Gravitational Analysis of Einstein-Non-Linear-Maxwell-Yukawa Black Hole under the Effect of Newman-Janis Algorithm

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In this paper, we analyze the rotating Einstein-non-linear-Maxwell-Yukawa black hole solution by Janis-Newman algorithmic rule and complex calculations. We investigate the basic properties (i.e., Hawking radiation) for the corresponding black hole solution. From the horizon structure of the black hole, we discuss the graphical behavior of Hawking temperature $T_H$ and analyze the effects of spin parameter (appears due to Newman-Janis approach) on the $T_H$ of black hole. Furthermore, we investigate the corrected temperature for rotating Einstein-non-linear-Maxwell-Yukawa black hole by using the vector particles tunneling strategy which is based on Hamilton-Jacobi method. We additionally study the graphical explanation of corrected $T_H$ through outer horizon to investigate the physical and stable conditions of black hole. Finally, we compute the corrected entropy and check that the effect of charged, rotation and gravity on entropy.

Keywords: Yukawa Black Hole; Newman-Janis algorithm; Semi-classical Phenomenon; Hawking temperature; corrected entropy.

I. INTRODUCTION

One of the main feature of Einstein’s theory of general relativity (GR) is the well-known conservation of the covariance of the energy momentum tensor (EMT) that according to the Noether symmetry postulate, prompts to a globally defined transformation of physical geometry. These moderated amounts show up in the form of integrals of the parts of the EMT on the corresponding spatial surface. These space-like surfaces concede at any rate some particular Killing vectors of the basic space-time as their typical. Therefore, the rest mass/total energy of the system is conserved with regards to GR. After this, some new generalized theories related about GR have been suggested that weaken the state of conservation of EMT. One of the feasible change of the overall theories of GR was presented by P. Rastall (1972) [1, 2]. The standard law of conversation can be determined from null divergence ($T^u_{\alpha}$; $u_\alpha=0$). At that point, the non-minimal connection of the matter with the geometry is considered in which the divergence ($T^u_{\alpha}$) is proportional to the Ricci scalar gradient ($T^u_{\alpha} \propto R^{u\alpha}$), so that, the standard conservation law is recovered within the flat space-time. The Rastall theory [3] may be an understandable theory within the idea that the fluctuation of any amplified gravity theory from the basic GR should be frail to overcome the sun oriented system evaluation. A few examinations on the different parts of this theory with regards to current speed up expansion period of the universe also other cosmological issues can be found [4]. The substance of the Rastall theory is related to the higher curvature situations and thus the black holes (BHs) physics give a suitable area in arrange to examine this theory in more subtleties. Other fascinating issues are that Rastall gravity appears to don’t endure from the entropy as well as age issues of usual cosmology [5] and it is also predictable with the gravitational lensing wonders [6]. Many other investigations on Rastall theory have been studied and references [7]-[10].

The tunneling phenomenon is observed for boson charged particles with electro-positive energy crosses the horizon of BH and these particles appear as Hawking radiation [11, 12]. The Hawking radiation and entropy of BH by applying the tunneling mechanism to the non-spherical Kerr and Kerr-Newman metric has been investigated [13]. The author calculated the entropy effects on the stability of BH.

The tunneling probability inside and outside the horizon as well as $T_H$ by taking into account the Newman-Penrose formalism and Hamilton Jacobi-method has been studied [14]. From their analysis, they concluded that the $T_H$ depends upon mass of a BH, magnetic charge and electric charge. The $T_H$ by incorporating fermions tunneling from squashed BH in the Gödel universe and charged Kaluza-Klein spacetime has been investigated [15]. The author
showed that the fermion and scalar particles with spin tunnel through the horizon give identical expressions for $T_H$. By taking into account the global mono-pole charge, Sharif and his fellow [16] have studied the Hawking radiation from Reissner-Nordström de Sitter BH. They extended their analysis by studying back reaction effects of fermions tunneling through horizon. The same authors also studied the thermodynamics of different BH solutions to the fermions. They also evaluated surface gravity, $T_H$ and first law of thermodynamics [17]. The tunneling radiation phenomenon was also discussed by Silva and Brito [18]. The advantage of discussing tunneling phenomenon is that, it is helpful to remove singularity as well to discuss thermodynamical properties of BHs. They also calculated the emission spectrum of self-dual BH. Quantum tunneling for Hawking radiation via static and dynamic BH was carried by Chakraborty and Saha [19]. They calculated quantum corrections upon the first order as the equations of motions for higher orders were complicated. They also discussed law of BH mechanics considering modified $T_H$. Darabi et al. [20] studied the Hawking radiations from generalized rotating and static BH. They have made use of WKB approximation for calculating tunneling. Liu [21] have analyzed the energy conservation of charged particles as well as quantum corrections via tunneling method for a modified version of Reissner BH. He concluded that the entropy is independent from the dispersion relations over matter fields. Ding et al. [22] studied the tunneling process of relativistic and non-relativistic particles for Killing and universal horizon of BH by using Hamilton-Jacobi process. They calculated the entropy effect on stability of BH.

The Hawking radiation and classical tunneling by applying the WKB approximation and the ray phase space process has been analyzed [23]. Moreover, they also determined the outgoing charged particles tunneling probability at event horizon of BH. Jusufi et al. [24] obtained the Hawking radiation of vector particle by applying the WKB-approximation in Friedman-Robertson-Walker (FRW) universe in a BH. They calculated the effects of gravity by radiation phenomenon of charged boson particle of BH and black ring. The BH radiation with modified dispersion relation in tunneling paradigm statical frame has been investigated [25]. The authors in [26] have calculated the geodesic equations for massive/massless particles in a very effective way. They also studied the radiation process with the help of tunneling strategy from cosmic BH. The particle dynamics around the Kerr MÖG BH with magnetic field has been discussed. Sharif and Shahzadi [27] concluded that the external magnetic field has strong influence on particle dynamics in MÖG depending upon the spin of a BH. Javed and his colleagues [28]-[30] have computed the charged boson particle tunneling with the help of couple of accelerating/rotating super gravity BH in 5D. In particular, they have used WKB approximation to study tunneling and Hawking temperature. Cvetković and Simić [31] investigated the static spherically symmetric solutions of Lovelock gravity by using torsion process. They found well-known solution, which is known as Boulware-Deser BH. Övgün et al. [32] investigated the $T_H$ of BH by calculating the tunneling rate of massive charge vector particles with the electromagnetic field.

The Hawking radiation process via tunneling method for 5D Myers-Perry BH has been investigated [33]. Generalized uncertainty principle (GUP) effects on temperature via geometry of BH was examined by Gecim and his fellow [34]. For this aim, the authors concluded that the $T_H$ increases with the increase of angular momentum. Javed and Babar [35] discussed charged fermion particles tunneling via Kerr-Newman-AdS BH. In order to study the required task, they used Hamilton-Jacobi ansatz to calculate the $T_H$ for spin-1/2 particles. Ali et al. [36]-[42] investigated the field equation for massive boson via WKB method and also discussed the gravity effects on radiation to check the stability and instability of BH.

The basic intention of this article is to derive the Einstein-non-linear Maxwell-Yukawa (ENLMY) BH in the background of Newman-Janis algorithm and to extend the ENLMY BH into rotating ENLMY BH. Furthermore, to discuss the stability conditions of RENLMY BH via graphical interpretation of its $T_H$. The paper is formatted in the following manner: In section II, we discuss a RENLMY BH solution in the Newman-Janis algorithm and also analyze the $T_H$ for the corresponding BH. The section III comprises the graphical analysis of $T_H$ with horizon and check the stable condition of RENLMY BH. The section IV study the $T'_H$ (corrected temperature) for RENLMY BH. The section V analyzes quantum gravity effects on RENLMY BH with graphical evaluation. Section VI study the corrected entropy for RENLMY BH and its graphical analysis. At last, Sec. VII contains the summary and conclusions.

II. ROTATING EINSTEIN-NON-LINEAR-MAXWELL-YUKAWA BLACK HOLE

By utilizing the Newman-Janis method, the rotation parameter $a$ can be calculated in a spherically symmetric explanation that gives the modification of Newman-Janis method. Here, we derive a metric for ENLMY with a modification of rotation parameter by utilizing Newman-Janis method. Furthermore, we compute the $T_H$ for the given solution of BH. The ENLMY BH with a spherically symmetric static metric can be written as [43, 44]

\[ ds^2 = -F(r)dt^2 + \frac{1}{F(r)}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2, \] (1)
where $F(r) \simeq 1 + \frac{2m}{r} - \frac{Qe}{r^{2}} + \frac{4Qe\beta}{r^{2}} - Qe\beta^{2} + O(\beta^{3})$. Here, $m$ represents the mass of BH, $C_{0}$ is an integration constant (dimensionless parameter), $Q$ depicts the charge of BH that is located at the origin and $\beta$ is a positive constant and it can be chosen as $\beta = 1$. The vector potential of the Yukawa black hole can be defined as

$$A = \frac{Q}{r^{2}e^{t}} dt$$

We consider the Eddington-Finkelstein (EF) coordinates transformations $(t, r, \theta, \phi)$ to the Boyer-Lindquist (BL) coordinates $(u, r, \theta, \phi)$, we get

$$du = dt - \frac{dr}{F(r)}$$

and by applying this coordinate transformation to metric Eq. (1), we have

$$ds^{2} = -F(r)du^{2} - 2du dr + r^{2}d\theta^{2} + r^{2}\sin^{2}\theta d\phi^{2}.$$  

The components of metric in the background of null framework can be re-written as

$$g^{\mu\nu} = -l^{\mu}n^{\nu} + m^{\mu}m^{\nu}.$$  

Here the evaluating elements are

$$l^{\mu} = \delta_{r}^{\mu}, \quad n^{\nu} = \delta_{u}^{\mu} - \frac{1}{2} F^{\nu}_{r} \delta_{r}^{\mu}.$$  

$$m^{\mu} = \frac{1}{\sqrt{2r}} \delta_{\theta}^{\mu} + \frac{i}{\sqrt{2r}\sin\theta} \delta_{\phi}^{\mu},$$  

$$\bar{m}^{\mu} = \frac{1}{\sqrt{2r}} \delta_{\theta}^{\mu} - \frac{i}{\sqrt{2r}\sin\theta} \delta_{\phi}^{\mu}.$$  

The following relation of the null tetrad are satisfied in $(u, r)$ plane $l_{\mu}l^{\mu} = n_{\mu}n^{\mu} = m_{\mu}m^{\mu} = l_{\mu}m^{\mu} = m_{\mu}n^{\mu} = 0$ and $l_{\mu}n^{\nu} = -m_{\mu}\bar{m}^{\nu} = 1$, we can choose the coordinate transformation as $r \rightarrow r + ia \cos \theta$, $u \rightarrow u - ia \cos \theta$, then we perform the transformation $F(r) \rightarrow \tilde{F}(r, a, \theta)$ and $\Sigma^{2} = a^{2}\cos^{2}\theta + r^{2}$. The vectors in given space become

$$l^{\mu} = \delta_{r}^{\mu}, \quad n^{\nu} = \delta_{u}^{\mu} - \frac{1}{2} \tilde{F}^{\nu}_{r} \delta_{r}^{\mu},$$  

$$m^{\mu} = \frac{1}{\sqrt{2r}} \delta_{\theta}^{\mu} + \frac{i}{\sqrt{2r}\sin\theta} \delta_{\phi}^{\mu},$$  

$$\bar{m}^{\mu} = \frac{1}{\sqrt{2r}} \delta_{\theta}^{\mu} - \frac{i}{\sqrt{2r}\sin\theta} \delta_{\phi}^{\mu}.$$  

From the definition of the null tetrad the metric tensor $g^{\mu\nu}$ in the EF coordinate can be given as

$$g^{uu} = \frac{a^{2}\sin^{2}\theta^{2}}{\Sigma^{2}}, \quad g^{rr} = -\frac{a^{2}\sin^{2}\theta^{2}}{\Sigma^{2}}, \quad g^{rr} = \tilde{F} + \frac{a^{2}\sin^{2}\theta^{2}}{\Sigma^{2}}, \quad g^{\theta\phi} = \frac{1}{\Sigma^{2}},$$  

$$g^{\phi\phi} = \frac{1}{\Sigma^{2}\sin^{2}\theta^{2}}, \quad g^{\nu\nu} = \frac{\sin\theta}{\Sigma^{2}}, \quad g^{\nu\nu} = -\frac{a}{\Sigma^{2}}.$$  

where

$$\tilde{F}(r, \theta) = 1 - \frac{2mr}{\Sigma^{2}} - \frac{QeC_{0}}{\Sigma^{2}} + \frac{4Qe\beta r}{3\Sigma^{2}} - \frac{Qe\beta^{2}r^{2}}{\Sigma^{2}} + O(\beta^{3}).$$

We do the coordinate transformation from EF to BL coordinates as

$$du = dt + Y(r)dr, \quad d\phi = d\phi + Z(r)dr,$$

where

$$Y(r) = \frac{r^{2} + a^{2}}{r^{2}F + a^{2}}, \quad Z(r) = -\frac{a}{r^{2}F + a^{2}}.$$
Finally, we get the EF coordinate transformation in the following from

\[ ds^2 = -\left[ \frac{r^2 - 2mr - QC_0 + \frac{4QC_0\beta r}{3} - Qr^2C_0\beta^2 + O(\beta^3)}{\Sigma^2} \right] dt^2 + \frac{\Sigma^2}{\Delta r} dr^2 \\
- 2a\sin\theta \left[ 1 - \frac{r^2 - 2mr - QC_0 + \frac{4QC_0\beta r}{3} - Qr^2C_0\beta^2 + O(\beta^3)}{\Sigma^2} \right] d\theta + \Sigma^2 d\phi^2 \\
+ a^2\sin^2\theta \left[ \Sigma^2 - a^2\sin^2\theta \left( r^2 - 2mr - QC_0 + \frac{4QC_0\beta r}{3} - Qr^2C_0\beta^2 + O(\beta^3) - 2 \right) \right] d\phi^2. \quad (12) \]

where

\[ \Delta_r = r^2 - 2mr - QC_0 + \frac{4QC_0\beta r}{3} - Qr^2C_0\beta^2 + O(\beta^3). \]

Thus the Eq. (12) gives the metric for rotating ENLMY BH with the spin parameter \( a \). In order to study the thermodynamical properties of ENLMY BH in rotating case, we compute the \( T_H \) by the given formula \( [45] \)

\[ T_H = \frac{\tilde{F}'(r_+)}{4\pi}. \quad (13) \]

The temperature \( T_H \) for rotating (RENLMY) BH can be derived as

\[ T_H = \frac{3mr_+^2 - 3C_0Qr_+ + 2C_0Qr_+^2\beta + a^2(3C_0Q\beta r_+ - 2C_0Q\beta - 3m)}{6\pi(r_+^2 + a^2)^2}. \quad (14) \]

From our calculation, we have computed that the \( T_H \) at which charged particles radiation both in and out through the horizons \( r \) is independent of the types of the particles, and \( T_H \) depends upon mass \( m \), BH charge \( Q \), spin parameter \( a \) and arbitrary constants \( \beta, C_0 \). It is concerning to note that for \( a = 0 \), we recover \( T_H \) with out Newman-Janis algorithm.

III. GRAPHICAL ANALYSIS OF \( T_H \) FOR RENLMY BH

The section give the graphical conduct of \( T_H \) with horizon \( r_+ \) for RENLMY BH. We investigate the effects of spin parameter \( a \) and charge \( Q \) of RENLMY BH on \( T_H \). We see the behavior of \( T_H \) by setting the fixed value of mass \( m = 1 \) and arbitrary constants \( \beta = 0.9 = C_0 \). Moreover, we analyze different parameters on the stability of RENLMY BH.

![Figure 1: \( T_H \) via horizon \( r_+ \).](image)

**Figure 1:** (i) states the interpretation of \( T_H \) for constant value of charge \( Q = 50 \) and different values of spin parameter \( a \). It is observable that at the initial stage the BH is unstable (due to negative \( T_H \)) but as the time passes the BH gets its stable form. After attaining a maximum height the \( T_H \) eventually drops down to get an asymptotically flat
The WKB approximation is expressed in the form

\[ g \phi(\tilde{r}) = \delta, \tilde{X} = \Sigma^2, \tilde{Y} = \sin \theta^2 \left[ \Sigma^2 - a^2 \sin^2 \theta(\tilde{F} - 2) \right], \tilde{Z} = 2a \left( \tilde{F}(r) - 1 \right) \sin^2 \theta. \]

where \( \tilde{r} \) is the particle charge and derivatives of co-variant, respectively. With the increasing value of spin parameter. (ii) shows the interpretation of purpose, we use the modified wave equation to study the tunneling behavior of vector particles from RENLMY BH. For this purpose, we use the modified wave equation to study the tunneling behavior of vector particles from RENLMY BH.

In order to meet our goal, the metric Eq. (12) can be written as

\[ ds^2 = -\tilde{V}dt^2 + \tilde{W}dr^2 + \tilde{X}d\theta^2 + \tilde{Y}d\phi^2 + 2\tilde{Z}dtd\phi, \]

where \( \tilde{V} = \tilde{F}(r) \), \( \tilde{W} = -\delta \), \( \tilde{X} = \Sigma^2 \), \( \tilde{Y} = \sin \theta^2 \left[ \Sigma^2 - a^2 \sin^2 \theta(\tilde{F} - 2) \right] \), \( \tilde{Z} = 2a \left( \tilde{F}(r) - 1 \right) \sin^2 \theta. \)

The generalized wave equation for vector particles motion can be expressed as [29]

\[ \partial_\mu (\sqrt{-g} \phi^{\mu}) + \sqrt{-g} \frac{m^2}{h^2} \phi^{\nu} + \sqrt{-g} \frac{i}{h} A_\mu \phi^{\nu} + \sqrt{-g} \frac{i}{h} \tilde{V}^{\nu} \phi^0 + \alpha \partial_\mu \partial_0 (\sqrt{-g} \phi^{00} \phi^0) h^2 - \alpha \partial_\mu \partial_0 (\sqrt{-g} \phi^{00} \phi^0) h^2 = 0, \]

(15)

here \( g \) is determinant of coefficient matrix, \( \phi^{\nu} \) is anti-symmetric tensor and \( m \) is particle mass.

The \( \phi_{\nu \mu} \) tensor can be expressed as

\[ \phi_{\nu \mu} = -(1 - \alpha h^2 \partial^2) \partial_\mu \phi_0 + (1 - \alpha h^2 \partial^2) \partial_\nu \phi_0 - (1 - \alpha h^2 \partial^2) \frac{i}{h} e A_\mu \phi_0 + (1 - \alpha h^2 \partial^2) \frac{i}{h} e A_\nu \phi_0, \quad F_{\nu \mu} = \nabla_\nu A_\mu - \nabla_\mu A_\nu, \]

where \( \alpha, A_\mu, e \) and \( \nabla_\mu \) are the quantum gravity parameter (correction parameter), Yukawa BH vector potential, the particle charge and derivatives of co-variant, respectively.

According to the above metric the non-zero components can be calculated as

\[ \phi^0 = \frac{-Y \phi_0 + \tilde{Z} \phi_3}{V Y + Z^2}, \quad \phi^1 = \frac{1}{W} \phi_1, \quad \phi^2 = \frac{1}{X} \phi_2, \quad \phi^3 = \frac{\tilde{Z} \phi_0 + \tilde{V} \phi_3}{V Y + Z^2}, \quad \phi^{01} = \frac{-D \phi_{01} + \tilde{Z} \phi_{13}}{W (V Y + Z^2)}, \quad \phi^{02} = \frac{-Y \phi_{02}}{X (V Y + Z^2)}, \]

\[ \phi^{03} = \frac{(\tilde{V} \tilde{Y} - V \tilde{Y}) \phi_{03}}{(V Y + Z^2)^2}, \quad \phi^{12} = \frac{1}{W X} \phi_{12}, \quad \phi^{13} = \frac{1}{W V Y + Z^2} \phi_{13}, \quad \phi^{23} = \frac{\tilde{Z} \phi_{02} + \tilde{V} \phi_{23}}{X (V Y + Z^2)}. \]

The WKB approximation is expressed in the form

\[ \phi_\nu = c_\nu \exp \left[ \frac{i}{h} I_0(t, r, \theta, \phi) + \Sigma h^m I_n(t, r, \theta, \phi) \right]. \]

(16)

The set of field equation are given below

\[ + \frac{\tilde{Y}}{W (V Y + Z^2)} \left[ c_1 (\partial_1 I_0) (\partial_1 I_0) + \alpha c_1 (\partial_0 I_0)^3 (\partial_1 I_0) - c_0 (\partial_1 I_0)^2 - \alpha c_0 (\partial_1 I_0)^4 + c_1 e A_0 (\partial_1 I_0) \right] + c_1 e A_0 (\partial_0 I_0)^2 (\partial_1 I_0) \]
\[- \frac{\dot{Y}}{W(VY + Z^2)} \left[ c_1(\partial_0 I_0)^2 + ac_1(\partial_0 I_0)^4 - c_0(\partial_0 I_0)(\partial_1 I_0) - ac_0(\partial_0 I_0)(\partial_1 I_0)^3 + c_1 eA_0(\partial_0 I_0) + ac_1 eA_0(\partial_0 I_0)^3 \right] \\
+ \frac{\dot{Z}}{W(VY + Z^2)} \left[ c_3(\partial_0 I_0)(\partial_1 I_0) + ac_3(\partial_0 I_0)(\partial_1 I_0)^3 - c_1(\partial_0 I_0)(\partial_3 I_0) - ac_1(\partial_0 I_0)(\partial_3 I_0)^3 \right] + \frac{1}{WX} \left[ c_2(\partial_1 I_0)(\partial_2 I_0) \\
+ ac_2(\partial_1 I_0)(\partial_2 I_0)^3 - c_1(\partial_2 I_0)^2 - ac_1(\partial_2 I_0)^4 \right] + \frac{1}{W(VY + Z^2)} \left[ c_3(\partial_1 I_0)(\partial_3 I_0) + ac_3(\partial_1 I_0)(\partial_3 I_0)^3 - c_1(\partial_3 I_0)^2 \right] \\
- ac_1(\partial_3 I_0)^4 \right] - \frac{m^2 c_1}{W} - \frac{c_1 eA_0}{W(VY + Z^2)} \left[ c_1(\partial_0 I_0) + ac_1(\partial_0 I_0)^3 - c_0(\partial_0 I_0) - ac_0(\partial_0 I_0)^3 + c_1 eA_0 c_1 \\
+ ac_1 eA_0(\partial_0 I_0)^2 \right] + \frac{c_1 eA_0}{W(VY + Z^2)} \left[ c_3(\partial_1 I_0) + ac_3(\partial_1 I_0)^3 - c_1(\partial_1 I_0) - ac_1(\partial_1 I_0)^3 \right] = 0, \quad (18) \\
+ \frac{\dot{Y}}{X(VY + Z^2)} \left[ c_2(\partial_0 I_0)^2 + ac_2(\partial_0 I_0)^4 - c_0(\partial_0 I_0)(\partial_2 I_0) - ac_0(\partial_0 I_0)(\partial_2 I_0)^3 + c_2 eA_0(\partial_0 I_0) + ac_2 eA_0(\partial_0 I_0)^3 \right] \\
+ \frac{1}{WX} \left[ c_2(\partial_1 I_0)^2 + ac_2(\partial_1 I_0)^4 - c_1(\partial_1 I_0)(\partial_2 I_0) - ac_1(\partial_1 I_0)(\partial_2 I_0)^3 \right] - \frac{\dot{Z}}{X(VY + Z^2)} \left[ c_2(\partial_0 I_0)(\partial_3 I_0) \\
+ ac_2(\partial_0 I_0)^3(\partial_3 I_0) - c_0(\partial_0 I_0)(\partial_3 I_0) - ac_0(\partial_0 I_0)^3(\partial_3 I_0) + c_2 eA_0(\partial_3 I_0) + ac_2 eA_0(\partial_3 I_0)^3 \right] \\
+ \frac{\dot{V}}{X(VY + Z^2)} \left[ c_2(\partial_1 I_0)(\partial_3 I_0) + ac_2(\partial_2 I_0)^3(\partial_3 I_0) - c_1(\partial_3 I_0)^2 - ac_2(\partial_3 I_0)^4 \right] - \frac{m^2 c_2}{X} \\
+ \frac{c_1 eA_0}{X(VY + Z^2)} \left[ c_2(\partial_1 I_0) + ac_2(\partial_0 I_0)^3 - (\partial_2 I_0) c_0 - ac_2(\partial_3 I_0)^3 c_0 + c_2 eA_0 c_2 + (\partial_2 I_0)^2 c_2 eA_0 \right] = 0, \quad (19) \\
+ \frac{(\dot{V}) - \dot{W}^2}{(VY + Z^2)^2} \left[ c_3(\partial_0 I_0)^2 + ac_3(\partial_0 I_0)^4 - c_0(\partial_0 I_0)(\partial_3 I_0) - ac_0(\partial_0 I_0)(\partial_3 I_0)^3 + c_2 eA_0 c_3(\partial_0 I_0) + ac_3 eA_0(\partial_0 I_0)^3 \right] \\
- \frac{\dot{Y}}{X(VY + Z^2)} \left[ c_3(\partial_1 I_0)^2 + ac_3(\partial_1 I_0)^4 - c_1(\partial_1 I_0)(\partial_3 I_0) - ac_1(\partial_1 I_0)(\partial_3 I_0)^3 \right] - \frac{\dot{Z}}{X(VY + Z^2)} \left[ c_2(\partial_0 I_0)(\partial_2 I_0) \\
+ ac_2(\partial_0 I_0)^3(\partial_2 I_0) - c_0(\partial_2 I_0)^2 + ac_0(\partial_2 I_0)^4 + c_2 eA_0 c_2(\partial_2 I_0) + ac_2 eA_0(\partial_2 I_0)^2(\partial_2 I_0) \right] - \frac{c_1 eA_0}{X(VY + Z^2)} \left[ c_3(\partial_2 I_0)^2 \right] \\
+ \frac{\dot{V} \dot{W}}{(VY + Z^2)^2} \left[ c_3(\partial_0 I_0)(\partial_3 I_0) - ac_2(\partial_0 I_0)(\partial_3 I_0) - ac_2(\partial_0 I_0)(\partial_3 I_0)^3 \right] - \frac{m^2 (\dot{Z} c_0 - \dot{V} c_3)}{(VY + Z^2)} + \frac{c_3 eA_0}{(VY + Z^2)^2} \left[ c_3(\partial_0 I_0) + ac_3(\partial_3 I_0)^3 \right] \\
- c_0(\partial_3 I_0)^3 - (\partial_3 I_0)^3 c_0 + c_2 eA_0 c_3 + (\partial_0 I_0)^2 c_2 eA_0 = 0, \quad (20) \\

By applying the variables separation technique, we can take \\
I_0 = -(E - j\omega)t + W(r) + \nu(\theta) + J\phi, \quad (21) \\

here \(\omega\) & \(J\) stands for angular momentums of BH and radiated particles, respectively, whereas \(\tilde{E} = E - j\omega\) is the particle energy.

\[ K(c_0, c_1, c_2, c_3)^T = 0, \]
which gives a $4 \times 4$ matrix $K^*$, whose elements are given as follows

$$K_{00} = \frac{-D}{W(VY + Z^2)}[W_1^2 + \alpha W_1^4] - \frac{\hat{Y}}{X(VY + Z^2)}[J^2 + \alpha J^4] - \frac{\hat{V}Y}{(VY + Z^2)^2}[\nu_1^2 + \alpha \nu_1^4] - \frac{m^2 \hat{Y}}{(VY + Z^2)},$$

$$K_{01} = \frac{-D}{W(VY + Z^2)}[\hat{E} + \alpha \hat{E}^3 + eA_0 + eA_0 \hat{E}^2]W_1 + \frac{\hat{Z}}{W(VY + Z^2)}[\nu_1 + \alpha \nu_1^3],$$

$$K_{02} = \frac{-D}{X(VY + Z^2)}[\hat{E} + \alpha \hat{E}^3 - eA_0 - eA_0 \hat{E}^2]J,$$

$$K_{03} = \frac{-E}{W(VY + Z^2)}[W_1^2 + \alpha W_1^4] - \frac{\hat{V}Y}{(VY + Z^2)^2}[\hat{E} + \alpha \hat{E}^3 - eA_0 - eA_0 \hat{E}^2][\nu_1 + \alpha \nu_1^3] + \frac{m^2 \hat{Z}}{(VY + Z^2)^2},$$

$$K_{10} = \frac{-D}{W(VY + Z^2)}[\hat{E}W_1 + \alpha \hat{E}W_1^3] - \frac{m^2}{W} - \frac{eA_0 \hat{Y}}{W(VY + Z^2)}[W_1 + \alpha W_1^3],$$

$$K_{11} = \frac{-D}{W(VY + Z^2)}[\hat{E}^2 + \alpha \hat{E}^4 - eA_0 \hat{E} - \alpha eA_0 \hat{E}^2 W_1^2] + \frac{\hat{Z}}{W(VY + Z^2)}[\nu_1 + \alpha \nu_1^3] - \frac{1}{W}[J^2 + \alpha J^4],$$

$$K_{12} = \frac{1}{W}[W_1 + \alpha W_1^3],$$

$$K_{13} = \frac{-E}{W(VY + Z^2)}[W_1 + \alpha W_1^3] \hat{E} + \frac{1}{W(VY + Z^2)^2}[W_1 + \alpha W_1^3][\nu_1 + \alpha \nu_1^3] W_1 + \frac{\hat{Z}eA_0}{W(VY + Z^2)}[W_1 + \alpha W_1^3],$$

$$K_{20} = \frac{\hat{Y}}{X(VY + Z^2)}[\hat{E}J + \alpha \hat{E}J^3] + \frac{\hat{Z}}{X(VY + Z^2)}[\hat{E} + \alpha \hat{E}^3 \nu_1^2] - \frac{\hat{Y}eA_0}{X(VY + Z^2)}[J + \alpha J^3],$$

$$K_{21} = \frac{1}{W}[J + \alpha J^3][W_1],$$

$$K_{22} = \frac{\hat{V}}{X(VY + Z^2)}[\hat{E}^2 + \alpha \hat{E}^4 - eA_0 \hat{E} - \alpha eA_0 \hat{E}^2] - \frac{1}{W}[J^2 + \alpha J^4],$$

$$K_{23} = \frac{\hat{V}}{X(VY + Z^2)}[\nu_1^2 + \alpha \nu_1^4] - \frac{m^2}{X} - \frac{eA_0 \hat{Y}}{X(VY + Z^2)}[\hat{E} + \alpha \hat{E}^3 - eA_0 - \alpha eA_0 \hat{E}^2],$$

$$K_{30} = \frac{(\hat{Y} - A^2)}{(VY + Z^2)^2}[\nu_1 + \alpha \nu_1^3] \hat{E} + \frac{\hat{Z}}{X(VY + Z^2)}[J^2 + \alpha J^4] - \frac{m^2 \hat{Z}}{(VY + Z^2)^2}[\nu_1 + \alpha \nu_1^3],$$

$$K_{31} = \frac{1}{W(VY + Z^2)}[\nu_1 + \alpha \nu_1^3] W_1,$$

$$K_{32} = \frac{\hat{Z}}{X(VY + Z^2)}[J + \alpha J^3] \hat{E} + \frac{\hat{V}}{X(VY + Z^2)}[\nu_1 + \alpha \nu_1^3] J,$$

$$K_{33} = \frac{(\hat{V} - A^2)}{(VY + Z^2)}[E^2 + \alpha E^4 - eA_0 \hat{E} - \alpha eA_0 \hat{E}^2] - \frac{1}{W(VY + Z^2)}[W_1^2 + \alpha W_1^4] - \frac{\hat{V}}{X(VY + Z^2)}[J^2 + \alpha J^4],$$

$$- \frac{m^2 \hat{V}}{(VY + Z^2)} - \frac{eA_0 (\hat{V} - A^2)}{(VY + Z^2)}[\hat{E} - \hat{E}^2 eA_0 + \hat{E}^3 \alpha],$$

where $J = \partial_\theta I_0$, $\nu_1 = \partial_\theta I_0$, $W_1 = \partial_r I_0$ and $\hat{E} = (E - j\omega)$. To get the non-trivial answer, we set the $\det(K) = 0$, so we obtain

$$ImW^\pm = \pm \int \sqrt{\frac{\hat{E} - A_0 e}{WY} \left[ 1 + \alpha \frac{X_1}{\hat{E} - A_0 e} \right]} dr,$$

$$= \pm i \pi \frac{(\hat{E} - A_0 e)}{2 \kappa (r_+)} \left[ 1 + \alpha \mathcal{Z} \right].$$

(22)
where \( \Xi = 6 \left( m^2 + \frac{(v_1^2 + J_0^2 \csc^2 \theta)}{r_+^2} \right) > 0 \) and

\[
X_1 = \frac{\tilde{W} \tilde{Z}}{(VY + Z^2)} [\tilde{E} - e A_0] \nu_1 + \frac{\tilde{V} \tilde{W}}{(VY + Z^2)} [\nu_1^2] - \tilde{W} m^2,
\]

\[
X_2 = \frac{\tilde{W} \tilde{Y}}{(VY + Z^2)} [\tilde{E}^4 - 2e A_0 \tilde{E}^3 + (e A_0)^2 \tilde{E}^2] + \frac{\tilde{W} \tilde{Z}}{X(Y + Z^2)} [\tilde{E}^3 - e A_0 \tilde{E}^2] \nu_1 - \frac{\tilde{V} \tilde{W}}{(VY + Z^2)} [\nu_1^2] - W_1^4.
\]

The particles tunneling from horizon is defined as

\[
\Gamma = \frac{\Gamma_{\text{emission}}}{\Gamma_{\text{absorption}}} = \exp \left[ -2\pi \frac{(\tilde{E} - e A_0 \nu_1)}{\kappa(r_+)} \right] \left[ 1 + \alpha \Xi \right].
\]

(23)

here

\[
\kappa(r_+) = \frac{3 m r_+^2 - 3 C_0 Q r_+ + 2 C_0 Q r_+^2 \beta + a^2 (3 C_0 Q \beta r_+ - 2 C_0 Q \beta - 3 m)}{3 \pi (r_+^2 + a^2)^2}.
\]

(24)

The \( T_H' \) of RENLMY BH is calculated by utilizing the Boltzmann formula \[28\] \( \Gamma_B = \exp \left[ (\tilde{E} - e A_0) / T_H' \right] \) in the form

\[
T_H' = \frac{3 m r_+^2 - 3 C_0 Q r_+ + 2 C_0 Q r_+^2 \beta + a^2 (3 C_0 Q \beta r_+ - 2 C_0 Q \beta - 3 m)}{6 \pi (r_+^2 + a^2)^2 \left[ 1 - \alpha \Xi \right]}.
\]

(25)

The \( T_H' \) of RENLMY BH depends upon gravity parameter \( \alpha \), BH charge \( Q \), particle mass \( m \), spin parameter \( a \).

It has worth to mention here that for \( \alpha = 0 \), we recover the temperature for Eq. (14), further when \( \alpha = 0 \), the \( T_H' \) reduced into ENLMY BH and when \( \alpha = 0 = Q \) the \( T_H' \) explicitely converts to the Schwarzschild BH temperature at \((r_+ = 2m)\) \[46\]. The temperature in terms of residual mass can be defined as

\[
T_H' = \frac{1}{8 \pi m} \left[ 1 - 6 \beta \left( m^2 + \frac{(v_1^2 + J_0^2 \csc^2 \theta)}{r_+^2} \right) \right],
\]

(26)

where \( \omega_e = \left( m^2 + \frac{(v_1^2 + J_0^2 \csc^2 \theta)}{r_+^2} \right) \) gives the component of kinetic energy for emitted particles. The gravity corrections reduce the increment in temperature throughout the radiation method. Due to the quantum corrections, the radiation stops at few particular temperature and a remnant mass left. A remnant is a last state of BH evaporation at a very small horizon. The increment in the temperature ceased whenever this condition satisfies \[47\]

\[
m \simeq (m - dm)(1 + \alpha \Xi),
\]

(27)

whereas \( \alpha = \frac{\omega_e}{M_p} \) and \( dm = \omega_e \) as well as \( \omega_e \simeq M_p \) here \( M_p \) represents the Planck’s mass and \( \alpha_0 \) stands for dimensionless parameter incorporating gravity effects with \( \alpha_0 < 10^5 \) \[48, 49\] and

\[
M_{Res} \simeq \frac{M_p^2}{\alpha_0 \omega_e} \geq \frac{M_p}{\alpha_0}, \quad T_{Res} \lesssim \frac{\alpha_0}{8 \pi M_p}.
\]

(28)

It is note worthy that the value of the corrected temperature \( T_H' \) is less than the original temperature (without corrections) and the radiation process ceased into BH, when the BH gets its minimum amount \( M_{Res} \). There are a couple of inspirations for BH remnants, one of them is that leftovers mass prevent BH from becoming so hot during the last phase of the evaporation. Chen and his colleagues have discussed in detail about the BH remnant in terms of information paradox as well as dark matter \[50, 51\].

V. GRAPHICAL ANALYSIS OF \( T_H' \) FOR RENLMY BH

This section describes the effects of different parameters on \( T_H' \) for RENLMY BH. We check the stability condition for RENLMY BH under quantum gravity effects by setting the values fixed for mass \( m = 1 \), arbitrary parameter \( \Xi = 1 \). In these plots, the Hawking’s strategy (the BH radius size reduces with the emission of high radiations) is clearly visible. We can observe in both plots when the \( T_H' \) is at its high value the horizon is very small. This condition states the physical form of BH and ensures its stability.
Figure 2: (i) depicts the conduct for $T_H'$ with horizon $r_+$ in the region $0 \leq r_+ \leq 15$ for fixed values of arbitrary constants $\beta = 0.9 = C_0$, spin parameter $a = 0.5$, BH charge $Q = 5$ and different values of $\alpha$. One can see that the BH gets its stable form after passage of some time and attains the asymptotically flat state up to $r_+ \rightarrow \infty$. The $T_H'$ increases as we increase the value of gravity parameter $\alpha$.

(ii) indicates the graphical significance for $T_H'$ with horizon $r_+$ various quantities of rotation parameter $a$ and constant values of arbitrary constants $\beta = 0.9 = C_0$, correction parameter $\alpha = 0.1$, BH charge $Q = 50$. One can observe, the $T_H'$ after getting its stable form with positive $T_H'$ becomes very high at very small horizon, at this stage the BH remnant left. Subsequently, attaining this maximum height the $T_H'$ eventually drops down and till $r_+ \rightarrow \infty$ it shows an asymptotically flat behavior. This behavior also states the stable form of BH. We can also check the effects of rotation parameter on $T_H'$, the $T_H'$ increases as we increase the values of rotation parameter.

VI. CORRECTED ENTROPY FOR RENLMY BH

Here, we evaluate the entropy corrections for RENLMY BH. Banerjee et al. [52-54] have investigated the temperature and entropy corrections by considering back-reaction effects via null geodesic phenomenon. We compute the entropy corrections for RENLMY BH in the background of Bekenstein-Hawking entropy technique for lowest order corrections. We calculate the logarithmic entropy corrections by utilizing the corrected temperature $T_H'$ and basic entropy $S_o$ for RENLMY BH. By utilizing the given formula, we can calculate the corrected entropy as follows

\[ S = S_o - \frac{1}{2} \ln \left| T_H'^2, S_o \right| + \ldots. \]  

(29)

The basic entropy for RENLMY BH can be derived from the given expression

\[ S_o = \frac{A_{r_+}}{4}, \]  

(30)

where

\[ A_{r_+} = \int_0^{2\pi} \int_0^{\pi} \sqrt{g_{\theta\theta} g_{\phi\phi}} d\theta d\phi, \]

\[ = 2\pi \left[ \left( r_+^2 + a^2 \right)^2 - a^2 \left( C_0 Q \left( 1 - \frac{4r_+}{3} + r_+^2 \beta^2 \right) + mr_+ \right) \right]. \]  

(31)

The basic entropy for RENLMY BH can be computed as

\[ S_o = \frac{\pi \left[ \left( r_+^2 + mr_+ + a^2 \right)^2 - a^2 \left( C_0 Q \left( 1 - \frac{4r_+}{3} + r_+^2 \beta^2 \right) \right) \right]}{2 \left( r_+^2 + a^2 \right)}. \]  

(32)
By putting the terms from Eq. (25) & (32) in the Eq. (29), we obtain the corrected entropy as

$$S = \frac{\pi \left[ (r_+^2 + a^2)^2 - a^2 \left( C_0 Q \left( 1 - \frac{4r_+\beta}{3} + r_+^2 \beta^2 \right) + mr_+ \right) \right]}{2(r_+^2 + a^2)} - \frac{1}{2} \ln \left[ \frac{\left( 3mr_+^2 - 3C_0 Q r_+ + 2C_0 Q r_+^2 \beta + a^2 (3C_0 Q \beta r_+ - 2C_0 Q \beta - 3m) \right) (1 - \alpha \Xi) \zeta}{72\pi (r_+^2 + a^2)^5} \right] + ... \quad (33)$$

where

$$\zeta = \left[ (r_+^2 + a^2)^2 - a^2 \left( C_0 Q \left( 1 - \frac{4r_+\beta}{3} + r_+^2 \beta^2 \right) + mr_+ \right) \right].$$

The Eq. (33) shows the entropy corrections for RENLMY BH. It depends upon charge $Q$, rotation parameter $a$, integration constant $C_0$, arbitrary parameters $\beta, \Xi$ and correction parameter $\alpha$.

**Figure 3**: (i) gives the conduct of $S$ for constant $\beta = 0.9 = C_0$, $\alpha = 0.1$, $Q = 5$ and different variations of gravity parameter $\alpha$ in the region $0 \leq r_+ \leq 5$. The $S$ exponentially decreases with the increasing horizon upto $r_+ \to \infty$. This kind of conduct states the stability of BH and the entropy increases within increase in $a$.

(ii) depicts the interpretation of entropy versus $r_+$ for different variations of charge $Q$ and fixed $\beta = 0.5 = \alpha$, $C_0 = 0.1, a = 2$. At first, the entropy slowly decreases and after getting a minima, it again starts to increase upto $r_+ \to \infty$. The entropy decreases for increasing values of charge $Q$.

**Figure 4**: (i) represents the graphical conduct of $S$ versus horizon for $Q = 0.5, m = 50, a = 15, C_0 = 0.1$ and for various values of $\alpha$ in the range $0 \leq r_+ \leq 15$. It is observable that the entropy slowly decreases for the increasing horizon and after some time it again starts to increase till $r_+ \to \infty$. This condition with positive entropy indicates the stability of BH. The entropy decreases for increasing values of correction parameter $\alpha$.

(ii) indicates the role of entropy for $m = 1, Q = 0.1, a = 2, C_0 = 0.5$ and for different variations of $\alpha$ in the domain
0 \leq r_+ \leq 5. Initially, the entropy decreases and then after attaining a minima it again starts to increase till \( r_+ \to \infty \). The initial behavior of entropy looks like a half cardioid.

It can be observe from both plots, the entropy is more stable in the large region of horizon \( 0 \leq r_+ \leq 15 \) as compared to small radii \( 0 \leq r_+ \leq 5 \) of BH for greater mass \( m = 50 \).

VII. SUMMARY AND DISCUSSION

This paper analyzed the RENLMY BH solution by applying the Janis-Newman algorithmic rule and complex calculations. Through the complex computation, we explored the Yukawa BH. If \( a \to 0 \), we obtained the solution of Yukawa BH without Newman-Janis algorithm. We also investigated the physical properties of BH (i.e., \( T_H \)) and analyzed the physical state of RENLMY BH via graphical interpretation of \( T_H \) with horizon. The \( T_H \) depends on the BH charge \( Q \), particle mass \( m \) and spin parameter \( a \). We have checked the spin parameter effects on \( T_H \). The \( T_H \) decreases with the increasing spin parameter values. Moreover, we have investigated the \( T_H' \) for RENLMY BH with the help of vector particles tunneling strategy by utilizing the Hamilton Jacobi strategy. For this investigation, we have utilized the wave equation of motion with the setting of quantum gravity parameter to study the vector tunneling of boson particles from RENLMY BH. In the wave equation, we have applied the WKB approximation that gives a set of field equations, and after this, by using the separation of variables technique, we have computed the field equations. The imagination can be found by using the coefficients of matrix, whose matrix determinant is equal to zero. We have formulated the tunneling probability and \( T_H' \) for the corresponding BH at the outer horizon by applying surface gravity. The \( T_H' \) of RENLMY BH depends upon gravity parameter \( \alpha \), BH charge \( Q \), particle mass \( m \), spin parameter \( a \). It has worth to point out that the both back-reaction and self-gravitating effects of particles on this Yukawa BH have been ignored, the computed \( T_H' \) are the parameters of Yukawa BH in Newman-Janis algorithm and quantum gravity. Furthermore, we have discussed the graphical interpretation of \( T_H' \) for RENLMY BH. We have analyzed the quantum gravity effects and spin parameter on \( T_H' \). We have concluded that the \( T_H' \) increases for the increasing values of both correction and rotation parameter. The \( T_H' \) at largest value with non-zero horizon depicts BH leftover mass. After largest values of the \( T_H' \) eventually drops down and obtain an asymptotically flat state till \( r_+ \to \infty \), that checks the stable BH condition. The graphical interpretation of \( T_H' \) with/without gravity parameter satisfy the Hawking’s strategy (the BH radius size reduces with the emission of high radiations). This behavior can be observed in both (Fig. 1 and 2). If \( \alpha = 0 \) in Eq. (25), we obtain the \( T_H' \) of the Eq. (14).

Furthermore, we have computed the logarithmic entropy corrections by considering the corrected temperature \( T_H' \) and basic entropy \( S_B \) for RENLMY BH and also checked the effects of charge, rotation and gravity parameters on entropy. The entropy increases with the increasing values of rotation parameter and it decreases with the increasing values of charge and gravity parameter. Moreover, we have concluded that the entropy is more stable in the large region of horizon \( 0 \leq r_+ \leq 15 \) as compared to small radii \( 0 \leq r_+ \leq 5 \) of BH for greater mass \( m = 50 \).

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