LOWER BOUNDS FOR WEIL’S EXPLICIT FORMULA VIA SELBERG’S TRACE FORMULA

TIAN AN WONG

Abstract. We show that the explicit formulae of Weil appear in the contribution of the continuous spectrum to the Arthur-Selberg trace formula for \( SL(2) \) in two ways: as a sum over zeroes, and as a sum of distributions. This follows from averaging the Maaß-Selberg relation for Eisenstein series in the \( t \)-aspect. As an application, we obtain an expression for a lower bound for the sums over zeroes with respect to the truncation parameter for Eisenstein series. We also observe that this technique holds for general reductive groups with continuous spectrum.

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1. Introduction

1.1. Background. In 1952, Weil wrote down an explicit formula to study the zeroes of Hecke \( L \)-functions, using certain sums over primes. Inspired by his proof of the Riemann hypothesis for curves over finite fields, he showed that the positivity of this sum over the zeroes is equivalent to the Riemann hypothesis for \( L(s, \chi) \). Later, in [Wei72], Weil extended his work to Artin-Hecke \( L \)-functions over a global field.

On the other hand, in 1956 Selberg introduced the trace formula [Sel56], which is a character identity expressing the spectrum of invariant differential operators as a sum of orbital integrals. When the operator is the Laplacian acting on a compact Riemann surface, the formula resembles the explicit formulae, where the prime closed geodesics behave in analogy with the prime numbers.

For a noncompact, finite volume Riemann surface, additional terms appear in the trace formula related to the continuous spectrum on the spectral side, and parabolic conjugacy classes on the geometric side. Their appearance is parallel in
the sense that by subtracting one from the other, Selberg was able to show that the result was absolutely convergent. But the additional terms, written in Hejhal [Hej76, 10.2] for the $SL_2(R)$ acting on $SL_2(Z) \backslash H_2$,

$$g(0) \log(\frac{\pi}{2}) - \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) \left( \frac{\Gamma'(\frac{1}{2} + ir)}{\Gamma} + \frac{\Gamma'(1 + ir)}{\Gamma} \right) dr + 2 \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} g(2 \log n),$$

look structurally similar to the explicit formula of Weil for the Riemann zeta function expressing the sum over zeroes as [Hej76, 6.7],

$$h(\frac{i}{2}) + h(-\frac{i}{2}) - g(0) \log \pi + \frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) \left( \frac{\Gamma'(\frac{1}{4} + \frac{1}{2} ir)}{\Gamma} + \frac{1}{2} \right) dr - 2 \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} g(\log n).$$

Here $h(r)$ is a function with certain analytic conditions, $g(u)$ its Fourier transform, and $\Lambda(n)$ the usual von Mangoldt function.

Hejhal [Hej76, p.478] points out this parallel, and asks the question: Does the Weil explicit formula have anything to do with the Selberg trace formula? He notes that ‘although there are structural similarities, one finds serious obstructions to interpreting [the explicit formula] as a trace formula.’

Could the explicit formulae be trace formulae? The work of others such as Goldfeld and Connes have suggested possible interpretations [Gol89, Con99, Mey95], giving weight to the conjecture of Hilbert and Pólya that the zeroes of $\zeta(s)$ may be interpreted as eigenvalues of a self-adjoint operator, not to mention the approach of random matrix theory initiated by Montgomery [Mon73], and extended most notably by Katz and Sarnak [KS99]. Also, related to this is work of Deninger, which proposes viewing the explicit formula as a Lefschetz trace formula [Den92, Den93, Den08], see also [Lei05].

1.2. Main results. In this paper, we present a different approach to this question: rather than interpret the entire trace formula as an explicit formula, we show that the sum over zeroes of $L$-functions of the explicit formula appears naturally within the spectral side of the trace formula. In particular, the continuous spectrum is described by Eisenstein series, and in the trace formula one requires the inner product of truncated Eisenstein series. The Maass-Selberg relation expresses this inner product using the logarithmic derivative of the constant term of the Eisenstein series, which can be written as a quotient of $L$-functions, and it is here that the sum over zeroes appear. Having achieved this, we are then in a position to study questions related to the explicit formula using our knowledge of the trace formula. It should be noted that while the zeroes of $L$-functions are obtained from the continuous spectrum, we do not claim any precise ‘spectral interpretation’ of these zeroes.

The main result of this paper can be illustrated by the following theorem.

**Theorem 1.1.** Let $g(x)$ be a smooth compactly supported function on $R^\times$, with $\hat{g}$ its Mellin transform. Consider the Selberg trace formula for $SL_2(R)$ acting on $SL_2(Z) \backslash H_2$. The contribution of the continuous spectral term

$$-\frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{m'}{m} (ir) \hat{g}(ir) dr$$

is given by

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} h(r) \left( \frac{\Gamma'(\frac{1}{2} + ir)}{\Gamma} + \frac{\Gamma'(1 + ir)}{\Gamma} \right) dr + 2 \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} g(2 \log n).$$
is equal to

\[
\sum_{\rho} \hat{g}(\rho) - \int_{0}^{\infty} \left\{ g(x) + \frac{1}{4} g^*(x) \right\} dx + \sum_{n=1}^{\infty} \Lambda(n) g(n) \\
+ \int_{1}^{\infty} \left\{ g(x) + g^*(x) - \frac{2}{x} g(1) \right\} \frac{x \, dx}{2(x^2 - 1)} + \frac{1}{2} (\log 4\pi - \gamma) g(1),
\]

where the sum \( \rho \) runs over zeroes of \( \zeta(s) \) with \( 0 < \text{Re}(\rho) < 1 \), and \( m(s) = \xi(s)/\xi(1+s) \), with \( \xi(s) \) the completed Riemann zeta function.

The shape of the formula obtained reflects that derived by Weil in 1952 [Wei52], though we give a version that follows that of Bombieri for the Riemann zeta function [Bom00]. In particular, the resulting expression is given solely in terms of the function \( g \), which is useful for applications. We also obtain a similar expression for Hecke \( L \)-functions, in the adelic language and over a number field, which we do not reproduce here as the formula is more complicated, and refer to Theorem 4.1 for the statement.

Having shown that the sum over zeroes appear in the trace formula, we next relate the formulae to the distributions in [Wei72]. This is particularly natural in the sense that the trace formula is an identity of distributions. The next theorem shows that we can also express the continuous spectral terms as distributions, as obtained in Weil’s 1972 form of the explicit formula, which we refer to as Weil-type distributions for their relation Weil’s formula:

**Theorem 1.2.** Let \( F \) be a number field, and \( \chi \) a Hecke character of \( F \). Also let \( g(x) \) be a smooth compactly supported function on \( \mathbb{R}_{\times}^{+} \), with \( \hat{g} \) its Mellin transform. Consider the Selberg trace formula for \( SL_2(A_F) \) acting on the space \( SL_2(F) \setminus SL_2(A_F) \). The contribution of the continuous spectral term (3.6)

\[-\frac{1}{4\pi} \int_{-i\infty}^{i\infty} m(\eta, s)^{-1} m'(\eta, s) \hat{g}(s) ds\]

can be expressed as a sum of Weil-type distributions

\[ g(0) \log |d_F| + \frac{1}{2} \int_{W_F} g(|w|) \chi(w) |w|^{-1} dw - \frac{1}{2} \sum_{\nu} \nu \int_{W_{\nu}} \hat{g}(|w|) \chi_{\nu}(w) \frac{|w|}{|1-w|} dw, \]

where the sum \( \nu \) runs over all places of \( F \), \( d_F \) is discriminant of \( F \), and \( W_F, W_{\nu} \) are the global and local Weil groups of \( F \) respectively. Finally, \( m(\eta, s) = L(s, \chi)/\epsilon(s, \chi, \psi)L(1+s, \chi) \), and the finite part \( \nu \) is described in Definition 2.6 below.

Having shown a relationship between the explicit formulae and trace formulae, one would hope that the connection will shed light on either of the two. As a first step, we show the following bound for the sums over zeroes, which follows as a straightforward corollary to Theorem 1.1 above:
Theorem 1.3. Let $g = g_0 * g_0^*$, for any $g_0$ in $C_\infty^\infty(\mathbb{R}_+^+)$. Then the sum over zeroes of $\zeta(s)$, where $0 < \text{Re}(\rho) < 1$, is bounded below by the following:

$$
\sum_{\rho} \hat{g}(\rho) \geq \int_0^\infty \left\{ g(x) + \frac{1}{4} g^*(x) \right\} dx - \sum_{n=1}^{\infty} \Lambda(n) g(n) - \int_1^{\infty} \left\{ g(x) + g^*(x) - \frac{2}{x} g(1) \right\} \frac{x}{2(x^2 - 1)} dx
$$

$$
- g(0) \log T - \frac{1}{4\pi i} \int_{-\infty}^{\infty} m(it) g(it) \frac{T^s}{t} dt.
$$

for any $T > \sqrt{3}/2$.

Finally, we note that this method holds in fact for the continuous spectral terms of trace formulae for any reductive group $G$ with continuous spectrum. In this paper we have confined ourselves to $SL(2)$ only for ease of exposition. In particular, given such a $G$, one considers the Eisenstein series attached to a maximal parabolic subgroup of $G$, from which one obtains Langlands-Shahidi $L$-functions in their constant terms (cf. [Gel11]). Then considering the Maass-Selberg relation for such Eisenstein series, one deduces the analogous results for the zeroes of the associated $L$-functions.

1.3. Relation to other work. It is well-known since [JS77] that the spectral theory of Eisenstein series, and in particular the Maass-Selberg relation, can be used to prove the nonvanishing of $L$-functions on the line $\text{Re}(s) = 1$. In [Sar04], Sarnak showed how to make this method effective, and in the spirit of de la Vallée Poussin obtain standard zero-free regions for $\zeta(s)$. This was extended to other $L$-functions, for example [GL06, GL16]. Using the method of trace formulae, we are essentially averaging the Maass-Selberg relation in the $t$-aspect, that is, integrating along the imaginary axis. Doing so, we get a hold of all possible zeroes in the critical strip.

As an application, the explicit formulae have been used recently to obtain striking results on low-lying zeroes of families of $L$-functions (see for example [ILS00, HM07, FM15, ST16] and the references therein). In order to match the predictions of random matrix theory [KS99], the compact support of the test function to be used in the explicit formula should be arbitrary. Indeed, this expectation is similar to the desiderata for test functions satisfying Weil’s criterion [Bom00, Bur00], which we discuss in detail in Remark 2.9 below. Yet, at present the support of the test function considered remains severely constrained. In Sect 5, we describe a new approach to bounding the sum over zeroes, which, if made effective, might in certain cases allow for functions with larger support.

This paper is organized as follows: In Sect. 2, we derive the explicit formula for Hecke $L$-functions classically, following the exposition of Bombieri [Bom00]. We also recall the adelic form of the explicit formula and the Weil criterion for Hecke $L$-functions. In Sect. 3, we recall the adelic trace formula for $SL(2)$, following Langlands and Labesse [LL79]. In Sect. 4, we relate the explicit formulae to trace formulae, and prove Theorem 1.1 and 1.2 above. Finally, in Sect. 5 we discuss the application to lower bounds, giving Theorem 1.3 as an example.

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2. The Weil explicit formula

In this section, we derive a version of Weil’s explicit formula for Hecke $L$-functions in the sense of [Wei52], then we state the second form as in [Wei72]. Finally, we describe Weil’s criterion for the associated $L$-functions.

**Definition 2.1.** To fix notation, we introduce the global Hecke $L$-function associated to $\chi$,

$$L(s, \chi) = \prod_v L_v(s, \chi_v),$$

the product taken over all places $v$ of $F$. It satisfies the functional equation

$$L(s, \chi) = \varepsilon(s, \chi, \psi) L(1 - s, \overline{\chi})$$

where the epsilon factor is defined as $\varepsilon(s, \chi, \psi) = W(\chi)|N_F/Q(f(\chi))|d_F|s - \frac{1}{2}|$ where $W(\chi)$ is the root number, $f(\chi)$ the conductor of $\chi$, and $\psi$ is a fixed additive character of $F$. We also define the completed $L$-function to be

$$\Lambda(s) = |N_F/Q(f(\chi))|d_F|s - \frac{1}{2}|L(s, \chi),$$

in which case the functional equation reads $\Lambda(s) = W(\chi)\Lambda(1 - s, \overline{\chi})$. The local $L$-factors are given as follows:

$$L_v(s, \chi_v) = \begin{cases} \Gamma_C(s) = (2\pi)^{1-s}\Gamma(s + w + |n|/2) & v \text{ complex} \\ \Gamma_R(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s+1+n}{2}\right) & v \text{ real} \\ (1 - \chi_v(p)N_F/Q(p)\overline{s})^{-1} & v = p \text{ finite, } \chi_v \text{ unramified} \\ 1 & v = p \text{ finite, } \chi_v \text{ ramified} \end{cases}$$

where $p$ is a prime ideal of $F$, and the gamma factors are explained below. We will denote by $\zeta_F(s, \chi)$ the product over all finite primes of $L_v(s, \chi_v)$.

**Remark 2.2.** (Ramified Gamma factors.) The archimedean $L$-factors occur as follows: If $v$ is a real place, the field $F_v \simeq \mathbb{R}$ has no nontrivial automorphisms, thus a character $\chi_v$ of $F_v^{\times}$ can be identified with one of $\mathbb{R}^{\times}$, hence necessarily of the form

$$\chi_v(t) = \text{sgn}(t)^n|t|^w,$$

where $n \in \{0, 1\}$, $w \in \mathbb{C}$, $\text{sgn}(t)$ is the usual sign of $t$, and one has

$$L_v(s, \chi_v) = \Gamma_R(s + w + n).$$

If $v$ is complex, the field $F_v \simeq \mathbb{C}$ has two possible identifications. Choosing one, the character $\chi_v$ will be a character of $\mathbb{C}^{\times}$, necessarily of the form

$$\chi_v(z) = \text{arg}(z)^n|z|^{2w},$$

where $n \in \mathbb{Z}$, $w \in \mathbb{C}$, $\text{arg}(z)$ is the complex argument $z/|z|_C = z^{1/2}\overline{z}^{-1/2}$, giving

$$L_v(s, \chi_v) = \Gamma_C(s + w + |n|/2).$$

We see that if we identify $F_v$ with the complex conjugate, we replace $\chi_v(z)$ with $\overline{\chi_v}(\overline{z})$ and thus $m$ with $-m$, and $L_v(s, \chi_v)$ remains well-defined. Finally, if $v_\infty$ is an archimedean place, we will write $\Gamma_k(s)$ for $\Gamma_R(s)$ or $\Gamma_C(s)$ depending on whether the completion $F_{v_\infty}$ is real or complex.
2.1. The explicit formula I. In 1952, Weil [Weil52] introduced an explicit formula for Hecke $L$-functions over a number field. To state the formula, we fix the following notation: given $f(x)$ a complex-valued function in $C^\infty_c(\mathbb{R}^+)$, define
\[ f^*(x) = \frac{1}{x}f\left(\frac{1}{x}\right) \]
and the Mellin transform
\[ \hat{f}(s) = \int_0^\infty f(x)x^{s-1}dx. \]
We will say that $f$ is even if $f = f^*$ and odd if $f = -f^*$.

Throughout we will fix a number field $F$ with $r_1$ real and $2r_2$ complex embeddings, and denote by $\mathcal{O}_F$ its ring of integers. Define the von Mangoldt function for number fields as follows: for any nonzero integral ideal $a$ of $\mathcal{O}_F$ we set
\[ \Lambda(a) = \begin{cases} \log N_{F/Q}(p) & \text{if } a = p^k \text{ for some } k \geq 1, \\ 0 & \text{otherwise} \end{cases} \]
not to be confused with the completed $L$-function $\Lambda(s, \chi)$ which we shall also use. To ease notation, we will write $N$ for the norm map $N_{F/Q}$, when there is no confusion.

We now derive a version of Weil’s explicit formula for $L(s, \chi)$, resembling the shape given to it by Bombieri for $\zeta(s)$. The techniques are standard, but we include the proof as the expression given to the archimedean contribution will be used again later.

**Theorem 2.3.** Let $g \in C^\infty_c(\mathbb{R}^+_\star)$, and $\chi$ a Hecke character of a number field $F$. Then we have
\[
\sum_{\rho} \hat{g}(\rho) = \delta_\chi \int_0^\infty (g(x) + g^*(x))dx - \sum_a \Lambda(a)\chi(a)g(N(a)) + g^*(N(a)) \\
- \log(|d_F|N(\chi))g(1) - \sum_{k=1}^{r_1+r_2} \frac{1}{2\pi i} \int_{-\infty}^\infty 2\text{Re}\left[\frac{\Gamma_k(1/2 + it)}{\Gamma_k(1/2)}\right] \hat{g}(1/2 + it) dt,
\]
where the sum $\rho$ is taken over zeroes of $L(s, \chi)$ with $0 < \text{Re}(\rho) < 1$, and $\delta_\chi$ is $1$ if $\chi$ is trivial and zero otherwise. Moreover, the last sum can be expressed in terms of $g$,
\[
\sum_{k=1}^{r_1+r_2} \left\{ \int_1^\infty \left\{ \frac{g(x) + g^*(x)}{x^{1/2}} \right\} \frac{M}{x^{M-\alpha+ib}} g(1) \right\} \frac{x^{M-\alpha+ib}}{x^{M-1}} dx + \left( \gamma + \frac{2}{M} \log(2\pi) \right) g(1) \}
\]
where $M = 2$ if $F_v$ is a real completion and $1$ if $F_v$ a complex completion.

**Proof.** Consider the integral
\[ I(g) = \frac{1}{2\pi i} \int_{(c)} \frac{\Lambda'}{\Lambda}(s)\hat{g}(s)ds \]
where the integration is taken over the line $(c - i\infty, c + i\infty)$ with $c > 1$. Since $g(x)$ is smooth with compact support, its Mellin transform $\hat{g}$ is an entire function of $s$ of order $1$ and exponential type, rapidly decreasing in every fixed vertical strip.

The logarithmic derivative of $\Lambda(s, \chi)$ is holomorphic for $\sigma > 1$, and has logarithmic growth on any vertical line $(c - i\infty, c + i\infty)$ with $c > 1$, hence the integral $I(g)$ is absolutely convergent.
For \( \sigma > 1 \), we have
\[
\frac{\Lambda'}{\Lambda}(s, \chi) = \frac{1}{2} \log(\|d_F\| N \tilde{h}(\chi)) + \sum_{k=1}^{r_1 + r_2} \frac{\Gamma'(s)}{\Gamma_k(s)} - \sum_a \frac{\Lambda(a) \chi(a)}{N(a)^s},
\]
whence
\[
I(g) = \frac{1}{2} \log(\|d_F\| N \tilde{h}(\chi)) \frac{1}{2\pi i} \int_{(c)} \hat{g}(s) ds + \sum_{k=1}^{r_1 + r_2} \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma'(s)}{\Gamma_k(s)} \hat{g}(s) ds
\]
\[
- \sum_a \Lambda(a) \chi(a) \frac{1}{2\pi i} \int_{(c)} \hat{g}(s) N(a)^{-s} ds,
\]
because term-by-term integration is justified by absolute convergence. The inverse Mellin transform being
\[
g(x) = \frac{1}{2\pi i} \int_{(c)} \hat{g}(s) x^{-s} ds,
\]
we arrive at the first expression (2.2)
\[
I(g) = \frac{1}{2} \log(\|d_F\| N \tilde{h}(\chi)) g(1) + \sum_{k=1}^{r_1 + r_2} \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma'(s)}{\Gamma_k(s)} \hat{g}(s) ds - \sum_a \Lambda(a) \chi(a) g(N(a)).
\]

Now we compute \( I(g) \) in a different way: starting with the initial expression, move the line of integration to the left to \((c' - i\infty, c' + \infty)\) for some \( c' < 0 \). This step is justified by integrating over a rectangle with vertices \((c' \pm i T, c \pm i T)\) and showing that the integral over the horizontal edges tends to 0 as we let \( T \) tend to infinity along a well-chosen sequence. This is a well-known method, described in [Ing32, Theorem 29] for \( \zeta(s) \), for example.

In our situation, the following bound in [Lan94, Prop 2.4] for completed Hecke \( L \)-functions will suffice: the number of zeroes of \( L(s, \chi) \) in a box \( 0 \leq \sigma \leq 1 \) and \( T \leq |t| \leq T + 1 \) is \( O(\log T) \). It follows that there exists constants \( c \) and \( T_m \) for every integer \( m \neq -1, 0, 1 \) such that \( L(s, \chi) \) has no zeroes on the horizontal strips
\[
|t \pm T_m| \leq \frac{c}{\log |m|} \quad m < T_m < m + t.
\]
Allowing then \( T \) to tend to infinity via the sequence \((T_m)\), the contribution from the horizontal edges also vanish.

Moving the line of integration to the left, we encounter the residues of \( L'/L(s, \chi) \) due to the zeroes of \( L(s, \chi) \) inside the critical strip \( 0 \leq \sigma \leq 1 \), and the simple poles of \( L(s, \chi) \) at \( s = 0, 1 \) in the case where \( \chi \) is trivial. It follows that
\[
(2.3) \quad I(g) = -\delta_{\chi}(\hat{g}(0) + \hat{g}(1)) + \sum_{\rho} \hat{g}(\rho) + \frac{1}{2\pi i} \int_{(c')} \frac{\Lambda'}{\Lambda}(s, \chi) \hat{g}(s) ds.
\]

Now we use the functional equation \( \Lambda(s, \chi) = W(\chi)\Lambda(1 - s, \bar{\chi}) \) to obtain the relation
\[
\frac{\Lambda'}{\Lambda}(s, \chi) = \frac{\Lambda'}{\Lambda}(1 - s, \bar{\chi}) = \frac{1}{2} \log(\|d_F\| N \tilde{h}(\chi)) - \sum_{k=1}^{r_1 + r_2} \frac{\Gamma'(s)}{\Gamma_k (1 - s)} + \sum_a \frac{\Lambda(a) \chi(a)}{N(a)^{1-s}}.
\]
Then substituting this into the integral in (2.3), we obtain as before
\[
\frac{1}{2} \log(|d_F Nf(\chi)|) \frac{1}{2\pi i} \int_{(\epsilon')} \hat{g}(s) ds - \sum_{k=1}^{r_1 + r_2} \frac{1}{2\pi i} \int_{(\epsilon')} \frac{\Gamma_k'}{\Gamma_k} (1-s) \hat{g}(s) ds \\
+ \sum_{a} \Lambda(a) \chi(a) \frac{1}{2\pi i} \int_{(\epsilon')} \hat{g}(s) N(a)^{s-1} ds,
\]
which is
\[
\frac{1}{2} \log(|d_F Nf(\chi)|) g(1) - \sum_{k=1}^{r_1 + r_2} \frac{1}{2\pi i} \int_{(\epsilon')} \frac{\Gamma_k'}{\Gamma_k} (1-s) \hat{g}(s) ds + \sum_{a} \Lambda(a) \chi(a) g^*(N(a)),
\]
where again term-by-term integration is justified because we are again in the region of absolute convergence, thanks to the functional equation.

Now equating the two expressions for \(I(g)\) we find
\[
\sum_\rho \hat{g}(\rho) = \delta_\chi (\hat{g}(0) + \hat{g}(1)) - \log(|d_F Nf(\chi)|) g(1) \\
- \sum_{a} \Lambda(a) \chi(a) (g(N(a)) + g^*(N(a))) \\
+ \sum_{k=1}^{r_1 + r_2} \frac{1}{2\pi i} \int_{(\epsilon')} \frac{\Gamma_k'}{\Gamma_k} (s) \hat{g}(s) ds + \sum_{k=1}^{r_1 + r_2} \frac{1}{2\pi i} \int_{(\epsilon')} \frac{\Gamma_k'}{\Gamma_k} (1-s) \hat{g}(s) ds.
\]
And observe that
\[
\hat{g}(0) = \int_0^\infty g^*(x) dx, \quad \hat{g}(1) = \int_0^\infty g(x) dx,
\]
thus we have the desired formula, save for the last two terms.

In order to obtain the explicit formula we compute the last two integrals as follows. First, we move the line of integration of both integrals to \(c = c' = \frac{1}{2}\), which we may do without encountering any pole of the integrand. Thus the sum of the two integrals becomes
\[
\sum_{k=1}^{r_1 + r_2} \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\Re \left[ \frac{\Gamma_k'}{\Gamma_k} \left( \frac{1}{2} + it \right) \right] \hat{g}\left( \frac{1}{2} + it \right) dt.
\]
Note that our \(\Gamma_k(s)\) here depends on the ramification of \(\chi\), and in fact the \(\Gamma_k(1-s)\) appearing in the second integral is associated to \(\bar{\chi}\), in the sense of Remark 2.2. We now treat the gamma factors in detail.

**Case 1:** \(\Gamma_k(s) = \Gamma_R(s) = \pi^{-s/2} \Gamma(\frac{s+w}{2})\), with ramification \(w = a + ib\). The logarithmic derivative is
\[
\frac{\Gamma_k'}{\Gamma_R}(s) = -\frac{1}{2} \log \pi + \frac{1}{2} \frac{\Gamma'}{\Gamma} \left( \frac{s}{2} \right),
\]
so we have
\[
\text{(2.4)} \quad -(\log \pi) g(1) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \Re \left[ \frac{\Gamma'}{\Gamma} \left( \frac{1}{2} + it + a + ib \right) \right] \hat{g}\left( \frac{1}{2} + it \right) dt.
\]
To treat the integral, we follow the exposition of \cite{Bom00, §2} closely. We first use the expressions \cite[Bom00, p.188]{Bom00}

\[
\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \left\{ \frac{1}{n} - \frac{1}{n + z} \right\}
\]

and

\[
\sum_{n=1}^{N} \frac{1}{n} = \log N + \gamma + O\left(\frac{1}{N}\right),
\]

to obtain

\[
(2.5) \quad \frac{\Gamma'(z)}{\Gamma(z)} = \log N - \sum_{n=0}^{N} \frac{1}{n + z} + O\left(\frac{1}{N}\right)
\]

uniformly for Re\(z\) > \(-\frac{N}{2}\) and \(z\) not equal to zero or a negative integer. This gives

\[
\text{Re}\left[\frac{\Gamma'(\frac{1}{2} + it + a + ib)}{\Gamma\left(\frac{1}{2}\right)}\right] = \log N - \sum_{n=0}^{N} \frac{2(2n + a + \frac{1}{2})}{(2n + a + \frac{1}{2})^2 + (t + b)^2} + O\left(\frac{1 + |t + b|}{N}\right)
\]

and so the integral in (2.4) becomes

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \log N - \sum_{n=0}^{N} \frac{2(2n + a + \frac{1}{2})}{(2n + a + \frac{1}{2})^2 + (t + b)^2} \right) \hat{g}(\frac{1}{2} + it) dt + O\left(\frac{1}{N}\right) + O\left(\frac{1+|t+b|}{N}\right)
\]

Since \(\hat{g}\) is rapidly decreasing on any vertical line, the last integral converges and the error term is \(O(1/N)\). Apply also Mellin inversion to the first term, whence

\[
(2.6) \quad -\sum_{n=0}^{N} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2(2n + a + \frac{1}{2})}{(2n + a + \frac{1}{2})^2 + (t + b)^2} \hat{g}(\frac{1}{2} + it) dt + (\log N) f(1) + O\left(\frac{1}{N}\right)
\]

We have by Fubini’s theorem

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2c}{c^2 + (t + b)^2} \left( \int_{0}^{\infty} g(x) x^{-\frac{1}{2} + it} dx \right) dt
\]

\[
= \int_{0}^{\infty} g(x) x^{-\frac{1}{2}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left( \frac{c}{t + ic} - \frac{c}{t - ic} \right) x^{i(t-b)} dt \ dx
\]

after making the change of variables \(t \mapsto t - b\). Applying the calculus of residues yields

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \left( \frac{c}{t + ic} - \frac{c}{t - ic} \right) x^{i(t-b)} dt = \min(x, \frac{1}{x}) \epsilon^{-ib}.
\]

Hence, taking \(c = 2n + a + \frac{1}{2}\), (2.6) is then

\[
\int_{0}^{\infty} \left( \sum_{n=0}^{N} x^{-2n-a+ib} \right) g(x) dx = \int_{0}^{1} \left( \sum_{n=0}^{N} x^{2n+a-b} \right) g(x) dx + (\log N) f(1) + O\left(\frac{1}{N}\right)
\]

\[
= \int_{0}^{\infty} \left( \sum_{n=0}^{N} x^{-2n-a+ib} \right) (g(x) + g^*(x)) \frac{dx}{x} + (\log N) f(1) + O\left(\frac{1}{N}\right).
\]
Finally, we write
\[
\int_{1}^{\infty} \left( \sum_{n=0}^{N} x^{-2n-a+ib} \right) (g(x) + g^*(x)) \frac{dx}{x}
\]
\[
= \int_{1}^{\infty} \left( \sum_{n=0}^{N} x^{-2n-a+ib} \right) (g(x) + g^*(x) - \frac{2}{x^{2-a+ib}g(1)}) \frac{dx}{x} + \sum_{m=1}^{N+1} \frac{1}{m} g(1)
\]
and substitute back to obtain
\[
- \int_{1}^{\infty} \frac{1 - x^{-2N\gamma}}{1 - x^{-2}} x^{-a+ib}(g(x) + g^*(x) - \frac{2}{x^{2-a+ib}g(1)}) \frac{dx}{x}
\]
\[+(\log N - \sum_{m=1}^{N+1} \frac{1}{m})f(1) + O\left(\frac{1}{N}\right).
\]
Now we take the limit as \(N \to \infty\) and deduce for the integral in (2.4)
\[
-(\gamma + \log \pi)g(1) - \int_{1}^{\infty} \left\{ g(x) + g^*(x) - \frac{2}{x^{2-a+ib}g(1)} \right\} \frac{x^{1-a+ib}}{x^2 - 1} \frac{dx}{x}
\]
where \(\gamma\) is the usual Euler-Mascheroni constant. This concludes the real archimedean case.

Case 2: \(\Gamma_{k}(s) = \Gamma_{C}(s) = (2\pi)^{1-w} \Gamma(s + w)\), with ramification \(w = a + ib\). The computations are almost identical to the first case. The logarithmic derivative is
\[
\frac{\Gamma'(\Gamma)}{\Gamma_{C}}(s) = -\log(2\pi) + \Gamma'(\Gamma)
\]
so we have
\[
2g(1) \log(2\pi) + \frac{1}{2\pi} \int_{-\infty}^{\infty} 2Re \left[ \frac{\Gamma'(1 + i(t + a + ib))}{\Gamma(1 + i(t + a + ib))} \right] dt.
\]
As with (2.6) we have the expression for the integral
\[
\sum_{n=0}^{N} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2(n + a + \frac{1}{2})}{(n + a + \frac{1}{2})^2 + (t + b)^2} \frac{1}{\Gamma(1 + i(t + a + ib))} dt + (\log N) f(1) + O\left(\frac{1}{N}\right).
\]
Observing that the difference between (2.8) and (2.6) is \(n\) and \(2n\), we see that the integral is
\[
-(\gamma + 2\log(2\pi))g(1) - \int_{1}^{\infty} \left\{ g(x) + g^*(x) - \frac{1}{x^{1-a+ib}g(1)} \right\} \frac{x^{-a+ib}}{x^2 - 1} dx,
\]
which completes the complex archimedean case.

\[\square\]

**Remark 2.4.** The shape of the distribution arising from the gamma factor is expressed in a slightly different form than in \[Wei52\]. By the above, we have generalized the expression given by Bombieri [Bom00, p.186] to Hecke L-functions.
2.2. The explicit formula II. In 1972, Weil [Wei72] wrote down a similar explicit formula for ‘Artin-Hecke’ $L$-functions, that is to say traces of complex $n$-dimensional representations of the Weil group. He observes that the positivity of this distribution for smooth, compactly supported functions $g(x)$ on the positive real numbers is equivalent to the Riemann hypothesis and also the Artin conjecture for the $L$-function. Importantly, this expression is also uniform in the sense that in positive characteristic the distribution is known to be positive-definite after the proof of the Weil conjectures.

Given a finite extension of number fields $K/F$, we may associate an $n$-dimensional representation $\rho$ of the relative Weil group of $K/F$, a dense subgroup of $\text{Gal}(K/F)$, defined explicitly below. Then taking the trace of this representation produces a representation of the relative Weil group of $K/F$. The associated $L$-function is what Weil refers to as an Artin-Hecke $L$-function, and derives an explicit formula for it.

To state the result, we recall some definitions to the Artin-Hecke $L$-functions:

**Definition 2.5.** Define the Weil group $W_{\mathcal{F}}$ of a local field $\mathcal{F}$ as follows: if $\mathcal{F}$ is nonarchimedean with residue field $\mathcal{F}$, then taking the trace of this representation produces a $\mathcal{F}$-characteristic distribution is known to be positive-definite after the proof of the Weil conjectures.

Furthermore, we denote for simplicity $v$ if $\mathcal{F}v$ is archimedean, and discrete if $\mathcal{F}v$ is nonarchimeean. Thus we can and do identify $W_{\mathcal{F}v}/W_{\mathcal{F}v}^0$ with $\mathbb{R}_+^\times$ or a discrete subgroup thereof.

Finally, the absolute Weil group $W_{\mathcal{F}}$ associated to the global field $\mathcal{F}$ is more complicated to define, instead we refer the reader to [Wei72, p.9] (also [Tat79]), and simply note that its abelianization can be identified with $\mathcal{F}_\infty^\times \backslash \mathbb{A}_\mathcal{F}^\times$. 

**Definition 2.6.** Define the functions $f_0(x) = \inf(x^{\frac{1}{2}}, x^{-\frac{1}{2}})$ and $f_1(x) = f_0(x)^{-1} - f_0(x)$ on $\mathbb{R}_+^\times$, and the principal value

$$pv \int_0^\infty f(x)d^\times x = \lim_{t \to \infty} \left( \int_0^\infty (1 - f_0(x)^2f(x)d^\times x - 2c\log t) \right)$$

where $c$ is a constant such that $f(x) - cf_1(x)^{-1}$ is an integrable function on $\mathbb{R}_+^\times$. Furthermore, we denote for simplicity

$$pv \int_0^\infty f(x)d^\times x = pv \int_0^\infty f(x) + 2c\log(2\pi).$$

Then the second form of Weil’s explicit formula is the following:
We can now generalize Bombieri’s strengthening of Weil’s criterion for $L(s, \chi)$ in this direction are by contradiction, namely, assuming that there exists a zero $\rho$.

Recall that the transform turns convolution into multiplication, where $g$ is a smooth, compactly supported function on $R_+^\times$. The novelty in Weil’s method is the following condition, referred to as Weil’s criterion. We will consider test functions in $C_c^\infty(R_+^\times)$ which are multiplicative convolutions of a function $g$ and its transpose conjugate $\hat{g}^*$, hence

$g * \hat{g}^* = \int_0^\infty g(xy^{-1})\hat{g}^*(y)\frac{dy}{y} = \int_0^\infty g(xy)\overline{g(y)}dy.$

Recall that the transform turns convolution into multiplication,

$\overline{g * \hat{g}^*(s)} = \hat{g}(s)\hat{g}^*(1-s).$

We can now generalize Bombieri’s strengthening of Weil’s criterion for $\zeta(s)$ [Bom00, p.191] to Hecke $L$-functions:

**Theorem 2.8.** Let $W(g)$ be the linear functional defined by (2.1), on the space $C_c^\infty(R_+^\times)$, so that

$W(g) = W(g^*) = \sum \hat{g}(\rho),$

the sum taken over complex zeroes of $L(s, \chi)$. Then the Riemann hypothesis for $L(s, \chi)$ is equivalent to the statement that

$W(g * \hat{g}^*) \geq 0$

with equality only if $g$ is identically zero. In short, we say that the functional is positive-definite on such functions.

**Proof.** The Riemann hypothesis for $L(s, \chi)$ is the statement that $1 - \rho = \bar{\rho}$ for every nontrivial zero $\rho$ of $L(s, \chi)$. Assuming this, we have

$\sum \hat{g}(\rho)\hat{g}(1-\rho) = \sum \hat{g}(\rho)\bar{\hat{g}(\rho)} = \sum |\hat{g}(\rho)|^2 \geq 0.$

It is also easy to show that equality holds only if $g(x)$ is identically zero. Indeed, equality can hold only if $\hat{g}(\rho) = 0$ for every $\rho$, whence $\hat{g}(s)$ has at least $(1/\pi + o(1))R\log R$ zeroes in a disk $|s| < R$. On the other hand, $\hat{g}(s)$ is an entire function of exponential type, thus if $\hat{g}$ is not identically zero it can have at most $O(R)$ zeroes in the disk $|s| < R$. This proves the first implication.

For the proof of the converse statement, see [Bom00, p.191-193] for the case of $\zeta(s)$, and [Wei52] or [Lan94, p.342] for a general $L(s, \chi)$. We note that all proofs in this direction are by contradiction, namely, assuming that there exists a zero
of $L(s, \chi)$ with real part different from $\frac{1}{2}$, then using this zero to produce a test function $g$ on which the functional $W(g)$ is negative. □

Remark 2.9. Weil’s criterion was originally proved for functions $g \ast \bar{g}^*$ where $g$ is a function on $\mathbb{R}$, written additively rather multiplicatively as we have done, such that

1. It is smooth everywhere except for a finite number of points $a_i$ where $g_0$ and $g'_0$ have at most a discontinuity of the first kind, and $g(a_i) = \frac{1}{2}[g(a_i + 0) + g(a_i - 0)].$

2. There exists a $b > 0$ such that $g$ and $g'$ are $O(e^{-\frac{1}{2} + b|x|}).$

The functional $W(g)$ was placed into a more tractable form by Barner, with the new conditions on the test functions later referred to as the Barner conditions [Lan94]. Moreover, one can formulate as we have an equivalent condition for smooth, compactly supported functions, and the criterion for $\zeta(s)$ is proved for functions whose support is restricted to a small neighborhood of 0. In particular, Yoshida proved the analogous criterion for smooth, compactly supported, even functions [Yos92], and verified positivity for functions supported on $[-t, t]$ with $t = \log 2 / 2$; Burnol proves positivity for $t = \sqrt{2}$ [Bur00, Théorème 3.7], and Bombieri for $t = \log 2$ [Bom00, Theorem 12]. One hopes that the lower bound obtained in Sect. 5 might be used to extend the support of functions for which positivity holds.

3. The Selberg trace formula

In this section we recall the adelic trace formula for $SL(2)$, and in particular the contribution of the continuous spectrum to the trace formula. The expert reader is encouraged to skip this, consulting only Theorem 3.9 which gives the expression to be analyzed in the next section.

3.1. The pre-trace formula. Begin with a locally compact group $G$ with a discrete subgroup $\Gamma$. Denote by $\rho(g)$ the representation of $G$ on $L^2(\Gamma \backslash G)$ acting by right translation. It is a unitary operator, and commutes with the action of the invariant differential operators on $G$. If $G = SL_2(\mathbb{R})$, the only invariant differential operator is the Laplacian $\Delta$.

Definition 3.1. Given a smooth, compactly supported function $f$ on $G$, one defines the integral operator

$$\rho(f) = \int_G f(x)\rho(x)dx$$

acting on functions $\varphi$ in $L^2(\Gamma \backslash G)$ by

$$(\rho(f)\varphi)(x) = \int_G f(y)\varphi(xy)dy$$

$$= \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} f(x^{-1}\gamma y)\varphi(y)dy$$

$$:= \int_{\Gamma \backslash G} K(x, y)\varphi(y)dy$$

where $K(x, y)$ is called the kernel of the integral operator $\rho(f)$. The sum over $\gamma$ is finite, taken over the intersection of the discrete $\Gamma$ with the compact subset $x\text{supp}(f)y^{-1}$.
Remark 3.2. Note that functions that are convolutions of the form
\[ f(x) = f_0(x) * f_0(x^{-1}) \]
are positive definite, where \( f_0 \in C_0^\infty(G) \), so that for such functions the operator \( \rho(f) \) is self-adjoint and positive definite [GGPS69, §2.4], and as a consequence its restriction to any invariant subspace is also positive definite. This property shall be useful to us when studying the explicit formulae.

If the quotient \( \Gamma \setminus G \) compact, the representation \( \rho(g) \) decomposes into a countable discrete sum of irreducible unitary representations with finite multiplicity. The trace of the operator exists, and can be computed in two ways, giving the pre-trace formula
\[
\text{tr}(\rho(f)) = \sum_{\pi} m(\pi) \text{tr}(\pi(f)) = \sum_{\{\gamma\}} \text{vol}(\Gamma_\gamma \setminus G_\gamma) \int_{G_\gamma \setminus G} f(x^{-1} \gamma x) dx
\]
where the first sum is over irreducible constituents \( \pi \) of \( \rho \) appearing with multiplicity \( m(\pi) \), while the second sum is over conjugacy classes \( \gamma \) of \( \Gamma \). We denote by the centralizer of \( \gamma \) in a group by the subscript \( \gamma \), for example \( G_\gamma \). The left hand side is referred to as the spectral side, consisting of characters of representations; whereas the right hand side is the geometric side, consisting of orbital integrals.

If instead the quotient were noncompact but of finite volume, as are the cases we shall consider, by the theory of Eisenstein series the \( L^2 \)-spectrum of \( \Gamma \setminus G \) decomposes into an orthogonal sum of discrete and continuous parts, and the discrete spectrum decomposes further into a direct sum of cuspidal and non-cuspidal subspaces:
\[
L^2_{\text{disc}}(G) \oplus L^2_{\text{cont}}(G) = L^2_{\text{cusp}}(G) \oplus L^2_{\text{res}}(G) \oplus L^2_{\text{cont}}(G)
\]
where for short we have written \( L^2(G) \) for \( L^2(\Gamma \setminus G) \). The cuspidal spectrum consists of cusp forms, which are certain functions vanishing at cusps of the fundamental domain, while the continuous spectrum is a direct integral of Hilbert spaces associated to certain principal series representations, which we shall discuss below.

Eisenstein series enter into the picture as the continuous spectrum is described by the inner product of Eisenstein series, and the non-cuspidal discrete spectrum is spanned by the residues of Eisenstein series, given as above. To each inequivalent cusp, or equivalently, to each non-conjugate parabolic subgroup of \( G \), one associates a different Eisenstein series.

The representation \( \rho \) acting on the discrete spectrum again gives a trace class operator, and the pre-trace formula is valid in the form
\[
\sum_{\pi_{\text{disc}}} m(\pi) \text{tr}(\pi(f)) = \sum_{\{\gamma\}} \text{vol}(\Gamma_\gamma \setminus G_\gamma) \int_{G_\gamma \setminus G} f(x^{-1} \gamma x) dx - \int_{\Gamma \setminus G} K_{\text{cont}}(x,x) dx
\]
where the left-hand sum is over irreducible subquotients of \( \rho \) occurring in the discrete spectrum, and \( K_{\text{cont}}(x,x) \) is the kernel of operator \( \rho(f) \) restricted to the continuous spectrum. We shall refer to the contribution of \( K_{\text{cont}}(x,x) \) to the trace formula as the continuous spectral terms.

The remarkable insight of Selberg is that while the trace \( \text{tr}(\rho(f)) \) does not converge in general, arranging the trace formula as (3.2) the divergent terms on the geometric and spectral sides cancel such that the formula converges. Adapting this
method, Arthur introduced the truncation operator $\Lambda^T$ with respect to some parameter $T$, such that the integral $\Lambda^T K(x, x)$ converges for 'sufficiently regular' $T$. (See Definition 5.1.)

In what follows we will want to consider this term on its own, for in the continuous spectral terms we will find the $L$-functions which we are interested in. In particular, we only consider the noncompact, finite volume setting where the theory of Eisenstein series is required to treat the resulting continuous spectrum.

3.2. The trace formula. Let now $F$ be an arbitrary number field, and $A_F$ the ring of adeles of $F$. We consider the adelic quotient

$$SL_2(F) \backslash SL_2(A_F)/\prod_v K_v,$$

the product taken over all places $v$ of $F$, and $K_v$ is a sequence of maximal compact subgroups taken with the restricted direct product. As $F^\times$ embeds discretely into $A^\times_F$, so $SL_2(F)$ is a discrete subgroup of $SL_2(A_F)$.

We now turn our attention to the adelic form of the trace formula. The first instance of the adelic trace formula was derived by Jacquet and Langlands in [JL70, §16] for the group $GL(2)$, and shortly afterwards Labesse and Langlands [LL79, §5] considered the trace formula for certain subgroups

$$SL(2) \subset G' \subset GL(2)$$

which we define below. Their motivation was to extend the Jacquet-Langlands correspondence, leading to a stabilized form of the trace formula.

We first recall the contribution of the continuous spectrum to the trace formula for $G'$, which will specialize to $SL(2)$. In order to do so we introduce some notation and definitions.

Definition 3.3. Let $A = \prod_v A_v$ be a closed subgroup of the ring of ideles $A^\times_F$, such that

1. $A_v$ is a closed subgroup of $F_v^\times$,
2. $F^\times \cdot A$ is closed in $A^\times_F$,
3. If $B$ is an open subgroup then $[A : A^2(\cap F^\times)B]$ is finite.

Then we define the intermediate group

$$G'_A = \{ g \in GL_2(A_F) : \text{det}(g) \in A \},$$

which, as a side remark, may not be the $F$ points of an algebraic group. Furthermore, let $Z'_0$ be a closed subgroup of the center $Z'_A$ of $G'_A$ with $Z'_0 F^\times$ closed and $Z'_0 (Z'_A \cap F^\times)/Z'_A$ compact, and let $\omega$ be a character of $Z'_0$ trivial on $Z'_0 \cap F^\times$ and absolute value one.

Definition 3.4. Consider measurable functions $\varphi$ on $G'_A$ modulo $G'_F := G'_A \cap GL_2(F)$, satisfying

1. $\varphi(\omega^{-1}(z)g) = \varphi(zg)$ for all $z \in Z'_0$,
2. $\int_{Z'_0} |\varphi(g)|^2 dg < \infty$,

and denote the space of such functions $L^2(G'_F \backslash G'_A, \omega)$, and let $\rho$ be the regular representation of $G'_A$ on this space. By the theory of Eisenstein series (cf. [Lan89]), it decomposes into a direct sum of irreducible representations, and a continuous direct integral of irreducible representations constructed by Eisenstein series, that is, the principal series representations.
Definition 3.5. Now we define the principal series for $G'_A$. Let $A'$ be the group of diagonal matrices in $G'_A$ and $A'_F = A' \cap GL_2(F)$. We consider the set $D^0$ of characters of $\eta$ of $A'_F \backslash A'$ such that

$$\eta|_{Z_0'} = \omega^{-1},$$

and each $\eta$ is again defined by the pair $(\mu, \nu)$ of idele class characters on $F^\times \backslash A'_F$. We also have the analogous height function

$$(3.3) \quad H : \left( \begin{array}{cc} a & 0 \\ 0 & b \end{array} \right) \mapsto \left| \begin{array}{c} a \\ 0 \end{array} \right|,$$

where $| \cdot |$ is taken to be the adelic norm. Putting these together we obtain the representation $\rho(g, \eta, s)$ acting by right translation on the induced representation space

$$\text{Ind}_{N'A'}^G N \otimes (\eta \otimes H^{1+2s})$$

so that the resulting space is that of smooth functions $\varphi_s$ on $N_A \backslash G'_A$ satisfying

$$\varphi_s \left( \left( \begin{array}{cc} a & s \\ 0 & b \end{array} \right) k \right) = \mu(a)\nu(b) \left| \begin{array}{c} a \\ 0 \end{array} \right|^{1+2s} \varphi(k),$$

where $k$ is an element of $K' = \prod K'_v$ where $K'_v$ are maximal compact subgroups, taken to be $G'(O_F)$ for almost all $v$. By the Iwasawa decomposition

$$G' = N'A'K',$$

this space of functions can be identified with those on $K'$. Moreover, since $A' \backslash G'_A = A_{GL(2)} \backslash GL(2)$, with $A_{GL(2)}$ the diagonal matrices in $GL(2)$, we may regard the space of functions on which $\rho(g, \eta, s)$ acts as a space of functions on $GL(2)$ by extending $\eta$ trivially to $A_{GL(2)}$.

Definition 3.6. If $\eta$ is associated to the pair $(\mu, \nu)$, then the intertwining operator

$$(M(\eta, s)\varphi)(g) = \prod_v M(\eta_v) = \prod_v \int_{N_v} \varphi(wn_v g_v) dn_v$$

intertwines the principal series of $(\mu, \nu)$ with that of $(\nu, \mu)$. We normalize it as

$$M(\eta, s) = \frac{L(1-s, \mu^{-1}\nu)}{L(1+s, \mu\nu^{-1})} \otimes_v R(\eta_v, s)$$

where $R(\eta_v)$ denotes the normalized local intertwining operator. We will refer to the quotient of completed $L$-functions as the scalar factor, and denote it by $m(\eta, s)$. By the functional equation, we may also write

$$(3.4) \quad m(\eta, s) = \frac{L(s, \mu\nu^{-1})}{\epsilon(s, \mu\nu^{-1}, \psi)L(1+s, \mu\nu^{-1})},$$

where $\psi$ is a fixed additive character of $F \backslash A_F$.

Definition 3.7. Finally, we define the Eisenstein series

$$E(g, \varphi, s) = \sum_{\gamma \in P_F \backslash G_F} \varphi_s(\gamma g)$$

for $g \in G'_A$ and $s \in \mathbb{C}$. It converges absolutely for $\text{Re}(s) > 1$, and the constant term can be expressed as

$$\int_{N_F \backslash N_A} E(ng, \varphi, s) dn = \varphi_s(g) + (M(s)\varphi_s)(g),$$
involving the intertwining operator.

Now to specialize to $SL(2)$, we observe that any character on the diagonal can be identified with the character
\[
\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mapsto \mu \nu^{-1}(a) = \eta^2(a)|a|^s
\]
of the idele class group of $F$. This is an immediate consequence of the definitions above.

**Definition 3.8.** Now consider the regular representation $\rho$ of $G'_A$ acting on the discrete spectrum
\[ L^2_{\text{disc}}(G'_F \backslash G'_A, \omega) \]
and a smooth function $f = \prod f_v$ in $G'_A$ that is compact modulo $Z'_0$ such that
\[
f(zg) = \omega(z)f(g)
\]
for any $z$ in $Z'_0$, and for almost all $v$, $f_v$ is supported on $G'_F \cap GL_2(O_F)$. Define a convolution operator $\rho_0(f)$ acting on $L^2_{\text{disc}}(G'_F \backslash G'_A, \omega)$ by
\[
\rho_0(f)(\phi(x)) = \int_{Z'_0 \backslash G'_A} f(g) \rho_0(g) \phi(x) dg.
\]
It is a Hilbert-Schmidt operator, and in particular trace class.

The trace formula now expresses the trace $\text{tr}(\rho_0(f))$ in two ways, first as a sum of characters of representations, and second as a sum of orbital integrals. We examine the portion of the trace formula arising from the noncuspidal spectrum of $L^2(SL_2(F) \backslash SL_2(A_F), \omega)$. For simplicity, we will sometimes assume that the central character $\omega$ is trivial.

**Theorem 3.9.** Let $f$ and $\rho$ be defined as above. Then the contribution of the continuous spectrum to the trace formula for $SL_2(A_F)$ are obtained by specializing the terms (5.5) and (5.6) in [LL79, p.754]:

\[
\sum_{\eta \in D^0} \frac{1}{4} \frac{1}{m(\eta, s)^{-1} m'(\eta, s)} \text{tr}(\rho(f, \eta, s))|ds|,
\]
the sum taken over characters in $D^0$ of terms involving the logarithmic derivative of the scalar factor of $M(s)$, and

\[
\sum_{\eta} \sum_v \frac{1}{4\pi} \frac{1}{m(\eta_v, s)^{-1} m'(\eta_v, s)} \text{tr}(R^{-1}(\eta_v, s)R'(\eta_v, s) \rho(f_v, \eta_v, s)) \prod_{w \neq v} \text{tr}(\rho(f_w, \eta_w, s))|ds|
\]
where $R^{-1}(\eta_v, s)$ is understood as the inverse operator of $R(\eta_v, s)$, and only finitely many nonzero terms appearing in each sum.
Remark 3.10. For comparison, we mention that these terms correspond to (vi), (vii), and (viii) in Jacquet-Langlands [JL70, §16], with the corrections indicated in [LL79, p.753]. If $\eta$ is trivial, then the quotient
\[
\lim_{s \to 0} \frac{L(1-s, \mu^{-1})}{L(1+s, \mu^{-1})} = -1
\]
and $R(\eta_v, 0) = 1$ for all $v$, and in fact the associated distribution in (3.5) is stable. If $\eta$ is a nontrivial quadratic character, then the scalar factor is equal to 1. Indeed, if $\eta_v = \eta_v^{-1}$, then $R(\eta_v, 0)$ intertwines the representation $\rho(g, \eta_v, 0)$ with itself, hence is the identity (cf. [LL79, Lemma 3.5]).

3.3. The test function. The following lemma shows that the character of the induced representation $\rho(f, \eta, s)$ defines the Mellin transform of a smooth compactly supported function on $R^+$. 

Lemma 3.11 (Test function). Let $f$ be a function in $C^\infty_c(Z_0' \setminus G_\Lambda')$. Then the character
\[
\text{tr}(\rho(f, \eta, s))
\]
is a Mellin transform of a function $g$ in $C^\infty_c(R^+ \times \mathbb{Z})$.

Proof. Using the Iwasawa decomposition, the trace $\text{tr}(\rho(f, \eta, s))$ is equal to
\[
\int_{K'} \int_{R^+_\mathbb{A}} \int_{\mathbb{A}^\times} \int_{N_\Lambda} f(k^{-1}a^{-1}n a^{-1}(t_0 t^{-1}) k)|t|^{1+s}dn \, da \, d^x t \, dk,
\]
for if we interchange $\nu = (1 n 1 \ a \ b)(1 0 1) = an'$ the measure on $N_\Lambda$ is multiplied by an element of norm 1, thus remains the same. Doing so, we obtain
\[
\text{vol}((\mathbb{A}^x)^1) \int_K \int_{R^+_\mathbb{A}} f(k^{-1}n(t_0 t^{-1}) k)|t|^{1+s}dn \, d^x t \, dk.
\]
Now denote the integration over the compact set $K'$ by an auxiliary function
\[
\Phi(g) = \int_{K'} f(k^{-1}gk)dk
\]
which allows us to write
\[
\text{vol}((\mathbb{A}^x)^1) \int_{R^+_\mathbb{A}} \int_{N_\Lambda} \Phi(n(t_0 1)|t|^{1+s}dn \, d^x t
\]
and denote by $S$ the Satake transform
\[
S\Phi(t) = \prod_v H_v(t_v)^{1/2} \int_{N_v} \Phi(n_v(t_v 0 1))dn_v
\]
with $H_v(t_v)^{1/2} = |t_v|$ the usual modulus character of $P = M \ltimes N$, related to the height function $H$ defined in (3.3), so that one has
\[
\text{vol}((\mathbb{A}^x)^1) \int_{R^+_\mathbb{A}} S\Phi(t)|t|^s d^x t
\]
Note that while we recognize the integral as the Satake transform, we are not in fact using the Satake isomorphism at unramified places. Finally, setting \( g = S\Phi \), we arrive at the Mellin transform
\[
\hat{g}(s) = \int_{\mathbb{R}_+} g(t) t^s dt = \int_0^\infty g(t) t^{s-1} dt,
\]
if we normalize measures such that \( \text{vol}((A^\times)^1) = 1 \). The condition is not a serious one; indeed, for our purposes it is enough to know that the measure is nonzero and finite, which is certainly true. (This is related to the constant \( c \) in \([JL70, \S 16]\).) So given a test function \( f \), we may view the trace \( \text{tr}(\rho(f, \eta, s)) \) as the Mellin transform \( \hat{g}(s) \) of a smooth compactly supported function defined on the positive real numbers. \( \square \)

**Remark 3.12.** It turns out that all functions \( g \) in \( C^\infty_c(\mathbb{R}^+ \times \mathbb{R}^+) \) can be obtained from \( f \) in this manner. For example, if \( G = SL_2(\mathbb{R}) \) and \( K = SO_2 \), given an element \( f \) of the spherical Hecke algebra \( C^\infty_c(K\backslash G/K) \) we have the Harish transform
\[
H(a)^{\frac{1}{2}} \int_{\mathbb{R}} f(an) du = |H(a)^{\frac{1}{2}} - H(a)^{-\frac{1}{2}}| \int_{A\backslash G} f(x^{-1}ax) dx.
\]
where \( A \) is the Cartan subgroup. The transform is invariant under the action of the Weyl group \( W \) of \( G \), and \( A/W \) can be represented by matrices \( a \) in \( A \) such that \( H(a) \geq 1 \). Thus changing variables \( y = e^u \) so that \( H(a) = e^{2u} \), we may write additively
\[
g(u) = |e^u - e^{-u}| \int_{A\backslash G} f(x^{-1} \begin{pmatrix} e^u & 0 \\ 0 & e^{-u} \end{pmatrix} x) dx,
\]
and observe that \( g(u) = g(-u) \). The Harish transform is an algebra isomorphism from the space \( C^\infty_c(K\backslash G/K) \) to \( C^\infty_c(A)^W \), where the superscript indicates functions invariant under the Weyl group \( W \), and the product given by convolution. Over a \( p \)-adic field, its analog is the Satake isomorphism.

Also, we note image of the Mellin transform defined for \( g \) in \( C^\infty_c(A)^W \),
\[
\hat{g}(s) = \int_{A} g(a) H(a)^s da,
\]
lies in the Paley-Wiener space, consisting of entire functions \( f \) for which there exists positive constants \( C \) and \( N \) such that
\[
|f(x + iy)| \ll C^{|x|}(1 + |y|)^{-N},
\]
which is to say \( f \) has at most exponential growth with respect to \( x \) and is uniformly rapidly decreasing in vertical strips.

### 4. The explicit formulae in trace formulae

We are now in a position to relate the explicit formulae to trace formulae. In this chapter we will show: (1) that the sums of zeroes of certain \( L(s, \chi) \) appear in the continuous spectral terms of the trace formula of \( SL_2 \), through the logarithmic derivative of the intertwining operator, first in the classical case of the Riemann zeta function and then in the general adelic setting, and (2) that this latter term can be rewritten as an expression similar to the distributions appearing in Weil’s explicit formula for Artin-Hecke \( L \)-functions \([Wei72]\).
4.1. **Sums of zeroes in the continuous spectral terms.** To illustrate the method, we first consider the classical trace formula for $SL_2(\mathbb{R})$ without ramification. The proof in the adelic setting will be similar. In particular, we consider $\rho_0$ the regular representation of $SL_2(\mathbb{R})$ on $L^2(SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/SO_2)$. In this case, the only character $\eta$ that appears is the trivial one, and the intertwining operator is

$$m(s) = \frac{\xi(s)}{\xi(1+s)}$$

Let us examine the contribution of (3.6), which involves the logarithmic derivative of $m(s)$. We can now prove Theorem 1.1 given in the Introduction.

**Proof of Theorem 1.1.** We begin with the term

$$-\frac{1}{4\pi i} \int_{-i\infty}^{i\infty} \frac{m'(s)}{m(s)} \hat{g}(s) ds.$$ 

By the functional equation, we express the logarithmic derivative of the scalar factor as

$$\frac{m'}{m}(s) = \frac{1}{2} \frac{\Gamma'(s)}{\Gamma(s)} + \frac{\zeta'}{\zeta}(s) - \frac{1}{2} \frac{\Gamma'(1+s)}{\Gamma(1+s)} - \frac{\zeta'}{\zeta}(1+s)$$

$$= \frac{1}{2} \frac{\Gamma'(s)}{\Gamma(s)} + \frac{\zeta'}{\zeta}(s) - \log \pi + \frac{1}{2} \frac{\Gamma'(\frac{s}{2})}{\Gamma(\frac{s}{2})} - \frac{\zeta'}{\zeta}(s) + \frac{\zeta'}{\zeta}(-s).$$

Then we substitute this into our integral above,

$$-\frac{1}{4\pi i} \int_{-i\infty}^{i\infty} \left\{ \frac{1}{2} \frac{\Gamma'(s)}{\Gamma(s)} + \frac{\zeta'}{\zeta}(s) - \log \pi + \frac{1}{2} \frac{\Gamma'(\frac{s}{2})}{\Gamma(\frac{s}{2})} - \frac{\zeta'}{\zeta}(s) \right\} \hat{g}(s) ds$$

$$= -\frac{1}{4\pi i} \int_{-i\infty}^{i\infty} \left\{ \text{Re} \left[ \frac{\Gamma'(s)}{\Gamma(s)} \right] + 2 \text{Re} \frac{\zeta'}{\zeta}(s) - \log \pi \right\} \hat{g}(s) dr$$

using the property that $\hat{g}(ir)$ is even. Applying Mellin inversion to the last term, we obtain

$$\frac{1}{2} \hat{g}(1) \log \pi - \frac{1}{4\pi i} \int_{-i\infty}^{i\infty} \text{Re} \left[ \frac{\Gamma'(s)}{\Gamma(s)} \right] \hat{g}(s) ds - \frac{1}{2\pi i} \int_{-\infty}^{i\infty} \text{Re} \left[ \frac{\zeta'}{\zeta}(s) \right] \hat{g}(s) ds.$$

To treat the third term, we use a variant of the method used, for example, in Ingham [Ing32, Theorem 29, p.77], and similar to the derivation of (2.3): Move the line of integration to the right to $(c-i\infty, c+\infty)$ for some $c > 1$. This step is justified by integrating over a rectangle $R$ with vertices $(c\pm iT, c\pm iT)$ and showing that the integral over the horizontal edges tends to 0 as we let $T$ tend to infinity along a well-chosen sequence $(T_m)$, $m = 2, 3, \ldots$ such that $m < T_m < m + 1$ and

$$\frac{\zeta'}{\zeta}(s) = O(\log^2 T)$$

for any $s$ with $-1 \leq \sigma \leq 2$ and $T = T_m$, and as $T$ tends to infinity (see [Ing32, Theorem 26, p.71]). Allowing then $t$ to tend to infinity via the sequence $(T_m)$, the contribution from the horizontal edges vanish, as the transform $\hat{g}(s)$ of a smooth, compactly supported function has rapid decay along a fixed vertical strip.

Moving the line of integration to the left, we obtain the residues of $\zeta'/\zeta(s)$ due to the zeroes of $\zeta(s)$ inside the critical strip $0 < \sigma < 1$, and the simple poles of $\zeta(s)$
at $s = 1$. Then allowing $T$ to go to infinity we obtain
\[-\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\zeta'(s)}{\zeta(s)} \hat{g}(s) ds = \sum_{\gamma} \hat{g}(\rho) - \hat{g}(1) - \frac{1}{2\pi i} \int_{(c)} \frac{\zeta'/\zeta(s)}{s} \hat{g}(s) ds \]
where the sum $\rho = \beta + i\gamma$ runs over the zeroes of $\zeta(s)$ with $0 < \beta < 1$. The integral now being in the region of absolute convergence, we may integrate term by term and apply Mellin inversion,
\[-\frac{1}{2\pi i} \int_{(c)} \frac{\zeta'(s)}{\zeta(s)} \hat{g}(s) ds = \sum_{n=1}^\infty \Lambda(n) \frac{1}{2\pi i} \int_{(c)} \hat{g}(s) n^{-s} ds = \sum_{n=1}^\infty \Lambda(n) g(n).\]

So we have the expression for (4.1)
\[
(4.2) \sum_{\gamma} \hat{g}(\rho) - \hat{g}(1) + \sum_{n=1}^\infty \Lambda(n) g(n) + \frac{1}{2} \hat{g}(1) \log \pi - \frac{1}{4\pi i} \int_{-i\infty}^{i\infty} \text{Re} \left[ \frac{\Gamma'(s)}{\Gamma(s/2)} \right] \hat{g}(s) ds.
\]

To treat the last integral, we first observe that
\[
\Gamma'(s)/\Gamma(s) = \log s - \frac{1}{2} + O\left(\frac{1}{|s|^2}\right)
\]
uniformly in any fixed angle $|\arg(s)| < \pi$ as $|s|$ tends to infinity. Now we move the line integration to the line $\text{Re}(s) = \frac{1}{4}$,
\[-\frac{1}{4\pi i} \int_{-i\infty}^{i\infty} \text{Re} \left[ \frac{\Gamma'(s)}{\Gamma(s/2)} \right] \hat{g}(s) ds = -\frac{1}{4} \hat{g}(0) - \frac{1}{4\pi i} \int_{(4)} \text{Re} \left[ \frac{\Gamma'(s)}{\Gamma(s/2)} \right] \hat{g}(s) ds.
\]
Here one half the residue of the pole of $\Gamma(s)$ at $s = 0$ is obtained as the initial line of integration is over $s = 0$. Then from the proof of Theorem 2.3, with $M = 2$ and $a, b = 0$ we can rewrite this as
\[-\frac{1}{4} \int_0^\infty g^\ast(x) dx + \frac{1}{2} (\log 4 + \gamma) g(1) + \int_1^\infty \left\{ g(x) + g^\ast(x) - \frac{2}{x} g(1) \right\} \frac{x dx}{2(x^2 - 1)},\]
(see also [Bom00, p.190]). Then substituting this last expression into (4.2) proves the claim.

Thus we see that the sum over zeroes as in Weil’s explicit formula appear in the continuous spectral terms of the trace formula for $SL_2(\mathbb{R})$. Having treated the basic case, we now discuss the adelic case: recall the term (3.6),
\[
\sum_{\eta} \frac{1}{4\pi} \int_{-i\infty}^{i\infty} m(\eta, s)^{-1} m'(\eta, s) \text{tr} \rho(f, \eta, s) ds
\]
where the sum $\eta$ is over idele class characters of $F$, $f$ is a function in $C_c^\infty(G'_{\mathbb{A}})$, and $m(\eta, s)$ is defined in (3.4). As in the previous theorem, we show that this term produces sums over zeroes of Hecke $L$-functions, providing the connection to explicit formulae.
Theorem 4.1. Fix an idele class character \( \eta \) of \( F \), and let \( \eta^2 = \chi \). With definitions given as above, the integral

\[
- \frac{1}{4\pi} \int_{-i\infty}^{i\infty} m(\eta, s)^{-1} m'(\eta, s) \text{tr}(\rho(f, \eta, s))ds
\]

is equal to

\[
\sum_{\rho, \rho'} \frac{1}{2} \{ \hat{g}(\rho) + \hat{g}(-\overline{\rho}) \} + \ldots
\]

the sum taken over zeroes of \( L(s, \chi) \) and \( L(s, \overline{\chi}) \) with real part between 0 and 1, and

\[
\ldots + \frac{1}{2} \sum_{a} \Lambda(a) \{ \chi(a) g(N(a)) + \overline{\chi}(a) g^*(N(a)) \} + \log(Nf(\chi)|d_F|)g(1)
\]

\[- \sum_{a \leq 0} \int_{0}^{\infty} g(x)(\delta x + x^{w-1})dx - \sum_{k=1}^{r_1 + r_2} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \text{Re} \left[ \Gamma_{k} \left( \frac{1}{2} + it + w \right) \right] \hat{g}(\frac{1}{2} + it)dt
\]

where the sum is over archimedean places \( v \) where \( \chi_v \) is ramified with ramification \( w = a + ib, a \leq 0 \). Moreover, the last term can be expressed as a function of \( g(x) \) only, the same as in Theorem 2.3.

Proof. The method of proof is similar to that of the previous theorem. By the functional equation, we have

\[
m(\eta, s) = \frac{L(s, \chi)}{\epsilon(s, \chi, \psi)L(1+s, \chi)} = \frac{\epsilon(-s, \overline{\chi}, \overline{\psi})}{\epsilon(s, \chi, \psi)} L(s, \chi),
\]

where \( \psi \) is a fixed additive character of \( F \). Its logarithmic derivative is

\[
- \frac{\epsilon'}{\epsilon}(-s, \overline{\chi}, \overline{\psi}) - \frac{\epsilon'}{\epsilon}(s, \chi, \psi) + \frac{L'}{L}(s, \chi) + \frac{L'}{L}(-s, \overline{\chi}).
\]

Writing \( \hat{g}(s) = \text{tr}(\rho(f, \chi, s)) \), the integral now becomes

\[
\frac{1}{4\pi i} \int_{-i\infty}^{i\infty} \left\{ \frac{\epsilon'}{\epsilon}(-s, \overline{\chi}, \overline{\psi}) + \frac{\epsilon'}{\epsilon}(s, \chi, \psi) - \frac{L'}{L}(s, \chi) - \frac{L'}{L}(-s, \overline{\chi}) \right\} \hat{g}(s)ds.
\]

We first treat the epsilon factors. Recall from Definition 2.1 that

\[
\epsilon(s, \chi, \psi) = W(\chi)|N(f(\chi))d_F|^{s-\frac{1}{2}}
\]

where the global Artin conductor \( f(\chi) = \prod p_v^\epsilon(\chi) \) is a product over all primes \( p_v \) which ramify in \( F \), of local conductors \( f_v(\chi) \) of the local character \( \chi_v \). Then

\[
\frac{1}{4\pi i} \int_{-i\infty}^{i\infty} \left\{ \frac{\epsilon'}{\epsilon}(-s, \overline{\chi}, \overline{\psi}) + \frac{\epsilon'}{\epsilon}(s, \chi, \psi) \right\} \hat{g}(s)ds
\]

\[
= \frac{1}{4\pi i} \left( \log(Nf(\chi)|d_F|) + \log(Nf(\overline{\chi})|d_F|) \right) \int_{-i\infty}^{i\infty} \hat{g}(s)ds
\]

\[
= \log(Nf(\chi)|d_F|)g(1),
\]

since \( f(\chi) = f(\overline{\chi}) \). Note that the additive character \( \psi \) is chosen as usual to give the self-dual measure on \( F \).
Now for the $L$-functions. Denoting by $w$ the ramification of $\chi_{v_k}$ at archimedean places (as in Definition 2.2), the logarithmic derivative is

$$\frac{L'}{L}(s, \chi) + \frac{L'}{L}(-s, \overline{\chi}) = \sum_{k=1}^{r_1 + r_2} \left\{ \frac{\Gamma'}{\Gamma_k}(s + w) + \frac{\Gamma'}{\Gamma_k}(-s + \overline{w}) \right\} + \frac{\zeta'}{\zeta}(s, \chi) + \frac{\zeta'}{\zeta}(-s, \overline{\chi}).$$

We then separate the integral, into the logarithmic derivatives of $\zeta_F(s)$ and $\Gamma_k(s)$ respectively. Consider first the zeta functions. Following the proof of Theorem 2.3 and Theorem 1.1, we move the line of integration to $c > 1$,

$$-\frac{1}{4\pi i} \int_{-i\infty}^{i\infty} \left\{ \frac{\zeta'}{\zeta_F}(s, \chi) + \frac{\zeta'}{\zeta_F}(-s, \overline{\chi}) \right\} \hat{g}(s) ds$$

picking up the residues of $\zeta_F(s, \chi)$ and $\zeta_F(-s, \overline{\chi})$ with $0 < \text{Re}(\rho) < 1$. Once again this step is justified by integrating over a box of height $T_m$, where $(T_m)$ is a suitably-chosen sequence so as to avoid crossing any zeroes. If $\chi$ is trivial, then so is $\overline{\chi}$ and hence $L(s, 1) = \zeta_F(s)$ has a pole at $s = 1$, contributing

$$-\hat{g}(1) = -\int_{0}^{\infty} g(x) dx.$$

The integral now being in the region of absolute convergence, we utilize the Euler product expansion of the $L$-function,

$$-\frac{1}{4\pi i} \int_{-i\infty}^{i\infty} \sum_{a} \left\{ \frac{\Lambda(a)\chi(a)}{N(a)^s} + \frac{\Lambda(a)\overline{\chi}(a)}{N(a)^{-s}} \right\} \hat{g}(s) ds$$

$$= \frac{1}{2} \sum_{a} \Lambda(a) \left\{ \chi(a)g(N(a)) + \overline{\chi}(a)g^*(N(a)) \right\}.$$

Thus our final expression for (4.6) is

$$\sum_{\rho} \frac{1}{2} \{ \hat{g}(\rho) + \hat{g}(-\rho') \} - \delta_\chi \int_{0}^{\infty} g(x) dx$$

$$+ \frac{1}{2} \sum_{a} \Lambda(a) \left\{ \chi(a)g(N(a)) + \overline{\chi}(a)g^*(N(a)) \right\},$$

where as before $\delta_\chi$ is 1 if $\chi$ is trivial and zero otherwise.

Finally, for the gamma factors, we begin with:

$$-\sum_{k=1}^{r_1 + r_2} \frac{1}{4\pi i} \int_{-i\infty}^{i\infty} \left\{ \frac{\Gamma'}{\Gamma_k}(s + w) + \frac{\Gamma'}{\Gamma_k}(-s + \overline{w}) \right\} \hat{g}(s) ds$$

$$= -\sum_{k=1}^{r_1 + r_2} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \text{Re} \left[ \frac{\Gamma'}{\Gamma_k}(s + w) \right] \hat{g}(s) ds,$$
since \( s \) here is pure imaginary. We now move the line of integration to \( \text{Re}(s) = \frac{1}{2} \),
encountering residues of poles of the gamma functions in the following scenarios: Recall that if \( w = a + ib \),
then for real places we have \( a = 0 \) or \( 1 \), whereas for complex
places we have \( a \in \mathbb{Z}[\frac{1}{2}] \). Since the poles of \( \Gamma(s) \) occur at \( s = 0, -1, -2, \ldots \),
we see that the integral of \( \Gamma_k/(\Gamma_k(s + w)) \) passes through a pole only if \( a \leq 0 \). Thus,

\[
- \sum_{a \leq 0} \tilde{g}(s + w) - \sum_{k=1}^{r_1 + r_2} \frac{1}{2\pi i} \int_{(\frac{1}{2})} \text{Re} \left[ \frac{\Gamma_k'}{\Gamma_k}(s + w) \right] \tilde{g}(s) ds,
\]

(4.8) \[= - \sum_{a \leq 0} \int_0^\infty g(x)x^{w-1} dx - \sum_{k=1}^{r_1 + r_2} \frac{1}{2\pi} \int_{-\infty}^\infty \text{Re} \left[ \frac{\Gamma_k'}{(1/2 + it + w)} \right] \tilde{g}(1/2 + it) dt,
\]

and the integral in this last expression is exactly the one studied previously in Theorem 2.3. Then putting these together with (4.5) and (4.7) proves the claim. \( \square \)

4.2. Weil distributions in the continuous spectral terms. Having shown
that the sum over zeroes appear in the contribution of the continuous spectrum
to the trace formula, we relate this to the distributions arising in Weil’s explicit
formula \( \text{(Weil72, p.18)} \). We will say the resulting distributions are of
Weil-type, for their resemblance to those appearing in the Weil formula (2.9); in our theorem
there appears a difference of certain exponents of \( \frac{1}{2} \), this is due to our choice of
normalization.

The rest of this chapter will be devoted to the proof of Theorem 1.2. We first
collect several lemmas, from which the theorem will follow quickly.

**Lemma 4.2** (\( \text{Weil72, p.13} \)). Let \( g \) be a function in \( C^\infty(\mathbb{R}_+^\times) \), and \( \tilde{g} \) its Mellin transform. Assume that there exists an \( A > \frac{1}{2} \) such that \( g(t) = O(t^A) \) as \( t \) tends to
0 and \( g(t) = O(t^{-A}) \) as \( t \) tends to infinity. Then for any \( \sigma \) such that \( |\sigma - \frac{1}{2}| < A \),
the following formula holds:

\[
\int_{\mathbb{R}_+^\times} \tilde{g}(\sigma - \frac{1}{2} + it) X^{\sigma + it} dt = 2\pi X^{\frac{1}{2}} g(X)
\]

by Mellin inversion.

**Proof.** The growth assumption on \( g \) implies that the transform \( \tilde{g}(\sigma - \frac{1}{2} + it) \) is
holomorphic in the region \( |\sigma - \frac{1}{2}| < A \), so that the inversion formula is independent
of \( \sigma \) in this range. \( \square \)

We combine this lemma with the next one:

**Lemma 4.3.** Let \( v \) be a nonarchimedean completion of \( F \), and \( q_v \) the cardinality
of the residue field of \( F_v \). Then

\[
\frac{d}{ds} \log L_v(s,\chi) = - \log q_v \sum_{n=1}^\infty q_v^{-ns} \int_{W_v^0} \chi_v(f^n w_0) dw_0,
\]

where \( f \) is a Frobenius element in \( W_{F_v} \).

**Proof.** We first justify interchanging the sum and integral: the character \( \eta_v \)
is assumed to be unitary, that is \( |\chi(w_0)| \leq 1 \) for all \( w_0 \) in \( W_v^0 \), so the sum converges
absolutely for \( \text{Re}(s) > 1 \), then apply Proposition 1 of [\( \text{Weil72, p.12} \)] \( \square \)

The following lemmas will allow us to treat the archimedean places:
Lemma 4.4 (Gauss-Weil identity). Let \(\text{Re}(s) > 0\). Then

\[-2 \frac{\Gamma'}{\Gamma}(s) = pv \int_0^\infty \frac{f_0(x)^{2s-1}}{f_1(x)} d^x x\]

with the definitions as above.

Proof. This statement is given in [Wei72, p.16], proved in [Mor05, p.160-162]. □

The next is a variant of that used in Weil’s explicit formula:

Lemma 4.5. The integral

\[
\frac{1}{2\pi i} \int \hat{g}(s - \frac{1}{2}) d \log \Gamma_k(\frac{1}{2} + s + a + ib) + \hat{g}(\frac{1}{2} - s) d \log \Gamma_k(\frac{1}{2} + s + a - ib)
\]

with \(a \geq 0, b \in \mathbb{R}\), taken over the line \(\text{Re}(s) = \frac{1}{2}\), can be expressed as

\[
- pv_0 \int_0^\infty g(\nu) \nu^{\frac{1}{2} + a + ib} f_0(\nu)^{2a+1-E} f_1(\nu^E) d^\times \nu
\]

where \(E = \dim_{F \otimes \mathbb{C}} C\), which is to say \(E = 2\) if \(F \otimes \mathbb{R} \simeq \mathbb{R}\) and \(E = 1\) if \(F \otimes \mathbb{C} \simeq \mathbb{C}\). Note that here we have used \(a + ib\) for \(w\) as in Remark 2.2.

Proof. A detailed proof of Weil’s statement is supplied in [Mor05, p.162-174]. We sketch the proof, indicating the necessary modifications. The transform \(\hat{h}(s)\) used in [Wei72, Mor05] is a shifted Mellin transform:

\[
\hat{h}(s) = \int_0^\infty h(\nu) \nu^{\frac{1}{2} - s} d^\times \nu,
\]

which relates to our function \(\hat{g}(s)\) by the relation

\[
\hat{h}(s) = \hat{g}(1 - s) = \hat{g}(s - \frac{1}{2}).
\]

The integral (4.9) is

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(i r) \Gamma_k^\prime(1 + i r + a + ib) + \hat{g}(-i r) \Gamma_k^\prime(1 + i r + a - ib) dr.
\]

We consider the two cases, first where \(k = \mathbb{R}\) the integral becomes after a change of variables \(r \mapsto r - b\):

\[-g(1) \log \pi + \frac{1}{4\pi} \int_{-\infty}^{\infty} \hat{g}(i(r - b))(\frac{\Gamma'}{\Gamma}(\frac{1 + i r + a}{2}) + \frac{\Gamma'}{\Gamma}(\frac{1 - i r + a}{2}) dr,
\]

where the first term follows from Mellin inversion. Considering next the integral, we shift the contour from \(ir\) to \(-\frac{1}{2} + ir\), noting that \(g(s)\) and \(\Gamma'/\Gamma(s)\) are holomorphic in this range. As a result we obtain

\[
\frac{1}{4\pi} \int_{-\infty}^{\infty} \hat{g}(\frac{1}{2} + i(r - b))(\frac{\Gamma'}{\Gamma}(\frac{1}{2} + i r + a}{2}) + \frac{\Gamma'}{\Gamma}(\frac{1}{2} - i r + a}{2}) dr;
\]

also apply the Gauss-Weil formula of Lemma 4.4 to get

\[
\frac{\Gamma'}{\Gamma}(\frac{1}{2} + i r + a}{2}) + \frac{\Gamma'}{\Gamma}(\frac{1}{2} - i r + a}{2}) = -pv \int_0^\infty \frac{f_0(\nu)^{\frac{1}{2} + a}}{f_1(\nu^E)} \nu^{ir/2} d^\times \nu.
\]
Making the change of variables $\nu \mapsto \nu^2$, the integral then becomes after interchanging the order of integration:

$$-rac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(-\frac{1}{2} + i(r-b)) \cdot \text{pv} \int_0^{\infty} \frac{f_0(\nu)^{2a-1}}{f_1(\nu^2)} \nu^{ir} d^\times \nu \; dr$$

$$= -\text{pv} \int_0^{\infty} \frac{f_0(\nu)^{2a-1}}{f_1(\nu^2)} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(-\frac{1}{2} + i(r-b)) \nu^{ir} dr \; d^\times \nu.$$ 

Then applying Lemma 4.2 to the inner integral, we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(-\frac{1}{2} + i(r-b)) \nu^{ir} dr = \nu^{\frac{1}{2}+ib} g(\nu).$$

Putting it together we obtain the desired form of (4.10)

$$-\text{pv} \int_0^{\infty} \frac{f_0(\nu)^{2a-1}}{f_1(\nu^2)} \nu^{\frac{1}{2}+ib} g(\nu) d^\times \nu.$$

Now for the second case where $k = \mathbb{C}$, we begin as before with the expression for (4.9),

$$-2g(1) \log(2\pi) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(-\frac{1}{2} + i(r-b)) (\frac{\Gamma'}{\Gamma}(\frac{1}{2} + a + ir) + \frac{\Gamma'}{\Gamma}(\frac{1}{2} + a - ir)) dr.$$ 

Applying Mellin inversion to get the first term, and shifting contours in the integral as before. Apply again the Gauss-Weil formula of Lemma 4.4 to the Gamma factors,

$$\frac{\Gamma'}{\Gamma}(\frac{1}{2} + a + ir) + \frac{\Gamma'}{\Gamma}(\frac{1}{2} + a - ir) = -\text{pv} \int_0^{\infty} \frac{f_0(\nu)^{2a}}{f_1(\nu)} \nu^{ir} d^\times \nu$$

and substitute back to the original expression, then interchange the orders of integration:

$$-\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(-\frac{1}{2} + i(r-b)) \cdot \text{pv} \int_0^{\infty} \frac{f_0(\nu)^{2a}}{f_1(\nu)} \nu^{ir} d^\times \nu \; dr$$

$$= -\text{pv} \int_0^{\infty} \frac{f_0(\nu)^{2a}}{f_1(\nu)} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(-\frac{1}{2} + i(r-b)) \nu^{ir} dr \; d^\times \nu.$$ 

Finally, apply again Lemma 4.2 to the inner integral

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(-\frac{1}{2} + i(r-b)) \nu^{ir} dr = \nu^{\frac{1}{2}+ib} g(\nu),$$

then putting it together we obtain the desired form of (4.10)

$$-\text{pv} \int_0^{\infty} \frac{f_0(\nu)^{2a}}{f_1(\nu^2)} \nu^{\frac{1}{2}+ib} h(\nu) d^\times \nu.$$

We apply this lemma to obtain an integral over the archimedean Weil group,

**Corollary 4.6** (Archimedean contribution). Let $\chi_v$ be a character of the local archimedean Weil group $W_v$, then the expression (4.10) can be rewritten as

$$(4.11) \quad -\text{pv} \int_{W_v} g(|w|) \chi_v(w) \frac{|w|_v}{|1-w|_v} dw.$$
Proof. The proof of this follows immediately from (4.10) by Lemma 3 of [Mor05, p.165], with the function $\varphi$ of the lemma replaced with

$$\varphi(w) = \frac{|w|_v}{|1 - w|_v}.$$ 

In particular, in the numerator we have $|w|_v$ instead of $|w_v|^2$. □

**Lemma 4.7** (Contribution of conductor). Consider the Herbrand distribution $H_v$ on the local Weil groups $W_v$, which is described in [Wei74, Ch.VIII, §3, XII, §4] and [Mor05, Ch.II, §6]. It is given by

$$H_v(\chi) = \int_{W^0_v} \chi_v(w_0)dH_v(w_0)$$

where $\chi_v$ is a character of the restriction to $W_v$ of a unitary representation of the Weil group $W_F$. The contribution of the conductor can be expressed as

$$g(1) \log |Nf| = -\sum_v \log q_v \int_{W^0_v} g(|w_0|)\chi_v(x)dH_v(w_0).$$

Proof. We follow [Wei72, p.17] and [Mor05, p.158]. The local Artin conductor $f_v(\chi)$ can be written as the integral

$$\int_{W^0_v} \chi(x)dH_v(x),$$

so that the contribution from the conductor for each place $v$ where $p_v$ ramifies is

$$\log p_v \int_{W^0_v} h(|w_0|)\chi_v(w_0)dH_v(w_0),$$

and zero otherwise.

By the product formula for the conductor, the contribution of the conductor is then

$$g(1) \log |Nf| = \sum_v g(1) \log (q_v^{-H_v(\chi)})$$

$$= -g(1) \sum_v (\log q_v)H_v(\chi)$$

$$= -g(1) \sum_v \log q_v \int_{W^0_v} \chi_v(w_0)dH_v(w_0)$$

$$= -\sum_v \log q_v \int_{W^0_v} g(|w_0|)\chi_v(w_0)dH_v(w_0).$$

as desired. □

Using this expression we are able to express the archimedean and nonarchimedean integrals in a uniform manner.

**Definition 4.8.** Let $g$ be a locally constant function on $W_v$ and suppose that the integral

$$\int_{W^0_v} g(w)\frac{dw}{|1 - w|}$$
exists. Then define the ‘principal value’ by

$$p v_0 \int_{W_v} \frac{g(w)}{|1 - w|} dw = \int_{W_v} \frac{g(w) - g(1)}{|1 - w|} dw + \int_{W - W_v} \frac{g(w)}{|1 - w|} dw.$$ 

Then the following is given in [Wei72, p.18].

**Corollary 4.9 (Nonarchimedean contribution).** The contribution of the nonarchimedean integrals and the conductor can be combined to obtain

$$pv_0 \int_{W_v} \hat{g}(|w|) \chi_v(w) \frac{|w|}{|1 - w|} dw.$$ 

where $1 - w$ and $|1 - w|$ are to have the same sense by the embedding of $W_v$ into the division algebra $A_v$, which follows from Shafarevitch (see [Mor83, Part I, Chap. VIII]).

Finally, we put all these together to prove the main theorem:

**Proof of Theorem 1.2.** First, from the logarithmic derivative of $m(\eta, s)$ as in (4.4), apply Lemma 4.7 to the contribution of the epsilon factors (4.5),

$$g(1) \log |d_F| + g(1) \log |N\eta| = g(1) \log |d_F| - \sum_v \log q_v \int_{W_v} g(|w_0|) \chi_v(w_0) dH_v(w_0).$$

Second, consider the individual summands involving the gamma factors, which converge as $\prod \hat{g}_k(r)$ has rapid decay at infinity. They take the form

$$\frac{1}{4\pi} \int_{-i\infty}^{i\infty} \left( \frac{\Gamma_k'(1 + ir + w)}{\Gamma_k(1 - ir + \bar{w})} \right) \hat{g}(ir) dr.$$ 

We use the property of $\hat{g}$ as an even function, and apply Lemma 4.5, with $w = a + ib$, to obtain

$$-\frac{1}{2} pv_0 \int_{W_v} \hat{g}(|w|) \chi_v(w) \frac{|w|}{|1 - w|} dw,$$

where the constants $a, b$ depending on the ramification are accounted for by the local character $\chi_v$.

Third, for the $\zeta_F(s, \chi)$ terms, write $L_v(s, \chi)$ for the local $L$-factor in the Euler product for $\zeta_F(s, \chi)$. For absolute convergence, we shift the contour slightly to the right of $\text{Re}(s) = 1$, where we may use the Euler product expansion for the logarithmic derivative:

$$\frac{1}{4\pi i} \int_{1 - i\infty}^{1 + i\infty} \frac{\zeta'_F(s, \chi)}{\zeta_F(s, \chi)} \hat{g}(s) ds = \sum_v \frac{1}{4\pi i} \int_{1 + \epsilon - i\infty}^{1 + \epsilon + i\infty} \frac{L'_v(s, \chi)}{L_v(s, \chi)} \hat{g}(s) ds + \frac{\delta \chi}{4} g(1)$$

for some $\epsilon > 0$, and $\delta \chi$ is 1 if $\chi$ is the trivial character, contributing half the residue at $\text{Re}(s) = 1$, and zero otherwise. As in [Wei72, p.12], the residue at 1 can be expressed as an integral over the global Weil group:

$$\frac{1}{4} \int_{W_F} g(|w|) \chi(w) \frac{dw}{|w|}.$$
To each local $L$-factor we apply Lemmas 4.2 and 4.3 as follows: set $\sigma = \frac{1}{2}$, and write
\[
\frac{1}{4\pi i} \int_{-\infty}^{\infty} \hat{g}(s) \frac{d}{ds} \log L_v(1 + s, \chi) \, ds = - \log q_v \int_{-\infty}^{\infty} \hat{g}(it) \left( \sum_{n=1}^{\infty} q_v^{-n(1+it)} \right) \chi_v(f^n w) \, dw \, dt
\]
\[
= - \frac{1}{2} \log q_v \sum_{n=1}^{\infty} g(q_v^{-n}) q_v^{-n} \int_{W_v^0} \chi_v(f^n w) \, dw.
\]

On the other hand, we have
\[
\frac{1}{4\pi i} \int_{-\infty}^{\infty} \hat{g}(s) \frac{d}{ds} \log L_v(1 - s, \chi) = - \frac{1}{2} \log q_v \sum_{n=1}^{\infty} g(q_v^n) q_v^{-n} \int_{W_v^0} \chi_v(f^n w) \, dw,
\]
plus the same contribution associated to the residue at 1, since $\chi$ is trivial if $\chi$ is.

Then using the fact that $\chi(w) = \chi(w^{-1})$ and
\[
W_v - W_v^0 = \bigcup_{n \in \mathbb{Z} - \{0\}} f^n W_v^0
\]
we combine the two terms two obtain
\[
- \frac{1}{2} \int_{W_v - W_v^0} \hat{g}(|w|) \chi_v(w) \inf(|w|, |w|^{-1}) \, dw = - \frac{1}{2} p v_0 \int_{W_v - W_v^0} \hat{g}(|w|) \chi_v(w) \frac{|w|}{|1 - w|} \, dw
\]
(cf. [Mor05, p.158]). Then we apply Corollary 4.9 to combine the contribution of the nonarchimedean $L$-factors and the conductor,
\[
- \frac{1}{2} p v_0 \int_{W_v} \hat{g}(|w|) \chi_v(w) \frac{|w|}{|1 - w|} \, dw,
\]
which is identical to the contribution of the archimedean factors.

Finally, putting this all together we have
\[
g(0) \log |d\mathcal{P}| + \frac{1}{2} \int_{W_v} g(|w|) \chi(w) \frac{dw}{|w|} - \frac{1}{2} p v_0 \int_{W_v} \hat{g}(|w|) \chi_v(w) \frac{|w|}{|1 - w|} \, dw
\]
as desired.

\section{Application to Lower Bounds for Sums over Zeroes}

The presence of the sums over zeroes, or equivalently, the Weil distributions gives us an approach to the zeroes of $L$-functions using the trace formula. In this section we give an example in the simplest case to illustrate how the distributions arising from Eisenstein series can be used to bound the sum over zeroes.

\subsection{Truncation}

The key to this analysis will be the following positivity result, proved by Arthur following an idea of Selberg, valid for reductive groups $G$. Though the case we are interested is also covered by Remark 3.2. First, we review the truncation operator, referring to the exposition of [Art05, §13] and onwards for details. We will work over $\mathbb{Q}$, though the discussion also holds for number fields, and readily extends to general reductive groups.
Definition 5.1. Let $G$ be a reductive group over $\mathbf{Q}$, and $G^1$ the elements of norm one. Let $\phi$ be a locally bounded, measurable function on $G \setminus G^1$, and $T$ a suitably regular point in the positive root space $a_0^+$ generated by the roots of the maximal torus $A$ of $G$. Define the truncation operator

$$\Lambda^T \phi = \sum_P (-1)^{\dim A^P/A} \sum_{\delta \in P_{\mathbf{Q}} \setminus G_{\mathbf{Q}}} \int_{N_P(\mathbf{Q}) \setminus N_P(A)} \phi(n\delta x) \hat{\tau}_P(\log(H_P(\delta x) - T)) \, dn$$

where the outer sum is taken over parabolic subgroups $P$ of $G$, $\hat{\tau}_P$ is the characteristic function of the positive Weyl chamber associated to $P$, and $H_P(x)$ is the usual height function, which in the case of $GL_2$ is

$$H_P(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}) = (|a|, |b|).$$

The inner sum is finite, while the integrand is a bounded function of $n$. In particular, if $\phi$ is in $L^2_{\text{cusp}}(G \setminus G^1)$ then $\Lambda^T \phi = \phi$; also $\Lambda^T E(g, \phi, s)$ is square integrable. It is self-adjoint and idempotent, hence an orthogonal projection.

Using this, Arthur shows that the spectral side

$$\sum_{\chi} J^T_\chi(f) = \sum_{\chi} \int_{G_{\mathbf{Q}} \setminus G_{\mathbf{A}}} \Lambda^T_2 K_\chi(x, x) \, dx$$

converges absolutely, where the subscript on $\Lambda^T_2$ indicates truncation with respect to the second variable, and the index $\chi$ corresponds to the $\eta$ in the case of $SL_2$ above. This gives what Arthur calls the coarse spectral expansion of the trace formula. We have the expression for the distribution $J^T_\chi(f)$ as

$$\int_{G_{\mathbf{Q}} \setminus G_{\mathbf{A}}} \Lambda^T_2 K_\chi(x, x) \, dx = \sum_P \frac{1}{n_P} \int_{G_{\mathbf{Q}} \setminus G_{\mathbf{A}}} \int_{\mathfrak{a}_P^*} \sum_{\phi} \Lambda^T E(x, \rho(f, \eta, s), \lambda) \Lambda^T E(x, \phi, \lambda) \, d\lambda \, dx$$

where $n_P$ is the number of chambers in $\mathfrak{a}_P$, and $\phi$ runs over an orthonormal basis of $\rho(f, \eta, s)$.

Example 5.2. When $G = SL(2)$, there is only one chamber, and the inner integral is one dimensional, that is, $i\mathfrak{a}_P^* = i\mathbf{R}$. Furthermore, by absolute convergence we may interchange the sum over $\phi$ with the integral, to obtain the expression for $SL_2$

$$J^T_\chi(f) = \sum_{\phi} \int_{-\infty}^{\infty} (\Lambda^T E(x, \rho(f, \eta, s), \phi, s), \Lambda^T E(x, \phi, s)) \, ds$$

thus we have an absolutely convergent sum-integrals of an inner product.

Now, the following is a consequence of Arthur’s method:

Lemma 5.3. $J^T_\chi(f)$ is a positive-definite distribution.

Proof. Using Arthur’s truncation $\Lambda^T$ with respect to the parameter $T$ and Arthur’s general notation, the intertwining operator is given as

$$(M^T_{P, \chi}(\lambda) \phi', \phi) = \int_{G_{\mathbf{Q}} \setminus G_{\mathbf{A}}} \Lambda^T E(x, \phi', \lambda) \Lambda^T E(x, \phi, \lambda) \, dx$$
for any vectors $\phi', \phi$ in the induced representation space. It is an integral of the usual $L^2$-inner product of truncated Eisenstein series (it is square integrable after truncation), so it follows that $M_{F,\chi}^T(\lambda)$ is positive-definite, self-adjoint operator.

Then, following the proof of the conditional convergence of $J_T^\chi$ in [Art05, §7], we see that for any positive-definite test function $f \ast f^\ast$ with $C_c^\infty(G_A)$, the resulting double integral is nonnegative, and the integrals can be expressed as an increasing limit of nonnegative functions. The integral converges, and $J_T^\chi(f)$ is positive-definite.

Alternatively, using the absolute convergence of the spectral side, we may interchange the sum and integrals to obtain the inner product expression as above, and observe again that the inner product is positive-definite.

As Arthur points out, a convenient choice of parameter $T$ for the group $G = GL(n)$ is $T = 1$. In particular, the expressions of the trace formula that we have considered in Sect. 3 can be considered to be evaluated at $T = 1$.

5.2. Application to lower bounds. Now the positivity of $J_\chi(f)$ gives immediately a lower bound for the sums over zeroes for any class $\chi$ and function $f$. We illustrate our method in the most basic case, that is, for the Riemann zeta function.

Proof of Theorem 1.3. We consider the continuous spectral terms in the trace formula (5.1), truncated at $T$, specializing in the case of $SL(2)$ to give

$$\int_{T \setminus G} \Lambda^T K_{cont}(x,x)dx = g(1) \log T - \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{m'}{m}(it)g(it)dt$$

$$+ \frac{1}{4\pi i} \int_{-\infty}^{\infty} m(it)g(it)\frac{Te^{it}}{t}dt.$$ 

From Lemma 5.3, for our choice of $g$ the above expression is nonnegative. The first term described in Theorem 1.1,

$$\sum_{\rho} \hat{g}(\rho) + \int_{0}^{\infty} \left\{ g(x) - \frac{1}{4} g^*(x) \right\} dx + \sum_{n=1}^{\infty} \Lambda(n)g(n)$$

$$+ \int_{1}^{\infty} \left\{ g(x) + g^*(x) - \frac{2}{x} g(1) \right\} \frac{x}{2(x^2-1)} + \frac{1}{2}(\log 4\pi + \gamma)g(1).$$

The requirement that $T > \sqrt{3}/2$ simply follows from the observation that the fundamental domain of $SL_2(\mathbb{Z}) \setminus \mathbb{H}_2$ has height at least $\sqrt{3}/2$ in $\mathbb{H}_2$. Then the claim follows immediately from rearranging the terms.

Remark 5.4. Certainly we may evaluate at the usual point $T = 1$, but it may be that for different test functions there will be more effective choices of $T$, which may be of interest to applications of the explicit formula. Finally, we leave it to the interested reader to examine the analogous result for Hecke $L$-functions.

Appendix A.

A.1. Sketch of proof of Theorem 3.9. The kernel of $\rho(f)$ restricted to the continuous spectrum can be expressed as

$$K_{cont}(g, h) = \frac{1}{8\pi} \sum_{\alpha, \beta} \int_{-\infty}^{\infty} (\rho(f, it) \varphi_\beta, \varphi_\alpha)E(g, \varphi_\alpha, it)E(h, \varphi_\beta, it)dt$$
where the $\varphi_\alpha, \varphi_\beta$ run over an orthonormal basis of the principal series $\rho(g, it)$, viewed as the representation $\rho(g)$ restricted to the continuous spectrum. This expression follows from the spectral decomposition of $L^2(G_F \backslash G_A, \chi)$ using the Eisenstein series and an orthogonal projection onto the continuous spectrum (cf. [GJ79, pp.232-234], [Kna97, p395]).

Then we compute the trace

$$\int_{Z_0'G_F \backslash G_A} \Lambda^T K_{\text{cont}}(g, g) dg$$

$$= \frac{1}{8\pi} \sum_{\alpha, \beta} \int_{-\infty}^{\infty} (\rho(f, it)\varphi_\beta, \varphi_\alpha) \int_{Z_0'G_F \backslash G_A} E(g, \varphi_\alpha, it) \overline{E(h, \varphi_\beta, it)} dg \, dt.$$  

Observe that the inner integral is an inner product of a pair of truncated Eisenstein series, which is given by the Maass-Selberg relation, which we prove in the following section as Theorem A.3. Assuming this, we use $\varphi_1(s) = \varphi_\alpha(s)$ and $\varphi_2(s) = \varphi_\beta(s)$ in the theorem to obtain

$$\int_{Z_0'G_F \backslash G_A} \Lambda^T K_{\text{cont}}(g, g) dg$$

$$= \log T \cdot \frac{1}{2\pi} \sum_{\alpha, \beta} \int_{-\infty}^{\infty} (\rho(f, it)\varphi_\beta, \varphi_\alpha)(\varphi_\alpha, \varphi_\beta) dt$$

$$- \frac{1}{4\pi} \sum_{\alpha, \beta} \int_{-\infty}^{\infty} (\rho(f, it)\varphi_\beta, \varphi_\alpha)(M^{-1}(it)M'(it)\varphi_\alpha, \varphi_\beta) dt$$

$$+ \frac{1}{4\pi} \sum_{\alpha, \beta} \int_{-\infty}^{\infty} (\rho(f, it)\varphi_\beta, \varphi_\alpha)\{(\varphi_\alpha, M(it)\varphi_\beta)T^{-it} - (M(it)\varphi_\alpha, \varphi_\beta)T^{-it}\} dt.$$  

The first term put back together as

$$\frac{\log T}{2\pi} \int_{-\infty}^{\infty} \text{tr}(\rho(f, it)) dt$$

cancels with a contribution from the geometric side, so it does not appear in the final expression of the trace formula. The second term can be broken into a sum over each principal series attached to $\eta$,

$$- \frac{1}{4\pi} \int_{-\infty}^{\infty} M^{-1}(it)M'(it)\rho(f, it) dt = \sum_{\eta} - \frac{1}{4\pi} \int_{-\infty}^{\infty} M^{-1}(\eta, it)M'(\eta, it)\rho(f, \eta, it) dt.$$  

Then we expand the logarithmic derivative

$$M(\eta, it)^{-1}M'(\eta, it) = m^{-1}(\eta, it)m'(\eta, it) \otimes I + \sum_{\nu} R^{-1}(\eta_u, it)R(\eta_u, it) \otimes_{\nu \neq u} I_v,$$

where $I_v$ is the identity operator on the space $B(\mu_v \alpha_v^{it/2}, \nu_v \alpha_v^{it/2})$, and substituting it into the integral gives the terms (3.6) and (3.7) directly.

The third term can be rewritten as the sum of two terms

$$\frac{1}{4\pi} \sum_{\beta} \int_{-\infty}^{\infty} (M^{-1}(it)\rho(f, it)\varphi_\beta, \varphi_\beta) \frac{e^{2Tit} - e^{-2Tit}}{2it} dt$$

$$= \frac{1}{4\pi} \sum_{\beta} \int_{-\infty}^{\infty} (M^{-1}(it)\rho(f, it)\varphi_\beta, \varphi_\beta) \frac{1}{2it} dt$$

$$= \frac{1}{4\pi} \sum_{\beta} \int_{-\infty}^{\infty} (\varphi_\beta, M^{-1}(it)M'(it)\varphi_\alpha) \frac{1}{2it} dt.$$
\[ \frac{1}{4\pi} \sum_{\beta} \int_{-\infty}^{\infty} \left\{ \left( \rho(f, it) \varphi_\beta, M(it) \varphi_\beta \right) - \left( \rho(f, it) \varphi_\beta, M^{-1}(it) \varphi_\beta, \varphi_\beta \right) \right\} e^{-2T|it|} \, dt. \]

The second term here is the Fourier transform of an integrable function, so by the Riemann-Lebesgue lemma it tends to zero as \( T \) tends to infinity. On the other hand, the first term by \([GJ79, \text{Lemma 6.31}]\) tends to

\[ \frac{1}{4} \sum_{\beta} (M(0) \rho(f, it) \varphi_\beta) = \frac{1}{4} \text{tr}(M(0) \rho(f, it)) = \frac{1}{4} \sum_{\eta \neq \chi} \text{tr}(M(\eta, 0) \rho(f, \eta, it)), \]

plus an \( O(1/T) \) term. This last sum is taken over characters \( \eta \) that square to \( \chi \), for the operator \( M(0) \) intertwines representations such that \( \eta = \chi \bar{\eta} \), so that \( M(0) \) is zero on characters \( \eta \) that do not square to \( \chi \).

### A.2. Proof of the Maaß-Selberg relation

We complete the proof of Theorem 3.6 reviewing the Maaß-Selberg relation. We first state the following lemma:

**Lemma A.1.** Let \( f \) be a measurable function on \( N_A P_F^I \backslash G_A' \) such that \( f(zg) = \chi(z) f(g) \) for all \( z \in Z_0^I \) and

\[ F := \sum_{\gamma \in P_F^I \backslash G_F^I} f(\gamma g) \]

is square integrable modulo \( Z_0^I G_F^I \). Also let \( f' \) be a function in \( L^2(G_F^I \backslash G_A', \chi) \), with constant term \( f_N \). Then

\[ (F, f')_{L^2(G_F^I \backslash G_A', \chi)} = (f, f'_N)_{L^2(Z_0^I N A P_F^I \backslash G_A', \chi)}, \]

where the equality depends on the normalization of Haar measures.

**Proof.** The proof of this statement for \( GL_2(\mathbb{A}_F) \) is given in \([Kna97, \text{Lemma 6.4}]\) and \( SL_2(\mathbb{R}) \) in \([Bor97, \text{p.125}]\), and one observes that the proof follows in the same manner for \( G_A' \) also. Note that choosing different Haar measures will preserve the equality up to a constant.

**Theorem A.2** (Maaß-Selberg relation). The inner product of two truncated Eisenstein series

\[ (A^T E(g, \varphi_1, s_1), A^T E(g, \varphi_2, s_2)) \]

can be expressed as

\[ \frac{2}{s_1 + s_2} \left\{ \left( \varphi_1, \varphi_2 \right) T^{s_1 + s_2} - \left( M(s_1) \varphi_1, M(s_2) \varphi_2 \right) T^{-(s_1 + s_2)} \right\} \]

\[ + \frac{2}{s_1 - s_2} \left\{ \left( \varphi_1, M(s_2) \varphi_2 \right) T^{s_1 - s_2} - \left( M(s_1) \varphi_1, \varphi_2 \right) T^{-(s_1 - s_2)} \right\} \]

for \( \text{Re}(s_1) > \text{Re}(s_2) > 0 \).

**Proof.** Given a function \( \varphi_\gamma \) in the principal series \( B(\mu \alpha^s/2, \nu \alpha^s/2) \) belonging to \( G_A' \), the Eisenstein series in this setting is

\[ E(g, \varphi, s) = \sum_{\gamma \in P_F^I \backslash G_F^I} \varphi_\gamma(\gamma g) \]
where $P'_c$ denotes the set of upper triangular matrices in $G'_F$. Applying the truncation operator of Arthur (see Chapter 1), we have for $T > 0$,

$$\Lambda^T E(g, \varphi, s) = E(g, \varphi, s) - \sum_{\gamma \in P'_c \setminus G'_F} E_N(\gamma g, \varphi, s) \chi_T(H(\gamma g)),$$

where $E_N(g, \varphi, s)$ is the constant term of the Fourier expansion,

$$E_N(g, \varphi, s) = \int_{F \setminus A} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx$$

$$= \varphi_s(g) + \int_{N_A} \varphi(s wng) dn$$

$$= \varphi_s(g) + (M(s) \varphi_s)(g)$$

(cf. [Kna97, p.392]). Then the truncated Eisenstein series can be written as

$$\Lambda^T E(g, \varphi, s) = \sum_{\gamma \in P'_c \setminus G'_F} \{\varphi_s(g) - \chi_T(H(\gamma g))(\varphi_s(g) - (M(s) \varphi_s)(g))\}. $$

The truncated Eisenstein series being square integrable, we now compute its inner product. Since $\Lambda^T$ is an orthogonal projection ([GJ79, p.210]), the inner product is equal to

$$(\Lambda^T(E(g, \varphi_1, s_1)), \Lambda^T(E(g, \varphi_2, s_2))) = \int_{Z_0'G'_F \setminus G'_A} \Lambda^T(E(g, \varphi_1, s_1))(E(g, \varphi_2, s_2)) dg$$

$$= \int_{Z_0'G'_F \setminus G'_A} \Lambda^T(E(g, \varphi_1, s_1))(E_N(g, \varphi_2, s_2)) dg.$$

Then unfold the sum and apply Lemma A.1 to get

$$\int_{Z_0'N_A P'_c \setminus G'_A} \{\varphi_1, s_1(g) - \chi_T(H(g))(\varphi_1, s_1(g) - (M(s_1) \varphi_1, s_1)(g))\} (E_N(g, \varphi_2, s_2)) dg.$$

Decomposing domain of integration into

$$Z_0'N_A P'_c \setminus G'_A \simeq A'_F Z_0' \setminus A' \times K \simeq \mathbb{R}^\infty_{+} \times (A^\times_F)^1 \times K,$$

we rewrite the integrand as

$$\{\varphi_1, s_1(g)(1 - \chi_T(H(g))) - (M(s_1) \varphi_1, s_1)(g) \chi_T(H(g))\} (E_N(g, \varphi_2, s_2))$$

$$\{\varphi_1, s_1(g)(1 - \chi_T(H(g))) - (M(s_1) \varphi_1, s_1)(g) \chi_T(H(g))\} (E_N(g, \varphi_2, s_2))$$

and then separate into two terms:

(A.1) $$\varphi_1, s_1(g)(1 - \chi_T(H(g))) (\varphi_2, s_2)(g) + (M(s_2) \varphi_2, s_2)(g))$$

and

(A.2) $$- (M(s_1) \varphi_1, s_1)(g) \chi_T(H(g)) (\varphi_2, s_2)(g) + (M(s_2) \varphi_2, s_2)(g)).$$

In both cases we shall use the transformation rule of the principal series,

$$\varphi_i, s_i(n \begin{pmatrix} pau & 0 \\ 0 & qbv \end{pmatrix} k) = \chi(bv)|ab^{-1}|^{\frac{1+\epsilon_i}{2}} \varphi_i(\begin{pmatrix} uv^{-1} & 0 \\ 0 & 1 \end{pmatrix} k)$$

for $p, q \in F^\times$, $a, b \in R^\times_{+}$, and $u, v \in (A^\times_F)^1$ (cf. [Kna97, p.390]). For simplicity we will write the latter function as $\varphi_i(uv^{-1}, k)$, as the function $\varphi_i(g)$ is determined
by its values on \((A^+_K)^1 \times K\). The integrals over this product space will be expressed as an inner product in the Hilbert space associated to the principal series representation.

In the first term \((A.1)\), the factor \(1 - \chi_T(H(g))\) is the characteristic function of \((0, T)\), thus truncates the integral over \(R^+_T\). The integral becomes

\[
\int_0^T \left( \int_{(A^+_K)^1} \int_K t^{1+(s_1+s_2)/2} \varphi_1(a,k) \overline{\varphi_2(a,k)} \right) dt \ dx
\]

and hence

\[
\int_0^T t^{1+(s_1+s_2)/2}(\varphi_1, \varphi_2) + t^{1+(s_1-s_2)/2}(\varphi_1, M(\bar{s})\varphi_2) \frac{dt}{T^2}
\]

\[
= \frac{2}{s_1 + s_2} T^{(s_1+s_2)/2}(\varphi_1, \varphi_2) + \frac{2}{s_1 - s_2} T^{(s_1-s_2)/2}(\varphi_1, M(\bar{s})\varphi_2).
\]

In the second term \((A.2)\), the factor \(\chi_T(H(g))\) is the characteristic function of \((T, \infty)\), and again truncates the integral over \(R^+_T\). Computing as before, we obtain

\[
- \int_T^\infty t^{1-(s_1-s_2)/2}(M(s_1)\varphi_1, \varphi_2) + t^{1-(s_1+s_2)/2}(M(s_1)\varphi_1, M(\bar{s})\varphi_2) \frac{dt}{t^2}
\]

\[
= \frac{2}{s_1 - s_2} T^{-(s_1-s_2)/2}(M(s_1)\varphi_1, \varphi_2) - \frac{2}{s_1 + s_2} T^{-(s_1+s_2)/2}(M(s_1)\varphi_1, M(\bar{s})\varphi_2).
\]

Then combining the two terms together gives the desired expression. \(\Box\)

The following special case of the Maaß-Selberg relation is the one encountered in the trace formula:

**Corollary A.3.** Let \(s_1 = s + h\), and \(s_2 = -s\) with \(h > 0\). Then the limit as \(h \to 0\), written as

\[
(A^T E(g, \varphi_1, s), \Lambda^T E(g, \varphi_2, -\bar{s}))
\]

is equal to

\[
4 \log T(\varphi_1, \varphi_2) + 2(M^{-1}(s)M'(s)\varphi_1, \varphi_s) + (\varphi_1, M(\bar{s})\varphi_2) T^s s - (M(s) \varphi_1, \varphi_2) T^{-s} s.
\]

**Proof.** By hypothesis \(s_1 + s_2 = h\), and \(s_1 - s_2 = 2s + h\). The Maaß-Selberg relation specializes in this case to

\[
\frac{2T^h}{h}(\varphi_1, \varphi_2) + \frac{2T^{-h}}{h}(M(s + h)\varphi_1, M(-\bar{s})\varphi_2)
\]

\[
+ \frac{2T^{2s+h}}{2s+h}(\varphi_1, M(-\bar{s})\varphi_2) - \frac{2T^{-2s-h}}{2s+h}(M(s + h)\varphi_1, \varphi_2)
\]

which is valid so long as \(s_1, s_2 \neq 0\).

The third and fourth terms are immediate. For the first two terms, we use the Taylor expansion of \(T^h\) at 0 in the variable \(h\),

\[
T^h = 1 + h \log T + \frac{1}{2} h^2 \log^2 T + \ldots,
\]

combined with the fact that the adjoint of \(M(s)\) is \(M(\bar{s})\) and the identity \(M(s)M(-s) = 1\) [Kna97, p.367], we write the second inner product as

\[
(M(-s)M(s + h)\varphi_1, \varphi_2).
\]
Then the limit of the first two terms now can be seen to evaluate to
\[4 \log T(\varphi_1, \varphi_2) + 2(M^{-1}(s)M'(s)\varphi_1, \varphi_s)\]
as desired. \[\Box\]

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E-mail address: twong@gradcenter.cuny.edu

Department of Mathematics, Graduate Center, City University of New York, NY 10016