Statistical mechanics of free particles on space with Lie-type noncommutativity

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Abstract
Effects of Lie-type noncommutativity on thermodynamic properties of a system of free identical particles are investigated. A definition for finite volume of the configuration space is given, and the grandcanonical partition function in the thermodynamic limit is calculated. Two possible definitions for the pressure are discussed, which are equivalent when the noncommutativity vanishes. The thermodynamic observables are extracted from the partition function. Different limits are discussed where either the noncommutativity or the quantum effects are important. Finally, specific cases are discussed where the group is SU(2) or SO(3), and the partition function of a nondegenerate gas is calculated.

1. Introduction

Noncommutative spacetime is recognized as the space whose coordinate operators do not commute. In the simplest case of canonical noncommutativity (the so-called Groenewold–Moyal space) the coordinates satisfy

\[ \{ \hat{x}_\mu, \hat{x}_\nu \} = i \theta_{\mu\nu} \mathbf{1}, \]

where \( \theta \) is an antisymmetric constant tensor and \( \mathbf{1} \) is the unit operator. The theoretical and phenomenological implications of such noncommutative coordinates have been extensively studied during the last decade [1], once it was understood that the longitudinal directions of D-branes in the presence of a constant B-field background appear to be noncommutative, as seen by the ends of open strings [2–5].

One direction to extend studies on noncommutative spaces is to consider spaces where the commutators of the coordinates are not constants. Examples of this kind are the noncommutative cylinder and the \( q \)-deformed plane (the Manin plane [6], the so-called \( \kappa \)-Poincaré algebra [7] (see also [8–11]) and linear noncommutativity of the Lie algebra type [12], see also [13, 14]). In the latter the dimensionless spatial position operators satisfy the commutation relations of a Lie algebra:

\[ \{ \hat{x}_a, \hat{x}_b \} = f^{c}_{\ ab} \hat{x}_c, \]
where $f^{a}_{\alpha \beta}$'s are the structure constants of a Lie algebra. One example of this kind is the algebra $SO(3)$, or $SU(2)$. A special case of this is the so-called fuzzy sphere [15] (see also [16]), where an irreducible representation of the position operators is used which makes the Casimir of the algebra, $(\hat{x}_1)^2 + (\hat{x}_2)^2 + (\hat{x}_3)^2$, a multiple of the identity operator (a constant, hence the name sphere). One can consider the square root of this Casimir as the radius of the fuzzy sphere. This is, however, a noncommutative version of a two-dimensional space (sphere).

In [17–19], a model was introduced in which the representation was not restricted to an irreducible one, instead the whole group was employed. In particular, the regular representation of the group was considered, which contains all representations. As a consequence in such models one is dealing with the whole space, rather than a sub-space, like the case of fuzzy sphere as a two-dimensional surface. In [17], basic ingredients for calculus on a linear fuzzy space, as well as basic notions for a field theory on such a space, were introduced. In [18, 19], basic elements for calculating the matrix elements corresponding to transition between initial and final states together with the explicit expressions for tree and one-loop amplitudes were given. It is observed that models based on Lie algebra-type noncommutativity enjoy three features.

- They are free from any ultraviolet divergences if the group is compact.
- There is no momentum conservation in such theories.
- In the transition amplitudes only the so-called planar graphs contribute.

The reason for the latter is that the non-planar graphs are proportional to $\delta$-distributions whose dimensions are less than their analogs coming from the planar sector, and so their contributions vanish in the infinite-volume limit usually taken in transition amplitudes [19].

The purpose of this work is to explore the effect of Lie-type noncommutativity on thermodynamics of physical systems. In particular, we consider the free particles which obey boson, fermion and classical statistics. First, we provide a recipe to give practical meaning to ‘a finite volume’ in a space with Lie-type fuzziness. Second, in the thermodynamical limit, we give the proper expression for the grand canonical partition function of free gas. It is explained how one can define in two inequivalent ways the pressure. Different limits are considered in calculation of the thermodynamical quantities.
2. The Hilbert space

The Hilbert space is defined as the space of $L^2$-distributions defined on the group manifold, where the integration measure is the Haar measure of the group. The group is assumed to be unimodular, so that the left- and right-Haar measures coincide, and also equal to its identity component, so that the exponential map is surjective. A completeness relation for the orthonormal kets $|U\rangle$ can be written as

$$\int dU |U\rangle\langle U| = 1.$$  

(3)

The element $|U\rangle$ corresponds the distribution $\delta_U$ with

$$\delta_U(U') = \delta(U^{-1}U'),$$

(4)

where $\delta$ is the Dirac distribution:

$$\int dU' \delta(U^{-1}U') f(U') = f(U).$$

(5)

It is clear that $|U\rangle$ does not belong to the Hilbert space, but to an extension of it. Yet the elements of the Hilbert space can be expanded in terms of $|U\rangle$’s.

The group element $U$ itself is written as a function of the coordinates $\hat{k}^a$ according to

$$U(\hat{k}) := [\exp(\hat{k}^a \hat{x}_a)] U(0),$$

(6)

where $U(\hat{k})$ is the group element corresponding to the coordinates $\hat{k}$. $U(0)$ is the identity and $\exp(\hat{x})$ is the flux corresponding to the vector field $\hat{x}$. The set of $\hat{x}_a$’s is a basis for the left-invariant vector fields. The action of $L_{\hat{x}_a}$ (the Lie derivative corresponding to the vector field $\hat{x}_a$) on an arbitrary scalar function $F$ can be written as

$$L_{\hat{x}_a}(F) = \hat{x}_a^b \frac{\partial F}{\partial \hat{k}^b},$$

(7)

where $\hat{x}_a^b$’s are scalar functions, and satisfy

$$\hat{x}_a^b(\hat{k} = 0) = \delta_a^b.$$  

(8)

The vector fields $\hat{x}_a^R$ are defined as right-invariant vector fields coinciding with their left-invariant analogs at the identity of the group:

$$\hat{x}_a^R(\hat{k} = 0) = \hat{x}_a(\hat{k} = 0).$$

(9)

Finally, the generators of the adjoint action are defined as

$$\hat{J}_a := \hat{x}_a - \hat{x}_a^R,$$

(10)

Dimensionalizing these as

$$p^a := (\hbar/\ell) \hat{k}^a,$$

(11)

$$x_a := i \ell \hat{x}_a,$$

(12)

$$x_a^b(p) := \hat{x}_a^b(\ell/\hbar)p,$$

(13)

$$J_a := i \hbar \hat{J}_a,$$

(14)

where $\ell$ is a constant of dimension length, one arrives at the following commutation relations [21, 22]:

$$[p^a, p^b] = 0.$$  

(15)
\[ [x_a, p^b] = i\hbar x_a^b, \quad (16) \]
\[ [x_a, x_b] = i\ell f_{ab}^c x_c, \quad (17) \]
\[ [J_a, x_b] = i\hbar f_{ab}^c x_c, \quad (18) \]
\[ [p^c, J_a] = i\hbar f_{ab}^c p^b, \quad (19) \]
\[ [J_a, J_b] = i\hbar f_{ab}^c J_c, \quad (20) \]

where \( x_a \)'s and \( p^b \)'s are the coordinate and momentum operators, respectively.

3. The partition function

The Hamiltonian corresponding to free particles in a space of infinite volume is a function of only momenta. Denoting this by \( H \), one can define another Hamiltonian corresponding to free particles in a space of finite volume. In the case of Lie-algebra-type noncommutative spaces, finiteness of the volume of the space can be implemented by restricting the representations of the coordinate operators. So, corresponding to any operator \( Q \) acting on the Hilbert space corresponding to the space of infinite volume, one constructs the operator \( Q_V \) acting on the Hilbert space corresponding to a space of volume \( V \) through

\[ Q_V := \Pi_V Q \Pi_V, \quad (21) \]

where \( \Pi_V \) is the Hermitian projection operator the image of which is the subspace of the Hilbert space corresponding to the desired representations. The aim is to find the partition function in the thermodynamic limit that the volume of the space becomes infinite (so that \( \Pi_V \) tends to \( I \)).

Denoting the grand canonical partition function of a system of free particles in a volume \( V \) by \( Z(V) \), one has

\[ \ln Z(V) = -\frac{1}{s} \text{tr}[\ln[1 - s \mathcal{Z} \exp(-\beta H_V)]], \quad (22) \]

where \( H \) is the one particle Hamiltonian, \( z \) is the fugacity, related to the temperature and the chemical potential \( \mu \) through

\[ z := \exp \left( \frac{\mu}{k_B T} \right), \quad (23) \]

and

\[ \beta := \frac{1}{k_B T}. \quad (24) \]

Here, \( k_B \) is Boltzmann’s constant, and \( T \) is the absolute temperature. Bosons, fermions and the fictitious classical particles (classons) correspond to the \( s = +1 \), \( s = -1 \) and the limit \( s \to 0 \), respectively. Also note that any identity operator on the right-hand side of (22) is the identity operator of the restricted Hilbert space.

To calculate the right-hand side of (22) in the thermodynamic limit, one notes that

\[ \text{tr}(H_V)^j = \int dU \langle U | \Pi_V (\Pi_V H \Pi_V)^j \Pi_V | U \rangle, \]

where

\[ \langle U_{j=1} | \Pi_V H | U_j \rangle \langle U_j | \Pi_V | U_{j-1} \rangle \cdots \]

\[ \times \langle U_1 | \Pi_V | U \rangle, \]

\[ \int dU \langle U | \Pi_V U_j | U_j | \Pi_V | U \rangle \cdots \]

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\[ \int dU \langle U | \Pi_V U_j | U_j | \Pi_V | U \rangle \cdots \]

\[ \times \langle U_1 | \Pi_V | U \rangle, \]

\[ \beta := \frac{1}{k_B T}. \quad (24) \]
where $E(U)$ is the eigenvalue of $H$ corresponding the eigenvector $|U\rangle$. In the thermodynamic limit the projection $\Pi_V$ tends to the identity; hence its matrix elements tend to the delta distribution. So on the right-hand side of (25), up to the leading order, one can substitute $E(U_i)$ by $E(U)$, arriving at

\[
\text{tr}(H_V) = \int dU dU_1 \cdots dU_j \langle U | \Pi_V | U_1 \rangle \langle U_1 | \Pi_V | U_2 \rangle \cdots \times \langle U_{j-1} | \Pi_V | U_j \rangle \langle U_j | \Pi_V | U \rangle |E(U)\rangle^j,
\]

\[
= \int dU \langle U | \Pi_V | U \rangle |E(U)\rangle^j. \tag{26}
\]

One has

\[
|U(k)\rangle = \exp[\hat{k}^a x_a/(i \ell)] |U(0\rangle, \tag{27}
\]

(where $\hat{k}^a$'s are numbers not operators). Assuming that the representations kept corresponding to the finite-volume space are determined by only the value of their corresponding Casimirs, it is seen that $\Pi_V$ is a function of only Casimirs. So it turns out that $\Pi_V$ commutes with the coordinate operators, from which one arrives at

\[
\langle U | \Pi_V | U \rangle = \langle U(0) | \Pi_V | U(0)\rangle. \tag{28}
\]

So

\[
\text{tr}(H_V) = \langle U(0) | \Pi_V | U(0)\rangle \int dU |E(U)\rangle^j. \tag{29}
\]

Using this, one arrives at

\[
\frac{1}{\langle U(0) | \Pi_V | U(0)\rangle} \ln Z(V) = -\frac{1}{s} \int dU \ln[1 - sz \exp[-\beta E(U)]] \tag{30}
\]

From (29) it is seen that the product of $\langle U(0) | \Pi_V | U(0)\rangle$ and the Haar measure is independent of the normalization choice for the Haar measure, so that (30) is in fact independent of the normalization choice for the Haar measure. From now on, the normalization of the Haar measure is chosen so that

\[
\lim_{\hat{k} \to 0} \left[ \frac{d^D \hat{k}}{(2\pi \ell)^D dU} \right] = 1, \tag{31}
\]

which is equivalent to

\[
\lim_{\hat{k} \to 0} \left[ \frac{d^D \hat{p}}{(2\pi \hbar)^D dU} \right] = 1, \tag{32}
\]

where $D$ is the dimension of the group, and

\[
\hat{k} := \sqrt{\delta_{ab} \hat{k}^a \hat{k}^b}. \tag{33}
\]

In the commutative limit ($\ell \to 0$), the limiting cases of (31) or (32) always apply, and the denominator on the left-hand side of (30) is the volume of the system. So one defines the volume of the noncommutative system as

\[
V := \langle U(0) | \Pi_V | U(0)\rangle. \tag{34}
\]

One can explicitly check the meaning of this definition for the groups $SU(2)$ and $SO(3)$. Suppose the spin of largest representation which is kept is $J$. Keeping in mind that the
representation with spin \( j \) (not greater than \( J \)) has dimension \((2j + 1)^2\) and appears \((2j + 1)\) times, it is seen that
\[
\text{tr } \Pi_V = \sum_{j=0}^{J} (2j + 1)^2, \tag{35}
\]
which (up to leading order) results in
\[
\text{tr } \Pi_V = \begin{cases} 
\frac{8}{3} J^3, & SU(2) \\
\frac{4}{3} J^3, & SO(3).
\end{cases} \tag{36}
\]
The difference between these two groups is that for \( SU(2) \) only \((2j)\) should be an integer, while for \( SO(3) \) the value of \( j \) itself should be an integer. For these groups, the Haar measure reads
\[
dU = \frac{4}{(2\pi \ell)^3} \sin^2 \frac{k}{2} d\Omega, \tag{37}
\]
where \( \Omega \) is the angular part of the spherical coordinates of \( \mathbf{k} \), and
\[
\begin{align*}
0 \leq k &\leq 2\pi, & SU(2) \\
0 \leq k &\leq \pi, & SO(3),
\end{align*} \tag{38}
\]
so that
\[
\int dU = \frac{4}{(2\pi \ell)^3} \begin{cases} 
16\pi^2, & SU(2) \\
8\pi^2, & SO(3).
\end{cases} \tag{39}
\]
Using (29) with \( j = 0 \), one arrives at
\[
\langle U(0) | \Pi_V | U(0) \rangle = \frac{4\pi}{3} (J \ell)^3, \tag{40}
\]
which is the volume of a sphere of radius \((J \ell)\). One also notes that the largest eigenvalue of \( \mathbf{x} \cdot \mathbf{x} \) is equal to \( J(J + 1)\ell^2 \), which is (to the leading order) the square of the same radius \((J \ell)\).

4. Thermodynamic quantities

Starting from the grand canonical partition function (30), one can easily obtain the number density and the internal energy density in a manner similar to the commutative case:
\[
\frac{\mathcal{N}}{V} = \frac{\partial}{\partial z} \left( \frac{\ln Z}{V} \right),
= \int dU \frac{z \exp[-\beta E(U)]}{1 - sz \exp[-\beta E(U)]}, \tag{41}
\]
and
\[
\frac{\mathcal{E}}{V} = -\frac{\partial}{\partial \beta} \left( \frac{\ln Z}{V} \right),
= \int dU \frac{z \exp[-\beta E(U)]}{1 - sz \exp[-\beta E(U)]} E(U), \tag{42}
\]
where \( \mathcal{N} \) and \( \mathcal{E} \) are the expectation values of the number of particles and the energy of the system, respectively. One can also define a number density in the phase space like
\[
n(U) := \frac{z \exp[-\beta E(U)]}{1 - sz \exp[-\beta E(U)]}, \tag{43}
\]
so that
\[ \frac{N}{V} = \int dU \, n(U), \]
\[ \frac{\mathcal{E}}{V} = \int dU \, n(U) E(U). \]  
(44)

These look exactly similar to the corresponding expressions in the commutative case, apart from the difference in the integration measure and the functions involved. Regarding the pressure, however, there arises a new concept. The point is that the way to change the volume of the system is not unique. One can change the noncommutativity length \( \ell \), or the largest representation involved. In the commutative limit \( (\ell \to 0) \), the energy function can be written so that it depends only on \( p \), or the combination \( (\hat{k}/\ell) \). The same is true for the integration measure and the integration region (which is infinite) on the right-hand side of (30). So the dependence of the grand canonical partition function on the representation and \( \ell \) is only through the volume \( V \), and as \( V \) is proportional to \( \ell D \), it is seen that
\[ \ln Z_V = \ell D V \frac{\partial}{\partial (\ell D)}. \]  
(45)

One then defines pressure as the left- or the right-hand side of the above equation, times \( (k_B T) \). In the noncommutative case, however, (45) does not hold, and one is faced with two different possible definitions for the pressure. The first, based on the left-hand side of (45), is
\[ P_1 := k_B T \frac{\ln Z}{V}. \]  
(46)

This is the pressure felt by something trying to move the boundaries of the system. One has
\[ P_1 = -\frac{k_B T}{s} \int dU \ln[1 - s z \exp\{-\beta E(U)]}. \]  
(47)

To obtain the form of the second definition of pressure, based on the right-hand side of (45), one notes that the product \( (V dU) \) does not involve \( \ell \). Hence,
\[ P_2 = -\frac{k_B T \ell D}{s} \int dU \frac{\partial}{\partial (\ell D)} \frac{\ln[1 - s z \exp\{-\beta E(U)]}{}, \]  
(48)

which results in
\[ P_2 = \int dU n(U) \left[ -\frac{\ell}{D} \frac{\partial E(U)}{\partial \ell} \right]. \]  
(49)

This is very similar to what obtained in the commutative limit, when one notes that
\[ -\ell \frac{\partial}{\partial \ell} = p \frac{\partial}{\partial p} \bigg|_{\hat{k}}, \]  
(50)

where \( \hat{k} \) in the superscript means differentiation with \( \hat{k} \) kept fixed.

5. Asymptotic behaviors

Let us consider a compact group. For such a group the energy function is bounded. The minimum of energy is taken to be zero, by convention. The maximum of energy \( (E_{\text{max}}) \) is decreasing in \( \ell \), and tends to infinity as \( \ell \) tends to zero. Another length parameter is the so-called thermal wavelength \( (\lambda) \), which is proportional to the quantum (but commutative)
wavelength of a particle of energy $k_B T$. Finally, there is a length parameter associated with the density of particles:

$$\sigma := \left( \frac{V}{N} \right)^{1/D}.$$  \hfill (51)

It is seen that of these three length parameters, $\ell$ is fixed, $\lambda$ is a decreasing function of the temperature and $\sigma$ is a decreasing function of the density.

5.1. High temperature limit

In this case,

$$\lambda \ll \sigma, \ell,$$  \hfill (52)

from which

$$\beta E_{\text{max}} \ll 1.$$  \hfill (53)

Putting this in the expressions for the number density, energy density and pressures, one arrives at

$$n(U) = \frac{z}{1 - sz},$$  \hfill (54)

$$\frac{N}{V} = \frac{z}{1 - sz} \int dU,$$

$$= \frac{z}{1 - sz} \text{vol}(G),$$  \hfill (55)

$$\frac{E}{V} = \frac{z}{1 - sz} \int dU E(U),$$

$$= \frac{N}{V} \langle E \rangle,$$  \hfill (56)

$$P_1 = -\frac{k_B T}{s} \ln(1 - sz) \text{vol}(G),$$  \hfill (57)

$$P_2 = \frac{N}{V} \left\{ \frac{\ell}{D} \frac{\partial E}{\partial \ell} \right\}. $$  \hfill (58)

It is seen that at this limit everything is temperature independent, apart from $P_1$ which diverges linearly (in temperature). It is also seen that the quantum behavior of the system (which is manifested in the value of $s$) is almost not important (again apart from $P_1$). One can define an effective fugacity $z_{\text{eff}}$ as

$$z_{\text{eff}} := \frac{z}{1 - sz}.$$  \hfill (59)

to see that $s$ is actually eliminated from the expressions. By almost it is meant that for the case of fermions, the density should not be greater than a critical limit:

$$\sigma_{\text{cr}}^{-D} := \text{vol } G.$$  \hfill (60)

In fact this condition is not restricted to high temperatures.
5.2. Low temperature limit

In this case,
\[ \lambda \gg \sigma, \ell, \]
from which
\[ \beta E_{\text{max}} \gg 1. \]

Here in all of the integrals involved in calculating the partition function and thermodynamic quantities, only small values of \( \hat{k} \) have significant contributions. So one can use the asymptotic forms of the integration measure based on (32), and also the commutative form of the energy function. This means that the effects of noncommutativity disappear in this limit. The quantum effects, however, are very strong. In fact in this limit one encounters a highly degenerate but commutative gas.

5.3. Moderate temperatures

If \( \sigma \) and \( \ell \) are much different, there is a region for temperature where \( \lambda \) is in between these two length scales and much different from these. There arise two cases.

5.3.1. Classical commutative behavior. In this case,
\[ \ell \ll \lambda \ll \sigma. \]

This case is, of course, possible only if
\[ \ell \ll \sigma. \]

Here (62) is satisfied so that one can eliminate the noncommutative parameter. For the effectively commutative system resulted, as the thermal wavelength is much smaller than the particle spacing, one has a nondegenerate (classical) gas.

5.3.2. Quantum noncommutative behavior. In this case,
\[ \sigma \ll \lambda \ll \ell. \]

This case is, of course, possible only if
\[ \sigma \ll \ell. \]

Here (53) holds, and it is seen that both quantum and noncommutative behaviors are pronounced. Note, however, that (66) and hence (65) cannot be satisfied for fermions, as for fermions (60) shows that
\[ \sigma > 2\pi \ell [\text{vol}_N(G)]^{-1/D}, \]
where
\[ \text{vol}_N(G) := (2\pi \ell)^D \text{ vol } (G), \]
and the left-hand side of (68) (the dimensionless volume of the group) is of the order of unit.
5.4. Degenerate gases

In this case,
\[ \sigma \ll \lambda. \]  
\(\text{(69)}\)

For bosons, the occurrence of Bose–Einstein condensation is similar to the commutative case. It depends on whether the right-hand side of (41) diverges for \( z = 1 \) or not, and this is determined by only the low-momentum behavior of the energy function and the integration measure. None of these depend on the noncommutative parameter \( \ell \). So exactly as it was in the commutative case, the condition for the occurrence of Bose–Einstein condensation is
\[ \lim_{p \to 0} \left( p \frac{\partial E}{\partial p} \right) < D. \]  
\(\text{(70)}\)

The left-hand side is 1 for a relativistic gas and 2 for a nonrelativistic gas. So one arrives at the familiar commutative result that there is a Bose–Einstein condensation (for a nonrelativistic gas) iff the dimension of the space is more than 2. The coexistence curve, however, does depend on the noncommutative parameter \( \ell \).

For fermions, it is seen from (41) that the fugacity diverges when the density approaches the critical density. One has
\[ \sigma^{-D} \sigma^{-D} = \int dU \frac{1}{1 + z \exp[-\beta E(U)]}. \]  
\(\text{(71)}\)

which results (up to leading order) in
\[ z = \left[ (\sigma^{-D} - \sigma^{-D}) \right] \left\{ \int dU \exp[\beta E(U)] \right\}^{-1}, \]  
\(\text{(72)}\)

showing that \( z \) diverges like \((\sigma - \sigma_c)^{-1}\). It is also seen that as the density approaches the critical density, \( P_1 \) diverges:
\[ P_1 = k_B T [\text{vol}(G)] \ln z \]  
\(\text{(73)}\)

(showing that \( P_1 \) diverges logarithmically), while \( P_2 \) tends to a finite value:
\[ P_2 = \int dU \left[ -\ell \frac{\partial E(U)}{\partial \ell} \right]. \]  
\(\text{(74)}\)

5.5. Nondegenerate gases

Here,
\[ \sigma \gg \lambda, \]  
\(\text{(75)}\)

so that the quantum behavior is not important. One then has
\[ \ln \frac{Z}{V} = z \int dU \exp[-\beta E(U)]. \]  
\(\text{(76)}\)

The noncommutative behavior manifests itself at high temperatures, where \( \lambda \) becomes comparable to (or less than) \( \ell \).
6. The group SU(2), and nondegenerate gases

As an example, let us obtain a closed form for function (76) for the groups SU(2) and SO(3). To do so, one uses (37) and (38), and needs the form of $E$. Examples are [18–22]

$$
E = \begin{cases} 
\frac{4\hbar^2}{\ell^2 m} \left(1 - \cos \frac{\hat{k}}{2}\right), & SU(2) \\
\frac{\hbar^2}{\ell^2 m} \left(1 - \cos \hat{k}\right), & SO(3).
\end{cases}
$$

(77)

One then arrives at

$$
\ln Z_{SU(2)} = \frac{2V}{\ell \lambda^2} \exp \left(-\frac{2\lambda^2}{\pi \ell^2}\right) I_1 \left(\frac{2\lambda^2}{\pi \ell^2}\right),
$$

(78)

and

$$
\ln Z_{SO(3)} = \frac{V}{\pi \ell^3} \exp \left(-\frac{\lambda^2}{2\pi \ell^2}\right) \left[ I_0 \left(\frac{\lambda^2}{2\pi \ell^2}\right) - I_1 \left(\frac{\lambda^2}{2\pi \ell^2}\right) \right],
$$

(79)

where $I_n$ is the modified Bessel function of order $n$.

7. Conclusion

Effects of noncommutativity on thermodynamic properties were explored in spaces with commutation relations of a Lie algebra. In particular, the case of a Lie algebra corresponding to a compact Lie group was investigated. In such cases the volume of the corresponding momentum space is finite. A finite volume for the configuration space was introduced in terms of a Hermitian projection the range of which covers the spaces of only certain representations in the regular representation. The grandcanonical partition function of a system of identical free particles was then expressed in terms of the noncommutativity parameter, the fugacity and the temperature. Regarding the concept of pressure, it turned out that two ways are possible to define the pressure, as there are two ways to change the volume of the system, either change the noncommutativity length parameter, or change the largest representation entering the truncated (finite volume) system. While these give identical results in the commutative limit, that is not the case for the noncommutative spaces. Different asymptotic behaviors of physical quantities were explored. It was seen that there are three length scales: the noncommutativity length scale, the thermal wavelength and the mean particle separation length. Of these, the last two are present in the commutative case. Quantum behavior is important when the thermal wavelength is large, and noncommutativity is important when the noncommutativity length scale is large. The effects of temperature and density on these were investigated. Finally, for the special groups SU(2) and SO(3) the partition function was explicitly calculated in the nondegenerate limit (where quantum effects are negligible).

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