The Laplacian Spectra of Graphs and Complex Networks*

Ya-Hong Chen
College of Education, Lishui University, Lishui, Zhejiang 323000, P.R. China

Rong-Ying Pan
Suzhou Vocational University Suzhou, Jiangsu, 210000, P.R. China

Xiao-Dong Zhang
Department of Mathematics, Shanghai Jiao Tong University
800 Dongchuan road, Shanghai, 200240, P.R. China

Email: xiaodong@sjtu.edu.cn

Dedicated to Professor Jiong-Sheng Li on the occasion of his 75th birthday

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Abstract

The paper is a brief survey of some recent new results and progress of the Laplacian spectra of graphs and complex networks (in particular, random graph and the small world network). The main contents contain the spectral radius of the graph Laplacian for given a degree sequence, the Laplacian coefficients, the algebraic connectivity and the graph doubly stochastic matrix, and the spectra of random graphs and the small world networks. In addition, some questions are proposed.

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Corresponding author: Xiao-Dong Zhang (xiaodong@sjtu.edu.cn)
1 Introduction

The study of graph Laplacian spectrum realized increasingly connections with many other areas. The objects arise in very diverse application, from combinatorial optimization to differential geometry, mathematical biology, computer science, machine learning, etc.) The Laplacian spectrum can be used to extract useful and important information about some graph invariants (for examples, expansion and vertex partition) that are hard to computer or estimate (see [1]). One of the most fascinating aspects of applications of eigenvalue methods in Combinatorics is the spectrum appear as a tool to prove results that appear to have nothing to do with the spectrum itself (see the excellent monographs [22]). The smallest nonzero and largest eigenvalues [39] can be expressed as solutions to a quadratic optimization problem. An important use of eigenvalues is Lovász’s notation of the \( \vartheta \) function ([30]) which was initiated by problems in communication networks.

The graph Laplacian can be used in a number of ways to provide interesting geometric representations of a graph. One of the most work ([14, 22]) is the Colin de Verdière number of a graph, which is regarded as one of the most important recent developments in graph theory. The parameter, which is minor-monotone, determines the embeddability properties of graphs.

Alon and Milman [1] proved that the Laplacian spectrum of graphs plays a crucial role from the explicit constructions of expander graphs and superconcentrators to the design of various randomized algorithms. Based on the Laplacian eigenvalues and isoperimetric properties of graphs, Lubotzky, Phillips and Sarnak in [31] gave several family explicit constructions of good expander graphs.

The Laplacian matrix of graphs can be regarded as the discrete Laplacian operator on the differential manifolds. There is an interesting bilateral link between spectral graph theory and spectral Riemannian geometry. The concepts and methods of spectral geometry make it possible to obtain new results in the study of graph Laplacian spectrum. Conversely, the results in spectral graph theory may be transferred to the Laplacian operator on manifolds. We refer to the excellent book [10].

At the dawn of the new century, the power law networks (the scale-free networks)
[3] and the small world networks [49] were discovered and studied. Since then, the analysis and modeling of networks, and networked dynamical system, have been the subject of considerable interdisciplinary interests, including physics, mathematics, computer science, biology, economics. The results have been called the ”new science of networks” (see [48]).

The Laplacian spectra of a various way of modeling of networks which represent the real networks can be used to give a tentative classification scheme for empirical networks and provide useful insight into the relevant structural properties of real networks. Spectral techniques and methods based on the analysis of the largest eigenvalues and eigenvectors of some complex networks (for example the web networks [43]) have proven algorithmically successful in detecting communities and clusters. Mihail and Papadimitriou [37] showed that the largest eigenvalue of a power law graph with exponent $\beta$ has power law distribution if the exponent $\beta$ of the power law graph satisfies $\beta > 3$. Chung etc. in [12, 13] established relationships between the spectrum of graphs and the structure of complex networks. It is showed that spectral methods are central in detecting clusters and finding patterns in various applications. For more information, we refer to the monograph [11] and a survey [43].

In this paper, we survey recent progress of the Laplacian spectra of graphs and complex networks. For basic information and earlier results, we refer several books and survey papers such as [10, 11, 33, 34, 38, 39, 52], for a detailed introduction and applications. The rest of this paper is organized as follows. In Section 2, some notations and results are presented. In Section 3, the Laplacian spectral radius of graphs with given degree sequences is extensively investigated. In Section 4, the relationship between the Laplacian coefficient and the ordering of graphs are studied, in particular, the Mohar’s problems [41] on the topic are discussed. In Section 5, we deeply studied the algebraic connectivity and structure and properties of the graph doubly stochastic matrix, in particular, on Merris’s problems [36] In Section 5, the Laplacian spectrum of random graphs and small world networks are discussed.

2 Preliminary

Let $G = (V(G), E(G))$ be a simple graph (no loops or multiple edges) with vertex set $V(G) = \{v_1, \cdots, v_n\}$ and edge set $E(G)$. The degree of vertex $v_i$, denote by $d(v_i)$ or $d_G(v_i)$, is the number of the edges incident with $v_i$. Let $D(G) = diag(d(u), u \in V)$ be the diagonal matrix of vertex degrees of $G$ and $A(G) = (a_{ij})$ be the $(0, 1)$ adjacency matrix of $G$. Let $G'$ be the matrix obtained from $G$ by adding an edge between two non-adjacent vertices $v_i$ and $v_j$. The Laplacian matrix $L(G)$ is obtained from $G$ by replacing $a_{ij}$ with $d(u)$ if $u = i$, $-1$ otherwise.

The Laplacian eigenvalues of $G$ are the eigenvalues of the Laplacian matrix $L(G)$.

$$L(G)v = \lambda Gv$$

where $v$ is the eigenvector corresponding to eigenvalue $\lambda$.

The largest eigenvalue of the Laplacian matrix $L(G)$ is called the spectral radius $\rho(G)$ of $G$.

$$\rho(G) = \max \{\lambda : L(G)v = \lambda v, v \neq 0\}$$

For more information, we refer to the monograph [11] and a survey [43].
matrix of \( G \), where \( a_{ij} = 1 \) for \( v_i \) adjacent to \( v_j \) and 0 for elsewhere. Then the matrix \( L(G) = D(G) - A(G) \) is called the Laplacian matrix of a graph \( G \). It is obvious that \( L(G) \) is positive semidefinite and singular \( M - \) matrix. Thus the set of all eigenvalues of \( L(G) \) are called the Laplacian spectrum of \( G \) and arranged in nonincreasing order:

\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = 0.
\]

When more than one graph is under discussion, we may write \( \lambda_i(G) \) instead of \( \lambda_i \). From the matrix-tree theorem, it is easy to see that \( \lambda_{n-1} > 0 \) if and only if \( G \) is connected. This observation led M. Fiedler [21] to define the algebraic connectivity of \( G \) by \( \alpha(G) = \lambda_{n-1}(G) \), which can be used as a quantitative measure of connectivity. Further, the characteristic polynomial of \( L(G) \) is called the Laplacian polynomial of \( G \) and is denoted by

\[
\Phi(G) = \det(\lambda I_n - L(G)) = \sum_{i=0}^{n} (-1)^i c_i \lambda^{n-i}.
\]

It is easy to see that \( c_0 = 1, c_1 = 2|E(G)|, c_n = 0 \) and \( c_{n-1} = n\tau(G) \), where \( \tau(G) \) is the number of spanning trees of \( G \). It is well known that

\[
\sum \lambda_{i_1} \cdots \lambda_{i_k} = c_k,
\]

where the sum is taken over all subsets of \( k \) elements in the set \{1, \cdots, n\}. Hence the spectrum \{\( \lambda_1, \cdots, \lambda_n \} \) and the set consisting of \( c_0, \cdots, c_n \) are mutually determined.

3 The Laplacian spectral radius and graphic degree sequences

A nonincreasing sequence of nonnegative integers \( \pi = (d_0, d_1, \cdots, d_{n-1}) \) is called graphic if there exists a simple graph with \( \pi \) as its vertex degree sequence. Moreover, \( \pi \) is called a unicyclic degree sequence, if there exists a unicyclic graph (i.e., the connected graph with only one cycle) with \( \pi \). From the point of view of linear algebra, many properties of the graph Laplacian spectrum can be characterized by simple graph invariants, such as vertex degree. There are a lot of results on the upper and lower bounds for the spectral radius of graphs in terms of the maximum degree, the minimum degree, etc.(see [52] for the detail). Motivated by the recent results in terms of vertex degrees, Zhang in [54] generally propose the following question.
Problem 3.1 [54] For a given graphic degree sequence \( \pi \), let

\[
G_\pi = \{ G \mid G \text{ is connected with } \pi \text{ as its degree sequence} \}.
\]

Study the sharp upper (lower) bounds for the Laplacian spectral radius of all graphs \( G \) in \( G_\pi \) and characterize all extremal graphs which attain the upper (lower) bounds.

It is natural to consider the tree degree sequence, since tree is a simplest connected graph. With aid of the properties of the eigenvector corresponding to the spectral radius, [54] gave the extremal tree in the set of given tree degree sequence which has the maximum Laplacian spectral radius.

Let \( \pi = (d_0, d_1, \ldots, d_{n-1}) \) with \( n \geq 3 \) be a given nonincreasing degree sequence of some tree. Now we construct a special tree \( T_\pi^* \) with degree sequence \( \pi \) by using a "breadth-first" scheme, which considers the \( n \) vertices of \( T^* \) to be partitioned into a sequence of layers starting with the 0th layer consisting of a single vertex \( v_{01} \) of maximum degree. Then recursively develop the layers, with the 1th layer having \( d_0 \) vertices each connected to \( v_{01} \). Denote the number of vertices in the \( m \)th layer by \( l_m \), and the number in all layers preceding layer \( m \) by \( l_{<m} \).

Theorem 3.2 [54] For a given degree sequence \( \pi \) of some tree, let

\[
T_\pi = \{ T \mid T \text{ is tree with } \pi \text{ as its degree sequence} \}.
\]

Then \( T_\pi^* \) is a unique tree with the maximum Laplacian spectral radius in \( T_\pi \).

Let \( \pi = (d_0, \ldots, d_{n-1}) \) and \( \pi' = (d'_0, \ldots, d'_{n-1}) \) be two nonincreasing sequences. If \( \sum_{i=0}^{k} d_i \leq \sum_{i=0}^{k} d'_i \) and \( \sum_{i=0}^{n-1} d_i = \sum_{i=0}^{n-1} d'_i \), then the sequence \( \pi' \) is said to major the sequence \( \pi \) and denoted by \( \pi \prec \pi' \). Further, the majorization theory on the two different tree degree sequences.

Theorem 3.3 [54] Let \( \pi \) and \( \tau \) be two different tree degree sequences with the same order. Let \( T_\pi^* \) and \( T_\tau^* \) have the maximum Laplacian spectral radii in \( T_\pi \) and \( T_\tau \), respectively. If \( \pi \prec \tau \), then \( \lambda(T_\pi^*) < \lambda(T_\tau^*) \).
Next, it will be interesting to characterize the extremal graphs with the set of all unicyclic graphs for given unicyclic graphic degree sequence $\pi$. But it seems to be more difficult. However, there are some partial results on the unicyclic graphs. For examples, Tan and Zhang in [46] presented a sharp bound for the spectral radius of the Laplacian Matrix of unicyclic graphs of order $n$ and the matching number $\beta$. Further, they characterized all extremal graphs which attained the upper bounds.

**Theorem 3.4** [46] Let $G$ be any unicyclic graph of order $n$ with the matching number $\beta$. Then the Laplacian spectral radius $\lambda_1(G)$ of $G$ is no more than the largest root of the following equation

$$
\lambda^3 - (n - \beta + 5)\lambda^2 + (3n - 3\beta + 7)\lambda - n = 0
$$

with equality if and only if $G$ is obtained from a triangle and $\beta - 2$ paths with length 2 and $n - 2\beta + 1$ edges by identifying one end vertex of them.

### 4 Graph Laplacian coefficients

It is an interesting and important problem how to order graphs for a given set of graphs, which has many application in computer science. There are many approaches to ordering these graphs. For example, the graphs can be ordered lexicographically according to their eigenvalues in nonincreasing order (for example, see [15], pp.268-269 and [16], p.70). Grone and Merris in [24] used the algebraic connectivity of a tree $T$ to order trees. Recently, Zhang in [53] further investigated ordering trees by their algebraic connectivity.

On the other hand, Another graph invariant, the Wiener index, can also be used to order graphs. The **Wiener index** of a connected graph is the sum of all distances between unordered pairs of vertices of a connected graph. In other words,

$$W(G) = \sum_{\{v_i, v_j\} \subseteq V(G)} d_G(v_i, v_j),$$

where $d_G(v_i, v_j)$ is the distance between vertices $v_i$ and $v_j$, i.e., the minimum number of edges between $v_i$ and $v_j$. The Wiener index was introduced in 1947 and extensively studied by chemists and mathematicians (for example, see [17, 50] for more detail). Since it is correlated with several graph properties and some physical and chemical properties of molecular graphs, the Wiener index can be used to order trees (see [18])
and extract useful information [61] from the structure of graphs. It is known that the Wiener index of a tree is equal to $c_{n-2}$ (for example, see [32] or [38]). Thus trees with the same Wiener index may be further ordered by other Laplacian coefficients.

Recently, Mohar in [40] and his homepage [41] proposed some problems on how to order trees with the Laplacian coefficients. Let $T_1$ and $T_2$ be two trees of order $n$. Let $r$ (resp. $s$) be the smallest (resp. largest) integer such that $c_r(T_1) \neq c_r(T_2)$ (respectively, $c_s(T_1) \neq c_s(T_2)$). Thus $r$ and $s$ exist if and only if $T_1$ and $T_2$ are not Laplacian cospectral. Hence two partial orderings can be defined as follows: If $c_r(T_1) < c_r(T_2)$, $T_1$ is "smaller than" $T_2$ and denote $T_1 \prec^1 T_2$. If $c_s(T_1) < c_s(T_2)$, $T_1$ is "smaller than" $T_2$ and denote $T_1 \prec^2 T_2$. Let $T_n$ be the set of all trees of order $n$. Another partial ordering in $T_n$ can be defined as follows: For any two trees $T_1, T_2 \in T_n$, if $(c_0(T_1), \ldots, c_n(T_1)) \leq (c_0(T_2), \ldots, c_n(T_2))$, i.e., $c_i(T_1) \leq c_i(T_2)$ with $i = 0, \ldots, n$, we say that $T_1$ is dominated by $T_2$ and denote $T_1 \preceq T_2$. Mohar in [41] proposed the following questions:

**Problem 4.1** [41] Do there exist two trees $T_1$ and $T_2$ of order $n$ such that $T_1 \prec^1 T_2$ and $T_2 \prec^2 T_1$?

**Problem 4.2** [41] Do there exist two trees $T_1$ and $T_2$ of order $n$ such that $T_1 \prec^1 T_2$ and $T_1 \prec^2 T_2$, but there is an index $i$ such that $c_i(T_1) > c_i(T_2)$?

**Problem 4.3** [41] Let $T_n$ be the set of all trees of order $n$. How large chains and anti-chains of pairwise non- Laplacian-cospectral trees are there?

**Problem 4.4** [41] Let $T_1$ and $T_2$ be two trees of order $n$ with $T_1 \preceq T_2$. Let $T(T_1, T_2)$ be the set of all trees $T$ of order $n$ with $T_1 \preceq T \preceq T_2$. For which trees $T_1$ and $T_2$ has $T(T_1, T_2)$ only one minimal element up to cospectrality, i.e., when are all minimal elements in $T(T_1, T_2)$ cospectral?

On the problems 4.1 and 4.2, Zhang in [56] gave a positive answer for problems 4.1 and 4.2 by presenting two examples, and investigated all majorization relationship among all trees of order $n$ with diameter 3. It may be more interesting to gave infinite family trees such that the results holds and to investigate structure of trees with the properties.

Gutman and Pavlovic in [26] proved there exists only maximal and minimal elements in the set of all trees of order $n$ with respective to $(T_n, \preceq)$. 


Theorem 4.5 [26] Let $T$, $K_{1,n-1}$ and $P_n$ be any tree, the star and path of order $n$. Then
\[ c_i(K_{1,n-1}) \leq c_i(T) \leq c_i(P_n). \]
In other words, $K_{1,n-1} \preceq T \preceq P_n$.

In addition, Ilić in [27] studied the maximal element for given a set of all trees of order $n$ with fixed the maximum degree.

Theorem 4.6 [27] Let $T$ be any tree of order $n$ with the maximum degree $\Delta$. Then
\[ c_i(T) \leq c_i(B_{n,\Delta}), \quad i = 0, \ldots, n \]
where $B_{n,\Delta}$ is the tree consisting of a star $K_{1,\Delta}$ and a path of length $n - \Delta - 1$ attached to an arbitrary pendant vertex of the star. Moreover, equality holds for some $2 \leq i \leq n - 2$ if and only if $T$ is $B_{n,\Delta}$.

Recently, Wang [47] and Zhang etc. [60] independently proved the minimum Wiener index in the set of all trees with given a tree degree sequence.

Theorem 4.7 [47, 60] For a given degree sequence $\pi$ of some tree, let
\[ T_\pi = \{ T \mid T \text{ is a tree with } \pi \text{ as its degree sequence} \}. \]
Then $T_\pi^*$ is a unique tree with the minimum Wiener index in $T(\pi)$.

Motivated Theorems 3.2, 4.7 and the related results, we proposed the following conjecture

Conjecture 4.8 For a given degree sequence $\pi$ of some tree, let
\[ T_\pi = \{ T \mid T \text{ is a tree with } \pi \text{ as its degree sequence} \}. \]
Then for any $T \in T_\pi$, we have
\[ c_i(T_\pi^*) \leq c_i(T), \quad i = 0, \ldots, n \]
with equality holding for some $2 \leq i \leq n - 2$ if and only if $T$ is $T_\pi^*$.

Remark: If Conjecture 4.8 holds, then Theorems 3.2 and 4.7 will be direct corollary of the conjecture.

In addition, Stevanović and Ilić [45] investigated the properties of the Laplacian coefficients of unicyclic graphs. Moreover, The related results on the Laplacian coefficients can be seen [28].
5 The algebraic connectivity and doubly stochastic matrix

In the study of chemical information processing, Golender et al. [23] introduced another important matrix: doubly stochastic graph matrix which is related to the graph Laplacian matrix and can be used to describe properties of topological structure of chemical molecular graphs. Let $I_n$ be the $n \times n$ identity matrix and $\Omega(G) = (I_n + L(G))^{-1} = (\omega_{ij})$. It is easy to see ([23] or [35]) that $\Omega(G)$ is a doubly stochastic matrix. Thus $\Omega(G)$ is called the doubly stochastic graph matrix. On the other hand, Chebotarev in [9] pointed out that the doubly stochastic graph matrix may be used to measure the proximity among vertices and evaluate the group cohesion in the construction of sociometric indices and represent a random walk. Moreover, with the entries of the doubly stochastic matrix, many topological measurements (such as dissociation, solitariness, provinciality, and etc) of small social groups on the basis of given relations on them can be quantified. Merris in [36] studied relationship between the the algebraic connectivity and the entries of the graph doubly stochastic matrix. In particular, he proposed two conjectures and two problems. One of the conjectures and one of the problems are follows:

**Conjecture 5.1** [36] Let $G$ be a graph on $n$ vertices. Then

$$\alpha(G) \geq 2(n + 1)\omega(G),$$

where $\omega(G)$ is the smallest entry of $\Omega(G) = (\omega_{ij})$, i.e., $\omega(G) = \min\{\omega_{ij}; 1 \leq i, j \leq n\}$.

**Question 5.2** [36] Let $G$ be a simple graph on vertex set $V = \{v_1, \ldots, v_n\}$ with doubly stochastic graph matrix $\Omega(G) = (\omega_{ij})$. Let $\rho(v_i, v_j) = \omega_{ii} + \omega_{jj} - 2\omega_{ij}$ and

$$z(G) = \min\{\sum_{j=1}^{n} \rho(v_i, v_j), \ 1 \leq i \leq n\}.$$

Does $d_k > d_i$ for all $i \neq k$, imply $r(k) = z(G)$?

Zhang and Wu in [59] presented an example to illustrate that Merris’ Conjecture 5.1 does not hold generally. Recently, Zhang in [57] carefully investigated when this conjecture still hold and gave many examples to illustrate that this conjecture does not holds. In particular,
**Theorem 5.3** [57] Let $T$ be a tree of order $n \geq 4$ with diameter $d$. If $d \geq \frac{\lg 3 + 3 \lg n}{\lg(3 + \sqrt{5}) - \lg 2} + 1$, then $\alpha(T) \geq 2(n + 1)\omega(T)$.

**Theorem 5.4** [57] Let $T$ be a tree of order $n$ with $p$ non- pendant vertices. Then

$$\alpha(T) \geq \frac{(n + p)\omega(T)}{1 - (n + p)\omega(T)}$$

with equality if and only if $T$ is the star graph $K_{1, n-1}$.

**Remark** From the above results, we may see that Conjecture 5.1 holds for many trees, since the diameter of any random trees is almost equal $O(\lg n)$. While Conjecture 5.1 does not holds for smaller diameter and larger order.

On Merris question 5.2, Zhang in [58] proved the following results

**Theorem 5.5** [58]

(i) There exists a family of graphs with $d(v_k) > d(v_i)$ for all $i \neq k$ but $r(k) > r(G)$.

(ii) Let $G$ be a simple graph. If $d(v_k) \geq 2d(v_i)$ for all $i \neq k$, then $r(k) = r(G)$.

**Remark:** There are many examples to illustrate the question 5.2 still holds for $d(v_k) \geq 2d(v_i) - 2$. Further Zhang in [58] proposed following question:

**Question 5.6** [58] Let $G$ be a simple connected graph on $n$ vertices $\{v_1, \ldots, v_n\}$ with the doubly stochastic graph matrix $\Omega(G) = (\omega_{ij})$. If $d(v_k) = \max\{d(v_1), d(v_2), \ldots, d(v_n)\}$ and $d(v_k) \geq 2d(v_i) - 2$ for all $i \neq k$, does $r(k) = r(G)$ hold?

The related works on the algebraic connectivity and the structure of the graph doubly stochastic matrix can be referred to [4, 8, 51, 59].

6 The algebraic connectivity of random graphs

Recently, there has been much interest in studying the small world networks and attempting to model their properties using random graphs. The study of complex systems in terms of random graphs was initiated by Erdős and Rényi [19]. Recently Watts and Strogatz [49] introduced the small-world network by interpolating order and random.

For given $0 \leq p \leq 1$, a ER random graph $G(n, p)$ is a graph of $n$ nodes and edges between nodes occurs independently with probability $p$. This random network
model has been intensively studied (see [6] and the references therein). Juhász in [29] established the asymptotic behavior of the algebraic connectivity of ER random graph.

**Theorem 6.1** [29] Let $G$ be an ER random graph of $n$ vertices with the probability $p$. Then for any $\varepsilon > 0$, we have

$$\alpha(G) = np + o(n^{1/2+\varepsilon}),$$

in probability.

Further, Chung etc. (see [13] or [11]) studied the semi-circle law for Laplacian eigenvalues of graphs.

The synchronization problem in networks of coupled oscillators is closely related to the consensus problem for network dynamics. The consensus problems are related to the connections between spectral properties of complex networks and ultrafast solution to distributed decision-making problems for interacting groups of agents. Hence the algebraic connectivity of networks is (locally) a measure of speed of synchronization.

From mathematical view, the measurement of convergence speed of solving the consensus problems in networks is used by the algebraic connectivity of a network. There are some strongly numerical evidences which support the conjecture that the network dynamics on the small-world networks would display enhanced global coordination compared to the regular lattices. In the recent paper of Olfati-Saber [44], it is observed that the algebraic connectivity of the small-world networks could be increased dramatically by more than 1000 times. Gu etc. in [25] gave a mathematical rigorous estimation of the lower bound for the algebraic connectivity of the small-world networks, which is much larger than the algebraic connectivity of the regular circle. This result explains why the consensus problems on the small-world network have a ultrafast convergence rate and how much it can be improved.

A $2k$ regular lattice $\mathcal{C}(n, k)$ with $n$ vertices can be constructed from a cycle of $n$ vertices and connected each vertex to its $2k$ nearest neighbors. Let $\mathcal{S}(n, c, k)$ be a small-world network that is the union of a random graph $\mathcal{G}(n, \frac{c}{n})$ and a $2k$ regular lattice $\mathcal{C}(n, k)$. Denote by $\lambda_2(n, c, k)$ the algebraic connectivity of $\mathcal{S}(n, c, k)$. If $c = 0$, then $\lambda_2(n, 0, k)$ is the algebraic connectivity of the $2k$ regular lattice $\mathcal{C}(n, k)$. The relationship between parameter $c$ and the average shortcut per node $s$ is $c = 2s$.

**Theorem 6.2** [25] Let $\mathcal{S}(n, c, k)$ be the small-world network with $n$ nodes, which is a union of an Erdős-Rényi random graph $\mathcal{G}(n, \frac{c}{n})$ and a $2k$ regular cycle. Then the
algebraic connectivity of \( S(n, c, k) \) is almost surely bounded below by

\[
\frac{k^2c^2 \log \log n}{2(k + 1)^2 \log^3 n}
\]

Olfati and Saber [44] defined

\[
\gamma_2(n, c, k) = \frac{\lambda_2(n, c, k)}{\lambda_2(n, 0, k)}
\]

to be the algebraic connectivity gain of \( S(n, c, k) \).

**Theorem 6.3** [25] The algebraic connectivity gain of the small-world network \( S(n, c, k) \) follows almost surely inequality

\[
\gamma_2(S(n, c, k)) \geq \frac{3kc^2n^2 \log \log n}{2(k + 1)^3(2k + 1)\pi^2 \log^3 n}.
\]

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