A cohomological Hasse principle over
two-dimensional local rings

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Abstract

Let $K$ be the fraction field of a two-dimensional henselian, excellent, equi-
characteristic local domain. We prove a local-global principle for Galois coho-
omology with finite coefficients over $K$. We use only classical machinery from étale
cohomology theory, drawing upon an idea in Saito’s work on two-dimensional lo-
cal class field theory. This approach works equally well over the function field of a
curve over an equi-characteristic henselian discrete valuation field, thereby giving
a different proof of (a slightly generalized version of) a recent result of Harbater,
Hartmann and Krashen. We also present two applications. One is the Hasse prin-
ciple for torsors under quasi-split semisimple simply connected groups without
$E_8$
factor. The other is a finiteness theorem for the Pythagoras number $p(k((x, y, z)))$
of a Laurent series field in three variables. This theorem yields an explicit, consid-
erably satisfactory upper bound for $p(k((x, y, z)))$ when $k$ is a finitely generated
extension of $\mathbb{R}$ or $\mathbb{Q}$.

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1 Introduction

Let $F$ be a field and $n \geq 1$ an integer that is invertible in $F$. For an integer $i \geq 0$,
let $\mathbb{Z}/n(i)$ be the tensor product of $i$ copies of $\mu_n$, where $\mu_n$ denotes the Galois module
(or étale sheaf in a more general context) of $n$-th roots of unity over varying bases. Let
$\Omega_F$ denote the set of normalized discrete valuations on $F$ and for each $v \in \Omega_F$, denote
by $F_v$ the completion of $F$ at $v$. If $F$ is a global function field, i.e. the function field
of a curve over a finite field, then the classical Albert–Brauer–Hasse–Noether theorem
for Brauer groups implies the Hasse principle for the cohomology group $H^2(F, \mathbb{Z}/n(1))$, i.e., the injectivity of the natural map

$$H^2(F, \mathbb{Z}/n(1)) \longrightarrow \prod_{v \in \Omega_F} H^2(F_v, \mathbb{Z}/n(1)).$$

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The same is true for a number field if we enlarge \( \Omega_F \) by adjoining the real places of \( F \).

In his influential work on higher dimensional class field theory [Kat86], Kato suggested that higher dimensional analogs of this Hasse principle should be concerned with the cohomology groups \( H^r(F, \mathbb{Z}/n(r-1)) \) for \( r > 1 \). In that paper he proved, among others, a theorem which implies the Hasse principle for \( H^3(F, \mathbb{Z}/n(2)) \) when \( F \) is the function field of a curve over a \( p \)-adic field. At almost the same time, Saito ([Sai86, Sai87]) studied local class field theory over two-dimensional local rings with finite residue fields. It can be shown that Saito’s work yields an analog of Kato’s theorem. Namely, if \( F \) is the fraction field of a two-dimensional, henselian, excellent, local domain with finite residue field, then the Hasse principle for the group \( H^3(F, \mathbb{Z}/n(2)) \) holds (cf. [Hu13a, Prop. 4.1]). In recent years, Kato’s and Saito’s theorems have been relied on in a couple of papers to derive Hasse principles for torsors under semisimple groups over the relevant fields. We may cite for example [CTPS12], [Hu12b] and [Pre13]. These results can be viewed as extensions of earlier work over two-dimensional geometric fields with algebraically closed residue fields (cf. [CTOP02], [CTGP04], [BKG04]).

Research interests in related problems have been summed up by Colliot-Thélène [CT11] to two local-global questions, which aim to generalize the aforementioned results by assuming no cohomological condition on the residue field. As we will work over the same base field as in his questions, let us now fix the setup and notation.

**Notation 1.1.** Let \( A \) be a henselian, excellent, normal local domain with residue field \( k \). Let \( X \) be a connected regular scheme equipped with a proper morphism \( \pi : X \to \text{Spec}(A) \) such that the reduced closed fiber \( Y \) of \( \pi \) is a simple normal crossing (snc) divisor on \( X \). Let \( K \) be the function field of \( X \). We assume one of the following two conditions is satisfied:

(a) \( A \) is a discrete valuation ring and the morphism \( \pi \) is flat. We call this case the **semi-global case**.

(b) \( A \) is two-dimensional and the morphism \( \pi \) is birational. This will be called the **local case**.

Let \( n \geq 1 \) be an integer that is invertible in the residue field \( k \). Let \( \Omega_K \) denote the set of (normalized, rank 1) discrete valuations on \( K \), and let \( K_v \) be the corresponding completion of \( K \) for each \( v \in \Omega_K \).

The two questions of Colliot-Thélène are the following:

1. Let \( r \geq 2 \) be an integer. Is the natural map
   \[
   H^r(K, \mathbb{Z}/n(r-1)) \longrightarrow \prod_{v \in \Omega_K} H^r(K_v, \mathbb{Z}/n(r-1))
   \]
   injective?

2. Let \( G \) be a smooth connected linear algebraic group over \( K \). Does the natural map
   \[
   H^1(K, G) \longrightarrow \prod_{v \in \Omega_K} H^r(K_v, G)
   \]
   have trivial kernel?
As mentioned above, quite a number of results on question (2) have been obtained when the residue field $k$ is assumed algebraically closed or finite. In the semi-global case, question (2) together with local-global problems in a different but closely related context has been studied in \cite{HHK09} and \cite{HHK11}. There certain $K$-rationality assumption on the group $G$ plays a special role in the arguments. Counterexamples have been given in \cite{CTPS13} to show that the answer to question (2) can be negative without rationality assumption.

Nevertheless, if $G$ is a semisimple simply connected group, the Hasse principle for $G$-torsors is expected even when $G$ is not $K$-rational. For groups of many types, question (2) can be treated by using a positive answer to question (1) via cohomological invariants, as was in most of the known results.

In the semi-global case, if $A$ is complete and equi-characteristic, the cohomological Hasse principle in question (1) has been proved by Harbater, Hartmann and Krashen in \cite[Thm. 3.3.6]{HHK12} using a patching method. This result makes no additional assumption on the residue field and hence yields Hasse principles for torsors under certain quasi-split groups in this generality (cf. \cite[§4.3]{HHK12}).

In this paper, we give a positive answer to question (1) in the equi-characteristic local case and discuss some applications to question (2).

The goal is to prove the following

**Theorem 1.2.** With notation as in Notation 1.1, assume $A$ is equi-characteristic. We identify the set $\mathcal{X}(1)$ of codimension 1 points of $\mathcal{X}$ with the subset of $\Omega_K$ consisting of discrete valuations defined by these points.

In both the semi-global case and the local case, the natural map

$$H^r(K, \mathbb{Z}/n(r-1)) \longrightarrow \prod_{v \in \mathcal{X}(1)} H^r(K_v, \mathbb{Z}/n(r-1))$$

is injective for every $r \geq 2$.

Note that no assumption (except for the restriction on the characteristic) is required on the residue field.

The $r = 2$ case of the above theorem was essentially observed in \cite{CTOP02}, as was explained in \cite[§3]{Hu12c} and \cite[Thm. 3.2.3]{Hu12a}.

Grosso modo, the proof of Harbater, Hartmann and Krashen in the semi-global case goes as follows. They first observe that if $R$ is an equi-characteristic regular local ring with fraction field $F$, then the Hasse principle for $H^r(F, \mathbb{Z}/n(r-1))$ follows from Panin’s work on Gersten’s conjecture for equi-characteristic regular local rings \cite{Pan03}. Thanks to their patching techniques, they are able to reduce the cohomological Hasse principle in the semi-global case to the regular local case.

Our proof also starts with the regular local case, which is built upon Panin’s work, whence the equi-characteristic hypothesis in the theorem. As in \cite{HHK12}, the Bloch–Kato conjecture has to be used. The major difference between the two methods is that we do not employ any patching machinery but only étale cohomology theory. While the patching method does not seem to apply to the local case, we treat the local case
and the semi-global case in a parallel way. A key idea, borrowed from Saito’s work on two-dimensional local class field theory ([Sai86], [Sai87]), is the utilisation of the hypercohomology of the complex $i^* R^j_* \mathbb{Z}/n(r - 1)$ (see §2 for details).

In the second part of the paper, we present applications of our main theorem to Hasse principles for torsors under semisimple simply connected groups. In particular, we will prove (at the end of §3) the following theorem.

**Theorem 1.3.** Let $K$ be as in Theorem 1.2 and let $G$ be a quasi-split, semisimple, simply connected group without $E_8$ factor over $K$. Assume that the order of the Rost invariant of every simple factor of $G$ is not divisible by the characteristic $\text{char}(K)$.

Then the natural map

$$H^1(K, G) \rightarrow \prod_{v \in \Omega_K} H^1(K_v, G)$$

has trivial kernel.

Note that even in the semi-global case this theorem covers several groups that go beyond the results in [HHK12, §4.3]. In the mixed characteristic case, if $A$ is a complete discrete valuation ring, the theorem is still true for quasi-split groups with only factors of classical types or of type $G_2$ (see Remark 3.7).

As another application of our cohomological Hasse principle, we will prove in section 4 a finiteness theorem for the Pythagoras number of Laurent series fields in three variables. As a notational convention, letters $x, y, z, t, t_1, \ldots$ will be used to denote independent variables unless otherwise stated.

For any field $F$ of characteristic different from 2, the Pythagoras number $p(F)$ of $F$ is the smallest integer $p \geq 1$ or infinity, such that every sum of (finitely many) squares in $F$ can be written as a sum of $p$ squares.

The study of Pythagoras number for Laurent series fields in more than one variables was initiated by Choi, Dai, Lam and Reznick [CDLR82]. They obtained the equality $p(k((x, y))) = 2$ for a real closed field $k$, and proved that for a general base field $k$ if $p(k(t))$ is bounded by a 2-power, then $p(k((x, y)))$ is bounded by the same 2-power. They asked the question about the finiteness and the exact value of the Pythagoras number $p(k((t_1, \ldots, t_n)))$ when $n \geq 3$ and $k$ is, for instance, the field $\mathbb{R}$ of real numbers. While further progress has been made in [Sch01] for two-dimensional regular local rings and their fraction fields, little was known for Laurent series fields in three variables until the equality $p(\mathbb{R}((x, y, z))) = 4$ is proved in [Hu13b].

In this paper we will prove the following theorem, generalizing results in [Hu13b].

**Theorem 1.4.** Let $k$ be a field of characteristic different from 2 and let $r \geq 2$ be an integer. Then

$$p(k((x, y))) \leq 2^r \quad \text{implies} \quad p(k((x, y, z))) \leq p(k((x, y))(t)) \leq p(k((x, y))(z, t)) \leq 2^r$$

and

$$p(k(x, y)) = 2^r \quad \text{implies} \quad p(k((x, y, z))) = p(k((x, y))(t)) = p(k((x, y))(z, t)) = 2^r.$$
There are two sample cases to which the theorem applies.

If $k$ is a finitely generated field extension of transcendence degree $d \geq 0$ over a real closed field, then we have $p(k(x, y)) \leq 2^{d+2}$ by a theorem of Pfister (cf. [Lam05 Thm. XI.4.10]), hence $p(k((x, y, z))) \leq 2^{d+2}$. This improves on the estimate in [Hu13b Coro. 6.5].

If $k$ is a finitely generated field extension of transcendence degree $d \geq 0$ over $\mathbb{Q}$, thanks to Jannsen’s work on Kato’s conjecture [Jan09] and the proof of Milnor’s conjecture by Orlov, Vishik and Voevodsky [OVV07], we can deduce $p(k((x, y))) \leq 2^{d+3}$ from [CTJ91, Thm. 4.1]. So in this case we obtain $p(k((x, y, z))) \leq 2^{d+3}$. Previously, it was even unknown whether $p(\mathbb{Q}((x, y, z)))$ is finite.

Together with earlier results in [CDLR82], Theorem 1.4 suggests that at least for $n \leq 3$, the finiteness of the Pythagoras number of a Laurent series field in $n$ variables can be reduced to that of a rational function field in $n-1$ variables. This is consistent in philosophy with [Hu13b Conjecture 7.2].

2 Proof of the main theorem

Let us first recall some important facts about hypercohomology, as will be used in our proof of Theorem 1.2.

Let $X$ be a noetherian scheme. Let $D(X)$ denote the derived category of complexes of étale sheaves of abelian groups on $X$, and let $D_+(X) \subseteq D(X)$ be the full subcategory of complexes that are bounded below. The global section functor on étale sheaves admits a derived functor $R\Gamma : D_+(X) \to D_+(\text{Ab})$, where $D_+(\text{Ab})$ denotes the derived category of complexes bounded below of abelian groups. For a complex $F \in D_+(X)$, the hypercohomology groups of $F$, denoted $H^i(X, F)$, are defined as the cohomology groups of the complex $R\Gamma(F)$, i.e.,

$$H^i(X, F) := H^i(R\Gamma(F)), \quad \forall i \in \mathbb{Z}.$$

If the complex $F$ is given by a sheaf concentrated in degree 0, then the hypercohomology groups are the usual cohomology groups.

Let $j : V \hookrightarrow X$ be an open immersion. The direct image functor $j_*$ has a derived functor $Rj_* : D_+(V) \to D_+(X)$. Since $j_*$ sends injectives to injectives, one has (by [Del77 p.308, Chap. C.D., §2.3, Prop. 3.1]) $R(\Gamma \circ j_*) = R\Gamma \circ Rj_*$. Thus, for a complex $\mathcal{M} \in D_+(V)$,

$$H^i(X, Rj_*(\mathcal{M})) = H^i(R(\Gamma \circ j_*)(\mathcal{M})) = H^i(R(\Gamma \circ j_*)\mathcal{M}) = H^i(V, \mathcal{M})$$

for all $i$.

Let $Z \subseteq X$ be a closed subscheme. For any sheaf $\mathcal{M}$ on $X$, put

$$\Gamma_Z(\mathcal{M}) := \text{Ker}(\Gamma(X, \mathcal{M}) \to \Gamma(X \setminus Z, \mathcal{M})).$$

This functor admits a derived functor $R\Gamma_Z : D_+(X) \to D_+(\text{Ab})$. One defines the hypercohomology groups with support in $Z$ of a complex $\mathcal{F} \in D_+(X)$ by

$$H^i_Z(X, \mathcal{F}) := H^i(R\Gamma_Z(\mathcal{F})).$$
For the same reason as above, for any $\mathcal{M} \in D_+(V)$ one has

$$H^i_Z(X, Rj_*\mathcal{M}) = H^i_{Z\cap V}(V, \mathcal{M})$$

(2.0.2)

for all $i \in \mathbb{Z}$.

Let $A, \mathcal{X}$ and so on be as in the introduction. To prove the main theorem we introduce some more notation:

- $U := \mathcal{X}\setminus Y$, where $Y \subseteq \mathcal{X}$ is the reduced closed fiber of $\pi : \mathcal{X} \to \text{Spec}(A)$. The natural inclusion $j : U \hookrightarrow \mathcal{X}$ is the complement of the closed immersion $i : Y \to \mathcal{X}$. In the semi-global case, $U$ is the generic fiber of $\pi$. In the local case, $\pi$ induces an isomorphism from $U$ to $\text{Spec}(A) \setminus \{m_A\}$, where $m_A$ denotes the closed point of $\text{Spec}(A)$.

- $Y_0$ denotes the set of closed points of $Y$ and $Y_1$ denotes the set of generic points (of the irreducible components) of $Y$.

- $P := \mathcal{X}(1)\setminus Y$, the set of codimension 1 points of $\mathcal{X}$ outside $Y$. One can identify $P$ with the set of closed points of $U$. In the local case, $\pi$ maps $P$ bijectively onto $(\text{Spec}(A))(1)$.

- For each $p \in P$, let $\bar{p}$ denote the scheme-theoretic closure of $p$ in $\mathcal{X}$. Each $\bar{p}$ meets $Y$ at one and only one point.

In the semi-global case, the structural morphism $\pi : \mathcal{X} \to \text{Spec}(A)$ induces a finite morphism $\bar{p} \to \text{Spec}(A)$. Since $A$ is henselian, $\bar{p} \cong \text{Spec}(B)$ for some henselian local domain $B$ with fraction field $\text{Frac}(B) = \kappa(p)$.

In the local case, $p$ can be viewed as a height 1 prime ideal of $A$ and $\pi$ induces a finite birational morphism $\bar{p} \to \text{Spec}(A/p)$. Since $A/p$ is henselian, $\bar{p} \cong \text{Spec}(B)$ for some henselian local domain $B$ with fraction field $\text{Frac}(B) = \text{Frac}(A/p) = \kappa(p)$.

- For each $x \in \mathcal{X}$, we denote by $A_{(x)}$ the henselization of the regular local ring $\mathcal{O}_{X,x}$ and $K_{(x)} := \text{Frac}(A_{(x)})$ the fraction field of $A_{(x)}$.

- For any $x \in Y_0 \subseteq \mathcal{X}$, put $P_x := \{p \in P \mid x \in \bar{p}\}$.

- For a fixed point $x \in Y_0 \subseteq \mathcal{X}$, let $\varphi = \varphi_x : \text{Spec}(A_{(x)}) \to \mathcal{X}$ be the canonical morphism. We say a codimension 1 point $p_x$ of $\text{Spec}(A_{(x)})$ is vertical if $\varphi(p_x) \in Y_1$, i.e., if the prime ideal $p_x \cap \mathcal{O}_{X,x}$ of $\mathcal{O}_{X,x}$ corresponds to an irreducible component of the reduced closed fiber $Y$. Otherwise we say $p_x \in (\text{Spec}(A_{(x)}))(1)$ is horizontal, namely, $p_x$ is horizontal if $\varphi(p_x) \in P$. We denote by $V_x$ (resp. $H_x$) the set of vertical (resp. horizontal) points of $\text{Spec}(A_{(x)})$.

Alternatively, vertical and horizontal points can be described as follows:

Let $R \mapsto R^h$ denote the henselization functor of local rings. If $I \subseteq \mathcal{O}_{X,x}$ is the ideal defining $Y$, then

$$\mathcal{O}^h_{Y,x} = (\mathcal{O}_{X,x}/I)^h = \mathcal{O}^h_{X,x}/I \mathcal{O}^h_{X,x} = A_{(x)} \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Y,x}.$$
by [GD67, IV.18.6.8]. So we have a cartesian diagram
\[
\begin{array}{ccc}
\text{Spec}(\mathcal{O}_{Y,x}^h) & \longrightarrow & \text{Spec}(A_{(x)}) \\
\downarrow & & \downarrow \varphi \\
Y & \longrightarrow & X
\end{array}
\]

If \( x \) is not a regular point of \( Y \), Spec(\( \mathcal{O}_{Y,x}^h \)) is not integral. Note however that, since \( Y \) is an snc divisor on the regular scheme \( X \), each irreducible component of Spec(\( \mathcal{O}_{Y,x}^h \)) is defined by a regular parameter in the regular local ring \( A_{(x)} = \mathcal{O}_{X,x}^h \).

Therefore, the cardinality of \( V_x \) is 1 or 2, accordingly as \( x \) is a regular point of \( Y \) or not. We may thus identify \( V_x \) with (Spec(\( \mathcal{O}_{Y,x} \)))(0), the set of generic points of Spec(\( \mathcal{O}_{Y,x} \)), which is mapped bijectively to (Spec(\( \mathcal{O}_{Y,x} \)))(0) via \( \varphi \).

Now assume \( p \in P_x \), i.e., \( x \in \bar{p} \). The affine coordinate ring \( B \) of the scheme \( \bar{p} \) is a henselian local domain. Considering \( p \) as a prime ideal of \( \mathcal{O}_{X,x} \), we have
\[
A_{(x)}/pA_{(x)} = \mathcal{O}_{X,x}^h/p\mathcal{O}_{X,x}^h \cong (\mathcal{O}_{X,x}^h/p)^h = B^h = B.
\]

It follows that \( A_{(x)} \) has a unique prime ideal lying over \( p \), which is given by \( pA_{(x)} \).

So \( \varphi \) induces a bijection
\[
H_x := \varphi^{-1}(P_x) \xrightarrow{\sim} P_x.
\]

Now consider \( \varphi^{-1}(U) = \text{Spec}(A_{(x)}) \times_X U \). Since the complement \( \varphi^{-1}(Y) = \text{Spec}(\mathcal{O}_{Y,x}^h) \) is a principal divisor (\( A_{(x)} \) being a regular local ring), \( \varphi^{-1}(U) \) is an affine scheme. We denote its affine ring by \( R_x \), so that there is a cartesian diagram
\[
\begin{array}{ccc}
\text{Spec}(R_x) & \longrightarrow & \text{Spec}(A_{(x)}) \\
\downarrow & & \downarrow \varphi \\
U & \longrightarrow & X
\end{array}
\]

The set \( H_x \) can also be viewed as the set of closed points of \( \varphi^{-1}(U) = \text{Spec}(R_x) \).

We shall now start the proof of Theorem [1.2]

First note that, we may replace each completion \( K_v \) by the corresponding henselization \( K_v^{(1)} \) (cf. [Jan09, Thm. 2.9] and its proof). With notation as above, we have \( X^{(1)} = Y_1 \cup P \). The kernel of the local-global map
\[
H^r(K, \mathbb{Z}/n(r - 1)) \longrightarrow \prod_{v \in X^{(1)}} H^r(K_v^{(1)}, \mathbb{Z}/n(r - 1))
\]
is contained in the kernel of the map
\[
(\partial, \rho) : H^r(K, \mathbb{Z}/n(r - 1)) \longrightarrow \bigoplus_{p \in P} H^{r-1}(\kappa(p), \mathbb{Z}/n(r - 2)) \oplus \bigoplus_{\eta \in Y_1} H^r(K_{(\eta)}, \mathbb{Z}/n(r - 1)),
\]
where the map \( \rho \) is induced by the natural maps
\[
H^r(K, \mathbb{Z}/n(r - 1)) \rightarrow H^r(K_{(\eta)}, \mathbb{Z}/n(r - 1))
\]
and the map \( \partial \) are induced by the residue maps

\[
\partial_p : H^r(K, \mathbb{Z}/n(r-1)) \to H^{r-1}(\kappa(p), \mathbb{Z}/n(r-2)).
\]

So Theorem 1.2 follows immediately from the following:

**Theorem 2.1.** With notation and hypotheses as in Theorem 1.2, the map

\[
(\partial, \rho) : H^r(K, \mathbb{Z}/n(r-1)) \to \bigoplus_{p \in P} H^{r-1}(\kappa(p), \mathbb{Z}/n(r-2)) \oplus \bigoplus_{\eta \in Y_1} H^r(K_\eta, \mathbb{Z}/n(r-1))
\]

is injective.

**Proof.** Write \( \Lambda = \mathbb{Z}/n(r-1) \) and consider the localization sequence of étale cohomology on \( U \):

\[
\bigoplus_{p \in P} H^r_p(U, \Lambda) \to H^r(U, \Lambda) \to H^r(K, \Lambda) \to \bigoplus_{p \in P} H^{r+1}_p(U, \Lambda) \quad (2.1.1)
\]

By the absolute purity for discrete valuation rings (see e.g. [Del77, p.139, Chap. Cycle, Prop. 2.1.4]), we have

\[
H^d_p(U, \Lambda) = H^{d-2}(\kappa(p), \Lambda(-1)), \quad \forall \, d \geq 0.
\]

Put \( \mathcal{F} = i^* R_{j_\Lambda} \Lambda \). We borrow from Saito’s work [Sai86] and [Sai87] the idea of using the hypercohomology of \( \mathcal{F} \). We have a commutative diagram with exact rows

\[
\begin{array}{cccccc}
\bigoplus_{p \in P} H^r_p(\mathcal{X}, R_{j_\Lambda}) & \to & H^r(\mathcal{X}, R_{j_\Lambda}) & \to & H^r(K, \Lambda) & \to & \bigoplus_{p \in P} H^{r+1}_p(\mathcal{X}, R_{j_\Lambda}) \\
\bigoplus_{x \in Y_0} H^r_x(\mathcal{Y}, \mathcal{F}) & \to & H^r(\mathcal{Y}, \mathcal{F}) & \to & \bigoplus_{\eta \in Y_1} H^r(\kappa(\eta), \mathcal{F}) & \to & \bigoplus_{x \in Y_0} H^{r+1}_x(\mathcal{Y}, \mathcal{F})
\end{array}
\]

where for each \( p \in P \), \( \bar{p} \) denotes its closure in \( \mathcal{X} \). In this diagram, the second vertical arrow is an isomorphism by the proper base change theorem (cf. [AGV73] Exp. XII, Coro. 5.5]), and by the functoriality of the functor \( R_{j_\Lambda} \) (cf. (2.0.1) and (2.0.2)), we may identify the first row with the exact sequence in (2.1.1). For each \( \eta \in Y_1 \), there are canonical isomorphisms

\[
H^d(\kappa(\eta), \mathcal{F}) \cong H^d(K_\eta, \Lambda), \quad \forall \, d \geq 0,
\]

again by the proper base change theorem and the functoriality of derived functors. Putting all these together, we obtain a commutative diagram with exact rows

\[
\begin{array}{cccccc}
\bigoplus_{p \in P} H^{r-2}(\kappa(p), \Lambda(-1)) & \to & H^r(U, \Lambda) & \to & H^r(K, \Lambda) & \to & \bigoplus_{p \in P} H^{r-1}(\kappa(p), \Lambda(-1)) \\
\bigoplus_{x \in Y_0} H^r_x(\mathcal{Y}, \mathcal{F}) & \to & H^r(\mathcal{Y}, \mathcal{F}) & \to & \bigoplus_{\eta \in Y_1} H^r(K_\eta, \Lambda) & \to & \bigoplus_{x \in Y_0} H^{r+1}_x(\mathcal{Y}, \mathcal{F})
\end{array}
\]

(2.1.2)
We need only to prove that the map $\rho$ is injective when restricted to $\operatorname{Ker}(\partial) = \operatorname{Im}(\phi)$. By an easy diagram chase, it suffices to show the surjectivity of the map

$$\gamma : \bigoplus_{p \in P} H^{r-2}(\kappa(p), \Lambda(\lambda)) \rightarrow \bigoplus_{p \in P} H^r_p(\mathcal{X}, Rj_*\Lambda) \rightarrow \bigoplus_{x \in Y_0} H^r_x(Y, \mathcal{F}).$$

Here for each pair $(p, x) \in P \times Y_0$, the $(p, x)$-component $\gamma_{p,x}$ of $\gamma$ is 0 if $x$ does not lie in the closure $\bar{p}$ of $p$ in $\mathcal{X}$. Thus the map $\gamma$ decomposes into a direct sum

$$\gamma = \bigoplus_{x \in Y_0} \left( \gamma_x := \sum_{p \in P_x} \gamma_{p,x} : \bigoplus_{p \in P_x} H^{r-2}(\kappa(p), \Lambda(\lambda)) \rightarrow H^r_x(Y, \mathcal{F}) \right).$$

(Recall that $P_x = \{ p \in P \mid x \in \bar{p} \}).$ Therefore, the theorem will follow from the surjectivity of $\gamma_x$ for every $x \in Y_0$.

Notice that by considering the localization sequences on $\operatorname{Spec}(R_x) = \operatorname{Spec}(A(x)) \times \mathcal{X}U$ and on $\operatorname{Spec}(\mathcal{O}_{Y,x}) = \operatorname{Spec}(A(x)) \times \mathcal{X}Y$, we have as an analog of (2.1.2) the following commutative diagram with exact rows

$$\begin{array}{c}
H^{r-1}(K_x, \Lambda) & \rightarrow & \bigoplus_{p \in H_x} H^{r-2}(\kappa(p), \Lambda(\lambda)) & \rightarrow & H^r(R_x, \Lambda) & \rightarrow & H^r_x(Y, \mathcal{F}) & \rightarrow & \cdots \\
\downarrow{\eta} & & \downarrow{\gamma_x} & & \downarrow{\phi_x} & & \downarrow{\beta_x} & & \downarrow{\tau_x} & & \cdots \\
\bigoplus_{\eta \in V_x} H^{r-1}(K_\eta, \Lambda) & \rightarrow & H^r_x(Y, \mathcal{F}) & \rightarrow & H^r(\mathcal{O}^1_{Y,x}, \mathcal{F}) & \rightarrow & \cdots \\
\downarrow{\phi_x} & & \downarrow{\partial_x} & & \downarrow{\theta_x} & & \cdots \\
\bigoplus_{\eta \in V_x} H^{r-1}(K_\eta, \Lambda) & \rightarrow & \bigoplus_{p \in H_x} H^{r-1}(\kappa(p), \Lambda(\lambda)) & \rightarrow & H^{r+1}(Y, \mathcal{F}) & \rightarrow & \cdots \\
\end{array}$$

(2.1.3)

By the Bloch-Kato conjecture, which becomes a theorem thanks to the work of Rost, Voevodsky et al. (see [VoeII]), for any field $F$ of characteristic not dividing $n$, one has a canonical isomorphism

$$H^{r-1}(F, \Lambda) = H^{r-1}(F, \mathbb{Z}/n(r - 1)) \cong K^M_{r-1}(F)/n$$

where $K^M_{r-1}(F)$ denotes the Milnor $K$-group of degree $r - 1$ of the field $F$.

For $r \geq 2$, the group $K^M_{r-1}(F)/n$ is generated by symbol classes, i.e. elements of the form

$$\tilde{a}_1 \otimes \cdots \otimes \tilde{a}_{r-1},$$

where $\tilde{a}_i \in K^M_1(F)/n = F^*/F^{*n}$. Given finitely many discrete valuations $v_1, \ldots, v_m$ on $F$, let $F_i = F_{(v_i)}$ be the corresponding henselization for each $v_i$. The natural map

$$g_1 : K^M_1(F)/n \rightarrow \bigoplus_i K^M_1(F_i)/n$$

is surjective. Indeed, for any $(a_i) \in \bigoplus_i F_i^*$, there exists an element $b \in F$ such that $c_i := a_i/b$ is a unit for $v_i$ for every $i$ (e.g. $b = \prod_i \pi_i^{n_i(v_i)}$, with $\pi_i \in F$ a uniformizer
for \( v_i \). By the weak approximation property for a finite number of discrete valuations, there exists \( c \in F \) such that \( v_i(c - c_i) > 0 \) for each \( i \). Then \( c/c_i \) is an \( n \)-th power in the henselian discrete valuation field \( F_i \). Taking \( a = bc \), we have \( a = a_i \in F_i^n/n \) for every \( i \), whence the claimed surjectivity.

For any \( \alpha = (\alpha_i) \in \bigoplus K_{r-1}(F_i)/n \), one can choose an integer \( N \geq 1 \) possibly depending on \( \alpha \), such that each \( \alpha_i \) is the sum of \( N \) symbols

\[
K_{r-1}(F_i)/n, \quad 1 \leq j \leq N.
\]

By the surjectivity of \( g_1 \), there exist symbols \( s_j \in K_{r-1}(F)/n \) for \( 1 \leq j \leq N \) such that for every \( i \) the image of \( s_j \) in \( K_{r-1}(F_i)/n \) coincides with \( s_{i,j} \). Thus, the map

\[
g_{r-1} : K_{r-1}(F)/n \to \bigoplus K_{r-1}(F_i)/n
\]

sends \( a := \sum_j s_i \) to \( \alpha \). Therefore, the map \( g_{r-1} \) is surjective.

Now return to diagram (2.1.3). For each \( \eta \in V_x \), \( K(\eta) \) is the henselization of \( K(x) \) with respect to a discrete valuation. It follows from the above that the map \( g \) in (2.1.3) is surjective when \( r \geq 2 \). By an easy diagram chase, the surjectivity of \( \gamma_x \) is implied by the injectivity of the induced map \( \rho_x : \text{Im}(\phi_x) \to \text{Im}(\tau_x) \). Since \( Y \) is an snc divisor on \( X \), each of its irreducible components is locally defined by a regular parameter. So we may apply Lemma 2.2 below to the local ring \( A(x) \). Thus, we obtain

\[
\text{Ker}(\rho_x) \cap \text{Im}(\phi_x) = \text{Ker}(\rho_x) \cap \text{Ker}(\partial_x) = 0.
\]

This completes the proof of the theorem. □

As already observed in [HHK12 Prop. 3.3.4], the following lemma is an easy consequence of Panin’s work ([Pan03]) on Gersten’s conjecture for equi-characteristic regular local rings.

**Lemma 2.2.** Let \( R \) be an equi-characteristic henselian regular local ring with fraction field \( F \) and \( m > 0 \) an integer invertible in \( R \). Let \( R^{(1)} \) denote the set of discrete valuations of \( F \) corresponding to codimension 1 points of \( \text{Spec}(R) \). For each \( v \in R^{(1)} \), denote by \( F_{(v)} \) the henselization of \( F \) at \( v \) and \( \kappa(v) \) the residue field of \( v \). Let \( \pi \in R \) be a regular parameter in \( R \) (i.e. \( \pi \) is an element of a regular system of parameters of the regular local ring \( R \)), identified with the element of \( R^{(1)} \) defined by the prime ideal \( \pi R \) of \( R \).

Then for any \( r, j \in \mathbb{Z} \), the natural map

\[
(\rho, \partial) : H^r(F, \mathbb{Z}/m(j)) \to H^r(F_{(\pi)}, \mathbb{Z}/m(j)) \bigoplus_{v \in R^{(1)}, v \neq \pi} H^{r-1}(\kappa(v), \mathbb{Z}/m(j - 1))
\]

is injective, where \( \rho : H^r(F, \mathbb{Z}/m(j)) \to H^r(F_{(\pi)}, \mathbb{Z}/m(j)) \) is the natural map induced by the inclusion \( F \subseteq F_{(\pi)} \) and

\[
\partial : H^r(F, \mathbb{Z}/m(j)) \to \bigoplus_{v \in R^{(1)}, v \neq \pi} H^{r-1}(\kappa(v), \mathbb{Z}/m(j - 1))
\]

is induced by the residue maps.
Proof. Put $M = \mathbb{Z}/m(j)$. Let $\alpha \in H^r(F, M)$ be such that $\rho(\alpha) = 0 = \partial(\alpha)$. By [Pan03, §5, Thm. C], we have an exact sequence

$$0 \to H^r_{\text{ét}}(R, M) \xrightarrow{-\iota} H^r(F, M) \to \bigoplus_{v \in R^{(1)}} H^{r-1}(\kappa(v), M(-1)).$$

For each $v \in R^{(1)}$, the map $H^r(F, M) \to H^{r-1}(\kappa(v), M(-1))$ factors through the natural map $H^r(F, M) \to H^r(F(v), M)$. So $\alpha$ maps to 0 in each $H^{r-1}(\kappa(v), M(-1))$ and thus comes from an element $\tilde{\alpha} \in H^r_{\text{ét}}(R, M)$.

Denote by $R^h_{\pi}$ the henselization of $R$ at $\pi$. Its residue field $\kappa(\pi)$ is the fraction field of the quotient ring $R/\pi$, which is a henselian regular local ring with the same residue field as $R$. Panin’s theorem ([Pan03, §5, Thm. C]) applied to $R_{\pi}$ implies in particular that the natural map $\tau_{\pi} : H^r(R_{\pi}, M) \to H^r(\kappa(\pi), M)$ is injective. On the other hand, by the proper base change theorem we have natural identifications

$$H^r(R, M) = H^r(k, M) = H^r(R^h_{\pi}, M),$$

where $k$ denotes the residue field of $R$. Now consider the following two commutative diagrams

$$
\begin{array}{ccc}
H^r(R, M) & \xrightarrow{\varphi_{\pi}} & H^r_{\text{ét}}(R^h_{\pi}, M) \\
\cong \downarrow & & \downarrow \\
H^r(R_{\pi}, M) & \xrightarrow{\tau_{\pi}} & H^r(\kappa(\pi), M)
\end{array}
$$

and

$$
\begin{array}{ccc}
H^r(R, M) & \xrightarrow{-\iota} & H^r(F, M) \\
\downarrow & & \downarrow \\
H^r(F_{\pi}, M) & \xrightarrow{\phi_{\pi}} & H^r(F_{\pi}(\kappa(\pi), M)
\end{array}
$$

From the left diagram we see that $\varphi_{\pi}$ is injective, and the right diagram shows that

$$0 = \phi_{\pi}(\alpha) = \phi_{\pi}(\iota(\tilde{\alpha})) = \tau_{\pi}(\varphi_{\pi}(\tilde{\alpha})).$$

But we know that $\tau_{\pi}$ is injective (by the discrete valuation ring case of Gersten’s conjecture, or by Panin’s theorem cited above). So we get $\tilde{\alpha} = 0$ and hence $\alpha = 0$. \hfill \square

**Remark 2.3.**

1. The statement of the main theorem (Theorem 1.2) does not hold if $r = 1$. Counterexamples can be found in [CTPS12, §6] in the semi-global case and in [Jaw01, Thm. 1.5] or [CTOP02, Remark 3.3] in the local case.

2. In the proof of Theorem 2.1, the only place where we have relied on the equi-characteristic assumption is the following version of Gersten’s conjecture: The complex

$$0 \to H^r_{\text{ét}}(A(x), \mathbb{Z}/n(r-1)) \xrightarrow{-\iota} H^r(K(x), \mathbb{Z}/n(r-1)) \to \bigoplus_{v \in A^{(1)}(x)} H^{r-1}(\kappa(v), \mathbb{Z}/n(r-2))$$

is exact for every closed point $x$ of $\mathcal{X}$.

So, if Gersten’s conjecture is true for the sheaf $\mathbb{Z}/n(r-1)$ over every 2-dimensional henselian excellent regular local ring, then Theorems 1.2 and 2.1 are still true without the equi-characteristic assumption.

3. The Galois module $\mathbb{Z}/n(r-1)$ in Theorem 1.2 cannot be substituted by an arbitrary finite Galois module. In fact, in both the semi-global case and the local case, even with an algebraically closed residue field, there exists a finite Galois module $\mu$ such that the Hasse principle for $H^2(K, \mu)$ fails [CTPS13, Corollaires 5.3 and 5.7].
3 Applications to torsors under semisimple groups

We keep the notation and hypotheses of Theorem 1.2. In particular, $A$ denotes an equi-characteristic, henselian, excellent, normal local domain with residue field $k$. $X$ is a two-dimensional regular integral scheme equipped with a proper morphism $\pi : X \to \text{Spec}(A)$ whose closed fiber is an snc divisor. $K$ is the function field of $X$. In the semi-global case, $A$ is a discrete valuation ring and $\pi$ is flat. In the local case, $A$ is two-dimensional and $\pi$ is birational.

In this section we give a number of Hasse principles for torsors under semisimple algebraic groups and for related structures. These Hasse principles are easy consequences of our cohomological Hasse principle (Thm. 1.2) thanks to injectivity properties of certain cohomological invariants. Since this strategy is certainly well-known to experts and has already been used by several authors in similar contexts (see e.g. [CTPS12], [HHK12], [Hu12b], [Pre13]), here we will only explain very briefly the results.

In the rest of this section, we denote by $G$ a semisimple simply connected algebraic group over $K$ and we consider the pointed set

$$\mathbb{III}(X, G) := \text{Ker} \left( H^1(K, G) \rightarrow \prod_{v \in \mathcal{X}(1)} H^1(K_v, G) \right).$$

When $G$ is absolutely simple, we denote by

$$R_G : H^1(K, G) \rightarrow H^3(K, \mathbb{Q}/\mathbb{Z}(2))$$

the Rost invariant of $G$ over $K$. We will always assume that the order of $R_G$ is invertible in $k$.

The following list of Hasse principles follows as were discussed in [HHK12, §4.3] in the semi-global case. These results do not require any assumption on the residue field $k$ (except for a few restrictions on the characteristic). We refer the reader to [Hu12b] and [Pre13] for more details when $k$ is assumed finite.

- (Groups of type $1^n A_n^*$.) Let $D$ be a central simple $K$-algebra of square-free index and let $G = \text{SL}_1(D)$. The Rost invariant of the group $G$ is injective by a theorem of Suslin [Sus83, Thm. 24.4]. (Our assumption on the order of the Rost invariant implies that the index of $D$ is invertible in $k$.) So we have $\mathbb{III}(X, G) = 1$ by Theorem 1.2.

- (Groups of type $G_2$.) Let $G$ be the automorphism group $\text{Aut}(C)$ of some Cayley algebra $C$ over $K$. If $\xi \in H^1(K, G)$ corresponds to a Cayley algebra $C'$, the Rost invariant $R_G$ maps $\xi$ to $e_3(N_C) - e_3(N_{C'})$, where $N_C$ and $N_{C'}$ denote the norm forms of $C$ and $C'$ respectively and $e_3$ is the Arason invariant for quadratic forms. Two Cayley algebras are isomorphic if and only if their norm forms are isomorphic. Since the norm form of a Cayley algebra is a 3-fold Pfister form and the Arason invariant is injective on Pfister forms by a well-known theorem of Merkurjev (cf. [Ara84, Prop. 2]), the Rost invariant $R_G$ has trivial kernel. Hence, $\mathbb{III}(X, G) = 1$. 

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• (Quasi-split groups.) For a quasi-split group $G$, we have $\text{III}(\mathcal{X}, G) = 1$ in the each of the following cases:

1. $G$ is exceptional not of type $E_8$;
2. $G$ is of type $B_n$ with $2 \leq n \leq 6$;
3. $G$ is of type $D_n$ with $3 \leq n \leq 6$ or split of type $D_7$;
4. $G$ is of type $2A_n$ with $n \leq 5$.

Indeed, the triviality of the Rost kernel in these case has been proved in [Gar01] (see also [Che03] in the first case).

• (Groups of type $F_4^{\text{red.}}$.) Recall that to each Albert algebra $J'$ over $K$ one can associate three cohomological invariants (cf. [Ser95, §9.4] or [KMRT98, §40])

$$f_3(J') \in H^3(K, \mathbb{Z}/2), \quad f_5(J') \in H^5(K, \mathbb{Z}/2), \quad g_3(J') \in H^3(K, \mathbb{Z}/3)$$

such that the following properties hold:

1. $J'$ is reduced if and only if $g_3(J') = 0$.
2. Two reduced Albert algebras are isomorphic if and only if they have the same $f_3$ and $f_5$ invariants.

Let $G = \text{Aut}(J)$ be the automorphism group of a reduced Albert algebra $J$ over $K$. Our cohomological Hasse principle yields $\text{III}(\mathcal{X}, G) = 1$ by means of the invariants $f_3, f_5$ and $g_3$.

We now discuss the Hasse principle for some other groups.

For any field $F$ of characteristic different from 2, we denote by $W(F)$ its Witt group of quadratic forms and for each $r \geq 1$, we denote by $I^r(F)$ the subgroup generated by the classes of $r$-fold Pfister forms. By the quadratic form version of Milnor’s conjecture (proved in [OVV07]), there is a canonical homomorphism

$$e_r : I^r(F) \longrightarrow H^r(F, \mathbb{Z}/2)$$

sending the class of a Pfister form

$$\langle a_1, \ldots, a_r \rangle := \langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_r \rangle$$

to the cup product

$$(a_1) \cup \cdots \cup (a_r) \in H^r(F, \mathbb{Z}/2),$$

whose kernel is $I^{r+1}(F)$.

**Proposition 3.1.** Assume $\text{char}(K) \neq 2$.

Then for each $r \geq 2$, the natural map

$$I^r(K) \longrightarrow \prod_{v \in \mathcal{X}(1)} I^r(K_v)$$

is injective.
Proof. Let $J_r$ be the kernel of the map in the proposition. The commutative diagram

$$
\begin{array}{ccc}
I^r(K) & \longrightarrow & \prod I^r(K_v) \\
\downarrow & & \downarrow \\
H^r(K, \mathbb{Z}/2) & \longrightarrow & \prod H^r(K_v, \mathbb{Z}/2)
\end{array}
$$

together with Theorem 1.2 shows that for every $r \geq 2$,

$$J_r \subseteq I^{r+1}(K) = \text{Ker} (I^r(K) \longrightarrow H^r(K, \mathbb{Z}/2)).$$

Therefore, we have for each $r \geq 2$,

$$J_r = J_r \cap I^{r+1}(K) = J_{r+1} = J_{r+2} = \cdots = \bigcap_{d \geq r} J_d.$$

Since $\bigcap_{d \geq r} I^d(K) = 0$, it follows immediately that $J_r = 0$. 

Theorem 3.2 (Groups of type $C_n^\ast$). Let $X$ and $K$ be as in Theorem 1.2. Assume char($K$) $\neq 2$. Let $D$ be a quaternion division algebra over $K$ with standard involution $\tau_0$ and $h$ a nonsingular hermitian form over $(D, \tau_0)$. Let $G = \text{U}(h)$ be the unitary group of the hermitian form $h$.

Then the natural map

$$H^1(K, G) \longrightarrow \prod_{v \in X(1)} H^1(K_v, G)$$

is injective. In particular, one has $\text{III}(X, G) = 1$.

Proof. The pointed set $H^1(K, G) = H^1(K, \text{U}(h))$ classifies up to isomorphism hermitian forms over $(D, \tau_0)$ of the same rank as $h$. Let $h_1$ and $h_2$ be two such hermitian forms. Put $h' = h_1 \perp (-h_2)$ and let $q_{h'}$ be the trace form of $h'$. Two hermitian forms over $(D, \tau_0)$ are isomorphic if and only if their trace forms are isomorphic as quadratic forms (cf. [Sch85, p.352, Thm. 10.1.7]). So we need only to show that $[q_{h'}] = 0$ in the Witt group $W(K)$. Note that the class $[q_{h'}] \in W(K)$ lies in the subgroup $I^3(K)$. The theorem thus follows from Prop. 3.1.

The following theorem has been proved in [Hu13a, Thm. 4.9] in the local case with finiteness assumption on the residue field (see also Remark 3.7 below).

Theorem 3.3. With notation as before, assume char($K$) $\neq 2$.

Then for any nonsingular quadratic form $\phi$ of rank $\geq 2$ over $K$, the natural map

$$H^1(K, \text{SO}(\phi)) \longrightarrow \prod_{v \in X(1)} H^1(K_v, \text{SO}(\phi))$$

is injective.
Proof. Let \( \psi, \psi' \) be nonsingular quadratic forms representing classes in \( H^1(K, \text{SO}(\phi)) \). As they have the same dimension, the forms \( \psi \) and \( \psi' \) are isometric if and only if they represent the same class in the Witt group. Since \( \psi \) and \( \psi' \) also have the same discriminant, we have \( [\psi] - [\psi'] \in I^2(K) \) by [Sch85, p.82, Chapt. 2, Lemma 12.10]. Then the theorem is immediate from Prop. 3.1.

**Theorem 3.4** (Special cases of groups of type \( ^2A_n \)). Let \( X \) and \( K \) be as in Theorem 1.2. Assume \( \text{char}(K) \neq 2 \). Let \( h \) be a nonsingular hermitian form of rank \( \geq 2 \) over a quadratic field extension \( L \) of \( K \) and let \( G = \text{SU}(h) \) be the special unitary group.

Then the natural map

\[
H^1(K, G) \longrightarrow \prod_{v \in \mathcal{X}^{(1)}} H^1(K_v, G)
\]

is injective. In particular, one has \( \text{III}(X, G) = 1 \).

Proof. The set \( H^1(K, G) \) classifies nonsingular hermitian forms \( h' \) which have the same rank and discriminant as \( h \) ([KMRT98 p.403, (29.19)]). Let \( q \) be the trace form of \( h \). There is a natural embedding of \( G \) into \( \text{SO}(q) \). The induced map

\[
H^1(K, G) = H^1(K, \text{SU}(h)) \longrightarrow H^1(K, \text{SO}(q))
\]

sending the class of a hermitian form \( h' \) to the class of its trace form, is injective by [Sch85 Thm. 10.1.1 (ii)]. The result thus follows from Theorem 3.3.

**Theorem 3.5** (Special cases of groups of type \( B_n \) or \( D_n \)). Assume \( \text{char}(K) \neq 2 \). Let \( q \) be a nonsingular quadratic form of dimension \( \geq 3 \) over \( K \) and let \( G = \text{Spin}(q) \). If \( q \) is isotropic over \( K \), then \( \text{III}(X, G) = 1 \).

Proof. Consider the exact sequence of algebraic groups

\[
1 \longrightarrow \mu_2 \longrightarrow \text{Spin}(q) \longrightarrow \text{SO}(q) \longrightarrow 1
\]

which gives rises to an exact sequence of pointed sets

\[
\text{SO}(q)(K) \xrightarrow{\delta} K^*/K^{*2} \longrightarrow H^1(K, \text{Spin}(q)) \longrightarrow H^1(K, \text{SO}(q)).
\]

Since \( q \) is isotropic, the spinor norm map \( \delta \) is surjective. It suffices to apply Theorem 3.3 the Hasse principle for the group \( \text{SO}(q) \).

Our Theorem 1.3 asserts that for a quasi-split semisimple simply connected group \( G/K \) without \( E_8 \) factor, the Hasse principle for \( G \)-torsors holds.

**Proof of Theorem 1.3** By a standard argument using Shapiro’s lemma (see e.g. [Hu12b, §3]), we may assume \( G \) is absolutely simple. Choosing a proper morphism \( X \to \text{Spec}(A) \) as in the introduction, we need only to show \( \text{III}(X, G) = 1 \).

The cases \( ^1A_n \) and \( C_n \) are trivial, since in these cases \( H^1(F, G) = 1 \) for a quasi-split group \( G \). The exceptional groups (not of type \( E_8 \)) have been discussed earlier in this section. The cases of classical groups of type \( ^2A_n \), \( B_n \) or \( D_n \) are covered by Theorems 3.4 and 3.5. This completes the proof.
**Remark** 3.6. There do exist quasi-split groups for which the Hasse principle holds but the Rost kernel is nontrivial. For example, the split group of type $B_7$ over $\mathbb{R}((x, y))$ has a nontrivial Rost kernel as shown in [Gar01, Example 1.6]. But the Hasse principle for torsors under this group holds according to Theorem 1.3.

**Remark** 3.7. Some results in this section extend to the mixed characteristic case when $A$ is a complete discrete valuation ring.

First note that in this case Thm. 3.3 is still true thanks to [CTPS12, Thm. 3.1]. This was explained in [Hu13a, Remark 4.2]. On the other hand, one can deduce Prop. 3.1 from Thm. 3.3. Indeed, it is sufficient to show the injectivity of the local-global map for $I^2(K)$. Given a nonsingular quadratic form $\psi$ over $K$ whose class lies in $I^2(K)$, the dimension of $\psi$ is even and the discriminant of $\psi$ is trivial. If $\psi$ is locally hyperbolic, Theorem 3.3 applied to the hyperbolic form $\phi$ with $\dim \phi = \dim \psi$ implies that $\psi$ is hyperbolic over $K$. Hence, the analogs of Theorems 3.2, 3.4 and 3.5 hold as our proofs show.

If $G$ is an absolutely simple simply connected group of type $G_2$ over $K$, and if 2 is invertible in the residue field $k$, then the Hasse principle for $G$-torsors follows from the Hasse principle for $I^3(K)$, because Cayley algebras are classified by their norm forms.

Finally, if $G = \text{SL}_1(D)$ for some quaternion algebra $D$ over $K$, we have $\text{III}(\mathcal{X}, G) = 1$ by [CTPS12, Thm. 3.1]. This is because saying an element $a \in K^*$ is a reduced norm for $D$ is equivalent to saying that the quadratic form $\langle -a \rangle$, $N_D$ is isotropic, where $N_D$ denotes the norm form of $D$.

From the above, we get the following Hasse principle:

With notation as in Notation 1.1, assume $A$ is a complete discrete valuation ring (which is possibly of mixed characteristic). Let $G$ be an absolutely simple simply connected group of $K$ such that the kernel of $R_G$ is invertible in the residue field $k$. Then we have $\text{III}(\mathcal{X}, G) = 1$ if $G$ is of type $G_2$ or quasi-split of classical type.

### 4 Applications to Pythagoras numbers

In this section we prove Theorem 1.4. The basic strategy is essentially the same as in the proof of [Hu13b, Thm. 6.1].

We first review some general facts on the Pythagoras number of fields.

Let $F$ be a field of characteristic $\neq 2$. Recall that the level $s(F)$ of $F$ is the smallest integer $s \geq 1$ or $+\infty$, such that $-1$ is the sum of $s$ squares in $F$. If $s(F) = +\infty$, the field $F$ is called (formally) real. Otherwise it is called nonreal. For any field $F$ one has $s(F(t)) = s(F) = s(F((t)))$.

If $F$ is a nonreal field, $s(F)$ is a power of 2 and one has

$$s(F) \leq p(F) \leq s(F) + 1, \quad p(F(t)) = s(F) + 1.$$

The following theorem will be frequently made use of and referred to as Pfister’s theorem.

**Theorem 4.1** (Pfister). Let $F$ be a real field and $r \geq 1$ an integer. Then the following two conditions are equivalent:

- $s(F) \geq r$
- $s(F(t^r)) = s(F) + r - 1$.
(i) \( p(F(t)) \leq 2^r \).
(ii) \( s(L) \leq 2^{r-1} \) for every finite nonreal extension \( L/F \).

Proof. See e.g. [Lam05] p.397, Examples XI.5.9 (3). □

Remark 4.2. Becher and Van Geel have shown in [BVG09] Thm. 3.5 that the two conditions in Pfister’s theorem are also equivalent to the following:

(iii) \( p(L) < 2^r \) for any finite real extension \( L/F \).

Given a field \( L \) of characteristic \( \neq 2 \) and an integer \( r \geq 1 \), the following assertions are equivalent:

(1) \( s(L) \leq 2^r - 1 \).

(2) The \( r \)-fold Pfister form \( \langle -1, \ldots, -1 \rangle = \langle 1, 1 \rangle^{\otimes r} \) is isotropic over \( L \).

(3) The \( r \)-fold Pfister form \( \langle -1, \ldots, -1 \rangle = \langle 1, 1 \rangle^{\otimes r} \) is hyperbolic over \( L \).

(4) The cohomology class \( (-1) \cup \cdots \cup (-1) \in H^r(L, \mathbb{Z}/2) \) vanishes.

Here the equivalences of (1), (2) and (3) are classical and that they are equivalent to (4) is guaranteed by the quadratic form version of Milnor’s conjecture.

Remark 4.3. What we really need in this paper is only the equivalence between properties (3) and (4) above, but not the full strength of Milnor’s conjecture. While a \( K \)-theoretic analog (cf. [Lam05] Thm. X.6.7) was known many years ago, this fact seems to have been proved only for \( r \leq 4 \) (cf. [JR89] and [MS90]) prior to the solution of Milnor’s conjecture.

We are now ready to prove Theorem 1.4, which states that if \( k \) is a field such that \( p(k(x, y)) \leq 2^r \), then

\[
p(k((x, y)(z))) \leq p(k((x, y)(t))) \leq p(k((x, y)(z, t))) \leq 2^r.
\]

Proof of Theorem 1.4. If \( k \) is nonreal, we have by [Hu13b] Prop. 3.3

\[
p(k((x, y, z))) = p(k((x, y)(t))) = s(k) + 1 = s(k(x)) + 1 = p(k(x, y))
\]

and

\[
p(k((x, y)(z, t))) = s(k((x, y))) + 1 = s(k) + 1 = p(k(x, y)).
\]

So we may assume \( k \) is real. It has been proved in [Hu13b] Lemma 3.5 and Coro. 5.5 that the inequalities

\[
p(k(x, y)) \leq p(k((x, y, z))) \leq p(k((x, y)(t)))
\]

hold in general. Moreover,

\[
p(k((x, y)(t))) \leq p(k((x, y)(z, t)))
\]

and the two Pythagoras numbers in this inequality are bounded by the same 2-powers whenever such bounds exist (cf. [CDLR82] Coro. 5.2 and [Hu13b] Example 4.3).

It is thus sufficient to show the inequality \( p(k((x, y)(t))) \leq 2^r \) under the assumption \( p(k(x, y)) \leq 2^r \). By Pfister’s theorem, we need only to show \( s(K) \leq 2^{r-1} \) for every finite nonreal extension \( K \) of the field \( k((x, y)) \). Since this property has an interpretation
in terms of the vanishing of a cohomology class in $H^r(K, \mathbb{Z}/2)$ and we have a Hasse principle for the cohomology group $H^r(K, \mathbb{Z}/2)$ (Theorem 4.4), it suffices to show that if $A$ is the integral closure of $k[x, y]$ in $K$ and if $X \to \text{Spec}(A)$ is chosen as in the introduction, then $s(K_v) \leq 2^r-1$ for every $v \in \mathcal{X}^{(1)}$.

For each $v \in X^{(1)}$, the residue field $k_v$ has the same level as $K_v$, and $k(v)$ is either isomorphic to $k'(t)$ for some finite nonreal extension $k'/k$, or a function field of transcendence degree $1$ over $k$. In the former case, Pfister’s theorem applied to the real field $k$ shows that $s(k(v)) = s(k') \leq 2^r-1$, since $p(k(t)) \leq p(k(x, y)) \leq 2^r$. In the latter case, $k(v)$ is (isomorphic to) a finite nonreal extension of $k(x)$. Again by Pfister’s theorem, applied to the real field $k(x)$ this time, we get $s(k(v)) \leq 2^r-1$. The theorem is thus proved.

Theorem 4.4 allows us to generalize the results in [Hu13b, §6].

**Corollary 4.4.** Let $k$ be an algebraic function field of transcendence degree $d \geq 0$ over a field $k_0$.

(i) If $k_0$ is a real closed field, then

$$p(k((x, y, z))) \leq p(k((x, y))(t)) \leq p(k((x, y))(z, t)) \leq 2^{d+2}.$$  

(ii) If $k_0$ is a number field, i.e., a finite extension of $\mathbb{Q}$, then

$$p(k((x, y, z))) \leq p(k((x, y))(t)) \leq p(k((x, y))(z, t)) \leq 2^{d+3}.$$  

**Proof.** As was explained in the introduction, in case (i) one has $p(k(x, y)) \leq 2^{d+2}$ and in case (ii) one has $p(k(x, y)) \leq 2^{d+3}$. So the corollary follows from Theorem 4.4.

Previously, even the finiteness of $p(\mathbb{Q}((x, y, z)))$ was unknown.

**Lemma 4.5.** Let $k_0$ be a field of characteristic $\neq 2$, $n \geq 1$ and $k = k_0((t_1)) \cdots ((t_n))$. Let $r \geq 2$ be an integer such that $p(k_0(x, y)) \leq 2^r$. Then $p(k(x, y)) \leq 2^r$.

**Proof.** By induction we may reduce to the case $n = 1$. Moreover, we may assume $k_0$ is real in view of [Hu13b, Prop. 3.3]. By Pfister’s theorem, it suffices to prove $s(K) \leq 2^r-1$ for every finite nonreal extension $K$ of $k(x) = k_0((t))(x)$. The argument for a similar statement given in our proof of Theorem 4.4 works verbatim, since our cohomological Hasse principle is also valid in the semi-global case. Alternatively, one can use [BGV12, Thm. 6.5].

**Corollary 4.6.** Let $k_0$ be a field, $n \geq 1$ and $k = k_0((t_1)) \cdots ((t_n))$.

(i) If $k_0$ is a function field of transcendence degree $d \geq 0$ over a real closed field, then

$$p(k((x, y, z))) \leq p(k((x, y))(t)) \leq p(k((x, y))(z, t)) \leq 2^{d+2}.$$  

(ii) If $k_0$ is a function field of transcendence degree $d \geq 0$ over $\mathbb{Q}$, then

$$p(k((x, y, z))) \leq p(k((x, y))(t)) \leq p(k((x, y))(z, t)) \leq 2^{d+3}.$$  

**Proof.** This follows by combining Lemma 4.5 and Theorem 4.4 using the corresponding upper bound for $p(k_0(x, y))$. 

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