Scaling limit of the random walk in random environment in the subdiffusive case

Loïc de Raphélis *

August 25, 2016

Abstract

We consider a random walk on a Galton–Watson tree in random environment, in the subdiffusive case. We prove the convergence of the renormalised height function of the walk towards the continuous-time height process of a spectrally positive strictly stable Lévy process, jointly with the convergence of the renormalised range of the walk towards the real tree coded by the latter continuous-time height process.

Mathematics Subject Classification (2010): 60J80, 60G50, 60K37, 60F17.

Key Words: Galton–Watson tree, Random walk, Random environment, Scaling limit.

1 Introduction

1.1 Definitions, background and main result

Let us consider $N$ a point process taking values in $\bigcup_{n\in N\cup\{0\}} \mathbb{R}^n$ (with the convention that $\mathbb{R}^0$ is the empty sequence, and $\mathbb{R}^N$ is the set of real sequences). Let $V := (T, (V(u))_{u \in T})$ be a branching random walk with reproduction law $N$, that is a random marked tree built by induction as follows:

- **Initialisation**
  Generation 0 of $T$ is only made up of the root, denoted by $\rho$. We set $V(\rho) = 0$.

- **Induction**
  Let $n \geq 0$, and suppose that the tree has been built up to generation $n$. If generation $n$ is empty, then generation $n + 1$ is empty. Otherwise, each vertex $u$ of generation $n$ gives progeny according to $N$ translated by $V(u)$, independently of other vertices, thus forming generation $n + 1$.

---

* Sorbonne Universités, UPMC Univ Paris 6, UMR 7599, Laboratoire de Probabilités et Modèles Aléatoires, 4 place Jussieu, F-75005 Paris.
For a vertex $u$ in the tree, we will denote by $\nu(u)$ its number of children (which may be infinite), children denoted by $u1, u2, \ldots$, and $|u|$ its generation. For any $u, v \in \mathbb{T}$, we will write $u \vdash v$ when $u$ is a strict ancestor of $v$. We let $\mathbb{P}$ be the measure of this branching random walk, $\mathbb{E}$ be the associated expectation, and $\mathbb{P}^*$ be the measure of the branching random walk conditioned to survive.

We are interested in the nearest-neighbour random walk $(X^V_n)_{n \geq 0}$ on $\mathbb{T}$ starting in $\rho$, and with transition probabilities depending on $(V(u))_{u \in \mathbb{T}}$, defined as follows for any $u \in \mathbb{T}$ and $n \geq 0$:

$$\mathbb{P}^V(X^V_{n+1} = \overset{\leftarrow}{u} \mid X^V_n = u) = \frac{e^{-V(u)}}{e^{-V(u)} + \sum_{i=1}^{\nu(u)} e^{-V(u)}}$$

$$\mathbb{P}^V(X^V_{n+1} = ui \mid X^V_n = u) = \frac{e^{-V(ui)}}{e^{-V(u)} + \sum_{i=1}^{\nu(u)} e^{-V(u)}} \quad \text{for any } 1 \leq i \leq \nu(u),$$

where $\overset{\leftarrow}{u}$ stands for the parent of $u$. In order to properly define the transition probabilities from the root, we artificially add a parent $\overset{\leftarrow}{\rho}$ to the root and we suppose that the random walk is reflected in $\overset{\leftarrow}{\rho}$. We denoted by $\mathbb{P}^V$ the law of $(X^V_n)_{n \geq 0}$ conditionally on $V$: this law is called the quenched law of the random walk. We associate the expectation $\mathbb{E}^V$ to this law. We denote by $\mathbb{P}$ the annealed law of the random walk (and by $\mathbb{E}$ the associated expectation), that is the quenched law averaged over $\mathbb{P}$; we will also denote by $\mathbb{P}^*$ the annealed probability averaged over $\mathbb{P}^*$. When there is no ambiguity on $V$, we will simply denote the walk by $(X_n)_{n \geq 0}$.

Let us introduce the Laplace transform of $V$, defined for all $t \geq 0$ by:

$$\psi(t) := \mathbb{E} \left[ \sum_{|u|=1} e^{-tV(u)} \right],$$

Notice that $\psi(0)$ is the mean of the offspring distribution of $\mathbb{T}$; in order the tree $\mathbb{T}$ to have a positive probability to be infinite, we will suppose from now on that $m := \psi(0) > 1$ (allowing $m = \infty$, as we allowed $\mathbb{P}(\sum_{|u|=1} 1 = \infty) > 0$). It was shown in this case by R. Lyons and R. Pemantle [24] that on the event of non-extinction, the walk is transient or positive recurrent, depending on whether $\min_{t \in [0;1]} \psi(t)$ is respectively $> 1$ or $< 1$. When $\min_{t \in [0;1]} \psi(t) = 1$, the random walk is recurrent. If $\psi$ is well-defined in a small neighbourhood of 1 and is differentiable in 1, then it was shown by G. Faraud [9] that the walk is null recurrent when $\psi'(1) < 1$. We will consider this case in this article; to sum up, we make the following hypotheses:

$$(H_e) \left\{ \begin{array}{l}
\bullet \ m := \psi(0) = \mathbb{E} \left[ \sum_{|u|=1} 1 \right] > 1, \\
\bullet \ \min_{t \in [0;1]} \psi(t) = 1, \\
\bullet \ \psi'(1) = -\mathbb{E} \left[ \sum_{|u|=1} V(u) e^{-V(u)} \right] < 0.
\end{array} \right.$$
limit theorem on the height function of the walk, \(|X_n|\) for \(n \geq 0\). Namely, they showed that when renormalised by a factor \(n^{1/2}\), the height function converges towards a reflected Brownian motion. This theorem has been extended later by A. Dembo and N. Sun in [5] to the case where the underlying tree is a multitype Galton–Watson tree, and under weaker hypotheses.

In the case where the environment is random, in order to further understand the behaviour of the random walk we need to introduce the following quantity:

\[
\kappa := \inf\{t > 1 : \psi(t) \geq 1\}
\]

(with the convention \(\inf\emptyset := \infty\)). Indeed, in [13], Y. Hu and Z. Shi showed that the height of the maximum of the random walk at time \(n\) is of order \(n^\gamma\), where \(\gamma := \min(\frac{1}{2}, 1 - \frac{1}{\kappa})\). This indicates a phase transition at \(\kappa = 2\).

When \(\kappa > 2\), the walk is therefore of order \(n^{1/2}\). In [9], G. Faraud generalised Y. Peres and O. Zeitouni’s result of [29] to the random-environment case, at least for \(\kappa > 5\) (resp. \(\kappa > 8\)) in the annealed case (resp. in the quenched case).

Then, in [1], we considered with E. Aïdékon the range of the walk at time \(n\), \(T^n := \{u \in T : \exists k \leq n, X_k = u\}\), which is the sub-tree of \(T\) made up of the vertices visited by the walk, regarded as a metric space (we will denote it by \(R_n\) when considered as such). We proved for \(\kappa > 2\) the convergence after renormalisation by a factor \(n^{1/2}\) of the range towards the real tree coded by the very reflected Brownian motion towards which converges the height function. Let us say a few words on the definition of the real tree coded by a given function say \(g : \mathbb{R}_+ \to \mathbb{R}_+\) a càdlàg function on \(\mathbb{R}_+\). Consider \(d\) on \(\mathbb{R}_+^2\) defined by \(d(t, s) = g(t) + g(s) - 2\inf_{r \in [t, s]} g(r)\). Then set \(t \sim s\) if \(d(t, s) = 0\); the real tree coded by \(g\) is the metric space \(T_g := \mathbb{R}_+ / \sim\) equipped with the distance \(d\). If \(g\) has compact support, then \(T_g\) is compact. The set of compact real trees can be endowed with the pointed Gromov-Hausdorff distance, which makes it a Polish space. See [20] for a detailed introduction to real trees.

We intend in this article to deal with the case \(\kappa \in (1; 2]\). More precisely, we will show that when renormalised by a factor \(n^{1-1/\kappa}\), the height function of the walk \(|X_n|\) for \(n \geq 0\) converges towards the continuous-time height process of a spectrally positive stable Lévy process of index \(\kappa\) (this kind of process will be defined in the next paragraph), jointly with the range \((R_n)_{n \geq 0}\) which converges towards the Lévy forest coded by this very continuous-time height process. We mention that several results were already obtained about the behaviour of the walk in the case \(\kappa \in (1; 2]\). Among others, in [2], P. Andreoletti and P. Debs showed that the largest entirely visited generation is of order \(\ln(n)\), and that the local time of the root at time \(n\) is of order \(n^{1/\kappa}\). This was refined in [13], where Y. Hu obtained the limit law of the local time of the root after renormalisation.

The proof of our theorem in [1] relied on a result of T. Duquesne and J.-F. Le Gall [7]: consider a critical Galton–Watson forest the offspring distribution of which has finite variance, then the associated height process (that is the sequence of heights of the vertices of the aforesaid forest taken in the lexicographical order induced by Neveu’s notation [28]) converges in law after renormalisation towards the reflected Brownian motion, which implies in some sense the convergence of the forest viewed as a metric space towards the real tree coded by the later reflected Brownian motion. Actually this theorem remains valid in the case of an offspring distribution with regularly varying tails, and we will use this version of the theorem in this
article. However the limit is different (if $\kappa < 2$), so let us introduce it. Suppose that the offspring distribution is regularly varying with index $-\theta$ where $\theta \in (1; 2]$. Let $(Y_t)_{t \geq 0}$ be a strictly stable spectrally positive Lévy process (that is with no negative jumps) of index $\theta$ (if $\theta = 2$, then $Y$ is a Brownian motion). Consider $I_t^\epsilon := \inf_{t \in [s, t]} Y_s$, and set for any $t \geq 0$

$$H_t := \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t 1_{\{Y_s < I_t^{\epsilon} + \epsilon\}} ds.$$ 

Somehow, $H_t$ is the local time at level 0 of $Y - I_t$ at time $t$. As explained in Subsection 4.3 of [21], the limit exists almost surely, and it is possible to consider a measurable version of the whole process $(H_t)_{t \geq 0}$. This process is called the continuous-time height process of $Y$, and was first introduced in [21]. If $\theta = 2$, then $H$ has the law of a reflected Brownian motion. Notice that given the definition, as $Y$ satisfies the scaling property $(Y_{ct})_{t \geq 0} \xrightarrow{law} (c^{1/\kappa}Y_t)_{t \geq 0}$ for any $c \geq 0$, so does $H$: $(H_{ct})_{t \geq 0} \xrightarrow{law} (c^{1/\kappa}H_t)_{t \geq 0}$. Moreover, as $Y$ is defined up to a constant, so is $H$ (to precisely give the law of $Y$ one could specify its Laplace transform, which is well-defined as it is spectrally positive). The main theorem of [7] says that the height process associated with a Galton–Watson forest with regularly varying offspring distribution converges towards the continuous-time height process $H$.

Let us add a few technical hypotheses:

\[(H_\kappa) \quad \psi(\kappa) = 1, \quad \mathbb{E}\left[\sum_{|u|=1} (-V(u), 0)e^{-\kappa V(u)}\right] < \infty \quad \text{and} \quad \mathbb{E}\left[\left(\sum_{|u|=1} e^{-V(u)}\right)^\kappa\right] < \infty.\]

The statement of our main theorem is the following.

**Theorem 1.** Suppose $(H_c)$ and $(H_\kappa)$ for a certain $\kappa \in (1; 2]$. Suppose also that the distribution of the point process $N$ is non-lattice. Then the following convergence holds in law under $\mathbb{P}^V$ for $\mathbb{P}^a$-almost every $V$:

$$\frac{1}{n^{1-\kappa}}\left((|X_{[nt]|})_{t \geq 0}, R_n\right) \xrightarrow{n \to \infty} (H_t)_{t \geq 0}, \mathcal{T}(H_{t0 \leq t \leq 1}) \quad \text{if } \kappa \in (1; 2),$$

or

$$\frac{1}{(n \ln^{-1}(n))^{1/\kappa}}\left((|X_{[nt]|})_{t \geq 0}, R_n\right) \xrightarrow{n \to \infty} (H_t)_{t \geq 0}, \mathcal{T}(H_{t0 \leq t \leq 1}) \quad \text{if } \kappa = 2,$$

where $(H_t)_{t \geq 0}$ is the continuous-time height process of a strictly stable spectrally positive Lévy process of index $\kappa$, and where $\mathcal{T}(H_{t0 \leq t \leq 1})$ is the real tree coded by $(H_t)_{0 \leq t \leq 1}$. The convergence holds in law for the Skorokhod topology on càdlàg functions and the pointed Gromov-Hausdorff topology on compact metric spaces.

By requiring the fact that the point process $N$ is non-lattice we actually mean that there shall not exist any $a > 0$ such that almost surely, for any $x \in N$, $x \in a\mathbb{Z}$. Notice that we do not specify the Laplace transform of $Y$, which is therefore defined up to a constant, thus making any multiplicative constant before $H$ superfluous. However we will as far as possible lead our computations specifying coefficients, and there is only one quantity (at the end of Subsection 4.3.3 see the remark) that we will be unable to explicitly compute.
1.2 An annealed theorem on random walks on forests

Inspired by the strategy of Y. Peres and O. Zeitouni in [29] who, in order to study $\lambda$-biased random walks on a Galton–Watson tree, studied random walks on Galton–Watson trees with infinite ray, we will first prove an annealed theorem on random walks on a Galton–Watson forest in random environment. Let us denote by $\mathbb{W} = (\mathbb{F}, V(u)_{u \in \mathbb{F}})$ a marked forest made up of a collection of i.i.d. branching random walks $(\mathbb{V}_i)_{i \in \mathbb{N}} = (\mathbb{T}_i, V(u)_{u \in \mathbb{T}_i})_{i \geq 1}$ (as defined in Section 1). The nearest-neighbour random walk $(X_n^W)_{n \geq 0}$ on $\mathbb{W}$ starts on $\rho_1$ the root of $\mathbb{V}_1$ and has transition probabilities defined as follows:

$$
P^W(X_{n+1}^W = u | X_n^W = u) = \frac{e^{-V(u)}}{e^{-V(u)} + \sum_{i=1}^{\nu(u)} e^{-V(u)}} \quad \text{for any } 1 \leq i \leq \nu(u),$$

$$
P^W(X_{n+1}^W = u | X_n^W = u) = \frac{e^{-V(u)}}{e^{-V(u)} + \sum_{i=1}^{\nu(u)} e^{-V(u)}} \quad \text{if } u \text{ is not a root},$$

$$
P^W(X_{n+1}^W = \rho_j | X_n^W = \rho_j) = \frac{e^{-V(u)}}{e^{-V(u)} + \sum_{i=1}^{\nu(u)} e^{-V(u)}} \quad \text{if for a certain } j, u = \rho_j,$$

where for $j \geq 1$ we denoted by $\rho_j$ the root of $\mathbb{V}_j$. The behaviour of the random walk on $\mathbb{W}$ is then similar to that on $\mathbb{V}$ except when on a root. Being recurrent, it will then visit each tree composing $\mathbb{W}$ for a finite time, and sometimes when on a root then jump to the next one. The sub-forest of $\mathbb{W}$ made up of the visited vertices $\mathbb{F} := \{u \in \mathbb{W} : \exists n \geq 0, X_n^W = u\}$ is then almost surely well-defined with finite component trees. When there is no ambiguity on $\mathbb{W}$ we will simply denote the walk by $(X_n)_{n \geq 0}$. Let us denote by $\mathcal{R}^W$ the forest $\mathbb{F}$ viewed as a metric space equipped with the graph distance. The following result suggests that a potential limit for $\mathcal{R}^W$ is the Lévy forest; however the latter is not locally compact and to our knowledge, the pointed Gromov-Hausdorff topology does not adjust well to non-locally compact objects. Therefore we prefer to state the result in terms of $\mathcal{R}_n$ rather than $\mathcal{R}^W$.

**Theorem 2.** Suppose $(\mathbf{H}_c)$ and $(\mathbf{H}_e)$ for a certain $\kappa \in (1; 2]$, and suppose that the distribution of the point process $N$ is non-lattice. Then the following convergence holds in law under $\mathbf{P}$:

$$
\frac{1}{n^{1-\frac{1}{\kappa}}} \left( |X_{[nt]}| \right)_{t \geq 0} \xrightarrow{\text{law}} \left( (H_t)_{t \geq 0}, T_{(H_t)_{0 \leq t \leq 1}} \right) \quad \text{if } \kappa \in (1; 2),
$$

or

$$
\frac{1}{(n \ln^{-1}(n))^{\frac{1}{\kappa}}} \left( |X_{[nt]}| \right)_{t \geq 0} \xrightarrow{\text{law}} \left( (H_t)_{t \geq 0}, T_{(H_t)_{0 \leq t \leq 1}} \right) \quad \text{if } \kappa = 2,
$$

where $(H_t)_{t \geq 0}$ is the continuous-time height process of a strictly stable spectrally positive Lévy process of index $\kappa$. The convergence holds in law for the Skorokhod topology on càdlàg functions and the pointed Gromov-Hausdorff topology on metric spaces.

Our paper is organised as follows. The next section (Section 2) gives the global strategy of the proof, which is similar to that used in [11]. It contains the proof of Theorem 2 provided that certain hypotheses are satisfied. Then, the whole Section 3 is dedicated to proving that these hypotheses are satisfied indeed. It contains the most important part of this paper, Subsection 3.3 in which we establish the regular variation of the tail of a key random variable. To do this, we carry out a subtle study of the random walk on a biased tree (inspired by the
strategy developed by H. Kesten, M.V. Kozlov and F. Spitzer in [16], on which we are led to use Kesten’s renewal theorem [15]. This is the main contribution of this paper, as in [1] we only had to establish the finiteness of the second moment of the aforesaid key random variable. Finally, we deduce Theorem 1 from Theorem 2 in Section 4.

2 Proof of Theorem 2

Let us give an overview of the organisation of this section. First, we will see in Subsection 2.1 that the range can be seen as a multitype Galton–Watson tree/forest, and we will introduce the notion of leafed Galton–Watson forest with edge lengths. We will then associate to \((|X_n|)_{n \geq 0}\) two leafed Galton–Watson forests with edge lengths, the first being such that its associated height process is equal to \((|X_n|)_{n \geq 0}\), and the second being such that its associated height process is equal to that of \(R^W\). This is what motivates Subsection 2.2, in which we will prove a result on the associated height processes of such forests, under certain hypotheses. Provided that these hypotheses are satisfied by the leafed Galton–Watson forest with edge lengths associated with \((X_n)_{n \geq 0}\), we conclude the proof of Theorem 2 in Subsection 2.3.

2.1 Reduction of trees

Recall that \(F\) is the sub-tree of \(G\) made up of the visited vertices. For any \(u \in F\) not a root, let us denote by \(\beta(u)\) the number of visits made to \(u\) by \(X_n\) from \(\leftarrow u\), that is

\[
\beta(u) := \#\{n \geq 0 : X_n = \leftarrow u \text{ and } X_{n+1} = u\}.
\]

If \(u\) is a root, we set \(\beta(u) = 1\). According to Lemma 3.1 of [1], \((F, \beta)\) is a multitype Galton–Watson forest with roots of initial type 1. We denote by \(\zeta = (\zeta_i)_{i \geq 1}\) its offspring distribution (that is for any \(i \geq 1\), \(\zeta_i\) stands for the offspring distribution of a vertex of type \(i\)). For any \(i \geq 1\), we denote by \(P_i\) the law of a multitype Galton–Watson tree \(T_i\) with such an offspring distribution, and with initial type \(i\).

Let us now introduce a new kind of branching process, that we introduced in [4]: leafed Galton–Watson forests with edge lengths. These forests are multitype Galton–Watson forests with edge lengths with two types, 0 and 1, such that only vertices of type 1 may give progeny (so vertices of type 0 are sterile). Formally, a leafed Galton–Watson forest consists in a triplet \((F, e, \ell)\) where for any \(u \in F\), \(e(u) \in \{0; 1\}\) stands for the type of \(u\) and \(\ell(u)\) for the length of the edge joining \(u\) with its parent. Let \(\zeta\) be a probability measure on \(\bigcup_{n \in \mathbb{N} \cup \{0\}} \{(0; 1) \times \mathbb{R}_+\}^n\) (with the convention that \((0; 1) \times \mathbb{R}_+\)^0 is the empty sequence). We build each component of the forest \((T, e, \ell)\) by induction on generations as follows:

- **Initialisation**: Generation 0 of \(T\) is only made up of the root, denoted by \(\rho\), such that \(e(\rho) = 1\) and \(\ell(\rho) = 0\).

- **Induction**: Let \(n \geq 0\), and suppose that the tree has been built up to generation \(n\). If generation \(n\) is empty, then generation \(n + 1\) is empty. Otherwise, each vertex \(u\) of generation \(n\) such that \(e(u) = 1\) gives progeny according to \(\zeta\), independently of other vertices, thus forming generation \(n + 1\). Vertices \(u\) of generation \(n\) such that \(e(u) = 0\) give no progeny.
The forest \((F, e, \ell)\) is then built as a collection of i.i.d. trees built as explained above. We define its associated weighted height process \(H_F^L\), which is such that for any \(n \geq 1\), if \(u\) is the \(n\)th vertex of \(F\) in the lexicographical order, then \(H_F^L(u) = \ell(u) + \sum_{v \in u} \ell(v)\).

We intend in this subsection to build from \((F, \beta)\) two leafed Galton–Watson forests with edge lengths: the first one will be such that its height process is equal to that of \(F\), and the second one will be such that its height process is equal to \((|X_n|)_{n \geq 0}\). Among others, we will apply to \((F, \beta)\) the transformation introduced in Section 3.2 of [27] (which is inspired by another transformation introduced in [27]). To this end, let us define the notion of optional line of a given type.

**Definition 1.** Let \(u \in F\).

- We denote by \(B^1_u\) the set of vertices descending from \(u\) in \(F\) having no ancestor of type 1 since \(u\). Formally,
  \[
  B^1_u := \{ v \in F : u \vdash v \text{ and } \beta(w) \neq 1 \ \forall w \in F \text{ such that } u \vdash w \vdash v \}.
  \]

- We denote by \(L^1_u\) the set of vertices of type 1 descending from \(u\) in \(F\) and having no ancestor of type 1 since \(u\). Formally,
  \[
  L^1_u := \{ v \in F : u \vdash v, \beta(v) = 1, \beta(w) \neq 1 \ \forall w \in F \text{ such that } u \vdash w \vdash v \}.
  \]

We will denote by \(L^1_u\) (resp. \(B^1_u\)) the cardinal of \(L^1_u\) (resp. \(B^1_u\)).

Figure 1 gives a representation of these sets, among others. This construction is also valid for trees. As for any \(u \in T\) the law of the random variables \(L^1_u\) (resp. \(B^1_u\)) only depends on the type of \(u\) under the annealed law, we will generically denote \(L^1_u\) (resp. \(B^1_u\)) by \(L^1\) (resp. \(B^1\)), and by \(L^1\) (resp. \(B^1\)) its cardinal. For \(u \in T\), we will write \(u < L^1\) if \(u \in B^1\) but \(u \notin L^1\).

### 2.1.1 The forests \(F^R\) and \(F^R^1\)

Let us build a leafed Galton–Watson forest with edge lengths the associated height process of which matches exactly that of \(F\). Let us explain how each component \(T^k\) of \(F^R\) is built from each component \(T_k\) of \(F\). We proceed by induction as follows:

- **Initialisation**
  Generation 0 of \(T^k\) is made up of the root \(\rho_k\) of \(T_k\) the \(k\)th component of \(F\), and we set \(\ell(\rho_k) = 0\) and \(e(\rho_k) = 1\). Let us build generation 1. Take, in the lexicographical order, the vertices \(v \in T_k\) such that \(v \in B^1_k\). Following their lexicographical ordering, for each \(v \in T_k\) among these vertices we add a vertex \(v'\) to the first generation of \(T^R_k\), setting \(e(v') = 1\) if \(\beta(v') = 1\) (that is if \(v \in L^1\)), \(e(v') = 0\) otherwise. Moreover, for each of these vertices \(v' \in T^R_k\), we set its edge length as \(\ell(v') = |v|\).

- **Induction**
  Let \(n \geq 1\), and suppose that generation \(n\) of \(T^R_k\) has been built. If generation \(n\) of \(T^R_k\) is empty then generation \(n+1\) of \(T^R_k\) is empty. Otherwise, for each \(u' \in T^R_k\) (associated with \(u \in T_k\)) of the \(n\)th generation of \(T^R_k\) such that \(e(u') = 1\), consider in the lexicographical order the vertices \(v \in T_k\) such that \(v \in B^1_k\). Proceeding in the lexicographical order, to each \(v \in T_k\) of these vertices, we associate a vertex \(v'\) as a child of \(u'\) in \(T^R_k\), thus forming the progeny of \(u'\). We set \(e(v') = 1\) if \(\beta(u') = 1\) (that is if \(v \in L^1\)) and \(e(v') = 0\) otherwise. Then, for each of these vertices \(v' \in T^R_k\), we set \(\ell(v') = |v| - |u|\).
We let $\mathbf{F}^R_1$ be the sub-forest of $\mathbf{F}^R$ made up of the vertices of type 1. By construction, we get the following lemma.

**Lemma 1.** Let $H_R^\ell$ be the weighted associated height process of $\mathbf{F}^R$, and $H_F$ be that of $\mathbf{F}$. Then for all $n \geq 0$,

$$H_R^\ell(n) = H_F(n).$$

![Diagram of forest construction](image)

Figure 1: Construction of $\mathbf{T}_k^R$ from $\mathbf{T}_k$.

Let us now introduce a leafed Galton–Watson forest with edge lengths associated with both $\mathbf{F}$ and $(|X_n|)_{n \geq 0}$.

### 2.1.2 The forests $\mathbf{F}^X$ and $\mathbf{F}^{X_1}$

Let us build a leafed Galton–forest with edge lengths the associated weighted height process of which is equal to $(|X_n|)_{n \geq 0}$. This forest, denoted by $\mathbf{F}^X$, will be made as follows. First, we consider the forest $\mathbf{F}^{R_1}$, and inside each set of siblings re-order the vertices according to their time of visit, thus yielding a forest that we denote by $\mathbf{F}^{X_1}$. More precisely, we proceed by induction on generations as follows:

- **Initialisation**
  The roots are already ordered according to their time of visit.

- **Induction**
  Suppose that generations $\leq n$ are ordered. If generation $n + 1$ is empty, then nothing is done. Otherwise, for each vertex $u' \in \mathbf{F}^{X_1}$ of generation $n$, we consider the progeny of the matching vertex in $\mathbf{F}^{R_1}$. It is made up of vertices matching vertices in $\mathcal{W}$ which have been visited from their parent by the walk only once. We then re-order these vertices according to these times of visit.
So this operation yields the forest $F^{x^1}$. Notice that the associated height process of this forest visits vertices in the same order as the corresponding vertices were chronologically visited by $(X_n)_{n \geq 0}$ in $W$, and that each vertex of $F^{x^1}$ corresponds to a vertex of $W$ which has only been visited once from its parent by the walk.

Now, let us explain the construction of $F^X$. The idea is to start from $F^{x^1}$, and then for each time $n \geq 0$ to add a vertex to $F^{x^1}$ in a way such that the forest finally obtained has its associated height process equal to $(|X_n|)_{n \geq 0}$. Formally, first, we consider $F^{x^1}$. Then we proceed chronologically: for each $n \geq 0$, we consider $X_n$ the vertex visited at time $n$,

- If $\beta(X_n) \neq 1$, then we attach a vertex $u$ of type 0 ($e(u) = 0$) to $v'$, where $v'$ is such that the corresponding vertex $v$ in $F$ is the most recent ancestor of $X_n$ such that $\beta(v) = 1$ (note that $v'$ is a vertex that was present initially in $F^{x^1}$). This vertex is attached at the right of the vertices which were previously attached. We set $\ell(u) = |u| - |v|$. Note that $\beta(X_n)$ stands well and truly for the total number of visits by the walk of the vertex $X_n$, and not only those before time $n$.

- If $\beta(X_n) = 1$, then either $n$ is the time of the first visit to the vertex $X_n$, in which case nothing is done (as there already existed a corresponding vertex in $F^{x^1}$). Or the vertex visited at time $n$ has already been visited before (that means that the walk was visiting a child of this vertex at time $n - 1$), and we add a vertex $u$ of type 0 to the corresponding vertex in $F^{x^1}$, at the right of the other vertices previously attached. We set $\ell(u) = 0$.

By construction, we get the following lemma:

**Lemma 2.** Let $H^t_X$ be the height process of $F^X$. Then for each $n \geq 0$ we have $|X_n| = H^t_X(n)$.

Thus, any result obtained on the scaling limit of $H^t_X$ will be valid for $(|X_n|)_{n \geq 0}$ too.

### 2.2 Convergence of the height processes associated with leaved Galton–Watson forests with edge lengths

Lemmas [1] and [2] indicate that the study of $F$ and $(|X_n^W|)_{n \geq 0}$ can boil down to that of certain leaved Galton–Watson forests with edge lengths. We intend in this section to give a result on the scaling limit of the height process associated with such forests. In [3], we proved under a condition of finite variance (Theorem 1) that their height process converges towards the reflected Brownian motion. Adjusting the proof of this theorem, we will be able to obtain an equivalent result in the infinite variance case. So let us consider the setting of [3]: let $(F, e, \ell)$ be a leaved Galton–Watson forest of reproduction law $\zeta$, recall that for any vertex $u \in F$ we let $\nu(u)$ (resp. $\nu^1(u)$) be the total number of children (resp. number of children of type 1) of $u$ in $F$, and recall that we denote by $F^1$ the forest $F$ limited to its vertices of type 1. We make the following hypotheses on $\zeta$:

\[
\begin{align*}
(i) \quad & E\left[\sum_{|u|=1,e(u)=1} 1\right] = E[\nu^1] = 1, \\
(ii) \quad & \text{There exists } \varepsilon > 0 \text{ s.t. } E\left[\left(\sum_{|u|=1} 1\right)^{1+\varepsilon}\right] = E[(\nu)^{1+\varepsilon}] < \infty, \\
(iii) \quad & \text{There exists } C_0 > 0 \text{ s.t. } P\left(\sum_{|u|=1,e(u)=1} 1 > x\right) = P(\nu^1 > x) \sim C_0 x^{-\kappa}, \\
(iv) \quad & \text{There exists } r > 1 \text{ s.t. } E\left[\sum_{|u|=1} r^{\ell(u)}\right] < \infty.
\end{align*}
\]
Following the notation of [4], we let \( m := \mathbb{E} \left[ \sum_{|u|=1} 1 \right] = \mathbb{E}[\nu] \), we let \( \mu := \mathbb{E} \left[ \sum_{|u|=1, \epsilon(u)=1} \ell(u) \right] \), and for each vertex \( u \in F \), we set \( h(u) := \sum_{k=1}^{\ell(u)} \ell(u_k) \). We define \( H^1 \) the height process of \( F^1 \) and \( H^\ell \) the weighted depth-first exploration process of \( F \) as follows:

\[
\forall n \in \mathbb{N}, \ H^1(n) := |u^1(n)| \quad \text{and} \quad H^\ell(n) := h(u(n)),
\]

where \( u(n) \) (resp. \( u^1(n) \)) stands for the \( n \)th vertex of \( F \) (resp. \( F^1 \)) taken in the lexicographical order. Our main result on leafed Galton–Watson forests with edge lengths is the following.

**Proposition 1.** Let \((F,e,\ell)\) be a leafed Galton–Watson forest with edge lengths, with offspring distribution \( \zeta \) satisfying hypothesis \((H_1)\). The following convergence in law holds for the Skorokhod topology on the space \( \mathbb{D}(\mathbb{R}^+,\mathbb{R}) \) of càdlàg functions:

\[
\frac{1}{n^{1-1/\kappa}} \left( \left( H^\ell([ns]) \right)_{s \geq 0}, \left( H^1([ns]) \right)_{s \geq 0} \right) \overset{d}{\underset{n \to \infty}{\to}} \left( \left( \mu H^{m-1}_s \right)_{s \geq 0}, \left( H_s \right)_{s \geq 0} \right)
\]

if \( 1 < \kappa < 2 \), or

\[
\frac{1}{(n \ln^{-1}(n))^2} \left( \left( H^\ell([ns]) \right)_{s \geq 0}, \left( H^1([ns]) \right)_{s \geq 0} \right) \overset{d}{\underset{n \to \infty}{\to}} \frac{1}{(2C_0)^{\frac{1}{2}}} \left( \left( \mu H^{m-1}_s \right)_{s \geq 0}, \left( H_s \right)_{s \geq 0} \right)
\]

if \( \kappa = 2 \),

where \( H \) is the continuous-time height process of a spectrally positive Lévy process \( Y \) of Laplace transform \( \mathbb{E}[\exp(-\lambda Y_t)] = \exp(t \lambda^\kappa) \) for any \( \lambda, t \geq 0 \), and where \( C_0 \) is the constant introduced in \((H_1)(iii)\).

Notice that this theorem could be stated in a weaker version by not specifying the Laplace transform of the Lévy process, thus making the multiplicative constants non-necessary. Notice also that hypotheses \((H_1)(ii)\) and \((H_1)(iv)\) can probably be weakened, but this would be unnecessary for the sequel of our paper and this would lengthen the proof.

As we said before, the proof of this proposition will follow that of Theorem 1 in [4]. First, Subsection 2.1 of [4] remains valid under our hypotheses, and we get the following two lemmas.

**Lemma 3.** There exist i.i.d. random variables \((\ell(w_i))_{i \geq 0}\) of measure \( \mathbb{P} \) such that for all \( n \in \mathbb{N} \), if \( g : \mathbb{R}^{n+1} \to \mathbb{R} \) is a measurable function,

\[
\mathbb{E} \left[ \sum_{|u|=n, u \in \Gamma^1} g(\ell(u_0), \ldots, \ell(u_{n-1}), \ell(u)) \right] = \tilde{\mathbb{E}} \left[ g(\ell(w_0), \ldots, \ell(w_{n-1}), \ell(w_n)) \right].
\]

The second lemma is an adaptation of Lemma 2 of [4].

**Lemma 4.** Let \( \Gamma^1_n := u^1(n) \) be the index of the tree in \( F^1 \) to which the \( n \)th vertex of \( F^1 \) belongs. Then under \((H_1)\), for all \( \varepsilon > 0 \), there exist \( M, M' > 0 \) such that for all sufficiently large \( n \in \mathbb{N} \),

\[
\mathbb{P} \left( \Gamma^1_n > Mn^{1/\kappa} \text{ or } \max_{0 \leq i \leq n} |u(i)| > M'n^{1-1/\kappa} \right) < \varepsilon.
\]

The proof of this lemma is exactly that of Lemma 2 in [4]. We give it for the sake of completeness.
Proof. According to Corollary 2.5.1 of [7],
\[ \mathbb{P}\left( \frac{\Gamma_n^1}{\ln 1/n} > M \right)_{n \to \infty} \mathbb{P}(L_1^0 > M) < \frac{\varepsilon}{3} \]
for \( M \) large enough, where \( L_1^0 \) is the local time at level 0 at time 1 of a strictly stable spectrally positive Lévy process \( Y \). Moreover,
\[ \mathbb{P}\left( \frac{\max_{0 \leq i \leq n} |u(i)|}{n^{1-1/\alpha}} > M' \right)_{n \to \infty} \mathbb{P}\left( \max_{0 \leq s \leq 1} |Y_s| > M' \right) < \frac{\varepsilon}{3} \]
for \( M' \) large enough. The union bound concludes the proof.

Let for \( i \in \mathbb{N} \), \( \varphi(i) \) be the index of \( u(i) \) in \( F^1 \) if \( e(u(i)) = 1 \), or of its parent in \( F^1 \) if \( e(u(i)) = 0 \); that is
\[ \varphi(i) := \begin{cases} k, & \text{where } u^i(k) = u(i) \quad \text{if } e(u(i)) = 1 \\ k, & \text{where } u^i(k) = u(i) \quad \text{if } e(u(i)) = 0. \end{cases} \]

Let us now explain how to extend Proposition 2 of [4] to the following proposition:

**Proposition 2.** Let \((F, e, \ell)\) be a leafed Galton–Watson forest with edge lengths satisfying hypothesis \((H_1)\). Then, for all \( \varepsilon > 0 \),
\[ \mathbb{P}\left( \max_{1 \leq i \leq n} H^1(i) - \mu H^1(\varphi(i)) > \varepsilon n^{1-1/\alpha} \right)_{n \to \infty} 0. \]

**Proof.** Let us follow the lines of the proof of Proposition 2 in [4]. As explained at the beginning, it suffices to show that
\[ (2.1) \]
\[ \mathbb{P}\left( \max_{1 \leq i \leq n} |H^1(i) - \mu H^1(\varphi(i))| > \varepsilon n^{1-1/\alpha} \right)_{n \to \infty} 0. \]

Now \((H_1)(iv)\) ensures that there exists \( c > 0 \) such that \( \mathbb{P}(\ell(w_1) > c \ln(n))_{n \to \infty} o(\frac{1}{n}) \). Similarly to how (2.4) is proved in [4], we get that
\[ \mathbb{P}\left( \exists i \leq n : \ell(u^i(i)) > c \ln(n) \right)_{n \to \infty} 0. \]
Considering for any \( n \geq 1 \) and any \( u \in F^1 \), \( \ell^{(u)}(u) := \ell(u) 1(\ell(u) < c \ln(n)) \), showing (2.1) boils down to show that
\[ \mathbb{P}\left( \exists i \leq n, \left| \sum_{u^i(i)} \ell^{(u)}(u) - \mu H^1(\varphi(i)) \right| > \varepsilon n^{1-1/\alpha} \right)_{n \to \infty} 0. \]

Now Lemma [4] allows us to get an equivalent of (2.8) of [4], and using the same arguments that follow it, we get that it suffices to show that
\[ (2.2) \]
\[ \mathbb{P}\left( \left| \sum_{i=1}^{k} \left( \ell^{(u)}(w_i) - \mathbb{E}[\ell^{(u)}(w_1)] \right) \right| > \frac{\varepsilon}{2} n^{1-1/\alpha} \right) = o(\frac{1}{n}) \]
uniformly in \( k \leq M' n^{1-1/\alpha} \). The many-to-one lemma (Lemma [3]) allows us to re-write \((H_1)(iv)\) as \( \mathbb{E}[\ell^{(u)}(w_1)] < \infty \). The \( \ell^{(u)}(w_i) - \mathbb{E}[\ell^{(u)}(w_1)] \) being i.i.d. centred random variables with some finite exponential moments, the Cramer-Chernoff theorem yields (2.2), which concludes the proof.
Let us now prove the equivalent of Proposition 3 of \cite{lukasiewicz} in our case.

**Proposition 3.** Recall that $m = \mathbb{E}[\nu]$. Under $(H_l)$, the function $\varphi([ns])\in\nu_s>0$ converges in probability to $(m^{-1}s)_s>0$ as $n$ tends to infinity, for the topology of uniform convergence over compact sets.

**Proof.** Let us follow the lines of the proof of Proposition 3 of \cite{lukasiewicz}. As explained, we just have to prove that $R_n:=\sum_{k=0}^{n-1}\#\{u\in\mathbb{F} : \frac{u}{v} = u^1(k), u^1(n) < u\}$ is such that $\mathbb{P}(R(n) > \varepsilon n) \rightarrow 0$ for any $\varepsilon > 0$. Let $c_n:=n^{1/\kappa}\ln(n)$. Hypothesis $(H_l)(iii)$ implies that $\mathbb{P}(\nu^1 > c_n) = o\left(\frac{1}{n}\right)$. Combining this with Lemma 4 and Markov’s inequality yields

\[
\mathbb{P}(R(n) > \varepsilon n) \leq \varepsilon' + \frac{1}{\varepsilon n} \mathbb{E}\left[\left( \sum_{u \sim u^1(n)} \nu(u) \right) \mathbb{1}_{\{\max_{i<n} \nu^1(u^1(i)) < c_n, |u^1(n)| \leq M[n^{1-1/\kappa}]\}} \right],
\]

and it is sufficient to show that $A_n$ is $o(n)$ to conclude the proof. Let us introduce the Lukasiewicz path of $F^1$, defined by $S_0 := 0$ and for all $k \geq 1$, $S_k := \sum_{i=0}^{k-1}(\nu^1(u^1(i)) - 1)$. Let us also set

\[
\tau_1 := \inf\{k \geq 1 : S_k > 0\} \quad \text{and} \quad \forall i \in \mathbb{N}, \quad \tau_{i+1} = \inf\{k > \tau_i : S_k > \max_{l<k} S_l\}
\]

as the stopping times at which record high are achieved. Following the lines of the proof of Proposition 3 of \cite{lukasiewicz} we get that

\[
A_n = \mathbb{E}\left[\left( \sum_{k \geq 1} \nu(u^1(\tau_k)) \mathbb{1}_{\{\tau_k \leq n\}} \right) \mathbb{1}_{\{\max_{i<n} \nu^1(u^1(i)) < c_n, \tau_{[Mn^{1-1/\kappa}]} \geq n\}} \right].
\]

Applying Markov's strong property to stopping times $\tau_1, \ldots, \tau_{[Mn^{1-1/\kappa}]}$, we obtain

\[
A_n \leq Mn^{1-1/\kappa} \mathbb{E}\left[\nu(u^1(\tau_1)) \mathbb{1}_{\{\nu^1(u^1(\tau_1)) < c_n\}} \right].
\]

Using Jensen’s inequality and then the same lines as in \cite{lukasiewicz} for $\mathbb{E}\left[\nu(u^1(\tau_1)) \mathbb{1}_{\{\nu^1(u^1(\tau_1)) < c_n\}} \right]$, we get for $\varepsilon > 0$ small enough

\[
\mathbb{E}\left[\nu(u^1(\tau_1)) \mathbb{1}_{\{\nu^1(u^1(\tau_1)) < c_n\}} \right] \leq \left( \mathbb{E}\left[\left( \nu(u^1(\tau_1)) \right)^{1+\varepsilon} \mathbb{1}_{\{\nu^1(u^1(\tau_1)) < c_n\}} \right] \right)^{\frac{1}{1+\varepsilon}}
\]

\[
\leq m_{\varepsilon}^{\frac{1}{1+\varepsilon}} \mathbb{E}\left[\sum_{k=0}^{\tau_{n-1}} \mathbb{1}_{\{S_k > c_n + 1\}} \right]^{\frac{1}{1+\varepsilon}},
\]

where $m_{\varepsilon} := \mathbb{E}[(\nu)^{1+\varepsilon}] < \infty$ according to $(H_l)(iii)$. Now the renewal theorem (p.360 of \cite{lukasiewicz}) ensures that the expectation in the last line is smaller than $c'c_n$ (where $c'$ is an adequate constant) and this yields

\[
A_n \leq (Mn^{1-1/\kappa})m_{\varepsilon}^{\frac{1}{1+\varepsilon}} (c'c_n)^{\frac{1}{1+\varepsilon}} \rightarrow \infty o(n),
\]

as required, thus concluding the proof. \hfill \Box
Proof of Proposition 1. The proof of the convergence of $n^{-(1-1/\kappa)}(H^\ell([ns]))$ is now similar to that in Section 2.4 of [4], let apart that we use Theorems 2.3.1 and 2.3.2 of [7] in the case of the infinite variance:

$$
\left( \frac{1}{n^{1-\frac{1}{\kappa}}} H^1([ns]) \right)_{s \geq 0} \overset{n \to \infty}{\Rightarrow} \left( \frac{1}{(C_0|\Gamma(1-\kappa))^{1/\kappa}} H_s \right)_{s \geq 0} \quad \text{if } 1 < \kappa < 2
$$

$$
\left( \frac{1}{(n \ln^{-1}(n))^\frac{1}{2}} H^1([ns]) \right)_{s \geq 0} \overset{n \to \infty}{\Rightarrow} \left( \frac{1}{(2C_0)^{1/2}} H_s \right)_{s \geq 0} \quad \text{if } \kappa = 2
$$

where $H$ is the continuous-time height process of a spectrally positive Lévy process of Laplace transform $\exp(t\lambda^c)$, and where $C_0$ is the constant introduced in $(H_1)(iii)$. The detail of the computation of the constants can be found in the proof of Theorem 1.1 of [18].

\[\square\]

2.3 Proof of Theorem 2

Let us introduce the following proposition, which will allow us to apply Proposition 1 to the forests $F^R$ and $F^X$.

Proposition 4. Suppose $(H_c)$ and $(H_\kappa)$ for a certain $\kappa \in (1;2]$, and suppose that the distribution of the point process $N$ is non-lattice. Then the leafed Galton–Watson forests with edge lengths $F^R$ and $F^X$ satisfy hypotheses $(H_1)$. Moreover, the associated constants $\mu, m$ are respectively

$$
\mu_R = \mu := (a_1b_1)^{-1} \quad \text{and} \quad m_R = a_1^{-1} \quad \text{for } F^R,
$$

$$
\mu_X = \mu = (a_1b_1)^{-1} \quad \text{and} \quad m_X = 2(a_1b_1)^{-1} \quad \text{for } F^X,
$$

where $a_1$ and $b_1$ are defined in Subsection 3.7.

Let us postpone the proof of this proposition to the next section, and give the proof of Theorem 2.

Proof of Theorem 2. Recall that we denote by $H^\ell_R$ and $H^\ell_X$ the weighted height process of the leafed Galton–Watson forests with edge lengths $F^R$ and $F^X$. Notice also that $H^1_R$ (resp. $H^1_X$) the height process restricted to vertices of type 1 associated with $F^R$ (resp. $F^X$) is actually the non-weighted height process associated with $F^R$ (resp. $F^X$). Let $C_0$ be the constant appearing in condition $(H_1)(iii)$ satisfied by $F^R$ and $F^X$, and let us denote for any $n \geq 1$,

$$
c_n := (C_0|\Gamma(1-\kappa))^{-1/\kappa} n^{1-1/\kappa} \quad \text{if } \kappa \in (1;2) \quad \text{or} \quad c_n := ((2C_0)^{-1} n \ln^{-1}(n))^{1/2} \quad \text{if } \kappa = 2
$$

Let $F^{R'}$ be the forest $F^R$ whose vertices have been re-ordered inside each set of siblings according to their first time of visit by the walk. It remains a leafed Galton–Watson tree with edge lengths, with same parameters as those of $F^R$. If we denote by $H^\ell_R$ (resp. $H^1_R$) its weighted height process (resp. its height process restricted to vertices of type 1), then according to Proposition 4 we can apply Proposition 1 to $F^{R'}$: for the Skorokhod topology on the space of càdlàg functions, we have the following convergence in law:

\begin{equation}
(2.3) \quad c_n^{-1} \left( (H^\ell_R([nt]))_{t \geq 0}, (H^1_R([nt]))_{t \geq 0} \right) \Rightarrow \left( (\mu(H_{m_R}^{-1})_{t \geq 0}, (H_t))_{t \geq 0} \right),
\end{equation}
where $H$ is the continuous-time height process of a spectrally positive Lévy process $Y$ of Laplace transform $E[\exp(-\lambda X_t)] = \exp(t e^\lambda)$ for any $t, \lambda > 0$.

Let us consider $R_n := \# \{X_k, k \leq n \}$ the number of different vertices visited by the walk at time $n$, and $R_n^1 := \# \{X_k, k \leq n : \beta(X_k) = 1 \}$ the number of vertices visited by the walk at time $n$ that will eventually be of type 1. For any $k \geq 1$, let us denote by $\chi^X(k)$ (resp. $\chi^R(k)$) the index for the lexicographical order of the $k$th vertex of type 1 in $F^X$ (resp. in $F^R$). Then $\chi^X(R_n^1)$ and $\chi^X(R_n^{1+})$ are lower and upper bounds of $n$, and $\chi^R(R_n^1)$ and $\chi^R(R_n^{1+})$ are lower and upper bounds of $R_n$. According to the equation below equation (2.10) in the proof of Proposition 5 of [21] (where $\chi$ is denoted by $\psi$), almost surely $\chi^X(R_n^1)/R_n^1 \to m_X$ and $\chi^R(R_n^1)/R_n^1 \to m_R$. Therefore, $R_n/n \to m_R/m_X = b_1/2$.

Now notice that for any $n \geq 1$, $R_n$ is coded by the contour function associated with $(H^R_{t}(k))_{1 \leq k \leq R_n}$ (we recall that when trees are seen as metric spaces, the ordering of vertices does not matter).

What precedes together with (2.3) yields the following convergence:

$$
   (2.4) \quad c_n^{-1} \left( R_n, T_{(H^R_{t}(\lfloor nt \rfloor))_{0 \leq t \leq 1}} \right) \Rightarrow \left( \mu T_{(H^X_{-1}t)}(\lfloor nt \rfloor), T_{(H^X_{t})_{0 \leq t \leq 1}} \right),
$$

where the convergence for the Gromov-Hausdorff topology is due to Lemma 2.4 of [20] (the convergence of the height function implying that of the contour function in our case, as we can use the same arguments as those of Theorem 2.4.1 of [21]). Now, still thanks to Proposition 4, let us apply Proposition 1 to $F^X$:

$$
   (2.5) \quad c_n^{-1} \left( (H^X_{t}(\lfloor nt \rfloor))_{t \geq 0}, T_{(H^X_{-1}t)}(\lfloor nt \rfloor), T_{(H^X_{t})_{0 \leq t \leq 1}} \right) \Rightarrow \left( \mu (H^X_{-1}t), T_{(H^X_{t})_{0 \leq t \leq 1}} \right).
$$

Notice that by construction, $F^X_{1} = F^R_{1}$.

Therefore, $T_{(H^R_{t}(\lfloor nt \rfloor))_{t \geq 0}} \Rightarrow T_{(H^X_{-1}t)}(\lfloor nt \rfloor), T_{(H^X_{t})_{0 \leq t \leq 1}}$ for all $n \in \mathbb{N}^*$, $t \geq 0$. Moreover, according to Lemma 2 $H^X_{t}(n) = |X_n|$ for any $n \geq 0$. So equation (2.5) yields:

$$
   (2.6) \quad c_n^{-1} \left( (|X_{nt}|)_{t \geq 0}, T_{(H^R_{t}(\lfloor nt \rfloor))_{0 \leq t \leq 1}} \right) \Rightarrow \left( \mu (H^X_{-1}t), T_{(H^X_{t})_{0 \leq t \leq 1}} \right).
$$

Now, according to Skorokhod’s representation theorem, there exists a probability space in which the convergence (2.6) holds a.s. and therefore in probability. It is then possible in this new probability space to build a sequence of random variable which has the same law as $(R_n)_{n \geq 1}$, which depends on $(|X_{nt}|)_{t \geq 0}$ and $F^R_{1}$ as initially. Now the convergence of (2.4) implies that in probability of $R_n$ in the new probability space towards the the real tree coded by the continuous time height process. This implies the convergence in probability (still in this space) of the couple $c_n^{-1} (|X_{nt}|, R_n)$ towards $\mu (H^X_{-1}t), T_{(H^X_{t})_{0 \leq t \leq 1}}$, which then holds in law for the initial couple (which was in the initial probability space). We mention that if we do not specify the Laplace transform of $Y$ as in the statement of Theorem 2 then $Y$ is defined up to a multiplicative constant and therefore we can get rid of the multiplicative constants $\mu, m_X, m_R$ and in $c_n$.

\[ \square \]

3 Proof of Proposition 4

Before tackling the proof of Proposition 4 that ensuring that $F^R$ and $F^X$ satisfy $(H_1)$, it is very important to notice that by construction:
For both $F^R$ and $F^X$, the law of the associated offspring distribution of vertices of type $1$, $\nu^1$, has the law of $L^1$ (the cardinal of the optional line $L^1$, introduced in Definition 1) under $P_1$.

The law of the associated total offspring distribution of $F^R$ (vertices of type 0 and 1) denoted by $\nu$ in $(H_1)$, has the law of $B^1$ (the cardinal of the optional line $B^1$, introduced in Definition 1) under $P_1$.

The law of the associated total offspring distribution of $F^X$ (vertices of type 0 and 1) has the law of $\sum_{u \in B^1} 2\beta(u)$ (the total time spent by the walk in $B^1$ in one excursion) under $P_1$.

Therefore, proving that hypothesis $(H_1)(i)$ is satisfied boils down to proving that $E_1[L^1] < \infty$, which will be done in a few lines at the beginning of Subsection 3.2. Then, to ensure that hypothesis $(H_1)(ii)$ is satisfied it is enough to prove that there exists an $\varepsilon > 0$ such that $E_1[(\sum_{u \in B^1} \beta(u))^{1+\varepsilon}] < \infty$, (we would also have to prove that $E_1[(B^1)^{1+\varepsilon}] < \infty$ but as $\sum_{u \in B^1} \beta(u) \geq B^1$ this will be automatic), which will be done in Lemma 7.

Still by construction, notice that to each vertex $u$ of the first generation of $F^R$ corresponds a vertex $u$ of $B^1$, and that $\ell(u') = |u|$. Moreover, the set of vertices of the first generation of $F^X$ matches that of the first generation of $F^R$, where each vertex of which $u$ has been replicated $2\beta(u) - 1$ times. Therefore, denoting by $\zeta^R$ (resp. $\zeta^X$) the law of the first generation of $F^R$ (resp. $F^X$), we have for any $r > 1$,  

$$E\left[\sum_{u \in \nu^R} r^{\ell(u)}\right] = E_1\left[\sum_{u \in B^1} r^{\ell(u)}\right] = E_1\left[\sum_{u \in B^1} 2\beta(u)r^{\ell(u)}\right] = E\left[\sum_{u \in \nu^X} r^{\ell(u)}\right].$$

Thus, it will be enough to show that there exists an $r > 1$ such that $E_1[\sum_{u \in B^1} \beta(u)r^{\ell(u)}] < \infty$ to prove that hypothesis $(H_1)(iv)$ is satisfied, and this is what we will do with Lemma 6.

Finally, hypothesis $(H_1)(iii)$ will be satisfied if we are able to show that there exists a positive constant $C_0$ such that $P_1(L^1 > x) \sim C_0e^{-x}$. This will be the subject of the whole Subsection 3.3 and this is actually the main contribution of this paper when compared to [1]. Indeed, in the latter, we only had to show the finiteness of the second moment of $\nu^1$, and this could be obtained by quite straightforward backbone decomposition techniques; whereas here proving the regular variation of the tails of $\nu^1$ will appear to be quite technical, and will require the fine understanding of $L^1$.

### 3.1 Study of the range and change of measure

Let us detail the probability law $P_i$ (introduced at the beginning of Subsection 2.1) conditionally on the environment. The law of the walk described in the introduction is such that, for any $u \in \mathbb{T}$, if $\nu(u) = n < \infty$, then for any $i \geq 1$, $k_1, \ldots, k_n \geq 1$,

$$P^V(\beta(u1) = k_1, \ldots, \beta(un) = k_n \mid \beta(u) = i) = \frac{(k_1 + \cdots + k_n + i - 1)}{1 + \sum_{j=1}^n e^{-V(u)}k_1+\cdots+k_n+i}.$$  

If $\nu(u) = \infty$, then for any sequence of integers with finite support $k_1, k_2, \ldots$, we have that

$$P^V(\beta(u1) = k_1, \beta(u2) = k_2, \cdots \mid \beta(u) = i) = \frac{\prod_{j \geq 1} e^{-k_jV(uj)}}{1 + \sum_{j \geq 1} e^{-V(uj)}}.$$
\[ (\beta(v))_{v \in u} \] is a negative multinomial random variable with parameters \( \beta(u) \) (which is the number of trials) and \( (e^{-V(v)}/(e^{-V(u)} + \sum_{v' \neq u} e^{-V(v')}))_{v \in u} \) (event probabilities).

This yields that the marginal law of any subset (potentially infinite) of children \( u' \), \( u'' \), \ldots of a given vertex \( u \) is given for any sequence of integers with finite support \( k_1, k_2, \ldots \) by

\[
P^V(\beta(u') = k_1, \beta(u'') = k_2, \ldots \mid \beta(u) = i) = \left( \frac{1}{1 + \sum_{v \not\in u} e^{-V(v)}} \right)^{\sum_{v \not\in u} e^{-V(v)}} \prod_{j \geq 1} e^{-k_j V(u_j')} \left( \frac{1}{1 + \sum_{v \not\in u} e^{-V(v)}} \right)^i \sum_{k \geq 1} k_j.
\]

In particular, conditionally on the environment and on the edge local time of the parent, the marginal for each vertex \( u \in T \) is therefore a negative binomial random variable of parameter \((\beta(u), e^{-V(u)}/(1 + e^{-V(u)})\)):

\[
P^V(\beta(u) = k \mid \beta(u) = i) = \left( \begin{array}{c} k + i - 1 \\ k \end{array} \right) \frac{e^{-k V(u)}}{(1 + e^{-V(u)})^{k+i}},
\]

and the sum of the edge local times of the children of a vertex \( u \) is a negative binomial random variable of parameter \((\beta(u), 1/(1 + (\sum_{v \not\in u} e^{-V(v)})^{-1}))\):

\[
P^V(\sum_{v \not\in u} \beta(v) = k \mid \beta(u) = i) = \left( \begin{array}{c} k + i - 1 \\ k \end{array} \right) \frac{(\sum_{v \not\in u} e^{-V(v)})^k}{(1 + \sum_{v \not\in u} e^{-V(v)})^{k+i}}.
\]

Considering the setting of multitype Galton–Watson forests used in [4], (3.2) yields that the mean matrix \((m_{i,j})_{i,j \geq 1}\) of \( F \) is given by

\[
m_{i,j} := E_i \left[ \sum_{|u|=1} 1_{\{\beta(u) = j\}} \right] = \left( \begin{array}{c} i + j - 1 \\ j \end{array} \right) E \left[ \sum_{|u|=1} \frac{e^{-j V(u)}}{(1 + e^{-V(u)})^{i+j}} \right].
\]

Let us introduce the random variable \( \hat{S}_1 \) with law defined by \( E[f(\hat{S}_1)] = E[\sum_{|u|=1} f(V(u)) e^{-V(u)}] \) for any bounded continuous real function \( f \). Let us define \((\hat{S}_k)_{k \geq 0}\) as the random walk starting from 0 with step distribution \( \hat{S}_1 \). Notice that under \((H_\rho), E[\hat{S}_1] = -\psi(1) > 0 \) so \( \sum_{k \geq 1} e^{-\hat{S}_k} \in (0; \infty) \) almost surely. Letting for all \( i \geq 1 \)

\[
a_i := E \left[ \left( \sum_{k \geq 1} e^{-\hat{S}_k} \right)^{i-1} \right] / E \left[ \frac{1}{1 + \sum_{k \geq 1} e^{-\hat{S}_k}} \right],
\]

\[
b_i := i E \left[ \frac{1}{1 + \sum_{k \geq 1} e^{-\hat{S}_k}} \right],
\]

then \((a_i)_{i \geq 1}\) and \((b_i)_{i \geq 1}\) are respectively left and right eigenvectors of the matrix \((m_{i,j})_{i,j \geq 1}\) associated with the eigenvalue 1 (this is proved in Lemma 6.2 of [1]).

The forest \( F \) being made up of i.i.d. trees \((\mathbf{T}_k)_{k \geq 1}\), the law of which is that of the range of \((X_{n,V}^\rho)_{n \geq 0}\) during an excursion from \( \rho \), let us introduce an important tool for the study of such trees (that we will generically denote by \( \mathbf{T} \)). We let for all \( n \geq 1 \)

\[
Z_n := \sum_{|u|=n} \beta(u)
\]
be the multitype additive martingale of \((T, \beta)\). If for \(n \geq 1\) we let \(\mathcal{F}_n\) be the sigma-algebra generated by \((u, \beta(u))_{u \in T, |u| \leq n}\) then for any \(i \geq 1\), \((Z_n/i)_{n \geq 0}\) is a \(\mathcal{F}_n\)-martingale under \(P_i\) of expectation 1 (this is due to the fact that \((b_i)_{i \geq 1}\) is a right eigenvector of \((m_{i,j})_{i,j \geq 1}\) associated to 1).

Let us introduce a new law \(\hat{P}_i\) on marked trees with distinguished path \((T, \beta, (w_n)_{n \geq 0})\). Let \(\hat{\zeta} = (\hat{\zeta}_i)_{i \geq 1}\) be a probability law on \(\bigsqcup_{n \in \mathbb{N}} \mathbb{N}^{*n}\) of Radon-Nikodym derivative \(Z_i^\ast\) with respect to \(\zeta\); that is that for any \(i \geq 1\), if \(X \sim \zeta_i\), then \(\hat{X} \sim \hat{\zeta}_i\) if and only if for any bounded real-valued function \(f\) on \(\bigsqcup_{n \in \mathbb{N}} \mathbb{N}^{*n}\), \(E[f(\hat{X})] = E[\sum_{1 \leq k \leq |X|} X_k f(X)]\) (where we denoted by \(|X|\) the length of \(X\), and by \(X_k\) its components). For any \(i \geq 1\), we construct \((T, \beta, (w_n)_{n \geq 0})\) under \(\hat{P}_i\) by induction as follows:

- **Initialisation**
  Generation 0 of \(T\) is only made up of the root \(\rho\) of given type \(\beta(\rho) = i\). We set \(w_0 = \rho\).

- **Induction**
  Let \(n \geq 0\). Suppose that the tree up to generation \(n\) and \((w_k)_{k \leq n}\) have been built. The vertex \(w_n\) has progeny according to \(\hat{\zeta}_{\beta(w_n)}\). Other vertices \(u\) of generation \(n\) have progeny according to \(\hat{\zeta}_{\beta(u)}\). Then, choose a vertex at random among children \(u\) of \(w_n\), each with probability \(\beta(u)/\left(\sum_{v = w_n} \beta(v)\right)\) and set \(w_{n+1}\) as this vertex.

We denote by \(\hat{P}_i\) the marginal law of \((T, \beta)\) under this construction, and \(\hat{E}_i\) the associated expectation. The following proposition is easily deduced from [19]:

**Proposition 5.** [19] Let \(i \geq 1\). We have the following links between \(P_i\) and \(\hat{P}_i\):

(i) Recall that for any \(n \geq 0\), \(\mathcal{F}_n\) stands for the sigma-algebra generated by the \((u, \beta(u))\) for \(u \in T, |u| \leq n\). Then \(\hat{P}_i|_{\mathcal{F}_n}\) is absolutely continuous with respect to \(P_i|_{\mathcal{F}_n}\) and is such that

\[
\frac{d\hat{P}_i}{dP_i}|_{\mathcal{F}_n} = \frac{1}{i} Z_n.
\]

(ii) Conditionally on \(\mathcal{F}_n\), for all \(u \in T\) such that \(|u| = n\),

\[
\hat{P}_i(w_n = u | \mathcal{F}_n) = \frac{\beta(u)}{Z_n}
\]

(iii) Under \(\hat{P}\), the process \((\phi_k)_{k \in \mathbb{N}} := (\beta(w_k))_{k \in \mathbb{N}}\) is a Markov chain taking its values in \(\mathbb{N}\) with initial state \(i\), and with transition probabilities denoted by \((\hat{p}_{i,j})_{i,j \geq 1}\) where for all \(i, j \geq 1\),

\[
\hat{p}_{i,j} = \frac{b_j}{b_i} m_{i,j} = \binom{i+j-1}{i} \mathbb{E}\left[\sum_{|u| = 1} \frac{e^{-\hat{Y}(u)}}{(1+e^{-\hat{Y}(u)})^{i+j}}\right].
\]

Notice that the Markov chain \((\phi_k)_{k \in \mathbb{N}}\) introduced in (iii) admits an invariant measure \((\pi_i)_{i \geq 1}\) where for all \(i \geq 1\),

\[
\pi_i = a_i b_i = \mathbb{E}\left[\frac{\left(\sum_{k \geq 1} e^{-\hat{S}_k}\right)^{i-1}}{(1 + \sum_{k \geq 1} e^{-\hat{S}_k})^{i+1}}\right]
\]

and that this measure is of total mass 1, thus making \((\phi_k)_{k \geq 0}\) a positive recurrent Markov chain. Proposition 5 yields the multitype many-to-one lemma:
Lemma 5. For all \( n \in \mathbb{N}^* \), let \( g : \mathbb{N}^n \to \mathbb{R}_+ \) be a measurable function and \( X_n \) a \( \mathcal{F}_n \)-measurable random function, then

\[
E_i \left[ \sum_{|u|=n} \beta(u)g(\beta(u_1), \beta(u_2), \ldots, \beta(u_n))X_n \right] = i\hat{E}_i \left[ g(\phi_1, \phi_2, \ldots, \phi_n)X_n \right].
\]

### 3.2 Hypotheses (H1) (i), (ii) and (iv)

Now recall that we suppose that hypotheses \( (H_c) \) and \( (H_s) \) are satisfied for a certain \( \kappa \in (1; 2] \). Let

\[
\hat{\tau}_1 := \min\{k \geq 1 : \phi_k = 1\}
\]

be the first non-null hitting time of 1 by \( (\phi_k)_{k \geq 0} \). Let us determine the first moment of \( \nu^1 \) for both \( \mathcal{F}^R \) and \( \mathcal{F}^X \), that is the first moment of \( L^1 \) under \( \mathbb{P}_1 \). More generally, for any \( i \geq 1 \), we have

\[
E_i \left[ L^1 \right] = E_i \left[ \sum_{u \in \mathcal{L}^1} 1 \right] = E_i \left[ \sum_{u \in \mathcal{L}^1} \beta(u) \right]
\]

\[
= \sum_{k \geq 1} E_i \left[ \sum_{|u|=k} 1_{\{\beta(u_1), \ldots, \beta(u_{k-1}) \neq 1 \text{ and } \beta(u_k) = 1\}} \beta(u) \right]
\]

\[
= \sum_{k \geq 1} i\hat{E}_i \left[ 1_{\{\phi_1, \ldots, \phi_{k-1} \neq 1 \text{ and } \phi_k = 1\}} \right] = i \sum_{k \geq 1} \hat{E}_i \left[ 1_{\{\hat{\tau}_1 = k\}} \right] = i,
\]

the Markov chain \( (\phi_k)_{k \geq 0} \) being recurrent, and where we used the multitype many-to-one lemma (Lemma 5) between the last two lines. The case \( i = 1 \) reads \( E_1[L^1] = 1 \), which proves that \( (H_1)(i) \) is satisfied by \( \mathcal{F}^R \) and \( \mathcal{F}^X \).

Let us prove now that \( \mathcal{F}^R \) and \( \mathcal{F}^X \) also satisfy hypothesis \( (H_1)(iv) \). As explained in the introduction of the section, it is enough to show the following lemma.

**Lemma 6.** There exists \( r > 1 \) such that \( E_i[\sum_{u \in \mathcal{L}^1} \beta(u)r^{||u||}] < \infty \).

**Proof.** Let \( r > 1 \), using the many-to-one lemma (Lemma 5), we get

\[
E_i \left[ \sum_{u \in \mathcal{L}^1} \beta(u)r^{||u||} \right] = \sum_{k \geq 1} E_i \left[ \sum_{|u|=k} \beta(u)1_{\{\beta(u_1), \ldots, \beta(u_{k-1}) \neq 1\}}r^k \right]
\]

\[
= \sum_{k \geq 1} i\hat{E}_i \left[ 1_{\{\phi_1, \ldots, \phi_{k-1} \neq 1\}}r^k \right] = \hat{E}_i \left[ \sum_{k=1}^{\hat{\tau}_1} r^k \right],
\]

Thus, proving the lemma boils down to proving that this last quantity is finished. To this end, let \( \alpha \in (0; \kappa - 1) \), and let us consider for any \( i \geq 1 \), \( F(i) := \frac{i(1+r^i)}{\Gamma(1+r^i)} \). As proved in the appendix (Lemma 15), there exists \( d \in (0; 1) \) such that for any \( i > i_0 \) large enough,

\[
\sum_{j \geq 1} \hat{p}_{ij}F(j) \leq dF(i),
\]

that is \( F \) satisfies condition (V4) introduced p.371 of [26]. As a consequence, Theorem 15.3.3 of [26] ensures that there exists a constant \( C_F > 0 \) (depending on \( d \), and then on \( F \)) such that
for any $r \in (1; d^{-1})$ and for any $i \geq 1$,

\[
(3.6) \quad \hat{E}_i \left[ \sum_{k=1}^{\hat{\tau}_i} F(\phi_k)r^k \right] \leq C_F F(i),
\]

and the function $F$ being greater than 1, this yields the finiteness of (3.5) and concludes the proof. \hfill \Box

Actually we can get from (3.6) a few more interesting estimates that will be useful later. Indeed, the function $F$ being greater than 1, and equivalent to $i^\alpha$ as $i \to \infty$, (3.6) implies that for any $\alpha \in (0; \kappa - 1)$, there exists $C_\alpha > 0$ such that for any $r \in (1; \psi(1 + \alpha)^{-1})$ and $i \geq 1$,

\[
(3.7) \quad \hat{E}_i \left[ \sum_{k=1}^{\hat{\tau}_i} \phi_k^\alpha r^k \right] \leq C_\alpha i^\alpha.
\]

Moreover, Jensen’s inequality yields that there exists a constant $C_1 > 0$ such that for any $p > 0$

\[
(3.8) \quad \hat{E}_i[\hat{\tau}_i^p] \leq C_1 \ln^p (1 + i).
\]

Let us now focus on hypothesis (H1)(ii). We are then interested in moments of order $1 + \varepsilon$ of $\sum_{u \in \mathcal{B}^1} \beta(u)$ under $\mathbf{P}_1$. For convenience, let us denote this quantity by $\tilde{B}^1$. We have the following lemma:

**Lemma 7.** For any $\alpha \in (0; \kappa - 1)$, $\varepsilon > 0$, there exists a constant $C_{\alpha + \varepsilon} > 0$ such that for any $i \geq 1$,

$$\mathbf{E}_i[(\tilde{B}^1)^{1+\alpha}] \leq C_{\alpha + \varepsilon} i^{1+\alpha + \varepsilon}. \quad (3.9)$$

**Proof.** Let us start by computing some estimates on the first moment of $\tilde{B}^1$ under $\mathbf{P}_i$ for $i \geq 1$:

\[
\mathbf{E}_i[\tilde{B}^1] = \mathbf{E}_i \left[ \sum_{u \in \mathbf{N}\setminus\{\rho\}} \beta(u) \mathbf{1}_{\{\beta(u_1), \beta(u_2), \ldots, \beta(u_{\hat{\tau}_i})\neq 1\}} \right] = \sum_{k \geq 1} \mathbf{E}_i \left[ \sum_{|u|=k} \beta(u) \mathbf{1}_{\{\beta(u_1), \ldots, \beta(u_{k-1})\neq 1\}} \right] = i \sum_{k \geq 1} \hat{E}_i \left[ \mathbf{1}_{\{\phi_1, \ldots, \phi_{k-1} \neq 1\}} \right] = i \hat{E}_i[\hat{\tau}_i],
\]

where we used the many-to-one lemma (Lemma 5). Now, equation (3.8) yields

\[
(3.9) \quad \mathbf{E}_i[\tilde{B}^1] \leq C_1 i \ln(1 + i).
\]

Let us now compute the $1+\alpha^{th}$ moment of $\tilde{B}^1$ under $\mathbf{P}_i$. Discussing on the generation to which vertices of $\mathcal{B}^1$ belong, we get

\[
\mathbf{E}_i[(\tilde{B}^1)^{1+\alpha}] = \sum_{k \geq 1} \mathbf{E}_i \left[ \left( \sum_{|u|=k} \beta(u) \mathbf{1}_{\{u \in \mathcal{B}^1\}} \right) \times (\tilde{B}^1)^{1+\alpha} \right].
\]
For \( k \geq 1 \), let us focus on the general term of the sum. When conditioning on \( \mathcal{F}_k \), it can be written as

\[
E_i \left[ \left( \sum_{|u|=k} \beta(u) \mathbf{1}_{\{u \in B^i_1\}} \right) (\tilde{B}^i)^{\alpha} \right] = E_i \left[ \left( \sum_{|u|=k} \beta(u) \mathbf{1}_{\{u \in B^i_1\}} \right) \tilde{B}^i \right].
\]

Let us apply the many-to-one lemma (Lemma 5) at generation \( k \) to this expectation, with the setting \( X_k = E_i \left[ (\tilde{B}^i)^{\alpha} \mid \mathcal{F}_k \right] \) (which is \( \mathcal{F}_k \)-measurable); we get

\[
(3.10) \quad E_i \left[ \left( \sum_{|u|=k} \beta(u) \mathbf{1}_{\{u \in B^i_1\}} \right) (\tilde{B}^i)^{\alpha} \right] = i \hat{E}_i \left[ \mathbf{1}_{\{k \leq \hat{\tau}_1\}} \tilde{B}^i \right].
\]

Decomposing \( B^1 \) according to generations we have for any \( k \leq \hat{\tau}_1 \),

\[
\tilde{B}^i = \sum_{|u|<k} \beta(u) \mathbf{1}_{\{u \in B^i_1\}} + \sum_{|u|=k, u \neq \phi_k} \mathbf{1}_{\{u \in B^i_1\}} (\beta(u) + \tilde{B}^i_1) + \beta(\phi_k) + \tilde{B}^i_{\phi_k},
\]

where for \( u \in T \) we denoted by \( \tilde{B}^i_u \) the quantity \( \sum_{v \in B^i_1} \beta(v) \), that we choose to be equal to 0 if \( \beta(u) = 1 \).

The fact that \( \alpha < 1 \) and equation (3.9) yield for any \( k \leq \hat{\tau}_1 \),

\[
E_i \left[ (\tilde{B}^i)^{\alpha} \mid \mathcal{F}_k \right] \leq E_i \left[ \left( \sum_{|u|<k} \beta(u) \mathbf{1}_{\{u \in B^i_1\}} + \sum_{|u|=k, u \neq \phi_k} \mathbf{1}_{\{u \in B^i_1\}} (\beta(u) + \tilde{B}^i_1) + \beta(\phi_k) \right)^{\alpha} \right] + E[\tilde{B}^i_{\phi_k} \mid \mathcal{F}_k]^\alpha,
\]

(3.11)

\[
\leq E_i \left[ \left( \sum_{|u|<k} \beta(u) \mathbf{1}_{\{u \in B^i_1\}} + \sum_{|u|=k, u \neq \phi_k} \mathbf{1}_{\{u \in B^i_1\}} (\beta(u) + \tilde{B}^i_1) + \beta(\phi_k) \right)^{\alpha} \right] + (C_1 \phi_k \ln(1 + \phi_k))^\alpha.
\]

Now notice that according to the construction of \( T \) in Subsection 3.1 for \( u \neq \phi_k \) the \( \tilde{B}^i_u \) have same law under \( \hat{\mathbb{P}}_i \) and \( \mathbb{P}_i \). The previous equation can therefore be written as

\[
E_i \left[ (\tilde{B}^i)^{\alpha} \mid \mathcal{F}_k \right] \leq \hat{E}_i \left[ \left( \sum_{|u|<k} \beta(u) \mathbf{1}_{\{u \in B^i_1\}} + \sum_{|u|=k, u \neq \phi_k} \mathbf{1}_{\{u \in B^i_1\}} (\beta(u) + \hat{\tilde{B}}^i_1) + \beta(\phi_k) \right)^{\alpha} \right] + (C_1 \phi_k \ln(1 + \phi_k))^\alpha
\]

Plugging this in (3.10) and summing over \( k \geq 1 \), we finally get a more convenient upper bound for the \( 1 + \alpha \)th moment:

\[
E_i \left[ (\tilde{B}^i)^{1+\alpha} \right] \leq i \hat{E}_i \left[ \sum_{k=1}^{\hat{\tau}_1} \hat{E}_i[(\tilde{B}^i)^{\alpha}] \right] + i \hat{E}_i \left[ \sum_{k=1}^{\hat{\tau}_1} (C_1 \phi_k \ln(1 + \phi_k))^{\alpha} \right]
\]

\[
= i \hat{E}_i \left[ \hat{\tau}_1(\tilde{B}^i)^{\alpha} \right] + i \hat{E}_i \left[ \sum_{k=1}^{\hat{\tau}_1} (C_1 \phi_k \ln(1 + \phi_k))^{\alpha} \right]
\]

(3.12)

\[
\leq i \hat{E}_i \left[ \hat{\tau}_1(\tilde{B}^i)^{\alpha} \right] + C_1 \alpha C_1 \epsilon \tau_1 \phi_k^{1+\alpha + \epsilon},
\]

where the last inequality was obtained thanks to (3.7) for any \( \epsilon > 0 \). We therefore only have to bound the first quantity to prove the lemma. Now we get, thanks to Hölder’s inequality, for any \( \epsilon' > 0 \) (we choose \( \epsilon' \) such that \( \alpha' := \alpha(1+\epsilon') < \kappa - 1 \))

\[
(3.13) \quad i \hat{E}_i \left[ \hat{\tau}_1(\tilde{B}^i)^{\alpha} \right] \leq i \hat{E}_i \left[ \hat{\tau}_1^{1+\epsilon'/\epsilon'} \right] \hat{E}_i \left[ (\tilde{B}^i)^{(\alpha+\epsilon')(1+\epsilon')} \right] \hat{E}_i \left[ (\tilde{B}^i)^{1+\epsilon'} \right] \leq C_1 i \ln(1+i) \hat{E}_i \left[ (\tilde{B}^i)^{\alpha(1+\epsilon')} \right] \hat{E}_i \left[ (\tilde{B}^i)^{1+\epsilon'} \right],
\]

20
where we used equation (3.8) in the last inequality. Now, computing this last quantity will require a decomposition more subtle. Under the biased law $\hat{P}$, $B^1$ is made up of the spine below $w_{\tilde{\tau}_1}$, together with the sets $B^1_u$ for any brother $u$ of the spine below $w_{\tilde{\tau}_1}$ such that $\beta(u) \neq 1$. That is, denoting by $\Omega(w_k)$ the set of brothers of $w_k$, and still for $u \in T$ by $B^1_u$ the quantity $\sum_{v \in B^1_u} \beta(v)$ (that we choose to be equal to 0 if $\beta(u) = 1$),

$$\hat{E}_i[(\hat{B}^1)^{\alpha'}] = \hat{E}_i\left[\left(\sum_{k=1}^{\tilde{\tau}_1} \beta(w_k) + \sum_{k=1}^{\tilde{\tau}_1} \sum_{u \in \Omega(w_k)} (\beta(u) + \hat{B}^1_u)\right)^{\alpha'}\right]$$

$$\leq \hat{E}_i\left[\sum_{k=1}^{\tilde{\tau}_1} \phi_k^{\alpha'}\right] + \hat{E}_i\left[\sum_{k=1}^{\tilde{\tau}_1} \left(\sum_{u \in \Omega(w_k)} (\beta(u) + \hat{B}^1_u)\right)^{\alpha'}\right]$$

$$\leq C_{\alpha'} \hat{E}_i\left[\sum_{k=1}^{\tilde{\tau}_1} \left(\sum_{u \in \Omega(w_k)} \hat{E}(\beta(u) + \hat{B}^1_u | \sigma(\beta(v), v \in \bigcup_{1 \leq k \leq \tilde{\tau}_1} \Omega(w_k)))\right)^{\alpha'}\right]$$

after this decomposition along the spine (we used (3.7) and Jensen’s inequality in the last inequality). Now (3.9) yields that

$$\hat{E}_i[(B^1)^{\alpha'}] \leq C_{\alpha'} \hat{E}_i\left[\sum_{k=1}^{\tilde{\tau}_1} \left(\sum_{u \in \Omega(w_k)} \beta(u) + C_1 \beta(u) \ln(1 + \beta(u))\right)^{\alpha'}\right],$$

where $C_1$ is a suitable constant. Conditioning with respect to $\sigma((w_k)_{k \in [0, \tilde{\tau}_1 - 1]})$, we get

$$\hat{E}_i\left[\sum_{k=1}^{\tilde{\tau}_1} \left(\sum_{u \in \Omega(w_k)} \beta(u) \ln(1 + \beta(u))\right)^{\alpha'}\right] \leq \hat{E}_i\left[\sum_{k=0}^{\tilde{\tau}_1 - 1} \hat{E}_{\Phi_k}\left[\left(\sum_{|u| = 1} \beta(u) \ln(1 + \beta(u))\right)^{\alpha'}\right]\right]$$

$$= \hat{E}_i\left[\sum_{k=0}^{\tilde{\tau}_1 - 1} \frac{1}{\phi_k} \phi_k \left(\sum_{|u| = 1} \beta(u) \ln(1 + \beta(u))\right)^{\alpha'}\right]$$

$$\leq \hat{E}_i\left[\sum_{k=0}^{\tilde{\tau}_1 - 1} \frac{1}{\phi_k} \phi_k \left(\sum_{|u| = 1} \beta(u) \ln(1 + \beta(u))\right)^{1 + \alpha'}\right],$$

where we used the branching property on each $w_k$ for $0 \leq k \leq \tilde{\tau}_1 - 1$ and where $\epsilon' > 0$ can be chosen as small as we want. We let $\alpha'' = \alpha(1 + \epsilon')$. Let for any $k \geq 1$

$$G_k := \sigma\left((u, V(u))_{u \in T, |u| \leq k}\right) \quad \text{and} \quad G := \bigvee_{k \geq 1} G_k$$

be the sigma-algebras generated by the knowledge of the environment. Recall from (3.3) that under $P_{\Phi_k}$, $\sum_{|u| = 1} \beta(u) \sim NB(\phi_k, \frac{\sum_{|u| = 1} e^{-V(u)}}{1 + \sum_{|u| = 1} e^{-V(u)}}$; Lemma 10 (given in the appendix) yields

$$\frac{1}{\phi_k} \mathbb{E}_{\Phi_k}\left[\left(\sum_{|u| = 1} \beta(u)\right)^{1 + \alpha''(1 + \epsilon')}\right] = \frac{1}{\phi_k} \mathbb{E}\left[\mathbb{E}_{\Phi_k}\left[\left(\sum_{|u| = 1} \beta(u)\right)^{1 + \alpha''} | G_k\right]\right]$$

$$\leq 16 \mathbb{E}\left[\sum_{|u| = 1} e^{-V(u)}\right] + \mathbb{E}\left[\left(\sum_{|u| = 1} e^{-V(u)}\right)^{1 + \alpha''}\right] + \phi_k^{1 + \alpha''} 2E\left[\left(\sum_{|u| = 1} e^{-V(u)}\right)^{\alpha''}\right].$$

21
Hypothesis \((H_{\kappa})\) ensures the finiteness of \(K_{\alpha'}^1\) and \(K_{\alpha''}^2\). Plugging this back in equation \((3.15)\) yields
\[
\hat{E}_i \left[ \sum_{k=1}^{\hat{\tau}_i} \left( \sum_{u \in \Omega(w_k)} \beta(u) \ln(1 + \beta(u)) \right)^{\alpha'} \right] \leq \hat{E}_i \left[ \sum_{k=1}^{\hat{\tau}_i} K_{\alpha''}^1 + K_{\alpha''}^2 \phi_k^1 \right]
\]
Now, together with inequality \((3.7)\) this yields
\[
\hat{E}_i \left[ \sum_{k=1}^{\hat{\tau}_i} \left( \sum_{u \in \Omega(w_k)} \beta(u) \ln(1 + \beta(u)) \right)^{\alpha'} \right] \leq \hat{E}_i \left[ \sum_{k=1}^{\hat{\tau}_i} \left( K_{\alpha''}^1 + K_{\alpha''}^2 \phi_k^1 \right) \right]^\alpha''
\]
where we recall that \(\alpha''\) can be chosen as close to \(\alpha\) as we want, and \(\alpha'\) as close to \(\alpha\) as we want. Plugging this into \((3.14)\) and then in \((3.12)\) (via \((3.13)\)), we get that for any \(\alpha'' > \alpha\) as close to \(\alpha\) as we want,
\[
E_i \left[ (B_1^1)^{1+\alpha} \right] \leq C_{\alpha''} 1^{1+\alpha''}
\]
where \(C_{\alpha''}\) is a suitable constant.

To conclude this subsection and before proving that \((H_l)(iii)\) is satisfied, let us compute the constants \(\mu_R, m_R, \mu_X\) and \(m_X\). According to the remark at the beginning of Section 3, we have
\[
\mu_X = \mu_R = E_i \left[ \sum_{u \in \mathcal{L}^1} |u| \right] = \hat{E}_i \left[ |w_{\hat{\tau}_1}| \right] = \hat{E}_i [\hat{\tau}_1] = 1/\pi_1 = (a_1 b_1)^{-1},
\]
where we used the many-to-one lemma (Lemma 5) in the third equality. We also have still using the remark at the beginning of Section 3 and the many-to-one lemma,
\[
m_R = E_i \left[ \sum_{u \in \mathcal{B}^1} \right] = \hat{E}_i \left[ \sum_{k=1}^{\hat{\tau}_1} \frac{1}{\phi_k} \right] = \sum_{i \geq 1} \frac{1}{\pi_1} = (a_1)^{-1}
\]
and
\[
m_X = E_i \left[ \sum_{u \in \mathcal{B}^1} 2\beta(u) \right] = 2\hat{E}_i \left[ \sum_{k=1}^{\hat{\tau}_1} 1 \right] = 2 \sum_{i \geq 1} \frac{\pi_i}{\pi_1} = 2(a_1 b_1)^{-1}.
\]

### 3.3 Hypothesis \((H_l)(iii)\)

Let us now show that hypothesis \((H_l)(iii)\) is satisfied by \(F^R\) and \(F^X\), i.e. that there exists a constant \(C_0 > 0\) such that
\[
\mathcal{P}_1(L_1 > x) \sim \frac{1}{C_0 x^{-\kappa}}.
\]
To this end, we will have to deal a subtle study on the behaviour of \(L_1\). First, according to Lemma 4.3 of \([22]\), the previous equation is equivalent to
\[
E_i [L_1 1_{(L_1 > x)}] \sim \frac{\kappa}{x \to \infty \kappa - 1} C_0 x^{-(\kappa - 1)}.
\]
Moreover, the many-to-one lemma (Lemma 5) yields
\[
E_1\left[L^1 1_{\{L^1 > x\}}\right] = \sum_{k \geq 1} E_1\left[\sum_{|u|=k} 1_{\{u \in L^1\}} 1_{\{L^1 > x\}}\right]
\]
\[
= \sum_{k \geq 1} \hat{E}_1\left[1_{\{k = \hat{n}\}} P_1(L^1 > x | F_k)\right]
\]
\[
= \sum_{k \geq 1} \hat{E}_1\left[1_{\{k = \hat{n}\}} \hat{P}_1(L^1 > x | F_k)\right] = \hat{P}_1(L^1 > x),
\]
where between the last two lines we used the fact that on the event \(\{k = \hat{n}\}\) we have \(P_1(L^1 > x | F_k) = \hat{P}_1(L^1 > x | F_k)\), and where the last equality comes from the fact that \((\phi_k)_{k \geq 0}\) is recurrent. Therefore showing that hypothesis \((H_1)(iii)\) is satisfied is equivalent to show that there exists \(C'_0 > 0\) such that
\[
(3.17) \quad \hat{P}_1(L^1 > x) \sim C'_0 x^{-(\kappa-1)},
\]
that is we have to understand the behaviour of \(L^1\) under \(\hat{P}\). To this end, let us give a new way of seeing the law of \((T, \beta)\) under \(\hat{P}\).

### 3.3.1 Understanding of the law of \((T, \beta)\) under \(\hat{P}\)

The idea is that \((T, \beta, (w_k)_{k \geq 0})\) under \(\hat{P}'\) can actually be seen as the range of a series of random walks on a certain tree with spine. Let us begin by introducing this tree.

Recall that \(N\) is the point process giving the offspring distribution of \(V\). Let us consider \(\hat{N}\) a point process defined as follows: for any bounded measurable function \(f : \bigcup_{n \in \mathbb{N} \cup \{\infty\}} \mathbb{R}^n \to \mathbb{R}\) we have \(E[f(\hat{N})] = E[\sum_{u \in \hat{N}} e^{-V(u)} f(N)]\). Hypothesis \((H_c)\) ensures that this defines a probability law indeed. Let us introduce a law on marked trees with spine \(\hat{P}'\). We build \((T, V, (w_k)_{k \geq 0})\) under \(\hat{P}'\) by induction as follows:

- **Initialisation**
  Generation 0 of \(T\) is only made up of the root \(\rho\) and we set \(V(\rho) = 0\). We set \(w_0 = \rho\).

- **Induction** Let \(n \geq 0\). Suppose that the tree up to generation \(n\) and \((w_k)_{k \leq n}\) have been built. The vertex \(w_n\) has progeny according to \(\hat{N}\) translated by \(V(w_n)\). Other vertices \(u\) of generation \(n\) have progeny according to \(N\) translated by \(V(u)\). Then, choose a vertex at random among children \(u\) of \(w_n\), each with probability \(e^{-V(u)}/\left(\sum_{v=w_n} e^{-V(v)}\right)\) and set \(w_{n+1}\) as this vertex.

We let \(\hat{P}\) be the marginal law \((T, V)\) under \(\hat{P}'\) (and \(\hat{E}\) the associated expectation). For any \(k \geq 0\) we let \(W_k := \sum_{|u|=k} e^{-V(u)}\). We get the following proposition from [23]:

**Proposition 6.** [23] Recall from (3.10) that for any \(k \geq 0\), \(\mathcal{G}_k\) stands for the sigma-algebra generated by the knowledge of \(V\) (but not \((w_k)_{k \geq 0}\)) up to generation \(k\).

- For any \(|u| \in T\), we have
  \[
  \hat{P}'(w_k = u | \mathcal{G}_k) = \frac{e^{-V(u)}}{W_k}
  \]
• The process \( (V(w_k))_{k \geq 0} \) under \( \hat{P}' \) has the same distribution as the random walk \( (S_k)_{k \geq 0} \) introduced in Subsection 3.1.

From this proposition we get the following version of the many-to-one lemma, which can be proved by induction:

**Lemma 8.** Let for any \( n \geq 0, g : \mathbb{R}^n \to \mathbb{R}_+ \) be a positive function. Recall also that we let \( (S_k)_{k \geq 0} \) be a random walk of step distribution \( S_1 \) (introduced in Subsection 3.1). Then for any \( k \geq 0 \), and for any \( \mathcal{G}_k \)-measurable random variable \( X_k \),

\[
\mathbb{E}[\sum_{|u|=k} e^{-V(u)}g(V(u_1), \ldots, V(u_k))X_k] = \hat{\mathbb{E}}[g(V(w_1), \ldots, V(w_k))X_k] = \hat{\mathbb{E}}[g(S_1, \ldots, S_k)X_k],
\]

where \( (S_k)_{k \geq 0} \) is the random walk with step distribution introduced in Subsection 3.1.

Let us now consider for each vertex of the spine \( w_i \) two i.i.d. truncated nearest-neighbour random walks with same law \( (X_{k \geq 0}^{1, w_i}) \) and \( (X_{k \geq 0}^{2, w_i}) \), each defined as follows. It starts on \( w_i \). If it is on a vertex \( u \in T \), then for each vertex \( v \) child of \( u \), it will jump to \( v \) with probability

\[
e^{-V(v)} \frac{e^{-V(v)}}{e^{-V(u)} + \sum_{z \in u} e^{-V(z)}}
\]

and towards \( u \) with probability

\[
e^{-V(u)} \frac{e^{-V(u)}}{e^{-V(u)} + \sum_{z \in u} e^{-V(z)}}.
\]

If it reaches \( w_{i-1} \), then it is killed instantly.

![Figure 2: Possible paths for the walks \( (X_{k \geq 0}^{1, w_i}) \) and \( (X_{k \geq 0}^{2, w_i}) \).](image-url)

Now for each \( u \in T \) we let

\[e_1^i(u) := \#\{n \geq 0 : X_{n \leq u}^{1, w_i} = \bar{u}, X_{n+1}^{1, w_i} = u\}\]

and

\[e_2^i(u) := \#\{n \geq 0 : X_{n \leq u}^{2, w_i} = \bar{u}, X_{n+1}^{2, w_i} = u\}\]

be the edge local times on \( u \) of the walks launched on \( w_i \), and we let

\[e^1(u) := \sum_{i=1}^{\infty} e_1^i(u)\]

and

\[e^2(u) := \sum_{i=1}^{\infty} e_2^i(u)\]

be the sum of the edge local times of all the walks. Finally, we let

\[e(u) := e^1(u) + e^2(u) + 1_{\exists n \geq 0 : u = w_i}.
\]
In other words, for any vertex \( u \in T \) not on the spine, \( e(u) \) corresponds to the sum of the edge local times of the walks launched on each vertex of the spine below; moreover, we add 1 for each vertex of the spine. Heuristically, one can also see \( e \) as the local time of the random walk on \( V \) conditioned on going infinitely high on the spine, and then on coming back (so for example, \( e^1 \) would be the edge local time of the ascent, and \( e^2 \) the edge local time of the descent).

We will denote by \( \bar{P}' \) the law of \( \{u \in T, e(u) > 0\} \), \( e, (w_k)_{k \geq 0} \) when \( (T, V, (w_k)_{k \geq 0}) \) is built under \( \bar{P}' \), and \( \bar{P} \) the marginal of \( \{u \in T, e(u) > 0\}, e \).

**Proposition 7.** Under \( \bar{P}'_1 \), \( (T, \beta, (w_k)_{k \geq 0}) \) has the same law as \( \{u \in T, e(u) > 0\}, e, (w_k)_{k \geq 0} \) under \( \bar{P}' \).

**Proof.** Notice that under \( \bar{P}'_1 \), \( \{u \in T, e'(u) > 0\}, e, (w_k)_{k \geq 0} \) is a multitype tree with spine, and can be built by recursion. Indeed,

- for any vertex \( u \in T \) not on the spine, conditionally on \( e(u) \), the progeny of \( u \) has the law of a negative multinomial random variable of parameter \( (e(u), 1/(1 + 1/N^u)) \), (where \( N^u \sim N \) is a random variable independent of everything else) independently of other vertices.

- for any \( i \geq 0 \), conditionally on \( e(w_i) \), the progeny of \( w_i \) has the law of a negative multinomial random variable of parameters \( e(w_i) + 1 \) (as \( e(w_i) - 1 \) stands for the edge local time of the walks launched below \( w_i \), to which we have to add the contribution of the two walks \((X_{k,w_i}^1, w_i)_{k \geq 0}\) and \((X_{k,w_i}^2, w_i)_{k \geq 0}\)) and \( (1/(1 + 1/N_{w_i}^u))_{k \leq |N_{w_i}|} \) (where \( N_{w_i} = (N_{w_i}^u)_{k \leq |N_{w_i}|} \sim N \) is independent of everything else), and after choosing \( w_{i+1} \) proportionally to \( N_{w_i} \), we add 1 to its type.

Since the offspring distribution outside the spine is the same as under \( \bar{P}' \), we only need to focus on the offspring distribution of vertices of the spine. Let \( f \) be a positive bounded function on \( \mathbb{N}^{|N|} \times \mathbb{N} \), and let \( i \geq 1 \),

\[
\hat{E}_i[f((\beta(u))_{|u|=1}, \beta(w_1))] = \hat{E}_i \left[ \sum_{|u|=1} \sum_{\beta(u)} f((\beta(u))_{|u|=1}, \beta(v)) \right] = \frac{1}{i} E_i \left[ \sum_{|u|=1} \beta(u) f((\beta(u))_{|u|=1}, \beta(v)) \right].
\]

Now decomposing according to the law of \( \beta(u)_{|u|=1} \), which is a negative multinomial random variable of parameter \( (i, (1/(1 + e^{V(u)}))_{|u|=1}) \) as explained in Subsection 3.1, we get (denoting 1, 2, \ldots, \nu the children of \( \rho \), and where for convenience in the writing we suppose \( \nu < \infty \) (but the proof is the same when considering the event \( \nu = \infty \)) according to equation (3.1),

\[
\hat{E}_i[f((\beta(u))_{|u|=1}, \beta(w_1))] = \frac{1}{i} \mathbb{E} \left[ \sum_{j=1}^{\nu} \mathbb{E}^T (\beta(j) f((\beta(1), \beta(2), \ldots, \beta(\nu)), \beta(j))) \right]
\]

\[
= \frac{1}{i} \mathbb{E} \left[ \sum_{j=1}^{\nu} \sum_{k_1, \ldots, k_{\nu} \geq 0} e^{-V(j)} \frac{k_1 + \cdots + k_{\nu} + i - 1}{k_1! \ldots (k_j - 1)! \ldots k_{\nu}!(i + 1 - 1)!} \times e^{-k_1 V(1)} \times \cdots \times e^{-k_j V(1)} \times \cdots \times e^{-k_{\nu} V(\nu)} \times \frac{1}{1 + \sum_{\ell=1}^{\nu} e^{-V(\ell)} k_1 + \cdots + k_{\nu} + i - 1} \right] \times f((k_1, \ldots, k_{\nu}), (k_j))
\]

\[
= \mathbb{E} \left[ \sum_{j=1}^{\nu} e^{-V(j)} \sum_{k_1, \ldots, k_{\nu} \geq 0, k_j \geq 1} e^{-k_1 V(1)} \times \cdots \times e^{-k_{j-1} V(1)} \times \cdots \times e^{-k_{\nu} V(\nu)} \times \frac{1}{1 + \sum_{\ell=1}^{\nu} e^{-V(\ell)} k_1 + \cdots + k_{j-1} V(1) + \cdots + k_{\nu} V(\nu)} \right] f((k_1, \ldots, k_{\nu}), (k_j)).
\]
Making the change of variable $k_j \leftarrow k_j + 1$, we get
\[
\hat{E}_i\left[f\left(\left(\beta(u)\right)_{|u|=1}, \beta(w)\right)\right] = \mathbb{E}\left[\sum_{j=1}^{\nu} e^{-V(j)} \sum_{k_1, \ldots, k_{\nu} \geq 0} \frac{(k_1 + \ldots + k_{\nu} + (i + 1) - 1)}{k_1! \ldots k_{\nu}! (i + 1 - 1)!} e^{-k_1 V(1)} \ldots e^{-k_{\nu} V(\nu)} (1 + \sum_{t=1}^{\nu} e^{-V(t)} k_1 + \ldots + k_{\nu} + (i + 1)) f\left((k_1, \ldots, k_j + 1, \ldots, k_{\nu}), k_j + 1\right)\right]
\]
\[
= \hat{E}_i\left[\sum_{k_1, \ldots, k_{\nu} \geq 0} \left((k_1 + \ldots + k_{\nu} + (i + 1) - 1)\right) \frac{e^{-k_1 V(1)} \ldots e^{-k_{\nu} V(\nu)}}{(1 + \sum_{t=1}^{\nu} e^{-V(t)} k_1 + \ldots + k_{\nu} + (i + 1))} f\left((k_1, \ldots, k_w + 1, \ldots, k_{\nu}), k_w + 1\right)\right].
\]
which is exactly the law of the progeny of the spine when of type $i$ under $\hat{\mathbb{P}}$, as described above.

From now on we will therefore consider $(T, \beta)$ under $\hat{\mathbb{P}}$ as the range of all these random walks on $\mathbb{V}$ under $\hat{\mathbb{P}}$, and we will use indifferently $\hat{\mathbb{E}}$ or $\mathbb{E}$, and $\beta$ or $\varepsilon$. Let us continue our investigation by describing the behaviour of $L^1$ after a large number of excursions from the root.

3.3.2 Behaviour of $L^1$ with large initial local time

Let us begin this subsection with the remark that the proof of Lemma 7 on the small moments of $\hat{B}^1$ can easily be adjusted to $L^1$, just by following its lines, and that we can get a sharper estimate. Indeed, there is not any need to shift from $\alpha$ to $\alpha'$, since we get $E_i\left[\left(\sum_{|u|=k} 1_{\{u \in L^1\}}\right) L^1\right] = iE_i\left[1_{\{\hat{I}_i = k\}} (L^1)^\alpha\right]$ in (3.10) and so $E_i[(L^1)^{1+\alpha}] \leq iE_i[(L^1)^\alpha]$ in (3.12). For the same reason, equation (3.13) will not be necessary and the $\ln(1 + \phi_k)$ will not appear in (3.11), so there will not be any need to shift from $\alpha''$ to $\alpha'''$. Moreover, there is not any need either to shift from $\alpha'$ to $\alpha(1 + \varepsilon) = \alpha''$ in equation (3.15) since the term $\ln(1 + \beta(u))$ will not appear in (3.14) (as we can use equation (3.4) instead of (3.9)). That is we finally get

(3.18)
\[
E_i[(L^1)^{1+\alpha}] \leq C_\alpha i^{1+\alpha},
\]
for any $\alpha \in (0; \kappa - 1)$ and $i \geq 1$, and where $C_\alpha$ is a suitable constant. Let us now state a proposition that describes the behaviour of $L^1$ under $\mathbb{P}_n$ when $n$ is large.

**Proposition 8.** Let $\alpha \in (0; \kappa - 1)$, we have the following convergence in mean of order $1 + \alpha$,

\[
E_n\left[\frac{L^1}{n} - W_\infty^{1+\alpha}\right] \stackrel{n \to \infty}{\to} 0,
\]
where $W_\infty$ is the almost sure limit of the positive martingale $(W_k)_{k \geq 1} := (\sum_{|u|=k} e^{-V(u)})_{k \geq 1}$.

**Proof.** Let us first establish the convergence in law of $L^1/n$. Actually we intend to prove a little more than this: for any $l \geq 1, M \geq 0$, we will prove the convergence in law of $L^1/n - W_{l,M}$, 26
where we recall that $W_t := \sum_{|u|=t} e^{-V(u)}$. To this end, we will establish the convergence of its Laplace transform (which exists as it is bounded from below).

Let us start with the lower bound. Recall that for any $k \geq 1$ we denote by $\mathcal{F}_k$ the sigma-algebra generated by $(\beta(u), u \in T, |u| \leq k)$, and recall from (3.16) that $\mathcal{G}_k$ is the sigma-algebra generated by the environment below generation $k$. For $u \in T$, recall that we denote by $L_u^1$ the cardinal of $\mathcal{L}_u^1$, and $W_t^M := W_{t \wedge M}$. Now for any $t > 0$, $k \geq l$,

$$
\mathbb{E}_n \left[ \exp(-t \frac{L_1^1}{n} - W_t^M) \right] = \mathbb{E}_n \left[ \mathbb{E}_n \left[ \exp(-\frac{t}{n} \left( \sum_{|u|=k} 1_{\{u < \mathcal{L}^1\}} L_u^1 + \sum_{|u|<k} 1_{\{u \in \mathcal{L}^1\}} \right)) \mid \mathcal{F}_k, \mathcal{G}_k \right] \exp(t W_t^M) \right] \geq \mathbb{E}_n \left[ \exp(-\frac{t}{n} \sum_{|u|=k} 1_{\{u < \mathcal{L}^1\}} \beta(u) [L^1] - \sum_{|u|<k} \frac{t}{n} 1_{\{u \in \mathcal{L}^1\}} \exp(t W_t^M) \right] = \mathbb{E}_n \left[ \exp(-\frac{t}{n} \sum_{|u|=k} 1_{\{u < \mathcal{L}^1\}} \beta(u) - \sum_{|u|<k} \frac{t}{n} 1_{\{u \in \mathcal{L}^1\}} \exp(t W_t^M) \right],
$$

where the last equality comes from (3.4). Now notice that conditionally on the environment, $\frac{1}{n} \sum_{|u|=k} \beta(u) \xrightarrow{n \to \infty} \sum_{|u|=k} e^{-V(u)}$ in probability, by the law of large number. Moreover, as for any $u \in T$ of generation $k$, $1_{\{u < \mathcal{L}^1\}}$ tends to 1 a.s. with $n$, this implies the convergence in probability of $\frac{1}{n} \sum_{|u|=k} 1_{\{u < \mathcal{L}^1\}} \beta(u)$ towards $\sum_{|u|=k} e^{-V(u)}$. These convergences are immediate if $\{u \in T : |u| = k\}$ is finite; otherwise, one way to see that is to introduce for any $p \geq 1$

$$
\mathcal{E}^p_k := \{u \in T : |u| = k, \#\{v \in T : e^{-V(v)} \geq e^{-V(u)}\} < p\},
$$

that is the set of the $p$ vertices of generation $k$ with lowest potential. Now

$$
\left| \frac{1}{n} \sum_{|u|=k} \beta(u) 1_{\{u < \mathcal{L}^1\}} - \sum_{|u|=k} e^{-V(u)} \right| \leq \frac{1}{n} \sum_{u \in \mathcal{E}_k^p} \beta(u) \left[ 1_{\{u < \mathcal{L}^1\}} - \sum_{u \in \mathcal{E}_k^p} e^{-V(u)} \right] + \frac{1}{n} \sum_{|u|=k, u \not\in \mathcal{E}_k^p} \beta(u) + \left| \sum_{|u|=k, u \not\in \mathcal{E}_k^p} e^{-V(u)} \right|,
$$

and since $\mathcal{E}_k^p$ is finite, the first member of this last sum tends to 0 with $n$, and the second tends a.s. to $| \sum_{|u|=k, u \not\in \mathcal{E}_k^p} e^{-V(u)} |$ by the law of large numbers, which can be made as small as desired with $p$ large, hence ensuring the convergence in probability of $\frac{1}{n} \sum_{|u|=k} \beta(u) 1_{\{u < \mathcal{L}^1\}}$. Moreover, the many-to-one lemma yields

$$
\frac{1}{n} \mathbb{E}_n \left[ \sum_{|u|<k} 1_{\{u \in \mathcal{L}^1\}} \right] = \hat{P}_n(\hat{\tau}_1 < k) \leq \left( \hat{P}(X_{1,u_0}^1 \text{ does not reach the level } k) \right)^n \xrightarrow{n \to \infty} 0.
$$

The inequality can be seen by the representation of the Markov chain $(\phi_k)_{k \geq 0}$ given in the previous subsection, and using the fact that $\hat{\tau}_1$ is the first level such that none of the walks launched below reach it. The latter equation implies the convergence in probability of $n^{-1} \sum_{|u|=k} 1_{\{u < \mathcal{L}^1\}}$ towards 0.

Finally, Lebesgue’s dominated convergence theorem yields

$$
\liminf_{n \to \infty} \mathbb{E}_n \left[ \exp(-t \frac{L_1^1}{n} + t W_t^M) \right] \geq \mathbb{E} \left[ \exp(-t \sum_{|u|=k} e^{-V(u)} + t W_t^M) \right].
$$
and \(k\) being arbitrary, \(\liminf_n \mathbb{E}_n[\exp(-t(L^1_n - W^M_i))] \geq \lim_{k \to \infty} \mathbb{E}[\exp(-t \sum_{|u| = k} e^{-V(u)} + tW^M_i)] = \mathbb{E}[\exp(-t(W_\infty - W^M_i))]\) (the equality is obtained by Lebesgue's dominated convergence theorem).

Let us now tackle the upper bound. For any \(t > 0\),

\[
\mathbb{E}_n[\exp(-t(L^1_n - W^M_i))] = \mathbb{E}_n \left[ \exp \left( -\frac{t}{n} \sum_{|u| < k} 1_{\{u \in \mathcal{L}_n \}} + \sum_{|u| = k, u \in \mathcal{L}_1} L^1(u) \right) \right] \exp(tW^M_i) \\
\leq \mathbb{E}_n \left[ \prod_{|u| = k, u \in \mathcal{L}_1} \mathbb{E}_n \left[ \exp \left( -\frac{t}{n} L^1(u) \right) 1_{\mathcal{F}_k} \right] \right] \exp(tW^M_i).
\]

Now as remarked in [25] between equations (B.4) and (B.5), the inequality \(e^{-x} \leq 1 - x + x^{1+\alpha} \leq e^{-x + x^{1+\alpha}}\) yields

\[
\leq \mathbb{E}_n \left[ \prod_{|u| = k, u \in \mathcal{L}_1} \left( 1 - \frac{t}{n} L^1(u) + \left( \frac{t}{n} L^1(u) \right)^{1+\alpha} \right) \right] \exp(tW^M_i) \\
\leq \mathbb{E}_n \left[ \left( \prod_{|u| = k, u \in \mathcal{L}_1} \mathbb{E}_n \left[ (1 + \frac{t}{n} \beta(u) + t^{1+\alpha} C_\alpha (\frac{\beta(u)}{n})^{1+\alpha}) \right] \right) \right] \exp(tW^M_i),
\]

where the last inequality results from (3.18). Now since \(\frac{1}{n} \sum_{|u| = k} 1_{\{u \in \mathcal{L}_1\}} \beta(u) n_{\to \infty} \sum_{|u| = k} e^{-V(u)}\) in probability and \(\frac{1}{n^{1+\alpha}} \sum_{|u| = k} 1_{\{u \in \mathcal{L}_1\}} \beta(u)^{1+\alpha} \to n_{\to \infty} \sum_{|u| = k} e^{-(1+\alpha)V(u)}\) (this can be proved using the same strategy as in (3.19)), by Lebesgue’s dominated convergence theorem we get

\[
\limsup_{n \to \infty} \mathbb{E}_n[\exp((-t(L^1_n - W^M_i))] \leq \mathbb{E} \left[ \left( \exp \left( -t \sum_{|u| = k} e^{-V(u)} + C_\alpha t^{1+\alpha} \sum_{|u| = k} e^{-(1+\alpha)V(u)} \right) \right) \right] \exp(tW^M_i)
\]

But according to our hypotheses, as \(\mathbb{E}[\sum_{|u| = 1} e^{-(1+\alpha)V(u)}] < 1\) since \(\alpha < \kappa - 1, \sum_{|u| = k} e^{-(1+\alpha)V(u)} \to 0\) in probability, and therefore the function in the expectation being bounded, and \(k\) arbitrary

\[
\limsup_{n \to \infty} \mathbb{E}_n[\exp((-t(L^1_n - W^M_i))] \leq \lim_{k \to \infty} \mathbb{E} \left[ \left( \exp\left( -t \sum_{|u| = k} e^{-V(u)} + C_\alpha t^{1+\alpha} \sum_{|u| = k} e^{-(1+\alpha)V(u)} \right) \right) \right] \exp(tW^M_i)
\]

\[
= \mathbb{E} \left[ \exp(-t(W_\infty - W^M_i)) \right].
\]

Thus, the convergence in law of \((L^1_n - W^M_i)\) is established, as its Laplace transform converges towards that of \(W_\infty - W^M_i\).

Let us now consider \(\alpha \in (0; \kappa - 1)\), we have

\[
\mathbb{E}_n[\frac{L^1_n}{n} - W^M_i]^{1+\alpha} \leq \mathbb{E}_n[\frac{L^1_n}{n}]^{1+\alpha} + \mathbb{E}[|W^M_i|]^{1+\alpha} \leq C + C',
\]

where we used equation (3.18) for the first inequality and Theorem 2.1 of [22] for the second (\(C'\) being then a constant independent of \(l\) and \(M\)). This ensures that \((|L^1_n - W^M_i|^{1+\alpha})^{n \geq 1}\) is uniformly integrable for any \(\alpha' < \alpha\), and \(\alpha\) being arbitrary \((|L^1_n - W^M_i|^{1+\alpha})^{n \geq 1}\) is uniformly integrable. This together with the convergence in law of \(L^1_n\) implies the convergence
Thus ensuring the convergence in mean of order $1 + k$ we get, as $L^3.3.3$ Tail of $E$
The convergence of the expectation). Therefore, as for any $k \geq 1$,

$$
E_n \left[ \frac{L^1}{n} - W_\infty^{1+\alpha} \right] \leq 2E_n \left[ \frac{L^1}{n} - W_i^{M|1+\alpha} \right] + 2E \left[ |W_i^{M} - W_\infty^{1+\alpha}| \right],
$$

we get, as $k$ and $M$ are arbitrary,

$$
\limsup_n E_n \left[ \frac{L^1}{n} - W_\infty^{1+\alpha} \right] \leq 4E \left[ |W_\infty^{M} - W_i^{M|1+\alpha}| \right]_{M \to \infty} \lim_{t \to \infty} 4E \left[ |W_\infty^{M} - W_i^{1+\alpha}| \right]_{t \to 0},
$$

thus ensuring the convergence in mean of order $1 + \alpha$ of $L^1/n$ under $P_n$ towards $W_\infty$. \qed

### 3.3.3 Tail of $L^1$ under $\hat{P}$

We now intend to show that (3.17) stands. In what follows, we will systematically consider $(T, V, (w_k)_{k \geq 0})$ under the law $\hat{P}$ and the walk under the law $\hat{P}_1$ (that we will denote by $\hat{P}$). Our proof will be strongly inspired from that of H. Kesten, M.V. Kozlov and F. Spitzer in [16]. In the latter the authors studied the behaviour of the random walks $(X_{k_1}^{w_k})_{k \geq 0}$ restricted to the spine. Letting $\hat{\tau}_1 := \min\{k \geq 0 : e_1(w_k) = 0\}$ and $\kappa' := \kappa - 1$, they proved that under our conditions,

$$
\hat{P} \left( \sum_{k=0}^{\hat{\tau}_1} e_1(w_k) > x \right) \sim Kx^{-\kappa'},
$$

where $K$ is a positive constant. Our strategy will rely on the fact that under $\hat{P}$, $L^1$ can be decomposed into smaller lines stemming from the spine (plus $w_{\hat{\tau}_1}$). In terms of counting, this yields

$$
(3.20) \quad L^1 = \sum_{k=1}^{\hat{\tau}_1} \sum_{u \in \Omega(w_k)} L^1_u + 1
$$

where we recall that for $k \geq 1$, we denoted by $\Omega(w_k)$ the set of the brothers of $w_k$, and that for $u \in T$, $L^1_u$ stands for the cardinal of the vertices of $L^1$ descending from $u$ (that we choose here to be equal to 1 if $u \in L^1$). Now let for any $u \in T$

$$
W^u_\infty := \lim_{n \to \infty} \sum_{u \in V, |v| - |u| = n} e^{-(V(v) - V(u))}
$$

be the the limit of the additive martingale stemming from $u$. We also let for any $A \geq 1$

$$
\sigma_A := \min\{k \geq 1 : e(w_k) > A\}
$$

be the first time that the edge local time of the spine is larger than $A$. The first thing that we will prove, in Lemma 9 is that actually if we want $L^1$ to be large, then $\sigma_A$ has to be smaller than $\hat{\tau}_1$. Then, still in Lemma 9 we will show that the number of vertices of $L^1$ which separated from the spine below $w_{\sigma_A}$ is negligible when $L^1$ is large; that is we will show that the two quantities

- $L^1 = \sum_{k=1}^{\hat{\tau}_1} \sum_{u \in \Omega(w_k)} L^1_u + 1$
\[ \sum_{k=\sigma_A+1}^{\sigma_A-1} \sum_{u \in \Omega(w_k)} L_u^1 \]

are close. Now, as the heuristics of Proposition 8 say that for \( u \in \mathbb{T} \), \( L_u^1 \approx e(u)W_\infty^u \) when \( e(u) \) is large, we expect that when \( L^1 \) is large,

\[ L^1 \approx \sum_{k=\sigma_A+1}^{\sigma_A-1} \sum_{u \in \Omega(w_k)} e(u)W_\infty^u. \]

This is what we will show at the end of this subsection, in Lemma 12. Before that, in Lemma 11, we will see that the contribution to \( L^1 \) of the walks launched above \( w_{\sigma_A} \) is negligible; that is if we denote for any \( u \in \mathbb{T} \) by

\[ e^{\sigma_A}(u) := \sum_{k=0}^{\sigma_A-1} e_k^1(u) + e_k^2(u) \]

the edge local time on \( u \) of the walks launched below \( w_{\sigma_A} \), then the quantities

- \( \sum_{k=\sigma_A+1}^{\sigma_A-1} \sum_{u \in \Omega(w_k)} e(u)W_\infty^u \)
- \( \sum_{k=\sigma_A+1}^{\sigma_A-1} \sum_{u \in \Omega(w_k)} e^{\sigma_A}(u)W_\infty^u \)

are close. And we will see in Lemma 10 that for large \( A \), the behaviour of this last quantity is dictated by \( e^{\sigma_A}(w_{\sigma_A}) \) together with the environment above \( w_{\sigma_A} \), namely that

- \( \sum_{k=\sigma_A+1}^{\sigma_A-1} \sum_{u \in \Omega(w_k)} e^{\sigma_A}(u)W_\infty^u \) and
- \( e^{\sigma_A}(w_{\sigma_A})W_\infty^{w_{\sigma_A}} \)

are close. Let us now say a few words about the law of \( W_\infty \) under \( \hat{P} \). Notice that hypotheses \((Hc), (H_\kappa)\) together with \( N \) being non-lattice ensure that Theorem 2.2 of [22] can be applied to \( W_\infty \) under \( \mathbb{P} \); this yields that there exists a constant \( C_\infty \) such that

\[ P(W_\infty > x) \sim x^{-1} \rightarrow \infty C_\infty x^{-\kappa}. \]

On the other hand, for any \( n \geq 0 \)

\[ \mathbb{E}[W_n 1_{\{W_n > x\}}] = \mathbb{E} \left[ \sum_{|u|=n} 1_{\{W_n > x\}} \right] = \hat{P}(W_n > x), \]

by Lemma 8. Since \( W_n \xrightarrow{L^1} W_\infty \) by the Kesten-Stigum theorem [17] (which we can apply thanks to \((H_\kappa))\), we have when \( n \) tends to infinity,

\[ \mathbb{E}[W_\infty 1_{\{W_\infty > x\}}] = \hat{P}(W_\infty > x). \]

But according to Lemma 4.3 of [22], equation (3.21) is equivalent to

\[ \mathbb{E}[W_\infty 1_{\{W_\infty > x\}}] \sim C_\infty^{-1} x^{-(\kappa-1)}, \]

so setting \( C_\infty := \frac{C_\infty}{\kappa-1} \) we have

\[ \hat{P}(W_\infty > x) \sim C_\infty x^{-\kappa'}, \]

and as by the branching property \( W_\infty^{w_{\sigma_A}} \) has same law as \( W_\infty \), this last equation will allow us to conclude. Let us now prove Lemmas 9, 10, 11 and 12.
Lemma 9. For all $\varepsilon > 0$, $A > 0$, 
\[ \hat{P}(L^1 > \varepsilon x, \hat{\tau}_1 \leq \sigma_A) = o(x^{-\kappa'}). \]
Moreover, we have for all $\varepsilon > 0$, $A > 0$, 
\[ \hat{P}(\sum_{k=1}^{\sigma_A} \sum_{u \in \Omega(w_k)} L^1_u > \varepsilon x, \sigma_A < \hat{\tau}_1) = o(x^{-\kappa'}). \]

Proof. Let us apply Markov’s inequality in both cases:
\[ \hat{P}(L^1 > \varepsilon x, \hat{\tau}_1 \leq \sigma_A) \leq (\varepsilon x)^{-\kappa'} \hat{E}[(L^1 1_{(\hat{\tau}_1 \leq \sigma_A)})^\kappa'] 1_{\{L^1 > \varepsilon x\}] \]
\[ = (\varepsilon x)^{-\kappa'} \hat{E}\left[(1 + \sum_{k=1}^{\hat{\tau}_1} \sum_{u \in \Omega(w_k)} L^1_u)^\kappa' 1_{\{\sigma_A \leq \hat{\tau}_1\}} 1_{\{L^1 > \varepsilon x\}] \right] \]
\[ \leq (\varepsilon x)^{-\kappa'} \hat{E}\left[(1 + \sum_{k=1}^{\hat{\tau}_1} 1_{\{e(w_{k-1}) < A\}} \sum_{u \in \Omega(w_k)} L^1_u)^\kappa' 1_{\{L^1 > \varepsilon x\}] \right] . \]
where the equality is due to (3.20). For the same reason,
\[ \hat{P}(\sum_{k=1}^{\sigma_A} \sum_{u \in \Omega(w_k)} L^1_u > \varepsilon x, \sigma_A < \hat{\tau}_1) \leq (\varepsilon x)^{-\kappa'} \hat{E}\left[(1 + \sum_{k=1}^{\sigma_A} \sum_{u \in \Omega(w_k)} L^1_u)^\kappa' 1_{\{\sigma_A < \hat{\tau}_1\}} 1_{\{L^1 > \varepsilon x\}] \right] \]
\[ \leq (\varepsilon x)^{-\kappa'} \hat{E}\left[(1 + \sum_{k=1}^{\hat{\tau}_1} 1_{\{e(w_{k-1}) < A\}} \sum_{u \in \Omega(w_k)} L^1_u)^\kappa' 1_{\{L^1 > \varepsilon x\}] \right] . \]
Now since $1_{\{L^1 > \varepsilon x\}}$ tends to 0 when $x$ tends to infinity, it is enough to show the finiteness of $\hat{E}\left[(\sum_{k=1}^{\hat{\tau}_1} 1_{\{e(w_{k-1}) < A\}} \sum_{u \in \Omega(w_k)} L^1_u)^\kappa' \right]$ to conclude. We have, $\kappa'$ being smaller than 1,
\[ \hat{E}\left[(\sum_{k=1}^{\hat{\tau}_1} 1_{\{e(w_{k-1}) < A\}} \sum_{u \in \Omega(w_k)} L^1_u)^\kappa' \right] \leq \hat{E}\left[\sum_{k=1}^{\hat{\tau}_1} 1_{\{e(w_{k-1}) < A\}} \left( \sum_{u \in \Omega(w_k)} L^1_u \right)^\kappa' \right] \]
\[ \leq \hat{E}\left[\sum_{k=1}^{\hat{\tau}_1} 1_{\{e(w_{k-1}) < A\}} \left( \sum_{u \in \Omega(w_k)} E_{e(u)} \left[L^1 \right] \right)^\kappa' \right] \]
\[ = \hat{E}\left[\sum_{k=1}^{\hat{\tau}_1} 1_{\{e(w_{k-1}) < A\}} \left( \sum_{u \in \Omega(w_k)} e(u) \right)^\kappa' \right] , \]
where the second inequality is obtained after applying Jensen’s inequality (as $\kappa' \leq 1$) and the branching property. The last equality is due to (3.1). Now conditioning on the edge local times
of the spine and using the branching property, we get that this is

\[
\begin{align*}
&\leq \mathbb{E}\left[\sum_{k=1}^{\hat{\tau}_1} \mathbf{1}_{e(w_{k-1}) < A} \mathbb{E}_e(w_{k-1}) \left( \sum_{|u|=1} e(u)^{\kappa'} \right)\right] \\
&= \mathbb{E}\left[\sum_{k=1}^{\hat{\tau}_1} \mathbf{1}_{e(w_{k-1}) < A} \mathbb{E}_e(w_{k-1}) \left( \sum_{|u|=1} e(u) \left( \sum_{|u|=1} e(u)^{\kappa'} \right) \right)\right] \\
&= \mathbb{E}\left[\sum_{k=1}^{\hat{\tau}_1} \mathbf{1}_{e(w_{k-1}) < A} \mathbb{E}_e(w_{k-1}) \left( \sum_{|u|=1} e(u)^{\kappa'} \right)\right]
\end{align*}
\]

where the first equality comes from Proposition 5. Finally, as noticed in (3.3), under \( P_e(w_{k-1}), \sum_{|u|=1} e(u) \sim NB(e(w_{k-1}), 1/(1 + (\sum_{|u|=1} e^{-V(u)})^{-1}) \), Lemma 16 yields that this is

\[
\begin{align*}
&\leq \mathbb{E}\left[\sum_{k=1}^{\hat{\tau}_1} \mathbf{1}_{e(w_{k-1}) < A} e(w_{k-1}) \left( 16e(w_{k-1}) \left( \mathbb{E}\left[ \sum_{|u|=1} e^{-V(u)} \right] + \mathbb{E}\left[ \left( \sum_{|u|=1} e^{-V(u)} \right)^{\kappa} \right] \right) + 2e(w_{k-1})^{\kappa} \mathbb{E}\left[ \left( \sum_{|u|=1} e^{-V(u)} \right)^{\kappa} \right] \right)\right] \\
&\leq \mathbb{E}\left[\sum_{k=1}^{\hat{\tau}_1} \left( 16 \mathbb{E}\left[ \sum_{|u|=1} e^{-V(u)} \right] + \mathbb{E}\left[ \left( \sum_{|u|=1} e^{-V(u)} \right)^{\kappa} \right] + 2A^{\kappa} \mathbb{E}\left[ \left( \sum_{|u|=1} e^{-V(u)} \right)^{\kappa} \right] \right) \right] < \infty,
\end{align*}
\]

by \( H_k \), and as \( \mathbb{E}[\hat{\tau}_1] \) is finite. \( \square \)

In light of (3.20), we get from this lemma that for any \( \varepsilon > 0 \), for sufficiently large \( A \) and \( x \),

\[
\begin{align*}
\hat{P}\left( \sum_{k=\sigma_{A}+1}^{\hat{\tau}_1} \sum_{u \in \Omega(w_k)} L_u^1 > x, \sigma_A < \hat{\tau}_1 \right) &\leq \hat{P}(L^1 > x) \leq \hat{P}\left( \sum_{k=\sigma_{A}+1}^{\hat{\tau}_1} \sum_{u \in \Omega(w_k)} L_u^1 > (1-\varepsilon)x, \sigma_A < \hat{\tau}_1 \right) + \varepsilon x^{-\kappa}'.
\end{align*}
\]

In the next lemma, we will consider a certain quantity, namely \( \mathbb{E}[e^{\sigma_A}(w_{\sigma_A}))^{\kappa'} \mathbf{1}_{\sigma_A < \hat{\tau}_1}] \). Let us prove that it is finite; to do this, we just have to follow the lines of Lemma 4 of [16]. For convenience, we will actually study \( e(w_{\sigma_A}) = e^{\sigma_A}(w_{\sigma_A}) + 1 \). Let \( A > 0 \), we have on the event \( \{\sigma_A < \hat{\tau}_1\} \),

\[
\begin{align*}
e(w_{\sigma_A}) = (e(w_{\sigma_A-1}) + 1) \frac{e(w_{\sigma_A})}{e(w_{\sigma_A-1}) + 1} &\leq (A + 1) \frac{e(w_{\sigma_A})}{e(w_{\sigma_A-1}) + 1} \leq (A + 1) \sum_{k=1}^{\hat{\tau}_1-1} \frac{e(w_k)}{e(w_{k-1}) + 1}.
\end{align*}
\]

Thus, as \( \kappa' \leq 1 \), and then by Jensen’s inequality, we have

\[
\begin{align*}
\mathbb{E}[\left( e(w_{\sigma_A}) \right)^{\kappa'} \mathbf{1}_{\{\sigma_A < \hat{\tau}_1\}}] &\leq (A + 1)^{\kappa'} \sum_{k \geq 1} \mathbb{E}\left[ \mathbf{1}_{\{k < \hat{\tau}_1\}} \left( \frac{e(w_k)}{e(w_{k-1}) + 1} \right)^{\kappa'} \right] \\
&\leq (A + 1)^{\kappa'} \sum_{k \geq 1} \mathbb{E}\left[ \mathbf{1}_{\{k < \hat{\tau}_1\}} \left( \frac{\mathbb{E}[e(w_k)] (e(w_1))_{l \in [1:k-1]} \right)^{\kappa'} \right].
\end{align*}
\]
Now recall from Subsection 3.3.1 that for any \( k \geq 1 \) \( e(w_k) \) is a negative binomial random variable of parameters \( (e(w_{k-1}) + 1, 1/(1 + e^V(w_k) - e^V(w_{k-1})) \) plus one. Thus,

\[
\mathbb{E}[(e(w_{\sigma A}))^{\kappa'} 1_{\{\sigma A < \hat{\tau}_1\}}] \leq (A + 1)^{\kappa'} \sum_{k \geq 1} \mathbb{E}[1_{\{k < \hat{\tau}_1\}} (\frac{(e(w_{k-1}) + 1)e^{-V(w_k) - V(w_{k-1})} + 1)}{e(w_{k-1}) + 1})^{\kappa'}] \\
\leq (A + 1)^{\kappa'} \sum_{k \geq 1} \mathbb{E}[1_{\{k < \hat{\tau}_1\}} (e^{-\kappa'V(w_k) - V(w_{k-1})} + 1)]
\]

The event \( \{k < \hat{\tau}_1\} = \{k - 1 \leq \hat{\tau}_1\} \) being independent of the environment above \( w_{k-1} \) (so independent of \( e^{-V(w_k) - V(w_{k-1})} \)), we get

\[
\mathbb{E}[(e(w_{\sigma A}))^{\kappa'} 1_{\{\sigma A < \hat{\tau}_1\}}] \leq (A + 1)^{\kappa'} \sum_{k \geq 1} \mathbb{E}[(\mathbb{E}[e^{-\kappa'V(w_k) - V(w_{k-1})}] + 1) \\
= (A + 1)^{\kappa'} \mathbb{E}[\hat{\tau}_1 - 1] (\psi(\kappa) + 1) < \infty,
\]

where we used the branching property and the many-to-one lemma (Lemma 5) in the last equality. The expectation \( \mathbb{E}[\hat{\tau}_1 - 1] \) is finite thanks to (3.3).

**Lemma 10.** For all \( \varepsilon > 0 \), there exists an \( A_0 \) such that for any \( A > A_0 \) we have

\[
\mathbb{P}\left( \left( \sum_{k=\sigma A + 1}^{\hat{\tau}_1} \sum_{u \in \Omega(w_k)} e^{\sigma A}(u)W_{\infty}^u - e^{\sigma A}(w_{\sigma A})W_{\infty}^{w_{\sigma A}} \right) > \varepsilon x, \ \sigma A < \hat{\tau}_1 \right) \leq \varepsilon x^{-\kappa'} \mathbb{E}[(e^{\sigma A}(w_{\sigma A}))^{\kappa'} 1_{\{\sigma A < \hat{\tau}_1\}}].
\]

**Proof.** Actually, we want to show a more general result, which will be helpful in the proof of Lemma 12: let us consider a family of i.i.d. random variables indexed by the set of the brothers of the spine, that we denote by \( (Z_u)_{u \in \Omega} \) (where we have set \( \Omega := \bigcup_{k \geq 1} \Omega(w_k) \)). Suppose that this family is independent of the walk and the environment on the spine and the brothers of the spine \( (e(w_k), V(w_k))_{k \geq 0}, (e(u), V(u))_{u \in \Omega} \), and suppose that these random variables admit a finite moment of order 1. Let for any \( k \geq 0 \)

\[
(3.24) \quad Z_{w_k} := \sum_{l \geq k+1} e^{-(V(w_{l-1}) - V(w_k))} \sum_{u \in \Omega(w_l)} e^{-V(u) - V(w_{l-1})} Z_u.
\]

Notice that when \( (Z_u)_{u \in \Omega} = (W_{\infty}^u)_{u \in \Omega} \), then \( Z_{w_k} = W_{\infty}^{w_k} \). As the \( Z_u \) have a finite moment of order \( \kappa' \leq 1 \), so do the \( \sum_{u \in \Omega(w_l)} e^{-(V(u) - V(w_{l-1}))} Z_u \) for \( l \geq 1 \) (this can be seen using the many-to-one lemma, Hölder’s inequality and (H_m)), and then Theorem B of (15) ensures that there exists a constant \( C_Z \) such that for any \( k \geq 0 \),

\[
(3.25) \quad \mathbb{P}\left( Z_{w_k} > x \right) \sim C_Z x^{-\kappa'}.
\]

We want to prove that for all \( \varepsilon > 0 \), there exists an \( A_0 \) such that for any \( A > A_0 \):

\[
(3.26) \quad \mathbb{P}\left( \left( \sum_{k=\sigma A + 1}^{\hat{\tau}_1} \sum_{u \in \Omega(w_k)} e^{\sigma A}(u)Z_u - e^{\sigma A}(w_{\sigma A})Z_{w_{\sigma A}} \right) > \varepsilon x, \ \sigma A < \hat{\tau}_1 \right) \leq \varepsilon x^{-\kappa'} \mathbb{E}[(e^{\sigma A}(w_{\sigma A}))^{\kappa'} 1_{\{\sigma A < \hat{\tau}_1\}}].
\]

The \( (W_{\infty}^u)_{u \in \Omega} \) being i.i.d. and independent of the walk and the environment on the spine and the brothers of the spine, and admitting a finite moment of order 1, proving (3.26) is enough.
to prove the lemma.

As \( e^{\sigma A}(v) = 0 \) for vertices \( v \in \mathbb{T} \) such that \( w_{\hat{v}} \models v \), and by definition of \( Z_{w_{\sigma A}} \), we have

\[
\sum_{k=\sigma A+1}^{\hat{v}} \sum_{u \in \Omega(w_k)} e^{\sigma A}(u) Z_u - e^{\sigma A}(w_{\sigma A}) Z_{w_{\sigma A}} = \sum_{k \geq \sigma A+1} \sum_{u \in \Omega(w_k)} \left( e^{\sigma A}(u) - e^{\sigma A}(w_{w_{k-1}}) e^{-(V(u)-V(w_{k-1}))} \right) Z_u \\
+ \sum_{k \geq \sigma A+1} \left( e^{\sigma A}(w_{k-1}) - e^{\sigma A}(w_{\sigma A}) e^{-(V(w_{k-1})-V(w_{\sigma A}))} \right) \sum_{u \in \Omega(w_k)} e^{-(V(u)-V(w_{k-1}))} Z_u.
\]

Hence, by the union bound, denoting \( Z_{\Omega(w_k)} := \sum_{u \in \Omega(w_k)} e^{-(V(u)-V(w_{k-1}))} Z_u \) for any \( k \geq 1 \), we just have to prove that the two following quantities

\[
f_A(x) := \hat{P}\left( \left| \sum_{k \geq \sigma A} \left( e^{\sigma A}(w_k) - e^{\sigma A}(w_{\sigma A}) e^{-(V(w_k)-V(w_{\sigma A}))} \right) Z_{\Omega(w_{k+1})} \right| > \frac{\varepsilon}{2} x, \quad \sigma_A < \hat{\tau}_1 \right) \quad \text{and}
\]

\[
g_A(x) := \hat{P}\left( \left| \sum_{k \geq \sigma A+1} \sum_{u \in \Omega(w_k)} \left( e^{\sigma A}(u) - e^{\sigma A}(w_{w_{k-1}}) e^{-(V(u)-V(w_{\sigma A}))} \right) Z_u \right| > \frac{\varepsilon}{2} x, \quad \sigma_A < \hat{\tau}_1 \right)
\]

are smaller than \( \frac{\varepsilon}{2} x^{-\kappa'} \hat{E}\left[ \left( e^{\sigma A}(w_{\sigma A}) \right)^{\kappa'} \mathbf{1}_{\{\sigma_A < \hat{\tau}_1\}} \right] \) for \( A \) large enough to get (3.26). Let us start with \( f_A(x) \). Let us decompose \( e^{\sigma A}(w_k) - e^{\sigma A}(w_{\sigma A}) e^{-(V(w_k)-V(w_{\sigma A}))} \) for \( k \geq \sigma_A \):

\[
(e^{\sigma A}(w_k) - e^{\sigma A}(w_{\sigma A}) e^{-(V(w_k)-V(w_{\sigma A}))}) = \sum_{l=\sigma A+1}^{k} \left( e^{\sigma A}(w_l) e^{-(V(w_l)-V(w_{l-1}))} - e^{\sigma A}(w_{l-1}) e^{-(V(w_{l-1})-V(w_{l-1}))} \right)
\]

\[
= \sum_{l=\sigma A+1}^{k} e^{-(V(w_l)-V(w_{l-1}))} \left( e^{\sigma A}(w_l) - e^{\sigma A}(w_{l-1}) e^{-(V(w_l)-V(w_{l-1}))} \right).
\]

Hence, inverting the sums on \( k \) and \( l \) we get

\[
\sum_{k \geq \sigma A} \left( e^{\sigma A}(w_k) - e^{\sigma A}(w_{\sigma A}) e^{-(V(w_k)-V(w_{\sigma A}))} \right) Z_{\Omega(w_{k+1})}
\]

\[
= \sum_{l \geq \sigma A+1} \left( e^{\sigma A}(w_l) - e^{\sigma A}(w_{l-1}) e^{-(V(w_l)-V(w_{l-1}))} \right) \sum_{k \geq l} e^{-(V(w_k)-V(w_l))} Z_{\Omega(w_{k+1})}
\]

\[
= \sum_{l \geq \sigma A+1} \left( e^{\sigma A}(w_l) - e^{\sigma A}(w_{l-1}) e^{-(V(w_l)-V(w_{l-1}))} \right) Z_{w_l}.
\]

This yields (using also the fact that \( \sum_{l \geq 1} l^{-2} = \frac{\pi^2}{6} < 2 \),

\[
f_A(x) = \hat{P}\left( \left| \sum_{l \geq \sigma A+1} \left( e^{\sigma A}(w_l) - e^{\sigma A}(w_{l-1}) e^{-(V(w_l)-V(w_{l-1}))} \right) Z_{w_l} \right| > \sum_{l \geq \sigma A+1} (l-\sigma_A)^{-2} \pi^{-2} \varepsilon x, \sigma_A < \hat{\tau}_1 \right) \]

\[
\leq \hat{E}\left[ \sum_{l \geq \sigma A+1} \left( \sum_{k \geq l} \left( l-\sigma_A \right)^{-2} \frac{\varepsilon}{2} x \right) \hat{P}\left( Z_{w_l} > \frac{(l-\sigma_A)^{-2} \varepsilon x}{2 \left| e^{\sigma A}(w_l) - e^{\sigma A}(w_{l-1}) e^{-(V(w_l)-V(w_{l-1}))} \right|}, \sigma_A < \hat{\tau}_1 \right) \right] \]

\[
\leq \hat{E}\left[ \sum_{l \geq \sigma A+1} C_2(\varepsilon x)^{-\kappa'} (l-\sigma_A)^{2\kappa'} \left| e^{\sigma A}(w_l) - e^{\sigma A}(w_{l-1}) e^{-(V(w_l)-V(w_{l-1}))} \right| \mathbf{1}_{\{\sigma_A < \hat{\tau}_1\}} \right],
\]

(3.27)
where the last inequality comes from equation (3.25), and where \( C_Z' \geq C_Z \) is a suitable constant. Now, as each \( e^{\sigma_A}(w_l) \) is a negative binomial random variable of parameter \( (e(w_{l-1}), 1/(1 + e(V(w_l)-V(w_{l-1})))) \), we have (using Jensen’s inequality first)

\[
\mathbb{E}\left[ \left| e^{\sigma_A}(w_l) - e^{\sigma_A}(w_{l-1})e^{-((V(w_l)-V(w_{l-1}))}\right|^{\kappa'}\right] \leq \mathbb{E}\left[ \left( e^{\sigma_A}(w_l) - e^{\sigma_A}(w_{l-1})e^{-((V(w_l)-V(w_{l-1}))}\right)^2\right]^{\kappa'/2} \\
= \mathbb{E}\left[ e^{\sigma_A}(w_{l-1})(e^{-2(V(w_l)-V(w_{l-1}))} + e^{-((V(w_l)-V(w_{l-1}))})\right] \left( e^{\sigma_A}(w_{l-1})\right)^{\kappa'/2} \\
\leq \mathbb{E}\left[ \left( e^{\sigma_A}(w_{l-1})\right)^{\kappa'/2} \right] \left( e^{-\kappa'(V(w_l)-V(w_{l-1}))} + e^{-\kappa'(V(w_l)-V(w_{l-1}))}\right)
\]

(we recall that \( \kappa' \leq 1 \)). The last equality is obtained by induction, using the fact that for \( l \geq 1 \), \( \mathbb{E}[e^{\sigma_A}(w_l)|e^{\sigma_A}(w_{l-1})] = e^{\sigma_A}(w_{l-1})e^{-((V(w_l)-V(w_{l-1}))} \). Plugging this into (3.27) yields

\[
f_A(x) \leq C_Z' 2^{\kappa'}(\varepsilon x) - e^{-\kappa'} \mathbb{E}\left[ (e^{\sigma_A}(w_{\sigma_A}))^{\kappa'/2} 1_{\{\sigma_A < \kappa^1\}} \right] \\
\times \mathbb{E}\left[ \sum_{l \geq 2^{\kappa'/2}} e^{-\kappa'(V(w_l)-V(w_{l-1}))}(e^{-\kappa'(V(w_l)-V(w_{l-1}))} + e^{-\kappa'(V(w_l)-V(w_{l-1}))})(l - \sigma_A)^{2\kappa'} \right]
\]

Now notice that by the branching property and then the many-to-one lemma (Lemma 3), the last expectation is equal to

\[
\mathbb{E}\left[ \sum_{l \geq 2^{\kappa'/2}} e^{-\kappa'(V(w_l)-V(w_{l-1}))} \mathbb{E}\left[ e^{-\kappa'(V(w_l)-V(w_{l-1}))} \right] \right] \leq \sum_{l \geq 2^{\kappa'/2}} \mathbb{E}\left[ e^{-\kappa'(V(w_l)-V(w_{l-1}))} \right] \mathbb{E}\left[ \sum_{l \geq 1} (e^{\sigma_A}(w_{l-1})e^{-((V(w_l)-V(w_{l-1}))}) \right] \\
\leq \mathbb{E}\left[ \sum_{l \geq 2^{\kappa'/2}} \mathbb{E}\left[ e^{-\kappa'(V(w_l)-V(w_{l-1}))} \right] \right] \mathbb{E}\left[ \sum_{l \geq 1} (e^{\sigma_A}(w_{l-1})e^{-((V(w_l)-V(w_{l-1}))}) \right] \\
= \sum_{l \geq 2^{\kappa'/2}} \mathbb{E}\left[ e^{-\kappa'(V(w_l)-V(w_{l-1}))} \right] \mathbb{E}\left[ \sum_{l \geq 1} (e^{\sigma_A}(w_{l-1})e^{-((V(w_l)-V(w_{l-1}))}) \right] \\
\leq C'
\]

which is finite, as \( \psi(1 + \kappa') < 1 \) (because \( 1 + \kappa' \in (1; \kappa) \)). Moreover, as \( e^{\sigma_A}(w_{\sigma_A}) > A \), (3.29) finally yields

\[
f_A(x) \leq C_Z'(\varepsilon x) - e^{-\kappa'} C' A - \frac{\varepsilon}{2}\mathbb{E}\left[ (e^{\sigma_A}(w_{\sigma_A}))^{\kappa'} 1_{\{\sigma_A < \kappa^1\}} \right] \leq \frac{\varepsilon}{2} - e^{-\kappa'} \mathbb{E}\left[ (e^{\sigma_A}(w_{\sigma_A}))^{\kappa'} 1_{\{\sigma_A < \kappa^1\}} \right]
\]

for \( A \) large enough, which is what we wanted on \( f_A(x) \). Let us now deal with \( g_A(x) \). Markov’s inequality together with the fact that \( \kappa' \leq 1 \) yields

\[
g_A(x) \leq \left( \frac{\varepsilon}{2} x \right)^{-\kappa'} \mathbb{E}\left[ \sum_{k \geq \sigma_A + 1} \sum_{u \in \Omega(w_{k-1})} (e^{\sigma_A}(u) - e^{\sigma_A}(w_{k-1})e^{-((V(u)-V(w_{k-1}))}) Z_u \right]^{\kappa'} 1_{\{\sigma_A < \kappa^1\}} \]

As for any \( k > \sigma_A \), \( (e^{\sigma_A}(u))_{u \in \Omega(w_k)} \) is a negative multinomial random variable of parameters \( e^{\sigma_A}(w_{k-1}) \) and \( e^{-((V(u)-V(w_{k-1}))} + \sum_{v \in \Omega(w_{k-1})} e^{-((V(v)-V(w_{k-1}))}) \), we have (after applying Jensen’s inequality)

\[
\mathbb{E}\left[ \left| \sum_{u \in \Omega(w_k)} (e^{\sigma_A}(u) - e^{\sigma_A}(w_{k-1})e^{-((V(u)-V(w_{k-1}))}) Z_u \right|^{\kappa'} \right] \leq \mathbb{E}\left[ \left| \sum_{u \in \Omega(w_k)} (e^{\sigma_A}(u) - e^{\sigma_A}(w_{k-1})e^{-((V(u)-V(w_{k-1}))}) Z_u \right|^{2\kappa'/2} \right] \\
= \mathbb{E}\left[ \left| \sum_{u \in \Omega(w_k)} e^{\sigma_A}(w_{k-1}) \left( \sum_{u \in \Omega(w_k)} e^{-((V(u)-V(w_{k-1}))}) Z_u + \left( \sum_{u \in \Omega(w_k)} e^{-((V(u)-V(w_{k-1}))}) Z_u \right)^2 \right) \right|^{\kappa'/2} \right]
\]

35
As for $l \geq 1$, $\hat{E}^V[e^{\sigma_A}(w_l) \mid e^{\sigma_A}(w_{l-1})] = e^{\sigma_A}(w_{l-1})e^{-(V(w_l) - V(w_{l-1}))}$, we have by induction that $\hat{E}^V[e^{\sigma_A}(w_{k+1})] = e^{\sigma_A}(w_{k})e^{-(V(w_{k+1}) - V(w_{k}))}$. So the previous equation yields

$$\hat{E}\left[\sum_{u \in \Omega(w_k)} (e^{\sigma_A}(u) - e^{\sigma_A}(w_{k-1})e^{-(V(u) - V(w_{k-1}))})Z_u^{\kappa'}e^{\sigma_A}(w_{\sigma_A}), \mathcal{G}\right]$$

$$\leq (e^{\sigma_A}(w_{\sigma_A}))^{\kappa'/2}e^{-\frac{\kappa'}{\tau'}(V(w_{\kappa}) - V(w_{\sigma_A}))}\left(\sum_{u \in \Omega(w_k)}e^{-(V(u) - V(w_{k-1}))}Z_u^{\kappa'} + \sum_{u \in \Omega(w_k)}e^{-(V(u) - V(w_{k-1}))}\hat{\tau}^{\kappa'}\right)$$

$$\leq (e^{\sigma_A}(w_{\sigma_A}))^{\kappa'/2}e^{-\frac{\kappa'}{\tau'}(V(w_{\kappa}) - V(w_{\sigma_A}))}\left(\hat{E}[Z]^{\kappa'} + \hat{E}[Z]^{\hat{\tau}^{\kappa'}}\right)$$

$$\times \left(\sum_{u \in \Omega(w_k)}e^{-(V(u) - V(w_{k-1}))}Z_u^{\kappa'} + \sum_{u \in \Omega(w_k)}e^{-(V(u) - V(w_{k-1}))}\hat{\tau}^{\kappa'}\right),$$

where we used Jensen’s inequality in the last equation, and where we generically denoted by $Z$ a random variable of same law as that of the $(Z_u)_{u \in \Omega}$. Hence, plugging this into (3.31), we get

$$g_A(x) \leq \left(\frac{\kappa'}{\tau'}\right)^{\kappa'}\left(\hat{E}[Z]^{\kappa'} + \hat{E}[Z]^{\hat{\tau}^{\kappa'}}\right)\hat{E}\left[(e^{\sigma_A}(w_{\sigma_A}))^{\kappa'/2}1_{\{\sigma_A < \tau\}}\right]$$

(3.33)

$$\times \hat{E}\left[\sum_{k \geq \sigma_A+1}e^{-\frac{\kappa'}{\tau'}(V(w_{\kappa}) - V(w_{\sigma_A}))}\left(\sum_{u \in \Omega(w_k)}e^{-(V(u) - V(w_{k-1}))}Z_u^{\kappa'} + \sum_{u \in \Omega(w_k)}e^{-(V(u) - V(w_{k-1}))}\hat{\tau}^{\kappa'}\right)\right],$$

The branching property and then the many-to-one lemma (Lemma 5) yield that this last expectation is equal to

$$\hat{E}\left[\sum_{k \geq 1}e^{-\frac{\kappa'}{\tau'}V(w_{\kappa})}\hat{E}\left[\left(\sum_{u \in \Omega(w_1)}e^{-V(u)}\right)^{\kappa'} + \left(\sum_{u \in \Omega(w_1)}e^{-V(u)}\hat{\tau}^{\kappa'}\right)\right]\right]$$

$$\leq \sum_{k \geq 1}\hat{E}\left[e^{-\frac{\kappa'}{\tau'}V(w_{\kappa})}\hat{E}\left[\left(\sum_{|u| = 1}e^{-V(u)}\right)^{\kappa'} + \left(\sum_{|u| = 1}e^{-V(u)}\hat{\tau}^{\kappa'}\right)\right]\right]$$

$$= \sum_{k \geq 1}\hat{E}\left[\sum_{|u| = 1}e^{-\left(1 + \frac{\kappa'}{\tau'}\right)V(u)}\right]\hat{E}\left[\left(\sum_{|u| = 1}e^{-V(u)}\right)^{\kappa'} + \left(\sum_{|u| = 1}e^{-V(u)}\hat{\tau}^{\kappa'}\right)\right]$$

$$= \sum_{k \geq 1}\psi(1 + \frac{\kappa'}{\tau'})\left(\hat{E}\left[\left(\sum_{|u| = 1}e^{-V(u)}\right)^{\kappa'}\right] + \hat{E}\left[\left(\sum_{|u| = 1}e^{-V(u)}\right)^{1 + \frac{\kappa'}{\tau'}}\right]\right) =: C''$$

(3.34)

which is finite according to (H$_{\kappa}$) and the fact that $\psi(1 + \frac{\kappa'}{\tau'}) < 1$ (as $1 + \frac{\kappa'}{\tau'} \in (1; \kappa)$). Plugging this into (3.33) we get

$$g_A(x) \leq C''\left(\frac{\kappa'}{\tau'}\right)^{\kappa'}\left(\hat{E}[Z]^{\kappa'} + \hat{E}[Z]^{\hat{\tau}^{\kappa'}}\right)\hat{E}\left[(e^{\sigma_A}(w_{\sigma_A}))^{\kappa'/2}1_{\{\sigma_A < \tau\}}\right]$$

$$\leq C''\left(\frac{\kappa'}{\tau'}\right)^{\kappa'}\left(\hat{E}[Z]^{\kappa'} + \hat{E}[Z]^{\hat{\tau}^{\kappa'}}\right)A^{-\frac{\kappa'}{\tau'}}\hat{E}\left[(e^{\sigma_A}(w_{\sigma_A}))^{\kappa'}1_{\{\sigma_A < \tau\}}\right],$$

which can be made smaller than $\frac{\kappa'}{\tau'}\hat{E}\left[(e^{\sigma_A}(w_{\sigma_A}))^{\kappa'}\right]1_{\{\sigma_A < \tau\}}$ for $A$ large enough, which is what we wanted on $g_A(x)$, thus concluding the proof. \[\square\]
Lemma 11. For all $\varepsilon > 0$, there exists an $A_0$ such that for any $A > A_0$ we have

$$
\hat{P}\left( \sum_{k=\sigma_A+1}^{\hat{\tau}_1} \sum_{u \in \Omega(w_k)} (e(u) - e^{\sigma_A}(u))W^u_\infty > \varepsilon x, \quad \sigma_A < \hat{\tau}_1 \right) \leq \varepsilon x^{-\kappa'} \hat{E}\left[ (e^{\sigma_A}(w_{\sigma_A}))^{\kappa'} 1_{\{\sigma_A < \hat{\tau}_1\}} \right].
$$

Proof. Just as in the proof of the previous lemma (Lemma 10), we intend to prove a little more than required. Denoting by $(Z_u)_{u \in \Omega}$ (with $\Omega := \bigcup_{k \geq 1} \Omega(w_k)$) a family of random variables independent of the walk and the environment on the spine and the brothers of the spine $((e(w_k), V(w_k)))_{k \geq 0}, (e(u), V(u))_{u \in \Omega}$ with a finite moment of order 1, we want to prove that for all $\varepsilon > 0$, there exists an $A_0$ such that for any $A > A_0$:

(3.35)

$$
h_A(x) := \hat{P}\left( \sum_{k=\sigma_A+1}^{\hat{\tau}_1} \sum_{u \in \Omega(w_k)} (e(u) - e^{\sigma_A}(u))Z_u > \varepsilon x, \quad \sigma_A < \hat{\tau}_1 \right) \leq \varepsilon x^{-\kappa'} \hat{E}\left[ (e^{\sigma_A}(w_{\sigma_A}))^{\kappa'} 1_{\{\sigma_A < \hat{\tau}_1\}} \right].
$$

As explained in the proof of Lemma 10, the properties of $(W^u_\infty)_{u \in \Omega}$ are such that proving (3.35) is enough to prove the lemma. So let $\varepsilon > 0$. Recall the definition of $e^1_i$ and $e^2_i$ from Subsection 3.3.1. Notice that on $\{\sigma_A < \hat{\tau}_1\}$, if $k \in [\sigma_A + 1; \hat{\tau}_1]$, then for any $u \in \Omega(w_k)$

$$
(e(u) - e^{\sigma_A}(u)) = \sum_{i=\sigma_A}^{\hat{\tau}_1-1} (e^1_i(u) + e^2_i(u)).
$$

This yields

$$
\sum_{k=\sigma_A+1}^{\hat{\tau}_1} \sum_{u \in \Omega(w_k)} (e(u) - e^{\sigma_A}(u))Z_u = \sum_{i=\sigma_A}^{\hat{\tau}_1-1} \left( \sum_{k=\sigma_A+1}^{\hat{\tau}_1} \sum_{u \in \Omega(w_k)} (e^1_i(u) + e^2_i(u))Z_u \right)
$$

$$
= \sum_{i=\sigma_A}^{\hat{\tau}_1-1} \left( \sum_{k=\sigma_A+1}^{\hat{\tau}_1} \sum_{u \in \Omega(w_k)} (e^1_i(u) + e^2_i(u))Z_u \right),
$$

where the last equality comes from the fact that $e^1_i(w_{k-1}) - e^1_i(w_{k-1}) = 0$ for $k \leq i$, as the walks launched on $w_i$ are killed when they reach $w_{i-1}$, and $e^2_i(w_{k-1}) - e^2_i(w_{k-1}) = 0$ for $k \geq \hat{\tau}_1$, as $\hat{\tau}_1$ is the first level such that none of the walk launched below reach it. Now as $\sum_{i=1}^{\infty} i^{-2} = \pi^2/6 < 2$, the union bound yields

$$
h_A(x) = \hat{P}\left( \sum_{i=\sigma_A}^{\hat{\tau}_1-1} \left( \sum_{k=\sigma_A+1}^{\hat{\tau}_1} \sum_{u \in \Omega(w_k)} (e^1_i(u) + e^2_i(u))Z_u \right) > \varepsilon x 6\pi^{-2} \sum_{i=\sigma_A}^{\infty} (i - \sigma_A + 1)^{-2} \right)
$$

$$
\leq \sum_{i=1}^{\infty} \hat{P}\left( 1_{\{\sigma_A \leq i < \hat{\tau}_1\}} \sum_{k=\sigma_A+1}^{\hat{\tau}_1} \sum_{u \in \Omega(w_k)} (e^1_i(u)Z_u > \frac{\varepsilon}{4} x (i - \sigma_A + 1)^{-2} \right)
$$

$$
+ \hat{P}\left( 1_{\{\sigma_A \leq i < \hat{\tau}_1\}} \sum_{k=\sigma_A+1}^{\hat{\tau}_1} \sum_{u \in \Omega(w_k)} (e^2_i(u)Z_u > \frac{\varepsilon}{4} x (i - \sigma_A + 1)^{-2} \right)
$$

But $e^1_i$, $e^2_i$ and $(Z_u)_{u \in \Omega(w_k)}$ (for $k > \sigma_A$) only depend on the walks launched above $w_i$ and on the environment above $w_i$, when the event $\{\sigma_A \leq i < \hat{\tau}_1\}$ only depends on the walks launched strictly below $w_i$ and on the environment of the spine below $w_i$. Thus $e^1_i$ and $e^2_i$ and the $(Z_u)_{u \in \Omega(w_k)}$ are independent of the event $\{\sigma_A \leq i < \hat{\tau}_1\}$, and since all the $\sum_{k=\sigma_A+1}^{\hat{\tau}_1} \sum_{u \in \Omega(w_k)} e^i_i(u)Z_u$
and \( \sum_{k \geq 1} \sum_{u \in \Omega(w_k)} e_i^j(u) Z_u \) have the distribution of \( \sum_{k \geq 1} \sum_{u \in \Omega(w_k)} e_i^0(u) Z_u \) we get (with the change of variable \( j = i - \sigma_A \)) that

\[
(3.36) \quad h_A(x) \leq \sum_{j \geq 1} 2\hat{P}(1 \leq j < \hat{\tau}_1 - \sigma_A) \hat{P}\left( \sum_{k \geq 1} \sum_{u \in \Omega(w_k)} e_i^0(u) Z_u > \frac{\varepsilon}{4} x j^{-2} \right).
\]

Notice that now, it would be enough to conclude that there exists a constant \( C \) such that

\[
(3.37) \quad \hat{P}\left( \sum_{k \geq 1} \sum_{u \in \Omega(w_k)} e_i^0(u) Z_u > x \right) \leq C x^{-\kappa'}.
\]

Indeed, if this inequality was satisfied, equation (3.36) would yield

\[
\begin{align*}
    h_A(x) & \leq C\left( \frac{4}{\varepsilon} \right)^{\kappa'} x^{-\kappa'} \sum_{j=1}^{\infty} 2\hat{P}(1 \leq j < \hat{\tau}_1 - \sigma_A) j^{2\kappa'} \\
    & = C\left( \frac{4}{\varepsilon} \right)^{\kappa'} x^{-\kappa'} E[(\hat{\tau}_1 - \sigma_A)^{2\kappa'+1}] \mathbf{1}_{\{\hat{\tau}_1 \geq \sigma_A\}} \\
    & = C\left( \frac{4}{\varepsilon} \right)^{\kappa'} x^{-\kappa'} E[\hat{E}_{w_{\sigma_A}}[\left( \hat{\tau}_1 \right)^{2\kappa'+1}] \mathbf{1}_{\{\hat{\tau}_1 \geq \sigma_A\}}] \\
    & \leq C\left( \frac{4}{\varepsilon} \right)^{\kappa'} x^{-\kappa'} C_A \ln^{1+2\kappa'}(1 + e(w_{\sigma_A})) \mathbf{1}_{\{\hat{\tau}_1 \geq \sigma_A\}} \\
    & \leq CC_1 \ln^{1+2\kappa'}(1 + A) \left( \frac{4}{\varepsilon} \right)^{\kappa'} x^{-\kappa'} E[(e(w_{\sigma_A}))^{\kappa'}] \mathbf{1}_{\{\hat{\tau}_1 \geq \sigma_A\}},
\end{align*}
\]

where we used the branching property in the second equality, and equation (3.3) in the one but one inequality. Now using the fact that \( e^{\sigma_A}(w_{\sigma_A}) = e(w_{\sigma_A}) + 1 \geq \frac{1}{2} e(w_{\sigma_A}) \), this last quantity could be made smaller than \( \varepsilon x^{-\kappa'} \hat{E}[\left( e^{\sigma_A}(w_{\sigma_A}) \right)^{\kappa'}] \mathbf{1}_{\{\hat{\tau}_1 \geq \sigma_A\}} \) for \( A \) large enough, thus proving the lemma. So let us prove that (3.37) stands. We have

\[
\hat{P}\left( \sum_{k \geq 1} \sum_{u \in \Omega(w_k)} e_i^0(u) Z_u > x \right) \leq \hat{P}\left( \sum_{k \geq 1} \sum_{u \in \Omega(w_k)} |e_i^0(u) - e_i^0(w_{k-1})| e^{-V(u) - V(w_{k-1})} |Z_u > x/2\right) + \hat{P}\left( \sum_{k \geq 1} e_i^0(w_{k-1}) \sum_{u \in \Omega(w_k)} e^{-V(u) - V(w_{k-1})}| Z_u > x/2 \right).
\]

Let us denote by \( f(x) \) and \( g(x) \) these last two quantities, and show that they are smaller than \( C x^{-\kappa'} \) for a certain \( C \). First, Markov’s inequality together with the fact that \( \kappa' \leq 1 \) yields

\[
(3.38) \quad f(x) \leq 2^{\kappa'} x^{-\kappa'} \hat{E}\left[ \sum_{k \geq 1} \sum_{u \in \Omega(w_k)} |e_i^0(u) - e_i^0(w_{k-1})| e^{-V(u) - V(w_{k-1})}| Z_u \right]^{\kappa'}.
\]

Using the fact that for \( k \geq 1 \), \( \sum_{u \in \Omega(w_k)} e_i^0(u) \sim NB(e_i^0(w_{k-1}), 1/(1 + \sum_{u \in \Omega(w_k)} e^{V(u) - V(w_{k-1})})) \), the same arguments as those used in (3.32) yield

\[
\begin{align*}
    & \hat{E}\left[ \sum_{u \in \Omega(w_k)} |e_i^0(u) - e_i^0(w_{k-1})| e^{-V(u) - V(w_{k-1})}| Z_u \right]^{\kappa'} \\
    \leq & \hat{E}[Z]^{\kappa'} \hat{E}\left[ e^{-\frac{1}{2} V(w_{k-1})} \left( \sum_{u \in \Omega(w_k)} e^{-V(u) - V(w_{k-1})} \right)^{\frac{\kappa'}{2}} + \left( \sum_{u \in \Omega(w_k)} e^{-V(u) - V(w_{k-1})} \right)^{\kappa'} \right],
\end{align*}
\]

38
where we denoted by \( Z \) a generic random variable of same law as that of the \((Z_u)_{u \in \Omega}\). Plugging this into (3.38) yields, when using the branching property and the same reasoning as in (3.34),

\[
f(x) \leq 2^{x'} \mathbb{E}[Z]^{x' - \kappa'} \times C',
\]
which yields that \( f(x) < Cx^{-\kappa'} \) for a certain \( C > 0 \) indeed. Now, let us deal with \( g(x) \). Recall the definition of \((Z_{w_k})_{k \geq 0}\) from (3.24). Now, for any \( k \geq 0 \), noticing that \( Z_{w_k} = (\sum_{u \in \Omega(w_{k+1})} e^{-{(V(u) - V(w_k))}^2} Z_u + e^{-{(V(w_{k+1}) - V(w_k))}^2} Z_{w_{k+1}}\) we get

\[
\sum_{k \geq 1} e_0^1(w_{k-1}) \sum_{u \in \Omega(w_k)} e^{-{(V(u) - V(w_k))}^2} Z_u = \sum_{k \geq 1} e_0^1(w_k) (Z_{w_k} - e^{-{(V(w_{k+1}) - V(w_k))}^2} Z_{w_{k+1}}) \\
= \sum_{k \geq 1} (e_0^1(w_k) - e_0^1(w_{k-1}) e^{-{(V(w_k) - V(w_{k-1}))}^2} Z_{w_k}),
\]

(we recall that \( e_0^1(w_0) = 0 \)). This yields by the union bound (and using the fact that \( \sum_{k \geq 1} k^{-2} = \pi^2/6 < 2 \))

\[
g(x) = \hat{P} \left( \sum_{k \geq 1} (e_0^1(w_k) - e_0^1(w_{k-1}) e^{-{(V(w_k) - V(w_{k-1}))}^2}) Z_{w_k} > \frac{x}{6} \pi^{-2} \sum_{k \geq 1} k^{-2} \right) \\
\leq \sum_{k \geq 1} \hat{P} \left( (e_0^1(w_k) - e_0^1(w_{k-1}) e^{-{(V(w_k) - V(w_{k-1}))}^2}) Z_{w_k} > \frac{1}{2} k^{-2} \right) \\
= \sum_{k \geq 1} \hat{E} \left[ \frac{1}{2} e_0^1(w_k) - e_0^1(w_{k-1}) e^{-{(V(w_k) - V(w_{k-1}))}^2} \right] \\
\leq C Z 2^{x' - \kappa'} k^{2\kappa'} \mathbb{E} \left[ e_0^1(w_k) - e_0^1(w_{k-1}) e^{-{(V(w_k) - V(w_{k-1}))}^2} \right]^{\kappa'},
\]

the last inequality being obtained thanks to (3.25). Using the same arguments as in (3.28) and then (3.30), we get

\[
g(x) \leq C Z 2^{x' - \kappa'} k^{2\kappa'} \mathbb{E} \left[ e^{-\frac{x}{2} V(w_k)} (e^{-\frac{x}{2} (V(w_k) - V(w_{k-1}))} + e^{-\kappa' (V(w_k) - V(w_{k-1}))}) \right] \\
\leq C Z 2^{x' - \kappa'} k^{2\kappa'} \psi(1 + \frac{\kappa'}{2}) \left( \psi(1 + \frac{\kappa'}{2}) + \psi(\kappa) \right) \leq C Z 2^{x' - \kappa'} \times C',
\]

concluding on \( g(x) \) and then the proof.

\[\square\]

**Lemma 12.** For all \( \varepsilon > 0 \), there exists an \( A_0 \) such that for any \( A > A_0 \) we have for \( x \) large enough

\[
\hat{P} \left( \sum_{k=\sigma_A+1}^{\sigma_A + \varepsilon} \sum_{u \in \Omega(w_k)} e(u) W_u - \sum_{k=\sigma_A+1}^{\sigma_A + \varepsilon} \sum_{u \in \Omega(w_k)} L_u \right) > \varepsilon x, \quad \sigma_A < \tilde{\tau}_1 \leq \varepsilon x^{-\kappa'} \hat{E} \left[(e^{\sigma_A (w_{\sigma_A})})^{\kappa'} 1_{\{\sigma_A < \tilde{\tau}_1\}} \right].
\]

39
Proof. First, the union bound yields

\[
\hat{\mathbb{P}}\left( \sum_{k=\sigma_A+1}^{\tilde{\tau}_1} \sum_{u \in \Omega(w_k)} e(u) W^u_\infty - \sum_{k=\sigma_A+1}^{\tilde{\tau}_1} \sum_{u \in \Omega(w_k)} L^1_u > \varepsilon x, \sigma_A < \hat{\tau}_1 \right) \\
\leq \hat{\mathbb{P}}\left( \sum_{k=\sigma_A+1}^{\tilde{\tau}_1} \sum_{u \in \Omega(w_k)} \mathrm{e}^{\sigma_A}(u) \left| W^u_\infty - \frac{L^1_u}{e(u)} \right| > \varepsilon x, \sigma_A < \hat{\tau}_1 \right) \\
+ \hat{\mathbb{P}}\left( \sum_{k=\sigma_A+1}^{\tilde{\tau}_1} \sum_{u \in \Omega(w_k)} \left( e(u) - \mathrm{e}^{\sigma_A}(u) \right) \left| W^u_\infty - \frac{L^1_u}{e(u)} \right| > \varepsilon x, \sigma_A < \hat{\tau}_1 \right).
\]

(3.39)

Now according to Proposition $\Box$ we can apply Lemma 17 (in the appendix) to the $|W^u_\infty - \frac{L^1_u}{e(u)}|$ for $u \in \Omega$: there exists a positive random variable $Y$ with a finite moment of order $1 + \alpha$ (with $\alpha \in (0; \kappa')$) and a decreasing sequence $(a_n)_{n \geq 1}$ which tends to zero (with $a_0 \leq 1$), such that for any $n$, $a_n Y$ is stochastically greater than $|W^u_\infty - \frac{L^1_u}{e(u)}|$ under $\mathbb{P}_k$ for any $k \geq n$. Let us set $(Y_u)_{u \in \Omega}$ a sequence of independent random variables of same law as $Y$, independent of the environment of the spine and the walk on the spine. This yields (using also the fact that the $(a_n)_{n \geq 1}$ are smaller than $1$) that the second quantity of (3.39) is smaller than

\[
\hat{\mathbb{P}}\left( \sum_{k=\sigma_A+1}^{\tilde{\tau}_1} \sum_{u \in \Omega(w_k)} \left( e(u) - \mathrm{e}^{\sigma_A}(u) \right) Y_u > \varepsilon x, \sigma_A < \hat{\tau}_1 \right),
\]

which according to (3.35) can be made smaller than $\varepsilon x^{-\kappa'} \hat{\mathbb{E}}[\left( \mathrm{e}^{\sigma_A}(w_{\sigma_A}) \right)^{\kappa'} 1_{\{\sigma_A < \hat{\tau}_1\}}]$ for $A$ large enough. Therefore, we just have to bound the first quantity of (3.39) to show the lemma. For the reasons explained above, we can write that

\[
\hat{\mathbb{P}}\left( \sum_{k=\sigma_A+1}^{\tilde{\tau}_1} \sum_{u \in \Omega(w_k)} \mathrm{e}^{\sigma_A}(u) \left| W^u_\infty - \frac{L^1_u}{e(u)} \right| > \varepsilon x, \sigma_A < \hat{\tau}_1 \right) \\
\leq \hat{\mathbb{P}}\left( \sum_{k=\sigma_A+1}^{\tilde{\tau}_1} \sum_{u \in \Omega(w_k)} Y_a \mathrm{e}(u) Y_u > \varepsilon x, \sigma_A < \hat{\tau}_1 \right). \\
\]

We will deal with this last quantity until the end of the proof, as it will be more convenient to bound as required. Let us introduce for any $k \geq 1$, $p \geq 1$,

\[
\hat{\mathbb{E}}_{\Omega(w_k)} := \{ u \in \Omega(w_k), \# \{ v \in T : e^{-V(v)} \geq e^{-V(u)} \} < p \},
\]

the set of the $p$ vertices of $\Omega(w_k)$ with lowest potential. Let $N \geq 1$, $p \geq 1$; discussing for each $u \in \Omega(w_k)$ ($k \geq \sigma_A + 1$) on whether $u \in \hat{\mathbb{E}}_{\Omega(w_k)}^p$ or not, and then on whether $e(u) > N$ or not, we get by the union bound (and the fact that $(a_n)_{n \geq 1} \leq 1$, and $e^{\sigma_A} \leq e$)

\[
\hat{\mathbb{P}}\left( \sum_{k=\sigma_A+1}^{\tilde{\tau}_1} \sum_{u \in \Omega(w_k)} \mathrm{e}^{\sigma_A}(u) a_{e(u)} Y_u > \varepsilon x, \sigma_A < \hat{\tau}_1 \right) \\
\leq \hat{\mathbb{P}}\left( \sum_{k=\sigma_A+1}^{\tilde{\tau}_1} \sum_{u \in \Omega(w_k) \setminus \hat{\mathbb{E}}_{\Omega(w_k)}^p} \mathrm{e}^{\sigma_A}(u) a_N Y_u > \frac{\varepsilon}{3} x, \sigma_A < \hat{\tau}_1 \right) + \hat{\mathbb{P}}\left( \sum_{k=\sigma_A+1}^{\tilde{\tau}_1} \sum_{u \in \hat{\mathbb{E}}_{\Omega(w_k)}^p} \mathrm{e}^{\sigma_A}(u) a_{e(u)} Y_u > \varepsilon x, \sigma_A < \hat{\tau}_1 \right) \\
\leq \hat{\mathbb{P}}\left( \sum_{k=\sigma_A+1}^{\tilde{\tau}_1} \sum_{u \in \Omega(w_k) \setminus \hat{\mathbb{E}}_{\Omega(w_k)}^p} \mathrm{e}^{\sigma_A}(u) a_N Y_u > \frac{\varepsilon}{3} x, \sigma_A < \hat{\tau}_1 \right) + \hat{\mathbb{P}}\left( \sum_{k=\sigma_A+1}^{\tilde{\tau}_1} \sum_{u \in \hat{\mathbb{E}}_{\Omega(w_k)}^p} N Y_u > \frac{\varepsilon}{3} x, \sigma_A < \hat{\tau}_1 \right) \\
= : f_{N,p}(x) + g_{N,p}(x)
\]

40
Therefore, to prove the lemma it suffices to show that there exist $p$ and $N$ such that $f_{N,p}(x)$, $g_{N,p}(x)$ and $h_p(x)$ can each be made smaller than $\varepsilon x^{-k'} \hat{E}[(e^{\sigma A}(w_{\sigma A}))' 1_{\sigma_A < \hat{\tau}}]$ for $A$ and $x$ large enough.

- Let us start with $h_p(x)$. We have that

$$h_p(x) \leq \hat{P}\left( \left| \sum_{k=\sigma_A+1}^{\hat{\tau}_1} \sum_{u \notin E^p_{\Omega(w_k)}} (e^{\sigma A}(u) - e^{\sigma A}(w_{\sigma A})) e^{-(V(u)-V(w_{\sigma A}))} Y_u \right| > \frac{\varepsilon}{6} x, \sigma_A < \hat{\tau}_1 \right)$$

$$+ \hat{P}\left( e^{\sigma A}(w_{\sigma A}) \sum_{k=\sigma_A+1}^{\hat{\tau}_1} e^{-(V(w_{k-1})-V(w_{\sigma A}))} \sum_{u \notin \Omega(w_k)} e^{-(V(u)-V(w_{k-1}))} Y_u > \frac{\varepsilon}{6} x, \sigma_A < \hat{\tau}_1 \right)$$

The first quantity of this last sum can be bounded by

$$\hat{P}\left( \left| \sum_{k=\sigma_A+1}^{\hat{\tau}_1} \sum_{u \in \Omega(w_k)} (e^{\sigma A}(u) - e^{\sigma A}(w_{\sigma A})) e^{-(V(u)-V(w_{\sigma A}))} Y_u \right| > \frac{\varepsilon}{6} x, \sigma_A < \hat{\tau}_1 \right),$$

which according to (3.26) can be bounded by $\frac{1}{6} x^{-k'} \hat{E}[(e^{\sigma A}(w_{\sigma A}))' 1_{\sigma_A < \hat{\tau}_1}]$ for $A$ large enough (the $(Y_u)_{u \in \Omega}$ satisfying the same conditions as the $(Z_u)_{u \in \Omega}$). For the second quantity of this sum, notice that as for any $k \geq 1$

$$\hat{E}\left[ ( \sum_{u \in \Omega(w_k)} e^{-(V(u)-V(w_{k-1}))} Y_u )^{k'} \right] \leq \hat{E}[Y_u]^{k'} \hat{E}\left[ ( \sum_{u = w_k} e^{-(V(u)-V(w_{k-1}))} )^{k'} \right] = \hat{E}[Y_u]^{k'} \hat{E}\left[ ( \sum_{|u|=1} e^{-V(u)} )^{k'} \right],$$

which is finite. We used Jensen’s inequality first, and then the branching property and the many-to-one lemma (Lemma S). This allows us to apply Theorem B of [15] to the random variable $\sum_{k \geq \sigma_A+1} e^{-(V(w_{k-1})-V(w_{\sigma A}))} \sum_{u \notin \Omega(w_k)} e^{-(V(u)-V(w_{k-1}))} Y_u$; there exists a constant $C_p$ such that

$$\hat{P}\left( \sum_{k \geq \sigma_A+1} e^{-(V(w_{k-1})-V(w_{\sigma A}))} \sum_{u \notin \Omega(w_k)} e^{-(V(u)-V(w_{k-1}))} Y_u > x \right) \sim C_p x^{-k'},$$

which implies that there exist positive constants $K_p, K_p'$ such that

$$\hat{P}\left( \sum_{k \geq \sigma_A+1} e^{-(V(w_{k-1})-V(w_{\sigma A}))} \sum_{u \notin \Omega(w_k)} e^{-(V(u)-V(w_{k-1}))} Y_u > x \right) \leq K_p 1_{\{x \leq K_p'\}} + C_p x^{-k'}.$$

Consequently,

$$\hat{P}\left( e^{\sigma A}(w_{\sigma A}) \sum_{k=\sigma_A+1}^{\hat{\tau}_1} e^{-(V(w_{k-1})-V(w_{\sigma A}))} \sum_{u \notin \Omega(w_k)} e^{-(V(u)-V(w_{k-1}))} Y_u > \frac{\varepsilon}{6} x, \sigma_A < \hat{\tau}_1 \right) \leq \hat{E}\left[ \hat{P}\left( \sum_{k \geq \sigma_A+1} e^{-(V(w_{k-1})-V(w_{\sigma A}))} \sum_{u \notin \Omega(w_k)} e^{-(V(u)-V(w_{k-1}))} Y_u > \frac{\varepsilon}{6} e^{\sigma A}(w_{\sigma A}) x \right) 1_{\{\sigma_A < \hat{\tau}_1\}} \right]$$

$$\leq K_p \hat{P}(e^{\sigma A}(w_{\sigma A}) \geq \frac{x}{6 R_p}, \sigma_A < \hat{\tau}_1) + C_p \hat{E}\left[ (e^{\sigma A}(w_{\sigma A}))' 1_{\{\sigma_A < \hat{\tau}_1\}} \right] x^{-k'}.$$
Now, as according to Corollary 4.3 of [11],
\[
C_p \leq \frac{1}{\kappa \mathbb{E} \left[ \sum_{u=1}^{V(u)} e^{-\kappa V(u)} \right]} \mathbb{P} \left[ \sum_{u \in \mathcal{E}^{(p)}_{\Omega(u)}} e^{-(V(u) - V(w_{k-1}))} Y_u \right]^{\kappa'}
\]
we have (by Lebesgue’s dominated convergence theorem) that for \( p \) large enough, for any \( A > 0 \), this last quantity is smaller than
\[
K_p \mathbb{P} \left[ e^{\sigma_A (w_{\sigma_A})} \right] \geq \frac{x}{6K_p^{\alpha}} \left( \sigma_A < \hat{\tau}_1 \right) + \frac{\varepsilon}{2} x^{-\kappa'} \mathbb{E} \left[ \left( e^{\sigma_A (w_{\sigma_A})} \right)^{\kappa'} 1_{\{\sigma_A < \hat{\tau}_1 \}} \right]
\]

\[
\leq K_p (6Kp)^{\kappa'} x^{-\kappa'} \mathbb{E} \left[ \left( e^{\sigma_A (w_{\sigma_A})} \right)^{\kappa'} 1_{\{\sigma_A < \hat{\tau}_1 \}} 1_{\{w_{k'} e^{\sigma_A (w_{\sigma_A})} \}} \right] + \frac{\varepsilon}{2} x^{-\kappa'} \mathbb{E} \left[ \left( e^{\sigma_A (w_{\sigma_A})} \right)^{\kappa'} 1_{\{\sigma_A < \hat{\tau}_1 \}} \right],
\]
by Markov’s inequality. This yields that for \( p \) large enough and \( x \) large enough, we have

\[(3.40)\]
\[
h_p(x) \leq \varepsilon x^{-\kappa'} \mathbb{E} \left[ \left( e^{\sigma_A (w_{\sigma_A})} \right)^{\kappa'} 1_{\{\sigma_A < \hat{\tau}_1 \}} \right].
\]

- Let us now deal with \( f_{N,p}(x) \) for \( p \) fixed such that (3.40) holds. We have

\[
f_{N,p}(x) \leq \mathbb{P} \left( \left| \sum_{k=\sigma_A+1}^{\hat{\tau}_1} \sum_{u \in \Omega(w_k)} e^{\sigma_A (u)} Y_u - e^{\sigma_A (w_{\sigma_A})} Y_{w_{\sigma_A}} \right| > \frac{\varepsilon}{6a_n} x, \sigma_A < \hat{\tau}_1 \right)
\]

\[
\quad + \mathbb{P} \left( e^{\sigma_A (w_{\sigma_A})} Y_{w_{\sigma_A}} > \frac{\varepsilon}{6a_n} x, \sigma_A < \hat{\tau}_1 \right),
\]
where we recall that following the notation introduced in the proofs of Lemmas 10 and 11, we denoted by \( Y_{w_{\sigma_A}} \) the random variable \( \sum_{k>\sigma_A+1} e^{-((V(w_{k-1})-V(w_{\sigma_A}))) \sum_{u \in \Omega(w_k)} e^{-((V(u)-V(w_{k-1})))} Y_u} \).
Now equation (3.26) and Theorem B of [13] applied to \( Y_{w_{\sigma_A}} \) yield that for \( A \) large enough

\[
f_{N,p}(x) \leq \varepsilon (6a_n)^{\kappa'} x^{-\kappa'} \mathbb{E} \left[ \left( e^{\sigma_A (w_{\sigma_A})} \right)^{\kappa'} 1_{\{\sigma_A < \hat{\tau}_1 \}} \right] + C_Y (6a_n)^{\kappa'} x^{-\kappa'} \mathbb{E} \left[ \left( e^{\sigma_A (w_{\sigma_A})} \right)^{\kappa'} 1_{\{\sigma_A < \hat{\tau}_1 \}} \right],
\]
where \( C_Y \) is to \( Y_{w_{\sigma_A}} \) what \( C_Z \) is to \( Z_{w_{\sigma_A}} \) in (3.25). Then, as \( a_n \to 0 \), we have for \( N \) large enough

\[(3.41)\]
\[
f_{N,p}(x) \leq \varepsilon x^{-\kappa'} \mathbb{E} \left[ \left( e^{\sigma_A (w_{\sigma_A})} \right)^{\kappa'} 1_{\{\sigma_A < \hat{\tau}_1 \}} \right].
\]

- Finally, let us consider \( g_{N,p}(x) \) for \( p \) and \( N \) fixed such that (3.40) and (3.41): we have by Markov’s inequality

\[(3.42)\]
\[
g_{N,p}(x) \leq \frac{3}{\varepsilon} x^{-\kappa'} \mathbb{E} \left[ \left( \sum_{k=\sigma_A+1}^{\hat{\tau}_1} \sum_{u \in \Omega(w_k)} NY_u \right)^{\kappa'} 1_{\{\sigma_A < \hat{\tau}_1 \}} 1_{\{\sum_{k=\sigma_A+1}^{\hat{\tau}_1} \sum_{u \in \Omega(w_k)} NY_u > \varepsilon x^\kappa \}} \right] \quad \text{.}
\]

Let us study the expectation appearing in this expression; we have as \( \kappa' \leq 1 \) and as \#\( \mathbb{E} \Omega(w_k) \leq p \),
\[
\mathbb{E} \left[ \left( \sum_{k=\sigma_A+1}^{\hat{\tau}_1} \sum_{u \in \Omega(w_k)} NY_u \right)^{\kappa'} 1_{\{\sigma_A < \hat{\tau}_1 \}} \right] \leq \mathbb{E} \left[ \sum_{k=\sigma_A+1}^{\hat{\tau}_1} \sum_{u \in \Omega(w_k)} N^{\kappa'} Y_u^{\kappa'} 1_{\{\sigma_A < \hat{\tau}_1 \}} \right] \leq C_1 \ln(1 + \mathbb{E}(w_{\sigma_A})) 1_{\{\sigma_A < \hat{\tau}_1 \}} p N^{\kappa'} \mathbb{E}[Y_u^{\kappa'}].
where the equality is obtained by the strong Markov property and where the last but one
inequality is obtained thanks to (3.8). This together with (3.42) (and the fact that \( e^{\sigma_A(w_{\sigma_A})} = e(w_{\sigma_A}) - 1 \geq \frac{1}{2} e(w_{\sigma_A}) \)) yields that for any \( A \), for \( x \) large enough,

\[
g_{N,p}(x) \leq \varepsilon x^{-\kappa'} \hat{E} \left[ (e^{\sigma_A(w_{\sigma_A})})^{\kappa'} 1_{\{\sigma_A < \hat{\tau}_1\}} \right]
\]
as required, thus concluding the proof of the lemma.

We can now prove that (3.17) stands. Let \( \varepsilon > 0 \). Combining (3.23) with Lemma 12 allows us to write that for \( A \) and \( x \) large enough,

\[
\hat{P}(L^1 > x) \geq \hat{P} \left( \sum_{k=\sigma_A+1}^{\hat{T}_1} \sum_{u \in \Omega(w_k)} e(u)W^u_\infty > (1 + \varepsilon)x, \quad \sigma_A < \hat{\tau}_1 \right) - \varepsilon x^{\kappa'} \hat{E} \left[ (e^{\sigma_A(w_{\sigma_A})})^{\kappa'} 1_{\{\sigma_A < \hat{\tau}_1\}} \right]
\]
and

\[
\hat{P}(L^1 > x) \leq \hat{P} \left( \sum_{k=\sigma_A+1}^{\hat{T}_1} \sum_{u \in \Omega(w_k)} e(u)W^u_\infty > (1-2\varepsilon)x, \quad \sigma_A < \hat{\tau}_1 \right) + \varepsilon x^{\kappa'} (2 + \hat{E} \left[ (e^{\sigma_A(w_{\sigma_A})})^{\kappa'} 1_{\{\sigma_A < \hat{\tau}_1\}} \right])
\]

Then, Lemmas 11 and 10 yield that for \( A \) and \( x \) large enough,

\[
\hat{P}(L^1 > x) \geq \hat{P} \left( e^{\sigma_A(w_{\sigma_A})}W^\infty_{\sigma_A} > (1 + 3\varepsilon)x, \quad \sigma_A < \hat{\tau}_1 \right) - 3\varepsilon x^{\kappa'} \hat{E} \left[ (e^{\sigma_A(w_{\sigma_A})})^{\kappa'} 1_{\{\sigma_A < \hat{\tau}_1\}} \right]
\]
and

\[
\hat{P}(L^1 > x) \leq \hat{P} \left( e^{\sigma_A(w_{\sigma_A})}W^\infty_{\sigma_A} > (1-4\varepsilon)x, \quad \sigma_A < \hat{\tau}_1 \right) + \varepsilon x^{\kappa'} (2 + 3\hat{E} \left[ (e^{\sigma_A(w_{\sigma_A})})^{\kappa'} 1_{\{\sigma_A < \hat{\tau}_1\}} \right])
\]

Therefore it is enough to show that there exists a \( K \geq 0 \) such that for any \( A > 0 \),

\[
\lim_{x \to \infty} x^{\kappa'} \hat{P} \left( e^{\sigma_A(w_{\sigma_A})}W^\infty_{\sigma_A} > x, \sigma_A < \hat{\tau}_1 \right) = K \hat{E} \left[ (e^{\sigma_A(w_{\sigma_A})})^{\kappa'} 1_{\{\sigma_A < \hat{\tau}_1\}} \right] \in (0; \infty)
\]
to conclude. This is the case indeed, as

\[
\lim_{x \to \infty} x^{\kappa'} \hat{P} \left( e^{\sigma_A(w_{\sigma_A})}W^\infty_{\sigma_A} > x, \sigma_A < \hat{\tau}_1 \right) = \lim_{x \to \infty} x^{\kappa'} \hat{E} \left[ \hat{P} \left( W^\infty_{\sigma_A} > \frac{x}{e^{\sigma_A(w_{\sigma_A})}} \mid \hat{F}_{\sigma_A} \right) 1_{\{\sigma_A < \hat{\tau}_1\}} \right] = \hat{C}_\infty \hat{E} \left[ (e^{\sigma_A(w_{\sigma_A})})^{\kappa'} 1_{\{\sigma_A < \hat{\tau}_1\}} \right],
\]

where we used (3.22) and Lebesgue’s dominated convergence theorem in the last equality.

**Remark:** The constant appearing in (3.17) is thus equal to \( \hat{C}_\infty \lim_{A \to \infty} \hat{E} \left[ (e^{\sigma_A(w_{\sigma_A})})^{\kappa'} 1_{\{\sigma_A < \hat{\tau}_1\}} \right] \), but we were unable to compute this last limit; this is what prevents us from giving a statement of Theorem 1 in which the multiplicative constants and the normalisation of the Lévy process are specified.
4 Proof of Theorem $[1]$

Let us now return to the random walk on a marked tree that we initially considered, $(X_n^V)_{n \geq 0}$. We intend to prove Theorem $[1]$ by linking this random walk to another one taking place on a forest as introduced in Subsection $[1.2]$. The strategy of the proof is the same as that presented in Section 5 of $[1]$, but we write it thoroughly for the sake of completeness.

For any $k \geq 1$, let

$$T_k := \inf \{ n \geq 1 : \sum_{1 \leq i \leq k} 1_{\{X_{i-1} = \hat{\rho}, X_i = \rho\}} = k \}.$$ 

be the time of the $k$th visit to $\rho$ from its parent. In other words, $T_n$ is the time after which $n$ excursions have been made from $\hat{\rho}$, and so the law of $T_{T_n}$ the visited tree at time $T_n$ is the same as that of $T$ under $P_n$. We extend the notation $\beta$ introduced in Subsection $[2.1]$ by setting for any $u \in T$, for any $k \geq 1$

$$\beta^{(k)}(u) := \sum_{i=1}^{T_k} 1_{\{X_{i-1} = \hat{u}, X_i = u\}}.$$ 

We have the following lemma, which is the analogue of Lemma 7.1 of $[1]$:

**Lemma 13.** $P^*$-almost surely, $T_{n^{1-1/n[\ln n(n)]}} \geq n$ for $n \geq 1$ large enough.

**Proof.** For any $k \geq 1$, let us set $R^{(k)} := \#T^T_k$ as the number of distinct vertices visited by $(X_n)_{n \geq 0}$ after $k$ excursions from $\hat{\rho}$. Observe that $\{T_{n^{1-1/n[\ln n(n)]}} \geq n\} \supset \{R^{(n^{1-1/n[\ln n(n)]})} \geq n\}$. Therefore, it is enough to show that $P^*$-almost surely,

$$R^{(k[\ln^{10}(k)])} \geq k^{\frac{n}{\ln^{10}(k)}}$$

for $k$ large enough.

Now notice that the probability $P^V(R^{(k[\ln^{10}(k)])} < k^{\frac{n}{\ln^{10}(k)}})$ is smaller than $P^V(R^{(1)} < k^{\frac{n}{\ln^{10}(k)}})^{k[\ln^{10}(k)]}$. On the event $\{P^V(R^{(1)} \geq k^{\frac{n}{\ln^{10}(k)}}) \geq \frac{1}{k[\ln^{10}(k)]}\}$, we get that $P^V(R^{(k[\ln^{10}(k)])} < k^{\frac{n}{\ln^{10}(k)}}) \leq e^{-[\ln^{5}(k)]}$ which is summable in $k$. Therefore, if we are able to show that $P^*$-almost surely, for $k$ large enough, $P^V(R^{(1)} \geq k^{\frac{n}{\ln^{10}(k)}}) \geq k^{\frac{n}{\ln^{10}(k)}}$, then the lemma will be proved.

Let us set $G_k := \{ u \in T : e^{V(u)} \geq k \ln^2(k) \text{ and } e^{V(u)} < k \ln^2(k) \text{ } \forall v \vdash u \}$ as the set of the vertices the exponential potential of which is the first of their ancestry line to overshoot $k \ln^2(k)$. According to the proof of Lemma 7.1 of $[1]$, there exist constants $C, c > 0$ such that if we set $G'_k := \{ u \in G_k : e^{V(u)} < \frac{2}{c}k \ln^2(k) \}$, then $P^*$-almost surely, there exists $\varepsilon > 0$ such that,

- for $k$ large enough, $\#G'_k \geq c\varepsilon k \ln^2(k)$, and
- for $k$ large enough, $|u| < [\ln^2(k)]$ for any $u \in G'_k$.

For any $\varepsilon > 0$, $k_0 > 0$, let $E_{\varepsilon,k_0} := \bigcup_{k \geq k_0} \{ \#G'_k \geq c\varepsilon k \ln^2(k) \text{ and } |u| < [\ln^2(k)] \text{ } \forall u \in G'_k \}$.

Let us now consider for any $u \in T$, $(X_n^{(u)})_{n \geq 0}$ as the first excursion from $u$ of the random walk $(X_n)_{n \geq 0}$. Let $E_{u,k}$ be the event that the walk $(X_n^{(u)})_{n \geq 0}$ hits more than $k^{\frac{n}{\ln^{10}(k)}}$ distinct vertices before hitting $\hat{u}$. Notice that the quenched probability $P^V(R^{(1)} \geq k^{\frac{n}{\ln^{10}(k)}})$ is larger than the probability that there exists a $u \in G'_{k}$ such that $u$ gets visited before $T_1$ and $E_{u,k}$ happens. Moreover, as remarked by A.O. Golosov $[12]$, the probability that the walks visits $u$ before
visiting $\tilde{\rho}$ is equal to $(1 + e^{V(u_1)} + \cdots + e^{V(u)})^{-1}$ and when $u \in G'_k$, $(1 + e^{V(u_1)} + \cdots + e^{V(u)})^{-1} \geq (2/C \times k[\ln^2(k)] \times (|u| + 1))^{-1} \geq 2([\ln(k)] \times k[\ln^2(k)] \times [\ln^2(k)])^{-1}$ for $k$ large enough. This yields that $P$-almost surely, for $k$ large enough,

$$P^V(R^{(1)} \geq k^{\frac{\kappa - 1}{\ln^2(k)}}) \geq \frac{2}{k[\ln^2(k)]}P^V(\bigcup_{u \in G'_k} E_{u,k}).$$

and therefore, for $k$ large enough,

$$(4.1) \quad \{P^V(R^{(1)} \geq k^{\frac{\kappa - 1}{\ln^2(k)}}) < \frac{1}{k[\ln^2(n)]}\} \supset \{P^V(\bigcup_{u \in G'_k} E_{u,k}) < \frac{1}{2}\} = \{P^V(\bigcap_{u \in G'_k} E_{u,k}) \geq \frac{1}{2}\}.$$

Now for any $\varepsilon > 0$, $k_0 \leq k$, we have by independence of the events $E_{u,k}$ for $k \in G'_k$,

$$P\left(\bigcap_{u \in G'_k} E_{u,k} \big| E_{\varepsilon,k_0}\right) \leq P\left(R^{(1)} < k^{\frac{\kappa - 1}{\ln^2(k)}}\right).$$

The quantity $R^{(1)}$ is smaller than $\#\{u \in T^{(1)} : \beta^{(1)}(u) = 1\}$, and since the tree made up of the vertices $u$ of type $\beta^{(1)}(u) = 1$ is a critical Galton-Watson tree with power-law tail of index $\kappa \leq 2$, Theorem 1 of [6] ensures that there exists a $\varepsilon'$ such that $P(R^{(1)} < k^{\frac{\kappa - 1}{\ln^2(k)}}) \leq 1 - \varepsilon' k$ if $1 < \kappa < 2$ and $P(R^{(1)} < k^{\frac{\kappa - 1}{\ln^2(k)}}) \leq 1 - \frac{\varepsilon}{k} \ln^2(k)$ if $\kappa = 2$. This yields in any case

$$P\left(\bigcap_{u \in G'_k} E_{u,k} \big| E_{\varepsilon,k_0}\right) \leq e^{-\varepsilon' \ln^2(k)},$$

and then Markov’s inequality gives

$$P^V\left(\bigcap_{u \in G'_k} E_{u,k} \big| E_{\varepsilon,k_0}\right) \leq 2e^{-\varepsilon' \ln^2(k)},$$

which is summable in $k$. This yields that for any $\varepsilon > 0$, $k_0 \geq 1$, on the event $E_{\varepsilon,k_0}$, $P^*$-almost surely for $k \geq k_0$ large enough $P^V\left(\bigcap_{u \in G'_k} E_{u,k}\right) \geq \frac{1}{2}$. In view of (L1) and by the fact that $P^*$-almost surely there exist $\varepsilon > 0$, $k_0 \geq 1$ such that $E_{\varepsilon,k_0}$ holds, this ensures that $P^*$-almost surely for $k$ large enough $P^V(R^{(1)} \geq k^{\frac{\kappa - 1}{\ln^2(k)}}) \geq \frac{1}{k[\ln^2(n)]}$, which is enough to conclude the proof as explained at the beginning.

Let us now prove the analogue of Lemma 5.2 of [1]. Let us denote for any $u \in T$ by $\|\rho, u\|$ the set of strict ancestors of $u$, excluding $\rho$.

**Lemma 14.** There exists a constant $c > 0$ such that for any $\ell, k \geq 1$,

$$P(\exists u \in T : |u| \geq \ell \quad \text{and} \beta^{(k)}(v) \geq 2 \quad \forall v \in \|\rho, u\| \leq 2k^2e^{-c\ell}$$

**Proof.** We just have to follow the lines of Lemma 5.2, let apart that at the end where we write instead of equation (5.6):

$$E\left[\sum_{|v| = \ell/2} 1_{(\beta^{(1)}(v) \geq 1, \beta^{(2)}(v) \geq 1)}\right] = E\left[\sum_{|v| = \ell/2} P^V(\beta^{(1)}(v) \geq 1)^2\right]$$

$$\leq E\left[\sum_{|v| = \ell/2} P^V(\beta^{(1)}(v) \geq 1)^{1 + \frac{\varepsilon}{2}}\right],$$

45
as \( \kappa \leq 2 \). Then as explained in the proof of Lemma 13, \( P^V(\beta(1)(v) \geq 1) = (1 + e^{V(\nu_1)} + \cdots + e^{V(\nu)})^{-1} \leq e^{-V(\nu)} \), and therefore the last equation is less than \( \psi(1) + 1/2 \) (with \( \psi(1/2) < 1 \)), which allows us to conclude as in the proof of Lemma 5.2 of \[1\].

We are now ready to give the proof of Theorem 1.

**Proof of Theorem 1.** Let us set for any \( n > 1 \), \( c_n := n^{-1/\kappa} \) if \( \kappa \in (1; 2) \) and \( c_n := (n \ln^{-1}(n))^{1/2} \) if \( \kappa = 2 \). We need to prove that for any \( a > 0 \), \( c_n^{-1}(|X_{\lfloor n \rfloor}|)_{0 \leq t \leq a}, R_n) \Rightarrow \left( (H_\ell)_{0 \leq \ell \leq a}, T(H_{\ell})_{0 \leq \ell \leq 1} \right) \). It is actually enough to prove this for \( a = 1 \); so let us now fix \( a \geq 1 \). Let for any \( n \geq 1 \), \( j_n := (an)^{1-1/\kappa} [\ln^{1/\kappa}(an)] \). According to Lemma 13, \( P^* \)-almost surely, \( T_{j_n} \geq an \) for \( n \) large enough. Let

\[
\mathcal{L}_n := \{ u \in \mathbb{T} : \beta(j_n)(u) = 1 \text{ and } \beta(j_n)(v) \geq 2 \ \forall v \ni u, |v| \geq 1/2 \ln^2(n) \}
\]

be the set of vertices which at time \( T_{j_n} \) are the first of their ancestry line since generation \( 1/2 \ln^2(n) \) to be of edge-local time 1. Let us also set

\[
\mathcal{Z}_n := \{ u \in \mathbb{T} : \beta(j_n)(u) \neq 1 \text{ and } \beta(j_n)(v) \geq 2 \ \forall v \ni u, \}
\]

as the set of the vertices "below" the line \( \mathcal{L}_n \) (see Figure 3 for a representation of \( \mathcal{Z}_n \)). According to Lemma 14, \( P \)-almost surely, for \( n \) large enough, \( \max_{u \in \mathcal{L}_n \cup \mathcal{Z}_n} |u| < \ln^2(n) \). Moreover, as for any \( u \in \mathbb{T}, k \geq 1 \), \( E^V[\beta(k)(u)] = k e^{-V(u)} \), Markov's inequality yields that

\[
P^V\left( \sum_{|u| \leq \ln^2(n)} \beta(j_n)(u) \geq j_n \ln^3(n) \right) \leq \frac{1}{\ln^2(n)} \sum_{k \leq \ln^2(n)} \sum_{|u| = k} e^{-V(u)}
\]

which goes to zero \( P \)-almost surely by the Kesten-Stigum theorem [17]. To sum up, for both the convergence under \( P^* \) and under \( P^V \) for \( P^* \)-almost every \( \mathbb{V} \), we just have to show that for any \( n_0 \geq 1 \), the theorem holds on the event

\[
\mathcal{E}_{n_0} := \left\{ \forall n \geq n_0, \ T_{j_n} \geq an, \ \max_{u \in \mathcal{L}_n \cup \mathcal{Z}_n} |u| < \ln^2(n), \text{ and } \sum_{|u| \leq \ln^2(n)} \beta(j_n)(u) < j_n \ln^3(n) \right\}.
\]

So let us fix an \( n_0 \) and restrict to the event \( \mathcal{E}_{n_0} \). Let \( n \geq n_0 \). Let us denote by \( \tilde{\rho}_1, \tilde{\rho}_2, \ldots \) the vertices of \( \mathcal{L}_n \) ordered according to their first time of visit by the walk \( (X_k)_{k \geq 0} \). For any \( 1 \leq i \leq \# \mathcal{L}_n \) we denote by \( \tilde{\mathbb{T}}_i \) the sub-tree of \( \mathbb{T} \) rooted in \( \tilde{\rho}_i \). See Figure 3 for a picture of the situation. We also let for any \( i \geq \# \mathcal{L}_n \), \( \tilde{\mathbb{W}}_i = (\tilde{\mathbb{T}}_i, V) \) be i.i.d. branching random walks with same law as that of \( \mathbb{V} \), and we denote by \( \tilde{\mathbb{W}} = (\tilde{\mathbb{F}}, V) \) the whole forest made up of the trees \( (\tilde{\mathbb{T}}_i)_{i \geq 1} \). For any \( i \geq 1 \),

\[
\eta_i^\text{in} := \min\{k \geq 1 : X_k = \tilde{\rho}_i\} \quad \text{and} \quad \eta_i^\text{out} := \min\{k \geq \eta_i^\text{in} : X_k = \tilde{\rho}_i\}
\]

be the entrance and exit times of the trees \( \tilde{\mathbb{T}}_i \) by the walk. The walk \( (X_k)_{k \geq 0} \) considered until time \( an \) will therefore behave as follows:

- It remains in the set \( \mathcal{Z}_n \) from time 0 to \( \eta_i^\text{in} - 1 \),
- for any \( i \geq 1 \), between times \( \eta_i^\text{in} \) and \( \eta_i^\text{out} - 1 \) it makes an excursion in the tree \( \tilde{\mathbb{T}}_i \),
• for any \( i \geq 1 \), between times \( \eta^\text{out}_i \) and \( \eta^\text{in}_i - 1 \) it remains in the set \( Z_n \).

\[
\sum_{1 \leq i \leq k} 1_{\{X_i \notin Z_n\}}
\]

Figure 3: The tree \( T \) (represented up to generation 9), the set \( Z_n \) and the trees \( \tilde{T}_i \).

We denote by \( (\tilde{X}_k)_{0 \leq k \leq \sum_{i=1}^{\#L_n}(\eta^\text{out}_i - \eta^\text{in}_i)} \) the random walk made up of the excursions of the walk on the trees \( (\tilde{T}_i)_{1 \leq i \leq \#L_n} \) (the roots of which are considered to be at height 0), that we extend for convenience as \( (\tilde{X}_k)_{k \geq 0} \) by adding excursions on the trees \( (\tilde{T}_i)_{i \geq \#L_n + 1} \). The random walk \( (\tilde{X}_k)_{k \geq 0} \) is therefore a random walk on a Galton–Watson forest in random environment, as introduced in Subsection 1.2.

For any \( k \leq an \), we set \( f_n(k) := \sum_{1 \leq i \leq k} 1_{\{X_i \notin Z_n\}} \) as the time spent by the walk in the forest \( \tilde{F} \) before time \( k \). On the event \( E_{no} \), uniformly in \( k \leq an \),

\[
0 \leq k - f_n(k) \leq n - f_n(n) \leq 2j_n \ln^3(n) = o(n),
\]

Let \( \tilde{R}_{f_n(n)} \) be the range of \( (\tilde{X}_k)_{k \geq 0} \) up to time \( f_n(n) \). Theorem 2 together with (4.2) (which also yields \( c_{f_n(n)} \sim c_n \)) says

\[
c^{-1}_n \left( (|\tilde{X}_{|f_n(n)|})_{0 \leq t \leq a}; \tilde{R}_{f_n(n)} \right) \xrightarrow{\text{a.s.}} \left( (H_t)_{0 \leq t \leq a}; T(H_t)_{0 \leq t \leq 1} \right)
\]

under the annealed law \( P^* \). This stands also under the quenched law \( P^V \) for \( P^* \)-almost every \( V \); we can use the same arguments as those used in Section 5 of [1] to prove it. The only
difference is in equation (5.9), but we can bound the probability to touch a vertex at generation
\( \frac{1}{2} \ln^2(n) \) by \( \max_{|u| = \frac{1}{2} \ln^2(n)} e^{-V(u)} \). Then, the upper bound of (5.9) of [1] can be replaced by
\( n j_n E \left[ \max_{|u| = \frac{1}{2} \ln^2(n)} e^{-V(u)} \right] \). Now we write
\[
 nj_n E \left[ \max_{|u| = \frac{1}{2} \ln^2(n)} e^{-V(u)} \right] \leq nj_n E \left[ \left( \sum_{|u| = \frac{1}{2} \ln^2(n)} e^{-\left( \frac{1 + \kappa}{2} \right)V(u)} \right)^{\frac{2}{1 + \kappa}} \right]
\]
\[
 \leq nj_n E \left[ \sum_{|u| = \frac{1}{2} \ln^2(n)} e^{-\left( \frac{1 + \kappa}{2} \right)V(u)} \right]^{\frac{2}{1 + \kappa}} = nj_n \psi \left( \frac{1 + \kappa}{2} \right) \ln^2(n),
\]
which is summable as \( \psi \left( \frac{1 + \kappa}{2} \right) \in (0; 1) \).

Now we just notice that by construction, for any \( k \leq an \), on the event \( \mathcal{E}_{n_0} \),
\[ (4.4) \quad \left| X_k - \tilde{X}_{f_n(k)} \right| \leq \max_{u \in Z_n \cup L_n} |u| \leq \ln^2(n). \]

Moreover, Lemma 2.3 of [8] says that if \((R, d)\) and \((R', d')\) are two real trees, and if \( \phi : R \mapsto R' \) is a surjective map which sends the root of \( R \) to that of \( R' \) (we mention that the roots of a forest are considered to be at a null distance from each other for the graph distance), then \( d_{GH}(R, R') \) the Gromov-Hausdorff distance between \( R \) and \( R' \) is smaller than \( \frac{1}{2} \sup \{ |d(x, y) - d'(\phi(x), \phi(y))|, (x, y) \in R \} \). Using this with the map \( \phi_1 : \mathcal{R}_n \mapsto \mathcal{R}_{f_n(n)} \) which is the identity on the trees \( \mathcal{T}_i \) and which sends all the vertices of \( Z_n \) to \( \tilde{\rho}_1 \), we get that
\[
 d_{GH}(\mathcal{R}_n, \mathcal{R}_{f_n(n)}) \leq \ln^2(n) = o(c_n).
\]
This and (4.4) used together with (4.3) concludes the proof. \( \square \)
Appendix

This appendix contains some technical lemmas which are useful in several proofs.

**Lemma 15.** Let \( \alpha \in (0; \kappa - 1) \), and let for any \( i \geq 1 \), \( F(i) := \frac{\Gamma(i + 1 + \alpha)}{\Gamma(i + 1)} \). There exists \( d \in (0; 1) \) such that for any \( i \) large enough,

\[
\sum_{j \geq 1} \hat{p}_{i,j} F(j) \leq d F(i).
\]

**Proof.** We have, for any \( i \geq 1 \),

\[
\sum_{j \geq 1} \hat{p}_{i,j} F(j) = \sum_{j \geq 1} \binom{i + j - 1}{i} E\left[ \sum_{|u| = 1} \frac{e^{-jV(u)}}{(1 + e^{-V(u)})^{i+j}} F(j) \right] = E\left[ \sum_{|u| = 1} \frac{1}{(1 + e^{-V(u)})^{i+1}} \times e^{-V(u)} \times \frac{1}{i!} \sum_{j \geq 0} \frac{\Gamma(j + 1 + \alpha) (j + i)!}{\Gamma(j + 1)} \left( \frac{e^{-V(u)}}{1 + e^{-V(u)}} \right)^j \right].
\]

Now for any \( x \in (0; 1) \), setting \( y = 1 - x \), we have

\[
\sum_{j \geq 0} \frac{\Gamma(j + 1 + \alpha) (j + i)!}{\Gamma(j + 1)} \frac{d^j}{dx^j} \left( \frac{x^i \Gamma(1 + \alpha)}{(1 - x)^{1+\alpha}} \right) = \Gamma(1 + \alpha)(-1)^i \frac{d^i}{dy^i} \left( \frac{(1 - y)^i}{y^{1+\alpha}} \right) = \Gamma(1 + \alpha)(-1)^i \frac{d^i}{dy^i} \left( \sum_{k=0}^{i} \binom{i}{k} (-1)^k y^{k-1-\alpha} \right) = \Gamma(1 + \alpha) \sum_{k=0}^{i} \binom{i}{k} (-1)^i (k - 1 - \alpha) \times \cdots \times (k - i - \alpha) (-1)^k y^{k-1-\alpha} = \frac{\Gamma(1 + \alpha)}{y^{i+1+\alpha}} \sum_{k=0}^{i} \binom{i}{k} (-1)^i (k - 1 - \alpha) \times \cdots \times (k - i - \alpha) (-1)^k y^k.
\]

Denoting the last sum by \( S \), notice that all its terms are negative for \( k > 1 \), and thus we can write

\[
S \leq \sum_{k=0}^{\lceil \ln(i) \rceil} \frac{\Gamma(i - k - 1 + \alpha)}{(i - k)!} \times \frac{\alpha \times \cdots \times (\alpha - (k - 1))}{k!} (-y)^k
\]

For \( k \leq \lceil \ln(i) \rceil \), noticing that \( \frac{\Gamma(i - k + 1 + \alpha)}{(i - k)!} \geq \frac{\Gamma(i - \lceil \ln(i) \rceil + 1 + \alpha)}{(i - \lceil \ln(i) \rceil)!} \rightarrow_{i \to \infty} \frac{\Gamma(i + 1 + \alpha)}{i!} + o\left( \frac{\Gamma(i + 1 + \alpha)}{i!} \right) \), we get uniformly in \( y \in (0; 1) \),

\[
S \leq_{i \to \infty} \left( \frac{\Gamma(i + 1 + \alpha)}{i!} + o\left( \frac{\Gamma(i + 1 + \alpha)}{i!} \right) \right) \sum_{k=\lceil \alpha \rceil + 2}^{\lceil \ln(i) \rceil} \alpha \times \cdots \times (\alpha - (k - 1)) (-y)^k.
\]
Plugging this inequality into (A.2) yields
\[
\sum_{j \geq 0} \frac{\Gamma(j + 1 + \alpha)}{\Gamma(j + 1)} \frac{(j + i)!}{i!} e^{-2y} \leq \frac{i!}{i!} \left( \frac{\Gamma(i + 1 + \alpha)}{i!} + o\left( \frac{\Gamma(i + 1 + \alpha)}{i!} \right) \right) (1 - y)^\alpha - \varepsilon(i(1 - y)),
\]
where \( \varepsilon(y) \) is a sequence of positive functions of \( y \) simply decreasing to 0 and smaller than \( (1 - y)^\alpha \). Plugging this latter inequality into (A.1) yields
\[
\sum_{j \geq 1} \hat{p}_i F(j) \leq E \left[ \sum_{|u| = 1} \frac{1}{(1 + e^{-V(u)})^{i+1}} \right] e^{-V(u)} - e^{-V(u)} (1 + e^{-V(u)})^{i+1} + \varepsilon \left( \frac{e^{-V(u)}}{1 + e^{-V(u)}} \right),
\]
where between these two lines we used the monotone convergence theorem on \( \left( \frac{e^{-V(u)}}{1 + e^{-V(u)}} \right)^\alpha - \varepsilon \left( \frac{e^{-V(u)}}{1 + e^{-V(u)}} \right) \). Now since \( 0 < 1 + \alpha < \kappa \), \( E \left[ \sum_{|u| = 1} e^{-(1+\alpha)V(u)} \right] < 1 \) so there exists \( d < 1 \) such that for all \( i > i_0 \) large enough,
\[
\sum_{j \geq 1} \hat{p}_i F(j) < d F(i),
\]
which is what we wanted indeed.

\[\square\]

**Lemma 16.** Let \( n \in \mathbb{N} \) and \( p \in (0; 1) \). Let \( X^n \sim NB(n, p) \) be a negative binomial random variable of parameter \((n, p)\); we have for any \( 0 < \alpha < 1 \),
\[
E[(X^n)^{1+\alpha}] \leq 16n \left( \frac{p}{1 - p} + \left( \frac{p}{1 - p} \right)^{1+\alpha} \right) + 2n^{1+\alpha} \left( \frac{p}{1 - p} \right)^{1+\alpha}.
\]

**Proof.** Recall that \( E[X^n] = n \frac{p}{1 - p} \) and \( \text{Var}(X^1) = \frac{p}{(1 - p)^2} \).

- If \( p \leq 1/2 \), then \( \text{Var}(X^1) = \frac{p}{(1 - p)^2} \leq 2 \frac{p}{1 - p} \). Therefore,
\[
E[|X^1 - \frac{p}{1 - p}|^{1+\alpha}] \leq E[|X^1 - \frac{p}{1 - p}|] + E[(X^1 - \frac{p}{1 - p})^2] \leq 4 \frac{p}{1 - p}.
\]

- On the other hand, if \( p > 1/2 \), then \( \text{Var}(X^1) = \frac{p}{(1 - p)^2} \leq 2 \frac{p^2}{(1 - p)^2} \), and then by Jensen’s inequality,
\[
E[|X^1 - \frac{p}{1 - p}|^{1+\alpha}] \leq E[(X^1 - \frac{p}{1 - p})^2] \leq 2 \frac{p^2}{(1 - p)^2}^{1+\alpha}.
\]
So in general, \( E[|X^1 - \frac{np}{1 - p}|^{1+\alpha}] \leq 4(\frac{p}{1 - p} + (\frac{p}{1 - p})^{1+\alpha}) \). Finally, according to B. von Bahr and C.-G. Esseen \( 3 \)
\[
E[|X^n - \frac{np}{1 - p}|^{1+\alpha}] \leq 2n E[|X^1 - \frac{p}{1 - p}|^{1+\alpha}] \leq 8n(\frac{p}{1 - p} + (\frac{p}{1 - p})^{1+\alpha}).
\]
The inequality \( (x + y)^{1+\alpha} \leq 2(x^{1+\alpha} + y^{1+\alpha}) \) for \( x, y \geq 0 \) concludes the proof. \[\square\]
Lemma 17. Let $p > 0$, and let $(X_n)_{n \geq 1}$ be a sequence of non-negative random variables which converges in mean of order $p$. Then for any $r \in (0; p)$, there exists a random variable $Y$ with a finite moment of order $r$ and a decreasing sequence $(a_n)_{n \geq 1}$ (with $a_1 = 1$) such that for any $n \geq 1$, $a_n Y$ is stochastically greater than $X_k$ for any $k \geq n$.

Proof. Let $r \in (0; p)$, and $s \in (r; p)$; let us consider $Y$ a random variable with distribution function defined for any $x > 0$ by

$$P(Y \leq x) = \left(1 - \frac{\max_{i \geq 1} \mathbb{E}[X_i]^s}{x^s}\right)^{+},$$

which is continuously increasing from 0 to 1 indeed, and defines a random variable with a finite moment of order $r$ as $s > r$. Let for any $n \geq 1$,

$$a_n := \left(\frac{\max_{i \geq n} \mathbb{E}[X_i]^s}{\max_{i \geq 1} \mathbb{E}[X_i]^s}\right)^{\frac{1}{s}}$$

(we do not consider the trivial case $X_n = 0$ a.s. for all $n \geq 1$). As $(X_n)_{n \geq 1}$ converges in mean of order $p > s$, this sequence tends to zero as required. Now Markov’s inequality together with the fact that distribution functions are positive yield that for any $k \geq n$,

$$P(X_k > x) \leq \left(\frac{\max_{i \geq n} \mathbb{E}[X_i]^s}{x^s}\right) \land 1 = \left(\frac{\max_{i \geq 1} \mathbb{E}[X_i]^s}{a_n^{-s} x^s}\right) \land 1 = P(a_n Y > x),$$

which proves that $a_n Y$ is stochastically greater than $X_k$ for any $k \geq n$ indeed. \qed
Acknowledgements: I thank my advisor Elie Aïdékon for guiding me throughout the development of this article.

References

[1] Aïdékon, E. and de Raphelis, L. (2015). Scaling limit of the recurrent biased random walk on a Galton-Watson tree. Probab. Theory Related Fields (to appear).

[2] Andreoletti, P. and Debs, P. (2014). The number of generations entirely visited for recurrent random walks on random environment. J. Theoret. Probab. 27, 518–538.

[3] von Bahr, B. and Esseen, C.-G. (1965). Inequalities for the rth Absolute Moment of a Sum of Random Variables, $1 \leq r \leq 2$. Ann. Math. Stat. 36, 299-303.

[4] de Raphélis, L. (2014). Scaling limit of multitype Galton-Watson trees with infinitely many types. Ann. Inst. Henri Poincaré Probab. Stat. (to appear).

[5] Dembo, A. and Sun, N. (2012). Central limit theorem for biased random walk on multi-type Galton–Watson trees. Electron. J. Probab. 17, 1–40.

[6] Doney, R. A. (1982). On the exact asymptotic behaviour of the distribution of ladder epochs. Stochastic Process. Appl. 12, 203–214.

[7] Duquesne, T. and Le Gall, J.-F. (2002). Random Trees, Lévy Processes and Spatial Branching Processes. Astérisque 281.

[8] Evans, S., Pitman, J and Winter, A. (2006). Rayleigh processes, real trees, and root growth with re-grafting. Probab. Theory Related Fields 134, 81–126.

[9] Faraud, G. (2011). A central limit theorem for random walk in a random environment on marked Galton-Watson trees. Electron. J. Probab. 16, 174–215.

[10] Feller, W. (1971). An introduction to Probability Theory and Its Applications II, 2nd ed. Wiley, New York.

[11] Goldie, C. M. (1991). Implicit Renewal Theory and Tails of Solutions of Random Equations. Ann. Appl. Probab. 1, 126–160.

[12] Golosov, A.O. (1984). Localization of random walks in one-dimensional random environments. Comm. Math. Phys. 92, 451–506.

[13] Hu, Y. (2014). Local times of subdiffusive biased walks on trees. J. Theoret. Probab. (to appear).

[14] Hu, Y. and Shi, Z. (2007). A subdiffusive behavior of recurrent random walk in random environment on a regular tree. Probab. Theory Related Fields 138, 521–549.

[15] Kesten, H. (1973). Random difference equations and Renewal theory for products of random matrices. Acta Math. 131, 208–248.

[16] Kesten, H., Kozlov, M.V. and Spitzer, F. (1975). A limit law for random walk in random environment Compos. Math. 30, 145–168.
[17] Kesten, H. and Stigum, B.P. (1966). A Limit Theorem for Multidimensional Galton-Watson Processes. *Ann. Math. Statist.* 5, 1211–1223.

[18] Kortchemski, I. (2012). Invariance principles for Galton-Watson trees conditioned on the number of leaves. *Stochastic Process. Appl.* 122, 3126–3172.

[19] Kurtz, T. G., Lyons, R., Pemantle, R. and Peres, Y. (1997). A Conceptual Proof of the Kesten-Stigum Theorem for Multi-type Branching Processes. In *Classical and Modern Branching Processes*. IMA Vol. Math. Appl. 84, 181–185.

[20] Le Gall, J.-F. (2005). Random trees and applications. *Probab. Surveys* 2005 2, 245–311.

[21] Le Gall, J.-F. and Le Jan, Y. (1998). Branching processes in Lévy processes: the exploration process. *Ann. Probab.* 26, 213–252.

[22] Liu, Q. (1999). On generalized multiplicative cascades. *Stochastic Process. Appl.* 86, 263–286.

[23] Lyons, R. (1997). A Simple Path to Biggins’ Martingale Convergence for Branching Random Walk. In: *Classical and Modern Branching Processes* (Eds.: K.B. Athreya and P. Jagers). IMA Volumes in Mathematics and its Applications 84, 217–221. Springer, New York.

[24] Lyons, R. and Pemantle, R. (1992). Random walk in a random environment and first-passage percolation on trees. *Ann. Probab.* 20, 125–136.

[25] Maillard, P. and Zeitouni, O. (2016). Slowdown in branching Brownian motion with inhomogeneous variance. *Ann. Inst. Henri Poincaré Probab. Stat.* 52, 1144–1160.

[26] Meyn, S.P. and Tweedie, R.L. (1993). *Markov Chains and Stochastic Stability*. Springer, New York.

[27] Miermont, G. (2008). Invariance principles for spatial multitype Galton–Watson trees. *Ann. Inst. Henri Poincaré Probab. Stat.* 44, 1128–1161.

[28] Neveu, J. (1986). Arbres et processus de Galton–Watson. *Ann. Inst. Henri Poincaré Probab. Stat.* 22, 199–207.

[29] Peres, Y. and Zeitouni, O. (2006). A central limit theorem for biased random walks on Galton–Watson trees. *Probab. Theory Related Fields* 140, 595–629.