REPRESENTATION THEORY, TOPOLOGICAL FIELD THEORY, AND THE ANDREWS-CURTIS CONJECTURE

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Abstract. We pose a representation-theoretic question motivated by an attempt to resolve the Andrews-Curtis conjecture. Roughly, is there a triangular Hopf algebra with a collection of self-dual irreducible representations $V_i$ so that the product of any two decomposes as a sum of copies of the $V_i$, and $\sum (\text{rank } V_i)^2 = 0$? This data can be used to construct a “topological quantum field theory” on 2-complexes which stands a good chance of detecting counterexamples to the conjecture.

The first section recalls the Andrews-Curtis conjecture and its analogy with diffeomorphisms of 4-manifolds. The second section suggests the relevance of topological field theory. In the third we precisely state the representation-theoretic question. The final section sketches how the data is used to construct a field theory on 2-complexes.

1. The Andrews-Curtis conjecture

The conjecture asserts that if two 2-dimensional CW complexes are simple homotopy equivalent then there is a deformation from one to the other through 2-complexes. In a deformation we allow expansions and collapses of cells of dimension $\leq 2$, and homotopy of the attaching maps of 2-cells. There is a combinatorial group theory formulation obtained by thinking of 2-complexes as presentations of groups.

The special case of contractible complexes is particularly provocative: can a contractible 2-complex be deformed to a point? As specific examples we note the presentations

$$\langle x, y \mid xyx = yxy, x^m = y^n \rangle$$

determine 2-complexes which are contractible if $m = n \pm 1$. This can be deformed to a point if $\{m, n\} = \{2, 3\}$ (S. Gersten), but is expected to be a counterexample for other values $[AK], [G]$.

The conjecture in the contractible case arose in an approach to the 4-dimensional smooth Poincaré conjecture [AC]. There is some literature on the question, eg. [M], [MH], but this considerably underrepresents the attention it has received because there has been almost no progress. If true the conjecture would have useful applications in the study of low dimensional smooth manifolds [AK], [Q1], [G], [C2]. If it is false as expected there do not seem to be direct consequences for manifolds [C1]. In this case its greatest attraction may be as a model problem for invariants of smooth 4-manifolds, as we explain next.
Exotic smooth structures on 4-manifolds have a strong nilpotence property: Suppose \( f : M \to N \) is a map which the high-dimensional theory predicts should be homotopic to a diffeomorphism (see eg. [FQ, Ch. 7]). Then for some \( k \) the map \( f \# \text{id} : M \# kS^2 \times S^2 \to N \# kS^2 \times S^2 \) is homotopic to a diffeomorphism [Q2]. For the usual sorts of algebraic-topological invariants one can recover invariants of \( f \) from those of \( f \# \text{id} \), so these invariants cannot detect exotic \( f \). Donaldson [DK] has shown exotic maps exist, using invariants defined with the space of solutions to anti-self-dual Yang Mills equations. These invariants are killed by connected sum with \( S^2 \times S^2 \). The proper context for these invariants is far from clear. Also they leave the smooth Poincaré conjecture untouched. It would be very helpful to have examples of similar invariants to study for clues on how to deal with these problems.

The Andrews-Curtis problem has a very similar nilpotence property: Suppose \( f : X \to Y \) is a simple homotopy equivalence of 2-complexes. Then for some \( k \) the map \( f \lor \text{id} : X \lor kS^2 \to Y \lor kS^2 \) is homotopic to a deformation through 2-complexes. Here \( X \lor kS^2 \) indicates the 1-point union with \( k \) 2-spheres. Again traditional invariants of \( f \) can be recovered from those of \( f \lor \text{id} \), so cannot detect counterexamples to the conjecture. Because of this similarity we expect invariants capable of detecting exotic 2-complexes to behave much like 4-manifold invariants. They might also be simpler, displaying algebra and topology without the heavy burden of analysis and geometry.

### 2. Topological field theory

Atiyah [A] has suggested that the Donaldson invariants might be some sort of “topological quantum field theory” (TQFT). Floer, Donaldson, and others have partly implemented this intuition, though profound mysteries remain.

Atiyah also suggested there might be interesting TQFT on 3-manifolds. Witten [W1] gave a much more detailed prediction which has been implemented in several cases ([Wa], [KM], [RT]). We feel this is not a particularly good model for the 4-dimensional case because 3-manifolds do not display any analog of the problematic nilpotence phenomenon.

Following this idea we consider TQFT on 2-complexes. Formally this is a functor from a category of 2-complex “bordisms” to a category of modules over a ring. The objects in the bordism category are graphs (1-complexes). A morphism \( G_1 \to G_2 \) is an equivalence class of 2-complexes containing the disjoint union \( G_1 \sqcup G_2 \). The equivalence relation is Andrews-Curtis deformation through 2-complexes, leaving \( G_1 \sqcup G_2 \) fixed.

Denote the functor by \( Z \), so a graph \( G \) has an associated \( R \)-module \( Z(G) \), and a 2-complex bordism \( X : G_1 \to G_2 \) has an associated homomorphism \( Z_X : Z(G_1) \to Z(G_2) \). These take disjoint unions to tensor products: \( Z(G_1 \sqcup G_2) \simeq Z(G_1) \otimes_R Z(G_2) \), and similarly for morphisms.

The empty set is taken to \( R \). To get invariants of a 2-complex without specified subgraphs regard it as a bordism from the empty graph to itself. This gives a homomorphism \( Z_X : R \to R \), or equivalently an element \( Z_X(1) \in R \).

We relate this back to the nilpotence problem. The class of TQFT we consider has the property that the homomorphisms \( Z_X \) are unchanged by attaching 1-complexes to \( X \). The 1-point union \( X \lor S^2 \) is equivalent to \( S^2 \) attached by an arc to \( X \). The induced homomorphism is unchanged by deleting the arc. But disjoint
unions are taken to products, so
\[ Z_{X\lor S^2} = Z_{X\cup S^2} = (Z_X)(Z_{S^2}). \]
If \( Z_{S^2} \) is a unit in \( R \) then we can recover \( Z_X \) from \( Z_{X\lor S^2} \), and \( Z \) cannot detect counterexamples to the conjecture. At the other extreme if \( Z_{X\lor S^2} = 0 \) then \( Z \) has the same sort of “sudden death” instability that the Donaldson invariants do with respect to connected sum with \( S^2 \times S^2 \).

Therefore we seek TQFT on 2-complexes for which \( Z_{S^2} = 0 \). In the ones constructed from representations \( Z_{S^2} = \sum (\text{rank } V_i)^2 \), which is why we want this identity. It is hard to imagine how such a TQFT could be nontrivial without detecting something new, but at present there is no general proof of this. It would have to be verified by calculations of examples.

3. Representations

Fix a commutative ring \( R \). The complex numbers is probably a good choice: we will end up with units \( \text{rank } V_i \) and want the sum of their squares to vanish. Let \( H \) be a Hopf algebra over \( R \) which is triangular in the sense of Drinfel’d [Dr1]. We really do mean triangular, and not quasitriangular.

Suppose \( V, W \) are representations of \( H \) (ie. modules which are finitely generated projective as \( R \)-modules). Then the coproduct in \( H \) gives a way to put an \( H \)-module structure on the product \( V \otimes_R W \). Denote this new representation by \( V \Box W \). The coassociativity required in a Hopf algebra makes this operation associative. The triangular structure makes it commutative in the sense there is a canonical isomorphism \( \Psi: V \Box W \to W \Box V \), and the square of this is the identity.

We now formulate the problem: Find \( H \) so that there are representations \( V_0, \ldots, V_n \) satisfying

(1) there are \( H \)-morphisms \( \lambda_i: V_i \Box V_i \to R \) which are symmetric in the sense \( \lambda_i = \lambda_i \Psi \), and nondegenerate as bilinear forms;
(2) each \( V_i \Box V_j \) is isomorphic to a sum of copies of the \( V_* \);
(3) the \( V_i \) are irreducible and distinct; and
(4) \( \sum (\text{rank } V_i)^2 = 0 \).

Remarks.

(i) \textit{Quasi-Hopf algebras.} These conditions are much like the definition of a “modular Hopf algebra” used in [RT], [Wa], etc. This reflects a basic similarity in the construction. The triangular Hopf algebra structure is used to give the category of representations a symmetric monoidal (tensor) category structure ([Mc], [DM]). For this it is sufficient to have a triangular quasi-Hopf algebra in the sense of [D2].

(ii) \textit{Ranks.} The rank used in (4) is the categorical one: the trace of the identity homomorphism. More explicitly let \( V^* \) denote the dual hom \( _R(V, R) \) with the induced \( H \) structure, then representations are reflexive in the sense that the natural map \( i: V \to V^{**} \) is an isomorphism. Evaluation is a morphism \( \text{ev}: V \Box V^* \to R \).

Form the composition
\[ R \stackrel{\text{ev}^*}{\longrightarrow} V^{**} \Box V^{**} \stackrel{\text{id} \Box i^{-1}}{\longrightarrow} V^* \Box V \stackrel{\Psi}{\longrightarrow} V \Box V^* \stackrel{\text{ev}}{\longrightarrow} R \]
then this is multiplication by some element of \( R \). That element is defined to be the rank of \( V \). It follows from (1) and (3) that \( \text{rank } V_i \) is a unit.
(iii) The functor $E$. Define a functor $E$ from $H$-representations to $R$-modules by
\[
E(V) = \text{hom}_H(R,V) / \{ x : R \to V \mid f x = 0 \text{ for every } f: V \to R \}.
\]
Roughly this is the trivial subrepresentation of $V$ modulo the elements which cannot be detected by an invariant function $V \to 1$. When $V$ has a nondegenerate form on it as in (1) then $E(V \boxtimes V) \simeq E(V \boxtimes V^*)$ is essentially the endomorphism ring of $V$.

(iv) Irreducibility. The condition in (3) is correct if $R$ is an algebraically closed field (eg. C). The general conditions are $E(V_i \boxtimes V_i) \simeq R$ as an algebra, and $E(V_i \boxtimes V_j) = 0$ for $i \neq j$. Since $V_i$ is self-dual $E(V_i \boxtimes V_i)$ is basically the endomorphism ring, so the first condition is “irreducibility.” The second condition asserts roughly that the $V_i$ are distinct.

(v) Essential isomorphisms. The construction uses the functor $E$, so it is sufficient for the conditions to hold “on the $E$ level.” Specifically we say two morphisms $f, g: V \to W$ are essentially equal if $E(f \circ \text{id}) = E(g \circ \text{id})$ for any identity $\text{id}: U \to U$. We caution that it is not enough simply to assume $E(f) = E(g)$.

Thus (1) can be weakened to “$\lambda_i$ essentially equal to $\lambda_i \Psi_i$,” and for (2) we need only an essential isomorphism $\sum n_i V_i \to V_i \boxtimes V_k$. The analog of (2) in [RT] is only true up to essential isomorphism.

(vi) Strategy. The form of the conditions suggests a strategy for finding examples somewhat like the construction of Drenfel’d in [D2].

Note that the associativity isomorphism $U \boxtimes (V \boxtimes W) \simeq (U \boxtimes V) \boxtimes W$ does not appear in the conditions; it only has to exist. The commutativity isomorphism $\Psi: U \boxtimes V \simeq V \boxtimes U$ appears in the symmetry hypothesis in (1) and the definition of the rank in (4). Perhaps one could begin with a Hopf algebra and representations $V_i$ satisfying the first three conditions. Then adjust the $R$-matrix in $H$, from which $\Psi$ is defined, to make (4) true. The forms in (1) may no longer be symmetric with respect to the new $\Psi$, but it seems likely that new ones can be found. Finally solve for a “coassociativity” structure as in [D2] to get a triangular quasi-Hopf algebra satisfying the conditions.

4. A sketch of the construction

We actually get a modular TQFT in the sense of [Q3], generalizing a definition of G. Segal. This means we have relative modules $Z(G,P)$ defined for a graph containing a finite set of points $P$. Graphs can be glued together along points in the specified subsets, and there are natural homomorphisms

\[
Z(G_1, P_1 \sqcup P_2) \otimes_R Z(G_2, P_2 \sqcup P_3) \to Z(G_1 \cup_{P_2} G_2, P_1 \sqcup P_3).
\]

Note if $G_1$ is the unit interval and is attached along one end then the union is equivalent to $G_2$. This gives a ring structure on $Z(I, \partial I)$, and module structures over this ring on $Z(G, P)$, one for each point in $P$. Putting these together gives a ring structure on $Z(P_2 \times I, P_2 \times \partial I)$ and a module structure over this on $Z(G_1, P_1 \sqcup P_2)$.

The modularity axiom then asserts that the homomorphism (1) factors through an isomorphism on the tensor product over this ring,

\[
Z(G_1, P_1 \sqcup P_2) \otimes_{Z(P_2 \times I, P_2 \times \partial I)} Z(G_2, P_2 \sqcup P_3) \to Z(G_1 \cup_{P_2} G_2, P_1 \sqcup P_3).
\]
We now begin the construction. Let $V = \sum V_i$ be the sum of the representations given by the data. If $G$ is contractible (a tree) then we define

$$Z(G, P) = E(\square^P V).$$

The right side of this is the functor $E$ of §3 applied to a product of copies of $V$, one copy for each point in $P$. In fact for this to make sense we must choose orderings and associations to describe the $P$-fold product as a sequence of 2-fold products. We obtain something independent of the choice by using an inverse limit. More precisely we define a little category whose objects are the possible orderings and associations of $P$, and morphisms generated by changing one association or changing the order of two adjacent objects. The associativity and commutativity isomorphisms in the representation category give a functor from this little category into the representations. Compose with $E$ and take the inverse limit (defined in [Mc]).

It is at this point that the triangularity of the Hopf algebra is needed. Up to appropriate equivalence we can think of the tree $G$ as the cone on $P$. The symmetries of the cone must be reflected in symmetries of $Z(G, P)$, so permuting points in $P$ must give an action of the permutation group. In the 3-manifold constructions the analogous step uses $G$ a punctured sphere and $P$ a union of circles. Symmetries of a punctured sphere which permute the boundary circles are given by the braid group rather than the permutation group. To get an action of the braid group on $Z(G, P)$ it is sufficient to take an inverse limit over a category whose morphisms are “braid equivalences” of orderings and associations. And to define a functor on this category one only needs a quasi-triangular structure on the Hopf algebra, ie. the isomorphisms $\Psi: U \square V \to V \square U$ need not have order two.

If $G = G_1 \sqcup G_2 \sqcup \cdots \sqcup G_n$ is a disjoint union of trees, then define $Z(G, P) = \otimes_{i=1}^n Z(G_i, P \cap G_i)$. This agrees with the disjoint union property expected of a TQFT.

Now define some of the natural homomorphisms (1). Suppose we are joining two contractible graphs at a single point $p$. Then define the homomorphism to be the composition

$$E(\square^{P_1 \sqcup P} V) \otimes E(\square^{P_2} V) \to E(\square^{P_1 \sqcup P_2 \sqcup P_2} V) \to E(\square^{P_1 \sqcup P_2} V)$$

where the first takes $x \otimes y$ to $x \square y$ (recall these are morphisms from $R$ into various representations), and the second uses the symmetric form $\lambda$ to contract the repeated copies of $V$ associated to the two copies of $p$.

This defines the ring structures on $Z(I, \partial I)$, and as remarked in §3 it is more or less the endomorphism ring of $V$. We can think of the actions of this ring on $Z(G, P) = E(\square^P V)$ as induced from the actions of the endomorphism ring on the components of $\square^P V$. The modularity condition (2) follows easily in these cases.

Now define $Z(G, P)$ for a general finite graph. Cut the graph at points $Q$ to make all components contra.

Note added by Greg Kuperberg on 2/18/01: Garbled material material has been removed from the TeX submission here. A discontinuity in the narrative remains.

References

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