Free Fractions: An Invitation to (applied) Free Fields

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Abstract
Long before we learn to construct the field of rational numbers (out of the ring of integers) at university, we learn how to calculate with fractions at school. When it comes to “numbers”, we are used to a commutative multiplication, for example $2 \cdot 3 = 6 = 3 \cdot 2$. On the other hand —even before we can write— we learn to talk (in a language) using words, consisting of purely non-commuting “letters” (or symbols), for example $xy \neq yx$ (with the concatenation as multiplication). Now, if we combine numbers (from a field) with words (from the free monoid of an alphabet) we get non-commutative polynomials which form a ring (with “natural” addition and multiplication), namely the free associative algebra. Adding or multiplying polynomials is easy, for example $(\frac{1}{3}xy + z) + \frac{1}{3}xy = xy + z$ or $2x(yx + 3z) = 2xyx + 6xz$. Although the integers and the non-commutative polynomials look rather different, they share many properties, for example the unique number of irreducible factors: $x(1 - yx) = x - xyx = (1 - x)yx$. However, the construction of the universal field of fractions (aka “free field”) of the free associative algebra is highly non-trivial (but really beautiful). Therefore we provide techniques (building on the work of Cohn and Reutenauer) to calculate with free fractions (representing elements in the free field or “skew field of non-commutative rational functions”) to be able to explore a fascinating non-commutative world.

Keywords: free associative algebra, universal field of fractions, minimal linear representation, admissible system, rational operations, non-commutative rational functions

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Introduction
Since most of the literature on free fields is almost inaccessible without a degree in mathematics and difficult without a specialization in algebra we want to provide an

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introduction with focus on the application. One of the main hurdles is the huge number of concepts and definitions (for precise formulations), needing a lot of time to digest. Even if non-commutativity (as we understand it here) is rather natural, one needs to get used to it. Just to test “non-commutative” awareness: \((x + y)^2 = \ldots ?\)

Here we restrict ourself to the simplest free fields, coming from the embedding of the ring of non-commutative polynomials (over a commutative field and a finite alphabet) into its universal field of fractions [Coh06, Chapter 7]. A “soft” introduction is [Coh03, Section 9.3]. The tools (or techniques) we are going to use are mainly based on the work of Cohn and Reutenauer [CR99]. We work directly with (a special form of) linear representations (aka “free fractions”) to add, multiply and invert (non-zero) elements in the free field. Usually one has to be careful and distinguish between an element and a representation (for it). We know that there are several different “classical” fractions for one element, for example \(\frac{1}{2} = \frac{3}{6} = \frac{2}{4} = r \in \mathbb{Q}\). What we learn in school is a test to check whether two fractions are “equal”, that is, representing the same element. This is the so-called word problem.

In the general case —for elements in the free field—, the word problem is rather difficult because of the much more complicated representations. Therefore it will take some effort, to learn a number of tools from [Sch18b] (word problem, minimal inverse), [Sch18c] (polynomial factorization), [Sch17] (general factorization theory) and [Sch18a] (constructing minimal linear representations) to be able to work with non-commutative fractions. However, these techniques enable also the implementation in computer algebra software. For further remarks on the latter (in German) we refer to [Sch18d, Section B.5]. The perfect theoretical introduction to fractions is [Coh84].

Section 1 is meant to get acquainted with the basic notation. The most important basic techniques are presented directly in Section 2 (calculating), Section 3 (factorizing) and Section 4 (minimizing). In a first reading, the (sub)sections marked with “⋆” can be skipped. Those marked with “⋆⋆” serve as a reference for further reading.

Remark. It should be noted that there are some minor differences (in notation and definitions) between the main publications due to the consecutive development. The main reference (and most coherent presentation) is [Sch18d] (in German), its structure of chapters and sections corresponds to sections and subsections here. Those who are mainly interested in polynomials should have a look on [Sch18a, Remark 1.10] before reading [Sch18c]. A very rich theoretical resource with focus on free associative algebras is [Coh74].

1 Representing Elements

First of all, we need a suitable representation for the elements in the free field \(F = \mathbb{K}((X))\) of the free associative algebra \(\mathbb{K}\langle X \rangle\) over the commutative field \(\mathbb{K}\) (for example the rational numbers \(\mathbb{Q}\) or the real numbers \(\mathbb{R}\)) and the (finite) alphabet \(X = \{x_1, x_2, \ldots, x_d\}\) (usually \(X = \{x, y, z\}\)). Here we use a special form of a linear representation of Cohn and Reutenauer [CR94], namely admissible linear systems.
To illustrate such a system, we consider a linear system of equations $A s = v$ of dimension $n \in \mathbb{N} = \{1, 2, 3, \ldots \}$, that is, we have $n$ unknown components $s_1, s_2, \ldots, s_n$ in the solution vector $s$ (and also $v$ is a column vector with $n$ rows). If $A$ is invertible, we can write $s = A^{-1}v$. Now let $n = 1$ with $A = a \in \mathbb{Z} \setminus \{0\}$ and $v \in \mathbb{Z}$ (integer entries). Then $s = a^{-1}v = \frac{v}{a}$ is a representation for a rational number $s \in \mathbb{Q}$. Now, given $s_1 = \frac{v_1}{a_1}$ and $s_2 = \frac{v_2}{a_2}$, we can compute the sum $s_1 + s_2 \in \mathbb{Q}$ by solving the linear system $A^t s' = v'$,

$$
\begin{bmatrix}
a_1 & -a_1 \\
0 & a_2
\end{bmatrix}
\begin{bmatrix}
s_1 + s_2 \\
s_2
\end{bmatrix}
= \begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}.
$$

(Notice the upper triangular form of the system matrix $A'$, the “blocks” in the diagonal —here they have size $1 \times 1$— are called pivot blocks.) Usually we are interested in the first component of the solution vector $s$. If $a_1$ and $a_2$ are invertible, then $A$ is invertible. In that case we call $A s = v$ an admissible linear system (ALS) for short. In other words: An ALS can represent a rational number. More general, one can view an ALS as a “generalized” fraction.

Important: $A$ has to be “invertible” (for $n = 1$ we need $A \neq 0$, for $n > 1$ we need to clarify the meaning). One can extract the first component using the first identity (row) vector $u = e_1^T = [1, 0, \ldots, 0]$, that is, the desired element $f = s_1 = u A^{-1} v$. The triple $\pi_f = (u, A, v)$ is called a linear representation of $f \in \mathbb{F}$. (Recall that usually we represent a rational number $r$ by a tuple of integers $(v, a)$, that is, $r = \frac{v}{a} = v a^{-1} = a^{-1} v = 1 \cdot a^{-1} v$. So here we could write $\pi_r = (1, a, v)$.)

For the polynomial $f = xy + yx - yz \in \mathbb{K}\langle X \rangle$ an ALS $A_f = (u, A, v)$ of dimension $n = 4$ is (the zeros are replaced by lower dots to emphasize the structure)

$$
\begin{bmatrix}
1 & -x & -y & . \\
1 & . & -y & . \\
. & 1 & z - x & . \\
. & . & 1 & 1
\end{bmatrix}
\begin{bmatrix}
s_1 + s_2 \\
s_2
\end{bmatrix}
= \begin{bmatrix}
v_1 \\
v_2
\end{bmatrix},
$$

$s = \begin{bmatrix}
xy + y(x - z) \\
y \\
x - z \end{bmatrix}$.

Let $A = (a_{ij})$. The solution can be easily computed (starting from the bottom): $s_4 = 1$ and $s_1 + a_{1,i+1}s_{i+1} + \ldots + a_{1,n}s_n = 0$ for $i = 3, 2, 1$. For this special form we have invertibility of $A$ (already over $\mathbb{K}\langle X \rangle$). Is it possible to represent $f = xy + yx - yz$ by a smaller system? And, if necessary, how could one construct a minimal ALS? These are fundamental questions here, their (general) answering needs some patience.

Later we will define the rank of an element $f \in \mathbb{F}$ by the dimension of a minimal admissible linear system (for $f$). For a word/monomial, for example $g = xyz$, an ALS can easily be stated:

$$
\begin{bmatrix}
1 & -x & . & . \\
1 & . & -y & . \\
. & 1 & -z & . \\
. & . & 1 & 1
\end{bmatrix}
\begin{bmatrix}
s_1 + s_2 \\
s_2
\end{bmatrix}
= \begin{bmatrix}
xyz \\
yz \\
z \\
1
\end{bmatrix}.$$

3
Intuitively here it is somehow clearer (compared to the system for $f$ before) that this ALS is minimal, but we have to make that more precise. The first goal will be to define “simple” rational operations on the level of these representations (systems), for example to scale, to add or to multiply elements (Proposition 2.10). That is not difficult but soon ponderous since the systems become bigger and bigger. And before we invert (take the reciprocal value of) an element, we have to ensure that this is allowed. If a system is minimal, also that is easy. An ALS for the sum of $f_1 = 2x$ and $f_2 = 3y$ is

$$
\begin{bmatrix}
1 & -x & -1 \\
1 & . & . \\
. & 1 & -y \\
. & . & 1
\end{bmatrix}
\begin{bmatrix}
s_1 \\
s_2 \\
s_3
\end{bmatrix}
= \begin{bmatrix}
2 \\
3
\end{bmatrix}.
$$

What is the solution vector $s$? Is that system minimal for $f = f_1 + f_2 = 2x + 3y$?

Let $R = \mathbb{K}(X)$. A (square) matrix $A \in R^{n \times n}$ is called full, if $A = PQ$ with $P \in R^{n \times m}$ and $Q \in R^{m \times n}$ implies $m \geq n$ [CR99]. To show that the full matrices over the free associative algebra are those which are invertible over the free field (and vice versa) is very difficult. For details we refer to [Coh06]. Important for us is that we can “address” each element $f$ in the free field via a linear representation [CR99], that is, $\pi_f = (u, A, v)$ with (for some $n \in \mathbb{N}$) $u \in \mathbb{K}^{1 \times n}$, full $A \in R^{n \times n}$ with entries of the form $\lambda_0 + \lambda_1 x_1 + \ldots + \lambda_d x_d$ with $\lambda_i \in \mathbb{K}$ and $x_i \in X$, $v \in \mathbb{K}^{n \times 1}$ and $f = uA^{-1}v$. If $u = [1, 0, \ldots, 0]$ we call $\pi_f$ an admissible linear system and write $A_f = \pi_f$.

Remark. The only non-invertible element in the rational numbers is zero. In our case, the non-invertible (square) matrices are the non-full matrices. Although the definition (of full matrices) is simple, testing fullness is very hard even for a linear matrix. An example for a non-full matrix is

$$
A = \begin{bmatrix}
z & . & . \\
x & . & . \\
y & -x & 1
\end{bmatrix} = \begin{bmatrix}
z & 0 \\
x & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
y & -x & 1
\end{bmatrix}.
$$

### 1.1 Free Fractions

The main idea (of free fractions) is as simple as in the usage of “classical” fractions (for elements in $\mathbb{Q}$): calculating, factorizing and minimizing (or cancelling), for example

$$
\frac{2}{3} \cdot \frac{3}{4} = \frac{6}{12} = \frac{2 \cdot 3}{2 \cdot 2 \cdot 3} = \frac{1}{2} \quad \text{or}
\frac{1}{2} + \frac{3}{4} = \frac{4}{2} = \frac{2 \cdot 2}{2} = 2.
$$

At some point one stops this loop and uses the fraction (with coprime numerator and denominator, that is, their greatest common divisor is 1 or $-1$). However, the
application (in our context) is not that easy. For a concrete expression like \( f = x^{-1}z^{-1}yz^{-1} = x^{-1}yz^{-1} \) one can find a simpler (and therefore a smaller ALS), for example
\[
\begin{bmatrix}
x & y \\
. & z
\end{bmatrix}
\begin{bmatrix}
. \\
1
\end{bmatrix}.
\]

But what should one do with \( g = x - (x^{-1} + (y^{-1} - x)^{-1})^{-1} \) from Example 2.14? (Hint: \( g \) is a polynomial.)

Additionally, we need minimal admissible linear systems for the factorization, therefore we would run into troubles if we need the factorization for the minimization.

The key idea to resolve this “dependencies” can be guessed already in the classical setting: One can remember the factorization of the numerator (for the product) and the denominator (for the sum and the product). The latter corresponds to the standard form (Definition 4.3).

There are a lot of definitions in Section 2.1 (and even more in [Sch18a, Section 1]). For an overview the mostly used will be introduced by examples. We take an element \( f \) in the free field \( \mathbb{F} \) given by the admissible linear systems \( A = (u, A, v) \) of dimension \( n = 4 \). (For a rational number \( r \in \mathbb{Q} \) we can write \( A_r = (1, a, v) \), that is, \( r = 1 \cdot a^{-1}v = \frac{a}{n} \).) Recall that \( f = uA^{-1}v \). If we write \( s = A^{-1}v \), then \( f \) is the first component of the solution vector \( s \) in the system of “row” equations \( As = v \). The \( n \)-tuple \((s_1, s_2, \ldots, s_n)\) of entries in \( s \) is called left family. (The column solution vector \( s \) and the left family are used synonymously.)

But before we take a closer look on this system of equations, we examine the “column” equations \( u = tA \), in which \( f \) can be expressed as a linear combination of the components of the row solution vector \( t = [t_1, t_2, \ldots, t_n] \). Here, underlined entries denote static entries, that is, they must not be changed. If we describe (elementary) transformations in the following, they always refer to the system matrix \( A \).

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
t_1 \\
t_2 \\
t_3 \\
t_4
\end{bmatrix}
= \begin{bmatrix}
1 - x & -x & -x \\
1 & y & 1 & -2 \\
. & 1 & . & -x \\
. & . & . & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
t_1 \\
t_2 \\
t_3
\end{bmatrix}
\]

The equations (starting from the left) are
\[
1 = t_1(1 - x) + t_2, \\
0 = t_2y + t_3, \\
0 = -t_1x + t_2 \quad \text{and} \\
0 = -t_1x - 2t_2 - t_3x + t_4.
\]

Instead of computing the solution \( t \) immediately, we will transform the system in such
a way that this will be easier. Now we take a look on the system $As = v$:

\[
\begin{bmatrix}
1 - x & -x & -x \\
1 & y & 1 - 2 \\
. & 1 & . -x \\
. & . & 1
\end{bmatrix}
\begin{bmatrix}
s_1 \\
s_2 \\
s_3 \\
s_4
\end{bmatrix}
= 
\begin{bmatrix}
. \\
-4 \\
. \\
2
\end{bmatrix}
\]

This is the system matrix $A$, left family $s = A^{-1}v$, right hand side $v$. One equation, namely $s_4 = 2$, is especially easy to solve. Here we have $\kappa_1 s_1 + \kappa_2 s_2 + \kappa_3 s_3 + \kappa_4 s_4 = 1$ for $\kappa_1 = \kappa_2 = \kappa_3 = 0$ and $\kappa_4 = \frac{1}{2}$, therefore we write $1 \in L(A)$, the linear span (over $\mathbb{K}$) of the left family. (If there were not such a linear combination, we would write $1 \not\in L(A)$.) We use an analogous notation for the linear span of the right family $R(A)$. Normally, we must distinguish between the element $f$ and the representation $A$. If $A$ is minimal (which is the case here), we can define the rank of $f$ as the dimension of $A$, rank $f := \dim A$. In this case we say “$f$ is of type $(*,1)$” or $1 \in L(f)$ if $1 \in L(A)$ respectively “$f$ is of type $(1,*)$” or $1 \in R(f)$ if $1 \in R(A)$.

Now we will transform this representation step by step such that the solution of both systems of equations, that is, the computation of $s$ and $t$, becomes easier. Those families play a crucial role in characterizing minimality of a linear representation. However, the goal in fact will be, that we do not have to compute these solutions at all because, in general, this would not help us. Usually we write $s$ and $t$ (without its components) in “generic” form. The look “inside” (into the representation) is only for explanation. After the following transformation one should not forget this “inspection” and the computation of the “new” solutions $s$ and $t$ because this helps to understand the naming in left respectively right family.

Firstly we add 2-times row 4 to row 2 (for the solution vector $t$ this means that we subtract 2-times $t_2$ from $t_4$). Then we exchange columns 2 and 3 (for $s$ this means to exchange $s_2$ and $s_3$) and subtract (the new) column 2 from column 1. We collect these elementary transformations in the admissible transformation $(P, Q)$, that is, the first component in the solution vector $s$ does not change, with

\[
P = \begin{bmatrix}
1 & . & . \\
. & 1 & 2 \\
. & . & 1 \\
. & . & 1
\end{bmatrix}
\quad \text{and} \quad
Q = \begin{bmatrix}
1 & 0 & 0 & 0 \\
. & . & 1 & . \\
-1 & 1 & . & . \\
. & . & . & 1
\end{bmatrix}.
\]

(Figure 1 on page 33 gives an overview of different transformation matrices.) Applying this transformation we obtain a new representation $A' = (u', A', v') = PAQ$,

\[
A' = (uQ, PAQ, P v) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
. & 1 & y & . \\
. & . & 1 & -x \\
. & . & . & 1
\end{bmatrix}.
\]
The first component of the (new) solution vector $s$ is (still) $f = 2x - 2xy$. Those who are not yet satisfied, can either subtract row 3 from row 1 or column 2 from column 4 and imagine our element alternatively as $x(2 - 2yx)$ or $(1 - xy)2x$. This will be closer investigated in Section 3. For polynomials we always find such a form with $n$ (scalar) “pivot blocks” of size $1 \times 1$. This is not possible in general, but we will try to obtain small pivot blocks. Either by factorization (Section 3) or by “abstract” refinement (Section 4). But we should not worry here. The examples in the beginning are such that we can easily minimize them by “hand” respectively check their minimality.

A last note concerning the system matrix $A$. We always write it in the compact form with (at most) linear entries (of non-commutative polynomials). In fact, $A$ can also be interpreted as linear matrix pencil $A = (A_0, A_1, \ldots, A_d)$ with coefficient matrices $A_i \in K^{n \times n}$ for an alphabet $X = \{x_1, \ldots, x_d\}$, also written as $A = A_0 + A_1x_1 + \ldots + A_dx_d$. For an implementation one can use a list of (square) matrices of size $n+1$. For the example $(x - xyx)^{-1}$ from the beginning of Section 3 with respect to the monomials $(1, x, y)$ we have

\[
\begin{pmatrix}
[1 & \ldots & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
\]

1.2 Left and Right Minimization Steps

For practical computations we repeatedly have to make admissible linear systems smaller. In concrete situations it is possible to minimize them. Later, in Section 4 we will see that there are some subtle details behind the rather simple looking (left and right) “minimization steps”. Let us take a closer look on the example $2x + 3y$ from before:

\[
\begin{pmatrix}
1 & -x & -1 \\
& 1 & 1 \\
& & 1 & 1 -y \\
& & & 1
\end{pmatrix}
\begin{pmatrix}
2 \\
2 \\
3
\end{pmatrix}
= 
\begin{pmatrix}
2x + 3y \\
2 \\
3
\end{pmatrix}

\]

First we try a “left” minimization step, that is, eliminate a component of the left family. For that we subtract $\frac{2}{3}$-times row 4 from row 2 and add $\frac{2}{3}$-times column 2 to column 4:

\[
\begin{pmatrix}
1 & -x & -1 & -\frac{2}{3}x \\
& 1 & 0 & 0 \\
& & 1 & 1 -y \\
& & & 1
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
3
\end{pmatrix}
= 
\begin{pmatrix}
2x + 3y \\
0 \\
3
\end{pmatrix}
\]
The second row reads \( s_2 = 0 \). That is, for the solution \( s_1 \) there is no contribution from (the new) \( s_2 \). Therefore we can remove the equation \( s_2 = 0 \) and the variable \( s_2 \) from our system of equations. Hence we get the following (not yet minimal) ALS for \( 2x + 3y \):

\[
\begin{bmatrix}
 1 & -1 & -\frac{2}{3}x \\
 1 & -y & \\
 . & 1 & 1 \\
 . & . & 1
\end{bmatrix}
= \begin{bmatrix}
 2x + 3y \\
 3 \\
 3
\end{bmatrix}.
\]

It is obvious that now it is possible to apply a “right” minimization step to eliminate \( t_2 \) (in the right family). In fact it is not necessary to compute the left or the right family at all to “minimize” (without checking minimality).

Minimality of a linear representation can be characterized by \( \mathbb{K} \)-linear independence of the entries of the column solution vector \( s = A^{-1}v \) (the left family) and \( \mathbb{K} \)-linear independence of the entries of the row solution vector \( t = uA^{-1} \) (the right family) [CR94, Proposition 4.7].

Since in general this is not easy to check we will investigate conditions (on the structure of the system matrix) in Section 4 such that we can guarantee minimality if no more (block) row and column minimization steps are possible.

**Example 1.1.** Sometimes a minimization is only possible in “blocks”. Now we consider the ALS \( \mathcal{A} = (u, A, v) \) for \( f f^{-1} = 1 \) with \( f = xy - z \),

\[
\mathcal{A} = \left( \begin{bmatrix} 1 \ . \ . \ . \ 1 \ -x \ z \ . \ . \ . \ 1 \ -y \ . \ . \ . \ 1 \ -1 \ . \ . \ . \ y \ -1 \ . \ . \ . \ -z \ x \ 1 \end{bmatrix}, \begin{bmatrix} 1 \ -x \ z \ . \ . \ . \ 1 \ -y \ . \ . \ . \ 1 \ -1 \ . \ . \ . \ y \ -1 \ . \ . \ . \ -z \ x \ 1 \end{bmatrix} \right).
\]

Here we can create an upper right block of zeros of size \( 3 \times 2 \) in \( A \) by (as a first step) adding column 3 to column 4 and row 4 to row 2:

\[
\mathcal{A}' = \left( \begin{bmatrix} 1 \ . \ . \ . \ 1 \ -x \ z \ z \ . \ . \ . \ 1 \ -y \ -1 \ . \ . \ . \ 1 \ 0 \ 0 \ . \ . \ . \ y \ -1 \ . \ . \ . \ -z \ x \ 1 \end{bmatrix}, \begin{bmatrix} 1 \ -x \ z \ z \ . \ . \ . \ 1 \ -y \ -1 \ . \ . \ . \ 1 \ 0 \ 0 \ . \ . \ . \ y \ -1 \ . \ . \ . \ -z \ x \ 1 \end{bmatrix} \right).
\]

\(^1\)This ALS can be constructed in the following way: One starts with a minimal ALS of dimension 3 for the monomial \( xy \) (Proposition 2.9). Since \( xy \) is of type \((1, 1)\), one can immediately “add” \( z \) in the upper right entry of the system matrix to get a minimal ALS for \( xy - z \). For the inverse we use the minimal inverse (Theorem 2.13). And finally, using the multiplication (Proposition 2.10) we obtain an ALS of dimension 5 for \( ff^{-1} \).
And (as a second step) adding column 2 to column 5 and row 5 to row 1:

$$
\mathcal{A}'' = \begin{pmatrix}
1 & \ldots & . \\
1 & \ldots & . \\
1 & \ldots & . \\
1 & \ldots & . \\
1 & \ldots & . \\
\end{pmatrix},
\begin{pmatrix}
1 & -x & z & 0 & 0 \\
. & 1 & -y & 0 & 0 \\
. & . & 1 & 0 & 0 \\
. & . & y & -1 & . \\
. & . & -z & x & 1 \\
\end{pmatrix},
\begin{pmatrix}
1 \\
\end{pmatrix}.
$$

Now we can invert the lower 2 × 2 diagonal block (over the free field \( F \)) and obtain \( t''_4 = t''_5 = 0 \) (due to the zeros in the corresponding entries in \( u \)). Hence we get the (non-minimal) ALS of dimension 3,

$$
\mathcal{A}''' = \begin{pmatrix}
1 & . & . \\
1 & . & . \\
1 & . & . \\
\end{pmatrix},
\begin{pmatrix}
1 & -x & z \\
. & 1 & -x \\
. & . & 1 \\
\end{pmatrix},
\begin{pmatrix}
1 \\
0 \\
0 \\
\end{pmatrix}.
$$

Notice, that the lower entries in the right hand side are zero. Therefore a left block minimization step yields immediately the minimal system \( \mathcal{A}''' = (1, [1], 1) \) for \( 1 \in F \).

**Notation.** Given an ALS \( \mathcal{A} = (u, A, v) \) with \( v = [0, \ldots, 0, \lambda]^\top \) we write also write \( \mathcal{A} = (1, A, \lambda) \).

**Remark.** The other case \( f^{-1}f = 1 \) is somewhat more difficult because we must not change the first component in the left family. The trick here is, to work with an “extended” ALS for \( 1 \cdot f^{-1}f \), using Proposition 2.10 to multiply “1 from the left”. For details and illustration see [Sch18a, Remark 4.3 respectively Example 4.5].

**Remark.** In some cases it is possible to do a left and a right minimization step simultaneously. This is used in [Sch18a, Example 5.4] to compute the left greatest common divisor of two polynomials \( p \) and \( q \) by minimizing an ALS for \( p^{-1}q \).

## 2 Calculating

One of the main parts of this section is the construction of a minimal admissible linear system for the inverse (of an element in the free field) in Section 2.5. The following (simple) construction (of an ALS for the inverse) is from Proposition 2.10. We assume that we have given the inverse of a monomial \( f = xyz \) by the ALS \( \mathcal{A}' = (u', A', v') \),

$$
\begin{bmatrix}
z & -1 \\
y & -1 \\
x & . \\
\end{bmatrix} s = \begin{bmatrix} . \\
. \\
1 \\
\end{bmatrix},
\begin{bmatrix} . \\
. \\
. \\
\end{bmatrix} s = \begin{bmatrix}
z^{-1}y^{-1}x^{-1} \\
y^{-1}x^{-1} \\
x^{-1} \\
\end{bmatrix}.
$$

Checking also the \( \mathbb{K} \)-linear independence of the right family, the minimality is clear immediately. A (minimal) ALS for \( f \) is given by

$$
\begin{bmatrix}
. & z & -1 \\
. & y & -1 \\
-1 & . & x \\
. & 1 & . \\
\end{bmatrix} s = \begin{bmatrix} . \\
. \\
. \\
1 \\
\end{bmatrix},
\begin{bmatrix} . \\
. \\
. \\
1 \\
\end{bmatrix} s = \begin{bmatrix} xyz \\
1 \\
z \\
yz \\
\end{bmatrix}.
$$
with $-v$ in the upper left and $u$ in the lower right part of the (new) system matrix.
To get the form from Proposition 2.9 we are already used to, we have to reverse the
rows $1, 2, 3$ and columns $2, 3, 4$ and multiply the rows $1, 2, 3$ by $-1$. As a new system
$\mathcal{A} = (u, A, v)$ for $f$ we obtain
\[
\mathcal{A} = \begin{pmatrix}
1 & \cdots & . \\
. & 1 & -y \\
. & . & 1 -z \\
. & . & . & 1
\end{pmatrix} \cdot \begin{pmatrix}
1 & -x & . & . \\
. & 1 & -y & . \\
. & . & 1 & -z \\
. & . & . & 1
\end{pmatrix}.
\]
Here it is immediate, that $1 \in L(f)$ and $1 \in R(f)$, that is, $f$ is of type $(1, 1)$. The
application of the inverse from Proposition 2.10 again yields an ALS for $f^{-1}$, however
with dimension 5 already. Therefore an important (technical) task will be to detect
“special” forms (of the system matrices).
To be able to minimize, we would like to have a very “simple” structure, that is,
the (diagonal) pivot blocks should be as small as possible. For the example here, an
ALS for a monomial, this is respected by the minimal inverse (Theorem 2.13).

**Notation.** The set of the natural numbers is denoted by $\mathbb{N} = \{1, 2, \ldots\}$, that
including zero by $\mathbb{N}_0$. Zero entries in matrices are usually replaced by (lower) dots to
emphasize the structure of the non-zero entries unless they result from transformations
where there were possibly non-zero entries before. We denote by $I_n$ the identity matrix
and $\Sigma_n$ the permutation matrix that reverses the order of rows/columns (of size $n$)
respectively $I$ and $\Sigma$ if the size is clear from the context.

### 2.1 Preliminaries

Let $\mathbb{K}$ be a commutative field, $\overline{\mathbb{K}}$ its algebraic closure and $X = \{x_1, x_2, \ldots, x_d\}$ be
a finite (non-empty) alphabet. $\mathbb{K}\langle X \rangle$ denotes the free associative algebra (or free
$\mathbb{K}$-algebra) and $\mathbb{F} = \mathbb{K}\langle \langle X \rangle \rangle$ its universal field of fractions (or “free field”) [Coh95],
[CR99]. An element in $\mathbb{K}\langle X \rangle$ is called (non-commutative or nc) polynomial. In our
examples the alphabet is usually $X = \{x, y, z\}$. Including the algebra of nc rational
series we have the following chain of inclusions:
\[
\mathbb{K} \subset \mathbb{K}\langle X \rangle \subset \mathbb{K}\text{rat}\langle \langle X \rangle \rangle \subset \mathbb{K}\langle \langle X \rangle \rangle =: \mathbb{F}.
\]
The free monoid $X^*$ generated by $X$ is the set of all finite words $x_{i_1}x_{i_2} \cdots x_{i_n}$ with
$i_k \in \{1, 2, \ldots, d\}$. An element of the alphabet is called letter, one of the free monoid
word. The multiplication on $X^*$ is the concatenation of words, that is, $(x_{i_1} \cdots x_{i_m}) \cdot
(x_{j_1} \cdots x_{j_n}) = x_{i_1} \cdots x_{i_m}x_{j_1} \cdots x_{j_n}$, with neutral element $1$, the empty word. The
length of a word $w = x_{i_1}x_{i_2} \cdots x_{i_m}$ is $m$, denoted by $|w| = m$ or $\ell(w) = m$. For a
detailed introduction see [BR11, Chapter 1].

**Definition 2.1** (Inner Rank, Full Matrix [Coh06, CR99]). Given a matrix $A \in \mathbb{K}\langle X \rangle^{n \times n}$, the inner rank of $A$ is the smallest number $m \in \mathbb{N}$ such that there exists
a factorization $A = TU$ with $T \in \mathbb{K}\langle X \rangle^{n \times m}$ and $U \in \mathbb{K}\langle X \rangle^{m \times n}$. The matrix $A$
is called full if $m = n$, non-full otherwise.
Definition 2.2 (Linear Representations, Dimension, Rank [CR94, CR99]). Let $f \in \mathbb{F}$. A linear representation of $f$ is a triple $\pi_f = (u, A, v)$ with $u \in \mathbb{K}^{1 \times n}$, full $A = A_0 \otimes 1 + A_1 \otimes x_1 + \ldots + A_d \otimes x_d$, that is, $A$ is invertible over $\mathbb{F}$, $A_\ell \in \mathbb{K}^{n \times n}$, $v \in \mathbb{K}^{n \times 1}$ and $f = uA^{-1}v$. The dimension of $\pi_f$ is $\dim (u, A, v) = n$. It is called minimal if $A$ has the smallest possible dimension among all linear representations of $f$. The “empty” representation $\pi = (\cdot, \cdot)$ is the minimal one of $0 \in \mathbb{F}$ with $\dim \pi = 0$. Let $f \in \mathbb{F}$ and $\pi$ be a minimal linear representation of $f$. Then the rank of $f$ is defined as $\text{rank } f = \dim \pi$.

Definition 2.3 (Left and Right Families [CR94]). Let $\pi = (u, A, v)$ be a linear representation of $f \in \mathbb{F}$ of dimension $n$. The families $(s_1, s_2, \ldots, s_n) \subseteq \mathbb{F}$ with $s_i = (A^{-1}v)_i$ and $(t_1, t_2, \ldots, t_n) \subseteq \mathbb{F}$ with $t_j = (uA^{-1})_j$ are called left family and right family respectively. $L(\pi) = \text{span}\{s_1, s_2, \ldots, s_n\}$ and $R(\pi) = \text{span}\{t_1, t_2, \ldots, t_n\}$ denote their linear spans (over $\mathbb{K}$).

Proposition 2.4 ([CR94, Proposition 4.7]). A representation $\pi = (u, A, v)$ of an element $f \in \mathbb{F}$ is minimal if and only if both, the left family and the right family, are $\mathbb{K}$-linearly independent. In this case, $L(\pi)$ and $R(\pi)$ depend only on $f$.

Notation. For any two minimal linear representations $\pi_1$ and $\pi_2$ of some element $f \in \mathbb{F}$ we have $1 \in L(\pi_1)$ if and only if $1 \in L(\pi_2)$ because otherwise they could not be transformed into each other by invertible matrices over $\mathbb{K}$. By $1 \in L(f)$ (respectively $1 \in R(f)$) we denote $1 \in L(\pi)$ (respectively $1 \in R(\pi)$) for any minimal $\pi$ of $f$.

Definition 2.5 (Element Types [Sch18a, Definition 1.8]). An element $f \in \mathbb{F}$ is called of type $(1, \ast)$ (respectively $(0, \ast)$) if $1 \in R(f)$ (respectively $1 \not\in R(f)$). It is called of type $(\ast, 1)$ (respectively $(\ast, 0)$) if $1 \in L(f)$ (respectively $1 \not\in L(f)$). Both subtypes can be combined.

Remark. The following definition is a special case of the more general admissible systems [Coh06, Section 7] and the slightly more general linear representations [CR94].

Definition 2.6 (Admissible Linear Systems, Admissible Transformations [Sch18b]). A linear representation $\mathcal{A} = (u, A, v)$ of $f \in \mathbb{F}$ is called admissible linear system (ALS) for $f$, written also as $As = v$, if $u = e_1 = [1, 0, \ldots, 0]$. The element $f$ is then the first component of the (unique) solution vector $s$. Given a linear representation $\mathcal{A} = (u, A, v)$ of dimension $n$ of $f \in \mathbb{F}$ and invertible matrices $P, Q \in \mathbb{K}^{n \times n}$, the transformed $PAQ = (uQ, PAQ, Pv)$ is again a linear representation (of $f$). If $\mathcal{A}$ is an ALS, the transformation $(P, Q)$ is called admissible if the first row of $Q$ is $e_1 = [1, 0, \ldots, 0]$.

Definition 2.7. Let $M = M_1 \otimes x_1 + \ldots + M_d \otimes x_d$ with $M_i \in \mathbb{K}^{n \times n}$ for some $n \in \mathbb{N}$. An element in $\mathbb{F}$ is called regular if it has a linear representation $(u, A, v)$ with $A = I - M$, that is, $A_0 = I$ in Definition 2.2, or equivalently, if $A_0$ is regular (invertible).
Definition 2.8 (Polynomial ALS and Transformation [Sch18c, Definition 24]). An ALS \( \mathcal{A} = (u, A, v) \) of dimension \( n \) with system matrix \( A = (a_{ij}) \) for a non-zero polynomial \( 0 \neq p \in \mathbb{K}\langle X \rangle \) is called polynomial, if

1. \( v = [0, \ldots, 0, \lambda]^\top \) for some \( \lambda \in \mathbb{K} \) and
2. \( a_{ii} = 1 \) for \( i = 1, 2, \ldots, n \) and \( a_{ij} = 0 \) for \( i > j \), that is, \( A \) is upper triangular.

A polynomial ALS is also written as \( \mathcal{A} = (1, A, \lambda) \) with \( 1, \lambda \in \mathbb{K} \). An admissible transformation \((P, Q)\) for an ALS \( \mathcal{A} \) is called polynomial if it has the form

\[
(P, Q) = \begin{pmatrix}
1 & \alpha_{1,2} & \ldots & \alpha_{1,n-1} & \alpha_{1,n} \\
& \ddots & \ddots & \ddots & \vdots \\
& & 1 & \alpha_{n-2, n-1} & \alpha_{n-2, n} \\
& & & 1 & \alpha_{n-1, n} \\
& & & & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
1 & \beta_{2,3} & \ldots & \beta_{2,n} \\
& \ddots & \ddots & \ddots & \vdots \\
& & \ddots & \ddots & \ddots \\
& & & \ddots & \beta_{n-1, n}
\end{pmatrix}.
\]

If additionally \( \alpha_{1,n} = \alpha_{2,n} = \ldots = \alpha_{n-1,n} = 0 \) then \((P, Q)\) is called polynomial factorization transformation. See also Figure 1 on page 33.

2.2 Minimal Systems

The main idea is to start with minimal admissible linear systems and construct minimal ones for the rational operations (scalar multiplication, sum, product, inverse). We already have seen the minimal monomial:

Proposition 2.9 (Minimal Monomial [Sch18b, Proposition 4.1]). Let \( k \in \mathbb{N} \) and \( f = x_{i_1} x_{i_2} \cdots x_{i_k} \) be a monomial in \( \mathbb{K}\langle X \rangle \subseteq \mathbb{K}(\langle X \rangle) \). Then

\[
\mathcal{A} = \begin{pmatrix}
1 & \cdots & \cdots & \cdots & \cdots \\
& \ddots & \ddots & \ddots & \vdots \\
& & 1 & \cdots & \cdots \\
& & & \ddots & \ddots \\
& & & & 1
\end{pmatrix},
\begin{pmatrix}
1 & -x_{i_1} \\
1 & -x_{i_2} \\
& \ddots & \ddots \\
& & 1 & -x_{i_k} \\
& & & 1
\end{pmatrix}
\] is a minimal (polynomial) ALS of dimension \( \dim \mathcal{A} = k + 1 \).

More general it is possible to state minimal systems for a class of polynomials by a (generalized) “companion” system [Sch18c, Section 3].

2.3 Rational Operations

“Basic” rational operations (on the level of admissible linear systems) are easy to formulate. For the multiplication we can provide alternative constructions yielding minimal admissible linear systems immediately in special cases, for example the minimal polynomial multiplication (Proposition 3.2).
Proposition 2.10 (Rational Operations [CR99]). Let $0 \neq f, g \in \mathbb{F}$ be given by the admissible linear systems $A_f = (u_f, A_f, v_f)$ and $A_g = (u_g, A_g, v_g)$ respectively and let $0 \neq \mu \in \mathbb{K}$. Then admissible linear systems for the rational operations can be obtained as follows:

The scalar multiplication $\mu f$ is given by

$$\mu A_f = (u_f, A_f, \mu v_f).$$

The sum $f + g$ is given by

$$A_f + A_g = \begin{pmatrix} u_f \\ A_f \\ v_f \end{pmatrix}, \begin{pmatrix} A_f \\ -A_f u_f \end{pmatrix}, \begin{pmatrix} v_f \\ v_g \end{pmatrix}.$$

The product $fg$ is given by

$$A_f : A_g = \begin{pmatrix} u_f \\ A_f \\ v_f \end{pmatrix}, \begin{pmatrix} A_f \\ -v_f u_g \end{pmatrix}, \begin{pmatrix} v_f \\ v_g \end{pmatrix}.$$

And the inverse $f^{-1}$ is given by

$$A_f^{-1} = \begin{pmatrix} 1 \\ A_f \\ 1 \end{pmatrix}, \begin{pmatrix} -v_f \\ A_f \\ u_f \end{pmatrix}, \begin{pmatrix} 1 \end{pmatrix}.$$

Lemma 2.11 (for Type $(1, *)$ [Sch18b, Lemma 4.12]). Let $A = (u, A, v)$ be a minimal ALS with $\dim A = n \geq 2$ and $1 \in R(A)$. Then there exists an admissible transformation $(P, Q)$ such that the first column of $PAQ$ is $[1, 0, \ldots, 0]^\top$ and $Pv = [0, \ldots, 0, \lambda]^\top$ for some $\lambda \in \mathbb{K}$.

Proposition 2.12 (Multiplication Type $(*, 1)$ [Sch18a, Proposition 2.10]). Let $f, g \in \mathbb{F} \setminus \mathbb{K}$ be given by the admissible linear systems $A_f = (u_f, A_f, v_f) = (1, A_f, \lambda_f)$ of dimension $n_f$ and $A_g = (u_g, A_g, v_g) = (1, A_g, \lambda_g)$ of dimension $n_g$ of the form

$$A_g = \begin{pmatrix} 1 \\ B \\ c \end{pmatrix}, \begin{pmatrix} 1 \\ B \end{pmatrix}, \begin{pmatrix} 1 \end{pmatrix}.$$

respectively. Then an ALS for $fg$ of dimension $n = n_f + n_g - 1$ is given by

$$A = \begin{pmatrix} u_f \\ A_f \\ \lambda_f b' \\ e_{n_f} \lambda_f b' \\ B \\ \lambda_g \end{pmatrix}, \begin{pmatrix} e_{n_f} \lambda_f b' \\ b'' \\ B \end{pmatrix}, \begin{pmatrix} \lambda_g \end{pmatrix}.$$

2.4 Disjoint Addition

For disjoint elements $f, g \in \mathbb{F}$ [CR99], that is, $\text{rank}(f + g) = \text{rank}(f) + \text{rank}(g)$, the addition from Proposition 2.10 is minimal. For further details we refer to the remarks after [Sch17, Definition 2.2]. An important result of Cohn and Reutenauer is the primary decomposition (of elements in the free field) [CR99, Theorem 2.3].
2.5 Minimal Inverse

The derivation of the minimal inverse in [Sch18b, Section 4] consists of two major steps (motivated in the beginning of this section): keeping the form for \( f = (f^{-1})^{-1} \) and distinguishing different cases to ensure minimality. Notice especially the remark before [Sch18b, Theorem 4.13] how to transfer admissible linear systems into the appropriate form.

**Theorem 2.13** (Minimal Inverse [Sch18b, Theorem 4.13]). Let \( f \in \mathbb{F} \setminus \mathbb{K} \) be given by the minimal admissible linear system \( A = (u, A, v) \) of dimension \( n \). Then a minimal ALS for \( f^{-1} \) is given in the following way:

\[
\text{for } \mathcal{A} = \begin{pmatrix} 1 & b' & b \\ \cdot & B & b'' \\ \cdot & \cdot & 1 \end{pmatrix}, \lambda.
\]

\( f \) of type \((1,1)\) yields \( f^{-1} \) of type \((0,0)\) with \( \dim(\mathcal{A}') = n - 1 \):

\[
\mathcal{A}' = \begin{pmatrix} 1, \begin{pmatrix} -\lambda \Sigma b'' & -\Sigma B \Sigma \\ -\lambda b & -b' \Sigma \end{pmatrix}, 1 \end{pmatrix}
\]

\( f \) of type \((1,0)\) yields \( f^{-1} \) of type \((1,0)\) with \( \dim(\mathcal{A}') = n \):

\[
\mathcal{A}' = \begin{pmatrix} 1, \begin{pmatrix} -\Sigma b'' & -\Sigma B \Sigma \\ -\Sigma b & -b' \Sigma \end{pmatrix}, 1 \end{pmatrix}
\]

\( f \) of type \((0,1)\) yields \( f^{-1} \) of type \((0,1)\) with \( \dim(\mathcal{A}') = n \):

\[
\mathcal{A}' = \begin{pmatrix} 1, \begin{pmatrix} -\lambda \Sigma b'' & -\Sigma B \Sigma \\ -\lambda b & -b' \Sigma \end{pmatrix}, 1 \end{pmatrix}
\]

\( f \) of type \((0,0)\) yields \( f^{-1} \) of type \((1,1)\) with \( \dim(\mathcal{A}') = n + 1 \):

\[
\mathcal{A}' = \begin{pmatrix} 1, \begin{pmatrix} -\Sigma A \Sigma \\ u \Sigma \end{pmatrix}, 1 \end{pmatrix}
\]

(Recall that the permutation matrix \( \Sigma \) reverses the order of rows/columns.)

2.6 Rational Identities

Using the minimal inverse (Theorem 2.13) and the rational operations (Proposition 2.10) one can already show non-trivial rational identities very systematically by “hand”. The following proof is from [Sch18a, Section 5].

**Example 2.14** (Hua’s Identity [Ami66]). We have:

\[
x - (x^{-1} + (y^{-1} - x)^{-1})^{-1} = xyx.
\]
Proof. Minimal admissible linear systems for $y^{-1}$ and $x$ are
\[
\begin{bmatrix} y \end{bmatrix} s = \begin{bmatrix} 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -x \\ . & 1 \end{bmatrix} s = \begin{bmatrix} . \end{bmatrix}
\]
respectively. The ALS for the difference $y^{-1} - x$,
\[
\begin{bmatrix} y & -y \\ . & 1 & -x \end{bmatrix} s = \begin{bmatrix} 1 \\ . & -1 \end{bmatrix}, \quad s = \begin{bmatrix} y^{-1} - x \\ -x \\ -1 \end{bmatrix}, \quad t = \begin{bmatrix} y^{-1} & -1 & y^{-1} - x \end{bmatrix}
\]
is minimal because the left family $s$ is $K$-linearly independent and the right family $t$ is $K$-linearly independent. Clearly we have $1 \in R(y^{-1} - x)$. Thus, by Lemma 2.11, there exists an admissible transformation
\[
(P, Q) = \left( \begin{bmatrix} . & 1 & . \\ 1 & 1 & . \\ . & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & . \\ . & 1 \\ 1 & . \end{bmatrix} \right),
\]
that yields the ALS
\[
\begin{bmatrix} 1 & 1 & -x \\ . & -y & 1 \\ . & . & 1 \end{bmatrix} s = \begin{bmatrix} . \\ . \\ -1 \end{bmatrix}.
\]
Now we can apply the inverse of type (1, 1):
\[
\begin{bmatrix} 1 & y \\ -x & -1 \end{bmatrix} s = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad s = \begin{bmatrix} (y^{-1} - x)^{-1} \\ -(1 - xy)^{-1} \end{bmatrix}.
\]
This system represents a regular element $(y^{-1} - x)^{-1} = (1 - xy)^{-1} y$, and therefore can be transformed into a regular ALS (Definition 2.7) by scaling row 2 by $-1$. Then we add $x^{-1} \text{ “from the left”}$:
\[
\begin{bmatrix} x & -x \\ . & 1 & y \\ . & x & 1 \end{bmatrix} s = \begin{bmatrix} 1 \\ . \\ -1 \end{bmatrix}, \quad s = \begin{bmatrix} x^{-1} + (y^{-1} - x)^{-1} \\ (y^{-1} - x)^{-1} \\ -(1 - xy)^{-1} \end{bmatrix}.
\]
This system is minimal and —after adding row 3 to row 1 (to eliminate the non-zero entry in the right hand side)— we apply the (minimal) inverse of type (0, 0):
\[
\begin{bmatrix} -1 & -1 & -x \\ . & -y & -1 \\ . & . & 0 & -x \end{bmatrix} s = \begin{bmatrix} . \\ . \\ 1 \end{bmatrix}.
\]
Now we multiply row 1 and the columns 2 and 3 by $-1$ and exchange column 2 and 3 to get the following system:
\[
\begin{bmatrix} 1 & -x & -1 \\ . & 1 & y \\ . & . & 1 & -x \end{bmatrix} s = \begin{bmatrix} . \\ . \\ 1 \end{bmatrix}, \quad s = \begin{bmatrix} x - xy \end{bmatrix}.
\]
The next step would be a scaling by $-1$ and the addition of $x$ (by Proposition 2.10). With two minimization steps we would reach again minimality. Alternatively we can add a linear term to a polynomial (in a polynomial ALS) ---depending on the entry $v_n$ in the right hand side--- directly in the upper right entry of the system matrix:

$$
\begin{bmatrix}
1 & -x & -1 & x \\
& 1 & y & . \\
& & 1 & -x \\
& & & 1
\end{bmatrix}
\begin{bmatrix}
s \\
1
\end{bmatrix}
= 
\begin{bmatrix}
-xyx \\
-yx \\
x \\
1
\end{bmatrix}.
$$



\begin{itemize}
\item
\end{itemize}

3 Factorizing

Since the whole factorization theory originated from a “small” problem of the minimization of linear representations, it should lead as a thread through this section. Somehow this theory has become independent and is interesting now from a purely algebraic point of view since it enables to view the free field as a “ring”. Not in the trivial sense, where each field is a ring, but using the richer “structure” by combining the non-commutative factorization theory and the embedding of non-commutative rings (to be more precise: free ideal rings, FIRs [Coh06]) into their respective universal field of fraction. There are a lot of open questions, for example, is the free field a “similarity unique factorization domain”? Or, is the extension of the “classical” factorization theory (in free associative algebras) to the free field —assuming that polynomial atoms (and their inverse) remain irreducible— unique?

To not loose the thread, we come back to a simple example: Assume that we have given an element $f$ by the admissible linear system $A_f$,

$$
\begin{bmatrix}
x & 1 & . \\
& y & -1 \\
& -1 & x
\end{bmatrix}
\begin{bmatrix}
s \\
1
\end{bmatrix}
= 
\begin{bmatrix}
. \\
1
\end{bmatrix}.
$$

By Proposition 2.12 we construct an ALS $A$ for $fx$, namely

$$
\begin{bmatrix}
x & 1 & . \\
& y & -1 \\
& -1 & x & -x \\
& & & 1
\end{bmatrix}
\begin{bmatrix}
s \\
1
\end{bmatrix}
= 
\begin{bmatrix}
. \\
1
\end{bmatrix}.
$$

Is $A$ minimal? Now we repeat this step for $f$ given by a different system $A'_f$ and construct again a system $A'$ for $fx$, namely

$$
\begin{bmatrix}
x & 1 & . \\
1 & y & -1 \\
& x & -x \\
& & & 1
\end{bmatrix}
\begin{bmatrix}
s \\
1
\end{bmatrix}
= 
\begin{bmatrix}
. \\
1
\end{bmatrix}.
$$
in which one can read \( A' \) directly in the upper left \( 3 \times 3 \) block of the system matrix. Here it is immediate that row/column 3 can be eliminated after adding column 3 to column 4. Therefore \( A' \) and hence \( A \) cannot be minimal. The connection to factorization will become much clearer in [Sch18c, Example 30], as soon as one verifies by the minimal inverse that \( f = (pq)^{-1} \) for \( p = x \) and \( q = 1 - xy \).

The \((\text{lower left})\) \( 2 \times 1 \) block of zeros in the system matrix of \( A_f \) becomes an upper right block of zeros in the system matrix of \( A_f^{-1} \), the standard inverse of \( A_f \),

\[
\begin{bmatrix}
1 & -x & 1 & 0 \\
. & 1 & -y & 0 \\
. & . & -1 & -x \\
. & . & . & -1
\end{bmatrix}
\begin{bmatrix}
s_1 \\
s_2 \\
s_3 \\
s_4
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix},
\]

which is minimal here because \( f \) is of type \((0,0)\) and \( A_f \) is minimal. And this upper right block of zeros is that one coming from multiplication \((1,*)\), see also [Sch18a, Proposition 2.7/2.10]. This yields a “natural” correspondence between factorizations and upper right zero block structure in the system matrix (assuming zero entries in the corresponding components of the right hand side).

In other words: One can find (non-trivial) factors of a polynomial by looking for “appropriate” transformations (of a minimal ALS). This is the main topic in Section 3.3 respectively [Sch18c, Section 2]. If one factorizes a polynomial in two \((\text{not necessarily irreducible})\) factors, “their” admissible linear systems are minimal. The converse — and that is the core of Section 3.2 — is also true. Although the minimal polynomial multiplication (Proposition 3.2) seems to be “obvious”, the proof is highly non-trivial. (A possible reason is that only minimality is assumed and not, for example, invertibility of the system matrix over the formal power series.)

[Sch18c, Example 50] could serve as an appetizer. There the polynomial factorization is used to compute the eigenvalues of a matrix via the factorization of its characteristic polynomial.

### 3.1 Preliminaries

For the main definitions we refer to [Sch18c, Section 1, Page 5]. The free associative algebra \( \mathbb{K}(X) \) is a “similarity” \textit{unique factorization domain} (UFD). For example, the polynomials \( p = 1 - xy \) and \( q = 1 - yx \) are \textit{similar} because there exist \( \tilde{p}, \tilde{q} \in \mathbb{K}(X) \) such that \( p\tilde{p} = \tilde{q}q \) with \( p, \tilde{q} \) \textit{left coprime} and \( \tilde{p}, q \) \textit{right coprime} [Coh63].

### 3.2 Minimal Polynomial Multiplication

As an introduction one could take the multiplication of \( x \) and \( 1 - yx \) using Proposition 2.12, see also [Sch18c, Example 30]. The following lemma is needed in Section 4.3 and (the proof of) the following proposition.

**Lemma 3.1** ([Sch17, Lemma 2.14]). \( \text{Let } A = (u, A, v) = (1, A, \lambda) \text{ be an ALS of dimension } n \geq 2 \text{ and } \mathbb{K}-\text{linearly dependent left family } s = A^{-1}v. \text{ Let } m \in \{2, 3, \ldots, n\} \)
be the minimal index such that the left subfamily $2 = (A^{-1}v)^n_{i=m}$ is $K$-linearly independent. Let $A = (a_{ij})$ and assume that $a_{ii} = 1$ for $1 \leq i \leq m$ and $a_{ij} = 0$ for $j < i \leq m$ (upper triangular $m \times m$ block) and $a_{ij} = 0$ for $j \leq m < i$ (lower left zero block of size $(n-m) \times m$). Then there exists matrices $T, U \in K^{1 \times (n+1-m)}$ such that

$$U + (a_{m-1,j})^n_{j=m} - T(a_{ij})^n_{i,j=m} = \begin{bmatrix} 0 & \ldots & 0 \end{bmatrix} \quad \text{and} \quad T(v_i)^n_{i=m} = 0.$$ 

**Proposition 3.2** (Minimal Polynomial Multiplication [Sch18c, Proposition 28]). Let $p, q \in K\langle X \rangle \setminus K$ be given by the minimal polynomial admissible linear systems $A_p = (1, A_p, \lambda_p)$ of dimension $n_f$ and $A_q = (1, A_q, \lambda_q)$ of dimension $n_g$ respectively. Then the ALS $A$ from Proposition 2.12 for $pq$ is minimal of dimension $n = n_p + n_q - 1$.

### 3.3 Polynomial Factorization

The polynomial factorization theory depends on minimal (polynomial) admissible linear systems. How to obtain such systems directly is discussed in Section 2.2. How to construct them in general is discussed in Section 4.3.

**Remark.** Notice that, although we use (general) admissible linear systems here to represent polynomials, the factorization does not depend on the construction of the free field. Indeed, the system matrix of a minimal linear representation of a polynomial is already invertible over the free associative algebra.

**Theorem 3.3** (Polynomial Factorization [Sch18c, Theorem 40]). Let $p \in K\langle X \rangle$ be given by the minimal polynomial admissible linear system $A = (1, A, \lambda)$ of dimension $n = \text{rank} p \geq 3$. Then $p$ factorizes in $p = q_1q_2$ with $\text{rank}(q_i) = n_i \geq 2$ if and only if there exists a polynomial factorization transformation $(P, Q)$ such that $PAQ$ has an upper right block of zeros of size $(n_1 - 1) \times (n_2 - 1)$.

As we have already seen in the beginning of this section, we have to find an admissible transformation (over the ground field $K$) to create upper right blocks of zeros (of appropriate size) in a minimal polynomial ALS to detect (non-trivial) factors of a polynomial. If $K$ is not algebraically closed it can be difficult to check if there is a solution. A simple case is illustrated in [Sch18c, Example 37]. The practical application is by [Sch18c, Proposition 42], a simple variant of [CR99, Theorem 4.1].

### 3.4 Factorization Theory**

The general factorization theory is somewhat difficult. Although it seems to be clear from the polynomials how it should be, the path to the divisibility equivalence (Theorem 3.4) is long and stony. One needs a notion of left (respectively right) divisibility on the level of minimal admissible linear systems. This is not straight forward (for details we refer to [Sch17, Section 3]). But in return one can "forget" the free associative algebra and factorize elements directly in the free field. And also here there are two sides of one coin, namely the (minimal) multiplication in Section 3.5 respectively [Sch17, Theorem 4.2] and the factorization via detecting zero blocks in Section 3.6 respectively [Sch17, Theorem 4.8].
**Theorem 3.4** (Divisibility Equivalence [Sch17, Theorem 3.10]). Let $p, q \in \mathbb{K}(X)$. Then $p$ left (respectively right) divides $q$ if and only if $p$ left (respectively right) divides $q$ in $\mathbb{F} = \mathbb{K}(\langle X \rangle)$.

### 3.5 Minimal Factor Multiplication**

Given two minimal admissible systems, under which conditions are the multiplications from Proposition 2.10 and 2.12 minimal? A special case is the minimal polynomial multiplication (Proposition 3.2). The general answer is given in [Sch17, Theorem 4.2] within the (framework of the) general factorization theory.

### 3.6 General Factorization**

Like in the general (minimal) multiplication in the previous subsection we have to distinguish several cases for the factorization [Sch17, Theorem 4.8]. Looking for zero (lower left and upper right) blocks (of appropriate size) in the system matrix of a minimal ALS (similar to the polynomial factorization) is rather natural when we want to “reverse” the multiplication. The main difficulties however are far from obvious and therefore one of the first steps in the general factorization theory [Sch17, Section 3] is to define, what we mean by a “factor” (since in a field there are no non-zero non-units, that is, each non-zero element is invertible).

### 3.7 Examples Factorization

Polynomial factorization is illustrated in detail (step by step) in [Sch18c, Section 4]. The general factorization (of a regular element) is discussed briefly in [Sch17, Example 4.9].

### 4 Minimizing

The basic idea of the minimization (of a linear representation) with left and right minimization steps is surprisingly simple. If the block structure becomes coarser and a “look” is not sufficient any more, row and column transformations can be found by solving a linear system of equations. That is the essential content of Section 4.2 (word problem), the foundation stone of the whole theory. The naive idea was to solve “local” word problems, producing plenty of questions which —among other things— led to the factorization theory . . .

But when, that is, under which conditions, is an admissible linear system (constructed out of two minimal ones by Proposition 2.10) minimal? If there are no more left or right “linear” minimization steps possible? Is it sufficient to find one “finest” structure such that the system matrix is an upper block triangular matrix with a maximal number of (quadratic) diagonal blocks?

For polynomials (given by polynomial admissible linear systems) this can be done by a relatively simple algorithm which is formulated in Section 4.3. If one knows “all”
factorizations of a polynomial, one also knows all “finest” pivot block structures of
the minimal admissible linear systems of its inverse and one can continue to calculate
“easily” because it is still rather simple to minimize.

Already in the beginning of Section 2 (calculating) we have discussed assumptions
on the construction of an ALS for the inverse of an element. In Section 4.4 we
investigate the connection between a factorization and the refinement of pivot blocks
in the system of the inverse a little more thoroughly and describe the approach of the
latter. One of the central question in Section 4.5 is that of a sufficient condition
for the minimization with linear techniques.

In fact one could develop a general minimization algorithm using polynomial sys-
tems of equations. However, these are usually difficult to solve. And if we do not
know anything about the existence of a solution, we do not know anything about mini-
mality. Therefore non-linear techniques should be avoided whenever this is possible
by “keeping” a fine block structure.

Since the main goal of this section is to “minimize” addition and multiplication,
some thoughts from this point of view should be summarized. That the factorization
of an element does make sense for the multiplication is immediately clear: In case one
can cancel factors. This is used for example to find the left greatest common divisor
of two polynomials [Sch18a, Example 5.4]. But it is not that trivial since an atom
might not necessarily lie “beside” its inverse, for example

\[ x(1 - xy) \cdot x^{-1} = (1 - xy)x \cdot x^{-1} = 1 - xy. \]

Additionally it can happen that two irreducible elements “fusion” to one [Sch17,
Section 3] and therefore we need a refinement of pivot blocks “inside” an atom (ir-
reducible element). But also from an additive point of view the factorization plays
a crucial role because one needs “common” left and right factors of two summands
only “once”. Notice that there are also linear techniques for refinement, for example
to bring an ALS to a suitable form for the minimal inverse (Theorem 2.13).

Recall that here we operate directly in the (system matrix of the) linear represen-
tation and therefore we are independent of its regularity (that is, invertibility over
the formal power series). And that has its price. The “classical” methods for the
minimization of linear representations for regular elements work mainly indirectly by
computing the left and right families, see for example [Sch18b, Section 3].

4.1 Preliminaries and a Standard Form

To be able to formulate statements—in particular for the minimization—in a con-
vienent way, we need some notation which formalizes what we have already used,
namely to describe an ALS (and admissible transformations) in terms of block rows
and columns instead of (single) rows and columns. Then it is possible to define a
standard form which plays an important role when we want to minimize admissible
linear systems coming from addition or multiplication (later in Section 4.5). This is
the first part in [Sch18a, Section 3]. To construct a standard admissible linear system
out of a minimal ALS we need to “refine” it. This is the goal of Section 4.4, the
second part in [Sch18a, Section 3].

**Definition 4.1** (Pivot Blocks, Pivot Block Transformation [Sch18a, Definition 3.1]). Let \( \mathcal{A} = (u, A, v) \) be an ALS and denote \( A = (A_{ij})_{i,j=1}^m \) the block decomposition (with square diagonal blocks \( A_{ii} \)) with maximal \( m \) such that \( A_{ij} = 0 \) for \( i > j \). The diagonal blocks \( A_{ii} \) are called pivot blocks, the number \( m \) is denoted by \( \#_{pb} \mathcal{A} \). The dimension (or size) of a pivot block \( A_{ii} \) for \( i \in \{1, 2, \ldots, m\} \) is \( n_i = \dim A \). For a pivot block \( k \) let \( I_{1:k-1} \) (respectively \( I_{k+1:m} \)) denote the identity matrix of size \( n_1 + \ldots + n_{k-1} \) (respectively \( n_{k+1} + \ldots + n_m \)). An admissible transformation \( (P, Q) \) of the form

\[
(P, Q)_k = \begin{pmatrix}
I_{1:k-1} & T & I_{k+1:m} \\
\vdots & \ddots & \vdots \\
& \ddots & I_{k+1:m}
\end{pmatrix}
\begin{pmatrix}
I_{1:k-1} \\
\vdots \\
& \ddots \\
& \ddots & I_{k+1:m}
\end{pmatrix}
\]

with \( T, U \in \mathbb{K}^{n_k \times n_k} \) is called (admissible) \( k \)-th pivot block transformation.

**Definition 4.2** (Refined Pivot Block and Refined ALS [Sch18a, Definition 3.3]). Let \( \mathcal{A} = (u, A, v) \) be an ALS with \( m = \#_{pb} \mathcal{A} \) pivot blocks of size \( n_i = \dim A \). A pivot block \( A_{kk} \) (for \( 1 \leq k \leq m \)) is called refined if there does not exist an admissible pivot block transformation \( (P, Q)_k \) such that \((PAQ)_{kk} \) has a lower left block of zeros of size \( i \times (n_k - i) \) for an \( i \in \{1, 2, \ldots, n_k - 1\} \). The admissible linear system \( \mathcal{A} \) is called refined if all pivot blocks are refined.

**Definition 4.3** (Standard Admissible Linear System [Sch18a, Definition 3.8]). A minimal and refined ALS \( \mathcal{A} = (u, A, v) = (1, A, \lambda) \), that is, \( v = [0, \ldots, 0, \lambda] \), is called standard.

**Remark.** For a polynomial \( p \) given by a standard ALS \( \mathcal{A} \) (of dimension \( n \geq 2 \)) the minimal inverse of \( \mathcal{A} \) (of dimension \( n - 1 \)) is refined if and only if \( \mathcal{A} \) is obtained by the minimal polynomial multiplication of its irreducible factors \( q_i \) in \( p = q_1q_2 \cdots q_m \). For a detailed discussion of polynomial factorization (in free associative algebras) we refer to [Sch18c].

### 4.2 The Word Problem*

One of the difficulties in free fields is (that of) the word problem, that is, to check whether two admissible linear systems represent the same element. A solution to the word problem is [CR99, Theorem 4.1]. Unfortunately it is hard to apply practically already for systems of dimension 3. If those systems are given by minimal admissible linear systems however, the word problem can be “linearized”, that is, it is equivalent to the solution of a linear system of equations. For a detailed discussion we refer to [Sch18b, Section 2].

**Theorem 4.4** (Linearized Word Problem [Sch18b, Theorem 2.4]). Let \( f, g \in \mathbb{F} \) be given by the minimal admissible linear systems \( \mathcal{A}_f = (u_f, A_f, v_f) \) and \( \mathcal{A}_g = (u_g, A_g, v_g) \) of dimension \( n \) respectively. Then \( f = g \) if and only if there exist matrices \( T, U \in \mathbb{K}^{n \times n} \) such that \( u_fU = 0, TA_g - A_fU = A_fu_f^T u_g \) and \( Tv_g = v_f \).
The techniques used for the minimization in Section 4.3 and 4.5 can be interpreted as solving “local” word problems. The other way around one can view the word problem as one “big” minimization step.

4.3 Minimizing a Polynomial ALS

To illustrate the main idea we (partially) minimize a non-minimal “almost” polynomial ALS \( A = (u, A, v) \) of dimension \( n = 6 \) for \( p = -xy + (xy + z) \). Note that we do not need knowledge of the left and right family at all. Let

\[
A = \begin{bmatrix}
1 & -x & \cdots & -1 & \cdots & 1 \\
. & 1 & -y & \cdots & . & . \\
. & . & 1 & \cdots & . & . \\
. & . & . & 1 & -x & -z \\
. & . & . & . & 1 & -y \\
. & . & . & . & . & 1
\end{bmatrix}
\]

(4.5)

First we do one “left” minimization step, that is, we remove (if possible) one element such that row \( k < n \) is \( \leq a_1 \) and column \( k \) in \( PAQ \) because \( (Q^{-1}s)_k = 0 \). (This is what we have already done in Section 1.) How can we find these blocks \( T, U \in \mathbb{K}^{1 \times (n-k)} \)? We write \( A \) in block form with respect to (block) row/column \( k \) and write \( A_{1:,m} \) for \( A_{1:k-1,k+1:m} \), etc. (Recall that here we have \( m = n \) pivot blocks. Block indices are underlined to distinguish them from component indices.)

\[
A[k] = \begin{bmatrix} u_1 & \cdots & A_{1:,1} & A_{1:,k} & A_{1:,m} & \cdots & u_k & \cdots & v_k & \cdots & v_m \end{bmatrix}
\]

(4.7)

and apply the transformation \((P, Q)\):

\[
PAQ = \begin{bmatrix} I_{k-1} & \cdots & . & . & . & \cdots & I_{k+1:m} \end{bmatrix} \begin{bmatrix} A_{1:,1} & A_{1:,k} & A_{1:,m} & . & . & . & . & . \end{bmatrix} \begin{bmatrix} I_{k-1} & \cdots & . & . & . & \cdots & I_{k+1:m} \end{bmatrix}
\]

\[
Pv = \begin{bmatrix} I_{k-1} & \cdots & . & . & . & \cdots & I_{k+1:m} \end{bmatrix} \begin{bmatrix} v_1 & \cdots & v_k & \cdots & v_m \end{bmatrix} = \begin{bmatrix} v_k + Tv_m \end{bmatrix}.
\]

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Now we can read of a sufficient condition for \((Q^{-1}s)_k = 0\), namely the existence of \(T, U \in \mathbb{K}^{1 \times (n-k)}\) such that

\[ U + A_{k,:m} + TA_{m,:,m} = 0 \quad \text{and} \quad v_{k} + T v_{m} = 0. \quad (4.8) \]

(Compare with the word problem, Theorem 4.4.) Let \(d\) be the number of letters in our alphabet \(X\). The blocks \(T = [\alpha_{k+1}, \alpha_{k+2}, \ldots, \alpha_n] \) and \(U = [\beta_{k+1}, \beta_{k+2}, \ldots, \beta_n] \) in the transformation \((P, Q)\) are of size \(1 \times (n-k)\), thus we have a linear system of equations (over \(\mathbb{K}\)) with \(2(n-k)\) unknowns (for \(k > 1\)) and \((d+1)(n-k) + 1\) equations:

\[
\begin{bmatrix}
\beta_{k+1} & \beta_{k+2} & \beta_{k+3} \\
\alpha_{k+1} & \alpha_{k+2} & \alpha_{k+3} \\
\end{bmatrix}
+
\begin{bmatrix}
0 & 0 & 0 \\
\end{bmatrix}
+
\begin{bmatrix}
1 & -x & -z \\
1 & -y & 1 \\
\end{bmatrix}
=
\begin{bmatrix}
0 & 0 & 0 \\
\end{bmatrix}.
\]

One solution is \(T = [0, 0, 1]\) and \(U = [0, 0, -1]\). We compute \(\tilde{A}_1 = PAQ\) and remove block row \(k\) and column \(k\) to get the new ALS

\[
\mathcal{A}_1 = (u, A, v) = \begin{pmatrix}
[1, \ldots, 1],
\begin{bmatrix}
1 & -x & -1 & . & . \\
1 & . & 1 & . & y \\
. & 1 & . & 1 & -x & -z & . & . \\
. & . & 1 & . & 1 & -y & . \\
. & . & . & 1 & . & 1
\end{bmatrix}
\end{pmatrix}.
\]

For a “right” minimization step, that is, removing (if possible) one element of the \(\mathbb{K}\)-linearly dependent right family \(t = u \mathcal{A}^{-1}\) we are looking for a transformation \((P, Q)\) of the form

\[
(P, Q) = \begin{pmatrix}
I_{k-1} & T & . \\
. & 1 & . \\
. & . & I_{n-k}
\end{pmatrix}, \begin{pmatrix}
I_{k-1} & U & . \\
. & 1 & . \\
. & . & I_{n-k}
\end{pmatrix}
\]

such that column \(k\) in \(PAQ\) is \([0, \ldots, 0, 1, 0, \ldots, 0]^\top\). A sufficient condition for \((tP^{-1})_k = 0\) is the existence of \(T, U \in \mathbb{K}^{(k-1) \times 1}\) such that

\[
A_{1:,1:} + A_{1:,k:} + T = 0.
\]

(4.10)

For the illustration we refer to [Sch18c, Section 2.2]. If a left (respectively right) minimization step with \(k = 1\) (respectively \(k = n\) and \(v = [0, \ldots, 0, \lambda]^\top\)) can be done, then the ALS represents zero and we can stop immediately.

The following is the only non-trivial observation: Recall that, if there exist row (respectively column) blocks \(T, U\) such that (4.8) (respectively (4.10)) has a solution
then the left (respectively right) family is \( \mathbb{K} \)-linearly dependent. To guarantee minimality we need the other implication, that is, the existence of appropriate row or column blocks for non-minimal polynomial admissible linear systems.

The following arguments can be found in the proof of [Sch18c, Proposition 28]: Let \( \mathcal{A} = (u, A, v) \) be a polynomial ALS of dimension \( n \geq 2 \) with left family \( s = (s_1, s_2, \ldots, s_n) \) and assume that there exists a \( 1 \leq k < n \) such that the subfamily \( (s_{k+1}, s_{k+2}, \ldots, s_n) \) is \( \mathbb{K} \)-linearly independent while \( (s_k, s_{k+1}, \ldots, s_n) \) is \( \mathbb{K} \)-linearly dependent. Then, by Lemma 3.1, there exist matrices \( T, U \in \mathbb{K}^{1 \times (n-k)} \) such that (4.8) holds. In other words: We have to start with \( k_s = n - 1 \) for a left and \( k_t = 2 \) for a right minimization step.

If we apply one minimization step, we must check the other family “again”, illustrated in the following example:

\[
\mathcal{A} = (u, A, v) = \begin{pmatrix} 1 & \ldots & 1 \end{pmatrix}, \begin{pmatrix} 1 & -x & -y & x + y & \cdot \\ 1 & 1 & -z & \cdot & \cdot \\ 1 & 1 & -z & \cdot & \cdot \\ 1 & 1 & -y & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}. \]

Clearly, the left subfamily \( (s_3, s_4, s_5) \) and the right subfamily \( (t_1, t_2, t_3) \) of \( \mathcal{A} \) are \( \mathbb{K} \)-linearly independent respectively. If we subtract row 3 from row 2 and add column 2 to column 3, we get the ALS

\[
\mathcal{A}' = (u', A', v') = \begin{pmatrix} 1 & \ldots & 1 \end{pmatrix}, \begin{pmatrix} 1 & -x & -x - y & x + y & \cdot \\ 1 & 1 & 0 & 0 & \cdot \\ 1 & 1 & -z & \cdot & \cdot \\ 1 & 1 & -y & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}. \]

The right subfamily \( (t'_1, t'_2, t'_3) \) of \( \mathcal{A}'' = \mathcal{A}'^{[-2]} \) is (here) not \( \mathbb{K} \)-linearly independent anymore, therefore we must check for a right minimization step for \( k = 3 \) again.

**Definition 4.11** (Minimization Equations [Sch18c, Definition 31]). Let \( \mathcal{A} = (u, A, v) \) be a polynomial ALS of dimension \( n \geq 2 \). Recalling the block decomposition (4.7), we denote by \( \mathcal{A}^{[-k]} \) the ALS \( \mathcal{A}^{[-k]} \) without (block) row/column \( k \) (of dimension \( n-1 \)):

\[
\mathcal{A}^{[-k]} = \left( \begin{bmatrix} u_{1:1-k} \\ \vdots \\ A_{1:1-k:n} \end{bmatrix}, \begin{bmatrix} \hat{v}_{1:1-k} \\ \vdots \\ \hat{v}_{n-k:n} \end{bmatrix} \right).
\]

For \( k \in \{1, 2, \ldots, n-1\} \) the equations \( U + A_{k:n} + TA_{n:n} = 0 \) and \( v_k + T\hat{v}_{n-k} = 0 \), see (4.8), with respect to the block decomposition \( \mathcal{A}^{[k]} \) are called left minimization equations, denoted by \( \mathcal{L}_k = \mathcal{L}_k(\mathcal{A}) \). A solution by the row block pair \( (T, U) \) is denoted by \( \mathcal{L}_k(T, U) = 0 \), the corresponding transformation by \( (P(T), Q(U)) \). For \( k \in \{2, 3, \ldots, n\} \) the equations \( A_{1:1-k}U + A_{1:k} + T = 0 \), see (4.10), with respect to the block decomposition \( \mathcal{A}^{[k]} \) are called right minimization equations, denoted by \( \mathcal{R}_k = \mathcal{R}_k(\mathcal{A}) \). A solution by the column block pair \( (T, U) \) is denoted by \( \mathcal{R}_k(T, U) = 0 \), the corresponding transformation by \( (P(T), Q(U)) \).
Algorithm 4.12 (Minimizing a polynomial ALS [Sch18c, Algorithm 32]).
Input: $A = (u, A, v)$ polynomial ALS of dimension $n \geq 2$ (for some polynomial $p$).
Output: $A' = (\cdot, \cdot, \cdot)$ if $p = 0$ or a minimal polynomial ALS $A' = (u', A', v')$ if $p \neq 0$.

1: \[ k := 2 \]
2: \[ \textbf{while } k \leq \text{dim } A \textbf{ do} \]
3: \[ n := \text{dim}(A) \]
4: \[ k' := n + 1 - k \]
5: \[ \text{lin. indep.} \]
6: \[ \text{Is the left subfamily } (s_{k'}, s_{k'+1}, \ldots, s_n) \text{ } K\text{-linearly dependent?} \]
7: \[ \textbf{if } \exists T, U \in K^{1 \times (k-1)} \text{ admissible : } L_{k'}(A) = L_{k'}(T, U) = 0 \textbf{ then} \]
8: \[ \textbf{endif} \]
9: \[ A := (P(T)AQ(U))^{-k'} \]
10: \[ \textbf{if } k > \max\{2, \frac{n+1}{2}\} \textbf{ then} \]
11: \[ k := k - 1 \]
12: \[ \textbf{endif} \]
13: \[ \textbf{continue} \]
14: \[ \text{lin. indep.} \]
15: \[ \text{Is the right subfamily } (t_1, \ldots, t_{k-1}, t_k) \text{ } K\text{-linearly dependent?} \]
16: \[ \textbf{if } \exists T, U \in K^{(k-1) \times 1} \text{ admissible : } R_k(A) = R_k(T, U) = 0 \textbf{ then} \]
17: \[ A := (P(T)AQ(U))^{-k} \]
18: \[ \textbf{if } k > \max\{2, \frac{n+1}{2}\} \textbf{ then} \]
19: \[ k := k - 1 \]
20: \[ \textbf{endif} \]
21: \[ \textbf{continue} \]
22: \[ k := k + 1 \]
23: \[ \text{done} \]
24: \[ \textbf{return } P.A, \text{ with } P, \text{ such that } P v = [0, \ldots, 0, \lambda]^{\top} \]

Remark. Notice that, compared to [Sch18a, Algorithm 4.13], the first row does not have to be treated separately (using an extended ALS), because for $\text{dim } A = 2$ $K$-linear independence of the left family is equivalent to $K$-linear independence of the right family. Hence the former is indirectly checked by the latter in line 16 and therefore the lines 12–15 (in the general algorithm) do not have a correspondence here.

Remark. Notice in particular Remark 4.15 for a correction of the general minimization algorithm, [Sch18a, Algorithm 4.13].

4.4 Pivot Block Refinement
To be able to minimize an ALS using linear techniques only the pivot blocks have to be refined, that is, none can be (admissibly) transformed such that it splits in two
Thus, additionally to det($P$) corresponding coefficient matrices for 1, $x$ and $\times$ (smaller) pivot blocks. For an illustration we consider the ALS

$$A = \begin{pmatrix} 1 & \ldots & . \\ \cdot & 2 + x & 1 \\ \cdot & 2y & -3 \\ x & 3x & \cdot \end{pmatrix},$$

with a $3 \times 3$ pivot block. Using the admissible block transformation

$$(P, Q) = \begin{pmatrix} 1 & \ldots & . \\ \cdot & 2 & 0 \\ \cdot & -3 & 0 \\ \cdot & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \ldots & . \\ \cdot & 2 & 0 \\ \cdot & -3 & 0 \\ \cdot & 1 & 1 \end{pmatrix}$$

we need to check if it is possible to create a lower left block of zeros of size $1 \times 2$ or $2 \times 1$ in the second pivot block of $PAQ$. First we need to ensure invertibility of $P$ and $Q$ by the conditions

$$0 \neq \det(P) = \alpha_{2,2} \alpha_{3,3} - \alpha_{2,3} \alpha_{3,2} \quad \text{and}$$

$$0 \neq \det(Q) = (\beta_{2,2} \beta_{3,3} - \beta_{2,3} \beta_{3,2}) \beta_{4,4} + (\beta_{2,4} \beta_{3,3} - \beta_{2,3} \beta_{4,3}) \beta_{4,3} + (\beta_{2,3} \beta_{3,4} - \beta_{2,4} \beta_{3,3}) \beta_{4,2}.$$

To (possibly) split the second pivot block into a $1 \times 1$ and $2 \times 2$ block we need to solve the equations obtained by applying the block transformation matrices to the corresponding coefficient matrices for 1, $x$ and $y$ (notice that there is no contribution with respect to $z$; irrelevant equations are marked with "*" on the right hand side)

$$\begin{pmatrix} \alpha_{2,2} & \alpha_{2,3} & 0 \\ \alpha_{3,2} & \alpha_{3,3} & 0 \\ \alpha_{4,2} & \alpha_{4,3} & 1 \end{pmatrix} \begin{pmatrix} 2 & . & 1 \\ . & -3 & . \\ . & . & . \end{pmatrix} \begin{pmatrix} \beta_{2,2} & \beta_{2,3} & \beta_{2,4} \\ \beta_{3,2} & \beta_{3,3} & \beta_{3,4} \\ \beta_{4,2} & \beta_{4,3} & \beta_{4,4} \end{pmatrix} = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

for 1, and

$$\begin{pmatrix} \alpha_{2,2} & \alpha_{2,3} & 0 \\ \alpha_{3,2} & \alpha_{3,3} & 0 \\ \alpha_{4,2} & \alpha_{4,3} & 1 \end{pmatrix} \begin{pmatrix} 1 & . & . \\ . & -3 & . \\ . & . & . \end{pmatrix} \begin{pmatrix} \beta_{2,2} & \beta_{2,3} & \beta_{2,4} \\ \beta_{3,2} & \beta_{3,3} & \beta_{3,4} \\ \beta_{4,2} & \beta_{4,3} & \beta_{4,4} \end{pmatrix} = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

for $x$, and

$$\begin{pmatrix} \alpha_{2,2} & \alpha_{2,3} & 0 \\ \alpha_{3,2} & \alpha_{3,3} & 0 \\ \alpha_{4,2} & \alpha_{4,3} & 1 \end{pmatrix} \begin{pmatrix} 1 & . & . \\ . & -3 & . \\ . & . & . \end{pmatrix} \begin{pmatrix} \beta_{2,2} & \beta_{2,3} & \beta_{2,4} \\ \beta_{3,2} & \beta_{3,3} & \beta_{3,4} \\ \beta_{4,2} & \beta_{4,3} & \beta_{4,4} \end{pmatrix} = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$$

for $y$.

Thus, additionally to $\det(P) = 1$ and $\det(Q) = 1$, we get the equations

$$\alpha_{3,2} \beta_{4,2} - 3\alpha_{3,3} \beta_{3,2} + 2\alpha_{3,2} \beta_{2,2} = 0,$$

$$\alpha_{4,2} \beta_{4,2} - 3\alpha_{4,3} \beta_{3,2} + 2\alpha_{4,2} \beta_{2,2} = 0,$$

$$\alpha_{3,2} \beta_{2,2} = 0,$$

$$3\beta_{3,2} + (\alpha_{4,2} + 1) \beta_{2,2} = 0,$$

$$\alpha_{3,3} \beta_{4,2} + 2\alpha_{3,3} \beta_{2,2} = 0 \quad \text{and}$$

$$\alpha_{4,3} \beta_{4,2} + 2\alpha_{4,3} \beta_{2,2} = 0.$$
with (at least one) solution

$$(P, Q) = \left(\begin{bmatrix} 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ \vdots & 1 & 0 & 0 & \cdots & \cdots \\ \vdots & 0 & 1 & 0 & \cdots & \cdots \\ \vdots & -1 & 0 & 1 & \cdots & \cdots \\ \end{bmatrix}, \begin{bmatrix} 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ \vdots & 1 & 0 & 0 & \cdots & \cdots \\ \vdots & 0 & 0 & \cdots & \cdots & \cdots \\ \vdots & -2 & 1 & 0 & \cdots & \cdots \\ \end{bmatrix}\right)$$

yielding the (refined) admissible linear system

$$P AQ = \left(\begin{bmatrix} 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ \vdots & x & 1 & \cdots & \cdots & \cdots \\ \vdots & y & -1 & \cdots & \cdots & \cdots \\ \vdots & -1 & x & \cdots & \cdots & \cdots \\ \end{bmatrix}, \begin{bmatrix} 1 - z & \cdots & \cdots & \cdots & \cdots \\ \vdots & x & \cdots & \cdots & \cdots \\ \vdots & y & \cdots & \cdots & \cdots \\ \vdots & -1 & \cdots & \cdots & \cdots \\ \end{bmatrix}\right)$$

representing $z(x^{-1}(1 - xy)^{-1})$. Notice that here it would also be possible to create a lower left $1 \times 2$ zero block in the second pivot block of $A$. This would correspond to the factorization $z((1 - yx)^{-1} x^{-1})$, while the original ALS could be interpreted as $z(x - xyx)^{-1}$.

Solving such polynomial systems of equations in general is very difficult, especially if the ground field $\mathbb{K}$ is not algebraically closed, that is, $\mathbb{K} \subsetneq \overline{\mathbb{K}}$. For further information we refer to [Sch18c, Section 4] and/or [Sch17, Example 4.9].

**Remark 4.13.** To ensure invertibility of the transformation matrices $P$ and $Q$ one can use additional (commuting) variables $P' = (\gamma_{ij})$, $Q' = (\delta_{ij})$ and equations $PP' = I$, $QQ' = I$ instead of $\det(P) = 1$, $\det(Q) = 1$. To say anything about the difference with respect to the computation of Gröbner bases, detailed investigations would be necessary. An introduction to the necessary concepts is [CLO15].

**Remark 4.14.** Since the computation of appropriate transformation matrices for the refinement of (unrefined) pivot blocks in general is difficult, one should try simpler techniques (before) to split pivot blocks. If the permutation of rows and/or columns is not successful, linear techniques could be used by avoiding “overlapping” of row and column transformations: As an example we take the following ALS for $(x - xyx)^{-1}$,

$$\begin{bmatrix} x & 1 & \cdots \\ 1 & y - 1 & -1 \\ \vdots & x & x \\ \end{bmatrix} s = \begin{bmatrix} \vdots \\ 1 \end{bmatrix},$$

and assume that we want to create a lower left block of zeros of size $2 \times 1$ in the system matrix. Then the ansatz

$$(P, Q) = \left(\begin{bmatrix} 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ \alpha_{2,1} & 1 & \cdots & \cdots & \cdots & \cdots \\ \alpha_{3,1} & 1 & \cdots & \cdots & \cdots & \cdots \\ \end{bmatrix}, \begin{bmatrix} 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ 0 & 1 & \cdots & \cdots & \cdots & \cdots \\ \beta_{3,1} & \cdots & \cdots & \cdots & \cdots & \cdots \\ \end{bmatrix}\right)$$

yields a linear system of equations with a solution $\alpha_{2,1} = 0$, $\alpha_{3,1} = -1$ and $\beta_{3,1} = 1$. This approach is also recommended for the factorization of polynomials (to create upper right blocks of zeros).
4.5 Minimizing a Refined ALS

The core of the minimization is to establish the equivalence of minimality and the non-existence of solutions of certain linear systems of equations. Firstly we need to formalize what we have already done, namely to apply (left and right) minimization steps (as “solutions” to linear systems of equations). This is somewhat technical (to implement) but rather simple. The other direction is difficult, namely to show that there is always a “linear” minimization step as long as the refined admissible linear system is not minimal. For the theoretical details we refer to [Sch18a, Section 4].

The basic procedure for the minimization is similar to that in Algorithm 4.12. Instead of \( n \) pivot blocks of size \( 1 \times 1 \) (with entry 1) we operate with respect to \( m \leq n \) (general) pivot blocks of size \( n_i \times n_i \) with \( n_1 + n_2 + \ldots + n_m = n \). To illustrate the setup of a linear system of equations to (possibly) eliminate a block we take (again) the ALS \( A = (u, A, v) \) from Example 1.1 for \( f f^{-1} = 1 \) with \( f = xy - z \), namely

\[
\begin{bmatrix}
1 & -x & z & \ldots & \\
\vdots & 1 & -y & \ldots & \\
\vdots & \vdots & 1 & -1 & \\
\vdots & \vdots & \vdots & y & -1 \\
\vdots & \vdots & \vdots & \vdots & -z & x
\end{bmatrix}
\begin{bmatrix}
s_1 \\
s_2 \\
s_3 \\
s_4 \\
s_5
\end{bmatrix}
= 
\begin{bmatrix}
. \\
. \\
. \\
. \\
1
\end{bmatrix}
\]

However here we are minimizing in a complete systematic way. The system matrix has \( m = 4 \) pivot blocks of size \( n_1 = n_2 = n_3 = 1 \) and \( n_4 = 2 \) respectively. We start with block \( k_s = m - 1 = 3 \) for a left minimization step. Notice that the left subfamily \((s_4, s_5)\) is \( \mathbb{K}\)-linearly independent because we obtained this (sub-)system by applying the minimal inverse (Theorem 2.13) on the minimal ALS

\[
\begin{bmatrix}
1 & -x & z \\
\vdots & 1 & -y \\
\vdots & \vdots & 1 \\
\vdots & \vdots & \vdots & -z & x
\end{bmatrix}
\begin{bmatrix}
s_1 \\
s_2 \\
s_3 \\
s_5
\end{bmatrix}
= 
\begin{bmatrix}
. \\
. \\
. \\
1
\end{bmatrix}
\]

for \( f = xy - z \). To check if the left subfamily \((s_3, s_4, s_5)\) is \( \mathbb{K}\)-linearly independent, we look for an admissible transformation

\[
(P, Q) = \begin{pmatrix}
1 & \mathbf{.} & \mathbf{.} & \mathbf{.} & \mathbf{.} \\
. & 1 & \mathbf{.} & \mathbf{.} & \mathbf{.} \\
. & \mathbf{.} & 1 & \alpha_{3,4} & \alpha_{3,5} \\
. & \mathbf{.} & \mathbf{.} & 1 & \beta_{3,4} \\
. & \mathbf{.} & \mathbf{.} & \mathbf{.} & 1
\end{pmatrix}
\begin{pmatrix}
1 & \mathbf{.} & \mathbf{.} & \mathbf{.} \\
. & 1 & \mathbf{.} & \mathbf{.} \\
. & \mathbf{.} & 1 & \beta_{3,5} \\
. & \mathbf{.} & \mathbf{.} & 1 \\
. & \mathbf{.} & \mathbf{.} & \mathbf{.} & 1
\end{pmatrix}
\]

such that \( PAQ \) has the form (“*” denotes an arbitrary entry)

\[
\begin{bmatrix}
1 & -x & z & \ast & \ast \\
\vdots & 1 & -y & \ast & \ast \\
\vdots & \vdots & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & y & -1 \\
\vdots & \vdots & \vdots & \vdots & -z & x
\end{bmatrix}
\begin{bmatrix}
s_1 \\
s_2 \\
s_3 \\
s_4 \\
s_5
\end{bmatrix}
= 
\begin{bmatrix}
. \\
. \\
. \\
0 \\
1
\end{bmatrix}
\]
by solving the linear system of equations
\[
\begin{align*}
-1 + \beta_{3,4} + \alpha_{3,4}y - \alpha_{3,5}z &= 0, \\
\beta_{3,5} - \alpha_{3,4} + \alpha_{3,5}x &= 0 \quad \text{and} \\
\alpha_{3,5} &= 0.
\end{align*}
\]
Notice that these are indeed 5 equations, namely \(\beta_{3,4} - 1 = 0, \beta_{3,5} - \alpha_{3,4} = 0\) and \(\alpha_{3,5} = 0\) (for 1), \(\alpha_{3,5} = 0\) (for \(x\)), \(\alpha_{3,4} = 0\) (for \(y\)) and \(-\alpha_{3,5} = 0\) (for \(z\)). Since there is a solution \((\alpha_{3,4} = \alpha_{3,5} = \beta_{3,5} = 0, \beta_{3,4} = 1)\) the left subfamily \((s_3, s_4, s_5)\) is \(\mathbb{K}\)-linearly dependent. Applying the transformation \((P, Q)\) with the appropriate entries on \(A\) and removing row 3 and column 3 yields the ALS \(A'\),
\[
\begin{bmatrix}
1 & -x & z \\
1 & -y \\
\vdots & \ddots & \ddots \\
1 & -z & x
\end{bmatrix} \quad \begin{bmatrix}
s'  \\
\end{bmatrix}
\]
with \(m' = 3\) pivot blocks. Now \(k'_1 = m' - 1 = 2\) and we check if the (new) left subfamily \((s'_2, s'_3, s'_4)\) is \(\mathbb{K}\)-linearly independent by looking for a transformation
\[
(P', Q') = \begin{pmatrix}
1 & \cdots & \cdots \\
1 & \alpha_{2,3} & \alpha_{2,4} \\
\vdots & \ddots & \ddots \\
1 & \beta_{2,3} & \beta_{2,4}
\end{pmatrix},
\]
such that \(P'A'Q'\) has the form
\[
\begin{bmatrix}
1 & -x & * & * \\
1 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots \\
1 & -z & x
\end{bmatrix} \quad \begin{bmatrix}
s'  \\
\end{bmatrix}
\]
Since such a transformation exists \((\alpha_{2,4} = \beta_{2,3} = 0, \alpha_{2,3} = \beta_{2,4} = 1)\), the left subfamily \((s'_2, s'_3, s'_4)\) is \(\mathbb{K}\)-linearly dependent. By removing row 2 and column 2 from \(P'A'Q'\) we obtain the ALS \(A'' = (P'A'Q')^{-k}\),
\[
\begin{bmatrix}
1 & z & -x \\
y & -1 \\
\vdots & \ddots & \ddots \\
-\frac{1}{z} & x
\end{bmatrix} \quad \begin{bmatrix}
s''  \\
\end{bmatrix}
\]
with \(m'' = 2\) pivot blocks. Now \(k''_1 = m'' - 1\) and it turns out that the (new) left subfamily \((s''_1, s''_2, s''_3)\) is \(\mathbb{K}\)-linearly independent because there is no solution to the corresponding linear system of equations. Therefore we switch to the right family and try a right minimization step for \(k''_1 = 2\), that is, looking for a transformation
\[
(P'', Q'') = \begin{pmatrix}
1 & \alpha_{1,2} & \alpha_{1,3} \\
1 & \beta_{1,2} & \beta_{1,3} \\
\vdots & \ddots & \ddots \\
1 & 1 & 1
\end{pmatrix}
\]
such that $P''A''Q''$ has the form

$$
\begin{bmatrix}
1 & 0 & 0 \\
. & y & -1 \\
. & -z & x
\end{bmatrix}
\begin{bmatrix}
s'' \\
. \cdot \cdot \\
. \cdot \cdot
\end{bmatrix}
= \begin{bmatrix}
\ast \\
1
\end{bmatrix}.
$$

Notice that the entries $\beta_{1,j}$ in the first row of $Q''$ have to be zero for $(P'', Q'')$ to be admissible and the corresponding entries in the left hand side of

$$
\begin{bmatrix}
1 & 0 & 0 \\
. & y & -1 \\
. & -z & x
\end{bmatrix}
\begin{bmatrix}
s'' \\
. \cdot \cdot \\
. \cdot \cdot
\end{bmatrix}
= \begin{bmatrix}
\ast \\
1
\end{bmatrix}
$$

are always zero. Thus the linear system of equations for checking $\mathbb{K}$-linear independence of the right (sub-)family $t'' = (t''_1, t''_2, t''_3)$ is

$$
z + \alpha_{1,2}y - \alpha_{1,3}z = 0 \quad \text{and} \\
-x - \alpha_{1,2} + \alpha_{1,3}x = 0.
$$

Since it has a solution for $\alpha_{1,2} = 0$ and $\alpha_{1,3} = 1$, the right family $t''$ is $\mathbb{K}$-linearly dependent. Removing block row 2 and block column 2 from $P''A''Q''$ yields the minimal ALS $A''' = (1, [1], 1)$ for $ff^{-1} = 1$. Although minimality is obvious here, it is the main result of the (general) minimization algorithm, given a refined admissible linear system. For the case $f^{-1}f = 1$ one has to treat the first block row seperately by using an extended ALS. This is illustrated in [Sch18a, Section 4].

**Remark 4.15** (Correction of [Sch18a, Algorithm 4.13], [Sch18d, Algorithm 4.5.15]). As the following example shows, it is neccessary (in the general case) to decrement the counter $k$ if a left block minimization step was successful for $k = m$ and $k > 2$, that is, to insert lines 9–10 after line 14. Since the left family $s$ of

$$
\begin{bmatrix}
1 & -1 & . & . \\
. & 1 & x & . \\
. & -y & 1 & -1 \\
. & . & . & 1
\end{bmatrix}
\begin{bmatrix}
s \\
. \cdot \cdot \\
. \cdot \cdot
\end{bmatrix}
= \begin{bmatrix}
. \\
. \cdot \cdot \\
. \cdot \cdot
\end{bmatrix}
$$

is $\mathbb{K}$-linearly dependent while the subfamily $(s_2, s_3) = (s_2, s_3, s_4)$ is $\mathbb{K}$-linearly independent, the first (block) row can be eliminated using an extended ALS. If $k$ is not decremented (to $k = 2$) in this case, the resulting ALS would be non-minimal, since it has only $m' = 2$ pivot blocks left:

$$
\begin{bmatrix}
1 & x & . \\
- y & 1 & -1 \\
. & . & 1
\end{bmatrix}
\begin{bmatrix}
s' \\
. \cdot \cdot \\
. \cdot \cdot
\end{bmatrix}
= \begin{bmatrix}
. \\
. \cdot \cdot \\
. \cdot \cdot
\end{bmatrix}.
$$
Epilogue

Learning to compute with fractions at school takes some time and needs “hard” work by hand. This will not be different for free fractions (but in general much more laborious). For those who want to experiment in computer algebra systems: An experimental implementation in [Fri18] should be available soon.

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Admissible Transformation
\[
\begin{pmatrix}
\alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} \\
\alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} \\
\alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \alpha_{3,4} \\
\alpha_{4,1} & \alpha_{4,2} & \alpha_{4,3} & \alpha_{4,4}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
\beta_{2,1} & \beta_{2,2} & \beta_{2,3} & \beta_{2,4} \\
\beta_{3,1} & \beta_{3,2} & \beta_{3,3} & \beta_{3,4} \\
\beta_{4,1} & \beta_{4,2} & \beta_{4,3} & \beta_{4,4}
\end{pmatrix}
\]

Block Transformation
\[
\begin{pmatrix}
\alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} \\
\alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} \\
0 & 0 & \alpha_{3,3} & \alpha_{3,4} \\
0 & 0 & \alpha_{4,3} & \alpha_{4,4}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
\beta_{2,1} & \beta_{2,2} & \beta_{2,3} & \beta_{2,4} \\
0 & 0 & \beta_{3,3} & \beta_{3,4} \\
0 & 0 & \beta_{4,3} & \beta_{4,4}
\end{pmatrix}
\]

Polynomial Transformation
\[
\begin{pmatrix}
1 & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} \\
1 & \alpha_{2,3} & \alpha_{2,4} \\
1 & \alpha_{3,4} \\
1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & \beta_{2,3} & \beta_{2,4} \\
1 & \beta_{3,4} \\
1
\end{pmatrix}
\]

Factorization Transformation
\[
\begin{pmatrix}
\alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & 0 \\
\alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & 0 \\
\alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & 0 \\
\alpha_{4,1} & \alpha_{4,2} & \alpha_{4,3} & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
\beta_{2,1} & \beta_{2,2} & \beta_{2,3} & \beta_{2,4} \\
\beta_{3,1} & \beta_{3,2} & \beta_{3,3} & \beta_{3,4} \\
\beta_{4,1} & \beta_{4,2} & \beta_{4,3} & \beta_{4,4}
\end{pmatrix}
\]

Block Factorization Transformation
\[
\begin{pmatrix}
\alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & 0 \\
\alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & 0 \\
0 & 0 & \alpha_{3,3} & 0 \\
0 & 0 & \alpha_{4,3} & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
\beta_{2,1} & \beta_{2,2} & \beta_{2,3} & \beta_{2,4} \\
0 & 0 & \beta_{3,3} & \beta_{3,4} \\
0 & 0 & \beta_{4,3} & \beta_{4,4}
\end{pmatrix}
\]

Polynomial Factorization Transformation
\[
\begin{pmatrix}
1 & \alpha_{1,2} & \alpha_{1,3} & 0 \\
1 & \alpha_{2,3} & 0 \\
1 & 0 \\
1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & \beta_{2,3} & \beta_{2,4} \\
1 & \beta_{3,4} \\
1
\end{pmatrix}
\]

Figure 1: The invertible transformation matrices \( P = (\alpha_{ij}) \in \mathbb{K}^{n \times n} \) and \( Q = (\beta_{ij}) \in \mathbb{K}^{n \times n} \) as pair \((P, Q)\), applied to a (not necessarily minimal) admissible linear system \( A = (u, A, v) \) of dimension \( n \) for an element \( f \) of the free field, yields an equivalent ALS \( A' = P.A.Q = (u.Q, P.A.Q, P.v) \) for \( f \). Here \( n = 4 \) (without loss of generality). To ensure invertibility, we need \( \text{det}(P) \neq 0 \) and \( \text{det}(Q) \neq 0 \).