Distribution of local density of states in disordered metallic samples: logarithmically normal asymptotics.

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Abstract

Asymptotical behavior of the distribution function of local density of states (LDOS) in disordered metallic samples is studied with making use of the supersymmetric $\sigma$–model approach, in combination with the saddle–point method. The LDOS distribution is found to have the logarithmically normal asymptotics for quasi–1D and 2D sample geometry. In the case of a quasi–1D sample, the result is confirmed by the exact solution. In 2D case a perfect agreement with an earlier renormalization group calculation is found. In 3D the found asymptotics is of somewhat different type: $P(\rho) \sim \exp(-\text{const} |\ln^3 \rho|)$.

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I. INTRODUCTION

Mesoscopic fluctuations of various physical quantities in disordered systems have been intensively investigated during the last years [1]. It was understood that a whole distribution function is to be studied to get the complete physical information concerning properties of a disordered system. In particular, statistical fluctuations of local quantities, such as eigenfunction intensity and local density of states (LDOS) have attracted a considerable research interest recently. Distributions of eigenfunction amplitudes and of LDOS are relevant for description of fluctuations of tunneling conductance across a quantum dot [4], of some atomic spectra properties [3], and of a shape of NMR line [3]. The microwave cavity technique [5] allows to simulate a disordered electronic system and to observe experimentally spatial fluctuations of the wave intensity [6].

Distribution of eigenfunction amplitudes $|\psi_i^2(r)|$, characterizes properly the spatial fluctuations in a closed system with well defined energy levels. On the other hand, in an open sample, connected to the leads, the states are broadened in an energy space. In this case, it is more appropriate to consider the distribution of LDOS,

$$\rho(E, r) = -\pi^{-1} \text{Im} G_R(r, r; E),$$

where $G_R$ is the retarded Green function. Distribution of LDOS in open metallic samples was studied by Altshuler, Kravtsov and Lerner [7,8] for the case of two− and $(2 + \epsilon)$−dimensional systems. The consideration was based on the derivation of the effective $\sigma$–model and its subsequent renormalization group analysis, following the earlier ideas by Wegner [9]. It was found that the LDOS distribution is close to the Gaussian one but has slowly decaying logarithmically normal (LN) asymptotics. When approaching the Anderson transition in $(2 + \epsilon)$ dimensions, the distribution was found to cross over to the completely LN one. Altshuler and Prigodin [10] studied the LDOS distribution in strictly 1D chains and also found the LN form of the distribution. It was conjectured on the basis of this similarity [7,10] that even in a metallic sample there is a finite probability to find “almost localized” eigenstates.

More recently, it was recognized [11–14] that the statistical properties of various quantities characterizing fluctuations of the wave functions can be very efficiently studied with use of the supersymmetric approach. In particular, it was shown in [11,14] that the distribution $P_y(y)$ of eigenfunction amplitudes $y = V|\psi(r_0)|^2$ (here $V$ is the system volume introduced for normalization purposes) can be written in terms of the supersymmetric $\sigma$–model in the following way. Let us define the function $Y(Q_0)$ as

$$Y(Q_0) = \int_{Q(r_0) = Q_0} DQ(r) \exp\{-S[Q]\},$$

where $S[Q]$ is the $\sigma$–model action

$$S[Q] = -\int d^d r \text{Str} \left[ \frac{\pi \nu D}{4} (\nabla Q)^2 - \pi \nu \eta \Lambda Q \right],$$

$Q(r)$ is a $4 \times 4$ supermatrix field, $\Lambda = \text{diag}\{1, 1, -1, -1\}$, $\text{Str}$ stands for the supertrace, $D$ is the diffusion constant, $\eta$ is the level broadening (imaginary frequency) and $\nu$ is the mean DOS [19]. We do not go into details of the supersymmetric formalism here, which can...
be found e.g. in [13–17]. For the invariance reasons, the function $Y(Q_0)$ turns out to be dependent on the two scalar variables $1 \leq \lambda_1 < \infty$ and $-1 \leq \lambda_2 \leq 1$, which are eigenvalues of the “retarded-retarded” block of the matrix $Q_0$. Moreover, in the limit $\eta \to 0$ (at a fixed finite value of the system volume) only the dependence on $\lambda_1$ persists:

$$Y(Q_0) \equiv Y(\lambda_1, \lambda_2) \to Y_a(2\pi \nu \eta \lambda_1)$$

(3)

The distribution of the eigenfunctions intensities is given by [14]

$$P_y(y) = \frac{d^2}{dy^2} Y_a(y/V) = \frac{d^2}{dy^2} Y(\lambda_1 = y/2\pi \nu \nu \eta)\bigg|_{\eta \to 0}$$

(4)

The distribution function $P_\rho(\rho)$ of LDOS can be also expressed through the function $Y(\lambda_1, \lambda_2)$ [14,18]. For convenience, we normalize $\rho$ by its mean value: $\tilde{\rho} = \rho/\langle \rho \rangle$, where $\langle \rho \rangle = \nu$, and write again $\rho$ instead of $\tilde{\rho}$ henceforth. Then,

$$P(\rho) = \frac{1}{4\pi} \frac{\partial^2}{\partial \rho^2} \left\{ \int_{2\rho}^{\infty} d\lambda_1 \overline{Y}(\lambda_1) \left( \frac{2\rho}{\lambda_1 - \rho^2 + 1} \right)^{1/2} \right\} ,$$

(5)

where

$$\overline{Y}(\lambda_1) = \int_{-1}^{1} d\lambda_2 \frac{Y(\lambda_1, \lambda_2)}{\lambda_1 - \lambda_2}$$

(6)

Let us note the important symmetry relation found in [14]:

$$P(\rho^{-1}) = \rho^3 P(\rho)$$

(7)

It follows from eq. (6) and is completely independent on a particular form of the function $Y(\lambda_1, \lambda_2)$. Obviously, eq. (7) relates the small–\rho asymptotical behavior of the distribution $P_\rho(\rho)$ to its large–\rho asymptotics.

Relations (6) and (7) gave the possibility to study the distributions of eigenfunctions amplitudes and of LDOS in various regimes. Exact calculation of the eigenfunction amplitudes distribution in a quasi–1D sample of arbitrary length was performed in [11]. LDOS distribution in a sample with given level width was calculated in [12,18] for the 0D and in [20] for the quasi–1D case. In Ref. [14], the distributions were studied in the vicinity of the Anderson metal–insulator transition, where $Y(Q)$ acquires the meaning of an order parameter function. In Ref. [21,22] the distribution $P_y(y)$ in a metallic sample was studied by a perturbative method which is valid for not too large $y$.

In a recent paper, Muzykantskii and Khmelnitskii [23] studied the long–time asymptotics of the current relaxation in a disordered metallic sample via the supersymmetric $\sigma$–model approach combined with the saddle–point method. Fal’ko and Efetov [24] employed the saddle–point approach suggested in [23] to study the asymptotical behavior of the distribution of eigenfunctions amplitude, $P_y(y)$, given by eq. (4). In the present paper, we apply this method to investigate the asymptotical behavior of the distribution function of LDOS
in an open sample (i.e. with leads being attached). Precisely this distribution function was studied by Altshuler, Kravtsov and Lerner \cite{7,8} in the renormalization group approach, so that the direct comparison is possible. We find the LN asymptotics of $P_\rho(\rho)$ in 2 dimensions, in agreement with the result of \cite{7}. As we discuss in Section 4, it is not completely obvious that this agreement should have been expected.

In the case of a quasi–1D geometry, we also calculate the LDOS distribution function using the exact solution of the 1D $\sigma$–model \cite{25}. In the metallic regime, the LN asymptotics obtained by the saddle–point method is reproduced. In the deeply localized regime, the distribution function $P_\rho(\rho)$ takes a completely LN form.

II. ASYMPTOTICAL BEHAVIOR OF THE LDOS DISTRIBUTION FUNCTION FROM THE SADDLE–POINT APPROXIMATION.

We consider a disordered sample in the weak localization regime, which means that

$$\xi = \int_{l-1}^{l-1} \frac{d^d q}{(2\pi)^d} \frac{1}{\pi \nu D q^2} \ll 1 \quad (8)$$

where $\xi$ is the usual parameter of the perturbation theory \cite{7}. In this situation, the distribution of normalized local DOS $\rho$ is mostly concentrated in a narrow Gaussian peak around its mean value $\rho = 1$, with the width $\langle (\rho - 1)^2 \rangle \sim \xi \ll 1$ \cite{7,8}. This close-to-Gaussian shape of $P_\rho(\rho)$ holds however in the region $|\rho - 1| \ll 1$ only. We will consider in contrast the “tails” of the distribution, $\rho \ll 1$ and $\rho \gg 1$, where a much slower decay of $P_\rho(\rho)$ will be found.

As is seen from eq.(5), the LDOS distribution $P_\rho(\rho)$ is determined by the function $Y(\lambda_1, \lambda_2)$ at $\lambda_1 \geq \frac{\rho^2 + 1}{\rho}$. Therefore, in the asymptotical region $\rho \gg 1$, only the form of $Y(\lambda_1, \lambda_2)$ at large values of $\lambda_1 \geq \rho/2 \gg 1$ is relevant. In this region, and under the condition (8), $Y(\lambda_1, \lambda_2)$ can be found with exponential accuracy with use of the saddle–point method \cite{23,24}. We consider the case of an open sample and neglect any inelastic processes in the bulk of the sample. The energy levels are then broadened solely due to the possibility for a particle to escape to the leads, and the action is given by eq.(2) at $\eta = 0$,

$$S[Q] = -\int d^d r \text{Str} \frac{\pi \nu D}{4} (\nabla Q)^2 \quad (9)$$

The saddle point equations have the same form as in Ref. \cite{23} for $\omega = 0$:

$$\Delta \theta_1 = \Delta \theta_2 = 0, \quad (10)$$

where $\cosh \theta_1 = \lambda_1; \cos \theta_2 = \lambda_2$, and $\Delta$ is the Laplace operator. Assuming the perfect coupling to the leads, the corresponding boundary condition takes the form $Q|_{\text{leads}} = \Lambda$, or in other words,

$$\theta_1, \theta_2|_{\text{leads}} = 0 \quad (11)$$

Another boundary condition which follows from the definition (1) of the function $Y(\lambda_1^{(0)}, \lambda_2^{(0)})$, fixes the value of $\theta_1, \theta_2$ in the observation point $r_0$, where the LDOS distribution is studied:
\[
cosh \theta_1(r_0) = \lambda_1^{(0)} \equiv \cosh \theta_1^{(0)}; \quad \cos \theta_2(r_0) = \lambda_2^{(0)} \equiv \cos \theta_2^{(0)} \tag{12}
\]

We start from the case of a quasi–1D sample. Let the observation point be at \(x = 0\), whereas the sample edges at \(x = -L_-\) and \(x = L_+\), so that the sample length is \(L = L_- + L_+\). The solution of the saddle–point equation (10) with the boundary conditions (11), (12) is

\[
\theta_1(x) = \begin{cases} 
\theta_1^{(0)} \left(1 - \frac{x}{L_-}\right); & x > 0 \\
\theta_1^{(0)} \left(1 - \frac{|x|}{L_+}\right); & x < 0 
\end{cases} \tag{13}
\]

and similarly for \(\theta_2\). The action on the saddle point solution is given by

\[
S = \frac{\pi \nu D}{2} \int \left[ (\nabla \theta_1)^2 + (\nabla \theta_2)^2 \right] d^d r = \frac{\pi \nu D}{2} \left[ \theta_1^{(0)2} + \theta_2^{(0)2} \right] \left( \frac{1}{L_+} + \frac{1}{L_-} \right) \tag{14}
\]

We find therefore that in the region \(\lambda_1^{(0)} \gg 1\),

\[
Y(\lambda_1^{(0)}, \lambda_2^{(0)}) \sim e^{-S} \sim \exp \left\{ -\frac{\pi \nu D}{2} \left( \frac{1}{L_+} + \frac{1}{L_-} \right) \ln^2(2\lambda_1^{(0)}) \right\}, \tag{15}
\]

with exponential accuracy. Dependence of \(Y\) on \(\lambda_2^{(0)}\) is not important within this accuracy, since it gives simply a prefactor after the \(\lambda_2\)–integration in eq.(3). Therefore, according to eq.(5) the asymptotics of the LDOS distribution function at \(\rho \gg 1\) is

\[
P_{\rho}(\rho) \approx \overline{Y}(\lambda_1^{(0)} = \rho/2) \sim \exp \left\{ -\frac{\pi \nu D}{2} \left( \frac{1}{L_+} + \frac{1}{L_-} \right) \ln^2 \rho \right\}, \tag{16}
\]

up to a preexponential factor. According to the symmetry relation (7), eq.(16) is equally describes the small–\(\rho\) asymptotics of \(P_{\rho}(\rho)\) at \(\rho \ll 1\).

The found LN asymptotics of the LDOS distribution in an open sample should be contrasted to the “square-root-exponential” behavior of \(P_{\rho}(\rho)\) and \(P_y(y)\) in a closed sample [11,17,20,22]. Let us show how the difference appears in the present saddle–point approach.

In a closed sample, the distribution of LDOS, \(P_{\rho}(\rho)\) is defined only if a finite level width \(\eta\) in the action, eq.(2), is introduced. It may account for some inelastic scattering processes in the bulk of the sample. The saddle point equation acquires then the same form, as in ref. [23]:

\[
D \Delta \theta_1 - 2\eta \sinh \theta_1 = 0 \tag{17}
\]

Eq.(17) should be supplemented by the boundary conditions

\[
\nabla_n \theta_1|_{\text{leads}} = 0, \tag{18}
\]

where \(\nabla_n\) is the normal derivative, and by the condition (12) in the observation point.

Let us first assume that the saddle-point solution satisfies the condition \(\theta_1(x) \gg 1\) throughout the sample. It will be seen below that this assumption is valid if \(E_c/\eta \gg 1\), where \(E_c = D/L^2\) is the Thouless energy. Then \(\sinh \theta_1\) in eq.(17) can be approximated by \(e^{\theta_1}/2\), and the solution for \(x > 0\) is found to be

\[
\theta_1(x) = \theta_1^{(0)} \left(1 - \frac{x}{L_-}\right); \quad x > 0
\]

\[
= \theta_1^{(0)} \left(1 - \frac{|x|}{L_+}\right); \quad x < 0
\]
\[ e^{-\theta_1(x)} = \frac{\eta}{DC} \left\{ 1 + \cos \left[ \sqrt{C}(L_+ - x) \right] \right\} , \quad (19) \]

where the constant \( C \) is defined from the condition
\[ L_+ \sqrt{C} = \pi - \arccos \left( 1 - \frac{CD}{2\eta \lambda_1^{(0)}} \right) \quad (20) \]

If \( \lambda_1^{(0)} \gg E_c/\eta \), the solution (19), (20) takes the form
\[ e^{-\theta_1(x)} = \frac{\eta}{D} \left( \frac{L_+}{\pi} \right)^2 \left\{ 1 - \cos \left[ \frac{\pi}{L_+} \left( x + \sqrt{\frac{D}{\eta \lambda_1^{(0)}}} \right) \right] \right\} \quad (21) \]

The action
\[ S \simeq \int dx \left\{ \frac{\pi \nu D}{2} \left( \theta'_1 \right)^2 + \pi \nu \eta e^{\theta_1} \right\} \]

is then found to be dominated by the vicinity of the observation point, where eq.(21) can be reduced to
\[ e^{-\theta_1(x)} = \frac{1}{2\lambda_1^{(0)}} \left( 1 + x \sqrt{\frac{\eta \lambda_1^{(0)}}{D}} \right)^2 \quad (23) \]

Substituting (23) in (22) and taking into account also an analogous contribution of the \( x < 0 \) region, we get
\[ S \simeq 8\pi \nu D \eta \lambda_1^{(0)} \quad (24) \]

In the opposite case \( 1 \ll \lambda_1^{(0)} \ll E_c/\eta \), the solution (19), (20) is reduced to
\[ \theta_1(x) = \theta_1^{(0)} - \frac{\eta \lambda_1^{(0)}}{D} \left( 2xL_+ - x^2 \right) \quad (25) \]

The corresponding action is
\[ S \simeq 2\pi \nu \eta \lambda_1^{(0)} L \quad (26) \]

Combining (24) and (26), we find therefore for \( E_c/\eta \gg 1 \)
\[ P_{\rho}(\rho) \sim Y(\lambda_1^{(0)} = \rho/2) \sim \begin{cases} \exp \left\{ -4\pi \nu \sqrt{2D\eta \rho} \right\} , & \rho \gg \frac{E_c}{\eta} \gg 1 \\ \exp \left\{ -\pi \nu \eta L \rho \right\} , & \frac{E_c}{\eta} \gg \rho \gg 1 \end{cases} \quad (27) \]

If \( E_c/\eta \ll 1 \), one has to use the exact equation (17), which has the following solution:
\[ x(\theta_1) = \left( \frac{D}{4\eta} \right)^{1/2} \int_{\theta_1}^{\theta_1^{(0)}} \frac{d\vartheta}{\cosh \vartheta - \cosh \theta_1(L)} \left[ \cosh \vartheta - \cosh \theta_1(L) \right]^{1/2} , \quad (28) \]

where \( \theta_1(L) \) is defined from the condition
\begin{equation}
L = \left( \frac{D}{4\eta} \right)^{1/2} \int_{\theta_1(L)}^{\theta_1(0)} \frac{d\theta}{\cosh \vartheta - \cosh \theta_1(L)}^{1/2},
\end{equation}

We find that for \( E_c \lesssim \eta \) the solution \((28)\) in the vicinity of the observation point \( x \ll \sqrt{D/\eta} \) has exactly the form \((23)\) for all \( \lambda_1(0) \gg 1 \), so that the action is given by eq.\((24)\). Thus, in this case the first of the asymptotics \((27)\) is applicable

\begin{equation}
P_\rho(\rho) \sim \exp\{ -4\pi \sqrt{2D\eta \rho}\}, \quad \frac{E_c}{\eta} \lesssim 1, \quad \rho \gg 1 \quad (30)
\end{equation}

Note that eq.(30) has exactly the same form as the asymptotics of \( P_\rho(\rho) \) in the infinitely long wire \([20]\). Eq.\((27)\) is also in agreement with the asymptotical behavior of the distribution function \( P_y(y) \) of eigenfunctions intensities \([21]\):

\begin{equation}
P_y(y) \sim \exp\{ -4\sqrt{2\pi \nu Dy/L}\}; \quad y \gg \frac{2\pi \nu D}{L} \quad (31)
\end{equation}

which is immediately reproduced from eqs.\((4), (31)\).

We return now to the case of an open sample and consider 2D geometry. As was suggested in \([23]\), we suppose the sample to be a disk of a radius \( L \) surrounded by a perfectly conducting media, with the observation point located in the center of the disk. We will find the result to be logarithmically dependent on \( L \), that justifies its application to a sample of more general 2D geometry with a characteristic size \( L \).

Eq.\((10)\) for \( \theta_1(r) \) with the boundary conditions \((11), (12)\) take now the form

\begin{equation}
\theta''_1(r) + \frac{\theta'_1(r)}{r} = 0 \quad (32)
\end{equation}

\begin{equation}
\theta_1(L) = \theta_1(0) \quad (33)
\end{equation}

\begin{equation}
\theta_1(L) = 0 \quad (34)
\end{equation}

The boundary condition \((33)\) has to be put not in the observation point \( r = 0 \), but rather at certain distance from it \( r = l_* \) \([23]\). This is because the \( \sigma \)–model approximation breaks down for momenta \( q > l^{-1} \), where \( l \) is of order of the mean free path. We will specify the value of \( l_* \) below. The solution of eq.\((32)\) with the boundary conditions \((33), (34)\) reads

\begin{equation}
\theta_1(r) = \theta_1(0) - \frac{\theta_1(0)}{\ln L/l_*} \ln r/l_* , \quad l_* < r < L , \quad (35)
\end{equation}

or, for \( \lambda_1(r) = \cosh \theta_1(r) \simeq \frac{1}{2} e^{\theta_1(r)} \),

\begin{equation}
\lambda_1(r) = \lambda_1(0) \left( \frac{l_*}{r} \right)^{\theta_1(0) \ln^{-1}(L/l_*)} \quad (36)
\end{equation}

The condition of applicability of the \( \sigma \)–model (diffusive) approximation is \([23] d\theta_1/dr < 1/l_* \). It leads to the following equation for the cut-off scale \( l_* \)

\begin{equation}
\frac{1}{l} = \left. \frac{d\theta_1}{dr} \right|_{r=l_*} = \frac{\theta_1(0)}{\ln(L/l_*) l_*} , \quad (37)
\end{equation}
The action (9) is now given by

\[ S \simeq \frac{\pi \nu D}{2} \int d^2r (\nabla \theta_1)^2 = \pi^2 \nu D \int_{l_*} L dr r [\theta'_1(r)]^2 \]
\[ = \frac{\pi^2 \nu D \theta_1^{(0)2}}{\ln L/l_*}. \]  

(38)

Therefore, the LDOS distribution has the asymptotics

\[ P_\rho(\rho) \sim \exp \left\{ -\frac{\pi^2 \nu D \ln^2 \rho}{\ln(L/l_*)} \right\}, \]

(39)

where \( l_* \) satisfies the condition

\[ l_* = l \frac{\ln \rho}{\ln(L/l_*)} \]  

(40)

The present consideration is meaningful provided \( l_* \ll L \), which holds if \( \ln \rho \ll L/l \). Under this condition, eq.(40) yields

\[ l_* \simeq l \frac{\ln \rho}{\ln \left( \frac{L}{l_1} \right)} \]

(41)

Taking into account the logarithmic dependence of the exponent in (39) on \( l_* \) and neglecting the corrections of the \( \ln(\ln \ldots) \) type, we can approximate \( l_* \) in eq.(39) by \( l \). The result is of the LN form, in exact agreement with the asymptotical behavior for \( P_\rho(\rho) \) found by the renormalization group method in [7]. We discuss this agreement in more detail in Conclusion. Let us also note that in contrast to the 1D case, in 2D the asymptotical form of the distribution \( P_\rho(\rho) \) in an open sample, eq.(39), is similar to that of \( P_\rho(\rho) \) in a closed sample [24].

Proceeding in the same way in 3D case, we get the equation

\[ \theta''_1(r) + \frac{\theta'_1(r)}{r^2} = 0, \]

(42)

with the same boundary conditions (33), (34). The solution has the form

\[ \theta_1(r) = \theta_1^{(0)} \frac{l_*}{r}, \quad l_* < r < L \]

(43)

The condition (37) for \( l_* \) now yields \( l_* \simeq l \theta_1^{(0)} \), and the action is estimated as

\[ S = 2\pi^2 \nu D \int_{l_*} L dr r^2 [\theta'_1(r)]^2 \]
\[ = 2\pi^2 \nu D l_* \theta_1^{(0)2} \]
\[ = 2\pi^2 \nu D l \theta_1^{(0)3} \]
\[ = \pi g(l) \theta_1^{(0)3}, \]

(44)
where \( g(l) = 2\pi^2 \nu D l \) is the conductance of a cube of the size \( l \). Therefore, the LDOS distribution asymptotics is

\[
P_\rho(\rho) \sim \exp\{ -g(l) \ln^2 \rho \}, \tag{45}
\]

Let us note that \( l \) is of order of mean free path, so that it is defined up to a numerical coefficient of order unity. This ambiguity does not affect the leading order behavior of the exponent in eq. (39) for the 2D case, due to its logarithmic dependence on \( l \). On the other hand, in 3D case, the exponent in (45) is proportional to \( l \). To fix the corresponding numerical coefficient, one should go beyond the long-wave-length \( \sigma \)–model approximation and consider the problem in the region \( r \ll l_\ast \).

To conclude this section, we have shown that the LDOS distribution \( P_\rho(\rho) \) has logarithmically normal asymptotics (16), (39) at \( \rho \gg 1 \) and \( \rho \ll 1 \) for quasi–1D and 2D samples, whereas for 3D systems a somewhat different result (45) was found. In the next section, we confirm this result for the quasi-1D case by the exact solution of the problem.

### III. LDOS DISTRIBUTION IN A QUASI–1D SAMPLE: EXACT SOLUTION.

In the case of a quasi–1D geometry of a sample, evaluation of the functional integral in (1) can be reduced to the solution of an evolution equation of the diffusion type for the function \( Y_1(\lambda_1, \lambda_2; x) \) [15]:

\[
2\pi \nu D \frac{\partial Y_1}{\partial x} = \left[ \frac{1}{J} \frac{\partial}{\partial \theta_1} J \frac{\partial}{\partial \theta_1} + \frac{1}{J} \frac{\partial}{\partial \theta_2} J \frac{\partial}{\partial \theta_2} \right] Y_1; \tag{46}
\]

\[
J(\theta_1, \theta_2) = \frac{\left| \sinh \theta_1 \sin \theta_2 \right|}{(\cosh \theta_1 - \cos \theta_2)^2}
\]

For the general case of a finite quasi–1D sample with distances \( L_-, L_+ \) from the observation point to the edges, the function \( Y(\lambda_1, \lambda_2) \) defined in the preceding section is given by the product

\[
Y(\lambda_1, \lambda_2) = Y_1(\lambda_1, \lambda_2; L_-)Y_1(\lambda_1, \lambda_2; L_+)
\]

For simplicity, we will concentrate on the case of a semi-infinite sample \( L_+ = \infty, L_- = L \). In this case \( Y(\lambda_1, \lambda_2) = Y_1(\lambda_1, \lambda_2; L) \), and we will omit the subscript “1” in the notation “\( Y_1 \)”. Results for the general case will be presented in the end of the section. Solution of eq. (46) with the boundary condition corresponding to a perfectly conducting lead was recently found to be [25]:

\[
Y(\lambda_1, \lambda_2; L) = 1 - (\lambda_1 - \lambda_2) \sum_{m=0}^{\infty} \int_0^{\infty} dk \ c_{mk} P_m(\lambda_2) P_{-1/2+i(k/2)}(\lambda_1) e^{-\epsilon_{mk} t};
\]

\[
t = \frac{L}{2\pi \nu D}; \quad c_{mk} = \frac{(2m + 1)k \tanh(\pi k/2)}{(2m + 1)^2 + k^2}; \quad \epsilon_{mk} = (m + 1/2)^2 + k^2/4 \tag{48}
\]

Substituting eq. (48) in (3), we find
\[ Y(\lambda_1) = \ln \frac{\lambda_1 + 1}{\lambda_1 - 1} - 2 \int_0^\infty \frac{dk}{1 + k^2} k \tanh(\pi k/2) P_{-1/2+ik/2}(\lambda_1)e^{-t(k^2+1)/4} \] (49)

The first term in eqs. (48) and (49) corresponds to the limit of a closed system with no level broadening, in which case \( Y(\lambda_1, \lambda_2) = 1 \) and \( P_\rho(\rho) = \delta(\rho) \). It can be really proven by direct substitution of the first term of eq. (49) into eq. (3) that the resulting contribution to \( P_\rho(\rho) \) is equal to zero at any \( \rho \neq 0 \). To evaluate the second term, we use the asymptotic expression for the Legendre functions \( P_{-1/2+\gamma}(z) \) at \( z \gg 1 \) [20]:

\[ P_{-1/2+i\gamma}(z) \simeq \frac{1}{\sqrt{2\pi z}} \left\{ e^{i\gamma\ln 2z} \frac{\Gamma(i\gamma)}{\Gamma(1/2+i\gamma)} + e^{-i\gamma\ln 2z} \frac{\Gamma(-i\gamma)}{\Gamma(1/2-i\gamma)} \right\} \] (50)

This allows us to reduce the second term in (49) to the form

\[ Y(2)(\lambda_1) = -4 \int_{-\infty}^\infty \frac{dk}{1 + k^2} \frac{1}{\sqrt{2\pi \lambda_1}} \frac{\Gamma(1/2-ik/2)}{\Gamma(-ik/2)} e^{i\gamma \ln 2\lambda_1} e^{-t(k^2+1)/4} \] (51)

At \( \ln \lambda_1 \gg 1 \) the integral can be evaluated via the saddle point method. The saddle point is \( k = i\frac{\ln 2\lambda_1}{t} \) and yields the following contribution:

\[ Y_{s.p.}(\lambda_1) \simeq -4\sqrt{\frac{2}{t\lambda_1}} \frac{1}{1 - \left( \frac{\ln 2\lambda_1}{2t} \right)^2} \frac{\Gamma \left( \frac{1}{2} + \frac{\ln 2\lambda_1}{2t} \right)}{\Gamma \left( \frac{\ln 2\lambda_1}{2t} \right)} \exp \left\{ -\frac{t}{4} - \frac{\ln^2 2\lambda_1}{4t} \right\} \] (52)

In fact, when deriving eq. (52), we should shift the integration contour in the complex plane in order that it would pass through the saddle point. However, if \( \ln 2\lambda_1 > t \), the contour crosses then the pole of the integrand, \( k = i \). Evaluating this pole contribution, we find that in cancels exactly the first term in eq. (48), (49). Thus, we get

\[ Y(\lambda_1) \simeq \ln \frac{\lambda_1 + 1}{\lambda_1 - 1} \theta(\ln 2\lambda_1 - t) + Y_{s.p.}(\lambda_1) \] (53)

where \( \theta(x) \) is the step function and \( Y_{s.p.} \) is given by eq. (52). Now we substitute this result into the formula (5) for the LDOS distribution. The leading contribution at \( \rho \gg 1 \) is found to be given by the second term, \( Y_{s.p.}(\lambda_1) \), in (53). It can be found by noticing that the factor

\[ \frac{1}{1 - \left( \frac{\ln 2\lambda_1}{2t} \right)^2} \frac{\Gamma \left( \frac{1}{2} + \frac{\ln 2\lambda_1}{2t} \right)}{\Gamma \left( \frac{\ln 2\lambda_1}{2t} \right)} \]

varies logarithmically slow with \( \lambda_1 \), and can be simply taken at \( \lambda_1 = \rho/2 \). The remaining integral is of the form

\[ \int_{\rho/2}^{\infty} d\lambda_1 \frac{1}{\sqrt{\lambda_1 (\lambda_1 - \rho/2)}} \exp \left\{ -\frac{t}{4} - \frac{\ln^2 2\lambda_1}{4t} \right\} \]

\[ = \int_1^\infty \frac{dy}{\sqrt{y(y-1)}} \exp \left\{ -\frac{t}{4} - \frac{\ln^2 \rho}{4t} - \frac{\ln \rho \ln y}{2t} - \frac{\ln^2 y}{4t} \right\} \]

\[ \simeq B \left( \frac{1}{2} \frac{\ln \rho}{2t} \right) \exp \left\{ -\frac{t}{4} - \frac{\ln^2 \rho}{4t} \right\} \] (54)
where $B(x, y)$ is the Euler’s beta function. When writing the last line, we assumed that $\ln \rho > \sqrt{t}$. It is not a restriction in the metallic (short sample) case, $t < 1$, and includes the region of characteristic LDOS, $\ln \rho \sim t$, in the insulating regime, $t \gg 1$.

Collecting now all the remaining factors in eqs. (5), (52), we finally get the distribution of LDOS:

$$P_{\rho}(\rho) \simeq \frac{1}{2\rho \sqrt{\pi t}} \exp \left\{ -\frac{1}{4t} (t + \ln \rho)^2 \right\}$$  \hspace{1cm} (55)

Let us remind that eq. (55) has been derived under the assumption $\rho \gg 1$. An additional condition $\ln \rho > \sqrt{t}$ in the insulating regime is not very restrictive, since the characteristic scale of $\ln \rho$ in this case is $\ln \rho \sim t$, as is seen from (55). Moreover, calculating the integral (54) in the opposite case $\ln \rho \ll t$, we arrive at the same LN behavior as in eq. (55); only the prefactor is different:

$$P_{\rho}(\rho) \simeq \frac{\pi |\ln \rho|}{4\rho t} \exp \left\{ -\frac{1}{4t} (t + \ln \rho)^2 \right\}$$  \hspace{1cm} (56)

Therefore, we have found the LN asymptotic behavior (55), (56) of the LDOS distribution $P_{\rho}(\rho)$ at $\rho \gg 1$. The symmetry relation (7) allows us to extend its validity to the region of $\rho \ll 1$ as well. Note that eqs. (55), (56) completely preserve their form under the transformation (7). Eqs. (55), (56) are in full agreement with the results of the saddle point approximation of section 2.

Let us remind that the above results have been derived for the case of a semiinfinite sample: $t_+ = \infty, t_- = 0$, where $t_{\pm} = L_{\pm}/2\pi \nu D$. Now we consider briefly the general case of finite $t_+, t_-$. In this case the function $Y(\lambda_1, \lambda_2)$ determining the LDOS distribution is defined by eq. (47) with $Y_1(\lambda_1, \lambda_2; L)$ given by eq. (48). Using the above analysis of the asymptotic behavior of $Y_1$ at $\lambda_1 \gg 1$, we find with an additional condition $\ln 2\lambda_1 > t_+, t_-,$

$$Y(\lambda_1; t_+, t_-) = \frac{\lambda_1}{2} Y_{s.p.}(\lambda_1, t_+) Y_{s.p.}(\lambda_1, t_-) ,$$  \hspace{1cm} (57)

with $Y_{s.p.}(\lambda_1, t)$ given by eq. (52). This leads to the following result for the LDOS distribution:

$$P_{\rho}(\rho) \simeq \frac{F \left( \frac{\ln \rho}{2t_+} \right) F \left( \frac{\ln \rho}{2t_-} \right)}{4\rho \sqrt{\pi t_+ t_-}} \frac{1}{4t_+ t_-} \exp \left\{ -\frac{1}{4} (t_+ + t_-) - \frac{1}{4} \left( \frac{1}{t_+} + \frac{1}{t_-} \right) \ln^2 \rho \right\}$$  \hspace{1cm} (58)

where

$$F(x) = \frac{\Gamma(x - 1/2)}{(x + 1/2) \Gamma(x)}$$

Eq. (58) holds in the metallic regime $t_+, t_- < 1$ for all $\rho \gg 1$ and is again in agreement with the result of the previous section, eq. (16). In the insulating regime eq. (58) holds provided the additional condition $\ln \rho > t_-, t_+$ is satisfied and represents therefore the far asymptotics. In this case, there is also an intermediate regime $1 < \ln \rho < t_+$, where we assumed for definitness $t_- < t_+$. Then $Y(\lambda_1, \lambda_2, t_+)$ can be approximated by 1, and we find $P_{\rho}(\rho)$ given by eq. (55) with $t = t_-$. In all the cases, the equality (7) allows to get the asymptotical behavior in the region of small LDOS, $\rho \ll 1$. 

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In conclusion of this section, let us note that in the insulating regime, $t_-, t_+ \gg 1$, the obtained results for $P_\rho(\rho)$ are completely analogous to what was found by Altshuler and Prigodin \cite{10} for the case of a strictly 1D sample by Berezinskii technique. This confirms once more the general conjecture \cite{17} that the statistics of smooth envelopes of the wave functions in 1D and quasi–1D samples are equivalent.

IV. DISCUSSION AND CONCLUSION.

In this article, we have studied the asymptotical behavior of the LDOS distribution function $P_\rho(\rho)$ in disordered metallic samples at $\rho \ll 1$ and $\rho \gg 1$. For this purpose, we used the supermatrix $\sigma$–model approach, which allows to express the distribution $P_\rho(\rho)$ in terms of the function $Y(\lambda_1, \lambda_2)$ defined as a certain integral over the supermatrix field. In the asymptotical regime, $\rho \ll 1$ and $\rho \gg 1$, this integral can be estimated via the saddle-point method, leading to the log-normal asymptotics of $P_\rho(\rho)$ for quasi–1D and 2D sample geometries, and to the result (45) in the 3D case.

In quasi–1D the result can be also found from the exact solution, which indeed yields the LN asymptotical behavior in the case of a short (metallic) sample. In the opposite case of a sample much longer than the localization length, the whole distribution function $P_\rho(\rho)$ has the log-normal form. This is completely analogous to the form of the LDOS distribution in a strictly 1D sample in the strongly localized regime studied in \cite{10} by the Berezinskii technique.

In 2D, the obtained asymptotics of the LDOS distribution is in full agreement with the result of renormalization group treatment \cite{7}. This agreement is highly non-trivial, for the following reason. The RG treatment is based on a resummation of the perturbation theory expansion and can be equally well performed within the replica (bosonic or fermionic) or supersymmetric formalism. At the same time, the present approach based on the supersymmetric formalism relies heavily on the topology of the saddle-point manifold combining non-compact ($\lambda_1$) and compact ($\lambda_2$) degrees of freedom. The asymptotic behavior of $P_\rho(\rho)$ at $\rho \gg 1$ and $\rho \ll 1$ is determined by the region $\lambda_1 \gg 1$ which is very far from the “perturbative” region of the manifold $Q \simeq \Lambda$ (i.e. $\lambda_1, \lambda_2 \simeq 1$). It is well known \cite{27} that for the problem of energy level correlation, the replica approach is not able to reproduce the correct result (which can be obtained by the supersymmetry method) even if all orders of perturbation theory are taken into account. This is related to the fact that within the replica method the topology of the saddle-point manifold is not properly reflected. This might suggest a conclusion that any result based on a resummation of the perturbative expansion (which can be therefore obtained within the replica approach) is incorrect for the same reason. The agreement of the results of supersymmetric and renormalization group treatments of the LDOS distribution shows however that this conclusion would be wrong. Apparently, this problem is of a different kind, as compared to the level correlation one, so that the replica trick combined with the RG is able to reproduce the correct result. The difference seems to be the following: in the case of level correlator both compact and non-compact sectors in the supersymmetric formulation are equally important, whereas for the present problem only the non-compact sector was essential, with compact one playing an auxiliary role. Let us note that the same situation appears in the vicinity of the Anderson transition \cite{28,14} where the function $Y(\lambda_1, \lambda_2)$ acquires a role of the order parameter function and depends
on the non-compact variable $\lambda_1$ only. The above agreement obtained for the LDOS distribution provides therefore support to other results \footnote{7} obtained with making use of the renormalization group approach and $2 + \epsilon$ expansion.

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