Contact bundle formulation of nonholonomic Maupertuis-Jacobi principle and a length minimizing property of nonholonomic dynamics

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Abstract

We prove a nonholonomic version of the classical Maupertuis-Jacobi principle which transforms an autonomous mechanical nonholonomic problem, determined by a kinetic minus potential energy and a distribution, in a kinetic nonholonomic problem over a fixed level set of the Lagrangian energy. To prove this result we introduce an appropriate contact bundle structure clarifying the geometric equivalence between both problems. By using the nonholonomic Maupertuis-Jacobi principle, we prove that the regular solutions of a mechanical nonholonomic problem starting from a fixed point and in the same level set of the
Lagrangian energy are reparametrizations of geodesics for a family of Riemannian metrics defined on the image of the nonholonomic exponential map. In particular, these trajectories minimize Riemannian length.

1 Introduction

One of the most fruitful ideas in mechanics is the intimate relationship between Riemannian geometry and Lagrangian and Hamiltonian mechanics. In particular, a classical and important result in mechanics, the Maupertuis-Jacobi principle (see, for instance, Arnold [1989]; Biesiada [1995]) establishes the relationship between solutions of a mechanical problem describing motion in a potential field and the geodesic motion for a modified Riemannian metric. More precisely, given a Riemannian (or semi-Riemannian) metric $g$ on a differentiable manifold and $V : Q \rightarrow \mathbb{R}$ a potential function then a mechanical Lagrangian system is determined by the Lagrangian $L_{(g,V)} : TQ \rightarrow \mathbb{R}$:

$$L_{(g,V)}(v_q) = \frac{1}{2} g(v_q, v_q) - V(q), \quad v_q \in T_q Q.$$  

In the region of the configuration space where $V(q) < e$ where $e$ is a constant we define the Riemannian metric (Jacobi metric) by:

$$g_e = (e - V) g$$

Then, it is possible to prove (see Abraham and Marsden [1978]; Godbillon [1969]) that the solutions of the Euler-Lagrange equations for the autonomous mechanical Lagrangian $L_{(g,V)}$ with energy $e$ are the same as the geodesics of the Jacobi metric $g_e$ with energy 1 up to reparametrization. This result is known as the Maupertuis-Jacobi principle. This important result opens the way to the use of well known techniques in Riemannian geometry to the study of the qualitative behaviour of the trajectories of mechanical systems as, for instance, in topological methods to find periodic trajectories of conservative dynamical systems, stability of trajectories, integrability, etc. (see Bolsinov et al. [1995] and references therein).

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However the problem for nonholonomic mechanics [Bloch 2015; Cortés Monforte 2002; Neimark and Fufaev 1972] has been less covered in the literature (see Koiller 1992). One of the reasons is that the equivalent theorem (see Theorem 4.1 or Proposition 8.1 in Koiller 1992 and Baks 1975) relates a nonholonomic problem given by a mechanical Lagrangian \( L(g,V) \) and a nonintegrable distribution \( D \) with the solutions of a nonholonomic kinetic problem determined by the Lagrangian \( L_{gc} \) and the distribution \( D \). It is clear that the new system is now determined by a Riemannian metric (without potential) but since the motion is nonholonomically constrained by the distribution \( D \) the obtained equations are also not of a variational type and, in principle, it seems impossible to use standard techniques of Riemannian geometry to analyse its qualitative behaviour. But, recently, we have shown in [Anahory Simoes et al. 2020c] that for kinetic nonholonomic systems, the nonholonomic solutions starting from a fixed point \( q \in Q \) are true geodesics for a family of Riemannian metrics on the image submanifold \( M_q^{nh} \) of the nonholonomic exponential map at \( q \). Therefore, by using these special Riemannian metrics combined with the construction of the Jacobi metric for nonholonomic systems, we can deduce that regular nonholonomic mechanical trajectories starting from a fixed point \( q \in Q \) and in the same level set of the Lagrangian energy are reparametrizations of true geodesics for a family of Riemannian metrics on the image submanifold of the nonholonomic exponential map. In particular, these nonholonomic trajectories minimize the Riemannian length for sufficiently small times. This is the main result of the paper. Note that, as a direct consequence, we could use Riemannian tools to analyse the qualitative behaviour of nonholonomic systems of the type \( (L(g,V),D) \).

The paper is structured as follows. In Section 2 we introduce the equations of motion of a kinetic nonholonomic system using tools of Riemannian geometry (see Lewis 1998; Synge 1928). Then, we recall the definition of the nonholonomic exponential map introduced in [Anahory Simoes et al. 2020a,b] and its main properties. In addition, section 2 also contains a result, Theorem 2.2 (proved in Anahory Simoes et al. 2020c), which shows that kinetic nonholonomic trajectories starting from the same initial point can be seen as geodesics of an appropriate family of Riemannian metrics. In Section 3 we move on to analyse the case of a mechanical nonholonomic system where we now consider an additional potential energy and we construct an associated kinetic nonholonomic problem introducing the associated Jacobi metric. With these elements, we state the main result of the paper, Theorem 3.2 showing the minimizing property of regular nonholonomic solutions on level sets of the Lagrangian energy. To give a complete proof of
this Theorem we develop in the next sections all the necessary mathematical tools. In particular, in Theorem 4.1 (Section 4), we prove a contact bundle formulation of the Maupertuis-Jacobi principle. For that purpose, we will use the symplectic bundle formulation of mechanical nonholonomic systems which was proposed in Bates and Śniatycki [1993]. With Theorem 4.1 and Theorem 2.2 the proof of Theorem 3.2 follows as a corollary.

2 Nonholonomic exponential map for kinetic nonholonomic systems

In this section, we will review the definition of the nonholonomic exponential map for a kinetic nonholonomic system and some results on this map (for the definition of the nonholonomic exponential map associated with an arbitrary nonholonomic system and its properties, see Anahory Simoes, Marrero, and Martín de Diego [2020a]).

First of all, we will see that the solutions of the equations of motion of a kinetic nonholonomic system are the geodesics of a constrained connection (the nonholonomic connection) on the configuration space restricted to initial conditions in Lewis [1998]. This construction seems to have been first made in Synge [1928].

As we have commented in the introduction, a kinetic nonholonomic system is determined by a triple \((Q, g, D)\), where \(Q\) is a finite dimensional smooth manifold, \(g\) is a Riemannian metric on \(Q\) and \(D\) is a nonintegrable distribution determining the nonholonomic constraints de León and de Diego [1996].

The nonholonomic connection \(\nabla^{nh}\) is defined as

\[
\nabla^{nh}_X Y := P(\nabla^g_X Y) + \nabla^g_X [P'(Y)], \quad \text{for } X, Y \in \mathfrak{X}(Q),
\]

(2.1)

where \(P : TQ \to D\) is the associated orthogonal projector onto the distribution \(D\) and \(P' : TQ \to D^\perp\) is the orthogonal projector onto \(D^\perp\), the orthogonal distribution.

This connection is not symmetric (that is, it is not torsion free, see Barbero-Liñán et al. [2012], for an alternative symmetry condition) and in general neither it is compatible with the metric. Nevertheless, it satisfies a more restricted condition of compatibility with the Riemannian metric \(g\) over sections of \(D\) (see Lewis [1998]), i.e.,

\[
X(g(Y, Z)) = g(\nabla^{nh}_X Y, Z) + g(Y, \nabla^{nh}_X Z), \quad \forall X, Y, Z \in \Gamma(D).
\]

(2.2)
It is interesting to note that if $Y \in \Gamma(\mathcal{D})$ then $\nabla_X^n Y = P(\nabla_X Y) \in \Gamma(\mathcal{D})$ for any vector field $X \in \mathcal{X}(Q)$.

The geodesics $c$ for this connection which satisfy the constraints, that is,

$$\nabla_{\dot{c}(t)} \dot{c}(t) = 0, \quad \dot{c}(0) \in \mathcal{D}_c(0)$$

are precisely the solutions of the nonholonomic problem given by $(Q, g, \mathcal{D})$ (see, for instance, Barbero-Liñán, de León, Martín de Diego, Marrero, and Muñoz-Lecanda, 2012; Lewis, 1998).

**Lemma 2.1** (Anahory Simoes et al. 2020). Let $c_v : I \to Q$ be a nonholonomic geodesic with initial velocity $v \in \mathcal{D}_q$, i.e.

$$c_v(t_0) = q \quad \text{and} \quad \dot{c}_v(t_0) = v.

1. We have that

$$\|\dot{c}_v(t)\|_{g(c_v(t))} = \|v\|_{g(q)}, \quad \text{for} \quad t \in I.$$  

2. If $v = 0$ then $c_v(t) = q$, for every $t \in I$.

3. If $v \neq 0$ then a reparametrization of $c_v$,

$$c_v \circ r : J \to Q, \quad s \to c_v(r(s))$$

is a nonholonomic geodesic if and only if

$$r(s) = as + b, \quad \text{with} \quad a, b \in \mathbb{R}.$$  

The tangent lifts of the nonholonomic geodesics of a kinetic nonholonomic system $(Q, g, \mathcal{D})$ are the integral curves of a vector field of $\Gamma(g, \mathcal{D}) \in \mathcal{X}(\mathcal{D})$, which is a second-order differential equation along the points of $\mathcal{D}$, considered as a vector subbundle of $TQ$ (see, for instance, de León and de Diego, 1996).

Denote by $\phi_t^{\Gamma(g, \mathcal{D})} : \mathcal{D} \to \mathcal{D}$ the flow of $\Gamma(g, \mathcal{D})$ and for a sufficiently small positive number $h$, we consider the open subset of $\mathcal{D}$ given by

$$M_h^{\Gamma(g, \mathcal{D})} = \{v \in \mathcal{D} \mid \phi_t^{\Gamma(g, \mathcal{D})}(v) \text{ is defined for } t \in [0, h]\}.$$  

Using the last part of Lemma 2.1 we can assume, without the loss of generality, that $h = 1$. Then, we will denote the open subset $M_1^{\Gamma(g, \mathcal{D})}$ of $\mathcal{D}$ by $M^{\Gamma(g, \mathcal{D})}$. In addition, from the second part of Lemma 2.1 we also have that the zero section in $\mathcal{D}$ is contained in $M^{\Gamma(g, \mathcal{D})}$.  

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From the flow of $\Gamma_{(g,D)}$, we can define the nonholonomic exponential map

$$\exp^{\Gamma_{(g,D)}} : M^{\Gamma_{(g,D)}} \subseteq D \to Q \times Q$$

$$v \mapsto (\tau_Q(v), \tau_Q \circ \phi^{\Gamma_{(g,D)}}_1(v))$$

(see Anahory Simoes, Marrero, and Martín de Diego, 2020a). We remark that if $c_v : [0,1] \to Q$ is the nonholonomic geodesic with $\dot{c}_v(0) = v$ then

$$\exp^{\Gamma_{(g,D)}}(v) = (\tau_Q(v), c_v(1)). \quad (2.5)$$

We will use in the sequel the restriction of this map to the open subset $M_q^{\Gamma_{(g,D)}} = M^{\Gamma_{(g,D)}} \cap D_q$ of $D_q$ with $q \in Q$ fixed, that is, we define

$$\exp_q^{nh} = \text{pr}_2 \circ \exp^{\Gamma_{(g,D)}}|_{M_q^{\Gamma_{(g,D)}}} : M_q^{\Gamma_{(g,D)}} \subseteq D_q \to Q$$

So, if $v_q \in D_q$ and $c_{v_q} : [0,1] \to Q$ is the nonholonomic geodesic with initial velocity $v_q$ then

$$\exp_q^{nh}(v_q) = c_{v_q}(1).$$

The reader is invited to compare the definition of $\exp_q^{nh}$ with that of the Riemannian exponential at $q$ (see do Carmo 1992).

In fact, the nonholonomic exponential map conserves many of the properties we may find in Riemannian exponential maps.

The most important result is the theorem showing that kinetical nonholonomic trajectories starting from the same initial point can be seen as geodesics of an appropriate family of Riemannian metrics (see Anahory Simoes et al. 2020d).

**Theorem 2.2.** Let $(Q, g, D)$ be a kinetic nonholonomic system and $q$ a fixed point in $Q$. Then:

i) There exists a submanifold $M_q^{nh}$ of $Q$, with $q \in M_q^{nh}$, and a diffeomorphism $\exp_q^{nh} : U_0 \subseteq D_q \to M_q^{nh} \subseteq Q$, where $U_0$ is a starshaped open subset of $D_q$ about $q \in U_0$ and $\exp_q^{nh}(0_q) = q$. The map $\exp_q^{nh}$ is the nonholonomic exponential map at $q$. Moreover, we have that:

(a) Under the canonical linear identification between $D_q$ and $T_0 U_0$, the linear monomorphism

$$T_0 \exp_q^{nh} : T_0 U_0 \simeq D_q \to T_q Q$$

is just the canonical inclusion of $D_q$ in $T_q Q$.  

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For every \( v_q \in U_0 \),

\[ \exp_q^{nh}(tv_q) = c_{v_q}(t), \quad t \in [0,1], \quad (2.6) \]

with \( c_{v_q} : [0,1] \to M_q^{nh} \subseteq Q \) the (unique) nonholonomic trajectory satisfying \( c_{v_q}(0) = q, \dot{c}_{v_q}(0) = v_q \).

ii) All the radial kinetic nonholonomic trajectories departing from the fixed point \( q \in Q \) are homothetic reparametrizations of nonholonomic trajectories given by equation \( (2.6) \). In addition, they are minimizing geodesics for a Riemannian metric \( g_q^{nh} \) on \( M_q^{nh} \) if and if only if the Riemannian metric \( G_0 = (\exp_q^{nh})^*(g_q^{nh}) \) on \( U_0 \) satisfies the Gauss condition, that is,

\[ G_0(v_q)(v_q, w_q) = G_0(0_q)(v_q, w_q), \quad \text{for } v_q \in U_0 \text{ and } w_q \in D_q. \]

iii) Such Riemannian metrics on \( M_q^{nh} \) always exist and if \( g_q^{nh} \) is one of them then the Riemannian exponential associated with \( g_q^{nh} \) at \( q \) is just \( \exp_q^{nh} \).

3 Mechanical nonholonomic trajectories

In this section, we state a nonholonomic version of the Maupertuis-Jacobi principle.

Then, using Theorem 2.2 we will immediately deduce that radial nonholonomic mechanical trajectories with fixed energy \( e \in \mathbb{R} \) are, for sufficiently small times, strictly increasing reparametrizations of minimizing Riemannian geodesics on a suitable Riemannian manifold.

Let \( g \) be a Riemannian metric on the \( n \)-dimensional manifold \( Q, V : Q \to \mathbb{R} \) be a smooth function called the potential energy and let \( D \) be a rank \( r \) distribution on \( Q \). Let \( L(g,V) : TQ \to \mathbb{R} \) be the mechanical Lagrangian function associated with the Riemannian metric \( g \) and potential energy \( V \), that is,

\[ L(g,V)(v) = \frac{1}{2}g(v,v) - V \circ \tau_Q(v), \quad v \in TQ. \]

The triple \((Q,(g,V),D)\) is called a nonholonomic mechanical system [Bloch 2015]. The trajectories of a nonholonomic mechanical system satisfy the equations:

\[ \nabla_{\dot{c}(t)}^{nh} \dot{c}(t) + \text{grad}^g V(c(t)) = 0 \ , \quad \dot{c}(0) \in D_{c(0)}, \quad (3.1) \]
where \( \text{grad}^g \) is the gradient vector field on \( Q \) associated with the potential energy \( V \) via the metric \( g \), that is,
\[
g(\text{grad}^g V, X) = \langle dV, X \rangle, \quad \forall X \in \mathfrak{X}(Q).
\]
Therefore, given \( v_q \in D \), denote by \( c_{v_q} : I \to Q \) the unique solution of (3.1) with initial conditions
\[
c_{v_q}(0) = q, \quad \dot{c}_{v_q}(0) = v_q.
\]
As it is well-known, \( (Q, (g, V), D) \) is a regular nonholonomic mechanical system and thus the tangent lift of the trajectories \( c_{v_q} \) (which we denote by \( \dot{c}_{v_q} : I \to TQ \)) are integral curves of a SODE denoted by \( \Gamma_{(g,V,D)} \in \mathfrak{X}(D) \).

The energy of the system \( (Q, L_{(g,V)}, D) \) is given by the function \( E_{(g,V)} : D \to \mathbb{R} \) defined by
\[
E_{(g,V)}(v) = \frac{1}{2}g(v,v) + V \circ \tau_Q(v), \quad v \in D.
\]
Recall that the energy is a first integral of the vector field \( \Gamma_{(g,V,D)} \), which implies that the energy is constant along the trajectories \( c_{v_q} \), i.e.,
\[
E_{(g,V)}(\dot{c}_{v_q}(t)) = e, \quad \forall t \in I,
\]
where \( e \in \mathbb{R} \) is some real number. Note that, for mechanical systems \( e \geq V(q) \).

Fixing a real number \( e \in \mathbb{R} \), it is possible to classify the mechanical trajectories into two different types:

(i) **Singular trajectories**: the energy of the trajectory \( c_{v_q} \) satisfies \( e = V(q) \), which automatically implies that the initial velocity is zero \( v_q = 0 \).

(ii) **Regular trajectories**: the energy of the trajectory \( c_{v_q} \) satisfies \( e > V(q) \) and the velocity of the trajectory may be written as
\[
\|\dot{c}_{v_q}(t)\|^2 = 2(e - V(c_{v_q}(t))), \quad \forall t \in I.
\]

So, there exists a real number \( \varepsilon > 0 \) such that the curve \( c_{v_q} : (-\varepsilon, \varepsilon) \to Q \) is a regular trajectory.

Now, if for a fixed \( e \in \mathbb{R} \), the curve \( c_{v_q} \) is a regular trajectory, that is \( e > V(q) \), then it is clear that the initial velocity is in the sphere centered at the zero vector \( 0_q \) and with radius \( \sqrt{2(e - V(q))} \), which we will denote by
\[
v_q \in S_g \left( \sqrt{2(e - V(q))} \right),
\]
where the subscript \( g \) indicates that the norm is measured relative to the Riemannian metric \( g \).

**Remark 3.1.** The set \( \{ q \in Q \mid e \geq V(q) \} \) is usually called the Hill region and the set \( \{ q \in Q \mid e = V(q) \} \) is called the Hill boundary or also sometimes called the zero velocity surface.

Now, take \( e \in \mathbb{R} \) such that

\[
U_e = \{ q \in Q \mid e > V(q) \}
\]

is a non-empty subset of \( Q \). Then, \( U_e \) is an open subset of \( Q \) and, if it is non-empty, it inherits the smooth manifold structure of \( Q \). We can consider on it the Jacobi metric

\[
ge_e = (e - V)g
\]

and the kinetic nonholonomic system \((U_e, g_e, D_e)\), where the distribution \( D_e \) is nothing but the fibers of \( D \) at the points in \( U_e \). In other words, \( D_e = (\tau_D)^{-1}(U_e) \), where \( \tau_D : D \to Q \) is the vector bundle projection.

Given a vector \( v_q \in D_e \), we will denote by \( c^e_{v_q} : I \to U_e \) the nonholonomic trajectory of \((U_e, g_e, D_e)\), with initial velocity \( v_q \), that is

\[
\nabla^{nh,e}_{c^e_{v_q}(t)}c^e_{v_q}(t) = 0, \quad c^e_{v_q}(0) \in D_{c^e_{v_q}(0)}
\]

where \( \nabla^{nh,e}_{X}Y := P(\nabla^{g_e}_{X}Y) + \nabla^{g_e}_{X}[P'(Y)], X, Y \in \Gamma(D_e) \). Observe that since \( g_e \) and \( g \) are in the same conformal class of metrics the orthogonal projectors are the same for both metrics.

Therefore, there exists a SODE \( \Gamma_{(g_e, D_e)} \in \mathfrak{X}(D_e) \) whose integral curve with initial velocity \( v_q \) is precisely the tangent lift of the trajectory \( c^e_{v_q} \).

Moreover, the energy of this system is simply given by the Lagrangian itself, that is, \( E_{g_e} : TU_e \to \mathbb{R} \) coincides with the Lagrangian function \( L_{g_e} : TU_e \to \mathbb{R} \) given by

\[
L_{g_e}(u) = \frac{1}{2}g_e(u, u), \quad u \in TU_e.
\]

Thus, \( L_{g_e}|_{D_e} \) is a first integral of \( \Gamma_{(g_e, D_e)} \).

Moreover, it is not difficult to prove that if the trajectory \( c^e_{v_q} \) has energy equal to 1, then the initial velocity \( v_q \) satisfies

\[
v_q \in S_g \left( \sqrt{\frac{2}{e - V(q)}} \right).
\]
using the same notation as before.

Now, let us introduce two projections and a suitable diffeomorphism between the two spheres mentioned above.

Let \( P_q : T_q Q \setminus \{0_q\} \to S_g \left( \sqrt{2(e - V(q))} \right) \) be the projection given by

\[
P_q(v_q) = \sqrt{2(e - V(q))} \frac{v_q}{\|v_q\|_g}
\]

and \( Q_q : T_q Q \setminus \{0_q\} \to S_g \left( \sqrt{\frac{2}{e - V(q)}} \right) \) be the projection given by

\[
Q_q(v_q) = \sqrt{\frac{2}{e - V(q)}} \frac{v_q}{\|v_q\|_g}.
\]

Consider the map \( \Psi_q : S_g \left( \sqrt{2(e - V(q))} \right) \to S_g \left( \sqrt{\frac{2}{e - V(q)}} \right) \) that makes the diagram of Figure 3.1 to commute. Observe that \( \Psi_q \) is a diffeomorphism

\[
\begin{array}{ccc}
T_q Q \setminus \{0_q\} & \xrightarrow{P_q} & S_g \left( \sqrt{2(e - V(q))} \right) \\
\Psi_q & \downarrow & \Psi_q \\
& S_g \left( \sqrt{\frac{2}{e - V(q)}} \right) & \xleftarrow{Q_q} \\
\end{array}
\]

Figure 3.1: Definition of the diffeomorphism \( \Psi_q \) between spheres with explicit expression

\[
\Psi_q(v_q) = \frac{1}{(e - V(q))} v_q.
\]

We are now in position to formulate the main result of this paper.

**Theorem 3.2.** Let \( (Q, (g, V), \mathcal{D}) \) be a mechanical nonholonomic system, \( q \in Q \) a fixed point of the manifold \( Q \) and let \( e \in \mathbb{R} \) such that \( e > V(q) \).

Then:

i) There exists \( \varepsilon > 0 \) and a submanifold \( \mathcal{M}^{nh,e}_q \subset Q \) with \( q \in \mathcal{M}^{nh,e}_q \) and a diffeomorphism

\[
\exp^{nh,e}_q : B_g \left( 0_q; \sqrt{\frac{2\varepsilon}{e - V(q)}} \right) \subset \mathcal{D}_q \to \mathcal{M}^{nh,e}_q,
\]

where the domain denotes the open ball in \( \mathcal{D}_q \) around \( 0_q \) with radius \( \sqrt{\frac{2\varepsilon}{e - V(q)}} \), with respect to the Riemannian metric \( g \).

Moreover we have that \( \exp^{nh,e}_q(0_q) = q \) and:
(a) The tangent map of $\exp_{q}^{nh,e}$ at $0_{q}$, under the canonical linear identification between $D_{q}$ and $T_{0_{q}} \left(B_{g}(0_{q}; \sqrt{\frac{2\varepsilon}{\varepsilon-V(q)}})\right)$,

$$T_{0_{q}}\exp_{q}^{nh,e} : D_{q} \rightarrow T_{q}Q,$$

is just the canonical inclusion of $D_{q}$ in $T_{q}Q$.

(b) For every non-zero vector $v_{q} \in B_{g}(0_{q}; \sqrt{\frac{2\varepsilon}{\varepsilon-V(q)}})$ the nonholonomic mechanical trajectory $c_{P_{q}(v_{q})} : [0, \lambda] \rightarrow Q$ satisfies

$$c_{P_{q}(v_{q})}(s) = \exp_{q}^{nh,e}(h(s)Q_{q}(v_{q})), \quad (3.4)$$

where $h : [0, \lambda] \rightarrow [0, \delta]$ is a strictly increasing reparametrization satisfying

$$\frac{dh}{ds} = e - V \circ c_{P_{q}(v_{q})}, \quad h(0) = 0$$

and $\lambda$ is sufficiently small in such a way that

$$h(s)Q_{q}(v_{q}) \in B_{g}(0_{q}; \sqrt{\frac{2\varepsilon}{\varepsilon-V(q)}}), \quad \forall s \in [0, \lambda].$$

ii) All the nonholonomic trajectories with starting point $q$ and energy $e$ are, for sufficiently small times, of the form $(3.4)$. In addition, if $g_{q}^{nh,e}$ is a Riemannian metric on $M_{q}^{nh,e}$ such that $G_{0}^{e} = (\exp_{q}^{nh,e})^{*}g_{q}^{nh,e}$ satisfies the Gauss condition, then the curves

$$t \in [0, 1] \mapsto \exp_{q}^{nh,e}(tv_{q}) \in M_{q}^{nh,e},$$

with $v_{q} \in B_{g}(0_{q}; \sqrt{\frac{2\varepsilon}{\varepsilon-V(q)}})$ are geodesics for $g_{q}^{nh,e}$ and, therefore, the nonholonomic trajectories

$$s \in [0, \lambda] \mapsto c_{P_{q}(v_{q})}(s) \in M_{q}^{nh,e}$$

are reparametrizations of minimizing geodesics for the metric $g_{q}^{nh,e}$. In particular, these nonholonomic trajectories minimize length in $M_{q}^{nh,e}$.

iii) The Riemannian metrics $g_{q}^{nh,e}$ on $M_{q}^{nh,e}$ always exist.

**Remark 3.3.** We have that the map

$$\exp_{q}^{nh,e} : B_{g}(0_{q}; \sqrt{\frac{2\varepsilon}{\varepsilon-V(q)}}) \subseteq D_{q} \rightarrow U_{e} \subseteq Q$$
is given by

$$\exp_{q}^{nh,e}(v_q) = \tau_{Q} \left( \phi_{1}^{\Gamma_{(g_{e}, D_{e})}}(v_q) \right)$$

for \( v_q \in B_{g} \left( 0_{q}; \sqrt{\frac{2e}{e-V(q)}} \right) \) and where \( \phi_{1}^{\Gamma_{(g_{e}, D_{e})}} \) is the flow of the SODE \( \Gamma_{(g_{e}, D_{e})} \) along \( D_{e} \). In other words, \( \exp_{q}^{nh,e} \) is the nonholonomic exponential map at \( q \) associated with the kinetic non-holonomic system \( (U_{e}, g_{e}, D_{e}) \).

**Remark 3.4.** In Theorem 3.2 (item ii)) we mention the notion of a metric on \( D_{q} \) satisfying the Gauss condition. This type of metrics were introduced in Anahory Simoes, Marrero, and Martín de Diego [2020c]. In fact, a Riemannian metric \( G_{0} \) on a finite-dimensional real vector space \( E \) is a Gauss metric if

$$G_{0}(u)(u, v) = G(0)(u, v), \quad \forall u, v \in E.$$  

In order to prove our main theorem we will need the following version of the nonholonomic Maupertuis-Jacobi principle relating nonholonomic mechanical trajectories with nonholonomic trajectories of an associated kinetical nonholonomic problem.

**Theorem 3.5** (Nonholonomic Maupertuis-Jacobi theorem). Let \((Q, (g, V), D)\) be a mechanical nonholonomic system, \( q \in Q \) a fixed point of the manifold and let \( e \in \mathbb{R} \) such that \( e > V(q) \). For a non-zero \( v_q \in T_{q}U_{e} \) denote by

$$c_{P_{q}(v_q)} : J \rightarrow U_{e} \quad \text{and} \quad c_{Q_{q}(v_q)} : I \rightarrow U_{e} \quad \text{with} \quad 0 \in I, J$$

the nonholonomic trajectories for the systems \( (U_{e}, L_{(g, V)}|_{TU_{e}}, D_{e}) \) and \( (U_{e}, L_{g_{e}}, D_{e}) \) with initial velocities \( P_{q}(v_q) \) and \( Q_{q}(v_q) \), respectively. Then, we have that

$$c_{P_{q}(v_q)}(s) = c_{Q_{q}(v_q)}(h(s)),$$

where \( h : J \rightarrow I \) is a strictly increasing reparametrization satisfying

$$\frac{dh}{ds} = e - V \circ c_{P_{q}(v_q)}, \quad h(0) = 0.$$  

### 4 Nonholonomic Maupertuis-Jacobi principle

In this section, we develop the machinery we will need to prove the nonholonomic Maupertuis-Jacobi Theorem 3.5.

#### 4.1 Symplectic bundle formulation of nonholonomic mechanical systems

Let \((Q, (g, V), D)\) be a mechanical nonholonomic system with rank \( D = r \).
4.1.1 The Lagrangian side

We will review the main ingredients of the construction given by Bates and Śniatycki \[1993\] (see also Cortés Monforte \[2002\]; J. Cortés \[2009\]). First of all, we will introduce the set

\[ \mathcal{T}^{D} = \bigcup_{v_{q} \in D} \{ X \in T_{v_{q}} D \mid (T_{v_{q}} \tau_{Q})(X) \in D_{q} \} \]

which is a symplectic vector bundle of rank 2 over \( D \), that is,

\[ (T_{v_{q}}^{D}, \omega_{L(g,V)}(v_{q})|_{T_{v_{q}}^{D}}) \]

is a symplectic vector space of dimension 2r, for all \( v_{q} \in D_{q} \), where \( \omega_{L(g,V)} \) is the Poincaré-Cartan 2-form associated with the mechanical Lagrangian \( L(g,V) \) (see de León and Rodrigues \[1989\]).

Let \( E(g,V) \) be the corresponding Lagrangian energy. Then we have that

\[ dE(g,V)(v_{q})|_{T_{v_{q}}^{D}} \in (T_{v_{q}}^{D})^{*} \]

for all \( v_{q} \in D_{q} \).

Moreover, we have that the nonholonomic vector field \( \Gamma_{(g,V,D)} \) defined in (3.1) is geometrically characterized by the equations

\[ \left( i_{\Gamma_{(g,V,D)}} \omega_{L(g,V)}|_{D} \right)|_{T^{D}} = \left( dE(g,V)|_{D} \right)|_{T^{D}} \]

\[ \Gamma_{(g,V,D)} \in \Gamma(T^{D}). \] (4.1)

As an immediate consequence, we deduce the preservation of energy for the nonholonomic trajectories:

\[ \Gamma_{(g,V,D)}(E(g,V)|_{D}) = 0. \] (4.2)

4.1.2 The Hamiltonian side

Given a Riemannian metric \( g \) and a potential energy function \( V \) on the manifold \( Q \), we may consider the Hamiltonian function \( H_{(g,V)} : T^{*}Q \rightarrow \mathbb{R} \) given by

\[ H_{(g,V)}(\alpha_{q}) = \frac{1}{2}q_{q}^{\tau}(\alpha_{q}, \alpha_{q}) + V(q), \quad \alpha_{q} \in T_{q}^{*}Q, \]
where we are denoting by $g^\sharp$ the co-metric associated to the Riemannian metric $g$. Indeed, given a Riemannian metric $g$, there is an isomorphism of modules $♭_g : \mathfrak{X}(Q) \to \Omega^1(Q)$ called the flat isomorphism given by
\[ ⟨♭_g(X(q)), Y(q)⟩ = g_q(X(q), Y(q)), \quad X, Y \in \mathfrak{X}(Q). \]
Then the co-metric is the map $g^\sharp : \Omega^1(Q) \times \Omega^1(Q) \to C^\infty(Q)$ given by
\[ g^\sharp_q(b_g(X(q)), b_g(Y(q))) = g_q(X(q), Y(q)), \quad X, Y \in \mathfrak{X}(Q). \]
It is also interesting to note that the Legendre transform of the mechanical Lagrangian function $L_{(g,V)}$, denoted by $\mathbb{F}L_{(g,V)} : TQ \to T^*Q$, coincides with the flat isomorphism, i.e.,
\[ \mathbb{F}L_{(g,V)} = b_g. \]
Moreover, we have that
\[ (\mathbb{F}L_{(g,V)})^*\omega_Q = \omega_{L_{(g,V)}} \quad \text{and} \quad (\mathbb{F}L_{(g,V)})^*H_{(g,V)} = E_{(g,V)} \]
where $\omega_Q$ is the canonical symplectic form on $T^*Q$.
If $D^\perp$ is the orthogonal complement of $D$ with respect to the metric $g$ and
\[ (D^\perp)^o = \bigcup_{q \in Q} \{ α_q \in T_q^*Q \mid ⟨α_q, v_q⟩ = 0, \quad \forall v_q \in D^\perp_q \} \]
then we have that
\[ \mathbb{F}L_{(g,V)}(D) = (D^\perp)^o. \]
It is clear that $(i_D^\perp)^o : (D^\perp)^o \to D^*$ is an isomorphism of vector bundles where $i_D : D \hookrightarrow TQ$ is the canonical inclusion. From now on, we will use the previous canonical identification between $(D^\perp)^o$ and $D^*$. We have that $T\mathbb{F}L_{(g,V)} = T♭_g$ is a vector bundle isomorphism over $\mathbb{F}L_{(g,V)} = b_g$. Hence, considering the following vector bundle over $D^*$
\[ T^{D^*}D^* = \bigcup_{\alpha_q \in D^*_q} \{ Y \in T_{\alpha_q}D^* \mid (T_{\alpha_q}\pi_Q)(Y) \in D_q \}, \]
where $\pi_Q : T^*Q \to Q$ is the cotangent bundle projection, we have that
\[ T♭_g(T^{D^*}D) = T^{D^*}D^*. \]
Hence, using the results in Subsection 4.1.1 we deduce that
is a symplectic vector bundle over $D^*$ of rank $2r$.

As a consequence, there exists a unique section $X_{(g,V,D)} \in \Gamma(T^*D^*)$ satisfying

$$\left(i_{X_{(g,V,D)}}\omega_Q\right)|_{T^*D^*} = \left(dH_{(g,V)}\right)|_{T^*D^*} \quad \text{and} \quad X_{(g,V,D)} \in \mathfrak{X}(D^*)$$

Moreover, from (4.1) and (4.3), we deduce that

$$X_{(g,V,D)} \circ \flat_g|_D = (T\flat_g)|_{T^*D} \circ \Gamma_{(g,V,D)}.$$  

So, if $\sigma : I \to D^*$ is an integral curve of $X_{(g,V,D)}$ then

$$\pi_Q \circ \sigma : I \to Q$$

is a trajectory of the nonholonomic mechanical system $(Q, (g,V,D))$.

### 4.2 A contact bundle formulation of the nonholonomic Maupertuis-Jacobi principle

Let $(Q, (g,V,D))$ be a mechanical nonholonomic system and consider the Hamiltonian function $H_{(g,V)} : T^*Q \to \mathbb{R}$ along with the corresponding Hamiltonian vector field $X_{(g,V,D)} \in \mathfrak{X}(D^*)$.

Suppose that $e \in \mathbb{R}$ is such that $U_e = \{q \in Q \mid e > V(q)\}$ is non-empty. Again consider the Jacobi metric $g_e$ defined in $U_e$ defined on (3.2) as well as the distribution $D_e$ and its dual distribution

$$D_e^* = \bigcup_{q \in U_e} D_q^* \subseteq T^*U_e.$$  

$D_e^*$ is a vector bundle over $U_e$ with vector bundle projection $\tau_e^* : D_e^* \to U_e$.

In the Hamiltonian side of the nonholonomic kinetic system $(U_e, g_e, D_e)$, we will denote by $X_{(g_e,D_e)} \in \mathfrak{X}(D_e^*)$ the corresponding Hamiltonian vector field.

As we know

$$(T^*D_e^*, \omega_Q|_{T^*D_e^* \times T^*D_e^*})$$

is a symplectic vector bundle over $D_e^*$ and also

$$\left(i_{X_{(g_e,D_e)}}\omega_Q\right)|_{T^*D_e^*} = \left(dH_{g_e}\right)|_{T^*D_e^*},$$

(4.5)
where $H_{g_e} : T^* U_e \to \mathbb{R}$ is the Hamiltonian function in the Hamiltonian side of the kinetic nonholonomic system $(U_e, g_e, D_e)$. It is important to note that the Hamiltonian function $H_{g_e}$ is given by

$$H_{g_e}(\alpha_q) = \frac{1}{2} g^e(\alpha_q, \alpha_q),$$

where $g^e$ is the Jacobi co-metric which is given by

$$g^e = \frac{1}{e - V^g}.$$

(4.6)

Let us introduce the subset $S^*_e$ of $D^*_e$ given by

$$S^*_e = \bigcup_{q \in U_e} \{ \alpha_q \in D^*_q \mid \| \alpha_q \|^2_{g^e} = 2(e - V(q)) \}.$$  

Then we may prove the following result:

**Theorem 4.1** (Contact bundle formulation of the nonholonomic Maupertuis-Jacobi principle). Using the notation we have introduced before, the following statements hold:

1. The subset $S^*_e$ satisfies

$$S^*_e = \left( (H_{g_e} |_{D^*_e})^{-1} (e) \right) = \left( (H_{g_{g^e}} |_{D^*_e})^{-1} (1) \right)$$

and, in addition, if $\alpha_q \in S^*_e$ then

$$\left( dH_{(g_e, V)}(\alpha_q) \right) |_{T^*_e D^*_e} = \left( dH_{g_e}(\alpha_q) \right) |_{T^*_e D^*_e} \neq 0,$$

so $S^*_e$ is a submanifold of codimension 1 in $D^*_e$. In fact,

$$T_{\alpha_q} S^*_e = \{ X \in T_{\alpha_q} D^*_e \mid \langle dH_{(g_e, V)}(\alpha_q), X \rangle = 0 \}$$

and $S^*_e$ is a bundle over $U_e$ with fiber at $q \in U_e$ the sphere centred at $0_q \in D^*_e$ and radius $\sqrt{2(e - V(q))}$, with respect to the Riemannian metric $g$.

2. If $C_e$ is defined by

$$C_e = \bigcup_{\alpha_q \in D^*_q} \left( T_{\alpha_q} S^*_e \cap T^{D^*_e}_{\alpha_q} D^*_e \right)$$

then $C_e$ is a vector bundle over $S^*_e$ which admits a contact bundle structure and the Reeb section $R_e$ is just $X_{(g_e, D_e)|D^*_e}$. 

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3. We have that
\[(e - V)|_{Ue} R_e = X_{(g,V,D)}|_{s^*_e}.\]

4. If \(v_q \in D_q\) is a non-zero vector with \(q \in U_e\) and \(c_{P_q}(v_q) : J \to U_e\), \(c_{Q_q}(v_q) : I \to U_e\) are the nonholonomic trajectories of the systems \((U_e(g,V)|_{U_e}, D_e)\), \((U_e, g_e, D_e)\) with initial velocities \(P_q(v_q)\) and \(Q_q(v_q)\), respectively, then
\[c_{P_q(v_q)}(s) = c_{Q_q(v_q)}(h(s)),\]
where \(h : J \to I\) is a strictly increasing reparametrization satisfying
\[
\frac{dh}{ds} = e - V \circ c_{P_q(v_q)}, \quad h(0) = 0.
\]

**Remark 4.2.** In the above theorem we used some notations introduced in the previous sections, namely the projections
\[P_q : D_q \setminus \{0_q\} \to \left(E_{(g,V)}\right)^{-1}(e) \cap D_q\]
given by
\[P_q(v_q) = \sqrt{2(e - V(q))} \frac{v_q}{\|v_q\|_g}\]
and \(Q_q : TqQ \setminus \{0_q\} \to (E_{g_e})^{-1}(1) \cap D_q\) given by
\[Q_q(v_q) = \sqrt{\frac{2}{e - V(q)}} \frac{v_q}{\|v_q\|_g}.

**Proof.** Let us prove each item in the theorem by order of appearance:

1. We have that
\[\|\alpha_q\|^2_g = 2(e - V(q))\]
is equivalent to
\[
\frac{\|\alpha_q\|^2_g}{2(e - V(q))} = 1
\]
and so, using the definition of the Jacobi co-metric \(g^e_g\) in (4.6), we have that
\[\frac{1}{2}\|\alpha_q\|^2_{g_e} = 1,
\]
which proves that \(\alpha_q \in \left(H_{(g,V)}|_{D^*_e}\right)^{-1}(e)\) if and only if \(\alpha_q \in \left(H_{g_e}|_{D^*_e}\right)^{-1}(1)\).
Now, let $\Delta^*$ be the Euler vector field of $D^*$ defined by

$$\Delta^*(\alpha_q) = (\alpha_q)_V = \left. \frac{d}{dt} \right|_{t=0} ((1 + t)\alpha_q) \in T_{\alpha_q}^* D_e^*.$$ 

Then, if $\alpha_q \in S_e^*$ we have that

$$\langle dH_{(g,V)}(\alpha_q), \Delta^*(\alpha_q) \rangle = \|\alpha_q\|_g^2 = 2(e - V(q)) > 0$$

as well as

$$\langle dH_{g_e}(\alpha_q), \Delta^*(\alpha_q) \rangle = \|\alpha_q\|_{g_e}^2 = 2 > 0.$$ 

Hence, $S_e^*$ is a submanifold of $D_e^*$ of codimension 1 and

$$T_{\alpha_q}S_e^* = \{ X \in T_{\alpha_q} D^*_e \mid \langle dH_{(g,V)}(\alpha_q), X \rangle = 0 \}$$

Thus,

$$T_{\alpha_q}D_e^* = T_{\alpha_q}S_e^* \oplus \langle \Delta^*(\alpha_q) \rangle.$$ 

Therefore, using that $\Delta^*$ is vertical with respect to the projection $\tau^*_e : D_e^* \to U_e$, it follows that the restriction of $\tau^*_e$ to $S_e^*$ is also a bundle with projection $\tau^*_e|_{S_e^*} : S_e^* \to U_e$. In addition, it is easy to prove that the fiber of $\tau^*_e|_{S_e^*}$ at $q \in U_e$ is just the sphere centred at $0_q \in D_e^*$ and radius $\sqrt{2(e - V(q))}$, with respect to the Riemannian metric $g$.

2. If $\alpha_q \in S_e^*$ then, from the previous item, we deduce that the set

$$T_{\alpha_q}^* D_e^* \cap T_{\alpha_q}S_e^*$$

is a vector subspace of codimension 1 of $T_{\alpha_q}^* D_e^*$. Therefore,

$$\mathcal{C}_e = \bigcup_{\alpha_q \in S_e^*, q \in U_e} (T_{\alpha_q}S_e^* \cap T_{\alpha_q}^* D_e^*)$$

is a vector bundle over $S_e^*$ with rank $2r - 1$ (we recall that $T_{\alpha_q}^* D_e^*$ is a $2r$-dimensional symplectic vector space).

Now, we consider the sections $(\theta_Q)_e$ and $(\omega_Q)_e$ of the vector bundles $\mathcal{C}_e \to S_e^*$ and $\Lambda^2(\mathcal{C}_e) \to S_e^*$, respectively, given by

$$(\theta_Q)_e(\alpha_q) = \frac{1}{2} \theta_Q(\alpha_q)|_{(\mathcal{C}_e)_{\alpha_q}}$$
and
\[(\omega_Q)_e(\alpha_q) = \frac{1}{2} \omega_Q(\alpha_q)|_{(c_e)_{\alpha_q} \times (C_e)_{\alpha_q}}\]

for \(\alpha_q \in S^*_e\).

We will see that \(((\theta_Q)_e, (\omega_Q)_e)\) is a contact bundle structure on the vector bundle \(C_e \to S^*_e\), that is,
\[(\theta_Q)_e \wedge (\omega_Q)_e^{r-1} \in \Gamma(\Lambda^2 r(C^*_e))\]
is non-vanishing at every point of \(S^*_e\). In fact, using that
\[X_{(g_e, D_e)}(\alpha_q) (H_{g_e}|_{D_e}) = 0\]
it follows that \(X_{(g_e, D_e)}(\alpha_q) \in (C_e)_{\alpha_q}\). Thus, we deduce that \(X_{(g_e, D_e)}|_{S^*_e} \in \Gamma(C_e)\). In addition,
\[\langle (\theta_Q)_e(\alpha_q), X_{(g_e, D_e)}(\alpha_q) \rangle = \frac{1}{2} \langle \alpha_q, T^{\tau^*_e}_e \cdot (X_{(g_e, D_e)}(\alpha_q)) \rangle,\]

where we used the definition of the canonical 1-form of the cotangent bundle and \(\tau^*_e : D^*_e \to U_e\) is the bundle projection. On the other hand, from (4.4), we have that
\[\langle (\theta_Q)_e(\alpha_q), X_{(g_e, D_e)}(\alpha_q) \rangle = \frac{1}{2} \langle \alpha_q, \#_{g_e}(\alpha_q) \rangle = \frac{1}{2} \|\alpha_q\|^2_{g_e} = 1.\]
Moreover, we have that
\[ \left. \left[ i_{X_{(g_e,D_e)}(\alpha_q)}(\omega_{q})_e(\alpha_q) \right] \right|_{(c_e)_{\alpha_q}} = dH_{g_e}(\alpha_q)|_{(c_e)_{\alpha_q}} = 0. \]

This implies that \((\theta_{q})_e, (\omega_{q})_e)\) is a contact bundle structure on the vector bundle \(C_e\) and that \(X_{(g_e,D_e)}|_{S^*_e} \in \Gamma(C_e)\) is the Reeb section of this contact structure, that is,
\[ i_{X_{(g_e,D_e)}|_{S^*_e}}(\theta_{q})_e = 1, \quad i_{X_{(g_e,D_e)}|_{S^*_e}}(\omega_{q})_e = 0. \]

3. Using that
\[ \left. \left( X_{(g,V,D)}|_{S^*_e} \right) \left( H_{(g,V)}|_{S^*_e} \right) = 0 \]

it follows that \(X_{(g,V,D)}|_{S^*_e} \in \Gamma(C_e)\). In addition, proceeding as in the previous item, one may prove that if \(\alpha_q \in S^*_e\) then
\[ ((\theta_{q})_e(\alpha_q), X_{(g,V,D)}(\alpha_q)) = \frac{1}{2} \|\alpha_q\|^2_g = e - V(q) \]

and
\[ \left. \left[ i_{X_{(g,V,D)}(\alpha_q)}(\omega_{q})_e(\alpha_q) \right] \right|_{(c_e)_{\alpha_q}} = dH_{(g,V)}(\alpha_q)|_{(c_e)_{\alpha_q}} = 0. \]

Therefore,
\[ (e - V(q))|v_q X_{(g_e,D_e)}|_{S^*_e} = X_{(g,V,D)}|_{S^*_e}. \quad (4.7) \]

4. It is easy to prove that the following diagram commutes:

\[\begin{array}{ccc}
E^{-1}_{(g,V)}(e) \cap D_q & \xleftarrow{\gamma_g} & (S^*_e)_{q} \\
\uparrow_{\beta_g} & & \downarrow_{\beta_{g,e}} \\
E^{-1}_{g_e}(1) \cap D_q & \xrightarrow{\beta_{g}} & D_q \backslash \{0_q\}
\end{array}\]

Figure 4.1: Commutative diagram.

Thus, if \(v_q \in D_q \backslash \{0_q\}\) then
\[ \beta_{g}(P_q(v_q)) = \beta_{g,e}(Q_q(v_q)) = \alpha_q \in S^*_e. \quad (4.8) \]
Now, we consider the integral curves \( \sigma_{\alpha q} : J \to S_e^* \) and \( \sigma^e_{\alpha q} : I \to S_e^* \) (with \( 0 \in I, J \)) of the vector fields \( X_{(g,V,D)}|_{S_e^*} \) and \( X_{(g_e,D_e)}|_{S_e^*} \), respectively, satisfying the initial conditions
\[
\sigma_{\alpha q}(0) = \sigma^e_{\alpha q}(0) = \alpha_q.
\]
Then, using Equation (4.7) in the previous item, it follows that there exists a strictly increasing reparametrization \( h : J \to I \) such that
\[
\frac{dh}{ds} = e - V \circ \pi_{D^*} \circ \sigma_{\alpha q}, \quad h(0) = 0
\]
and
\[
\sigma_{\alpha q}(s) = \sigma^e_{\alpha q}(h(s)), \quad \text{for } s \in J,
\]
with \( \pi_{D^*} : D^* \to Q \) the canonical projection. But, recall that, if \( v_q = \sharp q(\alpha_q) \) then using (4.4), Figure 4.1 and (4.8), we deduce that
\[
\pi_{D^*} \circ \sigma_{\alpha q} = cQ_q(v_q) \quad \text{and} \quad \pi_{D^*} \circ \sigma^e_{\alpha q} = cQ_q(v_q),
\]
which implies the result.

\[\square\]

**Remark 4.3.** A coordinate derivation of Maupertuis-Jacobi principle (see also Koiller [1992]) Having chosen a system of coordinates \( (q^i) \), \( 1 \leq i \leq n = \dim Q \) then we induce a system of coordinates \( (q^i, \dot{q}^i) \) on \( TQ \). In these coordinates, the Lagrangian \( L_{(g,V)} : TQ \to \mathbb{R} \) is written as
\[
L(q^i, \dot{q}^i) = \frac{1}{2} g_{ij}(q) \dot{q}^i \dot{q}^j - V(q)
\]
where \( g_{ij} = g(\partial/\partial q^i, \partial/\partial q^j) \). The linear velocity constraints are determined by the distribution \( D \) where \( \operatorname{rank} D = m \leq n \) and it is locally determined by its annihilator:
\[
D^\alpha = \operatorname{span}\{\mu^\alpha = \mu^\alpha_q(q) dq^i; m + 1 \leq \alpha \leq n\}
\]
However in the case of nonholonomic mechanics it can be better to adapt the coordinates on the tangent bundle to the linear velocity constraints and to the Riemannian metric. To this end, consider a local basis \( \{X_a, Y_\alpha\} \), \( 1 \leq a \leq m \) and \( m + 1 \leq \alpha \leq n \) of vector fields such that locally
\[
D_q = \operatorname{span}\{X_a(q)\} \quad \text{and} \quad D^\perp_q = \operatorname{span}\{Y_\alpha(q)\},
\]
where $\mathcal{D}^\perp_q$ is the Riemannian-orthogonal to $\mathcal{D}$, i.e.

$$g(X_a, Y_\alpha) = 0, \quad 1 \leq a \leq m \quad \text{and} \quad m + 1 \leq \alpha \leq n.$$  

Denote by $g_{ab} = g(X_a, X_b)$ and consider the Lie bracket:

$$[X_a, X_b] = C^c_{ab} X_c + C^\alpha_{ab} Y_\alpha$$  

Observe that the non-vanishing of some of functions $C^\alpha_{ab}$ implies the non-integrability of the distribution $\mathcal{D}$.

Obviously we have that $T_qQ = \mathcal{D}_q \oplus \mathcal{D}^\perp_q$. Therefore, the adapted basis $\{X_a, Y_\alpha\}$ induces a new set of coordinates on the tangent bundle $(q^i, y^a, y^\alpha)$. Observe that the elements $v_q \in \mathcal{D}_q$ are distinguished by the condition $y^\alpha = 0$. That is, the nonholonomic constraint are now $y^\alpha = 0$ and $\mathcal{D}$ is completely described by coordinates $(q^i, y^a)$.

Denote by $\{X^a, Y^\alpha\}$ the dual basis corresponding to $\{X_a, Y_\alpha\}$ inducing coordinates $(q^i, p^a, p^\alpha)$ on $T^*Q$ and $(q^i, p_a)$ on $D^*$. The Hamiltonian is now

$$H_{(g,V)}|D^*(q^i, p_a) = \frac{1}{2} g^{ab}(q)p_ap_b + V(q).$$

The equations of motion of a nonholonomic system are written in the system of adapted coordinates $(q^i, p_a)$ as follows (see, for instance, Celledoni et al. [2019]; Cortés et al. [2009]):

$$\dot{q}^i = X^i_b \frac{\partial H_{(g,V)}|D^*}{\partial p_b} = X^i_b g^{ab} p_a, \quad (4.9a)$$

$$\dot{p}_a = -C^c_{ab} p_c \frac{\partial H_{(g,V)}|D^*}{\partial p_b} - X^i_a \frac{\partial H_{(g,V)}|D^*}{\partial q^i}$$

$$\quad - C_{ad}^{\beta d} p_\beta p_d - X^i_a \left( \frac{1}{2} \frac{\partial g^{\alpha \beta}}{\partial q^i} p_\beta p_d + \frac{\partial V}{\partial q^i} \right), \quad (4.9b)$$

where $X_a = X^i_a \frac{\partial}{\partial q^i}$. The dynamics is precisely the given by the vector field $X_{(g,V,D)}$ intrinsically defined in Equation (4.3).

From the other hand, if we consider the Hamiltonian $H_{g_e}|\mathcal{D}_e^* : \mathcal{D}_e^* \to \mathbb{R}$:

$$H_{g_e}|\mathcal{D}_e^*(q^i, p_a) = \frac{1}{2(e - V(q))} g^{ab} p_ap_b.$$  

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Then the corresponding nonholonomic equations are:

\[
\begin{align*}
\dot{q}^i &= \frac{1}{e - V(q)} X^i_q g^{ab} p_a , \\
\dot{p}_a &= -\frac{1}{e - V(q)} C^c_{ab} g^{bd} p_c p_d \\
&\quad - X^i_a(q) \left( \frac{1}{2(e - V(q))} \frac{\partial g^{cb}}{\partial q^i} p_c p_b + \frac{1}{2(e - V(q))^2} \frac{\partial^2 V}{\partial q^i} g^{cb} p_c p_b \right)
\end{align*}
\] (4.10a, 4.10b, 4.10c)

These equations are precisely the ones defined by the integral curves of the vector field \(X_{(g,e,D_e)}\) given in Equation (4.5).

Therefore

\[
X_{(g,e,D_e)} - \frac{1}{e - V(q)} X_{(g,V,D)}|_{D_e^*} = X^i_a(q) \left( \frac{1}{2(e - V(q))} \frac{\partial V}{\partial q^i} g^{cb} p_c p_b - \frac{\partial V}{\partial q^i} \right) \frac{\partial}{\partial p_a}
\]

Along the set \(S^*_e = (H_{(g,V,D)}|_{D_e^*})^{-1}(e)\) we have that \(\frac{1}{2} g^{cb} p_c p_b = e - V(q)\) and in consequence,

\[
R_e = X_{(g,e,D_e)}|_{S^*_e} = \frac{1}{e - V(q)} X_{(g,V,D)}|_{S^*_e}
\]
as appears in Theorem 4.1.

5 Proof of the main Theorem [3.2]

Now we have all the ingredients to prove of Theorem [3.2] since it is a direct consequence combining first the nonholonomic Maupertuis-Jacobi principle stated in Theorem 4.1 and then Theorem 2.2. We just add a few reasons why we take the open ball \(B_g\left(0_q; \sqrt{\frac{2\varepsilon}{e - V(q)}}\right)\), with \(\varepsilon\) a sufficiently small positive number, as the domain of the map \(\exp^{n,h,e}_q\):

- It is clear that \(B_g\left(0_q; \sqrt{\frac{2\varepsilon}{e - V(q)}}\right)\) is a star-shaped open subset of \(D_q\) about \(0_q \in D_q\);

- If \(v_q \in D_q \setminus \{0_q\}\), then \(Q_q(v_q) \in B_g\left(0_q; \sqrt{\frac{2\varepsilon}{e - V(q)}}\right)\). So, if we fix \(\varepsilon > 0\) small enough, it is possible to choose a sufficiently small positive number \(\lambda\) such that

\[
h(s) Q_q(v_q) \in B_g\left(0_q; \sqrt{\frac{2\varepsilon}{e - V(q)}}\right), \forall s \in [0, \lambda]
\]

(note that \(h(0) = 0\);
• Using the previous facts, we can directly apply Theorem 2.2 to the
map \( \exp_{q}^{nh,e} : B_q \left( 0_q; \sqrt{\frac{2e}{e-V(q)}} \right) \subseteq D_q \to Q \).

6 Example

Example 6.1. Let us first consider a mechanical nonholonomic system
describing a particle with unitary mass in euclidean three dimensional space
\( Q = \mathbb{R}^3 \) equipped with the euclidean metric \( g \), subjected to a potential force
\( V : Q \to \mathbb{R} \) given by
\[
V(x, y, z) = z,
\]
and to the nonholonomic constraint determined by
\[
\mathcal{D} = \{(q, \dot{q}) \in TQ \mid \dot{z} = y\dot{x}\}.
\]

Let \( e \in \mathbb{R} \) be a fixed energy value and consider the set
\[
U_e = \{(x, y, z) \in Q \mid z < e\}
\]
where the Jacobi metric
\[
g_e = (e - z)g
\]
is defined. The kinetic nonholonomic system \((g_e, D_e)\) associated to the
mechanical nonholonomic system \((g, V, D)\) is associated to the kinetic La-
grangian \( L_{g_e} : TU_e \to \mathbb{R} \) given by
\[
L_{g_e}(q, \dot{q}) = \frac{e - z}{2} \left( \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right).
\]

To observe explicitly the results of Theorem 4.1, it is easier to work on the
Hamiltonian side and using a basis adapted to \( \mathcal{D} \), as in Remark 4.3. In that
sense, we will use the basis given by
\[
X_1 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial y}
\]
spanning \( \mathcal{D} \) and the vector
\[
Y_1 = -y \frac{\partial}{\partial x} + \frac{\partial}{\partial z}
\]
spanning the orthogonal complement \( \mathcal{D}^\perp \). Hence, we obtain the following
non-vanishing components of the Riemannian metric
\[
g_{11} = 1 + y^2, \quad g_{22} = 1.
\]
Finally, the non-vanishing structure functions \((C^c_{ab})\) relative to this basis are

\[ C^1_{12} = -\frac{y}{y^2 + 1} = -C^3_{23}, \quad C^1_{23} = -\frac{1}{y^2 + 1} = C^3_{12}. \]

The Hamiltonian function is written with respect to this basis as

\[ H_{(g,V)}|_{D^*}(q^i, p_a) = \frac{1}{2} \left( \frac{p_1^2}{y^2 + 1} + p_2^2 \right) + z \]

and the corresponding Hamiltonian equations in this adapted coordinates are

\[
\begin{align*}
\dot{x} &= \frac{p_1}{y^2 + 1} \\
\dot{y} &= p_2 \\
\dot{z} &= \frac{yp_1}{y^2 + 1} \\
\dot{p}_1 &= \frac{yp_2}{y^2 + 1} - y \\
\dot{p}_2 &= 0
\end{align*}
\]

On the other hand, the kinetic Hamiltonian function \(H_{g_e} : T^*U_e \to \mathbb{R}\) is given on these coordinates by

\[ H_{g_e}(q^i, p_a) = \frac{1}{2(e - z)} \left( \frac{p_1^2}{y^2 + 1} + p_2^2 \right) \]

implying the following Hamiltonian equations

\[
\begin{align*}
\dot{x} &= \frac{1}{e - z} \frac{p_1}{y^2 + 1} \\
\dot{y} &= \frac{1}{e - z} p_2 \\
\dot{z} &= \frac{1}{e - z} \frac{yp_1}{y^2 + 1} \\
\dot{p}_1 &= \frac{1}{e - z} \frac{yp_2}{y^2 + 1} - \frac{y}{2(e - z)^2} \left( \frac{p_1^2}{y^2 + 1} + p_2^2 \right) \\
\dot{p}_2 &= 0.
\end{align*}
\]

Then it is clear that if we restrict to the set \(S^*_e = \left( H_{(g,V)}|_{D^*} \right)^{-1} (e) \), we have that

\[
\frac{1}{2(e - z)} \left( \frac{p_1^2}{y^2 + 1} + p_2^2 \right) = e - z
\]

on \(S^*_e\), showing that

\[ X_{(g_e,D_e)}|S^*_e = \frac{1}{e - z} X_{(g,V,D)}|S^*_e. \]

It is now clear that the integral curves of \(X_{(g,V,D)}\) must be a reparametrization of the integral curves of the Hamiltonian vector field \(X_{(g_e,D_e)}\) on \(S^*_e\).
Example 6.2. The vertical rolling disk with harmonic potential in the steering angle. Consider the mechanical Lagrangian function $L_{(g,V)} : TQ \rightarrow \mathbb{R}$ in the manifold $Q = \mathbb{R}^2 \times S^1 \times S^1$ given by

$$L_{(g,V)}(q,\dot{q}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{\theta}^2 + \dot{\varphi}^2) - \frac{\varphi^2}{2},$$

subject to the constraint

$$D = \{(q,\dot{q}) \in TQ \mid \dot{x} = \dot{\theta} \cos \varphi, \quad \dot{y} = \dot{\theta} \sin \varphi\}.$$

It is not difficult to show that the general solution is

$$\begin{align*}
x(t) &= \int_0^t \cos(\varphi(s)) \, ds + x_0 \\
y(t) &= \int_0^t \sin(\varphi(s)) \, ds + y_0 \\
\theta(t) &= \Omega t + \theta_0 \\
\varphi(t) &= \varphi_0 \cos(t) + \omega \sin(t),
\end{align*}$$

with $q_0 = (x_0, y_0, \theta_0, \varphi_0) \in Q$, $(\Omega, \omega) \in \mathbb{R}^2$ a coordinate chart on $D_{q_0}$, representing the initial angular velocities. Then the nonholonomic exponential map, which is the map $\exp_{q_0}^{nh} : D_{q_0} \rightarrow Q$ given by

$$\exp_{q_0}^{nh}(v_{q_0}) = (\tau_Q \circ \phi_{(g,V,D)}^{(g,V,D)}(v_{q_0}),$$

where $\phi_t^{(g,V)}$ is the flow of the nonholonomic mechanical system $(L_{(g,V)}, D)$, is a local diffeomorphism onto its image and so its inverse map is $R_{q_0}^{nh} : \mathcal{M}_{q_0}^{nh} \rightarrow D_{q_0}$ given by

$$R_{q_0}^{nh}(\theta, \varphi) = \left(\theta - \theta_0, \frac{\varphi - \varphi_0 \cos(1)}{\sin(1)}\right).$$

The corresponding kinetic nonholonomic system is determined by the Lagrangian function $L : g_e : TU_e \rightarrow \mathbb{R}$ given by

$$L_{g_e}(q,\dot{q}) = \frac{e - \varphi^2}{2}(\dot{x}^2 + \dot{y}^2 + \dot{\theta}^2 + \dot{\varphi}^2).$$

After some computations, we may eliminate the Lagrange multipliers appearing in Lagrange-d’Alembert equations and find that the trajectories of the nonholonomic system $(L_{g_e}, D_e)$ must satisfy

$$\begin{align*}
\dot{x} &= \dot{\theta} \cos \varphi \\
\dot{y} &= \dot{\theta} \sin \varphi \\
\ddot{\theta} &= \frac{2\varphi \dot{\varphi}}{e - \frac{\varphi^2}{2}} \\
\ddot{\varphi} &= \frac{e \varphi^2 \sin \varphi}{(e - \frac{\varphi^2}{2})^2}.
\end{align*}$$
Then the trajectories of this system form the exponential map \( \exp^{(g_e, D_e)} : \mathcal{D}_{q_0} \to Q \).

Moreover, using Theorem 3.3, we know there is a strictly increasing function \( h : J \to I \) satisfying

\[
\frac{dh}{ds} = e - V \circ c_{P_q(v_q)}, \quad h(0) = 0.
\]

Solving the differential equations, we obtain that

\[
h(s) = es - \frac{1}{2} \left( \frac{\varphi_0^2 - \omega^2}{2} \cos s \sin s + \frac{(\varphi_0^2 + \omega^2)s}{2} + \varphi_0 \omega \sin^2 s \right).
\]

**Example 6.3.** The vertical rolling disk with linear potential in the steering angle. Consider the mechanical Lagrangian function \( L : TQ \to \mathbb{R} \) in the manifold \( Q = \mathbb{R}^2 \times S^1 \times S^1 \) given by

\[
L(q, \dot{q}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{\theta}^2 + \dot{\varphi}^2) = -\varphi,
\]

subject to the constraint

\[
\mathcal{D} = \{ (q, \dot{q}) \in TQ \mid \dot{x} = \theta \cos \varphi, \quad \dot{y} = \theta \sin \varphi \}.
\]

It is not difficult to show that the general solution is

\[
\begin{align*}
x(t) &= \int_0^t \cos(\varphi(s)) \, ds + x_0 \\
y(t) &= \int_0^t \sin(\varphi(s)) \, ds + y_0 \\
\theta(t) &= \Omega t + \theta_0 \\
\varphi(t) &= \omega t + \varphi_0 - \frac{t^2}{2},
\end{align*}
\]

with \( q_0 = (x_0, y_0, \theta_0, \varphi_0) \in Q, (\Omega, \omega) \in \mathbb{R}^2 \) a coordinate chart on \( \mathcal{D}_{q_0} \), representing the initial angular velocities. Then the nonholonomic exponential map \( \exp^{nh}_{q_0} : \mathcal{D}_{q_0} \to Q \) given by

\[
\exp^{nh}_{q_0}(v_{q_0}) = (\tau_Q \circ \phi^{(g,V,D)}_1)(v_{q_0}),
\]

where \( \phi^{(g,V,D)}_1 \) is the flow of the nonholonomic mechanical system \( (L_{(g,V)}, \mathcal{D}) \), is a local diffeomorphism onto its image and so its inverse map is \( R^{nh}_{q_0} : \mathcal{M}^{nh}_{q_0} \to \mathcal{D}_{q_0} \) given by

\[
R^{nh}_{q_0}(\theta, \varphi) = \left( \theta - \theta_0, \varphi - \varphi_0 + \frac{1}{2} \right).
\]
The corresponding kinetic nonholonomic system is determined by the Lagrangian function $L_{ge} : TU_e \to \mathbb{R}$ given by

$$L_{ge}(q, \dot{q}) = \frac{e - \varphi}{2}(\dot{x}^2 + \dot{y}^2 + \dot{\theta}^2 + \dot{\varphi}^2).$$

After some computations, we may eliminate the Lagrange multipliers appearing in the corresponding Lagrange-d’Alembert equations and find that the trajectories of the nonholonomic system $(L_{ge}, D_e)$ must satisfy

$$\begin{cases} \dot{x} = \dot{\theta} \cos \varphi \\ \dot{y} = \dot{\theta} \sin \varphi \\ \ddot{\theta} = \frac{\dot{\varphi}(\dot{\theta} + \sin(\varphi)\dot{y} + \cos(\varphi)\dot{x})}{2e - 2\varphi} \\ \ddot{\varphi} = \frac{\dot{\theta}^2 - 2\dot{y}^2}{2e - 2\varphi}. \end{cases}$$

Then the trajectories of this system form the exponential map $\exp^{(g_e, D_e)} : D_{q_0} \to Q$.

Moreover, using Theorem 3.5 we know there is a strictly increasing function $h : J \to I$ satisfying

$$\frac{dh}{ds} = e - V \circ c_{P_q(v_q)}, \quad h(0) = 0.$$ 

Solving the differential equations, we obtain that

$$h(s) = es + \frac{s^3}{6} - \frac{\omega s^2}{2} - \varphi_0 s.$$

Moreover, by the definition of nonholonomic exponential map we have that

$$\exp^{nh, e}_{q}(v_q) = c_{v_q}^e(1),$$

where $c_{v_q}^e$ is the trajectory of the kinetic nonholonomic system $(L_{ge}, D_e)$. In addition, note that every non-zero vector in $D$ might be uniquely written in the form

$$v_q = \lambda(v_q)Q_q(v_q), \quad \lambda(v_q) = \sqrt{\left(\frac{e - V(q)}{2}\right)^2 ||v_q||_g}.$$ 

Hence, by the homothetic property of kinetic nonholonomic trajectories we deduce

$$\exp^{nh, e}_{q}(v_q) = c_{Q_q(v_q)}^e(\lambda(v_q)).$$

Alternatively, using again Theorem 3.5 we may also write

$$\exp^{nh, e}_{q}(v_q) = c_{P_q(v_q)}(h^{-1}(\lambda(v_q))).$$
Let \((\Omega, \omega)\) be coordinates on \(\mathcal{D}\) associated to the basis
\[
\{\cos \varphi \frac{\partial}{\partial x} + \sin \varphi \frac{\partial}{\partial y} + \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi}\}
\]
and \((\theta, \varphi)\) coordinates on \(\mathcal{M}^{n_{h,e}}_{q_0} = \mathcal{M}^{n_{h}}_{q_0}\) under which
\[
c_{v_0}(t) = \left(\Omega t + \theta_0, \omega t + \varphi_0 - \frac{t^2}{2}\right).
\]
Now, \(E_{(g,V)}(\Omega, \omega) = e\) if and only if the initial velocity \(\Omega\) is equal to
\[
\Omega^\pm := \pm \sqrt{(e - \varphi_0) - \frac{\omega^2}{2}},
\]
so that
\[
c_{P(v_0)}(t) = \left(\Omega^\pm t + \theta_0, \omega t + \varphi_0 - \frac{t^2}{2}\right).
\]
If \(k = h^{-1} \circ \lambda\) then we have that
\[
\exp^{n_{h,e}}_q(\Omega^\pm, \omega) = \left(\Omega^\pm k(\Omega^\pm, \omega) + \theta_0, \omega k(\Omega^\pm, \omega) + \varphi_0 - \frac{k^2(\Omega^\pm, \omega)}{2}\right).
\]
Considering the flat metric in \(\mathcal{D}\) as a Gauss metric, i.e., the metric
\[
\mathcal{G}_0 = d\Omega \otimes d\Omega + d\omega \otimes d\omega
\]
then the reparametrization by the function \(h\) of the unit energy geodesics with respect to the metric
\[
g^{n_{h,e}}_q = \left((\exp^{n_{h,e}}_q)^{-1}\right)^* \mathcal{G}_0
\]
are just the mechanical nonholonomic trajectories with energy \(e\) with initial point \(q\). Therefore, the nonholonomic trajectories are reparametrizations of minimizing geodesics for the Riemannian metric \(g^{n_{h,e}}_q\). In particular, they minimize the Riemannian length associated with this metric.

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