Abstract: We study and clarify in a reduced dynamical model for \( QCD(SU(\infty)) \) called Bollini-Giambiagi model and defined by constant gauge fields Yang-Mills path integral, several concepts on the validity of string representations on \( QCD(SU(\infty)) \) and the confinement problem.

1. Introduction

In the last decades (since 1980’s years), approaches have been pursued to reformulate non supersymmetric quantum chromodynamics as a String Theory ([1], [2], [3]) and thus handle the compound hadron structure in the \( QCD \) model for strong interactions ([3]). The common idea of all those attempts is to represent the full quantum ordered non supersymmetric phase factor as a string path integral, which certainly takes into account more explicitly the geometrical setting of the non abelian gauge theory than its usual description by gauge potential.

Other main protocol to achieve such string representation for the wilson loop operator in \( QCD \) is to use the still not completely understand large number of colors of t’Hooft
for non supersymetric quantum Yang-Mills theory.

It is the purpose of this paper to evaluate the static potential between two static charges with opposite signal on the approach of an effective reduced quantum dynamics of Yang-Mills constant-gauge fields ([3]) these results surely are expected to be relevant for the validity of the old conjecture of E. Witten about a $QCD(SU(\infty))$ dynamics dominated by a infinite dimensional “matrix” constant gauge $SU(\infty)$ master field configuration ([4], [5], [6]). These studies are presented in section 2 of this paper. In section 3, we present the relevant $QCD(SU(\infty))$ loop wave equation for our reduced model of constant – gauge fields for $QCD(SU(\infty))$ and suggest that, a free bosonic string as solution for this reduced Loop Wave Equation ([7]). We continue with our study and present also a detailed calculation of the quark-antiquark static potential from a one-loop approximation on the Regge slope string constant directly from the well-known Nambu-Goto string path integral ([8]).

Finally in section 4, we present also some studies on the dynamical aspects of this framework of constant field model by presenting path integral studies on evaluation of vectorial-scalar color singlet quark currents ([6],[8]).

Before to proceed, let us firstly reproduce two enlighten discourses on the present day problem of handle quantitatively Yang-Millls fields outside the lattice approximation 1-Quoted from A. Jaffe and Eduard Witten.

"Classical properties of non abelian gauge theory are within the reach of established mathematical methods, and indeed, classical non abelian gauge theory has played a very important role in pure mathematics in the last twenty years, especially in the study of three- and four-dimensional ($C^\infty$-differentiable) manifolds.

On the other hand, one does not yet have a mathematically complete example of a quantum gauge theory in four-dimensional space-time, not even a non abelian quantum gauge theory in four-dimensional.

Related to the pure string holographic approach based on the Maldacena conjecture and Super String Theory it appears interesting to cite V. Rivasseu (math-ph/0006017) about the general philosophy underlying supersymmetric strings.

It is worth to call attention that our constant gauge fields at $SU(\infty)$ are not the rigorous continuum version of the one-plague the Eguchi-Kawai lattice model.
Today the main stream of theoretical physics holds the view that field theory is only an effective (approximated!) theory and that superstring or its variant, M-theory are the best candidate for a fundamental global theory of nature (including QCD).

However this superstring theory has not yet received direct experimental confirmation; it has (surely) opened up a new interface with mathematicians, mostly centered around concepts and ideas of geometry and topology (of $C^\infty$-manifolds), with algebra and geometry dominating over analysis and calculational aspects.

Fortunately there is Lattice Gauge Theory, which although has remained largely phenomenological, it has produced somewhat “precise” results on Experimental Hadron mass spectroscopy as it has been pointed out by F. Wilczek (Nature 456,449, 2008). Being enough for that by just taking some mesons ($\pi, \mu, \Sigma$) mass inputs, even if in the context of QED one needs as input only the fine structure $\alpha = \frac{e^2}{4\pi \hbar c} \sim \frac{1}{137}$.

In this paper we propose to implement the QED one universal protocol above pointed out by taking now our reduced model as the QCD effective theory at large $N_c$ and as universal input parameter, the Gluonic condensate $\langle 0 | \text{Tr}(F^2) | 0 \rangle_{SU(\infty)}$ ([7]), instead of the fine structure constant $\alpha = \frac{e^2}{4\pi \hbar c}$.

1. The Static Confining Potential for the Bollini-Giambiagi model on $D = 4$.

The basic gauge-invariant observable on probing the non-perturbative vacuum ([1]) of $SU(N)$ Euclidean Yang-Mills bosonic field theory on $R^4$ is the Wilson loop quantum average
\[
\langle W[C] \rangle = \frac{\int d\mu[A] W[C]}{\int d\mu[A]}
\] (2.1)
where the loop parallel transport $SU(N)$-valued matrix is given by
\[
W[C] = \frac{1}{N} \text{Tr}_{SU(N)} \mathbb{P} \left\{ \exp \left[ i g \oint_C A_\mu(x) \, dx_\mu \right] \right\}
\] (2.2)
and $d\mu[A]$ denotes the Yang-Mills path-integral measure given formally by the Feynman prescription
\[
d\mu[A] = \left( \prod_{x \in R^4} (dA(x)) \right) \times \exp \left( -\frac{1}{4} \int_{R^D} \text{tr}(F_{\mu\nu}^2(A))(x)d^4x \right).
\] (2.3)
The Gauge connection $A_{\mu}(x)$ and the field strength $F_{\mu\nu}(x)$ are explicitly given by
\begin{align*}
F_{\mu\nu}(x) &= \left( \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + ig[A_{\mu}, A_{\nu}] \right)(x) \\
A_{\mu}(x) &= A_{\mu}^{a}(x)\lambda_{a}
\end{align*}

(2.4)

The $SU(N)$ generators $\{\lambda_{a}\}_{a=1,\ldots,N^{2}-1}$ are supposed to be Hermitians and satisfying the well known matrix relationship below
\begin{align*}
[\lambda^{a}, \lambda^{b}] &= if_{abc}\lambda^{c} \\
\text{Tr}(\lambda_{a}, \lambda_{b}) &= \frac{\delta_{ab}}{2} \\
f^{abc}f^{dce} &= N\delta^{cd}
\end{align*}

(2.5)

In our proposal for the Euchi-Kawai model on continuum, we introduce the space-time trajectory $C(R,T)$ of a static quark-antiquark pair, separated apart a distance $R$ with a (Euclidean) temporal evolution $0 \leq x_{0} \leq T$ ($R^{4} = \{(x_{0}, \vec{x}), \vec{x} \in \mathbb{R}^{3}\}$).

We face thus the problem of evaluating the path integral eq(2-1) for constant gauge fields configurations at the large $N_{c}$ limit ([5], [6]) and at the physical space-time $R^{4}$.

Let us briefly review our previous framework on our $D=4$ highly non-trivial generalization of $D=2$ model of Bollini-Giambiagi ([3]).

We firstly consider the full infinite volume space-time $R^{4}$ reduced to a finite volume space-time $\Omega$. This step has the effect to turns our “reduced” path integrals mathematically well defined.

This finite volume space-time is supposed to be formed by the superposition of $p$ four-dimensional hypercubes of characteristic volume (“size”) $V = L^{4}$.

The area $S[C(R,T)]$ enclosed by our rectangle $C(R,T)$ is such that $S[C(R,T)] \sim q L^{2}$ for large $p$. Obviously $(S[C])^{2} \leq \text{vol}(\Omega)$. So ours integers $q$ and $p$ should satisfy $q \leq \sqrt{p}$, an important bound to be kept on mind on what follows.

Our improved large $N$-limit will be defined in such way by already taking into account the basic phenomena of QCD-Yang-Mills dimensional transmition of the strong coupling constant, a fully non perturbative phenomena similar to the Higgs mass mechanism on Weinberg-Salan theory. We, thus, define the effective $SU(\infty)$ coupling constant through the relationship for our space-time of finite volume
\begin{equation}
\lim_{N \rightarrow \infty} \left( \frac{g^{2}N}{L^{2}} \right) \left( \frac{q}{p} \right) < \infty.
\end{equation}

(2.6)
It is worth observe that for $p \to \infty$ the infinite volume limit should be taken according the underlying $N_c \to \infty$ limit. Namely

$$\frac{q}{p} \equiv a \to 0$$

$$\lim_{N \to \infty} g^2 \left( \frac{Na}{L^2} \right) = (g_{\infty})_{\text{dim}}^2 < \infty$$

Here $L^2$ is a physical finite area parameter somewhat to be related to the domain area size of the famous $QCD$ spaghetti vacuum ([7]), a topic to be discussed elsewhere.

After these preliminaries remarks, we must solve the invariant constant gauge field $SU(\infty)$ matrix path integral below written:

$$W_{SU(\infty)} [C(R,T)] = \left\{ \lim_{N \to \infty} \frac{1}{W(0)} \times \left[ \int_{-\infty}^{+\infty} \left( \prod_{a=1}^{N^2-N} \prod_{\mu=0}^{D-1} dA^a_\mu \right) \times \right. \right.$$

$$\left. \times \Delta_{Fp}[A_\mu] \exp \left( +g^2 \frac{V}{2} \text{Tr} \left( [A_\mu, A_\nu]_-^2 \right) \right) \times \right.$$  

$$\left. \times \left( \exp \frac{|g^2 S[C(R,T)]|^2}{2} \cdot \frac{1}{\text{Tr}_{SU(N)}} \left( [A_0, A_1]^2 \right) \right) \right\}. \quad (2.9)$$

In order to evaluate the $SU(N)$-invariant constant Gauge field path integral eq(2.9), we use the Bollini-Giambiagi Cartan matrix decomposition ([5])

$$A_\mu = B^a_\mu H_a + G^b_\mu E_b \quad (2.10)$$

where the Cartan basis $\{H_a, E_a\}$ of the $SU(N)$ Lie algebra possesses the special calculations properties ([5], [6])

a) For $a, b = 1, 2, \ldots, N - 1$

$$[H_a, H_b]_- = 0. \quad (2.11)$$

b) For $b = \pm 1, \ldots, \pm \frac{N(N-1)}{2}$

$$[H_a, E_b]_- = r_a(b) E_b. \quad (2.12)$$

c) For $a = 1, 2, \ldots, \frac{N(N-1)}{2}$.

$$[E_a, E_{-a}] = \sum_{\ell=1}^{N-1} r_c(a) H_a. \quad (2.13)$$
d) For $a \neq -b$, $a, b = \pm 1, \ldots, \pm \frac{N(N - 1)}{2}$
\[ [E_a, E_b] = N_{ab} E_{a+b}. \] (2.14)

In this distinguished Lie Algebra basis, one can easily fix the Gauge on the $SU(N)$-valued invariant matrix path integral by simply choosing all the $N$-abelian components $B^a_\mu$ on the connection eq(2-10) to be vanished. Namely
\[ B^a_\mu = 0. \] (2.15)

It is expected thus that the Faddev-Popov term $\Delta_{F_P}[A_\mu]$ should be quenched at the $N \to \infty$ limit eq(2-6)-eq(2-8) i.e. ([5])
\[ \lim_{N \to \infty} \Delta_{F_P}[A_\mu] \to 1 \] (2.16)

Any way we take the Faddev-Popov quenched determinant to unity in this sort of approximate evaluation of ours. $SU(\infty)$-matrix valued invariant path integral (note that procedure to evaluate degrees of Freedom reduced path integrals is usually implemented when one handles for instance, fermions degrees of Freedom on $SU(N)$ lattice path integral ([3]). We will adopt such procedure here).

By assembling all the above results one gets the following outcome eq(2.9) defined now by $SU(N)$ constant gauge field configurations for a general euclidean space-time $R^D$

\[ W[C_{(R,T)}] = \frac{1}{W[0]} \times \left\{ \int_{-\infty}^{+\infty} \left[ \prod_{a=1}^{N^2-N-1} \prod_{\mu=0}^{D-1} dG^a_\mu \right] \right\} \exp \left\{ +\frac{1}{2} G^a_\mu G^b_\nu G^c_\mu G^d_\nu \mathcal{L}_{abcd} \left[ g^2 V + \left( \delta_{\mu0} \delta_{\nu1} \frac{[g^2 S][C_{(R,T)}]}{N_T} \right) \right] \right\} \] (2.17)

The above matrix valued path integral can be easily exactly evaluated through rescalings, at large $N$: Namelly (see Appendix A for details).

\[ \mathcal{L}_{abcd} = \left( \sum_{i,\ell=1}^{N-1} r_i(a) r_\ell(c) \delta_{i\ell} \delta_{c_i-d} \delta_{c_i-b} \right) \]
\[ + (N_{ab} N_{cd} (1-\delta_{a_i-b})(1-\delta_{c_i-d}) \delta_{a+b,(c+d)}) \]
a) For $\mu \neq 0, 1$: 
\[ G_\mu^a \to G_\mu^a [g^2 V]^{-1/4}. \]

b) For $\mu = 0, 1$:
\[ G_\mu^a \to G_\mu^a \left[ g^2 + \frac{[g^2 S[C(R,T)]]^2}{(N/2)} \right]^{-1/4} \]

and leading thus to the exactly result
\[ W[C_{(R,T)}] = \lim_{N \to \infty} \left\{ \left[ \left( g^2 V + \frac{[g^2 S[C(R,T)]]^2}{N/2} \right) \right]^{-\left( \frac{N^2 - N}{2} \right)} \right\} \]
\[ \times \left[ (g^2 V) \frac{[N^2 - N)(D - 2)}{4} \right] \left/ (g^2 V) \frac{(N_2 - N)D}{4} \right\} \]

It yields thus, the following $N \to \infty$ limit
\[ W[C_{R,T}] = \lim_{N \to \infty} \left\{ 1 + \frac{\frac{[g^2 S[C_{R,T}]]}{(N/2)V}}{V} \right\}^{-\frac{N(N - 1)}{2}} \]
\[ = \lim_{N \to \infty} \left\{ \exp \left[ -\frac{g^2(N - 1) L^2}{L^2} \left( \frac{S^2}{V} \right) \right] \right\} \]

For $D = 4$, we have on the context of our proposed $SU(\infty)$ infinite volume limit eqs(2-6)-(2-8), our $R^4$ Wilson Loop “string” behavior.
\[ W[C_{(R,T)}] = \exp \left\{ -\frac{g^2(N - 1) L^2}{L^2} \left( \frac{q^2 L^4}{p L^4} \right) \right\}^{(N \to \infty)} \exp \left\{ -\frac{g^2N}{L^2} \left( \frac{q}{p} \right) (qL^2) \right\}^{(N \to \infty)} \exp \left\{ -(g_\infty)^2 RT \right\} \]

From eq(2-21), one obtains the confining quark-antiquark potential
\[ V(R) = \lim_{T \to \infty} \left[ -\frac{1}{T} \ell g(W[C_{R,T}]) \right] = (g_\infty)^2 R \]
leading to an attractive constant force “biding” the static pair of quarks as originally obtained by K. Wilson on his lattice gauge - modelling ([3]).
3. The Luscher correction to inter quark potential on the reduced model

In this section we intend to show that our proposed $SU(\infty)$ constant gauge field theory leads to a free string theory path-integral. We thus evaluate explicitly through the string path-integral the next non-confining corrections to the quark-antiquark potential eq(2-22).

In order to argument an effective low energy QCD string representation in this model, we are going to consider the loop have equation ([1]) for constant gauge fields already on the continuum at large $N$ limit.

Let us thus firstly consider general loops $C_{xx} = \{(X_\mu(\sigma))_{\mu=0,1,2,3}; 0 \leq \sigma \leq 2\pi\}$ on $R^4$. It is well-known that formally we have the functional loop derivative ([1])

\begin{equation}
\psi[C_{xx}, A_\mu(x)] = \frac{1}{N} \text{Tr} \mathbb{P} \left\{ \exp \left( ig \oint_{C_{xx}} A_\mu(X(\sigma)) dX^\mu(\sigma) \right) \right\}
\end{equation}

\begin{equation}
\frac{\delta}{\delta X_\mu(\sigma)} \psi[C_{xx}, A_\mu(x)] = \frac{ig}{N} \text{Tr} \mathbb{P} \left\{ F_{\mu\nu}(X(\sigma)) \frac{dX^\nu(\sigma)}{d\sigma} \right\}
\end{equation}

\begin{equation}
\times \exp \left( ig \oint_{C_{xx}} A_\mu(X(\sigma)) dX^\mu(\sigma) \right) \right\},
\end{equation}

\begin{equation}
\frac{\delta^2}{\delta X_\mu(\sigma) \delta X_\nu(\sigma)} \psi_{SU(N)}[C_{xx}, A_\mu(x)] = \delta_{\mu\nu} (F_{\mu\nu}(X(\sigma)))
\end{equation}

\begin{equation}
= \frac{1}{N} ig \text{Tr} \mathbb{P} \left\{ (\nabla F_{\mu\nu})(X(\sigma)) \frac{dX^\nu(\sigma)}{d\sigma}(\sigma) \exp \left( ig \oint_{C_{xx}} A_\mu dX^\mu \right) \right\]
\end{equation}

\begin{equation}
+ \left[ \frac{(ig)^2}{N} \text{Tr} \mathbb{P} \left\{ (F_{\mu\nu}F_{\mu\nu})(X(\sigma)) \times \left( \frac{dX^\rho}{d\sigma} \cdot \frac{dX^\rho}{d\sigma} \right)(\sigma) \exp \left( ig \oint_{C_{xx}} A_\mu dX^\mu \right) \right\} \right]
\end{equation}
For constant gauge fields configurations the first term of the right-hand side of eq(3.25)
\( \left( \frac{\delta}{\delta X_\mu(\sigma)} \left( \text{constant } F_{\mu\nu} \right) = 0! \right) \) vanishes identically. So, after taking the path integral average of eq(3.25) through the path integral of constant gauge fields configurations eq(2.9) and considering the usual path integral factorization of a product of gauge invariant observable at \( SU(\infty) \), together with the formation of non-vanishing value of the Yang-Mills energy on the non-trivial QCD vacuum one gets finally the following loop wave equation for the quantum Wilson Loop our Loop in our reduced \( SU(\infty) \) gauge theory on \( \mathbb{R}^4 \).

\[
\int_0^{2\pi} \left[ \left( \frac{\delta^2}{\delta X_\mu(\sigma) \delta X_\mu(\sigma)} \right) \psi_{SU(\infty)}[X^\mu(\sigma)] \right] d\sigma 
\]

\( \geq 0 \)

\[
= \left( - (g_\infty)^2(0) |F^2|_0 \right)_{SU(\infty)} \times \left( \int_0^{2\pi} |X'_\mu(\sigma)|^2 \times \psi_{SU(\infty)}[X^\mu(\sigma)] \right) d\sigma \quad (3.26)
\]

Here, we have the \( SU(\infty) \) Euclidean Gauge Theory parameters identification
a)**

\( (g_\infty)^2(0) = \lim_{N \to \infty} (g^2 N) < \infty \)

b)***

\[
\delta^{\rho\alpha} \langle 0 | F^2 | 0 \rangle_{SU(\infty)} = \lim_{N \to \infty} \left( \frac{1}{N} Tr \langle 0 | \int d^D x \left( F_{\rho\alpha}(x) F^{\mu\alpha}(x) \right) | 0 \rangle \right) < 0 \quad (3.27)
\]

By comparing the above parameters with those coupling constants of the static case, one has the following identification for the Spaghetti QCD non-perturbative broken scale

*Gerard ’t Hooft hypothesis

**One of the main points on the search for string representations on Q.C.D. (SU(\infty)) (formally a quantum field theory defined by only by “planar” Q.C.D., infrared regularized, Feynman diagrammas - G. ’t Hooft), is the non-perturbative condensate formation of the Yang-Mills SU(\infty) field strength:

\[
\lim_{x \to x'} \langle \Omega_{\text{vac}}^\infty | (F_{\mu\nu}(x) F^{\rho\sigma}(x')) | \Omega_{\text{vac}}^\infty \rangle = \delta^{\mu\rho} \delta^{\nu\sigma} \langle \Omega_{\text{vac}}^\infty | F^2(x) | \Omega_{\text{vac}}^\infty \rangle.
\]

This hypothesis has the same foundational importance of the non-zero formation of the Higgs field expectation value on Weinberg-Salan Weak interaction theory.

The (non-perturbative) non-vanishing of eq(3.27) is a fundamental hypothesis on all of ours works on the subject ([3]).
invariance vacuum effective area domain with the $QCD$ value condensate

$$\frac{1}{\alpha'^2} = - \langle 0 | F^2 | 0 \rangle > 0. \quad (3.28)$$

A result already expected ([7]).

At this point we point out that the reduced loop wave equation is the same of a free Bosonic string theory with the string Regge slope identification with the reduced Gauge theory at $SU(\infty)$

$$\frac{1}{(2\pi \alpha')^2} = - (g_{\infty})^2 (0 | F^2 | 0) \quad (3.29)$$

As a consequence, one should expects the phenomenological path-integral representation between the large $N_c$ and extreme low energy continuum $QCD(SU(\infty))$ (represented by constant $SU(\infty)$ gauge fields), with a free bosonic (creation process) string path integral on the light-cone gauge

$$\langle \psi[C_{xx}, A(x)] \rangle_{SU(\infty)_{\text{low-energy}}} = G(C_{xx}, A)$$

$$= \int_0^{\infty} dA \left\{ \int_{X^\mu(\sigma,0)=0}^{X^\mu(\sigma)=C_{xx}(\sigma)} DF[X^\mu(\sigma)] \right\}$$

$$\times \exp \left\{ - \frac{1}{2} \int_0^A dt \int_0^{2\pi} d\sigma \left[ (\partial_\zeta X^\mu)^2 + \frac{1}{(2\pi \alpha')^2} (\partial_\sigma X^\mu)^2 \right] \right\} \quad (3.30)$$

$$G_{\text{string}}(C_{xx}, 0) = G_{\text{string}}(C_{xx}, \infty) = 0 \quad (3.31)$$

At this point it is worth observe that our light-cone string path integral propagator

$$G_{\text{string}}(C_{xx}, A) = \int_{X^\mu(\sigma,0)=C_{xx}(\sigma)}^{X^\mu(\sigma)=C_{xx}(\sigma)} DF[X^\mu(\sigma, \pi)]$$

$$\times \exp \left\{ - \frac{1}{2} \int_0^A dt \int_0^{2\pi} d\sigma \left[ (\partial_\zeta X^\mu)^2 + \frac{1}{(2\pi \alpha')^2} (\partial_\sigma X^\mu)^2 \right] \right\} \quad (3.32)$$

satisfies the area-Difussion euclidean Schrodinger loop functional equation:

$$\frac{\partial G_{\text{string}}(C_{xx}, A)}{\partial A} = \left\{ \int_0^{2\pi} d\sigma \left( \frac{\delta^2}{\delta X^\mu(\sigma) \delta X^\mu(\sigma)} - \frac{1}{(2\pi \alpha')^2} |X^\mu(\sigma)| \right) G_{\text{string}}(C_{xx}, A) \right\} \quad (3.33)$$

(*)- Rigorously, one should expect the Polyakov’s covariant path integral for $D = 4$ (with Liouville quantum degrees of freedom as representing the QCD ($SU(\infty)$) Wilson Loop ([8])).
together with the boundary conditions:

\[ G_{\text{string}}(C_{xx}, 0) = G_{\text{string}}(C_{xx}, \infty) = 0. \] (3.34)

Let us now evaluate in details the quark-antiquark potential from the general Nambu-Goto string, path integral in \( R^D \)

\[ \psi[C_{(R,T)}] = \int D^F[X^\mu(\sigma, \zeta)] \times \exp \left\{ -\frac{1}{2\pi\alpha'} \left[ \int_0^T d\zeta \int_0^R d\sigma \sqrt{\text{det}(h(X^\mu(\sigma, \zeta)))} \right] \right\} \] (3.35)

Here the orthogonal dynamical string vector position is considered as closed quantum fluctuations from the static quark-antiquark trajected by \( C_{(R,T)} \) i.e.:

a) \[ X^\mu(\sigma, \zeta) = \zeta(1, 0) + \sigma(0, 1) + \sqrt{\pi\alpha'} Y^\mu(\sigma, \zeta) \]

b) \[ \begin{align*}
Y^\mu(\sigma, \zeta \pm T) &= Y^\mu(\sigma, \zeta) \\
Y^\mu(0, \pm T) &= Y^\mu(0, 0) \\
\mu &= 2, 3, \ldots, D - 2
\end{align*} \]

c) \[ h_{00}(X^\mu(\sigma, \zeta)) = (\partial_\zeta X^\mu \partial_\zeta X^\mu)(\sigma, \zeta) = 1 + \pi\alpha' \left( \partial_\zeta Y^\mu \partial_\zeta Y^\mu \right)(\sigma, \zeta) \]
d) \[ h_{01}(X^\mu(\sigma, \zeta)) = \pi\alpha' \left( \partial_\sigma Y^\mu \partial_\zeta X^\mu \right)(\sigma, \zeta) \]
e) \[ h_{11}(X^\mu(\sigma, \zeta)) = 1 + \pi\alpha' \left( \partial_\sigma Y^\mu \partial_\sigma X^\mu \right)(\sigma, \zeta) \] (3.36)

As a consequence, we have explicitly the following one-loop order approximation for the string path integral weight eq(3.35):

\[ \frac{1}{2\pi\alpha'} \sqrt{h(X^\mu(\sigma, \zeta))} = \frac{1}{2\pi\alpha'} \left[ 1 + \pi\alpha' \left( \frac{\partial Y^\mu}{\partial \zeta} \frac{\partial Y^\mu}{\partial \zeta} + \frac{\partial Y^\mu}{\partial \sigma} \frac{\partial Y^\mu}{\partial \sigma} \right) \right] \times O((\alpha')^2) \] (3.37)
As a result of substituting eq(3.36) on eq(3.37), one gets the following closed bosonic string Gaussian path-integral to evaluate

\[ \psi[C(R,T)] = \int_{Y^\mu(\sigma,\zeta)=Y^\mu(\sigma,\zeta)} D^F[Y^\mu(\sigma,\zeta)] \exp \left\{ \left( -\frac{RT}{2\pi\alpha'} \right) - \frac{1}{2} \int_0^T d\zeta \int_0^R d\sigma (Y^\mu(-\Delta')(R,T)Y_\mu)(\sigma,\zeta) \right\} \]

\[ = e^{-\frac{RT}{2\pi\alpha'}} \left( \det -\frac{D-2}{2}( -\Delta')(R,T) \right) \] (3.38)

where the Laplacian \(-\Delta'(R,T)\) on the rectangle \(C(R,\pm 1)\) has Dirichlet boundary conditions and considered projected out from the zero modes.

It has been evaluated fully on the literature ([8]):

\[ \det -\frac{D-2}{2}( -\Delta')(R,T) = \left[ \left( \frac{R}{T} \right)^{\frac{D-2}{2}} e^{\frac{\pi T}{\alpha'}} (D-2) \times \prod_{n=1}^{\infty} \left( 1 - e^{-\frac{2\pi n}{\alpha'}} \right)^{-2(D-2)} \right] \] (3.39)

At this point, one can easily verify the string result for the quark-antiquark potential with the Coulomb interaction L"uscher correction.

\[ V(R) = \lim_{T \to \infty} \left( -\frac{1}{T} \ln \psi[C(R,T)] \right) = \frac{1}{2\pi\alpha'} R - \frac{(D-2)\pi}{6} \cdot \frac{1}{R} \] (3.40)

4. Some path integral dynamical aspects of the reduced QCD as a path integral dynamics of euclidean strings

Let us start this section on dynamical aspects by writing firstly in details, the operational euclidean path integral expression for the non-relativistic Feynman propagator of a spinless particle in the presence of an external (euclidean) abelian gauge field and an external scalar potential.

As a first step let us write the Feynman propagator above cited in the euclidean space-time \(R^4(h=1)\)

\[ G(x,y,t) = \langle x \exp \left\{ -t \left[ \frac{1}{2m} \left( -i\nabla - \frac{e}{c} \vec{A} \right)^2 - ie\varphi + gV \right] \right\} y \rangle \] (4.41)
Then $\vec{A} = (A_1, A_2, A_3)$ denotes the time-dependent vectorial abelian field, $\varphi$ the field potential and $V(x, t)$, the external potential, also supposed time dependent.

The phase-space path integral is easily written as of as

$$
G(x, y, t) = \int_{\vec{X}(0) = y}^{\vec{X}(t) = x} D^F(\vec{X}(\sigma)) \int D^F[\vec{p}(\sigma)]
$$

$$
\times \exp \left\{ +i \int_0^t \left( \vec{P} \cdot \frac{d\vec{A}}{d\sigma} \right) (\sigma) d\sigma \right\}
$$

$$
\times \exp \left\{ - \left[ \int_0^t \frac{1}{2m} \left( \vec{P}(\sigma) - \frac{e}{c} \vec{A}(\vec{X}(\sigma)) \right)^2 - ie \varphi(\vec{X}(\sigma), \sigma) + g V(\vec{X}(\sigma), \sigma) \right] \right\}
$$

(4.42)

After the formal path-integral change of variable into eq (4.42)

$$
\vec{P}(\sigma) - \frac{e}{c} \vec{A}(\vec{X}(\sigma), \sigma) = \vec{Q}(\sigma)
$$

(4.43)

we get the following result:

$$
G(x, y, t) = \int_{\vec{X}(0) = y}^{\vec{X}(t) = x} D^F(\vec{X}(\sigma)) \int D^F[\vec{Q}(\sigma)]
$$

$$
\exp \left\{ i \int_0^t \left( \vec{Q}(\sigma) + \frac{e}{c} \vec{A}(\vec{X}(\sigma), \sigma) \right) \frac{d\vec{X}(\sigma)}{d\sigma} \right\}
$$

$$
\exp \left\{ - \frac{1}{2m} \int_0^t (\vec{Q}(\sigma))^2 \right\}
$$

$$
\exp \left\{ +ie \int_0^t \varphi(\vec{X}(\sigma), \sigma)d\sigma \right\}
$$

$$
\exp \left\{ - \int_0^t V(\vec{X}(\sigma), \sigma)d\sigma \right\}
$$

(4.44)

After realizing the Gaussian $\vec{Q}(\sigma)$ functional integral

$$
\int D^F[\vec{Q}(\sigma)] \exp \left[ - \frac{1}{2m} \int_0^t (\vec{Q}(\sigma))^2 d\sigma \right] \exp \left[ i \int_0^t \vec{Q}(\sigma) \frac{d\vec{X}(\sigma)}{d\sigma} \right] =
$$

$$
\exp \left\{ - \frac{m}{2} \int_0^t \left( \frac{d\vec{X}}{d\sigma} \right)^2 (\sigma) d\sigma \right\}, \quad (4.45)
$$

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we get our gauge invariant path integral expression for the euclidean Feynman propagator under study

\[ G(x, y, t) = \int_{X(0) = y}^{X(t) = x} D^F[X(\sigma)] \exp \left[ -\frac{m}{2} \int_0^t \left( \frac{d\bar{X}}{d\sigma} \right)^2 (\sigma) d\sigma \right] \exp \left[ -g \int_0^t V(X(\sigma), \sigma) d\sigma \right] \exp \left[ ie \left( \int_0^t \vec{A}(X(\sigma)) \cdot \frac{d\bar{X}}{d\sigma} d\sigma \right) + ie \left( \int_0^t \varphi(\bar{X}(\sigma), \sigma) d\sigma \right) \right] \]

It is curious to point out the gauge invariance of eq(4.46) is solely under all periodic Gauge transformation \((x \leq z \leq y ; 0 \leq t' \leq t)\)

\[
\begin{align*}
\vec{A}(z, t') &= (\vec{A} + \vec{\nabla}\Lambda)(z, t') \\
\varphi(z, t') &= (\varphi + \frac{\partial \Lambda}{\partial t})(z, t') \\
\Lambda(x, t) &= \Lambda(y, 0) + \frac{2\pi n}{\epsilon}
\end{align*}
\]

Namely

\[
\exp \left\{ ie \int_0^t (\vec{\nabla}\Lambda)(X(\sigma), \sigma) \frac{d\bar{X}}{d\sigma} d\sigma + ie \int_0^t \frac{d\Lambda}{d\sigma} d\sigma \right\}
= \exp \left\{ ie \int_0^t \frac{d}{d\sigma} (\Lambda(X(\sigma), \sigma)) d\sigma \right\}
= \exp \{ ie(\Lambda(x, t) - \Lambda(y, 0)) \}.
\]

In the euclidean quantum field case in \(R^D\) one must consider the generating fermionic case

\[
Z[A_\mu] = \frac{\det \frac{1}{2}(D^*(A) D(A))}{\det \frac{1}{2}(\theta^* \theta)},
\]

which can be re-write through well-known proportion loop space techniques as a loop
space $D$-dimensional non-relativistic propagator

$$
\ell g(Z[A]/Z(A = 0)) = -\frac{1}{2} \int_0^\infty \frac{dt}{t} \left\{ \int_{R^D} d^D x_\mu \int_{X_\mu(0) = x_\mu}^{X_\mu(t) = x_\mu} DF[X_\mu(\sigma)] \right. \\
\exp \left\{ -\frac{1}{2} \int_0^t \left( \frac{dX_\mu}{d\sigma} \right)^2 (\sigma) d\sigma \right\} \\
\left. \mathbb{P}_{\text{spin}} \mathbb{P}_{SU(N)} \left[ \exp \left( ie \int_0^t \left( A_\mu(X_\mu(\sigma)) \right) + \frac{ie}{4} [\gamma^\mu, \gamma^\nu] F_{\mu\nu}(X_\rho(\sigma)) d\sigma \right) \right] \right\},
$$

(4.50)

where the symbols $\mathbb{P}_{\text{spin}}$ and $\mathbb{P}_{SU(N)}$ means $\sigma$-ordered matrixes indexes of the spin-color gauge connection phase factor (the fermionic Wilson loop). At very low energy region, one could consider as an effective theory the situation that all degrees of Dirac spin of the particle, are non-dynamical (i.e.: frozen to scalar values), or equivalently one can disregard the spin orbit shengt field coupling on eq(4.50) \( \left( ig \hbar [\gamma^\mu, \gamma^\nu] F_{\mu\nu}(X^\rho(\sigma)) \cong 0 \right) \).

Let us now apply the above well-known remarks to evaluate approximately “scalar” composite operators quark-antiquark Green functions.

The effective connected generating functional for vectorial quark currents at very low energy (the strong coupling region of the underlying Massless Yang-Mills theory) is given by the following loop expression

$$
\ell g(Z_{QCD}^{\text{eff}}(J_\mu)) = \ell g\left( \det \frac{1}{2}(D^*(ig A_\mu + J_\mu) D(ig A_\mu + J_\mu)) \right)_{A_\mu}
$$

(4.51)

there \( \langle \; \rangle_{A_\mu} \) denotes the complete Yang-Mills path integral, $A_\mu$ the Yang-Mills field and $J_\mu(x)$ the external source of the vectorial quark currents \( (J_\mu(x)(\bar{\psi} \gamma^\mu \psi)(x)) \).

On the basis of the above discussions one has the following expression for eq(4.50), with the $QCD$ scale $\Lambda_{QCD}$ already built in a large $SU(\infty)$ limit in the proper-time gauge (or in the string light-cone gauge)

$$
\ell g(Z_{QCD}^{\text{eff}}[J_\mu]_{\Lambda_{QCD}}) = -\frac{1}{2} \left\{ \int_{1/\Lambda_{QCD}}^{\Lambda_{QCD}} \frac{dt}{t} \times \\
\mathbb{P}_{SU(N)} \left\{ \exp i g \int_0^t \left( A_\mu \frac{dX_\mu}{d\sigma} \right) (\sigma) d\sigma \right\} \exp \left( \frac{i}{4} \int_0^t \left( J_\mu \frac{dX_\mu}{d\sigma} \right) d\sigma \right) \right\},
$$

(4.52)
The vectorial $N$-point bilinear quark current is given by in momentum space

$$
\langle (\bar{\psi}\gamma^\mu_1 \psi)(x_1) \cdots (\bar{\psi}\gamma^\mu_N \psi)(x_N) \rangle_{A_\mu} = \frac{\delta^2}{\delta J_{\mu_1}(x_1), \cdots, \delta J_{\mu_N}(x_1)} \left[ \ell g \left( Z_{\text{QCD}}^{\text{eff}}(J_\mu) \right) \right] = G_{\mu_1 \cdots \mu_N}(x_1, \cdots, x_N) \tag{4.53}
$$

Or equivalently, after suitable Fourier momenta transforms.

$$
\tilde{G}_{A_{\text{QCD}}}(P_{\mu_1}, \cdots, P_{\mu_N}) = -\frac{1}{2}(i)^N \left\{ \int_{t/A_{\text{QCD}}}^t \frac{dt}{t} \int_0^t d\sigma_1 \cdots \int_0^t d\sigma_N \int d^Dz_\mu \times \int_{X_\mu(0)=z_\mu}^{X_\mu(t)=z_\mu} D^F[X(\sigma)] \right. \\
\left. \times \left[ \exp \left( -\frac{1}{2} \int_0^t \left( \frac{dX}{d\sigma} \right)^2 (\sigma) d\sigma \right) \right. \right. \\
\left. \times \left( \frac{dX_{\mu_1}}{d\sigma}(\sigma_1) \cdots \frac{dX_{\mu_N}}{d\sigma}(\sigma_N) \right) \right] \\
\left. \times \left( \exp \left( i \sum_{h=1}^N \frac{p_{\mu_h}}{\alpha'} X_{\mu_h}(\sigma) \right) \right) \right. \\
\left. \left. \times \left\langle \frac{ig}{\ell} \int_0^t A_\mu dX^\mu \right\rangle_{\text{SU}(\infty)} \right. \tag{4.54}
$$

On the basis of eq(4.54), one could envisage to try to evaluate eq(4.53) through an Gaussian (euclidean) string path integral. Let us take for granted such string representation as a workable sound hypothesis on basis of our previous studies.

The key point is to evaluate in terms of the loop variable $X^\mu(\sigma)$, the following annihilation string path integral

$$
W[X_\mu(\sigma), 0 \leq \sigma \leq t] = \int_0^\infty dA \left\{ \int_{Y^\mu(\sigma, 0) = X^\mu(\sigma)}^{Y^\mu(\sigma, A) = 0} D^F(Y^\mu(\sigma, s)) \right. \\
\exp \left\{ -\frac{1}{2} \int_0^A ds \int_0^t d\sigma \left[ (\partial_\sigma Y^\mu)^2 + \frac{1}{(\pi \alpha')^2} (\partial_\sigma Y^\mu)^2 \right](\sigma, s) \right\} \right\} \tag{4.55}
$$

If the action is $\int_0^A ds \int_0^t d\sigma [(\partial_\sigma Y^\mu)^2 + \frac{1}{(\pi \alpha')^2} (\partial_\sigma Y^\mu)^2](s, \sigma)$, then eq.(4.56) takes the form $\sigma \to \varphi = \sigma(\pi \alpha')$ and $\overline{Y}^\mu(\varphi, \beta) = Y^\mu(\sigma, s)$.  

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In order to evaluate eq(4.55) exactly, let us firstly consider the standard re-scale

\[
\sigma \rightarrow \sigma (\pi \alpha')^{1/2} = \overline{\sigma}
\]

\[s \rightarrow s
\]

\[Y^\mu (\sigma, s) \rightarrow \overline{Y}^\mu (\overline{\sigma}, s) \equiv (\pi \alpha')^{1/4} (Y^\mu (\sigma, s))\] (4.56)

which formally turns the string velocity into a overall factor into the path integral weight

\[
W[\overline{X}_\mu(\overline{\sigma})] = \int_0^\infty dA \left\{ \int_{\overline{Y}^\mu(\overline{\sigma},0)=\overline{X}_\mu(\overline{\sigma})}^{\overline{Y}^\mu(\overline{\sigma},A)=0} D_\Sigma^{\overline{\sigma}}[\overline{Y}^\mu(\overline{\sigma}, s)]
\right.
\]

\[
\times \exp \left\{ -\frac{1}{2 (2\pi \alpha')^2} \left[ \int_0^A ds \int_0^{t(\pi \alpha')^{1/2}} d\overline{\sigma} \left( \left( \frac{\partial \overline{Y}^\mu}{\partial s} \right)^2 + \left( \frac{\partial \overline{Y}^\mu}{\partial \overline{\sigma}} \right)^2 \right) \right] \right\}
\] (4.57)

After considering the “Brownian Bridge like” background loop-surface decomposition which has the meaning of considering a toroidal like fluctuating closed string world sheet \(Z_\mu(\overline{\sigma}, \zeta)\) bounded by the closed quark-antiquark trajectory \(\overline{X}_\mu(\overline{\sigma})\) (\(\overline{X}_\mu(\overline{\sigma} + t) = \overline{X}_\mu(\overline{\sigma})\), \(t\), fixed loop proper-time)

\[
\overline{Y}^\mu(\overline{\sigma}, s) = \overline{X}_\mu(\overline{\sigma}) \left( \frac{A - s}{A} \right) + \sqrt{\pi \alpha'} Z^\mu(\overline{\sigma}, s)
\]

\[
Z^\mu(\overline{\sigma}, A) = Z^\mu(\overline{\sigma}, 0) = 0
\]

\[
Z^\mu(\overline{\sigma} + t, s) = Z^\mu(\overline{\sigma}, s),
\] (4.58)

one gets the regularized proper-time string propagator

\[
W[\overline{X}_\mu(\overline{\sigma})] = \int_{\varepsilon}^\infty dA \exp \left\{ -\left( \frac{(A/3)}{2\pi \alpha'} \right) \left( \int_{0}^{(\pi \alpha')^{1/2} t} \left( \frac{d\overline{X}_\mu(\overline{\sigma})}{d\overline{\sigma}} \right)^2 d\overline{\sigma} \right) \right.
\]

\[
- \left( \frac{1}{2\pi \alpha' A} \right) \left( \int_{0}^{(\pi \alpha')^{1/2} t} \left( \overline{X}_\mu(\overline{\sigma}) \right)^2 d\overline{\sigma} \right) \left\} \times \left( \det -\frac{\partial}{\partial \overline{\sigma}} \left( \Delta_{(\pi \alpha')^{1/2} t, A} \right) \right) \right\}
\] (4.59)

Just for completeness, we not the following exactly expressions for the fluctuating world-sheet \(Z^\mu\) Laplacean determinant and its Green function on the rectangle
\[ \det \frac{D}{\pi} \left( -\Delta \left( \frac{\pi}{2} t, A \right) \right) = \left( \prod_{n,m} \left[ \int \left( \frac{2\pi n}{\left( \frac{\pi}{2} t \right)} \right)^2 + \left( \frac{2\pi m}{A} \right)^2 \right] \right)^{-D/2} \]

\[ = \left( \frac{(\pi \alpha')^{1/2} t}{A} \right)^{D/2} \exp \left( \frac{\pi}{6} \left( \frac{2\pi n A}{(\pi \alpha')^{1/2} t} \right) D \right) \]

\[ \times \left( \prod_{n=1}^{\infty} \left[ 1 - \exp \left( \frac{2\pi n A}{(\pi \alpha')^{1/2} t} \right) \right] \right)^{-2D} \] (4.60-a)

\[ (-\Delta)^{-1} \left( \frac{\pi}{2} t, A \right) (\sigma, \sigma', s, s') = \]

\[ = -\frac{1}{2} \left\{ \sum_{n,m} \left( e^{\frac{2\pi n}{(\pi \alpha')^{1/2} t}} \right) \right\} \times \]

\[ \times \left[ \cos \left( \frac{2\pi m}{A} (s - s') \right) - \cos \left( \frac{2\pi m}{A} (s + s') \right) \right] \] (4.60-b)

As a consequence we get for \( N \)-point euclidean scalar meson Green function after disregarding the contribution of the functional determinant eq(4.60-a) and by considering \( \pi \alpha' = 1 \) from now on

\[ \tilde{G}(t) (P_1^\mu, \ldots, P_N^\mu) \]

\[ = -\frac{1}{2} \times \left\{ \int_{1/\Lambda_{QCD}}^{\Lambda_{QCD}} dt \int_{0}^{t} d\sigma_1 \cdots \int_{0}^{t} d\sigma_N \left[ \int_{\epsilon}^{\infty} dA \times \right. \right. \]

\[ \times F\left( (P_k^\mu \cdot P_k^\mu), A, T, \{\sigma_1, \ldots, \sigma_N\} \right) \left. \right\} \] (4.61)

Where the quark-antiquark harmonic oscillator form factor coming from eqs(4.59), eq(4.54) (with for notation simplicity \( \pi \alpha' = 1 \) and by considering the scalar case \( \frac{dX^\mu_1(\sigma_1)}{d\sigma_1} \)
\[ \frac{dX^\mu_N(\sigma_N)}{d\sigma_N} \to 1 \] is given explicitly by the result:

\[
F\left( (P^\mu_k, \dot{P}^\mu_k), A, t, \{\sigma_1, \ldots, \sigma_\mu\} \right) =
\left( \frac{(3 + 2A)\sqrt{2}}{6\pi \sin h(\sqrt{\frac{2}{A}} t)} \right)^{D/2} \times \exp \left\{ -\frac{3\sqrt{A}}{\sqrt{2}(3 + 2A) \sin h(\sqrt{\frac{2}{A}} t)} \right\}
\times \sum_{k=1}^{N} \sum_{k'=1}^{N} (P^\mu_k \cdot P^\mu_{k'}) \left( \sin h\left( \sqrt{\frac{2}{A}} (t - \sigma_k) \right) \times \sin h\left( \sqrt{\frac{2}{A}} \sigma_{k'} \right) \right) \] (4.62)

It is very important to remark that our “toy model” given by eq(4.62) has the correct structure to generate a Lorentz-invariant scattering amplitude, after continuation to Minkowski space, on the light of the Hall and Wightman theorem ([9]) \((2\pi \alpha' = 1)\) a very important results obtained from these partially phenomenological studies on strings for QCD \((SU(\infty))\). Namely:

\[
\mathcal{G}(\Lambda(QCD), t) (P^\mu_1, \ldots, P^\mu_N) = F(\Lambda_{QCD}) (P^\mu_i \cdot P^\mu_k) \] (4.63)

One point now worth to be called the reader attention is that case should be taken in applying straightforwardly the Feynman path integral eq(4.50) to represent the propagator of a particle possessing fermionic degrees in the presence of an external Gauge field ([3]). One can avoid this operational path integral procedure by squaring the fermionic determinant and making use now of the well-defined proper-time formalism for bosonic coloured particles ([3]). Namely (see eq(4.51))

\[
\det^{\frac{1}{2}}(D(igA + J)D(igA + J)) = \\
= \det^{\frac{1}{2}}(D^*(igA + J)D(igA + J)) \times \\
\times \det \left( 1 - \frac{ig}{4} [\gamma^\mu, \gamma^\nu](D^*D)^{-1}(igA + J) \times F_{\mu\nu}(igA + J) \right) \] (4.64)

Since the Klein-Gordon bosonic propagator can be written in term of the \(SU(N)\)
normalized holonomy factor as of as

\[
(D^*(igA_\mu + J_\mu)D(igA_\mu + J_\mu))(x_1, x_2) = \]

\[
= N \left\{ \int_0^\infty dt \int_{X(0)=x_1}^{X(t)=x_2} D[X(\sigma)] e^{-\frac{1}{2} \int_0^t \dot{X}_\sigma^2(\sigma) d\sigma} \right\}
\]

\[
\times \psi_{x_1x_2}[C, A] \times \Phi_{x_1x_2}[C, J]
\]

Here the non-abelian dynamical and abelian vectorial external sources phase factors are defined explicitly by

\[
\psi_{x_1x_2}[C, A] = \left[ \frac{1}{N} \exp \left( \int_0^t A_\mu(X(\sigma)) dX^\mu(\sigma) \right) \right]
\]

\[
\Phi_{x_1x_2}[C, J] = \left[ \exp \left( \int_0^t J_\mu(X(\sigma)) dX^\mu(\sigma) \right) \right]
\]

The final expression for the generating functional eq(4.51) at large \(N\), is thus easily written in the proper-time formalism, before taking the Yang-Mills path integral average is

\[
\ell g \left\{ \det \frac{1}{N} \left( D^*(igA + J) D(igA + J) \right) \right\}
\]

\[
= -\frac{1}{N} \left\{ \int_0^\infty dt \int d^4x_1 \int d\mu[C_{x_1x_2}] \right\}
\]

\[
(\text{Tr}_{SU(N)} [\Psi_{x_1x_2}[C, A]] \Phi_{x_1x_2}[C, J])
\]

\[
- \left\{ \sum_{n=2}^\infty \left( \frac{1}{2} \right)^{n-1} \left( \frac{1}{1/N} \right)^n \int d^4x_1 \ldots d^4x_n \int_0^\infty (dt_1 \ldots dt_n) \int d\mu[C_{x_1x_2}] \right\}
\]

\[
d\mu[C_{x_nx_1}] \times \text{Tr}_{\text{Dirac}} ([\gamma_{\mu_1}, \gamma_{\nu_1}] \ldots [\gamma_{\mu_n}, \gamma_{\nu_n}])
\]

\[
\times \text{Tr}_{SU(N)} \left\{ \frac{\delta}{\delta \sigma_{\mu_1\nu_1}(x_1)} (\psi_{x_1x_2}[C, A] \Phi_{x_1x_2}[C, A]) \right\}
\]

\[
\ldots \frac{\delta}{\delta \sigma_{\mu_n\nu_n}(x_n)} (\psi_{x_nx_1}[C, A] \Phi_{x_nx_1}[C, A]) \right\}
\]

Here the Migdal-Makeenko loop derivative is introduced ([3])

\[
\frac{\delta}{\delta \sigma_{\mu\nu}(X(\sigma))} = \lim_{\epsilon \to 0} \int_{-\epsilon}^\epsilon d\zeta \zeta \frac{\delta^2}{\delta X_\mu(\sigma + \frac{\zeta}{2}) \delta X_\nu(\sigma - \frac{\zeta}{2})}
\]

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which by its turn has the geometrical meaning of dividing the path trajectory $C_{x_1x_1}$, quite closely analogous to the joining and splitting picture of the old theory of dual strings (after taking the $SU(\infty)$ limit into our constant gauge fields model as given by eq(3.31) or eq(3.35) in section 3 of this paper).

Another point worth to call attention is the expansion parameter on eq(4.67) is the color $N$, but appearing now as a Laurent power series on $\left(\frac{1}{1/N}\right)^{+n}$ (note that $\frac{1}{N} \sim 0$, for large $N$).

**Conclusion:** We can see from this work that another time in QCD physics it is raised hopes that on underlying string dynamics is in the way to handle correctly the mathematical – calculational aspects of Euclidean – abelian Gauge theories in theirs confining phase, signaled here by the explicit hypothesis of a non-vanishing energy for the non-perturbative vacuum $(\langle 0|tr(F^2)|0 \rangle \neq 0)$.

At this point let us remain that our string representation for the QCD-Eguchi-Kawai reduced model is a free bosonic one. However if one consider next non-constant full space-time variable corrections/fluctuations to the gauge connections entering into the full Yang-Mills path integrals, one is lead to the self-avoiding fermionic full structure of the QCD($SU(\infty)$) ([3]) with the extrinsic string as an effective bosonic string representation for QCD($SU(\infty)$).

Finally, we should roughly say that our path integral is at $SU(\infty)$, but surely we are in the context of a somewhat $\frac{1}{D}$ expansion for the pure quantum Yang-Mills field, with a non perturbative vacuum. Unfortunately, the famous $\frac{1}{D}$ expansion of Lattice QCD has not been generalized or even well-understood on the continuum. We hope that our work should be a step in this direction.

**Acknowledgments:** This research was completed under the financial support of a CNPq Visiting Senior Fellowship. The author is also thankful to Professor W. Rodrigues – IMECC/UNICAMP for discussions and support.
Appendix A

Let us consider the term

\[ J_1 = \exp \left\{ \frac{1}{4} G_0^a G_1^b G_0^c G_1^d \mathcal{L}_{abcd} \times \left[ g^2 V + \frac{(g^2 S)^2}{(N/2)} \right] \right\} \]  

(A.1)

After the re-scaling

\[ G_{0,1}^f = \tilde{G}_{0,1}^f \left[ g^2 V + \frac{(g^2 S)^2}{(N/2)} \right]^{-\frac{1}{4}} \]  

(A.2)

It terms out to be

\[ I_1 = \exp \left\{ \frac{1}{4} \tilde{G}_0^a \tilde{G}_1^b \tilde{G}_0^c \tilde{G}_1^d \mathcal{L}_{abcd} \right\} \]  

(A.3)

However a “mixed” term of the form

\[ I_2 = \exp \left\{ \frac{1}{4} G_0^a G_2^b G_0^c G_2^d \mathcal{L}_{abcd}(g^2 V) \right\} \]  

(A.4)

under the re-scaling

\[ G_2^f = \tilde{G}_2^f [g^2 V]^{-\frac{1}{4}} \]  

(A.5)

becomes now

\[ I_2 = \exp \left\{ \frac{1}{4} \tilde{G}_0^a \tilde{G}_2^b \tilde{G}_0^c \tilde{G}_2^d \mathcal{L}_{abcd}(g^2 V) \times \frac{(g^2 V) \times (g^2 V)^{-1/2}}{[g^2 V + (g^2 S)^2/N]} \right\} \]  

(A.6)

Note that \( N \to \infty \), we have the leading asymptotic limit

\[ \frac{(g^2 V)^{1/2}}{[g^2 V + (g^2 S)^2/N]^{1/2}} \sim \frac{(g^2 V)^{1/2}}{(g^2 V)^{1/2}} \to 1. \]  

(A.7)

It is worth recall that

\[ \prod_{a=1}^{N^2-N} (dG_0^a dG_1^a) = \left\{ g^2 V + \frac{(g^2 S)^2}{N/2} \right\}^{-\frac{(N^2-N)}{2}} \prod_{a=1}^{N^2-N} d\tilde{G}_0^a d\tilde{G}_1^a \]  

(A.8)

\[ \prod_{a=1}^{N^2-A} dG_\mu^a = \left\{ (g^2 V) \frac{(N^2-N)(D-2)}{} \right\} \times \prod_{a=1}^{N^2-N} d\tilde{G}_\mu^a \]  

(A.9)

\[ \text{22} \]
We added also the remark about the constant gauge field non-abelian Stokes theorem

\[
W[C] = \text{Tr}_{SU(N)} \left\{ \mathbb{P} \left( e^{ig \oint A_\mu \, dX^\mu} \right) \right\} \\
= \frac{1}{N} \text{Tr}_{SU(N)} \left\{ \mathbb{P} \left( e^{ig F_{\mu\nu} \left( \int ds^\nu \right)} \right) \right\} \\
= \frac{1}{N} \text{Tr}_{SU(N)} \left\{ \mathbb{P} \left( e^{ig [A_\mu, A_\nu] - S} \right) \right\} (\delta_{\mu 0} \delta_{\nu 1}) \\
\xrightarrow{N \to \infty} \frac{1}{N} \text{Tr}_{SU(N)} \left\{ 1 - g^2 S[A_0, A_1] - \frac{(g^2 S)^2}{2} [A_0, A_1]^2 + \ldots \right\} \\
\xrightarrow{N \to \infty} \exp \left\{ -\frac{(g^2 S)^2}{2} \frac{\text{Tr}_{SU(\infty)} ([A_0, A_1]^2)}{2N} \right\} 
\tag{A.10}
\]

As a last point of our Wilson loop evaluations at large \(N\) at the context of constant gauge field configurations, we point out that at \(D = 2\) (the two dimensional case, it is not need to consider the phenomena of the dimensional transmutation coupling constant and the evaluation above displayed leads directly to the area behaviour for the Wilson Loop. (Remark originally due Bollini-Giambiagi) ([3]). However it is important to keep in mind that such result can be obtained quite straightforwardly by using the axial gauge \(A_a^0 = 0\), and mostly important, it shows the non-dynamical behaviour of the pure Yang-Mills quantum (perturbative) theory at two-dimensions. At \(D = 3\), our \(SU(\infty)\)-constant gauge field model yields charge screening instead of color charge confinement (a length behaviour for the Wilson Loop). Finally for \(D > 4\), we have a infinite volume vanishing Wilson Loop, which by its term signals that the Yang-Mills theory in \(R^D\), \(D > 4\) is a trivial QFT, in place of the usual wrong, but always argued for non-renormalizability of Yang-Mills theory for \(D > 4\).

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