On a class of algebraic solutions to Painlevé VI equation, its determinant formula and coalescence cascade

Tetsu Masuda
Department of Mathematics, Kobe University,
Rokko, Kobe, 657-8501, Japan
masuda@math.kobe-u.ac.jp

Abstract

A determinant formula for a class of algebraic solutions to Painlevé VI equation (PVI) is presented. This expression is regarded as a special case of the universal characters. The entries of the determinant are given by the Jacobi polynomials. Degeneration to the rational solutions of PV and PIII is discussed by applying the coalescence procedure. Relationship between Umemura polynomials associated with PVI and our formula is also discussed.

1 Introduction

Enlarging the work by Yablonskii and Vorob’ev for PII [29] and Okamoto for PIV [23], Umemura has introduced a class of special polynomials associated with the algebraic solutions of each Painlevé equation PIII, PV and PVI [28]. These polynomials are generated by Toda equation that arises from Bäcklund transformations of each Painlevé equation. It is also known that the coefficients of the polynomials admit mysterious combinatorial properties [14, 27].

It is remarkable that some of such polynomials are expressed as a specialization of the Schur functions. Yablonskii-Vorob’ev polynomials are expressible by 2-reduced Schur functions, and Okamoto polynomials by 3-reduced Schur functions [6, 7, 15]. It is now recognized that these structures reflect the affine Weyl group symmetry as groups of the Bäcklund transformations [30]. The determinant formulas of Jacobi-Trudi type for Umemura polynomials of PIII and PV resemble each other. In both cases, they are expressed by 2-reduced Schur functions and entries of the determinant are given by the Laguerre polynomials [4, 16].

Furthermore, in a recent work, it has been revealed that the whole families of the characteristic polynomials for the rational solutions of PV, which include Umemura polynomials for PV as a special case, admit more general structures [12]. Namely, they are expressed in terms of the universal characters that are a generalization of the Schur functions. The latter are the characters of the irreducible polynomial representations of GL(n), while the former are introduced to describe the irreducible rational representations [11].

What kind of determinant structures do Umemura polynomials for PVI admit? Recently, Kirillov and Taneda have introduced a generalization of Umemura polynomials for PVI in the context of combinatorics and shown that their polynomials degenerate to the special polynomials for PV in some limit [8, 9, 10]. This result suggests that the special polynomials associated with the algebraic solutions of PVI are also expressible by the universal characters.

In this paper, we consider PVI

\[
\frac{d^2 y}{dt^2} = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) \left( \frac{dy}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) \frac{dy}{dt} \\
+ \frac{y(y-1)(y-t)}{2t^2(t-1)^2} \left[ \kappa_\infty^2 - \kappa_0^2 \frac{t}{y^2} + \kappa_1^2 \frac{t-1}{(y-1)^2} + (1-\theta^2) \frac{t(t-1)}{(y-t)^2} \right],
\]

(1.1)
where \( \kappa_{\infty} \), \( \kappa_0 \), \( \kappa_1 \) and \( \theta \) are parameters. As is well known \([21]\), \( P_{VI} \) is equivalent to the Hamilton system

\[
S_{VI} : \quad q' = \frac{\partial H}{\partial p}, \quad p' = -\frac{\partial H}{\partial q}, \quad \tau = t(t-1)\frac{d}{dt},
\]

with the Hamiltonian

\[
H = q(q-1)(q-t)p^2 - [\kappa_0 (q-1)(q-t) + \kappa_1 q(q-t) + (\theta-1)q(q-1)]p + \kappa(q-t),
\]

\( \kappa = \frac{1}{4}(\kappa_0 + \kappa_1 + \theta - 1)^2 - \frac{1}{4}\kappa_{\infty}^2. \) (1.3)

In fact, the equation for \( y = q \) is nothing but \( P_{VI} \).

The aim of this paper is to investigate a class of algebraic solutions to \( P_{VI} \) (or \( S_{VI} \)) that originate from the fixed points of the Bäcklund transformations corresponding Dynkin automorphisms and to present its explicit determinant formula.

Let us remark on the terminology of “algebraic solutions”. \( P_{VI} \) admits the several classes of algebraic solutions \([20, 2, 13, 1]\), and the classification has not established. In this paper, if we do not comment especially, we use “algebraic solutions” in the above restricted sense.

This paper is organized as follows. In Section 2, we present a determinant formula for the algebraic solutions of \( P_{VI} \) (or \( S_{VI} \)). This expression is also a specialization of the universal character \( s \) and the entries of the determinant formula.

In Section 6, we show that the algebraic solutions of \( P_{VI} \) degenerate to the rational solutions of \( P_V \) and \( P_{III} \) with preserving the determinant structures. Section 5 is devoted to discuss the relationship to the original Umemura polynomials for \( P_{VI} \).

2 A determinant formula

**Definition 2.1** Let \( p_k = p_k^{(c,d)}(x) \) and \( q_k = q_k^{(c,d)}(x) \), \( k \in \mathbb{Z} \), be two sets of polynomials defined by

\[
\sum_{k=0}^{\infty} p_k^{(c,d)}(x)\lambda^k = G(x; c, d; \lambda), \quad p_k^{(c,d)}(x) = 0 \text{ for } k < 0,
\]

\[
q_k^{(c,d)}(x) = p_k^{(c,d)}(x^{-1}),
\]

respectively, where the generating function \( G(x; c, d; \lambda) \) is given by

\[
G(x; c, d; \lambda) = (1 - \lambda)^{-d} (1 + x \lambda)^{-c}.
\]

For \( m,n \in \mathbb{Z}_{\geq 0} \), we define a family of polynomials \( R_{m,n} = R_{m,n}(x; c, d) \) by

\[
R_{m,n}(x; c, d) = \begin{cases}
q_1 & q_0 & \cdots & q_{-m+2} & q_{-m+1} & \cdots & q_{-m+n+1} & q_{-m+n+2} \\
q_3 & q_2 & \cdots & q_{-m+4} & q_{-m+3} & \cdots & q_{-m+n+3} & q_{-m+n+4} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
q_{2m-1} & q_{2m-2} & \cdots & q_m & q_{m-1} & \cdots & q_{m+n-1} & q_{m+n} \\
p_{n-m} & p_{n-m+1} & \cdots & p_{n-1} & p_n & \cdots & p_{2n-2} & p_{2n-1} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
p_{-n-m+4} & p_{-n-m+5} & \cdots & p_{-n+3} & p_{-n+4} & \cdots & p_2 & p_3 \\
p_{-n-m+2} & p_{-n-m+3} & \cdots & p_{-n+1} & p_{-n+2} & \cdots & p_0 & p_1
\end{cases}. \quad (2.3)
\]

For \( m, n \in \mathbb{Z}_{\leq 0} \), \( R_{m,n} \) are defined through

\[
R_{m,n} = (-1)^{m(m+1)/2} R_{-m-1,n}, \quad R_{m,n} = (-1)^{n(n+1)/2} R_{m,-n-1}. \quad (2.4)
\]
Theorem 2.2 We set
\[ R_{m,n}(x; c, d) = S_{m,n}(x; a, b), \] (2.5)
with
\[ c = a + b + n - \frac{1}{2}, \quad d = 2b - m + n. \] (2.6)
Then, for the parameters
\[ \kappa_\infty = b, \quad \kappa_0 = b - m + n, \quad \kappa_1 = a + m + n, \quad \theta = a, \] (2.7)
we have a family of algebraic solutions of the Hamilton system \( S_{VI} \),
\[ q = x \frac{S_{m,n-1}(x; a + 1, b)S_{m-1,n}(x; a + 1, b)}{S_{m-1,n}(x; a + 1, b - 1)S_{m,n-1}(x; a + 1, b + 1)}, \]
\[ p = \frac{1}{2} \left( a + b + n - \frac{1}{2} \right) x^{-1} \frac{S_{m-1,n}(x; a + 1, b - 1)S_{m,n-1}(x; a + 1, b + 1)S_{m,n-1}(x; a, b)}{S_{m-1,n-1}(x; a + 1, b - 1)S_{m,n-1}(x; a + 1, b)}, \] (2.8)
with \( x^2 = t \).

This Theorem means that a class of algebraic solutions of \( P_{VI} \) is expressed in terms of the universal characters \( [11] \), which also appear in the expression of the rational solutions of \( P_V \) \( [12] \). Note that the entries \( p_k \) and \( q_k \) are essentially the Jacobi polynomials, namely,
\[ p_k^{(c,d)}(x) = P_k^{(d-1,c-d-k)}(-1-2x). \] (2.9)

Applying some Bäcklund transformations, which can include outer transformations given in \( [7,14] \), to the above solutions, we can get other families of algebraic solutions of \( P_{VI} \). Some examples are presented in Corollary \( 6.2 \) and \( 6.8 \), so we omit detail here.

3 A symmetric description of Painlevé VI equation

Noumi and Yamada have introduced the symmetric form of Painlevé equations \( [17,18,19] \). This formulation provides us with a clear description of symmetry structures of Bäcklund transformations and a systematic tool to construct special solutions.

In this section, we present a symmetric description for the Bäcklund transformations of \( P_{VI} \) \( [14,20] \). After introducing the \( \tau \)-functions via Hamiltonians, we derive several sets of bilinear equations.

3.1 Bäcklund transformations of \( P_{VI} \)

We set
\[ f_0 = q - t, \quad f_3 = q - 1, \quad f_4 = q, \quad f_2 = p, \] (3.1)
and
\[ \alpha_0 = \theta, \quad \alpha_1 = \kappa_\infty, \quad \alpha_3 = \kappa_1, \quad \alpha_4 = \kappa_0. \] (3.2)
Then, the Hamiltonian \( [1,3] \) is written as
\[ H = f_2^4 f_0 f_3 f_4 - [(\alpha_0 - 1)f_3 f_4 + \alpha_3 f_0 f_4 + \alpha_4 f_0 f_3] f_2 + \alpha_2 (\alpha_1 + \alpha_2) f_0, \] (3.3)
with
\[ \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1, \] (3.4)
and the Hamilton equation \( [1,2] \) is written down as
\[ f'_4 = 2f_2 f_0 f_3 f_4 - (\alpha_0 - 1)f_3 f_4 - \alpha_3 f_0 f_4 - \alpha_4 f_0 f_3, \]
\[ f'_2 = -(f_0 f_3 + f_0 f_4 + f_3 f_4) f_2^2 + [(\alpha_0 - 1)(f_3 + f_4) + \alpha_3 (f_0 + f_4) + \alpha_4 (f_0 + f_3)] f_2 - \alpha_2 (\alpha_1 + \alpha_2). \] (3.5)
The Bäcklund transformations of PVI are described as follows [19],

\begin{align}
    s_i(\alpha_j) &= \alpha_j - a_{ij}\alpha_i, \quad (i, j = 0, 1, 2, 3, 4) \\
    s_2(f_i) &= f_i + \frac{\alpha_2}{f_2}, \quad s_i(f_2) = f_2 - \frac{\alpha_i}{f_i}, \quad (i = 0, 3, 4)
\end{align}

\begin{align}
    s_5 : \quad &\alpha_0 \leftrightarrow \alpha_1, \quad \alpha_3 \leftrightarrow \alpha_4, \\
    f_2 &\to -\frac{f_0(f_2f_0 + \alpha_2)}{t(t - 1)}, \quad f_0 \to \frac{t(t - 1)}{f_0}, \quad f_3 \to (t - 1)\frac{f_4}{f_0}, \quad f_4 \to \frac{f_3}{f_0}, \\
    s_6 : \quad &\alpha_0 \leftrightarrow \alpha_3, \quad \alpha_1 \leftrightarrow \alpha_4, \\
    f_2 &\to -\frac{f_4(f_1f_2 + \alpha_2)}{t}, \quad f_0 \to -\frac{t\frac{f_3}{f_4}}{f_4}, \quad f_3 \to -\frac{f_0}{f_4}, \quad f_4 \to \frac{t}{f_4}, \\
    s_7 : \quad &\alpha_0 \leftrightarrow \alpha_4, \quad \alpha_1 \leftrightarrow \alpha_3, \\
    f_2 &\to -\frac{f_3(f_3f_2 + \alpha_2)}{t - 1}, \quad f_0 \to -(t - 1)\frac{f_4}{f_3}, \quad f_3 \to -\frac{t - 1}{f_5}, \quad f_4 \to \frac{f_0}{f_3},
\end{align}

where $A = (a_{ij})_{i,j=0}^{4}$ is the Cartan matrix of type $D_4^{(1)}$:

\begin{align}
    a_{ii} &= 2 \quad (i = 0, 1, 2, 3, 4), \quad a_{ij} = a_{ji} = -1, \quad (j = 0, 1, 3, 4), \quad a_{ij} = 0 \quad \text{(otherwise)}.
\end{align}

These transformations commute with the derivation $'$, and satisfy the following relations

\begin{align}
    s_i^2 &= 1 \quad (i = 0, \ldots, 7), \quad s_is_2s_i = s_2s_is_2, \quad (i = 0, 1, 3, 4), \\
    s_5s_0s_1 &= s_1s_0s_1, \quad s_6s_1s_2 = s_2s_1s_2, \quad s_7s_1s_2 = s_2s_1s_2, \\
    s_5s_6 &= s_6s_5, \quad s_5s_7 = s_7s_5, \quad s_6s_7 = s_7s_6.
\end{align}

This means that transformations $s_i \ (i = 0, \ldots, 4)$ generate the affine Weyl group $W(D_4^{(1)})$, and $s_i \ (i = 0, \ldots, 7)$ generate its extension including the Dynkin diagram automorphisms.

### 3.2 The $\tau$-functions and bilinear equations

We add a correction term to the Hamiltonian (3.3) as follows,

\begin{equation}
    H_0 = H + \frac{t}{4} \left[ 1 + 4\alpha_1\alpha_2 + 4\alpha_2^2 - (\alpha_3 + \alpha_4)^2 \right] + \frac{1}{4} \left[ (\alpha_1 + \alpha_4)^2 + (\alpha_3 + \alpha_4)^2 + 4\alpha_2\alpha_4 \right].
\end{equation}

This modification gives a simpler behavior of the Hamiltonian with respect to the Bäcklund transformations. From the corrected Hamiltonian (3.13), we introduce a family of Hamiltonians $h_i (i = 0, 1, 2, 3, 4)$ as

\begin{equation}
    h_0 = H_0 + \frac{t}{4}, \quad h_1 = s_5(H_0) - \frac{t - 1}{4}, \quad h_3 = s_6(H_0) + \frac{1}{4}, \quad h_4 = s_7(H_0), \quad h_2 = h_1 + s_1(h_1).
\end{equation}

Then, we have

\begin{align}
    s_i(h_j) &= h_j, \quad (i \neq j, \ i, j = 0, 1, 2, 3, 4), \\
    s_0(h_0) &= h_0 - \alpha_0(t - 1)\frac{f_4}{f_0}, \quad s_1(h_1) = h_1 - \alpha_1f_3, \\
    s_3(h_3) &= h_3 + \alpha_3\frac{t - 1}{f_5}, \quad s_4(h_4) = h_4 + \alpha_4\frac{f_0}{f_4}.
\end{align}

Moreover, from (3.14), (3.16) and the equations (3.4), we obtain

\begin{align}
    [s_i(h_i) + h_i] - [s_1(h_1) + h_1] &= \frac{f_i'}{f_i}, \quad (i = 0, 3, 4) \\
    [s_2(h_2) + h_2] - (h_0 + h_1 + h_3 + h_4) &= \frac{f_2'}{f_2} - \frac{1}{2}(t - 1).
\end{align}
Next, we also introduce \( \tau \)-functions \( \tau_i \) \((i = 0, 1, 2, 3, 4)\) by
\[
h_i = \frac{\tau_i'}{\tau_i}. \tag{3.18}
\]
Implying that the action of \( s_i \)'s on \( \tau \)-functions also commute with the derivation \( \,' \), one can lift the Bäcklund transformations to the \( \tau \)-functions. From (3.15) and (3.17), we get
\[
s_i(\tau_j) = \tau_j, \quad (i \neq j, \ i, j = 0, 1, 2, 3, 4), \tag{3.19}
\]
and
\[
s_0(\tau_0) = f_0 \frac{\tau_2}{\tau_0}, \quad s_1(\tau_1) = \frac{\tau_2}{\tau_1}, \quad s_2(\tau_2) = t^{-\frac{1}{2}} f_2 \frac{\tau_0 \tau_1 \tau_2 \tau_4}{\tau_2}, \quad s_3(\tau_3) = f_3 \frac{\tau_2}{\tau_3}, \quad s_4(\tau_4) = f_4 \frac{\tau_2}{\tau_4}, \tag{3.20}
\]
respectively. The action of the diagram automorphisms \( s_5, s_6, s_7 \) are derived by using (3.14) as follows,
\[
s_5 : \quad \begin{align*}
\tau_0 & \rightarrow [t(t-1)]^{\frac{1}{4}} \tau_1, \quad \tau_1 \rightarrow [t(t-1)]^{-\frac{1}{4}} \tau_0, \\
\tau_3 & \rightarrow -t^{-\frac{1}{4}} (t-1)^{\frac{1}{4}} \tau_4, \quad \tau_4 \rightarrow t^{-\frac{3}{4}} (t-1)^{-\frac{1}{4}} \tau_3,
\end{align*}
\]
\[
t_2 \rightarrow [t(t-1)]^{-\frac{1}{4}} f_0 t_2, \tag{3.21}
\]
\[
s_6 : \quad \begin{align*}
\tau_0 & \rightarrow \pm it^{-\frac{1}{4}} \tau_3, \quad \tau_3 \rightarrow \mp it^{-\frac{1}{4}} \tau_0, \quad \tau_1 \rightarrow t^{-\frac{1}{4}} \tau_1, \quad \tau_4 \rightarrow t^{\frac{1}{4}} \tau_1, \quad \tau_2 \rightarrow t^{-\frac{1}{4}} f_4 \tau_2,
\end{align*}
\]
\[
s_7 : \quad \begin{align*}
\tau_0 & \rightarrow (1) t^{-\frac{1}{4}} (t-1)^{\frac{1}{4}} \tau_4, \quad \tau_4 \rightarrow (1) t^{\frac{1}{4}} (t-1)^{-\frac{1}{4}} \tau_0, \\
\tau_1 & \rightarrow (1) t^{-\frac{1}{4}} (t-1)^{\frac{1}{4}} \tau_3, \quad \tau_3 \rightarrow (1) t^{\frac{1}{4}} (t-1)^{-\frac{1}{4}} \tau_1,
\end{align*}
\]
\[
t_2 \rightarrow -i(t-1)^{-\frac{1}{4}} f_3 t_2. \tag{3.22}
\]
The algebraic relations of \( s_i \)'s are preserved in this lifting except for the following modification,
\[
s_i s_2(\tau_2) = -s_2 s_i(\tau_2) \quad (i = 5, 6, 7), \tag{3.24}
\]
and
\[
\begin{align*}
s_5 s_6 \tau_{(0,1,2,3,4)} &= \{i, -i, -1, -1, i\} s_5 s_6 \tau_{(0,1,2,3,4)}, \\
s_5 s_7 \tau_{(0,1,2,3,4)} &= \{i, -i, -1, -1, i\} s_5 s_7 \tau_{(0,1,2,3,4)}, \\
s_6 s_7 \tau_{(0,1,2,3,4)} &= \{i, -i, -1, -1, i\} s_6 s_7 \tau_{(0,1,2,3,4)}.
\end{align*}
\]
Note that one can regard (3.20) as the multiplicative formulas for \( f_i \) in terms of \( \tau \)-functions,
\[
f_0 = \frac{\tau_0 s_0(\tau_0)}{\tau_1 s_1(\tau_1)}, \quad f_3 = \frac{\tau_3 s_3(\tau_3)}{\tau_1 s_1(\tau_1)}, \quad f_4 = \frac{\tau_4 s_4(\tau_4)}{\tau_1 s_1(\tau_1)}, \quad f_2 = t^{\frac{1}{2}} \frac{\tau_1 s_1(\tau_1) s_2 s_1(\tau_1)}{\tau_0 \tau_3 \tau_4}. \tag{3.25}
\]
From these formulas, it is possible to derive various bilinear equations for \( \tau \)-functions. First, the constraints for \( f \)-variables
\[
f_0 = f_4 - t, \quad f_3 = f_4 - 1, \tag{3.26}
\]
yield to
\[
\begin{align*}
\tau_1 s_1(\tau_1) + \tau_3 s_3(\tau_3) - \tau_4 s_4(\tau_4) &= 0, \\
\tau_0 s_0(\tau_0) + t \tau_1 s_1(\tau_1) - \tau_4 s_4(\tau_4) &= 0, \\
\tau_1 s_2 s_1(\tau_1) + \tau_3 s_2 s_3(\tau_3) - \tau_4 s_2 s_4(\tau_4) &= 0, \\
\tau_0 s_2 s_0(\tau_0) + t \tau_1 s_2 s_1(\tau_1) - \tau_4 s_2 s_4(\tau_4) &= 0.
\end{align*}
\]
The Bäcklund transformations (3.14) are lead to the following sets of bilinear equations,
\[
\begin{align*}
\alpha_0 t^{-\frac{1}{2}} \tau_3 \tau_4 - s_0(\tau_0) s_2 s_1(\tau_1) + \tau_0 s_0 s_2 s_1(\tau_1) &= 0, \\
\alpha_0 t^{-\frac{1}{2}} (t-1) \tau_1 \tau_4 - s_0(\tau_0) s_2 s_3(\tau_3) + \tau_0 s_0 s_2 s_3(\tau_3) &= 0, \\
\alpha_0 t^{\frac{1}{2}} \tau_1 \tau_3 - s_0(\tau_0) s_2 s_4(\tau_4) + \tau_0 s_0 s_2 s_4(\tau_4) &= 0.
\end{align*}
\]
\begin{align*}
\alpha_1 t^{-\frac{1}{2}} \tau_3 \tau_4 + s_1(\tau_1)s_2 s_0(\tau_0) - \tau_1 s_1 s_2 s_0(\tau_0) = 0, \\
\alpha_1 t^{-\frac{1}{2}} \tau_0 \tau_4 + s_1(\tau_1)s_2 s_3(\tau_3) - \tau_1 s_1 s_2 s_3(\tau_3) = 0, \\
\alpha_1 t^{-\frac{1}{2}} \tau_0 \tau_3 + s_1(\tau_1)s_2 s_4(\tau_4) - \tau_1 s_1 s_2 s_4(\tau_4) = 0, \\
\alpha_3 t^{-\frac{1}{2}} \tau_0 \tau_4 - s_3(\tau_3)s_2 s_1(\tau_1) + \tau_3 s_3 s_2 s_1(\tau_1) = 0, \\
\alpha_3 t^{-\frac{1}{2}} (1-t) \tau_1 \tau_4 - s_3(\tau_3)s_2 s_0(\tau_0) + \tau_3 s_3 s_2 s_0(\tau_0) = 0, \\
\alpha_3 t^{-\frac{1}{2}} \tau_0 \tau_1 - s_3(\tau_3)s_2 s_4(\tau_4) + \tau_3 s_3 s_2 s_4(\tau_4) = 0,
\end{align*}

\begin{equation}
\frac{1}{2} \tau_1 \tau_3 - s_4(\tau_4)s_2 s_1(\tau_1) + \tau_4 s_4 s_2 s_1(\tau_1) = 0,
\end{equation}

\begin{equation}
-\alpha_4 t^{-\frac{1}{2}} \tau_1 \tau_3 - s_4(\tau_4)s_2 s_0(\tau_0) + \tau_4 s_4 s_2 s_0(\tau_0) = 0,
\end{equation}

\begin{equation}
-\alpha_4 t^{-\frac{1}{2}} \tau_0 \tau_1 - s_4(\tau_4)s_2 s_1(\tau_1) = 0,
\end{equation}

\begin{align*}
\alpha_2 t^{-\frac{1}{2}} \tau_1 \tau_4 - s_1(\tau_1)s_2 s_0(\tau_0) + s_0(\tau_0)s_2 s_1(\tau_1) = 0, \\
\alpha_2 t^{-\frac{1}{2}} \tau_0 \tau_4 - s_1(\tau_1)s_2 s_3(\tau_3) + s_3(\tau_3)s_2 s_1(\tau_1) = 0, \\
\alpha_2 t^{-\frac{1}{2}} \tau_0 \tau_3 - s_1(\tau_1)s_2 s_4(\tau_4) + s_4(\tau_4)s_2 s_1(\tau_1) = 0, \\
\alpha_2 t^{-\frac{1}{2}} \tau_0 \tau_1 - s_4(\tau_4)s_2 s_3(\tau_3) + s_3(\tau_3)s_2 s_4(\tau_4) = 0, \\
\alpha_2 t^{-\frac{1}{2}} \tau_1 \tau_3 - s_4(\tau_4)s_2 s_0(\tau_0) + s_0(\tau_0)s_2 s_4(\tau_4) = 0, \\
\alpha_2 t^{-\frac{1}{2}} (t-1) \tau_1 \tau_4 - s_1(\tau_1)s_2 s_0(\tau_0) + s_0(\tau_0)s_2 s_3(\tau_3) = 0.
\end{align*}

### 3.3 The \( \tau \)-functions on the weight lattice of type \( D_4 \)

Let us define the following translation operators

\begin{align*}
T_{03} &= s_{3s_0}s_{2s_4}s_{1s_2}s_{6}, & T_{14} &= s_{4s_1}s_{2s_3}s_{0}s_{2s_8}, \\
T_{34} &= s_{3s_2}s_{0}s_{1}s_{2s_3}s_{5}, & T_{34} &= s_{4s_3}s_{2s_1}s_{0}s_{2s_8}.
\end{align*}

\begin{align}
\text{which act on parameters } \alpha_i \text{ as}
T_{03}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &= (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) + (1, 0, -1, 1, 0), \\
T_{14}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &= (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) + (0, 1, -1, 0, 1), \\
T_{34}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &= (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) + (0, 0, 0, 1, -1), \\
T_{34}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &= (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) + (0, 0, 1, 1, 0),
\end{align}

and generate the weight lattice of type \( D_4 \). It is possible to derive Toda and Toda-like equations.

**Proposition 3.1** We have

\begin{align*}
T_{03}(\tau_0)T_{03}^{-1}(\tau_0) &= t^{-\frac{1}{2}} \left[ (t-1) \frac{d}{dt} \log(\tau_0)' - \log(\tau_0)' + \frac{1}{4}(1 - \alpha_0 - \alpha_3)^2 + \frac{1}{2} \right] \tau_0^2, \\
T_{14}(\tau_0)T_{14}^{-1}(\tau_0) &= -t^{-\frac{1}{2}} \left[ (t-1) \frac{d}{dt} \log(\tau_0)' - \log(\tau_0)' + \frac{1}{4}(1 + \alpha_0 + \alpha_4)^2 + \frac{1}{2} \right] \tau_0^2, \\
T_{34}(\tau_0)T_{34}^{-1}(\tau_0) &= \left( \frac{t-1}{t} \right)^{\frac{1}{2}} \left[ \frac{d}{dt} \log(\tau_0)' + \frac{1}{4}(\alpha_3 - \alpha_4)^2 - \frac{1}{2} \right] \tau_0^2, \\
T_{34}(\tau_0)T_{34}^{-1}(\tau_0) &= \left( \frac{t-1}{t} \right)^{\frac{1}{2}} \left[ \frac{d}{dt} \log(\tau_0)' + \frac{1}{4}(\alpha_3 + \alpha_4)^2 - \frac{1}{2} \right] \tau_0^2.
\end{align*}

**Proof.** Note that

\begin{equation}
\frac{d}{dt} h_0 = -f_2^2 f_3 f_4 + (\alpha_3 f_4 + \alpha_4 f_3) f_2 - \frac{1}{4}(\alpha_3 + \alpha_4)^2 + \frac{1}{2}.
\end{equation}
Using \((3.6), (3.7), (3.20), (3.22)\) and \((3.34)\), we have

\[
T_{03}(\tau_0)T_{03}^{-1}(\tau_0) = t^{-\frac{1}{2}} \left[ (t - 1) \frac{d}{dt} h_0 - h_0 + \frac{1}{4}(1 - \alpha_0 - \alpha_3)^2 + \frac{1}{2} \right] \tau_0^2,
\]

which gives the first equation in \((3.36)\). The other equations are obtained in similar way.

These translation operators commute each other except for the action on \(\tau\)-functions. Due to the algebraic relations \((3.24)\) and \((3.25)\), the commutation relations on \(\tau_0\) are described as

\[
\begin{align*}
T_{03}T_{14}(\tau_0) &= -T_{14}T_{03}(\tau_0), & T_{03}T_{34}(\tau_0) &= i\hat{T}_{34}T_{03}(\tau_0), \\
T_{03}T_{34}(\tau_0) &= iT_{34}T_{03}(\tau_0), & T_{14}T_{34}(\tau_0) &= -i\hat{T}_{34}T_{14}(\tau_0), \\
T_{14}T_{34}(\tau_0) &= -iT_{34}T_{14}(\tau_0), & T_{34}T_{34}(\tau_0) &= T_{34}\hat{T}_{34}(\tau_0).
\end{align*}
\]

We introduce \(\tau\)-functions on the weight lattice of type \(D_4\) as

\[
\tau_{k,l,m,n} = \tau_{34}^{-m}\tau_{14}^{-l}\tau_{03}^{-n}(\tau_0), \quad k, l, m, n \in \mathbb{Z}.
\]

In terms of this notation, the 24 \(\tau\)-functions in the bilinear equations \((3.28)-(3.33)\) are expressed as follows,

\[
\begin{align*}
\tau_{0,0,0,0} &= \tau_0, & \tau_{1,-1,0,1} &= -[t(t - 1)]\tau_1, & \tau_{1,0,-1,0} &= i[t(t - 1)]\tau_1, & \tau_{1,0,0,-1} &= i(t - 1)\tau_4, \\
\tau_{2,0,-1,1} &= -s_0(\tau_0), & \tau_{1,1,0,-1} &= [t(t - 1)]s_1(\tau_1), & \tau_{1,0,0,0} &= it\tau_{s_3(\tau_3)}, & \tau_{0,1,0,-1} &= (t - 1)\tau_{s_4(\tau_4)}, \\
\tau_{1,-1,-1,1} &= s_2s_0(\tau_0), & \tau_{0,0,0,-1} &= -[t(t - 1)]s_2s_1(\tau_1), & \tau_{0,-1,0,0} &= -it\tau_{s_2s_3(\tau_3)}, & \tau_{0,-1,-1,0} &= i(t - 1)\tau_{s_3s_4(\tau_4)}, \\
\tau_{1,1,0,-2} &= -s_1s_2s_0(\tau_0), & \tau_{1,-1,0,0} &= -s_3s_2s_0(\tau_0), & \tau_{1,-1,-2,0} &= s_4s_2s_0(\tau_0), \\
\tau_{2,0,-1,2} &= [t(t - 1)]s_0s_2s_1(\tau_1), & \tau_{0,0,1,0} &= [t(t - 1)]s_3s_2s_1(\tau_1), & \tau_{0,0,-1,0} &= -[t(t - 1)]s_4s_2s_1(\tau_1), \\
\tau_{2,1,-1,-1} &= -it\tau_{s_0s_2s_3(\tau_3)}, & \tau_{0,1,1,-1} &= it\tau_{s_1s_2s_3(\tau_3)}, & \tau_{0,-1,1,-1} &= -it\tau_{s_4s_2s_3(\tau_3)}, \\
\tau_{2,-1,-2,-1} &= i(t - 1)\tau_{s_0s_2s_4(\tau_4)}, & \tau_{0,1,0,-1} &= -i(t - 1)\tau_{s_1s_2s_4(\tau_4)}, & \tau_{0,-1,0,1} &= i(t - 1)\tau_{s_3s_2s_4(\tau_4)}.
\end{align*}
\]

Toda equations \((3.36)\) yield to

\[
\begin{align*}
\tau_{k+1,l,m,n}\tau_{k-1,l,m,n} &= t^{-\frac{1}{2}} \left[ (t - 1) \frac{d}{dt} \log \tau_{k,l,m,n} \right]' - \left( \log \tau_{k,l,m,n} \right)' + \left( 1 - \alpha_0 - \alpha_3 - 2k - m - n \right)^2 + \frac{1}{2} \right] \tau_{k,l,m,n}^2, \\
\tau_{k,l+1,m,n}\tau_{k,l-1,m,n} &= -t^{-\frac{1}{2}} \left[ (t - 1) \frac{d}{dt} \log \tau_{k,l,m,n} \right]' - \left( \log \tau_{k,l,m,n} \right)' + \left( \alpha_1 + \alpha_4 + 2l - m - n \right)^2 + \frac{1}{2} \right] \tau_{k,l,m,n}^2, \\
\tau_{k,l,m+1,n}\tau_{k,l,m-1,n} &= \left( \frac{t - 1}{t} \right) \left[ \frac{d}{dt} \log \tau_{k,l,m,n} \right]' + \left( \alpha_0 - \alpha_4 + k - l + 2m \right)^2 + \frac{1}{2} \right] \tau_{k,l,m,n}^2, \\
\tau_{k,l,m,n+1}\tau_{k,l,m,n-1} &= \left( \frac{t - 1}{t} \right) \left[ \frac{d}{dt} \log \tau_{k,l,m,n} \right]' + \left( \alpha_3 + \alpha_4 + k + l + 2n \right)^2 - \frac{1}{2} \right] \tau_{k,l,m,n}^2.
\end{align*}
\]
It is easy to see from (3.23), (3.40) and (3.41) that we have

\[
T_n^{34} T_{14}^{T^k} T_{03}^{T^k}(f_0) = t^{\frac{1}{2}} (t - 1)^{\frac{1}{2}} \frac{\tau_{k,l,m,n} \tau_{k+2,l,m-1,n-1}}{\tau_{k+1,l-1,m-1,n} \tau_{k+1,l+1,m,n}},
\]

\[
T_n^{34} T_{14}^{T^k} T_{03}^{T^k}(f_3) = (t - 1)^{\frac{1}{2}} \frac{\tau_{k+1,l,m-1,n} \tau_{k+1,l,m-1,n-1}}{\tau_{k+1,l-1,m-1,n} \tau_{k+1,l+1,m,n}},
\]

\[
T_n^{34} T_{14}^{T^k} T_{03}^{T^k}(f_4) = t^{\frac{1}{2}} \frac{\tau_{k+1,l,m,n-1} \tau_{k+1,l,m-1,n}}{\tau_{k+1,l-1,m-1,n} \tau_{k+1,l+1,m,n}},
\]

\[
T_n^{34} T_{14}^{T^k} T_{03}^{T^k}(f_2) = -(t - 1)^{-\frac{1}{2}} \frac{\tau_{k+1,l-1,m,n} \tau_{k+1,l,m-1,n-1}}{\tau_{k+1,l,m,n} \tau_{k+1,l+1,m,n-1}}.
\]

(3.44)

4 Construction of a family of algebraic solutions

It is known that one can get an algebraic solution of Painlevé equations by considering the fixed points with respect to the Bäcklund transformations corresponding to Dynkin automorphisms \([28, 15]\). Iteration of Bäcklund transformations to the seed solution gives a family of algebraic solutions, which are expressed by the ratio of some characteristic polynomials, such as Yablonskii-Vorob’ev, Okamoto and Umemura polynomials. These polynomials are defined as the non-trivial factors of \(\tau\)-functions and generated by Toda type recursion relations.

In this section, we construct a family of algebraic solutions to the symmetric form of \(P_{\text{VI}}\) by following the above recipe.

4.1 A seed solution

Consider the Dynkin diagram automorphism \(s_6\) to get a seed solution. By (4.9), the fixed solution is derived from

\[
\alpha_0 = \alpha_3, \quad \alpha_1 = \alpha_4, \quad f_4 = \frac{t}{f_4}, \quad f_2 = -\frac{f_4 f_2 + \alpha_2}{t}.
\]

(4.1)

Then, we obtain

\[
(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left( a, b, \frac{1}{2} - a - b, a, b \right),
\]

\[
f_0 = x - x^2, \quad f_3 = x - 1, \quad f_4 = x, \quad f_2 = \frac{1}{2} \left( a + b - \frac{1}{2} \right) x^{-1}, \quad x^2 = t,
\]

(4.2)

as a seed solution, which is equivalent to the following algebraic solution of \(S_{\text{VI}}\),

\[
q = x, \quad p = \frac{1}{2} \left( a + b - \frac{1}{2} \right) x^{-1},
\]

(4.3)

for the parameters

\[
\kappa_{\infty} = b, \quad \kappa_0 = b, \quad \kappa_1 = a, \quad \theta = a.
\]

(4.4)

**Remark 4.1** Equation (4.4) gives a plane in the parameter space. One can choose the other diagram automorphism, e.g., \(s_5\), to get a seed solution. Such a solution exists on the other plane and can be transformed to (4.2) by some Bäcklund transformations. The seed solution (4.2) is the simplest one.

Under the specialization of (4.2), the Hamiltonians \(h_i\) and \(\tau\)-functions \(\tau_i\) are calculated as

\[
h_0 = \frac{7}{16} x^2 + \frac{1}{8} (2a + 2b - 1)(2a - 2b - 1)x + \frac{1}{16} (8a^2 - 8a + 3 + 8b^2),
\]

\[
h_1 = -\frac{1}{16} x^2 + \frac{1}{8} (2a + 2b - 1)(2a - 2b + 1)x + \frac{1}{16} (8a^2 + 8b^2 - 8b + 7),
\]

8
\[
\begin{align*}
    h_2 &= \frac{1}{8}x^2 + \frac{1}{4}(4a^2 - 4b^2 - 1)x + \frac{1}{8}(8a^2 + 8b^2 + 7), \\
    h_3 &= \frac{3}{16}x^2 + \frac{1}{8}(2a + 2b - 1)(2a - 2b - 1)x + \frac{1}{16}(8a^2 - 8a + 7 + 8b^2), \\
    h_4 &= \frac{3}{16}x^2 + \frac{1}{8}(2a + 2b - 1)(2a - 2b + 1)x + \frac{1}{16}(8a^2 + 8b^2 - 8b + 3),
\end{align*}
\]

and
\[
\begin{align*}
    \tau_0 &= (x - 1)^{\alpha^2 - \frac{1}{2} + \frac{1}{2} - \alpha^2 - \frac{1}{2}}(x + 1)^{\beta^2 + \frac{1}{2}}, \\
    \tau_1 &= (x - 1)^{\alpha^2 + \frac{1}{2} - \alpha^2 - \frac{1}{2} - b^2 - \frac{1}{2}}(x + 1)^{b^2 - \frac{1}{2}}, \\
    \tau_2 &= (x - 1)^{2\alpha^2 + \frac{1}{2} - x - 2\alpha^2 - 2b^2 - \frac{1}{2}}(x + 1)^{2b^2 + 1}, \\
    \tau_3 &= (x - 1)^{\alpha^2 + \frac{1}{2} - \alpha^2 - \frac{1}{2} - b^2 - \frac{1}{2}}(x + 1)^{b^2 + 1}, \\
    \tau_4 &= (x - 1)^{\alpha^2 + \frac{1}{2} - \alpha^2 - \frac{1}{2} - b^2 - \frac{1}{2}}(x + 1)^{b^2 - \frac{1}{2}}.
\end{align*}
\]

up to the multiplication by some constants, respectively.

Using the multiplicative formulas (4.20) and the bilinear equations (4.23)-(4.32), we get the 24 \( \tau \)-functions in (3.41) and (3.42). These are expressed in the form of
\[
\tau_{k,l,m,n} = \sigma_{k,l,m,n}(x - 1)^{\tilde{a}^2 + \frac{1}{2} - \tilde{a}^2 - \frac{1}{2} - b^2 - \frac{1}{2} + \frac{1}{2} + \frac{1}{2}}(x + 1)^{\tilde{b}^2 + \frac{1}{2}},
\]

where \( \sigma_{k,l,m,n} \) are given as follows,

\[
\begin{align*}
    \sigma_{0,0,0,0} &= 1, & \sigma_{1,0,0,0} &= i, & \sigma_{0,-1,0,0} &= -i \left( \frac{1}{2} - a - b \right), & \sigma_{1,-1,0,0} &= -\frac{i}{2} \left( \frac{1}{2} - a + b \right), \\
    \sigma_{0,0,-1,0} &= \frac{1}{2} \left( \frac{1}{2} - a + b \right), & \sigma_{0,-1,-1,0} &= -\frac{i}{2} \left( \frac{1}{2} - a - b \right), & \sigma_{1,0,-1,0} &= i, & \sigma_{1,-1,-1,0} &= -1, \\
    \sigma_{0,0,0,-1} &= \frac{1}{2} \left( \frac{1}{2} - a - b \right), & \sigma_{0,1,0,-1} &= -\frac{i}{2} \left( \frac{1}{2} - a + b \right), & \sigma_{1,0,0,-1} &= i, & \sigma_{1,1,0,-1} &= 1, \\
    \sigma_{1,0,-1,-1} &= i, & \sigma_{2,0,-1,-1} &= 1, & \sigma_{1,-1,-1,-1} &= \frac{1}{2} \left( \frac{1}{2} - a - b \right), & \sigma_{2,-1,-1,-1} &= i \left( \frac{1}{2} - a - b \right), \\
    \sigma_{0,-1,0,1} &= \frac{i}{2} \left[ \left( \frac{1}{2} - a - b \right) x - \left( \frac{1}{2} + a - b \right) \right], \\
    \sigma_{0,0,1,0} &= -\frac{i}{2} \left[ \left( \frac{1}{2} + a + b \right) x - \left( \frac{1}{2} - a - b \right) \right], \\
    \sigma_{0,1,-1,-1} &= \frac{i}{2} \left[ \left( \frac{1}{2} - a + b \right) x + \left( \frac{1}{2} - a - b \right) \right], \\
    \sigma_{0,-1,-1,1} &= -\frac{i}{2} \left[ \left( \frac{1}{2} - a - b \right) x + \left( \frac{1}{2} - a + b \right) \right], \\
    \sigma_{1,1,0,-2} &= \frac{1}{2} \left[ \left( \frac{1}{2} - a - b \right) x + \left( \frac{1}{2} - a + b \right) \right], \\
    \sigma_{1,-1,-2,0} &= \frac{1}{2} \left[ \left( \frac{1}{2} - a + b \right) x + \left( \frac{1}{2} - a - b \right) \right],
\end{align*}
\]
\[ \sigma_{2,-1,-2,-1} = -\frac{i}{2} \left( \left( \frac{1}{2} + a - b \right) x - \left( \frac{1}{2} - a - b \right) \right), \]
\[ \sigma_{2,0,-1,-2} = \frac{1}{2} \left( \left( \frac{1}{2} - a - b \right) x - \left( \frac{1}{2} + a - b \right) \right). \]

### 4.2 Application of Bäcklund transformations

Assume that \( T_{t,l,m,n} \) are expressed as (4.7) for any \( k, l, m, n \in \mathbb{Z} \). Substituting \( \alpha_0 = \alpha_3 = a, \alpha_1 = \alpha_4 = b \) and (4.7) into (3.43), we obtain the Toda equations for \( \sigma_{k,l,m,n} \). The first two equations yield to
\[
4\sigma_{k+1,l,m,n}\sigma_{k-1,l,m,n} = [(x+1)^2D^2 - (\ddot{a} + \ddot{b})(\dddot{a} - \dddot{b})] \sigma_{k,l,m,n} \cdot \sigma_{k,l,m,n},
\]
\[
4\sigma_{k,l+1,m,n}\sigma_{k,l-1,m,n} = -[(x-1)^2D^2 - (\ddot{a} + \ddot{b})(\dddot{a} - \dddot{b})] \sigma_{k,l,m,n} \cdot \sigma_{k,l,m,n},
\]
where we denote as
\[
D^2 \sigma \cdot \sigma = x(\ddot{\sigma} - \dot{\sigma}^2) + \dot{\sigma} \sigma, \quad \ddot{\sigma} = \frac{d\sigma}{dx}.
\]
The others are reduced to
\[
4\sigma_{k,l,m+1,n}\sigma_{k,l,m-1,n} = (x-1)x(x+1)(\ddot{\sigma}_{k,l,m,n}\sigma_{k,l,m,n} - \dot{\sigma}_{k,l,m,n}^2) + (3x^2 - 1)\ddot{\sigma}_{k,l,m,n}\sigma_{k,l,m,n} + \left\{ \left[ (\ddot{a} - \ddot{b} + m - \frac{1}{2}) \left( \ddot{a} - \ddot{b} + 3m + \frac{1}{2} \right) - n(n+1) \right] x \right. \\
+ \left. \left[ (\ddot{a} - \ddot{b} + m - \frac{1}{2}) \left( \ddot{a} + \ddot{b} + n - \frac{1}{2} \right) \right] \right\} \sigma_{k,l,m,n}^2,
\]
\[
4\sigma_{k,l,m,n+1}\sigma_{k,l,m,n-1} = (x-1)x(x+1)(\ddot{\sigma}_{k,l,m,n}\sigma_{k,l,m,n} - \dot{\sigma}_{k,l,m,n}^2) + (3x^2 - 1)\ddot{\sigma}_{k,l,m,n}\sigma_{k,l,m,n} + \left\{ \left[ (\ddot{a} + \ddot{b} - n - \frac{1}{2}) \left( \ddot{a} + \ddot{b} + 3n + \frac{1}{2} \right) - m(m+1) \right] x \right. \\
+ \left. \left[ (\ddot{a} + \ddot{b} - n - \frac{1}{2}) \left( \dddot{a} - \dddot{b} + m - \frac{1}{2} \right) \right] \right\} \sigma_{k,l,m,n}^2,
\]
with
\[
\ddot{a} = a + k, \quad \ddot{b} = b + l.
\]
Toda equations (4.10) and (4.18) with the initial data (4.8)-(4.15) generate \( \sigma_{k,l,m,n} = \sigma_{k,l,m,n}(x; a, b) \) for \( k, l, m, n \in \mathbb{Z} \). From (3.44) and (4.7), we see that the ratio of \( \sigma_{k,l,m,n} \) gives a family of algebraic solutions to the symmetric form of \( P_{VI} \).

Noticing that we have
\[
T_{14}^1 T_{03}^1 (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left( \ddot{a}, \ddot{b}, \frac{1}{2} - \ddot{a}, \ddot{b}, , \ddot{a}, \ddot{b} \right),
\]
under the specialization (4.2), we see that the action of \( T_{03}^1 \) and \( T_{14}^1 \) on the parameter space is absorbed by shift of the parameters \( a \) and \( b \), respectively. This suggests that we do not need to consider the translations \( T_{03}^1 \) and \( T_{14}^1 \) in order to get a family of algebraic solutions of \( P_{VI} \).

To verify this, we put
\[
\sigma_{k,l,m,n} = \omega_{k,l,m,n} V_{k,l,m,n}, \quad \omega_{k,l,m,n} = \omega_{k,l,m,n}(a, b), \quad k, l, m, n \in \mathbb{Z}.
\]
The constants \( \omega_{k,l,m,n} \) are determined by recurrence relations as follows. With respect to the indices \( k \) and \( l \), \( \omega_{k,l,i,j} \) with \( (i, j) = (-1, -1), (-1, 0), (0, -1), (0, 0) \) are subject to
\[
4\omega_{k+1,l,i,j}\omega_{k-1,l,i,j} = -\left( \ddot{a} - \ddot{b} + i - \frac{1}{2} \right) \left( \ddot{a} + \ddot{b} + j - \frac{1}{2} \right) \omega_{k,l,i,j}^2,
\]
\[
4\omega_{k,l+1,i,j}\omega_{k,l-1,i,j} = \left( \ddot{a} - \ddot{b} + i - \frac{1}{2} \right) \left( \ddot{a} + \ddot{b} + j - \frac{1}{2} \right) \omega_{k,l,i,j}^2.
\]
The initial conditions are given by
\[
\begin{align*}
\omega_{0,0,0,0} &= 1, \quad \omega_{1,0,0,0} = i, \quad \omega_{0,-1,0,0} = -\frac{i}{2} \left( \frac{1}{2} - a - b \right), \quad \omega_{1,-1,0,0} = -\frac{i}{2} \left( \frac{1}{2} + a - b \right), \\
\omega_{0,-1,0} &= \frac{1}{2} \left( \frac{1}{2} - a + b \right), \quad \omega_{0,-1,-1} = \frac{i}{2} \left( \frac{1}{2} - a - b \right), \quad \omega_{1,0,-1,0} = i, \quad \omega_{1,-1,0} = -1, \\
\omega_{1,0,0,0} &= \frac{1}{2} \left( \frac{1}{2} - a - b \right), \quad \omega_{0,0,0,0} = \frac{i}{2} \left( \frac{1}{2} - a + b \right), \\
\omega_{1,0,-1} &= \omega_{1,0,0,0} = \frac{1}{2} \left( \frac{1}{2} - a - b \right), \quad \omega_{0,0,0,0} - 1 = \frac{i}{2} \left( \frac{1}{2} + a - b \right).
\end{align*}
\] (4.23)

Note that these imply
\[
V_{k,l,-1,-1} = V_{k,l,-1,0} = V_{k,l,0,-1} = V_{k,l,0,0} = 1, \quad k,l \in \mathbb{Z}.
\] (4.27)

For the indices \(m\) and \(n\), we set
\[
8\omega_{k,l,m+1,n} \omega_{k,l,m-1,n} = \left( \hat{a} - \hat{b} + m - \frac{1}{2} \right) \omega_{k,l,m,n}^2,
\] (4.28)

Thus, \(\omega_{k,l,m,n}\) are determined for any \(k,l,m,n \in \mathbb{Z}\). As a result, the functions \(V_{k,l,m,n} = V_{k,l,m,n}(x; a, b)\) introduced in (1.21) have a symmetry described in the following lemma.

**Lemma 4.2** We have
\[
V_{k,l,m,n}(x; a, b) = V_{0,0,m,n}(x; a + k, b + l).
\] (4.29)

**Proof.** Toda equations (4.18) are reduced to
\[
\begin{align*}
\frac{1}{2} \left( \hat{a} - \hat{b} + m - \frac{1}{2} \right) V_{k,l,m+1,n} V_{k,l,m-1,n} &= (x - 1)x(x + 1)(\hat{V}_{k,l,m,n} V_{k,l,m,n} - \hat{V}_{k,l,m,n}^2) + (3x^2 - 1) \hat{V}_{k,l,m,n} V_{k,l,m,n} \\
& \quad + \left\{ \left( \hat{a} - \hat{b} + m - \frac{1}{2} \right) \left[ \left( \hat{a} - \hat{b} + 3m + \frac{1}{2} \right) x + \left( \hat{a} + \hat{b} + n - \frac{1}{2} \right) \right] - n(n + 1)x \right\} V_{k,l,m,n}^2,
\end{align*}
\] (4.30)

Thus, we see that \(V_{k,l,m,n}(x; a, b)\) satisfy the same Toda equations as \(V_{0,0,m,n}(x; a + k, b + l)\). Since the initial conditions are given by (1.27), we obtain (1.29).

On the other hand, we have from (3.44), (1.7) and (1.21)
\[
T_{34}^{\sigma_{k,l,m,n} \sigma_{k+2,l,m-1,n-1}} (fo) = x(x - 1) \frac{\sigma_{k,l,m,n} \sigma_{k+2,l,m-1,n-1}}{\sigma_{k+1,l-1,m-1,n} \sigma_{k+1,l+1,m-1,n}} \frac{V_{k,l,m,n} V_{k+2,l,m-1,n-1}}{V_{k+1,l-1,m-1,n} V_{k+1,l+1,m-1,n}}.
\] (4.31)

The ratio of \(\omega\)'s is calculated as
\[
\frac{\omega_{k,l,m,n} \omega_{k+2,l,m-1,n-1}}{\omega_{k+1,l-1,m-1,n} \omega_{k+1,l+1,m-1,n}} = -1.
\] (4.32)
Similarly, for \( f_3, f_4 \) and \( f_2 \), we get

\[
\begin{align*}
\frac{\omega_{k+1,l,m-1,n-1}\omega_{k+1,l,m,n}}{\omega_{k+1,l-1,m-1,n}\omega_{k+1,l+1,m,n-1}} &= 1, \\
\frac{\omega_{k+1,l,m-1,n-1}\omega_{k+1,l,m,n}}{\omega_{k+1,l-1,m,n}\omega_{k+1,l+1,m,n-1}} &= 1, \\
\frac{\omega_{k+1,l-1,m-1,n+1}\omega_{k+1,l+1,m,n-1}}{\omega_{k,l,m,n}\omega_{k+1,l-1,m-1,n-1}^{-1}\omega_{k+1,l+1,m,n-1}} &= -\frac{1}{2} \left( \hat{a} + \hat{b} + n - \frac{1}{2} \right).
\end{align*}
\] (4.33)

The above discussion means that we can put \( k = l = 0 \) without loss of generality for our purpose. Hence, we denote \( V_{0,0,m,n} = V_{m,n} \).

Moreover, it is observed that \( V_{m,n} = V_{m,n}(x; a, b) \) \((m, n \in \mathbb{Z})\) are polynomials in \( a, b \) and \( x \) with coefficients in \( \mathbb{Z} \). We will show this in the next section by presenting the explicit expressions.

**Proposition 4.3** Let \( V_{m,n} = V_{m,n}(x; a, b) \) \((m, n \in \mathbb{Z})\) be polynomials generated by Toda equations,

\[
\begin{align*}
\frac{1}{2} \left( a - b + m - \frac{1}{2} \right) V_{m+1,n} &= (x - 1)x(x + 1)(V_{m,n} - \dot{V}_{m,n}) + (3x^2 - 1)\dot{V}_{m,n}V_{m,n} \\
&+ \left\{ \left( a - b + m - \frac{1}{2} \right) \left[ (a - b + 3m + \frac{1}{2}) x + \left( a + b + n - \frac{1}{2} \right) \right] - n(n + 1)x \right\} V_{m,n}^2, \\
\frac{1}{2} \left( a + b + n - \frac{1}{2} \right) V_{m,n+1} &= (x - 1)x(x + 1)(\dot{V}_{m,n} - \dot{V}_{m,n}) + (3x^2 - 1)V_{m,n}V_{m,n} \\
&+ \left\{ \left( a + b + n - \frac{1}{2} \right) \left[ (a + b + 3n + \frac{1}{2}) x + \left( a - b + m - \frac{1}{2} \right) \right] - m(m + 1)x \right\} V_{m,n}^2,
\end{align*}
\] (4.34)

with the initial conditions,

\[
V_{-1,-1} = V_{-1,0} = V_{0,-1} = V_{0,0} = 1.
\] (4.35)

Then,

\[
\begin{align*}
f_0 &= x(1 - x) \frac{V_{m,n}(x; a, b)V_{m-1,n-1}(x; a + 2, b)}{V_{m-1,n}(x; a + 1, b - 1)V_{m,n-1}(x; a + 1, b + 1)}, \\
f_3 &= (x - 1) \frac{V_{m-1,n-1}(x; a + 1, b)V_{m,n}(x; a + 1, b)}{V_{m-1,n}(x; a + 1, b - 1)V_{m,n-1}(x; a + 1, b + 1)}, \\
f_4 &= x \frac{V_{m,n-1}(x; a + 1, b)V_{m,n-1}(x; a + 1, b)}{V_{m-1,n}(x; a + 1, b - 1)V_{m,n-1}(x; a + 1, b + 1)}, \\
f_2 &= \frac{1}{2} \left( a + b + n - \frac{1}{2} \right) x^{-1} \frac{V_{m,n}(x; a + 1, b)V_{m,n-1}(x; a + 1, b)V_{m,n-1}(x; a, b)}{V_{m,n}(x; a, b)V_{m,n-1}(x; a + 1, b)V_{m,n-1}(x; a + 1, b)},
\end{align*}
\] (4.36)

satisfy the symmetric form of \( PV_1 \) for the parameters

\[
(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left( a, b, \frac{1}{2} - a - b - n, a + m + n, b - m + n \right).
\] (4.37)

Furthermore, the bilinear relations for \( V_{m,n} \) are derived from (3.28)-(3.33).

**Proposition 4.4** The polynomials \( V_{m,n}(x; a, b) \) satisfy the following bilinear relations,

\[
\begin{align*}
V_{m,n}(1-x) &+ \dot{V}_{m,n} = 0, \\
(x - 1)V_{m,n} &+ \dot{V}_{m,n} = 0, \\
V_{m,n} &+ \dot{V}_{m,n} = 0, \\
(x + 1)V_{m,n} &+ \dot{V}_{m,n} = 0,
\end{align*}
\] (4.38)
where we denote as systems, the consistency is guaranteed by construction. In order to show Theorem 2.2, we will prove not Toda

Conversely, by solving the bilinear relations (4.38)-(4.43) with the initial conditions (4.35), one can get

\[ 4aV_{m-1,n-1}^{(1,0)} + 2\nu_n(x-1)V_{m,n-1}^{(0,0)} - V_{m,n}^{(0,0)} = 0, \]

\[ 2aV_{m-1,n}^{(1,0)} - \nu_n V_{m-1,n-1}^{(2,0)} - \mu_{m+1}V_{m,n}^{(0,0)} - V_{m,n}^{(2,0)} = 0, \]

\[ 4axV_{m-1,n}^{(1,0)} + 2\nu_n(x-1)V_{m,n-1}^{(0,0)} - V_{m,n}^{(0,0)} = 0, \]

\[ 4bV_{m-1,n}^{(1,0)} - 2\nu_n(x+1)V_{m,n}^{(1,1)} - V_{m,n}^{(1,1)} + V_{m-1,n}^{(1,1)} = 0, \]

\[ 4bxV_{m,n}^{(0,0)} - 2\nu_n(x+1)V_{m,n}^{(0,0)} - V_{m,n}^{(0,0)} = 0, \]

\[ 2bV_{m,n}^{(0,0)} - \nu_n V_{m,n-1}^{(0,0)} + \mu_{m+1}V_{m,n}^{(0,0)} = 0, \]

\[ 2(b - m + n)V_{m-1,n}^{(0,0)} = 0, \]

\[ 4(b - m + n)V_{m,n-1}^{(0,0)} - 2\nu_n(x+1)V_{m,n}^{(0,0)} + V_{m,n}^{(0,0)} = 0, \]

\[ 2V_{m,n}^{(0,0)} - (x+1)V_{m,n}^{(0,0)} - (x-1)V_{m,n}^{(0,0)} = 0, \]

\[ 2xV_{m,n}^{(0,0)} - (x+1)V_{m,n}^{(0,0)} - (x-1)V_{m,n}^{(0,0)} = 0, \]

\[ 2V_{m,n}^{(0,0)} - (x+1)V_{m,n}^{(0,0)} - (x-1)V_{m,n}^{(0,0)} = 0, \]

\[ 2V_{m-1,n}^{(1,0)} - (x+1)V_{m-1,n}^{(1,0)} - (x-1)V_{m-1,n}^{(1,0)} = 0, \]

\[ 2xV_{m-1,n}^{(1,0)} - (x+1)V_{m-1,n}^{(1,0)} - (x-1)V_{m-1,n}^{(1,0)} = 0, \]

\[ 2V_{m,n}^{(1,0)} - (x+1)V_{m,n}^{(1,0)} - (x-1)V_{m,n}^{(1,0)} = 0, \]

\[ 2V_{m-1,n}^{(1,0)} - (x+1)V_{m-1,n}^{(1,0)} - (x-1)V_{m-1,n}^{(1,0)} = 0, \]

\[ \text{where we denote as} \]

\[ \mu_n = a - b + m - \frac{1}{2}, \quad \nu_n = a + b + n - \frac{1}{2}, \]

\[ V_{m,n}^{(k,l)} = V_{m,n}(x; a + k, b + l). \]

Conversely, by solving the bilinear relations (4.38)-(4.43) with the initial conditions (4.35), one can get the family of algebraic solutions (4.36) with (4.37). Even though these bilinear relations are overdetermined systems, the consistency is guaranteed by construction. In order to show Theorem 2.2, we will prove not Toda equations (4.34) but the bilinear relations (4.38)-(4.43).

5 Proof of the determinant formula

In this section, we give a proof of Theorem 2.2

Proposition 5.1 We have

\[ V_{m,n}(x; a, b) = (-2x)^{m(m+1)/2}(-2)^{n(n+1)/2}\xi_{m,n}S_{m,n}(x; a, b), \quad m, n \in \mathbb{Z}, \]

where \( S_{m,n} = S_{m,n}(x; a, b) \) is defined in Theorem 2.2 and \( \xi_n \) is the factor determined by

\[ \xi_n = (2n + 1)\xi_n^2, \quad \xi_0 = 1. \]

\[ \xi_n = 1. \]
From this Proposition, it is easy to verify that \( V_{m,n}(x; a, b) \) are indeed polynomials in \( a, b \) and \( x \) with coefficients in \( \mathbb{Z} \).

Substituting (5.1) into (4.36), we find that Theorem 2.2 is a direct consequence of Proposition 5.1. Taking (2.3), (2.4) and (5.1) into account, we obtain the bilinear relations for \( R_{m,n} \).

**Proposition 5.2** Let \( R_{m,n} \) be a family of polynomials given in Definition 2.1. Then, we have

\[
\begin{align*}
(2c - d - m - n)R_{m,n}^{(0,0)} &+ c(x - 1)R_{m,n}^{(1,0)} + (2n + 1)R_{m,n}^{(0,1)} = 0, \\
(2c - d - m - n)R_{m,n}^{(0,0)} &- cR_{m,n}^{(1,0)} - (c - d)R_{m,n}^{(0,1)} = 0, \\
(2c - d - m - n)xR_{m,n}^{(0,1)} &- c(x - 1)R_{m,n}^{(1,0)} + (2m + 1)xR_{m,n}^{(0,0)} = 0, \\
(d + m - n)R_{m,n}^{(1,1)} &- c(x + 1)R_{m,n}^{(1,2)} - (2n + 1)R_{m,n}^{(0,1)} = 0, \\
(d + m - n)xR_{m,n}^{(0,1)} &- c(x + 1)R_{m,n}^{(1,1)} - (2m + 1)xR_{m,n}^{(0,0)} = 0, \\
(d + m - n)R_{m,n}^{(0,0)} &- cR_{m,n}^{(1,1)} + (c - d)R_{m,n}^{(0,1)} = 0, \\
(2c - d + m + n)xR_{m,n}^{(0,1)} &- c(x - 1)R_{m,n}^{(1,1)} + (2m + 1)xR_{m,n}^{(0,0)} = 0, \\
(2c - d + m + n)R_{m,n}^{(0,0)} &- cR_{m,n}^{(1,1)} - (c - d)R_{m,n}^{(0,1)} = 0, \\
(2c - d + m + n)xR_{m,n}^{(0,1)} &- c(x - 1)R_{m,n}^{(1,1)} + (2n + 1)xR_{m,n}^{(0,0)} = 0, \\
(d + m - n)R_{m,n}^{(0,0)} &- cR_{m,n}^{(1,1)} + (c - d)R_{m,n}^{(0,1)} = 0, \\
(d + m - n)xR_{m,n}^{(0,1)} &- c(x + 1)R_{m,n}^{(1,2)} - (2m + 1)xR_{m,n}^{(0,0)} = 0, \\
(2c - d + m + n)xR_{m,n}^{(0,1)} &- c(x - 1)R_{m,n}^{(1,1)} + (2m + 1)xR_{m,n}^{(0,0)} = 0,
\end{align*}
\]



where we denote as

\[
R^{(i,j)}_{m,n} = R_{m,n}(c + i, d + j).
\] (5.9)

From the above discussion, now the proof of Theorem 2.2 is reduced to that of Proposition 5.2.

It is possible to reduce the number of bilinear relations to be proved by the following symmetries of \( R_{m,n} \).
Lemma 5.3 We have the relations for $m,n \in \mathbb{Z}_{\geq 0}$

\[
R_{m,n}(x^{-1}) = R_{m,n}(x),
\]
\[
R_{m,n}(-c,-d) = (-1)^{m(m+1)/2+n(n+1)/2} R_{m,n}(c,d),
\]
\[
R_{m,n}(-x;c,2c-d) = (-1)^{m(m+1)/2+n(n+1)/2} R_{m,n}(x;c,d).
\]

Proof. The first relation (5.10) is easily obtained from the definition (2.1) and (2.3). To verify the second relation (5.11), we introduce two sets of polynomials $\tilde{p}_k = \tilde{p}^{(c,d)}_k(x)$ and $\tilde{q}_k = \tilde{q}^{(c,d)}_k(x)$, $k \in \mathbb{Z}$, by

\[
\sum_{k=0}^{\infty} \tilde{p}^{(c,d)}_k(x)\lambda^k = G(x; -c,-d; -\lambda), \quad \tilde{p}^{(c,d)}_k(x) = 0 \text{ for } k < 0,
\]
\[
\tilde{q}^{(c,d)}_k(x) = \tilde{p}^{(c,d)}_k(x^{-1}),
\]
where $G$ is the generating function (2.2). Since we have

\[
\frac{G(x; -c,-d; -\lambda)}{G(x; c,d; \lambda)} = (1 - \lambda^2)^{-c+d} \left(1 - x^2 \lambda^2\right)^c,
\]
we see that

\[
\tilde{p}_k(x) = p_k(x) + \sum_{j=1}^{\infty} \rho_j(x)p_{k-2j}(x), \quad \tilde{q}_k(x) = q_k(x) + \sum_{j=1}^{\infty} \rho_j(x^{-1})q_{k-2j}(x),
\]
where $\rho_j(x) = \rho_j(x;c,d)$ are some functions. Therefore, $R_{m,n}$ for $m,n \in \mathbb{Z}_{\geq 0}$ can be expressed in terms of the same determinant as (2.3) with the entries $p_k$ and $q_k$ replaced by $\tilde{p}_k$ and $\tilde{q}_k$, respectively. Noticing that

\[
\tilde{p}^{(c,d)}_k(x) = (-1)^k \tilde{p}^{(c,-d)}_k(x), \quad \tilde{q}^{(c,d)}_k(x) = (-1)^k \tilde{q}^{(c,-d)}_k(x),
\]
we obtain the relation (5.11). The third relation (5.12) is verified similarly.

By the symmetries of $R_{m,n}$ described by (2.4) and Lemma 7.3, it is sufficient to prove the following bilinear relations for $m,n \in \mathbb{Z}_{\geq 0}$,

\[
(x + 1)R^{(1,1)}_{m,m-1}R^{(0,0)}_{m-1,m-1} - xR^{(1,0)}_{m,m-1}R^{(0,1)}_{m,m-1} - R^{(1,0)}_{m,m-1}R^{(0,0)}_{m-1,m-1} = 0,
\]
\[
(d + m - n)R^{(0,1)}_{m,m-1}R^{(0,0)}_{m-1,m-1} - c(x + 1)R^{(1,2)}_{m,m-1}R^{(1,1)}_{m-1,m-1} - (2n + 1)R^{(0,0)}_{m,m-1}R^{(0,1)}_{m-1,m-1} = 0,
\]
\[
(2R^{(0,0)}_{m,m-1}R^{(0,0)}_{m-1,m-1} - (x + 1)R^{(1,1)}_{m,m-1}R^{(1,1)}_{m-1,m-1} - R^{(1,1)}_{m,m-1}R^{(1,1)}_{m-1,m-1} = 0,
\]
\[
(2R^{(0,0)}_{m,m-1}R^{(0,0)}_{m-1,m-1} - (x + 1)R^{(1,1)}_{m,m-1}R^{(0,1)}_{m-1,m-1} - x - 1)R^{(1,0)}_{m,m-1}R^{(1,0)}_{m-1,m-1} = 0.
\]

From the symmetry (5.12) and the bilinear relation (5.17), we have

\[
(x - 1)R^{(0,0)}_{m,m-1}R^{(1,0)}_{m-1,m-1} - xR^{(0,1)}_{m,m-1}R^{(1,1)}_{m-1,m-1} + R^{(0,1)}_{m,m-1}R^{(0,0)}_{m-1,m-1} = 0.
\]

Then, it is possible to derive the bilinear relation (5.22) as follows,

\[
R^{(0,0)}_{m-1,m} \times (5.20)_{d=d-1} - R^{(1,1)}_{m-1,m} \times (5.17) + R^{(1,0)}_{m-1,m} \times (5.23)_{c=c-1,d=d-1} = R^{(0,1)}_{m-1,m} \times (5.24).
\]

Therefore, the bilinear relations to be proved are (5.17)-(5.21).

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In the following, we show that these bilinear relations (5.17)-(5.21) are reduced to Jacobi's identity of determinants. Let $D$ be an $(m + n + 1) \times (m + n + 1)$ determinant and $D \begin{bmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{bmatrix}$ the minor that are obtained by deleting the rows with indices $i_1, \ldots, i_k$ and the columns with indices $j_1, \ldots, j_k$. Then, we have Jacobi's identities
\begin{align}
D \cdot D \begin{bmatrix} m & m+1 \\ 1 & m+n+1 \end{bmatrix} &= D \begin{bmatrix} m \\ 1 \end{bmatrix} D \begin{bmatrix} m+1 \\ m+n+1 \end{bmatrix} - D \begin{bmatrix} m \\ m+n+1 \end{bmatrix} D \begin{bmatrix} m+1 \\ 1 \end{bmatrix}, \quad (5.25) \\
D \cdot D \begin{bmatrix} m & m+1 \\ 1 & m+n+1 \end{bmatrix} &= D \begin{bmatrix} m \\ 1 \end{bmatrix} D \begin{bmatrix} m+1 \\ 2 \end{bmatrix} - D \begin{bmatrix} m \\ 2 \end{bmatrix} D \begin{bmatrix} m+1 \\ 1 \end{bmatrix}, \quad (5.26) \\
D \cdot D \begin{bmatrix} 1 & m+1 \\ 2 & m+n+1 \end{bmatrix} &= D \begin{bmatrix} 1 \\ 2 \end{bmatrix} D \begin{bmatrix} m+1 \\ m+n+1 \end{bmatrix} - D \begin{bmatrix} 1 \\ m+n+1 \end{bmatrix} D \begin{bmatrix} m+1 \\ 2 \end{bmatrix}. \quad (5.27)
\end{align}
First, we give the proof of the bilinear relations (5.17)-(5.19). We have the following lemmas.

**Lemma 5.4** \[ \text{Put} \]
\[ D = \begin{bmatrix} -q_1 & q_1 & \cdots & q_1 \\ -q_3 & q_3 & \cdots & q_3 \\ \vdots & \vdots & \ddots & \vdots \\ -q_{m+n-1} & q_{m+n-1} & \cdots & q_{m+n-1} \\ x^{-m} p_{2n} & x^{-m+1} p_{2n} & \cdots & x^{-1} p_{2n} \\ x^{-m+n} p_{2n} & x^{-m+n+1} p_{2n} & \cdots & x^{-1} p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x^{-m+n} p_{0} & x^{-m+n+1} p_{0} & \cdots & x^{-1} p_{0} \end{bmatrix}, \quad (5.28) \]
Then, we have
\begin{align}
D &= (-1)^m (1 + x^{-1})^{m+n} R_{m,n}^{(0,0)} \\
D \begin{bmatrix} m \\ 1 \end{bmatrix} &= R_{m-1,n}^{(0,-1)} \\
D \begin{bmatrix} m+1 \\ m+n+1 \end{bmatrix} &= (-1)^{m+1} (1 + x^{-1})^{m+n+1} R_{m,n}^{(-1,-1)} \\
D \begin{bmatrix} m+1 \\ 1 \end{bmatrix} &= R_{m,n}^{(0,-1)}. \quad (5.29)
\end{align}

**Lemma 5.5** \[ \text{Put} \]
\[ D = \begin{bmatrix} q_1 & q_1 & \cdots & q_1 \\ q_3 & q_3 & \cdots & q_3 \\ \vdots & \vdots & \ddots & \vdots \\ q_{2n-1} & q_{2n-1} & \cdots & q_{2n-1} \\ x^{-m+1} p_{2n+1} & x^{-m+n} p_{2n+1} & \cdots & x^{-1} p_{2n+1} \\ x^{-m+n+1} p_{2n+1} & x^{-m+1} p_{2n+1} & \cdots & x^{-1} p_{2n+1} \\ \vdots & \vdots & \ddots & \vdots \\ x^{-m+n} p_{0} & x^{-m+n+1} p_{0} & \cdots & x^{-1} p_{0} \end{bmatrix}, \quad (5.30) \]
with

\[
\frac{1}{p_{2k}} = \frac{x^{m-n}}{2k+1} p_{2k-1}^{(c-m-n,d-m-n)}, \quad \frac{1}{q_{2k-1}} = \frac{q_{2k-1}^{(c-m-n,d-m-n)}}{(d-m-n+2k-2)x}.
\] (5.31)

Then, we have

\[
D = (-1)^{m+n}(1 + x)^{m+n} x^{-m} \prod_{j=1}^{m} (c - m - n + j - 1) \prod_{k=0}^{n} (d - m - n + 2i - 2) \prod_{k=0}^{m+n} (2k+1) R_{m,n}^{(0,0)}.
\]

\[
D \left[ \begin{array}{c} \frac{m}{m+n+1} \\ \end{array} \right] = (-1)^{m+n-1}(1 + x)^{m+n-1} x^{-m} \prod_{j=1}^{m+n-1} (c - m - n + j - 1) \prod_{k=0}^{n-1} (d - m - n + 2i - 2) \prod_{k=0}^{m+n-1} (2k+1) R_{m,n}^{(-1,-1)}.
\] (5.32)

\[
D \left[ \begin{array}{c} \frac{m+1}{m+n+1} \\ \end{array} \right] = (-1)^{m+n-1}(1 + x)^{m+n-1} x^{-m} \prod_{j=1}^{m+n-1} (c - m - n + j - 1) \prod_{k=0}^{n-1} (d - m - n + 2i - 2) \prod_{k=0}^{m+n-1} (2k+1) R_{m,n}^{(-1,-1)}.
\]

Lemma 5.6 Put

\[
D = \begin{bmatrix}
q_1^{(c,d-m-n)} & q_1^{(c-1,d-m-n-1)} & \cdots & q_1^{(c-1,d-3)} & q_1^{(c-1,d-2)} \\
q_1^{(c,d-m-n)} & q_1^{(c-1,d-m-n-1)} & \cdots & q_1^{(c-1,d-3)} & q_1^{(c-1,d-2)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
q_1^{(c,d-m-n)} & q_1^{(c-1,d-m-n-1)} & \cdots & q_1^{(c-1,d-3)} & q_1^{(c-1,d-2)} \\
q_1^{(c,d-m-n)} & q_1^{(c-1,d-m-n-1)} & \cdots & q_1^{(c-1,d-3)} & q_1^{(c-1,d-2)} \\
\end{bmatrix},
\] (5.33)

with

\[
\frac{1}{p_{2k}} = (-1)^{m+n} \frac{p_{2k}^{(c,d-m-n)}}{2k+1}, \quad \frac{1}{q_{2k-1}} = \frac{q_{2k-1}^{(c,d-m-n)}}{(d-m-n+2k-2)x}.
\] (5.34)

Then, we have

\[
D = (-1)^{m+n}(1 + x)^{m+n} x^{-m} \prod_{j=1}^{m} (c - d + m + n - j + 1) \prod_{k=0}^{n} (d - m - n + 2i - 2) \prod_{k=0}^{m+n} (2k+1) R_{m,n}^{(0,0)}.
\]
with \( p \) and \( q \), respectively. By Lemma 5.6, Jacobi’s identity (5.25) is reduced to

\[
D \left[ \begin{array}{c}
m \\ 
 m + n + 1
\end{array} \right] = (-1)^{m}(1 + x)^{m+n-1} x^{-m+1} \prod_{j=1}^{m+n-1} (c-d+m+n-j+1) \prod_{i=1}^{m-1} (d-m-n+2i-2) \prod_{k=0}^{n-1} (2k+1) \] 

\[ R_{m-1,n}^{(0,-1)}, \]

\[
D \left[ \begin{array}{c}
m + 1 \\ 
 m + n + 1
\end{array} \right] = (-1)^{m-1}(1 + x)^{m+n-1} x^{-m} \prod_{j=1}^{m+n-1} (c-d+m+n-j+1) \prod_{i=1}^{m+n-1} (d-m-n+2i-2) \prod_{k=0}^{n-1} (2k+1) \] 

\[ R_{m,n}^{(0,-1)}, \]

\[
D \left[ \begin{array}{c}
m \\ 
 m + 1 \\ 
 m + n + 1
\end{array} \right] = (-1)^{m} R_{m-1,n}^{(-1,-2)}, \quad D \left[ \begin{array}{c}
m + 1 \\ 
 m + 1 \\ 
 m + n + 1
\end{array} \right] = (-1)^{n} R_{m-1,n}^{(-1,-3)}. \quad (5.35)
\]

It is easy to see that the bilinear relations (5.17) and (5.18) follow immediately from Jacobi’s identity (5.25) by using Lemma 5.4 and 5.5, respectively. By the Lemma 5.6, Jacobi’s identity (5.25) is reduced to

\[
(d + m - n)(xR_{m-1,n}^{(1,1)} R_{m,n}^{(0,0)} + (c-d)(x+1)R_{m,n}^{(1,2)} R_{m-1,n}^{(0,-1)} + (2n+1)R_{m-1,n}^{(0,0)} R_{m,n-1}^{(1,1)} = 0. \quad (5.36)
\]

Then, the bilinear relation (5.19) is derived as follows,

\[
R_{m,n}^{(0,1)} \times (5.36) + R_{m,n-1}^{(1,1)} \times (5.18) + (d + m - n) R_{m,n}^{(0,0)} \times (5.17)_{d \rightarrow d+1} = (x+1)R_{m,n}^{(1,2)} \times (5.19). \quad (5.37)
\]

The proof of Lemmas 5.4–5.6 is given in Appendix II.

Next, we prove the bilinear relations (5.20) and (5.21). We have the following lemmas.

**Lemma 5.7** Put

\[
D = \begin{bmatrix}
-x^{-1} q_{-1}^- & x^{-1} q_1^+ & q_1 & \cdots & q_{m-n+1} \\
-x^{-1} (q_2^- + x^{-2} q_1^-) & x^{-1} q_3^+ & q_3 & \cdots & q_{m-n+3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-x^{-1} (p_{m-n+1}^+ + \cdots + x^{-2m+2} q_{m-1}^-) & x^{-1} q_{m-1}^+ & q_{m-1} & \cdots & q_{m+n-2} \\
p_{m-n+1}^- + \cdots + x^{2m-2} p_{m-n-1}^- & p_{m-n+1}^- & p_{m-n+2} & \cdots & p_{m-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
p_{m-n+m+5}^- + x^2 p_{m-n-3}^- & p_{m-n-m+5}^+ & p_{m-n-6} & \cdots & p_3 \\
p_{m-n-3}^- & p_{m-n-4}^+ & p_{m-n-4} & \cdots & p_1
\end{bmatrix}, \quad (5.38)
\]

with \( p_k^\pm = p_k^{(\pm 1,\pm 1)} \). Then, we have

\[
D \left[ \begin{array}{c}
m \\ 
 1
\end{array} \right] = x^{-m+1} R_{m-1,n}^{(1,1)}, \quad D \left[ \begin{array}{c}
m + 1 \\ 
 1
\end{array} \right] = x^{-m} R_{m,n}^{(1,1)}, \quad (5.39)
\]

\[
D \left[ \begin{array}{c}
m \\ 
 2
\end{array} \right] = -(x)^{-m+1} R_{m-1,n}^{(-1,1)}, \quad D \left[ \begin{array}{c}
m + 1 \\ 
 2
\end{array} \right] = -(x)^{-m} R_{m,n-1}^{(-1,-1)},
\]

\[
D \left[ \begin{array}{c}
m \\ 
 1 \\ 
 2
\end{array} \right] = R_{m-1,n}^{(0,0)}, \quad (5.40)
\]

and

\[
D = 2(-1)^{-m} x^{-2m+1} R_{m,n}^{(0,0)}. \quad (5.40)
\]
Lemma 5.8 Put

\[
D = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
-x^{-1} q_1^- & x^{-1} q_1^+ & q_1 & \cdots & q_m - n + 3 \\
-x^{-1} (q_2^- + x^{-2} q_1^-) & x^{-1} q_3^- & q_3 & \cdots & q_m - n + 5 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-p_n - m + 4 + x^2 p_n - m + 2 & p_n^+ - m + 2 & p_n^- - m + 2 & \cdots & p_3 \\
\end{pmatrix}
\]

Then, we have

\[
D = x^{-m} R_{m,n}^{(1,1)}, \quad D \left[ \begin{array}{c} m + 1 \\ m + n + 1 \end{array} \right] = (-x)^{-m+1} R_{m-1,n-1}^{(-1,-1)},
\]

\[
D \left[ \begin{array}{c} 1 \\ 2 \end{array} \right] = (-x)^{-m} R_{m,n}^{(-1,-1)}, \quad D \left[ \begin{array}{c} m + 1 \\ m + n + 1 \end{array} \right] = x^{-m+1} R_{m-1,n-1}^{(1,1)},
\]

\[
D \left[ \begin{array}{c} m + 1 \\ 2 \end{array} \right] = R_{m-1,n}^{(0,0)}, \quad D \left[ \begin{array}{c} 1 \\ m + n + 1 \end{array} \right] = 2(-1)^{-m} x^{-2m+1} R_{m-1,n-1}^{(0,0)}.
\]

From Lemma 5.7, Jacobi’s identity (5.20) is lead to the bilinear relation (5.21). Lemma 5.8 and Jacobi’s identity (5.27) give the bilinear relation (5.21). We also give the proof of Lemma 5.7 and 5.8 in Appendix B.

6 Degeneration of algebraic solutions

It is well known that, starting from \( P_{VI} \), one can obtain \( P_V, \ldots, P_I \) by successive limiting procedures in the following diagram (5.3).

\[
P_{VI} \rightarrow P_V \rightarrow P_{III} \\
\downarrow \quad \downarrow \\
P_{IV} \rightarrow P_{II} \rightarrow P_I.
\]

It is also known that each Painlevé equation, except for \( P_I \), admits particular solutions expressed by special functions, and that the coalescence diagram of these special functions is given as

hypergeometric \( \rightarrow \) confluent hypergeometric \( \rightarrow \) Bessel

\( \downarrow \)

Hermite-Weber \( \rightarrow \) Airy.

How is the degeneration diagram of algebraic (or rational) solutions that originate from the fixed points of Dynkin automorphisms? In this section, we show that, starting from the family of algebraic solutions to \( P_{VI} \) given in Section 2, we can obtain that of the rational solutions to \( P_V, P_{III} \) and \( P_{II} \) by degeneration in the following diagram,

\[
P_{VI} \rightarrow P_V \\
\downarrow \quad \downarrow \\
P_{IV} \rightarrow P_{II}.
\]

The horizontal arrows in (5.3) are naturally obtained as the coalescence of Painlevé equations (or corresponding Hamilton systems). We need to remark on the vertical arrows. The degeneration procedures themselves are obtained as the combination of those for equations or Hamilton systems. Nevertheless, the degeneration of the algebraic (or rational) solutions seems to be direct one. Namely, we believe that the rational solutions of \( P_V \) cannot degenerate to those of \( P_{III} \) and that the rational solutions of \( P_{IV} \) expressed in terms of Okamoto polynomials cannot be linked with the above diagram.
6.1 Degeneration from \( P_{VI} \) to \( P_V \)

As is known \([22]\), \( P_V \)

\[
\frac{d^2 y}{dt^2} = \left( \frac{1}{2y} + \frac{1}{y - 1} \right) \left( \frac{dy}{dt} \right)^2 - \frac{1}{2t} \frac{dy}{dt} + \frac{(y - 1)^2}{2t^2} \left( \kappa_0^2 y - \kappa_0^2 \right) - (\theta + 1) \frac{t}{2} \frac{y}{y+1}.
\]

(6.4)

is equivalent to the Hamilton system

\[
S_V : \quad q' = \frac{\partial H}{\partial p}, \quad p' = -\frac{\partial H}{\partial q}, \quad t' = \frac{d}{dt},
\]

(6.5)

with the Hamiltonian

\[
H = q(q - 1)^2 p^2 - \left[ \kappa_0(q - 1)^2 + \theta q(q - 1) + tq \right] p + \kappa(q - 1),
\]

\[\kappa = \frac{1}{4} (\kappa_0 + \theta)^2 - \frac{1}{4} \kappa_0^2.\]

(6.6)

This system can be derived from \( S_{VI} \) by degeneration \([3]\). The Hamilton equation (1.2) with the Hamiltonian (1.3) is reduced to (6.5) with (6.6) by putting

\[t \to 1 - \varepsilon t, \quad \kappa_1 \to \varepsilon^{-1} + \theta + 1, \quad \theta \to -\varepsilon^{-1},\]

(6.7)

and taking the limit of \( \varepsilon \to 0 \).

The rational solutions of \( S_V \) are expressed as follows \([12]\).

**Proposition 6.1** Let \( p_k = p_k^{(r)}(z) \) and \( q_k = q_k^{(r)}(z) \), \( k \in \mathbb{Z} \), be two sets of polynomials defined by

\[
\sum_{k=0}^{\infty} p_k^{(r)} \lambda^k = (1 - \lambda)^{-r} \exp \left( - \frac{z \lambda}{1 - \lambda} \right), \quad p_k^{(r)} = 0 \text{ for } k < 0,
\]

\[q_k^{(r)}(z) = p_k^{(r)}(-z).\]

(6.8)

We define the polynomials \( R_{m,n} = R_{m,n}^{(r)}(z) \) by

\[
R_{m,n}^{(r)}(z) = \begin{vmatrix}
q_1^{(r)} & q_0^{(r)} & \cdots & q_{-m+2}^{(r)} & q_{-m+1}^{(r)} & \cdots & q_{-m-n+3}^{(r)} & q_{-m-n+2}^{(r)} \\
q_2^{(r)} & q_1^{(r)} & \cdots & q_{-m+3}^{(r)} & q_{-m+2}^{(r)} & \cdots & q_{-m-n+4}^{(r)} & q_{-m-n+3}^{(r)} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
q_{2m-1}^{(r)} & q_{2m-2}^{(r)} & \cdots & q_{m-1}^{(r)} & q_m^{(r)} & \cdots & q_{m-n+1}^{(r)} & q_m^{(r)} \\
p_{n-m}^{(r)} & p_{n-m+1}^{(r)} & \cdots & p_{n-1}^{(r)} & p_n^{(r)} & \cdots & p_{2n-2}^{(r)} & p_{2n-1}^{(r)} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
p_{n-m-4}^{(r)} & p_{n-m-5}^{(r)} & \cdots & p_{n-3}^{(r)} & p_{n-2}^{(r)} & \cdots & p_2^{(r)} & p_3^{(r)} \\
p_{n-m-2}^{(r)} & p_{n-m-3}^{(r)} & \cdots & p_{n-1}^{(r)} & p_{n+1}^{(r)} & \cdots & p_0^{(r)} & p_1^{(r)}
\end{vmatrix},
\]

(6.9)

for \( m, n \in \mathbb{Z}_{\geq 0} \) and by

\[
R_{m,n} = (-1)^{m(m+1)/2} R_{-m-1,n}, \quad R_{m,n} = (-1)^{n(n+1)/2} R_{m,-n-1},
\]

(6.10)

for \( m, n \in \mathbb{Z}_{<0} \), respectively. Then, setting

\[
R_{m,n}^{(r)}(z) = S_{m,n}(t,s),
\]

(6.11)

with

\[
z = \frac{t}{2}, \quad r = 2s - m + n,
\]

(6.12)
we see that
\[
q = - \frac{S_{m,n-1}(t,s)S_{m-1,n}(t,s)}{S_{m-1,n}(t,s-1)S_{m,n-1}(t,s+1)},
\]
\[
p = - \frac{2n-1}{4} \frac{S_{m-1,n}(t,s-1)S_{m,n-1}(t,s+1)S_{m-1,n-2}(t,s)}{S_{m-1,n-1}(t,s)S_{m,n-1}(t,s)},
\]
(6.13)
give a family of rational solutions to the Hamilton system \(S_V\) for the parameters
\[
\kappa_\infty = s, \quad \kappa_0 = s - m + n, \quad \theta = m + n - 1.
\]
(6.14)

Let us consider the degeneration of the algebraic solutions of \(S_{V_1}\). Applying the Bäcklund transformation \(s_0\) to the solutions in Theorem 2.2, we obtain the following corollary.

**Corollary 6.2** Let \(S_{m,n} = S_{m,n}(x; a, b)\) be polynomials given in Theorem 2.2. Then, for \(m, n \in \mathbb{Z}\),
\[
q = x \frac{S_{m,n}^{(1,0)} S_{m-1,n}^{(1,0)}}{S_{m-1,n}^{(1,1)} S_{m,n-1}^{(1,1)}}, \quad p = \frac{2n-1}{2x(1-x)} \frac{S_{m-1,n}^{(1,0)} S_{m,n-1}^{(1,0)} S_{m,n}^{(2,0)}}{S_{m-1,n-1}^{(1,0)} S_{m,n-1}^{(2,0)}},
\]
(6.15)
where we denote as \(S_{m,n}^{(k,l)} = S_{m,n}(x; a + k, b + l)\), satisfy \(S_{V_1}\) for the parameters
\[
\kappa_\infty = b, \quad \kappa_0 = b - m + n, \quad \kappa_1 = a + m + n, \quad \theta = -a,
\]
(6.16)
with \(x^2 = t\).

It is easy to see that by putting
\[
t \to 1 - \varepsilon t, \quad a = \varepsilon^{-1},
\]
(6.17)
\(S_{V_1}\) with (6.16) is reduced to \(S_V\) with (6.14) in the limit of \(\varepsilon \to 0\).

Next, we investigate the degeneration of \(R_{m,n}^{(i,j)}\) given in Definition 2.1. Putting
\[
x \to -(1 - \varepsilon t)^{\frac{1}{2}}, \quad c = \varepsilon^{-1} + s + n - \frac{1}{2}, \quad d = 2s - m + n,
\]
(6.18)
we see that the generating function \(2.2\) degenerates as
\[
G = (1 - \lambda)^{-d} \exp \left\{ c \left[ \log(1 - \lambda) - \log(1 + x^{\pm 1} \lambda) \right] \right\}
\]
\[
= (1 - \lambda)^{-r} \exp \left( \frac{z \lambda}{1 - \lambda} + O(\varepsilon) \right),
\]
(6.19)
where we use (6.12). Then, we have
\[
\lim_{\varepsilon \to 0} p_k^{(r,c,d)}(x) = p_k^{(r)}(z), \quad \lim_{\varepsilon \to 0} q_k^{(r,c,d)}(x) = q_k^{(r)}(z),
\]
(6.20)
which gives
\[
\lim_{\varepsilon \to 0} R_{m,n}^{(r,i,j)}(x) = R_{m,n}^{(r + j)}(z).
\]
(6.21)

Finally, it is easy to see that (6.15) yields to (6.13).

**Remark 6.3** As we mentioned in Section 4, Kirillov and Taneda have introduced “generalized Umemura polynomials” for \(P_{V_1}\) in the context of combinatorics and shown that these polynomials degenerate to \(S_{m,n} = S_{m,n}(t,s)\) defined in Proposition 6.4 in some limit \(4 \pm 4 \pm 12\).

**Remark 6.4** The polynomials \(p_k^{(r)}\) (and \(q_k^{(r)}\)) defined by (6.3) are essentially the Laguerre polynomials, namely, \(p_k^{(r)}(z) = L_k^{(r-1)}(z)\). The above degeneration corresponds to that from the Jacobi polynomials to the Laguerre polynomials.
6.2 Degeneration from $P_{VI}$ to $P_{III}$

Next, we consider $P_{III}$

\[
\frac{d^2y}{dt^2} = \frac{1}{y} \left( \frac{dy}{dt} \right)^2 - \frac{1}{t} \frac{dy}{dt} - \frac{4}{t} \left[ \eta_\infty \theta_\infty y^2 + \eta_0(\theta_0 + 1) \right] + 4\eta_\infty^2 y^3 - \frac{4\eta_0^2}{y},
\]

which is equivalent to the Hamilton system

\[\begin{align*}
S_{III} : & \quad q' = \frac{\partial H}{\partial p}, \quad p' = -\frac{\partial H}{\partial q}, \quad t' = t \frac{d}{dt},
\end{align*}\]

with the Hamiltonian

\[H = 2q^2p^2 - [2\eta_\infty tq^2 + (2\theta_0 + 1)q + 2\eta_0 p] + \eta_\infty(\theta_\infty + \theta_0)tp.\]

This system can be also derived from $S_{VI}$, directly, by degeneration. This process is achieved by putting

\[t \to \varepsilon^2 t', \quad q \to \varepsilon q, \quad p \to \varepsilon^{-1}t^{-1}p,\]

\[\kappa_\infty \to \eta_\infty \varepsilon^{-1} + \theta_\infty^{(1)}, \quad \kappa_0 \to -\eta_0 \varepsilon^{-1} + \theta_0^{(1)} + 1, \quad \kappa_1 \to -\eta_\infty \varepsilon^{-1} + \theta_\infty^{(2)}, \quad \theta \to \eta_0 \varepsilon^{-1} + \theta_0^{(2)},\]

and taking the limit of $\varepsilon \to 0$. In fact, the system \[\text{(6.23)}\] with the Hamiltonian \[\text{(6.24)}\] is reduced to \[\text{(6.23)}\] with \[\text{(6.24)}\] by this procedure, where we put

\[\theta_\infty = \theta_\infty^{(1)} + \theta_\infty^{(2)}, \quad \theta_0 = \theta_0^{(1)} + \theta_0^{(2)}.\]

It is known that the rational solutions of $S_{III}$ are expressed as follows \[\text{(6.28)}\].

**Proposition 6.5** Let $p_k = p_k^{(r)}(t), \quad k \in \mathbb{Z}$, be polynomials defined by

\[\sum_{k=0}^{\infty} p_k^{(r)}\lambda^k = (1 + \lambda)^r \exp(-t\lambda), \quad p_k^{(r)} = 0 \text{ for } k < 0.\]

We define a family of polynomials $R_n^{(r)} = R_n^{(r)}(t)$ by

\[R_n^{(r)}(t) = \begin{vmatrix}
p_n^{(r)} & \cdots & p_{2n-2}^{(r)} & p_{2n-1}^{(r)} \\
\vdots & \ddots & \vdots & \vdots \\
p_{-n+4}^{(r)} & \cdots & p_2^{(r)} & p_1^{(r)} \\
p_{-n+2}^{(r)} & \cdots & p_0^{(r)} & p_1^{(r)}
\end{vmatrix},\]

for $n \in \mathbb{Z}_{\geq 0}$ and by

\[R_n = (-1)^{n(n+1)/2}R_{-n-1},\]

for $n \in \mathbb{Z}_{< 0}$, respectively. Then,

\[q = \frac{R_{n+1}^{(r+1)}R_{n-1}^{(r)}}{R_n^{(r+1)}R_{n-1}^{(r)}}, \quad p = \frac{2n - 1}{2} \frac{R_{n+1}^{(r+1)}R_{n-2}^{(r+1)}}{R_{n-1}^{(r+1)}R_{n-1}^{(r)}},\]

give the rational solutions of $S_{III}$ for the parameters

\[\theta_\infty = r + \frac{1}{2} + n, \quad \theta_0 + 1 = -r - \frac{1}{2} + n,\]

with

\[\eta_\infty = \eta_0 = \frac{1}{2}.\]
Before discussing the degeneration to the rational solutions of $S_{III}$, we slightly rewrite the determinant expression in Definition 2.1 for convenience.

**Lemma 6.6** Let $\tilde{p}_k = \tilde{p}_k^{(\tilde{\varepsilon},\tilde{d})}(x)$ and $\tilde{q}_k = \tilde{q}_k^{(\tilde{\varepsilon},\tilde{d})}(x)$, $k \in \mathbb{Z}$, be two sets of polynomials defined by

$$
\sum_{k=0}^{\infty} \tilde{p}_k^{(\tilde{\varepsilon},\tilde{d})}(x)\lambda^k = \tilde{G}(x;\tilde{c},\tilde{d};\lambda), \quad \tilde{p}_k^{(\tilde{\varepsilon},\tilde{d})}(x) = 0 \text{ for } k < 0, \\
\tilde{q}_k^{(\tilde{\varepsilon},\tilde{d})}(x) = \tilde{p}_k^{(\tilde{\varepsilon},\tilde{d})}(x^{-1}),
$$

(6.35)

respectively, where the generating function $\tilde{G}(x;\tilde{c},\tilde{d};\lambda)$ is given by

$$
\tilde{G}(x;\tilde{c},\tilde{d};\lambda) = (1 - \lambda)^{\tilde{d} - 1}(1 + x\lambda)^{-\tilde{\varepsilon}}.
$$

(6.36)

Define $\bar{R}_{m,n} = \bar{R}_{m,n}(x;\bar{c},\bar{d})$ in terms of the same determinant as (2.9) with entries $p_k$ and $q_k$ replaced by $\tilde{p}_k$ and $\tilde{q}_k$, respectively. Then, we have

$$
\bar{R}_{m,n}(x;\bar{c},\bar{d}) = S_{m,n}(x;a,b), \\
$$

(6.37)

with

$$
\bar{c} = a + b + n - \frac{1}{2}, \quad \bar{d} = a - b + m + \frac{1}{2}.
$$

(6.38)

**Remark 6.7** The polynomials $\tilde{p}_k$ and $\tilde{q}_k$ are also expressed by the Jacobi polynomials as

$$
\tilde{p}_k^{(\tilde{\varepsilon},\tilde{d})}(x) = (-1)^k P_k^{(\tilde{d} - 1 - k,\tilde{\varepsilon} \tilde{d})}(1 + 2x).
$$

(6.39)

Let us consider the degeneration of the algebraic solutions of $S_{V_1}$. Applying the Bäcklund transformation $s_1s_0$ to the solutions in Theorem 2.2, we obtain the following corollary.

**Corollary 6.8** Let $\bar{R}_{m,n} = \bar{R}_{m,n}(x;\bar{c},\bar{d})$ be polynomials defined in Lemma 6.6. Then, for $m, n \in \mathbb{Z},$

$$
q = x \frac{\bar{R}_{m,n-1}^{(1,0)} \bar{R}_{m-1,n}^{(1,0)}}{\bar{R}_{m-1,n-1}^{(1,0)}}, \quad p = \frac{2n - 1}{2x(1-x)} \frac{\bar{R}_{m-1,n}^{(0,1)} \bar{R}_{m,n-1}^{(1,0)} \bar{R}_{m-1,n-2}^{(0,1)}}{\bar{R}_{m-1,n-1}^{(0,1)} \bar{R}_{m-1,n-1}^{(1,1)}},
$$

(6.40)

where we denote as $\bar{R}_{m,n}^{(i,j)} = \bar{R}_{m,n}(x;\bar{c} + i,\bar{d} + j)$, satisfy $S_{V_1}$ for the parameters

$$
\kappa_\infty = -b, \quad \kappa_0 = b - m + n, \quad \kappa_1 = a + m + n, \quad \theta = -a,
$$

(6.41)

under the setting of (6.37), (6.38) and $x^2 = t$.

According to (6.26) and (5.41), we fix as (6.34) and put

$$
a = \frac{1}{2} \left( -\varepsilon^{-1} + r + \frac{1}{2} - m + \zeta \right), \quad b = \frac{1}{2} \left( -\varepsilon^{-1} - r - \frac{1}{2} + m + \zeta \right),
$$

(6.42)

where $\zeta$ is a quantity of $O(1)$, then, we have

$$
\theta^{(1)} = \frac{1}{2} \left( r + \frac{1}{2} - m - \zeta \right), \quad \theta^{(2)} = \frac{1}{2} \left( r + \frac{1}{2} + m + \zeta \right) + n,
$$

$$
\theta^{(1)}_0 + 1 = -\frac{1}{2} \left( r + \frac{1}{2} + m - \zeta \right) + n, \quad \theta^{(2)}_0 = -\frac{1}{2} \left( r + \frac{1}{2} - m + \zeta \right).
$$

(6.43)

Setting as (6.25) and (6.27), we see that $S_{V_1}$ with (6.41) is reduced to $S_{III}$ with (6.33) in the limit of $\varepsilon \to 0$. Note that $m$ vanishes in (6.33). Then, it is possible to put $m = 0$ without losing generality in this limiting procedure.
Next, we investigate the degeneration of \( \bar{R}_n^{(i,j)} = \bar{R}_{-1,n}^{(i,j)} = \bar{R}_{0,n}^{(i,j)} \). Putting \( x \rightarrow \varepsilon t, \quad \bar{c} = -\varepsilon^{-1} + \zeta + n - \frac{1}{2}, \quad \bar{d} = r + 1, \) we find that the generating function (6.36) degenerates as

\[
G = (1 - \lambda)^{\bar{d} - 1} \exp\left[-\bar{c}\log(1 + x\lambda)\right] = (1 - \lambda)^r \exp\left[t\lambda + O(\varepsilon)\right].
\]

Then, we have

\[
\lim_{\varepsilon \to 0} \bar{p}_k^{(\bar{c}, \bar{d})}(x) = (-1)^k p_k^{(r)}(t),
\]

which gives

\[
\lim_{\varepsilon \to 0} \bar{R}_n^{(i,j)}(x) = (-1)^{n(n+1)/2} \bar{R}_n^{(r+j)}(t).
\]

Finally, it is easy to see that (6.40) is lead to (6.32) in the above limit.

**Remark 6.9** The polynomials \( p_k^{(r)} \) defined by (6.29) are also the Laguerre polynomials, namely, \( p_k^{(r)}(t) = L_k^{(r-k)}(t) \). Then, the above degeneration also corresponds to that from the Jacobi polynomials to the Laguerre polynomials.

Similarly, the rational solutions of \( P_V \) and \( P_{III} \) given in Proposition 6.1 and 6.5, respectively, degenerate to those of \( P_{II} \). We give detail in Appendix A. Therefore, the coalescence cascade (6.3) is obtained.

### 7 Relationship to the original Umemura polynomials

In this section, we show that the original Umemura polynomials for \( P_{VI} \) are understood as a special case of our polynomials \( V_{m,n}(x; a, b) \) introduced in Section 4.

#### 7.1 Umemura polynomials associated with \( P_{VI} \)

First, we briefly summarize the derivation of the original Umemura polynomials for \( P_{VI} \) [28]. Set the parameters \( b_i (i = 1, 2, 3, 4) \) as

\[
b_1 = \frac{1}{2}(\kappa_0 + \kappa_1), \quad b_2 = \frac{1}{2}(\kappa_0 - \kappa_1), \quad b_3 = \frac{1}{2}(\theta - 1 + \kappa_\infty), \quad b_4 = \frac{1}{2}(\theta - 1 - \kappa_\infty),
\]

namely,

\[
b_1 = \frac{1}{2}(\alpha_4 + \alpha_3), \quad b_2 = \frac{1}{2}(\alpha_4 - \alpha_3), \quad b_3 = \frac{1}{2}(\alpha_0 - 1 + \alpha_1), \quad b_4 = \frac{1}{2}(\alpha_0 - 1 - \alpha_1).
\]

Umemura has shown that

\[
q = \frac{(\alpha + \beta)^2 t \pm (\alpha^2 - \beta^2) \sqrt{t(t - 1)}}{(\alpha - \beta)^2 + 4\alpha\beta t}, \quad p = \frac{\alpha q - (\alpha + \beta)/2}{q(q - 1)},
\]

give an algebraic solution of the Hamilton system \( S_{VI} \) for the parameters

\[
(b_1, b_2, b_3, b_4) = \left(\alpha, \beta, -\frac{1}{2}, 0\right).
\]

Substituting the solution of upper sign into the Hamiltonian [13], one obtain

\[
H = \frac{1}{4} \left[-(\alpha + \beta) + (\alpha + \beta)^2 + 2\alpha t - 2(\alpha^2 + \beta^2)t + 2(\alpha^2 - \beta^2)\sqrt{t(t - 1)}\right].
\]
Application of the translation

\[(b_1, b_2, b_3, b_4) \to (b_1, b_2, b_3, b_4) + n(0, 0, 1, 0), \quad n \in \mathbb{Z}, \quad (7.6)\]

to the seed solution (7.3) with (7.4) generates a sequence of algebraic solutions \((q_n, p_n)\). Let \(\tau_n\) be a \(\tau\)-function with respect to the solution \((q_n, p_n)\). Okamoto has pointed out that \(\tau_n\) satisfy the Toda equation \([21]\)

\[
\frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2} = \frac{d}{dt} \left(\log \tau_n\right)' + (b_1 + b_3 + n)(b_3 + b_4 + n), \quad t = (t-1) \frac{d}{dt}. \quad (7.7)
\]

Define a family of functions \(T_n\) for \(n \in \mathbb{Z}\) by

\[
\left(\log \tau_n\right)' = \left(\log T_n\right)' + H - n \left(\alpha t - \frac{\alpha + \beta}{2}\right). \quad (7.8)
\]

Then, Toda equation (7.7) yields to

\[
T_{n+1}T_{n-1} = t(t-1) \left[\frac{d^2T_n}{dt^2} T_n - \left(\frac{dT_n}{dt}\right)^2 + (2t-1)\frac{dT_n}{dt} T_n\right]
+ \left\{\frac{1}{4} \left[-2(\alpha^2 + \beta^2) + (\alpha^2 - \beta^2) \frac{2t-1}{\sqrt{t(t-1)}}\right] + \left(n - \frac{1}{2}\right)^2\right\} T_n^2. \quad (7.9)
\]

Moreover, introducing a new variable \(v\) as

\[
v = \sqrt{\frac{t}{t-1}} + \sqrt{\frac{t-1}{t}}, \quad (7.10)
\]

one find that \(T_n\) are generated by the recurrence relation

\[
T_{n+1}T_{n-1} = \frac{1}{4}(v^2 - 4) \left[(v^2 - 4)\frac{d^2T_n}{dv^2} + v\frac{dT_n}{dv}\right] T_n - \frac{1}{4}(v^2 - 4)^2 \left(\frac{dT_n}{dv}\right)^2
+ \left\{\frac{1}{4} \left[-2(\alpha^2 + \beta^2) + (\alpha^2 - \beta^2)v\right] + \left(n - \frac{1}{2}\right)^2\right\} T_n^2. \quad (7.11)
\]

with the initial conditions \(T_0 = T_1 = 1\). It is shown that \(T_n\) for \(n \in \mathbb{Z}_{\geq 0}\) are polynomials in \(\alpha, \beta\) and \(v\), and \(\deg_v T_n = n(n-1)/2\). These polynomials are called Umemura polynomials associated with \(PVI\).

### 7.2 Correspondence of the seed solution

We investigate how the Umemura’s seed solution (7.3) with (7.4) is related to ours,

\[q = f_4 = x, \quad p = f_2 = \frac{1}{2} \left(a + b - \frac{1}{2}\right) x^{-1}, \quad (7.12)\]

with

\[
(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(a, b, \frac{1}{2} - a - b, a, b\right). \quad (7.13)
\]
In addition to the Bäcklund transformations stated in Section 3, it is known that \( P VI \) admits the outer symmetry as follows [21],

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\sigma & a_0 & a_1 & a_2 & a_3 & a_4 & t & f_4 & f_2 \\
\hline
\sigma_{01} & a_1 & a_0 & a_2 & a_3 & a_4 & 1 - t & f_0 f_4 & f_2 \\
\sigma_{03} & a_3 & a_1 & a_0 & a_2 & a_4 & \frac{t}{t - 1} & \frac{f_0}{f_4} & f_2 \\
\sigma_{04} & a_4 & a_1 & a_0 & a_2 & a_3 & \frac{1}{t - 1} & \frac{f_0}{f_1} & (1 - t) f_2 \\
\sigma_{13} & a_0 & a_3 & a_1 & a_2 & a_4 & \frac{t}{t - 1} & \frac{f_3}{f_4} & -f_3 (f_3 f_2 + \alpha_2) \\
\sigma_{14} & a_0 & a_1 & a_2 & a_3 & a_1 & \frac{1}{t} & \frac{f_3}{f_1} & -f_4 (f_4 f_2 + \alpha_2) \\
\sigma_{34} & a_0 & a_1 & a_2 & a_4 & a_3 & 1 - t & -f_3 & -f_2 \\
\hline
\end{array}
\]

**Proposition 7.1** Umemura’s seed solution (7.3) with (7.4) is obtained by applying the Bäcklund transformation defined by

\[
\sigma = \sigma_{13} \sigma_{32} \sigma_{21},
\]

(7.15)
to ours (7.12) with (7.13), where we put

\[
\alpha = \frac{1}{2} - a, \quad \beta = b.
\]

(7.16)

Proof. First, we check for the parameters. Application of \( \sigma \) to (7.13) gives

\[
(a_0, a_1, a_2, a_3, a_4) = \left( \frac{1}{2}, -\frac{1}{2} a, \frac{1}{2} - a - b, \frac{1}{2} - a + b \right),
\]

(7.17)

which coincides with (7.4) by using (7.2) and (7.16).

Next, we verify the correspondence of \( q = f_4 \). We have

\[
\sigma (f_4) = \frac{f_2 f_4 + a_1 + a_2}{f_2 f_3 + a_1 + a_2} = \frac{1}{2} - a + b \left\{ \left( \frac{1}{2} - a + b \right) x^{-1} + \left( \frac{1}{2} - a - b \right) x^{-1} \right\}.
\]

(7.18)

Note that \( x \) is now given by

\[
x = \pm \sqrt{\frac{t}{t - 1}}.
\]

(7.19)

due to the action of \( \sigma_{13} \). Thus, the expression (7.18) is equivalent to the first of (7.3). It is possible to check for \( p = f_2 \) in similar way.

\[\Box\]

### 7.3 Relationship to the original Umemura polynomials

The above discussion on the seed solution suggests that the family of polynomials \( V_{m,n}(x; a, b) \) constructed in Section 4 corresponds to the original Umemura polynomials under the setting of (7.16) and

\[
x = -\sqrt{\frac{t}{t - 1}}.
\]

(7.20)

Notice that, from (7.9) and (7.20), \( T_n = T_n(x; \alpha, \beta) \) satisfy the recurrence relation

\[
4 T_{n+1} T_{n-1} = x^{-1} \left[ (x^2 - 1)^2 D^2 - \alpha^2 (x + 1)^2 + \beta^2 (x - 1)^2 + (2n - 1)^2 x \right] T_n \cdot T_n,
\]

(7.21)

with \( T_0 = T_1 = 1 \).
Theorem 7.2 We have
\[ T_n(x; \alpha, \beta) = 2^{-2n(n-1)}(-x)^{-n(n-1)/2}V_{-n-n}(x; a + n, b), \tag{7.22} \]
with (7.14).

We prove Theorem 7.2 by showing that both hand sides of (7.22) satisfy the same recurrence relation and initial conditions. Let \( \hat{T}_{30} \) be the translation operator defined by
\[ \hat{T}_{30} = T_{34}T_{34}^{-1}. \tag{7.23} \]
Then, we have a Toda equation
\[ \hat{T}_{30}(\tau_0)\hat{T}_{30}^{-1}(\tau_0) = t^{-\frac{1}{2}} \left[ (t - 1) \frac{d}{dt} \log \tau_0' - (\log \tau_0)' + \frac{1}{4}(\alpha_0 - \alpha_3)(\alpha_0 - \alpha_3 - 2) + \frac{3}{4} \right] \tau_0^2. \tag{7.24} \]
For simplicity, we denote as
\[ \bar{\tau}_n = \hat{T}_{30}(\tau_0), \quad n \in \mathbb{Z}, \tag{7.25} \]
namely \( \bar{\tau}_n = \tau_{-n,0,n,n} \) in the notation of (3.40). The above Toda equation is expressed as
\[ \bar{\tau}_{n+1}\bar{\tau}_{n-1} = t^{-\frac{1}{2}} \left[ (t - 1) \frac{d}{dt} \log \bar{\tau}_n' - (\log \bar{\tau}_n)' + \frac{(\alpha_0 - \alpha_3 - 2n)(\alpha_0 - \alpha_3 - 2n - 2)}{4} + \frac{3}{4} \right] \bar{\tau}_n^2. \tag{7.26} \]
In the following, we restrict our discussion to the algebraic solutions. According to (4.17) and (4.21), we introduce \( \bar{V}_n = \bar{V}_n(x; a, b) \) as
\[ \bar{\tau}_n = \omega_n \bar{V}_n(x - 1)(a - \frac{1}{2})^2 + \frac{1}{2} x - (a - \frac{1}{2})^2 - b^2 - n(n + 1) - \frac{1}{2} (x + 1)^2 + \frac{1}{2}, \tag{7.27} \]
where \( \omega_n = \omega_{-n,0,n,n} \). Substituting (7.27) and \( \alpha_0 = \alpha_3 = a \) into Toda equation (7.26) and noticing
\[ \omega_{n+1}\omega_{n-1} = -\frac{1}{16} \omega_n^2, \tag{7.28} \]
we find that \( \bar{V}_n = \bar{V}_n(x; a, b) \) are generated by the recurrence relation
\[ -\frac{1}{4} \bar{V}_{n+1}\bar{V}_{n-1} = \left[ (x^2 - 1)^2 \bar{D}^2 - \left( a - \frac{1}{2} \right)^2 (x + 1)^2 + b^2 (x - 1)^2 + 2n + 1 \right] \bar{V}_n \cdot \bar{V}_n, \tag{7.29} \]
with the initial conditions \( \bar{V}_{-1} = \bar{V}_0 = 1 \). By construction, it is easy to see that
\[ \bar{V}_n(x; a, b) = V_{n,n}(x; a - n, b). \tag{7.30} \]
Moreover, we introduce \( \bar{T}_n = \bar{T}_n(x; a, b) \) as
\[ \bar{T}_n = 2^{-2n(n-1)}(-x)^{-n(n-1)/2}V_{-n-n}. \tag{7.31} \]
Then, \( \bar{T}_n \) satisfy the recurrence relation
\[ 4\bar{T}_{n+1}\bar{T}_{n-1} = x^{-1} \left[ (x^2 - 1)^2 \bar{D}^2 - \left( a - \frac{1}{2} \right)^2 (x + 1)^2 + b^2 (x - 1)^2 + 2n + 1 \right] \bar{T}_n \cdot \bar{T}_n, \tag{7.32} \]
with \( \bar{T}_0 = \bar{T}_1 = 1 \).
Comparing (7.22) with (7.32), we find
\[ T_n(x; \alpha, \beta) = \bar{T}_n(x; a, b), \tag{7.33} \]
under the setting of (7.14), which is nothing but Theorem 7.2.
Remark 7.3 Toda equation (7.3) can be regarded as the recurrence relation with respect to the translation operator

\[ T_{01} = T_{34}^{-1} T_{14} T_{03}, \]  

which acts on the parameters as

\[ T_{01}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) + (1, 1, -1, 0, 0). \]  

(7.34)

Theorem 7.2, namely (7.33), is consistent with the relation

\[ T_{01}(\sigma) = \sigma T_{34}^{-1}. \]  

(7.36)

From the discussion of the previous sections and (7.31), it is clear that \( T_n \) for \( n \in \mathbb{Z} \geq 0 \) are polynomials in \( \alpha, \beta \) and \( v \), and \( \text{deg}_v T_n = n(n-1)/2 \) under the setting of \( v = -(x + x^{-1}) \).

(7.37)

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A Degeneration of rational solutions

In this section, we show that the rational solutions of \( P_V \) and \( P_{III} \) degenerate to those of \( P_{II} \),

\[ dq^2 \Bigg/ dt^2 = 2q^3 - 4tq + 4 \left( \alpha + \frac{1}{2} \right). \]  

(A.1)

As is known \( [23] \), \( P_{II} \) (A.1) is equivalent to the Hamilton system

\[ S_{II} : \quad q' = \frac{\partial H}{\partial p}, \quad p' = -\frac{\partial H}{\partial q}, \quad t' = \frac{d}{dt}, \]  

(A.2)

with the Hamiltonian

\[ H = -2p^2 - (q^2 - 2t)p + \alpha q. \]  

(A.3)

The rational solutions of \( S_{II} \) are expressed as follows \( [1] \).

Proposition A.1 Let \( q_k = q_k(t), \ k \in \mathbb{Z}, \) be polynomials defined by

\[ \sum_{k=0}^{\infty} q_k \lambda^k = \exp \left( t\lambda + \frac{\lambda^3}{3} \right), \quad q_k = 0 \text{ for } k < 0. \]  

(A.4)

We define \( R_n = R_n(t) \) by

\[ R_n = \begin{bmatrix} q_n & \cdots & q_{2n-2} & q_{2n-1} \\ \vdots & \ddots & \vdots & \vdots \\ q_{-n+4} & \cdots & q_2 & q_3 \\ q_{-n+2} & \cdots & q_0 & q_1 \end{bmatrix}, \]  

(A.5)

for \( n \in \mathbb{Z}_{\geq 0} \) and by

\[ R_n = (-1)^{n(n+1)/2} R_{-n-1}, \]  

(A.6)

for \( n \in \mathbb{Z}_{<0}, \) respectively. Then,

\[ q = \frac{d}{dt} \log \frac{R_n}{R_{n-1}}, \quad p = \frac{2n - 1}{2} \frac{R_n R_{n-2}}{R_{n-1}}, \]  

(A.7)

give the rational solutions of \( S_{II} \) for the parameters

\[ \alpha = n - \frac{1}{2}. \]  

(A.8)
A.1 From $P_V$ to $P_{II}$

It is possible to derive the Hamilton system $S_{II}$ from $S_V$, directly, by degeneration. Putting

$$t \to \varepsilon^{-3}(1 + 2\varepsilon^2 t), \quad q \to -1 + 2\varepsilon q, \quad p \to \frac{1}{2} \varepsilon^{-1} p,$$

(A.9)

$$\kappa_\infty \to \frac{\sigma}{4} \varepsilon^{-3} + \kappa_\infty^{(0)}, \quad \kappa_0 \to \frac{1}{4} \varepsilon^{-3} + \kappa_0^{(0)}, \quad \theta \to 2\theta^{(0)},$$

(A.10)

$$H \to \frac{1}{2} \varepsilon^{-2} H - \frac{1}{2} \varepsilon^{-3} \alpha, \quad \alpha = \theta^{(0)} + \frac{\kappa_0^{(0)} - \sigma \kappa_\infty^{(0)}}{2},$$

(A.11)

with $\sigma = \pm 1$ and taking the limit of $\varepsilon \to 0$, we find that the system (A.5) with the Hamiltonian (A.4) is reduced to (A.2) with (A.3).

We show that the rational solutions of $S_V$ given in Proposition B.1 degenerate to those of $S_{II}$ in Proposition A.1. According to (A.10) and (6.14), we put $\sigma = 1$ and

$$s = \frac{1}{4} \varepsilon^{-3}, \quad \kappa_\infty^{(0)} = 0, \quad \kappa_0^{(0)} = -m + n, \quad \theta^{(0)} = \frac{m + n - 1}{2}.$$

(A.12)

Then, after the replacement (A.9) and (A.11), we find that $S_V$ with (B.14) is reduced to $S_{II}$ with (A.8) in the limit of $\varepsilon \to 0$. Note that $m$ vanishes in (A.8). Then, it is possible to put $m = 0$ without loss of generality in this limiting procedure.

Next, we investigate the degeneration of $R_n^{(r)} = R_{n-1,n}^{(r)} = R_{0,n}^{(r)}$. It is obvious that we have the following lemma.

**Lemma A.2** Let $\bar{p}_k = \bar{p}_k^{(r)}(z), \ k \in \mathbb{Z}$, be polynomials defined by

$$\sum_{k=0}^{\infty} \bar{p}_k^{(r)} \lambda^k = \exp \left[ \sum_{j=1}^{\infty} \left( -\frac{r}{j} + \frac{r}{2} \lambda^2 \right) \lambda^j \right], \quad \bar{p}_k^{(r)} = 0 \ \text{for} \ k < 0.$$

(A.13)

Then, we have

$$R_n^{(r)}(z) = \begin{vmatrix} \bar{p}_n^{(r)} & \cdots & \bar{p}_{2n-2}^{(r)} & \bar{p}_{2n-1}^{(r)} \\ \vdots & \ddots & \vdots & \vdots \\ \bar{p}_{-n+4}^{(r)} & \cdots & \bar{p}_2^{(r)} & \bar{p}_3^{(r)} \\ \bar{p}_{-n+2}^{(r)} & \cdots & \bar{p}_0^{(r)} & \bar{p}_1^{(r)} \end{vmatrix}.$$  

(A.14)

Put

$$\lambda \to -\varepsilon \lambda, \quad \bar{q}_k^{(r)} = (-\varepsilon)^k \bar{p}_k^{(r)},$$

(A.15)

and

$$z \to \frac{1}{2} \varepsilon^{-3} (1 + 2\varepsilon^2 t), \quad r = \frac{1}{2} \varepsilon^{-3} + n.$$

(A.16)

Then, (A.13) yields to

$$\sum_{k=0}^{\infty} q_k^{(r+j)} \lambda^k = \exp \left( t \lambda + \frac{\lambda^3}{3} \right) \left[ 1 - \varepsilon \left( j \lambda + n \lambda + t \lambda^2 + \frac{3}{8} \lambda^4 \right) + O(\varepsilon^2) \right].$$

(A.17)

By using (A.4), we obtain

$$\bar{q}_k^{(r+j)} = q_k - \varepsilon j q_{k-1} - \varepsilon \left( n q_{k-1} + t q_{k-2} + \frac{3}{8} q_{k-4} \right) + O(\varepsilon^2).$$

(A.18)
Since it is easy to see that
\[ \frac{dq_k}{dt} = q_{k-1}, \]  
we have, from (A.3),
\[ R_n^{(r+j)} = (-\varepsilon)^{-n(n+1)/2} \left[ R_n - \varepsilon j \frac{dR_n}{dt} - \varepsilon Q_n + O(\varepsilon^2) \right], \]  
where \( Q_n \) denotes the contribution from the third term of (A.18).

Finally, we verify the degeneration of the variables \( q \) and \( p \). The above procedure gives
\[ -\frac{R_n^{(r-1)}}{R_n^{(r)}} \frac{R_n^{(r+1)}}{R_n^{(r-1)}} = -1 + 2\varepsilon \frac{d}{dt} \log \frac{R_n}{R_n^{(r-1)}} + O(\varepsilon^2), \]
\[ -\frac{R_n^{(r-1)}}{R_n^{(r)}} \frac{R_n^{(r+1)}}{R_n^{(r-2)}} = \varepsilon^{-1} \frac{R_n R_{n-2}}{R_n^{(r-1)}} + O(1). \]  
Thus, from (A.9), we get (A.7) in the limit of \( \varepsilon \to 0 \).

### A.2 From \( P_{III} \) to \( P_{II} \)

It is well known that the Hamilton system \( S_{II} \) is derived from \( S_{III} \) by degeneration \[ \boxed{3} \]. This process is achieved by putting
\[ t \to -\varepsilon^{-3} (1 - \varepsilon^2 t), \quad q \to 1 + \varepsilon q, \quad p \to \varepsilon^{-1} p, \]  
\[ \theta_{\infty} \to -\varepsilon^{-3} + \theta_{\infty}^{(0)}, \quad \theta_0 \to \varepsilon^{-3} + \theta_0^{(0)}, \]  
\[ H \to -\varepsilon^{-2} H - \varepsilon^{-3} \alpha, \quad \alpha = \frac{\theta_{\infty}^{(0)} + \theta_0^{(0)}}{2}, \]  
and taking the limit of \( \varepsilon \to 0 \).

We show that the rational solutions of \( S_{III} \) given in Proposition B.5 degenerate to those of \( S_{II} \) in Proposition A.1. From (A.23), we put
\[ r = -\varepsilon^{-3}, \quad \theta_{\infty}^{(0)} = n + \frac{1}{2}, \quad \theta_0^{(0)} = n - \frac{3}{2}. \]  
Then, after replacing as (A.22) and (A.24), we see that \( S_{III} \) with (6.33) is reduced to \( S_{II} \) with (A.8) in the limit of \( \varepsilon \to 0 \).

Next, we investigate the degeneration of \( R_n^{(r)} \) defined by (6.29) and (6.30). It is obvious that we have the following lemma.

**Lemma A.3** Let \( \tilde{p}_k = \tilde{p}_k^{(r)}(t), \ k \in \mathbb{Z}, \) be polynomials defined by
\[ \sum_{k=0}^{\infty} \tilde{p}_k^{(r)} \lambda^k = \exp \left[ \sum_{j=1}^{\infty} \frac{(-1)^{j-1} r}{j} \lambda^j - t \lambda + \frac{r}{2} \lambda^2 \right], \quad \tilde{p}_k^{(r)} = 0 \text{ for } k < 0. \]  
Then, we have
\[ R_n^{(r)}(t) = \begin{vmatrix} \tilde{p}_n^{(r)} & \cdots & \tilde{p}_{2n-2}^{(r)} & \tilde{p}_{2n-1}^{(r)} \\ \vdots & \ddots & \vdots & \vdots \\ \tilde{p}_{n+4}^{(r)} & \cdots & \tilde{p}_2^{(r)} & \tilde{p}_1^{(r)} \\ \tilde{p}_{n+2}^{(r)} & \cdots & \tilde{p}_0^{(r)} & \tilde{p}_1^{(r)} \end{vmatrix}. \]
Thus, we have
\[ \lambda \to -\varepsilon \lambda, \quad q_k^{(r)} = (-\varepsilon)^k p_k^{(r)}, \] (A.28)
and
\[ t \to -\varepsilon^{-3}(1 - \varepsilon^2 t), \quad r = -\varepsilon^{-3}. \] (A.29)

Then, (A.26) is written as
\[ \sum_{k=0}^{\infty} q_k^{(r+j)} \lambda^k = \exp \left( t\lambda + \frac{\lambda^3}{3} \right) \left[ 1 + \varepsilon \left( -j\lambda + \frac{1}{4} \lambda^4 \right) + O(\varepsilon^2) \right]. \] (A.30)

By using (A.4), we obtain
\[ q_k^{(r+j)} = q_k + \varepsilon \left( -j q_{k-1} + \frac{1}{4} q_{k-4} \right) + O(\varepsilon^2). \] (A.31)
Thus, we have
\[ R_n^{(r)} = (-\varepsilon)^{-n(n+1)/2} \left[ R_n + \varepsilon \left( -j \frac{dR_n}{dt} + Q_n \right) + O(\varepsilon^2) \right], \] (A.32)
where \( Q_n \) denotes the contribution from the term of \( q_{k-4} \) in (A.31).

Finally, it is easy to see that (A.32) is reduced to (A.7) by the above limiting procedures.

**B Proof of Lemma 5.4-5.8**

We first note that the following contiguity relations hold by definition [3.1] and [3.2],
\[ p_k^{(c-1,d-1)} = p_k^{(c,d)} + x p_{k-1}^{(c,d)}, \quad q_k^{(c-1,d-1)} = q_k^{(c,d)} + x^{-1} q_{k-1}^{(c,d)}, \] (B.1)
\[ p_k^{(c,d-1)} = p_k^{(c,d)} - p_{k-1}^{(c,d)}, \quad q_k^{(c,d-1)} = q_k^{(c,d)} - q_{k-1}^{(c,d)}, \] (B.2)
\[ (k+1)p_{k+1}^{(c,d)} = -(c-d)p_k^{(c,d+1)} - c x p_k^{(c+1,d+1)}, \] (B.3)
\[ (k+1)q_{k+1}^{(c,d)} = -(c-d)q_k^{(c,d+1)} - c x^{-1} q_k^{(c+1,d+1)}. \]

Let us prove Lemma 5.4. Adding the \((i+1)\)-th column multiplied by \(x^{-1}\) to the \(i\)-th column of \(R_{m,n}^{(0,0)}\) for \(i = 1, 2, \ldots, j, \ j = m + n - 1, m + n - 2, \ldots, 1\) and using (B.1), we get
\[
R_{m,n}^{(0,0)} = \begin{bmatrix}
q_1^{(c-m-n+1,d-m-n+1)} & q_2^{(c-m-n+2,d-m-n+2)} & \cdots & q_{m-n+3}^{(c-1,d-1)} & q_{m-n+2}^{(c,d)} \\
q_2^{(c-m-n+1,d-m-n+1)} & q_2^{(c-m-n+2,d-m-n+2)} & \cdots & q_{m-n+4}^{(c-1,d-1)} & q_{m-n+4}^{(c,d)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
q_{2m-1}^{(c-m-n+1,d-m-n+1)} & q_{2m-2}^{(c-m-n+2,d-m-n+2)} & \cdots & q_{m-n+1}^{(c-1,d-1)} & q_{m-n}^{(c,d)} \\
x_{m-n+1}^{(c-m-n+1,d-m-n+1)} & x_{m-n+2} p_{2n-1}^{(c-m-n+2,d-m-n+2)} & \cdots & x_{m-n+1} p_{2n-1}^{(c-1,d-1)} & p_{2n-1}^{(c,d)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
x_{m-n+1}^{(c-m-n+1,d-m-n+1)} & x_{m-n+2} p_{3}^{(c-m-n+2,d-m-n+2)} & \cdots & x_{m-n+1} p_{3}^{(c-1,d-1)} & p_{3}^{(c,d)} \\
x_{m-n+1}^{(c-m-n+1,d-m-n+1)} & x_{m-n+2} p_{1}^{(c-m-n+2,d-m-n+2)} & \cdots & x_{m-n+1} p_{1}^{(c-1,d-1)} & p_{1}^{(c,d)} \\
x_{m-n+1}^{(c-m-n+1,d-m-n+1)} & x_{m-n+2} p_{1}^{(c-m-n+2,d-m-n+2)} & \cdots & x_{m-n+1} p_{1}^{(c-1,d-1)} & p_{1}^{(c,d)} \\
\end{bmatrix}.
\] (B.4)
Noticing that \( p_0 = 1 \) and \( p_k = 0 \) for \( k < 0 \), we see that \( R_{m,n} \) can be rewritten as

\[
R_{m,n} = \begin{pmatrix}
q_1 & q_0 & \cdots & q_{-m-n+2} & q_{-m-n+1} \\
q_3 & q_2 & \cdots & q_{-m-n+4} & q_{-m-n+3} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
q_{2m-1} & q_{2m-2} & \cdots & q_m & q_{m-1} \\
p_{m} & p_{m+1} & \cdots & p_{2n-1} & p_{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
p_{-n-m+2} & p_{-n-m+3} & \cdots & p_1 & p_2 \\
p_{-n-m} & p_{-n-m+1} & \cdots & p-1 & p_0
\end{pmatrix}.
\] (B.5)

By the similar calculation to the above, we obtain

\[
R^{(0,0)}_{m,n} = \begin{pmatrix}
q_1^{(c-m-n,d-m-n)} & q_0^{(c-m-n+1,d-m-n+1)} & \cdots & q_{-m-n+2}^{(c-1,d-1)} & q_{-m-n+1}^{(c,d)} \\
q_3^{(c-m-n,d-m-n)} & q_2^{(c-m-n+1,d-m-n+1)} & \cdots & q_{-m-n+4}^{(c-1,d-1)} & q_{-m-n+3}^{(c,d)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
q_{2m-1}^{(c-m-n,d-m-n)} & q_{2m-2}^{(c-m-n+1,d-m-n+1)} & \cdots & q_m^{(c-1,d-1)} & q_{m-1}^{(c,d)} \\
x^{-m-n}p_{2n}^{(c-m-n,d-m-n)} & x^{-m-n}p_{2n-1}^{(c-m-n+1,d-m-n+1)} & \cdots & x^{-1}p_{2n}^{(c-1,d-1)} & x^{-1}p_{2n}^{(c,d)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
x^{-m-n}p_0^{(c-m-n,d-m-n)} & x^{-m-n}p_1^{(c-m-n+1,d-m-n+1)} & \cdots & x^{-1}p_0^{(c-1,d-1)} & x^{-1}p_0^{(c,d)}
\end{pmatrix}.
\] (B.6)

We have from (B.1) and (B.2)

\[
(1 + x)p_k^{(c,d)} = p_k^{(c-1,d-1)} + xp_k^{(c,d-1)}, \quad q_{k+1}^{(c,d-1)} + (1 + x^{-1})q_k^{(c,d)} = q_k^{(c-1,d-1)}.
\] (B.7)

Subtracting the \( j \)-th column multiplied by \( (1 + x^{-1})^{-1} \) from the \( (j+1) \)-th column of (B.4) for \( j = m + n, m + n - 1, \ldots, 1 \) and using (B.7), we get

\[
R^{(0,0)}_{m,n} = (-1)^m(1 + x^{-1})^{-m-n}
\]

\[
\begin{pmatrix}
-q_1^{(c-m-n,d-m-n)} & q_0^{(c-m-n+1,d-m-n+1)} & \cdots & q_{-m-n+2}^{(c-1,d-2)} & q_{-m-n+1}^{(c,d)} \\
-q_3^{(c-m-n,d-m-n)} & q_2^{(c-m-n+1,d-m-n+1)} & \cdots & q_{-m-n+4}^{(c-1,d-2)} & q_{-m-n+3}^{(c,d)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-q_{2m-1}^{(c-m-n,d-m-n)} & q_{2m-2}^{(c-m-n+1,d-m-n+1)} & \cdots & q_m^{(c-1,d-2)} & q_{m-1}^{(c,d)} \\
x^{-m-n}p_{2n}^{(c-m-n,d-m-n)} & x^{-m-n}p_{2n-1}^{(c-m-n+1,d-m-n+1)} & \cdots & x^{-1}p_{2n}^{(c-1,d-2)} & x^{-1}p_{2n}^{(c,d)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
x^{-m-n}p_0^{(c-m-n,d-m-n)} & x^{-m-n}p_1^{(c-m-n+1,d-m-n+1)} & \cdots & x^{-1}p_0^{(c-1,d-2)} & x^{-1}p_0^{(c,d)}
\end{pmatrix}.
\] (B.8)

From (B.6) and (B.8), we obtain Lemma 5.4.

Next, we prove Lemma 5.5. We have

\[
(k + 1)p_{k+1}^{(c,d)} = dp_k^{(c,d+1)} - c(1 + x)p_k^{(c+1,d+2)},
\]

\[
(d + k + 1)q_{k+1}^{(c,d)} = dq_{k+1}^{(c,d+1)} - (1 + x^{-1})q_k^{(c+1,d+2)}.
\] (B.9)

Subtracting the \( j \)-th column multiplied by \( \frac{d - m - n + j - 2}{(c - m - n + j - 1)(1 + x^{-1})} \) from the \( (j+1) \)-th column of (B.6)
for \( j = m + n, m + n - 1, \ldots, 1 \) and using (B.9), we get

\[
R_{m,n}^{(0,0)} = (-1)^{m+n}(1 + x)^{-m-n} x^m \prod_{k=1}^{m}(2k+1) \prod_{j=1}^{m+n} (c - m - n + j - 1)
\]

\[
\begin{pmatrix}
q_1 \quad q_1^{(c-m-n,d-m-n)} \\ q_2 \quad q_2^{(c-m-n,d-m-n)} \\ \vdots \\ q_{2m-1} \quad q_{2m-1}^{(c-m-n,d-m-n)} \\ p_n \quad p_n^{(c-m-n,d-m-n)} \\ \vdots \\ p_{n-m+1} \quad p_{n-m+1}^{(c-m-n,d-m-n)} \\ p_{n-m+2} \quad p_{n-m+2}^{(c-m-n,d-m-n)} \\ \vdots \\ p_{n+1} \quad p_{n+1}^{(c-m-n,d-m-n)}
\end{pmatrix}
\]

Lemma 5.5 follows from (B.4) and (B.10).

Note that we have

\[
(k + 1)p_k^{(c,d)} = -dxp_k^{(c+1,d+1)} - (c - d)(1 + x)p_k^{(c+1,d+2)},
\]

\[
(d + k + 1)q_{k+1}^{(c,d)} = dq_{k+1}^{(c+1,d+1)} - (c - d)(1 + x^{-1})q_{k+1}^{(c+1,d+2)}.
\]

It is easy to see that Lemma 5.6 is proved similarly to Lemma 5.5 by using (B.2) and (B.11).

The proof of Lemma 5.7 is given as follows. Adding the \((j - 1)\)-th column multiplied by \(x\) to the \(j\)-th column of \(R_{m,n}^{(1,1)}\) for \(j = m + n, m + n - 1, \ldots, 2\) and using (B.1), we get

\[
R_{m,n}^{(1,1)} = x^m 
\begin{pmatrix}
x^{-1}q_1^+ & q_1 & \cdots & q_{m-n+4} & q_{m-n+3} \\
x^{-1}q_3^+ & q_3 & \cdots & q_{m-n+6} & q_{m-n+5} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
x^{-1}q_{2m-1} & q_{2m-1} & \cdots & q_{m-n+2} & q_{m-n+1} \\
p_n & p_n+1 & \cdots & p_{2n-2} & p_{2n-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
p_{n-m+4} & p_{n-m+5} & \cdots & p_2 & p_3 \\
p_{n-m+2} & p_{n-m+3} & \cdots & p_0 & p_1
\end{pmatrix}
\]

We have from (B.1)

\[
p_k^{(c,d)} - x^n p_{k-2}^{(c,d)} = p_k^{(c-1,d-1)} - xp_k^{(c-1,d-1)},
\]

\[
q_k^{(c,d)} - x^n q_{k-2}^{(c,d)} = q_k^{(c-1,d-1)} - x^{-1}q_k^{(c-1,d-1)}.
\]

Then, subtracting the \((j - 1)\)-th column multiplied by \(x\) from the \(j\)-th column of \(R_{m,n}^{(1,-1)}\) for \(j = m + n, m + n - 1, \ldots, 2\) and using (B.1), we get
\( n - 1, \ldots, 2 \) and using (B.13), we get

\[
R_{m,n}^{(-1,-1)} = \begin{pmatrix}
-\frac{x^{-1}q_1}{q_1} & x^{-1}q_1 - xq_1 & \cdots & x^{-1}(q_{m-n-1} - xq_{m-n+3}) \\
-\frac{x^{-1}q_3}{q_3} & x^{-1}q_3 - xq_3 & \cdots & x^{-1}(q_{m-n-3} - xq_{m-n+5}) \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{x^{-1}q_{2m-1}}{q_{2m-1}} & x^{-1}q_{2m-3} - xq_{2m-1} & \cdots & x^{-1}q_{m-n-1} - xq_{m-n+1} \\
\vdots & \vdots & \ddots & \vdots \\
p^{-n-m+4} & p^{-n-m+5} - x^2p^{-n-m+3} & \cdots & p_3 \\
p^{-n-m+2} & p^{-n-m+3} - x^2p^{-n-m+1} & \cdots & p_1
\end{pmatrix} \tag{B.14}
\]

Noticing that \( p_k = q_k = 0 \) for \( k < 0 \), we obtain

\[
R_{m,n}^{(-1,-1)} = (-x)^m \begin{pmatrix}
-\frac{x^{-1}q_1}{q_1} & q_1 & q_0 & \cdots & q_{m-n+3} \\
-\frac{x^{-1}q_3}{q_3} & q_3 & q_2 & \cdots & q_{m-n+5} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-\frac{x^{-1}(q_{2m-1} + \cdots + x^{-2m+2}q_1)}{q_{2m-1}} & q_{2m-1} & q_{2m-2} & \cdots & q_{m-n+1} \\
p^{-n-m+4} + x^2p^{-n-m+2} & p^{-n-m+5} & p^{-n-m+6} & \cdots & p_1 \\
p^{-n-m+2} & p^{-n-m+3} & p^{-n-m+4} & \cdots & p_1
\end{pmatrix} \tag{B.15}
\]

The first half of Lemma 5.7 is obtained by (B.12) and (B.15). Moreover, we have

\[
D = \begin{pmatrix}
-\frac{x^{-1}q_1}{q_1} & x^{-1}q_1 - x^{-2}q_0 & q_1 - x^{-2}q_1 & \cdots & q_{m-n+4} - x^{-2}q_{m-n+2} \\
-\frac{x^{-1}q_3}{q_3} & x^{-1}q_3 - x^{-2}q_2 & q_3 - x^{-2}q_1 & \cdots & q_{m-n+6} - x^{-2}q_{m-n+4} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-\frac{x^{-1}q_{2m-1}}{q_{2m-1}} & x^{-1}q_{2m-3} - x^{-2}q_{2m-1} & q_{2m-1} - x^{-2}q_{2m-3} & \cdots & q_{m-n+2} - x^{-2}q_{m-n} \\
p^{-n-m+5} - xp^{-n-m+4} & p^{-n-m+6} - x^2p^{-n-m+4} & \cdots & p_3 - x^2p_1 \\
p^{-n-m+3} - xp^{-n-m+2} & p^{-n-m+4} - x^2p^{-n-m+2} & \cdots & p_1 - x^2p_1
\end{pmatrix} \tag{B.16}
\]

Subtracting the 2\textsuperscript{nd} column from the 1\textsuperscript{st} column using (B.1), we get

\[
D = 2(-1)^{m-x^{-2m+1}}P_{m,n}^{(0,0)} \tag{B.17}
\]

which is nothing but the second half of Lemma 5.7.

From the above discussion, it is easy to verify Lemma 5.8.
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