Minimum Tournaments with the Strong $S_k$-Property and Implications for Teaching

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Abstract

A tournament is said to have the $S_k$-property if, for any set of $k$ players, there is another player who beats them all. Minimum tournaments having this property have been explored very well in the 1960’s and the early 1970’s. In this paper, we define a strengthening of the $S_k$-property that we name “strong $S_k$-property”. We show, first, that several basic results on the weaker notion remain valid for the stronger notion (and the corresponding modification of the proofs requires only little extra-effort). Second, it is demonstrated that the stronger notion has applications in the area of Teaching. Specifically, we present an infinite family of concept classes all of which can be taught with a single example in the No-Clash model of teaching while, in order to teach a class $C$ of this family in the recursive model of teaching, order of $\log |C|$ many examples are required. This is the first paper that presents a concrete and easily constructible family of concept classes which separates the No-Clash from the recursive model of teaching by more than a constant factor. The separation by a logarithmic factor is remarkable because the recursive teaching dimension is known to be bounded by $\log |C|$ for any concept class $C$.

1 Introduction

A tournament is said to have the $S_k$-property if, for any set of $k$ players, there is another player who beats them all. In the 1960’s and early 1970’s, several researchers pursued the goal of finding the smallest number $n$ such that there exists a tournament with $n$ players which has the $S_k$-property. In the year 1963, an upper bound on this number was proven by Erdős [3] by means of the probabilistic method. Eight years later, Graham and Spencer [5] presented another upper bound, which is weaker than the bound proven by Erdős, but results from a concrete and easily constructible family of tournaments, namely the so-called quadratic-residue tournaments.

The purpose of this paper is twofold. First we bring into play a stronger version of the $S_k$-property. We demonstrate that it is surprisingly simple to transfer the afore-mentioned results to the new setting. Second, we show that the new setting has implications for teaching.
For instance, the quadratic-residue tournaments induce an infinite family of concept classes all of which can be taught with a single example in the No-Clash model of teaching while, in order to teach a class \( \mathcal{C} \) of this family in the recursive model of teaching, order of \( \log |\mathcal{C}| \) many examples are required. The family of concept classes induced by the quadratic-residue tournaments is the first concrete and easily constructible family which separates the two mentioned teaching models by more than a constant factor. The existence of concept classes like this had been shown before [7] only by means of the probabilistic method. The separation by a logarithmic factor is remarkable because the recursive teaching dimension is known to be bounded by \( \log |\mathcal{C}| \) for any concept class \( \mathcal{C} \).

The paper is organized as follows. Section 2 fixes some notation and terminology. In Section 3 it is shown that, for \( k \geq 1 \), there exists a tournament of relative small order which does have the strong \( S_k \)-property. In Section 4 the existence of such classes (though of somewhat larger order) is shown by construction. Here the QR-tournaments come into play. The final Section 5 is devoted to the implications for teaching. Here the concept classes induced by tournaments come into play.

2 Tournaments and Teaching Models

Section 2.1 reminds the reader to the definition of a tournament. The definition of the weak and strong \( S_k \)-property is postponed to Section 3. The definition of QR-tournaments will be given in Section 4. Section 2.2 calls into mind some models of teaching, including the No-Clash and the recursive model. The definition of a concept class induced by a tournament is postponed to Section 5.

2.1 Tournaments

A tournament \( G = (V, E) \) of order \( n \) is a complete oriented graph with \( n \) vertices. In other words, \( |V| = n \) and, for every choice of two distinct vertices \( x, y \in V \), exactly one of the edges \( (x, y) \) and \( (y, x) \) is contained in \( E \). Informally, we may think of \( V \) as set of players who compete against each other in pairs. An edge \( (x, y) \in E \) can be interpreted as “\( x \) has beaten \( y \)”.

2.2 Teaching Models

Readers familiar with teaching models may skip this section and proceed immediately to Section 3.

A concept over domain \( \mathcal{X} \) is a function from \( \mathcal{X} \) to \( \{0, 1\} \) or, equivalently, a subset of \( \mathcal{X} \). A set whose elements are concepts over domain \( \mathcal{X} \) is referred to as a concept class over \( \mathcal{X} \). The elements of \( \mathcal{X} \) are called instances. The powerset of \( \mathcal{X} \) is denoted by \( \mathcal{P}(\mathcal{X}) \).

We now call into mind the definition of several popular teaching models and the corresponding teaching dimensions. The definition of the No-Clash and the recursive model of teaching will later help us to fully articulate the implications of our results for teaching.
From a technical point of view, we will later require mainly the parameter $TD_{\text{min}}$, which is specified in the first part of the following definition:

**Definition 2.1** (Teaching Models [4, 9, 6]). Let $\mathcal{C}$ be a concept class over $\mathcal{X}$.

1. A teaching set for $C \in \mathcal{C}$ is a subset $D \subseteq \mathcal{X}$ which distinguishes $C$ from any other concept in $\mathcal{C}$, i.e., for every $C' \in \mathcal{C} \setminus \{C\}$, there exists some $x \in D$ such that $C(x) \neq C'(x)$. The size of the smallest teaching set for $C \in \mathcal{C}$ is denoted by $TD(C, \mathcal{C})$. The teaching dimension of $\mathcal{C}$ in the Goldman-Kearns model of teaching is then given by

$$TD(\mathcal{C}) = \max_{C \subseteq \mathcal{C}} |T(C, \mathcal{C})|.$$  

A related quantity is

$$TD_{\text{min}}(\mathcal{C}) = \min_{C \subseteq \mathcal{C}} |T(C, \mathcal{C})|.$$  

2. Let $T : \mathcal{C} \rightarrow \mathcal{P}(\mathcal{X})$ be a mapping that assigns to every concept in $\mathcal{C}$ a set of instances. $T$ is called an NC-teacher for $\mathcal{C}$ if for every $C \neq C' \in \mathcal{C}$, there exists $x \in T(C) \cup T(C')$ such that $C(x) \neq C'(x)$. The NC-teaching dimension of $\mathcal{C}$ is given by

$$\text{NCTD}(\mathcal{C}) = \min\{\max_{C \subseteq \mathcal{C}} |T(C)| : T \text{ is an NC-teacher for } \mathcal{C}\}.$$  

3. Let $C_{\min} \subseteq \mathcal{C}$ be the easiest-to-teach concepts in $\mathcal{C}$, i.e.,

$$C_{\min} = \{C \in \mathcal{C} : TD(C, \mathcal{C}) = TD_{\text{min}}(\mathcal{C})\}.$$  

The recursive teaching dimension of $\mathcal{C}$ is then given by

$$\text{RTD}(\mathcal{C}) = \begin{cases} TD_{\text{min}}(\mathcal{C}) & \text{if } \mathcal{C} = C_{\min} \\ \max\{TD_{\text{min}}(\mathcal{C}), \text{RTD}(\mathcal{C} \setminus C_{\min})\} & \text{otherwise} \end{cases}.$$  

Some remarks are in place here:

1. It was shown in [2] that

$$\text{RTD}(\mathcal{C}) = \max_{C' \subseteq \mathcal{C}} TD_{\text{min}}(C') \geq TD_{\text{min}}(\mathcal{C}). \quad (1)$$  

2. The set $T(C)$ in Definition [2, 1] is an unlabeled set of instances. Intuitively, one should think of the learner as receiving the correctly labeled instances i.e., the learner receives $T(C)$ plus the corresponding $C$-labels where $C$ is the concept that is to be taught.

3. We say that two concepts $C$ and $C'$ clash (with respect to $T : \mathcal{C} \rightarrow \mathcal{P}(\mathcal{X})$) if they agree on $T(C) \cup T(C')$, i.e., if they assign the same 0,1-label to all instances in $T(C) \cup T(C')$. NC-teachers for $\mathcal{C}$ are teachers who avoid clashes between any pair of distinct concepts from $\mathcal{C}$.

$^1$NC = No-Clash.
3 Tournaments with the $S_k$-Property

In this paper, the $S_k$-property will be called “weak $S_k$-property” so that it can be easier distinguished from its strong counterpart. Here are the formal definitions of the weak and the strong $S_k$-property:

**Definition 3.1 (Weak $S_k$-Property).** A tournament $G = (V, E)$ is said to have the weak $S_k$-property if the following holds: for any choice of $k$ distinct vertices $a_1, \ldots, a_k \in V$, there exists another vertex $x \in V$ such that $(x, a_j) \in E$ for $j = 1, \ldots, k$.

**Definition 3.2 (Strong $S_k$-Property).** A tournament $G = (V, E)$ is said to have the strong $S_k$-property if the following holds: for any choice of $k$ distinct vertices $a_1, \ldots, a_k \in V$ and any choice of $b_1, \ldots, b_k \in \{\pm 1\}^k$, there exists another vertex $x \in V$ such that the following holds:

$$\forall j = 1, \ldots, k : \begin{cases} (x, a_j) \in E & \text{if } b_j = +1 \\ (a_j, x) \in E & \text{if } b_j = -1 \end{cases}.$$  \hspace{1cm} (2)

Let $f(k)$ (resp. $F(k)$) be the smallest number $n \geq k$ such that there exists a tournament of order $n$ which has the weak (resp. the strong) $S_k$-property. The following is known about the function $f(k)$:

$$2^{k-1}(k + 2) - 1 \leq f(k) \leq \min \left\{ n : \binom{n}{k} (1 - 2^{-k})^{n-k} < 1 \right\} \leq (1 + o(1)) \ln(2)k^{2^k}.$$  

The lower bound is found in [8]. The upper bound is from [3]. It is an easy application of the probabilistic method. Note that the gap between the lower and the upper bound is of order $k$. Clearly $f(k) \leq F(k)$ so that each lower bound on $f(k)$ is a lower bound on $F(k)$ too. Moreover, an obvious application of the probabilistic method yields an upper bound on $F(k)$ that differs from the above upper bound on $f(k)$ only by inserting an additional factor $2^k$ in front of $\binom{n}{k}$. Hence we get

$$2^{k-1}(k + 2) - 1 \leq F(k) \leq \min \left\{ n : 2^k \binom{n}{k} (1 - 2^{-k})^{n-k} < 1 \right\} \leq (1 + o(1)) \ln(2)k^{2^k}.$$  

4 Construction of Tournaments with the $S_k$-Property

As outlined in Section[3] the probabilistic method yields good upper bounds on $f(k)$ or $F(k)$, however without providing us with a concrete tournament which satisfies this bound. As far as the function $f(k)$ is concerned, Graham and Spencer [5] have filled this gap. They defined and analyzed a tournament that is is based on the quadratic residues and non-residues in the prime field $\mathbb{F}_p$. It became known under the name quadratic-residue tournament (or briefly QR-tournament):

\footnote{This factor accounts for the possible choices of $b_1, \ldots, b_k$.}
**Definition 4.1** (QR-Tournament). Let $p$ be a prime such that $p \equiv 3 \pmod{4}$. The QR-tournament of order $p$ is the tournament $(V, E)$ given by

$$V = \{0, 1, \ldots, p - 1\} \quad \text{and} \quad E = \{(x, y) \in V \times V : x - y \text{ is a quadratic residue modulo } p\}$$

Let $\chi : \mathbb{F}_p \rightarrow \{-1, 0, 1\}$ be the function

$$\chi(x) = \begin{cases} 
+1 & \text{if } x \neq 0 \text{ is a quadratic residue modulo } p \\
-1 & \text{if } x \neq 0 \text{ is a quadratic non-residue modulo } p \\
0 & \text{if } x = 0
\end{cases}$$

In the sequel, $p$ always denotes a prime that is congruent to 3 modulo 4 (so that $-1$ is a quadratic non-residue). Note that the graph $(V, E)$ in Definition 4.1 is indeed a tournament because $\chi(y - x) = -\chi(x - y)$ so that exactly one of the edges $(x, y)$ and $(y, x)$ is included in $E$. Graham and Spencer have shown the following result:

**Theorem 4.2** ([5]). The QR-tournament of order $p$ has the weak $S_k$-property provided that $p > k^2 2^{2k-2}$.

As we show now, the same construction works for the strong $S_k$-property:

**Theorem 4.3.** The QR-tournament of order $p$ has the strong $S_k$-property provided that $p > k^2 2^{2k-2}$.

**Proof.** The proof will be a slight extension of the proof of Theorem 4.2 in [5], but it will have to deal with the variables $b_1, \ldots, b_k \in \{\pm 1\}$ that occur in the definition of the strong $S_k$-property (and are missing in the definition of the weak $S_k$-property).

Consider first the case $k = 1$. A tournament has the strong $S_1$-property iff no vertex has in- or outdegree $p - 1$. Every QR-tournament has this property because every vertex has in- and outdegree $\frac{p-1}{2} < p - 1$.

The remainder of the proof is devoted to the case $k \geq 2$. Let $G = (V, E)$ be the QR-tournament of order $p$. Let $a_1, \ldots, a_k \in V$ be $k$ distinct vertices and let $b_1, \ldots, b_k \in \{\pm 1\}$. Set $A = \{a_1, \ldots, a_k\}, a = (a_1, \ldots, a_k), b = (b_1, \ldots, b_k)$ and consider the auxiliary functions

$$g(a, b) = \sum_{x \in V \setminus A} \prod_{j=1}^k [1 + b_j \chi(x - a_j)] \quad \text{and} \quad h(a, b) = \sum_{x=0}^{p-1} \prod_{j=1}^k [1 + b_j \chi(x - a_j)] .$$

An inspection of $g(a, b)$ reveals that there exists an $x \in V$ which satisfies (2) if and only if $g(a, b) > 0$. It suffices therefore to show that $g(a, b) > 0$. To this end, we decompose $g(a, b)$ according to

$$g(a, b) = p + (h(a, b) - p) - (h(a, b) - g(a, b)) .$$

In order to show that $g(a, b) > 0$, it suffices to show that

$$|h(a, b) - p| \leq \sqrt{p} \cdot ((k - 2) 2^{k-1} + 1) \quad \text{and} \quad h(a, b) - g(a, b) \leq 2^k$$

(3)
because \( p - \sqrt{p}(k - 2)2^{k-1} + 1) - 2^k > 0 \) provided that \( p > k^22^{k-2} \), as an easy calculation shows.  

We still have to verify (3). In order to get \(|h(a, b) - p| \leq \sqrt{p}(k - 2)2^{k-1} + 1\), we apply the distributive law and rewrite \( h(a, b) \) as follows:

\[
    h(a, b) = \sum_{x=0}^{p-1} 1 + \sum_{x=0}^{p-1} \sum_{j=1}^{k} b_j \chi(x - a_j) + \sum_{r=2}^{k} S_r \tag{4}
\]

where

\[
    S_r = \sum_{x=0}^{p-1} \prod_{1 \leq j_1 < \ldots < j_r \leq k} b_{j_i} \chi(x - a_{j_i}) = \sum_{1 \leq j_1 < \ldots < j_r \leq k} \left( \prod_{i=1}^{r} b_{j_i} \right) \sum_{x=0}^{p-1} \prod_{i=1}^{r} \chi(x - a_{j_i}) . \tag{5}
\]

Since \( \sum_{x=0}^{p-1} 1 = p \) and

\[
    \sum_{x=0}^{p-1} \sum_{j=1}^{k} b_j \chi(x - a_j) = \sum_{j=1}^{k} b_j \sum_{x=0}^{p-1} \chi(x - a_j) = 0,
\]

we can bring (4) in the form

\[
    h(a, b) - p = \sum_{r=2}^{k} S_r .
\]

Burgess \[1\] has shown that

\[
    \left| \sum_{x=0}^{p-1} \prod_{i=1}^{r} \chi(x - a_{j_i}) \right| \leq (r - 1) \sqrt{p}
\]

holds for every fixed choice of \( 1 \leq j_1 < \ldots < j_r \leq k \). In combination with (3), it follows that

\[
    |h(a, b) - p| = \left| \sum_{r=2}^{k} S_r \right| \leq \sqrt{p} \cdot \sum_{r=2}^{k} \binom{k}{r} (r - 1) .
\]

A straightforward calculation shows that \( \sum_{r=2}^{k} \binom{k}{r} (r - 1) = (k - 2)2^{k-1} + 1 \). We may therefore conclude that the first inequality in (3) is valid. We finally have to show that \( h(a, b) - g(a, b) \leq 2^k \). Note first that

\[
    h(a, b) - g(a, b) = \sum_{i=1}^{k} \prod_{j=1}^{k} [1 + b_j \chi(a_i - a_j)] .
\]

We call \( \prod_{j=1}^{k} [1 + b_j \chi(a_i - a_j)] \) the contribution of \( i \) to \( h(a, b) - g(a, b) \). Set \( I_b = \{ i \in \{0, 1, \ldots, p - 1\} : b_i = b \} \) for \( b = \pm 1 \). The following observations are rather obvious:

\[3\]This calculation makes use of the case-assumption \( k \geq 2 \).
• Every $i$ makes a contribution of either 0 or $2^{k-1}$.

• If $i$ makes a non-zero contribution, then $\chi(a_i - a_j) = b_j$ for every $j \neq i$.

• For each $b \in \{\pm 1\}$, at most one $i \in I_b$ makes a non-zero contribution.

These observations imply that $h(a, b) - g(a, b) \leq 2^k$, which concludes the proof of the theorem.

5 Implications for Teaching

With each tournament $G = (V, E)$, we associate the concept class $C(G) = \{C_x : x \in V\}$ given by

$$C_x = \{a \in V : (x, a) \in E\}.$$ 

Intuitively, we can think of $x$ as a player in the tournament and of $C_x$ as the set of players who were beaten by $x$. Note that $TD_{\text{min}}(G) \leq k$ intuitively means that there exists a player $x \in V$ who can be uniquely identified from telling which of $k$ (appropriately chosen) players he has beaten, and which he has not beaten.

It is well known that concept classes induced by a tournament are easy to teach in the NC-model:

**Remark 5.1 (\[7\]).** For every tournament $G$, we have that $\text{NCTD}(C(G)) \leq 1$ (with equality for all tournaments of order at least 2).

Let $G$ be a tournament of order $n$. Since, as noted already in Section [2.2](#), the recursive teaching dimension is lower bounded by $TD_{\text{min}}$, we can show that $RTD(C(G))$ exceeds $\text{NCTD}(C(G))$ by a factor of order $\log(n)$ by proving logarithmic lower bounds on $TD_{\text{min}}(C(G))$ for appropriately chosen tournaments $G$. This is precisely what we will do in the sequel.

We first relate the parameter $TD_{\text{min}}(G)$ to a refinement of the strong $S_k$-property. A tournament $G = (V, E)$ is said to have the strong $S_{k,m}$-property if the following holds: for any choice of $k$ distinct vertices $a_1, \ldots, a_k$ and any choice of $b_1, \ldots, b_k \in \{\pm 1\}$, there exists a set $X \subseteq V$ of size $m$ such that every $x \in X$ satisfies (2). Let $F(k, m)$ be the smallest number $n$ such that there exists a tournament of order $n$ which has the strong $S_{k,m}$-property. It is rather obvious that the following holds:

**Remark 5.2.**

1. If $G$ has the strong $S_{k,2}$-property, then $TD_{\text{min}}(G) > k$.

2. The strong $S_{k,1}$-property coincides with the strong $S_k$-property. Consequently $F(k) = F(k, 1)$.

3. The strong $S_{k,m+1}$-property implies the strong $S_{k,m}$-property. Consequently $F(k, m) \leq F(k, m + 1)$.

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[3] If two distinct $i, i' \in I_b$ made a non-zero contribution, then we would get $\chi(a_i - a_i') = b = \chi(a_i' - a_i)$, which is in contradiction to $\chi(a_i' - a_i) = -\chi(a_i - a_i')$. 

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4. The strong $S_{k+1}$-property implies the strong $S_{k,2}$-property. Consequently $F(k,2) \leq F(k+1)$.\footnote{Using methods from [8], it can even be shown that the strong $S_{k+1}$-property implies the strong $S_{k,k+2}$-property. Consequently $F(k,k+2) \leq F(k+1)$.}

5. $F(k,m)$ is non-decreasing in both arguments.

The following is an immediate consequence of the first of these remarks:

**Corollary 5.3.** Let $G_k$ denote a tournament having the strong $S_{k,2}$-property and being of order $F(k,2)$. Then $TD_{\min}(C(G_k)) > k$.

Corollary 5.3 does not tell explicitly how $TD_{\min}(C(G_k))$ depends on $n = F(k,2)$. As already shown in [7], a more useful lower bound can be obtained via the probabilistic method:

**Theorem 5.4 ([7]).** For every sufficiently large $n$, there exists a tournament $G_n$ of order $n$ such that

$$TD_{\min}(C(G_n)) > \log(n) - 2 \log \log(2n) - 2.$$ 

**Proof.** The proof given here makes use of the implication “strong $S_{k+1} \Rightarrow$ strong $S_{k,2}$” and is slightly simpler than the proof given in [7]. If

$$2^{k+1} \binom{n}{k+1} (1 - 2^{-(k+1)} n^{-(k+1)} < 1,$$  \( \text{(6)} \)

then there is a strictly positive probability for the event that a random tournament of order $n$ has the strong $S_{k+1}$- and therefore also the strong $S_{k,2}$-property. In this case, we may conclude that there exists a tournament $G_n$ of order $n$ such that $TD_{\min}(C(G_n)) > k$. We may clearly assume that $n \geq 2(k+1)$ so that $n - (k+1) \geq n/2$. Making use of $n - (k+1) \geq n/2$, \( \binom{n}{k} \leq n^k \) and $1 + x \leq e^x$ with equality for $x = 0$ only, we get the following sufficient condition for (6):

$$(2n)^{k+1} \exp \left( -\frac{n}{2^{k+2}} \right) \leq 1.$$ 

After taking logarithm on both hand-sides and rearranging some terms, this becomes

$$(k+1)2^{k+2} \ln(2n) \leq n.$$ 

A straightforward calculation shows that the latter condition is satisfied whenever $k \leq \log(n) - 2 \log \log(2n) - 2$. From this discussion, the assertion of the theorem is immediate. $\square$

Our main implication for teaching is the fact that the quadratic-residue tournament of order $p$ induces a concept class whose $TD_{\min}$ grows logarithmically with $p$:

**Theorem 5.5.** Let $p$ be a prime that is congruent to 3 modulo 4. Let $G_p$ be the quadratic-residue tournament of order $p$. Then

$$TD_{\min}(G_p) > \frac{1}{2} \log(p) - \log \log(p) - 1.$$
Proof. We make again use of the implication “strong $S_{k+1} \Rightarrow$ strong $S_{k,2}$”. According to Theorem 4.3, the following holds: if

$$(k + 1)^2 2^{2k} < p$$

then $G_p$ has the strong $S_{k+1}$- and therefore also the strong $S_{k,2}$-property. In this case, we may conclude that $\text{TD}_{\text{min}}(G_p) > k$. A straightforward calculation shows that (7) holds whenever $k \leq \frac{1}{2} \log(p) - \log \log(p) - 1$. From this discussion, the assertion of the theorem is immediate.

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