ABSTRACT

We describe results from large numbers of $N$-body simulations containing from 250 to 1000 stars each. The distribution of stellar masses is a power law, and the systems are isolated. While the collapse of the core exhibits the expected segregation of different masses, we find that the post-collapse evolution is, at a first approximation, homologous. This is surprising because there is no reason for supposing that mass segregation should not continue to have a substantial effect on the evolution of the cluster. In fact, the spatial distribution of the mean stellar mass is nearly static throughout the post-collapse regime, except for the overall expansion of the systems, and this helps to explain why the post-collapse evolution is nearly self-similar. Self-similarity is also exhibited by the distribution of anisotropy and the profile of departures from equipartition, which show little change during the post-collapse phase. The departures from energy equipartition and isotropy are small in the core and increase with radius. During post-collapse evolution, massive stars (mainly) are removed from the system by binary activity. This effect dominates the preferential escape of low-mass stars owing to standard two-body relaxation processes.

Key words: celestial mechanics, stellar dynamics – globular clusters: general.

1 INTRODUCTION

This is the third in a series of papers in which we investigate the evolution of stellar systems using $N$-body simulations. This has often been done before, but one of the main drawbacks of most such studies is the poor statistical quality of much of the data. Even in a system with 1000 stars, the sampling noise in the results becomes overwhelming whenever measurements are taken from a small subset of the stars. Therefore it is difficult to carry out detailed quantitative studies of many interesting phenomena, for example the evolution of the core, the spatial distribution of anisotropy, the escape rate, and so on. This problem becomes much more acute whenever stars of different masses are present, if the aim is to investigate mass segregation, departures from equipartition, the differential escape rate, etc. This paper is concerned with precisely this kind of issue.

The main technical improvement in the present research is an extremely simple one. Instead of presenting results from one simulation, or only a small number, we combine statistically the results from several dozen simulations. The intention is to improve the signal-to-noise ratio of our results to the point at which reliable quantitative data can be presented, and compared with the results of simplified models (for example gaseous models, or Fokker–Planck models). An alternative method of improving the statistical quality would be to simulate a single, larger system. An advantage of that approach is that it gives results that are more realistic (if the intention is to understand the dynamical evolution of globular star clusters). However, a much greater increase in computational effort is required for a given improvement in the signal-to-noise ratio, compared with our strategy of simulating large numbers of more modest simulations (see Giersz & Heggie 1994a). This strategy is especially effective now that parallel computers have become readily available.

The present paper differs from the first two papers in this series (Giersz & Heggie 1994a,b, hereafter referred to as Papers I and II, respectively) by dealing with the case of unequal stellar masses. This greatly extends the range of parameters to be explored, but we have restricted ourselves to parameters comparable to those in the well-known Fokker–Planck survey carried out by Chernoff & Weinberg (1990). (One of the motivations for our research was the desire to test their results using $N$-body methods, which make fewer assumptions.) On the other hand, we still consider systems that are isolated (without tidal effects) and in
which the stellar masses are constant, i.e. we do not model systems in which the stars lose mass through internal evolution. These two processes are included in later sets of models, which we will discuss in two further papers.

The plan of the paper is as follows. We begin with a discussion of the initial parameters that we adopted, such as the slope and range of the stellar mass function. Then subsequent sections discuss, in turn, the spatial evolution of the systems, and in particular the segregation of masses. Next we discuss the distribution of velocities, and in particular the extent to which equipartition is obeyed in the systems at various radii. There then follow sections on the evolution of the core and of the binaries which mostly form there. Towards the end of the paper we discuss global parameters, for example the evolution of the total mass and energy, with a detailed description of the differential rate of escape (i.e. its dependence on stellar mass). The last section discusses a few points of interest and summarizes our conclusions.

2 ORGANIZATION OF THE CALCULATIONS

As in the earlier papers of this series, all models reported here started as Plummer models. The initial mass spectrum was chosen to be a power law of index 2.5, i.e. \( dN(m) \propto m^{-2.5} \, dm \), in a range such that \( m_{max}/m_{min} = 37.5 \). This choice was made in order to facilitate comparison (in later papers in this series) with results of Chernoff & Weinberg (1990), who used a very similar spectrum.

Although the spectrum of masses is continuous, for purposes of analysis the masses were grouped into 10 classes, along the lines of those used by Chernoff & Weinberg (1990) (see Table 1 in this paper). (In their Fokker–Planck calculations it is impractical to include a continuous spectrum of masses; therefore a finite number of masses is used. After considerable experimentation, Chernoff & Weinberg found that use of 10 masses yielded results quite similar to those using larger numbers of masses.)

The calculations were carried out on the Edinburgh Computing Surface (ECS), which was a transputer array containing about 400 processors (see Paper I). The code used was NBODY5 (Aarseth 1985). Three sets of calculations were carried out, in which the numbers of stars were \( N = 250, 500 \) and 1000, respectively. In each set, the respective number of models was 56, 56 and 40. Virtually all runs were completed to time \( t = 400 \).

The organization of the output was handled somewhat differently from that in earlier papers in this series. The reason for this was directly related to the use of a spectrum of masses, divided into the 10 classes described above. Because of the steepness of the mass spectrum, the average number of stars in the class with the heaviest stellar masses was less than one for \( N = 250 \). Since we wished to measure Lagrangian radii for each mass group in a uniform manner, and consistently with earlier papers where data for the inner 1 per cent Lagrangian radii were discussed, it was necessary to combine data from all models at the same \( N \)-body time, and also to combine data from three mass bins corresponding to the most massive stars. (Even so, with only 56 models in each series, there are still only of order 150 stars in the heaviest mass group.) Since the calculations produce data at different rates, however, this approach meant that it was necessary to store data on each star from each model at each output time. In order to conserve space, this was done by coding data in such a way as to allow it to be reconstructed to adequate accuracy in subsequent analysis.

| range of masses  | mean mass (units of \( m_{min} \)) | initial data mass fraction and number |
|------------------|----------------------------------|-------------------------------------|
| \( N = 250 \)    |                                  |                                     |
| 1.000            | 1.439                            | 0.1895 101.1                       |
| 1.439            | 2.067                            | 0.1722 63.63                       |
| 2.067            | 2.970                            | 0.1428 36.71                       |
| 2.970            | 4.267                            | 0.1161 20.84                       |
| 4.267            | 6.132                            | 0.09463 11.85                      |
| 6.132            | 8.810                            | 0.08664 7.446                      |
| 8.810            | 12.86                            | 0.06648 4.054                      |
| 12.86            | 18.19                            | 0.06046 2.554                      |
| 18.19            | 26.13                            | 0.04526 1.321                      |
| 26.13            | 37.50                            | 0.02588 0.8557                    |
| Note. The mass fractions and numbers were determined from the sets of initial conditions. For an analytical mass function, the mass fraction would be independent of \( N \), and the number would be proportional to \( N \).
The data stored (to limited accuracy) for each star were as follows:

(i) mass;
(ii) distance from the density centre (defined as in Casser
tano & Hut 1985, with a modification for the presence of a mass spectrum, described below); and
(iii) radial and tangential velocities (the radial direction being that from the density centre).

In addition, the following details of all regularized binaries, and all unregularized binaries with binding energy \( \geq 0.1 kT \) (where the current mean kinetic energy of all single stars is \( 3kT/2 \)), were output:

(iv) masses of components;
(v) separation of components, and distance of binary centre of mass from the density centre;
(vi) internal energy of the binary and its eccentricity;
(vii) radius of neighbour sphere and number of neighbours (defined as in NBODY4);
(viii) names of binary components.

The typical size of an output file from 40 runs at \( N = 1000 \) for 400 output times was of the order of 125 Mb. The analysis of such a file proved to be manageable, if awkward, even on a diskless workstation requiring access to files over a fairly busy ethernet.

In addition, each model outputed some pre-analysed data, as described for equal-mass models in Paper I. For example, the values of 10 Lagrangian radii (defined below) were output. As in our earlier work, the different sets of calculations did not all output every item in the list in the appendix to Paper I: the data selected for output were changed slightly as our interests and experience developed.

The previous discussion refers to ‘Lagrangian radii’, a term that requires careful definition in the context of these multimass models. First, these radii are referred to the ‘density centre’, which is defined as follows: for each star the density is defined to be proportional to 

\[ \rho \propto \frac{m}{r^3} \]

where \( r \) is the distance to the fifth-nearest neighbour (cf. Casertano & Hut 1985) and \( M_r \) is the total mass of the five nearest neighbours. Then the density centre is defined as in Aarseth & Heggie (1992). Incidentally, the density centres of different models were found gradually to move away from the origin (whose position and velocity coincide initially with those of the barycentre) in different directions. The rms value of each coordinate of the density centre (averaged over all systems in a given series) increases nearly linearly with time in the post-collapse regime, roughly as 0.054 for \( 50 \leq t \leq 200 \) for \( N = 250 \).

With the density centre thus defined, a Lagrangian radius is defined to be the radius of a sphere centred on the density centre and containing a fixed fraction of the total bound mass of the system. (When this does not correspond to a whole number of stars, interpolation of ordered radii is used as in Paper I.) Because of mass segregation and the presence of a spectrum of masses, it is possible that the innermost radius, which corresponds to a mass fraction of 1 per cent, may contain less than one star in a single model. For this reason, the Lagrangian radii discussed in this paper were actually determined differently. For each output time, the radii of all stars (measured from the density centre of the model to which it belongs) were aggregated from all models in a given series, and then Lagrangian radii were determined as above from this much larger set of stellar data. It was hoped that this would improve the determination of the inner Lagrangian radii, and avoid the difficulties associated with mass segregation, as described above.

### 3 SPATIAL EVOLUTION

#### 3.1 The entire mass distribution

The essential facts about the evolution of the mass distribution are apparent from Fig. 1. The collapse of the core is much more rapid than in the equal-mass cases considered in Papers I and II (cf. Table 2). Indeed, the ratio of \( t_{\text{coll}} \) (between the values for unequal and equal masses) is a decreasing function of \( N \) even for the largest value of \( N \) that we studied. At first sight, this appears to contradict the result expected from the standard theory of relaxation, which is that this ratio should be independent of \( N \). It was, however, argued long ago by Heggie (1975) that the coefficient \( \gamma \), which appears in expressions for the relaxation time in the Coulomb logarithm \( \ln \gamma N \), is generally smaller in systems with unequal masses than in those with equal masses, by as much as a factor of 5. In fact, his theory yields the result \( \gamma \geq 0.044 \) for the mass function used in our models.

Now we turn to the numerical data (Table 2). If it is assumed that the collapse time (as a function of \( N \) for unequal masses) is proportional to the initial half-mass relaxation time, then we find that the results of Table 2 are roughly consistent with each other if \( \gamma \sim 0.015 \), which is a factor of 7 smaller than the value found in Paper I for systems with equal masses.

An independent method of determining \( \gamma \) is to compare \( t_{\text{coll}} \) with the results of a Fokker–Planck computation with the same initial conditions and mass spectrum. This has been done using an isotropic code (Inagaki & Wiyanto 1984), from which the value of \( t_{\text{coll}} \) was obtained by finding the time at which the central potential reached the minimum value attained in the \( N \)-body results (Fig. 12). Scaling to the values of \( t_{\text{coll}} \) in Table 2 yielded values of \( \gamma \) in the range \( 0.016 \leq \gamma \leq 0.026 \).

Whatever the correct value of \( \gamma \), we have seen that there are three independent lines of evidence for a substantially smaller value than in the case of equal masses. This implies that considerable care is required if the results of models with small \( N \) are to be scaled to much larger systems. For example, when \( N = 1000 \) the relaxation times estimated by Heggie’s value and the value \( \gamma = 0.02 \) differ by about 25 per cent. Even the largest of our models is, therefore, a somewhat uncertain guide to the absolute time dependence of relaxation processes in larger \( N \)-body systems.

The post-collapse evolution is, to a first approximation, self-similar, as one can see from the fact that the curves in Fig. 1 are very nearly parallel, in the sense that all of them can be brought nearly into coincidence by a suitable vertical displacement. This is evidently not true in the early post-collapse phase for the outermost radii (mass fractions of 75 per cent or more). These aspects of the evolution are illustrated qualitatively in Fig. 2, which shows the surface-density profile scaled (radially) to the half-mass radius. In
post-collapse, the profiles could be brought into near-coincidence by a vertical shift. Nevertheless, even for the later profiles shown in Fig. 2, the outermost radius at which the density is plotted (which corresponds to a mass fraction of 90 per cent) moves outwards slowly as time advances, indicating a modest but sustained departure from homologous evolution.

Although there are departures from self-similarity, the half-mass radius still varies in approximate accord with naive theoretical expectations, i.e. $r_h \propto t^{2/3}$. There are, however, small departures from this formula. To clarify this point we carried out an exercise similar to that described in Paper II, i.e. we fitted the following functions to the $N$-body data:

$$
N = a(t - t_0)^{-1},
$$

$$
r_h = b(t - t_0)^{2/3 + v/3},
$$

$$
(Nr_h^2/\bar{m})^{1/2} = c(t - t_0),
$$

where $N$ is the number of bound stars, $r_h$ is the half-mass radius, $t$ is time, $\bar{m}$ is the mean individual mass of stars inside $r_h$, and $a$, $b$, $c$, $v$ and $t_0$ are fitting parameters. The values obtained are given in Table 3.

According to standard relaxation theory (Spitzer 1987), the coefficient $c$ should be proportional to the Coulomb logarithm, usually taken as $\ln(\gamma N)$ for some value of the constant $\gamma$. As can be seen from results presented in Table 3, however, it is a good approximation to assume that $c$ is constant, i.e. independent of $N$, and this is inconsistent with any plausible value of $\gamma$. A similar conclusion follows from a consideration of the values of $b$. We have no definitive explanation for these results, although it should not be concluded that the evolution of systems of stars with unequal masses behaves in a manner inconsistent with standard relaxation theory. At some level of detail, the mass distribution evolves differently in systems with different $N$, and our results have been obtained by fitting to the evolution over an $N$-dependent number of relaxation times; for example, the factor by which $r_h$ expanded varied greatly over the different models. These $N$-dependent effects may have conspired to mask the expected $N$ dependence of the Coulomb logarithm.

It is to be noted that Fig. 1 illustrates projected Lagrangian radii, i.e. those containing a given fraction of the projected mass. The corresponding picture for the conventional (3D) Lagrangian radii is qualitatively similar, at least in respect of the aspects to which attention has been drawn above. The 3D half-mass radius is larger in the post-collapse phase by about 30 per cent. In the same way, the space-density profiles exhibit very similar features to the surface-density profiles shown in Fig. 2. Also, the results for $N = 1000$ and 250 do not differ qualitatively from those shown here for $N = 500$. The main quantitative difference, which is the time-scale of the evolution, has already been described above.

3.2 Mass segregation

At first sight it is surprising that the structure of the cluster in post-collapse evolution appears to be so nearly self-similar, and that the naive theoretical expectation, equation (1), is so well satisfied, because there is no reason for supposing

![Figure 1. Evolution of projected Lagrangian radii for $N = 500$. Also shown is the (3D) core radius, defined in Section 5.](https://academic.oup.com/mnras/article-abstract/279/3/1037/967535/1037)
that mass segregation should have a negligible effect after core collapse. Therefore, even though it remains true that the time-scale for post-collapse expansion should be governed by a time of relaxation, it is not as clear as it is in the case of equal masses that this time-scale may be characterized by the half-mass relaxation time.

As conventionally defined (Spitzer 1987), this depends on the mean mass as well as the total mass and the half-mass radius, and one might have expected the different time-scale for mass segregation to complicate the evolution of the half-mass radius. In this section we shall discuss mass segregation, and we shall find an empirical explanation for the fact that the evolution of the half-mass radius is so simple.

What is still lacking, however, is a theoretical understanding of why mass segregation in post-collapse evolution takes the simple form observed.

The basic result is illustrated in Fig. 3. Extremely rapid segregation of masses takes place throughout core collapse. The remarkable conclusion to be drawn from this figure, however, is that the profile of mean masses evolves remarkably little throughout the remainder of the calculations, which amounts to about 20 collapse times. It is presumably this fact, i.e. the fact that the distribution of stellar masses is nearly static (in Lagrangian coordinates), that helps to explain why mass segregation does not, as one might expect, complicate the post-collapse evolution of the cluster.

While Fig. 3 illustrates the projected distribution of the mean mass, the spatial distribution shows very similar features. There is, however, a slight but noticeable drop in the mean mass inside the 20 per cent radius in the early post-collapse regime, and for all Lagrangian radii during the advanced post-collapse phase. This drop is more pronounced than the corresponding drop in the projected data (Fig. 3), but still no greater than about 0.1 dex. It is not clear that this is correctly attributed to mass segregation, since stars of high mass are removed from the core and the whole system by binary formation, evolution and escape (Section 6).

![Figure 2. Evolution of the projected (surface) mass-density profile for N=500. The radial coordinate is scaled by the instantaneous projected half-mass radius. The densities plotted are average values between each Lagrangian radius, and have been plotted at the mean radius in each Lagrangian shell.](https://academic.oup.com/mnras/article-abstract/279/3/1037/967535/1041)
Since the distribution of masses is almost constant in the post-collapse regime, except for the overall expansion of the radii and the corresponding decrease in the density, we illustrate the density distribution of the different masses, binned as stated in Section 2, at one point in the post-collapse phase (Fig. 4). Initially, of course, all density profiles would be identical, except for an overall vertical displacement from one species to another. As expected at this point in the post-collapse evolution, however, the effects of mass segregation are clearly visible. The most massive stars dominate the central parts of the systems, and the less massive stars dominate the halo. As a rule, the larger the stellar mass the greater the concentration to the centre. There is one exception to this: stars from the most massive bin are not, as one might have expected, confined to the central and middle parts of the system – they are present in the outer halo as well, at distances even greater than those of the least massive stars. (More precisely, their 90 per cent Lagrangian radius exceeds that of the least massive stars.) This behaviour is probably connected with the ejection of massive stars and binaries from the core to the outer parts of the system during interactions between binaries and field stars. This is evidence that binary activity can disturb the otherwise clear picture of mass segregation in these systems.

4 EVOLUTION OF THE VELOCITY DISTRIBUTION

4.1 The overall velocity dispersion

Many aspects of the overall evolution of the velocity distribution are shown in Fig. 5. After the relatively brief period of core collapse, the decline in the mass-weighted rms projected speed is nearly homologous. The profiles of the projected velocity dispersion behave in a manner very similar to that of the density profiles (cf. Fig. 2): if the radial coordinate is scaled by the projected half-mass radius, the profiles at different times differ from each other only by a vertical shift in a logarithmic plot, to a good approximation. This is also true of the 3D velocity dispersion profiles, although they are very slightly steeper: between the centre and the 90 per cent Lagrangian radius the difference in rms speed is about 0.36 dex for the 3D profile, and about 0.33 dex for the projected profile.

4.2 Equipartition

The most interesting kinematic distinction between the models discussed here and those analysed in Papers I and II

Table 3. Fitted parameters.

| N  | t₀  | a     | ν     | b     | c     |
|----|-----|-------|-------|-------|-------|
| 250| -5.688 | 316.94 | -0.067 | 0.159 | 16.95 |
| 500| -8.256 | 622.95 | -0.063 | 0.095 | 17.24 |
| 1000| +4.661 | 1236.8 | -0.061 | 0.064 | 17.25 |

Note. The symbols are explained in equation (1) and the accompanying discussion.

Figure 3. Evolution of the mean individual stellar mass within each projected Lagrangian shell for N = 500. Thus the curve labelled ‘2%’ gives the mean masses of stars whose projected distance from the density centre lies between projected radii which enclose 1 and 2 per cent of the projected mass, respectively. A binary was treated as a single star with a mass equal to the total mass of the components.
concerns the question of equipartition, or, more generally, how the velocity dispersion differs from one population to another. Just as we have attempted to summarize the distribution of masses by the mean mass profile (Fig. 3) we may begin by attempting to define a single measure of the extent to which different masses are in equipartition, as follows.

For a given Lagrangian shell let \( T_k \) be the mean kinetic energy of the stars in that shell in mass class \( k \), with \( 1 \leq k \leq 10 \) (Section 2), and \( T \) be the mean kinetic energy of all stars in the shell. If there are no stars in a given mass class in that shell then we define \( T_k = 0 \). Next we define the 'equipartition parameter' as

\[
B = \frac{\sum_k T_k}{N_s T}, \tag{2}
\]

where \( N_s \) is the number of mass classes for which \( T_k > 0 \). The obvious feature of \( B \) is that in the case of equipartition (of kinetic energies) we should have \( T_k = T \) for all \( k \), and so \( B = 1 \). Likewise, in the case of equipartition of velocities we would have \( B = \frac{\sum m_k \langle 10 \langle m \rangle \rangle}{\sum m_k \langle 10 \langle m \rangle \rangle} \neq 1 \), since the average on the left-hand side is not weighted by the number of stars in each mass class. For our initial mass function, for example, we would have \( B = 3.8 \) (i.e. \( \log B = 0.58 \)) in the case of equipartition of velocities.

Clearly, one may compute \( B \) either for 3D Lagrangian shells or for projected annuli, in which case \( T_k \) could be defined as the mean kinetic energy of motion in the line of sight. This has been adopted in Fig. 6, which shows profiles at various times. The curves are noisier than those previously shown, because the small number of stars of high mass have relatively high weight in the numerator. Nevertheless, one can easily observe an extremely rapid evolution (in the first five time units for \( N = 500 \)) from the initial flat profile to one which remains almost constant throughout the remainder of the core collapse and in the post-collapse phase, provided (as is the case here) that the radial coordinate is scaled by the projected half-mass radius. In the core the value of \( B \) is close to that expected for equipartition of energy, whereas in the outermost parts of the system the departure from energy equipartition, to the extent that this is measured by \( B \), is even greater than it was initially.

More detail is shown in Fig. 7, which gives the profiles of the mean-square projected velocity as a function of projected radius, for all mass groups. (We give this for only one time, since Fig. 6 suggests that there is little evolution after an initial short period of adjustment.) Unfortunately, because of the small number of stars in each mass bin (particularly for the massive bins) the data presented in Fig. 7 are very noisy. However, the main features of this picture are clearly visible. In the core, dominated by massive stars, the mean-square projected velocities follow closely the rule of energy equipartition. In the outer halo, dominated by less massive stars, the mean-square projected velocities behave differently: they are nearly the same for most of the mass bins. This reflects the initial conditions, for which all mass bins were in velocity equipartition. In the outer halo the relaxation time is very long (compared to that in the core, or

Figure 4. Profile of projected mass density of each mass class at time \( t = 300 \) for \( N = 500 \). The Lagrangian radii have been determined separately for each mass class, and since the plotted values of \( \Sigma \) represent average estimates for annuli separated by these projected Lagrangian radii, the innermost and outermost plotted points vary from one species to another.
even at the half-mass radius), and so the evolution is very slow and the initial conditions are well preserved. Therefore the equipartition parameter $B$ increases towards the outer parts of the system (see Fig. 6).

For the most massive bin, the velocity is nearly constant throughout the system. The explanation for this may be connected again with binary activity, which removes stars and binaries from the core and ejects them into the halo with velocities that are high compared with those of most other stars in the halo. Because massive stars are preferentially removed from the core, the behaviour of the most massive bin is markedly different from that of the others.

4.3 Anisotropy

In discussing the anisotropy we shall concentrate on projected values (Fig. 8). Thus each annulus is bounded by circles containing fixed fractions of the total mass (relative to the density centre of each model), and within each annulus we denote by $\langle v_r^2 \rangle$ and $\langle v_t^2 \rangle$, respectively, the mean-square radial and transverse components of the projected velocity. Then the anisotropy, defined by $\langle v_r^2 \rangle / \langle v_t^2 \rangle$, is plotted at a radius given by the mean of the radii of the bounding circles scaled by the projected half-mass radius.

It can be seen from Fig. 8 that the evolution of the anisotropy profile occurs on a time-scale longer than that of mass segregation (Fig. 3) and the tendency to equipartition (Fig. 6). (This difference of time-scales is not really surprising, since the main evolution of the anisotropy occurs at large radii, unlike the other two phenomena.) When $N = 500$ the profile continues to evolve in the outer parts until about $t \sim 100$, and only thereafter does it remain relatively constant (when the radius is scaled by the expanding projected half-mass radius).

As already stated, the values plotted in Fig. 8 are projected values. In the quasi-steady phase after $t \sim 100$, the 3D values of log($\langle v_r^2 \rangle / \langle v_t^2 \rangle$) are about $-0.6$ at the outermost radii shown in Fig. 8 (i.e. at a similar value of the 3D radius scaled by the 3D half-mass radius). Although this might be taken to imply that the 3D anisotropy is smaller, it must be recalled that the value of the logarithm in an isotropic distribution is log 2 in three dimensions, and 0 for projected data.

Another point to be made about these data concerns the $N$ dependence of the anisotropy. As in the case of equal masses (Giersz & Spurzem 1994), the anisotropy in the outer parts saturates at a value which is greater (i.e. more radially anisotropic) for smaller $N$. For $N = 250, 500$ and $1000$ the values of the logarithm (3D anisotropy) are about $-0.85, -0.63$ and $-0.48$, respectively. This may have an explanation similar to that of the faster expansion of the outer Lagrangian radii for smaller $N$ in systems with equal masses (see Paper II), i.e. the fact that the relatively shallower potential well of systems with small $N$ makes it easier for stars which are ejected from the core to populate the far halo.
Figure 6. Projected equipartition parameter $B$ (equation 2) at various times in a set of models with $N = 500$. The radial coordinate has been scaled by the projected half-mass radius.

Figure 7. Projected velocity dispersion profiles for each mass group for $N = 500$ at time $t = 300$.

The way in which the anisotropy is distributed among the different mass classes is illustrated in Fig. 9, which exhibits the projected anisotropy profile for $N = 500$ at $t = 300$, i.e. well into the phase in which the overall profile appears to have settled down into equilibrium (as a function of radius scaled by the half-mass radius). Generally, the core is isotropic for all mass bins (with very substantial fluctuations for massive bins). In the outer halo the anisotropy is bigger for...
the bins corresponding to the most massive stars; as with equipartition and the velocity dispersion, the most massive bin is exceptional. This is again consistent with the suggestion that the massive stars are ejected from the core into the halo by binary activity.

5 EVOLUTION OF THE CORE

In systems of stars of equal mass there are already two commonly used measures of the core radius. One, stemming from the work of King (1966), defines the core radius $r_c$ in
Figure 10. Mass-weighted rms central speed as a function of time. The result for \( N = 250 \) is plotted to scale, while those for \( N = 500 \) and 1000 have been displaced vertically by 0.3 and 0.6 units, respectively. The data have been smoothed over two time units.

terms of the central 1D velocity dispersion \( \sigma_c \) and the central mass density \( \rho_c \) by the formula

\[
r_c^2 = \frac{9\sigma_c^2}{4\pi G \rho_c}. \tag{3}
\]

The other, derived from the work of Casertano & Hut (1985), defines it as

\[
r_c = \frac{\sum \rho_i r_i^2}{\sum \rho_i}, \tag{4}
\]

where \( \rho_i \) is a measure of the density at the location of the \( i \)th star, and \( r_i \) is its distance from the density centre (Section 2). The formula proposed by Casertano & Hut differs from equation (4) in that all the terms are unsquared. In our work we have estimated \( \sigma_c \) and \( \rho_c \) from the innermost 1 per cent by mass (see below), while \( \rho_c \) was estimated as in Section 2. It is known (see Aarseth & Heggie 1992) that the definition in equation (3) gives a larger value than equation (4) when applied to a Plummer model, which is the starting configuration of our calculations. In our unequal-mass models, however, the value from equation (3) quickly becomes the smaller, exhibiting a much deeper collapse: for \( N = 500 \), for instance, the core radius defined by equation (3) decreases by a factor of approximately 3, whereas the decrease for \( r_c \) defined by equation (4) is only by a factor of about 1.4. Throughout the post-collapse evolution, equation (4) gives a core radius which is larger than that defined by equation (3) by a fairly consistent factor nearly equal to 2. We prefer equation (3), entirely on the grounds that it has a dynamical significance, which has not been demonstrated for equation (4).

As for the value of \( r_c \) itself, the simplest way of expressing the result for post-collapse is to give the ratio to the half-mass radius (Table 2), which is almost constant in this phase of the evolution. These results are quite interesting, as they contradict the theoretical expectation for equal masses, which implies that, for large \( N \), the ratio should vary as \( N^{-2/3} \) (Goodman 1987). Whether this implies that the models have not yet reached the asymptotic behaviour of systems of sufficiently large \( N \), that systems with stars of unequal mass behave in a qualitatively different way, or that our adopted definition of core radius is faulty in some way for systems with unequal masses, cannot be decided on the basis of our data. Certainly, our conclusion for systems of equal mass (Paper II) was that the size of the core was roughly in line with theoretical expectations, even for the small values of \( N \) that were studied.

The definition of core radius having been decided on, we can readily determine the number of stars within this radius (Table 2), from which it appears that \( N_c \approx N/60 \) throughout almost all of the post-collapse evolution. The mass of stars within \( r_c \) is approximately 0.05, independent of \( N \), and so the mean stellar mass in the core is approximately \( 3/N \), i.e. about three times the initial global mean mass (see also Fig. 3, which gives a similar value for the projected central mean mass).

Our estimate for the central velocity dispersion, which we now discuss, is obtained from the mass-weighted mean-square speed of stars within the innermost 1 per cent by mass. Even though we include all stars from all models (at given \( N \)), because of mass segregation the stars in this zone are relatively massive and few in number. Therefore the
The estimate of the mean-square speed is subject to quite large fluctuations, especially in relation to the rather small changes in this quantity during the evolution. Fig. 10 shows the results for all the values of \( N \) that we have studied. One reason why these results are interesting is the initial decline in the early phase of core collapse, although because of the fluctuations the result is convincing only for \( N = 1000 \). A similar drop was noticed in the equal-mass calculations discussed in Paper I, but it is more natural here to interpret it as a consequence of mass segregation and the tendency towards equipartition. As already stated, and discussed further below, the core quickly becomes dominated by massive stars, whose rms speed drops below that of the stars of lower mass. Therefore, the core collapse, which involves mainly the dominant massive stars, leads to the expected increase in the rms speed.

The fact that the central density rapidly becomes dominated by the most massive stars is responsible for the evolution of the mean mass within the innermost Lagrangian radius, as shown in Fig. 3. Fig. 11 gives more detail, showing the contribution to the central projected density from several mass bins. (Note, however, that the density is influenced also by the sizes of the mass groups, which are uniform in \( \log m \).)

Another core parameter of importance (for example for the escape of binaries and the efficiency of heating by three-body binaries) is the depth of the potential well at the centre of the system (Fig. 12). The minimum is less sharp than in the case of equal masses (Paper II), perhaps because of an even greater relative spread in the times of core collapse in the models with unequal masses, and perhaps because the presence of a few massive stars in the core causes larger fluctuations than in the case of equal masses. Note that the minimum depth of the potential increases with \( N \). This result is interesting as it contradicts the theoretical expectation (e.g. Goodman 1987) and numerical results for equal masses (Paper II), which imply that the minimum of the potential decreases with increasing \( N \). We interpret this as a richness effect: the maximum individual stellar mass present in a given small \( N \)-body system may be much less than the maximum allowed mass, if the mass spectrum is steep. (There is evidence for this in Table 1, which shows that the fraction of mass in the heaviest mass group is an increasing function of \( N \).) One consequence of this is that a small \( N \)-body system is unlikely to be able to form a binary from stars close to the maximum allowed mass, and therefore binary heating is likely to be less efficient than in a system with sufficiently large \( N \).

All the results on core evolution displayed so far concern the time dependence, but it is also instructive to display the evolution of two fundamental core parameters against each other. For Fig. 13 we have chosen the central mass density and the central mass-weighted rms speed. This bears some resemblance to the corresponding result for equal masses (Paper II). The dip in the central velocity dispersion during core collapse has already been discussed, but the post-collapse behaviour is another example of the surprisingly simple behaviour of these unequal-mass models in this phase of the evolution: if it is assumed that \( \rho_c \) and \( v^2 \) are

![Figure 11](image-url)
Figure 12. Evolution of the central potential (i.e. at the location of the density centre) for $N = 250, 500$ and 1000.

Figure 13. Evolution of core parameters $\rho_c$ and $v_c$ for $N = 500$. The straight line corresponds to $\rho_c \propto v_c^6$. The $N$-body data were smoothed using the standard routine SMOOT from Press et al. (1986).
proportional to values at the half-mass radius, then it follows that \( \rho_s r_e^{-6} \propto M^{-2} \). Since the total mass \( M \) varies slowly (see Section 7), this predicts an approximate power-law relation which is quite closely verified in Fig. 13. It is surprising that the presence of a spectrum of masses does not invalidate such simple arguments.

6 BINARIES

6.1 Definitions

For systems of stars with equal masses the notions of ‘hard’ and ‘soft’ binaries are relatively uncomplicated. When there is a spectrum of masses, however, it becomes less clear what definition to adopt. The simplest approach would be to classify binaries according to the ratio \( \varepsilon / (kT) \), where \( \varepsilon \) is the (internal) binding energy of the binary, and \( 1.5kT \) is the mean kinetic energy of all stars in the neighbourhood of the binary. On the other hand, it is not clear that this is the most useful definition when the masses could be very unequal. In the case of equal masses, the notions of ‘soft’ and ‘hard’ are important usually because they correspond, roughly, to the distinction between binaries that tend to be destroyed by encounters and those that become harder (Gurevich & Levin 1950). While work by Hills (1990) and by Sigurdsson & Phinney (1993) gives much useful information on the effect of encounters between a binary and a single star when the masses are different, it is still difficult to decide whether a binary, with a given energy and component masses, will tend to harden or disrupt under the action of encounters with a field of single stars, when the spectrum of stellar masses may be very time- and position-dependent.

In view of this difficulty we have adopted a definition which is much simpler to implement, but is more difficult to interpret physically. In what follows we define a ‘hard’ binary to be a regularized binary, in the sense of the code \textsc{nbODYs} (Aarseth 1985). We shall also refer more briefly to ‘soft’ binaries, which are defined to be unregularized binaries with binding energies greater than \( 0.1kT \) (see Section 2).

Clearly, many binaries fall into neither the ‘hard’ nor the ‘soft’ class, because there will in general be a number of binaries whose internal binding energy is too low to satisfy the criteria for regularization. There is no evidence, however, for the presence of long-lived binaries that do not fall into the class of regularized pairs.

6.2 Statistics of binaries

As with equal masses (Paper II), the onset of the formation of hard binaries is abrupt (Fig. 14). Although this is to be expected, as it is associated with the end of core collapse, what is perhaps more surprising is the result in the post-collapse phase, where the number of hard binaries is nearly constant in time, and independent of \( N \). In this phase the mean value is about 1.3. For systems of equal masses (Paper II) the corresponding result is about 2.4 (but there are some indications for an \( N \) dependence of the form: \( N_s \propto N^{1.5} \)).

The spatial distribution of these binaries is heavily centrally concentrated: in the early post-collapse phase about 80 per cent of the hard binaries lie within the radius contain-

![Figure 14. Number of hard binaries as a function of time, for \( N = 250, 500 \) and 1000. Only binaries that are not regarded as escapers are included.](https://academic.oup.com/mnras/article-abstract/279/3/1037/967535)
ing 20 per cent of the mass of all stars. The central concentration of hard binaries decreases somewhat in later phases of post-collapse expansion, reaching a nearly constant value (as measured by the number of binaries within the 20 per cent Lagrangian radius) for times greater than about 100 \( N \)-body time units (which happens to coincide roughly with the time at which the anisotropy levels off). This drop is accompanied by an increase in the number of hard binaries in the outer parts of the cluster. This is particularly clearly visible in data for \( N = 250 \), which extend to a larger number of collapse times than our data for the larger systems.

The total internal energy of all bound hard pairs also shows the expected sharp change towards the close of core collapse (Fig. 15), and in the early post-collapse phase it follows closely a power-law dependence, which can be interpreted very roughly as follows. It is known from the behaviour of the half-mass radius (Section 3.1) that the 'external' energy of the bound stars (i.e. excluding the internal energy of bound binaries) varies roughly as

\[
E^b \approx -C(t-t_0)^{-\frac{5}{3}},
\]

where \( E^b \) is the initial energy of the system, i.e. \(-0.25\) in our units.

In fact equation (5) provides a reasonable fit to the data early in the post-collapse phase, until one of the above assumptions begins to break down seriously. Indeed, the rise in \( E^b \) after this early phase indicates the escape of hard binaries (see Section 7), and then the derivation of equation (5) breaks down. Before this happens, however, we see from Fig. 15 that \( E^b \) falls below the minimum value predicted by equation (5), and the reason for this is that escape by two-body encounters (Section 7) tends to decrease the external energy of the system, i.e. to make the system more bound, thus decreasing the effective value of \( E^b \).

The mean mass of the hard binaries is also instructive (Fig. 16). In the innermost parts of the cluster the hard binaries tend to consist of the most massive stars, whose mean mass declines slowly with time as the most energetic (and most massive) binaries escape or are ejected to the outer parts of the cluster (Section 7). This gives rise to the perhaps rather surprising behaviour of the hard binaries in the outer parts, whose mean mass tends, if anything, to increase. It is in the intermediate radii that the mean masses of binaries are smaller, and indeed decline relatively rapidly as the evolution proceeds.

The assertion that the massive binaries in the outer parts are ejected from the core gains further support from an examination of the eccentricities of the hard binaries. Outside the half-mass radius the mean eccentricity of hard binaries, which is about 0.8, significantly exceeds that of the hard binaries inside \( r_h \) (about 0.7, i.e. close to the thermal average of 2/3 (Jeans 1929)).
Figure 16. Mean mass of hard binaries in four spatial zones for \( N = 500 \). \( M_{\text{tot}} \) is the total bound mass of the system.

Figure 17. Evolution of total number and mass of bound stars in systems with \( N = 500 \) stars initially, expressed as a fraction of the initial value.
Figure 18. Fractional escape of stars in different mass classes, for N = 1000. (a) shows details for the collapse phase and the start of the post-collapse evolution, while (b) shows the long-term trend.
7 GLOBAL EVOLUTION

7.1 Escape

In the previous section, attention was drawn to the behaviour of massive hard binaries, which tend to be ejected into the outer parts of the cluster, if not out of the cluster altogether. Surprisingly, perhaps, this escape mechanism is sufficiently effective that the mean mass of the remaining bound members of the cluster tends to decrease, whereas one might have thought that the preferential escape of low-mass stars in two-body encounters (Hénon 1969) would lead to an increase in the mean mass. This is illustrated in Fig. 17, which shows that the mass of the cluster decreases by 15 per cent at a time when the number of stars has decreased by less than 11 per cent.

These mean data actually mask a more complicated picture, which is given in Fig. 18. At early times, i.e. during core collapse (Fig. 18a) there is indeed a preferential escape of the stars of lowest mass, as might be expected from Hénon's theory and as a by-product of mass segregation. The actual fractional escape is less than 1 per cent, however. After core collapse (Fig. 18b), the mass dependence is reversed, as the most massive stars preferentially escape. As we have seen (Fig. 3) there is little further mass segregation after core collapse, and therefore part of the mechanism that led to the preferential escape of the lightest stars is suppressed. Now, however, interactions in the core, which consists predominantly of stars of high mass, some in binaries, lead to the escape of stars directly from the core, and this depletes the population of the stars of highest mass.

The number of binary escapers is very small. For example when $N=500$ the average number is about 0.5 at $t \sim 400$, $\approx 20t_{\text{coll}}$ (see Table 2); this number is about 1 per cent of all escapers (see Fig. 17). Simple theory implies that the possibility of escape of binaries depends, among other things, on the mass of stars that form the binary. The fraction of the kinetic energy that the centre of mass of the binary (of total mass $m_3$) can gain during an interaction with a field star of mass $m_1$ is approximately equal to $m_1/(m_1 + m_3)$. Therefore when $m_3 > 2m_1$, this fraction is smaller than in the case of equal masses. It follows that a massive binary has to experience more (or more energetic) interactions with field stars before its removal from the system. In a sense, therefore, it remains in the core longer than a binary that consists of stars with masses equal to that of the single stars.

7.2 The energy budget

Initially, the energy of the N-body systems consists entirely of the 'external' energy of the bound members, which we denote by $E_{\text{ext}}$. When binaries form, $E_{\text{ext}}$ increases (i.e. the cluster becomes less bound), and the 'internal' energy of the bound binaries $E_{\text{int}}$ decreases in compensation (Fig. 15). At the same time, the escape of single stars by two-body processes diverts energy into the external energy of the escapers $E_{\text{ext}}$, and this tends to decrease the energy $E_{\text{int}}$ of the remaining bound stars. Finally, as interactions with binaries become energetic enough to lead to their escape (the energy associated with the motion of their barycentres being denoted by $E_{\text{b}}$), the internal energy of escaping binaries ($E_{\text{int}}$) decreases (i.e. increases in magnitude), and the

![Figure 19. Energy balance for $N=1000$. The meanings of the symbols are given in the text.](https://academic.oup.com/mnras/article-abstract/279/3/1037/967535/68275)
internal energy of bound binaries $E_{\text{int}}^\text{b}$ increases (decreases in magnitude) in compensation. The balance between these various processes is shown in Fig. 19. Note that, although the number of binary escapers is small (Section 7.1), they have a very significant effect on the energy budget. It is worth comparing Fig. 19 with the corresponding diagrams published by Aarseth & Heggie (1992) for larger systems ($N \geq 2500$) containing primordial binaries, although in Fig. 19 we have split $E_{\text{int}}^\text{b}$ into two components, corresponding to the kinetic energies of single stars and of binaries. Although the initial value of $E_{\text{int}}^\text{b}$ is non-zero in that case, the trends in the data are remarkably similar, even though the calculations of Aarseth & Heggie used larger $N$ and equal masses.

8 DISCUSSION AND CONCLUSIONS

We have analysed the evolution of isolated $N$-body systems in which the spectrum of stellar masses is a power law. The systems contained between 250 and 1000 stars, and for each $N$ we have averaged results from a large number of runs, in order to improve the statistical quality of the data.

Our results confirm the extreme rapidity of core collapse in the presence of a mass spectrum (cf. Inagaki & Wiyanto 1984). Theoretically, the $N$ dependence of the rate of core collapse involves the Coulomb logarithm $\ln\gamma/N$, but our results do not constrain the value of $\gamma$ as strictly as in the case of equal masses (Paper I). Our results are consistent with a value of about $\gamma = 0.02$, which is much smaller than the corresponding result for equal masses, in qualitative agreement with the theory of Hénon (1975). For such relatively small values of $N$ as in our simulation, the interpretation of the time-scales is quite sensitive to the value of $\gamma$. For example, the value of the initial half-mass relaxation time $t_{\text{rel}}(0)$ for $N = 500$ varies from about 20 for $\gamma = 0.02$ to 11 for the standard value $\gamma = 0.4$. If the former value is adopted we find that core collapse is complete (Table 2) in about 0.9$t_{\text{rel}}(0)$. Lest this seem much too short, it may be noted that Chernoff & Weinberg (1990), using a Fokker–Planck code, also found that the time to core collapse was about 0.9$t_{\text{rel}}(0)$ for the same mass spectrum, although the initial model they adopted was a King model with dimensionless central potential $W_f = 3$, and a tidal cut-off was included.

One of our more curious findings (Section 3.2 and Fig. 3) is that the spatial distribution of the mean stellar mass is nearly constant (i.e. time-independent) throughout the post-collapse evolution, when the overall expansion of the system is scaled out. While this helps to explain why the time dependence of the post-collapse expansion takes the simple form of equation (1), it is only an empirical explanation. It remains to be understood on theoretical grounds why the extremely rapid segregation of masses during core collapse comes to such an abrupt halt. One way in which this question might be approached would be to try to find a self-similar, multimass, post-collapse solution of the Fokker–Planck or gas equations, in analogy with the equal-mass models of Goodman (1984, 1987) or Inagaki & Lynden-Bell (1983). It was suggested long ago by Lynden-Bell (private communication) that one might search for such a solution for core collapse, in which the distribution of stellar masses was a power law. Be that as it may, there is now empirical evidence that a search for such a post-collapse solution might be fruitful.

It is equally surprising, perhaps, that the departures from equipartition (Fig. 6) also follow a very simple behaviour in the post-collapse regime. Certainly it may be expected that departures from energy equipartition will be small in the core, where the relaxation time is short, and this is what is found. It might have been expected that, at radii where the departure from equipartition becomes significant, one should have observed a progressive tendency towards equipartition. We have found, however, that there is essentially no change in the profile of our equipartition parameter, except for that caused by the nearly homologous expansion of the entire system.

Another surprise, except in hindsight, is the evolution of the heaviest stars. We have seen (Section 7.1) that they escape at a relatively high rate, which would not necessarily be expected because it might be thought that they would tend to eject stars of low mass while settling, through mass segregation, into the core. Perhaps this does happen, but in the core the massive stars tend to form energetic binaries, whose further interactions tend to eject these binaries and the moderately massive stars in the core, thus enhancing their rate of escape.

Some hard binaries are ejected into the outer parts of the cluster without completely escaping, and we noticed (Section 6.2) that these tend to have higher eccentricities than average, presumably as a result of the ejecting encounter. Thus hard binaries beyond the half-mass radius tend to have large masses and large eccentricities. Although our models obviously differ in crucial respects from the galactic globular clusters, it is possible to see here an example of the dynamical behaviour that gives rise to such objects as the binary millisecond pulsar in M15 (Phinney 1993).

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