Boundary value problems for a third order equation of mixed-composite type

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Abstract. We study the two boundary value problems for a third order equations of mixed-composite type. Under the certain conditions on the coefficients and the right-hand side of the equation, the regular solvability of these problems is proved. For both problems, the convergence estimate of the approximate solutions is obtained.

1. Introduction
Equations of mixed type are important in the applications to gas dynamics (see, for example, [1, 2]). The well-posed formulation of the boundary value problem for mixed type equation with an arbitrary manifold of type switch was first proposed in [3]. The equations of composite type often arise in the mathematical models of real processes. The boundary value problems for such equations were studied in many publications (for example, [4, 5]).

Here, a Vragov boundary value problem for a third order equation of mixed-composite type will be studied. The uniquely solvability of the problem with local boundary conditions will be proved, using the nonstationary Galerkin method and the regularization method.

An error estimate for approximate solutions of this problem in terms of the regularization parameter and the eigenvalues of the Dirichlet spectral problem for the Laplace operator will be obtained. Moreover, the boundary value problem with an integral boundary condition will be studied. Replacing the desired function, this problem will be reduced to previous boundary value problem, but for differential-integral equation. The regular solvability of this auxiliary problem will be proved, using the method of consecutive approximations. For the auxiliary problem and the nonlocal boundary problem, the convergence estimate of the approximate solutions will be obtained.

2. Formulation of the boundary value problems
Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $S$. Denote $Q = \Omega \times (0, T)$, $S_T = S \times (0, T)$, $T = \text{const} > 0$, $\Omega_t = \Omega \times \{t\}$, $0 \leq t \leq T$.

In the cylindrical domain $Q$ consider the equation

$$Lu \equiv k(x,t)u_{tt} - \Delta u_t - \Delta u + a(x,t)u_t + c(x)u = f(x,t).$$

Assume that the coefficients of the equation (1) are sufficiently smooth. Introduce the sets

$$\Omega^+_0 = \{(x,0) : k(x,0) \geq 0, x \in \Omega\}, \quad \Omega^+_T = \{(x,T) : k(x,T) \geq 0, x \in \Omega\}.$$
Boundary value problem I. Find a solution of equation (1) in \(Q\), such that

\[ u|_{S_T} = 0, \]  
\[ u|_{t=0} = 0, \]  
\[ u_t|_{\Omega_0} = 0, \quad u_t|_{\Omega_T} = 0. \]

Boundary value problem II. Find a solution of equation (1) in \(Q\), such that the conditions (2), (4) are satisfied and

\[ u(x, 0) = \frac{T}{0} \int N(\tau)u(x, \tau)d\tau, \]

where \(N(t) \in L_2(0, T)\) is the known function, \(N_0 = \frac{T}{0} \int N(\tau)d\tau\), \(N_1 = (1 - N_0)^{-1}\).

In an anisotropic Sobolev space \(W_2^{m,s}(Q)\) introduce the norm

\[ \|u\|_{m,s}^2 = \int_Q \left[ \sum_{|\alpha| \leq m} (D^{\alpha}u)^2 + (D^{\alpha}u_t)^2 \right] dQ \]

with \(\|u\|_{m,m} = \|u\|_m\) for \(u \in W_2^{m,m}(Q) = W_2^m(Q)\).

Denote

\[ (u, v)_0 = \int_\Omega u(x)v(x)dx, \quad \forall \ u, v \in L_2(\Omega) \]

the scalar product in the space \(L_2(\Omega)\) and \((u, v) = \int_0^T (u, v)_0 dt\) for \(u, v \in L_2(Q)\), \(\|u\|^2 = (u, u)\).

3. Solvability of the boundary value problem I

Introduce the class of functions

\[ C_L = \{u(x, t) : u \in W_2^2(\Omega), \ u_{x_i, x_j,t} \in L_2(\Omega), \ i, j = 1, n \text{ and conditions (2)-(4) hold}\}. \]

Integration by parts proves the following assertion.

**Lemma 1** Assume that \(c(x) \geq 0, \ a - \frac{1}{2}k_t \geq \delta > 0\).

Then the following inequality holds:

\[ (Lu, u_t) \geq C_1 \left[ \|u\|^2_1 + \int_Q \sum_{i=1}^n u_{x_i}^2 dQ \right], \quad C_1 > 0 \]

for all functions \(u \in C_L\).

From Lemma 1 follows the uniqueness of a regular solution of the boundary value problem (1)-(4).

Let the functions \(\varphi_k(x)\) be the solutions of a spectral problem

\[ -\Delta \varphi = \lambda \varphi, \quad x \in \Omega, \quad \varphi|_S = 0. \]
The functions $\varphi_k(x)$ form an orthonormal basis in $L_2(\Omega)$, and their corresponding eigenvalues satisfy the conditions: $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ and $\lambda_k \rightarrow +\infty$ for $k \rightarrow \infty$. For $0 < \varepsilon \leq 1$ define $L_\varepsilon u = -\varepsilon u_{tt} + Lu$.

Approximate solution $u^{N,\varepsilon}(x, t)$ of the boundary value problem (1)-(4) has the form

$$u^{N,\varepsilon}(x, t) = v(x, t) = \sum_{k=1}^{N} c_k^{N,\varepsilon}(t) \varphi_k(x), \quad N \geq 1, \quad 0 < \varepsilon \leq 1,$$

where $c_k^{N,\varepsilon}(t)$ are defined as the solution of the following boundary value problem for the third order ordinary differential equation system:

$$(L_\varepsilon u^{N,\varepsilon}, \varphi_l)_0 = (f, \varphi_l)_0, \quad (6)$$

$$c_i^{N,\varepsilon}|_{t=0} = 0, \quad D_t c_i^{N,\varepsilon}|_{t=0} = 0, \quad D_t^2 c_i^{N,\varepsilon}|_{t=T} = 0, \quad l = 1, N, \quad (7^1)$$

when $k(x, 0) > 0$, $k(x, T) \geq 0$;

$$c_i^{N,\varepsilon}|_{t=0} = 0, \quad D_t^2 c_i^{N,\varepsilon}|_{t=0} = 0, \quad D_t^2 c_i^{N,\varepsilon}|_{t=T} = 0, \quad l = 1, N, \quad (7^2)$$

when $k(x, 0) \leq 0$, $k(x, T) \geq 0$.

With the help of Lemma 1 we prove the following assertion.

**Lemma 2** Assume the conditions of Lemma 1 hold, $f \in L_2(Q)$ and the function $k(x, t)$ satisfies either of the conditions: $k(x, 0) > 0$, $k(x, T) \geq 0$, or $k(x, 0) \leq 0$, $k(x, T) \geq 0$. Then for an approximate solution $u^{N,\varepsilon}(x, t)$ the following estimate holds:

$$\varepsilon \|u_{tt}^{N,\varepsilon}\| + \|u^{N,\varepsilon}\|_1^2 + \sum_{i=1}^{n} \|u_{txi}^{N,\varepsilon}\|_1^2 \leq C_2 \|f\|^2, \quad C_2 > 0.$$

From Lemma 2 follows the uniquely solvability of the boundary value problem (6), $(7^p), p = 1, 2$.

We prove a priory estimates for approximate solutions $u^{N,\varepsilon}(x, t)$ and, correspondingly, the following theorem.

**Theorem 1** Assume that $c(x) \geq 0$, $a - \frac{1}{2}|k_i| \geq \delta > 0$, $f(x, t) \in W^{0,1}_2(Q)$ and either of the two conditions $k(x, 0) > 0$, $k(x, T) \geq 0$, or $k(x, 0) \leq 0$, $k(x, T) \geq 0$ holds. Then the boundary value problem (1)-(4) has a unique solution $u(x, t)$ in $C_L$, and the following inequality holds:

$$\|u\|_2^2 + \|\Delta u\|^2 \leq C_3 \|f\|_{0,1}^2, \quad C_3 > 0.$$
4. Solvability of the auxiliary problem

Introduce the space

\[ W_L = \{ u(x,t) : u \in W^2_2(Q), \ u_{x_i x_j t} \in L^2(Q), \ i,j = 1,n \} \]

with the norm

\[ \| u \|_L = \| u \|_2 + \| \Delta u \|_L. \]

If the function \( u(x,t) \in W_L \) is a solution of the nonlocal boundary value problem (1), (2), (4), (5) then the function

\[ v(x,t) = u(x,t) - \int_0^T N(\tau)u(x,\tau)d\tau \]

will be a solution for the following problem.

**Auxiliary problem.** Find a solution of equation

\[ L v = \Phi(x,v) + f(x,t), \ (x,t) \in Q, \]

such that the boundary conditions (2)-(4) are satisfied, where

\[ \Phi(x,v) = N_1 \int_0^T N(\tau)A_x v(x,\tau)d\tau, \ A_x v = \Delta v - c(x) v. \]

Due to the Theorem 1, with \( N(t) \equiv 0 \) the boundary value problem (8), (2)-(4) has a unique solution \( v_0(x,t) \in W_L \) and the following estimate holds:

\[ \| v_0 \|_L \leq C_0 \| f \|_{0,1}, \ C_0 > 0. \]

For all functions \( v(x,t) \in W_L \) the following inequality holds:

\[ \| \Phi(x,v) \| \leq C_5 |N_1| T^{1/2} \| N \|_{L^2(0,T)} \| v \|_L. \]

We prove the following theorem, using the method of consecutive approximations with \( v_0(x,t) \) as initial approximation.

**Theorem 3** Assume all conditions of Theorem 1 are satisfied and

\[ q = C_0 C_5 |N_1| T^{1/2} \| N \|_{L^2(0,T)} < 1. \]

Then the boundary value problem (8), (2)-(4) has a unique solution \( v(x,t) \in W_L \) and the convergence estimate holds:

\[ \| v - v_m \|_L \leq C_0 \frac{2 + q}{1 - q} \| f \|_{0,1} q^m, \]

where \( v_m(x,t) \) is an approximation with number \( m \) for \( v(x,t) \).
5. Solvability of the boundary value problem II

It is proved above that the boundary value problem (8), (2)-(4) has a unique solution \( v(x,t) \in W_L \).

Denote

\[
\begin{align*}
  u(x,t) &= v(x,t) + N_1 \int_0^T N(\tau)v(x,\tau)d\tau, \\
  u_m(x,t) &= v_m(x,t) + N_1 \int_0^T N(\tau)v_m(x,\tau)d\tau.
\end{align*}
\]

Then it is easy to prove the following assertion.

**Theorem 4** Assume all conditions of Theorem 3 are satisfied and

\[
T^{1/2}\|N\|_{L^2(0,T)} < 1.
\]

Then the boundary value problem (1), (2), (4), (5) has a unique solution \( u(x,t) \in W_L \), and the convergence estimate holds:

\[
\|u - u_m\|_L \leq C_0 \frac{2 + q}{1 - q} (1 + |N_1|T^{1/2}\|N\|_{L^2(0,T)}) \|f\|_{0,1} q^m.
\]

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