Rates of Convergence for Chains of Expansive Markov Operators

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Abstract

We provide conditions that guarantee local rates of convergence in distribution of iterated random functions that are not nonexpansive mappings in locally compact Hadamard spaces. Our results are applied to stochastic instances of common algorithms in optimization, stochastic tomography for X-FEL imaging, and a stochastic algorithm for the computation of Fréchet means in model spaces for phylogenetic trees.

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1 Introduction

This work concerns abstract mathematical structures that guarantee convergence in distribution of Markov chains when the transition kernel is generated by mappings that are not nonexpansive. Our motivation for the abstract study pursued here is the very concrete application of X-ray free electron laser (X-FEL) imaging experiments [18, 27, 73]. Our framework provides conditions under which algorithms for the reconstruction of electronic densities or orbitals of a molecule from X-FEL measurements are guaranteed to be successful. Another motivating application comes from already established stochastic algorithms for computing the Fréchet means of phylogenetic trees in nonlinear model spaces. For this problem our analysis immediately yields improvements to the state of the art for these kinds of algorithms. While our focus is mainly mathematical, we briefly introduce the X-FEL imaging problem to set the scene.

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The goal of X-FEL imaging is to determine the electron density of a molecule from experimental samples of its scattering probability distribution (i.e. diffraction pattern). Without going into the intricate aspects of the measurement device or scattering theory, we imagine a three-dimensional measurement device that counts the occurrence of photons in a three-dimensional domain partitioned into voxels of a fixed size far away from (that is, in the far field of) a molecule that has been illuminated by a short X-FEL pulse. The molecule under observation is at a random orientation relative to the measurement device. The illuminating pulse is only long enough to cause a few (averaging between 10 and 100) scattering events from the interaction of a molecule’s electrons with the X-ray. The experiment is repeated about $10^9$ times, each time with the molecule at a different random orientation.

The physical model mapping the probability of a scattering event at a particular location on the molecule to the resulting electromagnetic field at large distances, is, to first order, simply the Fourier transform of the electronic density of the molecule. An X-FEL measurement is truly a random sample of the electromagnetic field far away from the molecule. The measurement device, however, can only measure the occurrence of photons, not their direction as indicated by the real and complex parts of the electromagnetic field. If both the probability of observing a photon at a certain location, and its direction – i.e. its phase – were known, then the electronic density of the molecule that produced those photons could be reconstructed from the observation by simply inverting the Fourier transform. When the probabilities of observing photons at each location in the far field of the molecule can be measured (as is the case in conventional coherent diffraction imaging [66] and wavefront sensing [52]), then the problem becomes the well-known phase retrieval problem, for which the mathematical analysis has been developed in [23, 52, 53, 56].

The algorithms that we envision for performing such reconstructions from X-FEL data proceed by constructing a projection mapping of the current estimate for the electron density of the molecule onto the nearest electron density that is consistent with each new X-FEL observation; in other words, the projection mapping is itself stochastic. We view this type of algorithmic procedure as a random function iteration generating a Markov chain whose transition kernel maps the current estimate to a new estimate based on new randomly sampled data.

In [38] it is shown that iterations of randomly selected $\alpha$-firmly nonexpansive mappings in a Euclidean space converge in the Prokhorov-Lévy metric to an invariant probability measure of the corresponding Markov operator. The analysis of Markov chains for the case of nonexpansive transition kernels in queuing theory can be found in [5, 6]. The transition kernels that we propose for the X-FEL application, however, are not nonexpansive; our analysis extends the more recent study of nonexpansive transition kernels in [38] to the expansive case using results from the deterministic theory of expansive fixed point mappings developed in [56]. We return to the specific application of X-FEL imaging in Section 4.

Random function iterations (RFI) [28] generalize deterministic fixed point iterations, and are a useful framework for studying a number of important applications. An RFI is a stochastic process of the form $X_{k+1} := T_{\xi_k} X_k$ ($k = 0, 1, 2, \ldots$) initialized by a random variable $X_0$ with distribution $\mu_0$ and values on some set $G$. This of course includes initialization from a deterministic point $x_0 \in G$ via the $\delta$-distribution. Here $\xi_k$ ($k = 0, 1, 2, \ldots$) is an element of a sequence of i.i.d. random variables that map from a probability space into a measurable space of indices $I$ (not necessarily countable) and $T_i$ ($i \in I$) are self-mappings on $G$. The iterates $X_k$ form a (time-homogeneous) Markov chain that takes values in the space $G$, which is, for our purposes, a Polish space. Deterministic fixed point iterations are included when the index set $I$ is just a singleton.

Notation, basic facts, and the main results are presented in Section 2. The assumptions on the mappings generating the Markov operators have been shown to be necessary and sufficient for
so that the regularity assumptions can be developed; namely, that

we present the statement of the main result, Theorem 3.4 which provides

where we

instance \[ and fundamental results, in particular regarding the convergence of Markov chains, see for
going existence of invariant measures and convergence of Markov chains require

This section follows the development of [38]. Our notation is standard. The natural numbers
are denoted by \( \mathbb{N} \). For \( G \), an abstract topological space, \( \mathcal{B}(G) \) denotes the Borel \( \sigma \)-algebra and

\((G, \mathcal{B}(G))\) is the corresponding measure space. We denote by \( \mathcal{P}(G) \) the set of all probability
measures on \( G \). The notation \( X \sim \mu \in \mathcal{P}(G) \) means that the law of \( X \), denoted \( \mathcal{L}(X) \), satisfies

\( \mathcal{L}(X) := \mathbb{P}(X \in \cdot) = \mu \), where \( \mathbb{P} \) is the probability measure on some underlying
probability space. All of these different ways of indicating a measure \( \mu \) will be used. We take
for granted a familiarity with probability theory and Markov chains. For general background
probability texts on Markov chain convergence are not directly applicable to our particular
setting, which includes the special conditions that are placed on the operators \( T_i \).

Throughout, the pair \((G, d)\) denotes a separable metric space with metric \( d \). Results concerning
existence of invariant measures and convergence of Markov chains require \emph{completeness} of
the metric space, that is that the space is \emph{Polish}. In characterizing the regularity of the building
blocks, however, completeness is not required. For our main result, Theorem 2.6, we restrict
the setting to self-mappings of a compact subset of a Hadamard space, a separable complete
uniformly convex metric space with nonpositive curvature. Our application examples (Section 4)
are mostly in Euclidean spaces, which are Hadamard spaces with zero curvature, but the example of computing mean phylogenetic trees (Section 4.4) requires the generality of Hadamard
spaces. The purpose of starting with abstract separable metric spaces is to make it easier to
see which results rely in part on the regularity of the space in addition to those properties
we require of the mappings. Markov operators generated from mappings on Hadamard spaces
represent the current limits of the direct applicability of the theory presented here.

The distance of a point \( x \in G \) to a set \( A \subset G \) is denoted by \( d(x, A) := \inf_{w \in A} d(x, w) \).
Continuing with the development initiated in the introduction, we will consider a collection of
mappings \( T_i : G \to G \), \( i \in I \), on \((G, d)\), where \( I \) is an arbitrary index set - not necessarily
countable. The measure space of indexes is denoted by \((I, \mathcal{I})\), and \( \xi \) is an \( I \)-valued random
variable on a probability space. The pairwise independence of two random variables \( \xi \) and
\( \eta \) is denoted \( \xi \independent \eta \). The random variables \( \xi_k \) in the sequence \((\xi_k)_{k \in \mathbb{N}}\) (abbreviated \((\xi_k)\)) are
independent and identically distributed (i.i.d.) with \( \xi_k \) distributed as \( \xi \) \((\xi_k \sim \xi)\). The method
of random function iterations is formally presented in Algorithm 1.

We will use the notation

\[ X_k^{X_0} := T_{\xi_0} \cdots T_{\xi_{k-1}} X_0 \quad (1) \]

to denote the sequence of the RFI initialized with \( X_0 \sim \mu_0 \). When characterizing sequences
Algorithm 1: Random Function Iterations (RFI)

Initialization: Set $X_0 \sim \mu_0 \in \mathcal{P}(G)$, $X_0 \perp \perp (\xi_k)$ with $\xi_k \sim \xi$ i.i.d.

1. for $k = 0, 1, 2, \ldots$ do
2. $X_{k+1} = T_{\xi_k}X_k$

We define the Markov chains corresponding to the random function iteration in terms of its transition kernel $p : G \times \mathcal{B}(G) \to [0, 1]$ where $p(\cdot, A)$ is measurable for all $A \in \mathcal{B}(G)$ and $p(x, \cdot)$ is a probability measure for all $x \in G$. A sequence of random variables $(X_k)$, $X_k : (\Omega, \mathcal{F}, \mathbb{P}) \to (G, \mathcal{B}(G))$ is called a Markov chain with transition kernel $p$ if for all $k \in \mathbb{N}$ and $A \in \mathcal{B}(G)$ $\mathbb{P}$-a.s. the following hold:

(i) $\mathbb{P}(X_{k+1} \in A \mid X_0, X_1, \ldots, X_k) = \mathbb{P}(X_{k+1} \in A \mid X_k)$;
(ii) $\mathbb{P}(X_{k+1} \in A \mid X_k) = p(X_k, A)$.

Under Assumption 2.1, the sequence of random variables $(X_k)$ generated by Algorithm 1 is a Markov chain with transition kernel $p$ given by

$$
(x \in G)(A \in \mathcal{B}(G)) \quad p(x, A) := \mathbb{P}(\Phi(x, \xi) \in A) = \mathbb{P}(T_{\xi}x \in A)
$$

for the measurable update function $\Phi : G \times I \to G$, $(x, i) \mapsto T_i x$ [38, Proposition 2.3].

The Markov operator $\mathcal{P}$ is defined pointwise for a measurable function $f : G \to \mathbb{R}$ via

$$
(x \in G) \quad \mathcal{P}f(x) := \int_G f(y)p(x, dy),
$$

when the integral exists.

The Markov operator $\mathcal{P}$ is Feller if $\mathcal{P}f \in C_b(G)$ whenever $f \in C_b(G)$, where $C_b(G)$ is the set of bounded and continuous functions from $G$ to $\mathbb{R}$. This property is central to the theory of existence of invariant measures. An application of Lebesgue’s dominated convergence theorem immediately yields that if $T_i$ is continuous for all $i \in I$ then the Markov operator $\mathcal{P}$ is Feller, see also [35, Theorem 4.22].

We denote the dual Markov operator $\mathcal{P}^* : \mathcal{P}(G) \to \mathcal{P}(G)$ acting on a measure $\mu$ by action on the right by $\mathcal{P}$ via

$$
(A \in \mathcal{B}(G)) \quad (\mathcal{P}^*\mu)(A) := (\mu\mathcal{P})(A) := \int_G p(x, A)\mu(dx).
$$

We can thus write the distribution of the $k$-th iterate of the Markov chain generated by Algorithm 1 as $\mathcal{L}(X_k) = \mu_0\mathcal{P}^k$. 

4
2.1 The Stochastic Fixed Point Problem

The stochastic feasibility problem is:

\[
\text{Find } x^* \in C := \{ x \in G \mid \mathbb{P}(x = T_k x) = 1 \}. \tag{3}
\]

This was first studied in [24, 25] in the context of stochastic convex set feasibility and has been extended more generally to finding the (almost sure) fixed points of multi-valued mappings in [37, 38]. A point \( x \) such that \( x = T_k x \) is a fixed point of the operator \( T_k \); the set of all such points is denoted by

\[
\text{Fix } T_k = \{ x \in G \mid x = T_k x \}.
\]

In [37] it was assumed that \( C \neq \emptyset \). If \( C = \emptyset \), following Butnariu [24] this is called the inconsistent stochastic feasibility.

One need only consider linear systems of equations, \( Ax = b \), to demonstrate an inconsistent feasibility problem. Each linear equation, \( \langle a_j, x \rangle = b_j \) represents a hyperplane; the solution to \( Ax = b \) is the intersection of the hyperplanes. Obviously, when the system is overdetermined, or underdetermined with noise, the intersection is likely empty - an inconsistent feasibility problem. Most readers will reflexively reformulate the problem as a least squares minimization problem, but there are compelling reasons why this is not always the best way around the dilemma of the inconsistent feasibility problem.

Avoiding debate about least-squares versus feasibility, what all can surely agree upon is that the inconsistency of the problem formulation is an artifact of asking the wrong question. A fixed point of the dual Markov operator \( \mathcal{P} \) is called an invariant measure and satisfies \( \pi \mathcal{P} = \pi \). The set of all invariant probability measures is denoted by \( \text{inv} \mathcal{P} \). The solution to the dilemma of the inconsistent feasibility problem is instead to solve the following stochastic fixed point problem:

\[
\text{Find } \pi \in \text{inv } \mathcal{P}. \tag{4}
\]

**Example 2.2** (inconsistent stochastic feasibility). Consider the (trivially convex, nonempty and closed) sets \( C_{-1} := \{-1\} \) and \( C_1 := \{1\} \) together with a random variable \( \xi \) such that \( \mathbb{P}(\xi = 1) = \mathbb{P}(\xi = -1) = 1/2 \). The mappings \( T_i x = P_{C_i} x = i \) for \( x \in \mathbb{R} \) and \( i \in I = \{-1,1\} \) are the projections onto the sets \( C_{-1} \) and \( C_1 \). The RFI iteration then amounts to just random jumps between the values \(-1\) and \(1\). So it holds that \( \mathbb{P}(T_i x = i) = 1/2 \) for all \( x \in \mathbb{R} \) and hence there is clearly no feasible fixed point to this iteration, that is, the set \( C \) defined in (3) is empty. Nevertheless, by disintegration we have

\[
\mathbb{P}(X_{k+1} = i) = \mathbb{E}[\mathbb{P}(T_{\xi_1} X_k = i \mid X_k)] = \frac{1}{2}
\]

for all \( k \in \mathbb{N} \). That means the unique invariant distribution to which the distributions of the iterates of the RFI (i.e. \( \mathbb{P}(X_k \in \cdot)_{\mathbb{R}} \)) converges is \( \pi = \frac{1}{2}(\delta_{-1} + \delta_1) \), and this is attained after one iteration. The least squares solution is the mean of this invariant distribution.

2.2 Modes of convergence

Algorithm 1 generates a sequence of random variables whose distributions, when conditions allow, converge to solutions to (4). For this we focus on convergence in distribution. Let \( (\mu_k) \) be a sequence of probability measures on \( G \). The sequence \( (\mu_k) \) is said to converge in distribution to \( \mu \) whenever \( \mu \in \mathcal{P}(G) \) and for all \( f \in C_b(G) \) it holds that \( \mu_k f \rightarrow \mu f \) as \( k \rightarrow \infty \), where \( \mu f := \int f(x) \mu(dx) \). Equivalently a sequence of random variables \( (X_k) \) is said to converge in distribution if their laws \( (\mathcal{L}(X_k)) \) do.
We consider convergence in distribution for the corresponding sequence of measures \( \mathcal{L}(X_k) \) to a probability measure \( \pi \in \mathcal{P}(G) \), i.e. for any \( f \in C_b(G) \)
\[
\mathcal{L}(X_k)f = E[f(X_k)] \to \pi f, \quad \text{as } k \to \infty.
\]
An elementary fact from the theory of Markov chains is that, if the Markov operator \( \mathcal{P} \) is Feller and \( \pi \) is a cluster point of the sequence of measures \( (\mu_k) := (\mu_0 P_k) \) with respect to convergence in distribution then \( \pi \) is an invariant probability measure \[36, \text{Theorem 1.10}\]. We will assume existence of invariant measures. For more on this theory readers are referred to \[15, 35, 36\].

Quantifying convergence is essential for establishing estimates for the distance of the iterates to the limit point, when this exists. A sequence \((x_k)\) in a metric space \((G, d)\) is said to converge \textit{R-linearly} to \( \tilde{x} \) with rate \( c \in [0, 1) \) when
\[
\exists \beta > 0 : \quad d(x_k, \tilde{x}) \leq \beta c^k \quad \forall k \in \mathbb{N}. \tag{5}
\]
The sequence \((x_k)\) is said to converge \textit{Q-linearly} to \( \tilde{x} \) with rate \( c \in [0, 1) \) if
\[
\exists c \in [0, 1) : \quad d(x_{k+1}, \tilde{x}) \leq cd(x_k, \tilde{x}) \quad \forall k \in \mathbb{N}. \tag{6}
\]
By definition, Q-linear convergence implies R-linear convergence with the same rate; the converse implication does not hold in general. Q-linear convergence is encountered with contractive fixed point mappings, and this leads to a priori and a posteriori error estimates. For more on these notions see \[62, \text{Chapter 9}\].

A common metric for spaces of measures is the \textit{Wasserstein metric}. For \( p \geq 1 \) let
\[
\mathcal{P}_p(G) = \left\{ \mu \in \mathcal{P}(G) \mid \exists x \in G : \int d^p(x, y) \mu(dy) < \infty \right\}. \tag{7}
\]
The Wasserstein \( p \)-metric on \( \mathcal{P}_p(G) \), denoted \( W_p \), is defined by
\[
W_p(\mu, \nu) := \left( \inf_{\gamma \in C(\mu, \nu)} \int_{G \times G} d^p(x, y) \gamma(dx, dy) \right)^{1/p} \quad (p \geq 1) \tag{8}
\]
where \( C(\mu, \nu) \) is the set of \textit{couplings} of \( \mu \) and \( \nu \):
\[
C(\mu, \nu) := \{ \gamma \in \mathcal{P}(G \times G) \mid \gamma(A \times G) = \mu(A), \gamma(G \times A) = \nu(A) \quad \forall A \in \mathcal{B}(G) \}. \tag{9}
\]
We will also refer in Proposition 4.1 to the \textit{Prokhorov-Lévy distance}, \( d_P \), metrizing weak convergence in distribution:
\[
d_P(\mu, \nu) = \inf \{ \epsilon > 0 \mid \mu(A) \leq \nu(\mathbb{B}(A, \epsilon)) + \epsilon, \nu(A) \leq \mu(\mathbb{B}(A, \epsilon)) + \epsilon \quad \forall A \in \mathcal{B}(G) \}. \tag{10}
\]

To conclude this section, we state without proof the elementary fact that, for the setting considered here, the set of invariant measures is closed. For proof see for example \[35, \text{Section 5}\].

\textbf{Lemma 2.3.} \textit{Let \( G \) be a Polish space and let \( \mathcal{P} \) be a Feller Markov operator, which is in particular the case under Assumption 2.1, if \( T_i \) is continuous for all \( i \in I \). Then the set of associated invariant measures \( \text{inv} \mathcal{P} \) is closed with respect to the topology of convergence in distribution.}
2.3 Regularity

The regularity of \( T_i \) depends on the application. In [38] the regularity of \( T_i \) is used to obtain generic convergence results for the corresponding Markov operator, but the regularity of the Markov operator is never explicitly defined. We make this explicit here by lifting the regularity of \( T_i \) to corresponding notions of regularity of Markov operators in probability spaces in the case that \( T_i \) is expansive following the development of such mappings in [56] in a Euclidean space setting. When working with expansive mappings, multi-valued mappings appear naturally. In the stochastic setting, to ease the notation and avoid certain technicalities, we will consider only single-valued mappings \( T_i \) that are only almost \( \alpha \)-firmly nonexpansive in expectation. Recent studies define the regularity of fixed point mappings in \( p \)-uniformly convex spaces \((p \in (1, \infty))\) with parameter \( c > 0 \) \([13, 48]\). These are uniquely geodesic metric spaces \((G,d)\) for which the following inequality holds \([60]\):

\[
(\forall t \in [0,1])(\forall x, y, z \in G) \quad d(z, (1-t)x \oplus ty)^p \leq (1-t)d(z, x)^p + td(z, y)^p - \frac{\kappa}{2}t(1-t)d(x, y)^p, \quad (11)
\]

where \( w = (1-t)x \oplus ty \) for \( t \in (0,1) \) denotes the point \( w \) on the geodesic connecting \( x \) and \( y \) such that \( d(w, x) = td(x, y) \). The definitions below hold formally in this nonlinear setting. The only object whose properties are strongly tied to the geometry of the space is the transport discrepancy \( \psi \) defined below in (14). We are limited so far to locally compact Hadamard spaces, which are locally compact complete \( \text{CAT}(0) \) spaces \((\text{Alexandrov} [1] \text{ and Gromov} [33])\). More generally, a \( \text{CAT}(\kappa) \) space is a geodesic metric space with sufficiently small triangles possessing comparison triangles with sides the same length as the geodesic triangle but for which the distance between points on the geodesic triangle are less than or equal to the distance between corresponding points on the comparison triangle. \( \text{CAT}(\kappa) \) spaces are separable, but not complete, and locally 2-uniformly convex with parameter \( c \leq 2 \). A \( \text{CAT}(0) \) space has \( p = c = 2 \) in (11).

The definition below conforms with the same objects defined in \([13, 48]\).

**Definition 2.4** (pointwise almost \((\alpha\)-firmly) nonexpansive mappings in \( \text{CAT}(0) \) spaces). Let \((G,d)\) be a \( \text{CAT}(0) \) metric space and \( D \subset G \) and let \( F : D \to G \).

(i) The mapping \( F \) is said to be **pointwise almost nonexpansive at** \( x_0 \in D \) **on** \( D \), abbreviated **pointwise ane**, whenever

\[
\exists \epsilon \in [0,1] : \quad d(Fx, Fx_0) \leq \sqrt{1 + \epsilon} d(x, x_0), \quad \forall x \in D. \quad (12)
\]

The **violation** is a value of \( \epsilon \) for which (12) holds. When the above inequality holds for all \( x_0 \in D \) then \( F \) is said to be **almost nonexpansive on** \( D \) (ane). When \( \epsilon = 0 \) the mapping \( F \) is said to be (pointwise) nonexpansive.

(ii) The mapping \( F \) is said to be **pointwise almost \( \alpha \)-firmly nonexpansive at** \( x_0 \in D \) **on** \( D \), abbreviated **pointwise afne** whenever

\[
\exists \epsilon \in [0,1) \text{ and } \alpha \in (0,1) : \quad d^2(Fx, Fx_0) \leq (1 + \epsilon)d^2(x, x_0) - \frac{1-\alpha}{\alpha}\psi(x, x_0, Fx, Fx_0) \quad x \in D, \quad (13)
\]

where the **transport discrepancy** \( \psi \) of \( F \) at \( x, x_0, Fx \) and \( Fx_0 \) is defined by

\[
\psi(x, x_0, Fx, Fx_0) := d^2(Fx, Fx_0) + d^2(Fx, Fx_0) + d^2(x_0, x_0) - d^2(Fx, Fx_0) - d^2(Fx_0, Fx_0). \quad (14)
\]
When the above inequality holds for all \( x_0 \in D \) then \( F \) is said to be \textit{almost }\( \alpha \)-\textit{firmly nonexpansive on }\( D \), \((\alpha\text{-fne})\). The \textit{violation} is the constant \( \epsilon \) for which (13) holds. When \( \epsilon = 0 \) the mapping \( F \) is said to be \textit{(pointwise) }\( \alpha \)-\textit{firmly nonexpansive}, abbreviated \textit{(pointwise) }\( \alpha \)-\textit{fne.}

Nonexpansive and \( \alpha \)-firmly nonexpansive mappings have been studied for decades under various names and in various settings \([3, 4, 7, 8, 20–22, 30, 32, 34, 47, 57]\). The violation \( \epsilon \) in (12) and (13) is a recently introduced feature in the analysis of fixed point mappings, first appearing in this form in \([56]\). Many are familiar with mappings for which (12) holds with \( \epsilon < 0 \) at all \( x_0 \in G \), i.e. contraction mappings. In this case, the whole technology of pointwise \( \alpha \)-fne mappings is not required since an appropriate application of Banach’s fixed point theorem delivers existence of fixed points and convergence of fixed point iterations at a linear rate. We will have more to say about this later; for the moment it suffices to note that the mappings associated with one of our target applications are expansive on all neighborhoods of fixed points and we will therefore require another property to guarantee convergence.

Our definition with \( \alpha = 1/2 \) and \( \epsilon = 0 \) is equivalent to the definition of \textit{firmly contractive} mappings given in \([20, \text{Definition 6}]\). In linear spaces the mappings with \( \epsilon = 0 \) are called “averaged” \([8]\). Ariza-Ruiz, Leuştean and López-Acedo \([3]\) defined \( \lambda \)-firmly nonexpansive operators on subsets \( D \) of \( \text{W-hyperbolic spaces} \), as those operators satisfying

\[
\exists \lambda \in (0, 1) : \ d(Fx, Fy) \leq d((1 - \lambda)x \oplus \lambda Fx, (1 - \lambda)y \oplus \lambda Fy) \quad \forall x, y \in D. \tag{15}
\]

Another notion of regularity in the context of Hadamard spaces that is equivalent to (15) for an operator \( F : H \to H \) and \( x, y \in H \) uses

\[
\phi_F(t) := d((1 - t)x \oplus tFx, (1 - t)y \oplus tFy), \quad \text{for } t \in [0, 1]. \tag{16}
\]

In \([32, \text{Chapter 24}]\) an operator \( F : H \to H \) is called \textit{firmly nonexpansive} whenever \( \phi_F \) is nonincreasing on \([0, 1]\) (see also \([12, \text{Definition 2.1.13}]\)). It is clear from the definition that \( F : D \to D \) satisfies (15) if and only if \( \phi_F \) is a nonincreasing function on \([0, 1]\) for all \( x, y \in D \). Banert \([9, \text{Remark pp.658}]\) shows that any mapping \( F \) satisfying (15) for all \( \lambda \in (0, 1] \) is \( \alpha \)-firmly nonexpansive with constant \( \alpha = 1/2 \).

The transport discrepancy \( \psi \) is the key to identifying the regularity required for convergence of fixed point iterations in metric spaces and the definition makes clear the contribution of the geometry of the space.

\textbf{Lemma 2.5} (\( \psi \) is nonnegative in \( \text{CAT}(0) \) spaces, Proposition 4 of \([13]\)). \textit{Let }\( (G, d) \text{ be a } \text{CAT}(0) \text{ metric space and } F : D \to G \text{ for } D \subset G. \text{ Then the transport discrepancy defined by (14) is nonnegative for all } x, y \in D. \text{ Moreover, if } F \text{ is pointwise } \alpha\text{-fne at } x_0 \in D \text{ with violation } \epsilon \text{ on } D, \text{ then } F \text{ is pointwise a\textae at } x_0 \text{ on } D \text{ with violation at most } \epsilon.\)

In \( \text{CAT}(\kappa) \) spaces the above statement does not hold. It is well known, for example, that in a \( \text{CAT}(\kappa) \) metric space the projector onto a convex set is \( \alpha \)-fne with \( \alpha = 1/2 \), but it is not nonexpansive \([3]\). In Hadamard space settings, when a mapping is firmly nonexpansive, it is clear that it is also nonexpansive. This implication was shown in \([3]\) to be a consequence of Busemann convexity and does not hold in general metric spaces. Nevertheless, the implication is recovered for \( \rho \)-uniformly convex spaces for \textit{pointwise} firmly nonexpansive mappings \textit{at their fixed points}, since the corresponding transport discrepancy \( \psi \) is nonnegative in this case \([13, \text{Proposition 4(i)}]\). It would be interesting to investigate these notions in Busemann spaces where, based on known extensions of the tools of variational analysis to Banach spaces, we conjecture that many of these notions of regularity carry over.
On normed linear spaces, when $\| \cdot \|$ is the norm induced by the inner product and $d(x, y) = \|x - y\|$, the transport discrepancy $\psi$ defined by (14) has the representation
\[
\psi(x, x_0, Fx, Fx_0) = \|(x - Fx) - (x_0 - Fx_0)\|^2. \tag{17}
\]
This representation shows the connection between our definition and more classical notions. Indeed, in a Hilbert space setting $(G, \| \cdot \|)$, a mapping $F : D \to G$ ($D \subset G$) is pointwise a.e.-finite at $x_0$ with constant $\alpha$ and violation at most $\epsilon$ on $D$ if and only if [56, Proposition 2.1]
\[
\|Fx - Fx_0\|^2 \leq (1 + \epsilon)\|x - x_0\|^2 - \frac{1 - \alpha}{\alpha} \|((x - Fx) - (x_0 - Fx_0))\|^2 \quad \forall x \in D. \tag{18}
\]

2.4 Main results

We can now state the main theorem concerning Markov operators $\mathcal{P}$ with update function $\Phi(x, i) = T_i(x)$ and transition kernel $p$ given by (2) for self mappings $T_i : G \to G$. For any $\mu_0 \in \mathcal{P}_2(G)$, we denote the distributions of the iterates of Algorithm 1 by $\mu_k = \mu_0^{\mathcal{P}^k} = \mathcal{L}(X_k)$, and we denote $d_{\mathcal{W}_2}(\mu_k, \text{inv } \mathcal{P}) := \inf_{\pi' \in \text{inv } \mathcal{P}} \mathcal{W}_2(\mu_k, \pi')$. It will be assumed that $\text{inv } \mathcal{P} \neq \emptyset$.

Convergence is quantified by an implicitly defined gauge function. Recall that $\rho : [0, \infty) \to [0, \infty)$ is a gauge function if $\rho$ is continuous, strictly increasing with $\rho(0) = 0$, and $\lim_{t \to \infty} \rho(t) = \infty$. This is defined in terms of another nonnegative mapping $\theta_{\tau, \epsilon} : [0, \infty) \to [0, \infty)$ with parameters $\tau > 0$ and $\epsilon > 0$:

\[
(i) \quad \theta_{\tau, \epsilon}(0) = 0; \quad (ii) \quad 0 < \theta_{\tau, \epsilon}(t) < t \ \forall t \in (0, \overline{T}) \text{ for some } \overline{T} > 0. \tag{19}
\]

The gauge then is given by
\[
\rho \left( \frac{(1 + \epsilon)t^2 - (\theta_{\tau, \epsilon}(t))^2}{\tau} \right)^{1/2} \Rightarrow \theta_{\tau, \epsilon}(t) = \left( (1 + \epsilon)t^2 - \tau \left( \frac{\rho^{-1}(t)}{\epsilon} \right)^2 \right)^{1/2} \tag{20}
\]
for $\tau > 0$ fixed.

The rate of convergence of the sequences of measures will be determined by $\theta_{\tau, \epsilon}$, with parameters $\tau$ and $\epsilon$ given by a certain characterization of the regularity of the Markov operator $\mathcal{P}$ (see Definition 3.1), which translates to a second regularity of the Markov operator $\mathcal{P}$, metric subregularity (see Definition 3.6) via the gauge $\rho$. In the case of linear convergence this becomes
\[
\rho(t) = rt \iff \theta_{\tau, \epsilon}(t) = \left( (1 + \epsilon) - \frac{\tau}{\rho^{-1}(t)} \right)^{1/2} t \quad (r \geq \sqrt{\frac{\tau}{1 + \epsilon}}).
\]
The conditions in (19) in this case simplify to $\theta_{\tau, \epsilon}(t) = \gamma t$ where
\[
0 < \gamma := 1 + \epsilon - \frac{\tau}{\rho^{-1}(t)} < 1 \iff \sqrt{\frac{\tau}{1 + \epsilon}} \leq r \leq \sqrt{\frac{\tau}{\epsilon}}. \tag{21}
\]

**Theorem 2.6** (convergence rates). Let $(H, d)$ be a separable Hadamard space and let $G \subset H$ be compact. Let $T_i : G \to G$ be continuous for all $i \in I$ and define $\Psi : \mathcal{P}_2(G) \to \mathbb{R}_+ \cup \{+\infty\}$ by
\[
\Psi(\mu) := \inf_{\pi \in \text{inv } \mathcal{P}} \inf_{\gamma \in C(\mu, \pi)} \left( \int_{G \times G} \mathbb{E} \left[ \psi(x, y, T_\xi x, T_\xi y) \right] \gamma(dx, dy) \right)^{1/2}. \tag{22}
\]
Assume furthermore:

(a) there is at least one $\pi \in \text{inv } \mathcal{P} \cap \mathcal{P}_2(G)$ where $\mathcal{P}$ is the Markov operator with update function $\Phi$ given by (2);
(b) $\Phi$ satisfies
\[
\exists \epsilon \in (0, 1), \alpha \in (0, 1) : \forall x, y \in G,
\]
\[
\mathbb{E} \left[ d^2(\Phi(x, \xi), \Phi(y, \xi)) \right] \leq (1 + \epsilon) d^2(x, y) - \frac{1 - \alpha}{\alpha} \mathbb{E} \left[ \psi(x, y, \Phi(x, \xi), \Phi(y, \xi)) \right],
\]
and
\[
(c) \Psi(\pi) = 0 \iff \pi \in \mathcal{P} \text{ and for all } \mu \in \mathcal{R}(G)
\]
\[
\inf_{\pi \in \mathcal{P}} W_2(\mu, \pi) = \inf_{\pi \in \Psi^{-1}(0)} W_2(\mu, \pi)
\]
\[
\leq \rho \left( d(0, \Psi(\mu)) \right) = \rho(\Psi(\mu)).
\]
with gauge $\rho$ given implicitly by (20) with $\tau = (1 - \alpha)/\alpha$ and the function $\theta_{r, \epsilon}$ satisfying (19).

Then for any $\mu_0 \in \mathcal{R}(G)$ the distributions $(\mu_k)$ of the iterates of Algorithm 1 converge in the $W_2$ metric to some $\pi^{\mu_0} \in \mathcal{P} \cap \mathcal{R}(G)$ with rate $O(s_k(t_0))$ where $t_0 = d_{W_2}(\mu_0, \mathcal{P} \cap \mathcal{R}(G))$ and $s_k(t_0) := \lim_{N \to \infty} \sum_{j=k}^{N} \theta_{r, \epsilon}(t_0)$; in other words
\[
d_{W_2}(\mu_{k+1}, \mathcal{P}) \leq \theta_{r, \epsilon}(d_{W_2}(\mu_k, \mathcal{P})) \forall k \in \mathbb{N}. \tag{24}
\]

An immediate corollary of this theorem is the following specialization to linear convergence.

**Corollary 2.7** (linear convergence rates). Under the same assumptions as in Theorem 2.6, if $\Psi$ satisfies (c) with gauge $\rho(t) = r \cdot t$ and constant $r$ satisfying $\sqrt{1 - \alpha}/\alpha < r < \sqrt{1 + \epsilon/\alpha}$, then the sequence of iterates $(\mu_k)$ converges $R$-linearly to some $\pi^{\mu_0} \in \mathcal{P} \cap \mathcal{R}(G)$:
\[
d_{W_2}(\mu_{k+1}, \mathcal{P}) \leq c d_{W_2}(\mu_k, \mathcal{P}) \tag{25}
\]
where $c := \sqrt{1 + \epsilon - \frac{1 - \alpha}{r \alpha}} < 1$ and $r \geq r'$ satisfies $r \geq \sqrt{(1 - \alpha)/(\alpha(1 + \epsilon))}$. If $\mathcal{P}$ consists of a single point then convergence is $Q$-linear.

### 3 Background theory and proofs of the main results.

As indicated by the discussion following Definition 2.4, assumption (b) of Theorem 2.6 has deep roots in fixed point theory. Assumption (c) has a similarly central significance in variational analysis. We develop each of these assumptions for the present setting in order.

#### 3.1 Almost $\alpha$-firm nonexpansive mappings in expectation

To begin, we develop assumption (b) of Theorem 2.6. The next definition uses the update function $\Phi$ defined in Assumption 2.1(b) and is an extension of [38, Definition 2.8].

**Definition 3.1** (pointwise almost ($\alpha$-firmly) nonexpansive in expectation). Let $(G, d)$ be a CAT(0) metric space, let $T_i : G \to G$ for $i \in I$, and let $\Phi : G \times I \to G$ be given by $\Phi(x, i) = T_i x$. Let $\psi$ be defined by (14) and let $\xi$ be an $I$-valued random variable.

(i) The mapping $\Phi$ is said to be pointwise almost nonexpansive in expectation at $x_0 \in G$ on $G$, abbreviated pointwise ane in expectation, whenever
\[
\exists \epsilon \in [0, 1) : \mathbb{E} \left[ d(\Phi(x, \xi), \Phi(x_0, \xi)) \right] \leq \sqrt{1 + \epsilon} d(x, x_0), \forall x \in G. \tag{26}
\]

When the above inequality holds for all $x_0 \in G$ then $\Phi$ is said to be $\text{almost nonexpansive - ane - in expectation}$ on $G$. As before, the violation is a value of $\epsilon$ for which (26) holds. When the violation is 0, the qualifier “almost” is dropped.
(ii) The mapping \( \Phi \) is said to be pointwise almost \( \alpha \)-firmly nonexpansive in expectation at \( x_0 \in G \) on \( G \), abbreviated pointwise \( \alpha \)-fne in expectation, whenever

\[
\exists \epsilon \in [0,1), \alpha \in (0,1): \quad \forall x \in G, \\
E \left[ d^2(\Phi(x, \xi), \Phi(x_0, \xi)) \right] \leq (1 + \epsilon) d^2(x, x_0) - \frac{1-\alpha}{\alpha} E \left[ \psi(x, x_0, \Phi(x, \xi), \Phi(x_0, \xi)) \right].
\]  

(27)

When the above inequality holds for all \( x_0 \in G \) then \( \Phi \) is said to be almost \( \alpha \)-firmly nonexpansive (\( \alpha \)-fne) in expectation on \( G \). The violation is a value of \( \epsilon \) for which (27) holds. When the violation is 0, the qualifier “almost” is dropped and the abbreviation \( \alpha \)-fne in expectation is used.

**Proposition 3.2.** Let \((G,d)\) be a CAT(0) space. The mapping \( \Phi: G \times I \to G \) given by \( \Phi(x,i) = T_i x \) is pointwise \( \alpha \)-fne in expectation at \( y \) on \( G \) with constant \( \alpha \) and violation at most \( \epsilon \) and pointwise \( \alpha \)-fne in expectation at \( y \) on \( G \) with violation at most \( \epsilon \) whenever \( T_i \) is pointwise \( \alpha \)-fne at \( y \) on \( G \) with constant \( \alpha \) and violation no greater than \( \epsilon \) for all \( i \).

**Proof.** By Lemma 2.5, whenever \((G,d)\) is a CAT(0) space \( \psi(x,y,\Phi(x,i), \Phi(y,i)) \geq 0 \) for all \( i \) and for all \( x,y \in G \), so the expectation \( E[\psi(x,y,\Phi(x,\xi), \Phi(y,\xi))] \) is well-defined and nonnegative for all \( x,y \in G \) (the value +\( \infty \) can be attained). This implies that, for all \( i \), \( T_i \) is pointwise ane at \( y \) on \( G \) with violation at most \( \epsilon \) on \( G \) whenever it is pointwise \( \alpha \)-fne at \( y \) with constant \( \alpha \) on \( G \) with violation at most \( \epsilon \) on \( G \) for all \( i \). It follows immediately from the definition, then, that \( \Phi \) is pointwise \( \alpha \)-fne in expectation at \( y \) on \( G \) with violation at most \( \epsilon \) on \( G \), and also pointwise ane in expectation at \( y \) on \( G \) with violation at most \( \epsilon \) on \( G \). \( \square \)

Following [38, Definition 2.10] we lift these notions of regularity to Markov operators. Denote the set of couplings where the distance \( W_2(\mu_1, \mu_2) \) is attained by

\[
C_\psi(\mu_1, \mu_2) := \left\{ \gamma \in C(\mu_1, \mu_2) \mid \int_{G \times G} d^2(x,y) \gamma(dx,dy) = W_2^2(\mu_1, \mu_2) \right\}.
\]  

(28)

Even though \( W_2(\mu_1, \mu_2) \) is defined as the infimum over all couplings, whenever this is finite the infimum is attained, and hence in this case \( C_\psi(\mu_1, \mu_2) \) is nonempty [38, Lemma A.7].

**Definition 3.3** (pointwise almost (\( \alpha \)-firmly) nonexpansive Markov operators). Let \((G,d)\) be a CAT(0) metric space, and let \( P \) be a Markov operator with transition kernel

\[
(x \in G)(A \in \mathcal{B}(G)) \quad p(x,A) := \mathbb{P}(\Phi(x,\xi) \in A)
\]

where \( \xi \) is an \( I \)-valued random variable and \( \Phi: G \times I \to G \) is a measurable update function. Let \( \psi \) be defined by (14).

(i) The Markov operator is said to be pointwise almost nonexpansive in measure at \( \mu_0 \in \mathcal{P}(G) \) on \( \mathcal{P}(G) \), abbreviated pointwise ane in measure, whenever

\[
\exists \epsilon \in [0,1): \quad W_2(\mu P, \mu_0 P) \leq \sqrt{1 + \epsilon} W_2(\mu, \mu_0), \quad \forall \mu \in \mathcal{P}(G).
\]

(29)

When the above inequality holds for all \( \mu_0 \in \mathcal{P}(G) \) then \( P \) is said to be almost nonexpansive (ane) in measure on \( \mathcal{P}(G) \). As before, the violation is a value of \( \epsilon \) for which (29) holds. When the violation is 0, the qualifier “almost” is dropped.
(ii) The Markov operator $P$ is said to be pointwise almost $\alpha$-firmly nonexpansive in measure at $\mu_0 \in G$ on $P(G)$, abbreviated pointwise $\alpha$-fne in measure, whenever

$$\exists \epsilon \in [0,1), \alpha \in (0,1) : \forall \mu \in P(G), \forall \gamma \in C_\ast(\mu, \mu_0)$$

$$W_2(\mu P, \mu_0 P)^2 \leq (1 + \epsilon)W_2(\mu, \mu_0)^2 - \frac{1 - \alpha}{\alpha} \int_{G \times G} \mathbb{E} [\psi(x, y, \Phi(x, \xi), \Phi(y, \xi))] \gamma(dx, dy).$$

(30)

When the above inequality holds for all $\mu_0 \in P(G)$ then $P$ is said to be $\alpha$-fne in measure on $P(G)$. The violation is a value of $\epsilon$ for which (30) holds. When the violation is 0, the qualifier “almost” is dropped and the abbreviation $\alpha$-fne in measure is employed.

**Remark 3.4:** By Lemma 2.5, when $(G, d)$ is a CAT(0) space the expectation on the right hand side of (30) is nonnegative, and the corresponding Markov operator is pointwise ane in measure at $\mu_0$ whenever it is pointwise $\alpha$-fne in measure at $\mu_0$ (Proposition 3.2). In particular, when $\mu = \mu_0 \in \text{inv} P$ the left hand side is zero and

$$\int_{G \times G} \mathbb{E} [\psi(x, y, T_{\xi} x, T_{\xi} y)] \gamma(dx, dy) = 0.$$

Here the optimal coupling is the diagonal of the product space $G \times G$ and $\psi(x, x, T_{\xi}x, T_{\xi}x) = 0$ for all $x \in G$.

**Proposition 3.5.** Let $(G, d)$ be a separable Hadamard space (a complete CAT(0) space) and $T_i : G \to G$ for $i \in I$, let $\Phi : G \times I \to G$ be given by $\Phi(x, i) = T_i x$ and let $\psi$ be defined by (14). Denote by $P$ the Markov operator with update function $\Phi$ and transition kernel $p$ defined by (2). If $\Phi$ is $\alpha$-fne in expectation on $G$ with constant $\alpha \in (0,1)$ and violation $\epsilon \in [0,1)$, then the Markov operator $P$ is $\alpha$-fne in measure on $P_2(G)$ with constant $\alpha$ and violation at most $\epsilon$, that is, $P$ satisfies

$$W_2^2(\mu_1 P, \mu_2 P) \leq (1 + \epsilon)W_2^2(\mu_1, \mu_2) - \frac{1 - \alpha}{\alpha} \int_{G \times G} \mathbb{E} [\psi(x, y, \Phi(x, \xi), \Phi(y, \xi))] \gamma(dx, dy)$$

$$\forall \mu_2, \mu_1 \in P_2(G), \forall \gamma \in C_\ast(\mu_1, \mu_2).$$

(31)

**Proof.** If $W_2(\mu_1, \mu_2) = \infty$ the inequality holds trivially with the convention $+\infty - (+\infty) = +\infty$. So consider the case where $W_2(\mu_1, \mu_2)$ is finite. Since $(G, d)$ is a separable, complete metric space, by [38, Lemma A.7] the set of optimal couplings $C_\ast(\mu_1, \mu_2)$ is nonempty. Since $\Phi$ is $\alpha$-fne in expectation on $G$ with constant $\alpha$ and violation $\epsilon$, we have

$$\int_{G \times G} \mathbb{E} [d^2(\Phi(x, \xi), \Phi(y, \xi))] \hat{\gamma}(dx, dy) \leq$$

$$\int_{G \times G} \left( (1 + \epsilon)d^2(x, y) - \frac{1 - \alpha}{\alpha} \mathbb{E} [\psi_c(x, y, \Phi(x, \xi), \Phi(y, \xi))] \right) \hat{\gamma}(dx, dy),$$

where $\hat{\gamma}$ is any coupling in $C(\mu_1, \mu_2)$, not necessarily optimal. In particular, since, for a random variable $X \sim \mu_1$, we have $\Phi(X, \xi) \sim \mu_1 P$, and for a random variable $Y \sim \mu_2$, we have $\Phi(Y, \xi) \sim \mu_2 P$, then, again for any optimal coupling $\gamma \in C_\ast(\mu_1, \mu_2)$,

$$W_2^2(\mu_1 P, \mu_2 P) \leq \int_{G \times G} \mathbb{E} [d^2(\Phi(x, \xi), \Phi(y, \xi))] \gamma(dx, dy)$$

$$\leq \int_{G \times G} \left( (1 + \epsilon)d^2(x, y) - \frac{1 - \alpha}{\alpha} \mathbb{E} [\psi_c(x, y, \Phi(x, \xi), \Phi(y, \xi))] \right) \gamma(dx, dy)$$

$$= (1 + \epsilon)W_2^2(\mu_1, \mu_2) - \frac{1 - \alpha}{\alpha} \int_{G \times G} \mathbb{E} [\psi_c(x, y, \Phi(x, \xi), \Phi(y, \xi))] \gamma(dx, dy).$$

Since the measures $\mu_2, \mu_1 \in P_2(G)$ were arbitrary, as was the optimal coupling $\gamma \in C_\ast(\mu_1, \mu_2)$, this completes the proof. □
3.2 Metric subregularity

We move now to assumption (c) of Theorem 2.6. In [56] a general quantitative analysis for iterations of expansive fixed point mappings is proposed consisting of two principle components: the constituent mappings are pointwise aа-fп, and the transport discrepancy of the fixed point operator is metrically subregular. Recall that, for any mapping $\Psi: A \to B$, the inverse mapping $\Psi^{-1}(y) := \{ z \in A | \Psi(z) = y \}$, which clearly can be set-valued.

**Definition 3.6** (metric subregularity). Let $(A,d_A)$ and $(B,d_B)$ be metric spaces and let $\Psi: A \to B$. The mapping $\Psi$ is called metrically subregular with respect to the metric $d_B$ for $y \in B$ relative to $U \subset A$ with gauge $\rho$ whenever

$$\inf_{z \in \Psi^{-1}(y) \cap U} d_A(x,z) \leq \rho(d_B(y,\Psi(x))) \quad \forall x \in U \cap \Lambda. \quad (32)$$

Our definition is modeled after [29], where the case where the gauge is just a linear function -- $\rho(t) = rt$ for $r > 0$ -- is developed. In this case, metric subregularity is one-sided Lipschitz continuity of the (set-valued) inverse mapping $\Psi^{-1}$. We will refer to the case when the gauge is linear as linear metric subregularity. For connections of this notion to the concept of transversality in differential geometry and its use in variational analysis see [41]. In the case of a finite dimensional linear transition kernel, the constant $\rho$ of metric subregularity characterizes the spectral gap, or the difference between the two largest eigenvalues of $\text{Id} - T_i$.

We apply metric regularity to the Markov operator $P$ on $\mathcal{P}(G)$ with the Wasserstein metric. In particular, the gauge of metric subregularity $\rho$ is constructed implicitly in (20) from $\theta_{\tau, \epsilon}: [0, \infty) \to [0, \infty)$ satisfying (19).

Metric subregularity plays a central role in the implicit function paradigm for solution mappings [17, 29]; it is also notoriously difficult to verify. Linear metric subregularity was shown in [37, Theorem 3.15] to be necessary and sufficient for $R$-linear convergence in expectation of random function iterations for consistent stochastic feasibility. This result is a stochastic analog of [55, Theorem 2] in the deterministic setting and all of this has been extended to nonlinear spaces in [48, Theorem 16]. It is an open problem whether metric subregularity is necessary in the present setting, though we see no reason why it should not be.

We apply this to the Markov operator $P$ on the metric space $(\mathcal{P}_2(G), W_2)$ in the following manner. Recall the transport discrepancy $\psi$ defined in (14). We construct the surrogate mapping $\Psi: \mathcal{P}(G) \to \mathbb{R}_+ \cup \{+\infty\}$ defined by (22). We call this the invariant Markov transport discrepancy. It is not guaranteed that both $\text{inv } P$ and $C_\epsilon(\mu, \pi)$ are nonempty; when at least one of these is empty, we define $\Psi(\mu) := +\infty$. It is clear that $\Psi(\pi) = 0$ for any $\pi \in \text{inv } P$.

Whether $\Psi(\mu) = 0$ only when $\mu \in \text{inv } P$ is a property of the space $(G,d)$. Indeed, as noted in the discussion after Lemma 2.5, in $\text{CAT}(\kappa)$ spaces with $\kappa > 0$ the transport discrepancy $\psi$ can be negative, and so by cancellation it could happen on such spaces that the invariant Markov transport discrepancy $\Psi(\mu) = 0$ for $\mu \notin \text{inv } P$. In $\text{CAT}(0)$ spaces, and hence Hadamard spaces, $\psi$ is nonnegative but this is still not enough to guarantee that the invariant Markov transport discrepancy $\Psi$ takes the value 0 at $\mu$ if and only if $\mu \in \text{inv } P$. The regularity we require of $P$ is that the invariant Markov transport discrepancy $\Psi$ takes the value 0 at $\mu$ if and only if $\mu \in \text{inv } P$, and is metrically subregular for 0 relative to $\mathcal{P}_2(G)$ on $\mathcal{P}_2(G)$ defined in (7).
3.3 Contractive Markov operators

Before moving to our main results, we put the more familiar contractive mappings into the present context. A survey of random function iterations for contractive mappings in expectation can be found in [69]. An immediate consequence of [69, Theorem 1] is the existence of a unique invariant measure and linear convergence in the Wasserstein metric from any initial distribution to the invariant measure.

Contraction Markov operators have been studied in [43, 61] using the parallel notion of the coarse Ricci curvature $\kappa(x, y)$ of the Markov operator $\mathcal{P}$ between two points $x$ and $y$:

$$\kappa(x, y) := 1 - \frac{W_1(\delta_x \mathcal{P}, \delta_y \mathcal{P})}{d(x, y)}.$$ 

Generalizing this definition to $W_\rho$ yields the coarse Ricci curvature with respect to $W_\rho$:

$$\kappa_\rho(x, y) := 1 - \frac{W_\rho(\delta_x \mathcal{P}, \delta_y \mathcal{P})}{d(x, y)^\rho}.$$ 

A few steps lead from this object for the Markov operator $\mathcal{P}$ with update function $\Phi(\cdot, \xi) = T_\xi$ and transition kernel defined by (2) to the violation $\epsilon$ in Proposition 3.5. Indeed, a formal adjustment of the proof of [61, Proposition 2] establishes that the property $\kappa_2(x, y) \geq \kappa \in \mathbb{R}$ for all $x, y \in G$ is equivalent to

$$W_2(\mu \mathcal{P}, \mu' \mathcal{P}) \leq \sqrt{1 - \kappa} W_2(\mu, \mu') \quad \forall \mu, \mu' \in \mathcal{P}_2(G).$$

When $\kappa > 0$, i.e. when the coarse Ricci curvature is bounded below by a positive number, this characterizes contractivity of the Markov operator. The negative of the violation in (29) is just a lower bound on the coarse Ricci curvature in $W_2$: $-\epsilon = \kappa \leq \kappa_2(x, y)$ for all $x, y \in G$. The consequences of Markov operators with Ricci curvature bounded below by a positive number have been extensively investigated. Error estimates for Markov chain Monte Carlo methods under the assumption of positive Ricci curvature in $W_1$ (i.e. negative violation) are explored in [43]. Applications to waiting queues, the Ornstein–Uhlenbeck process on $\mathbb{R}^n$ and Brownian motion on positively curved manifolds, as well as demonstrations of how to verify the assumptions on the Ricci curvature are developed in [61]. Our approach extends this to expansive mappings, which allows one to treat our target application of electron density reconstructions from X-FEL experiments (see Section 4).

The next result shows that update functions $\Phi$ that are contractions in expectation generate $\alpha$-fine Markov operators with metrically subregular invariant Markov transport discrepancy. In this context see [45].

**Theorem 3.7.** Let $(G, \| \cdot \|)$ be a Hilbert space, let $\mathcal{T}_i : G \to G$ for $i \in I$ and let $\Phi : G \times I \to G$ be given by $\Phi(x, i) := T_i(x)$. Denote by $\mathcal{P}$ the Markov operator with update function $\Phi$ and transition kernel $\rho$ defined by (2). Suppose that $\Phi$ is a contraction in expectation with constant $r < 1$, i.e. $\mathbb{E}[\|\Phi(x, \xi) - \Phi(y, \xi)\|^2] \leq r^2 \|x - y\|^2$ for all $x, y \in G$. Suppose in addition that there exists $y \in G$ with $\mathbb{E}[\|\Phi(y, \xi) - y\|^2] < \infty$. Then the following hold.

(i) There exists a unique invariant measure $\pi \in \mathcal{P}_2(G)$ for $\mathcal{P}$ and

$$W_2(\mu_0 \mathcal{P}^n, \pi) \leq r^n W_2(\mu_0, \pi)$$

for all $\mu_0 \in \mathcal{P}_2(G)$; that is, the sequence $(\mu_k)$ defined by $\mu_{k+1} = \mu_k \mathcal{P}$ converges to $\pi$ $Q$-linearly (geometrically) from any initial measure $\mu_0 \in \mathcal{P}_2(G)$. 


(ii) \( \Phi \) is \( \alpha \)-fine in expectation with constant \( \alpha = (1+r)/2 \), and the Markov operator \( \mathcal{P} \) is \( \alpha \)-fine on \( \mathcal{P}_2(G) \); that is, \( \mathcal{P} \) satisfies (31) with \( \epsilon = 0 \) and constant \( \alpha = (1+r)/2 \) on \( \mathcal{P}_2(G) \).

(iii) If \( \Psi \) defined by (22) satisfies

\[
\exists q > 0 : \quad \Psi(\mu) \geq qW_2(\mu, \mu) \quad \forall \mu \in \mathcal{P}_2(G),
\]

then \( \Psi \) is linearly metrically subregular for 0 relative to \( \mathcal{P}_2(G) \) on \( \mathcal{P}_2(G) \) with gauge \( \rho(t) = (q(1-r))^{-1}t \).

**Proof.** Note that for any pair of distributions \( \mu_1, \mu_2 \in \mathcal{P}_2(G) \) and an optimal coupling \( \gamma \in C_*\{\mu_1, \mu_2\} \) (nonempty by [38, Lemma A.7]) it holds that

\[
W_2^2(\mu_1 \mathcal{P}, \mu_2 \mathcal{P}) \leq \int_{G \times G} E[d^2(\Phi(x, \xi), \Phi(y, \xi))] \gamma(dx, dy)
\]

\[
\leq r^2 \int_{G \times G} d^2(x, y) \gamma(dx, dy) = r^2W_2^2(\mu_1, \mu_2),
\]

where \( \xi \) is independent of \( \gamma \). Moreover, \( \mathcal{P} \) is a self-mapping on \( \mathcal{P}_2(G) \). To see this let \( \mu \in \mathcal{P}_2(G) \) independent of \( \xi \) and let \( y \) be a point in \( G \) where \( E[\|\Phi(y, \xi) - y\|^2] < \infty \). Then by the triangle inequality and the contraction property

\[
\int_G E[\|\Phi(x, \xi) - y\|^2] \mu(dx)
\]

\[
\leq 4 \left( \int_G E[\|\Phi(x, \xi) - \Phi(y, \xi)\|^2] \mu(dx) + E[\|\Phi(x, \xi) - y\|^2] \right)
\]

\[
\leq 4 \left( \int_G r^2\|x - y\|^2 \mu(dx) + E[\|\Phi(x, \xi) - y\|^2] \right) < \infty.
\]

Therefore \( \mu \mathcal{P} \in \mathcal{P}_2(G) \). Altogether, this establishes that \( \mathcal{P} \) is a contraction on the separable complete metric space \( (\mathcal{P}_2(G), W_2) \) and hence Banach’s Fixed Point Theorem yields existence and uniqueness of inv \( \mathcal{P} \) and \( Q \)-linear convergence of the fixed point sequence.

To see (ii), note that, by (17),

\[
E[\psi_2(x, y, T_\xi x, T_\xi y)] = E \left[ \| (x - \Phi(x, \xi)) - (y - \Phi(y, \xi)) \| \right]
\]

\[
= \|x - y\|^2 + E \left[ \|\Phi(x, \xi) - \Phi(y, \xi)\|^2 - 2\langle x - y, \Phi(x, \xi) - \Phi(y, \xi) \rangle \right]
\]

\[
\leq (1 + r)^2\|x - y\|^2,
\]

where the last inequality follows from the Cauchy-Schwarz inequality and the fact that \( \Phi(\cdot, \xi) \) is a contraction in expectation. Again using the contraction property and (34) we have

\[
E \left[ \|\Phi(x, \xi) - \Phi(y, \xi)\|^2 \right] \leq \|x - y\|^2 - (1 - r^2)\|x - y\|^2
\]

\[
\leq \|x - y\|^2 - \frac{r^2}{(1+r)^2}E[\psi_2(x, y, T_\xi x, T_\xi y)].
\]

The right hand side of this inequality is just the characterization (27) of mappings that are \( \alpha \)-fine in expectation with \( \alpha = (1+r)/2 \). The rest of the statement follows from Proposition 3.5.

The proof of (iii) is modeled after the proof of [13, Theorem 32]. By the triangle inequality and part (i) we have

\[
W_2(\mu_{k+1}, \mu_k) \geq W_2(\mu_k, \pi) - W_2(\mu_{k+1}, \pi)
\]

\[
\geq (1-r)W_2(\mu_k, \pi) \quad \forall k \in \mathbb{N}.
\]
On the other hand, (33) implies that $\Psi$ takes the value zero only at invariant measures so that by the uniqueness of invariant measures established in part (i)

$$\Psi^{-1}(0) \cap \mathcal{P}(G) = \text{inv }\mathcal{P} \cap \mathcal{P}(G) = \{\pi\}.$$  

Combining this with (35) and (33) then yields for all $k \in \mathbb{N}$

$$|\Psi(\mu_k) - 0| = \Psi(\mu_k) \geq qW_2(\mu_{k+1}, \mu_k) \geq q(1-r)W_2(\mu_k, \Psi^{-1}(0) \cap \mathcal{P}(G)).$$

In other words,

$$(q(1-r))^{-1}|\Psi(\mu_k) - 0| \geq W_2(\mu_k, \Psi^{-1}(0) \cap \mathcal{P}(G)) \quad \forall k \in \mathbb{N}. \quad (36)$$

Since this holds for any sequence $\mu_k$ initialized with $\mu_0 \in \mathcal{P}(G)$, we conclude that $\Psi$ is metrically subregular for 0 relative to $\mathcal{P}(G)$ with gauge $\rho(t) = (q(1-r))^{-1}t$ on $\mathcal{P}(G)$, as claimed. \hfill \square

### 3.4 Proofs of the main results

We are now in a position to prove the main result.

**Proof of Theorem 2.6.** First note that by assumption (i) the Markov operator $\mathcal{P}$ is a self-mapping on $\mathcal{P}(G)$, hence $W_2(\mu, \mu \mathcal{P}) < \infty$, and for any $\mu_1, \mu_2 \in \mathcal{P}(G)$ the set of optimal couplings $C_s(\mu_1, \mu_2)$ is nonempty [38, Lemma A.7]. Since $(H, d)$ is a Hadamard space and $G \subset H$, the function $\Psi(\mu)$ defined by (22) is extended real-valued, nonnegative (see Lemma 2.5), and finite since $C_s(\mu, \pi)$ and inv $\mathcal{P}$ are nonempty. Moreover, by assumption (c) $\Psi^{-1}(0) = \text{inv }\mathcal{P}$ and

$$\frac{1 - \alpha}{\alpha} \left( \rho^{-1} \left( \inf_{\pi \in \text{inv }\mathcal{P}} W_2(\mu, \pi) \right) \right)^2 \leq \frac{1 - \alpha}{\alpha} \Psi^2(\mu). \quad (37)$$

On the other hand, note that assumption (b) is that $\Psi$ is a-o-fine in expectation, so by definition (22), assumption (b) and Proposition 3.5 (which applies because we are on a separable Hadamard space) we have

$$\frac{1 - \alpha}{\alpha} \Psi^2(\mu) \leq \int_{G \times G} \mathbb{E}[\psi(x, y, T_{\xi}x, T_{\xi}y)] \gamma(dx, dy) \leq (1 + \epsilon)W_2^2(\mu, \pi) - W_2^2(\mu \mathcal{P}, \pi) \quad \forall \pi \in \text{inv }\mathcal{P}, \forall \mu \in \mathcal{P}(G). \quad (38)$$

Incorporating (37) into (38) and rearranging the inequality yields

$$W_2^2(\mu \mathcal{P}, \pi) \leq (1 + \epsilon)W_2^2(\mu, \pi) - \frac{1 - \alpha}{\alpha} \left( \rho^{-1} \left( \inf_{\pi' \in \text{inv }\mathcal{P}} W_2(\mu, \pi') \right) \right)^2 \quad \forall \pi \in \text{inv }\mathcal{P}, \forall \mu \in \mathcal{P}(G).$$

Since this holds at any $\mu \in \mathcal{P}(G)$, it certainly holds at the iterates $\mu_k$ with initial distribution $\mu_0 \in \mathcal{P}(G)$ since $\mathcal{P}$ is a self-mapping on $\mathcal{P}(G)$. Therefore

$$W_2(\mu_{k+1}, \pi) \leq \sqrt{(1 + \epsilon)W_2^2(\mu_k, \pi) - \frac{1 - \alpha}{\alpha} \left( \rho^{-1} \left( \inf_{\pi' \in \text{inv }\mathcal{P}} W_2(\mu_k, \pi') \right) \right)^2} \quad \forall \pi \in \text{inv }\mathcal{P}, \forall k \in \mathbb{N}. \quad (39)$$
Equation (39) simplifies. Indeed, by Lemma 2.3, \( \text{inv} \mathcal{P} \) is closed with respect to convergence in distribution. Moreover, since \( G \) is assumed to be compact, \( \mathcal{P}_2(G) \) is locally compact ([2, Remark 7.19] so, for every \( k \in \mathbb{N} \) the infimum in (39) is attained at some \( \pi_k \). This yields

\[
W_2^2(\mu_{k+1}, \pi_{k+1}) \leq W_2^2(\mu_{k+1}, \pi_k) \leq (1+\varepsilon)W_2^2(\mu_k, \pi_k) - \frac{1-\alpha}{\alpha} \left( \rho^{-1}(W_2(\mu_k, \pi_k)) \right)^2 \quad \forall k \in \mathbb{N}. \tag{40}
\]

Taking the square root and recalling (19) and (20) yields (24).

To obtain convergence, note that for \( \mu_0 \in \mathcal{P}_2(G) \) satisfying \( W_2(\mu_0, \pi) < \infty \) and \( \mu_0 \mathcal{P} \in \mathcal{P}_2(G) \) (exists by compactness of \( G \)), the triangle inequality and (40) yield

\[
W_2(\mu_{k+1}, \mu_k) \leq W_2(\mu_{k+1}, \pi_k) + W_2(\mu_k, \pi_k) \leq \theta_{\tau, \varepsilon}(W_2(\mu_k, \pi_k)) + W_2(\mu_k, \pi_k).
\]

Using (24) and continuing by backwards induction yields

\[
W_2(\mu_{k+1}, \mu_k) \leq \theta_{\tau, \varepsilon}^{k+1}(d_0) + \theta_{\tau, \varepsilon}^{k}(d_0)
\]

where \( d_0 := \inf_{\pi \in \text{inv} \mathcal{P}} W_2(\mu_0, \pi) \). Repeating this argument, for any \( k < m \)

\[
W_2(\mu_m, \mu_k) \leq \theta_{\tau, \varepsilon}^m(d_0) + 2 \sum_{j=k+1}^{m-1} \theta_{\tau, \varepsilon}^j(d_0) + \theta_{\tau, \varepsilon}^k(d_0).
\]

By assumption, \( \theta_{\tau, \varepsilon} \) satisfies (19), so for any \( \delta > 0 \)

\[
W_2(\mu_m, \mu_k) \leq \theta_{\tau, \varepsilon}^m(d_0) + 2 \sum_{j=k+1}^{\infty} \theta_{\tau, \varepsilon}^j(d_0) + \theta_{\tau, \varepsilon}^k(d_0) < \delta
\]

for all \( k, m \) large enough; that is the sequence \( (\mu_k) \) is a Cauchy sequence in \( (\mathcal{P}_2(G), W_2) \) – a separable complete metric space ([71, Theorem 6.9] – and therefore convergent to some probability measure \( \pi^{\text{inv}} \in \mathcal{P}_2(G) \). The Markov operator \( \mathcal{P} \) is Feller since \( T_1 \) is continuous and by [38, Proposition 3.1] (see also [36, Theorem 1.10]) when a Feller Markov chain converges in distribution, it does so to an invariant measure: \( \pi^{\text{inv}} \in \text{inv} \mathcal{P} \). \( \square \)

**Remark 3.8:** The compactness assumption on \( G \) can be dropped if \( (H, d) \) is a Euclidean space.

**Proof of Corollary 2.7.** In the case that the gauge \( \rho \) is linear with constant \( r' \), then \( \theta_{\tau, \varepsilon}(t) \) is linear with constant

\[
e = \sqrt{1 + \varepsilon - \frac{1 - \alpha}{r'^2 \alpha}} < 1,
\]

where \( r \geq r' \) satisfies \( r^2 \geq (1 - \alpha)/(\alpha(1 + \varepsilon)) \). Specializing the argument in the proof above to this particular \( \theta_{\tau, \varepsilon} \) shows that, for any \( k \) and \( m \) with \( k < m \), we have

\[
W_2(\mu_m, \mu_k) \leq d_0 c^m + 2d_0 \sum_{j=k+1}^{m-1} c^j + d_0 c^k. \tag{41}
\]

Letting \( m \to \infty \) in (41) yields R-linear convergence (5) with rate \( c \) given above and leading constant \( \beta = \frac{1+c}{1-c}d_0 \).

If, in addition, \( \text{inv} \mathcal{P} \) is a singleton, then \( \{\pi^{\text{inv}}\} = \text{inv} \mathcal{P} \) in the above and convergence is actually Q-linear, which completes the proof. \( \square \)
4 Examples: Stochastic Optimization and Inconsistent Nonconvex Feasibility

To fix our attention we focus on the following optimization problem

$$\min_{\mu \in \mathcal{P}(\mathbb{R})} \int_{\mathbb{R}} \mathbb{E}_\xi [f_{\xi^f}(x) + g_{\xi^g}(x)] \mu(dx). \quad (42)$$

It is assumed throughout that $f_i : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable for all $i \in I_f$ and that $g_i : \mathbb{R}^n \to \mathbb{R} \cup +\infty$ is proper (not everywhere infinite), lower semicontinuous ($g_i(\tau) \leq \lim \inf_{x \to \tau} g(x)$ for all $\tau$) for all $i \in I_g$ and subdifferentially regular, i.e. the limiting subdifferential of $g_i$ at $\tau$, where the subdifferential of $g_i$ at $\tau$, denoted $\partial g_i(\tau)$, is defined by

$$\partial g_i(\tau) := \{v \mid g_i(x) \geq g_i(\tau) + \langle v, x - \tau \rangle + o(\|x - \tau\|)\}.$$

In particular, when $g_i$ is differentiable at $\tau$, the regular subdifferential is a singleton consisting of the gradient: $\partial g_i(\tau) = \{\nabla g_i(\tau)\}$. For more background on nonsmooth analysis see [67]. The random variable with values on $I_f \times I_g$ will be denoted $\xi = (\xi^f, \xi^g)$. This model covers deterministic composite optimization as a special case: $I_f$ and $I_g$ consist of single elements and the measure $\mu$ is a point mass.

The algorithms reviewed in this section rely on resolvents of the subdifferentials/gradients of the functions $f_i$ and $g_i$, denoted $\mathcal{J}_{\partial f_i, \lambda}$ and $\mathcal{J}_{\partial g_i, \lambda}$. The resolvent of a multi-valued mapping $F$ from elements in $G \subset \mathbb{R}^n$ to one or more elements in $\mathbb{R}^n$ is defined by

$$\mathcal{J}_F(x) := \left(\frac{1}{\lambda} \text{Id} + F\right)^{-1}(x) := \{z \in \mathbb{R}^n \mid x = \frac{1}{\lambda} z + F(z)\}.$$

For proper, lower semicontinuous convex functions $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, this is equivalent to the proximal mapping [59] (often just called the prox mapping) defined by

$$\text{prox}_{f, \lambda}(x) := \arg\min_y \{f(y) + \frac{1}{2\lambda} d(y, x)^2\}. \quad (43)$$

In general one has

$$\text{prox}_{f, \lambda}(x) \subset \mathcal{J}_{\partial f, \lambda}(x) \quad (44)$$

whenever the subdifferential is defined. When $\lambda = 1$ in the definitions above, we will just write $\text{prox}_f$ or $\mathcal{J}_{\partial f}$.

4.1 Stochastic (nonconvex) forward-backward splitting

A splitting algorithm applied to problem (42) is any algorithm that proceeds by taking steps with respect to $f_{\xi^f}(x)$ and $g_{\xi^g}(x)$ separately and combining these either through convex combinations or compositions of some sort. Classical examples of splitting methods are the Gauss-Seidel and Jacobi algorithms for solving linear systems. We present a general prescription of the corresponding fixed point mapping, and then specialize this to more concrete instances.

Algorithm 2 is called forward-backward because the step in the direction of the negative gradient $-t \nabla f_{\xi^f}(X_k)$ is interpreted as a “forward” step (in the context of differential equations, this would be an explicit Euler step), while the step taken by applying the resolvent $\mathcal{J}_{\partial g_{\xi^g}}$ is interpreted as a “backward” step (in the context of differential equations, this would be an implicit Euler step).
When \( f_{\xi}(x) = f(x) + \xi \cdot x \) and \( g_{\xi} \) is the zero function, then this is just steepest descents with linear noise discussed in Section 2.1. More generally, (45) with \( g_{\xi} \) the zero function models stochastic gradient descents, which is a central algorithmic paradigm for nonconvex problems. The next statement is the nonconvex analog to \cite[Proposition 4.1]{38}. The main difference here is that the violation given by (48) depends on the step size \( t \) in (45). We will have more to say about this below. Part (i) of the statement is the nonconvex analog to \cite[Proposition 4.1(ii)]{38}; part (iv) of the statement covers the convex case already established in \cite[Proposition 4.1(iii)]{38}; this is included for completeness.

**Proposition 4.1.** On the Euclidean space \((\mathbb{R}^n, \| \cdot \|)\) suppose the following hold:

(a) for all \( i \in I_f \), \( \nabla f_i \) is Lipschitz continuous with constant \( L \) on \( G \subset \mathbb{R}^n \) and hypomonotone on \( G \) with violation \( \tau_f > 0 \) on \( G \subset \mathbb{R}^n \):

\[
- \tau_f \| x - y \|^2 \leq \langle \nabla f_i(x) - \nabla f_i(y), x - y \rangle \quad \forall x, y \in G.
\]

(b) there is a \( \tau_g \) such that for all \( i \in I_g \), the (limiting) subdifferential \( \partial g_i \) satisfies

\[
- \frac{\tau_g}{2} \left( \| x^+ + z \| - (y^+ + w) \right) \| z - w, \ x^+ - y^+ \|
\]

at all points \((x^+, z) \in \text{gph} \partial g_i \) and \((y^+, w) \in \text{gph} \partial g_i \) where \( z = x - x^+ \) for \( \{x^+\} = J_{\partial g_i}(x) \) for any \( x \in \bigcup_{i \in I_f} (\text{Id} - t \nabla f_i) \) \((G)\) and where \( w = y - y^+ \) for \( \{y^+\} = J_{\partial g_i}(y) \) for any \( y \in \bigcup_{i \in I_f} (\text{Id} - t \nabla f_i) \) \((G)\).

(c) \( T_{i}^{\text{FB}} \) is a self-mapping on \( G \subset \mathbb{R}^n \) for all \( i \).

Then the following hold.

(i) \( T_{i}^{\text{FB}} \) is \( \alpha \)-ac-fne on \( G \) with constant \( \alpha = 2/3 \) and violation at most

\[
\epsilon = \max \{0, (1 + 2 \tau_g) \left( 1 + t(2 \tau_f + 2t L^2) \right) - 1 \}
\]

for all \( i \in I_f \).

(ii) \( \Phi(x, i) := T_i \) is \( \alpha \)-ac-fne in expectation on \( G \) with constant \( \alpha = 2/3 \) and violation at most \( \epsilon \) given in (48).

(iii) The Markov operator \( P \) corresponding to (45) is \( \alpha \)-ac-fne in measure on \( \mathcal{P}_2(G) \) with constant \( \alpha = 2/3 \) and violation no greater than \( \epsilon \) given in (48), i.e. it satisfies (31).
Suppose that assumption (a) holds with condition (46) being satisfied for \( \tau_f < 0 \) (that is, \( \nabla f_i \) is strongly monotone for all \( i \)), and that condition (47) holds with \( \tau_g = 0 \) (for instance, when \( g_i \) is convex). Then, whenever there exists an invariant measure for the Markov operator \( \mathcal{P} \) corresponding to (45), for any fixed step length \( t \in (0, 2\alpha/L) \) the distributions of the sequences of random variables converge to an invariant measure in the Prokhorov-Lévy metric.

(v) Let \( G \) be compact and \( \mathcal{P}_2(G) \cap \text{inv } \mathcal{P} \neq \emptyset \). If \( \Psi \) given by (22) takes the value 0 only at points in \( \text{inv } \mathcal{P} \) and is metrically subregular for 0 on \( \mathcal{P}_2(G) \) with gauge \( \rho \) given by (20) with \( \tau = 1/2 \), \( \epsilon \) satisfying (48), and \( \theta_{\tau,\epsilon} \) satisfying (19), then the Markov chain converges to an invariant distribution in the \( W_2 \) metric with rate given by (24).

Before proving the statement, some background for conditions (46) and (47) might be helpful. The inequality (46) is satisfied by functions \( f \) that are \textit{prox-regular} \cite{64}. This traces back to Federer’s study of curvature measures \cite{31} where such functions would be called functions whose epigraphs have \textit{positive reach}. Inequality (47) is equivalent to the property that \( \mathcal{J}_{\partial g_i} \) is \( \alpha \)-affine with constant \( \alpha_i = 1/2 \) and violation \( \tau_g \) on \( G \) \cite[Proposition 2.3]{56}. Any differentiable function \( g_i \) with gradient satisfying (46) with constant \( \tau_g/(2(1 + \tau_g)) \) will satisfy (47) with constant \( \tau_g \). In the present setting, if \( g_i \) is \textit{prox-regular} on \( G \), then \( \partial g_i \) is hypomonotone on \( G \) and therefore satisfies (47) \cite{56}. Convex functions are trivially hypomonotone with constant \( \tau = 0 \). If the functions \( g_i \) are convex, then the violation (48) can be made arbitrarily small by taking the step size \( t \) small enough.

\textbf{Proof.} (i). This is \cite[Proposition 3.7]{56}.

(ii). This follows immediately from Part (i) above and Proposition 3.2.

(iii). This follows immediately from Part (ii) above and Proposition 3.5.

(iv). This is \cite[Proposition 4.1(iii)]{38}.

(v). This follows from Part (iii) and Theorem 2.6.

The compactness assumption on \( G \) in part (v) is just to permit the application of Theorem 2.6. As noted in Remark 3.8 this assumption can be dropped for mappings \( T_i \) on Euclidean space. The result narrows the work of proving convergence of stochastic forward-backward algorithms to verifying existence of \( \text{inv } \mathcal{P} \). The case of convex stochastic gradient descent was presented in \cite[Proposition 4.2]{38}.

### 4.2 Stochastic Douglas-Rachford

Another prevalent algorithm for nonconvex problems is the Douglas-Rachford algorithm \cite{50}. This is based on compositions of \textit{reflected resolvents}:

\[
R_f := 2\mathcal{J}_{\partial f} - \text{Id}.
\]

Algorithm 3 has been studied for solving large-scale, convex optimization and monotone inclusions (see for example \cite{19, 26}). Proposition 4.3 below opens the analysis to nonconvex, nonmonotone problems; the setting for this statement is given by the following assumptions.

\textbf{Assumption 4.2.} On the Euclidean space \( (\mathbb{R}^n, \| \cdot \|) \) the following assumptions hold.

(a) There is a \( \tau_g \) such that for all \( i \in I_g \), the (limiting) subdifferential \( \partial g_i \) satisfies

\[
-\frac{\tau_g}{2} \| (x^+ + z) - (y^+ + w) \|^2 \leq \langle z - w, x^+ - y^+ \rangle
\]

at all points \( (x^+, z) \in \text{gph } \partial g_i \) and \( (y^+, w) \in \text{gph } \partial g_i \) where \( z = x - x^+ \) for \( \{x^+\} = \mathcal{J}_{\partial g_i}(x) \) for any \( x \in G \subset \mathbb{R}^n \) and where \( w = y - y^+ \) for \( \{y^+\} = \mathcal{J}_{\partial g_i}(y) \) for any \( y \in G \).
Algorithm 3: Stochastic Douglas-Rachford Splitting

Initialization: Set $X_0 \sim \mu_0 \in \mathcal{P}_2(G)$, $X_0 \perp (\xi_k)$ with $\xi_k := (\xi_k^f, \xi_k^g) \sim \xi := (\xi^f, \xi^g)$ i.i.d. with values in $I_f \times I_g$.

\begin{algorithm}
\begin{algorithmic}
\State \For{$k = 0, 1, 2, \ldots$ } \Do
\State $X_{k+1} = T^{DR}_i X_k := \frac{1}{2} \left( R_{f_k^i} \circ R_{g_k^i} + Id \right) (X_k)$ \EndDo
\end{algorithmic}
\end{algorithm}

(b) There is a $\tau_f$ such that for all $i \in I_f$, the (limiting) subdifferential $\partial f_i$ satisfies

\[
- \frac{\tau_f}{2} \| (x^+ + z) - (y^+ + w) \|^2 \leq \left\langle z - w, x^+ - y^+ \right\rangle
\]

at all points $(x^+, z) \in \text{gph} \partial f_i$ and $(y^+, w) \in \text{gph} \partial f_i$ where $z = x - x^+$ for $\{x^+\} = J_G f_i(x)$ for any $x \in \bigcup_{j \in I_g} \{J_G b_j(G)\}$ and where $w = y - y^+$ for $\{y^+\} = J_G f_i(y)$ for any $y \in \bigcup_{j \in I_g} \{J_G b_j(G)\}$.

(c) $T^{DR}_i$ is a self-mapping on $G \subset \mathbb{R}^n$ for all $i$.

Proposition 4.3. Under Assumption 4.2 the following hold.

(i) For all $i \in I_f \times I_g$ the mapping $T^{DR}_i$ defined by (50) is a.a.-fne on $G$ with constant $\alpha = 1/2$ and violation at most

\[
\epsilon = \frac{1}{2} ((1 + 2\tau_g)(1 + 2\tau_f) - 1)
\]

on $G$.

(ii) $\Phi(x, i) := T^{DR}_i x$ is a.a.-fne in expectation with constant $\alpha = 1/2$ and violation at most $\epsilon$ given by (53).

(iii) The Markov operator $\mathcal{P}$ corresponding to (50) is a.a.-fne in measure with constant $\alpha = 1/2$ and violation no greater than $\epsilon$ given by (53), i.e. it satisfies (31).

(iv) Suppose that parts (a) and (b) of Assumption 4.2 hold with conditions (51) and (52) being satisfied for $\tau_g = \tau_f = 0$ (i.e., when $f_i$ and $g_i$ are convex for all $i$). Then, whenever there exists an invariant measure for the Markov operator $\mathcal{P}$ corresponding to (50), the distributions of the sequences of random variables converge to an invariant measure in the Prokhorov-Lévy metric.

(v) Let $G$ be compact and $\mathcal{P}_2(G) \cap \text{inv} \mathcal{P} \neq \emptyset$. If $\Psi$ given by (22) takes the value 0 only at points in $\text{inv} \mathcal{P}$ and is metrically subregular for 0 on $\mathcal{P}_2(G)$ with gauge $\rho$ given by (20) with $\tau = 1/2$, $\epsilon$ satisfying (53), and $\theta_{\tau, \epsilon}$ satisfying (19), then the Markov chain converges to an invariant distribution with rate given by (24).

Proof. (i). By [56, Proposition 3.7] for all $j \in I_g$, $J_G b_j$ is a.a.-fne with constant $\alpha = 1/2$ and violation $\epsilon_g = 2\tau_g$ on $G$. Likewise, for all $i \in I_f$, $J_G f_i$ is a.a.-fne with constant $\alpha = 1/2$ and violation $\epsilon_f = 2\tau_f$ on $\bigcup_{j \in I_g} \{J_G b_j(G)\}$. By [56, Propositions 2.3-2.4], for all $i \in I_f \times I_g$ the Douglas-Rachford mapping $T^{DR}_i$ is therefore a.a.-fne with constant $\alpha = 1/2$ and violation at most $\frac{1}{2} ((1 + 2\tau_g)(1 + 2\tau_f) - 1)$ on $G$.

(ii) - (v) follow in the same way as their counterparts in Proposition 4.1. \qed

Here as in Proposition 4.1 the compactness assumption on $G$ in part (v) can be dropped since $T_i$ is a mapping on $\mathbb{R}^n$ (see Remark 3.8).
4.3 Application to X-FEL Imaging

We apply the Stochastic Forward-Backward Algorithm \(^{2}\) and the Stochastic Douglas-Rachford Algorithm \(^{3}\) to the problem of X-ray free electron laser imaging discussed in the introduction.

Here, a high-energy X-ray pulse illuminates molecules suspended in fluid that is streaming across the beam. A low-count diffraction image is recorded for each pulse. The goal is to reconstruct the three-dimensional electron density of the target molecules from the observed diffraction images. This is a stochastic tomography problem with a nonlinear model for the data - stochastic because the molecule orientations are random, and uniformly distributed on SO(3).

Computed tomography with random orientations has been studied for more than two decades \([10, 11]\) and been successfully applied for inverting the Radon transform (a linear operator) with unknown orientations \([63, 68]\). The model for the data in X-FEL imaging is nonlinear and nonconvex: Fraunhofer diffraction (mathematically equivalent to Fourier transformation) with missing phase \([16, 72]\). The problem of recovering an object from diffraction intensity data is the optical phase retrieval problem \([65, 66]\). The most successful and widely applied methods for solving this problem are fixed point algorithms where the fixed point mappings consist of compositions and averages of projection mappings onto nonconvex sets \([54]\); the connection to the general framework considered here is through the fact that the projector is the proximal mapping of the indicator function \([67]\), and the gradient of the squared distance of a point to a set is twice the difference between that point and its projection onto the set \([52, Eq(5.3)]\). With additional constraints ensuring that the reconstructions are confined to a certain region, or are real-valued, the feasibility model for the X-FEL problem, and phase retrieval in general, is an example of an inconsistent, nonconvex feasibility problem: there does not exist a point that simultaneously explains the measurement and satisfies the a priori constraints.

A theoretical framework for unifying and extending the first proofs of local convergence of projection methods for inconsistent, nonconvex feasibility, with rates, was established in \([56]\). This analysis accommodates iterations of averages and compositions of expansive mappings. Moreover, unlike many other approaches, the framework does not require that the constituent mappings have common fixed points. This has been applied to prove, for the first time, local linear convergence of a wide variety of fundamental algorithms for phase retrieval \([40, 53, 56, 70]\).

The discretized mathematical model for the physical experiment is

\[
\left\| \left( \mathcal{F}^\xi (\rho) \right) _i \right\| = \phi^\xi _i, \quad \forall \ i = 1, 2, \ldots, n.
\]

Here \(\mathcal{F}^\xi : \mathbb{C}^n \to \mathbb{C}^n\) is the Fourier transform composed with a rotation \(\xi \in SO(3)\) accounting for the propagation of an electromagnetic wave with incident angle \(\xi\); \(\rho \in \mathbb{C}^n\) is the unknown electron density in discretized voxels numbered from 1 to \(n\) (real-valuedness and nonnegativity will be applied in the qualitative constraints); this density interacts with the wave at one end (the pupil or object plane) of the instrument, and \(\phi^\xi _i \in [0, 1]\) is the probability of observing a scattered photon in the \(i^{th}\) voxel \((i = 1, 2, \ldots, n)\) of the imaging volume in the far field under the rotation \(\xi\). The problem is to determine \(\rho\) from \(\phi^\xi _i\). The set of possible vectors satisfying such measurements is given by

\[
C(\xi) := \left\{ \rho \in \mathbb{C}^n \left| \left\| \left( \mathcal{F}^\xi (\rho) \right) _i \right\| = \phi^\xi _i, \quad \forall \ i = 1, 2, \ldots, n \right. \right\}.
\]

This set has been studied extensively in \([23, 51–53, 56]\). Although this set is nonconvex, it is prox-regular \([51]\) and hence “weakly nonconvex”. Projectors onto \(C(\xi)\) are therefore pointwise

\(^1\)In this presentation, we ignore the additional complication that the rotation \(\xi\) is unobservable and must be estimated instead as a most likely orientation conditioned on the current estimate for the density.
almost \( \alpha \)-firmly nonexpansive at any point \( \bar{\rho} \in C(\xi) \) with violation \( \epsilon \) vanishing to zero as the neighborhood of \( \bar{\rho} \) collapses.

To give an idea of the size of this problem, in a typical experiment \( n \) is about \( 10^8 \). In practice, the observation described above for a randomly selected orientation \( \xi_k \) is repeated about \( 10^9 \) times, each time with a different orientation.

In addition to the sets generated by the data, there are certain a priori qualitative constraints that can (and should) be added depending on the type of experiment that has been conducted. Often these are support constraints, or real-valuedness, or nonnegativity; an electron density, for example is by definition real-valued and nonnegative. All of these are convex constraints for which we reserve the set \( C_0 \) for the qualitative constraints.

The problem is a specialization of (42) where \( I_f = \{1\}, I_g = SO(3), \xi^f_k = 1 \) for all \( k \), \( \xi^g_k \) is a uniformly distributed random variable on \( SO(3) \) for all \( k \), and

\[
(\forall k) \quad f^{\xi^g_k} (\rho) := \frac{\lambda}{2(1-\lambda)} \text{dist}^2(\rho, C_0) \\
g^{\xi^g_k} (\rho) := \iota_{\text{C}(\xi^g_k)}(\rho) := \begin{cases} 0 & \text{if } \rho \in \text{C}(\xi^g_k) \\ +\infty & \text{otherwise} \end{cases}
\]  

Both Algorithm 2 and Algorithm 3 can be applied to this problem. Assumptions (a) and (b) of both Proposition 4.1 and Proposition 4.3 hold on all neighborhoods of the sets \( C_0 \) and \( C(\xi) \) small enough [56, Example 3.6]. Indeed, for this application \( \tau_f = 0 \) since \( C_0 \) is convex and \( \tau_{g^{\xi^g_k}} \) in (51) can be estimated from the distance of \( \rho \) to the set \( C(\xi^g_k) \), which is easy to calculate. Deterministic versions of Algorithm 2 have been fully studied in [56], though this algorithm is not recommended for this problem in particular due to the frequent occurrence of local minima. In [53] the fixed points of the deterministic version of Algorithm 3 for the phase retrieval problem have been characterized, and metric subregularity of the transport discrepancy (14) has been determined for geometries applicable to cone and sphere problems [54] such as this. So for a majority of relevant instances, there is good reason to expect that Propositions 4.1 and 4.3 can be applied provably to X-FEL measurements. The determination of the local domain \( G \) in condition (c) of Proposition 4.1 and Assumption 4.2 is therefore key. There are some unresolved cases, however, that are relevant for optical phase retrieval (see [53, Example 5.4]), and this needs further study.

4.4 Stochastic Proximal Algorithms in Tree Space

We conclude with proximal splitting in locally compact Hadamard spaces. Our target application in this setting is the computation of Fréchet means in what is known as the BHV space, a model space for phylogenetic trees [14]. This is a Hadamard space consisting of stratified Euclidean spaces. Each strata of the space consists of all trees (connected, acyclic, undirected graphs) with the same sets of nodes and edges, but different edge lengths connecting the various nodes. The leaves of such trees are individual species, and the nodes connecting the species are points of common genetic ancestry; the lengths of the edges between nodes represents the time required for the observed genetic variation to occur. A particular tree relationship depends on what part of the genome is being compared; measurements from two different parts of the genome will generally lead to two different phylogenetic trees. If two trees belong to the same strata, i.e. if the trees differ only in edge lengths, then a geodesic connecting these trees is simply a line between two points in a Euclidean space; otherwise, the geodesic passes from one stratum to another by passing through the origin, i.e. by shrinking an edge length to zero and/or by extending an edge from the origin.
The distance between two points \( x \) and \( y \) in this space, \( d(x, y) \) is (obviously) the length of the geodesic connecting them, and this can be decomposed into the sum of piecewise Euclidean distances of the geodesic confined to individual strata. The distance function is clearly convex. The task we consider in this example is that of computing the average, or more precisely the Fréchet mean, of more than two trees in the BHV space: Given \( N \) trees, \( x_1, x_2, \ldots, x_N \) find \( x^* \) that solves

\[
\inf_x \sum_{i=1}^{N} d(x, x_i)^2.
\]  

Readers interested in learning more about the application to phylogenetic trees are referred to [14] and references therein; our primary interest is in the mathematical structure of the problem and demonstrating the theory developed in the previous sections.

Let \((H, d)\) be a Hadamard space, and let \( f_i : H \to \mathbb{R} \) be proper, lower semicontinuous convex functions for \( i = 1, 2, \ldots, N \). The Fréchet mean problem is an instance of the following convex optimization problem

\[
\inf_{x \in H} \sum_{i=1}^{N} f_i(x).
\]

In a Hadamard space the proximal mapping of a function \( f \) is defined by (43) with the only difference that the domain of \( f \) is \( H \). Note that the prox mapping is well-defined, even though the resolvent of the subdifferential is not since \( H \) is not a linear space. This has been studied in CAT(0) spaces in [3, 9, 42] and in the Hilbert ball in [46]. In these earlier works it was already known that proximal mappings of lower semicontinuous convex functions are (everywhere) \( \alpha \)-firmly nonexpansive with \( \alpha = 1/2 \). In [12] a randomized proximal splitting algorithm is studied under the assumption that the proximal parameters \( \lambda \) in (43) decay to zero, in which case the prox mapping converges to the identity mapping. The theory developed above allows us to conclude convergence to an invariant probability measure without the assumption that the prox mapping converges to the identity.

Letting \( \xi \) be a uniformly distributed random variable with values on \( \{1, 2, \ldots, N\} \) the backward-backward splitting method applied to this problem yields Algorithm 4. In the deterministic setting where the prox mappings are selected in a fixed sequential manner, convergence has been established already in [4, Theorem 4.1]. Local linear convergence was established in [13, Theorem 27] under the assumption of linear metric subregularity, which in this setting reduces to

\[
d(x, \text{Fix} \ T_m \cap G) \leq rd(T_m x, x) \quad \forall x \in G \subset H
\]

where

\[
T_m := \left( \text{prox}_{f_m, \lambda_m} \circ \cdots \circ \text{prox}_{f_2, \lambda_2} \circ \text{prox}_{f_1, \lambda_1} \right).
\]

A random version of this algorithm with diminishing constants \( \lambda_{\xi_k} \) was shown to converge in [12, Theorem 3.7].
The immediate contribution of our work to these earlier studies is algorithmic: by Corollary 2.7 our random implementation without diminishing constants converges linearly to a stationary distribution under the assumption $\Psi^{-1}(0) = \text{inv} P$ and $\Psi(\pi) = 0$ for all $\pi \in \text{inv} P$, and for all $\mu \in \mathcal{P}_2(G)$

$$\inf_{\pi \in \text{inv} P} W_2(\mu, \pi) \leq r d_2(0, \Psi(\mu)) = r \Psi(\mu).$$

In a Euclidean setting this assumption for this problem can be shown to hold by demonstrating that the discrete set of points $\{x_1, \ldots, x_N\}$ is (trivially) subtransversal [56, Definition 3.2 and Proposition 3.4]. We conjecture that this holds in the present setting of randomized algorithms on Hadamard spaces, but to show this would take more space than we have here.

This example does not use the full potential of our framework since this problem is convex, and hence the proximal mappings are all $\alpha$-firmly nonexpansive with constant $\alpha = 1/2$. One prominent nonconvex problem in this setting to which our framework can be applied is the $K$-means clustering problem: find the best assignment of a sample of $N$ trees $\{x_1, x_2, \ldots, x_N\}$ into $K$ clusters. In this problem one begins with $N$ clusters (the individual trees) and reduces the number of clusters by assigning nearest neighbors to a single cluster whose center is the Fréchet mean. The operation of assigning a tree to a cluster is expansive since two trees arbitrarily close to a point that is equidistant to more than one cluster will be assigned to distinct clusters with possibly very distant Fréchet means.

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