Kinematic Bias in Cosmological Distance Measurement

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ABSTRACT
Recent calculations using non-linear relativistic cosmological perturbation theory show biases in the mean luminosity distance and distance modulus at low redshift. We show that these effects may be understood very simply as a non-relativistic, and purely kinematic, Malmquist-like bias, and we describe how the effect changes if one averages over sources that are limited by apparent magnitude. This effect is essentially identical to the distance bias from small-scale random velocities that has previously been considered by astronomers, though we find that the standard formula overestimates the homogeneous bias by a factor 2.

Key words: Cosmology: theory, observations, distance scale, large-scale structure; galaxies: distances and redshifts

1 INTRODUCTION
It is well known that the local rate of expansion $H_0$ is significantly perturbed, at linear order, by peculiar velocities associated with the growth of density perturbations. The impact of this on cosmological parameter estimation is quantified theoretically by calculating the covariance of the first order velocity field which is given in terms of the power spectrum of density fluctuations (Hui & Greene 2006; Cooray & Caldwell 2006; Davis et al. 2011; Kaiser & Hudson 2014).

The subject of this paper, in contrast, is the systematic bias in distances, and therefore $H_0$, caused by velocities, and which is a second order effect. This has been studied using 2nd order relativistic cosmological perturbation theory in a number of recent papers (Vanderveld, Flanagan & Wasserman, 2007; Li & Schwarz 2008; Clarkson, Ananda & Larena 2009; Umei, Larena & Clarkson 2011; Gasperini et al. 2011; Wiegand & Schwarz 2012; Fanizza et al. 2013; Ben-Dayan et al. 2012a, 2012b, 2013a, 2013b, 2014).

These papers all compute the deviation of quantities such as the mean luminosity distance and distance modulus (log distance), averaged over a surface of constant redshift, from that which would apply in a homogeneous universe. Second order perturbation theory is being used in order to explore the regime of non-linear gravitational dynamics. Most of these papers describe the effect as backreaction from the formation of structure, though the term may be used in a relatively broad sense compared to the narrow definition as the effect of non-commutativity of spatial averaging and time evolution deriving from the non-linearity of Einstein’s equations.

Quantitative predictions in the context of conventional structure formation models are provided in e.g. figure 6 of Ben-Dayan et al. 2013b which shows that the bias falls off inversely as the square of the redshift; that the fractional perturbation to the mean distance $\delta_d \equiv \langle \delta d_L \rangle / d_L$ is positive, and that the perturbation to the mean flux density $\Phi$ is negative with $\delta \Phi \equiv \langle \delta \Phi \rangle / \Phi \approx -0.5 \delta d$. Further, according to Ben-Dayan et al. 2014 (hereafter BDMS14), for low redshift $z \ll 1$ the mean flux density perturbation is given in terms of $\langle v^2 \rangle$, the total variance of the first order line-of-sight peculiar velocity, by $\delta \Phi = -\langle v^2 \rangle / c^2 z^2$, and they give the bias in the distance modulus $\mu = 5 \log d_L = (5/\ln 10) \ln d_L$ as $\langle \delta \mu \rangle = (7.5/\ln 10) \langle v^2 \rangle / c^2 z^2$.

There are two surprising features of these results if they are assumed to be caused by inhomogeneity affecting the evolution of the averaged universe. First, a cosmological effect would be expected to grow with increasing redshift rather than decrease. Second, one would expect perturbations to distance, distance modulus and flux density to be related by $\delta \mu = (5/\ln 10) \langle \delta d_L \rangle / d_L$ and $\delta \Phi / \Phi = -2 \langle \delta d_L \rangle / d_L$, just as for an individual ‘standard candle’. The relations between these quantities obtained from perturbation theory are quite different, and suggest that the cause of these effects are fluctuations. In that case, the usual relations for a standard candle would not apply, simply because of the non-commutativity of averaging and non-linear transformations; the mean of the square of a fluctuating quantity, for example, is of course not the same as the square of the mean. The effect of fluctuations and the non-linearity of the relationships between $d_L$, $\mu$ and $\Phi$ was discussed by BDMS14 who noted that the bias in $H_0$ depends on the observable used, and by Ben-Dayan et al. 2013a, who argued for using the flux density $\Phi$ in $H_0$ measurements, claiming this to be the least sensitive to fluctuations.

Statistical biases in distance estimation, often asso-
associated with the names Eddington (1914) and Malmquist (1920), have been known and widely studied for a long time, in the context of both cosmological parameter estimation and measurements of large-scale peculiar motions or ‘cosmic flows’. Substantial biases may result from the typically ~20% uncertainty in luminosity distance estimators for galaxies such as are obtained from the Tully-Fisher (TF) relation for spirals (Tully & Fisher 1977) and from the ‘fundamental plane’ (FP) for elliptical galaxies (Djorgovski & Davis, 1987; Dressler et al. 1987). In particular, distance estimates to galaxies may suffer so-called ‘homogeneous Malmquist bias’ in that field galaxies in some range of estimated distance will tend to have true distances that are, on average, systematically enhanced as more galaxies are scattered inward from larger distances than outward from smaller distances (see Lynden-Bell et al. 1988; Willick 1994; and the reviews of Faber et al. 1994 and Strauss & Willick 1995 for more details). Lynden-Bell et al. 1988 showed that with a log-normal model for the distribution of distance errors the mean log-distance in a spatially homogeneous universe would be biased upward by \( \delta \ln d = 3 \Delta^2 \) where \( \Delta^2 \) is the fractional distance error variance.

This particular kind of bias may be avoided by considering the mean peculiar displacement in redshift-space (where neighbouring sources have, to a good approximation, the same distance) rather than the peculiar motion in estimated distance space (Schechter 1980). This bias is also not particularly relevant to the calculations above as in estimated distance space (Schechter 1980). This bias is also not particularly relevant to the calculations above as the effects of interest here are generally of order \( \sim (v/c)^2 \).

We first consider the bias in the distance and related quantities when averaged over the surface of constant redshift as this is simple, illustrates the key features of the phenomenon, and is what was considered in the relativistic perturbation theory studies. We then generalise the analysis to the more realistic case where we average these quantities over sources.

2 MALMIQUST BIASES FROM LARGE-SCALE COHERENT FLOWS

Here we will calculate the kinematic bias arising from ‘coherent flows’ or ‘streaming motions’; these being the focus of the relativistic perturbation theory calculations. We consider small-scale ‘thermal’ motions later. Since we are interested in the low redshift regime \( z \ll 1 \) we work in flat, empty space and, we will also ignore special relativistic effects as the effects of interest here are generally of order \( \sim (v/c)^2 \).

2.1 Area Averaged Bias

We imagine an ensemble of realisations of a smooth field of test particles that have a spatially continuous velocity field that consists of a Hubble flow \( H \mathbf{r} \) plus a statistically homogeneous random velocity perturbation field, and where one particle is selected at random as the observer and is taken to lie at the origin of spatial coordinate system. Let the velocity with respect to this observer be \( \mathbf{u}(\mathbf{r}) \) and define the peculiar velocity \( \mathbf{v} = \mathbf{u} - H \mathbf{r} \). Let us further assume, in the spirit of perturbation analysis, that the amplitude and scale length for perturbations in the peculiar velocity are such that there is a unique mapping from velocity (or redshift) space to real space; i.e. all particles in some region of redshift space have the same peculiar velocity. Working in units such that both the speed of light \( c \) and the expansion rate \( H \) are unity, the distance is \( d = |\mathbf{r}| = z - v \) where \( v \) is the line-of-sight component of the peculiar velocity.

Consider a cone of infinitesimal solid angle \( d\Omega \). In redshift space, the intersection of this cone and a constant-\( z \) surface has area \( dA_z = z^2 d\Omega \). That two dimensional surface maps to surface element in real space that will lie at a perturbed distance \( d = z - v = z(1 - v/z) \) and which will, in general, be slightly tilted relative to the line of sight as there will, in general, be some gradient of \( v \) transverse to the line of sight \( \nabla_z v \). The surface element area in real space is then

\[
dA_r = (1 - v/z)^2 (1 + |\nabla_z v|^2/2) dA_z.
\]
The average of the fractional perturbation to the distance \( \delta_d = (d - z)/z = -v/z \) over a solid angle \( \Delta \Omega \), weighted by real-space area, is then
\[
\overline{\delta_d} = \frac{\int d\Omega (1 - v/z)^2 (1 + |\nabla v|^2/2)(-v/z)}{\int d\Omega (1 - v/z)^2 (1 + |\nabla v|^2/2)}. \tag{2}
\]

We wish to evaluate \( \overline{\delta_d} \) accurately to second order in velocities. Since there is a factor \( v/z \) in the numerator, that means we need only keep first order terms in the denominator, and we can completely ignore the transverse derivative terms as they appear only at third order, to give
\[
\overline{\delta_d} = \frac{\int d\Omega}{\Delta \Omega} \left\{ -\frac{v}{z} + 2\frac{v^2}{z^2} - \frac{v}{z} \int \frac{d\Omega'}{\Delta \Omega} (v'v') \right\} \tag{3}
\]
the last factor here allowing for correlation between the numerator and denominator in (2).

The integrals here are evaluated on the surface \( z = \) constant i.e. on the perturbed surface in real space \( d = z - v \). Working to second order precision, \( \overline{\delta_d} \) is given in terms of quantities on the constant distance surface \( d = z \) using \( v(z - v) = v(d = z) - v|v|d + \ldots = v - (1/2)v|v|^2dz + \ldots \) (we can ignore the effect on the second order terms above as the change in these is third order in \( v \)).

Taking the ensemble average, which we will denote by \( \langle \ldots \rangle \), the expectation value of the first order term here vanishes as the velocity is equally likely to be positive as negative – this is equally true in real-space and redshift-space since, like \( v(d) \), \( dv^2/dz \) is equally likely to be positive or negative – with the result
\[
\overline{\langle \delta_d \rangle} = \frac{2\langle v^2 \rangle}{z^2} - \frac{2}{z^2} \int \frac{d\Omega}{\Delta \Omega} \int \frac{d\Omega'}{\Delta \Omega} (v'v') = \frac{1}{z^2} \int \frac{d\Omega}{\Delta \Omega} \int \frac{d\Omega'}{\Delta \Omega} (v - v')^2. \tag{4}
\]

The last expression above makes it clear that \( \overline{\langle \delta_d \rangle} > 0 \) so the mean distance is biased upwards. It also shows that, for an averaging area that subtends a small solid angle \( \Delta \Omega \ll 1 \), only velocities caused by density perturbations with scale comparable to or smaller than the averaging region contribute significantly to the bias; for perturbations much larger than the averaging region size the velocity will vary little within the area so \( v' \simeq v \) and the bias is strongly suppressed.

If instead of the perturbation to the distance, which is linear in \( v \) (for given \( z \)), we calculate the perturbation to some observable \( X \) that is a non-linear function of distance like the flux-density or the distance modulus then we need to include the second order term in the expansion of \( X \) as a function of \( v/z \). If the perturbation is \( \delta X = av/z + bn|z|^2 + \ldots \), then we simply replace the factor \( (-v/z) \) in (2) by \( av/z + bn|z|^2 \) and performing the same expansion – dropping terms that are cubic or higher in the velocity – and ensemble averaging that led to (3) and then to (4) now gives
\[
\overline{\langle \delta X \rangle} = \left( -2a + b \right) \frac{\langle v^2 \rangle}{z^2} + \frac{2a}{z^2} \int \frac{d\Omega}{\Delta \Omega} \int \frac{d\Omega'}{\Delta \Omega} (v'v'). \tag{5}
\]

We can use this to give the fractional perturbation to the flux density of standard sources. These have \( \Phi(d) \propto 1/d^2 \) so \( \Phi(d) = \Phi(z)(1 - v/z)^2 \) and \( \delta \Phi \equiv \langle \Phi(d) - \Phi(z) \rangle / \Phi(z) = (1 - v/z)^2 - 1 = 2v/z + 3v^2/z^2 + \ldots \), so the ensemble average of the area averaged flux density perturbation \( \delta \Phi \) is given by (4) with \( a = 2, b = 3 \) or
\[
\overline{\langle \delta \Phi \rangle} = -\frac{\langle v^2 \rangle}{z^2} + \frac{4}{z^2} \int d\Omega \int d\Omega' (v'v'). \tag{6}
\]

Similarly, the perturbation to the distance modulus (DM) \( \mu \equiv 5 \log_{10} d \) is \( \delta \mu = \alpha \ln(1 - v/z) = -\alpha (v/z + v^2/2z^2 + \ldots) \) with \( \alpha \equiv 5/\ln 10 \simeq 2.17 \), so the ensemble average of the area average of \( \delta \mu \) is given by (5) with \( a = -\alpha, b = -\alpha/2 \) or
\[
\overline{\langle \delta \mu \rangle} = \alpha \left[ \frac{3\langle v^2 \rangle}{2z^2} - \frac{2}{z^2} \int d\Omega \int d\Omega' (v'v') \right]. \tag{7}
\]

Note that in both of these cases, in contrast to (1), there is not complete suppression of the effect of perturbations on scales larger than the averaging area.

If we take the averaging area to cover the entire sky, and assume that the redshift is sufficiently large that the distance to this shell is much greater than the coherence scale for the velocity fluctuations then the second term involving \( \langle v'v' \rangle \) in each of equations (1), (5) & (6) will be much smaller than the first term and we have
\[
\overline{\langle \delta_d \rangle} = 2\langle v^2 \rangle/z^2 \hspace{1cm} \overline{\langle \delta \Phi \rangle} = -\langle v^2 \rangle/z^2 \hspace{1cm} \overline{\langle \delta \mu \rangle} = (7.5/\ln(10))\langle v^2 \rangle/z^2. \tag{8}
\]

These are identical to the low-z limit expressions of BDMS14. So the relatively large low-z effects are not in an essential way a result of non-linearity of gravitational dynamics (relativistic or Newtonian) as they are fully accounted for by kinematics and statistics. We believe, of course, that the velocities we observe are really caused by gravity, and non-linear structure is involved, but our point here is that the same bias would be found if one were observing test particles of negligible mass with peculiar motions caused by non-gravitational forces.

These Malmquist-like biases are easy to understand. The perturbation to the mean distance, for example, comes about because even though the velocity field on a sphere of constant-z is equally likely to be positive or negative, so as many areas (or solid angle elements at the observer) get pushed out as get pushed in in distance-space, those that get pushed out to larger \( d \) get pushed in the radial direction and so get expanded in area by a factor \( (1 - v/z)^2 \simeq 1 - 2v/z \) (see figure 1). Similarly those that get displaced inwards get compressed. The result is a rectification of the real-space area averaged distance. The different numerical factors for the other variables comes about simply because they are non-linear functions of the distance.

BDMS14 noted that the above imply that the bias in \( H_0 \) obtained from the area-averaged flux density is a factor 3 lower than that obtained from averaging the distance modulus. The above analysis shows that one can do even better by averaging \( \Phi(z)^2 \propto (1 - v/z)^{-2} \) since this gives \( a = 3 \) and \( b = 0 \) so \( -2a + b = 0 \) and, in the approximation that the depth is greater than the coherence scale used to obtain (8), the bias vanishes.

We would emphasise that, according to our analysis, the simple results (8) are only valid for velocity perturbations with coherence scale less than the distance. But at the same time the effects are really only significant at low redshift because of the \( 1/z^2 \) scaling. For realistic power spectra there is

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significant contribution to the velocity variance from quite large scales; certainly extending to tens if not hundreds of Mpc, so except for observations at much greater distance – where the effects rapidly become uninterestingly small – one should not use these formulae with the total velocity variance computed in the usual way from the matter power spectrum rather one should use equations (4), (6) and (7) that incorporate the terms involving the velocity correlation function \( \langle vv' \rangle \).

It is also important to realise that we have defined the peculiar velocity here such that the velocity of the observer vanishes. Thus the variances and co-variances in these equations are of velocities relative to the observer, which in practice is usually taken to mean relative to the velocity of the local group (LG), since it is the LG peculiar velocity, unlike the motion of the earth or the sun, that is thought to best reflect the gravitational acceleration from large scale structures. This eliminates the effect of perturbations on scales much greater than the survey depth which would otherwise give unphysical effects if the total velocity variance were used.

This is somewhat at odds with BDMS14, and deserves some clarification. Their equations 5, 6 give a bias that depends on the total velocity dispersion, including a contribution that comes from modes which are larger than the survey scale spanned by the target objects (in their case \( H_0 \) calibrators). This is the dispersion of one component of the velocity of a galaxy relative to the ‘cosmic-frame’, as is thought to be well approximated by the frame in which the CMB has zero dipole (since any intrinsic dipole is usually thought to be very small). In their discussion of this BDMS14 say that they remove the motion of the observer since the observations are usually quoted in the CMB frame, corresponding to \( v_0 = 0 \), and that a non-vanishing observer velocity would nearly double the effect. This doubling seems to us to be misleading. The observer velocity is not zero in the CMB frame – the LG is moving at about 600 km/s in that frame – but the CMB frame is not of much relevance here as the results should be independent of any frame that the observers choose to refer the observations to. Our formulae, including the correlation function \( \langle vv' \rangle \), refer to ensemble averages and, if one had no idea how the LG motion originated, then these should be in the LG frame. Working in the LG frame would indeed increase the co-variance from perturbations on scales smaller than the survey scale, though the effect of motions on larger scales would still be suppressed.

But there is a difference between the variance of the motions of different source regions and our motion, which has a variance in an ensemble sense, but we only sample one realisation of the ensemble (though it is a realisation of all three components of the velocity, not just one). The exact impact of the LG’s motion depends on the depth of the gravitational sources that are responsible for its motion: if these sources are deeper then the \( H_0 \) secondary calibrators themselves, then the \( H_0 \) calibrators and the LG motion share the same bulk velocity and so, by operating in the LG frame, these super-survey modes disappear, as noted above. If on the other hand, the source of the LG’s motion is very local to the LG itself (for example, a very nearby attractor such as Virgo), then, when operating in the LG frame, the LG motion induces a coherent dipole pattern (see Kaiser & Hudson 2014 and references therein). This coherent dipole is different in character to the less-coherent distortion due to the motions of the \( H_0 \) calibrators.

In practice, however, the LG’s motion arises from gravitational sources over a wide range of distances, so the true situation is more complicated than the two scenarios sketched above. Fortunately, by mapping out the distribution of nearby galaxies with an all-sky redshift survey and predicting peculiar velocities via linear perturbation theory, we now have a good idea of the gravitational sources responsible for much of the LG’s motion (e.g. Erdogdu et al. 2006; Lavaux & Hudson 2011, Carrick et al. 2014). Consequently, because in practice these surveyed volumes contain within them the secondary calibrators with which one is attempting to measure the local \( H_0 \), the bias in the local value of \( H_0 \) could be reduced by working in the frame of the redshift survey itself. In other words, the solution is to use the predicted peculiar velocities to correct for the redshifts of the calibrators (Neill et al. 2007, Riess et al. 2011), leaving...
perturbation has the expansion \( \delta X = av/z + bv^2/z^2 \ldots \) as before, the analogue of (\ref{eq:deltaX}) is

\[
\langle \delta X \rangle = \{-[2 + \gamma]a + b\} \frac{a^2}{z^2} + \frac{a}{z^2} \int \frac{d\Omega}{\Delta \Omega} \int \frac{d\Omega'}{\Delta \Omega'} \left\{ (2 + \gamma) \langle vv' \rangle + z(dv'/dz - \delta') \right\}. \tag{12}
\]

On dimensional grounds, one might expect these new terms appearing in the double integral to have a large contribution (as compared to the term involving \( vv' \)) from perturbations with wavelength \( \lambda \ll z \) since both \( \delta' \) and \( dv'/dz \sim v/\lambda \). But that is misleading for the following reason. That part of the velocity field which derives from waves in the Fourier spectrum with wave-number \( k = 2\pi/\lambda \) has a coherence scale of order \( \lambda \). So pairs of points that have significant correlation are restricted to have separation \( \sim \lambda \), and if \( \lambda \ll z \) these pairs have a separation whose direction is nearly perpendicular to the line-of-sight. This actually suppresses the contribution to \( \langle \delta d \rangle \) from the \( dv'/dz \) term to be smaller than that from the \( vv' \) term. The same is true for the term involving \( \langle vv' \rangle \). Thus the differences introduced by averaging over galaxies, as opposed to the simpler averaging over areas, are small.

As was the case of averaging weighting by area, if we average over the entire sky and assume that this covers many ‘coherence-areas’, then we can ignore the double integral in (\ref{eq:deltaX}) and we have, in analogy with (\ref{eq:deltaPhi}),

\[
\langle \delta d \rangle = (2 + \gamma) \langle v^2 \rangle /z^2 \tag{13}
\]

\[
\langle \delta \Phi \rangle = -(1 + 2\gamma) \langle v^2 \rangle /z^2
\]

\[
\langle \delta \mu \rangle = (5/\ln(10))(3/2 + \gamma) \langle v^2 \rangle /z^2.
\]

For distances of practical interest, the actual bias involves the additional terms in (\ref{eq:deltaX}). But the simpler expressions above are potentially useful in a situation where large-scale motions have been modelled and corrected for, as they would then describe any residual bias caused by un-modelled motions on smaller scales.

At any redshift the variable \( d^n \) is unbiased for \( n = -3 - 2\gamma(z) \). In terms of flux density \( \Phi \) (and selection function \( \phi \)) this is \( \Phi^{3/2} \text{d} \ln \phi / \text{d} \ln z \). At the distance at which the number of galaxies per logarithmic interval of distance is maximised – the distance where most of the galaxies reside, in some sense – the selection function is falling as \( \phi \propto d^{-3} \) and so the unbiased variable is \( d^n_3 \propto \Phi^{-3/2} \) (as compared to the \( \Phi^{3/2} \) that applies if there is no distance dependence selection).

### 3 MALMIQUEST BIAS FROM INCOHERENT SMALL-SCALE MOTIONS

The foregoing analysis was somewhat restricted in that it was assumed that at each point in real-space there is a single velocity – i.e. that the galaxies move like a fluid, thus ruling out application to bound virialised systems where there are multiple streams – and yet more restrictive in that it was assumed that there was a unique velocity at each point in redshift space; which rules out e.g. ‘triple valued’ regions in redshift space that exist around clusters. These assumptions are reasonable only for large scale motions.

At the other extreme, a useful and commonly used model for small scale motions within bound structures is
that these motions are spatially incoherent with peculiar velocities drawn from a distribution function $P_{\alpha}(v)dv$. As mentioned in the Introduction, the bias caused by small-scale motions with an assumed Maxwellian distribution (for which the distribution of the line-of-sight velocity is Gaussian) was considered by Lynden-Bell (1992) and by Willick et al. 1997, both of whom found an effect qualitatively similar to, but twice as large as, the bias obtained from perturbation analysis (for motions with coherence scale less than the size of the averaging region). This is puzzling. Why would the result care about whether the coherence scale is just much smaller than the averaging cell size or microscopically small?

We now show, at least in the limit that $v \ll z$, that the result for $\langle \delta \mu \rangle$ in (8) applies also to small-scale incoherent motions, and that the use of the standard formula for the bias with distance errors replaced by velocity errors, while entirely plausible, actually over-predicts the effect (by a factor 2 in the case that selection is ignorable).

At any $z$, $P(d|z) \propto P(d,z) = P(z|d)P(d)$. But $z = d + v$, so $P(z|d) = P_{\alpha}(z-d|d)$. If we assume that the distribution of peculiar velocities $P_{\alpha}$ is position independent, then $P(d,z) \propto P_{\alpha}(d)P_{\alpha}(z-d)$, from which we can compute expectation values for distance, distance modulus etc.

With $\delta \mu = \alpha \ln(d/z)$ and assuming galaxies are uniformly distributed in angle, but subject to some smoothly varying selection function $\phi(d)$, so $P_{\alpha}(d) = P_{\alpha}(z)(1 - (2 + \gamma)v/z + \ldots)$, the mean DM for galaxies at redshift $z$ is

$$\langle \delta \mu |z\rangle = \frac{\int dv \left(1 - (2 + \gamma)^{2}z^{2}\right) P_{\alpha}(v)\alpha \left(-v - \frac{v^{2}}{2z^{2}} + \ldots\right)}{\int dv \left(1 - (2 + \gamma)^{2}z^{2}\right) P_{\alpha}(v)}$$

or, keeping only terms up to second order in velocity in the numerator and only the leading order term in the denominator,

$$\langle \delta \mu |z\rangle = \alpha(-\langle v \rangle/z + (\gamma + 3/2)(\langle v^{2} \rangle/z^{2}).$$

(14)

For the assumed centred Gaussian distribution, $\langle v \rangle = 0$ and (15) agrees with the the third of (13) and, ignoring selection (i.e. setting $\gamma = 0$), we have

$$\langle \delta \mu |z\rangle = (7.5/\ln(10))\langle v^{2} \rangle/z^{2}$$

(16)
in accord with the third of (3) but in conflict with equation 15 of Willick et al. 1997 and at odds both with equation 9.17 of Lynden-Bell (1992) and with the seemingly reasonable analogy with Lynden-Bell et al. 1988, all of which would suggest that for a uniform spatial distribution of galaxies $\delta \ln d = 3(\langle v^{2} \rangle)/z^{2}$, which is twice as large as what we have here.

The reconciliation with Lynden-Bell (1992) is that the quantity he considers is the mode of $P(\ln(d)|v)$ the distribution of log-distances given an observed recession velocity $v$ and assuming a Gaussian scatter in $v$. That is the most probable log-distance. But what we are interested in here is the mean of the log-distance. The $\ln d$ probability distribution, under these conditions, is asymmetric, and the shift of the mean is half the shift of the mode. Using the shift of the mode, we would argue, overestimates the ‘homogeneous Malmquist bias’ caused by small scale velocity dispersion by a factor two.

Regarding the analogy with Lynden-Bell et al. 1988, what they assumed was a model for FP distance errors in which the probability distribution for the estimated log-distance $l_{e}$ given a true log-distance $l = \ln d$ was a Gaussian:

$$P(l_{e}|l) = (2\pi \Delta^{2})^{-1/2} \exp(-(l_{e} - l)^{2}/(2\Delta^{2}))$$

(17)

In the present context redshift $z$ plays the role of estimated distance, with $v$ the distance error. But the model (17) differs from that assumed above (with a Gaussian distribution for velocity errors) in two respects: First, this distribution implies an asymmetric distribution for the peculiar velocity, with a non-zero mean and asymmetric tails. Second, the fractional distance error is independent of distance, so in this model the absolute error grows with distance. This is appropriate for TF or FP distances, but not for errors produced by random motions. As we show in the appendix, the former does not, by itself, resolve the inconsistency; if one uses the moments of $v$ implied by this distribution in (15) this gives $\delta \mu = (5/\ln(10))\Delta^{2}$, which does not agree with (16) nor, for that matter, is it in accord with $\delta \ln d = 3\Delta^{2}$. The full resolution, again demonstrated in the appendix, is that one needs to modify the above argument to treat the case of distance independent fractional distance errors, and the bias is then given by (13) which is very similar to (15) but which has the numerical factor $\gamma + 3/2$ replaced by $7/2$. Using the first and second velocity moments implied by (17) in (13) gives $\langle \delta \mu |z\rangle = (15/\ln(10))\langle v^{2} \rangle/z^{2}$ in accordance with the usual formula $\delta \ln d = 3\langle v^{2} \rangle/z^{2}$. But this is not correct for distance errors from velocities, which is what we are considering here, where it is the absolute rather than fractional distance error that is independent of distance, and where the velocity distribution is symmetric.

The above argument is idealised in that it assumes both the density of galaxies and the velocity distribution function to be independent of position. Regarding the homogeneous Malmquist bias the effect of relaxing this is that the expectation of the sky-averaged $\Phi\mu$ involves the galaxy weighted velocity variance. For large-scale density perturbations there is also an inhomogeneous Malmquist bias term (whose expectation vanishes), just as found by Lynden-Bell (1992). In this regard, we note that the variable $\Phi^{3/2+4\ln d/4\ln z}$ is only unbiased with respect to the homogeneous Malmquist bias and is still affected by the inhomogeneous Malmquist bias.

4 SUMMARY

We have shown in (2) that the relatively large low-redshift perturbations to the mean distances, flux densities or distance moduli obtained from relativistic second order perturbation theory can be understood as a purely classical kinematic and statistical Malmquist-like effect and are not, in any essential way, a manifestation of non-linear dynamics. While gravity is involved in generating peculiar velocities, precisely the same bias would be found if one were observing test particles with non-gravitationally generated motions. The relativistic treatment may contain other effects that are essentially gravitational in nature, but as they are apparently extremely small they are of limited interest.

In (2) we generalised the analysis to obtain the bias when, as in reality, the distance is averaged over sources such as galaxies or supernovae that are subject to selection bias.

Our analysis provides formulae that could, in principle,
be used to correct for biases in distance, and hence in the ‘local’ value of $H_0$, from large-scale or small-scale motions. For the former, our results properly account for covariance and suppression of the effect of super-survey modes that is missing from the relativistic perturbation theory papers. But we emphasise that the effects on $H_0$ at least are very small and much smaller than the fluctuations in measurements of $H_0$ that arise in linear theory.

We have shown in [3] that small-scale incoherent velocities have essentially the same effect. They do not cause a perturbation to log-distance $\delta \ln d = 3\sigma_z^2/c^2 z^2$ as has previously been found, and as would seem reasonable by analogy with the commonly used formula for homogeneous Malmquist bias. The effect is a factor two smaller. The reason that the standard formula is not valid for bias from velocity dispersion is in part because the model implies an unrealistic distribution of distance whereas errors from motions are distance independent.

We showed that the average of $\Phi_3^{1/2} d\ln \phi / d\ln r$ does not suffer velocity dispersion induced homogeneous Malmquist bias.

We provide in appendix A a slightly generalised formula for the homogeneous Malmquist bias produced by errors in estimated luminosity distance – as from e.g. Tully-Fisher or fundamental plane techniques – when using the ‘forward’ method. This result is only valid for $\Delta^2 \ll 1$, but makes no assumption about the form for the distribution function for the distance errors.

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APPENDIX A: MALMQUIST BIAS FROM LUMINOSITY DISTANCE ERRORS

We now obtain the analogue of (15) for the situation where distances are estimated from the source flux density, rather than redshift, and the distance error scales linearly with distance as is appropriate, for TF or FP distances or for supernovae. Thus we assume sources with real distances $d$ and estimated distances $\hat{d}$ and distance error $\epsilon = d - \hat{d}$. I.e. we use the same notation as before, but with a different interpretation as the cause of the errors, and to obtain a distance independent distribution for fractional errors we take $P_e(v|d)dv = f(v|d)dv/d$ where $f(y)$ is some normalised bell-shaped function: $\int dy f(y) = 1$.

If we assume the fractional distance errors are small $v \ll d$, we have

$$P_e(v|d)dv = \frac{z}{z-v} f\left(\frac{v}{z-v}\right) \frac{dv}{z}$$

$$= \left(1 + \frac{v}{z} + \ldots \right) \left(f\left(\frac{v}{z}\right) + \frac{v^2}{z^2} f'\left(\frac{v}{z}\right) + \ldots \right) \frac{dv}{z}$$

where $f'(y) = df/dy$.

Our goal is to compute $\langle d|z \rangle$ from the conditional distribution of distance $P(d|z) \propto P(z|d)P(d)$. Previously we used $P(d) = d^2 \phi(d)$, but here the estimated distance is not the redshift, it is the inverse square root of the flux density, so a magnitude limit imposes a selection that is a function of the estimated distance ($z$ in our notation). The upshot, as explained by Strauss & Willick (1995), is that the selection function drops out when we compute $\langle d|z \rangle$ or, equivalently, the bias is the same as obtained without any selection.

Using (A1) in (11) we find two extra terms that produce significant contributions to the numerator (when multiplied...
by \(-v/z\) and integrated):

\[-z^2 \int \frac{dv}{z} \left( \frac{v^2}{z^2} f \left( \frac{v}{z} \right) + \frac{v^3}{z^3} f' \left( \frac{v}{z} \right) \right) = 2\langle v^2 \rangle\] (A2)

where we have integrated by parts and assumed that \(f\) falls to zero for large argument sufficiently fast that the boundary term is negligible.

As in (15) one finds, in addition to a term proportional to the variance in the distance error, the mean distance error \(-\langle v \rangle\). This is not necessarily zero – it is not zero, for instance, if the distance estimator is obtained by minimising residuals in magnitude – but it is reasonable to assume that the strength of any bias in \(\langle v \rangle\) is, to order of magnitude, at most proportional to the variance \(\langle v^2 \rangle\). Keeping terms up to quadratic order in the distance error \(v\) and ignoring the sub-dominant terms in the denominator in (14) yields the general result valid up to linear order in the fractional distance variance \(\Delta^2 = \langle v^2 \rangle/d^2\) for the homogeneous Malmquist bias

\[\langle d\mu | z \rangle = \alpha (-\langle v \rangle / z + 7/2 \langle v^2 \rangle / z^2)\] (A3)

where \(v\) is minus the distance error and \(z\) can be taken to be either the estimated distance \(z\) or the real distance \(d = z - v\). One can also obtain the perturbation to any other variables. The perturbation to the distance, for instance, is

\[\langle \delta d \rangle = -\langle v \rangle / z + 4\langle v^2 \rangle / z^2.\] (A4)

As a check, we can apply (A3) and (A4) to the log-normal model of Lynden-Bell et al. 1988 that is known to give exactly \(\delta \ln d = 3\Delta^2\) (or \(\langle d\mu | z \rangle = (15/\ln 10) \Delta^2\)). In this model the probability distribution for the estimated log-distance \(l_e\) given a true log-distance \(l = \ln d\) is a Gaussian:

\[P(l_e | l) = (2\pi \Delta^2)^{-1/2} \exp(- (l_e - l)^2 / (2\Delta^2)).\] (A5)

With \(d = e^l\) and \(z = e^{l_e}\) the moments of the estimated distance distribution are \(\langle z^n \rangle = d^n \exp(n^2 \Delta^2 / 2)\). The first moment is \(\langle z \rangle = d(1 + \Delta^2/2 + \ldots)\), so the mean of the distance error is \(\langle v \rangle = d\Delta^2/2 + \ldots\), which is non-zero, and the second moment is \(\langle z^2 \rangle = d^2(1 + 2\Delta^2 + \ldots)\) so the distance error variance is \(\langle v^2 \rangle = d^2\Delta^2 + \ldots\), where the notation \(\ldots\) indicates quantities that are of higher order in the assumed small logarithmic variance \(\Delta^2\). Using these in (A3) gives \(\langle d\mu | z \rangle = (15/\ln 10) \Delta^2\) in agreement with Lynden-Bell et al. 1988, while (A4) gives \(\langle \delta d \rangle = (7/2) \Delta^2\), in accord with equation 185 of Strauss & Willick (1995).

Equations (A3) and (A4) provide a generalisation of the standard results in that they do not assume a perfectly log-normal distribution, though they are limited to the regime where the fractional variance \(\Delta^2 \ll 1\). They apply only to the ‘forward’ method where one averages the peculiar velocity of objects as a function of estimated distance. The more popular ‘inverse’ methods do not suffer this bias. Instead they have the much smaller residual bias from random motions causing scatter in distance that is the focus of the main part of this paper.