Tree Amplitudes in Scalar Field Theories

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Abstract

The tree amplitudes in scalar field theories are presented at all $n$. The momentum routing of propagators is given at $n$-point in terms of a specified set of numbers, and the mass expansion of the massive theories is generated. A group structure on the diagrams is given. The tree amplitudes can be used to find the effective action.
Introduction

The tree amplitudes of gauge theories have been recently under a large amount of interest, in view of the simplified derivation using the weak-weak duality of the gauge theory with a twistor formulation. In general the tree amplitudes of quantum field theories are required in order to find the $L \geq 1$ loop amplitudes (the general form is necessary also in the quantum derivative expansion [2]-[11]). In this work, scalar field theories are examined in order to generate the coefficients of the scattering in the mass expansion, and the general form of the classical amplitudes.

The mass expansion of the scalar graphs is direct to obtain. An obstacle is a compact formula that spells out the momentum routing of the individual diagrams. One way of generating this formula is by iterating the amplitudes through attaching trees to the ladder diagrams. The means presented gives a mapping between a set of integers that parameterize the diagram to another set of numbers that label the propagators. (The ladders are effectively rooted trees and can be useful for other purposes, such as illustrating symmetry.)

The general momentum is sufficient to also specify massless scalar field classical scattering. The massless $\phi^3$ diagrams and their specification are required in order to notate all gauge theory and gravity theory tree-level amplitudes. The latter may be obtained in a non-spinor helicity basis, but in closed form via the known string-inspired tree-level rules. The complete specification of the spin-1 and spin-2 classical scattering is forthcoming [1].

In $d = 4$ the general form of the perturbatively renormalizable scalar field theory is

$$\mathcal{L} = \frac{1}{2}(\partial \phi \partial \phi + m^2 \phi^2) + \frac{\lambda_3}{3!} \phi^3 + \frac{\lambda_4}{4!} \phi^4.$$  \hspace{1cm} (1)

In general, within a momentum cutoff formalism, there may be further interactions,

$$\mathcal{L}_i = \lambda_6 \frac{\phi^6}{A^2} + \ldots,$$  \hspace{1cm} (2)

which require a further perturbative renormalization of the relevant terms in the theory [1].

The counting of the loop parameters for a $\phi^3$ diagram with insertions of $v_{2m}^{(2n)}$ operators $\partial^{2n} \phi^{2m}$ is
\[3v + \sum 2mv_{2m}^{(2n)} = 2I + E\]  

\[L = I - v + 1 - \sum v_{2m}^{(2n)} \quad L \equiv \text{Loop No},\]  

with \(v\) the number of \(\phi^3\) vertices, \(I\) the number of internal lines, and \(E\) the number of external lines. The case of \(\phi^4\) vertices is included by setting \(m = 2\) and \(n = 0\).

A derivation of loop amplitudes in either a coupling expansion involving the parameters \(\lambda_i\) or in momenta require the general form of the classical scattering, i.e. tree-level scattering. The general form at all \(n\)-point has not appeared in the literature and is presented here. The \(\phi^3\) and \(\phi^4\) scalar field theories pertinent to four dimensions are examined in this work, although general theories may be examined as well. The latter theories are of import to higher dimension operators (i.e. 'irrelevant' ones from the definition of the renormalization group flow) and to theories in various dimensions.

\(\phi^3\) Theory

The classical amplitudes considered are placed in a color ordered form; the primary ordering is \((1, \ldots, n)\); the non-colored theory’s amplitudes are derived by summing the sets permutations. A general scalar field theory diagram at tree-level is parameterized by the set of propagators at the momenta labeling them. The diagrams are labeled by

\[D_\sigma = \lambda^{n-2} \prod \frac{1}{t_{\sigma(i,p)} - m^2},\]  

with the Lorentz invariants \(t_{\sigma(i,p)}\) defined by,

\[t_{\sigma(i,p)} = (k_{\sigma(i)} + \ldots + k_{\sigma(i+p-1)})^2.\]  

The sets of permutations \(\sigma\) are what are required in order to specify the individual diagrams. The full sets of combinatorics \(\sigma(i,p)\) form all of the diagrams. These combinatorics, for a given uniform mass \(m\), change between \(\phi^3\) and \(\phi^4\) theory, but are necessary and sufficient to label all of the diagrams. An example 6-point graph has the collection of the numbers as:
\[ \sigma(1, 2) = \sigma(3, 2) = \sigma(5, 2) = \sigma(1, 4) = \sigma(3, 4) = \sigma(5, 4) = 1 . \] (7)

These numbers parameterize the diagrams with the momenta on the external legs following a cyclic ordering.

An additional vector \( \eta(i) \) is required in order to specify the color ordering. As only the primary ordering of \( 1, \ldots, n \) is considered, this vector is not relevant.

The individual terms in the scattering are parameterized by all of the individual momenta flows, found by the diagrams in (5); these terms are to be expanded in momenta, using the mass parameter. Such an expansion is described in the 'effective' action by terms such as

\[
L_{\text{eff}} = \frac{\lambda_3}{m^2} \phi^5 + \frac{\lambda_4}{m^3} (\partial^2 \phi^3) \phi^3 + \ldots .
\] (8)

In general all operators are found by expanding the classical scattering (i.e. the tree diagrams) in derivatives, or rather in the mass. The coefficients and the momenta flow are determined from the diagram’s momentum structure. The terms in the action (8) can be constrained in the placement of the derivatives in the individual terms of \( L_{\text{eff}} \); as an example, at five-point only the \( \phi^2 \partial^2 \phi^3 \) are allowed due to the momentum flow of the contributing diagrams. This restriction on the momentum structure becomes more apparent when the theory is quantized (classically) together with higher dimension operators.

The general term of the classical scattering written in momentum space is,

\[
D_\sigma = \lambda_3^{n-2} \prod \frac{1}{t_{\sigma(i,p)} - m^2} .
\] (9)

The mass expansion of the diagrams in (9), i.e. with only the three-point coupling, is

\[
L_{\sigma, \tilde{\sigma}}^n = C_{\sigma} \frac{\lambda_3^{n-2}}{m^{n-3}} \prod \frac{t_{\tilde{\sigma}(i,p)}}{m^{2\tilde{\sigma}(i,p)}} .
\] (10)

with the coefficient \( C_\sigma \) determined from the tree graphs. The additional set of permutations \( \tilde{\sigma} \) is required in order to specify the expansion of the propagators.

The momenta invariants \( t_{\sigma(i,p)} \) at \( n \)-point are defined in (10). In specifying the effective action comprised of all the terms in (10) the numbers and the orderings of
Figure 1: The ladder tree diagrams at seven-point.
\( \sigma(i,p) \) (and \( \tilde{\sigma}(i,p) \)) must be given. In general at \( n \)-point, the invariants are defined for \( i = 1 \) to \( i = n \) with a non-cyclic ordering of the numbers when \( p \) forces the numbers of \( \sigma(j) \) to go beyond \( \sigma(n) \) (these numbers pertain to an \( n \)-point graph). For example, at 6-point the sets of numbers of \( \sigma \) are labeled by a collecting the indices from 1 to \( n \) in particular orderings such as (e.g. \( 1, 3, 2, 6, 5, 4 \)).

The general \( \phi^3 \) \( n \)-point diagram, upon color ordering, is constructed from a preferred ladder diagram with \( n - 4 \) internal legs. At each of these legs is attached an internal \( m < n \) point tree diagram. The sum of all of these ladder diagrams from \( m = 1 \) to \( m = n - 4 \), with the currents attached, generates the complete sum of graphs at tree level. These ladder diagrams at a fixed ordering possess \( 2^m \) permutations; they are illustrated in Figure 2. The propagator structure of these diagrams can be found by iterating the lower-point ladder diagrams. The preferred basis in terms of the ladder trees can be avoided, but some symmetry will be lost in the process.

The internal vertices are labeled so that the outer numbers from a two-particle tree are carried into the tree diagram in a manner so that \( j > i \) is always chosen from the two numbers. The numbers are carried in from the \( n \) most external lines. An example diagram with the labeling is illustrated in Figure 3 in the case of four-point currents which are to be attached to a ladder tree, and in Figure 4 for a ladder tree with four internal lines.

The labeling of the vertices is such that in a current the unordered numbers are sufficient to reconstruct the current. For an \( m \)-point current there are \( m - 1 \) vertices and hence \( m - 1 \) numbers contained in \( \phi_m(j) \). These \( m - 1 \) numbers are such that the greatest number may occur \( m - 1 \) times, and must occur at least once, the next largest number is \( m - 2 \), and so on. The smallest number can not occur in the set contained in \( \phi_m(j) \).

Two example permutation sets are:

\[
\begin{pmatrix}
444 \\
443 \\
442 \\
433 \\
432
\end{pmatrix}
\]

(11)
with the 5(3) representing the (3)-permutation set attached to the 5 in the total count. There are 5 and 15 in the counts. The set of numbers in $\phi(j)$ is ordered from largest to least.

The $4 + m$ point ladder is labeled as $\kappa = (a_1; a_2; \ldots; a_{m+2})$. Each of the numbers $a_i$ are from the last node of the current; the $a_1$ and $a_{m+2}$ are the external 2-point trees. The vertex numbers are found by the previous clockwise $j > i$ labeling; the ladder node numbers $a_j$ are used in this definition. (The construction could also be made direct on an ordinary tree graph, without the use of currents, but some symmetry would be lost.)

The numbers $\kappa(i)$ and $\phi(j)$ are used to find the propagators in the labeled diagram. The procedure to determine the set of $t_i^{[p]}$, or the $\sigma(i, p)$, is as follows. First, label all momenta as $l_i = k_i$. Then, the invariants are found with the procedure,

1) $i = \phi(m - 1)$, $p = 2$, then $l_{am-1} + l_{am} \rightarrow l_{m-1}$

2) $i = \phi(m - 2)$, $p = 2$, then $l_{am-2} + l_{am-1} \rightarrow l_{m-2}$

\[ \ldots \]

$m - 1)$ $i = 1$, $p = m$

(13)

The labeling of the kinematics, i.e. $t_i^{[p]}$, is direct from the definition of the vertices.

The numbers $\phi_n(i)$ can be arranged into the numbers $(p_i; [p_i])$, in which $p_i$ is the repetition of the value of $[p_i]$. Also, if the number $p_i$ equals zero, then $[p_i]$ is not present in $\phi_n$. These numbers can be used to obtain the $t_i^{[q]}$ invariants without intermediate steps with the momenta. The branch rules are recognizable as, for a single $t_i^{[q]}$,
0) \( l_{\text{initial}} = [p_m] - 1 \)

1) 
\[ r = 1 \text{ to } r = p_m \]
if \( r + \sum_{j=1}^{m-1} p_j = [p_m] - l_{\text{initial}} \) then \( i = [p_m] \) \( q = [p_m] - l_{\text{initial}} + 1 \)
beginning conditions has no sum in \( p_j \)

2) 
else \( l_{\text{initial}} \rightarrow l_{\text{initial}} - 1 : \) decrement the line number
\( l_{\text{initial}} > [p_l] \) else \( l \rightarrow l - 1 : \) decrement the \( p \) sum

3) goto 1)

(14)

The branch rule has to be iterated to obtain all of the poles. This rule checks the number of vertices and matches to compare if there is a tree on it in a clockwise manner. If not, then the external line number \( l_{\text{initial}} \) is changed to \( l_{\text{initial}} \) and the tree is checked again. The \( i \) and \( q \) are labels to \( t_i^{[q]} \).

The algorithm in (14) has other forms. There should be a matrix transformation between the data in \( \phi_n(i) \) to the set of numbers in \( t_i^{[p]} \). The latter is of dimension \( n - 3 \) (could be twice that if the redundant invariants \( t_i^{[n-p]} \) are included in the set) and the former of \( n - 2 \).

The previous recipe pertains to currents, i.e. amplitudes with one leg off-shell and without a line factor. There are \( m - 1 \) poles in an \( m \)-point current (does not include the off-shell line in \( m \), but does include the pole). In order to apply the recipe to an amplitude, the three-point vertex is attached; the counting is clear when the attached vertex has two external lines with numbers less than the smallest external line number of the current. There are \( n - 3 \) poles in an \( n \)-point diagram, and these lines are accounted for in the amplitude with this formula; the ladder diagrams with their legs can be analyzed with this approach, or simply a current with the numbers \( \phi_m(i) \).

An example set of \( \sigma(i, p) \) pertains to several seven point diagrams, with the indices \( i \) and \( p \),
The ladder diagrams at $n + 4$ point; there are $2^n$ combinations. Some of the 'internal' lines are to be attached to an off-shell current to make the various $m > n + 4$ point diagrams.

\[
\begin{pmatrix}
t_i, 2 & t_i, 3 & t_i, 4 & t_i, 5 \\
16 & 15 & 14 & 13 \\
146 & 3 & 6 & 13 \\
15 & 47 & 37 & 37 \\
135 & 5 & 1 & 357 \\
\end{pmatrix}
\]  

(15)

The vertex labels are,

\[
\begin{pmatrix}
75432 \\
77552 \\
76662 \\
77642 \\
\end{pmatrix}
\]

(16)

The inclusion of the $\phi^4$ term requires the additional coupling $\lambda_4$ (in $d = 4$) that alters the coefficients $C_\sigma$. Higher dimension operators such as $\phi^6/\Lambda^2$ for example, require further modification of the tree results. The addition of these higher dimensional operators is relevant to model the full quantum field theory and its deformations, in general QFTs.

The derivation of the general term in (10) is obtained by expanding the general $n$-point graph using the standard propagators via

\[
\frac{1}{m^2 - p^2} = \sum_{q=0}^{\infty} \frac{1}{m^2} \left( \frac{p^2}{m^2} \right)^q,
\]  

(17)
Figure 3: The labeling of the 3-particle trees, which are to be sewn on to the ladder diagrams.

and collecting the terms in the power series. Due to the combinatorics $\sigma(i, p)$ which parameterize the momenta, the second factor $\tilde{\sigma}(i, p)$ is essentially arbitrary due to the infinite number of terms in the expansion (17); the entries are non-vanishing on the support of the $\sigma(i, p)$.

As an example, the zeroeth order term at $n$-point without the derivatives is simply found by collecting the number of coupling constants and mass terms from propagator expansions; the term at $n$-point is,

$$L^n_{\sigma, \tilde{\sigma}} = N_n \frac{\lambda^{n-2}}{m^2(n-3)},$$

with $N_n$ counting the number of graphs at $n$-point order. The parameters corresponding to these non-derivative terms are,

$$\tilde{\sigma} = (0, 0, \ldots, 0_n) \quad C_{\sigma, \tilde{\sigma}} = 1,$$

with $\sigma$ essentially undetermined for the $n$-point amplitude. The remaining terms require the expansions of the internal tree propagators, with momenta following the routings of these lines.
The crux then to finding the general term in (10) is in the orderings of $\sigma(i, p)$. The coefficients are direct, due to the unity coefficient of (17) in the expansion of the individual propagators. The powers in the mass expansion are labeled via all integers attached to the propagators, i.e. $\tilde{\sigma}(i, p)$. The symmetry factors of the diagrams are relevant, in addition to the signature of space-time due to the sign in $p^2 \pm m^2$. The set of $\sigma(i, p)$ are required to determine all of the terms in the classical effective action. The classical ‘quantization’ could be carried on the basis of the $\sigma(i, p)$, or $\phi_n(j)$, as all diagrams are constructable with these vectors.

The general $\sigma_n(i, p)$ can be determined via the collection of numbers in $\phi_n(i)$ in (13) or (14), and the latter can be labeled by the polynomials,

$$P(\sigma) = \sum \sigma_n(i, p)y^p x^i,$$

with the coefficients $(i, p)$ generating the expansion; the coefficients are unity as the propagator either exists or not (i.e. zero or one). The numbers $i$ are constrained between 1 and $n$, and those of $p$ are bounded by 2 to $n - 1$; the polynomial is essentially a matrix.

This polynomial in (20) should satisfy a differential equation due to its polynomial nature. For example, a two-dimensional harmonic oscillator generates solutions in
terms of Hermite polynomials, with a specified degeneracy at level \( n_1 + n_2 \). The expansion of the polynomials in the wavefunction \( \psi(x, y) = H_{n_1 n_2} e^{-x^2 - y^2} \) generate sets of coefficients; alternatively an OSp(\( n, n \)) wavefunction with the grassmann and base expansion could generate the \( P(\sigma) \) expansion. An example differential equation at level \( n \)-point could generate all of the coefficients, and one of the form

\[
\left( \partial_x^2 + \partial_y^2 + V_n(x, y) \right) \psi_n(x, y) = \varepsilon_n \psi_n(x, y) ,
\]

(21)
is desired (or an OSp(\( n, n \)) eigenoperator), with degeneracy and form of the wavefunction \( \psi_n(x, y) \) that of the tree-level field theory diagrams. The momenta associated with the operator in (21) is obtained with the ordering of the \( \sigma \) in a cyclic fashion, i.e. the principle color ordered series of 1, 2, \ldots, \( n \).

The degeneracy of diagrams in field theory for various field theories at \( n \)-point goes as \( n! \), and in gauge theory the number of independant ones at tree level taking into account Ward identities and color orderings follows approximately as \((n - 2)!\). The specific counting of color ordered \( \phi^3 \) graphs at \( n \)-point for low orders of \( n \) at tree level is,

\[
\begin{pmatrix}
  n = 4 & Q(4) = 2 \\
  n = 5 & Q(5) = 5 \\
  n = 6 & Q(6) = 15
\end{pmatrix}
\]

(22)
The count of \( \phi^4 \) graphs at low orders of \( n \) is,

\[
\begin{pmatrix}
  n = 4 & Q(3) = 1 \\
  n = 6 & Q(4) = 3 \\
  n = 8 & Q(5) = 6
\end{pmatrix}
\]

(23)
The general count of the tree diagrams follows from the various number theoretic combinations of \( \phi_n(i) \); that is, the count of sets of \( n \) numbers from \( n \) to 2 with the maximal occurrence of each number being \( n - 1 \) to 1. This is clearly a factorial. The count of the \( \phi^4 \) diagrams is obtained by eliminating propagators in \( \phi^3 \) graphs.

The algebra associated to tree diagrams can be made clear with the explicit parameterization in this section (see [12], [13] to find definitions of Hopf structures on rooted trees). The set of numbers \( \phi_n(i) \) form a basis with an OSp structure. The \( j \)th number may repeat at most \( j \) times in a set of \( n - 1 \) elements; the set of numbers and the combinations is more algebraic. The operations on trees have an action on these
Figure 5: The representation of a group structure on the ladder trees. There are $\mathbb{Z}_2$ factors at the nodes which orient the current.

vectors $\phi_n(i)$, such as eliminating nodes or propagators or exchanging the orders of legs. The sets of numbers $\phi_n(i)$ make the Hopf algebra of the rooted trees explicit, and the algebra can be obtained without reference to diagrams.

Another structure in the scalar theory is obtained from the currents attached to the ladder diagrams, which is not the most general one. These trees may be associated with elements $G_i$ and a $\mathbb{Z}_2$ which orients the branch, illustrated in Figure 5. Bracket operations such as $[G_i, G_j]$ can be placed on the trees, in accord with the labeling and dimension of the currents’ vector space.

Algebras in further classical scalar theories presumably may be obtained by pinching the $\phi^3$ diagrams to obtain their graphs, and then using the set of numbers $\phi_n(i)$. The classical quantization can be performed by using these numbers directly to obtain the tree diagrams. The discrete symmetry of the set of $\phi_n$, for all $n$, may be understood as an extension of the Poincare algebra (which generates the Lagrangian). It should be useful in the quantization.

$\phi^4$ Theory

In the case of $\phi^4$ theory the momentum routing of the individual diagrams is also modeled by a set of polynomials at $n$-point,
\[ P(\sigma_n) = \sum \sigma_4(i, p)x^iy^p. \]  

(24)

The determination of the general term in the scattering at tree-level follows in almost the same manner, except that the coupling constant has mass dimension in \( d = 4 \) and appears with a different factor at \( n \)-point (\( \lambda_n^{4/2-1} \) instead of \( \lambda_n^{3-2} \)). The color ordered \( \phi^4 \) diagrams can be obtained by pinching the propagators, i.e. removing non-adjacent propagators in certain \( \phi^3 \) trees.

Several ladder \( \phi^4 \) diagrams are illustrated in Figure 5. The pinching of \( \phi^3 \) diagrams is straightforward to obtain. These ladder diagrams are relevant for the same reason as the \( \phi^3 \) ones are: the potential sewing of currents to obtain amplitudes (which is not required), and the manifestation of an algebra of the currents and of the theory’s classical scattering.

The diagrams are defined by the \( \phi^{(4)}_n(i) \) numbers, which can be found by propagator pinching of \( \phi^{(3)}_n(i) \). The pinches occur on every other propagator, and this translates to an altering of the set of \( \phi^{(3)}_n(i) \).

The expansion in momenta (or in the mass) is accomplished via the power expansion of the individual propagators. The general term in the scattering as a function of the kinematic invariants is given by (with \( \sigma_4 \to \sigma \)),

\[ A_{\sigma, \tilde{\sigma}}^n = C_{\sigma, \tilde{\sigma}} \frac{\lambda^{n/2-1}}{m^{n/2-2}} \prod \frac{t^{\tilde{\sigma}(i, p)}}{m^{2\sigma(i, p)}}, \]  

(25)

The primary differences between the \( \phi^3 \) and \( \phi^4 \) theories are: 1) the powers of the coupling and mass differ due to the difference in the number of propagators, and 2) the combinatoric factor \( \phi^{(4)}_n(i) \) (or \( \sigma_4(i, p) \)) which labels the different momentum routing along the \( \phi^4 \) diagrams.

The \( t^{[p]}_i \) invariants which define the propagator define the vector space of the polynomials \( P(\sigma) = \sum \sigma(i, p)x^iy^j \). These functions are expected to solve, as in \( \phi^3 \) theory, a differential system,

\[ \left( \partial_x^2 + \partial_y^2 + V_n^{(3)}(x, y) \right) \psi_n(x, y) = E_n \psi_n(x, y). \]  

(26)

The degeneracy of the solutions should match the number of sets of vertex numbers \( \sigma_4(i, p) \), which generate the tree amplitudes of the theory.
Figure 6: The 3 tree 7-point ladder diagrams.
φ³ and φ⁴ Theory

The combination of φ³ and φ⁴ scalar interactions can be examined in the same context as φ⁴ theory. The momentum flow of diagrams in this theory, as in the φ³ theory, can be used to generate the propagator momentum flow in gauge theories although this theory is not required for that purpose. Tuning the couplings also interpolates between the two scalar examples.

The general form of the tree diagrams are more complicated as the number of propagators at a given n-point varies and is not a function only of the number of external lines. For example, at 6-point there may be one propagator in a pure φ⁴ graph or three propagators in a φ³ graph. The general form of the momenta routing is,

\[ D_\sigma = \lambda_3^{N_3} \lambda_4^{N_4} \prod \frac{1}{t_{\sigma(i,p)} - m^2}, \]  

(27)

with the propagators found via the σ₃,₄(i, p) combinatoric factors.

The momentum expansion of the diagrams is,

\[ A_{3,4,\sigma, \tilde{\sigma}}^n = C_{\sigma \tilde{\sigma}} \frac{\lambda_3^{N_3} \lambda_4^{N_4}}{m^{N_3-2}} \prod \frac{t_{\sigma(i,p)}^{\tilde{\sigma}(i,p)}}{m^{2\tilde{\sigma}(i,p)}}. \]  

(28)

The coefficients Cσ,tilde σ factors, i.e. the momentum tensor factors, differ from the individual φ³, φ⁴ theories. These may be obtained from pinching any numbers of propagators in a φ³ diagram.

The σ(i, p) combinatorics is again to be generated by the vector space of functions \( P_{\sigma, \tilde{\sigma}} = \sum \sigma_3,\sigma_4 x^i y^j \), with degeneracy at level n that of the number of n-point diagrams.

Conclusions

The set of tree amplitudes in φ³ and other scalar field theories is given. These amplitudes are obtained through sets of numbers φₙ which describe the propagator structure. The sets of numbers are quite simple; there are n − 1 of them from 2 to n and the jth one can occur more than j − 1 times. The φ⁴ and φ³ theories, with higher dimensional operators such as φ⁶ / Λ², are also number theoretic classically; the amplitudes of these theories can be found by pinching propagators of the φ³ theory in a systematic fashion, which generates a map to φₙ. Two dimensional models (and other dimensions) can also be examined in this context, in which the φₘ interactions are perturbatively renormalizable.
Classical scattering of these scalar theories are relevant examples for theories with non-vanishing spin. The routing of the propagators in these examples is necessary for the latter theories. These amplitudes can be covariantized in order to find the classical effective action.

The symmetries of the tree level scattering are obtainable through the vertex algebra associated with the diagrams, i.e. $\phi_n(i)$. The sets of numbers and their group aspect can be considered a discrete extension of the Poincare algebra. These numbers may be used for a direct classical quantization of the scattering, as given in the text. Further sets of numbers, which label the propagators, and their symmetries, are necessary for the quantization of higher loops by a direct writing down of the loop amplitudes without performing integrals.
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