Physical flavor neutrino states

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Abstract. The problem of representation for flavor states of mixed neutrinos is discussed. By resorting to recent results, it is shown that a specific representation exists in which a number of conceptual problems are resolved. Phenomenological consequences of our analysis are explored.

1. Introduction
The detailed study of neutrino mixing in the context of quantum field theory has led to the discovery of unexpected features associated to such a phenomenon. Indeed, it was found [1] that the Hilbert space where the mixed (flavor) field operators are defined is unitarily inequivalent, in the infinite volume limit, to the Hilbert space for the original (unmixed) field operators. This is due to the condensate structure of the vacuum state for the flavor fields, the flavor vacuum, which turns out to be a coherent state.

These results have been then found to have general validity independently of the type or number of field involved [2, 3, 4, 5, 6]. Flavor oscillation formulas were derived [7, 8, 9] exhibiting corrections with respect to the usual ones derived in quantum mechanics [10]. Effects of flavor vacuum structure have been discussed also in connection with cosmological constant [11] and Lorentz invariance [12]. Flavor states have also been studied as (relativistic) examples of single-particle entanglement [13]. More recently, a non-abelian gauge structure has been recognized to be associated to flavor mixing [14].

In this paper, we discuss a novel representation of flavor vacuum and flavor states which presents some advantages with respect to the one previously adopted [1]. Phenomenological consequences of this choice are explored.

2. Neutrino mixing in QFT
Let us denote by $\nu_e, \nu_\mu$ the neutrino fields with definite flavors and by $\nu_1, \nu_2$ the neutrino fields with definite masses $m_1, m_2$, respectively. We consider the lagrangian

$$L = \bar{\nu}_e (i \not \partial - m_e) \nu_e + \bar{\nu}_\mu (i \not \partial - m_\mu) \nu_\mu - m_{e\mu} (\bar{\nu}_e \nu_\mu + \bar{\nu}_\mu \nu_e) . \quad (1)$$

which is diagonalized by the transformation

$$\nu_e(x) = \nu_1(x) \cos \theta + \nu_2(x) \sin \theta , \quad (2)$$
$$\nu_\mu(x) = -\nu_1(x) \sin \theta + \nu_2(x) \cos \theta , \quad (3)$$
where $\theta$ is the mixing angle and $m_e = m_1 \cos^2 \theta + m_2 \sin^2 \theta$, $m_{\mu} = m_1 \sin^2 \theta + m_2 \cos^2 \theta$, $m_{e\mu} = (m_2 - m_1) \sin \theta \cos \theta$. The result is the sum of two free Dirac Lagrangians:

$$L = \bar{\nu}_1 (i \gamma^0 - m_1) \nu_1 + \bar{\nu}_2 (i \gamma^0 - m_2) \nu_2.$$

(4)

The expansions for $\nu_1$ and $\nu_2$ are:

$$\nu_i(x) = \sum_{r=1,2} \int \frac{d^3k}{(2\pi)^3} \left[u_{k,i}^r \nu_{k,i}^r(t) + v_{-k,i}^r \beta_{-k,i}^r(t)\right] e^{ikx}, \quad i = 1, 2,$$

(5)

with $\alpha_{k,i}^r(t) = e^{-i\omega_{k,i} t} \alpha_{k,i}(0)$, $\beta_{k,i}^r(t) = e^{-i\omega_{k,i} t} \beta_{k,i}(0)$ and $\omega_i = \sqrt{k^2 + m_i^2}$. Here and in the following we use $t = x_0$, when no misunderstanding arises. The vacuum for the $\alpha_i$ and $\beta_i$ operators is denoted by $|0\rangle_{1,2}$: $\alpha_{k,i}^r \equiv \beta_{k,i}^r \equiv |0\rangle_{1,2} = 0$. The anticommutation relations are the usual ones (see Ref. [1]). The orthonormality and completeness relations are:

$$v_{k,i}^r v_{k,i}^{r \dagger} = \delta_{rs}, \quad v_{k,i}^r v_{-k,i}^{r \dagger} = \delta_{i1}, \quad v_{-k,i} v_{k,i} = 0, \quad \sum_r (u_{k,i}^r v_{k,i}^{r \dagger} + v_{-k,i}^r v_{-k,i}^{r \dagger}) = \mathbb{1}.$$

(6)

The fields $\nu_e$ and $\nu_\mu$ are thus completely determined through Eq. (3), which can be rewritten in the following form (we use $(\sigma, j) = (e, 1), (\mu, 2))$:

$$\nu_\sigma(x) = G_{\theta}^{-1}(t) \nu_\mu(x) G_\theta(t) = \sum_{r=1,2} \int \frac{d^3k}{(2\pi)^3} \left[u_{k,i}^r \nu_{k,i}^r(t) + v_{-k,i}^r \beta_{-k,i}^r(t)\right] e^{ikx},$$

(7)

$$G_\theta(t) = \exp \left[\theta \int d^3x \left(\nu_1^r(x) \nu_2^r(x) - \nu_1^r(x) \nu_1^r(x)\right)\right],$$

(8)

where $G_\theta(t)$ is the generator of the mixing transformations (3) (see Ref. [1] for a discussion of its properties).

Eq.(7) gives an expansion of the flavor fields $\nu_e$ and $\nu_\mu$ in the same basis of $\nu_1$ and $\nu_2$. The flavor annihilation operators are then identified with

$$\begin{pmatrix} \alpha_{k,i,\sigma}^r(t) \\ \beta_{-k,i,\sigma}^r(t) \end{pmatrix} = G_{\theta}^{-1}(t) \begin{pmatrix} \alpha_{k,j}^r(t) \\ \beta_{-k,j}^r(t) \end{pmatrix} G_\theta(t).$$

(9)

The action of the mixing generator on the vacuum $|0\rangle_{1,2}$ is non-trivial and we have:

$$|0(t)\rangle_{e,\mu} \equiv G_{\theta}^{-1}(t) |0\rangle_{1,2}.$$  

(10)

$|0(t)\rangle_{e,\mu}$ is the flavor vacuum, i.e., the vacuum for the flavor fields[1].

The explicit expression of the flavor annihilation operators is (we choose $k = (0, 0, |k|)$):

$$\begin{pmatrix} \alpha_{k,1,\sigma}^r(t) \\ \alpha_{k,2,\sigma}^r(t) \\ \beta_{-k,1,\sigma}^r(t) \\ \beta_{-k,2,\sigma}^r(t) \end{pmatrix} = \begin{pmatrix} c_\theta & s_\theta & 0 & s_\theta e^r \\ -s_\theta & c_\theta & s_\theta e^r & 0 \\ 0 & -s_\theta & c_\theta & s_\theta e^r \\ 0 & 0 & -s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} \alpha_{k,1}^r(t) \\ \alpha_{k,2}^r(t) \\ \beta_{-k,1}^r(t) \\ \beta_{-k,2}^r(t) \end{pmatrix},$$

(11)

where $c_\theta \equiv \cos \theta$, $s_\theta \equiv \sin \theta$, $e^r \equiv (-1)^r$, and

$$|U_k\rangle \equiv u_{k,2}^r u_{k,1}^r = v_{-k,1}^r v_{-k,2}^r,$$  

(12)

$$|V_k\rangle \equiv e^r u_{k,1}^r v_{-k,2}^r = -e^r u_{k,2}^r v_{-k,1}^r.$$  

(13)
The number of particles with definite mass condensed in the flavor vacuum is given by
\[ e,\mu (0(t)|\alpha_{k,i}^{\dagger} \beta_{k,i} |0(t)\rangle_{e,\mu} = e,\mu (0(t)|\beta_{k,i}^{\dagger} \alpha_{k,i} |0(t)\rangle_{e,\mu} = \sin^2 \theta |V_{k}|^2, \quad i = 1, 2. \] (14)

Following the usual procedure, we define the flavor charges for the flavor neutrino fields [8]:
\[ Q_{\sigma}(t) = \int d^3 x \nu_{\sigma}^\dagger(x) \nu_{\sigma}(x), \quad \sigma = e, \mu, \] (15)

with \( Q_e(t) + Q_\mu(t) = Q \), where \( Q \) is the total (conserved) lepton charge.

The flavor charges are diagonal when expressed in terms of the flavor ladder operators Eq (11):
\[ Q_{\sigma}(t) = \sum_k \int d^3k \left( \alpha_{k,\sigma}^{\dagger}(t) \alpha_{k,\sigma}(t) - \beta_{-k,\sigma}^{\dagger}(t) \beta_{-k,\sigma}(t) \right). \] (16)

The eigenstates of such charge operators (flavor neutrino states) are consistently defined as:
\[ |\nu_{k,\sigma}(t)\rangle = \alpha_{k,\sigma}^{\dagger}(t)|0(t)\rangle_{e,\mu}. \] (17)

3. General expansion

It was noted [2], that in the expansion Eq. (7) one could use eigenfunctions with arbitrary masses \( \mu_{\sigma} \), and therefore not necessarily the same as the masses which appear in the (diagonalized) Lagrangian. On this basis, the flavor fields can be also written as [2, 15]
\[ \nu_{\sigma}(x) = \sum_{r=1,2} \int \frac{d^3k}{(2\pi)^2} \left[ u_{r,\sigma}^{\dagger}(t) + v_{r,\sigma}(t) \right] e^{ikx}, \] (18)

where \( u_{\sigma} \) and \( v_{\sigma} \) are the helicity eigenfunctions with mass \( \mu_{\sigma} \). We denote by a tilde the generalized flavor operators introduced in Ref. [2] in order to distinguish them from the ones defined in Eq. (9). The expansion Eq. (18) is more general than the one in Eq. (7) since the latter corresponds to the particular choice \( \mu_e \equiv m_1, \mu_\mu \equiv m_2 \).

The relation between the flavor and the mass operators is now:
\[ \left( \begin{array}{c} \tilde{\alpha}_{k,\sigma}^{\dagger}(t) \\ \beta_{-k,\sigma}^{\dagger}(t) \end{array} \right) = K^{-1}_{\theta,\mu}(t) \left( \begin{array}{c} \alpha_{k,\sigma}^{\dagger}(t) \\ \beta_{-k,\sigma}^{\dagger}(t) \end{array} \right) K_{\theta,\mu}(t), \] (19)

with \( (\sigma, j) = (e, 1), (\mu, 2) \) and where \( K_{\theta,\mu}(t) \) is the generator of the transformation (3) and can be written as
\[ K_{\theta,\mu}(t) = I_\mu(t) G_\theta(t), \] (20)
\[ I_\mu(t) = \prod_{k,r} \exp \left\{ i \sum_{(\sigma, j)} \xi_{k,\sigma,j} \left[ \alpha_{k,\sigma,j}^{\dagger}(t) \beta_{-k,\sigma,j}^{\dagger}(t) + \beta_{-k,\sigma,j}^{\dagger}(t) \alpha_{k,\sigma,j}^{\dagger}(t) \right] \right\}, \] (21)

with \( \xi_{k,\sigma,j} \equiv (\chi_{\sigma} - \chi_j)/2 \) and \( \cot \chi_{\sigma} = |k|/\mu_{\sigma}, \cot \chi_j = |k|/m_j \). For \( \mu_e \equiv m_1, \mu_\mu \equiv m_2 \) one has \( I_\mu(t) = 1 \). The explicit form of the flavor operators is [2]:
\[ \left( \begin{array}{c} \tilde{\alpha}_{k,\epsilon}^{\dagger}(t) \\ \tilde{\alpha}_{k,\mu}^{\dagger}(t) \\ \beta_{-k,\epsilon}^{\dagger}(t) \\ \beta_{-k,\mu}^{\dagger}(t) \end{array} \right) = \left( \begin{array}{cccc} c_\theta \rho_1^{k} & s_\theta \rho_2^{k} & i c_\theta \lambda_1^{k} & i s_\theta \lambda_2^{k} \\ -s_\theta \rho_1^{k} & c_\theta \rho_2^{k} & -i s_\theta \lambda_1^{k} & i c_\theta \lambda_2^{k} \\ i c_\theta \lambda_1^{k} & i s_\theta \lambda_2^{k} & c_\theta \rho_1^{k} & s_\theta \rho_2^{k} \\ -i s_\theta \lambda_1^{k} & i c_\theta \lambda_2^{k} & -s_\theta \rho_1^{k} & c_\theta \rho_2^{k} \end{array} \right) \left( \begin{array}{c} \alpha_{k,1}^{\dagger}(t) \\ \alpha_{k,2}^{\dagger}(t) \\ \beta_{-k,1}^{\dagger}(t) \\ \beta_{-k,2}^{\dagger}(t) \end{array} \right), \] (22)
where \( c_\theta \equiv \cos \theta, s_\theta \equiv \sin \theta \) and

\[
\rho_{ab}^k r_s = \frac{\cos (\chi_a - \chi_b)}{2} \delta_{rs} = u_{k,a}^T u_{k,b} = v_{k,a}^T v_{k,b},
\]

\[
i \lambda_{ab}^k r_s = \frac{i \sin (\chi_a - \chi_b)}{2} \delta_{rs} = u_{k,a}^T v_{k,b} = v_{k,a}^T u_{k,b},
\]

with \( a, b = 1, 2, e, \mu \). Since \( \rho_{12}^k = |U_k| \) and \( i \lambda_{12}^k = e^*|V_k| \), etc., the operators (22) reduce to the ones in Eqs. (11) when \( \mu_e \equiv m_1 \) and \( \mu_\mu \equiv m_2 \).

The generalized flavor vacuum, which is annihilated by the flavor operators given by Eq. (22), is now written as [2]:

\[
|\tilde{0}(t)\rangle_{e,\mu} \equiv K^{-1}_{\theta,\mu}(t)|0\rangle_{1,2}.
\]

For \( \mu_e \equiv m_1 \) and \( \mu_\mu \equiv m_2 \), this state reduces to the standard flavor vacuum \( |0(t)\rangle_{e,\mu} \) of Eq. (10).

4. Gauge structure and choice of the representation

We have seen how it is possible to define exact flavor charge eigenstates for mixed neutrinos. We have also seen that an apparent arbitrariness exists in the choice of the basis of free fields with respect to which the flavor fields are expanded. Such a choice cannot be arbitrary since the structure of the flavor vacuum and thus the physical results depend on it. A remarkable result was presented in Ref. [15], where it was shown that the oscillation formulas are insensitive to the choice of such a basis.

However, there are other aspects which need to be considered. One is that Lorentz invariance is broken, since the flavor vacuum is explicitly time-dependent. As a consequence, flavor states cannot be interpreted in terms of irreducible representations of the Poincaré group. A possible way to recover Lorentz invariance for mixed fields has been explored in Ref. [12] where non-standard dispersion relations for the mixed particles have been related to non-linear realizations of the Poincaré group [16].

A different way has been explored in Ref. [14], where it has been shown that a non-abelian gauge structure appears naturally in connection with flavor mixing. In this framework, it is then possible to account for the above-mentioned violation of Lorentz invariance due to the flavor vacuum having, at the same time, standard dispersion relations for flavor neutrino states.

To see how this is possible, let us note that the Lagrangian Eq. (1) can be rewritten as describing a doublet of Dirac fields in interaction with an external Yang–Mills field:

\[
\mathcal{L} = \bar{\nu}_f(i\gamma^\mu D_\mu - M_d)\nu_f,
\]

where \( \nu_f = (\nu_e, \nu_\mu)^T \) is the flavor doublet and \( M_d = \text{diag}(m_e, m_\mu) \) is a diagonal mass matrix.

We have introduced a covariant derivative and a gauge field as

\[
D_\mu = \partial_\mu + ig A_\mu, \quad A_\mu = \frac{1}{2} A^a_\mu \sigma_a = n_\mu \delta m \frac{\sigma_1}{2} \in su(2), \quad n^\mu \equiv (1, 0, 0, 0)^T,
\]

where \( m_\mu = \frac{1}{2} \tan 2\theta \delta m_0 \) and \( \delta m_0 \equiv m_\mu - m_e \). We also define \( g \equiv \tan 2\theta \) as the coupling constant for the mixing interaction. Flavor mixing can thus be seen as an interaction of the flavor fields with an \( SU(2) \) constant gauge field.

Here \( \alpha_i, i = 1, 2, 3 \) and \( \beta \) are the usual Dirac matrices in a given representation. For definiteness, we choose the following representation:

\[
\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix},
\]

1. Up to a minus sign, see Ref. [15].
where $\sigma_i$ are the Pauli matrices and $\mathbb{I}$ is the $2 \times 2$ identity matrix.

We now consider the energy momentum tensor associated with the flavor neutrino fields in interaction with the external gauge field \cite{14}:

$$
\tilde{T}_{\rho\sigma} = \bar{\nu}_f i \gamma_\rho D_\sigma \nu_f - \eta_{\rho\sigma} \bar{\nu}_f (i \gamma^\lambda D_\lambda - M_\delta) \nu_f ,
$$

(30)

which is to be compared with the canonical energy momentum tensor associated with the Lagrangian (1):

$$
T_{\rho\sigma} = \bar{\nu}_f i \gamma_\rho D_\sigma \nu_f - \eta_{\rho\sigma} \bar{\nu}_f (i \gamma^\lambda D_\lambda - M_\delta) \nu_f + \eta_{\rho\sigma} m_\mu \bar{\nu}_f \sigma_1 \nu_f .
$$

(31)

We then define a 4-momentum operator as $\tilde{P}^\mu \equiv \int d^3x \tilde{T}^{0\mu}$ and obtain a conserved 3–momentum operator:

$$
\tilde{P}^\mu = i \int d^3x \nu^\dagger_\beta \bar{\nu}_\alpha \nu_f = i \int d^3x \nu^\dagger_\beta \bar{\nu}_\alpha \nu_c + i \int d^3x \nu^\dagger_\mu \bar{\nu}_\alpha \nu_f = \tilde{P}^i_e(t) + \tilde{P}^i_\mu(t), \quad i = 1, 2, 3
$$

(32)

and a non-conserved Hamiltonian operator:

$$
\tilde{H}^0(t) = \tilde{H}(t) = \int d^3x \nu_r \nu_r (i \gamma_0 D_0 - i \gamma^\mu D_\mu + M_\delta) \nu_f \\
= \int d^3x \nu_\beta (-i \alpha \cdot \nabla + \beta m_\epsilon) \nu_c + \int d^3x \nu_\mu (-i \alpha \cdot \nabla + \beta m_\mu) \nu_f \\
= \tilde{H}_e(t) + \tilde{H}_\mu(t).
$$

(33)

Note that both the Hamiltonian and the momentum operators split in a contribution involving only the electron neutrino field and in another where only the muon neutrino field appears.

We remark that the tilde Hamiltonian is \textit{not} the generator of time translations. This role competes to the complete Hamiltonian $H = \int d^3x T^{00}$, obtained from the energy-momentum tensor Eq. (31).

We now show that it is possible to define flavor neutrino states which are simultaneously eigenstates of the 4-momentum operators above constructed and of the flavor charges. Such a non-trivial request requires a redefinition of the flavor vacuum. Indeed, one can show \cite{14} that this is achieved by means of the expansion Eq. (7), provided one sets $\mu_\epsilon = m_\epsilon$ and $\mu_\mu = m_\mu$. From now on we use the tilde to denote the flavor operators so defined.

The tilde flavor operators are connected to those of Section 2 by a Bogoliubov transformation:

$$
\begin{pmatrix}
\tilde{\alpha}^r_{k,\sigma}(t) \\
\tilde{\beta}^r_{-k,\sigma}(t)
\end{pmatrix}
= J^{-1}(t) \begin{pmatrix}
\alpha^r_{k,\sigma}(t) \\
\beta^r_{-k,\sigma}(t)
\end{pmatrix} J(t),
$$

(34)

$$
J(t) = \prod_{k,r} \exp \left\{ i \sum_{(\sigma,j)} \xi^k_{\sigma,j} \left[ \alpha^r_{k,\sigma}(t) \beta^{r*}_{-k,\sigma}(t) + \beta^r_{-k,\sigma}(t) \alpha^{r*}_{k,\sigma}(t) \right] \right\},
$$

(35)

with $(\sigma,j) = (e, 1), (\mu, 2)$, and with $\xi^k_{\sigma,j} = (\chi_\sigma - \chi_j)/2$ and $\cot \chi_\sigma = |k|/m_\sigma$, $\cot \chi_j = |k|/m_j$. The new (physical) flavor vacuum is given by

$$
|\tilde{0}(t)\rangle_{\epsilon \mu} = J^{-1}(t) |0(t)\rangle_{\epsilon \mu}.
$$

(36)
Notice that the flavor charges are invariant under the above Bogoliubov transformations \[15\], i.e., \( \tilde{Q}_\sigma = Q_\sigma \), with:

\[
\tilde{Q}_\sigma (t) = \sum_r \int d^3k \left( \tilde{\alpha}^\dagger_{k,\sigma} (t) \tilde{\alpha}_{k,\sigma} (t) - \tilde{\beta}^\dagger_{k,\sigma} (t) \tilde{\beta}_{k,\sigma} (t) \right) . \tag{37}
\]

In terms of the tilde flavor ladder operators, the Hamiltonian and momentum operators Eqs. (32) and (33) read:

\[
\hat{P}_\sigma (t) = \sum_r \int d^3k \ k \left( \tilde{\alpha}^\dagger_{k,\sigma} (t) \tilde{\alpha}_{k,\sigma} (t) + \tilde{\beta}^\dagger_{k,\sigma} (t) \tilde{\beta}_{k,\sigma} (t) \right) , \tag{38}
\]

\[
\hat{H}_\sigma (t) = \sum_r \int d^3k \omega_{k,\sigma} \left( \tilde{\alpha}^\dagger_{k,\sigma} (t) \tilde{\alpha}_{k,\sigma} (t) - \tilde{\beta}^\dagger_{k,\sigma} (t) \tilde{\beta}_{k,\sigma} (t) \right) . \tag{39}
\]

Since all the above operators are diagonal, we can define common eigenstates as follows:

\[
|\tilde{\alpha}^r_{k,\sigma} (t) \rangle = \tilde{\alpha}^\dagger_{k,\sigma} (t) |0(t)\rangle_{e,\mu} , \tag{40}
\]

and similar ones for the antiparticles. We have

\[
\begin{pmatrix} \hat{H}_\sigma (t) \\ \hat{P}_\sigma (t) \end{pmatrix} |\tilde{\alpha}^r_{k,\sigma} (t) \rangle = \begin{pmatrix} \omega_{k,\sigma} \\ k \end{pmatrix} |\tilde{\alpha}^r_{k,\sigma} (t) \rangle , \tag{41}
\]

making explicit the 4–vector structure.

Note that the above construction and the consequent Poincaré invariance holds at a given time \( t \). Thus, for each different time, we have a different Poincaré structure. Flavor neutrino fields behave (locally in time) as ordinary on-shell fields with definite masses \( m_\nu \) and \( m_\mu \), rather than those of the mass eigenstates of the standard approach, \( m_1 \) and \( m_2 \). Flavor oscillations then arise as a consequence of the interaction with the (constant) gauge field, which acts as a birefringent medium and can be seen as a neutrino aether.

The operator \( \hat{H} \) can be viewed as the sum of the kinetic energies of the flavor neutrinos, or equivalently as the energy which can be extracted from flavor neutrinos by scattering processes, the mixing energy being “frozen”. A thermodynamical picture is given in Ref. \[14\].

Finally we consider the condensation densities for the physical flavor vacuum Eq. (36). They are given by

\[
e_{\mu} (\tilde{0}(t)|\alpha^r_{k,1} |\tilde{0}(t)\rangle_{e,\mu} = e_{\mu} (\tilde{\alpha}^\dagger_{k,1} |\tilde{\alpha}_{k,1} \tilde{0}(t)\rangle_{e,\mu} = \cos^2 \theta \sin^2 \xi_{e,1} + \sin^2 \theta \sin^2 \xi_{e,2} , \tag{42}
e_{\mu} (\tilde{0}(t)|\alpha^r_{k,2} |\tilde{0}(t)\rangle_{e,\mu} = e_{\mu} (\tilde{\alpha}^\dagger_{k,2} |\tilde{\alpha}_{k,2} \tilde{0}(t)\rangle_{e,\mu} = \cos^2 \theta \sin^2 \xi_{\mu,2} + \sin^2 \theta \sin^2 \xi_{\mu,1} , \tag{43}
\]

which have to be compared with the result Eq.(14).

5. Phenomenology
A number of (possible) physical effects can be discussed starting from the results of last Section. Here we briefly consider some of them.

5.1. Neutrino oscillations
The flavor oscillation formulas are derived by computing, in the Heisenberg representation, the expectation value of the flavor charge operators on the flavor state \[7\].
In the physical representation above defined, we obtain\(^2\)
\[
\langle \tilde{\nu}_{k,\rho}^r | \tilde{Q}_\sigma(t) | \tilde{\nu}_{k,\rho}^r \rangle = \left| \left\{ \tilde{\alpha}_{k,\sigma}^r(t), \tilde{\alpha}_{k,\rho}^r(0) \right\} \right|^2 + \left| \left\{ \tilde{\beta}_{-k,\sigma}^r(t), \tilde{\beta}_{-k,\rho}^r(0) \right\} \right|^2, \quad \sigma, \rho = e, \mu. \quad (44)
\]
A similar result holds if we consider expectation values on antiparticle states.

For the case of two flavors, the above formula turns out to be identical\(^1\) to the one defined by means of the representation of Section 2. However, corrections arise for the case of three flavors, when CP violation is present\(^9\).

5.2. Beta decay
In the above picture, flavor neutrinos states have standard dispersion relations and the oscillations are due to the interaction with the external gauge field (neutrino aether). Consequently, in experiments for the direct measurement of the neutrino masses, as the ones based on tritium (beta) decay, flavor neutrinos are predicted to exhibit the behavior of ordinary free particles with masses \(m_e\) and \(m_\mu\) rather than superpositions of massive states with masses \(m_1\) and \(m_2\). This is discussed in detail in Ref.\(^14\).

5.3. Cosmological constant
The energy of the flavor vacuum can provide a non-standard contribution to the cosmological constant. In Ref.\(^11\) such a contribution was evaluated by means of the representation of Section 2. In the light of the above results, it needs to be recalculated by use of the correct flavor vacuum.

In Ref.\(^11\) the contribution \(\langle \rho_{\text{mix}}^{\text{vac}} \rangle\) of the neutrino mixing to the vacuum energy density is shown to be:
\[
\langle \rho_{\text{mix}}^{\text{vac}} \rangle_{00} = e,\mu \langle 0 | \sum_i T^{(i)}_{00} | 0 \rangle_{e,\mu},
\quad (45)
\]
where
\[
T^{(i)}_{00} = \sum_r \int d^3k \omega_{k,i} \left( \alpha_{k,i}^r \alpha_{k,i}^{r\dagger} + \beta_{-k,i}^r \beta_{-k,i}^{r\dagger} \right), \quad i = 1, 2. \quad (46)
\]
By using the result Eqs.\((42)\), \((43)\), we obtain
\[
\langle \rho_{\text{mix}}^{\text{vac}} \rangle_{00} = 4 \int d^3k \omega_{k,1} \left( \cos^2 \theta \sin^2 \xi_{e,1} + \sin^2 \theta \sin^2 \xi_{e,2} \right) \quad (47)
+ 4 \int d^3k \omega_{k,2} \left( \cos^2 \theta \sin^2 \xi_{e,2} + \sin^2 \theta \sin^2 \xi_{e,1} \right), \quad (48)
\]
which is to be compared with the result of Ref.\(^11\). Notice that the above contribution is zero in the no-mixing limit when the mixing angle \(\theta = 0\) and/or \(m_1 = m_2\).

When a cutoff \(K\) is introduced, the above integral can be evaluated and the result turns out to be proportional to \(K^2\), whereas the usual free-field zero-point energy contribution would be going like \(K^4\).

\(^2\) We use the notation \(\tilde{\nu}_{k,\rho}^r \equiv \tilde{\alpha}_{k,\rho}^r(0) | \tilde{0}(0) \rangle_{e,\rho}\) and \(\tilde{0}(0)_{e,\rho} \equiv | \tilde{0}(0) \rangle_{e,\rho}\).
5.4. Supersymmetry

It has been argued [17, 18] that the flavor vacuum structure can provide a mechanism for
supersymmetry breaking. This can be seen [17] by considering a Lagrangian of the form\(^3\)

\[
\mathcal{L} = -i \frac{1}{2} \bar{\psi}_f (\theta + M) \psi_f - \frac{1}{2} \partial_j S_f \partial^j S_f - \frac{1}{2} S_j^T M^2 S_f - \frac{1}{2} \partial_j P_f \partial^j P_f - \frac{1}{2} P_j^T M^2 P_f,
\]

\[
= -i \frac{1}{2} \bar{\psi} (\theta + M_d) \psi - \frac{1}{2} \partial_j S \partial^j S - \frac{1}{2} S^T M_d^2 S - \frac{1}{2} \partial_j P \partial^j P - \frac{1}{2} P^T M_d^2 P,
\]

with \( M = \left( \begin{array}{cc} m_a & m_{ab} \\ m_{ab} & m_b \end{array} \right) \) and \( M_d = \text{diag}(m_1, m_2) \). The fields are two free Majorana fermions \( \psi_i \), two free real scalars \( S_i \), two free real pseudoscalars \( P_i \): \( \psi = (\psi_1, \psi_2)^T, S = (S_1, S_2)^T, P = (P_1, P_2)^T \). The flavor fields are defined as:

\[
\psi_f = U \psi, \quad S_f = U S, \quad P_f = U P,
\]

where \( \psi_f = (\psi_a, \psi_b)^T, S = (S_a, S_b)^T, P = (P_a, P_b)^T, \) and \( U = \left( \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right) \). It is also \( m_a = m_1 \cos^2 \theta + m_2 \sin^2 \theta, m_b = m_1 \sin^2 \theta + m_2 \cos^2 \theta, \) and \( m_{ab} = (m_2 - m_1) \sin \theta \cos \theta \).

The Fourier expansion of the fields \( (i = 1, 2) \) are:

\[
\psi_i(x) = \sum_{r=1}^2 \int \frac{d^3k}{(2\pi)^3} e^{i kx} \left[ u_{k,i}^{\dagger} \alpha_{k,i}^r e^{-i \omega_k t} + v_{k,i}^{\dagger} \beta_{k,i}^r e^{i \omega_k t} \right],
\]

\[
S_i(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} e^{i kx} \left[ b_{k,i} e^{-i \omega_k t} + b_{-k,i}^\dagger e^{i \omega_k t} \right],
\]

\[
P_i(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} e^{i kx} \left[ c_{k,i} e^{-i \omega_k t} + c_{-k,i}^\dagger e^{i \omega_k t} \right],
\]

where \( v_{k,i}^{\dagger} = \gamma_0 C(v_{k,i}^*)^* \) and \( u_{k,i}^{\dagger} = \gamma_0 C(u_{k,i}^*)^* \) by the Majorana condition and the operators \( \alpha_{k,i}, b_{k,i} \) and \( c_{k,i} \) annihilate the vacuum \( |0\rangle = |0\rangle^\psi \otimes |0\rangle^S \otimes |0\rangle^P \). The expectation value of the Hamiltonian on this vacuum is zero

\[
\langle 0 \rangle (H^\psi + H^S + H^P) |0\rangle = 0,
\]

where \( H_B = H_S + H_P \). The generators of the mixing transformations (50) are given by [6]:

\[
G_\psi(\theta) = \exp \left[ i \theta \int d^3x \left( \psi_1^\dagger(x) \psi_2(x) - \psi_2^\dagger(x) \psi_1(x) \right) \right],
\]

\[
G_S(\theta) = \exp \left[ -i \theta \int d^3x \left( \pi_1^S(x) S_2(x) - \pi_2^S(x) S_1(x) \right) \right],
\]

\[
G_P(\theta) = \exp \left[ -i \theta \int d^3x \left( \pi_1^P(x) P_2(x) - \pi_2^P(x) P_1(x) \right) \right],
\]

where \( \pi_1^S(x) \) and \( \pi_1^P(x) \) are the conjugate momenta of the fields \( S_i(x) \) and \( P_i(x) \), respectively. The flavor vacuum (at \( t = 0 \)) is: \( |0\rangle_f \equiv |0\rangle_f^\psi \otimes |0\rangle_f^S \otimes |0\rangle_f^P \), where

\[
|0\rangle_f^\psi \equiv G_\psi^{-1}(\theta) |0\rangle^\psi, \quad |0\rangle_f^S \equiv G_S^{-1}(\theta) |0\rangle^S, \quad |0\rangle_f^P \equiv G_P^{-1}(\theta) |0\rangle^P,
\]

\(^3\) The symbol \( M_d \) introduced here should not be confused with the one of Section 4.
are the flavor vacua of the fields $\psi_\sigma(x)$, $S_\sigma(x)$, $P_\sigma(x)$, respectively.

In a similar way as done above, we now introduce the physical flavor vacuum:

$$|\tilde{0}\rangle_f \equiv |\tilde{0}\rangle_f^\psi \otimes |\tilde{0}\rangle_f^S \otimes |\tilde{0}\rangle_f^P,$$

(59)

The expectation value of the fermionic part of $H$ on $|\tilde{0}\rangle_f$ is given by:

$$f(\tilde{0}|H_\psi|\tilde{0})_f = - \int d^3 k \left( \omega_{k,1} + \omega_{k,2} \right)$$

$$+ 2 \int d^3 k \omega_{k,1} \left( \cos^2 \theta \sin^2 \xi_{a,1} + \sin^2 \theta \sin^2 \xi_{a,2} \right)$$

$$+ 2 \int d^3 k \omega_{k,2} \left( \cos^2 \theta \sin^2 \xi_{b,1} + \sin^2 \theta \sin^2 \xi_{b,2} \right),$$

(60)

while for the bosonic part we obtain:

$$f(\tilde{0}|H_B|\tilde{0})_f = \int d^3 k \left( \omega_{k,1} + \omega_{k,2} \right)$$

$$+ 2 \int d^3 k \omega_{k,1} \left( \cos^2 \theta \sin^2 \tilde{\xi}_{a,1} + \sin^2 \theta \sin^2 \tilde{\xi}_{a,2} \right)$$

$$+ 2 \int d^3 k \omega_{k,2} \left( \cos^2 \theta \sin^2 \tilde{\xi}_{b,1} + \sin^2 \theta \sin^2 \tilde{\xi}_{b,2} \right),$$

(61)

with $\tilde{\xi}_{\sigma,j} = \frac{1}{2} \ln \frac{\omega_{k,j}}{\omega_{k,j}}$ (see Ref. [3]).

Combining Eqs.(60) and (61) we finally have:

$$f(\tilde{0}|(H_\psi + H_B)|\tilde{0})_f = 2 \cos^2 \theta \int d^3 k \omega_{k,1} \left( \sin^2 \xi_{a,1} + \sin^2 \tilde{\xi}_{a,1} \right) + \omega_{k,2} \left( \sin^2 \xi_{b,1} + \sin^2 \xi_{b,2} \right)$$

$$+ 2 \sin^2 \theta \int d^3 k \omega_{k,1} \left( \sin^2 \xi_{a,2} + \sin^2 \tilde{\xi}_{a,2} \right) + \omega_{k,2} \left( \sin^2 \xi_{b,1} + \sin^2 \xi_{b,2} \right),$$

(62)

which exhibits supersymmetry breaking associated to flavor mixing. The above result differs from that of Ref. [17].

6. Conclusions

In the framework of the quantum field theory treatment of particle mixing, we have discussed flavor states for two flavor neutrino mixing. We have given arguments for selecting a physically relevant representation for the flavor states. Phenomenological consequences of our discussion have been explored, also in connection to previous results.

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