Highly Symmetric Neural Networks of Hopfield Type (exact results)

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Abstract

A set of fixed points of the Hopfield type neural network is under investigation. Its connection matrix is constructed with regard to the Hebb rule from a highly symmetric set of the memorized patterns. Depending on the external parameter the analytic description of the fixed points set has been obtained.

A set of fixed points of the Hopfield type neural network is under investigation. Its connection matrix is constructed with regard to the Hebb rule from a $(p \times n)$-matrix $S$ of memorized patterns:

$$
S = \begin{pmatrix}
1 - x & 1 & \ldots & 1 & 1 & \ldots & 1 \\
1 & 1 - x & \ldots & 1 & 1 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & \ldots & 1 - x & 1 & \ldots & 1 
\end{pmatrix}.
$$

Here $n$ is the number of neurons, $p$ is the number of memorized patterns $\vec{s}^{(l)}$, which are the rows of the matrix $S$, and $x$ is an arbitrary real number.

Depending on $x$ the memorized patterns $\vec{s}^{(l)}$ are interpreted as $p$ distorted vectors of the standard

$$
\vec{\varepsilon}(n) = (\underbrace{1,1,\ldots,1}_{n}).
$$

The problem is as follows: the network has to be learned by $p$-times showing of the standard (1), but a distortion has slipped in the learning process. How does the fixed points set depends on the value of this distortion $x$?

Depending on the distortion parameter $x$ the analytic description of the fixed points set has been obtained. It turns out to be very important that the memorized patterns $\vec{s}^{(l)}$ form a highly symmetric group of vectors: all of them correlate one with another in the same way:

$$
(\vec{s}^{(l)}, \vec{s}^{(l')}) = r(x),
$$

where $r(x)$ is independent of $l, l' = 1, 2, \ldots, p$. Namely this was the reason to use the words "highly symmetric" in the title.

It is known [1], that the fixed points of a network of our kind have to be of the form:

$$
\vec{\sigma}^* = (\sigma_1, \sigma_2, \ldots, \sigma_p, 1, \ldots, 1), \quad \sigma_i = \{\pm 1\}, \quad i = 1, 2, \ldots, p.
$$
Let’s join into one class $\Sigma^{(k)}$ all the configuration vectors $\vec{\sigma}^*$ given by Eq.(3), which have $k$ coordinates equal to ”–1” among the first $p$ coordinates. The class $\Sigma^{(k)}$ consists of $C_p^k$ configuration vectors of the form (3), and there are $p + 1$ different classes ($k = 0, 1, \ldots, p$). Our main result can be formulated as a Theorem.

**Theorem.** As $x$ varies from $-\infty$ to $\infty$ the fixed points set is exhausted in consecutive order by the classes of the vectors

$$\Sigma^{(0)}, \Sigma^{(1)}, \ldots, \Sigma^{(K)},$$

and the transformation of the fixed points set from the class $\Sigma^{(k-1)}$ into the class $\Sigma^{(k)}$ occurs when $x = x_k$:

$$x_k = p \frac{n - (2k - 1)}{n + p - 2(2k - 1)}, \quad k = 1, 2, \ldots, K.$$  

If $\frac{p-1}{n-1} < \frac{1}{3}$, according this scheme all the $p$ transformations of the fixed points set are realized one after another and $K = p$. If $\frac{p-1}{n-1} > \frac{1}{3}$, the transformation related to

$$K = \left\lceil \frac{n + p + 2}{4} \right\rceil$$

is the last. The network has no other fixed points.

The Theorem makes it possible to solve a number of practical problems. We would like to add that the Theorem can be generalized onto the case of arbitrary vector

$$\vec{u} = (u_1, u_2, \ldots, u_p, 1, \ldots, 1), \quad \sum_{i=1}^{p} u_i^2 = p$$

being a standard instead the standard (1). Here memorized patterns $\vec{s}^{(l)}$ are obtained by the distortion of the first $p$ coordinates of the vector $\vec{u}$ with regard to the fulfillment of Eqs.(2).

The obtained results can be interpreted in terms of neural networks, Ising model and factor analysis.

[1] L.B. Litinsky. Direct calculation of the stable points of a neural network. Theor. and Math. Phys. **101**, 1492 (1994)