A note on another approach on power sums

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Abstract

In this note, we first review the novel approach to power sums put forward recently by Muschielok in arXiv:2207.01935v1, which can be summarized by the formula

\[ S_m(n) = \sum_k c_{mk} \psi_k(n), \]

where the \( c_{mk} \)'s are the expansion coefficients and where the basis functions \( \psi_k(n) \) fulfil the recursive property \( \psi_k(n+1) = \sum_{i=1}^n \psi_k(i) \). Then, we point out a number of supplementary facts concerning the said approach not contemplated explicitly in Muschielok’s paper. In particular, we show that, for any given \( m \), the values of the \( c_{mk} \)'s can be obtained by inverting a matrix involving only binomial coefficients. This may be compared with the original approach of Muschielok, where the values of the \( c_{mk} \)'s can be obtained by inverting a lower triangular matrix involving the Stirling numbers of the first kind. Also, we make a conjecture about the functional form of the coefficients \( c_{m,m-k} \).

1 Introduction

For integers \( m, a \geq 0 \) and \( n \geq 1 \), the \( a \)-fold summation \( S_m(n) \) (or hyper-sum) over the first \( n \) positive integers to the \( m \)-th powers is defined recursively as

\[ S_m(n) = \begin{cases} \phantom{=} n^m & \text{if } a = 0, \\ \sum_{i=1}^n S_{m-1}(i) & \text{if } a \geq 1. \end{cases} \]

Several methods to obtain explicit formulas for \( S_m(n) \) have been described in the literature (see, e.g., [1, 2, 3, 4, 6, 7, 8]). Recently, in a very interesting paper, C. Muschielok [10] has developed another approach on the iterated power sums \( S_m(n) \). Briefly, Muschielok’s procedure is as follows.

First, define the polynomial sequence

\[ \psi_m(n) = n + (m-1)(n-1)B_{m-1,n-1}, \]

where the binomial coefficient \( B_{a,b} \) is given by

\[ B_{a,b} = \binom{a+b}{a} = \binom{a+b}{b}. \]

Note that \( \psi_m(n) \) can alternatively be expressed as

\[ \psi_m(n) = B_{1,n-1} + m(m-1)B_{m,n-2}, \]

where it is understood that \( B_{m,n-2} = 0 \) when \( n = 1 \). Similarly to \( S_m(n) \), we can define recursively the polynomial series of \( a \)-th order \( \psi_m(n) \) with respect to the sequence \( \psi_m(n) \) as

\[ \psi_m(n) = \begin{cases} \phantom{=} \psi_m(n) & \text{if } a = 0, \\ \sum_{i=1}^n \psi_{m-1}(i) & \text{if } a \geq 1. \end{cases} \]
Furthermore, using the property \( B_{a,b} = \sum_{\beta=1}^{b} B_{a-1,\beta} \), it can be shown that the polynomials \( \psi_m^{(a)}(n) \) have the closed form [10, Lemma 3]

\[
\psi_m^{(a)}(n) = B_{a+1,n-1} + \frac{m(m-1)}{m+a}(n-1)B_{m+a-1,n-1}.
\] (1)

Having established the \( \psi_m^{(a)}(n) \)'s, the next step of Muschielok’s procedure is to express the monomial \( n^m \) as a linear combination of the polynomial basis \( \psi_k(n) \)'s, that is,

\[
n^m = \sum_{k=0}^{m} c_{mk} \psi_k(n),
\]

for some certain coefficients \( c_{mk} \). Hence, from the preceding equation, it immediately follows that

\[
S_m^{(a)}(n) = \sum_{k=0}^{m} c_{mk} \psi_k^{(a)}(n).
\]

In order to find the coefficients \( c_{mk} \), one expands \( \psi_m(n) \) in terms of powers of \( n \), that is,

\[
\psi_m(n) = \sum_{i=0}^{m} a_{mi} n^i.
\]

As this point, it should be noted that, as shown in [10], \( a_{m,0} = a_{m,1} = 0 \). Therefore, in the following (as is done in [10]), and without loss of generality, we restrict ourselves to the case where \( m \geq 2 \). Clearly, the bases \( \{n^k\}_{k=2}^{m} \) and \( \{\psi_k(n)\}_{k=2}^{m} \) form a pair of dual bases, and thus we have

\[
\sum_{i=2}^{m} a_{mi} c_{ij} = \delta_{mj}, \quad j = 2, 3, \ldots, m,
\]

or, equivalently,

\[
c_{mk} = \begin{cases} 
1/a_{mm} = (m-2)!, & \text{if } k = m; \\
-(m-2)! \sum_{l=k}^{m-1} a_{ml} c_{lk}, & \text{if } 2 \leq k \leq m - 1; \\
0, & \text{else.}
\end{cases}
\] (2)

The recurrence equation (2) enables one to obtain the coefficients \( c_{mk} \) in terms of the \( a_{ml} \)'s and the earlier coefficients \( c_{lk} \). Alternatively, as pointed out in [10], we can build the truncated matrix \( A_m = (a_{ml}) \) and calculate its inverse. As we shall presently see, the \( a_{ml} \)'s can be expressed in terms of the Stirling numbers of the first kind. For completeness, below we write down the matrix \( C_{10} = (c_{mk})_{2 \leq m,k \leq 10} \) containing the coefficients \( c_{mk} \) for \( m = 2, 3, \ldots, 10 \), and where, for each involved \( m \), the index \( k \) ranges from 2 to \( m \):

\[
C_{10} = \begin{pmatrix}
1 & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad \\
0 & 1 & \quad & \quad & \quad & \quad & \quad & \quad & \quad & \quad \\
1 & -2 & 2 & \quad & \quad & \quad & \quad & \quad & \quad & \quad \\
0 & 5 & -10 & 6 & \quad & \quad & \quad & \quad & \quad & \quad \\
1 & -10 & 40 & -54 & 24 & \quad & \quad & \quad & \quad & \quad \\
0 & 21 & -140 & 336 & -336 & 120 & \quad & \quad & \quad & \quad \\
1 & -42 & 462 & -1764 & 3024 & -2400 & 720 & \quad & \quad & \quad \\
0 & 85 & -1470 & 8442 & -22176 & 29520 & -19440 & 5040 & \quad & \quad \\
1 & -170 & 4580 & -38178 & 144648 & -288000 & 313200 & -176400 & 40320 & \quad \\
\end{pmatrix}. 
\]
In this way, provided with the coefficients \( c_{mk} \), we can obtain \( S_m^{(a)}(n) \) from

\[
S_m^{(a)}(n) = \sum_{k=2}^{m} c_{mk} \psi_k^{(a)}(n), \quad m \geq 2,
\]

with the basis functions \( \psi_k^{(a)}(n) \) being given explicitly in (1).

**Remark 1.** The expansion coefficients \( c_{mk} \) correspond to the sequence A355570 in OEIS (The On-Line Encyclopedia of Integer Sequences).

In the next section, we present a number of additional facts concerning Muschielok’s approach not appearing in [10]. In particular, we show that, for any given \( m \), the numerical values of the coefficients \( c_{m2}, c_{m3}, \ldots, c_{mm} \) in the summation formula (3) can be obtained by inverting a matrix involving only binomial coefficients (see equation (9) below). Moreover, we make a conjecture about the functional form of the coefficients \( c_{mm-k} \), with \( k \geq 0 \) and \( m \geq k + 2 \).

## 2 Facts and Conjecture

In what follows, we point out several facts concerning Muschielok’s summation procedure, along with a conjecture.

**Fact 1.** When \( a = 0 \), equation (3) becomes

\[
S_m^{(a)}(n) = n^m = \sum_{k=2}^{m} c_{mk} \psi_k(n).
\]

Since \( \psi_k(1) = 1 \), it immediately follows that

\[
\sum_{k=2}^{m} c_{mk} = 1, \quad \text{for all } m \geq 2.
\]

Property (4) can be readily checked for each of the rows in the matrix \( C_{10} \) above.

**Fact 2.** When \( m = 2 \), from (3) and (1) we obtain

\[
S_2^{(a)}(n) = c_{22} \psi_2^{(a)}(n) = B_{a+1,n-1} + \frac{2}{a+2}(n-1)B_{a+1,n-1} = \frac{2n+a}{a+2}B_{a+1,n-1} = \frac{2n+a}{a+2} S_1^{(a)}(n),
\]

in accordance with the result for \( S_2^{(a)}(n) \) given in [8, p. 281].

**Fact 3.** For \( m \geq 2 \) and \( l = 2, 3, \ldots, m \), the coefficients \( a_{ml} \) are given explicitly by

\[
a_{ml} = \frac{1}{(m-2)!} \left( \left[ \begin{array}{c} m-1 \\ l-1 \end{array} \right] - \left[ \begin{array}{c} m-1 \\ l \end{array} \right] \right),
\]

or, equivalently,

\[
a_{ml} = \frac{1}{(m-2)!} \left( \left[ \begin{array}{c} m \\ l \end{array} \right] - m \left[ \begin{array}{c} m-1 \\ l \end{array} \right] \right),
\]

where \( \left[ \begin{array}{c} m \\ l \end{array} \right] \) denotes the (unsigned) Stirling numbers of the first kind. Using either (5) or (6), we can construct the matrix \( A_m = (a_{ml}) \). For example, the matrix \( A_{10} = (a_{ml})_{2 \leq m,l \leq 10} \) whose inverse is
$C_{10}$, is given by

$$A_{10} = \begin{pmatrix}
1 & 0 & 1 \\
-\frac{1}{2} & 1 & \frac{1}{2} \\
-\frac{5}{6} & 0 & \frac{5}{6} & \frac{1}{6} \\
-\frac{13}{12} & 5 & \frac{25}{24} & \frac{3}{8} & \frac{1}{24} \\
\frac{77}{60} & \frac{49}{120} & \frac{7}{6} & \frac{7}{12} & \frac{1}{60} & \frac{1}{120} \\
-\frac{29}{20} & \frac{7}{36} & \frac{889}{720} & \frac{7}{9} & \frac{77}{360} & \frac{1}{36} & \frac{1}{720} \\
-\frac{223}{140} & -\frac{1}{10} & \frac{101}{80} & \frac{229}{120} & \frac{13}{70} & \frac{3}{280} & \frac{1}{5040} \\
-\frac{481}{280} & -\frac{61}{288} & \frac{1274}{1008} & \frac{437}{384} & \frac{853}{1920} & \frac{19}{172} & \frac{17}{1344} & \frac{1}{1182} & \frac{1}{40320}
\end{pmatrix}.$$

Let us further note, incidentally, that the elements of the first column of $A_m$ are given by $a_{m2} = 1 - H_{m-2}$, $m \geq 2$, where $H_m$ denotes the $m$-th harmonic number.

**Fact 4.** By substituting the coefficient $a_{ml}$ in (2) by its expression in either (5) or (6), one can derive explicit formulas for the coefficients $c_{mm}, c_{m\,m-1}, c_{m\,m-2}, c_{m\,m-3}$, etc., in succession. It should be noticed, however, that the complexity of the calculation of $c_{m\,m-k}$ grows rapidly with $k$.

Next, we quote the exact formula of $c_{m\,m-k}$ for $k = 0, \ldots, 7$:

- $c_{mm} = (m - 2)!$, $m \geq 2$,
- $c_{m\,m-1} = (m - 3)! \left(3 - m\right) \frac{1}{2}m$, $m \geq 3$,
- $c_{m\,m-2} = (m - 4)! \left(\begin{pmatrix} m \\ 2 \end{pmatrix} \left[ \frac{1}{4}m^2 - \frac{23}{12}m + \frac{46}{12} \right] \right)$, $m \geq 4$,
- $c_{m\,m-3} = (m - 5)! (5 - m) \left(\begin{pmatrix} m \\ 3 \end{pmatrix} \left[ \frac{1}{8}m^2 - \frac{9}{8}m + \frac{11}{4} \right] \right)$, $m \geq 5$,
- $c_{m\,m-4} = (m - 6)! \left(\begin{pmatrix} m \\ 4 \end{pmatrix} \left[ \frac{1}{16}m^4 - \frac{11}{8}m^3 + \frac{553}{48}m^2 - \frac{1747}{40}m + \frac{1901}{30} \right] \right)$, $m \geq 6$,
- $c_{m\,m-5} = (m - 7)! (7 - m) \left(\begin{pmatrix} m \\ 5 \end{pmatrix} \left[ \frac{1}{32}m^4 - \frac{37}{48}m^3 + \frac{697}{96}m^2 - \frac{1489}{48}m + \frac{611}{12} \right] \right)$, $m \geq 7$,
- $c_{m\,m-6} = (m - 8)! \left(\begin{pmatrix} m \\ 6 \end{pmatrix} \left[ \frac{1}{64}m^6 - \frac{43}{64}m^5 + \frac{775}{64}m^4 - \frac{67513}{576}m^3 \\
+ \frac{1930}{3}m^2 - \frac{1916141}{1008}m + \frac{198721}{84} \right] \right)$, $m \geq 8$,
- $c_{m\,m-7} = (m - 9)! (9 - m) \left(\begin{pmatrix} m \\ 7 \end{pmatrix} \left[ \frac{1}{128}m^6 - \frac{47}{128}m^5 + \frac{2777}{384}m^4 - \frac{88093}{1152}m^3 \\
+ \frac{14669}{32}m^2 - \frac{425993}{288}m + \frac{16083}{8} \right] \right)$, $m \geq 9$. 
Motivated by the patterns exhibited by the above expressions for $c_{m,m-k}$, $k = 0, \ldots, 7$, we propose the following conjecture regarding the functional form of the coefficients $c_{m,m-k}$:

**Conjecture 1.**

- For all integers $k \geq 0$ and $m \geq 2k + 2$, we have
  
  \[ c_{m,m-2k} = (m - (2k + 2))! \left( \frac{m}{2k} \right)^{2k} \sum_{j=0}^{2k} \Gamma_{m,j} m^j, \]

  where the non-zero (rational) coefficients in the set $\{\Gamma_{m,0}, \Gamma_{m,1}, \ldots, \Gamma_{m,2k}\}$ have alternating signs, with the leading coefficient $\Gamma_{m,2k} = 1/2^{2k}$ being positive. Furthermore, the polynomial $\sum_{j=0}^{2k} \Gamma_{m,j} m^j$ is always a (rational) positive number, and the coefficient $c_{m,m-2k}$ is a positive integer for all $m \geq 2k + 2$. In particular, for $m = 2k + 2$, we have $c_{2k+2,2} = 1$ for all $k \geq 0$.

- For all integers $k \geq 0$ and $m \geq 2k + 3$, we have
  
  \[ c_{m,m-(2k+1)} = (m - (2k + 3))! (2k + 3 - m) \left( \frac{m}{2k + 1} \right)^{2k} \sum_{j=0}^{2k} \Upsilon_{m,j} m^j, \]

  where the non-zero (rational) coefficients in the set $\{\Upsilon_{m,0}, \Upsilon_{m,1}, \ldots, \Upsilon_{m,2k}\}$ have alternating signs, with the leading coefficient $\Upsilon_{m,2k} = 1/2^{2k+1}$ being positive. Furthermore, the polynomial $\sum_{j=0}^{2k} \Upsilon_{m,j} m^j$ is always a (rational) positive number, and the coefficient $c_{m,m-(2k+1)}$ is a negative integer for all $m \geq 2k + 4$, whereas, for $m = 2k + 3$, we have that $c_{2k+3,2} = 0$ for all $k \geq 0$.

**Fact 5.** By using (1) and (3), and taking into account the property (4) and the definition of $B_{a,b}$, we can express $S_m^{(a)}(n)$ in the form

\[ S_m^{(a)}(n) = \frac{n + a}{a + 1} \left[ 1 + (n - 1)(a + 1) \sum_{k=2}^{m} c_{mk} \binom{a + k}{k}^{-1} \binom{n + a + k - 2}{k - 2} \right], \quad m \geq 2. \quad (7) \]

In particular, when $a = 1$, the above expression yields the following formula for the ordinary power sums $S_m^{(1)}(n) = 1^n + 2^n + \cdots + n^m$:

\[ S_m^{(1)}(n) = \frac{1}{2} n(n + 1) \left[ 1 + 2(n - 1) \sum_{k=2}^{m} c_{mk} \binom{n + k - 1}{k - 2} \right], \quad m \geq 2. \]

Note that formula (7) gives us $S_m^{(a)}(n)$ as $S_m^{(a)}(n)$ times a polynomial in $n$ of degree $m - 1$. Furthermore, formula (7) tells us that $S_m^{(a)}(1) = 1$ for all integers $a \geq 0$ and $m \geq 2$. (By the way, $S_m^{(a)}(1) = 1$ for all integers $a \geq 0$ and $m \geq 0$.)

**Fact 6.** For $a = 0$, the formula (7) reduces to

\[ n^m = n \left[ 1 + (n - 1) \sum_{k=2}^{m} c_{mk} \binom{n + k - 2}{k - 2} \right], \]
which can be written in the form
\[ \sum_{k=2}^{m} c_{mk} \binom{n+k-2}{n} = \frac{n^{m-1} - 1}{n-1}, \]  
(8)

provided that \( m \geq 2 \) and \( n \geq 2 \). Letting successively \( n = 2, 3, \ldots, m \) in (8) yields the following linear system of \( m - 1 \) equations in the unknowns \( c_{m2}, c_{m3}, \ldots, c_{mm} \):
\[ \sum_{k=2}^{m} \binom{k}{2} c_{mk} = 2^{m-1} - 1, \]
\[ \sum_{k=2}^{m} \binom{k+1}{3} c_{mk} = \frac{3^{m-1} - 1}{2}, \]
\[ \vdots \]
\[ \sum_{k=2}^{m} \binom{m+k-2}{m} c_{mk} = \frac{m^{m-1} - 1}{m-1}. \]

Hence, solving for the coefficients \( c_{mk} \), we are left with the matrix equation
\[
\begin{pmatrix}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{pmatrix} \begin{pmatrix}
\binom{2}{2} \\
\binom{3}{3} \\
\binom{4}{4} \\
\vdots \\
\binom{m}{m} \\
\end{pmatrix} = \begin{pmatrix}
\binom{m-1}{2} \\
\binom{m}{3} \\
\vdots \\
\binom{2m-3}{m} \\
\binom{2m-2}{m} \\
\end{pmatrix},
\]  
(9)

from which we can obtain, for any given \( m \), the corresponding values of the \( c_{mk} \)'s. For example, for \( m = 10 \), the matrix equation (9) reads

\[
\begin{pmatrix}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{pmatrix} = \begin{pmatrix}
1 & 10 & 28 & 36 & 45 \\
3 & 35 & 64 & 120 & 165 \\
5 & 70 & 126 & 210 & 330 \\
7 & 126 & 252 & 462 & 792 \\
9 & 182 & 396 & 756 & 1374 \\
11 & 268 & 715 & 2002 & 5005 \\
13 & 360 & 792 & 1716 & 3432 \\
15 & 455 & 1001 & 2025 & 4045 \\
17 & 555 & 1201 & 2402 & 4804 \\
19 & 666 & 1401 & 2802 & 5605 \\
\end{pmatrix}^{-1} \begin{pmatrix}
511 \\
9841 \\
87381 \\
488281 \\
2015539 \\
6725601 \\
19173961 \\
48427561 \\
11111111 \\
\end{pmatrix},
\]

\[
\begin{pmatrix}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{pmatrix} = \begin{pmatrix}
1 \\
-170 \\
4580 \\
-38178 \\
144648 \\
-288000 \\
313200 \\
-176400 \\
403200 \\
\end{pmatrix},
\]
thus retrieving the values of $c_{10,2}, c_{10,3}, \ldots, c_{10,10}$ appearing in the last row of the matrix $C_{10}$.

**Fact 7.** Let us recall that $S_m^{(a)}(n)$ admits the following representation in terms of the Stirling numbers of the second kind $\{ \begin{array}{c} m \\ k \end{array} \}$:

$$S_m^{(a)}(n) = \sum_{k=1}^{m} k! \left\{ \begin{array}{c} m \\ k \end{array} \right\} \frac{(n + a)}{(a + k)} , \quad a \geq 0 , \ m \geq 1 ,$$

(see [6]). Therefore, writing the last formula as

$$S_m^{(a)}(n) = \frac{(n + a)}{(a + 1)} \left[ 1 + \left( \frac{n + a}{a + 1} \right)^{-1} \sum_{k=2}^{m} k! \left\{ \begin{array}{c} m \\ k \end{array} \right\} \frac{(n + a)}{(a + k)} \right] ,$$

and comparing it to (7), we obtain the identity

$$\sum_{k=2}^{m} c_{mk} \left( \frac{a + k}{k} \right)^{-1} \frac{(n + a - k + 2)}{(k - 2)} = \frac{1}{n(n-1)} \left( \frac{n + a}{a} \right)^{-1} \sum_{k=2}^{m} k! \left\{ \begin{array}{c} m \\ k \end{array} \right\} \frac{(n + a)}{(a + k)} ,$$

which holds for any integers $a \geq 0$, $m \geq 2$, and $n \geq 2$. In particular, when $a = 0$, the above identity reduces to

$$\sum_{k=2}^{m} c_{mk} \left( \frac{n + k - 2}{k - 2} \right) = \frac{1}{n(n-1)} \sum_{k=2}^{m} k! \left\{ \begin{array}{c} m \\ k \end{array} \right\} \frac{n}{(k)} ,$$

which, of course, is equivalent to (8).

**Fact 8.** Starting from $\psi_m(n) = \sum_{i=0}^{m} a_{mi} n^i$, one can readily obtain the recurrence relation

$$S_m^{(a)}(n) = (m - 2)! \left[ \psi_m^{(a)}(n) - \sum_{i=2}^{m-1} a_{mi} S_i^{(a)}(n) \right] ,$$

which applies to any integers $a \geq 0$ and $m \geq 2$, and where the summation on the right-hand side is zero if $m = 2$. For the case $m = 2$, as shown in Fact 2, we have $S_2^{(a)}(n) = \psi_2^{(a)}(n) = \frac{2n+a}{n+2} (n+a)$.

When $m > 2$, the above recurrence relation gives us $S_m^{(a)}(n)$ in terms of $\psi_m^{(a)}(n)$, the coefficients $a_{mi}$, and the earlier sums $S_i^{(a)}(n)$, $i = 2, 3, \ldots, m - 1$.

**Note added**

In a sequel to [10], Muschielok evaluated sums of the form

$$T_m^{\alpha} = \sum_{k=2}^{m} c_{mk} k^\alpha ,$$

for integer $\alpha \geq 1$. In particular, he showed that $T_m^{1} = \sum_{k=2}^{m} c_{mk} k = m$ ([11, Equation (35)]). Moreover, he obtained the relation ([11, Equation (48)])

$$n^m - n + m = \sum_{i=2}^{n} a_{ni} T_m^{i} ,$$

which completes the proof.
which we write as

\[ n^m - n = -T_m^1 + \sum_{l=2}^{n} a_{nl} T_m^l, \]  

(10)

with \( n \geq 2 \). On the other hand, it is to be noted that (8) can be expressed in the form

\[ n^m - n = \frac{1}{(n-2)!} \sum_{k=2}^{m} c_{nk}(k-1)^{\overline{n}}, \]

where \( n \geq 2 \) and \((k-1)^{\overline{n}} = (k-1)k\cdots(k+n-2)\). Hence, using that \((k-1)^{\overline{n}} = \sum_{j=0}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right] (k-1)^j \)

and applying the binomial theorem to \((k-1)^j\), it follows immediately that

\[ n^m - n = \sum_{l=0}^{n} A_{nl} T_m^l, \]  

(11)

where

\[ A_{nl} = \frac{1}{(n-2)!} \sum_{j=l}^{n} (-1)^{j-l} \left( \begin{array}{c} j \\ l \end{array} \right) \left[ \begin{array}{c} n \\ j \end{array} \right]. \]

Therefore, comparing (10) and (11), we are led to conclude that \( A_{n0} = 0, A_{n1} = -1 \), and, for \( l \geq 2 \), \( A_{nl} = a_{nl} \). Renaming \( n \) as \( m \), the latter means that

\[ a_{ml} = \frac{1}{(m-2)!} \sum_{j=l}^{m} (-1)^{j-l} \left( \begin{array}{c} j \\ l \end{array} \right) \left[ \begin{array}{c} m \\ j \end{array} \right]. \]  

(12)

Incidentally, in view of (5) and (12), we deduce the identity

\[ \sum_{j=l}^{m} (-1)^{j-l} \left( \begin{array}{c} j \\ l \end{array} \right) \left[ \begin{array}{c} m \\ j \end{array} \right] = \left[ \begin{array}{c} m-1 \\ l-1 \end{array} \right] - \left[ \begin{array}{c} m-1 \\ l \end{array} \right], \text{ with } 2 \leq l \leq m. \]

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