ON THE KATO SQUARE ROOT PROBLEM

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Abstract. In the infinite-dimensional separable complex Hilbert space we construct new abstract examples of unbounded maximal accretive and maximal sectorial operators $B$ for which $\text{dom } B^{\frac{1}{2}} \neq \text{dom } B^{\frac{1}{2}}$. New criterions for the equality are established.

1. Introduction

Let $\mathcal{H}$ be an infinite-dimensional separable complex Hilbert space and let $B$ be a maximal accretive operator [22]. As is well known (see, e.g., [13, 19, 23, 25, 28, 32]) for each $\gamma \in (0, 1)$ the maximal accretive fractional power $B^\gamma$ can be defined. For instance, one can use

- the Balakrishnan representation [11]

$$B^\gamma u = \frac{\sin(\gamma \pi)}{\pi} \int_0^\infty t^{\gamma-1}B(tI)^{-1}u \, dt, \quad u \in \text{dom } B;$$

- the Sz.-Nagy – Foias functional calculus for contractions [32, Chapter IV]:

$$B^\gamma = v^\gamma(T)(u^\gamma(T))^{-1}, \quad v(z) = 1 - z, \quad u(z) = 1 + z, \quad z \in \mathbb{D},$$

i.e., $B^\gamma = ((I - T)(I + T)^{-1})^\gamma = (I - T)^\gamma ((I + T)^\gamma)^{-1}$.

The fractional powers possess the properties [19, 32]:

- the operator $B^\gamma$ is regularly accretive with index $\leq \tan \frac{\pi \gamma}{2}$ [19], i.e., $B^\gamma$ is maximal sectorial with vertex at the origin and with semi-angle $\frac{\pi \gamma}{2}$ [22];

- $\text{dom } (B + \lambda I)^\gamma = \text{dom } B^\gamma$ for all $\lambda > 0$;

- $(B^*)^\gamma = (B^\gamma)^*$ (in the sequel $(B^*)^\gamma$ we will denote by $B^{*\gamma}$);

- if $\gamma \in (0, \frac{1}{2})$, then $\text{dom } B^\gamma = \text{dom } B^{*\gamma}$ and the operator $\text{Re } (B^\gamma) := \frac{1}{2}(B^\gamma + B^{*\gamma})$ is selfadjoint;

- for each $\gamma \in (\frac{1}{2}, 1)$ there is an example of $B$ such that $\text{dom } B^\gamma \neq \text{dom } B^{*\gamma}$;

- for the square root $B^{\frac{1}{2}}$ ($\gamma = \frac{1}{2}$) it is proved in [19, Theorem 5.1] that the intersection $\text{dom } B^{\frac{1}{2}} \cap \text{dom } B^{*\frac{1}{2}}$ is a core of both $B^{\frac{1}{2}}$ and $B^{*\frac{1}{2}}$ and the operator

$$\text{Re } (B^{\frac{1}{2}}) = \frac{1}{2}(B^{\frac{1}{2}} + B^{*\frac{1}{2}}), \quad \text{dom } \text{Re } (B^{\frac{1}{2}}) = \text{dom } B^{\frac{1}{2}} \cap \text{dom } B^{*\frac{1}{2}}$$

is selfadjoint.

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Other remarkable properties of powers \( B^\gamma \) for \( \gamma \in (0, \frac{1}{2}) \) can be found in [19, Theorem 1.1, Theorem 3.1] and in [32, Chapter IV, Theorem 5.1].

The following problem was formulated by Kato in [19]: is the equality

\[(1.2) \quad \text{dom } B^{\frac{1}{2}} = \text{dom } B^{\frac{1}{2} \ast}\]

true for an arbitrary unbounded maximal accretive operator \( B \)?

The negative answer was given in [24] by Lions. By means of the theory of interpolation spaces he showed that for the maximal accretive operator in \( L^2(\mathbb{R}_+) \)

\[(1.3) \quad B_0 = \frac{d}{dx} \quad \text{dom } B_0 = \{ f \in H^1(\mathbb{R}_+): f(0) = 0 \}\]

holds the inequality \( \text{dom } B_0^{\frac{1}{2}} \neq \text{dom } B_0^{\frac{1}{2} \ast} \). Note that the operator \( B_0 \) is not sectorial and, moreover, the operator \(-iB_0\) is maximal symmetric and non-selfadjoint.

The next results were established in [24] and [20].

**Theorem 1.1.** Let \( B \) be a maximal sectorial operator and let \( b \) be the closed quadratic form associated with \( B \). Then

1) the inclusion \( \text{dom } B^{\frac{1}{2}} \subseteq \text{dom } b \) is equivalent to the inclusion \( \text{dom } B^{\frac{1}{2} \ast} \supseteq \text{dom } b \); the same is true when \( B \) and \( B^\ast \) are exchanged;

2) the inclusion \( \text{dom } B^{\frac{1}{2}} \subseteq \text{dom } B^{\frac{1}{2} \ast} \) is equivalent to the inclusions \( \text{dom } B^{\frac{1}{2}} \subseteq \text{dom } b \subseteq \text{dom } B^{\frac{1}{2} \ast} \); the same is true when \( B \) and \( B^\ast \) are exchanged.

It follows [24, 20] that if both \( \text{dom } B^{\frac{1}{2}} \) and \( \text{dom } B^{\frac{1}{2} \ast} \) are subsets (or oversets) of \( \text{dom } b \), then

\[(1.4) \quad \text{dom } B^{\frac{1}{2}} = \text{dom } B^{\ast \frac{1}{2}} = \text{dom } b, \]

and therefore

\[b[u, v] = (B^{\frac{1}{2}} u, B^{\ast \frac{1}{2}} v), \quad u, v \in \text{dom } B^{\frac{1}{2}} = \text{dom } B^{\ast \frac{1}{2}} = \text{dom } b.\]

Besides, it is shown in [24, 20] that the equality \( \text{dom } B = \text{dom } B^\ast \) implies (1.4).

McIntosh in [26] (see also [8, Preliminaries, Theorem 6]) presented an abstract example of unbounded maximal sectorial operator \( B \) such that \( \text{dom } B^{\frac{1}{2}} \neq \text{dom } B^{\ast \frac{1}{2}} \). The operator constructed in [26] is the countable orthogonal sum of special finite-dimensional sectorial operators with a fixed semi-angle.

By means of another way and by using the operator \( B_0 \) in (1.3), Gomilko in [16] has proved the existence of maximal sectorial operators \( B \) with the property \( \text{dom } B^{\frac{1}{2}} \neq \text{dom } B^{\ast \frac{1}{2}} \). In [16] it is shown that if a maximal accretive operator \( B \) is boundedly invertible, then the operator \( B^{\frac{1}{2}} \) admits the representation

\[(1.5) \quad B^{\frac{1}{2}} f = S^{-1}(G^* + I) f \quad \forall f \in \text{dom } B^{\frac{1}{2}}, \]

where \( S = 2\text{Re}(B^{-\frac{1}{2}}), \quad G = B^{\ast \frac{1}{2}} B^{-\frac{1}{2}}, \quad \text{dom } G = B^{\frac{1}{2}} \left( \text{dom } B^{\frac{1}{2}} \cap \text{dom } B^{\ast \frac{1}{2}} \right) \). It is proved in [16, Theorem 1] that the operator \( G \) is maximal accretive and the equality \( \text{dom } B^{\frac{1}{2}} = \text{dom } B^{\ast \frac{1}{2}} \) holds if and only if the operator \( G \) is bounded and has bounded inverse. Moreover, the following theorem has been established:
Theorem 1.2. [16, Theorem 3] Let \( B_\gamma = (S^{-1}(G^* \gamma + I))^2 \), \( \gamma \in (0, 1) \). Then the operator \( B_\gamma \) is maximal sectorial with semi-angle \( \frac{\pi \gamma}{2} \) and the equality \( \text{dom } B_\gamma^{1/2} = \text{dom } B_\gamma^{*1/2} \) holds if and only if holds the equality \( \text{dom } B_\gamma^{1/2} = \text{dom } B_\gamma^{*1/2} \).

The Kato problem was positively solved in [13] for a class of abstract maximal sectorial operators, in [7, 8, 6, 9, 10, 24] for elliptic differential operators, in [30, 31] for some class of strongly elliptic functional-differential operators.

In this paper yet another approach to the abstract Kato square root problem is proposed. We essentially relay on the properties of contractions and especially of contractions which are linear fractional transformations of \( m \)-sectorial operators (see Section 3). The usage of our method allows

(a) to find new representations for maximal accretive and maximal sectorial operators and their square roots, see Section 4;
(b) to give new proofs for known and to establish new equivalent conditions of equality (1.2) for maximal accretive and maximal sectorial operators, see Theorem 4.8;
(c) to prove that for an arbitrary maximal accretive operator of the form \( B = iA \), where \( A \) is a maximal symmetric non-selfadjoint operator, holds the relation \( \text{dom } B^{1/2} \neq \text{dom } B^{*1/2} \) (Theorem 5.1);
(d) to give new proofs for main results established in [16];
(e) to construct a series of new abstract examples of maximal accretive and maximal sectorial operators \( B \) such that \( \text{dom } B^{1/2} \neq \text{dom } B^{*1/2} \) (Section 5), in particular, we construct abstract examples of countable and continuum families of unbounded maximal accretive non-sectorial and maximal sectorial operators for which equality (1.2) is violated or holds and which possess additional properties.

Notations. Throughout the paper we suppose that the Hilbert spaces are infinite-dimensional, complex and separable. We use the symbols \( \text{dom } T \), \( \text{ran } T \), \( \ker T \) for the domain, range, and the kernel of a linear operator \( T \), by \( \rho(T) \) and \( \sigma(T) \) we denote the resolvent set and the spectrum of the operator \( T \), \( \mathbb{D} \) is the open unit disc, the closures of \( \text{dom } T \), \( \text{ran } T \) are denoted by \( \overline{\text{dom } T} \), \( \overline{\text{ran } T} \), respectively, the identity operator in a Hilbert space is denoted by \( I \). If \( \mathcal{L} \subset \mathcal{H} \) is a subspace (closed linear manifolds), the orthogonal projection in \( \mathcal{H} \) onto \( \mathcal{L} \) is denoted by \( P_\mathcal{L} \), the orthogonal complement \( \mathcal{H} \ominus \mathcal{L} \) we denote by \( \mathcal{L}^\perp \). The notation \( T|_{\mathcal{N}} \) means the restriction of a linear operator \( T \) on the set \( \mathcal{N} \subset \text{dom } T \). \( \mathbb{N} \) is the set of natural numbers, \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \), \( \mathbb{R}_+ := [0, +\infty) \), \( \mathbb{C} \) is the field of complex numbers. By \( \text{Re } (A) \) and \( \text{Im } (A) \) we denote the real and imaginary parts, respectively, of an operator \( A \), i.e.,

\[
\text{Re } (A) = \frac{1}{2} (A + A^*), \quad \text{Im } (A) = \frac{1}{2i} (A - A^*), \quad \text{dom } \text{Re } (A) = \text{dom } \text{Im } (A) = \text{dom } A \cap \text{dom } A^*.
\]

2. Maximal sectorial operators

The numerical range \( W(A) \) of a linear operator \( T \) with domain \( \text{dom } A \) in a Hilbert space \( \mathcal{H} \) with the inner product \( (\cdot, \cdot) \) is given by

\[
W(A) = \{(Au, u) : u \in \text{dom } A, \|u\| = 1\}.
\]

By the Toeplitz-Hausdorff theorem (a short proof can be found in [17]) the numerical range is a convex set.
A linear operator $A$ in a Hilbert space $\mathcal{H}$ is called \textit{accretive} \cite{19, 22, 32} if $\text{Re}(Au, u) \geq 0$ for all $u \in \text{dom } A$, i.e.,

$$W(A) \subseteq \{ z \in \mathbb{C} : \text{Re } z \geq 0 \}.$$ 

An accretive operator $A$ is called \textit{maximal accretive}, or \textit{m-accretive} for short, if one of the following equivalent conditions is satisfied:

(i) $A$ has no proper accretive extensions in $\mathcal{H}$;
(ii) $A$ is densely defined and $\text{ran } (A - \lambda I) = \mathcal{H}$ for some $\lambda \in \mathbb{C}$ with $\text{Re } \lambda < 0$;
(iii) $A$ is densely defined and closed, and $A^\ast$ is accretive;
(iv) $-A$ generates contractive one-parameter semigroup $T(t) = \exp(-tA)$, $t \in \mathbb{R}_+$. 

If $A$ is $m$-accretive, then $\ker A = \ker A^\ast$ and hence $\ker A \subseteq \text{dom } A \cap \text{dom } A^\ast$.

An accretive operator $A$ is called \textit{coercive} if there is a positive constant $a$ such that

$$\text{Re } (Af, f) \geq a||f||^2 \ \forall f \in \text{dom } A.$$ 

If $A$ is a coercive, then $\text{ran } A$ is a subspace (closed linear manifold) $A^{-1}\lceil \text{ran } A$ is a bounded accretive operator. If $A$ is $m$-accretive and coercive in $\mathcal{H}$, then $\ker A = \mathcal{H}$. Clearly, if $A$ is accretive operator, then $A - \lambda I$ is coercive accretive for each $\lambda$, $\text{Re } \lambda < 0$.

A linear operator $A$ in a Hilbert space $\mathcal{H}$ is called \textit{sectorial} with semi-angle $\alpha \in [0, \pi/2)$, or \textit{$\alpha$-sectorial} for short, if its numerical range is contained in a closed sector with semi-angle $\alpha$, i.e.,

$$W(A) \subseteq \{ z \in \mathbb{C} : |\arg z| \leq \alpha \}.$$ 

or, equivalently, $|\text{Im } (Au, u)| \leq \tan \alpha \text{Re } (Au, u)$ for all $u \in \text{dom } A$. Clearly, an $\alpha$-sectorial operator is accretive; it is called \textit{maximal sectorial}, or \textit{m-$\alpha$-sectorial} for short, if it is $m$-accretive.

The basic definitions and results concerning sesquilinear forms can be found in \cite{22}. If $a$ is a closed densely defined sectorial form in the Hilbert space $\mathcal{H}$, then by the First Representation Theorem \cite{22}, there exists a unique $m$-sectorial operator $A$ in $\mathcal{H}$ associated with $a$ in the following sense: $(Au, v) = a[u, v]$, for all $u \in \text{dom } A$ and for all $v \in \text{dom } a$. The adjoint operator $A^\ast$ is associated with the adjoint form $a^\ast[u, v] := \overline{a[v, u]}$. Denote by $A_R$ the nonnegative selfadjoint operator associated with the real part $a_R[u, v] := (a[u, v] + a^\ast[u, v])/2$ of the form $a$. The operator $A_R$ is called the \textit{real part} of $A$. According to the Second Representation Theorem \cite{22} the equality $\text{dom } a = \text{dom } A_R^{1/2}$ holds. Moreover,

\begin{equation}
\tag{2.1}
a[u, v] = ((I + iG)A_R^{1/2}u, A_R^{1/2}v), \ u, v \in \text{dom } a,
\end{equation}

where $G$ is a bounded selfadjoint operator in the subspace $\text{ran } A_R$ and $||G|| \leq \tan \alpha$ iff $a$ is $\alpha$-sectorial. The operator $A_R$ one can consider as the half of the form–sum \cite{22}, i.e., $(+ \text{ denotes the form-sum }) A_R = 1/2(A + A^\ast)$.

By the first representation theorem \cite{22}, the associated $m$-sectorial operators $A$ and $A^\ast$ are given by

\begin{equation}
\tag{2.2}
A = A_R^{1/2}(I + iG)A_R^{1/2}, \ \text{dom } A = \left\{ u \in \text{dom } A_R^{1/2} : (I + iG)A_R^{1/2}u \in \text{dom } A_R^{1/2} \right\};
\end{equation}

$$A^\ast = A_R^{1/2}(I - iG)A_R^{1/2}, \ \text{dom } A^\ast = \left\{ \phi \in \text{dom } A_R^{1/2} : (I - iG)A_R^{1/2}\phi \in \text{dom } A_R^{1/2} \right\}.$$
These relations yield that
\[ \text{dom} A \cap \text{dom} A^* = \left\{ u \in \text{dom} A_R : GA_R^\frac{1}{2} u \in \text{dom} A_R^\frac{1}{2} \right\}, \]
\[ \text{Re} (A) u := \frac{1}{2} (A + A^*) u = A_R u \quad \forall u \in \text{dom} A \cap \text{dom} A^*, \]
\[ \text{dom} A_R \supseteq \text{dom} A \cap \text{dom} A^*, \quad A_R \supseteq \text{Re} (A), \]
and
\[ \text{ker} A = \text{ker} A^* = \text{ker} A_R. \]
If \( A \) is an \( m \)-accretive (respectively, \( m - \alpha \)-sectorial) operator and \( \text{ker} A = \{0\} \), then the inverse operator \( A^{-1} \) is \( m \)-accretive (respectively, \( m - \alpha \) sectorial) as well.

An \( m \)-sectorial operator \( A \) is coercive if and only if the operator \( A_R \) is positive definite.
Hence, for \( m - \alpha \)-sectorial \( A \) we get in the sense of quadratic forms [21, Chapter VI, § 2, Section 5, Theorem 2.21]
\[ A_R \leq (\text{Re} (A^{-1}))^{-1} \leq \frac{1}{\cos^2 \alpha} A_R \]
and then
\[ \text{dom} A_R^\frac{1}{2} = \text{dom} (\text{Re} (A^{-1}))^{-\frac{1}{2}} = \text{ran} (\text{Re} (A^{-1}))^\frac{1}{2}. \]

3. CONTRACTIONS OF THE CLASS \( \tilde{C}_\beta \)

As is well known (see e.g., [32]), if \( A \) is \( m \)-accretive operator, then the linear fractional transformation
\[ Z := (I - A)(I + A)^{-1} \]
is a contraction. Conversely, if \( Z \) is a contraction, then the linear relation [1]
\[ A := -I + 2(I + Z)^{-1} = \{(I + Z) h, (I - Z) h \} : h \in \mathcal{H} \]
is \( m \)-accretive. Moreover, \( A \) in (3.1) is operator if and only if \( \text{ker}(I + Z) = \{0\} \) and if this is the case, then
\[ (Af, f) = ((I + Z^*)(I - Z) h, h) = ((I - Z^* Z) h, h) - ((Z - Z^*) h, h), \quad f = (I + Z) h. \]
The operator \( A \) is bounded if and only if \( \text{ran} (I + Z) = \mathcal{H} \).

Note that for any contraction \( Z \) holds the equality \( \text{ker}(I - Z) = \text{ker}(I - Z^*) \) (see [32, Chapter I, Proposition 3.1]).

The lemma below will be used in the next sections.

**Lemma 3.1.** (1) Let \( Z \) be a bounded operator in the Hilbert space \( \mathcal{H} \). Suppose \( \text{ker}(I - Z) = \text{ker}(I + Z) = \{0\} \). Then
\[ \text{ran} (I - Z) \cap \text{ran} (I + Z) = \text{ran} (I - Z^2), \]
and, therefore the equality \( \text{ran} (I - Z^2) = \mathcal{H} \) is equivalent to the equalities \( \text{ran} (I - Z) = \text{ran} (I + Z) = \mathcal{H} \).
Moreover,
(a) if \( \text{ran} \left( I - Z \right) \subseteq \text{ran} \left( I + Z \right) \), then \( \text{ran} \left( I + Z \right) = \mathfrak{H} \),
(b) if \( \text{ran} \left( I - Z \right) = \text{ran} \left( I + Z \right) \), then \( \text{ran} \left( I - Z \right) = \text{ran} \left( I + Z \right) = \mathfrak{H} \).

(2) If \( X \) is a selfadjoint contraction and \( \ker(I - X) = \ker(I + X) = \{0\} \), then
\[
\text{ran} \left( I - X \right)^{\frac{1}{2}} \cap \text{ran} \left( I + X \right)^{\frac{1}{2}} = \text{ran} \left( I - X^2 \right)^{\frac{1}{2}}.
\]

Proof. (1) Since
\[
I - Z^2 = (I - Z)(I + Z) = (I + Z)(I - Z),
\]
the inclusion \( h \in \text{ran} \left( I - Z^2 \right) \) implies inclusions \( h \in \text{ran} \left( I - Z \right) \) and \( h \in \text{ran} \left( I + Z \right) \), i.e.,
\[
h \in \text{ran} \left( I - Z \right) \cap \text{ran} \left( I + Z \right).
\]
Assume that \( h \in \text{ran} \left( I - Z \right) \cap \text{ran} \left( I + Z \right) \), i.e., \( h = (I - Z)f = (I + Z)g \). Set
\[
\varphi := f - g, \quad \psi := f + g
\]
Then
\[
\varphi = Z\psi, \quad f = \frac{1}{2}(I + Z)\psi, \quad g = \frac{1}{2}(I - Z)\psi.
\]
It follows that \( h = \frac{1}{2}(I - Z^2)\psi \), i.e., \( h \in \text{ran} \left( I - Z^2 \right) \).

Suppose that \( \text{ran} \left( I - Z \right) \subseteq \text{ran} \left( I + Z \right) \), then
\[
\text{ran} \left( I - Z^2 \right) = \text{ran} \left( I + Z \right) \cap \text{ran} \left( I - Z \right) = \text{ran} \left( I - Z \right).
\]
Hence, from the equality \( \text{ran} \left( I - Z^2 \right) = (I - Z)\text{ran} \left( I + Z \right) \) we get \( \text{ran} \left( I + Z \right) = \mathfrak{H} \). Consequently, the equality \( \text{ran} \left( I - Z \right) = \text{ran} \left( I + Z \right) \) yields that \( \text{ran} \left( I - Z \right) = \text{ran} \left( I + Z \right) = \mathfrak{H} \).

(2) Let \( E_X(t), \ t \in [-1, 1] \) be the orthogonal spectral function of \( X \). Then due to the equality \( (1 - t^2)^{-1} = \frac{1}{2}((1 - t)^{-1} + (1 + t)^{-1}) \) we get the equivalence for a vector \( f \in \mathfrak{H} \setminus \{0\} \)
\[
\int_{-1}^{1} \frac{d(E_X(t)f, f)}{1 - t^2} < \infty \iff \begin{cases} 
\int_{-1}^{1} \frac{d(E_X(t)f, f)}{1 - t} < \infty \\
\int_{-1}^{1} \frac{d(E_X(t)f, f)}{1 + t} < \infty.
\end{cases}
\]
Hence
\[
f \in \text{dom} \left( I - X^2 \right)^{-\frac{1}{2}} \text{ if and only if } f \in \text{dom} \left( I - X \right)^{-\frac{1}{2}} \cap \text{dom} \left( I + X \right)^{-\frac{1}{2}},
\]
i.e., \( \text{ran} \left( I - X \right)^{\frac{1}{2}} \cap \text{ran} \left( I + X \right)^{\frac{1}{2}} = \text{ran} \left( I - X^2 \right)^{\frac{1}{2}} \). \hfill \Box

Remark 3.2. If \( Z \) is a bounded operator, then the factorization
\[
I - Z^{2n+1} = (I - Z)(I + Z + \ldots + Z^{2n}), \ n \in \mathbb{N},
\]
yields the implication
\[
\text{ran} \left( I - Z \right) \neq \mathfrak{H} \implies \text{ran} \left( I - Z^{2n+1} \right) \neq \mathfrak{H}.
\]

Further we need contractions which are linear fractional transformations of maximal sectorial linear operators or linear relations.

Definition 3.3. [2] Let \( \alpha \in (0, \pi/2) \). A linear operator \( Z \) on the Hilbert space \( \mathfrak{H} \) is said to belong to the class \( C_{\mathfrak{H}}(\alpha) \) if
\[
||Z \sin \alpha \pm i \cos \alpha I|| \leq 1.
\]

The next proposition immediately follows from Definition 3.3.
Proposition 3.4. (cf. [2, 3]) The following are equivalent:

(i) $Z \in C\beta(\alpha)$;
(ii) $\pm Z^* \in C\beta(\alpha)$;
(iii) $Z$ is a contraction and
\begin{equation}
2|\text{Im} (Zf, f)| \leq \tan \alpha (||f||^2 - ||Zf||^2), \quad f \in \mathfrak{H}; \tag{3.2}
\end{equation}
(iv) holds the inequality
\begin{equation*}
2|\text{Im} (Zf, f)| \leq \tan \alpha (||f||^2 - ||Z^* f||^2), \quad f \in \mathfrak{H}.
\end{equation*}
(v) the bounded operator $S := (I + Z)(I - Z^*) = I - ZZ^* + Z - Z^*$ is $\alpha$-sectorial;
(vi) the linear relation $A$ in (3.1) is $m - \alpha$-sectorial.

Due to (3.2), it is natural to denote the set of all selfadjoint contractions by $C\beta(0)$. Clearly, $C\beta(0) = \bigcap_{\alpha \in (0, \pi/2)} C\beta(\alpha)$. If $||Z|| = \delta < 1$, then $Z \in C\beta(\alpha_\delta)$, where $\alpha_\delta = 2 \tan^{-1} \delta$.

The numerical range $W(Z)$ of $Z \in C\beta(\alpha)$ is contained in the intersection $C\mathcal{C}(\alpha)$ of two disks on the complex plane:
\begin{equation}
C\mathcal{C}(\alpha) = \{ z \in \mathbb{C} : |z \sin \alpha + i \cos \alpha| \leq 1 \wedge |z \sin \alpha - i \cos \alpha| \leq 1 \}.
\end{equation}
Therefore, the operators $I + Z$ and $I - Z$ are sectorial with vertex at the origin and semi-angle $\alpha$. Besides, since $|\text{Im} z| \leq \tan \frac{\alpha}{2}$ for all $z \in C\mathcal{C}(\alpha)$, the operator $Z = iY$, where $Y$ is a selfadjoint contraction, belongs to $C\beta(\alpha)$ if and only if $||Y|| \leq \tan \frac{\alpha}{2}$.

Set
\[ \widetilde{C}_\beta := \bigcup_{\alpha \in [0, \pi/2]} C\beta(\alpha). \]
Note that due to (3.2), an isometric operator belongs to the class $\widetilde{C}_\beta$ if and only if it is additionaly selfadjoint.

Further for a contraction $Z$ we will use the notation $D_Z := (I - Z^* Z)^{1/2}$. As is known [32] the commutation relations $ZD_Z = D_ZZ^*, Z^*D_Z = D_Z Z^*$ hold. Note that one of the equalities $\text{ran} D_Z = \mathfrak{H}$ or $\text{ran} D_{Z^*} = \mathfrak{H}$ is equivalent to the condition $||Z|| < 1$.

Properties of operators of the class $\widetilde{C}_\beta$ were studied in [2, 3]. In particular, the following assertions are valid.

Theorem 3.5. [2]. (1) If $Z \in \widetilde{C}_\beta$, then subspaces $\ker D_Z$ and $\text{ran} D_Z$ reduce $Z$, the restriction $Z|_{\ker D_Z}$ is selfadjoint and unitary operator, the restriction $Z|_{\text{ran} D_Z}$ belongs to the class $C_{00}$ [32], i.e.,
\[ \lim_{n \to \infty} Z^n f = \lim_{n \to \infty} Z^{*n} f = 0 \forall f \in \text{ran} D_Z. \]
(2) If $Z \in C\beta(\alpha)$, then $Z^n \in C\beta(\alpha)$ and
\begin{equation}
\text{ran} D_{Z^n} = \text{ran} D_{Z^{*n}} = \text{ran} D_{\text{Re}(Z)} \quad \forall n \in \mathbb{N}. \tag{3.4}
\end{equation}
(3) If $Z_1, Z_2 \in C\beta(\alpha)$, then $\frac{1}{2}(Z_1 Z_2 + Z_2 Z_1) \in C\beta(\alpha)$, in particular, if $Z_1$ and $Z_2$ commute, then $Z_1, Z_2 \in C\beta(\alpha) \implies Z_1 Z_2 \in C\beta(\alpha)$.
(4) If $A$ is $m - \alpha$-sectorial operator, then $\exp(-tA) \in C\beta(\alpha)$ for all $t \in \mathbb{R}_+$.

Note that if $A$ is $m - \alpha$-sectorial operator then, for each $t \in \mathbb{R}_+$ the numerical range $W(\exp(-tA))$ is contained in the square of the set $C\mathcal{C}(\alpha)$ (see [4, Theorem 3.4]), i.e.,
\[ W(\exp(-tA)) \subseteq \Omega(\alpha) := \{ z^2 : z \in C\mathcal{C}(\alpha) \}. \]
The next theorem will be used in Section 4.

**Theorem 3.6.** The following statements are equivalent for \( Z \in \widetilde{C}_\beta \):

1. \( ||Z|| < 1 \);
2. \( \text{ran}(I + Z) \subseteq \text{ran}D_{Z^*} \) and \( \text{ran}(I - Z) \subseteq \text{ran}D_{Z^*} \);
3. \( \text{ran}(I + Z) \supseteq \text{ran}D_{Z^*} \) and \( \text{ran}(I - Z) \supseteq \text{ran}D_{Z^*} \);
4. \( \text{ran}(I + Z) = \text{ran}(I - Z) = \mathcal{H} \).

**Proof.** Clearly

\[
||Z|| < 1 \implies \text{ran}(I + Z) = \text{ran}(I - Z) = \text{ran}D_{Z^*} = \mathcal{H}.
\]

Suppose (ii) holds. Then by Douglas’ lemma \([12]\) we get

\[
||(I + Z^*)f|| \leq c_1 ||D_{Z^*}f||, \quad ||(I - Z^*)f|| \leq c_2 ||D_{Z^*}f|| \quad \forall f \in \mathcal{H}.
\]

Since

\[
||(I \pm Z^*)f||^2 + ||D_{Z^*}f||^2 = 2\text{Re} ((I \pm Z)f, f) = 2||(I \pm \text{Re}(Z))\frac{1}{2}f||^2 \quad \forall f \in \mathcal{H},
\]

we get

\[
||(I \pm \text{Re}(Z))\frac{1}{2}f|| \leq b ||D_{Z^*}f|| \quad \forall f \in \mathcal{H}.
\]

Hence

\[
\text{ran}(I + \text{Re}(Z))\frac{1}{2} \subseteq \text{ran}D_{Z^*}, \quad \text{ran}(I - \text{Re}(Z))\frac{1}{2} \subseteq \text{ran}D_{Z^*}.
\]

On the other side equalities \((3.5)\) and the Douglas’ lemma implies the inclusions

\[
\text{ran}D_{Z^*} \subseteq \text{ran}(I - \text{Re}(Z))\frac{1}{2}, \quad \text{ran}D_{Z^*} \subseteq \text{ran}(I + \text{Re}(Z))\frac{1}{2}.
\]

Thus

\[
\text{ran}D_{Z^*} = \text{ran}(I - \text{Re}(Z))\frac{1}{2} = \text{ran}(I + \text{Re}(Z))\frac{1}{2}.
\]

Because \( Z \in \widetilde{C}_\beta \), from \((3.4)\) and Lemma \(3.1\) we get the equalities

\[
\text{ran}D_{Z^*} = \text{ran}D_{\text{Re}(Z)} = \text{ran}(I + \text{Re}(Z))\frac{1}{2} = \text{ran}(I - \text{Re}(Z))\frac{1}{2}.
\]

Since

\[
\text{ran}D_{\text{Re}(Z)} = (I + \text{Re}(Z))\frac{1}{2}\text{ran}(I - \text{Re}(Z))\frac{1}{2} = (I - \text{Re}(Z))\frac{1}{2}\text{ran}(I + \text{Re}(Z))\frac{1}{2},
\]

equalities in \((3.6)\) imply

\[
\text{ran}(I + \text{Re}(Z))\frac{1}{2} = \text{ran}(I - \text{Re}(Z))\frac{1}{2} = \text{ran}D_{\text{Re}(Z)} = \mathcal{H}.
\]

Hence \( \text{ran}D_{Z^*} = \mathcal{H} \) and therefore \( ||Z|| < 1 \). Thus, (ii) implies (i).

Suppose that (iii) holds. Using \((3.5)\), Douglas’ lemma and arguing as above, we get the equalities

\[
\text{ran}(I + \text{Re}(Z))\frac{1}{2} = \text{ran}(I + Z), \quad \text{ran}(I - \text{Re}(Z))\frac{1}{2} = \text{ran}(I - Z).
\]

Because the operator \( Z \) belongs to the class \( \widetilde{C}_\beta \), the operators \( I \pm Z \) are sectorial and, therefore, admit the representations

\[
I + Z = (I + \text{Re}(Z))\frac{1}{2}(I + iF_+)(I + \text{Re}(Z))\frac{1}{2},
\]
\[
I - Z = (I - \text{Re}(Z))\frac{1}{2}(I + iF_-)(I - \text{Re}(Z))\frac{1}{2},
\]

where \( F_\pm \) are bounded selfadjoint operators. Hence and from \((3.7)\)

\[
\text{ran}(I \pm \text{Re}(Z))\frac{1}{2} = (I \pm \text{Re}(Z))\frac{1}{2}(I \pm iF_\pm)\text{ran}(I \pm \text{Re}(Z))\frac{1}{2}.
\]
Consequently, because \( \text{ran} (I + iF_\pm) = \mathcal{H} \), we obtain the equalities
\[
\text{ran} (I + \text{Re} (Z))^{\frac{1}{2}} = \text{ran} (I - \text{Re} (Z))^{\frac{1}{2}} = \mathcal{H}.
\]
It follows that \( \pm 1 \in \rho(\text{Re} (Z)) \). Therefore \( \text{ran} D_{\text{Re}(Z)} = \mathcal{H} \) and (3.3) implies that \( \text{ran} D_Z = \mathcal{H} \). Therefore \( ||Z|| < 1 \). Thus, (iii) \( \implies \) (i).

Suppose (iv) holds. Then there are \( c_\pm > 0 \) such that \( ||(I \pm Z^*) f|| \geq c_\pm ||f|| \) for all \( f \in \mathcal{H} \). Consequently, from (3.3) we get the equalities
\[
\text{ran} (I + \text{Re} (Z))^{\frac{1}{2}} = \text{ran} (I - \text{Re} (Z))^{\frac{1}{2}} = \mathcal{H}.
\]
Now Lemma 3.1 and (3.4) lead to the equalities \( \text{ran} D_{\text{Re}(Z)} = \text{ran} D_{Z^*} = \mathcal{H} \). Hence \( ||Z|| < 1 \). The proof is complete. □

**Proposition 3.7.** Let \( Z \) be a contraction in \( \mathcal{H} \). Then the operators
(3.8)
\[
Z(t) := Z \exp(-t(I - Z^2)), \quad t \in \mathbb{R}_+
\]
are contractions and if \( Z \in C_0(\alpha) \), then \( Z(t) \in C_0(\alpha) \) for each \( t \in \mathbb{R}_+ \).

Moreover, if \( \text{ran} (I - Z^2) \neq \mathcal{H} \), then \( \text{ran} (I - Z(t)^2) \neq \mathcal{H} \) for each \( t > 0 \).

**Proof.** Because \( Z^2 \) is a contraction, the operator \( I - Z^2 \) is accretive. It generates contractive one-parameter \( C_0 \)-semigroup \( \{\exp(-t(I - Z^2))\}_{t \in \mathbb{R}_+} \) [14]. Hence, for each \( t \in \mathbb{R}_+ \) the operator \( Z(t) \) is a contraction.

Assume \( Z \in C_0(\alpha) \). Then \( Z^2 \in C_0(\alpha) \) (see Theorem 3.5) and, hence, \( I - Z^2 \) is \( \alpha \)-sectorial. From Theorem 3.5, statement (3) (see also [2] Theorem 1, Theorem 2) it follows that
\[
\exp(-t(I - Z^2)) \in C_0(\alpha) \quad \forall t > 0.
\]
Because the operators \( Z \) and \( \exp(-t(I - Z^2)) \) commute and belong to the class \( C_0(\alpha) \), the operator \( Z(t) \) belongs to the same class \( C_0(\alpha) \) (see Theorem 3.5).

The operator \( I - Z(t)^2 \) can be represented as follows
\[
I - Z(t)^2 = I - Z^2 \exp(-2t(I - Z^2)) = (I - Z^2) \left( I - Z^2 \sum_{n=1}^{\infty} \frac{(-2t)^n}{n!}(I - Z^2)^{n-1} \right).
\]
Hence \( \text{ran} (I - Z^2) \neq \mathcal{H} \) implies \( \text{ran} (I - Z(t)^2) \neq \mathcal{H} \). □

**Remark 3.8.** Because the operator \( Z \) is bounded, the one-parameter group
\[
\exp(-t(I - Z^2)), \quad t \in \mathbb{R}
\]
is continuous on \( \mathbb{R} \) w.r.t. the operator-norm topology (see [14] Proposition 3.5). Hence, the same is true for the operator-valued function \( Z(t) \) defined in 3.8.

4. New representations of maximal accretive operators and of their square roots

In the following statements we characterize bounded sectorial operators whose squares are accretive and sectorial operators.

**Proposition 4.1.** Let \( \mathcal{H} \) be a Hilbert space, let \( Q \) be a bounded selfadjoint nonnegative operator in \( \mathcal{H} \), \( \ker Q = \{0\} \). Suppose that
(4.1)
\[
Z \text{ is a contraction and } QZ = -Z^*Q.
\]
Then
(1) the operator
\[ T := Q(I + Z) = (I - Z^*)Q \]
is \( \frac{\pi}{4} \)-sectorial, \( \ker T = \{0\} \) and
\[ T^* = Q(I - Z) = (I + Z^*)Q; \]

(2) hold the equalities
\[ \text{ran } T = Q \text{ran } (I + Z), \quad \text{ran } T^* = Q \text{ran } (I - Z), \]

(3) the following are equivalent
(i) \( \text{ran } T = \text{ran } T^* \),
(ii) \( \text{ran Re } (T) \subseteq \text{ran } T \cap \text{ran } T^* \),
(iii) \( \text{ran } (I + Z) = \text{ran } (I - Z) = \mathcal{H} \);

(4) the operator \( T^2 \) takes the form
\[ T^2 = Q(I + Z)(I - Z^*)Q = (I - Z^*)Q^2(I + Z) = Q ((I - ZZ^*) + (Z - Z^*))Q \]
and is accretive;

(5) the operator \( T^2 \) is \( \alpha \)-sectorial if and only if \( Z \in C_\mathcal{H}(\alpha) \);

(6) if \( \text{ran } Z \cap \text{ran } Q = \{0\} \), \( \ker Z = \{0\} \), then
\[ \text{ran } T^2 \cap \text{ran } T^{*2} = \{0\}. \]

Proof. The equality in (4.1) implies that the operator \( iQZ \) is selfadjoint. Therefore for the operator
\[ T := Q(I + Z) = (I - Z^*)Q \]
we have
\[ \text{Re } (T) = Q, \quad \text{Im } (T) = -iQZ = iZ^*Q. \]
Hence, the bounded operator \( T \) is accretive. Therefore, \( \ker T = \ker T^* \subseteq \ker \text{Re } (T) \). Because \( \text{Re } (T) = Q \) and \( \ker Q = \{0\} \) we get \( \ker T = \ker T^* = \{0\} \). Since \( T^* = Q(I - Z) = (I + Z^*)Q \), we get that \( \ker (I + Z) = \ker (I - Z) = \{0\} \) and equalities (4.2) hold. Since
\[ \text{ran } T = Q \text{ran } (I + Z), \quad \text{ran } T^* = Q \text{ran } (I - Z), \]
the equality \( \text{ran } T = \text{ran } T^* \) is equivalent to the equality \( \text{ran } (I + Z) = \text{ran } (I - Z) \). By Lemma 3.1 the latter is equivalent to the equalities \( \text{ran } (I + Z) = \text{ran } (I - Z) = \mathcal{H} \). Because of the inclusions \( \text{ran } Q = \text{ran } \text{Re } (T) \supseteq \text{ran } T, \text{ran } T^* \), the condition \( \text{ran } \text{Re } (T) \subseteq \text{ran } T \cap \text{ran } T^* \), is equivalent to the equalities \( \text{ran } T = \text{ran } T^* = \text{ran } \text{Re } (T) \).

From (4.1) it follows that the operator \( QZQ^{-1}|_{\text{ran } Q} \) is a contraction in \( \mathcal{H} \), then due to [21, Theorem 1] we get that the operator
\[ X := -iQ^\frac{1}{2}ZQ^{-\frac{1}{2}}|_{\text{ran } Q^\frac{1}{2}} \]
is contraction.

The equalities
\[ (XQ^\frac{1}{2}g, Q^\frac{1}{2}h) = -i(QZg, h) = i(Z^*Qg, h) = i(Qg, Zh) = i(Q^\frac{1}{2}g, Q^\frac{1}{2}Zh) = (Q^\frac{1}{2}g, XQ^\frac{1}{2}h), \]
imply that \( X \) is essentially selfadjoint.

Due to the equality
\[ iQ^\frac{1}{2}XQ^\frac{1}{2} = QZ \]
we obtain
\[ T = Q + QZ = Q - Z^*Q = Q + iQ^{1/2}XQ^{1/2}, \]
\[ T^* = Q - QZ = Q + Z^*Q = Q - iQ^{1/2}XQ^{1/2}, \]
and
\[ |\text{Im} (Tf, f)| = |(XQ^{1/2}f, Q^{1/2}f)| \leq ||Q^{1/2}f||^2 = \text{Re} (Tf, f), \ f \in \mathcal{H}. \]

It follows that the operator \( T \) is \( \frac{\pi}{4} \)-sectorial.

For the square \( T^2 \) one obtains
\[ T^2 = Q(I + Z)(I - Z^*)Q, \text{ Re} (T^2) = Q(I - ZZ^*)Q, \text{ iIm} (T^2) = Q(Z - Z^*)Q. \]

Since \( Z \) is a contraction, the operator \( T^2 \) is accretive. Clearly \( T^2 \) is \( \alpha \)-sectorial if and only if the operator \( (I + Z)(I - Z^*) \) is \( \alpha \)-sectorial and by Proposition 3.4 the latter is equivalent to \( Z \in C_S(\alpha) \).

Suppose that \( \text{ran} Z \cap \text{ran} Q = \{0\} \) and \( \ker Z = \{0\} \).

If \( T^2f = T^2g \) for some \( f, g \in \mathcal{H} \), then
\[ Q(I - ZZ^* + Z - Z^*)Qf = Q(I - ZZ^* - Z + Z^*)Qg. \]

Since \( \ker Q = \{0\} \), we get the equality
\[ (Z - Z^*)Q(f + g) = (I - ZZ^*)Q(g - f). \]

Set \( \psi := f + g, \varphi := g - f \). Then, using the equality \( QZ = -Z^*Q \), we have
\[ (Z - Z^*)Q\psi = (I - ZZ^*)Q\varphi \iff QZ\psi + ZZ^*Q\varphi = Q\varphi + Z^*Q\psi \iff QZ\psi = Q\varphi \iff QZ\psi = Q(Z\psi). \]

Taking into account that \( \text{ran} Z \cap \text{ran} Q = \{0\} \), we obtain \( \varphi = Z\psi \) and therefore
\[ \text{ker} Q = \{0\} \implies \text{ker} (I - Z^2) = \{0\}. \]

Therefore \( f = g = 0 \), i.e., \( \text{ran} T^2 \cap \text{ran} T^* = \{0\} \).

**Remark 4.2.** (1) The following statement follows from Proposition 3.1: if \( Q \) is a bounded nonnegative selfadjoint operator, \( \ker Q = \{0\} \), then there is no a nonzero bounded selfadjoint operator \( Z \) which anti-commutes with \( Q \) \( (QZ + ZQ = 0) \).

Actually, if \( Z \) is a such selfadjoint operator, then \( \widehat{Z} = ||Z||^{-1}Z \) is a selfadjoint contraction and \( Q\widehat{Z} = -\widehat{Z}Q \). The operator \( \widehat{T} = Q + Q\widehat{Z} \) is accretive and \( \widehat{T}^2 = Q(I - \widehat{T}^2)Q \geq 0 \), i.e. the operator \( \widehat{T}^2 \) is nonnegative selfadjoint. The uniqueness of maximal accretive square root of a maximal accretive operator (see [22, Chapter V, Theorem 3.35]) yields that \( \text{Im} (\widehat{T}) = 0 \), i.e., \( QZ = 0 \) and hence \( Z = 0 \).

On the other side the above statement can be derived from the well known Heinz inequality (see e.g. [15]): if \( Q_1 \) and \( Q_2 \) are bounded nonnegative selfadjoint operators, then for each bounded operator \( Z \) and for an arbitrary \( r \in [0, 1] \) holds
\[ ||Q_1Z + Q_2Z|| \geq ||Q_1^rZQ_2^{1-r} + Q_1^{1-r}ZQ_2^r||. \]
In fact, if \( Q_1 = Q_2 = Q \) and if \( r = \frac{1}{2} \), then \( ||QZ + QZ|| \geq 2 ||Q^{1/2}QZQ^{1/2}||. \) Hence, \( \ker Q = \{0\} \) and the equality \( QZ + QZ = 0 \) yield \( Z = 0 \).
(2) Suppose that the operator $Z$ is skew-selfadjoint, i.e., $Z = iY$, where $Y$ is a selfadjoint contraction. Then (4.1) means that $Y$ commutes with $Q$ and the operator $T = Q(I + iY)$ is a normal, $\frac{\pi}{4}$-sectorial, $T^* = Q(I - iY)$, and $\text{ran } T = \text{ran } T^* = \text{ran } Q$. The operator $T^2 = Q(I + iY)^2$ is normal and accretive. It is $\alpha$-sectorial if and only if $iY \in \mathcal{C}_\alpha(\alpha)$. Since $Y$ is a selfadjoint contraction, the latter means that $||Y|| \leq \tan \frac{\alpha}{2}$.

Observe, that condition (4.1) implies the equalities

$$QZ^{2n} = Z^{*2n}Q \quad \text{and} \quad Q(iZ^{2n}) = -(iZ^{2n})^*Q,$$

$$QZ^{2n-1} = -(Z^{2n-1})^*Q,$$

$$TZ^{2n} = Z^{*2n}T, \quad TZ^{2n-1} = -Z^{*(2n-1)}T \quad \forall n \in \mathbb{N}.$$ Moreover, since the contractive function $Z(t)$, defined in (3.8), admits the representation

$$Z(t) = Z \exp(-t(I - Z^2)) = \exp(-t) \sum_{n=0}^{\infty} \frac{t^n}{n!}Z^{2n+1}, \quad t \in \mathbb{R}_+,$$

one obtains the equalities

$$QZ(t) = -Z(t)^*Q, \quad TZ(t) = -Z(t)^*T \quad \forall t \in \mathbb{R}_+.$$ Thus, the operators $iQZ^{2n-1}$ and $iQZ(t)$ are selfadjoint for all $n \in \mathbb{N}$ and all $t \in \mathbb{R}_+$. It follows that the operators

$$T_n = Q(I + Z^{2n-1}), \quad T(t) = Q(I + Z(t))$$

are accretive and $\text{Re } (T_n) = \text{Re } (T_t) = Q \forall n \in \mathbb{N}, \forall t \in \mathbb{R}_+$. In addition, the operator-valued function $T(t)$ is continuous on $\mathbb{R}_+$ w.r.t. the operator-norm topology (see Remark 3.8).

Taking into account Proposition 3.7 we arrive to the following corollary.

Corollary 4.3. Let (4.1) be satisfied. Then

(1) assertions of Proposition 4.1 hold true for the operators

$$T_n = Q(I + Z^{2n-1}), \quad T_n^* = Q(I - Z^{2n-1}), \quad T^2_n = Q(I + Z^{2n-1})(I - Z^{*(2n-1)})Q,$$

$$\tilde{T}_n = Q(I + iZ^{2n}) = (I + iZ^{*2n})Q, \quad \tilde{T}^2_n = Q(I + iZ^{2n})(I + iZ^{*2n})Q, \quad n \in \mathbb{N}$$

$$T(t) = Q(I + Z(t)) = (I - Z(t)^*)Q, \quad T(t)^* = Q(I - Z(t)) = (I + Z(t)^*)Q,$$

$$T(t)^2 = Q(I + Z(t))(I - Z(t)^*)Q, \quad t \in \mathbb{R}_+.$$ Besides

$$\text{Re } (T_n) = \text{Re } (\tilde{T}_n) = \text{Re } (T(t)) = Q \quad \forall n \in \mathbb{N}, \quad \forall t \in \mathbb{R}_+$$

and the operator-valued functions $T(t)$ and $T(t)^2$ are continuous w.r.t. the operator-norm topology.

(2) The operators $\tilde{T}_n := Q(I + Z^{2n}) = (I + Z^{*2n})Q, \quad n \in \mathbb{N},$ are selfadjoint.

Proposition 4.4. Let $T$ be a bounded accretive operator in $\mathcal{H}, \ker T = \{0\}$. Suppose that $T^2$ is also accretive. Then

(1) $T$ is $\frac{\pi}{4}$-sectorial;

(2) the operator

$$Z := i(\text{Re } (T))^{-1}\text{Im } (T),$$
is a contraction and holds the relation

\[(\text{Re}(T))Z = -Z^*(\text{Re}(T)).\]

(3) the operators $T$, $T^*$ and $T^2$ admit the representations

\[T = (\text{Re}(T))(I + Z) = (I - Z^*)(\text{Re}(T)),\]

(4.4) \[T^* = (I + Z^*)(\text{Re}(T)) = (\text{Re}(T))(I - Z),\]

\[T^2 = (\text{Re}(T))(I + Z)(I - Z^*)(\text{Re}(T)) = (I - Z^*)(\text{Re}(T))^2(I + Z);\]

(4) the operator

\[\begin{cases}
\text{Re}(T^{-1}) = \frac{1}{2}(T^{-1} + T^{*-1}), \\
\text{dom Re}(T^{-1}) = \text{dom } T^{-1} \cap \text{dom } T^{*-1} = (\text{Re}(T))\text{ran } (I - Z^2)
\end{cases}\]

is selfadjoint;

(5) the operators $T^{-1}T^*$ and $T^{*-1}T$ are $m$-accretive and hold the equalities

\[T^{-1}T^* = (I - Z)(I + Z)^{-1}, \quad \text{dom } (T^{-1}T^*) = \text{ran } (I + Z) = T^{*-1}(\text{ran } T \cap \text{ran } T^*),\]

(4.5) \[T^{*-1}T = (I + Z)(I - Z)^{-1}, \quad \text{dom } (T^{*-1}T^*) = \text{ran } (I - Z) = T^{-1}(\text{ran } T \cap \text{ran } T^*);\]

(6) the following are equivalent:

(i) the operator $T^2$ is $\alpha$-sectorial,

(ii) the operator $T^{-1}T^*$ is $m - \alpha$-sectorial,

(iii) the operator $Z$ belongs to the class $C_{\alpha}(\alpha)$.

Proof. Let

\[T = \text{Re}(T) + i\text{Im}(T), \quad T^* = \text{Re}(T) - i\text{Im}(T),\]

be Cartesian decompositions of the operators $T$ and $T^*$. Then the Cartesian decomposition of the operator $T^2 = (\text{Re}(T) + i\text{Im}(T))^2$ takes the form

\[T^2 = (\text{Re}(T))^2 - (\text{Im}(T))^2 + i((\text{Re}(T))(\text{Im}(T) + (\text{Im}(T))(\text{Re}(T)))).\]

Since $T^2$ is accretive, we get

\[\text{Re}(T^2f, f) = \|\text{Re}(T)f\|^2 - \|\text{Im}(T)f\|^2 \geq 0 \quad \forall f \in \mathcal{H},\]

i.e., $\|\text{Im}(T)f\| \leq \|\text{Re}(T)f\|$ for all $f \in \mathcal{H}$. It follows that if $\text{Re}(T)f = 0$, then $\text{Im}(T)f = 0$ and, consequently, $Tf = 0$. Thus, the condition $\ker T = 0$ and accretiveness of $T^2$ imply $\ker \text{Re}(T) = \{0\}$. Moreover, because $\text{ran } \text{Re}(T) = \mathcal{H}$, there exists a contraction $V$ in $\mathcal{H}$ such that

\[\text{Im}(T) = V(\text{Re}(T)) = (\text{Re}(T))V^*.\]

Hence, the operator $Z$ given by (4.3) is a contraction, coincides with the operator $iV^*$, and $(\text{Re}(T))Z = -Z^*(\text{Re}(T)) = i\text{Im}(T)$. It follows that equalities (4.4) are valid and $\ker (I \pm Z) = \{0\}$. Besides, Proposition 4.1 yields that $T$ is a $\frac{\pi}{4}$-sectorial.

From expressions for $T$ and $T^*$ and Lemma 3.1 we get

\[\text{ran } T \cap \text{ran } T^* = (\text{Re}(T))\left(\text{ran } (I + Z) \cap \text{ran } (I - Z)\right) = (\text{Re}(T))\text{ran } (I - Z^2).\]
The operator $T^{-1}$ is $m - \frac{\pi}{4}$ sectorial operator and from (4.4)
\[
T^{-1} = (I + Z)^{-1}(\text{Re}(T))^{-1}, \quad T^{-1} = (I - Z)^{-1}(\text{Re}(T))^{-1}.
\]
\[
\text{dom} \ T^{-1} \cap \text{dom} \ T^{-1} = (\text{Re}(T))\text{ran} \ (I - Z^2).
\]
Let $f = (\text{Re}(T))(I - Z^2)g, \ g \in \mathcal{H}$. Then 
\[
\text{Re}(T^{-1})(\text{Re}(T))(I - Z^2)g = \frac{1}{2} (T^{-1} + T^{-1}) f
\]
\[
= \frac{1}{2} ((I + Z)^{-1}(\text{Re}(T))^{-1} + (I - Z)^{-1}(\text{Re}(T))^{-1}) (\text{Re}(T))(I - Z^2)g
\]
\[
= \frac{1}{2} ((I - Z)g + (I + Z)g) = g,
\]
\[
(\text{Re}(T^{-1}) + I) (\text{Re}(T))(I - Z^2)g = (I + \text{Re}(T) + Z^*(\text{Re}(T))Z)g.
\]
Since the Re$T$ and $Z^*(\text{Re}T)Z$ are bounded nonnegative selfadjoint operators, we get that 
\[
\text{ran} \ (I + \text{Re}(T) + Z^*(\text{Re}(T))Z) = \mathcal{H}.
\]
Hence 
\[
\text{ran} \ (\text{Re}(T^{-1}) + I) = \mathcal{H}.
\]
Therefore, the operator Re$T^{-1}$ is selfadjoint.

From (4.3) it follows the equality 
\[
(4.6) \quad T^*(I + Z)h = T(I - Z)h = \text{Re}(T)(I - Z^2)h \ \forall h \in \mathcal{H}.
\]
This leads to (4.5). Equivalences in (6) follow from Proposition 3.4 and equalities (4.4), (4.5).

**Remark 4.5.** (1) Set $B := T^{-2}$. Then $T = B^{-\frac{1}{2}}$ and statement (4) (a) means that the operator 
\[
\text{Re}(B^{\frac{1}{2}}) = \frac{1}{2} \left( B^{\frac{1}{2}} + B^{+\frac{1}{2}} \right), \quad \text{dom} \ \text{Re}(B^{\frac{1}{2}}) = \text{dom} \ B^{\frac{1}{2}} \cap \text{dom} \ B^{+\frac{1}{2}}
\]
is selfadjoint. This result was established in [19, Theorem 5.1] (see Introduction). The proof in [19] is based on the Yosida approximation $A_n := B^{\frac{1}{2}} \left( I + n^{-1}B^{\frac{1}{2}} \right)^{-1}, \ n = 1, 2, ...$ of the operator $B^{\frac{1}{2}}$. Using the same approximation, slightly different proof of this fact was given in [27, Theorem 3.1, Corollary 3.2].

(2) Set $G := T^{-1}T = B^{\frac{1}{2}}B^{-\frac{1}{2}}$. From (4.6) we get that 
\[
G = (I + Z)(I - Z)^{-1} = -I + 2(I - Z)^{-1}.
\]
Hence 
\[
G^* + I = 2(I - Z^{-1}), \ G(I - Z) = I + Z, \ G^*(I - Z^*) = I + Z^*.
\]
Then equalities in (4.4) yield that 
\[
G^*(\text{Re}(T))Gf = (\text{Re}(T))f \ \forall f \in \text{dom} \ G,
\]
\[
T = 2(G^* + I)^{-1}\text{Re}(T).
\]
Hence, if $S = 2\text{Re}(T) = B^{-\frac{1}{2}} + B^{+\frac{1}{2}}$, then $B^{\frac{1}{2}} = S^{-1}(G^* + I)$ and $G^*SGf = Sf$ for all 
f $f \in \text{dom} \ G = \text{ran} \ (I - Z)$. These relations have been established (by another way) in
Theorem 4.6. There is a one-to-one correspondence between
(a) pairs of bounded operators \( \{Q, Z\} \) in \( \mathcal{S} \) satisfying conditions
(1) \( Q \) is a nonnegative selfadjoint operator in \( \mathcal{S} \), \( \ker Q = \{ 0 \} \),
(2) \( Z \) is a contraction in \( \mathcal{S} \),
(3) \( QZ = -Z^*Q \),
and
(b) bounded sectorial operators \( T \) such that \( T^2 \) is accretive and \( \ker T = \{ 0 \} \).
The correspondence is given by the mappings
\[
\begin{align*}
\{Q, Z \} \quad &\mapsto \quad \begin{cases}
\text{Re} (T) = Q, \text{Im} (T) = -iQZ, \\
T = Q(I + Z) = (I - Z^*)Q, \quad T^* = Q(I - Z) = (I + Z^*)Q, \\
T^2 = Q(I + Z)(I - Z^*)Q = (I - Z^*)Q^2(I + Z) \\
\end{cases}, \\
\{ \text{Re} (T) + i\text{Im} (T) \} \quad &\mapsto \quad \{ Q = \text{Re} (T), \quad Z = iQ^{-1}\text{Im} (T) \}.
\end{align*}
\]

The operator \( T \) is normal if and only if \( Z \) is a skew-selfadjoint \((Z^* = -Z)\).
The operator \( T^2 \) is \( \alpha \)-sectorial if and only if \( Z \in C_\alpha(\alpha) \).

Further in Theorem 4.7 we give new representations of unbounded \( m \)-accretive and \( m \)-sectorial coercive operators and their square roots. The proof is based on Proposition 4.1 and Proposition 4.4.

Theorem 4.7. Let \( B \) be an \( m \)-accretive operator having bounded inverse. Then
(1) \( B \) admits the representations
\[
B = \mathcal{L}((I + Z)(I - Z^*))^{-1} \mathcal{L} = (I + Z)^{-1} \mathcal{L}^2(I - Z^*)^{-1},
\]
where a positive definite selfadjoint operator \( \mathcal{L} \) and a contraction \( Z \) are given by the relations
\[
\mathcal{L} := (\text{Re} (B^{-\frac{1}{2}}))^{-1}, \\
Z := i(\text{Re} (B^{-\frac{1}{2}}))^{-1}(\text{Im} (B^{-\frac{1}{2}})),
\]
and satisfy the condition
\[
\mathcal{L}Z^*f = -Z\mathcal{L}f, \quad \forall f \in \text{dom} \mathcal{L};
\]
(2) the square roots \( B^{\frac{1}{2}}, B^{* \frac{1}{2}} \) admit the representations
\[
B^{\frac{1}{2}} = (I + Z)^{-1}\mathcal{L}(I - Z^*)^{-1}, \quad B^{* \frac{1}{2}} = (I - Z)^{-1}\mathcal{L} = \mathcal{L}(I + Z^*)^{-1};
\]
(3) the operator \( B^{\frac{1}{2}}B^{* \frac{1}{2}} \) is \( m \)-accretive and takes the form
\[
B^{\frac{1}{2}}B^{* \frac{1}{2}} = (I - Z)(I + Z)^{-1};
\]
(4) the following are equivalent:
(i) operator \( B \) is \( m - \alpha \)-sectorial;
(ii) the operator \( B^{\frac{1}{2}}B^{* \frac{1}{2}} \) is \( m - \alpha \)-sectorial;
(iii) $Z \in C_B(\alpha)$.

In the next theorem we establish criterions for the equality $\text{dom } B_{1/2}^\downarrow = \text{dom } B_{*1/2}^\downarrow$.

**Theorem 4.8.** Let $B$ be an unbounded $m$-accretive operator having bounded inverse and let the operator $Z$ be given by (4.9). Then the following are equivalent

(i) $\text{dom } B_{1/2}^\downarrow = \text{dom } B_{*1/2}^\downarrow$,
(ii) the operator $B^{1/2}B_{*1/2}^{-1}$ is bounded and has bounded inverse
(iii) $\text{ran } (I + Z) = \text{ran } (I - Z) = \mathcal{H}$,
(iv) $\text{ran } (I - Z^2) = \mathcal{H}$,
(v) $\text{dom } (\text{Re } (B_{1/2}^-))^{-1} \subseteq \text{dom } B_{1/2}^\downarrow \cap \text{dom } B_{*1/2}^\downarrow$.

If $B$ is $m$-sectorial, then the following are equivalent:

(a) $\text{dom } B_{1/2}^\downarrow = \text{dom } (B_R_{1/2}^\downarrow)$;
(b) $\text{dom } B_{1/2}^\downarrow = \text{dom } (B_R_{*1/2}^\downarrow)$;
(c) $\text{dom } B_{*1/2}^\downarrow = \text{dom } (B_R_{1/2}^\downarrow)$;
(d) $\text{dom } B_{1/2}^\downarrow \subseteq \text{dom } (B_R_{*1/2}^\downarrow)$ and $\text{dom } B_{*1/2}^\downarrow \subseteq \text{dom } (B_R_{1/2}^\downarrow)$;
(e) $\text{dom } (B_R_{1/2}^\downarrow) \subseteq \text{dom } B_{1/2}^\downarrow \cap \text{dom } B_{*1/2}^\downarrow$;
(f) $\|Z\| < 1$;

(g) $\sup_{f \in \mathcal{H} \setminus \{0\}} \frac{\|\text{Re } (B_{1/2}^-) f\|^2}{\text{Re } (B_{-1} f, f)} < \infty$;

(h) $\sup_{f \in \mathcal{H} \setminus \{0\}} \frac{\|B_{1/2}^- f\|^2 + \|B_{*1/2}^- f\|^2}{\text{Re } (B_{-1} f, f)} < \infty$;

(i) $\text{dom } (\text{Re } (B_{1/2}^-))^{-1} = \text{dom } (\text{Re } (B_{*1/2}^-))^{-1}$.

**Proof.** Set $T := B_{1/2}^-$. Then $T^2 = B_{-1}$. We can apply Propositions 4.1, 4.4, and Theorem 4.6. Representations (4.7), (4.8), (4.10), (4.11) hold true with $Z$ given by (4.9). So we have

$$\text{dom } B_{1/2}^\downarrow = \mathcal{L}^{-1} \text{ran } (I + Z), \text{ dom } B_{*1/2}^\downarrow = \mathcal{L}^{-1} \text{ran } (I - Z),$$

$$\text{Re } (B_{-1} f, f) = \text{Re } (T^2 f, f) = (\mathcal{L}^{-1} (I - ZZ^*) \mathcal{L}^{-1} f, f) = \|D_Z \mathcal{L}^{-1} f\|^2, \ f \in \mathcal{H}.$$}

Hence, from (2.4) and from Douglas’s lemma [12] we get

$$\text{dom } (B_R_{1/2}^\downarrow) = \text{ran } (\text{Re } (B_{-1}^-)) = \mathcal{L}^{-1} \text{ran } D_Z.$$}

Equalities in (4.11) and (4.12) imply that (i) and (ii) are equivalent. The equivalences (i) $\iff$ (iii) and (i) $\iff$ (iv) follow from Lemma 3.1. Equalities (4.8), (4.12), and Proposition 4.1(3) yield the equivalence of (iii) and (v).

Assume now that $B$ is $m$-sectorial. From Theorem 3.6 and equalities (4.12), (4.14), (4.11) we get that (a), (b), (c), (d), (e), and (f) are equivalent.

Equality (4.14) implies that $\text{ran } (B_{-1}^-)^{1/2} = \mathcal{L}^{-1} = \text{ran } (B_{1/2}^-)$ is equivalent to $\|Z\| < 1$, i.e., (f) and (i) are equivalent.
Condition $||Z|| < 1$ and (4.13) imply that for some $0 < c \leq 1$ holds

$$\text{Re} \left( B^{-1}f, f \right) \geq c||L^{-1}f||^2 = c||\text{Re}(T)||^2 = c||\text{Re}(B^{-\frac{1}{2}}f)||^2 = c\left\| \frac{1}{2}(B^{-\frac{1}{2}} + B^{\ast -\frac{1}{2}})f \right\|^2$$

$$= \frac{c}{4} \left( \left\| B^{\ast -\frac{1}{2}}f \right\|^2 + \left\| B^{-\frac{1}{2}}f \right\|^2 + 2\text{Re} (B^{-\frac{1}{2}}f, B^{\ast -\frac{1}{2}}f) \right)$$

$$= \frac{c}{4} \left( \left\| B^{\ast -\frac{1}{2}}f \right\|^2 + \left\| B^{-\frac{1}{2}}f \right\|^2 + 2\text{Re} (B^{-1}f, f) \right) \forall f \in \mathcal{F}.$$ 

So, (f) $\implies$ (g), (f) $\implies$ (h).

Suppose (g) is fulfilled. Then there is $c > 0$ such that

$$\left\| (\text{Re} (B^{-\frac{1}{2}}))f \right\|^2 \leq c \text{Re} (B^{-1}f, f) \forall f \in \mathcal{F}.$$ 

From (4.13) and (4.8) we obtain

$$||L^{-1}h||^2 \leq c ||D_ZL^{-1}h||^2 = c (||L^{-1}h||^2 - ||Z^\ast L^{-1}h||^2), \forall h \in \mathcal{F}.$$ 

This means that $||Z^\ast|| = ||Z|| < 1$. Consequently, (g) $\implies$ (f).

Now assume that (h) holds, i.e., there is $c > 0$ such that

$$\left\| B^{\ast -\frac{1}{2}}f \right\|^2 + \left\| B^{-\frac{1}{2}}f \right\|^2 \leq c \text{Re} (B^{-1}f, f) \forall f \in \mathcal{F}.$$ 

Then

$$4 ||(\text{Re} (B^{-\frac{1}{2}}))f||^2 = \left\| B^{\ast -\frac{1}{2}}f \right\|^2 + \left\| B^{-\frac{1}{2}}f \right\|^2 + 2\text{Re} (B^{-\frac{1}{2}}f, B^{\ast -\frac{1}{2}}f)$$

$$= \left\| B^{\ast -\frac{1}{2}}f \right\|^2 + \left\| B^{-\frac{1}{2}}f \right\|^2 + 2\text{Re} (B^{-1}f, f) \leq (c + 2) \text{Re} (B^{-1}f, f) \forall f \in \mathcal{F}.$$ 

Hence

$$||L^{-1}f||^2 \leq \frac{c + 2}{4} ||D_ZL^{-1}f||^2 \forall f \in \mathcal{F}.$$ 

It follows that $||Z|| < 1$. Therefore (h) $\implies$ (f). The proof is complete. \qed

**Remark 4.9.** The equivalences of (i) and (ii) in Theorem 4.7 and of (i) and (ii) in Theorem 4.8 have been established in [16, Theorem 1]. We use another approach.

As has been mentioned in Introduction (see Theorem 1.1), equivalences of (a), (b), (c), (d), (e) were established in [24] and [20]. The proofs in [24] are based on the theory interpolation spaces and in [20] on the representations of m-sectorial operators and their associated quadratic forms [21, 22].

Now we formulate and prove an analogue of [16, Theorem 3] (see Introduction, Theorem 1.2).

**Theorem 4.10.** Let $B$ be an unbounded $m$-accretive operator having bounded inverse and let the operators $L$ and $Z$ be given by (4.8) and (4.9), respectively. Set

$$Z_\gamma := ((I + Z)^\gamma - (I - Z)^\gamma) ((I + Z)^\gamma + (I - Z)^\gamma)^{-1}, \gamma \in (0, 1).$$

Then

(4.15)
(1) $Z_\gamma \in C_\beta \left( \frac{\pi \gamma}{2} \right)$ and
\[ \mathcal{L}Z_\gamma^* f = -Z_\gamma \mathcal{L} f, \quad \forall f \in \text{dom } \mathcal{L}, \]

(2) the operator
\[ B_\gamma = \mathcal{L} \left( (I + Z_\gamma)(I - Z_\gamma^*) \right)^{-1} \mathcal{L} = (I + Z_\gamma)^{-1} \mathcal{L}^2 (I - Z_\gamma^*)^{-1}, \]
is $m - \frac{\pi \gamma}{2}$ sectorial and its square root is given by
\[ B_\gamma^{\frac{1}{2}} = (I + Z_\gamma)^{-1} \mathcal{L} = \mathcal{L}^2 (I - Z_\gamma^*)^{-1}, \]

(3) the following are equivalent:
(i) $\text{dom } B_\gamma^{\frac{1}{2}} = \text{dom } B_\gamma^{\frac{1}{2}}^*$,
(ii) $\text{dom } B_\gamma^{\frac{1}{2}} = \text{dom } B_\gamma^{\frac{1}{2}}^*$.

Proof. Let $Q := \mathcal{L}^{-1}$. By Theorem 1.7, the operator $R := B^{-1}$ takes the form
\[ R = Q(I + Z)(I - Z^*)Q \]
and
\[ R_\gamma^{\frac{1}{2}} = B^{-\frac{1}{2}} = Q(I + Z) = (I + Z^*)Q, \quad R_\gamma^{\frac{1}{2}} = B_\gamma^{\frac{1}{2}}^{-1} = Q(I - Z) = (I + Z^*)Q. \]

Since $QZ = -Z^*Q$, from (1.1), we obtain the equality
\[ Q(I \pm Z)\gamma = (I \mp Z^*)\gamma Q, \quad \gamma \in (0, 1). \]

Set
\[ F := (I - Z)(I + Z)^{-1}. \]

Then $F$ is an $m$-accretive operator, moreover, due to (4.11) we have the equality $F = B_\gamma^{\frac{1}{2}} B_\gamma^{\frac{1}{2}}^{-1}$.

If $\gamma \in (0, 1)$, then $F^\gamma$ is $m - \frac{\pi \gamma}{2}$ sectorial operator and hence
\[ (I - F^\gamma)(I + F^\gamma)^{-1} \in C_\beta \left( \frac{\pi \gamma}{2} \right). \]

Clearly
\[ (I - F^\gamma)(I + F^\gamma)^{-1} = ((I + Z)^\gamma - (I - Z)^\gamma)((I + Z)^\gamma + (I - Z)^\gamma)^{-1} = Z_\gamma. \]

Since
\[ Z_\gamma ((I + Z)^\gamma + (I - Z)^\gamma) = (I + Z)^\gamma - (I - Z)^\gamma, \]
we get
\[ QZ_\gamma = -Z_\gamma^* Q. \]

Hence, $\mathcal{L}Z_\gamma^* f = -Z_\gamma \mathcal{L} f, \quad f \in \text{dom } \mathcal{L}$. Besides
\[ \text{ran } (I \pm Z_\gamma) = \text{ran } (I \pm Z)^\gamma. \]

By Proposition 1.1, the operator
\[ R_\gamma := Q(I + Z_\gamma)(I - Z_\gamma^*)Q \]
is bounded $\frac{\pi \gamma}{2}$-sectorial and

$$
R_{\gamma}^{\frac{1}{2}} = Q(I + Z_{\gamma}) = (I - Z_{\gamma}^*)Q, \ R_{\gamma}^{\frac{1}{2}} = Q(I - Z_{\gamma}) = (I + Z_{\gamma}^*)Q.
$$

Because

$$
\text{ran} (I \pm Z_{\gamma}) = \mathcal{H} \iff \text{ran} (I \pm Z) = \mathcal{H},
$$

we get that

$$
\text{ran} R_{\gamma}^{\frac{1}{2}} = \text{ran} R_{\gamma}^{\frac{1}{2}} \iff \text{ran} R_{\gamma}^{\frac{1}{2}} = \text{ran} R_{\gamma}^{\frac{1}{2}}.
$$

□

**Remark 4.11.** The function

$$
w_{\gamma} := \frac{(1 + z)^{\gamma} - (1 - z)^{\gamma}}{(1 + z)^{\gamma} + (1 - z)^{\gamma}} = \frac{1 - \left(\frac{1 - z}{1 + z}\right)^{\gamma}}{1 + \left(\frac{1 - z}{1 + z}\right)^{\gamma}}, \ z \in \mathbb{D}, \ \gamma \in (0, 1)
$$

is odd, holomorphic in $\mathbb{D}$ and has values in the set $C_{c}(\frac{\pi \gamma}{2})$. By the Schwarz lemma it has the representation

$$
w_{\gamma}(z) = zw_{\gamma}(z), \ z \in \mathbb{D},
$$

where $w_{\gamma}(z)$ is even, holomorphic in $\mathbb{D}$ and $|w_{\gamma}(z)| < 1, \ z \in \mathbb{D},$

$$
w_{\gamma}(z) = \frac{w_{\gamma}(z)}{z}, \ z \in \mathbb{D} \setminus \{0\}, \ w_{\gamma}(0) = \gamma.
$$

Hence, by means of the functional calculus for contractions [32] the operator $Z_{\gamma}$ given by (1.15) can be represented as follows:

$$
Z_{\gamma} = Zw_{\gamma}(Z).
$$

Therefore, $\text{ran} Z_{\gamma} \subseteq \text{ran} Z$. In particular, $Z_{\gamma}^{\frac{1}{2}} = Z\left(I + (I - Z^2)^{\frac{1}{2}}\right)^{-1}$.

5. Examples

Here we present abstract (counter)examples related to the Kato square root problem. Recall that the Hilbert space $\mathcal{H}$ is supposed to be infinite-dimensional and separable.

5.1. Square roots of $m$-accretive operators. We begin with a general result related to an arbitrary maximal accretive operators of the form $iA$, where $A$ is a maximal symmetric non-selfadjoint operator. This is a generalization of Lions’ example [24] (see (1.3)). We apply our methods from Section [4]

**Theorem 5.1.** Let $A$ be a non-selfadjoint maximal symmetric operator in the Hilbert space $\mathcal{H}$, $i \in \rho(A)$. Then for maximal accretive operator $B = iA$ holds the inequality $\text{dom} B^{\frac{1}{2}} \neq \text{dom} B^{\frac{1}{2}}$. 

Proof. Because \( \text{Re}(Bf, f) = 0 \) for all \( f \in \text{dom} \, B \), the operator \( B \) is accretive. Since \( i \in \rho(A) \), we get \(-1 \in \rho(B)\). Therefore \( B \) is \( m \)-accretive, the square root \( B^{\frac{1}{2}} \) is \( m - \frac{\pi}{4} \)-sectorial. Set

\[
\mathcal{U} := (B - I)(I + B)^{-1} = (I - iA)(I + iA)^{-1}.
\]

Then \( \mathcal{U} \) is a non-unitary isometry because \( \text{ran} \, \mathcal{U} \) is a proper subspace of \( \mathcal{H} \). This means that \( 0 \in \sigma(\mathcal{U}) \). The operator \( I - \mathcal{U} \) is accretive and since

\[
I - \mathcal{U} = 2(I + B)^{-1},
\]

we obtain the equalities

\[
\text{dom} \, B^{\frac{1}{2}} = \text{dom} \, (I + B)^{\frac{1}{2}} = \text{ran} \, (I - \mathcal{U})^{\frac{1}{2}}, \quad \text{dom} \, B^{*\frac{1}{2}} = \text{dom} \, (I + B^{*})^{\frac{1}{2}} = \text{ran} \, (I - \mathcal{U}^{*})^{\frac{1}{2}}.
\]

Since \( \ker(I - \mathcal{U}) = \ker(I - \mathcal{U}^{*}) = \{0\} \) and the operator \( (I - \mathcal{U})^{\frac{1}{2}} \) is \( m - \frac{\pi}{4} \)-sectorial, we get (see (2.3))

\[
\ker \, \text{Re} ((I - \mathcal{U})^{\frac{1}{2}}) = \{0\}.
\]

Set

\[
Q = \text{Re} ((I - \mathcal{U})^{\frac{1}{2}}) = Q^{-1} \text{Im} ((I - \mathcal{U})^{\frac{1}{2}}).
\]

Then \( QZ = -Z^{*}Q \) and according to Proposition 4.4 the operators \( (I - \mathcal{U})^{\frac{1}{2}} \), \( (I - \mathcal{U}^{*})^{\frac{1}{2}} \), \( I - \mathcal{U} \), and \( I - \mathcal{U}^{*} \) admit the representations

\[
(I - \mathcal{U})^{\frac{1}{2}} = Q(I + Z) = (I - Z^{*})Q,
\]

\[
(I - \mathcal{U}^{*})^{\frac{1}{2}} = Q(I - Z) = (I + Z^{*})Q,
\]

\[
I - \mathcal{U} = (Q(I + Z))^{2} = ((I - Z^{*})Q)^{2} = Q(I + Z)(I - Z^{*})Q,
\]

\[
I - \mathcal{U}^{*} = (Q(I - Z))^{2} = ((I + Z^{*})Q)^{2} = Q(I - Z)(I + Z^{*})Q.
\]

Set

\[
\mathcal{M} := (I + Z)(I - Z^{*}),
\]

then \( I - \mathcal{U} = \mathcal{Q} \mathcal{M} Q \), \( I - \mathcal{U}^{*} = \mathcal{Q} \mathcal{M}^{*} Q \),

\[
\mathcal{U} = I - \mathcal{Q} \mathcal{M} Q, \quad \mathcal{U}^{*} = I - \mathcal{Q} \mathcal{M}^{*} Q.
\]

Hence,

\[
\mathcal{U}^{*} \mathcal{U} = I \iff \mathcal{M}^{*} \mathcal{Q} \mathcal{M} = \mathcal{M} + \mathcal{M}^{*} \iff \mathcal{M}^{*} (\mathcal{Q} \mathcal{M} - I) = \mathcal{M}.
\]

Now assume that \( \text{ran} \, (I - \mathcal{U})^{\frac{1}{2}} = \text{ran} \, (I - \mathcal{U}^{*})^{\frac{1}{2}} \). This yields (see Proposition 4.1) that \( \text{ran} \, (I + Z) = \text{ran} \, (I - Z) = \mathcal{H} \). It follows that \( 0 \in \rho(\mathcal{M}) \). Then the equality \( \mathcal{M}^{*} (\mathcal{Q} \mathcal{M} - I) = \mathcal{M} \) implies that \( 1 \in \rho(\mathcal{Q} \mathcal{M}) \). It follows that \( 1 \in \rho(\mathcal{Q} \mathcal{M} Q) \). Since \( I - \mathcal{Q} \mathcal{M} Q = \mathcal{U} \), we get that \( 0 \in \rho(\mathcal{U}) \). Contradiction, because \( \text{ran} \, \mathcal{U} \neq \mathcal{H} \). Therefore \( \text{ran} \, (I - \mathcal{U})^{\frac{1}{2}} \neq \text{ran} \, (I - \mathcal{U}^{*})^{\frac{1}{2}} \), i.e., \( \text{dom} \, B^{\frac{1}{2}} \neq \text{dom} \, B^{*\frac{1}{2}} \).

Note that for the operator \( B \) in Theorem 5.1 the inclusion \( \text{dom} \, B \subset \text{dom} \, B^{*} \) holds. In the next theorem we construct \( m \)-accretive non-sectorial operator \( B_{0} \) such that

\[
\text{dom} \, B_{0}^{\frac{1}{2}} \neq \text{dom} \, B_{0}^{*\frac{1}{2}}, \quad iB_{0}^{\frac{1}{2}} B_{0}^{-\frac{1}{2}} \text{ is maximal symmetric but non-selfadjoint,}
\]

\[
\text{dom} \, B_{0} \cap \text{dom} \, B_{0}^{*} = \{0\}.
\]
Further we will need a result established in [29, Theorem 5.1]: if $\mathcal{R}$ is an operator range (the domain of an unbounded selfadjoint operator or a dense linear manifold, which is the range of a bounded nonnegative selfadjoint), then there is a subspace $\mathcal{M}$ such that

$$\mathcal{M} \cap \mathcal{R} = \{0\} \quad \text{and} \quad \mathcal{M} \perp \cap \mathcal{R} = \{0\}.$$

**Theorem 5.2.** Let $Q$ be a bounded nonnegative selfadjoint operator, $\ker Q = \{0\}$ and $\text{ran} Q \neq \mathcal{H}$. Let $\mathcal{M}$ be a proper subspace in $\mathcal{H}$ such that $\mathcal{M} \cap \text{ran} Q = \{0\}$. Set

$$T_0 = Q + i (QP_{\mathcal{M}}Q)^{\frac{1}{2}}. \quad (5.2)$$

Then

- the operator $T_0$ is bounded and $\frac{\pi}{4}$-sectorial, $\ker T_0 = \{0\}$;
- $\mathcal{M} \perp \cap \text{ran} T_0 = \mathcal{M} \perp \cap \text{ran} T_0^* = \{0\}$;
- $\text{ran} T_0 \neq \text{ran} T_0^*$;
- the square

$$T_0^2 = QP_{\mathcal{M}}Q + i \left( Q (QP_{\mathcal{M}}Q)^{\frac{1}{2}} + (QP_{\mathcal{M}}Q)^{\frac{1}{2}} Q \right),$$

is accretive and non-sectorial.

Moreover, the following are equivalent:

1. $\mathcal{M} \cap \text{ran} Q = \{0\}$;
2. $\ker \text{Re} \left( T_0^2 \right) = \{0\}$;
3. $\text{ran} T_0^2 \cap \text{ran} T_0^{*2} = \{0\}$.

Set $B_0 := (T_0^2)^{-1}$. Then

1. $B_0$ is $m$-accretive but not sectorial,
2. $\text{dom} B_0^{\frac{1}{2}} \neq \text{dom} B_0^{\frac{1}{2}}$,
3. the operator $B_0^{\frac{1}{2}} B_0^{-\frac{1}{2}}$ is $m$-accretive,

$$\text{Re} \left( B_0^{\frac{1}{2}} B_0^{-\frac{1}{2}} h, h \right) = 0 \quad \forall h \in \text{dom} \left( B_0^{\frac{1}{2}} B_0^{-\frac{1}{2}} \right),$$

and $\text{dom} B_0^{\frac{1}{2}} B_0^{-\frac{1}{2}} \subsetneq \text{dom} \left( B_0^{\frac{1}{2}} B_0^{-\frac{1}{2}} \right)^*$,

4. $\text{dom} B_0 \cap \text{dom} B_0^* = \{0\}$ if and only if $\mathcal{M} \cap \text{ran} Q = \{0\}$.

**Proof.** Since $\text{Re} \left( T_0 \right) = Q \geq 0$, the bounded operator $T_0$ is accretive. The condition $\ker Q = \{0\}$ yields that $\ker T_0 = \ker T_0^* = \{0\}$. From (5.2) we get

$$\text{Re} \left( T_0^2 \right) = Q^2 - QP_{\mathcal{M}}Q = QP_{\mathcal{M}}Q \geq 0. \quad (5.3)$$

Therefore, the operator $T_0^2$ is accretive and $T_0$ is the accretive square root of $T_0^2$. By Proposition 4.4 the operator $T_0$ is $\frac{\pi}{4}$-sectorial.

Equality (5.3) yields that $\ker \text{Re} \left( T_0^2 \right) = \{0\}$ if and only if $\text{ran} Q \cap \mathcal{M} = \{0\}$. Because

$$\left\| (QP_{\mathcal{M}}Q)^{\frac{1}{2}} f \right\|^2 = \|P_{\mathcal{M}}Qf\|^2 \quad \forall f \in \mathcal{H},$$

the operator $(QP_{\mathcal{M}}Q)^{\frac{1}{2}}$ admits the representation

$$(QP_{\mathcal{M}}Q)^{\frac{1}{2}} = VP_{\mathcal{M}}Q = QV^*,$$
where \( V : \mathcal{M} \to \mathcal{N} \) is an isometry. Since \( \text{ran} \ Q \cap \mathcal{M}^\perp = \{0\} \), we get that \( \text{ran} \ (QP_{\mathcal{M}}) = \mathcal{N} \). Therefore, \( \text{ran} \ V = \mathcal{N} \) and, acting in \( \mathcal{N} \), the operator

\[
Z_0 := iV^* = iQ^{-1}(QP_{\mathcal{M}})^{\frac{1}{2}}
\]

isometrically maps \( \mathcal{N} \) onto \( \mathcal{M} \). Hence, \( Z_0 \) is a non-unitary isometry in \( \mathcal{N} \), \( \ker Z_0^* = \mathcal{M}^\perp \) and \( Z_0 \) satisfies the equality \( QZ_0 = -Z_0^*Q \). Because \( DZ_0 = 0 \) and \( DZ_0^* = P_{\mathcal{M}^\perp} \neq 0 \), the operator \( Z_0 \) does not belong to the class \( C_{\mathcal{N}} \) (see Theorem 3.3 and (3.4)). Therefore, by Proposition 4.4 the operator \( T_0^* \) is not sectorial.

From (5.2) it follows that the operators \( T_0 \) and \( T_0^* \) admit representations

\[
T_0 = Q(I + Z_0), \quad T_0^* = Q(I - Z_0).
\]

Therefore, \( \ker (I + Z_0) = \ker (I - Z_0) = \{0\} \). Applications of Lemma 3.1 gives

\[
\text{ran} \ T_0 \cap \text{ran} \ T_0^* = Q(\text{ran} (I + Z_0) \cap \text{ran} (I - Z_0)) = Q\text{ran} (I - Z_0^*).
\]

If \( \text{ran} (I - Z_0) = \mathcal{N} \), then the operator \( \widehat{S} = i(I + Z_0)(I - Z_0)^{-1} \) is a bounded selfadjoint and \( Z_0 = (\widehat{S} - iI)(\widehat{S} + iI)^{-1} \) is the Cayley transform of \( \widehat{S} \). Hence, \( \text{ran} Z_0 = \mathcal{N} \). This contradicts to the inclusion \( \text{ran} Z_0 = \mathcal{M} \not\subseteq \mathcal{N} \). Therefore \( \text{ran} (I - Z_0) \neq \mathcal{N} \) and similarly, \( \text{ran} (I + Z_0) \neq \mathcal{N} \).

From Lemma 3.1 it follows that

\[
\text{ran} (I + Z_0) \neq \text{ran} (I - Z_0).
\]

Thus, \( \text{ran} T_0 \neq \text{ran} T_0^* \).

The operators \( T_0^2 \) and \( T_0^{02} \) take the form

\[
T_0^2 = Q(I + Z_0)(I - Z_0^*)Q = Q(P_{\mathcal{M}^\perp} + (Z_0 - Z_0^*))Q,
\]

\[
T_0^{02} = Q(I - Z_0)(I + Z_0^*)Q = Q(P_{\mathcal{M}^\perp} - (Z_0 - Z_0^*))Q.
\]

Observe that \( \ker (Z_0 - Z_0^*) = \{0\} \). Actually, if \( Z_0 f = Z_0^* f \), then \( f = Z_0^{02} f \). This implies \( f = 0 \).

If \( \mathcal{M} \cap \text{ran} Q \neq \{0\} \), then there is \( f \neq 0 \) such that \( Qf \in \mathcal{M} \). Hence \( P_{\mathcal{M}^\perp} Qf = 0 \) and

\[
T_0^2 f = Q(Z_0 - Z_0^*)Qf = T_0^{02}(-f) \neq 0.
\]

Suppose \( \mathcal{M} \cap \text{ran} Q = \{0\} \). Because \( \text{ran} Z_0 = \mathcal{M} \), \( \ker Z_0 = \{0\} \) and \( \ker (I - Z_0^2) = \{0\} \) we can apply Proposition 4.1 (see statement (5)). So, \( \text{ran} T_0^2 \cap \text{ran} T_0^{02} = \{0\} \).

Since \( Z_0 \) is an isometry, relation (4.5) implies that the operator

\[
F_0 := T_0^{-1}T_0^* = B_0^\frac{1}{2}B_0^{*\frac{1}{2}}
\]

is \( m \)-accretive. \( \text{Re} (F_0 h, h) = 0 \) for all \( h \in \text{dom} F_0 = \text{ran} (I + Z_0) \). Besides the operator \( iF_0 \) is maximal symmetric but non-selfadjoint. Note that because \( iB_0^\frac{1}{2}B_0^{*\frac{1}{2}} \) is maximal symmetric and non-selfadjoint, the inclusion \( \text{dom} B_0^\frac{1}{2}B_0^{*\frac{1}{2}} \subseteq \text{dom} (B_0^\frac{1}{2}B_0^{*\frac{1}{2}})^* \) holds. \( \square \)

In the corollary below the countable set \( \{B_n\}_{n \in \mathbb{N}} \) of \( m \)-accretive operators satisfying (5.1) is constructed.
Corollary 5.3. Let an operator $Q$ and a subspace $\mathcal{M}$ be given as in Theorem 5.2 and let the operator $Z_0$ be defined by (5.4). Then for each $n \in \mathbb{N}$

1. the operators

$$T_n := Q(I + Z_0^{2n+1}) = (I - Z_0^{(2n+1)})Q = Q + (-1)^n i Q \left( Q^{-1} (Q P_{\mathcal{M}} Q)^{\frac{1}{2}} \right)^{2n+1}$$

are $\frac{\pi}{4}$-sectorial, $\text{Re}(T_n) = Q$, $\ker T_n = \ker (I + Z_0^{2n+1}) = \ker (I - Z_0^{(2n+1)}) = \{0\}$,

2. $\text{ran } T_n^* \neq \text{ran } T^*$,

3. the operators

$$T_n^2 = Q(I + Z_0^{2n+1})(I - Z_0^{(2n+1)})Q$$

are accretive and non-sectorial,

4. the operators $T_n^{-1} T_n^*$ are $m$-accretive, $\text{Re}(T_n^{-1} T_n^* h, h) = 0$ for all $h \in \text{dom}(T_n^{-1} T_n^*)$, and $\text{dom}(T_n^{-1} T_n^*) \subsetneq \text{dom}(T_n^{-1} T_n^*)^*$,

5. if $\mathcal{M} \cap \text{ran } Q = \{0\}$, then $\text{ran } T_n^2 \cap \text{ran } T_n^* = \{0\}$.

Therefore, the operator

$$B_n := (T_n)^{-2} = \left( Q(I + Z_0^{2n+1})(I - Z_0^{(2n+1)})Q \right)^{-1}$$

is $m$-accretive, non-sectorial, and for each $n \in \mathbb{N}$

- $\text{dom } B_n^\frac{1}{2} \neq \text{dom } B_n^{-\frac{1}{2}}$,

- the operator $B_n^\frac{1}{2} B_n^{-\frac{1}{2}}$ is $m$-accretive, $\text{Re}(B_n^\frac{1}{2} B_n^{-\frac{1}{2}} h, h) = 0$ for all $h \in \text{dom}(B_n^\frac{1}{2} B_n^{-\frac{1}{2}})$, and $\text{dom}(B_n^\frac{1}{2} B_n^{-\frac{1}{2}}) \subsetneq \text{dom}(B_n^\frac{1}{2} B_n^{-\frac{1}{2}})^*$,

- if $\mathcal{M} \cap \text{ran } Q = \{0\}$, then $\text{dom } B_n \cap \text{dom } B_n^* = \{0\}$.

Proof. The operators $Z_0^{2n+1}$ are isometries and $\text{ran } Z_0^{2n+1} \subset \mathcal{M}$ for all $n \in \mathbb{N}$. Since $Q Z_0^{2n+1} = -Z_0^{(2n+1)} Q$, $n \in \mathbb{N}$ (see Corollary 4.3), we can apply arguments in Theorem 5.2. \hfill \Box

The next corollary is proved similarly.

Corollary 5.4. Let an operator $Q$ and a subspace $\mathcal{M}$ be given as in Theorem 5.2 and let the operator $Z_0$ be defined by (5.4). Set

$$Z_0(t) := Z_0 \exp(-t(I - Z_0^2)), \ t \in \mathbb{R}_+.$$

Then for each $t \in \mathbb{R}_+$ the operator

$$B_0(t) := (Q(I + Z_0(t))(I - Z_0(t)^*))Q^{-1} = (Q(I + Z_0(t))Q(I + Z_0(t)))^{-1}$$

is $m$-accretive. The square root takes the form

$$B_0(t) = (Q(I + Z_0(t)))^{-1} = ((I - Z_0(t)^*)Q)^{-1},$$

and

$$\text{dom } B_0(t)^\frac{1}{2} \neq \text{dom } B_0(t)^* \forall t \in \mathbb{R}_+.$$

In addition, the operator-valued function $B_0(t)$ is norm-resolvent continuous on $\mathbb{R}_+$.

If $\mathcal{M} \cap \text{ran } Q = \{0\}$, then $\text{dom } B_0(t) \cap \text{dom } B_0(t)^* = \{0\}$ for each $t \in \mathbb{R}_+$.

Further we construct a continuum family of maximal accretive operators possessing specific properties.
Theorem 5.5. There exists a norm-resolvent continuous family \( \{A(\xi)\}_{\xi \in \mathbb{C}} \) of unbounded and boundedly invertible \( m \)-accretive and non-sectorial operators such that

- \( \text{dom} A(\xi) \cap \text{dom} A(\xi)^* = \{0\} \) for each \( \xi \in \mathbb{C} \);
- \( \text{dom} A(\xi)^{\frac{1}{2}} \neq \text{dom} A(\xi)^{\star \frac{1}{2}} \) for each \( \xi \in \mathbb{C} \);
- the operator \( \text{Re} (A(\xi)^{-\frac{1}{2}}) \) does not depend on \( \xi \in \mathbb{C} \), therefore, the domain of the closed sesquilinear form associated with \( m \)-sectorial operator \( A(\xi)^{\frac{1}{2}} \) does not depend on \( \xi \);
- the linear manifold \( \text{ran} \text{Im} (A(\xi)^{-\frac{1}{2}}) \) is dense in \( \mathcal{H} \) for each \( \xi \) and if \( \zeta \neq \xi \), then
  \[
  \text{ran} \text{Im} (A(\xi)^{-\frac{1}{2}}) \cap \text{ran} \text{Im} (A(\zeta)^{-\frac{1}{2}}) = \{0\},
  \]
  \[
  \text{ran} \text{Im} (A(\xi)^{-\frac{1}{2}}) \cap \text{ran} \text{Im} (A(\zeta)^{-\frac{1}{2}}) = \text{Const};
  \]
- for each \( \xi \in \mathbb{C} \) the operator \( A(\xi)^{\frac{1}{2}} A(\xi)^{\star \frac{1}{2}} \) is \( m \)-accretive and, moreover,
  \[
  \text{Re} \left( A(\xi)^{\frac{1}{2}} A(\xi)^{\star \frac{1}{2}} h, h \right) = 0 \quad \forall h \in \text{dom} \left( A(\xi)^{\frac{1}{2}} A(\xi)^{\star \frac{1}{2}} \right)
  \]
  and \( \text{dom} \left( A(\xi)^{\frac{1}{2}} A(\xi)^{\star \frac{1}{2}} \right) \subset \text{dom} \left( A(\xi)^{\frac{1}{2}} A(\xi)^{\star \frac{1}{2}} \right)^* \).

Proof. Let \( Q \) be a bounded nonnegative selfadjoint operator, \( \ker Q = \{0\} \) and \( \text{ran} Q \neq \mathcal{H} \). By [5] Theorem 3.9 there exists a family of subspaces \( \{\mathcal{M}(\xi)\}_{\xi \in \mathbb{C}} \) such that

\[
\begin{align*}
\mathcal{M}(\xi) \cap \text{ran} Q &= (\mathcal{M}(\xi))^{\bot} \cap \text{ran} Q = \{0\} \quad \forall \xi \in \mathbb{C}, \\
\mathcal{M}(\xi) \cap \mathcal{M}(\zeta) &= \{0\}, \quad \mathcal{M}(\xi) + \mathcal{M}(\zeta) = \mathcal{H}, \quad \xi \neq \zeta, \\
\mathcal{M}(\xi)^{\bot} &= \mathcal{M}(-1/\xi) \quad \forall \xi \in \mathbb{C} \setminus \{0\},
\end{align*}
\]
(5.5)

- the ortho-projector-valued function \( P_{\mathcal{M}(\xi)} \) is continuous on \( \mathbb{C} \) w.r.t. the operator-norm topology.

Set

\[
\mathcal{T}(\xi) := Q + i \left( Q P_{\mathcal{M}(\xi)} Q \right)^{\frac{1}{2}}, \quad \xi \in \mathbb{C}.
\]

Then \( \mathcal{T}(\xi) \) is of the form (5.2) and the operator-valued function \( \mathcal{T}(\xi), \xi \in \mathbb{C} \) is continuous on \( \mathbb{C} \) w.r.t. the operator-norm topology. The corresponding operator \( \mathcal{Z}(\xi) = i Q^{-1} \left( Q P_{\mathcal{M}(\xi)} Q \right)^{\frac{1}{2}} \) isometrically maps \( \mathcal{H} \) onto \( \mathcal{M}(\xi) \).

Set \( A(\xi) := \mathcal{T}(\xi)^{-2} \). Applying Theorem 5.2 we get that all statements of this theorem hold true.

By definition of \( \mathcal{T}(\xi) \) and \( A(\xi) \) we have that

1. holds the equality \( \text{Re} (A(\xi)^{-\frac{1}{2}}) = Q \), therefore [2.4] yields that the domain of the closed sesquilinear form associated with \( m \)-sectorial operator \( A(\xi)^{\frac{1}{2}} \) does not depend on \( \xi \);
2. \( \text{Im} (A(\xi)^{-\frac{1}{2}}) = \left( Q P_{\mathcal{M}(\xi)} Q \right)^{\frac{1}{2}} \), consequently, \( \text{ran} \text{Im} (A(\xi)^{-\frac{1}{2}}) = Q \mathcal{M}(\xi) \) and from (5.5)

\[
\begin{align*}
\text{ran} \text{Im} (A(\xi)^{-\frac{1}{2}}) &= \mathcal{H} \quad \forall \xi, \\
\text{ran} \text{Im} (A(\xi)^{-\frac{1}{2}}) \cap \text{ran} \text{Im} (A(\zeta)^{-\frac{1}{2}}) &= \{0\}, \\
\text{ran} \text{Im} (A(\xi)^{-\frac{1}{2}}) \cap \text{ran} \text{Im} (A(\zeta)^{-\frac{1}{2}}) &= \text{ran} Q = \text{ran} \text{Re} (A(\xi)^{-\frac{1}{2}}), \quad \xi \neq \zeta.
\end{align*}
\]
\[\square\]
5.2. Square roots of m-sectorial operators.

**Theorem 5.6.** Let \( Q \) be a bounded nonnegative selfadjoint operator, \( \ker Q = \{0\} \) and \( \operatorname{ran} Q \neq \mathbb{N} \). Suppose \( \mathcal{M} \) is a proper subspace in \( \mathcal{H} \) such that \( \mathcal{M}^\perp \cap \operatorname{ran} Q = \{0\} \) and let the contractive operator \( Z_0 \) be given by

\[
Z_0 = iQ^{-1} (QP_{\mathcal{M}}Q)^{\frac{1}{2}}.
\]

For \( \gamma \in (0, 1) \) set

\[
Z_{\gamma}^{(0)} := ((I + Z_0)^\gamma - (I - Z_0)^\gamma)((I + Z_0)^\gamma + (I - Z_0)^\gamma)^{-1}, \quad \gamma \in (0, 1),
\]

\[
Z_{\gamma}^{(0)}(t) := Z_{\gamma}^{(0)} \exp(-t(I - (Z_{\gamma}^{(0)})^2)), \quad t \in \mathbb{R}_+,
\]

\[
B_{\gamma,n} := \left(Q(I + (Z_{\gamma}^{(0)})^{2n+1})(I - (Z_{\gamma}^{(0)})^{*(2n+1)})Q\right)^{-1}, \quad n \in \mathbb{N}_0,
\]

\[
S_{\gamma}(t) := \left(Q(I + Z_{\gamma}^{(0)}(t))(I - Z_{\gamma}^{(0)}(t)^*)Q\right)^{-1}, \quad t \in \mathbb{R}_+.
\]

Then the operators \( B_{\gamma,n} \) and \( S_{\gamma}(t) \) are unbounded \( m - \frac{\pi \gamma}{2} \)-sectorial for each \( n \in \mathbb{N}_0 \) and for each \( t \in \mathbb{R}_+ \), the operator-valued function \( S_{\gamma}(t) \) is norm-resolvent continuous on \( \mathbb{R}_+ \), and

\[(5.6) \quad \operatorname{dom} B_{\gamma,n} \neq \operatorname{dom} B_{\gamma,n}^*, \quad \operatorname{dom} S_{\gamma}(t) \neq \operatorname{dom} S_{\gamma}(t)^* \frac{1}{2}.
\]

If \( \mathcal{M} \cap \operatorname{ran} Q = \{0\} \), then

\[(5.7) \quad \operatorname{dom} B_{\gamma,n} \cap \operatorname{dom} B_{\gamma,n}^* = 0, \quad \operatorname{dom} S_{\gamma}(t) \cap \operatorname{dom} S_{\gamma}(t)^* = \{0\} \ \forall n \in \mathbb{N}_0, \ \forall t \in \mathbb{R}_+.
\]

**Proof.** Due to Theorem 5.2, the operator

\[
B_0 := \left(QP_{\mathcal{M}}Q + i \left(Q(P_{\mathcal{M}}Q)^{\frac{1}{2}} + (QP_{\mathcal{M}}Q)^{\frac{1}{2}} Q\right)\right)^{-1} = (Q(I + Z_0)(I - Z_0^*)Q)^{-1}
\]

is \( m \)-accretive but non-sectorial, and the inequality \( \operatorname{dom} B_0^{\frac{1}{2}} \neq \operatorname{dom} B_0^{*\frac{1}{2}} \) holds. This is because the operator \( Z_0 \) is a non-unitary isometry, \( QZ_0 = -Z_0^*Q \), and

\[
\operatorname{ran} (I + Z_0) \neq \operatorname{ran} (I - Z_0)
\]

(see the proof of Theorem 5.2). By Theorem 4.10 the operator \( Z_{\gamma}^{(0)} \) belongs to the class \( C_\delta(\frac{\pi \gamma}{2}) \) and hence

\[
(Z_{\gamma}^{(0)})^{2n+1}, Z_{\gamma}^{(0)}(t) \in C_\delta(\frac{\pi \gamma}{2})
\]

for all natural numbers \( n \) and for all \( t \in \mathbb{R}_+ \) (see Theorem 3.5 and Proposition 3.7). Besides

\[
Q(Z_{\gamma}^{(0)})^{2n+1} = -(Z_{\gamma}^{(0)})^{*(2n+1)}Q, \quad QZ_{\gamma}^{(0)}(t) = -Z_{\gamma}^{(0)}(t)^*Q.
\]

Consequently, by Corollary 4.3 and Theorem 4.7 the operators \( B_{\gamma,n} \) and \( S_{\gamma}(t) \) and \( m - \frac{\pi \gamma}{2} \) sectorial for all \( n \in \mathbb{N} \) and all \( t \in \mathbb{R}_+ \).

Since \( B_{\gamma,0} = S_{\gamma}(0) \), due to Theorem 4.10 (statement (3)) we have the inequalities

\[
\operatorname{dom} B_{\gamma,0}^{\frac{3}{2}} = \operatorname{dom} S_{\gamma}(0)^{\frac{3}{2}} \neq \operatorname{dom} B_{\gamma,0}^{*\frac{1}{2}} = \operatorname{dom} S_{\gamma}(0)^{*\frac{1}{2}},
\]
and apply Theorem 5.6. Applying Lemma 3.1, Remark 3.2, and Proposition 3.7, one obtains for all \( n \in \mathbb{N} \) and for all \( t \in \mathbb{R}_+ \) the inequalities
\[
\text{ran} \left( I + Z^{(0)}_\gamma \right)^{2n+1} \neq \text{ran} \left( I - Z^{(0)}_\gamma \right)^{2n+1}, \quad \text{ran} \left( I + Z^{(t)}_\gamma \right) \neq \text{ran} \left( I - Z^{(t)}_\gamma \right).
\]
Now inequalities (5.6) follow from Proposition 4.1 and Corollary 4.3.

Assume \( \mathcal{M} \cap \text{ran} \, Q = \{0\} \). Taking into account that (see Remark 4.11)
\[
\text{ran} \, Z^{(0)}_\gamma \subseteq \mathcal{M}, \quad \ker \, Z^{(0)}_\gamma = \{0\}, \quad \ker (I - Z^{(0)}_\gamma)^2 = \{0\},
\]
and applying Proposition 4.11, we obtain (5.7).

Since the function \( S_\gamma(t)^{-1} \) is continuous w.r.t. operator-norm topology on \( \mathbb{R}_+ \), the function \( S_\gamma(t) \) is norm-resolvent continuous on \( \mathbb{R}_+ \).

**Corollary 5.7.** Let \( \alpha \in (0, \pi/2) \).

1. There exists a family \( \{S_\alpha(\xi)\}_{\xi \in \mathbb{C}} \) of unbounded and coercive \( m - \alpha \)-sectorial operators such that for each \( \xi \in \mathbb{C} \)
   - \( \text{dom} \, S_\alpha(\xi) \cap \text{dom} \, S_\alpha(\xi)^* = \{0\} \),
   - \( \text{dom} \, S_\alpha(\xi)^{1/2} \neq \text{dom} \, S_\alpha(\xi)^{1/2} \),
   - \( \Re \, (S_\alpha(\xi)^{-\frac{1}{2}}) = \text{Const} \),
   - \( \text{ran} \, \text{Im} \, (S_\alpha(\xi)^{-\frac{1}{2}}) \cap \text{ran} \, \text{Im} \, (S_\alpha(\xi)^{-\frac{1}{2}}) = \{0\}, \ \xi \neq \zeta \).

2. There exists a norm-resolvent continuous family \( \{X_\alpha(\xi)\}_{\xi \in \mathbb{C}} \) of unbounded and coercive \( m - \alpha \)-sectorial operators such that for each \( \xi \in \mathbb{C} \)
   - \( \text{dom} \, X_\alpha(\xi) \cap \text{dom} \, X_\alpha(\xi)^* = \{0\} \),
   - \( \text{dom} \, X_\alpha(\xi)^{1/2} = \text{dom} \, X_\alpha(\xi)^{1/2} = \text{Const} \) (does not depend on \( \xi \)),
   - the linear manifold \( \text{ran} \, \text{Im} \, (X_\alpha(\xi)^{-\frac{1}{2}}) \) is dense in \( \mathcal{H} \) for each \( \xi \) and if \( \xi \neq \zeta \), then
     \[
     \text{ran} \, \text{Im} \, (X_\alpha(\xi)^{-\frac{1}{2}}) \cap \text{ran} \, \text{Im} \, (X_\alpha(\zeta)^{-\frac{1}{2}}) = \{0\},
     \]
     \[
     \text{ran} \, \text{Im} \, (X_\alpha(\xi)^{-\frac{1}{2}}) + \text{ran} \, \text{Im} \, (X_\alpha(\zeta)^{-\frac{1}{2}}) = \text{Const}.
     \]

**Proof.** Let \( Q \) be a bounded nonnegative selfadjoint operator, \( \ker Q = \{0\} \) and \( \text{ran} \, Q \neq \mathcal{H} \) and let \( \{\mathcal{M}(\xi)\}_{\xi \in \mathbb{C}} \) be a family of subspaces (5.5). Set
\[
\gamma_\alpha = \frac{2\alpha}{\pi}, \quad a_\alpha = \tan \frac{\alpha}{2},
\]
\[
Z_0(\xi) := iQ^{-1} \left( QP_{\mathcal{M}(\xi)} \right)^{\frac{1}{2}},
\]
\[
Z^{(0)}_{\gamma_\alpha}(\xi) := \left( (I + Z_0(\xi))^{\gamma_\alpha} - (I - Z_0(\xi))^{\gamma_\alpha} \right) \left( (I + Z_0(\xi))^{\gamma_\alpha} + (I - Z_0(\xi))^{\gamma_\alpha} \right)^{-1},
\]
\[
Z_{a_\alpha}(\xi) := a_\alpha Z_0(\xi) = ia_\alpha Q^{-1} \left( QP_{\mathcal{M}(\xi)} \right)^{\frac{1}{2}}, \ \xi \in \mathbb{C}.
\]

1. Define
\[
S_\alpha(\xi) := \left( Q(I + Z^{(0)}_{\gamma_\alpha}(\xi))(I - Z^{(0)}_{\gamma_\alpha}(\xi)^*)Q \right)^{-1}
\]
\[
= \left( Q(I + Z^{(0)}_{\gamma_\alpha}(\xi)) \right)^{-2} = \left( (I - Z^{(0)}_{\gamma_\alpha}(\xi)^*)Q \right)^{-2}, \ \xi \in \mathbb{C}
\]
and apply Theorem 5.6.
(2) We have \( QZ_{a_0}(\xi) = -Z_{a_0}(\xi)^*Q \), \( \|Z_{a_0}(\xi)\| = a_0 < 1 \) and therefore \( Z_{a_0}(\xi) \in C_B(\alpha) \), for each \( \xi \in \mathbb{C} \). Set
\[
\mathcal{X}_\alpha(\xi) := \left( Q + ia_0 \left( QP_{\mathcal{M}(\xi)Q} \right)^{\frac{1}{2}} \right)^{-2} = (Q(I + Z_{a_0}(\xi))(I - Z_{a_0}(\xi)^*)Q)^{-1}.
\]
By Proposition 4.11 the operator \( \mathcal{X}_\alpha(\xi) \) is \( m - \alpha \)-sectorial. Theorem 4.8 and (4.2) yield the equalities \( \text{dom} (\mathcal{X}_\alpha(\xi))^{\frac{1}{2}} = \text{dom} (\mathcal{X}_\alpha(\xi))^*^{\frac{1}{2}} = \text{ran} Q \) for all \( \xi \in \mathbb{C} \). Since \( \text{ran} Z_{a_0}(\xi) = \mathcal{M}(\xi) \), from Proposition 4.11 we get that \( \text{dom} \mathcal{X}_\alpha(\xi) \cap \text{dom} \mathcal{X}_\alpha(\xi)^* = \{0\} \) for all \( \xi \in \mathbb{C} \).

By definition, we have that hold the following equalities:
\[
\text{Re}(\mathcal{S}_\alpha(\xi)^{-\frac{1}{2}}) = \text{Re}(\mathcal{X}_\alpha(\xi)^{-\frac{1}{2}}) = Q \quad \forall \xi.
\]
Therefore, from (2.4) we get that domains of the closed sesquilinear forms associated with \( \mathcal{S}_\alpha(\xi)^\frac{1}{2} \) and \( \mathcal{X}_\alpha(\xi)^\frac{1}{2} \) do not depend on \( \xi \).

From the relations
\[
\text{ran} Z_{a_0}(\xi) = \mathcal{M}(\xi), \quad \text{ran} Z_{a_0}^{(0)}(\xi) \subseteq \mathcal{M}(\xi),
\]
\[
\text{ran} \text{Im} (\mathcal{X}_\alpha(\xi)^{-\frac{1}{2}}) = Q\text{ran} \mathcal{M}(\xi), \quad \mathcal{M}(\xi)^\perp \cap \text{ran} Q = \{0\} \quad \forall \xi
\]
(see Remark 4.11) it follows that the linear manifold \( \text{ran} \text{Im} (\mathcal{X}_\alpha(\xi)^{-\frac{1}{2}}) \) is dense in \( \mathcal{H} \) for all \( \xi \) and hold the relations
\[
\text{ran} \text{Im} (\mathcal{S}_\alpha(\xi)^{-\frac{1}{2}}) \cap \text{ran} \text{Im} (\mathcal{S}_\alpha(\xi)^{-\frac{1}{2}}) = \{0\},
\]
\[
\text{ran} \text{Im} (\mathcal{X}_\alpha(\xi)^{-\frac{1}{2}}) \cap \text{ran} \text{Im} (\mathcal{X}_\alpha(\xi)^{-\frac{1}{2}}) = \{0\},
\]
\[
\text{ran} \text{Im} (\mathcal{X}_\alpha(\xi)^{-\frac{1}{2}}) \cap \text{ran} \text{Im} (\mathcal{X}_\alpha(\xi)^{-\frac{1}{2}}) = \text{ran} Q, \quad \xi \neq \xi.
\]

\[ \square \]

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