $M^I$ for any set $I$. The matrix subgroups of an injective module $M$ are precisely the annihilators in $M$ of subsets of $R$ (prove $[A, \alpha]M = \text{Ann}_M \text{Ann}_R[A, \alpha]M$ or consult [80, Beispiel 1.1]). Thus Theorem 6 shows (1) to be equivalent to the descending chain condition for $R$-annihilators in $M$. But the latter descending chain condition has an equivalent flip side, namely the ascending chain condition for $M$-annihilators in $R$. $\square$

**Corollary 8.** Any pure submodule of a $\Sigma$-algebraically compact module is a direct summand.

**Proof.** Let $M$ be $\Sigma$-algebraically compact and $U \subseteq M$ a pure submodule. By Proposition 4, $[A, \alpha]M \cap U = [A, \alpha]U$ for every finite matrix functor $[A, \alpha]$, and hence $U$ inherits the descending chain condition for finite matrix subgroups from $M$. Therefore, $U$ is in turn $\Sigma$-algebraically compact and consequently a direct summand of $M$. $\square$

The next result is part of the bridge taking us back to our global decomposition problems. We will obtain the first part as another consequence of Theorem 6, while the second can be most elegantly derived from the following category equivalence due to Gruson-Jensen [32] which provides a very general tool for extending properties of injective modules to algebraically compact ones: Let fin.pres-$R$ be the full subcategory of mod-$R$ consisting of the finitely presented right $R$-modules, and let (fin.pres-$R$, Ab) be the category of all additive covariant functors from fin.pres-$R$ to the abelian goups. This functor category is a Grothendieck category, and the functor

$$R\text{-Mod} \to (\text{fin.pres-}R, \text{Ab})$$

defined by $M \mapsto - \otimes_R M$ induces a category equivalence from the full subcategory of algebraically compact objects of $R\text{-Mod}$ to the full subcategory of injective objects of (fin.pres-$R$, Ab).
Proposition 9. (Further assets of (Σ-)algebraically compact modules)

1. Every Σ-algebraically compact module is a direct sum of indecomposable summands with local endomorphism rings.

2. If M is an algebraically compact left module with endomorphism ring S, then S/\text{rad}(S) is von Neumann regular and right self-injective; moreover, idempotents can be lifted modulo rad(S).

In particular, S is local in case M is indecomposable.

3. Each strongly invariant submodule M of an algebraically compact module has the exchange property, i.e., can be shifted inside direct sum grids as follows: Given any equality of left R-modules M ⊕ X = ⨁_{i ∈ I} Y_i, there exist submodules Y'_i ⊆ Y_i such that ⨁_{i ∈ I} Y'_i = M ⊕ ⨁_{i ∈ I} Y_i.

(Here a submodule M of a module N is said to be strongly invariant in case each homomorphism φ from M to N satisfies φ(M) ⊆ M.)

Proof. Part (1) [37]: Suppose that M is Σ-algebraically compact, and let \( (U_i)_{i ∈ I} \) be a maximal family of independent indecomposable submodules of M such that U = ⨁_{i ∈ I} U_i is pure in M. By Corollary 8, U is a direct summand of M, say M = U ⊕ V. If V were nonzero, we could pick a nonzero element v ∈ V and choose a pure submodule W ⊆ V which is maximal with the property of excluding v. Again we would obtain splitting, V = W ⊕ Z, which would provide us with an indecomposable summand Z of V. But the existence of such a Z is incompatible with the maximal choice of the family \( (U_i) \). That the \( U_i \) even have local endomorphisms rings, will follow from part (2).

For part (2), we use the above category equivalence and the well-known fact that endomorphism rings of injectives have the listed properties; indeed, the argument given by Osofsky in [57] for modules carries over to Grothendieck categories. (Note however that the direct argument given in [37] is a bit slicker than the original proof for injective modules, one of the reasons being that algebraic compactness is passed on to endomorphism rings by the last of the examples following Theorem 6.)

As for part (3), we will only sketch a proof for the finite exchange property and refer the reader to [38] for a complete argument. Let M be a strongly invariant submodule of an algebraically compact left R-module N, and let S be the endomorphism ring of M. From the last of the remarks following Theorem 6, we know that S = \text{Hom}_R(M, N) is an algebraically compact right S-module. Therefore S is an exchange ring in the sense of Warfield by part (2), which easily implies the finite exchange property of M (see [73]).

To conclude the section, we point to two examples to mark what we consider potentially dangerous curves: Namely, while any right artinian ring is necessarily Σ-algebraically compact on the left, on the right it need not even be algebraically compact. An example demonstrating this was given by Zimmermann in [81]. Moreover, while the descending chain condition for finite matrix subgroups of a module M entails the descending chain condition for arbitrary matrix subgroups, the analogous implication fails for the ascending chain condition as the following example shows: One of the consequences we derived from Theorem 6 says that the algebra \( R = K[X_i \mid i ∈ N]/(X_i \mid i ∈ N)^2 \) over a field K is Σ-algebraically compact. Hence, if E denotes the minimal injective cogenerator for R, then the
$R$-module $E = \text{Hom}_R(R, E)$ has the ascending chain condition for finite matrix subgroups (use Tool 17 below to see this). Note that the ascending chain condition for arbitrary matrix subgroups would amount to $E$ being noetherian over its endomorphism ring $S$, since all finitely generated $S$-submodules of $E$ are in fact matrix subgroups. But the $S$-module $E$ fails to be noetherian: Indeed, if it were, then the equality $A = \text{Ann}_R \text{Ann}_E(A)$ for any ideal $A$ of $R$ would make $R$ artinian, an obvious absurdity. On the other hand:

**Remark.** Suppose that $M$ is a direct sum of finitely presented objects in $R$-mod. Then the $R$-module $M$ satisfies the ascending chain condition for finite matrix subgroups if and only if it is noetherian over its endomorphism ring.

To see this, let $M = \bigoplus_{i \in I} M_i$ with finitely presented summands $M_i$, and let $S$ be the endomorphism ring of $M$. For the nontrivial implication, suppose that $M$ has the ascending chain condition for finite matrix subgroups. Clearly, $M$ is noetherian over $S$ in case all finitely generated $S$-submodules are finite matrix subgroups. By Observation 3(3), it thus suffices to prove that each cyclic $S$-submodule $Sm$ of $M$ is a finite matrix subgroup. Choose a finite subset $I' \subseteq I$ such that $m \in \bigoplus_{i \in I'} M_i$. Then $Sm = \text{Hom}_R(\bigoplus_{i \in I'} M_i, M)(m)$ is indeed a finite matrix subgroup of $M$ by Observation 3(1).

To learn about the module-theoretic impact of the maximum condition for finite matrix subgroups, consult [82], for generalizations of classical results to modules satisfying this condition, see [83].

4. Return to our global decomposition problems

As we mentioned in Section 2, the arguments settling the commutative case rest on the fact that certain large direct products tend to resist nontrivial direct sum decompositions. It is therefore not surprising that, also in the non-commutative situation, the decomposition properties of such products are crucial.

**Theorem 10.** For a left $R$-module $M$, the following conditions are equivalent:

1. There exists a cardinal number $\aleph$ such that every direct product of copies of $M$ is a direct sum of $\aleph$-generated modules.
2. Every direct product of copies of $M$ is a direct sum of submodules with local endomorphism rings.
3. $M$ is $\Sigma$-algebraically compact.

The implication ‘$1 \implies 3$’ is due to Gruson-Jensen [31], ‘$2 \implies 3$’ to the author of this article [35], whereas ‘$3 \implies 1$’ can already be found in work of Kielpinski going back to 1967 [46]. The implication ‘$3 \implies 2$’ is a consequence of Proposition 9. Credit should also go to Chase, since the ideas he developed in [12] play an essential role in the proofs. These ideas can be extracted and upgraded to the following format:

**Lemma 11.** (Chase) Let

$$f : \prod_{i \in N} U_i \longrightarrow \bigoplus_{j \in J} V_j$$


be a homomorphism of $R$-modules, and $P_1 \supseteq P_2 \supseteq P_3 \supseteq \cdots$ a descending chain of $p$-functors. Then there exists a natural number $n_0$ such that

$$f\left(P_{n_0} \prod_{i \geq n_0} U_i\right) \subseteq \bigoplus_{\text{finite}} V_j + \bigcap_{n \in \mathbb{N}} P_n\left(\bigoplus_{J} V_j\right).$$

(Here $\bigoplus_{\text{finite}}$ stands for a direct sum extending over some finite subset of $J$.)

Since the summand $\bigcap_{n \in \mathbb{N}} P_n\left(\bigoplus_{J} V_j\right)$ on the far right of the pivotal inclusion is a correction term which can be suppressed in most applications, the lemma says that, after a bit of trimming on both sides, a cofinite subproduct of the $U_i$'s maps to a finite subsum of the $V_j$'s.

Before we give some of the arguments to illustrate the techniques, we derive the following consequence of Theorem 10.

**Corollary 12.** (Chase [12])

- If there exists a cardinal number $\aleph$ such that all of the left $R$-modules $R^I$ are direct sums of $\aleph$-generated submodules (this being, e.g., the case if the projectives in $R$-$\text{Mod}$ are closed under direct products), then $R$ is left perfect.

- If there is a cardinal number $\aleph$ such that all left $R$-modules are direct sums of $\aleph$-generated modules, then $R$ is left artinian.

**Proof of Corollary 12.** From the hypothesis of the first assertion we deduce that $R$ has the descending chain condition for finitely generated right ideals, those being among the matrix subgroups of the right $R$-module $R$ by Section 3.
Now suppose that we have the global decomposition property of the second assertion. Then $R$ is left perfect by the first part. Moreover, all left $R$-modules have the descending chain condition for annihilators of subsets of $R$, and hence $R$ has the ascending chain condition for annihilators of subsets of arbitrary left $R$-modules; in other words, $R$ is left noetherian. This implies that $R$ is indeed left artinian. □

We will sketch a proof of Chase’s Lemma, as the argument is clarified by the use of $p$-functors.

**Proof of Lemma 11.** The natural projection $\bigoplus_{k \in J} V_k \to V_j$ will be denoted by $q_j$. Assume the conclusion to be false. Then a standard induction yields a sequence $(n_k)_{k \in \mathbb{N}}$ of natural numbers with $n_{k+1} > n_k$, together with sequences of of pairwise different elements $j_k \in J$, resp. $x_k \in P_{n_k} (\prod_{i \geq n_k} U_i)$, such that

$q_{j_k} f(x_k) \notin P_{n_{k+1}} V_{j_k}$ and $q_{j_k} f(x_l) = 0$ for $l < k$.

Note that the definition $x = \sum_{k \in \mathbb{N}} x_k \in \prod_{i \in \mathbb{N}} U_i$ makes sense (indeed, in view of $x_k \in \prod_{i \geq n_k} U_i$, the sum of the $x_k$ reduces to a finite sum in each $U_i$-component), and that for all $k \in \mathbb{N}$ we have

$q_{j_k} f(x) = q_{j_k} f(x_k) + q_{j_k} f(\sum_{l > k} x_l) \neq 0$,

because the first summand does not lie in $P_{n_{k+1}} V_{j_k}$, whereas the second does. But this contradicts the fact that $f(x)$ belongs to a finite subsum of $V_j$'s. □

Note that the implication ‘(3) $\implies$ (2)’ of Theorem 10 is an immediate consequence of Proposition 9. For the converse, we refer the reader to [35]. We include a proof for the equivalence of (1) and (3) however.

**Proof of ‘(1) $\iff$ (3)’ of Theorem 10.** First suppose that (1) is satisfied, and let $P_1 \supseteq P_2 \supseteq P_3 \supseteq \cdots$ be a chain of $p$-functors. Abbreviate the intersection $\bigcap_{n \in \mathbb{N}} P_n$ by $P$. We want to prove the existence of an index $n_0$ such that $P_{n_0} M = PM$. For that purpose, it is clearly harmless to assume that $PM = 0$. This implies that $PV = 0$ for arbitrary direct summands $V$ of direct powers of $M$, since $P$ is in turn a $p$-functor and, as such, commutes with direct sums and direct products.

Set $\tau = \max\{\aleph_0, \aleph_1, |R|\}$, and choose a set $I$ of cardinality at least $\tau$. By hypothesis, $M^I \cong \prod_{n \in \mathbb{N}} M^I$ has a direct decomposition into $\aleph_0$-generated summands $V_j$. Applying Chase’s Lemma to

$$\prod_{n \in \mathbb{N}} M^I \xrightarrow{id} \bigoplus_{j \in J} V_j,$$

we obtain a natural number $n_0$ with

$$\prod_{n \geq n_0} (P_{n_0} M)^I \hookrightarrow \bigoplus_{\text{finite}} V_j + P(\bigoplus_{j} V_j),$$

the final summand on the right being zero by our assumption on $M$. But this shows the right-hand side to have cardinality at most $\tau$, while the left-hand side has cardinality $> \tau$.
if $P_{n_0}M \neq 0$. To make the left-hand side small enough to fit into the right, we thus need to have $P_{n_0}M = 0$, which shows that our chain of $p$-functors becomes stationary on $M$. Now apply Theorem 6 to obtain (3).

For the converse, assume $M$ to be $\Sigma$-algebraically compact, and set $\aleph = \max(\aleph_0, |R|)$. It is straightforward to see that each $\aleph$-generated submodule $U$ of $M$ can be embedded into a pure $\aleph$-generated submodule (just close $U$ under solutions of finite linear systems with right-hand sides in $U$, repeat the process $\aleph_0$ times, and take the union of the successive closures), the latter being a direct summand of $M$ by Corollary 8. Another application of this corollary thus shows that any maximal family of independent $\aleph$-generated submodules, summing up to a pure submodule of $M$, must generate all of $M$. □

As a matter of course, Theorem 10 leads us to the following answer to the questions concerning global decompositions of modules posed in Section 2.

**Theorem 13.** (Gruson-Jensen, Huisgen-Zimmermann, [loc. cit.], Zimmermann [78]) For a ring $R$, the following statements are equivalent:

1. Every left $R$-module is a direct sum of finitely generated submodules.
2. Every left $R$-module is a direct sum of indecomposable submodules.
3. Every left $R$-module is algebraically compact.

**Proof.** Clearly ‘(1) $\implies$ (1’) $\implies$ (3)’ by Theorem 10. The implication ‘(3) $\implies$ (2)’ follows from Proposition 9.

To derive ‘(2) $\implies$ (3)’ from Theorem 10, keep in mind that each left $R$-module $M$ can be embedded as a pure submodule into an algebraically compact module $N$. By (2) all direct products $N^I$ are direct sums of indecomposable modules, all of which have local endomorphism rings by Proposition 9. Thus $N$ is $\Sigma$-algebraically compact by Theorem 10, and so is $M$ by Corollary 8.

‘(3) $\implies$ (1)’. By (3), all pure inclusions of left $R$-modules split, and therefore all left $R$-modules are pure projective. As we mentioned before, due to Warfield [72], this means that every module $M$ is a direct summand of a direct sum of finitely presented modules $U_i$, say $M \oplus M' = \bigoplus_{i \in I} U_i$. On the other hand, we already know that (3) implies decomposability of all left $R$-modules into submodules with local endomorphism rings, and hence the Krull-Remak-Schmidt-Azumaya Theorem, applied to $M \oplus M'$, yields the required decomposition property for $M$. □

In particular, we retrieve the following result of Fuller [25]: Condition (1) of Theorem 13 is equivalent to

4. Every left $R$-module $M$ has a direct sum decomposition $M = \bigoplus_{i \in I} M_i$ which complements direct summands (i.e., given any direct summand $N$ of $M$, there exists a subset $I' \subseteq I$ such that $M = N \oplus \bigoplus_{i \in I'} M_i$).

Due to the fact that every module can be purely embedded into an algebraically compact (= pure injective) one – go back to the first of the examples in Section 1 – one can build a homology theory based on pure injective resolutions, in analogy with the traditional homology theories. (Alternatively, one can use the fact that every module is an epimorphic
image of a pure projective module under an epimorphism with pure kernel and consider pure projective resolutions.) In particular, it makes sense to speak of the pure injective and pure projective dimensions of a module, and to define the left pure global dimension of a ring $R$ to be the supremum of the pure injective dimensions of its left modules. By playing off the two arguments of the Hom-functor against each other as in the case of the traditional global dimensions, one observes that the left pure global dimension of a ring equals the supremum of the pure projective dimensions of its left modules.

Thus the rings pushed into the limelight by Theorem 10 are precisely the ones for which the left pure global dimension is zero. Since they can be equivalently described by the requirement that all pure inclusions of left modules split, they are also referred to as the left pure semisimple rings. We already know from Corollary 12 that they are necessarily left artinian, but this of course does not tell much of the story.

5. Rings of vanishing left pure global dimension

On the negative side, the rings of the title are still not completely understood. Ironically, however, this fact also has an upside: Namely, as a host of inconclusive arguments on this theme appeared in circulation, numerous interesting insights resulted. The purpose of this section is to describe a representative selection of such insights and to delineate the status quo for further work on the subject.

The first milestone along the way was the recognition that vanishing of the pure global dimension on both sides takes us to a class of thoroughly studied rings. In fact, the solution to our problems in this left-right symmetric situation closely parallels the outcome in the commutative case.

**Theorem 14.** (Auslander [2], Ringel-Tachikawa [62], Fuller-Reiten [26]) A ring $R$ has finite representation type if and only if the left and right pure global dimensions of $R$ are zero.

The implication ‘only if’ was shown independently by Auslander and Ringel-Tachikawa in [2] and [26]; the following easy argument is due to Zimmermann [unpublished]. The converse was first established by Fuller-Reiten; we will re-obtain it as a consequence of the versatile Tool 17 at the end of this section.

**Proof of ‘only if’.** Suppose that $R$ has finite representation type, and set $M = \bigoplus_{1 \leq i \leq n} M_i$, where $M_1, \ldots, M_n$ represent the isomorphism types of the indecomposable finitely generated left $R$-modules. The ring $R$ being left artinian, the module $M$ has finite length, whence its endomorphism ring is semiprimary and so, in particular, left perfect. We denote the opposite of this endomorphism ring by $S$, thus turning $M$ into a right $S$-module. To prove that the left pure global dimension of $R$ is zero, it suffices to check that every left $R$-module $A$ is a direct sum of copies of the $M_i$. For that purpose, we start by choosing a pure projective presentation of $A$. Since in our present situation the pure projective modules are precisely the objects in $\text{Add}(M)$ (see the remark following the definition of algebraic compactness in Section 1), this amounts to the existence of a short exact sequence

\[
0 \to K \to M^{(I)} \to A \to 0,
\]
such that $K$ is pure in $M^I$. Our goal is to show that this sequence splits. The module $M^I$ being a direct sum of finitely generated modules with local endomorphism rings, the Crawley-Jónsson-Warfield theorem (see, e.g., [1]) will then tell us that $A$ is in turn a direct sum of copies of the $M_i$.

To show splitness of $(\dagger)$, we observe that the pure projectivity of $M$ guarantees the following sequence of left $S$-modules to be exact:

$$(\dagger) \quad 0 \to \text{Hom}_R(M, K) \to \text{Hom}_R(M, M^I) \to \text{Hom}_R(M, A) \to 0$$

In fact, this sequence is even pure exact. We will deduce this from the elementary fact that the pure exact sequences are precisely those short exact sequences whose exactness is preserved by all functors $\text{Hom}_S(B, -)$ with finitely presented first argument $B$. So let $B$ be a finitely presented left $S$-module, and consider the following commutative diagram, the columns of which reflect the adjointness of $\text{Hom}$ and tensor product; for compactness, $\text{Hom}$-groups are denoted by square brackets:

$$
\begin{array}{cccccc}
0 & \to & [B, [M, K]] & \to & [B, [M, M^I]] & \to & [B, [M, A]] & \to & 0 \\
& \downarrow \cong & \downarrow \cong & \downarrow \cong & & & & & \\
0 & \to & [M \otimes_S B, K] & \to & [M \otimes_S B, M^I] & \to & [M \otimes_S B, A] & \to & 0
\end{array}
$$

The lower row is exact, since $M \otimes_S B$ is a finitely presented $R$-module, and pure projective as such. Consequently, the upper row is exact as well.

The sequence $(\dagger)$ thus provides us with a pure inclusion of left $S$-modules,

$$0 \to \text{Hom}_R(M, K) \to \text{Hom}_R(M, M^I) \cong S^I.$$ 

Since $S$ is left perfect, this sequence actually splits by Observations 0 of Section 1. The splitness of $(\dagger)$ can now be gleaned from the following commutative diagram which results from the fact that $M$ is a generator for $R$-Mod and hence provides us with a functorial isomorphism $M \otimes_S \text{Hom}_R(M, -) \to \text{id}_{R\text{-Mod}}$:

$$
\begin{array}{cccccc}
0 & \to & M \otimes_S [M, K] & \to & M \otimes_S [M, M^I] & \to & M \otimes_S [M, A] & \to & 0 \\
& \downarrow \cong & \downarrow \cong & \downarrow \cong & & & & & \\
0 & \to & K & \to & M^I & \to & A & \to & 0
\end{array}
$$

Indeed, splitness of the upper row yields splitness of the lower. \(\square\)
Still open: The pure semisimplicity problem. Is every ring with vanishing one-sided pure global dimension of finite representation type?

In view of the preceding theorem, the question can be rephrased as to whether pure semisimplicity is a left-right symmetric property. In the sequel, we list a number of partial results which resolve the problem in the positive for classes of rings exhibiting some – even if faint – ‘commutativity symptoms’. Subsequently, we will see that in general the left pure semisimple rings at least come very close to having finite representation type, in a sense to be made precise.

The first statement of the next theorem is due to Auslander [3], while the second strengthened version was proved by Herzog [34]. The third assertion was first obtained by Herzog and later derived by Schmidmeier from a more general duality principle (see [34] and [63]).

**Theorem 15.** The answer to the pure semisimplicity question is ‘yes’ within the following classes of rings:

- Artin algebras.
- More generally, rings with self-duality.
- P.I. rings.

We will sketch an elementary proof for a slight generalization of the first assertion. It relates left-right symmetry of pure semisimplicity directly to the existence of almost split maps. Namely, as the author showed in [36], the following is true:

- If $R$-mod has left almost split maps, then vanishing of the left pure global dimension of $R$ implies finite representation type.

**Proof of the preceding statement.** The following argument is inspired by [4], where the notions of a preprojective/preinjective partition are introduced. Suppose that $R$ has left pure global dimension zero. By Corollary 12, this forces $R$ to be left artinian. Moreover, suppose that each indecomposable object $A$ in $R$-mod is the source of a left almost split map, i.e., of a nonsplit monomorphism $\phi : A \to B$ such that each homomorphism from $A$ to another object in $R$-mod, which is not a split monomorphism, factors through $\phi$. Our proof for representation-finiteness of $R$ uses Tool 17 below, as well as the following well-known duality discovered by Hullinger [41] and Simson [65]: If $E$ is the minimal injective cogenerator for $R$-Mod and $T$ the opposite of the endomorphism ring of $E$, then $T$ is twosided artinian, again has left pure global dimension zero, and the functor $\text{Hom}_{R}(-, E)$ induces a Morita duality $R$-mod $\to$ mod-$T$. As in the proof of Corollary 18, we exploit the descending chain condition for finite matrix subgroups, satisfied globally in $T$-mod, to obtain the ascending chain condition for finite matrix subgroups in arbitrary right $T$-modules. The remark at the end of Section 3 now shows that each direct sum of finitely generated right $T$-modules is in fact noetherian over its endomorphism ring.

Let $\mathcal{D}$ be a transversal of the finitely generated indecomposable right $T$-modules. In a first step we will establish a strong preprojective partition on $\mathcal{D}$, namely an ordinal-indexed partition $\mathcal{D} = \bigcup_{\alpha} \mathcal{D}_{\alpha}$ of $\mathcal{D}$ into pairwise disjoint finite subsets $\mathcal{D}_{\alpha}$ such that each $\mathcal{D}_{\alpha}$ is a minimal generating set for $\mathcal{D} \setminus \bigcup_{\beta < \alpha} \mathcal{D}_{\beta}$ and such that, for each $\alpha$, $\mathcal{D}_{\alpha}$ consists
precisely of those objects in $\mathcal{D} \setminus \bigcup_{\beta<\alpha} \mathcal{D}_\beta$ for which every epimorphism $\bigoplus_{\text{finite}} D_i \to D$ with $D_i \in \mathcal{D} \setminus \bigcup_{\beta<\alpha} \mathcal{D}_\beta$ splits. This claim will follow from an obvious transfinite induction if we can show that every nonempty subset $\mathcal{D}' \subseteq \mathcal{D}$ contains a finite generating set, and that any minimal such generating set $\mathcal{D}_0$ consists precisely of those $D \in \mathcal{D}'$ which have the property that all epimorphisms from add $\mathcal{D}'$ onto $D$ split. For simplicity, we denote the subset $\mathcal{D}'$ again by $\mathcal{D}$. Let $M = \bigoplus_{i \in I} M_i$ be the direct sum of the objects in $\mathcal{D}$, and $S$ the endomorphism ring of $M$. Since $M$ is noetherian over $S$, we obtain a finite subset $I' \subseteq I$ such that $M = S(\bigoplus_{i \in I'} M_i)$; this shows that the set $\{M_i \mid i \in I'\}$ generates $\mathcal{D}$. Let $\mathcal{D}_0 \subseteq \mathcal{D}$ be a minimal finite generating set for $\mathcal{D}$. Then, clearly, each object $D \in \mathcal{D}$, with the property that arbitrary epimorphisms from add $\mathcal{D}$ onto $D$ split, belongs to $\mathcal{D}_0$. For the converse, let $D \in \mathcal{D}_0$ and $f : X \to D$ be an epimorphism with $X \in \text{add} \mathcal{D}$. Moreover choose an epimorphism $g : D^l \oplus \bigoplus_{1 \leq i \leq m} D_i \to X$, where the $D_i$ are objects in $\mathcal{D}_0 \setminus \{D\}$. Note that splitting of $h = fg$ will imply splitting of $f$. In order to see that $h$ splits, denote the endomorphism ring of $D$ by $S(D)$, and let $h_1, \ldots, h_l$ be the restrictions of $h$ to the various copies of $D$ occurring as summands of the domain of $h$. Assume that all of the $h_i$ are non-isomorphisms. The ring $S(D)$ being local, this means that the $h_i$ belong to the radical $J(D)$ of $S(D)$. Let $D' \subseteq D$ be the trace of $\bigoplus_{1 \leq i \leq m} D_i$ in $D$. The fact that $J(D)D$ is superfluous in $D$ (keep in mind that $D$ is noetherian over $S(D)$), applied to the equality $D = h_1 D + \cdots + h_l D + D'$ yields $D = D'$. But this means that $\mathcal{D}_0 \setminus \{D\}$ is still a generating set for $\mathcal{D}$, a contradiction to our minimal choice of $\mathcal{D}_0$. Hence one of the $h_i$ is an isomorphism, and consequently $h$ splits.

Next we apply the duality $\text{Hom}_R(-, E)$ to $\mathcal{D}$, to obtain a strong preinjective partition of the transversal $\mathcal{E} = \text{Hom}_R(\mathcal{D}, E)$ of the indecomposable objects in $R$-mod. In other words, we obtain a partition $\mathcal{E} = \bigcup_{\alpha} \mathcal{E}_\alpha$, such that each $\mathcal{E}_\alpha$ is a minimal finite cogenerating set for $\mathcal{E} \setminus \bigcup_{\beta<\alpha} \mathcal{E}_\beta$, where the $\mathcal{E}_\alpha$ are pinned down by the following additional property (\dagger): Namely, each $\mathcal{E}_\alpha$ consists precisely of those objects $C$ in $\mathcal{E} \setminus \bigcup_{\beta<\alpha} \mathcal{E}_\beta$ for which all monomorphisms from $C$ to add$(\mathcal{E} \setminus \bigcup_{\beta<\alpha} \mathcal{E}_\beta)$ split.

The smallest ordinal $\tau$ such that $\mathcal{E}_\tau = \emptyset$ is called the length of the partition; clearly $\mathcal{E}_\beta = \emptyset$ for all $\beta > \tau$. Observe that $\tau$ is a successor ordinal. Indeed, if $S_1, \ldots, S_n$ are the simple modules in $\mathcal{E}$ — say $S_i \in \mathcal{E}_{\beta_i}$ — then $\beta = 1 + \max(\beta_1, \ldots, \beta_n)$ is the length of the partition. This is due to the fact that each object $C \in \mathcal{E}_\beta$ is non-simple indecomposable and hence gives rise to a nonsplit embedding of one of the $S_i$ into $C$; but such a nonsplit monomorphism is incompatible with property (\dagger) of $\mathcal{E}_{\beta_i}$.

Finally, we show that $\tau$ is bounded above by the first infinite ordinal number $\omega$. In light of the preceding paragraph, this will yield finiteness of $\tau$ and thus finiteness of $\mathcal{E}$, i.e., finiteness of the representation type of $R$. Suppose to the contrary that there exists an object $A \in \mathcal{E}_\omega$, and let $\phi : A \to B$ be a left almost split monomorphism with $B \in R$-mod. Moreover, for $\alpha < \omega$, let $B^\alpha$ be the reject of $\mathcal{E}_\alpha$ in $B$, and observe that $B^\beta \subseteq B^\alpha$ whenever $\beta < \alpha < \omega$. Due to the finite length of $B$, the resulting chain of rejects becomes stationary, say at $\gamma < \omega$. This shows that $B/B^\gamma$ is cogenerated by $\mathcal{E}_\alpha$ for all ordinal numbers $\alpha$ between $\gamma$ and $\omega$, which places a copy of $B/B^\gamma$ in add$(\bigcup_{\beta \geq \omega} \mathcal{E}_\beta)$. On the other hand, since each of the $\mathcal{E}_\alpha$ for $\alpha < \omega$ cogenerates $A$ — recall that $A \in \mathcal{E}_\omega$ — we obtain a family of non-split monomorphisms $f_\alpha : A \to C_\alpha$, where $C_\alpha \in \text{add} \mathcal{E}_\alpha$ and $\alpha$ again runs through the ordinals.
between $\gamma$ and $\omega$. Now left almost splitness of $\phi$ permits us to factor all of these maps through $\phi$, say $f_\alpha = g_\alpha \phi$ for a suitable homomorphism $g_\alpha : B \to C_\alpha$. By the choice of $\gamma$, the reject $B^\gamma$ is contained in the kernel of each $g_\alpha$, which shows that the homomorphism $\phi : A \to B/B^\gamma$ obtained by composing $\phi$ with the canonical map $B \to B/B^\gamma$ is still a monomorphism. It is in turn nonsplit because $\phi$ does not split. But in view of the above placement of $B/B^\gamma$ relative to our preprojective partition, this contradicts property (†) of $C_\omega$. Hence the assumption that $C_\omega$ be nonempty is absurd and our argument is complete. □

In the meantime, Simson has replaced the pure semisimplicity problem by a conjecture, to the effect that the answer is negative (see [68, 69]). His key to a potential class of counterexamples is the following:

**Connection with a strong Artin problem for division rings.** The answer to the pure semisimplicity question is positive if and only if the following is true:

For every simple $D$-$E$-bimodule $M$, where $D$ and $E$ are division rings, such that $\dim(DM) < \infty$ and $\dim(M_E) = \infty$, there exists a non-finitely generated indecomposable left module over the triangular matrix ring

$$
\begin{pmatrix}
D & M \\
0 & E
\end{pmatrix}.
$$

For a proof see [68].

On the other hand, the next theorem guarantees the rings of vanishing left pure global dimension to at least have a very sparse supply of indecomposable finitely generated modules. The result was independently proved by Prest [58] and Zimmermann and the author [39]. In particular, it relates the pure semisimplicity conjecture to the quest for a better understanding of those left artinian rings of infinite representation type which fail to satisfy the conclusion of the second Brauer-Thrall Conjecture. Recall that for a left artinian ring $R$ of infinite cardinality, the latter postulates the following: If $R$ has infinite representation type, then there exist infinitely many distinct positive integers $d_n$ such that, for each $n$, there are infinitely many isomorphism classes of indecomposable left $R$-modules of composition length $d_n$. While this conjecture has long been confirmed for finite dimensional algebras over algebraically closed fields, in [60] Ringel constructed a class of artinian rings with infinite center which violate this implication.

**Theorem 16.** Suppose that $R$ has left pure global dimension zero. Then:

- For each $d \in \mathbb{N}$, there exist only finitely many left $R$-modules of length $d$, up to isomorphism.
- For each $d \in \mathbb{N}$, there exist only finitely many length-$d$ modules in fin.pres-$R$, up to isomorphism.

The obstacle one meets in trying to strengthen the latter assertion for right $R$-modules to the level of that for the left lies in the fact that it is not known whether left pure semisimple rings are necessarily right artinian. In fact, it is known that a positive answer to this question would resolve the pure semisimplicity problem in the positive.
The following duality for finite matrix subgroups, proved in [39] and – in model-theoretic terms – also in [58], is one of the pivotal tools in proving Theorem 16. We include it here because it yields some by-products of independent interest.

**Tool 17.** If $M$ is an $R$-$S$-bimodule and $C$ an injective cogenerator for Mod-$S$, then the following lattices are anti-isomorphic:
- the lattice of finite matrix subgroups of the left $R$-module $M$
- the lattice of finite matrix subgroups of the right $R$-module $\text{Hom}_S(M, C)_R$.

**Sketch of proof.** Keep in mind that the finite matrix subgroups of $R^n$ are precisely the kernels of the maps $M \to Q \otimes_R M$, $m \mapsto q \otimes m$ with $Q$ finitely presented and $q \in Q$, and that they can alternately be given the form $\text{Hom}_R(P, M)(p)$ with $P$ finitely presented and $p \in P$ (see Observations 3(1), (2) – a proof can be found in [39, Lemma 1]).

First one checks that, given a finite matrix subgroup $U = \text{Hom}_R(P, M)(p)$ of $R^n$ with $P$ and $p$ as above, the $C$-dual $\text{Hom}_S(M/U, C)$ is a finite matrix subgroup of the right $R$-module $M^+ = \text{Hom}_S(M, C)$. From the exactness of the sequence

$$\text{Hom}_R(P, M) \xrightarrow{\psi} M \xrightarrow{\text{canon}} M/U \to 0,$$

where $\psi$ is evaluation at $p$, one obtains exactness of the sequence of induced maps

$$0 \to \text{Hom}_S(M/U, C) \to \text{Hom}_S(M, C) \xrightarrow{\psi^*} \text{Hom}_S(\text{Hom}_R(P, M), C).$$

Using the canonical isomorphism $\tau : \text{Hom}_S(M, C) \otimes_R P \to \text{Hom}_S(\text{Hom}_R(P, M), C)$ and the fact that the map $\tau^{-1}\psi^*$ sends $f \in \text{Hom}_S(M, C)$ to $f \otimes p$, one deduces the first claim. Clearly, this finite matrix subgroup of $M^+$ coincides with the annihilator $\text{Ann}_{M^+}(U)$.

Similarly, one verifies that, for any finite matrix subgroup $V$ of the right $R$-module $M^+$, the annihilator $\text{Ann}_M(V)$ is a finite matrix subgroup of the left $R$-module $M$.

It is now a question of routine to check that the lattice anti-homomorphisms $U \mapsto \text{Ann}_{M^+}(U)$ and $V \mapsto \text{Ann}_M(V)$ are inverse to each other. \(\square\)

It was shown by Crawley-Boevey that, among the finite dimensional algebras over an algebraically closed field, the ones of finite representation type are characterized by the nonexistence of generic modules, i.e., of non-finitely generated indecomposable endofinite modules. (As the term suggests, a module is endofinite if it has finite length over its endomorphism ring.) The following consequence of the above duality shows that the absence of generic objects is offset by the richest possible supply of endofinite modules in the representation-finite case.

**Corollary 18.** ([58, 39]) For any ring $R$, the following statements are equivalent:

1. All objects in $R$-$\text{Mod}$ are endofinite.
2. Same as (1) for $\text{Mod}$-$\text{R}$.
3. The left and right pure global dimensions of $R$ vanish.

**Proof.** It clearly suffices to show the equivalence of (1) and (3). If (1) holds, Tool 17, applied to $R$ and the opposites of the endomorphism rings of the modules considered,
yields the descending chain condition for finite matrix subgroups in all right \(R\)-modules. That the left modules satisfy this chain condition is immediate from our hypothesis. By Theorem 6, this shows that all \(R\)-modules, left and right, are algebraically compact, i.e. (3) holds.

Now assume (3). In view of the descending chain condition for matrix subgroups, satisfied globally for left and right \(R\)-modules, Tool 17 provides us with the ascending chain condition for finite matrix subgroups as well. Let \(M\) be any left \(R\)-module and \(S\) its endomorphism ring. We know that all finitely generated \(R\)-modules, Tool 17 provides us with the ascending chain condition for finite matrix subgroups, so if we can show that all matrix subgroups of \(RM\) arise from finite matrices, we are done. But this finiteness condition follows easily from our hypothesis as follows: Indeed, let \([A, \alpha]\) be any matrix functor on \(R\text{-Mod}\), where \(A = (a_{ij})_{i,j \in I}\). Moreover, for any finite subset \(I'\) of \(I\), let \(A(I')\) be the matrix consisting of those rows of \(A\) which are indexed by \(I'\). Since \(A\) is row-finite, the matrix \(A(I')\) is actually finite in effect. Now let \(I_1\) be any finite subset of \(I\) and note that the finite matrix subgroup \([A(I_1), \alpha]M\) of \(M\) contains \([A, \alpha]M\). If the inclusion is proper, there exists a finite subset \(I_2 \subseteq I\) containing \(I_1\) such that \([A(I_2), \alpha]M\) is properly contained in \([A(I_1), \alpha]M\). If \([A, \alpha]M\) is still strictly contained in \([A(I_2), \alpha]M\), we repeat the process. Eventually, it has to terminate by our chain condition, which shows that \([A, \alpha]M\) equals some finite matrix subgroup \([A(I_m), \alpha]M\), and our argument is complete. \(\square\)

We conclude this section by looping back to its beginning and filling in the proof of the as yet unjustified implication of Theorem 14. We prepare with the following proposition (not needed in full strength here) which generalizes Crawley-Boevey’s observation that an endofinite direct sum of modules with local endomorphism rings involves only finitely many isomorphism types of summands [17, Proposition 4.5].

**Proposition 19.** Suppose that \((M_i)_{i \in I}\) is a family of left \(R\)-modules with local endomorphism rings. Let \(M = \bigoplus_{i \in I} M_i\), and denote by \(S\) the endomorphism ring of \(M\), by \(J(S)\) the Jacobson radical of \(S\).

If there exists a natural number \(N\) with \(J(S)^N M = 0\) such that, moreover, \(J(S)^k M\) is finitely generated over \(S\) for all \(0 \leq k \leq N\), then the family \((M_i)_{i \in I}\) involves only finitely many isomorphism classes.

**Proof.** Suppose \(N\) is as in the claim, and start by noting that \(\text{Hom}_R(M_i, M_j) \subseteq J(S)\), whenever \(M_i \not\cong M_j\); this is due to the locality of the endomorphism rings of the \(M_i\). Next observe that, for any finite sequence \(x_1, \ldots, x_r\) in \(M\), we have

\[
\sum_{i=1}^{r} Sx_i \subseteq \bigoplus_{i \in I'} M_i + \sum_{i=1}^{r} J(S)x_i,
\]

where \(I' \subseteq I\) is the ‘closure of the supports of the \(x_l\), \(1 \leq l \leq r\), under isomorphism’; by this we mean that

\[
I' = \{i \in I \mid M_i \cong M_j \text{ for some } j \in \bigcup_{1 \leq l \leq r} \text{supp}(x_l)\}.
\]
Now pick generators $x_{k1}, \ldots, x_{kr_k}$ for each of the left $S$-modules $J(S)^k M$, where $k$ runs from 0 to $N - 1$. Moreover, for each $k$, let $I_k$ be the closure of $\bigcup_{1 \leq l \leq r_k} \text{supp}(x_{kl})$ under isomorphism. Then

$$M = \sum_{l=1}^{r_0} Sx_{0l} \subseteq \bigoplus_{i \in I_0} M_i + \sum_{l=1}^{r_0} J(S)x_{0l} \subseteq \bigoplus_{i \in I_0} M_i + \sum_{l=1}^{r_1} Sx_{1l}$$

$$\subseteq \bigoplus_{i \in I_0 \cup I_1} M_i + \sum_{l=1}^{r_1} J(S)x_{1l} \subseteq \bigoplus_{i \in I_0 \cup I_1} M_i + \sum_{l=1}^{r_2} Sx_{2l},$$

and since $J(S)x_{N-1,l} = 0$ for all $l$ by hypothesis, an obvious induction yields

$$M = \bigoplus_{i \in I_0 \cup \cdots \cup I_{N-1}} M_i.$$

But the modules $M_i$ with $i \in I_0 \cup \cdots \cup I_{N-1}$ fall into finitely many isomorphism classes by construction, and our argument is complete. □

On the side, we point out that, in Proposition 19, neither the hypothesis that $J(S)^N M = 0$ for some $N$, nor the condition that $J(S)^k M$ be finitely generated over $S$ for all $k$, suffices to yield the conclusion: If $R = \mathbb{Z}$ and $M = \bigoplus_{p \text{ prime}} \mathbb{Z}/(p)$, then clearly $J(S) = 0$. An example to justify the second remark is as follows: Let $R$ be the Kronecker algebra and $M_n$ the preprojective string module of dimension $2n + 1$. If we set $M = \bigoplus_{n \in \mathbb{N}} M_n$, then $J(S)^k M = \bigoplus_{n \geq k+1} M_n$ is finitely generated over $S$ for each $k$.

**Proof of the implication ‘if’ in Theorem 14.** Our hypothesis being condition (3) of Corollary 18, we see that all left $R$-modules are endofinite. Since we already know $R$ to be artinian (Corollary 12), we only need to show that any transversal $(M_i)_{i \in I}$ of the indecomposable objects in $R$-mod is finite. But this is an immediate consequence of Proposition 19. □

For the reader interested in picking up the threads that are still dangling, we list a number of additional references addressing the pure semisimplicity problem: [65], [66], [41], [67], [42], [76], [6], [19], [43], [44], [36], [5], [33], [52], [70].

### 6. Vanishing pure global dimension and product-completeness

From Theorem 10 we know that the $\Sigma$-algebraically compact modules $M$ are precisely those with the property that all direct products of copies of $M$ are direct sums of indecomposable components with local endomorphism rings. This naturally raises the question as to the structure of the indecomposable summands in such decompositions.

More generally, we ask: **What are the indecomposable summands of large $\Sigma$-algebraically compact direct products $\prod_{i \in I} M_i$?**

This problem is further motivated by the following points: (a) The question is intimately related to the pure semisimplicity problem. The connection first surfaced in a theorem of Auslander (Theorem 20 below), and is reinforced by results of Gruson, Garavaglia, and...
Krause-Saorín (also compare with Corollary 18). And (b), direct powers of non-generic \(\Sigma\)-algebraically compact modules provide prime hunting ground for generic objects, as evidenced by Theorems 2.11 and 2.12 (due to Krause and Ringel, respectively) in Zwara’s contribution to this volume.

Slightly extending the terminology of Krause-Saorín [51], we call a family \((M_i)_{i \in I}\) product-complete, if the direct product \(\prod_{i \in I} M_i\) belongs to \(\text{Add}(\bigoplus_{i \in I} M_i)\), the closure of \(\{M_i \mid i \in I\}\) in \(R\)-Mod under formation of direct summands and direct sums. As in [51], a module \(M\) will be labeled product-complete if all families consisting of copies of \(M\) have this property.

According to Gruson [29] and Garavaglia [27], an indecomposable module \(M\) is product-complete if and only if it is endofinite. This equivalence was generalized by Krause-Saorín as follows: For an arbitrary module \(M\), finite endolength is equivalent to the postulate that all direct summands of \(M\) be product-complete [51, Theorem 4.1].

This yields the following hierarchy within the class of algebraically compact modules:

\[
\begin{array}{c}
\text{generic} \\
\text{endofinite} \\
\mathbb{Q} \oplus \mathbb{Z}(p^{\infty}) \\
\mathbb{Z}(p^{\infty}) \\
\hat{\mathbb{Z}}_p \\
\Sigma\text{-algebraically compact} \\
\text{algebraically compact} \\
K[[X]]
\end{array}
\]

Each of the inclusions is proper: The algebra of power series \(K[[X]]\) is algebraically compact without being \(\Sigma\)-algebraically compact; and the same is true for the ring \(\hat{\mathbb{Z}}_p\) of \(p\)-adic integers, viewed either as \(\mathbb{Z}\)- or \(\hat{\mathbb{Z}}_p\)-module. The ring \(T = K[X_i \mid i \in \mathbb{N}] / (X_i \mid i \in \mathbb{N})^m\), where \(K\) is a field and \(m\) some integer \(\geq 2\), is \(\Sigma\)-algebraically compact (c.f. examples following Theorem 6), but not product-complete; indeed, \(T^\mathbb{N}\) is not projective, as \(T\) fails to be coherent. On the other hand, Schulz proved the non-projective summands in arbitrary direct products of copies of \(T\) to be all isomorphic to the unique simple \(T\)-module (see [64]), whence \(T^\mathbb{N}\) is product-complete without being endofinite. Furthermore, the direct sum of the group of rational numbers and the Prüfer group \(\mathbb{Z}(p^{\infty})\) for some prime \(p\) is a product-complete abelian group, but fails to be endofinite. Over any Artin algebra \(R\), finally, all objects in \(R\)-mod are endofinite without being generic.

As is to be expected in light of our previous discussion, global product-completeness of families of \(R\)-modules occurs only rarely. More precisely, we have:

**Theorem 20.** (Auslander [3]) For any ring \(R\), the following statements are equivalent:
(1) $R$ has finite representation type.
(2) All families of finitely generated indecomposable left $R$-modules are product-complete.

We give a proof relying only on the results established in the previous sections. Note that the first part of our argument is very similar to the reasoning of Krause-Saorín ([51, 3.8]).

Proof. ‘(1) $\implies$ (2)’. Given (1), we know from Theorem 14 that the left pure global dimension of $R$ is zero. Let $(M_i)_{i \in I}$ be a family of indecomposable objects in $R$-mod. To see that each indecomposable direct summand of $\prod_{i \in I} M_i$ is isomorphic to one of the $M_j$, it suffices to show this for the case where all $M_i$ are pairwise isomorphic since, by hypothesis, our family contains only finitely many isomorphism types. So suppose $M_i \cong M$ for all $i$. Denote the endomorphism ring of $M$ by $S$. From Corollary 18, we know that $M$ has finite length over $S$, and hence $S$ is left artinian: Indeed, letting $m_1, \ldots, m_n$ be a generating set for $M$ over $R$, we see that $S$ embeds into $M^n$ as a left $S$-module, via $s \mapsto (sm_k)$. In particular, this implies that $S^I$ is projective as a right $S$-module. The ring $S$ being local, this means that $S^I$ is free, say $S^I \cong S^{(J)}$. Using the fact that the tensor functor $- \otimes_S M$ commutes with direct products in our setting, we see that $M^I \cong S^I \otimes_S M \cong S^{(J)} \otimes M \cong M^{(J)}$, which yields (2).

For the converse, assume that (2) is true, and let $(M_i)_{i \in I}$ be a transversal of the isomorphism types of the indecomposable objects in $R$-mod. We will obtain finiteness of $I$ by showing that the direct product $\prod_{i \in I} M_i$ equals the direct sum $\bigoplus_{i \in I} M_i$. We start by observing that $\bigoplus_{i \in I} M_i$ is algebraically compact: Indeed, all direct powers of $\prod_{i \in I} M_i$ are direct sums of copies of the $M_i$ by hypothesis, whence $\prod_{i \in I} M_i$ is $\Sigma$-algebraically compact by Theorem 10; but by Theorem 6, this is tantamount to $\Sigma$-algebraic compactness of $\bigoplus_{i \in I} M_i$. In particular, the pure inclusion $\bigoplus_{i \in I} M_i \subseteq \prod_{i \in I} M_i$ splits, say $\prod_{i \in I} M_i = \bigoplus_{i \in I} M_i \oplus N$, and $N$ is in turn a direct sum of copies of $M_i$’s. If $N$ were nonzero, we could thus find a direct summand isomorphic to some $M_k$ in $N$. On the other hand, we may cancel $M_k$ from the above product-sum equality to obtain $\prod_{i \in I \setminus \{k\}} M_i \cong \bigoplus_{i \in I \setminus \{k\}} M_i \oplus N$ (keep in mind that $\text{End}(M_k)$ is local). This makes $M_k$ a direct summand of $\prod_{i \in I \setminus \{k\}} M_i$, thus contradicting our hypothesis. We conclude $N = 0$, which forces $I$ to be finite as required. □

The following result, due to Krause and Saorín [51, Proposition 4.2], characterizes the product-complete modules in terms of their matrix subgroups. It continues the line of Theorem 10, where we related direct sum decompositions of direct products $M^I$ to finiteness conditions on the lattice of matrix subgroups of $M$.

**Proposition 21.** An object $M \in R$-$\text{Mod}$ is product-complete if and only if $M$ has the descending chain condition for (finite) matrix subgroups and all (finite) matrix subgroups of $M$ are finitely generated over the endomorphism ring of $M$.

We include an elementary proof for one implication.

Proof of ‘only if’. Suppose $M$ is product-complete. Then, clearly, $M$ is $\Sigma$-algebraically compact by Theorem 10, which is tantamount to $M$ satisfying the descending chain condition for matrix subgroups by Theorem 6. To see that any matrix subgroup $[A, \alpha]M$
is finitely generated over the endomorphism ring $S$ of $M$, recall that

$$[A, \alpha]M = \text{Hom}_R(M^I, M)(m)$$

for some set $I$ and some element $m \in M^I$ (this was explained after the statement of Theorem 5). By hypothesis, $M^I$ is a direct summand of a suitable direct sum $M^{(J)}$, and hence $[A, \alpha]M = \text{Hom}_R(M^{(J)}, M)(m)$ for some $m = (m_j) \in M^{(J')}$, where $J'$ is a finite subset of $J$. This gives $[A, \alpha]M = \sum_{j \in J'} Sm_j$ as required. □

In view of the final remark of Section 3A, a ring $R$ is right coherent if and only if all finite matrix subgroups of the regular left module $R$ are finitely generated as right ideals. We can thus supplement Corollary 12 to retrieve Chase’s characterization of the rings whose left projective modules are closed under direct products.

**Corollary 22.** (Chase [12]) $R$ has the property that all direct products of projective left $R$-modules are again projective if and only if $R$ is left perfect and right coherent.

The following question appears of significantly lower importance than the one with which we opened the section. However, the fact that it is not yet answered shows the lacunary state of our present understanding of direct sum decompositions of large direct products.

*Given a $\Sigma$-algebraically compact module $M$, is $M^N$ product-complete?*

As was already observed by Krause and Saorín [51], there is some power $M^I$ which is product-complete; indeed, the fact that all powers of $M$ split into summands of cardinalities bounded above by $\max(|R|, \aleph_0)$ guarantees that, eventually, saturation with respect to the appearance of new direct summands is reached.

### 7. Concluding remarks on pure global dimension

From the previous section we know that a ring $R$ has finite representation type if and only if its left and right pure global dimensions are zero. Moreover, by Theorem 15, the latter condition is left-right symmetric for Artin algebras. What can one say about the pure global dimensions of $R$ when they do not vanish?

The question of how these invariants relate to other properties of the ring and its module categories is largely open. Some general facts of interest are available, however, as well as some classes of algebras where the connection is understood. We include only a few results to trigger interest in further investigation of the problem.

A very rough, but not unreasonable, answer to the above question is this: “That depends on the cardinality of $R$.” As a first step in justifying this response, we present an insight due to Gruson and Jensen [32]; for an alternate approach, see [47].

**Theorem 23.** The following implications hold for any ring $R$.

1. If $R$ is countable, but not of finite representation type, then the left (right) pure global dimension of $R$ equals 1.
2. If the cardinality of $R$ is bounded from above by $\aleph_t$ for some integer $t \geq 0$, the pure global dimensions of $R$ are at most $t + 1$. 
One can do far better for specialized classes of rings, however. For example, as was observed by Kępiński and Simson [47], if \( R = S[X] \) is a polynomial ring over a commutative ring \( S \) of cardinality \( \aleph_\alpha \), in a set \( X \) of indeterminates which has cardinality \( \aleph_\beta \) (with \( s, t \geq 0 \)), then the pure injective dimension of \( R \) equals \( \max(s, t) + 1 \). Our final result, due to D. Baer, Brune, and Lenzing [7], presents a smooth picture of pure homology for hereditary algebras over algebraically closed base fields subject to certain cardinality restrictions. In that case, the pure global dimension mirrors the representation type of \( R \) as follows:

**Theorem 24.** Suppose that \( R \) is a hereditary finite dimensional algebra over an algebraically closed field of cardinality \( \aleph_t \), where \( t \geq 2 \) is an integer. Then:

1. The left (right) pure global dimension of \( R \) equals 2 if and only if \( R \) has tame, but infinite, representation type.

2. The left (right) pure global dimension of \( R \) equals \( t + 1 \) if and only if \( R \) has wild representation type.

Finally, we refer the reader to [56], [8] and [7] for further instances in which the pure global dimension is understood. Motivated by the lectures of Benson at the 1998 conference in Bielefeld, we conclude with the following

**Problem.** Given a finite group \( G \) and a field \( K \) of suitable cardinality, how does the pure global dimension of the group algebra \( KG \) reflect the representation type?

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