The euclidean propagator in a model with two non-equivalent instantons

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We consider in detail how the quantum-mechanical tunneling phenomenon occurs in a well-behaved octic potential. Our main tool will be the euclidean propagator just evaluated between two minima of the potential at issue. For such a purpose we resort to the standard semiclassical approximation which takes into account the fluctuations over the instantons, i.e., the finite-action solutions of the euclidean equation of motion. As regards the one-instanton approach, the functional determinant associated with the so-called stability equation is analyzed in terms of the asymptotic behaviour of the zero-mode. The conventional ratio of determinants takes as reference the harmonic oscillator whose frequency is the average of the two different frequencies derived from the minima of the potential involved in the computation. The second instanton of the model is studied in a similar way. The physical effects of the multi-instanton configurations are included in this context by means of the alternate dilute-gas approximation where the two instantons participate to provide us with the final expression of the propagator.

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I. INTRODUCTION.

The tunneling phenomenon through classically forbidden regions represents one of the most outstanding effects in quantum theory. Starting from the pioneering work of Polyakov on the subject \[1\], the semiclassical treatment of the tunneling is usually presented via the euclidean version of the path-integral formalism. The basis of this approach relies on the so-called instanton calculus. In this scheme the instantons correspond to localised finite-action solutions of the euclidean equation of motion where the time variable is essentially imaginary. To sum up, one finds the appropriate classical configuration and subsequently evaluate the term associated with the quadratic fluctuations. In doing so the functional integration itself is solved by means of the gaussian scheme except for the zero-modes which appear in connection with the translational invariances of the system. Next one introduces a set of collective coordinates so that ultimately the gaussian integration is performed along the directions orthogonal to the zero-modes. In principle a functional determinant includes an infinite product of eigenvalues so that a highly divergent result is expected. However one can regularize these fluctuation factors by means of the ratio of determinants.

Let us describe in brief the instanton calculus for the one-dimensional particle as can be found for instance in \[2\]. We assume that our particle moves under the action of a confining potential \( V(x) \) which yields a pure discrete spectrum of energy eigenvalues. In addition the minima of the potential satisfy \( V(x) = 0 \). If the particle is located at the initial time \( t_i = -T/2 \) at the point \( x_i \) while one finds it when \( t_f = T/2 \) at the point \( x_f \), the functional version of the non-relativistic quantum mechanics allows us to write the transition amplitude in terms of a sum over all paths joining the world points with coordinates \((-T/2, x_i)\) and \((T/2, x_f)\). Performing the change \( t \rightarrow -i\tau \), known in the literature as the Wick rotation, the euclidean formulation of the path-integral reads

\[
<x_f|\exp(-HT)|x_i> = N(T) \int [dx] \exp\{-S_e[x(\tau)]\}
\]

(1)

where \( H \) represents the hamiltonian of the model, the factor \( N(T) \) serves to normalize the
amplitude conveniently while $[dx]$ indicates the integration over all functions which fulfil the boundary conditions at issue. As usual the euclidean action $S_e$ corresponds to

$$S_e = \oint_{-T/2}^{T/2} \left[ \frac{1}{2} \left( \frac{dx}{d\tau} \right)^2 + V(x) \right] d\tau$$

whenever the mass of the particle is set equal to unity. In the following we take care of the octic potential $V(x)$ given by

$$V(x) = \frac{\omega^2}{2} (x^2 - 1)^2 (x^2 - 4)^2$$

whose appearance can be seen in fig.1. Going to the regime in which $\omega^2 \gg 1$ the energy barriers are high enough to decompose the physical system into a sum of independent harmonic oscillators. In doing so the particle can execute small oscillations around each minima of the potential located at $x = \pm 1$ and $\tilde{x} = \pm 2$. As usual the second derivative of the potential just evaluated at these points, i.e. $V''(x = \pm 1) = 36 \omega^2$ and $V''(\tilde{x} = \pm 2) = 144 \omega^2$ characterizes the respective frequencies of the harmonic oscillators at issue.

![FIG. 1. Profile of the potential.](image)

In terms of the discrete symmetry $x \rightarrow -x$ which the potential $V(x)$ enjoys, one notices how the four minima are non-equivalent as a whole since no connection is possible between the two sets represented by $x = \pm 1$ and $\tilde{x} = \pm 2$. Next we would like to make the description of the tunneling phenomenon to explain how the symmetry cannot appear spontaneously.
broken at quantum level. In such a case the expectation value of the coordinate $x$ evaluated for the ground-state is zero as corresponds to the even character of the potential $V(x)$.

**II. THE ONE-INSTANTON CONTRIBUTION.**

In this section we carry things further to describe the tunneling phenomenon according to the tools of the euclidean version of the path-integral. At first glance it seems physically relevant to take into account the transition between the points $x_i = 1$ and $x_f = 2$. In doing so we need the explicit form of the topological configuration with $x_i = 1$ at $t_i = -T/2$ while $x_f = 2$ when $t_f = T/2$. It is customarily assumed that $T \to \infty$ since the explicit solution of the problem is much more complicated for finite $T$. Fortunately the difference is so small that can be ignored mainly because we are interested in such a limit to obtain information about the energy of the first levels of the model. To get the instanton $x_{c1}(\tau)$ which connects the points $x_i = 1$ and $x_f = 2$ with infinite euclidean time, we can resort to the well-grounded Bogomol’nyi condition [3]. The problem is solved just by integration of a first-order differential equation according to the zero-energy condition for the motion of a single particle under the action of $-V(x)$. To sum up

$$x_{c1}(\tau) = 2 \cos \left[ \frac{\pi}{3} - \frac{1}{3} \arccos \left( \frac{e^{-12\omega(\tau-\tau_c)}}{e^{-12\omega(\tau-\tau_c)} + 1} \right) \right]$$

(4)

where the parameter $\tau_c$ indicates the point at which the instanton makes the jump. As usual equivalent solutions are obtained by means of the transformations $\tau \to -\tau$ and $x_{c1}(\tau) \to -x_{c1}(\tau)$. It may be interesting at this point to remind how the instanton procedure allows only the connection between adjoint minima of the potential. In this regard we notice the existence of a second instanton just interpolating between $x_i = -1$ and $x_f = 1$ (more on this later). Going back to $x_{c1}(\tau)$, the classical euclidean action $S_1$ associated with a such configuration is computed according to [2] so that $S_1 = 22\omega/15$. Now the conventional description of the one-instanton amplitude between $x_i = 1$ and $x_f = 2$ takes over

$$< x_f = 2 | \exp(-HT) | x_i = 1 >= N(T) \left\{ \text{Det} \left[ \frac{d^2}{d\tau^2} + \nu^2 \right] \right\}^{-1/2}$$
\[
\left\{ \frac{\text{Det} \left[ -\left( \frac{d^2}{d\tau^2} + \nu^2 \right) \right]}{\text{Det} \left[ -\left( \frac{d^2}{d\tau^2} + \nu^2 \right) \right]} \right\}^{-1/2} \exp(-S_1) \tag{5}
\]

where we have multiplied and divided by the determinant of a generic harmonic oscillator of frequency \( \nu \). This regularization term can be interpreted as a new amplitude given by

\[
< x_f = 0 | \exp(-H_{ho} T) | x_i = 0 >= N(T) \left\{ \text{Det} \left[ -\frac{d^2}{d\tau^2} + \nu^2 \right] \right\}^{-1/2} \tag{6}
\]

Fortunately the explicit evaluation of (6) is made according to the procedure exposed in [4]. In short

\[
< x_f = 0 | \exp(-H_{ho} T) | x_i = 0 >= \left( \frac{\nu}{\pi} \right)^{1/2} \left( 2 \sinh \nu T \right)^{-1/2} \tag{7}
\]

Going back to the determinant of the stability equation over the instanton \( x_{c1}(\tau) \), the existence of a zero-mode \( x_o(\tau) \) requires the introduction of a collective coordinate. From a physical point of view this zero eigenvalue comes by no surprise since it reflects the translational invariance of the system as a whole. In other words, there is one direction in the functional space of the second variations which results incapable of changing the action. Including the right factor of normalization the explicit form of the zero-mode \( x_o(\tau) \) corresponds to the derivative of the topological configuration, i.e.

\[
x_o(\tau) = \frac{1}{\sqrt{S_1}} \frac{dx_{c1}}{d\tau} \tag{8}
\]

In addition the integral over the zero-mode itself becomes equivalent to the integration over the center of the instanton \( \tau_c \). When this change of variables is incorporated our ratio of determinants corresponds to [2]

\[
\left\{ \frac{\text{Det} \left[ -\left( \frac{d^2}{d\tau^2} + V''[x_{c1}(\tau)] \right) \right]}{\text{Det} \left[ -\left( \frac{d^2}{d\tau^2} + \nu^2 \right) \right]} \right\}^{-1/2} =
\]

\[
\left\{ \frac{\text{Det}' \left[ -\left( \frac{d^2}{d\tau^2} + V''[x_{c1}(\tau)] \right) \right]}{\text{Det} \left[ -\left( \frac{d^2}{d\tau^2} + \nu^2 \right) \right]} \right\}^{-1/2} \sqrt{\frac{S_1}{2\pi}} \ d\tau_c \tag{9}
\]

where \( \text{Det}' \) stands for the so-called reduced determinant once the zero-mode has been removed. To make an explicit computation of the quotient of determinants we take advantage
of the Gelfand-Yaglom where only the knowledge of the large-\(\tau\) behaviour of the classical solution \(x_{c1}(\tau)\) is necessary \[5\]. Starting from \(\hat{O}\) and \(\hat{P}\), which represent a couple of second order differential operators whose eigenfunctions vanish at the boundary, the quotient of determinants is given in terms of the respective zero-energy solutions \(f_o(\tau)\) and \(g_o(\tau)\) according to

\[
\frac{\text{Det} \hat{O}}{\text{Det} \hat{P}} = \frac{f_o(T/2)}{g_o(T/2)}
\]

whenever the eigenfunctions at issue fulfil the initial conditions

\[
f_o(-T/2) = g_o(-T/2) = 0, \quad \frac{df_o}{d\tau}(-T/2) = \frac{dg_o}{d\tau}(-T/2) = 1
\]

The zero-mode \(g_o(\tau)\) associated with the harmonic oscillator of frequency \(\nu\) corresponds to

\[
g_o(\tau) = \frac{1}{\nu} \sinh[\nu(\tau + T/2)]
\]

so that now we only need the form of the solution \(f_o(\tau)\) associated with the topological configuration written in \[4\]. From the aforementioned \(x_o(\tau)\) zero-mode we can write a second solution \(y_o(\tau)\) given by

\[
y_o(\tau) = x_o(\tau) \int_0^\tau \frac{ds}{x_o^2(s)}
\]

Accordingly we may summarize the asymptotic behaviour of \(x_o(\tau)\) and \(y_o(\tau)\) as follows

\[
x_o(\tau) \sim \begin{cases} 
C \exp(-12\omega\tau) & \text{if } \tau \to \infty \\
D \exp(6\omega\tau) & \text{if } \tau \to -\infty 
\end{cases}
\]

\[
y_o(\tau) \sim \begin{cases} 
\exp(12\omega\tau)/24\omega C & \text{if } \tau \to \infty \\
-\exp(-6\omega\tau)/12\omega D & \text{if } \tau \to -\infty 
\end{cases}
\]

where the constants \(C\) and \(D\) derive from the explicit form of the derivative of \(x_o(\tau)\). Now we investigate the particular solution \(f_o(\tau)\) which is the one we are really interested in. Starting from the linear combination of \(x_o(\tau)\) and \(y_o(\tau)\) given by
\( f_o(\tau) = Ax_o(\tau) + By_o(\tau) \) \hspace{1cm} (16)

the incorporation of the initial conditions at issue leads us to

\[ f_o(\tau) = x_o(-T/2)y_o(\tau) - y_o(-T/2)x_o(\tau) \] \hspace{1cm} (17)

From this expression, which is exact, we can extract the asymptotic behaviour of \( f_o(\tau) \), i.e.

\[ f_o(T/2) \sim \frac{D}{24\omega C} \exp(3\omega T) \text{ if } T \to \infty \] \hspace{1cm} (18)

Now we need to consider in detail the lowest eigenvalue of the stability equation to obtain the right value of the quotient of determinants. From a physical point of view we can explain the situation as follows: the derivative of the topological solution does not quite satisfy the boundary conditions for the interval \((-T/2, T/2)\). When enforcing such a behaviour, the eigenstate is compressed and the energy shifted slightly upwards. In doing so the zero-mode \( x_o(\tau) \) is substituted for the \( f_{\lambda}(\tau) \) which corresponds to

\[- \frac{d^2 f_{\lambda}(\tau)}{d\tau^2} + V''[x_{c1}(\tau)]f_{\lambda}(\tau) = \lambda f_{\lambda}(\tau)\] \hspace{1cm} (19)

whenever

\[ f_{\lambda}(-T/2) = f_{\lambda}(T/2) = 0 \] \hspace{1cm} (20)

To lowest order in perturbation theory we obtain that

\[ f_{\lambda}(\tau) \sim f_o(\tau) + \lambda \left. \frac{df_{\lambda}}{d\lambda} \right|_{\lambda=0} \] \hspace{1cm} (21)

so that ultimately we can write that

\[ f_{\lambda}(\tau) = f_o(\tau) + \lambda \int_{-T/2}^{\tau} [x_o(\tau)y_o(s) - y_o(\tau)x_o(s)] f_o(s) \, ds \] \hspace{1cm} (22)

The asymptotic behaviour of \( f_o'(\tau), \ x_o(\tau) \) and \( y_o(\tau) \), together with the condition \( f_{\lambda}(T/2) = 0 \) provides us with the lowest eigenvalue \( \lambda \), namely

\[ \lambda = 12\omega D^2 \exp(-6\omega T) \] \hspace{1cm} (23)
The final expression of the quotient of determinants requires a choice for the parameter \( \nu \) so that ultimately the frequency of the harmonic oscillator of reference is the average of the frequencies over the non-equivalent minima located at \( x = 1 \) and \( \tilde{x} = 2 \). In other words \( \nu = 9 \omega \). It may be interesting at this point to remark the differences with the well-grounded double-well model where the two minima of the potential are equivalent so that the aforementioned average is not necessary. However in this case the Gelfand-Yaglom method fixes the frequency \( \nu \) and subsequently the ratio of determinants is finite. Going back to the explicit form of \( x_{cl}(\tau) \) (see (4)) we find

\[
C = \frac{4\sqrt{3}\omega}{\sqrt{S_1}}, \quad D = \frac{16\omega}{3\sqrt{S_1}}
\]  

(24)

Armed with this information we can write the one-instanton amplitude between the points \( x_i = 1 \) and \( x_f = 2 \) according to

\[
<x_f = 2|\exp(-HT)|x_i = 1> = \left(\frac{9\omega}{\pi}\right)^{1/2} (2\sinh 9\omega T)^{-1/2} \sqrt{S_1} K_1 \exp(-S_1) \omega d\tau_c
\]  

(25)

where \( K_1 \) stands for a numerical factor given by

\[
K_1 = 16\sqrt{\frac{15\sqrt{3}}{11\pi}}
\]  

(26)

Apart from the first factor, which represents the contribution of the harmonic oscillator of reference, we get a transition amplitude just depending on the point \( \tau_c \) at which the instanton makes the jump. According to the values of the interval \( T \) the result seems plausible whenever

\[
\sqrt{S_1} K_1 \exp(-S_1) \omega T \ll 1
\]  

(27)

a nonsense condition if \( T \) is large enough. However in this regime we can accommodate configurations constructed of instantons and anti-instantons which mimic the behaviour of a trajectory strictly derived from the euclidean equation of motion. In doing so we get
an additional bonus since the integration over the centers of the string of instantons and anti-instantons is performed in a systematic way.

To finish this section we sketch the situation as far as the second instanton of the model concerns. For such a purpose we take into account the one-instanton amplitude between $x_i = -1$ and $x_f = 1$ which is based on the topological configuration $x_{c2}(\tau)$

$$x_{c2}(\tau) = 2 \cos \left[ \frac{\pi}{3} + \frac{1}{3} \arccos \left( \frac{e^{12\omega \tau} - 1}{e^{12\omega \tau} + 1} \right) \right]$$

(28)

whose classical euclidean action corresponds to $S_2 = 76\omega/5$. As expected the second instanton reminds the case of the double-well potential since connects equivalent minima of the potential. In doing so the $x_o(\tau), y_o(\tau)$ are symmetric so that the application of the Gelfand-Yaglom method is straightforward. The explicit form of the ratio of determinants at issue should be

$$\left\{ \frac{\text{Det}' \left[-(d^2/d\tau^2) + V''[x_{c2}(\tau)]\right]}{\text{Det} \left[-(d^2/d\tau^2) + 36 \omega^2 \right]} \right\}^{-1/2} = \sqrt{S_2} K_2 \omega \, d\tau_c$$

(29)

where $K_2$ is given by

$$K_2 = 12 \sqrt{\frac{15}{38\pi}}$$

(30)

### III. THE ALTERNATE DILUTE-GAS APPROXIMATION.

As all the above calculations were carried out over a single instanton, it remains to discuss the complete amplitude which incorporates the physical effect of a string of instantons and anti-instantons along the $\tau$ axis. The octic potential represents in this context a more complicated case since we need to include the whole scheme of non-equivalent instantons. It is customarily assumed that these combinations of topological solutions represent no strong deviations of the trajectories derived from the euclidean equation of motion without any kind of approximation. We wish to compute the functional integral by summing over all such configurations, with $n$ instantons and anti-instantons centered at points $\tau_1, ..., \tau_n$ whenever
\[-\frac{T}{2} < \tau_1 < \ldots < \tau_n < \frac{T}{2}\] (31)

Being narrow enough the regions where the instantons (anti-instantons) make the jump, the action of the proposed path is almost extremal. We can carry things further and assume that the action of the string of instantons and anti-instantons is given by the sum of the \(n\) individual actions. This scheme is well-known in the literature where it appears with the name of dilute-gas approximation [6]. In addition the translational degrees of freedom yield an integral of the form

\[
\int_{-T/2}^{T/2} \omega d\tau_n \int_{-T/2}^{\tau_n} \omega d\tau_{n-1} \ldots \int_{-T/2}^{\tau_2} \omega d\tau_1 = \frac{(\omega T)^n}{n!}
\] (32)

When evaluating the transition amplitude between \(x_i = 1\) and \(x_f = 2\) the total number \(n\) of topological configurations must be odd. As a matter of fact we can split \(n\) (odd) into the sum of two contributions \(n_1\) (odd) and \(n_2\) (even) which represent the different possibilities associated with the existence of non-equivalent instantons. Accordingly we have \(n_1\) topological configurations just interpolating between \(x = 1\) and \(\tilde{x} = 2\) or \(x = -1\) and \(\tilde{x} = -2\). Of course identical situation appears in connection with \(n_2\) where now the initial and final points of the trip are \(x = \pm 1\). In addition we need to include a combinatorial factor \(F\) to count the different possibilities that we have of distributing the \(n\) instantons. Except for the last step which corresponds to the instanton analyzed in the previous section, we deal with a closed path of topological configurations starting and coming back to the point \(x = 1\). As regards the instantons (anti-instantons) belonging to the first type we observe the formation of pairs in a systematic way due to the location of the four minima of the potential along the real axis. In doing so we have \((n_1 - 1)/2 + n_2\) holes to fill bearing in mind that once the \((n_1 - 1)/2\) pairs of instantons and anti-instantons are distributed no freedom at all remains to locate the topological configurations associated with \(n_2\). To sum up

\[
F = \binom{(n_1 - 1)/2 + n_2}{(n_1 - 1)/2}
\] (33)
As regards the quadratic fluctuations over the \( n \) topological configurations at issue the alternate dilute-gas approximation means that

\[
\left\{ \frac{\text{Det}' \left[ -(d^2/d\tau^2) + V''[x_{c1}(\tau)] \right]}{\text{Det} \left[ -(d^2/d\tau^2) + 81\omega^2 \right]} \right\}^{-1/2} \rightarrow \left[ \left\{ \frac{\text{Det}' \left[ -(d^2/d\tau^2) + V''[x_{c1}(\tau)] \right]}{\text{Det} \left[ -(d^2/d\tau^2) + 81\omega^2 \right]} \right\}^{-1/2} \right]^{n_1} \left[ \left\{ \frac{\text{Det}' \left[ -(d^2/d\tau^2) + V''[x_{c2}(\tau)] \right]}{\text{Det} \left[ -(d^2/d\tau^2) + 36\omega^2 \right]} \right\}^{-1/2} \right]^{n_2}
\]

(34)

With all this information we can discuss the complete transition amplitude we are looking for once for simplicity we introduce the so-called instanton density, i.e.

\[
d_i = \sqrt{S_i} K_i \exp(-S_i), \quad i = 1, 2
\]

(35)

In other words

\[
<x_f = 2| \exp(-HT)|x_i = 1 > = \left( \frac{9\omega}{\pi} \right)^{1/2} \left( 2 \sinh 9\omega T \right)^{-1/2}
\]

\[
\sum_{n_1,n_2} [d_1 \omega T]^{n_1} [d_2 \omega T]^{n_2} \frac{F}{n!}
\]

(36)

The sum \( S \) of (36) can be written in terms of

\[
n = 2r + 1, \quad r = 0, 1, ...
\]

(37)

\[
n_2 = 2q, \quad q = 0, 1, ...
\]

(38)

so that

\[
S = \sum_{r=0}^{\infty} \sum_{q=0}^{r} \frac{(r+q)!}{(r-q)! (2q)!} \frac{[d_1 \omega T]^{2(r-q)+1} [d_2 \omega T]^{2q}}{(2r+1)!}
\]

(39)

On the other hand the best way of organizing this double sum should be the following

\[
S = \sum_{r=0}^{\infty} \frac{[d_1 \omega T]^{2r+1}}{(2r+1)!} \sum_{q=0}^{r} \frac{(r+q)}{(r-q)} (d_2/d_1)^{2q}
\]

(40)

Next we can handle the sum \( \tilde{S} \) concerning the variable \( q \) taking advantage of [7]
\[
\sum_{q=0}^{r} (-1)^q \left( \frac{r+q}{2q} \right) x^{2q} = \sec[\arcsin(x/2)] \cos[2r+1) \arcsin(x/2)]
\] (41)

including the transformation \(x \rightarrow ix\) to obtain that

\[
\tilde{S} = \frac{\cosh[(2r+1) \arg \sinh(s/2)]}{\cosh[\arg \sinh(s/2)]}
\] (42)

where \(s\) stands for the relative instanton density given by \(s = d_2/d_1\). In terms of a new variable \(z\) defined as

\[
z = \arg \sinh(s/2)
\] (43)

it is the case that a typical value of \(r\) provides us with the final expression for \(\tilde{S}\), i.e.

\[
\tilde{S} = \frac{\exp[(2r+1)z]}{\sqrt{4+s^2}}
\] (44)

Now it suffices to introduce this result in (40) so that

\[
S = \frac{\sinh[d_1 \omega T \exp(z)]}{\sqrt{4+s^2}}
\] (45)

Next we can collect these partial results to write the final expression for the complete euclidean transition amplitude between the points \(x_i = 1\) and \(x_f = 2\), namely

\[
< x_f = 2 | \exp(-HT) | x_i = 1 > = 
\]

\[
\left( \frac{9 \omega}{\pi} \right)^{1/2} (2 \sinh 9 \omega T)^{-1/2} \frac{\sinh[d_1 \omega T \exp(z)]}{\sqrt{4+s^2}}
\] (46)

Our approach provides a new way of dealing with quantum-mechanical models which exhibit a more complicated structure of non-equivalent classical vacua in comparison with the well-grounded cases of the double-well or periodic sine-Gordon potentials where the equivalence of all the minima of \(V(x)\) is taken for granted [6]. As regards the octic potential the topological solutions of the system inherit the property of non-equivalence. In any case the quantum fluctuations can be evaluated by means of the Gelfand-Yaglom method. The global effect of the multi-instanton configurations is discussed via the different kinds of instantons that take part to provide us with the final expression of the euclidean propagator.
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