A FUNCTION ON THE HOMOLOGY OF 3-MANIFOLDS

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Abstract. In analogy with the Thurston norm, we define for an orientable 3-
manifold $M$ a numerical function on $H_2(M;\mathbb{Q}/\mathbb{Z})$. This function measures the
minimal complexity of folded surfaces representing a given homology class.
A similar function is defined on the torsion subgroup of $H_1(M;\mathbb{Z})$. These
functions are estimated from below in terms of abelian torsions of $M$.

1. Introduction

One of the most beautiful invariants of a 3-dimensional manifold $M$ is the
Thurston semi-norm on $H_2(M;\mathbb{Q})$, see [8]. The geometric idea leading to this
semi-norm is to consider the minimal genus of a surface in $M$ realizing any given 2-
homology class of $M$. Thurston’s definition of the semi-norm uses a suitably normalized
Euler characteristic of the surface rather than the genus. The Thurston semi-
norm is uninteresting for a rational homology sphere $M$, since then $H_2(M;\mathbb{Q}) = 0$.
However, a rational homology sphere may have non-trivial 2-homology with coefficients in $\mathbb{Q}/\mathbb{Z}$. Homology classes in $H_2(M;\mathbb{Q}/\mathbb{Z})$ can be realized by folded surfaces,
locally looking like unions of several half-planes in $\mathbb{R}^3$ with common boundary line.
It is natural to consider “smallest” folded surfaces in a given homology class.

We use this train of ideas to define for an arbitrary orientable 3-manifold $M$ (not
necessarily a rational homology sphere) a function

$$\theta : H_2(M;\mathbb{Q}/\mathbb{Z}) \to \mathbb{R}_+ = \{ r \in \mathbb{R} \mid r \geq 0 \}.$$ 

This function measures the “minimal” normalized Euler characteristic of a folded
surface representing a given class in $H_2(M;\mathbb{Q}/\mathbb{Z})$.

Using the boundary homomorphism

$$d : H_2(M;\mathbb{Q}/\mathbb{Z}) \to H_1(M) = H_1(M;\mathbb{Z}),$$

whose image is equal to Tors $H_1(M)$, we derive from $\theta$ a function

$$\Theta : \text{Tors} H_1(M) \to \mathbb{R}_+$$

by $\Theta(u) = \inf_{x \in d^{-1}(u)} \theta(x)$ for any $u \in \text{Tors} H_1(M)$. One can view $\Theta(u)$ as a
“normalized minimal genus” of oriented knots in $M$ representing $u$. If $M$ is a
rational homology sphere, then $d$ is an isomorphism and $\Theta = \theta \circ d^{-1}$.

We give an estimate of the function $\theta$ from above in terms of the Thurston
semi-norm on knot complements in $M$. This estimate implies that $\theta$ is bounded
from above and is upper semi-continuous with respect to a natural topology on
$H_2(M;\mathbb{Q}/\mathbb{Z})$. (I do not know whether $\theta$ is continuous.) The functions $\theta$ and $\Theta$ are
also estimated from below using abelian torsions of $M$. These estimates are parallel.
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to the McMullen \[6\] estimate of the Thurston semi-norm in terms of the Alexander polynomial.

In contrast to the Thurston semi-norm, the function \(\theta\) is non-homogeneous, that is in general \(\theta(kx) \neq k \theta(x)\) for \(k \in \mathbb{Z}\) and \(x \in H_2(M; \mathbb{Q}/\mathbb{Z})\). Examples show that the function \(\theta\) may not satisfy the triangle inequality.

The Thurston semi-norm of a 3-manifold \(M\) is fully determined by the Heegaard-Floer homology of \(M\), see \[7\], and by the Seiberg-Witten monopole homology of \(M\), see \[4\]. It would be interesting to obtain similar computations of the functions \(\theta\) and \(\Theta\).

The organization of the paper is as follows. We introduce the functions \(\theta\) and \(\Theta\) in Section 2 and estimate them from above in Section 3. In Section 4 these functions are estimated from below in the case where the first Betti number of the 3-manifold is non-zero. A similar estimate for rational homology spheres is given in Section 5. In Section 6 we describe a few examples. In Section 7 we make several miscellaneous remarks.

Throughout the paper, the unspecified group of coefficients in homology is \(\mathbb{Z}\).

2. Folded surfaces and the functions \(\theta, \Theta\)

2.1. Folded surfaces. By a \emph{folded surface} (without boundary), we mean a compact 2-dimensional polyhedron such that each point has a neighborhood homeomorphic to a union of several half-planes in \(\mathbb{R}^3\) with common boundary line. Such a neighborhood is homeomorphic to \(\mathbb{R} \times \Gamma_n\) where \(n\) is a positive integer and \(\Gamma_n\) is a union of \(n\) closed intervals with one common endpoint and no other common points.

The \emph{interior} \(\text{Int}(X)\) of a folded surface \(X\) consists of the points of \(X\) which have neighborhoods homeomorphic to \(\mathbb{R}^2\). Clearly, \(\text{Int}(X)\) is a 2-dimensional manifold. The \emph{singular set} \(\text{sing}(X) = X - \text{Int}(X)\) of \(X\) consists of a finite number of disjoint circles. A neighborhood of a component of \(\text{sing}(X)\) in \(X\) fibers over this component with fiber \(\Gamma_n\) for some \(n \neq 2\).

Cutting out \(X\) along \(\text{sing}(X)\) we obtain a compact 2-manifold (with boundary) \(X_{\text{cut}}\). Each component of \(\text{Int}(X)\) is the interior of a component of \(X_{\text{cut}}\). Set \(\chi_-(X) = \sum_Y \chi_-(Y)\), where \(Y\) runs over all components of \(X_{\text{cut}}\) and

\[\chi_-(Y) = \max(-\chi(Y), 0)\]

The number \(\chi_-(X) \geq 0\) measures the complexity of \(X\). It is equal to zero if and only if all components of \(X_{\text{cut}}\) are either spheres or tori or annuli or disks.

By an \emph{orientation} of a folded surface \(X\), we mean an orientation of the 2-manifold \(\text{Int}(X)\). An orientation of \(X\) allows us to view \(X\) as a singular 2-chain with integer coefficients. This 2-chain is denoted by the same letter \(X\). Its boundary expands as \(\sum_K i(K) \langle K \rangle\) where \(K\) runs over connected components of \(\text{sing}(X)\), the symbol \(\langle K \rangle\) denotes a 1-cycle on \(K\) representing a generator of \(H_1(K) \cong \mathbb{Z}\), and \(i(K) \in \mathbb{Z}\). Multiplying, if necessary, both \(\langle K \rangle\) and \(i(K)\) by \(-1\), we can assume that \(i(K) \geq 0\).

In this way the integer \(i(K)\) is uniquely determined by \(K\). It is called the \emph{index} of \(K\) in \(X\). For \(K\) with \(i(K) \neq 0\), the 1-cycle \(\langle K \rangle\) determines an orientation of \(K\).

We say that this orientation is \emph{induced} by the one on \(X\).

We call a folded surface \(X\) \emph{simple} if it is oriented, the set \(\text{sing}(X)\) is homeomorphic to a circle, and its index in \(X\) is non-zero. This index is denoted \(i_X\). Note
that $X$ is not required to be connected; however, all components of $X$ but one are closed oriented 2-manifolds.

2.2. Representation of 2-homology by folded surfaces. Let $M$ be an orientable 3-manifold. By a folded surface in $M$, we mean a folded surface embedded in $M$. Given a simple folded surface $X$ in $M$, the 2-chain $(i_X)^{-1}X$ with rational coefficients is a 2-cycle modulo $\mathbb{Z}$. This cycle represents a homology class in $H_2(M; \mathbb{Q}/\mathbb{Z})$ denoted $[X]$.

The short exact sequence of groups of coefficients $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ induces an exact homology sequence

$$
\cdots \to H_2(M; \mathbb{Q}) \to H_2(M; \mathbb{Q}/\mathbb{Z}) \to H_1(M) \to H_1(M; \mathbb{Q}) \to \cdots.
$$

The homomorphism $H_2(M; \mathbb{Q}/\mathbb{Z}) \to H_1(M)$ in this sequence will be denoted $d_M$ and called the boundary homomorphism. The exactness of (1) implies that the image of $d_M$ is equal to the group $\text{Tors} H_1(M)$ consisting of all elements of $H_1(M)$ of finite order.

For a simple folded surface $X$ in $M$, the homomorphism $d_M$ sends $[X]$ into the 1-homology class represented by the circle $\text{sing}(X)$ with orientation induced by the one on $X$.

For example, if $X \subset M$ is a compact oriented 2-manifold with connected nonvoid boundary, then $X$ is a simple folded surface with $\text{sing}(X) = \partial X$, $i_X = 1$, and $[X] = 0$. Another example: consider an unknotted circle $K$ lying in a 3-ball in $M$ and pick $n \neq 2$ closed 2-disks bounded by $K$ in this ball and having no other common points. We orient these disks so that the induced orientations on $K$ are the same. The union of these disks, $X = X(n)$, is a simple folded surface with $\text{sing}(X) = K$, $i_X = n$, and $[X] = 0$.

**Lemma 2.1.** Any homology class $x \in H_2(M; \mathbb{Q}/\mathbb{Z})$ can be represented by a simple folded surface.

**Proof.** Set $d = d_M : H_2(M; \mathbb{Q}/\mathbb{Z}) \to H_1(M)$. We can represent $d(x) \in \text{Tors} H_1(M)$ by an oriented embedded circle $K \subset \text{Int}(M) = M - \partial M$. Pick an integer $n \geq 1$ such that $n d(x) = 0$. The standard arguments, using the Poincaré duality and transversality, show that there is a simple folded surface $X$ in $M$ such that $\text{sing}(X) = K$ and $i_X = n$.

Since both $X$ and $M$ are orientable, the 1-dimensional normal bundle of $\text{Int}(X)$ in $M$ is trivial. Keeping $\text{sing}(X)$ and pushing $X - \text{sing}(X)$ in a normal direction, we obtain a “parallel” copy $X_1$ of $X$ such that $X \cap X_1 = \text{sing}(X_1) = \text{sing}(X) = K$. The orientation of $X$ induces an orientation of $X_1$ in the obvious way. Repeating this process $k \geq 1$ times, we can obtain $k$ parallel copies $X_1, X_2, \ldots, X_k$ of $X$ meeting each other exactly at $K$. Then $X^{(k)} = X_1 \cup X_2 \cup \ldots \cup X_k$ is a simple folded surface such that $\text{sing}(X^{(k)}) = K$ and $i_{X^{(k)}} = nk$. It follows from the construction that $[X^{(k)}] = [X] \in H_2(M; \mathbb{Q}/\mathbb{Z})$ for all $k \geq 1$.

The equalities $d(x) = [K] = d([X])$ imply that $x - [X] \in \text{Ker} d = \text{Im} j$, where $j$ is the homomorphism $H_2(M; \mathbb{Q}) \to H_2(M; \mathbb{Q}/\mathbb{Z})$ induced by the projection $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$. Pick $y \in j^{-1}(x - [X]) \subset H_2(M; \mathbb{Q})$. There is an integer $k \geq 1$ such that $ky$ lies in the image of the coefficient homomorphism $H_2(M; \mathbb{Q}) \to H_2(M; \mathbb{Q})$. *A fortiori*, the homology class $nky$ lies in this image. We represent $nky$ by a closed oriented (possibly non-connected) surface $\Sigma \subset M$. Since $d(x) \in \text{Tors} H_1(M)$, the intersection number $\Sigma \cdot K = \Sigma \cdot d(x)$ is 0. Applying if necessary surgeries of index
1 to $\Sigma$, we can assume that $\Sigma \cap K = \emptyset$. Then $y$ is represented by the 2-cycle $(nk)^{-1} \Sigma$ in $M - K$ and $x = [X] + j(y) = [X^{(k)}] + j(y)$ is represented by the 2-cycle $(nk)^{-1}(X^{(k)} + \Sigma)$ mod $\mathbb{Z}$. Applying to $X^{(k)}$ and $\Sigma$ the usual cut and paste technique, we can transform their union into a simple folded surface $Z$ such that $\text{sing}(Z) = \text{sing}(X^{(k)}) = K$ and $i_Z = nk$. Clearly, $[Z] = x$. \qed

2.3. Functions $\theta$ and $\Theta$. For an orientable 3-dimensional manifold $M$, we define a function $\theta = \theta_M : H_2(M; \mathbb{Q}/\mathbb{Z}) \to \mathbb{R}_+$ by
\begin{equation}
\theta(x) = \inf_X \frac{\chi_-(X)}{i_X},
\end{equation}
where $x \in H_2(M; \mathbb{Q}/\mathbb{Z})$ and $X$ runs over all simple folded surfaces in $M$ representing $x$. In particular, the class $x = 0$ can be represented by the simple folded surface $X = X(n) \subset M$ with $n \neq 2$, constructed before Lemma 2.1. The equality $\chi_-(X) = 0$ implies that $\theta(0) = 0$.

For a simple folded surface $X$, denote by $-X$ the same simple folded surface with opposite orientation in its interior. The obvious equalities
\begin{align*}
[-X] = -[X], & \quad \chi_-(X) = \chi_-(X), & \quad i_{-X} = i_X
\end{align*}
imply that $\theta(-x) = \theta(x)$ for all $x \in H_2(M; \mathbb{Q}/\mathbb{Z})$.

We define a function $\Theta = \Theta_M : \text{Tors} H_1(M) \to \mathbb{R}_+$ by
\begin{equation}
\Theta(u) = \inf_{x \in d(u)} \theta(x) = \inf_X \frac{\chi_-(X)}{i_X},
\end{equation}
where $u \in \text{Tors} H_1(M)$, $X$ runs over all simple folded surfaces in $M$ such that the circle $\text{sing}(X)$ represents $u$, and $d : H_2(M; \mathbb{Q}/\mathbb{Z}) \to H_1(M)$ is the boundary homomorphism. In (3), we can restrict ourselves to connected $X$. Indeed, all components of $X$ disjoint from $\text{sing}(X)$ are closed oriented surfaces. They may be removed from $X$ without increasing $\chi_-(X)$.

The properties of $\theta$ imply that $\Theta(0) = 0$ and $\Theta(-u) = \theta(u)$ for all $u \in \text{Tors} H_1(M)$. By the very definition of $\Theta$, for all $x \in H_2(M; \mathbb{Q}/\mathbb{Z})$,
\begin{equation}
\theta(x) \geq \Theta(d(x)).
\end{equation}

Using folded surfaces with boundary, we can similarly define relative versions
\begin{equation*}
H_2(M, \partial M; \mathbb{Q}/\mathbb{Z}) \to \mathbb{R}_+ \quad \text{and} \quad \text{Tors} H_1(M, \partial M; \mathbb{Z}) \to \mathbb{R}_+
\end{equation*}
of the functions $\theta$ and $\Theta$. We will not study them in this paper.

2.4. Constructions and examples. 1. Let $\Sigma$ be a closed connected 2-manifold embedded in an oriented 3-manifold $M$. Let $K \subset \Sigma$ be a simple closed curve such that $\Sigma - K$ has an orientation which switches to the opposite when one crosses $K$ in $\Sigma$. (Such an orientation exists when $\Sigma$ is orientable and $K$ splits $\Sigma$ into two surfaces or when $\Sigma$ is non-orientable and $K$ represents the Stiefel-Whitney class $w^i(\Sigma) \in H^i(\Sigma; \mathbb{Z}/2\mathbb{Z}) = H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$.) The orientations of $M$ and $\Sigma - K$ induce an orientation of the normal bundle of $\Sigma - K$ in $M$. Keeping $K$ and pushing $\Sigma - K$ in the corresponding normal direction, we obtain a copy $\Sigma'$ of $\Sigma$ such that $\Sigma'$ transversely meets $\Sigma$ along $K$. The union $X = \Sigma \cup \Sigma'$ is a simple folded surface such that $\text{sing}(X) = K$ and $i_X = 4$. Then $\theta([X]) \leq (1/4) \chi_-(X) = (1/2) \chi_-(\Sigma - K)$.

For example, we can apply this construction to the projective plane $\Sigma = \mathbb{R}P^2$ in $\mathbb{R}P^3$ taking as $K$ a projective circle on $\mathbb{R}P^2$. The resulting simple folded surface $X$ represents the only non-zero element $x$ of $H_2(\mathbb{R}P^3; \mathbb{Q}/\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ because $\text{sing}(X)$
represents the non-zero element of $H_1(\mathbb{R}P^3) = \mathbb{Z}/2\mathbb{Z}$. The equality $\chi_-(\Sigma - K) = 0$ implies that $\theta_{\mathbb{R}P^3} = 0$ and $\Theta_{\mathbb{R}P^3} = 0$.

2. Consider the 3-dimensional lens space $M = L(p,q)$, where $p,q$ are co-prime integers with $p \geq 2$. The manifold $M$ splits as a union of two solid tori with common boundary. It is easy to exhibit a folded surface $X \subset M$ such that $\text{sing}(X)$ is the core circle of one of the solid tori and $X - \text{sing}(X)$ is a disjoint union of $p$ open 2-disks. This implies that the function $\Theta_M$ annihilates the elements of $H_1(M)$ represented by the core circles of the solid tori. Under an appropriate isomorphism $H_1(M) \cong \mathbb{Z}/p\mathbb{Z}$, these elements correspond to $1$ (mod $p$) and $q$ (mod $p$). This implies that $\Theta_M = 0$ if $p = 2$ or $p = 3$ or $p = 5, q = 2$. For $p = 2$, we recover the previous example, since $L(2,1) = \mathbb{R}P^3$.

3. Let $K$ be an oriented homologically trivial knot in an oriented 3-manifold $N$. Let $M$ be obtained by a $(p,q)$-surgery on $K$ where $p,q$ are co-prime integers with $p \geq 2$. Thus, $M$ is obtained by cutting out a tubular neighborhood $U \subset N$ of $K$ and gluing it back along a homeomorphism $\partial U \to \partial U$ mapping the meridian $\mu \subset \partial U$ of $K$ onto a curve on $\partial U$ homological to $p\mu + q\lambda$, where $\lambda \subset \partial U$ is the longitude of $K$ homologically trivial in $N - K$. The element $u \in H_1(M)$ represented by the (oriented) core circle of the solid torus $U \subset M$ has finite order. This follows from the fact that the $p$-th power of the core circle is homotopic in $U \subset M$ to $\lambda \subset \partial U$. We claim that $\Theta(u) = 0$ if $K$ is a trivial knot in $N$ and $\Theta(u) \leq p^{-1}(2g - 1)$ if $K$ is a non-trivial knot of genus $g \geq 1$. Indeed, the longitude $\lambda$ bounds in $N - \text{Int}(U)$ an embedded compact connected oriented surface of genus $g$. This surface extends in the obvious way to a simple folded surface $X$ in $M$ such that $\text{sing}(X)$ is the core circle of $U \subset M$ and $i_X = p$. Clearly, $\chi_-(X) = \max(2g - 1, 0)$. This implies our claim. (For $p = 2$, one should “double” $X$ along $\text{sing}(X)$ as in Example 1.) As we shall see below, if $K$ is a non-trivial fibred knot and $p \geq 4g - 2$, then $\Theta(u) = p^{-1}(2g - 1)$.

3. Estimates from above and semi-continuity

In this section we estimate the function $\theta = \theta_M$ from above using the Thurston norm. Throughout this section, $M$ is a connected orientable 3-manifold (possibly, non-compact).

3.1. Comparison with the Thurston norm. Recall first the definition of the Thurston semi-norm $\|\cdot\|_M$ on $H_2(M;\mathbb{Q})$. The Poincaré duality (applied to compact submanifolds of $M$) implies that the abelian group $H_2(M) = H_2(M;\mathbb{Z})$ has no torsion. We shall view $H_2(M)$ as a lattice in the $\mathbb{Q}$-vector space $H_2(M;\mathbb{Q}) = \mathbb{Q} \otimes_{\mathbb{Z}} H_2(M)$. For any $x \in H_2(M;\mathbb{Q})$, there is an integer $n \geq 1$ such that $nx \in H_2(M)$. Then $\|x\|_M = n^{-1} \min_{\Sigma} \chi_-(\Sigma) \in \mathbb{Q}$, where $\Sigma$ runs over all closed oriented embedded surfaces in $M$ representing $nx$. The number $\|x\|_M$ does not depend on the choice of $n$ and is always realized by a certain $\Sigma$. Using surfaces in $M$ with boundary on $\partial M$, one similarly defines the Thurston semi-norm on $H_2(M,\partial M;\mathbb{Q})$.

Lemma 3.1. Let $j$ be the coefficient homomorphism $H_2(M;\mathbb{Q}) \to H_2(M;\mathbb{Q}/\mathbb{Z})$. Then $\theta(j(x)) \leq \|x\|_M$ for any $x \in H_2(M;\mathbb{Q})$.

Proof. Let $\Sigma$ be a closed oriented embedded surface in $M$ representing $nx \in H_2(M)$ with $n \geq 3$. The surface $\Sigma$ is an oriented folded surface with empty singular set. Consider a folded surface $X = X(n)$ inside a 3-ball in $M - \Sigma$, as constructed before.
Lemma 2.1 The union \( Z = X \cup \Sigma \) is a simple folded surface representing \( x \) and \( i_Z = i_X = n \). By the definition of \( \theta \),

\[
\theta(j(x)) \leq n^{-1} \chi_-(Z) = n^{-1} \chi_-(\Sigma).
\]

Therefore \( \theta(j(x)) \leq ||x||_M \).

\[
\Box
\]

Lemma 3.2. Let \( K \) be an oriented knot in \( M \). Set \( N = M - K \) and let \( \imath \) be the inclusion homomorphism \( H_2(N; \mathbb{Q}) \to H_2(M; \mathbb{Q}) \). Let \( j \) be the coefficient homomorphism \( H_2(M; \mathbb{Q}) \to H_2(M; \mathbb{Q}/\mathbb{Z}) \). Then for any simple folded surface \( X \) in \( M \) with \( \text{sing}(X) = K \) and any \( y \in H_2(N; \mathbb{Q}) \),

\[
\theta([X] + j_\imath(y)) \leq (i_X)^{-1} \chi_-(X) + ||y||_N.
\]

Proof. Set \( n = i_X \) and let \( k \) be a positive integer such that \( k\imath \in H_2(N) \subset H_2(N; \mathbb{Q}) \). It is enough to prove that for any closed oriented surface \( \Sigma \subset N \) representing \( n\imath \),

\[
\theta([X] + j_\imath(y)) \leq n^{-1} \chi_-(X) + (nk)^{-1} \chi_-(\Sigma).
\]

This can be reformulated in terms of the simple folded surface \( X^{(k)} \) as

\[
\theta([X^{(k)}] + j_\imath(y)) \leq (nk)^{-1} (\chi_-(X^{(k)}) + \chi_-(\Sigma)).
\]

Therefore it is enough to prove that for any simple folded surface \( T \) in \( M \) with \( \text{sing}(T) = K \) and \( i_T = nk \),

\[
\theta([T] + j_\imath(y)) \leq (nk)^{-1} (\chi_-(T) + \chi_-(\Sigma)).
\]

Suppose first that \( T \) is compressible in \( N = M - K \) in the sense that there is an embedded closed 2-disk \( D \subset N \) such that \( T \cap D = \partial D \subset T - K \) and the circle \( \partial D \) does not bound a 2-disk in \( T - K \). The surgery on \( T \) along \( D \) yields a simple folded surface \( T_D \) with \([T_D] = [T]\) and \( \chi_-(T_D) < \chi_-(T) \). Applying this procedure several times, we can reduce (4) to the case where \( T \) is incompressible, i.e., \( T \) admits no disks \( D \) as above. By the same reasoning, we can assume that \( \Sigma \) is incompressible in \( N \) (it may be compressible in \( M \)). The homology class \([T] + j_\imath(y) \in H_2(M; \mathbb{Q}/\mathbb{Z})\) is represented by the 2-cycle \((nk)^{-1} T \cup \Sigma \) (mod \( \mathbb{Z} \)). Deforming \( \Sigma \) in \( N \) so that it meets \( T \) transversely and applying to \( T \cup \Sigma \) the usual cut and paste technique, we can transform \( T \cup \Sigma \) into a simple folded surface \( Z \) with \( \text{sing}(Z) = \text{sing}(T) = K \) and \( i_Z = nk \). Clearly, \([Z] = [T] + j_\imath(y) \). The folded surface \( Z \) may have spherical components (that is components homeomorphic to \( S^2 \)) created from pieces of \( T - K \) and \( \Sigma \) by cutting and pasting. One of these pieces will necessarily be a 2-disk \( D \) such that either \( D \subset T - K \) and \( \partial D \subset \Sigma \) or \( D \supset \Sigma \) and \( \partial D \subset (T - K) \). In the first case the incompressibility of \( \Sigma \) implies that the circle \( \partial D \) bounds a disk on \( \Sigma \). The surgery on \( \Sigma \) along \( D \) yields a surface \( \Sigma_+ \approx \Sigma \amalg S^2 \) homological to \( \Sigma \) in \( N \). Then \( \chi_-(\Sigma_+) = \chi_-(\Sigma) \) and the 1-manifold \( T \cap \Sigma_+ \) has one component less than \( T \cap \Sigma \). Similarly, if \( D \subset \Sigma \), then the incompressibility of \( T - K \) implies that \( \partial D \) bounds a disk on \( T - K \). The surgery on \( T \) along \( D \) yields a simple folded surface \( T_+ \approx T \amalg S^2 \) such that \([T+] = [T] \), \( \chi_-(T+) = \chi_-(T) \), and the 1-manifold \( T_+ \cap \Sigma \) has one component less than \( T \cap \Sigma \). Continuing in this way, we can reduce ourselves to the case where \( Z \) does not have spherical components except the spherical components of \( T \) disjoint from \( \Sigma \) and the spherical components of \( \Sigma \) disjoint from \( T \). A similar argument allows us to assume that the components of \( Z - K \) are not disks except the disk components of \( T - K \) disjoint from \( \Sigma \).
the additivity of the Euler characteristic under cutting and pasting implies that 
\( \chi_-(Z) = \chi_-(T) + \chi_-(\Sigma) \). Therefore 
\[ \theta((T) + j_!y) \leq (nk)^{-1}\chi_-(Z) = (nk)^{-1}(\chi_-(T) + \chi_-(\Sigma)). \]
This proves (4), (5), and (6).

**Theorem 3.3.** If \( M \) is compact, then there is a number \( C > 0 \) (depending on \( M \)) such that \( \theta(x) \leq C \) for all \( x \in H_2(M; \mathbb{Q}/\mathbb{Z}) \).

**Proof.** Set \( d = d_M : H_2(M; \mathbb{Q}/\mathbb{Z}) \to H_1(M) \). Since the group \( \text{Im} \, d = \text{Tors} \, H_1(M) \) is finite, it is enough to prove that for every \( u \in \text{Tors} \, H_1(M) \), the values of \( \theta \) on the elements of the set \( d^{-1}(u) \) are bounded from above.

Consider first the case \( u = 0 \). Then \( d^{-1}(u) = \text{Im} \, j \) where \( j \) is the coefficient homomorphism \( H_2(M; \mathbb{Q}) \to H_2(M; \mathbb{Q}/\mathbb{Z}) \). We need to prove that the values of \( \theta \circ j \) are bounded from above. Since \( M \) is compact, the group \( H_2(M) \) is finitely generated. Pick a basis \( a_1, ..., a_n \) in \( H_2(M) \) and let \( Q \subset H_2(M; \mathbb{Q}) \) be the cube consisting of the vectors \( r_1a_1 + ... + r_na_n \) with rational non-negative \( r_1, ..., r_n \leq 1 \). The supremum \( s = \sup \, \|x\|_M \) is a finite number, because the Thurston semi-norm extends to a continuous semi-norm on \( H_2(M; \mathbb{R}) \) and the closure of \( Q \) in \( H_2(M; \mathbb{R}) \) is compact. We claim that \( \theta(j(x)) \leq s \) for any \( x \in H_2(M; \mathbb{Q}) \). Indeed, there is \( a \in H_2(M) \) such that \( x + a \in Q \). Then \( j(x) = j(x + a) \) and \( \theta(j(x)) = \theta(j(x + a)) \leq s \).

Consider now the case \( u \neq 0 \). Pick an oriented knot \( K \subset M \) representing \( u \) and a simple folded surface \( X \) in \( M \) with \( \text{sing}(X) = K \). Then \( d^{-1}(u) = \{[X] + j_!(y)\}_{y} \) where \( i \) is the inclusion homomorphism \( H_2(M - K; \mathbb{Q}) \to H_2(M; \mathbb{Q}) \) and \( y \) runs over \( H_2(M - K; \mathbb{Q}) \). The rest of the argument goes as in the case \( u = 0 \) using Lemma 3.2. \( \square \)

### 3.2. Semi-continuity
For compact \( M \), the group \( H_2(M; \mathbb{Q}/\mathbb{Z}) \) has a natural topology as follows. The image of the coefficient homomorphism \( j : H_2(M; \mathbb{Q}) \to H_2(M; \mathbb{Q}/\mathbb{Z}) \) can be identified with the quotient \( H_2(M; \mathbb{Q})/H_2(M) \). Provide \( \text{Im}(j) \) with the quotient topology induced by the standard topology in the finite dimensional \( \mathbb{Q} \)-vector space \( H_2(M; \mathbb{Q}) \). This extends to a topology in \( H_2(M; \mathbb{Q}/\mathbb{Z}) \) by declaring a set \( U \subset H_2(M; \mathbb{Q}/\mathbb{Z}) \) open if \( (a + U) \cap \text{Im}(j) \) is open in \( \text{Im}(j) \) for all \( a \in H_2(M; \mathbb{Q}/\mathbb{Z}) \). Recall that an \( \mathbb{R} \)-valued function \( f \) on a topological space \( A \) is upper semi-continuous if for any point \( a \in A \) and any real \( \epsilon > 0 \), there is is a neighborhood \( U \subset A \) of \( a \) such that \( f(U) \subset (-\infty, f(a) + \epsilon) \).

**Lemma 3.4.** For compact \( M \), the function \( \theta = \theta_M \) is upper semi-continuous.

**Proof.** Let \( a \in H_2(M; \mathbb{Q}/\mathbb{Z}) \) and \( \epsilon > 0 \). Let \( X \) be a simple folded surface in \( M \) representing \( a \) and such that \( (i_X^{-1})\chi_-(X) \leq \theta(a) + \epsilon/2 \). Set \( K = \text{sing}(X) \) and \( N = M - K \). Let \( i : H_2(N; \mathbb{Q}) \to H_2(M; \mathbb{Q}) \) be the inclusion homomorphism. Put
\[ V = \{ y \in H_2(N; \mathbb{Q}) \mid \|y\|_N < \epsilon/2 \}. \]
The set \( V \) is open in \( H_2(N; \mathbb{Q}) \) since the Thurston norm is continuous. The set \( i(V) \) is open in \( H_2(M; \mathbb{Q}) \) since \( i \) is an epimorphism. The set \( j_!(V) \) is open in \( \text{Im}(j) \) by definition of the topology in \( \text{Im}(j) \). Finally, the set \( U = a + j_!(V) \) is an open neighborhood of \( a \) in \( H_2(M; \mathbb{Q}/\mathbb{Z}) \) by definition of the topology in \( H_2(M; \mathbb{Q}/\mathbb{Z}) \). By (4), \( \theta(U) \subset (-\infty, \theta(a) + \epsilon) \). Hence \( \theta \) is upper semi-continuous. \( \square \)
4. Estimates from below: the case $b_1 \geq 1$

In this section we give an estimate from below for the functions $\theta = \theta_M$ and $\Theta = \Theta_M$ of a 3-manifold $M$ with non-zero first Betti number $b_1(M)$.

We begin with preliminaries on group rings and abelian torsions of 3-manifolds.

4.1. Preliminaries. Let $H$ be a finitely generated abelian group written in multiplicative notation. Any element $a$ of the group ring $\mathbb{Q}[H]$ expands uniquely in the form $a = \sum_{h \in H} a_h h$, where $a_h \in \mathbb{Q}$ and $a_h = 0$ for all but finitely many $h$. We say that an element $h \in H$ is $a$-basic if $a_h \neq 0$. The (finite) set of $a$-basic elements of $H$ is denoted $B_a$. The element $\sum_{h \in \text{Tors} H} h$ of $\mathbb{Q}[H]$ will be denoted $\Sigma_H$. Clearly, $B_{\Sigma_H} = \text{Tors} H$.

The classical ring of quotients of $\mathbb{Q}[H]$ that is, the (commutative) ring obtained by inverting all non-zero-divisors of $\mathbb{Q}[H]$ is denoted $\mathbb{Q}(H)$. It is known that $\mathbb{Q}[H]$ splits as a direct sum of domains. Therefore $\mathbb{Q}(H)$ splits as a direct sum of fields and the natural ring homomorphism $\mathbb{Q}[H] \to \mathbb{Q}(H)$ is an embedding. We identify $\mathbb{Q}[H]$ with its image under this embedding. Note that if $H$ is a finite abelian group, then $\mathbb{Q}(H) = \mathbb{Q}[H]$.

Let $M$ be a compact connected 3-manifold. From now on, we use multiplicative notation for the group operation in $H = H_1(M)$. In particular, the neutral element of $H$ is denoted $1$. The manifold $M$ gives rise to a maximal abelian torsion $\tau(M)$ which is an element of $\mathbb{Q}(H)$ defined up to multiplication by $-1$ and elements of $H$, see [10][11]. If $b_1(M) \geq 2$, then all representatives of $\tau(M)$ belong to $\mathbb{Z}[H] \subset \mathbb{Q}[H] \subset \mathbb{Q}(H)$. We express this by writing $\tau(M) \in \mathbb{Z}[H]$. If $b_1(M) = 1$ and $\partial M \neq \emptyset$, then $\tau(M) \in \mathbb{Z}[H] + \Sigma_H \cdot \mathbb{Q}(H)$. This implies that $(h-1)\tau(M) \in \mathbb{Z}[H]$ for all $h \in \text{Tors} H$ (indeed $(h-1)\Sigma_H = 0$).

If $M$ is oriented and $b_1(M) \geq 2$, then the Thurston semi-norm $\| \cdot \|_M$ on $H_3(M, \partial M; \mathbb{Q})$ can be estimated in terms of $\tau(M)$ as follows (see [10]): for any $s \in H_3(M, \partial M; \mathbb{Q})$ and any representative $a \in \mathbb{Z}[H]$ of $\tau(M)$,

$$\|s\|_M \geq \max_{h,h' \in B_a} |h \cdot s - h' \cdot s|,$$

(7)

where $h \cdot s \in \mathbb{Z}$ is the intersection index of $h$ and $s$. Note that the right hand side of (7) does not depend on the choice of $a$ in $\tau(M)$.

4.2. An estimate for $\theta_M$. The function $\theta$ will be estimated in terms of spans of subsets of $\mathbb{Q}/\mathbb{Z}$. The span $\text{spn}(A)$ of a finite set $A \subset \mathbb{Q}/\mathbb{Z}$ is a rational number defined as the minimal length of an interval in $\mathbb{Q}/\mathbb{Z}$ containing $A$, that is the minimal rational number $t \geq 0$ such that for some $r \in \mathbb{Q}$, the projection of the set $[r, r + t] \cap \mathbb{Q}$ into $\mathbb{Q}/\mathbb{Z}$ contains $A$. Clearly, $1 > \text{spn}(A) \geq 0$ and $\text{spn}(A) = 0$ if and only if $A$ is empty or has only one element.

Given an oriented 3-manifold $M$ and a homology class $x \in H_3(M; \mathbb{Q}/\mathbb{Z})$, we set for any $a \in \mathbb{Q}[H_1(M)]$,

$$\text{spn}_a(x) = \text{spn}(\{ h \cdot x \}_{h \in B_a}),$$

where $h \cdot x \in \mathbb{Q}/\mathbb{Z}$ is the intersection index of $h$ and $x$. Clearly, $1 > \text{spn}_a(x) \geq 0$.

**Theorem 4.1.** Let $M$ be a compact connected oriented 3-manifold with $b_1(M) \geq 1$. Set $H = H_1(M)$ and let $\tau \in \mathbb{Q}(H)$ be a representative of the torsion $\tau(M)$. Let $x \in H_3(M; \mathbb{Q}/\mathbb{Z})$ and $u = d_M(x) \in H$. Then $(u-1)\tau \in \mathbb{Z}[H]$ and

$$\theta_M(x) \geq \text{spn}_a((u-1)\tau),$$

(8)
Proof: If \( b_1(M) \geq 2 \), then \( \tau \in \mathbb{Z}[H] \) and \((u - 1) \tau \in \mathbb{Z}[H] \). The inclusion \( u \in \text{Tors} \) H and the remarks in Section \[\text{[4.4]}\] imply that \((u - 1) \tau \in \mathbb{Z}[H] \) for \( b_1(M) = 1 \) as well.

We prove \([8]\). Let \( X \) be a simple folded surface in \( M \) representing \( x \). The knot \( \text{sing}(X) \subset M \) endowed with orientation induced from the one on \( X \) represents \( u \in \text{Tors} \) H. Let \( E \) be the exterior of this knot in \( M \). The homological sequence of the pair \((M, E)\) and the inclusion \( u \in \text{Tors} \) H imply that \( b_1(E) \geq b_1(M) + 1 \geq 2 \). Therefore \( \tau(E) \in \mathbb{Z}[H_1(E)] \). Pick a representative \( a \in \mathbb{Z}[H_1(E)] \) of \( \tau(E) \). Denote by \( \iota \) the inclusion homomorphism \( H_1(E) \to H_1(M) = H \) and denote \( \iota_* \) the induced ring homomorphism \( \mathbb{Z}[H_1(E)] \to \mathbb{Z}[H] \). By \([10]\), Theorem VII.1.4, \( \iota_*(a) = (u - 1)b \) where \( b \) is a representative of \( \tau(M) \). Note that the right hand side of \([8]\) does not depend on the choice of \( \tau \) in \( \tau(M) \). Therefore without loss of generality we can assume that \( \tau = b \).

Deforming, if necessary, \( X \) in \( M \), we can assume that \( S = X \cap E \) is the complement in \( X \) of a regular neighborhood of \( \text{sing}(X) \). Then \( S \) is a proper surface in \( E \) and \( \chi_-(X) = \chi_-(S) \). The orientation of \( \text{Int}(X) \) induces an orientation of \( S \). The oriented surface \( S \) represents a relative homology class \( s \in H_2(E, \partial E) \). By \([7]\),

\[
\chi_-(X) = \chi_-(S) \geq \max_{h, h' \in B_a} |h \cdot s - h' \cdot s|
\]

where \( B_a \subset H_1(E) \) is the set of \( a \)-basic elements. Let \( r \in \mathbb{Q} \) be the minimal element of the set \( \{h \cdot s\}_{h \in B_a} \). Then

\[
\{h \cdot s\}_{h \in B_a} \subset [r, r + \chi_-(X)].
\]

Denote the projection \( \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \) by \( \pi \). Observe that for any \( h \in H_1(E) \),

\[
\iota(h) \cdot x = \pi \left( \frac{h \cdot s}{i_X} \right).
\]

Therefore

\[
\{\iota(h) \cdot x\}_{h \in B_a} \subset \pi \left( \left[ \frac{r}{i_X}, \frac{r + \chi_-(X)}{i_X} \right] \right).
\]

The equality \( \iota_* (a) = (u - 1) \tau \) implies that \( B_{(u-1)\tau} \subset \iota(B_a) \). Hence

\[
\{g \cdot x\}_{g \in B_{(u-1)\tau}} \subset \{\iota(h) \cdot x\}_{h \in B_a} \subset \pi \left( \left[ \frac{r}{i_X}, \frac{r + \chi_-(X)}{i_X} \right] \right).
\]

Therefore

\[
\text{spn}_x((u - 1) \tau) \leq (i_X)^{-1} \chi_-(X).
\]

Since this holds for all simple folded surfaces \( X \) representing \( x \), we have \([8]\). \( \square \)

4.3. An estimate for \( \Theta_M \). Let \( M \) and \( H \) be as in Theorem \([4.4]\). To estimate the function \( \Theta_M : \text{Tors} H \to \mathbb{Q}/\mathbb{Z} \), we need the linking form \( L_M : \text{Tors} H \times \text{Tors} H \to \mathbb{Q}/\mathbb{Z} \) of \( M \). It is defined by \( L_M(h, g) = h \cdot x \in \mathbb{Q}/\mathbb{Z} \) where \( x \) is an arbitrary element of \( H_2(M; \mathbb{Q}/\mathbb{Z}) \) mapped to \( g \) by the boundary homomorphism \( \partial : H_2(M; \mathbb{Q}/\mathbb{Z}) \to H \). The pairing \( L_M \) is well defined, bilinear, and symmetric.

Given \( u \in \text{Tors} H \) and \( a \in \mathbb{Q}[H] \), set

\[
\text{spn}_u(a) = \text{spn} \left( \{L_M(h, u)\}_{h \in B_a \cap \text{Tors} H} \right).
\]

Clearly, \( \text{spn}_x(a) \geq \text{spn}_{\iota(x)}(a) \) for any \( x \in H_2(M; \mathbb{Q}/\mathbb{Z}) \) and any \( a \in \mathbb{Q}[H] \). This and Theorem \([4.4]\) imply that, under the conditions of this theorem,

\[
\Theta_M(u) \geq \text{spn}_u((u - 1) \tau),
\]

(9)

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for any \( u \in \text{Tors} H \) and any representative \( \tau \) of \( \tau(M) \). Generally speaking, the right-hand side of (9) depends on the choice of \( \tau \).

4.4. **Remark.** Estimate (7) strengthens the McMullen [8] estimate of the Thurston norm via the Alexander polynomial. For recent more general estimates of this type, see Friedl [3].

5. **Estimates from below: the case of \( \mathbb{Q} \)-homology spheres**

For \( \mathbb{Q} \)-homology spheres, the functions \( \theta \) and \( \Theta \) contain the same information and it is enough to give an estimate for \( \Theta \). We begin with preliminaries on refined torsions and \( \mathbb{Q} \)-homology spheres, referring for details to [10], Chapters I and X.

5.1. **Refined torsions.** The maximal abelian torsion \( \tau(M) \) of a compact connected 3-manifold \( M \) admits a refinement \( \tau(M, e, \omega) \in \mathbb{Q}(H_1(M)) \) depending on an orientation \( \omega \) in the vector space \( H_*(M; \mathbb{Q}) = \oplus_{i \geq 0} H_i(M; \mathbb{Q}) \) and an Euler structure \( e \) on \( M \). An Euler structure on \( M \) is determined by a non-singular vector field on \( M \) directed outside on \( \partial M \). Two such vector fields determine the same Euler structure if for a point \( x \in \text{Int}(M) \), the restrictions of these fields to \( M - \{x\} \) are homotopic in the class of non-singular vector field on \( M - \{x\} \) directed outside on \( \partial M \). The set of Euler structures on \( M \) is denoted \( \text{Eul}(M) \). This set admits a canonical free transitive action of the group \( H_1(M) \). The torsion \( \tau(M, e, \omega) \) satisfies \( \tau(M, he, \pm \omega) = \pm h \tau(M, e, \omega) \) for any \( e \in \text{Eul}(M), h \in H_1(M) \). The refined torsion \( \tau(M) \) is just the set \( \{\pm \tau(M, e, \omega)\}_{e \in \text{Eul}(M)} \). If \( \partial M = \emptyset \), then the set \( \text{Eul}(M) \) can be identified with the set of \( \text{Spin}^c \)-structures on \( M \).

5.2. **Homology spheres.** Let \( M \) be an oriented 3-dimensional \( \mathbb{Q} \)-homology sphere. Denote \( \omega_M \) the orientation in \( H_*(M; \mathbb{Q}) = H_0(M; \mathbb{Q}) \oplus H_3(M; \mathbb{Q}) \) determined by the following basis: (the homology class of a point, the fundamental class of \( M \)).

The group \( H = H_1(M) \) is finite and the linking form \( L_M : H \times H \to \mathbb{Q}/\mathbb{Z} \) is non-degenerate in the sense that the adjoint homomorphism \( H \to \text{Hom}(H, \mathbb{Q}/\mathbb{Z}) \) is an isomorphism. Recall that we use multiplicative notation for the group operation in \( H \). Every Euler structure \( e \in \text{Eul}(M) \) determines a torsion \( \tau(M, e, \omega_M) \in \mathbb{Q}(H) = \mathbb{Q}[H] \). The linking form \( L_M \) can be computed from this torsion by

\[
L_M(h, g) = -\pi((1 - h)(1 - g) \tau(M, e, \omega_M))_1 \in \mathbb{Q}/\mathbb{Z}
\]

for all \( h, g \in H \), where \( \pi \) is the projection \( \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \) and for any \( a \in \mathbb{Q}[H] \), the symbol \( a_1 \in \mathbb{Q} \) denotes the coefficient of the neutral element \( 1 \in H \) in the expansion of \( a \) as a formal linear combination of elements of \( H \) with rational coefficients. The Euler structure \( e \) determines a function \( q_e : H \to \mathbb{Q}/\mathbb{Z} \) by

\[
q_e(u) = \pi(((1 - u) \tau(M, e, \omega_M))_1),
\]

for any \( u \in H \). It follows from (10), (11) that \( q_e \) is quadratic in the sense that \( q_e(hg) = q_e(h) + q_e(g) + L_M(h, g) \) for all \( h, g \in H \). Formula (11) also implies that

\[
q_{he}(u) = q_e(u) + L_M(h, u),
\]

for any \( h \in H \).

If \( u \in H \) has order \( n \) (i.e., \( n \) is the minimal positive integer such that \( u^n = 1 \)), then by [10], Section X.4.3 there is a unique residue \( K(e, u) \in \mathbb{Z}/2n\mathbb{Z} \) such that

\[
q_e(u) = \frac{K(e, u)}{2n} + \frac{1}{2} \pmod{\mathbb{Z}}.
\]

\[ q_e(u) = \frac{K(e, u)}{2n} + \frac{1}{2} \pmod{\mathbb{Z}}. \]
Formula \( \text{(12)} \) implies that the residue \( K(e,u) \pmod{2} \) does not depend on \( e \). We say that \( u \) is even if this residue is 0 and odd if it is 1.

Every homology class \( u \in H \) gives rise to a group

\[
G = G_u = \{ g \in H \mid L_M(u,g) = 0 \} \subset H.
\]

The non-degeneracy of \( L_M \) implies that the quotient \( H/G \) is a finite cyclic group whose order is equal to the order, \( n \), of \( u \) in \( H \). Moreover, there is an element \( v = v_u \in H \) such that \( L_M(u,v) = n^{-1} \pmod{Z} \). Such \( v \) is determined by \( u \) uniquely up to multiplication by elements of \( G \). The inclusion \( v^n \in G \) implies that the order, \( p \), of \( v \) is divisible by \( n \) (in particular, \( p \geq n \)). Set

\[
\alpha_v = \frac{1 + 2v + 3v^2 + \cdots + pv^{p-1}}{p} - \frac{p + 1}{2} \cdot \frac{1 + v + \cdots + v^{p-1}}{p}.
\]

This element of \( \mathbb{Q}[H] \) can be uniquely characterized by the following property: for any ring homomorphism \( \varphi \) from \( \mathbb{Q}[H] \) to a field, \( \varphi(v) = 1 \Rightarrow \varphi(\alpha_v) = 0 \) and \( \varphi(v) \neq 1 \Rightarrow \varphi(\alpha_v) = (\varphi(v) - 1)^{-1} \). In \( \text{[10]} \) we used the notation \((v-1)^{-1}_{\text{par}}\) for \( \alpha_v \).

**Theorem 5.1.** Let \( M \) be an oriented 3-dimensional \( \mathbb{Q} \)-homology sphere. Let \( u \) be an element of \( H = H_1(M) \) of order \( n \geq 1 \). Set \( G = \{ g \in H \mid L_M(u,g) = 0 \} \) and \( \Sigma_G = \sum_{g \in G} g \in Z[G] \subset Z[H] \). Pick any \( v \in H \) such that \( L_M(u,v) = n^{-1} \pmod{1} \). For \( e \in \text{Eul}(M) \), set

\[
a_e(u) = (u-1)^2 \tau(M,e,\omega_M) - \frac{v^{K(e,u)/2}(v+1)}{2} \alpha_v \Sigma_G \in \mathbb{Q}[H],
\]

if \( u \) is even and

\[
a_e(u) = (u-1)^2 \tau(M,e,\omega_M) - v^{(K(e,u)+1)/2} \alpha_v \Sigma_G \in \mathbb{Q}[H],
\]

if \( u \) is odd. Then for any \( e \in \text{Eul}(M) \),

\[
\Theta_M(u) \geq \text{spn}_u(a_e(u)) = \text{spn} (\{L_M(h,u)\}_{h \in B_{a_e(u)}}).
\]

**Proof:** If \( u \) is even (resp. odd), then \( K(e,u) \in \mathbb{Z}_{2n} \) is even (resp. odd). Therefore the power of \( v \) in the definition of \( a_e(u) \) is well defined up to multiplication by \( v^n \). However, \( v^n \in G \) and \( v^n \Sigma_G = \Sigma_G \). Therefore the right hand sides of the formulas for \( a_e(u) \) are well defined. If \( v' \) is another element of \( H \) such that \( L_M(u,v') = n^{-1} \pmod{1} \), then \( v' \in G \) and \( v^k \Sigma_G = (v')^k \Sigma_G \) for all \( k \in \mathbb{Z} \). Therefore \( a_e(u) \) does not depend on the choice of \( v \). It is easy to see that \( a_e(h \cdot u) = h \cdot a_e(u) \) for all \( h \in H \). Therefore the number \( \text{spn}_u(a_e(u)) \) does not depend on \( e \).

Let \( X \subset M \) be a simple folded surface representing the 2-homology class \( x = d_M^{-1}(u) \in H_2(M;\mathbb{Q}/\mathbb{Z}) \). The knot \( K = \text{sing}(X) \) with orientation induced from the one on \( X \) represents \( u \in H_1(M) \). Let \( E \) be the exterior of \( K \) in \( M \). Clearly \( b_1(E) = 1 \). Fix an orientation \( \omega \) in \( H_1(E;\mathbb{Q}) \) and an Euler structure \( e_K \) on \( E \). The torsion \( \tau(E,e_K,\omega) \in Q(H_1(E)) \) can be canonically expanded as a sum of a certain \( \tau = \tau(E,e_K,\omega) \in \mathbb{Q}[H_1(E)] \) with an element of \( Q(H_1(E)) \) given by an explicit formula using solely \( \omega \) and the Chern class of \( e_K \), see \( \text{[10]} \), Section II.4.5. The inclusion homomorphism \( Q(H_1(E)) \to \mathbb{Q}[H_1(M)] \) sends \( \tau \) to \( \pm_a(u) \) for some \( a \in \text{Eul}(M) \), see \( \text{[10]} \), Formula X.4.d. The inequality \( \text{(7)} \) holds for any \( s \in H_2(E,\partial E;\mathbb{Q}) \) and \( a = \tau \), see \( \text{[10]} \), Chapter IV. The rest of the argument goes as the proof of Theorem \( \text{[4]} \) with \( \tau \) replaced by \( \tau \). This gives \( (i_X)^{-1} \chi_-(X) \geq \text{spn}_u(a_e(u)) = \text{spn}_u(a_e(u)) \). Since this holds for all \( X \) representing \( x \), we have

\[
\Theta_M(u) = \theta_M(x) \geq \text{spn}_u(a_e(u)).
\]

\( \square \)
5.3. **Remarks.** Let \( \frac{1}{2} \mathbb{Z} \) be the additive group of integers and half-integers. In Theorem 5.1, \( a_e(u) \in \mathbb{Z}[H] \) if \( u \) is even and \( a_e(u) \in \frac{1}{2} \mathbb{Z}[H] \) if \( u \) is odd. This follows from the proof of this theorem and the inclusion \([\tau] \in \mathbb{Z}[H_1(E)]\) if \( u \) is even and \([\tau] \in \frac{1}{2} \mathbb{Z}[H_1(E)]\) if \( u \) is odd.

2. It is proven in [2] that the function \( q_e : H \to \mathbb{Q}/\mathbb{Z} \) derived from the torsion coincides with the quadratic function defined geometrically in [5], [1].

6. **Examples**

6.1. **Lens spaces.** The computation of the abelian torsions for the lens space \( M = L(p,q) \) goes back to K. Reidemeister, see, for instance, [9] for an introduction to the theory of torsions. Let \( t, t^b \) be the generators of \( H = H_1(M) \) represented by the core circles of the two solid tori forming \( M \). For an appropriate choice of an orientation on \( M \) and an Euler structure \( e \) on \( M \), we have \( \tau(M, e, \omega_M) = \alpha_t \alpha_v \), where \( \alpha_v \in \mathbb{Q}[H] \) is defined by (14) for any \( v \in H \). This allows us to compute \( a_e(u) \) for any \( u \in H \) and to apply Theorem 5.1. We give here a few examples.

Consider the lens space \( M = L(5,1) \). By Sections 2.3, \( \Theta(t^4) = \Theta(t) = \Theta(1) = 0 \) and \( \Theta(t^2) = \Theta(t^3) \). We show that \( \Theta(t^2) \geq 1/5 \). We have

\[
a_t = -2 - t^3 + 2t^4.
\]

Then

\[
\tau = \tau(M, e, \omega_M) = \alpha_t^2 = \frac{t + t^2 - 2t^4}{5}.
\]

A direct computation shows that \( L_M(t, t) = (-1 - t)^2 \tau_1 = 1/5 \) and \( q_e(t^2) = ((1-t) \tau)_1 = 0 \). Note that \( u = t^2 \) has order 5 in \( H \). From (13), \( K(e, u) = 5 \) (mod 10). Therefore \( u \) is odd. The associated group \( G_u \) is trivial, \( v = v_u = t^3 \), and

\[
a_e(u) = (u - 1) \tau - v^3 \alpha_v = t^4 - t.
\]

Since \( L_M(t^4, u) = 3/5 \) (mod 1) and \( L_M(t, u) = 2/5 \) (mod 1), the span of the set \( \{L_M(h, u) \}_{h \in G_u} \) is equal to 1/5. By Theorem 5.1, \( \Theta(t^2) \geq 1/5 \). In this example, the function \( \Theta : H \to \mathbb{R}_+ \) takes non-zero values only on \( t^2 \) and \( t^3 \). This function is non-homogeneous and does not satisfy the triangle inequality.

Consider the lens space \( M = L(6,1) \). Then

\[
\alpha_t = \frac{-5 - 3t - t^2 + t^3 + 3t^4 + 5t^5}{12},
\]

\[
\tau = \alpha_t^2 = \frac{-5 + 13t + 19t^2 + 13t^3 - 5t^4 - 35t^5}{72},
\]

and \( L_M(t, t) = 1/6 \). For \( u = t^2 \), the computations similar to the ones above give \( q_e(u) = 0 \) (mod 1), \( K(e, u) = 3 \) (mod 6), \( G_u = \{1, t^3\} \), \( v_u = t \), and \( a_e(u) = t^5 - t \). Theorem 5.1 yields \( \Theta(t^2) \geq 1/3 \). For \( u = t^3 \), we similarly obtain \( q_e(u) = 3/4 \) (mod 1), \( K(e, u) = 1 \) (mod 4), \( G_u = \{1, t^2, t^4\} \), \( v_u = t \), and \( a_e(u) = t^5 - t^2 \). Theorem 5.1 yields \( \Theta(t^3) \geq 1/2 \).

6.2. **Surgeries on knots.** Let \( L \) be an oriented knot in an oriented 3-dimensional \( \mathbb{Z} \)-homology sphere \( N \). Let \( M \) be the closed oriented 3-manifold obtained by surgery on \( N \) along \( L \) with framing \( p \geq 2 \). Let \( u \in H = H_1(M) \) be the homology class of the meridian of \( L \) whose linking number with \( L \) is +1. Clearly, \( H \) is a cyclic group of order \( p \) with generator \( u \) and \( L_M(u, u) = p^{-1} \) (mod 1). We explain now how to
estimate $\Theta(u)$ in terms of the Alexander polynomial of $L$. We will see that in some cases this estimate is exact.

Recall that the \textit{span} $\text{span}(\Delta)$ of a non-zero Laurent polynomial $\Delta = \sum a_i t^i \in \mathbb{Z}[t^{\pm 1}]$ is the number $\max \{i \mid a_i \neq 0 \} - \min \{i \mid a_i \neq 0 \}$. Let $\Delta = \Delta_L(t)$ be the Alexander polynomial of $L$ normalized so that $\Delta(t^{-1}) = \Delta(t)$ and $\Delta(1) = 1$. Expand $\Delta(t) = 1 + (t - 1) \beta(t)$ where $\beta(t) \in \mathbb{Z}[t^{\pm 1}]$. We claim that the expression $a_e(u) \in \mathbb{Q}[H]$ defined in Theorem 5.1 is equal to $\beta(u)$ for an appropriate Euler structure $e$ on $M$. By Theorem 5.1 this will imply that $\Theta(u) \geq \text{span}(\beta(u))$. For example, if $p \geq 2 \text{span}(\beta)$, then $\text{span}(\beta(u)) = p^{-1} \text{span}(\beta) = p^{-1}(\text{span}(\Delta) - 1)$.

Therefore $\Theta(u) \geq p^{-1}(\text{span}(\Delta) - 1)$. On the other hand, by Section 2.4.3, $\Theta(u) \leq p^{-1}(2g - 1)$, where $g$ is the genus of $K$. In particular, if $\text{span}(\Delta) = 2g > 0$ (for instance, if $K$ is a non-trivial fibred knot) and $p \geq 4g - 2$, then $\Theta(u) = p^{-1}(2g - 1)$.

We now verify the claim above. Set $\tau = \alpha^2 \Delta(u) \in \mathbb{Q}[H]$. It is easy to deduce from the multiplicativity of the torsions that $\tau(M, e, \omega_M) = \tau$ for a certain orientation on $M$ and a certain Euler structure $e$ on $M$ (for details, see [10], Formula X.5.e). Set $\sigma = 1 + u + u^2 + \cdots + u^{p-1} \in \mathbb{Z}[H]$. Clearly, $\sigma^k = \sigma$ for any integer $k$. Therefore for any integer 1-variable polynomial $f$, the product $\sigma^k f(u)$ is equal to $\text{aug}(f) \sigma$ where $\text{aug}(f) = f(1)$ is the sum of coefficients of $f$. Since $\text{aug}(\alpha_u) = 0$, we have $\sigma \alpha_u = 0$. A direct computation shows that $(1 - u) \alpha_u = \sigma/p - 1$. Hence

\[(1 - u) \tau = (1 - u) \alpha^2 \Delta = (\sigma/p - 1) \alpha \Delta = -\alpha \Delta(u)
= -\alpha + \alpha (1 - u) \beta = -\alpha + (\sigma/p - 1) \beta = -\alpha - \beta(u),\]

where we use the equality $\text{aug}(\beta) = 0$ which follows from the symmetry of $\Delta$. Thus,

\[q_e(u) = ((1 - u) \tau) = -\alpha_1 = (p - 1)/2p \pmod{1}.
\]

Formula (6.3) implies that $K(e, u) = -1 \pmod{2p}$. In particular, $u$ is odd.

We also have

\[(1 - u)^2 \tau = (1 - u)(-\alpha - \beta(u)) = 1 - \sigma/p - (1 - u)\beta(u).
\]

Hence

\[L_M(u, u) = -((1 - u)^2 \tau) = p^{-1} \pmod{1}.
\]

This shows that the orientation of $M$ chosen so that $\tau(M, e, \omega_M) = \tau$ is actually the one induced from the orientation on $N$. The equality $L_M(u, u) = p^{-1} \pmod{1}$ implies that $v_u = u$ and $G_u = 1$. We conclude that

\[a_e(u) = (u - 1) \tau - \alpha_u = \alpha_u + \beta(u) - \alpha_u = \beta(u).
\]

6.3. \textbf{Surgeries on 2-component links.} Let $M$ be a closed oriented 3-manifold obtained by surgery on a 2-component oriented link $L = L_1 \cup L_2$ in an oriented 3-dimensional $\mathbb{Z}$-homology sphere $N$. Suppose that the linking number of $L_1, L_2$ in $N$ is 0, the framing of $L_1$ is $p \neq 0$, and the framing of $L_2$ is 0. Then $H = H_1(M) = (\mathbb{Z}/p\mathbb{Z})u_1 \oplus \mathbb{Z}u_2$, where $u_i \in H$ is the homology class of the meridian of $L_i$ whose linking number with $L_i$ is +1, for $i = 1, 2$. The Alexander polynomial of $L$ has the form

\[\Delta_L(t_1, t_2) = f(t_1, t_2)(t_1 - 1)(t_2 - 1)\]

for some Laurent polynomial $f(t_1, t_2) \in \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}]$. Both $\Delta_L$ and $f$ are defined only up to multiplication by $-1$ and monomials on $t_1, t_2$. By [10], Formula VIII.4.e, the torsion $\tau(M)$ is represented by

\[\tau = f(u_1, u_2) \pm \Delta_L(u_2) u_2^u (u_2 - 1)^{-2} \Sigma_H \in \mathbb{Q}(H)\]
for an appropriate sign $\pm$ and an integer $n$, both depending on the choice of $f$. Here $\Delta_{L_2}$ is the Alexander polynomial of $L_2$ normalized as in Section 6.2. Pick $x \in H_2(M; \mathbb{Q}/\mathbb{Z})$ and set $u = d(x) \in \text{tors } H$. Since $(u - 1)\Sigma_H = 0$, Theorem 4.1 implies that
\begin{equation}
\theta(x) \geq \text{spn}_z((u - 1) f(u_1, u_2)).
\end{equation}
For sufficiently big $p$, the span on the right hand side does not depend on $p$.

Note another curious phenomenon. Suppose for simplicity that $f(t_1, t_2) = 1$ (a constant polynomial). Then $\theta(x) \geq \text{spn}_z(u - 1)$. If $u = d(x) \neq 1$, then the set $B_{u-1} \subset H$ consists of two elements $u, 1$ and
\[
\text{spn}_z(u - 1) = \text{spn}(\{u \cdot x, 0\}) = \text{spn}(\{L_M(u, u), 0\}).
\]
For $u = u_1^k$ with $k \in \{0, 1, \ldots, n - 1\}$, we have $L_M(u, u) = k^2/n (\text{mod } 1)$. For $k < \sqrt{n/2}$, we obtain $\text{spn}(\{L_M(u, u), 0\}) = k^2/n$. Thus $\Theta(u_1^k) \geq k^2/n$. This suggests that the number $\Theta(u_1^k)$, considered as a function of $k$, may behave like a quadratic function for small values of $k$.

7. Miscellaneous

7.1. Quasi-simple folded surfaces. One can use a larger class of folded surfaces to represent 2-homology classes. Let us call a folded surface $X$ quasi-simple if it is oriented, $\text{sing}(X) \neq \emptyset$, and the indices of all components of $\text{sing}(X)$ in $X$ are equal to each other and non-zero. Denote the common value of these indices $i_X$.

In particular, simple folded surfaces are quasi-simple.

For a quasi-simple folded surface $X$ in a 3-manifold $M$, the 2-chain $(i_X)^{-1}X$ is a 2-cycle mod $\mathbb{Z}$ representing a homology class $[X] \in H_2(M; \mathbb{Q}/\mathbb{Z})$. We claim that
\begin{equation}
\theta([X]) \leq i_X^{-1} \chi_-(X) + b_0(\text{sing}(X)) - 1
\end{equation}
where $b_0(\text{sing}(X))$ is the number of components of $\text{sing}(X)$. Indeed, $X$ can be modified in a neighborhood of $\text{sing}(X)$ so that each point of $\text{sing}(X)$ is adjacent to exactly $i_X$ local branches of $\text{Int}(X)$ (which then induce the same orientation on $\text{sing}(X)$). Let $\Gamma$ be a graph with two vertices and $i_X$ edges connecting these vertices. Given an embedded arc in $M$ with endpoints on different components of $\text{sing}(X)$ and with interior in $M - X$, we can modify $X$ by cutting it out along $\text{sing}(X)$ near the endpoints and gluing in $\Gamma \times [0, 1]$ along the arc. This gives a quasi-simple folded surface, $Z$, such that
\[
b_0(\text{sing}(Z)) = b_0(\text{sing}(X)) - 1, \quad i_Z = i_X, \quad [Z] = [X], \quad \text{and } \chi_-(Z) \leq \chi_-(X) + i_X.
\]
Modifying $X$ in this way, we can reduce ourselves to the case where $\text{sing}(X)$ is connected. In this case (17) follows from the definition of $\theta$. It may happen that there are no distinct components of $\text{sing}(X)$ connected by an arc with interior in $M - X$. This occurs if each arc joining distinct components of $\text{sing}(X)$ has to cross the closed 2-manifold $X_0$ formed by the components of $X$ disjoint from $\text{sing}(X)$.

To circumvent this obstruction, we first modify $X_0$ so that $X - X_0$ is contained in a connected component of $M - X_0$, cf. [10], p. 60.

Formula (16) implies that for any $x \in H_2(M; \mathbb{Q}/\mathbb{Z})$,
\begin{equation}
\theta(x) = \inf_X \left( \frac{\chi_-(X)}{i_X} + b_0(\text{sing}(X)) \right) - 1,
\end{equation}
where $X$ runs over all quasi-simple folded surfaces in $M$ representing $x$. 

7.2. Coverings. Let $M$ be a compact oriented 3-manifold and $p : \widehat{M} \to M$ be an $n$-fold (unramified) covering. Let $p^* : H_2(M; \mathbb{Q}/\mathbb{Z}) \to H_2(\widehat{M}; \mathbb{Q}/\mathbb{Z})$ be the following composition of the duality isomorphisms and the pull back

$$H_2(M; \mathbb{Q}/\mathbb{Z}) \cong H^1(M, \partial M; \mathbb{Q}/\mathbb{Z}) \to H^1(\widehat{M}, \partial \widehat{M}; \mathbb{Q}/\mathbb{Z}) \cong H_2(\widehat{M}; \mathbb{Q}/\mathbb{Z}).$$

Then for any $x \in H_2(M; \mathbb{Q}/\mathbb{Z})$,

$$\theta^{-1}_M(p^*(x)) + 1 \leq n(\theta_M(x) + 1).$$

This follows from \cite{17} and the fact that if a simple folded surface $X$ in $M$ represents $x$, then $p^{-1}(X) \subset \widehat{M}$ is a quasi-simple folded surface representing $p^*(x)$.

7.3. Norms associated with links. A link $L$ in an oriented 3-manifold $M$ determines a semi-norm $\| \cdot \|$ on $H_2(M; \mathbb{Q})$ as follows. Let $U \subset M$ be a regular neighborhood of $L$ and $E = \overline{M - U}$ the exterior of $L$. We can embed $H_2(M; \mathbb{Q})$ into $H_2(E, \partial E; \mathbb{Q})$ via the inclusion homomorphism

$$H_2(M; \mathbb{Q}) \hookrightarrow H_2(M, L; \mathbb{Q}) \cong H_2(M, U; \mathbb{Q}) \cong H_2(E, \partial E; \mathbb{Q}).$$

Restricting the Thurston semi-norm on $H_2(E, \partial E; \mathbb{Q})$ to $H_2(M; \mathbb{Q})$, we obtain the semi-norm $\| \cdot \|_{M,L}$. The arguments as above allow us to estimate the latter semi-norm from below for compact $M$. Namely, if $L$ has $m \geq 1$ components and $h_1, \ldots, h_m \in H = H_1(M)$ are their homology classes, then

$$\|x\|_{M,L} \geq \text{spn}_x \left( \prod_{i=1}^{m} (h_i - 1) \tau \right)$$

for any $x \in H_2(M; \mathbb{Q})$ and any $\tau \in Q(H)$ representing $\tau(M)$ in the case $b_1(M) \geq 2$ and representing $[\tau](M)$ in the case $b_1(M) = 1$. A similar construction can be used to derive a function on $H_2(M; \mathbb{Q}/\mathbb{Z})$ from the function $\theta$ on $H_2(E, \partial E; \mathbb{Q}/\mathbb{Z})$. It would be interesting to see whether these semi-norms and functions may be used to distinguish non-isotopic links.

7.4. Open questions. Is the infimum in \cite{2} realizable by a simple folded surface? Does $\theta$ take only rational values? A positive answer to the first question certainly implies a positive answer to the second one. Similar questions can be asked for $\Theta$.

It would be interesting to compute the function $\Theta$ for the lens spaces. Is it true that for the lens spaces, the inequality in Theorem 5.1 is an equality?

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