Min-Mid-Max Scaling, Limits of Agreement, and Agreement Score

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Abstract

In this paper, I present a new feature scaling technique that piecewise linearly maps random variables with compact supports to [-1,1] with a center of user’s choice. By using this feature scaling technique, I devise a new measure of agreement for contingency tables that compares the observed agreement with minimum, maximum, and chance-expected agreement levels. I also provide a simple formulation for the minimum agreement given row and column marginals by generating a new algorithm that minimizes the sum of diagonals in contingency tables.

Keywords: Agreement, contingency table, feature scaling

Min-Mid-Max Scaling

Feature scaling is an integral part of data pre-processing while working with multiple variables. Consider a sample with a set of features including housing price, age of house, and size of house. Imagine that housing prices vary between $40,000 and $80,000, ages of houses vary between 1 and 10, and sizes of houses vary between 80 and 400 square feet. If one uses a method with an objective function in the form of Euclidean distance, then age and size of house features play a smaller role as they are smaller in scale compared to the housing price feature. On the other hand, these features may contain valuable information for some tasks. Scaling all features into a common scale reduces the impact of features defined on large scales and allows the small-scale features to contribute equally in optimizing the objective function.

One of the most common scaling techniques is min-max scaling. It is defined as the difference between the feature and its minimum scaled by the difference between its maximum and minimum.

\[ f^{\text{min-max}}(x) = \frac{x - \min}{\max - \min} \]

Note that the min-max scaling maps random variables to [0,1]. However, it does not allow for centering the variable around a selected point, such as mean or median. Centering variables around their means allows us to interpret the transformed variables as deviations from a selected point, such as mean or median. The mean normalization centers a feature around its mean, but it fails to scale features on a single range.

\[ f^{\text{mean}}(x) = \frac{x - \text{mean}}{\max - \min} \]

As both scaling and centering features are essential parts of the pre-processing data stage for many algorithms, I propose a new scaling technique that can scale and center every feature with compact support. Consider random variable X defined on support [a,c], and a real number b strictly between a and c. The min-mid-max scaling is a mapping defined as follows.

\[ f^{\text{min-mid}}(x; a, b, c) = \begin{cases} x - b & x \leq b \\ b - a & b < x \leq c \\ x - b & x > b \end{cases} \]

The min-mid-max scaling is a piecewise linear mapping. As a particular case, it is linear when b = 0.5(a + c). Also, it is strictly increasing in x. Thus, it is bijective. Finally, middle point b can be freely selected by the user. It can be median, mean, or another benchmark. Therefore, min-mid-max scaling offers computational ease and flexibility.

By design, the transformed variable measures deviation of the original variable from b relative to two extremes, minimum and maximum. The transformed variable equals -1 when the original variable is at its minimum, a. Also, the transformed variable is 1 when the original variable is at its maximum, c. Finally, the transformed variable equals 0 when the original variable equals b. As the original variable gets close to b, the transformed variable gets close to 0. The sign of the transformed variable indicates the position of the original variable compared to b. The transformed variable is negative when the original variable is less than b, and positive when the original variable is greater than b.

Limits of Agreement

Consider a set of objects on which two raters pass a categorical judgment. The consensus between the raters often measures the reliability of these raters. Here, the agreement is defined as the fraction of observations for which both raters pass the same judgment.

In statistics, a contingency table is a matrix-form table used to display bivariate distributions.

| Rater-1 | Rater-2 |
|---------|---------|
| No      | 0.3     | 0.1    |
| Yes     | 0.2     | 0.4    |

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The contingency table above displays the bivariate distribution of raters’ decisions. For example, both raters categorize 30% of observations as ‘No’ and 40% of observations as ‘Yes’. The raters agree for 70% of observations. Is this level of agreement sufficiently high?

Assume that there are $K$ categories: $\{1, ..., K\}$. Let $f(i)$ and $g(i)$, respectively, denote the fraction of observations reported in category $i$ by rater-1 and rater-2. Also, let $p(i,j)$ denote the fraction of observations reported in category $I$ by rater-1 and category $j$ by rater-2.

Cohen (1960) proposes the chance-expected agreement as a benchmark to assess the degree of agreement. He formulates the chance-expected agreement as $\sum_{i=1}^{K} f(i)g(i)$. It is important to note that the chance-expected agreement is defined based on marginal distributions raters’ decisions. Thus, one can use it to control for differences in marginal distributions between contingency tables to a degree. Cohen’s kappa is a widely used measure to assess the degree of agreement for these reasons. Cohen’s kappa is defined as the fraction of chance agreement level $A$ which does not occur.

$$
k = \frac{\sum_{i=1}^{K} (p(i, i) - f(i)g(i))}{1 - \sum_{i=1}^{K} f(i)g(i)}$$

Cohen’s kappa can only be used to compare agreement with the chance-expected agreement. However, it does not inform us about the position of agreement relative to the minimum and maximum feasible agreement. These benchmarks are essential to compare agreement levels across contingency tables since they also depend on marginal distributions raters’ decisions.

**Definition 1.** For given $f(\cdot)$ and $g(\cdot)$, agreement level $A$ is feasible if there exist $\{p(i,j)\}$ such that

(a) $\sum_{j=1}^{K} p(i, j) = f(i)$ for all $i \in \{1, ..., K\}$,
(b) $\sum_{i=1}^{K} p(i, j) = g(j)$ for all $i \in \{1, ..., K\}$, and
(c) $\sum_{i=1}^{K} p(i, i) = A$.

**Definition 2.** For given $f(\cdot)$ and $g(\cdot)$,

(a) the excess agreement is the difference between maximum feasible agreement and chance-expected agreement, and
(b) the excess disagreement is the difference between chance-expected agreement and minimum feasible agreement.

The critical agreement values are denoted as follows.

$A^{obs}$: the observed agreement in a contingency table

$A^{rand}$: the chance-expected agreement

$A^{min}$: the minimum feasible agreement

$A^{max}$: the maximum feasible agreement

$A^{excess}$: the excess agreement

$D^{excess}$: the excess disagreement

The definition of excess disagreement may seem confusing. Cohen (1960) formulates the chance-expected disagreement as $1 - A^{rand}$. Similarly, the maximum feasible disagreement can be formulated as $1 - A^{min}$. In sync with the formulation of the excess agreement, I formulate the excess disagreement as the difference between the maximum feasible disagreement and the chance-expected disagreement and define the excess disagreement as it is stated in Definition 2.

$$
D_{\text{excess}} := \{1 - A^{\text{min}}\} - \{1 - A^{\text{rand}}\} = A^{\text{rand}} - A^{\text{min}}
$$

To formulate the excess agreement and disagreement, one first needs to formulate the maximum and minimum feasible agreement. The formulation of the maximum feasible agreement is well-known.

$$
A^{\text{max}} = \sum_{i=1}^{K} \min\{f(i), g(i)\}
$$

To formulate the minimum feasible agreement, without loss of generality, I assume that the categories are ordered in a way that the following condition holds.

$$
f(i)g(i) \leq f(i+1)g(i+1) \text{ for all } i \in \{1, ..., K-1\}
$$

**Theorem 1.** The minimum feasible agreement is formulated as follows.

$$
A^{min} = \max\{0, f(K) + g(K) - 1\}
$$

I prove Theorem 1 by devising an algorithm that generates a contingency table with the least feasible agreement for given $f(\cdot)$ and $g(\cdot)$.

**Algorithm 1. The off-diagonal matching algorithm**

Step-1 (Initialization):

$$
p(i,j) \leftarrow f(i)g(j) \text{ for all } i,j \in \{1, ..., K\}
$$

Step-2 (Update-1):

$$
\text{for i from 1 to K-1 }
\{ 
\begin{align*}
p(i, i+1) &\leftarrow p(i, i+1) + p(i,i) \\
p(i+1,i) &\leftarrow p(i+1,i) + p(i,i) \\
p(i+1,i+1) &\leftarrow p(i+1,i+1) - p(i,i) \\
p(i,i) &\leftarrow 0
\end{align*}
\}
$$

Step-3 (Update-2):

$$
B \leftarrow \sum_{i=1}^{K-1} \sum_{j=1}^{K-1} p(i,j)
$$

while $p(K, K) > 0$ and $B > 0$

$$
\text{for i from 1 to K-1 }
\{ 
\begin{align*}
p(i,K) &\leftarrow p(i,K) + \min\{p(i,j), p(K,K)\} \\
p(K,j) &\leftarrow p(K,j) + \min\{p(i,j), p(K,K)\} \\
p(i,j) &\leftarrow p(i,j) - \min\{p(i,j), p(K,K)\} \\
p(K,K) &\leftarrow p(K,K) - \min\{p(i,j), p(K,K)\} \\
B &\leftarrow B - \min\{p(i,j), p(K,K)\}
\end{align*}
\}
$$
If \( f(K) + g(K) \leq 1 \), this algorithm generates a contingency table with zero diagonals. When this condition does not hold, then it generates a contingency table with the least feasible agreement given \( f() \) and \( g() \). To understand this result, let \( C \) denote the total fraction of observations in row \( i \) and column \( j \) for all \( i, j < K \) after Step-1.

\[
C = \sum_{i=1}^{K-1} \sum_{j=1}^{K-1} f(i)g(j) = (1 - f(K))(1 - g(K))
\]

Until the last iteration in Step-2, \( C \) does not change. After the last iteration in Step-2, both \( C \) and \( p(K, K) \) decrease by \( \sum_{i=1}^{K-1}(-1)^{K-1-i}f(i)g(i) \). To show that at the end of Step-3, \( p(K, K) \) becomes zero iff \( f(K) + g(K) \leq 1 \), it suffices to prove that the following inequality holds iff \( f(K) + g(K) \leq 1 \).

\[
f(K)g(K) - \sum_{i=1}^{K-1}(-1)^{K-1-i}f(i)g(i) \\
\leq C - \sum_{i=1}^{K-1}(-1)^{K-1-i}f(i)g(i)
\]

**Proof.**

\[
(C - \sum_{i=1}^{K-1}(-1)^{K-1-i}f(i)g(i)) - (f_\text{Kg}_{\text{K}} - \sum_{i=1}^{K-1}(-1)^{K-1-i}f_{\text{g}_{i}})
\]

\[
= C - f_\text{Kg}_{\text{K}}
\]

\[
= (1 - f_\text{K})(1 - g_\text{K}) - f_\text{Kg}_{\text{K}} \text{ (by assumption } f_\text{K} \leq 1 - g_\text{K})
\]

\[
\geq (g_\text{K} - g_\text{K})(1 - g_\text{K})
\]

\[
\geq 0. \quad \blacksquare
\]

At the end of Step-3, \( p(K, K) \) equals the difference between \( f_\text{Kg}_{\text{K}} \) and \( C \) when \( f(K) + g(K) \geq 1 \).

\[
f_\text{Kg}_{\text{K}} - C = f_\text{Kg}_{\text{K}} - (1 - f_\text{K})(1 - g_\text{K}) = f(K) + g(K) - 1
\]

This discussion concludes the proof of Theorem 1 and establishes the minimum feasible agreement.

**Agreement Score**

Despite its extensive use, Cohen’s kappa is not a uniform measure of agreement, because its range depends on marginals \( f() \) and \( g() \).

### Rater-1 | Rater-2
---|---
No | No | 0.3 | 0.2 | 0.1 | 0.0 | 0.4 | 0.5

In the tables above, the chance-expected agreement is 0.5, and the observed agreement is 0.6. As a result, Cohen’s kappa is 0.2 in both tables. However, these tables have different ranges of Cohen’s kappa.

### Table 1

| \( A_{\text{min}} \) | \( A_{\text{max}} \) | Range of \( \kappa \) |
---|---|---|
Table 1 | 0 | 1 | [-1,1] |
Table 2 | 0.4 | 0.6 | [-0.2,0.2] |

Cohen’s kappa is at its maximum in table 2 and far from its maximum in Table 1. It is closer to the chance-expected agreement. This example illustrates that a simple comparison between the observed and chance-expected agreement levels is not sufficient to infer the position of the observed agreement relative to the chance-expected agreement. One can solve this problem by incorporating minimum and maximum feasible agreement levels into the calculation. To this end, I propose a new statistic called the agreement score, denoted by \( S \).

\[
S = \begin{cases} 
\frac{A_{\text{rand}} - A_{\text{obs}}}{A_{\text{rand}} - A_{\text{min}}} & A_{\text{obs}} \leq A_{\text{rand}} \\
\frac{A_{\text{obs}} - A_{\text{rand}}}{A_{\text{max}} - A_{\text{rand}}} & A_{\text{obs}} > A_{\text{rand}}
\end{cases}
\]

It is important to note that the agreement score is an example of the min-mid-max scaled version of the observed agreement with the middle point b being the chance-expected agreement. This structure offers a clear interpretation.

The interpretation of the agreement score depends only on two components: sign and magnitude. Its magnitude refers to the fraction of excess disagreement observed in the table when it is negative. Similarly, the magnitude refers to the fraction of excess agreement observed in the table when it is positive.

For the previous two tables, the agreement scores are 0.2 and 1. It implies that 20\% of the feasible excess agreement is observed in table 1, and it is fully realized in table 2.

**Conclusion**

The contribution of this paper is summarized here. First, it presents a new feature scaling algorithm that scales variables to [-1,1] centered around a point of user’s choice. Secondly, it offers a simple formulation of the least feasible agreement in a contingency table for given marginal distributions. Finally, it proposes a new agreement statistic that accounts for not only the chance-expected agreement but also the highest and the lowest feasible agreement levels. This statistic measures the fraction of feasible agreement/disagreement, which cannot be explained by chance.

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