Two-dimensional plan source, vortex and vortex source

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Abstract. Modern hydraulic structures require highly reliable water-supply channels, free-flow pipes and open spillways. Therefore, they must be built with considering dynamic properties of the affecting flow. The theory of one-dimensional open flows cannot answer many questions raised by the hydraulic engineering practice. Hence, this paper considers two-dimensional graphical open flows, with separate particular models following from the general theory, important for designing couplings, curves in more complex constructions (two different channels) required in hydraulic engineering (Hydraulics technical structures) (HTS). And the more such models are obtained, the more opportunities will appear for scientists and designers to synthesize more complex structures required in HTS.

This study identifies three elements, and three simplest models of two-dimensional graphic open water flows similar to flat models: a source, a vortex and a vortex source. The number of such individual models will continue to increase and expand the range of possibilities for designing more complex water technical objects (WTO).

The models are obtained analytically from solving the system of two-dimensional graphical open potential water flows in the plane of the velocity hodograph. The elementary construction of each model includes an analytical solution for the potential function and the stream function.

The paper considers the problem of determining the parameters of a two-dimensional graphical open water flow at any point in the flow: a source, a vortex, a vortex source. The practical significance of the models lies in the possibility to use the results by the designers of hydraulic structures both at the first stage of solving problems and at the subsequent ones, with flow resistance forces taken into account.

The study also represents the transition method from the plane of the flow velocity hodograph to the flow diagram by integrating the models in the plane of the velocity hodograph from the condition of the connection between the considered planes.

1. Introduction

The design of modern hydraulic structures is closely associated with comprehensive consideration of impacting dynamic properties of the flow. Continuously increasing dimensions and requirements for the reliability of water supply channels and free-flow pipes as well as open spillways require deep knowledge of the specific properties of open streams.

The classical one-dimensional flow theory cannot give answers to many questions raised by the hydraulic engineering practice. The two-dimensional graphical flow is a flow in an open channel, whose surface width is several times greater than the depth, the bottom relief is smooth and the current curvature in the graph is not great.
Scientists V.M. Makkaveev, N.M. Vernadsky, N.T. Meleschenko, G.I. Sukhomel, S.I. Numerov, A.T. Ippen, R.T. Knapp where the first developers of the 2D graphical open streams theory. Both Russian and foreign works mainly consider potential flows in a horizontal channel and this made it possible to use the well-known gas-hydraulic analogy and the method of characteristics associated with it. The results obtained in the works mentioned above are fundamental and retain their significance today.

The theory under consideration was later developed in the works of S.N. Numerov, who extended the calculation method based on characteristics to the case of vortex motion, and also for the first time took into account the friction forces and gave a numerical calculation method. In addition, S.N Numerov solved some problems of the two-dimensional theory in relation to slow graphical flows. A number of articles and a dissertation by I.A. Sherenkov are devoted to a detailed study of the problem of spreading a rapid current on the basis of the 2D theory. I.A. Sherenkov developed an original method for calculating a potential flow by characteristics as well as a numerical method that takes into account friction and bottom slope.

A number of foreign authors (A. Ippen, H. Rouse, D. Harleman, D. Dawson, R. Knapp) used the fundamentals of the theory of two-dimensional potential flows to construct calculating methods for transitional sections of rapid currents. They also carried out a number of experimental studies.

This study represents further research of previous studies of other authors [1-8]. Paper [9] gave solutions for plane problems of potential flow hydrodynamics which have both special theoretical and practical significance. However, similar problems can be solved for 2D graphical flows.

The aim of this work is to solve problems of spreading two-dimensional graphical, potential, open, stationary water flows. Namely, solving the problems of:

- two-dimensional graphical source;
- two-dimensional graphical vortex;
- a two-dimensional graphical vortex source.

The urgency of solving these problems is obvious, since their solution is to expand the capabilities of the two-dimensional plan flows' hydraulics [10-20].

2. Formulation of the problem
We proceed from the flow plane motion equations system of the velocity hodograph for the research. This system has the form [8, 19]:

\[
\begin{align*}
\frac{\partial \phi}{\partial \tau} &= \frac{h_0}{2H_0} \left(3\tau-1\right) \frac{\partial \psi}{\partial \theta}, \\
\frac{\partial \phi}{\partial \theta} &= 2 \frac{h_0}{H_0} \frac{\tau}{1-\tau} \frac{\partial \psi}{\partial \tau},
\end{align*}
\]

(1)

where \( \tau, \theta \) are the independent parameters: \( \tau = \frac{V^2}{2gH_0} \) – defines the speed-dependent parameter; \( V \) – is the local flow rate; \( \theta \) – is the angle characterizing the two-dimensional flow velocity vector direction; \( g \) – defines gravity acceleration; \( \phi = \phi(\tau, \theta) \) – is the potential function; \( \psi = \psi(\tau, \theta) \) – is the current function; \( H_0 = \frac{V^2}{2g} + h_0 \) – is constant for the entire flow - hydrodynamic head; \( V_0, h_0 \) – are the local velocity and depth values at some characteristic point of the flow.

The system (1) is also complemented by D. Bernoulli's integral for two-dimensional flows:
\[ \frac{V^2}{2g} + h = H_0, \]  

moreover

\[ V = \tau^{1/2} \sqrt{2gH_0}; \quad h = H_0 (1 - \tau). \]  

Figure 1. Flow diagram of a two-dimensional vortex

The system (1) has a whole range of analytical solutions [8, 19]. However, to solve the problems formulated in this article, we choose the following configuration from the spectrum of analytical solutions:

\[
\begin{align*}
\psi &= \frac{C_2 H_0}{2 h_0} [\ln \tau - \tau] + C_1 \theta; \\
\phi &= -\frac{C_1 h_0}{2 H_0} \left[ \ln \frac{\tau}{1 - \tau} - \frac{2}{1 - \tau} \right] + C_2 \theta.
\end{align*}
\]  

(4)

It is possible to verify by direct verification that the equation (4) is a solution to the system (1). This design is basic for the following tasks: source, vortex, vortex source.

In case of a two-dimensional source from the configuration of analytical solutions (4), we single out the following:

\[
\begin{align*}
\psi &= C_1 \theta; \\
\phi &= -\frac{C_1 h_0}{2 H_0} \left[ \ln \frac{\tau}{1 - \tau} - \frac{2}{1 - \tau} \right].
\end{align*}
\]  

(5)

Since the spreading of the flow is radial, the streamlines are rays emanating from the origin, see figure 1. This problem was solved and presented in [8, 9] for both quiet and turbulent flows.

The problem is solved from the general provisions of the two-dimensional potential plan flows theory for the first time for a two-dimensional vortex, and the configuration is singled out from (4).
\[ \psi = \frac{C_2 H_0}{2 h_0} \ln \left( \frac{\tau - \tau_0}{\tau - \tau_0} \right); \]
\[ \varphi = C_2 \theta, \]

where \( \psi = \text{const} \) – defines the streamlines; \( \varphi = \text{const} \) – defines the equipotential lines.

3. The solution of the problem

3.1. A two-dimensional vortex parameters’ determination

For definiteness, the flow is assumed to be turbulent:

\[ \frac{1}{3} \leq \tau \leq 1. \]  

(7)

3.2. Flow spreading scheme in the physical domain

From the condition of the connection between the flow plan \( \Phi(OXY) \) and the speed hodograph plane \( \Gamma(\tau, \theta) \) we have [7, 8]:

\[ d(x + iy) = \left( \frac{d\varphi + i}{H_0 (1 - \tau)} \right) \frac{e^{i\theta}}{\sqrt{\tau^2 + 2gH_0}}. \]  

\[ (8) \]

Since along the streamline \( d\psi = 0 \), the following comes from (8):

\[ dx = \frac{d\varphi \cdot \cos \theta}{\sqrt{\tau^2 + 2gH_0}}; \quad dy = \frac{d\varphi \cdot \sin \theta}{\sqrt{\tau^2 + 2gH_0}}. \]  

\[ (9) \]

Having determined the differential from (6)

\[ d\varphi = C_2 d\theta, \]  

(10)

from (9) follows the system of equations

\[ dx = \frac{C_2 \cdot \cos \theta \cdot d\theta}{2gH_0}; \quad dy = \frac{C_2 \cdot \sin \theta \cdot d\theta}{\sqrt{2gH_0}}; \quad \frac{dy}{dx} = \tan \theta. \]  

\[ (11) \]

Integrating (11), we obtain:

\[ x = \frac{C_2 \cdot \sin \theta}{\sqrt{2gH_0}}; \quad y = -\frac{C_2 \cdot \cos \theta}{\sqrt{2gH_0}}. \]  

\[ (12) \]

Raising both sides of the equalities in (12), adding them and using the main trigonometric identity, we obtain:

\[ r^2 = \frac{C_2^2}{2gH_0 \cdot \tau}. \]  

\[ (13) \]

3.3. The vortex streamline equation at a fixed \( \tau \) – circle equation

We define the constant \( C_2 \) from the condition that for \( r = r_0 \):

\[ V = V_0; \quad h = h_0; \quad \tau = \tau_0. \]  

\[ (14) \]

Therefore:

\[ C_2^2 = \frac{2gH_0 \cdot \tau_0 \cdot r_0^2}{\tau}. \]  

\[ (15) \]
\[
\begin{align*}
V &= \tau^{1/2} \sqrt{2gH_0}; \\
h &= H_0 (1 - \tau).
\end{align*}
\] (16)

It is also seen from (13), (16), that for \( \tau \to 1 \)

\[
V \to V_{\text{max}} = \sqrt{2gH_0}; \quad h \to 0; \quad r \to r_{\text{up}} = \frac{C_2}{\sqrt{2gH_0}}.
\]

Along the equipotential:

\[
d\phi = 0; \\
d\psi = \frac{C_2 H_0}{2h_0} \left( \frac{1 - \tau}{\tau} \right) d\tau.
\] (17)

And it follows from (8):

\[
\begin{align*}
dx &= -\frac{1}{2} \frac{\sin \theta}{\sqrt{2gH_0}} \frac{C_2}{\tau^{3/2}} d\tau; \\
dy &= \frac{1}{2} \frac{\cos \theta}{\sqrt{2gH_0}} \frac{C_2}{\tau^{3/2}} d\tau.
\end{align*}
\] (18)

It follows from (18):

\[
\frac{dy}{dx} = -\text{ctg} \theta.
\] (19)

Comparing last equation (11) and (19) we see that streamlines and equipotential lines are mutually orthogonal, since the condition \([21]\) is satisfied according to (11) and (19):

\[
tg \theta \cdot (-\text{ctg} \theta) = -1.
\] (20)

Integrating the system (18) and setting the integration constants to zero, since the constant \(C_2\) had already been defined earlier, we get the system:

\[
\begin{align*}
x &= \frac{C_2 \cdot \sin \theta}{\sqrt{2gH_0}} \frac{1}{\sqrt{\tau}}; \\
y &= \frac{C_2 \cdot \cos \theta}{\sqrt{2gH_0}} \frac{1}{\sqrt{\tau}}.
\end{align*}
\] (21)

\(y = -\text{ctg} \theta \cdot x\) is the equipotential equation.

It becomes obvious from the comparison of (12) and (21) that the circle intersection points coincide with the rays themselves. Therefore, since \(\tau_0 \leq \tau \leq 1\), then the system (21) describes the segments of rays emerging from the points on a circle with radius \(r_0\) and up to a circle with limiting radius \(r_{\text{lim}}\). Depths \(h\) on a circle of limiting radius \(r_{\text{lim}}\) tend to zero, the velocity tends to the maximum possible for turbulent flows:

\[
V_{\text{max}} = \sqrt{2gH_0}.
\]

For a vortex source with the origin at the coordinates center, it is necessary to choose a structure (4) in the form of absorbing both the source and the vortex.

The procedure for solving the problem in principle coincides with the solution of the source and vortex problems.
4. Conclusions
The two-dimensional potential flows problems in terms of potential flows can be solved using the analytical solutions of the velocity hodograph plane system. It is quite simple to find a solution to the problems: a source, a vortex and a source.

The research started in this article, with a detailed study of turbulent and quiet flows and with further consideration of the flow resistance forces, can be presented within the framework of the dissertation work.

The convenient, simple algorithms for calculating two-dimensional plan flows in terms of source, vortex, vortex source have been shown in the presented work.

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