We construct a Lax pair with spectral parameter for the elliptic Calogero-Moser Hamiltonian systems associated with each of the finite dimensional Lie algebras, of the classical and of the exceptional type. When the spectral parameter equals one of the three half periods of the elliptic curve, our result for the classical Lie algebras reduces to one of the Lax pairs without spectral parameter that were known previously. These Calogero-Moser systems are invariant under the Weyl group of the associated untwisted affine Lie algebra. For non-simply laced Lie algebras, we introduce new integrable systems, naturally associated with twisted affine Lie algebras, and construct their Lax operators with spectral parameter (except in the case of $G_2$).
I. INTRODUCTION

The recent exact solutions of four-dimensional supersymmetric gauge theories [1] (for a review, see for example [2]) have revealed a deep correspondence between these theories and integrable models [3-8]. While the existence of such a correspondence can now be established on very general grounds [4] (see also [9]), the fundamental nature of the correspondence itself has remained largely elusive. Precise matches between specific gauge theories and integrable models have been identified in special cases, on an ad hoc basis, and it is clearly desirable to have a more systematic classification. Since four-dimensional \( \mathcal{N} = 2 \) supersymmetric gauge theories are labelled by their gauge Lie algebra \( \mathcal{G} \) and the representation \( R \) of their matter hypermultiplets, it is natural to look for integrable models associated with Lie algebras to obtain such a classification.

Integrable models associated to the root system of a Lie algebra \( \mathcal{G} \) include the Toda and affine Toda systems [10]

\[
\ddot{X} = -\frac{1}{2} \sum_{\alpha \in \mathcal{R}^*(\mathcal{G})} M_{|\alpha|}^2 e^{-\alpha \cdot X} \alpha,
\]

\[
\dddot{X} = -\frac{1}{2} \sum_{\alpha \in \mathcal{R}^*(\mathcal{G}^{(1)})} M_{|\alpha|}^2 e^{-\alpha \cdot X} \alpha,
\]

the elliptic, trigonometric and rational Calogero-Moser systems [11,12]

\[
\ddot{x} = \frac{1}{2} \sum_{\alpha \in \mathcal{R}(\mathcal{G})} m_{|\alpha|}^2 \varphi'(\alpha \cdot x) \alpha,
\]

\[
\ddot{x} = -\sum_{\alpha \in \mathcal{R}(\mathcal{G})} m_{|\alpha|}^2 \frac{\cosh(\alpha \cdot x)}{\sinh^3(\alpha \cdot x)} \alpha,
\]

\[
\ddot{x} = -\sum_{\alpha \in \mathcal{R}(\mathcal{G})} m_{|\alpha|}^2 (\alpha \cdot x)^{-3} \alpha,
\]

and the Hitchin systems [13]. Here \( \mathcal{R}(\mathcal{G}) \) and \( \mathcal{R}^*(\mathcal{G}) \) denote respectively the set of roots and the set of simple roots of \( \mathcal{G} \), and \( \mathcal{G}^{(1)} \) is the untwisted affine Lie algebra associated with \( \mathcal{G} \). For general reviews, see [14]. Very early on, the spectral curves for the pure \( SU(2) \) Yang-Mills theory and for the affine \( SU(2) \) Toda system were recognized as identical [3]. Since then, many more correspondences have been established. Of particular interest are the one between pure Yang-Mills theories with gauge algebra \( \mathcal{G} \) and affine Toda systems for \( (\mathcal{G}^{(1)})^\vee \) [5], and the one between the \( SU(N) \) Yang-Mills theory with matter in the adjoint representation and the \( SU(N) \) Hitchin system [4], (or equivalently, the elliptic \( SU(N) \)
Calogero-Moser system [8]). The advantage of an explicit correspondence with integrable models is much in evidence throughout [8], where an exact renormalization group type relation is derived for the effective prepotential function of the gauge theory in terms of the Calogero-Moser Hamiltonian.

The purpose of the present paper and the two companion papers [15,16] is to identify/construct the integrable models corresponding to $\mathcal{N} = 2$ supersymmetric Yang-Mills with gauge algebra $\mathcal{G}$ and matter in the adjoint representation of $\mathcal{G}$, where $\mathcal{G}$ is an arbitrary simple Lie algebra. In analogy with the $SU(N)$ case, a natural candidate for the integrable model corresponding to the gauge theory with gauge algebra $\mathcal{G}$ is the elliptic Calogero-Moser system defined by the root system of $\mathcal{G}$, with the hypermultiplet mass $m$ and the coupling constants $g, \theta$ of the gauge theory corresponding respectively to the coupling parameter $m_{|\alpha|}$ and the modulus parameter

$$\tau = \frac{4\pi i}{g^2} + i \frac{\theta}{2\pi},$$

of the Calogero-Moser system. Our main results are

- This is indeed the case when $\mathcal{G}$ is simply laced, i.e., all roots of $\mathcal{G}$ have equal length, but not otherwise;

- When $\mathcal{G}$ is non-simply laced, i.e. has both long and short roots, denoted respectively by $\mathcal{R}_l(\mathcal{G})$ and $\mathcal{R}_s(\mathcal{G})$, the correct model is given rather by the following *twisted* version of the elliptic Calogero-Moser system

$$\ddot{x} = \frac{1}{2} \left( \sum_{\alpha \in \mathcal{R}_s(\mathcal{G})} m^2_{|\alpha|} \varphi'_\nu(\alpha \cdot x)\alpha + \sum_{\alpha \in \mathcal{R}_l(\mathcal{G})} m^2_{|\alpha|} \varphi'(\alpha \cdot x)\alpha \right), \quad (1.3)$$

where $\nu$ is the ratio of the length squared of the long to the short roots of $\mathcal{G}$, and $\varphi_\nu(z)$ is the twisted version of the Weierstrass $\wp(z)$-function defined by (2.4) below. When $\mathcal{G}$ is one of the classical algebras $B_n$ or $C_n$, the system (1.3) can be re-expressed as one of the systems introduced by Inozemtsev [17-18]. For the exceptional Lie algebras, the twisted Calogero-Moser systems appear not to have been considered before.

- Our considerations are based on two crucial consistency checks, corresponding to the limits $m \to 0$ and $m \to \infty$. When $m$ tends to 0, the $\mathcal{N} = 2$ gauge theory with matter in the adjoint representation acquires an $\mathcal{N} = 4$ supersymmetry. Since this theory receives no quantum corrections, its prepotential is the classical prepotential and we can verify directly that it agrees with the prepotential predicted by the Calogero-Moser systems at
zero coupling. The consistency check when \( m \to \infty \) is more subtle. From the viewpoint of the four-dimensional gauge theory, this limit corresponds to the decoupling of the hyper-multiplet as it becomes infinitely massive. On the basis of instanton considerations, the correct scaling law as \( m \to \infty \) is given by

\[
m = M q^{-\frac{1}{2} \delta^\vee}, \quad q = e^{2\pi i \tau},
\]  

(1.4)

where \( \delta^\vee \) is the dual Coxeter number of \( \mathcal{G} \). The limiting theory is pure Yang-Mills, and the corresponding integrable model is the affine Toda system for \( (\mathcal{G}^{(1)})^\vee \). Now the \( SU(N) \) elliptic Calogero-Moser system is known to scale under (1.4) to the affine \( SU(N) \) Toda system [18]. More generally, we find that the elliptic Calogero-Moser systems always tend to a finite limit under the rule

\[
m = M q^{-\frac{1}{2} \delta}
\]  

(1.5)

where \( \delta \) is the Coxeter number, and that the limit is the affine Toda system defined by \( \mathcal{G}^{(1)} \). When \( \mathcal{G} \) is simply-laced, the Coxeter number and the dual Coxeter numbers are the same, \( \mathcal{G}^{(1)} = (\mathcal{G}^{(1)})^\vee \), and the elliptic Calogero-Moser system satisfies the \( m \to \infty \) consistency check. However, when \( \mathcal{G} \) is not simply laced (i.e. when \( \mathcal{G} = B_n, C_n, F_4, \) or \( G_2 \)), the limit of the \( \mathcal{G} \) elliptic Calogero-Moser system under (1.4) is infinite. Thus new Calogero-Moser systems are required. The twisted systems defined by (1.3) are systems which admit finite limits under (1.4). These limits also turn out to be precisely the desired \( (\mathcal{G}^{(1)})^\vee \) affine Toda systems, and thus the twisted systems qualify as the models solving the \( \mathcal{G} \) gauge theory with matter in the adjoint representation.

- Although we have repeatedly referred to both the twisted and the untwisted \( \mathcal{G} \) Calogero-Moser systems as integrable models, the integrability of these models for general Lie algebras is a complex issue far less understood (see e.g. [31] for a recent discussion) than their Toda and affine Toda counterparts. (We discuss this in greater detail below). Nevertheless, we have succeeded in producing a Lax pair of operators \( L(z), M(z), \) with spectral parameter \( z \), satisfying the Lax equation \( \dot{L}(z) = [L(z), M(z)] \) for each of these models. (For the case of \( E_8 \), we have to make an extra assumption on the existence of a certain \( \pm 1 \)-valued cocycle. For the case of twisted \( G_2 \), we have been able to complete the proof only partially.) It is quite important for our considerations that the Lax pair be allowed to depend on a free external parameter. In particular, this is required for the construction of the corresponding spectral curves \( \Gamma \) and differentials \( d\lambda \)

\[
\Gamma = \{(k, z); \det(kI - L(z)) = 0\}, \quad d\lambda = k dz,
\]  

(1.6)
Taking the Lax pair for the untwisted Calogero-Moser when $G = A_n, D_n, E_6, E_7$, and the Lax pair for the twisted Calogero-Moser when $G = B_n, C_n, F_4$, we obtain in this way the candidate Seiberg-Witten spectral curves and differentials for the corresponding $\mathcal{N} = 2$ $G$ gauge theory with matter in the adjoint representation. In principle, the prepotential $\mathcal{F}(a, \tau)$ can then be evaluated explicitly, as was done in [8] for the $SU(N)$ theory, and in [19][20] for theories with classical gauge groups and matter in the fundamental representation. (A sample calculation will be given in [16] for $G = D_n$.)

In this first paper of the series, we shall concentrate on the construction of the Lax pairs with spectral parameter. A second paper [15] will be devoted to a detailed study of scaling limits for twisted and untwisted Calogero-Moser systems. We have already described above the outcome for the limits of the Hamiltonians of these systems. It turns out that the Lax pairs we construct in this paper also scale to appropriate finite limits, and produce a Lax pair for the affine Toda systems. We discuss the spectral curves themselves and the resulting physics of $\mathcal{N} = 2$ supersymmetric gauge theories in a third paper [16]. Finally, more severe divergences for Calogero-Moser systems than (1.4) and (1.5) will be discussed in a fourth paper, together with certain decoupling limits in the spirit of the results of [8].

We return to a detailed description of the main topic of the present paper, namely the construction of Lax pairs with spectral parameters for Calogero-Moser systems. For affine Toda systems, a very general prescription is available, with $L(Z)$ and $M(Z)$ given by [10][5]

$$L = \sum_{i=1}^{n} P_i h_i + \sum_{\alpha \in \mathcal{R}_+(G)} M|_{\alpha} e^{-\frac{i}{2} \alpha \cdot X}(E_{\alpha} - E_{-\alpha}) + M|_{\alpha_0} e^{+\frac{i}{2} \alpha_0 \cdot X}(-Z^{-1}E_{\alpha_0} + ZE_{-\alpha_0})$$

$$M = -\frac{1}{2} \sum_{\alpha \in \mathcal{R}_+(G)} M|_{\alpha} e^{-\frac{i}{2} \alpha \cdot X}(E_{\alpha} + E_{-\alpha}) + \frac{1}{2} M|_{\alpha_0} e^{+\frac{i}{2} \alpha_0 \cdot X}(Z^{-1}E_{\alpha_0} + ZE_{-\alpha_0}).$$

(1.7)

Here $\alpha_0$ is the highest root of $G$ (so that $-\alpha_0$ is the additional simple root for the affine algebra $G^{(1)}$ [Kac [21], Goddard-Olive [22]]), $Z$ is the spectral parameter, identifiable in this case with the loop variable of the loop group, and $E_{\alpha}$ are generators for $G$ in a Cartan-Weyl basis. The operators $L(Z), M(Z)$ become $N \times N$ matrices $\rho(L(Z)), \rho(M(Z))$ upon choosing a $N$-dimensional representation $\rho$ of $G$, and a ($\rho$-dependent) spectral curve $\Gamma$ can then be defined by

$$\Gamma = \{(k, Z); \det \left( kI - \rho(L(Z)) \right) = 0 \}.$$
Different curves corresponding to different choices of representations $\rho$ are related by syzygies, and it has been advocated in [5] that the corresponding prepotentials, and hence physics, should be the same. At the present time, there is no general formula of type (1.7) for Calogero-Moser systems. In fact, only in the $SU(N)$ case was a Lax pair $L(z), M(z)$ with spectral parameter known (this is a classical result going back to Krichever [23]). For the classical finite dimensional (simple) Lie algebras $G$ other than $A_{N-1}$, only a Lax pair without spectral parameter appears to be known. Its construction was due to Olshanetsky and Perelomov [14], and it was formulated in terms of the geometry and group theory of symmetric spaces. For the exceptional Lie algebras $E_8$, $E_7$, $E_6$, $F_4$ and $G_2$, no Lax pair appears to be known at all. These statements refer to the untwisted Calogero-Moser systems. The twisted Calogero-Moser systems had of course not even been considered.

Our approach is based on a general Ansatz for Lax operators $L(z)$ and $M(z)$ with spectral parameter, which is applicable to any finite-dimensional Lie algebra in an arbitrary representation. This Ansatz is a natural extension of the original Ansatze of Calogero-Moser, Krichever, and Olshanetsky-Perelomov. The conditions that the proposed Lax operators close onto the Calogero-Moser system for $G$ are reduced to purely algebraic equations. Nevertheless these algebraic equations can be quite difficult to solve, and they depend in an intricate manner on the choice of representation.

We show that these equations can be solved in the following cases. For the untwisted Calogero-Moser systems,

- in the case of $BC_n$, $B_n$, $C_n$, and $D_n$, by imbedding $B_n$ into $GL(2n+1, \mathbb{C})$;
- in the case of $A_n$, by taking the fundamental and the anti-symmetric rank $p$, $1 \leq p \leq n-1$, tensor representations;
- in the case of $B_n$ and $D_n$, by taking the spinor representation;
- in the case of $G_2$, by taking the 7 of $G_2$;
- in the case of $F_4$, by taking the 26 + 1 of $F_4$;
- in the case of $E_6$, by taking the 27 of $E_6$;
- in the case of $E_7$, by taking the 56 of $E_7$.
- in the case of $E_8$, by taking the 248 representation, we obtain a solution for the Lax pair upon making an extra assumption on the existence of a $\pm1$-values cocycle. Thus we expect our Ansatz will also produce a Lax pair, but we have no full proof at this time.

The dimension of the Lax pair is then the dimension of the imbedding. For the twisted Calogero-Moser systems, the equations can be solved to produce
• in the case of $B_n$, a $2n$-dimensional Lax pair;
• in the case of $C_n$, a $2n + 2$-dimensional Lax pair;
• in the case of $F_4$, a 24-dimensional Lax pair.
• in the case of $G_2$, the Ansatz for a 6-dimensional Lax pair seems consistent, but we have not yet found any full proof of it.

In the case of the classical Lie algebras $BC_n, D_n$, when the spectral parameter $z$ equals one of the three half periods of the torus $\Sigma = C/(2\omega_1 \mathbb{Z} + 2\omega_2 \mathbb{Z})$, our Lax pairs coincide with the Lax pairs without spectral parameter which were formulated previously for these groups by Olshanetsky and Perelomov [14]. Thus these previously known solutions have been in effect imbedded in a whole family of solutions. As mentioned previously, the Olshanetsky-Perelomov admit a symmetric space interpretation which depends heavily on the fact that the crucial function $\Phi(x, z)$ introduced by Krichever [23] becomes odd in $x$. It may be worth noting that a similar geometric interpretation of our Lax pair for general values of the spectral parameter $z$ is still possible, upon replacing the naive reflection symmetries $x \leftrightarrow -x, z \leftrightarrow -z$, by a twisted version (c.f. (4.17) below).

We also observe that the dimensions of the Lax pairs in the twisted $\mathcal{G}$ Calogero-Moser case may not even be dimensions of representations of $\mathcal{G}$. This is one of the difficulties in finding a systematic representation theoretic construction of the particular Lax pairs we found, say a construction analogous to (1.7) for affine Toda systems. It is also remarkable that the Lax pair in the spinor of $SO(2n + 1)$ has two free couplings, while the Lax pair in the fundamental of this group only had one. In practice, to find the Lax pairs, we are forced at this moment to proceed case by case, and it would certainly be very valuable to have a more general or more conceptual approach.

The remainder of the paper is organized as follows. The basic material is introduced in §II. This includes the description of both the twisted and untwisted elliptic Calogero-Moser Hamiltonians, their symmetries, and the explicit form of the Lax pair in the case of $A_n$.

In §III, we propose a general construction for the Lax operators $L(z)$ and $M(z)$, for any Lie algebra in an arbitrary representation. The conditions under which the Lax equation is the Hamilton-Jacobi equation of a Calogero-Moser system are reduced to a set of purely algebraic equations, the solutions of which are the central issue of the paper.

In §IV, we present the solutions, and thus the Lax pairs with spectral parameter for the classical Lie algebra series $B_n, C_n$ and $D_n$, corresponding to the complexified Lie algebras of $SO(2n + 1), Sp(2n)$, and $SO(2n)$. In some cases, solutions in several representations
are possible, as listed above. By introducing twisted actions and twisted Lie algebras, we show how our formulas can be viewed as natural extensions of the Olshanetsky-Perelomov formulas when \( z \) is at a half period.

In §V, we apply our construction to the case of the exceptional Lie algebras. For the cases of \( E_6 \) and \( E_7 \) a Lax pair with spectral parameter is constructed in the representations of dimension \( 27 \) and \( 56 \) respectively, each with a single free coupling. For the case of \( G_2 \), a Lax pair is constructed in the representations of dimensions \( 7 \) (the fundamental) and \( 8 \) (the fundamental plus a singlet), each with a spectral parameter, and with one and two free couplings respectively. For the case of \( F_4 \), no Lax pair appears to exist in the representation of dimension \( 26 \) (the fundamental), but we find a Lax pair with spectral parameter and two couplings in the representation of dimension \( 27 \) (the fundamental plus a singlet). For \( E_8 \), we summarize the key features of the \( 248 \) Lax pair we expect, leaving the detailed verification of some consistency checks to the appendices.

Finally, the twisted Calogero-Moser systems are treated in §VI. An important new feature of the Lax pairs is the emergence of new twisted versions of the function \( \Phi(x, z) \), notably the functions \( \Lambda(x, z) \), \( \Phi_2(x, z) \), and \( \Phi_2(x \pm \omega_2, z) \). These functions satisfy striking functional equations generalizing Landen’s doubling identities [24], without which the algebraic equations characterizing the Lax pair would be intractable. It is likely that similar functions should exist which satisfy the tripling identities needed to solve the \( G_2 \) case, but we have not succeeded in constructing them, and this case remains open.

In Appendix §A, we have collected for the convenience of the reader all the necessary group theoretical information, including weight systems and the Weyl orbit decompositions of various representations which we need for the analysis of exceptional Lie algebras. In Appendix §B, we derive all the identities for elliptic functions we need, including the doubling identities we mentioned above. In Appendix §C, we present the details of the consistency checks for \( E_8 \).

We conclude the introduction by pointing out that the Seiberg-Witten curves and differential may be constructed by string theory techniques. One method is by exploiting the appearance of enhanced gauge symmetries (of the A-D-E type) at certain singular compactifications. (See for example [26].) A second method is by obtaining supersymmetric Yang-Mills theory as an effective theory on a configuration of branes in string theory or in M-theory. This approach was pioneered in [27] (see also [28]) for SU(\( N \)) gauge group (and products thereof), and was extended to other classical groups in [29]. The relation between the string theory and M-theory approaches and integrable systems were proposed in [27] and [30]. Their interplay should produce substantial advances in both fields.
II. THE ELLIPTIC CALOGERO-MOSER SYSTEMS

The elliptic Calogero-Moser systems are integrable Hamiltonian models with $n$ dynamical degrees of freedom $x_i$, $i = 1, \ldots, n$ and associated canonical momenta $p_i$, $i = 1, \ldots, n$, which are complex valued, and denoted simply by vectors $x$ and $p$ respectively. They are parametrized by the periods $2\omega_1$ and $2\omega_2$ of an elliptic curve (or torus) $\Sigma$, whose modulus is $\tau = \omega_2/\omega_1$. Each system is naturally associated with a finite dimensional simple Lie algebra $G$ of rank $n$, whose set of roots is denoted by $\mathcal{R}(G)$. We are led to distinguish between two types of elliptic Calogero-Moser systems. The ordinary or untwisted elliptic Calogero-Moser system was introduced long ago in [14] for all simple Lie algebras $G$. The twisted Calogero-Moser system will be introduced in (b) below for all simple Lie algebras $G$. The twisted system coincides with the untwisted system when $G$ is simply laced (i.e. all roots of $G$ have the same length), but differs from it when $G$ is non-simply laced.

(a) The Untwisted Elliptic Calogero-Moser Systems

The untwisted elliptic Calogero-Moser system associated with a simple Lie algebra $G$ and with periods $2\omega_1$, $2\omega_2$ is defined by the Hamiltonian

$$H = \frac{1}{2} p \cdot p - \frac{1}{2} \sum_{\alpha \in \mathcal{R}(G)} m_{|\alpha|}^2 \wp(\alpha \cdot x; 2\omega_1, 2\omega_2). \quad (2.1)$$

Here, $\wp(z; 2\omega_1, 2\omega_2)$ is the Weierstrass elliptic function of $\Sigma$, whose definition and properties may be found in Appendix §B. Henceforth we shall suppress the $2\omega_1$ and $2\omega_2$ dependence of $\wp$. The inner product of vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$, such as $\alpha$ and $x$, is denoted by $\alpha \cdot x$. The Hamiltonian is required to be invariant under the Weyl group of the Lie algebra $G$, so that the constants $m_{|\alpha|}$ only depend upon the Weyl orbit, denoted by $|\alpha|$, of the root $\alpha$. When the orbits are uniquely labeled by the length of the roots, we set simply $|\alpha| = \alpha^2$.

The Hamiltonian of (2.1) is invariant under a large discrete symmetry group, generated by the following transformations on the periods $2\omega_1$, $2\omega_2$ and on $x$ and $p$.

(1) The Weyl group $W_G$ of the finite dimensional Lie algebra $G$, leaving $\omega_{1,2}$ unchanged, and acting on $x$ and $p$ by Weyl reflections $W_{\alpha}$

$$x \rightarrow W_{\alpha}(x) = x - 2\alpha \frac{x \cdot \alpha}{\alpha^2}, \quad p \rightarrow W_{\alpha}(p), \quad \alpha \in \mathcal{R}(G). \quad (2.2a)$$

(2) The modular group $SL(2, \mathbb{Z})$, leaving $x$ and $p$ unchanged and acting on $\omega_{1,2}$ by

$$\omega_1 \rightarrow a\omega_1 + b\omega_2, \quad \omega_2 \rightarrow c\omega_1 + d\omega_2, \quad a, b, c, d \in \mathbb{Z}; \quad ad - bc = 1. \quad (2.2b)$$
(3) Affine transformations \( \mathbb{Z}^{2n} \) leaving \( \omega_1, \omega_2 \) and \( p \) unchanged while shifting \( x \) by

\[
x \rightarrow x + \sum_{i=1}^{n} (2\omega_1 n_1^i + 2\omega_2 n_2^i) \lambda^i, \quad n_{1,2}^i \in \mathbb{Z},
\]

where \( \lambda^i \) are a set of generators of the dual lattice to \( \mathcal{R}(\mathcal{G}) \). (If \( \alpha^i \) is a set of simple roots of \( \mathcal{R}(\mathcal{G}) \), then \( \lambda^i \) may be defined by \( \alpha^i \cdot \lambda^j = \delta^{ij}, i, j = 1, \ldots, n. \))

The combined set of transformations (2.2a,b,c) contains the action of the Weyl group \( \hat{W}_\mathcal{G} \) of the affine extension \( \hat{\mathcal{G}} \) of \( \mathcal{G} \). This affine extension is untwisted, and will be denoted by \( \mathcal{G}^{(1)} \), following standard notation. Thus, it is natural to associate the Calogero-Moser system not just with a finite dimensional Lie algebra \( \mathcal{G} \), but instead with its full affine extension \( \mathcal{G}^{(1)} \). The automatic appearance of the affine Weyl group in Calogero-Moser systems seems to have gone unnoticed so far.

**(b) The Twisted Elliptic Calogero-Moser Systems**

The twisted elliptic Calogero-Moser system associated with a finite-dimensional simple Lie algebra \( \mathcal{G} \) and with periods \( 2\omega_1, 2\omega_2 \) is defined by the Hamiltonian

\[
H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 - \frac{1}{2} \sum_{\alpha \in \mathcal{R}(\mathcal{G})} m_{|\alpha|}^2 \varphi_{\nu(\alpha)}(\alpha \cdot x).
\]

The function \( \nu(\alpha) \) depends upon the length of the root \( \alpha \) only. For any simply laced \( \mathcal{G} \), we set \( \nu(\alpha) = 1 \) on all roots. For non-simply laced \( \mathcal{G} \), roots of only two different lengths appear: long roots and short roots. We set \( \nu(\alpha) = 1 \) for all long roots, \( \nu(\alpha) = 2 \) for the short roots of \( B_n, C_n \) and \( F_4 \), and \( \nu(\alpha) = 3 \) for the short roots of \( G_2 \). \( \varphi_\nu(z) \) is a twisted Weierstrass function

\[
\varphi_\nu(z) = \sum_{\sigma=0}^{\nu-1} \varphi(z + 2\omega_a \sigma \frac{\sigma}{\nu}).
\]

Here, \( \omega_a \) is any one of the half periods \( \omega_1, \omega_2 \) or \( \omega_3 = \omega_1 + \omega_2 \). Since we have \( \varphi_1(z) = \varphi(z) \), the twisted system coincides with the untwisted one for any simply laced \( \mathcal{G} \). It will turn out that the twisted elliptic Calogero-Moser systems for the Lie algebra \( \mathcal{G} \) is naturally associated with the affine Lie algebra \( (\mathcal{G}^{(1)})^\vee \). When \( \mathcal{G} \) is simply laced, we have \( (\mathcal{G}^{(1)})^\vee = \mathcal{G}^{(1)} \), and we recover the untwisted elliptic Calogero-Moser system. When \( \mathcal{G} \) is non-simply laced, \( (\mathcal{G}^{(1)})^\vee \) equals a twisted affine Lie algebra: \( (B_n^{(1)})^\vee = A_{2n-1}^{(2)}, (C_n^{(1)})^\vee = D_{n+1}^{(2)}, (F_4^{(1)})^\vee = E_6^{(2)} \) and \( (G_2^{(1)})^\vee = D_4^{(3)} \). †

† Dynkin diagrams of affine Lie algebras, as well as other group theoretic information is collected in Appendix §A; for general sources, see [21,22,25].
(c) Limits of the Elliptic Calogero-Moser Systems

Various limits of the elliptic Calogero-Moser system yield related integrable systems. By taking one of the periods $2\omega_1$ or $2\omega_2$ to infinity, we obtain the trigonometric Calogero-Moser system, with $\varphi(x)$ replaced by $1/\sin^2 x$. By taking both of the periods to infinity, we obtain the rational Calogero-Moser system, with $\varphi(x)$ replaced by $1/x^2$. Finally, a simultaneous limit of the periods and of $x$ yields the Toda integrable system. Specifically, the scaling of the untwisted and twisted elliptic Calogero-Moser systems yields the non-periodic (or ordinary) Toda system associated with the Lie algebras $\mathcal{G}$ and $\mathcal{G}^\vee$ respectively. A certain critical scaling of the untwisted and twisted elliptic Calogero-Moser systems yields the periodic Toda system associated with the affine Lie algebras $\mathcal{G}^{(1)}$ and $(\mathcal{G}^{(1)})^\vee$ respectively. These results are derived in a companion paper [15].

(d) Existence of a Lax Pair with Spectral Parameter

The Hamilton-Jacobi equations are $\dot{x} = p$ and

$$\dot{p} = \frac{1}{2} \sum_{\alpha \in R(\mathcal{G})} m_{|\alpha|}^2 \alpha \varphi'_{\nu}(\alpha)(\alpha \cdot x). \quad (2.5)$$

Here, a dot denotes time derivation, and $\varphi'_\nu(z)$ denotes the derivative with respect to $z$ of $\varphi_\nu(z)$.

Complete integrability of the (twisted or untwisted) elliptic Calogero-Moser system is guaranteed by the existence of a Lax pair of $N \times N$ matrix-valued functions of $x$ and $p$ (and also of $\omega_1$ and $\omega_2$), denoted by $L$, $M$, such that the Lax equation

$$\dot{L} = [L, M] \quad (2.6)$$

is equivalent to the Hamilton-Jacobi equations of (2.5). One manifestation of integrability is the existence of a maximal number of conserved integrals of motion. Assuming that we have a Lax pair, it is immediately clear that the quantities $\text{tr} L^r_r$, for $r = 0, 1, 2, \cdots, \infty$ are conserved integrals of motion, since they are time independent by (2.6). On general grounds, at most $N$ of these quantities are functionally independent of one another. In practice, we always have $N \geq n$, so that there are enough integrals of motion to completely separate the dynamics of the $n$ degrees of freedom in (2.5).

Many integrable systems admit a generalized Lax pair in which $L$ and $M$ are allowed to depend upon an arbitrary complex valued spectral parameter $z$. The spectral parameter $z$ does not enter the Hamilton-Jacobi equations of the system, and modifies the Hamiltonian
by at most an added term that depends on $z$ but is independent of $x$ and $p$. When this is the case, the Lax equation (2.6) for $L(z)$ and $M(z)$ yields (2.5) for every value of $z$. The existence of a Lax pair with spectral parameter allows one to associate a time independent spectral curve with the Calogero-Moser system (twisted or untwisted) for each finite dimensional Lie algebra $G$, given by the following equation

$$\det(kI - L(z)) = 0. \quad (2.7)$$

The Weyl group of $G$ leaves the spectral curve invariant and acts by permutation on the various leaves of the Riemann surface defined by the various roots for $k$ of (2.7).

For any given Hamiltonian, the Lax pair is not unique. The Lax equation (2.6) is covariant under conjugation of the operator $L$ by an arbitrary $N \times N$ matrix valued function $S$ of $x$, $p$, $z$ and $\omega_{1,2}$ and an accompanying gauge transformation on $M$. We have

$$\dot{L}^S = [L^S, M^S] \quad \text{with} \quad L^S = SLS^{-1}, \quad M^S = SMS^{-1} - \dot{S}S^{-1}. \quad (2.8)$$

The conserved integrals of motion $\text{tr}L^m$ and the spectral curve of (2.7) are invariant under these gauge transformations.

(e) **The Lax Pair for the $A_n$ System**

A generalized Lax pair was found for the elliptic Calogero-Moser system associated with the Lie algebras $A_{N-1}$, in terms of a spectral parameter $z$ that lives on the elliptic curve $\Sigma$. We parametrize the roots of $A_{N-1}$ by an orthonormal basis in $\mathbb{C}^N$ of vectors $e_i$, $i = 1, \cdots, N$. The set of all roots is then $R(A_{N-1}) = \{e_i - e_j; \ i \neq j\}$. We set $m_{|\alpha|} = m_2$, since all roots have the same length squared, 2. In this basis, the Lax pair is in terms of $N \times N$ matrices (with $i, j = 1, \cdots, N$)

$$L_{ij}(z) = p_1\delta_{ij} - m_2(1 - \delta_{ij})\Phi(x_i - x_j, z)$$
$$M_{ij}(z) = m_2\delta_{ij} \sum_{k \neq i} \varphi(x_i - x_k) + m_2(1 - \delta_{ij})\Phi'(x_i - x_j, z). \quad (2.9)$$

Here, $\Phi'(x, z)$ stands for the $x$-derivative of $\Phi(x, z)$. The function $\Phi(x, z)$ obeys

$$\Phi(x, z)\Phi'(y, z) - \Phi(y, z)\Phi'(x, z) = \begin{cases} (\varphi(x) - \varphi(y))\Phi(x + y, z), & x + y \neq 0; \\ \varphi'(x), & x + y = 0. \end{cases} \quad (2.10)$$

$\Phi(x, z)$ is doubly periodic in $z$ with periods $2\omega_1$ and $2\omega_2$, has monodromy in $x$, and has an essential singularity at $z = 0$. Properties of elliptic functions are given in Appendix §B.
III. THE GENERAL CONSTRUCTION OF LAX PAIRS

In this section, we present a general formalism for the construction of Lax pairs for the untwisted and twisted elliptic Calogero-Moser systems associated with an arbitrary simple Lie algebra \( \mathcal{G} \). We begin with a discussion of the Lie algebra theory needed to formulate the Ansatz for the Lax pairs in Theorems 1 and 2.

(a) Decomposing Roots of \( GL(N) \) under the Action of a Subalgebra \( \mathcal{G} \)

Let \( \mathcal{G} \) be a finite dimensional, complex simple Lie algebra of rank \( n \), and dimension \( d \). Let \( \Lambda \) be a representation of \( \mathcal{G} \) with dimension \( N < \infty \), and generators \( \Lambda_a, a = 1, \cdots, d \). The representation \( \Lambda \) embeds \( \mathcal{G} \) into the fundamental (\( N \)-dimensional) representation of \( GL(N, \mathbb{C}) \) as a subalgebra; the generators \( \Lambda_a \) are then linear combinations of the \( N^2 \) generators of \( GL(N, \mathbb{C}) \).

Since \( \mathcal{G} \) is a subalgebra of \( GL(N, \mathbb{C}) \), we choose a Cartan subalgebra \( \mathcal{H} \) that contains the Cartan subalgebra \( \mathcal{H}_\mathcal{G} \) chosen for \( \mathcal{G} \), so that \( \mathcal{H}_\mathcal{G} = \mathcal{H} \cap \mathcal{G} \). This choice allows us to split the set of Cartan generators of \( \mathcal{H} \) into a set of Cartan generators \( h_i, i = 1, \cdots, n \) of \( \mathcal{H}_\mathcal{G} \) and a complementary set of generators \( \tilde{h}_j, j = n + 1, \cdots, N \) in \( \mathcal{H} \) that commute with all \( h_i \) and mutually commute: \([h_i, \tilde{h}_j] = 0\) and \([\tilde{h}_j, \tilde{h}_j'] = 0\). Without loss of generality, we shall choose the \( h \) and \( \tilde{h} \) generators to be mutually orthogonal under the Cartan-Killing inner product, \( \text{tr}(\text{Ad}_h \text{Ad}_{\tilde{h}}) = 0 \).

The centralizer in \( GL(N, \mathbb{C}) \) of the Cartan subalgebra \( \mathcal{H}_\mathcal{G} \) of \( \mathcal{G} \) may be larger than \( \mathcal{H} \). We denote it by \( \mathcal{H} \oplus GL_0 \). The subspace \( GL_0 \) consists of the roots of \( GL(N, \mathbb{C}) \) which project to zero under the orthogonal projection of the weights of \( GL(N, \mathbb{C}) \) to \( \mathcal{G} \). When the representation \( \Lambda \) has at most one zero weight under \( \mathcal{G} \), we have \( GL_0 = 0 \). This is the case for all Lie algebras in their lowest dimensional faithful representation, except for \( F_4 \) and \( E_8 \), where \( \dim GL_0 = 2 \) and 56 respectively.

We now develop a general description for the embedding of the representation \( \Lambda \) of \( \mathcal{G} \) into the fundamental representation of \( GL(N, \mathbb{C}) \). The weights of the fundamental representation of \( GL(N, \mathbb{C}) \) may be chosen to be \( N \) orthonormal vectors in \( \mathbb{C}^N \), which we denote by \( u_I, I = 1, \cdots, N \). As usual, the root vectors of \( GL(N, \mathbb{C}) \) are then given by the differences \( u_I - u_J \). The fundamental representation of \( GL(N, \mathbb{C}) \) restricts to the representation \( \Lambda \), as \( GL(N, \mathbb{C}) \) is restricted to the subalgebra \( \mathcal{G} \). The weight vectors of the fundamental representation of \( GL(N, \mathbb{C}) \) admit a decomposition into weight vectors of \( \mathcal{G} \), (corresponding to the eigenvalues of the Cartan generators \( h_i, i = 1, \cdots, n \) of \( \mathcal{H}_\mathcal{G} \)) and into vectors in the complement of \( \mathcal{G} \) in \( GL(N, \mathbb{C}) \), (corresponding to the eigenvalues of the
Cartan generators $\tilde{h}_j, j = n + 1, \cdots, N)$. This decomposition is orthogonal in view of the orthogonality of $h$ and $\tilde{h}$.

The weight vectors of the representation $\Lambda$ of $\mathcal{G}$ are $N$ vectors in $\mathbb{C}^n$, which we shall view as vectors in $\mathbb{C}^N \supseteq \mathbb{C}^n$, by assigning coordinates 0 to the extra generators $\tilde{h}$. According to the preceding discussion, we have the orthogonal decomposition

$$su_I = \lambda_I + v_I \quad I = 1, \cdots, N \tag{3.1}$$

where any $v_I, I = 1, \cdots, N$ is orthogonal to any $\lambda_J, J = 1, \cdots, N$, and $s$ is a normalization factor. The vector space generated by all $\lambda_I$ is of dimension $n$, while that generated by all $v_I$ is of dimension $N - n$. The normalization factor $s$ is related to the second Dynkin index of the representation $\Lambda$:

$$s^2 = I_2(\Lambda) = \frac{1}{n} \sum_{I=1}^{N} \lambda_I \cdot \lambda_I. \tag{3.2}$$

Once $s$ has been determined, we parametrize the weight vectors $\lambda_I$ as well as the vectors $v_I$ in terms of the $N$-dimensional basis vectors $u_I$ in an explicit and unique way by

$$\lambda_I = \frac{1}{s} \sum_{J=1}^{N} (\lambda_I \cdot \lambda_J) u_J$$

$$v_I = s \cdot u_I - \frac{1}{s} \sum_{J=1}^{N} (\lambda_I \cdot \lambda_J) u_J. \tag{3.3}$$

The decomposition of the weight vectors of the fundamental representation of the algebra $GL(N, \mathbb{C})$ under the action of $\mathcal{G}$ allows us to obtain the corresponding decomposition for any representation of $GL(N, \mathbb{C})$. In particular, the generators of $GL(N, \mathbb{C})$ corresponding to the roots may be decomposed in this way. We shall label the generators associated with the roots of $GL(N, \mathbb{C})$ by $E_{IJ}, I \neq J$. The root decomposition is then obtained by evaluating the commutators of $h$ and $\tilde{h}$ with $E_{IJ}$. We find

$$[h, E_{IJ}] = (\lambda_I - \lambda_J) E_{IJ}$$

$$[\tilde{h}, E_{IJ}] = (v_I - v_J) E_{IJ}. \tag{3.4}$$

Under the action of $\mathcal{G}$, the adjoint representation of $GL(N, \mathbb{C})$ decomposes into the adjoint representation of $\mathcal{G}$, plus other representations, according to the tensor product $\Lambda \otimes \Lambda^*$. 

14
(b) Construction of Lax pairs for arbitrary Lie Algebra $G$

We make use of the results of (a) to provide a Lax pair construction for the elliptic Calogero-Moser systems of (2.1) and (2.3) associated with any Lie algebra $G$ in an arbitrary representation $\Lambda$ of dimension $N$, with weight system $\{\lambda_I\}_{I=1,\ldots,N}$. The Lax operators $L$ and $M$ are $N \times N$ dimensional matrices, given by the following Ansatz,

$$L = P + X, \quad P = p \cdot h,$$

$$M = D + Y, \quad D = d \cdot (h \oplus \tilde{h}) + \Delta.$$  \hspace{1cm} (3.5)

Here, $P \in \mathcal{H}_G$, $\Delta \in GL_0$ and $D \in \mathcal{H} \oplus GL_0$, so that $[P, D] = 0$, while $X$ and $Y$ are given by

$$X = \sum_{I,J=1; I \neq J}^{N} C_{I,J} \Phi_{I,J}(\alpha_{IJ} \cdot x, z) E_{IJ}$$

$$Y = \sum_{I,J=1; I \neq J}^{N} C_{I,J} \Phi'_{I,J}(\alpha_{IJ} \cdot x, z) E_{IJ},$$  \hspace{1cm} (3.6)

The combination $\alpha_{IJ} \equiv \lambda_I - \lambda_J$ is the weight under $G$ associated with the root $u_I - u_J$ of $GL(N, \mathbb{C})$, $C_{I,J}$ are constants, $\Phi'_{I,J}(x, z)$ are the $x$-derivatives of $\Phi_{I,J}(x, z)$ and the functions $\Phi_{I,J}$ are certain elliptic functions of the type of $\Phi$, which remain to be determined.

**Theorem 1 : The General Case**

The Lax equation $\dot{L} = [L, M]$ implies the elliptic Calogero-Moser system (2.1) or (2.3) if and only if the following conditions hold

(1) $s^2 \sum_{\alpha \in \mathcal{R}(G)} m_{2|\alpha|}^2 \psi_{\nu}^{(\alpha)}(\alpha \cdot x) \alpha = \sum_{I,J=1; I \neq J}^{N} C_{I,J} C_{J,I} \Phi'_{I,J}(\alpha_{IJ} \cdot x) \alpha_{IJ}$.  \hspace{1cm} (3.7)

(2) $0 = \sum_{I,J=1; I \neq J}^{N} C_{I,J} C_{J,I} \Phi'_{I,J}(\alpha_{IJ} \cdot x) (v_I - v_J)$.  \hspace{1cm} (3.8)

(3) There exists a vector $d \in \mathbb{C}^N$ and a matrix $\Delta$ with $D = d \cdot (h \oplus \tilde{h}) + \Delta \in \mathcal{H} \oplus GL_0$, such that for all $I \neq J$, we have

$$sC_{I,J} \Phi_{I,J}(\alpha_{IJ} \cdot x) d \cdot (u_I - u_J) + \sum_{K \neq I,J}^{\Delta_{IK}} C_{K,J} \Phi_{K,J}(\alpha_{KJ} \cdot x)$$

$$- \sum_{K \neq I,J} C_{I,K} \Phi_{IK}(\alpha_{IK} \cdot x) \Delta_{KJ}$$

$$= \sum_{K \neq I,J} C_{I,K} C_{K,J} \left\{ \Phi_{IK}(\alpha_{IK} \cdot x) \Phi'_{KJ}(\alpha_{KJ} \cdot x) - \Phi'_{IK}(\alpha_{IK} \cdot x) \Phi_{KJ}(\alpha_{KJ} \cdot x) \right\}$$  \hspace{1cm} (3.9)
Here, the Weierstrass functions are defined by

$$\Phi_{IJ}(x, z)\Phi'_{JI}(-x, z) - \Phi'_IJ(x, z)\Phi_{JI}(-x, z) = \varphi'_{IJ}(x).$$ \hspace{1cm} (3.10)

To prove this Theorem, we use the fact that $[P,D] = 0$, and that the Lax equation (2.6) decomposes into three parts upon separating out the $\dot{x}$ and $p$ dependence of various terms.

$$\dot{X} = [P, Y]$$
$$\dot{P} = [X, Y]_\mathcal{H}$$
$$[D, X] = [X, Y]_\mathcal{M}.$$ \hspace{1cm} (3.11)

Here, $\mathcal{M}$ is the complement to $\mathcal{H}$ in $GL(N, \mathbb{C}) = \mathcal{H} \oplus \mathcal{M}$, and the symbols $[,]_\mathcal{H}$ and $[,]_\mathcal{M}$ denote the projections of the commutator $[,]$ onto $\mathcal{H}$ and $\mathcal{M}$ respectively.

The first equation in (3.11) is guaranteed by the form of the Ansatz for $X$ and $Y$, and by the fact that $\alpha_{IJ} = \lambda_I - \lambda_J$. The second equation in (3.11) may be reduced to conditions (1) and (2), by using the algebra of $GL(N, \mathbb{C})$ generators

$$[E_{IJ}, E_{KL}] = \delta_{JK}E_{IL} - \delta_{IL}E_{KJ},$$ \hspace{1cm} (3.12)

as well as the identities between the $\Phi_{IJ}$ and $\varphi_{IJ}$ functions of (3.10). Indeed, the second equation in (3.11) is equivalent to

$$\dot{p} \cdot h = \sum_{I,J=1; I \neq J}^{N} C_{I,J} C_{J,I} \varphi'_{IJ}(\alpha_{IJ} \cdot x) E_{II},$$ \hspace{1cm} (3.13)

which decomposes into two parts. Its part along $\mathcal{H}_G$ yields the Hamilton-Jacobi equation for the Calogero-Moser system, while the remaining part is a further constraint on the couplings $C_{I,J}$. To disentangle the two, it suffices to obtain the decomposition of the generators $E_{II}$ in terms of $h$ and $\tilde{h}$. By comparing (3.4) and (3.12) for $I = J$, one finds

$$E_{II} = \frac{1}{s^2}(\lambda_I \cdot h + v_I \cdot \tilde{h}),$$ \hspace{1cm} (3.14)

where $s$ and $v_I$ were introduced in (3.1) and $s$ was evaluated in (3.2). Thus, (3.13) splits into

$$\dot{p} = \sum_{I,J=1; I \neq J}^{N} \frac{1}{2s^2} C_{I,J} C_{J,I} \varphi'_{IJ}(\alpha_{IJ} \cdot x) \alpha_{IJ}$$
$$0 = \sum_{I,J=1; I \neq J}^{N} C_{I,J} C_{J,I} \varphi'_{IJ}(\alpha_{IJ} \cdot x)(v_I - v_J).$$ \hspace{1cm} (3.15)
The first equation in (3.15) manifestly reproduces the Calogero-Moser Hamilton-Jacobi equations (2.5), provided (1) in (3.7) holds. The second equation is purely algebraic and coincides with (2) in (3.8). Finally, the third equation in (3.11) is easily reduced to condition (3), using (3.12).

**Theorem 2 : Untwisted Elliptic Calogero-Moser Systems**

For each of the untwisted Calogero-Moser systems with the Hamiltonians of (2.1), a Lax pair may obtained in which the functions $\Phi_{IJ}$ are all identical

$$
\Phi_{IJ}(x, z) = \Phi(x, z) \quad I, J = 1, \cdots, N \quad (3.16)
$$

and the matrix of constants $C_{IJ}$ is symmetric: $C_{IJ} = C_{JI}$. Except for the Lie algebra $E_8$, we may also set $\Delta = 0$. Under these conditions, very considerable simplifications take place, and the statements of Theorem 1 may be simplified, as follows.

The Lax equation $\dot{L} = [L, M]$ implies the untwisted elliptic Calogero-Moser system (2.1) if and only if conditions (1), (2) and (3) below are satisfied.

1. The constants $C_{IJ}$ are related to the coupling constants $m_{|\alpha|}$ of the Calogero-Moser system by

$$
\sum_{I, J=1; \alpha_IJ=\alpha}^N C_{IJ}^2 = s^2 m_{|\alpha|}^2 \quad (3.17)
$$

2. The constants $C_{IJ}$ for each weight $\alpha$ of $\Lambda \otimes \Lambda^*$ satisfy:

$$
0 = \sum_{I, J=1; \alpha_IJ=\alpha}^N C_{IJ}^2 (v_I - v_J) \quad (3.18)
$$

3. For all $G$, except $E_8$, there exists a vector $d$ with $D = d \cdot (h \oplus \tilde{h}) \in \mathcal{H}$, such that for all $I \neq J$, we have

$$
sC_{IJ} d \cdot (u_I - u_J) = \sum_{K \neq I, J} C_{IK} C_{KJ} \{\wp(\alpha_{IK} \cdot x) - \wp(\alpha_{KJ} \cdot x)\} \quad (3.19)
$$

For the case of $E_8$, $\Delta \neq 0$, and we should retain here the full statement of Theorem 1, (3), but with $\Phi_{IJ}$ given by (3.16).

4. Conditions (1) and (2) imply that whenever $\alpha_{IJ} \notin \mathcal{R}(G)$, we have

$$
C_{IJ} = 0 \quad (3.20)
$$
Conditions (1), (2) and (3) of Theorem 2 are readily derived from Theorem 1 by using the fact that all \( \Phi_{IJ} \) are equal as given in (3.16), that \( C_{I,J} \) is symmetric, that \( \Delta = 0 \) (for \( G \neq E_8 \)) and that the functions \( \wp(\alpha \cdot x) \) and \( \wp(\beta \cdot x) \) are linearly independent when \( \beta \neq \pm \alpha \).

To show (4), we use the fact that when \( \alpha \not\in \mathcal{R}(G) \), we have \( m_{|\alpha|} = 0 \). Projecting condition (2) onto a vector \( u_L \), and using (3.3), conditions (1) and (2) then reduce to

\[
0 = \sum_{I,J; \alpha_{I,J} = \alpha} C_{I,J}^2
\]

\[
0 = \sum_{I,J; \alpha_{I,J} = \alpha} C_{I,J}^2 (s^2(\delta_{I,L} - \delta_{J,L}) - \alpha \cdot \lambda_L)
\]  

(3.21)

The term proportional to \( \alpha \cdot \lambda_L \) in the second equation vanishes in view of the first one. In the remaining equation, let \( \alpha \not\in \mathcal{R}(G) \) be such that it can be written as \( \alpha = \lambda_I - \lambda_J \) for some weights \( \lambda_I \) and \( \lambda_J \). Now choose \( L = I \) : it follows that \( C_{I,J} = 0 \). Clearly this result holds for any \( I, J \) such that \( \alpha = \lambda_I - \lambda_J \), so that (4) immediately follows.

For the twisted elliptic Calogero-Moser systems associated with non-simply laced \( G \), the functions \( \Phi_{IJ} \) in (3.6) cannot be all equal. One is left to using the general Theorem 1, although we shall find that a solution may be obtained with \( C_{I,J} = C_{J,I} \), which we shall henceforth assume. The precise expressions will depend upon each case and will be obtained in \( \S \)VI, for \( G = B_n, C_n, F_4 \). For \( G = G_2 \), we have been able to solve the conditions of Theorem 1 only partially.

IV. THE CLASSICAL LIE ALGEBRAS : UNTWISTED CASES

We make use of Theorem 2, presented in \( \S \)III, to construct Lax pairs with spectral parameter for the untwisted elliptic Calogero-Moser systems associated with the classical Lie algebras \( B_n, C_n \) and \( D_n \). (The case of the algebra \( A_n \) was already discussed in \( \S \)II.) In \( \S (a) \), we find the Lax pairs for the \( BC_n \) root system (to be defined below), and deduce the Lax pairs for \( B_n, C_n \) and \( D_n \) in their fundamental representations. The results are summarized in Theorem 3. In \( \S (b) \), we indicate how the symmetric space construction of the Lax pairs, given in Perelomov [14], is recovered as a special case of our results. The general formalism developed in \( \S \)III is flexible enough to describe Lax pairs in representations other than the fundamental. To illustrate this result, we present three examples, in \( \S (c) \) : \( A_n \) in a rank \( p \) totally anti-symmetric tensor representations; in \( \S (d) \) : \( B_n \) and \( D_n \) in a spinor representation.
(a) Untwisted Elliptic Calogero-Moser Systems for Classical Lie algebras

To obtain the Lax pairs for the classical Lie algebras $B_n$, $C_n$ and $D_n$, it is convenient to derive the Lax pair for the untwisted elliptic Calogero-Moser system associated with the root system $\mathcal{R}(BC_n) = \mathcal{R}(B_n) \cup \mathcal{R}(C_n)$. While this root system is not properly associated with a finite-dimensional simple Lie algebra, Theorem 2 nonetheless still applies. We then deduce the Lax pair for each of the classical Lie algebras by setting one of the independent couplings in the $BC_n$ system to zero.

As a starting point, we take $G = B_n$ viewed as a subgroup of $GL(N, \mathbb{C})$, with $N = 2n+1$, by embedding the fundamental representation of $B_n$ into the fundamental representation of $GL(N, \mathbb{C})$. The weights of $G$ obtained by the decomposition of the adjoint representation of $GL(N, \mathbb{C})$ under $G$ then automatically contains all the root vectors of the $BC_n$ system.

We denote the weights of the fundamental representation of $GL(N, \mathbb{C})$ by $u_I$, $I = 1, \ldots, N$, and the weights of the fundamental representation of $B_n$ by $\lambda_I$, just as in (3.1). Since the rank of the weight system $\lambda$ is $n$, we may express all weights in an orthonormal basis of vectors $e_i$, $i = 1, \ldots, n$, so that

$$
\lambda_i = +e_i, \quad i = 1, \ldots, n \tag{4.1}
$$

$$
\lambda_{n+i} = -e_i, \quad i = 1, \ldots, n
$$

$$
\lambda_N = 0, \quad N = 2n + 1
$$

It is straightforward to work out the decomposition (3.1) : we find $s^2 = 2$, $v_{n+i} = v_i$, for all $i = 1, \ldots, n$ and

$$
\sqrt{2}u_i = + e_i + v_i
$$

$$
\sqrt{2}u_{n+i} = - e_i + v_i
$$

$$
\sqrt{2}u_N = v_N \tag{4.2}
$$

The decomposition of the roots of $GL(N, \mathbb{C})$ into weights of $B_n$ immediately follows from (4.2) and yields three orbits. We have weights of $B_n$ of length$^2$ = 2 (which may be viewed as roots of $D_n$)

$$
\sqrt{2}(u_i - u_j) = + e_i - e_j + v_i - v_j \quad i \neq j
$$

$$
\sqrt{2}(u_{n+j} - u_{n+i}) = + e_i - e_j - v_i + v_j \quad i \neq j \tag{4.3a}
$$

$$
\sqrt{2}(u_i - u_{n+j}) = + e_i + e_j + v_i - v_j \quad i \neq j
$$

$$
\sqrt{2}(u_{n+i} - u_j) = - e_i - e_j + v_i - v_j \quad i \neq j
$$

weights of length$^2$ = 4 (additional roots for $C_n$)

$$
\sqrt{2}(u_i - u_{n+i}) = + 2e_i
$$

$$
\sqrt{2}(u_{n+i} - u_i) = - 2e_i, \tag{4.3b}
$$

19
and weights of length \(2^2 = 1\) (additional roots for \(B_n\))

\[
\begin{align*}
\sqrt{2}(u_i - u_N) &= +e_i + v_i - v_N \\
\sqrt{2}(u_N - u_{n+i}) &= +e_i - v_i + v_N \\
\sqrt{2}(u_{n+i} - u_N) &= -e_i + v_i - v_N \\
\sqrt{2}(u_N - u_i) &= -e_i - v_i + v_N.
\end{align*}
\]

(4.3c)

Following the general construction of the Lax pair for this system in (3.5), (3.6) and (3.16), we have three couplings \(m_2, m_4\) and \(m_1\), namely one for each of the above orbits under the Weyl group of \(B_n\), and the Lax pair is given by (3.5), (3.6) and (3.16). It remains to satisfy the three conditions of Theorem 2 in order to guarantee closure of the Lax equation onto the Calogero-Moser system.

We begin by verifying conditions (1) and (2) of Theorem 2. There are two distinct roots of \(GL(N, \mathbb{C})\) that project to \(e_i - e_j\) for each \(i \neq j\): \(\sqrt{2}(u_i - u_j)\) and \(\sqrt{2}(u_{N+j} - u_{n+i})\). Conditions (1) and (2) on these roots read respectively

\[
\begin{align*}
2m_2^2 &= C^2_{i,j} + C^2_{n+j,n+i} \\
0 &= C^2_{i,j}(v_i - v_j) + C^2_{n+j,n+i}(-v_i + v_j).
\end{align*}
\]

(4.4)

Proceeding analogously for the roots \(e_i + e_j, i < j\), \(2e_i\) and \(e_i\), and solving by using the linear independence of the vectors \(v_i\), we find

\[
\begin{align*}
m_2^2 &= C^2_{i,j} = C^2_{n+i,n+j} = C^2_{n+i,j} & i \neq j \\
2m_4^2 &= C^2_{i,n+i} \\
m_1^2 &= C^2_{i,N} = C^2_{n+i,N}
\end{align*}
\]

(4.5)

We shall obtain a solution for the Lax pair in Theorem 2 by choosing all square roots of the above relations with the same sign,

\[
\begin{align*}
m_2 &= C_{i,j} = C_{n+i,n+j} = C_{n+i,j}, & i \neq j, \\
\sqrt{2}m_4 &= C_{i,n+i}, \\
m_1 &= C_{i,N} = C_{n+i,N}
\end{align*}
\]

(4.6)

Next, we verify condition (3) of Theorem 2, using the results of (4.6). Because of antisymmetry of (3.19) under the interchange of \(I\) and \(J\), there are 6 cases to be analyzed (here, \(i, j = 1, \ldots, n\)) : (1) \(I = i, J = j\); (2) \(I = i, J = N\); (3) \(I = i, J = n + j\) with \(i \neq j\); (4) \(I = i, J = n + i\); (5) \(I = N, J = n + j\); and (6) \(I = n + i, J = n + j\). A solution
may be obtained in which $d \cdot e_i = 0$. Then, case (4) is satisfied automatically, cases (3) and (6) yield the same equations as case (1), and case (5) yields the same equation as case (2). Thus, there remain just two cases: (1) and (2), which yield the following equations

$$m_2 d \cdot (v_i - v_j) = \sum_{k \neq i, j} m_2^2 [\varphi(x_i - x_k) - \varphi(x_k - x_j) + \varphi(x_i + x_k) - \varphi(x_k + x_j)]$$

$$+ m_1^2 [\varphi(x_i) - \varphi(x_j)] + \sqrt{2m_2}m_4 [\varphi(2x_i) - \varphi(2x_j)]$$

$$m_1 d \cdot (v_i - v_N) = \sum_{k \neq i} m_1 m_2 [\varphi(x_i - x_k) + \varphi(x_i + x_k) - 2\varphi(x_k)]$$

$$+ \sqrt{2m_1}m_4 [\varphi(2x_i) - \varphi(x_i)]$$

(4.7)

Without loss of generality, we assume that $m_2 \neq 0$, since for $m_2 = 0$, the system would decompose into a set of non-interacting one dimensional systems. Thus, the first equation in (4.7) is non-trivial, and its most general solution is given by

$$d \cdot v_i = d_0 + \frac{m_1^2}{m_2} \varphi(x_i) + \sqrt{2m_4} \varphi(2x_i) + \sum_{k \neq i} m_2 [\varphi(x_i - x_k) + \varphi(x_i + x_k)],$$

(4.8)

where $d_0$ is an arbitrary function of $x$ which is independent of $i$. Substituting this solution for $d \cdot v_i$ into the second equation in (4.7) yields

$$m_1 d \cdot v_N = m_1 d_0 + m_1 (-2m_2 + \sqrt{2m_4} + \frac{m_1^2}{m_2}) \varphi(x_i) + \sum_{k} 2m_1 m_2 \varphi(x_k).$$

(4.9)

The left-hand side is independent of $i$, hence the right-hand side must also be independent of $i$, which requires that $m_1 (m_1^2 - 2m_2^2 + \sqrt{2m_2}m_4) = 0$. Once this condition is satisfied, (4.9) yields the component of the vector $d$ along $v_N$, and integrability of the associated Calogero-Moser system is guaranteed. Henceforth, we choose $d_0$ so that $d \cdot v_N = 0$. The result may be summarized in the following Theorem 3, in which we make the matrix form of the Lax operators completely explicit.

**Theorem 3 : Lax pair for the $BC_n$, $B_n$, $C_n$ and $D_n$ Systems**

The untwisted elliptic Calogero-Moser Hamiltonian of (2.1) with root system $\mathcal{R}(BC_n)$ is integrable if

$$m_1 (m_1^2 - 2m_2^2 + \sqrt{2m_2}m_4) = 0.$$  

(4.10)

It reduces to the untwisted elliptic Calogero-Moser systems for the classical Lie algebras $B_n$, $C_n$ and $D_n$ by the following choice of couplings

$$B_n \quad m_4 = 0, \quad m_1^2 = 2m_2^2$$

$$C_n \quad m_1 = 0,$$

$$D_n \quad m_1 = 0, \quad m_4 = 0.$$  

(4.11)
In all of these cases, the Calogero-Moser system admits a Lax pair with spectral parameter, given by the following $(2n+1) \times (2n+1)$ matrix-valued functions

\[
L = P + X \quad P = \text{diag}(p_1, \ldots, p_n; -p_1, \ldots, -p_n; 0) \\
M = D + Y \quad D = \text{diag}(d_1, \ldots, d_n; +d_1, \ldots, +d_n; 0)
\]  

(4.12a)

The matrices $X$ and $Y$ are given by

\[
X = \begin{pmatrix} A & B_1 & C_1 \\ B_2 & A^T & C_2 \\ C_2^T & C_1^T & 0 \end{pmatrix} \quad Y = \begin{pmatrix} A' & B_1' & C_1' \\ B_2' & A'^T & C_2' \\ C_2'^T & C_1'^T & 0 \end{pmatrix},
\]

(4.12b)

where the superscript $T$ stands for transposition. The entries of the matrix $X$ are defined by (with $i, j = 1, \ldots, n$)

\[
A_{ij} = m_2(1 - \delta_{ij})\Phi(+x_i - x_j, z) \\
B_{1ij} = m_2(1 - \delta_{ij})\Phi(+x_i + x_j, z) + \sqrt{2}m_4\delta_{ij}\Phi(2x_i, z) \\
B_{2ij} = m_2(1 - \delta_{ij})\Phi(-x_i - x_j, z) + \sqrt{2}m_4\delta_{ij}\Phi(-2x_i, z) \\
C_{1i} = m_1\Phi(+x_i, z) \\
C_{2i} = m_1\Phi(-x_i, z),
\]

(4.12c)

while the entries of the matrix $Y$ are as in (4.12c), but with $A, B, C$ replaced by $A', B', C'$ and $\Phi$ replaced by $\Phi'$. The entries $d_i = d \cdot v_i$ of the matrix $D$ are as given in (4.8).

**Remarks**

(1) The Lax operators for the Lie algebras $C_n$ and $D_n$ are effectively of dimension $2n$ instead of $2n+1$, as is seen in (4.12) by setting $m_1 = 0$, so that $C_1 = C_2 = C_1' = C_2' = 0$.

(2) The Lax pairs with spectral parameter for the Calogero-Moser system associated with the root system $BC_n$ and hence with the Lie algebras $B_n$, $C_n$ and $D_n$ coincide with the known Lax pairs without spectral parameter [14] at the three half periods $\omega_1$, $\omega_2$ and $\omega_3$. This is seen by evaluating $\Phi$ at these values

\[
\Phi(x, \omega_1) = \rho \frac{\text{cn}(\rho x)}{\text{sn}(\rho x)} \\
\Phi(x, \omega_2) = \rho \frac{\text{dn}(\rho x)}{\text{sn}(\rho x)} \\
\Phi(x, \omega_3) = \rho \frac{1}{\text{sn}(\rho x)},
\]

(4.13)

where $\rho^2 = \varphi(\omega_1) - \varphi(\omega_2)$, and substituting these expressions into (4.12).  

22
(3) The three half periods \( \omega_a, a = 1, 2, 3 \) are the only values of the spectral parameter for which the function \( \Phi \) has an extra reflection symmetry: \( \Phi(-x, \omega_a) = -\Phi(x, \omega_a) \), and \( \Phi'(-x, \omega_a) = \Phi'(x, \omega_a) \), so that
\[
A^T = -A, \quad B_2 = -B_1, \quad C_2 = -C_1
\]
\[
A'^T = +A', \quad B'_2 = +B'_1, \quad C'_2 = +C'_1.
\]
(4.14)
At these values, \( X \) is anti-symmetric, and thus belongs to the Lie algebras \( B_n, C_n \) or \( D_n \) respectively.

(b) Construction from Symmetric Spaces of Affine Lie Algebras

A natural setting for the construction of the Lax pairs without spectral parameter was obtained in Perelomov [14] in terms of symmetric spaces of classical Lie algebras. That construction uses specific reflection symmetry properties of the Lax pair that do not continue to hold for arbitrary values of the spectral parameter. However, a generalization of these reflection symmetry properties, in which the sign of the spectral parameter \( z \) is reversed, does hold for all values of \( z \). To establish this, we use the reflection property of \( \Phi \) (see Appendix §B):
\[
\Phi(-x, z) = -\Phi(x, -z).
\]
(4.15)
Upon introducing the conjugation matrix \( S \), with \( S^2 = I_{2n+1} \),
\[
S \equiv \begin{pmatrix}
0 & I_n & 0 \\
I_n & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]
(4.16)
we have the following reflection relations
\[
P(z)^T = +P(-z) = -SP(z)S
\]
\[
D(z)^T = +D(-z) = +SD(z)S
\]
\[
X(z)^T = -X(-z) = +SX(z)S
\]
\[
Y(z)^T = +Y(-z) = +SY(z)S.
\]
(4.17)
Here, we have included a fictitious \( z \)-dependence for \( P \), for the sake of respecting the pattern exhibited by \( D, X, \) and \( Y \).

We shall now formulate the Lax pair in more geometrically intrinsic terms. We begin by defining the ring \( F[z] \) of elliptic functions that are holomorphic, except for a singularity at \( z = 0 \). This ring is generated by the functions \( \Phi(x, z) \) and its \( x \)-derivatives. We define an affine Lie algebra \( \mathcal{G}[z] \) by
\[
\mathcal{G}[z] \equiv \mathcal{G} \otimes F[z].
\]
(4.18)
\( \mathcal{G}[z] \) may be viewed as the space of elliptic functions with values in the Lie algebra \( \mathcal{G} \), that are holomorphic except for a singularity at \( z = 0 \).

The Lax operators \( L(z) \) and \( M(z) \) belong to \( \mathcal{G}[z] = SL(2n+1)[z] \) for \( B_n \), and \( \mathcal{G}[z] = SL(2n)[z] \) for \( C_n \) and \( D_n \), which may be decomposed into \( \mathcal{G}[z] = K \oplus N \), where

\[
M(z) \in K = \{ g(z) \in \mathcal{G}[z] : g(-z) = +Sg(z)S \}
\]
\[
L(z) \in N = \{ g(z) \in \mathcal{G}[z] : g(-z) = -Sg(z)S \}.
\]  

(4.19)

The coset \( N \) is the tangent space to a symmetric space, since we have

\[
[K, K] \in K; \quad [K, N] \in N; \quad [N, N] \in K.
\]

(4.20)

At one of the three half periods, this symmetric space may be identified in terms of classical Lie algebras and is given by

\[
N \sim \frac{SL(2n+1)}{SL(n) \times SL(n+1)}.
\]

(4.21)

This form of the Lax pair is familiar from the symmetric space construction of the \( BC_n \) root system in Perelomov [14]. However, at generic values of the spectral parameter, the space \( N \) is characterized by a reflection symmetry relation (4.19) that reverses the sign of the spectral parameter.

\( \text{(c) Calogero-Moser for } A_n \text{ in Anti-symmetric Tensor Representations} \)

The rank \( p \) anti-symmetric tensor representation of the Lie algebra \( A_n \) has dimension

\[
N = \binom{n+1}{p} \equiv \frac{(n+1)!}{p!(n+1-p)!}
\]

(4.22)

and will be denoted here by \( T_p \). Its complex conjugate is \( T_{n+1-p} \) and will also be denoted by \( T_p^* \). In the standard basis for the weight space of \( A_n \) of \( n+1 \) orthonormal vectors \( e_i, i = 1, \cdots, n+1 \), (with \( e_0 = (e_1 + e_2 + \cdots + e_{n+1})/(n+1) \)), the weights of \( T_p \) are

\[
T_p = \{ \lambda = e_{i_1} + e_{i_2} + \cdots + e_{i_p} - pe_0, \quad i_1 < i_2 < \cdots < i_p \}
\]

\[
\lambda \cdot \lambda = p(n+1-p)/(n+1).
\]

(4.23)

All weights of \( T_p \) have the same length and lie in a single Weyl orbit. The precise correspondence between the labels \( I \) and \( (i_1 \; i_2 \; \cdots \; i_p) \) is immaterial, * since the weights are permuted into one another under the action of the Weyl group of \( A_n \).

* For any representation \( \Lambda \) in which each weight vector \( \lambda \) occurs with multiplicity precisely 1, one may parametrize the labels \( I \) directly by the weights \( \lambda \) themselves, since the correspondence is one to one.
Theorem 4: Lax pairs for $A_n$ in anti-symmetric tensor representations

The Calogero-Moser system for the Lie algebra $A_n$ admits a Lax pair with spectral parameter in the anti-symmetric tensor representation $T_p$ given by (3.5), (3.6) and (3.16) with

$$C_{\lambda,\mu} = \begin{cases} m_2 & (\lambda - \mu)^2 = 2 \\ 0 & \text{otherwise}, \end{cases}$$

$$sd \cdot u_{\lambda} = \sum_{\lambda \cdot \delta = 1} m_2 q_\delta (\delta \cdot x),$$

(4.24)

where $\lambda$ and $\mu$ run over the weights of $T_p$, as explained in the last footnote.

To prove this Theorem, we show that the conditions of Theorem 2 are satisfied. We begin by describing the necessary group theory. The weights of $GL(N, \mathbb{C})$ are denoted by the $N$ orthonormal vectors $u_{\lambda}$, where $\lambda$ runs over all the weights of $T_p$. The decomposition (3.1) is given by

$$su_{\lambda} = \lambda + v_{\lambda} \quad \text{with} \quad s^2 = \binom{n-1}{p-1},$$

(4.25)

where the vectors $v_{\lambda}$ are orthogonal to the root space of $A_n$. The roots of $GL(N, \mathbb{C})$ decompose under $A_n$ as follows

$$s(u_{\lambda} - u_{\mu}) = \lambda - \mu + v_{\lambda} - v_{\mu} \quad \lambda \neq \mu.$$  

(4.26)

Under the Weyl group of $A_n$, the roots transform in different orbits $[U_q]$, which are precisely the Weyl orbits occurring in the tensor product $T_p \otimes T_p^*$. The orbits $[U_q]$ are defined by

$$[U_q] \equiv \{ \alpha = e_{i_1} + \cdots + e_{i_q} - e_{j_1} - \cdots - e_{j_q}, \text{ all } i_k, j_l \text{ different} \} \quad \alpha^2 = 2q.$$  

(4.27)

The representation $(U_1)$ is the adjoint representation of $A_n$. The decomposition of the roots of $GL(N, \mathbb{C})$ into orbits of the Weyl group of $A_n$ is given by

$$T_p \otimes T_p^* = \bigoplus_{q=0}^{p} \binom{n + 1 - 2q}{p - q} [U_q].$$  

(4.28)

In order to reproduce the Calogero-Moser system for $A_n$, only the coupling of the roots of $A_n$ can be non-zero. We denote this unique coupling by $m_2$. Thus, by (4) of Theorem 2, we see right away that $C_{\lambda,\mu} = 0$ unless $\lambda - \mu \in \mathcal{R}(A_n)$.

Conditions (1) and (2) for a root $\alpha \in \mathcal{R}(A_n)$, are given by

$$s^2 m_2^2 = \sum_{\alpha = \lambda - \mu} C_{\lambda,\mu}^2$$

$$0 = \sum_{\alpha = \lambda - \mu} C_{\lambda,\mu}^2 (v_\lambda - v_\mu)$$

(4.29)

25
If a given root $\alpha$ can be written as $\alpha = \lambda - \mu$, we take the inner product of the second equation in (4.29) with $u_\lambda$. To evaluate this, we use (3.3) and we find $C^2_{\lambda, \mu} = m^2_2$. Using the expression for $s^2$ of (4.25) and of the multiplicity of $[U_q]$ for $q = 1$ of (4.28), we see that the first equation in (4.29) holds for all roots. It remains to verify that the second condition in (4.29) holds when projected onto a vector $u_\sigma$ for an arbitrary weights $\sigma \in T_p$. Using again (3.3), we find

$$
\alpha \cdot \sigma m^2_2 = \sum_{\alpha = \lambda - \mu} C^2_{\lambda, \mu}(\delta_{\lambda, \sigma} - \delta_{\mu, \sigma}).
$$

(4.30)

Since $\alpha$ is now a root of $A_n$, and in view of (4.23), $\alpha \cdot \sigma$ can take only the values $\alpha \cdot \sigma = -1, 0, +1$. When $\alpha \cdot \sigma = 0$, $\sigma + \alpha$ is not a root, and all terms in (4.30) vanish separately. When $\alpha \cdot \sigma = +1$, $\sigma - \alpha$ is a weight $\beta$, while $\sigma + \alpha$ is not a weight. As a result, $\alpha = \sigma - \beta$, and we recover $C^2_{\sigma, \beta} = m^2_2$. Similar reasoning for $\alpha \cdot \sigma = -1$ yields again the same result, and (4.30) is satisfied in all cases. It is possible to find a Lax pair by choosing all the square roots to have the same sign, so that $C_{\lambda, \mu} = m_2^2$ when $\lambda - \mu \in R(A_n)$, whence the first equation of (4.24).

It remains to satisfy condition (3) of Theorem 2, i.e. equation (3.19), which may be re-expressed as

$$
sC_{\lambda, \mu} d \cdot (u_\lambda - u_\mu) = \sum_{\kappa \neq \lambda, \mu} C_{\lambda, \kappa} C_{\kappa, \mu}\{\varphi((\lambda - \kappa) \cdot x) - \varphi((\kappa - \mu) \cdot x)\}.
$$

(4.31)

Here, the weights $\lambda, \kappa, \mu$ all belong to $T_p$. Because $C_{\lambda, \mu}$ is non-zero only when $\lambda - \mu$ is a root of $A_n$, we may replace the summation over $\kappa$ in (4.31) by a summation over roots : $\delta = \lambda - \kappa$ in the first sum, and $\delta = \kappa - \mu$ in the second sum. It is convenient to express the conditions that $\lambda - \delta$ and $\mu + \delta$ are weights while $\lambda - \mu - \delta$ is a root, in terms of inner products of these weights :

$$
sC_{\lambda, \mu} d \cdot (u_\lambda - u_\mu) = \sum_{\delta^2 = 2, \mu \cdot \delta = 1 - \frac{1}{2}(\lambda - \mu)^2} m^2_2 \varphi(\delta \cdot x) - \sum_{\delta^2 = 2, \lambda \cdot \delta = 1 + \frac{1}{2}(\lambda - \mu)^2} m^2_2 \varphi(\delta \cdot x).
$$

(4.32)

We now analyze this equation for $\lambda - \mu$ belonging to each of the possible Weyl orbits $U_q$ occurring in (4.28), i.e. for each of the possible value of $(\lambda - \mu)^2 = 2q$, $q = 1, \cdots, p$. For $q \geq 3$, the l.h.s. vanishes because $C_{\lambda, \mu} = 0$ in this case by the first equation in (4.24). The r.h.s. of (4.32) also vanishes : no roots $\delta$ can satisfy the inner product conditions since $\mu \cdot \delta \leq -2$ in the first sum and $\lambda \cdot \delta \geq 2$ in the second sum. Indeed, if $\delta$ is a root, then it is clear from (4.23) that $\lambda \cdot \delta$ and $\mu \cdot \delta$ can only take on the values $-1, 0, +1$. 

26
For \( q = 2 \), the left hand side vanishes, and so does the right hand side since the first and second sum cancel one another. Finally, for \( q = 1 \), (4.32) is readily solved and we obtain the last equation in (4.24).

**(d) Calogero-Moser for \( B_n \) and \( D_n \) in Spinor Representations**

The spinor representation of \( B_n \) (denoted by \( S \), of dimension \( N = 2^n \)) has a single Weyl orbit of weights \( \lambda \)

\[
S = \{ \lambda = \frac{1}{2} \sum_{i=1}^{n} \epsilon_i e_i \}, \quad \lambda^2 = \frac{n}{4}. \tag{4.33}
\]

Here and below, \( \epsilon_i \) can take the values \( \pm 1 \). The spinor representations of \( D_n \) (denotes by \( S_\pm \), of dimension \( N = 2^{n-1} \)) each have a single Weyl orbit of weights of \( D_n \), given by

\[
S_\pm = \{ \lambda = \frac{1}{2} \sum_{i=1}^{n} \epsilon_i e_i, \quad \prod_{i=1}^{n} \epsilon_i = \pm 1 \}, \quad \lambda^2 = \frac{n}{4}. \tag{4.34}
\]

The Lax pairs in these spinor representations are described by the following result.

**Theorem 5 : Lax pairs for \( B_n \) and \( D_n \) in spinor representations**

1. The Calogero-Moser system for \( B_n \) admits a Lax pair, with spectral parameter and two independent couplings \( m_1 \) and \( m_2 \), in the spinor representation \( S \) given by (3.5), (3.6), (3.16) and

\[
C_{\lambda,\mu} = \begin{cases} 
\frac{m_1}{\sqrt{2}} & (\lambda - \mu)^2 = 1 \\
\frac{m_2}{2} & (\lambda - \mu)^2 = 2 \\
0 & \text{otherwise},
\end{cases} \quad sd \cdot u_\lambda = \sum_{\delta^2 = 2; \lambda \cdot \delta = 1} m_{2\delta}(\delta \cdot x). \tag{4.35}
\]

2. The Calogero-Moser system for \( D_n \) admits a Lax pair with spectral parameter in the spinor representations \( S_\pm \) given by (3.5), (3.6), (3.16) and

\[
C_{\lambda,\mu} = \begin{cases} 
m_2 & (\lambda - \mu)^2 = 2 \\
0 & \text{otherwise},
\end{cases} \quad sd \cdot u_\lambda = \sum_{\lambda \cdot \delta = 1} m_{2\delta}(\delta \cdot x). \tag{4.36}
\]

We begin by proving (1) in detail. The weights of \( GL(N, \mathbb{C}) \) are denoted by \( u_\lambda \) and the decomposition (3.1) is given by

\[
su_\lambda = \lambda + v_\lambda \quad \text{with} \quad s^2 = 2^{n-2}. \tag{4.37}
\]

The tensor product \( S \otimes S \) decomposes into the Weyl orbits of the anti-symmetric tensor representations of rank \( p \), which we denote by \( [T_p] \), and which have weight system

\[
[T_p] = \{ \alpha = \pm e_{i_1} \pm e_{i_2} \pm \cdots \pm e_{i_p}, \; i_1 < i_2 < \cdots < i_p \}; \quad \alpha^2 = p. \tag{4.38}
\]
The decomposition of the roots of $GL(N, \mathbb{C})$ into orbits of the Weyl group of $A_n$ is given by the tensor product decomposition of $S \otimes S$

$$S \otimes S = \bigoplus_{p=0}^{n} 2^{n-p}[T_p]. \quad (4.39)$$

In order to reproduce the Calogero-Moser system for $B_n$, only the couplings of the roots of $B_n$ can be non-zero. We denote the couplings associated with short and long roots by $m_1$ and $m_2$ respectively. Thus, by (4) of Theorem 2, we see right away that $C_{\lambda,\mu} = 0$ unless $\lambda - \mu \in R(B_n)$.

Conditions (1) and (2) for a root $\alpha \in [T_p]$, with $p = 1, 2$ are given by

$$s^2m_p^2 = \sum_{\alpha = \lambda - \mu} C_{\lambda,\mu}^2$$

$$0 = \sum_{\alpha = \lambda - \mu} C_{\lambda,\mu}^2(v_\lambda - v_\mu). \quad (4.40)$$

If a given root $\alpha \in [T_p]$ can be written as $\alpha = \lambda - \mu$, then taking the inner product of the second equation in (4.40) with $u_\lambda$ and using (3.3) yields

$$C_{\lambda,\mu}^2 = \frac{1}{2} pm_p^2, \quad (4.41)$$

for any $\lambda$ and $\mu$ such that $\alpha = \lambda - \mu$. Substituting this result into the first equation of (4.40) and using the formula for the multiplicity of the orbit $[T_p]$ of (4.39), we find that the equation is automatically satisfied for all roots. It remains to verify that the second condition in (4.40) holds for $p = 1, 2$ when projected onto a vector $u_\sigma$ for an arbitrary weight $\sigma \in S$. Using again (3.3) to evaluate this product, we find

$$\alpha \cdot \sigma \ m_\alpha^2 = \sum_{\alpha = \lambda - \mu} C_{\lambda,\mu}^2(\delta_{\lambda,\sigma} - \delta_{\mu,\sigma}). \quad (4.42)$$

Since $\alpha$ is a root of $B_n$, and $\sigma$ is a weight of $S$ as in (4.33), we can only have the values $\alpha \cdot \sigma = 0, \pm \alpha^2/2$. When $\alpha \cdot \sigma = 0$, $\sigma \pm \alpha$ is not a root, and all terms in (4.42) vanish separately. When $\alpha \cdot \sigma = \pm \alpha^2/2$, $\sigma \mp \alpha$ is a weight while $\sigma \pm \alpha$ is not. Thus, (4.42) is satisfied in all cases. In fact, a Lax pair may be found in which all square roots of the relation (4.41) are taken with the same sign, and this gives rise to the first equation in (4.35).

It remains to satisfy condition (3) of Theorem 2, i.e. equation (3.19) :

$$sC_{\lambda,\mu}d \cdot (u_\lambda - u_\mu) = \sum_{\kappa \neq \lambda, \mu} C_{\lambda,\kappa}C_{\kappa,\mu}\{\varphi((\lambda - \kappa) \cdot x) - \varphi((\kappa - \mu) \cdot x)\}. \quad (4.43)$$

28
Here, the weights \( \lambda, \kappa, \mu \) all belong to \( S \). Because \( C_{\lambda,\mu} \) is non-zero only when \( \lambda - \mu \) is a root of \( B_n \), we may replace the summation over \( \kappa \) in (4.43) by a summation over roots, \( \delta = \lambda - \kappa \) in the first sum, and \( \delta = \kappa - \mu \) in the second sum. We also separate the summation over the roots according to the value of \( C_{\lambda,\mu} \) and express the corresponding conditions on \( \delta \) in terms of inner products \( \lambda \cdot \delta \) and \( \mu \cdot \delta \).

\[
sC_{\lambda,\mu} d \cdot (u_\lambda - u_\mu) = \sum_{\delta^2 = 1, \ 2\lambda \cdot \delta = 1, \ 2\mu \cdot \delta = 2 - (\lambda - \mu)^2} \frac{1}{\sqrt{2}} m_1 m_2 \varphi(\delta \cdot x) + \sum_{\delta^2 = 1, \ 2\lambda \cdot \delta = 1, \ 2\mu \cdot \delta = 1 - (\lambda - \mu)^2} \frac{1}{\sqrt{2}} m_1 m_2 \varphi(\delta \cdot x) \\
+ \sum_{\delta^2 = 1, \ 2\lambda \cdot \delta = 1, \ 2\mu \cdot \delta = 1 - (\lambda - \mu)^2} \frac{1}{2} m_1^2 \varphi(\delta \cdot x) + \sum_{\delta^2 = 2, \ 2\lambda \cdot \delta = 1 - (\lambda - \mu)^2} m_2^2 \varphi(\delta \cdot x) - (\lambda \leftrightarrow \mu)
\]

(4.44)

We analyze this equation for each possible orbits of \( \lambda - \mu \in [T_p] \) with value of \((\lambda - \mu)^2 = p, p = 1, \cdots, n \). Since in each of the above sums, \( \delta \) is a root, it is clear from the form of the weights of \( S \) in (4.33) that \( \lambda \cdot \delta \) and \( \mu \cdot \delta \) can only take on the values 0, \( \pm \delta^2/2 \). For \( p \geq 4 \), there are no roots in any of the 8 sums in (4.44) that satisfy the inner product relations. As a result, for \( p \geq 4 \), the r.h.s. of (4.44) vanishes, and in view of the first equation in (4.35), the l.h.s. vanishes as well. For \( p = 3 \), the third and fourth sums in (4.44) vanish since the inner product relations cannot be satisfied for them. The first and second sums (and their \( \lambda \leftrightarrow \mu \) contribution) precisely cancel one another, the r.h.s. of (4.44) vanishes, and so does the l.h.s. in view of the first equation in (4.35). Finally, the remaining equations for \( p = 1 \) and \( p = 2 \) are found to be proportional to one another and to

\[
sd \cdot (u_\lambda - u_\mu) = \sum_{\delta^2 = 2, \ 2\lambda \cdot \delta = 1; \ 2\mu \cdot \delta = 0} m_2 \varphi(\delta \cdot x) - \sum_{\delta^2 = 2, \ 2\lambda \cdot \delta = 0; \ 2\mu \cdot \delta = -1} m_2 \varphi(\delta \cdot x).
\]

(4.45)

But, (4.45) is easily solved and we recover the second equation in (4.35).

The proof of the Lax pair in (4.36) in one of the two Weyl spinor representations of \( D_n \) is analogous to the case of \( B_n \), so we only give an outline here. The conjugates of the spinor representations \( S_{\pm} \) are obtained by reversing the sign of the weights in (4.34) and corresponds to \( S_{\pm} \) when \( n \) is even, but to \( S_{\mp} \) when \( n \) is odd. Henceforth, we concentrate on the case +. The tensor decomposition \( S_+ \otimes S_+^* \) is given by

\[
S_+ \otimes S_+^* = \sum_{p=0}^{\lfloor n/2 \rfloor} 2^{n-2p-1} [T_{2p}]
\]

(4.46)

Here, \([T_p]\) are again the Weyl orbits of the anti-symmetric tensor representations of rank \( p \), but this time of \( D_n \). Their weight system and length are just as given in (4.38). We
now find $s^2 = 2^{n-3}$ and solve conditions (1) and (2) of Theorem 2 by $C_{\lambda,\mu}^2 = m_2^2$, and by taking square roots with the same sign, we recover the first equation in (4.36). Condition (3) reduces to (4.44) but now with $m_1 = 0$, and is solved analogously.

V. THE EXCEPTIONAL LIE ALGEBRAS : UNTWISTED CASES

In this section, we apply the construction of §III, and obtain a Lax pair with spectral parameter and one independent coupling for each of the five (untwisted) elliptic Calogero-Moser systems for exceptional Lie algebras. The Lax pairs are built out of the following representations: for $E_6$, $E_7$ and $E_8$, in the representations of dimensions 27, 56 and 248 respectively. For the case of $E_8$, the analysis is complete only up to the determination of certain sign assignments which we have not constructed explicitly. For $G_2$, we construct Lax pairs in the representations of dimension 7 (the fundamental) and 8 (the fundamental plus a singlet). For $F_4$, we find a Lax pair in a 27-dimensional representation which is the direct sum of the fundamental and a singlet. For conventions and general information on the group theory used here, we refer to Appendix §A.

(a) Untwisted Elliptic Calogero-Moser System for $E_6$

We start by embedding the 27-dimensional representation of highest weight (100000) into the fundamental representation of $GL(27, \mathbb{C})$. This representation is denoted by $27$ for short; its complex conjugate, the $27^*$ has highest weight (000010). The weights of the $27$ are given in terms of 6 orthonormal vectors $e_i, i = 1, \cdots, 6$ by

$$\lambda = \left\{ \begin{array}{ll}
+ \frac{2}{\sqrt{3}}e_6 \\
+ \frac{1}{2\sqrt{3}}e_6 - \frac{1}{2} \sum_{i=1}^{5} \epsilon_i e_i & \text{with } \prod_{i=1}^{5} \epsilon_i = 1 \\
- \frac{1}{\sqrt{3}}e_6 ± e_i.
\end{array} \right. \quad (5.1)$$

Here, $\epsilon_i = \pm 1$. All weights belong to a single Weyl orbit of $E_6$, denoted by $[100000]$ and have the same length $\lambda^2 = 4/3$.

Theorem 6 : Lax pair for $E_6$ in the $27$

The untwisted elliptic Calogero-Moser Hamiltonian for $\mathcal{G} = E_6$ admits a Lax pair with spectral parameter and one independent coupling in the $27$ of $E_6$, given by (3.5), (3.6), (3.16) and

$$C_{\lambda,\mu} = \left\{ \begin{array}{ll}
m_2 & (\lambda - \mu)^2 = 2 \\
0 & \text{otherwise},
\end{array} \right. \quad \sqrt{6d} \cdot u_\lambda = \sum_{\lambda,\delta=1} m_{2\varphi} (\delta \cdot x). \quad (5.2)$$
To prove this Theorem, we show that the conditions of Theorem 2 are satisfied. The weights of the fundamental representation of $GL(27, \mathbb{C})$ are denoted by 27 orthogonal vectors $u_\lambda$, where $\lambda$ runs over the weights of the $27$ of $E_6$. The decomposition (3.1) is given by $s^2 = 6$ and

$$\sqrt{6}u_\lambda = \lambda + v_\lambda,$$  

(5.3)

where the vectors $v_\lambda$ are orthogonal to the weight space of $E_6$. The roots of $GL(27, \mathbb{C})$ decompose under $E_6$ as follows

$$\sqrt{6}(u_\lambda - u_\mu) = \lambda - \mu + v_\lambda - v_\mu \quad \lambda \neq \mu.$$  

(5.4)

Under the Weyl group of $E_6$, these roots transform in different orbits, which are precisely the Weyl orbits occurring in the tensor product $27 \otimes 27^*$ of $E_6$. They are given by

| Weyl Orbit | Multiplicity | # Weights | Length $^2$ |
|------------|--------------|-----------|-------------|
| $27 \otimes 27^*$ : | [000000] | 27 | 1 | 0 |
| [000001] | 6 | 72 | 2 |
| [100010] | 1 | 270 | 4 |

(5.5)

The orbits [000000], corresponding to the trivial representation of $E_6$, do not actually occur in (5.4), since we are restricting to the off-diagonal elements for which $\lambda \neq \mu$.

Condition (4) of Theorem 2 applies to the weights $\lambda - \mu$ in orbit [100010], and readily implies that $C_{\lambda,\mu} = 0$ when $(\lambda - \mu)^2 = 4$. Conditions (1) and (2) for the roots $\alpha = \lambda - \mu$ in orbit [000001], are given by a sum over the 6 possible orbits in which $\alpha$ can lie:

$$6m_2^2 = \sum_{\lambda-\mu=\alpha} C_{\lambda,\mu}^2$$

$$0 = \sum_{\lambda-\mu=\alpha} C_{\lambda,\mu}^2 (v_\lambda - v_\mu).$$  

(5.6)

Taking the inner product of the second equation with an arbitrary vector $u_\sigma$ with $\sigma \in \mathcal{R}(E_6)$ yields the equation

$$\alpha \cdot \sigma m_2^2 = \sum_{\lambda-\mu=\alpha} C_{\lambda,\mu}^2 (\delta_{\lambda,\sigma} - \delta_{\mu,\sigma}).$$  

(5.7)

Now, since $\alpha$ is a root, and $\lambda$ is in one of the fundamental representations of $E_6$, the combination $\alpha \cdot \sigma$ can take on only the values $\alpha \cdot \sigma = -1, 0, 1$. When $\alpha \cdot \sigma = 0$, $\sigma \pm \alpha$ are not weights of the $27$; thus all terms in (5.7) vanish separately. When $\alpha \cdot \sigma = \pm 1$, $\sigma \mp \alpha$ is a weight (but $\sigma \pm \alpha$ is not), so that (5.7) yields $C_{\lambda,\mu}^2 = m_2^2$ when $(\lambda - \mu)^2 = 2$. In fact,
we shall find that a solution exists where all square roots have the same sign, and we thus recover the first equation in (5.2).

It remains to satisfy condition (3) of Theorem 2, i.e. (3.19). When \( \alpha = \lambda - \mu \) belongs to the orbit \([100010]\) (for which \( \alpha^2 = 4 \) and \( \lambda \cdot \mu = -2/3 \)), (3.19) becomes

\[
0 = \sum_{\kappa \cdot \lambda = \kappa \cdot \mu = 1/3} m_2^2 \{ \varphi((\lambda - \kappa) \cdot x) - \varphi((\kappa - \mu) \cdot x) \}. \tag{5.8}
\]

By changing variables \( \delta = \kappa - \lambda \) in the first sum and \( \delta = \kappa - \mu \) in the second sum, this equation may be rewritten in terms of a sum over roots \( \delta \)

\[
0 = \sum_{\delta \cdot \lambda = -\delta \cdot \mu = -1} m_2^2 \varphi(\delta \cdot x) - \sum_{\delta \cdot \lambda = -\delta \cdot \mu = 1} m_2^2 \varphi(\delta \cdot x). \tag{5.9}
\]

Since the argument of the above sum is even in \( \delta \), this relation is automatically satisfied.

When \( \alpha = \lambda - \mu \) belongs to the orbit \([000001]\) (for which \( \alpha^2 = 2 \) and \( \lambda \cdot \mu = 1/3 \)), (3.19) becomes

\[
\sqrt{6}m_2d \cdot (u_\lambda - u_\mu) = \sum_{\kappa \cdot \lambda = \kappa \cdot \mu = 1/3} m_2^2 \{ \varphi((\lambda - \kappa) \cdot x) - \varphi((\kappa - \mu) \cdot x) \}. \tag{5.10}
\]

By changing variables to \( \delta = \kappa - \lambda \) in the first sum and to \( \delta = \kappa - \mu \) in the second sum, this equation may be rewritten in terms of a sum over roots \( \delta \)

\[
\sqrt{6}m_2d \cdot (u_\lambda - u_\mu) = \sum_{\delta \cdot \lambda = 1; \delta \cdot \mu = 0} m_2^2 \varphi(\delta \cdot x) - \sum_{\delta \cdot \lambda = 0; \delta \cdot \mu = 1} m_2^2 \varphi(\delta \cdot x). \tag{5.11}
\]

The solution to this equation is readily obtained by extending the first sum to include \( \delta \cdot \mu = \pm 1 \) and the second sum to include \( \delta \cdot \lambda = \pm 1 \) without affecting the left hand side. Doing so, we recover the second equation in (5.2).

(b) Untwisted Elliptic Calogero-Moser System for \( E_7 \)

We start by embedding the 56-dimensional representation of \( E_7 \) with highest weight \((0000010)\) (denoted 56 for short) into the fundamental representation of \( GL(56, \mathbb{C}) \). The weights of the 56 are given in terms of 7 orthonormal vectors \( e_i, i = 1, \cdots, 7 \) by

\[
\lambda_I = \begin{cases} 
\pm \frac{1}{2} \sum_{i=1}^{6} \epsilon_i e_i & \text{with} \quad \prod_{i=1}^{6} \epsilon_i = 1 \\
\pm \frac{1}{\sqrt{2}} e_7 + e_i, & \pm \frac{1}{\sqrt{2}} e_7 - e_i \\
\end{cases} 
\quad i = 1, \cdots, 6. \tag{5.12}
\]
All weights belong to a single Weyl orbit of \( E_7 \), denoted by \([0000010]\) and have the same length \( \lambda^2 = 3/2 \).

**Theorem 7 : Lax pair for \( E_7 \) in the 56**

The untwisted elliptic Calogero-Moser Hamiltonian for \( E_7 \) admits a Lax pair with spectral parameter and one independent coupling in the 56 of \( E_7 \), given by (3.5), (3.6), (3.16) and

\[
C_{\lambda,\mu} = \begin{cases} 
  m_2 & (\lambda - \mu)^2 = 2 \\
  0 & \text{otherwise},
\end{cases} \quad \sqrt{12}d \cdot u_\lambda = \sum_{\lambda,\delta=1} m_2 \phi(\delta \cdot x). \quad (5.13)
\]

The proof is completely analogous to the case of \( E_6 \). The weights of the fundamental representation of \( GL(56, \mathbb{C}) \) are denoted by 56 orthonormal vectors \( u_\lambda \), where \( \lambda \) runs over the weights of the 56 of \( E_7 \). The decomposition (3.1) is given by \( s^2 = 12 \) and

\[
\sqrt{12}u_\lambda = \lambda + v_\lambda, \quad (5.14)
\]

where the vectors \( v_\lambda \) are orthogonal to the weights space of the 56 of \( E_7 \). The roots of \( GL(56, \mathbb{C}) \) decompose under \( E_7 \) as follows

\[
\sqrt{12}(u_\lambda - u_\mu) = \lambda - \mu + v_\lambda - v_\mu \quad \lambda \neq \mu. \quad (5.15)
\]

Under the Weyl group of \( E_7 \), these roots transform under the different Weyl orbits that occur in the tensor product 56 \( \otimes \) 56 of \( E_7 \). They are given by

| Weyl Orbit | Multiplicity | # Weights | Length^2 |
|------------|--------------|-----------|-----------|
| 56 \( \otimes \) 56 : \[0000000]\ | 56 | 1 | 0 |
| \[1000000]\ | 12 | 126 | 2 |
| \[0000100]\ | 2 | 756 | 4 |
| \[0000020]\ | 1 | 56 | 6 |

(5.16)

Again, the orbits \([0000000]\) do not occur in (5.15).

Applying (4) of Theorem 2, we readily have \( C_{\lambda,\mu} = 0 \), whenever \((\lambda - \mu)^2 \neq 2\). Conditions (1) and (2) for roots \( \alpha = \lambda - \mu \) in the remaining 12 orbits \([1000000]\) with \( \alpha^2 = 2 \) and \( \lambda \cdot \mu = 1/2 \) give

\[
12m_2^2 = \sum_{\lambda - \mu = \alpha} C_{\lambda,\mu}^2 \quad 0 = \sum_{\lambda - \mu = \alpha} C_{\lambda,\mu}^2 (v_\lambda - v_\mu). \quad (5.17)
\]
Taking the inner product with an arbitrary vector \( u_\sigma \) with \( \sigma \in 27 \) yields the equation

\[
\alpha \cdot \sigma m_2^2 = \sum_{\lambda-\mu=\alpha} C_{\lambda,\mu}^2 (\delta_{\lambda,\sigma} - \delta_{\mu,\sigma}).
\] (5.18)

Since \( \alpha \) is a root, and \( \sigma \) is a weight of a fundamental representation, the product \( \alpha \cdot \sigma \) can take on only the values \( \alpha \cdot \sigma = -1, 0, +1 \). When \( \alpha \cdot \sigma = 0 \), \( \sigma \pm \alpha \) are not weights of the 56, while \( \sigma \pm \alpha \) are not. Hence, (5.18) yields \( C_{\lambda,\mu}^2 m_2^2 \) whenever \( (\lambda - \mu)^2 = 2 \). In fact, we shall find a solution where all the square roots may be taken with a positive sign, so that we recover the first equation in (5.13).

It remains to satisfy condition (3) of Theorem 2, i.e. (3.19). The cases \( \alpha = \lambda - \mu \) with either \( \alpha^2 = 4 \) or \( \alpha^2 = 6 \) are satisfied by arguments analogous to (5.8) of \( E_6 \). The remaining equation for roots \( \alpha = \lambda - \mu \) is given by

\[
\sqrt{12m_2d} \cdot (u_\lambda - u_\mu) = \sum_{\kappa.\lambda=\kappa.\mu=1/2} m_2^2 \{ \varphi((\lambda - \kappa) \cdot x) - \varphi((\kappa - \mu) \cdot x) \}. \] (5.19)

It is easily solved without any further restrictions, and we find recover the second equation in (5.13).

(d) Untwisted Elliptic Calogero-Moser System for \( E_8 \)

The lowest dimensional representation of \( E_8 \) is the adjoint of dimension 248, denoted by \( 248 \) for short. Its weights are given in terms of 8 orthonormal vectors \( e_i, i = 1, \ldots, 8 \), as follows. There are 8 zero weights, and 240 (non-zero) roots, given by

\[
\lambda = \begin{cases} 
\pm e_i \pm e_j, & \text{with } i \neq j \\
\frac{1}{2} \sum_{i=1}^{8} \epsilon_i e_i, & \text{with } \prod_{i=1}^{8} \epsilon_i = 1
\end{cases}
\] (5.20)

All roots belong to a single Weyl orbit of \( E_8 \), denoted by \([10000000]\) and have the same length \( \lambda^2 = 2 \). We embed the 248 of \( E_8 \) into the fundamental representation of \( GL(248, \mathbb{C}) \), whose weights are an orthonormal set of vectors \( u_I, I = 1, \ldots, 248 \). It is convenient to labels the first 240 of these weights by the 240 roots \( \lambda \) of \( E_8 : u_\lambda \), and the last 8 by an index \( a = 1, \ldots, 8 \) which distinguishes the zero weights of \( E_8 \).

The presence of the 8 zero weights in the 248 gives rise to serious complications in the construction of the Calogero-Moser Lax pair : the centralizer in \( GL(248, \mathbb{C}) \) of the
Cartan subalgebra of $E_8$ is now larger than the Cartan subalgebra of $GL(248, \mathbb{C})$. As a result, the space $GL_0$ of roots of $GL(248, \mathbb{C})$ that restrict to 0 in $E_8$ is non-trivial, with $\dim GL_0 = 56$, and on general grounds, we are led to include the term $\Delta \in GL_0$ in the construction of the Lax pair for $E_8$ in (3.5), (3.6). However, a Lax pair still involves only a single function $\Phi$, as in (3.16), and symmetric constants $C_{I,J} = C_{J,I}$.

The $E_8$ root system contains a maximal set of 8 mutually orthogonal roots, which we shall denote by $\beta_a$, $a = 1, \cdots, 8$, with $\beta_a \cdot \beta_b = 2\delta_{a,b}$. This set specifies a maximal $SU(2)^8$ subalgebra of $E_8$ that will play a special role in the sequel.

**Theorem 8 : Lax pair for $E_8$ in the 248**

The untwisted elliptic Calogero-Moser system for $E_8$ admits a Lax pair with spectral parameter and one independent coupling $m_2$ given by (3.5), (3.6), (3.16) and

\[
C_{\lambda,\mu} = \begin{cases} 
m_2 c(\lambda, \mu) & \lambda \cdot \mu = 1 \\
0 & \text{otherwise} \end{cases} \quad c(\lambda, \mu) = \pm 1 (5.21a)
\]

\[
C_{\lambda,c} = \frac{1}{2} \sum_{a=1}^{8} \frac{1}{2} (\lambda \cdot \beta_a) c(\lambda, \beta_a(\lambda \cdot \beta_a)) C_{\beta_a,c} \quad \lambda \neq \pm \beta_b \\
\pm C_{\beta_a,c} \quad \lambda = \pm \beta_b (5.21b)
\]

\[
\sqrt{60d} \cdot u_\lambda = \sum_{\delta, \lambda=1}^{8} m_2 \varphi(\delta \cdot x) + 2m_2 \varphi(\lambda \cdot x) (5.21c)
\]

\[
\Delta_{a,b} = \frac{m_2}{2} \left( c(\beta_a, \delta) c(\delta, \beta_b) + c(\beta_a, \beta_a - \delta) c(\beta_a - \delta, -\beta_b) \right) \varphi(\delta \cdot x) \\
- \sum_{\delta, \beta_a=1}^{8} \frac{m_2}{2} \left( c(\beta_a, \delta) c(\delta, -\beta_b) + c(\beta_a, \beta_a - \delta) c(\beta_a - \delta, \beta_b) \right) \varphi(\delta \cdot x) (5.21d)
\]

\[
\Delta_{aa} = \sum_{\beta_a, \delta=1}^{8} m_2 \varphi(\delta \cdot x) + 2m_2 \varphi(\beta_a \cdot x), (5.21e)
\]

provided there exists a solution to the following $\pm 1$ valued cocycle factors $c(\lambda, \mu)$

\[
c(\lambda, \lambda - \delta) c(\lambda - \delta, \mu) = c(\lambda, \mu + \delta) c(\mu + \delta, \mu) \quad \text{when } \delta \cdot \lambda = -\delta \cdot \mu = 1, \lambda \cdot \mu = 0 (5.22a)
\]

\[
c(\lambda, \mu) c(\lambda - \delta, \mu) = c(\lambda, \lambda - \delta) \quad \text{when } \delta \cdot \lambda = \lambda \cdot \mu = 1, \delta \cdot \mu = 0 (5.22b)
\]

\[
c(\lambda, \mu) c(\lambda, \lambda - \mu) = -c(\lambda - \mu, -\mu) \quad \text{when } \lambda \cdot \mu = 1. (5.22c)
\]

We conjecture that a solution exists to these cocycle condition of (5.22). (The matrix $C_{\beta_a,c}$, $b, c = 1, \cdots, 8$ is proportional to an arbitrary $8 \times 8$ orthogonal matrix, as will be discussed below.)
To prove this Theorem, we use the fact that only a single function $\Phi$ is involved by (3.16), and that $C_{I,J}$ is symmetric. Thus, we may resort to the simplifications of conditions (1), (2) and (4) of Theorem 2, since these are independent of $\Delta$. However, as argued previously, $\Delta \neq 0$ now, so we need to keep condition (3) of Theorem 1. We begin by decomposing the weights of $GL(248, \mathbb{C})$ under $E_8$ according to (3.1); we find $s^2 = 60$ and

$$\sqrt{60}u_\lambda = \lambda + v_\lambda \quad \lambda \in \mathcal{R}(E_8)$$

$$\sqrt{60}u_a = v_a \quad a = 1, \cdots, 8$$

(5.23)

The vectors $v_\lambda$ are orthogonal to $E_8$ roots, while vectors $v_a$ are orthogonal to both roots of $E_8$ and to all $v_\lambda$. The roots of $GL(248, \mathbb{C})$ decompose under $E_8$ as follows

$$\sqrt{60}(u_\lambda - u_\mu) = \lambda - \mu + v_\lambda - v_\mu, \quad \lambda \neq 0$$

$$\sqrt{60}(u_\lambda - u_a) = \lambda + v_\lambda - v_a$$

$$\sqrt{60}(u_a - u_\mu) = -\mu + v_a - v_\mu$$

$$\sqrt{60}(u_a - u_b) = v_a - v_b, \quad a \neq b,$$

where $\lambda, \mu \in \mathcal{R}(E_8)$ and $a, b = 1, \cdots, 8$. The last line of roots $u_a - u_b$ in (5.24) span the space $GL_0$, discussed previously. Under the Weyl group of $E_8$, the roots of (5.24) transform in the orbits occurring in the tensor product $248 \otimes 248$. They are given by

| Weyl Orbit | Multiplicity | # Weights | Length \(^2\) |
|------------|--------------|-----------|------------|
| $248 \otimes 248$ : [00000000] | 304 | 1 | 0 |
| [10000000] | 72 | 240 | 2 | (5.25) |
| [00000010] | 14 | 2160 | 4 |
| [01000000] | 2 | 6720 | 6 |
| [20000000] | 1 | 240 | 8 |

The orbits [00000000] do not occur in (5.24).

As argued previously, we may apply condition (4) of Theorem 2 to this case. We readily deduce that $C_{\lambda,\mu} = 0$ whenever $(\lambda - \mu)^2 \neq 2$, and this yields the second line in (5.21a). Conditions (1) and (2) of Theorem 2 for roots $\alpha = \lambda - \mu$ in the remaining 72 orbits [10000000] with $\alpha^2 = 2$ and $\lambda \cdot \mu = 1$ are given by

$$60m^2 = \sum_{\lambda-\mu=\alpha} C^2_{\lambda,\mu} + \sum_{b=1}^{8} \{C^2_{\alpha,b} + C^2_{b,-\alpha}\}$$

$$0 = \sum_{\lambda-\mu=\alpha} C^2_{\lambda,\mu}(v_\lambda - v_\mu) + \sum_{b=1}^{8} \{C^2_{\alpha,b}(v_\alpha - v_b) + C^2_{b,-\alpha}(v_b - v_\alpha)\}$$

(5.26)
Using linear independence of the vectors $v_b$ from $v_\lambda$, and symmetry of the coefficients $C_{I,J}$, we find that

$$C^2_{-\alpha,b} = C^2_{\alpha,b}$$  \hfill (5.27)

Taking the inner product with a vector $u_{\sigma}$ for an arbitrary root $\sigma$ of the second equation in (5.26), using (3.3) and combining with the first equation in (5.26), we obtain

$$\alpha \cdot \sigma m_2^2 = \sum_{\lambda-\mu=\alpha} C^2_{\lambda,\mu}(\delta_{\lambda,\sigma} - \delta_{\mu,\sigma}) + (\delta_{\alpha,\sigma} - \delta_{-\alpha,\sigma})C_{\alpha} \cdot C_{\alpha}. \hfill (5.28)$$

We solve this equation for each of the possible values of $\alpha \cdot \sigma$. Since both $\alpha$ and $\sigma$ are roots, the allowed values are $\alpha \cdot \sigma = 0, \pm 1, \pm 2$. When $\alpha \cdot \sigma = 0$, $\alpha \pm \sigma$ are not roots, so that all sides of (5.28) vanish separately. If $\alpha \cdot \sigma = \pm 1$, $\alpha \mp \sigma$ is a root, and we find

$$C^2_{\lambda,\mu} = m_2^2, \hfill (5.29)$$

which yields the first line in (5.21a), upon taking the square root, and introducing some as yet unspecified sign factors $c(\lambda, \mu)$. Finally, if $\alpha \cdot \sigma = \pm 2$, then $\sigma = \pm \alpha$ and (5.28) reduces to $C_{\alpha} \cdot C_{\alpha} = 2m_2^2$. With these results, the first equation of (5.26) is automatically satisfied, using the fact that a given root can be written as the difference between two (non-zero) roots in 56 different ways!

It remains to satisfy condition (3) of Theorem 1. Amongst other relations, this will give rise to a number of conditions on the inner products $C_{\lambda} \cdot C_{\mu}$, and it is convenient to record those here,

$$C_{\lambda} \cdot C_{\mu} = \left\{ \begin{array}{ll} \pm 2m_2^2 & \lambda \cdot \mu = \pm 2 \\
 \pm c(\lambda, \mu)m_2^2 & \lambda \cdot \mu = \pm 1 \\
 0 & \lambda \cdot \mu = 0. \end{array} \right. \hfill (5.30)$$

The first line of (5.30) with the + sign was already obtained above.

Condition (3) of Theorem 1 reduces to two sets of equations

$$sC_{\lambda,\mu}d \cdot (u_{\lambda} - u_{\mu}) = \sum_{\kappa \neq \lambda,\mu} C_{\lambda,\kappa}C_{\kappa,\mu}\{\varphi((\lambda - \kappa) \cdot x) - \varphi((\kappa - \mu) \cdot x)\}$$

$$+ C_{\lambda} \cdot C_{\mu}\{\varphi(\lambda \cdot x) - \varphi(\mu \cdot x)\} \hfill (5.31a)$$

$$sC_{\lambda,b}d \cdot u_{\lambda} - \sum_a C_{\lambda,a}\Delta_{ab} = \sum_{\kappa \neq \lambda} C_{\lambda,\kappa}C_{\kappa,b}\{\varphi((\lambda - \kappa) \cdot x) - \varphi(\kappa \cdot x)\} \hfill (5.31b)$$

# Henceforth, we shall assemble the 8 components of $C_{\lambda,a}$, for $a = 1, \cdots, 8$ into an 8-dimensional vector, simply denoted by $C_{\lambda}$. Inner products then stand for $C_{\lambda} \cdot C_{\mu} = \sum_{a=1}^{8} C_{\lambda,a}C_{\mu,a}$. 

37
To solve (5.31a), we recast the sums in terms of \( \delta = \lambda - \kappa \) in the first and \( \delta = \kappa - \mu \) in the second sum. As a result, we have

\[
\sum_{\delta = \lambda \mu = 1} \delta \cdot \lambda, \mu \cdot (u_{\lambda} - u_{\mu}) = \sum_{\delta = \lambda = \mu = 1} \delta \cdot \lambda, \mu \cdot (u_{\lambda} - u_{\mu}) \psi(\delta \cdot x) - \sum_{\delta = \lambda = 1 - \mu \mu = -1} \delta \cdot \lambda, \mu \cdot (u_{\lambda} - u_{\mu}) \psi(\delta \cdot x)
\]

\[+ C_{\lambda} C_{\mu} \{ \psi(\lambda \cdot x) - \psi(\mu \cdot x) \}. \]

(5.32)

The case \( \lambda \cdot \mu = 2 \) is excluded, while the case \( \lambda \cdot \mu = -2 \) is automatically satisfied, since both sides vanish identically. When \( \lambda \cdot \mu = 0 \), the l.h.s. of (5.32) vanishes, since \( \lambda - \mu \) is not a root. The two sums on the r.h.s. of (5.32) cancel one another provided we demand (5.22b), and \( C_{\lambda} C_{\mu} = 0 \), which is precisely the last equation in (5.22a). When \( \lambda \cdot \mu = -1 \), the l.h.s. of (5.32) vanishes, and each sum on the r.h.s. reduces to a single term, since \( \delta \cdot \mu = -2 \) (i.e. \( \delta = -\mu \)) in the first and \( \delta \cdot \lambda = 2 \) (i.e. \( \delta = \lambda \)) in the second sum. This condition reduces to

\[
C_{\lambda} C_{\mu} = m_{2}^{2} \, c(\lambda, \lambda + \mu) c(\lambda + \mu, \mu) \quad \lambda \cdot \mu = -1
\]

(5.33)

Finally, for \( \lambda \cdot \mu = 1 \), (5.32) is equivalent to

\[
\sum_{\delta = \lambda = \mu = 1} \delta \cdot \lambda, \mu \cdot (u_{\lambda} - u_{\mu}) = \sum_{\delta = \lambda = \mu = 1} \delta \cdot \lambda, \mu \cdot (u_{\lambda} - u_{\mu}) \psi(\delta \cdot x)
\]

\[+ \frac{1}{m_{2}} c(\lambda, \mu) C_{\lambda} C_{\mu} \{ \psi(\lambda \cdot x) - \psi(\mu \cdot x) \}. \]

(5.34)

The l.h.s. of (5.34) is a difference of the function \( sd \cdot u_{\alpha} \), evaluated at \( \lambda = \alpha \) and \( \alpha = \mu \), and the r.h.s. must also be such a difference. In order to split the first two terms, it is sufficient (and one can show also necessary) that the product \( c(\lambda, \mu) c(\lambda, \lambda - \delta) c(\lambda - \delta, \mu) \psi(\delta \cdot x) \) be independent of \( \lambda \) and \( \mu \). As a result, it is independent of \( \delta \) as well. Its values can be only \( \pm 1 \), and by choosing the sign of \( m_{2} \), we can choose this product to be +1 as in (5.22c). The last term must also split, so that the product \( c(\lambda, \mu) C_{\lambda} C_{\mu} \) must also be independent of \( \lambda \) and \( \mu \). Introducing a suitable new constant \( m \), we may express the last fact as

\[
C_{\lambda} C_{\mu} = (m^{2} - m_{2}^{2}) c(\lambda, \mu) \quad \lambda \cdot \mu = 1
\]

(5.35)

Once these conditions are fulfilled, the solution of (5.32) is readily obtained

\[
\sum_{\lambda \cdot \delta = 1} \delta \cdot \lambda, \mu \cdot (u_{\lambda} - u_{\mu}) = \sum_{\lambda \cdot \delta = 1} \delta \cdot \lambda, \mu \cdot (u_{\lambda} - u_{\mu}) \psi(\delta \cdot x) + \frac{m_{2}^{2}}{m_{2}^{2}} \psi(\lambda \cdot x).
\]

(5.36)
It remains to solve for \( m \) and to analyze (5.31b).

To determine \( m \), we make use of the fact that conditions (1) and (2) of Theorem 2 as well as condition (3) of Theorem 1 (i.e. equations (5.31)) are invariant under the following constant similarity transformation of \( L \) and \( M \), as given in (2.8), with \( \eta = \pm 1; \)

\[
C_{\lambda,\mu} \rightarrow \eta \eta \mu C_{\lambda,\mu} \\
C_{\lambda,a} \rightarrow \eta \eta \beta a C_{\lambda,a} \\
\Delta_{a,b} \rightarrow \eta \beta a \eta \beta b \Delta_{a,b}.
\] (5.37)

It may be checked explicitly that the equations (5.21) and (5.22) of Theorem 8 are indeed invariant under these transformations.

The solution of (5.27) is \( C_{-\alpha,b} = \pm C_{\alpha,b} \), and the sign in general depends upon the root \( \alpha \) and the zero weight label \( a \). Let us pick an orthonormal basis of roots \( \beta_a, a = 1, \cdots, 8, \) discussed before Theorem 8. We choose a basis for the vectors \( C_{\beta,a} \), in which only one component is non-vanishing. Then by the above argument, we must have that \( C_{-\beta,a} = \pm C_{\beta,a} \). Furthermore, by making a transformation of the form (5.37), we may choose the sign for each of the 8 vectors \( C_{\beta,a} \) at will. We shall choose \( C_{-\beta,a} = -C_{\beta,a} \) for all \( a = 1, \cdots, 8 \). Let \( \lambda \) be a root such that \( \lambda \cdot \beta_a = 1 \), for some \( a \). Since the vectors \( C_{\beta,a} \) and \( C_{-\beta,a} \) are now opposites of one another, we obtain a relation between equations (5.33) and (5.35).

\[
C_{\lambda} \cdot C_{-\beta,a} = -C_{\lambda} \cdot C_{\beta,a} = m_2^2 c(\lambda, \lambda - \beta_a)c(\lambda - \beta_a, \beta_a) = (m_2^2 - m_2^2)c(\lambda, \beta_a).
\] (5.38)

Since \( c = \pm 1 \), there are two possible solutions for the constant \( m : m_2^2 = 0 \) or \( m_2^2 = 2m_2^2 \). It turns out that (5.31b) is inconsistent for \( m_2^2 = 0 \), and we shall now proceed to show that the solution (5.21d,e) exists for

\[
m_2^2 = 2m_2^2 \\
C_{-\lambda} = -C_{\lambda} \\
c(-\lambda, -\mu) = c(\lambda, \mu).
\] (5.39)

Consistency of (5.33) and (5.35) with (5.39) requires that condition (5.22c) be satisfied, and this yields the middle equation in (5.30). Choosing \( C_{\lambda} \) to be odd in \( \lambda \) restricts the symmetry of (5.37) to the subgroup of transformations for which \( \eta = \eta_{-\lambda} \).

Finally, we show that to satisfy (5.31b), the equations of (5.21a,b,c) and (5.22) suffice, and produce (5.21d,e). This in itself is remarkable since (5.31b) is a set of 8 \times 240 equations,
for only $8 \times 8$ remaining unknowns $\Delta_{a,b}$. The arguments and the calculations are lengthy and are deferred to Appendix §C.

(d) Untwisted Elliptic Calogero-Moser System for $G_2$

The smallest non-trivial representation of the Lie algebra $G_2$ is of dimension 7, and denoted 7 for short. Its weights are given in terms of 3 orthonormal vectors $e_i$, $i = 1, 2, 3$. It is convenient to introduce $e_0 = \frac{1}{3}(e_1 + e_2 + e_3)$ so that

$$\lambda_I = \begin{cases} \pm \alpha_i = \pm(e_i - e_0) & (I = 1, \cdots, 6) \\ 0 & (I = 7). \end{cases}$$

(5.40)

The precise correspondence between the labels $\pm$, $i$ and $I$ is immaterial since the order is permuted by the Weyl group of $G_2$.

The 7 may naturally be embedded into the fundamental of $GL(7, \mathbb{C})$. However, one way to define $G_2$ is as the subalgebra of $B_3$ that leaves one of the spinor weights invariant; thus it is also natural to embed the $7 \oplus 1$ of $G_2$ into the 8-dimensional spinor representation of $B_3$. We shall treat both cases, and use the spinor embedding as an example of a case in which the untwisted elliptic Calogero-Moser Hamiltonian and Lax pair may be obtained by restriction to a subgroup of the Calogero-Moser Hamiltonian and Lax pair of a larger Lie algebra.

Theorem 9 : Lax pairs for $G_2$ in the 7 and in the $7 \oplus 1$

The untwisted elliptic Calogero-Moser Hamiltonian for $G_2$ admits a Lax pair with spectral parameter and one independent coupling

(1) in the 7 of $G_2$, given by (3.5), (3.6), (3.16) and

$$C_{\lambda, \mu} = \begin{cases} m_2 & \lambda \cdot \mu = \pm \frac{1}{3} \\ 0 & \text{otherwise,} \end{cases} \quad C_{\lambda, 7} = \sqrt{2}m_2 \quad (5.41a)$$

$$\sqrt{2}d \cdot u_\lambda = \sum_{\delta^2 - 2, \lambda \delta = 1} m_2 \varphi(\delta \cdot x) + m_2 \varphi(\lambda \cdot x), \quad (5.41b)$$

$$\sqrt{2}d \cdot u_7 = \frac{1}{2} \sum_{\kappa^2 = 2/3} m_2 \varphi(\kappa \cdot x); \quad (5.41c)$$

(2) in the $7 \oplus 1$ of $G_2$, embedded in the spinor representation of $B_3$, given in part (1) of Theorem 5 with

$$m_1 = m_2 \quad \text{and} \quad x \cdot \lambda_s = 0 \quad (5.42)$$

where $\lambda_s$ is any one spinor weight of the spinor of $B_3$. 40
Proof of (1)

We start by embedding the $7$ of $G_2$ into the fundamental representation of $GL(7, \mathbb{C})$, whose weights are $7$ orthonormal vectors $u_I, I = 1, \cdots, 7$. The decomposition (3.1) is given by $s^2 = 2$ and

\[
\sqrt{2}u_\lambda = \lambda + v_\lambda \quad \text{for} \quad \lambda^2 = 2/3,
\]

\[
\sqrt{2}u_\tau = v_\tau.
\]

Here, the vectors $v_\lambda$ and $v_7$ are orthogonal to one another and to the weights of (5.40). The roots of $GL(7, \mathbb{C})$ decompose under $G_2$ as follows

\[
\sqrt{2}(u_\lambda - u_\mu) = \lambda - \mu + v_\lambda - v_\mu, \quad \lambda \neq \mu
\]

\[
\sqrt{2}(u_\lambda - u_7) = \lambda - v_7
\]

\[
\sqrt{2}(u_7 - u_\mu) = -\mu - v_\mu + v_7
\]

Under the Weyl group of $G_2$, these roots transform in the Weyl orbits occurring in the tensor product $7 \otimes 7$ of $G_2$, given by

| Weyl Orbit | Multiplicity | # Weights in Orbit | Length$^2$ |
|------------|--------------|---------------------|------------|
| $7 \otimes 7$ : | [00] | 7 | 1 | 0 |
| [01] | 4 | 6 | 2/3 |
| [10] | 2 | 6 | 2 |
| [02] | 1 | 6 | 8/3 |

(5.45)

The orbits [00] do not occur in (5.43). Using (4) of Theorem 2, and the fact that weights of the orbit [02] are not roots of $G_2$, we readily find that $C_{\lambda,\mu} = 0$ whenever $\lambda \cdot \mu = -2/3$, i.e. $\mu = -\lambda$. The remaining orbits correspond to roots, and we have two independent couplings: $m_2$ for long roots, and $m_{2/3}$ for long roots.

Conditions (1) and (2) for the weights $\alpha = \lambda - \mu$ in the two Weyl orbits of type [10] and length $\alpha^2 = 2$ are given by

\[
2m_2^2 = C_{\lambda,\mu}^2 + C_{-\mu,-\lambda}^2
\]

\[
0 = C_{\lambda,\mu}^2(v_\lambda - v_\mu) + C_{-\mu,-\lambda}^2(v_{-\mu} - v_{-\lambda}).
\]

(5.46)

Taking the inner product of the second equation with an arbitrary vector $u_\sigma$ for $\sigma^2 = 2/3$, this set of equations reduces to

\[
\alpha \cdot \sigma m_2^2 = C_{\lambda,\mu}^2(\delta_{\lambda,\sigma} - \delta_{\mu,\sigma}) + C_{-\mu,-\lambda}^2(\delta_{-\mu,\sigma} - \delta_{-\lambda,\sigma}).
\]

(5.47)

Since $\alpha$ is a root with $\alpha^2 = 2$, the combination $\alpha \cdot \sigma$ can only assume integer values; since $|\alpha \cdot \sigma| < 2$, only three possible values $\alpha \cdot \sigma = -1, 0, +1$ are allowed. When $\alpha \cdot \sigma = 0$,
all terms in (5.47) vanish separately. When \( \alpha \cdot \sigma = \pm 1 \), then \( \sigma \mp \alpha \) is also a weight and (5.47) reduces to \( C^2_{\lambda, \mu} = m^2_2 \) when \((\lambda - \mu)^2 = 2\). It turns out that a Lax pair may be found in which all the square roots are taken with the same signs, which gives the statement of (5.41a) for \( \lambda \cdot \mu = -1/3 \).

Conditions (1) and (2) for the weights \( \alpha = \lambda - \mu \) with \( \lambda \cdot \mu = 1/3 \) or \( \alpha = \lambda \), in the four Weyl orbits of type [01] and length \( \alpha^2 = 2/3 \) lead to the equations

\[
2m^2_{2/3} = C^2_{\lambda, \mu} + C^2_{-\mu, -\lambda} + C^2_{\alpha, \tau} + C^2_{\tau, -\alpha} \\
0 = C^2_{\lambda, \mu}(v_\lambda - v_\mu) + C^2_{\lambda, -\lambda}(v_\mu - v_\lambda) \\
+ C^2_{\alpha, \tau}(v_\alpha - v_\tau) + C^2_{\tau, -\alpha}(v_\tau - v_\alpha).
\] (5.48)

The projection of the last equation in (5.48) onto \( v_\tau \) yields \( C^2_{\tau, -\alpha} = C^2_{\alpha, \tau} \). The remaining equations are analyzed by taking the inner product with a general vector \( u_\sigma \), with \( \sigma^2 = 2/3 \)

\[
\alpha \cdot \sigma m^2_{2/3} = C^2_{\lambda, \mu}(\delta_{\lambda, \sigma} - \delta_{\mu, \sigma}) + C^2_{-\mu, -\lambda}(\delta_{-\mu, \sigma} - \delta_{-\lambda, \sigma}) \\
+ C^2_{\alpha, \tau}(\delta_{\alpha, \sigma} - \delta_{-\alpha, \sigma}).
\] (5.49)

Since \( \alpha \) is a root with \( \alpha^2 = 2/3 \), the combination \( 3\alpha \cdot \sigma \) can take on only integer values; since \(|3\alpha \cdot \sigma| \leq 2\), only the values \( 3\alpha \cdot \sigma = -2, -1, 0, 1, 2 \) are allowed. For \( \alpha \cdot \sigma = 0 \), all terms in (5.49) vanish separately since \( \sigma \neq \pm \alpha \) and \( \sigma \pm \alpha \) are not weights in [01]. Taking respectively \( 3\alpha \cdot \sigma = \pm 2 \) and \( 3\alpha \cdot \sigma = \pm 1 \), we find for \( \lambda \cdot \mu = 1/3 \) and \( \alpha^2 = 2/3 \) that

\[
m^2_{2/3} = 3C^2_{\lambda, \mu} = \frac{3}{2} C^2_{\alpha, \tau}.
\] (5.50)

Finally, we must satisfy condition (3) of Theorem 2, i.e. equation (3.19). We may do this again Weyl orbit by Weyl orbit for the weights \( \alpha = \lambda_I - \lambda_J \) in (3.19). For the orbit [02] with \( \alpha^2 = 8/3 \), (3.9) is satisfied automatically. The remaining equations are as follows. For \( \lambda \cdot \mu = -1/3 \) (the two orbits [10]), we have

\[
sm_{2d} \cdot (u_\lambda - u_\mu) = \sum_{\kappa \cdot \lambda = \kappa \cdot \mu = \pm 1/3} m^2_2 \{ \wp((\lambda - \kappa) \cdot x) - \wp((\kappa - \mu) \cdot x) \} \\
+ \frac{2}{3} m^2_{2/3} \{ \wp(\lambda \cdot x) - \wp(\mu \cdot x) \}.
\] (5.51)

Here, the sum over \( \kappa \) is restricted to \( \kappa \cdot \lambda = \kappa \cdot \mu = \pm 1/3 \) for the following reasons. In general, \( \kappa \cdot \lambda \) (and analogously \( \kappa \cdot \mu \)) can take on the values \(-2/3, -1/3, 1/3, 2/3 \). The values \( 2/3 \) is ruled out since from (3.19), \( \kappa \neq \lambda \); similarly, \( \kappa \cdot \lambda = -2/3 \) is ruled out since the associated coupling \( C_{\lambda, -\lambda} \) that would appear on the right and side of (5.51) vanishes in
view of (5.41a). Now, since in this case, we have $\lambda \cdot \mu = -1/3$, the cases $\kappa \cdot \lambda = -\kappa \cdot \mu = \pm 1/3$ have no solutions $\kappa$ that are weights of the 7 of $G_2$. Thus, their contribution was dropped from the sums in (5.51).

For $\lambda \cdot \mu = 1/3$ (2 of the four orbits [01]) we have

$$s\sqrt{\frac{4}{3}}m_{2/3}d \cdot (u_\lambda - u_\mu) = \sum_{\kappa \cdot \lambda = \pm 1/3} \sqrt{\frac{4}{3}}m_{2/3}m_2 \{ \varphi((\lambda - \kappa) \cdot x) - \varphi((\kappa - \mu) \cdot x) \} + \frac{2}{3}m_{2/3}^2 \{ \varphi(\lambda \cdot x) - \varphi(\mu \cdot x) \}. \quad (5.52)$$

For reasons analogous to the ones explained after (5.50), the sums above have been restricted to $\kappa \cdot \lambda = -\kappa \cdot \mu = \pm 1/3$. For the remaining 2 orbits [01] we have

$$s\sqrt{\frac{4}{3}}m_{2/3}d \cdot (u_\lambda - u_7) = \sum_{\kappa \cdot \lambda = \pm 1/3} \sqrt{\frac{4}{3}}m_{2/3}m_2 \{ \varphi((\lambda - \kappa) \cdot x) - \varphi(\kappa \cdot x) \}. \quad (5.53)$$

The general solutions of (5.51) and (5.52) are respectively

$$sd \cdot u_\lambda = d_0 + \frac{1}{3m_2}m_{2/3}m_2^2 \varphi(\lambda \cdot x) + \sum_{\kappa \cdot \lambda = -1/3} m_2 \varphi((\lambda - \kappa) \cdot x) \quad (5.54)$$

$$sd \cdot u_\lambda = d_0 + \left( \frac{2}{\sqrt{3}}m_{2/3}m_2 - m_2 \right) \varphi(\lambda \cdot x) + \sum_{\kappa \cdot \lambda = -1/3} m_2 \varphi((\lambda - \kappa) \cdot x),$$

where $d_0$ is independent of $\lambda$. These solutions agree provided the coefficients of the $\varphi(\lambda \cdot x)$ terms agree and $m_{2/3} = \sqrt{3}m_2$. Under those conditions, (5.53) is compatible with (5.54). Combining these results with (5.50), we find all equations of (5.41a) and (5.41b).

**Proof of (2)**

Instead of repeating the construction using Theorem 2, we obtain the Lax pair by restricting the Lax pair of $B_3$ in the spinor representation (as derived in part (1) of Theorem 5) to the $G_2$ subgroup of $B_3$. This restriction is easy to carry out, since $G_2$ is the subgroup of $B_3$ that leaves any one of the spinor weights of $B_3$ invariant. We shall choose the weight

$$\lambda_s = \frac{1}{2}(e_1 + e_2 + e_3). \quad (5.55)$$

The Lax pair of $B_3$ in the spinor representation is given in terms of two independent couplings $m_1$ and $m_2$ by

$$C_{\lambda, \mu} = \begin{cases} m_1 & (\lambda - \mu)^2 = 1 \\ m_2 & (\lambda - \mu)^2 = 2 \\ 0 & \text{otherwise} \end{cases} \quad (5.56)$$
The restriction of $B_3$ to $G_2$ corresponds to $x$ and $p$ orthogonal to the weight $\lambda_s = \frac{1}{2}(e_1 + e_2 + e_3)$. It is easy to analyze under which conditions this restriction is consistent with the Hamilton-Jacobi equations of the Calogero-Moser system. Consistency requires that when $\lambda_s \cdot x = 0$, and thus $\lambda_s \cdot p = \lambda_s \cdot \dot{p} = 0$, the right hand side of (2.1) be orthogonal to $\lambda_s$ as well, so that

$$0 = \sum_{\alpha \in \mathcal{R}(B_3)} m_{|\alpha|}^2 \lambda_s \cdot \alpha \varphi'(\alpha \cdot x).$$  \tag{5.57}$$

The sum over $\alpha$ in (5.57) is even in $\alpha$, and may be restricted to range over positive roots only. Furthermore, the roots $e_i - e_j$ of $B_3$ are orthogonal to $\lambda_s$ and do not contribute to (5.57). The remaining sum reduces to

$$0 = m_2^2 \left[ \varphi'(x_1 + x_2) + \varphi'(x_2 + x_3) + \varphi'(x_3 + x_1) \right]$$

$$+ m_1^2 \left[ \varphi'(x_1) + \varphi'(x_2) + \varphi'(x_3) \right]$$ \tag{5.58}$$

Since $0 = \lambda_s \cdot x = x_1 + x_2 + x_3$, the right hand side of (5.58) cancels for all $x$ when $m_1 = m_2$, which is precisely the condition (5.42) of Theorem 9.

(e) Untwisted Elliptic Calogero-Moser System for $F_4$

The smallest non-trivial representation of $F_4$ is of dimension 26, and denoted $26$ for short. This representation has 2 zero weights. As a result, when the $26$ of $F_4$ is embedded into the fundamental representation of $GL(26, \mathbb{C})$, the centralizer of the Cartan subalgebra of $F_4$ is larger than the Cartan subalgebra of $GL(26, \mathbb{C})$. The space $GL_0$ has dimension 2, and the quantity $\Delta$ in (3.5) may not vanish. This situation presents serious complications, just as it did in the case of $E_8$. Fortunately, for $F_4$, there is an alternative where no such complications appear.

Instead, we consider the 27-dimensional representation $26 \oplus 1$ of $F_4$, which has 3 zero weights, and which may be viewed as the restriction of the 27-dimensional representation of $E_6$ to its $F_4$ subalgebra. It is a remarkable fact (see Appendix §A) that the 24 long roots of $F_4$ form a $D_4$ subalgebra of $F_4$, and that the 24 short roots of $F_4$ (which precisely coincide with the non-zero weights of the $26$ of $F_4$), may be viewed as the weights of the direct sum of the three 8-dimensional distinct (but equivalent) representations $8^v$, $8^s$ and $8^c$ of $D_4$. Thus, it makes sense to group the 24 short roots (i.e. the 24 non-zero weights of the $26$) into classes – which we shall call “8-classes” of $F_4$ – and which we shall denote by $8^v$, $8^s$ and $8^c$. The 8-class of a non-zero weight $\lambda$ of the $26$ will be denoted by $[\lambda]$. This underlying $D_4$ structure of $F_4$ will turn out to play a crucial role in the construction of the $F_4$ Lax pair.
The weights of the $26 \oplus 1$ together with their 8-class assignments are

\[
\begin{align*}
8^v & \quad \pm e_i \quad i = 1, 2, 3, 4 \\
8^s & \quad \frac{1}{2} \sum_{i=1}^{4} \epsilon_i e_i \quad \prod_{i=1}^{4} \epsilon_i = +1 \\
8^c & \quad \frac{1}{2} \sum_{i=1}^{4} \epsilon_i e_i \quad \prod_{i=1}^{4} \epsilon_i = -1 \\
\text{zero} & \quad 0 \\
\end{align*}
\]

which are readily established by inspection of (5.59).

Theorem 10 : Lax pair for $F_4$ in the $26 \oplus 1$

The untwisted elliptic Calogero-Moser Hamiltonian for $F_4$ admits a Lax pair with spectral parameter and one independent coupling, given by (3.5), (3.6), (3.16) and

\[
C_{\lambda, \mu} = \begin{cases} 
    m_2 & \lambda \cdot \mu = 0, \frac{1}{2} \\
    0 & \text{otherwise,}
\end{cases} \quad C_{\lambda, a} = \frac{m_2(1 - \delta_{[\lambda], a})}{C_{a, b} = 0} \quad (5.61a)
\]

\[
\sqrt{6}d \cdot u_{\lambda} = 2m_2 \varphi(\lambda \cdot x) + \sum_{\delta^2=2; \lambda \cdot \delta = 1} m_2 \varphi(\delta \cdot x) - \frac{1}{2} m_2 \sum_{\kappa \in [\lambda]} \varphi(\kappa \cdot x) \quad (5.61b)
\]

\[
\sqrt{6}d \cdot v_{a} = -m_2 \sum_{[\kappa] = a} \varphi(\kappa \cdot x) + \sum_{\kappa} \frac{1}{2} m_2 \varphi(\kappa \cdot x) \quad (5.61c)
\]

The notations are explained in detail below.

To prove this Theorem, we begin by embedding the $26 \oplus 1$ into the fundamental representation of $GL(27, \mathbb{C})$, whose weights are 27 orthonormal vectors $u_I, I = 1, \cdots, 27$. The decomposition (3.1) is given by $s^2 = 6$ and

\[
\begin{align*}
\sqrt{6}u_{\lambda} & = \lambda + v_{\lambda} \quad \lambda^2 = 1 \\
\sqrt{6}u_{a} & = v_{a} \quad a = v, s, c
\end{align*}
\]

Here, the vectors $v_{\lambda}$ and $v_a$ are orthogonal to one another and to the weight space of $F_4$. The roots of $GL(27, \mathbb{C})$ decompose under $F_4$ as follows

\[
\begin{align*}
\sqrt{6}(u_{\lambda} - u_{\mu}) & = \lambda - \mu + v_{\lambda} - v_{\mu}, \quad \lambda \neq \mu \\
\sqrt{6}(u_{\lambda} - u_{a}) & = \lambda + v_{\lambda} - v_{a} \\
\sqrt{6}(u_{a} - u_{\mu}) & = -\mu + v_{a} - v_{\mu} \\
\sqrt{6}(u_{a} - u_{b}) & = v_{a} - v_{b}, \quad a \neq b.
\end{align*}
\]
Under the Weyl group of $F_4$, these roots transform in the Weyl orbits occurring in the tensor product $(26 \oplus 1) \otimes (26 \oplus 1)$ and are given by

| Weyl Orbit | Multiplicity | # Weights | Length |
|------------|--------------|-----------|--------|
| $[0000]$  | 33           | 1         | 0      |
| $[0001]$  | 14           | 24        | 1      |
| $[1000]$  | 6            | 24        | 2      |
| $[0010]$  | 2            | 96        | 3      |
| $[0002]$  | 1            | 24        | 4      |

(5.64)

The orbits $[0000]$ do not occur in (5.63). Using (4) of Theorem 2, and the fact that only the orbits $[0001]$ and $[1000]$ are roots, we readily find that

$$C_{\lambda,\mu} = 0 \text{ whenever } (\lambda - \mu)^2 \neq 1, 2,$$

and recover the result in (5.61a) for $\lambda \cdot \mu \neq 0$. Conditions (1) and (2) for roots $\alpha = \lambda - \mu$ (with $\lambda \cdot \mu = 0$) in the 6 Weyl orbits of type $[1000]$ are given by

$$6m_2^2 = \sum_{\alpha=\lambda-\mu} C_{\lambda,\mu}^2,$$

$$0 = \sum_{\alpha=\lambda-\mu} C_{\lambda,\mu}^2 (v_{\lambda} - v_{\mu}).$$

(5.65)

Taking the inner product of the second equation with $u_{\sigma}$ for an arbitrary weights vector $\sigma$ of $26 \oplus 1$, and using the first equation in (5.65), we get

$$\alpha \cdot \sigma m_2^2 = \sum_{\lambda,\mu} C_{\lambda,\mu}^2 (\delta_{\lambda,\sigma} - \delta_{\mu,\sigma}).$$

(5.66)

The combination $\alpha \cdot \sigma$ can take only the values $\alpha \cdot \sigma = 0$, for which all sides of (5.66) vanish separately, and $\alpha \cdot \sigma = \pm 1$, from which we find that $C_{\lambda,\mu}^2 = m_2^2$. Taking square roots with a plus sign of this result, we obtain the expression (5.61a) for $\lambda \cdot \mu \neq 0$.

Conditions (1) and (2) for the 8 weights of the form $\alpha = \lambda - \mu$ with $\lambda \cdot \mu = \frac{1}{2}$, and the 6 weights of the form $\alpha = \lambda$, in orbits of type $[0001]$ with $\alpha^2 = 1$ yield

$$6m_1^2 = \sum_{\alpha=\lambda-\mu} C_{\lambda,\mu}^2 + \sum_a (C_{\alpha,\alpha}^2 + C_{\alpha,-\alpha}^2),$$

$$0 = \sum_{\alpha=\lambda-\mu} C_{\lambda,\mu}^2 (v_{\lambda} - v_{\mu}) + \sum_a (C_{\alpha,\alpha}^2 (v_{\alpha} - v_a) + C_{\alpha,-\alpha}^2 (v_a - v_{-\alpha})).$$

(5.67)

Using linear independence of $v_a$, we deduce that $C_{\alpha,\alpha}^2 = C_{-\alpha,\alpha}^2$; it turns out that we can find a Lax pair with $C_{\alpha,\alpha} = C_{-\alpha,\alpha}$, which we shall henceforth assume to hold. Taking
the inner product of the remaining part of the second equation in (5.67) with \( u_\sigma \) for an arbitrary weight \( \sigma \), and using the first equation in (5.67) yields

\[
\alpha \cdot \sigma m_1^2 = \sum_{\alpha = \lambda - \mu} C_{\lambda,\mu}^2 + C_{\alpha} \cdot C_{\alpha} (\delta_{\alpha,\sigma} - \delta_{-\alpha,\sigma}). \tag{5.68}
\]

Analyzing this equation according to the possible values of \( \alpha \cdot \sigma = 0, \pm \frac{1}{2}, \pm 1 \), we find

\[
C_{\lambda,\mu} = \frac{1}{\sqrt{2}} m_1 \quad C_{\alpha} \cdot C_{\alpha} = m_1^2. \tag{5.69}
\]

With these values, the first equation in (5.67) is satisfied automatically. Henceforth, we shall assume that \( m_1 \neq 0 \), since otherwise the system reduces to that of a \( D_4 \) algebra.

To complete the proof of Theorem 10, it remains to satisfy condition (3) of Theorem 2 for this case. Henceforth, we shall set \( C_{a,b} = 0 \) and \( \Delta_{a,b} = 0 \), since a consistent solution exists under these assumptions. Conditions (3) may then be split into two parts.

\[
C_{\lambda,\mu sd} \cdot (u_\lambda - u_\mu) = \sum_{\kappa \neq \lambda,\mu} C_{\lambda,\kappa} C_{\kappa,\mu} \left( \varphi((\lambda - \kappa) \cdot x) - \varphi((\kappa - \mu) \cdot x) \right) + C_{\lambda} \cdot C_{\mu} \left( \varphi(\lambda \cdot x) - \varphi(\mu \cdot x) \right) \tag{5.70a}
\]

\[
C_{\lambda,a sd} \cdot (u_\lambda - u_a) = \sum_{\kappa \neq \lambda} C_{\lambda,\kappa} C_{\kappa,a} \left( \varphi((\lambda - \kappa) \cdot x) - \varphi(\kappa \cdot x) \right). \tag{5.70b}
\]

We analyze (5.70a) according to the value of \((\lambda - \mu)^2\), i.e. the orbit type which \( \lambda - \mu \) belongs to. For \((\lambda - \mu)^2 = 4\), both sides of (5.70a) manifestly cancel. For \((\lambda - \mu)^2 = 3\), the l.h.s. still vanishes, the sums on the r.h.s. reduce to a single term proportional to \( \varphi(\lambda \cdot x) - \varphi(\mu \cdot x) \) and the resulting condition is

\[
C_{\lambda} \cdot C_{\mu} = \frac{1}{2} m_1^2, \quad \lambda \cdot \mu = -\frac{1}{2}. \tag{5.71}
\]

For \((\lambda - \mu)^2 = 2\), i.e. \( \lambda \cdot \mu = 0 \), \( \lambda \) and \( \mu \) must belong to the same 8-class. Thus, in the sum in (5.70a), \( \kappa \) either belongs to the common 8-class \([\lambda] = [\mu]\) or belongs to another class, so that only the cases \( \kappa \cdot \lambda = \kappa \cdot \mu = 0 \), or \( \frac{1}{2} \) remain. We find

\[
m_2 sd \cdot (u_\lambda - u_\mu) = \sum_{\kappa : \kappa \cdot \lambda = \kappa \cdot \mu = 0} m_2^2 \left( \varphi((\lambda - \kappa) \cdot x) - \varphi((\kappa - \mu) \cdot x) \right) + \sum_{\kappa : \kappa \cdot \lambda = \kappa \cdot \mu = \frac{1}{2}} \frac{1}{2} m_1^2 \left( \varphi((\lambda - \kappa) \cdot x) - \varphi((\kappa - \mu) \cdot x) \right) + C_{\lambda} \cdot C_{\mu} \left( \varphi(\lambda \cdot x) - \varphi(\mu \cdot x) \right). \tag{5.72}
\]
Upon changing variables in the second sum on the r.h.s. of (5.72), to \( \delta = \lambda - \kappa \) in the first term and to \( \delta = \kappa - \mu \) in the second term, we see that the second sum in (5.72) cancels identically. The remaining equation may be separated in terms of \( \lambda \) and \( \mu \) dependent terms provided \( C_\lambda \cdot C_\mu \) is independent of \( \lambda \) and \( \mu \) whenever \( \lambda \cdot \mu = 0 \). This means that \( \lambda \) and \( \mu \) belong to the same 8-class. If we assume (and this will be justified by the fact that we can find a consistent Lax pair satisfying this assumption) that the inner product \( C_\lambda \cdot C_\mu \) should only depend upon the 8-classes of \( \lambda \) and \( \mu \), then this value must coincide with that of \( \lambda = \mu \), and we have

\[
C_\lambda \cdot C_\mu = m_1^2, \quad [\lambda] = [\mu]. \tag{5.73}
\]

We then have

\[
sd \cdot u_\lambda = d_1([\lambda]) + \sum_{\delta^2 = 2; \delta \cdot \lambda = 1} m_2 \varphi(\delta \cdot x) + \frac{m_1^2}{m_2} \varphi(\lambda \cdot x). \tag{5.74}
\]

It will be very important to realize in the sequel that the difference \( d \cdot (u_\lambda - u_\mu) \) was evaluated for \( \lambda \) and \( \mu \) belonging to the same 8-class. Thus, upon separating the equation as is done in (5.74), we are left with an arbitrary function \( d([\lambda]) \) which depends only upon the 8-class of \( \lambda \), but not upon the representative of this class.

Finally, for \( (\lambda - \mu)^2 = 1 \), \( \lambda \) and \( \mu \) necessarily belong to different 8-classes, and (5.70a) in this case reduces to

\[
sd \cdot (u_\lambda - u_\mu) = \sum_{\kappa \cdot \lambda = 0; \kappa \cdot \mu = \frac{1}{2}} m_2 ( \varphi((\lambda - \kappa) \cdot x) - \varphi((\kappa - \mu) \cdot x) )
+ \sum_{\kappa \cdot \lambda = \frac{1}{2}; \kappa \cdot \mu = 0} m_2 ( - \varphi((\kappa - \mu) \cdot x) + \varphi((\lambda - \kappa) \cdot x) )
+ \sum_{\kappa \cdot \lambda = \kappa \cdot \mu = \frac{1}{2}} \frac{1}{\sqrt{2}} m_1 ( \varphi((\lambda - \kappa) \cdot x) - \varphi((\kappa - \mu) \cdot x) )
+ \frac{\sqrt{2}}{m_1} C_\lambda \cdot C_\mu ( \varphi(\lambda \cdot x) - \varphi(\mu \cdot x) ). \tag{5.75}
\]

The second terms in the first and second sums on the r.h.s. of (5.75) cancel one another. The remaining terms may be separated provided \( C_\lambda \cdot C_\mu \) is independent of \( \lambda \) and \( \mu \) as long as they satisfy \( \lambda \cdot \mu = \frac{1}{2} \), and we obtain

\[
sd \cdot u_\lambda = d_0 + \sum_{\delta^2 = 2; \delta \cdot \lambda = 1} m_2 \varphi(\delta \cdot x) - \sum_{\kappa \in [\lambda]} \frac{1}{2\sqrt{2}} m_1 \varphi(\kappa \cdot x) + \left( \frac{\sqrt{2}}{m_1} C_\lambda \cdot C_\mu + \frac{m_1}{\sqrt{2}} \right) \varphi(\lambda \cdot x). \tag{5.76}
\]

\[\text{48}\]
Here, \( d_0 \) is independent of \( \lambda \) (and of the 8-class of \( \lambda \)), and the second on the r.h.s. is over all \( \kappa \) in the 8-class of \( \lambda \).

We are now ready to put all the conditions obtained together and to solve them. First, the expressions (5.74) and (5.76) must agree, so that \( C_\lambda \cdot C_\mu = \frac{1}{2} m_1^2 \) whether \( \lambda \cdot \mu = -\frac{1}{2} \) or \( +\frac{1}{2} \). This implies that \( m_1 = \sqrt{2} m_2 \) and as a consequence

\[
d_1([\lambda]) = d_0 - \sum_{\kappa \in [\lambda]} \frac{1}{2} m_2 \varphi(\kappa \cdot x). \tag{5.77}
\]

Putting all inner product relations together, we have

\[
C_\lambda \cdot C_\mu = \begin{cases} 
2 m_2^2 & [\lambda] = [\mu] \\
m_2^2 & [\lambda] \neq [\mu]
\end{cases} \tag{5.78}
\]

whence the solution of (5.61a).

It now only remains to solve (5.70b); to do so, we use all the results already obtained so far. First, we separate the sum over \( \kappa \) according to the inner product \( \kappa \cdot \lambda \):

\[
C_{\lambda,a} sd \cdot (u_\lambda - u_a) = \sum_{\kappa \cdot \lambda = 0} m_2 C_{\kappa,a} \left( \varphi((\lambda - \kappa) \cdot x) - \varphi(\kappa \cdot x) \right) \\
+ \sum_{\kappa \cdot \lambda = \frac{1}{2}} m_2 C_{\kappa,a} \left( \varphi((\lambda - \kappa) \cdot x) - \varphi(\kappa \cdot x) \right). \tag{5.79}
\]

Upon using the fact that \( C_{\kappa,a} \) only depends upon the 8-class of \( \kappa \), and (5.77) to eliminate \( sd \cdot u_\lambda \), we get

\[
C_{\lambda,a} (d_0 - sd \cdot u_a) = - \sum_{\kappa \in [\lambda]} \frac{1}{2} m_2 C_{\lambda,a} \varphi(\kappa \cdot x) + \sum_{\kappa \notin [\lambda]} \frac{1}{2} m_2 C_{\lambda \pm \kappa,a} \varphi(\kappa \cdot x) \\
- \sum_{\kappa \notin [\lambda]} \frac{1}{2} m_2 C_{\kappa,a} \varphi(\kappa \cdot x). \tag{5.80}
\]

When \([\lambda] = a\), the l.h.s. cancels and so does the first sum on the r.h.s. of (5.80). The remaining two sums cancel one another. Thus, there only remains an equation for \([\lambda] \neq a\), which may be simplified with the help of (5.61), to obtain

\[
d_0 - sd \cdot u_a = - \sum_{\kappa \in [\lambda]} \frac{1}{2} m_2 \varphi(\kappa \cdot x) + \sum_{[\kappa] = a} \frac{1}{2} m_2 \varphi(\kappa \cdot x) - \sum_{\kappa \notin [\lambda], [\kappa] \neq a} \frac{1}{2} m_2 \varphi(\kappa \cdot x). \tag{5.81}
\]

\[
= - \sum_{\kappa} \frac{1}{2} m_2 \varphi(\kappa \cdot x) + \sum_{[\kappa] = a} m_2 \varphi(\kappa \cdot x).
\]
Setting $d_0 = 0$ in this equation, we recover (5.61c), completing the proof of Theorem 10.

*F*$_4$ Untwisted Elliptic Calogero-Moser Hamiltonian by Restriction from *E*$_6$

By comparing the root systems in Table 3 of Appendix §A of *E*$_6$ and *F*$_4$, it is clear that the subalgebra *F*$_4$ of *E*$_6$ is obtained by projection orthogonal to the basis vectors $e_5$ and $e_6$. The projection of the weights of the 27 of *E*$_6$ is immediately seen to reproduce the 24 non-zero weights of the 26 of *F*$_4$, together with 3 zero weights, which arise from the weights $2/\sqrt{3}e_6$ and $-1/\sqrt{3}e_6 \pm e_5$ of *E*$_6$. We may check directly that this restriction is consistent with the Hamilton-Jacobi equations for the Calogero-Moser system (2.1). Indeed, when $e_5 \cdot x = e_6 \cdot x = 0$, the right hand side of (2.1) is orthogonal to $e_5$ and $e_6$:

$$0 = \sum_{\alpha \in \mathcal{R}(E_6)} m^2_{\alpha} \wp'(\alpha \cdot x)(e_{5,6} \cdot \alpha).$$  \hspace{1cm} (5.82)

The contributions from the roots of *E*$_6$ of the form $\pm e_i \pm e_j$, $1 \leq i < j \leq 5$ are readily seen to cancel in (5.82), leaving just

$$0 = \sum_{\epsilon_i = \pm 1} m^2_{\epsilon_5} \wp'\left(\frac{1}{2} \sum_{i=1}^{4} \epsilon_i e_i \cdot x\right) \prod_{i=1}^{4} \epsilon_i = 1.$$  \hspace{1cm} (5.83)

The contribution for $\epsilon_5 = +1$ is a sum over the weights of the Weyl spinor $8^a$ of *D*$_4$, while that for $\epsilon_5 = -1$ is over the weights of the Weyl spinor $8^c$ of *D*$_4$. Since both representations are real, their weight lattices are even under sign reversal. Thus, the contributions to (5.83) from $\epsilon = \pm 1$ cancel separately in (5.83), and the Hamiltonian-Jacobi equations for the Calogero-Moser system of *E*$_6$ in the 27 projects consistently to the Hamiltonian for *F*$_4$ in the 26 $\oplus$ 1 of *F*$_4$. The relation of the couplings $m_1$ and $m_2$ is automatically guaranteed.

**VI. THE TWISTED ELLIPTIC CALOGERO-MOSER SYSTEMS**

The twisted elliptic Calogero-Moser systems associated with simply-laced *G* are identical to the untwisted ones, which were solved for in §IV and §V. In this section we propose to solve for the Lax pair conditions of Theorem 1 for non-simply laced *G*. There are thus only four cases left. For *G* = *B*$_n$, *C*$_n$ we shall derive Lax pairs of dimensions $2n$ and $2n + 2$ respectively. These dimensions happen to coincide with the dimensions in which the respective associated twisted affine Lie algebras $(B^{(1)}_n)^\vee = C^{(2)}_n$ and $(C^{(1)}_n)^\vee = D^{(2)}_{n+1}$ can be realized. For *G* = *F*$_4$, we shall obtain a Lax pair of dimension 24, which, surprisingly, is not equal to the dimension of any representation of *F*$_4$, but arises in relation with the
number of short roots of $F_4$. For $G = G_2$, we have not succeeded in proving the existence of a Lax pair. We strongly believe that there should exist a Lax pair of dimension 6 or 8, but the elliptic function analysis appears unwieldy at this point.

(a) Twisted Elliptic Calogero-Moser System for $B_n$

The twisted elliptic Calogero-Moser Hamiltonian for $B_n$ is defined by

$$H = \frac{1}{2} p \cdot p - \sum_{\alpha \in \mathcal{R}_l(B_n)} \frac{1}{2} m_2^2 \wp(\alpha \cdot x) - \sum_{\alpha \in \mathcal{R}_s(B_n)} \frac{1}{2} m_1^2 \wp_2(\alpha \cdot x),$$  \hfill (6.1)$$

where the roots are divided into long and short roots of $B_n$

$$\mathcal{R}_l(B_n) = \{ \pm (e_i - e_j), \pm (e_i + e_j), 1 \leq i < j \leq n \}$$

$$\mathcal{R}_s(B_n) = \{ \pm e_i \}. \hfill (6.2)$$

Here, we have chosen the twisted half period of $\wp_2$ of (2.4) to be $\omega_1$ for later convenience. It is possible (by a Landen transformation [24]) to express this Hamiltonian as a twisted elliptic Calogero-Moser system for the root system of the dual Lie algebra $B_n^\vee = C_n$. To see this, it suffices to use (B.12) and to perform a canonical transformation $x \rightarrow 2x$, $p \rightarrow p/2,$

$$H = \frac{1}{8} p \cdot p - \sum_{\alpha \in \mathcal{R}_l(B_n^\vee)} \frac{1}{8} m_2^2 \wp_2(\alpha \cdot x) - \sum_{\alpha \in \mathcal{R}_s(B_n^\vee)} \frac{1}{2} m_2^2 \{ \wp_2(\alpha \cdot x) + \wp_2(\alpha \cdot x + \omega_2) \}, \hfill (6.3)$$

where the roots are expressed as roots of $C_n$

$$\mathcal{R}_s(B_n^\vee) = \{ \pm (e_i - e_j), \pm (e_i + e_j), 1 \leq i < j \leq n \}$$

$$\mathcal{R}_l(B_n^\vee) = \{ \pm 2e_i \}. \hfill (6.4)$$

Theorem 11 : Lax pair for $B_n$ Twisted Calogero-Moser

The twisted elliptic Calogero-Moser Hamiltonian for $B_n$ admits a Lax pair of dimension $N = 2n$, with spectral parameter and two independent couplings $m_1$ and $m_2$. The Lax operators are given by (3.5), (3.6), $\Delta = 0$ and

$$\Phi_{IJ}(x, z) = \begin{cases} 
    \Phi(x, z) & I - J \neq 0, \pm n \\
    \Lambda(x, z) & I - J = \pm n
\end{cases} \hfill (6.5a)$$

$$C_{I,J} = \begin{cases} 
    m_2 & I - J \neq 0, \pm n \\
    m_1 & I - J = \pm n
\end{cases} \hfill (6.5b)$$

$$d \cdot v_i = \sum_{J - i \neq 0, n} m_2 \wp((e_i - \lambda_J) \cdot x) + \frac{1}{2} m_1 \wp_2(e_i \cdot x) \hfill (6.5c)$$
The function $\Lambda(x,z)$ is defined in (B.22) of Appendix §B, and the weight vectors $\lambda_I$ will be defined below.

To prove this Theorem, we verify that the conditions (1), (2) and (3) of Theorem 1, with $\Delta = 0$ are obeyed. The natural starting point for the construction of the Lax pair for the twisted elliptic Calogero-Moser system for $B_n$ appears to be the dual algebra $\mathcal{G} = B_n^\vee$. We begin by defining the weights $\lambda$ of Theorem 1 as the weights of the fundamental representation of $B_n^\vee$, which is of dimension $N = 2n$, and which are given by $\lambda_i = -\lambda_{n+i} = e_i$ for $i = 1, \ldots, n$. Following Theorem 1, we embed this representation into $GL(N, \mathbb{C})$ by (3.1), with $s^2 = 2$, and

$$\sqrt{2} u_i = e_i + v_i, \quad \sqrt{2} u_{n+i} = -e_i + v_i \quad i = 1, \ldots, n. \quad (6.6)$$

The roots of $GL(N, \mathbb{C})$ decompose into short roots of $B_n^\vee$

$$\sqrt{2}(u_i - u_j) = + e_i - e_j + v_i - v_j \quad i \neq j$$

$$\sqrt{2}(u_{\nu+j} - u_{\nu+i}) = + e_i - e_j - v_i + v_j \quad i \neq j \quad (6.7a)$$

$$\sqrt{2}(u_i - u_{\nu+j}) = + e_i + e_j + v_i - v_j \quad i \neq j$$

$$\sqrt{2}(u_{\nu+i} - u_j) = - e_i - e_j + v_i - v_j \quad i \neq j,$$

and long roots of $B_n^\vee$

$$\sqrt{2}(u_i - u_{\nu+i}) = + 2e_i$$

$$\sqrt{2}(u_{\nu+i} - u_i) = - 2e_i. \quad (6.7b)$$

Condition (2) of Theorem 1 is manifestly satisfied by the values of $C$ listed in (6.5b), because each short root of $B_n^\vee$ has two roots of $GL(N, \mathbb{C})$ as pre-images, and they come with opposite values of $v_i - v_j$, which automatically cancel in (3.8). Each long root of $B_n^\vee$ has no $v$-dependence at all and thus does not enter into (3.8). Satisfying condition (1) of Theorem 1 requires that the coefficients $C$ satisfy the relations of (6.5b), that the function $\Phi$ satisfy (2.10) and that the functions $\Lambda$ and $\wp_2$ obey

$$\Lambda(2x,z)\Lambda'(-2x,z) - \Lambda'(2x,z)\Lambda(-2x,z) = \frac{1}{2} \wp_2'(x). \quad (6.8)$$

This relation follows from the definitions and results of (B.22-25) in Appendix §B.

It remains to satisfy condition (3) of Theorem 1, for $\Delta = 0$. Using the anti-symmetry of the r.h.s. of condition (3) in (3.9) under $x \to -x$ and $I \to J$ on the l.h.s. of (3.9) implies that $d(-x) = d(x)$. Using now this symmetry, we may restrict attention to the cases $I < J$.
We now make use of the relations (B.20) and (B.23) for the functions \( \Phi \) and \( \Lambda \) to simplify the last sum in (6.10) vanishes identically, because the conditions \( I - K = \pm n \) and \( K - J = \pm n \) imply that \( I - J = 0, \pm 2n \), which is impossible since \( I \neq J \). By noticing that if \( I - K = \pm n \), we have that \( \lambda_I = -\lambda_K \) for all \( I \) and \( K \), we can easily make the second and third sums collapse to single terms. Thus, we obtain

\[
m_2 \Phi(\alpha_{IJ} \cdot x)sd \cdot (u_I - u_J) = \sum_{I-K \neq 0, \pm n \atop K-J \neq 0, \pm n} m_2^2 \{ \Phi(\alpha_{IK} \cdot x)\Phi'(\alpha_{KJ} \cdot x) - \Phi'(\alpha_{IK} \cdot x)\Phi(\alpha_{KJ} \cdot x) \} + m_1 m_2 \{ \Lambda(2\lambda_I \cdot x)\Phi'(-\lambda_I + \lambda_J \cdot x) - \Lambda'(2\lambda_I \cdot x)\Phi(-\lambda_I + \lambda_J \cdot x) \} + m_1 m_2 \{ \Phi((\lambda_I + \lambda_J) \cdot x)\Lambda'(-2\lambda_J \cdot x) - \Phi'((\lambda_I + \lambda_J) \cdot x)\Lambda(-2\lambda_J \cdot x) \}.
\]

We now make use of the relations (B.20) and (B.23) for the functions \( \Phi \) and \( \Lambda \) to simplify the r.h.s. of (6.11). Omitting also an overall factor of \( m_2^2 \Phi(\alpha_{IJ} \cdot x) \), (6.11) is reduced to

\[
\sum_{I-K \neq 0, \pm n \atop K-J \neq 0, \pm n} \{ \varphi(\alpha_{IK} \cdot x) - \varphi(\alpha_{KJ} \cdot x) \} + \frac{1}{2} m_1 \{ \varphi_2(\lambda_I \cdot x) - \varphi_2(\lambda_J \cdot x) \}, \quad (6.12)
\]

* Henceforth we use the abbreviation \( x_i = e_i \cdot x \).
from which (6.5c) readily follows, and this concludes the proof of Theorem 11.

(b) Twisted Elliptic Calogero-Moser System for \( C_n \)

The twisted elliptic Calogero-Moser Hamiltonian for \( C_n \) is given by

\[
H = \frac{1}{2} p \cdot p - \sum_{\alpha \in \mathcal{R}_l(C_n)} \frac{1}{2} m_2^2 \varphi(\alpha \cdot x) - \sum_{\alpha \in \mathcal{R}_s(C_n)} \frac{1}{2} m_2^2 \varphi_2(\alpha \cdot x),
\]

where the roots are divided into long and short roots of \( C_n \)

\[
\mathcal{R}_s(C_n) = \{ \pm(e_i - e_j), \pm(e_i + e_j), 1 \leq i < j \leq n \}
\]

\[
\mathcal{R}_l(C_n) = \{ \pm 2e_i \},
\]

and where we have chosen the twisted half period of \( \varphi_2 \) of (2.4) to be \( \omega_1 \) for later convenience. It is possible (by a Landen transformation [24]) to express this Hamiltonian as a twisted elliptic Calogero-Moser system for the root system of the dual Lie algebra \( C_n^\vee = B_n \) (given in (6.2)), by using (B.12),

\[
H = \frac{1}{2} p \cdot p - \sum_{\alpha \in \mathcal{R}_l(C_n^\vee)} \frac{1}{2} m_2^2 \varphi_2(\alpha \cdot x) - \sum_{\alpha \in \mathcal{R}_s(C_n^\vee)} \frac{1}{8} m_4^2 \{ \varphi_2(\alpha \cdot x) + \varphi_2(\alpha \cdot x + \omega_2) \}.
\]

Theorem 12 : Lax pair for \( C_n \) Twisted Calogero-Moser

The twisted elliptic Calogero-Moser Hamiltonian for \( C_n \) admits a Lax pair of dimension \( N = 2n + 2 \), with spectral parameter and one independent couplings \( m_2 \) given by (3.5), (3.6), \( \Delta = 0 \) and

\[
\Phi_{I,J}(x, z) = \Phi_2(x + \omega_{I,J}, z)
\]

\[
\omega_{I,J} = \begin{cases} 
0 & I \neq J, 1, \ldots, 2n + 1 \\
+\omega_2 & I = 1, \ldots, 2n; J = 2n + 2 \\
-\omega_2 & J = 1, \ldots, 2n; I = 2n + 2
\end{cases}
\]

\[
C_{I,J} = \begin{cases} 
\frac{1}{\sqrt{2}} m_4 = \sqrt{2} m_2 & I = 1, \ldots, 2n; J = 2n + 1, 2n + 2; I \leftrightarrow J \\
\frac{1}{2} m_2 & I = 2n + 1, J = 2n + 2; I \leftrightarrow J
\end{cases}
\]

\[
\text{sd} \cdot u_I = \sum_{J-I \neq 0, \pm n} m_2 \varphi_2((\lambda_I - \lambda_J) \cdot x) + 8m_2 \varphi(2\lambda_I \cdot x); I = 1, \ldots, 2n
\]

\[
\text{sd} \cdot u_{2n+1} = \sum_{J=1}^{2n} \varphi_2(\lambda_J \cdot x) + 2m_2 \varphi_2(\omega_2)
\]

\[
\text{sd} \cdot u_{2n+2} = \sum_{J=1}^{2n} \varphi_2(\lambda_J \cdot x + \omega_2) + 2m_2 \varphi_2(\omega_2).
\]
The function \( \Phi_2 \) is defined in (B.22) by \( \Phi_2(x, z) = \Lambda(2x, z) \), and satisfies the differential equation (B.23) and (B.24), which are crucial in establishing (6.16). The projected weight system is defined by \( \lambda_i = -\lambda_{n+i} = e_i, i = 1, \cdots, n \) and \( \lambda_a = 0, a = 2n + 1, 2n + 2 \). Notice that \( \omega_{IJ} \) as defined in (6.16b) satisfies

\[
\omega_{JI} = -\omega_{IJ} \\
\omega_{IJ} + \omega_{JK} + \omega_{KI} = 0.
\]

To prove Theorem 12, we verify conditions (1), (2) and (3) of Theorem 1, with \( \Delta = 0 \). The algebra \( \mathcal{G} \) in Theorem 1 is the dual algebra \( \mathcal{G} = C_{n}^{\vee} = B_n \). The weights \( \lambda_I, I = 1, \cdots, 2n+2 \), span the fundamental representation of \( B_n = C_{n}^{\vee} \) plus a singlet, of dimension \( N = 2n + 2 \). We embed this representation into \( \text{GL}(N, \mathbb{C}) \) by (3.1) with \( s^2 = 2 \) and

\[
\sqrt{2}u_i = + e_i + v_i \quad i = 1, \cdots, n \\
\sqrt{2}u_{n+i} = - e_i + v_i \\
\sqrt{2}u_a = v_a \quad a = 2n + 1, 2n + 2.
\]

The roots of \( \text{GL}(N, \mathbb{C}) \) decompose onto long roots of \( C_{n}^{\vee} \)

\[
\sqrt{2}(u_i - u_j) = + e_i - e_j + v_i - v_j \\
\sqrt{2}(u_i - u_{n+j}) = + e_i + e_j - v_i + v_j \\
\sqrt{2}(u_{n+i} - u_j) = - e_i - e_j + v_i - v_j \\
\sqrt{2}(u_{n+i} - u_{n+j}) = + e_i + e_j - v_i + v_j
\]

and onto short roots of \( C_{n}^{\vee} \)

\[
\sqrt{2}(u_i - u_a) = + e_i + v_i - v_a \\
\sqrt{2}(u_{n+i} - u_a) = - e_i + v_i - v_a \\
\sqrt{2}(u_a - u_a) = + e_i - v_i + v_a \\
\sqrt{2}(u_a - u_i) = - e_i - v_i + v_a.
\]

and zero roots \( \sqrt{2}(u_a - u_b) = v_a - v_b \). Here, we have \( i, j = 1, \cdots, 2n \) and \( a, b = 2n+1, 2n+2 \). Conditions (1) and (2) of Theorem 1 are satisfied provided the coefficients \( C \) are such that they match the couplings \( m_4 \) and \( m_2 \) for the long and short roots respectively. This yields

\[
m_2^2 = C_{i,j}^2 = C_{n+j,n+i}^2 = C_{i,n+j}^2 = C_{j,n+i}^2 \\
\frac{1}{2}m_4^2 = C_{i,a}^2 = C_{n+i,a}^2.
\]

Actually, a Lax pair exists when all square roots of the above relations are taken with the same sign. This gives rise to (6.16c), except for the fact that the relation between \( m_4 \) and \( m_2 \) remains to be established below. Notice that at this stage, the coefficient \( C_{2n+1,2n+2} \) is unconstrained.
It remains to verify condition (3) of Theorem 1. It turns out that this condition may be satisfied for $\Delta = 0$, which we shall henceforth assume. With the help of the properties of $\omega_{I,J}$, condition (3) then reduces to
\[ C_{I,J} s d \cdot (u_I - u_J) = \sum_{I \neq K \neq J} C_{I,K} C_{K,J} \{ \varphi_2(\alpha_{I} \cdot x + \omega_{IK}) - \varphi_2(\alpha_{KJ} \cdot x + \omega_{KJ}) \}. \] (6.22)

Making use of an argument analogous to the one used to study condition (3) for (b) in the preceding subsection, we find that $d(-x) = d(x)$ and that (3.9) is anti-symmetric under the interchange of $I$ and $J$, so that we may limit the analysis to the cases $I < J$.

The case $I = i$, $J = n + i$, $i = 1, \ldots, n$ is automatically satisfied, along the lines of (6.9) for the preceding case. The case $I - J \neq 0, \pm n$ with $i, j = 1, \ldots, 2n$ yields (with $K = 1, \ldots, 2n$)
\[ m_2 s d \cdot (u_I - u_J) = \sum_{I - K \neq 0, \pm n} m_2^2 \{ \varphi_2(\alpha_{IK} \cdot x) - \varphi(\alpha_{KJ} \cdot x) \} \]
\[ + 2m_2^2 \{ \varphi(2\lambda_I \cdot x) - \varphi(2\lambda_J) \}. \] (6.23)

This equation is easily solved and we find
\[ s d \cdot u_I = \sum_{I - K \neq 0, \pm n} \varphi_2(\alpha_{I} \cdot x) + 2 \frac{m_4}{m_2} \varphi(2\lambda_I \cdot x) \] (6.24)
up to an $I$-independent term which we omit.

The cases $I = 1, \ldots, 2n$, and $J = 2n + 1, 2n + 2$ are reduced to
\[ s d \cdot (u_I - u_{2n+1}) = \sum_{I - K \neq 0, \pm n} m_2 \{ \varphi_2(\alpha_{IK} \cdot x) - \varphi_2(\lambda_K \cdot x) \} \]
\[ + C_{2n+1,2n+2} \varphi_2(\lambda_I \cdot x + \omega_2) - \varphi_2(\omega_2) \]
\[ s d \cdot (u_I - u_{2n+2}) = \sum_{I - K \neq 0, \pm n} m_2 \{ \varphi_2(\alpha_{IK} \cdot x) - \varphi_2(\lambda_K \cdot x + \omega_2) \} \]
\[ + C_{2n+1,2n+2} \varphi_2(\lambda_I \cdot x) - \varphi_2(\omega_2) \}. \] (6.25)

Substituting the result of (6.24) into (6.25), we obtain a relation between $m_4$ and $m_2$. To see this, we shall just look at the first equation in (6.25); the second one is completely analogous. We obtain
\[ \frac{m_4}{2m_2} \{ \varphi_2(\lambda_I \cdot x) + \varphi_2(\lambda_I \cdot x + \omega_2) \} - s d \cdot u_{2n+1} \]
\[ = - \sum_{I - K \neq 0, \pm n} \varphi_2(\lambda_K \cdot x) + C_{2n+2,2n+1} \{ \varphi_2(\lambda_I \cdot x + \omega_2) - \varphi_2(\omega_2) \}. \] (6.26)
To make the \( I \)-dependence match, we need to require that the terms in \( \varphi_2(\lambda_I \cdot x + \omega_2) \) cancel one another, and that the remaining \( \varphi_2(\lambda_I \cdot x) \) term combine with the sum, so that all \( I \)-dependence can indeed disappear in \( d \cdot u_{2n+1} \). The conditions are
\[
m_4 = 2m_2 \quad \quad C_{2n+1,2n+2} = \frac{m_4^2}{2m_2} = 2m_2 \quad (6.27)
\]
and provide the missing identifications in (6.16c). The remaining equations for \( d \cdot u_{2n+1} \) and \( d \cdot u_{2n+2} \) yields (6.16d) and (6.16e). To complete the proof, one case remains: \( I = 2n+1, \ J = 2n+2 \). Using the value for \( C_{2n+1,2n+2} = 2m_2 \) obtained in (6.27), this condition takes the form
\[
2m_2sd \cdot (u_{2n+1} - u_{2n+2}) \sum_{j=1}^{2n} \{ \varphi_2(\lambda_J \cdot x) - \varphi_2(\lambda_J \cdot x + \omega_2) \}. \quad (6.28)
\]
But this condition follows directly from the solutions for \( d \cdot u_{2n+1} \) and \( d \cdot u_{2n+2} \) already obtained in (6.16e) and (6.16f).

(c) Twisted Elliptic Calogero-Moser System for \( F_4 \)

The twisted elliptic Calogero-Moser Hamiltonian for \( F_4 \) is given by
\[
H = \frac{1}{2} p \cdot p - \sum_{\alpha \in R_l(F_4)} \frac{1}{2} m_2^2 \varphi(\alpha \cdot x) - \sum_{\alpha \in R_s(F_4)} \frac{1}{2} m_1^2 \varphi_2(\alpha \cdot x), \quad (6.29)
\]
where the long and short roots of \( F_4 \) are given by
\[
R_l(F_4) = \{ \pm (e_i - e_j), \ \pm (e_i + e_j), 1 \leq i < j \leq 4 \}
\]
\[
R_s(F_4) = \{ \pm e_i; \ \frac{1}{2} \sum_{i=1}^{4} \epsilon_i e_i; \ \epsilon_i = \pm 1 \}. \quad (6.30)
\]
Letting \( x \to 2x \), \( p \to p/2 \) and using the duplication formula for the Weierstrass function of (B.12), which induces a Landen transformation [24], the Hamiltonian may be re-expressed in a dual form,
\[
H = \frac{1}{8} p \cdot p - \sum_{\alpha \in R_l(F_4)} \frac{1}{8} m_2^2 \{ \varphi_2(\alpha \cdot x) + \varphi_2(\alpha \cdot x + \omega_2) \} - \sum_{\alpha \in R_s(F_4)} \frac{1}{2} m_1^2 \varphi_2(2\alpha \cdot x). \quad (6.31)
\]
Since the Lie algebra \( F_4 \) is selfdual, the set \( \alpha \in R_l \) plays the role of short roots of \( F_4 \), while the set \( 2\alpha \) with \( \alpha \in R_s \) plays the role of long roots of \( F_4 \).

**Theorem 13 : Lax pair for \( F_4 \) Twisted Elliptic Calogero-Moser**
The twisted elliptic Calogero-Moser Hamiltonian for $F_4$ admits a Lax pair of dimension $N = 24$, with spectral parameter and two independent couplings $m_1$ and $m_2$, given by (3.5), (3.6), $\Delta = 0$ and

$$\Phi_{\lambda \mu}(x, z) = \begin{cases} 
\Phi(x, z) & \lambda \cdot \mu = 0 \\
\Phi_1(x, z) & \lambda \cdot \mu = \frac{1}{2} \\
\Lambda(x, z) & \lambda \cdot \mu = -1 
\end{cases} \quad (6.32a)$$

$$C_{\lambda \mu} = \begin{cases} 
m_2 & \lambda \cdot \mu = 0 \\
\frac{1}{\sqrt{2}} m_1 & \lambda \cdot \mu = \frac{1}{2} \\
0 & \lambda \cdot \mu = -\frac{1}{2} \\
\sqrt{2} m_1 & \lambda \cdot \mu = -1 
\end{cases} \quad (6.32b)$$

$$sd \cdot v_{\lambda} = \sum_{\delta \in R_1; \delta \cdot \lambda = 1} m_2 \wp(\delta \cdot x) - \sum_{\kappa \in [\lambda]} \frac{1}{\sqrt{2}} m_1 \wp_2(\kappa \cdot x) + \frac{m_1}{\sqrt{2}} \wp_2(\lambda \cdot x). \quad (6.32c)$$

Here, the entries of the 24-dimensional Lax pair are labeled by the 24 non-zero weights $\lambda$ of the 26 of $F_4$, which are also the 24 short roots of $F_4$, as given in (6.30). As discussed in §V (e), the 24 short roots of $F_4$ fall into three different 8-classes $8^v$, $8^s$ and $8^c$, which are defined in (5.59). The 8-class of a short root $\lambda$ is denoted by $[\lambda]$. In this way, the second sum on the r.h.s. of (6.32c) is over all short roots $\kappa$ in the same 8-class as $\lambda$. The elliptic functions $\Phi$, $\Phi_1$ and $\Lambda$ are defined in §B.

Before we start proving this Theorem, some comments are in order. As is manifest from (6.32a,b), we shall retain the weights $2\lambda$, which are obtained when $\mu = -\lambda$, in the construction of the Lax pair. The reasons for doing so are three-fold. First, since these weights are proportional to the short roots, it is certainly conceivable that with a suitable $\Phi_{IJ}$-function, these weights will enter into the Lax operators and thus into the Hamiltonian on the same footing with the short roots. Second, we have already observed this very phenomenon in the case of the twisted elliptic Calogero-Moser system for $B_n$ in §VI (a). Third, this scheme works!

To prove Theorem 13, we verify the conditions of Theorem 1, with $\Delta = 0$. The weights $\lambda$ are the short roots of $F_4$ and following Theorem 1, we embed these into $GL(24, \mathbb{C})$ by (3.1). We denote the weight vectors of the fundamental representation of $GL(24, \mathbb{C})$ by an orthonormal basis $u_\lambda$, and we have

$$su_\lambda = \lambda + v_\lambda \quad (6.33)$$

where $v_\lambda$ is orthogonal to all roots of $F_4$, and $s^2 = 6$. The roots of $GL(24, \mathbb{C})$ decompose under $F_4$ as

$$\sqrt{6}(u_\lambda - u_\mu) = \lambda - \mu + v_\lambda - v_\mu, \quad \lambda \neq \mu \quad (6.34)$$
which correspond to long roots in \( \mathcal{R}(F_4) \) provided \( \lambda \cdot \mu = 0 \), or to short roots in \( \mathcal{R}(F_4) \) provided \( \lambda \cdot \mu = \frac{1}{2} \). As discussed in the previous paragraph, we also retain the weights \( 2 \lambda \).

We are now ready to analyze conditions (1) and (2) of Theorem 1. We begin by decomposing the sums over all roots in (3.7) and (3.8) into sums over long roots and short root. For any long root \( \alpha \in \mathcal{R}_l \), we have

\[
s^2m_2^2\varphi'(\alpha \cdot x) = \sum_{\alpha = \lambda - \mu \in \mathcal{R}_l} C_{\lambda,\mu}^2 \varphi_{\lambda\mu}'(\alpha \cdot x) \tag{6.35a}
\]

\[
0 = \sum_{\alpha = \lambda - \mu \in \mathcal{R}_l} C_{\lambda,\mu}^2 (v_\lambda - v_\mu) \varphi_{\lambda\mu}'(\alpha \cdot x), \tag{6.35b}
\]

while for any short root \( \alpha \in \mathcal{R}_s \) (as well as their doubles \( 2\alpha \)), we have

\[
s^2m_1^2\varphi'_2(\alpha \cdot x) = \sum_{\alpha = \lambda - \mu \in \mathcal{R}_s} C_{\lambda,\mu}^2 \varphi_{\lambda\mu}'(\alpha \cdot x) + 2C_{\alpha,-\alpha}^2 \varphi_{\alpha,-\alpha}'(2\alpha \cdot x) \tag{6.36a}
\]

\[
0 = \sum_{\alpha = \lambda - \mu \in \mathcal{R}_s} C_{\lambda,\mu}^2 (v_\lambda - v_\mu) \varphi_{\lambda\mu}'(\alpha \cdot x)
+ C_{\alpha,-\alpha}^2 (v_\alpha - v_{-\alpha}) \varphi_{\alpha,-\alpha}'(2\alpha \cdot x). \tag{6.36b}
\]

We analyze first the case of long roots \( \alpha \in \mathcal{R}_l \) of (6.35). Taking the inner product of (6.35b) with \( u_\sigma \) where \( \sigma \) is an arbitrary short root, using (3.3), and (6.35a) gives

\[
m_2^2\alpha \cdot \sigma \varphi'(\alpha \cdot x) = \sum_{\alpha = \lambda - \mu} C_{\lambda,\mu}^2 (\delta_{\lambda,\sigma} - \delta_{\mu,\sigma}) \varphi_{\lambda\mu}'(\alpha \cdot x). \tag{6.37}
\]

Since \( \alpha \in \mathcal{R}_l \), the product \( \alpha \cdot \sigma \) can take only the values \(-1, 0, +1\). For \( \alpha \cdot \sigma = 0 \), both sides of (6.37) vanish separately, while for \( \alpha \cdot \sigma = \pm 1 \), only one term remains from the sum on the right. Setting \( C_{\lambda,\mu} = m_2 \), as in (6.32b), we obtain (omitting an irrelevant constant upon integrating)

\[
\varphi(x) = \varphi_{\lambda\mu}(x) \quad \lambda - \mu \in \mathcal{R}_l(F_4). \tag{6.38}
\]

For the short roots, we proceed analogously and take the inner product of (6.36b) with \( u_\sigma \) for any short root \( \sigma \). We find

\[
m_1^2\alpha \cdot \sigma \varphi'_2(\alpha \cdot x) = \sum_{\alpha = \lambda - \mu \in \mathcal{R}_s} C_{\lambda,\mu}^2 (\delta_{\lambda,\sigma} - \delta_{\mu,\sigma}) \varphi_{\lambda\mu}'(\alpha \cdot \sigma)
+ C_{\alpha,-\alpha}^2 (\delta_{\alpha,\sigma} - \delta_{-\alpha,\sigma}) \varphi_{\alpha,-\alpha}'(2\alpha \cdot x). \tag{6.39}
\]

Since \( \alpha \in \mathcal{R}_s \), the product \( \alpha \cdot \sigma \) can take only the values \(-1, -\frac{1}{2}, 0, +\frac{1}{2}, +1\). For \( \alpha \cdot \sigma = 0 \), both sides of (6.39) vanish separately. For \( \alpha \cdot \sigma = \pm \frac{1}{2} \), we have \( \sigma \neq \pm \alpha, \sigma \pm \alpha \) is a short
root, while $\sigma \pm \alpha$ is not a root. Thus, only the sum on the r.h.s. of (6.39) contributes. By choosing $C_{\lambda,\mu} = m_1/\sqrt{2}$, as in (6.32b), $\varphi_{\lambda\mu}$ is given in (6.40a) below. For $\alpha \cdot \sigma = \pm 1$, we have $\sigma = \pm \alpha$. We shall set $C_{\alpha,-\alpha} = \sqrt{2}m_1$, as in (6.32b), so that

$$\varphi_2(x) = \varphi_{\lambda\mu}(x) \quad \lambda - \mu \in \mathcal{R}_s(F_4)$$  \hspace{1cm} (6.40a)

$$\varphi_2(x) = \varphi_{\alpha,-\alpha}(2x) \quad \alpha \in \mathcal{R}_s(F_4),$$  \hspace{1cm} (6.40b)

where we have again omitted irrelevant additive integration constants. The functions $\Phi_{\lambda\mu}$ are related to the Weierstrass functions $\varphi_{\lambda\mu}$ by (3.10). We conclude from (6.38) and (6.40) that there should be three different kinds of $\Phi$-functions, $\Phi$, $\Phi_1$ and $\Lambda$, as in (6.32a), which are precisely those defined in Appendix §B : (B.16), (B.26) and (B.22) respectively.

It remains to satisfy condition (3) of Theorem 1. This condition holds provided the functions $\Phi$, $\Phi_1$ and $\Lambda$ obey the relations (B.21), (B.23), (B.24) (B.27) and (B.28). To see this, we notice that condition (3) of Theorem 1 reduces to

$$C_{\lambda,\mu} \Phi_{\lambda\mu}((\lambda - \mu) \cdot x)sd \cdot (u_\lambda - u_\mu)$$

$$= \sum_{\kappa \neq \lambda,\mu} C_{\lambda,\mu}C_{\kappa,\mu}\{\Phi_{\lambda\kappa}((\lambda - \kappa) \cdot x)\Phi'_{\kappa\lambda}((\kappa - \mu) \cdot x)$$

$$- \Phi'_{\lambda\kappa}((\lambda - \kappa) \cdot x)\Phi_{\kappa\mu}((\kappa - \mu) \cdot x)\}.$$  \hspace{1cm} (6.41)

Here, $\lambda$, $\mu$ and $\kappa$ run over $\mathcal{R}_s(F_4)$. We analyze this equation according to the values of $\lambda \cdot \mu = -1, -\frac{1}{2}, 0, +\frac{1}{2}$; the value $\lambda \cdot \mu = +1$ is excluded since $\lambda \neq \mu$.

For $\lambda \cdot \mu = -1$, we have $\mu = -\lambda$, and $\lambda$ and $\mu$ belong to the same 8-class. Thus the sum on the r.h.s. of (6.41) for this case can receive contributions only from $\kappa \cdot \lambda = \kappa \cdot \mu = 0, \frac{1}{2}$. However, the contributions $\kappa \cdot \lambda = \kappa \cdot \mu = \frac{1}{2}$ vanish since then at least one of the coefficients $C$ must vanish. The remaining r.h.s. of (6.41) vanishes by requiring equation (B.24). The l.h.s. then requires that $d \cdot u_\lambda = d \cdot u_\mu$.

For $\lambda \cdot \mu = -\frac{1}{2}$, the l.h.s. of (6.41) vanishes, since $C_{\lambda,\mu} = 0$ then. The contributions to the sum over $\kappa$ for which $\kappa \cdot \lambda = \kappa \cdot \mu = 0$ cannot contribute since this condition would imply that $\kappa$ belongs both to the 8-class of $\lambda$ and of $\mu$. But, by $\lambda \cdot \mu = -\frac{1}{2}$, $\lambda$ and $\mu$ belong to different 8-classes. The contributions to the sum from $\kappa \cdot \lambda = \frac{1}{2}$, $\kappa \cdot \mu = 0$ and $\kappa \cdot \lambda = 0$, $\kappa \cdot \mu = \frac{1}{2}$ cancel one another. Making use of (B.27a) on the remaining sum with $\kappa \cdot \lambda = \kappa \cdot \mu = \frac{1}{2}$ and of

$$\sum_{\kappa \cdot \lambda = \kappa \cdot \mu = \frac{1}{2}} \{\varphi_2((\lambda - \kappa) \cdot x) - \varphi_2((\kappa - \mu) \cdot x)\} = \{\varphi_2(\mu \cdot x) - \varphi_2(\lambda \cdot x)\},$$  \hspace{1cm} (6.42)
we find
\[ \frac{1}{2} m_1^2 \{ \varphi_2(\lambda \cdot x) - \varphi_2(\mu \cdot x) \} \Phi_1((\lambda - \mu) \cdot x) \]
\[ = m_1^2 \{ \Lambda(2\lambda \cdot x) \Phi'_1(\lambda + \mu) \cdot x) - \Lambda'(2\lambda \cdot x) \Phi_1((\lambda + \mu) \cdot x) \]
\[ - \Lambda(-2\mu \cdot x) \Phi'_1((\lambda + \mu) \cdot x) + \Lambda'(-2\mu \cdot x) \Phi_1((\lambda + \mu) \cdot x) \}. \]
(6.43)

This equation will be satisfied when \( \Phi_1 \) and \( \Lambda \) obey (B.28b).

For \( \lambda \cdot \mu = 0 \), \( \lambda \) and \( \mu \) belong to the same 8-class, so the sum in (6.43) reduces to contributions from \( \kappa \cdot \lambda = \kappa \cdot \mu = 0 \), \( \kappa = -\lambda \), \( \kappa = -\mu \) and \( \kappa \cdot \lambda = \kappa \cdot \mu = \frac{1}{2} \). The latter sum is proportional to
\[ \sum_{\kappa \cdot \lambda = \kappa \cdot \mu = \frac{1}{2}} \{ \Phi_1((\lambda - \kappa) \cdot x) \Phi'_1((\lambda - \mu) \cdot x) - \Phi'_1((\lambda - \kappa) \cdot x) \Phi_1((\lambda - \mu) \cdot x) \}
\[ = \Phi_1((\lambda - \mu) \cdot x) \sum_{\kappa \cdot \lambda = \kappa \cdot \mu = \frac{1}{2}} \{ \varphi_2((\lambda - \kappa) \cdot x) - \varphi_2((\kappa - \mu) \cdot x) \}, \]
(6.44)

and is easily seen to vanish by changing variables in the sum on the r.h.s. to \( \delta = \lambda - \kappa \) for the first term, and \( \delta = \kappa - \mu \) for the second term. In the remaining terms on the r.h.s. of (6.41), we use (B.24) for \( \Phi \) and \( \Lambda \), we simplify by the common overall factor of \( m_2 \Phi((\lambda - \mu) \cdot x) \) and obtain
\[ sd \cdot (u_\lambda - u_\mu) = \sum_{\delta \in \mathcal{R}_1, \delta \cdot \lambda = 1} m_2 \varphi(\delta \cdot x) - \sum_{\delta \in \mathcal{R}_1, \delta \cdot \mu = 1} m_2 \varphi(\delta \cdot x) \]
\[ + \frac{1}{\sqrt{2}} \sqrt{2} m_1 \{ \varphi_2(\lambda \cdot x) - \varphi_2(\mu \cdot x) \}. \]
(6.45)

This equation may be solved for \( d \cdot u_\lambda \), but care must be taken of the fact that in (6.45), \( \lambda \) and \( \mu \) belong to the same 8-class. Thus, (6.45) separates and is determined up to a function that only depends upon the 8-class of \( \lambda \). We find
\[ sd \cdot u_\lambda = d_0([\lambda]) + \sum_{\delta \in \mathcal{R}_1, \delta \cdot \lambda = 1} m_2 \varphi(\delta \cdot x) + \frac{1}{\sqrt{2}} m_1 \varphi_2(\lambda \cdot x) \]
(6.46)

Finally, for \( \lambda \cdot \mu = \frac{1}{2} \), we cannot have \( \kappa = -\lambda \) or \( \kappa = -\mu \) since then \( \kappa \cdot \mu = \frac{1}{2} \) and \( \kappa \cdot \lambda = \frac{1}{2} \) respectively, but the coefficients \( C \) corresponding to these values cancel. Also, the sums \( \kappa \cdot \lambda = \kappa \cdot \mu = 0 \) cannot occur since \( \lambda \) and \( \mu \) are in different 8-classes. In the remaining sums, we make use of (B.27) and (B.28), we simplify by a common factor of \( \frac{1}{\sqrt{2}} m_1 \Phi_1((\lambda - \mu) \cdot x) \) and we obtain
\[ sd \cdot (u_\lambda - u_\mu) = \sum_{\delta \in \mathcal{R}_1, \delta \cdot \lambda = 1} m_2 \varphi(\delta \cdot x) - \sum_{\delta \in \mathcal{R}_1, \delta \cdot \mu = 1} m_2 \varphi(\delta \cdot x) \]
\[ + \frac{1}{\sqrt{2}} m_1 \{ \varphi_2((\lambda - \kappa) \cdot x) - \varphi_2((\kappa - \mu) \cdot x) \}. \]
(6.47)
The last equation is solved by (6.46) with
\[
d_0(\lambda) = - \sum_{\kappa \in \lambda} \frac{1}{2\sqrt{2}} m_1 \varphi_2(\lambda \cdot x)
\]  
(6.48)
which together with (6.46) yields (6.32c). This completes the proof of Theorem 13.

(d) Twisted Elliptic Calogero-Moser System for \(G_2\)

The twisted elliptic Calogero-Moser Hamiltonian for \(G_2\) is given by
\[
H = \frac{1}{2} p \cdot p - \sum_{\alpha \in \mathcal{R}_l(G_2)} \frac{1}{2} m_2 \varphi(\alpha \cdot x) - \sum_{\alpha \in \mathcal{R}_s(G_2)} \frac{1}{2} m_2 \varphi_3(\alpha \cdot x),
\]  
(6.49)
where the long and short roots of \(G_2\) are given by
\[
\mathcal{R}_l(G_2) = \{ \pm (e_i - e_j); 1 \leq i < j \leq 3 \}
\]
\[
\mathcal{R}_s(G_2) = \{ \pm (e_i - e_0); i = 1, 2, 3, e_0 = \frac{1}{4}(e_1 + e_2 + e_3) \}.
\]  
(6.50)

We have only partially succeeded in solving for the Lax pair of the twisted elliptic Calogero-Moser system associated with \(G_2\). The difficulty arises principally from: (1) the fact that the dimension \(N\) of the Lax pair representation is not a priori known (even though some educated guesses may be made as to what \(N\) should be), (2) the fact that several different and unknown elliptic type functions should enter, (3) the fact that the unknown elliptic functions satisfy many coupled non-linear differential equations. Below we shall briefly discuss the equations involved and our best conjecture for what the solution should look like.

The dimension \(N\) of the Lax pair has two natural candidates: \(N = 8\), as the dimension in which \((G^{(1)}_2)^{\vee} = D_4^{\vee}\) can be realized and \(N = 6\), as the number of short roots, by analogy with \(F_4\). Indeed, group theoretically, \(F_4\) and \(G_2\) appear to be very similar in some respects. The set of long roots on the one hand and the set of short roots on the other hand each form the root system of mutually isomorphic subalgebras: \(D_4\) for \(F_4\) and \(A_2\) for \(G_2\). In either case, the representation on which the Lax pair is built will contain the short roots (which coincide with the non-zero weights of the \(7\) of \(G_2\)), and we shall leave the number \(\nu\) of zero weights undetermined. Notice that \(\nu\) will be allowed to vanish! Thus, the weights \(\lambda_I, I = 1, \cdots, 6 + \nu\) are given by \(\lambda_i = -\lambda_{3+i} = \alpha_i, i = 1, 2, 3\) and \(\lambda_a = 0, a = 7, \cdots, 6 + \nu\) and the embedding into \(GL(6 + \nu, \mathbb{C})\) as usual by (3.1): \(su_I = \lambda_I + v_I\) with \(s^2 = 2\). The
roots of $GL(6 + \nu, \mathbb{C})$ decompose under $G_2$ as follows

\[
s(u_\lambda - u_\mu) = \lambda - \mu + v_\lambda - v_\mu
\]
\[
s(u_\lambda - u_a) = \lambda + v_\lambda - v_a
\]
\[
s(u_a - u_\mu) = -\mu + v_a - v_\mu
\]
\[
s(u_a - u_b) = v_a - v_b.
\]

(6.51)

Long roots of $G_2$ arise in the first line when $\lambda \cdot \mu = -1/3$, while short roots arise in the second and third lines, and when $\lambda \cdot \mu = 1/3$. The double short roots $2\lambda$ arise when $\mu = -\lambda$, i.e. $\lambda \cdot \mu = -2/3$.

For long roots $\alpha = \lambda - \mu$ with $\lambda \cdot \mu = -1/3$, conditions (1) and (2) of Theorem 1 are

\[
2m_2^2 \varphi'(\alpha \cdot x) = \sum_{3\lambda,\mu = -1} C_{\lambda,\mu}^2 \varphi'_{\lambda\mu}(\alpha \cdot x)
\]
\[
0 = \sum_{3\lambda,\mu = -1} C_{\lambda,\mu}^2 \varphi'_{\lambda\mu}(\alpha \cdot x)(v_\lambda - v_\mu).
\]

(6.52)

This set of equations is simply solved as follows: whenever $\lambda \cdot \mu = -1/3$, we have

\[
C_{\lambda,\mu} = m_2
\]
\[
\Phi_{\lambda\mu}(x, z) = \Phi(x, z)
\]
\[
\varphi_{\lambda\mu}(x) = \varphi(x).
\]

(6.53)

For short roots $\alpha = \lambda - \mu$ with $\lambda \cdot \mu = 1/3$, those arising from the second and third lines in (6.51), and those arising from the double short roots, we have

\[
2m_{2/3}^2 \varphi'_3(\alpha \cdot x) = \sum_{3\lambda,\mu = -1} C_{\lambda,\mu}^2 \varphi'_{\lambda\mu}(\alpha \cdot x) + 2C_{\alpha,-a}^2 \varphi'_{\alpha,-a}(2\alpha \cdot x)
\]
\[
+ \sum_{a=7}^{6+\nu} \{C_{\alpha,a}^2 \varphi'_{\alpha,a}(\alpha \cdot x) + C_{-\alpha,a}^2 \varphi'_{-\alpha,a}(\alpha \cdot x)\}
\]
\[
0 = \sum_{3\lambda,\mu = -1} C_{\lambda,\mu}^2 \varphi'_{\lambda\mu}(\alpha \cdot x)(v_\lambda - v_\mu) + C_{\alpha,-a}^2 \varphi'_{\alpha,-a}(2\alpha \cdot x)(v_\alpha - v_{-\alpha})
\]
\[
+ \sum_{a=7}^{6+\nu} \{C_{\alpha,a}^2 \varphi'_{\alpha,a}(\alpha \cdot x)(v_\alpha - v_a) + C_{-\alpha,a}^2 \varphi'_{-\alpha,a}(\alpha \cdot x)(v_a - v_{-\alpha})\}.
\]

(6.54)

Using the linear independence of the vectors $v_a$ in the second line of (6.54), we readily find

\[
C_{\alpha,a}^2 \varphi_{\alpha,a}(x) = C_{-\alpha,a}^2 \varphi_{-\alpha,a}(x).
\]

(6.55)
Projecting the second equation of (6.54) on the remaining \( u_\lambda \) vectors, we obtain two sets of equations. Whenever \( \lambda \cdot \mu = 1/3 \), the equations are solved by

\[
C_{\lambda,\mu} = \frac{1}{\sqrt{3}} m_{2/3} \\
\Phi_{\lambda \mu}(x, z) = \Phi_3(x, z) \\
\varphi_{\lambda \mu}(x) = \varphi_3(x).
\]  

(6.56)

The functions \( \varphi_3 \) and \( \Phi_3 \) are defined in (B.13) and (B.29). We also obtain a set of coupled equations mixing the contributions from the short roots arising from the second and third lines in (6.51) and from the double short roots. These equations cannot be readily solved, unfortunately, so we shall leave them in their original form,

\[
\frac{2}{3} m_{2/3}^2 \varphi_3'(\alpha \cdot x) = C_{\alpha,-\alpha}^2 \varphi_{\alpha,-\alpha}'(2\alpha \cdot x) + \sum_a C_{\alpha,a}^2 \varphi_{\alpha,a}'(\alpha \cdot x).
\]  

(6.57)

To proceed further, we use \( G_2 \) Weyl invariance to set

\[
C_{\alpha,-\alpha} = m, \quad \Phi_{\alpha,-\alpha}(x, z) = \psi(x, z),
\]  

(6.58)

where \( m \) and \( \psi \) remain to be determined.

It remains to work out condition (3) of Theorem 1. For general values of \( \nu \geq 1 \), this condition splits into three sets: (1) \( I = \lambda, \ J = \mu, \ I = \lambda, \ J = b \) (and its symmetric image \( I = b, \ J = \lambda \)), and \( I = a, \ J = b \), where \( \lambda \) and \( \mu \) are short roots of \( G_2 \) and \( a, b = 7, \ldots, 6+\nu \) label the extra zero weights. The equations in set (2) are linear in the coefficients \( C_{\lambda,a} \), while the equations in set (3) are at least linear in the coefficients \( C_{\lambda,a} \) and \( C_{a,b} \). For the minimal value \( \nu = 0 \), only the conditions in set (1) remain. (This value of \( \nu \) is equivalent to setting \( C_{\lambda,a} = C_{a,b} = 0 \).)

Pursuing the analogy with \( F_4 \), we shall concentrate on the 6-dimensional Lax pair, with \( \nu = 0 \), which involves by far the smallest number of unknown elliptic functions as well as the smallest number of equations, given by (for \( \lambda \neq \mu \))

\[
sC_{\lambda,\mu} \Phi_{\lambda \mu}(\alpha \cdot x) d \cdot (u_\lambda - u_\mu) = \sum_{\kappa \neq \lambda, \mu} C_{\lambda,\kappa} C_{\kappa,\mu} \{ \Phi_{\lambda \kappa}((\lambda - \kappa) \cdot x) \Phi'_{\kappa \mu}((\kappa - \mu) \cdot x) \\
- \Phi'_{\lambda \kappa}((\lambda - \kappa) \cdot x) \Phi_{\kappa \mu}((\kappa - \mu) \cdot x) \}.
\]  

(5.59)

This set of equations may be separated into the parts corresponding to \( \lambda \cdot \mu = \pm 1/3 \). It is convenient to partially solve these equations by setting

\[
sd \cdot u_\lambda(x) = b(\lambda \cdot x) + \sum \binom{2}{2} m_2 \varphi_3(\delta \cdot x).
\]  

(5.60)
Then, making use of the differential equations satisfied by $\Phi$ and $\Phi_3$, given in (B.30), we are left with just two equations to be obeyed by $\psi$ and by $b$. From the $\lambda \cdot \mu = 1/3$ part, we have

$$\Phi_3(x - y)(b(x) - b(y)) = \frac{m^2}{3} \left\{ [\varphi_3(y) - \varphi_3(x)]\Phi_3(x - y) + [\varphi_3^+(y) - \varphi_3^-(x)]\Phi_3^+(x - y) \\
+ [\varphi_3^+(y) - \varphi_3^-(x)]\Phi_3^-(x - y) \right\}$$

$$+ \frac{\sqrt{3}mm}{m^{2/3}} \left\{ \psi(2x)\Phi_3'(-x - y) - \psi'(2x)\Phi_3(-x - y) \\
- \psi(-2y)\Phi_3'(x + y) + \psi'(-2y)\Phi_3(x + y) \right\},$$

where $\Phi_3^\pm$ and $\varphi_3^\pm$ are defined in Appendix §B. From the $\lambda \cdot \mu = -1/3$ part, we have

$$m_2 \Phi(x - y)(b(x) - b(y)) = \frac{m^2}{3} [\varphi_3(y) - \varphi_3(x)]\Phi_3(x - y)$$

$$+ \frac{m^{2/3}}{\sqrt{3}} \left\{ \psi(2x)\Phi_3'(x - y) - \psi'(2x)\Phi_3(x - y) \\
- \psi(-2y)\Phi_3'(x + y) + \psi'(-2y)\Phi_3(x + y) \right\}.$$  (5.62)

It is possible to show that condition (5.62) is a consequence of (5.61), provided we assume that the functions $b(x)$ and $\psi(x)$ are periodic with third period $2\omega_1/3$. We suspect that without such an assumption, (5.61) and (5.62) will be contradictary. To establish this fact, it suffices to shift the arguments $x$ and $y$ by third periods, and to use the definitions of $\Phi_3$-functions and $\varphi_3$-functions given in Appendix §B.

Thus, there remains a single equation (5.61) for $\psi(x)$ and $b(x)$. We believe, but we have not succeeded in proving, that a solution with the usual analyticity and monodromy properties exists.
A. APPENDIX : LIE ALGEBRA THEORY

In Table 1, we give the Dynkin diagrams for the finite dimensional simple Lie algebras; for the untwisted affine Lie algebras (left column) and for the twisted affine Lie algebras (right column). The simple roots are labeled following Dynkin notation, and are given in an orthonormal basis in Table 2, where we also list the dimension, the Coxeter and dual Coxeter numbers (to be defined below). We list the set of all roots in Table 3, and of the highest roots in Table 4. Below we provide additional notations and definitions [21,25].

Let $G$ be one of the finite dimensional simple Lie algebras of rank $n$, let $\alpha_i$, and $\alpha_i^\vee \equiv 2\alpha_i/\alpha_i^2$, $i = 1, \cdots, n$ be its simple roots and coroots respectively. The coroot $\alpha_i^\vee$ of any root is defined by $\alpha_i^\vee = 2\alpha_i/\alpha_i^2$. Any (co-)root admits a unique decomposition into a sum of simple (co-)roots, with integer coefficients $l_i$ and $l_i^\vee$.

\[ \alpha = \sum_{i=1}^{n} l_i \alpha_i \quad \alpha^\vee = \sum_{i=1}^{n} l_i^\vee \alpha_i^\vee. \] (A.1)

The coefficients $l_i$ and $l_i^\vee$ are either all positive or all negative according to whether $\alpha$ (or $\alpha^\vee$) is positive or negative respectively. They are related by

\[ l_i^\vee = \frac{\alpha_i^2}{\alpha_i^2} l_i, \quad i = 1, \cdots, n. \] (A.2)

The highest root $\alpha_0$ and co-root $\alpha_0^\vee$ play special roles. The extension of the simple root system of an algebra $G$ by $\alpha_0$ generates the untwisted affine Lie algebra $G^{(1)}$, while the extension of the simple coroot system of $G$ by $\alpha_0^\vee$ generates the dual affine Lie algebra $(G^{(1)})^\vee$. When $G$ is non-simply laced, $(G^{(1)})^\vee$ coincides with one of the twisted affine Lie algebras. The Dynkin diagrams of these various Lie algebras are given in Table 1. The decompositions of $\alpha_0$ and $\alpha_0^\vee$ onto roots or coroots

\[ \alpha_0 = \sum_{i=1}^{n} a_i \alpha_i \quad \alpha_0^\vee = \sum_{i=1}^{n} a_i^\vee \alpha_i^\vee. \] (A.3)

define the marks $a_i$ and the comarks $a_i^\vee$, which are given in Table 4. The Coxeter number $h_G$ and the dual Coxeter number $h_G^\vee$ are defined by

\[ h_G = 1 + \sum_{i=1}^{n} a_i \quad \quad h_G^\vee = 1 + \sum_{i=1}^{n} a_i^\vee, \] (A.4)

and their values are given in Table 2. For simply laced Lie algebras, for which all roots have the same length (normalized to $\alpha_i^2 = 2$), we have $a_i^\vee = a_i$ and $h_G = h_G^\vee$. The dual Coxeter number equals the quadratic Casimir operator in the adjoint representation, $h_G^\vee = C_2(G)$. 

66
The highest weight vectors $\lambda_j, j = 1, \cdots, n$ of the fundamental representations (also called fundamental weights) of $G$ are defined by

$\alpha_i^\vee \cdot \lambda_j = \delta_{ij}$. \hspace{1cm} (A.5)

The highest weight vector $\lambda$ of any finite dimensional representation $\Lambda$ of $G$ is then uniquely specified by positive or zero integers $q_i, i = 1, \cdots, n,$ with

$$\Lambda \equiv (q_1, \cdots, q_n) \quad \lambda = \sum_{i=1}^{n} q_i \lambda_i$$ \hspace{1cm} (A.6)

The Weyl orbit of the highest weight vector of $(q_1, \cdots, q_n)$ is denoted by $[q_1, \cdots, q_n]$.

The level $l(\lambda)$ and the co-level $l^\vee(\alpha)$ are defined by

$$l(\lambda) = \lambda \cdot \rho^\vee, \quad l(\alpha_i) = 1, \quad i = 1, \cdots, n;$$

$$l^\vee(\lambda) = \lambda \cdot \rho, \quad l^\vee(\alpha_i^\vee) = 1, \quad i = 1, \cdots, n.$$ \hspace{1cm} (A.7)

Here, the level vector $\rho^\vee$ is related to the Weyl vector $\rho$ by exchanging weights $\lambda_i$ and coweights $\lambda_i^\vee = 2\lambda_i/\alpha_i^2$. Both are uniquely determined by the above normalization, and may be expressed in terms of the fundamental weights and coweights by

$$\rho = \sum_{i=1}^{n} \lambda_i = \frac{1}{2} \sum_{\alpha \in R_+(G)} \alpha$$

$$\rho^\vee = \sum_{i=1}^{n} \lambda_i^\vee = \frac{1}{2} \sum_{\alpha^\vee \in R_+(G)^\vee} \alpha^\vee.$$ \hspace{1cm} (A.8)

Here, we have provided the relation between the Weyl vector and the half sum of all positive roots, and its dual relation. It is clear from (A.3), (A.4) and (A.8) that the Coxeter and dual Coxeter numbers are related to the level of the highest root and the co-level of the highest coroot

$$h_G = 1 + \alpha_0 \cdot \rho^\vee = 1 + l(\alpha_0)$$

$$h_G^\vee = 1 + \alpha_0^\vee \cdot \rho = 1 + l^\vee(\alpha_0^\vee).$$ \hspace{1cm} (A.9)

As $\alpha$ (resp. $\alpha^\vee$) ranges through $R(G)$ (resp. $R(G)^\vee$), the maxima of $l(\alpha)$ and $l^\vee(\alpha^\vee)$ are $h_G - 1$ and $h_G^\vee - 1$ respectively.

The exponents $\gamma_i, i = 1, \cdots, n$ are such that the numbers $\gamma_i + 1$ are the degrees of the independent Casimir operators of the algebra $G$. Their values are also listed in Table 4.
Table 1. Dynkin diagrams, with labeled simple roots, for Affine Lie Algebras: untwisted on the left; twisted on the right. Dynkin diagrams for finite dimensional Lie algebras are obtained from the untwisted algebras by deleting the affine root, with label 0 and indicated with a cross.
Table 2: Basic data on finite dimensional simple Lie algebras

| $g$   | dim $g$ | $h_g$ | $h_g^\vee$ | simple roots                                                                 |
|-------|---------|-------|-------------|-------------------------------------------------------------------------------|
| $A_n$ | $n^2 + 2n$ | $n + 1$ | $n + 1$     | $\alpha_i = e_i - e_{i+1}, i = 1, \ldots, n.$                               |
| $B_n$ | $2n^2 + n$ | $2n$   | $2n - 1$    | $\alpha_i = e_i - e_{i+1}, i = 1, \ldots, n - 1; \alpha_n = e_n.$          |
| $C_n$ | $2n^2 + n$ | $2n$   | $n + 1$     | $\alpha_i = e_i - e_{i+1}, i = 1, \ldots, n - 1; \alpha_n = 2e_n.$          |
| $D_n$ | $2n^2 - 2n$ | $2n - 2$ | $2n - 2$    | $\alpha_i = e_i - e_{i+1}, i = 1, \ldots, n - 1; \alpha_n = e_{n-1} + e_n.$ |
| $E_6$ | 78      | 12    | 12          | $\alpha_1 = \frac{1}{2} (e_1 - e_2 - e_3 - e_4 - e_5 + \sqrt{3} e_6);$ $\alpha_i = e_i - e_{i-1}, i = 2, \ldots, 5; \alpha_6 = e_1 + e_2.$ |
| $E_7$ | 133     | 18    | 18          | $\alpha_1 = \frac{1}{2} (e_1 - e_2 - e_3 - e_4 - e_5 - e_6 + \sqrt{2} e_7);$ $\alpha_i = e_i - e_{i-1}, i = 2, \ldots, 6; \alpha_7 = e_1 + e_2.$ |
| $E_8$ | 248     | 30    | 30          | $\alpha_1 = \frac{1}{2} (e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8);$ $\alpha_i = e_i - e_{i-1}, i = 2, \ldots, 7; \alpha_8 = e_1 + e_2.$ |
| $G_2$ | 14      | 6     | 4           | $\alpha_1 = e_1 - e_2; \alpha_2 = \frac{1}{4} (-2e_1 + e_2 + e_3).$        |
| $F_4$ | 52      | 12    | 9           | $\alpha_1 = e_2 - e_3; \alpha_2 = e_3 - e_4; \alpha_3 = e_4; \alpha_4 = \frac{1}{2} (e_1 - e_2 - e_3 - e_4).$ |

The set of vectors $e_i$ forms an orthonormal basis.

Table 3: Root system of finite dimensional simple Lie algebras

| $g$   | all roots                                                                 |
|-------|---------------------------------------------------------------------------|
| $A_n$ | $\pm (e_i - e_j), 1 \leq i < j \leq n + 1.$                               |
| $B_n$ | $\pm (e_i - e_j); \pm (e_i + e_j), 1 \leq i < j \leq n; \pm e_i, 1 \leq i \leq n.$ |
| $C_n$ | $\pm (e_i - e_j); \pm (e_i + e_j), 1 \leq i < j \leq n; \pm 2e_i, 1 \leq i \leq n.$ |
| $D_n$ | $\pm (e_i - e_j), \pm (e_i + e_j), 1 \leq i < j \leq n.$                  |
| $E_6$ | $\pm (e_i - e_j), \pm (e_i + e_j), 1 \leq i < j \leq 5; \pm \frac{1}{2} (\sqrt{3} e_6 + \sum_{i=1}^{5} e_i e_i), \prod_i e_i = +1.$ |
| $E_7$ | $\pm (e_i - e_j), \pm (e_i + e_j), 1 \leq i < j \leq 6; \pm \frac{1}{2} (\sqrt{2} e_7 + \sum_{i=1}^{6} e_i e_i), \prod_i e_i = -1.$ |
| $E_8$ | $\pm (e_i - e_j), \pm (e_i + e_j), 1 \leq i < j \leq 8; \frac{1}{2} \sum_{i=1}^{8} e_i e_i, \prod_i e_i = +1.$ |
| $G_2$ | $\pm (e_i - e_j), 1 \leq i < j \leq 3; \pm (e_i - \frac{1}{4} (e_1 + e_2 + e_3)), i = 1, 2, 3.$ |
| $F_4$ | $\pm (e_i - e_j), \pm (e_i + e_j), 1 \leq i < j \leq 4; \pm e_i, 1 \leq i \leq 4; \pm \frac{1}{2} \sum_{i=1}^{4} e_i e_i.$ |
### Table 4: Marks, Co-marks and Exponents

| $G$   | marks ($a_i$) | comarks ($a_i^\vee$) | exponents $\gamma_i$ |
|-------|---------------|----------------------|----------------------|
| $A_n$ | (1,1,1, ... ,1,1) | (1,1,1, ... ,1,1) | 1,2,3, ... ,n        |
| $B_n$ | (1,2,2, ... ,2,2) | (1,2,2, ... ,2,1) | 1,3,5, ... ,2n-1     |
| $C_n$ | (2,2,2, ... ,2,1) | (1,1,1, ... ,1,1) | 1,3,5, ... ,2n-1     |
| $D_n$ | (1,2, ... ,2,1,1) | (1,2, ... ,2,1,1) | 1,3,5, ... ,2n-3,n-1 |
| $E_6$ | (1,2,3,2,1,2) | (1,2,3,2,1,2) | 1,4,5,7,8,11         |
| $E_7$ | (2,3,4,3,2,1,2) | (2,3,4,3,2,1,2) | 1,5,7,9,11,13,17     |
| $E_8$ | (2,3,4,5,6,4,2,3) | (2,3,4,5,6,4,2,3) | 1,7,11,13,17,19,23,29 |
| $G_2$ | (2,3) | (2,1) | 1,5                  |
| $F_4$ | (2,3,4,2) | (2,3,2,1) | 1,5,7,11             |

### B. Appendix: Elliptic Functions

In this appendix we review some basic definitions and properties of elliptic functions on an elliptic curve of periods $2\omega_1$ and $2\omega_2$ and modulus $\tau = \omega_2/\omega_1$, $\text{Im}\tau > 0$. The half periods are $\omega_1$, $\omega_2$ and $\omega_3 = \omega_1 + \omega_2$. For a useful source, see [24].

#### (a) Basic Definitions and properties

The Weierstrass function is defined by

$$
\wp(z; 2\omega_1, 2\omega_2) \equiv \frac{1}{z^2} + \sum_{(m_1,m_2) \neq (0,0)} \left\{ \frac{1}{(z + 2\omega_1 m_1 + 2\omega_2 m_2)^2} - \frac{1}{(2\omega_1 m_1 + 2\omega_2 m_2)^2} \right\} \quad (B.1)
$$

and may alternatively be written as

$$
\wp(z; 2\omega_1, 2\omega_2) = -\frac{\eta_1}{\omega_1} + \left( \frac{\pi}{2\omega_1} \right)^2 \sum_{n=-\infty}^{\infty} \frac{1}{\sinh^2 i\pi n \tau} \frac{1}{2\omega_1(z - 2n\omega_2)}, \quad (B.2)
$$

where

$$
\frac{\eta_1}{\omega_1} = -\frac{1}{12} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{\sinh^2 i\pi n \tau}. \quad (B.3)
$$

The function $\wp$ is related to the Weierstrass functions $\zeta$ and $\sigma$ by

$$
\wp(z; 2\omega_1, 2\omega_2) = -\frac{d}{dz}\zeta(z; 2\omega_1, 2\omega_2) = -\frac{d^2}{dz^2} \log \sigma(z; 2\omega_1, 2\omega_2). \quad (B.4)
$$
These functions satisfy
\[
\begin{align*}
\wp(-z) &= \wp(z), & \wp(z + 2\omega_a) &= \wp(z) & a = 1, 2, 3 \\
\zeta(-z) &= -\zeta(z), & \zeta(z + 2\omega_a) &= \zeta(z) + 2\eta_a \\
\sigma(-z) &= \sigma(z), & \sigma(z + 2\omega_a) &= -\sigma(z)e^{2\eta_a(z+2\omega_a)},
\end{align*}
\]
(B.5)
where \(\eta_a = \zeta(\omega_a)\) and
\[
\begin{align*}
\sigma(z) &= z + O(z^3) \\
\zeta(z) &= \frac{1}{z} + O(z^3) \\
\wp(z) &= \frac{1}{z^2} + O(z^2).
\end{align*}
\]
The function \(\sigma\) may be expressed in terms of the Jacobi \(\vartheta\)-function
\[
\sigma(z; 2\omega_1, 2\omega_2) = 2\omega_1 \exp\left(\frac{\eta_1 z^2}{2\omega_1}\right) \frac{\vartheta_1\left(\frac{z}{2\omega_1}\right|\tau\right)}{\vartheta_1'(0|\tau)},
\]
(B.7)
which in turn is defined in terms of
\[
q = e^{2\pi i \tau} \quad v = \frac{z}{2\omega_1}
\]
(B.8)
by
\[
\vartheta_1(u|\tau) = 2q^{1/2} \sin \pi u \prod_{n=1}^{\infty} (1 - q^n e^{2\pi i u})(1 - q^n e^{-2\pi i u})(1 - q^n).
\]
(B.9)
The function \(\wp\) satisfies the differential equation
\[
\wp'(z)^2 = 4(\wp(z) - \wp(\omega_1))(\wp(z) - \wp(\omega_2))(\wp(z) - \wp(\omega_3)).
\]
(B.10)
These and additional properties of elliptic functions may be found in [24].

(b) Half and Third Period Functions

Elliptic functions at half and third period (which are the only ones that we shall need here, since the order of twisting is at most 3) are expressible in terms of the original periods using Landen’s transformations [24]. It is convenient to make a definite choice for the period that is to be twisted; we shall choose this period to be \(\omega_1\). It is straightforward to adapt these formulas when an arbitrary period \(2\omega_a, a = 1, \cdots, 3\) is twisted.

Formulas for Twists of Order 2 : Elliptic Functions with Half Periods

71
Henceforth, we shall reserve the notation \( \wp(z) \), \( \zeta(z) \) and \( \sigma(z) \) for the corresponding Weierstrass functions with periods \( 2\omega_1 \) and \( 2\omega_2 \), as defined in §B (a). The elliptic functions at half period \( \omega_1 \) are given by

\[
\wp_2(z) = \wp(z;\omega_1, 2\omega_2) = \wp(z) + \wp(z + \omega_1) - \wp(\omega_1)
\]

\[
= \frac{1}{\wp(\omega_1)} \left[ \wp(z)\wp(z + \omega_1) - (\wp(\omega_1) - \wp(\omega_2))(\wp(\omega_1) - \wp(\omega_3)) \right]
\]

\[
\zeta_2(z) = \zeta(z;\omega_1, 2\omega_2) = \zeta(z) + \zeta(z + \omega_1) + z\wp(\omega_1) - \eta_1
\]

\[
\sigma_2(z) = \sigma(z;\omega_1, 2\omega_2) = \frac{\sigma(z)\sigma(z + \omega_1)}{\sigma(\omega_1)} e^{\frac{1}{2}z^2\wp(\omega_1) - z\eta_1}
\]

This gives rise to the duplication formula

\[
4\wp(2z) = \wp(z) + \wp(z + \omega_1) + \wp(z + \omega_2) + \wp(z + \omega_1 + \omega_2). \tag{B.12}
\]

**Formulas for Twists of order 3 : Elliptic Functions at Third Periods**

Similarly, we have the following formulas for the third period elliptic functions

\[
\wp_3(z) = \wp(z;2\omega_1/3, 2\omega_2) = \wp(z) + \wp(z + 2\omega_1/3) + \wp(z + 4\omega_1/3)
\]

\[
- \wp(2\omega_1/3) - \wp(4\omega_1/3)
\]

\[
\zeta_3(z) = \zeta(z;2\omega_1/3, 2\omega_2) = \zeta(z) + \zeta(z + 2\omega_1/3) + \zeta(z + 4\omega_1/3)
\]

\[
+ z\wp(2\omega_1/3) + z\wp(4\omega_1/3) - \eta_1
\]

\[
\sigma_3(z) = \sigma(z;2\omega_1/3, 2\omega_2) = \frac{\sigma(z)\sigma(z + 2\omega_1/3)\sigma(z + 4\omega_1/3)}{\sigma(2\omega_1/3)\sigma(4\omega_1/3)} e^{\frac{1}{2}z^2\wp(\omega_1) - z\eta_1}
\]

This gives rise to the triplication formula

\[
9\wp(3z) = \sum_{j,k=0}^{2} \wp(z + j\frac{2\omega_1}{3} + k\frac{2\omega_2}{3}). \tag{B.14}
\]

All of the above formulas may be established by identifying singularities in \( z \) and establishing that the remainder must be independent of \( z \) by Liouville’s theorem.

The functions at half and third periods, defined above are related to one another in a way analogous to (B.4)

\[
\wp_\nu(z;2\omega_1, 2\omega_2) = -\frac{d}{dz} \zeta_\nu(z;2\omega_1, 2\omega_2) = -\frac{d^2}{dz^2} \log \sigma_\nu(z;2\omega_1, 2\omega_2), \tag{B.15}
\]

where \( \nu = 1, 2, 3 \).
(c) The Function $\Phi$

We define the function $\Phi$ by

$$\Phi(x, z) = \Phi(x, z; 2\omega_1, 2\omega_2) = \frac{\sigma(z - x)}{\sigma(z)\sigma(x)}e^{x\zeta(z)}$$  \hspace{1cm} (B.16)$$

where $\sigma(z)$ and $\zeta(z)$ are the Weierstrass functions of (B.4) and (B.7). As a function of $z$, $\Phi(x, z)$ is periodic with periods $2\omega_1$ and $2\omega_2$, is holomorphic except for an essential singularity at $z = 0$, and has a single zero at $z = x$. As a function of $x$, $\Phi(x, z)$ has multiplicative monodromy, given by

$$\Phi(x + 2\omega_a, z) = \Phi(x, z)e^{2\omega_a\zeta(z) - 2\eta_a z}.$$  \hspace{1cm} (B.17)$$

is holomorphic in $x$ except for a simple pole at $x = 0$, and has a single zero at $x = z$. Some useful asymptotics are given as follows

$$\Phi(x, z) = \frac{1}{x} - \frac{1}{2}x\phi(z) + O(x^2)$$

$$\Phi(x, z) = \{-\frac{1}{x} + \zeta(x) + O(z)\}e^{x\zeta(z)} \quad z \to 0.$$  \hspace{1cm} (B.18)$$

Products of the function $\Phi(x_\alpha, z)$, with $\sum_{\alpha=1}^{n} x_\alpha = 0$, are periodic in $z$, with periods $2\omega_1$ and $2\omega_2$, and meromorphic in $z$ since the essential singularities cancel. As a result, they satisfy

$$\prod_{\alpha=1}^{n} \Phi(x_\alpha, z) = P_n[\phi(z); x_\alpha] + \phi'(z)Q_n[\phi(z); x_\alpha],$$  \hspace{1cm} (B.19)$$

where $P_n$ and $Q_n$ are polynomials of degrees $\left[\frac{n}{2}\right]$ and $\left[\frac{n-3}{2}\right]$ in $\phi(z)$ respectively, with $x_\alpha$-dependent coefficients. The simplest case is

$$\Phi(x, z)\Phi(-x, z) = \phi(z) - \phi(x).$$  \hspace{1cm} (B.20)$$

In general, the polynomials $P$ and $Q$ may be determined by the fact that the r.h.s. of (B.19) has a simple zero at each point $z = x_\alpha$, and that the pole highest order in $z$ has coefficient $(-1)^n$.

The function $\Phi(x, z)$ satisfies a fundamental differential equation,

$$\Phi(x, z)\Phi'(y, z) - \Phi(y, z)\Phi'(x, z) = (\phi(x) - \phi(y))\Phi(x + y, z),$$  \hspace{1cm} (B.21)$$

where $\Phi'(x, z)$ denotes the derivative with respect to $x$ of $\Phi(x, z)$.
(d) The Functions $\Lambda$, $\Phi_1$, $\Phi_2$ and $\Phi_3$

The functions $\Lambda$ and $\Phi_2$ are defined by

$$\Lambda(2x, z) = \Phi_2(x, z) = \frac{\Phi(x, z)\Phi(x + \omega_1, z)}{\Phi(\omega_1, z)}.$$  \hfill (B.22)

The essential singularity in $z$ and the monodromy in $x$ of $\Lambda(2x, z)$ coincide with those of $\Phi(x, z)$. We shall need the following two basic differential equations,

$$\Lambda(2x, z)\Lambda'(2y, z) - \Lambda'(2x, z)\Lambda(2y, z) = \frac{1}{2}(\wp_2(x) - \wp_2(y))\Lambda(2x + 2y, z), \quad (B.23a)$$

$$\Phi_2(x, z)\Phi'_2(y, z) - \Phi_2(y, z)\Phi'_2(x, z) = (\wp_2(x) - \wp_2(y))\Phi_2(x + y, z), \quad (B.23b)$$

as well as differential equation that involves both $\Phi$ and $\Lambda$,

$$\Lambda(2x, z)\Phi'(-x - y, z) - \Lambda'(2x, z)\Phi(-x - y, z)$$

$$-\Lambda(-2y, z)\Phi'(x + y, z) + \Lambda'(-2y, z)\Phi(x + y, z) = \frac{1}{2}(\wp_2(x) - \wp_2(y))\Phi(x - y, z). \quad (B.24)$$

By letting $y \to x$ in (B.23), and taking into account the known zeros of $\Lambda$, we derive another useful formula

$$\Lambda(2x, z)\Lambda(-2x, z) = \wp_2(z) - \wp_2(x). \quad (B.25)$$

Actually, $\Phi_2(x, z)$ may be viewed as the function $\Phi(x, z)$ associated with a torus of periods $\omega_1$ and $2\omega_2$.

The function $\Phi_1(x, z)$ is defined by

$$\Phi_1(x, z) = \Phi(x, z) + f(z)\Phi(x + \omega_1, z)$$

$$f(z) = -e^{\pi i \zeta(z) + n_1 z}.$$  \hfill (B.26)

It obeys the monodromy relation $\Phi_1(x + \omega_1, z) = f(z)^{-1}\Phi_1(x, z)$, as well as the following differential equations

$$\Phi_1(x, z)\Phi'_1(y, z) - \Phi'_1(x, z)\Phi_1(y, z) = (\wp_2(x) - \wp_2(y))\Phi_1(x + y, z) \quad (B.27a)$$

$$\Phi_1(x, z)\Phi'(y, z) - \Phi(y, z)\Phi'_1(x, z) = \{\wp(x + \omega_1) - \wp(y)\}\Phi_1(x + y, z)$$

$$+ \{\wp(x) - \wp(x + \omega_1)\}\Phi(x + y, z). \quad (B.27b)$$
and
\[
\Phi(2x, z)\Phi'_1(-x - y, z) - \Phi'(2x, z)\Phi_1(-x - y, z)
\]
\[-\Phi(-2y, z)\Phi'_1(x + y, z) + \Phi'(-2y, z)\Phi_1(x + y, z)
\]
\[= (\wp(2x) - \wp(2y))\Phi_1(x - y, z), \quad (B.28a)\]
\[
\Lambda(2x, z)\Phi'_1(-x - y, z) - \Lambda'(2x, z)\Phi_1(-x - y, z)
\]
\[-\Lambda(-2y, z)\Phi'_1(x + y, z) + \Lambda'(-2y, z)\Phi_1(x + y, z)
\]
\[= \frac{1}{2}(\wp_2(x) - \wp_2(y))\Phi_1(x - y, z). \quad (B.28b)\]

Finally, we define functions of twist order 3. We introduce \(\gamma = 2\omega_1/3\) so that
\[
\Phi_3(x, z) = \Phi(x, z) + \Phi(x + \gamma, z) + \Phi(x + 2\gamma, z),
\]
\[
\Phi^\pm_3(x, z) = \Phi(x, z) + e^{\pm\gamma}\Phi(x + \gamma, z) + e^{\pm\gamma}\Phi(x + 2\gamma, z),
\]
\[
\wp_3(x) = \wp(x) + \wp(x + \gamma) + \wp(x + 2\gamma)
\]
\[
\wp^\pm_3(x) = \wp(x) + e^{\pm\gamma}\wp(x + \gamma) + e^{\pm\gamma}\wp(x + 2\gamma).
\]

These functions obey many natural differential equations, of which we shall only quote the one directly relevant here,
\[
\Phi(x, z)\Phi'_3(y, z) - \Phi'(x, z)\Phi_3(y, z) = \wp(x)\Phi_3(x + y, z) - \frac{1}{4}\wp_3(y)\Phi_3(x + y, z)
\]
\[-\frac{1}{4}\wp^+_3(y)\Phi^+_3(x + y, z) - \frac{1}{4}\wp^-_3(y)\Phi^-_3(x + y, z). \quad (B.30)\]

**C. APPENDIX : COMPLETING THE PROOF FOR \textit{E}_8**

In this appendix, we obtain (5.21b) and derive (5.21d) and (5.21e) from (5.31b) and the results already obtained in the main section : (5.21a), (5.21c) and the assumed solution to equations (5.22). Then, we show that the vastly overdetermined system (5.31b) is satisfied by the solution of (5.21d,e).

First, we obtain the general equations for \(\Delta_{a,b}\) from (5.31b), by substituting the solution (5.21c) into (5.31b). It is convenient to introduce the quantities
\[
\Delta_{\lambda,\mu} = \frac{1}{2m^2} \sum_{a,b} C_{\lambda,a}\Delta_{a,b}C_{b,\mu} \quad (C.1)
\]
Equation (5.31b) may be recast in terms of $\Delta_{\lambda,\mu}$ as follows

\[
\Delta_{\lambda,\mu} = \frac{1}{2m^2} \left( \sum_{\lambda,\delta=1}^{\varphi(\delta \cdot x) + 2\varphi(\lambda \cdot x)} C_{\lambda} \cdot C_{\mu} - \frac{1}{2m^2} \sum_{\kappa,\lambda=1}^{c(\lambda,\kappa)C_{\kappa} \cdot C_{\mu}} \left( \varphi((\lambda - \kappa) \cdot x) - \varphi(\kappa \cdot x) \right) \right) \quad (C.2)
\]

which is more symmetrical in $\lambda$ and $\mu$, as can be seen from using the symmetry properties of $C_{\lambda}$

\[
\Delta_{\lambda,\mu} = \Delta_{\mu,\lambda} = \Delta_{-\lambda,-\mu} = -\Delta_{\lambda,-\mu} = -\Delta_{-\lambda,\mu} \quad (C.3)
\]

Next, we make use of the basis of orthonormal roots $\beta_a$, $a = 1, \cdots, 8$, and label the zero weights of the $248$ in terms of this basis. As a result, the vectors $C_{\beta_a}$ form an orthogonal basis of the space of $C_{\lambda}$. To show this, use (5.30) between different $\beta_a$ and between any $\beta_a$ and any root $\lambda$ which does not equal any of the $\pm \beta_b$. We then have the following inner product relations

\[
C_{\beta_a} \cdot C_{\beta_b} = 2m^2 \delta_{a,b} \\
C_{\lambda} \cdot C_{\beta_a} = m^2 c(\lambda, \beta_a(\lambda \cdot \beta_a)) \quad (\lambda \neq \pm \beta_b) \quad (C.4)
\]

This gives us the projections of the vectors $C_{\lambda}$ onto the basis of $\beta_a$, and determines $C_{\lambda}$ up to an undetermined vector $V$ which is orthogonal to all $C_{\beta_a}$.

\[
C_{\lambda} = V_{\lambda} + \sum_{a=1}^{8} \frac{1}{2} \lambda \cdot \beta_a c(\lambda, \beta_a(\lambda \cdot \beta_a)) C_{\beta_a}. \quad (C.5)
\]

Using the relation $C_{\lambda} \cdot C_{\lambda} = 2m^2$ in (5.30) and then evaluating the same quantity directly from (C.5), we find that $V = 0$, so that the vectors $C_{\beta_a}$ indeed span a basis of the space of all $C_{\lambda}$. With $V = 0$, (C.5) precisely reproduces (5.21b) of Theorem 8. Using the orthogonality of $C_{\beta_a}$ in (C.4) and equation (C.1), we find

\[
\Delta_{\beta_a,\beta_b} = \Delta_{a,b}, \quad (C.6)
\]

evaluating (C.2) for $\lambda = \beta_a$ and $\mu = \beta_b$, with the help of (5.30), we recover the expressions in (5.21d) and (5.21e).

It remains to show that the results of (5.21d,e) consistently solve (5.31b) for all roots $\lambda$. Since the solution (5.21d,e) was derived for $\lambda = \pm \beta_a$ already, it suffices to study the cases $\lambda \neq \beta_a, a = 1, \cdots, 8$. The issue of consistency arises here because on the one hand, $\Delta_{\lambda,\beta_b}$ may be evaluated directly from (C.1) (we shall denote this quantity by $\Delta_{\lambda,\beta_b}$), while
on the other hand, the same quantity may be evaluated using the solution (5.21d,e) and the expression (C.4) above (we shall denote this quantity by \( \bar{\Delta}_{\lambda,\beta_b} \), and the two need to be equal for consistency. Following this notation, we evaluate

\[
\bar{\Delta}_{\lambda,\beta_b} = m_2^2 \lambda \cdot \beta_b \ c(\lambda, \beta_b(\lambda \cdot \beta_b)) \ \Delta_{b,b} + m_2^2 \sum_{a \neq b} \lambda \cdot \beta_a \ c(\lambda, \beta_a(\lambda \cdot \beta_a)) \ \Delta_{a,b}
\]  

(C.7)

The quantity \( \Delta_{\lambda,\beta_b} \), which is directly evaluated from (C.2), equals

\[
\Delta_{\lambda,\beta_b} = \frac{1}{2} m_2 \lambda \cdot \beta_b \ c(\lambda, \beta_b(\lambda \cdot \beta_b)) \left( \sum_{\lambda, -\beta_b = 1} \varphi(\delta \cdot x) + 2\varphi(\lambda \cdot x) \right)
\]  

- \[\frac{1}{2m_2} \sum_{\kappa, \lambda = 1} c(\lambda, \kappa) C_{\lambda} \cdot C_{\beta_b} (\varphi((\lambda - \kappa) \cdot x) - \varphi(\kappa \cdot x)) \]  

(C.8)

With the help of (5.22), we shall now show that (C.8) agrees with (C.7). Since we restricted to \( \lambda \neq \beta_a \), and using the symmetry properties of (C.3), we are left to consider only the cases \( \lambda \cdot \beta_a = 0 \) and \( \lambda \cdot \beta_a = 1 \).

The case \( \lambda \cdot \beta_a = 0 \)

Eq. (C.8) may be evaluated using (5.30) and reduces to

\[
\Delta_{\lambda,\beta_b} = -\frac{1}{2} m_2 \sum_{\kappa, \lambda = 1, \beta_b = 1} c(\lambda, \kappa) c(\kappa, \beta_b) \ (\varphi((\lambda - \kappa) \cdot x) - \varphi(\kappa \cdot x))
\]  

\[+ \frac{1}{2} m_2 \sum_{\kappa, \lambda = 1, \beta_b = -1} c(\lambda, \kappa) c(\kappa, -\beta_b) \ (\varphi((\lambda - \kappa) \cdot x) - \varphi(\kappa \cdot x)) \]  

(C.9)

Making a change of variables \( \lambda - \kappa = \delta \) in the first term of each sum, and letting \( \kappa = \delta \) in the second, and regrouping terms, we obtain

\[
\Delta_{\lambda,\beta_b} = + \frac{1}{2} m_2 \sum_{\delta, \lambda = 1, \beta_b = 1} (c(\lambda, \delta) c(\delta, \beta_b) + c(\lambda, \lambda - \delta) c(\lambda - \delta, -\beta_b)) \varphi(\delta \cdot x)
\]  

- \[\frac{1}{2} m_2 \sum_{\delta, \lambda = 1, \beta_b = -1} (c(\lambda, -\delta) c(\delta, \beta_b) + c(\lambda, \lambda + \delta) c(\lambda + \delta, \beta_b)) \varphi(\delta \cdot x). \]  

(C.10)

The product relations between \( \lambda, \beta_a \) and \( \delta \) are precisely such that we are allowed to use (5.22a) on the second term in each sum. The relations used are

\[
c(\lambda, \lambda - \delta) c(\lambda - \delta, -\beta_b) = c(\lambda, -\beta_b + \delta) c(\beta_b + \delta, \beta_b) \quad \text{first sum}
\]  

\[
c(\lambda, \lambda + \delta) c(\lambda + \delta, \beta_b) = c(\lambda, \beta_b - \delta) c(\beta_b - \delta, \beta_b) \quad \text{second sum,}
\]  

(C.11)
and yield
\[
\Delta_{\lambda, \beta_b} = + \frac{1}{2} m_2 \sum_{\delta \cdot \lambda = 1, \delta \cdot \beta_b = 1} \left( c(\lambda, \delta) c(\delta, \beta_b) + c(\lambda, \delta - \beta_b) c(\beta_b - \delta, \beta_b) \right) \phi(\delta \cdot x) \\
- \frac{1}{2} m_2 \sum_{\delta \cdot \lambda = -1, \delta \cdot \beta_b = 1} \left( c(\lambda, -\delta) c(\delta, \beta_b) + c(\lambda, \beta_b - \delta) c(\beta_b - \delta, \beta_b) \right) \phi(\delta \cdot x).
\]

Using (5.30) again, we rewrite the cocycle factors that involve \( \lambda \) as inner products with \( C_\lambda \).
\[
\Delta_{\lambda, \beta_b} = + \frac{1}{2} m_2 \sum_{\delta \cdot \beta_b = 1} \left( C_\lambda \cdot C_\delta c(\delta, \beta_b) + C_\lambda \cdot C_{\delta - \beta_b} c(\beta_b - \delta, \beta_b) \right) \phi(\delta \cdot x). 
\]

In establishing equivalence of (C.12) and (C.13), we use the fact that terms with \( \delta \cdot \lambda = \pm 2 \) cannot contribute since they would imply \( \delta = \pm \lambda \), but this cannot be realized with \( \lambda \cdot \beta_b = 0 \) and \( \delta \cdot \beta_b = 1 \). Next, we make use of (5.21b), and obtain
\[
\Delta_{\lambda, \beta_b} = \frac{1}{2} \sum_{a=1}^{8} c(\lambda, \beta_a(\lambda \cdot \beta_a)) 
\]
with
\[
\Delta'_{a,b} = + \frac{1}{2} m_2 \sum_{\delta \cdot \beta_b = 1} \left( C_{\beta_a} \cdot C_\delta c(\delta, \beta_b) + C_{\beta_a} \cdot C_{\delta - \beta_b} c(\beta_b - \delta, \beta_b) \right) \phi(\delta \cdot x). 
\]

It remains to show that \( \Delta'_{a,b} \) coincides with \( \Delta_{a,b} \) of (5.21). This is established by decomposing the sum over \( \delta \) according to the values of \( \delta \cdot \beta_a \). The values \( \delta \cdot \beta_a = \pm 2 \) are not allowed, since already \( \delta \cdot \beta_b = 1 \).
\[
\Delta'_{a,b} = + \frac{1}{2} m_2 \sum_{\delta \cdot \beta_b = 1} \left( c(\beta_a, \delta) c(\delta, \beta_b) + c(\beta_a, \delta - \beta_b) c(\beta_b - \delta, \beta_b) \right) \phi(\delta \cdot x) \\
- \frac{1}{2} m_2 \sum_{\delta \cdot \beta_b = -1} \left( c(\beta_a, -\delta) c(\delta, \beta_b) + c(\beta_a, \beta_b - \delta) c(\beta_b - \delta, \beta_b) \right) \phi(\delta \cdot x). 
\]

Using again the relations (5.22a), but this time for \( \beta_a, \beta_b \) and \( \delta \), we see that this expression precisely reproduces (5.21d,e).

The Case \( \lambda \cdot \beta_b = 1 \)

The spirit of the proof of this case is completely analogous to that of the case \( \lambda \cdot \beta_b = 0 \). We start with (C.2) for \( \mu = \beta_b \) and show that it is solved by (5.21d,e) for all roots \( \lambda \neq \beta_a \),
\[ a = 1, \ldots, 8. \] To do so, let \( \delta = \lambda - \kappa \) in the first term of the second sum on the r.h.s. of (C.7), and \( \delta = \kappa \) in the second term, then we decompose these sums according to the values of \( \delta \cdot \beta_b \) and evaluate the inner products using (5.30).

\[
\Delta_{\lambda, \beta_b} = \frac{1}{2} m_2 c(\lambda, \beta_b) \left( \sum_{\lambda, \delta = 1} \varphi(\delta \cdot x) + 2 \varphi(\lambda \cdot x) - 2 \varphi((\lambda - \beta_b) \cdot x) + 2 \varphi(\beta_b \cdot x) \right)
- \frac{1}{2} m_2 \sum_{\delta, \lambda = 1 \atop \delta \neq \lambda, \beta_b = 0} c(\lambda, \lambda - \delta) c(\lambda - \delta, \beta_b) \varphi(\delta \cdot x)
+ \frac{1}{2} m_2 \sum_{\delta, \lambda = 1 \atop \delta \neq \lambda, \beta_b = 1} c(\lambda, \delta) c(\delta, \beta_b) \varphi(\delta \cdot x) - \frac{1}{2} m_2 \sum_{\delta, \lambda = 1 \atop \delta \neq \lambda, \beta_b = -1} c(\lambda, \delta) c(\delta, -\beta_b) \varphi(\delta \cdot x)
+ \frac{1}{2} m_2 \sum_{\delta, \lambda = 1 \atop \delta \neq \lambda, \beta_b = 2} c(\lambda, \lambda - \delta) c(\lambda - \delta, -\beta_b) \varphi(\delta \cdot x).
\] (C.17)

We now also decompose the first sum on the r.h.s. of (C.17) according to the values of \( \delta \cdot \beta_b \),

\[
\sum_{\delta, \lambda = 1} \varphi(\delta \cdot x) = \varphi(\beta_b \cdot x) + \sum_{\delta, \lambda = 1 \atop \delta \neq \lambda, \beta_b = 0} \varphi(\delta \cdot x) + \sum_{\delta, \lambda = 1 \atop \delta \neq \lambda, \beta_b = 1} \varphi(\delta \cdot x).
\] (C.18)

Using relation (5.22b) on the terms with \( \delta \cdot \beta_b = 0 \), we find that these terms cancel between the sums in the first and second lines in (C.17). Rearranging the remaining terms, we find

\[
\Delta_{\lambda, \beta_b} = \frac{1}{2} m_2 c(\lambda, \beta_b) \left( \sum_{\lambda, \delta = 1 \atop \delta \neq \lambda, \beta_b = \pm 1} \varphi(\delta \cdot x) + 2 \varphi(\lambda \cdot x) - 2 \varphi((\lambda - \beta_b) \cdot x) + 2 \varphi(\beta_b \cdot x) \right)
+ \frac{1}{2} m_2 \sum_{\delta, \lambda = 1 \atop \delta \neq \lambda, \beta_b = 1} c(\lambda, \delta) c(\delta, \beta_b) \varphi(\delta \cdot x) - \frac{1}{2} m_2 \sum_{\delta, \lambda = 1 \atop \delta \neq \lambda, \beta_b = -1} c(\lambda, \delta) c(\delta, -\beta_b) \varphi(\delta \cdot x).
\] (C.19)

The last two sums in (C.19) are rewritten with the help of (5.30) as

\[-m_2 c(\lambda, \beta_b) \varphi(\lambda \cdot x) + \frac{1}{2 m_2} \sum_{\delta, \beta_b = 1} C_\lambda \cdot C_\delta c(\delta, \beta_b) \varphi(\delta \cdot x),\]

so that, using (5.30) also on the first term, we have

\[
\Delta_{\lambda, \beta_b} = \frac{1}{m_2} C_\lambda \cdot C_{\beta_b} \left( \sum_{\lambda, \delta = 1 \atop \delta \neq \lambda, \beta_b = \pm 1} \varphi(\delta \cdot x) - 2 \varphi((\lambda - \beta_b) \cdot x) + 2 \varphi(\beta_b \cdot x) \right)
+ \frac{1}{2 m_2} \sum_{\delta, \lambda = 1} C_\lambda \cdot C_\delta c(\delta, \beta_b) \varphi(\delta \cdot x).
\] (C.20)
Expressing $C_\lambda$ again with the help of (C.4), and decomposing the sums over $\delta$ according to the values of $\delta \cdot \beta_a$, we find that (C.20) reproduces (C.6), with $\Delta_{a,b}$ given in (5.21d,e). This completes the proof of Theorem 8.

ACKNOWLEDGEMENTS

We have benefited from useful conversations with Elena Caceres, Ron Donagi and Igor Krichever. The first author wishes to thank Edward Witten for a generous invitation to the Princeton Institute for Advanced Study, where this research was initiated, as well as the Aspen Center for Physics. He would also like to acknowledge David Gross and the members of the Institute for Theoretical Physics in Santa Barbara for the hospitality extended to him while most of this work was being carried out. Both authors would like to thank David Morrison, I.M. Singer and Edward Witten for inviting them to participate in the 1998 workshop on “Geometry and Duality”, at the Institute for Theoretical Physics.

REFERENCES

[1] Seiberg, N. and E. Witten, “Electro-magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory”, Nucl. Phys. B 426 (1994) 19, [hep-th/9407087].
Seiberg, N. and E. Witten, “Monopoles, duality, and chiral symmetry breaking in N=2 supersymmetric QCD”, Nucl. Phys. B431 (1994) 494, [hep-th/9410167].
[2] Lerche, W., “Introduction to Seiberg-Witten theory and its stringy origin”, Proceedings of the Spring School and Workshop in String Theory, ICTP, Trieste (1996), [hep-th/9611190]; Nucl. Phys. Proc. Suppl. 55B (1997) 83, and references therein.
[3] Gorski, A., I.M. Krichever, A. Marshakov, A. Mironov, A. Morozov, “Integrability and Seiberg-Witten Exact Solution”, Phys. Lett. B355 (1995) 466, [hep-th/9505037].
Matone, M., “Instantons and Recursion Relations in N=2 SUSY Gauge Theories”, Phys. Lett. B357 (1996) 342, [hep-th/9506102]; Nakatsu, T. and K. Takasaki, “Whitham-Toda Hierarchy and N=2 Supersymmetric Yang-Mills Theory”, Mod. Phys. Lett. A 11 (1996) 157-168, [hep-th/9509162]; “Isomonodromic Deformations and Supersymmetric Gauge Theories”, Int. J. Mod. Phys. A11 (1996) 5505, [hep-th/9603069].
[4] Donagi, R. and E. Witten, “Supersymmetric Yang-Mills theory and integrable systems”, Nucl. Phys. B460 (1996) 299-334, [hep-th/9510101].
[5] Martinec, E. and Warner, N., “Integrable systems and supersymmetric gauge theories”, Nucl. Phys. B459 (1996) 97-112, hep-th/9509161.
[6] Martinec, E., “Integrable structures in supersymmetric gauge and string theory”, hep-th/9510204.
[7] Sonnenschein, J., S. Theisen, and S. Yankielowicz, “On the Relation between the Holomorphic Prepotential and the Quantum Moduli in SUSY Gauge Theories”, Phys. Lett. B367 (1996) 145-150, hep-th/9510129.
Eguchi, T. and S.K. Yang, “Prepotentials of N=2 supersymmetric gauge theories and soliton equations”, hep-th/9510183.
Itoyama, H. and A. Morozov, “Prepotential and the Seiberg-Witten theory”, Nucl. Phys. B491 (1997) 529, hep-th/9512161; “Integrability and Seiberg-Witten theory”, hep-th/9601168; “Integrability and Seiberg-Witten Theory: Curves and Periods”, Nucl. Phys. B477 (1996) 855, hep-th/9511126.
Ahn, C. and S. Nam, “Integrable Structure in Supersymmetric Gauge Theories with Massive Hypermultiplets”, Phys. Lett. B387 (1996) 304, hep-th/9603028.
Krichever, I.M. and D.H. Phong, “On the integrable geometry of soliton equations and N=2 supersymmetric gauge theories”, J. Differential Geometry 45 (1997) 349-389, hep-th/9604199.
Bonelli, G., M. Matone, “Nonperturbative Relations in N=2 SUSY Yang-Mills WDVV Equation”, Phys. Rev. Lett. 77 (1996) 4712, hep-th/9605090.
Marshakov, A., A. Mironov, and A. Morozov, “WDVV-like equations in N=2 SUSY Yang-Mills theory”, Phys. Lett. B389 (1996) 43, hep-th/9607103.
Marshakov, A. “Non-perturbative quantum theories and integrable equations”, Int. J. Mod. Phys. A12 (1997) 1607, hep-th/9610242.
Nam, S. “Integrable Models, Susy Gauge Theories and String Theory”, Int. J. Mod. Phys. A12 (1997) 1243, hep-th/9612134.
Marshakov, A., A. Mironov, A. Morozov, “More Evidence for the WDVV Equations in N=2 SUSY Yang-Mills Theories”, hep-th/9701123.
Marshakov, A., “On Integrable Systems and Supersymmetric Gauge Theories”, Theor. Math. Phys. 112 (1997) 791, hep-th/9702083.
Krichever, I.M. and D.H. Phong, “Symplectic forms in the theory of solitons”, hep-th/9708170, to appear in Surveys in Differential Geometry, Vol. III.
[8] D’Hoker, E. and D.H. Phong, “Calogero-Moser Systems in SU(N) Seiberg-Witten Theory”, Nucl. Phys. B513 (1998) 405, hep-th/9709053.
[9] Donagi, R., “Seiberg-Witten Integrable Systems”, alg-geom/9705010.
Freed, D.S., “Special Kähler Manifolds”, hep-th/9712042.
Carroll, R., “Prepotentials and Riemann Surfaces”, hep-th/9802130.

[10] M. Adler and P. van Moerbeke, “Completely integrable systems, Euclidian Lie algebras, and curves”, Advances in Math. 38 (1980) 267-317;
M. Adler and P. van Moerbeke, “Linearization of Hamiltonian systems, Jacobi varieties, and representation theory”, Advances in Math. 38 (1980) 318-379;
M. Adler and P. van Moerbeke, “The Toda lattice, Dynkin diagrams, singularities and Abelian varieties”, Inventiones Math. 103 (1991) 223-278.

[11] Calogero, F., “Solution of the one-dimensional N-body problem with quadratic and/or inversely quadratic pair potentials”, J. Math. Physics 12 (1971) 419-436.
Moser, J., “Integrable systems of non-linear evolution equations”, in Dynamical Systems, Theory and Applications, J. Moser, ed., Lecture Notes in Physics 38 (1975) Springer-Verlag.

[12] Krichever, I.M., “Elliptic solutions of the Kadomtsev-Petviashvili equation and integrable systems of particles”, Funct. Anal. Appl. 14 (1980) 282-290.

[13] Hitchin, N., “Stable bundles and integrable systems”, Duke Math. J. 54 (1987) 91.

[14] M.A. Olshanetsky and A.M. Perelomov, “Classical integrable finite-dimensional systems related to Lie algebras”, Phys. Rep. 71C (1981) 313-400.
Perelomov, A.M., “Integrable Systems of Classical Mechanics and Lie Algebras”, Vol. I, Birkausser (1990), Boston; and references therein.
Leznov, A.N. and M.V. Saveliev, Group Theoretic Methods for Integration of Non-Linear Dynamical Systems, Birkhauser 1992.

[15] D’Hoker, E. and D.H. Phong, “Calogero-Moser and Toda Systems for Twisted and Untwisted Affine Lie Algebras”, April 1998 preprint, hep-th/9804125.

[16] D’Hoker, E. and D.H. Phong, “Spectral Curves for Super-Yang-Mills with Adjoint Hypermultiplet for General Lie Algebras”, April 1998 preprint, hep-th/9804126.

[17] V.I. Inozemtsev, “The finite Toda lattices”, Comm. Math. Phys. 121 (1989) 629-638.

[18] V.I. Inozemtsev, “Lax representation with spectral parameter on a torus for integrable particle systems”, Lett. Math. Phys. 17 (1989) 11-17.

[19] Klemm, A., W. Lerche, and S. Theisen, “Non-perturbative actions of N=2 supersymmetric gauge theories”, Int. J. Mod. Phys. A11 (1996) 1929-1974, hep-th/9505150.

[20] D’Hoker, E., I.M. Krichever, and D.H. Phong, “The effective prepotential for N=2 supersymmetric SU(Nc) and SO(Nc) gauge theories”, Nucl. Phys. B 489 (1997) 179-210, hep-th/9609041;
D’Hoker, E., I.M. Krichever, and D.H. Phong, “The effective prepotential for N=2 supersymmetric SO(Nc) and Sp(Nc) gauge theories”, Nucl. Phys. B 489 (1997) 211-222, hep-th/9609145.
D’Hoker, E., I.M. Krichever, and D.H. Phong, “The renormalization group equation for N=2 supersymmetric gauge theories”, Nucl. Phys. B 494 (1997), 89-104, [hep-th/9610156].

D’Hoker, E. and D.H. Phong, “Strong coupling expansions in SU(N) Seiberg-Witten theory”, [hep-th/9701151], Phys. Lett. B 397 (1997) 94-103.

[21] Kac, V., “Infinite-dimensional Lie algebras”, Birkhäuser (1983) Boston.

[22] Goddard, P. and D. Olive, “Kac-Moody and Virasoro algebras in relation to quantum physics”, International J. of Modern Physics A, Vol. I (1986) 303-414.

[23] Krichever, I.M., “Elliptic solutions of the Kadomtsev-Petviashvili equation and integrable systems of particles”, Funct. Anal. Appl. 14 (1980) 282-290.

[24] Erdelyi, A., ed., “Higher Transcendental Functions”, Bateman Manuscript Project, Vol. II, R.E. Krieger (1981) Florida.

[25] W.G. McKay, J. Patera and D.W. Rand, “Tables of Representations of Simple Lie Algebras”, Vol. I : Exceptional Simple Lie algebras, Centre de Mathématiques, Université de Montréal, 1990.

[26] Kachru, S., C. Vafa, “Exact Results for N=2 Compactifications of Heterotic Strings”, Nucl. Phys. B450 (1995) 69, [hep-th/9505103];

Bershadsky, M., K. Intriligator, S. Kachru, D.R. Morrison, V. Sadov and C. Vafa, “Geometric Singularities and Enhanced Gauge Symmetries”, Nucl. Phys. B481 (1996) 215, [hep-th/9605200];

Katz, S., A. Klemm, C. Vafa, “Geometric Engineering of Quantum Field Theories”, Nucl. Phys. B497 (1997) 173, [hep-th/9609239];

Katz, S., P. Mayr, and C. Vafa, “Mirror symmetry and exact solutions of 4D N=2 gauge theories”, Adv. Theor. Math. Phys. 1 (1998) 53, [hep-th/9706110].

[27] Witten, E., “Solutions of four-dimensional field theories via M-theory”, Nucl. Phys. B500 (1997) 3, [hep-th/9703166].

[28] Hanany, A., and E. Witten, “Type IIB Superstrings, BPS Monopoles, and Three-Dimensional Gauge Dynamics”, Nucl. Phys. B492 (1997) 152.

[29] Brandhuber, A., J. Sonnenschein, S. Theisen and S. Yankielowicz, “M-Theory and Seiberg-Witten Curves : Orthogonal and Symplectic Groups”, Nucl. Phys. B504 (1997) 175, [hep-th/9705232].

Landsteiner, K., E. Lopez, “New Curves From Branes”, [hep-th/9708118];

Landsteiner, K., E. Lopez, DA. Lowe, “N=2 Supersymmetric Gauge Theories, Branes and Orientifolds”, Nucl. Phys. B507 (1997) 197, [hep-th/9705199];

Uranga, A.M., “Towards Mass Deformed N=4 SO(N) and Sp(K) Gauge Theories from Brane Configurations”, [hep-th/9803054].
Yokono, T., “Orientifold four plane in brane configurations and N=4 USp(2N) and SO(2N) theory”, hep-th/9803123.

[30] Gorskii, A., “Branes and Integrability in the N=2 SUSY YM Theory”, Int. J. Mod. Phys. A12 (1997) 1243, hep-th/9612238;
Gorskii, A., S. Gukov and A. Mironov, “Susy Field Theories, Integrable Systems and their Stringy/Brane Origin”, hep-th/9710239;
Cherkis, S.A., A. Kapustin, “Singular Monopoles and Supersymmetric Gauge Theories in Three Dimensions”, hep-th/9711143.

[31] Braden, H.W. “R-Matrices, Generalized Inverses and Calogero-Moser-Sutherland Models”, to appear in the Proceedings of the Workshop on Calogero-Moser-Sutherland Models, in the CRM Series in Mathematical Physics, Springer-Verlag; available from http://www.maths.ed.ac.uk/preprints/97-017.