THE LEBESGUE DECOMPOSITION OF THE FREE ADDITIVE
CONVOLUTION OF TWO PROBABILITY DISTRIBUTIONS

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Abstract. We prove that the free additive convolution of two Borel probability
measures supported on the real line can have a component that is singular
continuous with respect to the Lebesgue measure on $\mathbb{R}$ only if one of the two
measures is a point mass. The density of the absolutely continuous part with
respect to the Lebesgue measure is shown to be analytic wherever positive
and finite. The atoms of the free additive convolution of Borel probability
measures on the real line have been described by Bercovici and Voiculescu in
a previous paper.

1. Introduction

The notion of freeness (or free independence) has been introduced by Voiculescu
in [19], with the main purpose of better understanding free group factors. As in
the classical case, the distribution of a sum of free random variables is uniquely
determined by the distributions of the summands, and the resulting distribution of
the sum is called the free additive convolution of the distributions of the summands.

Specifically, for the case of probability distributions on $\mathbb{R}$, let $\mu$ and $\nu$
be Borel probability measures on the real line. We can define the free additive convolution
$\mu \boxplus \nu$ of $\mu$ and $\nu$ in the following way. Denote by $F[a, b]$ the free group with
free generators $a$ and $b$, and consider the group von Neumann algebra $L(F[a, b])$
generated by the left regular representation of $F[a, b]$, endowed with the (unique)
normal faithful trace $\tau$. Choose two selfadjoint operators $X_\mu$ and $X_\nu$ affiliated
with the subalgebras of $L(F[a, b])$ generated by the images, via the left regular
representation, of $a$ and $b$, respectively, so that their distribution with respect to $\tau$
is $\mu$ and $\nu$, respectively. It has been shown in [6] by Bercovici and Voiculescu that
the distribution of the selfadjoint operator $X_\mu + X_\nu$ with respect to $\tau$ depends only
on the distributions $\mu$ and $\nu$ of $X_\mu$ and $X_\nu$, respectively. We denote it by $\mu \boxplus \nu$.

For an introduction to the field of free probability we refer to [21].

An analytic method for the computation of free additive convolutions has been
devised in [19] (for compactly supported probabilities) and in [6] (for the case of
probabilities with arbitrary support on $\mathbb{R}$). We will give below a brief outline of
this method.

For any finite positive measure $\sigma$ on $\mathbb{R}$, define its Cauchy transform

$$G_\sigma(z) = \int_\mathbb{R} \frac{d\sigma(t)}{z - t}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

and let $F_\sigma(z) = 1/G_\sigma(z)$. Since $G_\sigma(\overline{z}) = \overline{G_\sigma(z)}$, we shall consider from now on only
the restrictions of $F_\sigma$ and $G_\sigma$ to the upper half-plane $\mathbb{C}^+ = \{ z \in \mathbb{C} : \Im z > 0 \}$. 

For given $\alpha \geq 0$, $\beta > 0$, let us denote $\Gamma_{\alpha, \beta} = \{ z \in \mathbb{C}^+: \Im z > \alpha, |\Re z| < \beta \Im z\}$. The following two results appear in [6].

**Proposition 1.1.** Let $\mu$ be a probability on $\mathbb{R}$. There exists a nonempty domain $\Omega$ in $\mathbb{C}^+$ of the form $\Omega = \cup_{\alpha > 0} \Gamma_{\alpha, \beta}$, such that $F_\mu$ has a right inverse with respect to composition $F_\mu^{-1}$ defined on $\Omega$. In addition, we have $\Im F_\mu^{-1}(z) \leq \Im z$ and

$$\lim_{z \to \infty, z \in \Gamma_{\alpha, \beta}} \frac{F_\mu^{-1}(z)}{z} = 1$$

for every $\alpha, \beta > 0$.

Let $\phi_\mu(z) = F_\mu^{-1}(z) - z$, $z \in \Omega$. The basic property of the function $\phi_\mu$ is described in the following theorem of Voiculescu:

**Theorem 1.2.** Let $\mu, \nu$ be two probability measures supported on the real line. Then $\phi_{\mu \boxplus \nu}(z) = \phi_\mu(z) + \phi_\nu(z)$ for $z$ in the common domain of the three functions.

Thus, the map $\phi$, called the Voiculescu transform, is the free analogue of the logarithm of the Fourier transform from classical probability theory. This function is related to the $R$-transform by the equality $\phi_\mu(z) = R_\mu(1/z)$. (Historically, the $R$-transform, defined as $R_\mu(z) = G_\mu^{-1}(z) - (1/z)$ was first introduced by Voiculescu in [19], but the analysis in the context of measures with unbounded support turns out to be simpler when expressed in terms of the Voiculescu transform.)

Another important property for (Cauchy transforms of) free convolutions of probability measures is subordination. It has been shown that $G_{\mu \boxplus \nu}$ is subordinated to $G_\mu$, in the sense that there exists a unique analytic self-map $\omega$ of the upper half-plane $\mathbb{C}^+$, so that $G_{\mu \boxplus \nu}(z) = G_\mu(\omega(z))$, $z \in \mathbb{C}^+$, and $\lim_{y \to +\infty} \omega(iy)/iy = 1$. This result was proved first in [20] under a genericity assumption, then extended to full generality in [9]. A new proof based on the theory of fixed points of analytic self-maps of the upper half-plane has been given in [6].

Subordination has been until now the most powerful tool for proving regularity results for free convolutions. Pioneering work in this direction has been done by Voiculescu, alone in [20] (see, for example, Proposition 4.7), and together with Bercovici in [4] and [8]. Among the results proved in [8], we mention the description of the atoms of $\mu \boxplus \nu$ (Theorem 7.4): a number $a \in \mathbb{R}$ is an atom for $\mu \boxplus \nu$ if and only if there exist $b, c \in \mathbb{R}$ so that $a = b + c$ and $\mu(\{b\}) + \nu(\{c\}) > 1$. Moreover, $(\mu \boxplus \nu)(\{a\}) = \mu(\{b\}) + \nu(\{c\}) - 1$. For the special case when $\mu$ is the semicircular distribution (i.e. $d\mu(t) = \frac{1}{\pi} \chi_{[-2,2]}(t) \sqrt{4-t^2} dt$, where $\chi_A$ denotes the characteristic function of the set $A$), Biane [10] proved several properties of $\mu \boxplus \nu$, from which we mention that the singular continuous part with respect to the Lebesgue measure of $\mu \boxplus \nu$ is always zero, while the density of its absolutely continuous part with respect to the Lebesgue measure is bounded and analytic wherever positive. Similar results have been proved for measures belonging to partially defined semigroups with respect to free additive and multiplicative convolutions (see [5] and [4]). In [2] it has been shown that, roughly speaking, free convolutions of two probability measures can be purely singular only if at least one of the two measures is a point mass. Moreover, for compactly supported measures, the support of the singular part, if existing, must be of zero Lebesgue measure.

All these results seem to indicate that free convolutions do not favorize large singular parts. In Theorem 4.1 of this paper we show that when neither $\mu$ nor $\nu$ is a point mass, the singular continuous part of $\mu \boxplus \nu$ is zero, while the density of
the absolutely continuous part of $\mu \boxplus \nu$ with respect to the Lebesgue measure is analytic outside a closed set of zero Lebesgue measure.

The rest of the paper is organized as follows: in Section 2 we give without proof several results from complex analysis that we will use later, in Section 3 we analyze the boundary behaviour of the subordination functions, and in Section 4 we prove the main result of the paper.

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2. Preliminary results

In the following, unless otherwise specified, the attributes "singular", "singular continuous", and "absolutely continuous" will be considered with respect to the Lebesgue measure on $\mathbb{R}$. Given a finite positive Borel measure $\sigma$ on the real line, we denote by $\sigma^s$ (respectively $\sigma^{sc}$, $\sigma^{ac}$) the singular (respectively singular continuous and absolutely continuous) part of $\sigma$. All measures considered in this paper are assumed to be Borel measures.

The following results characterize the Cauchy transform of $\sigma$. For more details and proofs we refer to [1].

**Theorem 2.1.** Let $G: \mathbb{C}^+ \rightarrow \mathbb{C}^-$, where $\mathbb{C}^- = -\mathbb{C}^+$, be an analytic function. The following statements are equivalent:

1. There exists a unique positive measure $\sigma$ on $\mathbb{R}$ such that $G = G_\sigma$;
2. For any $\alpha, \beta > 0$, we have that
   \[ \lim_{z \to \infty, z \in \Gamma_{\alpha, \beta}} \frac{zG(z)}{z} \]
   exists and is finite ($\Gamma_{\alpha, \beta} = \{ z \in \mathbb{C}^+: |\Re z| < \alpha \Im z, \Im z > \beta \}$).
3. The limit $\lim_{y \to +\infty} i y G(iy)$ exists and is finite.

Moreover, the limits from 2 and 3 equal $\sigma(\mathbb{R})$.

Observe also that
\[ -\frac{1}{\pi} \Im G_\sigma(x + iy) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(x - t)^2 + y^2} d\sigma(t), \quad x \in \mathbb{R}, y > 0, \]

is the Poisson integral of $\sigma$.

As mentioned also in the introduction, it turns out that in many situations it is much easier to deal with the reciprocal $F_\sigma = 1/G_\sigma$ of the Cauchy transform of the measure $\sigma$. The following proposition is an obvious consequence of Theorem 2.1.

**Proposition 2.2.** Let $F: \mathbb{C}^+ \rightarrow \mathbb{C}^+$ be an analytic self-map of the upper half-plane. The following statements are equivalent:

1. There exists a positive measure $\sigma$ on $\mathbb{R}$ such that $F = 1/G_\sigma$;
2. For any $\alpha, \beta > 0$, the limit $\lim_{z \to \infty, z \in \Gamma_{\alpha, \beta}} \frac{F(z)}{z}$ exists and belongs to $(0, +\infty)$;
3. The limit $\lim_{y \to +\infty} \frac{F(iy)}{iy}$ exists and belongs to $(0, +\infty)$.

Moreover, both limits form 2. and 3. equal $\sigma(\mathbb{R})^{-1}$. 

In general, analytic self-maps of the upper half-plane can be represented uniquely by a triple \((a, b, \rho)\), where \(a\) is a real number, \(b \in [0, +\infty)\), and \(\rho\) is a positive finite measure on \(\mathbb{R}\). This representation is called the Nevanlinna representation (see [1]).

**Theorem 2.3.** Let \(F: \mathbb{C}^+ \rightarrow \mathbb{C}^+\) be an analytic function. Then there exists a triple \((a, b, \rho)\), where \(a \in \mathbb{R}\), \(b \geq 0\), and \(\rho\) is a positive finite measure on \(\mathbb{R}\) such that

\[
F(z) = a + bz + \int_{\mathbb{R}} \frac{1 + tz}{t - z} d\rho(t), \quad z \in \mathbb{C}^+.
\]

The triple \((a, b, \rho)\) satisfies \(a = \Re F(i)\), \(b = \lim_{y \to +\infty} \frac{F(iy)}{iy}\), and \(b + \rho(\mathbb{R}) = \Im F(i)\).

The converse of Theorem 2.3 is obviously true.

**Remark 2.4.** An immediate consequence of Proposition 2.2 and Theorem 2.3 is that for any finite measure \(\sigma\) on \(\mathbb{R}\), we have \(\Im F_\sigma(z) \geq \sigma(\mathbb{R})^{-1} \Re z\) for all \(z \in \mathbb{C}^+\), with equality for any value of \(z\) if and only if \(\sigma\) is a point mass. In this case, the measure \(\rho\) in the statement of Theorem 2.3 is zero.

As observed above, any finite measure \(\sigma\) on the real line is uniquely determined by its Cauchy transform. Moreover, regularity properties of \(\sigma\) can be deduced from the behaviour of \(G_\sigma\), and hence of \(F_\sigma\), near the boundary of its domain. In the following we shall state several classical theorems concerning analytic self-maps of the unit disc \(D = \{z \in \mathbb{C}: |z| < 1\}\) and their boundary behaviour, i.e. the behaviour near points belonging to the boundary \(T = \{z \in \mathbb{C}: |z| = 1\}\) of \(D\). Because the upper half-plane is conformally equivalent to the unit disc via the rational transformation \(z \mapsto \frac{z + i}{z - i}\), most of these theorems will have obvious formulations for self-maps of the upper half-plane.

We shall consider the set \(\mathbb{C} \cup \{\infty\}\) to be endowed with the usual topology: for any point \(z \in \mathbb{C}\), the family \(B_n(z) = \{w \in \mathbb{C}: |z - w| < 1/n\}\), \(n \in \mathbb{N}\), forms a basis of neighbourhoods of \(z\), while the family \(K_n = \{w \in \mathbb{C}: |w| > n\} \cup \{\infty\}\), \(n \in \mathbb{N}\), forms a basis of neighbourhoods for the point infinity. The notions of limit and continuity will be considered with respect to this topology and, when subsets of \(\mathbb{C} \cup \{\infty\}\) are involved, we consider on them the topology inherited from \(\mathbb{C} \cup \{\infty\}\), unless otherwise specified. For a function \(f: \mathbb{C}^+ \rightarrow \mathbb{C} \cup \{\infty\}\), and a point \(x \in \mathbb{R}\), we say that the nontangential limit of \(f\) at \(x\) exists if the limit \(\lim_{z \to x, z \in \Gamma_\alpha(x)} f(z)\) exists in \(\mathbb{C} \cup \{\infty\}\) for all \(\alpha > 0\), where \(\Gamma_\alpha(x) = \{z \in \mathbb{C}^+: |\Re z - x| < \alpha |\Im z|\}\). A similar definition holds for functions defined in the unit disc. We shall denote nontangential limits by \(\lim_{z \to x} f(z)\), or

\[
\lim_{z \to x, z \in \Gamma_\alpha} f(z).
\]

The nontangential limit of \(f\) at infinity is defined in a similar way: \(\lim_{z \to \infty} f(z)\) is said to exist in \(\mathbb{C} \cup \{\infty\}\) if the limit \(\lim_{z \to \infty, z \in \Gamma_\alpha(0)} f(z)\) exists in \(\mathbb{C} \cup \{\infty\}\) for all \(\alpha > 0\).

In the following three theorems are described some properties of meromorphic functions in the unit disc related to their nontangential boundary behaviour.

**Theorem 2.5.** Let \(f: D \rightarrow \mathbb{C}\) be a bounded analytic function. Then the set of points \(x \in T\) at which the nontangential limit of \(f\) fails to exist is of linear measure zero.

**Theorem 2.6.** Let \(f: D \rightarrow \mathbb{C}\) be an analytic function. Assume that there exists a set \(A\) of nonzero linear measure in \(T\) such that the nontangential limit of \(f\) exists at each point of \(A\), and equals zero. Then \(f(z) = 0\) for all \(z \in D\).
Theorem 2.7. Let $f: \mathbb{D} \to \mathbb{C} \cup \{\infty\}$ be a meromorphic function, and let $e^{i\theta} \in \mathbb{T}$. Assume that the set $(\mathbb{C} \cup \{\infty\}) \setminus f(\mathbb{D})$ contains at least three points. If there exists a path $\gamma: [0,1) \to \mathbb{D}$ such that $\lim_{t \to 1^-} \gamma(t) = e^{i\theta}$ and $\ell = \lim_{t \to 1^-} f(\gamma(t))$ exists in $\mathbb{C} \cup \{\infty\}$, then the nontangential limit of $f$ at $e^{i\theta}$ exists, and equals $\ell$.

Theorem 2.5 is due to Fatou, and Theorem 2.6 to Privalov. Theorem 2.7 is an extension of a result by Lindelöf. For proofs, we refer to [13], theorems 2.4, 8.1 and 2.20.

These three theorems have obvious reformulations for meromorphic functions defined on the upper half-plane. For the convenience of the reader we will provide them below.

1. Let $f: \mathbb{C}^+ \to \mathbb{C}^+ \cup \mathbb{R} \cup \{\infty\}$ be an analytic function. Then the set of points $x \in \mathbb{R}$ at which the nontangential limit of $f$ fails to exist in $\mathbb{C} \cup \{\infty\}$ is of Lebesgue measure zero.

2. Let $f: \mathbb{C}^+ \to \mathbb{C}$ be an analytic function. Assume that there exists a set $A \subseteq \mathbb{R}$ of nonzero Lebesgue measure such that the nontangential limit of $f$ exists at each point of $A$ and equals zero. Then $f(z) = 0$ for all $z \in \mathbb{C}^+$.

3. Let $f: \mathbb{C}^+ \to \mathbb{C} \cup \{\infty\}$ be a meromorphic function, and let $x \in \mathbb{R} \cup \{\infty\}$. Assume that the set $(\mathbb{C} \cup \{\infty\}) \setminus f(\mathbb{C}^+)$ contains at least three points. If there exists a path $\gamma: [0,1) \to \mathbb{C}^+$ such that $\lim_{t \to 1^-} \gamma(t) = x$ and $\ell = \lim_{t \to 1^-} f(\gamma(t))$ exists in $\mathbb{C} \cup \{\infty\}$, then the nontangential limit of $f$ at $x$ exists, and equals $\ell$.

The equivalence of statements (2) and (3) above to Theorems 2.6 and 2.7 follows by simply considering $f \circ T$, where $T$ is the conformal automorphism $T(w) = \frac{w + i\theta}{1 + iw}$, and recalling that $T$ preserves angles, carries a set $A \subseteq \mathbb{T}$ of zero linear measure into a set of zero Lebesgue measure, and vice-versa. The equivalence of Theorem 2.6 with statement (1) follows by considering the conjugation of $f$ with $T$ and, if necessary, a re-scaling. We would also like to mention here that in the context of the above three theorems the point infinity is not in any way a special point; for example, Theorem 2.5 forbids any nonconstant analytic function to have constant - finite as well as infinite - nontangential limit on a non-negligible set. Thus, given a non-constant analytic self-map of $\mathbb{C}^+$ and a countable set $C \subset \mathbb{C} \cup \{\infty\}$, we can always find a set $A \subseteq \mathbb{R}$ whose complement is negligible so that our map has nontangential limits at all points of $A$ which do not belong to $C$.

Consider a domain (i.e. an open connected set) $D \subseteq \mathbb{C} \cup \{\infty\}$ and a function $f: D \to \mathbb{C} \cup \{\infty\}$. Assume that $\Gamma \subseteq D$ and $x_0$ is an accumulation point for $\Gamma$. The cluster set $C^\Gamma(f,x_0)$ of the function $f$ at the point $x_0$ relative to $\Gamma$ is

$$\{ z \in \mathbb{C} \cup \{\infty\} \mid \exists \{z_n\}_{n \in \mathbb{N}} \subset \Gamma \setminus x_0 \text{ such that } \lim_{n \to \infty} z_n = x_0, \lim_{n \to \infty} f(z_n) = z \}.$$

If $\Gamma = D$, we shall write $C(f,x_0)$ instead of $C_D(f,x_0)$. The following result is immediate.

Lemma 2.8. Let $D \subseteq \mathbb{C} \cup \{\infty\}$ be a domain and let $f: D \to \mathbb{C} \cup \{\infty\}$ be continuous. If $D$ is locally connected at $x \in \partial D$, then $C(f,x)$ is connected.

This result appears in [13], as Theorem 1.1.

It will be useful for our purposes to understand the behaviour of analytic self-maps of $\mathbb{C}^+$ near open intervals in $\mathbb{R}$ on which their nontangential limits are real almost everywhere. The following theorem of Seidel can be used to describe the
behaviour of such analytic functions near the boundary of their domain of definition. For proof, we refer to [13], Theorem 5.4.

**Theorem 2.9.** Let \( f : \mathbb{D} \to \mathbb{D} \) be an analytic function such that the radial limit \( f(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta}) \) exists and has modulus 1 for almost every \( \theta \) in the interval \((\theta_1, \theta_2)\). If \( \theta \in (\theta_1, \theta_2) \) is such that \( f \) does not extend analytically through \( e^{i\theta} \), then \( C(f, e^{i\theta}) = \overline{\mathbb{D}} \).

This theorem can be applied to self-maps of the upper half-plane, via a conformal mapping, but in that case one must consider meromorphic, instead of analytic, extensions.

A second result referring to the behaviour of \( C(f, x) \) for bounded analytic functions \( f \) is the following theorem of Carathéodory. (This result appears in [13], Theorem 5.5.)

**Theorem 2.10.** Let \( f : \mathbb{D} \to \mathbb{C} \) be a bounded analytic function. Assume that for almost every \( \theta \in (\theta_1, \theta_2) \) the radial limit \( f(e^{i\theta}) \) belongs to a set \( W \) in the plane. Then, for every \( \theta \in (\theta_1, \theta_2) \) the cluster set \( C(f, e^{i\theta}) \) is contained in the closed convex hull of \( W \).

The following proposition is a consequence of the previous two theorems.

**Proposition 2.11.** Let \( f \) be an analytic self-map of \( \mathbb{C}^+ \) such that \( \lim_{y \to 0} f(x+iy) \) exists and belongs to \( \mathbb{R} \) for almost every \( x \in (a, b) \). Suppose that \( x_0 \in (a, b) \) is such that \( f \) cannot be continued meromorphically through \( x_0 \). Then for any \( c < d \) there is a set \( E \subseteq (a, b) \) of nonzero Lebesgue measure such that \( \lim_{y \to 0} f(x+iy) \) exists for all points \( x \in E \), and the set \( \{ \lim_{y \to 0} f(x+iy) : x \in E \} \) is dense in the interval \((c, d)\).

For proof we refer to [2], Proposition 1.9.

We will next focus on boundary behaviour of derivatives of analytic self-maps of the unit disk and of the upper half-plane. These results are described in detail by Nevanlinna [16] and Shapiro [17]; see also Exercises 6 and 7 in Chapter I of Garnett’s book [15].

**Theorem 2.12.** Let \( f : \mathbb{D} \to \mathbb{D} \) be an analytic function, and let \( w \in \mathbb{T} \). The following statements are equivalent:

1. We have
   \[
   \lim_{z \to w} \inf_{z \to w} \frac{|f(z)| - 1}{|z| - 1} < \infty;
   \]

2. There exists a number \( \zeta \in \mathbb{D} \) such that
   \[
   \lim_{z \to w} f(z) = \zeta,
   \]
   and the limit
   \[
   \ell = \frac{w}{\zeta} \lim_{z \to w} \frac{f(z) - \zeta}{z - w}
   \]
   exists and belong to \((0, +\infty)\).

Moreover, if the equivalent conditions above are satisfied, the limit \( \lim_{z \to w} f'(z) \) exists, and the following equality holds:

\[
\ell = \frac{w}{\zeta} \lim_{z \to w} f'(z) = \liminf_{z \to w} \frac{|f(z)| - 1}{|z| - 1}.
\]
If
\[ \liminf_{z \to w} \left| f(z) - c \right| = \infty \]
and
\[ \lim_{z \to w} f(z) = \zeta, \]
then the limit in equation \((1)\) exists and equals infinity.

The number \(\ell\) from the above theorem is called the Julia-Carathéodory derivative of \(f\) at \(w\). Since it will be useful in the third and fourth section, we discuss below in some detail the formulation of the Julia-Carathéodory Theorem for self-maps of the upper half-plane. In the following lemma we isolate the part of the Julia-Carathéodory Theorem for the upper half-plane that will be used in Sections 3 and 4.

**Lemma 2.13.** Let \(F: \mathbb{C}^+ \to \mathbb{C}^+\) be analytic, and let \(a \in \mathbb{R}\). Assume that \(\lim_{z \to a} F(z) = c \in \mathbb{R}\). Then
\[ \lim_{z \to a} \frac{F(z) - c}{z - a} = \liminf_{z \to a} \frac{\Re F(z)}{\Re z}, \]
where the equality is considered in \(\mathbb{C}\). Conversely, if
\[ \liminf_{z \to a} \frac{\Re F(z)}{\Re z} < \infty, \]
then \(\lim_{z \to a} F(z)\) exists and belongs to \(\mathbb{R} \cup \{\infty\}\). Moreover, if \(F\) is not constant, then we have \(\liminf_{z \to a} \Re F(z) / \Re z > 0\).

**Proof.** Observe that by replacing \(F\) with the function \(F(z) - c\) we may assume without loss of generality that \(c = 0\). Let \(T(z) = \frac{z}{1+z}, z \in \mathbb{C}\). \(T\) maps \(\mathbb{C}^+\) conformally onto the unit disc \(\mathbb{D}\), and its composition inverse is \(T^{-1}(w) = \frac{1+aw}{1-a}\). We consider \(f: \mathbb{D} \to \mathbb{D}\) defined by \(f(w) = T(F(T^{-1}(w)))\). Obviously, \(\lim_{w \to T(a)} f(w) = -1\). Denote \(T(a) = b\). We have
\[ \lim_{z \to a} \frac{F(z)}{z - a} = \lim_{w \to b} \frac{F(T^{-1}(w))}{T^{-1}(w) - T^{-1}(b)} \]
\[ = \lim_{w \to b} \frac{1 + f(w)}{1 - f(w)} \cdot \frac{(1 - w)(1 - b)}{2(w - b)} \]
\[ = \frac{(1 - b)^2}{4} \lim_{w \to b} \frac{f(w) - (-1)}{w - b} \]
\[ = \frac{(1 - b)^2}{4} \cdot \lim_{w \to b} \frac{-1}{b} \liminf_{w \to b} \frac{1 - |f(w)|}{1 - |w|} \]
\[ = \frac{1}{a^2 + 1} \liminf_{w \to b} \frac{1 - |f(w)|}{1 - |w|}. \]

We have used the Julia-Carathéodory Theorem in the next to last equality and the definition of \(b\) in the last one. (The situation in which all expressions in the equalities above are infinite is not excluded.)
On the other hand,
\[
\frac{\Re F(z)}{\Re z} = \frac{\Re F(T^{-1}(w))}{\Re T^{-1}(w)} = \frac{\Re \left( \frac{1+|f(w)|}{1-|f(w)|} \right)}{\Re \left( \frac{1+|w|}{1-|w|} \right)}
\]
\[
= \frac{1-|f(w)|^2}{1-2|f(w)|+|f(w)|^2} = \frac{1+|f(w)|}{1+|w|} \cdot \frac{|1-w|^2}{|1-f(w)|^2}
\]
By taking \( \lim \inf \) in the above equality we obtain
\[
\lim_{z \to a} \frac{\Re F(z)}{\Re z} = \lim_{w \to b} \frac{1-|f(w)|}{1-|w|} \cdot \frac{1+|f(w)|}{1+|w|} \cdot \lim_{w \to b} \frac{|1-w|^2}{|1-f(w)|^2}
\]
(2)

Observe that if \( |f(w)| \) does not tend to one, the first \( \lim \inf \) in the last row above is infinite. Thus, the first \( \lim \inf \) is realized on a sequence on which \( |f| \) tends to 1. The second \( \lim \inf \) is realized when \( w \) tends nontangentially to \( b \), and equals \( (1-b)^2/4 = 1/(a^2 + 1) \). (This follows trivially from the facts that \( |f(w)| < 1 \), \( w \in \mathbb{D} \) and \( \lim_{w \to b} f(w) = -1 \), since this implies directly that the denominator of our expression \( |1-w|^2/|1-f(w)|^2 \) cannot be greater than 2, value reached when \( w \) tends to \( b \) nontangentially.) Thus,
\[
\lim_{z \to a} \frac{\Re F(z)}{\Re z} \geq \frac{1}{a^2 + 1} \lim_{w \to b} \frac{1-|f(w)|}{1-|w|}
\]
We conclude that
\[
\lim_{z \to a} \frac{F(z)}{z-a} \leq \lim_{z \to a} \frac{\Re F(z)}{\Re z}
\]
(Again the case in which both sides of the inequality are infinite is included.) But
\[
\lim_{z \to a} \frac{F(z)}{z-a} = \lim_{y \to 0} \frac{F(a+iy)}{iy} = \lim_{y \to 0} \frac{\Re F(a+iy)}{iy} + \frac{\Im F(a+iy)}{y} \geq \lim_{z \to a} \frac{\Re F(z)}{\Re z},
\]
as our limit is real or infinite. Thus,
\[
\lim_{z \to a} \frac{F(z)}{z-a} = \lim_{z \to a} \frac{\Re F(z)}{\Re z}
\]
Assume now that \( \lim \inf_{z \to a} \Re F(z)/\Re z = d \in \mathbb{R}_+ \). Equation (2) above implies that the limit \( \lim_{w \to b}(1-|f(w)|)/(1-|w|) \) is also finite (recall that \( b = T(a) \neq 1 \), so the second \( \lim \inf \) in (2) is nonzero), so, by the Julia-Carathéodory Theorem \( f \) has nontangential limit at \( b \), and thus \( F \) has nontangential limit at \( a \). (Observe that this limit is infinite if and only if \( \lim_{w \to b} f(w) = 1 \).) Relation (2) together with the Julia-Carathéodory Theorem guarantees that \( d > 0 \).

Consider now an analytic function \( f : \mathbb{D} \to \overline{\mathbb{D}} \). A point \( w \in \overline{\mathbb{D}} \) is called a Denjoy-Wolff point for \( f \) if one of the following two conditions is satisfied:

1. \( |w| < 1 \) and \( f(w) = w \);
\( |w| = 1, \lim_{z \to w} f(z) = w, \) and
\[
\lim_{z \to w} \frac{f(z) - w}{z - w} \leq 1.
\]
The following result is due to Denjoy and Wolff.

**Theorem 2.14.** Any analytic function \( f : \mathbb{D} \to \mathbb{D} \) has a Denjoy-Wolff point. If \( f \) has more than one such point, then \( f(z) = z \) for all \( z \) in the unit disc. If \( z \in \mathbb{D} \) is a Denjoy-Wolff point for \( f \), then \( |f'(z)| \leq 1 \); equality occurs only when \( f \) is a conformal automorphism of the unit disc.

We refer to Shapiro’s book [17], Chapter 5, for a detailed introduction to this subject.

The Denjoy-Wolff point of a function \( f \) is characterized also by the fact that it is the uniform limit on compact subsets of the iterates \( f \circ f \circ \cdots \circ f \) of \( f \). We state the following theorem for the sake of completeness (for the original statements, see [14] and [22]):

**Theorem 2.15.** Let \( f : \mathbb{D} \to \mathbb{D} \) be an analytic function. If \( f \) is not a conformal automorphism of \( \mathbb{D} \), then the functions \( f \circ f \circ \cdots \circ f \) converge uniformly on compact subsets of \( \mathbb{D} \) to the Denjoy-Wolff point of \( f \).

The previous two theorems have been used in [5] to give a new proof for the subordination property for free additive and free multiplicative convolutions. We reproduce below the result for the free additive convolution (Theorem 4.1 in [5]):

**Theorem 2.16.** Given two Borel probability measures \( \mu, \nu \) on the real line, there exist unique analytic functions \( \omega_1, \omega_2 : \mathbb{C}^+ \to \mathbb{C}^+ \) such that

1. \( \Im \omega_j(z) \geq \Im z \) for \( z \in \mathbb{C}^+ \), and
\[
\lim_{y \to +\infty} \frac{\omega_j(iy)}{iy} = 1, \quad j = 1, 2.
\]
2. \( F_{\mu \oplus \nu}(z) = F_{\mu}(\omega_1(z)) = F_{\nu}(\omega_2(z)) \), and
3. \( \omega_1(z) + \omega_2(z) = z + F_{\mu \oplus \nu}(z) \), for all \( z \in \mathbb{C}^+ \).

For \( z \in \mathbb{C}^+ \), the point \( \omega_1(z) \) appears as the Denjoy-Wolff point of the function \( f_z : \mathbb{C}^+ \to \mathbb{C}^+ \) given by \( f_z(w) = F_{\mu}(F_{\mu}(w) - w + z) - F_{\mu}(w) + w \). The function \( f_z \) is well-defined for all \( z \in \mathbb{C}^+ \cup \mathbb{R} \) by Remark 2.4. An immediate consequence of Theorem 2.16 is the fact that free additive convolution can be defined equivalently by purely complex analytic methods, using equations (2) and (3) from the theorem above. This has been proved independently by different means in [12].

We shall use boundary properties of the subordination functions to describe the atomic, singular continuous, and absolutely continuous parts, with respect to the Lebesgue measure on \( \mathbb{R} \), of the free convolution \( \mu \boxplus \nu \) of two probability measures \( \mu, \nu \) on the real line. The following lemma describes the behaviour of the Cauchy transform \( G_{\mu} \) near points belonging to the support of the singular part of the probability measure \( \mu \). For proof, we refer to [2], Lemma 1.10, [8], Lemma 7.1, and [15], Theorem 3.16 of Chapter II.

**Lemma 2.17.** Let \( \mu \) be a Borel probability measure on \( \mathbb{R} \).

1. For \( \mu^* \)-almost all \( x \in \mathbb{R} \), the nontangential limit of the Cauchy transform \( G_{\mu} \) of \( \mu \) at \( x \) is infinite.
(2) We have \( \mu(\{x\}) = < \lim_{x \to \pm}(z-x)G_\mu(z) \).

(3) Denote by \( f \) the density of \( \mu^{ac} \) with respect to the Lebesgue measure on \( \mathbb{R} \). Then for almost all \( x \in \mathbb{R} \), we have \( -f(x) = < \lim_{x \to \pm} 3G_\mu(z) \).

Finally, we provide a technical lemma, whose proof can be also found in the proof of Theorem 2.3 of [2]. We shall give here a more conceptual proof.

**Lemma 2.18.** Let \( f: \mathbb{C}^+ \to \mathbb{C}^+ \) be a nonconstant analytic function, \( x \in \mathbb{R} \cup \{\infty\} \), and assume that \( C(f, x) \subseteq \mathbb{R} \cup \{\infty\} \) contains more than one point, and hence, by Lemma [2,8] a closed nondegenerate interval or the complement of an open interval. Then for all \( c \in C(f, x) \), with the possible exception of at most two points, there exists a sequence \( \{z_n^{(c)}\}_{n \in \mathbb{N}} \subset \mathbb{C}^+ \) so that

(i) \( \lim_{n \to \infty} z_n^{(c)} = x \);

(ii) \( \lim_{n \to \infty} f(z_n^{(c)}) = c \), and

(iii) \( \Re f(z_n^{(c)}) = c \) for all \( n \in \mathbb{N} \).

**Proof.** Observe first that, by replacing \( f(z) \) with \( f(\frac{-1}{2}z) \) if necessary, we may assume that \( x \in \mathbb{R} \). Now pick two arbitrary points \( c_1 < c_2 \in \mathbb{R} \cap C(f, x) \), and fix two arbitrary constants \( \varepsilon \in (0, |c_1 + c_2|/4) \) and \( M > \max\{2 + \varepsilon + |c_1|, 2 + \varepsilon + |c_2|\} \).

Define the compact region

\[
X_M = \{ z \in \mathbb{C}^+ : |Re(z)| \leq M, 1/M \leq \Im(z) \leq M \}.
\]

Let \( \{z_n^j\}_{n \in \mathbb{N}} \subset \mathbb{C}^+ \), \( j \in \{1, 2\} \), be two sequences with the following properties:

1. \( \lim_{n \to \infty} z_n^j = x \), \( j \in \{1, 2\} \);
2. \( 1 > |z_n^2 - x| > 2|z_n^1 - x| > 4|z_n^{1+1} - x| \) for all \( n \in \mathbb{N} \);
3. \( |f(z_n^j) - c_j| < \frac{1}{n} \) for all \( n \in \mathbb{N} \), \( n > 0 \) and \( j \in \{1, 2\} \).

Define a path \( \gamma: [0, 1] \to \mathbb{C}^+ \cup \{x\} \) so that \( \gamma(0) = i, \gamma(1) = x, \gamma(1 - \frac{1}{2n}) = z_n^2, \gamma(1 - \frac{1}{2n+1}) = z_n^1 \), and \( \gamma \) is linear on the intervals \([0, \frac{1}{2}], [1 - \frac{1}{2n}, 1 - \frac{1}{2n+1}]\) and \([1 - \frac{1}{2n+1}, 1 - \frac{1}{2n+2}]\) for all \( n \in \mathbb{N}, n > 0 \). We easily observe that \( \gamma \) is a simple curve in \( \mathbb{C}^+ \cup \{x\} \) and \( \lim_{t \to \gamma(t)} = x \). Thus, there exists \( n_M \in \mathbb{N} \) so that \( f(\gamma(1 - \frac{1}{2n_M} - 1)) \cap X_M = \emptyset \). Indeed, assume towards contradiction that for all \( n \in \mathbb{N} \) there exists \( t_n \in (1 - \frac{1}{2n}, 1) \) so that \( f(\gamma(t_n)) \in X_M \subset \mathbb{C}^+ \). Since \( X_M \) is compact, there exists a subsequence \( \{t_{n_k}\}_k \) of \( \{t_n\}_n \) with the property that \( \lim_{n \to \infty} f(\gamma(t_{n_k})) \) exists and belongs to \( X_M \). But \( t_{n_k} \in (1 - \frac{1}{2n_k}, 1) \) implies that \( \lim_{n_k \to \infty} t_{n_k} = 1 \) and thus \( \lim_{n_k \to \infty} \gamma(t_{n_k}) = x \), so that \( \lim_{n_k \to \infty} f(\gamma(t_{n_k})) \in C(f, x) \subseteq \mathbb{R} \cup \{\infty\} \). Contradiction. Thus indeed the path \( f \circ \gamma: [0, 1] \to \mathbb{C}^+ \) ultimately stays out of \( X_M \) as \( t \to 1 \).

For each \( n \in \mathbb{N}, n > n_M + \frac{1}{\varepsilon} \), define \( \Gamma_n \) to be the image of the restriction of \( f \circ \gamma \) to \([1 - \frac{1}{2n}, 1 - \frac{1}{2n+1}]\). By the construction of our sequences \( \{z_n^j\}_n \) and the path \( \gamma, \Gamma_n \subset \mathbb{C}^+ \setminus X_M \) unites through \( \mathbb{C}^+ \setminus X_M \) the balls \( B(c_2, 1/n) \) and \( B(c_1, 1/n) \) for all \( n \in \mathbb{N}, n > n_M + \frac{1}{\varepsilon} \). Thus, it must intersect at least one of the following three segments:

\[
S_0 = \left\{ \frac{c_1 + c_2}{2} + is : 0 < s < 1/M \right\},
\]

\[
S_1 = \{ c_1 - \varepsilon + is : 0 < s < 1/M \}, \quad \text{or}
\]

\[
S_2 = \{ c_2 + \varepsilon + is : 0 < s < 1/M \},
\]
at least once. We conclude that there exists $t_n \in \left(1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}\right)$ so that $f(\gamma(t_n)) \in S_r$ for some $r = r(n) \in \{0, 1, 2\}$. Since this holds for all $n \in \mathbb{N}$, $n > n_M + 1$, there is at least one of $S_0, S_1, S_2$; call it $S_{\nu_0}$, which is intersected by infinitely many paths $\Gamma_n$, and thus there exists a subsequence $\{t_{n_k}\}$ with $t_{n_k} \in \left(1 - \frac{1}{2^{n_k}}, 1 - \frac{1}{2^{n_k+1}}\right)$ so that $f(\gamma(t_{n_k})) \in S_{\nu_0}, k \in \mathbb{N}$. But $\lim_{k \to \infty} t_{n_k} = 1$, so $\lim_{k \to \infty} \gamma(t_{n_k}) = x$, and thus $\lim_{k \to \infty} f(\gamma(t_{n_k})) = c$, where $\{c\} = \overline{S_{\nu_0}} \cap \mathbb{R} \subset \{c_1 - \varepsilon, c_2 + \varepsilon, (c_1 + c_2)/2\}$.

For the point $c$, $\{c\} = \overline{S_{\nu_0}} \cap \mathbb{R}$, we have constructed the sequence $z_n^{(c)} = \gamma(t_{n_k})$ satisfying conditions (i), (ii) and (iii) from our lemma. We shall prove next that all points of $C(f, x)$, with at most two exceptions, can be realized in the form of a $c$, with $\{c\} = \overline{S_{\nu_0}} \cap \mathbb{R}$. Consider two separate cases:

**Case 1:** $C(f, x) = \mathbb{R} \cup \{\infty\}$. If for all $d \in \mathbb{R}$ we can find a sequence $\{z_n^{(d)}\}_n$ with the properties (i), (ii) and (iii), then we are done. Assume thus that there exists a point $d \in \mathbb{R}$ for which no sequence with property (iii) exists (observe that properties (i) and (ii) can always be satisfied, by the definition of the cluster set). We claim that for any $c \in \mathbb{R} \setminus \{d\}$ there is a sequence $\{z_n^{(c)}\}_n$ satisfying properties (i), (ii) and (iii). Indeed, choose such a point $c \neq d$ and pick $c_1 < c < c_2 < d$ so that $c = (c_1 + c_2)/2$. Construct sequences $\{z_n^{(c)}\}_n$, $j \in \{1, 2\}$, and a path $\gamma$ as above. We claim that there exists an $\eta > 0$ so that $f(\gamma([0, 1])) \cap \{d + is: 0 < s < \eta\} = \emptyset$. Indeed, assume towards contradiction that this is not the case. Then for any $n \in \mathbb{N}$ there exists a point $v_n \in f(\gamma([0, 1])) \cap \{d + is: 0 < s < 1/n\}$, and so there exists a point $t_n \in [0, 1)$ so that $f(\gamma(t_n)) = v_n$. Since $\gamma([0, 1)) \subset \mathbb{C}^+$ and $f$ is not constant by hypothesis, the sequence $\{t_n\}_n$ cannot have any accumulation points in $[0, 1)$, so $\lim_{n \to \infty} t_n = 1$. But then $\lim_{n \to \infty} \gamma(t_n) = x$, $\lim_{n \to \infty} f(\gamma(t_n)) = d$ and $\mathbb{R}f(\gamma(t_n)) = d$ for all $n \in \mathbb{N}$. So the sequence $\{\gamma(t_n)\}_n$ satisfies properties (i), (ii) and (iii) for $d$, a contradiction. This proves the existence of the required $\eta > 0$.

Choose now $\varepsilon \in (0, \min\{\eta, |c_1 + c_2|/4, 1\})$ and $M > \max\{1/\eta, |c_1| + 2 + \varepsilon, |c_2| + 2 + \varepsilon\}$ and define $X_M$ and $\Gamma_n$ as above. It follows immediately then that the paths $\Gamma_n$ must intersect the segment $S_0 = \left\{\frac{1}{1+\varepsilon} + is: 0 < s < 1/M\right\}$ infinitely often, since $c_2$ is separated from $c_1$ by $S_0$, the lower edge of $X_M$ and $\{d + is: 0 < s < \eta\}$, and $\Gamma_n$ does not intersect the lower edge of $X_M$ or $\{d + is: 0 < s < \eta\}$. Thus we have found a sequence $z_n^{(c)}$ as required for our $c = (c_1 + c_2)/2$. Since $c \in \mathbb{R} \setminus \{d\}$ is arbitrary, we have proved our lemma, with the exceptional points $d$ and $\infty$.

**Case 2:** $\mathbb{R} \setminus C(f, x) \neq \emptyset$. This case is straightforward; pick $c_1$ and $c_2$ to be, one of them the (or a) finite endpoint of $C(f, x)$, and the other any arbitrary point in $C(f, x)$ except the previously chosen one. It follows trivially that any point $c \in C(f, x)$ in between these two points accepts a sequence $\{z_n^{(c)}\}_n$ as in our lemma. We leave the details of the proof to the reader. Thus, the lemma holds, with the two exceptional points being the endpoints of the cluster set.

3. Boundary behaviour of the subordination functions

In the following we fix two Borel probability measures $\mu$ and $\nu$ on $\mathbb{R}$, neither of them a point mass. For technical reasons, we first study the case when $\mu$ and $\nu$ are both convex combinations of two point masses. (We denote by $\delta_a$ the probability which gives mass one to the point $a$.) Let $a \in \mathbb{R}$ be fixed and define the following two
self-maps of the upper half-plane: $h_\mu(w) = F_\mu(w) - w + a$, $h_\nu(w) = F_\nu(w) - w + a$, $w \in \mathbb{C}^+$. The following proposition generalizes Lemma 2.1 of [2].

**Proposition 3.1.** The function $f_\mu(w) = h_\nu(h_\mu(w))$ is a conformal automorphism of $\mathbb{C}^+$ if and only if each of $\mu$ and $\nu$ are convex combinations of two distinct point masses. In this case, $\omega_1(z)$ and $\omega_2(z)$, provided by Theorem [2.10] satisfy quadratic equations having $z$ and real numbers as coefficients.

**Proof.** Assume that $f_\mu$ is a conformal automorphism of $\mathbb{C}^+$. We claim that the analytic functions $h_\mu, h_\nu : \mathbb{C}^+ \to \mathbb{C}^+$, are both conformal automorphisms of the upper half-plane. Observe that by the definition of a conformal automorphism, $h_\mu$ must be injective, and $h_\nu$ surjective. Let $k$ be the inverse with respect to composition of $h_\nu \circ h_\mu$, so that

$$h_\nu(h_\mu(k(z))) = z, \quad z \in \mathbb{C}^+.$$  

Applying $h_\mu \circ k$ to both sides of the above equality gives $h_\mu(k(h_\nu(w))) = w$ for all $w$ in the open subset $(h_\mu \circ k)(\mathbb{C}^+)$ of the upper half-plane, so, by analytic continuation, for all $w \in \mathbb{C}^+$. This proves surjectivity of $h_\mu$, and injectivity of $h_\nu$ and thus our claim is proved.

Observe that, by Theorem [2.3] $\lim_{y \to +\infty} h_\mu(iy)/iy = \lim_{y \to +\infty} h_\nu(iy)/iy = 0$. Thus, there exist real numbers $b_\mu, c_\mu, d_\mu, b_\nu, c_\nu, d_\nu$ so that

$$h_\mu(z) = \frac{b_\mu z + c_\mu}{z + d_\mu}, \quad h_\nu(z) = \frac{b_\nu z + c_\nu}{z + d_\nu}, \quad z \in \mathbb{C}^+,$$

and

$$\det \begin{pmatrix} b_\mu & c_\mu \\ 1 & d_\mu \end{pmatrix} > 0, \quad \det \begin{pmatrix} b_\nu & c_\nu \\ 1 & d_\nu \end{pmatrix} > 0.$$  

By the definition of $h_\mu$, we obtain

$$F_\mu(z) = h_\mu(z) + z - a = \frac{z^2 + z(d_\mu + b_\mu - a) + c_\mu - d_\mu a}{z + d_\mu} = \left( t \frac{1}{z - u} + (1 - t) \frac{1}{z - v} \right)^{-1},$$

where

$$t = \frac{d_\mu b_\mu + a + \sqrt{(d_\mu b_\mu - a)^2 - 4c_\mu + 4d_\mu a}}{2 \sqrt{(d_\mu b_\mu - a)^2 - 4c_\mu + 4d_\mu a}},$$

and

$$u = \frac{a - d_\mu b_\mu + \sqrt{(d_\mu b_\mu - a)^2 - 4c_\mu + 4d_\mu a}}{2},$$

$$v = \frac{a - d_\mu b_\mu - \sqrt{(d_\mu b_\mu - a)^2 - 4c_\mu + 4d_\mu a}}{2}.$$  

Thus, $\mu$ is a convex combination of two point masses $\delta_u$ and $\delta_v$, with weights $t$ and $1 - t$, respectively. The result for $\nu$ follows the same way. Conversely, if $\mu = t\delta_u + (1 - t)\delta_v$, then a direct computation shows that $F_\mu(z) = (z - v)(z - u)(z - tv - (1 - t)u)^{-1}$, so that

$$h_\mu(z) = \frac{(a - (tu + (1 - t)v))z + uw - a(tu + (1 - t)u)}{z - (tv + (1 - t)u)}, \quad z \in \mathbb{C}^+,$$
and
\[
\det \begin{bmatrix} a - (tu + (1 - t)v) & wz - a(tv + (1 - t)u) \\ 1 & -tv - (1 - t)u \end{bmatrix} = t(1 - t)(u - v)^2 > 0,
\]
for all \(0 < t < 1, u \neq v\). This proves the first statement of the proposition.

Assume now that \(\mu = t \delta_u + (1 - t) \delta_v, \mu = s \delta_w + (1 - s) \delta_x\), with \(u \neq v, w \neq x\), and \(0 < s, t < 1\). The computations above provide real numbers \(b_\mu, c_\mu, d_\mu, b_\nu, c_\nu, d_\nu\) which depend polynomially on \(u, v, w, x, s, t\), so that
\[
h_\mu(z) = \frac{b_\mu z + c_\mu}{z + d_\mu}, \quad h_\nu(z) = \frac{b_\nu z + c_\nu}{z + d_\nu}, \quad z \in \mathbb{C}^+.
\]

On the other hand, by parts (2) and (3) of Theorem 2.16 we obtain that
\[
F_\mu(\omega_1(z)) = F_\nu(F_\mu(\omega_1(z)) - \omega_1(z) + z)
= h_\nu(F_\mu(\omega_1(z)) - \omega_1(z) + z + F_\mu(\omega_1(z)) - \omega_1(z) + z - a,
\]
so that
\[
\omega_1(z) = h_\nu(F_\mu(\omega_1(z)) - \omega_1(z) + z) + z - a
= h_\nu(h_\mu(\omega_1(z)) - a + z) + z + a
= \frac{b_\nu(\frac{b_\mu \omega_1(z) + c_\mu}{\omega_1(z)} + z - a)}{z + d_\mu} + z - a
= b_\nu(\frac{b_\mu \omega_1(z) + c_\mu + (z - a)(\omega_1(z) + d_\mu)}{\omega_1(z) + d_\mu}) + c_\nu(\omega_1(z) + d_\mu) + z - a.
\]

Thus, \(\omega_1\) satisfies an equation of degree two with coefficients that are polynomials of \(z, s, t, u, v, w, x\). The same argument shows the required statement for \(\omega_2\).

This proves the second statement of the proposition. \(\square\)

A consequence of the proposition above is the following

**Corollary 3.2.** With the notations from Theorem 2.16, if \(\mu\) and \(\nu\) are both convex combinations of two point masses, then:

1. \(\omega_1\) and \(\omega_2\) extend continuously to \(\mathbb{R}\) as functions with values in the extended complex plane \(\overline{\mathbb{C}}\);
2. If \(\omega_1(a) \in \mathbb{C}^+\) or \(\omega_2(a) \in \mathbb{C}^+\) for some \(a \in \mathbb{R}\), then \(\omega_1, \omega_2\), and \(F_{\mu \oplus \nu}\) extend analytically around \(a\);
3. \((\mu \oplus \nu)^{ac} = 0\), and \(\frac{d(\mu \oplus \nu)^{ac}(t)}{dt}\) is analytic wherever positive and finite.

Parts (1) and (2) of the corollary are obvious. For the part (3) observe that \(F_{\mu \oplus \nu}\) extends continuously to \(\mathbb{R}\) and the set \(F_{\mu \oplus \nu}^{-1}(\{(0)\})\) contains only finitely many points, by Theorem 2.16(3) and by Proposition 3.1. Thus, by Lemma 2.17(1), \((\mu \oplus \nu)^{ac} = 0\). The statement regarding \(\frac{d(\mu \oplus \nu)^{ac}(t)}{dt}\) follows immediately from Theorem 2.16(3), Proposition 3.1, and Lemma 2.17(3).

The next theorem describes the boundary behaviour of the subordination functions provided by Theorem 2.16.

**Theorem 3.3.** Let \(a \in \mathbb{R}\) be fixed. With the notations from Theorem 2.16, the following hold:

1. If \(C(\omega_1, a) \cap \mathbb{C}^+ \neq \emptyset\), or \(C(\omega_2, a) \cap \mathbb{C}^+ \neq \emptyset\), then the functions \(\omega_1, \omega_2, \) and \(F_{\mu \oplus \nu}\) extends analytically in a neighbourhood of \(a\).
(2) The functions $\omega_1$ and $\omega_2$ have nontangential limits at $a$.

(3) Assume in addition that the sets $I_1 = \mathbb{R} \setminus \text{supp}(\mu)$ and $I_2 = \mathbb{R} \setminus \text{supp}(\nu)$ are nonempty. Then $\lim_{z \to a} \omega_j(z)$, $j = 1, 2$, exist in the extended complex plane $\overline{\mathbb{C}}$.

A slightly less general version of part (3) of this theorem appears implicitly in the proof of Theorem 2.3 in [2].

Proof. Consider a sequence $\{z_n\}_{n \in \mathbb{N}} \subset \mathbb{C}^+$ and a number $\ell \in \mathbb{C}^+$ with the property that $\lim_{n \to \infty} z_n = a$ and $\ell = \lim_{n \to \infty} \omega_1(z_n)$. Define $f : (\mathbb{C}^+ \cup \mathbb{R}) \times \mathbb{C}^+ \to \mathbb{C}^+$ by $f(z, w) = F_\mu(F_\nu(w) - w + z) - F_\mu(w) + w$. As noted in the comments following Theorem 2.16, $\omega_1(z)$ is the Denjoy-Wolff point of the function $f_z = f(z, \cdot)$ whenever $z \in \mathbb{C}^+$. Thus, we have

$$\ell = \lim_{n \to \infty} \omega_1(z_n) = \lim_{n \to \infty} f(z_n, \omega_1(z_n)) = f(a, \ell).$$

If $f_a = f(a, \cdot)$ is not an automorphism of the upper half-plane, then we can use Theorem 2.14 to conclude that $|f'_a(\ell)| < 1$. It follows from Remark 2.4 that $f$ can be extended on a bidisc $B(a, \varepsilon) \times B(\ell, \varepsilon)$ for $\varepsilon > 0$ small enough, so by the Implicit Function Theorem there exists $\eta \in (0, \varepsilon)$ and an analytic function $\omega : B(a, \eta) \to B(\ell, \varepsilon)$ such that $f(z, \omega(z)) = \omega(z)$ for all $z \in B(a, \eta)$. Since $f(z, \omega_1(z)) = \omega(z)$ for $z \in \mathbb{C}^+$, we conclude by the uniqueness part of Theorem 2.16 that $\omega$ extends $\omega_1$ to $\mathbb{C}^+ \cup B(a, \eta)$. Since $F_\mu \circ \omega_1$ and $\ell = \omega_1(a) \in \mathbb{C}^+$, $F_\mu \circ \omega_1$ also extends analytically to some neighbourhood of $a$. The similar statement for $\omega_2$ follows from part (3) of Theorem 2.16.

The case when $f_a$ is a conformal automorphism of $\mathbb{C}^+$ is covered by Corollary 2.2. This proves (1).

Assume now that the hypothesis of (3) holds and yet $C(\omega_1, a)$ contains more than one point. By part (1), $C(\omega_1, a) \subseteq \mathbb{R} \cup \{\infty\}$, and by Lemma 2.8 either $C(\omega_1, a) \setminus \{\infty\}$ is a closed interval in $\mathbb{R}$ (possibly all of $\mathbb{R}$), or $\mathbb{R} \setminus C(\omega_1, a)$ is an open interval in $\mathbb{R}$.

By Lemma 2.18 for any $c$ in $C(\omega_1, a) \setminus \{\infty\}$, with the possible exception of two points, there exists a sequence $\{z_n(c)\}_{n \in \mathbb{N}}$ converging to $c$ such that $\lim_{n \to \infty} \omega_1(z_n(c)) = c$, and $\Re(\omega_1(z_n(c)) = c$ for all $n$ (i.e. $\omega_1(z_n(c))$ converges nontangentially - in fact approaches $c$ vertically).

Thus, if existing, $\lim_{n \to \infty} F_\mu(\omega_1(z_n(c))) = \lim_{z \to c} F_\mu(z)$. By Fatou’s theorem (Theorem 2.9), this limit exists for almost all $c \in C(\omega_1, a)$. Denote it by $F_\mu(c)$. We shall prove that for every $c \in C(\omega_1, a)$ for which $F_\mu(c)$ exists, there at most two exceptions, $F_\mu(c) \notin \mathbb{C}^+$. Indeed, suppose that $F_\mu(c) \in \mathbb{C}^+$ for some $c \in C(\omega_1, a)$ for which $\omega_1(z_n(c))$ as above can be constructed. Recall that $\omega_1(z_n(c))$ converges nontangentially to $c$. Then, using parts (2) and (3) of Theorem 2.16 Remark 2.4 and the fact that $\nu$ is not a point mass, we obtain

$$\exists F_\mu(c) = \lim_{n \to \infty} \exists F_\mu(\omega_1(z_n(c)))$$

$$= \lim_{n \to \infty} \exists F_\mu(F_\mu(\omega_1(z_n(c))) - \omega_1(z_n(c)) + z_n(c))$$

$$= \exists F_\mu(F_\mu(c) - c + a)$$

$$> \exists F_\mu(c) - c + a$$

$$= \exists F_\mu(c),$$
Applying again Privalov’s theorem, we obtain that
\[ F \circ g \subseteq \int_{\omega} C \subseteq \int_{\omega} \subseteq \int_{\omega} \cup \{\infty\}. \]
Suppose there were a point \( d \) that \( \lim_{n \to \infty} F(\omega_n) \) exists (we do not claim that \( \mu \) is not a point mass. So \( C(\omega_2, a) \) must be an infinite set.

Assume that \( c_0 \in \int C(\omega_1, a) \) is a point where \( F_\mu \) does not continue meromorphically. Proposition 2.11 shows that the set
\[ E = \{ c \in C(\omega_1, a) : a + F_\mu(c) = c \in I_2 \} \]
has nonzero Lebesgue measure. In particular, for all \( c \in E \),
\[ F_\mu(c) = \lim_{n \to \infty} F_\mu(\omega_n) \]
so that, by Theorem 2.6, \( F_\mu(z) = z - a + \lim_{n \to \infty} \omega_n(z) \) for all \( z \in \mathbb{C}^+ \). (We denote by \( \int A \) the interior of \( A \subseteq \mathbb{R} \) with respect to the usual topology on \( \mathbb{R} \).) This contradicts the fact that \( \mu \) is not a point mass. So \( C(\omega_2, a) \) leads to a contradiction.

Consider now \( \omega_2(z) = F_\mu(\omega_1(z)) - \omega_1(z) + z, \ z \in \mathbb{C}^+ \). We shall argue that the set \( C(\omega_2, a) \subseteq \mathbb{R} \cup \{\infty\} \) must be also infinite. Suppose this were not the case. Then for any sequence \( \{ z_n \} \subset \mathbb{N} \) constructed above, we may take a subsequence so that \( \lim_{n \to \infty} \omega_2(z_n) \) exists (we do not claim that \( \omega_2(z_n) \) converges non-tangentially to this limit). Suppose there were a point \( d \in C(\omega_2, a) \) and a set \( V_d \\subset \int C(\omega_1, a) \) of nonzero Lebesgue measure such that \( \lim_{n \to \infty} \omega_2(z_n) = d \) for all \( n \in V_d \). Taking limit as \( n \to \infty \) in the equality
\[ F_\mu(\omega_1(z_n)) + z_n = \omega_1(z_n) + \omega_2(z_n) \]
gives
\[ F_\mu(c) = c + d, \text{ for all } c \in V_d. \]
Applying again Privalov’s theorem, we obtain that \( F_\mu(z) = z - (a - d) \) for all \( z \in \mathbb{C}^+ \). This contradicts the fact that \( \mu \) is not a point mass. Thus, there exists a set \( E = \int C(\omega_1, a) \) of positive Lebesgue measure such that \( \{ c = \lim_{n \to \infty} \omega_2(z_n) : c \in E \} \subseteq \int C(\omega_2, a) \). Then, since \( F_\mu \) extends meromorphically through \( \int C(\omega_2, a) \),...
by Theorem\textsuperscript{2.16} we conclude that
\[ F_{\mu}(c) = \lim_{n \to \infty} F_{\mu}(\omega_1(z_n^{(c)})) \]
\[ = \lim_{n \to \infty} F_{\nu}(\omega_2(z_n^{(c)})) \]
\[ = \lim_{n \to \infty} F_{\nu}\left( F_{\mu}(\omega_1(z_n^{(c)})) - \omega_1(z_n^{(c)}) + z_n^{(c)} \right) \]
\[ = F_{\nu}(a + F_{\mu}(c) - c) \]
for all \( c \in E \). Privalov's theorem implies that \( F_{\mu}(z) = F_{\nu}(a + F_{\mu}(z) - z) \) for all \( z \in \mathbb{C}^+ \). As we have proved already, this implies that \( h_{\mu} \) and \( h_{\nu} \) are conformal automorphisms of the upper half-plane. Corollary 3.2 and Proposition\textsuperscript{3.1} provide again a contradiction. This proves part (3) of the theorem.

Consider now the case when at least one of the two sets \( I_1, I_2 \) is empty. Without loss of generality, assume \( I_1 = \emptyset \). We show that in this case, if any of the two sets \( C(\omega_1, a), C(\omega_2, a) \) contains more than one point, we must have \( C(\omega_1, a) = C(\omega_2, a) = \mathbb{R} \cup \{\infty\} \). We shall do this in four steps.

**Step 1:** We show that \( \text{Int}C(\omega_1, a) \cap \text{supp}(\mu^{ac}) = \emptyset \). Indeed, assume this is not the case. Then (by Lemmas\textsuperscript{2.17} and\textsuperscript{2.18}) for almost all \( c \) in \( \text{Int}C(\omega_1, a) \cap \text{supp}(\mu^{ac}) \) with respect to the Lebesgue measure, we have \( \lim_{z \to c} F_{\mu}(z) = \infty \), and there exists \( \{z_n^{(c)}\}_{n \in \mathbb{N}} \) so that \( \lim_{n \to \infty} z_n^{(c)} = a \), \( \Re(\omega_1(z_n^{(c)})) = c \), and \( \lim_{n \to \infty} \omega_1(z_n^{(c)}) = c \). Thus,
\[ \lim_{n \to \infty} \omega_2(z_n^{(c)}) = \lim_{n \to \infty} F_{\mu}\left( \omega_1(z_n^{(c)}) - \omega_1(z_n^{(c)}) + z_n^{(c)} \right) = a - c + \lim_{z \to c} F_{\mu}(z) \in \mathbb{C}^+, \]
contradicting part (1) of the theorem.

**Step 2:** We show that \( C(\omega_2, a) = \mathbb{R} \cup \{\infty\} \). One inclusion is an immediate consequence of part (1) of this theorem. Thus, it is enough to show that \( C(\omega_2, a) \) is dense in \( \mathbb{R} \). By Step 1 and our hypothesis \( I_1 = \emptyset \), we have \( \text{Int}C(\omega_1, a) \subseteq \text{supp}(\mu^{ac}) \setminus \text{supp}(\mu^{ac}) \). Thus, by Proposition\textsuperscript{2.11} for any \( x \in \text{Int}C(\omega_1, a) \) and any open interval \( I \) containing \( x \), the set
\[ \{ \lim_{z \to c} F_{\mu}(z) : c \in I, F_{\mu} \text{ has nontangential limit at } c \} \]
is dense in \( \mathbb{R} \). So, given \( x \) as above, \( \varepsilon > 0 \) and \( s \in \mathbb{R} \), there exists \( c \in I \cap (x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}) \) such that \( |s - x + a - \lim_{z \to c} F_{\mu}(z)| < \varepsilon/2 \), and thus
\[ |s - \lim_{n \to \infty} \omega_2(z_n^{(c)})| = |s - \lim_{n \to \infty} F_{\mu}(\omega_1(z_n^{(c)})) - \omega_1(z_n^{(c)}) + z_n^{(c)}| \]
\[ = |s - c + a - \lim_{z \to c} F_{\mu}(z)| \]
\[ \leq |s - x + a - \lim_{z \to c} F_{\mu}(z)| + |x - c| \]
\[ < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \]
\[ = \varepsilon. \]
We conclude that \( C(\omega_2, a) \) is dense in \( \mathbb{R} \). This proves step 2.

**Step 3:** We show that \( \nu = \nu^* \). This follows from Step 2 and the argument used in Step 1.

**Step 4:** We show that \( C(\omega_1, a) = \mathbb{R} \cup \{\infty\} \). If \( \nu \) is a convex combination of point
masses and there exist two consecutive atoms of \( \nu \) at \( \alpha \) and \( \beta \) (\( \alpha < \beta \)), then obviously \( G_x \) extends analytically to the interval \( (\alpha, \beta) \) and \( G_x((\alpha, \beta)) = \mathbb{R} \), so that \( F_x((\alpha, \beta)) = (\mathbb{R} \setminus \{0\}) \cup \{\infty\} \). For any \( x \in (\alpha, \beta) \) let \( z_n^{(x)} \) converge to \( a \) so that \( \omega_2(z_n^{(x)}) \to x \) as \( n \) tends to infinity. Then

$$\lim_{n \to \infty} \omega_1(z_n^{(x)}) = \lim_{n \to \infty} F_x(\omega_2(z_n^{(x)})) - \omega_2(z_n^{(x)}) + z_n^{(x)} = F_x(x) - x + a,$$

and thus

\[
C(\omega_1, a) \supseteq \{ \lim_{n \to \infty} \omega_1(z_n^{(x)}): x \in (\alpha, \beta) \} = \{ F_x(x) - x + a: x \in (\alpha, \beta) \} = \mathbb{R} \cup \{\infty\}.
\]

If either \( \nu \) is not purely atomic, or there exist no consecutive atoms of \( \nu \), then there exists at least one point \( x_0 \in \mathbb{R} \) so that \( F_x \) does not extend meromorphically through \( x_0 \). The argument used in the proof of Step 2, with \( I \) an open interval containing \( x_0 \), assures us that \( C(\omega_1, a) = \mathbb{R} \cup \{\infty\} \). This proves Step 4.

We have proved now that if there exists a point \( a \) where either \( C(\omega_1, a) \) or \( C(\omega_2, a) \) is nondegenerate (i.e., contains more than one point), then \( \mu = \mu^s \), \( \nu = \nu^s \), at least one has total support, and \( C(\omega_1, a) = C(\omega_2, a) = \mathbb{R} \cup \{\infty\} \). We assume without loss of generality that \( \mu \) has support equal to the whole real line. Choose a point \( c \in \mathbb{R} \) so that \( \lim_{z \to c} F_x(z) = 0 \) and there exists \( \{ z_n^{(c)} \}_{n \in \mathbb{N}} \) converging to \( a \) so that \( \Re \omega_1(z_n^{(c)}) = c \) and \( \lim_{n \to \infty} \omega_1(z_n^{(c)}) = c \). By Lemmas 2.17 and 2.13 we have

\[
\frac{1}{\mu(\{c\})} - 1 = \lim_{z \to c} \frac{F_x(z)}{z-c} - 1 = \liminf_{z \to c} \frac{F_x(z)}{z-c} - 1 \leq \liminf_{n \to \infty} \frac{F_x(\omega_1(z_n^{(c)}))}{\omega_1(z_n^{(c)})} + \frac{\Re z_n^{(c)}}{\Re \omega_1(z_n^{(c)})} - 1 \leq \liminf_{n \to \infty} \frac{\Re \omega_2(z_n^{(c)})}{\Re \omega_1(z_n^{(c)})}.
\]

By our assumption, \( (\mu(\{c\}))^{-1} - 1 \in (0, \infty) \), so that \( \liminf_{z \to a-c} F_x(z) < \infty \). Lemma 2.13 implies that \( F_x \) has a nontangential limit at \( a-c \) belonging to \( \mathbb{R} \cup \{\infty\} \). Denote it by \( l \). Moreover, we claim that \( \mu(\{c\}) > 0 \). Indeed, assume this is not the
case. Then the above chain of inequalities together with Lemma 2.13 imply that

\[
\liminf_{z \to a - c} \frac{3F_\nu(z)}{3z} = 1,
\]

so that \(h_\nu(z) = F_\nu(z) - z\) is zero. This contradicts the assumption that \(\nu\) is not a point mass and proves our claim. We conclude that \(\mu\) must be an infinite convex combination of point masses, densely distributed in \(\mathbb{R}\).

Assume towards contradiction that \(\omega_1\) has no nontangential limit at \(a\). Then there exists an angle \(\Gamma \subset \mathbb{C}^+\) with vertex at \(a\) and bisected by the line \(a + i\mathbb{R}_+\) with the property that \(C_\Gamma(\omega_1, a)\) is infinite. By Lemma 2.8 and the argument above, \(C_\Gamma(\omega_1, a)\) must contain a nondegenerate open subinterval \(J\) so that for any \(c \in J\) there exists a sequence \(\{z_n(c)\}_{n \in \mathbb{N}} \subset \Gamma\) converging to \(a\) such that \(\Re \omega_1(z_n(c)) = c\) and \(\Im \omega_1(z_n(c))\) converges to zero as \(n \to \infty\). Let \(m = \max_{x \in \mathbb{R}} \nu(\{x\})\). By hypothesis, \(0 \leq m < 1\). Since the set of atoms of \(\mu\) is dense in \(\mathbb{R}\), in particular infinite, there exists \(c \in J\) so that \(0 < \mu(\{c\}) < 1 - m\). Observe that since \(1 > \mu(\{c\}) > 0\), the non-constant function \(F_\nu(z) - z\) maps any nontangential path ending at \(c\) into a nontangential path. Indeed, since

\[
\lim_{z \to a - c} (F_\nu(z) - z + c) = \frac{1}{\mu(\{c\})} - 1 \in (0, \infty),
\]

this follows by simply analyzing the real and the imaginary part of the left-hand term above. Thus, the sequence \(\{\omega_2(z_n(c))\}_{n \in \mathbb{N}} = \{F_\nu(\omega_1(z_n(c))) - \omega_1(z_n(c))\}_{n \in \mathbb{N}}\) must also converge nontangentially to \(a - c\). But we have seen above that \(F_\nu\) has nontangential limit at \(a - c\). Thus,

\[
0 = \lim_{z \to a} F_\nu(z) = \lim_{n \to \infty} F_\nu(\omega_1(z_n(c))) = \lim_{n \to \infty} F_\nu(\omega_2(z_n(c))) = \lim_{z \to a - c} F_\nu(z),
\]

and

\[
\frac{1}{\mu(\{c\})} - 1 \leq \left(\liminf_{z \to a - c} \frac{3F_\nu(z)}{3z} - 1\right)^{-1} = \left(\lim_{z \to a - c} \frac{F_\nu(z)}{z - (a - c)} - 1\right)^{-1} = \left(\frac{1}{\nu(\{a - c\})} - 1\right)^{-1}.
\]

Multiplication with \(\frac{1}{\mu(a - c)} - 1\) in the above inequality gives

\[
(1 - \mu(\{c\}))(1 - \nu(\{a - c\})) \leq \mu(\{c\})\nu(\{a - c\}),
\]

so

\[
\mu(\{c\}) + \nu(\{a - c\}) \geq 1.
\]

This contradicts the choice \(\mu(\{c\}) < 1 - m\) and concludes the proof of part (2) of the theorem. \(\square\)
We can now prove the main result of this paper. For the sake of completeness, we state in the theorem below the result of Bercovici and Voiculescu describing the atoms of the free additive convolution of two probability distributions.

**Theorem 4.1.** Let \(\mu, \nu\) be two Borel probability measures on \(\mathbb{R}\), neither of them a point mass. Then

1. The point \(a \in \mathbb{R}\) is an atom of the measure \(\mu \ast \nu\) if and only if there exist \(b, c \in \mathbb{R}\) such that \(a = b + c\) and \(\mu(\{b\}) + \nu(\{c\}) > 1\). Moreover, \((\mu \ast \nu)(\{a\}) = \mu(\{b\}) + \nu(\{c\}) - 1\).

2. The absolutely continuous part of \(\mu \ast \nu\) is always nonzero, and its density is analytic wherever positive and finite. More precisely, there exists an open set \(U \subseteq \mathbb{R}\) so that the density function \(f(x) = \frac{d(\mu \ast \nu)^{ac}(x)}{dx}\) with respect to the Lebesgue measure in the real line is analytic on \(U\) and \((\mu \ast \nu)^{ac}(\mathbb{R}) = \int_U f(x)dx\).

3. The singular continuous part of \(\mu \ast \nu\) is zero.

**Proof.** Part (1) of the theorem is due to Bercovici and Voiculescu (see [8], Theorem 7.4). We shall proceed with the proof of part (2). Suppose that \(\mu \ast \nu\) is purely singular, and thus for almost all \(x \in \mathbb{R}\) with respect to the Lebesgue measure, we have

\[
\lim_{y \to 0} \Im F_{\mu \ast \nu}(x + iy) = \lim_{y \to 0} \Im G_{\mu \ast \nu}(x + iy) = 0.
\]

By part (1), we are assured that \(\mu \ast \nu\) cannot be purely atomic, so we must have \((\mu \ast \nu)^{ac} \neq 0\), and hence, by Lemma 2.17 applied to the function \(F_{\mu \ast \nu}\) yields a point \(x_0 \in \mathbb{R}\) such that \(C(F_{\mu \ast \nu}, x_0) = \mathbb{C}^+\). Using relations (2) and (3) from Theorem 2.16 we conclude that at least one of \(C(\omega_1, x_0), C(\omega_2, x_0)\), will intersect the upper half-plane. But now we can apply Theorem 3.3 (1) to obtain a contradiction. Thus, \(\mu \ast \nu\) cannot be purely singular.

Next we prove that there exists an open subset \(U\) of \(\mathbb{R}\) on which the density \(f(x) = \frac{d(\mu \ast \nu)^{ac}(x)}{dx}\) is analytic. By Theorem 2.5 there exists a subset \(E\) of \(\mathbb{R}\) of zero Lebesgue measure such that for all \(x \in \mathbb{R}\) \(\\setminus E\) the limits \(\lim_{y \to 0} F_{\mu \ast \nu}(x + iy), \lim_{y \to 0} \omega_j(x + iy), j \in \{1, 2\}\) exist and are finite. Also, by Lemma 2.17 (3), for almost all \(x \in \text{supp}(\mu \ast \nu)^{ac}\) \(\\setminus E\), with respect to \((\mu \ast \nu)^{ac}\), we have \(\lim_{y \to 0} F_{\mu \ast \nu}(x + iy) \in \mathbb{C}^+\). By relation (3) in Theorem 2.16 at least one of \(\lim_{y \to 0} \omega_j(x + iy), j \in \{1, 2\}\), must be in \(\mathbb{C}^+\). By definiteness, assume \(\omega_1(x) = \lim_{y \to 0} \omega_1(x + iy) \in \mathbb{C}^+\). Part (1) of Theorem 3.3 assures us that \(\omega_1, \omega_2\) and \(F_{\mu \ast \nu}\) extends analytically through \(x\). We conclude by part (3) of Lemma 2.17 that the density of \((\mu \ast \nu)^{ac}\) must be analytic in \(x\). Of course, the set \(U\) of points \(x\) where \(f(x) = \frac{d(\mu \ast \nu)^{ac}(x)}{dx}\) is analytic is open in \(\mathbb{R}\), and by Lemma 2.17 (3) we conclude that \(\int_U f(x)dx = (\mu \ast \nu)^{ac}(\mathbb{R})\).

To prove (3), assume that \((\mu \ast \nu)^{ac} \neq 0\); Lemma 2.17 provides an uncountable set \(H\) of points \(c \in \mathbb{R}\) such that \((\mu \ast \nu)(\{c\}) = 0\) and \(\lim_{z \to c} F_{\mu \ast \nu}(z) = 0\). Theorem 3.3 (2) assures us that \(\omega_1\) and \(\omega_2\) have nontangential limits at all points of \(\mathbb{R}\), in particular at each such point \(c \in H\). We claim that

(i) \(\lim_{z \to c} \omega_j(z) \in \mathbb{R}, j = 1, 2\). Denote those limits by \(\nu_j, j = 1, 2\);
(ii) The following equalities hold:

$$\lim_{z \to v_1} F_\mu(z) = 0 \quad \text{and} \quad \lim_{z \to v_2} F_\nu(z) = 0.$$ 

(iii) $\mu(\{v_1\}) + \nu(\{v_2\}) = 1$.

Parts (2) and (1) of Theorem 3.3 guarantee the existence of the limits in (i) and the fact that they do not belong to $\mathbb{C}^+$. If, say, $\lim_{z \to c} \omega_1(z) = \infty$, then Theorem 2.7 assures us that

$$0 = \lim_{y \downarrow 0} F_{\mu \boxplus \nu}(c + iy) = \lim_{y \downarrow 0} F_\mu(\omega_1(c + iy)) = \lim_{s \to \infty, s \in \omega_1(c + iR)} F_\mu(s) = \lim_{z \to v_2} F_\mu(z) = \infty,$$

which is of course a contradiction. Thus (i) holds. To prove (ii) just observe that, by the same Theorem 2.7

$$0 = \lim_{z \to c} F_{\mu \boxplus \nu}(z) = \lim_{z \to c} F_\nu(\omega_2(z)) = \lim_{z \to v_2, z \in \omega_2(c + iR)} F_\nu(z) = \lim_{z \to v_2} F_\nu(z).$$

The same argument provides the proof for $F_\mu$.

We now prove (iii). Recall that by Lemmas 2.17 and 2.18, for any $c$ as above we have

$$\frac{1}{\mu(\{c\})} = \lim_{z \to c} \left( z - c \right) \int_{\mathbb{R}} \frac{d\mu(t)}{z - t}^{-1} = \lim_{z \to c} \frac{F_\mu(z)}{z - c} = \lim inf_{z \to c} \frac{\Re F_\mu(z)}{\Im z}.$$
Thus the following chain of inequalities holds:

\[
\begin{align*}
\frac{1}{\nu(\{v_2\})} - 1 &= \liminf_{z \to v_2} \frac{\Im F_\nu(z)}{z} - 1 \\
&\leq \liminf_{y \downarrow 0} \frac{\Im F_\nu(\omega_2(c + iy))}{\Im \omega_2(c + iy)} - 1 \\
&\leq \limsup_{y \downarrow 0} \left( \frac{\Im F_\nu(\omega_2(c + iy))}{\Im \omega_2(c + iy)} + \frac{y}{\Im \omega_2(c + iy)} - 1 \right) \\
&= \limsup_{y \downarrow 0} \frac{\Im \omega_1(c + iy)}{\Im \omega_2(c + iy)} \\
&= \left( \liminf_{y \downarrow 0} \frac{3\omega_2(c + iy)}{\Im \omega_1(c + iy)} \right)^{-1} \\
&\leq \left( \liminf_{y \downarrow 0} \frac{3\omega_1(\omega_2(\omega_1(c + iy))}{3\omega_1(c + iy)} + \liminf_{y \downarrow 0} \frac{y}{3\omega_1(c + iy)} - 1 \right)^{-1} \\
&\leq \left( \liminf_{y \downarrow 0} \frac{3\omega_1(\omega_2(\omega_1(c + iy))}{3\omega_1(c + iy)} - 1 \right)^{-1} \\
&= \left( \liminf_{z \to v_1} \frac{3\omega_1(z)}{3z} - 1 \right)^{-1} \\
&= \left( \frac{1}{\mu(\{v_1\})} - 1 \right)^{-1}.
\end{align*}
\]

We have assumed that \( \mu \) and \( \nu \) are not point masses, so the above implies that

\[1 < \frac{1}{\mu(\{v_1\})} \cdot \frac{1}{\nu(\{v_2\})} < \infty.\]

Thus, multiplying the above inequality by \( \frac{1}{\mu(\{v_1\})} - 1 \) will give

\[(1 - \mu(\{v_1\}))(1 - \nu(\{v_2\})) \leq \mu(\{v_1\})\nu(\{v_2\}),\]

or, equivalently,

\[\mu(\{v_1\}) + \nu(\{v_2\}) \geq 1.\]

Using relation (3) from Theorem 2.16 and the fact that \( F_{\mu \boxplus \nu}(c) = 0 \), we obtain

\[v_1 + v_2 = c \quad \text{for all} \ c \in H.\]

Since \( c \) has been chosen so that \( (\mu \boxplus \nu)(\{c\}) = 0 \), part (1) of the theorem tells us that the inequality above must be an equality. This proves the last point of our claim.

Now, since any probability can have at most countably many atoms, this, together with part (iii) of our claim contradicts the fact that \( H \) is uncountable and concludes the proof. \( \square \)

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