The permutation entropy rate equals the metric entropy rate for ergodic information sources and ergodic dynamical systems

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Abstract

Permutation entropy quantifies the diversity of possible orderings of the values a random or deterministic system can take, as Shannon entropy quantifies the diversity of values. We show that the metric and permutation entropy rates—measures of new disorder per new observed value—are equal for ergodic finite-alphabet information sources (discrete-time stationary stochastic processes). With this result, we then prove that the same holds for deterministic dynamical systems defined by ergodic maps on \(n\)-dimensional intervals. This result generalizes a previous one for piecewise monotone interval maps on the real line (Bandt, Keller and Pompe, “Entropy of interval maps via permutations”, Nonlinearity 15, 1595-602, (2002)), at the expense of requiring ergodicity and using a definition of permutation entropy rate differing in the order of two limits. The case of non-ergodic finite-alphabet sources is also studied and an inequality developed. Finally, the equality of permutation and metric entropy rates is extended to ergodic non-discrete information sources when entropy is replaced by differential entropy in the usual way.

1 Introduction

The entropy rate is a key parameter associated with stochastic processes, information sources and dynamical systems. Roughly speaking, the entropy rate quantifies the average uncertainty, disorder or irregularity generated by a process or system per ‘time’ unit and, it is the primary subject of fundamental
results in information and coding theory (Shannon’s noiseless coding theorem) and statistical mechanics (second law of thermodynamics). It is not surprising, therefore, that this notion, appropriately generalized and transformed, is ubiquitous in many fields of mathematics and science when randomness or ‘random-like’ behavior is at the heart of the theory or model being studied.

For definiteness consider a stationary information source emitting a time-series of observed values \( x_1, \ldots, x_n \) in a continuous state space — formally, draws from the random variables \( X_1, \ldots, X_n \). Since the realization of a non-discrete random variable cannot be observed exactly (this would mean an infinite amount of information), the observer has to content himself with a finite degree of accuracy. Generally speaking, the metric or Shannon entropy rate of an information source is the rate of new information it generates per unit time (as the metric or Kolmogorov-Sinai entropy rate of a deterministic dynamical system is a measure of its pseudo-randomness or chaotic behavior).

Given a certain discretization scale \( \Delta \) of the state space, the metric (Shannon) entropy rate \( h_m \) of the discretized information source \( X^\Delta = (X_n^\Delta)_{n \in \mathbb{N}} \) is

\[
h_m(X^\Delta) = \lim_{L \to \infty} \frac{1}{L} H_m(X^\Delta L_1),
\]

with \( X^\Delta L_1 = X^\Delta_1 \ldots X^\Delta_L \), a length \( L \) word of symbols \( X^\Delta \) discretized at resolution \( \Delta \) from \( X_1^L = X_1 \ldots X_L \). We use \( H_m(Z) \) for the entropy of the discrete random variable \( Z \), i.e., \( H_m(Z) \equiv H_m(\Pr(Z)) = -\sum_z \Pr(z) \log_2 \Pr(z) \) for the probability distribution \( \Pr(z) \) of \( Z \). We come back to the metric entropy and entropy rate in the next section, where we set the conceptual background of this paper on a more formal footing.

Consider a length \( L \) word of observables \( X^\Delta L_1 \). Assuming there exists a natural order relation on the state space of the source \( X^\Delta \) (e.g., real scalars or vectors with a defined lexicographic ordering), each block of observations \( X^\Delta L_1 \) selects one particular permutation \( \Pi \) out of the \( L! \) possible permutations. For example, if \( X^\Delta_2 \leq X^\Delta_1 \leq X^\Delta_3 \), then the corresponding permutation can be expressed explicitly as \( \Pi(X^\Delta_3) = (2, 1, 3) \). Note that the mapping from \( X^\Delta \)-orderings to permutations can be many-to-one when there are repeated values; to overcome this shortcoming, we will use ‘ranks’ below (see Sect. 3), so that words defining the same permutation have the same rank variables which, in turn, can be identified with the corresponding permutation. Bandt and Pompe [3] defined the permutation entropy of order \( L \) as

\[
H^*_m(X^\Delta L_1) = \frac{1}{L-1} H_m(\Pi(X^\Delta L_1)),
\]

1 The factor \( 1/(L-1) \) is used instead of \( 1/L \) because \( \Pi(X^\Delta_1) = 1 \) contributes nothing to the entropy. This choice is, of course, inconsequential when \( L \to \infty \), but it is preferable for numerical simulations and the applications we discuss in the last section.
with \( \Pr(\Pi(X^\Delta_1)) \) being the probability of observing any particular permutation given a block of observables. In direct analogy to the Shannon entropy rate, the \textit{permutation entropy rate} at resolution \( \Delta \) is hence defined as (following the notation of [2])

\[
h^*_m(X^\Delta) := \lim_{L \to \infty} \bar{H}^*_m(X^\Delta_L).
\]

(1)

For \textit{deterministic} maps \( f \) of a proper interval \( I \subset \mathbb{R} \) with a finite number of monotony segments, Bandt, Keller and Pompe [2,3] analytically and numerically investigated a permutation entropy rate we denote by \( h^*_\text{BKP}(f) \), based on the entropy of certain partitions, proving that it exists and, in fact, equals the metric (Kolmogorov-Sinai) entropy rate \( h_m(f) \). They also prove this equality for the topological versions of permutation and ordinary entropy rates. Relative changes in \( h^*_\text{BKP} \) estimated numerically from time-series from the logistic map tended to track very well, over a wide range of varying nonlinearity parameter, the behavior of \( h_m \) (estimated from the positive Lyapunov exponent of the map directly). There remained a substantial bias, though it was nearly constant over parameters.

The correspondence observed in [3] between permutation entropy and metric entropy rates of time series is not coincidental, nor restricted to one-dimensional dynamics. Under only the assumption of ergodicity, we show that the permutation entropy rate of stationary, finite-alphabet random processes equals the metric entropy rate. A similar result follows for the permutation and metric \textit{differential} entropy rates of non-discrete sources. With these results on stochastic processes in the hand, we further show that for \textit{ergodic} maps on \( d \)-dimensional intervals \( I^d \) the two entropy rates are also equal. In doing so, we define the permutation entropy rate as \( h^*_m(f) = \lim_{\Delta \to 0} h^*_m(X^\Delta) \), where \( X^\Delta \) stands now for the ‘simple observations’ of \( f \) supplied by a discretization of \( I^d \) with resolution \( \Delta \) —a finite-state stochastic process. The generality of all these results gives a strong support to our approach, which provides a unified treatment for stochastic and deterministic dynamical systems.

This paper is organized as follows. For the reader’s convenience we review in Sect. II the theoretical background and fix the notation. Sect. III contains one of the main results of this paper, namely, \( h_m = h^*_m \) for \textit{ergodic} finite-alphabet stochastic processes (Theorem 1). This result is generalized in Sect. IV to non-discrete ergodic information sources using the differential entropy rate (Theorem 2) and, in Sect. V, to maps on \( d \)-dimensional intervals (Theorem 3). We also mention in Sect. III that \( h^*_m \geq h_m \) for \textit{non-ergodic} finite-alphabet sources; the proof can be found in Appendix B. Sect. V contains the main result on finite-dimensional maps, and Sect. VI, a discussion of the two definitions of permutation entropy. Finally, in Sect. VII we show some numerical examples and discuss open practical issues in using permutation entropies in time-series.
2 Theoretical framework

2.1 Stochastic processes and dynamical systems

Let $\mathbb{R}^{dN} = \{ x = (x_n)_{n \in \mathbb{N}} : x_n \in \mathbb{R}^d \}$, $\mathcal{B}$ the product sigma-algebra of $\mathbb{R}^{dN}$ generated by the Borel sets of $\mathbb{R}^d$, and $\sigma$ the (left) shift transformation on $\mathbb{R}^{dN}$, $(\sigma x)_n = x_{n+1}$. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, i.e., $\Omega$ is a nonempty set, $\mathcal{F}$ is a sigma-algebra of subsets of $\Omega$ and $\mu$ is a (positive) measure on $(\Omega, \mathcal{F})$. Any stationary stochastic (or random) process in discrete time $X = (X_n)_{n \in \mathbb{N}}$ on the probability space $(\Omega, \mathcal{F}, \mu)$ with values in $\mathbb{R}^d$ corresponds in a standard way to the shift dynamical system $(\mathbb{R}^{dN}, \mathcal{B}, m, \sigma)$ via the map $\phi : \Omega \rightarrow \mathbb{R}^{dN}$ defined by $(\phi \omega)_n = X_n(\omega)$, $n \in \mathbb{N}$. The probability measure $m$ is defined on the Borel sets $\mathcal{B}$ of $\mathbb{R}^{dN}$ by

$$m(B) := \mu(\phi^{-1}B)$$

($\phi^{-1}B \in \mathcal{F}$ because $X_n$ is $\mathcal{F}$-measurable for all $n$) and it is $\sigma$-invariant (i.e., $m \circ \sigma^{-1} = m$) because of the stationarity of $X$. The measure $m$ is sometimes called the induced probability measure or distribution on the space of possible outputs of the random process. Moreover, if $\pi_n : \mathbb{R}^{dN} \rightarrow \mathbb{R}^d$ is the projection onto the $n$th component, $\pi_n x = x_n = X_n(\omega)$ (or $\pi_n = X_n \circ \phi^{-1}$), then the ‘sampling function’ $\pi = (\pi_n)$ has the same joint distributions on $\mathbb{R}^{dN}$ as $X = (X_n)$ on $\Omega$, i.e., both processes are equivalent. Any point $x$ of the state space $\mathbb{R}^{dN}$ is a possible realization (or ‘sample path’) of the whole process. Such one-sided random processes provide better models than the two-sided processes $(X_n)_{n \in \mathbb{Z}}$ for physical information sources that must be turned on at some time and thus we will use both denominations interchangeably in this paper.

We will also refer to the shift dynamical system $(\mathbb{R}^{dN}, \mathcal{B}, m, \sigma)$ as the (sequence space) model of the stochastic process or information source $X$. Sometimes $\mathbb{Z}_+ = \{0, 1, \ldots\}$ is used instead of $\mathbb{N}$ to number the random variables $X_n$ and their samples $x_n$ (we do so in Sect. 4). Models allow to focus on the random process itself as given by the probability distribution on their outputs, dispensing with a perhaps complicated underlying probability space. As usual, we will also identify $X_n$ with $\pi_n = \pi_0 \circ \sigma^n$.

Finite-state or finite-alphabet sources $S = (S_n)_{n \in \mathbb{N}}$ on $(\Omega, \mathcal{F}, \mu)$, where $S_n : \Omega \rightarrow A$ with alphabet $A = \{a_1, \ldots, a_{|A|}\}$, are dealt with in a similar way to the previous, non-discrete sources and, as a matter of fact, most of the general setup, properties and observations above apply mutatis mutandis to
this simpler case. The sequence space of the corresponding model is now
\( A^\mathbb{N} = \{ s = (s_n)_{n \in \mathbb{N}} : s_n \in A \} \), \( A \) being endowed with the discrete topology; let \( \mathcal{Z} \) be the product sigma-algebra of \( A^\mathbb{N} \) generated by the elements of \( A \). Since no confusion will arise, we continue denoting by \( \sigma \) the shift on \( A^\mathbb{N} \), \((\sigma s)_n = s_{n+1}\), and by \( m \) the \( \sigma \)-invariant measure on \((A^\mathbb{N}, \mathcal{Z})\) defined as the pushforward of \( \mu \) by the map \( \phi : \Omega \to A^\mathbb{N} \), \((\phi \omega)_n = S_n(\omega)\). The finite order probability distribution of \( S \), \( \Pr(S_{i_1} = s_{i_1}, \ldots, S_{i_n} = s_{i_n}) =: \Pr(s_{i_1}, \ldots, s_{i_n}) \), can be alternatively expressed by means of the probability distribution on the outputs of \( S \),

\[
\Pr(s_{i_1}, \ldots, s_{i_n}) = m \{ \xi \in A^\mathbb{N} : s_{i_1}, \ldots, s_{i_n} \}
\]

for any \( i_1, \ldots, i_n \in \mathbb{N} \) and \( s_{i_1}, \ldots, s_{i_n} \in A \).

In this paper we will consider mostly finite-alphabet sources, although these will also occasionally arise as discretizations or quantizations \( X^\Delta \) of sources \( X \) taking values on a proper interval \( I^d \) of \( \mathbb{R}^d \) (in symbols) endowed with Lebesgue measure \( \lambda \). Formally, this means that there exists a (usually, uniform) partition \( \delta = \{ \Delta_1, \ldots, \Delta_{|\delta|} \} \) of \( I^d \) into a finite number of \( \lambda \)-measurable subsets such that \( X^\Delta_n \) is the discrete random variable defined by

\[
\Pr(X^\Delta_n = i) = \mu \{ \omega \in \Omega : X^\Delta_n(\omega) \in \Delta_i \}
= m \{ \xi \in A^\mathbb{N} : s_i = i \} = \int_{\Delta_i} dF(x),
\]

where \( F(x) = \Pr(X^\Delta_n \leq x) = \mu \{ \omega \in \Omega : X^\Delta_n(\omega) \leq x \} \) is the common distribution function to all \( X^\Delta_n \) (in case \( X^\Delta_n \) is a vector random variable, the inequality is understood component-wise), \( m \) is the induced probability measure on the outputs and \( A^\Delta = \{ 1, \ldots, |\delta| \} \) is the alphabet of \( X^\Delta \). If \( X^\Delta_n \) has a density function \( \rho(x) \) (formally, the Radon-Nykodim derivative of \( F \) with respect to \( \lambda \)), then \( \Pr(X^\Delta_n = i) = \int_{\Delta_i} \rho(x) dx \). Distribution functions and densities of higher finite order are analogously defined. For \( \Delta \), the ‘discretization scale’ or ‘resolution’ we referred to in the Introduction, one can take any measure of the ‘coarseness’ of \( \delta \), say, the largest diameter of its elements, also called the norm of \( \delta, ||\delta|| \).

**2.2 Entropy rate of dynamical systems and stochastic processes**

Let \((\Omega, \mathcal{F}, \mu)\) be a probability space and \( f : \Omega \to \Omega \) a \( \mu \)-preserving transformation, i.e., \( \mu(f^{-1}A) = \mu(A) \) for all \( A \in \mathcal{F} \). Given the dynamical system
\((\Omega, \mathcal{F}, \mu, f)\) and a finite partition \(\alpha = \{A_1, \ldots, A_{|\alpha|}\} \subset \mathcal{F}\) of \(\Omega\), the entropy of \(f\) with respect to \(\alpha\) is defined as

\[
h_{\mu}(f, \alpha) := \lim_{L \to \infty} \frac{1}{L} H_{\mu}\left(\bigvee_{i=0}^{L-1} f^{-i} \alpha\right),
\]

(3)

where \(\bigvee_{i=0}^{L-1} f^{-i} \alpha = \{\bigcap_{i=0}^{L-1} f^{-i} A_{j}\}\) is the least common refinement of the partitions \(\{\alpha, f^{-1} \alpha, \ldots, f^{-L+1} \alpha\}\) and \(H_{\mu}(\beta) := -\sum_{j=1}^{\beta} \mu(B_j) \log \mu(B_j)\) for any finite partition \(\beta = \{B_1, \ldots, B_{|\beta|}\} \subset \mathcal{F}\). The metric or Kolmogorov-Sinai entropy rate of map \(f\) is then defined as:

\[
h_{\mu}(f) := \sup_{\alpha} h_{\mu}(f, \alpha).
\]

(4)

The convergence in (3) can be proved to be monotonically decreasing [6]. Assuming logarithms base 2 everywhere herein, \(h_{\mu}(f)\) has units of bits per symbol or time unit, if \(n\) is interpreted as discrete time. By convention, \(0 \cdot \log 0 := \lim_{x \to 0^+} x \log x = 0\). In an information-theoretical setting, \(h_{\mu}(f, \alpha)\) represents the long-term average of the information gained per unit time with respect to a certain partition and \(h_{\mu}(f)\) the maximum information per unit time available from any stationary process generated by the source, typically equal to the sum of the positive Lyapunov exponents by the Pesin theorem. If there exists finite \(\gamma\) such that \(h_{\mu}(f, \gamma) = h_{\mu}(f)\), then \(\gamma\) is called a generator, or generating partition, of \(f\).

Given a discrete alphabet source \(S = (S_n)\) with model \((A^N, \mathcal{Z}, m, \sigma)\), the (Shannon) entropy of the random variables \(S_1^L := S_1 \ldots S_L\) is

\[
H_m(S_1^L) := H_m\left(\bigvee_{i=0}^{L-1} \sigma^{-i} \zeta\right),
\]

where \(\zeta = \{C_1, \ldots, C_{|A|}\}\) is the partition of \(A^N\) consisting of the basic ‘cylinder sets’ \(C_i = \{s \in A^N : s_1 = a_i\}\), \(1 \leq i \leq |A|\). According to (2),

\[
H_m(S_1^L) = -\sum \Pr(s_1, \ldots, s_L) \log \Pr(s_1, \ldots, s_L),
\]

and, correspondingly, the entropy rate (or uncertainty) of the source is defined as \(h_m(S) := h_m(\sigma, \zeta) = h_m(\sigma)\) since \(\zeta\) is a (one-sided) generator of \(\mathcal{Z}\), i.e.,

\[
h_m(S) = \lim_{n \to \infty} \frac{1}{L} H_m(S_1^L).
\]

In other words, the Shannon entropy rate of \(S\) is, by definition, the Kolmogorov-Sinai entropy rate of its sequence space model. This explains our using ‘metric’ to refer to both concepts, independently of the random or deterministic nature of the system considered. Sometimes we will also use the \(n\)th order entropy of
so that \( h_m(S) = \lim_{L \to \infty} \bar{H}_m(S^L) \). In general, \( S^j \) stands for the string \( S_i \ldots S_j \).

Other dynamical, statistical or information-theoretical concepts like conditional entropy, mutual information, ergodicity, mixing properties, etc., are also defined via the sequence space model. For example, \( S \) is said to be ergodic if \((A^n, \mathcal{Z}, m, \sigma)\) is ergodic, i.e., for \( C_1, C_2 \in \mathcal{Z} \) with \( m(C_1) > 0, m(C_2) > 0 \), there exists \( n > 0 \) such that \( m(C_1 \cap \sigma^{-n}C_2) > 0 \).

If, more generally, \( X \) is a non-discrete scalar or vector source with outcomes on an interval \( I^d \subset \mathbb{R}^d \), define its differential entropy rate as

\[
 h_m(X) := \lim_{\Delta \to 0} \left( h_m(X^\Delta) + \log \Delta \right),
\]

(5)

where \( X^\Delta \) is a uniform discretization of \( X \) with resolution scale \( \Delta \). The differential entropy shows how the average rate of information furnished by a quantization of resolution \( \Delta \) differs from \( |\log \Delta| \) when \( \Delta \to 0 \). If \( X^\Delta \) happens to have a density function \( \rho(x_1, \ldots, x_L) \) for every \( L \geq 1 \), then

\[
 h_m(X) = \int_{I^d} \rho(x_1, \ldots, x_L) \log \rho(x_1, \ldots, x_L) d^Lx.
\]

3 Permutations and the metric entropy rate of finite-alphabet sources

Given a finite-alphabet source \( S = (S_n) \) with model \((A^n, \mathcal{Z}, m, \sigma)\), each possible permutation of a block of length \( L \), e.g., \( S^L := S_1 \ldots S_L \), can be indexed as a word of ranks, each an integer in successively larger alphabets. In particular, define for \( n \geq 1 \) the rank variable \( R_n = |\{S_i, 1 \leq i \leq n : S_i \leq S_n\}| = \sum_{i=1}^{n} \delta(S_i \leq S_n) \), where, as usual, the \( \delta \)-function of a proposition is 1 if it holds and 0 otherwise. By definition, \( R_n \) is a discrete random variable on \( \Omega \) with range \( \{1, \ldots, n\} \) and the sequence \( \mathbf{R} = (R_n) \) builds a discrete-time non-stationary process. Then the permutation \( \Pi(S^L) \) in (1) can also be viewed as the word \( R^L = R_1 \ldots R_L \), the relation between both being one-to-one. The many-to-one relation between \( S^L \) and \( R^L \) is written as \( R^L = \varphi(S^L) \).

For example, consider a source \( S \) over the alphabet \( \{1, 2, 3\} \). Suppose we observe the word \( S^3 = 1, 3, 3 \). Then, \( R_1^3 = \varphi(S^3) = 1, 2, 3 \), (of course other strings, e.g., \( 1, 1, 1 \) or \( 2, 2, 2 \), also map to \( R_1^3 = 1, 2, 3 \) and \( \Pi(S^3) = (1, 2, 3) \). The string \( 1, 3, 3 \) could be counted as matching both the ordering \( S_1 \leq S_2 \leq S_3 \) and \( S_1 \leq S_3 \leq S_2 \). By using ranks, by contrast, the measure associated with each word is unambiguously associated with one permutation, and the rest of our development follows this approach.
The permutation entropy rate of $S$ is then defined as

$$h^*_m(S) := \lim_{L \to \infty} \bar{H}_m(R^L_1),$$

alternatively to the definition (1), with

$$\bar{H}_m^*(S^L_1) = \bar{H}_m(R^L_1)$$

$$= -\frac{1}{L-1} \sum \Pr(r_1, \ldots, r_L) \log \Pr(r_1, \ldots, r_L)$$

defined to be the permutation entropy of order $L \geq 2$ of $S$. Remember that the overbar notation $\bar{H}$ means that the relevant factor of $1/L$ or $1/(L-1)$ has been included for the entropy of a block of length $L$.

Let $\sigma_L$ denote the set of permutations of $\{1, \ldots, L\}$ for the time being. We say that the word $S^L_1$ is of type $\pi \in \sigma_L$ if $R^L_1 = \varphi(S^L_1)$ defines the permutation $\pi$. It follows $s_{\pi(1)} \leq \ldots \leq s_{\pi(L)}$. The cylinder sets

$$C_\pi := \{ s \in A^N : s^L_1 \text{ is of type } \pi \}$$

such that $C_\pi \neq \emptyset$ build a partition of $A^N$ with $m(C_\pi) = \Pr(R^L_1 = r^L_1)$, $1 \leq r_k \leq k$ for $k = 1, \ldots, L$. Therefore

$$\bar{H}_m^*(S^L_1) = -\frac{1}{L-1} \sum_{\pi \in \sigma_L} m(C_\pi) \log m(C_\pi). \quad (6)$$

That is, the permutation entropy is sensitive to the measures of non-trivial order relationships observed in a word, as the Shannon entropy is sensitive to the measures of the different word values themselves.

Observe as a technical point for later reference that, if

$$Q_\pi := \{ s \in A^N : s_{\pi(1)} \leq s_{\pi(2)} \leq \ldots \leq s_{\pi(L)} \},$$

then $C_\pi \not\subseteq Q_\pi$ due to words $s^L_i$ with repeated letters: if $s_i \neq s_j$ for every $1 \leq i, j \leq L$, then $s \in C_\pi$ if and only if $s \in Q_\pi$.

**Lemma 1** Given an ergodic information source $S$,

$$\lim_{k \to \infty} H_m(R^{k+l}_{k+1} | S^k_1) = \lim_{k \to \infty} H_m(S^{k+l}_{k+1} | S^k_1)$$

for all $l \geq 1$.

That is, given a sufficiently long tail of previously observed symbols, the later ranks can be predicted virtually as well as the symbols themselves. Heuristically, this is because the distribution of rank variable $R_{k+1}$ for $k$ sufficiently
large depends effectively on only the cumulative distribution function of
the source, approximated by the normalized sum of \( S_1^k \). In turn this means that
the information contained in \( R_{k+1} \) is the same as the information in \( S_{k+1} \). The
proof, and an elementary example, is given in Appendix A. With Lemma 1 in
hand, we turn to our first main result, the equality between permutation and
metric entropy for finite-alphabet stochastic processes.

**Theorem 2** For finite-alphabet ergodic sources \( S \) the permutation entropy
rate exists and equals the metric entropy rate: \( h^*_m(S) = h_m(S) \).

**PROOF.** We prove inequalities in both directions.
(a) \( \limsup_{L \to \infty} \bar{H}^*_m(L^L) \leq h_m(S) \). Given \( S^L_1 \), the corresponding rank variables
are uniquely determined via \( R^L_1 = \varphi(S^L_1) \). By [4] (Ch 2, exercise 5), \( H(\varphi(Z)) \leq
H(Z) \) for any discrete random variable \( Z \), so \( h_m(R^L_1) \leq h_m(S^L_1) \) and thus
\( \limsup_{L \to \infty} \bar{H}^*_m(R^L_1) \leq \limsup_{L \to \infty} \bar{H}^*_m(S^L_1) = h_m(S) \).

(b) \( \liminf_{L \to \infty} \bar{H}^*_m(L^L) \geq h_m(S) \). There are several ways to prove this in-
equality. Consider, for instance,

\[
\liminf_{L \to \infty} \bar{H}^*_m(S^L_1) \geq \liminf_{L \to \infty} \frac{1}{L} \left[ H_m(R_L|S^L_1) + \ldots + H_m(R_{L^*+1}|R^L_1) + H_m(R^L_1) \right]
\]

for any \( L^* < L \), where we have applied the chain rule for entropy. As \( R^L_1 =
\varphi(S^L_1) \) we apply the data processing inequality \( H(Y|\varphi(Z)) \geq H(Y|Z) \) [4] to
all elements of the first term on the rhs:

\[
\liminf_{L \to \infty} \bar{H}^*_m(S^L_1) \geq \liminf_{L \to \infty} \frac{1}{L} \left[ H_m(R_L|S^L_1) + \ldots + H_m(R_{L^*+1}|S^L_1) + H_m(R^L_1) \right].
\]

By Lemma 1, for any \( \varepsilon > 0 \) there is some \( L^* \) such that \( |H_m(S_L|S^L_1) - H_m(R_L|S^L_1)| < \varepsilon \)
for \( L > L^* \), so

\[
\liminf_{L \to \infty} \bar{H}^*_m(S^L_1) \geq \liminf_{L \to \infty} \frac{1}{L} \left[ H_m(S_L|S^L_1) + \ldots + H_m(S_2|S_1) + H_m(S_1) \right], \]

\[
+ \frac{1}{L} \left[ H_m(R^L_1^*) - H_m(S^L_1^*) \right] - \left( \frac{L - L^*}{L} \right) \varepsilon
\]

\[
= h_m(S) - \varepsilon.
\]

The existence of the limit and equality follows from (a) and (b). \( \square \)
More generally, we can only show an inequality for non-ergodic cases, namely,

\[ \lim_{L \to \infty} \inf \bar{H}^*_m(S^L_1) \geq h_m(S). \]  

(7)

The proof of (7) uses the ergodic decomposition of the entropy rate and is given in Appendix B.

4 Non-discrete information sources

Information sources can have also non-discrete alphabets, although their outcomes are only observable with a finite precision. In this case, it is well-known that Shannon’s entropy rate, defined as the limit over ever finer uniform quantizations of the source, diverges logarithmically with the quantization scale. In order to obtain a finite measure of the asymptotic behavior of such quantizations, one has to resort to the differential entropy rate (5) instead. It turns out that Theorem 2 can be extended to scalar and vector ergodic non-discrete sources if entropy is replaced by differential entropy.

Let \( X = (X_n) \) be a scalar or vector ergodic source taking values on an interval \( I^d \subset \mathbb{R}^d, \ d \geq 1 \). In case \( d > 1 \) (vector sources), \( I^d \) is supposed to be endowed with the product (or lexicographical) order: \( x \leq x' \) if \( x_k = x'_k \) for \( k = d, d - 1, ..., d - s > 1 \) and \( x_{d-s-1} < x'_{d-s-1} \) (other conventions are also possible).

With the equality between permutation and metric entropy rates for ergodic finite-alphabet sources, we now consider the source \( X \) uniformly discretized to an alphabet \( A^\Delta = \{1, \ldots, N\} \) by means of a partition \( \delta = \{\Delta_1, \ldots, \Delta_N\} \) of \( I^d \) with \( \lambda(\Delta_i) = \lambda(I^d)/N =: \Delta \) for \( 1 \leq i \leq N \), where \( \lambda \) is, as before, Lebesgue measure. One can then define the ranks \( R^\Delta_n : \Omega \to \{1, \ldots, N\} \) of blocks of discretized symbols \( X^\Delta_1^L \) in the known way: \( R^\Delta_n = \sum_{i=1}^n \delta(X^\Delta_i \leq X^\Delta_n), \ 1 \leq n \leq L \). If \( (A^{\Delta N}, Z^\Delta, m^\Delta, \sigma) \) is the sequence space model for \( X^\Delta \), we define the permutation entropy rate at resolution \( \Delta \) as usual: \( h^*_m(X^\Delta) := \lim_{L \to \infty} \bar{H}_m(R^\Delta_1^L) \). We can take now the limit \( \Delta \to 0 \) and, analogously to (5), define the differential permutation entropy rate of \( X \) as,

\[
h^*_m(X) := \lim_{\Delta \to 0} \left( h^*_m(X^\Delta) + \log \Delta \right)
= \lim_{\Delta \to 0} \left( \lim_{L \to \infty} \bar{H}_m(R^\Delta_1^L) + \log \Delta \right).
\]

This yields:

**Theorem 3** Suppose \( X \) is an ergodic non-discrete source. Then \( h^*_m(X) = h_m(X) \), that is, the differential permutation and metric entropy rates of \( X \) are equal.
PROOF. If \((\mathbb{R}^d, \mathcal{B}, m, \sigma)\) is ergodic, so is \((A^{\Delta N}, \mathcal{Z}^\Delta, m^\Delta, \sigma)\). By Theorem 2, 
\[ h^{*}_m(X^\Delta) = h_m(X), \]
so 
\[ h^*_m(X) = \lim_{\Delta \to 0} \left( h_m(X^\Delta) + \log \Delta \right) = h_m(X), \]
where \(h_m(X)\) is the metric differential entropy rate of \(X\). \(\square\)

5 Permutations and the metric entropy of ergodic maps

In this section we will use our result on finite-alphabet stochastic processes to show that the equality between permutation and Kolmogorov-Sinai entropy rate applies to ergodic maps on finite-dimensional intervals.

Let \(I^d\) be a proper interval of \(\mathbb{R}^d\) endowed with the sigma-algebra \(\mathcal{B}_{|I^d} = \mathcal{B} \cap I^d\), the restriction of Borel sigma-algebra of \(\mathbb{R}^d\) to \(I^d\), and let \(f : I^d \to I^d\) be a \(\mu\)-preserving transformation, with \(\mu\) being a measure on \((I^d, \mathcal{B}_{|I^d})\). In order to define the permutation entropy of \(f\), we consider first product partitions 
\[ t = \prod_{k=1}^{d} \{I_{1,k}, \ldots, I_{N_k,k}\} \]
of \(I^d\) into \(N^d := N_1 \cdots N_d\) subintervals of lengths \(\Delta_{j,k}\), \(1 \leq j \leq N_k\), in each coordinate \(k\), defining \(\|t\| = \max_{j,k} \Delta_{j,k}\). The intervals are lexicographically ordered in each dimension, i.e., points in \(I_{j,k}\) are smaller than points in \(I_{j+1,k}\), and for the multiple dimensions a lexicographic order is defined, \(I_{j,k} < I_{j,k+1}\), so there is an order relation between all the \(N^d\) partition elements, and we can enumerate them with a single index \(i \in [1, N^d]\):
\[ t = \{I^d_i: 1 \leq i \leq N^d\}, \quad I^d_i < I^d_{i+1} \]

Next define a collection of simple observations \(S^t = (S^t_n)\) with respect to \(f\) with precision \(\|t\|\): \(S^t_n(x) = i\) if \(f^n(x) \in I^d_i\), \(n = 0, 1, \ldots\). Then \(S^t\) is an ergodic stationary \(N^d\)-state random process or, equivalently, an ergodic source on \((I^d, \mathcal{B}_{|I^d}, \mu)\) with finite alphabet \(A^t = \{1, \ldots, N^d\}\) and output probability distribution \(m = \mu \circ \phi^{-1}\), with \(\phi(x) = (S^t_0(x), S^t_1(x), \ldots) \in A^{\Delta N}\), so that

\[
\text{Pr}(i_0, \ldots, i_{n-1}) \\
= \text{Pr}(S^t_0 = i_0, \ldots, S^t_{n-1} = i_{n-1}) \\
= m\{s \in A^{\Delta N} : s_0 = i_0, \ldots, s_{n-1} = i_{n-1}\} \\
= \mu(I^d_{i_0} \cap f^{-1} I^d_{i_1} \cap \ldots \cap f^{-n+1} I^d_{i_{n-1}}).
\]  

(8)
In fact, $f$ and the left shift $\sigma$ on the sequences $(S^i_n(x))$ are conjugate. A simple implementation of $S^i$ for $I = [0, 1]$ and $N = 10^k$ is the following: $S^i_n(x) = \left\lfloor f^n(x) \cdot 10^k \right\rfloor + 1 = \left\lfloor f^n(x) \cdot 10^k \right\rfloor$ with $I_i = [(i-1)10^{-k}, i10^{-k}]$ for $1 \leq i \leq N$. We see that using simple observations as a finite alphabet measurement with respect to $f$ provides a direct link between the entropies of $S^i$ and $f$. Accordingly, we define the permutation entropy rate of $f$ as

$$h^*_\mu(f) := \lim_{\|\iota\| \to 0} h^*_m(S^\iota)$$

provided the limit exists. With this definition, and Theorem 2, we may prove the principal result on ergodic dynamical systems.

**Theorem 4** If $f : I^d \to I^d$ is ergodic, then $h^*_\mu(f) = h_\mu(f)$. In words, the permutation entropy rate of ergodic maps equals the metric entropy rate.

**PROOF.** If $h_\mu(f) = \infty$, the statement follows in general (also for non-ergodic maps) from (7). If $h_\mu(f) < \infty$, we have (see (8))

$$h_m(S^\iota) = -\lim_{n \to \infty} \frac{1}{n} \sum \Pr (i_0, \ldots, i_{n-1}) \log \Pr (i_0, \ldots, i_{n-1})$$

$$= -\lim_{n \to \infty} \frac{1}{n} H_\mu \left( \bigvee_{i=0}^{n-1} f^{-i} \iota \right)$$

$$= h_\mu(f, \iota).$$

On the other hand, $h_m(S^\iota) = h^*_m(S^\iota)$ by Theorem 2 (since $S^\iota$ is ergodic with respect to the measure $m$).

Let $\gamma$ denote the finite generating partition of $f$ that, according to Krieger’s Theorem [13], must exist (due to $f$’s ergodicity and finite metric entropy), so that $h_\mu(f) = h_\mu(f, \gamma) = h_m(S^\gamma)$. We claim that

$$\lim_{\|\iota\| \to 0} h_m(S^\iota) = h_m(S^\gamma)$$

and, hence,

$$h^*_\mu(f) = \lim_{\|\iota\| \to 0} h^*_m(S^\iota) = \lim_{\|\iota\| \to 0} h_m(S^\iota) = h_\mu(f).$$

**Case 1.** Suppose that the elements of $\gamma$ are $(d$-dimensional) intervals or, more generally, that all elements of $\gamma$ consist of a finite number of intervals. In either case, taking if necessary a refinement of $\gamma$ (thus, also a generator that we call $\gamma$ as well) so that $\gamma$ becomes a product partition $\iota$ of $I^d$, we deduce $h_m(S^\iota) = h_m(S^\gamma) = h_\mu(f)$ and the same is true for any further refinement of $\iota$. 

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Case 2. If, otherwise, some component of \( \gamma \) consists (modulo 0) of infinitely many intervals, we can define a sequence of ever finer partitions \( (\iota_n)_{n \in \mathbb{N}} \) of \( I^d \) that, after an hypothetical refinement can be assumed without restriction to be a product partition (Case 1) such that \( \mathcal{A}(\iota_n) \), the finite sigma-algebras generated by the \( \iota_n \), build an increasing sequence and \( \mathcal{V}_{n=1}^\infty \mathcal{A}(\iota_n) = \mathcal{B}|_{I^d} \, (\mod 0) \). Then \( h_\mu(f) = \lim_{n \to \infty} h_\mu(f, \iota_n) \) [13].

This proves our claim and the theorem. \( \Box \)

6 On the definition of permutation entropy rate for dynamical systems

The original definition of Bandt, Keller and Pompe (BKP) [2] of the permutation entropy of maps on intervals \( I \subset \mathbb{R} \) involves partitions of the form

\[
P_\pi = \{ x \in I : f^{\pi(0)}(x) < f^{\pi(1)}(x) < \ldots < f^{\pi(L-1)}(x) \},
\]

where \( \pi \in \sigma_L \), here the set of permutations of \( \{0, 1, \ldots, L-1\} \), \( L \geq 2 \). In fact, if \( f \) is supposed to be piecewise monotone as in [2] or just ergodic, as in our case, it is easy to show that

\[
P^* \subset \{ P_\pi \neq \emptyset : \pi \in \sigma_L \}
\]

is a partition of \( I \) (except maybe for a set of points of measure zero). BKP define then the permutation entropy of order \( L \) as

\[
\bar{H}^{\text{BKP}}_\mu(f, L) := \frac{1}{L-1} H_\mu(P^*_L)
= - \frac{1}{L-1} \sum_{\pi \in \sigma_L} \mu(P_\pi) \log \mu(P_\pi)
\]

(compare to (6)) and their permutation entropy rate of \( f \) to be

\[
h^{\text{BKP}}_\mu(f) := \lim_{L \to \infty} \bar{H}^{\text{BKP}}_\mu(f, L),
\]

provided the limit exists. They prove \( h^{\text{BKP}}_\mu(f) = h_\mu(f) \) for piecewise monotone maps on intervals of \( \mathbb{R} \), but in the more general case, ergodic maps it seems that only the inequality \( \lim \inf_{L \to \infty} \bar{H}^{\text{BKP}}_\mu(f, L) \geq h_\mu(f) \)—formally similar to (7)—can be proved, which we have done in Appendix C for ergodic maps on \( d \)-dimensional intervals. Comparing such particular results to the generality of Theorem 4, we may conclude that our definition (9) of permutation entropy rate offers a substantial advantage.
Note that the central distinction, which makes our formulation easier and more natural, is that (9) takes the limit of infinite long conditioning \( (L \to \infty) \) first, and the discrete limit \( (\Delta \to 0) \) last, similarly to Kolmogorov-Sinai entropy rate, and as opposed to (11), where an explicit discretization was not taken. We conjecture that for non-pathological dynamical systems of the sort one might observe in Nature the two formulations are equivalent, but there are likely to be some non-trivial technicalities involved in a rigorous analysis. For example, [11] shows a 1-dimensional map with an infinite number of monotonicity intervals, where the topological entropy rate and the permutation version of the topological entropy rate (i.e., counting simply the number of distinct permutations with non-zero measure, and not weighting them by their measure) are unequal: 

\[
h^*_0(f) = \lim_{L \to \infty} \frac{1}{L-1} \log |P^*_L| \neq h_0(f).
\]

7 Numerical examples and Discussion

As a by-product of our result, the practitioner of time-series analysis will find an alternative way to envision or, eventually, numerically estimate the entropy rate of real sources. It is worth reminding that the entropy of information sources can be measured by a variety of techniques that go beyond counting word statistics and comprise different definitions of ‘complexities’ such as, for example, counting the patterns along a digital (or digitalized) data sequence [10,14,1]. Bandt and Pompe refer, in [3], to the permutation entropy of time series as complexity. That the entropy rate can also be computed by counting permutations shows once again that it is a so general concept that can be captured with different and seemingly blunt approaches.

We demonstrate numerical results on time series from the logistic map \( x_{n+1} = Ax_n(1-x_n) \). Figure 1 shows an estimate of the permutation entropy rate estimate on noise-free data as a function of \( A \), comparing the Lyapunov exponent (computed from the orbit knowing the equation of motion) to the permutation entropy. To be precise, we are estimating \( h^*_m(S) \) with \( S \) discretized from the logistic map iterated at the discretization of double-precision numerical representation, i.e., \( S \) is the output of a standard numerical iteration. The entropy estimator of the block ranks was the plug-in estimator (substituting observed frequencies for probabilities) plus the classical bias correction, first order in \( 1/N \). The key unresolved issue in using permutation entropies for empirical data analysis is, as with standard Shannon entropy rate estimation, balancing the tension between larger word lengths \( L \), to capture more dependencies, and the loss of sufficient sampling for good statistics in the ever larger discrete space. The finite \( L \) performance and convergence rate and bias of any specific computational method are key issues when it comes to accurately estimating the entropy rate of a source from observed data. It is now appreciated that numerically estimating the Shannon block entropy from finite data and, espe-
Fig. 1. (color online) Lyapunov exponent (black line, thick) of logistic map and permutation entropy rate estimates $\hat{h} = \bar{H}^*(X_1^L)$ for $N = 10^5, 10^6$ length time series from the map (red and black thin lines). The permutation entropy estimate tracks changes in the Lyapunov exponent (equal to the Kolmogorov-Sinai entropy rate where nonnegative) well, with a nearly constant bias. Periodic orbits give a finite permutation entropy, but the rate estimate would tend to zero given a sufficiently long word.

Theoretically, the asymptotic entropy rate, can be surprisingly tricky [12,9,1,7,8]. The theoretical definitions of entropy rate do not necessarily lead to good statistical methods, and superior alternatives have been developed over the many years since Shannon. We believe that some of these ideas may similarly be applicable to the permutation entropy situation. Figure 2 shows a very simple application of the part of the method of [12], fitting an empirical asymptotic scaling $\bar{H}^*(X_1^L) = h_{L=\infty} + C/L$ for $L = 13, 14$, comparing to the block estimate. This procedure shows a lower bias, but the specific choice of scaling region $L$ (as with block entropy) is a key empirical issue, and does not have a generally satisfactory resolution.

Also important for practical time-series analysis is the usual situation where observations of a predominantly deterministic source is contaminated with a small level of observational noise. Here, we recommend that the user fix some discretization level $\Delta$ characteristic of the noise, and evaluate the per-
mutation entropies via entropies of rank words evaluated from the discretized observables. Figure 3 shows analysis of permutations on significantly noise-contaminated signals, with no explicit $\Delta$ (i.e., it is the size of the numerical precision of the computations). The consequence is the permutation entropy is heavily dominated by the noise. Figure 4 shows the restoration of monotonic scaling with $h_\mu$ when an explicit, finite $\Delta = 0.2$ is used to discretize the data before rank variables are computed. Note that as computing ranks involves looking at the difference between noise contaminated variables, when the characteristic noise size is 0.1, as in this example, an appropriate discretization scale is 0.2.

For vector-valued sources, we applied lexicographic ordering and construction of outer product variables in the proof. For analyzing chaotic observed data, however, it may be acceptable to still use but one scalar projection, subject to the traditional caveats of time-delay embedology. We would expect that for appropriately mixing sources and generic observation functions, the Kolmogorov-Sinai entropy estimated through that scalar still equals the true value, and likewise so might permutation entropy rate. We have found that
numerically this appears to work in practice. With a direct higher-dimensional product space, the undersampling issue becomes even more difficult with increasing $L$, hence using scalars, as in a time-delay embedding, may turn out to be a superior approach for observed time-series of higher-dimensional sources.

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A Ergodic finite-alphabet information sources

Proof of Lemma 1 Given an ergodic information source $S$,

$$\lim_{k \to \infty} H_m(R^{k+1}_1|S_k) = \lim_{k \to \infty} H_m(S^{k+1}_1|S_k)$$
Fig. 4. (color online) Lyapunov exponent (black line, thick) of noise-free logistic map and permutation entropy rate estimates \( \hat{h} = \hat{H}^*(X_1^{\Delta 14}) \) for \( N = 10^4, 10^5, 10^6 \) length time series from the map (blue, red and black thin lines), contaminated with uniform zero-mean observational noise of width 0.1, and discretized to \( \Delta = 0.2 \). With this discretization the entropy estimate tracks the macroscopic entropy from the dynamics much better, though the bias is increased, as expected, since the entropy due to noise still has some effect.

for all \( l \geq 1 \). Consider \( R_{k+1} = \sum_{i=1}^{k+1} \delta(S_i \leq S_{k+1}) \). For \( a \in \{1, \ldots, N\} \) define the sample frequency of the letter \( a \) in the word \( S_1^{k+1} \) to be

\[
\vartheta_{k+1}^+(a) = \frac{1}{k + 1} \sum_{i=1}^{k+1} \delta(S_i = a).
\]

With the help of \( \vartheta_{k+1}(a) \) we may express \( R_{k+1} \) in terms of \( S_i, 1 \leq i \leq k + 1 \), namely,

\[
R_{k+1}(S_{k+1}) = (k + 1) \sum_{a=1}^{S_{k+1}} \vartheta_{k+1}(a),
\]

where we assume the outcomes \( S_1^{k+1} \) to be known. Then, the identity

\[
\Pr(R_{k+1} = y) = \sum_{q=1}^{N} \Pr(S_{k+1} = q) \delta(R_{k+1}(q) = y)
\]  \hspace{1cm} (A.1)
give us the probability for observing some $R_{k+1}$ with value $y \in \{1, \ldots, k + 1\}$ by means of $\Pr(S_{k+1} = q)$, $1 \leq q \leq N$. Since, given $S_k$, $R_{k+1}$ is a deterministic function of the random variable $S_{k+1}$, i.e., $\Pr(R_{k+1} = y | S_{k+1} = q) = \delta(R_{k+1}(q) = y)$, Eq. (A.1) can be seen as an application of the law of total probability.

Without loss of generality, we may first rearrange the sum in (A.1) to consider only those symbol values $q$ with non-zero $\Pr(S_{k+1} = q)$, summing to $N' \leq N$. Expand the sum,

$$
\Pr(R_{k+1} = y) = \Pr(S_{k+1} = 1) \delta[y = (k + 1)\vartheta_{k+1}(1)] 
+ \Pr(S_{k+1} = 2) \delta[y = (k + 1)(\vartheta_{k+1}(1) + \vartheta_{k+1}(2))]
+ \ldots + \Pr(S_{k+1} = N') \times \delta[y = (k + 1)(\vartheta_{k+1}(1) + \ldots + \vartheta_{k+1}(N'))].
$$

Suppose all the relevant sample frequencies $\vartheta_{k+1}(1), \ldots, \vartheta_{k+1}(N')$ are greater than zero. This means that for any $y$, only a single one of the $\delta$-functions can be nonzero, and hence we have a one-to-one transformation taking non-zero elements from the distribution $\Pr(S_{k+1})$ without change into some bin for $\Pr(R_{k+1})$. Since entropy is invariant to a renaming of the bins, and the remaining zero probability bins add nothing to the entropy, we conclude that, if $\vartheta_{k+1}(a) > 0$ for all $a$ where the true probability $\Pr(S_{k+1} = a) > 0$ (i.e., $a = 1, \ldots, N'$ after a hypothetical rearrangement), then $H_m(R_{k+1} | S_k^k) = H_m(S_{k+1} | S_k^k)$. Because of the assumed ergodicity, we can make the probability that $\vartheta_{k+1}(a) = 0$ when $\Pr(S_{k+1} = a) > 0$ to be arbitrarily small by taking $k$ to be sufficiently large, and the claim follows for $l = 1$.

This construction can be extended without change to words $S_{k+1}^{k+l}$ of arbitrary length $l \geq 1$ via

$$
\Pr(R_{k+1}^{k+l} = y_1 \ldots y_l) = \sum_{q_1, \ldots, q_l=1}^{N'} \left[ \Pr(S_{k+1}^{k+l} = q_1 \ldots q_l) \times \delta(R_{k+1}(q_1) = y_1) \times \ldots \times \delta(R_{k+l}(q_l) = y_l) \right].
$$

Observe that if $\vartheta_{k+1}(a) > 0$ for $1 \leq a \leq N'$, then the same happens with $\vartheta_{k+2}(a), \ldots, \vartheta_{k+l}(a)$ and $H(R_{k+1}^{k+l} | S_k^k) = H(S_{k+1}^{k+l} | S_k^k)$ follows. Again, ergodicity guarantees that there exist realizations $S_k^{k+l}$ whose sample frequencies fulfill the said condition. □

As way of illustration, suppose that $S_n = 0, 1$ are independent random vari-
ables with probability $\Pr(S_n = 0) = \Pr(S_n = 1) = \frac{1}{2}$. Given $S_1^k = s_1,\ldots,s_k \in \{0,1\}^k$, let $N_0 = \left| \{s_i = 0 \text{ in } S_i^k \} \right|$, $0 \leq N_0 \leq k$. Consider the case $L = 2$ in Lemma 1. There are two possibilities:

(i) $0 \leq N_0 \leq k$. Then

\[
S_{k+1}^{k+2} = 0, 0 \Rightarrow R_{k+1}^{k+2} = N_0 + 1, N_0 + 2 \\
S_{k+1}^{k+2} = 0, 1 \Rightarrow R_{k+1}^{k+2} = N_0 + 1, k + 2 \\
S_{k+1}^{k+2} = 1, 0 \Rightarrow R_{k+1}^{k+2} = k + 2, N_0 + 1 \\
S_{k+1}^{k+2} = 1, 1 \Rightarrow R_{k+1}^{k+2} = k + 1, k + 2
\]

Each of these events has the joint probability

\[
\Pr(N_0 = \nu, R_{k+1}^{k+2} = r_{k+1}^{k+2}) = \frac{\binom{k}{\nu}}{2^k} \cdot \frac{1}{4} = \frac{1}{2^k} \binom{k}{\nu}
\]

and conditional probability

\[
\Pr \left( R_{k+1}^{k+2} = r_{k+1}^{k+2} | N_0 = \nu \right) = \frac{1}{4},
\]

where $0 \leq \nu \leq k - 1$ and $r_{k+1}^{k+2} = (\nu + 1, \nu + 2), (\nu + 1, k + 2), (k + 2, \nu + 1)$ or $(k + 1, k + 2)$.

(ii) $N_0 = k$. Then

\[
S_{k+1}^{k+2} = 0, 0 \& S_{k+1}^{k+2} = 0, 1 \& S_{k+1}^{k+2} = 1, 1 \\
\Rightarrow R_{k+1}^{k+2} = k + 1, k + 2 \\
S_{k+1}^{k+2} = 1, 0 \Rightarrow R_{k+1}^{k+2} = k + 2, k + 1
\]

These events have the joint probabilities

\[
\Pr \left( N_0 = k, R_{k+1}^{k+2} = (k + 1, k + 2) \right) = \frac{1}{2^k} \cdot \frac{1}{4} \cdot 3 = \frac{3}{2^k}
\]
\[
\Pr \left( N_0 = k, R_{k+1}^{k+2} = (k + 2, k + 1) \right) = \frac{1}{2^k} \cdot \frac{1}{4} = \frac{1}{2^k}
\]

and conditional probabilities

\[
\Pr \left( R_{k+1}^{k+2} = (k + 1, k + 2) | N_0 = k \right) = \frac{3}{4}
\]
\[
\Pr \left( R_{k+1}^{k+2} = (k + 1, k + 2) | N_0 = k \right) = \frac{1}{4}.
\]
From (i) and (ii), we get

\[ H_m(R_{k+1}^{k+2}|S_1^k) = -4 \sum_{\nu=0}^{k-1} \frac{1}{2k+2} \binom{k}{\nu} \log \frac{1}{4} - \frac{3}{2k+2} \log \frac{3}{4} - \frac{1}{2k+2} \log \frac{1}{4} \]

\[ = 4 \left( \frac{2}{2k+2}(2k - 1) + \frac{8}{2k+2} - \frac{3}{2k+2} \log 3 \right) \]

\[ = 2 \left( 1 - \frac{3}{2k+3} \log 3 \right). \]

On the other hand,

\[ H_m(S_{k+1}^{k+2}|S_1^k) = H_m(S_{k+1}^{k+2}) = 2 \]

and so \( H_m(R_{k+1}^{k+2}|S_1^k) \) and \( H_m(S_{k+1}^{k+2}|S_1^k) \) are equal in the limit \( k \to \infty \), as guaranteed by Lemma 1.

\[ \text{B Non-ergodic finite-alphabet sources} \]

In order to deal with the general, non-ergodic case, we appeal to the theorem on ergodic decompositions [6]: If \( \Omega \) is a compact metrizable space and \( f : (\Omega, \mathcal{F}, \mu) \to (\Omega, \mathcal{F}, \mu) \) is continuous, then there is a partition of \( \Omega \) into \( f \)-invariant subsets \( \Omega_w \), each equipped with a sigma-algebra \( \mathcal{F}_w \) and a probability measure \( \mu_w \), such that \( f \) acts ergodically on each \( (\Omega_w, \mathcal{F}_w, \mu_w) \), the indexing set being another probability space \( (W, \mathcal{G}, \nu) \) (in fact, a Lebesgue space). Furthermore,

\[ \mu(E) = \int_W \int_E d\mu_w d\nu(w) = \int_W \mu_w(E) d\nu(w) \quad (E \in \mathcal{F}). \]

The family \( \{\mu_w : w \in W\} \) is called the ergodic decomposition of \( \mu \).

If \( \sigma \) is the shift on the (compact, metric) sequence space \((A^N, \mathcal{Z}, m)\), the indexing set can be taken to be itself, i.e.,

\[ m(C) = \int_{A^N} \int_C dm_s dm(s) = \int_{A^N} m_s(C) dm(s) \quad (C \in \mathcal{Z}), \quad (B.1) \]

where \( m_{\sigma(s)} = m_s \) [5]. This result shows that any source which is not ergodic can be represented as a mixture of ergodic subsources. The next lemma states that such a decomposition holds also for the entropy rate.

**Lemma 5 (Ergodic Decomposition of the Entropy Rate [5])** Let \((A^N, \mathcal{Z}, m, \sigma)\) be the sequence space model of a stationary finite alphabet source \( S = (S_n) \). Let
\{m_s : s \in A^N\} be the ergodic decomposition of \(m\). If \(h_m(s)\) is \(m\)-integrable, then

\[
h_m(S) = \int_{A^N} h_m(s) dm(s). \quad (B.2)
\]

**Theorem 6** Under the assumptions of Lemma 5, \(\liminf_{L \to \infty} \bar{H}^*_m(S^L_1) \geq h_m(S)\) holds for any finite alphabet source \(S\).

**PROOF.** Fix \(L \geq 2\). From (6) and (B.1),

\[
\begin{align*}
\bar{H}^*_m(S^L_1) &= -\frac{1}{L-1} \sum_{\pi \in \sigma_L} \left( \int_{A^N} m_s(C_\pi) dm(s) \right) \times \log \left( \int_{A^N} m_s(C_\pi) dm(s) \right) \\
&\geq -\frac{1}{L-1} \sum_{\pi \in \sigma_L} \left( \int_{A^N} m_s(C_\pi) \log m_s(C_\pi) dm(s) \right) \quad (B.3) \\
&= \int_{A^N} \left( -\frac{1}{L-1} \sum_{\pi \in \sigma_L} m_s(C_\pi) \log m_s(C_\pi) \right) dm(s) \\
&= \int_{A^N} h^*_m(S^L_1) dm(s),
\end{align*}
\]

where in (B.3) we have used Jensen’s inequality,

\[
\Phi \left( \int_{A^N} f d\mu \right) \leq \int_{A^N} \Phi \circ f d\mu,
\]

with \(\Phi(t) = t \log t\) convex in \([0, \infty)\) and \(f(s) = \mu_s(Q_\pi) \geq 0\).

Therefore,

\[
\begin{align*}
\liminf_{L \to \infty} \bar{H}^*_m(S^L_1) \\
&\geq \liminf_{L \to \infty} \int_{A^N} \bar{H}^*_m(S^L_1) dm(s) \quad (B.4) \\
&\geq \int_{A^N} \left( \liminf_{L \to \infty} \bar{H}^*_m(S^L_1) \right) dm(s) \quad (B.5) \\
&= \int_{A^N} h^*_m(S) dm(s), \quad (B.6)
\end{align*}
\]

where we have applied Fatou’s lemma in (B.5) to the sequence of positive and \(m\)-measurable functions \(\bar{H}^*_m(S^L_1)\). Observe that \(h^*_m(S)\) exists for all \(s \in A^N\) (and is \(m\)-integrable as a function of \(s\)) since \(h^*_m(S) = h_m(s)\) by Theorem 1.
(S acts ergodically on \((A_s^N, Z_s, m_s)\)). Therefore,

\[
\lim_{L \to \infty} \inf h^*_m(S_{L1}^t) \geq \int_{A^N} h_m(S)dm(s) = h_m(S)
\]

by (B.2). □

Theorem 6 and Eqs. (B.4) and (B.6) yield:

**Corollary 7** If \(h^*_m(S) = \lim_{L \to \infty} \bar{H}_m^*(S_{L1}^r)\) exists for a non-ergodic finite-alphabet source S, then \(h^*_m(S) \geq h_m(S)\) and \(h^*_m(S) \geq \int_{A^N} h^*_m(S)dm(s)\).

**C Interval maps**

Suppose first that \(I\) is a one-dimensional interval and \(f : I \to I\) an ergodic and \(\mu\)-preserving transformation, where \(\mu\) is a measure on \((I, \mathcal{B} \cap I), \mathcal{B}\) being Borel sigma-algebra of \(\mathbb{R}\).

**Lemma 8** If \(f : I \to I\) is ergodic and \(h_\mu(f) < \infty\), then \(\lim_{L \to \infty} \bar{H}_\mu^*(f, L) \geq h_\mu(f)\). See (10) for the definition of \(\bar{H}_\mu^*(f, L)\). It follows, \(h^*_{\mu, BKP}(f) \geq h_\mu(f)\).

**PROOF.** Let \(\gamma\) be a finite generator of \(f\) (Krieger’s Theorem, [13]). We split the proof in two parts. In the first part we follow the approach of [2, Sect. 3].

**Case 1.** Suppose that the elements of \(\gamma\) are connected sets (intervals) or, more generally, that all elements of \(\gamma\) consist of a finite number of intervals. In either case, taking if necessary a refinement of \(\gamma\) (thus, also a generator) that we call \(\gamma\) as well, we write without restriction \(\gamma = \{I_j, 1 \leq j \leq |\gamma|\}\), were \(I_j \subset I\) are intervals. This being the case, let \(c_1 < c_2 < ... < c_{|\gamma| - 1}\) be the points that subdivide the interval \(I = [a, b]\) into the \(|\gamma|\) intervals \(I_j\) of the generator \(\gamma\). We consider a fixed \(P_\pi \in \mathcal{P}_L^*\) and show that it can intersect at most \((L + 1)^{|\gamma| - 1}\) sets of the partition \(\gamma_0^{L-1} := \bigvee_{i=0}^{L-1} f^{-i}(I_{j_i})\) with \(I_{j_0}, ..., I_{j_{|\gamma| - 1}} \in \gamma\). For \(x \in P_\pi\), let \(\Delta_L[x]\) denote the set in \(\gamma_0^{L-1}\) that contains \(x\). Thus, \(\Delta_L[x]\) can be written as \(I_{j_0} \cap f^{-1}(I_{j_1}) \cap ... \cap f^{-(L-1)}(I_{j_{L-1}})\) with \(I_{j_0}, ..., I_{j_{L-1}} \in \gamma\), so that it can be specified by the \(n\)-tuple \(j[x] = (j_0, ..., j_{L-1}) \in \{1, ..., |\gamma|\}^L\).

Now, \(\pi\) is given by inequalities \(x_{k_1} < ... < x_{k_L}\) with \(\{k_1, ..., k_L\} = \{0, ..., L - 1\}\) and \(x_k = f^k(x)\). For each \(x \in P_\pi\) we can extend these inequalities so that they give the common order of the \(c_r\) and the \(x_{k_i}\), where \(1 \leq r \leq |\gamma| - 1\) and \(1 \leq l \leq L\). It follows that there are at most \((L + 1)^{|\gamma| - 1}\) possible extended orders since each \(c_r\) has \(L + 1\) possible bins to go among the \(x_{k_i}\) (as \(x\) varies in \(P_\pi\), the \(L\) points \(x_{k_i}\) defining the bins move but do not cross each other). Moreover, when
we know the common order of the $c_x$ and $x_{k_i}$, then $j[x]$ is uniquely determined (since $c_{j-1} < x_k < c_j$, implies $x_k \in I_j$ and thus $x \in f^{-k}(I_j)$, with $1 \leq j \leq |\gamma|$, $c_0 \equiv a$ and $c_{|\gamma|} \equiv b$).

Each $P_{x} \in \mathcal{P}_{L}^{*}$ is then the union of at most $(L + 1)^{|\gamma|-1}$ sets $V_k \in \gamma_0^{L-1} \lor \mathcal{P}_{L}^{*}$ with total measure $\mu(P_{x})$. Hence,

$$
- \frac{(L+1)^{|\gamma|-1}}{} \sum_{k=1}^{(L+1)^{|\gamma|-1}} \mu(V_k) \log \mu(V_k)
\leq - \frac{(L+1)^{|\gamma|-1}}{} \sum_{k=1}^{(L+1)^{|\gamma|-1}} \mu(P_{x}) \log \frac{\mu(P_{x})}{(L+1)^{|\gamma|-1}}
= - \mu(P_{x}) \log \mu(P_{x}) + (|\gamma| - 1) \mu(P_{x}) \log(L + 1)
$$

and therefore, summing over all $\pi \in \sigma_L$,

$$
H_{\mu}(\gamma_0^{L-1}) \leq H_{\mu}(\gamma_0^{L-1} \lor \mathcal{P}_{L}^{*}) \leq H_{\mu}(\mathcal{P}_{L}^{*}) + (|\gamma| - 1) \log(L + 1). \quad (C.1)
$$

It follows

$$
\frac{1}{L-1} H_{\mu}(\mathcal{P}_{L}^{*}) \geq \frac{1}{L-1} \left[ H_{\mu}(\gamma_0^{L-1}) - (|\gamma| - 1) \log(L + 1) \right]
$$

and

$$
\lim_{L \to \infty} \inf \frac{1}{L-1} H_{\mu}(\mathcal{P}_{L}^{*}) \geq h_{\mu}(f) \quad (C.2)
$$

since $\gamma$ is a generator of $f$. Definition (10) completes the proof in this case.

Case 2. If some component of $\gamma$ consists of infinitely many intervals, we can define a sequence of interval partitions $(\gamma_n)_{n \in \mathbb{N}}$ (Case 1) such that $\mathcal{A}(\gamma_n)$, the finite sigma-algebras generated by the $\gamma_n$, build an increasing sequence and $\lor_{n=1}^{\infty} \mathcal{A}(\gamma_n) = \mathcal{B} \ (\mod 0)$. Then $h_{\mu}(f) = \lim_{n \to \infty} h_{\mu}(f, \gamma_n)$ [13].

We claim that, also in this case, Eq. (C.2) holds. Otherwise, for every $\varepsilon > 0$ and for every $L \geq 2$, there exists $L' > L$ such that

$$
\frac{1}{L'-1} H_{\mu}(\mathcal{P}_{L'}^{*}) < h_{\mu}(f) - \varepsilon. \quad (C.3)
$$

Take now $n_0$ such that $|h_{\mu}(f) - h_{\mu}(f, \gamma_n)| < \varepsilon$ for all $n \geq n_0$. From (C.3) it follows

$$
\frac{1}{L'-1} H_{\mu}(\mathcal{P}_{L'}^{*}) < h_{\mu}(f, \gamma_{n_0}) \leq \frac{1}{L'-1} H((\gamma_{n_0})_{0}^{L'-1})
$$

because $\frac{1}{L} H((\gamma_{n_0})_{0}^{L'-1})$ decreases monotonically to $h_{\mu}(f, \gamma_{n_0})$. Use now (C.1) to deduce
\[
\frac{1}{L' - 1} H_\mu(\mathcal{P}_{L'}) < h_\mu(f, \gamma_{n_0}) \leq \frac{1}{L' - 1} H_\mu(\mathcal{P}_{L'}) + \frac{|\gamma_{n_0}| - 1}{L' - 1} \log(L' + 1).
\]

But the last term can be made arbitrarily small because the \( L' \) fulfilling (C.3) form an unbounded subsequence and \( n_0 \) is independent of \( L' \). This contradiction proves our claim and completes the proof. \( \square \)

More generally, let \( I^d \) be now a proper, lexicographical ordered interval of \( \mathbb{R}^d \).

**Theorem 9** Let \( f \) be an ergodic interval map in \( \mathbb{R}^d \) fulfilling the above assumptions. If \( h_\mu(f) < \infty \), then \( \lim \inf_{L \to \infty} \bar{H}_\mu(f, L) \geq h_\mu(f) \), where the permutation entropy is defined by means of the product order of \( \mathbb{R}^d \).

**Proof outline** As in Lemma 8, we split again its proof in two cases. If \( (\text{Case 1}) \) the generating partition is a product partition or can be refined to a product partition

\[
\gamma = \{ I^d_i, 1 \leq i \leq |\gamma| \}, \quad \prod_{i} = [a_{1}^{(i)}, b_{1}^{(i)}] \times \ldots \times [a_{d}^{(i)}, b_{d}^{(i)}],
\]

(whose elements are, without restriction, lexicographically ordered), then the same approach used for one-dimensional intervals works through to Eq. (C.2). Otherwise \( (\text{Case 2}) \), each element of \( \gamma \) is the countable union of disjoint intervals. They allow to define (after an eventual refinement) a sequence of product partitions \( (\gamma_n)_{n \in \mathbb{N}} \) \( (\text{Case 1}) \) such that \( h_\mu(f) = \lim_{n \to \infty} h_\mu(f, \gamma_n) \). The proof that \( \lim \inf_{L \to \infty} \bar{H}_\mu(f, L) \geq h_\mu(f) \) is then completed again by contradiction. \( \square \)

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