THE TRIANGLE OF SMALLEST AREA WHICH CIRCUMSCRIBES A SEMICIRCLE

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ABSTRACT. An interesting problem that determine a triangle of smallest area which circumscribes a semicircle is solved. Then a generalized golden right triangles sequence $T_n$ is obtained, and an interesting construction of the maximum generalized golden right triangle $T_2$ is also shown.

1. THE TRIANGLE OF SMALLEST AREA WHICH CIRCUMSCRIBES A SEMICIRCLE

In [1], DeTemple showed that the isosceles triangle of smallest perimeter which circumscribes a semicircle is made up of two congruent right triangles which have sides proportional to $(1, \sqrt{\phi}, \phi)$, and the right triangle is known as the Kepler triangle [2, p. 149][3]. Here, we’ll consider another interesting problem that determine a triangle of smallest area which circumscribes a semicircle.

![Figure 1. The triangle of smallest area which circumscribes a semicircle](image)

In Figure [1] Let a triangle $\triangle ABC$ circumscribe a semicircle of radius $R = 1$, with the diameter of the semicircle contained in the base $BC$, let $O$ denote the center of the semicircle, $OD$ and $OE$ are both the radii, then let $AB = x$ and $\angle A = \theta$, now, our problem is:

**Problem 1.1.** If the radius $R$ and the two sides $AB$ and $AC$ of $\triangle ABC$ can not be three edges of a triangle, determine $\triangle ABC$ of smallest area.

**Proposition 1.2.** If $R$, $AB$ and $AC$ can not be three edges of a triangle, then the of smallest area $\triangle ABC$ is a right triangle similar to the $(1, \phi, \sqrt{1 + \phi^2})$ triangle which forms half of a golden rectangle [5][4, p. 274][6 p. 115].

**Proof.** If $R$, $AB$ and $AC$ can not be three edges of a triangle, then first, we suppose the radius $R = 1$ is the longest segment, thus, we have $AC = R - AB = 1 - x$, where $0 < x < 1$, then in $\triangle ABC$, we have the area equation (1.1)

$$S_{ABC} = S_{AOB} + S_{AOC}$$ (1.1)
and
\[ S_{ABC} = \frac{1}{2} AB \cdot AC \sin \theta = \frac{1}{2} x(1-x) \sin \theta \]
\[ S_{AOB} + S_{AOC} = \frac{1}{2} x + \frac{1}{2}(1-x) \]
then we get
\[ \sin \theta = \frac{1}{x(1-x)} > 1 \]  \hspace{1cm} (1.2)
hence, we conclude that the radius \( R \) can not be the longest segment among the three segments.

Next, we suppose \( AC \) is the longest segment, then we have \( AC = R + AB = 1 + x \), using the similar method, we can get
\[ \sin \theta = \frac{2x + 1}{x^2 + x} \] \hspace{1cm} (1.3)
since \( 0 < \sin \theta \leq 1 \), we have
\[ 0 < \frac{2x + 1}{x^2 + x} \leq 1 \] \hspace{1cm} (1.4)
and with \( x > 0 \), we conclude that \( x \geq \phi \), where \( \phi = \frac{1 + \sqrt{5}}{2} \), hence
\[ S_{ABC} = \frac{2x + 1}{2} \geq \frac{\phi^3}{2} \] \hspace{1cm} (1.5)
The equality in (1.5) holds if and only if \( x = \phi \). Then we get \( AB = x = \phi \), \( AC = 1 + x = \phi^2 \), and in (1.3), we have \( \sin \theta = 1 \), which means \( \triangle ABC \) is a right triangle having sides proportional to \( (1, \phi, \sqrt{1 + \phi^2}) \).

2. A SEQUENCE OF GENERALIZED GOLDEN RIGHT TRIANGLES

**Figure 2.** A sequence of generalized golden right triangles \( T_n \)

There is an identity (2.1) of the golden ratio (2) and Fibonacci numbers (see, e.g., [4, p. 78]).
\[ \phi^{n+1} = F_{n+1} \phi + F_n, \quad (n = 0, 1, 2, \ldots) \] \hspace{1cm} (2.1)
If we rewrite (2.1) in the form of (2.2),
\[ 1 + \left( \frac{\phi F_{n+1}}{F_n} \right)^2 = \left( \frac{\phi^{n+1}}{F_n} \right)^2, \quad (n = 1, 2, 3, \ldots) \] \hspace{1cm} (2.2)
we will obtain a right triangles sequence $T_n$ with sides $(1, \sqrt{\frac{\phi^{n+1}}{F_n}}, \sqrt{\frac{\phi^n+1}{F_n}})$, see Figure 2. It’s easy to see that, the first right triangle $T_1$ with sides $(1, \sqrt{\phi}, \phi)$ is the Kepler triangle whose side lengths are in geometric progression, the second right triangle $T_2$ is a $(1, \sqrt{2\phi}, \phi\sqrt{\phi})$ triangle, and furthermore, let $n \to +\infty$, we find that the limiting right triangle of $T_n$ is just a $(1, \phi, \sqrt{1+\phi^2})$ triangle which forms half of a golden rectangle.

Since there are only two golden (see [5, p. 73]) right triangles (one is the Kepler triangle with sides $(1, \sqrt{\phi}, \phi)$, the other is the $(1, \phi, \sqrt{1+\phi^2})$ triangle), and they are both special cases of $T_n$, then, we can call $T_n$ a generalized golden right triangles sequence. In addition, we have the following simple area inequality (2.3) for $T_n$,

$$\Delta T_1 \leq \Delta T_n \leq \Delta T_2$$

(2.3)

where $\Delta T_n$ denoted as the area of $T_n$, and

$$\Delta T_n = \frac{\phi}{2} \sqrt{\frac{F_{n+1}}{F_n}}, \quad (n = 1, 2, 3, \ldots)$$

(2.4)

in the inequality (2.3), we also notice that the second triangle $T_2$ with sides $(1, \sqrt{2\phi}, \phi\sqrt{\phi})$ is just the maximum area triangle in $T_n$, therefore, we can call $T_2$ the maximum generalized golden right triangle.

Interestingly enough, similar to the construction of the Kepler triangle $T_1$ by first creating a golden rectangle shown in [3], we can also construct the maximum generalized golden right triangle $T_2$ by first constructing a golden rectangle, see Figure 3.

**Construction 2.1.** A simple and interesting 3-step construction of $T_2$:

1. construct a golden rectangle $ABCD$ with $BC = 1, AB = \phi$ (see, e.g., [6, p. 118])
2. construct $O$ dividing $BC$ in the golden ratio and $\frac{BO}{OC} = \phi$
3. draw an arc with the center at $O$ and the radius of length $AC$, cutting the extension of $BA$ at $E$, and join $E$ to $C$

Then $\triangle EBC$ is just $T_2$ having sides $(1, \sqrt{2\phi}, \phi\sqrt{\phi})$.

**Proof.** $BO = \frac{BC}{\phi} = \frac{1}{\phi}, EO = AC = \sqrt{1+\phi^2}$, thus, $BE = \sqrt{EO^2 - BO^2} = \sqrt{2\phi}$. □

Last, back to Figure 1 again, we give a problem to readers:
Problem 2.2. If the radius $R$ and the two sides $AB$ and $AC$ of $\triangle ABC$ can not be three edges of a triangle, determine $\triangle ABC$ of smallest perimeter. And if $\triangle ABC$ can not be an acute-angled triangle, determine $\triangle ABC$ of smallest perimeter.

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