1. Introduction

1.1. Overview. Let $F$ be a local or global field. In this paper we are going to define and study a functor $G \mapsto B(F, G)$ from the category of linear algebraic $F$-groups to the category of pointed sets. When $G$ is commutative (e.g. a torus) $B(F, G)$ will actually be an abelian group. For $p$-adic fields $F$ the functor $G \mapsto B(F, G)$ is naturally isomorphic to the functor $G \mapsto B(G)$ studied in [Kot97] (which agrees with $B(G)$ from [Kot85] when $G$ is connected). There is a natural inclusion

$$H^1(F, G) \hookrightarrow B(F, G),$$

so $B(F, G)$ can be thought of as an enlargement of $H^1(F, G)$.

The study of $B(F, G)$ breaks into two parts. To get the theory off the ground, one must first treat the case in which $G$ is an $F$-torus $T$. Let $K$ be a finite Galois extension of $F$ that splits $T$, and let $G(K/F)$ denote the Galois group of $K/F$. Then $H^1(F, T) = H^1(G(K/F), T(K))$, and Tate-Nakayama theory gives us a $G(K/F)$-module $X(K)$ (discussed in more detail later in this introduction) such that

$$H^1(G(K/F), T(K)) \xrightarrow{\sim} H^{-1}(G(K/F), X_*(T) \otimes X(K)),$$

where $X_*(T)$ is the dual of $X(T)$.
the isomorphism being given by cup product with a canonical class
\[ \alpha(K/F) \in H^2(G(K/F), \mathbb{D}_{K/F}(K)), \]
where \( \mathbb{D}_{K/F} \) is the protorus over \( F \) whose character group is \( X(K) \). Now the Tate cohomology group \( H^{-1}(G(K/F), X_\ast(T) \otimes X(K)) \) is by definition the subgroup of \( (X_\ast(T) \otimes X(K))_{G(K/F)} \) obtained as the kernel of the norm map
\[ N_{K/F} : (X_\ast(T) \otimes X(K))_{G(K/F)} \to (X_\ast(T) \otimes X(K))^{G(K/F)}, \]
and in this paper we will extend (1.2) to an isomorphism
\[ (1.3) \quad B(F, T) \xleftarrow{\sim} (X_\ast(T) \otimes X(K))_{G(K/F)}. \]

The part of this paper that treats tori is very much inspired by two sources. The first is Tate’s article [Tat66], which is used heavily throughout the early sections of the paper. The second is Satz 2.3 in [LR87]. In fact Satz 2.3 of Langlands and Rapoport can be viewed as a way of constructing certain special elements in \( B(F, T) \) (though they do not phrase things in this way), at least when \( F \) is a number field. In this paper we pursue such ideas systematically and end up with the isomorphism (1.3).

The second part of the study of \( B(F, G) \) consists in going from tori to general connected reductive groups. As usual [Kot85, Kot97, Kot84, Kot86] this is done in two steps. First one goes from tori to connected reductive groups with simply connected derived group, and then one uses \( z \)-extensions to go from these to general connected reductive groups. In several respects our treatment has been influenced by Borovoi’s work [Bor98].

The rest of this introduction will summarize the main results in the paper, but before doing so I want to express my gratitude to T. Kaletha and M. Rapoport for encouraging me to flesh out and write up the rough ideas I had on this topic, and for sharing with me their ideas about the relation between \( \kappa_G(b) \) and the Newton point of \( b \) (see subsection 11.7 as well as 1.4.3). I would also like to thank them, as well as T. Haines, for some very helpful comments on a preliminary version of this paper.

1.2. Definition of \( B(F, G) \). For any finite Galois extension \( K/F \) we consider the \( G(K/F) \)-module
\[ X(K) := \begin{cases} \mathbb{Z} & \text{if } F \text{ is local}, \\ \mathbb{Z}[V_K]_0 & \text{if } F \text{ is global}, \end{cases} \]
where \( \mathbb{Z}[V_K] \) is the free abelian group on the set \( V_K \) of places of \( K \), and \( \mathbb{Z}[V_K]_0 \) is the kernel of the homomorphism \( \mathbb{Z}[V_K] \to \mathbb{Z} \) defined by \( \sum_{v \in V_K} n_v v \mapsto \sum_{v \in V_K} n_v \). We define \( \mathbb{D}_{K/F} \) to be the \( F \)-group of multiplicative type whose character group is \( X(K) \). When \( F \) is local, \( \mathbb{D}_{K/F} = \mathbb{G}_m \), and when \( F \) is global, \( \mathbb{D}_{K/F} \) is an interesting protorus over \( F \).

The Tate-Nakayama isomorphisms are given by cup product with a canonical element
\[ \alpha(K/F) \in H^2(G(K/F), \mathbb{D}_{K/F}(K)). \]
We choose an extension
\[ 1 \to \mathbb{D}_{K/F}(K) \to \mathcal{E}(K/F) \to G(K/F) \to 1 \]
whose associated cohomology class is $\alpha(K/F)$. This extension is an example of a Galois gerb for $K/F$, as in [LR87].

Using this extension (and the protorus $D_{K/F}$), we define (see subsection 2.4), for each linear algebraic group $G$ over $F$, a pointed set $H^1_{\text{alg}}(\mathcal{E}(K/F), G(K))$. Up to canonical isomorphism, this pointed set is independent of the choice of a specific extension $\mathcal{E}(K/F)$ having $\alpha(K/F)$ as its associated cohomology class. This is due to the vanishing of $H^1(G(K/F), D_{K/F}(K))$. We have to prove many such vanishing theorems; this is the main purpose of Appendix A.

Given a larger finite Galois extension $L \supset K$ there are natural $G(L/F)$-maps $p : X(K) \to X(L)$ and $j : X(L) \to X(K)$. Moreover $j$ induces an isomorphism $\gamma : X(L)_{G(L/K)} \to X(K)$. Using $p$, one forms an inflation map $H^1_{\text{alg}}(\mathcal{E}(K/F), G(K)) \to H^1_{\text{alg}}(\mathcal{E}(L/F), G(L))$.

Using these inflation maps as transition morphisms, we form a pointed set $B(F,G)$ as the colimit of $H^1_{\text{alg}}(\mathcal{E}(K/F), G(K))$, with $K$ varying over the directed set of finite Galois extensions of $F$ in some fixed separable closure $\bar{F}$ of $F$.

Readers familiar with [LR87] (or [SR72]) will understand that, for $F = \mathbb{Q}_p$, the category of representations of the Galois gerb $\mathcal{E}(K/F)$ is equivalent to the category of isocrystals having all slopes in $[K: \mathbb{Q}_p]^{-1}\mathbb{Z} \subset \mathbb{Q}$.

In the same vein, for any $p$-adic field $F$, the pointed set $B(F,G)$ is naturally isomorphic to the pointed set $B(G)$ in [Kot97]. The Tannakian reasoning required to justify this last statement is standard enough to be left to the reader. In any case we start from scratch in this paper, proving everything we need about $B(F,G)$ in the $p$-adic case directly, without appealing to [Kot85, Kot97].

1.3. General discussion of $B(F,G)$ for linear algebraic groups $G$.

1.3.1. For any finite separable extension $E/F$ there is a restriction map

$$B(F,G) \to B(E,G)$$

and a Shapiro isomorphism (see section 12)

$$B(F, R_{E/F}G_0) = B(E, G_0).$$

1.3.2. For any place $u$ of a global field $F$ there is a localization map (see section 7)

$$B(F,G) \to B(F_u,G).$$

1.3.3. There is a Newton map (see subsection 10.4)

$$(1.4) \quad B(F,G) \to [\text{Hom}_F(D_F, G(\bar{F}))/G(\bar{F})]^\Gamma,$$

where $\Gamma := \text{Gal}(\bar{F}/F)$ and $D_F := \text{proj lim}_K D_{K/F}$, the limit being taken over the directed set of finite Galois extensions $K$ of $F$ in $\bar{F}$. The kernel of the Newton map is the image of $H^1(F,G)$ under the inclusion (1.1).

1.3.4. Inside the target of the Newton map is the subset $\text{Hom}_F(D_F, Z(G))$, where $Z(G)$ denotes the center of $G$. The preimage of $\text{Hom}_F(D_F, Z(G))$ under the Newton map is by definition the set $B(F,G)_{\text{bsc}}$ of basic elements in $B(F,G)$. Obviously $B(F,G)_{\text{bsc}}$ contains the image of $H^1(F,G)$ under the inclusion (1.1).
1.3.5. When $G$ is connected, the total localization map

$$B(F, G) \to \prod_{u \in V_F} B(F_u, G)$$

takes values in $\bigoplus_{u \in V_F} B(F_u, G)$, by which we mean the subset of the direct product consisting of families of elements that are trivial at all but finitely many places $u \in V_F$. See Corollary 14.3 for this.

1.3.6. Let

$$1 \to Z \to G' \to G \to 1$$

be a short exact sequence of linear algebraic $F$-groups in which $Z$ is a central torus in $G'$. Then the natural map

$$p : B(F, G') \to B(F, G)$$

is surjective (see Proposition 10.4). Moreover the map (1.5) induces a bijection between $B(F, G')$ and the quotient of $B(F, G)$ by the action of $B(F, G')_{bsc}$ by the action of $B(F, Z)$. These facts are needed whenever we use $z$-extensions to reduce results about $B(F, G)$ for general connected reductive groups to the special case of ones with simply connected derived group.

1.4. Discussion of $B(F, G)$ for connected reductive groups.

1.4.1. Now let $G$ be a connected reductive $F$-group. Then $Z(G)$ is a group of multiplicative type, and we denote by $C(G)$ the biggest torus in $Z(G)$. We write $\Lambda_G$ for Borovoi’s algebraic fundamental group of $G$. Restricted to basic elements, the Newton map yields

$$B(F, G)_{bsc} \to \text{Hom}_F(D_F, C(G)) = (\Lambda_{C(G)} \otimes X^*(D_F))^\Gamma.$$ 

1.4.2. Choose a finite Galois extension $K$ of $F$ in $\bar{F}$ such that $\text{Gal}(\bar{F}/K)$ acts trivially on $\Lambda_G$, and put

$$A(F, G) := (\Lambda_G \otimes X(K))_{G(K/F)},$$

this group being independent of the choice of $K$, up to canonical isomorphism. Then (see section 11.3) there is a natural map

$$\kappa_G : B(F, G) \to A(F, G).$$

1.4.3. Let $b \in B(F, G)_{bsc}$. Then the image of $b$ under the Newton map is determined by $\kappa_G(b)$. More precisely, the square (see Proposition 11.3)

$$B(F, G)_{bsc} \xrightarrow{\kappa_G} A(F, G) \xrightarrow{i} (\Lambda_G \otimes X^*(D_F))^\Gamma$$

commutes, and the map $i$ is injective. Here the vertical map $N$ is the composite

$$(\Lambda_G \otimes X(K))_{G(K/F)} \xrightarrow{N_{K/F}} (\Lambda_G \otimes X(K))^{G(K/F)} \to (\Lambda_G \otimes X^*(D_F))^\Gamma$$

and the bottom arrow $i$ is induced by the inclusion $C(G) \hookrightarrow G$. (We are taking $K$ large enough that $\text{Gal}(\bar{F}/K)$ acts trivially on $\Lambda_G$, and so $A(F, G) = (\Lambda_G \otimes X(K))_{G(K/F)}$.)
For arbitrary \( b \in B(F, G) \) there is a compatibility between \( \kappa_G(b) \) and the Newton point of \( b \). For this see Proposition 13.1 which generalizes part of Theorem 1.15 of Rapoport-Richartz \([RR96]\).

1.4.4. Since the map \( i \) in (1.6) is injective, we may view \((\Lambda_G \otimes X^*(\mathcal{D}_F))^\Gamma\) as a subset of \((\Lambda_G \otimes X^*(\mathcal{D}_F))^\Gamma\) and then form its preimage \( A_0(F, G) \) under \( N \).

1.4.5. Propositions 13.4 and 15.5 assert that the image of the total localization map \( B \) restricts to a bijection (see Proposition 13.1).

\[ B(F, G)_{bsc} \simeq A(F, G). \tag{1.7} \]

1.4.6. When \( F = \mathbb{R} \), the set \( B(F, G)_{bsc} \) can be understood in terms of \( B(F, T) \) for any fundamental maximal \( \mathbb{R} \)-torus \( T \) in \( G \) (see Lemma 13.2).

1.4.7. When \( F = \mathbb{C} \), we have (see subsection 13.4).

\[ B(\mathbb{C}, G)_{bsc} = \Lambda_C(G). \tag{1.8} \]

1.4.8. When \( F \) is global, the square

\[ \begin{array}{ccc}
B(F, G)_{bsc} & \longrightarrow & \prod_{u \in S_{\infty}} B(F_u, G)_{bsc} \\
\kappa_G & \downarrow & \downarrow \\
A(F, G) & \longrightarrow & \prod_{u \in S_{\infty}} A(F_u, G)
\end{array} \]

is cartesian (see Proposition 15.1). So \( B(F, G)_{bsc} \) is a fiber product involving three sets that are easy to understand, and therefore \( B(F, G)_{bsc} \) may itself be regarded as well understood. In the function field case \( S_{\infty} \) is empty, and so \( \kappa_G \) induces an isomorphism \( B(F, G)_{bsc} \simeq A(F, G) \), just as in the nonarchimedean local case.

The picture of \( B(F, G)_{bsc} \) given by the cartesian square (1.9) is further enhanced by Proposition 15.6 which tells us that the image of the total localization map

\[ B(F, G)_{bsc} \rightarrow \bigoplus_{u \in V_F} B(F_u, G)_{bsc} \]

is the kernel of a certain natural map

\[ \bigoplus_{u \in V_F} B(F_u, G)_{bsc} \rightarrow (\Lambda_G)^r. \]

1.5. Tori. For a torus \( T \) over a global field there is more to be said. Let \( K \) be a finite Galois extension of \( F \) that splits \( T \). We write \( M \) for the \( G(K/F) \)-module \( X_\cdot(T) \). Then there is a commutative diagram

\[ \begin{array}{ccc}
H^1(K/F, T(K)) & \longrightarrow & H^1(K/F, T(\mathbb{A}_K)/T(K)) \\
\simeq & \uparrow & \simeq \\
H^{-1}(K/F, M \otimes X_3) & \longrightarrow & H^{-1}(K/F, M \otimes X_2) & \longrightarrow & H^{-1}(K/F, M \otimes X_1)
\end{array} \]

with exact rows, in which the vertical arrows are Tate-Nakayama isomorphisms. Here \( X_1 = \mathbb{Z} \), \( X_2 = \mathbb{Z}[V_K] \) and \( X_3 = \mathbb{Z}[V_K]_0 \), and so there is a natural short exact sequence

\[ 0 \rightarrow X_3 \rightarrow X_2 \rightarrow X_1 \rightarrow 0 \]
of Galois modules. Earlier in this introduction we wrote $X(K)$ rather than $X$. We are now removing $K$ from the notation, in order to save space, and adding the subscript 3 in order to have uniform notation in the diagram.

In section 6 we enlarge all the groups in the diagram, obtaining

$$
\begin{align*}
B_3(F, T) & \longrightarrow B_2(F, T) & \longrightarrow B_1(F, T) & \longrightarrow 0 \\
\cong & & \cong & \\
(M \otimes X_3)_{G(K/F)} & \longrightarrow (M \otimes X_2)_{G(K/F)} & \longrightarrow (M \otimes X_1)_{G(K/F)} & \longrightarrow 0
\end{align*}
$$

in which again the rows are exact and the vertical maps are isomorphisms. Here

- $B_3(F, T) = H^1_{\text{alg}}(\mathcal{E}_3(K/F), T(K))$,

- $B_2(F, T) = H^1_{\text{alg}}(\mathcal{E}_2(K/F), T(\mathfrak{a}_K))$,

- $B_1(F, T) = H^1_{\text{alg}}(\mathcal{E}_1(K/F), T(\mathfrak{a}_K)/T(K))$,

all three being independent of the choice of $K$, up to canonical isomorphism.

The group $B_3(F, T)$ was denoted simply by $B(F, T)$ previously in this introduction. The group $B_2(F, T)$ is canonically isomorphic to the direct sum of all the local groups $B(F_{\sigma}, T)$. The group $B_1(F, T)$ is more interesting. It is defined using algebraic 1-cocycles of the group $\mathcal{E}_1(K/F)$ in $T(\mathfrak{a}_K)/T(K)$, and $\mathcal{E}_1(K/F)$ is the usual Weil group associated to $K/F$. Since $X_1 = \mathbb{Z}$, the rightmost vertical isomorphism is telling us that $B_1(F, T)$ is canonically isomorphic to the Galois coinvariants on $X_1(T)$, in perfect analogy to what happens in the local case. So these Galois coinvariants measure the failure of the total localization map for $T$ to be surjective. This is just a special case of Proposition 1.8.5 but with the new feature that these Galois coinvariants can also be interpreted as a group $B_1(F, T)$ built using a suitable notion of algebraic 1-cocycles for global Weil groups. For all this see section 6 and the sections preceding it. That all three groups $B_i(F, T)$ are independent of the choice of $K$ splitting $T$ is established in section 8.

In subsection 12.14 we define corestriction maps for tori. Let $E/F$ be a finite separable extension of global fields, and let $T$ be an $F$-torus. Then for $i = 1, 2, 3$ we define a corestriction map $\text{Cor} : B_i(E, T) \to B_i(F, T)$. Now let $K/E$ be a finite extension such that $K/F$ is Galois and $T$ is split by $K$. Put $Y_i(K) := X_i(T) \otimes X_i(K)$. Then, for $i = 1, 2, 3$ there is a commutative diagram

$$
\begin{align*}
Y_i(K)_{G(K/E)} & \overset{\cong}{\longrightarrow} B_i(E, T) & \longrightarrow & Y_i(K)^{G(K/E)} \\
\downarrow & & \downarrow \text{Cor} & \downarrow \\
Y_i(K)_{G(K/F)} & \overset{\cong}{\longrightarrow} B_i(F, T) & \longrightarrow & Y_i(K)^{G(K/F)}
\end{align*}
$$

The left vertical arrow is induced by the identity map on $Y_i(K)$. The middle vertical arrow is corestriction for $E/F$. The right vertical arrow is given by $y \mapsto \sum_{\sigma \in G(K/F)/G(K/E)} \sigma(y)$. Similarly, for $i = 1, 2, 3$ there is a commutative diagram

$$
\begin{align*}
Y_i(K)_{G(K/E)} & \overset{\cong}{\longrightarrow} B_i(E, T) & \longrightarrow & Y_i(K)^{G(K/E)} \\
\uparrow & & \uparrow \text{Res} & \uparrow \\
Y_i(K)_{G(K/F)} & \overset{\cong}{\longrightarrow} B_i(F, T) & \longrightarrow & Y_i(K)^{G(K/F)}
\end{align*}
$$
The left vertical arrow is given by \( y \mapsto \sum_{\sigma \in G(K/E) \setminus G(K/F)} \sigma(y) \). The middle vertical arrow is restriction for \( E/F \). The right vertical arrow is induced by the identity map on \( Y_i(K) \). There are two analogous commutative diagrams in the local case (see Lemma 12.12).

1.6. Comments on notation. In this introduction we have consistently used \( G(K/F) \) to denote the Galois group of \( K/F \), and \( G \) to denote a linear algebraic \( F \)-group. In the body of the text we often, but not always, follow the same conventions. In parts of the text in which an abstract finite group is being considered, it is usually denoted by \( G \). In parts where a single Galois extension \( K/F \) is in play, and a general linear algebraic group is not, we sometimes abbreviate \( G(K/F) \) to \( G \). Such conventions are spelled out at the beginning of sections or subsections, as appropriate.

For any global field \( F \) we denote by \( V_F \) the set of places of \( F \). When \( K/F \) is an extension of global fields, we typically denote places of \( K \) by \( v \), and places of \( F \) by \( u \). When \( v \in V_K \) lies over \( u \in V_F \), the local Galois group \( G(K_v/F_u) \) can be identified with the stabilizer \( G_v \) of \( v \) in \( G = G(K/F) \). When using the abbreviation \( G = G(K/F) \), we often employ the notation \( G_v \) rather than \( G(K_v/F_u) \).

We consistently write \( \overline{F} \) for a separable closure of a given field \( F \), and \( \Gamma \) for the absolute Galois group \( \text{Gal}(\overline{F}/F) \).

2. The set \( H_{\text{alg}}^1(\mathcal{E}, G(K)) \)

2.1. Goal of this section. Langlands-Rapoport [LR87] found a convenient way to make concrete the notion of a gerb over a field \( F \). In this section we begin by reviewing their definition of Galois gerb, but using slightly different conventions:

- We do not require that our base field \( F \) have characteristic 0.
- We only work with Galois gerbs \( \mathcal{E} \) for finite Galois extensions \( K/F \).
- We require that \( \mathcal{E} \) be bound by a group \( D \) of multiplicative type over \( F \).
- We do not assume that \( D \) is of finite type over \( F \).

Once the definition of Galois gerb \( \mathcal{E} \) has been reviewed, we introduce, for any linear algebraic group \( G \) over \( F \), the set \( H_{\text{alg}}^1(\mathcal{E}, G(K)) \) of equivalence classes of algebraic 1-cocycles. We then go on to develop some basic constructions involving \( H_{\text{alg}}^1(\mathcal{E}, G(K)) \).

2.2. Review of Galois gerbs \( \mathcal{E} \) for \( K/F \). Let \( K \) be a finite Galois extension of \( F \). We write \( G(K/F) \) for the Galois group of \( K/F \). Let \( X \) be a \( G(K/F) \)-module, and let \( D \) denote the group of multiplicative type having \( X \) as its module of characters.

(Over \( K \) the ring of regular functions on \( D \) is the group algebra \( K[X] \), and over \( F \) it is \( K[X]^{G(K/F)} \).) We are going to consider gerbs bound by \( D \).

For the purposes of this note, a Galois gerb for \( K/F \), bound by \( D \), is an extension of groups

\[
1 \to D(K) \to E \to G(K/F) \to 1.
\]

(The corresponding Tannakian category over \( F \) is then equipped with a fiber functor over \( K \), but we are not going to pursue the Tannakian point of view.) Associated to such an extension of groups is a class \( \alpha \in H^2(G(K/F), D(K)) \). We will typically denote an element in \( \mathcal{E} \) by the letter \( w \). (Using \( e \) for this purpose might be confusing, because it is often used to denote the identity element in a group. Besides, when
$F$ is a local field, the main Galois gerb of interest is the Weil group $W_{K/F}$, and it is natural to denote its elements by $w$.)

2.3. **Algebraic 1-cocycles.** Let $G$ be a linear algebraic group over $F$ (i.e. a smooth affine group scheme of finite type over $F$). The Galois group $G(K/F)$ acts on $G(K)$. The canonical surjection $E \to G(K/F)$ then yields an action of $E$ on $G(K)$, with the subgroup $D(K)$ acting trivially. So we can consider the set $Z^1(E, G(K))$ of abstract 1-cocycles of $E$ in $G(K)$. Such a 1-cocycle $x$ is a map $w \mapsto x_w$ from $E$ to $G(K)$ satisfying the cocycle condition

$$x_{w_1 w_2} = x_{w_1} w_1(x_{w_2}).$$

We need to take notice of two simple consequences of the cocycle condition. The first is that

- $d \mapsto x_d$ is a homomorphism $\nu_0$ from $D(K)$ to $G(K)$.

There is a natural action of $G(K/F)$ on the set of homomorphisms $\nu_1 : D(K) \to G(K)$, defined by $\sigma(\nu_1)(d) := \sigma(\nu_1(\sigma^{-1}(d)))$. The second simple consequence of the cocycle condition is that

- $\text{Int}(x_w) \circ \sigma(\nu_0) = \nu_0$ whenever $w \in E$ maps to $\sigma \in G(K/F)$.

(For an element $x$ in a group, we denote by $\text{Int}(x)$ the inner automorphism of that group defined by $g \mapsto xgx^{-1}$.)

An algebraic 1-cocycle of $E$ in $G(K)$ is a pair $(\nu, x)$ consisting of

- a homomorphism $\nu : D \to G$ over $K$, and
- an abstract 1-cocycle $x$ of $E$ in $G(K)$,

satisfying the following two compatibilities:

- $x_d = \nu(d)$ for all $d \in D(K)$,
- $\text{Int}(x_w) \circ \sigma(\nu) = \nu$ whenever $w \in E$ maps to $\sigma \in G(K/F)$.

2.4. **The pointed set $H^1_{\text{alg}}(E, G(K))$.** We write $Z^1_{\text{alg}}(E, G(K))$ for the set of algebraic 1-cocycles of $E$ in $G(K)$. There is an obvious action of $G(K)$ on the set of algebraic 1-cocycles: $g \in G(K)$ transforms an algebraic 1-cocycle $(\nu, x)$ into $(\text{Int}(g) \circ \nu, w \mapsto gx_w g^{-1})$. We write $H^1_{\text{alg}}(E, G(K))$ for the pointed set obtained as the quotient of $Z^1_{\text{alg}}(E, G(K))$ by the action of $G(K)$. The basepoint is of course the class of the pair consisting of the trivial homomorphism and the trivial abstract 1-cocycle.

The map $(\nu, x) \mapsto \nu$ induces a well-defined map

$$H^1_{\text{alg}}(E, G(K)) \to \left( \text{Hom}_K(D, G)/\text{Int}(G(K)) \right)^{G(K/F)},$$

which we call a Newton map.

2.5. **The situation when $G$ is a torus.** Suppose that $G$ is a torus $T$. Then the second compatibility in the definition of algebraic 1-cocycle just says that $\nu$ is defined over $F$. Moreover it is easy to check that the commutative square

$$\begin{array}{ccc}
H^1_{\text{alg}}(E, T(K)) & \longrightarrow & \text{Hom}_F(D, T) \\
\downarrow & & \downarrow \\
H^1(E, T(K)) & \longrightarrow & \text{Hom}(D(K), T(K))
\end{array}$$
is cartesian. So, for a torus $T$, we could equally well have defined $H^1_{\text{alg}}(E, T(K))$ as a fiber product. This observation will become relevant later, when we are working with $T(M_K)$ and $T(M_K)/T(K)$ instead of $T(K)$.

2.6. The $F$-group $J_b$. For any $K$-homomorphism $\nu : D \to G$ we denote by $G_\nu$ the $K$-group obtained as the centralizer in $G$ of $\nu$. Let $b = (\nu, x)$ be an algebraic 1-cocycle of $E$ in $G(K)$. Let $\sigma \in G(K/F)$ and choose $w \in E$ such that $w \mapsto \sigma$. Then the restriction of $\text{Int}(x_w)$ is a $K$-isomorphism, call it $f_\sigma$, from $\sigma(G_\nu) = G_\sigma(\nu)$ to $G_\nu$. Moreover $f_\sigma$ is independent of the choice of lifting $w$, and the family of isomorphisms $(f_\sigma)_{\sigma \in G(K/F)}$ is descent data for $K/F$. This descent data produces an $F$-form, call it $J_b$, of the $K$-group $G_\nu$. The action $\sigma J_b$ of $\sigma \in G(K/F)$ on $h \in J_b(K) = G_\nu(K)$ is given by $\sigma J_b(h) = x_w \sigma(h)x_w^{-1}$ (for any lift $w$ of $\sigma$). It follows that the group $J_b(F)$ coincides with the stabilizer of $b$ in $G(K)$.

It is a tautology that the $K$-homomorphism $\nu$ factors through the center of $G_\nu$. Moreover, $\nu$ is defined over $F$ for the $F$-form $J_b$ of $G_\nu$. Therefore we may view $\nu$ as a central $F$-homomorphism $\nu : D \to J_b$.

2.7. Algebraic 1-cocycles with a given first component $\nu$. Let us fix a $K$-homomorphism $\nu : D \to G$. There may or may not be an algebraic 1-cocycle having $\nu$ as its first component, but let us suppose that there does exist such a 1-cocycle $b = (\nu, x)$. It is then easy to see that $j \mapsto (\nu, jx)$ is a bijection from the set of 1-cocycles $j$ of $G(K/F)$ in $J_b(K)$ to the set of algebraic 1-cocycles having $\nu$ as their first component. In this way we obtain a bijection from $H^1(G(K/F), J_b(K))$ to the fiber of $(2.2)$ through the class of $b$.

In the special case that $\nu$ is trivial, we may take $b$ to be trivial as well. Then $J_b = G$ and we obtain a canonical injection

$$H^1(G(K/F), G(K)) \hookrightarrow H^1_{\text{alg}}(E, G(K))$$

whose image consists of classes $b$ whose image under the Newton map is trivial. So $H^1_{\text{alg}}(E, G(K))$ can be viewed as an enlargement of $H^1(G(K/F), G(K))$.

2.8. Changing the band $D$. Now let us consider two Galois gerbs

$$1 \to D_i(K) \to E_i \to G(K/F) \to 1$$

(for $i = 1, 2$) bound by two groups $D_1$, $D_2$ of multiplicative type over $F$ (and coming from $G(K/F)$-modules $X_1$, $X_2$ respectively). Suppose further that we are given a homomorphism $\phi : D_1 \to D_2$ and a homomorphism $\eta : E_1 \to E_2$ making

$$
\begin{array}{ccc}
1 & \longrightarrow & D_1(K) & \longrightarrow & E_1 & \longrightarrow & G(K/F) & \longrightarrow & 1 \\
\downarrow \phi & & \downarrow \eta & & \| & & \| & & \| \\
1 & \longrightarrow & D_2(K) & \longrightarrow & E_2 & \longrightarrow & G(K/F) & \longrightarrow & 1
\end{array}
$$

commute. Then there is a natural map

$$\eta^* : H^1_{\text{alg}}(E_2, G(K)) \to H^1_{\text{alg}}(E_1, G(K))$$

for any $G$, induced by the cocycle-level map sending $(\nu, x)$ to $(\nu \circ \phi, x \circ \eta)$.
2.9. Isomorphisms of Galois gerbs for $K/F$. Let us consider two Galois gerbs for $K/F$, both bound by $D$:

\[
\begin{align*}
1 & \to D(K) \to \mathcal{E}' \to G(K/F) \to 1 \\
1 & \to D(K) \to \mathcal{E} \to G(K/F) \to 1
\end{align*}
\]

An isomorphism from the first to the second is an isomorphism $\eta : \mathcal{E}' \to \mathcal{E}$ making the diagram

\[
\begin{array}{c}
1 \\
\downarrow \\
1
\end{array} \quad \begin{array}{ccc}
D(K) & \to & \mathcal{E}' \\
\| & \quad & \| \\
D(K) & \to & \mathcal{E} \\
\| & \quad & \|
\end{array} \quad \begin{array}{ccc}
\downarrow & \quad & \downarrow \\
G(K/F) & \to & 1 \\
\| & \quad & \|
\end{array}
\]

(2.5)

commute. Such an isomorphism exists if and only if the associated classes $\alpha, \alpha' \in H^2(G(K/F), D(K))$ for $\mathcal{E}, \mathcal{E}'$ are equal.

The map defined in the previous subsection is then an isomorphism

\[
\eta^* : H^1_{alg}(\mathcal{E}, G(K)) \to H^1_{alg}(\mathcal{E}', G(K)).
\]

If we assume that the group $H^1(G(K/F), D(K))$ vanishes, then the isomorphism $\eta^*$ is independent of the choice of $\eta$. It is for this reason that we will need to prove quite a number of such vanishing theorems.

2.10. Changing the Galois extension $K/F$. Consider a Galois gerb

\[
1 \to D(K) \to \mathcal{E} \to G(K/F) \to 1
\]

(2.6)

for $K/F$, and suppose that we are given another finite Galois extension $K'/F'$, as well as embeddings $F' \hookrightarrow F$ and $K \hookrightarrow K'$. We do not assume that the extensions $F'/F$ and $K'/K$ are algebraic, but we do assume that the square

\[
\begin{array}{ccc}
K & \to & K' \\
\uparrow & & \uparrow \\
F & \to & F'
\end{array}
\]

(2.7)

commutes. There is then a canonical homomorphism $\rho : G(K'/F') \to G(K/F)$.

First pulling back (2.6) along $\rho$, and then pushing forward along the inclusion $D(K) \hookrightarrow D(K')$, we obtain the following commutative diagram (with exact rows):

\[
\begin{array}{ccc}
1 & \to & D(K) & \to & \mathcal{E} & \to & G(K/F) & \to & 1 \\
& & \| & & \| & & \| & & \\
1 & \to & D(K) & \to & \mathcal{E}' & \to & G(K'/F') & \to & 1 \\
& & \| & & \| & & \| & & \\
1 & \to & D(K') & \to & \mathcal{E}' & \to & G(K'/F') & \to & 1.
\end{array}
\]

The homomorphism $\mathcal{E}' \to \mathcal{E}'$ is injective. Using it to identify $\mathcal{E}'$ with a subgroup of $\mathcal{E}'$, we then have $\mathcal{E}' = D(K')\mathcal{E}'$ and $D(K') \cap \mathcal{E}' = D(K)$.

For any linear algebraic group $G$ over $F$ there is a natural map

\[
H^1_{alg}(\mathcal{E}, G(K)) \to H^1_{alg}(\mathcal{E}', G(K')),
\]

(2.8)

induced by the cocycle-level map sending $(\nu, x)$ to $(\nu, x')$, where $x'$ is defined as follows. Write $w' \in \mathcal{E}'$ as a product $dw''$, with $d \in D(K')$ and $w'' \in \mathcal{E}'$, and then
define the value of $x'$ on $w'$ to be $\nu(d)x_w$, where $w$ denotes the image of $w''$ under $\mathcal{E}'' \to \mathcal{E}$. (It is easy to see that the product $\nu(d)x_w$ is independent of the choice of decomposition $w' = dw''$.)

We will use (2.8) in the following three situations.

**Example 2.1.** Let $F'$ be a field between $K$ and $F$, and take $K' = K$. Then (2.8) yields a restriction map

$$\text{Res} : \mathcal{H}^1_{\text{alg}}(\mathcal{E}, G(K)) \to \mathcal{H}^1_{\text{alg}}(\mathcal{E}', G(K)).$$

In this case $\mathcal{E}'$ is simply the preimage of $G(K/F')$ under $\mathcal{E} \to G(K/F)$.

**Example 2.2.** Let $K'$ be a finite Galois extension of $F$ such that $K' \supset K$, and take $F' = F$. Then (2.8) yields an inflation map

$$\text{Inf} : \mathcal{H}^1_{\text{alg}}(\mathcal{E}, G(K)) \to \mathcal{H}^1_{\text{alg}}(\mathcal{E}', G(K')).$$

In this situation we often write $\mathcal{E}\text{inf}$ instead of $\mathcal{E}'$.

**Example 2.3.** Suppose that $K/F$ is a finite Galois extension of global fields. Choose a place $u$ of $F$ and a place $v$ of $K$ lying over $u$. Take $F', K'$ to be $F_u, K_v$ respectively. The natural map $G(K_v/F_u) \to G(K/F)$ identifies $G(K_v/F_u)$ with the decomposition group of $v$, and (2.8) yields a localization map

$$\text{Loc} : \mathcal{H}^1_{\text{alg}}(\mathcal{E}, G(K)) \to \mathcal{H}^1_{\text{alg}}(\mathcal{E}', G(K_v)).$$

**2.11. Short exact sequences of linear algebraic groups.** In the remaining subsections of this section we study the behavior of $\mathcal{H}^1_{\text{alg}}(\mathcal{E}, G(K))$ with respect to short exact sequences of linear algebraic groups.

**Lemma 2.4.** Let $1 \to N \xrightarrow{i} G' \xrightarrow{p} G \to 1$ be a short exact sequence of linear algebraic $F$-groups, and assume that $p : G'(K) \to G(K)$ is surjective. Then

$$\mathcal{H}^1_{\text{alg}}(\mathcal{E}, N(K)) \xrightarrow{i} \mathcal{H}^1_{\text{alg}}(\mathcal{E}, G'(K)) \xrightarrow{p} \mathcal{H}^1_{\text{alg}}(\mathcal{E}, G(K))$$

is an exact sequence of pointed sets, i.e. the image of $i$ is equal to the kernel of $p$.

**Proof.** This follows easily from the definitions.

In the next lemma we are going to consider a short exact sequence

$$1 \to Z \xrightarrow{i} G' \xrightarrow{p} G \to 1$$

of linear algebraic $F$-groups with $Z$ central in $G'$. Of course $Z$ is necessarily commutative. In this situation there is a natural action of the abelian group $H^1_{\text{alg}}(\mathcal{E}, Z(K))$ on the set $H^1_{\text{alg}}(\mathcal{E}, G'(K))$: the class of $(\mu, x) \in Z^1_{\text{alg}}(\mathcal{E}, Z(K))$ transforms the class of $(\nu, y) \in Z^1_{\text{alg}}(\mathcal{E}, G'(K))$ into the class of $(\mu \nu, xy) \in Z^1_{\text{alg}}(\mathcal{E}, G'(K))$. It is clear that this action preserves the fibers of the map

$$H^1_{\text{alg}}(\mathcal{E}, G'(K)) \xrightarrow{p} H^1_{\text{alg}}(\mathcal{E}, G(K)).$$

**Lemma 2.5.** If $p : G'(K) \to G(K)$ is surjective, then $H^1_{\text{alg}}(\mathcal{E}, Z(K))$ acts transitively on the fibers of the map (2.12).

**Proof.** Suppose that $b', b'' \in H^1_{\text{alg}}(\mathcal{E}, G'(K))$ have the same image in $H^1_{\text{alg}}(\mathcal{E}, G(K))$. Because $G'(K) \to G(K)$ is surjective, we can choose algebraic $1$-cocycles $(\nu', x')$, $(\nu'', x'')$ in the classes $b', b''$ in such a way that they have the same image in $Z^1_{\text{alg}}(\mathcal{E}, G(K))$. It is then easy to check that there exists a unique element $(\mu, z) \in Z^1_{\text{alg}}(\mathcal{E}, Z(K))$ such that $\mu \nu' = \nu''$ and $zx' = x''$.

□
Remark 2.6. The surjectivity of $p : G'(K) \to G(K)$ is automatic when $Z$ is an $F$-torus split by $K$, because then $H^1(K, Z)$ vanishes by Hilbert’s Theorem 90.

2.12. Lemma on extensions of tori. The following result is the most basic special case of Prop. 7.1.1 in SGA 3, Tome II, Exp. XVII.

Lemma 2.7. Any extension of a torus by a torus is again a torus.

This result will be needed in the next subsection.

2.13. Stronger results under two additional hypotheses. Lemma 2.5 is especially useful when the map (2.12) is surjective, for then we may identify $H^1_{\text{alg}}(E, G(K))$ with the quotient of $H^1_{\text{alg}}(E, G'(K))$ by the natural action of $H^1_{\text{alg}}(E, Z(K))$. We are now going to prove a result of this kind, but only for Galois gerbs $E$ satisfying the following two assumptions (which will hold for all the specific local and global Galois gerbs studied later in this paper).

Assumption 1. The group $D$ is a protorus split by $K$. Equivalently, the group $X^*(D)$ is a $G(K/F)$-module that is torsion-free as abelian group.

Assumption 2. For every short exact sequence $1 \to T_1 \to T_2 \to T_3 \to 1$ of $F$-tori split by $K$ the natural map $H^1_{\text{alg}}(E, T_2(K)) \to H^1_{\text{alg}}(E, T_3(K))$ is surjective.

Proposition 2.8. Suppose that $E$ satisfies the two assumptions above. Let

(2.13) $1 \to Z \xrightarrow{i} G' \xrightarrow{p} G \to 1$

be a short exact sequence of linear algebraic $F$-groups in which $Z$ is a torus that splits over $K$ and is central in $G'$. Then the natural map

(2.14) $H^1_{\text{alg}}(E, G'(K)) \xrightarrow{p} H^1_{\text{alg}}(E, G(K))$

is surjective. Moreover the map (2.14) induces a bijection between $H^1_{\text{alg}}(E, G(K))$ and the quotient of $H^1_{\text{alg}}(E, G'(K))$ by the action of $H^1_{\text{alg}}(E, Z(K))$.

Proof. Let $b \in H^1_{\text{alg}}(E, G(K))$ and choose $(\nu, x) \in Z^1_{\text{alg}}(E, G(K))$ representing $b$. So $\nu \in \text{Hom}_{K}(D, G)$ and $x \in Z^1(E, G(K))$ satisfy

(1) $x_d = d^\nu$ for all $d \in D(K)$, and

(2) $\text{Int}(x_w) \circ \sigma(\nu) = \nu$ for any $\sigma \in G(K/F)$ and any $w \in E$ such that $w \mapsto \sigma$.

Here we are writing $d^\nu$ (instead of $\nu(d)$) for the value of $\nu$ on $d$.

By virtue of Assumption 1 the image of $D$ in $G$ is a split $K$-torus $T$ in $G$. Pulling back the extension (2.13) along the inclusion $T \hookrightarrow G$, we obtain an extension

(2.15) $1 \to Z \to T' \to T \to 1$

and an inclusion $T' \hookrightarrow G'$. By Lemma 2.7, $T'$ is a torus, and since $Z$ and $T$ are both $K$-split, so too is $T'$. Therefore the exact sequence (2.15) splits (non-canonically), from which it follows that the sequence

$$0 \to \text{Hom}_{K}(D, Z) \to \text{Hom}_{K}(D, T') \to \text{Hom}_{K}(D, T) \to 0$$

is exact.

We conclude that there exists $\nu' \in \text{Hom}_{K}(D, T')$ such that $\nu' \mapsto \nu$. We would like to lift our 1-cocycle $x$ to a 1-cocycle $x'$ in $G'(K)$, but there is an obvious obstruction to doing so. The map $p : G'(K) \to G(K)$ is surjective (see Remark 2.6), so we may choose a 1-cochain $x'$ of $E$ in $G'(K)$ such that $x' \mapsto x$. The coboundary of $x'$ is
then a 2-cocycle of $\mathcal{E}$ in $Z(K)$. We can choose $x'$ to be a 1-cocycle if and only if the cohomology class of this 2-cocycle vanishes. Unfortunately the group $H^2(\mathcal{E}, Z(K))$ is hard to work with, so it would be helpful if we could choose $x'$ in such a way that its coboundary is inflated from a 2-cocycle of $G(K/F)$. Implementing this idea will be our next task, but in doing so we will be led to enlarge the group $G'$.

For each $\sigma \in G(K/F)$ we choose $\tilde{\sigma} \in \mathcal{E}$ such that $\tilde{\sigma} \mapsto \sigma$, and then we choose $x'_{\tilde{\sigma}} \in G'(K)$ such that $x'_{\tilde{\sigma}} \mapsto x_{\tilde{\sigma}}$. We claim that $\text{Int}(x'_{\tilde{\sigma}}) \circ \sigma(\nu')$ is independent of the choice of $\tilde{\sigma}$ and $x'_{\tilde{\sigma}}$. Indeed, suppose we replace $\tilde{\sigma}$ by $\sigma'$. Then $\sigma' = \sigma d$ for some $d \in D(K)$, and so $x_{\sigma'} = x_{\sigma}(d') = x_{\sigma}(d)\sigma(\nu')$. Therefore one particular lifting of $x_{\sigma'}$ is $x_{\tilde{\sigma}}(d)(\nu')$, and any such lifting is of the form $x_{\tilde{\sigma}}(d)(\nu')$ for some $z \in Z(K)$. From this it is clear that $\text{Int}(x'_{\tilde{\sigma}}) \circ \sigma(\nu')$ is independent of the choice of $\tilde{\sigma}$ and $x'_{\tilde{\sigma}}$.

Now $\nu'$ is one lifting of $\nu$ to $G'$, and by item (2) above $\text{Int}(x'_{\tilde{\sigma}}) \circ \sigma(\nu')$ is another such lifting. So there exists a unique $\lambda_{\sigma} \in \text{Hom}_K(D, Z)$ such that

$$\text{Int}(x'_{\tilde{\sigma}}) \circ \sigma(\nu') = \nu' + \lambda_{\sigma},$$

and an easy calculation shows that $\lambda$ is a 1-cocycle of $G(K/F)$ in $\text{Hom}_K(D, Z)$.

In order to kill the cohomology class of $\lambda$ we are going to enlarge $G'$. There is a canonical $F$-embedding

$$(2.16) \quad Z \rightarrow Z'',$$

where $Z'' := R_{K/F}(Z)$. (Here we are applying Weil restriction of scalars $R_{K/F}$ to the $K$-torus obtained from $Z$ by extension of scalars from $F$ to $K$.) Pushing out the extension $$(2.13)$$ along the inclusion $$(2.16)$$, we obtain a commutative diagram

$$
\begin{array}{cccccc}
1 & \rightarrow & Z & \rightarrow & G' & \rightarrow & p & G & \rightarrow & 1 \\
& & \downarrow & & \downarrow & & \| & & \| & \\
1 & \rightarrow & Z'' & \rightarrow & G'' & \rightarrow & q & G & \rightarrow & 1.
\end{array}
$$

Of course $G''$ is a central extension of $G$ by the torus $Z''$.

Now the $G(K/F)$-module $\text{Hom}_K(D, Z'') = X^*(D) \otimes X_*(Z'')$ is coinduced from the $\mathbb{Z}$-module $X^*(D) \otimes X_*(Z)$, so $H^1(G(K/F), \text{Hom}_K(D, Z''))$ vanishes, which guarantees that there exists $\mu \in \text{Hom}_K(D, Z'')$ such that $\lambda_{\sigma} = \sigma(\mu) - \mu$. It is then clear that $\nu'' := \nu' - \mu$ is a $K$-homomorphism $D \rightarrow G''$ lifting $\nu$ such that

$$(2.18) \quad \text{Int}(x'_{\tilde{\sigma}}) \circ \sigma(\nu'') = \nu''$$

for all $\sigma \in G(K/F)$.

Next we define a 1-cocycle $x''$ of $\mathcal{E}$ in $G''(K)$ by putting $x''_{\sigma} := d''x'_{\sigma}$. Obviously $x''$ is a lifting of $x$ to $G''(K)$, so there exists a unique 2-cocycle $z$ of $\mathcal{E}$ in $Z''(K)$ such that

$$x''_{w_1 w_2} = z_{w_1, w_2} x''_{w_1 w_2}$$

for all $w_1, w_2 \in \mathcal{E}$. From $$(2.18)$$ it follows easily that $z$ is inflated from a (unique) 2-cocycle (still call it $z$) of $G(K/F)$ in $Z''(K)$. But $H^2(G(K/F), Z''(K))$ vanishes by Shapiro's lemma, and so there exists a 1-cocycle $y$ of $G(K/F)$ in $Z''(K)$ such that $x'' := x'' y_{\sigma}$ (where $\sigma$ is the image of $w$ in $G(K/F)$) is a 1-cocycle of $\mathcal{E}$ in $G''(K)$ such that $x'' \mapsto x$. By construction we have

- $x''_{\sigma} = d''$, 
- $\text{Int}(x''_{\sigma}) \circ \sigma(\nu'') = \nu''$ when $w \mapsto \sigma$, 

B(G) FOR ALL LOCAL AND GLOBAL FIELDS

13
and so \((\nu'', x'')\) is an algebraic 1-cocycle of \(E\) in \(G''(K)\) lifting \((\nu, x)\).

At this point we have constructed an element \(b'' \in H^1_{\text{alg}}(E, G''(K))\) such that \(b'' \rightarrow b\). What we really want is an element \(b' \in H^1_{\text{alg}}(E, G'(K))\) such that \(b' \rightarrow b\). To show that \(b'\) exists we are going to make use of Assumption 2.

It is easy to see that the inclusion \(Z'' \rightarrow G''\) induces a canonical isomorphism \(Z''/Z = G''/G'\). To simplify notation we put \(C := Z''/Z = G''/G'\). Obviously \(C\) is an \(F\)-torus split by \(K\), and there is a short exact sequence
\[
1 \rightarrow Z \rightarrow Z'' \rightarrow C \rightarrow 1.
\]

Moreover (2.17) can be enlarged to a commutative diagram
\[
\begin{array}{cccccc}
1 & \longrightarrow & Z & \overset{i}{\longrightarrow} & G' & \overset{p}{\longrightarrow} & G & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & Z'' & \overset{j}{\longrightarrow} & G'' & \overset{q}{\longrightarrow} & G & \longrightarrow & 1 \\
& & & \downarrow & & \downarrow & & & \\
& & & C & \underset{\cong}{\longrightarrow} & C & & & \\
\end{array}
\]

and this gives rise to another commutative diagram
\[
\begin{array}{cccccccc}
H^1_{\text{alg}}(E, Z(K)) & \overset{i}{\longrightarrow} & H^1_{\text{alg}}(E, G'(K)) & \overset{p}{\longrightarrow} & H^1_{\text{alg}}(E, G(K)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
H^1_{\text{alg}}(E, Z''(K)) & \overset{j}{\longrightarrow} & H^1_{\text{alg}}(E, G''(K)) & \overset{q}{\longrightarrow} & H^1_{\text{alg}}(E, G(K)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
H^1_{\text{alg}}(E, C(K)) & \underset{\cong}{\longrightarrow} & H^1_{\text{alg}}(E, C(K))
\end{array}
\]

We have constructed \(b''\) such that \(q(b'') = b\) and we seek \(b'\) such that \(p(b') = b\). We want to apply Lemma 2.4 to the short exact sequence
\[
1 \rightarrow G' \rightarrow G'' \rightarrow C \rightarrow 1,
\]
so we need to check that \(G''(K) \rightarrow C(K)\) is surjective. For this it suffices to prove that \(Z''(K) \rightarrow C(K)\) is surjective, and this follows from Hilbert’s Theorem 90 and the exactness of (2.19). From Lemma 2.4 we see that it is enough to produce \(b''_1 \in H^1_{\text{alg}}(E, G''(K))\) such that
- \(q(b''_1) = q(b'')\), and
- \(b''_1\) has trivial image in \(H^1_{\text{alg}}(E, C(K))\).

We now apply Assumption 2 to the short exact sequence (2.19) of \(F\)-tori split by \(K\). We conclude that there exists \(b_2 \in H^1_{\text{alg}}(E, Z''(K))\) whose image under
\[
H^1_{\text{alg}}(E, Z''(K)) \rightarrow H^1_{\text{alg}}(E, C(K))
\]
is equal to the image of \(b''\) under
\[
H^1_{\text{alg}}(E, G''(K)) \rightarrow H^1_{\text{alg}}(E, C(K)).
\]
It is then clear that \(b''_1 := b_2^{-1} b''\) does the job (see subsection 2.11 for a discussion of the natural action of \(H^1_{\text{alg}}(E, Z''(K))\) on \(H^1_{\text{alg}}(E, G''(K))\)).
This finishes the proof that (2.14) is surjective. To prove the last statement
of the proposition, we need only invoke Lemma 2.5 which applies by Remark 2.6. □

3. Key result in an abstract setting

Throughout this section \( G \) is a finite group. In our applications it will be a
Galois group. In this section, however, we work in an abstract setting, which will
allow us to prove the key result Lemma 3.5 in a way that brings out its simple,
general nature.

3.1. Notation. Let \( M \) be a \( G \)-module. There are then Tate cohomology groups
\( \hat{H}^r(G, M) \) for all \( r \in \mathbb{Z} \). We are going to simplify notation by writing
\( H^r(G, M) \) instead of \( \hat{H}^r(G, M) \). We write \( M^G \) for the
\( G \)-invariants in \( M \), and \( M^G \) for the
\( G \)-coinvariants of \( M \). We write
\[ N : M \rightarrow M \] for the map \( m \mapsto \sum_{g \in G} gm \). The
map \( N \) gives rise to maps
\[ M \rightarrow M^G \rightarrow M^G \rightarrow M, \]
all of which will
be denoted simply by \( N \). Recall that \( H^0(G, M) \) (resp. \( H^{-1}(G, M) \)) is the cokernel
(resp. kernel) of \( N : M^G \rightarrow M^G \).

Let \( A \) and \( B \) be abelian groups. We write \( \text{Hom}(A, B) \) for the group \( \text{Hom}(A, B) \)
of homomorphisms from \( A \) to \( B \). When \( A, B \) are \( G \)-modules, there is a natural
action of \( G \) on \( \text{Hom}(A, B) \), given by \( (gf)(a) = g(f(g^{-1}a)) \), and \( \text{Hom}(A, B)^G \) co-
incides with the group \( \text{Hom}_G(A, B) \) of \( G \)-maps from \( A \) to \( B \). We write \( A \otimes B \) for
the group \( A \otimes_Z B \). When \( A \) and \( B \) are \( G \)-modules, there is a natural action of \( G \)
on \( A \otimes B \), given by \( g(a \otimes b) = ga \otimes gb \).

3.2. The extension \( E \). We consider an extension
\[ 1 \rightarrow A \rightarrow E \rightarrow G \rightarrow 1 \]
of \( G \) by an abelian group \( A \). As usual, this extension gives rise to a cohomology
class \( \alpha \), which we now review.

There is a unique \( G \)-module structure on \( A \) for which \( ga = waw^{-1} \) for any
\( w \in E \) such that \( w \mapsto g \). Choose a set-theoretic section \( s : G \rightarrow E \) of the surjection
\( E \rightarrow G \). Define a 2-cochain \( a_{\sigma, \tau} \) of \( G \) in \( A \) by the rule
\[ s(\sigma)s(\tau) = a_{\sigma, \tau} = \sigma(\sigma). \]

Then \( a_{\sigma, \tau} \) is a 2-cocycle whose cohomology class is independent of the choice of
section \( s \). Let us denote this cohomology class by \( \alpha \).

The inflation-restriction sequence for a \( G \)-module \( M \) is simpler than usual, be-
because our normal subgroup \( A \) is abelian, and because \( A \) is acting trivially on \( M \).
The sequence boils down to
\[ 0 \rightarrow H^1(G, M) \overset{\text{inf}}{\rightarrow} H^1(E, M) \overset{\text{res}}{\rightarrow} \text{Hom}_G(A, M) \overset{\text{tran}}{\rightarrow} H^2(G, M). \]
The homomorphism at the right end of this sequence is the transgression homo-
morphism. In this simple situation it coincides with the composed map
\[ \text{Hom}_G(A, M) = \text{Hom}(A, M)^G \rightarrow H^0(G, \text{Hom}(A, M)) \rightarrow H^2(G, M), \]
the cup product being formed using the tautological pairing \( A \otimes \text{Hom}(A, M) \rightarrow M \).
3.3. Definition of $H^1_Y(E,M)$. We now consider a triple $(M,Y,\xi)$ consisting of a $G$-module $M$, a $G$-module $Y$, and a $G$-map $\xi : Y \to \text{Hom}(A,M)$. Taking $G$-invariants, we obtain $\xi^G : Y^G \to \text{Hom}(A,M)^G = \text{Hom}_G(A,M)$, and we define $H^1_Y(E,M)$ to be the fiber product of $H^1(E,M)$ and $Y^G$ over $\text{Hom}_G(A,M)$. In other words, we are forming the fiber product square

\[
\begin{array}{ccc}
H^1_Y(E,M) & \xrightarrow{r} & Y^G \\
\pi \downarrow & & \xi^G \downarrow \\
H^1(E,M) & \xrightarrow{\text{res}} & \text{Hom}_G(A,M),
\end{array}
\]

in which $r$ and $\pi$ are the two canonical projections.

3.4. Alternative description of $H^1_Y(E,M)$ using cocycles. It is sometimes convenient to think in terms of 1-cocycles when working with $H^1_Y(E,M)$. For this we now introduce groups $Z^1_Y(E,M)$ and $B^1_Y(E,M)$ of 1-cocycles and 1-coboundaries respectively. By definition, an element in $Z^1_Y(E,M)$ is a pair $(y,m)$ consisting of an element $y \in Y^G$ and a 1-cocycle $m$ of $E$ in $M$ such that the restriction of $m$ to $A$ (which is a $G$-map from $A$ to $M$) coincides with the image of $y$ under $\xi^G : Y^G \to \text{Hom}_G(A,M)$. By definition, $B^1_Y(E,M)$ is the subgroup of elements in $Z^1_Y(E,M)$ consisting of pairs $(0,m)$, where $m$ is a 1-coboundary for the $E$-module $M$; since $A$ acts trivially on $M$, the restriction of $m$ to $A$ is trivial. Writing $[m]$ for the cohomology class of $m$, we see that

\[(y,m) \mapsto (y,[m]) \in Y^G \times_{\text{Hom}_G(A,M)} H^1(E,M)\]

induces an isomorphism from $Z^1_Y(E,M)/B^1_Y(E,M)$ to $H^1_Y(E,M)$.

Remark 3.1. The group denoted by $H^1_{\text{alg}}(E,T(K))$ in subsection 2.5 can be identified with $H^1_Y(E,M)$, where $E = E$, $Y = \text{Hom}_K(D,T)$ and $M = T(K)$. So the results in this section apply to $H^1_{\text{alg}}(E,T(K))$. The advantage of the more general theory being developed now is that it can be used for other groups $M$ such as $T(\mathbb{A}_K)$ and $T(\mathbb{A}_K)/T(K)$.

3.5. Inflation-restriction sequence for $H^1_Y(E,M)$. The fiber square occurs in the middle of a bigger diagram

\[
\begin{array}{cccc}
0 & \longrightarrow & H^1(G,M) & \xrightarrow{i} H^1_Y(E,M) & \xrightarrow{r} Y^G & \xrightarrow{t} H^2(G,M) \\
\| & & \| & & \| & & \\
0 & \longrightarrow & H^1(G,M) & \xrightarrow{\inf} H^1(E,M) & \xrightarrow{\text{res}} \text{Hom}_G(A,M) & \xrightarrow{\text{tran}} H^2(G,M)
\end{array}
\]

that is obtained as follows. The map $i$ is the unique homomorphism such that

- the left square commutes, and
- $ri = 0$.

The map $t$ is the unique one making the right square commute.

We already know that the bottom row is exact. The top row will be referred to as the inflation-restriction sequence for $H^1_Y(E,M)$. It is easily seen to be exact.

3.6. Definition of $c : Y_G \to H^1_Y(E,M)$. Since $A$ has finite index in $E$, there is a corestriction map

\[\text{cor} : \text{Hom}(A,M) = H^1(A,M) \to H^1(E,M).\]
**Lemma 3.2.** There exists a unique map $c_0 : Y \to H^1_Y(E, M)$ such that

- $rc_0$ is equal to $N : Y \to Y^G$, and
- $\pi c_0$ is equal to the composed map $Y \xrightarrow{\xi} \text{Hom}(A, M) \xrightarrow{\text{cor}} H^1(E, M)$.

*Proof.* Because $H^1_Y(E, M)$ is a fiber product, we just need to check that $\text{res} \circ \text{cor} \circ \xi$ coincides with $Y \xrightarrow{N} Y^G \xrightarrow{\xi^G} \text{Hom}_G(A, M)$. This is true, because the composition $\text{Hom}(A, M) \xrightarrow{\text{cor}} H^1(E, M) \xrightarrow{\text{res}} \text{Hom}_G(A, M)$ coincides with the norm map $N : \text{Hom}(A, M) \to \text{Hom}_G(A, M)$ (see Lemma B.1(1)). □

**Lemma 3.3.** The homomorphism $c_0$ in the previous lemma factors through the quotient $Y_G$ of $Y$.

*Proof.* To show that $c_0$ factors through $Y_G$, it is enough to show that both $rc_0$ and $\pi c_0$ do so. In the case of $\pi c_0$, this is because the map $\text{cor}$ factors through the quotient $\text{Hom}(A, M)_G$ (see Lemma B.1(2)). In the case of $rc_0$, it is obvious, because $rc_0 = N$. □

**Definition 3.4.** We define $c : Y_G \to H^1_Y(E, M)$ to be the unique homomorphism such that $c_0$ is equal to the composed map $Y \to Y_G \xrightarrow{c} H^1_Y(E, M)$.

In our next lemma we will see when $c$ is an isomorphism.

3.7. **A key lemma.** There is a tautological pairing $A \otimes \text{Hom}(A, M) \to M$, which, combined with our given map $\xi : Y \to \text{Hom}(A, M)$, yields a pairing $A \otimes Y \to M$, and this in turn induces cup product pairings $H^i(G, A) \otimes H^j(G, Y) \to H^{i+j}(G, M)$. In particular cup product with $\alpha$ gives maps

\[
(3.2) \quad H^i(G, Y) \xrightarrow{\alpha \cdot} H^{i+2}(G, M).
\]

**Lemma 3.5.**

1. The diagram

\[
\begin{array}{ccc}
H^1(G, M) & \xrightarrow{i} & H^1_Y(E, M) \\
\alpha \cdot & \uparrow & c \\
H^{-1}(G, Y) & \xrightarrow{\alpha \cdot} & Y_G
\end{array}
\]

commutes.

2. The homomorphism $c : Y_G \to H^1_Y(E, M)$ is an isomorphism if and only if the map $\alpha \cdot$ is bijective for $i = -1$ and injective for $i = 0$.

*Proof.* (1) is proved by reducing to the case in which $Y$ is $\text{Hom}(A, M)$ and $\xi$ is the identity map. Then one needs to show the equality of two homomorphisms $H^{-1}(G, \text{Hom}(A, M)) \to H^1(G, M)$, one being cup product with $\alpha$, the other being induced by the restriction of $\text{cor}$ to the kernel of $N$ on $\text{Hom}(A, M)$. For this, see Lemma B.1.

(2) is proved by applying the 5-lemma to the diagram

\[
\begin{array}{ccccccc}
0 & \longrightarrow & H^1(G, M) & \xrightarrow{i} & H^1_Y(E, M) & \xrightarrow{r} & Y^G \\
& & \alpha \cdot & \uparrow & c & \uparrow & \alpha \cdot \\
0 & \longrightarrow & H^{-1}(G, Y) & \longrightarrow & Y_G & \xrightarrow{N} & Y^G \\
& & N & \longrightarrow & H^0(G, Y) & \longrightarrow & 0
\end{array}
\]
The commutativity of the right and middle squares is clear from the definitions of the maps involved, and the left square was handled in the first part of this lemma.

3.8. **Naturality with respect to** \((M, Y, \xi)\). In order to form \(H^1_Y(E, M)\) we need two \(G\)-modules \(M, Y\) and a \(G\)-map \(\xi : Y \to \text{Hom}(A, M)\). There is an obvious naturality with respect to \((M, Y, \xi)\). Suppose we are given two such triples \((M_i, Y_i, \xi_i)\) \((i = 1, 2)\).

A morphism from the first triple to the second is a pair \((f, g)\) of homomorphisms \(f : M_1 \to M_2, g : Y_1 \to Y_2\) such that the square

\[
\begin{array}{ccc}
Y_1 & \xrightarrow{\xi_1} & \text{Hom}(A, M_1) \\
g \searrow & & \swarrow f \\
Y_2 & \xrightarrow{\xi_2} & \text{Hom}(A, M_2)
\end{array}
\]

commutes. The right vertical map is \(h \mapsto f \circ h\). It can also be viewed as the map \(H^1_{\text{res}}(E, M_1) \to H^1_{\text{res}}(E, M_2)\) induced by \(f\).

When we have such a morphism \((f, g)\), there is an obvious map

\[\psi : H^1_{Y_1}(E, M_1) \to H^1_{Y_2}(E, M_2)\]

obtained from the vertical arrows in the commutative diagram

\[
\begin{array}{ccc}
Y_1^G & \xrightarrow{\xi_1^G} & \text{Hom}_G(A, M_1) \\
g^G \searrow & & \swarrow f^G \\
Y_2^G & \xrightarrow{\xi_2^G} & \text{Hom}_G(A, M_2)
\end{array}
\]

\[
\begin{array}{ccc}
& & \xleftarrow{\text{res}} \\
H^1_{Y_1}(E, M_1) & \xleftarrow{\text{res}} & H^1_{Y_2}(E, M_2)
\end{array}
\]

**Lemma 3.6.** The diagram

\[
\begin{array}{ccc}
(Y_1)_G & \xrightarrow{c_1} & H^1_{Y_1}(E, M_1) \\
gc \searrow & & \swarrow \psi \\
(Y_2)_G & \xrightarrow{c_2} & H^1_{Y_2}(E, M_2)
\end{array}
\]

commutes.

**Proof.** Unwind the definitions, and then use the naturality of corestriction with respect to \(M_1 \to M_2\).

3.9. **Naturality with respect to** \(E\). There is another kind of naturality, this time with respect to the extension \(E\). Suppose we are given a commutative diagram

\[
\begin{array}{cccccc}
1 & \longrightarrow & A' & \longrightarrow & E' & \longrightarrow & G & \longrightarrow & 1 \\
\downarrow h & & \downarrow \bar{h} & & \; & & \; & & \; \\
1 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1
\end{array}
\]

with exact rows. In other words we are considering a morphism from the extension in the top row to the one in the bottom row. Let \(M\) be a \(G\)-module. We consider a triple \((M, Y, \xi)\) of the kind relevant for \(E\). So \(\xi\) is a \(G\)-map from \(Y\) to \(\text{Hom}(A, M)\). We then obtain a triple \((M, Y, \xi')\) relevant for \(E'\) by setting \(\xi'\) equal to the composed map

\[Y \xrightarrow{\xi} \text{Hom}(A, M) \xrightarrow{h} \text{Hom}(A', M)\].
Moreover, there is an obvious homomorphism
\[ \psi' : H^1_Y(E, M) \to H^1_Y(E', M) \]
obtained from the pullback map \( \tilde{h}^* : H^1(E, M) \to H^1(E', M) \) together with the identity map on \( Y_G \).

**Lemma 3.7.** \( \psi' \) is an isomorphism and the diagram
\[
\begin{array}{ccc}
Y_G & \xrightarrow{c} & H^1_Y(E, M) \\
\downarrow \psi' & & \downarrow \psi' \\
Y_G & \xrightarrow{c'} & H^1_Y(E', M)
\end{array}
\]
commutes.

**Proof.** To see that the square commutes, one uses the functoriality of corestriction with respect to \((E, A)\) (see Lemma [133]). To see that \( \psi' \) is an isomorphism, one applies the 5-lemma to
\[
\begin{array}{cccccc}
0 & \rightarrow & H^1(G, M) & \xrightarrow{i} & H^1_Y(E, M) & \xrightarrow{r} & Y^G \rightarrow & H^2(G, M) \\
\downarrow & & \downarrow \psi' & & \downarrow & & \downarrow & \\
0 & \rightarrow & H^1(G, M) & \xrightarrow{i'} & H^1_Y(E', M) & \xrightarrow{r'} & Y'^G \rightarrow & H^2(G, M)
\end{array}
\]
\( \square \)

**3.10. Naturality with respect to \( G \).** Now we want to consider a form of naturality in which \( G \) is allowed to vary. As usual we start with an extension
\[
(3.5) \quad 1 \to A \to E \to G \to 1
\]
and a triple \((M, Y, \xi)\). We may form \( H^1_Y(E, M) \).

Now suppose we are given a finite group \( G' \), a group homomorphism \( \rho : G' \to G \), a \( G' \)-module \( A' \), and a \( G' \)-module map \( h : A \to A' \). First pulling back \((3.5)\) along \( \rho \) and then pushing forward along \( h \), we obtain a commutative diagram
\[
\begin{array}{cccccc}
1 & \rightarrow & A & \rightarrow & E & \rightarrow & G & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
1 & \rightarrow & A & \rightarrow & E'' & \rightarrow & G' & \rightarrow & 1
\end{array}
\]
(3.6)

with exact rows.

Consider a triple \((Y', M', \xi')\) of the kind relevant to the extension \( E' \) in the bottom row of our commutative diagram. We are going to define a natural homomorphism
\[
(3.7) \quad \Phi(f,g,h) : H^1_Y(E, M) \to H^1_{Y'}(E', M')
\]
that depends on the map \( h : A \to A' \) we already chose as well as on \( G' \)-module maps \( f : M \to M' \), \( g : Y \to Y' \) that we choose now. The map \( \Phi(f,g,h) \) is defined
only when the diagram
\[
\begin{array}{ccc}
Y & \xrightarrow{\xi} & \text{Hom}(A, M) \\
\| & f & \downarrow \\
Y & \text{Hom}(A, M') & \\
\downarrow h & & \uparrow \\
Y' & \xrightarrow{\xi'} & \text{Hom}(A', M')
\end{array}
\]
(3.8)
commutes.

The map \(\Phi(f, g, h)\) is induced by the cocycle level map \((\nu, x) \mapsto (g(\nu), x')\), where \(x'\) is the unique 1-cocycle of \(E'\) in \(M'\) such that
- the restriction of \(x'\) to \(A'\) is equal to the map \(\xi'(g(\nu)) : A' \to M'\), and
- the pullback of \(x'\) to \(E''\) (via \(E'' \to E'\)) is equal to the 1-cocycle obtained by applying \(f\) to the pullback of \(x\) to \(E''\) (via \(E'' \to E\)).

**Example 3.8.** When \(G'\) is a subgroup, \(\rho\) is the inclusion, and \((M', Y', \xi') = (M, Y, \xi)\), we obtain a restriction map
\[
\text{Res} : H^1_Y(E, M) \to H^1_Y(E', M),
\]
where \(E'\) is the preimage of \(G'\) in \(E\).

**Example 3.9.** When \(\rho\) is surjective, \(\Phi(f, g, h)\) is a very general kind of inflation map. Particular examples will arise in a later section on inflation in the context of local and global Tate-Nakayama triples.

4. **Abstract Tate-Nakayama Triples for a Finite Group**

We again consider a finite group \(G\). As we will now see, groups \(H^1_Y(E, M)\) of the type studied in the previous section arise naturally in any setting in which one has Tate-Nakayama isomorphisms. In fact there are four such settings, one local and three global. So, in order to avoid much tiresome repetition, we need an axiomatic version of Tate-Nakayama theory.

4.1. **Definition of Tate-Nakayama Triples.** Let \(X, A\) be \(G\)-modules, and let \(\alpha \in H^2(G, \text{Hom}(X, A))\). We say that \((X, A, \alpha)\) is a weak Tate-Nakayama triple for \(G\) if the following condition holds for every subgroup \(G'\) of \(G\):
- For all \(r \in \mathbb{Z}\) cup product with \(\text{Res}_{G/G'}(\alpha)\) induces isomorphisms
  \[
  H^r(G', X) \to H^{r+2}(G', A).
  \]

We say that \((X, A, \alpha)\) is rigid if
- \(H^1(G', \text{Hom}(X, A))\) is trivial for every subgroup \(G'\) of \(G\).

Finally, a Tate-Nakayama triple is a weak Tate-Nakayama triple that is also rigid.

For any weak Tate-Nakayama triple it is result of Nakayama \[\text{Nak57}\] (see \[\text{Ser68}\] p. 156] for a textbook reference) that cup product with \(\alpha\) induces isomorphisms
\[
H^r(G, M \otimes X) \to H^{r+2}(G, M \otimes A)
\]
for all \(r \in \mathbb{Z}\) and every \(G\)-module \(M\) that is torsion-free as abelian group.

In sections [5] and [6] we will review the standard examples of Tate-Nakayama triples. For the moment our goal is merely to explain how the theory in the last
section applies to Tate-Nakayama triples. Weak Tate-Nakayama triples will make an appearance only in Appendix A where we develop tools to show that certain weak Tate-Nakayama triples are rigid.

4.2. **The extension** \( \mathcal{E} \). Given a Tate-Nakayama triple \( (X, A, \alpha) \) for \( G \), we choose an extension

\[
1 \to \text{Hom}(X, A) \to \mathcal{E} \to G \to 1
\]

whose associated class in \( H^2(G, \text{Hom}(X, A)) \) is equal to \( \alpha \). Because our triple is assumed to be rigid, the group \( H^1(G, \text{Hom}(X, A)) \) vanishes, and so every automorphism of our extension (by which we mean an automorphism \( \theta \) of \( \mathcal{E} \) making

\[
\begin{array}{cccccc}
1 & \longrightarrow & \text{Hom}(X, A) & \longrightarrow & \mathcal{E} & \longrightarrow & G & \longrightarrow & 1 \\
& & \parallel & & \theta & & \parallel & & \\
1 & \longrightarrow & \text{Hom}(X, A) & \longrightarrow & \mathcal{E} & \longrightarrow & G & \longrightarrow & 1
\end{array}
\]

commute) is the inner automorphism \( \text{Int}(x) \) coming from some element \( x \) in the subgroup \( \text{Hom}(X, A) \) of \( \mathcal{E} \). Such automorphisms are harmless for our purposes, and so \( \mathcal{E} \) is canonical enough. (The situation is just like that for the Weil group.)

4.3. **Extending** \( H^{-1}(G, M \otimes X) \simeq H^1(G, M \otimes A) \) to \( (M \otimes X)_G \simeq H^1_{\text{alg}}(\mathcal{E}, M \otimes A) \).

Let \( (X, A, \alpha) \) be a Tate-Nakayama triple for \( G \), and choose \( \mathcal{E} \) as above. Let \( M \) be a \( G \)-module. There is a tautological pairing \( X \otimes \text{Hom}(X, A) \to A \). Tensoring this with \( M \), we obtain

\[
M \otimes X \otimes \text{Hom}(X, A) \to M \otimes A,
\]

adjoint to which is a homomorphism

\[
\xi : M \otimes X \to \text{Hom}(\text{Hom}(X, A), M \otimes A).
\]

Applying the discussion in subsection 3.3 to \( \mathcal{E} \) and the triple \( (M \otimes A, M \otimes X, \xi) \), we may form the group \( H^1_Y(\mathcal{E}, M \otimes A) \) with \( Y := M \otimes X \). The rigidity of \( (X, A, \alpha) \) ensures that this group is independent of the choice of \( \mathcal{E} \), up to canonical isomorphism.

We no longer need to choose \( Y \) and \( \xi \) (as was the case in subsection 3.3); they are determined by \( A, M \) and \( X \). For this reason it is now less useful to retain \( Y \) in the notation, and we will often write \( H^1_Y(\mathcal{E}, M \otimes A) \) in place of \( H^1_Y(\mathcal{E}, M \otimes X) \).

Our next result makes use of the canonical homomorphism

\[
c : Y_G \to H^1_Y(\mathcal{E}, M \otimes A)
\]

of Definition 3.4.

**Lemma 4.1.** Let \( M \) be any \( G \)-module that is torsion-free as abelian group. The canonical homomorphism \( c \) is then an isomorphism. Moreover, the diagram

\[
\begin{array}{ccc}
(M \otimes X)_G & \longrightarrow & H^1_{\text{alg}}(\mathcal{E}, M \otimes A) \\
\alpha \downarrow & & \downarrow \\
H^{-1}(G, M \otimes X) & \longrightarrow & H^1(G, M \otimes A)
\end{array}
\]

commutes, the two vertical maps being the canonical injections.

**Proof.** In view of the Nakayama isomorphism 4.1, this follows from Lemma 3.5.

\[ \square \]
4.4. Restriction for a subgroup $G'$ of $G$. Let $(X, A, \alpha)$ be a Tate-Nakayama triple for $G$, and let $E$ be an extension of $G$ by $\text{Hom}(X, A)$ with corresponding cohomology class $\alpha$. Let $G'$ be a subgroup of $G$, and put $\alpha' := \text{Res}_{G/G'}(\alpha) \in H^2(G', \text{Hom}(X, A))$. It is evident that $(X, A, \alpha')$ is a Tate-Nakayama triple for $G'$. For every $G$-module $M$ there is a restriction map

$$ \text{Res} : H^1_{\text{alg}}(E, M \otimes A) \to H^1_{\text{alg}}(E', M \otimes A), $$

where $E'$ denotes the preimage of $G'$ under $E \to G$ (see Example 3.8).

Define a homomorphism $M \otimes X \to (M \otimes X)_{G'}$ by sending $\mu \in M \otimes X$ to $\sum_{g \in G' \setminus G} g \mu$. This map factors through the coinvariants of $G$ on $M \otimes X$, yielding a natural homomorphism

$$(M \otimes X)_G \to (M \otimes X)_{G'}$$

Lemma 4.2. The square

$$
\begin{array}{ccc}
(M \otimes X)_G & \xrightarrow{\text{Res}} & (M \otimes X)_{G'} \\
\downarrow c & & \downarrow c \\
H^1_{\text{alg}}(E, M \otimes A) & \xrightarrow{\text{Res}} & H^1_{\text{alg}}(E', M \otimes A)
\end{array}
$$

commutes.

Proof. We enlarge the square to a diagram

$$
\begin{array}{ccc}
(M \otimes X)_G & \xrightarrow{\text{Res}} & (M \otimes X)_{G'} \\
\downarrow c & & \downarrow c \\
H^1_{\text{alg}}(E, M \otimes A) & \xrightarrow{\text{Res}} & H^1_{\text{alg}}(E', M \otimes A) \\
\downarrow r & & \downarrow r' \\
(M \otimes X)^G & \longrightarrow & (M \otimes X)^{G'}
\end{array}
$$

the bottom horizontal arrow being the obvious inclusion. The bottom square and outer rectangle are easily seen to commute. Consequently the top square commutes whenever $r'$ is injective. This is the case when $M$ is free of finite rank as $\mathbb{Z}[G]$-module, because then the kernel $H^1(G', M \otimes A)$ of $r'$ obviously vanishes.

Since the square we seek to prove commutative is functorial in $M'$, to prove its commutativity for a given $M$, it is sufficient to prove its commutativity for any $M'$ that dominates $M$ in the sense that there exists a $G$-map $M' \to M$ for which $(M' \otimes X)_G \to (M \otimes X)_G$ is surjective. This can obviously be achieved by using a suitable $M'$ that is free of finite rank as $\mathbb{Z}[G]$-module. \hfill \square

4.5. Naturality. We are now going to discuss the naturality of the construction $(X, A, \alpha) \mapsto H^1_{\text{alg}}(E, M \otimes A)$, and in order to do so we need a suitable notion of morphism.

Definition 4.3. A morphism $(X_2, A_2, \alpha_2) \to (X_1, A_1, \alpha_1)$ of Tate-Nakayama triples is a pair $(b, a)$ of $G$-maps $b : X_2 \to X_1$, $a : A_2 \to A_1$ such that $a(\alpha_2) = b(\alpha_1) \in H^2(G, \text{Hom}(X_2, A_1))$. 

Given such a morphism \((b, a)\), we are now going to define a natural map

\[
(4.7) \quad \rho : H^1_{\text{alg}}(E_2, M \otimes A_2) \to H^1_{\text{alg}}(E_1, M \otimes A_1)
\]

for any \(G\)-module \(M\) that is torsion-free as abelian group. Here of course we have chosen extensions

\[
1 \to \text{Hom}(X_i, A_i) \to E_i \to G \to 1 \quad (i = 1, 2)
\]

with associated cohomology classes \(\alpha_i \in H^2(G, \text{Hom}(X_i, A_i))\). In order to define the map \((4.7)\) we begin by choosing an extension

\[
1 \to \text{Hom}(X_2, A_1) \to F \to G \to 1
\]

with associated cohomology class \(a(\alpha_2) = b(\alpha_1)\), and then choosing homomorphisms \(\hat{a}, \hat{b}\) making the diagram

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & \text{Hom}(X_2, A_2) & \longrightarrow & E_2 & \longrightarrow & G & \longrightarrow & 1 \\
\downarrow a & & \downarrow \hat{a} & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \text{Hom}(X_2, A_1) & \longrightarrow & F & \longrightarrow & G & \longrightarrow & 1 \\
\uparrow b & & \uparrow \hat{b} & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \text{Hom}(X_1, A_1) & \longrightarrow & E_1 & \longrightarrow & G & \longrightarrow & 1
\end{array}
\]

commute. We have not assumed that \(H^1(G, \text{Hom}(X_2, A_1))\) vanishes, so \(\hat{a}, \hat{b}\) are far from unique. Fortunately, however, the map \((4.7)\) we are going to define will turn out to be independent of the choice of \(F, \hat{a}, \hat{b}\).

There is commutative diagram

\[
\begin{array}{ccccccccc}
X_2 \otimes \text{Hom}(X_2, A_2) & \longrightarrow & A_2 \\
\uparrow \text{id} & & \uparrow a & & \uparrow & & \uparrow & & \\
X_2 \otimes \text{Hom}(X_2, A_2) & \longrightarrow & A_1 \\
\downarrow \text{id} \otimes a & & \downarrow & & \downarrow & & \downarrow & & \\
X_2 \otimes \text{Hom}(X_2, A_1) & \longrightarrow & A_1 \\
\downarrow \text{id} \otimes b & & \downarrow & & \downarrow & & \downarrow & & \\
X_2 \otimes \text{Hom}(X_1, A_1) & \longrightarrow & A_1 \\
\downarrow b \otimes \text{id} & & \downarrow & & \downarrow & & \downarrow & & \\
X_1 \otimes \text{Hom}(X_1, A_1) & \longrightarrow & A_1
\end{array}
\]

in which the top, middle and bottom pairings are the tautological ones, and the other two are the unique ones making the diagram commute. Put \(Y_i := M \otimes X_i\). Tensoring the entire diagram with \(M\), and then applying adjointness of \(\otimes\) and \(\text{Hom}\).
we obtain another commutative diagram

\[
\begin{array}{c}
Y_2 \longrightarrow \text{Hom}(\text{Hom}(X_2, A_2), M \otimes A_2) \\
\| \\
Y_2 \longrightarrow \text{Hom}(\text{Hom}(X_2, A_2), M \otimes A_1) \\
\| \\
Y_2 \longrightarrow \text{Hom}(\text{Hom}(X_2, A_1), M \otimes A_1) \\
\| \\
Y_2 \longrightarrow \text{Hom}(\text{Hom}(X_1, A_1), M \otimes A_1) \\
\end{array}
\]

\[\text{id}_M \otimes b \downarrow \]

\[
Y_1 \longrightarrow \text{Hom}(\text{Hom}(X_1, A_1), M \otimes A_1)
\]

From Lemmas 3.6 and 3.7 we obtain a commutative diagram

\[
\begin{array}{c}
(Y_2)_G \longrightarrow H^1_Y(E_2, M \otimes A_2) \\
\| \\
(Y_2)_G \longrightarrow H^1_Y(E_2, M \otimes A_1) \\
\| \\
(Y_2)_G \longrightarrow H^1_Y(F, M \otimes A_1) \\
\| \\
\end{array}
\]

\[\text{id}_M \otimes b \downarrow \]

\[
(Y_1)_G \longrightarrow H^1_Y(E_1, M \otimes A_1),
\]

where the right vertical arrows are as follows (starting from the top):

1. \(a : H^1_Y(E_2, M \otimes A_2) \rightarrow H^1_Y(E_2, M \otimes A_1)\), induced by the \(G\)-map \(\text{id}_M \otimes a : M \otimes A_2 \rightarrow M \otimes A_1\).
2. \(\tilde{a}^* : H^1_Y(F, M \otimes A_1) \rightarrow H^1_Y(E_2, M \otimes A_1)\),
3. \(\tilde{b}^* : H^1_Y(F, M \otimes A_1) \rightarrow H^1_Y(E_1, M \otimes A_1)\),
4. \(b : H^1_Y(E_1, M \otimes A_1) \rightarrow H^1_Y(E_1, M \otimes A_1)\), induced by \(\text{id}_M \otimes b : Y_2 \rightarrow Y_1\).

Remembering that \(H^1_{\text{alg}}(E_i, M \otimes A_i)\) stands for \(H^1_{Y_i}(E_i, M \otimes A_i)\), we see that the composition of the right vertical arrows yields a homomorphism

\[
\rho : H^1_{\text{alg}}(E_2, M \otimes A_2) \rightarrow H^1_{\text{alg}}(E_1, M \otimes A_1)
\]

making the square

\[
\begin{array}{c}
(Y_2)_G \longrightarrow H^1_{\text{alg}}(E_2, M \otimes A_2) \\
\| \rho \downarrow \\
(Y_1)_G \longrightarrow H^1_{\text{alg}}(E_1, M \otimes A_1)
\end{array}
\]

\[(4.9)
\]

\[(4.10)\]
commute. Since the horizontal maps $c$ are isomorphisms, we conclude that $\rho$ is independent of the choice of $\mathcal{F}$, $\tilde{a}$, $\tilde{b}$, and it is $\rho$ that we take as the map (4.7) we wished to define.

4.6. Preview. The standard situations in which there are Tate-Nakayama isomorphisms are all associated with Tate-Nakayama triples. There is a canonical Tate-Nakayama triple associated with every finite Galois extension of local fields. It will be discussed in section 5. Associated to every finite Galois extension $K/F$ of global fields (and a suitable set $S$ of places of $F$) are three Tate-Nakayama triples, and there are canonical morphisms from the third to second, and from the second to the first. All this, together with a localization map (global to adelic), will be discussed in section 6.

5. Local Tate-Nakayama triples

5.1. Notation. In this section we consider a finite Galois extension $K/F$ of local fields, whose Galois group we denote by $G$. It is part of local class field theory that we get a Tate-Nakayama triple $(X, A, \alpha)$ for $G$ by taking

- $X$ to be $\mathbb{Z}$, with $G$ acting trivially,
- $A$ to be $K^\times$, with the natural $G$-action,
- $\alpha$ to be the fundamental class in $H^2(G, K^\times)$.

Observe that the group $E$ occurring in the extension (4.2) is a Weil group for $K/F$.

5.2. The group $H^1_{alg}(\mathcal{E}, T(K))$. Let $T$ be a torus over $F$ that splits over $K$. Its group $X_* (T)$ of cocharacters is a $G$-module that is finite free as $\mathbb{Z}$-module. Take $M = X_*(T)$ in our abstract theory. We then obtain from Lemma 4.1 the following result, the nonarchimedean case of which gives a slightly different perspective on the results in [Kot97, §8]. In the lemma we write $H^1_{alg}(\mathcal{E}, T(K))$ in place of $H^1_{Y}(\mathcal{E}, T(K))$, where $Y$ is $M \otimes X = X_*(T)$.

**Lemma 5.1.** The diagram

$$
\begin{array}{ccc}
X_*(T)_G & \longrightarrow & H^1_{alg}(\mathcal{E}, T(K)) \\
\uparrow & & \uparrow \\
H^{-1}(G, X_*(T)) & \longrightarrow & H^1(G, T(K))
\end{array}
$$

commutes, the two vertical maps being the canonical injections. Moreover, the two horizontal maps are isomorphisms, the bottom one being one of the Tate-Nakayama isomorphisms.

**Proof.** We just need to notice that $M \otimes A$ works out to $T(K)$. □

6. Global Tate-Nakayama triples

6.1. Notation. For any global field $F$ we write $V_F$ for the set of places of $F$. For any finite extension $E/F$ of global fields there is a natural map $f_{E/F} : V_E \to V_F$ sending a place of $E$ to the unique place of $F$ below it.

In this section $K/F$ is a finite Galois extension of global fields, with Galois group $G$. For any set $S$ of places of $F$ and any finite extension $E/F$, we write $S_E$ for the preimage of $S$ under $f_{E/F}$.

In this section we work with a set $S$ of places of $F$ satisfying the following conditions:
• $S$ contains all archimedean places.
• $S$ contains all finite places that ramify in $K$.
• For every intermediate field $E$ of $K/F$, every ideal class of $E$ contains an ideal with support in $S_E$.

6.2. Three global Tate-Nakayama triples. We need to recall the constructions Tate [Tate66] used to prove global Tate-Nakayama isomorphisms for tori over $F$. We will see that they yield Tate-Nakayama triples $(X_i, A_i, c_i)$ (for $i = 1, 2, 3$) as well as morphisms

$$(X_3, A_3, c_3) \to (X_2, A_2, c_2) \to (X_1, A_1, c_1).$$

Tate considers two short exact sequences of $G$-modules. The first is

$$(A) \quad 1 \to A_3 \xrightarrow{a} A_2 \xrightarrow{a} A_1 \to 1,$$

where

• $A_3$ is the group of $S_K$-units in $K^\times$, that is, elements of $K^\times$ that are units at all places not in $S_K$.
• $A_2$ is the group of $S_K$-ideles of $K$, that is, ideles whose $v$-component is a unit for each place $v$ not in $S_K$.
• $A_1 = A_2/A_3$ is the group of $S_K$-idele classes of $K$.

Our third assumption on $S$ tells us that the inclusion of $A_2$ in $\mathbb{A}_K^\times$ induces an isomorphism $A_2/A_3 \simeq \mathbb{A}_K^\times/K^\times$, so $A_1$ is in fact the group of idele classes of $K$.

Lemma 6.1. Let $G'$ be any subgroup of $G$. Then $H^1(G', A_1)$, $H^1(G', A_2)$ and $H^1(G', A_3)$ vanish, and the sequence

$$(6.1) \quad 1 \to A_3^{G'} \to A_2^{G'} \to A_1^{G'} \to 1$$

is short exact.

Proof. Let $E$ be the fixed field of $G'$ on $K$. The vanishing of $H^1(G', A_1)$ is part of global class field theory for $K/E$. The vanishing of $H^1(G', A_2)$ follows from Hilbert’s theorem 90 for local fields, together with the vanishing of the first Galois cohomology of $G_m$ and $\mathbb{G}_a$ over finite fields. See Tate’s article for more details. Since $H^1(G', A_2)$ vanishes, there is an exact sequence

$$(6.2) \quad 1 \to A_3^{G'} \to A_2^{G'} \to A_1^{G'} \to H^1(G', A_3) \to 1.$$ 

Now $A_1^{G'}$ is the idele class group for $E$, and $A_2^{G'}$ is the group of $S_E$-ideles of $E$. So our third assumption on $S$ tells us that $A_2^{G'} \to A_1^{G'}$ is surjective, and hence that $H^1(G', A_3)$ vanishes. □

The second short exact sequence considered by Tate is

$$(X) \quad 0 \to X_1 \xrightarrow{b'} X_2 \xrightarrow{b} X_1 \to 0,$$

where

• $X_1$ is the group of integers, with $G$ acting trivially.
• $X_2$ is the free abelian group on the set $S_K$, the $G$-action on $X_2$ being induced by the natural $G$-action on $S_K$. (Thus $X_2 = \mathbb{Z}[S_K]$).
• The homomorphism $b$ maps $\sum_{v \in S_K} n_v v \in X_2$ to $\sum_{v \in S_K} n_v$.
• $X_3$ is the kernel of $b$, and $b'$ the canonical inclusion. (Thus $X_3 = \mathbb{Z}[S_K]$).
Next Tate constructs a commutative diagram
\[(6.3) \quad \cdots \longrightarrow H^r(G, X_3) \longrightarrow H^r(G, X_2) \longrightarrow H^r(G, X_1) \longrightarrow \cdots \]
\[\alpha_3^- \downarrow \quad \alpha_2^- \downarrow \quad \alpha_1^- \downarrow \quad \cdots \longrightarrow H^{r+2}(G, A_3) \longrightarrow H^{r+2}(G, A_2) \longrightarrow H^{r+2}(G, A_1) \longrightarrow \cdots \]
in which the vertical arrows are isomorphisms given by cup product with certain cohomology classes \(\alpha_i \in H^2(G, \text{Hom}(X_i, A_i))\). The two rows in the commutative diagram are the long exact Tate-cohomology sequences for the short exact sequences (A) and (X).

Global class field theory is encoded in the arrows \(\alpha_i\), and in fact \(\alpha_1\) is nothing but the global fundamental class. Similarly, local class field theory is encoded in the arrows \(\alpha_i^\prime\), and in fact \(\alpha_2\) is built up from the various local fundamental classes.

We need to be more precise about this.

Let \(v \in S_K\). Then \(K_v^\times\) is, in an obvious way, a direct factor of \(A_2\). The canonical injection \(i_v : K_v^\times \hookrightarrow A_2\) and canonical projection \(\pi_v : A_2 \to K_v^\times\) are both \(G_v\)-equivariant, where \(G_v\) denotes the stabilizer in \(G\) of \(v\) (in other words, the decomposition group of \(v\)).

In order to specify \(\alpha_2\), Tate uses the following lemma (see page 714 of his article).

**Lemma 6.2.** For any \(G\)-module \(M\) there is a canonical isomorphism
\[H^r(G, \text{Hom}(X_2, M)) \to \prod_{u \in S} H^r(G_v, M),\]
where, for each place \(u \in S\), we choose a place \(v\) of \(K\) above \(u\), and write \(G_v\) for its decomposition group. The \(u\)-th component of this isomorphism is the composed map
\[(6.4) \quad H^r(G, \text{Hom}(X_2, M)) \xrightarrow{\text{Res}_{G/G_v}} H^r(G_v, \text{Hom}(X_2, M)) \xrightarrow{\text{eval}_v} H^2(G_v, M),\]
where \(\text{eval}_v\) is the \(G_v\)-map sending \(f \in \text{Hom}(X_2, M)\) to its value at the basis element \(v \in X_2\).

To define \(\alpha_2\) Tate applies the lemma with \(M = A_2\) and \(r = 2\). According to that lemma, giving \(\alpha_2\) is the same as giving a family of elements \(\alpha_2(u) \in H^2(G_v, A_2)\), one for each \(u \in S\). Tate takes \(\alpha_2(u)\) to be the image under \(i_v : K_v^\times \hookrightarrow A_2\) of the local fundamental class \(\alpha(K_v/F_u) \in H^2(G_v, K_v^\times)\).

Tate constructs \(\alpha_3\) as follows. First he remarks that in order to produce \(\alpha_1\), \(\alpha_2\), \(\alpha_3\) making diagram (6.3) commute, it would be enough to produce an element \(\alpha \in H^2(G, \text{Hom}(X, A))\) whose image under the \(i\)-th projection \(\pi_i : \text{Hom}(X, A) \to \text{Hom}(X_i, A_i)\) is equal to \(\alpha_i\). Here \(\text{Hom}(X, A)\) denotes the subgroup of \(\text{Hom}(X_3, A_3) \times \text{Hom}(X_2, A_2) \times \text{Hom}(X_1, A_1)\) consisting of all triples \((h_3, h_2, h_1)\) such that
\[X_3 \longrightarrow X_2 \longrightarrow X_1 \quad h_3 \downarrow \quad h_2 \downarrow \quad h_1 \downarrow \quad A_3 \longrightarrow A_2 \longrightarrow A_1\]
commutes.

The following lemma is proved in the course of the discussion on page 716 of Tate’s article.
Lemma 6.3. The diagrams

\[
\begin{array}{c}
\text{Hom}(X, A) \xrightarrow{\pi_1} \text{Hom}(X_1, A_1) \\
\pi_2 \downarrow \quad \quad \quad \downarrow b \\
\text{Hom}(X_2, A_2) \xrightarrow{a} \text{Hom}(X_2, A_1)
\end{array}
\]

and

\[
\begin{array}{c}
\text{Hom}(X, A) \xrightarrow{\pi_1} \text{Hom}(X_1, A_1) \\
\pi_2 \downarrow \\
\text{Hom}(X_2, A_2) \xrightarrow{a} \text{Hom}(X_2, A_1)
\end{array}
\]

are cartesian.

Tate observes that \(a(\alpha_2) = b(\alpha_1)\); this boils down to the statement that, for \(u \in S\) and a place \(v\) of \(K\) above \(u\), the restriction of the global fundamental class \(\alpha_1\) to the subgroup \(G_v\) is the image under

\[
K_v^\times \xrightarrow{i} A_2 \xrightarrow{a} A_1
\]

of the local fundamental class \(\alpha(K_v/F_u)\). From Lemma [6.3] he concludes that there exists unique \(\alpha \in H^2(G, \text{Hom}(X, A))\) such that \(\pi_i(\alpha) = \alpha_i\) for \(i = 1, 2\), and he then defines \(\alpha_3 \in H^2(G, \text{Hom}(X_3, A_3))\) to be \(\pi_3(\alpha)\).

6.3. Proof that the three triples \((X_i, A_i, \alpha_i)\) are Tate-Nakayama triples.

The maps

\[
H^r(G', X_i) \xrightarrow{\text{Res}_{G'/G}(\alpha_i)} H^{r+2}(G', A_i)
\]

are isomorphisms for every subgroup \(G'\) of \(G\) (see Tate’s article). Therefore the triples \((X_i, A_i, \alpha_i)\) are weak Tate-Nakayama triples. Here we used the following result of Tate (see the lemma on page 717 of his article).

Lemma 6.4 (Tate). Let \(G'\) be a subgroup of \(G\), and let \(E\) denote the fixed field of \(G'\) on \(K\). Then the canonical class \(\alpha \in H^2(G, \text{Hom}(X, A))\) restricts to the canonical class \(\alpha' \in H^2(G', \text{Hom}(X, A))\) for \(K/E\). Therefore, for \(i = 1, 2, 3\) the class \(\alpha_i\) for \(K/F\) restricts to the analogous class for \(K/E\).

To verify rigidity of our weak Tate-Nakayama triples we must prove the following lemma.

Lemma 6.5. \(H^1(G', \text{Hom}(X_i, A_i)) = 0\) for \(i = 1, 2, 3\) and every subgroup \(G'\) of \(G\).

Proof. We apply Lemma [A.3]. Our goal is to establish the rigidity of \((X_i, A_i, \alpha_i)\) for \(i = 1, 2, 3\). In the notation of that lemma, \(X_2 = \mathbb{Z}[S_K],\ X_3 = \mathbb{Z}[S_K|_0,\) and of course \(X_1\) is of the form \(\mathbb{Z}[T]\) with \(T\) any set having exactly one element. When \(i = 1, 2\) Lemma [A.3] shows directly that \((X_i, A_i, \alpha_i)\) is rigid. When \(i = 3\), we need to check that \(H^{-1}(G', X_3)\) vanishes for every subgroup \(G'\) of \(G\). In view of the isomorphism \((6.2)\), it is the same to check the vanishing of \(H^1(G', A_3)\), and this was done in Lemma [6.1]. \(\square\)

Remark 6.6. It follows from Lemma [6.4] that \(\text{Cor}_{E/F}(\alpha_i(K/E)) = [E:F]\alpha_i(K/F)\) for \(i = 1, 2, 3\), but we will have no occasion to use this.
6.4. The morphisms $(X_3, A_3, \alpha_3) \to (X_2, A_2, \alpha_2) \to (X_1, A_1, \alpha_1)$. It is evident that

- $(b', a')$ is a morphism from $(X_3, A_3, \alpha_3)$ to $(X_2, A_2, \alpha_2)$, and
- $(b, a)$ is a morphism from $(X_2, A_2, \alpha_2)$ to $(X_1, A_1, \alpha_1)$.

Here the notion of morphism is the one in Definition 3.3.

6.5. Main global result. We now apply the general results in section 4 to the three Tate-Nakayama triples $(X_i, A_i, \alpha_i)$ and the $G$-module $M$ obtained as the cocharacter group of a torus $T$ over $F$ that is split by $K$. We choose an extension $E_i$ corresponding to the class $\alpha_i \in H^2(G, \text{Hom}(X_i, A_i))$. Observe that $E_1$ is a global Weil group for $K/F$. As in section 4, we form the group

$$H^3_{\text{alg}}(E_i, M \otimes A_i) := H^3_{\text{alg}}(E_i, M \otimes A_i)$$

with $Y_i = X_*(T) \otimes X_i$.

Our general results on Tate-Nakayama triples then yield a commutative diagram

$$
\begin{array}{ccc}
(X_*(T) \otimes X_3)_G & \longrightarrow & (X_*(T) \otimes X_2)_G & \longrightarrow & X_*(T)_G \\
\downarrow & & \downarrow & & \downarrow \\
H^3_{\text{alg}}(E_3, M \otimes A_3) & \longrightarrow & H^3_{\text{alg}}(E_2, M \otimes A_2) & \longrightarrow & H^3_{\text{alg}}(E_1, M \otimes A_1)
\end{array}
$$

(6.6)

in which the vertical arrows are the isomorphisms $c$ in Definition 3.3, and the bottom horizontal maps are obtained by naturality (see subsection 4.5) from the morphisms of Tate-Nakayama triples that we discussed in the previous subsection.

6.6. Discussion of the top row in (6.6). The first two groups in the top row of (6.6) can be better understood by remembering that the exact sequence $(X)$ was defined to be

$$0 \to X_3 \to \bigoplus_{v \in S_K} \mathbb{Z} \to \mathbb{Z} \to 0.$$

Tensoring with $X_*(T)$ preserves exactness, and yields

$$0 \to X_*(T) \otimes X_3 \to \bigoplus_{v \in S_K} X_*(T) \to X_*(T) \to 0.$$

Taking coinvariants for $G$ is right exact, so the top row of (6.6) is part of the exact sequence

$$
(X_*(T) \otimes X_3)_G \to \bigoplus_{v \in S} X_*(T)_{G_v} \to X_*(T)_G \to 0,
$$

(6.7)

where $G_v$ is again the decomposition group of a place $v$ of $K$ that lies over $u$.

Lemma 6.7. The kernel of the map at the left end of (6.7) is canonically isomorphic to $\ker[H^1(G, M \otimes A_3) \to H^1(G, M \otimes A_2)]$.

Proof. The vertical isomorphisms in the commutative diagram (6.6) yield a canonical isomorphism from $\ker[(X_*(T) \otimes X_3)_G \to \bigoplus_u X_*(T)_{G_v}]$ to $\ker[H^1_{\text{alg}}(E_3, M \otimes A_3) \to H^1_{\text{alg}}(E_2, M \otimes A_2)]$. One sees that this last kernel coincides with $\ker[H^1(G, M \otimes A_3) \to H^1(G, M \otimes A_2)]$ by using the inflation-restriction sequences (see subsection 3.3) for both $H^1_{\text{alg}}(E_3, M \otimes A_3)$ and $H^1_{\text{alg}}(E_2, M \otimes A_2)$, bearing in mind that $Y_3^G \to Y_2^G$ is an injective map.

\qed
Corollary 6.8. The result of applying the functor of $G$-coinvariants to the short exact sequence $(X)$ is a short exact sequence

$$0 \to (X_3)_G \to (X_2)_G \to (X_1)_G \to 0.$$  

Proof. This can be viewed as the special case of the previous lemma in which $T = \mathbb{G}_m$ and $M = \mathbb{Z}$. The vanishing of $H^1(G, M \otimes A_3) = H^1(G, A_3)$ is part of Lemma 6.1. \qed

6.7. The special case in which $S = V_F$. The special case in which $S = V_F$ is especially important. The case in which $S$ is finite (but sufficiently large) is useful as well, but plays a more technical role. When $S = V_F$ it seems more natural to express things differently. For any torus $T$ we then have $M \otimes A_3 = T(K)$, $M \otimes A_2 = T(A_K)$, $M \otimes A_1 = T(A_K)/T(K)$. With $M = X_*(T)$, as usual.

6.8. Restriction. It follows from Lemma 6.4 and subsection 4.4 that there are natural restriction maps

$$\text{Res}_{G/G'} : H^1_{\text{alg}}(E_i, M \otimes A_i) \to H^1_{\text{alg}}(E'_i, M \otimes A_i)$$

for $i = 1, 2, 3$ and any subgroup $G'$ of $G$. Here $E'_i$ is the analog for $K/E$ of $E_i$ for $K/F$, where $E$ is the fixed field of $G'$ on $K$.

6.9. Discussion of the localization map (global to adelic). The discussion of naturality in subsection 4.5 yielded the maps in the bottom row of diagram (6.6). In particular, when $S = V_F$ we have constructed a localization map (global to adelic)

$$H^1_{\text{alg}}(E_3, T(K)) \to H^1_{\text{alg}}(E_2, T(A_K)).$$

We will see later that there is a more direct way to define localization maps (from global all the way to local), and this can even be done in a more general situation in which the torus $T$ is replaced by a linear algebraic group over $F$.

6.10. Preliminary discussion of inflation. The canonical isomorphism $c$ from $(X_*(T) \otimes X_3)_G$ to $H^1_{\text{alg}}(E_3, T(K))$ is a satisfying generalization of the Tate-Nakayama isomorphism. The reader may be troubled, however, that both the source and target of this isomorphism seem to depend on the choice of $K$, which can be any finite Galois extension of $F$ that splits $T$. Fortunately, the choice of $K$ is unimportant, in the following sense. Let $K'$ be a finite Galois extension of $F$ such that $K' \supset K$. Eventually we will see that there is a commutative diagram

$$\begin{array}{ccc}
(X_*(T) \otimes X_3)_G & \xrightarrow{c} & H^1_{\text{alg}}(E_3, T(K)) \\
\simeq & & \simeq \\
(X_*(T) \otimes X'_3)_G & \xrightarrow{c'} & H^1_{\text{alg}}(E'_3, T(K'))
\end{array}$$

in which the bottom row is the analog for $K'$ of the top row, and the vertical arrows are natural isomorphisms that we will define later.
7. Localization \( \mathcal{E}_3(K/F), G(K) \) → \( H^1_{\text{alg}}(\mathcal{E}_3(K/F), G(K)) \)

7.1. Notation. For the most part we retain the notation of section 6. There are a few differences however. In section 6 we worked with an arbitrary set \( S \) of places of \( F \) satisfying the conditions imposed in subsection 6.1 but in this section we will keep things simple by considering only the case in which \( S \) is the set \( V_F \) of all places of \( F \). Another difference is that we now denote the groups \( E_i \) (\( i = 1, 2, 3 \)) of Galois modules, we obtain the short exact sequence

\[
\begin{aligned}
0 & \to \mathcal{E}_3(K/F) \to \mathcal{E}_2(K/F) \to \mathcal{E}_1(K/F) \to 0
\end{aligned}
\]

of protori.

For every place \( w \) of \( K \) there are homomorphisms \( \lambda_w : \mathbb{Z} \to X_2 \) and \( \mu_w : X_2 \to \mathbb{Z} \) defined by \( \lambda_w(n) = nw \) and \( \mu_w(\sum_{v \in V_K} n_v v) = n_w \). Dually, we have homomorphisms

\[
\begin{aligned}
\mathbb{G}_m & \xrightarrow{\mu_w} \mathbb{T}_{K/F} \xrightarrow{\lambda_w} \mathbb{G}_m,
\end{aligned}
\]

defined over the fixed field of the decomposition group of \( w \). In particular, \( \mu_v \) and \( \lambda_u \) are defined over \( E \). We denote by \( \mu_v' \) the composition

\[
\begin{aligned}
\mu_v' : \mathbb{G}_m & \xrightarrow{\mu_v} \mathbb{T}_{K/F} \xrightarrow{b} \mathbb{T}_{K/F}.
\end{aligned}
\]

7.2. Goal of this section. In this section \( G \) denotes a linear algebraic group over \( F \). As in section 6 we may then consider the pointed set \( H^1_{\text{alg}}(\mathcal{E}_3(K/F), G(K)) \) as well as its local analog \( H^1_{\text{alg}}(\mathcal{E}_3(K/F), G(K_v)) \). The goal of this section is to define a localization map

\[
\begin{aligned}
\ell^F : H^1_{\text{alg}}(\mathcal{E}_3(K/F), G(K)) & \to H^1_{\text{alg}}(\mathcal{E}_3(K/F), G(K_v)),
\end{aligned}
\]

This map will be defined as the composition of two other maps. The first is very easy to define, the second a little less so.

7.3. Construction of the first map. As in Example 2.3, there is a localization map

\[
\begin{aligned}
\text{Loc} : H^1_{\text{alg}}(\mathcal{E}_3(K/F), G(K)) & \to H^1_{\text{alg}}(\mathcal{E}_3^\vee(K/F), G(K_v)),
\end{aligned}
\]

where \( \mathcal{E}_3^\vee(K/F) \) is obtained from \( \mathcal{E}_3(K/F) \) by first pulling back along \( G(K_v/F_v) \to G(K/F) \) and then pushing forward using \( \mathbb{T}_{K/F}(K) \to \mathbb{T}_{K/F}(K_v) \). The arrow (7.4) is the first of the two maps we need to define.
7.4. Construction of the second map. The second map we need to define is of the type considered in subsection 2.8; it involves a change in band. The Galois gerb $\mathcal{E}_3^0(K/F)$ for $K_v/F_v$ is bound by the protorus $\mathbb{T}_{K/F}$, but now viewed over $F_v$ rather than $F$. The local Galois gerb $\mathcal{E}(K_v/F_v)$ is bound by $\mathbb{G}_m$; the associated cohomology class is the fundamental class $\alpha(K_v/F_v)$. We have already defined (see (7.2)) a canonical $F_v$-homomorphism $\mu'_v : \mathbb{G}_m \to \mathbb{T}_{K/F}$. In order to invoke subsection 2.8 we need to extend $\mu'_v$ to $\tilde{\mu}'_v$, as in the next lemma.

**Lemma 7.1.**

1. The groups $H^1(G(K_v/F_v), \mathbb{T}_{K/F}(K_v))$ and $H^1(G(K_v/F_v), \mathbb{T}_{K/F}(K_v))$ vanish.

2. There exists a homomorphism $\tilde{\mu}'_v$ making the diagram

$$
\begin{array}{cccccc}
1 & \to & \mathbb{G}_m(K_v) & \to & \mathcal{E}(K_v/F_v) & \to & G(K_v/F_v) & \to & 1 \\
& & & & \mu'_v & & \tilde{\mu}'_v & & \\
1 & \to & \mathbb{T}_{K/F}(K_v) & \to & \mathcal{E}_3^0(K/F) & \to & G(K_v/F_v) & \to & 1
\end{array}
$$

commute, and $\tilde{\mu}'_v$ is unique up to conjugation by $\mathbb{T}_{K/F}(K_v)$.

**Proof.** (1) When we view $0 \to X_3 \xrightarrow{b'} X_2 \xrightarrow{b} X_1 \to 0$ as a short exact sequence of $G(K_v/F_v)$-modules, it has a canonical splitting, namely the homomorphism $\lambda_v$ defined in subsection 7.1.

From this we conclude that $H^1(G(K_v/F_v), \mathbb{T}_{K/F}(K_v))$ is a direct summand of $H^1(G(K_v/F_v), \mathbb{T}_{K/F}(K_v))$, a group that vanishes by Lemma 6.2 and Hilbert’s Theorem 90.

(2) The uniqueness assertion regarding $\tilde{\mu}'_v$ follows from part (1) of this lemma. In proving the existence statement it is harmless to replace $F$ by $E$, and so we may assume that $F = E$ (and hence that $G(K/F) = G(K_v/F_v)$). This is convenient notationally, because we may then write $\alpha_i$ without having to specify whether we are referring to $K/F$ or $K/E$. We do this for the rest of the proof, and, because the linear algebraic group is irrelevant at the moment, we temporarily revert to denoting $G(K/F)$ by $G$, and the decomposition group of a place $w \in V_K$ by $G_w$.

The existence of $\tilde{\mu}'_v$ is equivalent to the equality

$$
\mu'_v(\alpha(K_v/F_v)) = \pi_v a' \alpha_3,
$$

where $\pi_v$ is (induced by) the projection of $\mathbb{A}_K$ on its direct factor $K_v$. We claim that (7.6) is a consequence of

$$
\mu'_v(\alpha(K_v/F_v)) = \pi_v \alpha_2.
$$

Indeed, one obtains the first equation from the second by applying $b'$ to both sides, bearing in mind that $b'$ commutes with $\pi_v$ and that $b' \alpha_2 = a' \alpha_3$.

It remains to prove (7.7). By Lemma 6.2 in order to show that $\pi_v \alpha_2$ and $\mu'_v(\alpha(K_v/F_v))$ are equal, it suffices to show that, for every place $w$ of $K$, they have the same image under

$$
\rho_w : H^2(G, \mathbb{T}_{K/F}(K_v)) \xrightarrow{\text{Res}_{G/F}^w} H^2(G_w, \mathbb{T}_{K/F}(K_v)) \xrightarrow{\lambda_w} H^2(G_w, K_v^\times).
$$
There is a similarly defined map $\rho_w$ with $A_K$ replacing $K_v$, as well as a commutative diagram

\[
\begin{array}{ccc}
H^2(G, \mathbb{T}_{K/F}(A_K)) & \xrightarrow{\rho_w} & H^2(G, \mathbb{A}_K^\times) \\
\pi_v & & \pi_v \\
H^2(G, \mathbb{T}_{K/F}(K_v)) & \xrightarrow{\rho_w} & H^2(G, K_v^\times),
\end{array}
\]

so we conclude that $\rho_w \pi_v \alpha_2 = \pi_v \rho_w \alpha_2$.

It follows (see the discussion at the bottom of page 714 in [Tat66]) from the definition of $\alpha_2$ that $\rho_w \alpha_2 = i_w(\alpha(K_w/F_w))$, where $F_w$ is the completion of $F$ at the unique place of $F$ lying under $w$, and $i_w$ is the obvious inclusion of $K_w^\times$ as a direct factor of $\mathbb{A}_K^\times$. Therefore

\[
\rho_w \pi_v \alpha_2 = \pi_v i_w \alpha(K_w/F_w) = \begin{cases} 
\alpha(K_v/F_u) & \text{if } w = v, \\
0 & \text{if } w \neq v.
\end{cases}
\]

Furthermore

\[
\rho_w \mu_v(\alpha(K_v/F_u)) = \begin{cases} 
\alpha(K_v/F_u) & \text{if } w = v, \\
0 & \text{if } w \neq v,
\end{cases}
\]

because $\lambda_w \mu_v$ is the identity map when $w = v$ and is 0 otherwise. This concludes the proof. \(\square\)

We use the homomorphism $\tilde{\mu}_v$ in the lemma to obtain

\[
(\tilde{\mu}_v)^* : H^1_{\text{alg}}(\mathcal{E}_2(K/F), G(K_v)) \to H^1_{\text{alg}}(\mathcal{E}(K_v/F_u), G(K_v))
\]

as in subsection 2.8.

### 7.5. End of the definition of the localization map $\ell^F_w$.

We now define the localization map (7.8) to be the composition of (7.4) and (7.8).

**Remark 7.2.** It follows immediately from the equality (7.7) that there exists a homomorphism $\hat{\mu}_v$, making the diagram

\[
\begin{array}{cccc}
1 & \xrightarrow{\mu_v} & \mathbb{G}_m(K_v) & \xrightarrow{\tilde{\mu}_v} \mathbb{T}_{K/F}(K_v) \\
& & \downarrow & \downarrow \\
1 & \xrightarrow{} & \mathbb{E}_2(K/F) & \xrightarrow{} \hat{\mathbb{E}}_2(K/E) & \xrightarrow{} \hat{\mathbb{G}}(K) & \xrightarrow{} 1
\end{array}
\]

commute, where the bottom row is obtained by pushforward from the extension $\mathbb{E}_2(K/E)$ of $G(K/E)$ by $\mathbb{T}_{K/F}(A_K)$. Moreover, by the first part of the lemma, $\hat{\mu}_v$ is unique up to conjugation by $\mathbb{T}_{K/F}(A_K)$. Using $\hat{\mu}_v$ one can easily construct a localization map

\[
H^1_{\text{alg}}(\mathcal{E}_2(K/F), G(A_K)) \to H^1_{\text{alg}}(\mathcal{E}(K_v/F_u), G(K_v))
\]

once one takes the trouble to define the set $H^1_{\text{alg}}(\mathcal{E}_2(K/F), G(A_K))$.

Then one can go on to construct a commutative diagram

\[
\begin{array}{ccc}
H^1_{\text{alg}}(\mathcal{E}_3(K/F), G(K)) & \xrightarrow{} & H^1_{\text{alg}}(\mathcal{E}_2(K/F), G(A_K)) \\
\downarrow & & \downarrow \\
H^1_{\text{alg}}(\mathcal{E}_3(K/F), G(K)) & \xrightarrow{} & H^1_{\text{alg}}(\mathcal{E}(K_v/F_u), G(K_v)).
\end{array}
\]

We omit the details, as we will not make use of this diagram.
7.6. **Compatibility of localization with the Newton map.** The next result gives the compatibility between localization and the Newton map (7.2).

**Lemma 7.3.** Let \( G \) be any linear algebraic group over \( F \). Then the diagram

\[
\begin{array}{ccc}
H^1_{\text{alg}}(E_3(K/F), G(K)) & \xrightarrow{\mu_1} & H^1_{\text{alg}}(E(K_v/F_u), G(K_v)) \\
| \downarrow & & | \downarrow \\
| \downarrow & & | \downarrow \\
[\text{Hom}_K(T_{K/F}, G)/G(K)]^{G(K/F)} & \xrightarrow{\mu_1^{'}} & [\text{Hom}_{K_v}(G_m, G)/G(K_v)]^{G(K_v/F_u)}
\end{array}
\]

commutes.

**Proof.** Easy. \( \square \)

7.7. **A commutative square involving the localization map for tori.** Now consider an \( F \)-torus \( T \) split by \( K \), and write \( M \) for \( X^*_{\text{tor}}(T) \). Then form the square

\[
\begin{array}{ccc}
(M \otimes X_3)_{G(K/F)} & \xrightarrow{c} & M_{G(K_v/F_u)} \\
| \downarrow & & | \downarrow \\
| \downarrow & & | \downarrow \\
H^1_{\text{alg}}(E_3(K/F), T(K)) & \xrightarrow{\mu_1^{'}} & H^1_{\text{alg}}(E(K_v/F_u), T(K_v))
\end{array}
\]

The top arrow is obtained as follows. We begin with the composed map

\[
(\sum_{w \in V_K} w \in M \otimes X_3) : M \otimes X_3 \rightarrow V_u[1],
\]

where \( V_u \) denotes the set of places of \( K \) lying over \( u \), and the second arrow is projection onto the direct summand \( \sum_{w \in V_K} M \otimes X_3 \rightarrow \bigoplus_{w \in V_K} M \).

**Lemma 7.4.** The square (7.13) commutes.

**Proof.** Use the same method as in the proof of Lemma 4.2 the point being that the diagram

\[
\begin{array}{ccc}
(M \otimes X_3)_{G(K/F)} & \xrightarrow{c} & M_{G(K_v/F_u)} \\
| \downarrow & & | \downarrow \\
| \downarrow & & | \downarrow \\
(M \otimes X_3)_{G(K/F)} & \xrightarrow{c} & M_{G(K_v/F_u)}
\end{array}
\]

commutes. Here the left (resp. right) vertical arrow is the global (resp. local) norm map. The top arrow is the same as the top arrow in (7.13). The bottom arrow sends \( G(K/F) \)-invariant \( \sum_{w \in V_K} m_w w \in M \otimes X_3 \) to its \( v \)-component \( m_v \). \( \square \)
8. Inflation $H^1_{\text{alg}}(\mathcal{E}(K/F), G(K)) \to H^1_{\text{alg}}(\mathcal{E}(L/F), G(L))$

When we studied the four examples of Tate-Nakayama triples (one local, three global), we always fixed the Galois extension $K$ of $F$. Now we need to see what happens to $H^1_{\text{alg}}$ when we enlarge $K$ to $L$. The essential point is the relationship between the various fundamental classes for the two layers $K/F$ and $L/F$.

8.1. Local theory. We consider local fields $L \supset K \supset F$ with both $L$ and $K$ finite Galois over $F$. We keep track of which Galois group and fundamental class we are talking about by labeling them with $K/F$ or $L/F$, as appropriate. We then have (see section 5) the local Tate-Nakayama triple $(\mathbb{Z}, K^\times, \alpha(K/F))$ for $K/F$, as well as the Galois gerb for $K/F$ (and bound by $\mathbb{G}_m$)

\begin{equation}
(8.1) \quad 1 \to \mathbb{G}_m(K) \to \mathcal{E}(K/F) \to G(K/F) \to 1.
\end{equation}

When $F$ is $\mathbb{Q}_p$, this is the Dieudonné gerb of $[LR87]$ attached to $K$. When $K/F$ is $\mathbb{C}/\mathbb{R}$, it is the weight gerb of $[LR87]$.

Now let $G$ be a linear algebraic group over $F$. We may then consider the set $H^1_{\text{alg}}(\mathcal{E}(K/F), G(K))$, as well as its analog for $L/F$. Our goal is to define a natural map

\begin{equation}
(8.2) \quad H^1_{\text{alg}}(\mathcal{E}(K/F), G(K)) \to H^1_{\text{alg}}(\mathcal{E}(L/F), G(L)),
\end{equation}

and then to show it is an isomorphism when $G$ is a torus split by $K$.

We can inflate the Galois gerb $\mathcal{E}(K/F)$ to $L/F$ (see subsection 2.4), obtaining

\begin{equation}
(8.3) \quad 1 \to \mathbb{G}_m(L) \to \mathcal{E}(K/F)_{\text{inf}} \to G(L/F) \to 1.
\end{equation}

As in Example 2.2, we then have an inflation map

\begin{equation}
(8.4) \quad H^1_{\text{alg}}(\mathcal{E}(K/F), G(K)) \to H^1_{\text{alg}}(\mathcal{E}(K/F)_{\text{inf}}, G(L)).
\end{equation}

Define a homomorphism $p_{L/K} : \mathbb{G}_m \to \mathbb{G}_m$ by $x \mapsto x^{[L:K]}$. The fundamental classes $\alpha(K/F) \in H^2(G(K/F), K^\times)$ and $\alpha(L/F) \in H^2(G(L/F), L^\times)$ are related by the equation

\begin{equation}
(8.5) \quad \inf(\alpha(K/F)) = [L:K]\alpha(L/F),
\end{equation}

and therefore there exists a homomorphism $\eta_{L/K} : \mathcal{E}(L/F) \to \mathcal{E}(K/F)_{\text{inf}}$ making the diagram

\begin{equation}
(8.6) \quad \begin{array}{c}
1 \quad \xrightarrow{p_{L/K}} \quad \mathbb{G}_m(L) \quad \xrightarrow{\eta_{L/K}} \quad \mathcal{E}(L/F) \quad \xrightarrow{} \quad G(L/F) \quad \xrightarrow{} \quad 1
\end{array}
\end{equation}

commute. We then obtain the induced map

\begin{equation}
(8.7) \quad \eta_{L/K}^* : H^1_{\text{alg}}(\mathcal{E}(K/F)_{\text{inf}}, G(L)) \to H^1_{\text{alg}}(\mathcal{E}(L/F), G(L))
\end{equation}

defined in subsection 2.8. The composition of $[S3]$ and $[S7]$ is the map $[S2]$ that we wanted to define.

Now take $G$ to be an $F$-torus $T$ split by $K$ and consider the diagram

\begin{equation}
(8.8) \quad \begin{array}{ccc}
X_*(T)_{G(K/F)} & \xrightarrow{\epsilon} & H^1_{\text{alg}}(\mathcal{E}(K/F), T(K)) & \xrightarrow{r} & X_*(T)_{G(K/F)} \\
\uparrow & \enspace [L:K] & \downarrow \enspace [S3] & \enspace [L:K] & \\
X_*(T)_{G(L/F)} & \xrightarrow{\epsilon} & H^1_{\text{alg}}(\mathcal{E}(L/F), T(L)) & \xrightarrow{r} & X_*(T)_{G(L/F)}
\end{array}
\end{equation}
with $c$ as in subsection 6.2, $r$ as in subsection 5.3 and where the left vertical arrow is the obvious isomorphism (the one induced by the identity map on $X_*(T)$).

**Lemma 8.1.** The diagram \( \text{(8.8)} \) commutes, and all four arrows in the left square are isomorphisms.

**Proof.** Since three of the four arrows in the left square are already known to be isomorphisms, our only real task is to prove that the diagram commutes.

The right square commutes, as one sees easily from the definition of the arrow \( \text{(8.2)} \), the point being that $p_{L/K} : \mathbb{G}_m \to \mathbb{G}_m$ induces multiplication by $[L : K]$ on cocharacter groups. Moreover, we know (see Lemmas 6.2 and 8.3) that the composition $rc$ in the top row (resp., bottom row) is the norm map for $G(K/F)$ (resp., $G(L/F)$). It is therefore clear that the outer rectangle commutes. We conclude that the left square commutes whenever the restriction map $r$ in the bottom row is injective, and, by the inflation-restriction sequence of subsection 3.5, this happens if and only if $H^1(G(L/F), T(L))$ vanishes.

In particular, by Shapiro’s lemma and Hilbert’s Theorem 90, the left square does commute when $X_*(T)$ is free of finite rank as $\mathbb{Z}[G(K/F)]$-module. For general $T$, we choose a free $\mathbb{Z}[G(K/F)]$-module $M$ and a surjective $G(K/F)$-module map $f : M \to X_*(T)$. We then obtain $f : T_M \to T$, where $T_M$ is the torus with cocharacter group $M$. The left square is functorial in $T$, so its commutativity for $T$ follows from that for $T_M$, because $f : M_{G(L/F)} \to X_*(T)_{G(L/F)}$ is surjective. \( \square \)

### 8.2. New system of notation for our three global Tate-Nakayama triples.

Next we are going to study inflation for our three global Tate-Nakayama triples $(X_i, A_i, \alpha_i)$, so it is no longer feasible to omit the extension $K/F$ from the notation. We write $G(K/F)$ for the Galois group of $K/F$, and we now denote our three Tate-Nakayama triples by $(X_i(K), A_i(K), \alpha_i(K/F))$. In this more elaborate system of notation the Tate class $\alpha \in H^2(G, \text{Hom}(X, A))$ discussed near the end of subsection 6.2 becomes

$$\alpha(K/F) \in H^2(G(K/F), \text{Hom}(X(K), A(K))).$$

We also need to remember that $A_2$ and $A_3$ depend on a choice of subset $S \subset V_F$, and that the conditions imposed on $S$ become more and more stringent the bigger the top field $K$ gets.

The relevant extensions are now denoted by

$$(8.9) \quad 1 \to \text{Hom}(X_i(K), A_i(K)) \to E_i(K/F) \to G(K/F) \to 1$$

and the relevant cohomology groups by $H^1_{\text{alg}}(E_i(K/F), M \otimes A_i(K))$, where $M$ is the cocharacter group of an $F$-torus $T$ that is split by $K$. When $S = V_F$ and $i = 3$ the cohomology group is $H^1_{\text{alg}}(E_3(K/F), T(K))$ and is of the type considered in section 2, the relevant group of multiplicative type being the protorus $\mathbb{T}_{K/F}$ whose group of characters is $X_3(K/F)$.

The norm map for the finite group $G(K/F)$ will now be denoted by $N_{K/F}$. We recall that there are functorial maps

$$(8.10) \quad (M \otimes X_i(K))_{G(K/F)} \xrightarrow{c} H^1_{\text{alg}}(E_i(K/F), M \otimes A_i(K)) \xrightarrow{\sigma} (M \otimes X_i(K))^{G(K/F)}.$$

Here $c$ is the isomorphism of subsection 6.5 and $\sigma$ is the restriction map of subsection 3.5. Moreover $rc = N_{K/F}$ by Lemmas 6.2 and 8.3.
8.3. The setup in which to discuss inflation in the global situation. We now consider a finite Galois extension $L$ of $F$ with $L \supset K$. We then have a short exact sequence

$$1 \to G(L/K) \to G(L/F) \to G(K/F) \to 1.$$ 

We seek an analog of Lemma 8.1 for our three global Tate-Nakayama triples, in which $S \subset V_F$ is assumed to be big enough to satisfy the conditions (see subsection 6.1) needed for the extension $L/F$; it is then automatic that $S$ also satisfies these conditions for $K/F$. In order to get started, we need maps relating the groups $X_i(K), A_i(K)$ to the parallel objects for $L/F$. In the case of $A_i$, this is straightforward: there is an obvious isomorphism of $A_i(K)$ with $A_i(L)^{G(L/K)}$.

In the case of $X_i$, there are maps in both directions. In fact we are going to define two commutative diagrams

$$
\begin{array}{cccccc}
0 & \longrightarrow & X_3(L) & \longrightarrow & X_2(L) & \longrightarrow & X_1(L) & \longrightarrow & 0 \\
\downarrow j_3 & & \downarrow j_2 & & \downarrow j_1 & & \\
0 & \longrightarrow & X_3(K) & \longrightarrow & X_2(K) & \longrightarrow & X_1(K) & \longrightarrow & 0 \\
\end{array}
$$

and

$$
\begin{array}{cccccc}
0 & \longrightarrow & X_3(L) & \longrightarrow & X_2(L) & \longrightarrow & X_1(L) & \longrightarrow & 0 \\
\uparrow p_3 & & \uparrow p_2 & & \uparrow p_1 & & \\
0 & \longrightarrow & X_3(K) & \longrightarrow & X_2(K) & \longrightarrow & X_1(K) & \longrightarrow & 0 \\
\end{array}
$$

All we really need to do is to define maps $j_2, j_1, p_2, p_1$ making the two right squares commute; we are then forced to define $j_3, p_3$ by restriction. Now $X_1(L) = \mathbb{Z} = X_1(K)$. We take $j_1$ to be the identity map on $\mathbb{Z}$, and $p_1$ to be multiplication by $[L : K]$.

Recall that $X_2(K)$ is the free abelian group on the set $S_K$ of places of $K$ that lie over some place in $S$. The value of $j_2$ on the basis element $w \in S_L$ of $X_2(L)$ is defined to be $v$, where $v$ is the unique place of $K$ lying under $w$. The value of $p_2$ on the basis element $v \in S_K$ of $X_2(K)$ is defined to be

$$p_2(v) := \sum_{w \mid v} [L_w : K_v]w.$$ 

The desired commutativity of the two right squares is clear.

**Lemma 8.2.** The following statements hold for $i = 1, 2, 3$.

1. $p_i \circ j_i = N_{L/K}$.
2. $j_i \circ p_i = [L : K]$. Consequently $p_i$ is injective.
3. The map $j_i : X_i(L) \to X_i(K)$ factors through the coinvariants of $G(L/K)$ on $X_i(L)$, inducing an isomorphism

$$\gamma_i : X_i(L)^{G(L/K)} \to X_i(K).$$

**Proof.** The only thing that might not be obvious is that $\gamma_i$ is an isomorphism. This follows readily from the definitions when $i = 1, 2$. To handle $i = 3$ one appeals to Corollary 6.8 applied to $L/K$. \qed
8.4. Definition of global inflation maps for tori. As before we write \( M \) for the cocharacter group of an \( F \)-torus \( T \) split by \( K \). We need to define global inflation maps (for \( i = 1, 2, 3 \))

\[
H_{\text{alg}}^1(\mathcal{E}_i(K/F), M \otimes A_i(K)) \to H_{\text{alg}}^1(\mathcal{E}_i(L/F), M \otimes A_i(L)).
\]

As in the local case, we will do so in three steps, first defining two auxiliary maps and then taking their composition as the definition of \((8.13)\).

**Step 1.** We use \( G(L/F) \to G(K/F) \) and \( A_i(K) = A_i(L)^{G(L/K)} \) to define

\[
(8.14) \quad H_{\text{alg}}^1(\mathcal{E}_i(K/F), M \otimes A_i(K)) \to H_{Y_i(K)}^1(\mathcal{E}_i(K/F)^{\text{inf}}, M \otimes A_i(L)),
\]

where \( Y_i(K) := M \otimes X_i(K) \), and \( \mathcal{E}_i(K/F)^{\text{inf}} \) is obtained from \( \mathcal{E}_i(K/F) \) by first pulling back along \( G(L/F) \to G(K/F) \) and then pushing out along the inclusion of \( \text{Hom}(X_i(K), A_i(K)) \) as the set of \( G(L/K) \)-fixed points in \( \text{Hom}(X_i(K), A_i(L)) \). Thus our inflated extension sits in an exact sequence

\[
1 \to \text{Hom}(X_i(K), A_i(L)) \to \mathcal{E}_i(K/F)^{\text{inf}} \to G(L/F) \to 1.
\]

The map \( \xi' : Y_i(K) \to \text{Hom}(\text{Hom}(X_i(K), A_i(L)), M \otimes A_i(L)) \) used to form the group \( H_{Y_i(K)}^1(\mathcal{E}_i(K/F)^{\text{inf}}, M \otimes A_i(L)) \) is the obvious tautological one.

In the special case \( S = V_F \) and \( i = 3 \), the map \((8.14)\) is an instance of the inflation map in Example 3.9.

In all cases it is an instance of the very general inflation map in Example 8.2. More precisely it is of the form \( \Phi(f, g, h) \) where

- \( f : M \otimes A_i(K) \to M \otimes A_i(L) \) is induced by \( A_i(K) \to A_i(L) \),
- \( g \) is the identity map on \( Y_i(K) \), and
- \( h \) is the inclusion \( \text{Hom}(X_i(K), A_i(K)) \to \text{Hom}(X_i(K), A_i(L)) \) that we used to form the pushout.

**Step 2.** From \( p_i : X_i(K) \to X_i(L) \) we obtain an induced map

\[
p_i : \text{Hom}(X_i(L), A_i(L)) \to \text{Hom}(X_i(K), A_i(L))
\]

We want to choose a homomorphism \( \tilde{p}_i \) making the diagram

\[
\begin{array}{ccc}
1 & \longrightarrow & \text{Hom}(X_i(L), A_i(L)) \\
\downarrow p_i & & \downarrow \tilde{p}_i \\
1 & \longrightarrow & \text{Hom}(X_i(K), A_i(L))
\end{array}
\begin{array}{ccc}
& & \longrightarrow & \mathcal{E}_i(L/F) & \longrightarrow & G(L/F) & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & \longrightarrow & \mathcal{E}_i(K/F)^{\text{inf}} & \longrightarrow & G(L/F) & \longrightarrow & 1
\end{array}
\]

commute.

**Lemma 8.3.** For \( i = 1, 2, 3 \) such a homomorphism \( \tilde{p}_i \) exists and is unique up to conjugation by an element in the subgroup \( \text{Hom}(X_i(K), A_i(L)) \).

**Proof.** The existence of \( \tilde{p}_i \) is equivalent to the statement that

\[
p_i(\alpha_i(L/F)) = \inf(\alpha_i(K/F)) \in H^2(G(L/F), \text{Hom}(X_i(K), A_i(L))).
\]

In fact, we will prove a slightly stronger statement involving the Tate classes

- \( \alpha(K/F) \in H^2(G(K/F), \text{Hom}(X(K), A(K))) \),
- \( \alpha(L/F) \in H^2(G(L/F), \text{Hom}(X(L), A(L))) \).

The slightly stronger statement is that

\[
p(\alpha(L/F)) = \inf(\alpha(K/F)) \in H^2(G(L/F), \text{Hom}(X(K), A(L))).
\]
As the notation suggests, $\text{Hom}(X(K), A(L))$ is defined as the group of triples \((h_3, h_2, h_1)\) making 
\[
\begin{array}{ccc}
X_3(K) & \longrightarrow & X_2(K) & \longrightarrow & X_1(K) \\
h_3 & \downarrow & h_2 & \downarrow & h_1 \\
A_3(L) & \longrightarrow & A_2(L) & \longrightarrow & A_1(L)
\end{array}
\]
commute, and the map \(p\) is induced by \((p_3, p_2, p_1)\). Because the image of \(\alpha(K/F)\) under the \(i\)-th projection is \(\alpha_i(K/F)\), the new statement does in fact imply the three old statements.

Lemma 6.3 exhibits \(H^2(G(K/F), \text{Hom}(X(K), A(K)))\) as a fiber product, and the same goes with \(K/F\) replaced by \(L/F\). We are now going to prove an analog of Lemma 6.3 for \(H^2(G(L/F), \text{Hom}(X(K), A(L)))\).

**Claim.** The diagrams
\[
\begin{array}{ccc}
\text{Hom}(X(K), A(L)) & \xrightarrow{\pi_1} & \text{Hom}(X_1(K), A_1(L)) \\
\pi_2 \downarrow & & \downarrow b \\
\text{Hom}(X_2(K), A_2(L)) & \xrightarrow{a} & \text{Hom}(X_2(K), A_1(L))
\end{array}
\tag{8.19}
\]
and
\[
\begin{array}{ccc}
H^2(G(L/F), \text{Hom}(X(K), A(L))) & \xrightarrow{\pi_1} & H^2(G(L/F), \text{Hom}(X_1(K), A_1(L))) \\
\pi_2 \downarrow & & \downarrow b \\
H^2(G(L/F), \text{Hom}(X_2(K), A_2(L))) & \xrightarrow{a} & H^2(G(L/F), \text{Hom}(X_2(K), A_1(L)))
\end{array}
\tag{8.20}
\]
are cartesian.

We prove the claim by imitating Tate’s argument. It is clear that \(8.19\) is cartesian. In other words \(\text{Hom}(X(K), A(L))\) is the equalizer of the two obvious maps 
\[
\text{Hom}(X_1(K), A_1(L)) \oplus \text{Hom}(X_2(K), A_2(L)) \rightarrow \text{Hom}(X_2(K), A_1(L)).
\]
Equivalently, \(\text{Hom}(X(K), A(L))\) is the kernel of the difference \(\delta\) of these two obvious maps. Now \(\delta\) is surjective, as follows from the surjectivity of the bottom horizontal arrow in \(8.19\) (itself a consequence of the fact that \(X_2(K)\) is a free abelian group). The fact that \(8.20\) is cartesian now follows from the long exact \(G(L/F)\)-cohomology sequence for the short exact sequence 
\[
0 \rightarrow \ker(\delta) \rightarrow \text{source}(\delta) \rightarrow \text{target}(\delta) \rightarrow 0,
\]
together with the vanishing of \(H^1(G(L/F), \text{Hom}(X_2(K), A_1(L)))\), an easy consequence of Lemmas 6.2 and 6.1. This finishes the proof of the claim.

The claim shows that in order to prove \(8.18\) (and hence prove \(8.17\) for \(i = 3\)), it is enough to prove \(8.17\) for \(i = 1, 2\). Now for \(i = 1\) \(8.17\) reduces to the (standard) fact that the inflation to \(L/F\) of the global fundamental class for \(K/F\) is \([L : K]\) times the global fundamental class for \(L/F\).

To handle \(i = 2\) we are going to reduce to the local case. We need to prove that 
\[
p_2(\alpha_2(L/F)) = \inf(\alpha_2(K/F)) \in H^2(G(L/F), \text{Hom}(X_2(K), \mathbb{A}_L^\times)).
\tag{8.21}
\]
The first step is to use Lemma A.6 in order to analyze the maps
\[(8.22) \quad H^2(K/F, \text{Hom}(X_2(K), \mathbb{A}_K^\times)) \xrightarrow{\text{inf}} H^2(L/F, \text{Hom}(X_2(K), \mathbb{A}_L^\times))\]
and
\[(8.23) \quad H^2(L/F, \text{Hom}(X_2(L), \mathbb{A}_L^\times)) \xrightarrow{p_2} H^2(L/F, \text{Hom}(X_2(K), \mathbb{A}_L^\times)).\]

From that lemma we have identifications
- \(H^2(K/F, \text{Hom}(X_2(K), \mathbb{A}_K^\times)) = \prod_{v \in V_F} H^2(G(K/F)_v, \mathbb{A}_K^\times),\)
- \(H^2(L/F, \text{Hom}(X_2(K), \mathbb{A}_L^\times)) = \prod_{w \in V_F} H^2(G(L/F)_w, \mathbb{A}_L^\times),\)
- \(H^2(L/F, \text{Hom}(X_2(L), \mathbb{A}_L^\times)) = \prod_{u \in V_F} H^2(G(L/F)_w, \mathbb{A}_L^\times),\)

where, for each \(u \in V_F\), we have first chosen a place \(v\) of \(K\) lying over \(u\) and then chosen a place \(w\) of \(L\) lying over \(v\). With these identifications, the map (8.22)
becomes the product (over \(u \in V_F\)) of the inflation maps
\[(8.24) \quad H^2(G(K/F)_v, \mathbb{A}_K^\times) \xrightarrow{\text{inf}} H^2(G(L/F)_w, \mathbb{A}_L^\times)\]
coming from \(G(L/F)_w \rightarrow G(K/F)_v\). From the definition of \(p_2\) it follows fairly easily that the map (8.23)
becomes the product (over \(u \in V_F\)) of the maps
\[(8.25) \quad H^2(G(L/F)_w, \mathbb{A}_L^\times) \xrightarrow{[L_w:K_v]\text{Cor}} H^2(G(L/F)_v, \mathbb{A}_L^\times),\]

where \(\text{Cor}\) is the corestriction map for the subgroup \(G(L/F)_w\) of \(G(L/F)_v\).

At this point it is helpful to consider the diagrams (one for each \(u \in V_F\))
\[
\begin{align*}
H^2(G(K/F)_v, \mathbb{A}_K^\times) \xrightarrow{\text{inf}} H^2(G(L/F)_v, \mathbb{A}_L^\times) & \xleftarrow{[L_w:K_v]\text{Cor}} H^2(G(L/F)_w, \mathbb{A}_L^\times) \\
\uparrow {i_K} & \uparrow {i_L} & \uparrow {i_L} \\
H^2(G(K/F)_v, K_v^\times) \xrightarrow{\text{inf}} H^2(G(L/F)_v, L_v^\times) & \xleftarrow{[L_w:K_v]\text{Cor}} H^2(G(L/F)_w, L_v^\times) \\
\uparrow {\text{Sh}} & & \uparrow {j} \\
H^2(G(K/F)_v, K_v^\times) \xrightarrow{\text{inf}} H^2(G(L/F)_w, L_w^\times) & \xleftarrow{[L_w:K_v]\text{Cor}} H^2(G(L/F)_w, L_w^\times)
\end{align*}
\]
where \(L_w := L \otimes_K K_v = \prod_{w' | w} L_{w'}\), the map \(\text{Sh}\) is the Shapiro isomorphism, and the maps \(i_K, i_L, j\) are (induced by) the obvious inclusions \(K_v^\times \rightarrow \mathbb{A}_K^\times, L_v^\times \rightarrow \mathbb{A}_L^\times, L_w^\times \rightarrow L_v^\times\), respectively.

Unwinding the definition of the adelic fundamental class \(\alpha_2\), we see that (8.21)
is equivalent to the equality, for all \(u \in V_F\), of the elements
\[
\beta_u', \beta_u'' \in H^2(G(L/F)_v, \mathbb{A}_L^\times)
\]
defined by \(\beta_u' := \text{inf}(i_K(\alpha(K_v/F_u)))\) and \(\beta_u'' := [L_w : K_v]\text{Cor}(i_L j(\alpha(L_w/F_u))).\)
Now it is part of local classfield theory that \([L_w : K_v]\alpha(L_w/F_u)\) is equal to the inflation of \(\alpha(K_v/F_u)\). So (8.21) follows from the commutativity of the big diagram above. The commutativity of the left top square comes from the naturality of inflation. The commutativity of the right top square comes from the naturality of corestriction. The commutativity of the left bottom square is easy to check, using that the Shapiro isomorphism is given by restriction (for \(G(L/F)_w \subset G(L/F)_v\)) followed by the natural projection \(L_v^\times \rightarrow L_w^\times\). The commutativity of the right lower square follows from Lemma 3.2. So we are done proving the equality (8.21) for \(i = 2\).
Lemma 8.4.
in which the rows are instances of (8.10).

Proof. (recall that proved that the diagram commutes. When we just need to prove the vanishing of $H^1(G(L/F), \text{Hom}(X_3(K), A_3(L)))$ for $i = 1, 2, 3$.

For $i = 1$ we just need the vanishing of $H^1(G(L/F), H^3_{\text{alg}}(L/F^c))$, and this is one of the standard results of global classfield theory (see Lemma 6.1). For $i = 2$ the desired vanishing follows from Lemmas 8.2 and 6.1.

For $i = 3$ we must prove the vanishing of $H^1(G(L/F), \text{Hom}(X_3(K), A_3(L)))$. We claim that cup product with $\alpha_3(L/F)$ yields isomorphisms

$$H^r(G(L/F), \text{Hom}(X_3(K), X_3(L))) \simeq H^{r+2}(G(L/F), \text{Hom}(X_3(K), A_3(L)))$$

for all $r \in \mathbb{Z}$. Indeed, $X_3(K)$ lies in the class $C(X_3(L), A_3(L), \alpha_3(L/F))$ of Definition A.2 as one sees from parts (2) and (5) of Lemma A.3.

So we just need to check that $H^{-1}(G(L/F), \text{Hom}(X_3(K), X_3(L)))$ vanishes, and this follows from Lemma A.11(3). In that lemma we take $\epsilon$ to be the obvious surjection $S_L \to S_K$. To apply the lemma, we need the vanishing of $H^{-1}(G', X_3(L))$ for every subgroup $G'$ of $G(L/F)$, and this follows from Tate’s isomorphism (cup product with the restriction to $G'$ of $\alpha_3(L/F)$)

$$H^{-1}(G', X_3(L)) \simeq H^1(G', A_3(L))$$

together with Lemma 6.1. The proof of Lemma 8.3 is finally complete. \(\square\)

As a consequence of Lemma 8.3, we obtain a well-defined map

$$\tilde{p}_i : H^1_{Y_i(K)}(E_i(K/F)^{\text{inf}}, M \otimes A_i(L)) \to H^1_{\text{alg}}(E_i(L/F), M \otimes A_i(L)).$$

When $S = V_F$ and $i = 3$, this is an instance of the map (8.14). In general one uses the map $\text{id}_M \otimes p_i : Y_i(K) \to Y_i(L)$ to define the cocycle-level map

$$\tilde{p}_i : \tilde{p}_i : (\nu, x) \mapsto ((\text{id}_M \otimes p_i)(\nu), x \circ \tilde{p}_i).$$

Step 3. Define the arrow (8.13) as the composition of (8.14) and (8.26).

8.5. Global inflation isomorphisms. Now we are in a position to prove our main result on inflation in the global situation. We consider the diagram

$$
\begin{array}{ccc}
(M \otimes X_i(K))_{G(L/F)} & \xrightarrow{c} & H^1_{\text{alg}}(E_i(K/F), M \otimes A_i(K)) \\
\text{id}_M \otimes j_i \downarrow & & \downarrow \text{id}_M \otimes p_i \\
(M \otimes X_i(L))_{G(L/F)} & \xrightarrow{c} & H^1_{\text{alg}}(E_i(L/F), M \otimes A_i(L))
\end{array}
$$

(8.27)

in which the rows are instances of (8.10).

Lemma 8.4. For $i = 1, 2, 3$ the diagram (8.27) commutes, and all four arrows in the left square are isomorphisms.

Proof. The two horizontal arrows $c$ are isomorphisms. So too is the left vertical arrow, because it is obtained by applying the functor of $G(K/F)$-coinvariants to the isomorphism

$$(M \otimes X_i(L))_{G(L/K)} = M \otimes X_i(L)_{G(L/K)} \xrightarrow{id_M \otimes \gamma_i} M \otimes X_i(K)$$

(recall that $\gamma_i$ was defined in Lemma 8.2(3)). Since three of the four arrows in the left square are isomorphisms, we will know that the fourth one is too, once we have proved that the diagram commutes.
We are going to prove that the diagram commutes by the method we used in the local case. We must show that (8.27) commutes for all $G(K/F)$-modules $M$ that are free of finite rank as $\mathbb{Z}$-modules. One sees directly that the right square commutes, and it follows easily from Lemma 8.2(1) that the outer rectangle also commutes. It remains to show that the left square commutes. We are going to use that all the maps in the left square are functorial in $M$. First let us treat the special case when $M$ is free of finite rank as $\mathbb{Z}[G(K/F)]$-module.

In this special case it is easily checked that $H^1(G(K/F), M \otimes A_i(K))$ vanishes, and the same is true with $K/F$ replaced by $L/F$. Therefore the horizontal arrows $r$ (coming from the inflation-restriction sequence discussed in subsection 3.5) are both injective. So the commutativity of the right square and outer rectangle implies that of the left square in this special case.

In the general case we choose a finite free $\mathbb{Z}[G(K/F)]$-module $M'$ and a surjective Galois-equivariant map $M' \twoheadrightarrow M$. The desired commutativity for $M$ follows from the known commutativity for $M'$, simply because the natural map

$$(M' \otimes X_i(L))_{G(L/F)} \rightarrow (M \otimes X_i(L))_{G(L/F)}$$

is surjective. □

8.6. Definition of the groups $B_i(F,T)$. Let $T$ be a torus defined over $F$. We choose a separable algebraic closure $\bar{F}$ of $F$. For $i=1,2,3$ we put

$$(8.28) \quad B_i(F,T) := \operatorname{inj \lim}_K H^1_{\text{alg}}(\mathcal{E}_i(K/F), M \otimes A_i(K)),$$

the direct limit being taken over the set $K$ of finite Galois extensions $K$ of $F$ in $\bar{F}$ such that $T$ splits over $K$.

As an immediate consequence of Lemma 8.4 we obtain for $i=1,2,3$ a canonical isomorphism

$$(8.29) \quad \operatorname{proj \lim}_K (M \otimes X_i(K))_{G(K/F)} = B_i(F,T),$$

with $K$ as before. As we saw in that lemma, the transition morphisms in the projective system are all isomorphisms. So the real content of (8.29) is that $B_i(F,T)$ can be identified with any of the groups $(M \otimes X_i(K))_{G(K/F)}$ (for $K \in K$), these all being canonically isomorphic to each other.

8.7. Definition of global inflation maps for linear algebraic groups. We have already defined three global inflation maps

$$(8.30) \quad H^1_{\text{alg}}(\mathcal{E}_i(K/F), M \otimes A_i(K)) \rightarrow H^1_{\text{alg}}(\mathcal{E}_i(L/F), M \otimes A_i(L))$$

for any $F$-torus $T$ split by $K$ (with $M = X_*(T)$). We did so for any sufficiently big subset $S$ of $V_F$. In this subsection we take $S = V_F$. For $i=3$ the inflation map (8.30) then becomes

$$(8.31) \quad H^1_{\text{alg}}(\mathcal{E}_3(K/F), T(K)) \rightarrow H^1_{\text{alg}}(\mathcal{E}_3(L/F), T(L)).$$

We are now going to generalize (8.31) to an inflation map

$$(8.32) \quad H^1_{\text{alg}}(\mathcal{E}_3(K/F), G(K)) \rightarrow H^1_{\text{alg}}(\mathcal{E}_3(L/F), G(L))$$

defined for any linear algebraic $F$-group $G$. This is easy to do; all the real work was done in proving in Lemma 8.3. Just as in the local case the global inflation
map (8.32) is defined as the composed map
(8.33)
$$H^1_{\text{alg}}(E_3(K/F), G(K)) \to H^1_{\text{alg}}(E_3(K/F)^{\text{inf}}, G(L)) \xrightarrow{\beta} H^1_{\text{alg}}(E_3(L/F), G(L))$$
with the first arrow as in Example 2.2 and the second one as in subsection 2.8.

9. The natural transformations $\kappa_G$ and $\bar{\kappa}_G$ for $H^1_{\text{alg}}$

9.1. Assumptions and notation for this section. Let $K/F$ be a finite Galois extension of fields. In this section we consider a Tate-Nakayama triple $(X, A, \alpha)$ for the Galois group $G(K/F)$ such that

- $X$ is torsion-free as abelian group, and
- $A$ is the $G(K/F)$-module $K \times X$.

When $F$ is local or global we have already seen that there exists a canonical such Tate-Nakayama triple. (The $G(K/F)$-module $X$ is $\mathbb{Z}$ in the local case and $\mathbb{Z}[V_K]_0$ in the global case.) The point of working with $(X, A, \alpha)$ as above is that it allows us to treat the local and global cases simultaneously.

We write $D_X$ for the $F$-group of multiplicative type with $X^*(D_X) = X$. Then $\text{Hom}(X, A) = D_X(K)$, so $\alpha \in H^2(G(K/F), D_X(K))$ provides us with a Galois gerb $1 \to D_X(K) \to E \to G(K/F) \to 1$, and the pointed set $H^1_{\text{alg}}(E, G(K))$ is defined for each linear algebraic $F$-group $G$.

9.2. Review of the algebraic fundamental group. Let $G$ be a connected reductive $F$-group split by $K$. We write $\Lambda_G$ for Borovoi’s [Bor98] algebraic fundamental group of $G$. For any maximal $F$-torus $T$ in $G$ there is a canonical identification $\Lambda_G = X_*(T)/X_*(T_{\text{sc}})$ of $G(K/F)$-modules.

For any torus $T$ split by $K$, we have $\Lambda_T = X_*(T)$. When the derived group of $G$ is simply connected, the natural map $\Lambda_G \to \Lambda_D$ is an isomorphism, where $D$ denotes the quotient of $G$ by its derived group. When $1 \to Z \xrightarrow{\iota} G' \xrightarrow{\pi} G \to 1$ is a $z$-extension, the sequence $0 \to \Lambda_Z \xrightarrow{\iota} \Lambda_G \xrightarrow{\pi} \Lambda_G \to 0$ is easily seen to be exact.

9.3. Construction of $\kappa_G$. For any $F$-torus $T$ split by $K$, the inverse of the map $c$ appearing in Lemma 4.1 provides us with a canonical isomorphism

(9.1) $\kappa_T : H^1_{\text{alg}}(E, T(K)) \to (\Lambda_T \otimes X)_{G(K/F)}$.

In the next result we consider both $H^1_{\text{alg}}(E, G(K))$ and $(\Lambda_G \otimes X)_{G(K/F)}$ as functors from the category of connected reductive $F$-groups $G$ split by $K$ to the category of pointed sets.

Proposition 9.1. There exists a unique natural transformation

(9.2) $\kappa_G : H^1_{\text{alg}}(E, G(K)) \to (\Lambda_G \otimes X)_{G(K/F)}$

that agrees with (9.1) for $F$-tori split by $K$.

Proof. As usual we construct $\kappa_G$ in two stages. In the first stage we consider only those $G$ whose derived group is simply connected. We are then forced to define $\kappa_G$...
as the unique map making
\[
H^1_{\text{alg}}(\mathcal{E}, G(K)) \xrightarrow{\kappa_G} (\Lambda_G \otimes X)_{G(K/F)}
\]
(9.3)
\[
H^1_{\text{alg}}(\mathcal{E}, D(K)) \xrightarrow{\kappa_D} (\Lambda_D \otimes X)_{G(K/F)}
\]

commute, where \( D \) is the quotient of \( G \) by its derived group. It is easy to see that \( \kappa_G \) is functorial in \( G \) (for homomorphisms between groups with simply connected derived group).

In the second stage we use \( z \)-extensions. As is well-known, for any connected reductive \( F \)-group \( G \) split by \( K \), there exists an extension
\[
1 \to Z \xrightarrow{i} G' \xrightarrow{p} G \to 1
\]
such that
- \( Z \) is a central torus in \( G' \),
- \( Z \) is obtained by Weil restriction of scalars from a split \( K \)-torus, and
- \( G'_{\text{der}} \) is simply connected.

This is true even when \( F \) is not perfect. Indeed it is very easy to construct an extension as above if one only asks that \( Z \) be a central subgroup of multiplicative type (take \( G' \) to be the product of \( G_{\text{sc}} \) and the biggest central torus in \( G \)). Then push out along some embedding \( Z \hookrightarrow Z' \) such that
- \( Z' \) is a torus whose character group is a finitely generated free module over \( \mathbb{Z}[G(K/F)] \), and
- \( X^*(Z') \to X^*(Z) \) is surjective

in order to obtain the desired \( z \)-extension.

We contend that
- The map \( p : H^1_{\text{alg}}(\mathcal{E}, G'(K)) \to H^1_{\text{alg}}(\mathcal{E}, G(K)) \) identifies \( H^1_{\text{alg}}(\mathcal{E}, G(K)) \) with the quotient of \( H^1_{\text{alg}}(\mathcal{E}, G'(K)) \) by the action of \( H^1_{\text{alg}}(\mathcal{E}, Z(K)) \).
- The map \( (\Lambda_G \otimes X)_{G(K/F)} \xrightarrow{p} (\Lambda_{G'} \otimes X)_{G(K/F)} \) identifies \( (\Lambda_G \otimes X)_{G(K/F)} \) with the quotient of \( (\Lambda_{G'} \otimes X)_{G(K/F)} \) by the action of \( (\Lambda_Z \otimes X)_{G(K/F)} \).
- The map \( \kappa_{G'} \) constructed in the first stage is equivariant with respect to the group \( H^1_{\text{alg}}(\mathcal{E}, Z(K)) = (\Lambda_Z \otimes X)_{G(K/F)} \).

The first item will follow from Proposition 2.8 once we check that \( \mathcal{E} \) satisfies the two assumptions made in subsection 2.13. It is clear that Assumption 1 holds, because we are assuming in this section that \( X^*(D_X) = X \) is torsion-free. Assumption 2 holds due to the isomorphism (9.1) and the right-exactness of the functor \( M \mapsto (M \otimes X)_{G(K/F)} \).

The second item is a restatement of the exactness of
\[
(\Lambda_Z \otimes X)_{G(K/F)} \xrightarrow{i} (\Lambda_{G'} \otimes X)_{G(K/F)} \xrightarrow{p} (\Lambda_G \otimes X)_{G(K/F)} \to 0,
\]
its a consequence of the exactness of \( 0 \to \Lambda_Z \xrightarrow{i} \Lambda_{G'} \xrightarrow{p} \Lambda_G \to 0 \). The third item is evident from the way \( \kappa_{G'} \) was defined.
The three items we just verified imply that there exists a unique bottom horizontal arrow making the diagram
\[
\begin{align*}
H^1_{\text{alg}}(E, G'(K)) & \longrightarrow (\Lambda_{G'} \otimes X)_{G(K/F)} \\
\kappa' & \\
H^1_{\text{alg}}(E, G(K)) & \longrightarrow (\Lambda_{G} \otimes X)_{G(K/F)}
\end{align*}
\] (9.4)
commute, and clearly we are forced to take \(\kappa_G\) to be this arrow. It follows easily from Lemma 2.4.4 in [Kot84] that \(\kappa_G\) is well-defined (independent of the choice of \(z\)-extension \(G'\)) and that it is functorial in \(G\).

\[\square\]

9.4. Construction of \(\bar{\kappa}_G\). We continue with \((X, A, \alpha)\) as in subsection 9.1. For any intermediate field \(E\) (for \(K/F\)) we obtain a Tate-Nakayama triple \((X, A, \alpha_E)\) for \(K/E\), where \(\alpha_E\) denotes the restriction of \(\alpha\) to the subgroup \(G(K/E)\) of \(G(K/F)\). We denote by \(E'\) the preimage of \(G(K/E)\) under \(E \rightarrow G(K/F)\). We again consider a connected reductive \(F\)-group \(G\) split by \(K\).

For the definition of the restriction map occurring as the left vertical map in the next lemma see Example 2.1.

Lemma 9.2. The square
\[
\begin{align*}
H^1_{\text{alg}}(E, G(K)) & \longrightarrow (\Lambda_{G} \otimes X)_{G(K/F)} \\
\kappa_G & \\
\text{Res} & \\
H^1_{\text{alg}}(E', G(K)) & \longrightarrow (\Lambda_{G} \otimes X)_{G(K/E)}
\end{align*}
\] (9.5)
commutes. Here the right vertical arrow sends the class of \(y \in \Lambda_{G} \otimes X\) to the class of \(\sum_{\sigma \in G(K/E) \backslash G(K/F)} \sigma y\).

Proof. Using (9.4) (and its analog for \(E'\)), we reduce to the case in which the derived group of \(G\) is simply connected. Next, using (9.3) (for \(E\) and \(E'\)), we reduce to the case in which \(G\) is a torus. Finally, tori are handled by Lemma 4.2. \(\square\)

In the extreme case when the intermediate field \(E\) is \(K\), the bottom horizontal arrow in (9.5) is a map
\[
\text{Hom}_K(D_X, G)/G(K) \rightarrow \Lambda_{G} \otimes X.
\] (9.6)
We claim that the map (9.6) is \(G(K/F)\)-equivariant. Indeed, the strategy of the proof of the previous lemma reduces us to the case of tori, for which the claim is obvious. We now define
\[
\bar{\kappa}_G : [\text{Hom}_K(D_X, G)/G(K)]^{G(K/F)} \rightarrow (\Lambda_{G} \otimes X)^{G(K/F)}
\] to be the map obtained by applying the functor of \(G(K/F)\)-invariants to (9.6).

9.5. Compatibility of \(\kappa_G\) and \(\bar{\kappa}_G\). As an immediate consequence of Lemma 9.2 we obtain the following result.

Lemma 9.3. The square
\[
\begin{align*}
H^1_{\text{alg}}(E, G(K)) & \longrightarrow (\Lambda_{G} \otimes X)_{G(K/F)} \\
\kappa_G & \\
[\text{Hom}_K(D_X, G)/G(K)]^{G(K/F)} & \longrightarrow (\Lambda_{G} \otimes X)^{G(K/F)}
\end{align*}
\] (9.8)
commutes. The right vertical arrow $N$ is the norm map for $K/F$.

10. The set $B(F,G)$

10.1. Notation. In this section $F$ is a local or global field. We fix a separable closure $\bar{F}$ of $F$ and put $\Gamma = \text{Gal}(\bar{F}/F)$. For any finite Galois extension $K/F$ inside $\bar{F}$ there is a canonical Tate-Nakayama triple $(X, A, \alpha)$ in which $A$ is the $G(K/F)$-module $K^\times$. In this section we denote this triple by $(X(K), K^\times, \alpha(K/F))$ in order to keep track of its dependence on $K/F$, and we write $\mathbb{D}_{K/F}$ for the $F$-group of multiplicative type with character group $X(K)$.

In the local case the triple is the one considered in section 5, so $X(K) = \mathbb{Z}$, $\mathbb{D}_{K/F} = \mathbb{G}_m$ and $\alpha(K/F)$ is the fundamental class in $H^2(K/F, \mathbb{G}_m(K))$. In the global case $X(K)$, $\alpha(K/F)$ are the objects $X_3$, $\alpha_3$ from subsection 6.2, with $S$ chosen to be the set of all places of $F$. So, in the global case, $X(K) = \mathbb{Z}[V_K]_0$, $\mathbb{D}_{K/F} = T_{K/F}$ and $\alpha(K/F)$ is Tate’s canonical class $\alpha_3 \in H^2(K/F, T_{K/F}(K))$.

We choose an extension

$$1 \to \mathbb{D}_{K/F}(K) \to \mathcal{E}(K/F) \to G(K/F) \to 1$$

whose associated cohomology class is $\alpha(K/F)$. In this section we are using a unified system of notation for the local and global cases, so that we can give uniform statements and proofs. In the global case $\mathcal{E}(K/F)$ was previously denoted by $\mathcal{E}_3(K/F)$.

10.2. Definition of $B(F,G)$. For any linear algebraic $F$-group $G$ we define a pointed set $B(F,G)$ by

$$B(F,G) := \text{inj lim}_K H^1_{\text{alg}}(\mathcal{E}(K/F), G(K)).$$

The colimit is taken over the set of finite Galois extensions $K$ of $F$ in $\bar{F}$. For $L \supset K$, the transition map is the inflation map

$$(10.1)$$

$$H^1_{\text{alg}}(\mathcal{E}(K/F), G(K)) \to H^1_{\text{alg}}(\mathcal{E}(L/F), G(L))$$

defined in section 8 (see (8.2) in the local case and (8.32) in the global case). The transition maps are easily seen to be transitive.

In order to define $B(F,G)$ we had to choose a separable closure $\bar{F}$. Just as for Galois cohomology, this choice is unimportant: an isomorphism $\phi$ from $\bar{F}$ to another separable closure $\bar{F}'$ induces an isomorphism $\phi_*$ from the set $B(F,G)$ formed using $\bar{F}$ to the one formed using $\bar{F}'$, and this induced isomorphism is independent of the choice of $\phi$. (To see that $\phi_*$ is well-defined one uses the vanishing of the groups $H^1(G(K/F), \mathbb{D}_{K/F}(K))$, and to see that $\phi_*$ is independent of the choice of $\phi$, one uses that inner automorphisms by elements in $\mathcal{E}(K/F)$ induce trivial automorphisms of $H^1_{\text{alg}}(\mathcal{E}(K/F), G(K))$.)

For a torus $T$ over a global field $F$, it is clear from the definitions that

$$(10.2)$$

$$B(F,T) = B_3(F,T).$$

(The group $B_3(F,T)$ was defined in subsection 8.6.)
10.3. **The maps $p$ and $j$.** Let $K$ and $L$ be finite Galois extensions of $F$ in $\bar{F}$ with $K \subset L$. We are now going to define a canonical injection $p : X(K) \to X(L)$ and a canonical surjection $j : X(L) \to X(K)$. In the local case $X(K) = X(L) = \mathbb{Z}$, and we take
- $p$ to be multiplication by $[L : K]$,
- $j$ to be the identity map.

In the global case we take $p, j$ to be the maps $p_3, j_3$ defined in subsection 8.3.

**Lemma 10.1.** The following statements hold.

1. $p \circ j = N_{L/K}$.
2. $j \circ p = [L : K]$.
3. The map $j : X(L) \to X(K)$ factors through the coinvariants of $G(L/K)$ on $X(L)$, inducing an isomorphism

$$\gamma : X(L)_{G(L/K)} \to X(K).$$

**Proof.** The local case is obvious and the global case is part of Lemma 8.2. $\square$

10.4. **The protorus $\mathbb{D}_F$ over $F$.** We define $\mathbb{D}_F$ to be the protorus over $F$ whose character group is

$$X^*(\mathbb{D}_F) := \lim_{\rightarrow} X(K),$$

with transition maps $p : X(K) \hookrightarrow X(L)$. Thus $\mathbb{D}_F = \text{proj lim}_K \mathbb{D}_{K/F}$.

10.5. **Concrete description of $X^*(\mathbb{D}_F)$ in the local case.** In this subsection $F$ is a local field. We then have $X^*(\mathbb{D}_F) = Br^*(F)$, where

$$Br^*(F) := \begin{cases} \mathbb{Q} & \text{if } F \text{ is nonarchimedean,} \\ \frac{1}{2}\mathbb{Z} & \text{if } F = \mathbb{R}, \\ \mathbb{Z} & \text{if } F = \mathbb{C}. \end{cases}$$

In all cases $Br^*(F)$ is an extension of $Br(F)$ by $\mathbb{Z}$. In other words, there are natural short exact sequences

$$0 \to \mathbb{Z} \to Br^*(F) \to Br(F) \to 0.$$

10.6. **Concrete description of $X^*(\mathbb{D}_F)$ in the global case.** In this subsection $F$ is a global field. For each place $u$ of $F$ we have $X^*(\mathbb{D}_{F_u}) = Br^*(F_u)$. We are going to introduce an analogous global group $Br^*(F)$. First we recall that $Br(F)$ can be identified with

$$\text{ker} \left[ \bigoplus_{u \in V_F} Br(F_u) \to \mathbb{Q}/\mathbb{Z} \right],$$

the map to $\mathbb{Q}/\mathbb{Z}$ sending $(x_u)_{u \in V_F}$ to $\sum_{u \in V_F} x_u$. This suggests defining $Br^*(F)$ as

$$Br^*(F) := \text{ker} \left[ \bigoplus_{u \in V_F} Br^*(F_u) \to \mathbb{Q} \right],$$

the map to $\mathbb{Q}$ sending $(x_u)_{u \in V_F}$ to $\sum_{u \in V_F} x_u$. There is then an obvious short exact sequence

$$0 \to X(F) \to Br^*(F) \to Br(F) \to 0.$$
We usually consider $X(K)$ for some finite Galois extension $K/F$, but in the exact sequence above we are working with $K = F$. However, we can now apply these definitions to any finite Galois extension $K$ of $F$, obtaining

$$0 \to X(K) \to \text{Br}^*(K) \to \text{Br}(K) \to 0.$$ 

Now we need to see what happens when we vary $K$. Suppose we have $F \subset K \subset L \subset \bar{F}$, with both $K$ and $L$ finite Galois over $F$. There is then a commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & X(K) & \longrightarrow & \text{Br}^*(K) & \longrightarrow & \text{Br}(K) & \longrightarrow & 0 \\
p & & \downarrow{q} & & \downarrow{\text{Res}} & & \\
0 & \longrightarrow & X(L) & \longrightarrow & \text{Br}^*(L) & \longrightarrow & \text{Br}(L) & \longrightarrow & 0.
\end{array}
$$

Here $p$ is our usual map, and $\text{Res}$ is the restriction map for the extension $L/K$. The map $q$ sends $(x_v)_{v \in V_K} \in \text{Br}^*(K)$ to $(y_w)_{w \in V_L} \in \text{Br}^*(L)$, with $y_w = [L_w : K_v]x_v$ when $w$ lies over $v$.

Now pass to the colimit over $K$. Since any element in $\text{Br}(K)$ dies in $\text{Br}(L)$ for sufficiently large $L$, the colimit of the groups $\text{Br}(K)$ (with restriction maps as transition morphisms) is trivial. So we obtain a canonical isomorphism

$$X^*(\mathbb{D}_F) = \text{inj lim}_K X(K) \simeq \text{inj lim}_K \text{Br}^*(K).$$

It is easy to check that the natural map $\text{Br}^*(K) \to X^*(\mathbb{D}_F)$ is injective, and that its image consists of the fixed points of $\text{Gal}(\bar{F}/K)$ on $X^*(\mathbb{D}_F)$. Similarly, for any finite extension $E$ of $F$ in $\bar{F}$, there is a natural identification of $\text{Br}^*(E)$ with the fixed points of $\text{Gal}(\bar{F}/E)$ on $X^*(\mathbb{D}_F)$.

### 10.7. Newton map.

The Newton maps fit together to give a Newton map

$$H^1_{\text{alg}}(\mathcal{E}(K/F), G(K)) \to [\text{Hom}_K(\mathbb{D}_{K/F}, G(K))/G(K)]^{G(K/F)}$$

(10.4) $B(F,G) \to [\text{Hom}_F(\mathbb{D}_F, G(\bar{F}))]^\Gamma$.

The image of $b \in B(F,G)$ under the Newton map is called the Newton point of $b$.

The maps $H^1(G(K/F), G(K)) \hookrightarrow H^1_{\text{alg}}(\mathcal{E}(K/F), G(K))$ fit together to give an injective map

(10.5) $H^1(F,G) \hookrightarrow B(F,G)$,

whose image is the kernel of the Newton map (10.4).

### 10.8. Basic elements.

We denote by $Z(G)$ the center of $G$. The inclusion of $Z(G)$ in $G$ induces an injection

$$\text{Hom}_F(\mathbb{D}_F, Z(G)) \hookrightarrow [\text{Hom}_F(\mathbb{D}_F, G(\bar{F}))]^\Gamma.$$

**Definition 10.2.** We say that $b \in B(F,G)$ is basic if its Newton point lies in the image of (10.6). We write $B(F,G)_{\text{bas}}$ for the set of basic elements in $B(F,G)$.
10.9. Localization. Now suppose that $F$ is global, and consider a place $u$ of $F$. In order to define $B(F,G)$ and $B(F_u,G)$ we have to choose separable closures $\bar{F}$ and $\bar{F}_u$, although, as we have already mentioned, this choice is of no real importance. Now we choose some embedding $\bar{F} \hookrightarrow \bar{F}_u$ over $F$. This embedding gives us, for each finite Galois extension $K$ of $F$ in $\bar{F}$, a place $v$ of $K$ lying over $u$.

The localization maps obtained using these places $v$ are compatible with inflation and therefore yield a map

$$B(F,G) \rightarrow B(F_u,G). \quad (10.7)$$

Just as for ordinary Galois cohomology, the map $(10.7)$ is actually independent of the choice of $F$-embedding $\bar{F} \hookrightarrow \bar{F}_u$.

When $G$ is connected, Corollary 11.3 tells us that the components of any element in the image of the total localization map

$$B(F,G) \rightarrow \prod_{u \in V_F} B(F_u,G). \quad (10.8)$$

are trivial for all but finitely many $u \in V_F$.

The choice of $F$-embedding $\bar{F} \hookrightarrow \bar{F}_u$ yields an $F_u$-homomorphism $\mu' : \mathbb{D}_{F_u} \rightarrow \mathbb{D}_F$ (assembled out of the maps $\mu'_v$ in subsection 7.1), and the induced map

$$[\text{Hom}_{\bar{F}}(\mathbb{D}_F,G)/G(\bar{F})]^\Gamma \rightarrow [\text{Hom}_{\bar{F}_u}(\mathbb{D}_{F_u},G)/G(\bar{F}_u)]^{\Gamma(u)}$$

is easily seen to be independent of the choice of $F$-embedding $\bar{F} \hookrightarrow \bar{F}_u$. Here we are writing $\Gamma(u)$ for the Galois group of $\bar{F}_u/F_u$. Moreover the Newton map is compatible with localization: the diagram

$$
\begin{array}{ccc}
B(F,G) & \longrightarrow & B(F_u,G) \\
\downarrow & & \downarrow \\
[\text{Hom}_{\bar{F}}(\mathbb{D}_F,G)/G(\bar{F})]^\Gamma & \longrightarrow & [\text{Hom}_{\bar{F}_u}(\mathbb{D}_{F_u},G)/G(\bar{F}_u)]^{\Gamma(u)}
\end{array}
$$

commutes.

**Lemma 10.3.** An element $b \in B(F,G)$ is basic if and only if its image $b_u$ in $B(F_u,G)$ is basic for every place $u$ of $F$.

**Proof.** It is clear that a globally basic element is locally basic everywhere. So our real task is to prove the converse, and we now suppose that $b_u$ is basic for every place $u$ of $F$. There exists a finite Galois extension $K/F$ such that $b$ is represented by an algebraic 1-cocycle $(\nu, x)$ of $E_3(K/F)$ in $G(K)$. The Newton point for $b$ is the $G(\bar{F})$-conjugacy class of the $F$-homomorphism

$$\mathbb{D}_F \rightarrow \mathbb{T}_{K/F} \xrightarrow{\nu} G.$$

To show that $b$ is basic we must show that $\nu$ is central.

Now let $u$ be a place of $F$, and let $v$ be any place of $K$ lying over $u$. Choose an $F_u$-embedding $K_v \hookrightarrow \bar{F}_u$. By Lemma 7.3 the Newton point for $b_u$ is the $G(\bar{F}_u)$-conjugacy class of

$$\mathbb{D}_{F_u} \rightarrow \mathbb{G}_m \xrightarrow{\nu_v} G,$$

where $\nu_v$ is the composed homomorphism

$$\mathbb{G}_m \xrightarrow{\nu_v} \mathbb{T}_{K/F} \rightarrow \mathbb{T}_{K/F} \xrightarrow{\nu} G.$$

Because $b_u$ is basic, the homomorphism $\nu_v$ is central. To see that $b$ is basic, we now use the following observations.
• $\mathbb{T}_{K/F} \hookrightarrow G$ is central if and only if $\mathbb{T}_{K/F} \rightarrow \mathbb{T}_{K/F} \twoheadrightarrow G$ is central.
• A homomorphism $\mathbb{T}_{K/F} \rightarrow G$ is central if and only if its composition with $\mu_v : \mathbb{G}_m \rightarrow \mathbb{T}_{K/F}$ is central for every place $v$ of $K$.

\[ \square \]

10.10. **Central extensions by tori.** For any linear algebraic $F$-group $G'$ and central subgroup $Z$, there is an obvious action of $B(F, Z)$ on $B(F, G')$. It comes from the actions discussed just before Lemma 2.8.

**Proposition 10.4.** Let
\[ 1 \rightarrow Z \hookrightarrow G' \twoheadrightarrow G \rightarrow 1 \]
be a short exact sequence of linear algebraic $F$-groups in which $Z$ is a torus that is central in $G'$. Then the natural map
\[ p : B(F, G') \rightarrow B(F, G) \]
is surjective. Moreover the map \[ 10.11 \]
induces a bijection between $B(F, G)$ and the quotient of $B(F, G')$ by the action of $B(F, Z)$. Similarly \[ 10.11 \]
induces a bijection between $B(F, G)_{\text{buc}}$ and the quotient of $B(F, G')_{\text{buc}}$ by the action of $B(F, Z)$.

**Proof.** The first two statements follow from the analogous ones in Proposition 2.8. The statement concerning basic elements uses the additional fact that an element $b' \in B(F, G')$ is basic if and only if its image $b$ in $B(F, G)$ is basic. Indeed, it is clear that $b$ is basic if $b'$ is, and to prove the converse one just applies the next lemma to the central torus $T$ in $G$ obtained as the image of the Newton homomorphism $D_F \rightarrow Z(G)$ for $b \in B(F, G)_{\text{buc}}$.

In the next lemma $k$ is an arbitrary field.

**Lemma 10.5.** Let
\[ 1 \rightarrow Z \hookrightarrow G' \twoheadrightarrow G \rightarrow 1 \]
be a short exact sequence of linear algebraic $k$-groups in which $Z$ is a torus that is central in $G'$. Let $T$ be a central torus in $G$. Then the preimage $T'$ of $T$ under $p : G' \rightarrow G$ is a central torus in $G'$.

**Proof.** This is clear when $G'$ is connected reductive, but the general case requires an argument. It follows from Lemma 2.7 that $T'$ is a torus. In proving the lemma it is harmless to assume that $k$ is algebraically closed. Because $T$ is central in $G$, the commutator morphism $G' \times T' \rightarrow G'$ actually takes values in $Z \hookrightarrow G'$, and a simple computation shows that the morphism $G' \times T' \rightarrow Z$ (given by $(g', t') \mapsto g't'y^{-1}t'^{-1}$) is bimultiplicative. In other words we have a $Z$-valued pairing between $G'$ and $T'$, and, to show that $T'$ is central, we must show that the pairing is trivial.

This pairing can be viewed as a homomorphism $f$ from $G'$ to the constant group scheme over $k$ obtained from the abstract abelian group $A := \text{Hom}(X^*(Z), X^*(T'))$. For this we used Cor. 1.5 in SGA 3, Tome II, Exposé VIII. Of course $A$ is a free abelian group of finite rank, and is therefore torsion-free. We just need to show that $f$ is trivial, and this follows from the fact that $G'$ is a scheme of finite type over $k$. Indeed, if $f^{-1}(a)$ were nonempty for some nonzero $a \in A$, then $f^{-1}(na)$ ($n \in \mathbb{Z}$) would be an infinite disjoint collection of nonempty open and closed subsets of $G'$, contradicting the fact that $G'$ is noetherian. \[ \square \]
11. **The Natural Transformations $\kappa_G$ and $\overline{\kappa}_G$ for $B(F,G)$**

In this section we retain the notation of the previous one. We consider only connected reductive $F$-groups $G$.

11.1. **A preliminary discussion of $\kappa_G$.** For a given finite Galois extension $K/F$ Proposition [9.1](#) provides a functorial map

$$\kappa_G : H^1_{\text{alg}}(\mathcal{E}(K/F), G(K)) \to (\Lambda G \otimes X(K))_{G(K/F)}$$

for any connected reductive $F$-group $G$ split by $K$. Our next task is to show that (11.1) is compatible with inflation and then to define and study a map

$$\kappa_G : B(F,G) \to A(F,G)$$

obtained from (11.1) by passing to the limit over $K$.

11.2. **Compatibility of (11.1) with inflation.** Once more let $L \supset K$ be two finite Galois extensions of $F$ in $\overline{F}$.

**Lemma 11.1.** Let $G$ be a connected reductive group over $F$ that splits over $K$. Then the diagram

$$
\begin{array}{ccc}
H^1_{\text{alg}}(\mathcal{E}(K/F), G(K)) & \xrightarrow{\kappa_G} & (\Lambda G \otimes X(K))_{G(K/F)} \\
\downarrow & & \downarrow \\
H^1_{\text{alg}}(\mathcal{E}(L/F), G(L)) & \xrightarrow{\kappa_G} & (\Lambda G \otimes X(L))_{G(L/F)}
\end{array}
$$

commutes and the right vertical arrow is an isomorphism. Here the left vertical arrow is the inflation map (10.1).

**Proof.** Lemma [10.1](#) shows that the right vertical arrow is an isomorphism, so we need only prove that the diagram commutes. In other words we must show that $\kappa_G = \kappa'_G$, where $\kappa'_G$ is the map defined by going the long way around the square. We prove that $\kappa_G = \kappa'_G$ in three steps.

When $G$ is a torus, we simply appeal to Lemmas [8.1](#) and [8.4](#). When the derived group $G_{\text{der}}$ is simply connected, we write $D$ for the torus obtained as the quotient $G/G_{\text{der}}$. Then the square

$$
\begin{array}{ccc}
H^1_{\text{alg}}(\mathcal{E}(K/F), G(K)) & \xrightarrow{\kappa_G} & (\Lambda G \otimes X(K))_{G(K/F)} \\
\downarrow & & \downarrow \\
H^1_{\text{alg}}(\mathcal{E}(K/F), D(K)) & \xrightarrow{\kappa_D=D'} & (\Lambda D \otimes X(K))_{G(K/F)}
\end{array}
$$

commutes, and the same is true with $\kappa_G$ replaced by $\kappa'_G$. Therefore $\kappa'_G = \kappa_G$ when $G_{\text{der}}$ is simply connected.

In the general case we choose a $z$-extension $G' \to G$ whose kernel is a torus split by $K$. Then the square

$$
\begin{array}{ccc}
H^1_{\text{alg}}(\mathcal{E}(K/F), G'(K)) & \xrightarrow{\kappa_{G'}=\kappa'_G'} & (\Lambda G' \otimes X(K))_{G(K/F)} \\
\downarrow & & \downarrow \\
H^1_{\text{alg}}(\mathcal{E}(K/F), G(K)) & \xrightarrow{\kappa_G} & (\Lambda G \otimes X(K))_{G(K/F)}
\end{array}
$$

commutes, and the same is true with $\kappa_G$ replaced by $\kappa'_G$. Using that the left vertical arrow is surjective (Proposition [2.8](#), we conclude that $\kappa'_G = \kappa_G$. 


11.3. Discussion of $C(G)$. In this section we are concerned only with connected reductive $F$-groups $G$. So the center $Z(G)$ is an $F$-group of multiplicative type. The biggest torus in $Z(G)$ will be denoted by $C(G)$; it is the subgroup of $Z(G)$ corresponding to the quotient of $X^*(Z(G))$ by its torsion subgroup. The inclusion $C(G) \hookrightarrow G$ induces a natural injection

\begin{equation}
\Lambda_{C(G)} \to \Lambda_G.
\end{equation}

11.4. Discussion of $A(F,G)$. Now consider any $\Gamma$-module $\Lambda$ on which some open subgroup of $\Gamma$ acts trivially. Let $K \subset L$ be finite Galois extensions of $F$ contained in $\bar{F}$.

**Lemma 11.2.** Suppose that the action of $\Gamma$ on $\Lambda$ factors through $G(K/F)$. Then the square

\begin{equation}
\begin{array}{ccc}
\Lambda \otimes X(K)_{G(K/F)} & \xrightarrow{N_{K/F}} & (\Lambda \otimes X(K))_{G(K/F)} \\
\id \otimes j & \uparrow & \id \otimes p \\
(\Lambda \otimes X(L))_{G(L/F)} & \xrightarrow{N_{L/F}} & (\Lambda \otimes X(L))_{G(L/F)}
\end{array}
\end{equation}

commutes, and the left vertical arrow is an isomorphism.

**Proof.** This follows directly from Lemma 11.1. \hfill \Box

We are interested in the $\Gamma$-module $\Lambda_G$ with $G$ connected reductive over $F$. Let $K$ be the set of finite Galois extensions $K$ of $F$ in $\bar{F}$ such that the action of $\Gamma$ on $\Lambda_G$ factors through $G(K/F)$. We put

\begin{equation}
A(F,G) := \text{proj lim}_{K \in K} (\Lambda_G \otimes X(K))_{G(K/F)},
\end{equation}

where the transition maps are the isomorphisms $\id \otimes j$ appearing in the lemma above. Because these transition maps are isomorphisms, we can equally well say that

\begin{equation}
A(F,G) = \text{inj lim}_{K \in K} (\Lambda_G \otimes X(K))_{G(K/F)},
\end{equation}

where the transition maps are now the inverses of the isomorphisms $\id \otimes j$ in the lemma.

Recall that $D_F$ is the $F$-group of multiplicative type whose character group is

\begin{equation}
X^*(D_F) := \text{inj lim}_{K \in K} X(K)
\end{equation}

with transition maps $p : X(K) \to X(L)$. Observe that

\begin{equation}
(\Lambda_G \otimes X^*(D_F))^\Gamma = \text{inj lim}_{K \in K} (\Lambda_G \otimes X(K))_{G(K/F)},
\end{equation}

and that, for any torus $T$, there is a canonical bijection

\begin{equation}
(\Lambda_T \otimes X^*(D_F))^\Gamma = \text{Hom}_F(D_F, T).
\end{equation}
Definition 11.3. For any connected reductive $F$-group $G$ we define
\begin{equation}
N : A(F, G) \to (\Lambda_G \otimes X^*(\mathbb{D}_F))^\Gamma
\end{equation}
to be the map resulting from taking the injective limit (over $K \in \mathcal{K}$) of the norm maps $N_{K/F}$ appearing in Lemma 11.2. Here we are using (11.9) and (11.10) to view the source and target of $N$ as injective limits.

We will need the map $N$ in the next proposition.

11.5. The canonical map $\kappa_G : B(F, G) \to A(F, G)$. Lemma 11.1 shows that, in the injective limit over $K \in \mathcal{K}$, the natural transformations $\kappa_G$ of Proposition 9.1 fit together to give a natural transformation
\begin{equation}
\kappa_G : B(F, G) \to A(F, G).
\end{equation}

11.6. The canonical map $\bar{\kappa}_G$. For each $K \in \mathcal{K}$ there is a map (see (9.7))
\begin{equation}
\bar{\kappa}_G : \left[\text{Hom}_F(\mathbb{D}_F, G(\bar{F}))\right]^\Gamma \to (\Lambda_G \otimes X^*(\mathbb{D}_F))^\Gamma.
\end{equation}

We claim that the maps (9.7) are compatible as $K$ varies through $\mathcal{K}$. Indeed, using the usual procedure involving $z$-extensions, we reduce to the case of tori, for which the claim is obvious. So the maps (11.13) fit together to give a map
\begin{equation}
\bar{\kappa}_G : \left[\text{Hom}_F(\mathbb{D}_F, G(\bar{F}))\right]^\Gamma \to (\Lambda_G \otimes X^*(\mathbb{D}_F))^\Gamma.
\end{equation}

11.7. A relation between $\kappa_G(b)$ and the Newton point of $b$. The two propositions in this subsection were inspired by an exchange of email with T. Kaletha and M. Rapoport, who pointed out to me that such results might hold in the framework of this paper.

The next proposition generalizes part of Theorem 1.15 in [RR96], for which a reference to [Kot85] is given. The proof given here should make it clear which results from [Kot85] justify the relevant part of Theorem 1.15 of Rapoport-Richartz. The propositions make use of the map $N$ (see (11.12)).

Proposition 11.4. The square
\begin{equation}
\begin{array}{ccc}
B(F, G) & \xrightarrow{\kappa_G} & A(F, G) \\
\downarrow \text{Newton} & & \downarrow N \\
\left[\text{Hom}_F(\mathbb{D}_F, G(\bar{F}))\right]^\Gamma & \xrightarrow{\bar{\kappa}_G} & (\Lambda_G \otimes X^*(\mathbb{D}_F))^\Gamma
\end{array}
\end{equation}
commutes.

Proof. This follows from Lemma 9.3. $\square$

Before stating the next proposition, we observe that the Newton point of a basic element in $B(F, G)$ lies in $\text{Hom}_F(\mathbb{D}_F, Z(G)) = \text{Hom}_F(\mathbb{D}_F, C(G))$, a group that the isomorphism (11.11) identifies with $(\Lambda_{C(G)} \otimes X^*(\mathbb{D}_F))^\Gamma$. The proposition makes use of the inclusion $i : \Lambda_{C(G)} \hookrightarrow \Lambda_G$ (see (11.18)).

Proposition 11.5. The square
\begin{equation}
\begin{array}{ccc}
B(F, G)_{\text{basic}} & \xrightarrow{\kappa_G} & A(F, G) \\
\downarrow \text{Newton} & & \downarrow N \\
(\Lambda_{C(G)} \otimes X^*(\mathbb{D}_F))^\Gamma & \xrightarrow{i} & (\Lambda_G \otimes X^*(\mathbb{D}_F))^\Gamma
\end{array}
\end{equation}
commutes. Moreover the bottom arrow in the square is injective, so the square lets us read off the Newton point of \( b \in B(F,G)_{bas} \) from \( \kappa_G(b) \).

**Proof.** It follows easily from the previous proposition that the diagram commutes. Now we prove that the bottom arrow in the square is injective. We have already mentioned that \( i : \Lambda_{C(G)} \to \Lambda_G \) is injective. Tensoring with the torsion-free abelian group \( X^*(D_F) \) preserves injectivity, and so does taking \( \Gamma \)-invariants. So the bottom arrow is indeed injective. \( \square \)

**Remark 11.6.** It is clear from Proposition 11.5 that

\[
\text{im}[B(F,G)_{bas} \xrightarrow{\kappa_G} A(F,G)] \subset A_0(F,G),
\]

where \( A_0(F,G) \) denotes the preimage under \( N \) of the subset \( (\Lambda_G \otimes X^*(D_F))^F \) of \( (\Lambda_G \otimes X^*(D_F))^F \). In Propositions 13.4 and 15.5 it will be shown that the inclusion (11.17) is in fact an equality.

11.8. **Compatibility of \( \kappa_G \) with localization.** In this subsection we consider a global field \( F \), a place \( u \) of \( F \), and a connected reductive \( F \)-group \( G \). For convenience we fix an \( F \)-embedding \( \overline{F} \to \overline{F}_u \) of separable closures of \( F \) and \( F_u \).

There is a natural localization map

\[
A(F,G) \to A(F_u,G),
\]

defined as follows. Once again let \( K \) be the set of finite Galois extensions \( K \) of \( F \) in \( \overline{F} \) such that \( \text{Gal} (\overline{F}/K) \) acts trivially on \( \Lambda_G \). For \( K \in K \) there is a natural homomorphism

\[
(\Lambda_G \otimes X(K))_{G(K/F)} \to (\Lambda_G)_{G(K/F_u)},
\]

where \( v \) is the place of \( K \) determined by our chosen embedding \( \overline{F} \to \overline{F}_u \). When \( G \) is a torus \( T \) split by \( K \), then \( \Lambda_G \) is the cocharacter group \( M \), and the map (11.19) was defined in subsection 7.7. In general we define (11.19) in exactly the same way, simply replacing \( M \) by \( \Lambda_G \) everywhere. We then obtain (11.18) by taking the colimit over \( K \) (see (11.19)) of the maps (11.19).

We are now going to prove the following compatibility between \( \kappa_G \) and localization.

**Lemma 11.7.** For every connected reductive \( F \)-group \( G \) the square

\[
\begin{array}{ccc}
B(F,G) & \xrightarrow{(10.7)} & B(F_u,G) \\
\kappa_G \downarrow & & \kappa_G \downarrow \\
A(F,G) & \xrightarrow{(11.18)} & A(F_u,G)
\end{array}
\]

commutes.

**Proof.** All four maps in the square are functorial in \( G \). Therefore we may reduce to the case in which the derived group is simply connected, and from there to the case of a torus. (The two reduction steps follow the same pattern as in the proofs of Lemma 11.1 and Proposition 11.5) Tori can be handled using Lemma 7.3. \( \square \)
12. A generalization of Shapiro’s lemma

In this section we are going to prove a version of Shapiro’s lemma for sets $H^1_Y(E, M)$ like the ones studied before, but with $M$ now allowed to be nonabelian. Then we will give applications involving $B(F, G)$, including a discussion of corestriction maps in the case of tori.

12.1. Two definitions. The two definitions that follow are standard in the theory of nonabelian cohomology for a group $G$.

- A $G$-group $M$ is a group $M$ equipped with an action of $G$ by automorphisms of $M$.
- A $G$-action of a $G$-group $M$ on a $G$-set $Y$ is an action of $M$ on $Y$ such that the action map $M \times Y \to Y$ is $G$-equivariant.

Giving a $G$-action of a $G$-group $M$ on $Y$ is the same as giving an action of $M \rtimes G$ on $Y$.

12.2. The set $H^1_Y(E, M)$ for nonabelian $G$-groups $M$. In subsection 3.3 we defined sets $H^1_Y(E, M)$. Now we want to generalize the definition by allowing $M$ to be nonabelian. As before our starting point is an extension

$$1 \to A \to E \to G \to 1$$

of $G$ by a $G$-module $A$. We still insist that $A$ be abelian, but we are going to consider an arbitrary (possibly nonabelian) $G$-group $M$, which we also view as an $E$-group, with $A$ acting trivially.

We regard $\text{Hom}(A, M)$ as a $G$-set in the usual way: $\sigma \in G$ acts on $f \in \text{Hom}(A, M)$ by the rule $(\sigma f)(a) = \sigma(f(\sigma^{-1}a))$. There are natural $G$-actions of the $G$-group $M$ on itself and on the $G$-set $\text{Hom}(A, M)$. Equivalently, there are natural actions of $M \rtimes G$ on both $M$ and $\text{Hom}(A, M)$. These actions are spelled out in the following definition.

**Definition 12.1.**

1. The $G$-group $M$ acts on itself by conjugation. The corresponding action of $M \rtimes G$ on $M$ is as follows: $m\sigma \in M \rtimes G$ transforms $m_1 \in M$ into $m\sigma(m_1)m^{-1}$.
2. The $G$-group $M$ acts on the $G$-set $\text{Hom}(A, M)$ through its action on $M$. The corresponding action of $M \rtimes G$ is as follows: $m\sigma \in M \rtimes G$ transforms $f \in \text{Hom}(A, M)$ into $\text{Int}(m) \circ \sigma(f)$.

We will also use the canonical surjective homomorphism $M \rtimes E \to M \rtimes G$ (given by the identity on $M$ and the canonical surjection $E \to G$ on $E$) to make $M \rtimes E$ act on $M$ and $\text{Hom}(A, M)$.

In order to define $H^1_Y(E, M)$ we need two more ingredients, namely an $(M \rtimes G)$-set $Y$ and an $(M \rtimes G)$-map $\xi : Y \to \text{Hom}(A, M)$. We require that $(Y, \xi)$ satisfy the following condition:

$$\xi(y)(A) \subset M_y \text{ for all } y \in Y.$$  

Here we are writing $M_y$ for the stabilizer of $y$ in $M$.

**Example 12.2.** Suppose that $M$ is abelian. As in subsection 3.3 let us consider a $G$-module $Y$ and $G$-module map $\xi : Y \to \text{Hom}(A, M)$. Because $M$ is abelian, it acts trivially on $\text{Hom}(A, M)$, and so $\xi$ becomes an $(M \rtimes G)$-map if we make $M$ act trivially on $Y$. The condition (12.1) is then automatically satisfied.
Example 12.3. Consider a Galois gerb $1 \to D(K) \to E \to G(K/F) \to 1$ (see subsection 2.2). Let $G$ be a linear algebraic group over $F$. Then

- take $M = G(K)$,
- take $Y = \text{Hom}_K(D,G)$, and
- take $\xi$ to be the natural map $\text{Hom}_K(D,G) \to \text{Hom}(D(K),G(K))$.

The condition (12.1) is automatically satisfied.

Returning to the general discussion, we now want to define $H^1_{\nu}(E,M)$ in such a way that it agrees with the previously (in subsection 3.3) defined notion in the first example and with $H^1_{\text{alg}}(E,G(K))$ in the second example. It is clear how to do this. We begin by defining suitable 1-cocycles, the set of which will be denoted $Z^1_{\nu}(E,M)$. By definition, an element in $Z^1_{\nu}(E,M)$ is a pair $(\nu,x)$ consisting of $\nu \in Y$ and $x \in Z^1(E,M)$ satisfying the following two conditions:

1. The restriction $x_0$ of $x$ to $A$ is the homomorphism $A \to M$ obtained as the image of $\nu$ under $\xi$.
2. $x_w \sigma(\nu) = \nu$ for any $w \in E$, with $\sigma$ denoting the image of $w$ under $E \to G$.

Use the 1-cocycle $x$ to define a homomorphism $\varphi_x : E \to M \times E$ (thus $\varphi_x(w) := x_w w$ for all $w \in E$). The 1-cocycle condition for $x$ shows that the element $x_0 \in \text{Hom}(A,M)$ is fixed by the subgroup $\varphi_x(E)$ of $M \times E$. Observe that (2) can be reformulated as the condition that $\nu$ be fixed by $\varphi_x(E) \subset M \times E$, with $M \times E$ acting on $Y$ through the canonical surjection $M \times E \to M \times G$ defined earlier.

The group $M$ acts on $Z^1_{\nu}(E,M)$ in the obvious way (the action of $m \in M$ sends $(\nu,x)$ to $(mv, w \mapsto mx_w w(m)^{-1})$), and $H^1_{\nu}(E,M)$ is by definition the quotient of $Z^1_{\nu}(E,M)$ by the action of $M$.

Remark 12.4. We now comment on the significance of the condition (12.1) we imposed on $(Y,\xi)$. Suppose for a moment that we did not impose it. We could still define $Z^1_{\nu}(E,M)$ and $H^1_{\nu}(E,M)$, in exactly the same way. Now, for any $\nu \in Y$ for which there exists $x \in Z^1(E,M)$ with $(\nu,x) \in Z^1_{\nu}(E,M)$, the conditions (1) and (2) would force the inclusion $\xi(\nu)(A) \subset M_\nu$ to hold. In other words, elements $\nu \in Y$ for which $\xi(\nu)(A)$ is not contained in $M_\nu$ are irrelevant when forming $H^1_{\nu}(E,M)$. So we lose nothing by imposing condition (12.1), and in fact we even gain something, because doing so will make the discussion of the maps $\Phi(f,g,h)$ in subsection 12.7 a bit simpler. This is the main reason for imposing (12.1).

There is an abstract Newton map

$$H^1_{\nu}(E,M) \to (M \setminus Y)^G$$

in this context (induced by $(\nu,x) \mapsto \nu$), and it would be easy enough to analyze its fibers (as we did before in the special case of $H^1_{\text{alg}}(E,G(K))$). However, in proving a version of Shapiro’s lemma for $H^1_{\nu}(E,M)$, it is more useful to analyze the fibers of the map

$$(12.2) \quad H^1_{\nu}(E,M) \to H^1(E,M)$$

induced by $(\nu,x) \mapsto x$.

12.3. Fibers of the map (12.2). Let $x \in Z^1(E,M)$, and let $[x]$ denote its class in $H^1(E,M)$. As above we use $x$ to define $x_0 \in \text{Hom}(A,M)$ and a homomorphism...
\( \varphi_x : E \to M \times E \). For any \((M \times G)\)-set \(X\) we denote by \(X_*\) the twisted \(E\)-set obtained by making \(E\) act on \(X\) through the homomorphism

\[
E \xrightarrow{\varphi_x} M \times E \to M \times G.
\]

The examples we have in mind for \(X\) are \(M\) and \(\text{Hom}(A, M)\) (with the actions in Definition 12.1), as well as \(Y\). We may view \(\xi\) as an \(E\)-map \(\xi : Y_* \to \text{Hom}(A, M)_*\).

Observe that \(x_0\) lies in the fixed-point set \((\text{Hom}(A, M)_*)^E\), so the fiber \(\xi^{-1}(x_0)\) is stable under the action of \(E\) on \(Y_*\).

**Lemma 12.5.** The fiber of (12.2) over the class \([x]\) of \(x\) is equal to the quotient set \(M_x \setminus Y_x\), where

- \(M_x := (M_x)^E\),
- \(Y_x := (\xi^{-1}(x_0))^E \subset Y_*\).

**Proof.** From the definition of \(Z^1_Y(E, M)\), we see that the set of \(\nu \in Y\) such that \((\nu, x) \in Z^1_Y(E, M)\) is equal to \(Y_x\). Two such pairs \((\nu, x), (\nu', x)\) are cohomologous if and only if there exists \(m \in M\) such that \(m\nu = \nu'\) and \(mx_w(m)^{-1} = x_w\), and the second of these equalities just says that \(m\) is fixed by the (twisted) action of \(E\). \(\square\)

**12.4. Naturality of \(H^1_Y(E, M)\) with respect to \((M, Y, \xi)\).** Let \((M', Y', \xi')\) be another triple like \((M, Y, \xi)\), and suppose we have a \(G\)-homomorphism \(f : M \to M'\) and an \((M \times G)\)-map \(g : Y \to Y'\) such that

\[
\begin{array}{ccc}
Y & \xrightarrow{\xi} & \text{Hom}(A, M) \\
g \downarrow & & f \downarrow \\
Y' & \xrightarrow{\xi'} & \text{Hom}(A, M')
\end{array}
\]

commutes. Then there is an induced map

\[
(12.4) \quad H^1_Y(E, M) \to H^1_Y'(E, M')
\]

sending the class of \((\nu, x)\) to that of \((g(\nu), f(x))\).

**12.5. Restriction maps for \(H^1_Y(E, M)\).** Let \(H\) be a subgroup of \(G\), and form an extension

\[
1 \to A \to E' \to H \to 1
\]

by taking \(E'\) to be the preimage of \(H\) under \(E \to G\). We then obtain a restriction map

\[
(12.5) \quad \text{Res} : H^1_Y(E, M) \to H^1_Y(E', M)
\]

by sending the class of \((\nu, x)\) to the class of \((\nu, x')\), where \(x'\) is the restriction of \(x\) to the subgroup \(E'\).

**12.6. More forms of naturality.** The next three subsections will study more general forms of naturality in which \(E\) is allowed to vary. These will be needed in section 13.
12.7. The map $\Phi(f, g, \tilde{h})$. This subsection is a generalization to nonabelian $M$ of subsection 3.10. All the maps in that subsection generalize easily, but, to keep things a little simpler, here we consider only the situation in which the homomorphism $\rho : G' \to G$ in subsection 3.10 is the identity map on $G$. This special case suffices for the needs of section 14.

We consider $1 \to A \to E \to G \to 1$, a $G$-group $M$, an $(M \times G)$-set $Y$ and an $(M \times G)$-map $\xi : Y \to \text{Hom}(A, M)$ satisfying (12.1). We may then form the pointed set $H^1_Y(E, M)$. In addition we consider another such collection of objects: $1 \to A' \to E' \to G \to 1$, a $G$-group $M'$, an $(M' \times G)$-set $Y'$ and an $(M' \times G)$-map $\xi' : Y' \to \text{Hom}(A', M')$ satisfying (12.1). We may then form the pointed set $H^1_Y(E', M')$.

Given some additional data $f, g, \tilde{h}$, we are going to define a map

$$\Phi(f, g, \tilde{h}) : H^1_Y(E, M) \to H^1_Y(E', M').$$

These data are as follows:

- a $G$-homomorphism $f : M \to M'$,
- an $(M \times G)$-map $g : Y \to Y'$, where we are using $f$ to view $Y'$ as $M$-set,
- a homomorphism $\tilde{h} : E \to E'$ of extensions,

satisfying the requirement that the diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\xi} & \text{Hom}(A, M) \\
\downarrow & & \downarrow f \\
Y' & \xrightarrow{\xi'} & \text{Hom}(A', M')
\end{array}
$$

commute, where $h$ is the unique map $A \to A'$ such that

$$
1 \longrightarrow A \longrightarrow E \longrightarrow G \longrightarrow 1
$$

commutes.

We define $\Phi(f, g, \tilde{h})$ to be the map sending the class of $(\nu, x)$ to the class of $(g(\nu), x')$, where $x'$ is the unique 1-cocycle of $E'$ in $M'$ such that

- the restriction of $x'$ to $A'$ is equal to the map $\xi'(g(\nu)) : A' \to M'$, and
- the pullback of $x'$ to $E$ (via $\tilde{h}$) is equal to $f(x)$.

In checking that $(g(\nu), x')$ satisfies condition (2) in the definition of $Z^1_Y(E', M')$, one needs to use (12.1) (for $(Y', \xi')$) in addition to the fact that $(\nu, x)$ satisfies (2).

It is easy to see that the map $\Phi(f, g, \tilde{h})$ depends only on the $A'$-conjugacy class of $\tilde{h}$. So, when $H^1(G, A')$ vanishes, the dependence of $\Phi(f, g, \tilde{h})$ on $\tilde{h}$ is through $h$.

When $H^1(G, A), H^1(G, A')$ both vanish, the sets $H^1_Y(E, M), H^1_Y(E', M')$ depend (up to canonical isomorphism) only on the cohomology classes $\alpha \in H^2(G, A), \alpha' \in H^2(G, A')$ associated to $E, E'$, and, whenever we have $f, g, \tilde{h}$ such that

- (12.6) commutes, and
- $h(\alpha) = \alpha'$,
we obtain a well-defined map
\[ \Phi(f, g, h) : H^1_Y(E, M) \to H^1_{Y'}(E', M') \]
by putting \( \Phi(f, g, h) := \Phi(f, g, \tilde{h}) \) for any homomorphism \( \tilde{h} \) making (12.7) commute.

The next lemma concerns compositions of maps of type \( \Phi \). We consider triples \((f_1, g_1, \tilde{h}_1)\) and \((f_2, g_2, h_2)\) such that \( \Phi(f_1, g_1, \tilde{h}_1) : H^1_Y(E, M) \to H^1_{Y'}(E', M') \) and \( \Phi(f_2, g_2, h_2) : H^1_Y(E', M') \to H^1_{Y''}(E'', M'') \) are defined. It is easy to check that the triple \((f_2 \circ f_1, g_2 \circ g_1, \tilde{h}_2 \circ \tilde{h}_1)\) satisfies the requirement needed in order to define the map \( \Phi(f_2 \circ f_1, g_2 \circ g_1, \tilde{h}_2 \circ \tilde{h}_1) \).

**Lemma 12.6.** The composed map
\[ H^1_Y(E, M) \xrightarrow{\Phi(f_1, g_1, \tilde{h}_1)} H^1_Y(E', M') \xrightarrow{\Phi(f_2, g_2, h_2)} H^1_{Y''}(E'', M'') \]
is equal to \( \Phi(f_2 \circ f_1, g_2 \circ g_1, \tilde{h}_2 \circ \tilde{h}_1) \).

**Proof.** Easy. \( \square \)

12.8. The map \( \Psi(g, \tilde{p}) \). We continue to consider \( 1 \to A \to E \to G \to 1, \ M, \ Y \) and \( \xi : Y \to \text{Hom}(A, M) \), as well as their primed versions. In this subsection, however, we make the further assumption that \( M' \) coincides with \( M \).

Given some additional data \( g, \tilde{p} \), we are going to define a pullback map
\[ \Psi(g, \tilde{p}) : H^1_Y(E, M) \to H^1_{Y'}(E', M). \]
These data are as follows:
- an \((M \times G)\)-map \( g : Y \to Y' \),
- a homomorphism \( \tilde{p} : E' \to E \) of extensions,

satisfying the requirement that the diagram
\[
\begin{array}{ccc}
Y & \xrightarrow{\xi} & \text{Hom}(A, M) \\
\downarrow g & & \downarrow p \\
Y' & \xrightarrow{\xi'} & \text{Hom}(A', M)
\end{array}
\]
(12.8)
commute, where \( p \) is the unique map \( A' \to A \) such that
\[
\begin{array}{c}
1 \quad \to \\
\uparrow p & & \uparrow \tilde{p} \\
A' \quad \to \\
\to E' \quad \to \\
\to G \quad \to 1
\end{array}
\]
commutes.

We define \( \Psi(g, \tilde{p}) \) to be the map sending the class of \((\nu, x)\) to the class of \((g(\nu), x')\), where \( x' \) is the pullback of \( x \) to \( E' \) (via \( \tilde{p} \)). It is easy to see that the map \( \Psi(g, \tilde{p}) \) depends only on the \( A\)-conjugacy class of \( \tilde{p} \); checking this involves using (12.7) for \((Y, \xi)\). So, when \( H^1(G, A) \) vanishes, the dependence of \( \Psi(g, \tilde{p}) \) on \( \tilde{p} \) is through \( p \).

When \( H^1(G, A) \), \( H^1(G, A') \) both vanish, the sets \( H^1_Y(E, M), H^1_{Y'}(E', M') \) depend (up to canonical isomorphism) only on the cohomology classes \( \alpha \in H^2(G, A), \alpha' \in H^2(G, A') \) associated to \( E, E' \), and, whenever we have \( g, p \) such that
- (12.8) commutes, and
- \( \tilde{p}(\alpha') = \alpha \),

we obtain a well-defined map
\[ \Phi(f, g, h) : H^1_Y(E, M) \to H^1_{Y'}(E', M') \]
by putting \( \Phi(f, g, h) := \Phi(f, g, \tilde{h}) \) for any homomorphism \( \tilde{h} \) making (12.7) commute. 

we obtain a well-defined map

$$\Psi(g, p) : H^1_Y(E, M) \to H^1_Y(E', M)$$

by putting $$\Psi(g, p) := \Psi(g, \tilde{p})$$ for any homomorphism $$\tilde{p}$$ making (12.9) commute.

The next lemma concerns compositions of maps of type $$\Psi$$. We consider pairs $$(g_1, \tilde{p}_1)$$ and $$(g_2, \tilde{p}_2)$$ such that $$\Psi(g_1, \tilde{p}_1) : H^1_Y(E, M) \to H^1_Y(E', M)$$ and $$\Psi(g_2, \tilde{p}_2) : H^1_Y(E', M) \to H^1_Y(E'', M)$$ are defined. It is easy to check that $$(g_2 \circ g_1, \tilde{p}_1 \circ \tilde{p}_2)$$ is such that $$\Psi(g_2 \circ g_1, \tilde{p}_1 \circ \tilde{p}_2)$$ is defined.

**Lemma 12.7.** The composed map

$$H^1_Y(E, M) \xrightarrow{\Psi(g_1, \tilde{p}_1)} H^1_Y(E', M) \xrightarrow{\Psi(g_2, \tilde{p}_2)} H^1_Y(E'', M)$$

is equal to $$\Psi(g_2 \circ g_1, \tilde{p}_1 \circ \tilde{p}_2)$$.

**Proof.** Easy. □

12.9. *A compatibility between maps of type $$\Phi$$ and $$\Psi$*. In the next lemma we suppose that we are given a commutative diagram

$$
\begin{array}{ccc}
E & \xleftarrow{\tilde{p}} & E_1 \\
\downarrow{\tilde{h}} & & \downarrow{\tilde{h}_1} \\
E' & \xleftarrow{\tilde{p}'} & E'_1
\end{array}
$$

(12.10)

of extensions, a $$G$$-homomorphism $$f : M \to M'$$, and a commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{g'} & Y_1 \\
\downarrow{g} & & \downarrow{g_1} \\
Y' & \xrightarrow{g''} & Y'_1
\end{array}
$$

(12.11)

in which the top arrow is a map of $$(M \rtimes G)$$-sets, the bottom arrow is a map of $$(M' \rtimes G)$$-sets, and the two vertical arrows are maps of $$(M \rtimes G)$$-sets. We further assume that the triples $$(f, g, \tilde{h})$$ and $$(f, g_1, \tilde{h}_1)$$ satisfy the requirements needed to define $$\Phi(f, g, \tilde{h})$$ and $$\Phi(f, g_1, \tilde{h}_1)$$. Finally, we assume that the pairs $$(g', \tilde{p})$$ and $$(g'', \tilde{p}')$$ satisfy the requirements needed to define $$\Psi(g', \tilde{p})$$ and $$\Psi(g'', \tilde{p}')$$.

**Lemma 12.8.** Under the assumptions above the square

$$
\begin{array}{ccc}
H^1_Y(E, M) & \xrightarrow{\Psi(g', \tilde{p})} & H^1_Y(E_1, M) \\
\downarrow{\Phi(f, g, \tilde{h})} & & \downarrow{\Phi(f, g_1, \tilde{h}_1)} \\
H^1_Y(E', M') & \xrightarrow{\Psi(g'', \tilde{p}')} & H^1_Y(E'_1, M')
\end{array}
$$

(12.12)

commutes.

**Proof.** Easy. □
12.10. **Coinduction for \( H \)-sets and \( H \)-groups.** Let \( G \) be a group and \( H \) a subgroup. The forgetful functor from \( G\text{-}Sets \) to \( H\text{-}Sets \) has a left adjoint \( L \) and a right adjoint \( R \). It is customary to refer to \( L \) as induction and \( R \) as coinduction. The values of these two functors on an \( H \)-set \( Y \) are given by

- \( L(Y) := G^H \).
- \( R(Y) = \{ f : G \to Y : f(\tau \sigma) = \tau(f(\sigma)) \quad \forall \sigma \in G, \tau \in H \} \).

Here \( G \times Y \) is the quotient of \( G \times Y \) by the \( H \)-action \( \tau(\sigma, y) := (\sigma \tau^{-1}, \tau y) \), and \( \sigma_1 \in G \) acts by \( \sigma_1(\sigma, y) := (\sigma_1 \sigma, y) \). The action of \( \sigma_1 \) on \( R(Y) \) is given by right-translation, i.e. \( (\sigma_1 f)(\sigma) := f(\sigma \sigma_1) \). The more useful of the two adjunction morphisms for \( R \) is the \( H \)-map \( \epsilon : R(Y) \to Y \) given by evaluation at the identity element of \( G \). Clearly \( \epsilon \) restricts to a bijection

\[
(12.13) \quad R(Y)^G \to Y^H
\]

between fixed-point sets.

Now suppose that \( M \) is an \( H \)-group. Then of course \( M \) is an \( H \)-set and so we may form the \( G \)-set \( R(M) \). In fact \( R(M) \) becomes a \( G \)-group for the group structure given by pointwise multiplication of maps: \( (ff')(\sigma) := f(\sigma)f'(\sigma) \), the product on the right being taken in the group \( M \). The functor \( R \) from \( H\text{-Groups} \) to \( G\text{-Groups} \) is right adjoint to the forgetful functor. When \( G \) is a Galois group \( G(K/F) \), groups coinduced from \( H = G(K/E) \) appear naturally in the context of Weil restriction of scalars from \( E \) to \( F \), where \( E \) is an intermediate field for \( K/F \) that is finite over \( F \).

12.11. **Shapiro’s lemma for \( H^1_\text{et}(E,M) \).** Again let \( H \) be a subgroup of \( G \) and form \( E' \subset E \) as in subsection 12.5. Before we discuss Shapiro’s lemma, we need to analyze the following situation. Suppose we are given an \( H \)-action

\[
(12.14) \quad M \times X \to X
\]

of an \( H \)-group \( M \) on an \( H \)-set \( X \). Applying the functor \( R \) (of coinduction from \( H \) to \( G \)) to the action map \((12.14)\), we obtain a \( G \)-action

\[
(12.15) \quad R(M) \times R(X) \to R(X)
\]

of the \( G \)-group \( R(M) \) on the \( G \)-set \( R(X) \). (Here we used that coinduction preserves products. Indeed, because it is a right adjoint, it preserves all small limits.) In other words, \( R(X) \) is an \((R(M) \times G)\)-set. The \( H \)-action of \( M \) on \( X \) can be viewed as an action of \( M \rtimes H \) on \( X \), and the natural map \( R(M) \times H \to M \rtimes H \) (given by \( f \tau \mapsto \epsilon(f) \tau \)) lets us view \( X \) as an \((R(M) \times H)\)-set.

Suppose further that we are given a 1-cocycle \( x \) of \( E \) in \( R(M) \). We then obtain a 1-cocycle \( y \) of \( E' \) in \( M \) by putting

\[
y_{w'} := \epsilon(x_{w'}).
\]

The map \( x \mapsto y \) on 1-cocycles induces the classical Shapiro isomorphism

\[
H^1(E, R(M)) \to H^1(E', M).
\]

We use \( x \) (resp. \( y \)) to form a homomorphism \( \varphi_x : E \to R(M) \rtimes G \) (resp. \( \varphi_y : E' \to M \rtimes H \)). Using these homomorphisms, we obtain a twisted \( E \)-set \( R(X)_x \) and a twisted \( E' \)-set \( X_y \).
Lemma 12.9. The \( E \)-sets \( R(X) \) and \( R_{E'}^{E}(X_{*}) \) are canonically isomorphic. Here we are denoting coinduction from \( E' \) to \( E \) by \( R_{E}^{E} \) in order to distinguish it from coinduction from \( H \) to \( G \). Under this isomorphism \( f_{1} \in R(X) \) corresponds to \( f_{2} \in R_{E'}^{E}(X_{*}) \) when \( f_{2}(w) = \epsilon(xw)f_{1}(\sigma) \), with \( \sigma \) denoting the image of \( w \) under \( E \to G \).

Proof. Easy. \( \square \)

Now we are ready to tackle Shapiro’s lemma. We start with a triple \((M,Y,\xi)\) relevant to \( E' \) rather than \( E \). So \( M \) is an \( H \)-group, \( Y \) is an \( H \)-set equipped with an \( H \)-action of \( M \), and \( \xi : Y \to \text{Hom}(A,M) \) is \((M \rtimes H)\)-equivariant. We may therefore form the set \( H_{1}^{1}(E',M) \).

Applying the functor \( R \) of coinduction from \( H \) to \( G \) to the map \( \xi \), we obtain an \((R(M) \rtimes G)\)-equivariant map
\[
R(\xi) : R(Y) \to R(\text{Hom}(A,M)).
\]
Now observe that \( R(\text{Hom}(A,M)) \simeq \text{Hom}(A,R(M)) \) as \((R(M) \rtimes G)\)-sets. Here \( f_{1} \in R(\text{Hom}(A,M)) \) corresponds to \( f_{2} \in \text{Hom}(A,R(M)) \) when
\[
f_{1}(\sigma)(a) = f_{2}(\sigma^{-1}(a))(\sigma).
\]
Therefore we may equally well regard \( R(\xi) \) as an \((R(M) \rtimes G)\)-map
\[
R(\xi) : R(Y) \to \text{Hom}(A,R(M)),
\]
and so we may form the set \( H_{1}^{1}(E',R(M)) \).

We then have a restriction map
\[
(12.16) \quad H_{1}^{1}(E,R(M)) \to H_{1}^{1}(E',R(M)).
\]
Moreover, naturality with respect to the commutative diagram
\[
\begin{array}{ccc}
R(Y) & \xrightarrow{R(\xi)} & R(\text{Hom}(A,M)) \\
\epsilon \downarrow & & \epsilon \downarrow \\
Y & \xrightarrow{\xi} & \text{Hom}(A,M)
\end{array}
\]
provides us with a map
\[
(12.17) \quad H_{1}^{1}(E',R(M)) \to H_{1}^{1}(E',M).
\]

The next result is our generalized version of Shapiro’s lemma.

Lemma 12.10. The composed map
\[
H_{1}^{1}(E,R(M)) \to H_{1}^{1}(E',R(M)) \to H_{1}^{1}(E',M)
\]
is bijective.

Proof. Consider the commutative square
\[
\begin{array}{ccc}
H_{1}^{1}(E,R(M)) & \xrightarrow{\text{12.16}} & H_{1}^{1}(E',R(M)) \\
\downarrow & & \downarrow \\
H^{1}(E,R(M)) & \to & H^{1}(E',M)
\end{array}
\]
We must prove that the top arrow is bijective. Now the classical form of Shapiro’s lemma asserts that the bottom arrow is bijective. So we are reduced to proving
the following. Fix \( x \in Z^1(E, R(M)) \) and let \( y \) be its image under the cocycle-level Shapiro map

\[
Z^1(E, R(M)) \to Z^1(E', R(M)) \to Z^1(E', M),
\]

the first arrow being restriction from \( E \) to \( E' \), and the second being the map induced by \( \epsilon : R(M) \to M \). What we must prove is that the top arrow restricts to a bijection from the fiber of the left arrow over \([x]\) to the fiber of the right arrow over \([y]\).

These fibers were described in Lemma 12.5. The fiber on the left is \( R(M)_x \), and this is precisely the canonical bijection obtained from \( E \) \( \rightarrow \) \( E' \) \( \rightarrow \) \( M \). Let \( y_0 \) be a linear algebraic group over \( E \), \( \xi \) be an intermediate field for \( K/E \). Let \( G_0 \) be a linear algebraic group over \( E \). Let \( G \) be an intermediate field for \( K/F \), and let \( G_0 \) be the preimage of \( G(K/E) \) in \( G \).

For the second one we begin by applying the functor \( R_{E'}^E \) to the cartesian square

\[
\begin{array}{c}
\xi^{-1}(y_0) \downarrow \\
\{y_0\} \to \Hom(A,M)_x
\end{array}
\]

of \( E' \)-sets. Since \( R_{E'}^E \) is a right adjoint, it preserves cartesian squares and final objects, and we conclude that the \( E \)-set obtained as the fiber of \( R(\xi) \) over \( x_0 \) is coinduced from the \( E' \)-set \( \xi^{-1}(y_0) \). It follows that

\[
(R(\xi)^{-1}(x_0))_x = (R_{E'}^E(\xi^{-1}(y_0)))^E = (\xi^{-1}(y_0))^E
\]

and this is precisely the canonical bijection \( R(\xi)^{-1}(x_0)_x = Y_y \) we needed to construct. So the lemma is proved.

12.12. Application to \( H^1_{\text{alg}}(E,G(K)) \). Let

\[
1 \to D(K) \to E \to G(K/F) \to 1
\]

be a Galois gerb for \( K/F \). Let \( E \) be an intermediate field for \( K/F \), and let \( E' \) be the preimage of \( G(K/E) \) in \( E \).

Let \( G_0 \) be a linear algebraic group over \( E \) and put \( G = R_{E/F}G_0 \) (Weil restriction of scalars). Then \( G(K) = R(G_0(K)) \) and \( \Hom_K(D, G) = R(\Hom_K(D, G)) \), where \( R \) denotes coinduction from \( G(K/E) \) to \( G(K/F) \). So there is a Shapiro isomorphism

\[
H^1_{\text{alg}}(E,G(K)) = H^1_{\text{alg}}(E',G_0(K)).
\]

12.13. Application to \( B(F,G) \). Now let \( F \) be a local or global field, and let \( E/F \) be a finite separable extension. Again consider \( G = R_{E/F}(G_0) \) for some linear algebraic \( E \)-group \( G_0 \). Then there is a Shapiro isomorphism

\[
B(F,G) = B(E,G_0).
\]
12.14. **Application to** $B_i(F,T)$ **for** $i = 1,2,3$. Let $E/F$ be a finite separable extension of global fields, let $T_0$ be a torus over $E$, and put $T = R_{E/F}(T_0)$. Then for $i = 1,2,3$ there are Shapiro isomorphisms

$$B_i(F,T) = B_i(E,T_0).$$

For $i = 3$ this is just \[\text{(12.21)}\] in different notation. (The groups $B_i(F,T)$ were defined in subsection 8.6.)

12.15. **Corestriction and restriction for** $B_i(F,T)$. Let $E/F$ be a finite separable extension of global fields, and let $T$ be an $F$-torus. The Shapiro isomorphism makes it easy to define corestriction maps for $T$. Put $\bar{T} := R_{E/F}(T)$. Because we started with a torus $T$ over $F$ (not $E$), there is a norm map $N_{E/F} : \bar{T} \to T$. For $i = 1,2,3$ we define a corestriction map

$$\text{Cor} : B_i(E,T) \to B_i(F,T)$$

as the composed map

$$B_i(E,T) \xrightarrow{\text{(12.22)}} B_i(F,\bar{T}) \xrightarrow{N_{E/F}} B_i(F,T).$$

**Lemma 12.11.** Let $K/E$ be a finite extension such that $K/F$ is Galois and $T$ is split by $K$. Put $Y_i(K) := X_\ast(T) \otimes X_i(K)$.

1. For $i = 1,2,3$ there is a commutative diagram

$$
\begin{array}{cccc}
Y_i(K)^G(K/E) & \xrightarrow{\sim} & B_i(E,T) & \xrightarrow{\text{Cor}} & Y_i(K)^{G(K/E)} \\
\downarrow & & \downarrow \text{Cor} & & \downarrow \\
Y_i(K)^G(K/F) & \xrightarrow{\sim} & B_i(F,T) & \xrightarrow{\text{Cor}} & Y_i(K)^{G(K/F)}
\end{array}
$$

The left vertical arrow is induced by the identity map on $Y_i(K)$. The middle vertical arrow is corestriction for $E/F$. The right vertical arrow is given by $y \mapsto \sum_{\sigma \in G(K/F) \setminus G(K/E)} \sigma(y)$.

2. For $i = 1,2,3$ there is a commutative diagram

$$
\begin{array}{cccc}
Y_i(K)^G(K/E) & \xrightarrow{\sim} & B_i(E,T) & \xrightarrow{\text{Res}} & Y_i(K)^{G(K/E)} \\
\uparrow & & \uparrow \text{Res} & & \uparrow \\
Y_i(K)^G(K/F) & \xrightarrow{\sim} & B_i(F,T) & \xrightarrow{\text{Res}} & Y_i(K)^{G(K/F)}
\end{array}
$$

The left vertical arrow is given by $y \mapsto \sum_{\sigma \in G(K/E) \setminus G(K/F)} \sigma(y)$. The middle vertical arrow is restriction for $E/F$. The right vertical arrow is induced by the identity map on $Y_i(K)$.

**Proof.** Both parts of the lemma can be proved in the same way as Lemma 1.2. The outer rectangles and right squares clearly commute. Therefore the left squares also commute when $X_\ast(T)$ is free as $Z[G(K/F)]$-module. The general case is then reduced to this special one by choosing $T' \to T$ with $X_\ast(T') \to X_\ast(T)$ surjective and $X_\ast(T')$ free as $Z[G(K/F)]$-module.
12.16. **Corestriction and restriction for** \(B(F,T)\) **when** \(F\) **is local.** Let \(E/F\) be a finite separable extension of local fields, and let \(T\) be an \(F\)-torus. As in the global case we define a corestriction map as the composed map

\[
B(E,T) = B(F,\tilde{T}) \xrightarrow{N_{E/F}} B(F,T).
\]

**Lemma 12.12.** Let \(K/E\) be a finite extension such that \(K/F\) is Galois and \(T\) is split by \(K\). Put \(Y := X_*(T)\).

(1) There is a commutative diagram

\[
\begin{array}{ccc}
Y_{G(K/E)} & \xrightarrow{=} & B(E,T) \xrightarrow{\text{Cor}} Y^{G(K/E)} \\
\downarrow & & \downarrow \\
Y_{G(K/F)} & \xrightarrow{=} & B(F,T) \xrightarrow{\text{Cor}} Y^{G(K/F)}
\end{array}
\]

The left vertical arrow is induced by the identity map on \(Y\). The middle vertical arrow is corestriction for \(E/F\). The right vertical arrow is given by \(y \mapsto \sum_{\sigma \in G(K/F)/G(K/E)} \sigma(y)\).

(2) There is a commutative diagram

\[
\begin{array}{ccc}
Y_{G(K/E)} & \xrightarrow{=} & B(E,T) \xrightarrow{\text{Res}} Y^{G(K/E)} \\
\uparrow & & \uparrow \\
Y_{G(K/F)} & \xrightarrow{=} & B(F,T) \xrightarrow{\text{Res}} Y^{G(K/F)}
\end{array}
\]

The left vertical arrow is given by \(y \mapsto \sum_{\sigma \in G(K/E) \setminus G(K/F)} \sigma(y)\). The middle vertical arrow is restriction for \(E/F\). The right vertical arrow is induced by the identity map on \(Y\).

**Proof.** Same as in global case. \(\square\)

13. **\(B(F,G)_{\text{bsc}}\) in the local case**

13.1. **Notation.** Let \(F\) be a local field. We fix a separable closure \(\bar{F}\) of \(F\) and put \(\Gamma := \text{Gal}(\bar{F}/F)\). Let \(G\) be a connected reductive \(F\)-group. We are going to study \(B(F,G)_{\text{bsc}}\) (see section 10) and the map \(\kappa_G : B(F,G)_{\text{bsc}} \to A(F,G) = (\Lambda G)_{\Gamma}\) (see section 11.5). As in sections 10 and 11.5 we write \(Z(G)\) for the center of \(G\), and \(C(G)\) for the biggest torus in \(Z(G)\).

13.2. **The case of tori.** From Lemma 8.1 it follows that, for every \(F\)-torus \(T\), the map

\[
\kappa_T : B(F,T) \to A(F,T) = (X_*(T))_{\Gamma}
\]

is an isomorphism. In this simple case the colimit defining \(B(F,T)\) is already attained when \(K\) is big enough to split \(T\).

13.3. **\(B(F,G)\) in the nonarchimedean case.** Assume that \(F\) is nonarchimedean. For any linear algebraic group \(G\) over \(F\), there is a canonical identification of \(B(F,G)\) with the set denoted by \(B(G)\) in [Kot97]. Strictly speaking [Kot97] treats only the \(p\)-adic case, but the definition of \(B(G)\) given there makes sense for all nonarchimedean \(F\).

**Proposition 13.1.** Let \(G\) be a connected reductive \(F\)-group. Then the following statements hold.
(1) The map $\kappa_G : B(F,G) \to A(F,G)$ restricts to a bijection

$$\kappa_G : B(F,G)_{bsc} \to A(F,G)$$

(13.1)

(2) If $T$ is an elliptic maximal $F$-torus in $G$, then the natural map

$$B(F,T) \to B(F,G)_{bsc}$$

is surjective.

Proof. First we prove that (13.1) is injective. When the derived group of $G$ is simply connected, this follows easily from the vanishing of $H^1$ for simply connected semisimple groups (due to Kneser [Kne65a, Kne65b] in the $p$-adic case and Bruhat-Tits [BT87] in general). The general case is then treated using $\mathbb{z}$-extensions and Proposition 10.4. The reader who finds these indications too brief can look ahead to the proof of Proposition 15.1, where the corresponding steps are treated in much greater detail.

Next we prove part (2) of the proposition. The image of the natural map $B(F,T) \to B(F,G)_{bsc}$ is contained in the subset $B(F,G)_{bsc}$, simply because $T$ is elliptic. (The natural injection $\text{Hom}_F(\mathbb{D}_F, Z(G)) \hookrightarrow \text{Hom}_F(\mathbb{D}_F, T)$ is actually bijective, since the image of any $F$-homomorphism $\mathbb{D}_F \to T$ is a split subtorus of $T$.)

The functoriality of $\kappa_G$ guarantees that the diagram

$$B(F,T) \xrightarrow{\kappa_T} (\Lambda_T)_R$$

$$\downarrow \quad \downarrow$$

$$B(F,G)_{bsc} \xrightarrow{\kappa_G} (\Lambda_G)_R$$

(13.2)

commutes. Since $\kappa_T$ is an isomorphism, the surjectivity of the left vertical map follows from that of the right vertical map and the (already established) injectivity of (13.1).

Finally we recall that elliptic maximal $F$-tori $T$ in $G$ are known to exist. This is due to Kneser [Kne65b, §15] in the $p$-adic case and DeBacker [DeB06] in general. The surjectivity of (13.1) now follows from part (2).

13.4. $B(F,G)$ and $B(F,G)_{bsc}$ in the complex case. The complex case is very simple: the map (10.4) is bijective, which just says that $B(C,G)$ is the set of $G(\mathbb{C})$-conjugacy classes of homomorphisms from $\mathbb{G}_m$ to $G$. In particular we have

$$B(C,G)_{bsc} = \Lambda_C(G).$$

13.5. $B(F,G)_{bsc}$ in the real case. We can analyze $B(\mathbb{R},G)_{bsc}$ using some results of Shelstad [She79]. We choose a fundamental maximal $\mathbb{R}$-torus $T$ in $G$. We write $\Omega$ for its absolute Weyl group, $\Omega(\mathbb{R})$ for the fixed points of complex conjugation on $\Omega$, and $\Omega_\mathbb{R}$ for the subgroup of $\Omega(\mathbb{R})$ consisting of elements that can be represented by an element in the normalizer of $T$ in $G(\mathbb{R})$. As Shelstad shows,

- $T$ transfers to every inner form of $G$, and
- there is a natural bijection

$$\Omega_\mathbb{R} \backslash \Omega(\mathbb{R}) \xrightarrow{\sim} \ker[H^1(\mathbb{R}, T) \to H^1(\mathbb{R}, G)].$$

Borovoi [Bor88, Theorem 1] observes that Shelstad’s results lead to a useful description of $H^1(\mathbb{R}, G)$. For this Borovoi uses the following (right) action of $\Omega(\mathbb{R})$ on $H^1(\mathbb{R}, T)$. Given an element $\omega \in \Omega(\mathbb{R})$ and a 1-cocycle $t$ of $\text{Gal}(\mathbb{C}/\mathbb{R})$ in $T$, the action of $\omega$ sends the class of $t$ to the class of the 1-cocycle $t' = \ldots$
\[ \hat{\omega}^{-1} t_{a, \sigma}(\hat{\omega}), \] where \( \hat{\omega} \) is a representative for \( \omega \) in the normalizer of \( T \) in \( G(\mathbb{C}) \).

Obviously the \( \Omega(\mathbb{R}) \)-orbit of the class of \( t \) is equal to the quotient \( \Omega^b \backslash \Omega(\mathbb{R}) \), where \( \Omega^b \) denotes the twist of \( \Omega \) by \( t \). (So \( \Omega^b \) is the Weyl group of \( T \) in the pure inner form \( G^i \) of \( G \) obtained as the twist by \( t \).) Now \( (13.3) \) applied to the inner form \( G^i \), implies (by the usual twisting argument in Galois cohomology) that the fiber of \( H^1(\mathbb{R}, T) \rightarrow H^1(\mathbb{R}, G) \) through \( t \) is equal to the \( \Omega(\mathbb{R}) \)-orbit of \( t \). (When we twist, \( \Omega(\mathbb{R}) \) changes, but \( \Omega(\mathbb{R}) \) does not.) Moreover, the fact that \( T \) transfers to every inner form of \( G \) implies (see, e.g., [Kot86]) that \( H^1(\mathbb{R}, T) \rightarrow H^1(\mathbb{R}, G) \) is surjective. Putting these observations together, Borovoi concludes that \( H^1(\mathbb{R}, G) \) is the quotient of \( H^1(\mathbb{R}, T) \) by the above action of \( \Omega(\mathbb{R}) \).

We are now going to follow the same line of reasoning to describe \( B(\mathbb{R}, G)_{bsc} \) in terms of the subset \( B(\mathbb{R}, T)_{G-\text{bsc}} \) of \( B(\mathbb{R}, T) \) consisting of all elements whose Newton point \( \nu : G_m \rightarrow T \) is central in \( G \). There is a natural action of \( \Omega(\mathbb{R}) \) on \( B(\mathbb{R}, T) \), induced by the following action on algebraic 1-cocycles. Let \( \omega \in \Omega(\mathbb{R}) \), and choose a representative \( \hat{\omega} \) of \( \omega \) in the normalizer of \( T \) in \( G(\mathbb{C}) \). Let \( b = (\nu, x) \) be an algebraic 1-cocycle in \( T \). Then the action of \( \omega \) sends the class of \( b \) to the class of the algebraic 1-cocycle \( b' := (\omega^{-1}(\nu), w \mapsto \hat{\omega}^{-1} x_w w(\hat{\omega})) \). When \( b \) is basic, so that \( \nu \) is central, \( \omega^{-1}(\nu) \) is of course equal to \( \nu \). In particular, the action of \( \Omega(\mathbb{R}) \) preserves the subset \( B(\mathbb{R}, T)_{G-\text{bsc}} \) of \( B(\mathbb{R}, T) \).

**Lemma 13.2.** The natural map \( B(\mathbb{R}, T)_{G-\text{bsc}} \rightarrow B(\mathbb{R}, G)_{bsc} \) induces a bijection between \( B(\mathbb{R}, G)_{bsc} \) and the quotient of \( B(\mathbb{R}, T)_{G-\text{bsc}} \) by the action of \( \Omega(\mathbb{R}) \).

**Proof:** We claim that \( B(\mathbb{R}, T)_{G-\text{bsc}} \rightarrow B(\mathbb{R}, G)_{bsc} \) is surjective. Indeed, consider an element \( b \in B(\mathbb{R}, G)_{bsc} \). Its image in \( B(\mathbb{R}, G_{ad})_{bsc} = H^1(\mathbb{R}, G_{ad}) \) lies in the image of \( H^1(\mathbb{R}, T_{ad}) \), because \( H^1(\mathbb{R}, T_{ad}) \rightarrow H^1(\mathbb{R}, G_{ad}) \) is surjective. Therefore \( b \) can be represented by an algebraic 1-cocycle \( (\nu, x) \) for which the image of \( x \) in the adjoint group takes values in \( T_{ad} \). It follows that \( x \) itself takes values in \( T \). This, together with the fact that \( \nu \) is central in \( G \), shows that \( (\nu, x) \) is the image of an algebraic 1-cocycle in \( T \), and the claim follows.

It remains to examine the fibers of our surjection. Just as for Galois cohomology, the fiber of \( B(\mathbb{R}, T)_{G-\text{bsc}} \rightarrow B(\mathbb{R}, G)_{bsc} \) through the class in \( B(\mathbb{R}, T)_{G-\text{bsc}} \) represented by the algebraic 1-cocycle \( b = (\nu, x) \) in \( T \) (with \( \nu \) central in \( G \)) can be identified with the kernel of

\[ B(\mathbb{R}, T)_{G-\text{bsc}} \rightarrow B(\mathbb{R}, J_b)_{bsc}, \]

where \( J_b \) is the inner form of \( G \) obtained as the twist by \( b \). Now the kernel of \( B(\mathbb{R}, T) \rightarrow B(\mathbb{R}, J_b) \) is equal to the kernel of \( H^1(\mathbb{R}, T) \rightarrow H^1(\mathbb{R}, J_b) \), because \( \text{Hom}(G_m, T) \rightarrow \text{Hom}(G_m, J_b) \) is obviously injective. Therefore, by the second of Shelstad’s results reviewed above, the fiber of \( B(\mathbb{R}, T)_{G-\text{bsc}} \rightarrow B(\mathbb{R}, G)_{bsc} \) through the class of \( b \) can be identified with \( \Omega^b \backslash \Omega(\mathbb{R}) \), where \( \Omega^b \) is the Weyl group of \( T \) in the twist \( J_b \). Unwinding the definitions, one sees that \( \Omega^b \) is the stabilizer in \( \Omega(\mathbb{R}) \) of the class of \( b \), and we conclude that the fiber of \( B(\mathbb{R}, T)_{G-\text{bsc}} \rightarrow B(\mathbb{R}, G)_{bsc} \) through the class of \( b \) is the \( \Omega(\mathbb{R}) \)-orbit of the class of \( b \), as desired. \( \square \)

**Remark 13.3.** When the fundamental torus \( T \) is elliptic (equivalently, when elliptic maximal tori exist in \( G \)), any \( F \)-homomorphism \( G_m \rightarrow T \) is automatically central in \( G \), so \( B(\mathbb{R}, T)_{G-\text{bsc}} = B(\mathbb{R}, T) \), and the lemma tells us that \( B(\mathbb{R}, G)_{bsc} \) is the quotient of \( B(\mathbb{R}, T) \) by the above action of \( \Omega(\mathbb{R}) \).
13.6. **The image of** $B(F,G)_{\text{bsc}} \to A(F,G)$. The next result involves the subset $A_0(F,G)$ of $A(F,G)$ defined in Remark 11.6

**Proposition 13.4.** For any local field $F$ the image of $\kappa_G : B(F,G)_{\text{bsc}} \to A(F,G)$ is $A_0(F,G)$.

**Proof.** We already know from Remark 11.6 that the image of $B(F,G)_{\text{bsc}} \to A(F,G)$ is contained in $A_0(F,G)$. So we just need to check that any element in $A_0(F,G)$ lies in $\text{im}[B(F,G)_{\text{bsc}} \to A(F,G)]$. In the nonarchimedean case this is clear from Proposition 13.1, and in the complex case it is clear from the fact that $B(\mathbb{C},G)_{\text{bsc}} = \Lambda_{C(G)}$. The real case is more interesting.

Before tackling the real case we need to make a definition. We say that a commutative diagram

$$
\begin{array}{c}
Z \\
\downarrow \\
Y \\
\downarrow \\
S
\end{array}
$$

(13.4)

of sets is *semicartesian* if the induced map $Z \to X \times_S Y$ is surjective.

In the real case the commutative square (11.16) works out to

$$
\begin{array}{cc}
B(\mathbb{R},G)_{\text{bsc}} & \to (\Lambda_G)^\Gamma \\
\downarrow_{\text{Newton}} & \downarrow_{\text{Newton}} \\
(\Lambda_{C(G)})^\Gamma & \to (\Lambda_G)^\Gamma.
\end{array}
$$

(13.5)

To prove the proposition we must prove that this square is semicartesian. Let $T$ be a fundamental maximal $\mathbb{R}$-torus in $G$. The map $B(\mathbb{R},T)_{G-\text{bsc}} \to B(\mathbb{R},G)_{\text{bsc}}$ is surjective by Lemma 13.2. Moreover, the map $i : (\Lambda_{C(G)})^\Gamma \to (\Lambda_G)^\Gamma$ factors as $((\Lambda_{C(G)})^\Gamma) \to (\Lambda_T)^\Gamma \xrightarrow{p} (\Lambda_G)^\Gamma$, where $p$ is (induced by) the canonical surjection in the short exact sequence

$$0 \to X_*(T_{sc}) \to \Lambda_T \xrightarrow{p} \Lambda_G \to 0.
$$

(13.6)

So, to prove the proposition, it will suffice to show that the square

$$
\begin{array}{cc}
(\Lambda_T)^\Gamma & \to (\Lambda_G)^\Gamma \\
\downarrow_{\text{Newton}} & \downarrow_{\text{Newton}} \\
(\Lambda_T)^\Gamma & \to (\Lambda_G)^\Gamma
\end{array}
$$

(13.7)

is semicartesian.

Consider $g \in (\Lambda_G)^\Gamma$ and $t \in (\Lambda_T)^\Gamma$ such that $N_{C/R}(g) = p(t)$. We need to construct $\tilde{t} \in (\Lambda_T)^\Gamma$ such that $p(\tilde{t}) = g$ and $N_{C/R}(\tilde{t}) = t$. We begin by choosing any $t_1 \in (\Lambda_T)^\Gamma$ such that $p(t_1) = g$. Then $y := t - N_{C/R}(t_1)$ lies in the kernel $(X_*(T_{sc}))^\Gamma$ of the bottom horizontal arrow in our square. To finish the proof it suffices to construct an element $x \in (X_*(T_{sc}))^\Gamma$ such that $N_{C/R}(x) = y$, since we will then obtain the desired element $\hat{t}$ as the sum $t_1 + x$.

The existence of $x$ (for arbitrary $y$) is just the statement that the Tate cohomology group $H^0(\Gamma, X_*(T_{sc}))$ vanishes. This is indeed the case, because $T_{sc}$ is isomorphic to a product $T_a \times T_i$, with $T_a$ anisotropic and $T_i$ of the form $R_{C/R}(S)$ for some $\mathbb{C}$-torus $S$ (see, e.g., the proof of Lemma 10.4 in [Kot86]).

□
14. A finiteness theorem

14.1. Motivation. Let $K/F$ be a finite Galois extension of global fields. As usual we write $V_F$ for the set of all places of $F$. When $u$ is a finite place, we write $\mathcal{O}_u$ for the valuation ring in $F_u$. For every subset $S$ of $V_F$ we denote by $S_K$ the preimage of $S$ under the natural surjection $V_K \twoheadrightarrow V_F$. When $S$ contains $S_\infty$, the set of infinite places of $F$, we put

$$F_S := \{ x \in F : x \in \mathcal{O}_u \quad \forall u \in V_F \setminus S \},$$

$$K_S := \{ x \in K : x \in \mathcal{O}_v \quad \forall v \in V_K \setminus S_K \},$$

$$A_{K,S} := \{ x \in A_K : x_v \in \mathcal{O}_v \quad \forall v \in V_K \setminus S_K \}.$$ 

If $S = V_F$, then $F_S = F$. In the number field case, if $S = S_\infty$, then $F_S$ is the ring of integers in $F$.

Now let $G$ be a linear algebraic group over $F$. By way of motivation for this section we begin by reviewing a standard finiteness result for $H^1(G(K/F), G(K))$. To formulate the result we first need to choose an extension of $G$ to a smooth affine group scheme $\mathcal{G}$ over $F_{S(\mathcal{G})}$, where $S(\mathcal{G})$ is some finite set of places containing $S_\infty$.

Given two such extensions $\mathcal{G}_1, \mathcal{G}_2$, there exists a finite set $S$ of places such that

- $S$ contains both $S(\mathcal{G}_1)$ and $S(\mathcal{G}_2)$, and
- the identity morphism for $G$ extends (uniquely) to an $F_S$-isomorphism between $\mathcal{G}_1, \mathcal{G}_2$.

For such a set $S$ we have $\mathcal{G}_1(\mathcal{O}_v) = \mathcal{G}_2(\mathcal{O}_v)$ when $v \notin S_K$.

Here is the standard finiteness result, along with its easy proof. When dealing with localization maps (as in the next result), we make the following convention: $u$ denotes a place of $F$, and $v$ denotes some chosen place of $K$ lying over $u$. This will allow us to keep our statements a little more succinct.

**Proposition 14.1.** Let $x \in H^1(G(K/F), G(K))$ and write $x_u$ for the image of $x$ under the localization map

$$H^1(G(K/F), G(K)) \to H^1(G(K_v/F_u), G(K_v)).$$

Then there exists a finite set $S$ of places of $F$ such that

- $S$ contains $S(\mathcal{G})$, and
- for all $u \notin S$ the element $x_u$ lies in the image of the map

$$H^1(G(K_v/F_u), \mathcal{G}(\mathcal{O}_v)) \to H^1(G(K_v/F_u), G(K_v)).$$

**Proof.** Choose a cocycle representing $x$. Because $G(K/F)$ is finite, there exists finite $S \supset S(\mathcal{G})$ such that $x$ takes values in $\mathcal{G}(K_S)$. Clearly $S$ does the job. \qed

14.2. Goal of this section. We are going to generalize the finiteness result from $H^1(G(K/F), G(K))$ to the bigger set $H^1_{alg}(\mathcal{E}_3(K/F), G(K))$. For $u \notin S(\mathcal{G})$ we define

$$H^1(G(K_v/F_u), \mathcal{G}(\mathcal{O}_v)) \to H^1_{alg}(\mathcal{E}(K_v/F_u), G(K_v))$$

as the composition of the natural map (induced by $\mathcal{G}(\mathcal{O}_v) \hookrightarrow G(K_v)$)

$$H^1(G(K_v/F_u), \mathcal{G}(\mathcal{O}_v)) \to H^1(G(K_v/F_u), G(K_v))$$

and the canonical inclusion

$$H^1(G(K_v/F_u), G(K_v)) \hookrightarrow H^1_{alg}(\mathcal{E}(K_v/F_u), G(K_v)).$$
Proposition 14.2. Let \( b \in H^1_{\text{alg}}(\mathcal{E}_3(K/F), G(K)) \) and write \( b_u \) for the image of \( b \) under the localization map

\[
\ell_u^F : H^1_{\text{alg}}(\mathcal{E}_3(K/F), G(K)) \rightarrow H^1_{\text{alg}}(\mathcal{E}(K_u/F_u), G(K_u)).
\]

Then there exists a finite set \( S \) of places of \( F \) such that

- \( S \) contains \( S(G) \), and
- for all \( u \not\in S \) the element \( b_u \) lies in the image of the map \( \ell_u^F \).

Before proving the proposition, we make note of a simple corollary.

Corollary 14.3. Assume that \( G \) is connected. Let \( b \in H^1_{\text{alg}}(\mathcal{E}_3(K/F), G(K)) \) and again write \( b_u \) for the image of \( b \) under the localization map \( \ell_u^F \). Then there exists a finite set \( S \) of places of \( F \) such that \( b_u \) is trivial for all \( u \not\in S \).

Proof. For any finite place \( u \) of \( F \) we denote by \( \kappa(u) \) the residue field of the valuation ring \( \mathcal{O}_u \). From Proposition 3.7 in Exposé \( VI_B \) of SGA 3, Tome I it follows easily that there exists a finite set \( S \) of places with \( S \supset S(G) \) and such that \( G \otimes \kappa(u) \) is connected for all \( u \not\in S \). Enlarging \( S \) if need be, we may also assume that \( K/F \) is unramified outside \( S \). Then a standard argument involving Hensel’s Lemma and Lang’s Theorem shows that \( H^1(G(K_u/F_u), G(\mathcal{O}_u)) \) is trivial for all \( u \not\in S \). From this it is clear that Corollary 14.3 does follow from Proposition 14.2.

The next two subsections will provide lemmas that will be used in the proof of Proposition 14.2.

14.3. A vanishing theorem. In this subsection the linear algebraic group \( G \) will not appear, so we will temporarily lighten our notation by using \( G \) as an abbreviation for the Galois group \( G(K/F) \). Moreover we denote the decomposition group at a place \( w \) of \( K \) simply by \( G_w \). (So \( G_w \) is the stabilizer of \( w \) in \( G \).)

For any set \( S \) of places of \( F \) we consider the short exact sequence

\[
(X(S)) \quad 0 \rightarrow X_3(S) \rightarrow X_2(S) \rightarrow X_1(S) \rightarrow 0
\]

with \( X_1(S) := \mathbb{Z} \), \( X_2(S) := \mathbb{Z}[S_K] \) and \( X_3(S) = \mathbb{Z}[S_K]_0 \). (When we worked with this sequence before, in subsection (6.2) we did not include \( S \) in the notation. Now we do, because we need to keep track of which set \( S \) we are using.) Dual to this short exact sequence of \( G \)-modules is a short exact sequence

\[
1 \rightarrow \mathbb{G}_m \rightarrow \mathbb{T}_S \rightarrow \mathbb{T}_S \rightarrow 1
\]

of protori over \( F \) (that split over \( K \)).

Lemma 14.4. Let \( S \) be a set of places of \( F \) such that

\[
\{G_w : w \in S_K\} = \{G_w : w \in V_K\}.
\]

Then the following statements hold.

1. The group \( H^1(G', \mathbb{T}_S(K)) \) vanishes for every subgroup \( G' \) of \( G \).
2. For every place \( v \) of \( K \) the group \( H^1(G_v, \mathbb{T}_S(K_v)) \) vanishes.

Proof. (1) Notice that our hypothesis on \( S \) implies that

\[
\{G'_w : w \in S_K\} = \{G'_w : w \in V_K\}.
\]

We are going to use the long exact sequence of Tate cohomology for the short exact sequence

\[
1 \rightarrow \mathbb{G}_m(K) \rightarrow \mathbb{T}_S(K) \rightarrow \mathbb{T}_S(K) \rightarrow 1
\]
of $G'$-modules. From Lemma 6.2 it follows that

$$H^r(G', \tilde{T}_S(K)) = \prod_{v \in \mathcal{O}_v \setminus S_K} H^r(G'_v, K_v^\times).$$

So $H^1(G', \tilde{T}_S(K))$ vanishes by Hilbert’s Theorem 90. To prove that $H^1(G', T_S(K))$ vanishes, we just need to prove that the natural map

$$H^2(G', K^\times) \to \prod_{v \in \mathcal{O}_v \setminus S_K} H^2(G'_v, K_v^\times)$$

is injective. By 6.2 it is equivalent to prove that

$$H^2(G', K_v^\times) \to \prod_{v \in \mathcal{O}_v \setminus S_K} H^2(G'_v, K_v^\times)$$

is injective, and this follows from the well-known injectivity of

$$H^2(G', K^\times) \to \prod_{v \in \mathcal{O}_v \setminus S_K} H^2(G'_v, K_v^\times).$$

(2) We are going to use the long exact sequence of Tate cohomology for the short exact sequence

$$1 \to G_m(K_v) \to \tilde{T}_S(K_v) \to T_S(K_v) \to 1$$

of $G_v$-modules. From Lemma 6.2 we have

$$H^r(G_v, \tilde{T}_S(K_v)) = \prod_{w \in G_v \setminus S_K} H^r(G_{v,w}, K_{v,w}^\times),$$

where $G_{v,w} = G_v \cap G_w$. So $H^1(G_v, \tilde{T}_S(K_v))$ vanishes by Hilbert’s Theorem 90. To prove that $H^1(G_v, T_S(K_v))$ vanishes, we just need to prove that the natural map

$$H^2(G_v, K_v^\times) \to \prod_{w \in G_v \setminus S_K} H^2(G_{v,w}, K_{v,w}^\times)$$

is injective. This is clear because (by our hypothesis on $S$) there exists $w \in S_K$ such that $G_w = G_v$, and for this $w$ we have $G_{v,w} = G_v$. \hfill \Box

Remark 14.5. There exist finite subsets $S$ of $V_F$ satisfying the hypothesis of the last lemma. This is obvious, because the Galois group $G$ is finite and therefore has only finitely many subgroups.

14.4. Comparison of two cohomology classes. In this subsection the linear algebraic group $G$ again does not appear, so we continue to use $G$ to denote $G(K/F)$. In this subsection $S$ denotes any set of places of $F$ satisfying the three conditions in subsection 6.1. As in subsection 6.2 using $S$-adeles we obtain another short exact sequence of $G$-modules, namely

$$(A(S)) \quad 1 \to A_3(S) \to A_2(S) \to A_1(S) \to 1.$$

(But we did not include $S$ in the notation.) Recall that

$$A_2(S) = K_{K,S}^\times = \left( \prod_{v \in S_K} \mathcal{O}_v^\times \right) \times \left( \prod_{v \in S_K} K_v^\times \right), \quad A_3(S) = K_S^\times,$$

and our assumptions on $S$ imply that $A_1(S)$ is the group of idèle classes of $K$.

We denote by $\alpha(S)$ Tate’s canonical class in $H^2(G, \text{Hom}(X(S), A(S)))$. Its projections under $\pi_i : \text{Hom}(X(S), A(S)) \to \text{Hom}(X_i(S), A_i(S))$ will be denoted by
\(\alpha_i(S)\). When \(S = V_F\), we write \(X, X_1, A, A_i, \alpha, \alpha_i, T, \tilde{T}\) instead of \(X(V_F), X_i(V_F), A_i(V_F), \alpha_i(V_F), T_{V_F}, \tilde{T}_{V_F}\).

There is an obvious morphism \(p^S\) from the exact sequence \(X(S)\) to the exact sequence \(X = X(V_K)\). This morphism is given by the vertical maps in the commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & X_3(S) & \longrightarrow & X_2(S) & \longrightarrow & X_1(S) & \longrightarrow & 0 \\
\downarrow p_3^S & & \downarrow p_2^S & & \downarrow p_1^S & & \downarrow \\
0 & \longrightarrow & X_3 & \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & 0.
\end{array}
\]

The maps \(p_2^S\) and \(p_3^S\) are induced by the inclusion \(S_K \subset V_K\), and \(p_1^S\) is the identity map on \(X_1(S) = \mathbb{Z} = X_1\).

Dual to this is another commutative diagram

\[
\begin{array}{cccccc}
1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \tilde{T} & \longrightarrow & T & \longrightarrow & 1 \\
\downarrow & & \downarrow p_2^S & & \downarrow p_3^S & & \downarrow & & \\
1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & \tilde{T}_S & \longrightarrow & T_S & \longrightarrow & 1.
\end{array}
\]

with exact rows.

Similarly there is an obvious inclusion morphism \(k^S : A(S) \hookrightarrow A\), given by the vertical inclusions in the commutative diagram

\[
\begin{array}{cccccc}
1 & \longrightarrow & A_3(S) & \longrightarrow & A_2(S) & \longrightarrow & A_1(S) & \longrightarrow & 1 \\
\downarrow k_3^S & & \downarrow k_2^S & & \downarrow k_1^S & & \downarrow & & \\
1 & \longrightarrow & A_3 & \longrightarrow & A_2 & \longrightarrow & A_1 & \longrightarrow & 1.
\end{array}
\]

Concretely, these vertical inclusions are (reading from left to right) \(K_S^\times \hookrightarrow K^\times\), \(K_{K,S}^\times \hookrightarrow K_K^\times\), and \(K_{K,S}^\times/K_S^\times \hookrightarrow K_K^\times/K^\times\). Observe that the last of these inclusions is actually an isomorphism by virtue of one of the conditions imposed on \(S\) in 6.1.

The purpose of the next lemma is to compare the Tate classes \(\alpha(S)\) and \(\alpha\). The relevant groups, maps and Tate classes are shown in the following diagram.

\[
\begin{array}{cccccc}
\alpha(S) & \overset{\text{Hom}(X(S), A(S))}{\longrightarrow} & \text{Hom}(X(S), A) & \overset{\text{Hom}(X(A), A)}{\longleftarrow} & \text{Hom}(X, A) & \overset{\alpha}{\longrightarrow} \\
\downarrow \pi_i & & \downarrow \pi_i & & \downarrow \pi_i & & \downarrow \\
\alpha_i(S) & \overset{\text{Hom}(X_i(S), A_i(S))}{\longrightarrow} & \text{Hom}(X_i(S), A_i) & \overset{\text{Hom}(X_i(A), A_i)}{\longleftarrow} & \text{Hom}(X_i, A_i) & \overset{\alpha_i}{\longrightarrow} 
\end{array}
\]

The proximity of \(\alpha(S)\) to \(\text{Hom}(X(S), A(S))\) is meant as a reminder that \(\alpha(S)\) lies in \(H^2(G, \text{Hom}(X(S), A(S)))\), and so on.

**Lemma 14.6.** There are equalities

\[
(14.3) \quad k^S(\alpha(S)) = p^S(\alpha), \quad (14.4) \quad k_i^S(\alpha_i(S)) = p_i^S(\alpha_i) \quad (i = 1, 2, 3).
\]

**Proof.** Because of the way the Tate classes are defined, we need to prove (14.4) for \(i = 1, 2\), then deduce (14.3), and from that obtain (14.4) for \(i = 3\).

It is clear that (14.4) holds for \(i = 1\), because both \(\alpha_1(S)\) and \(\alpha_1\) are the fundamental class in \(K_{K,S}^\times/K_S^\times \simeq K_K^\times/K^\times\).
Next we show that (14.4) holds for $i = 2$. We must check that the elements $k^2_v (\alpha_2(S))$ and $p^2_v (\alpha_2)$ in $H^2(G, \text{Hom}(X_2(S), A_2))$ are equal. This follows from Lemma [6.2] since, for all $v \in S_K$, both $k^2_v (\alpha_2(S))$ and $p^2_v (\alpha_2)$ have the same image in $H^2(G_v, A_2)$, namely the image of the local fundamental class under $K^*_v \hookrightarrow A^*_v$.

Now we prove the equality (14.3). Applying $\pi_i$ to this equality (with $i = 1, 2$) we obtain the equalities (14.4) for $i = 1, 2$, and these have already been verified. To prove (14.3) it remains only to prove that the map

$$(\pi_1, \pi_2) : H^2(G, \text{Hom}(X(S), A)) \to H^2(G, \text{Hom}(X_1(S), A_1)) \oplus H^2(G, \text{Hom}(X_2(S), A_2))$$

is injective. We use essentially the same reasoning as in Tate’s proof of Lemma [6.3] the reader can also consult our proof that the square (8.20) is cartesian. The desired injectivity follows from the vanishing of $H^1(G, \text{Hom}(X_2(S), A_1))$, and this is an easy consequence of Lemmas [6.2] and [6.1].

The equality (14.4) for $i = 3$ is obtained by applying $\pi_3$ to (14.3). □

14.5. Proof of Proposition [14.2]. We begin with a definition. We say that a subset $S$ of $V_F$ is adequate if it satisfies the following list of conditions:

- $S$ is finite.
- $S$ contains $S(G)$ (and consequently all archimedean places).
- $S$ contains all finite places that ramify in $K$.
- For every intermediate field $E$ of $K/F$, every ideal class of $E$ contains an ideal with support in the preimage $S_E$ of $S$ under $V_E \to V_F$.
- $S$ satisfies the hypothesis of Lemma [14.4] i.e.

$$\{ G(K/F)_w : w \in S_K \} = \{ G(K/F)_w : w \in V_K \}.$$

It is clear that, if $S$ is adequate and $S'$ is a finite set of places with $S' \supset S$, then $S'$ is adequate. It is also clear that adequate sets $S$ exist.

When $S$ is adequate, the first and third conditions in the list above imply that the protorus $T_S$ is actually a torus, and that it extends uniquely to a torus $T_S$ over $F_S$. We will soon need the following vanishing theorem.

**Lemma 14.7.** Let $u$ be a place of $F$ outside $S$, and let $v$ be a place of $K$ lying over $u$. Then $H^r(G(K_v/F_u), T_S(O_v)) = 0$ for all $r \in \mathbb{Z}$.

**Proof.** This is a standard consequence of Hensel’s lemma and Lang’s Theorem. □

For adequate $S$ we are now going to construct a pointed set $H^1_{alg}(\mathcal{E}_3(S), G(K_S))$ together with canonical map

$$(14.5) \quad H^1_{alg}(\mathcal{E}_3(S), G(K_S)) \to H^1_{alg}(\mathcal{E}_3(K/F), G(K)).$$

Here $\mathcal{E}_3(S)$ denotes an extension

$$1 \to T_S(K_S) \to \mathcal{E}_3(S) \to G(K/F) \to 1$$

with corresponding cohomology class $\alpha_3(S)$. (In section 6 the set $S$ was fixed and this extension was denoted simply by $\mathcal{E}_3$, but now we need to keep track of $S$.)

The definition of the set $H^1_{alg}(\mathcal{E}_3(S), G(K_S))$ involves an obvious extension of our usual notion of $H^1_{alg}$ to a situation involving a Galois extension of rings (namely $K_S/F_S$) rather than fields. More precisely $H^1_{alg}(\mathcal{E}_3(S), G(K_S))$ is defined to be the set $H^1_{alg}(\mathcal{E}_3(S), G(K_S))$ from subsection [12.2] formed using $Y = \text{Hom}_{K_S}(T_S, G)$ and the obvious map $\xi : Y \to \text{Hom}(T_S(K_S), G(K_S))$. So an element in $Z^1_{alg}(\mathcal{E}_3(S), G(K_S))$
is a pair \( (\nu, x) \) consisting of a \( K\)-homomorphism \( \nu : T_S \to G \) and a 1-cocycle \( x \) of \( E_3(S) \) in \( G(K_S) \) satisfying the two conditions imposed in subsection \[12.2\]

The map \((14.6)\) will be defined as a composition

\[
H^1_{\text{alg}}(E_3(S), G(K_S)) \xrightarrow{BC} H^1_{\text{alg}}(\mathcal{E}^K_3(S), G(K)) \xrightarrow{\nu^*} H^1_{\text{alg}}(E_3(K/F), G(K)).
\]

Here \( \mathcal{E}^K_3(S) \) is the Galois gerb (for \( K/F \)) defined by pushing out \( E_3(S) \) along the map \( T_S(K_S) \to T_S(K) \) induced by the inclusion of \( K_S \) in \( K \). The map \( BC \) appearing in \((14.6)\) is the obvious base change map: on the level of cocycles it sends \( (\nu, x) \) to \( (\nu, x') \), where \( x' \) is the unique element of \( Z^1(\mathcal{E}^K_3(S), G(K)) \) that agrees with \( \nu \) on \( T_S(K) \) and with \( x \) on \( E_3(S) \to \mathcal{E}^K_3(S) \). (So the map \( BC \) is essentially an instance of \((2.8)\), though we are now working with a slightly extended notion of \( H^1_{\text{alg}} \).)

It remains to define the map \( p^* \) appearing in \((14.6)\). Taking \( i = 3 \) in the second equality in Lemma \((14.6)\) we see that the image of \( \alpha_3(S) \) under \( H^2(G(K/F), T_S(K_S)) \to H^2(G(K/F), T_S(K)) \) is equal to the image of \( \alpha_3 \) under \( p^S_3 : H^2(G(K/F), T(K)) \to H^2(G(K/F), T_S(K)). \)

Therefore there exists \( \tilde{p}_3^S \) making the diagram

\[
\begin{array}{c}
1 \\
\downarrow \tilde{p}_3^S \\
\end{array}
\begin{array}{cccc}
\mathbb{T}(K) & \longrightarrow & E_3(K/F) & \longrightarrow & G(K/F) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & & \\
T_S(K) & \longrightarrow & \mathcal{E}^K_3(S) & \longrightarrow & G(K/F) & \longrightarrow & 1
\end{array}
\]

(14.7)

commute. To lighten the notation we are now going to abbreviate \( p^S_3 \) to \( p \). This should cause no confusion since we will have no further use for \( p^S_1, p^S_2, p^S_3 \). Similarly we will abbreviate \( p^S_3 \) to \( \tilde{p} \).

As in subsection \((2.8)\) \( \tilde{p} \) induces a map

\[
\tilde{p}^* : H^1_{\text{alg}}(\mathcal{E}^K_3(S), G(K)) \to H^1_{\text{alg}}(E_3(K/F), G(K))
\]

that, by Lemma \((14.4)\), is independent of the choice of \( \tilde{p} \) extending \( p \). Therefore we further lighten our notation by writing \( p^* \) in place of \( \tilde{p}^* \); this is the second map appearing in \((14.6)\).

The next lemma is the first step towards proving Proposition \((14.2)\).

**Lemma 14.8.** Let \( b \in H^1_{\text{alg}}(E_3(K/F), G(K)) \). Then there exists an adequate subset \( S \) of \( V_F \) such that \( b \) lies in the image of the map \((14.5)\).

**Proof.** The proof is similar to that of the standard finiteness theorem in global Galois cohomology that we reviewed earlier. We choose an algebraic 1-cocycle \( (\nu, x) \) representing \( b \). Thus \( \nu \) is a \( K \)-homomorphism from \( T \) to \( G \). Now \( G \) is of finite type over \( F \) and \( T \) is the prtowerus obtained from the projective system \( T_S \) of tori (with \( S \) varying over all finite subsets of \( V_F \)). So there exists an adequate set \( S \) such that \( \nu \) comes from a \( K \)-homomorphism \( \nu : T_S \to G \). At this point we have refined \( (\nu, x) \) to an algebraic 1-cocycle of \( \mathcal{E}^K_3(S) \) in \( G(K) \).

Enlarging \( S \), we may assume that \( \nu \) is defined over \( K_S \). The restriction of the 1-cocycle \( x \) to \( T_S(K_S) \) agrees with \( \nu \) and therefore takes values in \( G(K_S) \). Since \( T_S(K_S) \) is of finite index in \( E_3(S) \), by enlarging \( S \) further we may assume that the restriction of \( x \) to \( E_3(S) \to \mathcal{E}^K_3(S) \) takes values in \( G(K_S) \). At this point we have
refined \( b \) to an algebraic 1-cocycle of \( \mathcal{E}_3(S) \) in \( \mathcal{G}(K_S) \), and we are done with the proof of the lemma.

□

To deduce Proposition 14.2 from the lemma we just proved, it clearly suffices to construct, for every adequate set \( S \) and every place \( u \notin S \), a localization map \( \ell_u^S \) making the square

\[
\begin{array}{ccc}
H^1_{\text{alg}}(\mathcal{E}_3(S), \mathcal{G}(K_S)) & \xrightarrow{\ell_u^S} & H^1_{\text{alg}}(\mathcal{E}_3(K/F), G(K)) \\
\downarrow \text{Loc} & & \downarrow \text{Loc} \\
H^1(\mathcal{G}(K_v/F_u), \mathcal{G}(\mathcal{O}_v)) & \xrightarrow{\ell_u^S} & H^1(\mathcal{G}(K_v/F_u), \mathcal{G}(K_v)).
\end{array}
\]

(14.8)

commute. Moreover, it is enough to do this in the special case when \( G(K_v/F_u) = G(K/F) \). Indeed, we may easily reduce to this special case by making use of the fixed field \( E \) of \( G(K_v/F_u) \) and the restriction maps from the two sets in the top row of (14.8) to their analogs for \( K/E \). So, for the rest of this section we assume that \( G(K_v/F_u) = G(K/F) \).

It remains only to explain how to construct \( \ell_u^S \) making the square (14.8) commute. To accomplish this we are going to construct a big commutative diagram

\[
\begin{array}{cccc}
H^1_{\text{alg}}(\mathcal{E}_3(S), \mathcal{G}(K_S)) & \xrightarrow{\mathcal{BC}} & H^1_{\text{alg}}(\mathcal{E}^K_3(S), G(K)) & \xrightarrow{p^*} & H^1_{\text{alg}}(\mathcal{E}^K_3(K/F), G(K)) \\
\downarrow \text{Loc} & & \downarrow \text{Loc} & & \downarrow \text{Loc} \\
H^1_{\text{alg}}(\mathcal{E}^0_3(S), \mathcal{G}(\mathcal{O}_v)) & \xrightarrow{\mathcal{BC}} & H^1_{\text{alg}}(\mathcal{E}^0_3(K_v), G(K_v)) & \xrightarrow{p^*} & H^1_{\text{alg}}(\mathcal{E}^0_3(K/F), G(K_v)) \\
\downarrow \mu_0 & & \downarrow \mu_0 & & \downarrow \mu_0^* \\
H^1(\mathcal{G}(K_v/F_u), \mathcal{G}(\mathcal{O}_v)) & \longrightarrow & H^1(\mathcal{G}(K_v/F_u), G(K_v)) & \longrightarrow & H^1(\mathcal{E}(K_v/F_u), G(K_v)).
\end{array}
\]

(14.9)

The top row of the diagram is (14.6). The composition of the two vertical maps at the right end is the localization map \( \ell_u^S \) (see section 7). The two bottom horizontal arrows are the ones we composed to obtain the map (14.1). So, once we have constructed this big commutative diagram, the composition of the two vertical maps at its left end will yield the desired map \( \ell_u^S \).

There are only two sets in the diagram that have not yet been defined, namely, the two at the left end of the second row. These two sets bear the same relation to the ones above them as the third set in the second row does to the one above it. More precisely, for each of the two rings \( R = \mathcal{O}_v, K_v \), we write \( \mathcal{E}_3^R(S) \) for the extension of \( G(K_v/F_u) \) obtained from \( \mathcal{E}_3(S) \) by pushing forward along \( T_S(K_S) \rightarrow T_S(R) \). Now \( \mathcal{E}_3^K(S) \) is a Galois gerb for \( K_v/F_u \), so the set \( H^1_{\text{alg}}(\mathcal{E}_3^K(S), G(K_v)) \) is defined. We define \( H^1_{\text{alg}}(\mathcal{E}_3^0(S), G(\mathcal{O}_v)) \) in the obvious way, as the set \( H^1_{\text{alg}}(\mathcal{E}^0_3(S), G(\mathcal{O}_v)) \) (see subsection 12.2) obtained from \( Y = \text{Hom}_{\mathcal{O}_v}(T_S, \mathcal{G}) \) together with the obvious map \( \xi : Y \rightarrow \text{Hom}(T_S(\mathcal{O}_v), G(\mathcal{O}_v)) \).
As a first step towards constructing all the maps in the big commutative diagram above, we consider the commutative diagram

\[
\begin{array}{cccc}
T_S(K_S) & \longrightarrow & T_S(K) & \longrightarrow^p T(K) \\
\downarrow & & \downarrow & \\
T_S(O_v) & \longrightarrow & T_S(K_v) & \longrightarrow^p T(K_v) \\
\uparrow & & \uparrow & \\
\{1\} & \longrightarrow & \{1\} & \longrightarrow^q G_m(K_v)
\end{array}
\]

in which the maps are as follows. The unlabeled maps in the top two rows of (14.10) are induced by the obvious injective ring homomorphisms

\[
\begin{array}{cccc}
K_S & \longrightarrow & K & \longrightarrow K \\
\downarrow & & \downarrow & \\
O_v & \longrightarrow & K_v & \longrightarrow K_v
\end{array}
\]

The map \(\mu'_v\) was defined in section 7 on localization. The homomorphisms \(\mu_0\) and \(q\) are of course trivial. The diagram commutes; indeed, \(p\mu'_v\) is trivial because \(u \notin S\), and it is clear that the other three squares commute.

Claim 1. We claim that \(H^1(G(K_v/F_u), A)\) vanishes for each of the nine groups \(A\) appearing in (14.10). (Earlier we used \(A\) as an abbreviation for the short exact sequence \(A(V_F)\), but we no longer need to reserve the notation \(A\) for this purpose.)

For the group \(A\) appearing in the lower right corner of (14.10), Claim 1 follows from Hilbert’s Theorem 90. For the remaining groups \(A\) it follows from the various vanishing theorems proved in Lemmas 6.5, 14.4 and 14.7.

It follows from Claim 1 that all the sets \(H^1_{\text{alg}}\) in the big commutative diagram (14.9) are well-defined up to canonical isomorphism (independent of the choice of extensions \(E\) having the right second cohomology classes).

Claim 2. The maps in diagram (14.10) can be extended to homomorphisms (of extensions of \(G(K_v/F_u)\))

\[
\begin{array}{cccc}
\mathcal{E}_3(S) & \longrightarrow & \mathcal{E}_3^K(S) & \longrightarrow \hat{\mathcal{E}}_3(K/F) \\
\downarrow & & \downarrow & \\
\mathcal{E}_3^{O_v}(S) & \longrightarrow & \mathcal{E}_3^{K_v}(S) & \longrightarrow \hat{\mathcal{E}}_3^{K_v}(K/F) \\
\uparrow & & \uparrow & \\
G(K_v/F_u) & \longrightarrow & G(K_v/F_u) & \longrightarrow \hat{\mathcal{E}}(K_v/F_u).
\end{array}
\]

(Of course the two arrows in the bottom row are necessarily the identity map on \(G(K_v/F_u)\) and the canonical surjection \(\mathcal{E}(K_v/F_u) \twoheadrightarrow G(K_v/F_u)\).) Moreover, for any choice of such a collection of extended homomorphisms, the diagram (14.12) is essentially commutative, by which we mean that each of its four squares commutes up to conjugacy under \(T_S(K_v)\).

The second part of Claim 2 follows from Claim 1. The first part of Claim 2 will follow from Claim 3 below.
Notice that there are canonical elements in $H^2(G(K_v/F_u), A)$ for each of the four modules $A$ appearing in the corners of \eqref{14.10}. They are as follows:

- $\alpha_3(S)$ when $A = T_S(K_S)$,
- $\alpha_3$ when $A = T(K)$,
- the local fundamental class $\alpha(K_v/F_u)$ when $A = \mathbb{G}_m(K_v)$,
- $1$ when $A = \{1\}$.

**Claim 3.** There is a unique collection of nine elements $\alpha_A \in H^2(G(K_v/F_u), A)$, one for each of the nine modules $A$ in diagram \eqref{14.10}, such that

- $\alpha_A$ is the canonical element when $A$ is one of the four corners, and
- each arrow $A \to A'$ in the diagram maps $\alpha_A$ to $\alpha_{A'}$.

Moreover these nine elements $\alpha_A$ are the 2-cohomology classes corresponding to the nine extensions of $G(K_v/F_u)$ appearing in diagram \eqref{14.12}.

To verify the first part of Claim 3 it is enough to check that, for each of the four outer edges

$$A' \to A \leftarrow A''$$

in diagram \eqref{14.10}, the image of $\alpha_{A'}$ under $A' \to A$ agrees with the image of $\alpha_{A''}$ under $A'' \to A$. This follows from

- Lemma \ref{14.6} for the top edge,
- Lemma \ref{7.1}(2) for the right edge,
- Lemma \ref{14.7} for the left edge,

and is trivially true for the bottom edge. The second part of Claim 3 is clear, once one remembers how the various extensions were defined.

Now we are going to define all the maps in \eqref{14.9}. Each of the nine sets in that diagram is of the form $H_1^Y(E,M)$ (see subsection \ref{12.2}) for suitable $E$, $M$, $Y$ and $\xi : Y \to \text{Hom}(A,M)$. The relevant groups $E$ and $M$ are shown explicitly in the diagram, and the relevant groups $A$ are the ones appearing in the corresponding locations in diagram \eqref{14.10}. The nine sets $Y$ (indicated only by the subscript “alg” in the diagram), together with certain natural maps linking them, are shown in the commutative diagram

\begin{equation}
\begin{array}{ccc}
\text{Hom}_{K_S}(T_S, \mathcal{G}) & \longrightarrow & \text{Hom}_K(T_S, G) \\
\downarrow & & \downarrow \\
\text{Hom}_{\mathcal{O}_v}(T_S, \mathcal{G}) & \longrightarrow & \text{Hom}_{K_v}(T_S, G)
\end{array}
\end{equation}

\begin{equation}
\begin{array}{ccc}
& \longrightarrow & \\
\mu_0 & & \mu_0 \\
\mu_v & & \mu_v \\
\text{Hom}_{\mathcal{O}_v}(\{1\}, \mathcal{G}) & \longrightarrow & \text{Hom}_{K_v}(\{1\}, G)
\end{array}
\end{equation}

The nine $G(K_v/F_u)$-groups $M$ are linked by obvious homomorphisms

\begin{equation}
\begin{array}{ccc}
\mathcal{G}(K_S) & \longrightarrow & G(K) \\
\downarrow & & \downarrow \\
\mathcal{G}(\mathcal{O}_v) & \longrightarrow & G(K_v)
\end{array}
\end{equation}

\begin{equation}
\begin{array}{ccc}
& \longrightarrow & \\
& & \| \\
& & \| \\
\mathcal{G}(\mathcal{O}_v) & \longrightarrow & G(K_v)
\end{array}
\end{equation}

\begin{equation}
\begin{array}{ccc}
& \longrightarrow & \\
& & \| \\
& & \| \\
G(K_v) & \longrightarrow & G(K_v)
\end{array}
\end{equation}
We use the maps shown in (14.14) (14.13), (14.12) to define the maps in (14.9). All of them are instances of the maps $\Phi$ or $\Psi$ defined in subsections 12.7 and 12.8. The ones for which the relevant arrow in (14.14) is an equality are the ones for which the map is of type $\Psi$, and the others are of type $\Phi$.

Some of the maps in (14.9) have already been defined earlier in this subsection, but it is easy to see that they agree with the ones we just described. The commutativity of the four squares in (14.9) follows from

- Lemma 12.6 for the upper left square,
- Lemma 12.7 for the lower right square,
- Lemma 12.8 for the remaining two squares.

The construction of diagram (14.9) is now complete, so we are finally finished with the proof of Proposition 14.2.

15. $B(F,G)_{bsc}$ in the global case

15.1. Goal. Let $F$ be a global field. We fix a separable closure $\bar{F}$ of $F$ and put $\Gamma := \text{Gal}(\bar{F}/F)$. Let $G$ be a connected reductive group over $F$. In this section we are going to study $B(F,G)_{bsc}$. The formulation of the main result is inspired by Borovoi’s Theorem 5.11 in [Bor98].

15.2. Main result. Let $S_{\infty}$ denote the set of infinite places of $F$. Consider the commutative square

$$
\begin{array}{ccc}
B(F,G)_{bsc} & \longrightarrow & \prod_{u \in S_\infty} B(F_u,G)_{bsc} \\
\kappa_G \downarrow & & \downarrow \\
A(F,G) & \longrightarrow & \prod_{u \in S_\infty} A(F_u,G),
\end{array}
$$

in which the right vertical arrow is the product over $S_{\infty}$ of the local maps $\kappa_G$, and the horizontal arrows have as $u$-components the localization maps appearing in Lemma 11.7.

Proposition 15.1. Diagram (15.1) is a cartesian square of sets.

The proof will be given in the next three subsections.

Corollary 15.2. In the function field case the map

$$\kappa_G : B(F,G)_{bsc} \to A(F,G)$$

is bijective.

15.3. Proof of Proposition [15.1] when $G_{der}$ is simply connected. In this subsection we assume that the derived group of $G$ is simply connected. We consider the short exact sequence

$$1 \to G_{der} \to G \to D \to 1,$$

where $D$ denotes the quotient of $G$ by its derived group $G_{der}$.

Proposition 15.1 is equivalent to the statement that

$$
\begin{array}{ccc}
B(F,G)_{bsc} & \longrightarrow & \prod_{u \in S_\infty} B(F_u,G)_{bsc} \\
\downarrow & & \downarrow \\
B(F,D) & \longrightarrow & \prod_{u \in S_\infty} B(F_u,D).
\end{array}
$$
is a cartesian square of sets, because $A(F, G) = B(F, D)$ and $A(F_u, G) = B(F_u, D)$.

We need to show that the map from $B(F, G)_{bsc}$ to the fiber product is bijective. We begin by showing that it is injective. For this we consider basic elements $b, b'$ in $B(F, G)$, and we assume that

- $b, b'$ become equal in $B(F, D)$, and
- $b, b'$ become equal in $B(F_u, G)$ for every infinite place $u$ of $F$.

We need to show that $b = b'$.

Let $Z(G)$ denote the center of $G$, and let $\nu, \nu' : D_F \to Z(G)$ be the Newton points of $b, b'$ respectively. We claim that $\nu = \nu'$. Indeed, this follows from the fact that $b, b'$ have the same image in $B(F, D)$, since the map $\text{Hom}_F(D_F, Z(G)) \to \text{Hom}_F(D_F, D)$ is injective (any homomorphism from a protorus to the finite group $\ker[Z(G) \to D]$ is obviously trivial).

Since $\nu = \nu'$, the discussion in 2.7 shows that the difference between $b$ and $b'$ is measured by an element $x$ in the pointed set $H^1(F, J_b)$. Here we have chosen an algebraic 1-cocycle representing $b$ and used it to obtain the inner form $J_b$ of $G$. We need to show that $x$ is trivial. Our second assumption implies that $x$ is locally trivial at every infinite place of $F$. Our first assumption, together with the fact that $B(F_u, G)_{bsc} \to B(F_u, D)$ is bijective for finite places $u$ (see Proposition 13.1(1)), tells us that $x$ is locally trivial at every finite place of $F$. Therefore $x$ is locally trivial everywhere. Again using the first assumption, we conclude that $x$ is an element in

$$\ker[\ker^1(F, J_b) \to \ker^1(F, D)].$$

To show that $x$ is trivial we just need to prove that the set (15.3) is trivial. In the number field case this follows from [Kot84, Lemma 4.3.1(b)] (closely related to Theorem 4.3 in [San81]). In the function field case it is even true that the set

$$\ker[H^1(F, J_b) \to H^1(F, D)]$$

is trivial. Indeed, the set (15.4) is the image of $H^1(F, (J_b)_{der})$, and this is trivial because $(J_b)_{der}$ is semisimple simply connected.

We are done proving injectivity of the map from $B(F, G)_{bsc}$ to the fiber product. Now we prove surjectivity. For this we consider

- an element $b_D \in B(F, D)$, and
- elements $b_u \in B(F_u, G)_{bsc}$, one for each $u \in S_\infty$,

such that

- for all $u \in S_\infty$, the elements $b_D$ and $b_u$ become equal in $B(F_u, D)$.

We must show that there exists a basic element $b \in B(F, G)$ such that

- $b$ maps to $b_D$ under $B(F, G) \to B(F, D)$, and
- for every infinite place $u$ the element $b$ maps to $b_u$ under $B(F, G) \to B(F_u, G)$.

Choose a finite set $S$ of places of $F$ such that

- $S$ contains all infinite places of $F$,
- $S$ contains some finite place $u_0$ of $F$, and
- $b_D$ comes from $S$.

Here we are using the following definition.
Definition 15.3. Say that \( b_D \in B(F, D) \) comes from \( S \) if \( b_D \) lies in the image of
\[
(X_*(D) \otimes X_3(K, S))_{G(K/F)} \to (X_*(D) \otimes X_3(K))_{G(K/F)} \simeq B(F, D)
\]
for some (equivalently, every) finite Galois extension \( K/F \) that splits \( D \), where
\( X_3(K, S) \) denotes \( \mathbb{Z}[S_K]_0 \).

To see the equivalence of “some” and “every” in this definition, use the surjectivity of
\( X_3(L, S) \to X_3(K, S) \) when \( L \) is a finite Galois extension of \( F \) with \( L \supset K \).

Next we choose (use [Har66, Lemma 5.5.3] in the number field case and [BW07, Prop. 3.2] in the function field case) a maximal \( F \)-torus \( T \) in \( G \) such that \( T \) is fundamental over \( F_u \) for every \( u \in S \). (In the nonarchimedean case we are using fundamental as a synonym for elliptic, so \( T \) is fundamental when \( T/Z(G) \) is anisotropic.) Finally, we choose a finite Galois extension \( K/F \) that splits \( T \). We then have an exact sequence
\[
1 \to T_{der} \to T \xrightarrow{q} D \to 1,
\]
where \( q \) denotes the restriction of \( G \) to \( D \), and \( T_{der} = T \cap G_{der} \). Observe that
\( D \) also splits over \( K \). Since we have chosen \( S \) large enough that \( b_D \) comes from \( S \), we can express \( b_D \) as the image of some element
\[
\sum_{v \in S_K} \mu_v \otimes v \in X_*(D) \otimes \mathbb{Z}[S_K]_0.
\]
Thus \( \mu_v \in X_*(D) \) for all \( v \in S_K \), and \( \sum_{v \in S_K} \mu_v = 0 \).

For any place \( u \) of \( F \) we now write \( V_u \) for the set of places of \( K \) lying over \( u \). Let \( u \) be an infinite place of \( F \). Because \( b_u \) is basic and \( T \) is fundamental over \( F_u \), Lemma 10.3 guarantees that there exists \( b_{T,u} \in B(F_u, T) \) such that \( b_{T,u} \) maps to our given element \( b_u \). We choose \( \sum_{v \in V_u} \mu'_v \otimes v \in X_*(T) \otimes \mathbb{Z}[V_u] \) whose image in
\( (X_*(T) \otimes \mathbb{Z}[V_u])_{G(K/F)} \simeq B(F_u, T) \) is equal to \( b_{T,u} \).

Now \( b_{T,u} \) and \( b_D \) become equal in \( B(F, D) \), so \( \sum_{v \in V_u} q(\mu'_v) \otimes v \) and \( \sum_{v \in V_u} \mu_v \otimes v \) represent the same class in \( (X_*(D) \otimes \mathbb{Z}[V_u])_{G(K/F)} = B(F_u, D) \). Since \( q : X_*(T) \to X_*(D) \) is surjective, we may modify \( \sum_{v \in V_u} \mu'_v \otimes v \) in such a way that
- it still represents \( b_{T,u} \), and
- \( q(\mu'_v) = \mu_v \) for all \( v \notin V_u \).

We do this for every infinite place \( u \).

Recall that there exists a finite place \( u_0 \) in \( S \). Choose some place \( v_0 \) of \( K \) lying over \( u_0 \). For every finite place \( v \in S_K \) except \( v_0 \) we choose \( \mu'_v \in X_*(T) \) such that \( q(\mu'_v) = \mu_v \). At this point we have chosen elements \( \mu'_v \) satisfying \( q(\mu'_v) = \mu_v \) for every \( v \in S_K \) except for \( v = v_0 \). We now define \( \mu_{v_0} \) to be the unique element of \( X_*(T) \) such that \( \sum_{v \in S_K} \mu'_v = 0 \). Applying \( q \) to this last equality, we see that
\( q(\mu_{v_0}) = \mu_{v_0} \).

It is then clear that \( \sum_{v \in S_K} \mu'_v \otimes v \) represents an element \( b_T \in B(F, T) \) such that
- \( q(b_T) = b_D \), and
- \( b_T \mapsto b_{T,u} \) under \( B(F, T) \to B(F_u, T) \) for all infinite places \( u \) of \( F \).

Therefore the image \( b \) of \( b_T \) in \( B(F, G) \) maps to \( b_D \) and to each \( b_u \). To conclude
the proof of surjectivity, it remains only to show that \( b \) is basic. By Lemma 10.3 it
suffices to show that \( b \) is locally basic everywhere. For infinite places \( u \), this follows
from our assumption that \( b_u \) is basic. For finite places \( u \in S \), it follows from the
fact that $T$ is elliptic at $u$. For finite places $u$ outside $S$, it is trivially true, since $b_T$ comes from $S$, hence is locally trivial outside of $S$.

15.4. An elementary lemma. We are going to need the following very easy lemma, whose proof is left to the reader. The lemma concerns the following situation. Suppose that we are given a cartesian square

$$
\begin{array}{ccc}
A_1 & \rightarrow & A_2 \\
\downarrow^{f_{12}} & & \downarrow^{f_{24}} \\
A_3 & \rightarrow & A_4
\end{array}
$$

(15.5)

in the category of groups, as well as a cartesian square

$$
\begin{array}{ccc}
X_1 & \rightarrow & X_2 \\
\downarrow^{g_{12}} & & \downarrow^{g_{24}} \\
X_3 & \rightarrow & X_4
\end{array}
$$

(15.6)

in the category of sets. Suppose further that

- for $i = 1, 2, 3, 4$ we are given an action of $A_i$ on $X_i$, and
- for $ij = 12, 13, 24, 34$ the map $g_{ij}$ is equivariant with respect to $f_{ij}$, i.e., $g_{ij}(a_i x_i) = f_{ij}(a_i) g_{ij}(x_i)$.

In this situation there is an obvious commutative square

$$
\begin{array}{ccc}
A_1 \setminus X_1 & \rightarrow & A_2 \setminus X_2 \\
\downarrow & & \downarrow \\
A_3 \setminus X_3 & \rightarrow & A_4 \setminus X_4
\end{array}
$$

(15.7)

of sets, with $A_i \setminus X_i$ denoting the quotient of $X_i$ by the action of $A_i$.

**Lemma 15.4.** The square (15.7) is cartesian if

- $A_4 = f_{24}(A_2) f_{34}(A_3)$, and
- $A_4$ acts freely on $X_4$, i.e., if $a_4 \in A_4$ fixes some element of $X_4$, then $a_4$ is the identity.

15.5. Proof of the general case of Proposition 15.1 using $z$-extensions. We have proved Proposition 15.1 when $G_{der}$ is simply connected. Now we use $z$-extensions to prove it in general. So, we begin by choosing a finite Galois extension $K/F$ splitting $G$ and a $z$-extension

$$
1 \rightarrow Z \rightarrow G' \rightarrow G \rightarrow 1
$$

with $Z = R_{K/F}(S)$ for some split $K$-torus $S$. Because $G'_{der}$ is simply connected, the square

$$
\begin{array}{ccc}
B(F, G')_{bsc} & \rightarrow & \prod_{u \in S_{sc}} B(F_u, G')_{bsc} \\
\downarrow & & \downarrow \\
A(F, G') & \rightarrow & \prod_{u \in S_{sc}} A(F_u, G').
\end{array}
$$

(15.8)

is cartesian.
The commutative diagram

\[
\begin{array}{ccc}
B(F, Z) & \longrightarrow & \prod_{u \in S_\infty} B(F_u, Z) \\
\downarrow & & \downarrow \\
A(F, Z) & \longrightarrow & \prod_{u \in S_\infty} A(F_u, Z).
\end{array}
\]

is trivially cartesian, because the vertical arrows are isomorphisms.

As in the lead-up to Lemma 15.4, the groups in diagram (15.9) act on the sets in diagram (15.8), and so we obtain a commutative square of quotient sets, and this boils down to

\[
\begin{array}{ccc}
B(F,G)_{bsc} & \longrightarrow & \prod_{u \in S_\infty} B(F_u,G)_{bsc} \\
\downarrow & & \downarrow \\
A(F,G) & \longrightarrow & \prod_{u \in S_\infty} A(F_u,G),
\end{array}
\]

by Proposition 10.4 and the fact that \(A(F, Z) \to A(F, G') \to A(F, G) \to 0\) is exact, locally as well as globally. (Use that \(0 \to \Lambda_Z \to \Lambda_{G'} \to \Lambda_G \to 0\) is exact.)

It follows from Lemma 15.4 that the square (15.10) is cartesian. The first hypothesis of that lemma is trivially satisfied, because the right vertical arrow in (15.9) is an isomorphism. To show that the second hypothesis is satisfied, we need to check that \(A(F_u, Z) \to A(F_u, G')\) is injective for every place \(u\).

So we need to check that

\[
(\Lambda_Z)|_{(K_v/F_u)} \to (\Lambda_{G'})|_{(K_v/F_u)}
\]

is injective for any \(v\) over \(u\). From the long exact sequence of group homology we see that the kernel of (15.11) is a torsion group (killed by \([K_v : F_u]\)). But, because \(Z = R_{K/F}(S)\), the group \((\Lambda_Z)|_{(K_v/F_u)}\) is torsion-free. So the kernel of (15.11) vanishes, and the proof of Proposition 15.1 is now complete.

15.6. The image of \(B(F,G)_{bsc} \to A(F,G)\). The next result involves the subset \(A_0(F,G)\) of \(A(F,G)\) defined in Remark 11.6.

**Proposition 15.5.** For any global field \(F\) the image of \(\kappa_G : B(F,G)_{bsc} \to A(F,G)\) is \(A_0(F,G)\).

**Proof.** We already know from Remark 11.6 that the image of \(B(F,G)_{bsc} \to A(F,G)\) is contained in \(A_0(F,G)\). So we just need to check that any element in \(A_0(F,G)\) lies in \(\text{im}[B(F,G)_{bsc} \to A(F,G)]\). This follows easily from Propositions 13.4 and 15.1. \(\square\)

15.7. Analysis of the total localization map. Let \(K\) be a finite Galois extension of \(F\) in \(\bar{F}\) such that \(\text{Gal}(\bar{F}/K)\) acts trivially on \(\Lambda_G\). For each finite place \(u\) of \(F\) we choose a place \(v\) of \(K\) lying over \(u\). We then have

\[
B(F_u,G)_{bsc} \simeq (\Lambda_G)|_{(K_v/F_u)}.
\]
So the fact that diagram (15.1) is cartesian can be reformulated as the fact that the diagram

\[
\begin{array}{c}
B(F,G)_{bsc} \\
\downarrow \\
(X_3(K) \otimes \Lambda_G)_{G(K/F)} \\
\downarrow \\
\bigoplus_{u \in V} B(F_u,G)_{bsc}
\end{array}
\]

(15.12)

is cartesian. Here we are using direct-sum notation in a nonstandard way by writing \(\bigoplus_{u \in V} B(F_u,G)_{bsc} \) for the subset of \(\prod_{u \in V} B(F_u,G)_{bsc} \) consisting of families of elements \(b_u \in B(F_u,G)_{bsc} \) such that \(b_u \) is trivial for all but finitely many places \(u \).

In order to understand the significance of this last cartesian diagram, we apply the right-exact functor \(X \mapsto X \otimes \Lambda_G \rightarrow G(K/F) \) to the short exact sequence

\[
0 \rightarrow X_3(K) \rightarrow X_2(K) \rightarrow X_1(K) \rightarrow 0,
\]

concluding that the cokernel of the bottom horizontal arrow in diagram (15.12) can be identified with \((\Lambda_G)_{G(K/F)} \) . The fact that (15.12) is cartesian then yields the following result.

**Proposition 15.6.** An element in \(\bigoplus_{u \in V} B(F_u,G)_{bsc} \) lies in the image of the localization map

\[
(15.13)
\]

if and only if its image under

\[
\bigoplus_{u \in V} B(F_u,G)_{bsc} \rightarrow \bigoplus_{u \in V} (\Lambda_G)_{G(K_u/F_u)} \rightarrow (\Lambda_G)_{G(K/F)}
\]

is trivial.

**Proof.** Clear. \(\square\)

The kernel of the localization map (15.13) is easily seen to coincide with \(\ker^1(F,G)\), and in [Kot84, §4] this is described in terms of \(Z(\hat{G})\). So we have a satisfactory understanding of the localization map.

**APPENDIX A. RIGIDITY OF WEAK TATE-NAKAYAMA TRIPLES**

**A.1. Review of the definition of Tate-Nakayama triple.** Let \(X, A\) be \(G\)-modules, and let \(\alpha \in H^2(G, \text{Hom}(X,A))\). Recall from section 4 that \((X, A, \alpha)\) is a **Tate-Nakayama triple for \(G\)** if the following two conditions hold for every subgroup \(G'\) of \(G\):

- For all \(r \in \mathbb{Z}\) cup product with Res\(_{G/G'}(\alpha)\) induces isomorphisms \(H^r(G',X) \rightarrow H^{r+2}(G',A)\).

- \(H^1(G',\text{Hom}(X,A))\) is trivial.

Weak Tate-Nakayama triples are ones for which the first condition holds (but possibly not the second). The second condition is referred to as rigidity. In this appendix our main goal is to show that weak Tate-Nakayama triples of a certain kind are automatically rigid.
A.2. **Review of Nakayama’s theorem.** Let $G$ be a finite group. For any $G$-module $M$ and any $r \in \mathbb{Z}$ the Tate cohomology group $H^r(G, M)$ is defined. For a subgroup $G'$ of $G$ there are restriction maps $\text{Res}_{G/G'}: H^r(G, M) \to H^r(G', M)$ (see [Ser68] for all this).

We now recall a special case of a result of Nakayama (see [Nak57, Ser68]).

**Theorem A.1.** Let $(X, A, \alpha)$ be a weak Tate-Nakayama triple. Then cup product with $\alpha$ is an isomorphism

$$H^r(G, M \otimes X) \to H^{r+2}(G, M \otimes A)$$

for every $r \in \mathbb{Z}$ and every $G$-module $M$ that is torsion-free as abelian group.

For any $G$-module there is an obvious pairing

$$\text{Hom}(X, A) \otimes \text{Hom}(M, X) \to \text{Hom}(M, A),$$

given by composition of mappings. So cup product with $\alpha \in H^2(G, \text{Hom}(X, A))$ also yields maps

(A.1) $$H^r(G, \text{Hom}(M, X)) \xrightarrow{\alpha \mapsto} H^{r+2}(G, \text{Hom}(M, A)).$$

**Definition A.2.** Let $(X, A, \alpha)$ be a weak Tate-Nakayama triple. Let $C = C(X, A, \alpha)$ be the class of $G$-modules for which (A.1) is an isomorphism for all $r \in \mathbb{Z}$.

The next lemma gives some simple observations about the class $C$, the fourth of which is a standard corollary of Nakayama’s theorem.

**Lemma A.3.**

1. The class $C$ is closed under arbitrary direct sums.
2. Let $0 \to M''' \to M'' \to M' \to 0$ be a short exact sequence of $G$-modules, and assume that $M'$ is free as abelian group. If two of $M', M'', M'''$ lie in the class $C$, then so does the third one.
3. The class $C$ contains all $\mathbb{Z}$-free $G$-modules $M$ which admit a chain

$$M_1 \subset M_2 \subset M_2 \subset \ldots$$

of submodules such that (i) each $M_n$ lies in $C$, and (ii) $M = \bigcup_{n=1}^{\infty} M_n$.
4. The class $C$ contains all $G$-modules $M$ that are free of finite rank as abelian groups.
5. The class $C$ contains all $G$-modules $M$ that are free as abelian groups and have a $\mathbb{Z}$-basis that is permuted by the action of $G$.
6. The class $C$ contains all $G$-modules $M$ that are free of countable rank as abelian groups.

**Proof.** (1) $\text{Hom}(\cdot, X)$ and $\text{Hom}(\cdot, A)$ convert direct sums into direct products, and these are preserved by Tate cohomology.

(2) Our assumption on $M'$ ensures that the sequences

$$0 \to \text{Hom}(M', X) \to \text{Hom}(M'', X) \to \text{Hom}(M'''', X) \to 0$$

$$0 \to \text{Hom}(M', A) \to \text{Hom}(M'', A) \to \text{Hom}(M'''', A) \to 0$$

are short exact. Now consider their long exact sequences of Tate cohomology and apply the 5-lemma.

(3) It follows from (1) that the module $N := \bigoplus_{n=1}^{\infty} M_n$ lies in $C$. We write elements $x \in N$ as sequences $(x_1, x_2, x_3, \ldots)$ with $x_n \in M_n$ and $x_n = 0$ for all
but finitely many \( n \). There is an obvious surjection \( g : N \to M \), defined by
\[
g(x_1, x_2, x_3, \ldots) = \sum_{n=1}^{\infty} x_n.
\]
We define an endomorphism \( f \) of \( N \) by the rule
\[
f(x_1, x_2, x_3, \ldots) = (x_1, x_2 - x_1, x_3 - x_2, \ldots, x_{n+1} - x_n, \ldots).
\]
It is easy to check that the sequence \( 0 \to N \xrightarrow{f} N \xrightarrow{g} M \to 0 \) is short exact. Since
\( N \) lies in \( C \) and \( M \) is \( \mathbb{Z} \)-free, we conclude from (2) that \( M \) lies in \( C \).

(4) Apply the theorem of Nakayama to the \( \mathbb{Z} \)-dual of \( M \).

(5) This follows from (1) and (4).

(6) This follows from (3) and (4).

\[
\square
\]

A.3. A sufficient condition for a weak Tate-Nakayama triple to be rigid.

We consider a weak Tate-Nakayama triple \((X, A, \alpha)\). We are going to give some simple conditions on \( X \) that imply the rigidity of \((X, A, \alpha)\). Before doing so we
introduce some notation. Let \( S \) be any \( G \)-set. Then we write \( \mathbb{Z}[S] \) for the free
abelian group on \( S \). There is an obvious \( G \)-module structure on \( \mathbb{Z}[S] \), for which the \( G \)-action permutes the basis elements \( s \in S \) according to the given action on
\( S \). There is an obvious \( G \)-map \( f \) from \( \mathbb{Z}[S] \) to the trivial \( G \)-module \( \mathbb{Z} \), defined by
\[
f(\sum_{s \in S} n_s s) = \sum_{s \in S} n_s. \]
We denote by \( \mathbb{Z}[S]_0 \) the \( G \)-module obtained as the kernel of \( f \). Thus there is a short exact sequence of \( G \)-modules
\[
(A.2) \quad 0 \to \mathbb{Z}[S]_0 \xrightarrow{i} \mathbb{Z}[S] \xrightarrow{j} \mathbb{Z} \to 0.
\]

**Lemma A.4.** Consider a weak Tate-Nakayama triple \((X, A, \alpha)\). Assume that \( X \) satisfies one of the following two conditions:

- There exists a \( G \)-set \( S \) such that \( X \) is isomorphic to \( \mathbb{Z}[S] \).
- There exists a \( G \)-set \( S \) such that \( X \) is isomorphic to \( \mathbb{Z}[S]_0 \), and, in addition, \( H^{-1}(G', X) \) vanishes for every subgroup \( G' \) of \( G \).

Then cup product induces an isomorphism
\[
(A.3) \quad H^r(G, \text{Hom}(X, X)) \to H^{r+2}(G, \text{Hom}(X, A))
\]
for all \( r \in \mathbb{Z} \). Moreover, \((X, A, \alpha)\) is rigid.

**Proof.** The statement that \((A.3)\) is an isomorphism for all \( r \in \mathbb{Z} \) is just the statement \( X \) lies in the class \( C(X, A, \alpha) \) of Definition \( A.2 \). When \( X = \mathbb{Z}[S] \), this follows from part (5) of Lemma \( A.2 \). When \( X = \mathbb{Z}[S]_0 \), it follows from parts (2) and (5)
of that lemma.

It remains to prove that \((X, A, \alpha)\) is rigid. So, for every subgroup \( G' \) of \( G \), we must show that \( H^1(G', \text{Hom}(X, A)) \) vanishes. In fact, we may as well take \( G' \) to be \( G \), since all the hypotheses of the lemma also hold for the Tate-Nakayama triple \((X, A, \text{Res}_{G/G'}(\alpha))\) for \( G' \). Because of the isomorphism \((A.3)\), we just need to check that \( H^{-1}(G, \text{Hom}(X, X)) \) vanishes.

When \( X = \mathbb{Z}[S] \), this vanishing is a special case of Corollary \( A.10 \) and, when \( X = \mathbb{Z}[S]_0 \), it is a special case of Lemma \( A.11(3) \). (In that lemma, when \( S = T \), we may take \( \epsilon \) to be the identity map.)

\[
\square
\]

In the rest of this appendix we make the calculations with Tate cohomology that were invoked in the proof of the previous lemma. Along the way we review some basic facts about Tate cohomology and prove some other technical results needed in the body of the text.
A.4. **Standard facts about Tate cohomology.** Let $G$ be a finite group. The following lemma reviews some of the most basic facts about Tate cohomology.

**Lemma A.5.**

1. $H^{-1}(G, \mathbb{Z}) = 0$.
2. $H^r(G, \prod_{i \in I} M_i) = \prod_{i \in I} H^r(G, M_i)$.
3. $H^r(G, \bigoplus_{i \in I} M_i) = \bigoplus_{i \in I} H^r(G, M_i)$.

**Proof.** (1) is clear. (2) follows formally from the fact that $H^r(G, M)$ is computed as the cohomology of the complex $\text{Hom}_G(P_n, M)$, where $P_n$ is the standard complete resolution of $G$. (3) follows formally from the fact that each of the $G$-modules $P_n$ in the standard resolution is finitely generated as abelian group, so that the functor $\text{Hom}(P_n, \cdot)$ preserves direct sums. □

A.5. **Modules induced from subgroups.** Again let $G$ be a finite group. Let $H$ be a subgroup of $G$, and let $M$ be a $G$-module. Suppose that there is a family $M_x$ of subgroups of $M$, one for each $x \in G/H$, such that

- $M = \bigoplus_{x \in G/H} M_x$, and
- $M_g = gM_x$ for all $g \in G$, $x \in G/H$.

Let $x_0$ denote the base-point in $G/H$ (in other words, $x_0$ is the trivial coset $H$ of $H$). The stabilizer $H$ of $x_0$ then acts on $M_0 := M_{x_0}$, and $M$ is both induced and coinduced from the $H$-module $M_0$.

Let us denote by $\pi_0$ the projection of $M$ onto the direct summand $M_0$. It is evident that $\pi_0$ is an $H$-map, and Shapiro’s lemma states that the composed map

$$H^r(G, M) \xrightarrow{\text{Res}_{G/H}} H^r(H, M) \xrightarrow{\pi_0} H^r(H, M_0)$$

is an isomorphism.

Now consider an arbitrary $G$-set $S$. Of course $S$ decomposes as

(A.4) $$S = \coprod_{s \in G \setminus S} Gs,$$

where $Gs$ denotes the orbit of $s \in S$ under $G$. For $s \in S$ the map $g \mapsto gs$ identifies $G/G_s$ with $Gs$. As a consequence of Lemma A.5 (2) and Shapiro’s lemma, one obtains the following lemma (see page 714 of [Tat66]).

**Lemma A.6.** For any $G$-set $S$ and any $G$-module $M$ there is a canonical isomorphism

$$\pi : H^r(G, \text{Hom}(\mathbb{Z}[S], M)) \rightarrow \coprod_{s \in G \setminus S} H^r(G_s, M),$$

in which, for any $s \in S$, the $s$-component of $\pi$ is given by the composed map

$$H^r(G, \text{Hom}(\mathbb{Z}[S], M)) \xrightarrow{\text{Res}_{G/G_s}} H^r(G_s, \text{Hom}(\mathbb{Z}[S], M)) \xrightarrow{\pi_s} H^r(G_s, M),$$

where $\pi_s$ is the map sending $f \in \text{Hom}(\mathbb{Z}[S], M)$ to its value at $s$.

The groups $H^r(G_s, M)$ and $H^r(G_t, M)$ are canonically isomorphic when $s, t \in S$ lie in the same $G$-orbit. Indeed, this isomorphism is induced by $\text{Int}(g) : G_s \rightarrow G_t$ and $m \mapsto gm$ for any $g \in G$ such that $gs = t$. The choice of $g$ is immaterial because inner automorphisms act trivially on Tate cohomology. This is why it is reasonable to write the target of the isomorphism in the previous lemma as a product over $G \setminus S$ (rather than going to the trouble of choosing a set of representatives for the orbits of $G$ on $S$).
**Lemma A.7.** Let $S$ be any $G$-set. Then

$$H^r(G, \mathbb{Z}[S]) = \bigoplus_{s \in G \setminus S} H^r(G_s, \mathbb{Z}).$$

**Proof.** Using the decomposition (A.4), we deduce this from Lemma A.5(3) and Shapiro’s lemma. □

When $S$ and $T$ are finite $G$-sets, the $G$-module $\text{Hom}(\mathbb{Z}[S], \mathbb{Z}[T])$ is canonically isomorphic to $\mathbb{Z}[S \times T]$, so there is a canonical isomorphism

$$H^r(G, \text{Hom}(\mathbb{Z}[S], \mathbb{Z}[T])) = \bigoplus_{(s,t) \in G \setminus (S \times T)} H^r(G_{s,t}, \mathbb{Z}),$$

where $G_{s,t}$ denotes the stabilizer of $(s,t)$ in $G$. The next lemma gives a similar result in the case that $S$ and $T$ are arbitrary $G$-sets.

**Lemma A.8.** Let $S$, $T$ be any $G$-sets. Then

$$H^r(G, \text{Hom}(\mathbb{Z}[S], \mathbb{Z}[T])) = \prod_{s \in G \setminus S} \bigoplus_{t \in G \setminus T} H^r(G_{s,t}, \mathbb{Z}).$$

**Proof.** This follows from the previous two lemmas. □

**Remark A.9.** The righthand side of the canonical isomorphism in the previous lemma can be viewed as a subgroup of

$$\prod_{(s,t) \in G \setminus (S \times T)} H^r(G_{s,t}, \mathbb{Z}).$$

The projection of $H^r(G, \text{Hom}(\mathbb{Z}[S], \mathbb{Z}[T]))$ onto $H^r(G_{s,t}, \mathbb{Z})$ is then given by the composed map

$$H^r(G, \text{Hom}(\mathbb{Z}[S], \mathbb{Z}[T])) \xrightarrow{\text{Res}_{G/G_{s,t}}} H^r(G_{s,t}, \text{Hom}(\mathbb{Z}[S], \mathbb{Z}[T])) \xrightarrow{\pi_{s,t}^{-1}} H^r(G_{s,t}, \mathbb{Z}),$$

where $\pi_{s,t} : \text{Hom}(\mathbb{Z}[S], \mathbb{Z}[T]) \to \mathbb{Z}$ is the $G_{s,t}$-map sending $f$ to the $t$-component of $f(s)$.

**Corollary A.10.** Let $S$, $T$ be any $G$-sets. Then

$$H^{-1}(G, \text{Hom}(\mathbb{Z}[S], \mathbb{Z}[T])) = 0.$$  

**Proof.** This follows from the previous lemma together with Lemma A.5(1). □

The next result again involves the short exact sequence (see (A.2))

$$0 \to \mathbb{Z}[S]_0 \xrightarrow{i} \mathbb{Z}[S] \xrightarrow{f} \mathbb{Z} \to 0.$$  

(A.5)

**Lemma A.11.** Let $S$, $T$ be any $G$-sets, and suppose that there exists a $G$-map $\epsilon : T \to S$. Then the following conclusions hold.

1. The map

$$H^0(G, \text{Hom}(\mathbb{Z}, \mathbb{Z}[T])) \xrightarrow{f} H^0(G, \text{Hom}(\mathbb{Z}[S], \mathbb{Z}[T]))$$

is injective.

2. The group $H^{-1}(G, \text{Hom}(\mathbb{Z}[S]_0, \mathbb{Z}[T]))$ vanishes.

3. Suppose further that $H^{-1}(G', \mathbb{Z}[T]_0) = 0$ for every subgroup $G'$ of $G$. Then the group $H^{-1}(G, \text{Hom}(\mathbb{Z}[S]_0, \mathbb{Z}[T]_0))$ vanishes.
Proof. To prove (1) we do the following. For each $t \in T$ we write $G_t$ for the stabilizer of $t$ in $G$, and we define a $G_t$-map

$$g_t : \text{Hom}(Z[S], Z[T]) \to Z$$

by sending $h$ to the $t$-component of $h(e(t))$ (thinking of $h$ as an $S \times T$-matrix satisfying a certain finiteness condition, we are sending $h$ to its entry $h_{e(t),t}$). It is clear that the composed map

$$Z[T] = \text{Hom}(Z, Z[T]) \xrightarrow{f} \text{Hom}(Z[S], Z[T]) \xrightarrow{g_t} Z$$

is nothing but the $G_t$-map $\pi_t$ projecting an element in $Z[T]$ onto its $t$-component. From this we see that any element $x$ in the kernel of the map \((A.6)\) has trivial image under

$$H^0(G, Z[T]) \xrightarrow{\text{Res}_{G/G_t}} H^0(G_t, Z[T]) \xrightarrow{\pi_t} H^0(G_t, Z)$$

for all $t \in T$. It then follows from Lemma \((A.7)\) that $x = 0$.

Now we prove (2). From the long exact sequence of Tate cohomology for the short exact sequence

$$0 \to \text{Hom}(Z, Z[T]) \xrightarrow{f} \text{Hom}(Z[S], Z[T]) \xrightarrow{i} \text{Hom}(Z[S]_0, Z[T]) \to 0$$

we see that the vanishing of $H^{-1}(G, \text{Hom}(Z[S]_0, Z[T]))$ follows from the first part of this lemma, together with the vanishing of $H^{-1}(G, \text{Hom}(Z[S], Z[T]))$ (see Corollary \((A.10)\)).

Finally, we prove (3). We use the long exact cohomology sequence for the short exact sequence

$$0 \to \text{Hom}(Z, Z[T]_0) \xrightarrow{f} \text{Hom}(Z[S], Z[T]_0) \xrightarrow{i} \text{Hom}(Z[S]_0, Z[T]_0) \to 0.$$ 

To prove that $H^{-1}(G, \text{Hom}(Z[S]_0, Z[T]_0))$ vanishes, it is enough to show that

\[(A.7)\]

$$H^{-1}(G, \text{Hom}(Z[S], Z[T]_0)) = 0$$

and that

\[(A.8)\]

$$H^0(G, \text{Hom}(Z, Z[T]_0)) \xrightarrow{f} H^0(G, \text{Hom}(Z[S], Z[T]_0))$$

is injective.

The vanishing of the group in \((A.7)\) follows from Lemma \((A.6)\) together with our assumption that $H^{-1}(G', Z[T]_0) = 0$ for every subgroup $G'$ of $G$. To prove that the map \((A.8)\) is injective, we consider the commutative square

\[(A.9)\]

$$\begin{array}{ccc}
H^0(G, \text{Hom}(Z, Z[T]_0)) & \xrightarrow{f} & H^0(G, \text{Hom}(Z[S], Z[T]_0)) \\
\downarrow & & \downarrow \\
H^0(G, \text{Hom}(Z, Z[T])) & \xrightarrow{f} & H^0(G, \text{Hom}(Z[S], Z[T])),
\end{array}$$

in which the two vertical maps are induced by the inclusion $Z[T]_0 \hookrightarrow Z[T]$. We want to prove that the top horizontal arrow is injective, and for this it will suffice to show that the left vertical arrow and bottom horizontal arrow are both injective. The vanishing of $H^{-1}(G, \text{Hom}(Z, Z)) = H^{-1}(G, Z)$ (see Lemma \((A.3)\)) implies the injectivity of the left vertical arrow. The injectivity of the bottom horizontal arrow was established in the first part of this lemma. \qed
APPENDIX B. REVIEW OF CORESTRICTION

In section 3 we used a number of simple results concerning corestriction in group cohomology. They are probably all standard, but I was not able to find a textbook reference that had everything I needed. For that reason I am including a reasonably complete exposition of corestriction in this appendix (with no claim of originality).

B.1. Notation. In this appendix $G$ is an arbitrary group, so that Tate cohomology is no longer defined, and we are free to write $H^r(G, M)$ ($r \geq 0$) for the ordinary cohomology groups of a $G$-module $M$. These can be computed using any $\mathbb{Z}[G]$-free resolution $P$ of the trivial $G$-module $\mathbb{Z}$. Indeed, $H^r(G, M)$ is the $r$-th cohomology group of the complex $\text{Hom}(P, M)^G = \text{Hom}_G(P, M)$. If one takes $P$ to be the standard resolution $\mathbb{P}$, one is led to standard cochains.

B.2. Automorphisms. Let $\theta$ be an automorphism of $G$. By a $\theta$-automorphism of a $G$-module $M$ we will mean an automorphism $\theta_M$ of the abelian group $M$ such that $\theta_M(gm) = \theta(g)\theta_M(m)$ for all $g \in G$, $m \in M$. Similarly for complexes of $G$-modules. Any $\theta$-automorphism of $M$ preserves the $G$-invariants $M^G$ in $M$.

There is an obvious $\theta$-automorphism $\theta_{\theta}$ of the standard resolution for $G$ (take the automorphism of $\mathbb{P}_r = \mathbb{Z}[G^{r+1}]$ induced by the automorphism $(g_0, \ldots, g_r) \mapsto (\theta(g_0), \ldots, \theta(g_r))$ of $G^{r+1}$).

Let $M$ be a $G$-module, and suppose that we are given a $\theta$-automorphism $\theta_M$ of $M$. There is then an obvious $\theta$-automorphism $\theta$ of the complex $\text{Hom}(\mathbb{P}, M)$ (sending $f : \mathbb{P}_r \to M$ to $\theta_M \circ f \circ \theta_{\mathbb{P}}^{-1}$). The induced automorphism on the cohomology of the complex $\text{Hom}(\mathbb{P}, M)^G$ then provides an automorphism $\theta$ of $H^r(G, M)$. However, it is not essential to use the standard resolution. It works equally well to take any free resolution $P$ (of $\mathbb{Z}$) equipped with a $\theta$-automorphism (of complexes) that induces the identity map on $\mathbb{Z} = H^0(P)$.

Now let $x \in G$, and consider the inner automorphism $\theta_x = \text{Int}(x)$ of $G$. Any $G$-module $M$ then admits a canonical $\theta_x$-automorphism, namely $\theta_M(m) = xm$. It is well-known (see [Ser08]) that the induced automorphism $\theta_x$ of $H^r(G, M)$ is trivial.

However, when one is given a normal subgroup $K$ of $G$, there are some interesting automorphisms (needed for the Hochschild-Serre spectral sequence). Again fix $x \in G$, but now write $\theta_x$ for the automorphism $k \mapsto xkx^{-1}$ of $K$. On any $G$-module $M$ we have the canonical $\theta_x$-automorphism $\theta_M(m) = xm$. So there is an induced automorphism $\theta_x$ on $H^r(K, M)$, and it is often non-trivial. This construction yields an action of $G$ on $H^r(K, M)$, and the normal subgroup $K$ acts trivially. We will refer to the resulting action of $G/K$ on $H^r(K, M)$ as the Hochschild-Serre action.

B.3. Restriction. Let $K$ be a subgroup of $G$. One way to think about restriction homomorphisms in group cohomology is as follows. Let $P$ be a $\mathbb{Z}[G]$-free resolution of $\mathbb{Z}$. Then $P$ is also a $\mathbb{Z}[K]$-free resolution of $\mathbb{Z}$.

For any $G$-module $A$ there is an obvious inclusion $A^G \subset A^K$. Applying this simple observation to the $G$-modules $\text{Hom}(P_r, M)$ ($M$ being some $G$-module), we obtain inclusions $\text{Hom}(P_r, M)^G \hookrightarrow \text{Hom}(P_r, M)^K$, and these give rise to restriction maps

$$\text{Res}_{G/K} : H^r(G, M) \to H^r(K, M)$$

for any $G$-module $M$. 
B.4. Corestriction. Let $K$ be a subgroup of $G$, and assume that the index $[G : K]$ is finite. For any $G$-module $A$ there is a norm map $N_{G/K} : A^K \to A^G$, defined by

$$N_{G/K}(a) = \sum_{g \in G/K} ga.$$  

Applying this construction to the $G$-modules $\text{Hom}(P_r, M)$, we obtain induced maps

$$H^r(K, M) \to H^r(G, M),$$

called corestriction maps, and denoted by $\text{Cor}_{G/K}$.

It is a standard result that $\text{Cor}_{G/K} \text{Res}_{G/K} = [G : K]$. When $K$ is normal in $G$, we have another standard result.

**Lemma B.1.** Assume that $K$ is a normal subgroup of finite index in $G$, and let $M$ be a $G$-module. We then have the Hochschild-Serre action of $G/K$ on $H^r(K, M)$.

1. The composed map $\text{Res}_{G/K} \text{Cor}_{G/K}$ coincides with the norm map $N_{G/K}$ formed using the action of $G/K$ on $H^r(G, M)$.
2. The corestriction map $\text{Cor}_{G/K} : H^r(K, M) \to H^r(G, M)$ factors through the canonical surjection $H^r(K, M) \to H^r(K, M)_{G/K}$ from $H^r(K, M)$ to the group of $G/K$-coinvariants for the Hochschild-Serre action of $G/K$.

**Proof.** For (1) see Corollary 9.2 on page 257 in Cartan-Eilenberg, though they are treating Tate cohomology and are therefore assuming that $G$ is finite. This makes no real difference.

For (2) we can reason as follows. Corestriction is functorial in the following sense. Suppose that we are given an automorphism $\theta$ of $G$ that preserves a subgroup $K$ of finite index in $G$. Suppose too that we are given a $\theta$-automorphism $\theta_M$ of some $G$-module $M$. We then obtain induced automorphisms (denoted by $\theta$) on the cohomology groups $H^r(G, M)$ and $H^r(K, M)$, and the square

$$\begin{array}{ccc}
H^r(K, M) & \xrightarrow{\text{Cor}_{G/K}} & H^r(G, M) \\
\downarrow \theta & & \downarrow \theta \\
H^r(K, M) & \xrightarrow{\text{Cor}_{G/K}} & H^r(G, M)
\end{array}$$

commutes. When $K$ is normal, we may take $\theta$ to be the inner automorphism $\text{Int}(x)$ obtained from some element $x \in G$. We may take $\theta_M$ to be $m \mapsto xm$. Then the induced automorphism of $H^r(G, M)$ is trivial, and the induced automorphism of $H^r(K, M)$ is precisely the Hochschild-Serre action of $x \in G/K$. The commutativity of our square for all $x \in G/K$ then tells us that $\text{Cor}_{G/K}$ factors through the coinvariants of $G/K$ on $H^r(K, M)$. \qed

B.5. Corestriction for coinduced modules. Again let $K$ be a subgroup of finite index in $G$. For any $K$-module $M$ we write $R(M)$ for the $G$-module coinduced from $M$. Thus an element in $R(M)$ is a function $f : G \to M$ such that $f(kx) = kf(x)$ for all $k \in K, x \in G$. The group $G$ acts by right translations. The adjunction morphism $\epsilon : R(M) \to M$ is the $K$-map given by $\epsilon(f) = f(1_G)$. There is a canonical $K$-map
$j : M \to R(M)$ such that $\epsilon j = \text{id}_M$. It sends $m \in M$ to the element $f_m \in R(M)$ defined by

$$f_m(x) := \begin{cases} xm & \text{if } x \in K, \\ 0 & \text{otherwise}. \end{cases}$$

We denote by $Sh : H^i(G, R(M)) \to H^i(K, M)$ the Shapiro isomorphism.

**Lemma B.2.** For any $K$-module $M$ and any $i \geq 0$ the composed map

$$(B.1) \quad H^i(K, M) \xrightarrow{j_*} H^i(K, R(M)) \xrightarrow{\text{Cor}_{G/K}} H^i(G, R(M)) \xrightarrow{Sh} H^i(K, M)$$

is equal to the identity map on $H^i(K, M)$.

**Proof.** Observe that the functor $R$ is exact and that all three arrows in (B.1) are actually morphisms of cohomological $\partial$-functors. Since the initial cohomological $\partial$-functor is universal, in order to check that (B.1) is always the identity map, it suffices to do so when $i = 0$. This is a simple computation. \hfill $\square$

**B.6. Corestriction of homogeneous cochains.** Again let $K$ be a subgroup of finite index in $G$. We want to give a cocycle-level formula for corestriction $\text{Cor}_{G/K}$. Of course this involves a choice. As we have seen, corestriction is most easily understood when one views a free $G$-resolution of $Z$ as also being a free $K$-resolution of $Z$. For this purpose we use the standard resolution $P_r(K) = \mathbb{Z}[G^{r+1}]$ for $G$. We will also need the standard resolution $P_r(K)$ for $K$, given by $P_r(K) = \mathbb{Z}[K^{r+1}]$. Both $\text{Hom}(P(G), M)^K$ and $\text{Hom}(P(K), M)^K$ compute the $K$-cohomology of $M$. To relate them, we can use any morphism $P(G) \to P(K)$ of complexes of $K$-modules that induces the identity on the module $Z$ that is being resolved. The most obvious way to get such a morphism is to choose a map $p : G \to K$ satisfying $p(kg) = kp(g)$ (for all $k \in K$, $g \in G$), and then to define $\tilde{p} : P_r(G) \to P_r(K)$ as the $\mathbb{Z}$-linear map induced by $(g_0, \ldots, g_r) \mapsto (p(g_0), \ldots, p(g_r))$.

Let us use homogeneous cochains. So, we start with a standard homogeneous cochain for $K$ with values in $M$. This is simply a map $f : K^{r+1} \to M$ satisfying $f(xk_0, \ldots, xk_r) = xf(k_0, \ldots, k_r)$ for all $x \in K$. From this we obtain a map $f_1 : G^{r+1} \to M$, defined by $f_1(g_0, \ldots, g_r) = f(p(g_0), \ldots, p(g_r))$. The corestriction $f_2$ of $f$ is obtained by applying $N_{G/K}$ to $f_1$. Thus

$$(B.2) \quad f_2(g_0, \ldots, g_r) = \sum_{x \in G/K} x(f(p(x^{-1}g_0), \ldots, p(x^{-1}g_r))).$$

This too is a homogeneous cochain.

**B.7. Compatibility of corestriction with pullback maps.** Suppose we are given a finite group $G$ and a commutative diagram

$$\begin{array}{cccccc}
1 & \to & A' & \to & E' & \to & G & \to & 1 \\
\downarrow h & & \downarrow \tilde{h} & & \downarrow & & \downarrow & & \downarrow \\
1 & \to & A & \to & E & \to & G & \to & 1
\end{array}$$

with exact rows. The homomorphisms $h, \tilde{h}$ induce pullback maps $h^*, \tilde{h}^*$ on group cohomology. These are inflation maps when $h, \tilde{h}$ are surjective.
Lemma B.3. Let $M$ be an $E$-module. Then the square

$$
\begin{array}{ccc}
H^r(A, M) & \xrightarrow{\text{Cor}_{E/A}} & H^r(E, M) \\
\downarrow h^r & & \downarrow h^r \\
H^r(A', M) & \xrightarrow{\text{Cor}_{E'/A'}} & H^r(E', M)
\end{array}
$$

commutes for all $r \geq 0$.

Proof. This follows easily from the explicit formula we gave for corestriction of homogeneous cochains.

B.8. Explicit formula for corestriction of inhomogeneous 1-cochains. In the case of 1-cochains let us now rewrite the formula for corestriction in terms of inhomogeneous cochains. So, we start with an inhomogeneous 1-cochain for $K$, i.e. a map $\phi : K \to M$. The corresponding homogeneous 1-cochain $f$ is given by $f(k_0, k_1) = k_0(\phi(k_0^{-1}k_1))$. Corestriction sends this to the homogeneous 1-cochain $f_2$. The corresponding inhomogeneous 1-cochain $\psi$ (which represents the corestriction of $\phi$) is given by

$$
\psi(g) = f_2(1, g) = \sum_{y \in G/K} yf(p(y^{-1}), p(y^{-1}g)) = \sum_{x \in K \setminus G} x^{-1}p(x)(\phi(p(x)^{-1}p(xg))).
$$

Having made this computation, we no longer have any use for homogeneous cochains, and we revert to our usual practice of referring to inhomogeneous cochains simply as cochains. The same goes for cocycles.

Let us apply the computation above to the following very special case. We consider an extension

$$
1 \to A \to E \to G \to 1
$$

of a finite group $G$ by an abelian group $A$. For any $G$-module $M$ we are interested in the corestriction map $H^1(A, M) \to H^1(E, M)$. Because $A$ is abelian, $H^1(A, M)$ is equal to $\text{Hom}(A, M)$.

As usual, inner automorphisms by elements in $G$ make $A$ into a $G$-module. Choose a set-theoretic section $s : G \to E$, and define a 2-cocycle $\alpha$ of $G$ in $A$ by the rule $s(\sigma)s(\tau) = \alpha_{\sigma, \tau}s(\tau)$. Our cochain level version of corestriction requires the choice of a map $p : E \to A$ such that $p(aw) = ap(w)$ for all $a \in A$, $w \in E$. The obvious way to get such a map $p$ is to put $p(as(\sigma)) := a$ for all $a \in A$, $\sigma \in G$.

A 1-cocycle of $A$ in $M$ is a homomorphism $\mu : A \to M$ of abelian groups. An easy computation shows that the corestriction of $\mu$ to $E$ is represented by the 1-cocycle $b$ of $E$ in $M$ defined by

$$
b_{as(\sigma)} = \sum_{\tau \in G} \tau^{-1}(\mu(\tau(a)) + \mu(\alpha_{\tau, \sigma}))
= (N_G\mu)(a) + \sum_{\tau \in G} \tau^{-1}(\mu(\alpha_{\tau, \sigma}))
$$

for all $a \in A$, $\sigma \in G$. (This provides a nice illustration of the principle that $\text{Res} \circ \text{Cor}$ is $N_G$ when we are dealing with a normal subgroup and the quotient group is $G$.)

In the special case when $N_G\mu = 0$, the formula above shows that $b$ is inflated from the 1-cocycle $b'$ of $G$ in $M$ given by

$$
b'_\sigma = \sum_{\tau \in G} \tau^{-1}(\mu(\alpha_{\tau, \sigma})).
$$
Now Tate cohomology makes sense for the finite group $G$, and, still assuming that $N_G\mu = 0$, we may view $\mu$ as a $(-1)$-cocycle of $G$ in $\text{Hom}(A,M)$, so we can also form the cup-product $c := \alpha \smallsmile \mu \in Z^1(G,M)$.

**Lemma B.4.** The cocycles $b'$ and $c$ are cohomologous.

*Proof.* This follows from the lemma in the next subsection. □

**B.9. Some formulas for cup products.** Let $G$ be a finite group and let $A$, $B$ be $G$-modules. We then have Tate cohomology groups and cup product pairings

$$H^p(G, A) \otimes H^q(G, B) \to H^{p+q}(G, A \otimes B).$$

We need a cochain level formula for this cup product when $p = 2$ and $q = -1$.

**Lemma B.5.** Let $a_{\sigma, \tau}$ be a 2-cocycle of $G$ in $A$ and let $b$ be a $(-1)$-cocycle of $G$ in $B$. Thus $b$ is an element of $B$ such that $Nb = 0$, where $Nb := \sum_{\sigma \in G} \sigma b$. Then the cup product $c = a \smallsmile b$ is represented by the 1-cocycle

$$c_{\sigma} = \sum_{\tau \in G} a_{\sigma, \tau} \otimes \sigma \tau b.$$ 

Moreover the 1-cocycle

$$d_{\sigma} = \sum_{\tau \in G} \tau^{-1} a_{\tau, \sigma} \otimes \tau^{-1} b$$

is cohomologous to $c_{\sigma}$, so it too represents $a \smallsmile b$.

*Proof.* The formula for $c$ comes from the article by Atiyah-Wall in [CF67]. It remains to prove that $c$ and $d$ are cohomologous. Now we will certainly get a 1-cocycle $c'_{\sigma}$ cohomologous to $c_{\sigma}$ if we replace $a_{\sigma, \tau}$ by a 2-cocycle $a'_{\sigma, \tau}$ cohomologous to $a_{\sigma, \tau}$, and in fact we will see that $d_{\sigma} = c'_{\sigma}$ for a suitable choice of $a'_{\sigma, \tau}$.

The right choice for $a'_{\sigma, \tau}$ turns out to be

$$a'_{\sigma, \tau} = -\sigma \tau a_{\tau^{-1}, \sigma^{-1}}.$$ 

Why is $a'$ cohomologous to $a$? Use the 2-cocycle $a$ to build an extension $p : E \to G$ of $G$ by $A$, equipped with a set-theoretic section $s : G \to E$ of $p$. By construction we have the multiplication rule $s_{\sigma} s_{\tau} = a_{\sigma, \tau} s_{\sigma \tau}$ in $E$. Any other set-theoretic section $s'$ gives rise to a cohomologous 2-cocycle. The 2-cocycle $a'$ is obtained in this way from the section $s'$ defined by $s'_{\sigma} = (s_{\sigma^{-1}})^{-1}$.

For this choice of $a'$ we find that $c'_{\sigma}$ is given by

$$c'_{\sigma} = -\sum_{\tau \in G} \sigma \tau a_{\tau^{-1}, \sigma^{-1}} \otimes \sigma \tau b.$$ 

The 1-cocycle property for $c'$ implies that $c'_{\sigma} + \sigma c'_{\sigma^{-1}} = 0$. Therefore we have

$$c'_{\sigma} = -\sigma c'_{\sigma^{-1}} = \sum_{\tau \in G} \tau a_{\tau^{-1}, \sigma} \otimes \tau b.$$ 

Replacing $\tau$ by $\tau^{-1}$ in this last sum, we find that $c'_{\sigma} = d_{\sigma}$, as claimed. □
References

[Bor88] M. V. Borovoi, Galois cohomology of real reductive groups and real forms of simple Lie algebras. Funktsional. Anal. i Prilozhen. 22 (1988), no. 2, 63–64.

[Bor98] Mikhail Borovoi, Abelian Galois cohomology of reductive groups, Mem. Amer. Math. Soc. 132 (1998), no. 626, viii+50.

[BT87] F. Bruhat and J. Tits, Groupes algébriques sur un corps local. Chapitre III. Compléments et applications à la cohomologie galoisienne, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 34 (1987), no. 3, 671–698.

[BW07] Kai-Uwe Bux and Kevin Wortman, Finiteness properties of arithmetic groups over function fields, Invent. Math. 167 (2007), no. 2, 355–378.

[CF67] Algebraic number theory, Proceedings of an instructional conference organized by the London Mathematical Society (a NATO Advanced Study Institute) with the support of the International Mathematical Union. Edited by J. W. S. Cassels and A. Fröhlich, Academic Press, London, 1967.

[DeB06] Stephen DeBacker, Parameterizing conjugacy classes of maximal unramified tori via Bruhat-Tits theory, Michigan Math. J. 54 (2006), no. 1, 157–178.

[Har66] Günter Harder, Über die Galoiskohomologie halbeinfacher Matrizengruppen. II, Math. Z. 92 (1966), 396–415.

[Har75] G. Harder, Über die Galoiskohomologie halbeinfacher algebraischer Gruppen. III, J. Reine Angew. Math. 274/275 (1975), 125–138, Collection of articles dedicated to Helmut Hasse on his seventy-fifth birthday, III.

[Kne65a] Martin Kneser, Galois-Kohomologie halbeinfacher algebraischer Gruppen über p-adischen Körpern. I, Math. Z. 88 (1965), 40–47.

[Kne65b] Martin Kneser, Galois-Kohomologie halbeinfacher algebraischer Gruppen über p-adischen Körpern. II, Math. Z. 89 (1965), 250–272.

[Kot84] Robert E. Kottwitz, Stable trace formula: cuspidal tempered terms, Duke Math. J. 51 (1984), no. 3, 611–650.

[Kot85] Robert E. Kottwitz, Isocrystals with additional structure, Compositio Math. 56 (1985), no. 2, 201–220.

[Kot86] Robert E. Kottwitz, Stable trace formula: elliptic singular terms, Math. Ann. 275 (1986), no. 3, 305–399.

[Kot97] Robert E. Kottwitz, Isocrystals with additional structure. II, Compositio Math. 109 (1997), no. 3, 255–339.

[LR87] R. P. Langlands and M. Rapoport, Shimuravarietäten und Gerben, J. Reine Angew. Math. 378 (1987), 113–220.

[Nak57] Tadasi Nakayama, Cohomology of class field theory and tensor product modules. I, Ann. of Math. (2) 65 (1957), 255–267.

[RR96] M. Rapoport and M. Richartz, On the classification and specialization of F-isocrystals with additional structure, Compositio Math. 103 (1996), no. 2, 153–181.

[San81] J.-J. Sansuc, Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres, J. Reine Angew. Math. 327 (1981), 12–80.

[Ser68] Jean-Pierre Serre, Corps locaux, Hermann, Paris, 1968, Deuxième édition, Publications de l’Université de Nancago, No. VIII.

[She79] D. Shelstad, Characters and inner forms of a quasi-split group over R, Compositio Math. 39 (1979), no. 1, 11–45.

[SR72] Neantro Saavedra Rivano, Catégories Tannakiennes, Lecture Notes in Mathematics, Vol. 265, Springer-Verlag, Berlin, 1972.

[Tat66] J. Tate, The cohomology groups of tori in finite Galois extensions of number fields, Nagoya Math. J. 27 (1966), 709–719.

Robert E. Kottwitz, Department of Mathematics, University of Chicago, 5734 University Avenue, Chicago, Illinois 60637

E-mail address: kottwitz@math.uchicago.edu