Cost-Efficient Numerical Schemes for the Boundary Integral Equation Solution in 2D Vortex Methods

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Abstract. The problem of the flow simulation around airfoils using Lagrangian vortex methods is considered. Numerical schemes of the second order of accuracy for free vortex sheet intensity distribution along the airfoil are developed for smooth and non-smooth airfoils. The schemes are based on the Galerkin approach with piecewise-constant and piecewise-linear basis functions. The finite element method ideas are used and the resulting piecewise-linear scheme has the same numerical complexity as the scheme with piecewise-constant numerical solution. The modification of a FEM-type scheme is developed for non-smooth airfoils which permits to take into account the discontinuity of the solution at the specified points; its computational cost increases insignificantly.

1. Introduction
The problem of simulation of two-dimensional viscous incompressible flow around the airfoil using vortex methods [1, 2, 3] is considered. In this paper, the pure Lagrangian modification of vortex methods is considered which doesn’t require construction of a mesh in the flow domain. It is well-known [1, 4] that an immovable airfoil in a flow can be replaced by a vortex sheet of unknown intensity \( \gamma(r) \). In the case of viscous flow this vortex sheet is free, it means that after being generated the vorticity becomes disjoined to the airfoil surface line and moves in the flow according to the Navier — Stokes equation. Free vortex sheet intensity can be found from no-slip boundary condition, which can be written down in the form of boundary integral equation, either of the first or the second kind [1, 5, 6]. If the airfoil is movable, in addition to the free vortex sheet, attached vortex and source sheets should be introduced with intensities \( \gamma^{at}(r) \) and \( q^{at}(r) \) equal to velocities’ components along the tangent and the normal line of the surface, respectively. For the given law of motion of the airfoil, the only unknown variable is free vortex sheet intensity distribution \( \gamma(r) \) along the airfoil surface line.

There are some commonly used for solving the integral equations numerical schemes [1, 6], however, their accuracy in the general case is not sufficient. In the present work a new approach is developed for construction of higher-order numerical schemes, which is based on the Galerkin-type method. In the framework of this paper, the airfoil is assumed to be approximated by
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contour integrals, which in this case become singular, we obtain the boundary integral equation

\[ v = \text{vortex and source sheets, respectively, with intensities} \]

\[ \text{points on the airfoil surface line. The last two integrals implement influences from the attached} \]

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\[ \text{vorticity generation; } n, \gamma \text{ are outward unit normal vectors with respect to the airfoil surface} \]

\[ \text{K: } V_K(\xi) \text{ is airfoil surface line velocity; } \alpha(r) \text{ is coefficient equal to 1 for points in the flow} \]

\[ \text{plane, equal to 0 for points inside the airfoil, and equal to the outer angle normalized to 2}\pi \text{ for} \]

\[ \text{points on the airfoil surface line. The last two integrals implement influences from the attached} \]

\[ \text{vortex and source sheets, respectively, with intensities} \]

\[ \gamma_{\text{att}}(\xi) = n(\xi) \times V_K(\xi), \quad q_{\text{att}}(\xi) = n(\xi) \cdot V_K(\xi), \quad \xi \in K. \]

\[ \text{V, } V(\xi) \text{ is velocity field; } \Omega(\xi) \text{ is vorticity field; } \Omega(r) \text{ is expressed as follows:} \]

\[ \alpha(r)V(r) = V_\infty + \int_S \frac{\Omega(\xi) \times (r - \xi)}{2\pi |r - \xi|^2} dS_\xi + \int_K \frac{\gamma(\xi) \times (r - \xi)}{2\pi |r - \xi|^2} dl_\xi + \]

\[ + \oint_K \frac{n(\xi) \cdot V_K(\xi)}{2\pi |r - \xi|^2} dl_\xi + \oint_K \frac{n(\xi) \times V_K(\xi)}{2\pi |r - \xi|^2} dl_\xi. \]

\[ \text{Here, } \gamma(\xi) = k_\gamma(\xi) \text{ is the free vortex sheet placed at the airfoil surface line which simulates new} \]

\[ \text{vorticity generation; } n(\xi, t) \text{ is an outward unit normal vector with respect to the airfoil surface} \]

\[ \text{line } K: V_K(\xi) \text{ is airfoil surface line velocity; } \alpha(r) \text{ is coefficient equal to 1 for points in the flow} \]

\[ \text{area, equal to 0 for points inside the airfoil, and equal to the outer angle normalized to } 2\pi \text{ for} \]

\[ \text{points on the airfoil surface line. The last two integrals implement influences from the attached} \]

\[ \text{vortex and source sheets, respectively, with intensities} \]

\[ \gamma_{\text{att}}(\xi) = n(\xi) \times V_K(\xi), \quad q_{\text{att}}(\xi) = n(\xi) \cdot V_K(\xi), \quad \xi \in K. \]

\[ \text{Considering the equation } (1) \text{ at the airfoil surface line } K \text{ and going to the limit values of} \]

\[ \text{contour integrals, which in this case become singular, we obtain the boundary integral equation} \]

\[ \oint_K \frac{k \times (r - \xi)}{2\pi |r - \xi|^2} \gamma(\xi) dl_\xi - \alpha(r)(k \times n(r)) \gamma(r) = f(r), \quad r \in K, \]

\[ \text{where right-hand side has the following form:} \]

\[ f(r) = \alpha(r)V_K(r) - V_\infty - \int_S \frac{\Omega(\xi) \times (r - \xi)}{|r - \xi|^2} dS_\xi - \int_K \frac{\gamma_{\text{att}}(\xi) \times (r - \xi)}{|r - \xi|^2} dl_\xi - \int_K \frac{q_{\text{att}}(\xi)(r - \xi)}{|r - \xi|^2} dl_\xi. \]

\[ \text{The equation } (2) \text{ is a vector integral equation with respect to the unknown scalar quantity} \]

\[ \gamma(r). \text{ According to [5], to solve it, considering one of the two scalar boundary integral equations} \]

\[ \text{obtained from } (2) \text{ by projection to a normal to the airfoil surface line is sufficient:} \]

\[ \oint_K Q_n(r, \xi) \gamma(\xi) dl_\xi = f_n(r), \]

\[ \text{or on a tangent to the airfoil surface line:} \]

\[ \oint_K Q_\tau(r, \xi) \gamma(\xi) dl_\xi - \alpha(r) \gamma(r) = f_\tau(r). \]

\[ \text{Here and further,} \]

\[ Q_n(r, \xi) = -\frac{\tau(r) \cdot (r - \xi)}{2\pi |r - \xi|^2}, \quad Q_\tau(r, \xi) = \frac{n(r) \cdot (r - \xi)}{2\pi |r - \xi|^2}. \]
\[ f_n(r) = f(r) \cdot n(r), \quad f_\tau(r) = f(r) \cdot \tau(r) \]

are the kernels and the right-hand sides of the corresponding equations, \( \tau(r) \) is the tangent vector to the airfoil surface line at the corresponding point which direction is chosen in such a way that \( n(r) \times \tau(r) = k \).

Both equations (3) and (4) have an infinite set of solutions. To select the unique solution that has a physical meaning, the value of the integral along the contour \( K \) is usually assigned:

\[ \oint_K \gamma(\xi) d\xi = \Gamma, \quad (5) \]

where \( \Gamma \) is the circulation of the velocity field along the airfoil surface line.

In this paper, we consider the integral equation (4) since this approach allows us to obtain results with higher accuracy [7].

3. Galerkin Approach

In paper [8], piecewise-constant and piecewise-linear numerical schemes for the approximate solution of the boundary integral equation (4) under condition (5) were constructed using the Galerkin method. Let us briefly describe the main ideas of constructing these schemes. The basis and projection functions are introduced, the numerical solution is considered as a linear combination of the basis functions with unknown coefficients, which can be found from orthogonality condition of the integral equation residual to the projection functions.

The commonly used numerical schemes [1, 6, 7, 8] can be treated as particular cases of this approach with Dirac delta functions or piecewise-constant functions selected as basis/projection functions in different combinations. The scheme with piecewise-constant basis and projection functions [7] seems to be rather useful, however, it provides only the 1-st order of accuracy with respect to vortex sheet intensity distribution. The scheme with piecewise-linear basis and projection functions [8] has the second order of accuracy, but the corresponding algorithm has much higher numerical complexity.

3.1. Scheme for piecewise-linear discontinuous solution

Let us introduce the following notations: \( K_i \) are the rectilinear panels, which approximate the surface line of the airfoil; \( c_i \) are the centers of the panels; \( L_i \) are their lengths, \( i = 1, \ldots, N \). We introduce two families of basis functions:

\[ \phi^0_i(r) = \begin{cases} 1, & r \in K_i, \\ 0, & r \notin K_i, \end{cases} \quad \phi^1_i(r) = \begin{cases} \frac{(r - c_i) \cdot \tau_i}{L_i}, & r \in K_i, \\ 0, & r \notin K_i, \end{cases} \quad (6) \]

The solution now is piecewise-linear function \( \gamma(r) = \sum_{i=1}^{N} (\gamma^0_i \phi^0_i(r) + \gamma^1_i \phi^1_i(r)) \). Assuming the projection functions to be equal to the basis ones, we obtain the system of linear algebraic equations, which approximates (4), and the equation \( \sum_{i=1}^{N} \gamma^0_i L_i = \Gamma \), which approximates the additional condition (5). This linear system is overdetermined, it is regularized as described in [6]. The resulting system has the following form:

\[ \begin{pmatrix} A^{00} + D^{00} & A^{01} + D^{01} & I \\ A^{10} + D^{10} & A^{11} + D^{11} & 0 \\ L^0 & L^1 & 0 \end{pmatrix} \begin{pmatrix} \gamma^0 \\ \gamma^1 \\ R \end{pmatrix} = \begin{pmatrix} b^0 \\ b^1 \\ \Gamma \end{pmatrix}, \quad (7) \]
where $A_{pq}$ are the square matrix blocks of the size $N \times N$, $D_{pq}$ are the diagonal matrix blocks of size $N \times N$, and $L^q$ are the vectors of integrals from the basis functions:

$$A_{ij}^p = \int_{K_i} \left( \int_{K_j} Q(r, \xi) \phi_j^p(\xi) d\xi \right) \phi_i^p(r) dr, \quad D_{ii}^p = -\frac{1}{2} \int_{K_i} \phi_i^p(r) \phi_i^j(\xi) d\xi, \quad (8)$$

$$L^q = \int_{K_i} \phi_i^p(r) dr, \quad p, q = 0, 1;$$

$\gamma^0$ and $\gamma^1$ are the vectors of unknown coefficients; $R$ is the regularization variable; $I$ is the column of ones; $b^0$ and $b^1$ are the parts of right-hand side vectors:

$$b_i^p = \int_{K_i} f_i(r) \phi_i^p(r) dr, \quad i = 1, \ldots, N, \quad p = 0, 1. \quad (9)$$

The expression for $D_{ii}^p$ in (8) is written taking into account that $\alpha(r) = 1/2$ at all points of the panels except for the endpoints, which do not affect the result of the calculation of integrals.

All the integrals in (8) and (9) can be calculated analytically; all the necessary formulae are given in [9].

The resulting vorticity distribution is piecewise-linear and has jump discontinuities between the panels (similar to Discontinuous Galerkin method). The described scheme provides the second order of accuracy in $L_1$-norm [8].

Note that if for practical purposes it is enough to achieve the first order of accuracy in $L_1$ norm (and the second order for the average value of vortex sheet intensity over the panels), the scheme can be simplified: the second column in the block matrix and the second raw should be excluded from the system (7). The resulting solution in this case is piecewise-constant, and the scheme itself coincides with [7]. The numerical complexity of the piecewise-linear scheme is 8 times higher due to the twofold increase in size over the piecewise-constant scheme (both matrices are fully populated and non-symmetric, so the Gauss elimination seems to be more suitable solving method).

Now, we suggest an approach for construction of the cost-efficient schemes providing the second order of accuracy with lower numerical complexity.

3.2. **FEM-type scheme for piecewise-linear continuous solution**

Note that in case of smooth airfoils the exact solution is continuous, so it seems to be convenient to use the finite element-type scheme: we choose basis and projection functions equal to the linear shape functions, which are referred to the nodes on the surface line. This functions can be expressed through the both families of the previously introduced functions $\phi_i^0(r)$ and $\phi_i^1(r)$:

$$\hat{\phi}_i(r) = \begin{cases} \frac{\phi_{i-1}^0(r)}{2} + \phi_{i-1}^1(r), & r \in K_{i-1}, \\ \frac{\phi_i^0(r)}{2} - \phi_i^1(r), & r \in K_i, \quad i = 1, \ldots, N, \quad K_0 \equiv K_N. \end{cases} \quad (10)$$

Then, approximate solution is expressed as

$$\gamma(r) = \sum_{i=1}^N \gamma_i \hat{\phi}_i(r),$$

and the resulting matrix has the following structure:

$$\begin{pmatrix} \hat{A} & I \\ \hat{L} & 0 \end{pmatrix} \begin{pmatrix} \hat{\gamma} \\ R \end{pmatrix} = \begin{pmatrix} \hat{b} \\ \Gamma \end{pmatrix}, \quad (11)$$
The matrix coefficients $\hat{A}_{ij}$ are expressed through the coefficients $A_{ij}^{00}$ in (8):

$$
\hat{A}_{ij} = \frac{A_{ij}^{00} + A_{i-1,j}^{00} + A_{i,j-1}^{00} + A_{i-1,j-1}^{00}}{4} + \frac{-A_{ij}^{01} - A_{i-1,j}^{01} + A_{i,j-1}^{01} + A_{i-1,j-1}^{01}}{2} + \frac{-A_{ij}^{10} + A_{i-1,j}^{10} - A_{i,j-1}^{10} + A_{i-1,j-1}^{10}}{2} + \frac{D_{ij}^{00} + D_{i-1,j}^{00} + D_{i,j-1}^{00} + D_{i-1,j-1}^{00}}{4} + (A_{ij}^{11} - A_{i-1,j}^{11} - A_{i,j-1}^{11} + A_{i-1,j-1}^{11}) + (D_{ij}^{11} - D_{i-1,j}^{11} - D_{i,j-1}^{11} + D_{i-1,j-1}^{11}).
$$

Components of $\hat{L}$ and $\hat{b}$ also can be expressed through coefficients described in the previous subsection:

$$
\hat{L}_i = \frac{1}{2}(L_i^0 + L_i^0), \quad \hat{b}_i = \frac{1}{2}(b_i^0 + b_i^0) - (b_i^0 - b_i^0).
$$

The matrix size is now only $(N + 1) \times (N + 1)$ (as for the above-mentioned scheme with piecewise-constant solution), however, it provides the second order of accuracy, but only for smooth airfoils.

3.3. FEM-type scheme with discontinuity extraction

In case of non-smooth airfoils with sharp edges or angle points, the scheme can be modified in order to take into account the solution discontinuity at the corresponding nodes. Let $D$ be the set of $d$ nodes where the exact solution is discontinuous. In these nodes, two basis functions $\tilde{\phi}_i^-(r)$ and $\tilde{\phi}_i^+(r)$ are introduced instead of one. Modified shape functions also can be expressed through $\tilde{\phi}_i^0(r)$ and $\tilde{\phi}_i^1(r)$:

$$
\begin{align*}
\tilde{\phi}_i^-(r) &= \frac{\phi_i^{0}(-1)}{2} + \phi_i^{1}(-1), \quad r \in K_{i-1}, \\
\tilde{\phi}_i^+(r) &= \frac{\phi_i^{0}(1)}{2} - \phi_i^{1}(1), \quad r \in K_i.
\end{align*}
$$

(12)

Basis functions for other nodes stay the same as for the previous case:

$$
\tilde{\phi}_i(r) = \hat{\phi}_i(r), \quad i \notin D.
$$

Approximate solution now has the following structure:

$$
\gamma(r) = \sum_{i \notin D} \tilde{\gamma}_i \tilde{\phi}_i + \sum_{i \in D} (\tilde{\gamma}_i^- \tilde{\phi}_i^- + \tilde{\gamma}_i^+ \tilde{\phi}_i^+).
$$

Coefficients $\tilde{\gamma}_i^-$ and $\tilde{\gamma}_i^+$ mean the limit values from the right and left at discontinuity points.

All the matrix coefficients are the same except the ones that correspond to the panels neighboring to the nodes with discontinuities. For simplicity let us consider only one angle point located between the panels $K_1$ and $K_N$ ($i = 1$). The resulting linear system

$$
\begin{pmatrix}
\hat{A} & I \\
\hat{L} & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{\gamma} \\
\hat{b}
\end{pmatrix}
= 
\begin{pmatrix}
\hat{R} \\
\Gamma
\end{pmatrix},
$$

has $(N + 2) \times (N + 2)$ size matrix; discontinuity-resolving matrix coefficients can be expressed...
through coefficients $A_{ij}^{pq}$ ($i, j = 1, \ldots, N$). For the first row

$$
\tilde{A}_{11} = \frac{A_{11}^{00}}{4} - \frac{A_{11}^{01}}{2} - \frac{A_{11}^{10}}{2} + A_{11}^{11} + \frac{D_{00}^{00}}{4} + D_{11}^{11};
$$

$$
\tilde{A}_{1j} = \frac{A_{1j}^{00}}{4} + A_{1j}^{01} + A_{1j-1}^{10} - \frac{A_{1j}^{10}}{2} + A_{1j}^{11} - \frac{A_{1j-1}^{10}}{2} + (A_{1j}^{11} + A_{1j-1}^{11}) + 
+ \frac{D_{00}^{00}}{4} + D_{1j-1}^{11};
$$

$$
\tilde{A}_{1,N+1} = \frac{A_{1N}^{00}}{4} + \frac{A_{1N}^{01}}{2} - \frac{A_{1N}^{11}}{2} - A_{1N};
$$

for the first and the last columns

$$
\tilde{A}_{i1} = \frac{A_{i1}^{00} + A_{i1}^{01} + A_{i1-1}^{10}}{4} - \frac{A_{i1}^{10}}{2} + (A_{i1}^{11} + A_{i1-1}^{11}) + 
+ \frac{D_{i1}^{00}}{4} + D_{i1}^{11},
$$

$$
\tilde{A}_{i,N+1} = \frac{A_{iN}^{00} + A_{iN}^{01} + A_{iN-1}^{10}}{4} + \frac{A_{iN}^{11}}{2} - (A_{iN}^{11} + A_{iN-1}^{11}) + 
+ \frac{D_{iN}^{00}}{4} + D_{iN}^{11};
$$

and for the last row

$$
\tilde{A}_{N+1,1} = \frac{A_{N+1,1}^{00}}{4} - \frac{A_{N+1,1}^{01}}{2} - A_{N+1,1};
$$

$$
\tilde{A}_{N+1,j} = \frac{A_{N+1,j}^{00} + A_{N+1,j}^{01} + A_{N+1,j-1}^{10}}{4} + \frac{A_{N+1,j}^{11} + A_{N+1,j-1}^{11}}{2} - (A_{N+1,j}^{11} + A_{N+1,j-1}^{11}) + 
+ \frac{D_{N+1,j}^{00}}{4} + D_{N+1,j-1}^{11};
$$

$$
\tilde{A}_{N+1,N+1} = \frac{A_{N+1,N+1}^{00}}{4} + \frac{A_{N+1,N+1}^{01}}{2} + \frac{A_{N+1,N+1}^{11}}{2} + D_{N+1,N}^{00} + D_{N+1,N}^{11}.
$$

The first and the last components of the right-hand side vector also should be modified:

$$
\tilde{L}_1 = \frac{L_1}{2}, \quad \tilde{L}_{N+1} = \frac{L_N}{2}, \quad \tilde{b}_1 = \frac{b_1^0}{2} - b_1^1, \quad \tilde{b}_{N+1} = \frac{b_N^0}{2} + b_N^1.
$$

The other coefficients of the matrix $\tilde{A}$, the right-hand side $\tilde{b}$, and the vector $\tilde{L}$ remain the same as in (11): $A_{ij} = \tilde{A}_{ij}$, $b_i = \tilde{b}_i$, $L_i = \tilde{L}_i$, $i, j = 2, \ldots, N$.

### 4. Test problem

For an example of the method’s application, we consider flows around an elliptic airfoil and a Zhukovsky airfoil; for these airfoils exact solutions are known [10]. We assume that the velocity of the incident flow is $|V_\infty| = 1$, the incidence angle is $\beta = \pi/6$, and there is no vorticity in the flow region.

In the figure 1, the approximate solutions obtained for the case of the elliptic airfoil with semi-axes $a_1 = 0.5$ and $b_1 = 0.05$ are given together with the exact solution. The change of the solution along the airfoil surface line, which is parameterized by the parameter $t \in [0, 2\pi)$, is shown: the values 0 and $2\pi$ of the parameter $t$ correspond to the right vertex of the ellipse, the path tracing is counterclockwise.
Vortex sheet intensity for elliptical airfoil, with parameters: $a_1 = 0.5, b_1 = 0.05, \beta = \pi/6$

Figure 1. Approximate and exact solution for elliptical airfoil

It is seen that at the sections with low curvature of the airfoil surface line approximate piecewise-linear solutions are almost coincident with the exact solution. Some discrepancy is observed at the left and right vertices of the ellipse. In order to reduce the error, one can reduce the lengths of the panels at the sections with high curvature or use schemes that allow taking into account the curvilinearity of the airfoil surface line [11].

In the figure 2, the approximate solutions are shown for Zhukovsky wing airfoil with the following parameters: length $a = 3.5$, width $d = 0.4$, and curvature $h = 0.3$. Airfoil surface line is parameterized by parameter $t \in [0, 2\pi)$; the values 0 and $2\pi$ of the parameter $t$ correspond to the sharp edge.

Vortex sheet intensity for Zhukovsky wing airfoil with parameters: $a = 3.5, d = 0.4, h = 0.3, \beta = \pi/6$

Figure 2. Numerical solutions and exact solution for Zhukovsky wing airfoil
It is seen that all the piecewise-linear schemes give approximately the same results, except the continuous FEM-type scheme, since it does not allow extracting the discontinuity of the solution at the sharp edge of the wing airfoil.

Conclusions
Cost-efficient numerical schemes are developed for the boundary integral equation solving, which arises in 2D vortex methods. They are based on the Galerkin approach and adapted for smooth and non-smooth airfoils. The schemes provide the second order of accuracy for vortex sheet intensity distribution along the airfoil. Modified basis functions permit to take into account discontinuities of the solution.

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