PRIMORDIAL BLACK HOLES AS A PROBE OF THE EARLY UNIVERSE AND A VARYING GRAVITATIONAL CONSTANT

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Abstract. We discuss recent developments in the study of primordial black holes, focussing particularly on their formation and quantum evaporation. Such studies can place important constraints on models of the early Universe. An especially interesting development has been the realization that such constraints may be severely modified if the value of the gravitational "constant" $G$ varies with cosmological epoch, a possibility which arises in many scenarios for the early Universe. The nature of the modification depends upon whether the value of $G$ near a black hole maintains the value it had at its formation epoch (corresponding to gravitational memory) or whether it tracks the background cosmological value. This is still uncertain but we discuss various approaches which might help to resolve the issue.

1. Introduction

It is well known that primordial black holes (PBHs) could have formed in the early Universe [1, 2]. A comparison of the cosmological density at any time after the Big Bang with the density associated with a black hole shows that PBHs would have of order the particle horizon mass at their formation epoch:

$$M(t) \approx \frac{c^3 t}{G} \approx 10^{15} \left( \frac{t}{10^{-23} \text{ s}} \right) \text{g.}$$

(1)

PBHs could thus span an enormous mass range: those formed at the Planck time ($10^{-43}$s) would have the Planck mass ($10^{-5}$g), whereas those formed at 1 s would be as large as $10^5 M_\odot$, comparable to the mass of the holes thought to reside in galactic nuclei. PBHs would could arise in various ways [3]. Since the early Universe is unlikely to have been exactly Friedmann,
they would form most naturally from initial inhomogeneities but they might also form through other mechanisms at a cosmological phase transition.

The realization that small PBHs might exist prompted Hawking to study their quantum properties. This led to his famous discovery [4] that black holes radiate thermally with a temperature

$$ T = \frac{\hbar c^3}{8\pi GMk} \approx 10^{26} \left(\frac{M}{g}\right)^{-1} \text{K} \approx 10^{-7} \left(\frac{M}{M_\odot}\right)^{-1} \text{K}. \quad (2) $$

This means that they evaporate on a timescale

$$ \tau(M) \approx \frac{\hbar c^4}{G^2 M^2} \approx 10^{64} \left(\frac{M}{M_\odot}\right)^3 \text{y}. \quad (3) $$

Only black holes smaller than $10^{15}$g would have evaporated by the present epoch, so eqn (1) implies that this effect could be important only for black holes which formed before $10^{-23}$s. Despite the conceptual importance of this result, it is bad news for PBH enthusiasts. For since PBHs with a mass of $10^{15}$g, which evaporate at the present epoch, would have a temperature of order 100 MeV, the observational limit on the $\gamma$-ray background intensity at 100 MeV immediately implies that their density could not exceed $10^{-8}$ times the critical density [5]. Not only does this render PBHs unlikely dark matter candidates, it also implies that there is little chance of detecting black hole explosions at the present epoch [6].

Despite this conclusion, PBH evaporations could still have interesting cosmological consequences. In particular, they might generate the microwave background [7] or modify the standard cosmological nucleosynthesis scenario [8] or contribute to the cosmic baryon asymmetry [9]. PBH evaporations might also account for the annihilation-line radiation coming from the Galactic centre [10] or the unexpectedly high fraction of antiprotons in cosmic rays [11]. PBH explosions occurring in an interstellar magnetic field might also lead to radio bursts [12]. Even if PBHs had none of these consequences, studying such effects leads to strong upper limits on how many of them could ever have formed and thereby constrains models of the early Universe. Indeed PBHs serve as a probe of times much earlier than that associated with most other “relicts” of the Big Bang. While photons decoupled at $10^6$y, neutrinos at 1 s and WIMPs at $10^{-10}$s, PBHs go all the way back to the Planck time. Therefore even if PBHs never formed, their non-existence gives interesting information.

We review the formation mechanisms and evaporation constraints on PBHs in Section 2. Much of this material is also contained in my contribution to the 1996 Chalonge School [13]. However, there have been several interesting developments since then and these are covered in Section 3. The
remaining sections will examine how the PBH constraints are modified if the value of the gravitational “constant” $G$ was different at early times.

As reviewed in Section 4, this idea has a long history and should no longer be regarded as exotic. It arises in various scalar-tensor theories of gravity and these are a natural setting for many currently popular models of the early Universe. Black hole formation and evaporation could be greatly modified in variable-$G$ cosmologies, since many of their properties (eg. their Hawking temperature) depend explicitly on $G$. However, as emphasized by Barrow [14] and discussed in Section 5, the nature of the modification depends upon whether the PBH preserves the value of $G$ at its formation epoch (corresponding to what is termed “gravitational memory”) or always maintains the changing background value. There would be interesting modifications to the cosmological consequences of PBH evaporation in both cases but they would be more dramatic in the first. Barrow & Carr [15] considered the implications of these two scenarios in detail and this work has been taken further with Goymer [16]. We will review the conclusions of these papers in Section 6 and highlight a particularly interesting development in Section 7.

2. PBH formation and constraints on the early Universe

One of the most important reasons for studying PBHs is that it enables one to place limits on the spectrum of density fluctuations in the early Universe. This is because, if the PBHs form directly from density perturbations, the fraction of regions undergoing collapse at any epoch is determined by the root-mean-square amplitude $\epsilon$ of the fluctuations entering the horizon at that epoch and the equation of state $p = \gamma \rho$ ($0 < \gamma < 1$). One usually expects a radiation equation of state ($\gamma = 1/3$) in the early Universe. In order to collapse against the pressure, an overdense region must be larger than the Jeans length at maximum expansion and this is just $\sqrt{\gamma}$ times the horizon size. On the other hand, it cannot be larger than the horizon size, else it would form a separate closed universe and not be part of our Universe [17].

This has two important implications. Firstly, PBHs forming at time $t$ should have of order the horizon mass given by eqn (1). Secondly, for a region destined to collapse to a PBH, one requires the fractional overdensity at the horizon epoch, $\delta$, to exceed $\gamma$. Providing the density fluctuations have a Gaussian distribution and are spherically symmetric, one can infer that the fraction of regions of mass $M$ which collapse is [18]

$$\beta(M) \sim \epsilon(M) \exp \left[ -\frac{\gamma^2}{2\epsilon(M)^2} \right]$$ (4)
where $\epsilon(M)$ is the value of $\epsilon$ when the horizon mass is $M$. The PBHs can have an extended mass spectrum only if the fluctuations are scale-invariant (i.e. with $\epsilon$ independent of $M$) but this is expected in many scenarios.

The fluctuations required to make the PBHs may either be primordial or they may arise spontaneously at some epoch. One natural source of fluctuations would be inflation [19, 20] and, in this context, $\epsilon(M)$ depends implicitly on the inflationary potential. PBHs formed before inflation would be drastically diluted but new ones could form from the fluctuations generated after inflation. Many people have studied PBH formation in this context [21, 22, 23, 24, 25, 26, 27, 28, 29] as an important way of constraining the inflationary potential. This was the focus of my 1996 Erice lecture, so it will not be covered again here. Note that the Gaussian assumption has been questioned in the inflationary context [30, 31], so eqn (4) may not apply, but one still finds that $\beta$ depends very sensitively on $\epsilon$.

In some situations eqn (4) would fail qualitatively. For example, PBHs would form more easily if the equation of state of the Universe were ever soft ($\gamma \ll 1$). This might apply if there was a phase transition which channelled the mass of the Universe into non-relativistic particles or which temporally reduced the pressure. In this case, only those regions which are sufficiently spherically symmetric at maximum expansion can undergo collapse; the dependence of $\beta$ on $\epsilon$ would then have the form [32]

$$\beta = 0.02 \epsilon^{13/2},$$

which is much weaker than indicated by eqn (4), but there would still be a unique relationship between the two parameters. Some formation mechanisms for PBHs do not depend on having primordial fluctuations at all. For example, at any spontaneously broken symmetry epoch, PBHs might form through the collisions of bubbles of broken symmetry [33, 34, 35]. PBHs might also form spontaneously through the collapse of cosmic strings [36, 37, 38, 39, 40] or domain walls [41]. In these cases $\beta(M)$ depends, not on $\epsilon(M)$, but on other cosmological parameters, such the bubble formation rate or the string mass-per-length. These mechanisms were discussed in more detail in my 1996 Erice contribution [13].

In all these scenarios, the current density parameter $\Omega_{PBH}$ associated with PBHs which form at a redshift $z$ or time $t$ is related to $\beta$ by [18]

$$\Omega_{PBH} = \beta \Omega_R (1 + z) \approx 10^6 \beta \left(\frac{t}{s}\right)^{-1/2} \approx 10^{18} \beta \left(\frac{M}{10^{15} g}\right)^{-1/2}$$

where $\Omega_R \approx 10^{-4}$ is the density parameter of the microwave background and we have used eqn (1). The $(1 + z)$ factor arises because the radiation density scales as $(1 + z)^4$, whereas the PBH density scales as $(1 + z)^3$. Any
limit on $\Omega_{\text{PBH}}$ therefore places a constraint on $\beta(M)$ and the constraints are summarized in Fig. 1. The constraint for non-evaporating mass ranges above $10^{15}$ g comes from requiring $\Omega_{\text{PBH}} < 1$. Stronger constraints are associated with PBHs smaller than this since they would have evaporated by now [42, 43, 44, 45]. The strongest one is the $\gamma$-ray limit associated with the $10^{15}$ g PBHs evaporating at the present epoch [5]. Other ones are associated with the generation of entropy and modifications to the cosmological production of light elements. The constraints below $10^6$ g are based on the (not necessarily secure) assumption that evaporating PBHs leave stable Planck mass relics, in which case these relics are required to have less than the critical density [22, 46, 47, 48].

The constraints on $\beta(M)$ can be converted into constraints on $\epsilon(M)$ using eqn (4) and these are shown in Fig. 2. Also shown here are the (non-PBH) constraints associated with the spectral distortions in the cosmic microwave background induced by the dissipation of intermediate scale density perturbations and the COBE quadrupole measurement, as well as lines corresponding to various slopes in the $\epsilon(M)$ relationship. This shows that one needs the fluctuation amplitude to decrease with increasing scale in order to produce PBHs.
3. Recent Developments

Recent hydrodynamical calculations for the $\gamma = 1/3$ case have refined the criterion $\delta > \gamma$ for PBH formation and this modifies the estimate for $\beta(M)$ given by eqn (4). Niemeyer & Jedamzik [49] find that one needs $\delta > 0.8$ rather than $\delta > 0.3$ to ensure PBH formation, and Shibata & Sasaki [50] reach similar conclusions. They also find that there is little accretion after PBH formation, as expected theoretically [17].

Another interesting development has been the application of “critical phenomena” to PBH formation. Studies of the collapse of various types of spherically symmetric matter fields have shown that there is always a critical solution which separates those configurations which form a black hole from those which disperse to an asymptotically flat state. The configurations are described by some index $p$ and, as the critical index $p_c$ is approached, the black hole mass is found to scale as $(p - p_c)^{\eta}$ for some exponent $\eta$. This effect was first discovered for scalar fields [51] but subsequently demonstrated for radiation [52] and then more general fluids with equation of state $p = \gamma \rho$ [53, 54].

In all these studies the spacetime was assumed to be asymptotically flat. However, Niemeyer & Jedamzik [55] have recently applied the same idea to study black hole formation in asymptotically Friedmann models.
and have found similar results. For a variety of initial density perturbation profiles, they find that the relationship between the PBH mass and the horizon-scale density perturbation has the form

$$M = K M_H (\delta - \delta_c)^\gamma$$

(7)

where $M_H$ is the horizon mass and the constants are in the range $0.34 < \gamma < 0.37$, $2.4 < K < 11.9$ and $0.67 < \delta_c < 0.71$ for the various configurations. Since $M \to 0$ as $\delta \to \delta_c$, this suggests that PBHs may be much smaller than the particle horizon at formation (although it is clear that a fluid description must break down if they are too small) and it also modifies the mass spectrum [56, 57].

There has been particular interest recently in whether PBHs could have formed at the quark-hadron phase transition at $10^{-5}$s. This is because the horizon mass is of order $1 M_\odot$ then, so such PBHs would naturally have the sort of mass required to explain the MACHO microlensing results [58]. This is discussed in more detail in my other lecture at this meeting. One might expect PBHs to form more easily at that epoch because of a temporary softening of the equation of state. If the QCD phase transition is assumed to be of 1st order, then hydrodynamical calculations show that the value of $\delta$ required for PBH formation is indeed reduced below the value which pertains in the radiation case [59]. This means that PBH formation will be strongly enhanced at the QCD epoch, with the mass distribution being peaked around the horizon mass.

One of the interesting implications of the PBH MACHO scenario is the possible existence of a halo population of binary black holes [60]. With a full halo of such objects, there could then be $10^8$ binaries inside 50 kpc and some of these could be coalescing due to gravitational radiation losses at the present epoch [61]. Current interferometers (such as LIGO) could detect such coalescences within 50 Mpc, corresponding to a few events per year. Future space-borne interferometers (such as LISA) might detect 100 coalescences per year. If the associated gravitational waves were detected, it would provide a unique probe of the halo distribution (eg. its density profile and core radius [62].

Kohri & Yokoyama [63] have recently improved the constraints on $\beta(10^8-10^{10} g)$ which come from cosmological nucleosynthesis considerations. Constraints from neutrino background have also been presented by Bugaev & Konischev [64]. The recent detection of a Galactic $\gamma$-ray background [65], measurements of the antiproton flux [66], and the discovery of very short period $\gamma$-ray burts [67] may even provide positive evidence for such PBHs. This is discussed in detail elsewhere [68].

Some people have emphasized the possibility of detecting very high energy cosmic rays from PBHs using air shower techniques [69, 70]. However,
recently these efforts have been set back by the claim of Heckler [71] that QED interactions could produce an optically thick photosphere once the black hole temperature exceeds $T_{\text{crit}} = 45 \text{ GeV}$. In this case, the mean photon energy is reduced to $m_e(T_{\text{BH}}/T_{\text{crit}})^{1/2}$, which is well below $T_{\text{BH}}$, so the number of high energy photons is much reduced. He has proposed that a similar effect may operate at even lower temperatures due to QCD effects [72]. This is discussed further in the contribution of Kapusta at this meeting [73]. However, these arguments should not be regarded as definitive: MacGibbon et al. [74] claim that Heckler has not included Lorentz factors correctly in going from the black hole frame to the centre-of-mass frame of the interacting particles; in their calculation QED interactions are never important.

4. Cosmology in varying-G theories

Most variable-G scenarios associate the gravitational “constant” with some form of scalar field $\phi$. This notion has its roots in Kaluza-Klein theory, in which a scalar field appears in the metric component $g_{55}$ associated with the 5th dimension. Einstein-Maxwell theory then requires that this field be related to $G$ [75]. Although this was assumed constant in the original Kaluza-Klein theory, Dirac [76] noted the the ratio of the electric to gravitational force between to protons ($e^2/Gm_p^2$) and the ratio of the age of the Universe to the atomic timescale ($t/t_a$) and the square-root of the number of particles in the Universe ($\sqrt{M/m_p}$) are all comparable and of order $10^{40}$. This unlikely coincidence led him to propose that these relationships must always apply, which requires

$$G \propto t^{-1}, \quad GM/R \sim 1,$$

where $R \sim ct$ is the horizon scale. The first condition led Jordan [77] to propose a theory in which the scalar field in Kaluza-Klein theory is a function of both space and time, and this then implies that $G \sim \phi^{-1}$ has the same property. The second condition implies the Mach-type relationship $\phi \sim M/R$, which suggests [78] that $\phi$ is a solution of the wave equation $\Box \phi \sim \rho$. This motivated Brans-Dicke (BD) theory [79], in which the Einstein-Hilbert Lagrangian is replaced by

$$L = \phi R - \frac{\omega}{\phi} \phi_{\mu\nu} \phi^{\mu\nu} + L_m,$$

where $L_m$ is the matter Langrangian and the constant $\omega$ is the BD parameter. The potential $\phi$ then satisfies

$$\Box \phi = \left( \frac{8\pi}{2\omega + 3} \right) T,$$
where $T$ is the trace of the matter stress-energy tensor, and this has the required Machian form. Since $\phi$ must have a contribution from local sources of the form $\Sigma_i(m_i/r_i)$, this entails a violation of the Strong Equivalence Principle. In order to test this, the PPN formalism was introduced. Applications of this test in a variety of astrophysical situations (involving the solar system, the binary pulsar and white dwarf cooling) currently require $|\omega| > 500$, which implies that the deviations from general relativity can only ever be small in BD theory [80].

The introduction of generalized scalar-tensor theories [81, 82, 83], in which $\omega$ is itself a function of $\phi$, led to a considerably broader range of variable-$G$ theories. In particular, it permitted the possibility that $\omega$ may have been small at early times (allowing noticeable variations of $G$ then) even if it is large today. In the last decade interest in such theories has been revitalized as a result of early Universe studies. Inflation theory [84] has made the introduction of scalar fields almost mandatory and extended inflation specifically requires a model in which $G$ varies [35]. In higher dimensional Kaluza-Klein-type cosmologies, the variation in the sizes of the extra dimensions also naturally leads to a variation in $G$ [85, 86, 87].

The currently popular low energy string cosmologies necessarily involve a scalar (dilaton) field [88] and bosonic superstring theory, in particular, leads [89] to a Lagrangian of the form (9) with $\omega = -1$.

The intimate connection between dilatons, inflatons and scalar-tensor theory arises because one can always transform from the (physical) Jordan frame to the Einstein frame, in which the Lagrangian has the standard Einstein-Hilbert form [90]

$$L = \bar{R} - 2\psi_{,\mu}\psi_{,\nu}\bar{g}^{\mu\nu} + L_m. \tag{11}$$

Here the new scalar field $\psi$ is defined by

$$d\psi = \left(\frac{2\omega + 3}{2}\right)^{1/2}\frac{d\phi}{\phi}. \tag{12}$$

and the barred (Einstein) metric and gravitational constant are related to the original (Jordan) ones by

$$g_{\mu\nu} = A(\phi)^2\bar{g}_{\mu\nu}, \quad G = [1 + \alpha^2(\phi)]A(\phi)^2\bar{G}, \quad \alpha \equiv A'/A, \tag{13}$$

where the function $A(\phi)$ specifies a conformal transformation and a prime denotes $d/d\phi$. Thus scalar-tensor theory can be related to general relativity plus a scalar field, although the theories are not identical because particles do not follow geodesics in the Einstein frame.

The behaviour of homogeneous cosmological models in BD theory is well understood [91]. Their crucial feature is that they are vacuum-dominated at
early times but always tend towards the general relativistic solution during the radiation-dominated era. This is a consequence of the fact that the radiation energy-momentum tensor is trace-free [i.e. $T = 0$ in eqn (10)]. This means that the full radiation solution can be approximated by joining a BD vacuum solution to a general relativistic radiation solution at some time $t_1$, which may be regarded as a free parameter of the theory. However, when the matter density becomes greater than the radiation density at $t_e \approx 10^{11}$ s, the equation of state becomes that of dust ($p = 0$) and $G$ begins to vary again. For a $k = 0$ model, one can show that in the three eras [15]

$$G = G_0(t_0/t_e)^n, \quad a \propto t^{(2-n)/3} \quad (t > t_e)$$

$$G = G_e \equiv G_0(t_0/t_e)^n, \quad a \propto t^{1/2} \quad (t_1 < t < t_e)$$

$$G = G_e(t/t_1)^{-n + \sqrt{4n + 1}}/2, \quad a \propto t^{(2-n-\sqrt{4n+1})/6} \quad (t < t_1)$$

where $G_0$ is the value of $G$ at the current time $t_0$, $n \equiv 2/(4 + 3\omega)$ and $(t_0/t_e) \approx 10^6$. Since the BD coupling constant is constrained by $|\omega| > 500$, which implies $|n| < 0.001$, eqns (14) to (16) imply that the deviations from general relativity are never large if the value of $n$ is always the same. However, as we now explain, it is also interesting to consider BD models in which $n$ and $\omega$ can vary and thus violate the current constraints.

The behaviour of cosmological models in more general scalar-tensor theories depends on the form of $\omega(\phi)$ but they still retain the feature that the general relativistic solution is a late-time attractor during the radiation era. Since one requires $G \approx G_0$ to 10% at the epoch of primordial nucleosynthesis [91], one needs the vacuum-dominated phase to end at some time $t_v < 1$ s. The theory approaches general relativity in the weak field limit only if $\omega \to \infty$ and $\omega'/\omega^3 \to 0$ but $\omega(\phi)$ is otherwise unconstrained. Barrow & Carr consider a toy model in which

$$2\omega + 3 = 2\beta(1 - \phi/\phi_c)^{-\alpha}$$

where $\alpha$ and $\beta$ are constants. This leads to

$$2\omega + 3 \propto t^{-\alpha/(2-\alpha)}, \quad \omega'/\omega^3 \propto t^{(1-2\alpha)/(1-\alpha)} \quad (t < t_v),$$

so one requires $1/2 < \alpha < 2$ in order to have $\omega \to \infty$ and $\omega'/\omega^3 \to 0$ as $t \to \infty$. In the $\alpha = 1$ case, one finds

$$G \propto t^{-2\lambda/(3-\lambda)}, \quad a \propto t^{(1-\lambda)/(3-\lambda)}, \quad \lambda \equiv \sqrt{3/(2\beta)} \quad (t < t_v).$$

During the vacuum-dominated era, such models can therefore be regarded as BD solutions in which $\omega$ is determined by the parameter $\beta$ and unconstrained by any limits on $\omega$ at the present epoch. After $t_v$, $G$ is constant and one has the standard radiation-dominated or dust-dominated behaviour.
The consequences of the cosmological variation of $G$ for PBH evaporation depend upon how the value of $G$ near the black hole evolves. Barrow [14] introduces two possibilities: in scenario A, $G$ everywhere maintains the background cosmological value (so $\phi$ is homogeneous); in scenario B, it preserves the value it had at the formation epoch near the black hole even though it evolves at large distances (so $\phi$ becomes inhomogeneous). On the assumption that a PBH of mass $M$ has a temperature and mass-loss rate

$$T = (8\pi GM)^{-1}, \quad \dot{M} \approx -\left(GM\right)^{-2},$$

with $G = G(t)$ in scenario A and $G = G(M)$ in scenario B, Barrow & Carr calculate the evaporation time $\tau$ for various values of the parameters $n$ and $t_1$ in BD theory [15]. The results are shown in Fig. 3(a) for scenario A and Fig. 3(b) for scenario B. Here $M_*$ is the mass of a PBH evaporating at the present epoch, $M_e$ is the mass of a PBH evaporating at time $t_e$ and $M_{\text{crit}}$ is the mass of a PBH evaporating at the present epoch in the standard (constant $G$) scenario. In scenario A with $n < -1/2$, there is a maximum mass of a PBH which can ever evaporate and this is denoted by $M_\infty$. The results for the scalar-tensor with $\omega(\phi)$ given by eqn (17) with $\alpha = 1$ are shown in Fig. 3(c) for scenario B with various values of the parameters $\lambda$ and $t_v$. The corresponding modifications to the constraints on $\beta(M)$ in all three cases are shown in Fig. 3(d), which should be compared to Fig. 1.

5. **Black holes in scalar-tensor theory**

Barrow & Carr considered both scenarios A and B but did not attempt to decide which was more plausible. In this section we address this question more carefully. The main argument for scenario A comes from an important result of Hawking [92]. He showed that in BD theory, providing the weak energy condition holds, the gradient of $\phi$ must be zero everywhere for stationary, asymptotically flat black holes. This means that such black holes are identical to those in general relativity. This result can be generalized to all scalar-tensor theories and suggests that such theories are in agreement with the “no-hair” theorem.

Numerical calculations support this theorem [93, 94, 95]. Collapse is accompanied by outgoing scalar gravitational radiation, which radiates away the scalar mass until the black hole settles down to the Schwarzschild form with a constant scalar field. In particular, Harada et al. [96] have investigated Oppenheimer-Snyder collapse in which a ball of dust described by a $k = +1$ Friedmann interior and a Schwarzschild exterior collapses to a black hole. In these calculations the scalar field is taken to be constant before the collapse and its back-reaction on the metric is assumed to be always negligible. It is found that, as the collapse proceeds, a scalar gravitational wave
propagates outwards before the scalar field settles down to being constant again.

It should be stressed that the scalar no hair theorem has only been proved for asymptotically flat spacetimes, so it is not clear that it also applies in the asymptotically Friedmann case. While the no hair theorem suggests that $\phi$ should tend to a \emph{locally} constant value (close to the black hole), it is not obvious that this needs to be the asymptotic cosmological value. Indeed, since the homogeneizing of $\phi$ is only ensured by scalar wave emission, one might infer that this can only be achieved on scales less than the particle horizon.

One way to determine what happens is to seek a precise mathematical model for a black hole in a cosmological background. For example, one can try to match a black hole and cosmological solution over some boundary
Such a matching is provided in general relativity by the Einstein-Straus or “Swiss cheese” model [97]. Here a Friedmann exterior is matched with a general spherically symmetric interior. If there is no scalar field, it turns out that the latter has to be the static Schwarzschild solution but the situation may be more complicated in the present context due to the presence of scalar gravitational radiation. In general one can show that the following continuity conditions must apply at Σ:

\[
\begin{align*}
[g_\mu\nu] &= 0, \\
[G_\mu\nu n^\mu n^\nu] &= 0, \\
[\phi] &= 0, \\
[\phi_\mu n^\mu] &= 0
\end{align*}
\]  

(21)

where \(n^\mu\) and \(u^\mu\) are 4-vectors normal and tangent to Σ, respectively [98]. Unfortunately, it turns out that an Einstein-Straus type solution does not exist in BD theory. This is because the only way to satisfy the junction conditions (21) is if \(\phi\) is spatially and temporally constant, which is just the general relativistic case.

Jacobsen [99] has addressed the problem analytically by looking for a spherically symmetric solution which represents a perturbation of the Schwarzschild solution near the origin but is asymptotically Friedmann at large distances, with \(\phi\) satisfying the appropriate cosmological conditions. He presupposes that the black hole event horizon is much smaller than the particle horizon, so that the cosmological timescale is much longer than the black hole timescale. In this case, he finds that there is little lag between the value of \(\phi\) at the event horizon and particle horizon, which suggests that memory can only be weak. However, it must be emphasized that this conclusion need not follow if the black hole has a size comparable to the particle horizon at formation and, as indicated in Section 2, this is expected for a PBH.

Another way to investigate the problem is to study the collapse of dust in a Tolman-Bondi background using the same approximation employed by Harada et al. [96], i.e. neglecting the back reaction of the scalar field, but requiring that \(\phi\) have the required cosmological time-dependence at large distances. One puts in an initial density perturbation for the dust but assumes that \(\phi\) is initially homogeneous. Our preliminary numerical calculations [100] use the characteristic method to determine the evolution of the scalar field perturbation along null and constant-time hypersurfaces. We find that \(\phi\) does initially build up near the centre but it then gets smoothed out, tending eventually to homogeneity. Although this suggests that there is no gravitational memory, it should be stressed that this conclusion only applies for dust and if one neglects the back reaction.

6. Variations of gravitational memory

Since we lack definite knowledge about the evolution of the scalar field when a black hole forms in a cosmological background, it is useful to consider
a range of scenarios which go beyond the two possibilities envisaged by Barrow. In general, the background scalar field will have a present value $\phi(t_0)$ and a value $\phi(t_f)$ when the black hole first formed. However, it is likely to develop inhomogeneities for at least some intervening period. In the following discussion, we will characterize the degree of gravitational memory by comparing the value of the scalar field at the black hole event horizon ($\phi_{EH}$) and the cosmological particle horizon ($\phi_{PH}$). We first consider the two extreme situations described by Barrow [14]:

**Scenario A**: \[ \phi_{EH}(t) = \phi_{PH}(t) \quad \text{for all } t \quad (22) \]

A Schwarzschild black hole forms at time $t_f$ with its event horizon radius being $R_f = 2G(t_f)M$. If $G(t)$ evolves with time, then the black hole adjusts quasi-statically through a sequence of Schwarzschild states approximated by $R = 2G(t)M$, see Fig. 4(a). In this scenario there is no gravitational memory.

**Scenario B**: \[ \phi_{EH}(t) = \phi_{EH}(t_f) \quad \text{for all } t \quad (23) \]

A Schwarzschild black hole of size $R_f$ forms at time $t_f$ and, while $G(t)$ equals the evolving background value beyond some scale-length $R_m \geq R_f$, it remains constant within $R_m$, see Fig. 4(b). In this case the black hole size is determined by $G(t_f)$ even at the present epoch and this means that the region $R < R_m$ has a memory of the gravitational “constant” at the time of its formation.

Neither of these scenarios can be completely realistic since they both assume that $\phi$ is homogeneous almost everywhere. However, even if $\phi$ were homogeneous initially, one would expect it to become inhomogeneous as collapse proceeds. Indeed, in the dust case, this is confirmed by the numerical calculations described above [100]. Therefore, if the background value is increasing (as usually applies), one would expect $\phi$ in the collapsing region to become first larger than the background value on a local dynamical timescale and then smaller than it on a cosmological timescale. Such behaviour would necessarily entail a variation of $\phi$ in space as well as time. We must also allow for the possibility that $\phi$ may vary interior to $R_m$ but on a slower or faster timescale than the background. We therefore propose two further scenarios:

**Scenario C**: \[ |\dot{\phi}_{EH}(t)| \geq |\dot{\phi}_{PH}(t)| \quad \text{for all } t \quad (24) \]

where the dot represents a time derivative. This implies that the scalar field evolves faster at the event horizon than at the particle horizon until it
eventually becomes homogeneous, see Fig. 4(c). We describe this as short-term gravitational memory and it reduces to scenario A as the timescale to become homogeneous tends to zero. This would apply, for example, if $\phi$ were to change on the dynamical timescales of the black hole since this is usually less than the cosmological timescale.

**Scenario D:** $|\dot{\phi}_{EH}(t)| < |\dot{\phi}_{PH}(t)|$ for all $t$ (25)

This implies that $\phi$ evolves faster at the particle horizon than the event horizon, see Fig. 4(d). We describe this as weak gravitational memory and it reduces to scenario B when the left-hand-side of eqn (25) is zero. In this case, the evolution of $\phi$ is again dominated by the black hole inside some length-scale $R_m$. Note that, in either this scenario or the last one, the length-scale $R_m$ need not be fixed, since it could either grow or shrink as scalar gravitational radiation propagates. A particular example of this, to which we return shortly, would be self-similar gravitational memory, in which the ratio of $\phi_{EH}$ to $\phi_{PH}$ always remains the same.

### 7. Gravitational memory and the accretion of a stiff fluid

In general relativity there is an equivalence between a scalar field and a stiff fluid and this can be exploited in studying gravitational memory. In the Einstein frame, the energy momentum tensor for a perfect fluid is

$$\bar{T}_{\mu\nu} = (\rho + p) u_\mu u_\nu + \bar{g}_{\mu\nu} p$$

(26)

where $u_\mu$ is the velocity of the fluid. If we define a velocity field by

$$u_\mu = \frac{\bar{\phi}_\mu}{(\bar{g}^{\rho\sigma} \bar{\phi}_\rho \bar{\phi}_\sigma)^{1/2}},$$

(27)

this gives

$$\bar{T}_{\mu\nu} = -\frac{(\rho + p) \bar{\phi}_\mu \bar{\phi}_\nu}{\bar{g}^{\rho\sigma} \bar{\phi}_\rho \bar{\phi}_\sigma} + p \bar{g}_{\mu\nu}.$$  

(28)

By comparing this to the energy-momentum tensor for a scalar field, we find that

$$p = \rho = -\frac{1}{2} \bar{g}^{\rho\sigma} \bar{\phi}_\rho \bar{\phi}_\sigma,$$

(29)

so we have a stiff fluid. This equivalence applies provided that the derivative of the scalar field is timelike. Otherwise the velocity field defined in (27) would be imaginary.

This is relevant to the gravitational memory problem because we can now interpret the various scenarios discussed in Section 6 in terms of the
accretion} of a stiff fluid. If the black hole does not accrete at all or accretes very little, this will correspond to strong or weak gravitational memory (scenarios B and D, respectively). However, if enough accretion occurs to homogenize \( \phi \), this will correspond to short-term gravitational memory (scenario C). The faster the accretion, the shorter the memory, so scenario A corresponds to the idealization in which homogenization is instantaneous.

A simple Newtonian treatment [1] for a general fluid suggests that the accretion rate in the Einstein frame should be

\[
\dot{M} = 4\pi \rho R_A^2 v_s, \tag{30}
\]

where \( R_A = GM/v_s^2 \) is the accretion radius and \( v_s \) is the sound-speed in the accreted fluid. For a stiff fluid, \( v_s = c \) and \( R_A = GM/c^2 \), while \( \rho \sim 1/(Gt^2) \) in a Friedmann universe at early times, so we have

\[
\frac{dM}{dt} \approx \frac{GM^2}{c^3 t^2}. \tag{31}
\]
This can be integrated to give

\[ M \approx \frac{c^3 t/G}{1 + \frac{t}{t_f} \left( \frac{c^3 t_f}{GM_f} - 1 \right)} \]  

(32)

where \( M_f \) is the black hole mass at the time \( t_f \) when it formed. If we define a parameter \( \eta = GM_f/c^3 t_f \), then eqn (32) implies

\[ M \rightarrow M_f (1 - \eta)^{-1} \text{ as } t \rightarrow \infty. \]  

(33)

If \( \eta \ll 1 \), the black hole could not grow very much. However, if \( \eta \) is close to 1, which must be the case if \( v_s \approx c \), then the black hole could grow significantly. In particular, in the limit \( \eta = 1 \), eqn (32) implies \( M \sim t \), so the black hole grows at the same rate as the universe. This simple calculation suggests that a black hole surrounded by a stiff fluid can accrete enough to grow at the same rate as the Universe.

Since the above calculation neglects the effects of the cosmological expansion, one needs a relativistic calculation to check this. The Newtonian result suggests that one should look for a spherically symmetric self-similar solution, in which every dimensionless variable is a function of \( z = r/t \), so that it is unchanged by the transformation \( t \rightarrow at, r \rightarrow ar \) for any constant \( a \). This problem has an interesting but rather convolved history. By looking for a black hole solution attached to an exact Friedmann solution via a sonic point, Carr & Hawking first showed that there is no such solution for a radiation fluid [17] and the argument can be extended to a general \( p = \gamma \rho \) fluid with \( 0 < \gamma < 1 \). Lin et al. [101] subsequently claimed that there is such a solution in the special case \( \gamma = 1 \). However, Bicknell & Henriksen [102] then showed that this solution is unphysical, in that the density gradient diverges at the event horizon. This suggests that the black hole must soon become much smaller than the particle horizon, after which eqn (32) implies there will be very little further accretion. Therefore the stiff fluid analysis suggests that there should be at least weak gravitational memory.

8. Conclusions

We have seen that studying the formation and evaporation of PBHs can place interesting constraints on models of the early universe even if they never existed. On the other hand, if they did exist, PBHs can provide unique information about times much earlier than those probed by any other relics of the Big Bang. In particular, they may provide information about the variation of \( G \) at early times. The precise signature of such a variation depends upon the degree to which a black hole can “remember” the value of \( G \) at its formation epoch. This is still unclear but various methods are being pursued to resolve this issue.
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