The Casimir Effect of Curved Space-time
(formal developments)

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Abstract

The free energy due to the vacuum fluctuations of matter fields on a classical gravitational background is discussed. It is shown explicitly how this energy is calculated for a non-minimally coupled scalar field in an arbitrary gravitational background, using the heat kernel method. The treatment of (self-)interacting fields of higher spin is outlined, using a meanfield approximation to the gaugefield when treating the gauge boson self interaction and the fermion-gauge boson interaction.

1 Introduction

When I first started, together with Frank Antonsen also of the NBI, on the study of the Casimir effect in curved space, I did so utterly ignorant about previous theoretical developments in the subject, thus our approach came to be different from the standard one. The motivation for the study was our interest in wormholes and time machines and the fact that a wormhole almost has the shape of a cylinder and a cylinder looks as an obvious candidate for a calculation of the Casimir effect. Thus the original idea was to study Casimir driven space-time evolution (to see if wormholes are stable) and because the Casimir energy is finite, we
figured out that it had to be incorporated in the Einstein equations (an insight, as prof. Grib kindly pointed out to us afterwards, other people had while we were still in kindergarten). This paper will describe the method we developed for determining the Casimir effect, taking as the starting point the flat space approach.

With the action given by

\[ S = S_{\text{Einstein-Hilbert}} + S_{\text{matter}} \]  

one obtains the classical Einstein equation

\[ \frac{\delta S_{\text{EH}}}{\delta g^{\mu\nu}} \sim G_{\mu\nu} = T_{\mu\nu} \sim -\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} \]  

which, upon treating matter fields quantum mechanically gives following, 'corrected' (or first quantized) Einstein equations

\[ \frac{\delta S_{\text{EH}}}{\delta g^{\mu\nu}} \sim G_{\mu\nu} = \langle T_{\mu\nu} \rangle \sim -\frac{\delta S_{\text{effective}}}{\delta g^{\mu\nu}} \]  

To get the right hand side of the first quantized Einstein equation in the case of a vacuous space-time one needs to determine the Casimir energy momentum tensor which in turn can be found from the Casimir free energy. Proceeding as in flat space one expects to renormalize this quantity as follows

\[ F_{\text{ren}} = F - F_{\text{Minkowski}} \]  

where one subtracts the infinite free energy of the Minkowski vacuum. This prescription for regularization however, only works in locally flat space-times such as the hyperspatial tube. From a practical point of view there is no need to worry, though. By choosing the method described below one gets expressions that are readily zeta-renormalizable

2 Determining the free energy from the zeta-function

The Helmholtz free energy is connected to the generating functional by the relationship

\[ F = -\frac{1}{\beta} \ln Z \]  

\[ 2 \]
where $\beta = (k_B T)^{-1}$ is the inverse temperature and where the partition function is given by the functional integral which, in the case of a free scalar field, is

$$Z = \int e^{iS} D\varphi = \int e^{i \int \sqrt{-g} d^4 x \frac{1}{2} \phi (\Box - m^2) \phi} D\phi = \left( \det (\Box - m^2) \right)^{-\frac{1}{2}}$$

with the curved space d’Alembertian is given by

$$\Box \equiv \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu)$$

where $g$ denotes the determinant of the metric. The use of a curved space d’Alembertian is the only change from the flat space approach.

The determinant of an operator, $A$, is, by definition, the product of its eigenvalues

$$\det(A) = \prod_\lambda \lambda$$

so with the zeta-function given by

$$\zeta_A(s) = \sum_\lambda \lambda^{-s} = \text{Tr} A^{-s}$$

these two quantities are related by the following equation

$$\frac{d \zeta_A}{ds} \bigg|_{s=0} = \sum_\lambda - \ln \lambda \cdot \lambda^{-s} \bigg|_{s=0} = - \sum_\lambda \ln \lambda = \ln \prod_\lambda \lambda = - \ln \det(A)$$

In the case of a minimally coupled scalar field, $A = \Box - m^2$, one thus gets the following expression for the free energy

$$F = -\frac{1}{2\beta} \frac{d}{ds} \bigg|_{s=0} \zeta_{\Box - m^2}$$

3 Determining the zeta-function from the heat kernel

Call the operator of interest $A$, which generally varies in space $A = A(x)$, and the corresponding eigenfunctions $\psi_\lambda$ so that

$$A \psi_\lambda = \lambda \psi_\lambda$$
The heat kernel, \( G_A(x, x', \sigma) \), is the function satisfying the heat kernel equation

\[
AG_A(x, x'; \sigma) = -\frac{\partial}{\partial \sigma}G_A(x, x'; \sigma)
\]

subject to the boundary condition \( G_A(x, x'; 0) = \delta(x - x') \).

Note that this equation is satisfied by

\[
G_A(\xi, \xi', \sigma) \equiv \sum_{\lambda} \psi_{\lambda}(x)\psi_{\lambda}^*(x') e^{-\lambda \cdot \sigma}
\]

Using this spectral representation of the heat kernel in turn makes it possible to show the following relationship between \( G_A \) and \( \zeta_A \)

\[
\int_0^\infty d\sigma \sigma^{s-1} \int \sqrt{-g}d^4x G_A(x, x, \sigma) = \int_0^\infty d\sigma \sigma^{s-1} \int \sqrt{-g}d^4x \sum_{\lambda} |\psi_{\lambda}(x)|^2 e^{-\lambda \cdot \sigma}
\]

\[
= \sum_{\lambda} \int_0^\infty d\sigma \sigma^{s-1} e^{-\lambda \cdot \sigma} \int \sqrt{-g}d^4x |\psi_{\lambda}(x)|^2
\]

\[
= \sum_{\lambda} \int_0^\infty d\sigma \sigma^{s-1} e^{-\lambda \cdot \sigma}
\]

\[
= \sum_{\lambda} \lambda^{-s} \int_0^\infty d(\lambda \cdot \sigma)(\lambda \cdot \sigma)^{s-1} e^{-\lambda \cdot \sigma}
\]

\[
= (\sum_{\lambda} \lambda^{-s}) \Gamma(s)
\]

\[
= \Gamma(s) \zeta_A(s)
\]

(14)

where the fact that the \( \psi_{\lambda} \)'s are eigenfunctions of a Hermitean operator, and thus can be chosen to be an orthonormal base, has been used to perform the space-time integral. This gives us

\[
\zeta_A(s) \equiv \frac{1}{\Gamma(s)} \int d\sigma \sigma^{s-1} \int G_A(x, x; \sigma) \sqrt{-g}d^4x
\]

and so, finally, for the minimally coupled scalar case

\[
F_{\Box - m^2} = -\frac{1}{2\beta} d_s \big|_{s=0} \zeta_{\Box - m^2}(s)
\]

\[
= -\frac{1}{2\beta} \int \sqrt{-g}d^4x \big|_{s=0} \int_0^\infty \frac{d\sigma \sigma^{s-1}}{\Gamma(s)} G_{\Box - m^2}(x, x; \sigma)
\]

\[
\equiv \int_0^\infty F_{\Box - m^2}(x)
\]

(15)
where $\mathcal{F}(x)$ is the free energy density. Note that the integral over $x$ is taken along the diagonal $x = x'$.

Thus to calculate the zero-point energy of a scalar field one has to determine the scalar field operator (the d’Alembertian) in the relevant space-time, solve the corresponding heat kernel equation and subsequently calculate the zeta-function from which the generating functional and thus all relevant quantities (including the zero-point energy) can be calculated.

### 4 Determining the heat kernel for a non-Minimally Coupled scalar field

Knowing how to relate the free energy (through the zeta-function) to the heat kernel let us proceed to actually determine the heat kernel for the scalar field operator, $\Box + \xi R$, with $R$ being the curvature scalar (one can of course use any other function, including a mass term; $-m^2$).

To this end, first rewrite the d’Alembertian in terms of the local vierbeins (i.e. the coordinate base of a freely falling observer, $e^a_\mu \equiv \frac{\partial x^a}{\partial \xi^\mu}$, where Greek indices refer to general coordinates while Latin indices refer to the local inertial frame (the metric is $g_{\mu \nu} = \eta_{ab} e^a_\mu e^b_\nu$ where $\eta_{ab}$ is the Minkowski metric, $g$ is the determinant of the metric and $e$ the vierbein determinant, $e = \sqrt{-g}$):

\[
\Box + \xi R = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu \nu} \partial_\nu) + \xi R
= \frac{1}{e} \partial_\mu (e \eta^{ab} e^a_\mu e^b_\nu \partial_\nu) + \xi R
= \frac{1}{e} e^m_\mu \partial_m (e \eta^{ab} e^a_\mu e^b_\nu e^n_\nu \partial_n) + \xi R
= \frac{1}{e} e^m_\mu \partial_m (e \eta^{ab} e^a_\mu e^b_\nu e^n_\nu \partial_n) + \eta^{ab} e^m_\mu e^a_\mu e^b_\nu e^n_\nu \partial_n + \xi R
= \Box_0 + \frac{1}{e} e^m_\mu (\partial_m (e e^n_\nu)) \partial^n + \xi R
\]

where $\Box_0 = \eta^{ab} \partial_a \partial_b$ is the flat space d’Alembertian the heat kernel of which is known to be

\[
G_0(x, x'; \sigma) = (4\pi \sigma)^{-\frac{\Delta^2(x, x')}{4\sigma}}
\]
where $\Delta(x,x') = \int_{x}^{x'} ds \; ( = x - x' \text{ in Cartesian coordinates})$ where $x, x'$ refer to freely falling coordinates which one eventually may have to express in terms of 'general' coordinates.

Now in the heat kernel equation

$$\left( \Box_0 + \frac{1}{e^\mu \partial_m (ee^\mu_a)} \partial^a + \xi R \right) G_{\Box_0 + \xi R}(x,x'; \sigma) = -\partial_\sigma G_{\Box_0 + \xi R}(x,x'; \sigma)$$

one can remove the first order term by substituting

$$G_{\Box_0 + \xi R} = \tilde{G}(x,x'; \sigma)e^{-\frac{1}{2} \int_{x}^{x'} \frac{1}{e^\mu \partial_m (ee^\mu_a)} dx^a}$$

(19)

to get the following equation (which is not quite a heat kernel equation because the solution must respect the boundary condition $G(x,x'; \sigma) = \delta(x - x')$ of the original equation, not $\tilde{G}(x,x'; \sigma) = \delta(x - x')$)

$$\left[ \Box_0 + \frac{1}{4} \left( \frac{1}{e^\mu \partial_m (ee^\mu_a)} \right)^2 - \frac{1}{2} \partial^a \left( \frac{1}{e^\mu \partial_m (ee^\mu_a)} \right) + \xi R \right] \tilde{G}(x,x'; \sigma) = -\partial_\sigma \tilde{G}(x,x'; \sigma)$$

(20)

written compactly as

$$(\Box_0 + f(x)) \tilde{G}(x,x'; \sigma) = -\partial_\sigma \tilde{G}(x,x'; \sigma)$$

(21)

To determine $\tilde{G}(x,x'; \sigma)$ from this equation, substitute

$$\tilde{G}(x,x'; \sigma) = G_0(x,x'; \sigma)e^T(x,x'; \sigma)$$

(22)

to obtain

$$\Box_0 T + \partial_\mu T \partial^\mu T + f + 2\partial_\mu \ln(G_0) \cdot \partial^\mu T = -\partial_\sigma T$$

(23)

where the last term on the left hand side vanishes on the diagonal, $x = x'$, because there, cf equation (17), the flat space heat kernel, $G_0 = \ldots$
$G_0(x; x; \sigma)$ is constant.

Then, Taylor expand $T$ in powers of $\sigma$;

$$T = \sum_{n=0}^{\infty} \tau_n \sigma^n$$ (24)

The coefficients, $\tau_n$, are easily determined:

From the boundary condition we demand that (cf equations (19,22))

$$G_{\Box + \xi R}(x, x'; 0) = G_0(x, x'; 0)e^T(x, x'; 0)e^{-\frac{1}{2} \int \frac{1}{e^\mu} (\partial_m (ee^\mu_m)) dx^n} = \delta(x - x')$$ (25)

Adding the fact that the flat space heat kernel already satisfies the boundary condition by itself, $G_0(x, x'; 0) = \delta(x - x')$, gives

$$\tau_0 = \frac{1}{2} \int \frac{1}{e^\mu} (\partial_m (ee^\mu_m)) dx^n$$ (26)

and collecting the $\sigma^1$-terms one obtains

$$\tau_1 = -f - \Box_0 \tau_0 - \partial_p \tau_0 \partial^p \tau_0$$ (27)

All higher order coefficients can be described by the recursion formula one obtains by inserting equation (24) into equation (23), remembering that the last term on the left hand side vanishes, and collecting the $\sigma^n$ terms

$$\tau_{n+1} = -\frac{1}{n+1} \left( \Box_0 \tau_n + \sum_{n'=0}^{n} \partial_p \tau_{n-n'} \partial^p \tau_{n'} \right) \quad : n > 1$$ (28)

yielding for the next two coefficients

$$\tau_2 = \frac{1}{2} \Box_0 f + \text{higher order terms}$$ (29)

and

$$\tau_3 = -\frac{1}{3} \partial_p f \partial^p f + \text{higher order terms}$$ (30)

and so, from equations (19,22,24,26,27,29,30)), we finally have

$$G_{\Box + \xi R}(x, x'; 0) = \frac{1}{(4\pi \sigma)^2} e^{-f\sigma - (\Box_0 \tau_0 + \partial_p \tau_0 \partial^p \tau_0) \sigma + \frac{1}{2} \Box_0 f \sigma^2 - \frac{1}{3} \partial_p f \partial^p f \sigma^3}$$

$$= \frac{1}{(4\pi \sigma)^2} e^{-f\sigma} \left( 1 - (\Box_0 \tau_0 + \partial_p \tau_0 \partial^p \tau_0) \sigma + \frac{1}{2} \Box_0 f \sigma^2 - \frac{1}{3} \partial_p f \partial^p f \sigma^3 \right)$$
giving the following renormalized expression for the Casimir free energy of a scalar field in an arbitrary gravitational background

\[
F_{\Box + \xi R} = \frac{1}{2\beta} \int \sqrt{-g} d^4x \, d_s|_{s=0} \int_0^{\infty} d\sigma \frac{\sigma^{s-1}}{\Gamma(s)} G_{\Box + \xi R}(x, x; \sigma)
\]

\[
\simeq \frac{1}{16\pi^2 \beta} \int \sqrt{-g} d^4x \left[ -\frac{1}{2} f^2 \ln(f) + \frac{3}{4} f^2 - (\Box_0 \tau_0 + \partial_\rho \tau_0 \partial^\rho \tau_0) f \ln(f) 
\right.
\]

\[
+ (\Box_0 \tau_0 + \partial_\rho \tau_0 \partial^\rho \tau_0) f - \frac{1}{2} (\Box_0 f) \ln(f) - \frac{1}{3} (\partial_\rho f \partial^\rho f) f^{-1}
\]

\[
\equiv \int \sqrt{-g} d^4x \, F_{\Box + \xi R}(x) \tag{31}
\]

One should note that in the expansion (22,24) and subsequently the free energy one gets terms of higher and higher order in the curvature. Thus the term \(-f\sigma\) in the heat kernel is the contribution from the classical gravitational background while the other terms that have been written out explicitly can be thought of as first order quantum corrections to that background. As we know general relativity to be renormalizable only to the one loop level it would thus probably be meaningless to continue the expansion (much) further.

5 Determining the heat kernel of (Spin 1) Non-abelian Gauge Bosons

In order to be able to carry out computations for higher spins as well, we want to relate the cases of vector bosons and spin \(\frac{1}{2}\) fermions to that of the scalar field case and as the treatment of gauge bosons carries some resemblance to that of the non-minimally coupled scalar field this case will be considered first.

When considering a Yang-Mills field in a curved space-time then, in order to obtain the field strength tensor the naive guess is to replace the derivatives of the Minkowski space field tensor with covariant derivatives which is not correct as this leads to a non-gauge covariant expression (eventhough, accidentally, it gives the right answer in the case of abelian fields). Instead proceed by considering the full theory of Dirac fermions interacting minimally with the gauge fields as well as with the gravitational field. In order to preserve local gauge and Lorentz covariance...
construct a covariant derivative of the form

$$D_m = e^\mu_m (\partial_\mu + \frac{i}{2} \omega^p_\mu (x) X_{pq} + ig A^a_\mu (x) T_a)$$  \hfill (32)$$

where $e^\mu_m$ is the vierbein (local base vectors), $\omega^p_\mu (x)$ is the spin connection being the gravitational analogue of the gauge field $A^a_\mu (x)$ and $X_{pq}$ the corresponding Lorentz group ($SO(3, 1)$) generators analogous to the gauge group generators $T_a$. As before, greek indices refer to curvilinear coordinates while latin indices refer to local Lorentz coordinates and are also used for gauge group indices (it should be clear from context: in general small Latin letters from the beginning of the alphabet will be used to denote gauge indices, reserving letters from the last half of the alphabet for use as Lorentz indices).

As in flat space field theory the gauge field tensor $F^{a}_{mn}$ is obtained from the commutator of the covariant derivatives

$$[D_m, D_n] = S^{q}_{mn}(x) D_q + \frac{i}{2} R^{pq}_{mn}(x) X_{pq} + i F^{a}_{mn} T_a$$  \hfill (33)$$

(where $S^{q}_{mn}(x)$ is the torsion and $R^{pq}_{mn}(x)$ is the Riemann curvature tensor). One obtains (after a lengthy calculation) the field strength tensor

$$F^{a}_{mn} = e^\mu_m e^n_\nu (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + ig f^a_{bc} A^b_\mu A^c_\nu)$$  \hfill (34)$$

The generating functional of the full theory of a fermion interacting with a non-abelian gauge field is

$$Z = \int DA_\mu \int D\psi D\bar{\psi} e^{S_{gauge\ field} + S_{fermion}}$$

$$= \int DA_\mu \int D\psi D\bar{\psi} e^{-\frac{i}{4} \int F^{a}_{mn} F^{mn}_a \sqrt{-g} dx^\mu + i \int \bar{\psi} \gamma^m D_m \psi \sqrt{-g} dx^\mu}$$  \hfill (35)$$

Refering back to equation (32) we see that the fermion part of the action contain reference to the gauge field so that one a priori cannot carry out the two integrations independently. One could probably treat the fermion action as a source term when doing the gauge field integration and then subsequently do the fermion integration. Instead however, I’m going to make a mean field approximation to $A_\mu$ in the fermionic integral, as well as in the higher order terms of the bosonic integral...
(see later) enabling one to consider the bosonic and the fermionic parts independently.

The bosonic part of the generating functional,

$$Z = \int DA_\mu e^{-\frac{1}{4} \int F_{mn} F^{mn} g dx^\mu}$$ \hspace{1cm} (36)$$
can (using commutation relations, suitable normalization and the Lorentz condition) be given the form

$$Z = \int DA_\mu \exp \left( -\int \sqrt{-g} d^4 x \frac{g^2}{4} \left[ -\delta^a_b \delta^m_n \partial_\mu \partial_\nu + \delta^a_b (\partial_\mu e^{mn} - \partial_\nu e^{m\mu}) e_\mu^p \partial_\nu \right.ight.$$

$$\left. + g f_{bc}^a (\partial_n A^{mc} - \partial_m A^{cn}) + \frac{1}{2} \delta^m_n g^2 f_{ebc}^a A^e_{pa} A^{pd} \right] A_a^n \right)$$ \hspace{1cm} (37)$$

In order to perform this path integral, make it Gaussian by choosing the following mean field approximation:

$$Z = \int DA_\mu \exp \left( -\int \sqrt{-g} d^4 x A^a_m \frac{g^2}{4} \left[ -\delta^a_b \delta^m_n \partial_\mu \partial_\nu + \delta^a_b (\partial_\mu e^{mn} - \partial_\nu e^{m\mu}) e_\mu^p \partial_\nu \right.$$

$$\left. + g f_{bc}^a (\partial_n A^{mc} - \partial_m A^{cn}) + \frac{1}{2} \delta^m_n g^2 f_{ebc}^a A^e_{pa} A^{pd} \right] A_a^n \right)$$ \hspace{1cm} (38)$$

We now proceed to determine the mean fields.

5.1 Determining the gauge mean field

By definition

$$\langle A^a_m(x) A^b_n(x') \rangle = \frac{\int A^a_m(x) A^b_n(x') e^{iS} DA}{\int e^{iS} DA}$$ \hspace{1cm} (39)$$

It can be shown [1] that

$$\langle A^a_m(x) A^b_n(x') \rangle = \frac{1}{2} \left( \frac{\delta^2 S}{\delta A^a_m(x) \delta A^b_n(x')} \right)^{-1} = \frac{1}{2} \delta^{ab} \eta_{mn} \delta(x - x') G(x, x')$$ \hspace{1cm} (40)$$

\footnote{Where ever the mean value of a any odd power of the gauge fields occur, we substitute \( \sqrt{A^2} \) for \( A \).}
where $S$ denotes the appropriate action and where $G(x, x')$ is a kind of Green's function related to the heat kernel $G(x, x'; \sigma)$ by the relationship

$$ G(x, x') = -\int_0^\infty G_A(x, x'; \sigma)d\sigma \quad (41) $$

The heat kernel is calculated using the same method as is done for the full theory in the following subsection so I just quote the lowest order result from [1]

$$ \langle A^a_m(x)A^b_n(x') \rangle = (\gamma - 1)\delta^a_b \left( -\partial e^{\mu}_m e^\mu_{n} + \frac{1}{2} e^{mp} e^k_{np} \right) + f^a_{bn} \quad (42) $$

where $\gamma$ is the Euler constant, $e^{mp} = -(\partial_n e^{m\mu} - \partial^m e_{n}^{\mu})e^p_{\mu}$ and where $f^a_{bn}$ is defined by equation (44) below. To explicitly find the meanfield start by putting it equal to zero in the action/formula (43) in order to calculate the first approximation to the meanfield. Then introduce this first approximation into the above formula to get the second approximation and so forth.

5.2 The heat kernel for gauge bosons in the mean field approximation (cont.)

The full heat kernel equation for the operator in the Gaussian path integral (39) is

$$ \frac{g^2}{4} \left[ \delta^a_b \delta^m_n \partial_p \partial^p - \delta^a_b (\partial_r e^{m\mu} - \partial^m e_{r}^{\mu}) e^p_{\mu} \partial_p - g f^a_{bc} (\partial_r A^{mc} - \partial^m A_{r}^c) > - \left. \frac{1}{2} \delta^m_n g^2 f^{ac} e_d A^p A^{pd} > \right] G^b_{bn}(x, x'; \sigma) = -\partial_\sigma G^a_{bn}(x, x'; \sigma) \quad (43) $$

or, in short notation

$$ \frac{g^2}{4} \left[ \delta^a_b \delta^m_n \partial_p \partial^p - \delta^a_b (\partial_r e^{m\mu} - \partial^m e_{r}^{\mu}) e^p_{\mu} \partial_p + f^a_{bc} (\langle A >) G^b_{bn}(x, x'; \sigma) = -\partial_\sigma G^a_{bn}(x, x'; \sigma) \quad (43) $$

The first order term is eliminated, as before, by the substitution

$$ G = \tilde{G} e^{\frac{1}{2} \int (\partial_n e^{m\mu} - \partial^m e_{n}^{\mu}) e^p_{\mu} dx_p} \quad (44) $$

leading to the following equation

$$ \frac{g^2}{4} \left[ \delta^a_b \delta^m_n \partial_p \partial^p + \frac{1}{2} \delta^a_b \partial_p ((\partial_r e^{m\mu} - \partial^m e_{r}^{\mu}) e^p_{\mu}) \right] $$
\[ + \frac{1}{4} \delta^a_b \left[ (\partial_k e^{m\mu} - \partial^m e_k^{\mu}) e^p_{\mu} \right] (\partial_r e^{k\nu} - \partial^k e_r^{\nu}) e_{p\nu} + g f^a_{\beta} \partial_r A^m - \partial^m A^r \right] > + \frac{1}{2} \delta^m_r g^2 f_{\nu\lambda} e_{d\nu} A_{\rho\lambda} A_{\rho d} > \]

\[ \tilde{g}_{bn}^{am}(x, x' ; \sigma) = - \partial_\sigma \tilde{G}_{bn}^{am}(x, x' ; \sigma) \] (45)

The heat kernel becomes a matrix-valued function so assume \( \tilde{G} \) to be of the form

\[ \tilde{G}_{bn}^{am}(x, x' ; \sigma) = G^o(x, x' ; \frac{4}{g^2} \sigma)(e^{T(x, x' ; \sigma)})^{am}_{bn} \] (46)

where \( G^o(x, x' ; \sigma) \) denotes the heat-kernel of \( \square_0 \) (thus \( G^o(x, x' ; \frac{4}{g^2} \sigma) \) is the heat kernel of \( \frac{2}{g^2} \square_0 \)) and \( T \) is some matrix \( (T)^{am}_{bn} = T^{am}_{bn} \). Inserting this expression for the heat-kernel into the heat equation we arrive at an equation for \( T^{am}_{bn} \)

\[ \square_0 T^{am}_{bn} + (\partial_p T^{am}_{bn}) (\partial^p T^{b'k}_{bn}) + O^{am}_{bn} = - \frac{\partial}{\partial \sigma} T^{am}_{bn} \] (47)

where a summation over repeated indices is understood and where the fact that, along the diagonal \( x = x' \), the flat space heat kernel is constant, and hence its derivatives vanish, has been used to eliminate the term \( 2 \partial_\nu G^o \partial^\nu T \)

We will furthermore write \( T^{am}_{bn} \) as a Taylor series

\[ T^{am}_{bn}(x, \sigma) = \sum_{\nu=0}^{\infty} \tau^{(\nu)am}_{bn}(x) \sigma^{\nu} \] (48)

Due to the boundary condition one has, from equations (45,47) and the fact that \( G^o \) satisfy the boundary condition by itself, that

\[ \partial^p \tau^{am}_{0 bn} = - \frac{1}{2} \delta^a_b \left[ (\partial_n e^{m\mu} - \partial^m e_n^{\mu}) e^p_{\mu} \right] \] (49)

and because of this and equation (48) the next coefficient becomes

\[ \tau^{1 am}_{bn} = \delta^a_b \partial_p [(\partial_n e^{m\mu} - \partial^m e_n^{\mu}) e^p_{\mu}] - \frac{1}{2} \delta^a_b [(\partial_k e^{m\mu} - \partial^m e_k^{\mu}) e^p_{\mu}] [(\partial_n e^{k\mu} - \partial^k e_n^{\mu}) e_{p\mu}] + f^{am}_{bn}(< A >) \] (50)
Finally, inserting the expansion (49) into the equation (48) for \(T\), yields a recursion relation for the higher order coefficients, \(\tau^{(\nu)am}_{bn}\),

\[
\tau^{(\nu)am}_{bn} = -\frac{1}{\nu + 1} (\Box r^{(\nu)am}_{bn} + \sum_{\nu' = 0}^{\nu} (\partial^\nu r^{(\nu-\nu')am}_{ck})(\partial^\nu r^{(\nu')bn}_{ck}) ; n > 1 (51)
\]

6 Heat Kernel Equation and Zeta-Function for spin 1/2 fermions

In order not to get calculational constipation, when relating the zeta-function of a fermionic field to that of a scalar one, I shall do the calculation the Quick 'n' Dirty way (in equation (54)).

6.1 Free spin 1/2 fermions

The zeta-function of the operator \(A\) is

\[
\zeta_A(s) = \sum_{\lambda} \lambda^{-s}
\]

where \(\lambda\) denotes the corresponding eigenvalues, \(A\psi_\lambda = \lambda\psi_\lambda\), and consequently, we also have the zeta-function for the operator \(A^2\) (with eigenvalues \(\lambda^2\))

\[
\zeta_{A^2}(s) = \sum_{\lambda} (\lambda^2)^{-s} = \zeta_A(2s)
\]

Now note that for a free fermion field the Dirac operator is

\[
\nabla = \gamma^m e^\mu_m (\partial_\mu + \frac{i}{2} \omega^{pq}_\mu(x)X_{pq})
\]

Representing the \(SO(3,1)\) generators in terms of the sigma matrices, \(\sigma_{pq} = \frac{i}{4} [\gamma_p, \gamma_q]\), one obtains for the derivative squared

\[
\mathbf{\nabla}^2 = (\Box + \xi_f R) \cdot 1_4
\]

(where \(1_4\) is the four dimensional unit matrix and \(\xi_f = 1/8\)) establishing the link between the scalar and the fermion cases if one remembers to include a factor of 4 (one for each spinor component) in the Dirac
case (corresponding to taking the trace over the unit matrix, cf a small
generalisation of equation (8)). Thus we find
\[
\zeta_\nu = \zeta_\nu \left( \frac{1}{2} s \right) = 4 \zeta_{\nu}^{\text{scalar}} \left( \frac{s}{2} \right)
\]
(56)
which, by use of section 4, makes it possible to determine the Casimir
free energy of a free fermionic field if one remembers to take into account
that for a Grassman field, the free energy is:
\[
F = -\frac{1}{\beta} \left( \ln(\det \nabla) \right)^{1/2}
\]
(57)

6.2 Fermions minimally coupled to a gauge field
For real world purposes one should consider the case where the fermion
field couples to a gauge boson field. The gauge invariant and space
covariant derivative is
\[
D_m = \epsilon_m^\mu (\partial_\mu + \frac{i}{2} \omega_{\mu}^{pq}(x) X_{pq} + ig A_\mu^a(x) T_a)
\]
(58)
one obtains, for the derivative squared;
\[
D^2 = \Box + \xi_f R + 2g \sigma^{pq} F_{pq} T_a + g \eta^{pq} A_\mu^a A_\mu^b T_a T_b + g (\partial^p A_p^a) T_a

+ \frac{g}{2} T_a \left( \sigma_{pq} \omega_{\mu}^{pq} e^x A^a_{\mu} + i \omega_{\mu}^{mn} (\epsilon_{\mu}^a A_n^a - \epsilon_{\mu}^n A_m^a) - 4 \omega_{\mu}^{pq} \epsilon_{\mu} A_n^a \sigma_{pq} \right)

\equiv \Box + \xi_f R + 2g \sigma^{pq} F_{pq} T_a + g \eta^{pq} A_\mu^a A_\mu^b T_a T_b + G(A)
\]
(59)
One can show that putting (the "carpet gauge")
\[
G(A) = 0
\]
(60)
is an allowed gauge condition (by demonstrating that \( \det(\frac{\delta G}{\delta A}) \neq 0 \)) [1].
Using the mean field approximation described in subsection 5.1, the
gauge field dependent terms simply becomes a function which is in prin-
ciple no different from the \( \xi_f R \) term to which it, for calculational pur-
poses, can be added.
6.3 Fermions minimally coupled to a gauge field in the presence of (background) torsion

The presence of (background) torsion, defined by equation (43), results in a small change in the calculation of the derivative squared of the preceding subsection. Explicitly:

\[ \mathcal{D}^2 = \gamma^m D_m \gamma^n D_n = \eta^{mn} D_m D_n + i \sigma^{mn} [D_m, D_n] + \gamma^m (D_m \gamma^n) D_n \quad (61) \]

which upon use of equation (43) and calculations as in the preceding subsection becomes

\[ \mathcal{D}^2 = \Box + \xi f R + 2g X X \sigma^{pq} F_{pq} + g \eta^{pq} A_q^a A_p^b T_a T_b + G(A) + i \sigma^{mn} S_{mn} D_p \quad (62) \]

From a calculational viewpoint the complication due to torsion thus consists of adding to the operator in the heat kernel equation a first order term which can be removed by the same token as it was done for the vector bosons (the presence of a sigma-matrix ensures the same complications) and two zero-order terms, one of them with a reference to the (meanfield of the) vector bosons.

7 Conclusion

I have outlined a method for explicitly calculating the Casimir energy for quantum fields on an arbitrary curved space background. The expansion of the heat kernel and consequently of the zeta-function and Casimir energy involves higher and higher order derivatives of vierbeins i.e. curvature (and of gauge fields, if present) which suggests that only the first few terms need be taken into account, in accordance with the results presented. In the case of interacting quantum fields I resorted to a meanfield approximation in order to be able perform the path integrals involved and a procedure for determining the meanfields was outlined. The first approximation to the mean field treats the quantum field as a free one, subsequent approximations do not. One would think this to be in line with what one would get calculating Feynman diagrams in curved space (could it be done) because interaction would involve a calculation to higher loop order and we therefore expect the results to be fairly reliable.
References
[1] F.Antonsen & K.Bormann: Casimir Driven Evolution of the Universe
(to be submitted to Phys.Rev.D)