State dependent correlations in the Vasicek default model

Abstract: This paper incorporates state dependent correlations (those that vary systematically with the state of the economy) into the Vasicek default model. Other approaches to randomizing correlation in the Vasicek model have either assumed that correlation is independent of the systematic risk factor (zero state dependence) or is an explicit function of the systematic risk factor (perfect state dependence). By contrast, our approach allows for an arbitrary degree of state dependence and includes both zero and perfect state dependence as special cases. This is accomplished by expressing the factor loading as a function of an auxiliary (Gaussian) variable that is correlated with the systematic risk factor. Using Federal Reserve data on delinquency rates we use maximum likelihood to estimate the parameters of the model, and find the empirical degree of state dependence to be quite high (but generally not perfect). We also find that randomizing correlation, without allowing for state dependence, does not improve the empirical performance of the Vasicek model.

Keywords: One-factor Gaussian Copula, Vasicek model, stochastic correlation; regulatory capital

MSC: 60E99, 62E99

1 Introduction

This paper incorporates state dependent correlations (those that vary systematically with the state of the economy) into the Vasicek default model. Other approaches to randomizing correlation in related models, such as [2, 7, 8, 15, 25, 28], have either assumed that correlation is independent of the systematic risk factor (zero state dependence) or is an explicit function of the systematic risk factor (perfect state dependence). By contrast, our approach allows for an arbitrary degree of state dependence and includes both zero and perfect state dependence as special cases. This is accomplished by expressing the factor loading as a function of an auxiliary (Gaussian) variable that is correlated with the systematic risk factor; the degree to which the two are correlated can be interpreted as the degree of state dependence.

We fit several different models to Federal Reserve data on delinquency rates, and compare their performance according to the Akaike Information Criteria (AIC). We find that a state-dependent model with two correlation regimes outperforms the traditional Vasicek model, and that the estimated degree of state dependence is very high across all loan types. Importantly, we also find the traditional Vasicek model outperforms a model with stochastic, but state-independent, correlations. In other words, randomizing correlation without allowing for state dependence does not improve the statistical performance of the Vasicek model.
1.1 Background and motivation

The Vasicek default model, originally proposed by [29] as a means to compute risk measures associated with loan portfolios, is one of the most influential models in quantitative finance. Indeed, it underpins the Basel risk-weight function used to determine regulatory capital requirements for internationally active banks (see Section 1.3). As does any model, the Vasicek model has its empirical shortcomings. For example, [14] finds that it tends to under-predict the severity of large default rates in corporate bond markets. Despite its shortcomings the model has considerable pedagogical value, evidently because of its clear intuition and computational simplicity, and for this reason it continues to play a central role in certain areas of quantitative risk management.

Its empirical shortcomings are often attributed to specific features of the Vasicek model, such as light-tailed risk factors (see [6, 21, 30] for extensions with heavier-tailed factors) and a limited correlation structure (see [23] for a clever workaround). One feature that has received comparatively little attention in the literature is the assumption of constant correlation. From a risk management perspective, correlation between the financial health of distinct exposures is the key parameter in the Vasicek model. This correlation is assumed constant, which is tantamount to assuming that the correlation that will prevail over the planning horizon is known with certainty in advance. This contrasts with empirical evidence for a wide variety of asset classes - correlations between returns on many assets tend to exhibit state dependence, by which we mean a tendency for correlations to rise during adverse economic scenarios ([9, 13, 18, 19]). It is natural to suspect that state dependence is also a feature of the latent "credit quality" variables that are the building blocks of the Vasicek model, and that ignoring this feature may be one (of several) reasons why the model under-estimates risk measures.

The remainder of Section 1 briefly discusses relevant literature on the Gaussian copula model (Section 1.2), regulatory capital (Section 1.3), and introduces notation (Section 1.4). Section 2 briefly reviews the Vasicek default model and Section 3 introduces our extension in general terms. Section 4 discusses model implementation in the case where correlation is a discrete random variable, and builds intuition for the impact of state dependence on important risk measures. Section 5 contains the empirical results, Section 6 briefly discusses the copula that is implicit in the proposed model and Section 7 concludes.

1.2 Gaussian copula model

The Vasicek model is a one-period (i.e. static) model used to construct default indicators of correlated exposures over a given time horizon. Closely related is the Gaussian copula model, a dynamic model used to construct the default times of correlated exposures. Like the Vasicek model, the Gaussian copula model is quite influential, in particular it played a major role in the development of credit derivative markets; see [22] for an engaging description of that process.

Although the Vasicek and Gaussian copula models do share the same dependence structure, there are subtle but important differences in the way that they are applied in practice. Specifically, the Gaussian copula model is typically used to price and hedge credit derivatives (which would be found on the bank's so-called "trading book"), whereas the Vasicek model is typically used to compute risk measures associated with portfolios of loans that the bank itself has extended (i.e. risk management of the so-called "banking book"). In the former case the relevant probability measure is risk-neutral (i.e. $Q$), in the latter case it is historical (i.e. $P$). Another important difference lies in the number of exposures. The number of exposures underlying a typical credit derivative pales in comparison to the number of loans in a typical banking book; in applications of the Vasicek model the number of exposures is typically large enough that so-called large portfolio approximation (described in the introduction to Section 2) is justified.

Because the Vasicek and Gaussian copula models share the same underlying dependence structure, it is important to consider the literature on the latter when proposing extensions to the former. Randomizing correlation has received considerable attention in the context of the Gaussian copula model; see [2, 7, 8, 15, 25, 28] for example. Each of these extensions assumes either zero state dependence, in which case correlation is...
independent of the systematic risk factor, or perfect state dependence, in which case correlation is a function of the systematic risk factor. Our approach is more general in the sense that it allows for an arbitrary degree of state dependence, including both extremes as special cases.

Because our interest in this paper lies with the Vasicek model, we work almost exclusively with the large portfolio approximation. That being said, Section 6 does consider the impact of the proposed extension on the dependence structure (conditional tail probabilities and hazard rates, specifically) in the case of two exposures. While this does give the reader some idea of how the proposed extension might affect the pricing and hedging of credit derivatives, a full treatment of that impact is beyond the scope of the this paper.

1.3 Regulatory capital

Regulatory capital is the capital that banks are legally compelled to raise as a cushion against the most adverse economic scenarios. Formally, the Basel Committee on Banking Supervision (BCBS) defines regulatory capital on a given portfolio as the difference between the (i) 99.9th percentile and (ii) mean, of the portfolio's loss distribution. The BCBS further specifies a particular formula, often called the risk-weight function, that banks must use in order to compute regulatory capital. See [5] for a detailed discussion.

The statistical model underpinning the risk-weight function is the Vasicek model. Specifically, the risk-weight function is an approximation to the exact value of regulatory capital in the Vasicek model. The approximation was originally introduced and justified in [11] in so-called Asymptotic Single Risk Factor (ASRF) framework, which includes a very wide range of models. See [24] for a more recent discussion of the ASRF framework.

1.4 Notation

Throughout the paper we let $\phi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$ denote the standard normal probability density function (pdf) and $\Phi(x) = \int_{-\infty}^{x} \phi(z) \, dz$ denote the associated cumulative distribution function (cdf). For $\mu \in \mathbb{R}$ and $\sigma > 0$ we let $\phi(x; \mu, \sigma^2) = \sigma^{-1} \phi((x - \mu)/\sigma)$ and $\Phi(x; \mu, \sigma^2) = \Phi((x - \mu)/\sigma)$ denote the pdf and cdf, respectively, of the normal distribution with mean $\mu$ and variance $\sigma^2$. Finally, we let $N(\mu, \sigma^2)$ denotes the normal distribution with mean $\mu$ and variance $\sigma^2$.

For $n \geq 1$ we let $\Phi_{n}(x; \mu, \Sigma)$ denote the multivariate normal cdf with mean vector $\mu$ and covariance matrix $\Sigma$, evaluated at the point $x \in \mathbb{R}^n$. Note that

$$
\Phi(x_1; \mu_1, \sigma_1^2) \cdot \Phi(x_2; \mu_2, \sigma_2^2) = \Phi_2(x; \mu, \Sigma),
$$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$. More generally, we have

$$
\Phi_n(x_1; \mu_1, \Sigma_1) \cdot \Phi_m(x_2; \mu_2, \Sigma_2) = \Phi_{n+m}(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_1 & 0_{n \times m} \\ 0_{m \times n} & \Sigma_2 \end{bmatrix}).
$$

where $0_{s \times t}$ is the $s \times t$ matrix with every entry equal to zero.

Suppose that $Z$ is normally distributed with zero mean and unit variance, and let $a$ and $b$ be $n \times 1$ vectors. Then

$$
E[1_{\{Z \leq c\}} \cdot \Phi_n(a + bZ; 0, \Sigma)] = \Phi_{n+1}(\begin{bmatrix} c \\ a \end{bmatrix}; 0, \begin{bmatrix} 1 & -b^T \\ -b & \Sigma + bb^T \end{bmatrix}),
$$

and

$$
E[\Phi_n(a + bZ, 0, \Sigma)] = \Phi_n(a; 0, \Sigma + bb^T).
$$
In order to see that this is true, note that the first expectation can be written as
\[
\frac{c}{a_1+b_1z} \cdots \frac{a_2+b_2z}{\phi_n(u, 0, \Sigma) \, dudz}.
\]

The display above is demonstrably equal to \( P(Z \leq c, U_1 - b_1 Z \leq a_1, \ldots, U_n - b_n Z \leq a_n) \), where \((U_1, \ldots, U_n)\) is multivariate normal with zero mean and covariance matrix \( \Sigma \), and \( Z \) is a standard normal independent of the vector \((U_1, U_2, \ldots, U_n)\). The result follows upon checking the covariance matrix. By symmetry the second expectation can be obtained from the first by replacing \( c \) with \(-c\) and \( b \) with \(-b\). Finally, the third equation is obtained by summing the first two.

## 2 Vasicek model

In order to model correlation between the default status of distinct exposures, the Vasicek model associates a “credit quality” variable \( X_i \) with every exposure (here \( i \) indexes exposures). Credit quality is decomposed into systematic and idiosyncratic risk sources as follows:

\[
X_i = a_i M + \sqrt{1-a_i^2} Y_i,
\]

where \( M, Y_1, Y_2, \ldots \) is a sequence of i.i.d. \( N(0, 1) \) random variables and \( a_1, a_2, \ldots \) is a sequence of constants such that \( a_i \in (0, 1) \) for every \( i \). \( M \) is referred to as the systematic risk factor, and is interpreted as representing the overall state of the economy. For instance if \( M = -1.2 \) then the economy is 1.2 standard deviations below average (whatever that means). The variable \( Y_i \) is an idiosyncratic risk factor that is specific to exposure \( i \). The parameter \( a_i \) is a factor loading that dictates how sensitive exposure \( i \) is to the overall economy. Alternatively, the factor loading governs the relative importance of systematic risk, as compared to idiosyncratic risk. It is clear that the correlation between distinct exposures \( i \neq j \) is \( a_i a_j \).

In the model, exposure \( i \) defaults if the realized value of credit quality is sufficiently low. Specifically, default occurs if and only if \( X_i \leq \Phi^{-1}(q_i) \), where \( q_i \in (0, 1) \) is a constant. Since \( X_i \) is standard normal (recall that \( M \) and \( Y_i \) are independent standard normal variables), it follows that \( q_i \) is the default probability associated with exposure \( i \). It is important to note that credit quality is latent, i.e. it is never observed directly. We only observe default indicator \( 1(X_i \leq \Phi^{-1}(q_i)) \), and not the credit quality variable itself. Here \( 1(A) \) denotes the indicator of event \( A \).

The default rate among the first \( N \) exposures is \( D_N := N^{-1} \sum_{i=1}^{N} 1(X_i \leq \Phi^{-1}(q_i)) \). The probability distribution of the default rate for large values of \( N \) is a fundamental object in quantitative risk management. The key insight in [29] is that, while the exact distribution might be complicated for finite \( N \), the limiting distribution as \( N \to \infty \) is eminently tractable. Indeed, under very mild conditions on the model parameters (to be discussed more formally in the next paragraph) the almost sure limit \( D := \lim_{N \to \infty} D_N \) is well-defined, depending only the realized value of the systematic risk factor (and not the realized values of the idiosyncratic risk factors). In other words, in essentially any case of practical interest we can write \( D = \nu(M) \) for some tractable function \( \nu \). As a tractable function of a single Gaussian variable, the probability distribution of \( D \) is easy to derive. In the remainder of this paper we will refer to the variable \( D \) is often called the large portfolio default rate.

We conclude this section with a more formal version of the discussion in the previous paragraph. To begin, Theorem A2 in [20] ensures that \( D_N - \mathbb{E}[D_N|M] \) converges to zero almost surely as \( M \to \infty \), for all values of the model parameters. If the model parameters are such that \( \mathbb{E}[D_N|M] \) converges to some limiting variable, then (i) that limiting variable is necessarily a (measurable) function of \( M \) and (ii) \( D_N \) converges to the same limit. In other words, if the model parameters are such that \( \mathbb{E}[D_N|M] \) converges almost surely as \( N \to \infty \), then (i) the variable \( D \) is well-defined and (ii) \( D = \nu(M) \) for some function \( \nu \). As discussed in [26], in practice it is common to place each exposure into one of \( G \) groups and to assume that exposures within a given group are homogeneous (i.e. all exposures in group \( g \) would have the common
default probability \( q_x \) and factor loading \( a_x \). Under these assumptions, and as discussed in [26] we have 
\[ \mathbb{E}[D_N | M = m] = \sum_{g=1}^{G} w_{g,N} \cdot \Phi(\Phi^{-1}(q_x); a_x m, 1 - a_x^2), \]
where \( w_{g,N} \) is the proportion of the first \( N \) exposures that fall into group \( g \). If, for each \( g \), \( w_{g,N} \) converges to some constant \( w_g \) as \( N \to \infty \), then \( \mathbb{E}[D_N | M = m] \) converges to \( \sum_{g=1}^{G} w_g \cdot \Phi(\Phi^{-1}(q_x); a_x m, 1 - a_x^2) \) and we have 
\[ D = \sum_{g=1}^{G} w_g \cdot \Phi(\Phi^{-1}(q_x); a_x M, 1 - a_x^2). \]
In the special case that \( G = 1 \) (homogeneous exposures) we have 
\[ D = \Phi(\Phi^{-1}(q); a M, 1 - a^2), \]
where \( q \) is the common default probability and \( a \) is the common factor loading. Note that in general (i.e. for arbitrary \( G \)), \( D \) is a monotone function of \( M \), whence the probability distribution of \( D \) is straightforward to obtain.

### 2.1 Homogeneous Vasicek model

In the case of homogeneous exposures, neither default probabilities nor factor loadings depend on \( i \). If \( a \in (0, 1) \) denotes the common factor loading and \( \rho := a^2 \), then \( \rho \) is the correlation between distinct credit qualities. And if \( q \) denotes the common default probability, then \( x := \Phi^{-1}(q) \) denotes the common default threshold. Mathematically it is more convenient to work with the variables \( x \) and \( a \), economically the variables \( q \) and \( \rho \) have more meaning.

Given that \( M = m \), credit qualities are i.i.d. Gaussian variables with mean \( a m \) and variance \( 1 - a^2 \), and the default indicators are i.i.d. Bernoulli variables with success probability \( \mathbb{P}(X_i \leq x | M = m) = \Phi(x; a m, 1 - a^2) \), where we recall that \( x = \Phi^{-1}(q) \) is the default threshold. By the strong law of large numbers, then, we have that 
\[ D_N \rightarrow \Phi(x; a m, 1 - a^2) \] as \( N \to \infty \). The implication is that the large portfolio default rate is given by
\[ D = \Phi(x; a M, 1 - a^2) = \Phi(\Phi^{-1}(q), \sqrt{\rho} M, 1 - \rho). \] (2.1)

For fixed \( x \in \mathbb{R} \) and \( a \in (0, 1) \) we define the function \( v_{x,a} : \mathbb{R} \to (0, 1) \) as follows:
\[ v_{x,a}(m) := \Phi(x; a m, 1 - a^2) = \Phi \left( \frac{x - a m \sqrt{1 - \rho}}{\sqrt{1 - a^2}} \right), \] (2.2)

We call this function a **Vasicek default curve**. Given a homogenous portfolio with correlation \( \rho \) and default probability \( q \), the relation between the state of the economy and the large portfolio default rate is described by the function \( v_{x,a}(\cdot) \) with \( x = \Phi^{-1}(q) \) and \( a = \rho^{1/2} \).

The relationship between the systematic risk factor and the large portfolio default rate (i.e. the graph of the function \( v_{x,a}(\cdot) \)) is illustrated in Figure 2.1 below. Important points are that (i) higher correlation exacerbates both good times and bad (i.e. higher correlation makes good times better and bad times worse) and (ii) higher correlation makes the default rate more sensitive to the state of the economy. For later use, it is worth noting that the inverse and derivative of a Vasicek default curve are given by
\[ v_{x,a}^{-1}(d) = \frac{x - \sqrt{1 - a^2} \Phi^{-1}(d)}{a}, \] (2.3)
and
\[ v_{x,a}'(d) = -a \Phi(x; a m, 1 - a^2), \] (2.4)
respectively.

Since \( D \) is a monotone (decreasing) function of \( M \) and \( M \) is standard normal, it is easy to obtain the probability distribution of \( D \). Indeed
\[ \mathbb{P}(D \geq d) = \mathbb{P}(M \leq v_{x,a}^{-1}(d)) = \Phi(v_{x,a}^{-1}(d)). \]

For fixed \( x \in \mathbb{R} \) and \( a \in (0, 1) \) we define the function \( s_{x,a} : (0, 1) \to (0, 1) \) as follows:
\[ s_{x,a}(d) := \Phi(v_{x,a}^{-1}(d)) = \Phi \left( \frac{x - \sqrt{1 - \rho} \Phi^{-1}(d)}{a} \right). \] (2.5)

We call \( s_{x,a}(\cdot) \) a **Vasicek survival function**. Given a homogenous portfolio with correlation \( \rho \) and default probability \( q \), the probability that the default rate exceeds a given threshold \( d \) is given by \( s_{x,a}(d) \), with \( x = \Phi^{-1}(q) \) and \( a = \rho^{1/2} \).
Differentiating $1 - s_{x,a}(d)$ we get the probability density function of the large portfolio default rate, which we denote by $f_{x,a}(d)$ and call a Vasicek density. It is readily verified that the Vasicek density is given by

$$f_{x,a}(d) = -\frac{\phi(v^{-1}_{x,a}(d))}{v'_{x,a}(v^{-1}_{x,a}(d))}$$  \hspace{1cm} (2.6)$$

where we recall that $v^{-1}_{x,a}$ and $v'_{x,a}$ are given by (2.3) and (2.4), respectively.

**Remark 2.1.** It is not hard to show that if $a_1 < a_2$ then

$$\lim_{d \to 1} \frac{s_{x,a_1}(d)}{s_{x,a_2}(d)} = \lim_{d \to 1} \frac{f_{x,a_1}(d)}{f_{x,a_2}(d)} = \lim_{d \to 0} \frac{f_{x,a_1}(d)}{f_{x,a_2}(d)} = \lim_{d \to 0} \frac{1 - s_{x,a_1}(d)}{1 - s_{x,a_2}(d)} = 0 .$$

The implication is that higher correlation makes extreme default rates (be they large or small) much more likely.

### 3 Incorporating state dependence in correlations

In order to introduce state dependent correlations into the homogeneous Vasicek model, we begin by writing credit qualities in the form

$$X_i = A \cdot M + \sqrt{1 - A^2} \cdot Y_i ,$$

where

- $M$, $Y_1$, $Y_2$, ... is an i.i.d. sequence of Gaussian variables with zero mean and unit variance.
- $A$ is a random variable supported on some subset of the unit interval $[0, 1]$, and independent of the sequence $Y_1, Y_2, ...$.

Note, importantly, that we have not insisted that $A$ and $M$ are independent. Indeed we have only insisted that the marginal of $M$ is Gaussian with mean zero and unit variance, and that the marginal of $A$ is supported on some subset of $[0, 1]$. Beyond this we have (intentionally) said nothing about the joint distribution of $A$ and $M$. There are myriad ways to construct the joint distribution of $A$ and $M$. For instance one could specify the marginal distribution of $A$ and the copula of the pair $(A, M)$, in which case the joint distribution of the pair
would be fully and uniquely specified. The construction that we use in this paper is described more fully in Section 3.1, for now we simply assume that the joint distribution is essentially arbitrary.

The marginal and joint distributions of credit qualities will depend on the joint distribution of $A$ and $M$, which we have yet to specify. That being said, certain insights are possible based on the limited assumptions we have made to this point. For example (justifications are given in Appendix A):

1. If $A$ and $M$ are uncorrelated then $X_i$ is mean zero, otherwise the mean of $X_i$ is positive or negative, according as $A$ and $M$ are positively or negatively correlated.
2. If $A$ and $M$ are independent then the variance of $X_i$ is one, otherwise the variance will depend (in an apparently complicated way) on the dependence structure of the pair $(A, M)$.
3. $\text{Cov}(X_i, X_j) = \text{Var}(AM)$. Credit qualities are positively correlated, and if $A$ and $M$ are independent then $\text{Cov}(X_i, X_j) = \mathbb{E}[A^2]$.

Now, for any $a$ in the support of $A$ we let

\[
\mu(a) := \mathbb{E}[M|A = a]
\]

and

\[
\sigma^2(a) := \text{Var}(M|A = a) = \mathbb{E}[M^2|A = a] - (\mu(a))^2.
\]

If $A$ and $M$ are independent then $\mu(a) \equiv 0$ and $\sigma^2(a) \equiv 1$, otherwise the exact forms of $\mu(\cdot)$ and $\sigma^2(\cdot)$ will depend on the joint distribution of the pair (which we have yet to specify). That being said, it is not hard to show that, regardless of the exact form of the joint distribution, we have

\[
\mathbb{E}[X_i|A = a] = a\mu(a),
\]

\[
\text{Var}(X_i|A = a) = 1 - a^2(1 - \sigma^2(a)), \quad (3.1)
\]

and

\[
\text{Cov}(X_i, X_j|A = a) = a^2 \sigma^2(a) \quad (3.2)
\]

Combining (3.1) and (3.2) we find that the conditional correlation between credit qualities, given the realized value of the factor loading, is

\[
\rho(a) := \text{Corr}(X_i, X_j|A = a) = \frac{a^2 \sigma(a)^2}{a^2 \sigma(a)^2 + (1 - a^2)} \quad (3.3)
\]

We can therefore think of the random variable $\rho(A)$ as a stochastic correlation between credit qualities. If $A$ and $M$ are independent then $\rho(a) = a^2$, otherwise the form of $\rho(\cdot)$ will depend on the form of the joint distribution of $A$ and $M$.

To complete the model, we assume that exposure $i$ defaults if and only if the realized value of credit quality is sufficiently low. In particular we assume that default occurs if and only if $X_i \leq x_q$, where $x_q$ is the $100q^{th}$ percentile of $X_i$ and $q$ is the marginal default probability.

### 3.1 Joint distribution of $A$ and $M$

In order for correlation to vary systematically with the state of the economy, we must allow for dependence between $A$ and $M$. Indeed if $A$ and $M$ are independent then correlation is stochastic but not state dependent. In order to create dependence between $A$ and $M$ we first introduce an auxiliary standard normal variable $T$, which we permit to be correlated with $M$, and then write the factor loading as a function of $T$. More specifically:

**Remark 3.1.** In this paper we construct the joint distribution of the pair $(A, M)$ as follows. First, we introduce an auxiliary variable $T$ such that (i) $T$ is Gaussian with mean zero and unit variance, (ii) $(M, T)$ is bivariate normal with correlation $\beta := \text{Corr}(M, T)$ and (iii) $T$ is independent of the sequence $Y_1, Y_2, \ldots$. Next, we write $A = g(T)$ for some deterministic function $g$. 
The role of the function $g$ is to govern the marginal behaviour of the factor loading $A$ (equivalently, the correlation $\rho(A)$). By choosing $g$ appropriately we can ensure that $A$ (or $\rho(A)$) has any desired probability distribution. The role of the parameter $\beta$ is to govern the degree of state dependence, by which we mean the tendency for correlation to vary systematically with the state of the economy.

If $\beta = 0$ then correlation is stochastic but independent of the state of the economy, and we obtain a model where credit qualities follow a Gaussian mixture. This is the approach taken in the stochastic correlation models proposed by [7] and [8] (see [15] for further discussion of these models). Although those approaches do allow for heterogenous factor loadings, the basic idea that factor loadings are randomized independently of the systematic risk factor is fundamentally the same. If $\beta = \pm 1$ then correlation is determined completely by the state of the economy, and we obtain both the random factor loading model proposed by [2] or the local correlation models proposed by [7] and [28] (see [15] for further discussion of these models). Intermediate values of $\beta$ interpolate between these two extremes, and allow us to consider the more realistic case of a non-trivial, but imperfect, relationship between correlations and the overall state of the economy.

**Remark 3.2.** If $g$ is a monotone function, then this approach effectively assumes a Gaussian copula between the systematic risk factor $M$ and the factor loading $A$. The marginal behaviour of the factor loading is governed by the function $g$, and the dependence between $M$ and $A$ is governed by the parameter $\beta$.

### 3.1.1 Discrete factor loading

With a view to developing intuition, we focus on the case where the factor loading is discrete. A discrete loading can be obtained by taking $g$ to be a simple function of the form

$$g(t) = \sum_{k=1}^{K} a_k \cdot 1(t_{k-1} < t \leq t_k),$$

where $K \geq 1$ is an integer, $a_1, a_2, \ldots, a_K$ are numbers in the unit interval and $-\infty = t_0 < t_1 < \ldots < t_{K-1} < t_K = \infty$. With this specification the factor loading $A$ is a discrete random variable taking on the value $a_k$ with probability

$$p_k := \mathbb{P}(t_{k-1} < T \leq t_k) = \Phi(t_k) - \Phi(t_{k-1}).$$

Given that $A = a_k$ (equivalently, $T \in (t_{k-1}, t_k]$), the correlation between credit qualities is

$$\rho_k := \rho(a_k) = \frac{a_k^2 \sigma_k^2}{a_k^2 \sigma_k^2 + (1 - a_k^2)},$$

where we have used (3.3), and where $\sigma_k^2 := \text{Var}(M|A = a_k) = \text{Var}(M|t_{k-1} < T \leq t_k)$. Note that $\sigma_k^2$ depends on $\beta$ (the correlation between $M$ and $T$), as well as the threshold values $t_{k-1}$ and $t_k$. An explicit formula for $\sigma_k^2$ will be given in Section 4.1.3.

Given the value of $\sigma_k$, the relation (3.6) is easily inverted to express $a_k$ in terms of $\rho_k$, indeed

$$a_k = \sqrt{\frac{\rho_k}{\sigma_k^2 + \rho_k(1 - \sigma_k^2)}}.$$

The implication is that the model can either be parametrized in terms of factor loadings or correlations. The former are easier to work with mathematically, the latter are (perhaps arguably) more natural at an intuitive level. In order to facilitate either parametrization, we introduce the variable

$$R := \sum_{k=1}^{K} k \cdot 1(t_{k-1} < T \leq t_k),$$

and call the event $\{R = k\} = \{t_{k-1} < T \leq t_k\}$ the $k^{th}$ (correlation) regime. Note that in the $k^{th}$ regime, the value of the factor loading is $a_k$ and the correlation between distinct credit qualities is $\rho_k$, and that $p_k =$
$\mathbb{P}(R = k)$ is the probability that the $k^{th}$ regime is realized. Finally, note that the threshold values $t_k$ can be recovered from regime probabilities $p_k$ via the relation $t_k = \Phi^{-1}(\sum_{j=1}^{k} \rho_j)$. The implication is that the model can be parametrized in terms of regime probabilities as opposed to threshold values, the former being far more natural to work with at an intuitive level.

In order to parametrize the version of the model with $K$ regimes, we must first specify (i) the marginal default probability $q$, (ii) the degree of state dependence $\beta$ and (iii) the probability mass function of the regime variable $R$ (i.e. any $K - 1$ of the probabilities $p_1, p_2, \ldots, p_K$). Having fixed the values of these quantities, we must then either specify (i) regime correlations $\rho_1, \rho_2, \ldots, \rho_K$, in which case factor loadings $a_k$ are then implied, or (ii) regime factor loadings $a_1, a_2, \ldots, a_K$, in which case correlations $\rho_k$ are then implied. In total, the $K$-regime version of the model has $2K + 1$ parameters.

It is not difficult to show that the joint distribution of the factor loading $A$ and the systematic risk factor $M$ is the same under the specification with parameters

$$(a_1, a_2, \ldots, a_K, p_1, p_2, \ldots, p_K, \beta, q)$$

as it is under the specification with parameters

$$(a'_1, a'_2, \ldots, a'_K, p'_1, p'_2, \ldots, p'_K, -\beta, q),$$

where $a'_k = a_{K-k+1}$ and $p'_k = p_{K-k+1}$. The implication is that there is a one-to-one correspondence between specifications with strictly negative state dependence, and those with strictly positive state dependence. For this reason we may, without loss of generality, restrict attention to either non-negative or non-positive values of $\beta$.

**Remark 3.3.** In the remainder of this paper we will, without loss of generality, assume that the degree of state dependence is non-negative. In other words, we assume that $\beta \geq 0$ in all instances.

## 4 Implementing the model

This section explains how to implement the discrete-loading version of the proposed model, and builds intuition for the impact of state dependence on important model-implied quantities. It culminates with closed form expressions for the probability density and survival functions of the large-portfolio default rate, and a discussion of important features of its distribution such as quantiles and tail expectations (which must in general be computed numerically).

In order to build intuition we consider a numerical example with two regimes. Parameter values are obtained by fitting the two-regime version of the proposed model to the All Loans data set described in Section 5. We compare important model outputs with those obtained from the Vasicek model fit to the same data set. For the two-regime model factor loadings are $(a_1, a_2) = (0.25, 0.16)$, regime probabilities are $(p_1, p_2) = (0.85, 0.15)$, the degree of state dependence is $\beta = 0.93$ and the marginal default probability is $q = 0.037$. For the Vasicek model, the factor loading is $a = 0.22$ and the marginal default probability is $q = 0.037$. Note that $p_1 a_1 + p_2 a_2 = a$, so that the mean factor loading in the state dependent model is approximately the same as the factor loading in the Vasicek model.

### 4.1 Correlation regime ($R$) and economic environment ($M$)

This section considers the relationship between the regime variable $R$ (a discrete variable) and the systematic risk factor $M$ (a continuous variable).
4.1.1 Correlation regime conditional on economic environment

Recall that \( p_k = \mathbb{P}(R = k) \) is the unconditional probability of regime \( k \), and let
\[
p_k(m) := \mathbb{P}(R = k|M = m)
\]
be the conditional probability that regime \( k \) is realized, given the realized value of the systematic risk factor. Then
\[
p_k(m) = \Phi(t_k; \beta m, 1-\beta^2) - \Phi(t_{k-1}; \beta m, 1-\beta^2),
\] (4.1)
where we have used the facts that \( \mathbb{P}(R = k|M = m) = \mathbb{P}(t_{k-1} < T \leq T_k|M = m) \) and that the conditional distribution of \( T \), given that \( M = m \), is Gaussian with mean \( \beta m \) and variance \( 1-\beta^2 \).

If \( \beta = 0 \) then \( p_k(m) \equiv p_k \) for all \( m \). This reflects the fact that in the absence of state dependence, correlation is stochastic but unrelated to the state of the economy. If \( \beta = 1 \) then \( p_k(m) = 1(t_{k-1} < m \leq t_k) \), which is either zero or one depending on the value of \( m \). This reflects the fact that under perfect state dependence, correlation is completely determined by the state of the economy. If \( 0 < \beta < 1 \) then the behaviour of \( p_k(\cdot) \) depends on \( k \). In particular:

- \( p_1(\cdot) \) is a decreasing function such that \( p_1(-\infty) = 1 \) and \( p_1(\infty) = 0 \). The more adverse the economic scenario, the more likely it is that the first regime is realized; during the most adverse scenarios it is virtually certain that the first regime prevails.

- \( p_K(\cdot) \) is an increasing function such that \( p_K(-\infty) = 0 \) and \( p_K(\infty) = 1 \). The more favourable the economic scenario, the more likely it is that regime \( K \) is realized; during the most favourable scenarios it is virtually certain that regime \( K \) prevails.

- For \( 1 < k < K - 1 \), \( p_k(\cdot) \) is a unimodal function (increasing and then decreasing) such that \( p_k(-\infty) = p_k(\infty) = 0 \). Middle regimes are most likely to be observed during moderate economic scenarios, and it is extremely unlikely that middle regimes prevail during extreme economic scenarios (be they favourable or adverse).

Figure 4.1(a) illustrates the graphs of \( p_1(\cdot) \) and \( p_2(\cdot) \) in our numerical example. We see that the first regime (where the factor loading is higher) is almost certain to prevail during below average economic scenarios, in the sense that \( p_1(m) = 1 \) for \( m < 0 \). We also see that the second regime (where the factor loading is lower) is certain to prevail during very good economic scenarios, specifically those that are two standard deviations above average (i.e. \( p_2(m) = 1 \) whenever \( m > 2 \)).

4.1.2 Economic environment given correlation regime

Let \( \phi_k(m) \) denote the density of the systematic risk factor in regime \( k \), i.e. the conditional density of \( M \) given that \( R = k \). Then
\[
\phi_k(m) = \phi(m) \cdot \frac{p_k(m)}{p_k},
\] (4.2)
where we have used Bayes’ rule and the fact that the unconditional distribution of the systematic risk factor is Gaussian with zero mean and unit variance.

If \( \beta = 0 \) then \( \phi_k(m) = \phi(m) \) for all \( k \) and \( m \). In the absence of state dependence, learning the correlation regime does not influence the likelihood of any economic scenario. If \( \beta = 1 \) then \( \phi_k(m) = \phi(m) \cdot 1(t_{k-1} < m \leq t_k)/p_k \), which is a truncated normal distribution concentrated on those values of \( m \) that correspond to regime \( k \). If \( 0 < \beta < 1 \) then \( \phi_k(m) = \phi(m) \) if and only if \( p_k(m) > p_k \). If scenario \( m \) increases the likelihood of regime \( k \), then \( \phi_k \) assigns relatively more weight to scenario \( m \) (as compared to the unconditional density \( \phi \)).

Figure 4.1(b) illustrates the conditional density of the systematic risk factor in each regime, in our numerical example. We see that the conditional density in the first regime concentrates its mass on adverse scenarios. The implication is that if the (random) factor loading takes on its higher value, then it is likely that the economic scenario will be below average. We also see that the conditional density in the second regime concentrates its mass on favourable scenarios, the implication being that if the factor loading takes on its lower value then it is likely that the economy is doing well.
Figure 4.1: Panel (a) depicts the relationship between the economic environment and conditional regime probabilities. More precisely, it depicts the graphs of $p_{1}(·)$ (solid) and $p_{2}(·)$ (dashed) in our numerical example. The horizontal axis represents the realized value of the systematic risk factor (a value of, say, $m = -1.2$ corresponds to an economy that is 1.2 standard deviations below average) and the vertical axis represents the conditional probability that a particular regime prevails, given the realized value of the systematic risk factor. Panel (b) depicts the conditional density of the systematic risk factor, given the prevailing regime. More precisely, it depicts the graphs of $\phi_{1}(·)$ (solid) and $\phi_{2}(·)$ (dashed) in our numerical example.

We conclude this section with the observation that if $h$ is some function, then

$$\mathbb{E}[h(M)|R = k] = p_k^{-1}\mathbb{E}[h(M)p_k(M)].$$  \hfill (4.3)

The identity (4.3) is easily verified by writing out and comparing the defining integrals, and will be used repeatedly in what follows.

### 4.1.3 Conditional mean and variance of $M$

The mean and variance of the conditional density $\phi_k$ are straightforward to compute. Using the tower property (recall that $R$ is a function of $T$) and the fact that the conditional distribution of $M$, given that $T = t$, is Gaussian with mean $\beta t$ and variance $1 - \beta^2$, we get that

$$\mathbb{E}[M|R = k] = \mathbb{E}[\mathbb{E}[M|T]|R = k] = \beta \mathbb{E}[T|R = k]$$

and

$$\mathbb{E}[M^2|R = k] = \mathbb{E}[\mathbb{E}[M^2|T]|R = k] = (1 - \beta^2) + \beta^2 \mathbb{E}[T^2|R = k].$$

Thus

$$\mu_k := \mathbb{E}[M|R = k] = \beta \mathbb{E}[T|R = k]$$  \hfill (4.4)

and

$$\sigma_k^2 := \text{Var}(M|R = k) = 1 - \beta^2 \left[ 1 - \text{Var}(T|R = k) \right].$$  \hfill (4.5)

Now, the conditional distribution of $T$, given that $R = k$, is a truncated Gaussian - specifically, the standard Gaussian truncated to the interval $[t_{k-1}, t_k]$. The moments of the truncated Gaussian are well known, and we get that

$$\mathbb{E}[T|R = k] = \frac{-\phi(t_k) - \phi(t_{k-1})}{p_k}$$  \hfill (4.6)

and

$$\text{Var}(T|R = k) = 1 - \frac{t_k \phi(t_k) - t_{k-1} \phi(t_{k-1})}{p_k} - \left( \frac{\phi(t_k) - \phi(t_{k-1})}{p_k} \right)^2,$$  \hfill (4.7)
where \( t\phi(t) = 0 \) whenever \( t = \pm \infty \). Inserting (4.6) into (4.4) and (4.7) into (4.6) we finally get that

\[
\mu_k = -\beta \cdot \frac{\phi(t_1) - \phi(t_{k-1})}{p_k} \tag{4.8}
\]

and

\[
\sigma_k^2 = 1 - \beta^2 \cdot \frac{t_k \cdot \phi(t_1) - t_{k-1} \cdot \phi(t_{k-1})}{p_k} - \mu_k^2. \tag{4.9}
\]

In our numerical example we have \((\mu_1, \mu_2) = (-0.26, 1.45)\) and \((\sigma_1, \sigma_2) = (0.83, 0.55)\).

**Remark 4.1.** Although it is not obvious from (4.7), well-known properties of the truncated Gaussian distribution ensure that \( \text{Var}(T|R = k) < 1 \). It follows from (4.5), then, that the conditional variance \( \sigma_k^2 \) is a strictly decreasing function of \( \beta \). In particular, we have that \( \sigma_k = 1 \) whenever \( \beta = 0 \) and \( \sigma_k^2 < 1 \) whenever \( \beta > 0 \).

### 4.2 Factor loadings and correlations

Recall that the correlation between distinct credit qualities in regime \( k \) is

\[
\rho_k = \frac{a_k^2 \sigma_k^2}{a_k^2 \sigma_k^2 + (1 - a_k^2)} = \frac{a_k^2 \sigma_k^2}{1 - a_k^2(1 - \sigma_k^2)}. \tag{4.10}
\]

In the absence of state dependence we have \( \sigma_k = 1 \) and \( \rho_k = a_k^2 \). In the presence of state dependence we have \( \sigma_k < 1 \), and it is readily verified that this implies \( \rho_k < a_k^2 \). In the presence of state dependence, then, the relationship between the factor loading and correlation is not the same as it is in the Vasicek model. In our numerical example we have \((\rho_1, \rho_2) = (0.044, 0.008)\).

It is readily verified that \( \rho_k \) is increasing in \( \sigma_k \), for fixed \( a_k \). And since \( \sigma_k \) is decreasing in \( \beta \) (recall Remark 4.1), it follows that \( \rho_k \) is a decreasing function of \( \beta \), for fixed \( a_k \). In other words, if the factor loading is held fixed then the correlation between distinct credit qualities decreases as the degree of state dependence becomes stronger. Although this may seem counter-intuitive at first, the intuition is straightforward. If factor loadings are held fixed but the degree of state dependence increases, then the variability of the idiosyncratic component of credit qualities \((1 - a_k^2)\) stays constant while the variability of the systematic component \((a_k^2 \sigma_k^2)\) decreases. The systematic component therefore becomes less important, relative to the idiosyncratic component, and since correlation here measures the relative importance of the systematic component, correlation will decrease. The upper half of Table 4.1 illustrates this phenomenon in the context of our numerical example. We see that the degree of state dependence has a non-linear impact on correlations. For example in the absence of state dependence a factor loading of \( a_1 = 0.25 \) would produce a first-regime correlation of 6.3%, in the presence of moderate state dependence (\( \beta = 0.5 \)) the same factor loading would produce a first-regime correlation of 5.7%, and in the presence of perfect state dependence it would produce a correlation of only 4.1%.
Table 4.1: The upper half of this table illustrates the impact of state dependence on correlations when factor loadings are held fixed, in the context of our numerical example. Recall that factor loadings are \((a_1, a_2) = (0.25, 0.16)\), regime probabilities are \((p_1, p_2) = (0.85, 0.15)\) and the marginal default probability is \(q = 0.037\). For each of the indicated values of \(\beta\), we compute \(\sigma_k\) using (4.9) and then compute \(\rho_k\) using (4.10), all other parameter values are held fixed. The lower half of the table illustrates the factor loading required to maintain correlations of \((\rho_1, \rho_2) = (0.044, 0.008)\) as we vary the degree of state dependence.

| Correlation \((\rho_k)\) with Fixed Factor Loading \((a_k)\) | First Regime \((k = 1)\) | Second Regime \((k = 2)\) |
|---|---|---|
| \(\beta = 0.00\) | 6.3% | 2.6% |
| \(\beta = 0.25\) | 6.1% | 2.4% |
| \(\beta = 0.50\) | 5.7% | 2.1% |
| \(\beta = 0.75\) | 5.1% | 1.4% |
| \(\beta = 1.00\) | 4.1% | 0.5% |

| Factor Loading \((a_k)\) with Fixed Correlation \((\rho_k)\) | \(\beta = 0.00\) | \(\beta = 0.25\) | \(\beta = 0.50\) | \(\beta = 0.75\) | \(\beta = 1.00\) |
|---|---|---|---|---|---|
| 0.21 | 0.21 | 0.22 | 0.23 | 0.26 |
| 0.09 | 0.09 | 0.10 | 0.12 | 0.20 |

The relationship (4.10) is easily inverted to express factor loadings in terms of correlations.

\[
a_k = \frac{\rho_k}{\sqrt{\sigma_k^2 + \rho_k(1 - \sigma_k^2)}}.
\]

(4.11)

It is of passing interest to consider how \(a_k\) varies with \(\beta\) when \(\rho_k\) is held fixed, i.e. the factor loading that would be required in order to maintain a given level of correlation as state dependence becomes stronger. If correlation is to be held fixed as the degree of state dependence increases, the factor loading must also increase. Indeed, if the factor loading were to stay constant then the variability of the systematic component \((a_k^2 \sigma_k^2)\) would decrease but the variability of the idiosyncratic component \((1 - a_k^2)\) would not change, leading to a decrease in correlation. In order to maintain a fixed level of correlation as the variability of the systematic risk factor \((\sigma_k^2)\) decreases, then, we must increase its relative importance \((a_k)\) in order to maintain a given level of correlation. The lower half of Table 4.1 illustrates this phenomenon in the context of our numerical example. In order to ensure a first-regime correlation of 4.4% in the absence of state dependence \((\beta = 0)\) a factor loading of \(a_1 = 0.21\), in the presence of moderate state dependence \((\beta = 0.5)\) a factor loading of \(a_1 = 0.22\) is required and in the presence of perfect state dependence \((\beta = 1.0)\) a factor loading of \(a_1 = 0.26\) is required.

### 4.3 Credit quality distributions

In this section we derive the marginal distribution of an individual credit quality, as well as the joint distribution of an arbitrary number of credit qualities. In preparation we note that if \(M = m\) and \(R = k\), then

\[
X_i = a_km + \sqrt{1 - a_k^2}Y_i.
\]

Since the \(Y_i\) are independent (and independent of \(R\) and \(M\)), it follows that the \(X_i\) are i.i.d. Gaussian variables with mean \(a_km\) and variance \(1 - a_k^2\). Thus

\[
\mathbb{P}(X_i \leq x | R = k, M = m) = \Phi(x; a_km, 1 - a_k^2),
\]

(4.12)
and if $X = (X_1, X_2, \ldots, X_n)^T$ and $x = (x_1, x_2, \ldots, x_n)^T$ then

$$
P(X \leq x | R = k, M = m) = F_n(x, ma_k, (1 - a_k^2)I_n),
$$

(4.13)

where $a_k = (a_k, a_k, \ldots, a_k)^T$ is $n \times 1$ and $I_n$ is the identity matrix of size $n$.

### 4.3.1 Marginal distribution of credit quality

Using (4.12) we get the marginal cdf of $X_i$ in regime $k$ is given by

$$
P(X_i \leq x | R = k) = \mathbb{E}[\Phi(x; a_k M, 1 - a_k^2) | R = k].
$$

(4.14)

In order to evaluate (4.14) we use (4.3), (1.1) and (1.5) to get that

$$
P(X_i \leq x | R = k) = p_k^{-1} \left[ \Phi_2(x_k; 0, \Sigma_k) - \Phi_2(x_{k-1}; 0, \Sigma_k) \right],
$$

(4.15)

where

$$
x_k = \begin{bmatrix} x \\ t_k \end{bmatrix}, \quad 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \Sigma_k = \begin{bmatrix} 1 & \beta a_k \\ \beta a_k & 1 \end{bmatrix}.
$$

(4.16)

Differentiating (4.15) we find that the conditional density of $X_i$ in regime $k$ is

$$
P(X_i \in dx | R = k) = p_k^{-1} \cdot \phi(x) \cdot r_k(x) \, dx,
$$

(4.17)

where

$$
r_k(x) := \Phi(t_k; \beta a_k x, 1 - \beta^2 a_k^2) - \Phi(t_{k-1}; \beta a_k x, 1 - \beta^2 a_k^2).
$$

(4.18)

The density appearing in (4.17) is a member of the parametric family considered by [3], which contains the skew-normal family introduced by [4] as a special case and is discussed in more detail in Appendix B. We will henceforth refer to this parametric family as the ABGM family, after the authors of [3].

In the absence of state dependence (i.e. if $\beta = 0$) then the density (4.17) reduces to $\phi(x)$, whence $X_i$ is standard normal in any regime. In the presence of state dependence (i.e. if $\beta > 0$) the distribution of $X_i$ in regime $k$ is a member of the ABGM family (with parameter values depending on $k$), and based on the discussion in Appendix B we can make the following general observations. In the first regime ($k = 1$) the density of $X_i$ is skewed to left, its left tail is proportional to $\phi(x)$ and its right tail is much thinner than $\phi(x)$. The exact opposite is true in the last regime ($k = K$), where the density is skewed to the right, the left tail is much thinner than $\phi(x)$ and the right tail is proportional to $\phi(x)$. For intermediate regimes ($1 < k < K$) the density of $X_i$ can be skewed in either direction and both tails are much thinner than $\phi(x)$. Panel (a) in Figure 4.2 illustrates the conditional densities (4.17) in our numerical example. In the first regime (where the factor loading is higher) the credit quality density is similar to a skew-normal density with negative skew, and for the second regime (where the factor loading is lower) it resembles a skew-normal with positive skew.
Figure 4.2: Panel (a) illustrates the conditional density of an individual credit quality in each regime, in our numerical example. The expression for the conditional density in regime \( k \) is given by (4.17). In the first regime (solid line) the density resembles a skew-normal density with negative skew, in the second regime (dashed line) the density resembles a skew-normal density with positive skew. Panel (b) illustrates the unconditional density of an individual credit quality in our numerical example and, for a point of reference, compares it to a standard normal density. Recall that the unconditional density here is of the form \( \phi(x) \cdot r(x) \), where \( r(x) = \sum_{k=1}^{K} r_k(x) \) and \( r_k(x) \) is given by (4.18). Note that \( r(x) \) is a non-negative function such that \( r(x) \to 1 \) as \( x \to \pm \infty \).

Recall that the default threshold, \( x_q \), is the \( 100q^{th} \) percentile of the unconditional distribution of \( X_i \). In order to compute the default threshold \( x_q \), we therefore require an expression for the unconditional cdf. Using (4.15) and the fact that \( \mathbb{P}(X_i \leq x) = \sum_{k=1}^{K} \mathbb{P}(X_i \leq x | R = k) \cdot \mathbb{P}(R = k) \) we get that

\[
\mathbb{P}(X_i \leq x) = \sum_{k=1}^{K} \left[ \Phi_2(x_k; \mathbf{0}, \Sigma_k) - \Phi_2(x_{k-1}; \mathbf{0}, \Sigma_k) \right],
\]

(4.19)

where we recall that \( x_k \) and \( \Sigma_k \) are given by (4.16).

**Remark 4.2.** *In the absence of state dependence (4.19) reduces to \( \Phi(x) \) and the default threshold is \( x_q = \Phi^{-1}(q) \), which only depends on the default probability \( q \) and none of the other model parameters. In the presence of state dependence the default threshold depends on all model parameters, and must be determined numerically (\( \Phi^{-1}(q) \) is typically a good initial guess).*

In our numerical example the Vasicek threshold is \( x_q = -1.79 \) and the state dependent threshold is \( x_q = -1.80 \).

Since \( \mathbb{P}(X_i \in dx) = \sum_{k=1}^{K} \mathbb{P}(X_i \in dx | R = k) \cdot \mathbb{P}(R = k) \), an expression for the unconditional density of \( X_i \) can be obtained by multiplying (4.17) by \( p_k \) and summing over \( k \). The end result is

\[
\mathbb{P}(X_i \in dx) = \phi(x) \cdot r(x) \cdot dx,
\]

(4.20)

where \( r(x) = \sum_{k=1}^{K} r_k(x) \). Note that (4.20) is a convex combination of ABGM densities (i.e. the unconditional distribution of \( X_i \) is a mixture of ABGM distributions); since the ABGM family of densities is not closed under convex combinations, (4.20) is not a member of that family in general. In the absence of state dependence we have \( r(x) \equiv 1 \) for all \( x \), whence (4.20) reduces to \( \phi(x) \) and \( X_i \) is standard normal. Otherwise the function \( r \) is not identically equal to one and \( X_i \) is not standard normal. That being said it will always be the case that \( r(x) \to 1 \) as \( x \to \pm \infty \), whence the density of \( X_i \) is asymptotically identical to that of a standard normal variable.
Remark 4.3. In the absence of state dependence credit qualities are standard normal (as they are in the Vasicek model). In the presence of state dependence credit qualities are not standard normal, but they do have standard normal tails.

Panel (b) in Figure 4.2 illustrates the density of $X_i$ in our numerical example and, for a point of reference, compares it to the standard normal density. The two densities are eminently similar, which suggests that the degree of state dependence does not have a material impact on the marginal behaviour of credit qualities.

4.3.2 Joint distribution of credit qualities

Using (4.13) we get the joint cdf of $X = (X_1, X_2, \ldots, X_n)^T$ in regime $k$ is given by

$$
\mathbb{P}(X \leq x | R = k) = \mathbb{E} \left[ \phi_n(x, Ma_k, (1 - a_k^2)I_n) | R = k \right]
$$

where we recall that $x = (x_1, x_2, \ldots, x_n)^T$, $a_k = (a_k, a_k, \ldots, a_k)^T$ is $n \times 1$ and $I_n$ is the identity matrix of size $n$. In order to evaluate (4.21) we use (3.1), (1.2) and (1.5) to get that

$$
\mathbb{P}(X \leq x | R = k) = p_k^{-1} \left[ \Phi_{n+1} \left( \begin{bmatrix} t_k \ x \ \ 0 \ \ \ b_k^T \\ \Sigma_k \end{bmatrix} \right) - \Phi_{n+1} \left( \begin{bmatrix} t_{k-1} \ x \ \ 0 \ \ \ b_k^T \\ \Sigma_k \end{bmatrix} \right) \right],
$$

where the $n \times 1$ vector $b_k$ and $n \times n$ matrix $\Sigma_k$ are given by

$$
b_k = \begin{bmatrix} \beta a_k \\ \beta a_k \\ \vdots \\ \beta a_k \end{bmatrix} \quad \text{and} \quad \Sigma_k = \begin{bmatrix} a_k^2 & a_k^2 & \cdots & a_k^2 \\ a_k^2 & 1 & \cdots & a_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_k^2 & a_k^2 & \cdots & 1 \end{bmatrix}.
$$

Differentiating (4.22) we get that the conditional density of $X$ in regime $k$ is given by

$$
\mathbb{P}(X \in dx | R = k) = p_k^{-1} \phi_n(x; 0, \Sigma_k) \cdot s_k(x) \, dx,
$$

where

$$
s_k(x) = \Phi \left( t_k; b_k^T \Sigma_k^{-1} x, 1 - b_k^T \Sigma_k^{-1} b_k \right) - \Phi \left( t_{k-1}; b_k^T \Sigma_k^{-1} x, 1 - b_k^T \Sigma_k^{-1} b_k \right).
$$

Multiplying (4.22) and (4.23) by $p_k$ and summing over $k$ we get that the unconditional cdf and pdf of $X$ are given by

$$
\mathbb{P}(X \leq x) = \sum_{k=1}^{K} \Phi_{n+1} \left( \begin{bmatrix} t_k \ x \ \ 0 \ \ \ b_k^T \\ \Sigma_k \end{bmatrix} \right) - \Phi_{n+1} \left( \begin{bmatrix} t_{k-1} \ x \ \ 0 \ \ \ b_k^T \\ \Sigma_k \end{bmatrix} \right)
$$

and

$$
\mathbb{P}(X \in dx) = \sum_{k=1}^{K} \phi_n(x; 0, \Sigma_k) \cdot s_k(x) \, dx,
$$

respectively. Note that in the absence of state dependence (i.e. if $\beta = 0$) then $s_k(x) = p_k$ for all $x$ and (4.25) reduces to a mixture of $n$-dimensional Gaussian densities.

4.4 Large portfolio default rate

Suppose that the realized value of the systematic risk factor is $m$, and that regime $k$ prevails. Then the default indicators $1(X_1 \leq x_q), 1(X_2 \leq x_q), \ldots$, are i.i.d. Bernoulli variables with success probability

$$
\mathbb{P}(X_i \leq x_q | M = m, R = k) = \Phi \left( x_q; a_k m, 1 - a_k^2 \right) = \nu_k(m),
$$

where
where
\[ v_k(\cdot) := v_{X_k, \alpha_k}(\cdot) \]  
(4.26)
is a Vasicek curve. The implication is that the large-portfolio default rate \( D \) is given by
\[
D := \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}(X_i \leq x_q) = \mathbb{P}(X_i \leq x_q | R, M) = v_p(M) = \sum_{k=1}^{K} v_k(M) \cdot \mathbf{1}(R = k).
\]
Note that the large portfolio default rate is a function of both \( M \) (a continuous variable) and \( R \) (a discrete variable). Note also that the unconditional (i.e. long-run, or through-the-cycle) default rate is
\[
\mathbb{E}[D] = \mathbb{E}[\mathbb{P}(X_i \leq x_q | M, R)] = \mathbb{P}(X_i \leq x_q) = q,
\]
(4.27)
which is as expected.

### 4.4.1 Default rate conditional on systematic risk factor

Given the realized value of the systematic risk factor \((m, \text{say})\), the large-portfolio default rate is a discrete variable taking one of the values \( v_1(m), v_2(m), \ldots, v_K(m) \) with respective probabilities \( p_1(m), p_2(m), \ldots, p_K(m) \). The mean default rate is therefore
\[
\mathbb{E}[D | M = m] = \sum_{k=1}^{K} v_k(m) \cdot p_k(m),
\]
(4.28)
The relationship between the systematic risk factor and the large-portfolio default rate is summarized by the function \( m \mapsto \mathbb{E}[D | M = m] \), and it is interesting to note that this function is a weighted average of Vasicek curves.

Recall that our numerical example consists of (i) the two-regime state dependent model and (ii) the Vasicek model, both fitted to the same set of data. Conditional on the realized value of the systematic risk factor, the large-portfolio default rate in (i) is a random variable taking on one of two possible values, whereas the large-portfolio default rate in (ii) is known with certainty. More precisely, if the realized value of the systematic risk factor is \( m \) then the large-portfolio default rate in the state dependent model is either \( v_1(m) \) or \( v_2(m) \), whereas it is certain to be \( v(m) \) in the state dependent model. Here \( v_1 \) and \( v_2 \) are Vasicek curves with factor loadings \( a_1 = 0.25 \) and \( a_2 = 0.16 \), respectively, and a common default threshold of \( x = -1.80 \), while \( v \) is a Vasicek curve with factor loading \( a = 0.22 \) and default threshold \( x = -1.79 \). Figure 4.3(a) illustrates the curves \( v_1 \) (solid line), \( v_2 \) (dashed line) and \( v \) (dash-dot line). Loosely speaking, the state dependent model will produce a higher default rate than the Vasicek model if either (i) the first regime (where correlation is higher) is realized during a sufficiently adverse scenario or (ii) the second regime (where correlation is lower) is realized during a sufficiently favourable scenario.

Figure 4.1(a) illustrates that \( p_1(m) = 1 \) for \( m < -1 \), and Figure 4.3(a) illustrates that \( v_1(m) > v(m) \) for \( m < -1 \). Thus, the state dependent model is virtually certain to produce a higher default rate if the economy is at least one standard deviation below average. Similarly, the state dependent model is virtually certain to produce a higher default rate if the economy is at least two standard deviations above average. In other words, the default rate will be higher in the state dependent model during extreme economic scenarios, be they good or bad. This is reflected in Figure 4.3(b), which compares the expected default rate \( \mathbb{E}[D | M = m] = p_1(m)v_1(m) + p_2(m)v_2(m) \) in the state dependent model to the default rate \( v(m) \) in the Vasicek model.

**Remark 4.4.** In “typical” economic environments (those that are no more than one standard deviation away from average), the average default rate is lower in the state dependent model than it is in the Vasicek model. Otherwise, if the economic environment is at least one standard deviation above or below average, the default rate is higher in the state dependent model.
Finally, Figure 4.3(c) illustrates how the two curves $v_1$ (the high-correlation curve) and $v_2$ (the low-correlation curve) are spliced together to produce the mean default rate $\mathbb{E}[D|M = m]$. The average default rate is indistinguishable from the high-correlation curve for $m < 0.5$, and indistinguishable from the low-correlation curve for $m > 1.75$. In between, the average default rate is relatively constant at approximately 2%.

![Figure 4.3](image)

**Figure 4.3:** Panel (a) illustrates the curves $v_1(\cdot)$ (solid line), $v_2(\cdot)$ (dashed line) and $v(\cdot)$ (dash-dot line) in our numerical example. The interpretations of these curves are as follows: given that the realized value of the systematic is $m$, the default rate in the state dependent model will either be $v_1(m)$ (with probability $p_1(m)$) or $v_2(m)$ (with probability $p_2(m)$), whereas in the Vasicek model it will be $v(m)$ with certainty. Here $v_1$ and $v_2$ are Vasicek curves with factor loadings $a_1 = 0.25$ and $a_2 = 0.16$, respectively, and common default threshold $x = -1.80$, while $v$ is a Vasicek curve with factor loading $a = 0.22$ and default threshold $x = -1.79$. Panel (b) compares the Vasicek default rate $v(m)$ (dashed line) to the expected default rate in the state dependent model, namely $\mathbb{E}[D|M = m] = p_1(m)v_1(m) + p_2(m)v_2(m)$ (solid line). Panel (c) illustrates how $v_1$ and $v_2$ are spliced together to produce $\mathbb{E}[D|M = m]$.

### 4.4.2 Default rate conditional on correlation regime

Given that regime $k$ prevails, the large-portfolio default rate is $D = v_k(M)$. Conditional on the correlation regime, then, the default rate is a continuous random variable and its probability density function in regime
is a Vasicek density.

Figure 4.4 illustrates the conditional density of the large-portfolio default rate in each regime, in the context of our numerical example. We see that the distribution of the default rate in the first regime is more concentrated on large default rates, whereas it is more concentrated on lower default rates in the second regime. The reason is that the systematic risk factor tends to take on more adverse values during the first regime (recall Figure 4.1(b)), and that the large-portfolio default rate tends to be higher when the systematic risk factor takes on adverse values (recall Figure 4.3(c)).

The mean of the default rate in regime $k$ can be computed in closed form. Indeed, we have

$$
\mathbb{E}[D|R = k] = \mathbb{E}[v_k(M)|R = k] = p_k^{-1}\mathbb{E}[v_k(M)p_k(M)],
$$

where we have used the identity (4.3). Now, $\mathbb{E}[v_k(M)p_k(M)]$ can be put in closed form using (1.1) and (1.5). The end result is

$$
\mathbb{E}[D|R = k] = p_k^{-1}[\Phi_2(x_k; 0, \Sigma_k) - \Phi_2(x_{k-1}; 0, \Sigma_k)],
$$

where

$$
\begin{align*}
x_k &= \begin{bmatrix} t_k \\
x_d \end{bmatrix}, \\
0 &= \begin{bmatrix} 0 \\
0 \end{bmatrix}, \\
\Sigma_k &= \begin{bmatrix} 1 & \beta \alpha_k \\
\beta \alpha_k & 1 \end{bmatrix}.
\end{align*}
$$

Recall that if $X$ is a random variable with density $f_X$ and $Y = h(X)$ for some decreasing function $h$, then the density of $Y$ is $f_Y(y) = -f_X(h^{-1}(y))/h'(h^{-1}(y))$.  

---

**Figure 4.4**: This figure illustrates the conditional density of the large-portfolio default rate in each regime, in the context of our numerical example. Specifically, it graphs $g_1(\cdot)$ (the conditional density in the first regime, solid line) and $g_2(\cdot)$ (the conditional density in the second regime, dashed line), where the expression for $g_k(\cdot)$ is given by (4.29). We see that the first-regime density is relatively more concentrated on large default rates.
In the absence of state dependence the mean default rate does not depend on the correlation regime, in particular \( \mathbb{E}[D|R = k] = q \) for every \( k \). By contrast, in the presence of state dependence mean default rates can vary widely across correlation regimes. In our numerical example the average default rate in the first regime is 4% (slightly higher than the long-run average default rate of 3.7%) whereas the default rate in the second regime is 2% (half the long-run rate). The default rate tends to be higher in the first regime, since the systematic risk factor tends to take on more adverse values there.

### 4.5 Default rate distribution

The unconditional density of the large-portfolio default rate is easily recovered from the conditional densities \( g_k \) given in (4.29). Indeed, if \( g(d) \) denotes the unconditional density of \( D \), then

\[
g(d) = \sum_{k=1}^{K} g_k(d) \cdot p_k = \sum_{k=1}^{K} f_k(d) \cdot p_k(v_k^{-1}(d)).
\]  

(4.31)

Important features of this distribution, such as tail probabilities and tail expectations, can be computed in closed form.

In order to compute tail probabilities, we use the fact that \( D = v_k(M) \) in regime \( k \) and the identity (4.3) to write

\[
P(D \geq d|R = k) = \mathbb{E}[\mathbf{1}(M \leq v_k^{-1}(d))|R = k] = p_k^{-1}\mathbb{E}[\mathbf{1}(M \leq v_k^{-1}(d)) \cdot p_k(M)].
\]

Now, \( \mathbb{E}[\mathbf{1}(M \leq v_k^{-1}(d)) \cdot p_k(M)] \) can be evaluated using (1.3), and the end result is

\[
P(D \geq d|R = k) = p_k^{-1}[\Phi_2(u_k; 0, \Delta) - \Phi_2(v_k; 0, \Delta)],
\]

where

\[
u_k = \begin{bmatrix} v_k^{-1}(d) \\ t_k \end{bmatrix}, \quad w_k = \begin{bmatrix} v_k^{-1}(d) \\ t_{k-1} \end{bmatrix}, \quad \Delta = \begin{bmatrix} 1 & \beta \\ \beta & 1 \end{bmatrix}.
\]

Finally, using the fact that \( P(D \geq d) = \sum_{k=1}^{K} P(D \geq d|R = k) \cdot P(R = k) \), we get

\[
P(D \geq d) = \sum_{k=1}^{K} [\Phi_2(u_k; 0, \Delta) - \Phi_2(v_k; 0, \Delta)].
\]  

(4.32)

Note that if \( \beta = 1 \), then (4.32) reduces to

\[
P(D \geq d) = \sum_{k=1}^{K} \left[ \Phi(\min(v_k^{-1}(d), t_k)) - \Phi(\min(v_k^{-1}(d), t_{k-1})) \right].
\]

In order to compute tail expectations, it remains to compute the quantity \( \mathbb{E}[D \cdot \mathbf{1}(D \geq d)] \), and we do use using an approach that is similar to the one used in the previous paragraph. We begin with

\[
\mathbb{E}[D \cdot \mathbf{1}(D \geq d)|R = k] = p_k^{-1}\mathbb{E}[v_k(M) \cdot p_k(M) \cdot \mathbf{1}(M \leq v_k^{-1}(d))],
\]

which can be evaluated in closed form using (1.1) and then (1.3). The end result is

\[
\mathbb{E}[D \cdot \mathbf{1}(D \geq d)|R = k] = p_k^{-1}[\Phi_3(y_k; 0, \Lambda_k) - \Phi_3(z_k; 0, \Lambda_k)],
\]

where

\[
y_k = \begin{bmatrix} v_k^{-1}(d) \\ x_q \\ t_k \end{bmatrix}, \quad z_k = \begin{bmatrix} v_k^{-1}(d) \\ x_q \\ t_{k-1} \end{bmatrix}, \quad \Lambda_k = \begin{bmatrix} 1 & a_k & \beta \\ a_k & 1 & a_k \beta \\ \beta & a_k \beta & 1 \end{bmatrix}.
\]

Thus,

\[
\mathbb{E}[D \cdot \mathbf{1}(D \geq d)] = \sum_{k=1}^{K} [\Phi_3(y_k; 0, \Lambda_k) - \Phi_3(z_k; 0, \Lambda_k)].
\]  

(4.33)

Combining (4.32) and (4.33), we finally get that tail expectations are given by

\[
\mathbb{E}[D|D \geq d] = \frac{\sum_{k=1}^{K} [\Phi_3(y_k; 0, \Lambda_k) - \Phi_3(z_k; 0, \Lambda_k)]}{\sum_{k=1}^{K} [\Phi_2(u_k; 0, \Delta) - \Phi_2(v_k; 0, \Delta)]}.
\]  

(4.34)
4.6 Quantiles and tail expectations

Let $d_p$ denote the 100th percentile of the default rate, i.e. $d_p$ satisfies $P(D \geq d_p) = 1 - p$. It does not appear possible to invert (4.32) in closed form, so $d_p$ must in general be computed numerically. Since $d_p$ is, for fixed $p$, the unique root of the function $d \mapsto P(D \geq d) - (1 - p)$, numerical computation of $d_p$ is straightforward. Figure 4.5 illustrates high percentiles of the default rate in our numerical example. As expected, percentiles in the state dependent model are considerably larger than those in the Vasicek model. For instance at the 99.9% level, the state dependent percentile is 14.4% as compared to a Vasicek percentile of 12.8%. In relative terms, difference is nearly 13%. At the 99.99% level the state dependent and Vasicek percentiles are 16.0% and 18.4%, respectively, a difference of more than 15% in relative terms.

Figure 4.5: Panel (a) illustrates percentiles of the large-portfolio default rate in our numerical example. The 100th percentile is denoted $d_p$ and satisfies $P(D \geq d_p) = 1 - p$. In the Vasicek model $d_p = v_x(z_{1-p})$, where $z_p$ is the 100th percentile of the standard Gaussian distribution. In the state dependent model $d_p$ is computed by numerically finding the root of the function $d \mapsto P(D \geq d) - (1 - p)$, and we use (4.32) to compute $P(D \geq d)$. Panel (b) illustrates tail expectations $E[D|D \geq d_p]$. Quantiles are first computed as in panel (a), tail expectations are then computed using (4.34).

Figure 4.5(b) compares the tail expectations $E[D|D \geq d_p]$ between the two models. Note that $E[D|D \geq d_p]$ represents the average default rate under the most adverse 100(1 - p)% default rate scenarios. At the 99.9% level (i.e. averaging over the worst 0.1% of default rate scenarios) the Vasicek and state dependent tail expectations are 14.2% and 16.2%, respectively, at the 99.99% level they are 17.3% and 20.2%. The Vasicek model considerably underestimates tail expectations relative to the state dependent model.

5 Empirical analysis

In this section we estimate the parameters of the proposed model, using publicly available Federal Reserve data on delinquency rates for various types of loans. There are eleven categories of loans for which data is available: all loans (AL), business loans (BL), loans secured by real estate (SRE), lease financing receivables (LFR), other consumer loans (OCL), commercial real estate (excluding farmland) loans (CRE), farmland loans (F), loans to finance agricultural production (AP), single-family residential mortgages (RM), consumer loans

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2 The data was obtained from https://www.federalreserve.gov/releases/chargeoff/
(CL) and credit card loans (CCL). Loans are considered delinquent if they are 30 more days over due, and
delinquency rates are aggregated across the largest 100 financial institutions. The data is seasonally adjusted,
and consists of quarterly delinquency rates expressed on an annual basis. One series (AL) ranges from the
first quarter of 1985 to the third quarter of 2019 (115 observations), five series (AP, BL, SRE, CL, LFR) range
from the first quarter of 1987 to the third quarter of 2019 (131 observations), and the remaining series (RM,
CCL, OCL, CRE, F) range from the first quarter of 1991 to the third quarter of 2019 (115 observations).

For each time series we use maximum likelihood to estimate the parameters of each of the following
models.

(i) The Vasicek model. This is a two-parameter model with parameter vector \( \theta = (x, a) \).
(ii) A two-regime model with stochastic, but state-independent, correlation. This is a four-parameter model
with parameter vector \( \theta = (x, a_1, a_2, p_1) \). Note that \( \beta = 0 \) here.
(iii) A two-regime model with state dependent correlation. This is a five-parameter model with parameter
vector \( \theta = (x, a_1, a_2, p_1, \beta) \).
(iv) A three-regime model with state dependent correlation. This is a seven-parameter model with parameter
vector \( \theta = (x, a_1, a_2, a_2, p_1, p_2, \beta) \).

### 5.1 Estimation procedure

We consider the observed sequence of delinquency rates \( d_1, d_2, \ldots, d_n \) as i.i.d. drawings from the large-
portfolio default rate density \( g(d) \) given in (4.31). The assumption of temporal independence is admittedly
strong, but temporal dynamics are well beyond the scope of this paper.

For the Vasicek model, maximum likelihood estimates can be obtained in closed form. Indeed, the vari-
able \( Z = \Phi^{-1}(D) \) is Gaussian with mean \( x/\sqrt{1 - a^2} \) and variance \( a^2/(1 - a^2) \). Given the observed sequence of
delinquency rates \( d_1, \ldots, d_n \) we let \( \hat{\mu} \) and \( \hat{\sigma} \) denote the sample mean and standard deviation of \( z_1, z_2, \ldots, z_n \),
where \( z_k = \Phi^{-1}(d_k) \). Estimates of \( x \) and \( a \) are then given by \( \hat{x} = \hat{\mu}/\sqrt{1 + \hat{\sigma}^2} \) and \( \hat{a} = \hat{\sigma}/\sqrt{1 + \hat{\sigma}^2} \). Maximum
likelihood estimates of default probability and correlation are then \( \hat{q} = \Phi^{-1}(x) \) and \( \hat{\rho} = \hat{\sigma}^2 \).

For the other models we use \textit{fmincon} in Matlab to numerically minimize the function

\[
\ell(\theta) = -\frac{1}{n} \sum_{i=1}^{n} \log(g_{\theta}(d_i)),
\]

where \( g_{\theta} \) is that version of the density (4.31) with parameter vector \( \theta \). The function \( \ell \) is simply \( n^{-1} \)
times the negative of the log-likelihood function (the normalization \( n^{-1} \) is introduced to ensure numerical stability
with respect to the number of observations). In an effort to identify the global minimum of (5.1) we run the
algorithm 500 different times, using 500 randomly generated initial points. This produces a set of candidate
estimates \( \hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_{500} \), where \( \hat{\theta}_k \) is the point to which the algorithm converged (or otherwise terminated)
after being initialized at the \( k \)th initial point. The maximum likelihood estimate is then taken to be that point
\( \hat{\theta}_k \) that produces the smallest value of \( \ell(\hat{\theta}_k) \).

When fitting model (ii) with \textit{fmincon} we constrain \( a_1, a_2 \) and \( p_1 \) to lie in the unit interval \([0, 1]\) but do
not impose any constraints on the default threshold \( x \). For model (iii) we impose the additional constraint
that \( \beta \) lie in the unit interval but we do not impose any restriction on the relationship between \( a_1 \) and \( a_2 \) (e.g.
we do not impose the restriction \( a_1 > a_2 \)). Finally, for model (iv) we impose the additional constraints that
\( p_2 \), \( a_3 \) and \( p_1 + p_2 \) lie in the unit interval, and we do not impose any restrictions on the relationship between
\( p_1, p_2 \) or \( a_1, a_2, a_3 \).

To protect against overfitting we also impose a non-linear constraint designed to ensure that the function
\( m \rightarrow E[D|M = m] \) is continuous when \( \beta = 1 \). Specifically, we insist that \( v_k(t_k) = v_{k+1}(t_k) \) for \( 1 \leq k \leq K - 1 \).
In order to understand why this might protect against overfitting, consider the two-regime case and let
\( d_1 = v_1(t_1) \) and \( d_2 = v_2(t_1) \). If \( d_1 > d_2 \) then according to the model it is never possible to observe default rates in
the interval \([d_2, d_1]\). And if \( d_2 > d_1 \) then the default rate is not a monotone function of the systematic risk
factor (which seems counterintuitive), and the effect is that “extra” mass is placed on the interval \([d_1, d_2]\). In
either case, this strikes us as an opportunity for the algorithm to overfit by placing too much or too little mass on ranges that are either over- or under-represented in the data.

5.2 Model performance

The Aikake Information Criteria (AIC) can be used to gauge the empirical performance of each model. AIC values for each model and time series are reported in Table 5.1. The two most striking features of the data are that (i) the model with stochastic, but state independent, correlation is the worst-performing model in every case except for one (credit card loans) and (ii) the state dependent model with two regimes outperforms the Vasicek model in every case.

**Remark 5.1.** Together, these observations suggest that simply randomizing correlation, without allowing for state dependence, does not improve the performance of the Vasicek model.

Table 5.1: This table reports Aikake Information Criteria (AIC) values for each model and time series. Recall that the model with the smallest AIC value is typically interpreted as the best-performing model. The column labelled Mixture corresponds to the model with stochastic, but state-independent, correlations.

| Series                                | Vasicek | Mixture | Two Regime | Three Regime |
|---------------------------------------|---------|---------|------------|--------------|
| Other Consumer Loans                  | −837    | −833    | −839       | −846         |
| Lease Financing Receivables           | −1026   | −1022   | −1027      | −1023        |
| Farmland                              | −622    | −618    | −626       | −621         |
| All Loans                             | −739    | −735    | −759       | −766         |
| Business Loans                        | −743    | −739    | −750       | −748         |
| Secured by Real Estate                | −590    | −586    | −602       | −605         |
| Commercial Real Estate                | −505    | −501    | −511       | −513         |
| Agricultural Production               | −626    | −622    | −634       | −632         |
| Residential Mortgages                 | −525    | −521    | −566       | −569         |
| Consumer Loans                        | −877    | −873    | −892       | −900         |
| Credit Card Loans                     | −685    | −694    | −695       | −703         |

Table 5.1 also suggests that the state dependent model with three regimes is not uniformly superior to the state dependent model with three regimes. Because it offers comparable performance and is more parsimonious, we would argue that the two-regime state dependent model is the foremost of the four models we consider. For this reason, and in the interest of brevity, we focus on the two-regime state dependent models in what follows.

The statistical significance of the AIC values reported in Table 5.1 can be assessed using the likelihood ratio test. Table 5.2 reports the p-values (in percentage points) associated with various hypotheses. For example, the first column of the table reports p-values corresponding to the test of the Vasicek (null) model against the state dependent (alternative) model. At the 10% level of significance, the null hypothesis (i.e. the hypothesis that the Vasicek model provides a superior fit to the data) would be rejected for every series, at the 5% level it would be rejected for all but one of the series and at the 1% level it would be rejected for all but three of the series. Overall the results strongly suggest that differences observed in Table 5.1 are statistically

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3 See Section 12.4.4 in [16] for a thorough and rigorous discussion of the likelihood ratio test.
significant. In particular, they support the observation that the state dependent model, with either two or three states, provides a superior fit to the data as compared to the Vasicek model.

Table 5.2: This table reports $p$-values, in percentage points, associated with the likelihood ratio test of various hypotheses. Each column corresponds to a different hypothesis. For example, in the first column the null (constrained) model is the Vasicek and the alternative (unconstrained) model is the state dependent model with two states. Recall that a small $p$-value suggests that the alternative (unconstrained) model is a better fit to the data than the null (constrained) model. For the Lease Financial Receivables and Farmland series, the two state model produced a likelihood that was at least as large as the three state model, resulting in the $p$-values of 100% observed in the last column for those series.

|                  | $H_0$ | Vasicek | Vasicek | Two State |
|------------------|-------|---------|---------|-----------|
|                  | $H_1$ | Two State | Three State | Three State |
| Other Consumer Loans |       | 3.8 | 0.2 | 0.5     |
| Lease Financing Receivables |   | 7.2 | 22.1 | 100.0 |
| Farmland          |       | 2.5 | 10.5 | 100.0 |
| All Loans         |       | $9.5 \times 10^{-4}$ | $6.0 \times 10^{-5}$ | $4.1 \times 10^{-1}$ |
| Business Loans    |       | 0.5 | 1.1 | 38.1 |
| Secured by Real Estate |   | $5.3 \times 10^{-2}$ | $2.0 \times 10^{-2}$ | $3.6 \times 10^{0}$ |
| Commercial Real Estate | | 0.8 | 0.3 | 4.2 |
| Agricultural Production | | 0.3 | 0.7 | 3.7 |
| Residential Mortgages | | $3.0 \times 10^{-8}$ | $2.2 \times 10^{-8}$ | $3.7 \times 10^{0}$ |
| Consumer Loans    |       | $1.2 \times 10^{-2}$ | $4.7 \times 10^{-4}$ | $2.8 \times 10^{-1}$ |
| Credit Card Loans |       | $1.7 \times 10^{-1}$ | $4.8 \times 10^{-3}$ | $2.2 \times 10^{-1}$ |

5.3 Parameter estimates

Table 5.3 illustrates the estimated parameters for our best-performing model, namely the state dependent model with two regimes. For comparative purposes, estimates for the Vasicek model are also presented. Three distinct groups are apparent in the table. The first group accounts for seven of the eleven series, and is characterized by the facts that correlation is higher in the first regime than it is in the second ($\rho_1 > \rho_2$) and that the first regime is (considerably) more likely than the second ($p_1 > p_2$). The second group (two series) is characterized by the facts that correlation is higher in the first regime ($\rho_1 > \rho_2$) and that the first regime is (considerably) less likely than the second ($p_1 < p_2$). The third group (two series) is characterized by the fact that correlation is lower in the first regime ($\rho_1 < \rho_2$) and that the two regimes are equally likely ($p_1 = p_2$).
were non-random and homogeneous across exposures, then the percentage difference in
the fitted two-regime state dependent model. More precisely, the table reports
regulatory capital might compare.

Recall from Section 1.3 that the regulatory capital on a given portfolio is defined as the difference between
the (i) 99th percentile of the large-portfolio default rate in the Vasicek model with parameters as reported in Table 5.3 and those from
the state dependent model with two regimes

The two most striking features of the data in Table 5.3 are that (i) the estimated degree of state dependence
is very high in every case (the median estimate is 93%, the smallest estimate is 72%) and that (ii) correlation
tends to be higher during adverse scenarios for a strong majority of the data (together the first two groups
account for over 80% of the data). In the introduction we noted that correlations between many asset classes
tend to rise during adverse economic scenarios, and wondered whether the same was true of the late credit
quality variables that underpin the Vasicek model. The results presented in Table 5.3 answer this question in
the affirmative.

5.4 Implications for risk management

Recall from Section 1.3 that the regulatory capital on a given portfolio is defined as the difference between
the (i) 99th percentile and (ii) mean, of the portfolio loss distribution. If we make the (admittedly strong)
assumptions that all exposures are the same size and that recovery rates are non-random and homogeneous
across exposures then portfolio loss is proportional to the portfolio’s default rate, in which case regulatory
capital on the portfolio is proportional to the difference between the (i) 99th percentile and (ii) mean, of
the default rate. Now, the mean default rate predicted by both the state dependent and Vasicek models are
nearly identical, so a comparison of model-implied high percentiles can give us a sense of how model-implied
regulatory capital might compare.

Table 5.4 reports the relative difference between percentiles in the fitted Vasicek model, and those from
the fitted two-regime state dependent model. More precisely, the table reports \( \left( d_{sd}(p) - d_{v}(p) \right)/d_{v}(p) \), where
\( d_{v}(p) \) is the 100\( p \)th percentile of the large-portfolio default rate in the Vasicek model with parameters as
reported in Table 5.3 and \( d_{sd}(p) \) is the corresponding percentile in the state dependent model with two regimes
(also with parameters as reported in Table 5.3). Note that if exposures were equally-sized and recovery rates
were non-random and homogeneous across exposures, then the percentage difference in 99.9th percentiles
would correspond exactly to the percentage difference in model-implied regulatory capital.
State dependent correlations in the Vasicek default model

| Series                        | 99% Confidence Level | 99.9% Confidence Level |
|-------------------------------|----------------------|------------------------|
| Other Consumer Loans          | 2.1                  | 2.9                    |
| Lease Financing Receivables   | 4.3                  | 6.3                    |
| Farmland                      | 12.2                 | 17.9                   |
| All Loans                     | 5.7                  | 8.4                    |
| Business Loans                | 9.0                  | 12.7                   |
| Secured by Real Estate        | 3.5                  | 5.0                    |
| Commercial Real Estate        | 2.4                  | 3.4                    |
| Agricultural Production       | 21.8                 | 34.1                   |
| Residential Mortgages         | 26.1                 | 50.9                   |
| Consumer Loans                | -12.2                | -16.9                  |
| Credit Card Loans             | -14.3                | -19.7                  |

The same three groups that emerged in Table 5.3 also emerge in Table 5.4. For the first group, the Vasicek model underestimates high percentiles (relative to the state dependent model) to a moderate degree. For the second group, Vasicek underestimates high percentiles to a high degree, and for the third group it actually overestimates high percentiles.

**Remark 5.2.** The most striking feature of the data in Table 5.4 is that Vasicek underestimates high percentiles whenever correlation tends to be higher during adverse scenarios (i.e. for every series for which \( a_1 > a_2 \)), and that the underestimate is severe if the high-correlation regime is relatively unlikely (i.e. if \( p_1 < p_2 \)). The clear implication is that if the state dependent model is a more faithful representation of the data (and Table 5.1 suggests that it is), then the Vasicek model will tend to underestimate regulatory capital, possibly to a very high degree.

### 6 Implied copula

Recall (Section 4.3.2) that the joint cdf of \( X = (X_1, X_2, \ldots, X_n)^T \) is

\[
F_n(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{K} \Phi_{n+1} \left( \begin{bmatrix} t_k & 0 & 1 & b_k^T \Sigma_k \end{bmatrix} ; \begin{bmatrix} 0 & 0 \end{bmatrix} \right) - \Phi_{n+1} \left( \begin{bmatrix} t_{k-1} & 0 & 1 & b_k^T \Sigma_k \end{bmatrix} ; \begin{bmatrix} 0 & 0 \end{bmatrix} \right),
\]

where

\[
x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b_k = \begin{bmatrix} \beta a_k \\ \beta a_k \\ \vdots \\ \beta a_k \end{bmatrix}, \quad \Sigma_k = \begin{bmatrix} 1 & a_k^2 & \cdots & a_k^2 \\ a_k^2 & 1 & \cdots & a_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_k^2 & a_k^2 & \cdots & 1 \end{bmatrix}.
\]

Note that \( b_k \) and \( \Sigma_k \) are \( n \times 1 \) and \( n \times n \), respectively. The copula of \( X \) is therefore

\[
C_n(u_1, u_2, \ldots, u_n) := F_n(F^{-1}(u_1), \ldots, F^{-1}(u_n)),
\]

where \( F = F_1 \) is the marginal cdf of \( X_i \). Recall (Section 4.3.1) that in the presence of state dependence (i.e. if \( \beta > 0 \)), \( F^{-1}(u) \) must be computed numerically and that \( \Phi^{-1}(u) \) typically provides a good initial guess.
Of particular interest in risk management and credit derivative pricing are the coefficients of tail dependence. Recall that in the bivariate case the coefficient of lower tail dependence is defined as

$$\lim_{u \to 0} P(X_i \leq F^{-1}(u)|X_j \leq F^{-1}(u)),$$

where $i \neq j$. Figure 6.1 illustrates the small-$u$ behaviour of

$$P(X_i \leq F^{-1}(u)|X_j \leq F^{-1}(u)) = \frac{C_2(u,u)}{u}$$

in the presence of state dependence. Numerical evidence suggests that (i) the coefficient of lower tail dependence is zero, but that (ii) the quantities $C_2(u,u)/u$ decay at a much slower rate in the presence of state dependence, than they do in the case of the Gaussian copula.

![Figure 6.1](image)

Figure 6.1: Panel (a) illustrates conditional tail probabilities of the form $C(u,u)/u$, where $C$ is a copula function. Recall that if $(X, Y)$ has copula $C$, then $C(u,u)/u = P(X \leq x_u|Y \leq y_u)$, where $x_u$ and $y_u$ are the 100$u^{th}$ percentiles of $X$ and $Y$, respectively. Solid line corresponds to the proposed model, in which case copula is given by (6.1) with $n = 2$. Parameters are $(a_1, a_2) = (0.25, 0.04), (p_1, p_2) = (0.25, 0.75)$ and $\beta = 0.90$. Dashed curve corresponds to two-dimensional Gaussian copula with correlation $0.23 = (a_1 p_1 + a_2 p_2)^2$. Panel (b) depicts the ratio of the conditional tail probabilities (i.e. the ratio of the solid curve to the dashed curve, from Panel (a)) on the log-log scale.

### 6.1 Conditional hazard function

Now let $h(t)$ be some pdf and let $H(t)$ be the corresponding cdf, and suppose we wish to construct a vector $\mathbf{T} = (T_1, T_2, \ldots, T_n)^T$ such that (i) the marginal distribution of $T_i$ is $H$ and (ii) the copula of $\mathbf{T}$ is $C_n$. Such an exercise is common in the credit derivatives literature, where the components of $\mathbf{T}$ represent the default times of $n$ exposures and the user is interested in pricing or hedging some derivative whose payoff is a function of $\mathbf{T}$. The desired vector $\mathbf{T}$ can be constructed by setting

$$T_i = H^{-1}(F(X_i)),$$

and the resulting joint density of $\mathbf{T}$ is

$$h_n(t_1, \ldots, t_n) := f_n(G(t_1), \ldots, G(t_n)) \cdot \prod_{i=1}^n G_i'(t_i),$$

where $G(t) := F^{-1}(H(t))$ and $G_i'(t) = h(t)/f(G(t))$. 
The conditional hazard function of $T_1$, given $T_2$, is of particular interest in credit derivatives modelling. It is defined as

$$\lambda(t_1 | t_2) := -\frac{d}{dt_1} \log \left( \mathbb{P}(T_1 > t_1 | T_2 = t_2) \right),$$

(6.3)

and we note that conditional survival probabilities can be recovered via

$$\mathbb{P}(T_1 > t_1 | T_2 = t_2) = \exp\left(-\int_{0}^{t_1} \lambda(t | t_2) \, dt\right).$$

(6.4)

It is not hard to show that in the present context, we have

$$\lambda(t_1 | t_2) = f(G(t_1), G(t_2)) \cdot G'(t_1) \int_{G(t_1)} f(x_1, G(t_2)) \, dx_1,$$

(6.5)

where $f(x_1, x_2)$ is the joint density of $(X_1, X_2)$.

Figure 6.2 illustrates the conditional hazard function (6.3) in both the proposed model and the Gaussian copula model, assuming the marginal distribution of $T_i$ is exponential. In panel (a) the second name defaults much earlier than expected, in the sense that the realized value of $T_2$ is equal to the 10th percentile of its distribution. We see (unsurprisingly) that in both models, an early default of one name causes the entire hazard curve for the other name to shift upwards. At the short end of the term structure (i.e. for sufficiently small values of $t_1$) the impact of the early default on the conditional hazard rate is larger in the proposed model, than it is in the Gaussian copula model. The implication is that the conditional survival probability (6.4) is smaller in the proposed model than it is in the Gaussian copula model, for sufficiently small values of $t_1$. Numerical integration of the conditional hazard curves appearing in the figure reveals that “sufficiently small” here means $t_1 \leq 7.2\ldots$, at which point the conditional survival probability in both models is approximately 1.14%; i.e. sufficiently small here is actually quite deep in the right tail of the conditional distribution of $T_1$. We conclude that the impact of the early default in the proposed model is more pronounced than it is in the Gaussian copula model.

In panel (b) of Figure 6.2 the second name defaults much later than expected, in the sense that the realized value of $T_2$ is equal to the 90th percentile of its distribution. Unsurprisingly, the late default of the second name causes the entire hazard curve of the first name to shift downwards in both models. As in panel (a) the impact at the short end of the term structure is larger in the proposed model, as compared to the Gaussian copula model. The implication is that conditional survival probabilities (6.4) are higher in the proposed model, than they are in the Gaussian model, for sufficiently small values of $t_1$. Numerical integration reveals that “sufficiently small” corresponds to $t_1 \leq 4.5\ldots$, at which point the conditional survival probability in both models is approximately 16.2%. As in the case of the early default, we conclude that the impact of the early default in the proposed model is more pronounced than it is in the Gaussian copula model.
Figure 6.2: This figure illustrates the conditional hazard function of $T_1$, given that $T_2 = t_2$, assuming that both $T_1$ and $T_2$ are exponentially distributed with rate $\lambda = 0.5$ (other parameter values are as in Figure 6.1). The conditional hazard function is defined in (6.3), and we compute it using (6.5). The solid line corresponds to the proposed model, the dashed line corresponds to the Gaussian copula model and the dash-dot line corresponds to the unconditional hazard rate (which is the same in both models). Panel (a) corresponds to the case where the second name defaults much earlier than expected, in the sense that the realized value of $T_2$ is equal to the 10th percentile of its marginal distribution (i.e. it depicts $\lambda(t_1|0.21)$). Panel (b) corresponds to the case where the second name defaults much later than expected, in the sense that the realized value of $T_2$ is equal to the 90th percentile of its marginal distribution (i.e. it depicts $\lambda(t_1|0.61)$).

7 Conclusion

This paper incorporates state dependent correlations (those that vary systematically with the state of the economy) into the Vasicek default model. This is accomplished by expressing the factor loading as a function of an auxiliary (Gaussian) variable that is correlated with the systematic risk factor; the degree to which the two are correlated can be interpreted as the degree of state dependence. We show how to implement the model in the special case where the factor loading is a discrete random variable, and illustrate that ignoring state dependence when it is in fact present will tend to underestimate important risk measures such as high percentiles and tail expectations.

We fit several different models to Federal Reserve data on delinquency rates, and compare their performance according to the Akaike Information Criteria (AIC). We find that a state-dependent model with two correlation regimes outperforms the traditional Vasicek model, and that the estimated degree of state dependence is very high across all loan types. Importantly, we also find the traditional Vasicek model outperforms a model with stochastic, but state-independent, correlations. In other words, randomizing correlation without allowing for state dependence does not improve the statistical performance of the Vasicek model. We also find that failure to incorporate state dependence, when it is in fact present, can lead one to underestimate important risk measures. The most pressing avenue for future research is to incorporate realistic time dynamics for the systematic risk factor.

Appendix A

In this appendix we derive the properties (P1), (P2) and (P3) that appear in the introductory paragraphs of Section 3.
Property (P1) states that \( \mathbb{E}[X_i] = \text{Cov}(A, M) \). To see this, first note that \( \mathbb{E}[\sqrt{1 - A^2} Y_i] = \mathbb{E}[\sqrt{1 - A^2}] \mathbb{E}[Y_i] = 0 \), where we have used the facts that (i) \( A \) and \( Y_i \) are independent and (ii) \( Y_i \) is mean zero. Thus \( \mathbb{E}[X_i] = \mathbb{E}[AM] \), and since \( M \) is mean zero we get that \( \mathbb{E}[AM] = \text{Cov}(A, M) \). This establishes (P1).

Property (P2) states that \( \text{Var}(X_i) = 1 + \text{Cov}(A^2, M^2) - (\text{Cov}(A, M))^2 \). To see this, first note that \( AM \) and \( \sqrt{1 - A^2} Y_i \) are uncorrelated, this because (i) \( Y_i \) is independent of \( (A, M) \) and (ii) \( Y_i \) is mean zero. Thus

\[
\text{Var}(X_i) = \text{Var}(AM + \sqrt{1 - A^2} Y_i) = \text{Var}(AM) + \text{Var}(\sqrt{1 - A^2} Y_i) .
\]

(A.1)

Now,

\[
\text{Var}(AM) = \mathbb{E}[A^2 M^2] - (\mathbb{E}[AM])^2 = \mathbb{E}[A^2 M^2] - (\text{Cov}(A, M))^2 ,
\]

(A.2)

where we have used (P1), and

\[
\text{Var}(\sqrt{1 - A^2} Y_i) = \mathbb{E}[(1 - A^2) Y_i^2] - (\mathbb{E}[\sqrt{1 - A^2} Y_i])^2 \\
= \mathbb{E}[1 - A^2] \mathbb{E}[Y_i^2] - (\mathbb{E}[\sqrt{1 - A^2}] \mathbb{E}[Y_i])^2 \\
= \mathbb{E}[1 - A^2] \\
= 1 - \mathbb{E}[A^2] \\
= 1 - \mathbb{E}[A^2] \mathbb{E}[M^2] ,
\]

(A.3)

where we have used independence of \( A \) and \( Y_i \), the fact that \( \mathbb{E}[Y_i] = 0 \) and the fact that \( \mathbb{E}[Y_i^2] = \mathbb{E}[M^2] = 1 \). Combining (A.1), (A.2) and (A.3), we get

\[
\text{Var}(X_i) = \left( \mathbb{E}[A^2 M^2] - (\text{Cov}(A, M))^2 \right) + \left( 1 - \mathbb{E}[A^2] \mathbb{E}[M^2] \right) = 1 + \text{Cov}(A^2, M^2) - (\text{Cov}(A, M))^2 ,
\]

as desired.

Finally, (P3) states that \( \text{Cov}(X_i, X_j) = \text{Var}(AM) \). We have already noted that \( AM \) and \( \sqrt{1 - A^2} Y_i \) are uncorrelated for any \( i \). Similarly, \( \sqrt{1 - A^2} Y_i \) and \( \sqrt{1 - A^2} Y_j \) are uncorrelated whenever \( i \neq j \). Thus

\[
\text{Cov}(X_i, X_j) = \text{Cov}(AM + \sqrt{1 - A^2} Y_i, AM + \sqrt{1 - A^2} Y_j) = \text{Cov}(AM, AM) = \text{Var}(AM) ,
\]

as required.

**Appendix B**

Suppose that \( (U, V) \) are bivariate normal with standard normal margins and correlation \( \rho \). For \( -\infty \leq v_1 < v_2 \leq \infty \), the conditional cdf of \( U \), given that \( V \in (v_1, v_2) \), is given by

\[
\mathbb{P}(U \leq u | v_1 < V < v_2) = \frac{\Phi_2 \left( \begin{bmatrix} u \\ v_2 \\ 1 \end{bmatrix} ; 0, 1 - \rho^2 , 1 - \rho \right) - \Phi_2 \left( \begin{bmatrix} u \\ v_1 \\ 1 \end{bmatrix} ; 0, 1 - \rho^2 , 1 - \rho \right)}{\Phi(v_2) - \Phi(v_1)} .
\]

(B.1)

The density of (B.1) is demonstrably

\[
g(u) := \frac{\rho(u)}{\Phi(v_2) - \Phi(v_1)} ,
\]

(B.2)

where

\[
r(u) := \Phi \left( v_2; \rho u, 1 - \rho^2 \right) - \Phi \left( v_1; \rho u, 1 - \rho^2 \right) .
\]

(B.3)

This construction defines a three-parameter family with parameters \((\rho, v_1, v_2)\). This family is discussed in [3], and we refer to it as the ABGM family after the authors of that paper. In general the ABGM family is asymmetric and unimodal, and its tails will either be proportional to, or thinner than, the standard normal density \( \phi(u) \). Tail behaviour is determined as follows.

- If both \( v_1 \) and \( v_2 \) are infinite, then \( r(u) \equiv 1 \) and \( g(u) \) reduces to the standard normal density.
• If both $v_1$ and $v_2$ are finite, then $r$ is a unimodal function such that $r(-\infty) = r(\infty) = 0$. In this case both tails are thinner than the standard normal, in the sense that $\lim_{u \to \pm \infty} g(u)/\phi(u) = 0$.

• If exactly one of $v_1$ or $v_2$ is infinite, then $r$ is a monotone function from $\mathbb{R}$ to $[0, 1]$ and either (i) $r(-\infty) = 0$ and $r(\infty) = 1$ or (ii) $r(-\infty) = 1$ and $r(\infty) = 0$. In the former case, the right tail of the density is proportional to a Gaussian and the left tail is much thinner than a Gaussian. In the latter case, the left tail is proportional to a Gaussian and the right tail is much thinner.

If $v_1 = -\infty$ and $v_2 = 0$ or $v_1 = 0$ and $v_1 = -\infty$ then $g(u)$ a skew-normal density (the skew-normal family is discussed in [4]). Finally, moments of the ABGM can be determined via an appeal to the truncated normal distribution.

Acknowledgements: This work was made possible through the generous financial support of the NSERC Discovery Grant program.

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