A Novel Dissipation Property of the Master Equation

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The time-decreasing property \( dF/dt \leq 0 \) of relative entropy \( F \) for the master equation is as important as the H-theorem for the Boltzmann equation. In this paper, we derive a non-zero upper bound for \( dF/dt \) and thereby provide new insights into the master equation without assuming the detailed balance. As a direct consequence, this new bound enables us to give a first and complete proof of the well-accepted fact that the solution of the master equation converges to the corresponding non-equilibrium steady state as time goes to infinity. More importantly, our results reveal a new dissipation property for Markov processes described by the master equation and thus leads to a strengthened version of the second law of thermodynamics.

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The master equation is of fundamental importance in statistical physics and describes the time evolution of the probability distribution of a stochastic system being among a set of states. It has been widely used in physics, chemistry, biology, and many other related fields [1]. Indeed, the master equation provides an effective way to model various stochastic processes, such as the birth-death processes, random walks, the Fokker-Planck equation, the Lindblad equation and so on [2].

In this paper, we are concerned with the master equation for finite Markov processes [2, 3]:

\[
\frac{dp_i}{dt} = \sum_{j \neq i} (q_{ij}p_j - q_{ji}p_i), \quad i = 1, \cdots, N, \tag{1}
\]

where \( p_i = p_i(t) \) is the probability for the system being in state \( i \) at time \( t \), and \( q_{ij} \geq 0 \) (\( i \neq j \)) is the transition rate from state \( j \) to state \( i \). It is clear that \( \sum_{i,j \neq i} (q_{ij}p_j - q_{ji}p_i) \equiv 0 \). In general, the transition matrix \( \{q_{ij}\} \) is required to satisfy the irreducibility condition

- for any two states \( i \neq j \), there is a sequence of indices \( j_1, j_2, \cdots, j_l \), such that \( j_1 = i, j_l = j \) and \( q_{j_m,j_{m+1}} > 0 \) for all \( m = 1, 2, \cdots, l - 1 \).

Namely, between each pair of states \( i \) and \( j \) there always exists a pathway with all positive transition steps.
Under the irreducible condition, it was shown in [4] that there exists a unique constant state \( \{ p_i^s \} \) satisfying

\[
\sum_i p_i^s = 1, \quad \sum_j q_{ij} p_j^s = \sum_j q_{ji} p_i^s \quad \text{and} \quad p_i^s > 0 \quad \forall i.
\]  

This state is called a non-equilibrium steady state (NESS) in the literature. The existence of such a state is typical for many biochemical processes. It serves as a fundamental concept to understand the long-time behavior of stochastic systems endowed with a Markovian dynamics. In contrast to the equilibrium state linked to several key notions like time reversibility, detailed balance and gradient-like potential functions, the NESS is usually correlated with the time irreversibility, breakdown of detailed balance, non-gradient-like dynamics, circular motions and positive entropy production rates [5, 6]. The NESS concept has been exploited to study stochastic resonances [7], single-molecule enzyme kinetics [8], chemically driven open systems [9] and so on. It has also been used in [10–12] to give new interpretations of the second law of thermodynamics.

With this NESS, we can define a Boltzmann-type relative entropy (or called Gibbs free energy) as

\[
F = \sum_i p_i \ln(p_i/p_i^s),
\]

where \( p_i = p_i(t) \) solves the master equation. It is well-known (see, e.g., [2]) that this relative entropy is non-negative and its time-derivative is non-positive. These properties of the master equation are as important as the H-theorem for the Boltzmann equation. They imply the ergodicity of the Markovian stochastic process and the convergence to the unique probability distribution [2, 13]. In [4], Schnakenberg further pointed out their connection to the Glansdorff-Prigogine criterion for the stability of a thermodynamic system in the steady state. In some recent attempts [10, 11] on formulating the second law of thermodynamics for non-equilibrium processes characterized through the master equation, the time evolution of the Boltzmann-type relative entropy is linked to the non-adiabatic part of the entropy production rate and the non-positiveness of \( dF/dt \) guarantees the right sign.

In this work, we derive a non-zero upper bound for \( dF/dt \). This result was inspired by the entropy-dissipation principle proposed by the third author in [14]. As a direct consequence of this upper bound, Theorem 3 shows that the NESS characterizes the long-time dynamics of the master equation. Most importantly, it reveals a new dissipation property for general non-equilibrium processes characterized by the master equation without assuming the detailed balance. This property seems unknown before and it suggests a strengthened version of the second law of thermodynamics. Supplementarily, we also establish analogous conclusions for relative entropies of Tsallis-type.

We start with the following fundamental fact established by Schnakenberg (1976) in [4] for the master equation.

**Lemma 1** (Schnakenberg [4]). Under the irreducible condition, solutions to the master equations with non-
negative initial data are strictly positive for \( t > 0 \).

Thanks to this lemma, we will assume throughout this paper that the initial data are non-negative and the corresponding solution is normalized to satisfy
\[
\sum_j p_j(t) = 1
\]
for \( t \geq 0 \).

For the sake of completeness, we first prove the following fact for the Boltzmann-type relative entropy.

**Theorem 1.** Solutions to the master equation satisfy the estimates
\[
\sum_i \left( \frac{p_i - p_i^*}{p_i + p_i^*} \right)^2 \leq F \leq \max_i \left( \frac{p_i - p_i^*}{p_i^*} \right).
\]

**Proof.** By using the elementary inequalities \( 2(\frac{x - 1}{x + 1}) \leq \ln x \leq x - 1 \) for \( x > 0 \), we have
\[
\frac{(p_i - p_i^*)^2}{p_i + p_i^*} + (p_i - p_i^*) = \frac{2p_i^2 - 2p_i p_i^*}{p_i + p_i^*} \leq p_i \ln \frac{p_i^*}{p_i} \leq p_i \frac{p_i - p_i^*}{p_i^*} \leq p_i \max_j \left( \frac{p_j - p_j^*}{p_j^*} \right).
\]
Recall that \( \sum_i p_i = \sum_i p_i^* = 1 \). We sum up the last inequalities over \( i \) to obtain the estimates in (4). This completes the proof.

Theorem 1 shows that \( F = 0 \) if and only if the system is in the NESS. Thus, the Boltzmann-type relative entropy can be well used as a characteristic quantity to measure how far the system is from the NESS. Our next theorem provides a sharp upper bound for \( dF/dt \), which is inspired by the entropy-dissipation principle proposed in [14].

**Theorem 2.** Under the irreducible condition, the Boltzmann-type relative entropy possesses the following dissipation property
\[
\frac{dF}{dt} \leq -\frac{c}{N - 1} \sum_i \left[ \sum_{j \neq i} (q_{ij} p_j - q_{ji} p_i) \right]^2,
\]
where \( c > 0 \) is completely determined with \( q_{ij} > 0 \).

**Proof.** First of all, we show that
\[
\theta(x, y) := \frac{(x - y) - y(\ln x - \ln y)}{(x - y)^2} \geq \frac{1}{2}
\]
for \( (x, y) \in (0, 1]^2 \) with \( x \neq y \). To do this, we set \( \varphi(x) = x \ln x - x \) for \( x > 0 \) and notice that \( \varphi'(x) = \ln x \) and \( \varphi''(x) = 1/x \geq 1 \) for \( x \leq 1 \). Then we may rewrite
\[
\theta(x, y) = \frac{\varphi(y) - \varphi(x) - \varphi'(x)(y - x)}{(x - y)^2} = \frac{1}{(y - x)^2} \int_x^y \int_x^z \varphi''(s) ds dz \geq \frac{1}{(y - x)^2} \int_x^y \int_x^z ds dz = \frac{1}{2},
\]
for \( (x, y) \in (0, 1]^2 \).
Now we use the master equations and \( \sum_i p_i = 1 \) to compute

\[
\frac{dF}{dt} = \sum_i \frac{dp_i}{dt} \left[ \ln \left( \frac{p_i}{p_i^*} \right) + 1 \right] = \sum_{i,j \neq i} (q_{ij} p_j - q_{ji} p_i) \ln \left( \frac{p_i}{p_i^*} \right) = \sum_{i,j \neq i} q_{ij} p_j \ln \left( \frac{p_i p_j^*}{p_j p_i^*} \right)
\]

\[
= \sum_{i,j \neq i} q_{ij} \left( p_i p_j^* - p_j p_i^* \right) / p_i^* - \sum_{i,j \neq i} q_{ij} \frac{(p_i p_j^* - p_j p_i^*) (\sigma_{ij} - p_j p_i^*)}{p_i^* \sigma_{ij}} \tag{7}
\]

with \( \sigma_{ij} = (p_i p_j^* - p_j p_i^*) / [\ln(p_i p_j^*) - \ln(p_j p_i^*)] \). Since \( \sum_{j \neq i} q_{ij} p_j^* = \sum_{j \neq i} q_{ji} p_i^* \), we have

\[
\sum_{i,j \neq i} q_{ij} p_j p_i / p_i^* = \sum_{i,j \neq i} q_{ji} p_i = \sum_{i,j \neq i} q_{ij} p_j
\]

and thereby

\[
\sum_{i,j \neq i} q_{ij} \frac{(p_i p_j^* - p_j p_i^*)}{p_i^*} = 0. \tag{8}
\]

On the other hand, for \( q_{ij} > 0 \) we deduce from (8) that

\[
q_{ij} \frac{(p_i p_j^* - p_j p_i^*) (\sigma_{ij} - p_j p_i^*)}{p_i^* \sigma_{ij}} = \frac{p_i^* (\sigma_{ij} - p_j p_i^*)}{q_{ij} \sigma_{ij} (p_i p_j^* - p_j p_i^*)} \left[ \frac{(p_i p_j^* - p_j p_i^*)}{p_i^*} \right]^2
\]

\[
= \frac{p_i^* \theta(p_i p_j^*, p_j p_i^*)}{q_{ij}} \left[ \frac{(p_i p_j^* - p_j p_i^*)}{p_i^*} \right]^2 \geq \frac{p_i^*}{2q_{ij}} \left[ \frac{(p_i p_j^* - p_j p_i^*)}{p_i^*} \right]^2 \tag{9}
\]

with \( q_i = \max\{q_{ij} > 0|j \neq i\} \). This obviously holds also for \( q_{ij} = 0 \). Combining (7–9), we arrive at

\[
\frac{dF}{dt} \leq - \sum_i \frac{p_i^*}{2q_i} \sum_{j \neq i} \left[ \frac{q_{ij} (p_i p_j^* - p_j p_i^*)}{p_i^*} \right]^2
\]

\[
\leq - \sum_i \frac{p_i^*}{2q_i (N-1)} \left[ \sum_{j \neq i} \frac{q_{ij} (p_i p_j^* - p_j p_i^*)}{p_i^*} \right]^2 = - \sum_i \frac{p_i^*}{2q_i (N-1)} \left[ \sum_{j \neq i} (q_{ij} p_j - q_{ji} p_i) \right]^2.
\]

Here the second inequality is due to the Cauchy-Schwartz inequality. Hence the proof is completed with \( c = \min \{p_i^*/q_i\}/2 \).

Remark that Theorem 2 is trivially true under the assumption of detailed balance. However, we do not need such an assumption here. Therefore, this theorem is applicable to general Markov processes. It shows that \( dF/dt = 0 \) if and only if the system is in the NESS. Thus, the non-equilibrium process will never stop evolving unless the system reaches the NESS. Moreover, even in the NESS, some kind of non-dissipative circular motions will still exist. This phenomenon is typical in many biochemical processes and constitutes a major difference between the NESS and the traditional equilibrium state.
In [10], Esposite and Broeck interpreted \(-dF/dt\) as the non-adiabatic part of the entropy production rates. Our result above points out a sharp lower bound for the non-adiabatic part and the bound is given in terms of the non-equilibrium fluxes \(J_i = \sum_{j\neq i} (q_{ij}p_j - q_{ji}p_i)\). With this bound, we can strengthen the second law of thermodynamics from the classical statement that the entropy production rate \(\sigma_s \geq 0\) into

\[
\sigma_s - \frac{c}{N-1} \sum_i |J_i|^2 = \sigma_s - \frac{c}{N-1} \sum_i \left[ \sum_{j\neq i} (q_{ij}p_j - q_{ji}p_i) \right]^2 \geq 0
\]

for non-equilibrium processes described with the master equation. As far as we know, this strengthened version has not been reported in the literature.

Theorem 2 provides new insights into the master equation. It could be used to improve some existing results. A simple example is about the long-time dynamics of the master equation, as shown with the following theorem.

**Theorem 3.** Under the irreducible condition, if the initial data \(p_i^0\) satisfy \(0 \leq p_i^0 \leq 1\) and \(\sum_i p_i^0 = 1\) for all \(i\), then

\[
\lim_{t \to \infty} (p_1(t), p_2(t), \cdots, p_N(t)) = (p_1^*, p_2^*, \cdots, p_N^*).
\]

**Proof:** From Lemma 1 it follows that \(p_i(t) > 0\), \(\sum_i p_i(t) = \sum_i p_i(0)\) for all \(t > 0\), and thereby \(p_i(t)\) is bounded on \([0, \infty)\). Thus, from the master equation we deduce that \(p_i(t)\) is Lipschitz continuous on \([0, \infty)\). On the other hand, we integrate the inequality in Theorem 2 to get

\[
F(t) + \frac{c}{N-1} \int_0^t \sum_i \left[ \sum_{j \neq i} (q_{ij}p_j(\tau) - q_{ji}p_i(\tau)) \right]^2 d\tau \leq F(0),
\]

meaning that the integrand, denoted by \(f(t)\), is integrable on \([0, \infty)\). Notice that \(f(t)\) is also bounded and Lipschitz continuous on \([0, \infty)\). Then it is not difficult to see that \(\lim_{t \to \infty} f(t) = 0\).

Having these preparations, we turn to prove the theorem by contradiction. Assume that, as \(t\) goes to infinity, \((p_1(t), p_2(t), \cdots, p_N(t))\) does not converge to \((p_1^*, p_2^*, \cdots, p_N^*)\). Since \(p_i(t)\) is bounded, there exist a state \(\vec{p}_* = (p_1^*, p_2^*, \cdots, p_N^*)\) and a time-sequence \(\{t_k : k = 1, 2, \cdots\}\) so that

\[
\lim_{k \to \infty} t_k = \infty \quad \text{and} \quad \lim_{t \to \infty} (p_1(t_k), p_2(t_k), \cdots, p_N(t_k)) = \vec{p}_* \neq (p_1^*, p_2^*, \cdots, p_N^*).
\]

Thanks to the uniqueness of NESS, it is obvious that

\[
C := \sum_i \left[ \sum_{j \neq i} (q_{ij}p_j^* - q_{ji}p_i^*) \right] > 0
\]

and thereby

\[
f(t_k) = \sum_i \left[ \sum_{j \neq i} (q_{ij}p_j(t_k) - q_{ji}p_i(t_k)) \right]^2 \geq C/2 > 0
\]

for all \(t_k\) in the above sequence.
for $k \gg 1$. This contradicts the already proved fact that $\lim_{t \to \infty} f(t) = 0$ and hence the proof is complete.

It is remarkable that Theorem 3 seems well-accepted in physical community. However, to our best knowledge, a mathematically complete proof is not available in the literature before.

Up to now, all of our discussions are about the Boltzamnn-type relative entropy for the master equation. Similar conclusions can also be established for relative entropies of Tsallis-type. Indeed, we refer to Tsallis’ statistics and define the generalized relative entropy

$$F_\alpha(p_i) = \frac{1}{\alpha(\alpha - 1)} \left[ \sum_{i=1}^{N} p_i \left( \frac{p_i}{p_s^i} \right)^{\alpha - 1} - 1 \right]$$

with a real parameter $\alpha \neq 0, 1$. It is known that, when $\alpha \to 1$, the Tsallis-type relative entropy converges to the Boltzmann-type one, namely, $\lim_{\alpha \to 1} F_\alpha = \sum_{i=1}^{N} p_i \ln(p_i/p_s^i)$. In [13], Shiino showed that $F_\alpha \geq 0$ and $dF_\alpha/dt \leq 0$ for the master equation. In contrast, we have the following conclusions.

**Theorem 4.** Under the irreducible condition, the Tsallis-type relative entropy possesses the following upper and lower bounds

$$\left\{ \begin{array}{c} \frac{1}{2} \sum_{i=1}^{N} \frac{(p_i - p_s^i)^2}{(p_s^i)^2} \leq F_\alpha \leq \max_i \left[ \frac{(p_i)^{\alpha} - (p_s^i)^{\alpha}}{\alpha(\alpha - 1)(p_s^i)^{\alpha - 1}} \right], \quad \alpha < 2, \alpha \neq 0, 1 \\ \frac{f_\alpha}{\alpha(\alpha - 1)} \sum_{i=1}^{N} |p_i - p_s^i|^\alpha \leq F_\alpha \leq \frac{\alpha}{2} \sum_{i=1}^{N} \frac{(p_i - p_s^i)^2}{(p_s^i)^{\alpha}}, \quad \alpha \geq 2 \end{array} \right.$$  

**Theorem 5.** Under the irreducible condition, the Tsallis-type relative entropy possesses the following dissipation property

$$\frac{dF_\alpha}{dt} \leq \left\{ \begin{array}{c} \frac{-c(\alpha)}{N} \sum_{i,j} (q_{ij}p_j - q_{ji}p_i)^2, \quad \alpha < 2, \alpha \neq 0, 1 \\ \frac{-c(\alpha)}{(N-1)} \sum_i |\sum_{j \neq i} (q_{ij}p_j - q_{ji}p_i)|^\alpha, \quad \alpha \geq 2 \end{array} \right.$$  

where $c(\alpha) > 0$ is independent of $p_i$ ($i = 1, \cdots, N$).

These two theorems will be proved in Appendix.

In summary, we have obtained new non-zero upper and lower bounds of both the Boltzmann-type and Tsallis-type relative entropies for the master equation not necessarily satisfying the detailed balance. These results provide new insights into the master equation and lead to a first and mathematically complete proof of the well-accepted fact that the solutions to the master equation converge to the NESS as time goes to infinity. Most importantly, they reveal a novel dissipation property for general non-equilibrium processes described by the master equation. This property leads to a new version of the second law of thermodynamics, that seemingly has never been reported before.
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Appendix

Here we present our detailed proofs of Theorems 4 and 5 by using the following two lemmas.

Lemma 2 (D.S. Mitrinovic and J.E. Pecaric [18]). Consider the function
\[ f_\alpha(x) = (x + 1)^\alpha - \alpha x - 1 - \frac{\alpha(\alpha - 1)}{2}(B + 1)^{\alpha - 2}x^2 \]
for \( \alpha \in (-\infty, \infty) \) and \( B \in (-1, \infty) \). Then, for \( x \in (-1, B) \) with \( B \geq 0 \), it holds that
(a). \( \alpha \in (-\infty, 0) \cup (1, 2) \) implies \( f_\alpha(x) \geq 0 \);
(b). \( \alpha \in (0, 1) \cup (2, +\infty) \) implies \( f_\alpha(x) \leq 0 \).

Lemma 3 (Leindler [19]). For any \( \alpha \geq \beta \geq 2 \), the inequality
\[ |1 + z|^\alpha \geq 1 + \alpha \text{Re}(z) + f_\alpha(\beta)|z|^{\beta} + g_\alpha(\beta)|z|^\alpha \]
holds, where
\[ 0 < f_\alpha(\beta) < \min_{x \geq 2}[(x - 1)^\alpha + \alpha x - 1]/x^\beta, \]
\[ 0 < g_\alpha(\beta) \leq \min_{x \geq 2}[(x - 1)^\alpha + \alpha x - 1 - f_\alpha(\beta)x^\beta]/x^\alpha. \]

Proof of Theorem 4.

Since \( x = \frac{p_i}{p_i^s} - 1 \in (-1, 1/p_i^s - 1) \), it follows from Lemma 2 that
\[
\sum_{i=1}^{N} p_i \left( \frac{p_i}{p_i^s} \right)^{\alpha - 1} = \sum_{i=1}^{N} p_i^s \left( \frac{p_i}{p_i^s} - 1 + 1 \right)^\alpha \leq \sum_{i=1}^{N} p_i^s \left[ 1 + \alpha \left( \frac{p_i}{p_i^s} - 1 \right) + \frac{\alpha(\alpha - 1)}{2} \left( \frac{p_i}{p_i^s} - 1 \right)^2 \right]
\]
\[
= 1 + \frac{\alpha(\alpha - 1)}{2} \sum_{i=1}^{N} \frac{(p_i - p_i^s)^2}{(p_i^s)^{\alpha - 1}}
\]
for \( \alpha \in (0, 1) \cup (2, +\infty) \) and the reverse holds for \( \alpha \in (-\infty, 0) \cup (1, 2) \). This leads directly to the first and fourth inequalities of Theorem 4.

For the third one, we have \( \alpha \geq 2 \) and use Lemma 3 with \( \beta = \alpha \) to obtain
\[
\sum_{i=1}^{N} p_i \left( \frac{p_i}{p_i^s} \right)^{\alpha - 1} = \sum_{i=1}^{N} p_i^s \left( \frac{p_i}{p_i^s} - 1 + 1 \right)^\alpha \geq \sum_{i=1}^{N} p_i^s \left[ 1 + \alpha \left( \frac{p_i}{p_i^s} - 1 \right) + f_\alpha(\alpha) \left( \frac{p_i}{p_i^s} - 1 \right)^\alpha \right] = 1 + f_\alpha(\alpha) \sum_{i=1}^{N} \frac{|p_i - p_i^s|^\alpha}{(p_i^s)^{\alpha - 1}},
\]
where \( 0 < f_\alpha(x) < \min_{x \geq 2}[(x - 1)^\alpha + \alpha x - 1]/x^\alpha \).

As to the second one, it is clear that

\[
F_\alpha = \frac{1}{\alpha(1 - \alpha)} \left[ \sum_{i=1}^{N} p_i^\alpha (p_i^\alpha)^{\alpha - (p_i^\alpha)^{\alpha}} \right] \leq \frac{1}{\alpha(1 - \alpha)} \max_{i} \left[ \frac{(p_i^\alpha)^{\alpha - (p_i^\alpha)^{\alpha}}}{(p_i^\alpha)^{\alpha}} \right]
\]

for \( \alpha \in (0, 1) \); and

\[
F_\alpha = \frac{1}{\alpha(\alpha - 1)} \left[ \sum_{i=1}^{N} p_i^\alpha (p_i^\alpha)^{\alpha - 1} - 1 \right] \leq \frac{1}{\alpha(\alpha - 1)} \max_{i} \left[ \frac{(p_i^\alpha)^{\alpha - (p_i^\alpha)^{\alpha}}}{(p_i^\alpha)^{\alpha}} \right].
\]

for \( \alpha \in (-\infty, 0) \cup (1, 2) \). This completes the proof of Theorem 4.

**Proof of Theorem 5.**

Set \( y_i = p_i/p_i^\alpha(> 0) \). From the master equation we deduce that

\[
\frac{dF_\alpha}{dt} = \frac{1}{(\alpha - 1)} \sum_{i} y_i^{\alpha - 1} \frac{dp_i}{dt} = \frac{1}{\alpha(\alpha - 1)} \sum_{i,j \neq i} q_{ij} [p_i \alpha(y_j^{\alpha - 1} - y_i^{\alpha - 1}) + p_i^{\alpha - 1}(1 - \alpha)(y_j^\alpha - y_i^\alpha)]
\]

\[
= \frac{-1}{\alpha(\alpha - 1)} \sum_{i,j \neq i} q_{ij} p_i^\alpha \left[ y_i^{\alpha} - \alpha y_i y_j^{\alpha - 1} + (\alpha - 1)y_j^\alpha \right]
\]

\[
\equiv \sum_{i,j \neq i} q_{ij} p_i^\alpha |y_i - y_j|^\beta R_\beta(y_i, y_j)
\]

with

\[
R_\beta(y_i, y_j) = \frac{y_i^{\alpha} - \alpha y_i y_j^{\alpha - 1} + (\alpha - 1)y_j^\alpha}{\alpha(1 - \alpha)|y_i - y_j|^\beta}.
\]

Assume that

\[
R_\beta(y_i, y_j) \leq -R = -R(\alpha, \beta) < 0.
\]

Then for \( \beta \geq 1 \) we have

\[
\frac{dF_\alpha}{dt} \leq -R \sum_{i,j \neq i} q_{ij} p_i^\alpha |y_i - y_j|^\beta = -R \sum_{i,j \neq i} q_{ij} p_i^\alpha (y_i - y_j)^{1-\beta}|q_{ij} p_j^\alpha(y_i - y_j)|^\beta
\]

\[
\leq -\tilde{q}^{1-\beta} \sum_{i,j \neq i} q_{ij} p_i^\alpha (y_i - y_j)^{1-\beta}
\]

\[
\leq -\frac{\tilde{q}^{1-\beta}}{(N-1)^{\beta-1}} \sum_{i} \left| \sum_{j \neq i} q_{ij} p_j^\alpha(y_i - y_j)^\beta \right| = -\frac{\tilde{q}^{1-\beta}}{(N-1)^{\beta-1}} \sum_{i} \left| \sum_{j \neq i} (q_{ij} p_j - q_{ij} p_i) \right|^{\beta}.
\]

Here \( \tilde{q} = \max_{i,j \neq i} \{q_{ij} p_i^\alpha\} \) and the third inequality is due to the Hölder inequality.

It remains to show the estimate in (11). For \( \alpha \geq 2 \), we take \( \beta = \alpha \). From Lemma 3 it follows that

\[
R_\alpha(y_i, y_j) = \frac{(x + 1)^\alpha - \alpha x - 1}{\alpha(1 - \alpha)|x|^\alpha} \leq -\frac{f_\alpha(x)}{\alpha(1 - \alpha)}
\]

with \( x = y_i/y_j - 1 \), where \( 0 < f_\alpha(x) < \min_{x \geq 2}[(x - 1)^\alpha + \alpha x - 1]/x^\alpha \).
For $\alpha < 2$, we take $\beta = 2$ and rewrite

$$R_2(y_i, y_j) = \frac{\alpha y_i \alpha \alpha y_j - (\alpha - 1) y_j^\alpha}{\alpha (1 - \alpha) |y_i - y_j|^2} = \frac{(x + 1)^\alpha - \alpha x - 1}{\alpha (1 - \alpha)x x^2 y_j^\alpha}. $$

Since $x \in (-1, (p_i^s y_j)^{-1} - 1)$ for $y_i \in (0, 1/p_i^s)$, we deduce from Lemma 2 when $p_i^s \leq p_j^s$ that

$$R_2(y_i, y_j) \leq -\frac{1}{2} (p_i^s)^{2-\alpha}, \quad \alpha \in (-\infty, 0) \cup (0, 1) \cup (1, 2).$$

When $p_i^s > p_j^s$, $x \in (-1, (p_j^s y_j)^{-1} - 1)$ for $y_i \in (0, 1/p_j^s)$ and we deduce from Lemma 2 that

$$R_2(y_i, y_j) \leq -\frac{1}{2} (p_j^s)^{2-\alpha}, \quad \alpha \in (-\infty, 0) \cup (0, 1) \cup (1, 2).$$

This completes the proof.

[1] L. E. Reichl, *A modern course in statistical physics*, University of Texas Press, Austin, 1980.

[2] N.G. van Kampen, *Stochastic Processes in Physics and Chemistry*, Elsevier, Singapore, 2009.

[3] J.R. Norris, *Markov Chains*, Cambridge, New York, 1998.

[4] J. Schnakenberg, Network theory of microscopic and macroscopic behavior of master equation systems. *Rev. Mod. Phys.*, 48: 571-585 (1976).

[5] X.J. Zhang, H. Qian, M. Qian, Stochastic theory of nonequilibrium steady states and its applications. Part I. *Phys. Rep.*, 510: 1-86 (2012).

[6] H. Ge, M. Qian, H. Qian, Stochastic theory of nonequilibrium steady states: Part II: Applications in chemical biophysics. *Phys. Rep.*, 510: 87-118 (2012).

[7] H. Qian, M. Qian, *Phys. Rev. Lett.* 84: 2271 (2000).

[8] M. Qian, X.J. Zhang, R.J. Wilson, J. Feng, *Europhys. Lett.* 84: 10014 (2008).

[9] H. Ge, H. Qian, Dissipation, generalized free energy, and a self-consistent nonequilibrium thermodynamics of chemically driven open subsystems. *Phys. Rev. E*, 87: 062125 (2013).

[10] M. Esposite, C.V.D. Broeck, Three faces of the second law. I. Master equation formulation. *Phys. Rev. E*, 82: 011143 (2010).

[11] M. Esposite, C.V.D. Broeck, Three faces of the second law. II. Fokker-Planck formulation. *Phys. Rev. E*, 82: 011144 (2010).

[12] H. Ge, H. Qian, Physical origins of entropy production, free energy dissipation, and their mathematical representations. *Phys. Rev. E*, 81: 051133 (2010).

[13] M. Shiino, H-theorem with generalized relative entropies and the Tsallis statistics, *J. Phys. Soc. Jap.* 67: 3658-3660 (1998).

[14] W.-A. Yong, *Entropy and global existence for hyperbolic balance laws*, Arch. Rational Mech. Anal., 172: 247–266 (2004).
[15] W.-A. Yong. *Conservation-dissipation structure of chemical reaction systems*, Phys. Rev. E, 49:033503 (2012).

[16] W.-A. Yong. *An interesting class of partial differential equations*, J. Math. Phys. 49:033503 (2008).

[17] C. Tsallis, *Possible generalization of Boltzmann-Gibbs statistics*, J. Stat. Phys. 52: 479-487 (1988).

[18] D.S. Mitrinovic, J.E. Pecaric, A.M. Fink, *Classical and New Inequalities in Analysis*, Ch. 3, (Kluwer Academic, Dordrecht, 1993).

[19] L. Leindler, On a generalization of Bernoulli’s inequality, *Acta Sci. Math. Hung.* 33: 225-230 (1972).