An integral generalization of the Gusein-Zade–Natanzon theorem

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One of the useful methods of the singularity theory, the method of real morsifications [AC^0, GZ] (see also [AGV]) reduces the study of discrete topological invariants of a critical point of a holomorphic function in two variables to the study of some real plane curves immersed into a disk with only simple double points of self-intersection. For the closed real immersed plane curves V.I.Arnold [Ar] found three simplest first order invariants \( J^\pm \) and \( St \). Arnold’s theory can be easily adapted to the curves immersed into a disk. In [GZN] S.M.Gusein-Zade and S.M.Natanzon proved that the Arf invariant of a singularity is equal to \( J^-/2 \mod 2 \) of the corresponding immersed curve. They used the definition of the Arf invariant in terms of the Milnor lattice and the intersection form of the singularity. However it is equal to the Arf invariant of the link of singularity, the intersection of the singular complex curve with a small sphere in \( \mathbb{C}^2 \) centered at the singular point. In knot theory it is well known (see, for example [Ka]) that Arf invariant is the \( \mod 2 \) reduction of some integer-valued invariant, the second coefficient of the Conway polynomial, or the Casson invariant [PV].

Arnold’s \( J^-/2 \) is also an integer-valued invariant. So one might expect a relation between these integral invariants.

A few years ago N.A’Campo [AC^1, AC^2] invented a construction of a link from a real curve immersed into a disk. In the case of the curve originating from the real morsification method the link is isotopic to the link of the corresponding singularity. But there are some curves which do not occur in the singularity theory. In this article we describe the Casson invariant of A’Campo’s knots as a \( J^\pm \)-type invariant of the immersed curves. Thus we get an integral generalization of the Gusein-Zade–Natanzon theorem. It turns out that this \( J^\pm_2 \) invariant is a second order invariant of the mixed \( J^+ \)- and \( J^- \)-types. To the best of my knowledge, so far nobody tried to study the mixed \( J^\pm \)-type invariants. It seems that our invariant is one of the simplest such invariants. The problem of describing all second order \( J^\pm \)-type invariants is open.

In section 1 we review the A’Campo construction and list the properties of the links obtained. In sections 2 and 3 we introduce the Casson invariant and Arnold’s type invariants of curves immersed into a disk. In section 4 we formulate our main result. The proof is based on M.Hirasawa’s construction [Hir] of a Seifert surface for the A’Campo links. We review this construction in section 5.
1 A’Campo divides and their links.

1.1. Definition ([AC1], [AC2]). A divide $D$ is the image of a generic immersion of a finite number of copies of the unit interval $I=[0,1]$ in the unit disk $B \subset \mathbb{R}^2$ such that $\partial I \subset \partial B$. Here the word “generic” means that double points are the only singularities allowed and $D$ is transversal to $\partial B$.

Similar object were known as long curves [Ta, GZN].

We consider divides up to isotopy of the disk $B$. The isotopy is not assumed to be identical on the boundary $\partial B$.

1.2. Example. The curve $x^3 + y^4 = 0$ has a singularity of type $E_6$ at the origin [AGV]. A small perturbation of it is a divide which looks as follows.

![Diagram]

1.3. Definition ([AC1], [AC2]). Let $x$ be the horizontal coordinate on the disk $B$ and $y$ be the vertical coordinate. A divide link $L_D$ is a link in the 3-sphere $S^3 = \{(x, y, u, v) \in \mathbb{R}^4 | x^2 + y^2 + u^2 + v^2 = 1 \}$ such that $(x, y)$ is a point on $D$ and $u$, $v$ are the coordinates of a tangent vector to $D$ at the point $(x, y)$.

So each interior point of $D$ has two corresponding points on $L_D$, whereas a boundary point of $D$ gives a single point on $L_D$.

The number of components of $L_D$ equals to the number of branches of the divide $D$ which is the number of the copies of the unit interval $I$ in definition 1.1. In particular, if $D$ consists of only one branch (like in 1.2) then $L_D$ will be a knot.

1.4. Properties of $L_D$.

1.4.1. Topological type of a divide link does not change under a regular transversal isotopy of the disk $B$. So it does not depend on the choice of coordinates in 1.3. Also it does not change under a moving of a piece of the curve $D$ through a triple point [CP]. In particular, the following two divides have the same knot type as in 1.2.

![Diagram]

1.4.2. The link $L_D$ has a natural orientation. Indeed, choose any orientation of every branch of $D$. Let $(u, v)$ be the tangent vector to $D$ at $(x, y)$ pointing to the direction of the chosen orientation of $D$. Then the orientation of $L_D$ is given by the vector $(\dot{x}, \dot{y}, \dot{u}, \dot{v})$. It
is easy to see that this orientation of $L_D$ does not depend on the choice of orientations of branches of $D$.

1.4.3. In [AC1] A’Campo proved that any singularity link is a divide link. But the divide in the picture on the right does not occur in singularity theory. Indeed, by the real morsification method one can find that the Milnor number of the corresponding singularity would be 4. However there are only two singularities with Milnor number 4, $A_4$ and $D_4$. But their intersection forms are different from the one which corresponds to this divide. The knot arising from divide is $10_{145}$ (see [AC2], Ch).

It is known for a long time [Bu] that all singularity knots are classified by the Alexander polynomial. N.A’Campo ([AC3]) found two different divide knots with the same Alexander polynomial. H.Morton [Mor], whom I have shown A’Campo’s example, found that these knots are mutant. So they cannot be distinguished by any classical polynomial invariants (Jones, HOMFLY, Kauffman). He distinguished them by a quantum invariant coming from the Lie algebra $gl_n$ in a certain higher (non standard) representation.

1.4.3. A’Campo [AC2] showed that the links $L_D$ corresponding to a connected divide $D$ are fibered and computed their monodromy in terms of combinatorics of the divide $D$. Not all fibered links have the form $L_D$. Figure eight knot $4_1$ is not a divide knot. It is not clear how large is the class of divide links in the class of all fibered links.

1.4.4. If we consider $\mathbb{R}^4$ with coordinates $x, y, u, v$ as a complex plane $\mathbb{C}^2$ with coordinates $z_1 = x + iu$ and $z_2 = y + iv$, then every tangent space to the unit sphere $S^3$ contains a unique complex line which is a two-dimensional real subspace in the tangent space. The distribution of these 2-planes forms the standard contact structure on $S^3$. Divide knots are transversal knots with respect to this contact structure. This fact was noticed in [AC2] but the arguments given there should be modified. For a one-branch divide $D$ the Bennequin number (self-linking number) of the knot $L_D$ is equal to $2\delta - 1$, where $\delta$ is the number of double points of $D$.

1.4.5. N.A’Campo [AC2] proved that the unknotting number of a one-branch divide knot $L_D$ and the genus of $L_D$ are equal to the number $\delta$ of double points of $D$. For a two-branch divide $D$ the link $L_D$ will have two components. Their linking number is equal to the number of common double points of the two branches of $D$.

2 Casson’s invariant of knots.

2.1. The Conway polynomial. The subject of this section is well known (see, for example [Ka, PV]). The Conway polynomial $C(\mathcal{L})$ of a link $\mathcal{L}$ a polynomial in a single variable $z$. It is one of the simplest invariants of links in $\mathbb{R}^3$, defined by the two properties: it is equal to one on the unknot and satisfies the skein relation

$$C(\ < >) - C(\ < >) = z \cdot C(\ < >),$$

3
where the three links are identical outside a small ball in $\mathbb{R}^3$ and look as shown inside the ball.

If $L$ is a knot then the Conway polynomial is even:

$$C(L) = 1 + C_2(L) \cdot z^2 + C_4(L) \cdot z^4 + \ldots + C_{2n}(L) \cdot z^{2n}.$$  

### 2.2. Casson’s invariant.

The coefficient $C_2(L)$ is called Casson’s knot invariant (see [PV]). It can be defined ([Ka]) by the initial condition $C_2(\text{unknot}) = 0$ and the skein relation:

$$C_2(L) - C_2(\text{unknot}) = lk(L),$$

where $lk$ means the linking number of the two component link in the right hand side of the relation.

The mod 2 reduction of Casson’s invariant $C_2(L)$ is called Arf invariant of a knot $L$.

### 3 Arnold’s invariants of immersed plane curves.

#### 3.1. Arnold’s invariants of divides.

In [Ar] V.I.Arnold defined three basic invariants $J^+, J^-, St$ of a generic closed immersed plane curve. See a survey of various explicit formulas for these invariants in [CD]. Following [GZN] we can define these invariants for a one-branch divide $D$ as the arithmetic mean of the corresponding Arnold’s invariants on the two closed curves obtained by smoothing the union of $D$ with either arc of the unit circle. Invariants of divides thus defined will be denoted by the same symbols.

#### 3.2. Example.

Let us compute $J^-$ on the divide of the example 1.5.2.

$$J^-(\text{example 1.5.2}) = \frac{J^-(\text{example 1.5.2}) + J^-(\text{example 1.5.2})}{2}.$$  

According to Arnold’s table [Ar, page 14], the first term of the right hand side equals $-4$ and the second term equals $-8$. So the result is $(-4 - 8)/2 = -6$.

#### 3.3. $J^\pm$ as invariants of order 1.

We are going to use a description of Arnold’s $J^\pm$ invariants as invariants of order 1 in the sense of the theory of finite type invariants.

Let us fix the images of the ends points of the interval $I$ at the boundary of the disk $\partial B$. A version of the classical theorem of H. Whitney states that the space of all smooth immersions of the interval $I$ in the disk $B$ mapping $\partial I$ into two fixed points is made up of a countable number of connected components differing by the absolute value of the rotation number.

Choose a standard divide $D_i$ for each nonnegative value of the rotation number $i$:

$$D_0 \quad D_1 \quad D_2 \quad D_3 \quad D_4 \quad D_5 \quad \ldots$$

So every one-branch divide can be deformed to one of $D_i$’s in the space of immersions.
During the deformation at certain moments non-divides may occur because of non-allowed singularities of the corresponding curves. For a generic deformation two types of such non-divides can occur: either a *triple point* on the curve or a *self-tangency* of the curve. In fact the self-tangency event can be split in two types: direct self-tangency, where the two tangent strings have coherent orientations (for any of the two possible orientations of the curve), and inverse self-tangency, where the two tangent strings have opposite orientations. A first order invariant is defined by its jumps on the events of the three types above, and its values on the standard divides

In particular, $J^-$ and $J^+$ are defined by the following relations:

\begin{align*}
(i)^- \quad & J^- \left( \begin{array}{c} \includegraphics[width=0.1\textwidth]{deformation1} \end{array} \right) - J^- \left( \begin{array}{c} \includegraphics[width=0.1\textwidth]{deformation2} \end{array} \right) = 2 ; \\
(ii)^- \quad & J^- \left( D_i \right) = -2i ; \\
(i)^+ \quad & J^+ \left( \begin{array}{c} \includegraphics[width=0.1\textwidth]{deformation1} \end{array} \right) = J^+ \left( \begin{array}{c} \includegraphics[width=0.1\textwidth]{deformation2} \end{array} \right) ; \\
(ii)^+ \quad & J^+ \left( D_i \right) = -i ; \\
(i)^* \quad & J^- \left( \begin{array}{c} \includegraphics[width=0.1\textwidth]{deformation1} \end{array} \right) = J^- \left( \begin{array}{c} \includegraphics[width=0.1\textwidth]{deformation2} \end{array} \right) ; \\
(ii)^* \quad & J^+ \left( \begin{array}{c} \includegraphics[width=0.1\textwidth]{deformation1} \end{array} \right) = J^+ \left( \begin{array}{c} \includegraphics[width=0.1\textwidth]{deformation2} \end{array} \right) ; \\
\end{align*}

Here in each equality we mean two divides identical outside a small fragments which are explicitly shown.

3.4. **Actuality tables for $J^\pm$.** According to V. Vassiliev [Va], actuality tables provide a way to organize the data necessary for the computation of a single finite order invariant.

For the first order invariants $J^\pm$ the actuality tables are essentially the same thing as the set of equations (i)\(^*\)–(ii). To describe the top row of the actuality table we use chord diagrams. Since in the $J^\pm$ theory we have two types of singular events, the inverse and direct self-tangencies, we need two types of chords. We use a dashed chord to depict the inverse self-tangency, and a solid chord to depict the direct self-tangency.

The actuality tables for $J^\pm$ look as follows

\begin{align*}
(i)^- \quad & J^- \left( \begin{array}{c} \includegraphics[width=0.1\textwidth]{deformation1} \end{array} \right) - J^- \left( \begin{array}{c} \includegraphics[width=0.1\textwidth]{deformation2} \end{array} \right) = 2 ; \\
(ii)^- \quad & J^- \left( D_i \right) = -2i ; \\
(i)^+ \quad & J^+ \left( \begin{array}{c} \includegraphics[width=0.1\textwidth]{deformation1} \end{array} \right) = J^+ \left( \begin{array}{c} \includegraphics[width=0.1\textwidth]{deformation2} \end{array} \right) ; \\
(ii)^+ \quad & J^+ \left( D_i \right) = -i ; \\
(i)^* \quad & J^- \left( \begin{array}{c} \includegraphics[width=0.1\textwidth]{deformation1} \end{array} \right) = J^- \left( \begin{array}{c} \includegraphics[width=0.1\textwidth]{deformation2} \end{array} \right) ; \\
(ii)^* \quad & J^+ \left( \begin{array}{c} \includegraphics[width=0.1\textwidth]{deformation1} \end{array} \right) = J^+ \left( \begin{array}{c} \includegraphics[width=0.1\textwidth]{deformation2} \end{array} \right) ; \\
\end{align*}

3.5. **Example.** Let us compute $J^-$ again for the example 1.5.2 but using now the relations (i)–(iv) only. We introduce the orientation as shown in order to distinguish between direct (i)\(^+\) and inverse (i)\(^-\) self-tangencies.

\begin{align*}
J^- \left( \begin{array}{c} \includegraphics[width=0.1\textwidth]{deformation1} \end{array} \right) & = J^- \left( \begin{array}{c} \includegraphics[width=0.1\textwidth]{deformation2} \end{array} \right) + J^- \left( \begin{array}{c} \includegraphics[width=0.1\textwidth]{deformation1} \end{array} \right) = J^- \left( \begin{array}{c} \includegraphics[width=0.1\textwidth]{deformation2} \end{array} \right) = J^- \left( \begin{array}{c} \includegraphics[width=0.1\textwidth]{deformation1} \end{array} \right) \\
& = -J^- \left( \begin{array}{c} \includegraphics[width=0.1\textwidth]{deformation1} \end{array} \right) + J^- \left( \begin{array}{c} \includegraphics[width=0.1\textwidth]{deformation1} \end{array} \right) = -2 + J^- \left( \begin{array}{c} \includegraphics[width=0.1\textwidth]{deformation1} \end{array} \right) \\
& = -2 + J^- \left( D_2 \right) \quad \text{(ii)} \\
& = -2 - 2 \cdot 2 = -6 .
\end{align*}
4 Main result

4.1. Theorem. For a one-branch divide $D$ the Casson invariant $C_2(\mathcal{L}_D)$ is equal to the invariant $J_2^\pm(D)$ defined below in 4.2.

Corollary (Gusein-Zade–Natanzon [GZN]). For a one-branch divide $D$, $\text{Arf}(\mathcal{L}_D) = J_2^-(D)/2(\mod 2)$.

4.2. Definition of the invariant $J_2^\pm$.

The invariant $J_2^\pm$ is a second order $J^\pm$-type invariant. In particular, it does not change under the triple point move. To define it we need to define its values on the chord diagrams with two chords and also to define its values on canonical divides with at most one self-tangency point. We have chosen the canonical divides without self-tangencies in 3.3. Now we must choose the canonical divides with one self-tangency in such a way that any divide with a single self-tangency can be deformed to the canonical one if we allow to pass through the codimension two strata in the space of immersions.

4.2.1. Canonical divides with a self-tangency point. Our choice of the canonical divides is:

$$D_{m,n}^- = \begin{array}{c}
\includegraphics[width=0.5\textwidth]{canonical_divide_left.png} \\
\text{n curls} \\
m \text{curls}
\end{array}$$

$$D_{m,n}^+ = \begin{array}{c}
\includegraphics[width=0.5\textwidth]{canonical_divide_right.png} \\
\text{n curls} \\
m \text{curls}
\end{array}$$

Here we omit the boundary circle (early depicted as a dashed circle) of the unit disk containing the divide. The parameters $m$ and $n$ in these divides run over all integral numbers. Under a negative curl we mean the curl which is going clockwise (instead of counterclockwise as above). Here is a couple of examples.

$$D_{-3,2}^- = \begin{array}{c}
\includegraphics[width=0.2\textwidth]{example_divide_left.png} \\
\end{array}$$

$$D_{2,-3}^+ = \begin{array}{c}
\includegraphics[width=0.2\textwidth]{example_divide_right.png} \\
\end{array}$$

4.2.2. Actuality table for $J_2^\pm$. The actuality table for $J_2^\pm$ looks as follows.

| $J_2^\pm(D_{m,n}^-)$ | $J_2^\pm(D_{m,n}^+)$ | $J_2^\pm(D_n^\pm)$ |
|----------------------|----------------------|----------------------|
| $\begin{array}{c}
\includegraphics[width=0.2\textwidth]{example.png} \\
\end{array}$ | $\begin{array}{c}
\includegraphics[width=0.2\textwidth]{example.png} \\
\end{array}$ | $\begin{array}{c}
\includegraphics[width=0.2\textwidth]{example.png} \\
\end{array}$ |
| $-6m - 3$, for $m \geq 0$ | $2m - 3$, for $m < 0$ | $-6m - 4$, for $m \geq 0$ |
| $2m - 3$, for $m < 0$ | $2m - 4$, for $m < 0$ | $n$ |
4.2.3. Comparing this actuality table with the one of 3.4 one can notice that the mod 2 reduction of $J_2^\pm$ is an invariant of the first order and $J_2^\pm = J^\pm / 2(mod 2)$. This proves the Corollary from 4.1.

4.2.4. It would be interesting to find a Polyak–Viro style formula (see [PV, CD]) for the invariant $J_2^\pm$ in terms of Gauss diagrams of the curve.

4.2.5. Example. Let us compute $J_2^\pm$ for the divide from 1.5.2 like it was done in 3.5. The first step is pretty much the same

\[ J_2^\pm (\begin{array}{c} \circ \circ \\ \circ \circ \end{array}) = J_2^\pm (\begin{array}{c} \circ \circ \\ \circ \circ \end{array}) + J_2^\pm (\begin{array}{c} \circ \circ \\ \circ \circ \end{array}) \]

Now we compute separately the two terms of the right-hand side.

\[ J_2^\pm (\begin{array}{c} \circ \circ \\ \circ \circ \end{array}) = J_2^\pm (\begin{array}{c} \circ \circ \\ \circ \circ \end{array}) = J_2^\pm (\begin{array}{c} \circ \circ \\ \circ \circ \end{array}) + J_2^\pm (\begin{array}{c} \circ \circ \\ \circ \circ \end{array}) \]

\[ = J_2^\pm (\begin{array}{c} \circ \circ \\ \circ \circ \end{array}) + J_2^\pm (\begin{array}{c} \circ \circ \\ \circ \circ \end{array}) = 2 - J_2^\pm (\begin{array}{c} \circ \circ \\ \circ \circ \end{array}) + J_2^\pm (\begin{array}{c} \circ \circ \\ \circ \circ \end{array}) \]

\[ = 2 - J_2^\pm (\begin{array}{c} \circ \circ \\ \circ \circ \end{array}) + J_2^\pm (\begin{array}{c} \circ \circ \\ \circ \circ \end{array}) = -4 \]

For the second term we have

\[ J_2^\pm (\begin{array}{c} \circ \circ \\ \circ \circ \end{array}) = J_2^\pm (\begin{array}{c} \circ \circ \\ \circ \circ \end{array}) = -J_2^\pm (\begin{array}{c} \circ \circ \\ \circ \circ \end{array}) + J_2^\pm (\begin{array}{c} \circ \circ \\ \circ \circ \end{array}) \]

\[ = -J_2^\pm (\begin{array}{c} \circ \circ \\ \circ \circ \end{array}) + J_2^\pm (\begin{array}{c} \circ \circ \\ \circ \circ \end{array}) = J_2^\pm (D^\pm) = 9 \]

Totally, summing up, we get $J_2^\pm (\begin{array}{c} \circ \circ \\ \circ \circ \end{array}) = 5$.

4.2.6. Idea of a proof. In the next section we describe how to draw a diagram of $L_D$ from the picture of the divide $D$. This allows us to trace what happens with the knot $L_D$ when the curve $D$ changes by a direct (inverse) self-tangency move. Applying the skein relation from 2.2 gives us the corresponding changings of the Casson invariant. All this information can be summarized in the actuality table of 4.2.2.

5 Hirasawa’s Seifert surface for $L_D$.

5.1. M.Hirasawa [Hir] suggested a procedure to draw a picture of the minimal genus Seifert surface for the link $L_D$ and so to draw a diagrams of the link $L_D$. The other ways to draw diagrams see in [CP, Ch]. In this section we describe the Hirasawa’s construction.

First let us prepare the divide $D$ for drawing the Seifert surface as follows. Choose an orientation of all branches of $D$. Then deform $D$ inside the unit disk so that:
• at every double point both branches are oriented from left to right;
• \(x\) coordinates of double points and points of \(D\) with vertical tangent are pairwise different.

Now to draw the Seifert surface we thicken every arc of \(D\) to form a band and then modify the bands near double points and near those points with vertical tangent where \(D\) is oriented down as shown on the pictures below.

\[\text{all arcs of } D \text{ located below the point with vertical tangent line}\]
\[\Rightarrow\]
\[\text{the bands corresponding to the arcs of } D \text{ below the point with vertical tangent line}\]

\[\text{all arcs of } D \text{ located below the point with vertical tangent line}\]
\[\Rightarrow\]
\[\text{the bands corresponding to the arcs of } D \text{ below the point with vertical tangent line}\]

5.2. Example. For the divide \(D\) with a single curl (corresponding to the singularity \(A_2\)) this construction gives the following Seifert surface.

\[\text{all arcs of } D \text{ located below the point with vertical tangent line}\]
\[\Rightarrow\]
\[\text{the bands corresponding to the arcs of } D \text{ below the point with vertical tangent line}\]
The corresponding knot $L_D$ is the trefoil.

\[ \includegraphics{trefoil.png} = 3_1 \]

References

[AC0] N. A’Campo, *Le groupe de monodromie du déploiement des singularités isolées de courbes planes* I, Math. Ann., 213 (1975) 1–32.

[AC1] N. A’Campo, *Real deformations and complex topology of plane curve singularities*, Ann. Fac. Sci. Toulouse Math. (6) 8 (1999), no. 1, 5–23 (see also [alg-geom/9710023](http://arxiv.org/abs/alg-geom/9710023)).

[AC2] N. A’Campo, *Generic immersions of curves, knots, monodromy and gordian number*, Inst. Hautes Etudes Sci. Publ. Math. No. 88 (1998), 151–169 (1999) (see also [math.GT/9803081](http://arxiv.org/abs/math.GT/9803081)).

[AC3] N. A’Campo, *Private communication*, January, 2001.

[Ar] V. I. Arnold, *Topological invariants of plane curves and caustics*, University Lecture Series, Vol. 5, AMS, Providence, RI, 1994.

[AGV] V. I. Arnold, S. M. Gusein-Zade, A. N. Varchenko, *Singularities of differentiable maps*, Vol. II, Birkhäuser, Boston, MA, 1988.

[Bu] W. Burau, *Kennzeichnung der Schlauchknoten*, Abh. Math. Sem. Univ. Hamburg 9 (1932) 125–133.

[CD] S. Chmutov, S. Duzhin, *Explicit formulas for Arnold’s generic curve invariants*, Arnold–Gelfand Mathematical Seminars, Birkhäuser, Boston, MA, (1997) 125-138.

[Ch] S. Chmutov, *Diagrams of divide links*, To appear in Proc. AMS (see also [math.GT/0205329](http://arxiv.org/abs/math.GT/0205329)).

[CP] O. Couture, B. Perron, *Representative braids for links associated to plane immersed curves*, Journal of Knot Theory and Its Ramifications, 9 (2000) 1–30.

[GZ] S. M. Gusein-Zade, *Intersection matrices for certain singularities of functions of two variables*, Functional Anal. and its Appl., 8 (1974) 10–13.

[GZN] S. M. Gusein-Zade, S. M. Natanzon, *The Arf-invariant and the Arnold invariants of plane curves*, The Arnold-Gelfand mathematical seminars, Birkhäuser, Boston, MA, (1997) 267–280.

[Hir] M. Hirasawa, *Visualization of A’Campo’s fibered links and unknotting operations*, to appear in Topology and its Applications.
[Ka] Louis H. Kauffman, *On knots*, Annals of Math. Studies 115, Princeton University Press (1987).

[Mor] H. Morton, *Private communication*, November, 2001.

[PV] M. Polyak, O. Viro, *On the Casson Knot Invariant*, preprint math.GT/9903158 to appear in Journal of Knot Theory and Its Ramifications.

[Ta] S. Tabachnikov, *Invariants of smooth triple point free plane curves*, Journal of Knot Theory and Its Ramifications, 5 (1996) 531–552.

[Va] V. A. Vassiliev, *Cohomology of knot spaces*, Theory of Singularities and Its Applications (ed. V. I. Arnold), Advances in Soviet Math., 1 (1990) 23–69.

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