Spin Gauge Theory of Gravity in Clifford Space

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Abstract.
A theory in which 16-dimensional curved Clifford space (C-space) provides a realization of Kaluza-Klein theory is investigated. No extra dimensions of spacetime are needed: “extra dimensions” are in C-space. We explore the spin gauge theory in C-space and show that the generalized spin connection contains the usual 4-dimensional gravity and Yang-Mills fields of the U(1)×SU(2)×SU(3) gauge group. The representation space for the latter group is provided by 16-component generalized spinors composed of four usual 4-component spinors, defined geometrically as the members of four independent minimal left ideals of Clifford algebra.

1. Introduction
Spacetime geometry can be elegantly described by means of geometric calculus in which basis vectors are generators of Clifford algebra [1]. This enables a description of geometric objects of different grades, i.e., multivectors or r-vectors associated with oriented r-dimensional surfaces. In previous works it has been shown [2, 3] that multivectors can sample extended objects, e.g., closed branes, and that in this respect they generalize the notion of center of mass. Instead of describing an extended object by an infinite number of degrees of freedom, which is one extreme, or only by the center of mass coordinates, which is another extreme, one can describe it by a finite number of multivector coordinates which take account of object’s extension and orientation. Such description works for macroscopic objects as well, if we assume that they are composed of branes, which is a reasonable assumption within the framework of a theory based on strings and branes. Then the multivectors associated with the constituent branes sum together to give an effective multivector describing the macroscopic object [2].

The basis multivectors span a 2^n-dimensional space, called Clifford space or C-space, n being the dimension of the underlying space that we start from. We will assume that the starting space is 4-dimensional spacetime. A point in C-space is described by a set of multivector coordinates (σ, x^μ, x^μν, ...) which altogether with the corresponding basis elements (1, γ_μ, γ_μν, ...) form a Clifford aggregate or polyvector X. It is well known that the elements of the right or left minimal ideals of Clifford algebra can be used to represent spinors. Therefore, a coordinate polyvector X automatically contains not only bosonic, but also spinor coordinates. In refs. [4, 5] it was proposed to formulate string theory in terms of polyvectors, and thus avoid usage of a higher dimensional spacetime. Spacetime can be 4-dimensional, whilst the extra degrees of freedom (“extra dimensions”) necessary for consistency of string theory are in C-space.
There is a fascinating possibility of a generalization from flat C-space, serving as an arena for physics, to curved C-space which is itself a part of the play [4, 6, 7, 8]. Since a dynamical (curved) Clifford space has 16 dimensions, it provides a realization of Kaluza-Klein idea. We do not need to assume that spacetime has more than four dimensions. The “extra dimensions” are in Clifford space, and they are all physical, because they are associated with the degrees of freedom related to extended objects (see also [2]). So we do not need to “compactify” or in whatever way to hide them.

2. Clifford space
In a series of preceding works [9]–[13] it has been proposed to construct the extended relativity theory in C-space by a natural generalization of the notion of spacetime interval:

$$dS^2 = d\sigma^2 + dx_\mu dx^\mu + dx_{\mu_1\mu_2} dx^{\mu_1\mu_2} + ... + dx_{\mu_1...\mu_n} dx^{\mu_1...\mu_n}$$

where $\mu_1 < \mu_2 < ... < \mu_n$. The Clifford valued polyvector

$$X = x^M \gamma_M = \sigma 1 + x^\mu \gamma_\mu + x^{\mu_1\mu_2} \gamma_{\mu_1\mu_2} + ... + x^{\mu_1...\mu_n} \gamma_{\mu_1...\mu_n}$$

denotes the position of a point in a manifold, called Clifford space or C-space. The series of terms in eq. (2) terminates at finite grade depending on the dimension $n$ of spacetime that we start from. A Clifford algebra $C_{r,q}$ with $r+q = n$ has $2^n$ basis elements. Here we keep $n$ and the signature arbitrary, but later we will take $n = 4$, and signature (+ − − −).

In flat C-space one can choose a basis so that the relation with wedge product

$$\gamma_M = \gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge ... \wedge \gamma_{\mu_r} = \frac{1}{r!} [\gamma_{\mu_1}, \gamma_{\mu_2}, ..., \gamma_{\mu_r}]$$

$\gamma_{\mu_1}, \mu = 1, 2, ..., n$, being the generators of Clifford algebra, holds globally for all points $X$ of C-space.

The interval (1) can be written as the scalar product

$$dS^2 = dX^\dagger \ast dX = dx^M dx^N G_{MN} = dx^M dx_M$$

of $dX = dx^M \gamma_M$ with its reverse $dX^\dagger$. Reversion is an operation which reverses the order of vectors, e.g., $(\gamma_1 \gamma_2 \gamma_3)^\dagger = \gamma_3 \gamma_2 \gamma_1$. The scalar product between two polyvectors $A$ and $B$ is defined according to $A \ast B = \langle AB \rangle_0$, where $\langle \rangle_0$ denotes the zero grade part.

The metric, entering the quadratic form (4) is given by

$$G_{MN} = \gamma_M^\ast \gamma_N$$

If the underlying spacetime $V_n$ has signature (+ − − − − − −), then the Clifford space has signature (+ + + ... − − − − − −) with equal number of plus and minus signs. This has some potentially far reaching consequences, discussed in refs. [4, 5].

As in the ordinary theory of relativity we generalize flat (Minkowski) spacetime to curved spacetime, so we now generalize flat C-space to curved C-space. We can use analogous concepts and techniques.

A set of $2^n$ linearly independent polyvector fields on a region $\mathcal{R}$ of C-space will be called a frame field. Of particular interest are:

1 If we do not restrict indices according to $\mu_1 < \mu_2 < ...$, then the factors $1/2!$, $1/3!$, ..., respectively, have to be included in front of every term in eq. (2).
(i) **Coordinate frame field** \{\gamma_M\}. Basis elements \(\gamma_\mu\), \(M = 1, 2, ..., 2^n\) depend on position \(X\) in \(C\)-space. The relation (3) with wedge product can hold only locally at a chosen point \(X\), but in general it cannot be preserved globally at all points \(X \in \mathcal{R}\) of curved \(C\)-space. The scalar product of two basis elements determines the metric tensor of the frame field \{\gamma_M\} according to eq. (5).

(ii) **Local flat frame field** \{\gamma_A\}. Basis elements \(\gamma_A\), \(A = 1, 2, ..., 2^n\) also depend on \(X\), but at every point \(X\) they determine flat metric

\[
\gamma_A^\dagger \ast \gamma_B = \eta_{AB} \tag{6}
\]

The relation between the two sets of basis elements is given in term of the \(C\)-space vielbein:

\[
\gamma_M = e_M^A \gamma_A \tag{7}
\]

All quantities in eq. (2.9) depend on position \(X\) in \(C\)-space. Reciprocal basis elements \(\gamma^M\) and \(\gamma^A\) are defined according to \((\gamma^M)^\dagger \ast \gamma_N = \delta^M_N\) and \((\gamma^A)^\dagger \ast \gamma_B = \delta^A_B\).

Let us now define a differential operator \(\partial_M \equiv \partial\), which will be called *derivative*, whose action depends on the quantity it operates on:

- \(\partial\) maps scalars \(\phi\) into scalars

\[
\partial_M \phi = \frac{\partial \phi}{\partial x^M} \tag{8}
\]

Then \(\partial\) is just the ordinary partial derivative.

- \(\partial_M\) maps Clifford numbers into Clifford numbers. In particular it maps a coordinate basis Clifford number \(\gamma_N\) into another Clifford number:

\[
\partial_M \gamma_N = \Gamma_M^J \gamma_J \tag{9}
\]

The above relation defines the *coefficients of connection* for the coordinate frame field \{\gamma_M\}. An analogous relation we have for the local flat frame field:

\[
\partial_M \gamma_A = -\Omega_A^B \gamma_B \tag{10}
\]

where \(\Omega_A^B\) are the coefficients of connection for the local flat frame field \{\gamma_A\}.

Expanding an arbitrary polyvector field according to \(A = A^M \gamma_M = A^A \epsilon^A_M \gamma_A\) and using eqs. (9),(10) we have

\[
\partial_M e^C_M - \Gamma_M^J e^C_J - e^A_M \Omega_A^C N = 0 \tag{11}
\]

which is analogous to the well known relation in an ordinary curved spacetime, and \(\Omega_A^C\) extends the notion of spin connection \(\omega_A^C\).

From (11) we obtain

\[
\partial_M e^C_N - \partial_N e^C_M + e^A_M \Omega_A^C N - e^A_N \Omega_A^C M = T_{MN}^J e^C_J \tag{12}
\]

where \(T_{MN}^J = \Gamma_{MN}^J - \Gamma_{JM}^N\) is the \(C\)-space torsion.

Taking the commutators of derivatives we have

\[
[\partial_M, \partial_N] \gamma_J = R_{MNJ}^K \gamma_K \quad \text{and} \quad [\partial_M, \partial_N] \gamma_A = R_{MNA}^B \gamma_B \tag{13}
\]

where

\[
R_{MNJ}^K = \partial_M \Gamma_{NJ}^K - \partial_N \Gamma_{MJ}^K + \Gamma_{NJ}^R \Gamma_{MR}^K - \Gamma_{MJ}^R \Gamma_{NR}^K \tag{14}
\]

\[
R_{MNA}^B = -(\partial_M \Omega_A^B - \partial_N \Omega_A^B M + \Omega_A^C N \Omega_C^B M - \Omega_A^C M \Omega_C^B N) \tag{15}
\]

are the coefficients of curvature for the frame field \{\gamma_M\} and \{\gamma_A\}, respectively.

\(^2\) This operator is the \(C\)-space analogue of the derivative \(\partial_\mu \equiv \partial_\nu\) which operates in an \(n\)-dimensional curved space \(V_\nu\), and was defined by Hestenes [1] (who used a different symbols, namely \(\nabla_\mu\)).
3. The generalized Dirac equation in $C$-space

We will leave aside further discussion of a classical general relativity in $C$-space (see [12, 14, 6, 7]) and go directly to quantum theory. We will assume that wave functions are polyvector valued fields.

Let $\Phi(X)$ be a polyvector valued field over coordinates polyvector field $X = x^M \gamma_M$:

$$\Phi = \phi^A \gamma_A$$

where $\gamma_A$, $A = 1, 2, ..., 16$, is a local (flat) basis of $C$-space (see eq.(2)) and $\phi^A$ the projections (components) of $\Phi$ onto the basis $\{\gamma_A\}$. We will suppose that in general $\phi^A$ are complex-valued scalar quantities. We will assume [12, 7] that the imaginary unit $i$ is the bivector of phase space, and that it commutes with all elements of $C\ell_{1,3}$, since it is not an element of the latter algebra.

Instead of the basis $\{\gamma_A\}$ one can consider another basis, which is obtained after multiplying $\gamma_A$ by 4 independent primitive idempotents [15] $P_i = \frac{1}{4} (1 + a_i \gamma_A)(1 + b_i \gamma_B)$, $i = 1, 2, 3, 4$. Here $a_i$, $b_i$ are complex numbers chosen so that $P_i^2 = P_i$. For explicit and systematic construction see [15].

By means of $P_i$ we can form minimal ideals of Clifford algebra. A basis of left (right) minimal ideal $I_i^L$ ($I_i^R$) is obtained by taking $P_i$ and multiplying it from the left (right) with all 16 elements $\gamma_A$ of the algebra:

$$\gamma_A P_i \in I_i^L, \quad P_i \gamma_A \in I_i^R$$

For a fixed $i$ there are 16 elements $\gamma_A P_i \in I_i^L$ (or $P_i \gamma_A \in I_i^R$, but only 4 amongst them are different, the remaining elements are just repetition—apart from constant factors—of those 4 different elements.

Let us denote those different element that form a basis of the i-th left ideal by symbol $\xi_{\alpha i}$, $\alpha = 1, 2, 3, 4$. Altogether, for $i = 1, 2, 3, 4$, there are 16 different basis elements $\xi_{\alpha i}$. Every Clifford number can be expanded either in terms of $\gamma_A$, or in terms of $\xi_{\alpha i} = (\xi_{11}, \xi_{12}, \xi_{13}, \xi_{14})$ according to

$$\Psi = \psi^A \xi_{\alpha} = \psi^{\alpha 1} \xi_{11} + \psi^{\alpha 2} \xi_{12} + \psi^{\alpha 3} \xi_{13} + \psi^{\alpha 4} \xi_{14}$$

Eq.(18) represents a sum of four independent 4-component spinors, each in a different left ideal $I_i^L$. We have introduced a single spinor index $\hat{A}$ which runs over all 16 basis elements $\xi_{\hat{A}}$ that span 4 independent left minimal ideals of $C\ell_{1,3}$. The set $\{\xi_{\hat{A}}\}$ of 16 linearly independent fields $\xi_{\hat{A}}(X)$ will be called generalized spinor frame field. An explicit relation between the two basis is given in refs. [15, 6, 7].

In refs. [11, 12] it was proposed that the polyvector valued wave function satisfies the Dirac equation in $C$-space:

$$\partial \Psi = \gamma^M \partial_M \Psi = 0$$

The derivative $\partial_M$ is the same derivative introduced in eqs. (8)–(10). Now it acts on the object $\Psi$ which, according to eq. (18), is expanded in terms of the 16 basis elements $\xi_{\hat{A}}$ which, in turn, can be written as a superposition of basis elements $\gamma_A$ of $C\ell_{1,3}$. The action of $\partial_M$ on $\gamma_A$ is given in (10). An analogous expression holds if $\partial_M$ operates on the spinor basis elements $\xi_{\hat{A}}$:

$$\partial_M \xi_{\hat{A}} = \Gamma_M \hat{\partial}_{\hat{B}} \xi_{\hat{B}}$$

where $\Gamma_M \hat{\partial}_{\hat{B}}$ are components of the generalized spin connection, i.e., the components of the connection of curved $C$-space for the generalized spinor frame field $\{\xi_{\hat{A}}\}$. Thus eq.(19) can be written as

$$\gamma^M \partial_M (\psi^{\hat{A}} \xi_{\hat{A}}) = \gamma^M (\partial_M \psi^{\hat{A}} + \Gamma_M \hat{\partial}_{\hat{B}} \psi^{\hat{B}}) \xi_{\hat{A}} \equiv \gamma^M (D_M \psi^{\hat{A}}) \xi_{\hat{A}} = 0$$

We see that in the geometric form of the generalized Dirac equation (19) spin connection is automatically present through the operation of the derivative $\partial_M$ on a polyvector $\Psi$. 
An action which leads to eq. (19) is (for a more detailed treatment see ref. [7]):

\[
I[\Psi, \Psi^\dagger] = \int d^{4n} x \sqrt{|G|} i \Psi^\dagger \partial \Psi = \int d^{4n} x \sqrt{|G|} i \psi^* \gamma^\dagger \xi_B^\dagger \dot{\xi}_A D_M \psi^A
\]  
(22)

where \(d^{4n} x \sqrt{|G|}\) is the invariant volume element of the \(2^n\)-dimensional \(C\)-space, \(G \equiv \det G_{MN}\) being the determinant of the \(C\)-space metric.

A generic transformation in the tangent \(C\)-space \(T_X C\) which maps a polyvector \(\Psi\) into another polyvector \(\Psi'\) is given by (\(\Sigma_{AB}\) being generators defined in ref. [6, 7])

\[
\Psi' = R \Psi S
\]  
(23)

where \(R = e^{i \Sigma_{AB} \alpha^{AB}}\) and \(S = e^{i \Sigma_{AB} \beta^{AB}}\), with \(\alpha^{AB}\) and \(\beta^{AB}\) being parameters of the transformation. It can be shown [6, 7] that in matrix form the transformation (23) reads

\[
\psi'A = U \dot{A} B \psi'B \ \text{or} \ \psi' = U \psi, \ \ U = R \otimes S^T
\]  
(24)

where \(R\) and \(S\) are \(4 \times 4\) matrices representing the Clifford numbers \(R\) and \(S\). We see that the matrix \(U\) is the direct product of \(R\) and the transpose \(S^T\) of \(S\), and it belongs, in general, to the group \(GL(4,C) \times GL(4,C)\), which is then subjected to further restrictions resulting from the requirement that \(\Psi^\dagger \star \Psi\) be invariant, which implies \(R^\dagger R = 1\) and \(S^T S = 1\). Then it can be shown [7] that the scalar part of the action (22) is invariant under local transformations (23),(24). Besides ordinary Lorentz transformations the latter group contains the “internal” transformations. The group is large enough to contain the subgroup \(U(1) \times SU(2) \times SU(3)\). Whether this indeed provides a description of the standard model remains to be fully investigated. But there is further evidence in favor of the above hypothesis in the fact that a polyvector field \(\Psi = \psi^A \xi_A\) has 16 complex components. Altogether it has 32 real components. This number matches, for one generation, the number of independent states for spin, weak isospin and color, i.e., \((e, \nu), (u, d)_{b, r, g}\), together with the corresponding antiparticle states, in the standard model.

Under a transformation (24) the covariant derivative and the spin connection transform, respectively, according to

\[
D_M' \psi^A = U \dot{A} \dot{B} D_M \psi^B, \ \ \ \text{i.e.,} \ \ D_M' \psi' = U D_M \psi
\]  
(25)

\[
\Gamma_{MA}^B = U_D^B U^{\dot{C}}_{\dot{A}} \Gamma_{MC}^\dot{C} \dot{A} \partial_{MB} U_D \partial_B + \partial_{MB} U_D^B \partial_B, \ \ \ \text{i.e.,} \ \ \ \Gamma_M = U \Gamma'_M U^{-1} + U \partial_M U^{-1}
\]  
(26)

where \(D_M' \psi^A = \partial_M \psi^A + \Gamma_M^\dot{A} \dot{B} \psi^B\) and \(D_M \psi^A = \partial_M \psi^A + \Gamma_M^\dot{A} \dot{B} \psi^B\). We see that \(\Gamma_M\) transforms as a non abelian gauge field. We have thus demonstrated that the generally covariant Dirac equation in 16-dimensional curved \(C\)-space contains the coupling of spinor fields \(\psi^A\) with non abelian gauge fields \(\Gamma_M^\dot{A} \dot{B}\) which altogether form components of connection in the generalized spinor basis.

4. The gauge field potentials and gauge field strengths

We can express the spin connection in terms of the generators \(\Sigma_{AB} = f_{AB}^C \gamma_C\):

\[
\Gamma = \frac{1}{4} \Omega^{AB} N \Sigma_{AB} = A_M^A \gamma_A, \ \ \ \ A_M^A = \frac{1}{4} \Omega^{CD} N f_{CD}^A
\]  
(27)

The matrices representing the Clifford numbers \(\gamma^M\) and \(\Gamma_M\) can be calculated according to [7]

\[
\langle \xi^A \gamma^M \xi_B \rangle_S = (\gamma^M)_{\dot{A} \dot{B}} = \gamma^M \ \ \text{and} \ \ \langle \xi^A \Gamma_M \xi_B \rangle_S = \Gamma_M^\dot{A} \dot{B} = \Gamma_M
\]  
(28)
The $C$-space Dirac equation, written in matrix form, can be split according to
\[ \gamma^M (\partial_M + \Gamma_M) = [\gamma^\mu (\partial_\mu + \Gamma_\mu) + \gamma^\tilde{M} (\partial_{\tilde{M}} + \Gamma_{\tilde{M}})] \psi = 0 \] (29)
where $M = (\mu, \tilde{M})$, $\mu = 0, 1, 2, 3$; $\tilde{M} = 5, 6, \ldots, 16$.

From eq. (27) we read that the gauge fields $\Gamma_M$ contain: (i) The connection of the 4-dimensional gravity $\Gamma^\mu = \frac{1}{2} \Omega^{ab}_{\mu} [\gamma^a, \gamma^b]$. (ii) The Yang-Mills fields $A_\mu \hat{\gamma}_A$, where we have split the local index according to $A = (a, \tilde{A})$. For the scalar part $A^\mu_\nu \gamma_a \equiv A_\mu^a 1$ we have just the U(1) gauge field. (iii) The antisymmetric potentials $A^\mu_{\tilde{A}} \equiv A_M = (A_\mu, A_{\mu\nu}, A_{\mu\nu\rho}, A_{\mu\nu\rho\sigma})$, if we take indices $A = g$ (scalar) and $M = \mu, \mu\nu, \mu\nu\rho, \mu\nu\rho$; (iv) Non abelian generalization of the antisymmetric potentials $A^{\mu_\nu \ldots}$.

The $C$-space spin connection thus contains all physically interesting fields, including the antisymmetric gauge field potentials which occur in string and brane theories.

Using (20) we can calculate the curvature according to
\[ [\partial_M, \partial_N] \xi_A = R_{MN} \tilde{B}_A \xi_B, \]
where $R_{MN} \tilde{B}_A = \partial_M \Gamma_N \tilde{B}_A - \partial_N \Gamma_M \tilde{B}_A + \Gamma_N \tilde{B}_C \Gamma^C \tilde{A}_A - \Gamma_M \tilde{B}_C \Gamma^C \tilde{A}_A$, or, in matrix notation, $R_{MN} = \partial_M \Gamma_N - \partial_N \Gamma_M + [\Gamma_M, \Gamma_N]$. Using (27), and renaming $R$ into $F$ we have
\[ F_{MN}^A = \partial_M A_N^A - \partial_N A_M^A + A_M^B A_N^C C_{BC}^A \] (30)
where $C_{BC}^A$ are the structure constants of $C\ell_{1,3}$ satisfying $[\gamma_A, \gamma_B] = C_{AB}^C \gamma_C$.

5. Conclusion
In current approaches to quantum gravity the starting point is often in assuming that at short distances there exists an underlying structure, based, e.g., on string and branes, or spin networks and spin foams. It is then expected that the smooth spacetime manifold of classical general relativity emerges as a sufficiently good approximation at large distances. However, it is feasible to assume that what will emerge is in fact not just spacetime, but spacetime with certain additional structure. The approach discussed in this contribution suggests that the long distance approximation to a more fundamental structure is Clifford space. This is the space of the degrees of freedom that describe extended objects. Since Clifford space is a higher dimensional space, it can serve for a realization of Kaluza-Klein theory, and since all its dimensions are physically observable to us, there is no need for compactification of extra dimensions.

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