SOLUTIONS OF $x_1^2 + x_2^2 - x_3^2 = n^2$ WITH SMALL $x_3$

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ABSTRACT. Friedlander and Iwaniec investigated integral solutions $(x_1, x_2, x_3)$ of the equation $x_1^2 + x_2^2 - x_3^2 = D$, where $D$ is square-free and satisfies the congruence condition $D \equiv 5 \mod 8$. They obtained an asymptotic formula for solutions with $x_3 \ll M$, where $M$ is much smaller than $\sqrt{D}$. To be precise, their condition is $M \geq D^{1/2 - 1/1332}$. Their analysis led them to averages of certain Weyl sums. The condition of $D$ being square-free is essential in their work. We investigate the "opposite" case when $D = n^2$ is a square of an odd integer $n$. This case is different in nature and leads to sums of Kloosterman sums. We obtain an asymptotic formula for solutions with $x_3 \ll M$, where $M \geq D^{1/2 - 1/16 + \varepsilon}$.

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1. INTRODUCTION

Friedlander and Iwaniec \[3\] considered representations of square-free positive numbers $D$ satisfying the congruence condition $D \equiv 5 \mod 8$ by the indefinite ternary quadratic form

$$x_1^2 + x_2^2 - x_3^2 = D.$$ 

One may refer to solutions of this equation as inhomogeneous Pythagorean triples. They proved that

$$\sum_{(x_1, x_2, x_3) \in \mathbb{Z}^3} F(x_3) = \frac{48}{\pi} L(1, \chi_D) \int_{\mathbb{R}} F(x_3) + O \left( D^{1/6 - 1/3996} M^{2/3} \right),$$

where $\chi_D$ is the real character of conductor $D$ and $F(x_3)$ is any smooth function supported in $[M, 2M]$ satisfying $|F^{(\alpha)}(x)| \leq M^{-\alpha}$ for $\alpha = 0, ..., 6$. This gives an asymptotic formula if $M > D^{1/2 - 1/1332}$.

The essential point is that they were able to break the $\sqrt{D}$ barrier for the size of $M$. Their analysis led them to averages of Weyl sums of the form

$$\sum_{b^2 \equiv D \mod c} e \left( \frac{2hb}{c} \right)$$

2020 Mathematics Subject Classification. 11E20, 11L05, 11D09.

Key words and phrases. indefinite ternary quadratic forms, Pythagorean triples, Kloosterman sums.
over \( c \). In [2], they managed to obtain non-trivial savings in such averages, using spectral methods. However, if \( D \) is a square rather than a square-free number, this result does not allow to break the \( \sqrt{D} \) barrier. In this article, we consider this "opposite" situation when \( D = n^2 \) is a square, i.e. we investigate the equation
\[
 x_1^2 + x_2^2 - x_3^2 = n^2
\]
and aim to obtain an asymptotic formula for the number of its integral solutions \((x_1, x_2, x_3)\) with \( x_3 \ll M \), where \( M \) is much smaller than \( n \). Our method starts out in a similar way as in [3], but here we are naturally led to averages of Kloosterman sums in place of the above-mentioned Weyl sums. We achieve an asymptotic estimate if
\[
 M > D^{1/2-1/16+\varepsilon}.
\]

2. Setup of the problem and initial transformations

Throughout this article, we shall adopt the following convention: For any integers \( k, l, \) not both 0, we denote by \((k, l)\) the greatest common positive divisor. We also assume \( \varepsilon \) to be an arbitrarily small but fixed positive number. We only consider the case when \( n \) is an odd number. In the situation investigated by Friedlander and Iwaniec [3], the congruence condition \( D \equiv 5 \mod 8 \) forces \( x_3 \) to be even. Here, for convenience, we want to keep the same condition. In [2], they managed to obtain non-trivial savings in such averages, using spectral methods. However, if \( n \) is a number, we investigate the equation
\[
 x_1^2 + x_2^2 - x_3^2 = n^2
\]
and take \( \Phi \) to be an arbitrarily small but fixed positive number. We further set \( \Phi_1 := \Phi_1(n, X) \) if \( X < Y \), \( \Phi_1 := 1 \) if \( X = Y \), \( \Phi_1 := 0 \) if \( X > Y \)

and
\[
 (2.2) \quad \Phi_1^{(j)}(x) \ll_j Y^{-j}
\]
for all \( j \in \mathbb{N} \). In addition, we assume that \( \Phi_1 \) is monotonically increasing in \([Y/2, Y]\) and monotonically decreasing in \([X, 2X]\). We further set \( \Phi_2 := \Phi_1 \) and take \( \Phi_3 \) to be of the form
\[
 (2.3) \quad \Phi_3(x) = \Phi \left( \frac{x}{M} \right),
\]
where \( \Phi : \mathbb{R} \to [0, 1] \) is a smooth function supported in \([1, 2]\) which is not constant 0. Since \( x_3 \) is supposed to be even and \( n \) is odd, precisely one of \( x_1 \) and \( x_2 \) is odd. By symmetry in \( x_1 \) and \( x_2 \), we therefore have
\[
 (2.4) \quad S = 2S_1,
\]
where
\[
 S_1 := \sum_{(x_1, x_2, x_3) \in \mathbb{Z}^3} \Phi_1(x_1) \Phi_2(x_2) \Phi_3(x_3).
\]
In the following, we transform the sum \( S_1 \).

Similarly as in [3], we first make a change of variables, writing
\[
 (2.5) \quad a = x_2 + x_3, \quad b = x_2 - x_3. \]
Then
\[
 (ab, 2) = 1
\]
and
\[
 x_2 = \frac{a + b}{2}, \quad x_3 = \frac{a - b}{2}.\]
Hence,

\[ S_1 = \sum_{(x_1, a, b) \in \mathbb{Z}^3} \Phi_1(x_1) \Phi_2 \left( \frac{a + b}{2} \right) \Phi_3 \left( \frac{a - b}{2} \right). \]

We have in mind the case when \( x_3 \) is small compared to \( x_2 \), so \( a \) and \( b \) will be of about the same size. We note that in contrast to [3], we may here break \( D - x_2^2 \) into \((n - x_1)(n + x_1)\) which significantly changes the nature of this problem and is advantageous.

Next, recalling that \( x_1 \) is necessarily even, we set \( x_1 = 2c \) and write \( S_1 \) in the form

\[ (2.6) \quad S_1 = \sum_{a \in \mathbb{Z}} \sum_{c \in \mathbb{Z}} \Phi_1(2c) \Phi_2 \left( \frac{a}{2} + \frac{n^2 - 4c^2}{2a} \right) \Phi_3 \left( \frac{a}{2} - \frac{n^2 - 4c^2}{2a} \right). \]

We further write

\[ a_1 = (a, n - 2c), \quad a_2 = (a, n + 2c). \]

Then the congruence relation \((n - 2c)(n + 2c) \equiv 0 \mod a\) is equivalent to \(a|a_1a_2\). Hence, our sum turns into

\[ S_1 = \sum_{a_1, a_2 \in \mathbb{N}} \sum_{c \in \mathbb{Z}} \Phi_1(2c) \Phi_2 \left( \frac{a}{2} + \frac{n^2 - 4c^2}{2a} \right) \Phi_3 \left( \frac{a}{2} - \frac{n^2 - 4c^2}{2a} \right). \]

It is desirable to break this sum into two nearly symmetric parts accounting for the contributions of \( a_1 > a_2 \) and \( a_1 \leq a_2 \). (This is reminiscent of Dirichlet’s parabola trick.) We thus write

\[ (2.7) \quad S_1 = S_1^\sharp + S_1^\flat, \]

where

\[ S_1^\flat := \sum_{a_1, a_2 \in \mathbb{N}} \sum_{c \in \mathbb{Z}} \Phi_1(2c) \Phi_2 \left( \frac{a}{2} + \frac{n^2 - 4c^2}{2a} \right) \Phi_3 \left( \frac{a}{2} - \frac{n^2 - 4c^2}{2a} \right) \chi_{>1} \left( \frac{a_1}{a_2} \right) \]

and

\[ S_1^\sharp := \sum_{a_1, a_2 \in \mathbb{N}} \sum_{c \in \mathbb{Z}} \Phi_1(2c) \Phi_2 \left( \frac{a}{2} + \frac{n^2 - 4c^2}{2a} \right) \Phi_3 \left( \frac{a}{2} - \frac{n^2 - 4c^2}{2a} \right) \chi_{\geq1} \left( \frac{a_2}{a_1} \right). \]

Here \( \chi_C \) denotes the characteristic function with respect to a condition \( C \). We deal only with the part \( S_1^\flat \). The treatment of \( S_1^\flat \) is similar and leads to the same asymptotic estimate as for \( S_1^\sharp \). We shall later indicate which modifications need to be made when dealing with \( S_1^\flat \).

It will be useful to bound \( \chi_{>1} \) from below and above by fixed smooth functions in the form

\[ \Phi^\pm \leq \chi_{>1} \leq \Phi^\pm. \]

For convenience, we define \( \Phi^+ \) and \( \Phi^- \) on whole of \( \mathbb{R} \) and assume that these functions are even with values in \([0, 1]\), increase monotonically in \( \mathbb{R}^+ \) and satisfy

\[ \Phi^-(x) = \begin{cases} 0 & \text{if } |x| \leq 1 \\ 1 & \text{if } |x| \geq 2 \end{cases} \]

and

\[ \Phi^+(x) = \begin{cases} 0 & \text{if } |x| \leq 1/2 \\ 1 & \text{if } |x| \geq 1. \end{cases} \]

Hence,

\[ S_1^{\sharp-} \leq S_1^\flat \leq S_1^{\sharp+}, \]
where
\[ S_1^{c,\pm} := \sum_{a_1, a_2 \in \mathbb{Z}} \sum_{a_1 | a, a_2 | a} \sum_{\substack{e \in \mathbb{Z} \\ (e, \frac{a_1 a_2}{a}) = 1}} \Phi_1(2c) \Phi_2 \left( \frac{a}{2} + \frac{n^2 - 4c^2}{2a} \right) \Phi_3 \left( \frac{a}{2} - \frac{n^2 - 4c^2}{2a} \right) \Phi^{\pm} \left( \frac{a_1}{a_2} \right). \]

Again, we deal only with \( S_1^{c, -} \) since the treatment of \( S_1^{c, +} \) will be similar and leads to the same asymptotic, giving an asymptotic for \( S_1^{c} \). Similarly as above, \( S_1^{c} \) can be bounded in the form
\[ S_1^{c, -} \leq S_1^{b} \leq S_1^{c, +}, \]
and \( S_1^{b, \pm} \) can be treated along essentially the same lines as \( S_1^{c, \pm} \).

To simplify the situation, we take out a common factor from \( a_1 \) and \( a_2 \), writing
\[ a_1 = d_1 e, \quad a_2 = d_2 e, \quad e = (a_1, a_2). \]
Then \( (d_1, d_2) = 1 \). We also note that we then have
\[ n - 2c \equiv 0 \mod e, \quad n + 2c \equiv 0 \mod e \]
which forces \( e | n \) and \( e | c \). We may therefore write \( c = ef \) and get
\[ S_1^{c, -} = \sum_{e \in \mathbb{N}} \sum_{d_1, d_2 \in \mathbb{N}} \sum_{\substack{f \in \mathbb{Z} \\ (d_1, d_2) = 1}} \sum_{\substack{e \in \mathbb{N} \\ (d_1, d_2, e) = 1}} \Phi_1(2ef) \Phi_2 \left( \frac{d_1 d_2 e g}{2} + \frac{n^2 - 4e^2 f^2}{2d_1 d_2 e g} \right) \Phi_3 \left( \frac{d_1 d_2 e g}{2} - \frac{n^2 - 4e^2 f^2}{2d_1 d_2 e g} \right) \Phi^{-} \left( \frac{d_1}{d_2} \right). \]

Now the coprimality condition \( (d_1, d_2) = 1 \) and the summation conditions on \( a \) force \( a \) to be of the form \( a = d_1 d_2 eg \), where \( g | e \). We therefore get
\[ S_1^{c, -} = \sum_{e \in \mathbb{N}} \sum_{d_1, d_2 \in \mathbb{N}} \sum_{\substack{f \in \mathbb{Z} \\ (d_1, d_2) = 1}} \sum_{\substack{e \in \mathbb{N} \\ (d_1, d_2, e) = 1}} \mu(h_1) \mu(h_2) \sum_{\substack{e \in \mathbb{N} \\ (d_1, d_2, e) = 1}} \Phi_1(2ef) \Phi_2 \left( \frac{d_1 d_2 e g}{2} + \frac{n^2 - 4e^2 f^2}{2d_1 d_2 e g} \right) \Phi_3 \left( \frac{d_1 d_2 e g}{2} - \frac{n^2 - 4e^2 f^2}{2d_1 d_2 e g} \right) \Phi^{-} \left( \frac{d_1}{d_2} \right). \]

Next, we remove the coprimality conditions in the last sum using Möbius inversion, getting
\[ (2.9) \]
\[ \Phi^{-} \left( \frac{d_1}{d_2} \right). \]
We write the first congruence in the last sum in the form
\[ 2f = n/e + kd_1 h_1, \]
which forces \( k \) to be odd. Then the second congruence is equivalent to
\[ 2n/e + kd_1 h_1 \equiv 0 \mod d_2 h_2. \]
Let
\[ j = (d_1 h_1, d_2 h_2). \]
Then we have
\[ \left( \frac{d_1 h_1}{j}, \frac{d_2 h_2}{j} \right) = 1. \]
Moreover, the above congruence forces \( j | n/e \) and can therefore be turned into
\[ k \equiv -\frac{2n}{e j} \frac{d_1 h_1}{j} \mod \frac{d_2 h_2}{j}. \]
Thus we obtain
\[
S_1^{+,-} = \sum_{e \in \mathbb{N}} \sum_{g \in \mathbb{Z}} \sum_{j \in \mathbb{N}} \sum_{h_1, h_2 \in \mathbb{N}} \mu(h_1)\mu(h_2) \sum_{d_1, d_2 \in \mathbb{N}} \sum_{k \in \mathbb{Z}} \Phi_1 \left( e \left( \frac{n}{e} + kd_1 h_1 \right) \right) \times
\]
\[
\Phi_2 \left( \frac{d_1 d_2 e g}{2} + \frac{n^2 - e^2 (n/e + kd_1 h_1)^2}{2d_1 d_2 e g} \right) \Phi_3 \left( \frac{d_1 d_2 e g}{2} - \frac{n^2 - e^2 (n/e + kd_1 h_1)^2}{2d_1 d_2 e g} \right) \Phi^-(d_1/d_2)
\]
\[
= \sum_{e \in \mathbb{N}} \sum_{g \in \mathbb{Z}} \sum_{j \in \mathbb{N}} \sum_{h_1, h_2 \in \mathbb{N}} \mu(h_1)\mu(h_2) \sum_{d_1, d_2 \in \mathbb{N}} \sum_{k \in \mathbb{Z}} \Phi_1 \left( n + kd_1 h_1 e \right) \times
\]
\[
\Phi_2 \left( \frac{d_1 d_2 e g}{2} + \frac{2nkh_1 + k^2 d_1 h_1^2 e}{2d_2 g} \right) \Phi_3 \left( \frac{d_1 d_2 e g}{2} + \frac{2nkh_1 + k^2 d_1 h_1^2 e}{2d_2 g} \right) \Phi^-(d_1/d_2).
\]

Alternatively, we could have written the second congruence in the last sum in (2.9) in the form \(2f = -n/e + kd_2 h_2\) and plugged this into the first. This is favorable when dealing with \(S_1^{+,-}\) and leads to similar contributions. In particular, the main term contributions will be the same as for \(S_1^{+,-}\).

To be able to apply Poisson summation in two variables later on, we continue with taking out common factors, writing

\[
g_1 = (h_1, g), \quad g_2 = (h_2, g), \quad h_1 = m_1 g_1, \quad h_2 = m_2 g_2.
\]

Then

\[
(m_1, g/g_1) = 1, \quad (m_2, g/g_2) = 1
\]

and the conditions

\[
h_1|d_2 g, \quad h_2|d_1 g
\]

turn into

\[
m_1|d_2, \quad m_2|d_1.
\]

Let

\[
d_2 = q_1 m_1, \quad d_1 = q_1 m_2.
\]

Then the condition

\[
(d_1 h_1, d_2 h_2) = j
\]

turns into

\[
(q_1 m_1 m_2 g_1, q_2 m_1 m_2 g_2) = j
\]

which implies

\[
m_1 m_2|j.
\]

Suppose

\[
j = m_1 m_2 r.
\]

Then it follows that

\[
(q_1 g_1, q_2 g_2) = r.
\]

Hence, we get

\[
S_1^{+,-} = \sum_{e \in \mathbb{N}} \sum_{g \in \mathbb{Z}} \sum_{g_1, g_2 \in \mathbb{N}} \sum_{m_1, m_2 \in \mathbb{N}} \mu(m_1 g_1)\mu(m_2 g_2) \sum_{r \in \mathbb{N}} \sum_{q_1, q_2 \in \mathbb{N}} \sum_{q_1 g_2 \in \mathbb{N}} \sum_{q_1 g_2 \in \mathbb{N}} \sum_{k \in \mathbb{Z}} \Phi_1 \left( n + kq_1 m_1 m_2 g_1 e \right) \times
\]
\[
\Phi_2 \left( \frac{q_1 q_2 m_1 m_2 e g}{2} + \frac{2nk g_1 + k^2 q_1 m_1 m_2 g_1^2 e}{2q_2 g} \right) \Phi_3 \left( \frac{q_1 q_2 m_1 m_2 e g}{2} + \frac{2nk g_1 + k^2 q_1 m_1 m_2 g_1^2 e}{2q_2 g} \right) \Phi^-(q_1 m_2/q_2 m_1).
\]

Suppose further that

\[
(g_1, g_2) = s, \quad g_1 = t_1 s, \quad g_2 = t_2 s.
\]

Then

\[
(t_1, t_2) = 1
\]
and the condition

\( (q_1q_1, q_2q_2) = r \)

turns into

\( (q_1q_1, q_2q_2) = r/s =: u. \)

It follows that

\[
\Phi_2 \left( \frac{q_1q_2m_1m_2eg}{2} \right) = \frac{2nk_1s + k^2q_1m_1m_2t_1^2s^2e}{2q_2g} \Phi_3 \left( \frac{q_1q_2m_1m_2eg}{2} + \frac{2nk_1s + k^2q_1m_1m_2t_1^2s^2e}{2q_2g} \right) \Phi^{-1} \left( \frac{q_1m_2}{q_2m_1} \right).
\]

Finally, set

\( (q_1, t_2) = v_1, \quad (q_2, t_1) = v_2, \quad q_1 = \alpha_1 v_1, \quad t_2 = \beta_2 v_1, \quad q_2 = \alpha_2 v_2, \quad t_1 = \beta_1 v_2. \)

Then taking \((t_1, t_2) = 1 = (q_1, q_2)\) into account, we have

\( v_1 v_2 = u, \quad (v_1, v_2) = 1, \quad (\alpha_1 \beta_1, \alpha_2 \beta_2) = 1, \)

and hence,

\[
\Phi_2 \left( \frac{\alpha_1 \alpha_2 m_1m_2v_1v_2eg}{2} \right) = \frac{2nk_1s + k^2\alpha_1 v_1m_1m_2\beta_1^2 v_2^2s^2e}{2\alpha_2 g} \Phi_3 \left( \frac{\alpha_1 \alpha_2 m_1m_2v_1v_2eg}{2} + \frac{2nk_1s + k^2\alpha_1 v_1m_1m_2\beta_1^2 v_2^2s^2e}{2\alpha_2 g} \right).
\]

We still separate the double sum over \( \alpha_1 \) and \( \alpha_2 \) into two separate sums and replace the conditions \( g \in \mathbb{Z} \) and \( \alpha_1 \in \mathbb{N} \) by \( g \in \mathbb{N} \) and \( \alpha_1 \in \mathbb{Z} \), which does not change the sum since the contribution of \((g, -\alpha_1, -k)\) equals the contribution of \((g, \alpha_1, k)\). Here we recall that we assumed the function \( \Phi^{-1} \) to be even. Thus we write

\[
\Phi_2 \left( \frac{\alpha_1 \alpha_2 m_1m_2v_1v_2eg}{2} \right) \Phi_3 \left( \frac{\alpha_1 \alpha_2 m_1m_2v_1v_2eg}{2} + \frac{2nk_1s + k^2\alpha_1 v_1m_1m_2\beta_1^2 v_2^2s^2e}{2\alpha_2 g} \right).
\]

3. Poisson summation

Now we perform Poisson summation over \( k \) and \( \alpha_1 \). To simplify matters, we set

\[
(3.1) \quad A := \beta_1 m_1m_2v_1v_2se, \quad B := m_1m_2v_1v_2eg, \quad C := \beta_1 s, \quad D := m_1m_2v_1v_2\beta_1^2 s^2e, \quad E := \frac{2n}{m_1m_2v_1v_2se}.
\]
Then the double sum over \( \alpha_1 \) and \( k \) becomes

\[
T = \sum_{\alpha_1 \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \Phi_1 \left( n + k\alpha_1 \right) \Phi_2 \left( \frac{\alpha_1 \alpha_2 B}{2} - \frac{2nkC + k^2 \alpha_1 D}{2\gamma\alpha_2} \right) \times \\
\Phi_3 \left( \frac{\alpha_1 \alpha_2 B}{2} + \frac{2nkC + k^2 \alpha_1 D}{2\gamma\alpha_2} \right) \Phi^{-} \left( \frac{\alpha_1 v_1 m_2}{\alpha_2 v_2 m_1} \right).
\]

We divide this into four sums,

\[
T = \sum_{\mu=1}^{2} \sum_{\nu=1}^{2} (-1)^{\mu + \nu} T_{\mu,\nu},
\]

where

\[
T_{\mu,\nu} := \sum_{\alpha_1 \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \Phi_1 \left( n + k\alpha_1 \mu \bar{\alpha} \right) \Phi_2 \left( \frac{\alpha_1 \alpha_2 \mu B}{2} - \frac{2nk\nu C + k^2 \alpha_1 \mu^2 D}{2\gamma\alpha_2} \right) \times \\
\Phi_3 \left( \frac{\alpha_1 \alpha_2 \mu B}{2} + \frac{2nk\nu C + k^2 \alpha_1 \mu^2 D}{2\gamma\alpha_2} \right) \Phi^{-} \left( \frac{\alpha_1 \nu_1 m_2}{\alpha_2 v_2 m_1} \right).
\]

We write

\[
(3.2)
\]

\[
T_{\nu,\mu} = \sum_{\alpha_1 \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \Phi_{\mu,\nu}(\alpha_1, k),
\]

where

\[
(3.3)
\]

\[
\Phi_{\mu,\nu}(x, y) := \Phi_1 \left( F_1(x, y) \right) \Phi_2 \left( F_2(x, y) \right) \Phi_3 \left( F_3(x, y) \right) \Phi^{-} \left( F_4(x) \right)
\]

with

\[
F_1(x, y) := n + \mu \nu Axy, \quad F_2(x, y) := \frac{\alpha_2 \mu Bx}{2} - \frac{2\mu \nu Cy + \mu^2 Dxy^2}{2\gamma\alpha_2}, \quad F_3(x, y) := \frac{\alpha_2 \mu Bx}{2} + \frac{2\mu \nu Cy + \mu^2 Dxy^2}{2\gamma\alpha_2}
\]

and

\[
F_4(x) := \frac{\mu \nu_1 m_2 x}{\alpha_2 v_2 m_1}.
\]

Poisson summation for the sum over \( k \) gives

\[
T_{\mu,\nu} = \frac{1}{\alpha_2 \beta_2} \sum_{\alpha_1 \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} e \left( -\frac{lE\alpha_1 \mu \nu \beta_1}{\alpha_2 \beta_2} \right) \int_{\mathbb{R}} \Phi_{\mu,\nu} \left( \alpha_1, y \right) e \left( -\frac{ly}{\alpha_2 \beta_2} \right) dy
\]

\[
= \frac{1}{\alpha_2 \beta_2} \sum_{l \in \mathbb{Z}} \sum_{\gamma \mod \alpha_2 \beta_2} e \left( -\frac{lE\gamma \mu \nu \beta_1}{\alpha_2 \beta_2} \right) \int_{\mathbb{R}} \left( \sum_{\alpha_1 \equiv \gamma \mod \alpha_2 \beta_2} \Phi_{\mu,\nu} \left( \alpha_1, y \right) \right) e \left( -\frac{ly}{\alpha_2 \beta_2} \right) dy.
\]

Now Poisson summation for the sum over \( \alpha_1 \) gives

\[
T_{\mu,\nu} = \frac{1}{\alpha_2 \beta_2} \sum_{l \in \mathbb{Z}} \sum_{\gamma \mod \alpha_2 \beta_2} e \left( -\frac{lE\gamma \mu \nu \beta_1}{\alpha_2 \beta_2} \right) \int_{\mathbb{R}} \left( \sum_{w \in \mathbb{Z}} e \left( \frac{w \gamma}{\alpha_2 \beta_2} \right) \int_{\mathbb{R}} \Phi_{\mu,\nu} \left( x, y \right) e \left( -\frac{wx + ly}{\alpha_2 \beta_2} \right) dx \right) dy
\]

\[
= \frac{1}{\alpha_2 \beta_2} \sum_{w \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} S(w, -lE\mu \nu \beta_1; \alpha_2 \beta_2) I_{\mu,\nu}(w, l; \alpha_2 \beta_2),
\]

where

\[
S(w, -lE\mu \nu \beta_1; \alpha_2 \beta_2) = \sum_{\gamma \mod \alpha_2 \beta_2} e \left( \frac{w \gamma - lE\mu \nu \beta_1 \gamma}{\alpha_2 \beta_2} \right)
\]

is a Kloosterman sum and

\[
(3.4)
\]

\[
I_{\mu,\nu}(w, l; \alpha_2 \beta_2) := \int_{\mathbb{R}} \int_{\mathbb{R}} \Phi_{\mu,\nu} \left( x, y \right) e \left( -\frac{wx + ly}{\alpha_2 \beta_2} \right) dxdy
\]
a Fourier integral. We keep in mind that \( \Phi^-(F_4(x)) = 0 \) if
\[
\alpha_2 v_2 m_1 \geq \mu v_1 m_2 |x|
\]
by definition of \( \Phi^- \). Hence, only \( x \) such that
\[
(3.5) \quad \alpha_2 \leq \frac{\mu v_1 m_2 |x|}{v_2 m_1}
\]
contribute to the integral.

4. Splitting into main and error terms

We further split \( T_{\mu,\nu} \) into
\[
(4.1) \quad T_{\mu,\nu} = \sum_{i=0}^1 \sum_{j=0}^1 T_{\mu,\nu}^{i,j},
\]
where
\[
(4.2) \quad T_{\mu,\nu}^{0,0} := \frac{1}{(\alpha_2 \beta_2)^2} S(0, 0; \alpha_2 \beta_2) I_{\mu,\nu}(0, 0; \alpha_2 \beta_2) = \frac{\varphi(\alpha_2 \beta_2)}{\varphi(\alpha_2 \beta_2)^2} \cdot I_{\mu,\nu}(0, 0; \alpha_2 \beta_2) dx dy,
\]
\[
T_{\mu,\nu}^{1,0} := \frac{1}{(\alpha_2 \beta_2)^2} \sum_{l \in \mathbb{Z}\setminus\{0\}} S(0, -l E \mu v_1; \alpha_2 \beta_2) I_{\mu,\nu}(0, l; \alpha_2 \beta_2)
\]
\[
= \frac{\varphi(\alpha_2 \beta_2)}{\varphi(\alpha_2 \beta_2)^2} \sum_{l \in \mathbb{Z}\setminus\{0\}} \mu(\alpha_2 \beta_2 / (l E \alpha_2 \beta_2)) \cdot I_{\mu,\nu}(0, l; \alpha_2 \beta_2),
\]
\[
T_{\mu,\nu}^{1,1} := \frac{1}{(\alpha_2 \beta_2)^2} \sum_{w \in \mathbb{Z}\setminus\{0\}} S(w, 0; \alpha_2 \beta_2) I_{\mu,\nu}(w, 0; \alpha_2 \beta_2)
\]
\[
= \frac{\varphi(\alpha_2 \beta_2)}{\varphi(\alpha_2 \beta_2)^2} \sum_{w \in \mathbb{Z}\setminus\{0\}} \mu(\alpha_2 \beta_2 / (w, \alpha_2 \beta_2)) \cdot I_{\mu,\nu}(w, 0; \alpha_2 \beta_2)
\]
and
\[
(4.5) \quad T_{\mu,\nu}^{1,1} := \frac{1}{(\alpha_2 \beta_2)^2} \sum_{w \in \mathbb{Z}\setminus\{0\}} \sum_{l \in \mathbb{Z}\setminus\{0\}} S(w, -l E \mu v_1; \alpha_2 \beta_2) I_{\mu,\nu}(w, l; \alpha_2 \beta_2),
\]
where we use the properties of Ramanujan sums for \( T_{\mu,\nu}^{0,0} \) and \( T_{\mu,\nu}^{1,0} \). The term \( T_{\mu,\nu}^{0,0} \) is the main term contribution. The three remaining terms contribute to the error.

5. Computation of \( I_{\mu,\nu}(0, 0; \alpha_2 \beta_2) \)

Now we recall the conditions on the weight functions \( \Phi_1, \Phi_2, \Phi_3 \) and \( \Phi^- \) set at the beginning of section 2 and calculate the integral \( I_{\mu,\nu}(0, 0; \alpha_2 \beta_2) \). Let
\[
V_1 := \{(x, y) \in \mathbb{R}^2 : Y/2 \leq F_1(x, y) \leq Y, \ 0 \leq F_2(x, y) \leq 2X, \ M \leq F_3(x, y) \leq 2M \},
\]
\[
V_2 := \{(x, y) \in \mathbb{R}^2 : 0 \leq F_1(x, y) \leq 2X, \ Y/2 \leq F_2(x, y) \leq Y, \ M \leq F_3(x, y) \leq 2M \},
\]
\[
V_3 := \{(x, y) \in \mathbb{R}^2 : X \leq F_1(x, y) \leq 2X, \ 0 \leq F_2(x, y) \leq 2X, \ M \leq F_3(x, y) \leq 2M \},
\]
\[
V_4 := \{(x, y) \in \mathbb{R}^2 : 0 \leq F_1(x, y) \leq 2X, \ X \leq F_2(x, y) \leq 2X, \ M \leq F_3(x, y) \leq 2M \},
\]
\[
V_5 := \{(x, y) \in \mathbb{R}^2 : 0 \leq F_1(x, y) \leq 2X, \ 0 \leq F_2(x, y) \leq 2X, \ M \leq F_3(x, y) \leq 2M, \ |x| \leq 2\alpha v_2 m_1 / (\mu v_1 m_2) \},
\]
\[
V := \{(x, y) \in \mathbb{R}^2 : 0 \leq F_1(x, y) \leq 2X, \ 0 \leq F_2(x, y) \leq 2X, \ M \leq F_3(x, y) \leq 2M \}.
\]
Then
\[
I_{\mu,\nu}(0, 0; \alpha_2 \beta_2) = \iint_{V} \Phi_3(F_3(x, y)) \, dx dy + O(\text{Vol}(V_1) + \text{Vol}(V_2) + \text{Vol}(V_3) + \text{Vol}(V_4) + \text{Vol}(V_5)).
\]

Now we change variables in the form
\[
\sigma := F_2(x, y) \quad \text{and} \quad \tau := F_3(x, y).
\]
We aim to calculate \( F_1(x, y) \) in terms of \( \sigma \) and \( \tau \). First, we observe that
\[
x = \frac{\sigma + \tau}{\alpha_2 \mu B}
\]
It follows that
\[
\frac{\sigma + \tau}{2} + \frac{2n\nu C y + \nu^2 D (\sigma + \tau)y^2}{2g \alpha_2} = \tau.
\]

Using the definitions of \(B, C, D\), this is equivalent to
\[
\frac{\sigma - \tau}{2} + \frac{2n\nu \beta_1 s y + \nu^2 \beta_1^2 s (\sigma + \tau)y^2}{2g \alpha_2} = 0.
\]

Multiplying by \(2(g \alpha_2)^2\), this turns into a quadratic equation
\[
\nu^2 \beta_1^2 s^2 (\sigma + \tau)y^2 + 2n\nu \beta_1 s g \alpha_2 y + (\sigma - \tau)(g \alpha_2)^2 = 0
\]
in the variable \(y\), which has a discriminant of
\[
\Delta = (2n\nu \beta_1 s g \alpha_2)^2 - 4\nu^2 \beta_1^2 s^2 (\sigma - \tau)(g \alpha_2)^2 = (2\nu \beta_1 s g \alpha_2)^2(n^2 - \sigma^2 + \tau^2)
\]
and solutions
\[
y = \frac{-2n\nu \beta_1 s g \alpha_2 \pm \sqrt{(2\nu \beta_1 s g \alpha_2)^2(n^2 - \sigma^2 + \tau^2)}}{2\nu^2 \beta_1^2 s^2 (\sigma + \tau)} = \frac{g \alpha_2(-n \pm \sqrt{n^2 - \sigma^2 + \tau^2})}{\nu \beta_1 s (\sigma + \tau)},
\]
provided that
\[\begin{align*}
(\sigma, \tau) &\in W := \{(\sigma, \tau) \in \mathbb{R}^2 : n^2 - \sigma^2 + \tau^2 \geq 0\}. \\
\end{align*}\]

In this case, we obtain
\[(5.1) \quad F_1(x, y) = n + \nu Axy = n + \nu A \frac{\sigma + \tau}{\alpha_2 \mu B} \cdot \frac{g \alpha_2(-n \pm \sqrt{n^2 - \sigma^2 + \tau^2})}{\nu \beta_1 s (\sigma + \tau)} = \pm \sqrt{n^2 - \sigma^2 + \tau^2}.
\]

Since \(F_1(x, y) \geq 0\) in our setting, we have
\[(5.2) \quad F_1(x, y) = \sqrt{n^2 - \sigma^2 + \tau^2}
\]
and only the solution
\[(5.3) \quad y = \frac{g \alpha_2(-n + \sqrt{n^2 - \sigma^2 + \tau^2})}{\nu \beta_1 s (\sigma + \tau)}.
\]

Now for any region \(\mathcal{R} \subset \mathbb{R}^2\) and smooth function \(\Phi : \mathcal{R} \to \mathbb{R}^2\), we have, by the general substitution rule,
\[
\int\int_{\Psi^{-1}(\mathcal{R})} \Phi(x, y)dxdy = \int\int_{\mathcal{R}} \Phi(\Psi^{-1}(\sigma, \tau)) \det(\text{Jac } \Psi^{-1})(\sigma, \tau)d\sigma d\tau,
\]
where
\[\Psi(x, y) = (\sigma, \tau).
\]

We here consider \(\Phi(x, y) \equiv 1\) or \(\Phi(x, y) = \Phi_3(F_3(x, y))\) and \(\mathcal{R} = \mathcal{U}_1, \ldots, \mathcal{U}_5, \mathcal{U}\), where
\[
\begin{align*}
\mathcal{U}_1 &:= \{ (\sigma, \tau) \in W : Y/2 \leq \sqrt{n^2 - \sigma^2 + \tau^2} \leq Y, \ 0 \leq \sigma \leq 2X, \ M \leq \tau \leq 2M \}, \\
\mathcal{U}_2 &:= \{ (\sigma, \tau) \in W : \sqrt{n^2 - \sigma^2 + \tau^2} \leq 2X, \ Y/2 \leq \sigma \leq Y, \ M \leq \tau \leq 2M \}, \\
\mathcal{U}_3 &:= \{ (\sigma, \tau) \in W : X \leq \sqrt{n^2 - \sigma^2 + \tau^2} \leq 2X, \ 0 \leq \sigma \leq 2X, \ M \leq \tau \leq 2M \}, \\
\mathcal{U}_4 &:= \{ (\sigma, \tau) \in W : \sqrt{n^2 - \sigma^2 + \tau^2} \leq 2X, \ X \leq \sigma \leq 2X, \ M \leq \tau \leq 2M \}, \\
\mathcal{U}_5 &:= \{ (\sigma, \tau) \in W : \sqrt{n^2 - \sigma^2 + \tau^2} \leq 2X, \ 0 \leq \sigma \leq 2X, \ M \leq \tau \leq 2M, \ \sigma + \tau \leq 2\alpha_3^2 \nu^2 \beta_1 m^2 e \}, \\
\mathcal{U} &:= \{ (\sigma, \tau) \in W : \sqrt{n^2 - \sigma^2 + \tau^2} \leq 2X, \ \sigma \leq 2X, \ M \leq \tau \leq 2M \}
\end{align*}
\]
so that \(\mathcal{V}_i = \Psi^{-1}(\mathcal{U}_i)\) for \(i = 1, \ldots, 5\), \(\mathcal{V} = \Psi^{-1}(\mathcal{U})\) and hence
\[
\text{Vol}(\mathcal{V}_i) = \int_{\mathcal{U}_i} \det(\text{Jac } \Psi^{-1})(\sigma, \tau)d\sigma d\tau
\]
and
\[
\int\int_{\mathcal{V}} \Phi_3(F_3(x, y))dxdy = \int\int_{\mathcal{U}} \Phi_3(\tau) \det(\text{Jac } \Psi^{-1})(\sigma, \tau)d\sigma d\tau.
\]
At this point, we recall that only \( x \) satisfying (3.5) contribute to the integral \( I_{\mu,\nu}(0,0;\alpha_2 \beta_2) \). After the above change of variables, this translates into saying that only \( \sigma,\tau \) satisfying

\[
\sigma \leq \frac{\sqrt{\sigma + \tau}}{v_2 m_1 \sqrt{e_g}}
\]

contribute. Since \( \sigma + \tau \leq 2X + 2M \leq 4X \), this sets a general restriction on \( \alpha_2 \), namely

\[
\alpha_2 \leq \frac{2 \sqrt{X}}{v_2 m_1 \sqrt{e_g}}.
\]

Now we calculate the Jacobian. We have

\[
\det(\text{Jac} \; \Psi^{-1})(\sigma,\tau) = \det(\text{Jac} \; \Psi)(x,y)^{-1} = \begin{vmatrix}
\partial_x F_2(x,y) & \partial_y F_2(x,y) \\
\partial_x F_3(x,y) & \partial_y F_3(x,y)
\end{vmatrix}^{-1}
\]

\[
= \begin{vmatrix}
\frac{\alpha_2 B}{2} - \frac{\mu^2 D_{\nu}^2}{2g_{\alpha \tau}} & -\frac{\mu C + \mu^2 D_{\tau \nu}}{g_{\alpha \tau}} \\
\frac{\alpha_2 B}{2} + \frac{\mu^2 D_{\nu}^2}{2g_{\alpha \tau}} & \frac{\mu C + \mu^2 D_{\tau \nu}}{g_{\alpha \tau}}
\end{vmatrix}^{-1}
\]

\[
=g(n\mu nBC + \mu^2 \nu BD_{\tau \nu})^{-1}
\]

\[
= \left(G\sqrt{n^2 - \sigma^2 + \tau^2}\right)^{-1}
\]

with

\[
G := \mu \nu m_1 m_2 v_1 v_2 \beta_1 se,
\]

where we use (5.1) and (5.2). In summary, we get

\[
\text{Vol}(\mathcal{V}_i) = \frac{1}{G} \int\int_{U_i} \frac{d\sigma d\tau}{\sqrt{n^2 - \sigma^2 + \tau^2}} \quad \text{for } i = 1, 2, 3, 4
\]

and

\[
\int\int_{\mathcal{V}} \Phi_3(F_3(x,y))dxydy = \frac{1}{G} \int\int_{U} \frac{\Phi_3(\tau)}{\sqrt{n^2 - \sigma^2 + \tau^2}} d\sigma d\tau.
\]

It is easily calculated that

\[
\frac{1}{G} \int\int_{U_{1,2}} \frac{d\sigma d\tau}{\sqrt{n^2 - \sigma^2 + \tau^2}} \ll \frac{MY}{Gn}
\]

and \( U_{3,4} = \emptyset \). The integral on the right-hand side of (5.6) was exactly calculated in [3, integral \( I(F) \) on page 162] and it turns out that

\[
\frac{1}{G} \int\int_{U} \frac{\Phi_3(\tau)}{\sqrt{n^2 - \sigma^2 + \tau^2}} d\sigma d\tau = \frac{2\pi \hat{\Phi}_3(0)}{G} = \frac{2\pi M \hat{\Phi}(0)}{G}.
\]

Along similar lines as (5.7), we find that

\[
\frac{1}{G} \int\int_{U_5} \frac{d\sigma d\tau}{\sqrt{n^2 - \sigma^2 + \tau^2}} \ll \frac{\alpha_2^2 \nu_2^2 \beta_1^2 e_g M}{Gn}
\]

where we use that \( \sigma \leq 2 \alpha_2^2 \nu_2^2 \beta_1^2 e_g \) in the case of region \( U_5 \). So altogether, we get

\[
I_{\mu,\nu}(0,0;\alpha_2 \beta_2) = \frac{2\pi \hat{\Phi}(0) M}{G} + O \left( \frac{MY}{Gn} + \frac{\alpha_2^2 \nu_2^2 \beta_1^2 e_g M}{Gn} \right).
\]

We note that for large \( \alpha_2 \) satisfying (5.4), the second \( O \)-term is of comparable size as the main term. This, however, is not a problem since we will see that the most significant contribution to the main term comes from small \( \alpha_2 \).
6. Evaluation of the main term

The main term contribution to $S_{1}^{4-}$ comes from the term $T_{\mu,\nu}^{0,0}$ in (1.2). Recalling (5.21), (5.24) and (5.34), this contribution takes the form

$$S_{1,0,0}^{4-} := 2 \sum_{\mu=1}^{2} \sum_{\nu=1}^{2} (-1)^{\mu+\nu} \sum_{e \in \mathbb{N}} \sum_{g \in \mathbb{N}} \sum_{s \in \mathbb{N}} \sum_{e'[\nu, \nu] = 1} \sum_{\beta_{1}, \beta_{2} \in \mathbb{N}} \sum_{\beta_{1} \gamma(\nu, \nu') = 1} \sum_{m_{1}, m_{2} \in \mathbb{N}} \mu(m_{1} \beta_{1} \nu) \mu(m_{2} \beta_{2} \nu) \times$$

$$(6.1) \sum_{\alpha_{2} \in \mathbb{N}} \frac{\varphi(\alpha_{2} \beta_{2})}{\alpha_{2} \beta_{2}} \left( \frac{2 \pi \tilde{\Phi}(0) M}{\mu \nu m_{1} \nu m_{1} \beta_{1} \nu \beta_{1} \nu} + O \left( \frac{M Y}{m_{1} m_{2} \nu m_{1} \beta_{1} \nu \beta_{1} \nu} \right) + O \left( \frac{\alpha_{2}^{2} \nu m_{1} \nu}{\nu m_{1} \beta_{1} \nu} \right) \right).$$

It will turn out that the contribution of the second $O$-term in (6.1) is by essentially a factor of logarithm smaller than the main term. The first $O$-term in (6.1) will give a small contribution if $Y$ is chosen sufficiently small. We will optimize $Y$ later.

Just using $\varphi(\alpha_{2} \beta_{2}) \leq \alpha_{2} \beta_{2}$, bounding the Möbius function trivially and ignoring all coprimality conditions, our first $O$-term contribution is bounded by

$$\leq \sum_{c, g, s, v_{1}, v_{2}, \beta_{1}, \beta_{2}, m_{1}, m_{2} \in \mathbb{N}} \frac{M Y}{\alpha_{2} m_{1} m_{2} \nu m_{1} \beta_{1} \nu \beta_{1} \nu} \leq \sum_{f, g, s, v_{1}, v_{2}, \beta_{1}, \beta_{2}, m_{1}, m_{2} \in \mathbb{N}} \frac{M Y}{\alpha_{2} m_{1} m_{2} \nu m_{1} \beta_{1} \nu \beta_{1} \nu} \ll MY n^{-1} (\log n) (\log \log n)^{9},$$

where we write $e = fg$ and use the well-known bound

$$\sum_{d \mid n} \frac{1}{d} \ll \log \log n.$$

Similarly, our second $O$-term contribution is bounded by

$$\leq \sum_{c, g, s, v_{1}, v_{2}, \beta_{1}, \beta_{2}, m_{1}, m_{2} \in \mathbb{N}} \frac{\alpha_{2} \beta_{2} \nu m_{1} \beta_{1} \nu \beta_{1} \nu}{\alpha_{2} \beta_{2} \nu m_{1} \beta_{1} \nu \beta_{1} \nu} \ll \sum_{f, g, s, v_{1}, v_{2}, \beta_{1}, \beta_{2}, m_{1}, m_{2} \in \mathbb{N}} \frac{M}{m_{1} m_{2} \nu m_{1} \beta_{1} \nu \beta_{1} \nu} \ll M (\log \log n)^{9},$$

Thus, if

$$Y \leq n (\log n)^{-1},$$

which we want to assume throughout the sequel, we get

$$S_{1,0,0}^{4-} = M + O \left( M (\log \log n)^{9} \right),$$

where

$$M := \sum_{\mu=1}^{2} \sum_{\nu=1}^{2} (-1)^{\mu+\nu} \sum_{e \in \mathbb{N}} \sum_{g \in \mathbb{N}} \sum_{s \in \mathbb{N}} \sum_{e'[\nu, \nu] = 1} \sum_{\beta_{1}, \beta_{2} \in \mathbb{N}} \sum_{\beta_{1} \gamma(\nu, \nu') = 1} \sum_{m_{1}, m_{2} \in \mathbb{N}} \mu(m_{1} \beta_{1} \nu) \mu(m_{2} \beta_{2} \nu) \times$$

$$(6.4) \sum_{\alpha_{2} \in \mathbb{N}} \frac{\varphi(\alpha_{2} \beta_{2})}{\alpha_{2} \beta_{2}} \left( \frac{2 \pi \tilde{\Phi}(0) M}{\mu \nu m_{1} \nu m_{1} \beta_{1} \nu} \right).$$
To evaluate the inner-most sum, we use Perron’s formula to write

\[
\sum_{\alpha_2 \in \mathbb{N}, \alpha_2 \leq Z} \frac{\varphi(\alpha_2 \beta_2)}{\alpha_2^s \beta_2^s} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left( \sum_{\alpha_2 \in \mathbb{N}, \alpha_2 \leq Z} \frac{\varphi(\alpha_2 \beta_2)}{\alpha_2^s \beta_2^s} \right) \frac{Z^s}{s} ds + O\left( \frac{\log 2Z}{T} + Z^{-1} \right)
\]

if \( Z \geq 1 \), where \( c := 1/\log 2Z \) and \( T \geq 1 \). If \( (\beta_2, 2\beta_1) = 1 \), then the above Dirichlet series can be written as an Euler product in the form

\[
\sum_{\alpha_2 \in \mathbb{N}, (\alpha_2, 2\beta_1) = 1} \frac{\varphi(\alpha_2 \beta_2)}{\alpha_2^s \beta_2^s} = \prod_{p \nmid 2\beta_1} \left( 1 + \sum_{k=1}^{\infty} \frac{(p-1)/p}{p^{k(s+1)}} \right) \prod_{p | \beta_2} \left( \sum_{k=0}^{\infty} \frac{1}{p^{k(s+1)}} \right) \frac{\varphi(\beta_2)}{\beta_2}
\]

\[
= \prod_{p \nmid 2\beta_1} \left( 1 + \frac{1}{p(p^{s+1}-1)} \right) \prod_{p | \beta_2} \left( 1 - p^{-s(s+1)} \right)^{-1} \frac{\varphi(\beta_2)}{\beta_2}
\]

\[
= \Pi_{\beta_1, \beta_2}(s+1) \zeta(s+1),
\]

where

\[
\Pi_{\beta_1, \beta_2}(s) := \prod_{p \nmid 2\beta_1} (1 - p^{-s}) \prod_{p | \beta_2} (1 - p^{-s}) \prod_{p | 2\beta_1 \beta_2} (1 - p^{-s})
\]

We note that \( \Pi(s) \) is analytic for \( \Re s > 0 \) and

\[
\Pi_{\beta_1, \beta_2}(s) \ll \tilde{\Pi}_{\beta_1}(1/2) := \prod_{p \nmid 2\beta_1} \left( 1 + p^{-1/2} \right) \quad \text{if} \ \Re s \geq 1/2.
\]

Now we proceed as usual by contour integration, shifting the line of integration to \( \Re s = -1/2 \). In this way, using the residue theorem, we get

\[
\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left( \sum_{\alpha_2 \in \mathbb{N}, (\alpha_2, 2\beta_1) = 1} \frac{\varphi(\alpha_2 \beta_2)}{\alpha_2^s \beta_2^s} \right) \frac{Z^s}{s} ds = \text{Res}_{s=0} \frac{\Pi_{\beta_1, \beta_2}(s+1)\zeta(s+1)Z^s}{s} + \frac{1}{2\pi i} \left( \int_{c-iT}^{c+iT} + \int_{-1/2-iT}^{-1/2+iT} + \int_{-1/2+iT}^{1/2} \right) \Pi_{\beta_1, \beta_2}(s+1)\zeta(1+s)Z^s ds.
\]

Using the convexity bound

\[
\zeta(\sigma + iT) \ll T^{(1-\sigma)/2+\varepsilon} \quad \text{for} \ 1/2 \leq \sigma \leq 1
\]

and

\[
\zeta(\sigma + iT) \ll T^\varepsilon \quad \text{for} \ \sigma > 1,
\]

we conclude that

\[
\int_{-1/2+iT}^{1/2} \Pi_{\beta_1, \beta_2}(s+1)\zeta(1+s)Z^s ds \ll \tilde{\Pi}_{\beta_1}(1/2)T^{-1/2} \int_{1/2}^{1+c} Z^{\sigma-1} T^{\max\{0, (1-\sigma)/2\} + \varepsilon} d\sigma
\]

\[
\ll \tilde{\Pi}_{\beta_1}(1/2)T^{-\varepsilon-1} \left( 1 + Z^{-1} T^{1/2} \int_{1/2}^{1} \left( Z T^{-1/2} \right)^{\sigma} d\sigma \right)
\]

\[
\ll \tilde{\Pi}_{\beta_1}(1/2)T^{-\varepsilon-1},
\]

provided that \( T \leq Z^2 \). Similarly,

\[
\int_{c-iT}^{-1/2-iT} \Pi_{\beta_1, \beta_2}(s+1)\zeta(1+s)Z^s ds \ll \tilde{\Pi}_{\beta_1}(1/2)T^{\varepsilon-1}
\]
under the same condition. Using Cauchy-Schwarz and the second moment bound for the zeta function on the critical line, we get

\[-1/2+iT \int_{-1/2-iT}^{1/2+iT} \Pi_{\beta_1, \beta_2}(s+1)\zeta(1+s)\frac{Z^s}{s} ds \ll \tilde{\Pi}_{\beta_1}(1/2)Z^{-1/2}T^\varepsilon.\]

Choosing \(T := Z^2\), we therefore obtain

\[\sum_{\alpha_2 \in \mathbb{N}} \frac{\varphi(\alpha_2 \beta_2)}{\alpha_2^2 \beta_2} = \text{Res}_{s=0} \frac{\Pi_{\beta_1, \beta_2}(s+1)\zeta(s+1)Z^s}{s} = O \left( \tilde{\Pi}_{\beta_1}(1/2)Z^{\varepsilon-1/2} \right).\]

We use the Laurent series expansions

\[\Pi_{\beta_1, \beta_2}(s+1) := \Pi_{\beta_1, \beta_2}(1) + s\Pi'_{\beta_1, \beta_2}(1) + \ldots,\]

and

\[\zeta(s+1) = \frac{1}{s} + \gamma + \ldots\]

to get

\[\text{Res}_{s=0} \frac{\Pi_{\beta_1, \beta_2}(s+1)\zeta(s+1)Z^s}{s} = \Pi'_{\beta_1, \beta_2}(1) + \Pi_{\beta_1, \beta_2}(1)(\log Z + \gamma).\]

Then we calculate the logarithmic derivative of \(\Pi_{\beta_1, \beta_2}(s)\) to be

\[\frac{\Pi'_{\beta_1, \beta_2}(s)}{\Pi_{\beta_1, \beta_2}(s)} = \sum_{p \mid 2^\beta_1} \frac{p^{-s} \log p}{1 - p^{-s}} + \sum_{p \mid 2^\beta_1 \beta_2} \frac{p^{-s+1} \log p}{1 - p^{-s+1}} =: \Sigma_{\beta_1, \beta_2}(s),\]

which implies

\[\frac{\Pi'_{\beta_1, \beta_2}(1)}{\Pi_{\beta_1, \beta_2}(1)} = \sum_{p \mid 2^\beta_1} \frac{\log p}{p - 1} + \sum_{p \mid 2^\beta_1 \beta_2} \frac{\log p}{p^2 - 1} = \Sigma_{\beta_1, \beta_2}(1).\]

We conclude that

\[\sum_{\alpha_2 \in \mathbb{N}} \frac{\varphi(\alpha_2 \beta_2)}{\alpha_2^2 \beta_2} = \Pi_{\beta_1, \beta_2}(1)(\Sigma_{\beta_1, \beta_2}(1) + \gamma + \log Z) + O \left( \tilde{\Pi}_{\beta_1}(1/2)Z^{\varepsilon-1/2} \right)\]

with

\[\Pi_{\beta_1, \beta_2}(1) = \prod_{p \mid 2^\beta_1} (1 - p^{-1}) \prod_{p \mid 2^\beta_1 \beta_2} (1 - p^{-2}) = \frac{1}{\zeta(2)} \prod_{p \mid 2^\beta_1 \beta_2} (1 + p^{-1})^{-1}.\]

Plugging (6.5) into (6.4), and bounding the \(O\)-term contribution using the bound \(d(k) \ll k^\varepsilon\) for the divisor function (which we shall employ frequently in this article), we get

\[\mathcal{M} = \sum_{\nu=1}^2 \sum_{\nu=1}^2 (-1)^{\mu + \nu} \sum_{\epsilon \mid \mu} \sum_{g \mid n} \sum_{e \mid n} \sum_{s \mid n} \sum_{(v_1, v_2) = 1} \sum_{\beta_1, \beta_2 \in \mathbb{N}} \sum_{\delta_1, \delta_2 \in \mathbb{N}} \mu(m_1 \beta_1 v_2 s)\mu(m_2 \beta_2 v_1 s) \times\]

\[\prod_{p \mid 2^\beta_1 \beta_2} (1 - p^{-1}) \prod_{p \mid 2^\beta_1 \beta_2} (1 - p^{-2}) = \frac{1}{\zeta(2)} \prod_{p \mid 2^\beta_1 \beta_2} (1 + p^{-1})^{-1}.\]

Here we note the additional summation condition \(v_2 m_1 \sqrt{eg} \leq 2\sqrt{X}\). The contribution arising from the term \(\Sigma_{\beta_1, \beta_2}(1) + \gamma\) is dominated by

\[\ll \left( 1 + \sum_{p \mid n} \frac{\log p}{p} \right) \left( \sum_{d \mid n} \frac{1}{d} \right)^9 M \ll M(\log \log n)^{10}\]
by a similar calculation as in (6.2). For the remaining contribution, we note that the additional summation condition \( v_2 m_1 \sqrt{fg} \leq 2 \sqrt{X} \) can be discarded at the cost of an error of size \( O(Mn^{-1/4}) \). (This is possible due to the denominator \( \mu m_1 m_2 v_1 v_2 \beta_1 \beta_2 s e \) on the right-hand side of (6.6) and on writing \( e = fg \) again.) Hence, we get

\[
\mathcal{M} = \mathcal{M}_1 - \mathcal{M}_2 + O(M(\log \log n)^{10}),
\]

where

\[
\mathcal{M}_1 := \frac{\pi}{2} \hat{\Phi}(0) M \log(2 \sqrt{X}) \sum_{\substack{e \in \mathbb{N} \\ g \in \mathbb{N} \\ e \mid n \\ g \mid e \mid s \mid g}} \sum_{v_1, v_2 \in \mathbb{N}} \sum_{(e_1, \beta_1, \beta_2) = 1} \sum_{(e_2, \beta_1, \beta_2) = 1} \Pi_{\beta_1, \beta_2} (1) \sum_{m_1, m_2 \in \mathbb{N}} \mu(m_1 \beta_1 v_2 s) \mu(m_2 \beta_2 v_1 s) \frac{1}{m_1 m_2 v_1 v_2 \beta_1 \beta_2 s e}
\]

and

\[
\mathcal{M}_2 := \frac{\pi}{2} \hat{\Phi}(0) M \sum_{\substack{e \in \mathbb{N} \\ g \in \mathbb{N} \setminus e \mid n \setminus g \mid e \mid s \mid g}} \sum_{v_1, v_2 \in \mathbb{N}} \sum_{(e_1, \beta_1, \beta_2) = 1} \sum_{(e_2, \beta_1, \beta_2) = 1} \Pi_{\beta_1, \beta_2} (1) \sum_{m_1, m_2 \in \mathbb{N}} \mu(m_1 \beta_1 v_2 s) \mu(m_2 \beta_2 v_1 s) \times
\]

\[
\frac{\log(v_2 m_1 \sqrt{fg})}{m_1 m_2 v_1 v_2 \beta_1 \beta_2 s e}.
\]

In the following, we evaluate \( \mathcal{M}_1 \). We observe that the appearance of the \( \mu \)-function creates the extra coprimality conditions \((m_1, \beta_1 v_2 s) = 1 = (m_2, \beta_2 v_1 s)\). Hence, the last two coprimality conditions in the sum over \( m_1 \) and \( m_2 \) simplify into \((m_1 m_2, g) = 1\). Hence, we get

\[
\sum_{m_1, m_2 \in \mathbb{N}} \mu(m_1 \beta_1 v_2 s) \mu(m_2 \beta_2 v_1 s) = \mu(\beta_1 v_2 s) \mu(\beta_2 v_1 s) \sum_{m_1, m_2 \in \mathbb{N}} \mu(m_1) \mu(m_2) \frac{1}{m_1 m_2} \prod_{p \mid n/e} \left(1 - \frac{2}{p}\right).
\]

(6.7)

where we use that \( s, v_1, v_2 \mid g \). Thus,

\[
\mathcal{M}_1 = \frac{\pi}{2} \hat{\Phi}(0) M \log(2 \sqrt{X}) \sum_{\substack{e \in \mathbb{N} \\ g \in \mathbb{N} \setminus e \mid n \setminus g \mid e \mid s \mid g}} \sum_{v_1, v_2 \in \mathbb{N}} \sum_{(e_1, \beta_1, \beta_2) = 1} \sum_{(e_2, \beta_1, \beta_2) = 1} \Pi_{\beta_1, \beta_2} (1) \cdot \mu(\beta_1 v_2 s) \mu(\beta_2 v_1 s) \frac{1}{v_1 v_2 \beta_1 \beta_2 s e}.
\]
Again, the $\mu$-factors create the extra coprimality conditions $(\beta_1, v_2 s) = 1 = (\beta_2, v_1 s)$. Hence, we get

\[
\sum_{\beta_1, \beta_2 \in \mathbb{N}} \Pi_{\beta_1, \beta_2}(1) \cdot \frac{\mu(\beta_1 v_2 s) \mu(\beta_2 v_1 s)}{v_1 v_2 \beta_1 \beta_2 s e} = \frac{\mu(v_1 s) \mu(v_2 s)}{v_1 v_2 s e} \sum_{\beta_1, \beta_2 \in \mathbb{N}} \Pi_{\beta_1, \beta_2}(1) \cdot \frac{\mu(\beta_1) \mu(\beta_2)}{\beta_1 \beta_2} \cdot 1 + \frac{1}{\beta_1 \beta_2} (1) - \frac{1}{p + 1} = \frac{1}{\zeta(2)} \cdot \frac{\mu(v_1 s) \mu(v_2 s)}{v_1 v_2 s e} \sum_{\beta_1, \beta_2 \in \mathbb{N}} \mu(\beta_1) \mu(\beta_2) \prod_{p|\beta_1 \beta_2} (1 - \frac{2}{p + 1}) = \frac{4}{\pi^2} \cdot \frac{\mu(v_1 s) \mu(v_2 s)}{v_1 v_2 s e} \cdot \prod_{p|g \atop p
ot| v_1 v_2 s} \left(1 - \frac{2}{p + 1}\right).
\]

and thus

\[
\mathcal{M}_1 = \frac{2}{\pi} \Phi(0) M \log(2\sqrt{X}) \sum_{e \in \mathbb{N}} \sum_{g \in \mathbb{N}} \prod_{p|g \atop p \not| e \not| n} \left(1 - \frac{2}{p}ight) \sum_{s \in \mathbb{N}} \sum_{\frac{v_1 v_2 s e}{s|g}} \frac{\mu(v_1 s) \mu(v_2 s)}{v_1 v_2 s e} \cdot \prod_{p|g \atop p
ot| v_1 v_2 s} \left(1 - \frac{2}{p + 1}\right).
\]

Again, the $\mu$-factors create the extra coprimality condition $(v_1 v_2, s) = 1$. Hence, we get

\[
\sum_{v_1, v_2 \in \mathbb{N}} \frac{\mu(v_1 s) \mu(v_2 s)}{v_1 v_2 s e} \cdot \prod_{p|g \atop p \not| v_1 v_2 s} \left(1 - \frac{2}{p + 1}\right) = \frac{\mu^2(s)}{se} \prod_{p|g \atop p \not| s} \left(1 - \frac{2}{p + 1}\right) \prod_{p|n \atop p \not| s} \left(1 - \frac{2(p + 1)}{p(p - 1)}\right)
\]

\[
= \frac{\mu^2(s)}{se} \prod_{p|g \atop p \not| s} \left(1 - \frac{2}{p + 1}\right) \prod_{p|n \atop p \not| s} \left(1 - \frac{2(p + 1)}{p(p - 1)}\right)
\]
and thus
\[ \mathcal{M}_1 = \frac{2}{\pi} \Phi(0) M \log(2\sqrt{X}) \sum_{e \in \mathbb{N}} \sum_{g \in \mathbb{N}_{p|n/e}} \prod_{s \in \mathbb{N}_{s|g}} \left( 1 - \frac{2}{p} \right) \prod_{p \nmid g} \left( 1 - \frac{2}{p} \right) \prod_{p \nmid g} \left( 1 - \frac{2p+1}{p(p-1)} \right) \] 
\[ = \frac{2}{\pi} \Phi(0) M \log(2\sqrt{X}) \sum_{e \in \mathbb{N}} \sum_{g \in \mathbb{N}_{p|n/e}} \prod_{s \in \mathbb{N}_{s|g}} \left( 1 - \frac{2}{p} \right) \prod_{p \nmid g} \left( 1 - \frac{2p+1}{p(p-1)} \right) \] 
\[ \sum_{s \in \mathbb{N}_{s|g}} \left( \frac{2}{p} \right) \prod_{p \nmid g} \left( 1 - \frac{2p+1}{p(p-1)} \right) \] 
\[ = \frac{2}{\pi} \Phi(0) M \log(2\sqrt{X}) \sum_{e \in \mathbb{N}} \sum_{g \in \mathbb{N}_{p|n/e}} \prod_{s \in \mathbb{N}_{s|g}} \left( 1 - \frac{2}{p} \right) \prod_{p \nmid g} \left( 1 - \frac{2p+1}{p(p-1)} \right) \] 
\[ \left( 1 + \frac{1}{p} \left( 1 - \frac{2}{p+1} \right)^{-1} \right) \prod_{p \nmid g} \left( 1 - \frac{2p+1}{p(p-1)} \right) \] 
which simplifies into
\[ \mathcal{M}_1 = \frac{2}{\pi} \Phi(0) M \log(2\sqrt{X}) \sum_{e \in \mathbb{N}} \sum_{g \in \mathbb{N}_{p|n/e}} \prod_{s \in \mathbb{N}_{s|g}} \left( 1 - \frac{2}{p} \right) \prod_{p \nmid g} \left( 1 - \frac{2p+1}{p(p-1)} \right) \] 
\[ = \frac{2}{\pi} \mathcal{P}(n) \Phi(0) M \log(2\sqrt{X}) \] 
with
\[ \mathcal{P}(n) := \sum_{f \in \mathbb{N}_{f|n}} \sum_{g \in \mathbb{N}_{g|n/f}} \prod_{p \nmid g} \left( 1 - \frac{2}{p} \right) \prod_{p \nmid g} \left( 1 - \frac{p-1}{p(p+1)} \right) \prod_{p \nmid g} \left( 1 - \frac{p-1}{p(p+1)} \right) \] 
where we have written $e = fg$. We note that the above implies the estimate
\[ \left( \frac{M \log n}{(\log \log n)^2} \right) \ll \mathcal{M}_1 \ll M(\log n)(\log \log n)^2. \]

It seems to be difficult to simplify $\mathcal{P}(n)$ for general $n$. However, if $n$ is square-free, then $p|g$ implies $p \nmid n/(fg)$ and $p|n/(fg)$ implies $p \nmid g$, and $\mathcal{P}(n)$ therefore simplifies into
\[ \mathcal{P}(n) = \sum_{f \in \mathbb{N}_{f|n}} \sum_{g \in \mathbb{N}_{g|n/f}} \prod_{p \nmid g} \left( 1 - \frac{2}{p} \right) \prod_{p \nmid g} \left( 1 - \frac{p-1}{p(p+1)} \right) = \prod_{p \nmid n} \left( 1 - \frac{p-1}{p^2(p+1)} \right), \] 
where we use the multiplicativity of $\mathcal{P}(n)$.

Next, we analyse $\mathcal{M}_2$. Writing $e = fg$, we obtain the rough bound
\[ \mathcal{M}_2 \ll M \sum_{f,g,s,v_1,v_2,t_1,t_2,m_1,m_2} \frac{\log(v_2m_1eg)}{m_1m_2v_1v_2t_1t_2sfg} \ll M \left( \sum_{d|n} \frac{1}{d} \right)^8 \sum_{d|n} \log d \ll M(\log \log n)^8 \sum_{d|n} \frac{\log d}{d}. \] 
To estimate the last sum on the right-hand side, we note that
\[ \sum_{d|n} \frac{\log d}{d} = -F'(1), \] 
where
\[ F(s) := \sum_{d|n} d^{-s} = \prod_{p|n} \left( 1 + p^{-s} + p^{-2s} + \cdots + p^{-v_p(n)s} \right), \]
where \( p^{v_p(n)} \) is the largest power of \( p \) dividing \( n \). Taking logarithmic derivative, we get
\[
\frac{F'(s)}{F(s)} = -\sum_{p|n} \frac{p^{-s} + 2p^{-2s} + \cdots + v_p(n)p^{-v_p(n)s}}{1 + p^{-s} + p^{-2s} + \cdots + p^{-v_p(n)s}} \cdot \log p.
\]

Hence,
\[
-F'(1) = F(1) \cdot \sum_{p|n} \frac{p^{-1} + 2p^{-2} + \cdots + v_p(n)p^{-v_p(n)}}{1 + p^{-1} + p^{-2} + \cdots + p^{-v_p(n)}} \cdot \log p \ll (\log \log n) \cdot \left( 1 + \sum_{p|n} \frac{\log p}{p} \right) \ll (\log \log n)^2.
\]

It follows that
\[
\mathcal{M}_2 \ll M(\log \log n)^{10}.
\]

Combining everything in this section and noting that
\[
\log(2\sqrt{X}) = \frac{\log n}{2} + O(1),
\]
we arrive at the asymptotic estimate
\[
(6.11) \quad S_{p,0,1}^{\leq \cdot} = \frac{2}{\pi} \mathcal{P}(n) \Phi(0) M \log n + O(M(\log \log n)^{10}).
\]

In view of (6.11), the \( O \)-term above is a true error term.

7. Estimations on \( I_{\mu,\nu}(w, l; \alpha_2 \beta_2) \)

Now we analyse the Fourier integral \( I_{\mu,\nu}(w, l; \alpha_2 \beta_2) \) defined in (3.14). Integration by parts \( I \) times in \( x \) gives
\[
I_{\mu,\nu}(w, l; \alpha_2 \beta_2) = \iint \Phi_{\mu,\nu}(x, y) e \left(-\frac{wx + ly}{\alpha_2 \beta_2}\right) dxdy
\]
(7.1)
and integration by parts \( J \) times in \( y \) gives
\[
I_{\mu,\nu}(w, l; \alpha_2 \beta_2) = \iint \Phi_{\mu,\nu}(x, y) e \left(-\frac{wx + ly}{\alpha_2 \beta_2}\right) dxdy
\]
(7.2)
We want to find out under which conditions on \( w, l, \alpha_2 \) the right-hand sides of the equations above become negligible if \( I \) or \( J \) are large enough, respectively. By the generalized product rule, the partial derivatives of the weight function turn out to be
\[
\frac{\partial^I}{\partial x^I} \Phi_{\mu,\nu}(x, y) = \sum_{C_1+C_2+C_3+C_4=I} \frac{I!}{C_1!C_2!C_3!C_4!} \cdot \frac{\partial^{C_1}}{\partial x^{C_1}} \Phi_1(F_1(x,y)) \frac{\partial^{C_2}}{\partial x^{C_2}} \Phi_2(F_2(x,y)) \frac{\partial^{C_3}}{\partial x^{C_3}} \Phi_3(F_3(x,y)) \frac{\partial^{C_4}}{\partial x^{C_4}} \Phi^-(F_4(x))
\]
and
\[
\frac{\partial^J}{\partial y^J} \Phi_{\mu,\nu}(x, y) = \Phi^-(F_4(x)) \cdot \sum \frac{J!}{D_1!D_2!D_3!} \cdot \frac{\partial^{D_1}}{\partial y^{D_1}} \Phi_1(F_1(x,y)) \frac{\partial^{D_2}}{\partial y^{D_2}} \Phi_2(F_2(x,y)) \frac{\partial^{D_3}}{\partial y^{D_3}} \Phi_3(F_3(x,y)).
\]
Moreover, by Faà di Bruno’s formula,
\[
\frac{\partial^{C_1}}{\partial x^{C_1}} \Phi_1(F_1(x,y)) = \sum_{E_1+2E_2+\cdots+C_1E_C_1=C_1} \frac{C_1!}{E_1!E_1!E_2!E_2! \cdots E_C!C_1!E_{C_1}} \cdot \Phi_1^{(E_1+\cdots+E_{C_1})}(F_1(x,y)) \cdot \prod_{k=1}^{C_1} \left( \frac{\partial^k}{\partial x^k} F_1(x,y) \right)^{E_k},
\]
\[
\frac{\partial^{C_4}}{\partial x^{C_4}} \Phi^-(F_4(x)) = \sum_{E_1+2E_2+\cdots+C_4E_C_4=C_4} \frac{C_4!}{E_1!E_1!E_2!E_2! \cdots E_{C_4}!C_4!E_{C_4}} \cdot \Phi^-(E_1+\cdots+E_{C_4})(F_4(x)) \cdot \prod_{k=1}^{C_4} \left( F_4^{(k)}(x) \right)^{E_k},
\]
\[
\frac{\partial^{D_1}}{\partial y^{D_1}} \Phi_1(F_1(x,y)) = \sum_{E_1+2E_2+\cdots+D_1E_{C_1}=D_1} \frac{D_1!}{E_1!E_1!E_2!E_2! \cdots E_{D_1}!D_1!E_{D_1}} \cdot \Phi_1^{(E_1+\cdots+E_{D_1})}(F_1(x,y)) \cdot \prod_{k=1}^{D_1} \left( \frac{\partial^k}{\partial y^k} F_1(x,y) \right)^{E_k},
\]
\[
\frac{\partial^{D_3}}{\partial y^{D_3}} \Phi_3(F_3(x,y)) = \sum_{E_1+2E_2+\cdots+D_3E_{C_3}=D_3} \frac{D_3!}{E_1!E_1!E_2!E_2! \cdots E_{D_3}!D_3!E_{D_3}} \cdot \Phi_3^{(E_1+\cdots+E_{D_3})}(F_3(x,y)) \cdot \prod_{k=1}^{D_3} \left( \frac{\partial^k}{\partial y^k} F_3(x,y) \right)^{E_k}.
\]
where \( i = 1, 2, 3 \). This simplifies into

\[
\frac{\partial^C_1}{\partial x^C_1} \Phi_1(F_1(x, y)) = \Phi_1^{(C_1)}(F_1(x, y)) \cdot (\mu \nu A \gamma)^{C_1},
\]

\[
\frac{\partial^C_2}{\partial x^C_2} \Phi_2(F_2(x, y)) = \Phi_2^{(C_2)}(F_2(x, y)) \cdot \left( \frac{\alpha_2 \mu B}{2} - \frac{\mu^2 D y^2}{2 g \alpha_2} \right)^{C_2},
\]

\[
\frac{\partial^C_3}{\partial x^C_3} \Phi_3(F_3(x, y)) = \Phi_3^{(C_3)}(F_3(x, y)) \cdot \left( \frac{\alpha_2 \mu B}{2} + \frac{\mu^2 D y^2}{2 g \alpha_2} \right)^{C_3},
\]

\[
\frac{\partial^C_4}{\partial x^C_4} \Phi_4^{-}(F_4(x)) = \left( \Phi_4^{-}\right)^{(C_4)}(F_4(x)) \cdot \left( \frac{F_4(x)}{x} \right)^{C_4},
\]

\[
\frac{\partial^D_1}{\partial y^D_1} \Phi_1(F_1(x, y)) = \Phi_1^{(D_1)}(F_1(x, y)) \cdot (\mu \nu A x)^{D_1},
\]

\[
\frac{\partial^D_2}{\partial y^D_2} \Phi_2(F_2(x, y)) = \sum_{E_1 + 2E_2 = D_2} \frac{D_2^1}{E_1^1E_2^2E_2^1} \cdot \Phi_2^{(E_1 + E_2)}(F_2(x, y)) \cdot \left( -\frac{\nu C + \mu^2 D y x y}{g \alpha_2} \right)^{E_1} \left( -\frac{\mu^2 D x}{g \alpha_2} \right)^{E_2},
\]

\[
\frac{\partial^D_3}{\partial y^D_3} \Phi_3(F_3(x, y)) = \sum_{E_1 + 2E_2 = D_3} \frac{D_3^1}{E_1^1E_2^2E_2^1} \cdot \Phi_3^{(E_1 + E_2)}(F_3(x, y)) \cdot \left( \frac{\nu C + \mu^2 D y x y}{g \alpha_2} \right)^{E_1} \left( \frac{\mu^2 D x}{g \alpha_2} \right)^{E_2}.
\]

To turn the above equalities into upper bounds for the partial derivatives, we need to localize the pairs \((x, y) \in \mathbb{R}^2\) for which the weight functions do not vanish. We recall the formulas

\[
x = \frac{\sigma + \tau}{\alpha_2 \mu B}
\]

and

\[
y = \frac{g \alpha_2 \left( -n + \sqrt{n^2 - \sigma^2 + \tau^2} \right)}{\nu \beta_1 s}.
\]

from section 5 and the inequalities

\[
0 \leq \sigma \leq 2X \quad \text{and} \quad 0 \leq \tau \leq M.
\]

Using the binomial formula \( x^2 - y^2 = (x - y)(x + y) \), \([4, 5]\) implies

\[
|y| \leq \frac{g \alpha_2 \left| \sigma - \tau \right|}{\nu \beta_1 s} \leq \frac{g \alpha_2 \left( \sigma + \tau \right)}{\nu \beta_1 s}.
\]

From this, we also deduce that

\[
|xy| \leq \frac{g(\sigma + \tau)^2}{n \beta_1 s B}.
\]

Since \( X \simeq n \) and \( M \leq n \), it follows that

\[
|\sigma| \leq \frac{n}{\alpha_2 B}, \quad |\tau| \leq \frac{g \alpha_2}{\beta_1 s} \quad \text{and} \quad |xy| \leq \frac{g n}{\beta_1 s B}.
\]

Now using our definitions of \( A, B, C, D \) in \([10, 11, \sigma, \theta]\), our conditions on the weight functions from section 2 and \( Y \leq M \), we obtain the bounds

\[
\frac{\partial^C_i}{\partial x^C_i} \Phi_i(F_i(x, y)) \leq C_i \left( \frac{m_1 m_2 v_1 v_2 e \gamma \alpha_2}{Y} \right)^{C_i}, \quad \text{for } i = 1, 2, 3,
\]

\[
\frac{\partial^C_4}{\partial x^C_4} \Phi_4^{-}(F_4(x)) \leq C_4 \left( \frac{m_1 m_2 v_1 v_2 e \gamma \alpha_2}{Y} \right)^{C_4},
\]

\[
\frac{\partial^D_j}{\partial y^D_j} \Phi_j(F_j(x, y)) \leq D_j \left( \frac{\beta_1 s n}{g \alpha_2 Y} \right)^{D_j}, \quad \text{for } j = 1, 2, 3.
\]

Therefore, the integral \( I_{\mu, \nu}(w, l; \alpha_2 \beta_2) \) is negligible if

\[
|w| \geq n \epsilon \cdot \frac{m_1 m_2 v_1 v_2 e \beta_1 \beta_2 \alpha_2}{Y} \quad \text{or} \quad |l| \geq n \epsilon \cdot \frac{\beta_1 \beta_2 s n}{g Y}.
\]

Otherwise, we may estimate the integral \( I_{\mu, \nu}(w, l; \alpha_2 \beta_2) \) trivially: Using \([11, 10, 14, \sigma, \theta]\), we have

\[
|I_{\mu, \nu}(w, l; \alpha_2 \beta_2)| \leq I_{\mu, \nu}(0, 0; \alpha_2 \beta_2) \leq \int_{\mathbb{R}^2} \Phi_3(F_3(x, y)) d\sigma d\tau = \frac{2\pi \delta(0) M}{\mu \nu m_1 m_2 v_1 v_2 \beta_1 s e}.
\]
To prepare calculations involving averages of Kloosterman sums, we estimate higher-order partial derivatives of \( I_{\mu,\nu}(w; l; \alpha \beta_2) \) as well. These estimates will not be needed to establish our main result, however. Therefore, we keep the calculations in the rest of this section brief. By similar considerations as above, it can be seen that

\[
\frac{\partial^{i+j+k}}{\partial w^i \partial l^j \partial \alpha_{2}^{k}} I_{\mu,\nu}(w; l; \alpha \beta_2)
\]

is negligible if \( w, l \) satisfy (7.8). We now bound these partial derivatives in the complementary case when

\[
(7.10) \quad |w| \leq n^c \cdot \frac{m_1 m_2 v_1 v_2 e g \beta_2 \alpha_2^2}{Y} \quad \text{and} \quad |l| \leq n^c \cdot \frac{\beta_1 \beta_2 s n}{gY}.
\]

Using the Leibniz rule for integration under the integral sign, we have

\[
(7.11) \quad \leq \left( \frac{2 \pi}{\beta_2} \right)^{i+j} \left| \int y^n x^{j} \Phi_1(n + \mu Axy) \frac{\partial^k}{\partial \alpha_{2}^{k}} \left( \frac{\partial^{i+j}}{\partial w^i \partial l^j} \Phi_2 \left( \frac{\mu \nu Axy}{2} - \frac{2n \nu Cy + \mu \nu^2 Dxy^2}{2g \alpha_2} \right) \right) \right|.
\]

Again using Faà di Bruno’s formula, our definitions of \( A, B, C, D \) in (3.1), (7.6), (7.10), our conditions on the weight functions from section 2 and \( Y \leq M \), we obtain the bounds

\[
\frac{\partial^{i+j}}{\partial \alpha_{2}^{i+j} \alpha_{2}^{i+j+c_1}} \leq c_1 \frac{1}{\alpha_{2}^{i+j+c_1}},
\]

\[
\frac{\partial^c_2}{\partial \alpha_{2}^{c_2}} \Phi_2 \left( \frac{\mu v_1 m_2 x}{2 \alpha_2^{c_2} v_2 m_1} \right) \leq c_2 \left( \frac{n}{\alpha_2 Y} \right)^{c_2},
\]

\[
\frac{\partial^c_3}{\partial \alpha_{2}^{c_3}} \Phi_3 \left( \frac{\mu v_1 m_2 x}{2 \alpha_2^{c_3} v_2 m_1} \right) \leq c_3 \left( \frac{n}{\alpha_2 Y} \right)^{c_3},
\]

\[
\frac{\partial^c_4}{\partial \alpha_{2}^{c_4}} \Phi_4 \left( \frac{\mu v_1 m_2 x}{2 \alpha_2^{c_4} v_2 m_1} \right) \leq c_4 \left( \frac{1}{\alpha_2} \right)^{c_4},
\]

\[
\frac{\partial^c_5}{\partial \alpha_{2}^{c_5}} e \left( \frac{wx + ly}{\alpha_2 \beta_2} \right) \leq c_5 \left( \frac{n^{1+c_5}}{\alpha_2 Y} \right)^{c_5}.
\]
Altogether, we get

\[
\sum_{c_1+c_2+c_3+c_4=0} \frac{k!}{c_1!c_2!c_3!c_4!} \frac{\partial^{c_1} 1}{\partial \alpha_1^{c_1}} \frac{\partial^{c_2}}{\partial \alpha_2^{c_2}} \Phi_3 \left( \frac{\alpha_2 \mu B x}{2} + \frac{2 n \nu C y + \mu^2 D x y^2}{2 g \alpha_2} \right) \times \\
\left( \frac{\alpha_2 \mu B x}{2} + \frac{2 n \nu C y + \mu^2 D x y^2}{2 g \alpha_2} \right)^{c_4} \Phi^- \left( \frac{\mu_1 \nu_1 m_2}{\alpha_2 \nu_2 m_1} \right) \frac{\partial^{c_4}}{\partial \alpha_2^{c_4}} \left( - \frac{w x + l y}{\alpha_2 \beta_2} \right) \right]_< \frac{1}{\alpha_2^{c_4+j}} \left( \frac{n^{1+e}}{\alpha_2 Y} \right)^k.
\]

Plugging (7.12) into the last line of (7.11), using the bound

\[
x^i y^j < \frac{n}{\alpha_2 m_1 m_2 v_1 v_2 e g} \left( \frac{\alpha_3}{\beta_1 s} \right)^j
\]

following from the definition of \( B \) in (3.1) and the bounds for \( x \) and \( y \) in (7.6), and noting the formula (cf. (7.6) and (5.8) in section 5)

\[
\text{Vol}(V) = \frac{1}{G} \int \frac{1}{\sqrt{n^2 - \sigma^2 + \tau^2}} d\sigma d\tau = \frac{2 \pi M}{\mu m_1 m_2 v_1 v_2 \beta_1 s e}
\]

we obtain

\[
\frac{\partial^{i+j+k}}{\partial \omega^i \partial \nu^j \partial \alpha_2^k} I_{\mu,\nu}(w, l; \alpha_2 \beta_2) < \frac{M}{m_1 m_2 v_1 v_2 \beta_1 s e} \cdot \left( \frac{n}{\alpha_2 \beta_2 m_1 m_2 v_1 v_2 e g} \right)^i \left( \frac{g}{\beta_1 s} \right)^j \left( \frac{n^{1+e}}{\alpha_2 Y} \right)^k
\]

\[
\ll \frac{M}{m_1 m_2 v_1 v_2 \beta_1 s e} \cdot \left( \frac{n^{1+e}}{Y} \right)^{i+j+k} \cdot \frac{1}{|w|^{|l|} |\alpha_2|^k},
\]

where we use (7.10) for the last line. This indicates that \( I_{\mu,\nu}(w, l; \alpha_2 \beta_2) \) is (weakly) oscillating.

8. An improved bound for \( I_{\mu,\nu}(0, l; \alpha_2 \beta_2) \)

In this section, we supply a particular bound for \( I_{\mu,\nu}(0, l; \alpha_2 \beta_2) \) which will be used in the next section where we handle the contribution of \( T_{\mu,1}^0 \) to \( S_{1}^0 \).

Using (7.2) with \( J = 1 \) and the last bound in (7.1) with \( D_j = 1 \), we obtain

\[
I_{\mu,\nu}(0, l; \alpha_2 \beta_2) < \frac{n}{Y} \cdot \frac{\beta_1 \beta_2 s}{|l| g} \cdot \frac{M}{m_1 m_2 v_1 v_2 \beta_1 s e}.
\]

Our goal is to replace the factor \( n/Y \) on the right-hand side by \( \log n \), i.e., to establish the sharper bound

\[
I_{\mu,\nu}(0, l; \alpha_2 \beta_2) < (\log n) \cdot \frac{\beta_1 \beta_2 s}{|l| g} \cdot \frac{M}{m_1 m_2 v_1 v_2 \beta_1 s e}.
\]

Changing variables as in section 5 we obtain

\[
I_{\mu,\nu}(0, l; \alpha_2 \beta_2) = \frac{1}{G} \int_M^{2M} \int_{Y/2}^{Y/2} \Phi_1 \left( \sqrt{n^2 - \sigma^2 + \tau^2} \right) \Phi_2(\sigma) \Phi_3(\tau) \Phi^- \left( \frac{\sigma + \tau}{\nu m_2 v_1 v_2 e g} \right) \cdot e(f(\sigma, \tau)) d\sigma d\tau,
\]

where

\[
f(\sigma, \tau) := \frac{g (\nu n + \sqrt{n^2 - \sigma^2 + \tau^2}) l}{\nu \beta_1 \beta_2 s (\sigma + \tau)}.
\]

Using integration by parts in \( \sigma \), it follows that

\[
I_{\mu,\nu}(0, l; \alpha_2 \beta_2) = \frac{1}{G} \int_M^{2M} \int_{Y/2}^{Y/2} \frac{\partial}{\partial \sigma} \left( \Phi_1 \left( \sqrt{n^2 - \sigma^2 + \tau^2} \right) \Phi_2(\sigma) \Phi_3(\tau) \Phi^- \left( \frac{\sigma + \tau}{\nu m_2 v_1 v_2 e g} \right) \right) \cdot e(f(\sigma, \tau)) \frac{d\sigma d\tau}{2\pi i \frac{\partial}{\partial \sigma} f(\sigma, \tau)}.
\]

A short calculation reveals that

\[
\frac{1}{|\frac{\partial}{\partial \sigma} f(\sigma, \tau)|} = \frac{\beta_1 \beta_2 s}{|l| g} \cdot \sqrt{n^2 - \sigma^2 + \tau^2} \cdot \left( 1 + \frac{\sigma \sqrt{n^2 - \sigma^2 + \tau^2} + \tau n}{\sigma n + \tau \sqrt{n^2 - \sigma^2 + \tau^2}} \right) \ll \frac{\beta_1 \beta_2 s}{|l| g} \cdot \sqrt{n^2 - \sigma^2 + \tau^2}.
\]
Further,

\[
\frac{\partial}{\partial \sigma} \left( \Phi_1 \left( \sqrt{n^2 - \sigma^2 + \tau^2} \right) \Phi_2(\sigma) \Phi_3(\tau) \Phi^-(\frac{\sigma + \tau}{\alpha_2 m_1^2 v_2^2 e g}) \right) \bigg|_{n^2 - \sigma^2 + \tau^2 = 0} = \frac{\partial}{\partial \sigma} \left( \Phi_1 \left( \sqrt{n^2 - \sigma^2 + \tau^2} \right) \Phi_2(\sigma) \Phi_3(\tau) \Phi^-(\frac{\sigma + \tau}{\alpha_2 m_1^2 v_2^2 e g}) \right) + \sigma \cdot \Phi_1 \left( \sqrt{n^2 - \sigma^2 + \tau^2} \right) \Phi_2(\sigma) \Phi_3(\tau) \Phi^-(\frac{\sigma + \tau}{\alpha_2 m_1^2 v_2^2 e g}).
\]

Combining the above relations yields the bound

\[
I_{\mu, \nu}(0, l; \alpha_2 \beta_2) \ll \frac{\beta_1 \beta_2 s}{|l| g} \cdot \frac{1}{G} \left( \int_M^{2M} \int_{Y/2}^{2X} \left| \frac{\partial}{\partial \sigma} \left( \Phi_1 \left( \sqrt{n^2 - \sigma^2 + \tau^2} \right) \Phi_2(\sigma) \Phi_3(\tau) \Phi^-(\frac{\sigma + \tau}{\alpha_2 m_1^2 v_2^2 e g}) \right) \right| d\sigma d\tau \right) + \int \int_{U} \frac{\sigma}{n^2 - \sigma^2 + \tau^2} d\sigma d\tau \ll M.
\]

The monotonicity properties of \( \Phi_1, \Phi_2 \) and \( \Phi^- \) assumed in section 2 together with the fundamental theorem of calculus imply

\[
\int \int_U \frac{\sigma}{n^2 - \sigma^2 + \tau^2} d\sigma d\tau \ll M \log n.
\]

The claimed bound \( \text{(8.1)} \) follows.

9. Error Contribution of \( T_{\mu, \nu}^{0,1} \)

Now we deal with the contribution of the term \( T_{\mu, \nu}^{0,1} \) in \( S_1^{-} \) to \( S_1^{-} \). Taking our restriction \( (5.4) \) into account, this contribution equals

\[
S_{1,0,1}^{-} = \sum_{\mu=1}^{2} \sum_{\nu=1}^{2} (-1)^{\mu+\nu} \sum_{e \in \mathbb{N}} \sum_{g \in \mathbb{N}} \sum_{s \in \mathbb{N}} \sum_{v_1, v_2, \ldots, v_8 \in \mathbb{N}} \sum_{n \in \mathbb{N}} \sum_{\beta_1, \beta_2, \ldots, \beta_8} \sum_{m_1, m_2 \in \mathbb{N}} \mu(m_1 \beta_1 v_2 s) \mu(m_2 \beta_2 v_1 s) \times
\]

\[
\sum_{\alpha_2 \in \mathbb{N}} \varphi(\alpha_2 \beta_2) \left( \frac{\alpha_2 \beta_2}{\varphi(\alpha_2 \beta_2)} \right) \sum_{\ell \in \mathbb{Z} \setminus \{0\}} \mu(\alpha_2 \beta_2 / (\ell E, \alpha_2 \beta_2)) \varphi(\alpha_2 \beta_2 / (\ell E, \alpha_2 \beta_2)) \cdot I_{\mu, \nu}(0, l; \alpha_2 \beta_2),
\]

where we recall that

\[
E = \frac{2n}{m_1 m_2 v_1 v_2 s e}.
\]

We will obtain cancellation in the \( l \)-summation by applying Poisson summation backward. To this end, we write

\[
\tilde{E} := (E, \alpha_2 \beta_2)
\]
and
\[ \sum_{l \in \mathbb{Z} \setminus \{0\}} \mu(\alpha_2^2 / (lE, \alpha_2^2)) \cdot I_{\mu, \nu}(0, l; \alpha_2^2) = \sum_{t | \alpha_2^2 / E} \mu(\alpha_2^2 / (t \bar{E})) \sum_{l \in \mathbb{Z} \setminus \{0\}} \phi(\alpha_2^2 / (t \bar{E})) \cdot I_{\mu, \nu}(0, l; \alpha_2^2) \]
\[ = \sum_{t | \alpha_2^2 / E} \mu(\alpha_2^2 / (t \bar{E})) \sum_{l' \in \mathbb{Z} \setminus \{0\}} \phi(\alpha_2^2 / (t \bar{E})) \cdot I_{\mu, \nu}(0, l' t; \alpha_2^2) \]
\[ = \sum_{t | \alpha_2^2 / E} \mu(\alpha_2^2 / (t \bar{E})) \sum_{l' \in \mathbb{Z} \setminus \{0\}} \mu(u) \cdot I_{\mu, \nu}(0, l'' t u; \alpha_2^2). \]

Recalling the definition of the Fourier integral in (3.4), we obtain
\[ \sum_{l' \in \mathbb{Z} \setminus \{0\}} I_{\mu, \nu}(0, l'' t u; \alpha_2^2) = \sum_{l' \in \mathbb{Z}} I_{\mu, \nu}(0, l'' t u; \alpha_2^2) - I_{\mu, \nu}(0, 0; \alpha_2^2) \]
\[ = \sum_{l' \in \mathbb{Z}} \int \int_{\mathbb{R}^2} \Phi_{\mu, \nu}(x, y) e\left(-\frac{l'' t u y}{\alpha_2^2}\right) d x d y - I_{\mu, \nu}(0, 0; \alpha_2^2). \]

Let
\[ \omega(y) := \int_{\mathbb{R}} \Phi_{\mu, \nu}(x, y) d x. \]

Then
\[ \int \int_{\mathbb{R}^2} \Phi_{\mu, \nu}(x, y) e\left(-\frac{l'' t u y}{\alpha_2^2}\right) d x d y = \int_{\mathbb{R}} \Phi_{\mu, \nu}(x, y) \left(\frac{l'' t u y}{\alpha_2^2}\right) d x. \]

the Fourier transform. Applying Poisson summation backward, we get
\[ \sum_{l' \in \mathbb{Z}} \int_{\mathbb{R}} \Phi_{\mu, \nu}(x, y) e\left(-\frac{l'' t u y}{\alpha_2^2}\right) d x d y = \sum_{l' \in \mathbb{Z}} \int_{\mathbb{R}} \Phi_{\mu, \nu}(x, y) e\left(\frac{k \alpha_2^2}{t u}\right) d x d y = \int_{\mathbb{R}} \Phi_{\mu, \nu}(x, \frac{k \alpha_2^2}{t u}) d x. \]

Combining this with (9.6), we obtain
\[ \sum_{l' \in \mathbb{Z} \setminus \{0\}} I_{\mu, \nu}(0, l'' t u; \alpha_2^2) = \frac{\alpha_2^2}{t u} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \Phi_{\mu, \nu}(x, \frac{k \alpha_2^2}{t u}) d x. \]

Due to our constraint \( M \leq F_\beta(x, y) \leq 2M \), for every given \( k \), the \( x \)-integral on the right-hand side of (9.6) ranges over an interval of length
\[ \ll \frac{M}{\alpha_2 B}. \]

(Outside this interval, the integrand vanishes.) Moreover, in view of the inequality for \( |y| \) in (7.6), the length of the relevant \( k \)-interval on the right-hand side of (9.7) is
\[ \ll \frac{\alpha_2}{\beta_1 s} \cdot \frac{t u}{\alpha_2^2} = \frac{t u q}{\beta_1 \beta_2 s}, \]

and hence the number of \( k \)'s which contribute is
\[ \ll \frac{t u q}{\beta_1 \beta_2 s} + 1. \]

Using the definition of \( B \) in (3.1), the product of the two quantities in (8.7) and (8.8) equals
\[ \frac{M}{\alpha_2 B} \cdot \left(\frac{t u q}{\beta_1 \beta_2 s} + 1\right) = \frac{M}{m_1 m_2 v_1 v_2 \beta_1 s} \cdot \frac{t u}{\alpha_2^2} + \frac{M}{m_1 m_2 v_1 v_2 e \alpha_2}. \]

From this, (7.8) and (9.6), we obtain the bound
\[ \sum_{l' \in \mathbb{Z} \setminus \{0\}} I_{\mu, \nu}(0, l'' t u; \alpha_2^2) \ll \frac{M}{m_1 m_2 v_1 v_2 \beta_1 s} + \frac{M \beta_2}{m_1 m_2 v_1 v_2 e \alpha_2} = \frac{M}{m_1 m_2 v_1 v_2 \beta_1 s} \left(1 + \frac{\beta_1 s}{q t u}\right). \]
Plugging \((9.9)\) into the last line of \((9.4)\), we see that

\[
\sum_{t \in \mathbb{Z} \setminus \{0\}} \frac{\mu(\alpha_2 \beta_2/(lE, \alpha_2 \beta_2))}{\varphi(\alpha_2 \beta_2/(lE, \alpha_2 \beta_2))} \cdot I_{t, \nu}(0, l; \alpha_2 \beta_2) \leq M \frac{\beta_1 \beta_2 s}{g} \cdot (\log n)^2 \cdot \sum_{t | \alpha_2 \beta_2/E \left( \alpha_2 \beta_2/(tE) \right)} \frac{1}{t \varphi(\alpha_2 \beta_2/(tE))}
\]

\[(9.10)\]

where we make use of the fact that \(\beta_1 s/g \leq 1\) since \(\beta_1|g/s\). Here we recall the notations

\[d(m) = \sum_{a|m} 1 \quad \text{and} \quad \sigma_{-1}(m) = \sum_{a|m} \frac{1}{a},\]

The above bound \((9.10)\) will be useful for small \(\alpha_2\). For large \(\alpha_2\), we estimate the above quantity in the different way: Using \((5.1)\) and being mindful of the restriction on \(l\) in \((7.10)\), we deduce that

\[
\sum_{t \in \mathbb{Z} \setminus \{0\}} \frac{\mu(\alpha_2 \beta_2/(lE, \alpha_2 \beta_2))}{\varphi(\alpha_2 \beta_2/(lE, \alpha_2 \beta_2))} \cdot I_{t, \nu}(0, l; \alpha_2 \beta_2) \leq M \frac{\beta_1 \beta_2 s}{g} \cdot (\log n)^2 \sum_{t | \alpha_2 \beta_2/E \left( \alpha_2 \beta_2/(tE) \right)} \frac{1}{t \varphi(\alpha_2 \beta_2/(tE))}
\]

\[(9.11)\]

again making use of \(\beta_1 s/g \leq 1\).

In the following, we will usually use the bounds

\[\sum_{r|m} \frac{1}{r} \leq \log \log m, \quad \sum_{r|m} \frac{d(r)}{r} \leq \left( \sum_{r|m} \frac{1}{m} \right)^2 \ll (\log \log m)^2, \quad \sum_{r|m} \frac{\sigma_3(r)}{r} \leq \left( \sum_{r|m} \frac{1}{m} \right)^3 \ll (\log \log m)^3.\]

Let \(T > 0\) be a parameter, to be optimized later. Recall \((9.1)\) and \((9.3)\). Employing \((9.10)\), the contribution of \(\alpha_2 \leq T\) to \(S_{1,0,1}^{T}\) is bounded by

\[
\ll M \sum_{e, s, v_1, \nu, m_1, m_2, \beta_1 | n} \sum_{e_1 | g} \frac{1}{m_1 m_2 v_1 v_2 \beta_1} \sum_{E_1, E_2} \frac{\beta_1}{\alpha_2} \sum_{\alpha_2 \in \mathbb{N}} \left( \frac{d(\alpha_2 \beta_2)}{\alpha_2} + \frac{E_1 E_2}{\alpha_2^2} \right) (\log \log n)^2
\]

\[
\ll M (\log \log n)^9 \sum_{g | n} \left( \left( \sum_{E_1, E_2} \frac{\beta_1}{\alpha_2} \sum_{\alpha_2 \in \mathbb{N}} \frac{d(\alpha_2 \beta_2)}{\alpha_2} \right) \left( \sum_{E_1, E_2} \frac{E_1}{\alpha_2} \sum_{\alpha_2 \in \mathbb{N}} \frac{E_2}{\alpha_2} \right) \left( \sum_{E_1, E_2} \frac{E_1 E_2}{\alpha_2^2} \right) \right),
\]

where we have written \(e = fg\) and proceeded similarly as in section \(9\) to obtain the factor \((\log \log n)^9\). For the above sums involving \(\alpha_2\) and \(\beta_2\), we have the estimates

\[
\sum_{E_1 | g} \frac{d(\beta_2 \beta_1)}{g} \ll \frac{(\log \log n)^4}{g}, \quad \sum_{E_2 | n} \frac{d(\alpha_2)}{\alpha_2} \ll (\log T)^2 (\log \log n)^2,
\]

\[
\sum_{E_1 | g} \frac{E_1}{g} \ll \frac{(g)(\log \log n)}{g}, \quad \sum_{E_2 | n} \frac{E_2}{\alpha_2} \ll \log \log n.
\]
So altogether, the contribution of $\alpha_2 \leq T$ to $S_{1,0,1}^{\tau}$ is dominated by
\begin{equation}
(9.12) \quad \ll (\log T)^2 M (\log \log n)^{16}. \nonumber
\end{equation}
Employing (9.11), the contribution of $\alpha_2 > T$ to $S_{1,0,1}^{\tau}$ is bounded by
\begin{equation}
\ll M (\log n)^2 (\log \log n)^2 \sum_{g \in \mathcal{E}} \frac{1}{m_1 m_2 v_1 v_2 (\beta_1 \beta_2)} \sum_{E_1[g] \beta \in \mathcal{E}} \sum_{E_2[n] \beta_2} \sum_{\beta_2 \beta_2} \frac{1}{\alpha_2} \sum_{\alpha_2 \alpha_2 > T} \frac{d(\alpha_2 \beta_2)}{\alpha_2} E_1 E_2 \nonumber
\end{equation}
\begin{equation}
\ll M (\log n)^2 (\log \log n)^9 \sum_{g \in \mathcal{E}} \left( \sum_{E_1[g] \beta \in \mathcal{E}} \frac{E_1 d(\beta_2)}{g \beta_2} \right) \left( \sum_{E_2[n] \beta_2 \in \mathcal{E}} \frac{E_2 d(\alpha_2)}{\alpha_2^2} \right), \nonumber
\end{equation}
where we have again written $e = fg$ and proceeded similarly as in section 6 to obtain the factor $(\log \log n)^9$. For the above sums involving $\alpha_2$ and $\beta_2$, we have the estimates
\begin{equation}
\sum_{E_1[g] \beta \in \mathcal{E}} E_1 d(\beta_2) \ll \frac{d_3(g) (\log \log n)^2}{g}, \quad \sum_{E_2[n] \beta_2 \in \mathcal{E}} E_2 d(\alpha_2) \ll \frac{(\log T) (\log \log n)^2}{T}. \nonumber
\end{equation}
So altogether, the contribution of $\alpha_2 > T$ to $S_{1,0,1}^{\tau}$ is dominated by
\begin{equation}
(9.13) \quad \ll \frac{M (\log n)^2 (\log T) (\log \log n)^{16}}{T}. \nonumber
\end{equation}
Now we choose $T := (\log n)^2$ to balance the above bounds (9.12) and (9.13) and obtain the final estimate
\begin{equation}
(9.14) \quad S_{1,0,1}^{\tau} \ll M (\log \log n)^{18}. \nonumber
\end{equation}

10. Error contribution of $T_{\mu, \nu}^{1,0}$

Next we deal with the contribution of the term $T_{\mu, \nu}^{1,0}$ in (4.4) to $S_{1}^{\tau}$. Taking our restriction (5.4) into account, this contribution equals
\begin{equation}
S_{1,1,0}^{\tau} = \sum_{\mu=1}^{2} \sum_{\nu=1}^{2} (-1)^{\mu + \nu} \sum_{e \in \mathcal{E}} \sum_{g \in \mathcal{G}} \sum_{s \in \mathcal{S}} \sum_{l \in \mathcal{L}} \sum_{v_1, v_2 \in \mathcal{V}} \sum_{m_1, m_2 \in \mathcal{M}} \mu(m_1 b_1 s_1) \mu(m_2 b_2 s_2) \times \sum_{\alpha_2 \in \mathcal{A}} \sum_{\alpha_2 \beta_2 \in \mathcal{A}} \varphi(\alpha_2 \beta_2) \varphi(\alpha_2 \beta_2) I_{\mu, \nu}(w, 0; \alpha_2 \beta_2). \nonumber
\end{equation}
Using the inequality
\begin{equation}
\frac{1}{\varphi(\alpha_2 \beta_2)} \leq \frac{(w, \alpha_2 \beta_2)}{\varphi(\alpha_2 \beta_2)}, \nonumber
\end{equation}
it follows that
\begin{equation}
S_{1,1,0}^{\tau} \ll \sum_{e \in \mathcal{E}} \sum_{g \in \mathcal{G}} \sum_{s \in \mathcal{S}} \sum_{l \in \mathcal{L}} \sum_{v_1, v_2 \in \mathcal{V}} \sum_{m_1, m_2 \in \mathcal{M}} \frac{1}{\alpha_2 \beta_2} \sum_{\alpha_2 \in \mathcal{A}} \frac{1}{\alpha_2^2} \sum_{w \in \mathcal{W} \setminus \{0\}} (w, \alpha_2 \beta_2) I_{\mu, \nu}(w, 0; \alpha_2 \beta_2). \nonumber
\end{equation}
We recall from section 7 that the integral $I_{\mu, \nu}(w, 0; \alpha_2 \beta_2)$ is negligible if $w$ satisfies the relevant inequality in (7.8). Otherwise, we use (7.9) to estimate it and again write $e = fg$ to get
\begin{equation}
S_{1,1,0}^{\tau} \ll M \sum_{f, g \in \mathcal{F}} \sum_{s, v_1, v_2, \beta_1, \beta_2, m_1, m_2 \in \mathcal{M}} \frac{1}{m_1 m_2 v_1 v_2 (\beta_1 \beta_2)} \sum_{\beta_2 \beta_2} \frac{1}{\alpha_2 \beta_2} \sum_{\alpha_2 \in \mathcal{A}} \frac{1}{\alpha_2^2} \times \sum_{0 < |w| \leq n^2 m_1 m_2 v_1 v_2 f g^2 \beta_2 \beta_2 / Y} (w, \alpha_2 \beta_2) + O(n^{-2022}). \nonumber
\end{equation}
This implies
\[ S_{1,1,0}^{t,-} \ll \frac{M n^{1/2+2\varepsilon}}{Y}, \]
where we have used the well-known bound
\[ \sum_{0 < r \leq R} (N, r) \ll R N^{\varepsilon} \]
which is valid for all \( R > 0 \) and \( N \in \mathbb{N} \). The above bound \((10.1)\) beats \( S_{1,1,0}^{t,-} \ll M \) if
\[ Y \geq n^{1/2+2\varepsilon}. \]

11. Error contribution of \( T_{\mu,\nu}^{1,1} \) via Weil bound for Kloosterman sums

We are left with the contribution of the term \( T_{\mu,\nu}^{1,1} \) in \((1.5)\) to \( S_{1}^{t,-} \). Again taking our restriction \((5.4)\) into account, this contribution equals
\[
S_{1,1,1}^{t,-} = \sum_{\mu=1}^{2} \sum_{\nu=1}^{2} (-1)^{\mu+\nu} \sum_{e \in \mathbb{N}} \sum_{g \in \mathbb{N}} \sum_{s \in \mathbb{N}} \sum_{v_1, v_2, v_3 \in \mathbb{N}} \sum_{(v_1, v_2, v_3) = 1} \sum_{\beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{N}} \mu(m_1 \beta_1 v_2 s) \mu(m_2 \beta_2 v_1 s) \times
\]
\[
\sum_{w \in \mathbb{Z} \setminus \{0\}} \sum_{l \in \mathbb{Z} \setminus \{0\}} S(w, -lE \mu v_2 s; \alpha_2 \beta_2) I_{\mu,\nu}(w, l; \alpha_2 \beta_2),
\]
where we recall \((7.2)\) again. The easiest way to estimate the above is to employ the Weil bound
\[ S(a, b; c) \ll (a, b, c)^{1/2+\varepsilon} \]
for Kloosterman sums and then add trivially. Using this bound in conjunction with \((7.9)\) and our restrictions in \((7.10)\), we obtain
\[
S_{1,1,1}^{t,-} \ll M \frac{1}{m_1 m_2 v_1 v_2 s e | n} \sum_{\beta_1 \beta_2} \frac{1}{\beta_2 | g} \sum_{\alpha_2 \in \mathbb{N}} \frac{1}{\alpha_2^{3/2-\varepsilon}} \times
\]
\[
\sum_{0 < |w| \leq n^{r} m_1 m_2 v_1 v_2 s g \beta_2 \alpha_2^2 \leq Y} \sum_{0 < |l| \leq \beta_1 \beta_2 s n^{1+r} / (g Y)} 1 + O \left(n^{-2022}\right).
\]
This implies
\[ S_{1,1,1}^{t,-} \ll M n^{7/4+10\varepsilon} / Y^2, \]
where we have used \((10.2)\) again. The above bound \((11.2)\) beats \( S_{1,1,1}^{t,-} \ll M \) if
\[ Y \geq n^{7/8+5\varepsilon}. \]

12. Error contribution of \( T_{\mu,\nu}^{1,1} \) via averages of Kloosterman sums

It is interesting to see how far we can get by utilizing the averaging over \( w, l, \alpha_2 \) in estimating the error term \( S_{1,1,1}^{t,-} \) involving Kloosterman sums. Unfortunately, it will turn out that we are not able to beat the bound \((11.2)\) obtained in the previous section. One difficulty is that the parameter \( E \) can be as large as \( 2n \), the other is the occurrence of the factor
\[
\left(\frac{n^{1+\varepsilon}}{Y}ight)^{i+j+k}
\]
on the right-hand side of \((7.13)\). To exploit the said averaging, a most suitable tool appears to be \([5, \text{Lemma } 9]\) which is a generalization of \([11, \text{Theorem } 9]\). However, we need to make a number of adjustments. In fact, we shall use the following slight generalization of \([5, \text{Lemma } 9]\).
Lemma 1. Let r, s and d be positive pairwise coprime integers with r and s square-free. Let M, N and C be positive numbers, \( U \geq 1 \) and \( g \) a real-valued infinitely differentiable function supported on \([M, 2M] \times [N, 2N] \times [C, 2C] \) such that

\[
\partial^{j+k+l} \frac{g}{\partial m^j \partial n^k \partial c^l} \leq U^{j+k+l} M^{-j} N^{-k} C^{-l} \quad \text{for } 0 \leq j, k, l \leq 2.
\]

Let \( X_d := \sqrt{dMN/(sC^r)} \). Then for \( \eta > 0 \) and any complex sequences \( a_m \) and \( b_n \) one has

\[
\sum_m a_m \sum_n b_n \sum_{(c,r)=1} g(m, n, c) S(dmn; \pm n, sc) \leq \left( U^{7/2} sC + U^3 C^n d^7 / 64 sC \sqrt{\frac{1 + X_d^{-1} X^7/32}{1 + X_d}} \right) \left( 1 + X_d + \sqrt{\frac{M}{rs}} \left( 1 + X_d + \sqrt{\frac{N}{rs}} \right) \right) \times \left( \sum_m \left| a_m \right|^2 \right)^{1/2} \left( \sum_n \left| b_n \right|^2 \right)^{1/2}.
\]

Proof. The case \( U = 1 \) is exactly [5 Lemma 9]. The case \( d = 1 \) of [5 Lemma 9], in turn, is the content of [1 Theorem 9]. First, we assume that \( d = 1 \). To understand which change the introduction of the factor \( U^{j+k+l} \) causes, we need to go to [1 page 269 in section 7] where the proof of [1 Theorem 13] starts. This theorem implies [1 Theorem 9].

Here the authors reduce general weight functions \( g(m, n, c) \) to special weight functions of the form

\[
g(m, n, c) = f \left( \frac{4 \pi \sqrt{mn}}{c} \right),
\]

where \( f \) is a \( C^2 \) class function satisfying

\[
supp(f) \subseteq [X, 8X], \quad \|f\|_\infty \leq 1, \quad \|f'\|_1 \leq 2, \quad \|f''\|_1 \leq 6MX^{-1}
\]

(see [1 page 234]). In their setting, the function \( g(m, n, c) \) satisfies

\[
\partial^{j+k+l} \frac{g}{\partial m^j \partial n^k \partial c^l} \leq M^{-j} N^{-k} C^{-l} \quad \text{for } 0 \leq j, k, l \leq 2.
\]

Their reduction leads to a function \( f \) satisfying the conditions in (12.3) with \( X = 2\pi \sqrt{MN/C} \) and \( M = 1 \). Now they apply [1 Theorem 8] which is formulated for these special weight functions to derive [1 Theorem 13].

To be precise, they define

\[
G(t_1, t_2; x) := \int \int g \left( x_1, x_2, \frac{4 \pi \sqrt{x_1 x_2}}{x} \right) e(-t_1 x_1 - t_2 x_2) dx_1 dx_2
\]

which implies

\[
g \left( x_1, x_2, \frac{4 \pi \sqrt{x_1 x_2}}{x} \right) = \int \int G(t_1, t_2; x) e(t_1 x_1 + t_2 x_2) dt_1 dt_2
\]

by the Fourier inversion formula. If \( t_1, t_2 \neq 0 \), they integrate \( G(t_1, t_2; x) \) by parts \( p \) times with respect to \( x_1 \) and \( p_2 \) times with respect to \( x_2 \) and then differentiate \( p \) times with respect to \( x \), getting

\[
\frac{\partial^p}{\partial x^p} G(t_1, t_2; x) = (2\pi it_1)^{-p_1} (2\pi it_2)^{-p_2} \int \int \left( \frac{\partial^{p_1+p_2+p}}{\partial x_1^{p_1} \partial x_2^{p_2} \partial x^p} g \left( x_1, x_2, \frac{4 \pi \sqrt{x_1 x_2}}{x} \right) \right) \cdot e(-t_1 x_1 - t_2 x_2) dx_1 dx_2.
\]

Under the condition in (12.3), this implies

\[
\frac{\partial^p}{\partial x^p} G(t_1, t_2; x) \ll (t_1 M)^{-p_1} (t_2 M)^{-p_2} (\sqrt{MN/C})^{-p} MN.
\]

Now if \( \epsilon > 0 \) is a small enough absolute constant, the function

\[
f(x) = \epsilon \cdot (t_1 M)^{p_1} (t_2 N)^{p_2} (MN)^{-1} G(t_1, t_2; x)
\]

indeed satisfies the conditions in (12.3) with \( X = 2\pi \sqrt{MN/C} \) and \( M = 1 \). As already said, the authors continue by applying [1 Theorem 8] on averages of Kloosterman sums, which is formulated for the said special weight functions. Then they integrate the resulting bound over \( t_1 \) and \( t_2 \). In their setting, assuming the condition (12.3), they take \( p_1 = 0 \) if \( |t_1| \leq 1/M \), \( p_2 = 0 \) if \( |t_2| \leq 1/N \) and \( p_1 = p_2 = 2 \) otherwise. This proves [1 Theorem 13] from which [1 Theorem 9] follows.

We need to adjust their procedure to weight functions \( g(m, n, c) \) which satisfy our more general condition (12.1). Under this condition, we get

\[
\frac{\partial^p}{\partial x^p} G(t_1, t_2; x) \ll U^{p_1+p_2+p} (t_1 M)^{-p_1} (t_2 M)^{-p_2} (\sqrt{MN/C})^{-p} MN.
\]
Now we define
\[ f(x) = \epsilon \cdot U^{-(p_1 + p_2 + 1)}(t_1 M)^{p_1} (t_2 M)^{p_2} (MN)^{-1} G(t_1, t_2; x). \]
Then \( f \) satisfies the conditions in (12.23) with the parameters \( X = 2\pi \sqrt{MN/C} \) and \( M = U \). Due to the change of the parameter \( M \), the application of [1] Theorem 8 now creates an additional term of size
\[
\sqrt{M} \left( \sum_m |a_m|^2 \right)^{1/2} \left( \sum_n |b_n|^2 \right)^{1/2} sC.
\]
(Here we need to scale by a factor of \( sC \).) As in [1], we integrate the bound resulting from [1] Theorem 8 over \( t_1 \) and \( t_2 \), where we now take \( p_1 = 0 \) if \( |t_1| \leq U/M \), \( p_2 = 0 \) if \( |t_2| \leq U/N \) and \( p_1 = p_2 = 2 \) otherwise. This creates an extra factor of \( U^2 \). After multiplying by this factor, we obtain the result in our Lemma 9 for \( d = 1 \). (Note that we also have to multiply the term in (12.23) by \( U^2 \).) The extension to general \( d \) is then achieved along similar lines as in the proof of [5] Lemma 9.

To apply Lemma 9 to our inner triple sum
\[
\sum_{\alpha_2 \in \mathbb{N}} \sum_{\alpha_2 \leq 2\sqrt{X/(2m_1 \epsilon g)}} \sum_{\alpha_2 \leq 2\sqrt{X/(2m_1 \epsilon g)}} S(w, -lE\mu_1\alpha_2) I_{\mu, \nu}(w, l; \alpha_2)
\]
in (11.1), we need to check the conditions in this lemma and make further small adjustments. First, we re-write our Kloosterman sum in the form \( S(wE\mu_1\beta_1, -l; \alpha_2\beta_2) \). Now we require that the parameters \( E\mu, \nu \beta_1 \) and \( \beta_2 \) are pairwise coprime and square-free. Recall that if \( \mu = 1, 2 \) and the definition of \( E \) in (12.2). For \( E \) to be square-free, we need to make an extra assumption, namely that \( n \) is square-free. Further, \( \beta_1 \) and \( \beta_2 \) may be assumed to be square-free because they appear inside a Möbius function. Since \( \mu = 1, 2 \) and \( \beta_1 \) is odd, \( \nu \beta_1 \) is square-free as well. Further, \( (\nu \beta_1, \beta_2) = 1 \) by our summation conditions on \( \beta_1 \) and \( \beta_2 \). Finally, since \( n \) is square-free and we have \( \beta_1, \beta_2 | e \) and \( e | n \), it follows that \( (n/e, \beta_1) = 1 = (n/e, \beta_2) \) and hence \( (E\mu, \beta_1) = 1 = (E\mu, \beta_2) \). Therefore, the only problematic case occurs when \( \mu = 1 \) and \( \nu = 2 \) in which case we have \( (E\mu, \nu \beta_1) = 2 \). However, in this case we can replace the triple of parameters by \( E\mu, \beta_1 \) and \( \beta_2 \). So all requirements are fulfilled in each case.

Henceforth, for simplicity, we deal only with the case \( \mu = 1 = \nu \), all other cases being similar. Also, we consider only the contribution of positive \( w \) and \( l \). The contributions of the cases i) \( w > 0 \), \( l < 0 \), ii) \( w < 0 \), \( l > 0 \), iii) \( w < 0 \), \( l < 0 \) can be dealt with in a similar way. Next, to apply Lemma 9 we need to use a smooth partition of unity to divide the sum in question into \( O(n^\epsilon) \) subsums of the form
\[
(12.6) \quad \Sigma(W, L, Q) = \sum_w \sum_l \sum_{\alpha_2} \sum_{(2\beta_1) = 1} \frac{1}{(\alpha_2 \beta_2)^2} S(wE\mu_1, -l; \alpha_2 \beta_2) \psi(w, l, \alpha_2) I_{1,1}(w, l; \alpha_2 \beta_2),
\]
where \( \psi(w, l, \alpha_2) \) is supported in \([W, 2W] \times [L, 2L] \times [Q, 2Q]\) with
\[
1/2 \leq W \ll n^\epsilon m_2 l_2 g_2 e_{g_2} e^2, \quad 1/2 \leq L \ll \beta_4 \beta_2 g_2 m_1 + e, \quad 1/2 \leq Q \ll \sqrt{X}/v_2 m_1 \epsilon g
\]
and satisfies
\[
\frac{1}{\alpha_2} \frac{\partial^{i+j+k}}{\partial w^i \partial \beta_1^j \partial \alpha_2^k} \psi(w, l, \alpha_2) \ll_{i, j, k} W^{-i} L^{-j} Q^{-k}.
\]

A small issue is the occurrence of the summation condition \((2, 2) = 1\) on the right-hand side of (12.6). This can be rectified by writing
\[
\Sigma(W, L, Q) = \sum_w \sum_l \sum_{\alpha_2} \sum_{(2\beta_1) = 1} \frac{1}{(\alpha_2 \beta_2)^2} S(wE\mu_1, -l; \alpha_2 \beta_2) \psi(w, l, \alpha_2) I_{1,1}(w, l; \alpha_2 \beta_2) - \sum_l \sum_w \sum_{\alpha_2} \sum_{(2\beta_1) = 1} \frac{1}{(\alpha_2 \beta_2)^2} S(wE\mu_1, -l; 2\alpha_2 \beta_2) \phi(w, l, 2\alpha_2) I_{1,1}(w, l; 2\alpha_2 \beta_2).
\]

In the following, we will only deal with the contribution of
\[
\Sigma'(W, L, Q) = \sum_w \sum_l \sum_{\alpha_2} \sum_{(2\beta_1) = 1} \frac{1}{(\alpha_2 \beta_2)^2} S(wE\mu_1, -l; \alpha_2 \beta_2) \psi(w, l, \alpha_2) I_{1,1}(w, l; \alpha_2 \beta_2),
\]
the other contribution being similar.
Now we are in a position to apply Lemma 1 to $\Sigma'(W, L, Q)$ above. Taking
\[ U := \frac{n^{1+\varepsilon}}{Y} \quad \text{and} \quad g(w, l, \alpha_2) := \frac{Q^2 \psi(w, l, \alpha_2) H_{11}(w, l; \alpha_2 \beta_2)}{C_0 \alpha_2^2 M/m_1 m_2 v_1 v_2 \beta_1 s e} \]
for a suitable absolute constant $C_0 > 0$, we have
\[ \left| \frac{\partial^i j + k}{\partial w^i \partial v^j \partial \alpha_2} g(w, l, \alpha_2) \right| \leq U^{i+j+k} w^{-i} l^{-j} \alpha_2^{-k} \quad \text{for } 0 \leq i, j, k \leq 2 \]
using (7.13), and therefore Lemma 1 yields
\[
\Sigma'(W, L, Q) \ll \frac{M}{m_1 m_2 v_1 v_2 \beta_1 s e} \left( \frac{n^{1+\varepsilon}}{Y} \right)^{7/2} \cdot \frac{1}{Q \beta_2} \cdot (WL)^{1/2} + \frac{M}{m_1 m_2 v_1 v_2 \beta_1 s e} \left( \frac{n^{1+\varepsilon}}{Y} \right)^3 \cdot \frac{1}{Q \beta_2} \cdot n^\varepsilon E^{7/64} \sqrt{\beta_1} \sqrt{1 + Z^{-1}} \sqrt{1 + Z} \times \left( 1 + Z + \sqrt{\frac{W}{\beta_1 \beta_2}} \right) \left( 1 + Z + \sqrt{\frac{L}{\beta_1 \beta_2}} \right) (WL)^{1/2},
\]
(12.7)
where
\[ Z := \sqrt{EWL} / \beta_2 Q \sqrt{\beta_1}. \]
The right-hand side above reaches its maximum when
\[ W \asymp \frac{n^2 m_1 m_2 v_1 v_2 e g \beta_2 Q^2}{Y}, \quad L \asymp \frac{\beta_1 \beta_2 s n^{1+\varepsilon}}{g Y}, \quad Q \asymp \frac{\sqrt{n}}{v_2 m_1 \sqrt{eg}} \]
in which case we calculate
\[ WL \asymp \frac{n^2 + 2 \varepsilon m_2 v_1 \beta_1 \beta_2^2 s}{Y^2 v_2 m_1 g}, \quad Z \asymp \frac{n^{1+\varepsilon}}{Y}, \quad \sqrt{\frac{W}{\beta_1 \beta_2}} \asymp \sqrt{\frac{n^{1+\varepsilon} m_2 v_1}{Y v_2 m_1 \beta_1}} \quad \sqrt{\frac{L}{\beta_1 \beta_2}} \asymp \sqrt{\frac{n^{1+\varepsilon} s}{Y g}}. \]
From $s \mid g$, it follows that
\[ \sqrt{\frac{L}{\beta_1 \beta_2}} \ll Z. \]
Using $E \leq n$, (12.7) simplifies into
\[ \Sigma'(W, L, Q) \ll \frac{M}{m_1 m_2 v_1 v_2 \beta_1 s e} \left( \frac{n^{1+\varepsilon}}{Y} \right)^{7/2} \cdot \frac{v_2 m_1 \sqrt{eg}}{\beta_2 \sqrt{n}} \cdot \frac{n^{1+\varepsilon}}{Y} \cdot \sqrt{\frac{m_2 v_1 \beta_1 \beta_2^2 s}{v_2 m_1 g}} + \frac{M}{m_1 m_2 v_1 v_2 \beta_1 s e} \left( \frac{n^{1+\varepsilon}}{Y} \right)^3 \cdot \frac{v_2 m_1 \sqrt{eg}}{\beta_2 \sqrt{n}} \cdot n^{7/64 + \varepsilon} \sqrt{\beta_1} \left( \frac{n^{1+\varepsilon}}{Y} + \sqrt{\frac{n^{1+\varepsilon} m_2 v_1}{Y v_2 m_1 \beta_1}} \right) \cdot \frac{n^{1+\varepsilon}}{Y} \cdot \sqrt{\frac{m_2 v_1 \beta_1 \beta_2^2 s}{v_2 m_1 g}} \ll Mn^{6\varepsilon} \left( \frac{n^4}{Y^{9/2}} + \frac{n^{9/2 + 7/64}}{Y^5} \right)
\]
and hence
\[ (12.9) \]
and hence
\[ (12.9) \]
\[ S_{1,1,1}^4 \ll Mn^{10\varepsilon} \left( \frac{n^4}{Y^{9/2}} + \frac{n^{9/2 + 7/64}}{Y^5} \right). \]
We note that the exponent $7/64$ comes from the Kim-Sarnak bound for the least eigenvalue associated to Maass forms and can be dropped under Selberg’s eigenvalue conjecture. The above bound (12.9) beats $S_{1,1,1}^4 \ll M$ if
\[ Y \geq n^{9/10 + 7/320 + 2\varepsilon}. \]
This is a stronger condition than (11.3).

**Comment:** To get into a range where one beats (11.3) (at least conditionally under Selberg’s eigenvalue conjecture) one would need to reduce the above bound for $S_{1,1,1}^4$ by a factor of more than $U = n^{1+\varepsilon}/Y$. At this point, the author is not able to achieve this.
13. Main result

In view of the estimates (6.11) and (9.14) and the conditions (6.3), (10.3) and (11.3), our work thus far yields the asymptotic formula

\[ S_1^{\pm} = \frac{2}{\pi} P(n) \hat{\Phi}(0) M \log n + O \left( M \log \log n \right) \]

under the conditions

\[ n^{7/8 + \varepsilon} \leq Y \leq n \left( \log n \right)^{-1} \quad \text{and} \quad Y \leq M. \]

A parallel treatment for \( S_1^{\pm} \) gives the same asymptotic under the above condition on \( M \). Recalling (2.8), we then get the same asymptotic for \( S_1 \). As remarked in section 2, the same asymptotic will also hold for \( S_1 \), and using (2.4) and (2.7), we thus have

\[ S = \frac{8}{\pi} P(n) \hat{\Phi}(0) M \log n + O \left( M \log \log n \right) \]

with \( S \) as defined in (2.1). Using the properties of the weight functions laid out at the beginning of section 2 we also find that

\[ \left| S - \sum_{\substack{(x_1, x_2, x_3) \in \mathbb{N}^3 \\colon \ x_1^2 + x_2^2 - x_3^2 = n^2 \ \text{and} \ \|x_3\| = 1}} \Phi(x_3) \right| \ll \sum_{1 \leq x \leq Y} d(n^2 - x^2) \ll \sum_{1 \leq x \leq Y} d(n - x) d(n + x) \ll \left( \sum_{1 \leq x \leq Y} d^2(n - x) \right)^{1/2} \left( \sum_{1 \leq x \leq Y} d^2(n + x) \right)^{1/2}. \]

It is due to Ramanujan [6] that

\[ \sum_{r \leq R} d^2(r) = R \cdot \text{Pol}(\log R) + O \left( R^{3/5 + \varepsilon} \right), \]

where \( \text{Pol}(y) \) is a cubic polynomial. Hence, for \( Y \) in the above range, we get

\[ \left( \sum_{1 \leq x \leq Y} d^2(n - x) \right)^{1/2} \left( \sum_{1 \leq x \leq Y} d^2(n + x) \right)^{1/2} \ll Y \log^3 n. \]

Therefore, if

\[ M \geq Y \log^3 n, \]

then the asymptotic formula

\[ \sum_{\substack{(x_1, x_2, x_3) \in \mathbb{N}^3 \\colon \ x_1^2 + x_2^2 - x_3^2 = n^2 \ \text{and} \ \|x_3\| = 1}} \Phi(x_3) = \frac{2}{\pi} P(n) \hat{\Phi}(0) M \log n + O \left( M \log \log n \right) \]

holds. Thus, we fix

\[ Y := \frac{M}{\log^3 n}. \]

Adjusting \( \varepsilon \), we now get the following final result. Below, to be consistent with the main result in [3], we also allow negative \( x_1 \) and \( x_2 \), which enlarges the number of solutions by another factor of 4.

**Theorem 1.** Fix \( \varepsilon \in (0, 1) \). Assume that \( n \) is odd and

\[ n^{7/8 + \varepsilon} \leq M \leq n. \]

Let \( \Phi : \mathbb{R} \to [0, 1] \) be a fixed smooth function supported in \([1, 2]\) which is not constant 0. Then we have

\[ \sum_{\substack{(x_1, x_2, x_3) \in \mathbb{Z}^3 \\colon \ x_1^2 + x_2^2 - x_3^2 = n^2 \ \text{and} \ \|x_3\| = 1}} \Phi \left( \frac{x_3}{M} \right) = \frac{32}{\pi} P(n) (\log n) \int_{\mathbb{R}} \Phi \left( \frac{x}{M} \right) dx + O \left( M \log \log n \right) \]
as \( n \to \infty \), where
\[
P(n) := \sum_{f \in \mathbb{N}} \sum_{g \in \mathbb{N}} \sum_{\substack{p \mid f(g) \mid n \mid f \mid n}} \left( 1 - \frac{2}{p} \right) \prod_{p \mid g} \left( 1 - \frac{p - 1}{p(p + 1)} \right) \prod_{p \mid g} \left( 1 - \frac{2}{p} - \frac{p - 1}{p(p + 1)} \right).
\]

If \( n \) is square-free, then
\[
P(n) = \prod_{p \mid n} \left( 1 - \frac{p - 1}{p^2(p + 1)} \right).
\]

Acknowledgement: The author would like to thank the Ramakrishna Mission Vivekananda Educational and Research Institute for excellent working conditions.

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