PARABOLIC EQUATIONS WITH EXPONENTIAL NONLINEARITY AND MEASURE DATA

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Abstract. Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ and $T > 0$. We study the problem

\begin{equation}
\begin{cases}
  u_t - \Delta u \pm g(u) = \mu & \text{in } Q_T := \Omega \times (0, T) \\
  u = 0 & \text{on } \partial\Omega \times (0, T) \\
  u(., 0) = \omega & \text{in } \Omega,
\end{cases}
\end{equation}

where $\mu$ and $\omega$ are bounded Radon measures in $Q_T$ and $\Omega$ respectively and $g(u) \sim e^{a|u|^{q}}$ with $a > 0$ and $q \geq 1$. We provide a sufficient condition in terms of fractional maximal potentials of $\mu$ and $\omega$ for solving (0.1).

Keywords: semilinear parabolic equations, exponential nonlinearity, parabolic Wolff potential, Radon measures.

1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ ($N \geq 2$), $T > 0$ and $Q_T := \Omega \times (0, T)$. Denote by $\mathcal{M}_b(\Omega)$ ($\mathcal{M}_b(Q_T)$) the space of bounded Radon measures on $\Omega$ (resp. $Q_T$) and $\mathcal{M}_b^+(\Omega)$ (resp. $\mathcal{M}_b^+(Q_T)$) the positive cone of $\mathcal{M}_b(\Omega)$ (resp. $\mathcal{M}_b(Q_T)$). For $a > 0$, $q \geq 1$, $\ell \geq 1$, define

\begin{equation}
\mathcal{E}_\ell(s) = e^s - \sum_{j=0}^{\ell-1} \frac{s^j}{j!}, \quad s \in \mathbb{R} \quad \text{and} \quad g_\ell(u) = \mathcal{E}_\ell(a|u|^{q}).
\end{equation}

In the present paper, we deal with the question of existence and uniqueness of solution to

\begin{equation}
\begin{cases}
  u_t - \Delta u + \text{sign}(u) g_\ell(u) = \mu & \text{in } Q_T \\
  u = 0 & \text{on } \partial\Omega \times (0, T) \\
  u(., 0) = \omega & \text{in } \Omega
\end{cases}
\end{equation}

where $\omega \in \mathcal{M}_b^+(\Omega)$ and $\mu \in \mathcal{M}_b^+(Q_T)$. This study is inspired by recent works on elliptic equations with exponential absorption and measure data. In particular, in [1], D. Bartolucci et al. proved that under the conditions $N > 2$, $\nu \in \mathcal{M}_b^+(\Omega)$, $\nu \leq 4\pi \mathcal{H}^{N-2}$ (here $\mathcal{H}^{N-2}$ is $(N-2)$-dimensional Hausdorff measure in $\mathbb{R}^N$) there exists a unique solution.
of

\begin{equation}
\begin{aligned}
-\Delta u + e^u - 1 &= \nu \quad \text{in } \Omega \\
\quad u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\end{equation}

When \( N = 2 \), a characterization of the set of measures for which the problem (1.3) has a solution was given by J. L. Vázquez (see [9]).

Concerning the case of nonlinear operators, due to delicate estimates on Wolff potentials and fractional maximal operators (see [2] for the definitions), M. F. Bidaut Véron et al. [3] established a sufficient condition on \( \lambda \in \mathcal{M}^{b}(\Omega) \) for which the problem

\[\begin{aligned}
-\Delta_p u + \text{sign}(u) g_{\ell}(u) &= \lambda \quad \text{in } \Omega \\
\quad u &= 0 \quad \text{on } \partial \Omega
\end{aligned}\]

admits a renormalized solution where \( \Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u) \) with \( 1 < p < N \).

Recently, M. F. Bidaut Véron and Q. H. Nguyen have considered the parabolic problem

\begin{equation}
\begin{aligned}
\quad u_t - \Delta_p u + \text{sign}(u) g_{\ell}(u) &= \mu \quad \text{in } Q_T \\
\quad u &= 0 \quad \text{on } \partial \Omega \times (0, T) \\
\quad u(., 0) &= u_0 \quad \text{in } \Omega
\end{aligned}
\end{equation}

where \( 1 < p < N, u_0 \in L^1(\Omega), \mu \in \mathcal{M}^{b}(Q_T) \). Because of lack of necessary tools concerning parabolic Wolff potentials, they only focused on the case where \( \mu \) satisfies

\begin{equation}
|\mu| \leq \lambda \otimes \vartheta
\end{equation}

with \( \lambda \in \mathcal{M}^{b}_{+}(\Omega) \) and \( \vartheta \in L^1_1((0, T)) \) (here the notation \( \otimes \) denotes the tensorial product). Under the condition (1.5), instead of dealing with \( \mu \), they were concerned with \( \lambda \), which enables them to employ results developed by themselves on elliptic Wolff potentials to point out the existence of solutions to (1.4).

In this paper, by limiting ourselves to the case of linear operator, we show that the condition (1.5) can be removed. More precisely, when \( p = 2 \), by adapting techniques used in [3] to parabolic framework, we obtain a sufficient condition on \( \mu \in \mathcal{M}^{b}(Q_T) \) and \( u_0 \in \mathcal{M}^{b}(\Omega) \) respectively in terms of parabolic and elliptic fractional maximal operators for solvability of (1.4). In order to state the results, we first introduce some notations.

**Notations and terminology.** For \( \alpha > 0 \) and \( \beta \geq 0 \), set

\[ h_{1,\alpha}(s) = (-\ln(s \wedge 2^{-1}))^{\frac{1}{\beta}}, \quad h_{2,\beta}(s) = (\ln(2ds^{-1} \vee 2))^{-\beta}, \quad \forall s > 0 \]
where $d = \text{diam}(\Omega) + T$ (here $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$).

For $0 < R \leq \infty$, we denote the $R$-truncated $\alpha$-fractional maximal potential of $\omega$ by

$$M^1_{\alpha,R}[\omega](x) = \sup_{0<s\leq R} \left( \frac{\omega(B_s(x))}{s^\alpha h_{1,\alpha}(s)} \right) \text{ for a.e. } x \in \mathbb{R}^N$$

where $B_s(x)$ is the ball of center $x$ and radius $s > 0$. The parabolic $R$-truncated $\beta$-fractional maximal potential of $\mu$ is defined by

$$M^2_{\beta,R}[\mu](x,t) = \sup_{0<s\leq R} \left( \frac{\mu(Q_s(x,t))}{s^\beta h_{2,\beta}(s)} \right) \text{ for a.e.}(x,t) \in \mathbb{R}^{N+1}$$

where $Q_s(x,t) = B_s(x) \times (t - s^2/2, t + s^2/2)$. Finally, the parabolic $R$-truncated Wolff potential of $\mu$ is defined by

$$W_R[\mu](x,t) = \int_{0}^{R} \frac{\mu(Q_s(x,t))}{s^\beta} \frac{ds}{s} \text{ for a.e.}(x,t) \in \mathbb{R}^{N+1}.$$  

**Definition 1.1.** Let $f \in C(\mathbb{R})$, $\mu \in \mathcal{M}^b(Q_T)$ and $\omega \in \mathcal{M}^b(\Omega)$. A function $u$ is a solution of

$$\begin{cases}  
  u_t - \Delta u + f(u) = \mu & \text{in } Q_T \\
  u = 0 & \text{on } \partial \Omega \times (0,T) \\
  u(.,0) = \omega & \text{in } \Omega
\end{cases}$$

if $u \in L^1(Q_T)$, $f(u) \in L^1(Q_T)$ and

$$\int_{Q_T} (-u(\zeta_t + \Delta \zeta) + f(u)\zeta) \, dx \, dt = \int_{Q_T} \zeta \, d\mu + \int_{\Omega} \zeta(.,0) \, d\omega$$

for every $\zeta \in X(Q_T)$, which is the space of functions in $C^{2,1}(\overline{Q}_T)$ vanishing on $(\partial \Omega \times [0,T]) \cup (\Omega \times \{T\})$.

In the sequel, if $\mu \in \mathcal{M}(Q_T)$ ($\omega \in \mathcal{M}(\Omega)$ resp.), we will consider $\mu$ (resp. $\omega$) as a measure in $\mathbb{R}^{N+1}$ (resp. $\mathbb{R}^N$) vanishing outside of $Q_T$ (resp. $\Omega$). The first result in the paper is the following

**Theorem 1.2.** Let $\Omega$ be a bounded domain with $C^2$ boundary. Assume $a > 0$, $q \geq 1$, $\ell \geq 1$, $\alpha \geq q$, $\beta \in [\frac{q-1}{q}, 1)$, $f_1 \in L^1(\Omega)$, $f_2 \in L^1(Q_T)$, $\omega \in \mathcal{M}^b(\Omega)$ and $\mu \in \mathcal{M}^b(Q_T)$. There exist $M_1 = M_1(N,\alpha,a)$ and $M_2 = M_2(N,\beta,a)$ such that if $\|M^1_{\alpha,\infty}[\omega^+]\|_{L^\infty(\mathbb{R}^N)} < M_1$ and $\|M^2_{\beta,\infty}[\mu^+]\|_{L^\infty(\mathbb{R}^{N+1})} < M_2$ then the problem

$$\begin{cases}  
  u_t - \Delta u + \text{sign}(u)g_\ell(u) = \mu + f_2 & \text{in } Q_T \\
  u = 0 & \text{on } \partial \Omega \times (0,T) \\
  u(.,0) = \omega + f_1 & \text{in } \Omega
\end{cases}$$

admits a unique solution $u$ satisfying $e^{a|u|^q} \in L^1(Q_T)$. 
We also consider the problem associated to equation with source terms
\begin{equation}
\begin{cases}
  u_t - \Delta u = g_\ell(u) + \mu & \text{in } Q_T \\
  u = 0 & \text{on } \partial\Omega \times (0,T) \\
  u(.,0) = \omega & \text{in } \Omega
\end{cases}
\end{equation}
where \( \omega \in \mathcal{M}_b^+(\Omega) \), \( \mu \in \mathcal{M}_b^+(Q_T) \) and \( g_\ell \) is defined as in (1.1) \( a > 0, p \geq 1, \ell p > 1 \).

Let \( G(x,t) \) be the heat kernel in \( \mathbb{R}^N \) which is defined by
\[
G(x,t) = \left(4\pi t\right)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}} \quad \text{if } x \in \mathbb{R}^N, t > 0 \text{ and } G(x,t) = 0 \quad \text{if } x \in \mathbb{R}^N, t \geq 0.
\]
For any \( y \in \Omega \), denote by \( G^\Omega(x,t,y) \) the fundamental solution of the heat equation in \( \Omega \) with zero Dirichlet condition on \( \partial\Omega \) and initial condition \( \delta_y \) (Dirac measure concentrated at \( y \)). Clearly \( G^\Omega(x,t,y) \leq G(x-y,t) \) for every \( x,y \in \Omega, t > 0 \). If \( \omega \in \mathcal{M}_b^+(\Omega) \), we denote \( G[\omega](x,t) = \int_{\Omega} G^\Omega(x,t,y) d\omega(y) \).

Existence result for (1.9) is stated in the following theorem

**Theorem 1.3.** Let \( \Omega \) be a bounded domain with \( C^2 \) boundary. Assume \( a > 0, q \geq 1, \ell q > 1, \alpha \geq q, \beta \geq \left[\frac{q-1}{q},1\right), \omega \in \mathcal{M}_b^+(\Omega) \) and \( \mu \in \mathcal{M}_b^+(Q_T) \). There exist \( c = c(N), b_0 = b_0(N,d,\ell,q) \in (0,1], M_1 = M_1(N,a,\alpha,q,\ell,d) \) and \( M_2 = M_2(N,a,\beta,q,\ell,d) \) such that if \( \|M_{a,\infty}[\omega]\|_{L_\infty(\mathbb{R}^N)} \leq M_1 \) and \( \|M_{\beta,\infty}[\mu]\|_{L_\infty(\mathbb{R}^{N+1})} \leq M_2 \) then the problem (1.9) admits a nonnegative solution \( u \) which satisfies
\begin{equation}
u \leq G[\omega] + c\mathcal{W}_{2d}[\mu] + cb_0.
\end{equation}
\begin{equation}g_\ell(2G[\omega] + 2c\mathcal{W}_{2d}[\mu] + 2cb_0) \in L^1(Q_T).
\end{equation}

The paper is organized as follows. In Section 2, we establish estimates on parabolic Wolff potentials. Section 3 is devoted to the study of linear parabolic equations with measure data. In section 4 we apply results obtained in Section 2 and Section 3 to prove existence of solution to equation (4.1) and (1.9).

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2. ESTIMATES ON PARABOLIC WOLFF POTENTIALS

We start this section with some notations. If $A$ is a measurable set in $\mathbb{R}^{N+1}$, we denote by $|A|$ the Lebesgue measure of $A$. If $f$, $g$ are functions defined in $\mathbb{R}^{N+1}$ and $a, b \in \mathbb{R}$ then set \( \{f > a\} := \{(x, t) \in \mathbb{R}^{N+1} : f(x, t) > a\} \), \( \{f > a, g \leq b\} := \{f > a\} \cap \{g \leq b\} \). Finally $\chi_A$ denote the characteristic function of $A$.

**Proposition 2.1.** Assume $\beta \in [0, 1)$ and $r > 0$. There exist $c_1 = c_1(N, \beta)$ and $\epsilon_1 = \epsilon_1(N, \beta, d, r)$ such that, for any $\mu \in \mathcal{M}^+_{\beta}(\mathbb{R}^{N+1})$ satisfying $\text{supp}(\mu) \subset Q_r(x^*, t^*)$ for some $(x^*, t^*) \in \mathbb{R}^N \times \mathbb{R}$ and for any $R \in (0, \infty)$, $\epsilon \in (0, \epsilon_1]$, $\lambda > \mu(\mathbb{R}^{N+1}) l(r, R)$ there holds

\[
\left| \{ W_R[\mu] > 3\lambda, M^2_{\beta,R}[\mu] \leq \epsilon \lambda \} \right| 
\leq c_1 \exp \left( -2^{-\frac{r}{1-\beta}} (1 - \beta)^{\frac{1}{1-\beta}} \epsilon^\beta \ln 2 \right) \left| \{ W_R[\mu] > \lambda \} \right|
\]

where $l(r, R) = N^{-1}((r \wedge R)^{-N} - R^{-N})$ if $R < \infty$ and $l(r, \infty) = N^{-1}r^{-N}$. If $\beta = 0$ then $\epsilon_0$ depends only on $N$, $\beta$ and (2.1) holds true for every $\mu \in \mathcal{M}^+_{\beta}(\mathbb{R}^{N+1})$ with compact support in $\mathbb{R}^{N+1}$, $R \in (0, \infty)$, $\epsilon \in (0, \epsilon_1]$, $\lambda > 0$.

**Proof.** We adapt the ideas used in [3] to parabolic setting. Denote the parabolic distance by

$$d_p((x, t), (y, \tau)) = |x - y| + |t - \tau|^{1/2} \quad \forall (x, t), (y, \tau) \in \mathbb{R}^N \times \mathbb{R}.$$ 

If $A, B \subset \mathbb{R}^{N+1}$, we denote

$$\text{diam}(A) = \sup \{d_p((x, t), (y, \tau)) : (x, t), (y, \tau) \in A\},$$

$$\text{dist} (A, B) = \inf \{d_p((x, t), (y, \tau)) : (x, t) \in A, (y, \tau) \in B\}.$$ 

For any $x = (x^1, \ldots, x^N) \in \mathbb{R}^N$, $t \in \mathbb{R}$, $r > 0$, the parabolic cube of center $x$ and edge $r$ is defined as follows

$$K_r(x, t) = \left[ x^1 - \frac{r}{2}, x^1 + \frac{r}{2} \right] \times \cdots \times \left[ x^N - \frac{r}{2}, x^N + \frac{r}{2} \right] \times \left[ t - \frac{r^2}{2}, t + \frac{r^2}{2} \right].$$ 

Notice that $\text{diam}(K_r(x, t)) = (\sqrt{N} + 1)r$ for every $(x, t) \in \mathbb{R}^N \times \mathbb{R}$.

**Case 1:** $R = \infty$. Let $\lambda > 0$ and $D_\lambda = \{W_\infty[\mu] > \lambda\}$. By Whitney covering lemma (see [1]), there exists a countable family $\mathcal{K} := \{K_i\}$, where $K_i = K_{r_i}(x_i, t_i)$, such that $\bigcup_i K_i = D_\lambda$, $K_i \cap K_j = \emptyset$ if $i \neq j$ and there exists a positive constant $C_w = C_w(\mathbb{N}) > 1$ such that

$$C_w^{-1} \text{diam}(K_i) \leq \text{dist}(K_i, D^C_\lambda) \leq C_w \text{diam}(K_i) \quad \forall i.$$

Let $\epsilon > 0$ and denote $F_{\epsilon, \lambda} = \{W_\infty[\mu] > 3\lambda, M^2_{\beta,\infty}[\mu] \leq \epsilon \lambda\}$. We will show that there exist $c_2 = c_2(N, \beta) > 0$ and $\epsilon_1 = \epsilon_1(N, \beta, r, d)$ such
that for any $K \in \mathcal{K}$, $\epsilon \in (0, \epsilon_1]$ and $\lambda > (\mu(\mathbb{R}^{N+1}))l(r,\infty)$ there holds
\[
(2.2) \quad |F_{\epsilon,\lambda} \cap K| \leq c_2 \exp \left( -2^{-\frac{\beta}{1-\beta}} (1-\beta) \frac{1}{1-\beta} \epsilon^{-\frac{1}{1-\beta}} \ln 2 \right) |K|.
\]

In order to do that we prove

**Assertion 1:** There exists $\epsilon_2 = \epsilon_2(N, \beta)$ such that for any $K \in \mathcal{K}$, $\epsilon \in (0, \epsilon_2]$, $\lambda > 0$, there holds $F_{\epsilon,\lambda} \cap K \subset E_{\epsilon,\lambda}$ where

$$E_{\epsilon,\lambda} = \{(x, t) \in K : \mathbb{W}_{(1+C_w)diam(K)}[\mu](x, t) > \lambda, \mathbb{M}_{\beta,\infty}[\mu](x, t) \leq \epsilon \}.$$

Take $K \in \mathcal{K}$ such that $F_{\epsilon,\lambda} \cap K \neq \emptyset$ and take $(\tilde{x}, \tilde{t}) \in D^\epsilon_{\lambda}$ satisfying $\text{dist}((\tilde{x}, \tilde{t}), K) \leq C_wdiam(K)$. Denote $r_0 = (1+C_w)diam(K)$. For any $k \in \mathbb{N}$ and $(x, t) \in F_{\epsilon,\lambda} \cap K$, we denote

$$A_k = \int_{2^k r_0}^{2^{k+1} r_0} \frac{\mu(Q_s(x, t))}{s^N} \frac{ds}{s}, \quad B_k = \int_{2^k r_0}^{2^{k+1} r_0} \frac{\mu(Q_s(x, t))}{s^N} \frac{ds}{s}.$$

Note that $B_k \leq c_3 \epsilon \lambda 2^{-k}$ where $c_3 = c_3(\beta)$. Set $\delta = \left( \frac{2^k}{1+2^k} \right)^N$ then $1-\delta < c_42^{-k}$ with $c_4 = c_4(N)$. Consequently, $(1-\delta)A_k \leq c_5 \epsilon \lambda 2^{-k}$ with $c_5 = c_5(N, \beta)$. For any $(x, t) \in F_{\epsilon,\lambda} \cap K$ and $s \in [(1+2^k)r_0, (1+2^{k+1})r_0]$, we have $Q_{\frac{2^k}{1+2^k} s}(x, t) \subset Q_s(\tilde{x}, \tilde{t})$, from which it follows

$$\delta A_k \leq \int_{(1+2^k)r_0}^{(1+2^{k+1})r_0} \frac{\mu(Q_s(\tilde{x}, \tilde{t}))}{s^N} \frac{ds}{s}.$$

As a consequence,

$$\int_{2^k r_0}^{2^{k+1} r_0} \frac{\mu(Q_s(x, t))}{s^N} \frac{ds}{s} = A_k + B_k \leq c_6 2^{-k} \epsilon \lambda + \int_{(1+2^k)r_0}^{(1+2^{k+1})r_0} \frac{\mu(Q_s(\tilde{x}, \tilde{t}))}{s^N} \frac{ds}{s}$$

where $c_6 = c_6(N, \beta)$. Therefore

$$\int_{2^k r_0}^{\infty} \frac{\mu(Q_s(x, t))}{s^N} \frac{ds}{s} \leq 2c_6 \epsilon \lambda + \int_{2r_0}^{\infty} \frac{\mu(Q_s(\tilde{x}, \tilde{t}))}{s^N} \frac{ds}{s} \leq (1 + 2c_6 \epsilon)\lambda.$$

Put $\epsilon_2 = (2c_6)^{-1}$. If $\epsilon \in (0, \epsilon_2]$ then

$$\int_{r_0}^{\infty} \frac{\mu(Q_s(x, t))}{s^N} \frac{ds}{s} \leq 2\lambda,$$

which implies Assertion 1.

**Assertion 2:** There exists $c_7 = c_7(N, \beta)$ such that

\[
(2.3) \quad |E_{\epsilon,\lambda}| \leq c_7 \exp \left( -2^{-\frac{\beta}{1-\beta}} (1-\beta) \frac{1}{1-\beta} \epsilon^{-\frac{1}{1-\beta}} \ln 2 \right) |K|.
\]
Denote $Q_1^r := Q_r(x^*, t^*)$ and $Q_2^r := Q_2r(x^*, t^*)$ for some $r > 0$ and $(x^*, t^*) \in \mathbb{R}^{N+1}$. Let $\lambda > (\mu(\mathbb{R}^{N+1})l(r, \infty)$. If $(x, t) \in (Q_2^r)^c$ and $s < r$ then $Q_s(x, t) \cap Q_1^r = \emptyset$. Hence
\[
\mathbb{W}_\infty[\mu](x, t) = \int_r^\infty \frac{\mu(Q_s(x, t))}{s^N} ds \leq \mu(\mathbb{R}^{N+1})l(r, \infty).
\]
Therefore $D_\lambda \subset Q_2^r$, which in turn implies $r_0 \leq 5(1 + C_w)r$. Next, we set $m_0 = (\ln 2)^{-1} \max(1, \ln(5d^{-1}(1 + C_w)r))$ then $2^{-m}r_0 \leq d$ for every $m \geq m_0$. For any $(x, t) \in E_{\epsilon, \lambda}$ and $m > m_0^{1/\beta}$,
\[
\int_{2^{-m}r_0}^{r_0} \frac{\mu(Q_s(x, t))}{s^N} ds \leq \frac{\epsilon \lambda}{1 - \beta}((m - m_0) \ln 2)^{1/\beta} + m_0 \epsilon \lambda \leq \frac{2\epsilon \lambda}{1 - \beta} m_0^{1/\beta}.
\]
If we define
\[
h_i(x, t) = \int_{2^{-m}r_0}^{2^{-m+1}r_0} \frac{\mu(Q_s(x, t))}{s^N} ds, \quad i \in \mathbb{N},
\]
then for any $m \geq m_0^{1/\beta}$,
\[
\mathbb{W}_{r_0}[\mu](x, t) \leq \frac{2\epsilon \lambda}{1 - \beta} m_0^{1/\beta} + \sum_{i=m+1}^{\infty} h_i(x, t).
\]
Consequently, for $0 < \gamma < 2$,
\[
|E_{\epsilon, \lambda}| \leq \sum_{i=m+1}^{\infty} \left| \left\{ (x, t) \in K : h_i(x, t) > 2^{-\gamma(i-m-1)}(1 - 2^{-\gamma})(1 - \frac{2}{1 - \beta} m_0^{1/\beta} \epsilon) \right\} \right|.
\]
After a long computation, we get
\[
|\{ (x, t) \in K : h_i(x, t) > s \}| \leq \frac{2^{2+2N}(\ln 2)^{-\beta}}{s} 2^{-2i} r_0^{N+2} \epsilon \lambda \leq c_8 \frac{2^{-2i}}{s} |K| \epsilon \lambda \quad \forall i
\]
which leads to
\[
(2.4) \quad |E_{\epsilon, \lambda}| \leq c_9 2^{-2m} \frac{\epsilon}{1 - \frac{2}{1 - \beta} m_0^{1/\beta} \epsilon} |K| \quad \forall m > m_0^{1/\beta}
\]
where $c_9 = c_9(N, \beta)$. Set $\epsilon_1 = \min\left\{ \frac{1}{4(1 - \beta)^{-1}m_0}, \epsilon_2 \right\}$. For any $\epsilon \in (0, \epsilon_1]$, we choose $m \in \mathbb{N}$ such that
\[
\left( \frac{1 - \beta}{2} \right)^{1/\beta} \left( \frac{1}{\epsilon} - 1 \right)^{1/\beta} - 1 < m \leq \left( \frac{1 - \beta}{2} \right)^{1/\beta} \left( \frac{1}{\epsilon} - 1 \right)^{1/\beta}.
\]
Then
\[
(2.5) \quad \frac{\epsilon}{1 - \frac{2}{1 - \beta} m_0^{1/\beta} \epsilon} \leq 1 \quad \text{and} \quad 2^{-2m} \leq 4 \exp \left( -2^{-\beta/\beta} \ln 2(1 - \beta)^{1/\beta} \epsilon^{-1/\beta} \right).
\]
Combining (2.4)-(2.5) yields to Assertion 2. Finally (2.2) follows straightforward.

If $\beta = 0$ then for any $m \in \mathbb{N}$, $\epsilon > 0$, $\lambda > 0$ and $(x,t) \in E_{\epsilon,\lambda}$

$$\mathbb{W}_{\epsilon_0}[\mu](x,t) \leq \epsilon \lambda m + \sum_{i=m+1}^{\infty} h_i(x,t).$$

Consequently, with $m \epsilon < 1$, $|E_{\epsilon,\lambda}| \leq c_9 2^{-2m} \epsilon (1 - m \epsilon)^{-1}|K|$. Put $\epsilon_1 = \min\{\frac{1}{2}, \epsilon_2\}$ then for any $\epsilon \in (0, \epsilon_1]$ and $\epsilon^{-1} - 2 < m < \epsilon^{-1} - 1$, we obtain

$$|E_{\epsilon,\lambda}| \leq 16c_9 \exp(-2\epsilon^{-1} \ln 2)|K|,$$

which leads to (2.2).

Case 2: $R < \infty$. For $\lambda > 0$, $D_{\lambda}^R = \{\mathbb{W}_R[\mu] > \lambda\}$ is an open subset of $\mathbb{R}^{N+1}$. By Whitney covering lemma, there exists a countable family of closed cubes $\mathcal{K} := \{K_i\}$ such that $\cup_i K_i = D_{\lambda}^R$, $K_i \cap K_j = \emptyset$ if $i \neq j$ and $\text{dist}(K_i, (D_{\lambda}^R)^c) \leq C_w \text{diam}(K_i)$. If $K \in \mathcal{K}$ is such that $\text{diam}(K) > \frac{R}{2C_w}$, there exists a finite number $n_K$ of closed dyadic cubes $\{P_{i,K}\}_{j=1}^{n_K}$ satisfying $\cup_{j=1}^{n_K} P_{i,K} = K$, $P_{i,K} \cup P_{j,K} = \emptyset$ if $i \neq j$ and $\frac{R}{4C_w} < \text{diam}(P_{i,K}) < \frac{R}{2C_w}$. We set $\mathcal{K}' = \{K \in \mathcal{K} : \text{diam}(K) \leq \frac{R}{2C_w}\}$, $\mathcal{K}'' = \{P_{i,K} : 1 \leq i \leq n_K, K \in \mathcal{K}, \text{diam}(K) > \frac{R}{2C_w}\}$ and $\tilde{\mathcal{K}} = \mathcal{K}' \cup \mathcal{K}''$.

For $\epsilon > 0$, we denote $F_{\epsilon,\lambda}^R = \{\mathbb{W}_R[\mu] > 3\lambda, \mathbb{M}_{\beta,R}[\mu] \leq \epsilon \lambda\}$. Let $K \in \tilde{\mathcal{K}}$ such that $F_{\epsilon,\lambda}^R \cap K \neq \emptyset$ and set $r_0 = (1 + C_w) \text{diam}(K)$.

Case 2.i: $\text{dist} ((D_{\lambda}^R)^c, K) \leq C_w \text{diam}(K)$. Let $(\tilde{x}, \tilde{t}) \in (D_{\lambda}^R)^c$ such that $\text{dist} ((\tilde{x}, \tilde{t}), K) \leq C_w \text{diam}(K)$ and $\mathbb{W}_R[\mu](\tilde{x}, \tilde{t}) \leq \lambda$. By using the same argument as in Case 1, we deduce that for any $(x,t) \in K \cap F_{\epsilon,\lambda}^R$,

$$\int_{r_0}^{R} \frac{\mu(Q_s(x,t))}{s^N} ds \leq (1 + c_{10}\epsilon) \lambda$$

where $c_{10} = c_{10}(N, \beta)$.

Case 2.ii: $\text{dist} ((D_{\lambda}^R)^c, K) > C_w \text{diam}(K)$. Then $K \in \mathcal{K}''$ and hence $\frac{R}{4C_w} < \text{diam}(K) \leq \frac{R}{2C_w}$. Therefore, for any $(x,t) \in K \cap F_{\epsilon,\lambda}^R$,

$$\int_{r_0}^{R} \frac{\mu(Q_s(x,t))}{s^N} ds \leq \int_{\frac{1}{4C_w}R}^{R} \frac{\mu(Q_s(x,t))}{s^N} ds \leq \epsilon \lambda (\ln 2)^{-\beta} \ln \left( \frac{4C_w}{1+4C_w} \right) \leq 2\epsilon \lambda.$$

Put $\epsilon_3 = \min\{1, c_{10}^{-1}\}$ then for any $\epsilon \in (0, \epsilon_3]$, $K \cap F_{\epsilon,\lambda}^R \subset E_{\epsilon,\lambda}^R$ where

$$E_{\epsilon,\lambda}^R = \{(x,t) \in K : \mathbb{W}_{r_0}[\mu](x,t) > \lambda, \mathbb{M}_{\beta,R}^2[\mu](x,t) \leq \epsilon \lambda\}.$$
By proceeding as in case 1, we can derive (2.3) with $E_{\epsilon, \lambda}$ replaced by $E_{\epsilon, \lambda}^R$ and with another constant. Thus (2.2) follows straightforward. Finally, the case $\beta = 0$ is treated as in case 1. 

\[ \Box \]

**Theorem 2.2.** Assume $0 \leq \beta < 1$ and $r > 0$. Set $\delta_1 = 2\left(\frac{1-\beta}{\beta}\right) \frac{1}{1-\beta} \ln 2$. There exists $c_{11} = c_{11}(N, \beta, d, r)$ such that for any $R \in (0, \infty)$, $\delta \in (0, \delta_1)$, $\mu \in \mathcal{M}_+([\mathbb{R}^N]^+, r') \in (0, r]$, $(x^*, t^*) \in \mathbb{R}^N \times \mathbb{R}$, there holds

\[ (2.6) \]

\[ \frac{1}{|Q_{2r'}(x^*, t^*)|} \int_{Q_{2r'}(x^*, t^*)} \exp \left( \delta M_2^{-\frac{1}{1-\beta}} \mathcal{W}_R[\mu^*]^{\frac{1}{1-\beta}} \right) \, dx \, dt \leq \frac{c_{11}}{\delta_1 - \delta} \]

where $\mu^* = \mu \chi_{Q_{2r'}(x^*, t^*)}$ and $M_2 = \|\mathcal{M}^2_{\beta, R}[\mu^*]\|_{L^\infty([\mathbb{R}^N]^+)}$. If $\beta = 0$ then $c_{11}$ is independent of $r$.

**Proof.** Let $\mu \in \mathcal{M}_+([\mathbb{R}^N]^+)$ satisfy $M_2 < \infty$. Due to Proposition 2.1 there exist $c_1 = c_1(N, \beta)$ and $\epsilon_1 = \epsilon_1(N, d, \beta, r)$ such that for any $R \in (0, \infty)$, $\epsilon \in (0, \epsilon_1)$, $\lambda > \mu([\mathbb{R}^N]^+)(r', R)$ there holds

\[ (2.7) \]

\[ \{\mathcal{W}_R[\mu^*] > 3\lambda, M_2^{-\frac{1}{1-\beta}} [\mu^*] \leq \epsilon \lambda \} \leq c_1 \exp \left( -2^{-\frac{1}{1-\beta}} (1 - \beta) \frac{1}{1-\beta} \ln 2 \right) \|\mathcal{W}_R[\mu^*] > \lambda \| \]

Since $\mu^*([\mathbb{R}^N]^+)(r', R) < N^{-1}(\ln 2)^{-\beta} M_2$, we can choose $\epsilon$ and $\lambda$ in (2.7) such that $\epsilon = \lambda^{-1} M_2$ with $\lambda > \max\{\epsilon_1^{-1}, N^{-1}(\ln 2)^{-\beta}\} M_2$. By using similar argument as in Proposition 2.1 we deduce that $D'_x \subset Q'_2$ where $D'_x = \{\mathcal{W}_R[\mu^*] > \lambda\}$ and $Q'_2 = Q_{2r'}(x^*, t^*)$. Hence

\[ (2.8) \]

\[ \{|\mathcal{W}_R[\mu^*] > 3\lambda\} \cap Q'_2 \leq c_1 \exp \left( -2^{-\frac{1}{1-\beta}} (1 - \beta) \frac{1}{1-\beta} M_2^{\frac{1}{1-\beta}} \ln 2 \lambda^{\frac{1}{1-\beta}} \right) |Q'_2| \]

Therefore $|\{\Psi > \theta\} \cap Q'_2| \leq |Q'_2| \chi_{(0, \theta_0)} + c_1 \epsilon \delta \theta |Q'_2| \chi_{(\theta_0, \infty)}$ where $\Psi = M_2^{-\frac{1}{1-\beta}} \mathcal{W}_R[\mu^*]^{\frac{1}{1-\beta}}$ and $\theta_0 = (3 \max\{\epsilon_1^{-1}, N^{-1}(\ln 2)^{-\beta}\})^{\frac{1}{1-\beta}}$. Thus, for each $\delta \in (0, \delta_1)$,

\[ \int_{Q'_2} e^{\delta \Psi} \, dx \, dt \leq (e^{\delta \theta_0} - 1) |Q'_2| + \frac{c_1 \delta}{\delta_1 - \delta} |Q'_2| \]

which implies the desired estimate.

The next result is crucial for proving existence of solution to (4.1) and (1.9) in section 4.

**Theorem 2.3.** Assume that $0 \leq \beta < 1$, $R > 0$ and $\mu \in \mathcal{M}_+([\mathbb{R}^N]^+)$ satisfies $\|\mathcal{M}^2_{\beta, \infty}[\mu]\|_{L^\infty([\mathbb{R}^N]^+)} \leq M_2$. Then there exist $\delta_2 = \delta_2(\beta)$ and
\[ c_i = c_i(N, \beta, R, d) \ (i = 12, 13) \] such that for any \( r \in (0, R) \) and any \( (x, t) \) \( \in \mathbb{R}^{N+1} \), there holds

\[ \int_{Q_r(x,t)} \exp(\delta_2 M_2^{-\frac{1}{1-\beta}} (\mathcal{W}_R[\mu])^{\frac{1}{1-\beta}})dyd\tau < c_{12}, \tag{2.9} \]

\[ \left\| \mathcal{W}_R[\exp(\delta_2 M_2^{-\frac{1}{1-\beta}} (\mathcal{W}_R[\mu])^{\frac{1}{1-\beta}})] \right\|_{L^\infty(\mathbb{R}^{N+1})} < c_{13}. \tag{2.10} \]

**Proof.** Fix \( (x, t) \in \mathbb{R}^N \times \mathbb{R} \). For every \( (y, \tau) \in \mathbb{R}^N \times \mathbb{R} \), we have

\[ \mathcal{W}_R[\mu](y, \tau) = \mathcal{W}_r[\mu](y, \tau) + \int_r^R \frac{\mu(Q_s(y, \tau))}{s^{N-\beta}} \frac{ds}{s} \]

\[ \leq \mathcal{W}_r[\mu](y, \tau) + M_2 \int_{\tau \wedge d}^d (\ln(2s^{-1}))^{-\beta} \frac{ds}{s} + M_2 \int_{\tau \wedge d}^{R \wedge d} (\ln 2)^{-\beta} \frac{ds}{s} \]

\[ \leq \mathcal{W}_r[\mu](y, \tau) + M_2 (\ln 2)^{-\beta} \ln \left( \frac{R}{d} \vee 1 \right) + M_2 (1 - \beta)^{-1} \left( \ln \left( \frac{d}{r} \vee 1 \right) \right)^{1-\beta}. \]

Consequently,

\[ \mathcal{W}_R[\mu](y, \tau) \leq 3^{1-\beta} M_2^{-\frac{1}{1-\beta}} (\ln 2)^{-\frac{\beta}{1-\beta}} \ln \left( \frac{R}{d} \vee 1 \right) \]

\[ + 3^{1-\beta} \mathcal{W}_r[\mu](y, \tau) + 3^{\frac{\beta}{1-\beta}} M_2^{\frac{\beta}{1-\beta}} (1 - \beta)^{-\frac{1}{1-\beta}} \ln \left( \frac{d}{r} \vee 1 \right). \]

Let \( \kappa \in (0, 1] \) (to be made precise later on). It follows from the above estimate that

\[ \exp \left( \frac{\kappa \delta_1}{4.3^{r^{-\beta}}} M_2^{-\frac{1}{1-\beta}} \mathcal{W}_R[\mu]^{\frac{1}{1-\beta}} \right) \]

\[ \leq 2^{-1} \exp \left( \frac{\delta_1}{2} M_2^{-\frac{1}{1-\beta}} \mathcal{W}_r[\mu]^{\frac{1}{1-\beta}} \right) + c_{14} \left( \frac{d}{r} \vee 1 \right)^{\kappa c_{15}} \]

where \( c_{14} = c_{14}(\beta, R, d), c_{15} = c_{15}(\beta) \) and \( \delta_1 \) is defined in Theorem 2.2.

For every \( s \in (0, r] \), \( (y, \tau) \in Q_r(x, t) \), we get \( Q_s(y, \tau) \subset Q_{2r}(x, t) \). Therefore \( \mathcal{W}_r[\mu](y, \tau) = \mathcal{W}_r[\mu \chi_{Q_{2r}(x,t)}](y, \tau) \) for every \( (y, \tau) \) \( \in Q_r(x, t) \).

Thanks to Theorem 2.2 we get

\[ \int_{Q_r(x,t)} \exp \left( \frac{\delta_1}{2} M_2^{-\frac{1}{1-\beta}} (\mathcal{W}_r[\mu])^{\frac{1}{1-\beta}} \right) dyd\tau \leq c_{16} r^{N+2} \]

where \( c_{16} = c_{16}(N, \beta, d, R) \). Thus

\[ \int_{Q_r(x,t)} \exp \left( \frac{\kappa \delta_1}{4.3^{r^{-\beta}}} M_2^{-\frac{1}{1-\beta}} \mathcal{W}_R[\mu]^{\frac{1}{1-\beta}} \right) dyd\tau \]

\[ \leq 2^{-1} c_{16} r^{N+2} + c_{17} \left( \frac{d}{r} \vee 1 \right)^{\kappa c_{15}} r^{N+2} \]
with \( c_{17} = c_{17}(N, \beta, R, d) \). By taking \( \kappa = 1 \wedge (2c_{15})^{-1} \) and \( \delta_2 = 2^{-23} \frac{\beta}{1-\beta} \kappa \delta_1 \), we derive (2.9). Finally, (2.10) follows from (2.11). □

3. Estimates on solutions to linear parabolic equation

In this section, let \( \Omega \) be a bounded domain with \( C^2 \) boundary. We first give some estimates on solutions to homogeneous linear equations with initial measure data.

**Theorem 3.1.** Assume \( \alpha \geq 1 \), \( \delta > 0 \) and \( \omega \in M^1_\alpha(\Omega) \). There exists a positive constant \( M_1 = M_1(N, \alpha, \delta) \) such that if \( \| M_{\alpha, \infty}[\omega] \|_{L^\infty(\mathbb{R}^N)} \leq M_1 \) then the unique solution \( u \) to the problem

\[
\begin{cases}
  u_t - \Delta u = 0 & \text{in } Q_T \\
  u = 0 & \text{on } \partial \Omega \times (0, T) \\
  u(., 0) = \omega & \text{in } \Omega.
\end{cases}
\]

satisfies

\[
\exp(\delta u^\alpha(x, t)) \leq c_{18} t^{-\frac{1}{2}} + 2 \quad \forall (x, t) \in Q_T
\]

where \( c_{18} = c_{18}(N) \).

**Proof.** The unique solution of (3.1) is represented by (see [11])

\[
(3.3) \quad u(x, t) = G[\omega](x, t) \leq \int_{\mathbb{R}^N} (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4t}} d\omega(y) \quad \forall (x, t) \in Q_T.
\]

Fix \((x, t) \in Q_T\). Using Fubini Theorem we get

\[
\int_{\mathbb{R}^N} (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x-y|^2}{4t}} d\omega(y) = \int_{\mathbb{R}^N} (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-r} dr d\omega(y)
\]

\[
= \int_{\mathbb{R}^N} \int_0^\infty (4\pi t)^{-\frac{N}{2}} \omega(B_{\sqrt{4t}}(x)) e^{-r} dr d\omega(y)
\]

\[
= \int_0^\infty (4\pi t)^{-\frac{N}{2}} \omega(B_{\sqrt{4t}}(x)) e^{-r} dr
\]

Let \( M_1 > 0 \) be made precise later on. If \( \| M_{\alpha, \infty}[\omega] \|_{L^\infty(\mathbb{R}^N)} \leq M_1 \) then by combining the assumption and (3.3), we get

\[
(3.4) \quad u(x, t) \leq M_1 \int_0^\infty (4\pi t)^{-\frac{N}{2}} (4tr)^{\frac{N}{2}} \left( -\ln \left( (4tr)^{\frac{1}{2}} \wedge 2^{-1} \right) \right)^{\frac{1}{\alpha}} e^{-r} dr
\]

\[
= M_1 \int_0^\infty \pi^{-\frac{N}{2}} \left( -\ln \left( (4tr)^{\frac{1}{2}} \wedge 2^{-1} \right) \right)^{\frac{1}{\alpha}} r^{\frac{N}{2}} e^{-r} dr
\]

\[
= M_1 \int_0^\infty c_{19} \left( -\ln \left( (4tr)^{\frac{1}{2}} \wedge 2^{-1} \right) \right)^{\frac{1}{\alpha}} \varphi(r) dr
\]
where \( c_{19} = \int_0^\infty \pi^{-\frac{N}{2}} r^\frac{N}{2} e^{-r} dr \) and \( \varphi(r) = c_{19}^{-1} \pi^{-\frac{N}{2}} r^\frac{N}{2} e^{-r} \). Since \( \alpha \geq 1 \) and \( \int_0^\infty \varphi(r) dr = 1 \), thanks to Jensen’s inequality we get

\[
\exp(\delta u^\alpha(x, t)) \leq \exp\left( \delta \left( c_{19} M_1 \left( -\ln \left( (4tr)^{\frac{1}{2}} \land 2^{-1} \right) \right) \right)^{\frac{1}{\alpha}} \varphi(r) dr \right).
\]

If \( M_1 = c_{19}^{-1} \delta^{-\frac{1}{\alpha}} \) then

\[
\exp(\delta u^\alpha(x, t)) \leq \int_0^\infty \left( (4tr)^{\frac{1}{2}} \land 2^{-1} \right)^{-1} \varphi(r) dr.
\]

Notice that \( (4tr)^{\frac{1}{2}} \land 2^{-1} \leq (4tr)^{-\frac{1}{2}} + 2 \), therefore

\[
\exp(\delta u^\alpha(x, t)) \leq t^{-\frac{1}{2}} \int_0^\infty (4r)^{-\frac{1}{2}} \varphi(r) dr + 2,
\]

which is (3.2) with \( c_{18} = \int_0^\infty (4r)^{-\frac{1}{2}} \varphi(r) dr \). \qed

**Theorem 3.2.** Assume \( \mu \in \mathfrak{M}^b_+(Q_T) \). There exists a positive constant \( c_{20} = c_{20}(N) \) such that the unique solution \( u \) of

\[
\begin{aligned}
\partial_t u - \Delta u &= \mu & \text{in } Q_T \\
u(t) &= 0 & \text{on } \partial \Omega \times (0,T) \\
u &= 0 & \text{in } \Omega
\end{aligned}
\]

satisfies \( u(x, t) \leq c_{20} \mathcal{W}_{2\alpha}[\mu](x, t) \) for every \( (x, t) \in Q_T \).

**Proof.** The unique solution of (3.5) is represented by (see [8])

\[
u(x, t) = \int_{Q_T} G^\Omega(x, t-s, y) d\mu(y, s) \quad \forall (x, t) \in Q_T.
\]

Due to Fubini theorem, we obtain

\[
u(x, t) \leq \int_{Q_T} G(x-y, t-s) d\mu(y, s)
\]

\[
= (4\pi)^{-\frac{N}{2}} \int_{Q_T} \left( \frac{N}{2} \int_{t-s}^\infty \tau^{-\frac{N+2}{2}} d\tau \right) \left( \int_{|x-y|^2}^{\infty} e^{-r} dr \right) d\mu(y, s)
\]

\[
= 2^{-N-1} N \pi^{-\frac{N}{2}} \int_0^\infty \int_{0}^\infty \tau^{-\frac{N+2}{2}} e^{-r} \mu(B_{\sqrt{4t\tau}}(x) \times (t-\tau, t)) dr d\tau
\]

where

\[
I_1 := 2^{-N-1} N \pi^{-\frac{N}{2}} \int_0^\infty \int_{0}^\infty \tau^{-\frac{N+2}{2}} e^{-r} \mu(B_{\sqrt{4t\tau}}(x) \times (t-\tau, t)) dr d\tau,
\]
By change of variables, we deduce
\[ I_1 \leq 2^{-\frac{N}{T}} N \pi^{-\frac{N}{T}} (1 - e^{-\frac{1}{2}}) W_\infty[\mu](x,t), \]
\[ I_2 \leq N \pi^{-\frac{N}{T}} \left( \int_{\frac{1}{2}}^\infty r^\frac{N}{T} e^{-r} dr \right) W_\infty[\mu](x,t). \]

Therefore
\[ (3.7) \quad u(x,t) \leq c_{21} W_\infty[\mu](x,t) \quad \forall (x,t) \in Q_T \]
where \( c_{21} = 2^{-\frac{N}{T}} N \pi^{-\frac{N}{T}} (1 - e^{-\frac{1}{2}}) + N \pi^{-\frac{N}{T}} \left( \int_{\frac{1}{2}}^\infty r^\frac{N}{T} e^{-r} dr \right) \). By combining \([3.7]\) and the estimate \( W_\infty[\mu](x,t) < \frac{2^N}{2^{N-1}} W_d[\mu](x,t) \), we finish the proof. \( \square \)

**Theorem 3.3.** Assume \( q \geq 1, \delta > 0, \alpha \geq q, \beta \in \left[\frac{q-1}{q}, 1\right) \), \( \omega \in \mathcal{M}_+^b(\Omega) \) and \( \mu \in \mathcal{M}_+^b(Q_T) \). There exist \( M_1 = M_1(N,\alpha,\delta) \), \( M_2 = M_2(N,\beta,\delta) \) and \( c_{22} = c_{22}(N,T,\Omega,d,\delta) \) such that if \( \|M_1^a[\omega]\|_{L^\infty(\mathbb{R}^N)} \leq M_1 \) and \( \|M_2^b[\mu]\|_{L^\infty(\mathbb{R}^{N+1})} \leq M_2 \) then the unique solution \( u \) of
\[ (3.8) \quad \begin{cases} u_t - \Delta u = \mu & \text{in } Q_T \\ u = 0 & \text{on } \partial \Omega \times (0,T) \\ u(.,0) = \omega & \text{in } \Omega \end{cases} \]
satisfies
\[ (3.9) \quad u \leq G[\omega] + c_{20} W_2d[\mu] \quad \text{in } Q_T, \]
\[ (3.10) \quad \int_{Q_T} \exp(\delta u^q(x,t)) dx dt \leq c_{22} \]
where \( c_{20} \) is the constant in Theorem 3.2. When \( \alpha = \frac{1}{1-\beta} = q \) then \( c_{22} \) is independent of \( \delta \).

**Proof.** Let \( v \) and \( w \) be the solution of (3.1) and (3.5) in \( Q_T \) respectively. The function \( u := v + w \) is the unique solution of (3.8) in \( Q_T \). Hence estimate (3.9) follows from Theorem 3.1 and Theorem 3.2.

We next prove (3.10). By taking into account the fact that \( e^{a+b} \leq 2^{-1}(e^{2a} + e^{2b}) \) for every \( a, b \in \mathbb{R} \), from Theorem 3.1 and Theorem 3.2.
we get
\[ \int_{Q_T} \exp (\delta u^q) \, dx \, dt \leq \int_{Q_T} \exp [\delta 2^{q-1} (v^q + w^q)] \, dx \, dt \]
\[ \leq \frac{1}{2} \int_{Q_T} (\exp [\delta (2v)^q] + \exp [\delta (2w)^q]) \, dx \, dt \]
\[ \leq \frac{1}{2} \int_{Q_T} (\exp [\delta (2v)^q] + \exp [\delta (2c_{20} \mathcal{W}_{2d}[\mu])^q]) \, dx \, dt. \]

Next we set \( M_1 = 2^{-1} c_{20}^{-1} \delta^{-\frac{1}{q}} \). It follows from Theorem 3.1 that if \( \|M_1^{1/\alpha} [\omega]\|_{L^\infty(\mathbb{R}^N)} \leq M_1 \) then
\[ \int_{Q_T} \exp (\delta (2v)^\alpha) \, dx \, dt < c_{23} \]
where \( c_{23} = c_{23}(N, T, \Omega) \). Put \( M_2 = 2^{-1} c_{20}^{-1} \delta^{-\frac{1}{\beta}} \delta^{-\frac{1}{\beta}} \) where \( \delta \) is the constant in Theorem 2.3. By Theorem 2.3 if \( \|M_2^{1/\beta} [\mu]\|_{L^\infty(\mathbb{R}^{N+1})} \leq M_2 \) then
\[ \int_{Q_T} \exp (\delta (2v)^\beta) \, dx \, dt < c_{24} \]
where \( c_{24} = c_{24}(N, \beta, d) \).

Since \( \alpha \geq q \) and \( \frac{1}{1-\beta} \geq q \), by combining Young inequality with (3.11), (3.12) and (3.13), we derive (3.10). Notice that if \( \alpha = \frac{1}{1-\beta} = q \) then \( c_{22} \) is independent of \( \delta \). \( \square \)

4. Applications

Let \( \Omega \) be a bounded domain with \( C^2 \) boundary. This section is devoted to the proof of Theorem 1.2 and Theorem 1.3.

4.1. Equations with absorption terms. We first study the existence and uniqueness of solution to the following problem
\[
\begin{align*}
\left\{ \begin{array}{ll}
\partial_t u - \Delta u + \text{sign}(u) g_\ell(u) = \mu & \text{in } Q_T \\
u = 0 & \text{on } \partial \Omega \times (0, T) \\
u(\cdot, 0) = \omega & \text{in } \Omega
\end{array} \right.
\]
where \( g_\ell \) is defined as in (1.1) with \( a > 0, q \geq 1, \ell \geq 1 \).

**Theorem 4.1.** Assume \( q \geq 1, a > 0, \alpha \geq q, \beta \in [\frac{q-1}{q}, 1), \omega \in \mathcal{M}^b(\Omega) \) and \( \mu \in \mathcal{M}^b(Q_T) \). There exist positive constants \( M_1 = M_1(a, \alpha, N) \) and \( M_2 = M_2(a, \beta, N) \) such that if \( \|M_1^{1/\alpha} [\omega]\|_{L^\infty(\mathbb{R}^N)} \leq M_1 \) and \( \|M_2^{1/\beta} [\mu]\|_{L^\infty(\mathbb{R}^{N+1})} \leq M_2 \) then the problem (4.1) admits a unique solution \( u \) satisfying \( e^{a|u|^q} \in L^1(\Omega) \).
Proof. Step 1: Uniqueness. If \( u_1 \) and \( u_2 \) are two solution of (4.1) with the same data \((\omega, \mu) \in \mathcal{M}_b(\Omega) \times \mathcal{M}_b(Q_T)\) then \( u = u_1 - u_2 \) is a solution to problem

\[
\begin{cases}
  u_t - \Delta u + \text{sign}(u_1)g_\ell(u_1) - \text{sign}(u_2)g_\ell(u_2) = 0 & \text{in } Q_T \\
  u = 0 & \text{on } \partial\Omega \times (0, T) \\
  u(., 0) = 0 & \text{in } \Omega.
\end{cases}
\]

By [3, Lemma 1.6 iii)], for every nonnegative function \( \zeta \in X(Q_T) \),

\[
\int_{Q_T} -(\zeta_t + \Delta \zeta) |u| dx dt \\
+ \int_{Q_T} \zeta \text{sign } (u_1 - u_2)(\text{sign}(u_1)g_\ell(u_1) - \text{sign}(u_2)g_\ell(u_2)) dx dt \leq 0.
\]

Since the second term on the right-hand side in (4.2) is nonnegative, it follows that \( \int_{Q_T} -(\zeta_t + \Delta \zeta) |u| dx dt \leq 0 \). By choosing \( \zeta = \psi \) which satisfies

\[
\begin{cases}
  -\psi_t - \Delta \psi = 1 & \text{in } Q_T \\
  \psi = 0 & \text{on } \partial\Omega \times (0, T) \\
  \psi(., T) = 0 & \text{in } \Omega
\end{cases}
\]

we deduce that \( u \equiv 0 \), namely \( u_1 \equiv u_2 \).

It remains to deal with the question of existence.

Step 2: Approximating solutions. Put \( \omega_{1,n} = \rho_n \ast \omega^+, \omega_{2,n} = \rho_n \ast \omega^- \), \( \omega_n = \omega_{1,n} - \omega_{2,n}, \mu_{1,n} = \tilde{\rho}_n \ast \mu^+, \mu_{2,n} = \tilde{\rho}_n \ast \mu^-, \mu_n = \mu_{1,n} - \mu_{2,n} \) where \( \{\rho_n\} \) and \( \{\tilde{\rho}_n\} \) are sequences of mollifiers in \( \mathbb{R}^N \) and \( \mathbb{R}^{N+1} \) respectively. We may assume that \( \omega_{i,n} \in C_0^\infty(\Omega) \) and \( \mu_{i,n} \in C_0^\infty(Q_T) \) for every \( n \) and \( i = 1, 2 \). For each \( n > 0 \), let \( u_n, u_{i,n}, v_{i,n} \) \((i = 1, 2)\) be respectively solutions to

\[
\begin{cases}
  (u_n)_t - \Delta u_n + \text{sign}(u_n)g_\ell(u_n) = \mu_n & \text{in } Q_T \\
  u_n = 0 & \text{on } \partial\Omega \times (0, T) \\
  u_n(., 0) = \omega_n & \text{in } \Omega
\end{cases}
\]

\[
\begin{cases}
  (u_{i,n})_t - \Delta u_{i,n} + g_\ell(u_{i,n}) = \mu_{i,n} & \text{in } Q_T \\
  u_{i,n} = 0 & \text{on } \partial\Omega \times (0, T) \\
  u_{i,n}(., 0) = \omega_{i,n} & \text{in } \Omega
\end{cases}
\]

\[
\begin{cases}
  (v_{i,n})_t - \Delta v_{i,n} = \mu_{i,n} & \text{in } Q_T \\
  v_{i,n} = 0 & \text{on } \partial\Omega \times (0, T) \\
  v_{i,n}(., 0) = \omega_{i,n} & \text{in } \Omega
\end{cases}
\]

By the maximum principle, \( -v_{2,n} \leq -u_{2,n} \leq u_n \leq u_{1,n} \leq v_{1,n} \) in \( Q_T \). Therefore, \( |u_n| \leq \max\{u_{1,n}, u_{2,n}\} \leq \max\{v_{1,n}, v_{2,n}\} \) in \( Q_T \).
Step 3: End of proof. Since \( \{ \omega_n \} \) and \( \{ \mu_n \} \) converge weakly to \( \omega \) and \( \mu \) respectively, there exists a function \( u \) and a subsequence, still denoted by \( \{ u_n \} \), such that \( \{ u_n \} \) and \( \{ g(u_n) \} \) converge to \( u \) and \( g(u) \) a.e. in \( QT \).

By \( \Box \), for any \( p \in [1, \frac{N+2}{N}] \), there exists a constant \( c_25 = c_25(\Omega, T, p) \) such that

\[
\|v_{i,n}\|_{L^p(Q_T)} \leq c_25(\|\mu_{i,n}\|_{L^1(Q_T)} + \|\omega_{i,n}\|_{L^1(\Omega)}) \leq c_25(\|\mu_i\|_{M(Q_T)} + \|\omega_i\|_{M(\Omega)}),
\]

from which it follows that

\[
\|u_n\|_{L^p(Q_T)} \leq c_25(\|\mu_i\|_{M(Q_T)} + \|\omega_i\|_{M(\Omega)}).
\]

Therefore, due to H"older inequality, the sequence \( \{ u_n \} \) is equi-integrable. By Vitali theorem, the sequence \( \{ u_n \} \) converges to \( u \) in \( L^1(Q_T) \).

Notice that if \( \|M_{1,\infty}[\omega^+]\|_{L^\infty(\mathbb{R}^N)} \leq M_1 \) for some \( M_1 > 0 \) then for every \( n \in \mathbb{N} \), \( \|M_{1,\infty}[\omega_{1,n}]\|_{L^\infty(\mathbb{R}^N)} \leq M_1 \). Indeed, for every \( x \in \mathbb{R}^N \) and \( s > 0 \), by Fubini theorem, we get

\[
\omega_{1,n}(B_s(x)) = \int_{B_s(x)} \int_{\mathbb{R}^N} \rho_n(y - z)d\omega^+(z)dy = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \chi_{B_s(x-z)}(y)\rho_n(y)d\omega^+(z) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \chi_{B_s(x-z)}(y)\omega^+(z)\rho_n(y)dy = \int_{\mathbb{R}^N} \omega^+(B_s(x-y))\rho_n(y)dy.
\]

Since \( \omega^+(B_s(x-y)) \leq M_1 s^N h_{1,\alpha}(s) \), we get \( \omega_{1,n}(B_s(x)) \leq M_1 s^N h_{1,\alpha}(s) \).

Hence \( \|M_{1,\infty}[\omega_{1,n}]\|_{L^\infty(\mathbb{R}^N)} \leq M_1 \). Similarly, if \( \|M_{2,\infty}[\mu^+]\|_{L^\infty(\mathbb{R}^{N+1})} \leq M_2 \) for some \( M_2 > 0 \) then for every \( n \in \mathbb{N} \), \( \|M_{2,\infty}[\mu_{i,n}]\|_{L^\infty(\mathbb{R}^{N+1})} \leq M_2 \).

Therefore, by setting \( M_1 = 2^{-\frac{a+1}{a}} c_1^{-1} a^{-\frac{1}{a}} \) and \( M_2 := 2^{\beta - 2} c_20^{\delta_2 - \beta} a^{\beta - 1} \), by Theorem 3.3, if \( \|M_{1,\infty}[\omega^+]\|_{L^\infty(\mathbb{R}^N)} \leq M_1 \) and \( \|M_{2,\infty}[\mu^+]\|_{L^\infty(\mathbb{R}^{N+1})} \leq M_2 \) then \( \int_{Q_T} \exp(2av_{i,n}^q)dxdt \leq c_{26} \) where \( c_{26} = c_{26}(N, T, \Omega, d, a) \). It follows that \( \int_{Q_T} \exp(2av_{u,n}^q)dxdt \leq c_{26} \). Consequently, \( \{\text{sign}(u_n)g_{\ell}(u_n)\} \) is equi-integrable. Hence, by Vitali theorem, up to a subsequence, \( \{\text{sign}(u_n)g_{\ell}(u_n)\} \) converges to \( \text{sign}(u)g_{\ell}(u) \) in \( L^1(Q_T) \).

The solution \( u_n \) satisfies, for every \( \zeta \in X(Q_T) \),

\[
\int_{Q_T} (-u_n(\zeta_t + \Delta \zeta) + \text{sign}(u_n)g_{\ell}(u_n)\zeta)dxdt = \int_{Q_T} \zeta d\mu_n + \int_{\Omega} \zeta(\,, 0)d\omega_n
\]

By letting \( n \to \infty \) in (4.7), we deduce that \( u \) is a solution to (4.1). \( \Box \)
Lemma 4.2. Assume $\omega \in \mathcal{M}^1_+(\Omega)$ and $\mu \in \mathcal{M}^1_+(Q_T)$. Let $\{\omega_n\}$, $\{\mu_n\}$ and $\{u_n\}$ be defined as in step 1 of the proof of Theorem 4.1. There holds
\begin{equation}
\left\|g_\epsilon(u_n)\right\|_{L^1(Q_T)} \leq \|\mu\|_{2\mathcal{M}(Q_T)} + \|\omega\|_{2\mathcal{M}(\Omega)} \quad \forall n \in \mathbb{N}.
\end{equation}

Proof. For any $k > 0$, define $T_k(s) = \min\{k, \max\{-k, s\}\}$ for $s \in \mathbb{R}$ and $T_k(s) = \int_0^s T_k(\sigma) d\sigma$. For any $n \in \mathbb{N}$, $\epsilon > 0$, the function $\epsilon^{-1}T_\epsilon(u_n)$ $(i = 1, 2)$ can be employed as a test function for the problem (4.4), i.e.
\begin{equation}
\int_{Q_T} \epsilon^{-1}(T_\epsilon(u_n), t) dx dt + \epsilon^{-1} \int_{Q_T} |\nabla T_\epsilon(u_n)|^2 dx dt \leq \int_{Q_T} g_\epsilon(u_n) \epsilon^{-1}T_\epsilon(u_n) dx dt \quad \forall n \in \mathbb{N}.
\end{equation}

Since
\begin{equation}
\int_{Q_T} \epsilon^{-1}T_\epsilon(u_n) dx dt = \int_\Omega \epsilon^{-1}T_\epsilon(u_n(T)) dx - \int_\Omega \epsilon^{-1}T_\epsilon(u_n) dx \geq - \|\omega_n\|_{L^1(\Omega)},
\end{equation}
it follows that
\begin{equation}
\int_{Q_T} g_\epsilon(u_n) \epsilon^{-1}T_\epsilon(u_n) dx dt \leq \|\mu_n\|_{L^1(Q_T)} + \|\omega_n\|_{L^1(\Omega)} \leq \|\mu\|_{2\mathcal{M}(Q_T)} + \|\omega\|_{2\mathcal{M}(\Omega)}.
\end{equation}

By letting $\epsilon \to 0$, we derive (4.8). \qed

Proof of Theorem 1.2. For each $k > 0$, $n \in \mathbb{N}$, denote by $u := u_{i,k,n}^{f_1,f_2}$ the solution of
\begin{equation}
\begin{cases}
  u_t - \Delta u + \text{sign}(u)g_\epsilon(u) &= \bar{\rho}_n \ast \mu + \bar{\rho}_n \ast (T_k(f_2)) & \text{in } Q_T \\
  u &= 0 & \text{on } \partial \Omega \times (0,T) \\
  u(.,0) &= \rho_n \ast \omega + \rho_n \ast (T_k(f_1)) & \text{in } \Omega
\end{cases}
\end{equation}
where $\{\rho_n\}$ and $\{\bar{\rho}_n\}$ are sequences of mollifiers in $\mathbb{R}^N$ and $\mathbb{R}^{N+1}$ respectively. Let $u := u_{i,k,n}^{f_1,f_2}$ be the solution of
\begin{equation}
\begin{cases}
  u_t - \Delta u + g_\epsilon(u) &= \bar{\rho}_n \ast (\mu^\pm) + \bar{\rho}_n \ast (T_k(f_2^\pm)) & \text{in } Q_T \\
  u &= 0 & \text{on } \partial \Omega \times (0,T) \\
  u(.,0) &= \rho_n \ast (\omega^\pm) + \rho_n \ast (T_k(f_1^\pm)) & \text{in } \Omega
\end{cases}
\end{equation}
By the comparison principle, $-u_{i,k,n}^{f_1,f_2} \leq u_{i,k,n}^{f_1,f_2} \leq u_{i,k,n}^{f_1,f_2}$ for any $n$. Using similar argument as in Theorem 4.1, we deduce that there exist $M_1 = M_1(N,\alpha,a)$ and $M_2 = M_2(N,\beta,a)$ such that if $\|\mathcal{M}_{\alpha,\infty}[\omega^\pm]\|_{L^\infty(\mathbb{R}^N)} <
\( M_1 \) and \( \| M_2^2 \beta, \infty [M^\pm] \|_{L^\infty (\mathbb{R}^{N+1})} < M_2 \), there holds
\[
\int_{Q_T} \exp(2a|u_{\pm, k,n}^f|^q) dx dt \leq c_{27}
\]
where \( c_{27} = c_{27}(N, T, \Omega, \beta, a, d, k) \). Hence we can find a subsequence, still denoted by \( \{u_{\pm, k, n}^f\} \), and a function \( u_{\pm, k}^f \), such that \( \{u_{\pm, k, n}^f\} \) and \( \{g_{\ell}(u_{\pm, k, n}^f)\} \) converge to \( u_{\pm, k}^f \) and \( g_{\ell}(u_{\pm, k}^f) \) respectively in \( L^1(Q_T) \) as \( n \to \infty \). Therefore, \( u_{\pm, k}^f \) is the solution of (4.1) with \( \mu \) replaced by \( \mu^\pm + T_k(f_2^\pm) \) and \( \omega \) replaced by \( \omega^\pm + T_k(f_1^\pm) \). By a similar argument, we can show that there exists a unique solution \( u_{k, f^2}^f \) of (4.1) with \( \mu \) replaced by \( \mu + T_k(f_2) \) and \( \mu \) replaced by \( \omega + T_k(f_1) \). Moreover, by the comparison principle, \( -u_{\pm, k}^f \leq u_{k, f^2}^f \leq u_{\pm, k}^f \), and the sequences \( \{u_{\pm, k}^f\} \) are increasing with respect to \( k \). Denote \( u_{\pm, k}^f := \lim_{k \to \infty} u_{\pm, k}^f \). Thanks to Lemma 4.2 that for every \( k > 0 \),
\[
\int_{Q_T} g_{\ell}(u_{\pm, k}^f) dx dt \leq \| \omega \|_{M(\Omega)} + \| \mu \|_{M(Q_T)} + \| f_1 \|_{L^1(\Omega)} + \| f_2 \|_{L^1(Q_T)}.
\]
Therefore, by monotone convergence theorem, \( \{g_{\ell}(u_{\pm, k}^f)\} \) converges to \( g_{\ell}(u_{\pm, k}^f) \) in \( L^1(Q_T) \).

Since \( -u_{\pm, k}^f \leq u_{k, f^2}^f \leq u_{\pm, k}^f \), it follows that \( g_{\ell}(u_{k, f^2}^f) \leq g_{\ell}(u_{\pm, k}^f) + g_{\ell}(u_{\pm, k}^f) \). Therefore the sequence \( \{\mu + T_k(f_2) - \text{sign}(u_{k, f^2}^f)g_{\ell}(u_{k, f^2}^f)\} \) is bounded in \( M(Q_T) \). Notice that the sequence \( \{\omega + T_k(f_1)\} \) is also bounded in \( M(\Omega) \). Hence, up to a subsequence, \( \{u_{k, f^2}^f\} \) converges to a function \( u_{k, f^2}^f \) in \( L^1(Q_T) \) and a.e. in \( Q_T \). Moreover, by dominated convergence theorem, the sequence \( \{\text{sign}(u_{k, f^2}^f)g_{\ell}(u_{k, f^2}^f)\} \) converges to \( \text{sign}(u_{k, f^2}^f)g_{\ell}(u_{k, f^2}^f) \) in \( L^1(Q_T) \) as \( k \to \infty \). By passing to the limit, we deduce that \( u_{k, f^2}^f \) is a solution of (1.8). The uniqueness is obtained by using similar argument as in Theorem 4.1.

4.2. Equations with source terms. In this section we deal with the existence of solutions to problem (1.9).

**Proof of Theorem 1.3** Let \( u_0 \) be a solution of
\[
\begin{cases}
\partial_t u_0 - \Delta u_0 = \mu & \text{in } Q_T \\
u_0 = 0 & \text{on } \partial \Omega \times (0, T) \\
u_0(\cdot, 0) = \omega & \text{in } \Omega.
\end{cases}
\]
For each $n \in \mathbb{N}$, let $u_{n+1}$ be a solution of
\[
\begin{cases}
\partial_t u_{n+1} - \Delta u_{n+1} = g_{\ell}(u_n) + \mu & \text{in } Q_T \\
u_{n+1} = 0 & \text{on } \partial \Omega \times (0, T) \\
u_{n+1}(\cdot, 0) = \omega & \text{in } \Omega
\end{cases}
\]namely, for every $\zeta \in X(Q_T),$
\[
\int_{Q_T} u_{n+1}(\zeta_t + \Delta \zeta) dx dt = \int_{Q_T} g_{\ell}(u_n) \zeta dx dt + \int_{Q_T} \zeta d\mu + \int_{\Omega} \zeta(\cdot, 0) d\omega
\]
We need the following lemma

**Lemma 4.3.** Under the assumptions of Theorem 1.3, there exist positive constants $b_0 = b_0(N, d, \ell, q) \in (0, 1)$, $M_1 = M_1(N, a, \alpha, q, \ell, d)$, $M_2 = M_2(N, a, \beta, q, \ell, d)$ such that if $\|M_1[\omega]\|_{L^\infty(\mathbb{R}^N)} \leq M_1$ and $\|M_2^2[\mu]\|_{L^\infty(\mathbb{R}^{N+1})} \leq M_2$ then
\[
u_n \leq G[\omega] + c_20 W_2 d[\mu] + c_20 b_0 \quad \forall n \in \mathbb{N}.
\]
where $c_20$ is the constant in Theorem 3.2

**Proof of Lemma 4.3.** We prove (4.12) by recurrence. Indeed, (4.12) holds true if $n = 0$ by the previous results. Assume now (4.12) holds true when $n = m$. We shall show that (4.12) remains true when $n = m + 1$. By Theorem 3.3
\[
u_{m+1} \leq G[\omega] + c_20 W_2 d[g_{\ell}(u_m) + \mu] = G[\omega] + c_20 W_2 d[\mu] + c_20 W_2 d[g_{\ell}(u_m)].
\]
Therefore, it’s sufficient to prove that
\[
\|W_2 d[g_{\ell}(u_m)] \| \leq b_0.
\]
Since (4.12) is valid when $n = m$, it follows that
\[
g_{\ell}(u_m) \leq 3^{-1} g_{\ell}(3G[\omega]) + 3^{-1} g_{\ell}(3c_20 W_2 d[\mu]) + 3^{-1} g_{\ell}(3c_20 b_0).
\]
Keeping in mind that $g_{\ell}(s) \leq \varepsilon^{\ell q} g_{\ell}(\varepsilon^{-1} s)$ for every $s \geq 0$ and $\varepsilon \in (0, 1],$
we get for $\varepsilon_1, \varepsilon_2 \in (0, 1],$
\[
g_{\ell}(u_m) \leq 3^{-1} \varepsilon_1 g_{\ell}(3\varepsilon_1^{-1} G[\omega]) + 3^{-1} \varepsilon_2 g_{\ell}(3\varepsilon_2^{-1} c_20 W_2 d[\mu]) + 3^{-1} b_0 \varepsilon_1^{\ell q} g_{\ell}(3c_20).
\]
Hence
\[
\|W_2 d[g_{\ell}(u_m)] \| \leq 3^{-1} \varepsilon_1 W_2 d[g_{\ell}(3\varepsilon_1^{-1} G[\omega])]
\]
\[
+ 3^{-1} \varepsilon_2 W_2 d[g_{\ell}(3\varepsilon_2^{-1} c_20 W_2 d[\mu])]
\]
\[
+ 3^{-1} b_0 \varepsilon_1^{\ell q} g_{\ell}(3c_20) W_2 d[1].
\]
We choose \( b_0 \) such that

\[
\tag{4.17} b_0^g g_\ell (3c_{20}) \mathbb{W}_{2d}[1] = b_0 \iff b_0 = \left( \frac{16}{3} \omega_N d^3 g_\ell (3c_{20}) \right)^{\frac{1}{q-1}}
\]

where \( \omega_N \) is the volume of unit ball in \( \mathbb{R}^N \).

**Step 1:** We show that if \( \varepsilon_1 \) is small enough then

\[
\tag{4.18} \varepsilon_1 \mathbb{W}_{2d}[g_\ell (3\varepsilon_1^{-1}G[\omega])] \leq b_0.
\]

When \( q = \alpha \), by Theorem 3.1 if \( \|M_1^1[\omega]\|_{L^\infty(\mathbb{R}^N)} \leq 3^{-1}c_{19}a^{-\frac{1}{2}}\varepsilon_1 \), then by Theorem 3.1 for any \((y, \tau) \in Q_T\),

\[
\tag{4.19} g_\ell (3\varepsilon_1^{-1}G[\omega](y, \tau)) \leq \exp(a3^q\varepsilon_1^{-q}(G[\omega](y, \tau))^{q}) \leq c_{18} \tau^{-\frac{1}{2}} + 2
\]

where \( c_{18} \) is the constant in Theorem 3.1. Therefore, for and \( s \geq 0 \) and fixed \((x, t) \in Q_T\),

\[
\tag{4.20} \int_{Q_s(x, t)} g_\ell (3\varepsilon_1^{-1}G[\omega](y, \tau)) dyd\tau = \int_{B_s(x)} \int_{(t-\frac{s^2}{2}) \wedge 0} g_\ell (3\varepsilon_1^{-1}G[\omega](y, \tau)) dyd\tau \\
\leq 2c_{18}\omega_N s^{N+1} + \omega_N s^{N+2}.
\]

Consequently,

\[
\tag{4.21} \mathbb{W}_{2d}[g_\ell (3\varepsilon_1^{-1}G[\omega])](x, t) \leq 4c_{18}\omega_N d^2 + 4\omega_N d^3 =: c_{28}
\]

Thus, if \( \varepsilon_1c_{28} \leq b_0 \), namely \( \varepsilon_1 \leq c_{28}^{-1}b_0 \), then (4.18) holds true. When \( q < \alpha \), by Young inequality and Theorem 3.1 if \( \|M_1^1[\omega]\|_{L^\infty(\mathbb{R}^N)} \leq 3^{-\frac{q}{2}}c_{19}a^{-\frac{1}{2}}\varepsilon_1 \) then \( \mathbb{W}_{2d}[g_\ell (3\varepsilon_1^{-1}G[\omega])](x, t) \leq c_{28}e^{a3^q} \). Hence if \( \varepsilon_1c_{28}e^{a3^q} \leq b_0 \), namely \( \varepsilon_1 \leq c_{28}e^{a3^q}b_0 \) then (4.18) holds true. Thus, by putting \( \varepsilon_1 = (c_{28}e^{a3^q}b_0) \wedge 1 \) and \( M_1 = 3^{-1}c_{19}a^{-\frac{1}{2}}((c_{28}e^{a3^q}b_0) \wedge 1) \), we derive (4.18) for every \( \alpha \geq q \).

**Step 2:** We show that if \( \varepsilon_2 \) small enough then

\[
\tag{4.22} \varepsilon_2 \mathbb{W}_{2d}[g_\ell (3\varepsilon_2^{-1}c_{20}\mathbb{W}_{2d}[\mu])] \leq b_0.
\]

When \( q = (1 - \beta)^{-1} \), thanks to Theorem 2.3 if

\[
a3^q\varepsilon_2^{-q}c_{20}^q \leq \delta_2 M_2^{-\frac{1}{\alpha - \beta}} \iff M_2 \leq a^{\beta - 3^{-1}c_{20}^{-1}\varepsilon_2^{-\frac{1}{\alpha - \beta}}}
\]

then for any \( 0 < s < 2d \), there holds

\[
\tag{4.23} \int_{Q_s(x, t)} g_\ell (3\varepsilon_2^{-1}c_{20}\mathbb{W}_{2d}[\mu]) dyd\tau \leq c_{29}.
\]
where $c_i = c_i(N, \beta, d)$ with $i = 29, 30$. Hence it’s sufficient to choose $\varepsilon_2$ such that $\varepsilon_2 c_{30} \leq b_0$, i.e. $\varepsilon_2 \leq c_{30}^{-1}b_0$. When $q < (1 - \beta)^{-1}$, by Young inequality and Theorem 2.3 if
\[
\varepsilon_2 c_{30} e^{a_3 \varepsilon_2^g} \leq b_0 \iff \varepsilon_2 \leq c_{30}^{-1} e^{-a_3 \varepsilon_2^g} b_0
\]
then (4.22) follows. Thus, by setting $\varepsilon_2 = (c_{30}^{-1} e^{-a_3 \varepsilon_2^g} b_0) \land 1$ and $M_2 = a_3^{-1} 3^{-1} c_{20}^{-1} b_0^{-1} ((c_{30}^{-1} e^{-a_3 \varepsilon_2^g} b_0) \land 1)$, we obtain (4.22).

**Step 3:** End of proof. By combining (4.17), (4.18) and (4.22), we deduce that if $\|M_{1, n}^1[\omega]\|_{L^\infty(\Omega)} \leq M_1$ and $\|M_{2, n}^1[\mu]\|_{L^\infty(\Omega)} \leq M_2$ then (4.15) and (4.12) hold true.

Moreover, by convexity, for any $\gamma \in (0, 1)$, we have
\[
g_t(2G[\omega] + 2c_{20}\mathcal{W}_{2d}[\mu] + 2c_{20}b_0) \leq \frac{\gamma}{4(1 + \gamma)} g_t\left(\frac{8(1 + \gamma)}{\gamma} G[\omega]\right) + \frac{\gamma}{4(1 + \gamma)} g_t\left(\frac{8(1 + \gamma)}{\gamma} c_{20} \mathcal{W}_{2d}[\mu]\right) + \frac{2 + \gamma}{2(1 + \gamma)} g_t\left(\frac{4(1 + \gamma)}{2 + \gamma} c_{20} b_0\right).
\]

We choose $\gamma$ such that
\[
\frac{8(1 + \gamma)}{\gamma} = 3(\varepsilon_1^{-1} \land \varepsilon_2^{-1}) \iff \gamma = \frac{8}{3(\varepsilon_1^{-1} \land \varepsilon_2^{-1}) - 8}.
\]

Then
\[
g_t(2G[\omega] + 2c_{20}\mathcal{W}_{2d}[\mu] + 2c_{20}b_0) \leq g_t(3\varepsilon_1^{-1} G[\omega]) + g_t(3\varepsilon_2^{-1} c_{20} \mathcal{W}_{2d}[\mu]) + g_t(4c_{20} b_0),
\]

which, together with (4.20) and (4.23), implies (4.13).

**Let us now return to the proof of Theorem 1.3**

By comparison principle, $\{u_n\}$ is increasing and converges to a function $u$ a.e. in $\Omega$. Moreover, it follows from (4.13) that the sequences $\{u_n\}$ and $\{g_t(u_n)\}$ are uniformly bounded in $L^1(\Omega_T)$. Thanks to monotone convergence theorem, $\{u_n\}$ and $\{g_t(u_n)\}$ converge to $u$ and $g_t(u)$ respectively in $L^1(\Omega_T)$. By letting $n \to \infty$ in (4.11), we derive that $u$ is a solution of (1.9).
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