Functions of Noncommuting Operators in an Asymptotic Problem for a 2D Wave Equation with Variable Velocity and Localized Right-Hand Side

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Dedicated to Vladimir Rabinovich

Abstract. In the present paper, we use the theory of functions of noncommuting operators, also known as noncommutative analysis (which can be viewed as a far-reaching generalization of pseudodifferential operator calculus), to solve an asymptotic problem for a partial differential equation and show how, starting from general constructions and operator formulas that seem to be rather abstract from the viewpoint of differential equations, one can end up with very specific, easy-to-evaluate expressions for the solution, useful, e.g., in the tsunami wave problem.

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1. Introduction

In the present paper, we use the theory of functions of noncommuting operators [1,3], aka noncommutative analysis (which can be viewed as a far-reaching generalization of pseudodifferential operator calculus), to solve an asymptotic problem

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for a partial differential equation and show how, starting from general constructions and operator formulas that seem to be rather abstract from the viewpoint of differential equations, one can end up with very specific, easy-to-evaluate expressions for the solution, useful, e.g., in the tsunami wave problem.

We consider the Cauchy problem with zero initial data for a 2D wave equation with variable velocity and with right-hand side localized near the origin in space and decaying in time. One physical interpretation of this problem is that it describes, in the linear approximation, the propagation of tsunami waves generated by local vertical displacements of the ocean bottom (see [4,9,17,19] and also [11,16] and the bibliography therein). Normally, the diameter of the region where these displacements occur (some tens to a hundred of kilometers) is much smaller than the distance traveled by the waves (thousands of kilometers), and their ratio, \( \mu \), can serve as a small parameter. Accordingly, we are interested in the asymptotics of the solution as \( \mu \to 0 \). In the simplest piston model of tsunami generation, the bottom displacement occurs instantaneously at \( t = 0 \). This corresponds to a right-hand side of the form \( \delta'(t)v(x) \), where \( \delta(t) \) is the Dirac delta function, and the problem is immediately equivalent, via Duhamel’s principle, to the Cauchy problem for the homogeneous wave equation with initial data \( v(x) \) for the unknown function itself and zero initial data for its \( t \)-derivative. Fairly explicit asymptotic solution formulas suitable for easy implementation in Wolfram Mathematica [27] were constructed and analyzed for the latter problem in [11,16] on the basis of a generalization of Maslov’s canonical operator [1,20]. Now assume we wish to take into account the fact that the ocean bottom displacement evolves in time rather than happens instantaneously. Then it is natural to consider a right-hand side of the form \( g'(t)v(x) \), where \( g(t) \) is some smooth approximation to the delta function. An analysis shows that the solution can be represented as the sum of two parts, a propagating part, which travels along the characteristics, and a transient part, which is localized in the vicinity of the origin and decays in time. The propagating part can further be represented as the solution of the Cauchy problem for the homogeneous wave equation with initial data obtained from \( v(x) \) by application of certain functions \( f(L) \) of the spatial part \( L \) of the wave operator, where the corresponding symbols \( f(\xi) \) are given by simple formulas expressing them via the Fourier transform of \( g(t) \). These initial data, also localized near the origin, will be referred to as the equivalent source functions. The transient part of the solution is given by a formula similar to those for the equivalent source functions with the only difference that the function \( f(\xi) \) additionally depends on time as a parameter. The transient part is apparently not so important in tsunami wave analysis, but nevertheless it might be useful from the viewpoint of satellite registration of tsunami waves [17,19]. Since, as was mentioned above, the asymptotic formulas for the solution of the Cauchy problem with localized initial data for the homogeneous wave equation are already known from [11,16], we see that the only remaining thing is to compute \( f(L)v \) for all these functions \( f(\xi) \). It is here that noncommutative analysis comes fully into play. Note that \( L \) is an operator with variable coefficients, and so computing the function \( f(L) \) efficiently may prove quite a challenging task. However, all we actually need is the asymptotics of \( f(L)v \), and methods of noncommutative analysis permit one to prove that \( f(L)v = f(L_0)v + \text{an asymptotically small remainder} \), where \( L_0 \) is obtained from \( L \) by freezing the coefficients at the origin. Now computing \( f(L_0)v \)
is a breeze, because $f(L_0)$ is conjugate by the Fourier transform to the operator of multiplication by the function $f(\sigma_{L_0}(p))$, where $\sigma_{L_0}(p)$ is the symbol of $L_0$.

The one-dimensional counterpart of the problem studied in the present paper was considered in [21]. In the two-dimensional case, the results were announced in [22], where the proofs were partly only sketched and partly absent altogether. Here we develop and refine these results and give complete proofs. Finally, note that we deal with the setting in which the wave propagation velocity is assumed to vanish nowhere. The case in which it vanishes (as it happens on the coastline in the tsunami run-up problem) is much more complicated. The asymptotics of solutions of such degenerate problems in some special cases was considered in the spirit of the approach of [11–16] in [23–25] (see also references therein); in the present paper, we restrict ourselves to wave propagation in open ocean.

The outline of the paper is as follows. In Sec. 2, we give a detailed statement of the mathematical problem and write out well-known formulas expressing the solution in operator form. Using these formulas, we split the solution into the sum of the propagating and transient parts. Section 3 presents simple formulas for the asymptotics of the solution. The proofs of the theorems stated in this section depend on the results presented in Sec. 4, which is the most important part of the paper and where the asymptotics of the equivalent source functions and the transient part of the solution are computed with the use of the noncommutative analysis machinery. Finally, Sec. 5 provides two simple examples; all computations and visualizations in these examples have been done with Wolfram Mathematica.

2. Exact solution

2.1. Statement of the problem. Consider the Cauchy problem for the wave equation

\[
\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial}{\partial x_1} \left( c^2(x) \frac{\partial \eta}{\partial x_1} \right) - \frac{\partial}{\partial x_2} \left( c^2(x) \frac{\partial \eta}{\partial x_2} \right) = Q, \quad t \geq 0,
\]

with the initial conditions

\[
\eta|_{t=0} = 0, \quad \eta_t|_{t=0} = 0,
\]

where $x = (x_1, x_2) \in \mathbb{R}^2$, $\eta = \eta(x, t)$ is the unknown function, $c(x)$ is an everywhere positive smooth function stabilizing at infinity\(^1\) and the right-hand side $Q = Q(x, t)$ depends on two parameters $\lambda, \mu > 0$ and has the form

\[
Q(x, t) = \lambda^2 g_0(\lambda t)V \left( \frac{x}{\mu} \right)
\]

with some smooth real functions $V(y)$, $y \in \mathbb{R}^2$, and $g_0(\tau)$, $\tau \in [0, \infty)$, such that

\[
|V^{(\alpha)}(y)| \leq C_\alpha (1 + |y|)^{-|\alpha|-\zeta}, \quad |\alpha| = 0, 1, 2, \ldots ,
\]

\[
g_0(0) = 0, \quad \int_0^\infty g_0(\tau) \, d\tau = 1, \quad \left| g_0^{(k)}(\tau) \right| \leq C_k e^{-\nu \tau}, \quad k = 0, 1, 2, \ldots ,
\]

for some $\zeta > 1$, $\nu > 0$, and positive constants $C_\alpha$ and $C_k$.

**Remark 2.1.** One can also consider the case in which $g_0(\tau)$ decays as some (sufficiently large) negative power of $\tau$ as $\tau \to \infty$. In this case, the estimates are

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\(^1\)That is, $c(x) = \text{const} > 0$ for sufficiently large $|x|$. 

somewhat more awkward, and we restrict ourselves to the case of the physically natural exponential decay (2.5) for the sake of clarity.

Our aim is to find the asymptotics as $\mu \to 0$ of the solution of problem (2.1) on an arbitrary finite time interval uniformly with respect to $\lambda$ in the region

$$\lambda \mu > \text{const} > 0.$$  

This will be done in Secs. 3 and 4 and in the present section we deal with the exact solution of the problem.

2.2. Physical interpretation and examples of right-hand sides. First, speaking in terms of the physical interpretation given in the introduction, let us explain the meaning of the parameters $\lambda$ and $\mu$ and condition (2.6). The right-hand side $Q(x, t)$ describes the time evolution (the factor $\lambda^2 g_0'(\lambda t)$) and the spatial shape (the factor $V(x/\mu)$) of the perturbation (the tsunami source). In view of (2.5), $\lambda$ characterizes the decay rate of the perturbation, so that $1/\lambda \sim t_0$, where $t_0$ is the mean lifetime of the perturbation. The small parameter $\mu$ characterizes the source size $r_0$, $\mu \sim r_0$. We see that the product $\lambda \mu = r_0/t_0$ has the dimension of velocity and rewrite condition (2.6) in the form

$$\frac{c_0}{\lambda \mu} \equiv \frac{c_0 t_0}{r_0} \leq \omega_0,$$

where $c_0 = c(0)$, the wave propagation velocity at the origin, is taken to represent the typical wave propagation velocity in the problem and $\omega_0$ is some dimensionless constant. This has a very clear meaning: the waves excited by the perturbation cannot travel too far before the perturbation dies out; they only cover a distance ($c_0 t_0$) of the same order of magnitude as the diameter $r_0$ of the perturbation region.

We introduce the ratio

$$\omega = \frac{c_0}{\lambda \mu},$$

so that condition (2.7) (and hence (2.6)) becomes

$$\omega < \omega_0.$$  

Mathematically, condition (2.9) means that the parameter $\lambda$ is large (at least of the order of $\mu^{-1}$) as $\mu \to 0$. Note that, in view of the first two conditions in (2.5), $\lambda g_0(\lambda t) \to \delta(t)$ and $\lambda^2 g_0'(\lambda t) \to \delta'(t)$ as $\lambda \to \infty$.

In what follows, the dependence on the parameters $\lambda$ and $\mu$ is sometimes not immediately important to the argument, and in such cases we often “hide” these parameters by using the notation

$$g(\tau) = \lambda g_0(\lambda \tau), \quad v(x) = V\left(\frac{x}{\mu}\right),$$

so that $Q(x, t) = g'(t)v(x)$.

Next, let us give specific examples of right-hand sides $Q(x, t)$. In practice, the actual ocean bottom displacement is known neither in much detail nor very precisely, because the corresponding measurements are impractical or impossible (cf. [17, 19]). This results in certain freedom, which can be turned into an advantage. Namely, when constructing the function $Q(x, t) = g'(t)v(x)$ to be used in the analytical-numerical simulation according to the model (2.1), one should take ansatzes that, on the one hand, fit the general information available about the source shape and evolution and, on the other hand, can be handled efficiently in the computations. (The latter includes the requirement that these functions, as well as...
Figure 1. The function $V(y)$ with $b_1 = 1$ and $b_2 = 4$ rotated by the angle $\theta = \pi/10$ (left) and its Fourier transform $\tilde{V}(p)$ (right).

their Fourier transforms, be given by closed-form expressions, which permits one to reduce the amount of numerical computations in favor of the less time-consuming analytical transformations.)

A useful class of functions $V(y)$ satisfying (2.4) is described by the expression [10,14,16], generalizing [5,8,9],

$$V(y) = A \left( 1 + \left( \frac{y_1}{b_1} \right)^2 + \left( \frac{y_2}{b_2} \right)^2 \right)^{-3/2},$$

where $A$, $b_1$, and $b_2$ are real parameters. The Fourier transform of this function is remarkably simple,

$$\tilde{V}(p) = Ab_1b_2e^{-\sqrt{b_1^2p_1^2 + b_2^2p_2^2}},$$

One can further apply a differential operator

$$\hat{P} = P\left( \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2} \right)$$

with constant coefficients to the function $V$ and then rotate the coordinate system by some angle $\theta$, thus obtaining a broad variety of functions of the form $V_{P,\theta}(y)$ satisfying (2.4). Such functions model elliptic-shaped sources of various eccentricity and various direction of axes with a wavy relief depending on the differential operator $\hat{P}$ (see [11,15]). Figure 1 shows the graph of $V(y)$ rotated by an angle of $\pi/10$ and of its Fourier transform.

Let us also give two examples of functions $g_0(t)$ satisfying (2.5),

(a) $g_0(\tau) = ae^{-\tau}(\sin(\alpha \tau + \phi_0) - \sin \phi_0)$,  
(b) $g_0(\tau) = e^{-\tau}P(\tau),$

where $\alpha > 0$ and $\phi$ are real parameters, $a = (\alpha^2 + 1)/(\alpha \cos \phi_0 - \alpha^2 \sin \phi_0)$ is a normalizing factor, and $P(\tau) = \sum_{k=1}^{n}(k!)^{-1}P_k \tau^k$ is a polynomial of degree $n$ with $\sum_{k=1}^{n}P_k = 1$ (see Fig.2).
2.3. Operator solution formulas and energy estimates. We denote the spatial part of the wave operator in (2.1) by \( L \); thus, \[
Lu = -\frac{\partial}{\partial x_1} \left( c^2(x) \frac{\partial u}{\partial x_1} \right) - \frac{\partial}{\partial x_2} \left( c^2(x) \frac{\partial u}{\partial x_2} \right) \equiv -\langle \nabla, c^2(x) \nabla \rangle u.
\]
The operator (2.14) (with domain \( W^2_2(\mathbb{R}^2) \)) is a nonnegative self-adjoint operator on \( L^2(\mathbb{R}^2) \). Let \( D = \sqrt{L} \) be the positive square root of \( L \). Problem (2.1) becomes \[
(2.15) \quad \eta''(t) + D^2 \eta(t) = g'(t)v, \quad \eta|_{t=0} = \eta_t|_{t=0} = 0.
\]
Duhamel's formula represents the solution of (2.15) as the integral \[
(2.16) \quad \eta(t) = \int_0^t w(t, \tau) d\tau,
\]
where \( w(t, \tau) \) is the solution of the problem \[
(2.17) \quad w''_{tt}(t, \tau) + D^2 w(t, \tau) = 0, \quad w|_{t=\tau} = g(\tau)v, \quad w_t|_{t=\tau} = 0.
\]
Indeed, the function (2.16) satisfies (2.15), because \[
\eta''(t) + D^2 \eta(t) = \int_0^t \left( w''_{tt}(t, \tau) + D^2 w(t, \tau) \right) d\tau + \frac{d}{dt}(w(t, t)) + w'_t(t, t) = g'(t)v, \quad \eta(0) = 0, \quad \eta'(0) = g(0)v = 0
\]
in view of (2.15). Now we can use the general solution formula (e.g., see [26, p. 191]) \[
(2.18) \quad u(t) = \cos(Dt)u_0 + D^{-1} \sin(Dt)u_1
\]
for the abstract hyperbolic Cauchy problem \[
(2.19) \quad u''(t) + D^2 u(t) = 0, \quad u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1
\]
and write \[
(2.20) \quad \eta(t) = \left[ \int_0^t \cos(D(t-\tau))g(\tau) d\tau \right] v = \Re \left[ \int_0^t e^{iD(t-\tau)}g(\tau) d\tau \right] v.
\]

\[\text{Figure 2. Examples of } g(t) = \lambda g_0(\lambda t) \text{ for } \lambda = 1, 2, 3, 4: g_0(\tau) = e^{-\tau}(\sin(\alpha \tau + \varphi_0) - \sin \varphi_0) \text{ (upper diagram); } g(\tau) = e^{-\tau}(0.2\tau + 0.4\tau^2) \text{ (lower diagram).} \]
Here we have used the fact that \( g(\tau) \) is real-valued; the real part of an operator \( A \) is defined as usual by \( \Re A = \frac{1}{2}(A + A^*) \). Formula (2.20) is the desired abstract operator formula for the solution of problem (2.1).

**Remark 2.2.** Since the operator \( D \) is self-adjoint, it follows that the expressions \( \cos(Dt) \), \( \sin(Dt)/D \), and \( e^{iDt} \), occurring in (2.18) and (2.20), are well defined in the framework of functional calculus for self-adjoint operators as functions \( f(D) \) with bounded continuous symbols \( f(\xi) = \cos \xi t \), \( f(\xi) = \xi^{-1} \sin \xi t \), and \( f(\xi) = e^{i\xi t} \), respectively. Moreover, \( e^{iDt} \) is none other than the strongly continuous group generated by \( D \), \( \Re(f(D)) = (\Re f)(D) \), and, for “good” functions \( f(\xi) \), the operator \( f(D) \) can be defined not only via the integral over the spectral measure but also via the Fourier transform as

\[
f(D)u = \frac{1}{\sqrt{2\pi}} \langle \tilde{f}(\tau), e^{i\tau D}u \rangle, \quad u \in L^2(\mathbb{R}^2),
\]

where \( \tilde{f} \) is the Fourier transform of \( f \) and the angle brackets stand for the value of the distribution \( \tilde{f}(\tau) \) on the \( L^2(\mathbb{R}^2) \)-valued function \( e^{i\tau D}u \).

**Remark 2.3.** The **energy** of the solution of the Cauchy problem (2.19) is defined by the formula [26, p. 191]

\[
\mathcal{E}[u](t) = \frac{1}{2}(\|u'(t)\|^2 + \|Du(t)\|^2) \equiv \frac{1}{2}(\|u'(t)\|^2 + (u(t), Lu(t)))
\]

(where \( \| \cdot \| \) stands for the \( L^2 \) norm and \( (\cdot, \cdot) \) for the \( L^2 \) inner product) and is conserved in the course of time. Hence, in view of (2.10) and the estimates (2.4), the solution of problem (2.19) (with \( \tau \) viewed as a parameter) satisfies

\[
\mathcal{E}[u](t, \tau) = \frac{g^2(\tau)}{2} \| D(V(x/\mu)) \|^2 = \frac{g^2(\tau)}{2\mu^2} \int_{\mathbb{R}^2} \left| c(x) \nabla V \left( \frac{x}{\mu} \right) \right|^2 \, dx_1dx_2
\]

\[
= \frac{g^2(\tau)}{2} \int_{\mathbb{R}^2} \left| \mu y \right|^2 |\nabla V(y)|^2 \, dy_1dy_2 = \frac{\lambda^2}{2} \| g_0(\lambda \tau) \|^2 \| V \|^2 \quad (1 + O(\mu))
\]

as \( \mu \to 0 \), where \( c_0 = c(0) \). Now it follows from (2.19) that, with some constant \( C \),

\[
\mathcal{E}[u](t) \leq \left\{ \int_0^t \sqrt{\mathcal{E}[u](\tau, \tau)} \, d\tau \right\}^2 + \frac{1}{2} \| w(t) \|^2
\]

\[
\leq C \left\{ \lambda \int_0^t |g_0(\lambda \tau)| \, d\tau \right\}^2 + \frac{\mu^2 \lambda^2}{2} \| g_0(\lambda \tau) \|^2 \| V \|^2
\]

\[
\leq C \left\{ \int_0^\infty |g_0(\tau)| \, d\tau \right\}^2 + \frac{\mu^2 \lambda^2}{2} \| g_0(\lambda t) \|^2 \| V \|^2
\]

\[
= O(1) + O(\mu^2 \lambda^2 e^{-2\nu \lambda t}) = O(1) + O(\omega^2 e^{-2\nu \lambda t});
\]

i.e., the energy of the solution is uniformly bounded as \( \mu \to 0 \) for all \( t > \varepsilon > 0 \). (However, it may have a “spike” of the order of \( \omega^{-2} \) for \( t \sim 1/\lambda \); of course, this is only important if \( \omega \ll 1 \).) In other words, we have chosen a physically natural normalization of the right-hand side of our problem.

**Remark 2.4.** For the inhomogeneous wave equation

\[
u''(t) + D^2 u(t) = F(t),
\]
one has the energy identity
\[ E[u](t) = E[u](0) + \text{Re} \int_0^t (F(\tau), u'(\tau)) \, d\tau, \]
which implies the well-known estimates\[ (2.23) \quad \|u(t)\|_{s+1} + \|u'(t)\|_s \leq C(t)(\|u(0)\|_{s+1} + \|u'(0)\|_s + \sup_{\tau \in [0,t]} \|F(\tau)\|_s), \]
where \(\| \cdot \|_s\) stands for the norm on the Sobolev space \(H^s = W^s_2(\mathbb{R}^2)\); in particular, \(\| \cdot \|_0 = \| \cdot \|\). Of the estimates (2.23), the most important for us is the one with \(s = 0\) (corresponding to the sum of the energy integral and the \(L^2\) norm), in which the main estimates for the norms of remainders in asymptotic formulas will be obtained. However, occasionally our argument involves estimates with different \(s\).

### 2.4. Solution splitting into propagating and transient components.

Let us further transform formula (2.20) to reveal the structure of the solution and represent it in a form suitable for subsequent computations. We have
\[ (2.24) \quad \int_0^t e^{iD(t-\tau)}g(\tau) \, d\tau = e^{iDt} \int_0^\infty e^{iD(\tau-t)}g(\tau) \, d\tau - \int_0^\infty e^{iD(\tau-t)}g(\tau) \, d\tau - \int_0^\infty e^{-iD\tau}g(\tau + t) \, d\tau. \]

Let \(H(\tau)\) be the Heaviside step function \((H(\tau) = 1\) for \(\tau \geq 0\) and \(H(\tau) = 0\) for \(\tau < 0\)), and, for \(t \geq 0\), let
\[ G(\xi, t) = \int_0^\infty e^{-i\xi\tau}g(\tau + t) \, d\tau \equiv \int_{-\infty}^\infty e^{-i\xi\tau}H(\tau)g(\tau + t) \, d\tau \]
be the Fourier transform of \(\sqrt{2\pi} H(\tau)g(\tau + t)\) with respect to the variable \(\tau\). (Note that \(G(\xi, 0) = \sqrt{2\pi} \tilde{g}(\xi)\), where the function \(g(\tau)\) is assumed to be extended by zero for the negative values of \(\tau\).) Then formula (2.24) can be rewritten as
\[ \int_0^t e^{iD(t-\tau)}g(\tau) \, d\tau = e^{iDt}G(D, 0) - G(D, t) = \sqrt{2\pi} e^{iDt} \tilde{g}(D) - G(D, t), \]
and accordingly
\[ (2.25) \quad \eta(t) = \eta_{\text{prop}}(t) + \eta_{\text{trans}}(t), \]
where
\[ (2.26) \quad \eta_{\text{prop}}(t) = \sqrt{2\pi} \text{Re}(e^{iDt} \tilde{g}(D))v \]
\[ \equiv \sqrt{2\pi} \cos(Dt) \text{Re} \tilde{g}(D)v - \sqrt{2\pi} \sin(Dt) \text{Im} \tilde{g}(D)v, \]
\[ (2.27) \quad \eta_{\text{trans}}(t) = -\text{Re}(G(D, t))v. \]

The function \(\eta_{\text{prop}}(t)\) given by (2.26) is the solution of the Cauchy problem for the homogeneous wave equation
\[ (2.28) \quad u''(t) + D^2u(t) = 0 \]
with the initial data
\[ (2.29) \quad u_0 = \sqrt{2\pi} \text{Re} \tilde{g}(D)v, \quad u_1 = -\sqrt{2\pi} \text{Im} \tilde{g}(D)Dv. \]

\[ ^3 \]
Their derivation takes into account the fact that the norm \(\|u\|_s\) is equivalent to the norm \(\|(1 + L)^{s/2}u\|\) by virtue of the conditions imposed on the velocity \(c(x)\).
where \( \nu \) that \( \tilde{g} \) operator \( G \) estimates. Formulas (2.27) and (2.29) involve the real and imaginary parts of the solution, because it exponentially decays as \( \lambda t \to \infty \), as shown by the following proposition. (We shall also see in Sec. 3.3 that \( \eta_{\text{trans}}(t) \) always remains localized near the origin.)

**Proposition 2.5.** As \( \mu \to 0 \), the propagating component satisfies the estimates

\[
\| \eta_{\text{prop}}(t) \|_1 = O(1), \quad \| \eta_{\text{prop}}'(t) \| = O(1),
\]

and the transient component satisfies the estimates

\[
\| \eta_{\text{trans}}(t) \|_1 = O(e^{-\nu \lambda t}), \quad \| \eta_{\text{trans}}'(t) \| = O(\omega^{-1} e^{-\nu \lambda t}),
\]

where \( \nu \) is the constant in condition (2.5).

**Proof.** We will estimate the transient part (2.27) of the solution directly and the propagating part (2.26) via the Cauchy data (2.29) by using the energy estimates. Formulas (2.27) and (2.29) involve the real and imaginary parts of the operator \( G(D, t) \) applied to the original right-hand side source function \( v \). (Recall that \( \tilde{g}(D) \) is a special case of \( G(D, t) \) for \( t = 0 \).) Thus, we need to estimate the operator \( G(D, t) \). Note that, for an arbitrary bounded measurable function \( f(\xi) \), one has

\[
(2.30) \quad \| f(D) : H^0 \to H^0 \| \leq \sup_{\xi \in \mathbb{R}} |f(\xi)|, \quad \| f(D) : H^1 \to H^1 \| \leq C \sup_{\xi \in \mathbb{R}} |f(\xi)|
\]

with some constant \( C \) independent of \( f \).

Thus, we need estimates for the function \( G(\xi, t) \). Since \( g(\tau) = \lambda g_0(\lambda \tau) \), we have

\[
(2.31) \quad G(\xi, t) = G_0(\xi/\lambda, \lambda t),
\]

where

\[
(2.32) \quad G_0(\xi, t) = \int_{0}^{\infty} e^{-i\xi \tau} g_0(\tau + t) \, d\tau
\]

is the Fourier transform of the function \( \sqrt{2\pi} H(\tau) g_0(\tau + t) \) with respect to \( \tau \). By Lemma 2.6 below, we have

\[
|\sqrt{2\pi} \tilde{g}(\xi)| = |G_0(\xi/\lambda, 0)| \leq C_0
\]

and hence, by (2.29) and (2.30),

\[
\| u_0 \|_1 = \sqrt{2\pi} \| \text{Re} \tilde{g}(D) v \|_1 \leq C C_0 \| v \|_1 = O(1),
\]

\[
\| u_1 \| = \sqrt{2\pi} \| \text{Im} \tilde{g}(D) Dv \| \leq C C_0 \| Dv \| \leq \tilde{C} \| v \|_1 = O(1),
\]

Indeed, the first estimate is obvious, because the operator \( D \) is self-adjoint on \( H^0 = L_2(\mathbb{R}^2) \). To obtain the second estimate, we replace the norm on \( H^1 \) by the equivalent Hilbert norm \( \| u \| = (u, (1 + L)u)^{1/2} \) (cf. Remark 2.3), then the operator \( D \) becomes self-adjoint on \( H^1 \), and the second estimate follows.
because \( v = V(x/\mu) \) and hence \( \|v\|_1 = O(1) \) (cf. the computation in Remark 2.3).

Now the energy estimates (2.23) for \( s = 0 \) give the desired estimates for \( \eta_{\text{prop}}(t) \).

The estimates for the transient part go as follows, again with the use of Lemma 2.6

\[
\|\eta_{\text{trans}}(t)\|_1 = \|\Re G(D, t)v\|_1 \leq C \sup_\xi |G_0(\xi/\lambda, \lambda t)| \|v\|_1 \\
\leq CC_0 e^{-\nu \lambda t} \|v\|_1 = O(e^{-\nu \lambda t}),
\]

\[
\|\eta_{\text{trans}}'(t)\| = \left\| \Re \frac{\partial G}{\partial t}(D, t)v \right\| \leq \sup_\xi \|\lambda \frac{\partial G_0}{\partial t}(\xi/\lambda, \lambda t)\| \|v\| \\
\leq C_01 e^{-\nu \lambda t} \|v\| = O(\mu \lambda e^{-\nu \lambda t}) = O(\omega^{-1} e^{-\nu \lambda t}),
\]

because \( v = V(x/\mu) \) and hence \( \|v\| = O(\mu) \). This completes the proof. \( \Box \)

The following lemma establishes the estimates for \( G_0(\xi, t) \) used in the proof given above and also estimates that will be useful below.

**Lemma 2.6.** The function \( G_0(\xi, t) \) satisfies the estimates

\[
\left| \frac{\partial^{m+k} G_0}{\partial t^m \partial \xi^k}(\xi, t) \right| \leq C_{km} e^{-\nu t} (1 + |\xi|)^{-k-1}, \quad k, m = 0, 1, 2, \ldots ,
\]

with some constants \( C_{km} \). For \( t = 0 \) and \( m = 0 \), one has the better estimates

\[
\left| \frac{\partial^k G_0}{\partial \xi^k}(\xi, 0) \right| \leq C_{{\mathfrak{k}}} (1 + |\xi|)^{-k-2}, \quad k = 0, 1, 2, \ldots .
\]

**Proof.** First, let us prove the estimates (2.33) and (2.34) for \( |\xi| \leq 1 \). Then we have

\[
\left| \frac{\partial^m G_0}{\partial t^m}(\xi, t) \right| = \left| \frac{i}{\xi} \right|^N \int_0^\infty \frac{d^N}{d\tau^N} (e^{-i \xi \tau}) g_0^{(m)}(\tau + t) d\tau 
\]

by (2.25), whence the claim follows. Now let \( |\xi| > 1 \). Then we write

\[
\frac{\partial^m G_0}{\partial t^m}(\xi, t) = \sum_{l=1}^N (i \xi)^{-l} g_0^{(m+l-1)}(t) + (i \xi)^{-N} \int_0^\infty e^{-i \xi \tau} g_0^{(m+N)}(\tau + t) d\tau.
\]

Next, we differentiate both sides of (2.35) \( k \) times with respect to \( \xi \), which gives

\[
\frac{\partial^{m+k} G_0}{\partial t^m \partial \xi^k}(\xi, t) = i^{-k} \sum_{l=1}^N \frac{(l + k - 1)!}{(l - 1)!} (i \xi)^{-l-k} g_0^{(m+l-1)}(t) \\
+ i^{-k} \sum_{s=0}^k \binom{k}{s} \frac{(l + s - 1)!}{(l - 1)!} (i \xi)^{-N-s} \int_0^\infty \tau^{k-s} e^{-i \xi \tau} g_0^{(m+N)}(\tau + t) d\tau.
\]

Here all factors \( g_0^{(m+l-1)}(t) \) and the integral are bounded in modulus by \( \text{const} \cdot e^{-\nu t} \) by virtue of (2.23), and the smallest power of \( \xi^{-1} \) on the right-hand side is \( \xi^{-k-1} \), which implies the estimate (2.33). For \( t = 0 \) and \( m = 0 \), the smallest power of \( \xi^{-1} \) on the right-hand side is \( \xi^{-k-2} \), since \( g_0(0) = 0 \), and we have the estimate (2.34). \( \Box \)
3. Asymptotics of the solution

In this section, we describe the asymptotics as $\mu \to 0$ of the solution $\eta(t) = \eta_{\text{prop}}(t) + \eta_{\text{trans}}(t)$ of problem (2.1), (2.2). In all theorems in this section, we assume that all conditions stated in Sec. 2.1 are satisfied. Recall that the problem also contains the large parameter $\lambda$, which is related to $\mu$ by the condition $\omega < \omega_0$ (see (2.9)), where $\omega = \lambda_0(\lambda\mu)^{-1}$ (see (2.8)). If $\omega$ can be treated as a second small parameter (i.e., the distance traveled by the waves in the lifetime of the source is much smaller than the source diameter), then additional Taylor series expansions in $\omega$ lead to further simplifications in the asymptotic formulas.

### 3.1. Asymptotics of the transient component.

The asymptotics of the transient component $\eta_{\text{trans}}(t)$ of the solution as $\mu \to 0$ is given by the following theorem.

**Theorem 3.1.** One has

\begin{equation}
\eta_{\text{trans}}(x,t) = -\frac{1}{2\pi} \iint \Re G_0(\omega,\lambda \mu) \tilde{V}(p) e^{ipx/\mu} dp_1 dp_2 + R(t),
\end{equation}

or, in the polar coordinates $(r, \varphi)$, $x = r\mathbf{n}(\varphi)$, where $\mathbf{n}(\varphi) = (\cos \varphi, \sin \varphi)$,

\begin{equation}
\eta_{\text{trans}}(rn(\varphi)) = -\frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} \rho \Re G_0(\omega,\lambda) \times \tilde{V}(\rho \mathbf{n}(\varphi)) e^{i\rho \cos(\psi - \varphi)/\mu} d\rho d\psi + R(t),
\end{equation}

where the remainder $R(t)$ satisfies the estimates

\begin{equation}
\|R(t)\|_1 = O(\mu e^{-\rho\lambda t}), \quad \|R'(t)\| = O(\mu^{-1} e^{-\rho\lambda t}), \quad \mu \to 0.
\end{equation}

**Proof.** Consider the operators

\begin{equation}
L^{(0)} = -\frac{\lambda_0^2}{\mu} \nabla^2, \quad D^{(0)} = (L^{(0)})^{1/2}.
\end{equation}

Thus, $L^{(0)}$ is obtained by freezing the coefficients of the operator $L$ at the origin, and $D^{(0)}$ is just the positive square root of the positive self-adjoint operator $L^{(0)}$.

**Lemma 3.2.** One has

\begin{equation}
[\Re G(D^{(0)}, t) - \Re G(D, t)] V\left(\frac{x}{\mu}\right) = R(t),
\end{equation}

where $R(t)$ satisfies the estimates (3.3).

The proof of this lemma will be given in Sec. 4. Thus, the operator $D$ in the expression (2.27) for $\eta_{\text{trans}}(x,t)$ can be replaced by the operator $D^{(0)}$ with constant coefficients. Now it remains to compute $\Re G(D^{(0)}, t)V(x/\mu)$. Since $D^{(0)}$ is an operator with constant coefficients and with symbol $c_0|p|$, it follows that

\begin{equation}
F(D^{(0)}) = F^{-1} \circ F(c_0|p|) \circ F
\end{equation}

for any function $F(\xi)$, where $F$ is the Fourier transform and the middle factor on the right-hand side is the operator of multiplication by $F(c_0|p|)$. Thus we obtain

\begin{equation}
G(D^{(0)}, t)V\left(\frac{x}{\mu}\right) = \frac{1}{2\pi} \iint \Re G_0(\omega|p|, \lambda \mu) \tilde{V}(p) e^{ipx/\mu} dp_1 dp_2
\end{equation}

(we have used the formula for $G(\xi, t)$ and made obvious changes of variables), which proves Theorem 3.1.

\[\square\]
3.2. Asymptotics of the equivalent source functions. Now we proceed
3.3. Asymptotics of the equivalent source functions. Now we proceed
to the computation of the propagating component of the solution. It satisfies the
Cauchy problem (2.28), (2.29), and so a good starting point would be to compute
the equivalent source functions (2.29). Once we compute them (asymptotically) and
prove that they are localized near the origin, we can use the methods developed in
\[11\]–\[16\] to obtain the asymptotics of the propagating solution component. However,
we would like to apply ready-to-use formulas from these papers rather than to write
out new formulas based on the same ideas. The formulas in \[11\]–\[16\] were obtained
for the case in which \(u_1\), the initial data for the
\(t\)-derivative of the solution, is zero.

So we resort to the following trick.

Proposition 3.3. The propagating solution component \(\eta_{\text{prop}}(t)\) can be repre-
sented in the form
\[
\eta_{\text{prop}}(t) = \eta_1(t) + \eta_2'(t),
\]
where \(\eta_1(t)\) and \(\eta_2(t)\) are the solutions of the Cauchy problems
\[
\begin{align*}
\eta_1''(t) + D^2\eta_1 &= 0, & \eta_1|_{t=0} &= \sqrt{2\pi} \text{Re} \tilde{g}(D)v, & \eta_1'|_{t=0} &= 0, \\
\eta_2''(t) + D^2\eta_2 &= 0, & \eta_2|_{t=0} &= \sqrt{2\pi D^{-1}} \text{Im} \tilde{g}(D)v, & \eta_2'|_{t=0} &= 0.
\end{align*}
\]

Proof. The sum (3.7) obviously satisfies the wave equation (2.28). Next,
\[
\eta_2'|_{t=0} = 0, & \ (\eta_2'|_{t=0} = 0 = -D^2\eta_2|_{t=0} = -\sqrt{2\pi D} \text{Im} \tilde{g}(D)v),
\]
which shows that the initial conditions (2.29) are satisfied and hence completes the
proof.

Thus, let us compute the asymptotics of the new equivalent source functions
\[
\begin{align*}
\eta_10 &= \sqrt{2\pi} \text{Re} \tilde{g}(D)v, & \eta_20 &= \sqrt{2\pi D^{-1}} \text{Im} \tilde{g}(D)v.
\end{align*}
\]

Theorem 3.4. The equivalent source functions (3.10) have the following as-
3.4. The equivalent source functions (3.10) have the following as-
3ymptotics as \(\mu \to 0\):
\[
\begin{align*}
\eta_10 &= U_1 \left(\frac{x}{\mu}\right) + R_1, & \quad \eta_20 &= U_2 \left(\frac{x}{\mu}\right) + R_2, \\
\end{align*}
\]
where the Fourier transforms of the functions \(U_1(y)\) and \(U_2(y)\) are given by the
formulas
\[
\begin{align*}
\tilde{U}_1(p) &= \sqrt{2\pi} \text{Re} \tilde{g}_0(\omega|p|)\tilde{V}(p), & \quad \tilde{U}_2(p) &= \sqrt{2\pi} \lambda^{-1} \text{Im} \frac{\tilde{g}_0(\omega|p|)}{\omega|p|} \tilde{V}(p)
\end{align*}
\]
and the remainders satisfy the estimates
\[
\|R_1\|_1 = O(\mu), & \quad \|R_2\|_2 = O(\mu).
\]

Proof. The proof goes along the same lines as that of Theorem 3.1. Namely,
we prove that the operator \(D\) in formulas (3.10) can asymptotically be replaced by
\(D^{(0)}\) and then compute the Fourier transforms of \(U_1\) and \(U_2\) using formula (3.6).
The latter computation is trivial, and we omit it altogether. As for the first part,
it is given by the following lemma, which will be proved together with Lemma 3.2
in Sec. 4.
LEMMA 3.5. One has
\[
\sqrt{2\pi} \left[ \text{Re} \tilde{g}(D) - \text{Re} \tilde{g}(D^{(0)}) \right] V \left( \frac{x}{\mu} \right) = R_1,
\]
\[
\sqrt{2\pi} \left[ D^{-1} \text{Im} \tilde{g}(D) - (D^{(0)})^{-1} \text{Im} \tilde{g}(D^{(0)}) \right] V \left( \frac{x}{\mu} \right) = R_2,
\]
where \( R_1 \) and \( R_2 \) satisfy the estimates \((3.13)\).

This completes the proof of Theorem 3.4.

REMARK 3.6. If we replace \( \eta_10 \) and \( \eta_20 \) in the Cauchy problems for \( \eta_1 \) and \( \eta_2 \) by \( U_1(x/\mu) \) and \( U_2(x/\mu) \), respectively, then the resulting error \( \delta(t) \) in the computation of \( \eta_{\text{prop}}(t) \) will satisfy the estimates
\[
\| \delta(t) \|_1 = O(\mu), \quad \| \delta'(t) \| = O(\mu), \quad \mu \to 0,
\]
uniformly on any finite time interval. Indeed, let us write \( \delta(t) = \delta_1(t) + \delta_2(t) \), where \( \delta_1 \) and \( \delta_2 \) are the errors in \( \eta_1 \) and \( \eta_2 \), respectively. Then, by virtue of the energy estimates \((2.28)\), we have
\[
\| \delta_1(t) \|_1 + \| \delta'_1(t) \| \leq C(t) \| \delta_1(0) \|_1 = C(t) \| R_1 \|_1 = O(\mu),
\]
\[
\| \delta_2(t) \|_1 + \| \delta'_2(t) \| \leq C(t) \| \delta_2(0) \| = C(t) \| D^2 \delta_2(0) \| = C(t) \| D^2 \delta_2(0) \| = \leq C(t) \| D^2 R_2 \| \leq C_1(t) \| R_2 \|_2 = O(\mu).
\]
Thus, the accuracy provided by Theorem 3.4 permits computing the propagating part \( O(\mu) \) in the energy norm.

3.3. Asymptotics of the propagating part. Remark 3.6 shows that, to compute the asymptotics of the propagating part of the solution of problem \((2.1)\) modulo \( O(\mu) \) in the energy norm, it suffices to solve problems \((3.8)\) and \((3.9)\) asymptotically with the initial data replaced by the functions \( U_1(x/\mu) \) and \( U_2(x/\mu) \) indicated in Theorem 3.4. Thus, we need to solve the problems
\[
\eta_1''(t) + D^2 \eta_1 = 0, \quad \eta_1|_{t=0} = U_1(x/\mu), \quad \eta_1'|_{t=0} = 0,
\]
\[
\eta_2''(t) + D^2 \eta_2 = 0, \quad \eta_2|_{t=0} = U_2(x/\mu), \quad \eta_2'|_{t=0} = 0.
\]
(We denote the new unknown functions by the same letters \( \eta_1, \eta_2 \); this will not lead to a misunderstanding.) The initial data in these problems are localized near the origin, and hence the asymptotics of solutions of these problems modulo \( O(\mu) \) in all spaces \( H^s \) can be obtained with the use of the approach developed in \([11, 14]\) and based on the Maslov canonical operator \([1, 20]\). Let us briefly recall this construction.

3.3.1. Bicharacteristics, canonical operator and solution formulas. In the phase space \( \mathbb{R}^4_{x,p} \) with the coordinates \( (x, p) = (x_1, x_2, p_1, p_2) \), consider the Hamiltonian system
\[
\dot{p} = -\frac{\partial \mathcal{H}}{\partial x}, \quad \dot{x} = \frac{\partial \mathcal{H}}{\partial p}
\]
corresponding to the Hamiltonian function \( \mathcal{H} = |p|c(x) \). This system determines the Hamiltonian phase flow \( g^t \mathcal{H} \). Let \( n(\psi) = \left( \cos \psi, \sin \psi \right) \). Consider the Lagrangian manifold \( \Lambda_0 = \{ p = n(\psi), x = \alpha n(\psi) \} \), isomorphic to the two-dimensional cylinder, where \( \psi \in [0, 2\pi) \) and \( \alpha \in \mathbb{R} \) are coordinates on \( \Lambda_0 \). By shifting this manifold along the flow \( g^t \mathcal{H} \), we obtain the family of Lagrangian manifolds \( \Lambda_t = g^t \mathcal{H} \Lambda_0 \), each of which
is equipped with the same coordinate system \((\psi, \alpha)\) as \(\Lambda_0\). We take the point with coordinates \((\psi, \alpha) = (0, 0)\) for the distinguished point on \(\Lambda_0\) and construct the Maslov canonical operator \(K_{\Lambda_0}^{h_0}\) on each of the manifolds \(\Lambda_t\). (Here \(h \to 0\) is the small parameter occurring in the construction of the canonical operator; all Jacobians in the definition of \(K_{\Lambda_t}^h\) are taken with respect to the coordinates \((\psi, \alpha)\).

It follows from the results in \([11, 14]\) that the asymptotics of the solutions \(\eta_{1,2}\) of problems (3.15) and (3.16) can be obtained as follows. Using the Fourier transforms (3.12) of the equivalent source functions computed in Theorem 3.4, we introduce the following two smooth functions on \(\Lambda_t\) independent of \(t\) and \(\alpha\) but depending on the coordinate \(\psi\) and an additional parameter \(\rho\):

\[
\begin{align*}
\varphi_1(\psi, \rho) &= \bar{U}_1(\rho \mathbf{m}(\psi)) = \sqrt{2\pi} \text{Re} \tilde{g}_0(\omega \rho) \bar{V}(\rho \mathbf{m}(\psi)), \\
\varphi_2(\psi, \rho) &= \bar{U}_2(\rho \mathbf{m}(\psi)) = \sqrt{2\pi} \lambda^{-1} \text{Im} \tilde{g}_0(\omega \rho) \bar{V}(\rho \mathbf{m}(\psi)).
\end{align*}
\]

Then the formulas in \([11, 13]\) give

\[(3.18) \quad \eta_{1,2}(t) = \sqrt{\frac{\mu}{2\pi}} \text{Re} \left( e^{-i\pi/4} \int_0^\infty K_{\Lambda_t}^{\mu/\rho}(\sqrt{\rho} \varphi_{1,2}(\psi, \rho)) \, d\rho \right) + O(\mu).\]

Let us find the derivative \(\eta'_2(t)\). By the commutation formula \([20]\) for the canonical operator, we have

\[
\begin{align*}
\eta'_2(t) &= \sqrt{\frac{\mu}{2\pi}} \text{Re} \left( e^{-i\pi/4} \int_0^\infty \frac{\partial}{\partial t} K_{\Lambda_t}^{\mu/\rho}(\sqrt{\rho} \varphi_2(\psi, \rho)) \, d\rho \right) \\
&= \sqrt{\frac{\mu}{2\pi}} \text{Re} \left( e^{-i\pi/4} \int_0^\infty K_{\Lambda_t}^{\mu/\rho} \left[ -\frac{i\rho}{\mu} H_{\Lambda_t} \sqrt{\rho} \varphi_2(\psi, \rho) \right] \, d\rho \right) + O(\mu).
\end{align*}
\]

But the Hamiltonian \(H\) is preserved along the trajectories of the Hamiltonian system, and hence \(H_{\Lambda_0} = H_{\Lambda_t} = c(\mathbf{m}(\psi))\). It was shown in \([11]\) that, modulo lower-order terms, one can set \(\alpha = 0\) in the functions on \(\Lambda_t\). Taking into account the definition of \(\varphi_2\), we obtain

\[
\eta'_2(t) = \sqrt{\mu} \text{Re} \left( e^{-\frac{i\pi}{4}} \int_0^\infty K_{\Lambda_t}^{\mu/\rho} \left[ (\sqrt{\rho} \text{Im} \tilde{g}_0(\omega \rho) \bar{V}(\rho \mathbf{m}(\psi))) \right] \, d\rho \right) + O(\mu).
\]

Finally, we use the formula \(\eta_{\text{prop}}(t) = \eta_1(t) + \eta'_2(t)\) and arrive at the following theorem.

**Theorem 3.7.** The propagating part of the solution has the following asymptotics:

\[(3.19) \quad \eta_{\text{prop}}(t) = \sqrt{\mu} \text{Re} \left( e^{-\frac{i\pi}{4}} \int_0^\infty K_{\Lambda_t}^{\mu/\rho} \left[ (\sqrt{\rho} \tilde{g}_0(\omega \rho) \bar{V}(\rho \mathbf{m}(\psi))) \right] \, d\rho \right) + R(t),\]

where the bar stands for complex conjugation and the remainder satisfies the estimates

\[(3.20) \quad \|R(t)\|_1 = O(\mu), \quad \|R'(t)\| = O(\mu)\]

uniformly on any finite interval of time \(t\).
3.3.2. Asymptotics near the front. Now let us compute the propagating part \((3.19)\) of the solution in more explicit terms. To this end, we need some geometry. Let \((P(t, \psi), X(t, \psi)), \psi \in [0, 2\pi)\), be the family of solutions of the Hamiltonian system \((3.17)\) with the initial conditions

\[
p|_{t=0} = n(\psi), \quad x|_{t=0} = 0.
\]

For each \(t\), the equations \(p = P(t, \psi), x = X(t, \psi), \psi \in [0, 2\pi)\), define a smooth closed curve \(\Gamma_t\) in the four-dimensional phase space \(\mathbb{R}^4_{x,p}\); this curve is called the wave front in \(\mathbb{R}^4_{x,p}\). The projection \(\gamma_t = \{x = X(t, \psi) : \psi \in [0, 2\pi)\}\) of \(\Gamma_t\) into \(\mathbb{R}^2_x\) is called the front in the configuration space. In contrast to \(\Gamma_t\), the curve \(\gamma_t\) may well be nonsmooth; namely, it may have turning (or focal) points (in this case, \(X_0 = 0\) for some \(\psi\)) and points of self-intersection. Moreover, the front \(\gamma_0\) at the initial time \(t = 0\) is just the point \(x = 0\).

For each \(t\), the function \(3.19\) is localized in a neighborhood of the front \(\gamma_t\) \([11][16]\). Formula \(3.19\) provides the global asymptotics of the propagating part of the solution; i.e., this formula holds both near regular and near focal points of the front. The formula can be simplified in a neighborhood of any point of the front, but the simplified expression depends on whether the point is regular or focal. Here we restrict ourselves to the case of a neighborhood of a regular point.

Take some time \(t\) and angle \(\psi^0\) and assume that the point \(X(t, \psi^0) \in \gamma_t\) is not focal; i.e., \(X_0(t, \psi^0) \neq 0\). In some neighborhood of \(X(t, \psi^0)\), we can introduce the local coordinates \((\psi, y)\), where \(y = y(x, t)\) is the (signed) distance between the point \(x\) and the front and \(\psi = \psi(x, t)\) is determined by the condition that the vector \(x - X(t, \psi(x, t))\) is orthogonal to the vector tangent to the wave front at the point \(X(t, \psi(x, t))\); in other words

\[
\langle x - X(t, \psi(x, t)), X_0(t, \psi(x, t)) \rangle = 0.
\]

Set

\[
S(x, t) = \langle P(t, \psi(x, t)), x - X(t, \psi(x, t)) \rangle.
\]

Next, we introduce the Morse index \(m(t, \psi^0)\) of the trajectory \(X(\tau, \psi^0), \tau \in (0, t]\), which is the number of zeros of the function \(|X_0(\tau, \psi^0)|\) on the half-open interval \(\tau \in (0, t]\) \([20]\).

It may happen that some region of points \(x\) where we intend to write out the asymptotics simultaneously belongs to several neighborhoods of the above-mentioned type, where the corresponding points \(X(t, \psi^0)\) lie on several (but finitely many!) distinct arcs of the front \(\gamma_t\). (For example, this is the case if we study the asymptotics near a point of self-intersection of the front \(\gamma_t\).) Then all these arcs contribute to the asymptotics at such points \(x\), and we use an additional subscript \(j\) to distinguish these neighborhoods as well as all associated objects \((\psi^0, \psi(x, t), S(x, t), \text{Morse index}, \text{etc.})\). Now from the results in \([11][12][15][16]\) we obtain the following theorem.

**Theorem 3.8. In a neighborhood of the front \(\gamma_t\) but outside a neighborhood of the focal points, the asymptotic formula \((3.19)\) for the propagating part of the**
solution can be rewritten in the form

\[
\eta_{\text{prop}}(t) = \sqrt{\mu} \text{Re} \sum_j \left[ \frac{e^{-i\pi m(\psi_j^0, t)/2}}{\sqrt{c(X(\psi, t))}} \frac{c_0}{\sqrt{c(X(\psi, t))}} F \left( \frac{S_j(x, t)}{\mu}, \psi \right) \right]_{\psi=\psi_j(x, t)} + R(t),
\]

where

\[
F(z, \psi) = e^{-iz/4} \int_0^\infty \sqrt{\rho} \tilde{g}_0(\omega \rho) \tilde{V}(\rho \mu(\psi)) e^{i\omega \rho} d\rho,
\]

\[R(t)\]
satisfies the estimate (3.20), and the sum with respect to \(j\) is taken over all distinct arcs of \(\gamma_t\) contributing to the asymptotics at \(x\).

Remark 3.9. The factor

\[
\frac{1}{\sqrt{|X_\psi(\psi, t)|}} \sqrt{\frac{c_0}{c(X(\psi, t))}}
\]

includes the two-dimensional analog of the so-called Green law and the trajectory divergence related to the velocity \(c(x)\) (with height \(c^2(x)\) describing the bottom topography). The function \(F\) depends on the time and space shape of the source generating the waves \[11–16\]. Formulas (3.19), (3.22), and (3.23) apply to any localized perturbation.

4. Obtaining asymptotic expansions by noncommutative analysis

The aim of this section is to prove Lemmas 3.2 and 3.5. Vaguely speaking, these lemmas state that the replacement of the operator \(D\) by the operator \(D^{(0)}\) with constant coefficients in certain expressions results in an \(O(\mu)\) error. However, it is much easier to deal with functions of the differential operators \(L\) and \(L^{(0)}\) than with functions of their square roots, the pseudodifferential operators \(D = \sqrt{L}\) and \(D^{(0)} = \sqrt{L^{(0)}}\). Hence in Sec. 4.1 we represent the latter functions via the former and accordingly restate the lemmas. In Sec. 4.2 we make all noncommutative computations.

4.1. Eliminating the square roots.

Lemma 4.1. The functions \(\text{Re} G_0(\xi, t)\) and \(\xi^{-1} \text{Im} G_0(\xi, t)\) are smooth even functions of \(\xi\) and hence smooth functions of \(\xi^2\).

Proof. The function \(g_0(\tau)\) is real-valued, and \(\overline{G_0(\xi, t)} = G_0(-\xi, t)\) by (2.52). Thus, \(\text{Re} G_0(-\xi, t) = \text{Re} G_0(\xi, t)\) and \(\text{Im} G_0(-\xi, t) = -\text{Im} G_0(\xi, t)\): i.e., the real part of \(G_0\) is an even function of \(\xi\), and the imaginary part of \(G_0\) is an odd function of \(\xi\). Hence the desired claim follows. \(\square\)

Now let us introduce the functions

\[
f_1(\xi) = \text{Re} G_0(\xi^{1/2}, 0), \quad f_2(\xi) = \xi^{-1/2} \text{Im} G_0(\xi^{1/2}, 0),
\]

\[
f_3(\xi, t) = \text{Re} G_0(\xi^{1/2}, t).
\]

More formally, for example, fix an \(\varepsilon > 0\); the intersection of \(\gamma_t\) with the \(\varepsilon\)-neighborhood of \(x\) can be covered by finitely many arcs of length \(\leq \varepsilon\); take the contributions of all these arcs.
By Lemma 4.1, these functions are smooth for all \( \xi \), including \( \xi = 0 \). Formulas (3.10) for the equivalent sources and (2.27) for the transient solution component can now be rewritten as

\[
\eta_{10} = f_1(\lambda^{-2}L)V\left(\frac{x}{\mu}\right), \quad \eta_{20} = \lambda^{-1}f_2(\lambda^{-2}L)V\left(\frac{x}{\mu}\right),
\]

\[
\eta_{\text{trans}}(t) = -f_3(\lambda^{-2}L, \lambda t)V\left(\frac{x}{\mu}\right).
\]

(4.2)

Indeed, for example,

\[
\lambda^{-1}f_2(\lambda^{-2}L) = \lambda^{-1}(\lambda^{-2}L)^{-1/2} \text{Im} G_0((\lambda^{-2}L)^{1/2}, 0) = D^{-1} \text{Im} G_0(\lambda^{-1}D, 0) = D^{-1} \text{Im} G(D, 0) = \sqrt{2\pi} D^{-1} \text{Im} \tilde{g}(D).
\]

The following theorem is an equivalent restatement of Lemmas 3.2 and 3.5 in terms of functions of \( L \) and \( L^{(0)} \). (We write \( R_3(t) = -R(t) \) to unify the notation.)

**Theorem 4.2.** One has

\[
f_1(\lambda^{-2}L)V\left(\frac{x}{\mu}\right) = f_1(\lambda^{-2}L^{(0)})V\left(\frac{x}{\mu}\right) + R_1,
\]

\[
\lambda^{-1}f_2(\lambda^{-2}L)\left(\frac{x}{\mu}\right) = \lambda^{-1}f_2(\lambda^{-2}L^{(0)})\left(\frac{x}{\mu}\right) + R_2,
\]

\[
f_3(\lambda^{-2}L, \lambda t)V\left(\frac{x}{\mu}\right) = f_3(\lambda^{-2}L^{(0)}, \lambda t)V\left(\frac{x}{\mu}\right) + R_3(t),
\]

where the remainders satisfy the estimate

\[
\|R_1\|_1 = O(\mu), \quad \|R_2\|_2 = O(\mu),
\]

\[
\|R_3(t)\|_1 = O(\mu e^{-\lambda t}), \quad \|R_3(t)\| = O(\mu^2 \lambda e^{-\lambda t}).
\]

The proof will be given below in Sec. 4.2.

We need some estimates for the symbols (4.1). These are provided by the following lemma.

**Lemma 4.3.** The following estimates hold for the functions (4.1):

\[
|f_1^{(k)}(\xi)| \leq C_{k0}(1 + |\xi|)^{-1-k}, \quad |f_2^{(k)}(\xi)| \leq C_{km}(1 + |\xi|)^{-3/2-k},
\]

(4.5)

\[
\left| \frac{\partial^{k+m} f_3(\xi, t)}{\partial x^k \partial t^m} \right| \leq C_{km} e^{-\mu t}(1 + |\xi|)^{-1/2-k},
\]

\( k = 0, 1, 2, \ldots \), where the \( C_{km} \) are some constants (in general, different from those introduced earlier).

**Proof.** For \( k = 0 \), the desired estimates (4.5) readily follow from (2.33) and (2.34); it suffices to replace \( \xi \) by \( \xi^{1/2} \) (and use the fact that the functions \( f_j \) given by (4.1) are smooth and in particular continuous at \( \xi = 0 \)). Next, note that if \( f(\xi) = F(\xi^{1/2}) \), where \( F(\zeta) \) is a smooth even function, then \( f'(\xi) = \Psi(\xi^{1/2}) \), where \( \Psi(\zeta) = \frac{1}{\zeta} F'(\zeta)/\zeta \) is again a smooth even function. Thus, it suffices to prove that if a smooth even function \( F \) satisfies estimates of the form

\[
|F^{(k)}(\zeta)| \leq d_k(1 + |\zeta|)^{-k-k_0}, \quad k = 0, 1, 2, \ldots,
\]
for some $k_0$, then $\Psi$ satisfies the same estimates but with $k_0$ increased by 2 and with new constants $d_k$, each of which is a finite linear combination of the old ones. This is trivial for $|\zeta| \geq 1$, and in the region $|\zeta| < 1$ one can use the identity

$$\zeta^{-1} F'(\zeta) = \zeta^{-1} (F'(\zeta) - F'(0)) = \int_0^1 F''(\theta \zeta) \, d\theta.$$  

\[ \square \]

### 4.2. Computation of the transient part and the equivalent sources.

Now we will prove Lemmas 3.2 and 3.5 by proving the equivalent Theorem 4.2. Let $f(\xi)$ be any of the functions $f_1(\xi)$, $f_2(\xi)$, and $f_3(\xi, t)$ given by (4.1) or the function $f_4(x, t) = \partial f_3(\xi, t)/\partial t$. We need to compute the difference

$$\mathcal{R} = (f(\lambda^{-2} L) - f(\lambda^{-2} L^{(0)}))V \left( \frac{x}{\mu} \right)$$

and estimate it in an appropriate norm. Let us make the change of variables $x = \mu y$. In the new variables, (4.7) becomes

$$\mathcal{R} = (f(L_y) - f(L_y^{(0)}))V(y),$$

where

$$L_y = -\omega^2 \nabla_y \frac{c^2(\mu y)}{c_0} \nabla_y, \quad L_y^{(0)} = -\omega^2 \nabla_y^2, \quad \nabla_y = \left( \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2} \right),$$

and $\omega = c_0/(\lambda \mu)$ is bounded by condition (2.6).

To compute this difference, we use the machinery of noncommutative analysis. We refer the reader to [12] for details concerning the definition and properties of functions of noncommuting operators and only recall that a function $F(A_0, \ldots, A_n)$ of (possibly, noncommuting) operators $A_0, \ldots, A_n$ can be defined as follows in the particular case where the $A_j$ are the generators of uniformly bounded strongly continuous one-parameter operator groups $e^{iA_j t}$, $t \in \mathbb{R}$, on a Hilbert space $H$:

$$F(A_0, \ldots, A_n)u = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathcal{F}(t_1, t_2, \ldots, t_n) e^{iA_{n+1} t_1} \cdots e^{iA_1 t_n} du \cdot \cdots \cdot du,$$

$u \in H$, where $\mathcal{F}$ is the Fourier transform of the symbol $F$, which is assumed to satisfy certain conditions (e.g., see [2]) guaranteeing that the integral on the right-hand side is well defined. The numbers (Feynman indices) over operators indicate the order of their action: of any two operators, the operator with the smaller Feynman index stands to the right of the operators with the larger Feynman index in products.

It follows by the zero-order Newton formula of noncommutative analysis (see [1] and [2] Theorem I.8) that

$$f(L_y) - f(L_y^{(0)}) = \frac{\delta f}{\delta \xi}(L_y, L_y^{(0)})(L_y - L_y^{(0)}) = \frac{\delta f}{\delta \xi}(L_y, L_y^{(0)}) T,$$

where

$$\frac{\delta f}{\delta \xi}(\xi_1, \xi_2) = \frac{f(\xi_1) - f(\xi_2)}{\xi_1 - \xi_2}$$

is the first difference quotient of $f$ and

$$T = L_y - L_y^{(0)} = \omega^2 \left\langle \nabla_y, \left( 1 - \frac{c^2(\mu y)}{c_0^2} \right) \nabla_y \right\rangle \equiv \omega^2 \sum_{j=1}^{2} \frac{\partial}{\partial y_j} \frac{\phi(\mu y)}{\partial y_j}.$$
Here we have denoted
\[
\phi(z) = 1 - \frac{c^2(z)}{c_0};
\]
this function is uniformly bounded together with all of its derivatives for \(z \in \mathbb{R}^2\), and \(\phi(0) = 0\).

We further transform the right-hand side of (4.8) as follows.

**Proposition 4.4.**

\[
\delta f \delta \xi \left(3L_y, 1 \frac{L_y^{(0)}}{} \right) T = \delta f \delta \xi \left(3L_y, 2 \frac{L_y^{(0)}}{} \right) T + \delta^2 f \delta \xi \left(4L_y, 3 \frac{L_y^{(0)}}{} \right) \left[ T, L_y^{(0)} \right],
\]

where
\[
\delta^2 f \delta \xi^2 (\xi_1, \xi_2, \xi_3) = \frac{\delta f \delta \xi (\xi_1, \xi_2) - \delta f \delta \xi (\xi_1, \xi_3)}{\xi_2 - \xi_3}
\]
is the second difference quotient of \(f\) and
\[
\left[ T, L_y^{(0)} \right] = TL_y^{(0)} - L_y^{(0)} T
\]
is the commutator of \(T\) and \(L_y^{(0)}\).

**Proof.** The proof of (4.9) mimics the derivation of the general commutation formula [2] Proposition I.3]

\[
A f(B) - A f(B) = \delta f \frac{1}{\delta \xi} (B, B) \left[ A, B \right]
\]
of noncommutative analysis, with \(T\) playing the role of \(A\) and \(L_y^{(0)}\) playing the role of \(B\). Recall this derivation (e.g., see [2] pp. 52–53). We need to compute

\[
[A, f(B)] = A f(B) - f(B) A = A f(B) - 1 f(B).
\]

The Feynman indices can be chosen independently for either summand on the right, and we can write

\[
[A, f(B)] = 2 A f(B) - A f(B) = 2 A f(B) - A f(B) = A( f(B) - f(B)) = A(B - B) \delta f \frac{1}{\delta \xi} (B, B).
\]

(Here we have used the identity \(f(x) - f(x) = (x - x) \delta f \frac{1}{\delta \xi} (x, x)\), which is in fact the definition of \(\delta f / \delta \xi\).) Next, we move apart the Feynman indices over the \(B\)'s, thus obtaining

\[
2 A(B - B) \delta f \frac{1}{\delta \xi} (B, B) = A(B - B) \delta f \frac{1}{\delta \xi} (B, B) = A(B - B) \delta f \frac{1}{\delta \xi} (B, B).
\]

(In the middle, we have written \(2 A(B - B) = [A, B]\) using the fact that no other operators in the expression have Feynman indices in the interval [1, 3].) Thus, we arrive at the desired commutation formula (4.10).

The derivation of (4.10) differs from this only in that now, instead of \(f(B)\), we have \(\delta f \frac{1}{\delta \xi} (L_y, L_y^{(0)});\) i.e., there is an additional operator argument, \(L_y\), but this argument does not invalidate the computation, because its Feynman number does not lie between those of \(A = T\) and \(B = L_y^{(0)}\).

Let us evaluate the commutator \([T, L_y^{(0)}]\).
for every \( s \) with some constants \( C_s \) independent of \( \mu \) as \( \mu \to 0 \).

**Proof.** We have

\[
[T, L_y^{(0)}] = \omega^4 \langle \nabla_y, [\nabla_y^2, \phi(\mu y)] \rangle = \mu \omega^4 \sum_{j=1}^{2} \left\langle \nabla_y, \left( 2\phi_{z_j}(\mu y) \frac{\partial}{\partial y_j} + \mu \phi_{z_j z_j}(\mu y) \right) \nabla_y \right\rangle,
\]

and it remains to recall that \( \phi(z) \) is uniformly bounded together with all derivatives. \( \square \)

By Propositions 4.4 and 4.5, we can write

\[
f(L_y) - f(L_y^{(0)}) = \frac{\delta f}{\delta \xi}(L_y, L_y^{(0)}) T + \mu \frac{\delta^2 f}{\delta \xi^2}(L_y, L_y^{(0)}) T_1.
\]

Accordingly,

\[
R = (f(L_y) - f(L_y^{(0)})) V = AW + \mu BV,
\]

where

\[
W = TV, \quad A = \frac{\delta f}{\delta \xi}(L_y, L_y^{(0)}), \quad B = \frac{\delta^2 f}{\delta \xi^2}(L_y, L_y^{(0)}) T_1.
\]

Let us estimate the expression \( 4.11 \) for \( f = f_j, j = 1, 2, 3, 4 \).

**Proposition 4.6.** One has \( V \in H^s(\mathbb{R}^2) \) for every \( s \).

**Proof.** This follows from the estimates \( 2.3 \). \( \square \)

**Proposition 4.7.** For every \( s \), one has \( W \in H^s(\mathbb{R}^2) \) and

\[
\|W\|_{H^s(\mathbb{R}^2)} = O(\mu), \quad \mu \to 0.
\]

**Proof.** We have

\[
\phi(\mu y) = \mu \langle F(\mu y), y \rangle,
\]

where the vector function

\[
F(z) = \int_0^1 \frac{\partial \phi}{\partial z}(\theta z) d\theta
\]

is bounded together with all derivatives, and hence for the function \( W = TV \) we obtain

\[
W(y) = \mu \omega^2 \left( \frac{\partial}{\partial y_1} \langle F(\mu y), y \rangle \frac{\partial V(y)}{\partial y_1} + \frac{\partial}{\partial y_2} (F(\mu y), y) \frac{\partial V(y)}{\partial y_2} \right).
\]

Since, by virtue of the estimates \( 2.4 \), the function \( y_j \partial V(y)/\partial y_k \) lies in \( H^s(\mathbb{R}^2) \) for every \( s \), we arrive at the desired assertion. \( \square \)

**Proposition 4.8.** Let \( f = f_j, j = 1, 2, 3, 4 \). Then for each \( s \in \mathbb{R} \) there exists a constant \( C_s \) independent of \( \mu \) such that

\[
\|A: H^s(\mathbb{R}^2) \to H^s(\mathbb{R}^2)\| \leq C_s, \quad \|B: H^s(\mathbb{R}^2) \to H^{s-3}(\mathbb{R}^2)\| \leq C_s
\]
for \( j = 1, 2, \)
\[
\| A : H^s(\mathbb{R}^2_y) \to H^s(\mathbb{R}^2_y) \| \leq C_s e^{-\nu t}, \quad \| B : H^s(\mathbb{R}^2_y) \to H^{s-3}(\mathbb{R}^2_y) \| \leq C_s e^{-\nu t}
\]
for \( j = 3, 4. \)

**Proof.** We make use of the following representation of the \( k \)th difference quotient:
\[
\frac{\delta^k f}{\delta \xi^k}(\xi_1, \ldots, \xi_{k+1}) = \int_{\Delta_k} f^{(k)}(\theta_1 \xi_1 + \cdots + \theta_{k+1} \xi_{k+1}) \, d\theta_1 \cdots d\theta_k
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_{\Delta_k} \left( \int_{-\infty}^{\infty} \tilde{f}^{(k)}(p)e^{ip(\theta_1 \xi_1 + \cdots + \theta_{k+1} \xi_{k+1})} \, dp \right) \, d\theta_1 \cdots d\theta_k,
\]
where \( \tilde{f}^{(k)}(p) \) is the Fourier transform of the \( k \)th derivative \( f^{(k)}(\xi) \) and
\[
\Delta_k = \{(\theta_1, \ldots, \theta_{k+1}) \in \mathbb{R}^{k+1} : \theta_1 + \cdots + \theta_{k+1} = 1, \theta_j \geq 0, j = 1, \ldots, k+1 \}
\]
is the standard \( k \)-simplex. Hence
\[
\frac{\delta f}{\delta \xi^2}(L_y, L_y) = \frac{1}{\sqrt{2\pi}} \int_{\Delta_1} \left( \int_{-\infty}^{\infty} \tilde{f}'(p)e^{ip\theta_1 L_y} \, dp \right) \, d\theta_1
\]
\[
= \frac{1}{\sqrt{2\pi}} \int_{\Delta_2} \left( \int_{-\infty}^{\infty} \tilde{f}''(p)e^{ip\theta_1 L_y} \, dp \right) \, d\theta_1 \, d\theta_2.
\]
(4.13)

Let us estimate the operators (4.13). To this end, we use the following lemma.

**Lemma 4.9.** For each \( s \), there exists a constant \( \tilde{C}_s \) independent of \( \mu \to 0 \) such that
\[
\| e^{itL_y} : H^s(\mathbb{R}^2_y) \to H^s(\mathbb{R}^2_y) \| \leq \tilde{C}_s, \quad \| e^{itL_y} : H^s(\mathbb{R}^2_y) \to H^s(\mathbb{R}^2_y) \| \leq \tilde{C}_s
\]
for all \( t \in \mathbb{R}. \)

**Proof.** For \( s = 0 \), the claim is obvious, because the operators \( L_y(0) \) and \( L_y \) are self-adjoint in \( L^2(\mathbb{R}_y) \). For other values of \( s \), one equips \( H^s(\mathbb{R}^2_y) \) with the equivalent norm \( \| (1 + L_y)^{s/2} u \| \), so that the operator \( L_y \) becomes self-adjoint. This norm depends on the parameter \( \mu \), but it is not hard to prove (for positive integer \( s \) by a straightforward computation, and for other \( s \) by duality and interpolation) that the constants in the inequalities specifying the equivalence of norms remain bounded as \( \mu \to 0 \). The argument for \( L_y(0) \) is simpler, because the parameter \( \mu \) is not involved. The proof of Lemma 4.9 is complete. \( \square \)

Now we can finish the proof of Proposition 4.8. If \( f = f_1, f_2, f_3, \) or \( f_4 \), then it follows from Lemma 4.3 that the Fourier transforms of \( f' \) and \( f'' \) belong to \( L^1(\mathbb{R}) \), and in the case of \( f_3 \) and \( f_4 \) the \( L^1 \)-norm decays as \( e^{-\nu t} \). By combining this with Lemma 4.9 and with the estimate for \( T_1 \) in Proposition 4.5 we arrive at the assertion of Proposition 4.8. \( \square \)

By applying Propositions 1.6, 1.7, and 4.8 to formulas (4.11) and (4.12), we find that \( \mathcal{R} = O(\mu) \) in all \( H^s(\mathbb{R}^2_y) \) for \( f = f_1 \) and \( f = f_2 \) and \( \mathcal{R} = O(\mu e^{-\nu t}) \) in all \( H^s(\mathbb{R}^2_y) \) for \( f = f_3 \) and \( f = f_4 \). Let us finally estimate the remainders \( R_j \) in (4.13).
We should take into account the additional factor $\lambda^{-1}$ for $j = 2$ and pass from the variables $y$ to the original variables $x = \mu y$. Since

\begin{equation}
\|u\| \equiv \|u\|_{H^s(R^2)} \leq \mu^{1-s} \|u\|_{H^s(R^2)} \quad \text{for } \mu \leq 1 \text{ and } s > 0,
\end{equation}

we arrive at the desired estimates (4.14). For example, for $R_2$ we obtain

$$
\|R_2\|_2 \leq C\mu \lambda^{-1} \mu^{1-2} = C\lambda^{-1} \leq \frac{C\omega}{c_0} \mu
$$

(where the factor $\lambda^{-1}$ comes from (4.2) and the factor $\mu^{1-2} = \mu^{-1}$ from (4.14) for $s = 2$). The estimates for $R_1$ and $R_3$ are similar. The proof of Theorem 4.2 and hence of Lemmas 3.2 and 3.5 is complete.

\section{Examples}

In conclusion, let us present two simple examples in which the asymptotics of the solution of the Cauchy problem (2.1), (2.2) with a special right-hand side will be demonstrated. Namely, we use the right-hand side $Q(x, t) = \lambda^2 g_0(\lambda t) V(x/\mu)$, where $V(y) = A(1 + (y_1/b_1)^2 + (y_2/b_2)^2)^{-3/2}$ is the simplest spatial shape factor (2.11) and the function $g_0(\tau)$ is given by one of formulas (a) (a sine source) and (b) (a polynomial source) in Eq. (2.13).

Recall that the asymptotics of the solution is given by Theorem 3.1, Eqs. (3.1) and (3.2) (the transient solution component) and by Theorem 3.8, Eq. (3.22) (the propagating solution component away from the focal points). The transient component $\eta_{trans}(x, t)$ and the wave profile $F(z, \psi)$ (see (3.23)) of the propagating component depend only on the right-hand side and on the parameters $\lambda, \mu$, and $\omega$; they are represented by integrals which, for our choice of the right-hand side, can be evaluated (or, in the case of the transient component, considerably simplified) analytically. The other ingredients of the asymptotic formula (3.22) for the propagating component (the phase functions $S_j(x, t)$, the Lagrangian coordinates $\psi_j(x, t)$, the Morse index $m(\psi_0, t)$, and the factors responsible for the Green law and for the trajectory divergence) depend on the solution of the Cauchy problem (3.21) for the Hamiltonian system (3.17), which, except for the simplest cases, should be solved numerically.

Accordingly, our exposition in both examples is as follows. First, we find the function $G_0(\xi, t)$ (2.32), which plays a crucial role in all subsequent calculations. Then we write out the wave profile $F(z, \psi)$ and finally present the expression for the transient component $\eta_{trans}(x, t)$ of the solution. In the second example, we also numerically compute the trajectories and display snapshots of the solution obtained with the use of Wolfram Mathematica.

The calculations are mostly carried out in polar coordinates, so let us rewrite formula (2.12) for the Fourier transform of $V$ in the polar coordinates $(\rho, \psi)$, where $p = \rho n(\psi)$ with $n(\psi) = (\cos \psi, \sin \psi)$:

\begin{equation}
\tilde{V}(\rho n(\psi)) = Ab_1b_2e^{-\rho \beta(\psi)}, \quad \text{where } \beta(\psi) \equiv \sqrt{b_1^2 \cos^2 \psi + b_2^2 \sin^2 \psi}.
\end{equation}

\subsection{5.1. The case of a sine source}

Let

$$
g_0(\tau) = ae^{-\tau}(\sin(\alpha \tau + \phi_0) - \sin \phi_0)
$$
where \( a = (\alpha^2 + 1)/(\alpha \cos \phi_0 - \alpha^2 \sin \phi_0) \) is a normalizing factor. By evaluating the integral in (2.32), we obtain

\[
G_0(\xi, t) = ae^{-t} \left( \frac{ie^{-i(\alpha t + \phi_0)/2}}{1 + i\alpha + i\xi} - \frac{ie^{i(\alpha t + \phi_0)/2}}{1 - i\alpha + i\xi} - \sin \phi_0 \right) e^{i\phi_0/2} (1 + i\xi).
\]

We see that \( G_0(\xi, t) \) is a rational function of \( \xi \). Moreover, a routine computation (which we omit) shows that it can be represented in the form

\[
G_0(\xi, t) = \sum_m q_m(t) (R_m(\xi^2) + i\xi Q_m(\xi^2)),
\]

where \( R_m(\zeta) \) and \( Q_m(\zeta) \) are rational functions with real coefficients and with denominators nonvanishing for \( \zeta \geq 0 \). This is, of course, consistent with the assertion in Lemma 4.1 concerning the parity of the real and imaginary parts of \( G_0 \). As to \( \tilde{g}_0(\xi) \), we have

\[
\tilde{g}_0(\xi) = \frac{1}{\sqrt{2\pi}} G_0(\xi, 0) = \frac{a}{\sqrt{2\pi}} \left( \frac{ie^{-i\phi_0/2}}{1 + i\alpha + i\xi} - \frac{ie^{i\phi_0/2}}{1 - i\alpha + i\xi} - \sin \phi_0 \right).
\]

To evaluate the wave profile \( F(z, \psi) \) of the propagating solution component, we substitute the functions (5.1) and (5.4) into formula (3.23) and obtain

\[
F(z, \psi) = \frac{a Ab_1 b_2 e^{-i\pi/4}}{\sqrt{2\pi \omega^{3/2}}} \times \int_0^\infty \sqrt{\beta} \left( \frac{ie^{-i\phi_0/2}}{1 + i\alpha - i\beta} - \frac{ie^{i\phi_0/2}}{1 - i\alpha - i\beta} - \sin \phi_0 \right) e^{-\rho\omega^{-1}(\beta(\psi) - iz)} d\rho
\]

\[
= \frac{a Ab_1 b_2 e^{-i\pi/4}}{\sqrt{2\pi \omega^{3/2}}} \left\{ \frac{i}{2} e^{-i\phi_0} I_0(\omega^{-1}(\beta(\psi) - iz), 1 + i\alpha) - \frac{i}{2} e^{i\phi_0} I_0(\omega^{-1}(\beta(\psi) - iz), 1 - i\alpha) - I_0(\omega^{-1}(\beta(\psi) - iz), 1) \sin \phi_0 \right\},
\]

where the integral

\[
I_0(C_1, C_2) = \int_0^\infty \frac{\sqrt{\rho e^{-C_1 \rho}}}{C_2 - i\rho} d\rho, \quad C_1, C_2 \in \mathbb{C}, \quad \text{Re} C_1 > 0, \quad \arg C_2 \neq \frac{\pi}{2},
\]

can be expressed via the complementary error function

\[
\text{erfc}(w) = \frac{2}{\sqrt{\pi}} \int_w^\infty e^{-v^2} dv
\]

by the formula

\[
I_0(C_1, C_2) = \frac{i\sqrt{\pi}}{\sqrt{C_1}} + e^{-i\pi/4} \pi \sqrt{C_2 e^{iC_1} C_2} \text{erfc} \left( e^{i\pi/4} \sqrt{C_1 C_2} \right).
\]
To evaluate the transient term of the solution, we substitute the functions (5.1) and (5.2) into (5.2) and obtain
\begin{align*}
\eta_{\text{trans}}(r_n(\varphi)) &= \frac{a A b_1 b_2 e^{-\lambda t}}{2 \pi \omega^2} \int_{0}^{2\pi} \text{Re} \left[ \frac{i}{z} \left( \sin(\alpha \lambda t + \phi_0) - \sin \phi_0 \right) + \sin \phi_0 \frac{i}{2} e^{-i z} \left( \pi + 2 i \text{Ci}(z) - 2 \text{Si}(z) \right) \right. \\
&\quad + \frac{\alpha - i e^{-i(\alpha t + \phi_0)}}{2} e^{i(\alpha t + \phi_0)} \left. \int_{0}^{\infty} \frac{e^{-\rho^2} d\rho}{\rho + \alpha - i} \right] d\psi + O(\mu) \\
&= \frac{a A b_1 b_2 e^{-\lambda t}}{2 \pi \omega^2} \int_{0}^{2\pi} \text{Re} \left[ \frac{i}{z} \left( \sin(\alpha \lambda t + \phi_0) - \sin \phi_0 \right) + \sin \phi_0 \frac{i}{2} e^{-i z} \left( \pi + 2 i \text{Ci}(z) - 2 \text{Si}(z) \right) \right. \\
&\quad + \frac{\alpha - i e^{-i(\alpha t + \phi_0)}}{2} e^{i(\alpha t + \phi_0)} \left. \int_{0}^{\infty} \frac{e^{-\rho^2} d\rho}{\rho + \alpha - i} \right] d\psi + O(\mu),
\end{align*}
where \( z = z(\tau, \varphi, \psi) = \omega^{-1} (\beta(\psi) - i r \mu^{-1} \cos(\psi - \varphi)) \), \( \text{Re}(z) > 0 \), and
\[ E_1(z) = \int_{z}^{1} e^{-t} \frac{e^{-t}}{t} dt, \quad \text{Ci}(z) = -\int_{z}^{\infty} \cos(t) \frac{e^{-t}}{t} dt, \quad \text{Si}(z) = \int_{0}^{z} \sin(t) \frac{e^{-t}}{t} dt. \]

5.2. The case of a polynomial source. Now let
\[ g_0(\tau) = e^{-\tau} P(\tau), \]
where
\[ P(\tau) = \sum_{k=1}^{n} P_k \tau^k \]
is a polynomial of degree \( n \) with coefficients \( P_k \) such that \( P_0 = 0 \) and \( \sum_{k=1}^{n} P_k = 1 \). Let us use formula (3.24) for \( G_0(\xi, \tau) \). Since
\[ \int_{0}^{\infty} e^{-t-\tau-i\xi \tau} (t + \tau)^k d\tau = e^{-t} \left( t + i \frac{\partial}{\partial \xi} \right)^k \frac{1}{1 + i \xi}, \]
it follows that
\[ G_0(\xi, t) = e^{-t} P \left( t + i \frac{\partial}{\partial \xi} \right) \frac{1}{1 + i \xi}, \quad \bar{g}_0(\xi) = \frac{1}{\sqrt{2\pi}} P \left( \frac{\partial}{\partial \xi} \right) \frac{1}{1 + i \xi}, \]
and we see that \( G_0(\xi, t) \) again has the form (5.3). Using (5.1), (5.6) and (3.24), we evaluate the wave profile of the propagating part of the solution as follows:
\[ F(z, \psi) = \frac{A b_1 b_2 e^{-i \pi/4}}{\sqrt{2\pi \omega^{3/2}}} \left[ P \left( - \frac{\partial}{\partial C_2} \right) I_0(\rho(\beta(\psi) - i z)/\omega, C_2) \right] \bigg|_{C_2=1} \]
\[ = -i \frac{A b_1 b_2 \sqrt{\pi}}{\sqrt{2\omega^{3/2}}} e^{i C_1} \left[ P \left( \frac{2}{C_1} \left( i + \frac{1}{2 C_1} + \frac{d}{d C_1} \right) \right) \text{erfc}(\sqrt{i C_1}) \right] \bigg|_{C_1=\frac{\beta(\psi) - i z}{2}}, \]
where \( I_0(C_1, C_2) \) is the integral (5.5).
Remark 5.1. In both examples, one can prove that the following asymptotic formulas hold for the functions $F(z, \psi)$ for small $\omega$: \[ F(z, \psi) = \frac{ib_1 b_2}{2\sqrt{2}(z + i\beta(\psi))^{3/2}} + O(\omega). \] This means that for small $\omega$ the solution of the inhomogeneous problem (corresponding to "sources stretched in time") passes into the solution of the homogeneous problem (corresponding to "instantaneous sources").

Let us compute the transient term of the solution for the case in which $P(\tau)$ is a second-order polynomial; then \[ G_0(\xi, t) = e^{-t} \left( \frac{P_2 t^2/2 + (P_1 - P_2)t - P_1}{1 + \xi^2} + \frac{2P_2 t + 2P_1 - 3P_2}{(1 + \xi^2)^2} + \frac{4P_2}{(1 + \xi^2)^3} \right) \]
\[ - i\xi e^{-t} \left( \frac{P_2 t^2/2 + P_1 t}{1 + \xi^2} + \frac{2P_2 t + 2P_1 - P_2}{(1 + \xi^2)^2} + \frac{4P_2}{(1 + \xi^2)^3} \right). \]
For the transient term, we find \[ \eta_{trans} = -\lambda^2 e^{-\lambda t} \left[ \left( P_2 \lambda^2 t^2/2 + (P_1 - P_2)\lambda t - P_1 \right) \Theta_1 \left( \frac{x}{\mu} \right) \right. \]
\[ \left. + \left(2P_2 \lambda^3 t + (2P_1 - 3P_2)\lambda^2 \right) \Theta_2 \left( \frac{x}{\mu} \right) + 4P_2 \lambda^4 \Theta_3 \left( \frac{x}{\mu} \right) \right], \]
where \[ \Theta_k(y, \mu) = \frac{Ab_1 b_2}{2\pi \lambda^2 k} \int_{\mathbb{R}^2} \frac{e^{i(p, \psi)} e^{-\sqrt{(b_1 p_1)^2 + (b_2 p_2)^2}}}{(1 + (\omega|p|^2)^k} dp dp. \]
If we pass to the polar coordinates by setting $y = r \mathbf{n}(\varphi)$ and $p = \rho \mathbf{n}(\psi)$, then we obtain \[ \Theta_k(r \mathbf{n}(\varphi), \mu) = \frac{Ab_1 b_2}{2\pi \lambda^2 k} \int_0^2 \int_0^{2\pi} \frac{e^{i(p, \psi)} e^{-\sqrt{(b_1 p_1)^2 + (b_2 p_2)^2}}}{(1 + (\omega^2 \rho^2)^k} \rho d\rho d\psi. \]
Here one can evaluate the integral over $\rho$. For $k = 1, 2, 3$, we obtain
\[ \Theta_1(r \mathbf{n}(\varphi), \mu) = \frac{Ab_1 b_2}{2\pi \lambda^2 \omega^2} \int_0^{2\pi} d\psi \left( -\cos(z) \text{Ci}(z) + \frac{1}{2} \sin(z) (\pi - 2 \text{Si}(z)) \right), \]
\[ \Theta_2(r \mathbf{n}(\varphi), \mu) = \frac{Ab_1 b_2}{8\pi \lambda^2 \omega^2} \int_0^{2\pi} d\psi \left( 2 - 2z \sin(z) \text{Ci}(z) - z \cos(z) (\pi - 2 \text{Si}(z)) \right), \]
\[ \Theta_3(r \mathbf{n}(\varphi), \mu) = \frac{Ab_1 b_2}{32\pi \lambda^2 \omega^2} \int_0^{2\pi} d\psi \left( 4 - z \sin(z) (\pi z + 2 \text{Ci}(z) - 2z \text{Si}(z)) \right. \]
\[ \left. + z \cos(z) (-\pi + 2z \text{Ci}(z) + 2z \text{Si}(z)) \right), \]
where $z(\psi) = \omega^{-1}(\beta - i \rho \cos(\psi - \varphi))$.

An illustration of the solution given by the sum of propagating and transient terms in the second example is shown in Fig. Here the propagating part is calculated for the constant velocity $c(x) \equiv c_0 = 1$, and other constants are $b_1 = 1, b_2 = 2, \Lambda = 1, \mu = 0.1, P_1 = 0, P_2 = 1$. The first four snapshots are taken at small times $t = 0.3, 0.7, 1.0, 1.5$ to show how the transient term behaves, and the last three snapshots are taken at large times $t = 1.5, 4.0, 6.5$. At $t = 6.5$, the transient term practically disappears, while the propagating part continues its motion. The function $g_0$ and the wave profile for $P_1 = -2, P_2 = 3$, and various $\lambda$ are
Figure 3. Sum of waves $\eta_{\text{prop}} + \eta_{\text{trans}}$.

compared in Fig. 4. For small $\lambda$, the wave profile has the form that “reproduces” the shape of the function $g_0$, while for large $\lambda$ the wave profile is almost the same as for $g_0 = \delta(t)$. 
\[ g = \lambda e^{-\frac{\lambda}{2}(P_1 \lambda t + \frac{P_2}{2} \lambda^2 t^2)}, \quad \lambda = 0.5, 1, 5, 100, \quad P_1 = -2, P_2 = 3 \]

Figure 4. Examples of profiles of propagating waves.

References

[1] V. P. Maslov, Operational Methods. Mir, 1973.
[2] V. E. Nazaikinskii, B. Yu. Sternin, and V. E. Shatalov, Methods of Noncommutative Analysis. Walter de Gruyter, 1996.
[3] V. P. Maslov and M. V. Karasev, Nonlinear Poisson Brackets, Geometry and Quantization. Amer. Math. Soc., 1993.
[4] C. C. Mei, The Applied Dynamics of Ocean Surface Waves. World Scientific, 1989.
[5] E. N. Pelinovski, Hydrodynamics of Tsunami Waves. Nizhni Novgorod, 1996.
[6] Yu. I. Shokin, L. B. Chubarov, A. G. Marchuk, and K. V. Simonov, Numerical Experiment in Tsunami Problem. Nauka, Siberian Branch, 1989.
[7] B. V. Levin, Tsunami and Seetquake in the Ocean. Nature 5 (1996), 48–61.
[8] S. F. Dotsenko, B. Yu. Sergievskii, and L. V. Cherkasov, *Space Tsunami Waves Generated by Alternating Displacement of the Ocean Surface*. Tsunami Research 1 (1986), 7–14.

[9] S. Wang, *The Propagation of the Leading Wave*, ASCE Specialty Conference on Coastal Hydrodynamics, University of Delaware, June 29–July 1, 1987, pp. 657–670.

[10] S. Ya. Sekerzh-Zenkovich, *Simple Asymptotic Solution to the Cauchy–Poisson Problem for Leading Waves*. Russ. J. Math. Phys. 16:2 (2009), 215–222.

[11] S. Dobrokhotov, S. Sekerzh-Zenkovich, B. Tirozzi, and T. Tudorovski, *Description of Tsunami Propagation Based on the Maslov Canonical Operator*. Doklady Math. 74:1 (2006), 592–596.

[12] S. Yu. Dobrokhotov, A. I. Shafarevich, and B. Tirozzi, *Localized Wave and Vortical Solutions to Linear Hyperbolic Systems and Their Application to the Linear Shallow Water Equations*. Russ. J. Math. Phys. 15:2 (2008), 192–221.

[13] S. Yu. Dobrokhotov, B. Tirozzi, and C. A. Vargas, *Behavior Near the Focal Points of Asymptotic Solutions to the Cauchy Problem for the Linearized Shallow Water Equations with Initial Localized Perturbations*. Russ. J. Math. Phys. 16:2 (2009), 228–245.

[14] S. Yu. Dobrokhotov, R. Nekrasov, and B. Tirozzi, *Asymptotic Solutions of the Linear Shallow-Water Equations with Localized Initial Data*. J. Engng. Math. 69:2 (2011), 225–242.

[15] S. Yu. Dobrokhotov, B. I. Volkov, S. Ya. Sekerzh-Zenkovich, and B. Tirozzi, *Explicit Asymptotics for Tsunami Waves in Framework of the Piston Model*. Russ. J. Earth Sci. 8 (ES403), 1–12 (2006).

[16] S. Yu. Dobrokhotov, B. I. Volkov, S. Ya. Sekerzh-Zenkovich, and B. Tirozzi, *Asymptotic description of tsunami waves in a frame of piston model: General construction and explicitly solvable models*. Fund. Applied Geophysics 2:4 (2009), 15–29.

[17] B. Levin, V. Kaistrenko, A. Kharlamov, M. Chepareva, and V. Kryshny, *Physical Processes in the Ocean as Indicators for Direct Tsunami Registration from Satellite*. Tsunami ’93, Wakayama, Japan, August 23–27, 1993. In *Proceedings of the IUGG / IOC International Tsunami Symposium*, Wakayama, Japan, 1993, pp. 309–320.

[18] B. W. Levin and V. M. Kaistrenko, *Recent Tsunamis and Earthquakes, its Physical Modelling and Satellite Registration*. In Intern. Union of Geodesy and Geophys. XXI General Assembly, Boulder, Colorado, July 2–14, 1995: Abstr. Week A., Boulder, 1995, p. A337.

[19] H. Yeh, V. Titov, V. Gusyakov, E. Pelinovsky, V. Khramushib, and V. Kaistrenko, *The 1994 Shikotan Earthquake Tsunamis*. PAGEOPH. 144:3–4 (1995), 856–874.

[20] V. P. Maslov and M. V. Fedoriuk, *Semi-Classic Approximation in Quantum Mechanics*. Reidel, 1981.

[21] D. Bianchi, S. Dobrokhotov, and B. Tirozzi, *Asymptotics of Localized Solutions of the One-Dimensional Wave Equation with Variable Velocity. II: Taking into Account a Source on the Right-Hand Side and a Weak Dispersion*. Russ. J. Math. Phys. 15:4 (2008), 427–446.

[22] S. Yu. Dobrokhotov, V. E. Nazarkin, V. Khramushib, and B. Tirozzi, *Asymptotic Solutions of 2D Wave Equations with Variable Velocity and Localized Right-Hand Side*. Russ. J. Math. Phys. 17:1 (2010), 66–76.

[23] S. Yu. Dobrokhotov, V. E. Nazarkin, and B. Tirozzi, *Asymptotic Solution of the One-Dimensional Wave Equation with Localized Initial Data and with Degenerating Velocity: I*. Russ. J. Math. Phys. 17:4 (2010), 428–444.

[24] S. Yu. Dobrokhotov, V. E. Nazarkin, and B. Tirozzi, *Asymptotic Solutions of the Wave Equation with Degenerating Velocity and Localized Initial Data (Two-Dimensional Case)*. Algebra Analiz 22:6 (2010), 67–90.

[25] S. Yu. Dobrokhotov and B. Tirozzi, *Localized Solutions of the One-Dimensional Nonlinear System of Shallow Water Equations with Velocity c = \( \sqrt{3} \)*. Uspekhi Mat. Nauk 65:1 (391) (2010), 185–186.

[26] M. Sh. Birman and M. Z. Solomyak, *Spectral Theory of Self-Adjoint Operators in Hilbert Space*. Reidel, 1987.

[27] Wolfram Mathematica®, [www.wolfram.com](http://www.wolfram.com).
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