Probing $f(R)$ gravity using the post-reionization HI 21-cm signal

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We propose the intensity mapping of the redshifted H \textsc{i} 21-cm signal from the post-reionization epoch as a cosmological probe of $f(R)$ gravity. We consider the Hu-Sawicki family of $f(R)$ gravity models characterized by a single parameter $f_{\text{m0}}$. The $f(R)$ modification to gravity affects the post-reionization 21-cm power spectrum through the change in the growth rate of density fluctuations. The quantity of interest is the redshift space distortion parameter $\beta_{\text{r}}(k,z)$ which imprints the change. We find that a radio interferometric observation at an observing frequency 710MHz with a SKA-1-Mid like radio telescope may measure the binned $\beta_{\text{r}}(k)$ at a level of sensitivity to distinguish $f(R)$ models with $\log_{10}|f_{\text{m0}}| > -5$ at a $\sigma$ level in the $k$ range $k > 0.4MPc^{-1}$. We find that using a Fisher matrix analysis the 68% bound obtained on the parameter is $-5.62 < \log_{10}|f_{\text{m0}}| < -4.38$ which is competitive with other probes of $f(R)$ gravity. Thus the future observation of the post-reionization H \textsc{i} signal holds the potential to put robust constraints on $f(R)$ gravity models and enrich our understanding of late time cosmic evolution and structure formation.

Einstein’s general theory of relativity (GR) has endured a complete century of intensive scrutiny, and has emerged as the most successful theory of gravitation. Several tests on solar system scales have proved its consistency on small scales [1]. However, modifications to the theory of gravity have often been proposed as a way to explain the observed cosmic acceleration [2]. Several observational evidences like Galaxy redshift surveys, Cosmic Microwave Background Radiation (CMBR) observations and supernovae surveys strongly indicate that the energy budget of our universe is dominated by dark energy- a fluid with energy-momentum tensor that violates the strong energy condition [3–6]. The cosmological constant ($\Lambda$) treated as a fluid with an equation of state $p = -\rho$ is the most popular candidate for dark energy in the framework of classical general relativity [7]. However, the $\Lambda$CDM model suffers from several theoretical and observational difficulties [6, 8–10]. In the matter sector, scalar fields have often been used to model various properties of dynamical and clustering dark energy [11–18]. Extensive literature is available on the diversity of such models and their general treatment using model-independent parametrizations [19–21].

Alternatively, a modification of Einstein’s theory can mimic dark energy without requiring an exotic fluid [22]. In $f(R)$ theory, the Ricci scalar $R$ appearing in the Einstein-Hilbert action, is replaced by a general function of $R$ [23–26] as

$$S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R) + S_m$$

where $\kappa^2 = \frac{8\pi G_N}{c^4}$ and $S_m$ is the action for matter. The $f(R)$ modification naturally has its imprint on the background cosmological evolution and growth of structures.

Tomographic intensity mapping of the neutral hydrogen H \textsc{i} distribution [27–29] using radio observations of the redshifted 21-cm radiation is a powerful probe of cosmic evolution and structure formation in the post reionization epoch [30–38]. The epoch of reionization is believed to be completed by redshift $z \sim 6$ [39]. Following the complex phase transition characterizing the epoch of reionization (EoR), some remnant neutral hydrogen remained clumped in the dense self shielded Damped Ly-\alpha (DLA) systems [40]. These DLA systems are the dominant source of the HI 21-cm signal in the post-reionization era. Intensity mapping experiments aim to map out the collective HI 21-cm radiation without resolving the individual gas clouds. The redshifted 21-cm signal from the post-reionization epoch as a biased tracer [41–44] of the dark matter distribution imprints a host of astrophysical and cosmological information. It is, thereby a direct probe of large scale matter distribution, growth of perturbations and the expansion history of the Universe. Observationally the post-reionization signal has two key advantages - Firstly, since the Galactic synchrotron foreground scales as $\sim (1+z)^{2.6}$, the lower redshifts are far less affected by the galactic foreground. Secondly, in the redshift range $z \leq 6$ the astrophysical processes of the EoR are absent whereby the background UV radiation field does not have any feature imprinted on the 21-cm signal.

The $f(R)$ modification to gravity will affect the 21-cm power spectrum through its signature on cosmic distances, the Hubble parameter and the growth rate of density perturbations. We consider a Hu-Sawicki form of $f(R)$, and investigate the possibility of differentiating such a modification from the standard $\Lambda$CDM model. We investigate observational strategies with upcoming radio telescopes towards constraining $f(R)$ theories at precession levels competitive if not significantly better than the next generation of supernova Ia observations,
galaxy surveys, and CMB experiments.

COSMOLOGY WITH $f(R)$ GRAVITY:

We consider a spatially flat Universe comprising of radiation (density $\rho_r$) and non-relativistic matter (density $\rho_m$). In a $f(R)$ gravity theory, the Einstein’s field equation and its trace for a Friedman-Lemitre-Robertson-Walker metric (FLRW) with a scale factor $a(t)$ and Hubble parameter $H = \frac{\dot{a}(t)}{a(t)}$ reduces to [45]

$$3H^2 f_R - \frac{1}{2} (Rf_R - f) + 3H \dot{f}_R = \kappa^2 (\rho_m + \rho_r)$$

(2)

$$Hf_R - 2f_R \dot{H} - \dot{f}_R = \kappa^2 \left( \rho_m + \frac{4}{3} \rho_r \right)$$

(3)

where $f_R = \partial f(R)/\partial R$ and the “.” denotes a differentiation with respect to the cosmic time $t$. The Ricci scalar $R$ is given by $R = 6(2H^2 + \dot{H})$. It is convenient to express the above equations in terms of dimensionless variables $x_1 \equiv -\frac{f_R}{H^2 f_R - f}$, $x_2 \equiv -\frac{\dot{f}_R}{H^2 f_R - f}$, $x_3 \equiv \frac{R}{6H^2}$, $x_4 \equiv \frac{\kappa^2 \rho_r}{3H^2 f_R}$. In terms of these quantities the dynamical evolution of the density parameters can be obtained by solving the following set of autonomous first order differential equations [45]

$$x'_1 = -1 - x_3 - 3x_2 + x_1^2 - x_1 x_3 + x_4$$

(4)

$$x'_2 = \frac{x_1 x_3}{m} - x_2(2x_3 - 4 - x_1)$$

(5)

$$x'_3 = -\frac{x_1 x_3}{m} - 2x_3(x_3 - 2)$$

(6)

$$x'_4 = -2x_3 x_4 + x_1 x_4$$

(7)

where $' = d/d\ln(a)$ and $m$ measures the deviation from $\Lambda$CDM model defined as $m \equiv 2\ln\frac{f_R}{\dot{a}/H^2} - \frac{Rf_{RR}}{f_R}$. These equations form a 4-dimensional coupled dynamical system which can be integrated numerically for a given $f(R)$ and with suitable initial conditions. Solution to the above coupled ODEs can be used to determine the dynamics of the density parameters and map a $f(R)$ gravity theory to a dark energy with an effective equation of state (EoS) $w_{\text{eff}}(z)$ as

$$\Omega_m \equiv \frac{\kappa^2 \rho_m}{3H^2 f_R} = 1 - (x_1 + x_2 + x_3 + x_4)$$

(8)

$$\Omega_r \equiv x_4$$

(9)

$$\Omega_{DE} \equiv x_1 + x_2 + x_3$$

(10)

$$w_{\text{eff}} \equiv -\frac{1}{3}(2x_3 - 1)$$

(11)

A wide variety of $f(R)$ models have been proposed [23, 46-48]. The functional form of $f(R)$ is chosen so that the model is phenomenologically satisfactory. We expect the $f(R)$ cosmology to be indistinguishable from the $\Lambda$CDM at high redshifts where the latter is well constrained from CMBR observations. At low redshifts the accelerated expansion history should be close to the $\Lambda$CDM predictions and on Solar system scales the proposed $f(R)$ model should be consistent with the $\Lambda$CDM model as a limiting case.

We consider the $f(R)$ gravity model proposed by Hu-Sawicki (HS), where the functional form of $f(R)$ is given by [22, 46]

$$f(R) = R - \mu R_c \frac{(R/R_c)^2}{(R/R_c)^2 + 1}$$

(12)

Here $\mu$ and $R_c$ are two non-negative parameters in the model where $R_c$ is the present day value of the Ricci scalar. The expansion rate $H$ for a viable $f(R)$ gravity theories is expected to be close to the concordance $\Lambda$CDM [49] predictions. The quantity $f_R$ plays a crucial role to quantify the deviation of $f(R)$ gravity models from GR whereby $f_R$ behaves like an extra degrees of freedom that acts similar to a scalar field. We may write

$$f_R = -2f_0 \frac{R}{H_0^2} \left[ 1 + \left( \frac{R}{R_c} \right)^2 \right]^{-2}$$

(13)

with $|f_0| \equiv (\mu R_0^2)/R_c$ as the only free parameter.

To recover standard GR results in Solar system tests, the present day value of $f_R$ is restricted to $\log_{10}|f_{R0}| < -6$ [46]. Further, the second derivative $f_{RR} = \frac{d^2 f(R)}{dR^2} > 0$ in order to avoid ghost and tachyonic solutions [50]. Weak Lensing peak abundance studies have provided strong constraints on $\log_{10}|f_{R0}| < -4.82$ and $< -5.16$ with WMAP9 and Planck15 priors, respectively. Tight constraints are also obtained from weak lensing peak statistics study with $\log_{10}|f_{R0}| < -4.73$ (WMAP9) and $\log_{10}|f_{R0}| < -4.79$ (Planck2013) [51]. In our work we adopt the fiducial value $f_0 = 10^{-5}$ from observations [51, 52].

Growth of large scale structure (LSS) offers a unique possibility to constrain cosmological models. The quantity of interest is the growth rate of matter density perturbations $\delta_g(k, z) \equiv \frac{d \ln \delta_m}{d \ln a}$ which is sensitive to the expansion history of the Universe. In the linear perturbation theory, and on sub-horizon scales ($k/a >> H$) the evolution of matter density perturbations $\delta_m(k, z)$ is dictated by the differential equation [45, 53-55]

$$\delta_m + 2H\delta_m - 4\pi G_{\text{eff}}(a,k)\rho_m \delta_m \simeq 0$$

(14)

where $G_{\text{eff}}$ is an effective gravitational constant which is related to standard Newtonian gravitational constant $(G_N)$ as

$$G_{\text{eff}}(a,k) = \frac{G_N}{f_R} \left[ 1 + \frac{(k^2/a^2)(f_{RR}/f_R)}{1 + 3(k^2/a^2)(f_{RR}/f_R)} \right]$$

(15)

In $f(R)$ theories $G_{\text{eff}}$ is a scale dependent function [56]. The scale dependence of the growing mode of density fluctuations is widely exploited to differentiate the
structure formation beyond standard model of cosmology. In obtaining the approximate equation (14) we have incorporated the assumption that oscillating modes are negligible compared to the modes induced by matter perturbations and also $f_R \approx 0$ on sub-horizon scales of interest [45].

\begin{equation}
\frac{\partial f_R^\Lambda}{\partial \eta} = - \frac{6}{\Omega} f_R^\Lambda(1 + 5\frac{\Omega}{\Lambda} - 3f_R^\Lambda) - \frac{3f_R^\Lambda}{\Omega}
\end{equation}

where $\Omega$ is the contribution of $\Omega/\Lambda$ to the total density of the visible universe, and $f_R$ is the fractional change in the density of radiation.

FIG. 1. The figure shows the departure of the growth rate $f_R(z, k)$ for the $f(R)$ theory with $\log_{10} |f_R| = -5$ from the $\Lambda CDM$ prediction. We plot the quantity $|\Delta f_R^\Lambda| / |f_R^\Lambda|$ in the $(z, k)$ plane.

\begin{equation}
\Delta T(\hat{n})_{\text{ISW}} = 2T \int_{\eta_{\text{LSS}}}^{\eta_0} d\eta \Phi'(\hat{n}, \eta)
\end{equation}

where $T$ is the present CMBR temperature, $\eta_{\text{LSS}}$ and $\eta_0$ are the conformal times at the last scattering surface and the present epoch respectively and $\Phi' = d\Phi/d\eta$. To quantify this effect we define a diagnostic

\begin{equation}
f^\Phi(z) = \frac{d}{dz} \ln \Phi = (f_R(z, k) - 1)\mathcal{H}(z)
\end{equation}

where $\mathcal{H} = \frac{1}{a} da/d\eta$. It is known that $f \sim 1$ in pure matter dominated epoch at high redshift and thus in the absence of $\Lambda$ or any other dark energy candidate any departure from $f_0 = 1$ will quantify $f(R)$ modification of gravity. We note that the function $f^\Phi(z)$ quantifies the interplay of two time scales - the background expansion rate (contained in $H(z)$) and the growth rate of cosmological structure (contained in $f_R(z, k)$).

Figure (2) shows the variation of $f^\Phi(z, k)$ in the $(z, k)$ plane. At low redshifts $z < 0.5$ the growth $f_R(z, k)$ is small at very low redshifts and also very high redshifts and increases monotonically with $k$ for a given redshift. We find that a $> 12\%$ departure is seen in the redshift window $0.5 < z < 1.5$ for $k > 0.5 Mpc^{-1}$.

A late time decay of the gravitational potential (the scalar perturbation in the Newtonian conformal gauge) generates a weak anisotropy in the CMB temperature fluctuation, through the Integrated Sachs Wolfe (henceforth ISW) effect [57, 58]. The gravitational potential is expected to remain constant in a purely matter dominated cosmology. In the absence of dark energy any late-time evolution of the gravitational potential [59–62] will be an imprint of modification of gravity. The ISW anisotropy is a line of sight integral [63, 64]

\begin{equation}
\Delta T(\hat{n})_{\text{ISW}} = 2T \int_{\eta_{\text{LSS}}}^{\eta_0} d\eta \Phi'(\hat{n}, \eta)
\end{equation}

FIG. 2. The figure shows the ISW diagnostic $f^\Phi(z, k)$ in the $(z, k)$ plane for the $f(R)$ theory with $\log_{10} |f_R| = -5$.

The 21-CM Signal from the Post-Reionization Era

Bulk of the low density hydrogen gets completely ionized by the end of the reionization epoch around $z \sim 6$ [39]. A small fraction of HI that survives the process of reionization is believed to remain confined in the overdense regions of the IGM. These clumped, dense damped Lyman-α systems (DLAs) [40] remain neutral as they are self shielded from the background ionizing radiation. They store $\sim 80\%$ of the HI at $z < 4$ [65] with
H I column density greater than $2 \times 10^{20} \text{atoms/cm}^2$ [66–68] and are the dominant source of the 21-cm radiation in the post-reionization epoch. The clustering properties of these DLA clouds suggest that they are associated with galaxies and located in regions of highly non-linear matter over densities [69–71]. The 21-cm signal from the post-reionization epoch has been extensively studied [30–32, 36–38, 72]. The emitted flux from individual clouds is extremely weak ($< 10 \mu$Jy). These individual DLA clouds are unlikely to be detected in radio observations, even with futuristic telescopes. However, in an intensity mapping experiment one does not aim to resolve the individual sources. The collective emission forms a diffused background in all radio-observations at the observation frequencies less than 1420MHz. Fluctuations of this signal on the sky plane and across redshift, maps out the three dimensional tomographic image of the Universe.

Several assumptions simplify the modeling of the post-reionization H I signal. These are either motivated from implicit observations or from numerical simulations.

- In the post-reionization epoch there is an enhancement of population of the triplet state of H I due to the Wouthuysen field coupling. This makes the spin temperature $T_s$ much greater than the CMB temperature $T_c$. Thus, the 21-cm radiation is seen in emission in this epoch against the background CMBR [73–75]. For $z \leq 6$ the spin temperature and the gas kinetic temperature remains strongly coupled through Lyman-α scattering or collisional coupling [76].

- Extensive study of the Lyman-α absorption lines in quasar spectra indicates that in the redshift range $1 \leq z \leq 3.5$ the cosmological density parameter of the neutral gas has a value $\Omega_{gas} \sim 10^{-3}$ [65]. Thus the mean neutral fraction is $\bar{\xi}_\text{HI} = \Omega_{gas}/\Omega_b \sim 2.45 \times 10^{-2}$, which does not evolve in the entire redshift range $z \leq 6$.

- On the large cosmological scales of interest, H I peculiar velocities are assumed to be determined by the dark matter distribution. Thus, peculiar velocity manifests as a redshift space distortion anisotropy in the 21-cm power spectrum.

- The discrete nature of DLA sources is not considered. The corresponding Poisson noise owing to this discrete sampling is neglected assuming that the number density of the DLA emitters is very large [38].

- H I perturbations are generated by a Gaussian random process. We do not consider any non-gaussianity and thereby the statistical information is contained in the two-point correlation or the power spectrum.

- Galaxy redshift surveys and numerical simulations show that the galaxies are a biased tracers of the underlying dark matter distribution [77–79]. If we assume that H I in the post-reionization epoch is housed predominantly in dark matter haloes, we may expect the gas to trace the underlying dark matter density field with a bias $b_T(k, z)$ defined as $b_T(k, z) = \left[ P_{HI}(k, z) / P(k, z) \right]^{1/2}$ where $P_{HI}(k, z)$ and $P(k, z)$ denote the H I and dark matter power spectra respectively. The bias function quantifies the nature of H I clustering in the post-reionization epoch. Further, the fluctuations in the ionizing background may also contribute to $b_T(k, z)$ [30]. On scales below the Jean’s length, the linear density contrast of H I gas is related to the dark matter density contrast though a scale dependent function [80]. However, on large scales the bias is known to be scale-independent, though the scales above which the bias is linear, is sensitive to the redshift being probed. Several authors have now demonstrated the nature of H I bias using N-body simulations [42–44, 81]. The simulations are based on the principle of populating dark matter halos in a certain mass range with gas and thereby identifying them as DLAs. These simulations show that the large scale linear bias grows monotonically with redshift for $1 < z < 4$ [82]. This feature is shared by galaxy bias as well [78, 83, 84]. There is a steep rise of the 21-cm bias on small scales. This is because of the absence of small mass halos as is expected from the CDM power spectrum and consequently the H I being distributed only in larger halos. A fitting formula for the bias $b_T(k, z)$ as a function of both redshift $z$ and scale $k$ has been obtained from numerical simulations [42, 43] of the post-reionization signal. We have used these simulation results in our modeling of the post-reionization epoch.

Adopting all the assumptions discussed above, the power spectrum of post-reionization H I 21-cm brightness temperature fluctuations from redshift $z$ is given by [74, 85]

$$P_{HI}(k, z) = \bar{T}(z)^2 \bar{\xi}_\text{HI}^2 b_T(k, z)^2 (1 + \beta_T(k, z) \mu^2) P(k, z)$$

(18)

where $\mu = \hat{k} \cdot \hat{n}$, $\beta_T(k, z) = f_g(k, z) / b_T(k, z)$, and

$$\bar{T}(z) = 4.0 \text{mK} \left(1 + z\right)^2 \left(\frac{\Omega_{b0} h^2}{0.02}\right) \left(\frac{0.7}{\bar{H}}\right) \frac{H_0}{\bar{H}(z)}$$

(19)

The term $f_g(z, k) \mu^2$ has its origin in the H I peculiar velocities [32, 74] which, as we mentioned, is also sourced by the dark matter fluctuations.

The $f(R)$ modification affects the 21-cm power spectrum through the change in the redshift space distortion parameter $\beta_T(k, z)$, and $P(k, z)$. Figure (3) shows the 21-cm power spectrum at $z = 1$ in the $(k_\parallel, k_\perp)$ space. The
The measured visibility can be written as a sum of signal and noise

\[ V(\mathbf{u}, \tau) = s(\mathbf{u}, \tau) + n(\mathbf{u}, \tau) \]

where \( s(\mathbf{u}, \tau) \) is the visibility \( v(\mathbf{u}, \tau) \) as a function of delay channel \( \tau \). The measured visibility can be written as a sum of signal and noise \( s(\mathbf{u}, \tau) = s(\mathbf{u}, \tau) + n(\mathbf{u}, \tau) \). The signal \( s(\mathbf{u}, \tau) \) can be written as

\[ s(\mathbf{u}, \tau) = \frac{2k_B}{\lambda^2} \int \frac{d^3 k}{(2\pi)^3} G(k, \mathbf{u}, \tau) \tilde{\delta T}_b(k) \] (20)

where \( \tilde{\delta T}_b(k) \) denotes the fluctuations of the 21-cm brightness temperature in Fourier space. The transformation kernel \( G \) is given by

\[ G(k, \mathbf{u}, \tau) = \tilde{A}(\mathbf{k} + \mathbf{u}) \tilde{B}(\mathbf{k} + \mathbf{u}) \]

where \( \tilde{A}(\mathbf{u}) \) and \( \tilde{B}(\mathbf{u}) \) denote the Fourier transform of the telescope beam \( A(\theta) \) and the frequency response window function \( B(\Delta \nu) \) respectively. We use \( r \) to denote comoving distance to the observing redshift \( z = (1420 \text{ MHz}/\nu) - 1 \) and \( r' = dr'/(d\nu) \).

The signal covariance matrix is defined as \( \langle s(\mathbf{u}_a, \tau_m) s^*(\mathbf{u}_b, \tau_n) \rangle = C^{(a, m), (b, n)} \) and is given by

\[ C^S = \left( \frac{2k_B}{\lambda^2} \right)^2 \frac{1}{r^2 r'} \int d^2 u d\tau G(k, \mathbf{u}_a, \tau_m) G^*(k, \mathbf{u}_b, \tau_n) \times P_{HI}(k) \]

where \( k = (2\pi u, 2\pi v) \). The noise in the visibilities measured at different baselines and frequency channels are uncorrelated. If we define the noise covariance matrix as \( C^N = \langle n(\mathbf{u}_a, \tau_m) n^*(\mathbf{u}_b, \tau_n) \rangle \), we have

\[ C^N = \left( \frac{2k_B}{\lambda^2} \right)^2 \left( \frac{\lambda^2 T_{sys}}{\lambda^2 - \lambda^2_{sys}} \right)^2 \frac{B}{T_0} \sum_{m,n} \delta_{a,b} \] (21)

We shall now investigate the possibility of constraining the scale dependent function \( \beta_T(k, z_{fid}) \). We divide the observational range \( k_{min} \) to \( k_{max} \) into \( N_{bin} \) bins and constrain the values of \( \beta_T(k) \) at the middle of the bin \( k_i \) using a Fisher matrix analysis. The departure from the \( \Lambda CDM \) model for the fiducial log \( \Omega_0 \) is \( \lesssim -5 \) model for a range of \( k \) values, peaks around \( z \sim 1 \). We choose the observational central frequency to be 710 MHz corresponding to this redshift. We first consider an OWFA [86–88] like array which is the upgraded version of the Ooty radio telescope and is expected to operate as an linear radio-interferometric array. The OWFA is a 530 m long and 30 m wide parabolic cylindrical reflector that is placed along the north-south direction on a hill that has the same slope (\( \sim 11^\circ \)) as the latitude of the place. This makes it possible to track a given patch of sky by rotating the cylinder about the long axis of the telescope. The OWFA has 1056 dipoles in total that are equally placed 5 m apart from each other along the long axis of the telescope. OWFA is capable of operating in two independent simultaneous radio-interferometric modes - PI and PII. The OWFA PII has 264 antennas in total, the radio signals from 4 consecutive dipoles have been combined to form a single antenna element. The OWFA PII has the smallest baseline length, \( d = 1.92 \text{ m} \) that corresponds to the distance between the two consecutive antennas in the array. The OWFA PII has an operating bandwidth, \( B = 39 \text{ MHz} \) (for detailed specifications [88]). The full covariance matrix is given by \( C_{ab} = C^S + C^N \). where \( N_r = 264 - a \) is the redundancy of the baselines. The Fisher matrix is given by

\[ F_{ij} = \sum_m C^{-1}(m)_{ab} C(m)_{bc,i} C^{-1}(m)_{cd} C(m)_{dc,j} \] (22)
where \(i\) and \(j\) runs over the parameters \(\beta_T(k_1), \beta_T(k_2), \ldots, \beta_T(k_{N_{\text{vis}}})\). The error on the \(i^{th}\) parameter is obtained from the Cramer Rao bound as \(\sqrt{F_{ii}^{-1}}\). We find that in the \(k\)-range 0.06 < \(k\) < 1.32 \(\beta_T(k)\) can be measured in 4 bins at > 9% for 500 \times 50\text{hrs}\) observation with 50 independent pointings. Since the maximum departure of \(\beta_T(k)\) from the \(\Lambda CDM\) is ~ 11% in the \(k\)-range of interest, such an observation will at its best be able to distinguish between a log\(_{10} [f, \rho_0] = -5\) at a \(\sim 1 - \sigma\) level and log\(_{10} [f, \rho_0] = -4\) at \(\sim 2 - \sigma\) level. For stronger constraints we now consider a

SKA1-mid type of radio array. We consider a binning in visibility \(\Delta U\), and a total observing time \(T_0\) causing a reduction of noise variance by a factor \(N_p\) where \(N_p\) is the number of visibility pairs in the bin given by \(N_p = N_{\text{vis}}(N_{\text{vis}} - 1)/2 \approx N_{\text{vis}}^2/2\) where \(N_{\text{vis}}\) is the number of visibilities in the bin measured in time \(T_0\). We may write

\[
N_{\text{vis}} = \frac{N_{\text{ant}}(N_{\text{ant}} - 1)}{2} \frac{T_0}{\tau} \rho(U) \delta^2 U
\]  

(23)

where \(N_{\text{ant}}\) is the total number of antennas in the array and \(\rho(U)\) is the baseline distribution function. In general, the baseline distribution function is given by a convolution

\[
\rho(U) = c \int d^2 r \rho_{\text{ant}}(r) \rho_{\text{ant}}(r - \lambda U)
\]  

(24)

Where \(c\) is fixed by normalization of \(\rho(U)\) and \(\rho_{\text{ant}}\) is the distribution of antennas. Further, if we assume a uniform frequency response over the entire observation bandwidth \(B\) and a Gaussian beam for the telescope the signal covariance matrix is given by

\[
\int d\tau \tilde{B}(\tau - \tau_m) \tilde{B}^*(\tau - \tau_n) = B \delta_{mn}
\]  

and

\[
\int d^2 U A(U - U_a) A^*(U - U_b) = \frac{\lambda^2}{A_e} \delta_{a,b}
\]  

(25)

where \(A_e\) is the effective area of the antenna dishes. With these simplifications we may then write

\[
C^S \approx \left( \frac{2 k_B}{\lambda^2} \right)^2 \frac{B \lambda^2}{r^2 r' \lambda^2} P_{HI} \left( \frac{2 \pi U_a}{r}, \frac{2 \pi U_m}{r'} \right) \delta_{m,n} \delta_{a,b}
\]  

The 21 cm power spectrum is not spherically symmetric, due to redshift space distortion but is symmetric around the polar angle \(\phi\). Using this symmetry, we would want to sum all the Fourier cells in an annulus of constant \((k, \mu = \cos \theta = k_{||}/k)\) with radial width \(\Delta k\) and angular width \(\Delta \theta\) for a statistical detection with improved SNR.

The number of independent cells in such an annulus is

\[
N_c = 2 \pi k^2 \Delta k \Delta \mu \frac{\text{Vol}}{(2\pi)^3}
\]  

(26)

where the volume \(\text{Vol}\) of the intensity mapping survey is given by \(\text{Vol} = \frac{r^2 \lambda^2}{4 A_e}\). Thus, the full covariance matrix may be written as

\[
C^{Tot} = \frac{1}{\sqrt{N_c}} \left[ C^S + \frac{C^N}{N_p} \right]
\]  

(27)

The covariance matrix is diagonal owing to the binning in \(U\) since different baselines which get correlated due to the telescope beam are now uncorrelated. Further, to increase the sensitivity we consider the angle averaged power spectrum by averaging over \(\mu\). Thus we have

\[
P_{HI}(k) = \bar{T}(z)^2 \tilde{x}^2_{HI} \delta^2 (1 + \frac{2}{3} \beta_T + \frac{1}{5} \beta_T^2) P(k, z)
\]  

(28)

and the corresponding variance is obtained by summing

\[
\delta P_{HI}(k) = \left[ \sum_{\mu} \frac{1}{\delta P_{HI}(k, \mu)} \right]^{-1/2}
\]  

(29)

where \(\delta P_{HI}(k, \mu) = \frac{A_e r^2}{\lambda^2} C^{Tot}\).

The fisher matrix for parameters \(\lambda_i\) may be written as

\[
F_{ij} = \sum_k \frac{1}{\delta^2 P_{HI}(k)} \frac{\partial P_{HI}(k, \mu)}{\partial \lambda_i} \frac{\partial P_{HI}(k, \mu)}{\partial \lambda_j}
\]  

(30)

We consider a radio telescope with an operational frequency range of 350MHz to 14 GHz. We consider 250 dish antennae each of diameter 15m and efficiency 0.7. To calculate the normalized baseline distribution function We assume that baselines are distributed such that the antenna distribution falls off as \(1/r^2\). We also assume that there is no baseline coverage below 30m. We assume \(T_{sys} = 60K\) and an observation bandwidth of 128 MHz. We assume \(\Delta U = U_{\text{min}} = 50\) over which the signal is averaged.
RESULTS AND DISCUSSION

Figure 4 shows the variation of $\beta_T(k,z_{fid})$ at the fiducial redshift $z = 1$ corresponding to the observing central frequency of 710 MHz. The monotonic rise of $\beta_T(k,z_{fid} = 1.0)$ owes its origin to both the monotonic growth of $f_0(k)$ and also a slow decrease of $b_T(k,z_{fid} = 1.0)$ in the $k$- range of interest. The behaviour is similar for different values of $\log_{10}|f_{R0}|$. The $\Lambda CDM$ result is seen to coincide with the $f(R)$ prediction on large scales. We note that the $\log_{10}|f_{R0}| = -6$ matches with the $\Lambda CDM$ model for $k < 0.15 \text{ Mpc}^{-1}$. We consider a fiducial $\log_{10}|f_{R0}| = -5$ for our analysis. The $k$- range between the smallest and largest baselines in binned as $\Delta k = \alpha k$ where $\alpha = \frac{1}{N_{\text{bin}}} \ln(U_{\text{max}}/U_{\text{min}})$, with $(U_{\text{min}}, U_{\text{max}}) = (50, 550)$. We consider $400 \times 50$ hrs observation in 50 independent pointings. The $1 - \sigma$ errors on $\beta_T(k_i)$ are obtained from the Fisher matrix analysis where the overall normalization of the power-spectrum is marginalized over. We find that for $k > 0.4 \text{ Mpc}^{-1}$, the $\log_{10}|f_{R0}| = -5$ can be differentiated from the $\Lambda CDM$ model at a sensitivity of $> 5 \sigma$ if we consider 6 $k$- bins. On larger scales $k < 0.4 \text{ Mpc}^{-1}$ the $f(R)$ models with $-6 < \log_{10}|f_{R0}| < -4$ remain statistically indistinguishable from the $\Lambda CDM$ model. Thus, it appears that 21-cm observations of the post-reionization epoch may only be able to constrain $f(R)$ theories on relatively small scales.

Instead of constraining the binned function $\beta_T(k)$, we investigate the possibility of putting bounds on $\log_{10}|f_{R0}|$ from the given observation. Marginalizing over the overall amplitude of the power spectrum, we are thus interested in two parameters ($\Omega_m, \log_{10}|f_{R0}|$). The $1 - \sigma$ bounds on $\log_{10}|f_{R0}|$ obtained from the marginalized Fisher matrix is given in the table below. Our error projection may be compared with constraints obtained from other observational probes. We find that our projected constraints are competitive with constraints obtained from diverse probes.

The radio-interferometric observation of the post-reionization H I 21-cm signal, thus holds the potential of providing robust constraints on $f(R)$ models. Several observational aspects, however, plague the detection of the 21-cm signal. We have evaded the key observational challenge arising from large astrophysical foregrounds that plague the signal. Astrophysical foregrounds from both galactic and extra galactic sources plague the signal and significant amount of foreground subtraction is required before one may detect the signal. Several methods of subtracting foregrounds have been suggested (see [92] and citations in this work) Cross-correlation of the 21-cm signal has also been proposed as a way to mitigate the issue of large foregrounds [93, 94]. The cosmological origin of the 21-cm signal may only be ascertained in a cross-correlation. The foregrounds appear as noise in the cross-correlation and may be tackled by considering larger survey volumes. Further, man made radio frequency interferences (RFIs), calibration errors and other observational systematics inhibits the sensitive detection of the HI 21-cm signal. A detailed study of these observational aspects shall be studied in a future work. We conclude by noting that future observation of the redshifted H I 21-cm signal shall be an important addition to the different cosmological probes aimed towards measuring possible modifications to Einstein’s gravity. This shall enhance our understanding of late time cosmological evolution and structure formation.

| TABLE I. The 68% (1 - $\sigma$) marginalized errors on $\log_{10}|f_{R0}|$ and $\Omega_m$ |
|-----------------|-----------------|-----------------|
| Model           | $\log_{10}|f_{R0}|$ | $\Omega_m$       |
| $f(R)$          | $-5 \pm 0.62$   | $0.315 \pm 0.005$ |

| TABLE II. Bounds on $p = \log_{10}|f_{R0}|$ from other probes |
|-----------------|-----------------|
| Probe of $f(R)$ gravity | Bound on $\log_{10}|f_{R0}|$ |
| GW Merger GW170817 | $p < -2.52$ [89] |
| Suyao Zeldovich clusters PLANCK | $-5.81 < p < -4.40$ [90] |
| Weak lensing Peak Statistics | $-5.16 < p < -4.82$ [51] |
| CMB + Cluster + SN + $H_0$ + BAO | $p < -3.89$ [91] |

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