Wajsberg algebras arising from binary block codes

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Abstract. In this paper we presented some connections between BCK-commutative bounded algebras, MV-algebras, Wajsberg algebras and binary block codes. Using connections between these three algebras, we will associate to each of them a binary block code and, in some circumstances, we will prove that the converse is also true.

Keywords: BCK bounded commutative algebras, MV-algebras, Wajsberg algebras, block codes.

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1. Introduction

BCK-algebras were first introduced in mathematics by Y. Imai and K. Iseki, in 1966, through the paper [II; 66], as a generalization of the concept of set theoretic difference and propositional calculi. These algebras form an important class of logical algebras and have many applications to various domains of mathematics (group theory, functional analyses, sets theory, etc.). Because of the necessity to establish certain rational logic systems as a logical foundation for uncertain information processing, various types of logical systems have been proposed. For this purpose, some logical algebras appeared and have been researched.([WDH; 17]) One of these algebras are MV-algebras, where MV is referred to "many valued"([GA; 90]), which were originally introduced by Chang in [CHA; 58]. He tried to provide a new proof for the completeness of the Lukasiewicz axioms for infinite valued propositional logic. These algebras appeared in the specialty literature under equivalent names: bounded commutative BCK-algebras or Wajsberg algebras, ([CT; 96]). Wajsberg algebras were introduced in 1984, by Font, Rodriguez and Torrens, through the paper [FRT; 84] as an alternative model for the infinite valued Lukasiewicz propositional logic.

In the following, we present some connections between BCK-commutative bounded algebras, MV-algebras, Wajsberg algebras and binary block codes and we gave an algorithm to find all finite partial ordered Wajsberg algebras. These new approach allows us to find for these structures new and interesting properties.
2. Preliminaries

**Definition 2.1.** An algebra \((X, *, \theta)\) of type \((2, 0)\) is called a \textit{BCI-algebra} if the following conditions are fulfilled:

1) \((x * y) * (x * z) * (z * y) = \theta\), for all \(x, y, z \in X\);
2) \((x * (x * y)) * y = \theta\), for all \(x, y \in X\);
3) \(x * x = \theta\), for all \(x \in X\);
4) For all \(x, y, z \in X\) such that \(x * y = \theta, y * x = \theta\), it results \(x = y\).

If a BCI-algebra \(X\) satisfies the following identity:

5) \(\theta * x = \theta\), for all \(x \in X\), then \(X\) is called a \textit{BCK-algebra}.

In a BCK-algebra we have the following order relation:

\[ x \leq y \text{ if and only if } x * y = \theta. \]

If the algebra \((X, *, \theta)\) has an element 1 such that \(x * 1 = \theta\), for all \(x \in X\) (that means \(x \leq 1\), for all \(x \in X\)), then the BCK-algebra \(X\) is called \textit{bounded}. If \(x \wedge y = y \wedge x\), for all \(x, y \in X\), where \(x \wedge y = y * (y * x)\), for all \(x, y \in X\), then \(X\) is called a \textit{commutative} BCK-algebra. For other details regarding BCK algebras, the readers are referred to [AAT; 96], [Me-Ju; 94].

**Definition 2.2.** ([CHA; 58]) An abelian monoid \((X, \theta, \oplus)\) is called \textit{MV-algebra} if and only if we have an operation "ⅅ" such that:

i) \((x')' = x;\)
ii) \(x \oplus \theta' = \theta';\)
iii) \((x' \oplus y)' \oplus y = (y' \oplus x)' \oplus x\), for all \(x, y \in X\). ([Mu; 07].)

We denote it by \((X, \oplus', \theta')\).

**Remark 2.3.** ([Mu; 07]) In an MV-algebra the constant element \(\theta'\) is denoted with 1, that means

\[ 1 = \theta', \]

and the following multiplications are also defined:

\[ x \circ y = (x' \oplus y')', \]

\[ x \oplus y = x \circ y' = (x' \oplus y)''. \]

**Remark 2.4.** ([COM; 00], Theorem 1.7.1, p. 30)

i) Let \((X, *, \theta, 1)\) be a bounded commutative BCK-algebra. If we define

\[ x' = 1 * x, \]

\[ x \oplus y = 1 * ((1 * x) * y) = (x' * y)', \]

\(x, y \in X,\)
we obtain that the algebra \((X, \oplus, ', \theta)\) is an \(MV\)-algebra, with
\[ x \oplus y = x * y. \]

ii) If \((X, \oplus, \theta')\) is an \(MV\)-algebra, then \((X, \ominus, \theta, 1)\) is a bounded commutative BCK-algebra.

**Definition 2.5.** ([COM; 00], Definition 4.2.1) An algebra \((W, \circ, 1)\) of type 
\((2, 1, 0)\) is called a \(Wajsberg\) algebra (or \(W\)-algebra) if and only if for every \(x, y, z \in W\), we have:
\[ \begin{align*}
  &i) \quad 1 \circ x = x; \\
  &ii) \quad (x \circ y) \circ [(y \circ z) \circ (x \circ z)] = 1; \\
  &iii) \quad (x \circ y) \circ y = (y \circ x) \circ x; \\
  &iv) \quad (x \circ y) \circ (y \circ x) = 1.
\end{align*} \]

**Remark 2.6.** ([COM; 00], Lemma 4.2.2 and Theorem 4.2.5)

i) If \((W, \circ, 1)\) is a Wajsberg algebra, defining the following multiplications
\[ x \circ y = (x \circ y) \quad \text{and} \quad x \oplus y = x \oplus y, \]
for all \(x, y \in W\), we obtain that \((W, \circ, \oplus, , 0, 1)\) is an \(MV\)-algebra.

ii) If \((X, \oplus, \circ, ', \theta, 1)\) is an \(MV\)-algebra, defining on \(X\) the operation
\[ x \circ y = x' \oplus y, \]
it results that \((X, \circ, ', 1)\) is a Wajsberg algebra.

**Example 2.7.** We consider the following set \(X = \{\theta, a, b, c, d, e\}\) and the multiplication \(\ast\) given in the below table:

\[
\begin{array}{ccccccc}
\ast & \theta & a & b & c & d & e \\
\hline
\theta & \theta & \theta & \theta & \theta & \theta & \theta \\
a & a & \theta & a & \theta & \theta & \theta \\
b & b & b & \theta & b & \theta & \theta \\
c & c & a & c & \theta & a & \theta \\
d & d & b & a & b & \theta & \theta \\
e & e & d & c & b & a & \theta \\
\end{array}
\]

Then \((X, \ast, \theta, 1)\) becomes a BCK commutative bounded algebra. The associated \(MV\)-algebra is \((X, \oplus, ', \theta)\), with the multiplication \(\oplus\) and the operation \(\prime\) given in the below tables:

\[
\begin{array}{ccccccc}
\oplus & \theta & a & b & c & d & e \\
\hline
\theta & \theta & a & b & c & d & e \\
a & a & c & d & e & e & e \\
b & b & d & e & d & e & e \\
c & c & c & e & e & e & e \\
d & d & e & d & e & e & e \\
e & e & e & e & e & e & e \\
\end{array}
\]

\[
\begin{array}{ccccccc}
\prime & \theta & a & b & c & d & e \\
\hline
\theta & \theta & a & b & c & d & e \\
e & e & d & c & b & a & \theta \\
\end{array}
\]
The associated Wajsberg algebra \((X, \circ', 1)\) is:

\[
\begin{array}{cccccc}
\circ & \emptyset & a & b & c & d & e \\
\emptyset & e & e & e & e & e & e \\
a & d & e & d & e & e & e \\
b & c & c & c & e & e & e \\
c & b & d & b & e & d & e \\
d & a & c & d & c & e & e \\
e & \emptyset & a & b & c & d & e \\
\end{array}
\]

Proposition 2.8. ([Bu; 06]) Let \((X, \oplus', \emptyset)\) be an MV-algebra. We have that

\[x \oplus x' = 1.\]

**Proof.** Indeed, from Definition 2.2 ii) and iii), it results that

\[1 = (x' \oplus 1)' \oplus 1 = (1' \oplus x)' \oplus x = x' \oplus x.\]

Proposition 2.9. ([Mu; 07]) Let \((X, \oplus', \emptyset)\) be an MV-algebra. For \(x, y \in X\), the following statements are equivalent:

i) \(x' \oplus y = 1;\)

ii) \(x \circ y' = 0;\)

iii) \(y = x \oplus (y \oplus x) = x \oplus (y' \oplus x)';\)

iv) There is an element \(z \in X\) such that \(x \oplus z = y.\)

Definition 2.10. ([COM; 00]) Let \((X, \oplus', \emptyset)\) be an MV-algebra and \(x, y \in X\). On \(X\), we define the following order relation:

\[x \leq y\] if and only if \(x' \oplus y = 1.\]

Remark 2.11. It is clear that \(x \leq y\) if and only if \(x\) and \(y\) satisfy one of the equivalent conditions i)-iv) from Proposition 2.9.

Definition 2.12. ([FRT; 84] If \((W, \circ, 1)\) is a Wajsberg algebra, on \(W\) we define the following binary relation

\[x \leq y\] if and only if \(x \circ y = 1.\]

This relation is an order relation, called the natural order relation on \(W\).

For other details regarding MV-algebras and Wajsberg algebras, the readers are referred to [Io; 08] and [Pi; 07].

3. Block codes associated to MV-algebras and Wajsberg algebras
In [JUN; 11], were introduced binary block-codes over finite BCK-algebras. In a similar way, code over MV-algebras and Wajsberg algebras can be established.

Let $A$ be a nonempty set and let $(X, \oplus', \theta)$ be an MV-algebra.

**Definition 3.1.** A mapping $f : A \rightarrow X$ is called an $MV$-function on $A$. A cut function of $f$ is a map $f_r : A \rightarrow \{0, 1\}, r \in X$, such that

$$f_r (x) = 1,$$

if and only if $r' \oplus f (x) = 1, \forall x \in A$.

A cut subset of $A$ is the following subset of $A$

$$A_r = \{ x \in A / r' \oplus f (x) = 1\}.$$

**Remark 3.2.** If $y \in A_r \cup A_s$, therefore $y \in A_r \oplus s$. Indeed, for $y \in A_r \cup A_s$, we suppose that $y \in A_r$. From here, we have $y \in A_r \oplus s$, since $A_r \oplus s = \{ x \in A / (r \oplus s) \oplus f (x) = 1\}$ and $(r \oplus s) \oplus f (y) = (r' \oplus s') \oplus f (y) = r' \oplus (s' \oplus f (y)) = r' \oplus 1 = 1$.

**Remark 3.3.** Let $f : A \rightarrow X$ be an $MV$-function on $A$. We define on $X$ the following binary relation

$$\forall r, s \in X, r \sim s \text{ if and only if } A_r = A_s.$$

This relation is an equivalence relation on $X$ and we denote with $\bar{r}$ the equivalence class of an element $r \in X$.

**Proposition 3.4.** Let $f : A \rightarrow X$ be an $MV$-function on $A$. Therefore

$$f (x) = \sup \{ r \in X / f_r (x) = 1\}.$$

**Proof.** For a chosen element $x \in A$, we denote $f (x) = s, s \in X$. From here we have $s' \oplus f (x) = s' \oplus s = 1$, therefore $f_s (x) = 1$. If there is an element $r \in X$ such that $r' \oplus f (x) = r' \oplus s = 1$, we obtain that $r \leq s$. We consider the set $M = \{ r \in X / f_r (x) = 1\}$. Since $f_s (x) = 1$, it results that $s \in M$, therefore $f (x) = s = \sup M$. $\square$

The above proposition extends to $MV$-algebras results obtained in Proposition 3.4 from [JUN; 11]

**Proposition 3.5.** Let $f : A \rightarrow X$ be an $MV$-function on $A$. Therefore, for $r, s \in X$, we have that $r \leq s$ implies $A_s \subseteq A_r$. If $A = X$ and $f : X \rightarrow X$ is the identity function, $f (x) = x$, then the converse of this statement is also true.

**Proof.** Let $r, s \in X$ such that $r \leq s$. From Proposition 2.9., iv), it results that $s = r \oplus t, t \in X$. For $x \in A_s$, we have $s' \oplus f (x) = 1$, which is equivalent with $s \leq f (x)$, that means $f (x) = s \oplus q, q \in X$. It results that $f (x) = (r \oplus t) \oplus q = r \oplus (t \oplus q)$, therefore $r \leq f (x)$. From here, we obtain that $r' \oplus f (x) = 1$, then $x \in A_r$.  


For the converse, if $A_s \subseteq A_r$, we have that $s \in A_s \subseteq A_r$, therefore $s \in A_r$. It results that $r' \oplus f(s) = r' \oplus s = 1$, which implies $r \leq s$. □

The above proposition extends to MV-algebras results obtained in Proposition 3.6 from [JUN; 11].

Let $A$ be a set with $n$ elements. We consider $A = \{1, 2, \ldots, n\}$ and let $X$ be an MV-algebra. Using above notations, to each equivalence class $\tilde{x}, \tilde{x} \in \tilde{X}$, will correspond the codeword $w_{\tilde{x}} = f_{\tilde{x}} = x_1 x_2 \ldots x_n$, with $x_i = j$, if and only if $f_x(i) = j, i \in A, j \in \{0, 1\}$. We denote this code with $V_X$. In this way, each MV-function $f : A \rightarrow X$ has associated a binary block-code of length $n$ and $V_X = \{f_x, x \in X\}$.

Let $V$ be a binary block-code of length $n$ and $w_x = x_1 x_2 \ldots x_n \in V$. Let $w_y = y_1 y_2 \ldots y_n \in V$ be two codewords. On $V$ we can define the following partial order relation:

$$w_x \preceq w_y \text{ if and only if } y_i \leq x_i, i \in \{1, 2, \ldots, n\}. \quad (3.1.)$$

Proposition 3.6. Let $X$ be an MV-algebra. With the above notations, relation $A_s \subseteq A_r$ is equivalent with $f_r \preceq f_s$.

Proof. Assuming $A_s \subseteq A_r$, we have $f_x(x) = f_r(x) = 1$, for all $x \in A_s$. It results $f_s(x) \leq f_r(x)$, for all $x \in X$, that means $f_r \preceq f_s$.

Conversely, if $f_r \preceq f_s$, we have $f_s(x) \leq f_r(x)$, for all $x \in X$. We obtain $s' \oplus f(x) = 1$, which is equivalent with $f_s(x) = 1$. This implies that $f_r(x) = 1$, which is equivalent with $r' \oplus f(x) = 1$. From here, we obtain $x \in A_r$. □

Proposition 3.7. Let $(X, \oplus', \theta)$ be a finite MV-algebra. To algebra $X$ corresponds a block-code $V$ such that $(X, \leq)$ is isomorphic to $(V_X, \preceq)$ as ordered sets.

Proof. Let $(X, \oplus', \theta)$ be a finite MV-algebra and, for $A = X$, let $f : X \rightarrow X$ be the identity function which is an MV-function. The function $f$ generates the following set $M = \{f_r / r \in X\}$ of cuts functions. Then the set $M$ with the order $\preceq$ is the associated block-code to the MV-algebra $X$, with the order relation defined in (3.1). Let $g : X \rightarrow M, g(r) = f_r$, for all $r \in X$. We will prove that this map is bijective and $r \leq s$ is equivalent with $f_r \preceq f_s$. By definition, the map is surjective. The map $g$ is injective. Indeed, if $r, s \in X$ with $g(s) = g(r)$, we have $f_r = f_s$, therefore $A_r = A_s$. It results that $1 = r' \oplus f(s) = r' \oplus s$ and $1 = s' \oplus f(r) = s' \oplus r$, therefore, $r \leq s$ and $s \leq r$. From here, we obtain that $r = s$, therefore $g$ is a bijective map. Let $r, s \in X$ such that $r \leq s$. From Proposition 3.5, this is equivalent with $A_s \subseteq A_r$, which is equivalent with $f_r \preceq f_s$, from Proposition 3.6. □

Theorem 3.8. Let $A = (X, * , \theta, 1)$ be a bounded commutative BCK-algebra and $B = (X, \oplus', \theta)$ be the associated MV-algebra. Therefore $A$ and $B$ are code equivalent, that means determine the same binary block-code.

Proof. With the above notations, let $V_A$ be the code associated to the BCK algebra $A$ and $V_B$ be the code associated to the MV-algebra $B$. Let
\( \varphi : V_A \rightarrow V_B, \varphi (f_r) = f_r', \) where \( f_r \) is a cut function associated to BCK-algebra as in [JUN; 11], \( f_r' \) is the corresponded cut function associated to MV-algebra and \( r' \) is the complement of the element \( r \in X \). The map \( \varphi \) is injective, therefore bijective. Indeed, if \( \varphi (f_r) = \varphi (f_s) \), it results \( f_r' = f_s' \), therefore \( r' = s' \) and \( r = s. \square \)

**Definition 3.9.** ([CHA; 19]) Let \( (X, \oplus', \theta) \) be an MV-algebra. The distance function defined on the algebra \( A \) is:

\[
\begin{align*}
d & : X \times X \rightarrow X, d(x, y) = (x \ominus y) \ominus (y \ominus x) \\
& = (x \ominus y') \ominus (y \ominus x') \\
& = (x' \ominus y') \ominus (y' \ominus x').
\end{align*}
\]

In the following, inspired from the above definition, we will define a new distance on a finite MV-algebra \( X \) with \( n \) elements. Let \( (X, \oplus', \theta) \) be an MV-algebra, \( f : A \rightarrow X \) be an MV-function on \( A \), \( f_r : A \rightarrow \{0, 1\} \), \( r \in X \), be a cut function and \( A_r \) be the associated cut subset. Let \( \varphi : X \rightarrow \mathcal{P}(A), \varphi (r) = A_r. \) Since \( \mathcal{P}(A, \cup, \cap, \emptyset, 1), A \) is an MV-algebra, let \( d \) the distance on \( \mathcal{P}(A) \) defined as above. We define on \( X \) the following map

\[
D : X \times X \rightarrow \mathbb{R}_+,
\]

\[
D(r, s) = |d(\varphi (r), \varphi (s))|.
\]

**Proposition 3.10.**

1) \( D(r, s) = 0 \) if and only if \( r = s. \)

2) \( D(r, s) \leq D(r, t) + D(t, s). \)

3) \( D(r, \emptyset) = |A_r| \) and \( D(r, 1) = |A_r'|, \) where \( A_r' \) is the complement of the set \( A_r. \)

**Proof.**

1) Indeed, if \( D(r, s) = |d(\varphi (r), \varphi (s))| = 0, \) we have \( (A_r' \cap A_s) \cup (A_r \cap A_s') = \emptyset. \) From here, we have that \( A_r' \cap A_s = A_r \cap A_s' = \emptyset, \) therefore \( A_r = A_s \) which is equivalent with \( r = s. \)

2) Indeed, \( D(r, s) = |d(\varphi (r), \varphi (s))| \leq |d(\varphi (r), \varphi (t)) + d(\varphi (t), \varphi (s))| \leq |d(\varphi (r), \varphi (t))| + |d(\varphi (t), \varphi (s))|. \square \)

**Definition 3.11.** The distance \( D \) is called the Hamming distance between the elements \( r \) and \( s \in X \), and it is the really Hamming distance between their associated code-words \( f_r \) and \( f_s. \)

In a similar way as above, we can introduce binary block-codes over finite Wajsberg algebra.

Let \( A \) be a nonempty set and let \( (W, \circ, 1) \) be a Wajsberg algebra.

**Definition 3.12.** A mapping \( f : A \rightarrow W \) is called a W-function on \( A \). A cut function of \( f \) is a map \( f_r : A \rightarrow \{0, 1\}, r \in W, \) such that

\[
f_r(x) = 1, \text{ if and only if } r \circ f(x) = 1, \forall x \in A.
\]
A cut subset of \( A \) is the following subset of \( A \)
\[
A_r = \{ x \in A / r \circ f(x) = 1 \}.
\]

Let \( f : A \to W \) be an \( W \)-function on \( A \). We define on \( W \) the following binary relation
\[
\forall r, s \in W, r \sim s \text{ if and only if } A_r = A_s.
\]

This relation is an equivalence relation on \( W \) and we denote with \( \overline{r} \) the equivalence class of an element \( r \in W \).

**Proposition 3.13.** Let \( f : A \to W \) be a \( W \)-function on \( A \). Therefore, for \( r, s \in W \), we have that \( r \leq s \) implies \( A_s \subseteq A_r \). If \( A = W \) and \( f : W \to W \) is the identity function, \( f(x) = x \), then the converse of this statement is also true.

**Proof.** Assuming that \( r \leq s \), we have \( r \circ s = 1 \). Let \( w \in A_s \), therefore \( s \circ f(w) = 1 \). We will prove that \( r \circ f(w) = 1 \). Since for all \( x, y, z \in W \), from \( (x \circ y) \circ (y \circ z) = 1 \), we have \( (r \circ s) \circ ([s \circ f(w)) \circ (r \circ f(w))] = 1 \), therefore \( 1 \circ [1 \circ (r \circ f(w))] = 1 \). From here, since \( 1 \circ x = x \), it results \( 1 = r \circ f(w) \), therefore \( A_s \subseteq A_r \).

For the converse, if \( A_s \subseteq A_r \), we have that \( s \in A_r \), therefore \( 1 = r \circ f(s) = r \circ s \). We obtain \( r \leq s \). □

The above proposition extends to \( W \)-algebras results obtained in Proposition 3.6 from [JUN; 11].

Let \( A \) be a set with \( n \) elements. We consider \( A = \{1, 2, \ldots, n\} \) and let \( W \) be a \( W \)-algebra. Using above notations, to each equivalence class \( \overline{x}, x \in W \), will correspond the codeword \( w_x = f_x = x_1 x_2 \ldots x_n \), with \( x_i = j \), if and only if \( f_x(i) = j, i \in A, j \in \{0, 1\} \). We denote this code with \( V_W \). In this way, each \( W \)-function \( f : A \to W \) has associated a binary block-code of length \( n \) and \( V_W = \{f_x, x \in W\} \).

**Proposition 3.14.** Let \( W \) be a \( W \)-algebra. With the above notations, relation \( A_s \subseteq A_r \) is equivalent with \( f_r \leq f_s \).

**Proof.** By straightforward calculation. □

**Proposition 3.15.** Let \( (W, \circ, 1) \) be a finite \( W \)-algebra. To algebra \( W \) corresponds a block-code \( V \) such that \( (W, \leq) \) is isomorphic to \( (V_W, \leq) \) as ordered sets.

**Proof.** By straightforward calculation. □

**Theorem 3.16.** Let \( A = (X, \oplus', \theta) \) be an \( MV \)-algebra and \( B = (X, \circ, 1) \) be the associated \( Wajsberg \) algebra. Therefore \( A \) and \( B \) are code equivalent, that means determine the same binary block-code.

**Proof.** With the above notations, let \( V_A \) be the code associated to the \( MV \)-algebra \( A \) and \( V_B \) be the code associated to the \( W \)-algebra \( B \). Let \( \varphi : V_A \to \)
$V_B, \varphi(f_r) = f'_r$, where $f_r$ is a cut function associated to MV-algebra and $f'_r$ is the corresponded cut function associated to W-algebra. The map $\varphi$ is injective, therefore bijective. Indeed, if $\varphi(f_r) = \varphi(f_s)$, it results $f'_r = f'_s$, therefore $r = s$. □

**Theorem 3.17.** Let $A = (X, *, \theta, 1)$ be a bounded commutative BCK-algebra, $B = (X, \oplus', \theta)$ be the associated MV-algebra and $C = (X, \circ, 1)$ be the associated Wajsberg algebra. Therefore $A, B$ and $C$ are code equivalent, that means determine the same binary block-code.

**Proof.** The result is obtained from Theorem 3.8 and Theorem 3.16. □

**Example 3.18.** Using algebras from Example 2.7, we can see that the code associated to BCK commutative bounded algebra $(X, *, \theta, 1)$ is

$V_1 = \{111111, 010111, 001011, 000101, 000011, 000001\}$, the code associated to MV-algebra $(X, \oplus', \theta), V_2$, is the same with $V_1$ and the code associated to Wajsberg algebra $(X, \circ, 1), V_3$, is also the same with $V_1$.

i) The BCK commutative bounded algebra and the associated code are given in the below tables:

| $\theta$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|----------|-----|-----|-----|-----|-----|
| $\theta$ | $\theta$ | $a$ | $a$ | $a$ | $a$ |
| $a$      | $a$ | $\theta$ | $\theta$ | $\theta$ | $\theta$ |
| $b$      | $b$ | $b$ | $b$ | $\theta$ | $\theta$ |
| $c$      | $c$ | $a$ | $c$ | $\theta$ | $a$ |
| $d$      | $d$ | $b$ | $a$ | $\theta$ | $\theta$ |
| $e$      | $e$ | $d$ | $c$ | $b$ | $a$ |

| $\theta$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|----------|-----|-----|-----|-----|-----|
| $\theta$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $a$      | $0$ | $1$ | $0$ | $1$ | $1$ |
| $b$      | $0$ | $0$ | $1$ | $0$ | $1$ |
| $c$      | $0$ | $0$ | $1$ | $0$ | $1$ |
| $d$      | $0$ | $0$ | $0$ | $0$ | $1$ |
| $e$      | $0$ | $0$ | $0$ | $0$ | $0$ |

ii) The MV-algebra algebra and the associated code are given in the below tables:

| $\oplus$ | $\theta$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|----------|----------|-----|-----|-----|-----|-----|
| $\theta$ | $\theta$ | $a$ | $b$ | $c$ | $d$ | $e$ |
| $a$      | $a$ | $c$ | $d$ | $c$ | $e$ |
| $b$      | $b$ | $d$ | $b$ | $e$ | $d$ |
| $c$      | $c$ | $c$ | $e$ | $c$ | $e$ |
| $d$      | $d$ | $e$ | $d$ | $e$ | $e$ |
| $e$      | $e$ | $e$ | $e$ | $e$ | $e$ |

| $\theta$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$      | $0$ | $0$ | $0$ | $0$ | $0$ |
| $b$      | $0$ | $0$ | $0$ | $0$ | $1$ |
| $c$      | $0$ | $0$ | $1$ | $0$ | $1$ |
| $d$      | $0$ | $0$ | $1$ | $0$ | $1$ |
| $e$      | $1$ | $1$ | $1$ | $1$ | $1$ |

iii) The Wajsberg algebra and the associated code are given in the below tables:
We remark that the attached code for each of these algebras is as a skeleton for these algebras( the same for all three algebras), on which we can insert different structures, as for example: a BCK commutative bounded algebra or an MV-algebra or a Wajsberg algebra. We consider the algebra \( A \in \{ \text{BCK, MV, Wajsberg} \} \), finite with \( n \) elements. The skeleton of such an algebra is a matrix of order \( n \) in which the elements of this matrix is black or white squares, with black square on the position \((i, j)\), if and only if \( x_i \leq x_j \) in \( A \), \( x_i, x_j \in A \). The associated skeleton of such a finite algebra \( A \in \{ \text{BCK, MV, Wajsberg} \} \) is nothing else than a representation of the associated order relation on the algebra \( A \).

As we can see in the below two tables, the skeleton is the same for the BCK bounded commutative algebra and for the attached Wajsberg algebra. For the attached MV algebra, the skeleton is the same but symmetric in respect to the Ox axis. The skeleton generates the same order relation on \( A \) as the attached binary block code on \( A, V_A \).

4. Some remarks regarding Wajsberg algebras

**Remark 4.1.** Let \( (X, \leq) \) be a finite totally ordered set, \( X = \{x_0, x_1, ..., x_n\} \), with \( x_0 \) the first element and \( x_n \) the last element. Using this order relation, we define the following multiplication " \( \circ \) " on \( X \):

\[
\begin{align*}
x_i \circ x_j &= 1, \text{ if } x_i \leq x_j; \\
x_i \circ x_j &= x_{n-i+j}, \text{ otherwise}; \\
x_0 = \theta, x_n = 1, x \circ \theta = \overline{x}. 
\end{align*}
\]
Therefore, \((X, \circ, 1)\) is a Wajsberg algebra. We remark that this is the only way to define a W-algebra structure on a finite totally ordered set such that the induced order relation on this algebra is given in (2.1). We also remark that \(\mathfrak{T}_i = x_{n-1}\). ([FRT; 84], Theorem 19).

**Definition 4.2.** Let \((W_1, \circ, 1)\) and \((W_2, \cdot', 1)\) be two finite Wajsberg algebras. We define on the Cartesian product of these algebras, \(W = W_1 \times W_2\), the following multiplication "\(\nabla\)",

\[
(x_1, x_2) \nabla (y_1, y_2) = (x_1 \circ y_1, x_2 \cdot y_2).
\]  

The complement of the element \((x_1, x_2)\) is \(\uparrow (x_1, x_2) = (\mathfrak{T}_1, x_2')\) and \(1 = (1, 1)\). Therefore, by straightforward calculation, we obtain that \((W, \nabla, \uparrow, 1)\) is also a Wajsberg algebra.

**Remark 4.3.** If \(x = (x_1, x_2), y = (y_1, y_2) \in W\), then the order relation corresponded to the algebra \((W, \nabla, \uparrow, 1)\) is given as follow:

\[
x \leq_W y \text{ if and only if } x_1 \leq_{W_1} y_1 \text{ and } x_2 \leq_{W_2} y_2.
\]  

**Definition 4.4.** Let \((W_1, \circ, 1)\) and \((W_2, \cdot', 1)\) be two Wajsberg algebras. A map \(f : W_1 \rightarrow W_2\) is a morphism of Wajsberg algebras if and only if:

1) \(f (0) = 0\);
2) \(f (x \circ y) = f (x) \cdot f (y)\);
3) \(f (\mathfrak{T}) = (f (x))'\).

**Proposition 4.5.** The algebras \(W_1 \times W_2\) and \(W_2 \times W_1\) are isomorphic.

**Proof.** Let \(f : W_1 \times W_2 \rightarrow W_2 \times W_1, f ((x_1, x_2)) = (x_2, x_1)\) is a bijective morphism. □

**Remark 4.6.** 1) ([HR; 99], Theorem 5.2, p. 43) An MV-algebra is finite if and only if it is isomorphic to a finite product of totally ordered MV algebras.

2) If \(M = (X, \oplus, \odot', \theta, 1)\) is a totally ordered MV-algebra, then the obtained Wajsberg algebra, \(W = (X, \circ', 1)\), is also totally ordered. The converse is also true. Indeed, since \(x \circ y = x' \oplus y\), we have that \(x \leq_M y\) if and only \(x \leq_W y\).

3) Using connections between MV-algebras and Wajsberg algebras, from the above, we have that a Wajsberg algebra is finite if and only if it is isomorphic to a finite product of totally ordered Wajsberg algebras.

4) If an MV-algebra or a Wajsberg algebra are finite with a prime number of elements, therefore these algebras are totally ordered.

5) If two Wajsberg algebras are isomorphic, then these algebras are also isomorphic as ordered sets.

**Definition 4.7.** Let \(n\) be a natural number, \(n \geq 4\). We consider the decomposition of the number \(n\) in factors:

\[
n = q_1 q_2 ... q_t, q_i \in \mathbb{N}, 1 < q_i < n, i \in \{1, 2, ..., t\}.
\]

This decomposition is not unique. We will count only one time the decompositions with the same terms but with other order of them in the product. We denote with \(\pi_n\) the number of all such decompositions.
From the above, we obtain the following Theorem.

**Theorem 4.8.** Let $n$ be a natural number, $n \geq 2$. There are exactly $\pi_n$ non-isomorphic (as ordered sets) Wajsberg algebras with $n$ elements. These algebras are obtained as a finite product of totally ordered Wajsberg algebras.$\square$

We denote these algebras with $(W^n_i, \nabla_i, 1_i, \leq^n_i)$, where $\leq^n_i$ is the corresponding order relation on $W^n_i$, $i \in \{1, 2, \ldots, \pi_n\}$.

**Remark 4.9.** We will denote with $(W^n_{ij}, \nabla_{ij}, 1_{ij}, \leq_{ij})$ the Wajsberg algebras isomorphic to $W^n_i$, as ordered sets, where $\leq^n_{ij}$ is the corresponding order relation on $W^n_{ij}$. Let $f^n_{ij}: W^n_i \to W^n_{ij}$ be such an isomorphism of ordered sets. The Wajsberg structure on the algebra $W^n_{ij}$ is given as follows. Let $x, y \in W^n_{ij}$ and $a, b \in W^n_i$ such that $f^n_{ij}(a) = x$ and $f^n_{ij}(b) = y$. We define

$$x \nabla_{ij} y = f^n_{ij}(a) \nabla_{ij} f^n_{ij}(b) \overset{def}{=} f^n_{ij}(a \nabla b).$$

We remark that this is the only way to define a Wajsberg algebra structure on $W^n_{ij}$ such that the induced order relation on this algebra is $\nabla_{ij}$. Algebras $W^n_i$ and $W^n_{ij}$ are isomorphic as ordered sets and are not always isomorphic as Wajsberg algebras.

From the above we obtain an algorithm to find all finite Wajsberg algebras of order $n$.

**Example 4.10.**

i) Let $n = 6$. We have that $n = 2 \cdot 3 = 3 \cdot 2$, therefore $\pi_6 = 1$.

ii) Let $n = 8$. We have that $n = 2 \cdot 4 = 4 \cdot 2 = 2 \cdot 2 \cdot 2$. Therefore, $\pi_8 = 2$.

iii) Let $n = 12$. We have that $n = 2 \cdot 2 \cdot 3 = 2 \cdot 3 \cdot 2 = 3 \cdot 2 \cdot 2 = 4 \cdot 3 = 3 \cdot 4 = 2 \cdot 6 = 6 \cdot 2$. Then, $\pi_{12} = 3$.

**Example 4.11.** There is only one type of partially ordered Wajsberg algebra with 4 elements, up to an isomorphism. Indeed, let $(W_1 = \{0, 1\}, o, 1)$ and $(W_2 = \{0, e\}, \cdot', e)$ be two finite totally ordered Wajsberg algebras. We consider $W_1 \times W_2 = \{(0, 0), (0, e), (1, 0), (1, e)\} = \{O, A, B, E\}$. On $W_1 \times W_2$ we obtain a Wajsberg algebra structure by defining the multiplication as in relation (4.2). We give this multiplication in the following table:

| $\nabla$ | O   | A   | B   | E   |
|---------|-----|-----|-----|-----|
| O       | E   | E   | E   | E   |
| A       | B   | E   | B   | E   |
| B       | A   | A   | E   | E   |
| E       | O   | A   | B   | E   |

**Example 4.12.** There is only one type of partially ordered Wajsberg algebra with 6 elements, up to an isomorphism. Indeed, $\pi_6 = 1$. Let $(W_1 = \{0, 1\}, o, 1)$ and $(W_2 = \{0, b, e\}, \cdot', e)$ be two finite totally ordered Wajsberg algebras. Using relation (4.1), on $W_2$ we have that $b' = b$. We consider $W_1 \times W_2 = \{(0, 0), (0, b), (0, e), (1, 0), (1, b), (1, e)\} =$
= \{O, A, B, C, D, E\}. On \( W_1 \times W_2 \) we obtain a Wajsberg algebra structure by defining the multiplication as in relation \((4.2)\). We give this multiplication in the following table:

\[
\begin{array}{c|cccccc}
\nabla_{11} & O & A & B & C & D & E \\
\hline
O & E & E & E & E & E & E \\
A & D & E & E & D & E & E \\
B & C & D & E & C & D & E \\
C & B & B & E & E & E & E \\
D & A & B & B & D & E & E \\
E & O & A & B & C & D & E \\
\end{array}
\]

(4.4.)

We remark that \( A \leq B, A \leq D, C \leq D \), and the other elements can’t be compared in the algebra \( W_{11} = W_{11} = (W_1 \times W_2, \nabla_{11}) \). We denote this order relation with \( \leq_{11} \).

If we consider the isomorphism

\[
f_{12}^6 : (W_1 \times W_2, \nabla_{11}) \rightarrow (W_1 \times W_2, \nabla_{12}),
\]

\[
f_{12}^6 (A) = A, f_{12}^6 (B) = C, f_{12}^6 (C) = B, f_{12}^6 (D) = D, f_{12}^6 (O) = O, f_{12}^6 (E) = E,
\]

we obtain on \( W_1 \times W_2 \) a new Wajsberg algebra structure, with the multiplication \( \nabla_{12} \) given in the below table:

\[
\begin{array}{c|cccccc}
\nabla_{12} & O & A & B & C & D & E \\
\hline
O & E & E & E & E & E & E \\
A & D & E & E & D & E & E \\
B & C & C & E & C & E & E \\
C & B & D & B & E & D & E \\
D & A & C & D & C & E & E \\
E & O & A & B & C & D & E \\
\end{array}
\]

(4.5.)

These two algebras, \( W_{11}^6 = (W_1 \times W_2, \nabla_{11}^6) \) and \( W_{12}^6 = (W_1 \times W_2, \nabla_{12}^6) \), are isomorphic. We remark that \( A \leq C, A \leq D, B \leq D \) and the other elements can’t be compared in the algebra \( W_{12}^6 \). We denote this order relation with \( \leq_{12}^6 \). This algebra is the Wajsberg algebra given in Example 3.18. The isomorphism \( f_{12}^6 \) is in the same time isomorphism of ordered sets and isomorphism of Wajsberg algebras.

If we consider the isomorphism

\[
f_{13}^6 : (W_1 \times W_2, \nabla_{11}^6) \rightarrow (W_1 \times W_2, \nabla_{13}^6),
\]

\[
f_{13}^6 (A) = B, f_{13}^6 (B) = D, f_{13}^6 (C) = C, f_{13}^6 (D) = A, f_{13}^6 (O) = O, f_{13}^6 (E) = E,
\]

we obtain on \( W_1 \times W_2 \) a new Wajsberg algebra structure, with the multiplication
\( \nabla_{13}^6 \) given in the below table:

| \( \nabla_{13}^6 \) | O | A | B | C | D | E |
|-----------------|---|---|---|---|---|---|
| O               | E | E | E | E | E | E |
| A               | B | E | D | A | D | E |
| B               | A | E | E | A | E | E |
| C               | D | E | D | E | D | E |
| D               | C | A | A | C | E | E |
| E               | O | A | B | C | D | E |

(4.6.)

In \( \mathcal{W}_{13}^6 = (W_1 \times W_2, \nabla_{13}^6) \), we have \( B \leq A, C \leq A, B \leq D \) and the other elements can’t be compared. We denote this order relation with \( \leq_{13}^6 \). The isomorphism \( f_{13}^6 \) is only isomorphism of ordered sets and is not isomorphism of Wajsberg algebras.

If we consider \( W_2 \times W_1 = \{(0,0), (0,1), (b,0), (b,1), (e,0), (e,1)\} = \{O, A, B, C, D, E\} \), on \( W_2 \times W_1 \) we obtain a Wajsberg algebra structure by defining the multiplication as in relation (4.2). We give this multiplication in the following table:

| \( \nabla_{14}^6 \) | O | A | B | C | D | E |
|-----------------|---|---|---|---|---|---|
| O               | E | E | E | E | E | E |
| A               | D | E | D | E | D | E |
| B               | C | C | E | E | E | E |
| C               | B | C | D | E | D | E |
| D               | A | A | C | C | E | E |
| E               | O | A | B | C | D | E |

(4.7.)

The algebras \( (W_1 \times W_2, \nabla_{11}^6) \) and \( (W_2 \times W_1, \nabla_{14}^6) \) are also isomorphic, by taking the map

\[
\begin{align*}
f_{14}^6 : (W_1 \times W_2, \nabla_{11}^6) &\rightarrow (W_2 \times W_1, \nabla_{14}^6), \\
f_{14}^6 (A) &= B, f_{14}^6 (B) = D, f_{14}^6 (C) = A, f_{14}^6 (D) = C, f_{14}^6 (O) = O, f_{14}^6 (E) = E.
\end{align*}
\]

In \( \mathcal{W}_{14}^6 = (W_2 \times W_1, \nabla_{14}^6) \), we have \( A \leq C, B \leq C, B \leq D \) and the other elements can’t be compared. We denote this order relation with \( \leq_{14}^6 \). The isomorphism \( f_{14}^6 \) is in the same time isomorphism of ordered sets and isomorphism of Wajsberg algebras.

**Example 4.13.** There is only two types of partially ordered Wajsberg algebras with 8 elements, up to an isomorphism. Indeed, \( \pi_8 = 2 \). Let \( (W_1 = \{0, a, b, e\}, \circ, 1) \) and \( (W_2 = \{0, 1\}, \cdot', e) \) be two finite totally ordered Wajsberg algebras. Using relation (4.1), on \( W_1 \) we have that \( \overline{b} = a \) and \( \overline{a} = b \).

We consider \( W_1 \times W_2 = \{(0,0), (0,1), (a,0), (a,1), (b,0), (b,1), (e,0), (e,1)\} = \{O, X, Y, Z, T, U, V, E\} \). On \( W_1 \times W_2 \) we obtain a Wajsberg algebra structure by defining the multiplication as in relation (4.2), namely \( \mathcal{W}_{11}^8 = (W_1 \times W_2, \nabla_{11}^8) \).
The multiplication $\nabla_{11}^{8}$ is given in the following table:

| $\nabla_{11}^{8}$ | $O$ | $X$ | $Y$ | $Z$ | $T$ | $U$ | $V$ | $E$ |
|-------------------|-----|-----|-----|-----|-----|-----|-----|-----|
| $O$               | $E$ | $E$ | $E$ | $E$ | $E$ | $E$ | $E$ | $E$ |
| $X$               | $V$ | $E$ | $V$ | $E$ | $V$ | $E$ | $V$ | $E$ |
| $Y$               | $U$ | $U$ | $E$ | $E$ | $E$ | $E$ | $E$ | $E$ |
| $Z$               | $T$ | $U$ | $V$ | $E$ | $V$ | $E$ | $V$ | $E$ |
| $T$               | $Z$ | $Z$ | $U$ | $E$ | $E$ | $E$ | $E$ | $E$ |
| $U$               | $Y$ | $Z$ | $T$ | $U$ | $T$ | $E$ | $V$ | $E$ |
| $V$               | $X$ | $X$ | $Z$ | $Z$ | $U$ | $U$ | $E$ | $E$ |
| $E$               | $O$ | $X$ | $Y$ | $Z$ | $T$ | $U$ | $V$ | $E$ |

(4.8.)

In $W_{11}^{8}$ we have that $O \leq X \leq Z \leq U \leq V \leq E$, $O \leq Y \leq Z \leq U \leq E$, $O \leq Y \leq T \leq U \leq E$ and the other elements can’t be compared in this algebra. We denote this order relation with $\leq_{11}^{8}$.

Now, we consider $W_{2} \times W_{1} = \{(0,0),(0,a),(0,b),(0,e),(1,0),(1,a),(1,b),(1,e)\} = \{O,X,Y,Z,T,U,V,E\}$. On $W_{2} \times W_{1}$ we obtain a Wajsberg algebra structure by defining the multiplication as in relation (4,2), namely $W_{12}^{8} = (W_{2} \times W_{1}, \nabla_{12}^{8})$.

The multiplication $\nabla_{12}^{8}$ is given in the following table:

| $\nabla_{12}^{8}$ | $O$ | $X$ | $Y$ | $Z$ | $T$ | $U$ | $V$ | $E$ |
|-------------------|-----|-----|-----|-----|-----|-----|-----|-----|
| $O$               | $E$ | $E$ | $E$ | $E$ | $E$ | $E$ | $E$ | $E$ |
| $X$               | $V$ | $E$ | $E$ | $E$ | $V$ | $E$ | $E$ | $E$ |
| $Y$               | $U$ | $V$ | $E$ | $U$ | $V$ | $E$ | $E$ | $E$ |
| $Z$               | $T$ | $U$ | $V$ | $E$ | $T$ | $U$ | $V$ | $E$ |
| $T$               | $Z$ | $X$ | $Z$ | $Z$ | $E$ | $E$ | $E$ | $E$ |
| $U$               | $Y$ | $Z$ | $Z$ | $V$ | $E$ | $E$ | $E$ | $E$ |
| $V$               | $X$ | $Y$ | $Z$ | $U$ | $V$ | $E$ | $E$ | $E$ |
| $E$               | $O$ | $X$ | $Y$ | $Z$ | $T$ | $U$ | $V$ | $E$ |

(4.8.)

We have that $O \leq X \leq Y \leq Z \leq U \leq V \leq E$, $O \leq X \leq Y \leq V \leq E$, $O \leq T \leq U \leq V \leq E$. These two structures, $W_{11}^{8}$ and $W_{12}^{8}$, are isomorphic. The morphism is

$$f_{12}^{8} : (W_{1} \times W_{2}, \nabla_{11}^{8}) \rightarrow (W_{2} \times W_{1}, \nabla_{12}^{8}),$$

$$f_{12}^{8}(X) = T, f_{12}^{8}(Y) = X, f_{12}^{8}(Z) = U, f_{12}^{8}(T) = Y, f_{12}^{8}(U) = V, f_{12}^{8}(V) = Z, f_{12}^{8}(O) = O, f_{12}^{8}(E) = E.$$

The isomorphism $f_{12}^{8}$ is in the same time isomorphism of ordered sets and isomorphism of Wajsberg algebras.

If we take $W_{2} \times W_{2} \times W_{2} = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0), (1,0,1),(1,1,0),(1,1,1)\} = \{O,X,Y,Z,T,U,V,E\}$, on $W_{2} \times W_{2} \times W_{2}$ we obtain a Wajsberg algebra structure by defining the multiplication as in relation (4,2), namely $W_{21}^{8} = (W_{2} \times W_{2} \times W_{2}, \nabla_{21}^{8})$. The multiplication $\nabla_{21}^{8}$ is given in the
The following table:

| $\Sigma_{23}^8$ | $O$ | $X$ | $Y$ | $Z$ | $T$ | $U$ | $V$ | $E$ |
|-----------------|-----|-----|-----|-----|-----|-----|-----|-----|
| $O$             | $E$ | $E$ | $E$ | $E$ | $E$ | $E$ | $E$ | $E$ |
| $X$             | $V$ | $E$ | $V$ | $E$ | $V$ | $E$ | $V$ | $E$ |
| $Y$             | $U$ | $U$ | $E$ | $E$ | $U$ | $U$ | $E$ | $E$ |
| $Z$             | $T$ | $U$ | $V$ | $E$ | $T$ | $U$ | $V$ | $E$ |
| $T$             | $Z$ | $Z$ | $Z$ | $E$ | $E$ | $E$ | $E$ | $E$ |
| $U$             | $Y$ | $Z$ | $Y$ | $Z$ | $V$ | $E$ | $V$ | $E$ |
| $V$             | $X$ | $X$ | $Z$ | $Z$ | $U$ | $U$ | $E$ | $E$ |
| $E$             | $O$ | $X$ | $Y$ | $Z$ | $T$ | $U$ | $V$ | $E$ |

We have that $X \leq Z, X \leq U, Y \leq Z, T \leq V, Y \leq V, T \leq U$. These two structures, $\Sigma_{21}^8$ and $\Sigma_{21}^8$, are not isomorphic as ordered sets, therefore are not isomorphic as Wajsberg algebras.

**Remark 4.14.**

1) There is only one type of partially ordered Wajsberg algebra with 9 elements, up to an isomorphism as ordered sets.

2) There is only one type of partially ordered Wajsberg algebra with 10 elements, up to an isomorphism, up to an isomorphism as ordered sets.

3) There is only three types of partially ordered Wajsberg algebra with 12 elements, up to an isomorphism, up to an isomorphism as ordered sets.

### 5. Special algebras arising from block codes

In [FL; 15], was developed an algorithm which provide conditions to attach a BCK algebra to a given block code. In the following, we will use those ideas to obtain a similar algorithm in the case of MV-algebras and Wajsberg algebras. The difference is that, in the first case, the BCK algebra arising from a block code is a non-commutative, a non-implicative, but a positive implicative BCK algebra ([FL; 17]). In the second case, of MV-algebras and Wajsberg algebras, we must obtain from a given block-code a BCK commutative bounded algebra. Our idea is to obtain a Wajsberg finite algebra associated to a given block code and from here the desired MV-algebra and the desired bounded commutative BCK-algebra.

Let $V$ be a binary block-code with $n + 1$ codewords of length $n + 1$, lexicographically ordered, $V = \{w_0, w_1, ..., w_n\}$ and $w_x = x_1x_2...x_nx_{n+1} \in V$, $w_y = y_1y_2...y_ny_{n+1} \in V$ be two codewords. On $V$ we can define the following partial order relation:

$$w_x \leq w_y \text{ if and only if } y_i \leq x_i, i \in \{0, 1, 2, ..., n\}. \quad (5.1.)$$

We consider the matrix $M_V = (m_{i,j})_{i,j \in \{1,2,...,n+1\}} \in \mathcal{M}_{n+1}(\{0,1\})$ with the rows consisting of the codewords of $V$. This matrix is called the matrix associated to the block-code $V$. 

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Theorem 5.1. With the above notations, if the matrix $M_V$ has the first line and the last column of the form $11...1_{n+1}$, the first column of the form $100...0_1$, $m_{ii} = 1$, for all $i \in \{1, 2, ..., n+1\}$, and if the order relation given by (5.1) coincide with one of the order relations $\leq_{ij}^{n+1}$ given by the Remark 4.9, then there are a set $A$ with $n+1$ elements, a Wajsberg algebra $W_{ij}^{n+1}$ and a W-function $f : A \rightarrow W_{ij}^{n+1}$ such that $f$ determines $V$.

Proof. We consider on $V$ the lexicographic order, denoted by $\leq_{lex}$. It results that $(V, \leq_{lex})$ is a totally ordered set. Let $V = \{w_0, w_1, ..., w_n\}$, with $w_n \geq_{lex} w_{n-1} \geq_{lex} ... \geq_{lex} w_0$. We denote $w_n = 11...1_1$ and $w_0 = 00...0_1$. We remark that $w_0 = 1$ is the "one" element and $w_n = \theta$ is the "zero" element in $(V, \leq)$, considered with order relation (5.1). If this order relation coincides with one of the order relation $\leq_{ij}^{n+1}$, given in Remark 4.9, it results that on $(V, \leq)$ we can obtain a Wajsberg algebra structure, which is isomorphic to a Wajsberg algebra $W_{ij}^{n+1}$, with attached order relation $\leq_{ij}^{n+1}$. If we consider $A = V$ and the identity map $f : A \rightarrow V$, $f(w) = w$ as a W-function, the decomposition of $f$ provides a family of maps $V_{W_{ij}^{n+1}} = \{f_r : A \rightarrow \{0,1\} / \text{ } f_r(x) = 1 \text{ if and only if } r \ast f(x) = w_0, \forall x \in A, r \in X\}$. This family is the binary block-code $V$ relative to the order relation $\leq$. □

The above Theorem extends to W-algebras results obtained in Theorem 3.2 from [FL: 15].

Proposition 5.2. Let $A = (a_{i,j})_{i\in\{1,2,...,n\}, j\in\{1,2,...,m\}} \in M_{n,m}(\{0,1\})$ be a matrix. Starting from this matrix, we can find a matrix $B = (b_{i,j})_{i,j\in\{1,2,...,q\}} \in M_{q}(\{0,1\})$, $q \geq \max\{m,n\}$, such that $B$ has the first line and the last column of the form $11...1_q$, the last line of the form $00...0_1$, the first column of the form $100...0_1$, the order relation given by (5.1) coincide with one of the order relations $\leq_{ij}^{n+1}$ given by the Remark 4.9, then there are a set $A$ with $n+1$ elements, a Wajsberg algebra $W_{ij}^{n+1}$ and a W-function $f : A \rightarrow W_{ij}^{n+1}$ such that $f$ determines $V$.

Proof. We consider on $V$ the lexicographic order, denoted by $\leq_{lex}$. It results that $(V, \leq_{lex})$ is a totally ordered set. Let $V = \{w_0, w_1, ..., w_n\}$, with $w_n \geq_{lex} w_{n-1} \geq_{lex} ... \geq_{lex} w_0$. We denote $w_n = 11...1_1$ and $w_0 = 00...0_1$. We remark that $w_0 = 1$ is the "one" element and $w_n = \theta$ is the "zero" element in $(V, \leq)$, considered with order relation (5.1). If this order relation coincides with one of the order relation $\leq_{ij}^{n+1}$, given in Remark 4.9, it results that on $(V, \leq)$ we can obtain a Wajsberg algebra structure, which is isomorphic to a Wajsberg algebra $W_{ij}^{n+1}$, with attached order relation $\leq_{ij}^{n+1}$. If we consider $A = V$ and the identity map $f : A \rightarrow V$, $f(w) = w$ as a W-function, the decomposition of $f$ provides a family of maps $V_{W_{ij}^{n+1}} = \{f_r : A \rightarrow \{0,1\} / \text{ } f_r(x) = 1 \text{ if and only if } r \ast f(x) = w_0, \forall x \in A, r \in X\}$. This family is the binary block-code $V$ relative to the order relation $\leq$. □

Theorem 5.3. With the above notations, let $V$ be a binary block-code with $n$ codewords of length $m, n \neq m$. Let $q$ be a natural number, $q \geq \max\{m,n\}$, and $B = (b_{i,j})_{i,j\in\{1,2,...,q\}} \in M_{q}(\{0,1\})$ be a matrix such that the matrix $M_V$ is a submatrix of the matrix $B$. If $B$ is the matrix attached to a code $C$ which satisfies the conditions from Theorem 5.1, therefore there is a set $A$ with $m$ elements, a Wajsberg algebra $W_{ij}^{n+1}$ and a W-function $f : A \rightarrow W_{ij}^{n+1}$ such that the obtained block-code $C_{nm}$ contains the block-code $V$ as a subset.

Proof. Let $V$ be a binary block-code, $V = \{w_1, w_2, ..., w_n\}$, with codewords of length $m$. Let $M \in M_{n,m}(\{0,1\})$ be its associated matrix. Using Proposition 5.2, we can extend the matrix $M$ to a square matrix $M' \in M_{q}(\{0,1\})$ and we
can apply Theorem 5.1 for the matrix $M'$. Assuming that the initial columns of the matrix $M$ have in the new matrix $M'$ positions $i_{j_1}, i_{j_2}, ..., i_{j_m} \in \{1, 2, ..., q\}$, let $A = \{x_{j_1}, x_{j_2}, ..., x_{j_m} \} \subseteq C$. The W-function $f : A \rightarrow C, f(x_{j_i}) = x_{j_i}, i \in \{1, 2, ..., m\}$, determines the binary block-code $C_{nm}$ such that $V \subseteq C_{nm}$. □

The above Theorem extends to W-algebras results obtained in Theorem 3.9 from [FL; 15].

**Remark 5.4.** In the above theorem, the associated Wajsberg algebra is not unique, as we can see in the Example 6.10.

**Theorem 5.5.** If a block-code $V$ satisfies the conditions from Theorem 5.1, there is an MV-algebra $X$ and a commutative bounded BCK-algebra $Y$ associated to this code.

**Proof.** We use Theorem 5.1, Remark 2.4 and Remark 2.6.

6. Examples

**Example 6.1.** We consider the code $V_1 = \{111111, 011011, 001001, 000111, 000011, 000001\} = \{\theta, a, b, c, d, e\}$, with elements lexicographically ordered. The associated matrix is

|    | $\theta$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|----|----------|-----|-----|-----|-----|-----|
| $\theta$ | 1 1 1 1 1 1 |
| $a$    | 0 1 1 0 1 1 |
| $b$    | 0 0 1 0 0 1 |
| $c$    | 0 0 0 1 1 1 |
| $d$    | 0 0 0 0 1 1 |
| $e$    | 0 0 0 0 0 1 |

Using order given by the relation (5.1), we have that $a \preceq b, c \preceq d, a \preceq d$ and the other elements can’t be compared. Therefore this order relation is $\leq_{11}^6$ and the associated Wajsberg algebra is algebra $W_{11}^{6}$, given by relation (4.4).

**Example 6.2.** We consider the code $V_2 = \{111111, 010111, 001001, 000101, 000011, 000001\} = \{\theta, a, b, c, d, e\}$, with elements lexicographically ordered. The associated matrix is

|    | $\theta$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|----|----------|-----|-----|-----|-----|-----|
| $\theta$ | 1 1 1 1 1 1 |
| $a$    | 0 1 0 1 1 1 |
| $b$    | 0 0 1 0 1 1 |
| $c$    | 0 0 0 1 0 1 |
| $d$    | 0 0 0 0 1 1 |
| $e$    | 0 0 0 0 0 1 |
Using order given by the relation (5.1), we have that $a \preceq c$, $a \preceq d$, $b \preceq d$ and the other elements can’t be compared. Therefore this order relation is $\leq_{12}$ and the associated Wajsberg algebra is $\mathcal{W}_{12}$, given by relation (4.5). Algebra $\mathcal{W}_{12}$ is the Wajsberg algebra given in the Example 3.18. From here, we can see the MV-algebra and the BCK commutative bounded algebra associated to the block-code $V_2$.

**Example 6.3.** We consider the code

$V_3 = \{111111, 010001, 011011, 010101, 000011, 000001\} = \{\theta, a, b, c, d, e\}$, with elements lexicographically ordered. The associated matrix is

|     | $\theta$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|-----|---------|-----|-----|-----|-----|-----|
| $\theta$ | 1 1 1 1 1 1 |
| $a$ | 0 1 0 0 0 1 |
| $b$ | 0 1 1 0 1 1 |
| $c$ | 0 1 0 1 0 1 |
| $d$ | 0 0 0 0 1 1 |
| $e$ | 0 0 0 0 0 1 |

Using order given by the relation (5.1), we have that $b \preceq a$, $c \preceq a$, $b \preceq d$ and the other elements can’t be compared. Therefore this order relation is $\leq_{13}$ and the associated Wajsberg algebra is $\mathcal{W}_{13}$, given by relation (4.6).

The skeleton of this algebra is

|     | $\theta$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|-----|---------|-----|-----|-----|-----|-----|
| $\theta$ | ■ ■ ■ ■ ■ ■ |
| $a$ | ■ ■ ■ ■ |
| $b$ | ■ ■ ■ ■ |
| $c$ | ■ ■ ■ |
| $d$ | ■ ■ |
| $e$ | ■ |

**Example 6.4.** We consider the code

$V_4 = \{111111, 010101, 001111, 000101, 000011, 000001\} = \{\theta, a, b, c, d, e\}$, with elements lexicographically ordered. Using order given by the relation (5.1), we have that $a \preceq c$, $b \preceq c$, $b \preceq d$ and the other elements can’t be compared. Therefore this order relation is $\leq_{14}$ and the associated Wajsberg algebra is $\mathcal{W}_{14}$, given by relation (4.7).

The skeleton of this algebra is

|     | $\theta$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|-----|---------|-----|-----|-----|-----|-----|
| $\theta$ | ■ ■ ■ ■ ■ ■ |
| $a$ | ■ ■ ■ ■ |
| $b$ | ■ ■ ■ ■ |
| $c$ | ■ ■ ■ |
| $d$ | ■ ■ |
| $e$ | ■ |

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**Example 6.5.** We consider the code $V_5 = \{011, 101, 100, 001, 000\}$. Using Theorem 5.3, we have the matrix $B$

$$B = \begin{bmatrix} \theta & a & b & c & d & e \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

the set $A = \{b, c, d\}$, the Wajsberg algebra $W_{12}^6$ and the W-function $f : A \rightarrow W_{12}^6$, $f(b) = b$, $f(c) = c$, $f(d) = d$. We have the code $C_{53} = \{111, 011, 101, 010, 001, 000\}$. The code $V_5 = \{011, 101, 100, 001, 000\}$ is a subcode of the code $C_{53}$.

**Example 6.6.** We consider the code $V_6 = \{111111, 011101, 001101, 000111, 000011, 000001\} = \{\theta, a, b, c, d, e\}$, with elements lexicographically ordered. The associated matrix is

$$B = \begin{bmatrix} \theta & a & b & c & d & e \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and the skeleton is

$$\begin{array}{cccccc} \theta & a & b & c & d & e \\ \hline \theta & \text{■} & \text{■} & \text{■} & \text{■} & \text{■} \\ a & \text{□} & \text{□} & \text{□} & \text{□} & \text{□} \\ b & \text{□} & \text{□} & \text{□} & \text{□} & \text{□} \\ c & \text{□} & \text{□} & \text{□} & \text{□} & \text{□} \\ d & \text{□} & \text{□} & \text{□} & \text{□} & \text{□} \\ e & \text{□} & \text{□} & \text{□} & \text{□} & \text{□} \end{array}$$

Using order given by the relation (5.1), we have that $a \preceq b, c \preceq d$ and the other elements can’t be compared. If we consider the binary relation given by the skeleton: $i \preceq_s j$ if and only if in the position $(i, j)$ we have a black square, this relation is not an order relation. Indeed, we do not have the transitivity: we have $a \preceq_s b, b \preceq_s c, c \preceq_s d$ but $a$ and $d$ can’t be compared, therefore the sets $(V_6, \preceq_s)$ and $(V_6, \preceq)$ are not isomorphic as ordered sets, as in Proposition 3.15. Even if the associated matrix to this code is on the form asked in Theorem 5.1, to this code we can’t associate a Wajsberg algebra, since the order relation $\preceq$ is not on the form $\preceq_{ij}$, given by Theorem 4.8 and Remark 4.9.

We must remark that, from [FL; 15], Theorem 3.2, to the code $V_6$ we can
attached a BCK-algebra, namely

\[
B = \begin{array}{cccccc}
\theta & a & b & c & d & e \\
\theta & \theta & \theta & \theta & \theta & \theta \\
a & a & \theta & \theta & a & a \\
b & b & b & \theta & b & b \\
c & c & c & c & \theta & \theta \\
d & d & d & d & \theta & \theta \\
e & e & e & e & e & \theta \\
\end{array}
\]

but the block code associated to the BCK-algebra \(B\) is

\[
V_B = \{111111, 011001, 001001, 000111, 000011, 000001\}
\]
and it is different from \(V_6\). Therefore, the above mentioned Theorem needs to be understood as follows:

i) In some circumstances, we can associate a BCK-algebra to a binary block code.

ii) The BCK-algebra \(B\) associated to a block code \(V\) generates the same code \(V\) if and only if the order relation \(\leq\), generated by the skeleton associated to the code \(V\), is the same with the order relation \(\preceq\), defined on the obtained BCK-algebra \(B\).

Example 6.7. We consider the code \(V_7 = \{1111111, 0101010, 0011111, 0001010, 0000111, 0000011, 0000001\} = \{O, X, Y, Z, T, U, V, E\}\), with elements lexicographically ordered. The associated matrix is

\[
\begin{array}{cccccccc}
& O & X & Y & Z & T & U & V & E \\
O & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
X & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
Y & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
Z & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
T & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
U & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
V & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
E & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\]

Using the order given by the relation (5.1), we have that \(O \preceq X \preceq Z \preceq U \preceq E\), \(O \preceq Y \preceq T \preceq V \preceq E\), \(O \preceq Y \preceq Z \preceq U \preceq E\), \(O \preceq Y \preceq T \preceq U \preceq E\) and the other elements can’t be compared. Therefore this order relation is \(\preceq_{11}\) and the associated Wajsberg algebra is \(W_{11}\), given by relation (4.7).

Example 6.8. We consider the code \(V_8 = \{1111111, 0110101, 0011111, 0001010, 0000111, 0000011, 0000001\} = \{O, X, Y, Z, T, U, V, E\}\), with elements lexicographically ordered. Using order given by the relation (5.1), we have that \(O \preceq X \preceq Y \preceq Z \preceq T \preceq U \preceq V \preceq E\), \(O \preceq X \preceq U \preceq V \preceq E\), \(O \preceq T \preceq U \preceq V \preceq E\), and the other elements can’t be compared. Therefore this order relation is \(\preceq_{12}\) and the associated Wajsberg algebra is \(W_{12}\), given by relation (4.8).
Example 6.9. We consider the code \( V_9 = \{11111111, 01010101, 00100011, 00010001, 00001111, 00000101, 00000011, 00000001\} = \{O, X, Y, Z, T, U, V, E\} \), with elements lexicographically ordered. Using order given by the relation (5.1), we have that \( O \leq X \leq Y \leq Z \leq E \), \( O \leq X \leq Y \leq V \leq E \), \( O \leq X \leq U \leq V \leq E \), and the other elements can’t be compared. Therefore this order relation is \( \leq^{8}_{21} \) and the associated Wajsberg algebra is \( \mathcal{W}^{8}_{21} \), given by the relation (4.9).

Example 6.10. We consider the code \( V_{10} = \{011, 101\} \). Using Theorem 5.3, we have the matrix

\[
\begin{array}{c|cccccccc}
& O & X & Y & Z & T & U & V & E \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\end{array}
\]

the set \( A = \{U, V, E\} \), the Wajsberg algebra \( \mathcal{W}^{8}_{11} \) and the W-function \( f : A \rightarrow \mathcal{W}^{8}_{11}, f(U) = U, f(V) = V, f(E) = E \). We have the code \( C_{23} = \{111, 101, 011, 111, 001\} \). The code \( V_{10} = \{011, 101\} \) is a subcode of the code \( C_{23} \). We remark that the associated Wajsberg is not unique. Indeed, we can consider the set \( A = \{b, c, d\} \), the Wajsberg algebra \( \mathcal{W}^{6}_{12} \) and the code \( C'_{23} = \{111, 011, 101, 010, 001, 000\} \), as in Example 6.5. The code \( V_{10} \) is a subcode of the code \( C'_{23} \).

Conclusions

In this paper, we presented some connections between BCK-commutative bounded algebras, MV-algebras, Wajsberg algebras and binary block codes. By studying these connections were identified two types of approaches. First of them is if these algebras can generate good codes? At a first glance, the answer can be no. Indeed, the codes generated by the BCK-commutative bounded algebras, MV-algebras and Wajsberg algebras have, in general, the minimum Hamming distance equal with 1, that means are not so good codes. Therefore, we turned our attention to the second approach: if we use the attached codes, we can find new and interesting properties for these algebras? The answer is yes and that is exactly what we did in this paper.

We found an algorithm to generate all finite partially ordered Wajsberg algebras and, from here, all finite partially ordered MV algebras and all finite partially ordered BCK commutative bounded algebras. In Section 6, we gave examples of the above mentioned algebras associated to a binary block codes.
We remarked that to a two isomorphic algebras correspond different codes, but we can find different algebras which can generate the same code.

Even if, for the moment, the answer is no, we will not give up the first approach and we hope that in a future research to achieve important results in this direction.

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