CONVERGENCE OF THE FINITE DIFFERENCE SCHEME FOR A GENERAL CLASS OF THE SPATIAL SEGREGATION OF REACTION-DIFFUSION SYSTEMS

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Abstract. In this work we prove convergence of the finite difference scheme for equations of stationary states of a general class of the spatial segregation of reaction-diffusion systems with \( m \geq 2 \) components. More precisely, we show that the numerical solution \( u^h_l \), given by the difference scheme, converges to the \( l \)th component \( u_l \), when the mesh size \( h \) tends to zero, provided \( u_l \in C^2(\Omega) \), for every \( l = 1, 2, \ldots, m \). In particular, our proof provides convergence of a difference scheme for the multi-phase obstacle problem.

1. Introduction

1.1. The setting of the problem. In recent years there have been intense studies of spatial segregation for reaction-diffusion systems. The existence of spatially inhomogeneous solutions for competition models of Lotka-Volterra type in the case of two and more competing densities have been considered in \[ 18, 19, 20, 21, 22, 23, 30 \]. Aforementioned segregation problems led to an interesting class of multi-phase obstacle-like free boundary problems. These problems have growing interest due to their important applications in the different branches of applied mathematics. To see the diversity of applications we refer \[ 5, 14, 15 \] and the references therein.

Nowadays, the theory of the one- and two-phase obstacle-like problems (elliptic and parabolic versions) is well-established and for a reference we address to the books \[ 27, 29 \] and references therein. For two-phase problems the interested reader is also referred to the recent works \[ 9, 28 \].

There is a vast literature devoted to the numerical analysis of one-phase obstacle-like problems, and we refer some of well-known papers \[ 16, 24, 25, 26 \]. For the numerical treatment of the two-phase problems we refer to the works \[ 1, 6, 8, 13, 31 \].

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The present work concerns to prove the convergence of the difference scheme for a certain class of the spatial segregation of reaction-diffusion system with \( m \) components.

Let \( \Omega \subset \mathbb{R}^n, n \geq 2 \) be a connected and bounded domain with smooth boundary and \( m \) be a fixed integer. We consider the steady-states of \( m \) competing species coexisting in the same area \( \Omega \). Let \( u_i(x) \) denotes the population density of the \( i \)th component with the internal dynamic prescribed by \( F_i(x, u_i) \).

We call the \( m \)-tuple \( U = (u_1, \cdots, u_m) \in (W^{1,2}(\Omega))^m, a \) segregated state if
\[
    u_i(x) \cdot u_j(x) = 0, \text{ a.e. for } i \neq j, x \in \Omega.
\]

The problem amounts to
\[
(1) \quad \text{Minimize } E(u_1, \cdots, u_m) = \int_\Omega \sum_{i=1}^m \left( \frac{1}{2} |\nabla u_i|^2 + F_i(x, u_i) \right) dx,
\]
over the set
\[
S = \{(u_1, \ldots, u_m) \in (W^{1,2}(\Omega))^m : u_i \geq 0, u_i \cdot u_j = 0, u_i = \phi_i \text{ on } \partial \Omega\},
\]
where \( \phi_i \in H^2(\partial \Omega), \phi_i \cdot \phi_j = 0, \text{ for } i \neq j \) and \( \phi_i \geq 0 \) on the boundary \( \partial \Omega \).

We assume that
\[
F_i(x, s) = \int_0^s f_i(x, v) dv,
\]
where \( f_i(x, s) : \Omega \times \mathbb{R}^+ \to \mathbb{R} \) is Lipschitz continuous in \( s \), uniformly continuous in \( x \) and \( f_i(x, 0) \equiv 0 \).

Remark 1. Functions \( f_i(x, s) \)'s are defined only for non negative values of \( s \) (recall that our densities \( u_i \)'s are assumed non negative); thus we can arbitrarily define such functions on the negative semiaxis. For the sake of convenience, when \( s \leq 0 \), we will let \( f_i(x, s) = -f_i(x, -s) \). This extension preserves the continuity due to the conditions on \( f_i \) defined above. In the same way, each \( F_i \) is extended as an even function.

Remark 2. We emphasize that for the case \( f_i(x, s) = f_i(x) \), the assumption is that for all \( i \) the functions \( f_i(x, s) \) are nonnegative and uniformly continuous in \( x \). Also for simplicity, throughout the paper we shall call both \( F_i(x, u_i) \) and \( f_i(x, u_i) \) internal dynamics.

We would like to point out that the only difference between our minimization problem (1) and the problem discussed in [19], is the sign in front of the internal dynamics \( F_i \).

In our case, the plus sign of \( F_i \) allows to get rid of some additional conditions, which are imposed in [19, Section 2]. Those conditions are important to provide coercivity of a minimizing functional in [19]. But in our case the above given conditions together with convexity assumption on \( F_i(x, s) \), with respect to the variable \( s \) are enough to conclude
\[
F_i(x, u_i(x)) \geq 0,
\]
which in turn implies coercivity of a functional (1).
In order to speak on the local properties of the population densities, let us introduce the notion of multiplicity of a point in $\Omega$.

**Definition 1.** The multiplicity of the point $x \in \overline{\Omega}$ is defined by:

$$m(x) = \text{card} \left\{ i : \text{measure}(\Omega_i \cap B(x, r)) > 0, \forall r > 0 \right\},$$

where $\Omega_i = \{u_i > 0\}$.

For the local properties of $u_i$ the same results as in [19] with the opposite sign in front of the internal dynamics $f_i$ hold. Below, for the sake of clarity, we write down these results from [19] with appropriate changes.

**Lemma 1.** (*Proposition 6.3 in [19]*) Assume that $x_0 \in \Omega$, then the following holds:

1) If $m(x_0) = 0$, then there exists $r > 0$ such that for every $i = 1, \cdots, m$;

$$u_i \equiv 0 \text{ on } B(x_0, r).$$

2) If $m(x_0) = 1$, then there are $i$ and $r > 0$ such that in $B(x_0, r)$

$$\Delta u_i = f_i(x, u_i), \quad u_j \equiv 0 \quad \text{for } j \neq i.$$ 

3) If $m(x_0) = 2$, then there are $i, j$ and $r > 0$ such that for every $k$ and $k \neq i, j$, we have $u_k \equiv 0$ and

$$\Delta(u_i - u_j) = f_i(x, (u_i - u_j))\chi_{\{u_i > u_j\}} - f_j(x, -(u_i - u_j))\chi_{\{u_i < u_j\}} \text{ in } B(x_0, r).$$

**Lemma 2.** (*Theorem 5.1 in [19]*) For every minimizer $(u_1, \cdots, u_m) \in S$ to the functional (1), the following inequality holds

$$\Delta \left( u_l(x) - \sum_{p \neq l} u_p(x) \right) \leq f_l(x, u_l),$$

for all $l = 1, 2, \ldots, m$.

Next, we state the following uniqueness theorem due to Conti, Terrachini and Verzini, by observing that in our case the plus sign in front of $F_i$ requires convexity condition on $F_i(x, s)$ rather than concavity condition given in [19].

**Theorem 1.** (*Theorem 4.2 in [19]*) Let the functional in minimization problem (1) is coercive and moreover each $F_i(x, s)$ is convex in the variable $s$, for all $x \in \Omega$. Then, the problem (1) has a unique minimizer.
1.2. Notation. We will work in two-dimensional space $\mathbb{R}^2$. For the sake of simplicity, we will assume that $\Omega = (0, a) \times (0, a)$. It should be remarked that the same results can be obtained rigorously also for more complicated domains.

Let $N \in \mathbb{N}$ be a positive integer, $h = a/N$ and

$$x_i = ih, \quad y_i = ih, \quad i = 0, 1, \ldots, N.$$ 

We use the notation $u_{lh}(x, y)$ for the finite difference scheme approximation to $u_l(x, y)$. We will heavily use the shorthand notations $u_{lh}(z)$ and $u_l(z)$, where $z = (x, y) \in \Omega$.

Concerning the boundary functions $\phi_l$, we assume they are extended to be zero everywhere outside the boundary $\partial \Omega$, for all $l = 1, 2, \ldots, m$. The discrete approximation for these functions will be $\phi_{lh}$.

Denote

$$\Omega_h = \{(x_i, y_j) = (ih, jh) : 0 \leq i, j \leq N\},$$

$$\Omega^o_h = \{(x_i, y_j) = (ih, jh) : 1 \leq i, j \leq N - 1\},$$

and

$$\partial \Omega_h = \Omega_h \setminus \Omega^o_h.$$ 

In two-dimensional case we introduce the following 5-point stencil approximation for Laplacian:

$$L_h v(x, y) = \frac{v(x - h, y) + v(x + h, y) - 4v(x, y) + v(x, y - h) + v(x, y + h)}{h^2}$$

for any $(x, y) \in \Omega$.

2. Finite difference scheme

We start this section by defining the finite difference scheme, which convergence analysis will be the subject of the study in the present work. We denote it by $(u_{lh}^1, u_{lh}^2, \ldots, u_{lh}^m)$. This vector solves the following system:

$$\begin{cases}
    u_{lh}^l(z) = \max \left( \frac{-f_l(z, u_{lh}^l(z))h^2}{4} + \overline{u}_{lh}^l(z) - \sum_{p \neq l} \overline{u}_{lh}^p(z), \ 0 \right), \quad z \in \Omega^o_h, \\
    u_{lh}^l(z) = \phi_{lh}^l(z) = \phi_l(z), \quad z \in \partial \Omega_h.
\end{cases}$$

for every $l = 1, 2, \ldots, m$ and $z = (x, y) \in \Omega_h$. Here for a given uniform mesh on $\Omega \subset \mathbb{R}^2$, we define $\overline{u}_{lh}^l(z)$ to be the average of $u_{lh}^l(z)$ for all neighbor points of $z = (x, y) \in \Omega^o_h$:

$$\overline{u}_{lh}^l(z) = \frac{1}{4}[u_{lh}^l(x_{i-1}, y_j) + u_{lh}^l(x_{i+1}, y_j) + u_{lh}^l(x_i, y_{j-1}) + u_{lh}^l(x_i, y_{j+1})].$$
Throughout the paper the following notations will play a crucial role:

\[ \hat{u}_l(z) := u_l(z) - \sum_{p \neq l} u_p(z), \]

and

\[ \hat{u}_h^l(z) := u_h^l(z) - \sum_{p \neq l} u_h^p(z). \]

It is easy to verify that the solution \((u_h^1, u_h^1, \ldots, u_h^m)\) to a difference scheme (2) for every \(l = 1, 2, \ldots, m\), satisfies the following properties, provided that all functions \(f_i(z, s)\) are nondecreasing with respect to the variable \(s\):

\[
\begin{cases}
L_h(\hat{u}_h^l(z)) \leq f_i(z, u_h^l(z)) & z \in \Omega_h, \\
L_h(\hat{u}_h^l(z)) = f_i(z, u_h^l(z)) & z \in \{u_h^l(z) > 0\}, \\
u_h^l(z) \geq 0 & z \in \Omega_h, \\
u_h^p(z) \cdot u_h^q(z) = 0, & p \neq q, \\
u_h^l(z) = \phi_h^l(z) = \phi(z), & z \in \partial \Omega_h.
\end{cases}
\]

The difference system (2), when the internal dynamics \(f_i(z, s) = 0, z \equiv (x, y) \in \mathbb{R}^2\), has been suggested in [7]. The author only implemented plausible numerical figures by this scheme, without its analysis. This finite difference method has been generalized in [11] for the case of non-negative internal dynamics \(f_i(z, s) = f_i(z)\). In [11] the authors give a numerical consistent variational system with strong interaction, and provide disjointness condition of populations during the iteration of the scheme. In this case the proposed algorithm is lack of deep analysis, especially for the case of three and more competing populations. In the recent work by the current author in collaboration [2] the existence and uniqueness of the scheme, which solves the system (2), have been proven, provided all \(f_i(z, s)\) are nonnegative and nondecreasing with respect to \(s\). It is noteworthy, that the difference schemes with the same spirit as the system (2), have been successfully applied in quadrature domains theory (see [12]) and in optimal partitions theory (see [10]). This makes us to strongly believe that the ideas behind the difference scheme (2) have great opportunities to be applied in different problems, where the segregated geometry arise.

3. Auxiliary lemmas

In this section we prove two technical lemmas, which will be used for the convergence analysis of the scheme. To this aim, for the sake of convenience we denote by \(nbr(z)\) the set of all closest neighbor points corresponding to a mesh point \(z = (x, y) \in \Omega_h\). We will
need also the following barrier function:

\[ V_h(z) = V_h(x, y) = \frac{1}{4}(x^2 + y^2 + 1) \sum_{l=1}^{m} ||L_h u_l - \Delta u_l||_{L^\infty(\Omega)}. \]

For simplicity, we set by Lemma 3.

\[ \alpha_h := L_h(V_h(z)) = \sum_{l=1}^{m} ||L_h u_l - \Delta u_l||_{L^\infty(\Omega)}. \]

**Lemma 3.** Let the functions \( f_l(z, s) \) be nondecreasing with respect to the variable \( s \). We set \((u_1, u_2, \ldots, u_m) \in S \cap (C^2(\Omega))^m \) to be an exact minimizer of (1) subject to \( S \), and by \((u^1_h, u^2_h, \ldots, u^m_h) \) we define the vector, which solves the finite difference system (2). Then the following statements are true:

\[ \max_{\Omega_h} (\hat{u}_l(z) - \hat{u}^i_h(z) + V_h(z)) = \max_{\{u_l(z) \leq u^i_h(z)\}} (\hat{u}_l(z) - \hat{u}^i_h(z) + V_h(z)), \]

and

\[ \max_{\Omega_h} (\hat{u}^i_h(z) - \hat{u}_l(z) + V_h(z)) = \max_{\{u^i_h(z) \leq u_l(z)\}} (\hat{u}^i_h(z) - \hat{u}_l(z) + V_h(z)), \]

for all \( l = 1, 2, \ldots, m. \)

**Proof.** We argue by contradiction. Suppose for some \( l_0 \) we have

\[ \hat{u}_{l_0}(z_0) - \hat{u}^{l_0}_h(z_0) + V_h(z_0) = \max_{\Omega_h} (\hat{u}_{l_0}(z) - \hat{u}^{l_0}_h(z) + V_h(z)) = \]

\[ = \max_{\{u_{l_0}(z) > u^{l_0}_h(z)\}} (\hat{u}_{l_0}(z) - \hat{u}^{l_0}_h(z) + V_h(z)) > \max_{\{u_{l_0}(z) \leq u^{l_0}_h(z)\}} (\hat{u}_{l_0}(z) - \hat{u}^{l_0}_h(z) + V_h(z)). \]

Then taking into account the following simple chain of inclusions

\[ \{u_l(z) > u^i_h(z)\} \subset \{\hat{u}_l(z) > \hat{u}^i_h(z)\} \subset \{u_l(z) \geq u^i_h(z)\}, \]

we obviously see that \( u_{l_0}(z_0) > u^{l_0}_h(z_0) \geq 0 \) implies \( \hat{u}_{l_0}(z_0) > \hat{u}^{l_0}_h(z_0) \). On the other hand, the discrete system (2) and Lemma 1 gives us

\[ \Delta \hat{u}_{l_0}(z_0) = \Delta u_{l_0}(z_0) = f_{l_0}(z_0, u_{l_0}(z_0)) \quad \text{and} \quad L_h \hat{u}^{l_0}_h(z_0) \leq f_{l_0}(z_0, u^{l_0}_h(z_0)). \]
Therefore

\[ L_h \left( \hat{u}_{t_0}(z_0) - \hat{u}_h^{t_0}(z_0) + V_h(z_0) \right) = \]
\[ = \left( L_h \hat{u}_{t_0}(z_0) - \Delta \hat{u}_{t_0}(z_0) + \alpha_h \right) + (\Delta \hat{u}_{t_0}(z_0) - L_h \hat{u}_h^{t_0}(z_0)) \geq \]
\[ \geq (\Delta \hat{u}_{t_0}(z_0) - L_h \hat{u}_h^{t_0}(z_0)) \geq f_{t_0}(z_0, u_{t_0}(z_0)) - f_{t_0}(z_0, u_h^{t_0}(z_0)) \geq 0. \]

Thus,

\[ \hat{u}_{t_0}(z_0) - \hat{u}_h^{t_0}(z_0) + V_h(z_0) \leq \frac{1}{4} \sum_{\{z \in \text{nbr}(z_0)\}} (\hat{u}_{t_0}(z) - \hat{u}_h^{t_0}(z) + V_h(z)), \]

which implies that \( \hat{u}_{t_0}(z_0) - \hat{u}_h^{t_0}(z_0) + V_h(z_0) = \hat{u}_{t_0}(z) - \hat{u}_h^{t_0}(z) + V_h(z) \), for all \( z \in \text{nbr}(z_0) \).

Since \( \hat{u}_{t_0}(z_0) > \hat{u}_h^{t_0}(z_0) \), then we apparently have

\[ \hat{u}_{t_0}(z) - \hat{u}_h^{t_0}(z) > V_h(z_0) - V_h(z), \]

for all \( z \in \text{nbr}(z_0) \). We take a particular neighbor point \( \hat{z} = (x_{i_0-1}, y_{j_0}) \), provided \( z_0 = (x_{i_0}, y_{j_0}) \in \Omega_h \). We obtain

\[ \hat{u}_{t_0}(\hat{z}) - \hat{u}_h^{t_0}(\hat{z}) > V_h(z_0) - V_h(\hat{z}) = \frac{1}{4}(x_{i_0}^2 - x_{i_0-1}^2)\alpha_h \geq 0. \]

In view of chain (5) we get \( u_{t_0}(\hat{z}) \geq u_h^{t_0}(\hat{z}) \). According to our assumption (4), the only possibility is \( u_{t_0}(\hat{z}) > u_h^{t_0}(\hat{z}) \). Now we can proceed the previous steps for this neighbor point \( \hat{z} = (x_{i_0-1}, y_{j_0}) \in \text{nbr}(z_0) \), and obtain the same strict inequality for \( (x_{i_0-2}, y_{j_0}) \) and so on. Continuing this along an \( x \) axis, we will finally approach to the boundary \( \partial \Omega_h \), where as we know \( u_{t_0}(z) = u_h^{t_0}(z) = \phi_{t_0}(z) \), for all \( z \in \partial \Omega_h \). Hence, the strict inequality fails, which implies that our initial assumption (4) is false. Observe that the same arguments can be applied if we interchange the role of \( u_l(z) \) and \( u_h^l(z) \). In this case we need to use the reversed chain of inclusions given below

\[ (6) \quad \{u^l_{t_0}(z) > u_{t_0}(z)\} \subset \{\hat{u}^l_{t_0}(z) > \hat{u}_l(z)\} \subset \{u^l_h(z) \geq u_l(z)\}, \]

and Lemma [2]. Thus, we also have

\[ \max_{\Omega_h} (\hat{u}^l_h(z) - \hat{u}_l(z) + V_h(z)) = \max_{\{u^l_h(z) \leq u_l(z)\}} (\hat{u}^l_h(z) - \hat{u}_l(z) + V_h(z)), \]

for every \( l = 1, 2, \ldots, m \). This completes the proof of Lemma. □

In the sequel and thanks to Lemma [3] we will use the following notations:
Lemma 4. Let the functions $f_1(x,s)$ be nondecreasing with respect to the variable $s$. We also set $(u_1, u_2, \ldots, u_m) \in S \cap (C^2(\Omega))^m$ to be an exact minimizer of \[1\] subject to $S$, and $(u^h_1, u^h_2, \ldots, u^h_n)$ to be the difference scheme, which solves the discrete system \[2\]. For these two elements we set $M_h$ and $R_h$ as defined above. If $M_h > \max_{\Omega_h} V_h(z)$ (respectively $R_h > \max_{\Omega_h} V_h(z)$) and it is attained for some $l_0$, then $M_h = R_h > \max_{\Omega_h} V_h(z)$. Moreover, there exists some $t_0 \neq l_0$, and $z_0 \in \Omega_h$, such that

$$M_h = \max_{\{u_{l_0}(z) = u^l_{h}(z) = 0\}} (\hat{u}_{l_0}(z) - \hat{u}^l_{h}(z) + V_h(z)) = u^l_{h}(z_0) - u_{t_0}(z_0) + V_h(z_0).$$

(R espectively,

$$R_h = \max_{\{u_{l_0}(z) = u^l_{h}(z) = 0\}} (\hat{u}^l_{h}(z) - \hat{u}_{l_0}(z) + V_h(z)) = u_{t_0}(z_0) - u^l_{h}(z_0) + V_h(z_0)).$$

Proof. Due to Lemma 3 we have

$$M_h = \max_{\{u_{l_0}(z) \leq u^l_{h}(z)\}} (\hat{u}_{l_0}(z) - \hat{u}^l_{h}(z) + V_h(z)).$$

It is easy to verify that $(\hat{u}_{l_0}(z) - \hat{u}^l_{h}(z))$ might be strictly positive only on the set $\{u_{l_0}(z) = u^l_{h}(z) = 0\}$ (for the other cases $(\hat{u}_{l_0}(z) - \hat{u}^l_{h}(z)) \leq 0$). Hence,

$$(\hat{u}_{l_0}(z) - \hat{u}^l_{h}(z) + V_h(z)) \leq \max_{\Omega_h} V_h(z) < M_h,$$

provided $z \notin \{u_{l_0}(z) = u^l_{h}(z) = 0\}$, which yields

$$M_h = \max_{\{u_{l_0}(z) = u^l_{h}(z) = 0\}} (\hat{u}_{l_0}(z) - \hat{u}^l_{h}(z) + V_h(z)).$$
Using the latter equality, one can prove that $M_h > \max_{\Omega_h} V_h(z)$ implies $R_h > \max_{\Omega_h} V_h(z)$. Indeed, it is easy to see that if the maximum $M_h$ is attained at the mesh point $z_0 \in \Omega_h$, then there exists $t_0 \neq l_0$ such that

\[ (7) \quad \max_{\Omega_h} V_h(z) < M_h = \max_{\{u_0(z) = u_{l_0}^h(z) = 0\}} \left( \hat{u}_{t_0}(z) - \hat{u}_{l_0}^h(z) + V_h(z) \right) = \]

\[ = \hat{u}_{t_0}(z_0) - \hat{u}_{l_0}^h(z_0) + V_h(z_0) = \sum_{l \neq l_0} u_l(z_0) + V_h(z_0) \leq \hat{u}_{l_0}^h(z_0) - \hat{u}_{t_0}(z_0) + V_h(z_0) \leq R_h. \]

In the same way we will obtain that $\max_{\Omega_h} V_h(z) < R_h \leq M_h$, and therefore

\[ M_h = R_h > \max_{\Omega_h} V_h(z). \]

On the other hand, the above computation (7) gives us

\[ u_{l_0}^h(z_0) - \sum_{l \neq l_0} u_l(z_0) = \hat{u}_{l_0}^h(z_0) - \hat{u}_{t_0}(z_0). \]

This leads to $2 \sum_{l \neq l_0} u_l(z_0) = 0$, and therefore $u_l(z_0) = 0$, for all $l \neq l_0$. Hence,

\[ M_h = u_{l_0}^h(z_0) - \sum_{l \neq l_0} u_l(z_0) + V_h(z_0) = u_{l_0}^h(z_0) - u_{t_0}(z_0) + V_h(z_0). \]

For $R_h$ the proof can be done in a similar way. This completes the proof.

\[ \square \]

4. Convergence of scheme

In this section we prove the main result of the paper. Next proposition shows the estimate between the exact and numerical solutions. Then the pointwise convergence of the scheme follows immediately.

**Proposition 1.** Let the functions $f_l(x, s)$ be nondecreasing with respect to the variable $s$. We set $(u_1, u_2, \ldots, u_m) \in S$ to be an exact minimizer of (1) subject to $S$. If $(u_{l_1}^h, u_{l_2}^h, \ldots, u_{l_m}^h)$ is the difference scheme, which solves the discrete system (2), then the following estimate holds:

\[ ||u_l - u_{l_0}^h||_{L^\infty(\Omega_h)} \leq C_\Omega \cdot \sum_{l=1}^m ||L_h u_l - \Delta u_l||_{L^\infty(\Omega)}, \]

for every $l \in \{1, 2, \ldots, m\}$, provided $(u_1, u_2, \ldots, u_m) \in S \cap (C^2(\Omega))^m$. Here $C_\Omega > 0$ is a constant depending only on $\Omega$. 
Proof. For the vectors \( (u_1, u_2, \ldots, u_m) \) and \( (u_h^1, u_h^2, \ldots, u_h^m) \) we set the definition of \( M_h \) and \( R_h \). We are going to prove that \( M_h \leq \max_{\Omega_h} V_h(z) \). As a consequence we will obtain that \( R_h \leq \max_{\Omega_h} V_h(z) \) holds as well.

Suppose \( M_h > \max_{\Omega_h} V_h(z) \). Our aim is to prove that this case leads to a contradiction. Let the value \( M_h \) is attained for some \( l_0 \in \overline{1, m} \), then due to Lemma 4 we have \( M_h = R_h \), and there exist \( z_0 \in \Omega_h \) and \( t_0 \neq l_0 \) such that:

\[
M_h = R_h = \max_{\{u_{l_0}(z)u_{l_0}'(z) = 0\}} \left( \hat{u}_{l_0}(z) - \hat{u}_{l_0}'(z) + V_h(z) \right)
\]

\[
= u_{l_0}^0(z_0) - u_{l_0}(z_0) + V_h(z_0).
\]

This yields

\[
u_{l_0}^0(z_0) - u_{l_0}(z_0) = M_h - V_h(z_0) > \max_{\Omega_h} V_h(z) - V_h(z_0) \geq 0,
\]

and therefore due to (3) and Lemma 2 we clearly obtain

\[
L_h \hat{u}_{l_0}^0(z_0) = f_{l_0}(z_0, u_{l_0}^0(z_0)) \quad \text{and} \quad \Delta \hat{u}_{l_0}(z_0) \leq f_{l_0}(z_0, u_{l_0}(z_0)).
\]

By proceeding similar steps as in the proof of Lemma 3 and recalling that \( f_l(x, s) \) are nondecreasing with respect to the variable \( s \), we conclude

\[
L_h \left( \hat{u}_{l_0}^0(z_0) - \hat{u}_{l_0}(z_0) + V_h(z_0) \right) \geq 0.
\]

Thus,

\[
\hat{u}_{l_0}^0(z_0) - \hat{u}_{l_0}(z_0) + V_h(z_0) \leq \frac{1}{4} \sum_{\{\gamma \in \text{nbr}(z_0)\}} \left( \hat{u}_{l_0}^0(\gamma) - \hat{u}_{l_0}(\gamma) + V_h(\gamma) \right),
\]

which implies that

\[
M_h = \hat{u}_{l_0}^0(z_0) - \hat{u}_{l_0}(z_0) + V_h(z_0) = \hat{u}_{l_0}^0(\gamma) - \hat{u}_{l_0}(\gamma) + V_h(\gamma) > \max_{\Omega_h} V_h(z),
\]

for all \( \gamma \in \text{nbr}(z_0) \). Hence, \( \hat{u}_{l_0}^0(\gamma) > \hat{u}_{l_0}(\gamma) \) and this along with the chain (6) gives that for all \( \gamma \in \text{nbr}(z_0) \), we have \( \hat{u}_{l_0}^0(\gamma) \geq u_{l_0}(\gamma) \). For the neighbor mesh points \( \gamma \) we proceed as follows: If \( u_{l_0}^0(\gamma) > u_{l_0}(\gamma) \), for some \( \gamma_0 \in \text{nbr}(z_0) \), then obviously

\[
L_h \left( \hat{u}_{l_0}^0(\gamma_0) - \hat{u}_{l_0}(\gamma_0) + V_h(\gamma_0) \right) \geq 0.
\]

This, as we saw a few lines above, leads to

\[
M_h = \hat{u}_{l_0}^0(\gamma_0) - \hat{u}_{l_0}(\gamma_0) + V_h(\gamma_0) = \hat{u}_{l_0}^0(\theta) - \hat{u}_{l_0}(\theta) + V_h(\theta) > \max_{\Omega_h} V_h(z),
\]

for all \( \theta \in \text{nbr}(\gamma_0) \).

If \( u_{l_0}^0(\gamma) = u_{l_0}(\gamma) \), for some \( \gamma \in \text{nbr}(z_0) \), then due to

\[
\hat{u}_{l_0}^0(\gamma_0) - \hat{u}_{l_0}(\gamma_0) = M_h - V_h(\gamma_0) \geq \max_{\Omega_h} V_h(z) - V_h(\gamma_0) \geq 0,
\]
the only case is $u^{t_0}_h(\gamma_0) = u_{t_0}(\gamma_0) = 0$. Hence, there exists some $\lambda_0 \neq t_0$, such that

$$M_h = \hat{u}^{\lambda_0}_h(\gamma_0) - \hat{u}_{t_0}(\gamma_0) + V_h(\gamma_0) =$$

$$= \sum_{l \neq t_0} (u_l(\gamma_0) - u^{l}_h(\gamma_0)) + V_h(\gamma_0) = u_{\lambda_0}(\gamma_0) - \sum_{l \neq t_0} u^{l}_h(\gamma_0) + V_h(\gamma_0).$$

Recalling that $M_h = R_h$, we write the following inequality

$$u_{\lambda_0}(\gamma_0) - \sum_{l \neq \lambda_0} u^{l}_h(\gamma_0) + V_h(\gamma_0) = M_h \geq \hat{u}_{\lambda_0}(\gamma_0) - \hat{u}^{\lambda_0}_h(\gamma_0) + V_h(\gamma_0),$$

which in turn gives $2 \sum_{l \neq \lambda_0} u^{l}_h(\gamma_0) \leq 0$, and therefore $u^{l}_h(\gamma_0) = 0$, for all $l \neq \lambda_0$. Hence,

$$M_h = u_{\lambda_0}(\gamma_0) - \sum_{l \neq t_0} u^{l}_h(\gamma_0) + V_h(\gamma_0) = u_{\lambda_0}(\gamma_0) - u^{\lambda_0}_h(\gamma_0) + V_h(\gamma_0).$$

This suggests us to apply the same approach as above and using the fact that

$$L_h(\hat{u}_{\lambda_0}(\gamma_0) - \hat{u}^{\lambda_0}_h(\gamma_0) + V_h(\gamma_0)) \geq 0,$$

we obtain

$$M_h = \hat{u}_{\lambda_0}(\gamma_0) - \hat{u}^{\lambda_0}_h(\gamma_0) + V_h(\gamma_0) = \hat{u}_{\lambda_0}(\theta) - \hat{u}^{\lambda_0}_h(\theta) + V_h(\theta) \geq \max_{\Omega_h} V_h(z),$$

for all $\theta \in nbr(\gamma_0)$.

Thus, continuing this process all the time for the neighbor points, in view of \(8\) and \(9\), we observe that for every mesh point $\gamma$ there always exists some $l_{\gamma} \in \overline{1, m}$ such that:

- either $\hat{u}_{t_{\gamma}}(\gamma) - \hat{u}^{l_{\gamma}}_h(\gamma) = M_h - V_h(\gamma) > 0$, or $\hat{u}_{t_{\gamma}}(\gamma) - \hat{u}^{l_{\gamma}}_h(\gamma) = V_h(\gamma) - M_h < 0$.

On the other hand, it is clear that sooner or later, we will reach the boundary $\partial \Omega_h$ and this will give a contradiction, because for every $\gamma \in \partial \Omega_h$, and $l = 1, m$ we have

$$\hat{u}_{t_{\gamma}}(\gamma) - \hat{u}^{l_{\gamma}}_h(\gamma) = \hat{u}^{l_{\gamma}}_h(\gamma) - \hat{u}_{t_{\gamma}}(\gamma) = 0.$$

From this we conclude that the only possibility is

$$M_h \leq \max_{\Omega_h} V_h(z).$$

In the light of Lemma \(4\) the inequality $M_h \leq \max_{\Omega_h} V_h(z)$ implies $R_h \leq \max_{\Omega_h} V_h(z)$ and vice-versa. Recalling the definition of $M_h$ and $R_h$ for arbitrary $l \in \overline{1, m}$, and $z \in \Omega_h$ we get

$$\begin{align*}
\begin{cases}
\hat{u}_t(z) - \hat{u}^l_h(z) + V_h(z) \leq \max_{\Omega_h} V_h(z), \\
\hat{u}^l_h(z) - \hat{u}_t(z) + V_h(z) \leq \max_{\Omega_h} V_h(z).
\end{cases}
\end{align*}$$

(10)
This leads to
\[ |\hat{u}_l(z) - \hat{u}_h^l(z)| \leq \max_{\Omega_h} V_h(z) - \min_{\Omega_h} V_h(z). \]

In view of a function \( V_h(z) \) we obtain
\[ |\hat{u}_l(z) - \hat{u}_h^l(z)| \leq D_{\Omega} \cdot \sum_{l=1}^{m} ||L_h u_l - \Delta u_l||_{L^\infty(\Omega)}, \]
for all \( l = 1, 2, \ldots, m \), where \( D_{\Omega} = \frac{a^2}{2} \). This in turn implies that for every \( z \in \Omega_h \) and \( l = 1, m \) we have
\[ |u_l(z) - u_h^l(z)| \leq 2D_{\Omega} \cdot \sum_{l=1}^{m} ||L_h u_l - \Delta u_l||_{L^\infty(\Omega)}. \]

Finally, we can write
\[ ||u_l - u_h^l||_{L^\infty(\Omega_h)} \leq 2D_{\Omega} \cdot \sum_{l=1}^{m} ||L_h u_l - \Delta u_l||_{L^\infty(\Omega)}, \]
for every \( l = 1, 2, \ldots, m \). This completes the proof. \( \Box \)

**Corollary 1.** It is clear that due to Proposition 7, we have \( u_h^l \to u_l \), for every \( l = 1, 2, \ldots, m \), whenever \( h \to 0 \), provided each component \( u_l \in C^2(\Omega) \).

**Corollary 2.** Assume \( u_l \in C^4(\Omega) \), for all \( l = 1, 2, \ldots, m \), then the Taylor expansion for the Laplacian operator yields \( L_h u_l - \Delta u_l = O(h^2) \). This together with Proposition 7 implies the following asymptotic decay:
\[ ||u_l - u_h^l||_{L^\infty(\Omega_h)} = O(h^2). \]

Similar convergence rates have been obtained in [3, 17, 25] for the difference schemes of one-phase obstacle-like problems.

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