ON FRACTIONAL QUADRATIC OPTIMIZATION PROBLEM
WITH TWO QUADRATIC CONSTRAINTS

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Abstract. In this paper, we study the problem of minimizing the ratio of two quadratic functions subject to two quadratic constraints in the complex space. Using the classical Dinkelbach method, we transform the problem into a parametric nonlinear equation. We show that an optimal parameter can be found by employing the S-procedure and semidefinite relaxation technique. A key element to solve the original problem is to use the rank-one decomposition procedure. Finally, within the new algorithm, semidefinite relaxation is compared with the bisection method for finding the root on several examples. For further comparison, the solution of fmincon command of MATLAB also is reported.

1. Introduction. In this paper, we consider the following fractional optimization problem:

\[
\min_{x \in \mathcal{G}} \frac{f_1(x)}{f_2(x)}
\]

where

\[
f_i(x) = x^H A_i x - 2 \text{Re}(b_i^H x) + c_i, \quad i = 1, 2,
\]

\[
\mathcal{G} = \{ x \in \mathbb{C}^n : g_i(x) = x^H B_i x - 2 \text{Re}(d_i^H x) + e_i \leq 0, \quad i = 1, 2 \},
\]

\(A_i, B_i \in \mathbb{C}^{n \times n}\) are complex matrices, \(b_i, d_i \in \mathbb{C}^n\) are complex vectors, \(c_i, e_i \in \mathbb{R}\) are real constants, \(\forall i = 1, 2\). The superscript \(H\) also denotes the conjugate transpose. Furthermore, we require that there exists \(\alpha > 0\) such that \(f_2(x) > \alpha\) for all \(x \in \mathcal{G}\), namely problem (1) is well defined. It is worth to note that in general problem (1) is not convex. Two commonly used approaches to transform (1) to a non-fractional problem, are generalized Charnes-Cooper transformation [9] and Dinkelbach method [11, 16], where the classical complex optimization duality theory

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and optimality conditions can be applied [1, 2]. In this paper, we use the second approach.

Complex optimization problem was established by Levinson [19]. Then, Swarup, et al. [25] developed linear fractional optimization problems in the complex space. This class of problems appears in several real world modelings, such as signal processing including radar detection, design of doppler filters, steering vector estimation and beamforming [3, 4, 5, 6, 8, 10, 14, 28]. For example, Aubry, et al. [3, 4] focused on some generalized fractional optimization and designed a robust filter based on polynomial time solution technique for radar pulse-doppler processing. Baleshan, et al. [5] presented a relationship between the power minimization and signal to noise ratio (SNR) maximization for a wireless network with distributed beamforming.

In what follows, we briefly describe an application where a problem of the form (1) arises [7, 26]. In cooperative beamforming in cognitive radio network with hybrid relay, consider SNR as shown in Figure 1. It includes two parts as: the PN with the PT and the PD, the SN with the ST and the SD and N relays \( \{SR_i\}_{i=1}^N \). All nodes in network have been configured with one antenna and it is further assumed that there is no direct link between the transmitters, the help of relays is necessary to establish the communication link. The dotted lines in Figure 1 represent the interference channels between different transmitters and receivers. Moreover, it is assumed that ST is far from PD and PT is far from SR, thus the interference between them is ignored. We denote the channel between ST and \( SR_i \) (the \( i \)th relay) as \( h_{SR_i} \in C \), and the channel between PT and \( SR_i \) as \( g_{PR_i} \in C \), the channel between \( SR_i \) and SD as \( h_{R_d} \in C \), and the channel between \( SR_i \) and PD as \( g_{R_D} \in C \), \( i = 1, \cdots, N \). The channel between PT and PD is denoted by \( h_{PD} \in C \). Now according to Figure

![Figure 1. Multi-relays cooperative cognitive communication model [7].](image-url)
and the power of the interference and the noise in PD can be written as

\[ I = \sum_{i=1}^{L} \sqrt{P_S} \beta_i w_i g_{PR,i} g_{R,i} \] ^2 + \sum_{i=1}^{N} \beta_i^2 |w_i|^2 |h_{R,i}|^2 N_R + \sum_{i=L+1}^{N} |w_i g_{R,i}|^2 N_D. \]

Equations (2) and (3) are equivalent to:

\[ SNR(SD) = \frac{w^H \hat{h} \hat{h}^H w}{w^H \tilde{g} \tilde{g}^H w + w^H \Phi \Phi^H w + N_d}, \]

\[ I = w^H g g^H w + w^H \tilde{h} \tilde{h}^H w + w^H \Sigma \Sigma^H w + N_D, \]

where

\[ w = [w_1, w_2, \ldots, w_N], \ h = [h_1, h_2, \ldots, h_N]^H, \ \hat{h} = [\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_N]^H, \]

\[ g = [g_1, g_2, \ldots, g_N]^H, \ \tilde{g} = [\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_N]^H, \]

\[ \Phi = \text{diag} [h_{11}, h_{22}, \ldots, h_{NN}]^H, \ \Sigma = \text{diag} [g_{11}, g_{22}, \ldots, g_{NN}]^H, \]

\[ h_i = \begin{cases} \sqrt{P_S} \beta_i h_{SR,i} h_{R,i}, & \text{if } 1 \leq i \leq L \\ h_{R,i}, & \text{if } L + 1 \leq i \leq N \end{cases}, \]

\[ \hat{h}_i = \begin{cases} \sqrt{P_S} \beta_i h_{SR,i} g_{R,i}, & \text{if } 1 \leq i \leq L \\ g_{R,i}, & \text{if } L + 1 \leq i \leq N \end{cases}, \]

\[ g_i = \begin{cases} \sqrt{P_P} \beta_i g_{PR,i} g_{R,i}, & \text{if } 1 \leq i \leq L \\ 0, & \text{if } L + 1 \leq i \leq N \end{cases}, \]

\[ \tilde{g}_i = \begin{cases} \sqrt{P_P} \beta_i g_{PR,i} h_{R,i}, & \text{if } 1 \leq i \leq L \\ 0, & \text{if } L + 1 \leq i \leq N \end{cases}, \]

\[ h_{ii} = \begin{cases} \sqrt{N_R} \beta_i h_{R,i}, & \text{if } 1 \leq i \leq L \\ 0, & \text{if } L + 1 \leq i \leq N \end{cases}, \]

\[ g_{ii} = \begin{cases} \sqrt{N_R} \beta_i g_{R,i}, & \text{if } 1 \leq i \leq L \\ 0, & \text{if } L + 1 \leq i \leq N \end{cases}. \]

Also, the power of the retransmit signal is \(|w_i|^2 \leq P\) for \(i = 1, 2, \ldots, N\) and the total power of the relays is \(\|w\|^2 \leq P_T\), where \(P_T\) is the maximum total transmitted power of relays. Indeed, SNR has been defined as the ratio of the signal power \(S_p\) to the noise power \(N_p\) where the signal power and the noise power of SD is obtained as:

\[ S_p = w^H \hat{h} \hat{h}^H w, \]

\[ N_p = w^H \tilde{g} \tilde{g}^H w + w^H \Phi \Phi^H w + N_d. \]
Then, the problem is formulated as the following optimization problem:

\[
\begin{align*}
\max_{w} & \quad w^Hhh^Hw \\
\text{s.t} & \quad \|w\|^2 \leq P_T, \\
& \quad I \leq I_{th},
\end{align*}
\]

where \(I_{th}\) is the maximum interference power at PD.

Since the set of quadratically constrained quadratic optimization problems is NP-hard, thus the quadratically constrained ratio of two quadratic functions as a special case of it also belongs to the class of NP-hard problems. Cai, et al. [7] first changed the complex fractional problem into a non-fractional one. Then, they used SDO relaxation and obtained the exact optimal solution. The authors in [22] studied quadratic fractional optimization problem with a quadratic constraint. They have applied Dinkelbach method and proposed a bisection method and a generalized Newton algorithm to solve the parametric problem. In [12], the problem of minimizing the ratio of two complex indefinite quadratic functions subject to a strictly convex quadratic constraint is studied. First, After reformulating the fractional problem as a univariate equation, to find the root of the univariate equation, the generalized Newton method is utilized that requires solving a nonconvex quadratic optimization problem at each iteration. To solve these nonconvex quadratic problems, the authors have presented an efficient algorithm by a diagonalization scheme that requires solving a univariate minimization problem at each iteration. It is worth to note also that Nguyen, et al. [20] considered the problem in which the ratio of two indefinite quadratic functions in real space is minimized subject to a two-sided quadratic constraint. Using the relationship between fractional and parametric optimization problems, they proposed the stronger version of the extended S-Lemma to achieve the optimal parameter.

In this paper, we use the classical Dinkelbach method and transform the main problem into a non-fractional one in Section 2. Using the S-Lemma, we show that the optimal parameter can be computed by solving an SDO relaxation problem. Then by solving another SDO relaxation problem and using the rank-one decomposition procedure of [15], the optimal solution of the original problem is extracted. We present the new algorithm and the bisection method in Section 3. Finally in Section 4, we give some numerical results for two sets of examples.

Throughout the paper, \(H^n\) and \(H^n_+\) denote the space of \(n \times n\) complex Hermitian matrices and complex Hermitian positive semidefinite matrices, respectively. The notation \(C \succeq 0(\succ)\) means \(C\) is a positive semidefinite (definite) matrix. For two complex matrices \(E\) and \(D\), their inner product is defined as

\[
E \cdot D = \text{Re}(\text{tr} \ E^H D) = \text{tr} \ [(\text{Re} \ E)^T (\text{Re} \ D) + (\text{Im} \ E)^T (\text{Im} \ D)],
\]

where \(\text{tr} (\cdot)\) denotes the trace of a matrix, \(T\) denotes the transpose of a matrix and \(\text{Re} \ Y\) and \(\text{Im} \ Y\) stand for the real and imaginary parts of \(Y \in H^n\).

2. Preliminaries. Let \(\mathcal{G} \subset \mathbb{C}^n\) be a given set. We denote \(C_+(\mathcal{G})\) as all Hermitian matrices which are co-positive over \(\mathcal{G}\), i.e.

\[
C_+(\mathcal{G}) = \{ \Lambda \in H^n : w^H \Lambda w \geq 0, \ \forall w \in \mathcal{G} \}.
\]
Obviously, \( C_+ (G) \) is a closed convex cone in \( H^n \). We define \( FC_+ (G) \), the cone of complex quadratic functions which are non-negative on \( G \) as follows:

\[
FC_+ (G) = \left\{ \begin{pmatrix} e & -d^H \\ -d & B \end{pmatrix} : w^H B w - 2 \text{Re}(d^H w) + e \geq 0, \forall w \in G \right\}.
\]

Moreover, for a quadratic function \( g(w) = w^H B w - 2 \text{Re}(d^H w) + e \), we denote its matrix representation by

\[
g(w) = M(g(.)) \bullet \begin{pmatrix} 1 \\ w \end{pmatrix} w^H w,
\]

where

\[
M(g(.)) = \begin{pmatrix} e & -d^H \\ -d & B \end{pmatrix}.
\]

Clearly, \( g(w) \geq 0 \) for all \( w \in G \) if only if \( M(g(.)) \in FC_+ (G) \).

For our further development, the following theorem which is an extended version of the S-procedure [15] plays an important role. One may see [17, 18, 21] for some recent developments of S-procedure.

**Theorem 2.1.** ([15]) Let

\[
G = \{ w \in \mathbb{C}^n : g_j(w) = w^H B_j w - 2 \text{Re}(d^H_j w) + e_j \leq 0, \ j = 1, 2 \}.
\]

Suppose that there exists \( w_0 \in \mathbb{C}^n \) such that \( g_1(w_0) < 0 \) and \( g_2(w_0) < 0 \), and there is no \( w \neq 0 \) such that \( w^H B_1 w = 0 \) and \( w^H B_2 w = 0 \). Then

\[
FC_+ (G) = \{ W : \exists \lambda_1, \lambda_2 \geq 0; W + \lambda_1 M(g_1(.)) + \lambda_2 M(g_2(.)) \succeq 0 \}.
\]

2.1. **Parametric approach.** The following proposition is a key element for our algorithmic devolvement.

**Proposition 2.1.** ([27]) The following two statements are equivalent:

1.

\[
\inf_{x \in \mathbb{R}} \frac{f_1(x)}{f_2(x)} = \lambda^*.
\]

2.

\[
\mathcal{F}(\lambda^*) := \inf_{x \in \mathbb{R}} \{ f_1(x) - \lambda^* f_2(x) \} = 0.
\] (7)

Thus we focus on the parametric optimization problem (7) instead of the main fractional optimization problem (1). In the following theorem we outline some properties of the univariate function \( \mathcal{F}(\lambda) \).

**Theorem 2.2.** ([27]) The following statements hold.

1. \( \mathcal{F} \) is concave over \( \mathbb{R} \).
2. \( \mathcal{F} \) is continuous at any \( \alpha \in \mathbb{R} \).
3. \( \mathcal{F} \) is strictly decreasing.
4. \( \mathcal{F}(\alpha) = 0 \) has a unique solution.
3. Main results. First, we give the following lemmas generalizing some primary results in fractional optimization. Then we apply the S-procedure and SDO relaxation to compute $\lambda^*$. Moreover we show strong duality holds for (7).

**Lemma 3.1.** [boundedness] ([23]) Suppose that problem (1) is well defined. It is bounded below if and only if there exists $\lambda \in \mathbb{R}$ such that $F(\lambda) \geq 0$. Furthermore, if $\lambda^* > -\infty$; then $\lambda^* = \max_{F(\lambda) \geq 0} \lambda$.

**Lemma 3.2.** [attainment] ([23]) Suppose that problem (1) is well defined. Then, $\lambda^*$ is attained at $x \in \mathcal{G}$ if and only if $\lambda^*$ is a root of $F(\lambda)$ and $x^*$ is an optimal solution to (7).

In order to be able to apply Theorem 2.1, we make the following assumption.

**Assumption A.** There is no $x \neq 0$ such that $x^H B_1 x = 0$ and $x^H B_2 x = 0$.

**Remark 1.** A large set of $B_1$ and $B_2$ matrices satisfy Assumption A. For example when one of the matrices is positive definite or negative definite.

**Lemma 3.3.** If $B_1$ and $B_2$ are semidefinite matrices and $\mathcal{N}(B_1) \cap \mathcal{N}(B_2) = \{0\}$, then Assumption A holds.

**Proof.** In contrary, suppose Assumption A does not hold. Then

$$\exists x \neq 0: x^H B_i x = 0 \Rightarrow B_i x = 0 \Rightarrow x \in \mathcal{N}(B_i), \ i = 1, 2,$$

so $x \in \mathcal{N}(B_1) \cap \mathcal{N}(B_2)$ which is a contradiction. \qed

**Theorem 3.4.** For any well defined problem (1) satisfying the strict feasibility condition and Assumption A

$$\lambda^* = \sup_{\lambda \in \mathbb{R}, \mu_j \geq 0, j = 1, 2} \left\{ \lambda : \left( \begin{array}{c} c_1 - \lambda c_2 + \sum_{j=1}^{2} \mu_j e_j \ - b_1^H + \lambda b_2^H - \sum_{j=1}^{2} \mu_j d_j^H \\ -b_1 + \lambda b_2 - \sum_{j=1}^{2} \mu_j d_j \end{array} \right) \mathcal{A}_1 - \lambda \mathcal{A}_2 + \sum_{j=1}^{2} \mu_j \mathcal{B}_j \right\} \geq 0 \right\}.$$

**Proof.** We have

$$\lambda^* = \inf_{f_1(x), f_2(x)} \left\{ \lambda : f_1(x) \leq 0, \ j = 1, 2 \right\},$$

$$= \sup_{\lambda \in \mathbb{R}} \left\{ \lambda : \{ x \in \mathbb{C}^n \mid f_1(x) < \lambda, g_j(x) \leq 0, j = 1, 2 \} = \emptyset \right\},$$

$$= \sup_{\lambda \in \mathbb{R}} \left\{ \lambda : \{ x \in \mathbb{C}^n \mid f_1(x) - \lambda f_2(x) < 0, g_j(x) \leq 0, j = 1, 2 \} = \emptyset \right\},$$

$$= \sup_{\lambda \in \mathbb{R}} \left\{ \lambda : \exists \mu_j \geq 0, j = 1, 2, \forall x \in \mathbb{C}^n, f_1(x) - \lambda f_2(x) + \mu_1 g_1(x) + \mu_2 g_2(x) \geq 0 \right\},$$

$$= \sup_{\lambda \in \mathbb{R}} \left\{ \lambda : \left( \begin{array}{c} c_1 - b_1^H \\ -b_1 \end{array} \right) + \lambda \left( \begin{array}{c} e_1 - b_2^H \\ -b_2 \end{array} \right) + \mu_1 \left( \begin{array}{c} e_1 - d_1^H \\ -d_1 \end{array} \right) + \mu_2 \left( \begin{array}{c} e_2 - d_2^H \\ -d_2 \end{array} \right) \geq 0 \right\},$$

$$= \sup_{\lambda \in \mathbb{R}} \left\{ \lambda : \left( \begin{array}{c} c_1 - b_1^H \\ -b_1 \end{array} \right) + \lambda \left( \begin{array}{c} e_1 - b_2^H \\ -b_2 \end{array} \right) + \mu_1 \left( \begin{array}{c} e_1 - d_1^H \\ -d_1 \end{array} \right) + \mu_2 \left( \begin{array}{c} e_2 - d_2^H \\ -d_2 \end{array} \right) \geq 0 \right\}.$$
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\[
\sup_{\lambda \in \mathbb{R}, \mu_j \geq 0, j = 1, 2} \left\{ \lambda \left| \begin{array}{ccc} c_1 - \lambda c_2 + \sum_{j=1}^{2} \mu_j e_j & -b_1^H + \lambda b_2^H - \sum_{j=1}^{2} \mu_j d_j^H \\ -b_1 + \lambda b_2 - \sum_{j=1}^{2} \mu_j d_j & A_1 - \lambda A_2 + \sum_{j=1}^{2} \mu_j B_j \end{array} \right| \right\} \geq 0,
\]

where (10) follows from Theorem 2.1.

Note that \( \lambda^* \) can be efficiently computed by solving problem (8), using CVX [13].

Now to find the optimal solution of (1) using Proposition 2.1, it is sufficient to solve (7). The classical SDO relaxation of (7) is

\[
\inf_{X \in \mathbb{H}_{n+1}} M_0 \cdot X \quad \text{s.t.} \\
M_1 \cdot X \leq 0, \\
M_2 \cdot X \leq 0, \\
M_3 \cdot X = 1, \\
X \succeq 0,
\]

where

\[
M_0 = \left( \begin{array}{cc} c_1 - \lambda c_2 & -(b_1 - \lambda b_2)^H \\ -(b_1 - \lambda b_2) & A_1 - \lambda A_2 \end{array} \right), \\
M_1 = \left( \begin{array}{cc} e_1 & -d_1^H \\ -d_1 & B_1 \end{array} \right), \\
M_2 = \left( \begin{array}{cc} e_2 & -d_2^H \\ -d_2 & B_2 \end{array} \right), \\
M_3 = \left( \begin{array}{ccc} 1 & 0_{1 \times n} & 0_{n \times n} \\ 0_{n \times 1} & 0_{n \times n} & 0_{n \times n} \end{array} \right), \\
X = \left( \begin{array}{ccc} 1 & x_0^H & x_0 \end{array} \right).
\]

The dual of (11) is given by

\[
\max_{y_3} y_3 \quad \text{s.t.} \\
Z = M_0 + y_1 M_1 + y_2 M_2 + y_3 M_3 \succeq 0_{n+1 \times n+1}, \\
y_1, y_2 \geq 0.
\]

Note that if \( A_1 - \lambda A_2 = 0 \) and Lemma 3.3 holds then Assumption A is satisfied while the strict feasibility condition for dual problem (12) may be violated. Consequently, solving problem (11) is hard in these cases, so we consider the following assumption.

**Assumption B.** There exist nonnegative \( \eta_1, \eta_2 \) such that

\[
A_1 - \lambda^* A_2 + \eta_1 B_1 + \eta_2 B_2 > 0.
\]

**Theorem 3.5.** Suppose that problem (1) has a strictly feasible solution \( x_0 \) and Assumption B holds. Then both problems (11) and (12) also satisfy the strict feasibility condition. Hence, both problems attain their optimal values and the duality gap is zero.

**Proof.** Let \( X_0 \) be as follows:

\[
X_0 = \left( \begin{array}{ccc} 1 & x_0^H & x_0 \end{array} \right),
\]

where \( Q = \text{diag}(q_1, \cdots, q_n) \) with all \( q_j > 0 \) and sufficiently small. Obviously by the Schur complement theorem, \( X \) is positive definite. Moreover,

\[
M_i \cdot X_0 < 0
\]

\[\iff M_i \cdot X_0 = \left( \begin{array}{cc} e_i & -d_i^H \\ -d_i & B_i \end{array} \right) \cdot \left( \begin{array}{ccc} 1 & x_0^H & x_0 \end{array} \right) \]

\[
= \left( \begin{array}{ccc} e_i & -d_i^H & x_0^H \\ -d_i & B_i & x_0 \end{array} \right) \cdot \left( \begin{array}{ccc} x_0 \\ x_0^H \end{array} \right).
\]
\[ e_i = 2 \text{Re}(d_i^H x_0) + x_0^H B_i x_0 + \sum_{j=1}^{n} (B_i)_{ij} q_j < 0, \ i = 1, 2. \]

Also
\[ M_3 \cdot X_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ x_0 \end{pmatrix} \begin{pmatrix} x_0^H \\ x_0 x_0^H + Q \end{pmatrix} = 1. \]

Then, \( X_0 \) is a strictly feasible solution for the problem (1). For the dual problem we have
\[ Z = M_0 + y_1 M_1 + y_2 M_2 + y_3 M_3 \]
\[ = \begin{pmatrix} c_1 - \lambda^* c_2 \\ -(b_1 - \lambda^* b_2) \end{pmatrix} \begin{pmatrix} A_1 - \lambda^* A_2 \\ \lambda^* \end{pmatrix} + y_1 \begin{pmatrix} e_1 \\ -d_1 \end{pmatrix} \begin{pmatrix} e_1 \\ -d_1 \end{pmatrix} B_1 \\
+ y_2 \begin{pmatrix} e_2 \\ -d_2 \end{pmatrix} B_2 + y_3 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \]

We further consider
\[ A = A_1 - \lambda^* A_2, \quad b = b_1 - \lambda^* b_2, \quad c = c_1 - \lambda^* c_2, \]
\[ Z = \begin{pmatrix} c + e_1 y_1 + e_2 y_2 + y_3 \\ -(b + d_1 y_1 + d_2 y_2) \end{pmatrix} \begin{pmatrix} A + y_1 B_1 + y_2 B_2 \\ \lambda^* \end{pmatrix}. \]

Now by Schur complement theorem
\[ Z > 0 \iff (A + y_1 B_1 + y_2 B_2) - \frac{1}{c + e_1 y_1 + e_2 y_2 + y_3} \]
\[ \times (b + d_1 y_1 + d_2 y_2) (b + d_1 y_1 + d_2 y_2)^H > 0. \]

Since
\[ (b + d_1 y_1 + d_2 y_2) (b + d_1 y_1 + d_2 y_2)^H \succeq 0 \]
and
\[ A + \eta_1 B_1 + \eta_2 B_2 \succ 0, \]
then by choosing \( y_3 \) sufficiently large and \( y_1 = \eta_1, y_2 = \eta_2, Z \) is positive definite which implies the strict feasibility condition of (11).

From Theorem 3.5, \( X^* \) and \( (y_1^*, y_2^*, y_3^*, Z^*) \) are optimal solutions of (11) and (12), respectively if and only if the following conditions are satisfied

(a) \( M_1 \cdot X^* \leq 0, \)  \hspace{1cm} (b) \( M_2 \cdot X^* \leq 0, \)
(c) \( M_3 \cdot X^* = 1, \)  \hspace{1cm} (d) \( X^* \succeq 0, \)
(e) \( y_1^* (M_1 \cdot X^*) = 0, \)  \hspace{1cm} (f) \( y_2^* (M_2 \cdot X^*) = 0, \)
(g) \( Z^* \cdot X^* = 0, \)  \hspace{1cm} (h) \( Z^* \succeq 0, \)
(i) \( y_1, y_2 \geq 0. \)  \hspace{1cm} (i) \( y_1, y_2 \geq 0. \)

Next theorem shows that there exists a rank-one decomposition of \( X^* \) that enables us to obtain an optimal solution for (7).
Thus we have

\[ X = \sum_{j=1}^{r} x_j x_j^H, \]

such that

\[ x_j^H M_1 x_j = \frac{M_1 \cdot X}{r}, \quad x_j^H M_2 x_j = \frac{M_2 \cdot X}{r}, \quad j = 1, \ldots, r. \]

**Theorem 3.6.** ([15]) Suppose that \( X \in \mathbb{H}_n^\times \) is a complex Hermitian positive semi-definite matrix of rank \( r \), and \( M_1, M_2 \in \mathbb{H}_n^\times \) be two given Hermitian matrices. Then, there is a rank-one decomposition of \( X \),

\[ X = \sum_{j=1}^{r} x_j x_j^H, \]

\[ x_j^H M_1 x_j = \frac{M_1 \cdot X}{r}, \quad x_j^H M_2 x_j = \frac{M_2 \cdot X}{r}, \quad j = 1, \ldots, r. \]

**Lemma 3.7.** Suppose the strict feasibility and Assumptions A and B hold. If \( X^* \) is the optimal solution of (11), then there exists \( \hat{x}_j^* \) from the rank-one decomposition of \( X^* \), such that \( \hat{x}_j^* \) is the optimal solution of (7).

**Proof.** Let \( X^* \) and \( (y_1^*, y_2^*, y_3^*, Z^*) \) be the optimal solutions of (11) and (12). From Theorem 3.6, there exists a rank-one decomposition of \( X^* \) i.e,

\[ X^* = \sum_{j=1}^{r} x_j^{*H} x_j^* \]

such that

\[ x_j^{*H} M_1 x_j^* = \frac{M_1 \cdot X^*}{r} \leq 0 \text{ and } x_j^{*H} M_2 x_j^* = \frac{M_2 \cdot X^*}{r} \leq 0. \]

Since

\[ M_3 \cdot X^* = \sum_{j=1}^{r} M_3 \cdot x_j^{*H} x_j^* = 1, \]

there must exist a \( j \) such that \( M_3 \cdot x_j^{*H} x_j^* > 0 \). Let \( \hat{x}_j = \frac{x_j^*}{\sqrt{x_j^{*H} M_3 x_j^*}} \), then we have

\[ M_1 \cdot \hat{x}_j^H = M_1 \cdot \frac{x_j^*}{\sqrt{x_j^{*H} M_3 x_j^*}} = \frac{x_j^*}{x_j^{*H} (M_3 x_j^*)} \leq 0, \quad i = 1, 2. \]

According to (e) and (f), we have

\[ y_i^* (M_1 \cdot \hat{x}_j^H) = y_i^* \left( \frac{1}{x_j^{*H} M_3 x_j^*} M_1 \cdot x_j^{*H} x_j^* \right) \frac{1}{r(x_j^{*H} M_3 x_j^*)} [y_i^* (M_1 \cdot X^*)] = 0, \quad i = 1, 2. \]

In addition, with regard to (g)

\[ Z^* \cdot X^* = \sum_{j=1}^{r} x_j^{*H} Z^* x_j^* = 0 \implies \forall j, \quad x_j^{*H} Z^* x_j^* = 0 \implies Z^* \cdot x_j^{*H} x_j^* = 0. \]

Thus we have

\[ Z^* \cdot \hat{x}_j^H = 0. \]

As a result \( (y_1^*, y_2^*, y_3^*, \hat{x}_j^H) \) satisfies the complementary conditions. Hence \( \hat{x}_j^H \) is the optimal solution of (11). Furthermore, let \( \hat{x}_j = \left( \frac{1}{\hat{x}_j^*} \right) \), then

\[ \hat{x}_j^{*H} B_l \hat{x}_j^* - 2 \text{Re}(d_l^H \hat{x}_j^*) + e_i = \left( \frac{e_i}{-d_i} \right) \cdot \left( \frac{1}{\hat{x}_j^*} \hat{x}_j^{*H} \right) = M_1 \cdot \hat{x}_j^H \leq 0, \quad i = 1, 2. \]

Then \( \hat{x}_j^* \) is the optimal solution of (7). \( \square \)
Algorithm 1. Rank-one decomposition

**Input.** $X, A, B \in \mathbb{H}^n$ and $X$ is a complex Hermitian positive semidefinite matrix with $r = \text{rank}(X)$.

**Output.** $X = \sum_{j=1}^{r} x_j^* x_j^H$, a rank-one decomposition of $X$ such that $x_j^H Ax_j^* = \frac{A \cdot X}{r}$, $x_j^H Bx_j^* = \frac{B \cdot X}{r}$, $j = 1, \ldots, r$.

1. Apply Corollary 4 [24] to obtain $X = \sum_{j=1}^{r} x_j x_j^H$ such that $x_j^H Ax_j = \frac{A \cdot X}{r}$, $j = 1, \ldots, r$.
2. If $x_j^H Bx_j = \frac{B \cdot X}{r}$, $j = 1, \ldots, r$, then $x_j^* = x_j$ break and terminate. Otherwise, let $j$ and $k$ be two indices such that $x_j^H Bx_j > \frac{B \cdot X}{r}$, $x_k^H Bx_k < \frac{B \cdot X}{r}$.
3. Let
   
   
   
   
   
   

   where ‘Arg’ denotes the principal argument of a complex number (which means that $x_j^H Ax_k = \gamma_1 e^{i \alpha_1}$, $x_j^H Bx_k = \gamma_2 e^{i \alpha_2}$), then calculate roots of following equation
   
   
   

   4. Let $\gamma$ be the positive root of (13), $\alpha := \alpha_1 + \frac{\pi}{2}$ and $\omega := \gamma e^{i \alpha}$, then
   
   
   

   5. Set $x_j := z_j$ and $x_k := z_k$ and return to Step 2.

In the following, we obtain the optimal solution of (11) by Algorithm 2 and apply the rank-one decomposition algorithm [15] (Algorithm 1) on it to find the optimal solution of (1).

Algorithm 2.

1. Solve the SDO relaxation (8) to get $\lambda^*$.
2. Solve the SDO relaxation (11) with $\lambda = \lambda^*$ to obtain $X^*$.
3. Apply Algorithm 1 on $X^*$ to obtain an optimal solution $x^*$ of (7).

Assume that there exists a lower bound and an upper bound of the problem (1) ($l_0$, $u_0$), i.e. $l_0 \leq \inf_{x \in \mathbb{R}^n} \frac{f_1(x)}{f_2(x)} \leq u_0$. Then the bisection method also can be used for solving the problem (7) as follows.

Algorithm 3. Bisection method

1. Choose $l_0$ and $u_0$ such that $l_0 \leq \inf_{x \in \mathbb{R}^n} \frac{f_1(x)}{f_2(x)} \leq u_0$ holds. Set $k := 1$.
2. Let $\lambda_k := \frac{l_0 + u_0}{2}$. Then, calculate $\mathcal{F}(\lambda_k)$ by solving problem (11).
3. If $|F(\lambda_k)| \leq \epsilon$, then terminate. Otherwise, update $l_k$ and $u_k$ as follows:

$$
\begin{cases}
    l_k := l_{k-1} & \text{if } F(\lambda_k) \leq 0, \\
    u_k := \lambda_k & \\
    l_k := \lambda_k & \text{if } F(\lambda_k) > 0.
\end{cases}
$$

4. Let $k := k + 1$ and return to Step 1.

4. **Numerical Results.** In this section, we consider two sets of examples of dimensions 50 to 400 for different densities to compare the algorithms. In the algorithms, SDO relaxation problems are solved by SeDuMi 1.34. For the bisection method we choose $\epsilon = 10^{-6}$ as the tolerance of the optimality. As there are no other algorithms to solve the problem, we also use ‘fmincon’ command of MATLAB to solve problem (1) directly when the starting point is considered zero and random. Results are summarized in Tables 1 to 4. In all tables, blank space means the running time of the algorithm for solving problem is exceeded 2000 seconds, so it is neglected and “−” means the algorithm is not able to solve the problem. Moreover, we generate 5 test problems for each dimension and report the average CPU time and roots. All computations are performed on MATLAB 9.2 using a PC with Intel(R) Core Duo CPU 2.40 GHz and 8.00 GB of RAM.

**Example.** Consider the following problems

1. $$\min_{x \in \mathbb{C}^n} \frac{x^H A_1 x - 2\text{Re}(b_1^H x) + c_1}{\|x\|^2 + 1}$$
   s.t. $$x^H B_1 x - 2\text{Re}(d_1^H x) + e_1 \leq 0,$$
   $$x^H B_2 x - 2\text{Re}(d_2^H x) + e_2 \leq 0,$$

2. $$\min_{x \in \mathbb{C}^n} \frac{x^H A_1 x - 2\text{Re}(b_1^H x) + c_1}{x^H A_2 x - 2\text{Re}(b_2^H x) + c_2}$$
   s.t. $$x^H B_1 x - 2\text{Re}(d_1^H x) + e_1 \leq 0,$$
   $$x^H B_2 x - 2\text{Re}(d_2^H x) + e_2 \leq 0,$$

where $A_j, B_j \in \mathbb{H}^n$ are complex Hermitian matrices, $d_j, b_j \in \mathbb{C}^n$ and $c_j, e_j \in \mathbb{R}$ for $j = 1, 2$. Matrices and vectors are generated using the following MATLAB code:

1. `n=input(' enter the size of the problem = ');`
2. `F1=randn(n); F2=randn(n); A1=[(F1+i*F2)+(F1+i*F2)']/2;`
3. `f1=randn(n,1); f2=randn(n,1); b1=f1 + i * f2;`
4. `c1=randn;`
5. `F3=randn(n); F4=randn(n); A2=[(F3+i*F4)+(F3+i*F4)']/2 + eye(n);`
6. `f3=randn(n,1); f4=randn(n,1); b2 = f3 + i * f4;`
7. `c2=randn;`
8. `F5=randn(n); F6=randn(n); B1=[(F5+i*F6)+(F5+i*F6)']/2;`
9. `f5=randn(n,1); f6=randn(n,1); d1 = f5 + i * f6;`
10. `c1=randn;`
11. $F_7 = \text{randn}(n)$; $F_8 = \text{randn}(n)$; $B_2 = \frac{(F_7 + iF_8) + (F_7 + iF_8)'}{2}$.
12. $f_7 = \text{randn}(1)$; $f_8 = \text{randn}(1)$; $d_2 = f_1 + i f_2$.
13. $c_2 = \text{randn}$.

The numerical results of Example 1 are provided in Tables 1 and 2. As indicated in Table 1, by considering zero as the starting point for `fmincon` command in MATLAB, although it stops at the shortest possible time in comparison with other algorithms, but failed to reach the optimal solution. In Table 2, by choosing a random starting point, the `fmincon` command cannot solve the problems and the bisection algorithm is not affordable because it extremely slow, thus, it is not applicable for dimensions 200 to 400 problems. While our algorithm solves all the problems within an acceptable period of time. Tables 3 and 4 also show the results of Example 2 which has similar analysis as Example 1.

| a | density | b Method | Bisection Method | fmincon |
|---|---------|----------|-----------------|---------|
| 50 | 1       | 1.29183e-01 | 1.29183e-01 | 0.1517 |
| 100 | 1       | 1.88141e-01 | 1.88141e-01 | 0.0548 |
| 150 | 1       | 1.64356e-01 | 1.64356e-01 | 0.0563 |
| 200 | 1       | 1.58441e-01 | 1.58441e-01 | 0.1534 |
| 250 | 1       | 2.36696e-01 | 2.36696e-01 | 0.2250 |
| 50 | 0.5     | 1.19232e-01 | 1.19232e-01 | 0.2473 |
| 100 | 0.5     | 1.44456e-01 | 1.44456e-01 | 0.0963 |
| 150 | 0.5     | 1.02433e-01 | 1.02433e-01 | 0.0431 |
| 200 | 0.5     | 2.13459e-01 | 2.13459e-01 | 0.5840 |
| 250 | 0.5     | 2.13397e-01 | 2.13397e-01 | 0.7982 |
| 300 | 0.5     | 1.59773e-01 | 1.59773e-01 | 0.3227 |
| 50 | 0.25    | 1.09724e-01 | 1.09724e-01 | 0.2444 |
| 100 | 0.25    | 3.69036e-01 | 3.69036e-01 | 0.9523 |
| 150 | 0.25    | 1.81565e-01 | 1.81565e-01 | 2.1760 |
| 200 | 0.25    | 2.12397e-01 | 2.12397e-01 | 0.5037 |
| 250 | 0.25    | 2.32511e-01 | 2.32511e-01 | 0.7562 |
| 300 | 0.25    | 1.39439e-01 | 1.39439e-01 | 1.3163 |

5. Conclusions. In this paper, we proposed a new method to solve a quadratic fractional optimization problem with two quadratic constraints in complex space. The method is based on S-procedure, parametrization approach of Dinkelbach and the rank-one decomposition. Computational results show that the method solves all test problems up to global optimality.

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### Table 2. Numerical Results for Example 1

| n  | density | $\lambda^*$ | Value | time(s) | $\lambda^*$ | Value | time(s) | $\lambda^*$ | Value | time(s) |
|----|---------|-------------|-------|---------|-------------|-------|---------|-------------|-------|---------|
| 50 | 1       | 9.903074e-02| 0.2313| 9.903074e-02| 0.2313| 9.903074e-02| 0.2313| 9.903074e-02| 0.2313| 9.903074e-02| 0.2313|
| 100| 1       | 2.954031e-02| 1.0632| 2.954031e-02| 1.0632| 2.954031e-02| 1.0632| 2.954031e-02| 1.0632| 2.954031e-02| 1.0632|
| 150| 1       | 1.079332e-01| 2.7307| 1.079332e-01| 2.7307| 1.079332e-01| 2.7307| 1.079332e-01| 2.7307| 1.079332e-01| 2.7307|
| 200| 1       | 1.596868e-01| 5.8439| 1.596868e-01| 5.8439| 1.596868e-01| 5.8439| 1.596868e-01| 5.8439| 1.596868e-01| 5.8439|
| 250| 1       | 2.374876e-01| 8.5416| 2.374876e-01| 8.5416| 2.374876e-01| 8.5416| 2.374876e-01| 8.5416| 2.374876e-01| 8.5416|

### Table 3. Numerical Results for Example 2

| n  | density | $\lambda^*$ | Value | time(s) | $\lambda^*$ | Value | time(s) | $\lambda^*$ | Value | time(s) |
|----|---------|-------------|-------|---------|-------------|-------|---------|-------------|-------|---------|
| 50 | 1       | 1.673855e-01| 0.6089| 1.673855e-01| 0.6089| 1.673855e-01| 0.6089| 1.673855e-01| 0.6089| 1.673855e-01| 0.6089|
| 100| 1       | 1.373146e-01| 1.4294| 1.373146e-01| 1.4294| 1.373146e-01| 1.4294| 1.373146e-01| 1.4294| 1.373146e-01| 1.4294|
| 150| 1       | 2.910353e-01| 4.2401| 2.910353e-01| 4.2401| 2.910353e-01| 4.2401| 2.910353e-01| 4.2401| 2.910353e-01| 4.2401|
| 200| 1       | 1.110109e+01| 6.5731| 1.110109e+01| 6.5731| 1.110109e+01| 6.5731| 1.110109e+01| 6.5731| 1.110109e+01| 6.5731|
| 250| 1       | 3.834175e-01| 1.8774| 3.834175e-01| 1.8774| 3.834175e-01| 1.8774| 3.834175e-01| 1.8774| 3.834175e-01| 1.8774|
| 300| 1       | 1.301979e-01| 19.2763| 1.301979e-01| 19.2763| 1.301979e-01| 19.2763| 1.301979e-01| 19.2763| 1.301979e-01| 19.2763|
| 350| 1       | 2.705206e-01| 23.9422| 2.705206e-01| 23.9422| 2.705206e-01| 23.9422| 2.705206e-01| 23.9422| 2.705206e-01| 23.9422|
Table 4. Numerical Results for Example 2

| n | density | λ* | value | time(s) | λ* | value | time(s) |
|---|---------|----|-------|---------|----|-------|---------|
| 50 | 1 | 1.579353e-01 | 1.579353e-01 | 0.2742 | 1.579353e-01 | 5.1885 | --- |
| 100 | 2 | 2.051871e-02 | 2.051871e-02 | 2.5028 | 2.051871e-02 | 4.5211 | --- |
| 150 | 1 | 4.537966e-01 | 4.537966e-01 | 5.2411 | 5.2411 | 7.3028 | --- |
| 250 | 1 | 1.302296e-01 | 1.302296e-01 | 7.5028 | 7.5028 | --- | --- |
| 50 | 0.5 | 2.984386e-02 | 2.984386e-02 | 0.2527 | 0.2527 | --- | --- |
| 100 | 0.5 | 2.547812e-02 | 2.547812e-02 | 0.7391 | 0.7391 | --- | --- |
| 150 | 0.5 | 1.302714e-01 | 1.302714e-01 | 2.2318 | 2.2318 | --- | --- |
| 200 | 0.5 | 4.219078e-01 | 4.219078e-01 | 5.2065 | 5.2065 | --- | --- |
| 250 | 0.5 | 2.903107e-01 | 2.903107e-01 | 7.1589 | 7.1589 | --- | --- |
| 50 | 0.5 | 3.692972e-02 | 3.692972e-02 | 1.3320 | 1.3320 | --- | --- |
| 100 | 0.25 | 1.742665e-02 | 1.742665e-02 | 0.2375 | 0.2375 | --- | --- |
| 150 | 0.25 | 6.999637e-02 | 6.999637e-02 | 0.7539 | 0.7539 | --- | --- |
| 200 | 0.25 | 2.181801e-02 | 2.181801e-02 | 1.2246 | 1.2246 | --- | --- |
| 250 | 0.25 | 1.989546e-01 | 1.989546e-01 | 4.9963 | 4.9963 | --- | --- |
| 50 | 0.25 | 4.199379e-01 | 4.199379e-01 | 8.6883 | 8.6883 | --- | --- |
| 100 | 0.25 | 2.342168e-01 | 2.342168e-01 | 1.2971 | 1.2971 | --- | --- |
| 150 | 0.25 | 2.354174e-02 | 2.354174e-02 | 0.2192 | 0.2192 | --- | --- |
| 200 | 0.25 | 1.947376e-02 | 1.947376e-02 | 0.6087 | 0.6087 | --- | --- |
| 250 | 0.25 | 1.596201e-01 | 1.596201e-01 | 1.9342 | 1.9342 | --- | --- |
| 50 | 0.1 | 2.030211e-01 | 2.030211e-01 | 4.5136 | 4.5136 | --- | --- |
| 100 | 0.1 | 2.274598e-01 | 2.274598e-01 | 6.2734 | 6.2734 | --- | --- |
| 150 | 0.1 | 2.575936e-01 | 2.575936e-01 | 12.4938 | 12.4938 | --- | --- |
| 200 | 0.1 | 1.974754e-02 | 1.974754e-02 | 0.7276 | 0.7276 | --- | --- |
| 250 | 0.1 | 6.123552e-02 | 6.123552e-02 | 1.5383 | 1.5383 | --- | --- |
| 50 | 0.01 | 9.402653e-02 | 9.402653e-02 | 4.2944 | 4.2944 | --- | --- |
| 100 | 0.01 | 2.350105e-01 | 2.350105e-01 | 8.8472 | 8.8472 | --- | --- |
| 150 | 0.01 | 6.199347e-01 | 6.199347e-01 | 11.8967 | 11.8967 | --- | --- |
| 200 | 0.01 | 2.482896e-01 | 2.482896e-01 | 20.1709 | 20.1709 | --- | --- |
| 250 | 0.01 | 2.519739e-01 | 2.519739e-01 | 25.0846 | 25.0846 | --- | --- |

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