Full counting statistics for the Kondo dot.

A. Komnik$^1$ and A.O. Gogolin$^2$

$^1$ Physikalisches Institut, Albert-Ludwigs-Universität, D-79104 Freiburg, Germany
$^2$ Department of Mathematics, Imperial College London, 180 Queen's Gate, London SW7 2AZ, United Kingdom

(Dated: December 30, 2021)

The generating function for the cumulants of charge current distribution is calculated for two generalised Majorana resonant level models: the Kondo dot at the Toulouse point and the resonant level embedded in a Luttinger liquid with the interaction parameter $g = 1/2$. We find that the low-temperature non-equilibrium transport in the Kondo case occurs via tunnelling of electronic particles as well as by coherent transmission of electron pairs. We calculate the third cumulant ('skewness') explicitly and analyse it for different couplings, temperatures, and magnetic fields. For the $g = 1/2$ set-up the statistics simplifies and is given by a modified version of the Levitov–Lesovik formula.

PACS numbers: 72.10.Fk, 71.10.Pm, 73.63.-b

Since Schottky’s realisation that the shot noise in a conductor contains invaluable information about the physical properties of the charge carriers, the question about the noise spectra of different circuits became as important as the knowledge of their current–voltage characteristics. The noise constitutes the second moment of the current distribution function (which is the probability of measuring a given value of the current) and is supposed to contain information about the change of current carrying excitations at weak transmission (reflection). That is indeed the case for $S–S$ and $S–N$ junctions but not proven for a generic interacting model. As has been pointed out in \cite{5, 7}, the third cumulant also contains valuable information about the charge of the current carriers. Therefore it is natural to investigate the full current distribution function. This was an academic question for a very long time as even the measurement of the second cumulant remained on the frontier of experimental physics. Only after the work by Reulet and coworkers the measurement of the third cumulant became possible \cite{6}. Inspired by this remarkable achievement, the full current distribution function (more often referred to as ‘full counting statistics’ or FCS) has been theoretically analysed in recent year for a wide range of systems.

In their seminal work \cite{7}, Levitov and Lesovik derived the exact formula for the FCS for the electron tunneling set–up (single channel):

$$\ln \chi_{\omega}(\lambda; V; \{T(\omega)\}) = T \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \ln \left\{ 1 + T(\omega) \right\}$$

$$\times \left\{ n_L(1-n_R)(e^{i\lambda} - 1) + n_R(1-n_L)(e^{-i\lambda} - 1) \right\},$$

where $T$ is the waiting time, $\lambda$ is the measuring field (we will explain this notation in more detail shortly), $n_R(L)(\omega) = n_F(\omega \pm V/2)$ are the electron filling factors in the (right and left) leads, $V$ is the bias voltage, and $T(\omega)$ is the single electron transmission coefficient. The knowledge of $T(\omega)$ thus fully defines the FCS for non–interacting systems.

While the Levitov–Lesovik approach can be relatively easily generalised to various multi–terminal (multi–channel) set–ups, it is notoriously difficult to include electron–electron interactions. Up to now most works in this direction relied upon various perturbative expansions, in the tunnelling amplitude or in the interaction strength \cite{6} (for a recent review see \cite{11}), as well as calculations at zero temperature \cite{11}. Certainly no paradigm for an interacting FCS emerged as of yet. Notable exceptions are the contribution \cite{12} as well as recent works by Andreev and Mishchenko and Kindermann and Trauzettel \cite{13, 14}, where the exact FCS was calculated for the (single–channel) Coulomb Blockade (CB) set–up of Matveev and Furusaki \cite{15, 16}. We shall establish the precise connection of these works to our results.

The purpose of this Letter is to contribute to our understanding of interacting FCSs by means of obtaining the exact FCS for two particular experimentally relevant set–ups: the Kondo dot and the resonant tunneling (RT) between two $g = 1/2$ Luttinger liquids (LL).

We start with a brief description of the method. Our goal is the calculation of the generating function $\chi(\lambda) = \sum e^{i\lambda q} P_q$ for the probabilities $P_q$ of $q$ electrons being transmitted through the system over time $T$. We first define operators $T_i$ transferring one electron through the system in the direction of the current ($i = R$) and in the reversed direction ($i = L$). The electron counting operator on the Keldysh contour $C$ can then be written down in its canonical form as \cite{6, 17}

$$T_\lambda = e^{i\lambda t/2} T_R + e^{-i\lambda t/2} T_L,$$

where the measuring field $\lambda(t)$ is explicitly time dependent, $\lambda(t) = \lambda\theta(t)(T - t)$ on the forward path and $\lambda(t) = -\lambda\theta(t)(T - t)$ on the backward path. According to \cite{6, 17}, the generating function is then given by the expectation value

$$\chi(\lambda) = \left\langle T_C e^{-i \int_C T_\lambda(t) dt} \right\rangle.$$

In order to calculate $\chi(\lambda)$ we define a more general functional $\chi[\lambda_-(t), \lambda_+(t)]$ formally given by the same Eq. \cite{6, 17}
but where $\lambda(t)$ is now understood to be an arbitrary function on the Keldysh contour, $\pm$ referring to two different functions on the time and anti-time ordered halves of the contour. Next we assume that both $\lambda_{\pm}(t)$ change slowly in time. Then, neglecting switching effects, one obtains at large $T$

$$
\chi[\lambda_{-}(t), \lambda_{+}(t)] = \exp \left\{ -i \int_0^T \mathcal{U}[\lambda_{-}(t), \lambda_{+}(t)] dt \right\},
$$

(3)

where $\mathcal{U}(\lambda_{-}, \lambda_{+})$ is the adiabatic potential. Once the adiabatic potential is computed, the statistics is recovered from $\ln \chi(\lambda) = -i \mathcal{T} \mathcal{U}(\lambda, -\lambda)$. As $\lambda_{\pm}$ are external parameters, performing the derivative of both (3) and (2) with respect to, say, $\lambda_{-}$, with help of the Feynman–Hellmann theorem (3) we immediately obtain

$$
\frac{\partial}{\partial \lambda_\pm} \mathcal{U}(\lambda_{-}, \lambda_{+}) = \left\langle \frac{\partial \mathcal{T}_\lambda(t)}{\partial \lambda_\pm} \right\rangle_\lambda,
$$

where we use notation

$$
\langle A(t) \rangle_\lambda = \frac{1}{\chi(\lambda_{-}, \lambda_{+})} \left\langle T_C \left\{ A(t) e^{-i \oint \mathcal{T}_\lambda(t) dt} \right\} \right\rangle.
$$

This is somewhat more complicated than the usual Hamiltonian formalism for a quasi–stationary situation, we’ll give further technical details in the long version [18]; in particular, we have verified that Eq. (4) comes out correctly in the non-interacting case.

In order to study the FCS for the Kondo dot we use the bosonization and refermionization approach, originally applied to this problem by Emery and Kivelson [19] (see also [20]) and refined by Schiller and Hershfield (SH), see [21]. The starting point is the two-channel Kondo Hamiltonian (we set $\hbar = v_F = e = k_B = 1$ throughout),

$$
H = H_0 + H_J + H_M + H_V,
$$

where, with $\psi_{\alpha, \sigma}$ being the electron field operators in the R,L channels,

$$
H_0 = i \sum_{\alpha=R,L, \sigma=\uparrow, \downarrow} \int dx \psi_{\alpha, \sigma}^\dagger(x) \partial_x \psi_{\alpha, \sigma}(x),
$$

$$
H_J = \sum_{\alpha=\uparrow, \downarrow} \sum_{\beta=\uparrow, \downarrow} J_{\alpha, \beta} \sum_{\nu=x, y, z} \sum_{\nu'}^N \frac{\partial^2}{\partial x_{\nu \nu'}},
$$

$$
H_V = (V/2) \sum_{\sigma} \int dx \left( \psi_{L, \sigma}^\dagger \psi_{R, \sigma} - \psi_{R, \sigma}^\dagger \psi_{L, \sigma} \right),
$$

$$
H_M = -\mu_B g_i H_{\tau_z} = -\Delta \tau_z.
$$

(4)

Here $\nu=x,y,z$ are the Pauli matrices for the impurity spin and $(\alpha, \beta = R, L; \sigma = \uparrow, \downarrow; \sigma''_{\sigma'})$ are the components of the $\nu$th Pauli matrix

$$
\sigma^{\nu}_{\alpha, \beta}(\lambda_{-}, \lambda_{+}) = \sum_{\sigma, \sigma'} \psi_{\alpha, \sigma}(0) \sigma^{\nu}_{\sigma\sigma'}(0) \psi_{\beta, \sigma'}(0),
$$

are the electron spin densities in (or across) the leads, biased by a finite $V$. The last term in Eq. (4) stands for the magnetic field, $\Delta = \mu_B g_i h$. Following SH, we assume $J_{\nu}^{\alpha, \beta} = J_{\nu}^{\beta', \alpha}$, $J_{\nu}^{L,R} = J_{\nu}^{R,L}$ and $J_{\nu}^{L,L} = J_{\nu}^{R,R} = 0$. The only transport process then allowed is the spin-flip tunnelling, so that we obtain for the $T_\lambda$ operator

$$
T_\lambda = \frac{J_{RL}^{\nu}}{2} \left( \tau^{+} e^{i\lambda(t)/2} \psi_{R, \sigma}^\dagger \psi_{L, \sigma} + \tau^{-} e^{-i\lambda(t)/2} \psi_{R, \sigma}^\dagger \psi_{L, \sigma} \right),
$$

After bosonization, Emery-Kivelson rotation, and refermionization (see details in [21]) and going over to the Toulouse point $J_z = 2\pi$, which is the only approximation we make, we obtain

$$
H' = H_0' - i(J_+ b \xi_f + J_- a \eta_f) - i\Delta a b + T_\lambda,
$$

(5)

where the counting term is given by

$$
T_\lambda = -i J_+ b \left[ \xi \cos(\lambda/2) - \eta \sin(\lambda/2) \right],
$$

(6)

with $J_\pm = (J_{\nu}^{L,L} \pm J_{\nu}^{R,R})/\sqrt{2\pi a_0}$, $J_\perp = J_{\nu}^{L,L}/\sqrt{2\pi a_0}$ ($a_0$ is the lattice constant of the underlying lattice model) and $a$ and $b$ being local Majorana operators originating from the impurity spin. The fields $\eta_f$ and $\xi_f$ in the spin–flavour sector are equilibrium Majorana fields, whereas $\eta$ and $\xi$ in the the charge–flavour sector are biased by $V$,

$$
H_0' = i \int dx \left[ \eta_f(x) \partial_x \eta_f(x) + \xi_f(x) \partial_x \xi_f(x) \right]
$$

$$
+ \eta(x) \partial_x \eta(x) + \xi(x) \partial_x \xi(x) + V_\xi(x) \eta(x) \right].
$$

(7)

Using Eqs. (5)-(7) one can straightforwardly evaluate the adiabatic potential $\mathcal{U}(\lambda_{-}, \lambda_{+})$ as the problem has become quadratic in the Majorana fields.

Skipping details of the calculation, we report the resulting exact formula for the FCS of the Kondo dot, which is the main result of this paper:
\[
\ln \chi(\lambda) = T \int_0^\infty \frac{d\omega}{2\pi} \ln \left\{ 1 + T_1(\omega) \left[ n_L(1 - n_R)(e^{2i\lambda} - 1) + n_R(1 - n_L)(e^{-2i\lambda} - 1) \right] \right. \\
+ T_2(\omega) \left[ n_F(1 - n_R) + n_L(1 - n_F)(e^{i\lambda} - 1) + n_F(1 - n_L) + n_R(1 - n_F)(e^{-i\lambda} - 1) \right] \left. \right\},
\]

where now the filling factors are \( n_{R,L} = n_F(\omega \pm V) \) [\( n_F(\omega) \) being the conventional Fermi function], not to be confused with notation in Eq. (1). The 'transmission coefficients' are

\[
T_1 = \frac{\Gamma_+^2(\omega^2 + \Gamma_+^2)}{[\omega^2 - \Delta^2 - \Gamma_+^2 (\Gamma_+ + \Gamma_-)^2]^2 + \omega^2 (\Gamma_+ + \Gamma_-)^2}, \quad T_2 = \frac{2T_1\Gamma_+^2(\omega^2 + \Gamma_+^2) + 2\Delta^2 \Gamma_+^2}{[\omega^2 - \Delta^2 - \Gamma_+^2 (\Gamma_+ + \Gamma_-)^2]^2 + \omega^2 (\Gamma_+ + \Gamma_-)^2},
\]

where \( \Gamma_i = J_i^2/2 \) (\( i = \pm, \perp \)).

For small voltages and zero temperature the generating function turns out to be quite simple,

\[
\chi(\lambda) = \left[ 1 + T_\varepsilon(e^{i\lambda} - 1) \right]^N,
\]

where \( N = TV/\pi \) is the number of the incoming particles during the time interval \( T \) and \( T_\varepsilon = \sqrt{T_1(0)} \) is the effective transmission coefficient. We analysed explicitly the behaviour of the system around the Toulouse point. The trivialisation \( T \) turns out to be robust against departure from this special point in parameter space \( \varepsilon \).

While \( T \) is the standard Levitov–Lesovik result for spinful systems, the physical content of Eq. (3) is more interesting, since it cannot be reduced to binomial statistics as in Eq. (3) at finite \( T \). One can see that the charge current is carried by two different quasi–particles with charges \( q_1 = 2e \) and \( q_2 = e \). We identify the corresponding transmission coefficients as \( T_1 \) and \( T_2 \), respectively. It was realised by SH that at least in the case of finite magnetic field \( \Delta \neq 0 \) it can become energetically favourable to tunnel electron pairs through the impurity rather than single electrons \( 2e \). The presence of the term containing \( 2\lambda \) in Eq. (3) can be interpreted as a signature of this effect. The full transport coefficient \( T \) as calculated by SH turns out to be a composite one and is recovered from \( T_1, T_2 \) through a very simple relation: \( T = T_1 + T_2/2 \). From the point of view of the Kondo physics, the case when \( T_2 = 0 \) and the statistics reduces to a modified Levitov–Lesovik formula, \( \chi(\lambda) = \chi_0^{1/2}(2\lambda, 2V; \{ T_1(\omega) \}) \) (binomial statistics at \( T = 0 \), is the symmetric model in zero field (the other, unphysical, case of \( T_2 = 0 \) is when \( J_\perp = 0 \)). We have evaluated the first and the second cumulant of the Kondo FCS Eq. (3) which are the same as calculated by SH at all \( V \) and \( T \). We shall not reproduce these two cumulants here and concentrate instead on new results.

The full analytic expression for the third cumulant exists but is too lengthy to be given here. We shall rather investigate various limits and use numerics for the general case. So, at \( T = 0 \) we obtain:

\[
\langle \delta q^3 \rangle = T \int_0^V \frac{d\omega}{2\pi} \left[ T_2 + 8T_1 - 3(T_2 + 2T_1)(T_2 + 4T_1) \right. \\
+ \left. 2(T_2 + 2T_1)^3 \right].
\]
1) in comparison to \( \langle \delta q^2 \rangle / \langle \delta q \rangle \), which is indeed in accordance with Eq.\( \text{[1]} \).

We now briefly turn to the \( g = 1/2 \) RT set–up. This set–up has caused much interest recently, see Ref.\( \text{[22]} \) and references therein. The Hamiltonian now is

\[
H = H_0 + \gamma (\psi_L^d \psi_R^d + d \psi_R^d + \text{H.c.}) + \Delta d^d d + H_C, \tag{13}
\]

where \( H_0 \) stands for two biased LLs at \( g = 1/2 \), \( d \) is the electron operator on the dot, \( \gamma \) is the tunneling amplitude and \( H_C \) is an electrostatic interaction we do not write explicitly here (see Eq.\( \text{[22]} \)). Introducing \( \lambda \) as standard, and carrying out the bosonization–refermionisation analysis, we find the same set of equations as for the Kondo dot, Eq.\( \text{[5]} \) and Eq.\( \text{[6]} \), but with \( \lambda \to \lambda/2 \) and \( J_\perp = 2\gamma \), \( J_\parallel = 0 \), when the Kondo statistics simplifies to binomial (unphysical case). Consequently, the FCS is given by a modification of the Levitov–Lesovik formula:

\[
\chi_{1/2}(\lambda) = \chi_0^{1/2}(\lambda; 2V; \{T_\Delta(\omega)\}) , \tag{14}
\]

with the effective transmission coefficient \( T_\Delta(\omega) = 4\lambda^4 \omega^2 / [4\lambda^4 \omega^2 + (\omega^2 - \Delta^2)] \) of the RT set–up in the symmetric case\( \text{[22]} \) (the contact asymmetry is unimportant). All the cumulants are thus obtainable from those of the non-interacting statistics Eq.\( \text{[1]} \).

The \( \Delta = 0 \) RT set–up is equivalent to the model of direct tunneling between two \( g = 2 \) LLs\( \text{[22]} \). The latter model is connected by the strong to weak coupling \( (1/g \to g) \) duality argument to the \( g = 1/2 \) Kane and Fisher model\( \text{[22]} \), which is, in turn, equivalent to the CB set–up studied by KT. Therefore their FCS must be related to our Eq.\( \text{[14]} \) at \( \Delta = 0 \) by means of the transformation: \( T_0 \to 1 - T_0 \) and \( V \to V/2 \). Indeed after some algebraic manipulation with KT’s Eq.\( \text{(12)} \), we find that the FCS for the CB set–up can be re–written as:

\[
\chi_{CB}(\lambda) = \chi_0^{1/2}(-\lambda; V; \{1 - T_0(\omega)\}) .
\]

To summarise, we derived the generating function for the charge transfer statistics for the Kondo dot in the Toulouse limit and analysed the third cumulant in detail. At low temperatures the transport is accomplished by electrons as well as electron pairs in the generic case whereas at \( T = 0 \) the conventional binomial statistics is restored.

We wish to thank H. Grabert, R. Egger, B. Trauzettel, M. Kindermann and W. Belzig for inspiring discussions. This work was supported by the Landesstiftung Baden-Württemberg (Germany) and by the EU RTN DIENOW.

[1] W. Schottky, Ann. Phys. (Leipzig) 57, 541 (1918).
[2] B. A. Muzykantskii and D. E. Khmelnitskii, Phys. Rev. B 50, 3082 (1994).
[3] J. C. Cuevas and W. Belzig, Phys. Rev. Lett. 91, 187001 (2003).
[4] X. Jehl, M. Sanquer, R. Calemczuk, and D. Mailly, Nature 405, 50 (2000).
[5] L. S. Levitov and M. Reznikov, Phys. Rev. B 70, 115305 (2004).
[6] B. Reulet, J. Senzier, and D. E. Prober, Phys. Rev. Lett. 91, 196601 (2003).
[7] L. S. Levitov and G. B. Lesovik, JETP Lett. 58, 230 (1993).
[8] D. A. Bagrets and Y. V. Nazarov, Phys. Rev. B 67, 085316 (2003).
[9] M. Kindermann and Y. V. Nazarov, Phys. Rev. Lett. 91, 136802 (2003).
[10] L. S. Levitov, in Quantum Noise in Mesoscopic Systems, edited by Y. V. Nazarov (Kluwer, Dordrecht, 2003).
[11] H. Saleur and U. Weiss, Phys. Rev. B 63, 201302 (2001).
[12] I. Safi and H. Saleur, Phys. Rev. Lett. 93, 126602 (2004).
[13] A. V. Andreev and E. G. Mishchenko, Phys. Rev. B 64, 233346 (2001).
[14] M. Kindermann and B. Trauzettel, cond-mat/0408666 (2004).
[15] K. A. Matveev, Phys. Rev. B 51, 1743 (1995).
[16] A. Furusaki and K. A. Matveev, Phys. Rev. Lett. 75, 709 (1995).
[17] R. P. Feynman, Phys. Rev. 56, 340 (1939).
[18] A. Komnik and A. O. Gogolin, [in preparation] (2005).
[19] V. J. Emery and S. Kivelson, Phys. Rev. B 46, 10812 (1992).
[20] A. O. Gogolin, A. A. Nersesyan, and A. M. Tsvelik, Bosonization and Strongly Correlated Systems (Cambridge University Press, 1998).
[21] A. Schiller and S. Hershfield, Phys. Rev. B 58, 14978 (1998).
[22] A. Komnik and A. O. Gogolin, Phys. Rev. Lett. 90, 246403 (2003).
[23] A. Komnik and A. O. Gogolin, Phys. Rev. B 68, 235323 (2003).
[24] C. L. Kane and M. P. A. Fisher, Phys. Rev. B 46, 15233
(1992).