Study of a parabolic cylinder elastic-plastic contact model

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Abstract
Based on Hertz contact theory, two parabolic cylinder normal contact models are established. The effect of contact angle on normal approach, actual contact area, and normal contact stiffness are investigated, and the effect of the distance from the focus to the directrix (focus distance) on the mechanical characteristics of the models is further analyzed. The parabolic cylinder contact model was verified by simulation analysis and comparison with cylinder contact model. The results demonstrated that the contact angle, focal distance, and load have significant effects on the mechanical properties of the model. The simulation data are basically consistent with the contact model data, and the parabolic cylinder contact model and cylinder contact model have the same change trend. The results verify the correctness of the parabolic cylinder contact model and reveal the variation of the mechanical properties of the contact model.

Keywords
Parabolic cylinder, elastic-plastic, contact angle, Hertz theory, normal approach

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Introduction
Since the Hertz contact theory was introduced,1 many scholars have performed detailed research on the contact problem of two micro-convex bodies. In general, the study of this problem considers mainly the macroscopic properties of the asperity elastic modulus and geometric parameters, while microscopic properties, such as the surface morphology and surface roughness of the asperity, are less taken into consideration.2

In recent years, scholars have conducted a lot of research on the surface morphology of asperities. The most typical contact model is the cylinder contact model. Based on the Hertz theory, Johnson3 and Radzimovshy4 proposed two cylindrical elastic contact models and carried out detailed research on the internal and external contact of their models. Li et al.5 proposed an elastic-plastic contact model between rigid sphere and elastic-plastic plane by modifying Johnson’s contact model, which was verified by finite element analysis. Goldsmith6 also proposed a cylindrical contact model, which is suitable for internal contact collisions between cylinders. Pereira et al.7 conducted a comparative study on the above three models, and indicated that these cylindrical models are associated with three main disadvantages, including low computational efficiency of the program, they consider only elastic contact, and they include logarithmic function, which may affect the scope of model application. Cinar and Sinclair8 investigated the elastic-
plastic contact of an infinitely long cylinder under frictionless and fully-bonded contact conditions, and improved the computational efficiency of the model using proper mesh expansion techniques. Sharma and Jackson employed the FEM to research the elastic-plastic contact between a cylinder and a rigid plane under plane stress, and analyzed the convergence of the FE mesh. Guo et al. proposed the elastic-plastic contact model of cylinder and cosine wave under different contact angles. More specifically, they investigated the effect of normal contact deformation, normal contact stiffness, and actual contact area on the mechanical characteristics of the contact model under different axis contact angles. Huang et al. developed a fractal contact model of two cylinders based on the Hertz theory and the MB fractal theory. Their model analyzed the effect of fractal dimensions, cylinder radius, and contact mode on load and contact area. Li et al. studied the normal contact stiffness of the joint surfaces of two cylinders by modifying the MB fractal model. They developed a simulation model to analyze the effect of contact area, load, and friction factor on contact stiffness.

In addition, there are relevant important studies on spherical and ellipsoidal contact models. The research on spherical elastic contact was first conducted by Hertz, who obtained a closed-form solution for the spherical elastic contact problem. Bondareva obtained an orthogonal analytical solution of the elastic sphere displacement problem by studying the general solution of the elastic sphere contact problem. Zhupanska investigated the normal contact problem of an elastic sphere, and analyzed the contact stress and surface displacement of the sphere. Argatov studied the symmetry and nonlinear contact problems of an elastic sphere, and obtained an approximate solution of the elastic sphere contact under a small contact area condition. Yuan et al. explored the friction properties of spherical contact surfaces, and suggested that high-frequency and large-fluctuation loads are the key factors affecting spherical contact sliding. Halling and Nur proposed a solution to the deformation of spherical or ellipsoidal convex bodies under the effect of work hardening, assuming independent deformation between asperities. Ghaednia et al. studied the contact problem between an elastic-plastic sphere and an elastic-plastic plane through a large number of finite element analysis examples, which gave the solution formulas of relevant contact parameters. Considering the inapplicability of Hertz elastic contact theory to elastic-plastic deformation materials, Rodriguez et al. studied the effect of reducing modulus on indenter deformation by finite element method. Kogut and Etsion proposed an elastic-plastic contact finite element model of deformable sphere and rigid plane, which analyzed the mechanical characteristics of contact load, actual contact area and average contact pressure in elastic stage, elastic-plastic stage, and complete plastic stage. In order to use particle method to simulate granular materials, Olsson and Larsson proposed a method to calculate the contact load and contact area of two different elastic-plastic spheres. The results of finite element analysis are in good agreement with the results of the model. Ghaednia et al. reviewed the research progress of single asperity contact model in recent years. Through comparative analysis of cylinder, sphere, sinusoidal, and wavy contact models, it was found that the average contact pressure of contact model in the elastic-plastic stage changes with the change of boundary conditions and geometry, which is not affected by the ratio of hardness and yield strength.

It is well known that parabolic cylinder contact models are also typical contact models. Steuermann proposed an even-order parabolic cylinder elastic contact model, which considered the parabolic cylinder model as an elastic half-space. However, Steuermann studied only the elastic deformation stage of the parabolic cylinder model.

In machining processes, and especially in precision machining, the machining surface often has regular and periodic features. For instance, in the grinding process, regular waves are formed on the machined surface. Corresponding surface equations can be developed based on the shape of these ripples. These surface ripples can be mainly divided into cylindrical, elliptical, and parabolic waves. By studying the regular elastic-plastic contact model, the mechanical properties of regular joint surfaces can be further analyzed, providing theoretical guidance for the contact stiffness between regular joints. Thus, the development of an elastic-plastic contact model of two regular bodies is the basis for studying the mechanical properties of regular joint surfaces.

In the above-mentioned studies, the research on elastic-plastic contact models of cylinders, spheres, and ellipsoids has been very mature, while, the research on parabolic cylinder contact models has been stagnant. Although Steuermann studied the elastic parabolic contact model, it did not analyze the mechanical properties of the contact model in the elastic-plastic stage and the fully plastic stage, and did not consider the influence of the contact angle. Therefore, in this paper, a parabolic cylinder elastic-plastic contact model with different contact angles is proposed to explore the change rules of normal approach, normal contact stiffness, and actual contact area. The research results are compared with the cylindrical contact model, as well as FEM analysis results.

**Parabolic cylinder model development**

**Elastic stage**

If the focus distance of two parabolic cylinders is \( p \), the equation of the parabolic cylinder surface in parallel contact with two axes can be expressed as:
\[
\begin{align*}
\alpha &= \omega_1 + \omega_2 \\
L &= h + \omega_1 + \omega_2
\end{align*}
\]

Considering the initial gap between any two points on the parabolic cylinder surface, the distance \( L \) between any two points on the contact surface can be expressed as:

\[
\sum_{i=1}^{2} \int_0^h f' R \, dx' \, dy' = L - \frac{1}{2p} x^2 \sin^2 \theta + \frac{1}{2p} y^2 (1 + \cos^2 \theta)
\]

where \( k_1 = \frac{1-u_1^2}{E_1}, \quad k_2 = \frac{1-u_2^2}{E_2}, \quad u_1 \) and \( u_2 \) are the Poisson’s ratios, \( E_1 \) and \( E_2 \) are the Young’s moduli, \( F' dx' dy' \) is the load acting on the contact element, and \( R \) is the distance parameter.

Calculating the integral of equation (7), \( F' dx' dy' \) under the action of the contact load \( F_c \), the normal approach of any two points on the parabolic cylinder contact model can be expressed as:

\[
\alpha = \frac{3(k_1 + k_2) K_e}{2} \left[ \frac{2p(E_{(c)} - (1 - e^2)K_{(c)})}{e^2(1-e^2)} (k_1 + k_2) \right]^{-1/2} F_c'^2
\]

where \( K_{(c)} = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - e^2 \sin^2 \varphi}}, \quad E_{(c)} = \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \varphi} \, d\varphi \).

The radius of the contact ellipse curvature can be obtained by:

\[
\frac{\sin^2 \theta}{1 + \cos^2 \theta} = \frac{E_{(c)} - (1 - e^2)K_{(c)}}{(1 - e^2)^3 K_{(c)} - E_{(c)}}
\]

At the same time, the contact state of the parabolic cylinder contact model will eventually become surface contact, and the final actual contact area \( S_e \) of the contact model can be obtained based on equation (7) as:

\[
S_e = \frac{4\pi p (E_{(c)} - (1 - e^2)K_{(c)})}{3e^2(1 - e^2)^3 K_{(c)}} \alpha
\]

In addition, the relationship between the average contact pressure \( P_c \) and the normal approach \( \alpha \) is as follows:

\[
P_c = \frac{e}{\pi (k_1 + k_2)} \left\{ \frac{\alpha}{3p K_{(c)} [E_{(c)} - (1 - e^2)K_{(c)}]} \right\}^{1/2}
\]

The relationship between the normal contact stiffness \( K_e \) and the normal approach \( \alpha \) is as follows:
\[ K_c = \left[ \frac{4p(E(\varepsilon) - (1 - \varepsilon^2)K(c))\alpha}{3\varepsilon^2(1 - \varepsilon^2)(k_1 + k_2)^2K(c)^3} \right]^{\frac{1}{2}} \] (12)

With the increase of the normal approach, the parabolic cylinder model will begin to enter into the plastic stage. The von Mises yield condition indicates that the critical point of fully plastic deformation, as well as the monotonicity in the elastic-plastic deformation stage, should be guaranteed throughout the entire deformation process. Hence, the normal contact stiffness \( K_p \) is
\[ K_p = \frac{8\pi P_p [E(\varepsilon) - (1 - \varepsilon^2)K(c)]}{3\varepsilon^2(1 - \varepsilon^2)^2K(c)} \] (20)

Tabor\(^{35} \) shows that the load increases by about 300 times from the beginning of yield to the full plastic deformation. When the contact model has just reached the full plastic stage, the normal approach \( \alpha_2 \) is
\[ \alpha_2 = \frac{75e}{\pi P_p (k_1 + k_2)^2} \left( \frac{\alpha_1^3}{3K(c)P [E(\varepsilon) - (1 - \varepsilon^2)K(c)]} \right)^{\frac{1}{2}} \] (21)

**Elastic-plastic stage**

With the increase of the normal approach, the parabolic cylinder contact model will enter into the fully elastic stage, elastic-plastic stage, and fully plastic stage in turn. When the normal approach reaches the initial yield point \( \alpha_1 \), the parabolic cylinder model enters the elastic-plastic deformation stage. When the normal approach increases to a certain level, the model reaches the critical point \( \alpha_2 \) of the fully plastic stage. According to equation (17), in the fully plastic stage, the average contact pressure \( P_p \) of the contact model is equal everywhere. At the same time, the continuity and smoothness of the contact model at the initial yield point and critical point of fully plastic deformation, as well as the monotonicity in the elastic-plastic deformation stage, should be guaranteed throughout the entire deformation process.

Zhao et al.\(^{36} \) proposed the use of a cubic spline interpolation function to describe the elastic-plastic deformation of asperities. This function determines the relationship of the actual contact area and the average contact pressure with the normal approach. The present study intends to use the cubic spline interpolation function and discuss the applicability of this method to parabolic cylinder contact models.

The cubic spline interpolation function is as follows:
\[ f(\alpha) = -2 \left( \frac{\alpha - \alpha_1}{\alpha_2 - \alpha_1} \right)^3 + 3 \left( \frac{\alpha - \alpha_1}{\alpha_2 - \alpha_1} \right)^2 \] (22)

To ensure the continuity and smoothness of the deformation process, the actual contact area and average contact pressure should satisfy the following relationships at the critical point\(^{37} \):
\[
\begin{align*}
S_c(\alpha_1) &= S_{cp}(\alpha_1); \\
S_c'(\alpha_1) &= S_{cp}'(\alpha_1) \\
P_e(\alpha_1) &= P_{cp}(\alpha_1); \\
P_e'(\alpha_1) &= P_{cp}'(\alpha_1)
\end{align*}
\] (23)
\[
\begin{align*}
S_p(\alpha_2) &= S_{cp}(\alpha_2); \\
S_p'(\alpha_2) &= S_{cp}'(\alpha_2) \\
P_p(\alpha_2) &= P_{cp}(\alpha_2); \\
P_p'(\alpha_2) &= P_{cp}'(\alpha_2)
\end{align*}
\] (24)
where $S_{ap}$ is the actual contact area and $P_{ap}$ is the average contact pressure in the elastic-plastic stage.

Based on the above conditions, a relational expression including the cubic spline interpolation function can be established:

$$\begin{align*}
S_{ap} & = S_e(\alpha) + \left[ S_p(\alpha) - S_e(\alpha) \right] f(\alpha) \\
P_{ap} & = P_e(\alpha) + \left[ P_p(\alpha) - P_e(\alpha) \right] f(\alpha)
\end{align*}$$

(25)

By substituting the relationship between actual and average contact area with the normal approach in the elastic and plastic stages of the parabolic cylinder contact model into equation (25), the following equation can be obtained:

$$
S_{ap} = \frac{4\pi p \alpha [E_e(\alpha) - (1 - \epsilon^2)K_e]}{3e^2(1 - \epsilon^2)^3K_e}
\left[ 1 - 2 \left( \frac{\alpha - \alpha_1}{\alpha_2 - \alpha_1} \right)^3 + 3 \left( \frac{\alpha - \alpha_1}{\alpha_2 - \alpha_1} \right)^2 \right]
$$

(26)

$$
P_{ap} = \frac{e}{\pi(k_1 + k_2)} \left\{ \frac{\alpha}{3pK_e[E_e(\alpha) - (1 - \epsilon^2)K_e]} \right\}^2
+ \left\{ P_p - \frac{e}{\pi(k_1 + k_2)} \left\{ \frac{\alpha}{3pK_e[E_e(\alpha) - (1 - \epsilon^2)K_e]} \right\}^2 \right\}
\left[ -2 \left( \frac{\alpha - \alpha_1}{\alpha_2 - \alpha_1} \right)^3 + 3 \left( \frac{\alpha - \alpha_1}{\alpha_2 - \alpha_1} \right)^2 \right]
$$

(27)

Figures 2 and 3 display the relationship between actual and average contact pressure obtained by the cubic spline interpolation function with the normal approach. The selected material parameters were as follows: $p = 5$ mm, $E = 2.1 \times 10^6$ MPa, $u = 0.3$, $\sigma_p = 350$ MPa, and $\theta = 60^\circ$. As it can be seen in Figure 2, the variation curve of the actual contact area with normal approach satisfies the conditions of smoothness, continuity, and monotonicity. Figure 3 indicates that the variation curve of the average contact pressure with normal approach satisfies only the smoothness and continuity conditions, but not that of monotonicity. In fact, the higher the power interpolation function, the more apparent the oscillation. In this paper, the maximum power of the relationship between average contact pressure and normal approach was 3.5, which explains the oscillation of the interpolation function curve of the average contact pressure. Consequently, in the parabolic cylinder contact model, the cubic spline interpolation function is applicable to the relationship between actual contact area and normal approach, but not to the average contact pressure.

The curve of the average contact pressure of the parabolic cylinder contact model in the elastic-plastic deformation stage obtained by the cubic spline interpolation function can ensure only continuity and smoothness, but not monotonicity, which results in oscillations. Therefore, in order to tackle the above problems, this study intends to use an elliptic curve of lower order to further model the changes in the elastic-plastic average contact pressure with normal approach.

The coordinate of the parabolic cylinder contact model at the initial yield point is $(\alpha_1, P_1)$ and when it just enters the full plasticity stage it is $(\alpha_2, P_2)$. The average contact pressure $P_p$ of the parabolic cylinder contact model in the fully plastic stage is the same everywhere. Therefore, the short half axis of the elliptic curve is selected to coincide with the vertical line passing through the $\alpha_2$ point, ensuring that the value at the critical point $\alpha_2$ is $P_p$ and the slope is 0. At the same time, it should be ensured that the curve is monotone convex in the interval, and the slope can change from 0 to $+\infty$.

Based on the above constraints, the elliptic curve equation can be expressed as follows:

$$
\frac{(\alpha - \alpha_2)^2}{a_p^2} + \frac{(P_{ap} - b_p)^2}{(P_p - b_p)^2} = 1
$$

(28)
where \( a_p \) represents the length of the long half axis of the elliptic curve, \((P_p - b_p)\) is the length of the short half axis, and \( b_p \) represents the offset of the ellipse center relative to the \( \alpha - axis \).

The average contact pressure and slope of the contact model at the initial yield point can be respectively obtained by:

\[
P_1 = \alpha_{ep} C \tag{29}
\]

\[
P'_1 = \frac{e}{2\pi(k_1 + k_2)} \left\{ 3pK_\text{ep} \left[ E_c - \left( 1 - e^2 \right) K_\text{ep} \right] \alpha_{1} \right\}^{-\frac{1}{2}} \tag{30}
\]

According to the continuity and smoothness requirements, the following results can be obtained:

\[
\begin{align}
\left( \frac{\alpha_1 - \alpha_2}{a_p} \right)^2 + \frac{(P_1 - b_p)^2}{(P_p - b_p)^2} &= 1 \\
\alpha_1 - \alpha_2 + \frac{(P_1 - b_p)P'_1}{(P_p - b_p)^2} &= 0.
\end{align}
\tag{31}
\]

The result of equation (31) can be obtained as follows:

\[
\begin{align}
ap &= \frac{(P_p - P_1)^2 - P'_1(P_p - P_1)(\alpha_1 - \alpha_2)}{P'_1(\alpha_1 - \alpha_2) + 2(P_p - P_1)} \\
b_p &= \frac{P_p^2 - P_1^2 + P'_1(\alpha_1 - \alpha_2)}{P'_1(\alpha_1 - \alpha_2) + 2(P_p - P_1)}
\end{align}
\tag{32}
\]

Finally, the relationship between the average contact pressure \( P_{ep} \) and the normal approach \( \alpha \) of the parabolic cylinder contact model can be obtained:

\[
P_{ep} = b_p + |P_p - b_p| \left[ 1 - \left( \frac{\alpha - \alpha_2}{a_p} \right)^2 \right]^{\frac{1}{2}} \tag{33}
\]

where \( a_p \) and \( b_p \) have been determined by equation (32).

\[
\begin{align}
K_{ep} &= b_k + |K_p - b_k| \left[ 1 - \left( \frac{\alpha - \alpha_2}{a_k} \right)^2 \right]^{\frac{1}{2}} \\
a_k &= \frac{\left( K_p - K_1 \right)^2 - K'_1(K_p - K_1)(\alpha_1 - \alpha_2)}{K'_1(\alpha_1 - \alpha_2) + 2(K_p - K_1)} \\
b_k &= \frac{K_p^2 - K_1^2 + K_1K'_1(\alpha_1 - \alpha_2)}{K'_1(\alpha_1 - \alpha_2) + 2(K_p - K_1)} \\
K'_1 &= \frac{3}{4} \left[ \frac{16p \left[ E_c - \left( 1 - e^2 \right) K_\text{ep} \right]}{27e^2(1 - e^2)(k_1 + k_2)^2K_\text{ep}} \right]^{\frac{1}{2}} \tag{34}
\end{align}
\]

Figure 4 demonstrates the relationship between the average contact pressure obtained using the elliptic curve and the normal approach. The parameters of the parabolic cylinder model were consistent with those presented in Figure 2. It can be seen in Figure 4 that the average contact pressure versus normal approach curve satisfies the conditions of smoothness, continuity, and monotonicity. Therefore, the elliptic curve is suitable for calculating the average contact pressure.

Considering that the normal contact stiffness of the parabolic cylinder contact model has the same change trend with the normal approach, the elliptic curve was used to model the relationship between normal contact stiffness and normal approach in the elastic-plastic deformation stage.

By following again the above-described solution equations (28)–(33), the relationship between the normal contact stiffness \( K_{ep} \) and the normal approach \( \alpha \) of the parabolic cylinder contact model in the elastic-plastic stage can be expressed as:

\[
\begin{align}
K_{ep} &= b_k + |K_p - b_k| \left[ 1 - \left( \frac{\alpha - \alpha_2}{a_k} \right)^2 \right]^{\frac{1}{2}} \\
a_k &= \frac{\left( K_p - K_1 \right)^2 - K'_1(K_p - K_1)(\alpha_1 - \alpha_2)}{K'_1(\alpha_1 - \alpha_2) + 2(K_p - K_1)} \\
b_k &= \frac{K_p^2 - K_1^2 + K_1K'_1(\alpha_1 - \alpha_2)}{K'_1(\alpha_1 - \alpha_2) + 2(K_p - K_1)} \\
K'_1 &= \frac{3}{4} \left[ \frac{16p \left[ E_c - \left( 1 - e^2 \right) K_\text{ep} \right]}{27e^2(1 - e^2)(k_1 + k_2)^2K_\text{ep}} \right]^{\frac{1}{2}} \tag{34}
\end{align}
\]
parabolic cylinder model were consistent with those in Figure 2. As it can be seen in Figure 5, the curve of normal contact stiffness with normal approach meets the smoothness, continuity, and monotonicity conditions, indicating that the elliptic curve is applicable also to normal contact stiffness.

**Analysis and comparison**

The Matlab and ABAQUS software were used for simulation analysis and comparison to verify the correctness of the proposed parabolic cylinder contact model.

**Comparative analysis between the parabolic cylinder and cylindrical models**

Figures 6 and 7 compare respectively the normal approach and normal contact stiffness between the parabolic cylinder and cylindrical models. The selected material parameters were as follows: $E = 2.1 \times 10^5$ MPa, $u = 0.3$, $p = 0.25$ mm, and $\sigma_p = 350$ MPa. As it can be seen in Figures 6 and 7, the normal approach and normal contact stiffness curves of the two models exhibited the same trend. In the elastic and elastic-plastic stages, the normal approach and normal contact stiffness increase with the increase of load, showing a power exponential relationship. When $F = 0.12$ N, the two contact model enters the complete plastic stage, the normal approach increases exponentially with the increase of load, and the normal contact stiffness is independent of load. This indicated that parabolic cylinder and cylindrical models with similar geometric structure and contact mode have similar elastic-plastic properties in the normal approach. Therefore, the parabolic cylinder model is credible.

**Finite element simulation analysis**

In this paper, the ABAQUS software was used to analyze the elastic-plastic contact of the parabolic cylinder model. The selected material parameters were as follows: $E = 2.1 \times 10^5$ MPa, $u = 0.3$, $p = 0.25$ mm, and $\sigma_p = 350$ MPa. In addition, the mechanical properties of the parabolic cylinder model at contact angles of 45°, 60°, and 90° were analyzed. Figure 8 shows the meshed parabolic cylinder model. The mesh type was C3d8R, and was composed of 42,480 elements and 46,624 nodes. The contact model adopted the surface to surface contact mode, while the friction force was not considered. The minimum element size was $4.75 \times 10^{-4}$ mm. The load setting and boundary conditions were set as follows: A uniformly distributed load is applied to the top of the upper parabolic cylinder model in the vertical direction, the other directions were set as fixed constraints, and a fully fixed constraint was set at the bottom of the lower parabolic cylinder model.

The deformation contours along the $Y$-direction of the parabolic cylinder model obtained by the post-processing module are shown in Figure 9, where the model is at a contact angle of 45° and the load is 0.1 N. It can be observed in Figure 9(b) that the deformation of the contact surface was symmetrically distributed.
and the maximum deformation occurred at the central position. During the contact process, the contact surface formed a face-to-face contact which originated from the initial point-to-point contact.

Figure 10 compares the simulation and theoretical values of the normal approach under different contact angles. As it can be observed, there was a certain error between the simulation values and the theoretical values of the normal approach; however, their change trend was consistent. The normal approach increased with increasing load, and exhibited an exponential relationship in the elastic stage and a linear relationship in the plastic stage.

To further analyze the error between the simulation and theoretical values of the normal approach, the simulation and theoretical values of the normal approach under a load of 0.05 and 0.15 N were compared (Table 1). The maximum normal approach error was 7.61% and the minimum was only 4.83%. Furthermore,
it can be seen that, under the same load, the normal approach increased with increasing contact angle.

**Effect of normal approach on normal contact stiffness**

Figure 11 shows the effect of normal approach on normal contact stiffness. The selected material parameters were as follows: \( p = 0.25 \text{ mm}, \ E = 2.1 \times 10^5 \text{MPa}, \ u = 0.3, \) and \( \sigma_p = 350 \text{MPa}. \) As it can be seen in Figures 5 and 11, in the elastic and elastic-plastic stages, the normal contact stiffness increased with increasing load, and presented an exponential relationship. In the plastic stage, the normal contact stiffness was independent of the normal approach. Under the same normal approach, the normal contact stiffness decreased with increasing contact angle.

**Effect of normal approach on actual contact area**

Figure 12 shows the effect of normal approach on actual contact area. The selected material parameters were as follows: \( p = 0.25 \text{ mm}, \ E = 2.1 \times 10^5 \text{MPa}, \ u = 0.3, \) and \( \sigma_p = 350 \text{MPa}. \) It can be seen in Figures 12 and 13 that in the elastic and elastic-plastic stage, the actual contact area increased with increasing normal approach, presenting an exponential relationship. In the plastic stage, the actual contact area increased with increasing normal approach, presenting a linear relationship. In the entire deformation stage, the actual contact area decreased with increasing contact angle.

**Effect of focus distance**

Figures 13 to 15 show the effect of focus distance on normal approach, normal contact stiffness, and actual contact area, respectively. The selected material parameters were as follows: \( p = 0.25 \text{ mm}, \ E = 2.1 \times 10^5 \text{MPa}, \ u = 0.3, \) \( \sigma_p = 350 \text{MPa}, \) \( \theta = 45^\circ, \) and a load of \( P_{el} = 0.1 \text{ N} \) in the elastic stage and \( P_{pl} = 0.2 \text{ N} \) in the plastic stage. Considering that the parabolic cylinder contact model presented the same change trend in the elastic and elastic-plastic stages, only the effect of focus distance on the contact model in the elastic and plastic stages needs to be analyzed. As it can be seen in Figure 13, the amount of normal approach decreased with increasing focal length, and gradually tended to zero. In Figure 14, it can be seen that the normal contact stiffness increased with increasing focal length, and exhibited an exponential relationship in the elastic stage and a linear relationship in the plastic stage. Finally, it can be seen in Figure 15 that, in the elastic phase, the actual contact area increased with increasing focal length.

**Table 1.** Comparison between simulation and theoretical values of the normal approach.

| \( \theta^\circ \) | \( P, \text{N} \) | \( \alpha \) (simulation value), mm | \( \alpha \) (theoretical value), mm | Error % |
|---|---|---|---|---|
| 45 | 0.05 | 0.00006785 | 0.00006269 | 7.61 |
| 45 | 0.15 | 0.00015051 | 0.00014324 | 4.83 |
| 60 | 0.05 | 0.00007679 | 0.00007161 | 6.75 |
| 60 | 0.15 | 0.00018456 | 0.00017540 | 4.96 |
| 90 | 0.05 | 0.00009091 | 0.00008590 | 5.52 |
| 90 | 0.15 | 0.00028155 | 0.00026044 | 7.50 |
with the focus distance, while in the plastic phase, the actual contact area was not affected by the focus distance.

Conclusions

In this paper, based on the Hertz contact theory, an elastic-plastic contact model of parabolic cylinder convex bodies with different contact angles was developed, and the change rules of the normal approach, normal contact stiffness, and actual contact area in the parabolic cylinder contact model were investigated. In the elastic-plastic contact stage, the relationship between actual contact area and normal approach was established through the cubic spline interpolation function, while the relationship of the average contact pressure and normal contact stiffness with the normal approach was established through the elliptic curve. The establishment of the elastic-plastic contact model of two parabolic cylinders provides the theoretical basis for further research on the contact stiffness of regular parabolic cylinders. The following conclusions can be drawn:

(1) The simulation value of the normal approach of the contact model is compared with the theoretical value, the maximum normal approach error was 7.61% and the minimum was only 4.83%.

(2) The actual contact area increased with increasing normal approach. In the elastic and elastic-plastic stages, the normal contact stiffness increased with increasing normal approach; in the plastic stage, the normal contact stiffness was independent of the normal approach.

(3) The normal approach increased with increasing contact angle. The normal contact stiffness and actual contact area decreased with increasing contact angle.

(4) The normal approach increased with increasing focus distance, and the normal contact stiffness decreased with increasing focus distance. In the elastic and elastic-plastic stages, the actual contact area increased with increasing focus distance; in the plastic stage, the actual contact area was not affected by the focus distance.

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| Symbol | Definition |
|--------|------------|
| \( M_1 \) and \( M_2 \) | any two normal points on the surface of the parabolic cylinder model |
| \( p \) | distance from the focus to the directrix, mm |
| \( P_1 \) | maximum average contact pressure, N/mm\(^2\) |
| \( P_2 \) | average contact pressure of the critical point in the plastic stage, N/mm\(^2\) |
| \( P_{e_1} \), \( P_{e_2} \) and \( P_p \) | average contact pressure in the elastic, elastic-plastic, and plastic stage, respectively, N/mm\(^2\) |
| \( R \) | distance parameter, mm |
| \( S_1 \) | actual contact area at the initial yield point, mm\(^2\) |
| \( S_e \) | actual contact area in the elastic stage, mm\(^2\) |
| \( S_{e_1} \) | actual contact area in the elastic-plastic stage, mm\(^2\) |
| \( S_p \) | actual contact area in the plastic stage, mm\(^2\) |
| \( x \) | value on the \( X \)-axis, mm |
| \( y \) | value on the \( Y \)-axis, mm |
| \( z_1 \) and \( z_2 \) | values on the \( Z \)-axis, mm |
| \( z_1 \) | value on the \( Z \)-axis, mm |
| \( \alpha \) | normal approach, mm |
| \( \alpha_1 \) | normal approach at the initial yield point, mm |
| \( \alpha_2 \) | normal approach of the critical point in the plastic stage, mm |
| \( u, u_1, \) and \( u_2 \) | Poisson’s ratio |
| \( \theta \) | rotation angle or contact angle, \(^\circ\) |
| \( \omega_1 \) and \( \omega_2 \) | initial gap of the upper and lower parabolic cylinder, respectively, mm |
| \( \sigma_1, \sigma_2, \) and \( \sigma_3 \) | first principal stress, second principal stress, and third principal stress, Mpa |
| \( \sigma_f \) | the von Mises yield stress, MPa |
| FE | finite element |
| FEM | finite element method |