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SELF-ADJOINT EXTENSIONS OF DISCRETE MAGNETIC SCHRÖDINGER OPERATORS

OGNJEN MILATOVIC, FRANÇOISE TRUC

Abstract. Using the concept of intrinsic metric on a locally finite weighted graph, we give sufficient conditions for the magnetic Schrödinger operator to be essentially self-adjoint. The present paper is an extension of some recent results proven in the context of graphs of bounded degree.

1. Introduction and the main results

1.1. The setting. Let \( V \) be a countably infinite set. We assume that \( V \) is equipped with a measure \( \mu: V \to (0, \infty) \). Let \( b: V \times V \to [0, \infty) \) be a function such that

(i) \( b(x, y) = b(y, x) \), for all \( x, y \in V \);
(ii) \( b(x, x) = 0 \), for all \( x \in V \);
(iii) \( \deg(x) := \sharp \{y \in V: b(x, y) > 0\} < \infty \), for all \( x \in V \). Here, \( \sharp S \) denotes the number of elements in the set \( S \).

Vertices \( x, y \in V \) with \( b(x, y) > 0 \) are called neighbors, and we denote this relationship by \( x \sim y \). We call the triple \((V, b, \mu)\) a locally finite weighted graph. We assume that \((V, b, \mu)\) is connected, that is, for any \( x, y \in V \) there exists a path \( \gamma \) joining \( x \) and \( y \). Here, \( \gamma \) is a sequence \( x_0, x_2, \ldots, x_n \in V \) such that \( x = x_0, y = x_n \), and \( x_j \sim x_{j+1} \) for all \( 0 \leq j \leq n - 1 \).

1.2. Intrinsic metric. Following [15] we define a pseudo metric to be a map \( d: V \times V \to [0, \infty) \) such that \( d(x, y) = d(y, x) \), for all \( x, y \in V \); \( d(x, x) = 0 \), for all \( x \in V \); and \( d(x, y) \) satisfies the triangle inequality. A pseudo-metric \( d = d_\sigma \) is called a path pseudo-metric if there exists a map \( \sigma: V \times V \to [0, \infty) \) such that \( \sigma(x, y) = \sigma(y, x) \), for all \( x, y \in V \); \( \sigma(x, y) > 0 \) if and only if \( x \sim y \); and

\[
d_\sigma = \inf \{ l_\sigma(\gamma): \gamma = (x_0, x_1, \ldots, x_n), n \geq 1, \text{is a path connecting } x \text{ and } y \},
\]

where the length \( l_\sigma \) of the path \( \gamma = (x_0, x_1, \ldots, x_n) \) is given by

\[
l_\sigma(\gamma) = \sum_{i=0}^{n-1} \sigma(x_i, x_{i+1}). \tag{1.1}
\]

As in [15] we make the following definitions.

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**Definition 1.3.** (i) A pseudo metric \(d\) on \((V, b, \mu)\) is called *intrinsic* if
\[
\frac{1}{\mu(x)} \sum_{y \in V} b(x, y)(d(x, y))^2 \leq 1, \quad \text{for all } x \in V.
\]
(ii) An intrinsic path pseudo metric \(d = d_\sigma\) on \((V, b, \mu)\) is called *strongly intrinsic* if
\[
\frac{1}{\mu(x)} \sum_{y \in V} b(x, y)(\sigma(x, y))^2 \leq 1, \quad \text{for all } x \in V.
\]

**Remark 1.4.** On a locally finite graph \((V, b, \mu)\), the formula
\[
\sigma_1(x, y) = b(x, y)^{-1/2} \min \left\{ \frac{\mu(x)}{\deg(x)}, \frac{\mu(y)}{\deg(y)} \right\}^{1/2}, \quad \text{with } x \sim y,
\]
where \(\deg(x)\) is as in property (iii) of \(b(x, y)\), defines a strongly intrinsic path metric; see [15, Example 2.1].

1.5. **Cauchy boundary.** For a path metric \(d = d_\sigma\) on \(V\), we denote the metric completion by \((\hat{V}, \hat{d})\). As in [15] we define the Cauchy boundary \(\partial_C V\) as follows: \(\partial_C V := \hat{V} \setminus V\). Note that \((V, d)\) is metrically complete if and only if \(\partial_C V\) is empty. For a path metric \(d = d_\sigma\) on \(V\) and \(x \in V\), we define
\[
D(x) := \inf_{z \in \partial_C V} d_\sigma(x, z). \tag{1.3}
\]

1.6. **Inner product.** In what follows, \(C(V)\) is the set of complex-valued functions on \(V\), and \(C_c(V)\) is the set of finitely supported elements of \(C(V)\). By \(\ell^2_\mu(V)\) we denote the space of functions \(f \in C(V)\) such that
\[
\|f\|^2 := \sum_{x \in V} \mu(x)|f(x)|^2 < \infty, \tag{1.4}
\]
where \(|\cdot|\) denotes the modulus of a complex number.

In particular, the space \(\ell^2_\mu(V)\) is a Hilbert space with the inner product
\[
(f, g) := \sum_{x \in V} \mu(x)f(x)\overline{g(x)}. \tag{1.5}
\]

1.7. **Laplacian operator.** We define the formal Laplacian \(\Delta_{b, \mu}: C(V) \to C(V)\) on \((V, b, \mu)\) by the formula
\[
(\Delta_{b, \mu} u)(x) = \frac{1}{\mu(x)} \sum_{y \in V} b(x, y)(u(x) - u(y)). \tag{1.6}
\]
1.8. **Magnetic Schrödinger operator.** We fix a phase function $\theta: V \times V \to [-\pi, \pi]$ such that $\theta(x, y) = -\theta(y, x)$ for all $x, y \in V$, and denote $\theta_{x,y} := \theta(x, y)$. We define the formal magnetic Laplacian $\Delta_{b,\mu;\theta}: C(V) \to C(V)$ on $(V, b, \mu)$ by the formula

$$\langle \Delta_{b,\mu;\theta} u \rangle(x) = \frac{1}{\mu(x)} \sum_{y \in V} b(x, y)(u(x) - e^{i\theta_{x,y}}u(y)).$$

We define the formal magnetic Schrödinger operator $H: C(V) \to C(V)$ by the formula

$$Hu := \Delta_{b,\mu;\theta} u + W u,$$

where $W: V \to \mathbb{R}$.

1.9. **Statements of the results.** We are ready to state our first result.

**Theorem 1.** Assume that $(V, b, \mu)$ is a locally finite, weighted, and connected graph. Let $d = d_\sigma$ be an intrinsic path metric on $V$ such that $(V, d)$ is not metrically complete. Assume that there exists a constant $C$ such that

$$W(x) \geq \frac{1}{2(D(x))^2} - C, \quad \text{for all } x \in V,$$

where $D(x)$ is as in (1.3). Then $H$ is essentially self-adjoint on $C_c(V)$.

**Remark 1.10.** It is possible to find $\mu, b$, and a potential $W$ satisfying $W(x) \geq \frac{k}{2(D(x))^2}$ with $0 < k < 1$, such that $H = \Delta_{b,\mu} + W$ is not essentially self-adjoint; see [2, Section 5.3.2].

If the graph $(V, b, \mu)$ has a special type of covering, the condition (1.9) on $W$ can be relaxed with the help of “effective potential,” as seen in the next theorem. First, we give a description of this special type of covering. In what follows, for a graph $(V, b, \mu)$, we define the set of unoriented edges as $E := \{\{x, y\}: x, y \in V \text{ and } b(x, y) > 0\}$. Sometimes, when we want to emphasize the set $E$, instead of $G = (V, b, \mu)$ we will use the notation $G = (V, E)$.

**Definition 1.11.** Let $m \in \mathbb{N}$. A **good covering of degree** $m$ of $G = (V, E)$ is a family $G_l = (V_l, E_l)_{l \in L}$ of finite connected sub-graphs of $G$ so that

(i) $V = \bigcup_{l \in L} V_l$;
(ii) for any $\{x, y\} \in E$, $0 < \#\{l \in L \mid \{x, y\} \in E_l\} \leq m$.

**Remark 1.12.** It is known that a graph with bounded vertex degree admits a good covering; see [3, Proposition 2.2]. The graph in Example 5.1 below does not have a bounded vertex degree. Note that this graph has a good covering of degree $m = 2$.

Assume that $(V, b, \mu)$ has a good covering $(V_l, E_l)_{l \in L}$. Let $\theta_l$ be the restriction of $\theta$ to $V_l \times V_l$. Let $\Delta_{1,\mu;\theta}$ be as in (1.7) with $V = V_l$, $\theta = \theta_l$, and $b \equiv 1$. Then $\Delta_{1,\mu;\theta}$ is a bounded and non-negative self-adjoint operator in $L^2(\mu(V_l))$. Let $p_l$ denote the lowest eigenvalue of $\Delta_{1,\mu;\theta}$. With
these notations, for a graph $(V, b, \mu)$ and the phase function $\theta$, we define the effective potential corresponding to a good covering $(V_l, E_l)_{l \in L}$ of degree $m$ as follows:

$$W_e(x) := \frac{1}{m} \sum_{\{l \in L \mid x \in V_l\}} p_l \inf_{\{y,z\} \in E_l} b(y, z). \quad (1.10)$$

We now state our second result.

**Theorem 2.** Assume that $(V, b, \mu)$ is a locally finite, weighted, and connected graph. Assume that $(V, b, \mu)$ has a good covering $(V_l, E_l)_{l \in L}$. Let $d = d_\sigma$ be an intrinsic path metric on $V$ such that $(V, d)$ is not metrically complete. Assume that there exists a constant $C$ such that

$$W_e(x) + W(x) \geq \frac{1}{2(D(x))^2} - C, \quad \text{for all } x \in V, \quad (1.11)$$

where $W_e$ is as in (1.10) and $D(x)$ is as in (1.3). Then $H$ is essentially self-adjoint on $C_c(V)$.

In the setting of metrically complete graphs, we have the following result:

**Theorem 3.** Assume that $(V, b, \mu)$ be a locally finite, weighted, and connected graph. Let $d_\sigma$ be a strongly intrinsic path metric on $V$. Let $q: V \to [1, \infty)$ be a function satisfying

$$|q^{-1/2}(x) - q^{-1/2}(y)| \leq K\sigma(x, y), \quad \text{for all } x, y \in V \text{ such that } x \sim y, \quad (1.12)$$

where $K$ is a constant. Let $H$ be as in (1.8) with $W: V \to \mathbb{R}$ satisfying

$$W(x) \geq -q(x), \quad \text{for all } x \in V. \quad (1.13)$$

Let

$$\sigma_q(x, y) = \min\{q^{-1/2}(x), q^{-1/2}(y)\} \cdot \sigma(x, y) \quad (1.14)$$

and let $d_{\sigma_q}$ be the path metric corresponding to $\sigma_q$. Assume that $(V, d_{\sigma_q})$ is metrically complete. Then $H$ is essentially self-adjoint on $C_c(V)$.

1.13. **Some comments on the existing literature.** The notion of intrinsic metric allows us to remove the bounded vertex degree assumption present in [2, 3, 20]. More specifically, Theorem 1 extends [2, Theorem 4.2], which was proven in the context of graphs of bounded vertex degree for the operator $\Delta_{b,\mu} + W$, with $\Delta_{b,\mu}$ as in (1.6). Theorem 2 is an extension of [3, Theorem 3.1], which was proven in the context of graphs of bounded vertex degree for the operator $\Delta_{b,\mu;\theta}$. In this regard, the first two results of the present paper answer a question posed in [3, Section 5]. Theorem 3 extends [20, Theorem 1], which was proven in the context of graphs of bounded vertex degree for the operator $\Delta_{b,\mu;\theta} + W$ with $W$ as in (1.13). We should also mention that in the context of locally finite graphs (with an assumption on $b$ and $\mu$ originating from [17]), a sufficient condition for the essential self-adjointness of a semi-bounded from below operator $\Delta_{b,\mu;\theta} + W$ is given in [19, Theorem 1.2]. Another sufficient condition for the essential self-adjointness of $\Delta_{b,\mu;\theta} + W$ is given in [9, Proposition 2.2]: Let $(V, b, \mu)$ be a locally finite weighted graph. Let $W: V \to \mathbb{R}$ and $\delta > 0$. Take $\lambda \in \mathbb{R}$ so that

$$\{x \in V : \lambda + \text{Deg}(x) + W(x) = 0\} = \emptyset, \quad (1.15)$$
where $\text{Deg}(x)$ denotes the “weighted degree”

$$
\text{Deg}(x) := \frac{1}{\mu(x)} \sum_{y \in V} b(x, y), \quad x \in V.
$$

(1.16)

Suppose that for every sequence of vertices \( \{y_1, y_2, \ldots \} \) such that \( y_j \sim y_{j+1}, j \geq 1 \), the following property holds:

$$
\sum_{n=1}^{\infty} (a_n^2 \mu(y_n)) = \infty, \quad \text{where} \quad a_n := \prod_{j=1}^{n-1} \left( \frac{\delta}{\text{Deg}(y_j)} + \left| 1 + \frac{\lambda + W(y_j)}{\text{Deg}(y_j)} \right| \right), \quad n \geq 2,
$$

(1.17)

and \( a_1 := 1 \). Then \( \Delta_{b,\mu;b} + W \) is essentially self-adjoint on \( C_c(V) \).

Note that [9, Proposition 2.2] allows potentials that are unbounded from below. We mention that Example 5.1 below describes a situation where Theorem 2 is applicable, while neither [19, Theorem 1.2] nor [9, Proposition 2.2] is applicable. Additionally, Example 5.2 below describes a situation where Theorem 3 is applicable, while neither [19, Theorem 1.2] nor [9, Proposition 2.2] is applicable.

The recent study [15] is concerned with the operator \( \Delta_{b,\mu} \) as in (1.6), with property (iii) of \( b \) (see Section 1.1 above) replaced by the following more general condition:

$$
\sum_{y \in V} b(x, y) < \infty, \quad \text{for all} \quad x \in V.
$$

Using the notion of intrinsic distance \( d \) with finite jump size, the authors of [15] show that if the weighted degree (1.16) is bounded on balls defined with respect to any such distance \( d \), then \( \Delta_{b,\mu} \) is essentially self-adjoint. In the context of a locally finite graph, the authors of [15] show that if the graph is metrically complete in any intrinsic path metric with finite jump size, then \( \Delta_{b,\mu} \) is essentially self-adjoint. In the metrically incomplete case, one of the results of [15] shows that if the Cauchy boundary has finite capacity, then \( \Delta_{b,\mu} \) has a unique Markovian extension if and only if the Cauchy boundary is polar (here, “Cauchy boundary is polar” means that the Cauchy boundary has zero capacity). Another result of [15] shows that if the upper Minkowski codimension of the Cauchy boundary is greater than 2, then the Cauchy boundary is polar.

Additionally, we should mention that the authors of [15] prove Hopf–Rinow-type theorem for locally finite weighted graphs with a path pseudo metric.

In recent years, various authors have developed independently the concept of intrinsic metric on a graph. The definition given in the present paper can be traced back to the work [8]. For applications of intrinsic metrics in various contexts, see, for instance, [1, 5, 6, 7, 10, 12, 13, 14, 18].

With regard to the problem of self-adjoint extensions of adjacency, (magnetic) Laplacian and Schrödinger-type operators on infinite graphs, we should mention that there has been a lot of interest in this area in the past few years. For references to the literature on this topic, see, for instance, [2, 3, 9, 11, 15, 17, 20, 24].
2. Proof of Theorem 1

In this section, we modify the proof of [2, Theorem 4.2]. Throughout the section, we assume that the hypotheses of Theorem 1 are satisfied. We begin with the following lemma, whose proof is given in [3, Lemma 3.3].

**Lemma 2.1.** Let $H$ be as in (1.8), let $v \in L^2_\mu(V)$ be a weak solution of $Hv = 0$, and let $f \in C_c(V)$ be a real-valued function. Then the following equality holds:

$$
(f, Hv) = \frac{1}{2} \sum_{x \in V} \sum_{y \sim x} b(x, y) \text{Re} \left[ e^{-i\theta(x,y)} v(x) \overline{v(y)} (f(x) - f(y))^2 \right].
$$

(2.1)

The key ingredient in the proof of Theorem 1 is the Agmon-type estimate given in the next lemma, whose proof, inspired by an idea of [21], is based on the technique developed in [4] for magnetic Laplacians on an open set with compact boundary in $\mathbb{R}^n$.

**Lemma 2.2.** Let $\lambda \in \mathbb{R}$ and let $v \in L^2_\mu(V)$ be a weak solution of $(H - \lambda)v = 0$. Assume that there exists a constant $c_1 > 0$ such that, for all $u \in C_c(V)$,

$$
(u, (H - \lambda)u) \geq \frac{1}{2} \sum_{x \in V} \max \left( \frac{1}{D(x)^2}, 1 \right) \mu(x) |u(x)|^2 + c_1 \|u\|^2.
$$

(2.2)

Then $v \equiv 0$.

**Proof.** Let $\rho$ and $R$ be numbers satisfying $0 < \rho < 1/2$ and $1 < R < +\infty$. For any $\epsilon > 0$, we define the function $f_\epsilon : V \to \mathbb{R}$ by $f_\epsilon(x) = F_\epsilon(D(x))$, where $D(x)$ is as in (1.3) and $F_\epsilon : \mathbb{R}^+ \to \mathbb{R}$ is the continuous piecewise affine function defined by

$$
F_\epsilon(s) = \begin{cases}
0 & \text{for } s \leq \epsilon \\
\rho(s - \epsilon)/(\rho - \epsilon) & \text{for } \epsilon \leq s \leq \rho \\
s & \text{for } \rho \leq s \leq 1 \\
1 & \text{for } 1 \leq s \leq R \\
R + 1 - s & \text{for } R \leq s \leq R + 1 \\
0 & \text{for } s \geq R + 1
\end{cases}
$$

We first note that by the definition of $F_\epsilon$ and continuity of $D(x)$, the support of $f_\epsilon$ is compact. Now by [15, Lemma A.3(b)] it follows that the support of $f_\epsilon$ finite. Using Lemma 2.1 with $H - \lambda$ in place of $H$, the inequality

$$
\text{Re} \left[ e^{-i\theta(x,y)} v(x) \overline{v(y)} \right] \leq \frac{1}{2} (|v(x)|^2 + |v(y)|^2),
$$

and Definition 1.3(i) we have

$$
(f_\epsilon v, (H - \lambda)(f_\epsilon v)) \leq \frac{1}{2} \sum_{x \in V} \sum_{y \sim x} b(x, y) |v(x)|^2 (f_\epsilon(x) - f_\epsilon(y))^2
$$

$$
\leq \frac{\rho^2}{2(\rho - \epsilon)^2} \sum_{x \in V} \sum_{y \sim x} |v(x)|^2 b(x, y) (d(x, y))^2 \leq \frac{\rho^2}{2(\rho - \epsilon)^2} \sum_{x \in V} \mu(x) |v(x)|^2,
$$

(2.3)

where the second inequality uses the fact that $f_\epsilon$ is a $\beta$-Lipschitz function with $\beta = \rho/(\rho - \epsilon)$. 
We now combine (2.4) and (2.3) to get

\[ \langle f_\varepsilon v, (H - \lambda)(f_\varepsilon v) \rangle \geq \frac{1}{2} \sum_{\rho \leq D(x) \leq R} \mu(x) |v(x)|^2 + c_1 \|f_\varepsilon v\|^2. \]  

(2.4)

We now combine (2.4) and (2.3) to get

\[ \frac{1}{2} \sum_{\rho \leq D(x) \leq R} \mu(x) |v(x)|^2 + c_1 \|f_\varepsilon v\|^2 \leq \frac{\rho^2}{2(\rho - \varepsilon)^2} \sum_{x \in V} \mu(x) |v(x)|^2. \]

We fix \( \rho \) and \( R \), and let \( \varepsilon \to 0^+ \). After that, we let \( \rho \to 0^+ \) and \( R \to +\infty \). As a result, we get \( v \equiv 0 \).

**Conclusion of the proof of Theorem 1.** Since \( \Delta_{b,\mu,\theta}|_{C_c(V)} \) is a non-negative operator, for all \( u \in C_c(V) \), we have

\[ \langle u, Hu \rangle \geq \sum_{x \in V} \mu(x)W(x)|u(x)|^2, \]

and, hence, by assumption (1.9) we get:

\[ \langle u, (H - \lambda)u \rangle \geq \frac{1}{2} \sum_{x \in V} \frac{1}{D(x)^2} \mu(x)|u(x)|^2 - (\lambda + C) \|u\|^2 \]

\[ \geq \frac{1}{2} \sum_{x \in V} \max \left( \frac{1}{D(x)^2}, 1 \right) \mu(x)|u(x)|^2 - (\lambda + C + 1/2) \|u\|^2. \]  

(2.5)

Choosing, for instance, \( \lambda = -C - 3/2 \) in (2.5) we get the inequality (2.2) with \( c_1 = 1 \).

Thus, \( (H - \lambda)|_{C_c(V)} \) with \( \lambda = -C - 3/2 \) is a symmetric operator satisfying \( \langle u, (H - \lambda)u \rangle \geq \|u\|^2 \), for all \( u \in C_c(V) \). In this case, it is known (see [22, Theorem X.26]) that the essential self-adjointness of \( (H - \lambda)|_{C_c(V)} \) is equivalent to the following statement: if \( v \in \ell^2_\mu(V) \) satisfies \( (H - \lambda)v = 0 \), then \( v = 0 \). Thus, by Lemma 2.2, the operator \( (H - \lambda)|_{C_c(V)} \) is essentially self-adjoint. Hence, \( H|_{C_c(V)} \) is essentially self-adjoint. \( \square \)

3. PROOF OF THEOREM 2

Throughout the section, we assume that the hypotheses of Theorem 2 are satisfied. We begin with the following lemma.

**Lemma 3.1.** Let \((V_l, E_l)_{l \in L}\) be a good covering of degree \( m \) of \((V, b, \mu)\), let \( H \) be as in (1.8), and let \( W_\varepsilon \) be as in (1.10). Then, for all \( u \in C_c(V) \) we have

\[ \langle u, Hu \rangle \geq \sum_{x \in V} \mu(x)(W_\varepsilon(x) + W(x))|u(x)|^2. \]  

(3.1)

**Proof.** It is well known that

\[ \langle u, Hu \rangle = \sum_{(x, y) \in E} b(x, y)|u(x) - e^{i\theta(x,y)}u(y)|^2 + \sum_{x \in V} \mu(x)W(x)|u(x)|^2, \]

\[ = \sum_{(x, y) \in E} \mu(x)|u(x) - e^{i\theta(x,y)}u(y)|^2 + \sum_{x \in V} \mu(x)W_\varepsilon(x)|u(x)|^2 + \sum_{x \in V} \mu(x)W(x)|u(x)|^2, \]

\[ \geq \sum_{x \in V} \mu(x)(W_\varepsilon(x) + W(x))|u(x)|^2, \]

\[ \geq \sum_{x \in V} \mu(x)|u(x)|^2 \geq \sum_{x \in V} \mu(x)|u(x)|^2. \]

\[ \geq \sum_{x \in V} \mu(x)(W_\varepsilon(x) + W(x))|u(x)|^2. \]
where $E$ is the set of unoriented edges of $(V, b, \mu)$. Thus, using the definition of the covering $(V_l, E_l)_{l \in L}$ of degree $m$ and the definition of $p_l$ we have

$$\langle u, Hu \rangle \geq \frac{1}{m} \sum_{l \in L} \sum_{\{x, y\} \in E_l} b(x, y)|u(x) - e^{i\theta(x, y)}u(y)|^2 + \sum_{x \in V} \mu(x)W(x)|u(x)|^2$$

$$\geq \frac{1}{m} \sum_{l \in L} \left( \inf_{\{y, z\} \in E_l} b(y, z) \right) p_l \sum_{x \in V_l} \mu(x)|u(x)|^2 + \sum_{x \in V} \mu(x)W(x)|u(x)|^2,$$

which together with (1.10) gives (3.1).

**Conclusion of the proof of Theorem 2.** By Lemma 3.1 and assumption (1.11), for all $u \in C_c(V)$ we have

$$\langle u, (H - \lambda)u \rangle \geq \sum_{x \in V} \mu(x)(W_e(x) + W(x) - \lambda)|u(x)|^2$$

$$\geq \frac{1}{2} \sum_{x \in V} \max \left( \frac{1}{D(x)^2}, 1 \right) \mu(x)|u(x)|^2 - (C + \lambda + 1/2)||u||^2.$$

From hereon we proceed in the same way as in the the proof of Theorem 1. \hfill \square

### 4. Proof of Theorem 3

In this section we modify the proof of [20, Theorem 1], which is based on the technique of [23] in the context of Riemannian manifolds. Throughout the section, we assume that the hypotheses of Theorem 3 are satisfied.

We begin with the definitions of minimal and maximal operators associated with the expression (1.8). We define the operator $H_{\text{min}}$ by the formula $H_{\text{min}}u := Hu$, for all $u \in \text{Dom}(H_{\text{min}}) := C_c(V)$. As $W$ is real-valued, it follows easily that the operator $H_{\text{min}}$ is symmetric in $\ell^2_\mu(V)$. We define $H_{\text{max}} := (H_{\text{min}})^*$, where $T^*$ denotes the adjoint of operator $T$. Additionally, we define $D := \{u \in \ell^2_\mu(V): Hu \in \ell^2_\mu(V)\}$. Then, the following hold: $\text{Dom}(H_{\text{max}}) = D$ and $H_{\text{max}}u = Hu$ for all $u \in D$; see, for instance, [20, Section 3] or [24, Section 3] for details. Furthermore, by [16, Problem V.3.10], the operator $H_{\text{min}}$ is essentially self-adjoint if and only if

$$\langle H_{\text{max}}u, v \rangle = \langle u, H_{\text{max}}v \rangle,$$

for all $u, v \in \text{Dom}(H_{\text{max}}).$ \hfill (4.1)

In the setting of graphs of bounded vertex degree, the following proposition was proven in [20, Proposition 12].

**Proposition 4.1.** If $u \in \text{Dom}(H_{\text{max}})$, then

$$\sum_{x, y \in V} b(x, y) \min\{q^{-1}(x), q^{-1}(y)\}|u(x) - e^{i\theta(x, y)}u(y)|^2 \leq 4(||Hu||||u|| + (K^2 + 1)||u||^2),$$

where $H$ is as in (1.8) and $K$ is as in (1.12).
Before proving Proposition 4.1, we define a sequence of cut-off functions. Let $d_\sigma$ and $d_{\sigma_q}$ be as in the hypothesis of Theorem 3. Fix $x_0 \in V$ and define
\[ \chi_n(x) := \left( \left( \frac{2n - d_\sigma(x_0, x)}{n} \right) \lor 0 \right) \land 1, \quad x \in V, \quad n \in \mathbb{Z}_+. \tag{4.3} \]

Denote
\[ B_n^\sigma(x_0) := \{ x \in V : d_\sigma(x_0, x) \leq n \}. \tag{4.4} \]
The sequence $\{\chi_n\}_{n \in \mathbb{Z}_+}$ satisfies the following properties: (i) $0 \leq \chi_n(x) \leq 1$, for all $x \in V$; (ii) $\chi_n(x) = 1$ for $x \in B_n^\sigma(x_0)$ and $\chi_n(x) = 0$ for $x \notin B_n^\sigma(x_0)$; (iii) for all $x \in V$, we have $\lim_{n \to \infty} \chi_n(x) = 1$; (iv) the functions $\chi_n$ have finite support; and (v) the functions $\chi_n$ satisfy the inequality
\[ |\chi_n(x) - \chi_n(y)| \leq \frac{\sigma(x, y)}{n}, \quad \text{for all } x \sim y. \]

The properties (i)–(iii) and (v) can be checked easily. By hypothesis, we know that $(V, d_\sigma)$ is a complete metric space and, thus, balls with respect to $d_\sigma$ are finite; see, for instance, [15, Theorem A.1]. Let $B_{2n}^{\sigma_q}(x_0)$ be as in (4.4) with $d_\sigma$ replaced by $d_{\sigma_q}$. Since $q \geq 1$ it follows that $B_{2n}^{\sigma_q}(x_0) \subseteq B_{2n}^{\sigma}(x_0)$. Thus, property (iv) is a consequence of property (ii) and the finiteness of $B_{2n}^{\sigma_q}(x_0)$.

**Proof of Proposition 4.1.** Let $u \in \text{Dom}(H_{\text{max}})$ and let $\phi \in C_c(V)$ be a real-valued function. Define
\[ I := \left( \sum_{x, y \in V} b(x, y)|u(x) - e^{i\theta_{x,y}}u(y)|^2((\phi(x))^2 + (\phi(y))^2) \right)^{1/2}. \tag{4.5} \]

We will first show that
\[ I^2 \leq 4|\langle \phi^2 Hu, u \rangle| + 4|\langle \phi^2 qu, u \rangle| \]
\[ + \sqrt{2} I \left( \sum_{x, y \in V} b(x, y)(\phi(x) - \phi(y))^2|\langle u(x) + e^{i\theta_{x,y}}u(y) \rangle|^2 \right)^{1/2}. \tag{4.6} \]

To do this, we first note that
\[ I^2 = 4(\phi^2 Hu, u) - 4(\phi^2 W u, u) \]
\[ + \sum_{x, y \in V} b(x, y)(e^{i\theta_{x,y}}u(y) - u(x))(e^{-i\theta_{x,y}}u(y) + \overline{u(x)})(\phi(x))^2 - (\phi(y))^2, \tag{4.7} \]
which can be checked by expanding the terms under summations on both sides of the equality and using the properties $b(x, y) = b(y, x)$ and $\theta(x, y) = -\theta(y, x)$. The details of this computation can be found in the proof of [20, Proposition 12].

The inequality (4.6) is obtained from (4.7) by using (1.13), the factorization
\[ (\phi(x))^2 - (\phi(y))^2 = (\phi(x) - \phi(y))(\phi(x) + \phi(y)), \]
Cauchy–Schwarz inequality, and
\[(\phi(x) + \phi(y))^2 \leq 2(\phi^2(x) + \phi^2(y)).\]

Let \(\chi_n\) be as in (4.3) and let \(q\) be as in (1.13). Define
\[\phi_n(x) := \chi_n(x)q^{-1/2}(x).\]  \hspace{1cm} (4.8)

By property (iv) of \(\chi_n\) it follows that \(\phi_n\) has finite support. By property (i) of \(\chi_n\) and since \(q \geq 1\), we have
\[0 \leq \phi_n(x) \leq q^{-1/2}(x) \leq 1, \quad \text{for all } x \in V.\]  \hspace{1cm} (4.9)

By property (iii) of \(\chi_n\) we have
\[\lim_{n \to \infty} \phi_n(x) = q^{-1/2}(x), \quad \text{for all } x \in V.\]  \hspace{1cm} (4.10)

By (1.12), properties (i) and (v) of \(\chi_n\), and the inequality \(q \geq 1\), we have
\[|\phi_n(x) - \phi_n(y)| \leq \left(\frac{1}{n} + K\right)\sigma(x,y), \quad \text{for all } x \sim y,\]  \hspace{1cm} (4.11)

where \(K\) is as in (1.12). We will also use the inequality
\[|e^{i\theta x \cdot y}u(y) + u(x)|^2 \leq 2(|u(x)|^2 + |u(y)|^2).\]  \hspace{1cm} (4.12)

By (4.11), (4.12), and Definition 1.3(ii), we get
\[
\left(\sum_{x,y \in V} b(x, y)(\phi_n(x) - \phi_n(y))^2|u(x) + e^{i\theta x \cdot y}u(y)|^2\right)^{1/2}
\leq \sqrt{2}\left(\frac{1}{n} + K\right) \left(\sum_{x,y \in V} b(x, y)(\sigma(x,y))^2(|u(x)|^2 + |u(y)|^2)\right)^{1/2}
\leq 2\left(\frac{1}{n} + K\right) \left(\sum_{x,y \in V} b(x, y)(\sigma(x,y))^2|u(x)|^2\right)^{1/2}
\leq 2\left(\frac{1}{n} + K\right) \left(\sum_{x \in V} \mu(x)|u(x)|^2\right)^{1/2}
\]  \hspace{1cm} (4.13)

By (4.6) with \(\phi = \phi_n\), (4.13), and (4.9), we obtain
\[I_n^2 \leq 4\|Hu\||u| + 4\|u\|^2 + 2\sqrt{2}I_n \left(\frac{1}{n} + K\right)\|u\|,\]  \hspace{1cm} (4.14)

for all \(u \in \text{Dom}(H_{\text{max}})\), where \(I_n\) is as in (4.5) with \(\phi = \phi_n\).
Using the inequality $ab \leq \frac{a^2}{4} + b^2$ with $a = \sqrt{2}I_n$ in the third term on the right-hand side of (4.14) and rearranging, we obtain

$$I_n^2 \leq 8 \left( \|H_u\|\|u\| + \left( \frac{1}{n} + K \right)^2 \|u\|^2 \right). \quad (4.15)$$

Letting $n \to \infty$ in (4.15) and using (4.10) together with Fatou’s lemma, we get

$$\sum_{x:y \in V} b(x, y)\min\{q^{-1}(x), q^{-1}(y)\} |u(x) - e^{i\theta_{x,y}}u(y)|^2 (q^{-1}(x) + q^{-1}(y)) \leq 8 \left( \|H_u\|\|u\| + (K^2 + 1)\|u\|^2 \right). \quad (4.16)$$

Since

$$2 \min\{q^{-1}(x), q^{-1}(y)\} \leq q^{-1}(x) + q^{-1}(y), \quad \text{for all } x, y \in V,$$

the inequality (4.2) follows directly from (4.16). \hfill \Box

**Continuation of the proof of Theorem 3.** Our final goal is to prove (4.1). Let $d_{\sigma_q}$ be as in the hypothesis of Theorem 3. Fix $x_0 \in V$ and define

$$P(x) := d_{\sigma_q}(x_0, x), \quad x \in V. \quad (4.17)$$

In what follows, for a function $f: V \to \mathbb{R}$ we define $f^+(x) := \max\{f(x), 0\}$. Let $u, v \in \text{Dom}(H_{\max})$, let $s > 0$, and define

$$J_s := \sum_{x \in V} \left( 1 - \frac{P(x)}{s} \right)^+ \left( (Hu)(x)(Hv)(x) - u(x)(Hv)(x) - u(x)(Hu)(x) \right) \mu(x), \quad (4.18)$$

where $P$ is as in (4.17) and $H$ is as in (1.8).

Since $(V, d_{\sigma_q})$ is a complete metric space, by [15, Theorem A.1] it follows that the set

$$U_s := \{x \in V: P(x) \leq s\}$$

is finite. Thus, for all $s > 0$, the summation in (4.18) is performed over finitely many vertices.

The following lemma follows easily from the definition of $J_s$ and the dominated convergence theorem; see the proof of [20, Lemma 13] for details.

**Lemma 4.2.** Let $J_s$ be as in (4.18). Then

$$\lim_{s \to +\infty} J_s = (Hu, v) - (u, Hv). \quad (4.19)$$

In what follows, for $u \in \text{Dom}(H_{\max})$, define

$$T_u := \left( \sum_{x:y \in V} b(x, y)\min\{q^{-1}(x), q^{-1}(y)\}|u(x) - e^{i\theta_{x,y}}u(y)|^2 \right)^{1/2}. \quad (4.20)$$

Note that $T_u$ is finite by Proposition 4.1.
Lemma 4.3. Let $u, v \in \text{Dom}(H_{\text{max}})$, let $T_u$ and $T_v$ be as in (4.20), and let $J_s$ be as in (4.18). Then

$$|J_s| \leq \frac{1}{2s} (\|v\|T_u + \|u\|T_v).$$

Proof. A computation shows that

$$2J_s = \sum_{x,y \in V} \left( (1 - P(x)/s^+) - (1 - P(y)/s^+) \right) b(x,y) \left( (e^{-i\theta_{x,y}}v(y) - v(x))u(x) - (e^{i\theta_{x,y}}u(y) - u(x))v(x) \right),$$

which, together with the triangle inequality and property

$$|f^+(x) - g^+(x)| \leq |f(x) - g(x)|,$$

leads to the following estimate:

$$2|J_s| \leq \frac{1}{s} \sum_{x,y \in V} b(x,y)|P(x) - P(y)| \left| e^{i\theta_{x,y}}v(y) - v(x) \right| \left| u(x) \right|$$

$$+ \left| e^{i\theta_{x,y}}u(y) - u(x) \right| \left| v(x) \right|.$$  

(4.22)

By (4.17) and (1.14), for all $x \sim y$ we have

$$|P(x) - P(y)| \leq d_s(x,y) \leq \sigma_q(x,y) = \min\{q^{-1/2}(x), q^{-1/2}(y)\} \cdot \sigma(x,y).$$  

(4.23)

To obtain (4.21), we combine (4.22) and (4.23) and use Cauchy–Schwarz inequality together with Definition 1.3(ii).

The end of the proof of Theorem 3. Let $u \in \text{Dom}(H_{\text{max}})$ and $v \in \text{Dom}(H_{\text{max}})$. By the definition of $H_{\text{max}}$, it follows that $Hu \in \ell^2_{\mu}(V)$ and $Hv \in \ell^2_{\mu}(V)$. Letting $s \rightarrow +\infty$ in (4.21) and using the finiteness of $T_u$ and $T_v$, it follows that $J_s \rightarrow 0$ as $s \rightarrow +\infty$. This, together with (4.19), shows (4.1).

5. Examples

In this section we give some examples that illustrate the main results of the paper. In what follows, for $x \in \mathbb{R}$, the notation $\lceil x \rceil$ denotes the smallest integer $N$ such that $N \geq x$. Additionally, $\lfloor x \rfloor$ denotes the greatest integer $N$ such that $N \leq x$.

Example 5.1. In this example we consider the graph $G = (V, E)$ whose vertices $x_{j,k}$ are arranged in a “triangular” pattern so that the first row contains $x_{1,1}$; for $2 \leq j \leq 4$, the $j$-th row contains $x_{j,1}$ and $x_{j,2}$; for $5 \leq j \leq 9$, the $j$-th row contains $x_{j,1}$, $x_{j,2}$, and $x_{j,3}$; for $10 \leq j \leq 16$, the $j$-th row contains $x_{j,1}$, $x_{j,2}$, $x_{j,3}$, and $x_{j,4}$; and so on. There are two types of edges in the graph: (i) for every $j \geq 1$, we have $x_{j,1} \sim x_{j+1,k}$ for all $1 \leq k \leq \lceil (j + 1)^{1/2} \rceil$; (ii) for every $j \geq 2$, we have the “horizontal” edges $x_{j,k} \sim x_{j,k+1}$, for all $1 \leq k \leq \lceil j^{1/2} \rceil - 1$. Clearly, $G$ does not have a bounded vertex degree.
Let $T = (V_T, E_T)$ be the subgraph of $G$ whose set of edges $E_T$ consists of type-(i) edges of $G$ described above. Note that $T$ is a spanning tree of $G$. Additionally, note that for every type-(ii) edge $e$ of $G$ the following are true: (i) $e \notin E_T$ and (ii) there is a unique 3-cycle (a cycle with 3 vertices) that contains $e$. Thus, by [3, Lemma 2.2], the corresponding 3-cycles, which we enumerate by $\{C_l\}_{l \in \mathbb{Z}_+}$, form a basis for the space of cycles of $G$. Furthermore, by Definition 1.11, the family $\{C_l = (V_l, E_l)\}_{l \in \mathbb{Z}_+}$ is a good covering of degree $m = 2$ of $G$. Following [3, Proposition 2.4(i)] and [3, Lemma 2.9], we define the phase function $\theta: V_l \times V_l \to [-\pi, \pi]$ satisfying the following properties: (i) if an edge $\{x, y\}$ belongs to $E_l \setminus E_T$, we have $\theta(x, y) = -\theta(y, x)$; (ii) if $\{x, y\} \in E_T$, we have $\theta(x, y) = 0$; and (iii) $p_1 = |1 - e^{i\pi/3}|^2 = 1$, where $p_1$ is as in (1.10) with $G_l$ replaced by $C_l$.

With this choice of $p_1$ and using the good covering $\{C_l\}_{l \in \mathbb{Z}_+}$ of degree $m = 2$, the definition of the effective potential (1.10) simplifies to

$$W_\sigma(x) := \frac{1}{2} \sum_{\{l \in L \mid x \in V_l\}} \inf_{\{y, z\} \in E_l} b(y, z). \quad (5.1)$$

Let $\{b_j\}_{j \in \mathbb{Z}_+}$ be an increasing sequence of positive numbers. We define (i) $b(x, y) = b_j$ if $x \sim y$ and $x$ is in the $j$-th row and $y$ is in the $(j+1)$-st row; (ii) $b(x, y) = b_j$ if $x \sim y$ and $x$ and $y$ are both in the $(j+1)$-st row; (iii) $b(x, y) = 0$, otherwise. With this choice of $b(x, y)$, we have $W_\sigma(x_{1,1}) = b_1/2$. Additionally, since $b_j$ is an increasing sequence of positive numbers, using (5.1) it is easy to see that if a vertex $x$ is in the $j$-th row, then

$$W_\sigma(x) \geq \frac{1}{2} b_{j-1}, \quad \text{for all } j \geq 2. \quad (5.2)$$

Let $0 < \beta < 3/4$, and set $\mu(x) := j^{-2\beta}$ if the vertex $x$ is in the $j$-th row. Let $\alpha > 0$ satisfy $\alpha + 2\beta > 3/2$, and set $b_j := j^\alpha$, for all $j \in \mathbb{Z}_+$. With this choice of $b(x, y)$ and $\mu(x)$, let $\sigma_1(x, y)$ be as in (1.2) and let $d_{\sigma_1}$ be the intrinsic path metric associated with $\sigma_1$ as in Section 1.2. As there are $\lfloor \sqrt{j} \rfloor + 3$ edges departing from the vertex $x_{j,1}$, we have

$$\sigma_1(x_{j,1}; x_{j+1,1}) = j^{-\alpha/2}(j + 1)^{-\beta}([\sqrt{j} + 1] + 3)^{-1/2}, \quad \text{for all } j \in \mathbb{Z}_+. \quad (5.3)$$

Additionally, note that the path $\gamma = (x_{1,1}; x_{2,1}; x_{3,1}; \ldots)$ is a geodesic with respect to the path metric $d_{\sigma_1}$, that is, $d_{\sigma_1}(x_{1,1}; x_{j,1}) = t_{\sigma_1}(x_{1,1}; x_{2,1}; \ldots; x_{j,1})$ for all $j \in \mathbb{Z}_+$, where $t_{\sigma_1}$ is as in (1.1). Since $\alpha + 2\beta > 3/2$, it follows that

$$\sum_{j=1}^\infty j^{-\alpha/2}(j + 1)^{-\beta}([\sqrt{j} + 1] + 3)^{-1/2} < \infty;$$

hence, by [15, Theorem A.1] the space $(V, d_{\sigma_1})$ is not metrically complete. Let $D(x)$ be as in (1.3) corresponding to $d_{\sigma_1}$. If a vertex $x$ is in the $n$-th row, using (5.3) and

$$[\sqrt{j} + 1] + 3 \leq 3\sqrt{j + 1}, \quad \text{for all } j \in \mathbb{Z}_+,$$

we have

$$D(x) \geq \frac{1}{\sqrt{3}} \sum_{k=n}^\infty (j + 1)^{-\beta-\alpha/2-1/4} \geq \frac{(n + 1)^{-\beta-\alpha/2+3/4}}{\sqrt{3}(\beta + \alpha/2 - 3/4)},$$
which leads to
\[
\frac{1}{2D(x)^2} \leq \frac{3(4\beta + 2\alpha - 3)^2(n + 1)^{2\beta + \alpha - 3/2}}{32},
\]
(5.4)
for all vertices \(x\) in the \(n\)-th row, where \(n \geq 1\). Define \(W(x) = -n^{2\beta + \alpha - 3/2}\) for all vertices \(x\) in the \(n\)-th row, where \(n \geq 1\). Using (5.2) and \(W(x,1) = b_1/2\), together with (5.4) and the assumption \(0 < \beta < 3/4\), it follows that there exists a constant \(C > 0\) (depending on \(\alpha\) and \(\beta\)) such that (1.11) is satisfied. Thus, by Theorem 2 the operator \(\Delta_{b,\mu,\theta} + W\) is essentially self-adjoint on \(C_c(V)\). Clearly, Theorem 2 is also applicable in the case \(W(x) = 0\) for all \(x \in V\), that is, the operator \(\Delta_{b,\mu,\theta}\) is essentially self-adjoint on \(C_c(V)\). A calculation shows that \(\mu\) and \(b\) in this example do not satisfy [19, Assumption A]; hence, we cannot use [19, Theorem 1.2].

We will now show that under more restrictive assumption \(1/2 < \beta < 3/4\), we cannot apply [9, Proposition 2.2] to this example with \(W(x) \equiv 0\). To see this, using (1.16) and the fact that among the \(\sqrt{j}\) + 3 edges departing from the vertex \(x_{j,1}\), there are \(\sqrt{j}\) + 1 edges with weight \(b_j\) and 2 edges with weight \(b_{j-1}\), we first note that
\[
\text{Deg}(x_{1,1}) = 2, \quad \text{Deg}(x_{j,1}) = j^{2\beta}((\sqrt{j} + 1)j^\alpha + 2(j - 1)\alpha), \quad \text{for all } j \geq 2.
\]
Let \(\lambda \in \mathbb{R}\) be such that (1.15) is satisfied, with \(W(x) \equiv 0\). Let \(\delta > 0\) and let \(a_n\) be as in (1.17) corresponding to the path \(\gamma = (x_{1,1}; x_{2,1}; x_{3,1}; \ldots)\), the potential \(W \equiv 0\), \(\delta > 0\), and \(\lambda\). Then \(a_1 = 1\),
\[
(a_2)^2 = \left(\frac{\delta}{2} + 1 + \frac{\lambda}{2}\right)^2 = \frac{(\delta + |2 + \lambda|)^2}{4},
\]
and
\[
(a_n)^2 = \frac{(\delta + |2 + \lambda|)^2}{4} \prod_{j=2}^{n-1} \left(\frac{\delta + j^{2\beta}((\sqrt{j} + 1) + 2j^{2\beta}(j - 1)\alpha + \lambda)}{j^{2\beta}((\sqrt{j} + 1) + 2j^{2\beta}(j - 1)\alpha)}\right)^2, \quad n \geq 3.
\]
Therefore,
\[
\sum_{n=1}^{\infty} (a_n)^2 \mu(x_{n,1}) = 1 + \frac{(\delta + |2 + \lambda|)^2}{4(2)^{2\beta}} + \sum_{n=3}^{\infty} \frac{(a_n)^2}{n^{2\beta}}.
\]
Using Raabe’s test, it can be checked that the series on the right hand side of this equality converges. (Here, we used the more restrictive assumption \(1/2 < \beta < 3/4\).) Hence, looking at (1.17), we see that [9, Proposition 2.2] cannot be used in this example.

**Example 5.2.** Consider the graph whose vertices are arranged in a “triangular” pattern so that \(x_{1,1}\) is in the first row, \(x_{2,1}\) and \(x_{2,2}\) are in the second row, \(x_{3,1}\), \(x_{3,2}\), and \(x_{3,3}\) are in the third row, and so on. The vertex \(x_{1,1}\) is connected to \(x_{2,1}\) and \(x_{2,2}\). The vertex \(x_{2,i}\), where \(i = 1,2\), is connected to every vertex \(x_{3,j}\), where \(j = 1,2,3\). The pattern continues so that each of \(k\) vertices in the \(k\)-th row is connected to each of \(k + 1\) vertices in the \((k + 1)\)-st row. Note that for all \(k \geq 1\) and \(j \geq 1\) we have \(\text{deg}(x_{k,j}) = 2k\), where \(\text{deg}(x)\) is as in (1.2). Let \(\mu(x) = k^{1/2}\) for every vertex \(x\) in the \(k\)-th row, and let \(b(x,y) \equiv 1\) for all vertices \(x \sim y\). Following (1.2), for
every vertex \( x \) in the \( k \)-th row and every vertex \( y \) in the \((k + 1)\)-st row, define
\[
\sigma(x, y) := \min \left\{ \frac{k^{1/2}}{2k}, \frac{(k + 1)^{1/2}}{2(k + 1)} \right\}^{1/2} = 2^{-1/2}(k + 1)^{-1/4}.
\]
For all vertices \( x \) in the \( k \)-th row, define \( W(x) = -2^{k^{1/2}} \) and \( q(x) = 2k \). Clearly, the inequality (1.13) is satisfied. With this choice of \( q \), following (1.14), for every vertex \( x \) in the \( k \)-th row and every vertex \( y \) in the \((k + 1)\)-st row, define
\[
\sigma_q(x, y) := \min\{(2k)^{-1/2}, (2(k + 1))^{-1/2}\} \cdot \sigma(x, y) = 2^{-1}(k + 1)^{-3/4}.
\]
Since
\[
\sum_{j=1}^{\infty} 2^{-1}(j + 1)^{-3/4} = \infty,
\]
by [15, Theorem A.1] it follows that the space \((V, d_{\sigma_q})\) is metrically complete. Additionally, it is easily checked that (1.12) is satisfied with \( K = 1 \). Therefore, by Theorem 3 the operator \( \Delta_{b,\mu} + W \) is essentially self-adjoint on \( C_c(V) \). Furthermore, it is easy to see that for every \( c \in \mathbb{R} \), there exists a function \( u \in C_c(V) \) such that the inequality
\[
((\Delta_{b,\mu} + W)u, u) \geq c\|u\|^2
\]
is not satisfied. Thus, the operator \( \Delta_{b,\mu} + W \) is not semi-bounded from below, and we cannot use [19, Theorem 1.2].

It turns out that [9, Proposition 2.2] is not applicable in this example. To see this, using (1.16) we first note that \( \text{Deg}(x_{k,j}) = 2^{k^{1/2}} \), for all \( k \geq 1 \) and all \( j \geq 1 \). Let \( \lambda \in \mathbb{R} \) be such that (1.15) is satisfied, with \( W \) as in this example. Let \( a_n \) be as in (1.17) corresponding to the path \( \gamma = (x_{1,1}; x_{2,1}; x_{3,1}; \ldots) \), the potential \( W(x_{k,1}) = -2^{k^{1/2}}, \delta > 0 \), and \( \lambda \). Then \( a_1 = 1 \), and for \( n \geq 2 \) we have
\[
(a_n)^2 = \prod_{k=1}^{n-1} \left( \frac{\delta}{2^{k^{1/2}}} + \left| 1 + \frac{\lambda - 2^{k^{1/2}}}{2^{k^{1/2}}} \right| \right)^2 = \prod_{k=1}^{n-1} \frac{(\delta + |\lambda|)^2}{4k} = \frac{(\delta + |\lambda|)^{2n-2}}{4^{n-1}(n-1)!}.
\]
Therefore,
\[
\sum_{n=1}^{\infty} (a_n)^2 \mu(x_{n,1}) = 1 + \sum_{n=2}^{\infty} \frac{\sqrt{n} \cdot (\delta + |\lambda|)^{2n-2}}{4^{n-1}(n-1)!}.
\]
Using ratio test, it can be checked that the series on the right hand side of this equality converges. Hence, looking at (1.17), we see that [9, Proposition 2.2] cannot be used in this example.

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