FINITENESS PROPERTIES OF NON-UNIFORM LATTICES ON CAT(0) POLYHEDRAL COMPLEXES

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Abstract. We show that the homological finiteness length of a non-uniform lattice on a locally finite CAT(0) $n$-dimensional polyhedral complex is less than $n$. As a corollary, we obtain an upper bound for the homological finiteness length of arithmetic groups over function fields. This gives an easier proof of a result of Bux and Wortman that solved a long-standing conjecture.

1. Introduction

The notion of a lattice in a locally compact group arises naturally in modern mathematics and has its roots in the study of Lie groups. A semisimple algebraic group over a local field can be realised as a group of automorphisms of its Bruhat–Tits building, and their lattices, called arithmetic lattices, are studied since the early 70’s. Other examples are given by tree lattices, which were introduced in the beginning of the 90’s by Bass and Lubotzky. Tree lattices are lattices in the isometry group of a locally finite tree [2]. More recently, lattices in isometry groups of higher dimensional locally finite cell complexes have appeared in literature [13, 8].

Let $X^n$ be $S^n$, $\mathbb{R}^n$ or $H^n$ with Riemannian metrics of constant curvature 1, 0 and $-1$ respectively. A finite-dimensional cell complex $X$ is a polyhedral complex if each $n$-dimensional open cell is isometric to the interior of a compact convex polyhedron in $X^n$ and the restrictions of the attaching maps are isometries onto its open cells. Suppose now that $X$ is a locally finite CAT(0) polyhedral complex, then its group of cellular isometries $\text{Aut}(X)$ has a natural structure of a locally compact topological group. We are interested in non-uniform lattices on $X$. We recall that a lattice on $X$ is a discrete subgroup $\Gamma$ of $\text{Aut}(X)$ with finite covolume. $\Gamma$ is said to be uniform if $\Gamma \backslash \text{Aut}(X)$ is compact.

The homological finiteness length $\phi(G)$ of a group $G$ is a generalisation of the concepts of finite generability and finite presentability. The main result of this paper is a bound for the homological finiteness length of certain lattices on $X$.

Theorem. If $\Gamma$ is a non-uniform lattice on a locally finite CAT(0) polyhedral complex of dimension $n$, then $\phi(\Gamma) < n$.

The finiteness properties of arithmetic groups have been widely investigated by Abels, Abramenko, Behr, Bux, Serre and Wortman just to mention a few. An upper bound for $\phi(\Gamma)$ in the case of arithmetic groups over function fields is given in [7]. Our result provides the same bound as a corollary.
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2. Finiteness properties and CAT(0) polyhedral complexes

A cohomological finiteness condition is a group-theoretical property that is satisfied by any group admitting a finite $K(G,1)$. Since every non-trivial finite group does not admit a finite-dimensional $K(F,1)$, being torsion-free is a cohomological finiteness condition, but not a finiteness condition in the usual group-theoretical sense. On the other hand the property of being locally finite is a classical but not cohomological finiteness condition. However there are finiteness conditions that agree, for example being finitely generated, being finitely presented...

A generalisation of these properties brings us to the concepts of cohomological conditions of finite type. More precisely, a group $\Gamma$ is of type $FP_n$ if the trivial $\mathbb{Z}\Gamma$-module $\mathbb{Z}$ admits a resolution of finitely generated projective $\mathbb{Z}\Gamma$-modules up to dimension $n$. If $\Gamma$ is of type $FP_n$ for every $n \geq 0$, then $\Gamma$ is said to be of type $FP_\infty$. A group is of type $F_\infty$ if it admits a $K(G,1)$ with finite $n$-skeleton; and $\Gamma$ is of type $F_n$ if it is of type $F_m$ for every $n \geq 0$. For a group being finitely generated is equivalent to being of type $FP_1$. A group is finitely presented if and only if it is of type $F_2$. For $n \geq 2$ a group is of type $F_n$ if and only if it is finitely presented and of type $FP_2$. Bestvina and Brady show the existence of non-finitely presented groups of type $FP_2$ [3].

The homological finiteness length of $\Gamma$ is defined as

$$\phi(\Gamma) = \sup\{m | \Gamma \text{ is of type } FP_m\}.$$ 

It is worth mentioning that Abels and Tiemeyer generalise the above finiteness conditions for discrete groups to compactness properties of locally compact groups [1].

We begin by recalling the terminology and in doing so we follow closely [3] and [5].

Let $X^n$ be $S^n$, $\mathbb{R}^n$ or $\mathbb{H}^n$ with Riemannian metrics of constant curvature $1$, $0$ and $\pm 1$ respectively. A finite-dimensional CW-complex $X$ is a polyhedral complex if it satisfies the followings:

- each open cell of dimension $n$ is isometric to the interior of a compact convex polyhedron in $X^n$;
- for each cell $\sigma$ of $X$, the restriction of the attaching map to each open $\sigma$-face of codimension one is an isometry onto an open cell of $X$.

Let $\text{Aut}(X)$ be the full group of cellular isometries of $X$. A subgroup $H \leq \text{Aut}(X)$ acts admissibly on $X$ if the set-wise stabiliser of each cell coincides with its point-wise stabiliser.

Remark 2.1. Every subgroup $G \leq \text{Aut}(X)$ acts admissibly on the barycentric subdivision of $X$. Furthermore, if $G$ acts admissibly on a CAT(0) polyhedral complex, then the fixed point set $X^G$ forms a subcomplex of $X$.
A subgroup $\Gamma$ of a locally compact topological group $G$ with left-invariant Haar measure $\mu$ is a lattice if:

- $\Gamma$ is discrete, and
- $\mu(\Gamma \backslash G) < \infty$.

Moreover, $\text{Aut}(X)$ is locally compact whenever $X$ is locally finite and so it makes sense to talk about lattices on locally finite CAT(0) polyhedral complexes. Let $G$ be a locally compact group with left-invariant Haar measure $\mu$. Let $\Gamma$ be a discrete subgroup of $G$ and $\Delta$ be a $G$-set with compact and open stabilisers. The $\Delta$-covolume, denoted by $\text{Vol}(\Gamma \backslash \Delta)$, is defined to be $\sum_{\sigma \in \Gamma \backslash \Delta} \frac{1}{|\sigma|} \leq \infty$.

**Lemma 2.2** ([2], Chapter 1). Let $X$ be a locally finite CAT(0) polyhedral complex with vertex set $V(X)$. If $\Gamma$ is a subgroup of $G = \text{Aut}(X)$, then:

- $\Gamma$ is discrete if and only if the stabiliser $\Gamma_x$ is finite for each $x \in V(X)$;
- $\mu(\Gamma \backslash G) < \infty$ if and only if $\text{Vol}(\Gamma \backslash \Delta) < \infty$. Moreover, the Haar measure $\mu$ can be normalised in such a way that for every discrete $\Gamma \subseteq G$, $\mu(\Gamma \backslash G) = \text{Vol}(\Gamma \backslash \Delta)$.

3. Homological finiteness length of non-uniform lattices on CAT(0) polyhedral complexes

The next lemma is a well-known criterion for finiteness that follows from Theorems 2.2 and 3.2 in [5].

**Lemma 3.1** ([5]). Let $\Gamma$ be a group that acts on an $n$-dimensional contractible CW-complex with stabilisers of type $F_x$. Then, $\Gamma$ is of type $F_x$ if and only if it is of type $F_n$.

**Definition 3.1.** The cohomological dimension of $\Gamma$ over a ring $R$ is defined as

$$\text{cd}_R \Gamma = \inf \{n \mid R \text{ admits an } RT\Gamma\text{-projective resolution of length } n\}$$

$$= \sup \{n \mid H^n_{R\Gamma}(G; M) \neq 0, \text{ for some } R\Gamma\text{-module } M\}.$$ 

The next result, due to Kropholler, is the key ingredient in the proof of Theorem 3.3.

**Theorem 3.2** (Proposition, [10]). Every group of finite rational cohomological dimension and of type $FP_x$ has a bound on the orders of its finite subgroups.

Furthermore, Kropholler in [10] shows that every $\mathfrak{H}$-group of type $FP_x$ has a bound on the orders of its finite subgroups.

**Theorem 3.3.** If $\Gamma$ is a non-uniform lattice on a locally finite CAT(0) polyhedral complex of dimension $n$, then $\phi(\Gamma) < n$.

**Proof.** Let $\Gamma$ be a non-uniform lattice on a CAT(0) polyhedral complex $X$ of dimension $n$. By Lemma 2.2, $\mu(\Gamma \backslash \text{Aut}(X)) = \sum_{\sigma \in \Gamma \backslash X} \frac{1}{|\sigma|}$, where $\sigma \supseteq \{\bar{\sigma}\}$. Since $\Gamma$ is non-uniform, the set $\Gamma \backslash X$ is infinite and so for any $n$ there is some $\sigma \in \Gamma \backslash X$ such that $\frac{1}{|\sigma|} < \frac{1}{n}$. Therefore, there is no bound on the orders of the stabilisers (which are finite), and so there is no bound on the orders of the finite subgroups of $\Gamma$. 

In view of Lemma 3.1 and Theorem 3.2, it only remains to argue that the rational cohomological dimension of $\Gamma$ is at most $n$. Since every CAT(0) space is contractible \cite{1}, $\Gamma$ acts on an $n$-dimensional contractible CW-complex with finite stabilisers. The augmented cellular chain complex of $X$ is an exact sequence of the form:
\[
\bigoplus_{i_n \in I_n} \mathbb{Z}[\Gamma_{i_n} \backslash \Gamma] \to \bigoplus_{i_n \in I_n} \mathbb{Z}[\Gamma_{i_{n-1}} \backslash \Gamma] \to \cdots \to \bigoplus_{i_0 \in I_0} \mathbb{Z}[\Gamma_{i_0} \backslash \Gamma] \to \mathbb{Z},
\]
where $\Gamma_{i_j}$ are finite subgroups of $\Gamma$ for every $0 \leq j \leq n$. Now, if $Q$ is flat over $\mathbb{Z}$ and $Q \otimes \mathbb{Z}[H/\Gamma] \cong Q[H/\Gamma]$ for any $H \leq \Gamma$, tensoring this sequence with $Q$ over $\mathbb{Z}$ leads to the exact sequence:
\[
\bigoplus_{i_n \in I_n} Q[\Gamma_{i_n} \backslash \Gamma] \to \bigoplus_{i_n \in I_n} Q[\Gamma_{i_{n-1}} \backslash \Gamma] \to \cdots \to \bigoplus_{i_0 \in I_0} Q[\Gamma_{i_0} \backslash \Gamma] \to Q.
\]
Now, $\bigoplus_{i_j \in I_j} Q[\Gamma_{i_j} \backslash \Gamma]$ is a $Q\Gamma$-projective module for every $0 \leq j \leq n$, and so $\text{cd}_Q \Gamma \leq n$.

Theorem 3.2 implies that $\Gamma$ is not of type FP-fl. Finite groups are of type F-fl, and Lemma 3.1 completes the proof. Note that this final step can be also achieved by applying Proposition 1 in \cite{8}.

\begin{remark}
Note that if a finite group acts on a locally finite CAT(0) polyhedral complex, then it is contained in the stabiliser of some cell. Now, let $F$ be a finite subgroup of a non-uniform lattice $\Gamma$ acting admissibly on a locally finite CAT(0) polyhedral complex $X$. Since $F$ acts admissibly on $X$, $X^F$ is contractible \cite{1}. In particular, $X$ is a model for $E\Gamma$.
\end{remark}

There are not many results that hold for all non-uniform lattices on CAT(0) polyhedral complexes. As a first immediate application we obtain a classical result.

\begin{corollary}
If $X$ is a tree, then every non-uniform lattice in $\text{Aut}(X)$ is not finitely generated. More generally, a non-uniform lattice on a product of $n$ trees is not of type FP-fl.
\end{corollary}

\begin{corollary}
Every non-uniform lattice on a locally finite 2-dimensional CAT(0) polyhedral complex is not finitely presented.
\end{corollary}

Before the last corollary we need to recall some more standard nomenclature. Let $K$ be a global function field, and $S$ be a finite non-empty set of pairwise inequivalent valuations on $K$. Let $\mathcal{O}_S \leq K$ be the ring of $S$-integers. Denote a reductive $K$-group by $G$. Given a valuation $v$ of $K$, $K_v$ is the completion of $K$ with respect to $v$. If $L/K$ is a field extension, the $L$-rank of $G$, $\text{rank}_L G$ is the dimension of a maximal $L$-split torus of $G$. The $K$-group $G$ is $L$-isotropic if $\text{rank}_L G \neq 0$. As in \cite{2}, to any $K$-group $G$, there is associated a non-negative integer $k(G, S) = \sum_{v \in S} \text{rank}_K G_v$. We are now ready to state and reprove the Theorem of Bux and Wortman.

\begin{corollary}[Theorem 1.2, \cite{1}]
Let $H$ be a connected non-commutative absolutely almost simple $K$-isotropic $K$-group. Then $\phi(H(\mathcal{O}_S)) \leq k(H, S) - 1$.
\end{corollary}

\begin{proof}
Let $H$ be a connected non-commutative absolutely almost simple $K$-isotropic $K$-group. Let $H$ be $\prod_{v \in S} H(K_v)$, there is a $k(H, S)$-dimensional Euclidean building $X$ associated to $H$. $X$ is a locally finite CAT(0) polyhedral complex. The
arithmetic group $H(O_S)$ becomes a lattice of $H$ via the diagonal embedding. $H$ is $K$-isotropic if and only if $H(O_S)$ is non-uniform by [9]. An application of Theorem 3.3 completes the proof.

Remark 3.8. Theorem 3.3 gives the upper bound on the homological finiteness length of arithmetic groups over function fields, a historical overview can be found in [7]. In a recent remarkable paper [6] Bux, Gramlich and Witzel showed that $\phi(H(O_S)) = k(H, S) - 1$. Calculating the homological finiteness length of non-uniform lattices on CAT(0) polyhedral complexes is an ambitious open problem. We conclude by mentioning that Thomas and Wortman exhibit examples of non-finitely generated non-uniform lattices on regular right-angled buildings [14]. This shows that the upper bound of Theorem 3.3 is not sharp and in particular, that the Theorem of Bux, Gramlich and Witzel does not hold for all non-uniform lattices on locally finite CAT(0) polyhedral complexes.

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