ROUGH STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT. We establish a simultaneous generalization of Itô’s theory of stochastic and Lyons’ theory of rough differential equations. The interest in such a unification comes from a variety of applications, including pathwise stochastic filtering, control and the conditional analysis of stochastic systems with common noise.

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1. INTRODUCTION

Itô’s important theory of stochastic integration gives meaning and well-posedness of multidimensional stochastic differential equations (SDEs) of the form

\[ dY_t = b_t(Y_t)dt + \sigma_t(Y_t)dB_t. \]

Here \( B = B(\omega) \) is a Brownian motion, the process \( Y = Y(\omega) \) is known as Itô diffusion and constitutes an important example of a continuous semimartingale, e.g. [RY99]. Crucially in Itô’s theory, coefficient fields like \( \sigma_t(x) = \sigma(\omega, t, x) \) must be nonanticipating to enable the use of martingale methods. In contrast, the purely deterministic theory of rough paths [Lyo98] gives well-posedness to rough differential equations (RDEs) of the form

\[ dY_t = b_t(Y_t)dt + f_t(Y_t)dX_t. \]
Here \( X = (X, \mathbb{X}) \) is a Hölder rough path, the solution \( Y = Y^X \) is an example of a controlled rough path [Gub04], or simply controlled (w.r.t. \( X \)), in the sense that it looks like \( X \) on small scales: \( Y_t \approx Y_s + \dot{Y}_s^X(X_t - X_s) \), with \( \dot{Y}_s := f_s(Y_s) \). Crucially, the definition of \( f(Y) dX \), and then integral meaning to (1.2), requires \( f(Y) \) itself to be controlled. Many works on this subject, including [FH20], consider \( f_t(\cdot) \equiv f(\cdot) \) so let us point out that non-regular time dependence requires extra considerations. A direct way to do so is to assume a controlled structure \( f_t(\cdot) \approx f_s(\cdot) + f'_s(\cdot)(X_t - X_s) \). RDE solutions then come with a refined expansion

\[
Y_t = Y_s + \dot{Y}_s^X(X_t - X_s) + \ddot{Y}_s^X \mathbb{X}_{s,t} + o(t - s), \quad \dddot{Y}_s^X = ((Df_s)f_s + f'_s)(Y_s).
\]

In quantified form, cf. [Dav07, FH20], this characterizes RDE solutions.

**Brownian rough paths.** Important examples of rough paths come from the typical realization of a multidimensional Brownian motion enhanced with iterated (Itô) integrals,

\[
X = (X, \mathbb{X}) = \left(B(\omega), \left(\int B \otimes dB\right)(\omega)\right) =: \mathbb{B}^{\text{Itô}}(\omega).
\]

As is well-known, e.g. [FH20, Ch.9], under natural conditions, \( \bar{Y}(\omega) := Y^X|_{X = \mathbb{B}^{\text{Itô}}(\omega)} \) yields a beneficial version of the Itô solution to (1.1); the Stratonovich case is similar.

It has been an open problem for some time to provide a unified approach to SDEs and RDEs, such as to give intrinsic meaning and well-posedness to rough stochastic differential equations (RSDEs), aiming for an adapted solution process \( Y = Y^X(\omega) \) to (1.3)

\[
dY_t = b_t(Y_t)dt + \sigma_t(Y_t)dB_t + f_t(Y_t)dX_t.
\]

At this stage, (1.3) is entirely formal. Making this equation meaningful and providing a satisfactory general solution theory under natural conditions is the main purpose of this work.

**Why RSDEs?** The interest in such a construction comes from a variety of applications and is the raison d’être of several ad-hoc approaches, reviewed (together with their limitations) below. For instance, recent progress on fast-slow systems, cf. [PIX21, HL21], involves mixed dynamics of the form

\[
dY_t = b(Y_t)dt + \sigma(Y_t)dB_t + f(Y_t)\circ dW^H,
\]

for some independent fractional Brownian noise \( W^H \). This fits into (1.3) provided \( W^H \) has a (canonical) rough path lift, which is well understood. Such dynamics also arise in quantitative finance [GJR18, BFG16] where one naturally mixes \( (dt, dB) \)-modeled semimartingale dynamics (for tradable assets) with “rough” fractional \( dW^H \)-dynamics for volatility. Perhaps the strongest case for (1.3), which relates to works spanning over 4 decades (selected references below), comes in the form of “doubly SDEs” under conditioning; that is,

\[
dY_t = b_t(Y_t)dt + \sigma_t(Y_t)dB_t + f_t(Y_t)\circ dW_t = \bar{b}_t(Y_t)dt + \sigma_t(Y_t)dB_t + f_t(Y_t)dW_t
\]

conditionally on some independent Brownian motion \( W = W(\omega) \). Let us describe some concrete situations.

(a) In the “Markovian” case of deterministic coefficient fields, with \( \mathcal{F}_T^W = \{W_t : t \leq T\} \),

\[
u(s, y; \omega) = \mathbb{E} \left(h \left(Y^s_{T, y}\right) \exp \left(\int_s^T c(Y^s_{t, y}) dt + \int_s^T \gamma \left(Y^s_{t, y}\right) dW_t\right) \bigg| \mathcal{F}_t^W\right),
\]

yields the Feynman-Kac solution for the (terminal value) SPDE problem, \( u(T, \cdot) = h \),

\[
-\frac{1}{2} Tr \left((\sigma \sigma^T)(x) D^2 u\right) + b(x) \cdot Du + c(x) u dt + f(x) \cdot Du + \gamma(x) u dW,
\]

\[\text{Cf. [BR19] for a rough flow - and [KM17] for a Banach RDE perspective.}\]
with $\omega dW$ understood in backward Stratonovich sense. See e.g. [Par80, Kun97, DFS17] and many references therein.

(b) As noted explicitly by Bismut–Michel [BM82], the conditional process $Y|W$ is not a semimartingale (since $W$ has a.s. locally infinite variation). Their analysis then relies on auxiliary stochastic flows, obtained from $d\Phi_t(x) = f(\Phi_t(x)) \circ dW_t, \Phi_0(x) = x$, to remove the troublesome “(...)$dW$ conditioned on $W$” term in (1.5).

Conditional Itô diffusions are also at the heart of stochastic filtering theory, e.g. [EKPY12, BC09] and many references therein. Here $Y$ describes the signal dynamics, not necessarily Markovian, affected through the observation $W$ which, after a Girsanov change of measure, has Brownian statistics. The celebrated Kallianpur–Scribble formula expresses the filter (the conditional expectation of some observable $h$ of the signal, given the observation $W$) as the ratio $\pi_t(h)/\pi_t(1)$, where $\pi_t(h)$ has a similar form to (1.6), with the exponential term coming from the Girsanov theorem. Understanding the robustness of the filter with respect to $W$ is a classical question in filtering theory [Dav11, CDFO13].

(c) Stochastic flow transformations are also employed by [BM07] where the authors have controlled Itô characteristics, $g \in \{b, \sigma\} : g_t(Y_t) = g(Y_t, \eta(\omega))$, for suitably non-anticipating controls $\eta(\omega)$, and study the random value function, for $0 \leq s \leq T$,

$$ (1.8) \quad v(s, y; \omega) := \text{essinf}_{\eta(\omega)} \mathbb{E} \left( h(Y_T^{s,y;\eta}) + \int_s^T \ell(t, Y_t^{s,y;\eta}; \eta_t) dt | \mathcal{F}^W_T \right). $$

This pathwise stochastic control problem was first suggested in [LS98], albeit with constant $f(.) \equiv f$ in dynamics (1.5), as motivation for stochastic viscosity theory: According to [LS98, BM07, BCO23] the value function defined in (1.8) is a “stochastic viscosity solution” for a nonlinear stochastic PDE of the form

$$ (1.9) \quad -dv = \text{inf}_{\eta(\omega)} \left( \frac{1}{2} \text{Tr} \left( (\sigma \sigma^T)(x, \eta) D^2v \right) + b(x, \eta) \cdot Dv \right) + (f(x) \cdot Dv) \circ dW. $$

Even in the noise-free case of classical HJB (viscosity) solutions, generically $v(s, \cdot; \omega) \notin C^1$ in space, so there is no hope that (1.9) has a bona fide (Itô/Stratonovich) stochastic integral meaning. Accordingly, [LS98] propose a pathwise theory; non-constant $f$ in (1.9) requires rough paths [CFO11, AC20, CHT24].

(d) Another motivating example comes from weakly interacting particle systems, driven by independent Brownian motions $B^1, \ldots, B^N$, subjected additionally to environmental (a.k.a. common) Brownian noise $W$. Under suitable assumptions, one has conditional propagation of chaos, cf. [CF16], with the effective dynamics (1.5) of such a system a governed by conditional McKean–Vlasov dynamics, with $g \in \{b, \sigma, f\} : g_t(Y_t) = g(Y_t, \text{Law}(Y_t | \mathcal{F}^W_T))$. In a Markovian situation, the law of this process follows a non-linear, non-local stochastic Fokker–Planck equation [CG19, CN21]. The case of controlled McKean-Vlasov dynamics, $g_t(Y_t) = g(Y_t, \eta(\omega), \text{Law}(Y_t | \mathcal{F}^W_T))$ arises in the important area of mean-field games, e.g. [CD18], with $W$ viewed as common noise. The conditional analysis of such equations, with common noise and progressive coefficients, is also central to [LSZ22].

Whether $W$ is interpreted as noise, observation, environment, or common noise, the importance of quantifying its impact on some stochastic model $Y|W$, or predication based thereon, is evident. We shall see that RSDEs, as developed in this work, do this in a satisfying way. Our (in SDE terminology) “strong” analysis not only removes tedious measure theoretical issues inherit to the conditional problems, but yields a fundamental partial decomposition of Itô-map: writing $\hat{Y} \equiv Y$ for the SDE solution to (1.5), driven by $(B, W)$ and with given initial data, we can decompose

$$ (1.10) \quad \hat{Y}(\omega) = (Y^\bullet) \circ W(\omega), \quad (Y^\bullet) : X \mapsto Y^X(\omega), $$
into a (well understood) universal lifting map \( \mathcal{L} : W \mapsto W(\omega) \), and a robust RSDE solution \( Y^\bullet \). (For completeness, we show in Appendix A how this leads to the first equality in (1.10), together with a robust disintegration of \( \text{Law}(\tilde{Y}) \), given \( W \).) This picture is reminiscent of Lyons' original work, aiming to decompose SDE solutions as (deterministic) RDE solutions driven \( \omega \)-wise by a lifted Brownian motion. Yet, existing rough path tools are quite insufficient for our goals. Before commenting on the new techniques involved, we give a loose statement of our main result.

**Theorem 1.1.** Under (minimal) regularity assumptions (on possibly progressive) coefficients fields \( b, \sigma, f \), consistent with those from Itô SDE and RDE theory, there is a unique strong RSDE solution to (1.3), to which we give intrinsic local and integral sense. The solution is exponentially integrable and comes with precise local Lipschitz estimates with respect to \( \{Y_0, b, \sigma, f, X\} \).

Theorem 1.1 allows to treat a variety of situations (ranging from mixed SDEs, pathwise filtering and stochastic control to common noise McKean–Vlasov and its particle approximations) in the desired (rough)pathwise fashion, that is, with \( W^H \) or \( W \) replaced by a deterministic rough path \( X \). Using the language of diffusions in random environments, we offer a quenched theory, with \( X \) seen as frozen environmental noise. At any stage, one can return to the annealed (“doubly stochastic”) setting by randomization of \( X \), as discussed in Appendix A.

**Theorem 1.1** is the loose summary of **Theorem 4.6** (existence, uniqueness), **Corollary 4.8** (exponential integrability) and **Theorem 4.9** (stability and local Lipschitz estimates).

Central to our analysis is a new class of processes, stochastic controlled rough paths (s.c.r.p.), conceptually related to rough semimartingales [FZK23] in their ability to mix martingales and adapted controlled processes, but analytically very different, based on an extension of stochastic sewing [Lê20] to mixed \( L_{m,n}(\Omega) \)-spaces, cf. Section 2.3. The resulting s.c.r.p.’s crucially involve two \( \mathbb{P} \)-integrability parameters which allow us to detangle an (inevitably) loss of integrability (of the sort \( L_{m,n} \to L_{m,n/2} \)) upon composition of a s.c.r.p. with spatially regular (non-linear) \( f \) and more general stochastic controlled vector fields (s.c.v.f.). While \( m = n = \infty \) does not even accommodate Brownian motion, leave alone other reasonable classes of solution processes, it turns out that \( m < n = \infty \) does. After developing a rough integration theory for s.c.r.p. (Section 3.2) we can close the loop in a fixed-point argument in our construction of a unique solution.

Like s.c.r.p.’s, we should remark that s.c.v.f.’s (Section 3.3) have no counterpart in the deterministic rough paths literature. While natural, our motivation for this kind of generality is rooted in the application to interacting particle systems with rough common noise with rough (“quenched”) McKean–Vlasov limit, subject of forthcoming work [FHL24]. In this case \( f \) not only depends on \( y = Y_t^X \) but comes with a non-regular time dependence induced \( t \mapsto \text{Law}(Y_t^X) \), \(^2\) or a random approximation thereof, namely the empirical measure of the particle cloud. There is an obvious need for such estimates when applied to RSDEs, as is clear when looking at the rough counterpart (replace \( W \) by \( X \)) of (1.6) or related expressions in filtering theory.

RSDE well-posedness is complemented with precise estimate of local Lipschitz type in the data (Theorem 4.9). In the so-called critical case Lipschitz estimates are lost, but the problem remains well-posed (Section 4.3), thanks to a “stochastic, rough Davie–Grönwall”-type lemma (Section 4.3.1) which may be useful in its own right.\(^3\)

\(^2\) with \( Y \equiv Y^X \) also written as \( \text{Law}(Y_t; X) \); with \( X \) deterministic, we do not wish to write \( \text{Law}(Y_t|X) \).

\(^3\) Readers familiar with previous (arXiv) versions of this article may note a simplified direct proof of the Lipschitz estimates, without reliance on the technical Davie–Grönwall lemma.
In Section 4.4 rough Itô processes are introduced, which provide a flexible class, beyond the semi-martingale world, for which one has an Itô-type formula. A rough stochastic calculus emerges, of which we can here only scratch at the surface: we introduce the rough martingale problem and further give an effective rough Fokker–Planck equation for RSDE, in a generality that also applies immediately to solutions of McKean–Vlasov SDEs with (rough) common noise, as provided by [FHL24].

Closely related to the rough martingale problem, our final section Section 4.5 makes the point that the “strong” RSDEs theory of Theorem 1.1 also has a “weak” counterpart. Many natural questions emerge, starting with well-posedness for non-degenerate low regularity coefficients à la Stroock–Varadhan, with accompanying analytic questions for rough PDEs. We finally mention the possibility of a localized RSDE theory, a systematic study of which is left for a future note.\textsuperscript{4}

Previously on RSDEs. Assume $d\Phi_t^X(x) = f(\Phi_t^X(x))dX_t$, $\Phi_0^X(x) = x$ is well posed, $X$ is a rough geometric path. The flow transformation (FT) method for RSDEs amounts to define $Y_t^X := \Phi^X(Y)$, in terms of a distorted Itô SDE for $\tilde{Y}(\omega) = (\Phi^X)^{-1}(Y_t^X)$,

$$d\tilde{Y}_t = \tilde{b}_t(Y_t; X)dt + \tilde{\sigma}_t(Y_t; X)dB_t.$$

This construction, classical in the SDE case, goes back to [CDF013] for RSDEs, where it was seen that, for $X$ of Brownian regularity, it is necessary $f \in C^5$ to have local Lipschitz dependence of $X \mapsto Y_t^X$. (In contrast, Theorem 1.1 gives this under the expected minimal $f \in C^2$ condition.) Excessive regularity demands aside, FT methods are rather rigid and do not cope well with general $f = f(y, \omega)$, as is possible in Theorem 1.1, and needed for instance in the common noise McKean–Vlasov situation described above. Even if one consents to a structural restriction like $f = f(Y_t)$, a flow-based definition of solution lacks the local description of solution that is relevant for natural discretizations of RSDE dynamics.

As a concrete example, let $s, t$ be consecutive points in some partition $\pi$ of $[0, T]$ and consider the “Euler-in-B, Milstein-in-$X$” scheme

\begin{equation}
Y_t^X = Y_s^X + b_s(Y_s^X)(t - s) + \sigma_s(Y_s^X)(B_t - B_s) + f_s(Y_s^X)(X_t - X_s) + F_s(Y_s^X)X_{s,t},
\end{equation}

with $F = (Dyf)f + f'$ where $f'$ accounts for possible $X$-controlled time dependence of $f$. A convergence analysis of this scheme would be tedious to carry out from a FT perspective. In contrast, local RSDE estimates as provided in Proposition 4.3 make it at least plausible that this can be done efficiently in the framework of this work, [FLZ24]. When applied in the (rough) pathwise control setting, that is, the rough counterpart of (1.8) with

$$v^X(s, y) := \inf_{\eta} \mathbb{E}\left(h(Y_T^{X,s,y,\eta}) + \int_s^T \ell(t, Y_t^{X,s,y,\eta}; \eta_t) dt\right),$$

this opens up to the possibility to study the finite difference of nonlinear stochastic PDEs of the form (1.9), not implied (unless $f$ is constant) by presently available theory [See20]. We also note in passing that the above expression for $v^X$ in conjunction with precise RSDE estimates (Theorem 4.9) gives a direct approach to estimating Hölder space time regularity of such SPDEs, valid for every (rough path) realization of the driving noise. This is a powerful way to obtain regularity results for stochastic HJB equations (problem left open in [BM07]) and can also be compared with recent work [CS21].

\textsuperscript{4}Partial results are contained in previous (arXiv) versions of this article.
A second previous approach, dubbed random rough path (RRP) method, amounts to define $Y_{\text{RPP}}(\omega) := \hat{Y}Z(\omega)$ as the $\omega$-wise solution to the RDE
\begin{equation}
\begin{aligned}
d\hat{Y}_t &= b_t(\hat{Y}_t)dt + (\sigma_t, f_t)(\hat{Y}_t)dZ_t(\omega), \\
\end{aligned}
\end{equation}
driven by the random rough path $Z(\omega)$ over $Z(\omega) := (B(\omega), X)$, where the second level $Z(\omega)$ is naturally specified via 4 blocks, given by
\begin{align*}
\int B \otimes dB, \quad \int X \otimes dB, \quad \int B \otimes dX := B \otimes X - \int (dB) \otimes X.
\end{align*}
Here, all $dB$-integrals are in Itô sense, $X$ is the second level component of $X$. This construction is due to [DOR15], see [FZK23] for the case of càdlàg martingale and $p$-rough paths. It was also used in [DFS17] for intrinsic well-posedness of the RPDEs counterpart of (1.7), with $W$ replaced by $X$, and most recently for McKean–Vlasov equations with (rough) common noise [CN21], to be distinguished from rough McKean–Vlasov (or mean field) equations [BCD20, BCD21] which also have a RRP flavor. As a general remark, after the construction of a suitable joint lift $Z = Z(\omega)$, RPP methods rely on deterministic analysis and cannot benefit from the (partial) martingale structure inherent in RSDEs. This becomes a serious issue for integrability, e.g. for exponential terms as seen in the rough counterpart (replace $W$ by $X$) of (1.6), and an insurmountability when it comes to general progressive randomness in coefficients, a situation that cannot be dealt with by RPP methods. Indeed, the Itô coefficient field $\sigma = \sigma_t(\cdot, \omega)$ is now subject to the stringent space-time regularity and rigid controlledness conditions of vector fields in RDE theory. This entails that the (minimal) Lipschitz-condition one expects from Itô SDE theory has to be replaced by a suboptimal $C^{1/\alpha}$-condition, and further rules out general (progressive) time-dependence, as would be required to incorporate stochastic control aspects in (1.12). (All these limitations are removed by our Theorem 1.1.)

**Summary and outlook:** Based on a complete intertwining of stochastic and rough analysis, the RSDE framework put forward in this work offers a powerful approach to many problems previously treated with flow transformation and/or random rough path methods. Immediate benefits include the removal of excessive regularity demands seen in (all) such previous works, intrinsic (local) meaning to the equations of interest, and a significant relaxation of previously imposed structural assumptions (e.g. progressive vs. deterministic coefficients fields, as required in stochastic control). Concerning the outlook, our results and techniques are of direct interest for stochastic analysis (“partial” Malliavin calculus, Hörmander theory, random heat kernels ... ) of conditional processes, as studied by Bismut, Kunita, Nualart, and many others in the 80/90ties. We also have first evidence that our framework enables a “robust” conditional analysis of doubly stochastic backward SDEs [PP94, DF12]. Further uses can be expected in the vast field of mean field games (with common noise). Last not least, we envision extensions from rough SDEs to rough SPDEs, as may arise from the filtering of non-linear SPDEs. The present work is of foundational nature.

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5. In these works, $Z(\omega)$ is a joint lift of $X(\omega')$ and $X(\omega'')$ for a suitable random rough path $X$. No martingale structure is assumed.

6. Update August 2024: a first study of Malliavin calculus for RSDEs is due to [BCN24].
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**Frequently used notation.** For two extended real numbers \( a, b \in \mathbb{R} \cup \{ \infty \} \) we write \( a \wedge b = \min\{a, b\} \) and \( a \vee b = \max\{a, b\} \). If more parameters are presented, \( \min\{\ldots\} \) and \( \max\{\ldots\} \) will be used instead. The Borel-algebra of a topological space \( \mathcal{T} \) is denoted by \( \text{Bor}(\mathcal{T}) \). Throughout the manuscript we fix a finite deterministic time horizon \( T > 0 \). Accordingly the notation \( I \subset [0, T] \) refers to a (generic) compact interval and we denote by \( |I| \) its width. The notation \( F \lesssim G \) means that \( F \leq CG \) for some positive constant \( C \). For two Banach spaces \((X, \| \cdot \|_X), (Y, \| \cdot \|_Y)\) with \( X \subset Y \), we write \( X \hookrightarrow Y \) if \( X \) is continuously embedded into \( Y \), in the sense that \( \| \cdot \|_Y \leq \| \cdot \|_X \). By \( V, \bar{V}, W, \bar{W} \) we denote real finite-dimensional Banach spaces. Their norms are denoted indistinctly by \( | \cdot | \). The Banach space of linear maps from \( V \) to \( W \), endowed with the induced norm \( |K| := \sup_{v \in V, |v| \leq 1} |Kv| \), is denoted by \( \mathcal{L}(V, W) \). Tensor products are equipped with a norm such that \( V \otimes W \cong \mathcal{L}(V, W) \) isometrically and, accordingly, we shall blur the difference between \( \mathcal{L}(V, \mathcal{L}(V, W)), \mathcal{L}(V \otimes V; W) \) and bilinear maps from \( V \times \bar{V} \rightarrow W \). (We nonetheless point out that all instances of \( \otimes \) in this manuscript pertain to finite-dimensional spaces.) For each \( K \in V \otimes W \), we let \( K^\dagger \in W \otimes V \) be the corresponding transpose. If \( K \in V \otimes V \), we further let \( \text{Sym}(K) := \frac{1}{2}(K + K^\dagger) \) and \( \text{Anti}(K) = \frac{1}{2}(K - K^\dagger) \).

2. Preparations

2.1. Framework. Herein we introduce further concepts and notations which will be extensively used in the paper. We start with functions.

2.1.1. Functional spaces. We denote by \( (\mathcal{C}_b, | \cdot |_\infty) \) the Banach space of continuous and bounded maps, namely

\[
\mathcal{C}_b = \mathcal{C}_b(\mathcal{T}; W) = \{ f : \mathcal{T} \rightarrow W \text{ continuous and s.t. } |f|_\infty < \infty \},
\]

where

\[
|f|_\infty := \sup_{x \in \mathcal{T}} |f(x)|.
\]

For every \( \alpha \in (0, 1) \) and every function \( g : V \rightarrow W \), we denote by \( [g]_\alpha \) its Hölder seminorm, i.e.

\[
[g]_\alpha = \sup_{x, y \in V : x \neq y} \frac{|g(x) - g(y)|_W}{|x - y|_V^\alpha}.
\]

For \( \kappa = N + \alpha \) where \( N \) is a non-negative integer and \( 0 < \alpha \leq 1 \), \( \mathcal{C}_b^\kappa(V; W) \) denotes the set of bounded functions \( f : V \rightarrow W \) such that \( f \) has Fréchet derivatives up to order \( N \), \( D^j f, j = 1, \ldots, N \) are bounded functions and \( D^N f \) is globally Hölder continuous with exponent \( \alpha \). Recall that for each \( v \in V \), \( D f(v) \in \mathcal{L}(V, W) \), \( D^2 f(v) \in \mathcal{L}(V \otimes V, W) \) and so on. Note that \( \mathcal{C}_b^N \) differs from the usual one in that we do not require the derivatives of order \( N \) to be continuous. Whenever clear from the context, we simply write \( \mathcal{C}_b^\kappa \) for \( \mathcal{C}_b^\kappa(V; W) \). For each \( f \) in \( \mathcal{C}_b^\kappa \), we denote

\[
[f]_\kappa = \sum_{k=1}^N |D^k f|_\infty + [D^N f]_\alpha \quad \text{and} \quad |f|_\kappa = |f|_\infty + [f]_\kappa.
\]
2.1.2. Rough paths. Given a compact interval \( I \subset [0, T] \) we shall work with the simplices \( \Delta(I) \) and \( \Delta(\bar{I}) \), defined as
\[
\Delta(I) := \{(s, t) \in \bar{I}^2, \min I \leq s \leq t \leq \max I\},
\]
\[
\Delta(\bar{I}) := \{(s, u, t) \in \bar{I}^3, \min I \leq s \leq u \leq t \leq \max I\}.
\]
We write \( \Delta = \Delta(I) \) and \( \Delta = \Delta(\bar{I}) \) whenever clear from the context. As is common in the rough path literature, given a path \( Y = (Y_t) : I \to W \), we denote by \( (\delta Y_{s,t})(s,t) \in \Delta \) the increment of \( Y \), which is the two-parameter map
\[
(2.1)
\delta Y_{s,t} := Y_t - Y_s, \quad \text{for every } (s, t) \in \Delta.
\]
If \( A : \Delta \to W \) has two parameters, the increment of \( A \) is a three-parameter map
\[
\delta A : \Delta \to W \text{ defined as }
\]
\[
(2.2)
\delta A_{s,u,t} := A_{s,t} - A_{s,u} - A_{u,t}, \quad \text{for every } (s, u, t) \in \Delta.
\]
In that case we also define the quantity
\[
(2.3)
|A|_\alpha := \sup_{(s,t) \in \Delta, s \neq t} \frac{|A_{s,t}|}{(t - s)^\alpha}
\]
For \( Y : I \to W \), the Hölder norm of \( Y \) is defined as the quantity
\[
|Y|_\alpha := \sup_{t \in I} |Y_t| + |\delta Y|_\alpha,
\]
the right-most term being understood as (2.3) with \( A_{s,t} = \delta Y_{s,t} \).

We now recall the definition of a two-step, \( \alpha \)-Hölder rough path (as encountered for instance in [FH12]).

**Definition 2.1.** Let \( \alpha \) be a fixed number in \( \left(\frac{1}{3}, \frac{1}{2}\right] \). We call \( X = (X, \bar{X}) \) a two-step, \( \alpha \)-Hölder rough path on \( I \subset [0, T] \) with values in \( V \), in symbols \( X \in \mathcal{C}^\alpha(I; V) \), if and only if
\[(a) \ (X, \bar{X}) \text{ belongs to } C^\alpha(I; V) \times C^{2\alpha}_2(I; V \otimes V),
(b) \text{ for every } (s, u, t) \in \Delta, \bar{X} \text{ satisfies the Chen's relation }
\]
\[
(2.4)
\bar{X}_{s,t} - \bar{X}_{s,u} - \bar{X}_{u,t} = \delta X_{s,u} \otimes \delta X_{u,t}.
\]
A natural distance in this context is defined for any \( X, \bar{X} \in \mathcal{C}^\alpha \) as
\[
(2.5)
\rho_{\alpha,\alpha'}(X, \bar{X}) := |\delta X - \delta \bar{X}|_\alpha + |X - \bar{X}|_{\alpha + \alpha'}
\]
and we write \( \rho_\alpha = \rho_{\alpha,\alpha} \), which is the classical \( \alpha \)-Hölder rough path metric. Another useful quantity associated to \( X \in \mathcal{C}^\alpha \) is that of
\[
\|X\|_{\alpha,\alpha'} = \sqrt{|\delta X|_\alpha |\delta \bar{X}|_{\alpha'} + |X|_{\alpha + \alpha'}}
\]
for \( \alpha' \in (0, \alpha) \). Accordingly, we write \( \|X\|_\alpha = \|X\|_{\alpha,\alpha} \). Every smooth path \( X : [0, T] \to V \) gives rise to a canonical rough path lift, with \( \bar{X}_{s,t} = \int_s^t \delta X_{s,r} \otimes dX_r \); we write \( \mathcal{C}^{0,\alpha}_g \) for the closure of such canonically lifted smooth paths in \( \mathcal{C}^\alpha \).

2.1.3. Stochastic setting. From now on, we work on a fixed complete probability space \( (\Omega, \mathcal{G}, \mathbb{P}) \) equipped with a filtration \( \{\mathcal{F}_t\} \) with index set \( [0, T] \), such that \( \mathcal{F}_0 \) contains the \( \mathbb{P} \)-null sets. We also denote by \( \Omega = (\Omega, \mathcal{G}, \mathbb{P}; \{\mathcal{F}_t\}) \) and call it a stochastic basis. Expectation with respect to \( \mathbb{P} \) is denoted by \( \mathbb{E} \).

In the sequel, we fix a generic, non-necessarily separable Banach space \( (\mathcal{X}, | \cdot |_{\mathcal{X}}) \).
Random variables and stochastic processes. We call \( \xi: \Omega \to \mathcal{X} \) a random variable if it is strongly \( \mathcal{G}/\text{Bor}(\mathcal{X}) \)-measurable. This means that \( \xi \) is measurable relative to \( \mathcal{G}/\text{Bor}(\mathcal{X}) \) and separably valued. Accordingly, we call \( Y = Y(\omega, t) \in \mathcal{X} \) a stochastic process if \( Y_t = Y(\cdot, t) \) forms a family of \( \mathcal{X} \)-valued random variables. We call it adapted if for every \( t \geq 0 \), \( Y_t \) is \( \mathcal{F}_t \)-measurable, progressive measurable if \( Y \), restricted to \( \Omega \times [0, t] \), is strongly \( \mathcal{F}_t \otimes \text{Bor}([0, t]) \)-measurable.

Lebesgue spaces. The usual Lebesgue space \( L^m(\Omega, \mathcal{G}, \mathbb{P}; \mathcal{X}) \) of \( \mathcal{X} \)-valued \( L_m \)-integrable random variables (in the previous sense) is denoted by \( L_m(\mathcal{X}) \) or \( L_m(\mathcal{G}; \mathcal{X}) \). Its norm is given by

\[
\xi \mapsto \|\xi\|_m = \|\xi|\mathcal{X}\|_m.
\]

Accordingly, the vector space of \( \mathcal{G} \)-random variables with values in \( \mathcal{X} \) is denoted by \( L_0(\mathcal{G}; \mathcal{X}) \) or \( L_0(\mathcal{X}) \).

Hölder (semi)norms. The notations (2.1) and (2.2) extend to stochastic processes \( Y: \Omega \times I \to \mathcal{X} \) and \( A: \Omega \times \Delta(I) \to \mathcal{X} \). The processes \( Y, A \) are called integrable \( (L_m \text{-integrable}) \) if \( Y_t, A_{s,t} \) are integrable \( (L_m \text{-integrable}) \) for every \( (s, t) \in \Delta \). For such two-parameter process \( (A_{s,t})_{(s,t) \in \Delta} \), we introduce the \( \alpha \)-Hölder norm\(^7\)

\[
\|A\|_{\alpha,m} := \sup_{(s,t) \in \Delta, \ s \neq t} \frac{\|A_{s,t}\|_m}{(t - s)^{\alpha}}.
\]

For \( Y: \Omega \times I \to \mathcal{X} \) we write correspondingly

\[
\|Y\|_{\alpha,m} := \sup_{t \in I} \|Y_t\|_m + \|\delta Y\|_{\alpha,m}.
\]

2.2. Spaces of mixed integrability. The main purpose of this subsection is to introduce a family of integrable two-parameter processes with suitable regularity and integrability properties with respect to a fixed filtration. The linear spaces formed by these stochastic processes are the foundation of our analysis in later sections.

To this aim, an important concept that needs to be discussed is that of “mixed integrability”. Even though it will be mostly used in the context of stochastic processes and a filtration, it is first better understood at the level of random variables, for which a sole sub-sigma algebra \( \mathcal{F} \subset \mathcal{G} \) suffices. It is very likely that these spaces have been introduced before, however finding appropriate references turns out to be difficult. For instance, the \( L_{1,\infty} \)-norm appears implicitly in [SV79, App. A], but not exactly (compare the left hand side of [SV79, eq. Appendix A(1.1)] with the quantity introduced in Definition 2.4(c) below). There, the authors rely on comparable tools to prove \( L_p \)-estimates for certain singular integral operators.

2.2.1. Random variables of mixed integrability. Let \( m \geq 1 \) be a real number and fix a sub-sigma-field \( \mathcal{F} \subset \mathcal{G} \). For an \( m \)-integrable random variable \( \xi(\omega) \in \mathcal{X} \), we define the conditional \( L_m \)-norm as the random variable

\[
(2.6) \quad \|\xi|\mathcal{F}\|_m = \[\mathbb{E}(\|\xi|\mathcal{F}\|^m )\]^{1/m}.
\]

For an extended real number \( n \in [1, \infty] \), we then define a space \( L_{m,n} \) consisting of all \( \xi \in L_{m,n} \), such that \( \|\xi\|_{m,n} := \|\xi|\mathcal{F}\|_m < \infty \).

As shown below in Proposition 2.3, the space \( L_{m,n} \) lies somewhere in between \( L_{m\wedge n} \) and \( L_{m\vee n} \) (hence the term “mixed integrability”). Loosely speaking, the size of the sub-sigma-algebra \( \mathcal{F} \) modulates the level of integrability.

---

\(^7\)Meaning (cf. eg. [LT91, Chap. 2]) that a closed separable subspace \( \mathcal{Y} \subset \mathcal{X} \) exists s.t. \( \mathbb{P} \circ \xi^{-1} \) is supported in \( \mathcal{Y} \). If \( \mathcal{X} \) is separable, there is no difference to \( \mathcal{G}/\text{Bor}(\mathcal{X}) \)-measurability.

\(^8\)Write \( \| \cdot \|_{\alpha,m, I} \), to emphasize dependence on \( I \subset [0, T] \).
Remark 2.2. There are some cases where “nothing happens” in (2.6), in the sense that \( \| \xi \|_{\mathcal{F}} m = \| \xi \|_{\mathcal{F}} m \) \( \mathbb{P} \)–a.s. (for instance if \( \xi \) is \( \mathcal{F} \)-measurable). In these cases the resulting mixed \( L_{m,n} \)-norm reduces to the classical \( L_n \)-norm when \( m = n < \infty \). This simple fact will be used tacitly in the rest of the paper.

With this definition at hand, we record the following basic properties.

**Proposition 2.3.** Let \( 1 \leq m \leq n \leq \infty \) such that \( m < \infty \).

- \( (L_{m,n}, \| \cdot \|_{m,n}) \) forms a Banach space, which coincides with \( L_m \) whenever \( m = n < \infty \). Namely, \( L_{m,m} = L_m \) as vector spaces and \( \| \cdot \|_{m,m} = \| \cdot \|_m \). We also have the particular cases:
  - \( L_{m,n} = L_m \) if \( \mathcal{F} = \{ \emptyset, \Omega \} \),
  - \( L_{m,n} = L_n \) if \( \mathcal{F} = \mathcal{G} \).

- The following continuous embeddings hold for \( n \geq m \):

\[
L_n \hookrightarrow L_{m,n} \hookrightarrow L_m .
\]

- The map

\[
(L_{m}, \| \cdot \|_m) \to [0, \infty], \quad \xi \mapsto \begin{cases} \| \xi \|_{m,n} & \text{if } \xi \in L_{m,n} \\ \infty & \text{otherwise} \end{cases}
\]

is lower semi-continuous.

**Proof.** The fact that \( \| \cdot \|_{m,m} = \| \cdot \|_m \) follows by \( \| \| \xi \|_\mathcal{F} \|_m \|_m \) = \( \mathbb{E}(\mathbb{E}(\| \xi \|_{\mathcal{F}} m)^{\frac{1}{m}})^{\frac{1}{m}} = \| \xi \|_m \).

The cases \( \mathcal{F} = \{ \emptyset, \Omega \} \) or \( \mathcal{F} = \mathcal{G} \) are also clear, hence omitted.

Next, suppose that \( n \geq m \). We establish the two-sided estimate

\[
\| \xi \|_m \leq \| \xi \|_{m,n} \leq \| \xi \|_n , \quad \text{for all } \xi \in L_n
\]

which, in particular, implies (2.7).

**Proof of (2.8).** The left-most inequality is a simple consequence of Jensen Inequality, which asserts that

\[
\| \xi \|_m = \mathbb{E}(\| \xi \|_{\mathcal{F}} m) \leq \mathbb{E}(\| \xi \|_{\mathcal{F}} m^\frac{1}{m}) \leq \mathbb{E}(\| \xi \|_{\mathcal{F}})^\frac{1}{m} = \| \xi \|_{m,n}
\]

(since \( n \geq m \)).

As for the right-most inequality, we also use Jensen inequality but in conditional form. We obtain:

\[
\| \xi \|_{m,n} = \mathbb{E}(\| \xi \|_{\mathcal{F}} m) \leq \mathbb{E}(\| \xi \|_{\mathcal{F}})^n = \| \xi \|_n
\]

and this yields (2.8).

**Banach property.** The space \( L_{m,n} \) is clearly linear. To show completeness, suppose that \( \{ \xi^k \}_k \) is a Cauchy sequence in \( L_{m,n} \). Since \( L_m \) is complete, the left part of (2.8) shows that we can find \( \xi \in L_m \) such that \( \lim_k \xi^k = \xi \in L_m \). For each \( \varepsilon > 0 \), let \( M_\varepsilon > 0 \) be such that

\[
\mathbb{E}(\mathbb{E}(\| \xi^k - \xi^l \|_{\mathcal{F}} m)^{\frac{1}{m}}) < \varepsilon \quad \forall k, l \geq M_\varepsilon .
\]

Next, we choose a subsequence \( \{ l_i \} \) such that \( l_i \geq M_\varepsilon \) and \( \lim i \xi^{l_i} = \xi \) \( \mathbb{P} \)–a.s. Applying Fatou’s lemma twice, the previous inequality implies that for each \( k \geq M_\varepsilon \),

\[
\mathbb{E}(\mathbb{E}(\| \xi^k - \xi^{l_i} \|_{\mathcal{F}} m)^{\frac{1}{m}}) \leq \lim \inf i \mathbb{E}(\mathbb{E}(\| \xi^k - \xi^{l_i} \|_{\mathcal{F}} m)^{\frac{1}{m}}) < \varepsilon .
\]

Since \( \varepsilon \) is arbitrary small, we conclude that \( \lim_k \| \xi^k - \xi \|_{\mathcal{F}} m = \| \xi \|_{m,n} = 0 \). This also shows that \( \xi \) belongs to \( L_{m,n} \), hence completeness and (2.7).

**Lower-semicontinuity.** At last, we prove lower-semicontinuity of the \( L_{m,n} \)-norm in \( L_m \), assuming without loss of generality that \( \xi(\omega) \in [0, \infty) \) (otherwise replace \( \xi \) by \( |\xi|_\mathcal{X} \)). To this aim, let \( \xi^k, k = 0, 1, \ldots \) be any sequence of real-valued random variables such
that $\xi^k \to \xi$ in $L_m$ for some $\xi \in L_m$. It is a well-known fact of measure theory that 
$\liminf_k \xi^k = \xi$ $\mathbb{P}$-almost surely. Thus, applying Fatou Lemma twice yields:

$$
\|\xi\|_{m,n} \leq \liminf_k \|\xi^k\|_{m,n} \\
\leq \liminf_k \|\xi^k\|_{m,n}.
$$

Hence our conclusion. \hfill $\Box$

### 2.2.2. Two parameter stochastic processes with mixed integrability.

Recall that $\{\mathcal{F}_t\}$ is a filtration on a fixed complete probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and that we denote by $\Omega = (\Omega, \mathcal{G}, \mathbb{P}; \{\mathcal{F}_t\})$. For computational ease, we introduce the following shorthand notation for the rest of the paper:

$$
E_s = E(\cdot | \mathcal{F}_s) \quad \text{for all } s \in [0, T].
$$

In keeping with the previous considerations on random variables, we introduce a space of two parameter stochastic processes as follows.

**Definition 2.4.** Fix $I \subset [0, T]$. For each $m, n \in [1, \infty]$, $m \leq n$, let

$$
C_2L_{m,n}(I, \omega; \mathcal{X})
$$

be the space of $\mathcal{X}$-valued, 2-parameter stochastic processes $(s, t) \mapsto A_{s,t}$ such that

- (a) $A : \Omega \times \Delta(I) \to \mathcal{X}$ is strongly $\mathcal{G} \otimes \text{Bor}(\Delta(I)) / \text{Bor}(\mathcal{X})$-measurable,
- (b) $A : \Delta(I) \to L_m(\Omega; \mathcal{X})$ is continuous,
- (c) $\|A\|_{\infty,m,n} := \sup_{(s,t) \in \Delta(I)} \|A_{s,t} \mathcal{F}_s\|_{m,n} < \infty$.

For notational ease, we will sometimes abbreviate this space as $C_2L_{m,n}$. Clearly, changing the filtration $\{\mathcal{F}_t\}$ changes the corresponding space. When $m = n$ however, the choice of filtration makes no difference (see Remark 2.2). In that case we will contract the two integrability indices and further abbreviate by $C_2L_n := C_2L_{n,n}$.

Similarly, we introduce further subclasses of such two-parameter stochastic processes as follows.

**Definition 2.5.** Fix $I \subset [0, T]$, let $\kappa \in (0, 1]$ and $1 \leq m \leq n \leq \infty$, $m < \infty$.

- The space $C_2^\kappa L_{m,n}(I, \omega; \mathcal{X})$ consists of two-parameter processes $(A_{s,t})_{(s,t) \in \Delta}$ in $C_2L_{m,n}$ such that

  $$
  \|A\|_{\kappa;m,n} := \sup_{s \leq t} \frac{\|A_{s,t} \mathcal{F}_s\|_{m,n}}{|t - s|^\kappa} < \infty.
  $$

- Similarly, the space $C_2^\kappa L_{m,n}(I, \omega; \mathcal{X})$ contains all stochastic processes $Y : \Omega \times I \to \mathcal{X}'$ such that $t \mapsto Y_t$ belongs to $C(I; L_m(\mathcal{X}))$ and $(s, t) \mapsto \delta Y_{s,t}$ belongs to $C_2^\kappa L_{m,n}$.

  It is equipped with the norm

  $$
  \|Y\|_{\kappa;m,n} := \sup_{t \in [0,T]} \|Y_t\|_m + \|\delta Y\|_{\kappa;m,n},
  $$

  which makes it Banach (proof omitted).

Similar to Proposition 2.3 and the spaces $L_{m,n}$, we now record a few additional properties.

**Proposition 2.6.** Let $1 \leq m \leq n \leq \infty$ and fix $\kappa \geq 0$. Then:

- $C_2^\kappa L_{m,n}$ is a Banach space.
- $C_2^{m,n} L_{m',n'} \hookrightarrow C_2^\kappa L_{m,n}$ for every $m, n, m', n' \in [1, \infty]$ such that $m' \geq m$, $n' \geq n$, $m \leq n$ and $m' \leq n'$.
- For each $A \in C_2^\kappa L_{m,n}$ such that $A_{s,t}$ is $\mathcal{F}_s$-measurable $(s, t) \in \Delta$, then $A \in C_2L_{m,n}$ and $\|A\|_{\kappa;m,n} = \|A\|_{\kappa;m,n}$. 

• the map $A \mapsto \|A\|_{\kappa;m,\infty}$ is lower-semicontinuous from $C^2_\kappa L_m \to [0, \infty]$.

The properties stated above remain true if each occurrence of $C^2_\kappa L_m$ is replaced by $C^2 L_{m,n}$ and $\|\cdot\|_{\kappa;m,n}$ by $\|\cdot\|_{\kappa;m,n}$.

**Proof.** The proofs for the first three properties are omitted as they essentially follow Proposition 2.3 and Remark 2.2.

Showing the claimed lower-semicontinuity is easy. Observe indeed from Proposition 2.3 that the map $A \mapsto \|A_{s,t}|_{F_s}\|_{m,\infty}$ from $L_m \to [0, \infty]$, is itself lower-semicontinuous for each $(s, t) \in \Delta$. From here, we divide both sides of the resulting inequality by $|t - s|^{\kappa}$ and use that $\sup_{\Delta} \liminf_{k} \leq \liminf_{k} \sup_{\Delta}$, which yields the desired property. □

Lastly, we record an interesting exponential inequality, known as John–Nirenberg inequality.

**Proposition 2.7** (John–Nirenberg inequality). Let $Y: \Omega \times I \to X$ be adapted process such that $\delta Y$ belongs to $C^2_\kappa L_{1,\infty}(I, \Omega; X)$ for some $\kappa \in (0, 1]$. Assume that $Y$ is a.s. continuous. Then there are finite constants $C, c > 0$ which are independent from $\Gamma, \kappa, |I|, X$ such that

$$\mathbb{E}e^{\lambda \sup_{t \in [0, T]} \|\delta Y_{t}\|_X} \leq Ce^{C(\lambda \delta Y_{1,\infty})^{1/\kappa}T} \quad \text{for every} \quad \lambda > 0. $$

While the classical John–Nirenberg inequality (see e.g. [SV06, Exercise A.3.2]) implies that $\mathbb{E}e^{\lambda \sup_{t \in [0, T]} \|\delta Y_{t}\|_X}$ is finite for some $\lambda > 0$, the explicit right-hand side of (2.10) follows from a more recent argument from [Lê22b, Lê22a]. For the reader’s convenience, we include a self-contained proof in Appendix B.

### 2.3. Stochastic sewing revisited

The stochastic sewing lemma was introduced previously in [Lê20, Theorem 2.1]. In Theorem 2.8 we provide an extension compatible with the mixed $L_{m,n}$-norm required in our analysis and also show that stochastic sewing limits are uniform (on compacts) in time. Below we let $I = [0, T]$ to fix ideas (extension to the general case is trivial).

**Theorem 2.8** (Stochastic Sewing Lemma). Let $2 \leq m \leq n \leq \infty$ be fixed, $m < \infty$. Let $A = (A_{s,t})_{(s,t) \in \Delta}$ be a stochastic process in $W$ such that $A_{s,s} = 0$ and $A_{s,t}$ is $F_t$-measurable for every $(s, t) \in \Delta$.

(i) Suppose that there are finite constants $\Gamma_1, \Gamma_2 \geq 0$ and $\varepsilon_1, \varepsilon_2 > 0$ such that for any $(s, u, t) \in \Delta$,

$$\|\mathbb{E}_s[\delta A_{s,u,t}]\|_n \leq \Gamma_1(t - s)^{1+\varepsilon_1} $$

and

$$\|\|\delta A_{s,u,t}|_{F_s}\|_m\|_n \leq \Gamma_2(t - s)^{1+\varepsilon_2}. $$

Then, there exists a unique stochastic process $A$ with values in $W$ satisfying the following properties

- $A_0 = 0$, $A$ is $\{F_t\}$-adapted, $A_t - A_{0,t}$ is $L_m$-integrable for each $t \in [0, T]$;
- there are positive constants $C_1 = C_1(\varepsilon_1), C_2 = C_2(\varepsilon_2)$ such that for every $(s, t) \in \Delta$,

$$\|A_t - A_s - A_{s,t}|_{F_s}\|_m\|_n \leq C_1\Gamma_1(t - s)^{1+\varepsilon_1} + C_2\Gamma_2(t - s)^{1+\varepsilon_2} $$

and

$$\|\mathbb{E}_s(A_t - A_s - A_{s,t})\|_n \leq C_1\Gamma_1(t - s)^{1+\varepsilon_1}. $$
Suppose furthermore that for each \( s \in [0, T] \), the map \( t \mapsto A_{s,t} \) is a.s. càdlàg (resp. continuous) on \([s, T] \), and there are finite constants \( \varepsilon_3 > 0 \) and \( \Gamma_3 \geq 0 \) such that for any \((s, t) \in \Delta \)

\[
\left\| \left\{ \sup_{u \in [s+(t+2)/2, t]} |\delta A_{s,(s+t)/2, u}| \right\} \mathcal{F}_s \right\|_m \leq \Gamma_3 (t-s)^{\frac{1}{m} + \varepsilon_3} \quad \forall (s, t) \in \Delta
\]

Let \( \mathcal{P} = \{ 0 = t_0 < t_1 < \cdots < t_N = T \} \) be a partition of \([0, T] \) and define for each \( t \in [0, T] \),

\[
A^\mathcal{P}_t := \sum_{i:t_i \leq t} A_{t_i, t_{i+1} \wedge t}.
\]

Then \( \mathcal{A} \) has a càdlàg (resp. continuous) version, denoted by the same notation, and for this version, we have

\[
\left\| \left\{ \sup_{t \in [0,T]} |A^\mathcal{P}_t - A_t| \right\} \mathcal{F}_0 \right\|_m \leq C |\mathcal{P}|^{\varepsilon_1 + \varepsilon_2 + \varepsilon_3} (\Gamma_1 + \Gamma_2 + \Gamma_3)
\]

where \( |\mathcal{P}| := \sup_i |t_i - t_{i+1}| \) and \( C \) is some constant depending on \( T, m, \varepsilon_1, \varepsilon_2, \varepsilon_3 \).

Even in case \( m = n \), where the convergence of \( A^\mathcal{P}_t \) to \( A_t \) for fix \( t \) is well-understood [Lé20], the (locally) uniformly in \( t \) convergence in Theorem 2.8(ii) is new. We also note that in Theorem 2.8, integrability is imposed on \( \delta A \), but not on \( A \). Relative to equation (4.1), this fact effectively removes all integrability conditions on the initial datum \( \xi \), cf. Theorem 4.6.

**Corollary 2.9.** Let \( n, m, \Gamma_1, \Gamma_2, \varepsilon_1, \varepsilon_2, A \) and \( \mathcal{A} \) be as in Theorem 2.8(ii). For each integer \( k \), let \( A^k = (A^k_{s,t})_{(s,t) \in \Delta} \) be a stochastic process that satisfies the hypotheses of Theorem 2.8(ii) and let \( A^k \) be the corresponding process. Suppose that for each \( s \in [0, T] \),

\[
\lim_k \left\| \sup_{t \in [s, T]} |A^k_{s,t} - A_{s,t}| \right\|_m = 0.
\]

Then \( \lim_k \left\| \sup_{t \in [0, T]} |A^k_t - A_t| \right\|_m = 0 \).

**Proof.** Let \( \mathcal{P} \) be a partition of \([0, 1] \). From Theorem 2.8(ii), we have

\[
\sup_k \left\| \sup_{t \in [0, T]} |A^k_t - A_t - \sum_{[u,v] \in \mathcal{P}, u \leq t} (A^k_{u,v \wedge t} - A_{u,v \wedge t})| \right\|_m \lesssim |\mathcal{P}|^{\varepsilon}
\]

for some \( \varepsilon > 0 \). By assumption, we have

\[
\lim_k \left\| \sup_{t \in [0, T]} \sum_{[u,v] \in \mathcal{P}, u \leq t} (A^k_{u,v \wedge t} - A_{u,v \wedge t}) \right\|_m = 0.
\]

By triangle inequality, we have

\[
\left\| \sup_{t \in [0, T]} |A^k_t - A_t| \right\|_m \lesssim |\mathcal{P}|^{\varepsilon} + \left\| \sup_{t \in [0, T]} \sum_{[u,v] \in \mathcal{P}, u \leq t} (A^k_{u,v \wedge t} - A_{u,v \wedge t}) \right\|_m
\]

From here, we send first \( k \to \infty \) then \( |\mathcal{P}| \to 0 \) to obtain the result. \( \Box \)

**Preparatory lemmas.** For the proof of Theorem 2.8 we prepare a few intermediate estimates based on Theorem 2.8(i). All implicit constants in the following depend only on \( m, \varepsilon_1, \varepsilon_2 \) and \( T \).

**Lemma 2.10.** \( \| \sup_{\mathcal{P}} |A^\mathcal{P}_t - A_t| \mathcal{F}_0 \|_m \|_n \leq |\mathcal{P}|^{\varepsilon_1 + \varepsilon_2} (\Gamma_1 + \Gamma_2) \).

**Proof.** For each \( j \geq 1 \), we write

\[
A^\mathcal{P}_t - A_t = \sum_{i \leq j} Z_i = \sum_{i \leq j} (Z_i - \mathbb{E}_{t_{i-1}} Z_i) + \sum_{i \leq j} \mathbb{E}_{t_{i-1}} Z_i,
\]
where for each \( i \), \( Z_i = A_{t_i-1, t_i} - \delta A_{t_i-1, t_i} \in \mathcal{F}_t \). Note that the former sum is a discrete martingale indexed by \( j \). Applying the conditional BDG inequality and the Minkowski inequality (see [Le20, Eq. (2.5)]), we have
\[
\| \sup_j |A_j^P - A_j| \|_{\mathcal{F}_0|m|n} \lesssim \left( \sum_i \|Z_i|\mathcal{F}_0|_{m|n}^2 \right)^{1/2} + \sum_i \|\mathbb{E}_{t_{i-1}} Z_i|\mathcal{F}_0|_{m|n}.
\]

Using (2.13) and (2.14), we can estimate the series on the right-hand side above, which yields the stated estimate. \( \square \)

**Lemma 2.11.** Under the setting of Theorem 2.8(ii), for every \( (s, t) \in \Delta \), we have
\[
(2.17) \quad \sup_{r \in D(s, t)} |\delta A_{s, r} - A_{s, r}| \|_{\mathcal{F}_s|m|n} \lesssim \Gamma_1(t - s)^{1+\varepsilon_1} + \Gamma_2(t - s)^{\frac{1}{2}+\varepsilon_2} + \Gamma_3(t - s)^{\frac{1}{10}+\varepsilon_3},
\]
where \( D(s, t) \) is the set of all dyadic points of \([s, t]\).

**Proof.** Fix \((s, t) \in \Delta\). Let \( P^k = \{r_i^k\}_{i=0}^{2^k-1} \) be the dyadic partition of \([s, t]\) with uniform mesh size \( 2^{-k} (t-s) \). Define \( |r|_k = \sup_{\tau \in [s, t]} |r| \), \( A^k_{r, s, r} = \sum_{i:r^k_i \leq \tau} A_{t_i-1, t_i}^k, u^k_{s, r} = (r^k_s + r^k_{s-1})/2 \) and \( Z^k = -\delta A^k_{r^k_s, r^k_s, r^k_s} \). For each \( r \in [s, t] \), each integers \( k \geq h \geq 0 \), we write
\[
A^h_{s, [r]_h} - A^h_{s, [r]_{h-1}} = \sum_{i:r^k_i \leq [r]_{h-1}} Z^h_{i, r} + A_{[r]_{h-1}, [r]_h},
\]
so that
\[
(2.18) \quad A^h_{s, [r]_h} - A^h_{s, [r]_h} = \sum_{h=0}^{k-1} \sum_{i:r^h_i \leq [r]_h} Z^h_{i, r} + \left( \sum_{h=0}^{k-1} A_{[r]_h, [r]_{h+1}} - A_{s, [r]_h} \right).
\]

For the former sum, we further decompose
\[
\sum_{h=0}^{k-1} \sum_{i:r^h_i \leq [r]_h} Z^h_{i, r} = \sum_{h=0}^{k-1} \left[ I^h_1([r]_h) + I^h_2([r]_h) \right]
\]
where
\[
I^h_1(r) = \sum_{i:r^h_i \leq r} \mathbb{E}_{r^h_{i-1}} Z^h_{i, r} \quad \text{and} \quad I^h_2(r) = \sum_{i:r^h_i \leq r} (Z^h_{i, r} - \mathbb{E}_{r^h_{i-1}} Z^h_{i, r}).
\]

Using triangle inequality, we have
\[
\sup_{r \in [s, t]} |I^h_1([r]_h)| \|\mathcal{F}_s|m|n \leq \sum_{i:r^h_i \leq t} \|\mathbb{E}_{r^h_{i-1}} Z^h_{i, r} |\mathcal{F}_s|m|n \lesssim \Gamma_1(t - s)^{1+\varepsilon_1} 2^{-h\varepsilon_1}.
\]

Note that for each \( h \), \( (I^h_2(u))_{u \in P^h} \) is a discrete martingale. Applying the BDG inequality and the Minkowski inequality, we have for every \( h < k \),
\[
\sup_{r \in [s, t]} |I^h_2([r]_h)| \|\mathcal{F}_s|m|n \leq \sup_{u \in P^h} |I^h_2(u)| \|\mathcal{F}_s|m|n \lesssim \left( \sum_{i:r^h_i \leq t} \|Z^h_{i, r} \|\mathcal{F}_s|m|n \right)^{1/2} \lesssim \Gamma_2(t - s)^{1/2+\varepsilon_2} 2^{-h\varepsilon_2}.
\]
For the later sum in (2.18), we have
\[
I_3^h(r) := \left| \sum_{h=0}^{k-1} A[r_{h+1}, r_{h+1}+1] - A_{r_k}[r_{k+1}] \right| = \left| \sum_{h=0}^{k-1} \delta A[r_{h+1}, r_{h+1}+1] \right| \\
\leq \sum_{h=0}^{k-1} \sup_{i=0, \ldots, \frac{2^h}{u^h+1}} \sup_{u^h} |\delta A_{r_i}^{h, u_i}|.
\]
We note that
\[
\mathbb{E}_s \sup_{i=0, \ldots, \frac{2^h}{u^h+1}} \left| \delta A_{r_i}^{h, u_i} \right|^m \leq \sum_{i=0}^{2^h-1} \mathbb{E}_s \sup_{u^h} |\delta A_{r_i}^{h, u_i}|^m.
\]
Hence using (2.15) and the fact that \( m \leq n \), we obtain that
\[
\left\| \left\| \sup_i \sup_{u^h} |\delta A_{r_i}^{h, u_i}|F_s \right\|_m \right\|_n \leq \left( \sum_{i=0}^{2^h-1} \mathbb{E}_s \sup_{u^h} |\delta A_{r_i}^{h, u_i}|^m \right)^{\frac{1}{m}} \leq \Gamma_3 2^{\frac{h}{m} - \delta_3} (t-s)^{h/m + \delta_3},
\]
which implies that
\[
\left\| \left\| \sup_r I_3(r)F_s \right\|_m \right\|_n \leq \Gamma_3 (t-s)^{h/m + \delta_3}.
\]
Applying the previous estimates altogether in (2.18), we obtain that
\[
\left\| \left\| \sup_{r \in [s,t]} A_{r_i}^{k} - A_{r_k} \right\|_m \right\|_n \leq \Gamma_1 (t-s)^{1+\delta_1} + \Gamma_2 (t-s)^{\frac{1}{2} + \delta_2} + \Gamma_3 (t-s)^{h/m + \delta_3},
\]
uniformly for every \( k \geq 1 \). Hence, for every \( k \leq l \),
\[
\left\| \left\| \sup_{r \in [s,t]} A_{r_i}^{l} - A_{r_k} \right\|_m \right\|_n \leq \Gamma_1 (t-s)^{1+\delta_1} + \Gamma_2 (t-s)^{\frac{1}{2} + \delta_2} + \Gamma_3 (t-s)^{h/m + \delta_3}.
\]
Sending \( l \to \infty \) then \( k \to \infty \), using the fact that \( \lim A_{r_i}^{l} = \delta A_{s,r} \), in probability for each \( s, r \) ([L20]), we obtain (2.17).

\[\text{Lemma 2.12.} \text{ Let } \mathcal{P}^k \text{ be the dyadic partition of } [0,T] \text{ with uniform mesh size } 2^{-k}T \text{ and put } D = \cup_k \mathcal{P}^k. \text{ Then under the setting of Theorem 2.8(ii), we have}
\]
\[
\left\| \left\| \sup_{t \in D} \left\| \frac{A_{r_i}^{P^k} - A_t}{m} \right\| \right\|_m \leq 2^{-k(\epsilon_1 + \epsilon_2 + \delta_3)} (\Gamma_1 + \Gamma_2 + \Gamma_3).
\]
\[\text{Proof.} \text{ For any } t, \text{ define } \lfloor t \rfloor_k = \sup\{r \in \mathcal{P}^k : r \leq t\}. \text{ For any } l \geq k, \text{ we have}
\]
\[
\sup_{t \in \mathcal{P}^l} \left\| \frac{A_{r_i}^{P^k} - A_t}{m} \right\| \leq \sup_{t \in \mathcal{P}^l} \left\| \frac{A_{\lfloor t \rfloor_k}^{P^k} - A_{\lfloor \frac{t}{k} \rfloor} + \sup_{t \in \mathcal{P}^l} |A_{t}^{\lfloor \frac{t}{k} \rfloor} - \delta A_{t, \lfloor \frac{t}{k} \rfloor,t}|}{m} \right\| \leq \sup_{t \in \mathcal{P}^k} \left\| \frac{A_{r_i}^{P^k} - A_t}{m} \right\| + \sup_{t \in \mathcal{P}^k} \sup_{s \in \mathcal{P}^k} \left\| \frac{A_{s,t} - \delta A_{s,t}}{m} \right\|.
\]
By Lemma 2.10, we have \( \left\| \sup_{t \in \mathcal{P}^k} |A_{r_i}^{P^k} - A_t| \right\| \leq 2^{-k(\epsilon_1 + \epsilon_2)} (\Gamma_1 + \Gamma_2). \) For the second term, we put \( \zeta_s := \sup_{t \in \mathcal{P}^k} \left\| \frac{A_{s,t} - \delta A_{s,t}}{m} \right\| \) and use Lemma 2.11 to obtain that
\[
\mathbb{E} \left[ \left\| \sup_{s \in \mathcal{P}^k} \zeta_s \right\| m \right] \leq 2^{-k \min \{1+\epsilon_1, m-1\} \epsilon_2} \min \{1, \frac{1}{2} + \epsilon_2 - m \epsilon_3 \} \Gamma_1 + \Gamma_2 + \Gamma_3.
\]
Since \( m \geq 2 \), we have \( \left\| \sup_{s \in \mathcal{P}^k} \zeta_s \right\| m \leq 2^{-k(\epsilon_1 + \epsilon_2 + \delta_3)} (\Gamma_1 + \Gamma_2 + \Gamma_3). \) These estimates yield
\[
\left\| \left\| \sup_{t \in \mathcal{P}^l} |A_{r_i}^{P^k} - A_t| \right\| \leq 2^{-k(\epsilon_1 + \epsilon_2 + \delta_3)} (\Gamma_1 + \Gamma_2 + \Gamma_3).
\]
Since \( l \geq k \) is arbitrary, this implies the result. \qed
**Proof of Theorem 2.8.** Part (i), i.e. convergence of $A_P^t$ to $A_t$ for fix $t$, is obtained by using analogous arguments as in [Le20, Theorem 2.1] in combination with a conditional BDG inequality. (The first arXiv version of this article has more details; see also [Le23, Theorem 3.1].)

We turn to Part (ii) and establish (locally) uniform convergence. From Lemma 2.12, using the Borel–Cantelli lemma, we see that the sequence $\{A_P^k\}_k$ converges a.s. uniformly to $A$ on $D$. By assumptions, for each $k$, $A_P^k$ is cadlag on $[0, T]$. Consequently, $A$ is cadlag on $D$ and it makes sense to define

$$\tilde{A}_t(\omega) = \lim_{r \to D_{r,t}} A_r(\omega), \quad t \in [0, T].$$

It is standard to verify that $\tilde{A}$ is a version of $A$ and that $\tilde{A}$ is cadlag on $[0, T]$. By Fatou’s lemma and (2.17), we have

$$(2.19) \quad \left\| \sup_{r \in [s, t]} |\delta \tilde{A}_{s,r} - A_{s,r}| \left\| F_s \right\| \right\|_m \leq \Gamma_1 |t - s|^{1+\varepsilon_1} + \Gamma_2 |t - s|^{\frac{1}{2} + \varepsilon_2}.$$  

We then repeat the argument in the proof of Lemma 2.12. Define $[t]_P = \sup\{r \in P : r \leq t\}$ and $[\tilde{t}]_P = \inf\{r \in P : r > t\}$. We have

$$\sup_{t \in [0, T]} |A_P^t - \tilde{A}_t| \leq \sup_{t \in [0, T]} |A_P^t - \tilde{A}_{[t]_P}| + \sup_{t \in [0, T]} |A_{[t]_P, t} - \delta \tilde{A}_{[t]_P, t}|$$

$$= \sup_{t \in P} |A_P^t - \tilde{A}_t| + \sup_{s \in P, t \in [s, [s]_P]} |A_{s, t} - \delta \tilde{A}_{s, t}|.$$

By Lemma 2.10, we have $\left\| \sup_{t \in P} |A_P^t - \tilde{A}_t| \| F_0 \|_m \right\|_0 \leq |P|^{|\varepsilon_1 + \varepsilon_2|} (\Gamma_1 + \Gamma_2)$. For the second term, we put $\tilde{c}_s = \sup_{t \in [s, [s]_P]} |A_{s, t} - \delta \tilde{A}_{s, t}|$ and use (2.19) to obtain

$$\left\| \sup_{s \in P} \tilde{c}_s \| F_0 \|_m \right\|_n \leq \left( \sum_{s \in P} \left\| \tilde{c}_s \| F_0 \|_m \right\|_n \right)^{1/m} \leq |P|^{\frac{1}{2} + \varepsilon_2 - \frac{1}{m}} (\Gamma_1 + \Gamma_2).$$

Since $m \geq 2$, we have $\left\| \sup_{s \in P} \tilde{c}_s \| F_0 \|_m \right\|_n \leq |P|^{\varepsilon_2} (\Gamma_1 + \Gamma_2)$. These estimates yield (2.16). 

---

**3. Rough stochastic analysis**

In the current section, we define and establish basic properties of the integration $\int ZdX$ where $Z$ is an adapted process and $X = (X, X)$ is an $\alpha$-Hölder rough path. The frequently used class of $X$-controlled rough paths in rough-path theory ([FH20]) turns out to be too restrictive to contain solutions to RSDEs (as introduced later in (4.1)). This has led us to the concept of stochastic controlled rough paths (introduced in Definition 3.1) and rough stochastic integrations, which are described herein. Throughout the section, $\Omega = (\Omega, \mathcal{G}, \mathbb{P}; \{\mathcal{F}_t\})$ is a stochastic basis whose underlying probability space is complete. We assume, as usual, that $\mathcal{F}_0$ contains $\mathbb{P}$-null sets, which ensures that modifications of adapted processes are still adapted.

**3.1. Stochastic controlled rough paths.** In the sequel we let $2 \leq m < \infty$, while $m \leq n \leq \infty$. The parameters $\alpha, \beta, \beta' \in (0, 1]$ are subject to $\alpha + \beta + \beta' > 1$. Unless stated otherwise, $I \subset [0, T]$ is an arbitrary compact interval and we let for convenience $o = \min I$.

For any 2-parameter stochastic process $A_{s,t}(\omega)$, we introduce the quantity

$$(3.1) \quad E_s A = (s, t; \omega) \mapsto E_s(A_{s,t})(\omega),$$
where we recall that $\mathbb{E}_s = \mathbb{E}(\cdot | \mathcal{F}_s)$. A progressive measurable stochastic process $Z_s(\omega) \in W$ is called \textit{stochastically controlled} with respect to $X$ if there exists another such $Z'_s(\omega) \in \mathcal{L}(V, W)$ so that for every $(s, t) \in \Delta(I)$, with probability one
\begin{equation}
|\mathbb{E}_s \delta Z_{s,t} - Z'_s \delta X_{s,t}| \leq C_{s,t} |t - s|^{\beta + \beta'}, \tag{3.2}
\end{equation}
where $C_{s,t} = C_{s,t}(\omega)$ is a uniformly $L_n$-integrable, two-parameter family of random variables. By a common abuse of language, we call $Z'$ the (generalized) Gubinelli derivative of $Z$ even though it is not unique in general.

It turns out that rough stochastic integrals can be defined for stochastically controlled processes that are subject to additional regularity assumptions in the spaces $C^\alpha L_{m,n}$.

One of these subclasses is that of stochastic controlled rough paths, as defined here. As will be seen in Section 4, it contains solutions to RSDEs of the form (4.1) for reasonable coefficients.

\textbf{Definition 3.1} (Stochastic controlled rough paths). We say that $(Z, Z')$ is a stochastic controlled rough path of $(m, n)$-integrability and $(\beta, \beta')$-Hölder regularity with values in $W$ with respect to $\{\mathcal{F}_t\}$ if the following are satisfied
(a) $Z: \Omega \times I \to W$ and $Z': \Omega \times I \to \mathcal{L}(V, W)$ are $\{\mathcal{F}_t\}$-progressive measurable;
(b) $\delta Z$ belongs to $C^{\alpha}_2 L_{m,n}(I, \Omega; \mathcal{L}(V, W))$;
(c) $Z$ is stochastically controlled with Gubinelli derivative $Z'$. Said otherwise, putting
\begin{equation}
R^Z_{s,t} = \delta Z_{s,t} - Z'_s \delta X_{s,t}, \quad \text{for } (s, t) \in \Delta,
\end{equation}
we have that $\mathbb{E}_s R^Z$ belongs to $C^{\alpha + \beta'}_2 L_n(I, \Omega; W)$;
(d) $Z'$ belongs to $C^{\alpha, \beta'} L_{m,n}(I, \Omega; \mathcal{L}(V, W))$ and $\delta Z'$ belongs to $C^{\alpha + \beta'}_2 L_{m,n}(I, \Omega; \mathcal{L}(V, W))$.

The class of such processes is denoted by $D^\alpha_{X, \beta, \beta'} L_{m,n}(I, \Omega; W)$, or simply $D^\alpha_{X, \beta, \beta'} L_{m,n}$ whenever clear from the context.

For a process $(Z, Z')$ in $D^\alpha_{X, \beta, \beta'} L_{m,n}$, its semi-norm is defined by
\begin{equation}
\| (Z, Z') \|_{X, \beta, \beta', m,n} = \| \delta Z \|_{\beta, m,n} + \| Z' \|_{\beta', m,n} + \| \mathbb{E}_s R^Z \|_{\beta + \beta', n}. \tag{3.3}
\end{equation}

Similarly if $(Y, Y') \in D^\alpha_{X, \beta, \beta'} L_{m,n}$ for another such $X \in C^\alpha(V)$, we define the quantity
\begin{equation}
\| Z, Z'; \bar{Z}, \bar{Z}' \|_{X, \bar{X}, \beta, \beta, m,n} = \| \delta \bar{Z} - \delta Z \|_{\beta, m,n} + \| \bar{Z}' - Z' \|_{\beta', m,n} + \| \mathbb{E}_s R^Z - \mathbb{E}_s R^Z \|_{\beta + \beta', m,n}, \tag{3.4}
\end{equation}
where $\bar{R}^Z_{s,t} := \delta \bar{Z}_{s,t} - Z'_s \delta \bar{X}_{s,t}$. We will simply write $D^\alpha_{X, \beta, \beta'} L_{m,n}$ for $D^\alpha_{X, \beta, \beta'} L_{m,n}(I, \Omega; W)$ when there is no risk of confusion. In addition, we write $D^\alpha_{X} L_{m,n} = D^\alpha_{X} L_{m,n}$.

The quantity defined in (3.4) is, at best, a pseudometric (i.e. separation axiom fails), moreover “triangle inequality” only holds in the sense that
\begin{equation}
\| Z, Z'; \bar{Z}, \bar{Z}' \|_{X, \bar{X}, \beta, \beta, m,n} \leq \| Z, Z'; \bar{Z}, \bar{Z}' \|_{X, \bar{X}, \beta, \beta, m,n} + \| \bar{Z}, \bar{Z}' \|_{X, \bar{X}, \beta, \beta, m,n}, \tag{3.5}
\end{equation}
for another such $X \in C^\alpha(V)$ and each $(\bar{Z}, \bar{Z}') \in D^\alpha_{X} L_{m,n}$. It is also worth noticing the translation invariance property:
\begin{equation}
\| Z + S, Z' + S'; \bar{Z} + S, \bar{Z}' + S' \|_{X, \bar{X}, \beta, \beta, m,n} = \| Z, Z'; \bar{Z}, \bar{Z}' \|_{X, \bar{X}, \beta, \beta, m,n}, \tag{3.6}
\end{equation}
for any $(S, S') \in D^\alpha_{X} L_{m,n} \cap D^\alpha_{X} L_{m,n}$. These facts will prove useful in the sequel.

\textbf{Remark 3.2}. Being stochastically controlled is fundamentally a statement about the increments $\delta Z$ rather than the path $Z$ itself. No integrability assumption is required on the ground value $Y_0 \in L_0(\mathcal{F}_0)$, as long as property (3.2) is not altered. This is the reason why working with semi-norms and pseudometrics (instead of norms and metrics)
is useful. When we want to think of \( D^{\beta,\beta'}_{X} L_{m,n} \) as a metric space, a convenient choice is to introduce the distance
\[
d_{l}(Z, Z'; \bar{Z}, \bar{Z}') = \| \|Z_{o} - \bar{Z}_{o}\| \wedge 1\|_{m} + \|Z, Z'; \bar{Z}, \bar{Z}'\|_{X,\beta,\beta';m,n}
\]
(and similarly in the context of a pair \( X, \bar{X} \)). With this definition, it is easily seen that \( (D^{\beta,\beta'}_{X} L_{m,n}, d_{l}) \) forms a complete space (the proof is omitted).

**Remark 3.3.** Martingales “have” zero generalized Gubinelli derivative, in the sense that letting \( M'_{s} = 0 \) yields \( \mathbb{E}_{s} R^{M}_{s,t} = \mathbb{E}_{s} \delta M_{s,t} \equiv 0 \), which is in \( C^{2} \). Additionally \((M,0)\) forms a stochastic controlled rough path in \( D^{\beta,\beta'}_{X} L_{m,n} \) if and only if \( t \mapsto \delta M_{t} \) lies in \( C^{\beta} L_{m,n} \) (with identical semi-norms). This is clear from Definition 3.1 and (3.4).

**Doob Meyer.** Other examples of stochastic controlled rough paths would be \( X\)-controlled rough paths taking values in the Banach space \( X = L_{m} \), as defined in [FI20] (denoted by \( \mathcal{D}^{\beta,\beta'}(X) \) therein). Recall that for two continuous paths \( Y: I \to X \) and \( Y': I \to \mathcal{L}(V, X) \), the pair \((Y,Y')\) is called \( X\)-controlled whenever
\[
\| \delta Y_{s,t} - Y'_{t} \delta X_{s,t} \|_{X} \lesssim (t-s)^{\beta+\beta'}.
\]
Now, because of the contraction property for conditional expectation, we have that \( \mathcal{D}^{\beta,\beta'}(L_{m}) \hookrightarrow D^{\beta,\beta'}_{X} L_{m,m} \).

Under mild regularity and integrability assumptions, it is clear from Remark 3.3 and linearity of that the sum of a martingale and an \( X\)-controlled process of that kind will yield a stochastic controlled rough path as in Definition 3.1. Similar to Doob-Meyer, we show a converse statement. It asserts that a stochastic controlled rough path can be written as a martingale (endowed with zero Gubinelli derivative as above) plus an additional path subject to (3.7). This is formulated in the following result.

**Theorem 3.4** (Doob–Meyer decomposition). Suppose that \((Z, Z')\) is a stochastic controlled rough path with respect to \( \{\mathcal{F}_{t}\} \) in \( D^{\beta,\beta'}_{X} L_{m,n} \) with \((\alpha \wedge \beta) + \beta' > \frac{1}{2}, m \in [2, \infty) \) and \( n \in [m, \infty] \). Then, there are uniquely characterized processes \( M, Y \) such that
\[
\begin{align*}
(i) & \quad Z_{t} = M_{t} + Y_{t} \quad \text{a.s. for every } t \in I; \\
(ii) & \quad M \text{ is an } \{\mathcal{F}_{t}\}\text{-martingale, } M_{0} = 0; \\
(iii) & \quad Y \text{ is } \{\mathcal{F}_{t}\}\text{-adapted and } X\text{-controlled in the sense that}
\end{align*}
\]
\[
\| \delta Y_{s,t} - Z'_{t} \delta X_{s,t} \|_{\mathcal{F}_{s}} \|_{m} \|_{n} \lesssim (\| \delta Z' \|_{\beta',m,n} \| \delta X \|_{\alpha} + \| \mathbb{E}_{s} R^{Z}_{s,t} \|_{\beta+\beta',m,n}) |t-s|^{(\alpha \wedge \beta) + \beta'}
\]
for every \((s, t) \in \Delta\).

Letting \( Y' = Z' \), estimate (3.8) implies moreover that \((Y,Y')\) belongs to \( D^{(\alpha \wedge \beta),\beta'}_{X} L_{m,n} \) (in particular, \( M \) belongs to \( C^{(\alpha \wedge \beta) L_{m,n}} \)).

**Proof.** We rely on [Lê23, Theorem 3.3]. Consider \( A_{s,t} = \delta Z_{s,t} \), which is integrable. Since \( \delta A \equiv 0 \), conditions (2.11) and (2.12) of Theorem 2.8 are trivially satisfied, and \( A_{t} = Z_{t} - Z_{0} \) for each \( t \). From the definition of the spaces \( D^{\beta,\beta'}_{X} L_{m,n} \) and the fact that \( Z_{s}' \) is \( \mathcal{F}_{s}\)-measurable, we have
\[
(\mathbb{E}_{s} - \mathbb{E}_{u}) \delta Z_{s,t} = (\mathbb{E}_{s} - \mathbb{E}_{u}) [Z'_{u} \delta X_{u,t}] + (\mathbb{E}_{s} - \mathbb{E}_{u}) R^{Z}_{u,t}
= (\mathbb{E}_{s} - \mathbb{E}_{u}) [\delta Z'_{s,u}] \delta X_{u,t} + (\mathbb{E}_{s} - \mathbb{E}_{u}) R^{Z}_{u,t}.
\]
Hence, we infer that
\[
\| (\mathbb{E}_{s} - \mathbb{E}_{u}) \delta Z_{s,t} \|_{\mathcal{F}_{s}} \|_{m} \|_{n} \lesssim (\| \delta Z' \|_{\beta',m,n} \| \delta X \|_{\alpha} + \| \mathbb{E}_{s} R^{Z}_{s,t} \|_{\beta+\beta',m,n}) |t-s|^{(\alpha \wedge \beta) + \beta'}.
\]
But since \((\alpha \wedge \beta) + \beta' > \frac{1}{2}\), the conditions of [Lê23, Theorem 3.3] are met. Hence, \( A = M + J \) where \( M, J \) satisfy the conclusions of [Lê23, Theorem 3.3]. We set
$M = M$ and $Y = J + Z_0$ so that $Z = M + Y$. We see from [Lé23, Eqn. (3.9)] that
\[
\|\delta Y_{s,t} - \mathbb{E}_s \delta Z_{s,t} \| \leq (\| \delta Z' \|_{\beta';m,n} \| \delta X \|_{\alpha} + \| \mathbb{E} R Z' \|_{\beta + \beta';n}) |t - s|^{(\alpha \land \beta) + \beta'}.
\]
Next, writing $\mathbb{E}_s \delta Z_{s,t} = Z'_s \delta X_{s,t} + \mathbb{E}_s R Z_{s,t}$ and applying the triangle inequality, we obtain
\[
\|\delta Y_{s,t} - Z'_s \delta X_{s,t} \| \mathcal{F}_s \| \leq \| \mathbb{E}_s R Z_{s,t} \| + \| \delta Y_{s,t} - \mathbb{E}_s \delta Z_{s,t} \| \mathcal{F}_s \|.
\]
Combined with the former estimate, we obtain (3.8). Uniqueness of $(M, Y)$ given $(Z, Z')$ follows from [Lé23, Theorem 3.3(vi)].

### 3.2. Rough stochastic integrals

We now tackle the heart of the matter by defining the rough stochastic integral of a progressive measurable process $Z: \Omega \times I \to W$ against a rough path $X = (X, X) \in \mathcal{C}^\alpha(I; V)$, assuming the former is stochastically controlled with respect to $X$. For that purpose though, extra regularity assumptions are required and we shall see in particular that these are fulfilled whenever the corresponding pair $(Z, Z')$ forms a stochastic controlled rough path in $D_X^{3,\beta} L_{m,n}(I, \Omega; \mathcal{L}(V, W)))$. Its rough stochastic integral $\int_0^t ZdX$ is then well-defined as the limit in probability of the Riemann sums
\[
\sum_{[u,v] \in \mathcal{P}, 0 \leq u < t} (Z_u \delta X_{u,v} \land t + Z'_u X_{u,v} \land t)
\]
as the mesh-size of $\mathcal{P}$ goes to 0, for each $t \in I$ (here $\mathcal{P}$ is any partition of $I$). Here, in writing $Z'_u X_{u,v}$, we have used the isomorphism $\mathcal{L}(V, \mathcal{L}(V, W)) \simeq \mathcal{L}(V \otimes V, W)$ (recall that $V, W$ are finite-dimensional). The resulting integration theory is self-consistent in the sense that $(\int ZdX, Z)$ shares all of the properties of stochastic controlled rough paths, except for the fact that its second component, namely $Z$, is not necessarily bounded uniformly in $L_n$.

Although (3.9) has the same form as the defining Riemann sums for rough integrals ([FH20]), the convergence of (3.9) only takes place in probability. This is due to the fact that the class of stochastic controlled rough paths contains not only controlled rough paths, but also nontrivial martingales (for which (3.9) fails to converge a.s.). This alludes that the usual sewing lemma is not applicable. Instead, we rely on the stochastic sewing lemma, [Theorem 2.8], to obtain such convergence.

We now state our main result on rough stochastic integration in which the reader may assume $\beta = \beta' = \alpha$ at the first reading. For any $X = (X, X) \in \mathcal{C}^\alpha(V) \oplus \mathcal{C}_2^{3\beta}(V \otimes 2)$ we introduce the quantities
\[
\Gamma_1^{\beta,\beta';m,n}(X; Z, Z') := |\delta X|_{\alpha} \| \mathbb{E}_s R Z' \|_{\alpha \land \beta + \beta';n} + |X|_{\alpha + \alpha \land \beta} \| \mathbb{E}_s \delta Z' \|_{\beta';m,n},
\]
\[
\Gamma_2^{\beta,\beta';m,n}(X; Z, Z') := |\delta X|_{\alpha} \| \delta Z \|_{\alpha \land \beta; m,n} + \| X \|_{\alpha, \alpha \land \beta} \sup_t \| Z'_t \|_m,
\]
where we denote by $\mathbb{E}_s R Z := (s, t; \omega) \mapsto \mathbb{E}_s (\delta Z_{s,t} - Z'_s \delta X_{s,t})(\omega)$, and we also recall the notation $\| X \|_{\alpha, \alpha'} = \sqrt{\| \delta X \|_{\alpha} \| \delta X \|_{\alpha'} \vee \| X \|_{\alpha + \alpha'}}$.

**Theorem 3.5** (Rough stochastic integral). Let $\alpha \in (\frac{1}{2}, 1]$, $\beta, \beta' \in (0, 1)$, $\alpha + \beta > \frac{1}{2}$, $\alpha + (\alpha \land \beta) + \beta' > 1$, $m \in [2, \infty)$, $n \in [m, \infty]$ and $X = (X, X) \in \mathcal{C}^\alpha([0, T]; V)$. Suppose that $Z, Z'$ are $\mathcal{F}_t$-progressive measurable and such that
\[
\max_{i=1,2} \Gamma_i^{\beta,\beta';m,n}(X; Z, Z'; I) < \infty.
\]
Then $A_{s,t} := Z_s \delta X_{s,t} + Z'_s X_{s,t}$ defines a two-parameter stochastic process which satisfies the hypotheses of the stochastic sewing lemma. We call
\[
\int_0^t ZdX =: \mathcal{A}.
\]
the continuous process supplied by Theorem 2.8(ii). In particular, \( \int_0^t Z dX \) is the continuous process which corresponds to the limit in probability of (3.9) uniformly in time. Moreover, the corresponding integral remainder \( J_{s,t} = \int_s^t Z dX - Z_s \delta X_{s,t} - Z_t' \bar{X}_{s,t} \) depends on \( (X; Z, Z') \) in a Lipschitz fashion. More precisely, let \( (\bar{X}; \bar{Z}, \bar{Z}') \) denote another tuple subject to \( \max_{i=1,2} \Gamma_i^{\beta,\beta';m,n}(\bar{X}, \bar{Z}, \bar{Z}') < \infty \). Then, defining \( \bar{J} \) accordingly, we have the estimates

\[
\|E\|_{\alpha+\alpha\wedge\beta+\beta';m} \lesssim \Gamma_1 (X - \bar{X}; \bar{Z}, \bar{Z}'; I) + \Gamma_1 (X; Z - \bar{Z}', Z' - \bar{Z}'; I)
\]

\[
(J - \bar{J})_{\alpha+\alpha\wedge\beta+\beta';m} \lesssim (|I|^\beta \Gamma_1 + \Gamma_2) (X - \bar{X}; \bar{Z}, \bar{Z}'; I) + (|I|^\beta \Gamma_1 + \Gamma_2) (X; Z - \bar{Z}, Z' - \bar{Z}'; I)
\]

and similarly

\[
\| \sup_r (J - \bar{J})_{\alpha,r} \|_{m} \lesssim \left( \Gamma_1 + \Gamma_2 \right) (X - \bar{X}; \bar{Z}, \bar{Z}'; I) + \left( \Gamma_1 + \Gamma_2 \right) (X; Z - \bar{Z}, Z' - \bar{Z}'; I) |I|^{\alpha + \alpha \wedge \beta + \frac{1}{2}(1 - \beta')}.
\]

All the hidden constants above depend on \( \alpha, \beta, \beta', m \) and \( |I| \), but are independent of \( X, Z, Z' \).

**Proof of Theorem 3.5.** Herein we let \( \Gamma_i = \Gamma_i^{\beta,\beta';m,n}(X; Z, Z'; I) \) for \( i = 1, 2 \). Using the Chen’s relation (2.4), we easily arrive at the identity

\[
-\delta A_{s,u,t} = \bar{R}_{s,u} X_{t} + \delta Z'_{s,u} X_{u,t}
\]

for every \( (s, u, t) \in \Delta \). This implies

\[
\|E_s \delta A_{s,u,t} \|_{m} \leq (t - s)^{\alpha + \alpha \wedge \beta + \beta'} \left( \| \delta X \|_{\alpha} R^\alpha \| \delta Z' \|_{\alpha \wedge \beta + \beta'} + |X|_{\alpha+\alpha\wedge\beta} \| E_\delta Z' \|_{\beta';m} \right),
\]

\[
\lesssim (t - s)^{\alpha + \alpha \wedge \beta + \beta'} \Gamma_1
\]

showing the first condition (2.11) with \( \varepsilon_1 = \alpha + \alpha \wedge \beta + \beta' - 1 > 0 \). For the second bound, we rely on the brute force estimate

\[
\| \sup_r |R_{s,u}^\alpha F_s \|_{m} \|_{n} \leq \| \| \delta Z'_{s,u} \|_{m} \|_{n} + |\delta X|_{\alpha+\alpha\wedge\beta} \sup_r \| Z'_{s} \|_{n} (u - s)^{\alpha+\beta}
\]

as well as

\[
\| \| \delta Z'_{s,u} \|_{m} \|_{n} \leq 2 \sup_r \| Z'_{s} \|_{n}.
\]

Plugging into (3.15) establishes (2.12) with \( \varepsilon_2 = \alpha + \alpha \wedge \beta - \frac{1}{2} > 0 \). Similarly, we observe that

\[
\| \sup_{\tau \in [u,t]} \| \delta A_{s,u,\tau} \|_{m} \|_{n} \lesssim \left[ \| \delta X \|_{\alpha} \| \delta Z' \|_{\alpha+\alpha\wedge\beta + m,\beta + \alpha+\alpha\wedge\beta} + 2 |\delta X|_{\alpha} \| \delta X \|_{\alpha+\alpha\wedge\beta} \| \delta Z' \|_{\beta';m} \right] (t - s)^{\alpha + \alpha \wedge \beta}
\]

and this shows (2.15) with \( \Gamma_3 \lesssim \Gamma_2 \) and \( \varepsilon_3 = \alpha + \alpha \wedge \beta - \frac{1}{m} > 0 \). Now, the desired conclusion follows from the fact that \( t \mapsto A_{s,t} \) is a.s. continuous for each \( s \). Indeed, we can apply Theorem 2.8 to see that \( \mathcal{A} = \int_0^t Z dX \) is well-defined as the limit of (3.9) in probability, uniformly in time, and that it has a continuous version. This shows well-posedness of the rough stochastic integral under the condition that \( \max_{i=1,2} \Gamma_i < \infty \), as claimed.

Moreover, when \( Z = 0 \) and \( \bar{X} = X \), the claimed estimates follow directly by (2.13), (2.14), (2.16) and the previous bounds (observing that \( \Gamma_1(0; Z, Z'; I) = 0 \)). The general case follows similar arguments based this time on the identity
\[
\delta(A - \tilde{A})_{s,u,t} = \bar{R}_{s,u}^{\bar{Z}}(\delta X - \delta \bar{X})_{u,t} + \delta \bar{Z}'_{s,u}(X - \bar{X})_{u,t} + (R\bar{Z} - \bar{R}\bar{Z})_{s,u}\delta X_{u,t} + (\delta \bar{Z}' - \delta Z')_{s,u}X_{u,t},
\]

where \(\tilde{A}_{s,t} = \bar{Z}_s\delta \bar{X}_{s,t} + \bar{Z}'_s\bar{X}_{s,t}\). We leave the details of these bounds to the reader. \(\square\)

We now state an important corollary concerning integrability of rough stochastic controlled paths as per Definition 3.1, as well as the continuity of the integration map in that context.

**Corollary 3.6** (Continuity of the integration map). Fix parameters \(\alpha, \beta, \beta'\) as in Theorem 3.5. For each \(X \in \mathcal{C}^\alpha(V)\), the integration map

\[
D^\beta,\beta'_{X}L_{m,n} \cap \{ (Z, Z') : \sup_t \|Z_t\|_n < \infty \} \rightarrow D^\alpha,\beta,\beta'_{X}L_{m,n}
\]

\[
(Z, Z') \mapsto \left( \int_0^\cdot ZdX, Z \right)
\]

is well-defined, linear and bounded in the sense that

\[
\left\| \left( \int_0^\cdot ZdX, Z \right) \right\|_{X;\alpha,\beta,\beta';m,n} \leq (1 + C)\|Z\|_{\beta;m,n} + C\|Z, Z'\|_{X;\beta,\beta';m,n}\|I\|_{\alpha,\beta},
\]

where we can take \(C = \|X\|_\alpha (1 + \|X\|_\alpha)\). More generally, the joint map \((X; Z, Z') \mapsto (\int_0^\cdot ZdX; Z)\) is locally Lipschitz continuous in the sense that

\[
\left\| \int_0^\cdot ZdX, Z; \int_0^\cdot \bar{Z}d\bar{X}, \bar{Z} \right\|_{X;\alpha,\beta,\beta';m,n} \leq C' \rho_{\alpha,\alpha,\beta}(X, \bar{X}) + (1 + C)\|Z - \bar{Z}\|_{\beta;m,n} + C\|Z, Z', \bar{Z}, \bar{Z}'\|_{X;\alpha,\beta,\beta';m,n}\|I\|_{\alpha,\beta},
\]

where \(C\) is as above and \(C' = \sup_t \|Z_t\|_n + \|\bar{Z}_t\|_n\).

**Proof.** Let \((Z, Z')\) be a stochastic controlled rough path in \(D^\beta,\beta'_{X}L_{m,n}\) and observe that \(\|RZ\|_{\alpha,\beta,\beta'} \leq \|RZ\|_{\beta,\beta'}\) while \(\|\delta Z\|_{\alpha,\beta,\beta;m,n} \leq \|\delta Z\|_{\beta;m,n}\). Moreover, the properties of conditional expectation imply that \(\|E,\delta Z'\|_{\beta;m,n} \leq \|\delta Z'\|_{\beta;m,n}\). Therefore, it follows from Definition 3.1 that

\[
\max_{i=1,2} \Gamma_{i}^{\beta,\beta';m,n}(X; Z, Z'; I) \leq C\|Z, Z'\|_{X;\beta,\beta';m,n}.
\]

In particular, Theorem 3.5 asserts that \((Z, Z')\) has a well-defined rough stochastic integral. Assuming in addition that \(\sup_t \|Z_t\|_n < \infty\), we now show that the pair \((\int_0^\cdot ZdX, Z)\) shares indeed all the properties of a stochastic controlled rough path in \(D^\alpha,\beta,\beta'_{X}L_{m,n}\). To this aim, it suffices to establish (3.17), as taking \(X = \bar{X}\) and \((\bar{Z}, \bar{Z}') = (0, 0)\) therein entails (3.17) and the desired property.

Define \((Y, Y') = (\int ZdX, Z)\) and similarly for \((\bar{Y}, \bar{Y}')\). In the notations of Theorem 3.5, we have \(\|E, (R^Y - \bar{R}^Y)_{s,t}\|_n \leq \|Z'(X_{s,t} - \bar{X}_{s,t})\|_n + \|(Z' - \bar{Z}')\bar{X}_{s,t}\|_n + \|E, (J - \bar{J})_{s,t}\|_n\).

Estimating the constants in the right hand side of (3.12) now yields:

\[
\|E, (R^Y - \bar{R}^Y)\|_{\alpha,\alpha,\beta;n} \leq C' \rho_{\alpha,\alpha,\beta}(X, \bar{X}) + C \sup_r \|Z'_r - \bar{Z}'_r\|_n
\]

\[
+ C\|Z, Z', \bar{Z}, \bar{Z}'\|_{X,\alpha,\beta,\beta;m,n}.
\]

We proceed similarly for the difference between increments, starting this time from

\[
\|(\delta Y - \delta \bar{Y})_{s,t}\|_{m,n} \leq \|Z_s(\delta X_{s,t} - \delta \bar{X}_{s,t}) + (Z_s - \bar{Z}_s)\delta \bar{X}_{s,t}\|_{m,n}
\]

\[
+ \|Z'_s(\bar{X}_{s,t} - \bar{X}_{s,t}) + (Z'_s - \bar{Z}'_s)\bar{X}_{s,t}\|_{m,n} + \|J_{s,t} - \bar{J}_{s,t}\|_{m,n}.
\]

It follows that
\[ \|\delta Y - \delta \tilde{Y}\|_{\alpha,m,n} \leq C' \rho_{\alpha,\alpha^\land,\beta}(X, \tilde{X})(1 + |I|^\alpha^\land\beta) \]
\[ + C \left( \sup_r \|Z_r - \tilde{Z}_r\|_n + \sup_r \|Z'_r - \tilde{Z}'_r\|_n |I|^\alpha + \|Z, Z'; \tilde{Z}, \tilde{Z}'\|_{X, \tilde{X}; \beta, \beta', m, n} |I|^\alpha^\land\beta \right) \]

Recalling that \( \|Y' - \tilde{Y}'\|_{\alpha^\land,\beta, m, n} = \sup_{r \in I} \|Z_r - \tilde{Z}_r\|_n + \|\delta Z - \delta \tilde{Z}\|_{\alpha^\land,\beta, m, n} \) by definition, we arrive at the desired conclusion after summing all these contributions. \( \Box \)

**Remark 3.7.** The averaged form in which the Gubinelli derivative appears in (3.10) shows that integration makes sense for a larger class of “extended” stochastic controlled rough path, as introduced later in Section 4.4. Since that class is not stable by composition (contrary to stochastic controlled rough paths, at least when \( n = \infty \), see Proposition 3.13 below), it does not play a prominent role as far as we are concerned with RSDEs of the form (4.1). That is why we postpone its introduction until later sections.

### 3.3. Stochastic controlled vector fields.

Unless stated otherwise, in the sequel we work with a Hölder path \( X \) in \( C^\alpha(I; V) \) for some \( \alpha \in (0, 1] \) and a compact interval \( I \subset [0, T] \). Recall that \( V, W \) are finite-dimensional Banach spaces, we also set here \( \hat{W} := \mathcal{L}(V, W) \), another (finite-dimensional) Banach space.

Herein we introduce a class of random, time-dependent and progressively measurable vector fields\(^{10} \)

\[
\begin{align*}
  f &: \Omega \times I \to \mathcal{X} \hookrightarrow C_b(W, \hat{W}) \\
  f' &: \Omega \times I \to \mathcal{Y} \hookrightarrow C_b(W, \mathcal{L}(V, \hat{W}))
\end{align*}
\]

for well-chosen Banach spaces of functions \( \mathcal{X}, \mathcal{Y} \), and where strong\(^{11} \) \( G/\text{Bor}(\mathcal{X}) \)-measurability (resp. strong \( G/\text{Bor}(\mathcal{Y}) \)-measurability) is assumed, see Section 2.1. Our main purpose here is to investigate a natural composition operation of that class with stochastic controlled rough paths and show that it yields a similar object, subject to explicit local-Lipschitz estimates.

As is well-known in the absence of time and sample parameters, the pair \((f^\circ(Y), Df^\circ(Y)Y')\) forms an \( X \)-controlled rough path if \((Y, Y')\) shares that property (with common Hölder regularity exponent \( \alpha \), say) provided that

\[
f^\circ \in C^\gamma_b(W; \hat{W}), \quad \text{for some } \gamma > \frac{1}{\alpha}.
\]

Given the functional analytic viewpoint laid out in (3.19), it is then tempting to let \( \mathcal{X} = C^\gamma_b(W, \hat{W}), \mathcal{Y} = C^\gamma_b(W, \mathcal{L}(V, \hat{W})) \simeq \mathcal{L}(V, \mathcal{X}) \) and simply define stochastic controlled vector fields as stochastic controlled rough paths with values in \( \mathcal{X} \) (note that Definition 3.1 extends trivially to infinite-dimensional state spaces). Although doing the job, this description would be too demanding regularity-wise as it fails to capture possible tradeoffs between space and time regularities at the level of the vector fields. A much better definition, which we employ in the rest of the paper, is the following.

**Definition 3.8** (Stochastic controlled vector fields). Let \( \beta, \beta' \in (0, 1] \) and \( \gamma > 1 \) be some fixed parameters. We call \((f, f')\) stochastic controlled vector field on \( W \) and write \((f, f') \in D^\beta,\beta'_{\mathcal{X}} L_{m,n} C^\gamma_b \) if the following conditions are satisfied.

(a) The pair

\[
(f, f') : \Omega \times I \to C^\gamma_b(W, \hat{W}) \times C^{\gamma-1}_b(W, \mathcal{L}(V, \hat{W}))
\]

\(^{10}\)Our terminology comes from viewing \( f \) as collection of \( d \) (time-dependent, random) vector fields when \( \text{dim} V = d \).

\(^{11}\)This particular detail matters in the discussion since none of the natural target spaces \( C_b \) or \( C^\gamma_b \) for \( \gamma > 0 \) is separable (see however Remark 3.9).
is progressively measurable in the strong sense and uniformly $n$-integrable i.e.

$$\sup_{s \in I} ||f_s||_n + \sup_{s \in I} ||f'_s||_{n-1} < \infty.$$  

(b) Letting

$$[Z]_{k;m,n} := \sup_{(s,t) \in \Delta(I): s \neq t} \frac{\left\| \sup_{x \in W} |Z_{s,t}(x)| \right\|_m}{(t-s)^{\kappa}},$$

the quantities $[\delta f]_{\beta;m,n}$, $[\delta f']_{\beta';m,n}$, $[\delta Df]_{\beta;m,n}$ are finite.

(c) The map $(s,t) \mapsto E_{\gamma} R_{t,s}^{\gamma} = E_f f_t - f_s - f'_s \delta X_{s,t}$ belongs to $C^{\alpha+\beta'}_2 L_n(C_b)^{12}$, namely

$$\left\| \sup_{y \in W} |E_{\gamma} R_{t,s}^{\gamma}(y)| \right\|_n < \infty.$$  

We call $(f, f') L_{m,\infty}$-integrable, $(\gamma, \alpha, \alpha')$-space-time-regular stochastic controlled vector fields if $(f, f') \in D_X^{\alpha} L_{m,\infty} C_0^\gamma$, and $(Df, Df') \in D_X^{\alpha,\alpha'} L_{m,\infty} C_0^{\gamma-1}$. (Write $2\alpha$ instead of $\alpha, \alpha'$ in case they are equal.)

For stochastic controlled vector fields as above, we introduce moreover the quantities

$$[f, f']_{\gamma;n} := \sup_{s \in I} \left( ||f_s||_n + ||f'_s||_{n-1} \right),$$

$$\left\| [f, f'] \right\|_{\gamma;n} := \left\| [(f, f')]_{\gamma;n} + \sup_{s \in I} ||f_s||_n \right\|_n,$$

$$\left\| [f, f'] \right\|_{X;\beta;\beta;m,n} := ||\delta f||_{\beta;m,n} + ||\delta Df||_{\beta';m,n} + ||\delta f'||_{\beta';m,n} + \left\| \left[ E_{\gamma} R^{\gamma}_t \right]_{\beta+\beta';m,n} \right\|_n,$$

which is abbreviated as $[(f, f')]_{X;\beta;\beta;m,n}$ if $m = n$, as $[(f, f')]_{X;2\beta;\beta;m,n}$ if $\beta = \beta'$ and as $[(f, f')]_{X;2\beta;\beta;m,n}$ when both conditions are met.

Similarly, if $(\tilde{f}, \tilde{f}') \in D_{\tilde{X}}^{\beta,\beta'} L_{m,\infty} C_0^\gamma$ for another such $\tilde{X} \in C^{\alpha}(W)$, we define

$$\left\| [f, f', \tilde{f}, \tilde{f}'] \right\|_{X;\tilde{X};\beta;\beta;m,n} = ||\delta f - \delta \tilde{f}||_{\beta;m,n} + ||\delta f' - \delta \tilde{f}'||_{\beta';m,n} + \left\| \left[ E_{\gamma} R^{\gamma}_t - E_{\tilde{\gamma}} R^{\tilde{\gamma}}_t \right]_{\beta+\beta';m,n} \right\|_n.$$

Remark 3.9. Definition 3.8 could be simplified if the “Lipschitz” space $C_0^\gamma(W)$ defined on $W$ (finite-dimensional Banach) is replaced by the “little Lipschitz” space $C_0^{0,\gamma}(W)$, defined as the closure of $C_0^{\infty}$ (bounded and all derivatives bounded). In practice, when one has “open” conditions, like $\gamma > 1/\alpha$, this can always be achieved by tweaking the exponent, using for instance that $C_0^\gamma \subset C_b^{0,\gamma-\varepsilon}$ for any $\varepsilon > 0$ (Theorems 4.6, 4.9, 4.13 and 4.19 below contain an open condition of this form; this is however not the case of Theorem 4.10 where $\gamma = \frac{1}{\alpha}$). The resulting little Lipschitz space is then separable and the standard measurability assumption is equivalent to the strong measurability assumption inherit or definition.

The reason we have not taken this popular short-cut, is our follow-up paper [FHL24] in which case we will encounter s.c.v.f. on infinite-dimensional spaces (through dependence of a Lions lift); in such situation there is no way to avoid non-separability. As most authors, we define r.v. with values in non-separable Banach space as strongly measurable.

\[12\] Similar to Definition 3.1, it is equivalent to say that $E_{\gamma} R^{\gamma}_t$ belongs to $C_2^{\alpha+\beta'} L_{m,n}(C_b)$.

\[13\] Reduced to the essence, consider a map $f : \Omega \mapsto C_0^\alpha(H,\mathbb{R})$ for some moment (separable Hilbert) space $H = L_2$. In case of integer regularity, say $\gamma = 2$, the corresponding “little” Lipschitz $C_0^2$ is precisely the space of twice continuously Fréchet differentiable function from $H$ to $\mathbb{R}$. But then, identifying $H$ with its dual,

$$DF : \Omega \mapsto C_0^{0,1}(H, H), \quad D^2 F : \Omega \mapsto C_0^{0,0}(H, L(H, H))$$

and by (well-known) non-separability of $L(H, H)$, we cannot avoid non-separability.
maps, for otherwise even the simplest properties (e.g. have a vector space of such r.v.) can fail. We refer for instance to [Coh13, Appendix E] and the references therein for more details about strong measurability.

The concept introduced in Definition 3.8 seems new even when the underlying pair \((f, f')\) is deterministic (to our best’s knowledge), see the forthcoming Section 4.5 for an application. It is also worth noticing that the condition \((f, f') \in D^\beta_\infty L_m,\infty C^\gamma_b\) reduces to “\(f \in C^\gamma_b\)” when there is no time nor sample parameter. In the next example, we give a natural recipe to build genuinely random elements.

Example 3.10. Let \((f, f') \in D^\beta_\infty L_m,\infty C^\gamma_b(W)\). Suppose that \(W = W_1 \times W_2\) and take \((Y, Y')\) a stochastic controlled rough path in \(D^\beta_\infty L_m,\gamma C^\gamma_b\). We can construct another stochastic controlled vector field on \(W_1\) through the formula

\[
\begin{align*}
  (3.23) \\
  g_t(\cdot) &= f_t(Y_t), \\
  g'_t(\cdot) &= D_2 f_t(Y_t) Y_t' + f'_t(Y_t),
\end{align*}
\]

where \(D_2\) is the derivative with respect to the argument in \(W_2\). It can be observed that \((g, g')\) belongs to \(D^\beta_\infty L_m,\infty C^\gamma_b\), for any \(\gamma < \gamma\) (see [FHL24] for a proof).

Lemma 3.11. Let \((f, f')\) be a stochastic controlled vector field in \(D^\beta_\infty L_m,\infty C^\gamma_b\), \(\gamma \in [2, 3]\), and let \((Y, Y')\) be a stochastic controlled rough path in \(D^\beta_\infty L_m,\gamma\). Define \(\beta'' = \min\{(\gamma - 2)\beta, \beta'\}\) and \((Z, Z') = (f(Y), D f(Y) Y' + f'(Y))\). Then \((Z, Z')\) is a stochastic controlled rough path in \(D^\beta_\infty L_m,\beta''\) with

\[
(3.24) \quad \|(Z, Z')\|_{X;\beta, \beta'', \gamma} \lesssim \left(\|f\|_{X;\beta, \beta''} + \|D f\|_{X;\beta, \beta''} \right) (1 + \|(Y, Y')\|_{X;\beta, \beta''}),
\]

for an implicit constant depending only on \(|I|\).

Proof. More explicitly, we establish the following estimates for each \((s, t) \in \Delta(I)\):

\[
(3.25) \quad \|\delta Z_{s,t}\|_{\mathcal{F}_s} \lesssim \left(\|\delta f\|_{\beta, \beta''} + \|D f\|_{\beta, \beta''} \right) |t - s|^{\beta'},
\]

\[
(3.26) \quad \|\mathbb{E}_s R^t_{s,t}\|_{X;\beta, \beta''} \lesssim \left(\|D f\|_{\beta, \beta''} + \|D f\|_{\beta, \beta''} \right) |t - s|^{\beta'},
\]

and

\[
(3.27) \quad \|\delta Z'_{s,t}\|_{\mathcal{F}_s} \lesssim \left(\|D f\|_{\beta, \beta''} + \|D f\|_{\beta, \beta''} \right) |t - s|^{\beta'},
\]

Since we have \(\|D f(Y_t) Y'_t\|_{\gamma} \leq \|D f\|_{\gamma} \|Y'_t\|_{\gamma} \) and \(\|f'_t(Y_t)\|_{\gamma} \leq \|f'_t\|_{\gamma} \), it is obvious on the other hand that \(Z'\) is uniformly \(L_{\gamma}\)-integrable. Thus, estimates (3.25)-(3.27) will suffice to show that \((Z, Z') \in D^\beta_\infty L_m,\gamma\), as claimed.

Now, the first of these inequalities is trivial, since by triangle inequality

\[
|f_t(Y_t) - f_s(Y_s)| \leq |f_t(Y_t) - f_s(Y_t)| + |f_s(Y_t) - f_s(Y_s)| \leq |\delta f_{s,t}| + |D f_s| \|\delta Y_{s,t}\|,
\]
leading to (3.25). To treat \( R^Z \), we write
\[
R^Z_{s,t} = f_s(Y_t) - f_s(Y_s) - Df_s(Y_s)Y_t' \delta X_{s,t}^s \\
+ f_t(Y_s) - f_s(Y_s) - f'_s(Y_s) \delta X_{s,t}^s \\
+ f_t(Y_t^s) - f_t(Y_s) - f_s(Y_t) + f_s(Y_s)
\]
(3.28)
\[
= R^f_{s,t}(Y) + R^f_{s,t}(Y) + (\delta f_{s,t}(Y_t) - \delta f_{s,t}(Y_s)).
\]
By the fundamental theorem of calculus,
\[
R^f_{s,t}(Y) = \left( \int_0^1 [Df_s(Y_s + \theta \delta Y_{s,t}) - Df_s(Y_s)] \, d\theta \right) \delta Y_{s,t} + Df_s(Y_s)R^Y_{s,t},
\]
which yields
(3.29)
\[
|E_s R^f_{s,t}(Y)| \leq \|Df_s\|_{\gamma - 2} |E_s |\delta Y_{s,t}|^{\gamma - 1} + |Df_s|_{\infty} |E_s |R^Y_{s,t} |.
\]
Applying the \( L_{n/\gamma - 1} \)-norm and triangle inequality gives
\[
\|E_s R^f_{s,t}(Y)| \leq \|Df_s\|_{\gamma - 2} \|\delta Y_{s,t}| \|F_s \|_{\gamma - 1} - 1 + |Df_s|_{\infty} \|E_s |R^Y_{s,t}| \|\frac{n}{n - 1}.
\]
From here, we obtain
(3.30)
\[
\|E_s R^f_{s,t}(Y)| \leq \|Df_s\|_{\gamma - 2} \|\delta Y_{s,t}| \|F_s \|_{\gamma - 1} - 1 + |Df_s|_{\infty} \|E_s |R^Y_{s,t}| \|\frac{n}{n - 1}.
\]
The second term in (3.28) is easily estimated by
\[
\|E_s R^f_{s,t}(Y)| \leq \|E_s R^f_{s,t}| \|R^f_{s,t}| \|F_s \|_{\gamma - 1} - 1 + |Df_s|_{\infty} \|E_s |R^Y_{s,t}| \|\frac{n}{n - 1}.
\]
For the last term in (3.28), we use the Lipschitz estimate \( |\delta f_{s,t}(Y_t) - \delta f_{s,t}(Y_s)| \leq |\delta Y_{s,t}| \|F_s \|_{\gamma - 1} - 1 + |Df_s|_{\infty} \|E_s |R^Y_{s,t}| \|\frac{n}{n - 1}.
\]
Putting these estimates in (3.28), we obtain (3.26).

Next, from the identity
\[
Df_t(Y_t)Y_t' - Df_s(Y_s)Y_s' = [Df_t(Y_t) - Df_s(Y_s)]Y_s' \\
+ [Df_t(Y_t) - Df_s(Y_t)]Y_s' + Df_t(Y_t)\delta Y_{s,t}^t
\]
we deduce that
\[
\|Df(Y_t)Y_t' - Df(Y_s)Y_s' \| \leq \|[Df_t]_{\gamma - 2} \|Y_t'||\|\delta Y_{s,t}| \|F_s \|_{\gamma - 1} - 1 + |Df_s|_{\infty} \|E_s |R^Y_{s,t}| \|\frac{n}{n - 1}.
\]
To treat the first two terms on the above right-hand side, we apply \( L_{n/\gamma - 1} \)-norm and use Hölder inequalities
\[
\|AB\|_{n - 1} \leq \|A\|_n \|B\|_{n - 2}, \quad \|AB\|_{n - 1} \leq \|A\|_n \|B\|_{n - 2}.
\]
This yields
\[ \| Df_t(Y_t)Y_t' - Df_s(Y_s)Y'_s \|_{\mathcal{F}_s} \|_{m, n} \leq \| Df_s \|_{r, \gamma - 2, \infty} \sup_r \| Y'_r \|_{n, \gamma} \| Y \|_{\beta(m, \gamma - 2), n} |t - s|^{(\gamma - 2)\beta} \]

+ \| \delta f \|_{r, \gamma - 2, \infty} \sup_r \| Y'_r \|_{n, \gamma} \| \delta Y \|_{\beta(m, \gamma - 2), n} |t - s|^{(\gamma - 2)\beta}.

Similarly

\[ | f'_t(Y_t) - f'_s(Y_s) | \leq |(f'_t - f'_s)(Y_t)| + |f'_s(Y_t) - f'_s(Y_s)| \leq |\delta f'_{s, t}|_{\infty} + |f'_s|_{\gamma - 2} |\delta Y_{s, t}|^{\gamma - 2} \]

and hence,

\[ \| f'_t(Y_t) - f'_s(Y_s) \|_{\mathcal{F}_s} \|_{m, n} \leq \| \delta f' \|_{r, \gamma - 2, \infty} \| \delta Y \|_{\beta(m, \gamma - 2), n} |t - s|^{(\gamma - 2)\beta} \]

+ \| f'_s \|_{r, \gamma - 2, \infty} \| \delta Y \|_{\beta(m, \gamma - 2), n} |t - s|^{(\gamma - 2)\beta}.

We arrive at (3.27) after observing that \( \| \delta Y \|_{\beta(m, \gamma - 2), n} \leq \| \delta Y \|_{\beta, m, n} \) and \( \| \delta Y' \|_{r, \gamma - 2, \infty} \leq \| \delta Y' \|_{\beta(m, \gamma - 2), n} \).

**Remark 3.12.** Unless \( n = \infty \) or \( Df \equiv 0 \), the estimate (3.29), and more precisely the term \( \mathbb{E}_s|\delta Y_{s, t}|^{\gamma - 1} \) therein, inevitably causes a loss of integrability from \( L_{m, n} \) to \( L_{m, \gamma - 1} \) in the composition map \( (Y, Y') \mapsto (f(Y), Df(Y)Y' + f'(Y)) \).

We now discuss in more detail the stability of stochastic controlled rough paths under compositions, so as to obtain local-Lipschitz estimates. Let \( X \) and \( \tilde{X} \) be two \( \alpha \)-Hölder paths, \( \alpha \in (\frac{\gamma}{3}, \frac{1}{2}) \).

**Proposition 3.13** (Stability of composition). Let \( m \geq 2; \gamma \in [2, 3]; \alpha \in (\frac{1}{3}, \frac{1}{2}); \alpha', \alpha'', \beta, \beta' \in (0, 1) \) be fixed numbers. Let \( X \) and \( \tilde{X} \) be two \( \alpha \)-Hölder paths. Let \( (Y, Y') \) and \( (\tilde{Y}, \tilde{Y}') \) be two elements in \( D_X^{\alpha, \beta} L_{m, \infty} \) and \( D_{\tilde{X}}^{\alpha', \beta'} L_{m, \infty} \) respectively. Assume that

\[ \| (Y, Y') \|_{X; \beta, \beta'; m, \infty} \vee \| (\tilde{Y}, \tilde{Y}') \|_{\tilde{X}; \beta, \beta'; m, \infty} \leq M < \infty. \]

Let \( \kappa \in (0, \min\{\alpha, \alpha', \beta\}) \) and \( \kappa' \in (0, \min\{\alpha'', \alpha', \beta, (\gamma - 2)\beta, \beta'\}) \). Let \( (f, f') \) and \( (\tilde{f}, \tilde{f}') \) be controlled vector fields in \( D_X^{\alpha, \alpha'} L_{m, \infty} C^{\gamma} \) and \( D_{\tilde{X}}^{\kappa', \kappa} L_{m, \infty} C^{\gamma - 1} \) respectively, with \( \gamma \in [2, 3] \). Assume that \( \langle Df, Df' \rangle \) belongs \( D_X^{\alpha', \beta''} L_{m, \infty} C^{\gamma - 1} \). Define

\[ (Z, Z') = (f(Y), Df(Y)Y' + f'(Y)) \]

and similarly for \( (\tilde{Z}, \tilde{Z}') \).

Then, recalling notations (3.4) and (3.22), we have the estimate

\[ \| Z, Z' \|_{X; \tilde{X}, \kappa, \kappa', m} \lesssim \| Y_0 - \tilde{Y}_0 \|_{m} \vee \| Y, Y' \|_{X, X; \kappa, \kappa'; m} \]

+ \| f, f'; \tilde{f}, \tilde{f}' \|_{X, X; \kappa, \kappa'; m} + \| (f - \tilde{f}, f' - \tilde{f}') \|_{\gamma - 2, m}, \]

for an implicit constant which only depends on \( M \).

**Proof.** Despite its length, the proof is elementary. We put \( \tilde{Y} = Y - \tilde{Y}, \tilde{Z} = Z - \tilde{Z}, \)

\( \tilde{f} = f - \tilde{f} \) and similarly for \( \tilde{Y}', \tilde{Z}', \tilde{f}' \).
Step 1. We show that
\begin{equation}
\|Z - \bar{Z}\|_{\kappa;m} \lesssim \|\delta\bar{f}\|_{\kappa;m} + \sup_s \|\bar{f}_s|_1\|_m + \|\bar{Y}_0\|_1 + \|\delta\bar{Y}\|_{\kappa;m},
\end{equation}
(3.32)
\begin{equation}
\|Df(Y) - D\bar{f}(\bar{Y})\|_{\kappa';m} \lesssim \|\delta Df\|_{\kappa';m} + \sup_s \|D\bar{f}_s|_{\gamma - 2}\|_m + \|\bar{Y}_0\|_1 + \|\delta\bar{Y}\|_{\kappa;m},
\end{equation}
(3.33)
\begin{equation}
\|f'(Y) - \bar{f}'(\bar{Y})\|_{\kappa';m} \lesssim \|\delta\bar{f}\|_{\kappa';m} + \sup_s \|\bar{f}_s|_{\gamma - 2}\|_m + \|\bar{Y}_0\|_1 + \|\delta\bar{Y}\|_{\kappa;m}.
\end{equation}
(3.34)

By triangle inequality
\begin{equation}
|f_s(Y_s) - \bar{f}_s(\bar{Y}_s)| \leq |f_s(Y_s) - f_s(\bar{Y}_s)| + |\bar{f}_s(\bar{Y}_s)|
\leq |f_s|_1(|Y_s - \bar{Y}_s|_1 + 1) + |\bar{f}_s|_\infty,
\end{equation}
which gives
\begin{equation}
\|Z_s - \bar{Z}_s\|_m \leq C\|f_s|_\infty\|_m + \|\bar{Y}_s\|_1 \|m\|_m.
\end{equation}
(3.35)

From \(\bar{Z}_t = (f_t(Y_s) - f_t(\bar{Y}_s)) + (\delta f_t(Y_s,t) - \delta f_t(\bar{Y}_s,t) + \bar{f}_t(\bar{Y}_t))\), we have
\begin{equation}
\delta \bar{Z}_{s,t} = \left(\delta f_{s,t}(Y_s) - \delta f_{s,t}(\bar{Y}_s)\right)
\end{equation}
\begin{equation}
+ \left(f_{t}(Y_t) - f_{t}(\bar{Y}_t) - f_t(Y_s) + f_t(\bar{Y}_s)\right)
\end{equation}
\begin{equation}
+ \left(f_{t}(Y_t) - \bar{f}_t(\bar{Y}_s)\right) =: I_1 + I_2 + I_3.
\end{equation}
(3.36)

It is easy to see that \(|I_1| \leq |\delta D f_{s,t}|_\infty \|\bar{Y}_s\|_1\) and \(|I_1| \leq 2|\delta f_{s,t}|_\infty\) so that
\begin{equation}
|I_1| \leq 2(|\delta D f_{s,t}|_\infty + |\delta f_{s,t}|_\infty)(\|\bar{Y}_s\|_1 + 1).
\end{equation}
(3.37)

Since \(\bar{Y}_s\) is \(\mathcal{F}_s\)-measurable, we have
\begin{equation}
\|I_2\|_m \leq 2\|\delta D f_{s,t}|_\infty + |\delta f_{s,t}|_\infty\|\mathcal{F}_s\|_m\|\bar{Y}_s\|_1 \|m\|_m \lesssim \|\bar{Y}_s\|_1 \|m\|_m(t - s)^{\alpha \wedge \alpha'}.
\end{equation}
Using the elementary estimate
\begin{equation}
|g(a) - g(b) - g(c) + g(d)| \leq |Dg|_1(|a - c| + |b - d|)(|c - d| + 1) + |Dg|_\infty|a - b - c + d|,
\end{equation}
we see that
\begin{equation}
|I_2| \leq |Df_t|_1(|\delta Y_s,t|_1 + |\delta \bar{Y}_s,t|_1) + |Df_t|_\infty|\delta \bar{Y}_s,t|_1.
\end{equation}
(3.38)

Hence, \(|I_2| \leq \|\bar{Y}_s\|_1 \|m\|_m(t - s)^{\beta} + \|\delta \bar{Y}_s,t\|_m\).

It is easy to see that
\begin{equation}
|I_3| \leq |\delta \bar{f}_{s,t}|_\infty + |\bar{f}_s|_{\text{Lip}}|\delta \bar{Y}_s,t|_1.
\end{equation}
(3.39)

Since \(\bar{f}_s|_{\text{Lip}}\) is \(\mathcal{F}_s\)-measurable, we have
\begin{equation}
\|\bar{f}_s|_{\text{Lip}}|\delta \bar{Y}_s,t\|_m \leq \|\bar{f}_s|_{\text{Lip}}\|_m\|\delta \bar{Y}_s,t\|_m \|\mathcal{F}_s\|_m\|_\infty,
\end{equation}
and hence
\begin{equation}
|I_3| \|m\| \lesssim \|\delta \bar{f}\|_{\kappa;m}|t - s|^\kappa + \|\bar{f}_s|_{\text{Lip}}\|_m|t - s|^\beta.
\end{equation}

Combining the estimates for \(I_1, I_2, I_3\) and (3.35), we obtain that
\begin{equation}
\|\delta \bar{Z}_{s,t}\| \lesssim \|\delta f\|_{\alpha;m} + |\delta D f|_{\alpha';m} + |\delta f_t|_1 \|\infty\|_m \|\bar{Y}_s\|_1 \|m\|_m(t - s)^{\alpha \wedge \alpha' + \beta}
\end{equation}
\begin{equation}
+ |\delta f_t|_\infty \|\delta \bar{Y}_s,t\|_m + |\delta \bar{f}\|_{\kappa;m}|t - s|^\kappa + \|\bar{f}_s|_{\text{Lip}}\|_m|t - s|^\beta.
\end{equation}
(3.34)

Noting that for every \(s\)
\begin{equation}
\|\bar{Y}_s\|_1 \|m\| \leq \|\bar{Y}_0\|_1 \|m\| + |\delta \bar{Y}|_{\kappa;m}|I|_1^\kappa,
\end{equation}
we derive (3.32) from the previous estimate. The estimates (3.33), (3.34) are obtained analogously. (Here, it is necessary to replace $|Df_s|_1$ in (3.38) by $\|D^2 f_s\|_{\gamma - 2}$ and $\|D f'_s\|_{\gamma - 2}$ respectively; replace $|f_s|_{1,p}$ in (3.39) by $|D f_s|_{\gamma - 2}$ and $|f'_s|_{\gamma - 2}$ respectively. This also justifies the restriction $\kappa' \leq \min\{\alpha', \alpha'', \beta, (\gamma - 2)/\beta\}$.)

**Step 2.** We show that

$$\| \tilde{Z}' \|_{\kappa';m} \lesssim \| \delta D \tilde{f} \|_{\kappa';m} + \| \delta \tilde{f}' \|_{\kappa';m} + \sup_s (\| D f_s \|_{\gamma - 2} + \| f'_s \|_{\gamma - 2}) m + \| Y_0 \| \wedge 1 \| m + \| \delta Y \|_{\kappa;m} + \| \tilde{Y}' \|_{\kappa';m}.$$  

(3.40)

It is elementary to verify that

$$\| \eta \zeta \|_{\kappa';m} \lesssim \| \eta \|_{\kappa';m} \sup_s \| \zeta_s \|_{\infty} + \| \delta \zeta \|_{\kappa';m,\infty}.$$  

(3.41)

From the identity

$$Df(Y)Y' - Df(\bar{Y})\bar{Y}' = (Df(Y) - Df(\bar{Y}))\bar{Y}' + Df(\bar{Y})(Y' - \bar{Y}'),$$

applying (3.33), (3.41) and the fact that $\sup_s \| \bar{Y}_s \|_{\infty} + \| \delta \bar{Y}' \|_{\kappa';m,\infty}$ and $\sup_s \| D f_s(\bar{Y}_s') \|_{\infty} + \| \delta D f(\bar{Y}') \|_{\kappa';m,\infty}$ are finite (by assumptions and Lemma 3.11), we obtain

$$\| D f(Y)Y' - D f(\bar{Y})\bar{Y}' \|_{\kappa';m} \lesssim \| \delta D \tilde{f} \|_{\kappa';m} + \sup_s \| D f_s \|_{\gamma - 2} \| m + \| Y_0 \| \wedge 1 \| m + \| \delta Y \|_{\kappa;m} + \| \tilde{Y}' \|_{\kappa';m}.$$  

This estimate and (3.34) yield (3.40).

**Step 3.** We show that

$$\| \mathbb{E}_s R^Z - \mathbb{E}_s \tilde{R}^Z \|_{\kappa + \kappa';m} \lesssim \| Y_0 \| \wedge 1 \| m + \| \delta \bar{Y} \|_{\kappa;m} + \sup_s \| \bar{Y}' \|_m$$

$$+ \| \mathbb{E}_s R^Y - \mathbb{E}_s \tilde{R}^Y \|_{\kappa + \kappa';m} + \sup_s \| D f_s \|_{\gamma - 2} \| m + \| \delta D \tilde{f} \|_{\kappa';m} + \| \mathbb{E}_s R' - \mathbb{E}_s \tilde{R}' \|_{\kappa + \kappa';m}.$$  

(3.42)

Similar to (3.28), we write

$$R^Z_{s,t} = T f_s(Y_s, Y_t) + D f_s(Y_s)[R^Y_{s,t}] + R^f_{s,t}(Y_s) + (\delta f_{s,t}(Y_t) - \delta f_{s,t}(Y_s))$$

(3.43)

where

$$Th(\xi, \eta) = h(\eta) - h(\xi) - Dh(\xi)[\eta - \xi].$$

We decompose $R^Z_{s,t}$ in an analogous way. We estimate separately the differences of the corresponding terms on the right-hand sides of the two decompositions.

We have

$$T f_s(Y_s, Y_t) - T \tilde{f}_s(\bar{Y}_s, \bar{Y}_t) = T \tilde{f}_s(\bar{Y}_s, \bar{Y}_t) + T f_s(Y_s, Y_t) - T f_s(\bar{Y}_s, \bar{Y}_t).$$

By Taylor’s expansion, it is evident that

$$| T \tilde{f}_s(\bar{Y}_s, \bar{Y}_t) | \leq | \tilde{f}_s |_{\gamma - 1} | \delta \bar{Y}_{s,t} |^{-1},$$

and hence,

$$\| \mathbb{E}_s T \tilde{f}_s(\bar{Y}_s, \bar{Y}_t) \|_m \leq \| \tilde{f}_s \|_{\gamma - 1} \| m(t - s)^{(\gamma - 1)/\beta}.$$
Next, we put \( Y^\theta = \theta Y + (1 - \theta)\bar{Y} \). When \( \gamma > 2 \), we apply the fundamental theorem of calculus to get that

\[
Tf_s(Y_s, Y_t) - Tf_s(\bar{Y}_s, \bar{Y}_t) = \int_0^1 \frac{d}{d\theta} \left( f_s(Y^\theta_s) - f_s(Y^\theta_s) - Df_s(Y^\theta_s)\delta s,t \right) d\theta
\]

\[
= \int_0^1 \left( Df_s(Y^\theta_s)[\bar{Y}_s] - Df_s(Y^\theta_s)[\bar{Y}_s] - Df_s(\bar{Y}_s)[\delta \bar{Y}_s,t] - D^2f_s(Y^\theta_s)[\bar{Y}_s, \delta Y^\theta_s,t] \right) d\theta
\]

\[
= \int_0^1 \left( Df_s(Y^\theta_s)[\bar{Y}_s] - Df_s(Y^\theta_s)[\bar{Y}_s] - D^2f_s(Y^\theta_s)[\bar{Y}_s, \delta Y^\theta_s,t] \right) d\theta
\]

Using regularity of \( Df_s \), we get that

\[
|Tf_s(Y_s, Y_t) - Tf_s(\bar{Y}_s, \bar{Y}_t)| \lesssim |Df_s|_{\gamma - 1} \int_0^1 \left( |\bar{Y}_s| ||\delta Y^\theta_s,t||^{\gamma - 1} + ||\delta \bar{Y}_s,t||^{\gamma - 1} + ||\delta \bar{Y}_s,t|| + ||\bar{Y}_s|| \right) d\theta
\]

On the other hand, we also have \(|Tf_s(Y_s, Y_t) - Tf_s(\bar{Y}_s, \bar{Y}_t)| \lesssim ||\delta Y^\theta_s,t||^{\gamma - 1} + ||\delta \bar{Y}_s,t||^{\gamma - 1} \). Thus, we have

\[
|Tf_s(Y_s, Y_t) - Tf_s(\bar{Y}_s, \bar{Y}_t)| \lesssim (||\bar{Y}_s|| + 1)(||\delta Y^\theta_s,t||^{\gamma - 1} + ||\delta \bar{Y}_s,t||^{\gamma - 1} + ||\delta \bar{Y}_s,t|| + ||\bar{Y}_s||)
\]

which implies that

\[
||E_s(Tf_s(Y_s, Y_t) - Tf_s(\bar{Y}_s, \bar{Y}_t))|| \lesssim ||\bar{Y}_s|| + 1||m(t-s)^{(\gamma - 1)} + ||\delta \bar{Y}_s,t||^\gamma m(t-s)^\beta.
\]

When \( \gamma = 2 \), \( D^2f(s) \) does not exist and we treat the first two terms in \( Tf_s \) separately. Indeed, similar to (3.38), we have

\[
||f_s(Y_s) - f_s(\bar{Y}_s) - (f_s(Y_t) - f_s(\bar{Y}_t))|| \lesssim ||\bar{Y}_s|| + 1||m(t-s)^{\beta} + ||\delta \bar{Y}_s,t||^\gamma m.
\]

Using regularity of \( f(s) \), it is evident that

\[
|Df_s(Y_s)[\delta Y^\theta_s,t] - Df_s(\bar{Y}_s)[\delta \bar{Y}_s,t]| \lesssim (||\bar{Y}_s|| + 1)||\delta Y^\theta_s,t|| + ||\delta \bar{Y}_s,t||
\]

which implies that

\[
||Df_s(Y_s)[\delta Y^\theta_s,t] - Df_s(\bar{Y}_s)[\delta \bar{Y}_s,t]|| \lesssim ||\bar{Y}_s|| + 1||m(t-s)^{\beta} + ||\delta \bar{Y}_s,t||^\gamma m
\]

and hence,

\[
||E_s(Tf_s(Y_s, Y_t) - Tf_s(\bar{Y}_s, \bar{Y}_t))|| \lesssim ||\bar{Y}_s|| + 1||m(t-s)^{\beta} + ||\delta \bar{Y}_s,t||^\gamma m.
\]

Thus, for each \( \gamma \in [2, 3] \), we have

\[
||E_s(Tf_s(Y_s, Y_t) - Tf_s(\bar{Y}_s, \bar{Y}_t))|| \lesssim ||\bar{Y}_s|| + 1||m(t-s)^{(\gamma - 1)} + ||\delta \bar{Y}_s,t||^\gamma m(t-s)^{\min(\beta,(\gamma-2))^{\beta}}.
\]

It follows that

\[
||E_s(Tf_s(Y_s, Y_t) - Tf_s(\bar{Y}_s, \bar{Y}_t))|| \lesssim (||\bar{Y}_s|| + 1)||m(t-s)^{(\gamma - 1)} + ||\delta \bar{Y}_s,t||^\gamma m(t-s)^{\min(\beta,(\gamma-2))^{\beta}}.
\]

For the difference corresponding to the second term in (3.43), we note that

\[
E_s(Df_s(Y_s)[R^Y_{s,t}] - Df_s(\bar{Y}_s)[\bar{R}^Y_{s,t}]) = Df_s(\bar{Y}_s)[E_s\bar{R}^Y_{s,t}] + (Df_s(Y_s)[E_sR^Y_{s,t}] - Df_s(\bar{Y}_s)[E_s\bar{R}^Y_{s,t}])
\]

Using regularity of \( f(s) \), we have
We note that which implies (3.43). Taking into account the regularity of \( Y, \bar{Y} \), we have

\[
\|E_s(Df_s(Y_s)[R^Y_{s,t}] - D\bar{f}_s(\bar{Y}_s)[\bar{R}^Y_{s,t}])\|_m \\
\lesssim (\|Df_s\|_\infty \|E_s\bar{R}^Y_{s,t}\| + \|Df_s\|_\infty \|E_s(R^Y_{s,t} - \bar{R}^Y_{s,t})\|) \tag{3.43}.
\]

For the difference corresponding to the last term in (3.43), we write

\[
R^f_{s,t}(Y_s) - R^f_{s,t}(\bar{Y}_s) = (R^f_{s,t} - R^f_{s,t})(Y_s) + R^f_{s,t}(Y_s) - R^f_{s,t}(\bar{Y}_s).
\]

We note that

\[
|E_s(R^f_{s,t}(Y_s) - R^f_{s,t}(\bar{Y}_s))| \leq 2\|E_sR^f\|_{\alpha+\alpha':\infty} (t-s)^{\alpha+\alpha'}
\]

and by the fundamental theorem of calculus that

\[
|E_s(R^f_{s,t}(Y_s) - R^f_{s,t}(\bar{Y}_s))| = \left| \int_0^1 R^{Df}_{s,t}(\theta Y_s + (1-\theta)\bar{Y}_s)[\bar{Y}_s]d\theta \right| \\
\leq \|E_sR^{Df}\|_{\alpha' + \alpha''; \infty} (t-s)^{\alpha' + \alpha''} |\bar{Y}_s|.
\]

Combining the previous inequalities, we obtain that

\[
\|\|E_s(R^f_{s,t}(Y_s) - R^f_{s,t}(\bar{Y}_s))\|_\infty \|_m \lesssim \|E_s(R^f_{s,t} - R^f_{s,t})\|_m + \|\bar{Y}_s\|_\infty 1 \|m(t-s)^{\alpha'+\alpha''}.
\]

For the difference corresponding to the last term in (3.43), we write

\[
\delta f_{s,t}(Y_t) - \delta f_{s,t}(\bar{Y}_t) - (\delta \bar{f}_{s,t}(Y_t) - \delta \bar{f}_{s,t}(\bar{Y}_t)) \\
= \delta \bar{f}_{s,t}(\bar{Y}_t) - \delta \bar{f}_{s,t}(\bar{Y}_t) + \left[ \delta f_{s,t}(Y_t) - \delta f_{s,t}(Y_t) - (\delta f_{s,t}(Y_t) - \delta f_{s,t}(\bar{Y}_t)) \right].
\]

Similar to (3.38), we have

\[
|\delta f_{s,t}(Y_t) - \delta f_{s,t}(Y_t) - (\delta \bar{f}_{s,t}(Y_t) - \delta \bar{f}_{s,t}(\bar{Y}_t))| \\
\lesssim |\delta \bar{f}_{s,t}|_1 |\delta \bar{Y}_{s,t}| + |\delta Df_{s,t}|_1 (|\delta Y_{s,t}| + |\delta \bar{Y}_{s,t}|) (|\bar{Y}_s| + 1) + |\delta Df_{s,t}|_\infty |\delta \bar{Y}_{s,t}|.
\]

Taking into account the regularity of \( f \) and \( Y, \bar{Y} \), we obtain that

\[
\|\|E_s[\delta f_{s,t}(Y_t) - \delta f_{s,t}(Y_t) - (\delta \bar{f}_{s,t}(Y_t) - \delta \bar{f}_{s,t}(\bar{Y}_t))]|_m \|_m \\
\lesssim \|\|\delta \bar{f}_{s,t}|_1 |\delta \bar{Y}_{s,t}| + |\delta Df_{s,t}|_1 (|\delta Y_{s,t}| + |\delta \bar{Y}_{s,t}|) (|\bar{Y}_s| + 1) + |\delta Df_{s,t}|_\infty |\delta \bar{Y}_{s,t}| |m(t-s)^{\beta} + \|\bar{Y}_s\|_\infty 1 \|m(t-s)^{\alpha''+\beta} + |\delta \bar{Y}_{s,t}| |m(t-s)^{\alpha'}.
\]

Summing up the estimates for all the differences, we obtain that

\[
\|E_sR^Z_{s,t} - E_s\bar{R}^Z_{s,t}|_m \lesssim \|\|\bar{Y}_s\|_\infty 1 \|m(t-s)^{\beta} + \|\bar{Y}_s\|_\infty 1 \|m(t-s)^{\alpha''+\beta} + |\delta \bar{Y}_{s,t}| |m(t-s)^{\alpha'}.
\]

(3.44)

which implies (3.42).

\textbf{Conclusion.} Combining (3.32), (3.40), (3.42) we obtain (3.31). \qed
4. Rough stochastic differential equations

Let $\Omega = (\Omega, \mathcal{G}, \mathbb{P}; \{\mathcal{F}_t\})$ be a stochastic basis, $B$ be a standard $\{\mathcal{F}_t\}$-Brownian motion in $\hat{V}$, $X = (X, \mathcal{K})$ be a deterministic rough path in $\mathcal{C}^\alpha(V)$ with $\alpha \in (\frac{1}{2}, \frac{1}{2})$. We consider the rough stochastic differential equation

\[(4.1) \quad dY_t(\omega) = b_t(\omega, Y_t(\omega))dt + \sigma_t(\omega, Y_t(\omega))dB_t(\omega) + (f_t, f'_t)(\omega, Y_t(\omega))dX_t, \quad t \in [0, T].\]

We are given a drift nonlinearity $b$ when the starting position $Y_0 = \xi$ is specified, we say that $Y$ is a solution starting from $\xi$. We assume further that $b, \sigma, f, f'$ are random bounded continuous functions, in the sense of the next definition.

**Definition 4.1.** Let $W, \hat{W}$ be some finite dimensional Euclidean spaces and fix a Borel set $S \subset W$. Let $(t, \omega) \mapsto g_t(\omega, \cdot)$ be a progressively measurable stochastic process from $\Omega \times [0, T] \rightarrow \mathcal{C}_b(S; \hat{W})$ and bounded as defined in Section 2. We say that:

(a) $g$ is **random bounded continuous** if is uniformly bounded, namely, there exists a deterministic constant $\|g\|_\infty$ such that

$$\sup_{t \in [0, T]} \sup_{\omega \in \Omega} \sup_{x \in S} |g_t(\omega, x)| \leq \|g\|_\infty.$$

(b) $g$ is **random bounded Lipschitz** if it is random bounded continuous, progressively measurable from $\Omega \times [0, T] \rightarrow \mathcal{C}_b(S; \hat{W})$ and uniformly bounded in the sense that

$$\sup_{t \in [0, T]} \sup_{\omega \in \Omega} \sup_{x, \bar{x} \in S} \frac{|g_t(\omega, x) - g_t(\omega, \bar{x})|}{|x - \bar{x}|} \leq \|g\|_{Lip}$$

for some constant $\|g\|_{Lip}$.

We give the definition of $L_{m,n}$-integrable solutions, make in particular use of the space of stochastic controlled rough paths from Definition 3.1.

**Definition 4.2** (Integrable solutions). Let $m, n$ be (extended) real numbers such that $m \in [2, \infty)$ and $n \in [m, \infty)$. An $L_{m,n}$-integrable solution of (4.1) over $[0, T]$ is a continuous $\{\mathcal{F}_t\}$-adapted process $Y$ such that the following conditions are satisfied

(a) $\int_0^T |b_r(Y_r)|dr$ and $\int_0^T |(\sigma^\alpha)_r(Y_r)|dr$ are finite a.s.;

(b) $(f(Y), Df(Y)f(Y) + f'(Y))$ belongs to $\mathbf{D}_X^{\tilde{\alpha}, \tilde{\alpha}'} L_{m,n}([0, T], \Omega; \mathcal{L}(V, \hat{W}))$ for some $\tilde{\alpha}, \tilde{\alpha}' \in (0, 1) : \alpha + (\alpha \wedge \tilde{\alpha}) > \frac{1}{2}, \alpha + (\alpha \wedge \tilde{\alpha}) + \tilde{\alpha}' > 1$;

(c) $Y$ satisfies the following stochastic Davie-type expansion

\[(4.2) \quad \|\|J_{s,t}\|_{\mathcal{F}_s}\|_n = o(t - s)^{1/2} \quad \text{and} \quad \|\mathbb{E}_s J_{s,t}\|_n = o(t - s)\]

for every $(s, t) \in \Delta$, where

\[J_{s,t} = \delta Y_{s,t} - \int_s^t b_r(Y_r)dr - \int_s^t \sigma_r(Y_r)dB_r - f_s(Y_s)\delta X_{s,t} - (Df_s(Y_s)f_s(Y_s) + f'_s(Y_s))X_{s,t}.\]

\[(4.3) \quad - \int_s^t b_r(Y_r)dr - \int_s^t \sigma_r(Y_r)dB_r - f_s(Y_s)\delta X_{s,t} - (Df_s(Y_s)f_s(Y_s) + f'_s(Y_s))X_{s,t}.\]

When the starting position $Y_0 = \xi$ is specified, we say that $Y$ is a solution starting from $\xi$. 

\[\text{Take } \alpha = \tilde{\alpha} = \tilde{\alpha}' \text{ at first reading.}\]
We begin by showing that a solution to (4.1) satisfies an integral equation, therefore, providing a dynamical description which is equivalent to the local description of Definition 4.2. The deterministic counterpart of this characterization appears in [Dav07].

**Proposition 4.3.** $Y$ is an $L_{m,n}$-integrable solution of (4.1) if and only if (a)-(b) of Definition 4.2 hold and for a.s. $\omega$,

\begin{equation}
Y_t = Y_0 + \int_0^t b_r(Y_r)dr + \int_0^t \sigma_r(Y_r)dB_r + \int_0^t f_r(Y)dX_r \text{ for all } t \in [0,T].
\end{equation}

Furthermore, in this case, we have for any $(s,t) \in \Delta$

\begin{equation}
\|J_{s,t}|F_\omega\|_n \lesssim |t-s|^{\alpha + (\alpha \wedge \bar{a})} \text{ and } \|E_s J_{s,t}\|_n \lesssim |t-s|^{\alpha + (\alpha \wedge \bar{a}) + \bar{a}'}.
\end{equation}

**Proof.** Assume first that (a) and (b) of Definition 4.2 hold. Define $A_{s,t} := f_s(Y_s)\delta X_{s,t} + (Df_s(Y_s)f_s(Y_s) + f'_s(Y_s))X_{s,t}$ and

\[Z_t = Y_t - Y_0 - \int_0^t b_r(Y_r)dr - \int_0^t \sigma_r(Y_r)dB_r.\]

Since $(f(Y), Df(Y)f(Y) + f'(Y))$ belongs to $D^\alpha_{\bar{a}, \bar{a}'} L_{m,n}$, we can apply Theorem 3.5 to define the rough stochastic integral $\mathcal{A} := \int_0^t f(Y)dX$ which then satisfies

\begin{equation}
\|\delta A_{s,t} - A_{s,t}|F_\omega\|_n \lesssim |t-s|^{\alpha + (\alpha \wedge \bar{a})} \text{ and } \|E_s(\delta A_{s,t} - A_{s,t})\|_n \lesssim |t-s|^{\alpha + (\alpha \wedge \bar{a}) + \bar{a}'}
\end{equation}

for every $(s,t) \in \Delta$. Now, suppose that $Y$ is a $L_{m,n}$-integrable solution. We can combine (4.6) with (4.2) and (4.3) to obtain that

\[\|\delta Z_{s,t} - \delta A_{s,t}|F_\omega\|_n = o(|t-s|^{\frac{1}{2}}) \text{ and } \|E_s(\delta Z_{s,t} - \delta A_{s,t})\|_n = o(|t-s|).\]

The previous estimates imply (by [Lê23, Lemma 3.5]) that $Z_t = A_t$ a.s. for every $t \in [0,T]$. Since both $Z$ and $\mathcal{A}$ are continuous, they are indistinguishable, which means that (4.4) holds. This shows the necessity.

Sufficiency is evident from the fact that if (4.4) holds then together with (4.6), it implies (4.2). That $Y$ is a.s. continuous is evident from (4.4). Hence, we have shown that $Y$ is an $L_{m,n}$-integrable solution. At last, observing that (4.6) implies (4.5), we conclude the proof. \qed

**Remark 4.4.** When $X$ belongs to $\mathcal{C}^\alpha$, $\alpha \in \left(\frac{1}{2}, \frac{1}{2}\right]$ and

\[\sup_{t \in [0,T]} \|g_s(Y_s)\|_n < \infty, \forall g \in \{b, \sigma \sigma^t, f, Dff, f'f\},\]

the estimates in (4.5) and (4.3) imply that

\begin{equation}
\|\delta Y_{s,t}|F_\omega\|_n \lesssim |t-s|^{\alpha} \text{ and } \|E_s(\delta Y_{s,t} - f_s(Y_s)\delta X_{s,t})\|_n \lesssim |t-s|^{2\alpha}.
\end{equation}

In this case, any $L_{m,n}$-integrable solution to (4.1) satisfying $f(Y) \in C^3 L_{m,n}$ for some $\beta \in (0, \alpha]$ necessarily belongs to $D_{X}^{\alpha + \beta} L_{m,n}$.

Each $L_{m,\infty}$-solution is bounded in the following sense.

**Proposition 4.5 (A priori estimates).** Suppose that $b, \sigma$ are random bounded continuous and $(f, f')$ belongs to $D^{\alpha + \beta}_{L_{m,\infty}} C^{\gamma-1}_{\frac{1}{2}}$ with $\beta \in (0, \alpha]$, $\beta' \in (0, 1]$ and $\gamma \in (2, 3]$ such that $\alpha + (\gamma - 1)\beta > 1$ and $\alpha + \beta + \beta' > 1$. Let $Y$ be an $L_{m,\infty}$-solution to (4.1) and take any finite constant $M$ such that

\[\|(f, f')\|_{\gamma-1; \infty; [0,T]} + \|[f, f']\|_{X; \beta, \beta'; m, \infty; [0,T]} \leq M.\]
Then, there exists a constant $C$ depending only on $T, m, \alpha, \beta, \beta', \gamma$ such that

\begin{equation}
\|\delta Y\|_{\alpha;m,\infty;[0,T]} + \|\mathbb{E}_s R^Y\|_{\alpha+\beta';\infty;[0,T]} \leq C(1 + \|b\|_{\infty} + \|\sigma\|_{\infty} + M\|X\|_{\alpha})^{-1/\beta''},
\end{equation}

where $\beta'' = \min\{\beta, (\gamma - 2)\beta, \beta'\}$ and $R^Y = \delta Y - f(Y)\delta X$. Furthermore, we have

\begin{align}
\|Y, Y'\|_{X^\beta;\beta', m, \infty} &\leq C M(1 + \|b\|_{\infty} + \|\sigma\|_{\infty} + M\|X\|_{\alpha})^{-1/\beta''}, \\
\|Z, Z'\|_{X^\beta;\beta', m, \infty} &\leq C M^2(1 + \|b\|_{\infty} + \|\sigma\|_{\infty} + M\|X\|_{\alpha})^{-1/\beta''},
\end{align}

where $Z = f(Y)$, $Z' = (Df f(Y) + f'(Y))$ and $\tilde{C}$ is a constant depending only on $T, m, \alpha, \beta, \beta', \gamma$.

**Proof.** We observe that (4.9) and (4.10) are direct consequences of (4.8) and (3.24), hence it suffices to show (4.8). To this aim, our strategy is to obtain a closed argument from Theorem 3.5 and Proposition 3.13. Without loss of generality, we can and will assume that $\beta' \leq \beta$. Moreover, by working with $(MX, M^2X)$ and $f/M$ instead of $(X, X)$ and $f$, we can also assume that $M = 1$. All implicit constants herein depend only on $T, m, \alpha, \beta, \beta', \gamma$.

**Step 1: local estimate** As in Remark 4.4, we observe that for every $(s, t) \in \Delta$,

\[\|\delta Y_{s,t}|F_s\|_{m,\infty} \lesssim (t-s)^{\alpha} \quad \text{and} \quad \|\mathbb{E}_s(\delta Y_{s,t} - f_t(Y)\delta X_{s,t})\|_{\infty} \lesssim (t-s)^{2\alpha}.\]

while

\[\|\delta(f(Y))_{s,t}|F_s\|_{m,\infty} \lesssim (t-s)^{\beta}.\]

These estimates show that $(Y, f(Y))$ belongs to $D^\alpha_\beta L_{m,\infty} \subset D^\beta_\beta L_{m,\infty}$. Applying Lemma 3.11, we see that $(Z, Z') = (f(Y), Df f(Y) + f'(Y))$ forms indeed a stochastic controlled rough path in $D^\beta_\beta L_{m,\infty}$, where $\beta'' = \min((\gamma - 2)\beta, \beta')$. Moreover, our assumptions on $\alpha, \beta, \beta''$ ensure that $\alpha + \beta > 1/2$ and $\alpha + \beta + \beta'' > 1$. By Theorem 3.5 and Proposition 4.3 with $\tilde{\alpha} = \beta$ and $\tilde{\alpha}' = \beta''$, for every $(s, t) \in \Delta$, one has

\begin{equation}
\|J_{s,t}|F_s\|_{m,\infty} \lesssim \Gamma^{\beta,\beta'';m,\infty}(X; Z, Z'; [s, t])|t-s|^{\alpha+\beta+\beta''} + \Gamma^{\beta,\beta'';m,\infty}(X; Z, Z'; [s, t])|t-s|^{\alpha+\beta}
\end{equation}

and

\begin{equation}
\|\mathbb{E}_s J_{s,t}\|_{\infty} \lesssim \Gamma^{\beta,\beta'';m,\infty}(X; Z, Z'; [s, t])|t-s|^{\alpha+\beta+\beta''}.
\end{equation}

Observe further that from (4.3),

\[\mathbb{E}_s R^Y_{s,t} = \mathbb{E}_s J_{s,t} + \mathbb{E}_s \int_s^t b_r(Y_r)dr + (Df_s(Y_s)f_s(Y_s) + f'_s(Y_s))X_{s,t}.\]

Thus, taking into account (4.12), we obtain from the above identity and the proof of Lemma 3.11 that

\begin{align}
\|\mathbb{E}_s R^Y_{s,t}\|_{\infty} &\lesssim \left[\|X\|_{\alpha}(1 \vee \|\delta Y\|_{\beta; m, \infty; [s, t]})^{\gamma-1} + \|X\|_{\alpha}\|\mathbb{E}_s R^Y\|_{\beta+\beta'; \infty; [s, t]} \right. \\
&\quad + \|X\|_{\alpha}^2(1 \vee \|\delta Y\|_{\beta; m, \infty; [s, t]})^{-2}||t-s|^{\alpha+\beta+\beta''} + \|b\|_{\infty}|t-s| + |X|_{\alpha}|t-s|^{2\alpha}.
\end{align}

We choose $t > 0$ sufficiently small such that

\begin{equation}
\|X\|_{\alpha} t^{\beta''} \ll 1.
\end{equation}

Since $\alpha \geq \beta \geq \beta' \geq \beta''$, we derive from (4.13) and (4.14) that

\begin{equation}
\|\mathbb{E}_s R^Y\|_{\beta+\beta'; \infty; [s, t]} \lesssim (\|\delta Y\|_{\beta; m, \infty; [s, t]} \vee 1)^{\gamma-1} + \|X\|_{\alpha}(\|\delta Y\|_{\beta; m, \infty; [s, t]} \vee 1)^{\gamma-2} + \|b\|_{\infty} + |X|_{\alpha}.
\end{equation}
for all \((s, t)\) satisfying \(0 \leq t - s \leq \ell\). From Proposition 3.13, we apply (4.14), (4.15) and the trivial bound \((1 \vee \|\delta Y\|_{\beta;m,\infty|[s,t]})^{\gamma-2} \leq (1 \vee \|\delta Y\|_{\beta;m,\infty|[s,t]})^{\gamma-1} + \|b\|_\infty + |X|_{2\alpha} |t - s|^{\alpha + \beta'} \) to obtain that
\[
\Gamma^{\beta,\beta'}_1 (X; Z, Z'; [s, t]) |t - s|^{\alpha + \beta'} \lesssim \left[ (1 \vee \|\delta Y\|_{\beta;m,\infty|[s,t]})^{\gamma-1} + \|b\|_\infty + |X|_{2\alpha} |t - s|^{\alpha + \beta'} \right] \\
\left( \|X\|_\alpha \ell^{\beta''} \right) \|\delta Y\|_{\beta;m,\infty|[s,t]}) |t - s|^{\alpha}
\]
and
\[
\Gamma^{\beta,\beta'}_2 (X; Z, Z'; [s, t]) |t - s|^{\alpha + \beta} \lesssim \left( \|X\|_\alpha \ell^{\beta''} \right) \|\delta Y\|_{\beta;m,\infty|[s,t]}) |t - s|^{\alpha + \beta} + \|X\|_\alpha |t - s|^{\alpha}.
\]
From the identity
\[
\delta Y_{s,t} = J_{s,t} + \int_s^t b_r(Y_r)dr + \int_s^t \sigma_r(Y_r)dB_r + f_s(Y_s)\delta X_{s,t} + (Df_s(Y_s)f_s(Y_s) + \delta f_s(Y_s))\delta Y_{s,t},
\]
we apply (4.11), the previous estimates for \(\Gamma^{\beta,\beta';m,\infty}_1, \Gamma^{\beta,\beta';m,\infty}_2\) and the bounds
\[
\left| \int_s^t b_r(Y_r)dr \right| \leq \|b\|_\infty |t - s|, \quad \left| \int_s^t \sigma_r(Y_r)dB_r \right| \lesssim \|\sigma\|_\infty |t - s|^{1/2}
\]
to obtain that
\[
\|\delta Y_{s,t} \| F_s \|_{m,\infty} \lesssim \left( \|\delta Y\|_{\alpha;m,\infty|[s,t]}) \vee 1 \right)^{\gamma-1} |t - s|^{\alpha + \beta'} \]
\[
+ \left( \|X\|_\alpha \ell^{\beta''} \|\delta Y\|_{\beta;m,\infty|[s,t]}) \vee 1 \right) + \|b\|_\infty + \|\sigma\|_\infty + \|X\|_\alpha |t - s|^{\alpha}
\]
and hence, using (4.14) again, to see that
\[
\left( \|\delta Y\|_{\alpha;m,\infty|[s,t]}) \vee 1 \right) \lesssim \left( \|\delta Y\|_{\alpha;m,\infty|[s,t]}) \vee 1 \right)^{\gamma-1} \ell^{\beta''} + 1 + \|b\|_\infty + \|\sigma\|_\infty + \|X\|_\alpha
\]
for every \((s, t)\) \(\in \Delta\) satisfying \(0 \leq t - s \leq \ell\). Reasoning as in [FH20, Chapter 8.4], we have
\[
\|\delta Y\|_{\alpha;m,\infty|[s,t]}) \vee 1 \lesssim 1 + \|b\|_\infty + \|\sigma\|_\infty + \|X\|_\alpha
\]
whenever \(0 \leq t - s \leq \ell\). Plugging (4.17) in (4.15) (noting that \(\gamma \in (2, 3]\)), then back into (4.13), we obtain
\[
\|E.R^Y\|_{\alpha + \beta;\infty|[s,t]} \lesssim (1 + \|b\|_\infty + \|\sigma\|_\infty + \|X\|_\alpha) \|X\|_\alpha \|^{(1-\alpha)/\beta''},
\]
which implies the first part of (4.8).

**Step 2: extension over the whole interval** \([0, T]\). If \(s \leq t\) are fixed, then for any partition \(\{\tau_i\}_{i=0}^N\) of \([s, t]\), we have by a telescopic argument (using \(\delta Y_{s,u} = -\delta Y_{s,u} \delta X_{u,t}\)) that
\[
R^Y_{s,t} = \sum_{i=0}^{N-1} \left( R^Y_{\tau_i, \tau_{i+1}} - \delta Y_{\tau_i, \tau_{i+1}} \delta X_{\tau_i, \tau_{i+1}} \right).
\]
Using triangle inequality and bounding the conditional expectations in an obvious way, we have
\[
\|E.R^Y_{s,t}\| \leq \sum_{i=0}^{N-1} \left( \|E.\delta Y_{\tau_i, \tau_{i+1}} \|_\infty + \|F_{\tau_i}|m\|_{\infty} |X|_{\alpha;[0,T]} |t-s|^{\alpha}\right).
\]
From here, the estimates on small intervals (4.17) and (4.18) can be combined to obtain
\[
\|E.R^Y\|_{\alpha + \beta;\infty|[s,t]} \lesssim (1 + \|b\|_\infty + \|\sigma\|_\infty + \|X\|_\alpha) \|X\|_\alpha \|^{(1-\alpha)/\beta''} + \|X\|_\alpha \|^{(1-2\alpha)/\beta''} |t-s|^{\alpha + \beta}.
\]
This yields the estimate for \(\|E.R^Y\|_{\alpha + \beta;\infty}\) in (4.8), completing the proof. \(\square\)
4.1. Existence and uniqueness. In this section, we construct a solution to (4.1) by a fixed-point argument.

**Theorem 4.6.** Let $m$ be in $[2, \infty)$ and $X \in \mathcal{C}^\alpha$ with $\frac{1}{3} < \alpha \leq \frac{4}{5}$. Let $b, \sigma$ be random bounded Lipschitz functions, assume that $(f, f')$ belongs to $\textbf{D}^{2\alpha} L_{m, \infty} \mathcal{C}_b^{\gamma - 1}$ while $(Df, DF')$ belongs to $\textbf{D}_X^{\alpha''} L_{m, \infty} \mathcal{C}_b^{\gamma - 1}$. Assume moreover that $\gamma > \frac{1}{\alpha}$ and $2\alpha + \alpha'' > 1$. Then for every $\xi \in L_0(F_0; W)$, there exists a unique $L_{m, \infty}$-integrable solution to (4.1) starting from $\xi$ over any finite time interval.

**Remark 4.7.** No integrability condition is required on $Y_0 = \xi$. Also, since $L_{m, \infty}$-integrability implies $L_{2, \infty}$-integrability, it is clear that uniqueness of $L_{m, \infty}$-solutions also holds within the wider class of $L_{2, \infty}$-solutions.

**Corollary 4.8.** Let $Y$ be the solution of Theorem 4.6. Then $Y$ satisfies the exponential estimate (2.10) (with $X = W$).

**Proof.** From Remark 4.4, $\delta Y$ belongs to $\mathcal{C}^\alpha L_{m, \infty}$. Being a solution, $Y$ is a.s. continuous and hence the result is a direct consequence of Proposition 2.7. □

Our method deviates from the familiar one for rough differential equations (e.g. [FH20, Ch.8]) in several ways. The highly non-trivial part is to identify a suitable metrics on the space of stochastic controlled rough paths for which a fixed-point theorem can be applied. All estimates, e.g. those for obtained in Theorem 3.5 for rough stochastic integrals, have already been prepared in this way. As already alluded in Remark 3.12, unless $n = \infty$, a loss of integrability (from $(\gamma - 1)n$ down to $n$) appears in the estimates of Lemma 3.11. For this reason, the invariance property of the fixed point map needs to be established on a bounded set of $\textbf{D}^{2\alpha} L_{m, \infty}$. As is quickly realized, however, the corresponding distance is too strong to yield any contraction property, which leads us to a weaker metric.

**Proof of Theorem 4.6.** Replacing $\gamma$ by $\gamma \wedge 3$ if necessary, have $2 \leq 1/\alpha < \gamma \leq 3$. We first construct a local solution, on $[0, T]$ for $T$ small. It suffices to construct a process $(Y, f(Y))$ in $\textbf{D}^{\beta, \beta'} L_{m, \infty}$ such that $Y$ is $\mathbb{P}$-a.s. continuous and the integral equation (4.4) is satisfied for some $\beta \in (\frac{1}{3}, \alpha)$, $\beta' \in (0, \beta)$ with\textsuperscript{15}

$$2\beta + \beta' > 1 \quad \text{and} \quad 2\beta + \alpha'' > 1.$$ 

Indeed, if $(Y, f(Y))$ is such process, then from Lemma 3.11, we see that $(f(Y), Df(Y) f(Y) + f'(Y))$ belongs to $\textbf{D}^{\beta, \beta''} L_{m, \infty}$, where $\beta'' = \min\{\beta', (\gamma - 2)\beta\}$. The conditions on $\beta, \beta'$ ensure that $\alpha + (\alpha \wedge \beta) > 1/2$ and $\alpha + (\alpha \wedge \beta + \beta' > 1$. Hence, by Proposition 4.3, $Y$ is an $L_{m, \infty}$-integrable solution to (4.1). From the conditions on $\alpha, \alpha''$, we can further assume that $\beta' \leq \min\{\alpha'', (\gamma - 2)\beta\}$.

Having $\beta, \beta'$ chosen as previously, we pick a constant

$$M > \|b\|_{\infty} + \|\sigma\|_{\infty} + \|(f, f')\|_{2; \infty} + \|(f, f')\|_{X; \beta, \beta'; m, \infty}$$

and define $\mathcal{B}_T$ as the collection of processes $(Y, Y')$ in $\textbf{D}^{\beta, \beta''} L_{m, \infty}([0, T], \Omega; W)$ such that $Y_0 = \xi, Y'_0 = f_0(\xi)$,

$$\|(Y, Y')\|_{X; \beta, m, \infty} \leq M.$$ 

It is easy to see that for $T$ sufficiently small, the set $\mathcal{B}_T$ contains the process $t \rightarrow (\xi + f_0(\xi) \delta X_{0, t}, f_0(\xi))$, and hence, is non-empty. For each $(Y, Y')$ in $\mathcal{B}_T$, define

$$\Phi(Y, Y') = (\xi + \int_0^1 b_t(Y_r)dr + \int_0^1 \sigma_t(Y_r)dB_r + \int_0^2 f(Y)dX_t, f(Y)).$$

\textsuperscript{15}We note that $\beta' < \beta$ is necessary for the final contraction estimates.
It is evident that both terms above are a.s. continuous and hence progressively measurable. We will now show that if $T$ is sufficiently small, $\Phi$ has a unique fixed point in $B_T$, which is a solution to (4.1).

**Invariance.** We show that there is a choice of $T^* = T^*(M, \|X\|_\alpha)$ such that $\Phi$ maps $B_T$ into itself, for any $T \leq T^*$. Let $(Y, Y')$ be an element in $B_T$ and for simplicity put $(Z, Z') = (f(Y), Df(Y)Y + f'(Y))$ (this belongs to $D_X^{\beta, \beta'} L_{m, \infty}$ by Lemma 3.11). Applying the BDG inequality and standard bounds for Riemann integrals, we have for $\delta := min\{\alpha - \beta, 1 - \beta - \beta', 1/2 - \beta\} > 0$. The above right hand side is indeed bounded above by $M$ provided that $T \leq T^* := (\frac{M}{C(1+M)}\delta).$ This proves the desired property.

**Contraction.** We suppose that $M, T$ are chosen as in the previous step. Taking $T$ smaller if necessary, we now show that $\Phi$ is a contraction on $B_T$, but for the associated $L_{m,m^*}$ metric (as opposed to $L_{m,\infty}$ as in the above proof of invariance), that is:

$$\|\Phi(Y, Y')\|_{X;\beta,\beta';m} \leq \|f\|_\infty + C(1 + M^\delta)T^{\delta'}$$

for a constant $C = C(T, \|X\|_\alpha)$ which is non-decreasing in $T$, and where $\delta' := min\{\alpha - \beta, 1 - \beta - \beta', 1/2 - \beta\} > 0$. The above right hand side is indeed bounded above by $M$ provided that $T \leq T^* := (\frac{M}{C(1+M)}\delta)$. This proves the desired property.

**Lemma 3.11.**
For the last term, we estimate
\[ \left\| \left( \int_0^T (Z - \bar{Z}) dX, Z - \bar{Z} \right) \right\|_{X; \beta, \beta'; m} \lesssim \left\| \left( \int_0^T (Z - \bar{Z}) dX, Z - \bar{Z} \right) \right\|_{X; \alpha, \beta, m} T^\delta \]
(\delta = (\alpha - \beta) \wedge (\beta - \beta') is as before), which easily follows from \( Z_0 = f_0(\xi) = \bar{Z}_0 \), by definition of \( B_T \). Combining with Corollary 3.6 (take \( n := m \) therein), and Proposition 3.13 with \( \alpha' = \alpha, \kappa = \beta, \kappa' = \beta' \), we arrive at
\[ \left\| \left( \int_0^T (Z - \bar{Z}) dX, Z - \bar{Z} \right) \right\|_{X; \beta, \beta'; m} \lesssim \left( \left\| Z - \bar{Z} \right\|_{\beta, m} + \left\| Z, Z' \right\| \bar{Z}, \bar{Z}' \right\|_{X; \beta, \beta'; m} T^\delta \right) T^\delta \]
\[ \lesssim \left\| Z, Z' ; \bar{Z}, \bar{Z}' \right\|_{X; \beta, \beta'; m} T^\delta \]
\[ \lesssim \left\| Y, Y', \bar{Y}, \bar{Y}' \right\|_{X; \beta, \beta'; m} T^\delta \]
To go from first to second line, we have used the fact that \( Z_0 = \bar{Z}_0 \) and thus \( \left\| Z - \bar{Z} \right\|_{\beta, m} \lesssim \left\| \delta Z - \delta \bar{Z} \right\|_{\beta, m} (1 + T^\delta) \lesssim \left\| Z, Z' ; \bar{Z}, \bar{Z}' \right\|_{X; \beta, \beta'; m} \), by definition. Next, the drift and diffusion terms are estimated as in the proof of invariance, noting this time that the right hand sides are proportional to the corresponding Lipschitz norms (as introduced in Definition 4.1). Finally, inserting these contributions in (4.22) entails
\[ \| \Phi(Y, Y') ; \Phi(\bar{Y}, \bar{Y}') \|_{X; \beta, \beta'; m} \leq CT^\delta \left\| Y, Y' ; \bar{Y}, \bar{Y}' \right\|_{X; \beta, \beta'; m} \]
where \( \delta' > 0 \) is the same as in the invariance step, and here the constant \( C \) depends only on \( \beta, \gamma, T_0, \left\| (Df, Df') \right\| \gamma, \left\| (Df, Df') \right\| X, \alpha, \alpha', \gamma, \right\| \left\| \delta X \right\| \beta, T_0 \], \right\| b \right\| \text{Lip} \) and \( \right\| \sigma \right\| \text{Lip} \). This proves that \( \Phi \) is indeed a contraction if \( T \) is sufficiently small.

**Concluding the proof.** Picard’s fixed point theorem asserts that we can find a unique process \( (Y', Y') \) in \( B_T \) such that \( \Phi(Y, Y') = (Y', Y') \). In particular, \( (Y, f(Y)) \) is a stochastic controlled rough path in \( D^\beta, \beta' \) \( L_{m, \infty} \) which satisfies equation (4.4). Because the smallness of \( T \) only depends on \( \| X \|_\alpha \) and the norms of the coefficients \( b, \sigma, f, f' \) but not on \( \xi \), the previous procedure can be iterated to construct a unique solution in \( D^\beta, \beta' \) \( L_{m, \infty} \) over \([0, T_0] \), for any \( T_0 > 0 \). To show uniqueness, we observe from Remark 4.4 that if \( \bar{Y} \) is a \( L_{m, \infty} \)-solution, then \( (\bar{Y}, f(\bar{Y})) \) belongs to \( D^\beta, \beta' \) \( L_{m, \infty} \). It follows that \( \bar{Y} \) belongs to \( B_T \) for \( T \) sufficiently small. Since \( \Phi \) is a contraction on \( B_T \), this shows that the \( L_{m, \infty} \)-solution is unique on small time intervals, which implies uniqueness on any finite time intervals. This proves the theorem.

4.2. Continuous dependence on data. We now establish the continuity of the solution to (4.1) with respect to its full inputs data. At first reading, the reader may assume \( \alpha = \beta = \beta' \) and \( \gamma = 3 \), with possible focus on time-independent \( f \) (which renders harmless all \( \beta \) exponents). In general, these exponents are needed to allow finer spatial regularity assumption on the vector fields, in interplay with their temporal regularity (and that of \( X \)).

**Theorem 4.9.** Let \( \xi, \bar{\xi} \) be in \( L_0(\mathcal{F}_0); X, \bar{X} \) be in \( \mathcal{C}^\alpha \), \( \alpha \in (\frac{1}{3}, \frac{1}{2}) \); \( \sigma, \bar{\sigma}, b, \bar{b} \) be random bounded continuous functions; fix \( m \geq 2 \), and parameters \( \gamma \in (2, 3], \beta \in (0, \alpha] \) such that \( \alpha + (\gamma - 1) \beta > 1 \). Consider \( (f, f') \in D^\beta X^\gamma L_{m, \infty}^\gamma \) such that \( (Df, Df') \) belongs to \( D^\beta, \beta' \) \( L_{m, \infty} \) \( C^\gamma_0 \) \( \gamma_0 \) where \( \beta' > 0 \) is taken so that
\[ 1 - \alpha - \beta < \beta' \leq 1, \]
and fix another stochastic controlled vector field \( (\tilde{f}, \tilde{f}') \in D^\beta, \beta' \) \( L_{m, \infty} \). Let \( Y \) be an \( L_{m, \infty} \)-integrable solution to (4.1) starting from \( \xi \), and similarly denote by \( \bar{Y} \) an \( L_{m, \infty} \)-integrable solution to (4.1) starting from \( \xi \) with associated coefficients \( (\bar{\sigma}, \bar{f}, \bar{f}', \bar{b}, \bar{X}) \). Let \( M \) be a constant such that
\[ \|X\|_\alpha + \|\bar{X}\|_\alpha + \|b\|_{\text{Lip}} + \|\sigma\|_{\text{Lip}} + \|(f, f')\|_{\gamma; \infty} + \|(f, f')\|_{X; 2\beta, m, \infty} + \|(Df, Df')\|_{X; \beta, \beta', m, \infty} \leq M. \]

Then, denoting by \(R_{s,t}^Y = \delta Y_{s,t} - f_s(Y_s)\delta X_{s,t} - \bar{R}_{s,t}^Y = \delta \bar{Y}_{s,t} - \bar{f}_s(\bar{Y}_s)\delta \bar{X}_{s,t} \) and recalling the notations \( (2.5), (3.21), (3.22) \), we have the estimate\(^\text{16}\)

\[(4.24) \qquad \sup_{t \in [0, T]} |\delta Y_{0,t} - \delta \bar{Y}_{0,t}|_m + \|\delta Y - \delta \bar{Y}\|_{\alpha; m} + \|\delta f(Y) - \delta \bar{f}(\bar{Y})\|_{\beta; m} + \|\mathbb{E}_t R^Y - \mathbb{E}_t \bar{R}^Y\|_{\alpha + \beta; m} \leq \|\theta\|_{\infty} + \|\mathbb{E}_t R^Y - \mathbb{E}_t \bar{R}^Y\|_{\alpha + \beta; m} \]

where the implied constant depends on \(\alpha, \beta, \beta', \gamma, T\) and \(M\).

**Proof.** Without loss of generality, we can and will assume that \(1 - \alpha - \beta < \beta' \leq \min\{\beta, (\gamma - 2)\beta\} \). We introduce the stochastic controlled rough path

\[ Z = Y' = f(Y), \quad Z' = Df(Y)f(Y) + f'(Y) \]

and similar for \((\bar{Z}, \bar{Z}')\). Thanks to Remark 4.4, we have that \((Y, f(Y))\) and \((\bar{Y}, \bar{f}(\bar{Y}))\) both belong to \(D^{2,\beta}_X L_{m,\infty}\). Consequently, Lemma 3.11 implies that \((Z, Z')\) and \((\bar{Z}, \bar{Z}')\) belong to \(D^{2,\beta}_X L_{m,\infty}\) and \(D^{\beta',\beta''}_X L_{m,\infty}\) respectively, where \(\beta'' = \min\{(\gamma - 2)\beta, \beta'\} \).

For convenience, we introduce the notations

\[ \theta = \rho_{\alpha, \beta}(X, \bar{X}) + \|\xi - \bar{\xi}\|_m + \sup_{t \in [0, T]} \|\sigma_t(x) - \bar{\sigma}_t(x)\|_m + \|\bar{f}_t(x) - \bar{\bar{f}}_t(x)\|_m + \|\delta Y - \delta \bar{Y}\|_{\gamma - 1; m} \]

and \(G_t = \sup_{t \in [0, T]} |\delta Y_{0,t} - \delta \bar{Y}_{0,t}|_m \). We will make use of the following fact concerning any \(g \in \{b_s, \sigma_s, f_s, f'_s\}\) with \(s\) fixed: we have the bound

\[(4.25) \qquad \|g(Y_s) - g(\bar{Y}_s)\|_m \leq (2\|g\|_\infty + \|g\|_{\text{Lip}})\|Y_s - \bar{Y}_s\|_1 + 1\|m \leq \|\xi - \bar{\xi}\|_m + \|\delta Y - \delta \bar{Y}\|_{0, s} m \leq \theta + G_s. \]

This stems from boundedness and Lipschitz property.

\(^{16}\)We note that if \(ma > 1\), then by Kolmogorov continuity theorem

\[ \sup_{t \in [0, T]} |\delta Y_{0,t}|_m \leq \|\delta Y\|_{\alpha; m}. \]
At first, we estimate the difference between rough stochastic integrals thanks to Corollary 3.6 and Proposition 3.13. Using additionally (4.25), we arrive at

\begin{equation}
\left\| \int_0^t ZdX, Z; \int_0^t \bar{Z}d\bar{X}, \bar{Z} \right\|_{X, \bar{X}; \beta, \beta'; m}
\lesssim \left( \|Z_s - \bar{Z}_s\|_m + \rho_{\alpha, \beta}(X, \bar{X}) + \|Z, Z'\|_{X, \bar{X}; \beta, \beta'; m} \right) \left\| t - s \right\|^{(\alpha - \beta) \wedge (\beta - \beta')}
\lesssim G_s + \theta + \|Z, Z'\|_{X, \bar{X}; \beta, \beta'; m} \left\| t - s \right\|^{\alpha \wedge (2\beta - \beta')}
\lesssim G_s + \theta + \|Y - \bar{Y}\|_{X, \bar{X}; \beta, \beta'; m} \left\| t - s \right\|^{\alpha \wedge (2\beta - \beta')}.
\end{equation}

Now, let \( S = \int_0^t b_r(Y_r)dr + \int_0^t \sigma_r(Y_r)dB_r \) and define \( \bar{S} \) accordingly. We may think of these as stochastic controlled rough paths with zero generalized Gubinelli derivatives. Proceeding as in the proof of Theorem 4.6, thanks to triangle inequality and translation invariance for \( \|\cdot\|_{X, \bar{X}; \beta, \beta'; m} \) (see (3.6), (3.5)), we obtain from (4.26) that

\begin{equation}
\|Y, Y'\|_{X, \bar{X}; \beta, \beta'; m} \leq \left\| \int ZdX, Z; \int \bar{Z}d\bar{X}, \bar{Z} \right\|_{X, \bar{X}; \beta, \beta'; m} + \|S, 0; \bar{S}, 0\|_{X, \bar{X}; \beta, \beta'; m}
\lesssim G_s + \theta + \|Y, Y'\|_{X, \bar{X}; \beta, \beta'; m} \left\| t - s \right\|^\delta
\end{equation}

for \( \delta = \min\{\alpha \wedge (2\beta - \beta'), 1 - \beta - \beta', \frac{1}{2} - \beta\} > 0 \), where to estimate the term \( \|S, 0; \bar{S}, 0\|_{X, \bar{X}; \beta, \beta'; m} = \|(S - \bar{S}), 0\|_{X, \bar{X}; \beta, \beta'; m} \) we used similar computations as in the proof of Theorem 4.6.

Next, using (3.14) of Theorem 3.5, as well as the integral equation satisfied by \( \delta(Y - \bar{Y})_{0, t} \), we can estimate the increments of \( G_t = \| \sup_{r \leq t} |\delta(Y - \bar{Y})_{0, r}| \| \) simply as follows:

\begin{equation}
G_t - G_s \lesssim \theta + \left\| t - s \right\|^\gamma G_s
\lesssim \theta + \left\| t - s \right\|^\delta G_s
\end{equation}

for \( \delta = \min\{\alpha, \alpha + \beta - \frac{1}{2} \vee (1 - \beta')\} > 0 \), for any \( |t - s| \ll 1 \), where in the last line we used Proposition 3.13 and (4.27). Now, (4.28) and a standard argument imply that \( G_T \lesssim \theta \), as desired.

Finally, we plug this into (4.27) to obtain (4.24), locally on any time interval \( I \subset [0, T] \) subject to \( |I| \leq \ell \), where \( \ell \) only depends on the hidden constants inside (4.27). To show that a similar bound holds on \( I = [0, T] \), it suffices to repeat the second step in the proof of Proposition 4.5 (details are omitted). This completes the proof.

4.3. Uniqueness at criticality. The result can be seen as extension of [Dav07, Theorem 3.6] to the setting of RSDEs. Since deterministic RDEs with time-independent vector fields, as considered in [Dav07], the counter examples given therein show that no improvement is possible.

**Theorem 4.10** (Uniqueness). Suppose that \( b, \sigma \) are random bounded Lipschitz functions, \( (f, f') \) belongs to \( D_b^{(2)} L_{2, \infty} C_{\gamma} \) and \( (Df, Df') \) belongs to \( D_b^{(\alpha, (\gamma - 2)\alpha)} C_{\gamma}^{-1} \), where \( \gamma = 1/\alpha \). Let \( \xi \) be in \( L_0(F_0; W) \). Let \( Y \) be an \( L_{2, \infty} \)-integrable solution on \( [0, T] \) starting from \( \xi \). Then \( Y \) is unique in following the sense. If \( \bar{Y} \) is another \( L_{2, \infty} \)-integrable solution on \( [0, T] \) starting from \( \xi \) defined on the same filtered probability space \( (\Omega, \mathcal{G}, \{F_t\}, \mathbb{P}) \), then \( Y \) and \( \bar{Y} \) are indistinguishable.
4.3.1. Davie–Grönwall-type lemma. We record an auxiliary result allowing to compare integral remainders as in Theorem 3.5 when the value of the exponents therein is critical. It is inspired by [Dav07, Thm 3.6] and [Lê20, Thm 2.1]. In essence, Davie’s inductive argument is replaced by a decomposition which allows exploiting BDG inequality.

**Lemma 4.11.** Let $T, \alpha, \eta, \varepsilon$ be positive numbers and $C, G, \Gamma_1, \Gamma_2$ be nonnegative numbers such that $\eta \in \left(\frac{1}{2}, 1\right]$ and $\alpha + \eta > 1$. Assume that $J$ is an $L_m$-integrable process indexed by $\Delta$ such that

\begin{align}
\|J_{s,t}\|_m &\leq C|t-s|^\eta, \quad \|E_s J_{s,t}\|_m \leq C|t-s|^{1+\varepsilon}, \\
\|\delta J_{s,u,t}\|_m &\leq G\left(\sup_{[r,v] \subseteq [s,t]} \|J_{r,v}\|_m\right)|t-s|^\alpha + \Gamma_2|t-s|^\eta
\end{align}

and

\begin{equation}
\|E_s \delta J_{s,u,t}\|_m \leq G\left(\sup_{[r,v] \subseteq [s,t]} \|J_{r,v}\|_m\right)|t-s|^\alpha + \Gamma_1|t-s|
\end{equation}

for every $(s,u,t)$ in $\Delta$. Then, there exist positive constants $c = c(\varepsilon, \eta, \alpha, m)$ and $\ell = \ell(\varepsilon, \eta, \alpha, m, G)$ such that for every $(s,t) \in \Delta$ with $|t-s| \leq \ell$

\begin{equation}
\|J_{s,t}\|_m \leq c\Gamma_1 \left(1 + |\log \frac{\Gamma_1}{C}| + |\log (t-s)|\right) (t-s) + \Gamma_2(t-s)^\eta.
\end{equation}

**Proof.** All the implicit constants in our estimates below depend only on $\varepsilon, \eta, \alpha, m$.

For each integer $k \geq 0$, let $P_k$ denote the dyadic partition of $[s,t]$ of mesh size $2^{-k}|t-s|$. By triangle inequality, we have

\begin{equation}
\|J_{s,t}\|_m \leq \| \sum_{[u,v] \in P_j} J_{u,v}\|_m + \|J_{s,t} - \sum_{[u,v] \in P_j} J_{u,v}\|_m.
\end{equation}

We estimate the first term using BDG inequality and condition (4.29). This yields

\begin{equation}
\| \sum_{[u,v] \in P_j} J_{u,v}\|_m \lesssim \sum_{[u,v] \in P_j} \|E_u J_{u,v}\|_m + \left(\sum_{[u,v] \in P_j} \|J_{u,v}\|_m^2\right)^{1/2} \\
\lesssim C 2^{-j\varepsilon}(t-s)^{1+\varepsilon} + C 2^{-j(\eta - \frac{1}{2})}(t-s)^\eta.
\end{equation}

For the second term, we derive from [Lê23, id. (3.17)] (see also [Lê20, id. (2.47)]) and BDG inequality that for $j \geq 1$

\begin{equation}
\|J_{s,t} - \sum_{[u,v] \in P_j} J_{u,v}\|_m = \| \sum_{k=0}^{j-1} \sum_{[u,v] \in P_k} \delta J_{u,(u+v)/2,v}\|_m.
\end{equation}

\begin{equation}
\lesssim \sum_{k=0}^{j-1} \sum_{[u,v] \in P_k} \|E_u \delta J_{u,(u+v)/2,v}\|_m + \sum_{k=0}^{j-1} \left(\sum_{[u,v] \in P_k} \|\delta J_{u,(u+v)/2,v}\|_m^2\right)^{1/2}.
\end{equation}

Applying (4.31) and (4.30), we have

\begin{equation}
\sum_{[u,v] \in P_k} \|E_u \delta J_{u,(u+v)/2,v}\|_m \lesssim G 2^{-k(\alpha + \eta - 1)}\|J\|_{\eta;m}(t-s)^{\alpha + \eta} + 2^{-k\kappa}\Gamma_1(t-s)^{1+\kappa}
\end{equation}

and

\begin{equation}
\left(\sum_{[u,v] \in P_k} \|\delta J_{u,(u+v)/2,v}\|_m^2\right)^{1/2} \lesssim G 2^{-k(\alpha + \eta - \frac{1}{2})}\|J\|_{\eta;m}(t-s)^{\alpha + \eta} + 2^{-k(\eta - \frac{1}{2})}\Gamma_2(t-s)^\eta.
\end{equation}

Summing in $k$, noting that $\alpha + \eta > 1$ and $\eta > \frac{1}{2}$, we have

\begin{equation}
\|J_{s,t} - \sum_{[u,v] \in P_j} J_{u,v}\|_m \lesssim G\|J\|_{\eta;m}(t-s)^{\alpha + \eta} + j\Gamma_1(t-s) + \Gamma_2(t-s)^\eta.
\end{equation}

From here, we obtain that for every integer $j \geq 1$ and every $(s,t) \in \Delta$

\begin{equation}
\|J\|_{\eta;m} \lesssim C 2^{-j\varepsilon}(t-s)^{1+\varepsilon-\eta} + C 2^{-j(\eta - \frac{1}{2})} + G\|J\|_{\eta;m}(t-s)^{\alpha} + j\Gamma_1(t-s) + \Gamma_2(t-s)^{1+\kappa-\eta}.
\end{equation}
For $|t - s| \leq \ell$ with sufficiently small $\ell$, this gives
\begin{equation}
(4.34) \\
\|J\|_{\eta,m} \lesssim C 2^{-j(\varepsilon \wedge (\eta - \frac{1}{2}))} + j \Gamma_1 (t - s)^{1+\kappa - \eta} + \Gamma_2.
\end{equation}

To conclude, we consider two cases. If $j_0 := \varepsilon^{-1} \lor (\eta - \frac{1}{2})^{-1} \log_2 \frac{C}{\Gamma(t-s)^{1-\eta}} < 2$, we have $C \lesssim \Gamma_1 (t - s)^{1-\eta}$ and choose $j = 1$. Then from (4.34), we have
\begin{equation}
\|J\|_{\eta,m} \lesssim \Gamma_1 (t - s)^\eta + \Gamma_2,
\end{equation}
which yields (4.32). If $j_0 \geq 2$, we choose $j = \lfloor j_0 \rfloor$ so that $C 2^{-j(\varepsilon \wedge (\eta - \frac{1}{2}))} \leq 2\Gamma_1 (t - s)^{1-\eta}$ and thus from (4.34),
\begin{equation}
\|J\|_{\eta,m} \lesssim \Gamma_1 (1 + j_0)(t - s)^{1-\eta} + \Gamma_2
\end{equation}
\begin{equation}
\lesssim \Gamma_1 \left(1 + |\log \frac{1}{C}| + |\log (t - s)|\right) (t - s)^{1-\eta} + \Gamma_2,
\end{equation}
which implies (4.32). This finishes the proof. \hfill \square

4.3.2. **Proof of Theorem 4.10.** We hinge on Lemma 4.11.

From Remark 4.4 and boundedness of the coefficients, we see that $(Y, f(Y))$ belongs to $D_X^{20}L_{2,\infty}$. Similarly, $(\bar{Y}, f(\bar{Y}))$ belongs to $D_X^{20}L_{2,\infty}$.

We denote $\bar{Y} = Y - \bar{Y}$ and
\begin{equation}
\bar{Z} := f(Y) - f(\bar{Y}), \quad \bar{Z}' = Df(Y)f(Y) - Df(\bar{Y})f(\bar{Y}) + f'(Y) - f'(\bar{Y})
\end{equation}
and $R\bar{Z}' = \delta \bar{Z} - \bar{Z}' \delta X$. For each $s \leq t$, we further denote
\begin{equation}
A_{s,t} = \bar{Z}_s \delta X_{s,t} + \bar{Z}'_s X_{s,t}
\end{equation}
and
\begin{equation}
J_{s,t} = \delta \bar{Y}_s - \int_s^t [b_r(Y_r) - b_r(\bar{Y}_r)] dr - \int_s^t [\sigma_r(Y_r) - \sigma_r(\bar{Y}_r)] dB_r - A_{s,t}
\end{equation}
\begin{equation}
= R\bar{Z}_s - \int_s^t [b_r(Y_r) - b_r(\bar{Y}_r)] dr - \int_s^t [\sigma_r(Y_r) - \sigma_r(\bar{Y}_r)] dB_r - \bar{Z}'_s X_{s,t}.
\end{equation}

We now verify that $J$ satisfies the hypotheses of Lemma 4.11 with $m = 2$ and every fixed but arbitrary $T > 0$. First, it follows from Lemma 3.11 that $(Z, Z')$, $(\bar{Z}, \bar{Z}')$ belong to $D_X^{20}L_{2,\infty}$. Hence, the inequalities in (4.5) hold with $\bar{\alpha} = \bar{\alpha}' = \alpha$, showing that (4.29) holds with $\eta = 2\alpha$. Define $\Gamma_{s,t} = \sup_{r \in [s,t]} \|\bar{Y}_r\|_2$. Reasoning as in steps 2 and 3 in the proof of Proposition 3.13, we see that
\begin{equation}
\|\delta \bar{Y}_{s,t}\|_2 + \|\delta \bar{Z}_{s,t}\|_2 + \|R\bar{Z}_{s,t}\|_2 \lesssim \|J_{s,t}\|_2 + \Gamma_{s,t}(t-s)^{\alpha},
\end{equation}
\begin{equation}
\|\delta \bar{Z}'_{s,t}\|_2 \lesssim \|J_{s,t}\|_2 + \Gamma_{s,t}(t-s)^{(\gamma-2)\alpha}
\end{equation}
and
\begin{equation}
\|E_s R\bar{Z}_{s,t}\|_2 \lesssim \|J_{s,t}\|_2 + \Gamma_{s,t}(t-s)^{(\gamma-1)\alpha}.
\end{equation}

Since $\delta J_{s,u,t} = -\delta A_{s,u,t} = R\bar{Z}_{s,u} \delta X_{u,t} + \delta \bar{Z}'_{s,u} X_{u,t}$, it follows from the above inequalities that
\begin{equation}
\|\delta J_{s,u,t}\|_2 \lesssim \|J_{u,t}\|_2 |t-s|^{\alpha} + \Gamma_{s,t}|t-s|^{2\alpha},
\end{equation}
\begin{equation}
\|E_s \delta J_{s,u,t}\|_2 \lesssim \|J_{u,t}\|_2 |t-s|^{\alpha} + \Gamma_{s,t}|t-s|.
\end{equation}
This shows that $J$ satisfies (4.30) and (4.31).

We view $\bar{Y}$ as an element in $CL_2$ and suppose that $\bar{Y} \neq 0$ on $[0,T]$. Since $\bar{Y}_0 = 0$ and $\bar{Y}$ belongs to $C^\alpha L_2$, for $k_0$ sufficiently large, we can find a strictly decreasing sequence $\{t_k\}_{k \geq k_0}$ in $[0,T]$ such that for each $k$, $\|\bar{Y}_t\|_2 < 2^{-k}$ for $0 < t < t_k$ and $\|\bar{Y}_{t_k}\|_2 = 2^{-k}$. Since $\bar{Y}$ is $L_2$-integrable, we have that $\Gamma_0 := \sup_{t \in [0,T]} \|\bar{Y}_t\|_2$ is finite. The previous
argument shows that for each \( k \), \( J \) satisfies (4.29)-(4.31) on \( \Delta([t_{k+1}, t_k]) \) with \( m = 2 \), \( \eta = 2\alpha \) and \( \Gamma = \Gamma_{t_{k+1}, t_k} = \sup_{t \in [t_{k+1}, t_k]} \| \tilde{Y}_t \|_2 \). Hence, by Lemma 4.11, we can find an \( \ell > 0 \), which may depend on \( N \), such that
\[
\| J_{t_{k+1}, t_k} \|_2 \lesssim \Gamma_{t_{k+1}, t_k} \left( 1 + | \log \Gamma_{t_{k+1}, t_k} | + | \log (t_k - t_{k+1}) | \right) (t_k - t_{k+1}) + \Gamma_{t_{k+1}, t_k} (t_k - t_{k+1})^{2\alpha}
\]
for every \( k \) sufficiently large so that \( t_k - t_{k+1} \leq \ell \). We now observe that \( \Gamma_{t_{k+1}, t_k} \leq 2^{-k} \), \( \| \delta \tilde{Y}_{t_{k+1}, t_k} \|_2 \geq \| \tilde{Y}_{t_k} \|_2 - \| \tilde{Y}_{t_{k+1}} \|_2 = 2^{-k-1} \) and take into account (4.37) to obtain that
\[
2^{-k-1} \| \delta \tilde{Y}_{t_{k+1}, t_k} \|_2 \lesssim 2^{-k} \left( 1 + k + | \log (t_k - t_{k+1}) | \right) (t_k - t_{k+1}) + 2^{-k} (t_k - t_{k+1})^\alpha.
\]
This implies \( t_k - t_{k+1} \geq C (1+k)^{-1} \) for some constant \( C > 0 \). Hence, we have \( \sum_{k \geq k_0} (t_k - t_{k+1}) = \infty \), which is a contradiction. It follows that \( Y_t = \tilde{Y}_t \) a.s. for each \( t \in [0, T] \). Since both processes are a.s. continuous, they are indistinguishable. 

4.4. Rough Itô formula. Let us start with a digression on the main integrability result Theorem 3.5. Herein we let \( \beta \in (0, \alpha] \).

4.4.1. Extended stochastic controlled rough paths. While the space \( D_{X}^{\beta, \beta'} L_{m,n} \) was needed to address solvability results for SRDEs, when dealing with sole integration purposes it is enough to use a slightly larger class of stochastic processes, obtained simply by replacing \( \delta Z' \) in Definition 3.1-(d) by its averaged-type analogue, that is \( \mathbb{E} \delta Z' \).

Suppose that \( Z : \Omega \times I \to W \) and \( Z' : \Omega \times I \to L(V, W) \) are \( \{ \mathcal{F}_t \} \)-progressive measurable and such that
\[
(4.42) \quad \Gamma(Z, Z') := \| \delta Z' \|_{\beta;m,n} + \sup_{r \in I} \| Z'_r \|_n + \| \mathbb{E} R^Z \|_{\beta+\beta';n} + \| \mathbb{E} \delta Z' \|_{\beta;m,n} < \infty
\]
where \( R^Z_{s,t} = \delta Z_{s,t} - Z'_t \delta X_{s,t} \). recalling the notations (3.10)-(3.11), it is clear from that definition that
\[
\Gamma_1^{\beta, \beta';m,n}(X; Z, Z'; I) \lor \Gamma_2^{\beta, \beta';m,n}(X; Z, Z'; I) \lesssim \Gamma(Z, Z')
\]
for an implicit constant which only depends on \( \| X \|_\alpha \). This asserts in particular that \( \int ZdX \) is well-defined, in the sense of Theorem 3.5. Our preliminary discussion motivates the next definition.

Definition 4.12 (Extended stochastic controlled rough paths). We say that \( (Z, Z') \) is an extended stochastic controlled rough path of \( (m, n) \)-integrability and \( (\beta, \beta') \)-Hölder regularity with values in \( W \) with respect to \( \{ \mathcal{F}_t \} \) if (a), (b), (c) of Definition 3.1 hold together with

\( (d') \sup_{r \in I} \| Z'_r \|_n \) is finite and \( \mathbb{E} \delta Z' \) belongs to \( C_2^{\beta'} L_n([0, T], \Omega; L(V, W)); \)

The class of such processes is denoted by \( \check{D}_{X}^{\beta, \beta'} L_{m,n}([0, T], \Omega; W) \), or simply \( \check{D}_{X}^{\beta, \beta'} L_{m,n} \).

4.4.2. Main result and discussion. We now prove a rough (stochastic) Itô formula, to be compared with the classical Itô formula and the rough Itô formula \([\text{FH20, Ch.7}].\)

We call rough Itô process any continuous adapted process with dynamics,
\[
dY_t(\omega) = b_t(\omega) dt + \sigma_t(\omega) dB_t + (Y', Y'')_{t, \omega} dX_t,
\]
provided this makes sense (in integral form), with the final term is understood in the sense of rough stochastic integration (Theorem 3.5). Our aim is to show, for \( t \in [0, T] \)
and with probability one,
\[
\varphi(Y_t) - \varphi(Y_0) - \int_0^t D\varphi(Y_s) \sigma_{s,\omega} dB_s - \int_0^t (\mathcal{L}_{s,\omega}\varphi)(Y_s) \, ds \\
= \int_0^t D\varphi(Y_s) Y'_s dX_s + \frac{1}{2} \int_0^t D^2\varphi(Y_s) (Y'_s, Y''_s) \, d[X]_s,
\]
(4.44)
\[
= \int_0^t D\varphi(Y_s) Y''_s d[X]_s,
\]
(4.45)
for sufficiently regular \(\varphi\), with
\[
(\mathcal{L}_{s,\omega}\varphi)(y) := b_{s,\omega} \cdot D\varphi(y) + \frac{1}{2}(\sigma_{s,\omega}^\top \sigma_{s,\omega}) : D^2\varphi(y) = b_{s,\omega} \cdot D\varphi(y) + \frac{1}{2} a_{s,\omega} : D^2\varphi(y),
\]
and rough path bracket \([X] \equiv (\delta X)^{\otimes 2} - 2\text{Sym}X\), as defined in [FH20, Ex. 2.11]. We also wrote \(odX \equiv dX^g\) to denote (stochastic) rough integration against the “geometric” of \(X = (X, X)\), explicitly given by \(X^g := (X, \text{Anti}(X) + (\delta X)^{\otimes 2}/2)\). (In case of geometric \(X\), we have \([X] \equiv 0\) and there is no difference between (4.44) and (4.45).)

**Theorem 4.13** (Rough Itô). Let \(b, \sigma\) be bounded progressive, \(X = (X, X) \in \mathcal{C}\) for some \(\alpha \in (\frac{1}{3}, \frac{1}{2})\), and consider a test-function \(\varphi \in \mathcal{C}\) for some \(\gamma \in (\frac{1}{3}, 3]\). Suppose that \(\|Y_0\|_4 < \infty\) and let the pair \((Y', Y'')\) be an extended stochastic controlled rough path in \(\mathcal{D}_{X,\beta,\beta'} L_{4,n}\), for some parameters \(n > 4, 0 < \beta' \leq \beta \leq 1\) subject to the conditions\(^{17}\)
\[
\alpha + \alpha \wedge \beta > \frac{1}{2} \quad \text{and} \quad \alpha + \alpha \wedge \beta + \min\{\alpha(\gamma - 2), \alpha(\frac{n}{4} - 1), \beta'\} > 1.
\]
(4.46)
Then the rough stochastic Itô formulas (4.44), (4.45) hold, with fully specified rough stochastic integral \(\int D\varphi(Y) Y' dX \equiv \int (T\varphi, T'\varphi)(Y) dX\) where\(^{18}\)
\[
(T\varphi, T'\varphi)(y) := (D\varphi(y) Y', D^2\varphi(y) (Y', Y'') + D\varphi(y) Y'')) \in \mathcal{D}_{X,\beta,\beta'} L_{2,2}.
\]
The bracket integrals in (4.44), (4.45) are Young integrals, with mesh limit taken in \(L_2\).

Examples of rough Itô processes, to which this Itô formula is applicable, includes general RSDEs solutions (as provided by **Theorem 4.6**) with
\[
b_{t,\omega} = b_t(\omega, Y_t(\omega)), \quad \sigma_{t,\omega} = \sigma_t(\omega, Y_t(\omega)),
\]
\[
(Y', Y'')_{t,\omega} = \left( f_t(\omega, Y_t(\omega)), (Df_t f + f'_t)(\omega, Y_t(\omega)) \right).
\]
This setting also accommodates McKean–Vlasov equations with rough common noise, in which case \(b_{t,\omega} = \bar{b}_t(\omega, Y_t(\omega)) = b_t(\omega, Y_t(\omega), \mu_t)\) where \(\mu_t\) is the law of \(Y_t\), denoted by \(\text{Law}(Y_t; X)\), and similar for the other coefficient fields. (Well-posedness of such rough McKean–Vlasov equations is treated in [FHL24]; our point here is only that solutions are rough Itô processes, hence amenable to Itô’s formula.)

We see many potential applications of (4.44), and various extensions thereof, notably in the area of rough (stochastic) PDEs and rough (doubly stochastic) BSDE, also in mean-field situations. That said, we cannot resist the temptation here to make a concrete use of **Theorem 4.13**, in revisiting some concepts (martingale problem, mimicking theorems) that will be familiar to many readers with stochastic analysis background. For any sufficiently nice test function \(\varphi\) **Theorem 4.13** allows to define a martingale,
\[
M_t^\varphi := \varphi(Y_t) - \varphi(Y_0) - \int_0^t (\mathcal{L}_{s,\omega}\varphi)(Y_s) \, ds - \int_0^t (T\varphi, T'\varphi)(Y_s) \, d[X]_s,
\]
(4.47)
where we have assumed that \(X\) is geometric (for simplicity only, otherwise carry along a \(d[X]\)-integral). We say that \(Y = Y(\omega)\) solves the rough martingale problem, RMP\((\mathcal{L}; T, T'\); \(X)\).

\(^{17}\)At first reading, take \(\gamma = 3, n = \infty\) and \(\alpha = \beta = \beta'\); then (4.46) is implied by the condition \(\alpha > \frac{1}{3}\).

\(^{18}\)Also write \((T\varphi, T'\varphi)\) to emphasize the progressive nature of this process.
Mind that all coefficients fields are progressive and we are far from a Markovian situation.

Even so, we can see that the flow of probability laws of $Y_t$ is measure-valued solution to an effective rough Fokker–Planck equation. To this end, define effective Markovian characteristics, i.e. (measurable) functions given by

$$\bar{b}_t(y) := \mathbb{E}(b_t,\omega|Y_t = y), \quad \bar{a}_t(y) := \mathbb{E}(a_t,\omega|Y_t = y),$$

with effective $(\mathcal{L}_t\varphi)(y) = \mathbb{E}(\mathcal{L}_{t,\omega}\varphi)(y)|Y_t = y)$, equivalently defined as $\mathcal{L}_{t,\omega}\varphi$ above, but using the effective data $(\bar{b}_t, \bar{a}_t)$. We further define $(\bar{T}_t\varphi)(y) = \mathbb{E}((\bar{T}_{t,\omega}\varphi)(y)|Y_t = y)$ and similarly $\bar{T}'$.

**Theorem 4.14.** Let $Y$ be a rough Itô process of the form (4.43), for some geometric rough path $X$, subject to the condition of Theorem 4.13. Then the flow of (deterministic) probability measures $\mu_t = \text{Law}(Y_t; X)$ satisfies the measure-valued rough partial differential (forward) equation

$$d\mu_t = \mathcal{L}_t^* \mu_t dt + \bar{T}_t^* \mu_t dX, \quad \mu_0 = \text{Law}(Y_0; X),$$

understood in analytically weak and integral sense. More precisely, for all $\varphi \in C^\gamma_b$,

$$(4.48) \quad \langle \mu_t, \varphi \rangle = \langle \mu_0, \varphi \rangle + \int_0^t \langle \mu_s, (\mathcal{L}_s\varphi) \rangle ds + \int_0^t \langle \mu_s, \bar{T}_s^* \varphi \rangle \langle \mu_s, \bar{T}'_s \varphi \rangle dX_s.$$

**Remark 4.15.** The rough forward equation of Theorem 4.14 is not valid as written for non-geometric rough paths. Indeed, let us introduce random first and second order differential operators $\mathcal{T}_{1,t,\omega}\varphi(y) = D\varphi(y)Y_t''(\omega)$, $\mathcal{T}_{2,t,\omega}\varphi(y) = D^2\varphi(y)(Y_t', Y_t')(\omega)$ so that $\mathcal{T}' = \mathcal{T}'_1 + \mathcal{T}'_2$. A look at (4.45) reveals that the correct equation involves a Young drift correction and reads

$$d\mu_t = \bar{\mathcal{L}}_t^* \mu_t dt + \bar{T}_t^* \mu_t dX + \frac{1}{2} (\bar{T}'_1^*)^* \mu_t d[X]$$

$$= \mathcal{L}_t^* \mu_t dt + \bar{T}_t^* \mu_t dX - \frac{1}{2} (\bar{T}'_1^*)^* \mu_t d[X]$$

where for $i = 1, 2$, we wrote $\bar{T}'_i \varphi(y) = \mathbb{E}[\mathcal{T}'_i \varphi(y)|Y_t = y]$.

**Example 4.16.** Assume rough McKean–Vlasov dynamics with progressively measurable coefficients, with $\mu_t = \text{Law}(Y_t; X)$ where

$$dY_t(\omega) = b_t(\omega, Y_t(\omega), \mu_t) dt + \sigma_t(\omega, Y_t(\omega), \mu_t) dB_t + f_t(\omega, Y_t(\omega), \mu_t) dX_t.$$

The stochastic rough integral is understood as $\int (Y''')dX$ with $Y'' := f_t(\omega, Y_t(\omega), \mu_t)$, and $Y'''$ given as sum of $((Df_t)f_t)(Y_t(\omega))$ and a term that captures the controlled structure (in $t$) induced by the dependence on $\text{Law}(Y_t; X)$, the full specification of which is left to [FHL.24]. There, the reader can also find existence and uniqueness of such equations, together with propagation of chaos results with fixed (rough) common noise. Upon randomization of $X$, similar to Appendix A, this yields a (common noise) robustification of an important class of equations, e.g. [LSZ.22], where the authors also emphasizes the importance of random coefficients. (This seems out of reach of previous work on rough McKean–Vlasov which however dealt with a different problem: the case of random rough paths $X = X(\omega)$ which is not at all our goal; a more detailed literature review is left to [FHL.24].)

In the “Markovian” McKean–Vlasov situation when coefficient dependence $(\omega, Y_t(\omega), \mu)$ is replaced by $(Y_t(\omega), \mu)$, the conditioning procedure for the coefficients is trivial, i.e.

$$\bar{b}_t(y) = b_t(y, \mu_t), \quad \bar{\sigma}_t(y) = \sigma_t(y, \mu_t),$$

and one easily arrives at the rough forward equation

$$d\mu_t = (\mathcal{L}_t[\mu_t])^* \mu_t dt + (\bar{T}_t[\mu_t])^* \mu_t dX,$$
with second order differential operator $\mathcal{L}[\nu]$ given by

$$(\mathcal{L}_t[\nu])\varphi(y) = b_t(y, \nu) \cdot D\varphi(y) + \frac{1}{2}\sigma_t \sigma_t^\top (y, \nu) : D^2\varphi(y).$$

**Remark 4.17.** In the setting of the random rough approach to RSDEs, it is seen in [DFS17, CN21] that uniqueness results for such rough forward equations can be obtained by forward-backward duality, more specifically if one has a (spatially) regular solution to the rough Kolmogorov backward equation in duality with (4.48).

**Proof.** Take expectation in (4.47), interchange $ds$-integration with $\mathbb{E}$ and use the tower property of conditional expectations to deal with all the terms, other than the final (and new) rough term. The interesting part is to justify (the first equality in)

$$
\mathbb{E} \int_0^t (T_{s, \omega}\varphi, T_{s, \omega}'\varphi)(Y_s) dX_s = \int_0^t \mathbb{E}[T_{s, \omega}\varphi(Y_s), T_{s, \omega}'\varphi(Y_s)] dX_s = \int_0^t \mathbb{E}[(\bar{T}_s\varphi, \bar{T}_s'\varphi)(Y_s)] dX_s
$$

(Since $\mu_s$ is precisely the law of $Y_s$ it is then easy to conclude.) The basic remark is that the first expectation on the left-hand side has form $\mathbb{E}(I) = \mathbb{E}(\lim P^n)$, where the rough stochastic integral $I$ is the limit (in some moment space, and certainly in $L^1$) of discrete approximations, along any deterministic sequence of partitions $P_n$ with mesh tending to zero. But then $\mathbb{E}I^n$ must also converge. Strictly speaking, this arguments implies that the compensated Riemann-sum with germ (for $s, t$ consecutive points in $P_n$)

$$
\mathbb{E}[T_{s, \omega}\varphi(Y_s)]X_{s, t} + \mathbb{E}[T_{s, \omega}'\varphi(Y_s)]X_{s, t} = \mathbb{E}[(\bar{T}_s\varphi(Y_s))X_{s, t} + \mathbb{E}[(\bar{T}_s\varphi(Y_s))]X_{s, t}
$$

converge; the limits then certainly deserve the rough integration notation. We can be more precise here in checking that $(\mathbb{E}[(\bar{T}_s\varphi(Y_s)]), \mathbb{E}[\bar{T}_s'\varphi(Y_s)])$ satisfy precisely the axioms of (deterministic) controlled rough paths, which identifies the rough integrals in (4.48) as bona fide (deterministic) controlled rough path integrals.

**Proof of the rough Itô formula.** We now address the proof of Theorem 4.13, assuming without loss of generality that $n \in (4, 4(\gamma - 1))$.

Let us first check that the right hand side of (4.44) is meaningful, which will be the case if the stochastic rough integral is well-defined. We note that our assumptions on the coefficients ensure that $(Y, Y') \in D^{2, \alpha, \beta} \mathcal{L}_{4, 4}$ from the stability of compositions (Lemma 3.11), we have that

$$(Z, Z') = (D\varphi(Y), D^2\varphi(Y)Y') \in \bar{D}^{\alpha, \beta, \alpha', \beta'}_{2, 2},$$

where $\gamma = \frac{1}{2} + 1$. Moreover, because of the algebraic identity $R_{s, t}^2 Y'_{s, t} = R_{s, t}^2 Y_{s, t} + \delta Y_{s, t} \delta Y'_{s, t}$ (and similar for $\delta(Y Y'')_{s, t}, \delta(Z Y'')_{s, t}$) we see using conditional Cauchy–Schwarz inequality that $(T \varphi, T' \varphi)(Y) = (Y', Y'' + Y')$ defines an extended stochastic controlled rough path such that

$$(\mathcal{T} \varphi, \mathcal{T}' \varphi)(Y) \in \bar{D}^{\alpha, \beta, \alpha', \beta'}_{2, 2},$$

where $\beta' = \min((\gamma - 2) \alpha, \beta')$. Next, Taylor theorem shows that

$$(4.50) \quad \varphi(Y_t) - \varphi(Y_0) = \langle D\varphi(Y_0), \delta Y_{s, t} \rangle + \frac{1}{2} \langle D^2\varphi(Y_0), \delta Y_{s, t} \rangle + O(|\delta Y_{s, t}|)$$

$$= \langle D\varphi(Y_0), \delta Y_{s, t} \rangle + \langle D^2\varphi(Y_0), Y_{s, t} \delta Y_{s, t} \rangle + \frac{1}{2} \langle D^2\varphi(Y_0), \delta Y_{s, t} \delta Y_{s, t} - 2 Y_{s, t} \delta Y_{s, t} \rangle + O(|\delta Y_{s, t}|)$$

$$=: A_{s, t} + O(|\delta Y_{s, t}|).$$

For any partition $P$ of $[0, t]$, we find in particular that

$$\varphi(Y_t) - \varphi(Y_0) = \sum_{[u, v]} P \varphi(Y_v) - \varphi(Y_u) = \sum_{[u, v]} P \varphi(Y_v) - \varphi(Y_u) + O(|P|^\alpha - 1)$$

and this establishes the fact that

$$\varphi(Y_t) - \varphi(Y_0) = L_2^\gamma \lim_{|P| \to 0} \sum_{[u, v]} P \varphi(Y_v) - \varphi(Y_u).$$
Let
\[ I_{s,t} := \int_s^t D\varphi(Y_r)\sigma_r dB_r + \int_s^t (L_r\varphi)(Y_r)dr + \int_s^t T\varphi(Y_r)dX_r + \frac{1}{2} \int_s^t D^2\varphi(Y_r)(Y'_r, Y'_r) d[X]_r \]
which is a well-defined adapted quantity in \( L_2 \) for each \((s, t) \in \Delta \) (by (4.49)). If we can show the existence of \( \lambda > \frac{1}{2} \) and \( \mu > 1 \) such that
\[
\|I_{s,t} - A_{s,t}\|_2 \lesssim (t-s)^\lambda \tag{4.51}
\]
and
\[
\|\mathbb{E}_s(I_{s,t} - A_{s,t})\|_2 \lesssim (t-s)^\mu, \tag{4.52}
\]
then the desired conclusion will be a consequence of the uniqueness part of Theorem 2.8.

Step 1: proof in the case when \( X \) is geometric. To obtain the bounds (4.51)-(4.52), we write
\[
I_{s,t} - A_{s,t} = \int_s^t (D\varphi(Y_r) - D\varphi(Y_s))\sigma_r dB_r + \int_s^t (D\varphi(Y_r) - D\varphi(Y_s))b_r dr
\]
\[
+ \left( \int_s^t (D\varphi(Y_r) - D\varphi(Y_s))Y'_r dX_r - \langle D^2\varphi(Y_s), Y'_s \otimes 2\mathbb{X}_{s,t} \rangle \right)
\]
\[
+ \left( \int_s^t a_r(Y_r)D^2\varphi(Y_r) dr - \frac{1}{2} \langle D^2\varphi(Y_s), \delta Y'_s \otimes 2 - 2Y'_s \otimes 2\text{Sym}\mathbb{X}_{s,t} \rangle \right)
\]
\[
= J^1_{s,t} + \cdots + J^4_{s,t}
\]
(where \( a_r = \frac{1}{2} \sigma_r \sigma_r^\perp \)) and estimate each term separately. The first term is easily estimated through Itô isometry, indeed
\[
\|J^1_{s,t}\|_2 \lesssim (t-s)^{\frac{1}{2} + \alpha},
\]
for an implicit constant only depends on \(|\varphi|_2\) and moreover \(\|\mathbb{E}_s J^1_{s,t}\|_2 = 0\). Similarly, we find
\[
\|J^2_{s,t}\|_2 \lesssim (t-s)^{1+\alpha}, \quad \|\mathbb{E}_s J^2_{s,t}\|_2 \lesssim (t-s)^{1+\alpha}.
\]
For the third term, we have
\[
J^3_{s,t} = \int_s^t D\varphi(Y_r)Y'_r dX_r - \langle D\varphi(Y_s), Y'_s \rangle \delta X_{s,t} - \langle D\varphi(Y_s), Y''_s \mathbb{X}_{s,t}\rangle - \langle D^2\varphi(Y_s), Y'_s \otimes 2\mathbb{X}_{s,t}\rangle
\]
so that (3.13), (3.12) and (4.49) imply
\[
\|J^3_{s,t}\|_2 \lesssim (t-s)^{\alpha + \alpha \wedge \beta}, \quad \|\mathbb{E}_s J^3_{s,t}\|_2 \lesssim (t-s)^{\alpha + \alpha \wedge \beta + \beta''},
\]
where this time the implied constants depend on \(|\varphi|_3\).

Finally, we can write
\[
\frac{1}{2} \langle D^2\varphi(Y_s), \delta Y'_s \otimes 2 - 2Y'_s \otimes 2\text{Sym}\mathbb{X}_{s,t}\rangle
\]
\[
= \langle D^2\varphi(Y_s), (\int_s^t \sigma_t dB_t) \otimes 2 + (\int_s^t Y'_r dX_r) \otimes 2 - Y'_s \otimes 2\mathbb{X}_{s,t} \rangle + \langle \bar{J}^4_{s,t}\rangle
\]
for some remainder term \( \bar{J}^4_{s,t} \) such that \( \|\bar{J}^4_{s,t}\|_2 \lesssim (t-s)^{\alpha + \frac{1}{2}} \) while \( \|\mathbb{E}_s \bar{J}^4_{s,t}\|_2 \lesssim (t-s)^{3\alpha} \). Consequently, it follows from Itô Isometry and standard arguments that
\[
\|J^4_{s,t}\|_2 \lesssim (t-s)^{\alpha + \frac{1}{2}}, \quad \|\mathbb{E}_s J^4_{s,t}\|_2 \lesssim (t-s)^{3\alpha}.
\]
Hence our conclusion.
Step 2: general case. In the notation of (4.45), we remark that (4.43) is equivalent to

\[ dY_t(\omega) = b_t,\omega dt - \frac{1}{2}Y''_t,\omega d[X]_t + \sigma_{t,\omega} dB_t + (Y', Y'')_t,\omega \circ dX_t \]

where the second integral is a Young one. Indeed, we have the Davie-type expansion

\[ \delta Y_{s,t} - \int_s^t b_r dr - \int_s^t \sigma_r dB_r = Y'_s \delta X_{s,t} + Y''_s (X^g - \frac{1}{2} \delta [X])_{s,t} + J_{s,t} \]

moreover the term

\[ \tilde{J}_{s,t} := -\frac{1}{2} \int_s^t Y''_r d[X]_r + Y''_s \frac{1}{2} \delta [X]_{s,t} \]

satisfies

\[ \| \tilde{J}_{s,t} \|_2 \lesssim (t - s)^{2\alpha + \beta'}, \quad \| E_n \tilde{J}_{s,t} \|_2 \lesssim (t - s)^{2\alpha + \beta'} \]

and so (4.55) is also a consequence of the uniqueness part of Theorem 2.8.

Now, the claimed formula follows by the same argument as in Step 1, where the drift term is replaced by a mixed Lebesgue/Young integral. (In this case the estimate (4.54) has to be replaced by the inequalities \( \| J_{s,t}^2 \|_2 \lesssim (t - s)^{1+\alpha} + (t - s)^{3\alpha} \), \( \| E_n J_{s,t}^2 \|_2 \lesssim (t - s)^{1+\alpha} + (t - s)^{3\alpha} \).) Applying the geometric rough Itô formula then shows that

\[ \varphi(Y_t) - \varphi(Y_0) - \int_0^t D\varphi(Y_s)\sigma_{s,\omega} dB_s - \int_0^t (L_{s,\omega,\varphi})(Y_s) ds \]

\[ = -\frac{1}{2} \int_0^t D\varphi(Y_s)Y''_s d[X]_s + \int_0^t D\varphi(Y_s)Y'_s dX^g_s \]

\[ = \int_0^t D\varphi(Y_s)Y'_s d(X^g - (0, \frac{1}{2} \delta [X]))_s + \frac{1}{2} \int_0^t D^2\varphi(Y_s)(Y'_s, Y'_s) d[X]_s, \]

as claimed.

\[ \square \]

Remark 4.18. Theorem 4.13 provides an explicit decomposition of \( f(Y) \) in terms of a martingale (stochastic integral) and a rough stochastic integral. For lower regularity exponents e.g. when \( \gamma \in (1, \frac{1}{\alpha}] \), it is still true that \( \varphi(Y) \) is the sum of a martingale and a random controlled rough path. This is indeed a consequence of the decomposition Theorem 3.4, which holds even more generally when \( \varphi = \varphi_t(\omega, \cdot) \) has the structure of a stochastic controlled vector field.

Moreover, let \( (f, f') \) be in \( D^{\beta, \beta'}_{X} L_{m, \infty} C_0 \), for some \( \gamma \in (1, 2) \), \( m \in [2, \infty) \). Let \( n \in [\gamma m, \infty) \) and \((Y, Y')\) be a stochastic controlled rough path in \( D^{\beta, \beta'}_{X} L_{m, n} \), \( Y \) being integrable. We assume that

\[ (\alpha \wedge \beta) + \beta'' > \frac{1}{2}, \quad \text{where} \quad \beta'' = \min\{(\gamma - 1) \beta, \beta'\}. \]

Then, there exist processes \( M^f, Y'^f \) such that

(i) \( f_t(Y_t) - f_0(Y_0) = M^f_t + Y'^f_t \) a.s. for every \( t \in [0, T] \);
(ii) \( M^f \) is an \( \{F_t\} \)-martingale, \( M^f_0 = 0 \);
(iii) \( Y'^f \) is \( \{F_t\} \)-adapted and satisfies

\[ \| Y'^f_t - Y'^f_s - (Df(Y_s)Y'_s + f'_s(Y_s)) \delta X_{s,t} |F_s|_m \|_{\gamma m} \]

\[ \lesssim (1 \vee |\delta X| \alpha)(1 \vee \|Y, Y'\|_{X; \beta', \beta' ; m, n})|t - s|^{(\alpha \wedge \beta) + \beta''} \]

for every \((s, t) \in \Delta \).

Furthermore, given \((Y, Y')\) in \( D^{\alpha, \alpha'}_{X} L_{m, n} \), the pair of processes \((M^f, Y'^f)\) is characterized uniquely by (i)-(iii).
Proof. Putting \((Z, Z') = (f(Y) - f_0(Y_0), DF(Y)Y' + f'(Y))\), we see from Lemma 3.11 that \((Z, Z')\) is a stochastic controlled rough path in \(D^{\beta,\beta'}_{\overline{X}}L_{m,n}\). From the estimate
\[
|f_t(Y_t) - f_0(Y_0)| \leq \sup_y |\delta f_0(t)(y)| + \sup_y |Df_0(y)||\delta Y_{0,t}|,
\]
using the regularity of \(f\), we see that \(Z\) is integrable. Our assumptions ensure that \((\alpha \wedge \beta) + \beta'' > \frac{1}{2}\). An application of Theorem 3.4 gives the result.

4.5. Weak solutions. Herein, we study weak solutions of (4.1). These are defined in such a way that is transparent from the corresponding classical notion for SDEs. Namely, given an initial probability distribution \(\mu\) on \(W\) and \(m \geq 2\), a weak solution to (4.1) starting from \(\mu\) consists of a filtered probability space \((\Omega, \mathcal{G}, \mathbb{P}, \{\mathcal{F}_t\})\) together with a pair \((\tilde{Y}, \tilde{B})\) such that \(\tilde{B}\) is an \(\{\mathcal{F}_t\}\)-Brownian motion in \(V_1\), \(\text{Law}(\tilde{Y}_0) = \mu\), and \(\tilde{Y}\) is an \(L_{m,\infty}\)-solution to
\[
(4.56) \quad d\tilde{Y}_t = b_t(\tilde{Y}_t)dt + \sigma_t(\tilde{Y}_t)d\tilde{B}_t + (f_t, f'_t)(\tilde{Y}_t)dX_t, \quad t \in [0, T].
\]
Weak solution is \(L_{m,\infty}\)-integrable if \(\tilde{Y}\) is an \(L_{m,\infty}\)-solution on the stochastic basis \((\Omega, \tilde{G}, \tilde{P}, \{\tilde{F}_t\})\).

In contrast to other sections, we assume here that the coefficients in (4.56) are deterministic. Namely, \(\omega \mapsto g_t(\omega, \cdot)\) is constant for every \(t \in I\) and each \(g \in \{b, \sigma, f, f'\}\). Likewise, we will call \((f, f')\) a deterministic controlled vector field and write
\[
(4.57) \quad (f, f') \in \mathcal{D}^{\beta,\beta'}_{\overline{X}}C^\gamma_b
\]
if \((f, f')\) is deterministic and belongs to \(\mathcal{D}^{\beta,\beta'}_{\overline{X}}L_{m,n}C^\gamma_b\) for some \(\beta, \beta' > 0, \gamma > 1\), and (irrelevant) parameters \(n\) and \(m\). We will abbreviate for convenience
\[
\|(f, f')\|_{X; \beta, \beta'} := \|(f, f')\|_{X;m,n;\beta,\beta'}, \quad \|(f, f')\|_{\gamma} = \|(f, f')\|_{\gamma;n}
\]
(and so on).

The first result is concerned about the existence of weak solutions in this setting.

Theorem 4.19. Suppose that \(b, \sigma\) are bounded continuous and \((f, f')\) be a deterministic controlled vector field in \(\mathcal{D}^{\beta,\beta'}_{\overline{X}}C^\gamma_b\) with \(\frac{1}{2} < \beta \leq \alpha, 2\beta + \beta' > 1\) and \(\gamma \in (\frac{1}{\beta} - 1, 2]\). Let \(\mu\) be a probability measure on \(W\). Then for every \(m \geq 2\), there exists a weak solution \((\Omega, \tilde{G}, \tilde{P}, \{\tilde{F}_t\}; \tilde{Y}, \tilde{B})\) to (4.1) starting from \(\mu\) which is \(L_{m,\infty}\)-integrable for every \(m \geq 2\).

We need the following intermediate result.

Lemma 4.20. Let \(\beta, \beta', m, n\) be as in Theorem 3.5. Let \((Z, Z'), \{(Z^k, Z'^k)\}_{k \geq 0}\) be extended stochastic controlled processes such that
\[
\Gamma^{\beta,\beta';m,n}(Z, Z') \vee \sup_{k \geq 0} \Gamma^{\beta,\beta';m,n}(Z^k, Z'^k) < \infty
\]
and for each \(s \in [0, T]\), \(\lim_k (Z^k_s, Z'^k_s) = (Z_s, Z'_s)\) in \(L_m\). Then
\[
\lim_k \sup_{t \in [0, T]} \left| \int_0^t Z^k dX - \int_0^t Z dX \right| = 0 \text{ in } L_m.
\]

Proof. Define for each \((s, t) \in \Delta\), \(A^k_{s,t} = Z^k_s \delta X_{s,t} + Z'^k_s \gamma X_{s,t}\) and similarly for \(A_{s,t}\). By assumptions, we have for each \(s \in [0, T]\), \(\lim_k \sup_{t \in [s, T]} |A^k_{s,t} - A_{s,t}| = 0 \text{ in } L_m\). Applying Theorem 3.5 and Corollary 2.9 yields the result. \(\square\)
Theorem 4.19. Using mollifiers, we can find a sequence of functions \( \{b^n, \sigma^n, f^n, (f^n)\}' \) such that \( b^n, \sigma^n \) are bounded Lipschitz functions (with respect to spatial variables), while \( (f^n, (f^n)') \) belongs to \( \mathcal{D}^{\beta,\beta'}_X \mathcal{C}_b^3 \),

\[
\lim_n \sup_{t \in [0,T]} \left( \|f^n_t - f_t\|_{\gamma-1} + \|(f^n)'_t - f'_t\|_{\gamma-2} + |b^n_t - b_t|_\infty + |\sigma^n_t - \sigma_t|_\infty \right) = 0
\]

and additionally:

\[
\sup_{t \in [0,T]} \left( |b^n(t, x) - b^n(t, \bar{x})| + |\sigma^n(t, x) - \sigma^n(t, \bar{x})| \right) \leq |x - \bar{x}| \quad \forall x, \bar{x},
\]

\[
\sup_n \left( \|(f^n, (f^n)')\|_{\gamma-1;[0,T]} + \|(f^n, (f^n)')\|_{X;\beta,\beta';[0,T]} + |b^n|_\infty + |\sigma^n|_\infty \right)
\leq C \left( \|(f, f')\|_{\gamma-1;[0,T]}, |b|_\infty, |\sigma|_\infty \right),
\]

uniformly in \( n \geq 0 \).

Let \( (\bar{\Omega}, \bar{\mathcal{G}}, \bar{\mathbb{P}}, \{\bar{F}_t\}) \) be a probability space which support an \( \{\bar{F}_t\} \)-Brownian motion \( \bar{B} \) and a random variable \( \xi \) with law \( \mu \). For each \( n \), let \( Y^n \) be the unique solution on \([0, T]\) to the rough stochastic differential equation

\[
dY^n = b^n(r, Y^n)dr + \sigma^n(r, Y^n)dB + (f^n, (f^n)')(r, Y^n)dB, \quad Y^n_0 = \xi.
\]

From Theorem 4.6, \( Y^n \) exists and is an \( L_{m, \infty} \)-solution for every \( m \geq 2 \). From Proposition 4.5, we see that for every \( m \geq 2 \),

\[
\sup_n \|\delta Y^n\|_{\alpha; m} \leq \sup_n \|\delta Y^n\|_{\alpha; m, \infty} < \infty.
\]

This in turn implies that the law of \( \{Y^n\} \) is tight on \( C([0, T]) \). By Skorokhod embedding, we can find a subsequence \( \{n_k\} \), a complete filtered probability space \((\tilde{\Omega}, \tilde{\mathcal{G}}, \tilde{\mathbb{P}}, \{\tilde{F}_t\})\) such that \( \tilde{B} \) is an \( \{\tilde{F}_t\} \)-Brownian motion, \( \text{Law}(\tilde{Y}^{n_k}, \tilde{B}) = \text{Law}(Y^{n_k}, B) \) and \( \lim_{k \to \infty} Y^{n_k} = Y \) a.s.

for every \( s \in [0, T] \). In particular, \( \tilde{Y}^{n_k} \) is an \( L_{m, \infty} \)-solution to (4.1) with coefficients \((b^{n_k}, \sigma^{n_k}, f^{n_k}, (f^{n_k})')\) and Brownian motion \( \tilde{B} \). Sending \( n_k \to \infty \) and using Lemma 4.20, we see that \( \tilde{Y} \) is a solution to (4.1) with coefficients \((b, \sigma, f, f')\) and Brownian motion \( \tilde{B} \).

We now turn our attention to uniqueness.

**Theorem 4.21 (Uniqueness in law).** Let \( b, \sigma \) be bounded Lipschitz functions and suppose that both \((f, f'), (Df, Df')\) are deterministic controlled vector fields in \( \mathcal{D}^{\alpha} \mathcal{C}_b^\gamma \) and \( \mathcal{D}^{\alpha-2n} \mathcal{C}_b^{\gamma-1} \) respectively, where \( \gamma \geq 1/\alpha \). Let \((Y, B, \{F_t\})\) and \((\bar{Y}, \bar{B}, \{\bar{F}_t\})\) be two integrable solutions to (4.1) defined respectively on stochastic bases \((\Omega, \mathcal{G}, \mathbb{P})\) and \((\bar{\Omega}, \bar{\mathcal{G}}, \bar{\mathbb{P}})\) such that \( \text{Law}(Y_0) = \text{Law}(\bar{Y}_0) \). Then \( Y \) and \( \bar{Y} \) have the same law on \( C([0, T]; W) \).

**Proof.** When \( \gamma > 1/\alpha \), for \( T > 0 \) small enough, we have

\[
(Y, f(Y)) = \lim_{n \to \infty} \Phi^{T, B}_{T} \circ \Phi^{T, B}_{T} \circ \cdots \circ \Phi^{T, B}_{T}
\]

where \( \Phi^{T, B} \) is the fixed point map given by (4.20) (we emphasize here its dependency on the underlying Brownian motion). In particular, there is a measurable map \( \Psi: C([0,T];\bar{V}) \to C([0,T];W) \) such that \( Y|_{[0,T]} = \Psi(B) \). In the critical case when \( \gamma = 1/\alpha \), we reason in the following way. We choose a sequence \( \{(f^n, (f^n)')\} \) as in the proof of Theorem 4.19. Let \( Y^n \) be the solution to (4.1) corresponding to the coefficients \((\sigma, f^n, b)\). By a tightness argument similar to the one in the proof of Theorem 4.19, we can find a complete filtered probability space \((\tilde{\Omega}, \tilde{\mathcal{G}}_t, \tilde{\mathbb{P}})\), an \( \{\tilde{F}_t\} \)-Brownian motion \( \tilde{B} \) and processes \( \tilde{Y}^n \) on it so that

- \((\tilde{Y}^n, \tilde{B}) \overset{\text{law}}{=} (Y^n, B)\).
there is a subsequence \( \{k_n\} \) so that \( \lim_{n} \tilde{Y}^{k_n} = \tilde{Y}^{(k)} \) in \( C([0, T]; W) \) a.s.

- \( \tilde{Y}^{(k)} \) as above is a solution to (4.1) (with coefficients \((\sigma, f, b)\)).

Let \( \{k_n\} \) and \( \{l_n\} \) be two subsequences such that \( \lim_{n} \tilde{Y}^{k_n} = \tilde{Y}^{(k)} \) and \( \lim_{n} \tilde{Y}^{l_n} = \tilde{Y}^{(l)} \) in \( C([0, T]; W) \) a.s. Since \( \tilde{Y}^{(k)} \) and \( \tilde{Y}^{(l)} \) are solutions to (4.1) on the same stochastic basis, by Theorem 4.10, it is necessary that \( \tilde{Y}^{(k)} = \tilde{Y}^{(l)} \).

Then it is a strong solution to the “doubly” Itô SDE (A.1) as always, \( \tilde{Y} \) satisfies the integral equation (A.1) given on a product stochastic basis, \( \tilde{Y} \in \mathcal{C}^{\alpha}([0, T]) \), \( 1/3 < \alpha < 1/2 \), for all \( \omega \). The theorem below can be seen as extension from related statements in [CDFO13, FZK23] that dealt with (flow-transformed and random rough path, respectively) based approaches, as discussed in the introduction.

**Theorem A.1.** (i) Assume the process \( \tilde{Y}(\omega^B, \omega^W) := Y^X(\omega^B)\mid_{X=W(\omega^W)} \) is adapted. Then it is a strong solution to the “doubly” Itô SDE (1.5) on \( \Omega \).

(ii) The law of \( Y^X \) provides a regular conditional distribution of the conditional law of \( \tilde{Y} \) given \( \sigma(W_t : 0 \leq t \leq T) = \sigma(W_t : 0 \leq t \leq T) \), continuous in \( X = W \).

We note that the adaptedness assumption is void when we are in the (existence / uniqueness) setting of Theorem 4.6, this is the same argument as e.g. in [FH20, Thm 9.1]. We also note that (ii) as an easy consequence of the definition of \( \tilde{Y} \), noting that \( W \) is independent or \( \mathcal{F}^B_T \), upon which \( Y^X(\omega^B), [0, T] \), depends measurably. Having the theory of RSDE in place, the proof of (i) is also not difficult: In view of Proposition 4.3 it suffices to show that the randomization \( X \rightsquigarrow W(\omega^W) \) of,

\[
\int_0^T f_s(Y_s^X) dX_s \big|_{X=W(\omega^W)} = \int_0^T f_s(\bar{Y}_s) dW_s.
\]

The right-hand side is a classical Itô-integral, on \( \Omega \), and hence the limit of

\[
\sum_{[s,t] \in \mathcal{P}} f_s(\bar{Y}_s)(W_t - W_s)
\]

for a.e. \( (\omega^B, \omega^W) \), along a suitable sequence \( \mathcal{P} = \mathcal{P}^n \) of partitions (and then trivially for any further subsequence). On the other hand, we know that for every \( \omega^W \), and thus every fixed rough path realization \( X = (W, \bar{W})(\omega^W) \),

\[
(A.1) \int_0^T f_s(Y_s^X) dX_s \big|_{X=W(\omega^W)} = \lim \sum_{[s,t] \in \mathcal{P}} f_s(\bar{Y}_s)(W_t - W_s) + ((Df_s)f_s + f'_s)(\bar{Y}_s)\bar{W}_{s,t}
\]

with limit in probability (on \( \Omega^B \)) and hence also for a.e. \( \omega^B \) along a suitable (and then any further sub-)sequence of partitions (which may depend on \( \omega^W \)). On the other hand, one sees as in [FH20, Prop. 5.1] or [FZK23, Lem. 6.1] that \( \sum((Df_s)f_s + f'_s)(\bar{Y}_s)\bar{W}_{s,t} \to 0 \).
in probability\(^\text{19}\) (on \(\Omega\)), and hence for a.e. \((\omega^B, \omega^W)\), along a suitable sequence of partitions. Passing to another subsequence if necessary, we see that the contribution of \(\sum(...)W_{s,t}\) in (A.1) disappears in the limit. The conclusion is then clear.

**Appendix B. John Nirenberg inequality**

We present a self-contained proof of Proposition 2.7. The main argument relies on Proposition B.1, we need the following elementary result.

**Proposition B.1.** Let \(V\) be a continuous adapted process. Suppose that for every \(s \leq t\), we have

\[
\|E|\delta V_{s,t}|\|_\infty \leq \Gamma(t - s)^\kappa
\]

Then there are universal finite constants \(C, c > 0\) which are independent from \(\Gamma, \kappa, T\) such that

\[
Ee^{\lambda \sup_{t \in [0,T]}|\delta V_{0,t}|} \leq Ce^{c(\lambda \Gamma)^{1/\kappa}T} \quad \text{for every } \lambda > 0.
\]

**Proof of Proposition 2.7.** Define \(V_t = |Y_t - Y_0|X\). Then \(V\) is a.s. continuous and satisfies

\[
\|E(|\delta V_{s,t}|\|_\infty \leq \|E(|\delta Y_{s,t}|X|\|_\infty \leq \|\delta Y\|_{\kappa;1,\infty}(t - s)^\kappa,
\]

for every \((s, t) \in \Delta\). From here, Proposition 2.7 is a direct consequence of Proposition B.1. \(\square\)

To show Proposition B.1, we need the following elementary result.

**Lemma B.2.** If \(X\) and \(Y\) are nonnegative random variables satisfying

\[
\mathbb{P}(Y > \alpha + \beta) \leq \theta \mathbb{P}(Y > \alpha) + \mathbb{P}(X > \theta \beta)
\]

for every \(\alpha > 0, \beta > 0\) and \(\theta \in (0, 1)\); then for every \(m \in (0, \infty)\),

\[
\|Y\|_m \leq c_m m \|X\|_m
\]

where the constant \(c_m\) is given by \((c_m)^m = m(1 + 1/m)^{(m+1)^2}\). (Note that \(\sup_m c_m < \infty\).)

**Proof.** We choose \(\beta = h\alpha\) for some \(h > 0\) and integrate the inequality with respect to \(ma^{-1}d\alpha\) over \((0, k/(1 + h))\) to get that

\[
(1 + h)^{-m} \int_0^k ma^{-1} \mathbb{P}(Y > \alpha) d\alpha \leq \theta \int_0^k ma^{-1} \mathbb{P}(Y > \alpha) d\alpha + \int_0^\infty ma^{-1} \mathbb{P}(X > \theta h\alpha) d\alpha.
\]

Sending \(k \to \infty\) and using the layer cake representation \(EX^m = \int_0^\infty ma^{-1} \mathbb{P}(X > \alpha) d\alpha\), we obtain that

\[
[(1 + h)^{-m} - \theta] EY^m \leq (\theta h)^{-m} EX^m.
\]

We now choose \(h = \frac{1}{m}\) and \(\theta = \left(\frac{m}{m+1}\right)^{m+1}\) to obtain the result. \(\square\)

**Proof of Proposition B.1.** Let \(\lambda > 0\) be fixed. For each \((s, t) \in \Delta\), define

\[
V^*_t = \sup_{r \in [0,t]} |\delta V_{0,r}| \quad \text{and} \quad M_{s,t} = \|Ee^{\lambda(V^*_t - V^*_s)}\|_\infty.
\]

Following [Lê22b], it is sufficient to establish that

\[
M_{s,t} \leq M \quad \text{whenever } 2\lambda \Gamma(t - s)^\kappa \leq e^{-3}
\]

\[^{19}\text{And in fact, in } L^2 \text{ in view of (ii) of Definition 4.2.}\]
and
\begin{equation}
M_{s,t} \leq M_{s,u} M_{u,t} \quad \text{whenever } s \leq u \leq t
\end{equation}
for some universal finite constant $M$. Indeed, assume for the moment that (B.3)-(B.4) hold. Partitioning $[0, T]$ by points $0 = t_0 < t_1 < \ldots < t_n = T$ so that $\lambda \Gamma(t_k - t_{k-1})^\kappa \leq e^{-3}$ for each $k$, one sees that
\[ M_{0,T} \leq \prod_{k=1}^{n} M_{t_{k-1},t_k} \leq M^n. \]

Omitting details, one can then choose $\{t_k\}$ efficiently so that $n$ is approximately $1 + T(e^{3\lambda \Gamma})^{\frac{1}{\kappa}}$. With such choice, the above estimate for $M_{0,T}$ implies (B.2).

Being a simple consequence of conditioning, the proof of (B.4) is left to the reader. Inequality (B.3) is a variant of the classical John–Nirenberg inequality for continuous processes (see [SV79, Exercise A.3.2]). Its proof is divided into several steps below.

Step 1. We show that for every $(s,t) \in \Delta$ and every stopping time $\mu$ satisfying $s \leq \mu \leq t$, one has
\begin{equation}
\|\mathbb{E}(\delta V_{\mu,t} | \mathcal{F}_\mu)\|_\infty \leq \Gamma(t-s)^\kappa.
\end{equation}
Indeed, fix $(s,t) \in \Delta$ and put $C = \Gamma(t-s)^\kappa$. Let $\mu$ be a stopping time, $s \leq \mu \leq t$, and suppose that $\mu$ takes finitely many values $\{s_1 < \ldots < s_k\}$. We have
\[ \mathbb{E}_{\mu}|V_t - V_\mu| = \sum_j \mathbb{E}_{\mu}[|V_t - V_\mu| \mathbf{1}_{(\mu=s_j)}] = \sum_j \mathbf{1}_{(\mu=s_j)} \mathbb{E}_{s_j}[|V_t - V_{s_j}|] \leq C, \]
where we used (B.1) to obtain the last inequality. For a general stopping time $\mu$, $s \leq \mu \leq t$, define for each $n$, the stopping time $\mu^n$,
\[ \mu^n = 0 \quad \text{if} \quad \mu = 0, \]
\[ \mu^n = j2^{-n}t \quad \text{if} \quad (j-1)2^{-n}t < \mu \leq j2^{-n}t, \quad j \leq 2^n t. \]
It is obvious that $\{\mu^n\}$ is decreasing to $\mu$ and $\mu^n \leq t$. Then by triangle inequality
\[ \mathbb{E}_{\mu}[|V_t - V_\mu| \wedge N] \leq \mathbb{E}_{\mu}\mathbb{E}_{\mu^n}[|V_t - V_{\mu^n}|] + \mathbb{E}_{\mu}[|V_{\mu^n} - V_\mu| \wedge N] \leq C + \mathbb{E}_{\mu}[|V_{\mu^n} - V_\mu| \wedge N]. \]

Note that $\lim_n V_{\mu^n} = V_\mu$ a.s. so that by Fatou lemma and Lebesgue dominated convergence theorem, we have $\mathbb{E}_{\mu}[|V_t - V_\mu| \wedge N] \leq C$. Sending $N \to \infty$ yields (B.5).

Step 2. We show that
\begin{equation}
\|\| \sup_{s \leq r \leq t} |V_r - V_s| \|_{\mathcal{F}_s} \|_m \|_\infty \leq 2c_m m \Gamma(t-s)^\kappa.
\end{equation}
Fix $s,t$. Without loss of generality, we can assume that $2\Gamma(t-s)^\kappa = 1$ so that by the previous step, for every stopping time $\mu$ with $s \leq \mu \leq t$, we have
\begin{equation}
\|\mathbb{E}_\mu|\delta V_{\mu,t}||_\infty \leq 1/2.
\end{equation}
We put $V^* = \sup_{r \in [s,t]} |V_r - V_s|$. Let $\alpha, \beta$ be two positive numbers and define
\[ \mu = t \wedge \inf\{r \in [s,t] : |V_r - V_s| > \alpha\}, \quad \nu = t \wedge \inf\{r \in [s,t] : |V_r - V_s| > \alpha + \beta\}, \]
with the standard convention that $\inf(\emptyset) = \infty$ (so $\mu = t$ and $\nu = t$ when these sets are empty). Clearly $\mu$ and $\nu$ are stopping times and $s \leq \mu \leq \nu \leq t$.

On the event $\{V^* > \alpha + \beta\}$, we have $|V_\mu - V_s| \geq \alpha + \beta$ and $|V_\nu - V_s| \geq \alpha$. In view of the triangle inequality $|V_\nu - V_s| \leq |V_\nu - V_\mu| + |V_\mu - V_s|$, this implies that
\[ \{V^* > \alpha + \beta\} \subset \{|V_\nu - V_\mu| \geq \beta, \quad V^* > \alpha\}. \]
It follows that for every $G \in \mathcal{F}_s$ and every $\theta \in (0, 1)$,
\[
\mathbb{P}(V^* > \alpha + \beta, G) \leq \mathbb{P}(|V^*_\nu - V^*_\mu| \geq \beta, V^* > \alpha, G) \\
\leq \mathbb{P}(|V^*_\nu - V^*_\mu| \geq \theta^{-1}, V^* > \alpha, G) + \mathbb{P}(1 > \theta \beta, V^* > \alpha, G).
\]

By conditioning, noting that $\{V^* > \alpha\}$ is $\mathcal{F}_\mu$-measurable, and applying Markov inequality we have
\[
\mathbb{P}(|V^*_\nu - V^*_\mu| \geq \theta^{-1}, V^* > \alpha, G) \leq \theta \|\mathbb{E}_\mu[\delta V_{\mu,\nu}]\|_\infty \mathbb{P}(V^* > \alpha, G).
\]
The conditional expectation is estimated using (B.7), this yields
\[
\mathbb{P}(|V^*_\nu - V^*_\mu| \geq \theta^{-1}, V^* > \alpha, G) \leq \theta \mathbb{P}(V^* > \alpha, G).
\]
Hence, we obtain from the above that
\[
\mathbb{P}(V^* > \alpha + \beta, G) \leq \theta \mathbb{P}(V^* > \alpha, G) + \mathbb{P}(1 > \theta \beta, V^* > \alpha, G).
\]

Applying Lemma B.2, we get $\|V^*_1\|_m \leq c_m \|1_G\|_m$. Given that $G$ is arbitrary in $\mathcal{F}_s$, a classical argument entails (B.6).

**Step 3.** Fix $\lambda > 0$. For $(s, t)$ such that $2\lambda \Gamma(t - s)\kappa \leq e^{-3}$, we have by Taylor’s expansion and (B.6) that
\[
\left\| \mathbb{E}_s \exp \left( \lambda \sup_{r \in [s, t]} |V_r - V_s| \right) \right\|_\infty \leq \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \left\| \mathbb{E}_s \left( \sup_{r \in [s, t]} |V_r - V_s| \right)^m \right\|_\infty \\
\leq 1 + \sum_{m=1}^{\infty} \frac{\lambda^m}{m!} (c_m m^m (2\Gamma(t - s)\kappa)^m) \leq M
\]
where $M = 1 + \sum_{m=1}^{\infty} a_m$ and $a_m = \frac{(c_m m^m)}{m} e^{-3m}$. Because $0 < m \to a_m \frac{a_{m+1}}{a_m} = e^{-1}$, $M$ is finite by the ratio test. Since $V^*_1 - V^*_s \leq \sup_{r \in [s, t]} |V_r - V_s|$, the previous estimate also implies that $\|\mathbb{E}_s e^{\lambda(V^*_r - V^*_s)}\|_\infty \leq M$ whenever $2\lambda \Gamma(t - s)\kappa \leq e^{-3}$, which is equivalent to (B.3). \(\square\)

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