Lens Model Degeneracy and Cosmological Tests by Strong Gravitational Lensing

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We estimate the sensitivity of lensing observables to the parameters in the lens model (isothermal sphere/Navarro-Frenk-White profile) parameters and to cosmological parameters. We find that the observables are primarily dependent on the lens model parameters, while the dependence on cosmological parameters is minor (especially so for the dark energy parameters). We demonstrate the lens model degeneracy by deriving both the projected mass density profile and the circular velocity profile. We also identify a possible source of the problem of fitting the averaged mass profile of CL 0024+1654 with the Navarro-Frenk-White profile.

1. Introduction. Gravitational lensing has been extensively studied as a useful cosmological tool to probe the high redshift universe. In particular, it has been used to provide limits on cosmological parameters. For example, the statistics of gravitational lensing of QSOs by intervening galaxies provide a powerful tool to set constraints on the cosmological constant. Lensed images of distant galaxies in cluster, so called arcs or rings, may provide a bound on the cosmological constant or even on the equation of state of dark energy. This is mainly because the distance relation $D_{LS}/D_{S}$, where $D_{LS}$ is the distance from the lens to the source and $D_{S}$ is that from the observer to the source, has a strong dependence on the cosmological constant.

However, results obtained regarding gravitational lensing also depend on the lens model. Hence, in using it as a tool to obtain bounds on cosmological parameters, we must be careful about the uncertainties inherent in the lens model itself. If derived values of gravitational lensing observables are sensitive to the lens model, constraints obtained on cosmological parameters may have a meaning whose implication will only become clear after resolving issues concerning the lens model. Only recently have these uncertainties begun to be systematically studied for the case of gravitational lens statistics. In this letter, we elucidate the problem involving the model uncertainties by estimating the sensitivity of observables to the parameters in the lens model and cosmological parameters. We consider the giant arc system observed in CL 0024+1654 as an example. As an aside, we point out that the NFW profile fit may not be excluded by the current data.

2. Lens model and observables. Let $\eta$ and $\xi$ be the source position and the impact position in the lens plane, respectively. Define a length scale $\xi_0$ in the lens plane and a corresponding length scale $r_0 = \xi_0 D_{S}/D_{L}$. Here $D_{L}$ and $D_{S}$ are the angular diameter distance to the lens and source, respectively. Then in terms of the
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dimensionless quantities \( x = \xi/\xi_0 \) and \( y = \eta/\eta_0 \), the lens equation is given by

\[
y = x - \frac{m(x)}{x},
\]

where \( m(x) = 2 \int_0^x (\Sigma(x')/\Sigma_{cr}) x' dx' \), and \( \Sigma_{cr} = D_S/4\pi G D_L D_{LS} \). Here \( D_{LS} \) is the angular diameter distance between the lens and the source. The determinant of the Jacobian \( A \equiv \partial y/\partial x \) of the mapping Eq. (0.1) is calculated as

\[
det A = \left( 1 - \frac{m}{x^2} \right) \left( 1 - \frac{d}{dx} \left( \frac{m}{x} \right) \right). \tag{0.2}
\]

The critical curves for axisymmetric lenses are those for which \( \det A = 0 \). Circles satisfying \( m/x^2 = 1 \) are called “tangential critical curves”, while those satisfying \( d(m/x)/dx = 1 \) are called “radial curves”.

For an isothermal model with a core, the mass profile is \( \rho(r) = \sigma^2/2\pi G (r^2 + r_c^2) \), with \( \sigma \) being the one-dimensional velocity dispersion and \( r_c \) the core radius. The radius of the tangential critical curve, \( \theta_E \), is then given by

\[
\sqrt{\theta_E^2 + \theta_c^2} + \theta_c = 4\pi \sigma^2 D_{LS}/D_S, \tag{0.3}
\]

where \( \theta_c = r_c/D_L \). The circular velocity at radius \( r \) is

\[
v^2(r) = \frac{GM(\leq r)}{r} = 2\sigma^2 \left( 1 - \frac{r_c}{r} \tan^{-1} \frac{r}{r_c} \right). \tag{0.4}
\]

For the Navarro-Frenk-White (NFW) model, \(^9\) the mass profile is given by \( \rho(r) = \rho_s (r/r_s)^{-1} ((r/r_s) + 1)^{-2} \). Taking \( \xi_0 = r_s \), the lens equation becomes \(^{10,11}\)

\[
y = x - \frac{4\rho_s r_s g(x)}{\Sigma_{cr} x}, \tag{0.5}
\]

where \( g(x) \) is defined by

\[
g(x) = \ln \frac{x}{2} + \frac{2}{\sqrt{1-x^2}} \tanh^{-1} \sqrt{\frac{1-x}{1+x}}, \tag{0.6}
\]

for \( x < 1 \) and by

\[
g(x) = \ln \frac{x}{2} + \frac{2}{\sqrt{x^2-1}} \tan^{-1} \sqrt{\frac{x-1}{x+1}}, \tag{0.7}
\]

for \( x > 1 \). The circular velocity is

\[
v^2(r) = \frac{4\pi G \rho_s r_s^3}{r} \left( \ln(1 + r/r_s) - \frac{r/r_s}{1 + r/r_s} \right). \tag{0.8}
\]

The truncated isothermal sphere (TIS) model is a particular solution of the Lane-Emden equation that results from the collapse and virialization of a top-hat density perturbation. \(^{12}\) The mass profile is fit well by

\[
\rho(r) = \rho_0 \left( \frac{A}{(r/r_c)^2 + a^2} - \frac{B}{(r/r_c)^2 + b^2} \right), \tag{0.9}
\]
where \((A, a^2, B, b^2) = (21.38, 9.08, 19.81, 14.62)\). Shapiro et al.\(^{12}\) found that this fitting formula is accurate within 3% over \(0 \leq r/r_c \lesssim 30\) for both the Einstein de Sitter model and low-density models \((\Omega_M \gtrsim 0.3)\). The circular velocity is

\[
v^2(r) = 4\pi G \rho_0 r_c^2 \left( A - B - \frac{aA}{x} \tan^{-1} \frac{x}{a} + \frac{bB}{x} \tan^{-1} \frac{x}{b} \right), \tag{0.10}
\]

where \(x = r/r_c\).

At this stage, it may be illuminating to estimate the sensitivity of the observable \((\theta_E)\) to the parameters in lens model and the cosmological parameters. Using Eq.\((0.3)\), for the case of an isothermal sphere with a core, the sensitivity is evaluated around \(r_c = 0, \Omega_M = 0.3, \Omega_A = 0.7, w = -1\) as

\[
\frac{\delta \theta_E}{\theta_E} \simeq -\frac{\theta_c}{\theta_E} \frac{\delta \theta_c}{\theta_c} + 2 \frac{\delta \sigma}{\sigma} - 7.5 \times 10^{-2} \frac{\delta \Omega_M}{\Omega_M} + 2.9 \times 10^{-2} \frac{\delta w}{w}, \tag{0.11}
\]

where \(w\) is the equation of state of dark energy \((w = -1\) for the cosmological constant\), and the source redshift and the lens redshift are taken as \(z_s = 1.675\) and \(z_l = 0.39\), respectively, and a flat FRW model is assumed. It should be noted that in reality there should be contribution from the eccentricity of the lens profile added to the above relation. Eq.\((0.11)\) and its counterpart for the NFW profile \((Eq.(0.13))\) indicate clearly that the lensing observable \(\theta_E\) is primarily dependent on the lens model parameters, \(\theta_c\) and \(\sigma\), and less dependent on the cosmological parameters, \(\Omega_M\) and \(w\). (This sensivity was noted previously in Ref.\(^{18}\) for the case of an isothermal sphere.) This is essentially because strong lensing is sensitive only to the mass inside the Einstein radius. For example, in order to put a constraint on \(w\), one needs to measure the Einstein radius and the velocity dispersion within \(O(1)\)% accuracy and determine \(\Omega_M\) within \(O(10)\)% accuracy. The former requirement would be an observational challenge,\(^4\) while the latter could be accomplished.\(^{13}\) Another interesting observable is the location of radial arc,\(^{14}\) -\(^{16}\) which depends on the angular gradient of the projected mass, as expressed by Eq.\((0.2)\).

3.Constraining the lens model by measurements of the velocity dispersion. We now show that there exists a degeneracy of lens models to a certain extent with regard to the projected mass density and that this degeneracy persists even when measurements of the velocity dispersion are included. We have in mind an arc or Einstein ring system, and thus one of the observables is the critical radius. Of course, the length of arc is another important observable,\(^{17}\) but we do not consider it here.

As an illustration, we consider the well-known lensing system CL 0024+1654, although our argument is not limited to cluster lenses. Bright multiple arcs were discovered in the cluster CL 0024+1654 at \(z = 0.39\) by Koo\(^{19}\) photographically. Five arcs are clearly seen in the HST image (see Fig. 1 in Ref. 23)). The redshift of the source galaxy was recently determined spectroscopically as \(z_s = 1.675\).\(^{20}\) The distance to the arc from the center of the cluster is \(\theta_E = 34.6''\), corresponding to \(110h^{-1}\)kpc for the Einstein-de Sitter model. Here \(h\) is the Hubble parameter in units of 100km/s/Mpc.
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Tyson et al.\(^{21}\) attempted to construct a high-resolution mass map of the cluster CL 0024+1654 using the Hubble Space Telescope. They found that the total mass profile within the arc radius is approximately represented by a power-law model,\(^{1}\)

\[
\Sigma(x) = \frac{K(1 + \gamma x^2)}{(1 + x^2)^{2-\gamma}},
\]

where \(x = r/r_{\text{core}}\), \(K = 7900\pm100h\,M_\odot\,\text{pc}^{-2}\), \(r_{\text{core}} = 35\pm3h^{-1}\text{kpc}\) and \(\gamma = 0.57\pm0.02\). They also noted that the asymmetry in the mass distribution inside the arcs for CL 0024+1654 is very small (less than 3%).

Recently, however, Broadhurst et al.\(^{20}\) have suggested that the mass profile of CL 0024+1654 is consistent with the NFW profile with \(r_s \approx 400h^{-1}\text{kpc}\) and \(\delta_c = \rho_s/\rho_{\text{crit}} \approx 8000\). Here \(\rho_{\text{crit}}\) is the critical density. Similarly to Eq.(0.11), for the case of the NFW profile, using Eqs.(0.5-0.6), we obtain the following relation for the above set of parameter values:

\[
\frac{\delta \theta_E}{\theta_E} \approx 2.0 \frac{\delta \rho_s}{\rho_s} + 3.0 \frac{\delta r_s}{r_s} - 0.23 \frac{\delta \Omega_M}{\Omega_M} + 0.19 \frac{\delta w}{w},
\]

Shapiro and Iliev found that the projected mass density profile obtained by Tyson et al.\(^{21}\) is fit well by that obtained from the TIS profile with \(\rho_0 \approx 0.064h^2\,M_\odot\,\text{pc}^{-3}\) and \(r_c \approx 20h^{-1}\text{kpc}\).\(^{22}\) They also claimed that the mass profile obtained by Broadhurst et al. implies a velocity dispersion (> 2230\,km/s) that is much larger than the measured value.

In Fig. 1, we show the projected mass density profiles for these three models. We assume the Einstein-de Sitter model and use two critical densities (\(\rho_{\text{crit}}(z = 0)\) and \(\rho_{\text{crit}}(z = z_L)\)) for the fit with the NFW profile. We note that the angular resolution of the Hubble Space Telescope is 0.1"\(^{23}\) corresponding to 0.32\,h^{-1}\text{kpc}. The shaded region is the two-sigma interval of the mass profile determined by Tyson et al. We assume that the parameters \((K, r_{\text{core}}, \gamma)\) are Gaussian-distributed with dispersions equal to the error bars. Within the uncertainties on the fitting parameters, the power-law profile and the TIS profile look similar,\(^{\ast}\) while the NFW profile with \(\rho_{\text{crit}}(z = 0)\) deviates slightly from the power-law profile.

As is clear from Fig. 1, the problem with the NFW mass profile fitted by Broadhurst et al. may not result from a problem with the NFW model itself but a problem with the definition of \(\rho_{\text{crit}}\) used in the analysis. In the original NFW fit of the cold dark matter halo profile, \(\rho_{\text{crit}}\) in the relation \(\rho_s = \delta_c\rho_{\text{crit}}\) should be evaluated at the redshift of the object. However, within a fitting model, it is not necessary to do this, and we can treat \(\rho_s\) just as a free parameter of the model. If we use \(\rho_{\text{crit}}(z = z_L)\) as Shapiro and Iliev\(^{22}\) did, then the projected mass density is much higher than that obtained from the Tyson’s fit of the data from the beginning.\(^{\ast\ast}\)

As is evident from the lens equation Eq.(0.1), gravitational lensing provides information regarding the projected two-dimensional mass density. Therefore one

\(^{\ast}\) Using the Davidon-Fletcher-Powell method, we independently fit Tyson’s profile with the TIS model and found that Tyson’s profile is also fit well (within 100kpc) by the TIS profile with \(\rho_0 \approx 0.0837h^2\,M_\odot\,\text{pc}^{-3}\) and \(r_c \approx 15.4h^{-1}\text{kpc}\).

\(^{\ast\ast}\) In fact, T. Broadhurst informed us that they used \(\rho_{\text{crit}}(z = 0)\) to normalize \(\rho_s\).
Fig. 1. The projected mass density profile for CL 0024+1654. The dotted line is the NFW profile with $\rho_s = \delta_c \rho_{\text{crit}}(z_L)$ used by Shapiro and Iliev, and the short-dashed line is the same with $\rho_s = \delta_c \rho_{\text{crit}}(0)$, where, $z_L = 0.39$ and we assume the Einstein-de Sitter model. The solid line is the mass profile fitted by Tyson et al. with the 2σ uncertainty by shades. The long-dashed line is the TIS profile fitted by Shapiro and Iliev.

may wonder if the degeneracy of the lens model could be lifted by combining with three-dimensional data, for example, the velocity dispersion. However, we believe that this is unlikely. The reason is the following. From Eq.(0.4) the sensitivity of the velocity dispersion on the parameters of the lens model is evaluated as

$$\frac{\delta v}{v} \simeq -\frac{\pi \delta r_c}{4r} + \frac{\delta \sigma}{\sigma}. \quad (0.14)$$

Comparing with Eq.(0.11), this indicates that the sensitivity of the velocity dispersion is less dependent than the sensitivity of $\theta_E$ on the parameters of the lens model. The size of the system at which the velocity dispersion is measured is larger than the Einstein radius.

In Fig. 2, we plot the circular velocity profiles divided by $\sqrt{2}$ for each mass model, although it is known that $v(r)/\sqrt{2}$ exactly coincides with the velocity dispersion only for a singular isothermal lens. The velocity profile of the power-law model was calculated using the Abel integral. As suggested by Fig. 1, the NFW fit with $\rho_{\text{crit}}(z = z_L)$ predicts a velocity profile that is much higher than the measured value, in accordance with the claim of Shapiro and Iliev. We find that the velocity profiles are all similar. The average velocity dispersion of CL 0024+1654 has been measured to be 1150km/s within a radius $r \approx 600h^{-1}$kpc, based on 107 galaxy redshifts, to an accuracy of roughly $\pm 100$km/s. For 33 galaxy redshifts, it has also been measured to be 1390km/s.
4. Summary. We have examined the relation between lensing observables and model parameters (the lens model and the cosmological parameters). We have found that observables are primarily dependent on the lens model parameters and have assessed the accuracy required to determine cosmological parameters [as expressed by Eq.(0.11) and Eq.(0.13)]. It is the surface density of a lens that can be determined from the observations of gravitational lensing, and therefore there exists a degeneracy among lens models given the observational uncertainties. This degeneracy cannot be broken even by combining measurements of the velocity dispersion. We also have identified possible source of the problem of fitting the averaged mass profile of CL 0024+1654 with the NFW profile. The reconstruction of the mass profile using detailed shear maps with weak lensing observations\textsuperscript{27},\textsuperscript{28}) may provide more accurate information regarding the mass profile of lenses. In any case, it is not possible to put meaningful constraints on cosmological parameters using gravitational lensing until we obtain more complete understanding of the lens model.

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