Quantum Trajectories and Quantum Measurement Theory

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Abstract

Beyond their use as numerical tools, quantum trajectories can be ascribed a degree of reality in terms of quantum measurement theory. In fact, they arise naturally from considering continuous observation of a damped quantum system. A particularly useful form of quantum trajectories is as linear (but non-unitary) stochastic Schrödinger equations. In the limit where a strong local oscillator is used in the detection, and where the system is not driven, these quantum trajectories can be solved. This gives an alternate derivation of the probability distributions for completed homodyne and heterodyne detection schemes. It also allows the previously intractable problem of real-time adaptive measurements to be treated. The results for an analytically soluble example of adaptive phase measurements are presented, and future developments discussed.

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1 Introduction

The concept of stochastic Schrödinger equations (sses) [1] or quantum trajectories [2] has generated an impressive degree of activity in the quantum optics community over the past few years

1.1 Non-Real Quantum Trajectories

The “non-real” interpretation of quantum trajectories treats the trajectories simply as numerical tools for solving the master equation. This is essentially the attitude taken by Dalibard, Castin and Mølmer [1] who introduced the concept, although they do mention that their “Monte Carlo Wavefunction” approach also provides “new physical pictures”. Note the emphasis on wavefunctions in their terminology. The value of quantum trajectories for pure states as a computational tool is that

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1Related ideas have also been developed by workers outside quantum optics. Space limitations prevent me from discussing these.
it only takes $2N - 2$ real numbers to store a state vector in an $N$-dimensional Hilbert space, compared to $N^2 - 1$ real numbers for a state matrix. Of course, this gain is offset by the fact that many trajectories are required to obtain a reliable ensemble average. However, for a large Hilbert space, quantum trajectories may offer a real advantage for numerical computation. One very practical application is computing the centre-of-mass motion of a fluorescent atom [5, 6, 7, 8, 9].

1.2 Subjectively Real Quantum Trajectories

In the body of this paper, quantum trajectories are attributed some degree of reality, in that they accurately model the evolution of a system conditioned on continuous observation. This can be called “subjective reality”, because the nature of the trajectories depend on the measurement scheme chosen by the observer. Once a particular measurement scheme has been chosen, the conditioned evolution of the system (as modeled by quantum trajectories) is objectively real in the sense that all observers will agree on the results. The subjectivity is present in the sense that it is not permissible to make a statement such as “an excited atom in free space will spontaneously emit a photon at a random point in time” unless there is a photodetector to record the emitted photon. A different detection scheme, such as heterodyne detection, will register a continuous flow of radiation from the atom, not a jump. In other words, the stochastic system evolution is only real in so far as the system is a subject of observation; it is not real for the system as an object in an observerless universe. As well as trajectories arising from different types of measurements, trajectories which do not preserve pure states (because of imperfect detection or bath preparation) also arise in this interpretation.

It is the subjective interpretation of quantum trajectories, in addition to their computational convenience, which is the motivation for the work by Carmichael and co-workers [2, 10, 11, 12, 13, 14]. Most of this work was numerical, but analytic results were obtained for some particular systems and particular measurement schemes [12, 13]. Both quantum jump (direct detection) and quantum diffusion (homodyne detection) trajectories were used. The measurement theory interpretation was also pointed out by Gardiner, Parkins, and Zoller [15] for the case of direct detection (which was formulated much earlier by Srinivas and Davies [16]).

There are a number of reasons for wishing to consider the measurement interpretation of quantum trajectories. Firstly, the simulated photocurrents can be used to calculate such quantities as the waiting time distribution for photodetections, which are quite difficult to calculate by traditional methods [17]. Another example is the all-or-nothing fluorescence exhibited in the electron shelving of a single atom [18, 19]. The technique of linear quantum trajectories recently published by Goetsch and Graham [21] has opened the possibility for at least partly analytical solutions of measurement theory problems using quantum trajectories. This is the subject of the body of this paper, and I will thus defer further discussion until there.

A second motivation is that subjectively real quantum trajectories give valuable physical insight into irreversible quantum processes, different from that offered by the master equation. For example, a system as simple as a stationary two-level atom shows strikingly different quantum trajectory evolution depending on the chosen method of detection, and that this evolution sheds light on the respective measurement results [20]. Another example is that coherences which are lost in the non-selective (master equation) picture are seen to be merely distributed among different quantum trajectories [2, 10, 13, 22, 23]. These reconstructed coherences can restore noisy oscillatory dynamics to a system which would otherwise exhibit only damped oscillations [24].

Thirdly, the measured results can be used to select certain individual systems from the ensemble. In this way, it may be possible to see the coherences which are lost in the non-selective ensemble evolution [13, 14]. This could be looked upon as a very basic sort of feedback, in which the measured result is used to eliminate some members of the ensemble. The use of quantum trajectories in more active feedback schemes is a fourth motivation for subjectively real quantum trajectories, which has been pursued by myself and Milburn and others [25, 26, 27, 28, 29, 30, 31].

1.3 Objectively Real Quantum Trajectories

The third attitude towards quantum trajectories is a belief in their objective existence, independent of any measurement scheme. This belief is presumably founded on mistrust of the standard
interpretations of quantum mechanics, and a desire to replace subjective collapses due to measurements with objectively real collapses. The roots of this approach are the dynamical state reduction models of Gisin [32], Diosi [33], Pearle [34] and Ghirardi-Rimini-Weber [35], which in turn hark back to the original objective interpretation of the wavefunction by Schrödinger [36]. In this desire, proponents of objectively real quantum trajectory theory are akin to hidden variable theorists, but unlike the latter, they accept the wavefunction as a representation of reality. It is only when irreversible processes occur (described by a master equation) that stochasticity enters quantum mechanics, giving rise to quantum trajectories. Because the state vector is considered a basic element of reality in this interpretation, it only makes sense to consider quantum trajectories which preserve pure states. Thus it is that stochastic Schrödinger equations (as opposed to the more general stochastic master equations of subjective quantum trajectories) are essential only if one regards them as not real or objectively real.

At least three pairs of authors have published on this theme. The first is Teich and Mahler (TM). They published a single paper [37] presenting a theory of quantum jumps in fluorescent atoms. It has been shown to be nonphysical by myself and Milburn in reference [20]. The second pair, Gisin and Percival (GP), have published many papers [38, 39, 40], also with other authors [41], and have received considerable attention. As pointed out by myself and Milburn, the quantum trajectory which GP consider can be given an interpretation in terms of unit-efficiency heterodyne detection (or ambi-quadrature homodyne detection) of the electromagnetic field radiated by the system [20]. However GP attribute an objective reality to their SSE, saying that it “represents the evolution of an individual quantum system in interaction with its environment”. By adding *ad hoc* irreversible processes to a model of a three level atom, they are able to induce their equation to exhibit jump-like behaviour [41], as seen in electron shelving experiments [18]. In general, however, GP cannot associate their trajectories with the detection of photons for the very good reason that they actually correspond to heterodyne, not direct detection. A third pair, Breuer and Petruccione (BP), have very recently published two papers [42, 43] in which they claim to show that the quantum jump unraveling of reference [15] is the SSE for a Markovian open quantum system. Although it is in many ways the most natural unraveling, the quantum jump SSE (which under some circumstances corresponds to direct detection [15, 20]) does not have any fundamental privileged status. BP derive it by choosing a particular basis in which to diagonalize the bath, namely its energy eigenstates.

The basic problem with the objective interpretation of quantum trajectories lies with the origin of irreversible evolution. In standard quantum optics, the irreversibility of the master equation for a system is an approximation to the exact reversible evolution for the system and its environment [44]. None of the above authors try to define in what contexts the master equation evolution is meant to be truly irreversible. If there were some fundamental physical mechanism by which irreversibility entered the world, for example at some sufficiently large scale, then it is conceivable that one of the three schemes outlined above could be correct 2. The quantum measurement “problem” would then be solved, as measurement results would be determined by the irreducibly stochastic quantum evolution. Irrespective of this speculation, the models of TM, GP and BP cannot be objectively real for the systems to which the authors apply them 3. Irreversible stochastic evolution, in the manner of TM, GP or BP, does not take place within an individual atom. The entanglement between an atom and its outgoing field is a fact which is not represented by the models of TM, GP and BP. This entanglement is most simply shown by the fact that one can choose different detection schemes to observe the atom, as discussed in [20]. Gisin and Percival’s scheme only gives the correct results for heterodyne detection, Breuer and Petruccione’s for direct detection, while Teich and Mahler’s scheme will not give the correct results for any detection scheme [20].

1.4 Outline

Having discussed the non-real and objectively real interpretations of quantum trajectories, the remainder of this paper is devoted to quantum trajectories as a part of quantum measurement theory. I begin with a discussion of traditional quantum measurement theory, and why this is inadequate to describe practical measurements. This leads to an outline of the theory of operations

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2Of course, these are not the only three conceivable dynamical state reduction schemes; two which have gained considerable attention are those of GRW [35] and Penrose [45].

3These criticism now appears to be accepted by Percival, as evidenced by reference [46].
and effects, and the different ways in which probability can be incorporated in the theory. Sec. 3 introduces quantum trajectories as an application of quantum measurement theory to continuous observation. The description of such trajectories as linear stochastic Schrödinger equations is covered in Sec. 4. Sec. 5 presents the general solution for the linear SSE for the case of an undriven system with a large local oscillator used in the detection process. This allows a new derivation of the operations and effects for completed homodyne and heterodyne measurements. In Sec. 6 I apply the results of Sec. 5 to a measurement scheme which has not previously been considered and which is probably intractable without linear quantum trajectories. The measurements which I am describing here are adaptive measurements, in which the past photocurrent is used to modify the present conditions of the measurement. Here I discuss the general approach, and present results for the simple but non-trivial example of adaptive phase measurements of a field containing at most one photon. Sec. 7 concludes.

2 Quantum Measurement Theory

Traditional quantum measurement theory is founded on projective measurements. Consider a measurement of fixed duration $T$ with a set of $N$ discrete possible measurement results labeled by $r$. With each result $r$ there is an associated projection operator $\Pi_r(T)$, such that $\Pi_r(T)\Pi_s(T) = \Pi_r(T)\delta_{rs}$. If the initial state of the system (assumed pure for simplicity) is $|\psi(t)\rangle$, then the probability for obtaining the result $r$ is

$$P_r(T) = \langle \psi(t) | \Pi_r(T) | \psi(t) \rangle.$$ (1)

This can also be written

$$P_r(T) = \langle \tilde{\psi}_r(t+T) | \tilde{\psi}_r(t+T) \rangle,$$ (2)

where the tilde denotes an unnormalized state vector

$$|\tilde{\psi}_r(t+T)\rangle = \Pi_r(T) |\psi(t)\rangle.$$ (3)

Furthermore, the final state of the system is simply this state, normalized if desired

$$|\psi_r(t+T)\rangle = \frac{|\tilde{\psi}_r(t+T)\rangle}{\sqrt{P_r(T)}}.$$ (4)

Although this theory may be fundamentally correct in some sense, it is inadequate to describe practical measurements such as those in quantum optics. That is because the experimenter never makes a direct measurement on the system of interest. Rather, the system of interest (such as an atom) interacts with its environment (the continuum of electromagnetic field modes), and the experimenter observes the radiated field. Of course one could argue that the experimenter does not observe the radiated field, rather that the field interacts with a photodetector, which triggers a current in a circuit, which is coupled to a CRO, which radiates more photons, which interact with the experimenter’s retina, and so on up the von-Neumann chain [47]. The point is that at some stage one has to consider the measurement as being made. If one attempts to apply a projection postulate directly to the atom, one will obtain nonsensical predictions. However, it can be shown that assuming a projective measurement of the field will yield results negligibly different from those assuming a projective measurement at any later stage, because of the rapid decoherence of macroscopic objects [48]. For this reason, it is sufficient to consider the field as being measured. Because the field has interacted with the system, their quantum states are entangled. If the initial state of the field is assumed known (which is not unreasonable in practice, because it is often close to the vacuum state), then the projective measurement of the field results in a measurement of the atom. Such a measurement however is not projective. Instead, we need a more general formalism to describe such measurements. This is called the theory of operations and effects [50], which I will now present in brief.

Say the initial system state vector is $|\psi(t)\rangle$ as above, and that the measurement results are again labeled by $r$. Each $r$ is again associated with an operator $\Omega_r(T)$, but it need not be a projection operator. I will call these measurement operators. The effect [50] for the result $r$ is defined by

$$F_r(T) = \Omega_r(T)\dagger\Omega_r(T).$$ (5)

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4This is why the original quantum Zeno paradox of Misra and Sudarshan [49] can be avoided in practice.
This operator is used to generate the probability for the result via
\[ P_r(T) = \langle \psi(t)|F_r(T)|\psi(t)\rangle. \]  
(6)

Conservation of probability yields the single restriction which exists for the measurement operators [50], namely
\[ \sum_r F_r(T) = 1. \]  
(7)

The set of all effects \{F(r)\} constitutes a positive-operator-valued measure (POVM) on the space of results \( r \) [50]. The state of the system given the result \( r \) is
\[ |\psi_r(t+T)\rangle = \frac{\Omega_r(T)|\psi(t)\rangle}{\sqrt{P_r(T)}}. \]  
(8)

If one were doing only a single measurement, then the conditioned state \(|\psi_r\rangle\) would be irrelevant. However one often wishes to consider a sequence of measurements, in which case the conditioned system state is vital. In terms of the state matrix \( \rho \), which allows the possibility of mixed initial states, the conditioned state is
\[ \rho_r(t+T) = \frac{\mathcal{J}[\Omega_r(T)]\rho(t)}{P_r(T)}, \]  
(9)

where \( P_r(T) = \text{Tr}[\rho(t)F_r(T)] \) and for an arbitrary operator \( a \),
\[ \mathcal{J}[a] \rho = a \rho a^\dagger. \]  
(10)

The superoperator \( \mathcal{J}[\Omega_r(T)] \) is known as the operation for \( r \) [50]. If the measurement were performed but the result ignored, the final state of the system would be
\[ \rho(t+T) = \sum_r P_r(T)\rho_r(t+T). \]  
(11)

Consider a hypothetical experiment involving a sequence of measurements for which one knows the set of measurement operators \{\Omega_r(T)\}. As a theorist, one may be interested in knowing a) the probability for obtaining the possible measurement sequences, and b) the state of the system during and at the end of the measurement, conditioned on the results obtained. If initially \( \rho(t) = |\psi(t)\rangle\langle\psi(t)| \) then the above formalism shows that one could obtain this information as follows. Start with \(|\psi(t)\rangle\), and generate the probabilities for the results using equation (6). One would then choose results at random using these probabilities, and see what the new conditioned state of the system is using equation (8). This is the input into the next measurement. The probability for each sequence of results to actually occur is simply the probability that it was generated by the simulation procedure. The simulation is repeated as may times as necessary to obtain good statistics. Call this method A.

As in the case for projective measurements, this theory can be alternatively stated as follows. The final state of the system conditioned on the result \( r \) is represented by an unnormalized state vector
\[ |\tilde{\psi}_r(t+T)\rangle = \Omega_r(T)|\psi(t)\rangle. \]  
(12)

The probability for this result having been obtained is then
\[ P_r(T) = \langle \tilde{\psi}_r(t+T)|\tilde{\psi}_r(t+T)\rangle. \]  
(13)

The spirit of this formulation is that one could simulate the measurement process by randomly generating a result \( r \) with probability \( 1/N \), then calculating the appropriate unnormalized conditioned state. This is the input into the next measurement step. The actual probability assigned to each sequence of measurement results is the final norm squared of the state vector (13), where here \( T \) would be the total time for the sequence of measurements. This simulation would be less efficient because all measurement results are given equal computational time, even those with vanishingly small probabilities. However it would have the advantage that probabilities need to be calculated only once (at the end). Call this method B.
In fact, it is possible to formulate the theory in any number of ways which include these two as special cases. Define an (in general unnormalized) conditioned state vector by

$$|\tilde{\psi}_r(t + T)\rangle = \frac{\Omega_r(T)|\psi(t)\rangle}{\sqrt{\Lambda_r(T)}}.$$  (14)

Here the $\Lambda_r(T)$ are positive numbers chosen such that

$$\sum_r \Lambda_r(T) = 1.$$  (15)

They can thus be interpreted as probabilities in some sense, and I will call $\Lambda_r$ the ostensible probability for the result $r$. The actual probability for getting result $r$ is

$$P_r(T) = \Lambda_r(T)\langle \tilde{\psi}_r(t + T)|\tilde{\psi}_r(t + T)\rangle.$$  (16)

If the $\Lambda_r(T)$ are chosen such that $\Lambda_r(T)$ happens to equal $\langle \psi(t)|F_r(T)|\psi(t)\rangle$, then this formulation is equivalent to the method A given above, with $|\tilde{\psi}_r(t + T)\rangle = |\psi_r(t + T)\rangle$. If they are chosen such that $\Lambda_r(T) = 1/N$, then this formulation is equivalent to the method B, with $|\tilde{\psi}_r(t + T)\rangle = |\psi_r(t + T)\rangle \sqrt{1/N}$. In general, one could do the simulation of the measurement by choosing the results $r$ with probability $\Lambda_r(T)$, then calculating the conditioned state vector $|\tilde{\psi}_r(t + T)\rangle$. Each result would then be weighted by $\langle \tilde{\psi}_r(t + T)|\tilde{\psi}_r(t + T)\rangle$, where again $T$ would be the total time. Because of the probability of generation, this weighting gives the correct overall weighting to each member of the ensemble as given in equation (16). Call this method C.

This flexibility of formulation has been known for some time (see Goetsch and Graham [21] and references therein). As presented here it must appear rather pointless. However we are about to consider quantum trajectories, as an application of measurement theory for continuously (in time) monitored systems. In this context, it turns out that the ability to separate out the probability of each possible result into two components (a probability of generation $\Lambda_r(T)$ and a weighting $\langle \psi_r(t + T)|\tilde{\psi}_r(t + T)\rangle$), will be of enormous use in solving the quantum trajectories.

Before turning to the topic of quantum trajectories, there is one more useful fact which can be explained in terms of the general formalism. Recall that the final state of the system if all measurement results are ignored is given by equation (11). This can also be written as

$$\rho(t + T) = \sum_r J[\Omega_r]\rho(t).$$  (17)

Now if we define a unitary rearrangement of of the measurement operators $\Omega_r$ by

$$\Omega'_r = \sum_s U_{r,s} \Omega_s,$$  (18)

where $U$ is a $c$–number matrix satisfying $\sum_{r,s} U_{r,s}U^*_{r,q} = \delta_{s,q}$. Then the unconditioned final state under the new measurement operators $\{\Omega'_r\}$ is

$$\sum_r \Omega'_r \rho(t)\Omega'_r = \sum_{r,s,q} U_{r,s} \Omega_s \rho(t)\Omega_q^\dagger U_{r,q}^*,$$  (19)

$$= \sum_{s,q} \delta_{s,q} \Omega_s \rho(t)\Omega_q^\dagger = \sum_s \Omega_s \rho(t)\Omega_s^\dagger.$$  (20)

which is the same as that under the old measurement operators $\{\Omega_r\}$. That is to say, one can unitarily rearrange the measurement operators without changing the non-selective system evolution.

### 3 Quantum trajectories

One case of the above measurement theory which is of great importance is continuous observation, with an infinitesimal measurement interval $T = dt$. The form of such a measurement theory is restricted by the requirement that the non-selective evolution generated by the measurements
should give a valid evolution equation for the state of the system. Explicitly, this non-selective evolution is

$$\rho(t + dt) = \sum_r \Omega_r(dt) \rho(t) \Omega_r^\dagger(dt). \quad (21)$$

This is obviously Markovian [51], with the increment in the system state from time \( t \) to time \( t + dt \) depending only on the state of the system at time \( t \). A Markovian evolution equation for the state matrix of a system is called a master equation. Physically, it can be derived from the interaction of the system with its environment, if the environment can be treated as a bath. That is to say, it is necessary for the environment to irreversibly carry away information from the system. Alternatively, one can view the environment as a continuous stream of independently prepared apparatuses which are wheeled up to the system, interact for an infinitesimal time \( dt \), and are wheeled away again to be measured. Although this sounds unrealistic, the electric-dipole interaction between a system and the electromagnetic vacuum behaves like this to a good approximation [15, 31].

The most general form of master equation is the so-called Lindblad form [52, 44]

$$\dot{\rho} = -i[H, \rho] + \sum_{\mu} \mathcal{D}[c_\mu] \rho. \quad (22)$$

Here \( H \) is an Hermitian operator and \( \mathcal{D} \) is a superoperator taking an operator argument (enclosed in square brackets). It is defined for an arbitrary operator \( r \) by

$$\mathcal{D}[r] \rho \equiv r \rho r^\dagger - \frac{1}{2}(r^\dagger r \rho + \rho r^\dagger r). \quad (23)$$

The operators \( c_\mu \) are completely arbitrary.

Consider now a special case where there is but one Lindblad operator \( c \) for the system, so that

$$\dot{\rho} = -i[H, \rho] + \mathcal{D}[c] \rho \equiv \mathcal{L} \rho. \quad (24)$$

Then it is possible to write the master equation as the nonselective evolution due to a measurement [as in equation (21)] with just two measurement operators \( \Omega_r(dt) \) by choosing

$$\Omega_1(dt) = \sqrt{dt} c, \quad (25)$$
$$\Omega_0(dt) = 1 - (iH + \frac{1}{2}c^\dagger c) \ dt. \quad (26)$$

This is because

$$\rho(t + dt) = \sum_{r=0,1} \mathcal{J}[\Omega_r(dt)] \rho(t) = (1 + \mathcal{L} dt) \rho(t), \quad (27)$$

where \( \mathcal{L} \) is as given in equation (24).

With the measurement operators \( \Omega_0(dt), \Omega_1(dt) \), it is evident that the measurement record will be a point process [53], as I will now explain. For almost all infinitesimal time intervals, the measurement result is \( r = 0 \), because \( \mathcal{P}_0(dt) = 1 - O(dt) \). The result \( r = 0 \) is thus regarded as a null result. In the case of no result, the system state changes infinitesimally, but not unitarily, via the operator \( \Omega_0 \). At randomly determined (but not necessarily Poisson distributed) times, there is a result \( r = 1 \), which I will call a detection. When this occurs, the system undergoes a finite evolution induced by the operator \( \Omega_1 \). This change can validly be called a quantum jump, although it must be remembered that it represents a sudden change in the observer’s knowledge, not an objective physical event as in Bohr’s original conception [54]. Real measurements which correspond approximately to this ideal measurement model are made routinely in experimental quantum optics. If \( c \) is the lowering operator for the dipole of the quantum system, multiplied by the square root of the damping rate, then this theory describes the system evolution in terms of photodetections [31].

Now these measurement operators can be unitarily rearranged by the matrix

$$\begin{pmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{2} |\gamma|^2 dt & \gamma \sqrt{dt} \\ -\gamma^* \sqrt{dt} & 1 - \frac{1}{2} |\gamma|^2 dt \end{pmatrix}. \quad (28)$$

This is equivalent to the transformation

$$c \rightarrow c + \gamma; \quad H \rightarrow H - i\frac{\gamma}{2}(\gamma^* c - \gamma c^\dagger). \quad (29)$$
under which the master equation (24) is of course invariant. This transformation preserves the property that the collapse operator \( \Omega_t \) is infinitesimal, and the smooth evolution operator \( \Omega_0 \) is close to unity. Explicitly,

\[
\begin{align*}
\Omega_1(dt) & = \sqrt{dt} (c + \gamma), \\
\Omega_0(dt) & = 1 - dt \left[ iH + \frac{1}{2}(c\gamma^* - c^\dagger \gamma) + \frac{1}{2}(c^\dagger + \gamma^*)(c + \gamma) \right].
\end{align*}
\]

Physically, it amounts to adding a coherent field to the output radiation field before it is detected by the system. This can be done using a low-reflectance beam-splitter. In the limit \( |\gamma|^2 \gg \langle c^\dagger c \rangle \), this procedure is known as homodyne detection.

The specification of the measurement operators is all that is necessary to define the measurement. However, it is useful to give a more explicit description of the evolution of the system under this scheme. This is done by introducing a stochastic increment \( dN(t) \) which equals one if the result 1 is obtained in the time interval \((t, t + dt)\) and 0 otherwise. Formally, \( dN(t) \) is defined by

\[
\begin{align*}
dN(t)^2 & = dN(t), \\
E[dN(t)] & = \langle \Omega_1(dt) \Omega_1(dt) \rangle = dt \langle \psi(t) | (c + \gamma)^\dagger (c + \gamma) | \psi(t) \rangle,
\end{align*}
\]

where a classical expectation value is denoted by \( E \) and the quantum expectation value by angle brackets. From the measurement operators (30,31), the stochastic evolution equation for the conditioned state vector is

\[
d|\psi(t)\rangle = \left[ dN(t) \left( \frac{c + \gamma}{\sqrt{\langle (c^\dagger + \gamma^*)(c + \gamma) \rangle(t)}} - 1 \right) \\
+ dt \left( \frac{c^\dagger c(t)}{2} - \frac{c^\dagger c + (\langle c^\dagger \gamma + \gamma^* c \rangle(t))}{2} - \gamma^* c - iH \right) \right] |\psi(t)\rangle.
\]

This gives the quantum trajectory for the system explicitly as a non-linear stochastic Schrödinger equation. Putting \( \gamma = 0 \) gives the form generally used for numerical solutions of the master equation.

## 4 Linear Quantum Trajectories

The evolution described by equation (34) corresponds to the method A of Sec. 2. That is to say, the results \( dN(t) \) are generated with the distribution they would have in actuality. No other weighting is necessary. In this case, method B would be outstandingly inefficient. It would imply either generating or not generating a photodetection with equal probability for every instant of time \((E[dN] = \frac{1}{2})\). The norms of the vast majority of states would rapidly become vanishingly small, and their contribution to the ensemble would be negligible. The only states which would significantly contribute to the ensemble would be those in which photodetections were sparse, as they would be in actuality. However, there is a half-way house, corresponding to method C of Sec. 2. This would be achieved by keeping \( E[dN] = O(dt) \). Specifically, a convenient choice is

\[
\Lambda_1(dt) = E[dN] = |\gamma|^2 dt,
\]

so that by necessity \( \Lambda_0(dt) = 1 - |\gamma|^2 dt \). Then the unnormalized conditioned state system would evolve via

\[
\begin{align*}
|\tilde{\psi}_1(t + dt)\rangle & = \frac{\Omega_1(dt)|\psi(t)\rangle}{\sqrt{\Lambda_1(dt)}} = \left( 1 + \frac{c}{\gamma} \right) |\psi(t)\rangle, \\
|\tilde{\psi}_0(t + dt)\rangle & = \frac{\Omega_0(dt)|\psi(t)\rangle}{\sqrt{\Lambda_0(dt)}} = \left[ 1 - dt \left( \frac{1}{2} c^\dagger c + c\gamma^* + iH \right) \right] |\psi(t)\rangle
\end{align*}
\]

The ostensible statistics for \( N(t) \) are those of a Poisson process. The actual probability for a detection varies in time according to

\[
P_1(dt) = \Lambda_1(dt) \langle \tilde{\psi}_1(t + dt) | \tilde{\psi}_1(t + dt) \rangle = dt \langle \psi(t) | (c + \gamma)^\dagger (c + \gamma) | \psi(t) \rangle,
\]
as before.

Writing out this evolution explicitly, we have

$$d|\psi(t)\rangle = \left[ dN(t)\frac{c}{\gamma} - dt \left( iH + \frac{1}{2}c^\dagger c + \gamma^* c \right) \right] |\psi(t)\rangle. \tag{39}$$

This is considerably simpler than equation (34), and, more importantly, it is linear in $|\psi(t)\rangle$. This has been achieved by transferring some of the information about the probability of each measurement record into the norm of $|\psi(t)\rangle$. The choice of $|\gamma|^2 dt$ for $\Lambda_1(dt)$ is obviously not a good one if the local oscillator amplitude $\gamma$ vanishes. In the remainder of this paper I am concerned with the opposite limit, $|\gamma| \to \infty$. Consider a short time $\delta t \ll 1$ in which the system changes negligibly, but the number of detections $\delta N(t) \sim |\gamma|^2 \delta t$ is very large. This allows the Poisson process $\delta N(t)$ to be approximated by a Gaussian process:

$$\delta N(t) \simeq |\gamma|^2 \delta t + |\gamma| \delta W(t), \tag{40}$$

where $\delta W(t)$ is a Gaussian random variable of zero mean and variance $\delta t$. Since this time is infinitesimal as far as the system is concerned, we can substitute this into equation (39) to obtain

$$d|\psi(t)\rangle = \left[ dW(t)e^{-i\Phi}c - dt \left( iH + \frac{1}{2}c^\dagger c \right) \right] |\psi(t)\rangle, \tag{41}$$

where $\Phi = \arg \gamma$ and $dW(t)$ can be treated as an infinitesimal Wiener increment [51] satisfying

$$\mathbb{E}[dW(t)] = 0, \quad dW(t)^2 = dt. \tag{42}$$

This equation has been previously derived by Goetsch and Graham, but in a different way. Their method involved considering ideal measurements of one quadrature of the radiated field, without involving photodetection or a local oscillator. As there is no known physical mechanism for undertaking such an ideal measurement, I believe that it is valuable to be able to derive the linear SSE (41) as a limit of a physical detection scheme.

5 Solving the Linear Quantum Trajectory

It is now convenient to introduce a new notation. Let the recorded photocurrent be defined

$$I(t) = \lim_{|\gamma| \to \infty} \frac{\delta N(t) - |\gamma|^2 \delta t}{|\gamma| \delta t} = \frac{dW(t)}{dt}, \tag{43}$$

and let the complete record of the photocurrent from time 0 to just before time $t$ be denoted $I_{(0,t)}$. Then as a reminder that $|\psi(t)\rangle$ is conditioned on the photocurrent, denote it by $|\psi^I_t\rangle$. Goetsch and Graham [21] have solved the linear SSE (41) for the case of homodyne detection, where the local oscillator phase $\Phi$ is held constant, and for the case of heterodyne detection where it varies rapidly: $\Phi(t) = \Phi_0 + \Delta t$. This was done for some particularly simple internal Hamiltonians $H$. Here I show that a solution may be found in a similar manner for any variation of $\Phi$ with time. I consider only the case $H = 0$, which amounts to a freely damped cavity. That is because I am interested in describing a single measurement of some property of the initial state of the cavity, in which all of the light is allowed to leak out and be detected.

Consider an initial coherent state $|\psi_0\rangle = |\alpha\rangle$ where $a|\alpha\rangle = \alpha|\alpha\rangle$, where $a = c$ is the annihilation operator for the damped mode, obeying $[a, a^\dagger] = 1$. Now coherent states are unique with respect to the damping master equation (24) with $H = 0$, in that they are the only states which remain pure under that evolution equation. Specifically, they have the solution $|\psi_t\rangle = |\alpha e^{-t/2}\rangle$. This means that the solution to the linear SSE (39) must be of the form

$$|\psi^I_t\rangle = \psi^I_t|\alpha e^{-t/2}\rangle. \tag{44}$$

Now using the Itô calculus, equation (41) can also be written as

$$|\psi^I_{t+dt}\rangle = \exp(-\frac{1}{2}a^\dagger adt) \exp \left( dW(t)e^{-i\Phi(t)}a - \frac{1}{2}e^{-2i\Phi(t)}a^2 dt \right) |\psi^I_t\rangle. \tag{45}$$
Now using the fact that
\[ \exp(-\frac{1}{2}a^\dagger at)|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2(1-e^{-t})\right)|ae^{-t}\rangle, \] (46)
one has
\[ \tilde{\psi}_t^A = \exp\left(-\frac{1}{2}|\alpha|^2e^{-t}dt - \frac{1}{2}e^{-2i\Phi(t)}e^{-t}dt + e^{-2i\Phi(t)}a^\dagger e^{-t}I(t)dt\right)\tilde{\psi}_0^A, \] (47)
where it is to be remembered that \( dW(t) = I(t)dt \). The total solution satisfying \( \tilde{\psi}_0 = 1 \) is thus
\[ \tilde{\psi}_t^A = \exp\left(-\frac{1}{2}|\alpha|^2(1-e^{-t}) - \frac{1}{2}\alpha^2\int_0^t e^{-2i\Phi(s)}e^{-s}ds + \alpha\int_0^t e^{-i\Phi(s)}I(s)ds\right). \] (48)

Now it follows from this that the solution for the initial coherent state \(|\psi_0\rangle = |\alpha\rangle\) can also be written
\[ |\tilde{\psi}_t^A\rangle = \exp(-\frac{1}{2}a^\dagger at)\exp\left(\frac{1}{2}S^*\langle t)a^2 + R^*(t)a\rangle|\psi_0\rangle, \] (49)
where
\[ R(t) = \int_0^t e^{2i\Phi(s)}e^{-s/2}I(s)ds; \quad S(t) = -\int_0^t e^{2i\Phi(s)}e^{-s}ds. \] (50)
That is to say, the solution at time \( t \) depends not on the full photocurrent record \( I_{[0,t]} \) (which is a continuous infinity of real numbers), but rather on that record only through the two complex integrals \( R(t) \) and \( S(t) \), regardless of how the local oscillator phase varies.

The measurement is complete in the limit \( t \to \infty \), when there is no light in the cavity and hence no information left to be obtained. In that limit, \( \lim_{t \to \infty}\exp(-\frac{1}{2}a^\dagger at) = |0\rangle\langle 0| \), so that the final conditioned state \(|0\rangle\) is obviously of no interest. What is of interest is the probability for having obtained the photocurrent record \( I_{[0,t]} \). As we have seen above, the only relevant aspects of this record are \( R(\infty) \) and \( S(\infty) \), which I will denote \( A \) and \( S(\infty) \), which I will denote \( B \). From Sec. 3, the probability for obtaining the results \( A \) and \( B \) is
\[ P(A, B) = P_0(A, B)|\tilde{\psi}_\infty^{A,B}|\tilde{\psi}_\infty^{A,B}\rangle, \] (51)
where \( |\tilde{\psi}_\infty^{A,B}\rangle \) is the unnormalized conditioned state of equation (49) and \( P_0(A, B) \) is the ostensible distribution for \( A \) and \( B \), given by
\[ P_0(A, B) = \int dI_{[0,\infty]}P_0(I_{[0,\infty]})\delta^{(2)}(A - R(\infty))\delta^{(2)}(B - S(\infty)). \] (52)
Here \( R(\infty) \) and \( S(\infty) \) are the functionals of \( I_{[0,\infty]} \) defined above, and \( P_0(I_{[0,\infty]}) \) is the ostensible distribution for \( I_{[0,\infty]} \), which as seen above, is simply that of Gaussian white noise.

From equation (49) we can rewrite the probability for obtaining results \( A, B \) as
\[ P(A, B) = \text{Tr}[F(A, B)\rho_0], \] (53)
where \( \rho_0 \) is the initial system state (which is allowed to be mixed), and \( F(A, B) \) is the effect for the results \( A, B \) defined by
\[ F(A, B) = P_0(A, B)\tilde{\Omega}^\dagger(A, B)\tilde{\Omega}(A, B), \] (54)
where the (unnormalized) measurement operators are
\[ \tilde{\Omega}(A, B) = |0\rangle\langle 0| \exp\left(\frac{1}{2}B^*a^2 + A^*a\right). \] (55)
This is a completely general result describing the sorts of measurements which can be made on an optical mode by photodetection in the large local oscillator limit. From it can be derived the standard results for homodyne and heterodyne detection (as shown below), and also some more
\footnote{It might be thought that \( S(t) \) does not even depend on the record \( I_{[0,t]} \), but it might if the local oscillator phase \( \Phi(t) \) is adjusted according to \( I_{[0,t]} \), so as to make an adaptive measurement as will be considered below.}
interesting results for adaptive measurements (as I will discuss in Sec. 6). At finite times one has less than maximal information about the system and as a result the effect cannot be factorized in terms of measurement operators. This is a minor generalization of the theory of operations and effects as stated in Sec. 2 [44]; the same condition (7) for the effects applies. Explicitly, equation (49) shows that the POVM for the measurement up to time \( t \) is

\[
F_1(R(t), S(t)) = P_0(R(t), S(t)) \exp \left( \frac{1}{2} S(t) a^\dagger a + R(t) a^\dagger \right) \exp(-a^\dagger a t) \exp \left( \frac{1}{2} S^*(t) a^2 + R^*(t) a \right),
\]

(56)

where \( P_0(R(t), S(t)) \) is defined analogously to (52).

## 5.1 Homodyne Detection

For homodyne detection of a given quadrature of the system, one desires the local oscillator phase \( \Phi \) to be constant. This means that \( B \) is simply given by \(-e^{2i\Phi} \int_0^\infty e^{-t} dt = -e^{2i\Phi} \). Thus it can be ignored as a measurement result, and we need consider only

\[
A = e^{i\Phi} \int_0^\infty e^{-t/2} dW(t).
\]

(57)

Since \( dW(t) \) is ostensibly white noise, it follows from equation (50) that we can write \( A = e^{i\Phi} X \) where the ostensible distribution for \( X \) is

\[
P_0(X) dX = \frac{1}{\sqrt{2\pi}} \exp(-X^2/2) dX.
\]

(58)

Thus the effect for such a measurement is

\[
F(X) = \frac{1}{\sqrt{2\pi}} \exp(-X^2/2) \tilde{E}(X) |0\rangle \langle 0| \tilde{E}^\dagger(X),
\]

(59)

where \( \tilde{E}(X) \) is an operator defined by

\[
\tilde{E}(X) = \exp \left( -\frac{1}{2} e^{2i\Phi} a^\dagger a \right) \left[ X e^{i\Phi} a \right]
\]

(60)

Now using some simple boson operator algebra [55], it is easy to show that

\[
(a e^{-i\Phi} + a^\dagger e^{i\Phi}) \tilde{E}(X) |0\rangle = \tilde{E}(X) (a e^{-i\Phi} + X) |0\rangle = X \tilde{E}(X) |0\rangle.
\]

(61)

That is to say, \( \tilde{E}(X) |0\rangle \) is an unnormalized eigenstate of of \( a e^{-i\Phi} + a^\dagger e^{i\Phi} \) with eigenvalue \( X \). The other factor in equation (59) supplies the normalization, so we can write

\[
F(X) = |X\rangle \langle X|,
\]

(62)

where \( |X\rangle \) is an eigenstate of \( a e^{-i\Phi} + a^\dagger e^{i\Phi} \) satisfying the delta-function normalization \( \langle X|X'\rangle = \delta(X - X') \). This is as has been known for a long time; that homodyne detection can measure a quadrature of the field arbitrarily accurately (see reference [56] for a “traditional” derivation using quantum Langevin equations). Specific cases of this result have been derived using the theory of quantum trajectories (see for example reference [13]). However, as far as I know, this is the first time it has been derived as a completely general result using this theory.

## 5.2 Heterodyne Detection

Under heterodyne detection, the local oscillator phase \( \Phi(t) \) varies rapidly at a rate \( \Delta \gg 1 \). This means that \( B \) averages over time to zero. The remaining measurement result \( A \) is defined by

\[
A = \int_0^\infty e^{i\Phi(0) + i\Delta t - t/2} dW(t).
\]

(63)

It is easy to verify that in the limit \( \Delta \gg 1 \), this has the ostensible distribution

\[
P_0(A) d^2 A = \frac{1}{\pi} \exp(-|A|^2) d^2 A
\]

(64)
Thus the effect for a completed heterodyne measurement is

\[ F(A) = \frac{1}{\pi} \exp(-|A|^2) \exp(a\dagger A)|0\rangle\langle 0| \exp(aA\dagger). \quad (65) \]

Using the techniques of reference [55], it is again simple to show

\[ \exp(-|A|^2/2 + a\dagger A) = \exp(a\dagger A - aA\dagger) \exp(aA\dagger), \quad (66) \]

and thus that

\[ \exp(-|A|^2/2 + a\dagger A)|0\rangle = \exp(a\dagger A - aA\dagger)|0\rangle = |A\rangle, \quad (67) \]

where this last state is the normalized coherent state \( a|A\rangle = A|A\rangle \). Thus we have

\[ F(A) = \frac{1}{\pi}|A\rangle\langle A|. \quad (68) \]

In this case the factor of \( \pi^{-1} \) is necessary because the coherent states are overcomplete. This result, that the distribution function for completed heterodyne measurements is the \( Q \) function \( \pi^{-1}\langle A|\rho|A\rangle \), has also been known for some time (and again there is a succinct derivation in reference [56]). It has not previously been derived using quantum trajectories as here.

### 6 Adaptive Measurements

In the two examples considered above, homodyne and heterodyne measurement, the quantum trajectory method for finding the finite time effect simply confirmed already known results. A type of optical measurement for which the effects have never before been calculated, and for which the quantum trajectory method is perhaps the only tractable approach, is continuously adaptive measurements. These are measurements in which the results so far (the photocurrent record \( I_{[0,t]} \) up to time \( t \)) are used to alter the unitary matrix which determines the measurement operators (28) at time \( t \). Note that, as expressed in equation (20), this does not change the overall evolution of the system as given by the master equation (24), and so is distinct from feedback onto the system dynamics as explored in Refs. [25, 26, 27]. In the infinite local oscillator amplitude limit being considered here, there is only one parameter which determines (28); that is the local oscillator phase \( \Phi \). The feedback-control of this phase is just what is required for a measurement which is of particular interest: that of the optical phase of the system.

A standard measurement of the phase of an optical system consists of making a completed heterodyne measurement yielding the result \( A \) as above, and then discarding the amplitude information in \( A \). That is to say, the effect for such a standard phase measurement is

\[ F_{\text{std}}(\phi) = \int |A|dA|\frac{1}{\pi}|A\rangle\langle A|, \quad (69) \]

where \( A = |A|e^{i\phi} \). Because it discards information, this effect cannot be factorized into measurement operators as in equation (5). In the language of reference [57], it is an unsharp measurement of phase. By contrast, an ideal measurement of phase [58, 59, 60] has the effect

\[ F_{\text{ideal}}(\phi) = \frac{1}{2\pi}|\phi\rangle\langle \phi|, \quad |\phi\rangle = \sum_{n=0}^{\infty} e^{in\phi}|n\rangle, \quad (70) \]

defined in terms of the unnormalized phase eigenstates \( |\phi\rangle \). Ideal measurements are the optimal way to extract information which has been encoded in the phase of the field [61].

Standard measurements of the phase are suboptimal because they measure both the phase and amplitude quadratures of the field, but use only the former information. It is intuitively obvious that one could make a better measurement (i.e., closer to ideal) by concentrating on measuring the phase quadrature. However assuming that one knows the phase quadrature before making the measurement is not in the spirit of a true phase measurement. Rather, one would have to estimate the phase quadrature during the course of the measurement, from the photocurrent so far. This is the essence of the adaptive measurement of phase. Precisely what algorithm one used
to estimate the phase quadrature would depend on the knowledge one had about the initial state of the system. Nevertheless, the analysis given above shows that whatever algorithm one used, the estimated phase at time \( t \), \( \hat{\phi}_t \), should depend on the photocurrent \( I_{[0,t)} \) only through the two complex numbers \( S(t) \) and \( R(t) \). In order to measure the phase quadrature, the local oscillator phase \( \Phi(t) \) should therefore be set equal to

\[
\Phi(t) = \hat{\phi}_t[R(t), S(t)] + \pi/2.
\]  

(71)

When the measurement is over (at \( t = \infty \)), one’s estimate for the original phase of the system is \( \phi = \hat{\phi}_\infty \).

Exactly how \( \hat{\phi}_t \) should be chosen as a function of \( R(t) \) and \( S(t) \) is a complicated issue. As stated above, it depends on the knowledge the observer has regarding the preparation of the system. This is evident in figure 1, which shows the Wigner function of the effect \( F_i(R(t), S(t)) \) (56) for some typical values of \( R \) and \( S \) which might occur in an adaptive phase measurement. The position and shape of this positive operator in phase space (\( q = a + a^\dagger \) and \( p = -ia + ia^\dagger \)) indicates the inherent uncertainties in the results of the single measurement up to time \( t \) yielding results \( R \) and \( S \). As this figure shows, the state \( F_i(R(t), S(t)) \) is a Gaussian state. As \( t \to \infty \) it becomes a minimum uncertainty squeezed state. For heterodyne detection this reduces to a coherent state of complex numbers \( R \) and \( S \) which might occur in an adaptive phase measurement. The displacements \( x \) and \( y \) are defined in terms of the measurement results \( R \) and \( S \) by

\[
x = \frac{2Re^{-i\theta}}{1 - e^{-t} - |S|} ; \quad y = \frac{2Im(e^{-i\theta})}{1 - e^{-t} + |S|}.
\]  

(72)

The variances in the \( x \) and \( y \) quadrature are

\[
V_x = \frac{1 - e^{-t} + |S|}{1 - e^{-t} - |S|} ; \quad V_y = \frac{1 - e^{-t} - |S|}{1 - e^{-t} + |S|}.
\]  

(73)

Evidently, the phase of this state is not unambiguously defined in general. One reasonable estimate would be the phase of the centre of the Wigner function, which would be

\[
\hat{\phi}_t = \theta_t + \arctan(y_t/x_t) = \arg [R(1 - e^{-t}) + SR^*].
\]  

(74)

However, depending on one’s initial knowledge of the state of the system, a different estimate may be more appropriate.

There is one example for which a different estimate is definitely more appropriate, and for which there is no ambiguity about the estimated phase of the system. This is when there is at most one photon in the system (and the observer knows this). First, I wish to present the results in this case for ideal and standard phase measurements. Projecting the optimal mode effects into the subspace spanned by \( \{|0\rangle, |1\rangle\} \) gives the ideal phase measurement effect

\[
F_{\text{ideal}}(\phi) = \frac{1}{2\pi} |\phi\rangle\langle \phi| , \quad |\phi\rangle = |0\rangle + e^{i\phi}|1\rangle,
\]  

(75)

and the marginal heterodyne effect

\[
F_{\text{std}}(\phi) = \frac{\sqrt{\pi}}{2} F_{\text{ideal}}(\phi) + \left( 1 - \frac{\sqrt{\pi}}{2} \right) \frac{1}{2\pi}.
\]  

(76)

That is to say, the standard technique has an efficiency of \( \sqrt{\pi}/2 \simeq 88\% \), in the sense that the same POVM would arise from a ideal measurement which worked 88\% of the time, and which gave a random answer on the interval \([0,2\pi]\) the other 22\% of the time.

I have considered the adaptive phase measurement of a field with at most one photon in detail in reference [62]. The lack of ambiguity in the estimated phase quadrature \( \hat{\phi}_t \) is due to the fact that with at most one photon in the field, the measurement result \( S(t) \) is irrelevant to the estimation of its state, which is obvious from equation (49). The estimated phase quadrature in this case is \( \hat{\phi}_t = \arg R(t) \). Substituting this into equation (71) leads to an analytic solution for \( P_0(A) \):

\[
P_0(A)d^2A = \pi^{-1} \delta(|A|^2 - 1)d^2A.
\]  

(77)
In other words, $A$ can contain only phase information, not amplitude information. This is precisely what is desired for a phase measurement, and indeed the POVM for this adaptive measurement is

$$ F_{\text{adapt}}(\phi) = \int_0^\infty |A|dA|P_0(A)(1 + |1\rangle\langle 0|A)\langle 0|(1 + |0\rangle\langle 1|A^* ) = F_{\text{ideal}}(\phi), \quad (78) $$

Here I have again used the result (54) for an optical mode, projected into the lowest two levels $\{ |0\rangle, |1\rangle \}$. Thus for any field with at most one photon, a simple adaptive measurement scheme can produce an optimal measurement of the phase, which is significantly better than the result with no feedback (76).

7 Conclusion

In this paper I have investigated the reciprocal links between quantum trajectories and quantum measurement theory. The first link is that one interpretation of quantum trajectories is as a special case of quantum measurement theory, that of continuous observation. Here each measurement has a duration which is infinitesimal on the time scale of the non-trivial system evolution, and is described in terms of operations and effects. This leads naturally to discrete measurement results (photodetections). The equations are actually easy to solve in the limit of a very large local oscillator, in which case the discrete photodetections are replaced by a noisy photocurrent. This is the limit which was considered in detail here.

The second link is that by solving the quantum trajectories over a finite time one can obtain the operations and effects for measurements of finite duration. These results would be difficult to obtain were it not for a formulation of quantum trajectories as linear stochastic Schrödinger equations due to Goetsch and Graham which I have rederived here in a new way. For the case where the system is not driven, but simply allowed to radiatively damp, the finite measurement time is naturally extended to infinity in order to detect all of the radiated light. Then it is possible to rederive the standard results for homodyne and heterodyne detection, and also to derive new results which would be very difficult to derive by other means. These new results are those pertaining to adaptive measurements, in which, for example, the photocurrent up to time $t$ is used to control the local oscillator phase at time $t$. Here I presented the results for one case which can be completely solved analytically: adaptive phase measurements on a system with an upper photon number bound of one.

The topic of real-time adaptive measurements begins rather than ends with phase measurements. Feedback-control of measurements can only improve the quality with which properties are measurable compared with standard non-adaptive techniques. For measuring quantities other than phase, one would not wish to work in the infinite local oscillator amplitude limit. Rather, one would have a finite local oscillator, in which case there would be two parameters to be controlled: its amplitude and phase. This is obviously a more difficult problem, and it may be that only a numerical solution using stochastic Schrödinger equations would be possible. Even using numerical techniques, it is not obvious what the general algorithm would be for optimizing such measurements, nor even how optimization should be defined. These are problems for future work, continuing the fruitful interaction between formal quantum measurement theory and its principal realm of application in quantum optics.

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Figure 1: One-standard-deviation contour for the Wigner function of the effect $F_t(R, S)$ (56) for typical values of $R$ and $S$ which would occur in the adaptive measurement of the phase of a state with a moderate coherent amplitude. Here $q = a + a^\dagger$ and $p = -ia + ia^\dagger$, $x$ and $y$ are defined by equation (72), and the variances in these quadratures by equation (73).