Jacobi spectral collocation method for the approximate solution of multidimensional nonlinear Volterra integral equation

Yunxia Wei¹*, Yanping Chen², Xiulian Shi³ and Yuanyuan Zhang⁴

Abstract
We present in this paper the convergence properties of Jacobi spectral collocation method when used to approximate the solution of multidimensional nonlinear Volterra integral equation. The solution is sufficiently smooth while the source function and the kernel function are smooth. We choose the Jacobi–Gauss points associated with the multidimensional Jacobi weight function 
\[ \omega(x) = \prod_{i=1}^{d} (1-\alpha_i x_i)^{\alpha_i} (1+\alpha_i x_i)^{\beta_i}, \quad -1 < \alpha_i, \beta_i < 1 - \frac{1}{d}, \quad -1 < \frac{1}{d} \]
(d denotes the space dimensions) as the collocation points. The error analysis in \( L^\infty \)-norm and \( L^2 \)-norm theoretically justifies the exponential convergence of spectral collocation method in multidimensional space. We give two numerical examples in order to illustrate the validity of the proposed Jacobi spectral collocation method.

Keywords: Multidimensional nonlinear Volterra integral equation, Jacobi collocation discretization, Multidimensional Gauss quadrature formula, Error estimates

Mathematics Subject Classification: 65R20, 45J05, 65N12

Background
We observe that there are many numerical approaches for solving one-dimensional Volterra integral equation, such as Runge–Kutta method (Brunner 1984; Yuan and Tang 1990), polynomial collocation method (Brunner 1986; Brunner et al. 2001; Brunner and Tang 1989), multistep method (Mckee 1979; Houwen and Riele 1985), hp-discontinuous Galerkin method (Brunner and Schötzau 2006) and Taylor series method (Goldfine 1977). The spectral collocation method is the most popular form of the spectral methods among practitioners. It is convenient to implement for one-dimensional problems and generally leads to satisfactory results an long as the problems possess sufficient smoothness. In the literature (Tang et al. 2008), the authors proposed a Legendre spectral collocation method for Volterra integral equation with a regular kernel in one-dimensional space. Subsequently, Chen and Tang (2009, 2010), Chen et al. (2013), developed the spectral collocation method for one-dimensional weakly singular Volterra integral equation. The proofs of the convergence properties of spectral collocation method for
Volterra integro-differential equation with a single spatial variable are given in Wei and Chen (2012a, b, 2013, 2014). Nevertheless, to the best of our knowledge, there have been no works regarding the theoretical analysis of the spectral approximation for multi-dimensional Volterra integral equation (Atdev and Ashirov 1977; Beesack 1985; Pachpatte 2011; Suryanarayana 1972), even for the case with smooth kernel.

We shall extend to several space dimensions the approximation results in Tang et al. (2008) for a single spatial variable. The expansion of Jacobi will be considered. We will be concerned with Sobolev-type norms that are most frequently applied to the convergence analysis of spectral methods. We get the discrete scheme by using multidimensional Gauss quadrature formula for the integral term. We will provide a rigorous verification of the exponential decay of the errors for approximate solution.

We study the multidimensional nonlinear Volterra integral equation of the form

\[
y(t_1, t_2, \ldots, t_d) + \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_d} K(t_1, s_1, t_2, s_2, \ldots, t_d, s_d, y(s_1, s_2, \ldots, s_d))
\]

by the Jacobi spectral collocation method. Here, \( g : [0, T_1] \times [0, T_2] \times \cdots \times [0, T_d] \rightarrow \mathbb{R} \) and \( K : D \times R \rightarrow R \) (where \( D := \{(t_1, s_1, t_2, s_2, \ldots, t_d, s_d) : 0 \leq s_i \leq t_i \leq T_i, i = 1, 2, \ldots, d\} \) are given smooth functions. If the given functions are smooth on their respective domains, the solution \( y \) is also the smooth function (see Brunner 2004). This fact will be the standing point of this paper.

**Discretization scheme**

We consider now the domain \( \Omega = (-1, 1)^d \) and we denote an element of \( \mathbb{R}^d \) by \( x = (x_1, x_2, \ldots, x_d) \). Let \( -1 < \alpha, \beta < \frac{1}{2} - \frac{1}{2^d} \) if \( \omega = \omega(x) = \prod_{i=1}^{d} (1 - x_i)^\alpha (1 + x_i)^\beta \) denotes a d-dimensional Jacobi weight function on \( \Omega \), we denote by \( L^2_\omega(\Omega) \) the space of the measurable functions \( u : \Omega \rightarrow \mathbb{R} \) such that \( \int_\Omega |u(x)|^2 \omega(x)dx < +\infty \). It is a Banach space for the norm

\[
\|u\|_{L^2_\omega(\Omega)} = \left( \int_\Omega |u(x)|^2 \omega(x)dx \right)^{1/2}.
\]

The space \( L^2_\omega(\Omega) \) is a Hilbert space for the inner product

\[
(u, v)_\omega = \int_\Omega u(x)v(x)\omega(x)dx.
\]

\( L^\infty(\Omega) \) is the Banach space of the measurable functions \( u : \Omega \rightarrow \mathbb{R} \) that are bounded outside a set of measure zero, equipped with the norm

\[
\|u\|_{L^\infty(\Omega)} = \text{ess sup}_{x \in \Omega} |u(x)|.
\]

Given a multi-index \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \) of nonnegative integers, we set

\[
|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d
\]

and
We define $H^m_{\omega}(\Omega) = \{ v \in L^2_{\omega}(\Omega) : \text{for each nonnegative multi-index } \alpha \text{ with } |\alpha| \leq m, \text{the distributional derivative } D^\alpha v \text{ belongs to } L^2_{\omega}(\Omega) \}$. This is a Hilbert space for the inner product

$$(u, v)_{m, \omega} = \sum_{|\alpha| \leq m} \int_{\Omega} D^\alpha u(x) D^\alpha v(x) \omega(x) \, dx,$$

which induces the norm

$$\| v \|_{H^m_{\omega}(\Omega)} = \left( \sum_{|\alpha| \leq m} \| D^\alpha v \|_{L^2_{\omega}(\Omega)}^2 \right)^{\frac{1}{2}}.$$

Let $\tilde{x}_j, 0 \leq j \leq N$ denote the Jacobi Gauss points on the one-dimensional interval $(-1, 1)$ (see Canuto et al. 2006; Shen and Tang 2006). We now consider multidimensional Jacobi interpolation. Let $P_N(x)$ be the space of all algebraic polynomials of degree up to $N$ in each variable $x_i$ for $i = 1, 2, \ldots, d$. Let us introduce the Jacobi Gauss points in $\Omega$:

$$\tilde{x}_j = (\tilde{x}_{j1}, \tilde{x}_{j2}, \ldots, \tilde{x}_{jd}) \text{ for } j = (j_1, j_2, \ldots, j_d) \in \mathbb{N}^d, \quad |j| = \max_{1 \leq i \leq d} j_i \leq N,$$

and denote by $I_N$ the interpolation operator at these points, i.e., for each continuous function $u$, $I_N u \in P_N$ satisfies

$$(I_N u)(\tilde{x}_j) = u(\tilde{x}_j) \text{ for all } j \in \mathbb{N}^d, \quad |j| \leq N.$$

We can represent $I_N u$ as follows:

$$I_N u(x) = \sum_{|j| \leq N} u(\tilde{x}_j) F_j(x),$$

where $F_j(x) = F_{j_1}(x_1) F_{j_2}(x_2) \cdots F_{j_d}(x_d)$, $\{F_j\}_{j=0}^N$ is the Lagrange interpolation basis function associated with the Jacobi collocation points $\{\tilde{x}_j\}_{j=0}^N$. The multidimensional Jacobi Gauss quadrature formula is

$$\int_{\Omega} f(x) \, dx \approx \sum_{|j| \leq N} f(\tilde{x}_{j1}, \tilde{x}_{j2}, \ldots, \tilde{x}_{jd}) \omega_{j1} \omega_{j2} \cdots \omega_{jd}. \quad (2)$$

We use the variable transformations $t_i = \frac{1}{2}(1 + x_i), \ x_i \in [-1, 1]$ and $s_i = \frac{1}{2}(1 + \tau_i), \ \tau_i \in [-1, 1], \ i = 1, 2, \ldots, d$ to rewrite (1) as follows

$$u(x_1, x_2, \ldots, x_d) + \int_{-1}^{x_1} \int_{-1}^{x_2} \cdots \int_{-1}^{x_d} \hat{K}(x_1, x_2, x_3, \ldots, x_d, \tau_d, u(\tau_1, \tau_2, \ldots, \tau_d)) \, d\tau_d \cdots d\tau_2 d\tau_1 = f(x_1, x_2, \ldots, x_d). \quad (3)$$

Here,
\[
\begin{align*}
f(x_1, x_2, \ldots, x_d) &= g\left( \frac{T_1}{2} (1 + x_1), \frac{T_2}{2} (1 + x_2), \ldots, \frac{T_d}{2} (1 + x_d) \right), \\
\tilde{K}(x_1, \tau_1, x_2, \tau_2, \ldots, x_d, \tau_d, u) &= \frac{T_1}{2} \frac{T_2}{2} \ldots \frac{T_d}{2} K \left( \frac{T_1}{2} (1 + x_1), \frac{T_1}{2} (1 + \tau_1), \frac{T_2}{2} (1 + x_2), \right. \right.
\end{align*}
\]

and \( u(x_1, x_2, \ldots, x_d) = g\left( \frac{T_1}{2} (1 + x_1), \frac{T_2}{2} (1 + x_2), \ldots, \frac{T_d}{2} (1 + x_d) \right) \) is the smooth solution of problem (3).

Firstly, Eq. (3) holds at the collocation points \( \tilde{x}_j = (\tilde{x}_{j_1}, \tilde{x}_{j_2}, \ldots, \tilde{x}_{j_d}) \) on \( \Omega \), i.e.,

\[
\begin{align*}
u(\tilde{x}_{j_1}, \tilde{x}_{j_2}, \ldots, \tilde{x}_{j_d}) + \int_{-1}^{\tilde{x}_{j_1}} \int_{-1}^{\tilde{x}_{j_2}} \cdots \int_{-1}^{\tilde{x}_{j_d}} \tilde{K}(\tilde{x}_{j_1}, \tau_1, \tilde{x}_{j_2}, \tau_2, \ldots, \tilde{x}_{j_d}, \tau_d, u(\tau_1, \tau_2, \ldots, \tau_d)) \\
d\tau_d \cdots d\tau_2 d\tau_1 = f(\tilde{x}_{j_1}, \tilde{x}_{j_2}, \ldots, \tilde{x}_{j_d}).
\end{align*}
\]

In order to obtain high order accuracy for the problem (4), we transfer the integral domain \([-1, \tilde{x}_{j_1}] \times [-1, \tilde{x}_{j_2}] \times \ldots \times [-1, \tilde{x}_{j_d}]\) to a fixed interval \( \Omega \)

\[
\begin{align*}
u(\tilde{x}_{j_1}, \tilde{x}_{j_2}, \ldots, \tilde{x}_{j_d}) + \int_{-1}^{1} \int_{-1}^{1} \cdots \int_{-1}^{1} \tilde{K}(\tilde{x}_{j_1}, \tau_1(\tilde{x}_{j_1}, \theta_1), \tilde{x}_{j_2}, \tau_2(\tilde{x}_{j_2}, \theta_2), \ldots, \tilde{x}_{j_d}, \tau_d(\tilde{x}_{j_d}, \theta_d)) d\theta_d \cdots d\theta_2 d\theta_1 = f(\tilde{x}_{j_1}, \tilde{x}_{j_2}, \ldots, \tilde{x}_{j_d}),
\end{align*}
\]

by using the following transformation

\[
\tau_i = \tau_i(\tilde{x}_{j_i}, \theta_i) = \frac{1 + \tilde{x}_{j_i}}{2} \theta_i + \frac{\tilde{x}_{j_i} - 1}{2}, \quad i = 1, 2, \ldots, d,
\]

where

\[
\tilde{K}(\tilde{x}_{j_1}, \tau_1, \tilde{x}_{j_2}, \tau_2, \ldots, \tilde{x}_{j_d}, \tau_d, u) = \frac{1 + \tilde{x}_{j_1}}{2} \frac{1 + \tilde{x}_{j_2}}{2} \ldots \frac{1 + \tilde{x}_{j_d}}{2} \tilde{K}(\tilde{x}_{j_1}, \tau_1, \tilde{x}_{j_2}, \tau_2, \ldots, \tilde{x}_{j_d}, \tau_d, u).
\]

Next, let \( u_{j_1j_2 \ldots j_d} \) be the approximation of the function value \( u(\tilde{x}_j) \) and use Legendre Gauss quadrature formula, (5) becomes

\[
\begin{align*}
u_{i_1i_2 \ldots i_d} + \sum_{|k| \leq N} \tilde{K}(\tilde{x}_{j_1}, \tau_1(\tilde{x}_{j_1}, \theta_{k_1}), \tilde{x}_{j_2}, \tau_2(\tilde{x}_{j_2}, \theta_{k_2}), \ldots, \tilde{x}_{j_d}, \tau_d(\tilde{x}_{j_d}, \theta_{k_d})) \omega_{k_1} \omega_{k_2} \cdots \omega_{k_d} = f(\tilde{x}_{j_1}, \tilde{x}_{j_2}, \ldots, \tilde{x}_{j_d}).
\end{align*}
\]

Here, \( \{\theta_k, |k| \leq N\} \) denotes the Legendre Gauss points on the multidimensional space \( \Omega \) and \( \{\omega_k, |k| \leq N\} \) denotes the corresponding weights. Let \( u_N(x_1, x_2, \ldots, x_d) = \sum_{|k| \leq N} u_{j_1j_2 \ldots j_d} F_{j_1}(x_1) F_{j_2}(x_2) \ldots F_{j_d}(x_d) \). Now, we use \( u_N \) to approximate the solution \( u \). Then, the Jacobi spectral collocation method is to seek \( u_N \) such that \( u_{i_1i_2 \ldots i_d} \) satisfy the following collocation equation:
where
\begin{equation}
\left.\begin{aligned}
\hat{K}(\tilde{x}_{j_1}, \tau_1(\tilde{x}_{j_2}, \theta_{j_2}), \tilde{x}_{j_2}, \tau_2(\tilde{x}_{j_3}, \theta_{j_3}), \ldots, \tilde{x}_{j_m}, \tau_m(\tilde{x}_{j_d}, \theta_{j_d})), \\
\sum_{|i| \leq N} u_{i_1i_2 \ldots i_d} F_{i_1}(\tau_1(\tilde{x}_{j_2}, \theta_{j_2})) F_{i_2}(\tau_2(\tilde{x}_{j_3}, \theta_{j_3})) \ldots F_{i_d}(\tau_d(\tilde{x}_{j_d}, \theta_{j_d}))) \omega_{i_1} \omega_{i_2} \ldots \omega_{i_d}
\end{aligned}\right) \omega_x(x) = f(\tilde{x}_{j_1}, \tilde{x}_{j_2}, \ldots, \tilde{x}_{j_d}).
\end{equation}

We can get the values of $u_{i_1i_2 \ldots i_d}$ by solving (8) and obtain the expressions of $u_N(x)$ accordingly.

Let the error function of the solution be written as $e_u(x) := u(x) - u_N(x)$. Since the exact solution of the problem (1) can be written as $y(t) = u(x)$ ($t_i = \frac{T_i}{T_d}(1 + x_i)$, $t_i \in [0, T_i]$, $x_i \in [-1, 1]$), we can define its approximate solution $y_N(t) = u_N(x)$. Then the corresponding error function satisfy

$$e_y(t) := y(t) - y_N(t) = e_u(x) = e_u \left( \frac{2}{T_1} t_1 - 1, \frac{2}{T_2} t_2 - 1, \ldots, \frac{2}{T_d} t_d - 1 \right).$$

Remark In our work, we let the multidimensional Jacobi weight function \( \omega(x) = \prod_{i=1}^{d} (1 - x_i)^{\alpha} (1 + x_i)^{\beta} \), $-1 < \alpha, \beta < \frac{1}{2} - \frac{1}{2d}$. So \( \omega(x) = (1 - x)^{d \alpha} (1 + x)^{d \beta} \), $-1 < \alpha, \beta < \frac{1}{2}$ for $d = 1$. In Tang et al. (2008), the authors choose $\alpha = \beta = 0$.

### Some lemmas

The following result can be found in Canuto et al. (2006).

**Lemma 1** Assume that Gauss quadrature formula is used to integrate the product $u \phi$, where $u \in H^m(\Omega)$ for some $m > \frac{d}{2}$ and $\phi \in \mathbb{P}_N(\Omega)$. Then there exists a constant $C$ independent of $N$ such that

$$|\langle u, \phi \rangle - \langle u, \phi \rangle_N| \leq C N^{-m} |u|_{H^{m,N}(\Omega)} \|\phi\|_{L^2(\Omega)},$$

where $\langle \cdot, \cdot \rangle$ represents the continuous inner product in $L^2(\Omega)$ space and

$$\langle u, \phi \rangle_N = \sum_{|i| \leq N} u(\theta_{j_1}, \theta_{j_2}, \ldots, \theta_{j_d}) \phi(\theta_{j_1}, \theta_{j_2}, \ldots, \theta_{j_d}) \omega_{j_1} \omega_{j_2} \ldots \omega_{j_d}.$$

The seminorm is defined as

$$|u|_{H^{m,N}(\Omega)} = \left( \sum_{k \geq \min(m, N+1)} \sum_{i=1}^{d} \left\| \frac{\partial^k u}{\partial x_i^k} \right\|_{L^2(\Omega)}^{2} \right)^{\frac{1}{2}}.$$

Note that only pure derivatives in each spatial direction appear in this expression.

From Fedotov (2004), we have the following result on the Lebesgue constant for the Lagrange interpolation polynomials associated with the Jacobi-Gauss points.

**Lemma 2** Let $\|L_N\|_{\infty} := \max_{x \in \Omega} \sum_{|i| \leq N} \left| F_{i_1}(x_1) F_{i_2}(x_2) \ldots F_{i_d}(x_d) \right|$, we have
Lemma 3 Assume that \( u(x) \in H^m_\omega(\Omega) \) for \( m > \frac{d}{2} \) and denote \((I_N u)(x)\) its interpolation polynomial associated with the multidimensional Jacobi Gauss points \( \{\tilde{x}_j, \|j\| \leq N\} \). Then the following estimates hold

\[
\|u - I_N u\|_{L^2_\omega(\Omega)} \leq CN^{-m}\|u\|_{H^m_\omega(\Omega)},
\]

\[
\|u - I_N u\|_{L^\infty(\Omega)} \leq CN^{d+2-m}\|u\|_{H^m_\omega(\Omega)}.
\]

Proof The inequality (11) can be found in Canuto et al. (2006). We now prove (12). From Canuto et al. (2006), we have

\[
\|u - I_N u\|_{H^l_\omega(\Omega)} \leq CN^{2l-m}\|u\|_{H^m_\omega(\Omega)}, \quad 0 \leq l \leq m.
\]

We know that \( H^l_\omega(\Omega) \) is embedded in \( C(\Omega) \) for \( l > \frac{d}{2} \), namely,

\[
\|u - I_N u\|_{L^\infty(\Omega)} \leq C\|u - I_N u\|_{H^l_\omega(\Omega)} \leq CN^{2l-m}\|u\|_{H^m_\omega(\Omega)}
\]

\[
\leq \begin{cases} 
CN^{d+2-m}\|u\|_{H^m_\omega(\Omega)}, & \text{when } d \text{ is an even number}, \\
CN^{d+1-m}\|u\|_{H^m_\omega(\Omega)}, & \text{when } d \text{ is an odd number}.
\end{cases}
\]

\[
\leq CN^{d+2-m}\|u\|_{H^m_\omega(\Omega)}.
\]
Lemma 5  For every bounded function \(v(x)\), there exists a constant \(C\) independent of \(v\) such that

\[
\sup_{N} \left\| \sum_{|\beta| \leq N} v(\bar{x}) F_{\beta}(x) \right\|_{L_{1}^{1}(\Omega)} \leq C \max_{x \in \Omega} |v(x)|. \tag{15}
\]

For \(r \geq 0\) and \(\kappa \in (0, 1)\), \(C^{r, \kappa}(\Omega)\) will denote the space of functions whose \(r\)-th derivatives are H"older continuous with exponent \(\kappa\), endowed with the norm:

\[
\|v\|_{C^{r, \kappa}(\Omega)} = \max_{|\alpha| \leq r} \max_{x \in \Omega} \left| \frac{\partial^{\alpha} v(x)}{\partial x_{1}^{\alpha_1} \partial x_{2}^{\alpha_2} \cdots \partial x_{d}^{\alpha_d}} \right| + \max_{|\alpha| \leq r} \max_{x \neq x' \in \Omega} \left| \frac{\partial^{\alpha} v(x')}{{\partial x_{1}^{\alpha_1} \partial x_{2}^{\alpha_2} \cdots \partial x_{d}^{\alpha_d}} - \partial^{\alpha} v(x)} \right| \left( (x_1' - x_1)^2 + (x_2' - x_2)^2 + \cdots + (x_d' - x_d)^2 \right)^{\kappa/2}.
\]

We shall make use of a result of Ragozin (1970, 1971) in the following lemma.

Lemma 6  For nonnegative integer \(r\) and \(\kappa \in (0, 1)\), there exists a constant \(C_{r, \kappa} > 0\) such that for any function \(v \in C^{r, \kappa}(\Omega)\), there exists a polynomial function \(T_{N}v \in \mathbb{P}_{N}\) such that

\[
\|v - T_{N}v\|_{L_{\infty}(\Omega)} \leq C_{r, \kappa} N^{-(r+\kappa)} \|v\|_{C^{r, \kappa}(\Omega)}.
\]

Actually, \(T_{N}\) is a linear operator from \(C^{r, \kappa}(\Omega)\) into \(\mathbb{P}_{N}\).

Lemma 7  Assume there are constants \(L_{0}, L_{1}, L_{2}, \ldots, L_{d}\) such that

\[
|\tilde{K}(x_1, \tau_1, x_2, \tau_2, \ldots, x_d, \tau_d, v_1) - \tilde{K}(x_1, \tau_1, x_2, \tau_2, \ldots, x_d, \tau_d, v_2)| \leq L_{0}|v_1 - v_2|, \\
|\tilde{K}_{i}(x_1, \tau_1, x_2, \tau_2, \ldots, x_d, \tau_d, v_1) - \tilde{K}_{i}(x_1, \tau_1, x_2, \tau_2, \ldots, x_d, \tau_d, v_2)| \leq L_{i}|v_1 - v_2|, \quad i = 1, 2, \ldots, d.
\]

Let \(M_{v_1, v_2}\) be defined by

\[
M_{v_1, v_2}(x) = \int_{x_1}^{x_1} \int_{-1}^{x_2} \cdots \int_{-1}^{x_d} \left[ \tilde{K}(x_1, \tau_1, x_2, \tau_2, \ldots, x_d, \tau_d, v_1(\tau_1, \tau_2, \ldots, \tau_d)) \\
- \tilde{K}(x_1, \tau_1, x_2, \tau_2, \ldots, x_d, \tau_d, v_2(\tau_1, \tau_2, \ldots, \tau_d)) \right] d\tau_d \cdots d\tau_2 d\tau_1.
\tag{17}
\]

Then, for any \(\kappa \in (0, 1)\) and \(v_1, v_2 \in C(\Omega)\), there exists a positive constant \(C \sim L_{0}, L_{1}, L_{2}, \ldots, L_{d}\) such that

\[
\frac{|M_{v_1, v_2}(x') - M_{v_1, v_2}(x'')|}{(x'_1 - x''_1)^2 + (x'_2 - x''_2)^2 + \cdots + (x'_d - x''_d)^2} \leq C \max_{x \in \Omega} |v_1(x) - v_2(x)|, \tag{18}
\]

for any \(x', x'' \in \Omega\) and \(x' \neq x''\). This implies that
\[ \|M_{v_1,v_2}\|_{C^0(\Omega)} \leq C \max_{x \in \Omega} |v_1(x) - v_2(x)|. \]

**Proof** For ease of exposition, and without essential loss of generality, we will proof this lemma for \( d = 2 \) and assume \( x'^1 \prec x'^1_1, x''_2 \prec x''_2 \).

\[
|M_{v_1,v_2}(x') - M_{v_1,v_2}(x'\prime\prime)| \\
= \left| \int_{-1}^{x'^1_1} \int_{-1}^{x'^1_2} [\hat{K}(x_1', \tau_1, x'_2, \tau_2, v_1(\tau_1, \tau_2)) - \hat{K}(x_1', \tau_1, x'_2, \tau_2, v_2(\tau_1, \tau_2))]d\tau_2d\tau_1 \\
- \int_{-1}^{x'^1_1} \int_{-1}^{x''_2} [\hat{K}(x_1''_1, \tau_1, x''_2, \tau_2, v_1(\tau_1, \tau_2)) - \hat{K}(x_1''_1, \tau_1, x''_2, \tau_2, v_2(\tau_1, \tau_2))]d\tau_2d\tau_1 \right| \\
\leq E_1 + E_2. \tag{20}
\]

Here,

\[
\begin{align*}
E_1 &= \left| \int_{-1}^{x'^1_1} \int_{-1}^{x'^1_2} [\hat{K}(x_1', \tau_1, x'_2, \tau_2, v_1(\tau_1, \tau_2)) - \hat{K}(x_1', \tau_1, x'_2, \tau_2, v_2(\tau_1, \tau_2))]d\tau_2d\tau_1 \\
&\quad - \int_{-1}^{x'^1_1} \int_{-1}^{x''_2} [\hat{K}(x_1''_1, \tau_1, x''_2, \tau_2, v_1(\tau_1, \tau_2)) - \hat{K}(x_1''_1, \tau_1, x''_2, \tau_2, v_2(\tau_1, \tau_2))]d\tau_2d\tau_1 \right| \\
&\leq CL_0||v_1 - v_2||_{L^\infty(\Omega)}(x'_2 - x''_2) \\
&\leq CL_0||v_1 - v_2||_{L^\infty(\Omega)}(x'_2 - x''_2)^{1+\kappa} (x'_2 - x''_2)^{-\kappa} \\
&\leq C||v_1 - v_2||_{L^\infty(\Omega)}[(x'_1 - x''_1)^2 + (x'_2 - x''_2)^2]^{-\frac{\kappa}{2}}, \tag{21}
\end{align*}
\]

\[
\begin{align*}
E_2 &= \left| \int_{-1}^{x'^1_1} \int_{-1}^{x''_2} [\hat{K}(x_1', \tau_1, x'_2, \tau_2, v_1(\tau_1, \tau_2)) - \hat{K}(x_1', \tau_1, x'_2, \tau_2, v_2(\tau_1, \tau_2))]d\tau_2d\tau_1 \\
&\quad - \int_{-1}^{x'^1_1} \int_{-1}^{x''_2} [\hat{K}(x_1''_1, \tau_1, x''_2, \tau_2, v_1(\tau_1, \tau_2)) - \hat{K}(x_1''_1, \tau_1, x''_2, \tau_2, v_2(\tau_1, \tau_2))]d\tau_2d\tau_1 \right| \\
&\leq \int_{-1}^{x'^1_1} (P_1 + P_2 + P_3)d\tau_2 \leq C \max_{x \in \Omega}(P_1 + P_2 + P_3), \tag{22}
\end{align*}
\]

where

\[
\begin{align*}
P_1 &= \left| \int_{-1}^{x'^1_1} [\hat{K}(x_1', \tau_1, x'_2, \tau_2, v_1(\tau_1, \tau_2)) - \hat{K}(x_1', \tau_1, x'_2, \tau_2, v_2(\tau_1, \tau_2))]d\tau_1 \\
&\quad - \int_{-1}^{x'^1_1} [\hat{K}(x_1''_1, \tau_1, x''_2, \tau_2, v_1(\tau_1, \tau_2)) - \hat{K}(x_1''_1, \tau_1, x''_2, \tau_2, v_2(\tau_1, \tau_2))]d\tau_1 \right| \\
&\leq CL_1||v_1 - v_2||_{L^\infty(\Omega)}(x'_2 - x''_2) \\
&\leq C||v_1 - v_2||_{L^\infty(\Omega)}(x'_2 - x''_2)^{1+\kappa} (x'_2 - x''_2)^{-\kappa} \\
&\leq C||v_1 - v_2||_{L^\infty(\Omega)}[(x'_1 - x''_1)^2 + (x'_2 - x''_2)^2]^{-\frac{\kappa}{2}}, \quad \exists \xi \in (x'_2, x''_2). \tag{23}
\end{align*}
\]
similarly,

\[
P_2 = \left| \int_{1}^{\xi_i} [\tilde{K}(x'_1, \tau_1, x''_2, \tau_2, v_1(\tau_1, \tau_2)) - \tilde{K}(x'_1, \tau_1, x''_2, \tau_2, v_2(\tau_1, \tau_2))]d\tau_1 \\
- \int_{1}^{\xi_i} [\tilde{K}(x''_1, \tau_1, x''_2, \tau_2, v_1(\tau_1, \tau_2)) - \tilde{K}(x''_1, \tau_1, x''_2, \tau_2, v_2(\tau_1, \tau_2))]d\tau_1 \right| \\
= \int_{1}^{\xi_i} [\tilde{K}_{x_1}(\eta, \tau_1, x''_2, \tau_2, v_1(\tau_1, \tau_2)) - \tilde{K}_{x_1}(\eta, \tau_1, x''_2, \tau_2, v_2(\tau_1, \tau_2))](x'_1 - x''_1)d\tau_1 \\
\leq CL_1|v_1 - v_2|_{L^\infty(\Omega)}(x'_1 - x''_1) \\
\leq C|v_1 - v_2|_{L^\infty(\Omega)}[(x'_1 - x''_1)^2 + (x''_2 - x'')^2]^{\frac{1}{2}}, \quad \exists \, \eta \in (x''_1, x'_1). \tag{24}
\]

\[
P_3 = \left| \int_{1}^{\xi_i} [\tilde{K}(x''_1, \tau_1, x''_2, \tau_2, v_1(\tau_1, \tau_2)) - \tilde{K}(x''_1, \tau_1, x''_2, \tau_2, v_2(\tau_1, \tau_2))]d\tau_1 \\
- \int_{1}^{\xi_i} [\tilde{K}(x''_1, \tau_1, x''_2, \tau_2, v_1(\tau_1, \tau_2)) - \tilde{K}(x''_1, \tau_1, x''_2, \tau_2, v_2(\tau_1, \tau_2))]d\tau_1 \right| \\
\leq \int_{1}^{\xi_i} L_0|v_1 - v_2|d\tau_1 \\
\leq C|v_1 - v_2|_{L^\infty(\Omega)}[(x'_1 - x''_1)^2 + (x''_2 - x'')^2]^{\frac{1}{2}}. \tag{25}
\]

The estimate (18) for \( d = 2 \) is obtained by combining (20)–(24).

\[
\]

**Error estimates**

**Theorem 1** Let \( u(x) \) be the exact solution of the multidimensional nonlinear Volterra integral equation (3), which is smooth. \( u_N(x) \) is the approximate solution, i.e., \( u(x) \approx u_N(x) \). Assume that

\[
\left| \frac{\partial^k}{\partial t^k} \tilde{K}(x_1, \theta_1, x_2, \theta_2, \ldots, x_d, \theta_d, v_1) - \frac{\partial^k}{\partial t^k} \tilde{K}(x_1, \theta_1, x_2, \theta_2, \ldots, x_d, \theta_d, v_2) \right| \\
\leq L_{ik}|v_1 - v_2|, \quad i = 1, 2, \ldots, d; \quad k = 1, 2, \ldots, m, \quad L = \max_{1 \leq i \leq d, 1 \leq k \leq m} L_{ik}.
\]

Then there is a constant \( C \) such that the errors satisfy for \( m > d + 2 \),

\[
|u - u_N|_{L^\infty(\Omega)} \leq CN^{-m}
\]

\[
\left\{ \begin{array}{ll}
(\log N)^d K^* + N^{d + 2}|u|_{H_0^m(N)} & \text{if } -1 < \alpha, \beta \leq -\frac{1}{2}, \\
(N^{\max(\alpha, \beta) + \frac{1}{2}})^d K^* + N^{d + 2}|u|_{H_0^m(N)} & \text{if } -\frac{1}{2} < \alpha, \beta < \frac{1}{2} - \frac{1}{2}, \\
(N^{\alpha + \frac{1}{2}})^d K^* + N^{d + 2}|u|_{H_0^m(N)} & \text{if } -1 < \beta \leq -\frac{1}{2}, -\frac{1}{2} < \alpha < \frac{1}{2} - \frac{1}{2}, \\
(N^{\beta + \frac{1}{2}})^d K^* + N^{d + 2}|u|_{H_0^m(N)} & \text{if } -1 < \alpha \leq -\frac{1}{2}, -\frac{1}{2} < \beta < \frac{1}{2} - \frac{1}{2}.
\end{array} \right. \tag{26}
\]
where
\[ K^* = \max_{\|\beta\| \leq N} \{K(\tilde{x}_j, \theta_1, \tilde{x}_j, \theta_2, \ldots, \tilde{x}_j, \theta_d, u(\theta_1, \theta_2, \ldots, \theta_d)) \}_{|j| = N} \}, \]
\[ C \sim d, L_0, L_1, L_2, \ldots, L_d. \]

Proof We subtract (8) from (5) to get the error equation
\[ u(\tilde{x}_j, \tilde{x}_j, \ldots, \tilde{x}_j) - u_{j_{l-1}, \ldots, j_d} + \int_{-1}^{1} \int_{-1}^{1} \cdots \int_{-1}^{1} [\tilde{K}(\tilde{x}_j, \tilde{x}_j, \theta_1), \tilde{x}_j, \tilde{x}_j, \theta_2, \ldots, \tilde{x}_j, \theta_d] \]
\[ \tau_d(\tilde{x}_j, \theta_d), \tau_d(\tilde{x}_j, \theta_d), u(\tau_1(\tilde{x}_j, \theta_1), \tau_2(\tilde{x}_j, \theta_2), \ldots, \tau_d(\tilde{x}_j, \theta_d)) - \tilde{K}(\tilde{x}_j, \tilde{x}_j, \theta_1, \tilde{x}_j, \tilde{x}_j, \theta_2, \ldots, \tilde{x}_j, \theta_d, \tau_d(\tilde{x}_j, \theta_d), u_N(\tau_1(\tilde{x}_j, \theta_1), \tau_2(\tilde{x}_j, \theta_2), \ldots, \tau_d(\tilde{x}_j, \theta_d)))|d\tau_d \cdots d\tau_2 d\tau_1 = I(\tilde{x}_j, \tilde{x}_j, \ldots, \tilde{x}_j), \]

where
\[ I(\tilde{x}_j, \tilde{x}_j, \ldots, \tilde{x}_j) = \sum_{\|\beta\| \leq N} \tilde{K}(\tilde{x}_j, \tilde{x}_j, \theta_1, \tilde{x}_j, \tilde{x}_j, \theta_2, \ldots, \tilde{x}_j, \theta_d) \]
\[ \tau_d(\tilde{x}_j, \theta_d), u_N(\tau_1(\tilde{x}_j, \theta_1), \tau_2(\tilde{x}_j, \theta_2), \ldots, \tau_d(\tilde{x}_j, \theta_d))| \omega_1 \omega_2 \cdots \omega_d \]
\[ = \int_{-1}^{1} \int_{-1}^{1} \cdots \int_{-1}^{1} [\tilde{K}(\tilde{x}_j, \tilde{x}_j, \theta_1), \tilde{x}_j, \tilde{x}_j, \theta_2, \ldots, \tilde{x}_j, \theta_d, \tau_d(\tilde{x}_j, \theta_d), u_N(\tau_1(\tilde{x}_j, \theta_1), \tau_2(\tilde{x}_j, \theta_2), \ldots, \tau_d(\tilde{x}_j, \theta_d)))|d\tau_d \cdots d\tau_2 d\tau_1 = I(\tilde{x}_j, \tilde{x}_j, \ldots, \tilde{x}_j). \]

Using the variable transformation (6), we have
\[ u(\tilde{x}_j, \tilde{x}_j, \ldots, \tilde{x}_j) - u_{j_{l-1}, \ldots, j_d} \]
\[ + \int_{-1}^{1} \int_{-1}^{1} \cdots \int_{-1}^{1} [\tilde{K}(\tilde{x}_j, \tilde{x}_j, \theta_1), \tilde{x}_j, \tilde{x}_j, \theta_2, \ldots, \tilde{x}_j, \theta_d, u(\tau_1(\tilde{x}_j, \theta_1), \tau_2(\tilde{x}_j, \theta_2), \ldots, \tau_d(\tilde{x}_j, \theta_d)))]d\tau_d \cdots d\tau_2 d\tau_1 \]
\[ = I(\tilde{x}_j, \tilde{x}_j, \ldots, \tilde{x}_j). \] (27)

Multiplying $F_j(x_1)F_j(x_2) \ldots F_j(x_d)$ on both sides of Eq. (27) and summing up $\|j\| \leq N$ yield
\[ e_u(x_1, x_2, \ldots, x_d) + \int_{-1}^{1} \int_{-1}^{1} \cdots \int_{-1}^{1} [\tilde{K}(x_1, x_2, x_2, \ldots, x_d, \tau_1, \tau_2, \ldots, \tau_d) \]
\[ - \tilde{K}(x_1, x_1, x_2, x_2, \ldots, x_d, \tau_d, u_N(\tau_1, \tau_2, \ldots, \tau_d))]|d\tau_d \cdots d\tau_2 d\tau_1 = f_1(x_1, x_2, \ldots, x_d) \]
\[ + f_2(x_1, x_2, \ldots, x_d) + f_3(x_1, x_2, \ldots, x_d). \] (28)

Consequently,
\[ |e_u(x_1, x_2, \ldots, x_d)| \leq L_0 \int_{-1}^{1} \int_{-1}^{1} \cdots \int_{-1}^{1} |e_u(\tau_1, \tau_2, \ldots, \tau_d)|d\tau_d \cdots d\tau_2 d\tau_1 \]
\[ + |f_1(x_1, x_2, \ldots, x_d)| + |f_2(x_1, x_2, \ldots, x_d)| + |f_3(x_1, x_2, \ldots, x_d)|, \] (29)

where
\[ J_1(x) = \sum_{|j| \leq N} I(\tilde{x}_{j_1}, \tilde{x}_{j_2}, \ldots, \tilde{x}_{j_d}) F_{j_1}(x_1) F_{j_2}(x_2) \ldots F_{j_d}(x_d), \]

\[ J_2(x) = u(x_1, x_2, \ldots, x_d) - (I_N u)(x_1, x_2, \ldots, x_d), \]

\[ J_3(x) = \int_{-1}^{1} \int_{-1}^{1} \cdots \int_{-1}^{1} \left[ \tilde{K}(x_1, x_2, x_d, x_d, u(x_1, x_2, \ldots, x_d)) \right] d\tau_d \ldots d\tau_2 d\tau_1 \]

It follows from the Gronwall inequality in Lemma 4 that
\[ \| e_u \|_{L^\infty(\Omega)} \leq C \left( \| J_1 \|_{L^\infty(\Omega)} + \| J_2 \|_{L^\infty(\Omega)} + \| J_3 \|_{L^\infty(\Omega)} \right). \] (30)

Using (9) and (10), we have
\[ \| J_1 \|_{L^\infty(\Omega)} \leq C \| I_N \|_{\infty} \left( \max_{|j| \leq N} |I(\tilde{x}_{j_1}, \tilde{x}_{j_2}, \ldots, \tilde{x}_{j_d})| \right) \]
\[ \leq C \| I_N \|_{\infty} N^{-m} \left( \| \tilde{K}(\tilde{x}_{j_1}, \tilde{x}_{j_2}, \ldots, \tilde{x}_{j_d}, u_N(\theta_1, \theta_2, \ldots, \theta_d)) \|_{H^{m-N}(\Omega)} \right) \]
\[ \leq C \| I_N \|_{\infty} N^{-m} \left( \| \tilde{K}(\tilde{x}_{j_1}, \tilde{x}_{j_2}, \ldots, \tilde{x}_{j_d}, u_N(\theta_1, \theta_2, \ldots, \theta_d)) \|_{H^{m-N}(\Omega)} \right). \] (31)

A straightforward computation shows that
\[ |\tilde{K}(\tilde{x}_{j_1}, \tilde{x}_{j_2}, \ldots, \tilde{x}_{j_d}, \theta_d, u_N(\theta_1, \theta_2, \ldots, \theta_d)) \]
\[ - \tilde{K}(\tilde{x}_{j_1}, \tilde{x}_{j_2}, \ldots, \tilde{x}_{j_d}, \theta_d, u(\theta_1, \theta_2, \ldots, \theta_d)) \|_{H^{m-N}(\Omega)} \]
\[ \leq \left( \sum_{k=1}^{d} \sum_{i=1}^{d} \left\| \frac{\partial^k}{\partial \theta_i^k} \tilde{K}(\tilde{x}_{j_1}, \tilde{x}_{j_2}, \ldots, \tilde{x}_{j_d}, \theta_d, u_N(\theta_1, \theta_2, \ldots, \theta_d)) \right\|_{L^2(\Omega)} \right)^{\frac{1}{2}} \]
\[ \leq \left( \sum_{k=1}^{d} \sum_{i=1}^{d} \left\| L_{ik} (u_N - u) \right\|_{L^2(\Omega)} \right)^{\frac{1}{2}} \]
\[ \leq L^{\frac{m}{2}} \| u_N - u \|_{L^2(\Omega)} \leq C \| e_u \|_{L^\infty(\Omega)}. \] (32)

Due to Lemma 3,
\[ \| J_2 \|_{L^\infty(\Omega)} \leq C N^{d+2-m} \| u \|_{H^{m-N}(\Omega)}. \] (33)

By virtue of Lemmas 6 and 7,
\[ ||J_3||_{L^\infty(\Omega)} = ||(I - I_N)(M_{u,N^*} - T_N M_{u,N^*})||_{L^\infty(\Omega)} \]
\[ \leq (1 + ||I_N||_{L^\infty})||M_{u,N^*} - T_N M_{u,N^*}||_{L^\infty(\Omega)} \]
\[ \leq C ||I_N||_{L^\infty} N^{-k} ||M_{u,N^*}||_{C^{\infty}(\Omega)} \]
\[ \leq C ||I_N||_{L^\infty} N^{-k} ||e_u||_{L^\infty(\Omega)} \]

where in last step we have used the following assumption,

\[
\begin{cases}
C(\log N)^{d} N^{-k} ||e_u||_{L^\infty(\Omega)}, & \text{if } -1 < \alpha, \beta \leq -\frac{1}{2}, \\
C(N^{\max(\alpha,\beta)} + \frac{1}{2})^d N^{-k} ||e_u||_{L^\infty(\Omega)}, & \text{if } -\frac{1}{2} < \alpha, \beta < \frac{1}{2}, \\
C(N^{\alpha+\frac{1}{2}})^d N^{-k} ||e_u||_{L^\infty(\Omega)}, & \text{if } -1 < \beta \leq -\frac{1}{2}, -\frac{1}{2} < \alpha < \frac{1}{2}, \\
C(N^{\beta+\frac{1}{2}})^d N^{-k} ||e_u||_{L^\infty(\Omega)}, & \text{if } -1 < \alpha \leq -\frac{1}{2}, -\frac{1}{2} < \beta < \frac{1}{2}.
\end{cases}
\]

We now obtain the estimate for \( ||e_u||_{L^\infty(\Omega)} \) by using (30)–(34),

\[ ||e_u||_{L^\infty(\Omega)} \leq CN^{-m} \left( ||I_N||_{L^\infty} K^* + N^{d+2} ||u||_{H^m_{H,N}(\Omega)} \right), \]

where in last step we have used the following assumption,

\[
\begin{cases}
0 < \kappa < 1, & \text{if } -1 < \alpha, \beta \leq -\frac{1}{2}, \\
\max(\alpha, \beta) + \frac{1}{2} < \kappa < 1, & \text{if } -\frac{1}{2} < \alpha, \beta < \frac{1}{2}, \\
(\alpha + \frac{1}{2})^d < \kappa < 1, & \text{if } -1 < \beta \leq -\frac{1}{2}, -\frac{1}{2} < \alpha < \frac{1}{2}, \\
(\beta + \frac{1}{2})^d < \kappa < 1, & \text{if } -1 < \alpha \leq -\frac{1}{2}, -\frac{1}{2} < \beta < \frac{1}{2}.
\end{cases}
\]

This completes the proof of the theorem. \( \square \)

**Theorem 2**  
If the hypotheses given in Theorem 1 hold and \( \kappa \) satisfies (35), then

\[ ||u - u_N||_{L^2(\Omega)} \leq CN^{-m} \]
\[
\begin{cases}
(1 + N^{-\kappa}(\log N)^{d}) K^* + (1 + N^{d+2-k}) ||u||_{H^m_{H,N}(\Omega)}', \\
(1 + N^{-k}) \left( N^{\max(\alpha,\beta)} + \frac{1}{2} \right)^d K^* + (1 + N^{d+2-k}) ||u||_{H^m_{H,N}(\Omega)}', \\
(1 + N^{-k}) \left( N^{\alpha+\frac{1}{2}} \right)^d K^* + (1 + N^{d+2-k}) ||u||_{H^m_{H,N}(\Omega)}', \\
(1 + N^{-k}) \left( N^{\beta+\frac{1}{2}} \right)^d K^* + (1 + N^{d+2-k}) ||u||_{H^m_{H,N}(\Omega)}',
\end{cases}
\]

**Proof**  
By using (28) and Gronwall inequality in Lemma 4, we obtain that

\[ ||e_u||_{L^2(\Omega)} \leq C \left( ||I_1||_{L^2(\Omega)} + ||I_2||_{L^2(\Omega)} + ||I_3||_{L^2(\Omega)} \right). \]

Using Lemmas 1, 5 and (32) we have for

\[ ||I_1||_{L^2(\Omega)} \leq C \max_{x \in \Omega} |I(x)| \leq CN^{-m} (K^* + ||e_u||_{L^2(\Omega)}). \]

Due to Lemma 3,
By virtue of Lemmas 6 and 7, the desired estimate (36) is obtained by combining (37)–(40) and using the same technique as in the proof of Theorem 1.

\[ \|Jz\|_{L^2(\Omega)} \leq CN^{-m}\|u\|_{H^{m,N}(\Omega)}, \]  

(39)

Numerical results

We give two numerical examples to confirm our analysis. To examine the accuracy of the results, \(L^2\) and \(L^\infty\) errors are employed to assess the efficiency of the method. All the calculations are supported by the software Matlab.

Example 1

We consider the following two-dimensional Volterra integral equation

\[ u(x, y) + \int_{-1}^{x} \int_{-1}^{y} \cos(x + y)e^{\xi\eta}u(\xi, \eta)d\eta d\xi = e^{x+y} \] 

(41)

The corresponding exact solution is given by \(u(x, y) = e^{-x-y}\). We select \(\alpha = -\frac{3}{4}, \beta = -\frac{1}{2}\). Table 1 shows the errors \(\|u - u_N\|_{L^2(\Omega)}\) and \(\|u - u_N\|_{L^\infty(\Omega)}\) obtained by using the spectral collocation method described above. Furthermore, the numerical results are plotted for \(2 \leq N \leq 12\) in Fig. 1. It is observed that the desired exponential rate of convergence is obtained.

Example 2

Consider the equation with

\[ u(x, y) + \int_{-1}^{x} \int_{-1}^{y} \cos(x + y)e^{\xi\eta}u(\xi, \eta)d\eta d\xi = e^{x+y} \] 

(41)

The corresponding exact solution is given by \(u(x, y) = e^{-x-y}\). We select \(\alpha = -\frac{3}{4}, \beta = -\frac{1}{2}\). Table 1 shows the errors \(\|u - u_N\|_{L^2(\Omega)}\) and \(\|u - u_N\|_{L^\infty(\Omega)}\) obtained by using the spectral collocation method described above. Furthermore, the numerical results are plotted for \(2 \leq N \leq 12\) in Fig. 1. It is observed that the desired exponential rate of convergence is obtained.

| \(N\)   | 2          | 4          | 6          |
|--------|------------|------------|------------|
| \(L^\infty\)-error | 9.3273e−003 | 3.3409e−005 | 5.1698e−008 |
| \(L^2\)-error      | 1.8151e−003 | 1.4154e−006 | 1.0899e−009 |

| \(N\) | 8    | 10   | 12   |
|-------|------|------|------|
| \(L^\infty\)-error | 4.5534e−011 | 6.5281e−014 | 6.7390e−014 |
| \(L^2\)-error      | 6.0859e−013 | 1.3022e−013 | 1.3124e−013 |
The corresponding exact solution is given by
\[ v(x, y) = \sin(x + y). \]
We select \( \alpha = -\frac{2}{3}, \beta = -\frac{3}{4} \). Table 2 shows the errors \( \|v - v_N\|_{L^2_\omega(\Omega)} \) and \( \|v - v_N\|_{L^\infty(\Omega)} \). The numerical results are plotted for \( 2 \leq N \leq 12 \) in Fig. 2.

### Conclusions
In this paper, we proposed a spectral collocation method based on Jacobi orthogonal polynomials to obtain approximate solution for multidimensional nonlinear Volterra integral equation. The most important contribution of this work is that we are able to

\[ v(x, y) + \int_{-1}^{x} \int_{-1}^{y} \cos(x + \xi)v(\xi, \eta)d\eta d\xi = \sin(x + y) - \frac{1}{4} \sin(3x + y) + \frac{1}{4} \sin(x + y - 2) - \frac{1}{2} (x + 1) \cos(x - y) + \frac{1}{2} (x + 1) \cos(x + 1) + \frac{1}{4} \sin(3x - 1) - \frac{1}{4} \sin(x - 3). \]  

(42)

\[ \text{Fig. 1 The errors } u - u_N \text{ versus the number of collocation points in } L^\infty \text{ and } L^2_\omega \text{ norms} \]

| N   | 2     | 4     | 6     |
|-----|-------|-------|-------|
| L^\infty-error | 1.0746e−001 | 1.2307e−003 | 7.3430e−006 |
| L^2_\omega-error | 4.7972e−002 | 3.5805e−004 | 1.3925e−006 |

| N   | 8     | 10    | 12    |
|-----|-------|-------|-------|
| L^\infty-error | 2.5031e−008 | 5.6992e−011 | 1.2992e−013 |
| L^2_\omega-error | 3.6864e−009 | 6.7815e−012 | 7.9865e−014 |
demonstrate rigorously that the errors of spectral approximations decay exponentially in both $L^\infty(\Omega)$ norm and $L^2_\omega(\Omega)$ norm on $d$-dimensional space, which is a desired feature for a spectral method.

Authors’ contributions
YW and YC carried out the spectral collocation method studies, performed the error analysis and drafted the manuscript. XS participated in the numerical experiments. YZ helped to draft the manuscript. All authors read and approved the final manuscript.

Author details
1 College of Mathematic and Information Science, Shandong Institute of Business and Technology, Yantai 264005, China. 2 School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China. 3 School of Mathematics and Statistics, Zhaqing University, Zhaqing 526061, China. 4 Department of Mathematics and Information Science, Yantai University, Yantai 264005, China.

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