MUMFORD CURVES WITH MAXIMAL AUTOMORPHISM GROUP

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ABSTRACT. It is known that a Mumford curve of genus $g \notin \{5, 6, 7, 8\}$ over a non-archimedean valued field of characteristic $p > 0$ has at most $2\sqrt{g}(\sqrt{g}+1)^2$ automorphisms. In this note, the unique family of curves which attains this bound, and their automorphism group are determined.

1. Introduction

It is well-known (cf. Conder [2]) that if a compact Riemann surface of genus $g \geq 2$ attains Hurwitz’s bound $84(g-1)$ on its number of automorphisms, then its automorphism group is a so-called Hurwitz group, i.e., a finite quotient of the triangle group $\Delta(2,3,7)$. Equivalently, the Riemann surface is an étale cover of the Klein quartic $X(7)$. However, it is hitherto unknown which finite groups can occur as Hurwitz groups. It is even true that, for every integer $n$, there exists a $g$ such that there are more than $n$ non-isomorphic Riemann surfaces of genus $g$ which attain Hurwitz’s bound (Cohen, [1]). In this note, we want to show that the corresponding questions for Mumford curves of genus $g$ over non-archimedean valued fields of positive characteristic have a very easy answer: the maximal automorphism groups can be explicitly described, and they occur for an explicitly given 1-parameter family of curves (at least for $g \notin \{5, 6, 7, 8\}$).

The set-up for our result is as follows. Let $(k, |·|)$ be a non-archimedean valued field of positive characteristic, and $X$ a Mumford curve ([7], [5]) of genus $g$ over $k$. This means that the stable reduction of $X$ over the residue field $\overline{k}$ of $k$ is a union of rational curves intersecting in $\overline{k}$-rational points. Equivalently, as a rigid analytic space over $k$, the analytification $X_{an}$ of $X$ is isomorphic to an analytic space of the form $\Gamma\backslash(\mathbb{P}^1_{k, an} - L)$, where $\Gamma$ (the so-called Schottky group of $X$) is a discrete free subgroup of $PGL(2, k)$ of rank $g$ (acting in the obvious way on $\mathbb{P}^1_{k, an}$) with $L$ as its set of limit points.

In [3], it was shown that a Mumford curve of genus $g \geq 2$ satisfies

$$|\text{Aut}(X)| \leq \max\{12(g-1), F(g)\},$$

where $F(g) := 2\sqrt{g}(\sqrt{g} + 1)^2$.

Remark 1. Note that $F(g) \leq 12(g-1)$ precisely when $g \in \{5, 6, 7, 8\}$. If the genus is in this range, it seems to be a difficult (though manageable) task to find all curves with $12(g-1)$ automorphisms, but the problem is of a different nature than the one under consideration here. Let us mention that there are at least three

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one-dimensional families of curves of genus 6 that have $12(g-1)$ automorphisms.

For $g \notin \{5, 6, 7, 8\}$, it was shown in loc. cit. that the so-called Artin-Schreier-
Mumford curves $X_{t,c}$ attain the bound $\Gamma$. Here, $t$ is a non-negative integer, $c \in k^*$
satisfies $|c| < 1$, and the affine equation of $X_{t,c}$ is given by

$$X_{t,c} : (y^{p^t} - y)(x^{p^t} - x) = c.$$  

The genus of $X_{t,c}$ is $g_t := (p^t - 1)^2$, and its automorphism group is isomorphic to
$A_t := \mathbb{Z}_p^{2t} \rtimes D_{p^t-1}$, where $\mathbb{Z}_n$ denotes the cyclic group of order $n$, and $D_n$ is the
dihedral group of order $2n$. The action of the automorphisms on the coordinates is
explicitly given by interchanging $x$ and $y$, adding an element of $\mathbb{F}_{p^t}$ to $x$ or $y$, and
multiplying $x$ and $y$ with $a \in \mathbb{F}_{p^t}$ and $a^{-1}$ respectively. For fixed $t$ and varying $c$,
the family $X_{t,c}$ over the punctured unit disc $\Delta^* := \{0 < |c| < 1\}$ is non-constant
(since it extends to a semistable family over the unit disk with a singular fiber at $c = 0$).
Actually, the curve $X_{t,c}$ is exactly Mumford if and only if $|c| < 1$, since
otherwise, its reduction is irreducible over $\bar{k}$. The aim of this paper is to show the
following:

**Theorem.** Let $g \notin \{5, 6, 7, 8\}$. Every Mumford curve over a non-archimedean
valued field $k$ of positive characteristic with the maximal number $F(g)$ of automor-
phisms is isomorphic to an Artin-Schreier-Mumford curve, i.e., an element of the
family $\{X_{t,c}\}_{c \in \Delta^*}$ for $t = \log_p(\sqrt{g} + 1)$, which, in particular, has to be an integer.

2. **Proof of the theorem**

Let $X$ be a Mumford curve of genus $g$ with $F(g)$ automorphisms. Remark that
$\text{Aut}(X) = N/\Gamma$, where $\Gamma$ is the Schottky group of $X$ and $N$ is its normalizer
in $PGL(2, k)$ (cf. VII.1).  Let us denote $A = \text{Aut}(X)$, and recall that the only
information given about $A$ is that its order is $2(\sqrt{g} + 1)^2$, where $g$ is the rank of
$\Gamma$. Let $\bar{\gamma} : N \to A$ denote the reduction map modulo $\Gamma$.

For integers $t, n$ such that $n|p^t - 1$, let $B(t,n) = \mathbb{Z}_p'^t \rtimes \mathbb{Z}_n$, where the semi-direct
action is that of the Borel subgroup of $PGL(2, k)$. Let $D_n = \mathbb{Z}_n \rtimes \mathbb{Z}_2$ denote the
dihedral group of order $2n$.

The first part of the proof consists in showing the following, which is the (much
easier) non-archimedean analogue of finding all Hurwitz groups:

**Proposition 1.** If $X$ is a Mumford curve of genus $g \notin \{5, 6, 7, 8\}$ with $F(g)$ au-
tomorphisms, then its genus is of the form $g = (p^t - 1)^2$ for some integer $t$, its
automorphism group $A$ is isomorphic to $\mathbb{Z}_p'^t \rtimes D_{p^t-1}$, $A \setminus X = \mathbb{P}^1$ and
$X \to A \setminus X$ is ramified above 3 points with ramification groups $(\mathbb{Z}_2, \mathbb{Z}_2, B(t,n))$ if $p \neq 2$ and
ramified above 2 points with groups $(\mathbb{Z}_2, B(t,n))$ if $p = 2$.

**Proof.** Let $*$ denote amalgamation product. The computations in paragraph 6 of
(6.9)–(6.12)) show the following:

**Lemma.** If $|A| = F(g)$ then, as abstract groups, $N \cong B(t,n) *_{\mathbb{Z}_n} D_n$ for $n = p^t - 1$,
and $g = (p^t - 1)^2$. Furthermore, if $\gamma$ denotes an involution in $D_n - \mathbb{Z}_n$, $E$ denotes
$\mathbb{Z}_p'^t$ (seen as a normal subgroup of $B(t,n)$), and $E' = \gamma E\gamma$ then $E \cap E' = \{1\}$.

Now observe that, since $\Gamma$ is free, any finite subgroup of $N$ is mapped by $\bar{\gamma}$
to an isomorphic image. Also, the image of any relation in $N$ holds in $A$ – this applies in particular to the relations given by the semi-direct product structure in
B(t, n) and D_n. In the end, A is a group of order 2p^{2t}(p^{t} − 1) which contains the following subgroups: two elementary abelian groups E, E′ which are disjoint, both of which are normalized by a cyclic group Z_n of order p^t − 1, which in its turn is normalized by an element ̃u of order 2. Now A is generated by the disjoint groups ⟨E, E′, Z_n, Z_2⟩, and the order of A equals the product of the order of these groups. Hence E and E′ generate a subgroup of A (namely, E · E′) of order p^{2t}. We will now rely on the following group-theoretical lemma:

**Lemma.** Let q = p^t be a power of a prime number, G a group of order q^2, generated by two disjoint subgroups E, E′ of order q. Assume that Z_{q−1} acts by automorphisms on G, stabilizing E and E′ such that the restriction of this action to E − {1} and E′ − {1} is simply transitive. Then G = E × E′.

**Proof.** Since G is a p-group, it has a non-trivial center Z. Let z ∈ Z be a non-trivial element. We claim that no element of Z_{q−1} fixes z (we will write the action exponentially).

Indeed, write z = ab for a ∈ E and b ∈ E′. This way of writing is unique, since G = E · E′ and E ∩ E′ = {1}. Let σ ∈ Z_{q−1}, then if z^σ = z, ab = a^σb^σ, so a = a^σ and b = b^σ by the uniqueness of writing. However, since the cyclic group is assumed to act simply transitively on E − {1} and E′ − {1}, this can only happen if a = b = 1, so z = 1, a contradiction.

Hence the set the set Z_{q−1} · z has q − 1 elements, and it belongs to Z, since Z_{q−1} acts by automorphisms on G.

Since Z is a subgroup of G, its order has to divide q^2, so it is at least q. Assume that Z ∩ E = {1}. Then G = Z · E, so |Z| = q, and since Z is normal in G, G = Z × E, so actually G = Z × E, contradicting the fact that Z is the center of G.

Hence we can find a non-trivial element ε ∈ Z ∩ E, so that [ε, E′] = 1. Acting on this with Z_{q−1} (which is transitive on E) shows that [E, E′] = 1, so that G = E × E′ as desired.

This lemma implies immediately that A = (E × E′) × D_n. Since the statement about A′X is in (2) (cf. (2.4)), the proof of proposition is finished.

**Remark 2.** From the above proof, we see that Γ = ker(̃u) contains [E, γEγ], and actually has to equal it, since (E * E′)/[E, E′] = E × E′. Up to conjugation, one can embed E into PGL(2, k) only as upper triangular matrices

\[ E = \left\{ \begin{pmatrix} 1 & V \\ 0 & 1 \end{pmatrix} \right\} \]

for some F_p-vector space V of dimension t in k, and actually, since a cyclic group Z_n (n = p^t − 1) has to act semidirectly on it, this cyclic Z_n has to be embedded as diagonal matrices over F_p^t, and hence V has to be of the form F_p^t · x for some x ∈ k^*, as a little matrix calculation shows. We can assume x = 1 by conjugating again with a suitable diagonal matrix. Then γ can only be embedded as

\[ γ = \begin{pmatrix} 0 & 1 \\ C & 0 \end{pmatrix} \]

for some C ∈ k^*. Thus, up to PGL(2, k)-conjugation (viz., isomorphism of the corresponding Mumford curves, cf. [3], IV.3.10), the group Γ is completely characterized by specifying the number C ∈ k^*, which should satisfy |C| > 1 so that Γ is indeed a discrete subgroup of PGL(2, k). The latter fact follows, e.g., via the
method of isometric circles, cf [4, §8]: the isometric circle of any element of $E$ is (in our normalization) the unit circle $\{|z| = 1\}$, whereas the isometric circle of any element of $E'$ is $\gamma \cdot \{|z| = 1\} = \{|z - C| = 1\}$. For $E \ast E'$ to be discretely embedded in $PGL(2, k)$, these isometric circles should not intersect, leading to $|C| > 1$.

It would be interesting to find the relation between the parameters $c$ of the algebraic description $X_{t,c}$ and $C$ in this description of the Schottky group.

The theorem will now follow from the following algebraic fact:

**Proposition 2.** If $X$ is an algebraic curve over a field $k$ of characteristic $p > 0$ whose automorphism group is isomorphic to $A = Z_p^r \times D_{p-1}$. Assume that, for $p \neq 2$, the quotient by $A$ is of the form $X \to A \setminus X = P^1$ and is ramified above 3 points, say, $(P_1, P_2, P_3)$ with ramification groups $(Z_2, Z_2, B(t, n))$. If $p = 2$ suppose that two points $(P_1, P_2)$ are ramified with groups $(Z_2, B(t, n))$. Then $X$ is isomorphic to an Artin-Schreier curve $X_{t,c}$ for some $c \in k^*$.

**Proof.** Let $p \neq 2$. The automorphism cover $X \to A \setminus X = P^1$ has to decompose into a tower of successive Galois extensions as follows:

$$X \to X_1 := Z_p^2 \setminus X \to X_2 := Z_n \setminus X_1 \to X_3 = Z_2 \setminus X_2.$$  

For this tower to have the correct ramification behaviour, the following should hold. In $X_2 \to X_3$, exactly $P_1$ and $P_2$ should ramify, hence $X_2 = P^1$; let $P_{3,1}$ and $P_{3,2}$ denote the points of $X_2$ above $P_3$. Since $X_1 \to X_2$ is a tame cover in which only the two points $P_{3,i}$ should ramify, they have to ramify completely, and $X_1 = P^1$. Denote the points in $X$ above $P_{3,i}$ in $X_1$ by the same symbol. Now $X \to X_1 = P^1$ is an elementary abelian $p$-cover in which two points should ramify with ramification groups $Z_p^r$. We can reorder the situation so that $X$ admits two quotients $X^{(1)} = E_1 \setminus X$ and $X^{(2)} = E_2 \setminus X$ for two groups $E_1 = E_2 = Z_p^r$ such that $P_{3,i}$ branches completely in $X^{(i)}$ and $P_{3,j}$ is unbranched in $X^{(i)}$ for $i \neq j$. Let $z$ be a coordinate on $X_1 = P^1$ and set $z(P_{3,1}) = 0$ and $z(P_{3,2}) = \infty$. The equation of $X^{(1)}$ has to be a succession of $p$-covers of $P^1$ in which exactly one point ramifies completely. If we decompose it into $t$ totally ramified $p$-covers $X_{t}^{(1)} \to X_{t-1}^{(1)}$, one can see inductively that each of these is given by an equation $y^p - y = y^{i-1}$ with $y_1 = z$ (by suitable normalization), which implies inductively that all $X_1^{(1)}$ are isomorphic to $P^1$, and in the end we find that $X^{(1)}$ is given by $x^{p^t} - x = z$. A similar argument applies to $X^{(2)}$, but an equation of the form $y^{p^i} - y = \frac{z}{2}$ comes out for some $c \in k^*$ (since $\infty$ is supposed to ramify in $X^{(2)}$). Note that we cannot normalize $c = 1$, since the coordinate $z$ was already normalized by the tower of $X^{(1)}$. Finally, the fiber product

$$(3) \quad X_1^{(1)} \times_{X_1} X^{(2)}$$

dominate $X$, and it has the same degree over $X_1$ as $X$, hence it equals $X$. But the curve (3) is exactly equal to $X_{t,c}$, which finishes the proof for $p \neq 2$.

For $p = 2$, except for the fact that $X_2 \to X_3$ is Artin-Schreier of order two with a unique totally ramified point (instead of Kummer with two ramified points), the proof is entirely analogous, and we refrain from presenting the details. 

**Remark 3.** If the moduli space of Mumford curves of genus $g$ over $k$ is stratified according to automorphism groups, then this proposition shows that the stratum
with maximal automorphism group is exactly equal to the locus of Artin-Schreier-Mumford curves, and hence it is rigid analytically connected. This connectedness statement can fail to hold for more general strata, see for example [6]. Nevertheless, the dimension of more general equivariant first order deformation spaces of Mumford curves can be computed in term of the ramification data associated to the automorphism group (or the tree of groups associated with the normalizer of the Schottky group), cf. [3].

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