Local Well-Posedness for Free Boundary Problem of Viscous Incompressible Magnetohydrodynamics

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Abstract: In this paper, we consider the motion of incompressible magnetohydrodynamics (MHD) with resistivity in a domain bounded by a free surface. An electromagnetic field generated by some currents located on a fixed boundary keeps an MHD flow in a bounded domain. On the free surface, free boundary conditions for MHD flow and transmission conditions for electromagnetic fields are imposed. We proved the local well-posedness in the general setting of domains from a mathematical point of view. The solutions are obtained in an anisotropic space $H^1_p((0, T), H^1_q) \cap L^p((0, T), H^2_q)$ for the velocity field and in an anisotropic space $H^1_p((0, T), L^p) \cap L^p((0, T), H^2_q)$ for the magnetic fields with $2 < p < \infty, N < q < \infty$ and $2/p + N/q < 1$. To prove our main result, we used the $L^p-L^q$ maximal regularity theorem for the Stokes equations with free boundary conditions and for the magnetic field equations with transmission conditions, which have been obtained by Frolova and the second author.

Keywords: free boundary problem; transmission condition; magnetohydrodynamics; local well-posedness; $L^p-L^q$ maximal regularity

MSC: 35K59; 76W05

1. Introduction

In this paper, we prove the local well-posedness of a free boundary problem for the viscous non-homogeneous incompressible magnetohydrodynamics. The problem here is formulated as follows: Let $\Omega_+ = \Omega$ be a domain in the $N$-dimensional Euclidean space $\mathbb{R}^N (N \geq 2)$, and let $\Gamma$ be the boundary of $\Omega_+$. Let $\partial \Omega$ be also a domain in $\mathbb{R}^N$ whose boundary is $\Gamma$ and $\partial \Omega$. We assume that $\partial \Omega \cap \Omega = \emptyset$. Throughout the paper, we assume that $\Omega_\pm$ are uniform $C^2$ domains, that the weak Dirichlet problem is uniquely solvable in $\Omega_+$. (The definition of uniform $C^2$ domains and the weak Dirichlet problem will be given in Section 3 below.) and that $\text{dist}(\Gamma, S_-) \geq 2d_-$ with some positive constants $d_-$. Here, $\text{dist}(A, B)$ denotes the distance of any two subsets $A$ and $B$ of $\mathbb{R}^N$ defined by setting $\text{dist}(A, B) = \inf \{|x - y| \mid x \in A, y \in B\}$. Let $\Omega = \Omega_+ \cup \Gamma \cup \Omega_-$ and $\hat{\Omega} = \Omega_+ \cup \Omega_-$. The boundary of $\Omega$ is $\partial \Omega$. We may consider the case that $\partial \Omega$ is an empty set, and in this case $\Omega = \mathbb{R}^N$. Physically, we consider the case where $\partial \Omega$ is filled by a non-homogeneous incompressible magnetohydrodynamic (MHD) fluid and $\partial \Omega$ is filled by an insulating gas. We consider a motion of an MHD fluid in a time dependent domain $\Omega_\pm$ whose boundary is $\Gamma$ subject to an electromagnetic field generated in a domain $\Omega_\pm = \Omega \setminus (\Omega_\pm \cup \Gamma_\pm)$ by some currents located on a fixed boundary $\partial \Omega$ of $\Omega_\pm$. Let $\mathbf{n}$ be the unit outer normal to $\Gamma$, oriented from $\omega_\pm$ into $\Omega_\pm$, and let $\mathbf{n}_-$ be respective the unit outer normals to $\partial \Omega$. Given
any functions, \( v_{\pm} \), defined on \( \Omega_{\pm} \), \( v \) is defined by \( v(x) = v_+(x) \) for \( x \in \Omega_{t+} \) for \( t \geq 0 \), where \( \Omega_0 = \Omega_{\pm} \). Moreover, what \( v = v_\pm \) denotes that \( v(x) = v_+(x) \) for \( x \in \Omega_{t+} \) and \( v(x) = v_-(x) \) for \( x \in \Omega_{t-} \). Let

\[
[[v]](x_0) = \lim_{x \to x_0^{+}} v_+(x) - \lim_{x \to x_0^{-}} v_-(x)
\]

for every point \( x_0 \in \Gamma_t \), which is the jump quantity of \( v \) across \( \Gamma \).

The purpose of this paper is to prove the local well-posedness of the free boundary problem formulated by the set of the following equations:

\[
\begin{align*}
\rho(\partial_t v + v \cdot \nabla v) - \text{Div} (T(v, p) + T_M(H_+)) &= 0, \quad \text{div} v = 0 \quad \text{in} \quad \Omega^T_t, \\
\mu_+ \partial_t H_+ + \text{Div} \{\alpha^{-1}_+ \text{curl} H_+ - \mu_+ (v \otimes H_+ - H_+ \otimes v)\} &= 0, \quad \text{div} H_+ = 0 \quad \text{in} \quad \Omega^T_t, \\
\mu_- \partial_t H_- + \text{Div} \{\alpha^{-1}_- \text{curl} H_-\} &= 0, \quad \text{div} H_- = 0 \quad \text{in} \quad \Omega^T_t, \\
(T(v, p) + T_M(H_+))n_+ &= 0, \quad V_{n_+} = v \cdot n_+ \quad \text{on} \quad \Gamma^T_t, \\
\left[([\alpha^{-1}_- \text{curl} H_-]n_-) - \mu_+ (v \otimes H_+ - H_+ \otimes v)n_+\right] &= 0 \quad \text{on} \quad \Gamma^T_t, \\
\left[[\mu H \cdot n_-]\right] &= 0, \quad \left[[H_- < H, n_- > n_0]\right] = 0 \quad \text{on} \quad \Gamma^T_t, \\
\left(n_- \cdot H_- = 0\right. & \quad \left.\left((\text{curl} H_-) \cdot n_- = 0 \quad \text{on} \quad S^T_t, \right)ight. \\
\left(v, H_+\right)|_{t=0} &= (v_0, H_0^+) \quad \text{in} \quad \Omega^T_0, \quad H_-|_{t=0} = H_0^- \quad \text{in} \quad \Omega^T_0.
\end{align*}
\]

Here,

\[
\Omega^T_\pm = \bigcup_{0 < t < T} \Omega^T_{t\pm} \times \{t\}, \quad \Gamma^T = \bigcup_{0 < t < T} \Gamma_t \times \{t\}, \quad S^T = S^- \times (0, T);
\]

\( v = (v_1(x, t), \ldots, v_N(x, t))^T \) is the velocity vector field, where \( M^T \) stands for the transposed \( M, \ p = p(x, t) \) the pressure fields, and \( H = H_\pm = (H_{1\pm}(x, t), \ldots, H_{N\pm}(x, t))^T \) the magnetic vector field. The \( v, p, \) and \( H \) are unknowns, while \( v_0 \) and \( H_0 \) are prescribed \( N \)-component vectors of functions. As for the remaining symbols, \( T(v, p) = vD(v) - pI \) is the viscous stress tensor, \( D(v) = \nabla v + (\nabla v)^T \) is the doubled deformation tensor whose \((i, j)\)th component is \( \partial_i v_j + \partial_j v_i \) with \( \partial_i = \partial/\partial x_i \), \( I \) the \( N \times N \) unit matrix, \( T_M(H_\pm) = \mu_+(H_\pm \otimes H_\pm - \frac{1}{2}H_\pm^2 I) \) the magnetic stress tensor, \( \text{curl} v = (\nabla v)^T - \nabla v \) the doubled rotation tensor whose \((i, j)\)th component is \( \partial_j v_i - \partial_i v_j \), \( V_{n_+} \) the velocity of the evolution of \( \Gamma_t \) in the direction of \( n_+ \). Moreover, \( \rho, \mu_\pm, \nu, \) and \( \alpha_\pm \) are positive constants describing respectively the mass density, the magnetic permeability, the kinematic viscosity, and the conductivity. Finally, for any matrix field \( K \) with \((i, j)\)th component \( K_{ij} \), the quantity \( K \) is an \( N \)-vector of functions with the \( i \)th component \( \sum_{j=1}^N \partial_j K_{ij} \), and for any \( N \)-vectors of functions \( u = (u_1, \ldots, u_N)^T \) and \( w = (w_1, \ldots, w_N)^T \), \( \text{div} u = \sum_{j=1}^N \partial_j u_j \) \( u \cdot \nabla w \) is an \( N \)-vector of functions with the \( i \)th component \( \sum_{j=1}^N \partial_j w_j \), and \( u \otimes w \) an \( N \times N \) matrix with the \((i, j)\)th component \( u_i w_j \). We notice that in the three dimensional case

\[
\Delta v = -\text{Div} \text{curl} v + \nabla \text{div} v, \quad \text{Div} (v \otimes H - H \otimes v) = v \text{div} H - H \text{div} v + H \cdot \nabla v - v \cdot \nabla H, \\
\text{rot rot} H = \text{Div} \text{curl} H, \quad \text{rot} (v \times H) = \text{Div} (v \otimes H - H \otimes v),
\]

where \( \times \) is the exterior product. In particular, in the three dimensional case, the set of equations for the magnetic vector field in Equation (1) is written by

\[
\begin{align*}
\mu_+ \partial_t H_+ + \text{rot} (\alpha^{-1}_+ \text{rot} H_+ - \nu_+ v \times H_+) &= 0, \quad \text{div} H_+ = 0 \quad \text{in} \quad \Omega^T_t, \\
\mu_- \partial_t H_- + \text{rot} (\alpha^{-1}_- \text{rot} H_-) &= 0, \quad \text{div} H_- = 0 \quad \text{in} \quad \Omega^T_t, \\
n_+ \times (\alpha^{-1}_- \text{rot} H_-) - n_- \times (\mu_+ v \times H_+) &= 0, \quad \left]\|\mu H \cdot n_-\right]\| = 0, \quad \left]\|H_- < H, n_- > n_0\right]\| = 0 \quad \text{on} \quad \Gamma^T_t.
\end{align*}
\]

This is a standard description and so the set of equations for the magnetic field in Equation (1) is the \( N \)-dimensional mathematical description of equations for the magnetic vector field with transmission conditions.
In Equation (1), there is one equation for the magnetic fields $H_{\pm}$ too many, so that in this paper instead of (1), we consider the following equations:

$$
\rho (\partial_t v + v \cdot \nabla v) - \text{Div} (T(v, p) + T_M(H_{\pm})) = 0, \quad \text{div } v = 0 \quad \text{in } \Omega^+_T,
$$

$$
\mu_+ \partial_t H_+ - \alpha^+_1 \Delta H_+ - \text{Div} \mu_+ (v \otimes H_+ - H_+ \otimes v) = 0 \quad \text{in } \Omega^+_T,
$$

$$
\mu_- \partial_t H_- - \alpha^-_1 \Delta H_- = 0 \quad \text{in } \Omega^-_T,
$$

$$
(T(v, p) + T_M(H_{\pm}))n_t = 0, \quad V_{\Gamma_t} = v \cdot n_t \quad \text{on } \Gamma^T.
$$

\[ (\alpha^+ \text{curl } H)n_t - \mu_+(v \otimes H_+ - H_+ \otimes v)n_t = 0, \quad [[\mu \text{div } H]] = 0 \quad \text{on } \Gamma^T, \]

\[ [[\mu H \cdot n_t]] = 0, \quad [[H^- < H, n_t > n_t]] = 0 \quad \text{on } \Gamma^T, \]

$$
\n_+ - \cdot H_+ = 0, \quad (\text{curl } H_-)n_- = 0 \quad \text{on } S^T, \quad (v, H_+)_{t=0} = (v_0, H_{0+}) \quad \text{in } \Omega^+, \quad H_{t-}|_{t=0} = H_{0-} \quad \text{in } \Omega^-.
$$

Namely, two equations: $\text{div } H_{\pm} = 0$ in $\Omega^\pm_T$ are replaced with one transmission condition: $[[\mu \text{div } H]] = 0$ on $\Gamma^T$. Employing the same argument as in Frolova and Shibata ([1], Appendix), we see that in Equation (3) if $\text{div } H = 0$ initially, then $\text{div } H = 0$ in $\Omega$ follows automatically for any $t > 0$ as long as solutions exist. Thus, the local well-posedness of Equation (1) follows from that of Equation (3) provided that the initial data $H_{0\pm}$ satisfy the divergence zero condition: $\text{div } H_0\pm = 0$. This paper is devoted to proving the local well-posedness of Equation (3) in the maximal $L^p$-$L^q$ regularity framework.

The MHD equations can be found in [2,3]. The solvability of MHD equations was first obtained by Ladyzhenskaya and Solonnikov [4]. The initial-boundary value problem for MHD equations with non-slip conditions for the velocity vector field and perfect wall conditions for the magnetic vector field was studied by Sermange and Temam [5] in a bounded domain and by Yamaguchi [6] in an exterior domain. In their studies [5,6], the boundary is fixed. On the other hand, in the field of engineering, when a thermonuclear reaction is caused artificially, a high-temperature plasma is sometimes subjected to a magnetic field and held in the air, and the boundary of the fluid at this time is a free one. From this point of view, the free boundary problem for MHD equations is important. The local well-posedness for free boundary problem for MHD equations was first proved by Padula and Solonnikov [7] in the case where $\Omega^+_1$ is a bounded domain surrounded by a vacuum area, $\Omega^-$. In [7], the solution was obtained in Sobolev-Slobodetski spaces in the $L^2$ framework of fractional order greater than 2. Later on, the global well-posedness was proved by Frolova [8] and Solonnikov and Frolova [9]. Moreover, the $L^p$ approach to the same problem was done by Solonnikov [10,11]. When $\Omega_{\pm}$ is a bounded domain, which is surrounded by an electromagnetic field generated in a domain, $\Omega_{\pm}$, Kacprzyk proved the local well-posedness in [12] and global well-posedness in [13]. In [12,13], the solution was also obtained in Sobolev-Slobodetski spaces in the $L^2$ framework of fractional order greater than 2.

Recently, the $L^p$-$L^q$ maximal regularity theorem for the initial boundary value problem of the system of parabolic equations with non-homogeneous boundary conditions has been studied by using $R$-solver in [14] and references therein and by using $H^\infty$ calculus in [15] and references therein. They are completely different approaches. In particular, Shibata [16,17] proved the $L^p$-$L^q$ maximal regularity for the Stokes equations with non-homogeneous free boundary conditions by $R$-solver theory and Frolova and Shibata [1] proved it for linearized equations for the magnetic vector fields with transmission conditions on the interface and perfect wall conditions on the fixed boundary arising in the study of two phase problems for the MHD flows also by using the $R$-solver. The results in [1,16,17] enable us to prove the local well-posedness for Equation (3) in the $L^p$-$L^q$ maximal regularity class.

Aside from dynamical boundary conditions on $\Gamma_t$, a kinematic condition, $V_{\Gamma_t} = v \cdot n_t$, is satisfied on $\Gamma_t$, which represents $\Gamma_t$ as a set of points $x = x(\xi, t)$ for $\xi \in \Gamma$, where $x(\xi, t)$ is the solution of the Cauchy problem:
\[
\frac{dx}{dt} = \mathbf{v}(x, t), \quad x|_{t=0} = \xi.
\] (4)

This expresses the fact that the free surface \( \Omega \) of the same fluid particles, which do not leave it and are not incident on it from inside \( \Omega_{++} \), satisfies the conditions:

\[
\Omega \text{ solvable in } \Omega_{++} \text{ connected with } \Omega \text{ by } (4). \quad \text{Since the velocity field, } u_+ (\xi, t) = \mathbf{v}(x, t), \text{ is given only in } \Omega_{++}, \text{ we extend it to } u_\pm \text{ defined on } \Omega_\pm \text{ in such a way that}
\]

\[
\lim_{\xi\to 0} \frac{\partial^2 u_+ (\xi, t)}{\partial \xi^2} = \lim_{\xi\to 1} \frac{\partial^2 u_- (\xi, t)}{\partial \xi^2} \quad \text{for } |\xi| \leq 3 \text{ and } (\xi_0, t) \in \Gamma \times (0, T),
\]

\[
\|u_- (\cdot, t)\|_{H^1_0(\Omega_\pm)} \leq C_q \|u_+ (\cdot, t)\|_{H^1_0(\Omega_\pm)} \quad \text{for } i = 0, 1, 2, 3 \text{ and } t \in (0, T). \quad (5)
\]

Let \( \varphi (\xi) \) be a \( C^\infty (\mathbb{R}^N) \) function, which equals 1 when \( |\xi| \leq 2d_\pm \) and equals 0 when \( |\xi| > d_\pm \). The connection between Euler coordinates \( x \) and Lagrangian coordinates \( \xi \) is defined by setting

\[
x = \xi + \varphi (\xi) \int_0^t u(\xi, s) \, ds = X_u(\xi, t).
\] (6)

Define \( q_\pm (\xi, t) := p(x, t) \) and \( H(\xi, t) = H_\pm (\xi, t) = H_\pm (x, t) \). Problem (3) is transformed by (6) to the following equations:

\[
\rho \partial_t u_+ - \text{Div } T(u_+, q_+) = N_1 (u_+, H_+) \quad \text{in } \Omega_+ \times (0, T),
\]

\[
\text{div } u_+ = N_2 (u_+) = \text{div } N_3 (u_+) \quad \text{in } \Omega_+ \times (0, T),
\]

\[
T(u_+, q_+) \cdot n = N_4 (u_+, H_+) \quad \text{on } \Gamma \times (0, T),
\]

\[
\mu \partial_t H - A^{-1} \Delta H = N_5 (u_+, H_+) \quad \text{in } \Omega \times (0, T),
\]

\[
[[\alpha^{-1} \text{curl } H]] n = N_6 (u_+, H_+) \quad \text{on } \Gamma \times (0, T),
\]

\[
[[\mu \text{div } H]] n = N_7 (u_+, H_+) \quad \text{on } \Gamma \times (0, T),
\]

\[
[[\mu H \cdot n]] n = N_8 (u_+, H_+) \quad \text{on } \Gamma \times (0, T),
\]

\[
[[H_\Gamma]] n = N_9 (u_+, H_+) \quad \text{on } \Gamma \times (0, T),
\]

\[
\mathbf{n}_- \cdot H_- = 0, \quad (\text{curl } H_-) \mathbf{n}_- = 0 \quad \text{on } S_- \times (0, T),
\]

\[
u_+ \big|_{t=0} = \mathbf{u}_{0+} \quad \text{in } \Omega_+, \quad H_\big|_{t=0} = H_0 \quad \text{in } \Omega_+.
\]

Here, \( n \) is the unit outer normal to \( \Gamma \) oriented from \( \Omega_+ \) into \( \Omega_- \), \( -d_\xi = -d \mathbf{n} \) for any \( N \)-vector \( d \), and the \( N_1 (u_+, H_+) \), \ldots, \( N_9 (u_+, H_+) \) are nonlinear terms defined in Section 2 below.

Our main result is the following theorem.

**Theorem 1.** Let \( 1 < p, q < \infty \) and \( B \geq 1 \). Assume that \( 2/p + N/q < 1 \), that \( \Omega_+ \) is a uniform \( C^3 \) domain and \( \Omega = \Omega_+ \cup \Gamma \cup \Omega_- \) a uniform \( C^2 \) domain, and that the weak problem is uniquely solvable in \( \Omega_+ \) for \( q \) and \( q' = q/(q - 1) \). Let initial data \( u_{0+} \) and \( H_{0+} \) with

\[
u_{0+} \in B^{3-2/p}_{p,q} (\Omega_+), \quad H_{0+} \in B^{2(1-1/p)}_{p,q} (\Omega_+)
\]

satisfy the conditions:

\[
\|u_{0+}\|_{B^{3-2/p}_{p,q}(\Omega_+)} + \|H_{0+}\|_{B^{2(1-1/p)}_{p,q}(\Omega_+)} \leq B \quad \text{(8)}
\]
and compatibility conditions

\[
\begin{align*}
\text{div } u_{0+} &= 0 \quad \text{in } \Omega_+,
\end{align*}
\]

\[
\begin{align*}
((u \cdot D(u_{0+}) + T_M(H_{0+})n) n) &= 0 \quad \text{on } \Gamma, \\
([a^{-1} \text{curl } \tilde{H}_0] - \mu_+(u_{0+} \otimes \tilde{H}_{0+} - \tilde{H}_{0+} \otimes u_{0+})) n &= 0 \quad \text{on } \Gamma, \\
\mu \text{div } \tilde{H}_0 &= 0, \quad \mu \bar{H}_0 \cdot n = 0, \quad \text{curl } \tilde{H}_0 n = 0 \quad \text{on } \Gamma, \\
\n \cdot \tilde{H}_{0-} &= 0, \quad \text{curl } \tilde{H}_{0-} n = 0 \quad \text{on } S_-.
\end{align*}
\]

Then, there exists a time \( T > 0 \) for which problem (7) admits unique solutions \( u_+ \) and \( \tilde{H}_\pm \) with

\[
\begin{align*}
u_+ &\in \mathcal{L}_p((0, T), H^2_q(\Omega_+)^N) \cap H^3_p((0, T), H^1_q(\Omega_+)^N), \\
\tilde{H}_\pm &\in \mathcal{L}_p((0, T), H^2_q(\Omega_\pm)^N) \cap H^3_p((0, T), L_q(\Omega_\pm)^N)
\end{align*}
\]

possessing the estimate:

\[
\begin{align*}
\| u_+ \|_{\mathcal{L}_p((0, T), H^2_q(\Omega_+))} + \| \partial_t u_+ \|_{\mathcal{L}_p((0, T), H^1_q(\Omega_+))} \\
+ \| \tilde{H} \|_{\mathcal{L}_p((0, T), H^2_q(\Omega))} + \| \partial_t \tilde{H} \|_{\mathcal{L}_p((0, T), L_q(\Omega))} &\leq f(B)
\end{align*}
\]

with some polynomial \( f(B) \) with respect to \( B \).

**Remark 1.** As was mentioned after Equation (3), if we assume that div \( \tilde{H}_{0\pm} = 0 \) in \( \Omega_\pm \) in addition, then div \( \tilde{H}_0 = 0 \) in \( \Omega_\pm \), and so \( v \) and \( \tilde{H} \) are solutions of Equation (1). Thus, we obtain the local well-posedness of Equation (1) from Theorem 1.

Finally, we explain some symbols used throughout the paper.

**Notation** We denote the set of all natural numbers, real numbers, complex numbers by \( \mathbb{N}, \mathbb{R}, \) and \( \mathbb{C} \), respectively, and set \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). For any multi-index \( \kappa = (\kappa_1, \ldots, \kappa_N) \), \( \kappa_j \in \mathbb{N}_0 \), we set \( \partial_\kappa \mathbf{g} = \partial_1^{\kappa_1} \cdots \partial_N^{\kappa_N} \mathbf{g} \) and \( |\kappa| = \sum_{j=1}^N \kappa_j \). For a scalar function, \( f \), and an \( N \)-vector of functions, \( \mathbf{g} = (g_1, \ldots, g_N) \), we set \( \nabla^n f = \{ \partial_\kappa f \mid \kappa = \kappa \} \) and \( \nabla^n \mathbf{g} = \{ \partial_\kappa \mathbf{g} \mid \kappa = \kappa \} \). In particular, \( \nabla^0 f = f, \nabla^0 \mathbf{g} = \mathbf{g}, \nabla^1 f = \nabla f, \) and \( \nabla^1 \mathbf{g} = \nabla \mathbf{g} \). For notational convenience, \( \nabla \mathbf{g} \) and \( \nabla^2 \mathbf{g} \) are sometimes considered as \( N^2 \) and \( N^3 \) column vectors, respectively, in the following way:

\[
\begin{align*}
\nabla \mathbf{g} &= (\partial_1 g_1, \ldots, \partial_N g_1, \ldots, \partial_1 g_N, \ldots, \partial_N g_N)^T, \\
\nabla^2 \mathbf{g} &= (\partial_2 \partial_1 g_1, \ldots, \partial_N \partial_1 g_1, \ldots, \partial_2 \partial_N g_N, \ldots, \\
&\quad \partial_N \partial_N g_N)^T
\end{align*}
\]

for \( \ell = 1, \ldots, N, \) and \( 1 \leq i \leq j \leq N \).  

For \( 1 \leq q \leq \infty, m \in \mathbb{N}, s \in \mathbb{R}, \) and any domain \( D \subset \mathbb{R}^N \), we denote the standard Lebesgue space, Sobolev space, and Besov space by \( L_q(D), H^m_q(D), \) and \( B^s_q(D) \) respectively, while \( \| \cdot \|_{L_q(D)}, \| \cdot \|_{H^m_q(D)}, \) and \( \| \cdot \|_{B^s_q(D)} \) denote their norms. We write \( W^m_q(D) = B^m_q(D) \) and \( H^m_q(D) = L_q(D) \). What \( f = f_\pm \) means that \( f(x) = f_\pm(x) \) for \( x \in \Omega_\pm \). For \( H = \{ L_q, H^m_q, B^s_q \} \), the function spaces \( \mathcal{H}(\Omega) (\Omega = \Omega_+ \cup \Omega_-) \) and their norms are defined by setting

\[
\mathcal{H}(\Omega) = \{ f = f_\pm \mid f_\pm \in \mathcal{H}(\Omega_\pm) \}, \quad \| f \|_{\mathcal{H}(D)} = \| f_+ \|_{\mathcal{H}(D_+)} + \| f_- \|_{\mathcal{H}(D_-)}.
\]

For any Banach space \( X, \| \cdot \|_X \) being its norm, \( X^d \) denotes the \( d \) product space defined by \( \{ x = (x_1, \ldots, x_d) \mid x_i \in X \} \), while the norm of \( X^d \) is simply written by \( \| \cdot \|_X \). which is defined by setting \( \| x \|_X = \sum_{i=1}^d \| x_i \|_X \). For any time interval \( (a, b) \), \( L_p((a, b), X) \) and \( H^m_p((a, b), X) \) denote respective the standard \( X \)-valued Lebesgue space and \( X \)-valued Sobolev space, while \( \| \cdot \|_{L_p((a, b), X)} \) and \( \| \cdot \|_{H^m_p((a, b), X)} \) denote their norms. Let \( \mathcal{F} \) and \( \mathcal{F}^{-1} \)
be respectively the Fourier transform and the Fourier inverse transform. Let $H^s_p(\mathbb{R}, X)$, $s > 0$, be the Bessel potential space of order $s$ defined by
\[
H^s_p(\mathbb{R}, X) = \{ f \in S'(\mathbb{R}, X) \mid \| f \|_{H^s_p(\mathbb{R}, X)} = \| \mathcal{F}^{-1}[(1 + |\tau|^2)^{s/2}\mathcal{F}[f](\tau)] \|_{L_p(\mathbb{R}, X)} < \infty \}.
\]
where $S'$ denotes the set of all $X$-valued tempered distributions on $\mathbb{R}$.

Let $a \cdot b = \langle a, b \rangle = \sum_{i=1}^{N} a_i b_i$ for any $N$-vectors $a = (a_1, \ldots, a_N)$ and $b = (b_1, \ldots, b_N)$. For any $N$-vector $a$, let $a^T := a = \langle a, n \rangle$. For any two $N \times N$-matrices $A = (A_{ij})$ and $B = (B_{ij})$, the quantity $A : B$ is defined by
\[
A : B = \sum_{i,j=1}^{N} A_{ij} B_{ij}.
\]
For any domain $G$ with boundary $\partial G$, we set
\[
(u, v)_G = \int_{G} u(x) \cdot \overline{v(x)} \, dx, \quad (u, v)_{\partial G} = \int_{\partial G} u(x) \cdot \overline{v(x)} \, d\sigma,
\]
where $\overline{v(x)}$ is the complex conjugate of $v(x)$ and $d\sigma$ denotes the surface element of $\partial G$. Given $1 < q < \infty$, let $d_q = q/(q-1)$. Throughout the paper, the letter $C$ denotes generic constants and $C_{a,b,...}$ the constant which depends on $a, b, \ldots$. The values of constants $C, C_{a,b,...}$ may be changed from line to line.

When we describe nonlinear terms $N_1(u_1, \dot{H}_1), \ldots, N_n(u, \dot{H})$ in (7), we use the following notational conventions. Let $u_i$ $(i = 1, \ldots, m)$ be $n_i$-vectors whose $j$th component is $u_{ij}$, and then $u_1 \otimes \cdots \otimes u_m$ denotes an $n = \prod_{i=1}^{m} n_i$ vector whose $(j_1, \ldots, j_m)$th component is $\Pi_{i=1}^{m} u_{ij_i}$ and the set $\{(j_1, \ldots, j_m) \mid 1 \leq j_i \leq n_i, i = 1, \ldots, m \}$ is rearranged as $\{k \mid k = 1, 2, \ldots, n\}$ and $k$ is the corresponding number to some $(j_1, \ldots, j_m)$. For example, $u \otimes \nabla v$ is an $N + N^2$ vector whose $(i, j, k)$ component is $u_i \partial_j v_k$ and $u \otimes \nabla v \otimes \nabla w$ is an $N + N^2 + N^3$ vector whose $(i, j, k, \ell, m, n)$ component is $u_i \partial_j \partial_k \partial_\ell v_m w_n$. Here, the sets $\{(i, j, k) \mid 1 \leq i, j, k \leq N\}$ and $\{(i, j, k, \ell, m, n) \mid 1 \leq i, j, k, \ell, m, n \leq N\}$ are rearranged as $\{k \mid 1 \leq k \leq N + N^2\}$ and $\{k \mid 1 \leq k \leq N + N^2 + N^3\}$, respectively. Let $u_i^f$ $(i = 1, \ldots, m, \ell = 1, \ldots, n)$ be $n_i^f$-vectors, let $A^f$ be $n^f \times N$ matrices, where $n^f = \prod_{i=1}^{m} n_i^f$, and set $A = \{A^1, \ldots, A^m\}$. Then, we write
\[
A(u_1^f \otimes \cdots \otimes u_{m_1}^f, \ldots, u_n^f \otimes \cdots \otimes u_{m_n}^f) = \sum_{\ell=1}^{n} A^\ell u_1^f \otimes \cdots \otimes u_{m_1}^f.
\]
When there are two sets of matrices $A = \{A^1, \ldots, A^m\}$ and $B = \{B^1, \ldots, B^n\}$, we write
\[
(A - B)(u_1^f \otimes \cdots \otimes u_{m_1}^f, \ldots, u_n^f \otimes \cdots \otimes u_{m_n}^f) = \sum_{\ell=1}^{n} (A^\ell - B^\ell) u_1^f \otimes \cdots \otimes u_{m_1}^f.
\]

2. Derivation of Nonlinear Terms

Let $u_+(\xi, t)$ be the velocity field with respect to the Lagrange coordinates $\xi \in \Omega_+$, and let $u_-(\xi, t)$ be the extension of $u_+$ to $\xi \in \Omega_-$ satisfying the conditions given in (5). Let
\[
\psi_u(\xi, t) = \varphi(\xi) \int_0^t u(\xi, s) \, ds
\]
and we consider the correspondence: $x = \xi + \psi_u(\xi, t)$ for $\xi \in \Omega$, which has been already given in (6). Let $\delta \in (0, 1)$ be a small constant and we assume that
\[
\sup_{t \in (0, T)} \| \psi_u(t) \|_{H^s_p(\Omega)} \leq \delta.
\]
(10)

Then, the correspondence: $x = \xi + \psi_u(\xi, t)$ is one to one. Since $\psi_u(\xi, t) = 0$ when $\text{dist}(\xi, S_{\pm}) \leq d_0$, if $u$ satisfies the regularity condition:
\[
u(\xi, t) \in H^s_p((0, T), H^1_q(\Omega)) \cap L_p((0, T), H^2_q(\Omega)),
\]
(11)
then the correspondence \( x = \xi + \psi_u(\xi, t) \) is a bijection from \( \Omega \) onto \( \Omega \), and so we set
\[
\Omega_{t\pm} = \{ x = \xi + \psi_u(\xi, t) \mid \xi \in \Omega_{t\pm} \}, \quad \Gamma_t = \{ x = \xi + \psi_u(\xi, t) \mid \xi \in \Gamma \}. \tag{12}
\]
In the following, for notational simplicity we set
\[
\Psi_u = \int_0^t \nabla (\varphi(\xi)u(\xi, s)) \, ds,
\tag{13}
\]
where \( \nabla = (\partial/\partial \xi_1, \ldots, \partial/\partial \xi_N) \). The Jacobian matrix of the correspondence: \( x = \xi + \psi_u(\xi, t) \) is
\[
\frac{\partial x}{\partial \xi} = I + \Psi_u.
\tag{14}
\]
Notice that
\[
\| \Psi_u \|_{L_{\infty}(0, T), L_{\infty}(\Omega)} \leq \delta
\tag{15}
\]
as follows from the assumption (10). Thus, we have
\[
\frac{\partial \xi}{\partial x} = (\frac{\partial x}{\partial \xi})^{-1} = I + \sum_{k=1}^{\infty} (-\Psi(\xi, t))^k = I + V_0(\Psi_u).
\tag{16}
\]
Here and in the following, we set
\[
V_0(k) = \sum_{k=1}^{\infty} (-k)^k
\tag{17}
\]
for \( |k| \leq \delta < 1 \). The \( k = (k_1, \ldots, k_N) \in \mathbb{R}^N \) with \( k_i = (k_{i1}, \ldots, k_{iN}) \in \mathbb{R}^N \) denotes the independent variables corresponding to \( \Psi_u = (\Psi_{u1}, \ldots, \Psi_{uN}) \) with \( \Psi_{iu} = \int_0^t \nabla (\varphi(\xi)\nu_i(\xi, s)) \, ds \). The \( V_0(k) \) is a matrix of analytic functions defined on \( |k| \leq \delta \) with \( V_0(0) = 0 \). Using this symbol, we have
\[
\nabla x = (I + V_0(\Psi_u))\nabla \xi, \quad \frac{\partial}{\partial x_i} = \frac{\partial}{\partial \xi_i} + \sum_{j=1}^{N} V_{0ij}(\Psi_u) \frac{\partial}{\partial \xi_j},
\tag{18}
\]
where \( V_{0ij} \) is the \((i, j)\)th component of the \( N \times N \) matrix \( V_0 \).

For any \( N \)-vector of functions, \( \mathbf{w}(x, t) = (w_1(x, t), \ldots, w_N(x, t))^T \), we set \( \tilde{\mathbf{w}}(\xi, t) = \mathbf{w}(x, t) \) and \( \mathbf{v}(x, t) = \mathbf{u}(\xi, t) \). Then, by (18)
\[
\partial_t \mathbf{w}(x, t) + \mathbf{v}(x, t) \cdot \nabla \mathbf{w}(x, t) = \partial_t \tilde{\mathbf{w}}(\xi, t) + (1 - \varphi(\xi)) \mathbf{u}(\xi, t) \cdot ((I + V_0(\Psi_u))\nabla \tilde{\mathbf{w}}(\xi, t)). \tag{19}
\]
By (18),
\[
D(\mathbf{w}) = D(\tilde{\mathbf{w}}) + D(\Psi_u)\nabla \tilde{\mathbf{w}}, \quad \text{curl} \, \mathbf{w} = \text{curl} \, \tilde{\mathbf{w}} + C(\Psi_u) \nabla \tilde{\mathbf{w}},
\tag{20}
\]
with
\[
D(\Psi_u) \nabla \tilde{\mathbf{w}} = V_0(\Psi_u) \nabla \tilde{\mathbf{w}} + (V_0(\Psi_u) \nabla \tilde{\mathbf{w}})^T, \quad C(\Psi_u) \nabla \tilde{\mathbf{w}} = V_0(\Psi_u) \nabla \tilde{\mathbf{w}} - (V_0(\Psi_u) \nabla \tilde{\mathbf{w}})^T.
\]
By (18),
\[
\text{div} \, \mathbf{w} = \text{div} \, \tilde{\mathbf{w}} + V_0(\Psi_u) : \nabla \tilde{\mathbf{w}},
\tag{21}
\]
with \( V_0(\Psi_u) : \nabla \tilde{\mathbf{w}} = \sum_{i,j=1}^{N} V_{0ij}(\Psi_u) \frac{\partial \xi_j}{\partial x_i} \). Analogously, for any \( N \times N \) matrix of functions, \( A = (A_{ij}) \), we set \( \tilde{A}(\xi, t) = A(x, t) \) and \( \tilde{A}_{ij}(\xi, t) = A_{ij}(x, t) \). Let \( V_0(\Psi_u) : \nabla \tilde{A} \) be an \( N \)-vector of functions whose \( i \)-th component is \( \sum_{j=1}^{N} V_{0jk}(\Psi_u) \frac{\partial \xi_j}{\partial x_i} \), and then
\[
\text{Div} \, A = \text{Div} \, \tilde{A} + V_0(\Psi_u) : \nabla \tilde{A}.
\tag{22}
\]
On the other hand, using the dual form, we see that

\[
\text{div } \mathbf{w} = \text{div } ( (I + \mathbf{V}_0(\Psi_u)^\top) \mathbf{w} ) = \text{div } \mathbf{\tilde{w}} + \text{div } (\mathbf{V}_0(\Psi_u)^\top \mathbf{\tilde{w}}).
\]  

Let \( q(\zeta, t) = p(x, t), \mathbf{u}_+(\zeta, t) = \mathbf{v}(x, t) \), and \( \mathbf{\tilde{H}}_\pm = \mathbf{H}(x, t) \). By (19), (20), and (22), we see that the first equation in Equation (3) is transformed to

\[
\rho \partial_t \mathbf{u}_+ + \rho (1 - \varphi) \mathbf{u}_+ : (I + \mathbf{V}_0(\Psi_u)) \nabla \mathbf{u}_+ - \text{Div} (\nu \mathbf{D}(\mathbf{u}_+) + \nu \mathbf{D}(\Psi_u) \nabla \mathbf{u}_+)
\]

\[
- \mathbf{V}_0(\Psi_u) : \nabla (\nu \mathbf{D}(\mathbf{u}_+) + \nu \mathbf{D}(\Psi_u) \nabla \mathbf{u}_+) + (I + \mathbf{V}_0(\Psi_u)) \nabla q - \text{Div} \mathbf{T}_M(\mathbf{H}_+) - \mathbf{V}_0(\Psi_u) : \nabla \mathbf{T}_M(\mathbf{H}_+) = 0.
\]

By (17) and (16), \((I + \Psi_u)(I + \mathbf{V}_0(\Psi_u)) = I\), and so we have

\[
\rho \partial_t \mathbf{u}_+ - \text{Div} \mathbf{T}(\mathbf{u}_+, q) = \mathbf{N}_1(\mathbf{u}_+, \mathbf{H}_+),
\]

with

\[
\mathbf{N}_1(\mathbf{u}_+, \mathbf{H}_+) = -\Psi_u \{ \rho \partial_t \mathbf{u}_+ - \text{Div} (\nu \mathbf{D}(\mathbf{u}_+))
\]

\[
+ (I + \mathbf{V}_0(\Psi_u)) (\rho(1 - \varphi) \mathbf{u}_+ : (I + \mathbf{V}_0(\Psi_u)) \nabla \mathbf{u}_+ + \nu \text{Div} (\mathbf{D}(\Psi_u) \nabla \mathbf{u}_+) + \nu \mathbf{D}(\Psi_u) \nabla \mathbf{u}_+)
\]

\[
+ \mathbf{V}_0(\Psi_u) : \nabla (\nu \mathbf{D}(\mathbf{u}_+) + \nu \mathbf{D}(\Psi_u) \nabla \mathbf{u}_+) + \text{Div} \mathbf{T}_M(\mathbf{H}_+) + \mathbf{V}_0(\Psi_u) : \nabla \mathbf{T}_M(\mathbf{H}_+).
\]

using the notational convention given in Notation, we may write

\[
\mathbf{N}_1(\mathbf{u}_+, \mathbf{H}_+) = \mathcal{A}_1(\Psi_u)(\Psi_u \otimes (\partial_t \mathbf{u}_+ + \nabla^2 \mathbf{u}_+), \mathbf{u}_+ \otimes \nabla \mathbf{u}_+, \nabla \Psi_u \otimes \nabla \mathbf{u}_+, \mathbf{H}_+ \otimes \nabla \mathbf{H}_+),
\]

where \( \mathcal{A}_1(k) \) is a set of matrices of smooth functions defined for \(|k| \leq \delta\). Combining (21) and (22), we see that the condition \( \text{div } \mathbf{v} = 0 \) in \( \Omega \) is transformed to

\[
\text{div } \mathbf{u}_+ = -\mathbf{V}_0(\Psi_u) : \nabla \mathbf{u}_+ = -\text{div} (\mathbf{V}_0(\Psi_u)^\top \mathbf{u}_+).
\]

Thus, we set \( \mathbf{N}_2(\mathbf{u}_+) = -\mathbf{V}_0(\mathbf{u}) : \nabla \mathbf{u}_+ \) and \( \mathbf{N}_3(\mathbf{u}_+) = -\mathbf{V}_0(\mathbf{k}) \mathbf{u}_+ \), and then by using notational convenience defined in Notation, we may write

\[
\mathbf{N}_2(\mathbf{u}_+) = \mathcal{A}_2(\Psi_u) \mathbf{V}_u \otimes \nabla \mathbf{u}_+, \quad \mathbf{N}_3(\mathbf{u}_+) = \mathcal{A}_3(\Psi_u) \mathbf{V}_u \otimes \mathbf{u}_+,
\]

where \( \mathcal{A}_i(k) (i = 2, 3) \) are matrices of smooth functions defined for \(|k| \leq \delta\). By (18),

\[
\Delta \mathbf{H}_\pm = \nabla \cdot \nabla \mathbf{H}_\pm = \Delta \mathbf{H}_\pm + \nabla (\mathbf{V}_0(\Psi_u) \nabla \mathbf{H}_\pm) + \mathbf{V}_0(\Psi_u) \nabla ((I + \mathbf{V}_0(\Psi_u)) \nabla \mathbf{H}_\pm).
\]

and so noting (19) we see that the second and third equations in (3) are transformed to

\[
\mu_+ \partial_t \mathbf{H}_+ - \alpha_+^{-1} \mathbf{\Delta H}_+ = \mathbf{N}_{5+}(u, H) \quad \text{in } \Omega_+ \times (0, T),
\]

\[
\mu_- \partial_t \mathbf{H}_- - \alpha_-^{-1} \mathbf{\Delta H}_- = \mathbf{N}_{5-}(u, H) \quad \text{in } \Omega_- \times (0, T),
\]

with

\[
\mathbf{N}_{5+}(u, H) = \mu_+ \varphi \mathbf{u} \cdot (I + \mathbf{V}_0(\Psi_u)) \nabla \mathbf{H}_+
\]

\[
+ \alpha_+^{-1} \{ \text{Div} (\mathbf{V}_0(\Psi_u) \nabla \mathbf{H}_+) + \mathbf{V}_0(\Psi_u) \nabla ((I + \mathbf{V}_0(\Psi_u)) \nabla \mathbf{H}_+) \}
\]

\[
+ \text{Div} (\mu_+ (\mathbf{u}_+ \otimes \mathbf{H}_+ - \mathbf{\Delta H}_+ \otimes \mathbf{u}_+) + \mathbf{V}_0(\Psi_u) : \nabla (\mu_+ (\mathbf{u}_+ \otimes \mathbf{H}_+ - \mathbf{\Delta H}_+ \otimes \mathbf{u}_+)));
\]

\[
\mathbf{N}_{5-}(u, H) = \mu_- \varphi \mathbf{u} \cdot (I + \mathbf{V}_0(\Psi_u)) \nabla \mathbf{H}_-
\]

\[
+ \alpha_-^{-1} \{ \text{Div} (\mathbf{V}_0(\Psi_u) \nabla \mathbf{H}_-) + \mathbf{V}_0(\Psi_u) \nabla ((I + \mathbf{V}_0(\Psi_u)) \nabla \mathbf{H}_-) \}.
\]

Thus, using the notational convention given in Notation, we may write

\[
\mathbf{N}_{5\pm}(u, H) = \mathcal{A}_{5\pm}(\Psi_u)(\mathbf{H}_\pm \otimes \nabla \mathbf{H}_\pm, \Psi_u \otimes \nabla^2 \mathbf{H}_\pm, \nabla \Psi_u \otimes \nabla \mathbf{H}_\pm, \partial_\pm \nabla \mathbf{u}_\pm \otimes \mathbf{H}_\pm, \mathbf{u}_\pm \otimes \nabla \mathbf{H}_\pm),
\]
where $\delta_+ = 1$ and $\delta_- = 0$, where $A_{6, \pm}(k)$ are two sets of matrices of smooth functions defined for $|k| \leq \delta$. In particular, we have the fourth equation in (7).

We now consider the transmission conditions. The unit outer normal, $\mathbf{n}_t$, to the $\Gamma_t$ is represented by

$$\mathbf{n}_t = \frac{(I + V_n(\Psi_u))\mathbf{n}}{|(I + V_n(\Psi_u))^\top \mathbf{n}|}.$$  

Choosing $\delta > 0$ small enough, we may write

$$\mathbf{n}_t = (I + V_n(\Psi_u))\mathbf{n},$$  \hspace{1cm} (27)

where $V_n(k)$ is a matrix of smooth functions defined on $|k| \leq \delta$ such that $V_n(0) = 0$. By (20)

$$(T(v, p) + T_M(\mathbf{H}_+))\mathbf{n}_t = v(D(\mathbf{u}_+) + D(\Psi_u))\mathbf{n} - (I + V_n(\Psi_u))\mathbf{q} + T_M(\mathbf{H}_+)\mathbf{n} = 0.$$

Choosing $\delta > 0$ small if necessary, we may assume that $(I + V_n(k))^{-1}$ exists and we may write $(I + V_n(k))^{-1} = I + V_{n-1}(k)$, where $V_{n-1}(k)$ is a matrix of smooth functions defined on $|k| \leq \delta$ such that $V_{n-1}(0) = 0$. Thus, setting

$$\mathbf{n}_d(u_+, \mathbf{H}_+) = -(v(D(\mathbf{u}_+)V_n(\Psi_u))\mathbf{n} + (I + V_n(\Psi_u))u^{-T}(I + V_n(\Psi_u))\mathbf{n}) + (I + V_{n-1}(\Psi_u))(v(D(\Psi_u))\mathbf{n} + T_M(\mathbf{H}_+)\mathbf{n}),$$

we have

$$T(u_+, q) = \mathbf{n}_d(u_+, \mathbf{H}_+) \quad \text{on } \Gamma \times (0, T).$$

Using the notational convention defined in Notation, we may write

$$\mathbf{n}_d(u_+, \mathbf{H}_+) = \mathcal{A}_4(\Psi_u)(\Psi_u \otimes \mathbf{u}_+, \mathbf{H}_+),$$  \hspace{1cm} (28)

where $\mathcal{A}_4(k)$ is a set of matrices of functions consisting of products of elements of $\mathbf{n}$ and smooth functions defined for $|k| \leq \delta$.

By (20) and (27),

$$[((\alpha^{-1}\operatorname{curl}\mathbf{H})\mathbf{n}_t)] - \mu_+(\mathbf{v} \otimes \mathbf{H}_+ - \mathbf{H}_+ \otimes \mathbf{v})\mathbf{n}_t = 0.$$

Thus, setting

$$\mathbf{n}_b(u, \mathbf{H}) = -\{[((\alpha^{-1}\operatorname{curl}\mathbf{H})\mathbf{v})\mathbf{n} + (((\alpha^{-1}\mathbf{C}(\Psi_u)\nabla\mathbf{H})(I + V_n(\Psi_u))\mathbf{n})) + \mu_+(\mathbf{u}_+ \otimes \mathbf{H}_+ - \mathbf{H}_+ \otimes \mathbf{u}_+)(I + V_n(\Psi_u))\mathbf{n}],$$

we have

$$[((\alpha^{-1}\operatorname{curl}\mathbf{H})\mathbf{n}_t)] = \mathbf{n}_b(u, \mathbf{H}) \quad \text{on } \Gamma \times (0, T).$$

Using the notational convention defined in Notation and noting that $[\Psi_u] = 0$ on $\Gamma$ as follows from (5), we may write

$$\mathbf{n}_b(u, \mathbf{H}) = \mathcal{A}_{61}(\Psi_u)[\alpha^{-1}\nabla\mathbf{H}] + (\mathcal{B} + \mathcal{A}_{62}(\Psi_u))\mathbf{u}_+ \otimes \mathbf{H}_+,$$  \hspace{1cm} (29)

where $\mathcal{A}_{61}(k)$ and $\mathcal{A}_{62}(k)$ are a matrix and a set of matrices of functions consisting of products of elements of $\mathbf{n}$ and smooth functions defined for $|k| \leq \delta$, and $\mathcal{B}$ is a set of matrices of functions such that $B\mathbf{u}_+ \otimes \mathbf{H}_+ = \mu_+(\mathbf{u}_+ \otimes \mathbf{H}_+ - \mathbf{H}_+ \otimes \mathbf{u}_+).$ In particular,

$$\|\mathcal{A}_{61}(\Psi_u)\|_{\mathcal{L}_1^0(\Omega)} \leq C\|\Psi_u\|_{\mathcal{H}_0^1(\Omega)}.$$  \hspace{1cm} (30)
By (23),
\[ |\mu \text{div} \mathbf{H}| = |\mu \text{div} \mathbf{H} + \mu \mathbf{V}_0(\Psi_u) : \nabla \mathbf{H}| = 0, \]
and so setting
\[ N_7(u, \mathbf{H}) = -|\mu \mathbf{V}_0(\Psi_u) : \nabla \mathbf{H}| = A_7(\Psi_u)(|\mu \nabla \mathbf{H}|) \]
we have
\[ |\mu \text{div} \mathbf{H}| = N_7(u, \mathbf{H}) \quad \text{on } \Gamma \times (0, T), \]
where \( A_7(k) \) is a matrix of functions consisting of products of elements of \( n \) and smooth functions defined for \( |k| \leq \delta \). Notice that
\[ \|A_7(\Psi_u)\|_{H^1(|\cdot|)} \leq C \|\Psi_u\|_{H^1(|\cdot|)}. \]

By (27), we have
\[ |\mu \mathbf{H} \cdot n| = |\mu \mathbf{H}(I - \mathbf{n}^\top \mathbf{n})| = 0, \]
and so, setting
\[ N_8(u, \mathbf{H}) = -|\mu \mathbf{H}| \mathbf{V}_n(\Psi_u) n, \]
we have
\[ |\mu \mathbf{H} \cdot n| = N_8(u, \mathbf{H}) \quad \text{on } \Gamma \times (0, T). \]

Finally, by (27)
\[ |H < h, n > n_1| = |H < h, (I + \mathbf{V}_n(\Psi_u)) n > (I + \mathbf{V}_n(\Psi_u)) n| \]
\[ = |H| - |< h, n > \mathbf{V}_n(\Psi_u) n| - |< h, \mathbf{V}_n(\Psi_u) n > (I + \mathbf{V}_n(\Psi_u)) n|, \]
and so, setting
\[ N_9(u, \mathbf{H}) = < |H|, n > \mathbf{V}_n(\Psi_u) n + < |H|, \mathbf{V}_n(\Psi_u) n > (I + \mathbf{V}_n(\Psi_u)) n, \]
we have
\[ |H| = N_9(u, \mathbf{H}) \quad \text{on } \Gamma \times (0, T). \]

For notational simplicity, we set
\[ (N_8(u, \mathbf{H}), N_9(u, \mathbf{H}) = A_8(\Psi_u)(|\mathbf{H}|), \]
where \( A_8(k) \) is a set of matrices of functions consisting of products of elements of \( n \) and smooth functions defined for \( |k| \leq \delta \). Notice that
\[ \|A_8(\Psi_u)\|_{H^1(|\cdot|)} \leq C \|\Psi_u\|_{H^1(|\cdot|)}. \]

3. Linear Theory

Since the coupling of the velocity field and the magnetic field in (7) is semilinear, the linearized equations are decoupled. Namely, we consider the two linearized equations: one is the Stokes equations with free boundary conditions on \( \Gamma \), and another is a system of heat equations with transmission conditions on \( \Gamma \) and the perfect wall conditions on \( S_\pm \). Recall that \( \bar{\Omega} = \Omega_+ \cup \Omega_- \) and \( \Omega = \bar{\Omega} \setminus \Gamma \).

3.1. The Stokes Equations with Free Boundary Conditions

This subsection is devoted to presenting the \( L_p-L_q \) maximal regularity theorem for the Stokes equations with free boundary conditions. The problem considered here is formulated by the following equations:
Here, we have set

\begin{align}
\rho \partial_t \mathbf{v} - \text{Div} \, \mathbf{T}(\mathbf{v}, q) &= f_1 \quad \text{in } \Omega_+ \times (0, T), \\
\text{div } \mathbf{v} &= g = \text{div } g \quad \text{in } \Omega_+ \times (0, T), \\
\mathbf{T}(\mathbf{v}, q) \mathbf{n} &= h \quad \text{on } \Gamma \times (0, T), \\
\mathbf{v}|_{t=0} &= \mathbf{u}_0+ \quad \text{in } \Omega_+.
\end{align}

To state assumptions for Equation (37), we make two definitions.

**Definition 1.** Let \( \Omega_+ \) be a domain given in the introduction. We say that \( \Omega_+ \) is a uniform \( \mathbb{C}^3 \) domain, if there exist positive constants \( a_1, a_2, \) and \( A \) such that the following assertion holds: For any \( x_0 = (x_{01}, \ldots, x_{0N}) \in \Gamma \) there exist a coordinate number \( j \) and a \( \mathbb{C}^3 \) function \( h(x') \) defined on \( B_{a_1}(x_0') \) such that \( \|h\|_{H^k(B_{a_1}(x_0'))} \leq A \) for \( k \leq 3 \) and

\[
\Omega_+ \cap B_{a_2}(x_0) = \{ x \in \mathbb{R}^N \mid x_j < h(x') \ (x' \in B_{a_1}(x_0')) \} \cap B_{a_2}(x_0), \\
\Gamma \cap B_{a_2}(x_0) = \{ x \in \mathbb{R}^N \mid x_j = h(x') \ (x' \in B_{a_1}(x_0')) \} \cap B_{a_2}(x_0).
\]

Here, we have set

\[
y' = (y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_N) \quad (y \in \{ x, x_0 \}), \]

\[
B_{a_1}(x_0') = \{ x' \in \mathbb{R}^{N-1} \mid |x' - x_0'| < a_1 \},
\]

\[
B_{a_2}(x_0) = \{ x \in \mathbb{R}^N \mid |x - x_0| < a_2 \}.
\]

Let \( \hat{H}^1_{q,0}(\Omega_+) \) be an homogeneous Sobolev space defined by letting

\[
\hat{H}^1_{q,0}(\Omega_+) = \{ \varphi \in L_{q,\text{loc}}(\Omega_+) \mid \nabla \varphi \in L_q(\Omega_+)^N, \quad \varphi|_{\Gamma} = 0 \}.
\]

Let \( 1 < q < \infty \). The variational equation:

\[
(\nabla u, \nabla \varphi)_{\Omega_+} = (f, \nabla \varphi)_{\Omega_+} \quad \text{for all } \varphi \in \hat{H}^1_{q,0}(\Omega_+)
\]

is called the weak Dirichlet problem, where \( q' = q/(q-1) \).

**Definition 2.** We say that the weak Dirichlet problem (39) is uniquely solvable for an index \( q \) if for any \( f \in L_q(\Omega)^N \), problem (39) admits a unique solution \( u \in \hat{H}^1_{q,0}(\Omega) \) possessing the estimate:

\[
\|\nabla u\|_{L_q(\Omega)} \leq C\|f\|_{L_q(\Omega)}.
\]

We say that \( u \in L_q(\Omega_+) \) is solenoidal if \( u \) satisfies

\[
(\mathbf{u}, \nabla \varphi)_{\Omega_+} = 0 \quad \text{for any } \varphi \in \hat{H}^1_{q,0}(\Omega_+).
\]

Let \( J_q(\Omega_+) \) be the set of all solenoidal vector of functions.

In this paper, we assume that

1. \( \Omega_+ \) is a uniform \( \mathbb{C}^3 \) domain.
2. The weak Dirichlet problem is uniquely solvable in \( \Omega_+ \) for indices \( q \in (1, \infty) \) and \( q' = q/(q-1) \).

By assumption (2), we see that \( L_q(\Omega_+)^N = J_q(\Omega_+) \oplus G_q(\Omega_+) \), where \( G_q(\Omega_+) = \{ \nabla \varphi \mid \varphi \in \hat{H}^1_{q,0}(\Omega_+) \} \) and the symbol \( \oplus \) here denotes the direct sum of \( J_q(\Omega_+) \) and \( G_q(\Omega_+) \).
Theorem 2. Let $1 < p, q < \infty$ with $2/p + N/q \neq 1$, and $T > 0$. Let $u_{0+} \in B^{3-2/p}_{q,p} (\Omega)$ and let $f, g, h$ be functions appearing in Equation (37) satisfying the following conditions:

$$f \in L_p((0, T), H^1_0(\Omega))^N, \quad g \in L_p((0, T), H^2_0(\Omega))^N \cap H^1_0((0, T), L_q(\Omega)), \quad g \in H^1_0((0, T), H^1_0(\Omega))^N, \quad h \in L_p((0, T), H^2_0(\Omega))^N \cap H^1_0((0, T), L_{q,0}(\Omega))^N).$$

Assume that $u_0, g$ and $h$ satisfy the following compatibility conditions:

$$\text{div } u_{0+} = g|_{t=0} \quad \text{on } \Omega, \quad u_{0+} - g|_{t=0} \in J_q(\Omega),$$

$$\left( \nu D(u_{0+}) \mathbf{n} \right)_t = h|_{t=0} \quad \text{on } \Gamma, \quad \text{provided } 2/p + 1/q < 1,$$  \hspace{1cm} (41)

where $d_r = d - <d, n>n$. Then, problem (37) admit unique solutions $v$ and $q$ with

$$v \in L_p((0, T), H^2_0(\Omega))^N \cap H^1_0((0, T), H^1_0(\Omega))^N, \quad q \in L_p((0, T), H^2_0(\Omega) + H^1_{q,0}(\Omega)), \quad \nabla^2 q \in L_p((0, T), L_{q,0}(\Omega)^N)$$

and $\nabla^2 q \in L_p((0, T), L_{q,0}(\Omega)^N)$ possessing the estimates:

$$\|\partial_t v\|_{L_p((0,T),H^1_0(\Omega))} + \|v\|_{L_p((0,T),H^2_0(\Omega))} \leq C e^{\gamma_1 T} \{ \|u_0\|_{B^{3-2/p}_{q,p}(\Omega)} + F_v(f, g, h) \}$$

with

$$F_v(f, g, h) = \|f\|_{L_p((0,T),L^2(\Omega_0))} + \|g\|_{L_p((0,T),L^2(\Omega_0))} + \|h\|_{L_p((0,T),L^2(\Omega_0))}$$

for some positive constants $C$ and $\gamma_1$ are independent of $T$.

Remark 2. (1) Theorem 2 has been proved by Shibata [16] in the standard case where

$$v \in H^1_0((0, T), L_q(\Omega))^N \cap L_p((0, T), H^2_0(\Omega))^N).$$

But, in Theorem 2 one more additional regularity is stated, which is necessary for our approach to prove the well-posedness of Equation (3). The idea of proving how to obtain third order regularity of the fluid vector field will be given in Appendix A below.

(2) The uniqueness holds in the following sense. Let $v$ and $q$ with

$$v \in L_p((0, T), H^2_0(\Omega))^N \cap H^1_0((0, T), L_q(\Omega))^N, \quad q \in L_p((0, T), H^1_0(\Omega) + H^1_{q,0}(\Omega))$$

satisfy the homogeneous equations:

$$\rho \partial_t v - \text{Div}(v, q) = 0, \quad \text{div } v = 0 \quad \text{in } \Omega_+ \times (0, T),$$

$$\text{T}(v, q) \mathbf{n} = 0 \quad \text{on } \Gamma \times (0, T),$$

$$v|_{t=0} = 0 \quad \text{in } \Omega_+,$$

then $v = 0$ and $q = 0$.

3.2. Two Phase Problem for the Linear Electro-Magnetic Vector Field Equations

This subsection is devoted to presenting the $L^p-L^q$ maximal regularity due to Frolova and Shibata [1] for the linear electromagnetic vector field equations. The problem is formulated by a set of the following equations:

$$\mu \partial_t \mathbf{H} - \alpha^{-1} \Delta \mathbf{H} = \mathbf{f} \quad \text{in } \hat{\Omega} \times (0, T),$$

$$[[\alpha^{-1} \text{curl } \mathbf{H}] \mathbf{n} = \mathbf{h}', \quad [[\mu \text{div } \mathbf{H}]] = h_N \quad \text{on } \Gamma \times (0, T),$$

$$[[\mathbf{H} - < \mathbf{H}, \mathbf{n} > \mathbf{n}] = \mathbf{k}', \quad [[\mu \mathbf{H} \cdot \mathbf{n}]] = k_N \quad \text{on } \Gamma \times (0, T),$$

$$\mathbf{n}_- \cdot \mathbf{H}_- = 0, \quad (\text{curl } \mathbf{H}_-) \mathbf{n}_- = 0 \quad \text{on } S_- \times (0, T),$$

$$\mathbf{H}|_{t=0} = \mathbf{H}_0 \quad \text{in } \hat{\Omega}.$$
To state the main result, we make a definition.

**Definition 3.** Let $\Omega = \Omega_+ \cup \Gamma \cup \Omega_-$ be a domain given in the introduction. We say that $\Omega$ is a uniform $C^2$ domain with interface $\Gamma$ if there exist positive constants $a_1$, $a_2$, and $A$ such that the following assertion holds: For any $x_0 = (x_{01}, \ldots, x_{0N}) \in \Gamma$ there exist a coordinate number $j$ and a $C^2$ function $h(x')$ defined on $B_{a_1}(x_0')$ such that $\|h\|_{H^2_h(B_{a_1}(x_0'))} \leq A$ for $k \leq 2$ and

$$
\begin{align*}
\Gamma \cap B_{a_2}(x_0) &= \{ x \in \mathbb{R}^N \mid x_j = h(x') \ (x' \in B_{a_1}(x_0')) \} \cap B_{a_2}(x_0), \\
\Omega_+ \cap B_{a_2}(x_0) &= \{ x \in \mathbb{R}^N \mid \pm x_j > h(x') \ (x' \in B_{a_1}(x_0')) \} \cap B_{a_2}(x_0),
\end{align*}
$$

and for any $x_0 = (x_{01}, \ldots, x_{0N}) \in S_-$ there exists a coordinate number $j$ and a $C^2$ function $h(x')$ defined on $B_{a_1}(x_0')$ such that $\|h\|_{H^2_h(B_{a_1}(x_0'))} \leq A$ for $k \leq 2$ and

$$
\begin{align*}
\Omega \cap B_{a_2}(x_0) &= \{ x \in \mathbb{R}^N \mid x_j > h(x') \ (x' \in B_{a_1}(x_0')) \} \cap B_{a_2}(x_0), \\
S_- \cap B_{a_2}(x_0) &= \{ x \in \mathbb{R}^N \mid x_j = h(x') \ (x' \in B_{a_1}(x_0')) \} \cap B_{a_2}(x_0).
\end{align*}
$$

Here, we have set

$$
\begin{align*}
y' &= (y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_N) \ (y \in \{x, x_0\}), \\
B_{a_1}(x_0') &= \{ x' \in \mathbb{R}^{N-1} \mid |x' - x_0'| < a_1 \}, \\
B_{a_2}(x_0) &= \{ x \in \mathbb{R}^N \mid |x - x_0| < a_2 \}.
\end{align*}
$$

**Theorem 3.** Let $1 < p, q < \infty$, $2/p + N/q \neq 1, 2$, and $T > 0$. Assume that $\Omega$ is a uniform $C^2$ domain with interface $\Gamma$. Then, there exists a $\gamma_2$ such that the following assertion holds: Let $H_0 = B^{\frac{1}{2}-1/p}_p(\Omega)$ and let $f \in L_p((0, T), L_q(\Omega)^N)$, and let $h = (h', h_N)$, and $k = (k', k_N)$ be functions such that $\hat{h}' = h'$, $\hat{h}_N = h_N$, $\hat{k}' = k'$, and $\hat{k}_N = k_N$ for $t \in (0, T)$, where $h', h_N, k'$, and $k_N$ are functions given in the right side of (42), and the following conditions hold:

$$
e^{-\gamma t}\hat{h} \in L_p(\mathbb{R}, H^1_p(\Omega)^N) \cap H^{1/2}_p(\mathbb{R}, L_q(\Omega)^N), \quad e^{-\gamma t}\hat{k} \in L_p(\mathbb{R}, H^1_p(\Omega)^N) \cap H^{1/2}_p(\mathbb{R}, L_q(\Omega)^N)$$

for any $\gamma \geq \gamma_2$. Moreover, we assume that $H_0$, $h$, and $k$ satisfy the following compatibility conditions:

$$
|\partial_n h|_{L^1(0, T;\mathbb{R})} = h'|_{t=0}, \quad |\partial_n k|_{L^1(0, T;\mathbb{R})} = k'|_{t=0}, \quad \mu_{H_0} \cdot n = 0 \quad \text{on} \quad \Gamma, \quad \mu_{H_0} \cdot n = 0 \quad \text{on} \quad S_-
$$

provided $2/p + N/q < 1$;

$$
|H_0 - h, n > n| = k'|_{t=0}, \quad |\partial_n H_0 \cdot n| = k_N|_{t=0} \quad \text{on} \quad \Gamma, \quad \partial_n H_0 \cdot n = 0 \quad \text{on} \quad S_-
$$

provided $2/p + N/q < 2$. Then, problem (42) admits a unique solution $H$ with

$$
H \in L_p((0, T), H^2(\Omega)^N) \cap H_1^0((0, T), L_q(\Omega)^N)
$$

possessing the estimate:

$$
\|\partial_t H\|_{L_p((0, T); L_q(\Omega)^N)} + \|H\|_{L_p((0, T); H^2_q(\Omega)^N)} \leq C\gamma T \{\|H_0\|_{L^2_p(\Omega)^N} + F_H(f, \hat{h}, \hat{k})\},
$$

with

$$
\begin{align*}
F_H(f, \hat{h}, \hat{k}) &= \|f\|_{L_p(\mathbb{R}, L_q(\Omega)^N)} + \|e^{-\gamma t}\hat{h}\|_{L_p(\mathbb{R}, H^1_q(\Omega)^N)} + \|e^{-\gamma t}\hat{k}\|_{L_p(\mathbb{R}, H^1_q(\Omega)^N)}/2 \quad + \gamma^{1/2}\|e^{-\gamma t}\hat{h}\|_{L_p(\mathbb{R}, L_q(\Omega)^N)} + \|e^{-\gamma t}\hat{k}\|_{L_p(\mathbb{R}, L_q(\Omega)^N)} + \|e^{-\gamma t}\partial_t H\|_{L_p(\mathbb{R}, L_q(\Omega)^N)}
\end{align*}
$$

for any $\gamma \geq \gamma_2$ with some constant $C > 0$ independent of $\gamma$. 
Remark 3. (1) Theorem 3 was proved by Frolova and Shibata [18].

(2) The uniqueness holds in the following sense. Let \( \mathbf{H} \) with

\[
\mathbf{H} \in L_p((0, T), H^2_q(\Omega)^N) \cap H^1_p((0, T), L_q(\Omega)^N)
\]
satisfy the homogeneous equations:

\[
\mu \partial_t \mathbf{H} - \alpha^{-1} \Delta \mathbf{H} = 0 \quad \text{in } \Omega \times (0, T),
\]
\[
[[\alpha^{-1} \text{curl} \mathbf{H}]] \mathbf{n} = 0, \quad [[\mu \text{div} \mathbf{H}]] = 0 \quad \text{on } \Gamma \times (0, T),
\]
\[
[[\mathbf{H} - \mathbf{H}]], \mathbf{n} > 0] = 0, \quad [[\mu \mathbf{H} \cdot \mathbf{n}]] = 0 \quad \text{on } \Gamma \times (0, T),
\]
\[
\mathbf{n}_+ \cdot \mathbf{H}_+ = 0, \quad (\text{curl} \mathbf{H}_+) \mathbf{n}_+ = 0 \quad \text{on } S_+ \times (0, T),
\]
\[
\mathbf{H}|_{t=0} = 0 \quad \text{in } \Omega.
\]

then \( \mathbf{H} = 0 \) in \( \Omega \times (0, T) \).

4. Estimate of Non-Linear Terms

Let \( \mathbf{u}_+ \) and \( \mathbf{H}_\pm \) be \( N \)-vectors of functions such that

\[
\mathbf{u}_+ \in H^1_p((0, T), H^2_q(\Omega_+)^N) \cap L_p((0, T), H^3_q(\Omega_+)^N), \quad \mathbf{u}_+|_{t=0} = \mathbf{u}_0+,
\]
\[
\mathbf{H}_\pm \in H^1_p((0, T), L_q(\Omega_\pm)^N) \cap L_p((0, T), H^2_q(\Omega_\pm)^N), \quad \mathbf{H}_\pm|_{t=0} = \mathbf{H}_0\pm,
\]

and we shall estimate nonlinear terms \( N_1(\mathbf{u}_+, \mathbf{H}_+), \ldots, N_0(\mathbf{u}, \mathbf{H}) \) appearing in the right side of Equation (7). Here, \( \mathbf{w} = \mathbf{w}_+ \) for \( x \in \Omega_+ \) and \( \mathbf{w} = \mathbf{w}_- \) for \( x \in \Omega_- \) \( (\mathbf{w} \in \{ \mathbf{u}_+, \mathbf{H} \}) \) and \( \mathbf{u}_- \) is an extension of \( \mathbf{u}_+ \) defined in (5). For notational simplicity, we set

\[
E_T^1(\mathbf{u}_+) = \| \partial_t \mathbf{u}_+ \|_{L_p((0, T), H^1_q(\Omega_+))} + \| \mathbf{u}_+ \|_{L_p((0, T), H^2_q(\Omega_+))},
\]
\[
E_T^{2 \pm}(\mathbf{H}_\pm) = \| \partial_t \mathbf{H}_\pm \|_{L_p((0, T), L^2_q(\Omega_\pm))} + \| \mathbf{H}_\pm \|_{L_p((0, T), H^1_q(\Omega_\pm))},
\]
\[
E_T^2(\mathbf{H}) = E_T^{2+}(\mathbf{H}_+) + E_T^{2-}(\mathbf{H}_-).
\]

Moreover, let \( \mathbf{u}_+^i \) and \( \mathbf{H}_\pm^i \) \((i = 1, 2)\) be \( N \)-vectors of functions such that

\[
\mathbf{u}_+^i \in H^1_p((0, T), H^2_q(\Omega_+)^N) \cap L_p((0, T), H^3_q(\Omega_+)^N), \quad \mathbf{u}_+^i|_{t=0} = \mathbf{u}_0+,
\]
\[
\mathbf{H}_\pm^i \in H^1_p((0, T), L_q(\Omega_\pm)^N) \cap L_p((0, T), H^2_q(\Omega_\pm)^N), \quad \mathbf{H}_\pm^i|_{t=0} = \mathbf{H}_0\pm.
\]

We also consider the differences \( N_1^i = N_1(\mathbf{u}_+^i, \mathbf{H}_+^i) - N_1(\mathbf{u}_+^1, \mathbf{H}_+^1), \ldots, N_0^i = N_0(\mathbf{u}_+^i, \mathbf{H}^i) - N_0(\mathbf{u}_+^1, \mathbf{H}^1) \). Here, \( \mathbf{w} = \mathbf{w}_+ \) for \( x \in \Omega_+ \) and \( \mathbf{w} = \mathbf{w}_- \) for \( x \in \Omega_- \) \( (\mathbf{w} \in \{ \mathbf{u}_+^i, \mathbf{H}_+^i, \mathbf{H}_-^i \}) \) and \( \mathbf{u}_-^i \) is an extension of \( \mathbf{u}_+^i \) defined in (5). For notational simplicity, we assume that

\[
\| \mathbf{u}_0+ \|_{L^2_{T+}((0, T), H^2_q(\Omega_+))} + \| \mathbf{H}_0+ \|_{L^2_{T+}((0, T), H^1_q(\Omega_+))} + \| \mathbf{H}_0- \|_{L^2_{T-}((0, T), H^1_q(\Omega_-))} \leq B
\]

for some constant \( B > 0 \). In what follows, we assume that \( 2 < p < \infty, N < q < \infty \) and \( 2/p + N/q < 1 \). To estimate nonlinear terms, we use the following inequalities which follows from Sobolev’s inequality.

\[
\| f \|_{L^2_q(\Omega_\pm)} \leq C \| f \|_{H^2_q(\Omega_\pm)},
\]
\[
\| g \|_{H^2_q(\Omega_\pm)} \leq C \| g \|_{H^1_q(\Omega_\pm)} \| g \|_{H^2_q(\Omega_\pm)},
\]
\[
\| g \|_{H^3_q(\Omega_\pm)} \leq C \| f \|_{H^1_q(\Omega_\pm)} \| g \|_{H^2_q(\Omega_\pm)} + \| f \|_{H^2_q(\Omega_\pm)} \| g \|_{H^2_q(\Omega_\pm)},
\]
\[
\| f \|_{L^2_{T+}((0, T), W^{1/q}_q(\Gamma))} \leq C \| f \|_{L^2_{T+}((0, T), W^{1/q}_q(\Gamma))},
\]
\[
\| g \|_{L^2_{T+}((0, T), W^{1/q}_q(\Gamma))} \leq C \| f \|_{L^2_{T+}((0, T), W^{1/q}_q(\Gamma))} \| g \|_{L^2_{T+}((0, T), W^{1/q}_q(\Gamma))}.
\]
By Hölder’s inequality and (5), we have
\[
\|\Psi_w\|_{L^q((0,T),H^2_\pm(\Omega))} \leq CT^{1/p'} E_T^1(w_+) \quad \text{for } w \in \{u, u^1, u^2\}, \\
\|\Psi_{u^i} - \Psi_{u^j}\|_{L^q((0,T),H^2_\pm(\Omega))} \leq CT^{1/p'} E_T^1(u^i_+ - u^j_+). 
\]  
(50)

In view of (49) and (50), choosing \( T > 0 \) so small, we may assume that
\[
\|\Psi_w\|_{L^q((0,T),L^\infty(\Omega))} \leq \delta \quad \text{for } w \in \{u, u^1, u^2\}. 
\]  
(51)

In the following, for simplicity, choosing \( T > 0 \) small, we also assume that
\[
T^{1/p'} E_T^1(w) + B \leq 1 \quad \text{for } w \in \{u_+, u^1_+, u^2_+\}. 
\]  
(52)

We may assume that the unit outer normal \( n \) to \( \Gamma \) is defined on \( \mathbb{R}^N \) and \( \|n\|_{H^\infty(\mathbb{R}^N)} < \infty \) because \( \Omega_+ \) is a uniform \( C^3 \) domain. Thus, setting
\[
[A_i(\Psi_w)]_{T,2} := \|A_i(\Psi_w)\|_{L^q((0,T),L^\infty(\Omega))} + \|\nabla A_i(\Psi_w)\|_{L^q((0,T),H^2(\Omega))} 
\]
by (51), (49), (50), and (52) we have
\[
[A_i(\Psi_w)]_{T,2} \leq C, 
\]  
(53)

with some constant \( C > 0 \) for \( i = 1, \ldots, 9 \) and \( w \in \{u, u^1, u^2\} \), where we have set \( A_5(\Psi_w) = (A_{5+}(\Psi_w), A_{5-}(\Psi_w)) \) and \( A_6(\Psi_w) = (A_{61}(\Psi_w), A_{62}(\Psi_w)) \).

We first consider \( N_1(u_+, H_+) \) and \( N_1' = N_1(u^1_+, H^1_+) - N_1(u^2_+, H^2_+) \). Recall (24). Applying (49) and using (50), we have
\[
\|\Psi_u \otimes (\partial_t u_+, \nabla u_+)\|_{L^p((0,T),H^2_\pm(\Omega_+))} \leq CT^{1/p'} E_T^1(u_+)^2, \\
\|u_+ \otimes \nabla u_+\|_{L^p((0,T),H^2_\pm(\Omega_+))} \leq CT^{1/p'} \|u_+\|_{L^q((0,T),H^2_\pm(\Omega_+))}^2, \\
\|\nabla \Psi_u \otimes \nabla u_+\|_{L^p((0,T),H^2_\pm(\Omega_+))} \leq CT^{1/p'} E_T^1(u_+)^2, \\
\|H_+ \otimes \nabla H_+\|_{L^p((0,T),H^2_\pm(\Omega_+))} \leq C\|H_+\|_{L^q((0,T),H^2_\pm(\Omega_+))} \|H_+\|_{L^p((0,T),H^2_\pm(\Omega_+))}. 
\]  
(54)

By real interpolation theory, we see that
\[
\sup_{t \in (0,T)} \|v_{\pm}(\cdot, t)\|_{\dot{H}^{\ell+2/p}_{6,p}(\Omega_+)} \\
\leq C(\|v|_{t=0}\|_{\dot{H}^{\ell+2/p}_{6,p}(\Omega_+)} + \|\partial_t v_{\pm}\|_{L^p((0,T),H^\ell_{6,p}(\Omega_+))} + \|v_{\pm}\|_{L^p((0,T),H^\ell_{6,p}(\Omega_+))}) 
\]  
(55)

(\( \ell = 0, 1 \)). In order to prove this, we make a few preparations. For a \( X \)-valued function \( f(\cdot, t) \) defined for \( t \in (0, T) \), where \( X \) is a Banach space, we set
\[
e_T[f](\cdot, t) = \begin{cases} 
0 & \text{for } t < 0, \\
f(\cdot, t) & \text{for } 0 < t < T, \\
f(\cdot, 2T - t) & \text{for } T < t < 2T, \\
0 & \text{for } t > 2T.
\end{cases}
\]  
(56)

Then, \( e_T[f](\cdot, t) = f(\cdot, t) \) for \( t \in (0, T) \). If \( f|_{t=0} = 0 \), then
\[
\partial_t e_T[f](\cdot, t) = \begin{cases} 
0 & \text{for } t < 0, \\
\partial_t f(\cdot, t) & \text{for } 0 < t < T, \\
-(\partial_t f)(\cdot, 2T - t) & \text{for } T < t < 2T, \\
0 & \text{for } t > 2T.
\end{cases}
\]  
(57)
In particular, we have
\[
\|e_T[f]\|_{L_p((0,T),X)} \leq 2\|f\|_{L_p((0,T),X)},
\]
\[
\|\partial_t e_T[f]\|_{L_p((0,T),X)} \leq 2\|\partial_t f\|_{L_p((0,T),X)}.
\]  
(58)

Let \( w_\pm \) be a \( N \)-vector of function defined on \( \Omega_\pm \) and let \( E_\pm[w_\pm] \) be an extension of \( w_\pm \) to \( \Omega_\mp \) for which
\[
E_\pm[w_\pm] = w_\pm \quad \text{for} \quad x \in \Omega_\pm,
\]
\[
\lim_{x \to x_0} \partial^\alpha_x E_\pm[w_\pm](x,t) = \lim_{x \to x_0} \partial^\alpha_x w_\pm(x,t) \quad \text{for} \quad |\alpha| \leq 1 \quad \text{and} \quad x_0 \in \Gamma,
\]
\[
\|E_\pm[w_\pm](\cdot,t)\|_{H^q_0(\Omega)} \leq C\|w_\pm(\cdot,t)\|_{H^q_0(\Omega_\pm)} \quad \text{for} \quad i = 0,1,2.
\]  
(59)

Let \( E_{\mathbb{R}^N}[E_\pm[w_\pm]] \) be an extension of \( E_\pm[w_\pm] \) to \( \mathbb{R}^N \) for which
\[
E_{\mathbb{R}^N}[E_\pm[w_\pm]] = E_\pm[w_\pm] \quad \text{on} \quad \Omega,
\]
\[
\|E_{\mathbb{R}^N}[E_\pm[w_\pm]]\|_{\bar{B}^{p(1+2(1\rho))))}_{\mathbb{R}^N}} \leq C\|w_\pm\|_{\bar{B}^{p(1+2(1\rho))))}_{\Omega_\pm}} \quad (\ell = 0,1).
\]  
(60)

For \( v_0 \in \bar{B}^{p(1+2(1\rho))))_{\mathbb{R}^N} \), let
\[
T(t)v_0 = e^{(-1+\Delta)t}v_0
\]  
(61)

be a \( C^0 \) analytic semigroup satisfying the condition: \( T(0)v_0 = v_0 \) and possessing the estimate:
\[
\|T(\cdot)v_0\|_{L_p((0,\infty),H^{1+2(1\rho))})} + \|\partial_t T(\cdot)v_0\|_{L_p((0,\infty),H^{1\rho})} \leq C\|v_0\|_{\bar{B}^{p(1+2(1\rho))}}_{\mathbb{R}^N})}
\]  
(62)

for \( \ell = 0,1,2 \). Let \( w_\pm \) be defined on \( \Omega_\pm \times (0,T) \) and set \( w_\pm = w_\pm |_{t=0} \). Let \( \psi(t) \) be a function that equals one for \( t > -1 \) and zero for \( t < -2 \) and let
\[
T[E_\pm[w_\pm]](t) = \psi(t)T(|t|)E_{\mathbb{R}^N}[E_\pm[w_\pm]],
\]  
(63)

Then, by (60) and (62)
\[
T[E_\pm[w_\pm]](0) = E_\pm[w_\pm] \quad \text{in} \quad \Omega,
\]
\[
\|T[E_\pm[w_\pm]]\|_{L_p((0,T),H^{1+2(1\rho))})} + \|\partial_t T[E_\pm[w_\pm]]\|_{L_p((0,T),H^{1\rho})} \leq C\|w_\pm\|_{\bar{B}^{p(1+2(1\rho))}_{\Omega_\pm}}.
\]  
(64)

Set
\[
E[E_\pm[w_\pm]] = T[E_\pm[w_\pm]] + eT[E_\pm[w_\pm]] - T[E_\pm[w_\pm]]].
\]  
(65)

Obviously, \( E[E_\pm[w_\pm]] = E_\pm[w_\pm] \) for \( t \in (0,T) \). Then, (55) is guaranteed by (58), (59) and (64) as follows:
\[
\sup_{t \in (0,T)} \|v_\pm(\cdot,t)\|_{\bar{B}^{p(1+2(1\rho))}_{\Omega_\pm}}} = \sup_{t \in (0,T)} \|E[E_\pm[v_\pm]](\cdot,t)\|_{\bar{B}^{p(1+2(1\rho))}_{\mathbb{R}^N)} 
\]
\[
\leq C(\|v\|_{t=0})_{\bar{B}^{p(1+2(1\rho))}_{\Omega_\pm}} + \|\partial_t v_\pm\|_{L_p((0,T),H^{1\rho})} + \|v_\pm\|_{L_p((0,T),H^{1+2(1\rho))}}.
\]

Combining (55) with (48) leads to
\[
\|w_\pm\|_{L_p((0,T),H^{1\rho})} \leq C(B + E_\pm^1(w_\pm)) \quad \text{for} \quad w \in \{u,u^1,u^2\},
\]
\[
\|z_\pm\|_{L_p((0,T),H^{1\rho})} \leq C(B + E_\pm^2(z_\pm)) \quad \text{for} \quad z \in \{H,H^1,H^2\},
\]  
(66)
because $p_{g_0}^{\ell+2/p}((\Omega_\pm))$ is continuously imbedded into $H^{\ell+1}_q((\Omega_\pm))$ as follows from $2 - 2/p > 1$, that is, $2 < p < \infty$. Moreover,

$$\|H_+\|_{L_\infty((0,T),H^1_\ell(\Omega_+))} \leq C(B + T^{s/p'}(1+s)E_{T}^{2+}(H_+)),$$

(67)

provided $0 < T < 1$. In fact, we write $H_+ = \hat{H}_{0+} + v$ with $v = H_+ - \hat{H}_{0+}$. Since $v_{|t=0} = 0$, we have

$$\|v(\cdot,t)\|_{L_2(\Omega_+)} \leq \int_0^t \|\partial_s H_+(\cdot,s)\|_{L_2(\Omega_+)} \leq T^{1/p'}E_{T}^{2+}(H_+)$$

for $t \in (0,T)$. On the other hand, choosing $s \in (0,1-2/p)$ yields that $B_{g_0}^{2(1-1/p)}((\Omega_+))$ is continuously imbedded into $W^{1,s}_q((\Omega_+))$, and so by (55)

$$\|v\|_{W^{1,s}_q((\Omega_+))} \leq C\|v\|_{B_{g_0}^{2(1-1/p)}((\Omega_+))} \leq C(B + E_{T}^{2+}(H_+)).$$

Since $\|v\|_{H_q^1((\Omega_+))} \leq C\|v\|_{L^{s/(1+s)}_q(\Omega_+)}\|v\|_{W^{1,s}_q((\Omega_+))}$, we have (67) provided $0 < T < 1$. Combining (54), (53), (66), and (67), we have

$$\|N_1(u_+,H_+)\|_{L_p((0,T),H^1_\ell(\Omega_+))} \leq C(T^{1/p'}E_{T}^1(u_2)^2 + T^{1/p}(B + E_{T}^1(u_+))^2 + (B + T^{s/p'}(1+s)E_{T}^{2+}(H_+))E_{T}^{2+}(H_+)).$$

(68)

We next consider $N_1 = N_1(u_+,H_+) - N_1(u_+^2,H_+)$, which is represented by $N_1 = N_{11} + N_{12}$ with

$$N_{11} = A_1(\Psi_{u^1}) - A_1(\Psi_{u^2})(K_1^1, K_2^1, K_3^1, K_4^1),$$

$$N_{12} = A_1(\Psi_{u^2})(K_1^1 - K_1^2, K_2^1 - K_2^2, K_3^1 - K_3^2, K_4^1 - K_4^2),$$

$$K_1^1 = \Psi_{u^1} \otimes (\partial_s u_+^1 \otimes \nabla^2 u_+^1), \quad K_2^1 = u_+^1 \otimes \nabla u_+^1, \quad K_3^1 = \nabla \Psi_{u^1} \otimes \nabla u_+^1, \quad K_4^1 = H_+^1 \otimes \nabla H_+^1.$$

Representing

$$A_1(\Psi_{u^1}) - A_1(\Psi_{u^2}) = \int_0^1 (d_k A_1)(\Psi_{u^2} + \theta(\Psi_{u^1} - \Psi_{u^2})) d\theta(\Psi_{u^1} - \Psi_{u^2}),$$

(69)

by (49), (51), (50), and (52), we have

$$\|A_1(\Psi_{u^1}) - A_1(\Psi_{u^2})\|_{L_\infty((0,T),H^1_\ell(\Omega_+))} \leq C T^{1/p'}\|u_+^1 - u_+^2\|_{L_p((0,T),H^1_\ell(\Omega_+))}$$

(70)

for $i = 1, \ldots, 9$. Estimating $\|(K_1^1, \ldots, K_4^1)\|_{L_p((0,T),H^1_\ell(\Omega_+))}$ in the same manner as in proving (68), we have

$$\|N_{11}\|_{L_p((0,T),H^1_\ell(\Omega_+))} \leq C T^{1/p'}(B^2 + E_{T}^1(u_+^1)^2 + E_{T}^{2+}(H_+^1)^2)E_{T}^1(u_+^1 - u_+^2),$$

(71)

where we have used $0 < T < 1$ and $(B + T^{s/p'}(1+s)E_{T}^{2+}(H_+))E_{T}^{2+}(H_+) \leq (1/2)B^2 + (3/2)E_{T}^{2+}(H_+)^2$. To estimate $N_{12}$ we use the fact:

$$\|f_1 f_2 - g_1 g_2\| \leq \|f_1 - g_1\|\|f_2\| + \|f_2 - g_2\|\|g_1\|.$$

(72)
Thus, by (49), (50), (52), (66), and (67), we have
\[
\|K_1 - K_2^2\|_{L^p((0,T),H^2_0(\Omega))} 
\leq C(\|\mathbf{\Psi}_{\text{u}^1} - \mathbf{\Psi}_{\text{u}^2}\|_{L^p((0,T),H_0^2(\Omega))} + \|\mathbf{\Xi}_{\text{u}^1} - \mathbf{\Xi}_{\text{u}^2}\|_{L^p((0,T),H_0^2(\Omega))}) \\
+ \|\mathbf{\Xi}_{\text{u}^1} - \mathbf{\Xi}_{\text{u}^2}\|_{L^p((0,T),H_0^2(\Omega))} E_{L^2}(u_1^1 - u_2^1) \\
\leq CT^{1/p}(E_{L^2}(u_1^1) + E_{L^2}(u_2^1)) E_{L^2}(u_1^1 - u_2^1);
\]
\[
\|K_2^3 - K_3^2\|_{L^p((0,T),H^2_0(\Omega))} 
\leq CT^{1/p}(E_{L^2}(u_1^1) + E_{L^2}(u_2^1)) E_{L^2}(u_1^1 - u_2^1);
\]
\[
\|K_3^3 - K_2^3\|_{L^p((0,T),H^2_0(\Omega))} 
\leq CT^{1/p}(E_{L^2}(u_1^1) + E_{L^2}(u_2^1)) E_{L^2}(u_1^1 - u_2^1);
\]
\[
\|K_4 - K_3^2\|_{L^p((0,T),H^2_0(\Omega))} 
\leq C(\|H_1^1 - H_2^1\|_{L^p((0,T),H^2_0(\Omega))} E_{L^2}(H_1^1) + \|H_2^1\|_{L^p((0,T),H^2_0(\Omega))} E_{L^2}(H_2^1) + \|H_1^2 - H_2^2\|_{L^p((0,T),H^2_0(\Omega))} E_{L^2}(H_1^2 - H_2^2)) \\
\leq C(T^{1/(1+p)} E_{L^2}(H_1^1) + \|H_2^1\|_{L^p((0,T),H^2_0(\Omega))} E_{L^2}(H_2^1) + (B + T^{1/(1+p)} E_{L^2}(H_2^1)) E_{L^2}(H_1^2 - H_2^2)) \\
\leq C\{B + T^{p/(1+p)} E_{L^2}(H_1^1) + E_{L^2}(H_2^1))\} E_{L^2}(H_1^1 - H_2^2),
\]
which, combined with (53) and (71), gives that
\[
\|N_1\|_{L^p((0,T),H^2_0(\Omega))} \leq C\{T^{1/p}(B^2 + E_{L^2}(u_1^1)^2 + E_{L^2}(u_2^1)^2) + \|H_2^1\|_{L^p((0,T),H^2_0(\Omega))} E_{L^2}(H_2^1) + \|H_1^2 - H_2^2\|_{L^p((0,T),H^2_0(\Omega))} E_{L^2}(H_1^2 - H_2^2)) \\
(73) + (B + T^{p/(1+p)} E_{L^2}(H_1^1) + E_{L^2}(H_2^1))\} E_{L^2}(H_1^1 - H_2^2).
\]
We now estimate $N_2$, $N_3$, $N_2'$, and $N_3'$. By (49), (50), (66), and (53), to the formula of $N_2(u_+, H_+)$ given in (25), we have
\[
\|N_2(u_+)\|_{L^p((0,T),H^2_0(\Omega))} \leq C\{A_2(\mathbf{\Psi}_{u})\} T_{\Omega}^2 \|\mathbf{\Psi}_{u}\|_{L^p((0,T),H^2_0(\Omega))} \|\nabla u_+\|_{L^p((0,T),H^2_0(\Omega))} \\
\leq CT^{1/p} E_{L^2}(u_+)^2. 
(74)
\]
By (51), (50), (69), (52), and (66),
\[
\begin{align*}
\|\partial_t A_1(\mathbf{\Psi}_{u})\|_{L^p((0,T),H^2_0(\Omega))} & \leq C\|u_+\|_{L^p((0,T),H^2_0(\Omega))} \leq C(B + E_{L^2}(w_+)) \quad \text{for } w \in \{u, u^1, u^2\}; \\
\|\partial_t A_1(\mathbf{\Psi}_{u}) - A_1(\mathbf{\Psi}_{u})\|_{L^p((0,T),H^2_0(\Omega))} & \leq C\|u_+\|_{L^p((0,T),H^2_0(\Omega))} \\
& + \|u_+\|_{L^p((0,T),H^2_0(\Omega))} \|\nabla u_+\|_{L^p((0,T),H^2_0(\Omega))} \|\nabla u_+\|_{L^p((0,T),L^2(\Omega))} \|\mathbf{\Psi}_{u} - \mathbf{\Psi}_{u}\|_{L^p((0,T),H^2_0(\Omega))} \\
& \leq CE_1(u_+ - u_+). 
\end{align*}
(75)
\]
By (49), (52), (50), (66), (53), and (75), we have
\[
\|\partial_t N_2(u_+)\|_{L^p((0,T),L^2(\Omega))} \\
\leq C\{\|\partial_t A_2(\mathbf{\Psi}_{u})\|_{L^p((0,T),H^2_0(\Omega))} \|\mathbf{\Psi}_{u}\|_{L^p((0,T),H^2_0(\Omega))} T^{1/p} \|\nabla u_+\|_{L^p((0,T),L^2(\Omega))} \\
+ T^{1/p} A_2(\mathbf{\Psi}_{u})\|_{L^p((0,T),H^2_0(\Omega))} \|\nabla u_+\|_{L^p((0,T),L^2(\Omega))} + \|A_2(\mathbf{\Psi}_{u})\|_{T_{\Omega}^2} \|\mathbf{\Psi}_{u}\|_{L^p((0,T),H^2_0(\Omega))} \|\nabla u_+\|_{L^p((0,T),L^2(\Omega))}\} \\
\leq C\{T^{1/p} E_{L^2}(u_+)^2 + T^{1/p}(B + E_{L^2}(u_+)^2)\};
\]
\[
\|N_3(u_+)\|_{L^p((0,T),H^2_0(\Omega))} \\
\leq CT^{1/p} A_3(\mathbf{\Psi}_{u}) T_{\Omega}^2 \|\mathbf{\Psi}_{u}\|_{L^p((0,T),H^2_0(\Omega))} \|u_+\|_{L^p((0,T),H^2_0(\Omega))} \\
\leq CT^{1/p} (B + E_{L^2}(u_+)^2); \\
\|\partial_t N_3(u_+)\|_{L^p((0,T),H^2_0(\Omega))}.
\]
which, combined with (74), yields that
\[
\|N_2(u_+ - u_2)\|_{L_t^p(0,T),H^2_t(\Omega)} + \|\partial_t N_2(u_+)\|_{L_t^p(0,T),L^q_t(\Omega)} + \|N_3(u_+)\|_{H^2_t((0,T),H^1_t(\Omega))} \leq C\{T^{1/p}(E_1^1(u_+)) + T^{1/p}(B + E_1^1(u_+))^2\},
\]

(76)

To estimate $N_2$ and $N_3$, we write $N_2 = N_{21} + N_{22}$ and $N_3 = N_{31} + N_{32}$ with
\[
N_{21} = (A_2(\Psi u_1) - A_2(\Psi u_2))(\Psi u_1 \otimes \nabla u_1^1 - \Psi u_2 \otimes \nabla u_2^1), \quad N_{22} = A_2(\Psi u_2)(\Psi u_1 \otimes \nabla u_1^2 - \Psi u_2 \otimes u_2^2)
\]
\[
N_{31} = (A_3(\Psi u_1) - A_3(\Psi u_2))(\Psi u_1 \otimes u_1^1 - \Psi u_2 \otimes u_2^1), \quad N_{32} = A_3(\Psi u_2)(\Psi u_1 \otimes u_1^2 - \Psi u_2 \otimes u_2^2).
\]

Employing the same argument as in proving (76) and using (52), (70) and (75), we have
\[
\|N_{21}\|_{L_t^p(0,T),H^2_t(\Omega))} \leq CT^{1/p}(E_1^1(u_+))E_1^1(u_+^1 - u_+^2),
\]
\[
\|\partial_t N_{21}\|_{L_t^p(0,T),L^q_t(\Omega)} \leq C(T^{1/p}(E_1^1(u_+)) + T^{1/p}(B + E_1^1(u_+)))E_1^1(u_+^1 - u_+^2);
\]
\[
\|N_{31}\|_{L_t^p(0,T),H^2_t(\Omega))} \leq CT(B + E_1^1(u_+))E_1^1(u_+^1 - u_+^2),
\]
\[
\|\partial_t N_{31}\|_{L_t^p(0,T),L^q_t(\Omega)} \leq C(T^{1/p}(E_1^1(u_+)) + T^{1/p}(B + E_1^1(u_+)))E_1^1(u_+^1 - u_+^2).
\]

By (52), (50), (53), (72), (75), and (66), we have
\[
\|N_{22}\|_{L_t^p(0,T),L^q_t(\Omega)} \leq C(A_2(\Psi u_1) - A_2(\Psi u_2))(\Psi u_1 - \Psi u_2)^2 + \|\nabla u_1^1\|_{L_t^p(0,T),H^2_t(\Omega)})^2 \leq C(T^{1/p}(E_1^1(u_+)) + E_1^1(u_+^1 - u_+^2);
\]
\[
\|\partial_t N_{22}\|_{L_t^p(0,T),L^q_t(\Omega)} \leq C((\Psi u_1 - \Psi u_2)^2 + \|\nabla u_1^1\|_{L_t^p(0,T),H^2_t(\Omega)})^2 \leq C(T^{1/p}(E_1^1(u_+)) + E_1^1(u_+^1 - u_+^2)).
\]

(78)

Employing the same argument as in proving the second inequality in (78), we also have
\[
\|N_{32}\|_{L_t^p(0,T),H^2_t(\Omega))} \leq CT^{1/p}(E_1^1(u_+^1) + E_1^1(u_+^2))E_1^1(u_+^1 - u_+^2),
\]
\[
\|\partial_t N_{32}\|_{L_t^p(0,T),H^2_t(\Omega))} \leq C(T^{1/p}(E_1^1(u_+^1) + E_1^1(u_+^2)) + T^{1/p}(B + E_1^1(u_+^1) + E_1^1(u_+^2)))E_1^1(u_+^1 - u_+^2),
\]

(79)
which, combined with (77) and (78), yields that

\[
\|\mathcal{N}_2\|_{L_p((0,T),H^2_{\Omega}(\Omega))} + \|\partial_t \mathcal{N}_2\|_{L_p((0,T),L_2(\Omega))} \|\partial_t \mathcal{N}_2\|_{L_p((0,T),H^1_{\Omega}(\Omega))} 
\leq C(T^{1/p}E_T^1(u^1) + T^{1/p}((B + E_T^1(u^1)))E_T^1(u^1 - u_0^1)). \tag{79}
\]

We now estimate \( \mathcal{N}_4 \) and \( \mathcal{N}_4 \). Applying (49) to the formula given in (28), we have

\[
\left\| \mathcal{N}_4(u^1, H^1) \right\|_{L_p((0,T),H^2(\Omega))} 
\leq C \{ \|\mathcal{A}_4(\Psi u)\|_{L_2((0,T),H^2(\Omega))} \|\nabla u^1\| + \|\Psi u\|_{L_2((0,T),H^2(\Omega))} \|\nabla u^1\| + \|H^1\|_{L_2((0,T),H^2(\Omega))} \|H^1\|_{L_2((0,T),H^2(\Omega))} \}.
\]

Thus, by (50), (53), (67), (52), and (75), we have

\[
\left\| \partial_t \mathcal{N}_4(u^1, H^1) \right\|_{L_p((0,T),L_2(\Omega))} \leq C \{ (T^{1/p}E_T^1(u^1))^2 + (B + T^{1/p}E_T^1(u^1))(B + E_T^1(u^1))^2 \}
\]

To estimate \( \mathcal{N}_4 \), we write \( \mathcal{N}_4 = \mathcal{N}_{41} + \mathcal{N}_{42} \) with

\[
\mathcal{N}_{41} = \mathcal{A}_4(\Psi u^1) - \mathcal{A}_4(\Psi w), \Psi u^1 \odot \nabla u^1, H^1 \odot H^1,
\]

\[
\mathcal{N}_{42} = \mathcal{A}_4(\Psi u^1) - \mathcal{A}_4(\Psi w) \odot \nabla u^1 \odot \nabla (u^1 - u_0^1), (H^1 - H^2) \odot H^1 + H^2 \odot (H^1 - H^2).
\]

By (49), we have

\[
\|\mathcal{N}_4\|_{L_p((0,T),H^2(\Omega))} \leq C \|\mathcal{A}_4(\Psi u^1) - \mathcal{A}_4(\Psi w)\|_{L_2((0,T),H^2(\Omega))} \|\nabla u^1\|_{L_p((0,T),H^2(\Omega))} \|\nabla u^1\|_{L_p((0,T),H^2(\Omega))} 
\leq C \left\{ \|\mathcal{A}_4(\Psi u^1) - \mathcal{A}_4(\Psi w)\|_{L_2((0,T),H^2(\Omega))} \|\nabla u^1\|_{L_p((0,T),H^2(\Omega))} \right\}.
\]

Thus, by (50), (52), (53), (66), (70), and (75), we have

\[
\left\| \partial_t \mathcal{N}_4(u^1, H^1) \right\|_{L_p((0,T),L_2(\Omega))} \leq C \left\{ (T^{1/p}E_T^1(u^1))^2 + (B + E_T^1(u^1))(B + E_T^1(u^1))^2 \right\}
\]

\[
\|\mathcal{N}_4\|_{L_p((0,T),H^2(\Omega))} \leq C \left\{ (T^{1/p}E_T^1(u^1))^2 + (B + E_T^1(u^1))(B + E_T^1(u^1))^2 \right\}.
\]
On the other hand, by \((49)\),
\[
\|N_{52}\|_{L_p(\Omega)} \leq C \|A_4(\Psi_\omega')\|_{L_2(\Omega)} \|\nabla u_1^1\|_{L_p(\Omega)} \|\nabla u_1^2\|_{L_p(\Omega)} + \|\Psi_\omega\|_{L_p(\Omega)} \|\nabla (u_1^1 - u_1^2)\|_{L_p(\Omega)} + \|H_1 - H_2\|_{L_p(\Omega)} \|H_1 + H_2\|_{L_p(\Omega)} + \|H_1 - H_2\|_{L_p(\Omega)} \|H_1 + H_2\|_{L_p(\Omega)}
\]

Thus, by \((50)\), \((52)\), \((53)\), \((66)\), \((67)\),
\[
\|\partial_t N_{52}\|_{L_p(\Omega)} \leq C \|A_4(\Psi_\omega')\|_{L_2(\Omega)} \|\nabla u_1^1\|_{L_p(\Omega)} \|\nabla u_1^2\|_{L_p(\Omega)} + \|\Psi_\omega\|_{L_p(\Omega)} \|\nabla (u_1^1 - u_1^2)\|_{L_p(\Omega)} + \|H_1 - H_2\|_{L_p(\Omega)} \|H_1 + H_2\|_{L_p(\Omega)} + \|H_1 - H_2\|_{L_p(\Omega)} \|H_1 + H_2\|_{L_p(\Omega)}
\]

which, combined with \((81)\), yields that
\[
\|N_{42}\|_{L_p(\Omega)} \leq C \|T^{1/p}(E_1^T(u_1^1) + E_1^T(u_1^2))\| + (B + T^{p/(1-p)}(E_1^T(H_1^1) + E_1^T(H_1^2)))\|E_1^T(H_1^1 - H_1^2)\|
\]

where we have used \(0 < T < 1\).

We now estimate \(N_5\) and \(N_5\). Applying \((49)\) to the formula given in \((26)\), we have
\[
\|N_{52}\|_{L_p(\Omega)} \leq C \|A_5(\Psi_\omega)\|_{L_2(\Omega)} \|\nabla u_1^1\|_{L_p(\Omega)} \|\nabla u_1^2\|_{L_p(\Omega)} + \|\Psi_\omega\|_{L_p(\Omega)} \|\nabla (u_1^1 - u_1^2)\|_{L_p(\Omega)} + \|H_1 - H_2\|_{L_p(\Omega)} \|H_1 + H_2\|_{L_p(\Omega)} \]

To estimate \(N_5\), we write \(N_{52} = N_{512} + N_{52}\) with
\[
N_{512} = (A_{52}(\Psi_\omega') - A_{52}(\Psi_\omega'))(K_{11}^5, K_{21}^5, K_{31}^5, K_{41}^5, K_{51}^5),
N_{52} = A_{52}(\Psi_\omega')(K_{11}^5 - K_{12}^5, K_{21}^5 - K_{22}^5, K_{31}^5 - K_{32}^5, K_{41}^5 - K_{42}^5, K_{51}^5 - K_{52}^5),
\]
where \( K^5_{\delta} = \nabla u \otimes \nabla H^2 \), \( K^5 = \Psi \otimes \nabla^2 H^2 \), \( K^5 = \nabla \Psi \otimes \nabla H^2 \), \( K^5 = \delta \cdot \nabla H^2 \), and \( K^5 = u \otimes H^2 \). By (70) and (83), we have

\[
\|\mathcal{N}_{51+}L_{p,T}(\Omega)\|_{L_p(\Omega)} \leq CT^{1/p}(B + E_T^2(H^1))(B + E_T^2(u_1, u_2)) + E_T(u_1, u_2), \tag{84}
\]

where we have used \( 0 < T < 1 \). Furthermore, by (49), (50), and (66)

\[
\|K^5 - K_{\delta}^5\|_{L_p(\Omega)} \leq CT^{1/p}(B + E_T^2(H^1)) + E_T^2(H^1 - H^2);
\]

\[
\|K^5 - K_{\delta}^5\|_{L_p(\Omega)} \leq CT^{1/p}(B + E_T^2(H^1)) + E_T^2(H^1 - H^2);
\]

\[
\|K^5 - K_{\delta}^5\|_{L_p(\Omega)} \leq CT^{1/p}(B + E_T^2(H^1)) + E_T^2(H^1 - H^2);
\]

\[
\|K^5 - K_{\delta}^5\|_{L_p(\Omega)} \leq CT^{1/p}(B + E_T^2(H^1)) + E_T^2(H^1 - H^2),
\]

which, combined with (53), yields that

\[
\|\mathcal{N}_{52\pm}L_{p,T}(\Omega)\|_{L_p(\Omega)} \leq C(T^{1/p}(B + E_T^2(H^1)) + E_T^2(H^1)) + E_T^2(H^1 - H^2) + C(T^{1/p}(B + E_T^2(H^1)) + E_T^2(H^1 - H^2)). \tag{85}
\]

Combining (84) and (85) gives that

\[
\|\mathcal{N}_5\|_{L_p(\Omega)} \leq C\{T^{1/p}(B + E_T^2(H^1)) + E_T^2(H^1)\} + T^{1/p}(B + E_T^2(H^1)) + E_T^2(H^1) + E_T^2(H^1 - H^2).
\]

We now consider \( N_6 \) and \( N_6 \). We have to extend them to \( t < 0 \). For this purpose, Let \( E_+ \) be an extension operator satisfying (59). In view of (29), we have

\[
N_6(u, H) = A_6(\Psi_u)\alpha^E - \nabla E_+[H_] - \alpha^{-1}\nabla E_+[H_-] = (B + A_6(\Psi_u))\nabla E_+[u_\pm] \otimes \nabla E_+[H_\pm];
\]

\[
N_6 = (A_6(\Psi_u) - A_6(\Psi_u))(\alpha^E - \nabla E_+[H_\pm]) + (A_6(\Psi_u) - A_6(\Psi_u))\nabla E_+[u_\pm] \otimes E_+[H_\pm]
\]

\[
+ A_6(\Psi_u)(\alpha^E - \nabla E_+[H_\pm] - \alpha^{-1}\nabla E_+[H_\pm]) + (B + A_6(\Psi_u))(\nabla E_+[u_\pm] - \nabla E_+[u_\pm]) \otimes E_+[H_\pm] + E_+[u_\pm] \otimes (\nabla E_+[H_\pm] - E_+[H_\pm]). \tag{87}
\]

on \( \Gamma \times (0, T) \). Define the extension operator \( e_T \) by (56) and let \( E_+ \) and \( E_{1,2} \) be extension operators satisfying (59) and (60), respectively. Let \( \gamma_1 \) and \( \gamma_2 \) be the positive constants appearing in respective Theorem 2 and Theorem 3 and set \( \gamma_0 = \max(\gamma_1, \gamma_2) \). Instead of (61), we set

\[
T(t)v_0 = e^{(-\gamma_0 - 1)\Delta t}v_0.
\]

Then let \( T \) and \( E \) be extension operators satisfying (63) and (65), respectively. For \( \ell = 0, 1 \), we obtain

\[
\|e^{\eta t}T(\cdot)v_0\|_{L_p(\Omega)} + \|e^{\eta t}T(\cdot)v_0\|_{L_p(\Omega)} \leq C\|v_0\|_{B_{\ell,p}(\Omega)}.
\]

and

\[
T(E_+[\omega_\pm])(0) = E_+[\omega_\pm] \quad \text{in } \Omega,
\]

\[
\|e^{\eta t}T(E_+[\omega_\pm])\|_{L_p(\Omega)} + C\|\omega_\pm\|_{B_{\ell,p}(\Omega)} \leq C\|\omega_\pm\|_{B_{\ell,p}(\Omega)}. \tag{88}
\]

We now define an extension \( \tilde{N}_6(u, H) \). Let \( \tilde{N}_6(u, H) = \tilde{N}_61(u, H) + \tilde{N}_62(u, H) \) with

\[
\tilde{N}_61(u, H) = A_6(e_T[\Psi_u])(\alpha^{-1}\nabla E_+[H_] - \alpha^{-1}\nabla E_+[H_-]).
\]
\[ \mathcal{N}_{62}(\mathbf{u}, \mathbf{H}) = (\mathcal{B} + \mathcal{A}_{62}(\mathbf{r}^2[\mathbf{Y}_{\mathbf{u}}])) \mathcal{E}[\mathcal{E}_- [\mathbf{u}_+]] \otimes \mathcal{E}[\mathcal{E}_- [\mathbf{H}_+]]. \]

To estimate \( H^{1/2}_p(\mathbb{R}, L^q(\Omega)) \) norm, we use the following lemma,

**Lemma 1.** Let \( 1 < p < \infty \) and \( N < q < \infty \). Let

\[ f \in L_\infty(\mathbb{R}, H^1_0(\Omega)) \cap H^1_0(\mathbb{R}, L^q(\Omega)), \quad g \in H^{1/2}_p(\mathbb{R}, H^1_0(\Omega)) \cap L_p(\mathbb{R}, H^1_q(\Omega)). \]

Then, we have

\[ \|fg\|_{H^{1/2}_p(\mathbb{R}, L^q(\Omega))} + \|fg\|_{L_p(\mathbb{R}, H^1_q(\Omega))} \leq C(\|\partial_t f\|_{L_\infty(\mathbb{R}, L^q(\Omega))} \|\partial_t g\|_{L_p(\mathbb{R}, H^1_q(\Omega))} + \|f\|_{L_\infty(\mathbb{R}, H^1_q(\Omega))} \|g\|_{L_p(\mathbb{R}, H^1_q(\Omega))}). \]

**Proof.** To prove Lemma 1, we use the fact that

\[ H^{1/2}_p(\mathbb{R}, L^q(\Omega)) \cap L_p(\mathbb{R}, H^1_q(\Omega)) = (L_p(\mathbb{R}, L^q(\Omega)), H^{1/2}_p(\mathbb{R}, L^q(\Omega)) \cap L_p(\mathbb{R}, H^1_q(\Omega)))[1/2], \]

where \( (\cdot, \cdot)[1/2] \) denotes a complex interpolation functor of order 1/2. We have

\[ \|fg\|_{L_p(\mathbb{R}, L^q(\Omega))} \leq C\|f\|_{L_p(\mathbb{R}, H^1_q(\Omega))} \|g\|_{L_p(\mathbb{R}, L^q(\Omega))}. \]

Thus, by complex interpolation, we have

\[ \|fg\|_{L_p(\mathbb{R}, H^1_q(\Omega))} \leq C\|f\|_{L_p(\mathbb{R}, H^1_q(\Omega))} \|g\|_{L_p(\mathbb{R}, H^1_q(\Omega))}. \]

Moreover, we have

\[ \|fg\|_{L_p(\mathbb{R}, H^1_q(\Omega))} \leq C\|f\|_{L_p(\mathbb{R}, H^1_q(\Omega))} \|g\|_{L_p(\mathbb{R}, H^1_q(\Omega))}. \]

Thus, combining these two inequalities gives the required estimate, which completes the proof of Lemma 1. \( \square \)

**Lemma 2.** Let \( 1 < p, q < \infty \). Then,

\[ H^1_p(\mathbb{R}, L^q(\Omega)) \cap L_p(\mathbb{R}, H^2_q(\Omega)) \subset H^{1/2}_p(\mathbb{R}, H^1_q(\Omega)) \]

and

\[ \|u\|_{H^{1/2}_p(\mathbb{R}, H^1_q(\Omega))} \leq C(\|u\|_{L_p(\mathbb{R}, H^1_q(\Omega))} + \|\partial_t u\|_{L_p(\mathbb{R}, L^q(\Omega))}). \]

**Proof.** For a proof, see Shibata ([17], Proposition 1). \( \square \)

By \((51)\), we may assume that

\[ \|\mathbf{r}^2[\mathbf{Y}_{\mathbf{w}}]\|_{L_\infty(\mathbb{R}, L^q(\Omega))} \leq \delta \quad \text{for} \ \mathbf{w} \in \{\mathbf{u}, \mathbf{u}^1, \mathbf{u}^2\}. \quad (89) \]
By (50), (30) and the same argument as in proving (70), we have
\[
\|A_{66}(e^t[Y_u])\|_{L^1_t(H^1_0(\Omega))} \leq CT^{1/\gamma} T^1_1(w_+) \quad \text{for} \ w \in \{u, u_1, u_2\};
\]
\[
\|A_{66}(e^t[Y_u^1]) - A_{66}(e^t[Y_u^2])\|_{L^1_t(H^1_0(\Omega))} \leq CT^{1/\gamma} T^1_1(u_1 - u_2^2).
\]
\[
(90)
\]
Employing the same as in proving (75) yields
\[
\|\partial_t(A_{66}(e^t[Y_u])\|_{L^1_t(H^1_0(\Omega))} \leq C(B + E_1^1(u_+)),
\]
\[
\|\partial_t(A_{66}(e^t[Y_u^1]) - A_{66}(e^t[Y_u^2])\|_{L^1_t(H^1_0(\Omega))} \leq CE_1^1(u_1 - u_2^2),
\]
\[
(91)
\]
and so noting that \(e^t[Y_u]\) vanishes for \(t \notin (0,2T)\), by Lemma 1, (90) and (91), we have
\[
\|e^{-\gamma t}N_{61}(u, H)\|_{L^1_t(H^1_0(\Omega))} + \gamma^{1/2}\|e^{-\gamma t}N_{61}(u, H)\|_{L^1_t(L_2(\Omega))} \leq CT^{1/\gamma}(B + E_1^1(u_+))
\]
\[
\times (\|\nabla[E[H^k]])\|_{L^1_t(L_2(\Omega))} + \|\partial_t(E[H^k]))\|_{L^1_t(L_2(\Omega))}
\]
\[
(92)
\]
Thus, applying Lemma 2, (65), (88), (58), and (59), we have
\[
\|e^{-\gamma t}N_{61}(u, H)\|_{L^1_t(H^1_0(\Omega))} + \gamma^{1/2}\|e^{-\gamma t}N_{61}(u, H)\|_{L^1_t(L_2(\Omega))} \leq CT^{1/\gamma}(B + E_1^1(u_+))(B + E_1^2(H)).
\]
\[
(93)
\]
We now estimate \(N_{62}^2(u, H)\). For this purpose we use the following estimate:
\[
\|f\|_{L^1_t(H^2_0(\Omega))} \leq C\|f\|_{L^1_t(L_2(\Omega))} \quad \text{for} \quad \gamma \in \mathbb{R}
\]
\[
(93)
\]
which follows from complex interpolation theory. Let
\[
A_1^2 = E[-[u_+]] \otimes E[-[H^+]], \quad A_2^2 = A_{62}(Y_u)A_1^1.
\]
And then, \(N_{62}^2(u, H) = BA_1^2 + A_2^2\). We further divide \(A_1^2\) into \(A_1^2 = A_1^{11} + A_1^{12} + A_1^{21} + A_1^{22}\) with
\[
A_1^{11} = T[-[u_+] + \cdot T[-[H_0^+)),
\]
\[
A_1^{12} = T[-[u_+] + e_T[-[H^k]] - T[-[H_0^+]]),
\]
\[
A_1^{21} = e_T[-[u_+] + T[-[u_+] + e_T[-[H_0^+]]),
\]
\[
A_1^{22} = e_T[-[u_+] - T[-[u_+] + e_T[-[H^k]] - T[-[H_0^+]]).
\]
By (48), (49), (66), and (88),
\[
\|e^{-\gamma t}A_1^2\|_{L^1_t(L_2(\Omega))} \leq C\|e^{-\gamma t}T[-[u_+] + e_T[-[H^k]] - T[-[H_0^+]])
\]
\[
\|
\]
\[ T^{1/p} \|T[E_{\cdot}(-u_{0+})]\|_{L_p(\mathbb{R},H^1_0(\Omega))} \leq \|e^{-\gamma t}T\|_{L_p(\mathbb{R},H_0^1(\Omega))} \|T[E_{\cdot}(-\tilde{H}_{0+})]\|_{L_\infty((0,2T),L_2(\Omega))} \]

\[ \leq CBT \|\partial_t e_tT[E_{\cdot}(-H_{+})] - T[E_{\cdot}(-\tilde{H}_{0+})]\|_{L_\infty((0,2T),L_2(\Omega))} \]

\[ \leq CBT(B + E_T^2(H_+)). \]

Using (93), we have

\[ \|e^{-\gamma t}A^2_{11}\|_{H^1_p(\mathbb{R},L_2(\Omega))} \leq C(2\gamma)^2 \]

\[ \leq C(2\gamma)^2 B^2. \]

And also, by (49), (66), (88), and (48),

\[ \|e^{-\gamma t}A^2_{11}\|_{L_p(\mathbb{R},H^1_0(\Omega))} \leq C\|e^{-\gamma t}T[E_{\cdot}(-u_{0+})]\|_{L_p(\mathbb{R},H^1_0(\Omega))} \|T[E_{\cdot}(-\tilde{H}_{0+})]\|_{L_\infty((0,2T),H^1_0(\Omega))} \]

\[ \leq C(2\gamma)^2 B^2. \]

\[ \|e^{-\gamma t}A^2_{12}\|_{L_p(\mathbb{R},H^1_0(\Omega))} \leq C\|T[E_{\cdot}(-u_{0+})]\|_{L_p(\mathbb{R},H^1_0(\Omega))} \|e_tT[E_{\cdot}(-H_{+})] - T[E_{\cdot}(-\tilde{H}_{0+})]\|_{L_p((0,2T),H^1_0(\Omega))} \]

\[ \leq CBT^{1/p}\|\partial_t e_tT[E_{\cdot}(-H_{+})] - T[E_{\cdot}(-\tilde{H}_{0+})]\|_{L_\infty((0,2T),H^1_0(\Omega))} \]

\[ \leq CB(B + E_T^2(H_+))T^{1/p}. \]

\[ \|e^{-\gamma t}A^2_{21}\|_{L_p(\mathbb{R},H^1_0(\Omega))} \leq CB(B + E_T^2(u_+))T^{1/p}; \]

\[ \|e^{-\gamma t}A^2_{22}\|_{L_p(\mathbb{R},H^1_0(\Omega))} \leq \|c\|_{H^1_p(\mathbb{R},L_2(\Omega))} \|\|\partial_t e_tT[E_{\cdot}(-H_{+})] - T[E_{\cdot}(-\tilde{H}_{0+})]\|_{L_\infty((0,2T),H^1_0(\Omega))} \]

\[ \|\partial_t e_tT[E_{\cdot}(-H_{+})] - T[E_{\cdot}(-\tilde{H}_{0+})]\|_{L_\infty((0,2T),H^1_0(\Omega))} \]

\[ \leq CBT(B + E_T^2(u_+))T^{1/p}; \]

and so we have

\[ \|e^{-\gamma t}A^2_{22}\|_{L_p(\mathbb{R},H^1_0(\Omega))} \leq C(2\gamma)^2 B^2 + T^{1/p}(B + E_T^2(u_+)). \]

which, combined with (94), yields that

\[ \|e^{-\gamma t}A^2_{11}\|_{H^1_p(\mathbb{R},L_2(\Omega))} + \|e^{-\gamma t}A^2_{12}\|_{L_p(\mathbb{R},H^1_0(\Omega))} + \|e^{-\gamma t}A^2_{21}\|_{L_p(\mathbb{R},L_2(\Omega))} + \|e^{-\gamma t}A^2_{22}\|_{L_p(\mathbb{R},L_2(\Omega))} \]

\[ \leq C\{2\gamma B^2 + (T^{1/2} + T^{1/p})(B + E_T^2(u_+)) + (B + E_T^2(H_+)) \}. \]

By (90), (91), and Lemma 1,

\[ \|e^{-\gamma t}A^2_{11}\|_{H^1_p(\mathbb{R},L_2(\Omega))} + \|e^{-\gamma t}A^2_{12}\|_{L_p(\mathbb{R},H^1_0(\Omega))} + \|e^{-\gamma t}A^2_{21}\|_{L_p(\mathbb{R},H^1_0(\Omega))} + \|e^{-\gamma t}A^2_{22}\|_{L_p(\mathbb{R},L_2(\Omega))} \]

\[ \leq CB(T^{1/2}(B + E_T^2(u_+))) + (B + E_T^2(u_+))(B + E_T^2(H_+)). \]
Combining (92), (95), and (96) yields that

\[ \| \mathbf{N}_6(u, H) \|_{L^{1/2}(R^d, L^2_0(\omega))} + \| \mathbf{N}_6(u, H) \|_{L^p(R^d; \Omega)} + \gamma^{1/2} \| \mathbf{N}_6(u, H) \|_{L^p(R^d, \Omega)} \]
\[ \leq C(1 + (B + E_1(u, u)) (B + E_1(H^1))) \]
\[ + (1 + (B + E_1(u, u)) (B + E_1(H^1))) (2^{(2 - \gamma \omega)} B^2 + (T^{1/2} + T^{1/4})(B + E_1(u, u)) (B + E_1(H^1))) \]
(97)

Recalling the formula of \( \mathbf{N}_6 \) in (87), we define an extension, \( \mathbf{N}_6^\ast \), of \( \mathbf{N}_6 \) by setting \( \mathbf{N}_6 = \mathbf{N}_6^\ast + \mathbf{N}_6^\ast + \mathbf{N}_6^\ast + \mathbf{N}_6^\ast \)

\[ \mathbf{N}_6^\ast = (A_6_1(\mathcal{E}^\ast [\mathbf{u}_1^2]), A_6_1(\mathcal{E}^\ast [\mathbf{u}_3^2]))(\alpha_+^2 \nabla \mathcal{E}[E [H^1_1]] + \alpha_+ \nabla \mathcal{E}[E [H^1_1]]) \]

\[ \mathbf{N}_6^\ast = (A_6_2(\mathcal{E}^\ast [\mathbf{u}_1^2]), A_6_2(\mathcal{E}^\ast [\mathbf{u}_3^2]))(\mathcal{E}[E [H^1_1]] \otimes \mathcal{E}[E [H^1_1]]) \]

\[ \mathbf{N}_6^\ast = A_6_1(\mathcal{E}^\ast [\mathbf{u}_1^2]) \{ \nabla \mathcal{E}[E [H^1_1]] - \nabla \mathcal{E}[E [H^1_1]] - \nabla \mathcal{E}[E [H^1_1]] - \nabla \mathcal{E}[E [H^1_1]] \} \]

\[ \mathbf{N}_6^\ast = (B + A_6_2(\mathcal{E}^\ast [\mathbf{u}_1^2]))(\mathcal{E}[E [H^1_1]] - \mathcal{E}[E [H^1_1]] - \mathcal{E}[E [H^1_1]] - \mathcal{E}[E [H^1_1]]) \]

\[ \mathbf{N}_6^\ast = \mathbf{N}_6^\ast + \mathbf{N}_6^\ast + \mathbf{N}_6^\ast + \mathbf{N}_6^\ast \]

Setting \( D_{6i} = A_6_2(\mathcal{E}^\ast [\mathbf{u}_1^2]) - A_6_2(\mathcal{E}^\ast [\mathbf{u}_3^2]) \) for notational simplicity, by (90) and (91) we have

\[ (\| \partial_t D_{6i} \|_{L^2(\Omega)} + \| \mathbf{D}_{6i} \|_{L^2(\Omega)}) \]
\[ \leq C T^{1/4} (u_1^* - u_3^*) \]
(98)

And, noting Lemma 2, we have

\[ \| \nabla \mathcal{E}[E [H^1_1]] \|_{L^{1/2}(\Omega)} + \| \nabla \mathcal{E}[E [H^1_1]] \|_{L^p(\Omega)} \]
\[ \leq C(B + E_1^2(H^1)) \]
which, combined with Lemma 1 and (98), yields that

\[ \| e^{-\gamma t} \mathbf{N}_6 \|_{L^{1/2}(\Omega)} \]
\[ + \| e^{-\gamma t} \mathbf{N}_6 \|_{L^p(\Omega)} \]
\[ + \gamma^{1/2} \| e^{-\gamma t} \mathbf{N}_6 \|_{L^p(\Omega)} \]
\[ \leq C T^{1/4} (B + E_1^2(H^1)) \]
(99)

Analogously, by (98), Lemma 1 and (95), we have

\[ \| e^{-\gamma t} \mathbf{N}_6 \|_{L^{1/2}(\Omega)} \]
\[ + \| e^{-\gamma t} \mathbf{N}_6 \|_{L^p(\Omega)} \]
\[ + \gamma^{1/2} \| e^{-\gamma t} \mathbf{N}_6 \|_{L^p(\Omega)} \]
\[ \leq C T^{1/2} \{ 2^{(2 - \gamma \omega)} B^2 + (T^{1/2} + T^{1/4})(B + E_1^2(u_1^*)) (B + E_1^2(H^1)) \} \]
(100)

Since

\[ \mathcal{E}[E [H^1_1]] - \mathcal{E}[E [H^1_3]] = \mathcal{E}[H^1_1] - \mathcal{E}[H^1_3], \]
\[ \mathcal{E}[E [u_1^2]] - \mathcal{E}[E [u_3^2]] = \mathcal{E}[u_1^2] - \mathcal{E}[u_3^2] \]
(101)
as follows from (65), by Lemma 1, Lemma 2, (90), and (91), we have

\[ \| e^{-\gamma t} \mathbf{N}_6 \|_{L^{1/2}(\Omega)} \]
\[ + \| e^{-\gamma t} \mathbf{N}_6 \|_{L^p(\Omega)} \]
\[ + \gamma^{1/2} \| e^{-\gamma t} \mathbf{N}_6 \|_{L^p(\Omega)} \]
\[ \leq C T^{1/2} (B + E_1^2(u_1^*)) E_1^2(H^1 - H^2) \]
(102)

In view of (101), by (57) we have

\[ \| e^{-\gamma t} \mathcal{E}[E [H^1_1]] - \mathcal{E}[E [H^1_3]] \|_{L^p(\Omega)} \]
\[ \leq C T \| \partial_t \mathcal{E}[u_1^2] - \mathcal{E}[u_3^2] \|_{L^p(\Omega)} \leq C T E_1^2(u_1^* - u_3^*) \]
(103)

Analogously,

\[ \| e^{-\gamma t} \mathcal{E}[E [u_1^2]] - \mathcal{E}[E [u_3^2]] \|_{L^p(\Omega)} \leq C T E_1^2(u_1^* - u_3^*) \]

\[ \| e^{-\gamma t} \mathcal{E}[E [H^1_1]] - \mathcal{E}[E [H^1_3]] \|_{L^p(\Omega)} \leq C T \| \partial_t [H^1_1] - [H^1_3] \|_{L^p(\Omega)} \]

\[ \leq C T E_1^2(H^1_1 - H^1_3) \]

Employing the same argument as in proving (92), (99) and (102), we have

\[
\|e^{-\gamma t}(E_-[u^+]_T) - E_-[u^+]_T\|_{L_p(R,H) \cap \Omega)} \leq CT^{1/p} E^{22}_T(H^1_+ - H^2_+);
\]

which, combined with Lemma 1, (90) and (91), yields that

\[
\|e^{-\gamma t}(E_-[u^+]) - E_-[u^+]_T\|_{L_p(R,H) \cap \Omega)} + \gamma^{1/2}\|e^{-\gamma t} N_6_4\|_{L_p(R,H) \cap \Omega}) \leq C(1 + 1/2)(B + E^2_T(H^1_+)) \leq (T^{1/2}(B + E^2_T(H^1_+)) E^2_T(u^+_T) - u^+_T) + (B + E^2_T(u^+_T) E^2_T(H^1_+ - H^2_+)).
\]
Recalling that \(\tilde{\alpha}\) be a solution of equations:
\[\begin{align*}
\mu \partial_t H - \alpha^{-1}\Delta H &= N_5(u, \tilde{H}) \quad \text{in } \Omega \times (0, T), \\
\\mu^{-1} \text{curl } H &\quad \text{on } \Gamma \times (0, T), \\
\\mu^{-1} \text{div } H &\quad \text{on } \Gamma \times (0, T), \\
\\mu^{-1} H \cdot n &\quad \text{on } \Gamma \times (0, T), \\
\text{curl } H - \text{curl } \tilde{H} &\quad \text{on } \Gamma \times (0, T), \\
\text{curl } H &\quad \text{on } \Gamma \times (0, T), \\
\n\end{align*}\]

Next, let \(N_1(u, H_+), N_2(u, H_+), N_3(u, H_+)\) and \(N_4(u, H_+)\) be respective non-linear terms given in (24), (25), and (28) by replacing \(H_+\) with \(H_+\), where \(H_+ = H|_{\Omega_+}\) and \(H\) is a solution of Equation (110). And then, let \(v\) be a solution of equations:
\[\begin{align*}
\rho \partial_t v_+ - \text{Div } T(v_+, q) &= N_1(u_+, H_+) \quad \text{in } \Omega_+ \times (0, T), \\
\text{div } v_+ &= N_2(u_+) \quad \text{in } \Omega_+ \times (0, T), \\
\text{div } N_3(u_+) &= \text{div } N_3(u_+) \quad \text{in } \Omega_+ \times (0, T), \\
T(v_+, q) n &= N_4(u_+, H_+) \quad \text{on } \Gamma \times (0, T), \\
v_+|_{t=0} &= u_0 \quad \text{in } \Omega_+.
\end{align*}\]

Recalling that \(E_T^f(u_+) \leq L\), in view of (49), (50), and (51) we choose \(T > 0\) so small that
\[\|\Psi u\|_{L^\infty((0,T),L^6(\Omega))} \leq C\|\Psi u\|_{L^\infty((0,T),H^2_0(\Omega))} \leq CT^{1/p'} L \leq \delta.\]

Moreover, in view of (52), we choose \(T > 0\) in such a way that
\[T^{1/p'}(E_T^f(u_+) + B) \leq T^{1/p'}(L + B) \leq 1.\]

Let \(\tilde{h} = (\tilde{N}_6(u, \tilde{H}), \tilde{N}_7(u, \tilde{H}), \tilde{N}_8(u, \tilde{H}), \tilde{N}_9(u, \tilde{H})\), and \(\tilde{k} = (\tilde{N}_6(u, \tilde{H}), \tilde{N}_9(u, \tilde{H})\), and let \(F_H(f, \tilde{h}, \tilde{k})\) a symbol given in Theorem 3. By (83), (97), (105), and (107), we have
\[\begin{align*}
F_H(N_5(u, \tilde{H}), \tilde{h}, \tilde{k}) &\leq C[T^{1/p}(B + L)^2 + T^{1/p'}(B + L)^2 + T^{1/(2p')}(B + L)^2 \\
&\quad + (1 + (B + L)T^{1/(2p')})(\nu^{2(\gamma - \gamma_0)}B^2 + (T^{1/2} + T^{1/p})(B + L)^2)]
\end{align*}\]
for any \( \gamma \geq \gamma_0 \). We fix \( \gamma = \gamma_0 \). Let \( \alpha = \min(1/p, 1/p', 1/2p', 1/2, s/p'(1+s)) \). Since \( 0 < t < 1 \), there exist positive constants \( M_1 \) and \( M_2 \) for which

\[
F_H(N_5(u, \hat{H}), \hat{h}, \hat{k}) \leq M_1B^2 + M_2((L + B)^2 + (L + B)^3)T^a.
\]

Applying Theorem 3 to Equation (110) yields that \( E_T^2(H) \leq C_e^{\gamma_0T}(B + F_H(N_5(u, \hat{H}), \hat{h}, \hat{k})) \) for some constant \( C_e \). Choosing \( T > 0 \) in such a way that \( \gamma_0T \leq 1 \) gives that

\[
E_T^2(H) \leq C_1e[B + M_1B^2 + M_2((L + B)^2 + (L + B)^3)T^a].
\]

In particular, we choose \( T > 0 \) so small that \( M_2((L + B)^2 + (L + B)^3)T^a \leq B + M_1B^2 \) and \( L > 0 \) so large that \( 4C_1e(B + M_1B^2) \leq L \), and then by (112)

\[
E_T^{2\pm}(H^\pm) \leq L/2.
\]

We next consider Equation (111). Let \( F_\nu \) be a symbol given in Theorem 2. By (68), (76), (80), (122), and (113),

\[
F_\nu(N_1(u_+, H_+), N_2(u_+), N_3(u_+), N_4(u_+, H)) \leq C(T^{1/p'}L^2 + T^{1/p}(L + B)^2 + T^{1/p(1+s)}(L^2 + (L + B)2) + (1 + B)BE_t^2(U_+))
\]

\[
\leq C[T^{1/p'}L^2 + T^{1/p}(L + B)^2 + T^{1/p(1+s)}(L^2 + (L + B)L^2)
\]

\[
+ (1 + B)BCe\{M_1B^2 + (L + B)^2 + (L + B)^3T^a\},
\]

which yields that

\[
F_\nu(N_1(u_+, H_+), N_2(u_+), N_3(u_+), N_4(u_+, H)) \leq M_3(1 + B)^2B^2 + M_4(1 + B)^2((L + B)^2 + (L + B)^3)T^a
\]

for some constants \( M_3 \) and \( M_4 \). Thus, applying Theorem 2 with \( 0 < T < 1 \) to Equation (111), we have

\[
E_T^3(v) \leq C_2e^{\gamma_1T}\{B + M_3(1 + B)^2B^2 + M_4(1 + B)^2((L + B)^2 + (L + B)^3)T^a\}
\]

for some constant \( C_2 \). Recalling that \( \gamma_1 \leq \gamma_0 \) and \( \gamma_0T \leq 1 \), choosing \( T > 0 \) so small that \( M_1(1 + B)^2((L + B)^2 + (L + B)^3)T^a \leq B + M_2(1 + B)^2B^2 \) and choosing \( L > 0 \) so large that \( L \geq 4C_2e(B + M_3(1 + B)^2B^2) \), we have \( E_T^3(v) \leq L/2 \), which implies that \((v, H_+) \in U_{T,L} \). In particular, we set \( L = 4e\max(C_1(B + M_3(1 + B)^2B^2), C_2(B + M_3(1 + B)^2B^2)) \). Let \( \Phi \) be a map acting on \((u, \hat{H}) \in U_{T,L}\) by setting \( \Phi(u, \hat{H}) = (v, \hat{H}) \), and then \( \Phi \) is a map from \( U_{T,L} \) into itself.

We now prove that \( \Phi \) is a contraction map. Let \((u_+, \hat{H}) \in U_{T,L} (i = 1, 2) \) and set \((v_+, \hat{H}) = \Phi(u_+, \hat{H}) \). In view of (51), (52), and (89), choosing \( T > 0 \) as smaller if necessary, we may assume that

\[
\|\Psi_v\|_{L_{\infty}(0,T), L_{\infty}(\Omega)} \leq \delta, \quad \|e^T[\Psi_v]\|_{L_{\infty}(0,T), L_{\infty}(\Omega)} \leq \delta, \quad T^{1/p'}(E_T^3(u) + B) \leq 1
\]

for \( i = 1, 2 \). Set

\[
v_+ = v_1^i - v_2^i, \quad H = H^1 - H^2, \quad N_1 = N_1(u_1^i, H_+^i) - N_1(u_2^i, H_+^i), \quad N_2 = N_2(u_1^i) - N_2(u_2^i), \quad N_3 = N_3(u_1^i) - N_3(u_2^i), \quad N_4 = N_4(u_1^i, H_+^i) - N_4(u_2^i, H_+^i), \quad N_5 = N_5(u_1^i, H_+^i) - N_5(u_2^i, H_+^i)
\]

for \( i = 1, 4 \), \( j = 5, 6, 9 \) and \( k = 7, 8 \). Noticing that \( v_1^i \|_{l=0} = v_2^i \|_{l=0} = u_0^+, \) and \( H_+^i \|_{l=0} = H_0^+ \) by (110), (111) we see that \( H \) satisfies the following equations:
\[ \mu \partial_t \mathbf{H} - \alpha^{-1} \Delta \mathbf{H} = N_5 \] in \( \Omega \times (0, T) \),
\[ [[\alpha^{-1} \text{curl} \mathbf{H}]] = N_6, \quad [[\mu \text{div} \mathbf{H}]] = N_7 \] on \( \Gamma \times (0, T) \),
\[ [[\mu \mathbf{H} \cdot \mathbf{n}]] = N_8, \quad [[\mathbf{H} - \mathbf{H} \cdot \mathbf{n} \mathbf{n}]] = N_9 \] on \( \Gamma \times (0, T) \),
\[ n_+ \cdot \mathbf{H}_+ = 0, \quad \text{(curl} \mathbf{H}_+)n_\pm = 0 \] on \( S_\pm \times (0, T) \),
\[ \mathbf{H}|_{t=0} = 0 \] in \( \Omega \).

and that \( \mathbf{v}_+ \) satisfies the following equations:
\[ \rho \partial_t \mathbf{v}_+ - \text{Div} T(\mathbf{v}_+, q) = N_1 \quad \text{in} \ \Omega_+ \times (0, T), \]
\[ \text{div} \mathbf{v}_+ = N_2 = \text{div} N_3 \quad \text{in} \ \Omega_+ \times (0, T), \]
\[ T(\mathbf{v}_+, q) = N_4 \quad \text{on} \ \Gamma \times (0, T), \]
\[ \mathbf{v}_+ = 0 \quad \text{on} \ S_+ \times (0, T), \]
\[ \mathbf{v}_+|_{t=0} = 0 \quad \text{in} \ \Omega_+. \]

Set \( \mathcal{H} = (N_6, N_7) \) and \( \mathcal{K} = (N_8, N_9) \). By (86), (104), (106), and (108), we have
\[ F_\mathcal{H}(N_5, \mathcal{H}, \mathcal{K}) \leq C \{ (T^{1/p}(B + L)^2 + (B + L)^2) + T^{1/p}(B + L) \} E_1^\alpha(u_1^1 - u_2^1) + E_2^T(\mathbf{H}_1^1 - \mathbf{H}_2^1) \]
for any \( \gamma \geq \gamma_0 \). Thus, choosing \( \gamma = \gamma_0 \) and noting \( 0 < T < 1 \), we have
\[ F_\mathcal{H}(N_5, \mathcal{H}, \mathcal{K}) \leq C(B + L + (B + L)^2) T^\alpha(E_1^\alpha(u_1^1 - u_2^1) + E_2^T(\mathbf{H}_1^1 - \mathbf{H}_2^1)). \]

Applying Theorem 3 to Equation (114) and using (116) gives that
\[ E_1^T(\mathbf{H}_1^1 - \mathbf{H}_2^1) \leq M_5(B + L + (B + L)^2) T^\alpha(E_1^\alpha(u_1^1 - u_2^1) + E_2^T(\mathbf{H}_1^1 - \mathbf{H}_2^1)) \]
for some constant \( M_5 \), where we have used \( \gamma_2 \leq \gamma_0 \) and \( \gamma_0 T \leq 1 \). Moreover, by (73), (79), and (82)
\[ F_0(N_1, N_2, N_3, N_4) \leq C \{ (T^{1/p'} + T^{1/p})(L + B) + (L + B)^2 \} E_1^\alpha(u_1^1 - u_2^1) + C \{ (T^{1/p}(B + L)^2 + (1 + T^{1/p}(B + L))(B + T^{1/p}(1+s)L) \} E_1^\alpha(\mathbf{H}_1^1 - \mathbf{H}_2^1), \]
which, combined with (117), leads to
\[ F_0(N_1, N_2, N_3, N_4) \leq C(B + L + (B + L)^2) T^\alpha E_1^\alpha(u_1^1 - u_2^1) + CM_5(B + L + (B + L)^2) T^\alpha(E_1^\alpha(u_1^1 - u_2^1) + E_2^T(\mathbf{H}_1^1 - \mathbf{H}_2^1)). \]

Thus, applying Theorem 2 to Equation (115) leads to
\[ E_1^T(u_1^1 - u_2^1) \leq M_6 \{ (B + L + (B + L)^2) + (B + L + (B + L)^2) \} T^\alpha \leq 1/4 \text{ in (117) and } M_6 \{ (B + L + (B + L)^2) + (B + L + (B + L)^2) \} T^\alpha \leq 1/4 \text{ in (118)} \]
gives that
\[ E_1^T(u_1^1 - u_2^1) + E_2^T(\mathbf{H}_1 - \mathbf{H}_2) \leq (1/2)(E_1^T(u_1^1 - u_2^1) + E_2^T(\mathbf{H}_1 - \mathbf{H}_2)). \]
which shows that the $\Phi$ is a contraction map. Thus, the Banach fixed point theorem yields the unique existence of a fixed point, $(u_+, \mathcal{H}_+) \in U_{L,L}$ of the map $\Phi$, which is a unique solution of Equation (7). This completes the proof of Theorem 1.

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**Appendix A. A Proof of Theorem 2**

**Appendix A.1**

To prove the maximal $L_p$-regularity, according to Shibata [14,16,17] a main step is to prove the existence of $\mathcal{R}$-solver for the following model problem:

$$
\begin{align*}
\lambda u - \text{Div} (\mu D (u) - p I) &= f \quad \text{in } \mathbb{R}^N_+,
\div u &= g = \div g \quad \text{in } \mathbb{R}^N_+,
(\mu D (u) - p I) n &= h \quad \text{on } \mathbb{R}^N_+,
\end{align*}
$$

(A1)

where $\mathbb{R}^N_+ = \{ x = (x_1, \ldots, x_N) \mid x_N > 0 \}$, $\mathbb{R}^N_0 = \{ x = (x_1, \ldots, x_{N-1}, 0) \mid (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1}_0 \}$, and $n = (0, \ldots, 0, -1)$. The $\lambda$ is a complex parameter ranging in $\Sigma_{\epsilon_0} = \{ \lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| \leq \pi - \epsilon, |\lambda| \geq \lambda_0 \}$ with $0 < \epsilon < \pi/2$ and $\lambda_0 > 0$. In [14,16], the existence of $\mathcal{R}$-solvers, $\mathcal{S}(\lambda), \mathcal{P}(\lambda)$, were proved for Equation (A1), which satisfy the following properties:

1. $\mathcal{S}(\lambda) \in \text{Hol}(\Sigma_{\epsilon_0}, \mathcal{L}(X^q_0(\mathbb{R}^N_0), H^2_q(\mathbb{R}^N_0)^N))$, $\mathcal{P}(\lambda) \in \text{Hol}(\Sigma_{\epsilon_0}, \mathcal{L}(X^q_0(\mathbb{R}^N_0), H^1_q(\mathbb{R}^N_0) + \mathcal{H}^0_{\mathcal{Q}}(\mathbb{R}^N_0)))$.

2. Problem (A1) admits unique solutions $u = \mathcal{S}(\lambda) F_3 (f, g, h)$ and $p = \mathcal{P}(\lambda) F_3 (f, g, h)$ for any $(f, g, h) \in X_0$ and $\lambda \in \Sigma_{\epsilon_0}$, where $F_3 (f, g, h) = (f, g, \lambda^{1/2} g, \lambda^{1/2} h)$.  

3. $\mathcal{R}_{\mathcal{L}(X^q_0(\mathbb{R}^N_0), L^2_q(\mathbb{R}^N_0)^N)}(\{ (\tau \delta_0)^{1/2} \mathcal{S}(\lambda) \} \mid \lambda \in \Sigma_{\epsilon_0}) \leq r_b$ \quad (j = 1, 2), $\mathcal{R}_{\mathcal{L}(X^q_0(\mathbb{R}^N_0), L^2_q(\mathbb{R}^N_0)^N)}(\{ (\tau \delta_0)^{1/2} \mathcal{P}(\lambda) \} \mid \lambda \in \Sigma_{\epsilon_0}) \leq r_b$ for $\ell = 0, 1$ with some constant $r_b$ depending on $\lambda_0$ and $\epsilon$.

Here, $\lambda = \gamma + i \tau \in \mathbb{C}$, $\mathcal{R}_{\mathcal{L}(X,Y)} T$ denotes the $\mathcal{R}$ norm of an operator family $T \subset \mathcal{L}(X,Y)$, $\mathcal{L}(X,Y)$ being the set of all bounded linear operators from $X$ into $Y$, $X_0(\mathbb{R}^N_0) = \{ (f_1, \ldots, f_k) \mid f_1, f_2, f_3, f_4, f_5 \in L^q(\mathbb{R}^N)^N, \quad f_2 \in H^1_q(\mathbb{R}^N), \quad f_3 \in L^q(\mathbb{R}^N), \quad f_5 \in H^1_q(\mathbb{R}^N)^N \}$; $X_0^q(\mathbb{R}^N) = \{ f \mid f \in L^q(\mathbb{R}^N)^N, \quad g \in H^1_q(\mathbb{R}^N), \quad g \in L^q(\mathbb{R}^N), \quad h \in H^1_q(\mathbb{R}^N)^N, \quad h = \text{div } g \}$.

The $F_1, F_2, F_3, F_4, F_5$, and $F_6$ are corresponding variables to $f$, $g$, $\lambda^{1/2} g$, $\lambda g$, $h$, and $\lambda^{1/2} h$, respectively. The norm of $X_0^q(\mathbb{R}^N)$ is defined by setting

$$
\| (F_1, \ldots, F_6) \|_{X_0^q(\mathbb{R}^N)} = \| (F_1,F_3,F_4,F_5) \|_{L^q(\mathbb{R}^N)} + \| (F_2,F_6) \|_{H^1_q(\mathbb{R}^N)^N}).
$$

In particular, we know an unique existence of solutions $u \in H^2_q(\mathbb{R}^N)^N$ and $p \in H^1_q(\mathbb{R}^N) + \mathcal{H}^0_q(\mathbb{R}^N)$ (Here, we just give an idea of obtaining third order regularities. An idea also is found in [19], Appendix 6.2). To prove Theorem 2 exactly from the $\mathcal{R}$-bounded solution operators point of view, we have to start returning the non-zero $f, g$ and $g$ situation to
the situation where \( f = g = 0 \), which needs an idea. We will give an exact proof of Theorem 2 in a forthcoming paper.) of Equation (A1) possessing the estimate:

\[
\begin{align*}
\|\lambda u\|_{L^q(\mathbb{R}^N)} + \|u\|_{H^2_q(\mathbb{R}^N)} + \|\nabla p\|_{L^q(\mathbb{R}^N)} & \leq C(\|f\|_{L^q(\mathbb{R}^N)} + \|g\|_{H^1_q(\mathbb{R}^N)} + \|\lambda^{1/2}(g, h)\|_{L^q(\mathbb{R}^N)} + \|\lambda g\|_{L^q(\mathbb{R}^N)}).
\end{align*}
\]

(A2)

We now prove that \( u \in H^2_q(\mathbb{R}^N) \) and \( \nabla p \in H^1_q(\mathbb{R}^N) \) provided that \( f \in H^2_q(\mathbb{R}^N), g \in H^2_q(\mathbb{R}^N), \) and \( h \in H^2_q(\mathbb{R}^N). \) Moreover, \( u \) and \( p \) satisfy the estimate:

\[
\begin{align*}
\|\lambda u\|_{H^1_q(\mathbb{R}^N)} + \|u\|_{H^2_q(\mathbb{R}^N)} + \|\nabla p\|_{H^1_q(\mathbb{R}^N)} & \leq C(\|f\|_{H^1_q(\mathbb{R}^N)} + \|g\|_{H^1_q(\mathbb{R}^N)} + \|\lambda g\|_{L^q(\mathbb{R}^N)} + \|\lambda g\|_{L^q(\mathbb{R}^N)}).
\end{align*}
\]

(A3)

In fact, differentiating Equation (A1) with respect to tangential variables \( x_j (j = 1, \ldots, N - 1) \) and noting that \( \partial_j u \) and \( \partial_j p \) satisfy equations replacing \( f, g = \text{div} \ u, \) and \( h \) with \( \text{div} f, \text{div} g, \text{div} h, \) by (A2) and the uniqueness of solutions we see that \( \partial_j u \in H^2_q(\mathbb{R}^N) \) and \( \nabla \partial_j p \in L^q(\mathbb{R}^N). \) Moreover, \( u \) and \( p \) satisfy:

\[
\begin{align*}
\|\lambda \partial_j u\|_{L^q(\mathbb{R}^N)} + \|\partial_j u\|_{H^2_q(\mathbb{R}^N)} + \|\nabla \partial_j p\|_{L^q(\mathbb{R}^N)} & \leq C(\|\partial_j f\|_{L^q(\mathbb{R}^N)} + \|\partial_j g, \partial_j h\|_{H^1_q(\mathbb{R}^N)} + \|\lambda^{1/2}(\partial_j g, \partial_j h)\|_{L^q(\mathbb{R}^N)} + \|\lambda g\|_{L^q(\mathbb{R}^N)}).
\end{align*}
\]

(A4)

for \( j = 1, \ldots, N - 1. \) To estimate \( \partial_N u \) and \( \partial_N p, \) we start with estimating \( \partial_N u \). In fact, from the divergence equations it follows that \( \partial_N u = -\sum_{j=1}^{N-1} \partial_j u + g, \) and so

\[
\lambda \partial_N u = -\sum_{j=1}^{N-1} \lambda \partial_j u + \lambda g, \quad \partial_N^2 u = -\sum_{j=1}^{N-1} \partial^2_N \partial_j u + \partial^2_N g,
\]

which, combined with (A4) yields that

\[
\begin{align*}
\|\lambda \partial_N u\|_{L^q(\mathbb{R}^N)} + \|\partial_N^2 u\|_{L^q(\mathbb{R}^N)} & \leq C\left\{ \sum_{j=1}^{N-1} (\|\partial_j f\|_{L^q(\mathbb{R}^N)} + \|\partial_j g, \partial_j h\|_{H^1_q(\mathbb{R}^N)} + \|\lambda^{1/2}(\partial_j g, \partial_j h)\|_{L^q(\mathbb{R}^N)} + \|\lambda g\|_{L^q(\mathbb{R}^N)}) \right\}.
\end{align*}
\]

(A5)

From the the \( N \)-th component of the first equation of Equation (A1) and \( \text{div} u = g, \) we have

\[
\lambda u_N - \mu \Delta u_N - \mu \partial_N g + \partial_N p = f_N,
\]

and so, we see that \( \partial_N^2 p \in L^q(\mathbb{R}^N) \) and

\[
\|\partial_N^2 p\|_{L^q(\mathbb{R}^N)} \leq \|\partial_N f\|_{L^q(\mathbb{R}^N)} + \|\partial_N^2 g\|_{L^q(\mathbb{R}^N)} + \|\lambda \partial_N u\|_{L^q(\mathbb{R}^N)} + \|\mu u\|_{H^2_q(\mathbb{R}^N)}. \quad (A6)
\]

From Equation (A1), we have

\[
\begin{align*}
\lambda u_j - \mu \Delta u_j &= f_j - \partial_j p + \mu \partial_N g \quad \text{in} \ \mathbb{R}^N, \\
\partial_N u_j &= -\partial_j u_N + \mu^{-1} h_j \quad \text{on} \ \mathbb{R}^N.
\end{align*}
\]

Differentiating the first equation of the above set of equations with respect to \( x_N \) and setting \( \partial_N u_j = v, \) we have

\[
\begin{align*}
\lambda v - \mu \Delta v &= \partial_N f_j - \partial_j \partial_N p + \mu \partial_N \partial_N g \quad \text{in} \ \mathbb{R}^N, \\
v &= -\partial_j u_N + \mu^{-1} h_j \quad \text{on} \ \mathbb{R}^N.
\end{align*}
\]
Thus, setting $w = v + \partial_j u_N - \mu^{-1} h_j$, we have

$$\lambda w - \mu \Delta w = \partial_N f_j - \partial_j \partial_N p + \mu \partial_N \partial_j g + (\lambda - \Delta) (\partial_j u_N - \mu^{-1} h_j) \quad \text{in} \quad \mathbb{R}^+ \times \\
\quad w = 0 \quad \text{on} \quad \mathbb{R}^0_+.$$

Thus, by a known estimate for the Dirichlet problem, we have

$$\|\lambda \partial_N u_j\|_{L^4(\mathbb{R}^+_N)} + \|\partial_N u_j\|_{H^3_0(\mathbb{R}^+_N)} \leq C \left\{ \|\partial_N f_j\|_{L^4(\mathbb{R}^+_N)} + \|\partial_j \partial_N p\|_{L^4(\mathbb{R}^+_N)} + \|\partial_j \partial_N g\|_{L^4(\mathbb{R}^+_N)} + \|\lambda \partial_j u_N\|_{L^4(\mathbb{R}^+_N)} + \|h_j\|_{L^4(\mathbb{R}^+_N)} \right\} \quad (A7)$$

Noting that $\|\lambda^{1/2} (\partial_j g, \partial_j h)\|_{L^4(\mathbb{R}^+_N)} \leq C (\|\lambda (g, h)\|_{L^4(\mathbb{R}^+_N)} + \|g, h\|_{H^3_0(\mathbb{R}^+_N)})$ and combining (A2) and (A4)–(A7), we have (A3).

Localizing the problem and using the argument above, we can show Theorem 2.

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