Matrix Dufresne identities

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Abstract

We prove a version of the classical Dufresne identity for matrix processes. More specifically, we show that the inverse Wishart laws on the space of positive definite $r \times r$ matrices can be realized by $\int_0^\infty M_s M_s^T ds$ in which $t \mapsto M_t$ is a drifted Brownian motion on $GL_r(\mathbb{R})$. This solves a problem in the study of spiked random matrix ensembles which served as the original motivation for this result. Various known extensions of the Dufresne identity (and their applications) are also shown to have analogs in this setting. In particular, we identify matrix valued diffusions built from $M_t$ which generalize in a natural way the scalar processes figuring into the geometric Lévy and Pitman theorems of Matsumoto and Yor.

1 Introduction

For $t \mapsto b_t$ a standard Brownian motion denote the associated geometric Brownian motion with drift, along with its (square) running integral by

$$m_t = m_t^{(\mu)} = e^{b_t + \mu t}, \quad a_t^{(\mu)} = \int_0^t m_s^2 ds.$$ (1)

We will use the convention that $\mu > 0$, with the choice of sign in the superscript of $a_t^{(\pm \mu)}$ reserved to produce an integral of $(m_t)^2 = (m_t^{(\pm \mu)})^2$ either converging or diverging (almost surely) as $t \to \infty$. In certain situations (if it does not cause confusion), we will not denote the dependence on $\mu$ explicitly.

The functional $a_t^{(\mu)}$ arises in a number of contexts including mathematical finance, diffusions in random environment, Brownian motion on hyperbolic spaces, and continuum models

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of 1 + 1 dimensional polymers (see [23] and references therein). Connected to the valuation of a certain perpetuity, Dufresne [13] established the fundamental identity in law,

$$a_t^{(-\mu)} \overset{\text{law}}{=} \frac{1}{2\xi},$$

(2)

in which $\xi$ has the Gamma($\mu$) distribution, with density function $\frac{1}{\Gamma(\mu)}x^{\mu-1}e^{-x}$ on the positive half line.

The Dufresne result provides one possible starting point to what is a vast collection of beautiful distributional identities for integrated geometric Brownian motion, much of which was pioneered by the work of Matsumoto and Yor. For instance, the following process level version of (2) was proved in [22]:

\[
\left\{ \frac{1}{a_t^{(-\mu)}}, \ t > 0 \right\} \overset{\text{law}}{=} \left\{ \frac{1}{a_t^{(\mu)}} + \frac{1}{a_t^{(-\mu)}}, \ t > 0 \right\}.
\]

(3)

Here $a_t^{(-\mu)}$ is a copy of $a_t^{(-\mu)}$, independent of the original Brownian motion $b_t$. It is important to note that Dufresne himself had earlier established (3) at fixed times in [14]. Further afield extensions include “geometric” versions of Lévy’s $M - X$ and Pitman’s $2M - X$ theorem [21], a Brownian Burke property [27], and the integrability of the O’Connell-Yor polymer model [28].

Motivated by a problem in random matrix theory one of the authors and J. Ramírez were led to a conjectured Dufresne type identity for matrix processes [31]. Here we prove that conjecture, and begin a program to extend the various results connected to the Dufresne identity to these matrix processes. In Section 1.1 we state our matrix analogs of (2) and (3). In Section 1.2 we introduce matrix diffusions which provide a possible generalization of those appearing in the just alluded to geometric Lévy and Pitman theorems, and discuss their asymptotics and intertwining properties. Section 1.3 states a partial Burke-type property for our matrix process. Finally in Section 1.4 we go back and describe the motivating spiked random matrix connection, and Section 1.5 discusses some open problems and related results in the literature.

1.1 Dufresne for matrix processes

The natural matrix extension $M_t = M_t^{(\mu)}$ of the geometric Brownian motion which arises in [31] is defined by the $r \times r$ matrix Itô equation

$$dM_t = M_t dB_t + \left( \frac{1}{2} + \mu \right) M_t dt, \quad M_0 = I, \quad t \geq 0$$

(4)
where \( t \mapsto B_t \) is the matrix valued Brownian motion comprised of independent standard Brownian motions \( \{b_{ij}(t)\}_{1 \leq i,j \leq r} \). Certainly this coincides with \( m_t \) when \( r = 1 \). Note that \( M_t \) is rotational invariant: if \( O \) is a fixed orthogonal matrix then \( OM_tO^T \) has the same law as a process as \( M_t \).

As we will point out below in Section 2, \( M_t \) is almost surely invertible for all time and for any \( s > 0 \), the process \( t \mapsto M_{s}^{-1}M_{t+s}, t \geq 0 \) has the same law as \( M_t, t \geq 0 \) and is independent of \( \{M_r, 0 \leq r \leq s\} \). Using the independent multiplicative increment property it is easy to extend \( M_t \) for all \( t \in \mathbb{R} \). Either version of the process may be referred to as the Brownian motion (with drift \( \mu \)) on the general linear group \( GL_r \).

Along with \( M_t \) we also define the additive functional \( A_t(\mu) = \int_0^t M_sM_s^Tds \) which is the matrix analog of the running integral \( a_t(\mu) \) from (1). Our basic matrix Dufresne identity is the following.

**Theorem 1.** If \( 2\mu > r - 1 \), the \( r \times r \) random matrix

\[
A_\infty(\mu) = \int_0^\infty M_sM_s^Tds
\]  

has the standard inverse Wishart distribution with parameter \( 2\mu \).

As Lemma 11 below shows, \( \lim_{t \to \infty} \frac{1}{t} \log \|M_t(-\mu)\| = -\mu + \frac{r-1}{2} \) with any matrix norm \( \|\cdot\| \). The condition \( 2\mu > r - 1 \) ensures that \( A_\infty(\mu) \) is almost surely finite. That condition is also necessary for the nondegeneracy of underlying Wishart distribution.

Recall that the standard \( r \times r \) (real) Wishart distribution with parameter \( p > r - 1 \) is the law on the cone of symmetric positive definitive matrices \( \mathcal{P} \) prescribed by:

\[
\gamma_p(dX) = \frac{1}{\Gamma_r(p/2)}(\det X)^{\frac{p-r-1}{2}}e^{-\frac{1}{2}trX}1_{\mathcal{P}}(X)dX.
\]  

(6)

Here \( \Gamma_r(p/2) \) the multivariate gamma function \( \Gamma_r(p/2) = \pi^{\frac{r(r-1)}{4}}\prod_{k=1}^r \Gamma\left(\frac{p-k+1}{2}\right) \). When \( p \) is also an integer \( \gamma_p \) can be realized by the random sample covariance matrix \( GG^T \) for \( G \) an \( r \times p \) matrix with independent standard normal entries. In either case it is the natural multivariate generalization of the gamma distribution. In symbols then Theorem 1 reads \( A_\infty(\mu) \overset{\text{(law)}}{=} \gamma_{2\mu}^{-1} \), the latter having density proportional to \( (\det X)^{-\mu + \frac{r+1}{2}}e^{-\frac{1}{2}trX^{-1}} \) on \( \mathcal{P} \). There are of course complex and quaternion Wishart distributions. Corollary 10 below provides a version of Theorem 1 for these settings.

\(^1\)Since all matrix variables will reside in \( GL_r \), we often omit the dependence on \( r \) from the notation for the various distributions as well as their support, e.g., \( \mathcal{P} \).
The original Dufresne identity (2) has a number of different proofs, not all of which appear extendable beyond the scalar case. To highlight those ideas that do carry over to the matrix case, we give two different proofs of Theorem 1. Both appear in Section 2. The first uses an inversion strategy employed by several authors. The second mimics an argument of Baudoin-O’Connell [3] which is likely the most succinct proof of the one dimensional identity and which we briefly summarize now.

Let \( y_t = y_0 e^{2b_t - 2\mu t} \), which is to say that \( y_t \) is \( m_t^2 \) with a variable starting point and the convergent choice of the sign of \( \mu \). The observation in [3] is that

\[
u(y) = E\left[ e^{-1/2 \int_0^\infty y_t dt} | y_0 = y \right] = E\left[ e^{-1/2 \int_0^\infty y_t dt} | y_0 = 1 \right] = E\left[ e^{-1/2 y_0 (\zeta - \mu)} \right],
\]

is on one hand the Laplace transform of the desired distribution, and on the other, courtesy of Feynman-Kac, a solution of

\[
2y \frac{d}{dy} y \frac{d}{dy} u(y) - 2\mu y \frac{d}{dx} u(y) - \frac{1}{2} y u(y) = 0, \quad u(0) = 1. \tag{7}
\]

The unique bounded solution of (7) is then shown to be \( u(y) = \frac{2^{1-\mu}}{1/\mu} y^{-\mu/2} K_{\mu}(\sqrt{y}) \), where

\[
K_s(a) = \frac{1}{2} \int_0^\infty x^{s-1} e^{-a(x^{1/2} + x^{-1/2})} dx
\]

is the Macdonald function (or modified Bessel function of the second kind). After a change of variables \( u \) is recognized as the Laplace transform of the (scaled) inverse gamma distribution.

Bessel functions of a matrix argument first appear in the 1955 work of Herz [16], and we introduce what is effectively his “\( K \)-Bessel” function:

\[
K_r(s|A, B) = \frac{1}{2} \int_P (\det X)^{s-\frac{1}{2}(r+1)} e^{-\frac{1}{4} trAX - \frac{1}{4} trBX} dX, \tag{9}
\]

for \( A, B \in P \).\(^2\) Note this reproduces the regular Macdonald function in the form \( K_1(s|a, b) = (ab)^{s/2} K_s(\sqrt{ab}) \) upon setting \( r = 1 \) and \( A, B = a, b \in \mathbb{R}_+ \). Both functions are well defined for all \( s \in \mathbb{C} \). It is also clear that, up to a normalization, \( K_r(-\mu|A, I) \) is the Laplace transform (in the variable \( A \)) of the \( \gamma_{2\mu}^{-1} \) distribution.

Picking up on the basic idea in [3] we set \( Y_t = M_t M_t^T \) with \( M_t = M_t^{(-\mu)} \) and \( 2\mu > r - 1 \), so that \( A_{\infty}^{(-\mu)} = \int_0^\infty Y_t dt \).

\(^2\)Herz actually denotes what is effectively this function by \( B_r \). We follow more closely the notation of Terras [34, §4.2.2], where this is referred to as the \( K \)-Bessel function of the second kind. We choose a slightly different normalization here (as in [17]) by introducing the extra 1/2 constants in the exponential term to better align with the standard \((r = 1)\) Macdonald function.
Theorem 2. The process \( t \mapsto Y_t \in \mathcal{P} \) is Markovian with generator,

\[
G_Y = 2\text{tr}(Y \frac{\partial}{\partial Y})^2 - 2\mu \text{tr}(Y \frac{\partial}{\partial Y}),
\]

expressed here through the matrix-valued operator \( \left[ \frac{\partial}{\partial Y} \right]_{ij} = (\frac{1}{2} + \frac{1}{2} \delta_{i,j}) \frac{\partial}{\partial Y_{ij}} \).

Furthermore, for \( 2\mu > r - 1 \) the unique bounded solution of

\[
G_Y U(Y) - \frac{1}{2}(\text{tr}Y) U(Y) = 0, \quad U(0) = 1, \quad Y \in \mathcal{P}
\]

is the normalized \( K \)-Bessel function \( U(Y) = \frac{K_r(-\mu|Y,I)}{2^{\mu-1} \Gamma_r(\mu)} \).

Theorem 1 then follows from considerations similar to those above:

\[
E[e^{-\frac{1}{2}\text{tr}(Y A(\tau)))}] = E[e^{-\frac{1}{2}\text{tr}(\int_0^\tau Y_t dt)}|Y_0 = I] = E[e^{-\frac{1}{2}\text{tr}(\int_0^\tau Y_t dt)}|Y_0 = Y] = U(Y).
\]

The middle equality uses that \( Y_t \) started from \( Y \in \mathcal{P} \) is equal in law to \( \sqrt{Y} Y_t \sqrt{Y^T} \), with now \( Y_t \) started from the identity, along with the trace being cyclic.

The theory of matrix Bessel functions has been developed considerably since [16], in part due to applications to multivariate statistics as well as to the harmonic analysis of symmetric spaces. See for example [25] (particularly Chapter 7) and [34], respectively. Both references include a number of differential operator characterizations of various matrix Bessel functions. Still, the present characterization of \( K_r(-|A,I) \) appears new despite the obvious similarities of (7) and (11).

Remark 3. The process \( Y_t \) (modulo drift) was previously studied in [26] as one of two canonical “Brownian motion on ellipsoids”. Its Markov property, along with that of its joint process of eigenvalues, was already remarked upon there. Because of the rotational invariance of \( M_t \), the function \( U(Y) \) is actually determined by the eigenvalues \( \Lambda = \Lambda(Y) = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \) alone. The eigenvalue process has generator

\[
G_\Lambda = \sum_{k=1}^r (2\lambda_k \frac{\partial}{\partial \lambda_k} \lambda_k \frac{\partial}{\partial \lambda_k} - (r - 1 + 2\mu) \lambda_k \frac{\partial}{\partial \lambda_k}) + \sum_{k<\ell} \frac{1}{\lambda_k - \lambda_\ell} (2\lambda_k^2 \frac{\partial}{\partial \lambda_k} - 2\lambda_\ell^2 \frac{\partial}{\partial \lambda_\ell}),
\]

and thus (11) can be expressed instead by \( (G_\Lambda - \frac{1}{2} \sum_{k=1}^r \lambda_k) U(\Lambda) = 0 \).

Last, we also have the exact matrix analog of the process level Dufresne identity (3).

Theorem 4. There is the following identity in distribution:

\[
\left\{ (A_t^{(\mu)})^{-1}, t \geq 0 \right\} \overset{\text{(law)}}{=} \left\{ (A_t^{-\mu})^{-1} - (A_\infty^{(-\mu)})^{-1}, t \geq 0 \right\}.
\]

Again it is assumed that \( 2\mu > r - 1 \).
Theorem 4 is proved in Section 3 by an enlargement of filtration argument, similar to the proof in [22] for the scalar case. Note the slightly different, but equivalent, presentation of the identity compared with (3). In this form (12) can actually be made an almost sure identity by an appropriate construction of the underlying matrix Brownian motions (see Proposition 15).

1.2 Geometric Lévy and Pitman theorems

Connected to their study of the functional \( a_t^{(\mu)} \), Matsumoto-Yor introduced the pair of processes,

\[
x_t = m_t^{-2} \int_0^t m_s^2 ds, \quad z_t = m_t^{-1} \int_0^t m_s^2 ds,
\]

(with \( m_t = m_t^{(\mu)} \)), both of which turn out to be diffusions [19, 20, 21]. For \( x_t \) the Markov property is immediate. An application of Itô’s formula produces the following simple sde for \( x_t \) for any \( \mu \in \mathbb{R} \):

\[
dx_t = 2x_t db_t + dt + (2 - 2\mu)x_t dt.
\]

Plainly, the same procedure applied to \( z_t \) cannot produce a closed equation. Nonetheless, \( z_t \) is a Markov process (for any \( \mu \in \mathbb{R} \)) with law described by,

\[
dz_t = z_t \bar{b}_t dt + \left( \frac{1}{2} - \mu \right) z_t dt + \frac{K_{\mu+1}}{K_\mu} \left( \frac{1}{z_t} \right) dt.
\]

Here \( K_\mu \) is the Macdonald function (8) and \( \bar{b}_t \) is a new Brownian motion (the subtlety is explained momentarily). An important property of \( z_t \) is its invariance under the transformation \( \mu \mapsto -\mu \) which follows from identity \( K_{\mu-1}(a) = K_{\mu+1}(a) - (2\mu/a)K_\mu(a) \) along with the more transparent fact \( K_\mu(a) = K_{-\mu}(a) \).

The interest in \( x_t \) and \( z_t \) is that they encode generalizations of the classical \( M - X \) theorem of Lévy, as well as the \( 2M - X \) theorem of Pitman, as was discovered by Matsumoto-Yor [19]. In particular, rescaling time by \( c^2 \) and taking \( \mu \) into \( \gamma/c \) yields:

\[
x_t^{\gamma/c} \overset{\text{(law)}}{=} c^2 \int_0^t e^{c(2b_s^\gamma - 2b_t^\gamma)} ds, \quad z_t^{\gamma/c} \overset{\text{(law)}}{=} c^2 \int_0^t e^{c(2b_s^\gamma - b_t^\gamma)} ds,
\]

in which \( b_t^\gamma \) is shorthand for the drifted Brownian motion. Simple Laplace asymptotics yield that, as \( c \to \infty \), \( \frac{1}{2c^2} \log \int_0^t e^{c(2b_s^\gamma - 2b_t^\gamma)} ds \) and \( \frac{1}{c} \log \int_0^t e^{c(2b_s^\gamma - b_t^\gamma)} ds \) converge pathwise to \( \max_{s<\ell}(b_s^\gamma - b_\ell^\gamma) \) and \( \max_{s<\ell}(2b_s^\gamma - b_\ell^\gamma) \). Working on the sde side (that is, with (14) and (15)), shows that \( \frac{1}{2c^2} \log x_t^{\gamma/c} \) and \( \frac{1}{c} \log z_t^{\gamma/c} \) have limiting processes that are equivalent in law to the diffusions with respective generators

\[
G_x = \frac{1}{2} \frac{d^2}{dx^2} - \gamma \text{sgn}(x) \frac{d}{dx}, \quad G_z = \frac{1}{2} \frac{d^2}{dz^2} + \gamma \cot(\gamma z) \frac{d}{dz}.
\]
The former is understood to be equipped with a Neumann boundary condition at the origin: the limiting \(x\)-process is reflected at the origin while the \(z\)-process has an entrance boundary at that point.

Letting \(\gamma \downarrow 0\), from the processes (16) we recover the reflected Brownian motion and the 3-d Bessel process occurring in the celebrated results of Lévy and Pitman identifying the distributions of the processes \(t \to \max_{s \leq t} (b_s - b_t)\) and \(t \to \max_{s \leq t} (2b_s - b_t)\). Taking the point of view that ‘\(\exp \int \log\)’ has replaced the running maximum, that the Brownian functionals in (13) are diffusions is now referred to as the “geometric” \(M - X\) or \(2M - X\) theorem.

Similar to the original Dufresne identity (2), there are various ways to identify the law of \(z_t\) with (15). The one relevant here is again due to Matsumoto-Yor [21], and rests on properties of the Generalized Inverse Gaussian (GIG) distribution. The GIG is a three-parameter distribution on the positive half line with density proportional to \(x^{p-1}e^{-\frac{1}{2}ax - \frac{1}{2}bx^{-1}}\), with arbitrary \(p\) and \(a, b > 0\). As such it is intimately connected to the Macdonald function \(K_p\): the ratio appearing in the drift (15) being the mean of a GIG with parameters \((\mu, 1/z_t, 1/z_t)\).

What is proved in [21] is that the law of \(m_t\) conditional on the field \(\{z_s, s \leq t\}\) is exactly this GIG, and the closed equation (15) is produced by a projection (and so the indicated \(\bar{b}_t\) is measurable with respect to \(\sigma(z_s, s \leq t)\)).

By analogy with (13) we introduce

\[
X_t = M_t^{-1}(\int_0^t M_s M_s^T ds)M_t^{-T}, \quad Z_t = M_t^{-1} \int_0^t M_s M_s^T ds,
\]

for our matrix process \(t \mapsto M_t\) as defined originally in (4). Note that from now on we use the shorthanded \(M^{-T} = [M^T]^{-1}\). That \(X_t\) is Markovian is again straightforward using Itô’s formula. For \(Z_t\), as can be anticipated at this point, there is a matrix GIG distribution on \(\mathcal{P}\), more or less defined by having normalizer given by the K-Bessel function (9). That is, it is the law

\[
\eta_{p,A,B}(dX) = c(\det X)^{p-\frac{1}{2}(r+1)}e^{-\frac{1}{2}(\text{tr}AX + \text{tr}BX^{-1})}1_p(X) dX,
\]

with \(A, B \in \mathcal{P}\) and \(c = \frac{1}{2}K_r(p|A, B)^{-1}\).

**Theorem 5.** For all \(\mu\), the process \(X_t\) is the diffusion defined by the Itô equation

\[
dX_t = Idt - 2\mu X_t dt + \text{tr}X_t Idt - dB_t X_t - X_t dB_t^T,
\]

run on the same Brownian motion \(B_t\) as \(M_t\) (4).

If \(|\mu| > \frac{r-1}{2}\) the process \(Z_t\) is also a diffusion. It satisfies

\[
dZ_t = dB_t Z_t + \left(\frac{1}{2} - \mu\right)Z_t dt + \kappa_\mu(I, (Z_t Z_t^T)^{-1})Z_t dt
\]

7
where now $\tilde{B}_t$ is a matrix valued Brownian motion adapted to $\sigma(Z_s, s \leq t)$ and $\kappa_p(A, B)$ denotes the mean of the $\eta_{p,A,B}$ distribution (18). In addition, the law of $t \mapsto Z_t$ is unchanged by taking $\mu$ into $-\mu$.

While both $Z_t$ and the right hand side of (20) are sensible for all $\mu$, our method uses Theorem 1 as input and so requires the same condition. One assumes this gap might be filled by other means.

In general the mean of a matrix GIG does not appear to have a particularly nice expression. Though if $A$ and $B$ are diagonal, one can see that $\kappa_p(A, B)$ is diagonal as well. And by bringing in a (well known) generalization of the $K$-Bessel functions introduced thus far one can get a reasonable handle on these diagonal components. See the proof of Theorem 7 below for both points. Note that the invariance of $Z_t$ under the map $\mu \mapsto -\mu$ implies the identity $$\kappa_{\mu}(I, A) = 2\mu I + \kappa_{-\mu}(I, A),$$ for $|\mu| > \frac{r-1}{2}$. (A standard analytic continuation argument extends the identity to all $\mu \in \mathbb{R}$. A direct verification of this identity seems laborious (and non-trivial).

The key to (20) is that the conditional distribution of $M_t$ given $\{Z_s, s \leq t, Z_t = Z\}$ is $Z^T \Xi$ for $\Xi \sim \eta_{\mu,I,(ZZ^T)^{-1}}$. This hinges on a characterization of the matrix GIG law due to Bernadac [4], which in turn builds on earlier work of Letac-Wesolowski [17]. An immediate consequence of this is the following.

**Corollary 6.** Let $|\mu| > \frac{r-1}{2}$. The laws of $M_t$ and $Z_t$ intertwine. Denote by $T^M_t$ and $T^Z_t$ the corresponding semigroups and define the Markov kernel $\Lambda$ as

$$\Lambda h(Z) = \int P h(Z^T X) \eta_{\mu,I,(ZZ^T)^{-1}}(dX),$$

for all suitable test functions $h : GL_r \mapsto \mathbb{R}$. Then it holds that $\Lambda T^M_t = T^Z_t \Lambda$. Since $X_t = Z_t M^{-T}_t$, it follows that $X_t$ also intertwines with $Z_t$. In this case,

$$\tilde{\Lambda} h(Z) = \int P h(X^{-1}) \eta_{\mu,I,(ZZ^T)^{-1}}(dX),$$

defines the corresponding kernel for which $\tilde{\Lambda} T^X_t = T^Z_t \tilde{\Lambda}$.

This intertwining has had far reaching implications in the scalar case. A remaining question here is whether the matrix processes contain either an $M - X$ or $2M - X$ type theorem. We show this occurs at the level of the eigenvalues (or singular values) of $X_t$ and $Z_t$, each of which comprise their own Markov process.
Theorem 7. Denote by $x_\mu^t(t) \leq \cdots \leq x_1^t(t)$ the eigenvalues of $X_t$. Denote the ordered singular values of $Z_t$ similarly by $z_\mu^t(t)$. Speeding up time by a factor of $c^2$ and rescaling $\mu$ as in $\mu = \frac{r-1}{2} + \frac{\gamma}{c}$ for a fixed $\gamma > 0$ we have that

$$\lim_{c \to \infty} \frac{1}{2c} \log x_1^\gamma/c(c^2t) \Rightarrow |b_t^{\gamma \text{sgn}(\cdot)}|,$$

and

$$\lim_{c \to \infty} \frac{1}{c} \log z_\mu^\gamma/c(c^2t) \Rightarrow b_t^{\gamma \text{coth}(\gamma \cdot)}.$$

The notations indicate a reflected Brownian motion with constant drift $-\gamma$, and a Brownian motion with variable drift $\gamma \text{ coth}(\gamma \cdot)$, respectively. In both cases the convergence takes place in the usual Skorohod topology.

In either case, the reminder of the spectrum has a relatively trivial limit in the chosen scaling. For $X_t$, each of the similarly scaled lower eigenvalues converge to the zero process. For $Z_t$, the larger singular values escape to infinity at increasing exponential rates. Note as well that Theorem 7 provides an analog for just half of the Matsumoto-Yor result — one might like at the same time to have path-wise identities by applying some sort of Laplace asymptotics to the definitions (17).

The proofs of Theorems 5 and 7, along with that of Corollary 6, are found in Section 4.

1.3 Burke properties

O'Connell-Yor [27] proved the following “Brownian Burke property”. Let $b_t$ and $c_t$ be independent Brownian motions, and set

$$r_t = \log \int_{-\infty}^{t} e^{b(s,t) + c(s,t) - \mu(t-s)} ds,$$

where $b(s,t) = b_t - b_s$ and $c(s,t) = c_t - c_s$. Then

\[
\begin{align*}
\{b_t + r_0 - r_t, t \in \mathbb{R}\} & \text{(law)} \quad \{b_t, t \in \mathbb{R}\} \quad \text{and are independent.}
\end{align*}
\]

The analogy with the classical Burke property is made by considering $t \mapsto r_t$ as a generalized queue, where ‘$\log \int \exp$’ again replaces ‘$\sup$’, with $t \mapsto b_t$ and $t \mapsto \mu t - c_t$ the respective arrival and departure processes. This result is key in the construction of the semi-directed Brownian polymer (also introduced in [27]) which is now understood to be a member of the KPZ universality class [8]. A similar Burke type property lies behind the integrability
of Seppäläinen’s log-gamma polymer [32] which has also subsequently been shown to have Tracy-Widom fluctuations [9].

The above scheme constructs two new independent Brownian motions from two independent input Brownian motions. As a preliminary step, a similar statement is established in [27] that shows that

\[ \{b^\mu_t + \alpha_t - \alpha_0, t \geq 0\} \overset{\text{law}}{=} \{b^\mu_t, t \geq 0\} \quad \text{where} \quad \alpha_t = \log \int_{-\infty}^{t} e^{2b^\mu_s - 2b^\mu_t} ds. \tag{25} \]

Here we are reusing notation from before, \(b^\mu_t\) denoting a Brownian motion with drift \(\mu\). Also, for fixed \(\tau \in \mathbb{R}\), the field generated by the new drifted Brownian motion \(b^\mu_t + \alpha_t - \alpha_0\) over \(t \leq \tau\) is independent of \(\{\alpha_t, t \geq \tau\}\).

The following provides a matrix extension of (25). It requires the full line version of the process \(M_t\), the details of which are again described at the beginning of Section 2.

**Theorem 8.** Fix \(2\mu > r - 1\) and let \(M_t = M_t^{(\mu)}\) be the solution of (4) extended over \(t \in (-\infty, \infty)\). Then,

\[ \left( \int_{-\infty}^{0} M_s M_s^T ds \right)^{-1} M_t \left( \int_{-\infty}^{t} M_t^{-1} M_s M_s^T M_t^{-T} ds \right) \overset{\text{law}}{=} M_t \quad \text{(law)} \tag{26} \]

as processes for \(t \geq 0\). In addition, for any fixed \(\tau > 0\), the process defined by the left hand side of (26) up to time \(\tau\) is independent of \(t \mapsto M_t^{-1}(\int_{-\infty}^{t} M_s M_s^T ds)M_t^{-T}\) for \(t \geq \tau\).

Note that \(M_t^{-1}(\int_{-\infty}^{t} M_s M_s^T ds)M_t^{-T}\) reduces to \(e^{\alpha_t}\) (and so (26) reduces precisely to (25)) for \(r = 1\).

The next result is a matrix version of (24), the Brownian Burke property of O’Connell-Yor.

**Theorem 9.** Let \(B_t\) and \(C_t\) be independent two-sided matrix Brownian motions and set \(2\mu > r - 1\). Consider the strong solution of

\[ dH_t = H_t(dB_t + dC_t) + (2\mu + 1)H_t dt, \quad H_0 = I, \]

extended to the whole line, again using the (multiplicative) independence stationary increment property as described in Section 2. Now define the processes \(A_{(-\infty, t]} = \int_{-\infty}^{s} H_u H_u^T du\) and

\[ F_t = B_t + 2\mu t - \frac{1}{2} \int_{0}^{t} H_s A_{(-\infty, s]} H_s ds, \quad G_t = C_t + 2\mu t - \frac{1}{2} \int_{0}^{t} H_s A_{(-\infty, s]} H_s ds. \]

Then \((F_t, G_t, t \geq 0) \overset{\text{law}}{=} (B_t, C_t, t \geq 0)\)
Here the analogy to the one-dimensional case is not as immediate, but the process \( t \mapsto 2\mu t - \frac{1}{2} \int_0^t H_s A(s) H_s ds \) can be seen to correspond to \( r_0 - r_t \) by first differentiating and then integrating back up in the definition (23). Note as well that \( H_{t/2} \overset{\text{law}}{=} M_t^{(\mu)} \).

Theorems 8 and 9 are proved in Section 3.

1.4 Connection to spiked random matrices

An important problem in mathematical statistics is to describe the law of the largest eigenvalue of sample covariance (or Wishart) matrices of the form \( G \Sigma G^\dagger \). In the basic setting \( G \) is \( p \times q \) and comprised independent unit Gaussians in \( F = \mathbb{R}, \mathbb{C}, \) or \( \mathbb{H} \) and \( \dagger \) is the associated conjugate transpose. One is typically interested in the limit as \( p \) and \( q \) tend to \( \infty \) with \( \Sigma \) some deterministic sequence of symmetric population matrices. When \( \Sigma = I \), this so-called soft-edge limit is well known to be given by the \( \beta = 1, 2 \) or 4 Tracy-Widom laws (for the case of \( \mathbb{R}, \mathbb{C}, \) or \( \mathbb{H} \) entries respectively). Moving toward the more general problem, the spiked ensembles in which \( \Sigma = \Sigma_r \oplus I_{q-r} \) and \( r \) remains fixed as \( p \) and \( q \) grow have generated considerable interest.

Using the determinantal framework at \( \beta = 2 \), \[2\] proved there exists a phase transition. Below criticality one sees Tracy-Widom in the limit, above criticality there are Gaussian effects (the limit given by the law of the largest eigenvalue of a finite rank GUE), with a new one parameter family of spiked soft-edge laws in the crossover regime. Subsequent analytic work was carried at \( \beta = 1 \) and \( \beta = 4 \) by \[24\] and \[35\], among others.

In another direction, \[5, 6\] proved that the \( \beta = 1, 2, \) or 4 soft-edge spiked laws can be characterized in a unified way through the eigenvalue problem for the (random) operator \( H \) acting on functions \( f \in L^2([0, \infty), \mathbb{F}' \) defined by

\[
H = -\frac{d^2}{dt^2} + rt + \sqrt{2} B_r', \quad f'(0) = C f(0).
\]

Here \( B_r \) is the standard \( \mathbb{F} \)-invariant Brownian motion, that is, for \( U \in U_r(\mathbb{F}) = \{ V \in \mathbb{F}^{r \times r} : VV^\dagger = I \} \) it holds that \( U B_r U^\dagger \overset{\text{law}}{=} B_r \), and \( C \) is the scaling limit of the matrix \( \Sigma_r \). At \( r = 1 \) the result holds for all \( \beta > 0 \). In that case the noise term reduces to \( \frac{2}{\sqrt{\beta}} b_x' \), and \( H \) is recognized as the Stochastic Airy Operator from \[29\], but with a Robin (rather than Dirichlet) boundary condition.

The authors of \[31\] asked whether one could similarly spike the hard edge, or smallest eigenvalue laws for general \( \beta \) (though see \[12\] for earlier work specific to \( \beta = 2 \)). This regime is defined by setting \( q = p + a \) for \( a > -1 \) remaining fixed as \( p \to \infty \). A primary motivation
was to confirm that the resulting spiked hard edge laws recover all known spiked soft edge laws via the familiar hard-to-soft transition.

To describe the spiked hard-edge operator, we set \( \mu = \frac{a+\alpha}{2} \) and introduce

\[
\begin{align*}
    dM_t &= M_t dW_t + \left( \frac{1}{2\beta} - \mu \right) M_t dt, \quad \mathcal{M}_t = M_t M_t^\dagger, \quad (27)
\end{align*}
\]

with the usual \( M_0 = I \). Here \( t \mapsto W_t \) is again an \( r \times r \) matrix of independent Brownian motions, but now with each off-diagonal entry a unit \( \mathbb{F} \)-valued Brownian motion and each diagonal entry a real Brownian motion with mean-square zero and mean-square \( \beta^{-1} \). The relevant result from [31] is that, under the condition that \((p\Sigma_r)^{-1} \to C^{-1}\) in norm, the limiting smallest eigenvalues of \( pG\Sigma G^\dagger \) are described by the eigenvalue problem for:

\[
Gf(t) = \int_0^\infty \left( \int_0^{t\wedge s} e^{ru} \mathcal{M}_u^\dagger du \right) \mathcal{M}_s f(s) ds + C^{-1} \int_0^\infty \mathcal{M}_s f(s) ds, \quad (28)
\]

acting on \( f \in L^2[\mathcal{M}] \), the space of functions \( f : \mathbb{R}_+ \mapsto \mathbb{F}^r \) with \( \int_0^\infty (f^\dagger \mathcal{M}_t f)(t) dt < \infty \). The operator \( G \) is positive compact and actually describes the limiting inverse Wishart eigenvalues, that is, the limiting Wishart eigenvalues are the spectral points \( \lambda \) for the problem \( \lambda Gf = f \).

As in the soft edge case, when \( r = 1 \) the result is valid for all \( \beta > 0 \). Also when \( r = 1 \), both \( \mathcal{M}_t \) and \( e^{rt} \mathcal{M}_t^{-1} \) reduce to geometric Brownian motions and \( G \) has the interpretation of the Green’s function for a Brownian motion in a Brownian potential. And again at \( r = 1 \) with \( C = c \in \mathbb{R} \), the limit \( c \to \infty \) recovers the basic \( \beta \) hard edge operator introduced in [30].

While there is no critical point at the hard edge, the supercritical regime refers to choosing \( C = cI \) and taking \( c \to 0 \) in (28). At one level the outcome is easy to describe: \( cG_{cI} \) converges (almost surely in operator norm) to the finite rank operator defined by integration against \( \mathcal{M}_t \). By analogy with the \( r \times r \) Gaussian invariant ensemble supercritical limit at the spiked soft edge, the obvious conjecture was that \( \text{spec} \left( \int_0^\infty \mathcal{M}_t dt \right) \) should be described by the inverse Wishart law(s). And of course for \( r = 1 \) the conjecture was known to be correct due to the original Dufresne identity.

**Corollary 10.** With the appropriate choice of \( t \mapsto W_t \), the eigenvalues of the \( r \times r \) random matrix \( \int_0^\infty \mathcal{M}_t dt \) have joint law given by the eigenvalues of the inverse \( \mathbb{F} \)-Wishart distribution with parameter \( 2\mu \). Using isotropic matrix Brownian motions and replacing the underlying process in (27) by

\[
\begin{align*}
    dM_t &= M_t dB_t + \left( \frac{1}{\beta} - \frac{1}{2} - \mu \right) M_t dt, \quad M_0 = I, \quad (29)
\end{align*}
\]
with \( B_t \) now comprised completely of independent unit \( \mathbb{F} \)-valued Brownian motions, the full \( \mathbb{F} \)-Wishart distribution is recovered by \( \int_0^\infty M_t M_t^\dagger dt \). In both cases, the natural condition on \( \mu \) remains \( 2\mu > r - 1 \).

Of course, when \( \beta = 1 \) the equations (27) and (29) agree and the above is a repeat of Theorem 1. While the structured noise in (27) is what arises in the spiked random matrix problem, we mention the result for (29) for \( \beta = 2 \) and 4 as it seems a more natural construction and readily produces the full matrix law. The proof of Corollary 10 is sketched alongside the proof of Theorem 1 in Section 2.

### 1.5 Further questions

The most obvious question is whether exists a (solvable) polymer model in matrix variables. The semi-directed Brownian polymer (or O’Connell-Yor polymer) alluded to above, can be defined by the partition function

\[
Z_{n,t} = \int_{0<s_1<\cdots<s_{n-1}<t} ds_1 \cdots ds_{n-1} \exp \left( b_{(0,s_1)}^{(1)} + b_{(s_1,s_2)}^{(2)} + \cdots + b_{(s_{n-1},t)}^{(n)} \right),
\]

where \((b^{(1)}, \ldots, b^{(n)})\) is a standard Brownian on \( \mathbb{R}^n \). The stationary version defined earlier in [27] has the first level not started at zero, but instead distributed over the negative half-line by the measure \( e^{bt-\mu t} \). That partition function was in fact arrived at, and aspects of its law understood, by iterating the Brownian Burke property described in Section 1.3 (recall (23) and (24)). What is missing in our case is a matrix Burke property that can be iterated through the non-commutativity in the same fashion.

In [28] O’Connell shows that the \( t \mapsto Z_{n,t} \) process has the same law as the top component of a diffusion on \( \mathbb{R}^n \) whose generator is a conjugation of the quantum Toda Hamiltonian. Remarkably, when \( n = 2 \) this result is exactly the Matsumoto-Yor 2M - X theorem (\( z_t \) is \( Z_{2,t} \) up to a change of variables). In both cases, there is an intertwining (between \( Z_{n,t} \) or \( z_t \) and the driving \( \mathbb{R}^n \) or \( \mathbb{R} \) Brownian motion) which provides a fairly explicit formula for the Laplace transform of the “\( z \)” processes. While we have an analogous intertwining (Corollary 6), the semigroup of Brownian motion on \( GL_r \) does not have a sufficiently concrete expression to afford a better characterization of the law of \( Z_t \). Potentially one might be able to bypass the intertwining, and find some description of the joint law \( (M_t, A_t) \), and so \( Z_t \), by more direct means (again, there are several such routes at \( r = 1 \) [20]).

One might also consider various parts of the above program for different groups. In this general spirit, but from different directions, we point out the very recent papers of
Chhaibi [10] and Bougerol [7]. In the second reference, geometric considerations lead to a process similar in structure to our $Z_t$, but constructed from a Brownian motion on the group of complex lower triangular matrices with positive diagonal. The singular values of this object are then shown to be Markov with generator given by a conjugation (by a polynomial function in copies of the Macdonald function and their derivatives) of quantum Toda on a Weyl chamber. Any direct link to the formulas derived here — to the generator of $Z_t$ (Theorem 5) or that for its singular values (see Lemma 22 below) — is not immediately transparent. More simply, it is natural to ask what matrix laws beyond the Wishart can be constructed from of a “Dufresne procedure” (back in the vein of Theorem 1).

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2 The matrix Dufresne identity

We prove the basic matrix Dufresne identity in two different ways. Stated above as Theorems 1 and 2, they fall below under the headings “Diffusion” and “Feyman-Kac” proof. We also provide a sketch of a “diffusion” proof of Corollary 10.

First we summarize some of the properties of $M_t = M_t^{(\mu)}$. Using the Taylor expansion of the determinant near $I$ and Itô’s formula one finds that,

$$d\det M_t = \det M_t (trdB + r(\frac{1}{2} + \mu)dt),$$

for $t \geq 0$. Hence, $\det M_t = \exp(trB_t + \mu rt)$, and $M_t$ is almost surely invertible for all time. Then, by the linearity of the sde (4) it follows that, for any $s > 0$, the process $t \rightarrow M_{s,t} = M_s^{-1}M_t, t \geq s$ satisfies the same equation subject to $M_{t,t} = I$. But that means that for fixed $s \geq 0$:

$$\{M_{s,s+t}, t \geq 0\} \overset{\text{law}}{=} \{M_t, t \geq 0\}, \quad \{M_{s,s+t}, t \geq 0\} \text{ is independent of } \{M_r, r \leq s\}. \quad (31)$$

These properties allow a natural extension of $M_t$ to all $t \in \mathbb{R}$. First extend the matrix Brownian motion $B_t$ for all $t \in \mathbb{R}$, and consider the strong solution $\tilde{M}_t$ of the sde:

$$d\tilde{M}_t = \tilde{M}_t dB_t + \tilde{M}_t(\frac{1}{2} + \mu)dt, \quad \tilde{M}_{-1} = I, \quad t \in [-1, 0].$$
Setting \( M_t = \tilde{M}_0^{-1} \tilde{M}_t \) for \( t \in [-1, 0) \), the extended \( M_t, t \geq -1 \) process satisfies (31) for any \( s \geq -1 \). Repeating this procedure for earlier starting points defines a version of \( M_t \) over the whole line. Importantly, the resulting process has properties (31) for each \( s \in \mathbb{R} \) and \( t \to M_{s,t} \) satisfies (4) for \( t \geq s \) with \( M_{t,t} = I \).

To close, we state the following lemma on the norm growth of \( M_t \) which we will use repeatedly. The proof is deferred to the very end of the section.

**Lemma 11.** Let \( m_1 \leq m_2 \leq \ldots \leq m_r \) be the singular values of \( M_t = M_t^{(\mu)} \). It holds that
\[
\lim_{t \to \infty} \frac{1}{t} \log m_i(t) = \mu + i - 1 \quad \text{with probability one for each } i = 1, \ldots, r.
\]

### 2.1 Diffusion proof

We actually prove Theorem 1 in two different ways as well. For completeness we first indicate how everything works directly through the matrix coordinates. The proof is somewhat more transparent in eigenvalue/eigenvector coordinates, and we carry out that approach afterwards.

**Via matrix coordinates**

Recall the definition of \( M_t = M_t^{(-\mu)} \) with the convergent choice of sign for the drift:
\[
dM_t = M_t dB_t + (\frac{1}{2} - \mu) M_t dt, \quad M_0 = I, \quad t \geq 0
\]
and of course \( 2\mu > r - 1 \).

Consider the version of this process extended to the whole line (as described just above), and then introduce the time reversed process \( N_t = M_{-t} \). We claim that \( N_t \) is also a Brownian motion on \( GL_r \), but with drift \( \mu \) instead of \( -\mu \). In particular, for \( t \geq 0 \) it solves the SDE
\[
dN_t = N_t d\tilde{B}_t + (\frac{1}{2} + \mu) N_t dt, \quad N_0 = I, \quad t \geq 0,
\]
where \( d\tilde{B}_t = -dB_{-t} \). A quick (but formal) explanation for this statement would follow from
\[
(I + dB_t + (\frac{1}{2} - \mu) Idt)^{-1} - I \approx -dB_t - (\frac{1}{2} - \mu) Idt + dB_t dB_t \quad \text{and} \quad dB_t dB_t = Idt.
\]
For the precise proof one needs to first verify that \( N_t \) also satisfies the stationary and independent increment property as \( M_t \), and then to show that if \( M_t \) solves (32) on say \( t \in [0, 1] \), the process \( \tilde{N}_t = M_1^{-1} M_{1-t}, t \in [0, 1] \) will solve the same sde with \(+\mu\) instead of \(-\mu\) and with \(-dB_{1-t}\) playing the role of \( dB_t \).

From this point (with a small abuse of notation) we will drop the tilde from \( d\tilde{B}_t \) and just use \( dB_t \) for the noise in \( N_t \).
Next consider \( N_{s,t} = N_t^{-1}N_{s+t} \) for any fixed \( t \). Then \( N_{s,t} \overset{\text{(law)}}{=} N_s \), again as processes in \( s \in \mathbb{R} \), and if we further define

\[
Q_t := \int_{-\infty}^0 N_{s,t}N_{s,t}^Tds = N_t^{-1}\left( \int_{-\infty}^t N_sN_s^Tds \right)N_t^{-T},
\]

we have at last a process that is stationary with marginal law given by that of

\[
Q_0 = \int_{-\infty}^0 N_sN_s^Tds \overset{\text{(law)}}{=} \int_0^{\infty} M_tM_t^Tdt.
\]

The proof of Theorem 1 then comes down to the following.

**Proposition 12.** Let \( 2\mu > r - 1 \). The process \( t \mapsto Q_t \) is a diffusion corresponding to the matrix sde

\[
dQ_t = Idt + (1 - 2\mu)Q_tdt + \text{tr}Q_tIdt - dB_tQ_t - Q_tdB_t^T.
\] (35)

If \( Q_0 \in \mathcal{P} \) then \( Q_t \) remains in \( \mathcal{P} \) for all \( t > 0 \) with \( \gamma_{2\mu}^{-1} \) as its unique invariant measure.

**Proof.** Applying Itô’s formula in (34) we find that

\[
dQ_t = Idt + dN_t^{-1}\left( \int_{-\infty}^t N_sN_s^Tds \right)N_t^{-T} + N_t^{-1}\left( \int_{-\infty}^t N_sN_s^Tds \right)dN_t^{-T}
\]

\[
+ dN_t^{-1}\left( \int_{-\infty}^t N_sN_s^Tds \right)dN_t^{-T}.
\]

We also have that,

\[
dN_t^{-1} = -N_t^{-1}dN_tN_t^{-1} + N_t^{-1}dN_tN_t^{-1}dN_tN_t^{-1}
\]

\[
= -dB_tN_t^{-1} + \left( \frac{1}{2} - \mu \right)N_t^{-1}dt,
\]

after substituting in (33) and using \( dB_tdB_t = Idt \) in the second term of line one. Combined, this produces,

\[
dQ_t = Idt - (dB + (\mu - \frac{1}{2})Idt)Q_t - Q_t(dB + (\mu - \frac{1}{2})Idt)^T + dBQ_tdB^T,
\]

which simplifies to (35) on account of the rule \( dB_tCdB_t^T = (\text{tr}C)Idt \) for any matrix \( C \).

To see that \( Q_t \), defined by (35) and a fixed starting point \( Q_0 \in \mathcal{P} \), remains in \( \mathcal{P} \) for all time, apply Itô’s formula yet again, now to \( t \mapsto \det(Q_t) \):

\[
d\det(Q_t) = \det(Q_t)\left( \text{tr}[Q_t^{-1}dQ_t] + \frac{1}{2}(\text{tr}[Q_t^{-1}dQ_t])^2 - \frac{1}{2}\text{tr}[Q_t^{-1}dQ_tQ_t^{-1}dQ_t] \right)
\]

\[
= \det(Q_t)\left(-2\text{tr}dB_t + \text{tr}Q_t^{-1}dt + (2r - 2\mu)dt \right).
\]
For the second line we used \( \text{tr}[Q_t^{-1}dQ_t Q_t^{-1}dQ_t] = \text{tr}[Q_t^{-1}(dB)^2Q_t] + \text{tr}(dB)^2 + 2\text{tr}[Q_t^{-1}dBdB^T] \).

Introducing \( b_t = -\text{tr}B_t/\sqrt{r} \) and

\[
dz_t = 2\sqrt{r}z_t db_t + (2r - 2\mu)z_t dt, \quad z_0 = \det(Q_0) > 0,
\]

we see that \( t \mapsto \det(Q_t) \) is bounded below by the geometric Brownian motion \( z_t \) up to the first passage of \( \det(Q_t) \) to zero. But \( z_t \) never vanishes, and so that passage time must be infinite.

The next ingredients are to show that \( Q_t \) admits a smooth positive transition density on \( P \), and then to identify the inverse Wishart law \( \gamma^{-1}_2 \) as an invariant measure. Since the latter also has a positive density with respect to Lebesgue measure on \( P \), it will follow that this is the unique invariant measure for \( Q_t \).

The generator of \( Q_t \) can be succinctly expressed in matrix coordinates as in,

\[
G_Q = 2\text{tr} \left( Q^2 \left( \frac{\partial}{\partial Q} \right)^2 \right) + (1 - 2\mu)\text{tr} \left( Q \frac{\partial}{\partial Q} \right) + (1 + \text{tr}Q)\text{tr} \left( \frac{\partial}{\partial Q} \right),
\]

where again \( \frac{\partial}{\partial Q} \) is the matrix-valued operator \( \left[ \frac{\partial}{\partial Q} \right]_{ij} = (\frac{1}{2} + \frac{1}{2}\delta_{ij}) \frac{\partial}{\partial Q_{ij}} \). The second order part of (37) is verified by writing out

\[
(dQ + QdB^T)_{ij}(dQ + QdB^T)_{k\ell} = [Q^2]_{ij\ell} \delta_{ik} + [Q^2]_{ik\ell} \delta_{jk} + [Q^2]_{ik} \delta_{ij}\delta_{\ell},
\]

and summing over \( i \leq j \) and \( k \leq \ell \). One then checks that the adjoint takes the form,

\[
G^*_Q = 2\text{tr} \left( Q^2 \left( \frac{\partial}{\partial Q} \right)^2 \right) + (2\mu + 2r + 3)\text{tr} \left( Q \frac{\partial}{\partial Q} \right) + (\text{tr}Q - 1)\text{tr} \left( \frac{\partial}{\partial Q} \right) + \mu r(r + 1) + \frac{1}{2} r(r + 1)^2,
\]

when restricted to act on functions of a symmetric matrix variable.

To invoke the necessary regularity estimates we temporarily consider the “vectorized” \( Q_t \), or \( \text{vect}(Q_t) = (Q_{11}(t), Q_{22}(t), \ldots) \in \mathbb{R}^{r(r+1)/2} \). In particular, we show that the diffusion matrix written in these coordinates is positive definite on the open set \( P \subset \mathbb{R}^{r(r+1)/2} \). This comes down to proving: if \( Q \) is a positive definite matrix and \( Z \) is an \( r \times r \) matrix normal with iid entries then the covariance matrix of \( \text{vect}(ZQ + QZ^T) \) is positive definite. To see this, take any \( r \times r \) orthogonal matrix \( O \) and let \( K_O \) be the linear map on \( \mathbb{R}^{r(r+1)/2} \) defined by \( \text{vect}(O^TQO) = K_O \text{vect}(Q) \). It is easy to verify that \( |\det K_O| = 1 \). Next write the spectral decomposition of \( Q = O^T \Lambda O \), and note that the desired covariance matrix satisfies:

\[
E\text{vect}(ZQ + QZ^T)\text{vect}(ZQ + QZ^T)^T
= EK_O\text{vect}(Z\Lambda + \Lambda Z^T)\text{vect}(Z\Lambda + \Lambda Z^T)^TK_O^T.
\]
Thus, by the first remark, we may assume that $Q$ is diagonal with entries $Q_{ii} > 0$. But in that case the covariance matrix is diagonal with entries $4[Q^2]_ii, 1 \leq i \leq r$ and $[Q^2]_{ii} + [Q^2]_{jj}, 1 \leq i < j \leq r$ which is clearly positive definite.

At last then, Theorem 3.4.1 of [33] (though see also Remark 3.4.2 there) implies that $\partial_t - G_Q^*$ is hypoelliptic on $P$. At the same time a straightforward but tedious calculation will show that $G_Q^* f(Q) = 0$ for $f(Q) = (\det Q)^{-\mu - \frac{r+1}{2}} e^{-\frac{1}{2} \text{tr} Q^{-1}}$, the $\gamma_{2\mu}^{-1}$ density. For smooth test functions $h$ of compact support on $P$ and $T_t$ the semigroup of $Q_t$, we have

$$\frac{\partial}{\partial t} \int_P (T_t h)(Q) f(Q) dQ = \int_P G_Q (T_t h)(Q) f(Q) dQ.$$ 

An integration by parts would continue the equality as $\int_P (T_t h)(Q) (G_Q^* f(Q)) dQ = 0$ and complete the proof. To justify this, that is, that there are no boundary terms, requires two facts. The first is that $T_t h(Q)$ and its normal derivative are bounded along the boundary $\det(Q) = 0$. This can be established by writing $G_Q$ in local coordinates in the vicinity of $\det(Q) = 0$, and working by comparison with one-dimensional process (36) whose semigroup is readily seen to have the desired property at the origin. The second is to check by a simple computation that $\frac{\partial}{\partial Q_{ij}} f(Q)|_{\det(Q)=0} = 0$ for all $i, j$, noting that $f(Q)|_{\det(Q)=0} = 0$ is obvious.

**Via the eigenvalue law**

One could alternately argue that, since the law of $Q_t$ defined in (35) is invariant under rotations by the orthogonal group, it is enough to consider the motion of the eigenvalues. More convenient still, is to identify the Wishart law $\gamma_{2\mu}$ itself by considering instead the eigenvalues of $P_t = Q_t^{-1}$.

We have that,

$$dP_t = -P_t^2 dt + (1 + 2\mu) P_t dt + \text{tr} P_t dt - dB_t^T P_t - P_t dB_t, \quad (39)$$

the solution of which, by similar reasoning as above, also remains in $P$ for all time after starting from any point in the interior. Further, $p_r \geq p_{r-1} \geq \cdots \geq p_1 \geq 0$, the ordered eigenvalues perform the joint diffusion,

$$dp_i = -p_i^2 dt + (2\mu + 2)p_i dt + p_i \sum_{j \neq i} \frac{p_i + p_j}{p_i - p_j} dt + 2p_i db_i, \quad (40)$$

with $\{b_i\}_{i=1,...,r}$ independent standard Brownian motions. The above can be derived from (39) by computing the Itô differential of the corresponding spectral representation. We will
do a sample of such a calculation below in a slightly more complicated context. It is by now standard that system \((40)\) possesses a strong solution, that the paths \(p_i = p_i(t)\) do not intersect for \(t > 0\), and any initial condition \(P_0\) with some \(p_i(0) = p_{i+1}(0)\) is an entrance point, see for example \([1, \S 4]\).

The action of the corresponding generator \(G_p\) can be expressed in the form,

\[
G_p f = \sum_{i=1}^{r} 2p_i^2 \partial_i^2 f + \sum_{i} \left( -p_i^2 + (2\mu + 2)p_i + p_i \sum_{j \neq i} \frac{p_i + p_j}{p_i - p_j} \right) \partial_i f \\
= \sum_{i=1}^{r} \partial_i (\phi(p_i) \partial_i f) - (\phi(p_i) \partial_i V) \partial_i f,
\]

where,

\[
\phi(p_i) = 2p_i^2, \quad V(p) = -\sum_{i > j} \log(p_i - p_j) + \frac{1}{2} \sum_{i=1}^{r} p_i - (\mu - \frac{r+1}{2}) \sum_{i=1}^{r} \log p_i,
\]

restricted to the Weyl chamber \(\Sigma_r = \{p_r \geq \cdots \geq p_1 \geq p_0 = 0\} \subset \mathbb{R}^r\). Now \(e^{-V(p)} \mathbf{1}_{\Sigma_r}(p)\) is recognized as (after a suitable normalization) the joint density of eigenvalues for the real Wishart ensemble \(\gamma_{2\mu}\), and the form of \(G_p\) in line two of \((41)\) is particularly suited for verifying that \(G_p^* (e^{-V(p)}) = 0\).

To identify \(e^{-V(p)}\) as the invariant measure, we only have to deal with the same integration by parts issue that came up when working with matrix coordinates. In this case Appendix A of \([15]\) explains why for smooth \(h\) of compact support in \(\Sigma_r\) we have that \(x \mapsto E_x[h(p_1(t), \ldots, p_r(t))]\) along with its normal derivative are bounded at the seams \(p_{i+1} = p_i\) (or boundary of \(\Sigma_r\)). After that a quick calculation yields \(\lim_{p_{i+1} \to p_i} \frac{\partial}{\partial(p_{i+1} + p_i)} e^{-V(p)} = 0\).

**Proof of Corollary 10**

For the complex and quaternion cases it is a bit more constructive to go through eigenvalue/eigenvector coordinates. The starting point remains the same: a matrix diffusion \(t \mapsto Q_t\) is constructed which has the desired distribution as its invariant measure (assuming the latter exists and is unique).

Corollary 10 considers two setting. The analogs of \((35)\) are:

\[
dQ_t = \Id t + \left( \frac{2}{\beta} - 1 - 2\mu \right) Q_t dt + \text{tr} Q_t dt + dB_t Q_t + Q_t dB_t^\dagger,
\]

for the \(U_r(\mathbb{F})\)-invariant noise \(t \mapsto B_t\), and

\[
dQ_t = \Id t + \left( \frac{1}{\beta} - 2\mu \right) Q_t dt + \left( \frac{1}{\beta} - 1 \right) \text{diag}(Q_t) dt + \text{tr} Q_t dt + dW_t Q_t + Q_t dW_t^\dagger,
\]
for the case of the particular structured $\beta = 1, 2, 4$ noise $t \mapsto W_t$ appearing in the original spiked random matrix problem. Here $\text{diag}(Q_t)$ is the diagonal matrix with the same diagonal as $Q_t$. This corresponding term in the sde is shows up because of $dW_t Q_t dW_t^\dagger = (\frac{1}{\beta} - 1) \text{diag}(Q_t) dt + \text{tr} Q_t dt$.

The key observation is:

**Proposition 13.** In either setting (43) or (44), the eigenvalues of $P_t = Q_t^{-1}$ are Markovian with common sde:

$$dp_i = -p_i^2 dt + (2\mu + \frac{2}{\beta}) p_i dt + p_i \sum_{j \neq i} \frac{p_i + p_j}{p_i - p_j} dt + \frac{2}{\sqrt{\beta}} p_i db_i,$$

for $\beta = 1, 2, 4$.

Compare (40), noting the overlap at $\beta = 1$. That the eigenvalues of (43) are Markov is self-evident. For (44) we only see it by going through the calculation, which we defer to the end of the proof. Note that one can use (45) after that fact to see that $P_t$ (and so $Q_t$) remains in $\mathcal{P}$ for all time.

The upshot is that, for the eigenvalue motion(s), the argument is now precisely the same as in the $\beta = 1$ case. The corresponding $\beta = 2$ or 4 generator $G_{\beta,p}$ has the same form as (41), with (42) replaced by:

$$\phi(p_i) = \frac{2}{\beta} p_i^2, \quad V(p) = -\beta \sum_{i > j} \log(p_i - p_j) + \frac{\beta}{2} \sum_{i=1}^r p_i - \beta \left( \mu - \frac{r + 2/\beta - 1}{2} \right) \sum_{i=1}^r \log p_i.$$

It follows that $G_{\beta,p}^*(e^{-V(p)}) = 0$. And, as it has to be, $e^{-V(p)}1_{\Sigma_r}(p)$ is proportional to the complex/quaternion Wishart eigenvalue density. For the isotropic setting one then has the full $\mathbb{F}$-Wishart law, as the eigenvector process of (43) clearly has the Haar measure on $U_r(\mathbb{F})$ as its unique invariant measure.

**Proof of Proposition 13.** Itô’s formula shows that if $Q$ solves (43) or (44) then $P$ solves the SDE analogue to (39), with $\dagger$ in place of the transpose and an extra $(\frac{1}{\beta} - 1) \text{diag}(P) dt$ term in the second case.

At this point we make two simplifications. For clarity we carry out the computation for $\beta = 2$ only (that the $\beta = 4$ case will go through in the same way will be clear). Also, we consider the simplified matrix sde:

$$dP_t = F(P_t) dt + dW_t P_t + P_t dW_t^\dagger, \quad F(P) = -\frac{1}{2} \text{diag}(P),$$

(46)
where again $W_t$ has independent real Brownian motions with variance $\frac{1}{2}t$ on the diagonal and independent unit complex Brownian motions elsewhere. The point is that (46) retains everything “non isotropic” in (44). That the corresponding isotropic case (with $B_t$ replacing $W_t$ and no $\text{diag}(\cdot)$ term in the drift) produces the same answer will also become clear in the course of the proof.

Either way, the strategy is standard. Write $P_t = U_t^\dagger \Lambda_t U_t$ for $\Lambda_t$ the diagonal matrix of eigenvalues $(\lambda_{1,t}, \ldots, \lambda_{r,t})$ and a unitary matrix $U_t$. Also introduce the notation,

$$W_t = U_t W_t U_t^\dagger, \quad dU_t U_t^\dagger = d\Gamma_t + dG_t.$$ (47)

Note that the former is that it is not simply a copy of $W_t$. The latter is the Doob-Meyer decomposition, with $\Gamma_t$ a local martingale and $G_t$ of finite variation. Since $d(U_tU_t^\dagger) = 0$ one finds that

$$d\Gamma_t = -d\Gamma_t, \quad \text{and} \quad dG_t + dG_t^\dagger = -d\Gamma_t d\Gamma_t^\dagger,$$ (48)

in particular $d\Gamma_{ii} = 0$. With this in hand an application of Itô’s formula produces

$$d\Lambda = d\mathcal{W}\Lambda + \Lambda d\mathcal{W}^\dagger + UF(U^\dagger \Lambda U)U^\dagger dt + (d\Gamma \Lambda + \Lambda d\Gamma^\dagger) + (dG\Lambda + \Lambda dG^\dagger)$$

$$+ d\Gamma \Lambda d\Gamma^\dagger + d\Gamma d\mathcal{W}\Lambda + d\Gamma \Lambda d\mathcal{W}^\dagger + d\mathcal{W}\Lambda d\Gamma^\dagger + \Lambda d\mathcal{W}^\dagger d\Gamma^\dagger.$$ (49)

As the martingale part of the right hand side must vanish off the diagonal we infer that,

$$d\Gamma_{ij} = \frac{\lambda_j d\mathcal{W}_{ij} + \lambda_i d\overline{\mathcal{W}}_{ji}}{\lambda_i - \lambda_j},$$ (50)

for $i \neq j$. Next, we write out (49) on the diagonal:

$$d\lambda_i = \lambda_i (d\mathcal{W}_{ii} + d\overline{\mathcal{W}}_{ii}) + \sum_j u_{ij} F_{jj}(U^\dagger \Lambda U) \overline{u}_{ij} dt$$

$$- \sum_{j \neq i} (\lambda_i - \lambda_j) d\Gamma_{ij} d\overline{\Gamma}_{ij}$$

$$+ \sum_{j \neq i} (\lambda_i d\mathcal{W}_{ji} + \lambda_j d\overline{\mathcal{W}}_{ij}) d\Gamma_{ij} + \sum_{j \neq i} (\lambda_i d\overline{\mathcal{W}}_{ji} + \lambda_j d\mathcal{W}_{ij}) d\overline{\Gamma}_{ij},$$

having used (48). We also record that,

$$\sum_j u_{ij} F_{jj}(U^\dagger \Lambda U) \overline{u}_{ij} = -\frac{1}{2} \sum_{j=1}^r \sum_{\ell=1}^r u_{ij} \overline{u}_{ij} \lambda_\ell u_{\ell j} \overline{u}_{ij} = -\frac{1}{2} \sum_{j=1}^r \sum_{\ell=1}^r \lambda_\ell |u_{ij}|^2 |u_{\ell j}|^2,$$ (52)

$$= -\frac{1}{2} \lambda_i + \frac{1}{2} \sum_{j \neq i} (\lambda_i - \lambda_j) \sum_{\ell=1}^r |u_{i\ell}|^2 |u_{j\ell}|^2,$$

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where \( u_{ab} \) are the entries of \( U \).

To finish, first note that \( (W_{ii} + \overline{W}_{ii}, i = 1, \ldots, r) \) \((\text{law}) = (\sqrt{2}b_i, i = 1, \ldots, r)\) for independent standard real Brownian motions \( b_i \). Next, one may check that for \( i \neq j \)

\[
dW_{ij}dW_{ij} = dt - \frac{1}{2} \sum_{\ell=1}^{r} |u_{i\ell}|^2 |u_{j\ell}|^2 dt, \quad dW_{ij}dW_{ji} = \frac{1}{2} \sum_{\ell=1}^{r} |u_{i\ell}|^2 |u_{j\ell}|^2 dt. \tag{53}
\]

These rules may then be used along with (50) in (51) to find that the contribution of the final two lines there is equal to:

\[
\sum_{j \neq i} (\lambda_i - \lambda_j)^{-1} \left( \lambda_i^2 |dW_{ji}|^2 + \lambda_j^2 |dW_{ij}|^2 + \lambda_i \lambda_j (dW_{ij}dW_{ji} + d\overline{W}_{ij}d\overline{W}_{ji}) \right)
= \sum_{j \neq i} \frac{\lambda_i^2 + \lambda_j^2}{\lambda_i - \lambda_j} - \frac{1}{2} \sum_{j \neq i} (\lambda_i - \lambda_j) \sum_{\ell=1}^{r} |u_{i\ell}|^2 |u_{j\ell}|^2. \tag{54}
\]

The last term here now cancels with the last term in (52). This produces the system \( d\lambda_i = \sqrt{2}\lambda_i db_i - \frac{1}{2} \lambda_i \mu dt + \sum_{j \neq i} \frac{\lambda_i^2 + \lambda_j^2}{\lambda_i - \lambda_j} dt \). Now putting back the additional drift terms \((\text{tr}P - P^2 + \left( \frac{1}{2} - 2\mu \right) P)\) will produce (45) and complete the proof.

Note also that in case of the isotropic noise the cross variations in (53) simplify as \( dt \) and 0 respectively, and thus in (54) only the term \( \sum_{j \neq i} \frac{\lambda_i^2 + \lambda_j^2}{\lambda_i - \lambda_j} \) remains (as it should). \( \square \)

### 2.2 Feynman-Kac proof

We prove Theorem 2.

Taking \( M_t \) with the convergent choice of drift, as in (32) and again with \( 2\mu > r - 1 \), the task is to show that: with \( Y_t = M_t M_t^T \),

\[
E \left[ e^{-\frac{1}{2} \int_0^\infty Y_t dt \mid Y_0 = Y} \right] = \frac{K_r(-\mu|Y, I)}{2^{\mu-1} \Gamma_r(\mu)}, \tag{55}
\]

via the partial differential equation that characterizes the left hand side.

This relies on the Markov property of \( t \mapsto Y_t \in \mathcal{P} \). Applying Itô’s formula gives

\[
dY_t = M_t(dB_t + dB_t^T)M_t^T + (r + 1 - 2\mu)Y_t dt, \tag{56}
\]

which does not appear to close. But a check of the matrix entry covariances produces the generator,

\[
G_Y = \sum_{i \leq j} \sum_{k \leq \ell} (y_{ik}y_{j\ell} + y_{i\ell}y_{jk}) \frac{\partial^2}{\partial y_{ij} \partial y_{k\ell}} + (r + 1 - 2\mu) \sum_{i \leq j} y_{ij} \frac{\partial}{\partial y_{ij}},
\]

\( \text{22} \)
which can then be put into the abbreviated form (10): $G_Y = 2\text{tr}(Y \frac{\partial}{\partial Y})^2 - 2\mu \text{tr}(Y \frac{\partial}{\partial Y})$. (One can also argue that by the rotation invariance of $dB_t$ each appearance of $M_t$ in (56) can be replaced by a positive square root $\sqrt{Y_t}$.)

The standard martingale argument used to derive the Feynman-Kac formula shows that the left hand side := $U(Y)$ of (55) solves $(G_Y - \frac{1}{2} \text{tr} Y)U(Y) = 0$. We have $U(0) = 1$ since $Y_t = 0$ for all time if $Y_0 = 0$. As for uniqueness, we claim that if $\tilde{U}(Y)$ is bounded and satisfies $(G_Y - \frac{1}{2} \text{tr} Y)\tilde{U}(Y) = 0$ and $\tilde{U}(0) = 0$, then $\tilde{U}(Y)$ is identically zero. Consider the bounded martingale $S_Y t = \tilde{U}(Y_t) e^{-\frac{1}{2} \int_0^t \text{tr}(Y_s) ds}$. By Lemma 11 the matrix norm of $Y_t \to 0$ almost surely as $t \to \infty$. Hence $S_Y t$ also converges to zero as $t \to \infty$. But then $S_Y t \equiv 0$ as $S_Y t$ is a regular martingale, and so $\tilde{U}(Y) \equiv 0$.

We are left to check that the $K$-Bessel function on the right hand side of (55) satisfies the PDE. First note,

$$e^{\frac{1}{2} \text{tr}(xy)}(2G_y - \text{tr}(y))e^{-\frac{1}{2} \text{tr}(xy)} = \text{tr}(xyxy) + (2\mu - r - 1)\text{tr}(xy) - \text{tr}(y),$$

where from here on we revert to lower case for matrix variables. Differentiating under the integral defining $K_r(-\mu|y,I)$, we are left to show that

$$0 = \int_{\mathcal{P}} (\text{tr}(xyxy) + (2\mu - r - 1)\text{tr}(xy) - \text{tr}(y)) \det(x)^{-\mu} e^{-\frac{1}{2} \text{tr}(xy + x^{-1})} d\mu_r(x).$$

(57)

Here we have introduced the invariant measure on $\mathcal{P}$:

$$d\mu_r(x) = (\det x)^{-\frac{r+1}{2}} dx,$$

(58)

so called because

$$z = t^T xt \sim d\mu_r(z) = d\mu_r(x), \quad z = x^{-1} \sim d\mu_r(z) = d\mu_r(x),$$

(59)

with any invertible $t$ in the first relation, see [34]. Unfortunately, integrating back up inside the integral (57) appears cumbersome. Instead we rely on the uniqueness of the associated Mellin transform.

**Proposition 14.** (See Theorem 1 of [34, §4.3.1]) Introduce the power function,

$$p_s(y) = \prod_{k=1}^r \det(y_k)^{s_k} \quad s = (s_1, \ldots, s_r) \in \mathbb{C}^r,$$

(60)

where $y_k \in \mathcal{P}_k$ (the $k \times k$ positive definite matrices) is the $k$th minor of $y \in \mathcal{P} = \mathcal{P}_r$. Then the Mellin transform,

$$\hat{h}(s) := \int_{\mathcal{P}} p_s(y) h(y) \mu_r(dy),$$

defines an invertible map on the subspace of rotation invariant functions — those $h$ for which $h(x) = h(O^T x O)$ for orthogonal $O$ — of $L^2(\mathcal{P},\mu_r)$. 23
The necessary isometry is actually stated in the cited theorem for the more general Helgason-Fourier transform on \( P \), but this reduces to the above Mellin transform on rotation invariant functions. The right hand side of (57) is clearly rotation invariant in \( y \) — it is a function of the eigenvalues of \( y \) alone.

Setting
\[
V(y) = \int_P \det(x)^{-\frac{1}{2}} e^{-\frac{1}{2} \text{tr}(xy+x^{-1})} d\mu_\tau(x) = \int_P \det(x)^{-\frac{1}{2}} e^{-\frac{1}{2} \text{tr}(x^{-1}y+x)} d\mu_\tau(x),
\]
that is, \( V(y) = 2K_r(-\mu|y, I) = 2K_r(\mu|I, y) \), we compute first \( \hat{V}(s) \) and then track the multipliers to this result produced after multiplying by, or integrating against, the additional factors \( \text{tr}(y), \text{tr}(\cdot y), \) and \( \text{tr}(\cdot y \cdot y) \).

\textbf{Step 1:} This formula can be found in [34], but it guides the later computations so we record it here. The trick is to introduce the change of variables \( x = t^T t \) for \( t \) upper triangular with \( t_{i,j} \in \mathbb{R}, \ t_{ii} \in \mathbb{R}_+ \). This results in the rules:
\[
d\mu_\tau(x) = 2^r \prod_{i=1}^r t_{ii}^{-1} \prod_{i \leq j} dt_{ij}, \quad p_s(x) = \prod_{j=1}^r t_{jj}^{r_j}, \quad \text{for } r_j = 2(s_j + \cdots + s_n). \tag{62}
\]
Using two such changes of variables \( y = t^T t \) and \( x = q^T q \) produces
\[
2^{-2r} \hat{V}(s) = \int \int p_s(t^T t) \det(q^T q)^{\frac{1}{2}} e^{-\frac{1}{2} \text{tr} t^T q^{-1} t} \prod_j t_{jj}^{-j} q_{jj}^{-j} dt dq = \int \int p_s(t^T t) p_s(q^T q) \det(q^T q)^{\frac{1}{2}} e^{-\frac{1}{2} \text{tr} q^{-1} t} \prod_j t_{jj}^{-j} q_{jj}^{-j} dt dq
\]
\[
= \int \int \prod_{j=1}^r t_{jj}^{r_j-j} q_{jj}^{r_j+2\mu_j-j} e^{-\frac{1}{2} \sum_{i \leq j} (q_{ii}+t_{ii})} dt dq. \tag{63}
\]
In going from line one to line two, we replaced \( y = t^T t \) with \( q^T q \), used the first invariance in (59), and then also the fact \( p_s(q^T q) = p_s(y)p_s(q^T q) \), see Proposition 1 of [34, §4.2.1]. The final line can be computed explicitly (each diagonal component producing a gamma function, the Gaussian integral over each off diagonals producing a factor of \( 2\pi \)), but for what we do here it is better to leave the answer in this form.

\textbf{Step 2:} Denote \( V_1(y) = \text{tr}(y)V(y) \). By comparison to the last step we easily find that
\[
2^{-2r} \hat{V}_1(s) = \int \int \text{tr}(t^T t q^T q) \prod_{j=1}^r t_{jj}^{r_j-j} q_{jj}^{r_j+2\mu_j-j} e^{-\frac{1}{2} \sum_{i \leq j} (q_{ii}+t_{ii})} dq dt. \tag{64}
\]
Now expanding out, \( \text{tr}(t^T t q^T q) = \sum_{k \leq i,j \leq \ell} t_{ki} t_{kj} q_{il} q_{je} \), we see that any term with \( i < j \) will vanish by producing a factor of the form \( \int u e^{-\frac{1}{2} u^2} du = 0 \). Hence, we can replace the
trace inside the integral (64) with \( \sum_{k \leq i \leq \ell} t_k^2 q_{k\ell}^2 \). Each integral in the resulting sum (over \( k \leq i \leq \ell \)) can be reduced to (63) after an integration by parts (or two), yielding a different multiplicative factor of \( \hat{V} \) for different pairings of indices (\( k < i < \ell \) versus \( k = i = \ell \), for example). Those multipliers are as follows.

| Multiplier: | \( k < i < \ell \) | \( k = i = \ell \) |
|------------|-----------------|-----------------|
| \( 1 \)    |                 |                 |
| \( r_i + 2\mu - i + 1 \) |                 |                 |
| \( r - i + 1 \) |                 |                 |
| \( (r_i - i + 1)(r_i + 2\mu - i + 1) \) | \( k = i < \ell \) | \( k = i = \ell \) |

Summing up we find a total multiplier of

\[
c_1 = \sum_{i=1}^{r} r_i^2 + (2\mu + r + 1) \sum_{i=1}^{r} r_i - 2 \sum_{i=1}^{r} ir_i, \tag{65}
\]

that is, \( \hat{V}_1 = c_1 \hat{V} \).

**Step 3:** Now let \( \hat{V}_2(s) = \int p_s(y) \int (\det x)^{-\mu} \text{tr}(xy) e^{-\frac{1}{2} \text{tr}(yx + x^{-1})} d\mu(x)d\mu(y) \), for which we have that

\[
2^{-2r} \hat{V}_2(s) = \int \int \text{tr}(t^T t) \prod_{j=1}^{r} t_{jj}^{-j} q_{jj}^{r_j + 2\mu - j} e^{-\frac{1}{2} \sum_{i \leq j} (q_{ij}^2 + t_{ij}^2)} dq dt. \tag{66}
\]

With \( \text{tr}(t^T t) = \sum_{k \leq i, k \leq j} t_{ki} t_{kj} \), the considerations are even simpler than above. Comparing (66) to (63), there are just two different cases.

| Multiplier: \( \) | \( i < j \) | \( i = j \) |
|------------------|-----------------|-----------------|
| \( 1 \)          |                 |                 |
| \( r - i + 1 \)  | \( i = j \)     |                 |

Summing over all possible \( i, j \) we find that \( \hat{V}_2 = c_2 \hat{V} \) with

\[
c_2 = \sum_{i=1}^{r} r_i, \tag{67}
\]

**Step 4:** Finally set \( \hat{V}_3(s) = \int p_s(y) \int (\det x)^{-\mu} \text{tr}(xyxy) e^{-\frac{1}{2} \text{tr}(yx + x^{-1})} d\mu(x)d\mu(y) \), and write

\[
2^{-2r} \hat{V}_3(s) = \int \int \text{tr}(t^T t t^T t) \prod_{j=1}^{r} t_{jj}^{-j} q_{jj}^{r_j + 2\mu - j} e^{-\frac{1}{2} \sum_{i \leq j} (q_{ij}^2 + t_{ij}^2)} dq dt. \tag{68}
\]

Now the expansion is

\[
\text{tr}(t^T t t^T t) = \sum_{k \leq i, j} t_{ki} t_{kj} t_{ti} t_{tj},
\]
and similar to the $\hat{V}_1$ calculation, all terms corresponding to $k \neq \ell < i \neq j$ terms will vanish.

The six remaining choices yield:

| Multiplier | Terms |
|------------|-------|
| $2 \times 1$, | $k = \ell < i < j$ or $k < \ell < i = j$, |
| $2 \times (r - i + 1)$, | $k = \ell = i < j$ or $k < \ell = i = j$, |
| $3$, | $k = \ell < i = j$, |
| $(r_i - i + 3)(r_i - i + 1)$, | $k = \ell = i = j$. |

The additional factor of two in lines one and two count the ordering of $(i, j)$ or $(k, \ell)$. A little algebra shows that then $\hat{V}_3 = c_3 V$ with

$$c_3 = \sum_{i=1}^{r} r_i^2 + 2(r + 1) \sum_{i=1}^{r} r_i - 2 \sum_{i=1}^{r} ir_i.$$  \hfill (69)

The proof of Theorem 2 is finished by checking that

$$c_3 + (2\mu - r - 1)c_2 - c_1 = 0,$$

recall (65) and (67).

To finish the proof of Theorem 2, we now return to the:

**Proof of Lemma 11.** This calculation can basically be found in [26]. It would be nice to have a way to control at least the matrix norm as sharply without going to eigenvalue coordinates.

Staying in the setting just considered, we let $y_1 \leq y_2 \leq \cdots \leq y_r$ be the eigenvalues of $Y_t$ and show that $\lim_{t \to \infty} \frac{1}{t} \log y_i(t) = -2\mu + i - 1$ with probability one for each $i = 1, \ldots, r$.

With $\gamma_i = \log y_i$ we find from (56) and considerations similar to those behind Proposition 13 that,

$$d\gamma_i = 2db_i - 2\mu dt + \sum_{j \neq i} \frac{e^{\gamma_i} + e^{\gamma_j}}{e^{\gamma_i} - e^{\gamma_j}} dt.$$  

One checks that $\sum_{j \neq i} \frac{e^{\gamma_i} + e^{\gamma_j}}{e^{\gamma_i} - e^{\gamma_j}} \geq r - 1$ and $\sum_{j \neq i} \frac{e^{\gamma_i} + e^{\gamma_j}}{e^{\gamma_i} - e^{\gamma_j}} \leq 1 - r$. Moreover, if we change $i$ to $i + 1$ then the interaction term will change by at most $2\frac{e^{\gamma_i + 1} + e^{\gamma_i}}{e^{\gamma_i + 1} - e^{\gamma_i}}$. Thus, $\gamma_{i+1} - \gamma_i$ is bounded above by the solution to

$$dz_i = 2d(b_{i+1} - b_i) + 2 \left( \frac{1 + e^{-z_i}}{1 - e^{-z_i}} \right) dt.$$  

The proof is finished by remarking that $P(\lim_{t \to \infty} \frac{z_i}{t} = 2, i = 1, \ldots, r - 1) = 1$. \hfill \Box

### 3 Process level identities

We prove Theorem 4 and the Burke property statements of Theorems 8 and 9.
3.1 $A_t^{(\mu)}$ and $A_t^{(-\mu)}$

Here we again start by taking $M_t = M_t^{(-\mu)}$ defined by (32) for $t \geq 0$ with driving matrix Brownian motion $t \mapsto B_t$. Throughout this section we drop the superscript on the corresponding additive functional, $A_t = A_t^{(-\mu)}$, and it is always assumed that $2\mu > r - 1$.

We need the following two facts.

**Proposition 15.** Denote by $B_t = \sigma(B_s, s \leq t)$ and by $\hat{B}_t$ the initial enlargement $B_t \vee \sigma(A_\infty)$. Then,

$$\hat{B}_t := B_t - \int_0^t (2\mu I - M_T s (A_\infty - A_s)^{-1} M_s) ds,$$  \hspace{1cm} (70)

is a standard matrix Brownian motion with respect to $\hat{B}_t$ and is independent of $A_\infty$.

**Proposition 16.** Almost surely,

$$(A_t^{-1} - A_\infty^{-1})^{-1} = \int_0^t N_s N_s^T ds, \quad \text{for } N_t := A_\infty (A_\infty - A_t)^{-1} M_t,$$ \hspace{1cm} (71)

Furthermore, conditioned on the value of $A_\infty$, the process $N_t$ satisfies

$$dN_t = N_t d\hat{B}_t + (\frac{1}{2} + \mu) N_t dt, \quad N_0 = I,$$ \hspace{1cm} (72)

where $\hat{B}_t$ is as defined in (70).

Granted these propositions the result is immediate:

**Proof of Theorem 4.** By (72), $\int_0^t N_s N_s^T ds$ has the same distribution as $A_t^{(\mu)}$ as a process. Thus conditioned on $A_\infty$, we have that $\{A_t^{-1} - A_\infty^{-1}, t \geq 0\}$ is equal in law to $\{(A_t^{(\mu)})^{-1}, t \geq 0\}$. But then this is also true unconditionally, which is precisely the desired statement. Afterward, one can read (71) as an almost sure version of the identity (12). \hfill \Box

**Proof of Proposition 15.** We start with the representation

$$A_\infty = A_t + M_t \left( \int_0^\infty M_t^{-1} M_{s+t} M_t^{-T} M_{s+t}^T ds \right) M_t^T,$$

noting that by the matrix Dufresne identity (Theorem 1) the integral on the right hand side has the $\gamma_{2\mu}^{-1}$ distribution. Then, with $f$ denoting the corresponding density function,

$$P(A_\infty \in dA \mid \mathcal{B}_t) = f (M_t^{-1} (A - A_t) M_t^{-T}) (\det M_t)^{-(r+1)} dA.$$  \hspace{1cm} (73)
Here we have used that the Jacobian of the map $A \mapsto MAM^T$ on symmetric matrices is given by $(\det M)^{r+1}$, see for example Lemma 2.2 of [17].

Next, by the tower property of conditional expectations the Radon-Nikodym derivative
\begin{equation}
R_{t,A} = \frac{dP(A_{\infty} \in dA \mid B_t)}{dP(A_{\infty} \in dA)}
= c_A \det(A - A_t)^{-\mu - \frac{r+1}{2}} \det(M_t)^{2\mu} \exp[-\frac{1}{2}tr((A - A_t)^{-1}M_tM_t^T)]
\end{equation}
is a $B_t$-martingale. The formula in line two follows from writing out the appearance of $f(M_t^{-1}(A - A_t)M_t^{-T})$ in (73). The factor $c_A$ is then the density $f(A)$ without the normalizing constant. As noted in (30), $t \mapsto \det M_t^{-\mu}$ is the geometric Brownian motion $\det M_t = e^{\text{tr}B_t - \mu t}$. Substituting this in the preceding formula we record
\begin{equation}
dR_{t,A} = R_{t,A} \left(2\mu \text{tr}B_t - \text{tr}[dB_tM_t^T(A - A_t)^{-1}M_t]\right)
\end{equation}
for later use.

Next we will show that for any bounded continuous test function $h$ and event $\Lambda_s \in B_s, s < t$:
\begin{equation}
E[1_{\Lambda_s}h(A_{\infty})(B_t - B_s)] = E[1_{\Lambda_s}h(A_{\infty}) \int_s^t (2\mu - M_u^T(A_{\infty} - A_u)^{-1}A_u) du].
\end{equation}
Granted (76) the monotone class theorem will imply that
\begin{equation}
E[B_t - B_s \mid \hat{B}_s] = \int_s^t (2\mu - M_u^T(A_{\infty} - A_u)^{-1}A_u) du,
\end{equation}
or in other words that $\hat{B}_t$ defined in (70) is a local martingale (with respect to the filtration $\hat{B}_t$). To see that it is actually a matrix Brownian motion, and so complete the proof, one checks the quadratic covariation of the entries and invokes Lévy’s theorem.

Returning now to (76) we introduce $\lambda_t(h) := E[h(A_{\infty})\mid B_t]$ and write the left hand side of that equality as in
\begin{equation}
E[1_{\Lambda_s}h(A_{\infty})(B_t - B_s)] = E[1_{\Lambda_s}(\lambda_t(h)B_t - \lambda_s(h)B_s)]
\end{equation},
as follows by conditioning separately with respect to both $B_t$ and then $B_s$. By (74) we have that,
\begin{align}
\lambda_t(h) &= \int h(A)f(M_t^{-1}(A - A_t)[M_t]^T)(\det M_t)^{-2}dA \\
&= \int h(A)R_{t,A}f(A)dA,
\end{align}
28
where again $f$ is the $\gamma_{\gamma_{2\mu}}^{-1}$ density function. And then (75) implies that,

$$
\lambda_t(h) - \lambda_s(h) = \int_P \int_s^t h(A) R_{u,A} \left[ 2\mu trdB_u - tr[dB_u M_u^T (A - A_u)^{-1} M_u] \right] f(A) dA.
$$

To continue we compute the cross variation of $\lambda_t(h)$ and $[B_t]_{i,j}$, for which we just need to check the coefficient of $dB_{i,j}$ in the previous integral with the result that

$$
\langle \lambda_t(h), [B_t]_{i,j} \rangle = \int_P h(A) R_{A,u} \left[ 2\mu 1_{i=j} - [M_u^T (A - A_u)^{-1} M_u]_{i,j} \right] f(A) dA.
$$

Then writing $B_t - B_s = \int_s^T dB_u$ leads to

$$
E \left[ \langle \lambda_t(h)B_t - \lambda_s(h)B_s \rangle \bigg| B_s \right] = \int_P \int_s^t h(A) R_{A,u} \left[ 2\mu I - M_u^T (A - A_u)^{-1} M_u \right] du f(A) dA
$$

$$
= \int_s^t E \left[ h(A) \lambda_t(h) - M_u^T (A - A_u)^{-1} M_u \right] |B_u| du.
$$

From here we see that $E \left[ 1_{A_t} \lambda_t(h)B_t - \lambda_s(h)B_s \right] = E \left[ 1_{A_t} E \left[ \langle \lambda_t(h)B_t - \lambda_s(h)B_s \rangle \bigg| B_s \right] \right]$ is equal to the right hand side of (76), as required.

**Proof of Proposition 16.** Since $t \mapsto A_t^{-1}$ is almost surely once differentiable and $||A_t - tI|| = o(t)$ as $t \rightarrow 0$ (as $M_0 = I$ and $M_t$ is continuous), we have that

$$
(A_t^{-1} - A_0^{-1})^{-1} = \int_0^t d \left[ (A_s^{-1} - A_0^{-1})^{-1} \right], \tag{77}
$$

also almost surely. On the other hand,

$$
dl [ (A_t^{-1} - A_0^{-1})^{-1} ] = dA_t (A_0^{-1} - A_t)^{-1} A_0^{-1} A_t (A_0^{-1} - A_t)^{-1} dA_t (A_0^{-1} - A_t)^{-1} A_0^{-1}
$$

$$
= M_t M_t^T (A_0^{-1} - A_t)^{-1} A_0^{-1} dt + A_t (A_0^{-1} - A_t)^{-1} M_t M_t^T (A_0^{-1} - A_t)^{-1} A_0^{-1} dt
$$

$$
= A_0^{-1} (A_0^{-1} - A_t)^{-1} M_t M_t^T (A_0^{-1} - A_t)^{-1} A_0^{-1} dt.
$$

Here we have used the matrix identity $(C^{-1} - D^{-1})^{-1} = C(D - C)^{-1}D$ in line one, and $I + C(D - C)^{-1} = D(D - C)^{-1}$ to go from line two to line three. Substituting back into (77) yields (71), identifying $N_t = A_0^{-1} (A_0^{-1} - A_t)^{-1} M_t$ at the same time.

To see (72) note that $N_0 = A_0^{-1} A_0^{-1} M_0 = I$ and compute,

$$
d [ A_0^{-1} (A_0^{-1} - A_t)^{-1} M_t ]
$$

$$
= A_0^{-1} (A_0^{-1} - A_t)^{-1} M_t M_t^T (A_0^{-1} - A_t)^{-1} M_t dt + A_0^{-1} (A_0^{-1} - A_t)^{-1} M_t dB_t + \left( \frac{1}{2} - \mu \right) dt
$$

$$
= N_t \left[ M_t^T (A_0^{-1} - A_t)^{-1} M_t dt + dB_t + \left( \frac{1}{2} - \mu \right) dt \right].
$$

By Proposition 15, the quantity inside the final bracket equals $d\hat{B}_t + \left( \frac{1}{2} + \mu \right) dt$, as desired.
3.2 Burke properties

Both results are consequences of Propositions 15 and 16. Let $N_t$ be defined as in the latter statement and introduce the temporary shorthand $\hat{A}_t = \int_0^t N_s N_s^T ds$, reserving $A_t$ for $\int_0^t M_s M_s^T ds$. Importantly, $M_t$ is also chosen as throughout the proofs of those same propositions. In particular the underlying matrix Brownian motions $\hat{B}_t$ and $B_t$ stand in the same relationship.

**Proof of Theorem 8.** We start by rewriting the almost sure identity (71) as in

$$\hat{A}_t^{-1} = A_t^{-1} - A_\infty^{-1}.$$  

Repeated use of this along with the resolvent identity then produces

$$A_\infty(A_\infty - A_t)^{-1} = (I - (\hat{A}_t^{-1} + A_\infty^{-1})A_\infty^{-1})^{-1} = (I - (I + A_\infty^{-1}A_t^{-1})^{-1})^{-1} = \hat{A}_t A_\infty^{-1} (I + A_\infty^{-1}A_t^{-1}) = (A_\infty + A_t)A_\infty^{-1}. \quad (78)$$

The important observations are: the left hand side of (78) times $M_t$ (from the right) is the definition of $N_t$, and, on the right hand side, we have that $A_\infty$ is independent of $\hat{A}_t$ (on account of being independent of $\hat{B}_t$ and so $N_t$).

To exploit the second point, we extend $N_t$ to $t \in (-\infty, 0)$ as in Section 2. By the same reasoning used in the proof of Theorem 1, $\int_{-\infty}^0 N_s N_s^T ds$ is an independent copy of $A_\infty$.

Now, modifying the introduced notation to let $\hat{A}_{(-\infty,t)}$ denote $\int_{-\infty}^t N_s N_s^T ds$, we conclude from the above comments and (78) that

$$\hat{A}_{(-\infty,0)}(\hat{A}_{(-\infty,t)})^{-1} N_t \overset{(law)}{=} M_t,$$

as processes for $t \geq 0$. In order to recognize this as equivalent to the identity (26) announced in Theorem 8, take inverse-transposes throughout the above to find that,

$$(\hat{A}_{(-\infty,0)})^{-1} N_t (\hat{A}_{(-\infty,t)}^{-1} N_t^{-1}) \overset{(law)}{=} M_t^{-T}. \quad (79)$$

An application of Itô’s formula will then show that $M_t^{-1}$ is a copy of $N_t$. In particular, it satisfies $dM_t^{-1} = M_t^{-1}d(-B_t^T) + (\frac{1}{2} + \mu)M_t^{-1}dt$, for $t \geq 0$.

Finally, the independence statement follows from the same trick used in [27] for the one-dimensional case (see the proof of Theorem 6 there). Bringing in yet more notation, let $L_t$ denote the left hand side of (79). We will show that

$$N_t^{-1}\hat{A}_{(-\infty,t)} N_t^{-T} = L_t^T (\int_t^\infty L_s^T L_s^{-1} ds)L_t, \quad (80)$$

as processes for $t \geq 0$. This follows from the independence statement and the identity (79).
with probability one. The independence of \( \{L_s, s \leq t\} \) and \( \{N_s^{-1}\hat{A}_{(-\infty,s)}N_s^{-T}, s > t\} \) being made clear by writing the right hand side of (80) as \( \int_t^{\infty}(L_t^{-1}L_s)^{-T}(L_t^{-1}L_s)^{-1}ds \). On account of (79) the process \( L_t \) inherits the independence of multiplicative increments from \( M_t \). To verify (80) notice that:

\[
L_t^T(\int_t^{\infty}L_s^{-T}ds)L_t = N_t^{-1}\hat{A}_{(-\infty,t)} \left( \int_t^{\infty}\hat{A}_{(-\infty,s)}^{-1}N_sN_s^TN_t^{-T}ds \right)\hat{A}_{(-\infty,t)}N_t^{-T}.
\]

The proof is finished upon integrating and using the fact that \( ||\hat{A}_{(-\infty,t)}^{-1}|| \to 0 \) as \( t \to \infty \) with probability one (which follows by the computation behind Lemma 11).

As for Theorem 9, the identity (78) together with the definition of \( N_t \) from (71) gives that

\[
M_s^T(A_\infty - A_s)^{-1}M_s = N_s^T(A_\infty + \hat{A}_s)^{-1}N_s = N_s^T\hat{A}_{(-\infty,s)}^{-1}N_s,
\]

where we continue using the notation introduced in the previous proof, and hence Proposition 15 can be rewritten as in:

\[
B_t = \hat{B}_t + \int_0^t (2\mu I - N_s^T\hat{A}_{(-\infty,s)}^{-1}N_s)ds.
\]

Since the right hand side only depends on \( \hat{B}_t \), this identity provides a nonlinear transformation producing one matrix Brownian motion from another. Reversing the roles of the Brownian motions and reverting to our original notation yields:

**Corollary 17.** Let \( 2\mu > r - 1 \) and now take \( M_t = M_t^{(\mu)} \), extended to \( t \in \mathbb{R} \) as described in Section 2. Denote the (two-sided) driving matrix Brownian motion for \( M_t \) by \( B_t \). Then

\[
\hat{B}_t = B_t + 2\mu It - \int_0^t M_s^T\left( \int_{-\infty}^s M_uM_u^Tdu \right)^{-1}M_sds \overset{\text{law}}{=} B_t
\]

as processes for \( t \geq 0 \).

From this the proof of Theorem 9 is straightforward.

**Proof of Theorem 9.** The introduced process \( H_{t/2} \) can be equated with \( M_t \), though driven by the standard matrix Brownian motion \( B_{t/2} + C_{t/2} \). Then by the corollary we have that

\[
\hat{B}_t = B_{t/2} + C_{t/2} + 2\mu It - \int_0^{t/2} M_s^T\left( \int_{-\infty}^s M_uM_u^Tdu \right)^{-1}M_sds
\]

\[
= B_{t/2} + C_{t/2} + 2\mu It - \int_0^{t/2} H_s^T\left( \int_{-\infty}^s H_uH_u^Tdu \right)^{-1}H_sds
\]
is another standard matrix Brownian motion. Checking the definitions we see that $F_t = \hat{B}_t^2 + \frac{B_t - C_t}{2}$ and $G_t = \hat{B}_t^2 - \frac{B_t - C_t}{2}$. Since $\hat{B}_t$ is constructed from $B_t + C_t$, it is independent (as a process) from the process $B_t - C_t$. But this means that $\{F_t, t \geq 0\}$ and $\{G_t, t \geq 0\}$ are independent of each other and they are both standard matrix Brownian motions.

4 The matrix $X_t$ and $Z_t$ processes

The process $X_t$ is actually equivalent to the $Q_t$ encountered in the proof of Theorem 1, and hence its sde has already been recorded in (35). As for $Z_t$, we will again rely in part on the technology developed in the last section. Recall $M_t$ and $N_t$ from Propositions 15 and 16 and define two versions of the $Z_t$ process:

$$Z_t = M_t^{-1} \int_0^t M_s M_s^T ds, \quad \hat{Z}_t = N_t^{-1} \int_0^t N_s N_s^T ds.$$  \hspace{1cm} (81)

That is, $Z_t$ corresponds to $-\mu$ and is driven by $B_t$, $\hat{Z}_t$ to $+\mu$ and $\hat{B}_t$, and $B_t$ and $\hat{B}_t$ are related by Proposition 15.\(^3\) Itô’s formula yields

$$dZ_t = (\frac{1}{2} + \mu)Z_t dt + M_t^T dt - dB_t Z_t, \quad d\hat{Z}_t = (\frac{1}{2} - \mu)\hat{Z}_t dt + N_t^T dt - d\hat{B}_t \hat{Z}_t.$$  \hspace{1cm} (82)

We first show how to close these equations using properties of the matrix GIG distribution (proving Theorem 5 and then Corollary 6). After that we consider the asymptotics of the underlying eigenvalue processes (proving Theorem 7).

4.1 The role of the GIG

The following rather surprising fact already implies the invariance in law of $Z_t$ under the map $\mu \mapsto -\mu$.

**Proposition 18.** $Z_t = \hat{Z}_t$ almost surely for $t \geq 0$.

**Proof.** From Proposition 16 we have that $\hat{Z}_t = N_t^{-1} \int_0^t N_s N_s^T ds = N_t^{-1}(A_t^{-1} - A_\infty^{-1})^{-1}$ as well as $N_t = A_\infty (A_\infty - A_t)^{-1} M_t$. Hence,

$$\hat{Z}_t = (A_\infty (A_\infty - A_t)^{-1} M_t)^{-1} (A_t^{-1} - A_\infty^{-1})^{-1} = M_t^{-1} (A_\infty - A_t) A_\infty^{-1} (A_t^{-1} - A_\infty^{-1})^{-1} A_t^{-1} A_t = M_t^{-1} A_t,$$

which is the definition of $Z_t$. \hspace{1cm} \(\Box\)

\(^3\)Note that in the discussion of Section 1.2, $Z_t$ was defined through $M_t$ with the $+\mu$ drift term. The choice to flip things around here is natural given the course of the previous arguments.
For the Markov property we need:

**Proposition 19.** The conditional distribution of $M_t^T A_t^{-1} M_t$ given $\{Z_s, s \leq t, Z_t = Z\}$ is the matrix GIG law $\eta_{-\mu, I, (Z^T Z)^{-1}}$. The conditional distribution of $Z_t (A_t^{-1} - A_\infty^{-1}) Z_t^T$ given the same $\sigma$-field is $\eta_{-\mu, (Z^T Z)^{-1}, I}$.

We go ahead with the consequences of Proposition 19, and return to its proof at the end of the section.

**Proof of Theorem 5.** We follow the strategy of [21], employing Theorem 7.12 of [18] to close the equation for $Z_t$ in (82) as follows. Taking conditional expectation throughout with respect to $Z_t = \sigma(Z_s, s \leq t)$, the ideas there show that

$$dZ_t = \left(\frac{1}{2} + \mu\right) Z_t dt + \text{E}[M_t^T | Z_s, s \leq t] dt - dW_t Z_t, \tag{83}$$

where $W_t$ is matrix Brownian motion adapted to $Z_t \subsetneq B_t = \sigma(B_s, s \leq t)$. Next, since

$$M_t^T A_t^{-1} M_t = M_t^T Z_t^{-1} \sim \eta_{-\mu, I, (Z_t^T Z_t)^{-1}},$$

by Proposition 19, there is the formula $\text{E}[M_t^T | Z_s, s \leq t] = \kappa_{-\mu}(I, (Z_t Z_t^T)^{-1}) Z_t$. We can then rewrite the above as in

$$dZ_t = \left(\frac{1}{2} + \mu\right) Z_t dt + \kappa_{-\mu}(I, (Z_t Z_t^T)^{-1}) Z_t dt - dW_t Z_t, \tag{84}$$

and believing that one can let $\mu \mapsto -\mu$, $Z_t$ is the stated diffusion (in its own filtration).

Note that above construction entails that

$$W_t = B_t + \int_0^t [M_s^T Z_s^{-1} - \left(\frac{1}{2} + \mu\right) - \kappa_{-\mu}(I, (Z_s Z_s^T)^{-1})] ds.$$}

Similar to the case considered in [18], one checks that this is a matrix Brownian motion by using of Itô’s formula to find an ordinary differential equation for $t \mapsto \text{E}[e^{t \text{tr}(CW_t)}], C \in GL_r$.

To verify that the formulas themselves hold up under the change of sign, consider $\hat{Z}_t$ for which the above procedure leads to the following analog of (83):

$$d\hat{Z}_t = \left(\frac{1}{2} - \mu\right) \hat{Z}_t dt + \text{E}[N_t^T | \hat{Z}_s, s \leq t] dt - d\hat{W}_t \hat{Z}_t dt, \tag{85}$$

with a similar considerations for the new matrix Brownian motion $\hat{W}_t$. Proposition 19 again applies after noting that

$$N_t^T = M_t^T (A_\infty - A_t)^{-1} A_\infty = M_t^T A_t^{-1} A_t (A_\infty - A_t)^{-1} A_\infty = Z_t^{-T} (A_t^{-1} - A_\infty^{-1})^{-1}.$$
Proof of Corollary 6. Proposition 2.1 of [11] provides a soft criteria for two processes $X_t$ and $Y_t$ defined on the same probability space to intertwine. With $X_t$ taking values in $E$ and $Y_t$ taking values in $F$ (possibly separate measure spaces), it is assumed that:

(i) $X_t$ is Markovian with respect to a filtration $\mathcal{F}_t$, and $Y_t$ is Markovian with respect to a filtration $\mathcal{G}_t$ such that $\mathcal{G}_t \subset \mathcal{F}_t$,

(ii) There exists a Markov kernel $\Lambda : E \mapsto F$ for which $\mathbb{E}[h(X_t)|\mathcal{G}_t] = (\Lambda h)(Y_t)$ for all $t > 0$ and integrable $h : E \mapsto \mathbb{R}_+$.

Given this the outcome is that $T_t^{Y} \Lambda = \Lambda T_t^{X}$ as operators under additional “mild continuity assumptions”.

From what we have just shown the above applies directly to $X_t = N_t$ and $Y_t = \hat{Z}_t$. The mild continuity assumptions being easily satisfied as both choices are continuous pathed Feller processes on $\mathbb{R}^r \times r$. (Note since the original statement takes the positive $\mu$ drift, it is consistent to consider here $(N_t, \hat{Z}_t)$ rather than $(M_t, Z_t)$ for which there is a corresponding result.)

For (i): $N_t$ is Markovian with respect to $\hat{\mathcal{B}}_t = \sigma(\hat{B}_s, s \leq t)$ and $\hat{Z}_t$ is Markovian with respect to its own filtration $\mathcal{Z}_t$ which the proof of Theorem 5 shows is contained strictly inside $\hat{\mathcal{B}}_t$. And (ii) is the second point of Proposition (19): $\hat{Z}_t^{-T} N_t \sim \eta_{\mu, I,(ZZ^T)^{-1}}$ and so,

$$\mathbb{E}[h(N_t)|\hat{Z}_t, \hat{Z}_t = Z] = \int_{GL_r} h(Z^TX)\eta(dX) := \Lambda h(Z),$$

where $\eta = \eta_{\mu, I,(ZZ^T)^{-1}}$.

For the second part, take $X_t = \hat{Z}_t N_t^{-T}$. Then the representation of the intertwining kernel by $\Lambda h(Z) = \int_{\mathcal{P}} h(X^{-1})\eta(dX)$, now viewed as from $\mathcal{P}$ into $GL_r$, follows from $(N_t^T \hat{Z}_t^{-1})^{-1}$ having conditional law $\eta_{\mu, I,(ZZ^T)^{-1}, I} = \eta^{-1}$. \qed

We now return to the proof of Proposition 19. The first step, Lemma 20 below, is designed to implement Bernadac’s characterization of the matrix GIG [4, Theorem 5.1] to this end. The notation $\gamma_{p,A}$ introduced in the statement refers to the non-central Wishart law, which has density proportional $(\det X)^{p-\frac{r+1}{2}}e^{-\frac{1}{2}\text{tr}(A^{-1}X)}1_{\mathcal{P}}(X)$ for $A \in \mathcal{P}$. It is worth pointing out that the posited independence structure in the statement (of $(X+Y)^{-1}$ and $X^{-1}-(X+Y)^{-1}$ given that of $X$ and $Y$) is now commonly referred to as the Matsumoto-Yor property. The proof of Proposition 19 is then completed by specifying our particular choice of $X$ and $Y$ in Lemma 21.

Lemma 20. Suppose that the $\mathcal{P}$-valued random variables $X$ and $Y$ are independent, and that $(X+Y)^{-1}$ and $X^{-1}-(X+Y)^{-1}$ are also independent. Suppose further that $Y \sim \gamma_{2p,A^{-1}}$ and $X^{-1}-(X+Y)^{-1} \sim \gamma_{2p,B^{-1}}$ for $A, B \in \mathcal{P}$. Then $X \sim \eta_{p,A,B}$ and $(X+Y)^{-1} \sim \eta_{p,B,A}$. 34
Proof. Set \( U = (X + Y)^{-1} \) and \( V = X^{-1} - (X + Y)^{-1} \). Let \( Y' \) be a copy of \( V \) (that is, \( Y' \sim \gamma_{2p,B^{-1}} \)) independent of both \( X \) and \( Y \). Then

\[
X = (U + V)^{-1} \overset{\text{law}}{=} (Y' + U)^{-1} = (Y' + (Y + X)^{-1})^{-1}.
\]

Bernadac’s result is that the above distributional identity holds with \( X, Y, \) and \( Y' \) all independent and \( Y \) and \( Y' \) having the corresponding Wishart distributions if and only if \( X \sim \eta_{p,A,B} \).

Alternatively, if we let \( V' \) be a copy of \( Y \) (so \( V' \sim \gamma_{2p,A^{-1}} \)) independent of \( U \) and \( V \) we will have that

\[
U = (Y + (V + U)^{-1})^{-1} \overset{\text{law}}{=} (V' + (V + U)^{-1})^{-1},
\]

and the result is that \( U = (X + Y)^{-1} \sim \eta_{p,B,A} \).

Lemma 21. Set

\[
X = M_t^T Z_t^{-1} = M_t^T A_t^{-1} M_t, \quad Y = M_t^T (A_\infty - A_t)^{-1} M_t.
\]

Then,

\[
(X + Y)^{-1} = Z_t (A_t^{-1} - A_\infty^{-1}) Z_t^T, \quad X^{-1} - (X + Y)^{-1} = Z_t A_\infty^{-1} Z_t^T.
\]

Conditioned on the \( \sigma \)-field generated by \( \{Z_s, s \leq t\} \) the random matrices \( X \) and \( Y \) are independent, as are \( (X + Y)^{-1} \) and \( X^{-1} - (X + Y)^{-1} \). Further, the conditional distribution of \( Y \) is \( \gamma_{2\mu,A^{-1}} \) where \( A = I \) and that of \( X^{-1} - (X + Y)^{-1} \) is \( \gamma_{2\mu,B^{-1}} \) where \( B = (Z_t Z_t^T)^{-1} \).

Proof. The formulas for \( (X + Y)^{-1} \) and \( X^{-1} - (X + Y)^{-1} \) follow from some simple algebra.

Clearly \( Y \) is independent of \( \sigma(B_s, s \leq t) \), and thus is independent of both \( X \) and \( \{Z_s, s \leq t\} \). For the independence of \( (X + Y)^{-1} \) and \( X^{-1} - (X + Y)^{-1} \) we need to prove the conditional independence of \( A_t^{-1} - A_\infty^{-1} \) and \( A_\infty \). By Proposition 16 we know that \( A_t^{-1} - A_\infty^{-1} = \int_0^T N_s N_s^T ds \) is measurable \( \sigma(\hat{B}_s, s \leq t) \) and so is independent of \( A_\infty \). But \( Z_t \) is in this \( \sigma \)-field as well (\( Z_t = \hat{Z}_t \)) implying there is also conditional independence.

4.2 Asymptotics

We start by identifying the underlying eigenvalue processes:

Lemma 22. Denote the (ordered, nonintersecting) eigenvalues of \( X_t \) by \( 0 \leq x_r \leq x_{r-1} \leq \cdots \leq x_1 \). These perform the joint diffusion:

\[
dx_i = 2x_i db_i + \left( 1 + (2 - 2\mu)x_i + x_i \sum_{j \neq i} \frac{x_i + x_j}{x_i - x_j} \right) dt. \tag{86}
\]
For \( Z_t \) consider instead the similarly ordered singular values \( \{z_i\} \) of \( Z_t \). This family is also Markov, and is governed by

\[
dz_i = z_idb_i + \left( \left( \frac{r}{2} - \mu \right) z_i + \frac{1}{z_i} [\kappa_\mu(\Lambda z, I)]_{ii} + z_i \sum_{j \neq i} \frac{z_j^2}{z_i^2 - z_j^2} \right) dt,
\]

where \( \Lambda z \) denotes the diagonal matrix \([\Lambda_z]_{ii} = z_i^{-2}\).

**Proof.** As \( X_t \) is equivalent in law to \( Q_t \) used throughout Section 2, Proposition 13 describes the process for the inverse eigenvalues of \( X_t \). That is to say that (86) follows from (45) upon making the substitution \( x_i = \rho_i^{-1} \) in the \( \beta = 1 \) instance of that equation.

The argument for \( Z_t \) requires only a couple of additional observations. Write

\[
d(Z_t^T Z_t) = -Z_t^T (dB_t + dB_t^T) Z_t + (1 - 2\mu + r) Z_t^T Z_t dt + 2Z_t^T \kappa_\mu(I, (Z_t Z_t^{-1})^{-1}) Z_t dt,
\]

which is now rotation invariant. In particular, setting \( Z_t = V_t \Lambda_t^{1/2} U_t^T \) for orthogonal \( U, V \) and \( \Lambda_t \) the diagonal of square-singular values of \( Z_t \), the right hand side of (88) equals

\[
U_t \left( \Lambda_t^{1/2} dB_t \Lambda_t^{1/2} + (1 - 2\mu + r) \Lambda_t dt + 2 \Lambda_t^{1/2} \kappa_\mu(I, \Lambda_t^{-1}) \Lambda_t^{1/2} dt \right) U_t^T.
\]

Here \( B_t = V_t^T (B_t + B_t^T) V_t \) is equal in law to twice a symmetric (or “Dyson”) Brownian motion, and we have used that \( \kappa_\mu(I, UA^TU^T) = U \kappa_\mu(I, A) U^T \), for any symmetric \( A \). From here the standard method used before will yield the system,

\[
d\lambda_i = 2\lambda_i db_i + [(1 - 2\mu + r) \lambda_i + 2 \kappa_\mu(\Lambda^{-1}, I)]_{ii} + 2 \sum_{j \neq i} \frac{\lambda_i \lambda_j}{\lambda_i - \lambda_j} dt,
\]

having employed the identity \( \Lambda^{1/2} \kappa_\mu(I, \Lambda^{-1}) \Lambda^{1/2} = \kappa_\mu(\Lambda^{-1}, I) \) along the way. Setting \( z_i = \sqrt{\lambda_i} \) completes the proof. \( \square \)

The proof of Theorem 7 is now split into two parts.

**Proposition 23.** Set \( \mu = \frac{r - 1}{2} + \gamma \) in (86) and denote the maximal eigenvalue by \( x^* \). Then, as processes, \( \frac{1}{2c} \log x^{\gamma/c} \) converges as \( c \to \infty \) to the Brownian motion with drift \(-\gamma\) reflected at the origin. (The lower eigenvalues converge to the zero process in this scaling).

**Proof.** Changing to logarithmic coordinates, \( y_i = \log x_i \), and introducing the scaling as in \( \gamma \mapsto \gamma/c \) (after putting \( \mu = \frac{r - 1}{2} + \gamma \) and \( y_i(t) \mapsto y_i(c^2 t)/(2c) \), we can work with the system

\[
dy_i = 2db_i + \left( -\gamma + \frac{c}{2} e^{-cy_i} + c \sum_{j \neq i} \frac{e^{cy_j}}{e^{cy_i} - e^{cy_j}} \right) dt.
\]
Here the drift was simplified ahead of time by using 
\[-(r - 1) + \sum_{j \neq i} \frac{x_i + x_j}{x_i - x_j} = \sum_{j \neq i} \frac{2x_j}{x_i - x_j}.
\]

On the other hand, back in the original coordinates we have that,
\[
P \left( \lim_{t \to \infty} (\log x_i(t) - \log x_j(t)) = +\infty \right) = 1,
\]
for any pair \(i > j\). This follows again by the proof of Lemma 11. From here we see that
the top point in (90) shares whatever \(c \to \infty\) process limit it may have with that for \(y_c(t)\)
defined by
\[
y_c(t) = b_t^{-\gamma} + L_c(t), \quad L_c(t) = \frac{c}{2} \int_0^t e^{-cy_c(s)}ds,
\]
and we want to show that \(L_c(t)\) produces a local time contribution in the limit.

For \(\epsilon > 0\), decompose the path \(t \mapsto y_c(t)\) at the time \(s \leq t\) at which it was last beneath
level \(\epsilon\) to find that
\[
y_c(t) \leq \epsilon + \max_{s \leq t} (b_t^{-\gamma} - b_s^{-\gamma}) + \frac{c}{2} e^{-\epsilon t}.
\]

Thus,
\[
L_c(t) \leq \epsilon + \max_{s \leq t} (b_t^{-\gamma} - b_s^{-\gamma}) + a_t - b_t^{-\gamma},
\]
and for each fixed \(t\) it holds that \(\sup_{c > 0} L_c(t) < \infty\) with probability one. Decomposing instead
at the last time that the path exceeds \(-\epsilon\) similar reasoning shows that \(\liminf_{c \to \infty} \inf_{t \leq s \leq t} y_c(s) \geq 0\). With both sequences bounded above and below, by passing to a subsequence
if needed there exist \((y(t), L(t))\) with \(y_c(t) \to y(t)\) and \(L_c(t) \to L(t)\) at all but countably
many \(t\) for which \(y(t) = b_t^{-\gamma} + L(t)\). Since \(\int_0^t \mathbf{1}_{[c, \infty)}(y_c(s))dL_c(s) \to 0\), any limiting \(L(t)\)
is non-decreasing and increases only when \(y(t) = 0\). As \(y(t) \geq 0\), we see that any such pair
\((y(t), L(t))\) is the (unique) solution to the Skorohod problem for \(b_t^{-\gamma}\). This precisely what we
wanted to show. \(\Box\)

**Proposition 24.** Now set \(\mu = \mathbb{e}^{-1} + \gamma\) in (87) and denote by \(z_t^\gamma\) the minimal singular value
there. Then, again in the Skorohod topology, \(\lim_{c \to \infty} \frac{1}{c} \log z_t^\gamma \Rightarrow r_t\), where \(t \mapsto r_t\) is the
diffusion on the positive half-line with generator \(\frac{1}{2} \frac{d^2}{dr^2} + \gamma \coth(\gamma r) \frac{d}{dr}\).

**Proof.** In order to get a workable formula for the matrix GIG mean, we bring in the more
general \(K\)-Bessel functions. For \(s = (s_1, \ldots, s_r) \in \mathbb{C}^r\) recall the power function \(p_s(X)\) from
(60) and the invariant measure \(\mu_r\) on \(\mathcal{P}\) from (58), and set
\[
K_r(s|A, B) = \frac{1}{2} \int_\mathcal{P} p_s(X)e^{-\frac{1}{2}trAX-\frac{1}{2}trBX^{-1}}d\mu_r(X).
\]
This is actually how Terras introduces the \(K\)-Bessel functions from the start (see §4.2.2 of
[34], though keep in mind our inclusion of various factors of \(\frac{1}{2}\) not used there), and reduces
to our earlier defined \(K_r(s|A, B)\) when \(s = (0, \ldots, 0, s)\).
In terms of the above we have: for any (positive) diagonal matrix \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_r) \),

\[
[k_\mu(\Lambda, I)]_{11} = \frac{K_r(s|\Lambda, I)}{K_r(s'|\Lambda, I)}, \quad (92)
\]

where now \( s = (1, 0, \ldots, 0, \mu) \) and \( s' = (0, 0, \ldots, 0, \mu) \). And likewise,

\[
[k_\mu(\Lambda, I)]_{ii} = \frac{K_r(s|\Lambda_{\sigma_i}, I)}{K_r(s'|\Lambda, I)}, \quad (93)
\]

in which \( \Lambda_{\sigma_i} \) is the matrix arrived at from \( \Lambda \) by swapping \( \lambda_1 \) and \( \lambda_i \). This uses (again) that \( k_\mu(I, \Lambda) = U^T k_\mu(I, U \Lambda U^T) U \) for orthogonal \( U \), here with the choice of \( U \) being the corresponding permutation matrix. Note that taking \( U \) to be a diagonal orthogonal matrix with \( \pm 1 \) entries so that \( U_i U_{jj} = -1 \) for given \( i \neq j \), yields \([k_\mu(I, \Lambda)]_{i,j} = -[k_\mu(I, \Lambda)]_{i,j}\), so we get that \( k_\mu(I, \Lambda) \) is actually diagonal.

The ratios in (92) and (93) can then be expanded with the help of Terras’ induction formula (see Exercise 20 of [34, §4.2.2] though note the typo, \( \frac{m-x}{2} \) should be \( \frac{m-x}{2} \)). In the present setting this implies that

\[
K_r(s|\Lambda, I) = \int_{\mathbb{R}^{r-1}} K_1 \left( \mu - \frac{r-3}{2} \right) \{x_1 + \sum_{i=2}^r \lambda_i x_i^2, 1\} \times K_{r-1} (s'|\Lambda^{(1)}, I + xx^T) \, dx_2 \ldots dx_r, \quad (94)
\]

with \( \Lambda^{(1)} = \text{diag}(\lambda_2, \ldots, \lambda_r) \) and \( s'' = (0, \ldots, 0, \mu - \frac{1}{2}) \in \mathbb{R}^{(r-1)} \). Applied to \( K_r(s'|\Lambda, I) \) the result is of course similar, the only difference that the \( K_1 \) factor on the right hand side of (94) is replaced by \( K_1(\mu - \frac{r-1}{2} | \lambda_1 + \sum_{i=2}^r \lambda_i x_i^2, 1) \).

Writing both occurrences of \( K_1(\cdot, \cdot) \) in terms of the standard Macdonald function we have that \( K_1(\mu - \frac{r-3}{2} | \psi^2, 1) = \psi^{-\mu + \frac{r-3}{2}} K_{\mu - \frac{r-3}{2}}(\psi) \) and \( K_1(\mu - \frac{r-1}{2} | \psi^2, 1) = \psi^{-\mu + \frac{r-1}{2}} K_{\mu - \frac{r-1}{2}}(\psi) \). So, with the shorthand,

\[
\psi = \psi(x, \Lambda) = \sqrt{\lambda_1 + \sum_{i=2}^r \lambda_i x_i^2}, \quad \mathcal{K}(x, \Lambda^{(1)}) = K_{r-1} (s''|\Lambda^{(1)}, I + xx^T)
\]

we record the new expression for (92): making the substitution \( \mu = \frac{r-1}{2} + \gamma \),

\[
[k_{\frac{r-1}{2}+\gamma}(\Lambda, I)]_{11} = \frac{\int_{\mathbb{R}^{r-1}} \psi^{-\gamma} K_{1+\gamma}(\psi) \mathcal{K}(x, \Lambda^{(1)}) \, dx}{\int_{\mathbb{R}^{r-1}} \psi^{-\gamma} K_\gamma(\psi) \mathcal{K}(x, \Lambda^{(1)}) \, dx}, \quad (95)
\]

\[
:= \frac{1}{\sqrt{\lambda_1}} \frac{\int_{\mathbb{R}^{r-1}} \psi_0^{-\gamma} K_{1+\gamma} (\sqrt{\lambda_1} \psi_0) \rho^{(1)}(dx)}{\int_{\mathbb{R}^{r-1}} K_\gamma (\sqrt{\lambda_1} \psi_0) \rho^{(1)}(dx)},
\]

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In line two we have made the change of variables $T_\Lambda: x_i \mapsto \sqrt{\frac{\Lambda_i}{\lambda}}x_i$, and have introduced

$$\psi_0(x) = \sqrt{1 + \sum_{i=2}^r x_i^2}, \quad \rho^{(1)}(x) = \psi_0(x)^{-\gamma}K(T_\Lambda x, \Lambda^{(1)}_1)dx.$$  \hfill (96)

By way of (93) there are allied expressions for the other diagonal components of the mean.

Finally returning to (87) and setting $y_i = \log z_i$ that equation becomes

$$dy_i = db_i + \left( -\gamma + e^{-y_i} \frac{\int_{R^{r-1}} \psi_0^{-1} K_{1+\gamma} (e^{-y_i} \psi_0) \rho^{(i)}(x) dx}{\int_{R^{r-1}} K_\gamma (e^{-y_i} \psi_0) \rho^{(1)}(x) dx} \right) dt$$

$$+ \left( r - 1 + \sum_{j \neq i} \frac{e^{2y_i + e^{2y_j}} - e^{2y_i - e^{2y_j}}}{2} \right) dt. \hfill (97)$$

Here $\Lambda$ is now the diagonal matrix $\Lambda_{ii} = e^{-2\gamma}$, and we have employed (95) while being a little fluid with notation: $\rho^{(i)}(x)$ stands for the the comparable object to $\rho^{(1)}(x)$ defined in the same way as in (96) but for the $i^{th}$ coordinate.

The strategy from this point is:

(i) Show yet again a separation of scales. That is, for long time it holds that $y_i \ll y_j$ for all $i < j$ with probability tending to one. Without the presence of the Macdonald function term in the drift, the same calculation used in Lemma 11 would (yet again) show that the solution of (97) satisfies $\frac{1}{t} \log y_i \to (r - i) - \gamma$ for all $i$ with probability one. The claim is that the added drift doesn’t affect this appraisal too much

(ii) Show that

$$\frac{\int_{R^{r-1}} \psi_0^{-1} K_{1+\gamma} (e^{-y_i} \psi_0) \rho^{(i)}(x) dx}{\int_{R^{r-1}} K_\gamma (e^{-y_i} \psi_0) \rho^{(1)}(x) dx} = \frac{K_{1+\gamma}(e^{-y_1})}{K_\gamma(e^{-y_1})} (1 + o(1))$$

uniformly in $y_1$ as $y_2, \ldots y_r \to \infty$.

The estimate for (i) follows from known bounds for the $K$-Bessel function at $\infty$. For (ii), using the explicit formulas it is easy to see that the measure $\rho^{(1)}_\Lambda$ has Gaussian concentration at the point $x = 0$ (where one notes that $\psi_0 = 1$).

Put together, and after the required scaling, the drift in the equation for $y = \frac{1}{c} y_1(c^2 t, \gamma/c)$ equals $-\gamma + e^{-c y} \frac{K_{1+\gamma}(e^{-c y})}{K_\gamma(e^{-c y})} \frac{1}{c} (e^{-c y})$ up to vanishing errors as $c \to \infty$. That is, we recover the same calculation needed by Matsumto-Yor in the one dimensional case, and so also the same limit.

\[ \square \]
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