Birationally rigid
Fano complete intersections. II

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We prove that a generic (in the sense of Zariski topology) Fano complete intersection $V$ of the type $(d_1, \ldots, d_k)$ in $\mathbb{P}^{M+k}$, where $d_1 + \ldots + d_k = M + k$, is birationally superrigid if $M \geq 7$, $M \geq k + 3$ and $\max\{d_i\} \geq 4$. In particular, on the variety $V$ there is exactly one structure of a Mori fibre space (or a rationally connected fibre space), the groups of birational and biregular self-maps coincide, $\text{Bir} V = \text{Aut} V$, and the variety $V$ is non-rational. This fact covers a considerably larger range of complete intersections than the result of [J. reine angew. Math 541 (2001), 55-79], which required the condition $M \geq 2k + 1$.

Bibliography: 14 titles.

To the memory of Eckart Viehweg

1. Formulation of the main result. Fix a set of integers $d_i \geq 2$, $i = 1, \ldots, k$, where $k \geq 2$, satisfying the following conditions:

- $\max_{i=1,\ldots,k}\{d_i\} \geq 4$,
- $d_1 + \ldots + d_k \geq \max\{2k + 3, k + 7\}$.

Set $M = d_1 + \ldots + d_k - k$. Obviously, $M \geq 7$ and $M \geq k + 3$. By the symbol $\mathbb{P}$ we denote the complex projective space $\mathbb{P}^{M+k}$. The integers $d_i$ are assumed to be non-decreasing: $d_i \leq d_{i+1}$.

Consider a smooth complete intersection

$$V = F_1 \cap \ldots \cap F_k \subset \mathbb{P}$$

of the type $(d_1, \ldots, d_k)$, that is, $\deg F_i = d_i$, $\dim V = M$. Obviously, $V$ is a smooth Fano variety, $\text{Pic} V = \mathbb{Z}H$, where $H$ is the class of a hyperplane section and $K_V = -H$, that is, $V$ is a Fano variety of index one. The main result of the present note is the following

Theorem 1. A generic (in the sense of Zariski topology) complete intersection $V$ of the type $(d_1, \ldots, d_k)$ is birationally superrigid.
Corollary 1. There are no other structures of a Mori fiber space on $V$, apart from $V/\text{pt}$: if $\chi: V \to V'$ is a birational map on the total space of a Mori fibre space $V'/S'$, then $\chi$ is a biregular isomorphism and $S'$ is a point. There exist no rational dominant maps $\alpha: V \to Y$ onto positive-dimensional varieties $Y$, the generic fibre of which is rationally connected. The groups of birational and biregular self-maps of the variety $V$ coincide: $\text{Bir } V = \text{Aut } V$.

The claim of Theorem 1 is proved in [1] for complete intersections, satisfying the condition $V \geq 2k + 1$. The present paper essentially improves that result: birational superrigidity remains an open question only for complete intersections of quadrics and cubics $(2, \ldots, 2, 3, \ldots, 3)$ and for one exceptional series $(2, \ldots, 2, 4)$; for a fixed dimension $M$ there are $[M/2] + 1$ such families. All the other Fano complete intersections of index one are covered by [1] and the present paper.

Birational superrigidity in the formulation of Theorem 1 is understood in the sense of [1,2] and other papers of the author: for any mobile linear system $\Sigma \subset |nH|$, $n \geq 1$, the equality

$$c_{\text{virt}}(\Sigma) = c(\Sigma, V) = n$$

holds, where $c(\cdot)$ is the threshold of canonical adjunction, and $c_{\text{virt}}(\cdot)$ is the virtual threshold of canonical adjunction. Obtaining Corollary 1 from Theorem 1 is a well known elementary exercise, see, for instance [1-3].

There are other definitions of birational (super)rigidity. Often (see, for instance, [4]) the first claim of Corollary 1 (the uniqueness of the structure of a Mori fibre space) is called birational rigidity, which together with the equality $\text{Bir } V = \text{Aut } V$ is called birational superrigidity. For Fano varieties of index one with the Picard group $\mathbb{Z}$ all the definitions that are currently in use are equivalent.

Now let us formulate the standard conjectures on birational geometry of Fano varieties.

**Conjecture 1.** A smooth Fano complete intersection $X \subset \mathbb{P}(a_0, \ldots, a_{M+k})$ of codimension $k$ and index one in a weighted projective space is birationally superrigid for $M \geq 5$ and birationally rigid for $M \geq 4$. A generic (in the sense of Zariski topology) complete intersection is birationally superrigid for $M \geq 4$.

Conjecture 1 is confirmed by all the known results on birational rigidity of Fano varieties, see [5-7] and the bibliography in [2]. Note that there is an example of a higher-dimensional Fano variety of index one with the Picard group $\mathbb{Z}$, which is birational to a Fano variety of high index and for that reason is not birationally rigid [8]. In that example the Fano variety has the numerical Chow group $A^2$ of codimension two cycles larger than $\mathbb{Z}$, and this difference between the example and the complete intersections seems to be crucial.

The paper is organized in the following way. In Sec. 2 we give a precise meaning to the assumption that the complete intersection $V \subset \mathbb{P}$ is generic and make the formulation of the main result more precise (Theorem 3). In Sec. 3 we prove birational superrigidity of complete intersections.
2. **Generic Fano complete intersections.** For a point \( o \in \mathbb{P} \) fix the standard affine set \( \mathbb{C}^{M+k} \subset \mathbb{P} \) with a system of linear coordinates \( (z_1, \ldots, z_{M+k}) \), where \( o = (0, \ldots, 0) \). Set
\[
 f_i = q_{i,0} + q_{i,1} + \ldots + q_{i,d_i},
\]
where the polynomials \( q_{i,a}(z) \) are homogeneous of degree \( a \). Obviously, \( o \in V(f_1, \ldots, f_k) \), if and only if \( q_{1,0} = \ldots = q_{k,0} = 0 \).

Let us introduce two sets of pairs of indices:
\[
 J = \{(i,j) \mid 1 \leq i \leq k, 1 \leq j \leq d_i, (i,j) \neq (k,d_k)\} \subset \mathbb{Z}_+ \times \mathbb{Z}_+
\]
and
\[
 J^+ = \{(i,j) \mid 1 \leq i \leq k, 2 \leq j \leq d_i, (i,j) \neq (k,d_k)\} \subset \mathbb{Z}_+ \times \mathbb{Z}_+.
\]
Recall [1] the following

**Definition 1.** The point \( o \in V(f_1, \ldots, f_k) \) is regular, if the set of homogeneous polynomials
\[
 Q = \{q_{i,j}, (i,j) \in J\}
\]
forms a regular sequence in \( \mathcal{O}_{o,\mathbb{C}^{M+k}} = \mathcal{O}_{o,\mathbb{P}} \).

A regular point is non-singular: the tangent space
\[
 T_o V = \mathbb{T} = \{q_{1,1} = \ldots = q_{k,1} = 0\} \subset \mathbb{C}^{M+k}
\]
is a linear subspace of codimension \( k \). Therefore, the point \( o \in V(f) \) is regular, if and only if it is non-singular and the set of homogeneous polynomials
\[
 Q^+ = \{q_{i,j}|_\mathbb{T}, (i,j) \in J^+\}
\]
forms a regular sequence in \( \mathcal{O}_{o,\mathbb{T}} \). The latter condition is equivalent to the closed algebraic set
\[
 \{q_{i,j}|_\mathbb{T} = 0, (i,j) \in J^+\}
\]
being a finite set of points in \( \mathbb{P}(\mathbb{T}) = \mathbb{P}^{M-1} \).

**Definition 2.** A regular point \( o \in V(f_*) \) is correct in quadratic terms, if none of the irreducible components of the closed set
\[
 \{q_{1,2}|_\mathbb{T} = \ldots = q_{k,2}|_\mathbb{T} = 0\} \tag{1}
\]
is contained in a linear subspace of codimension two in \( \mathbb{T} \).

In other words, the regular point \( o \) is correct in quadratic terms, if for any irreducible component \( W \) of the closed set \( \mathbb{H} \) its linear span \( < W > \) is either a hyperplane in \( \mathbb{T} \), or the whole space \( \mathbb{T} \). Since all polynomials \( q_{i,j} \) are homogeneous, Definition 2 can be understood in terms of the projective space \( \mathbb{P}(\mathbb{T}) = \mathbb{P}^{M-1} \).

**Theorem 2.** For a generic (in terms of Zariski topology on the space of \( k \)-uples \( (f_1, \ldots, f_k) \)) complete intersection \( V(f_1, \ldots, f_k) \) every point \( o \in V \) is regular and correct in quadratic terms.
By Theorem 2, the main result of the present paper (Theorem 1) is implied by the following fact.

**Theorem 3.** A complete intersection \( V \subset \mathbb{P} \), which is regular and correct in quadratic terms at every point is birationally superrigid.

**Proof** of Theorem 3 is given below in Sec. 3.

**Proof of Theorem 2.** Fix a point \( o \in \mathbb{P} \), an affine subset \( \mathbb{C}^{M+k} \) with coordinates \((z_1, \ldots, z_{M+k})\), where \( o = (0, \ldots, 0) \), and a set of linearly independent linear forms \( q_{1,1}(z), \ldots, q_{k,1}(z) \), so that \( \mathbb{T} = \{ q_{1,1} = \ldots = q_{k,1} = 0 \} \) is a linear subspace of codimension \( k \). Set

\[
\mathcal{L}_1 = \{ (q_{i,j}|_{\mathbb{T}}, (i, j) \in J^+) \}
\]

to be the space of tuples of \((M - 1)\) homogeneous polynomials on \( \mathbb{P}^{M-1} = \mathbb{P}(\mathbb{T}) \),

\[
\mathcal{L}_2 = \{ (q_{i,2}|_{\mathbb{T}}, i = 1, \ldots, k) \}
\]

to be the space of tuples of \( k \) quadratic polynomials on \( \mathbb{P}^{M-1} \). Define the closed set \( Y_1 \subset \mathcal{L}_1 \) by the condition that the set \( \{ q_{i,j}|_{\mathbb{T}}, (i, j) \in J^+ \} \) does not form a regular sequence (that is, the set of zeros of those polynomials is of positive dimension).

Define the closed set \( Y_2 \subset \mathcal{L}_2 \) as the closure of the set \( Y_2^o \) of such tuples of quadratic polynomials \((q_{i,2}|_{\mathbb{T}}, i = 1, \ldots, k)\), that

- the set \( Z(q_{s,2}) \) of their common zeros is of codimension \( k \) in \( \mathbb{P}^{M-1} \),
- there is an irreducible component \( W \) of the set \( Z(q_{s,2}) \), the linear span \( < W > \) of which is of codimension \( \geq 2 \) in \( \mathbb{P}^{M-1} \).

It is easy to see (as in [1, Sec. 3.2]), that Theorem 2 follows immediately from

**Proposition 1.** The following inequalities hold:

\[
\text{codim}(Y_1 \subset \mathcal{L}_1) \geq M + 1, \quad \text{codim}(Y_2 \subset \mathcal{L}_2) \geq M + 1.
\]

**Proof.** Let us obtain the first estimate. Order the set of polynomials \( q_{i,j}|_{\mathbb{T}} \) in some way, so that

\[
(q_{i,j}|_{\mathbb{T}}, (i, j) \in J^+) = (p_1, \ldots, p_{M-1}),
\]

we may assume that \( \deg p_i \leq \deg p_{i+1} \), the polynomials \( p_i \) are considered on \( \mathbb{P}^{M-1} \).

Let \( \mathcal{L}_{1,1} = \{ (p_1, \ldots, p_l) \} \) be the space of truncated tuples, \( Y_{1,1,a} \) the closure of the set \( Y_{1,1,a}^o \subset \mathcal{L}_{1,1} \), defined by the following conditions:

- the set \( Z(p_1, \ldots, p_{l-1}) = \{ p_1 = \ldots = p_{l-1} = 0 \} \) is of codimension \( l - 1 \) in \( \mathbb{P}^{M-1} \),
- there is an irreducible component \( W \) of the set \( Z(p_1, \ldots, p_{l-1}) \), on which \( p_l \) vanishes, and the linear span \( < W > \) is of codimension \( a \leq l - 1 \) in \( \mathbb{P}^{M-1} \).
It is easy to see that the first inequality of Proposition 1 follows from the estimates

$$\text{codim}(Y_{1,l,a} \subset L_{1,l}) \geq M + 1$$  \hspace{1cm} (2)

for any $l \leq M - 1$ and $a \leq l - 1$. Let us prove the inequality (2). Consider first the case $a = l - 1$. Here $W = \langle W \rangle$ is a linear subspace of codimension $l - 1$, and

$$p_1|_W \equiv 0, \ldots, p_l|_W \equiv 0.$$

Since $\deg p_i \geq 2$ and the dimension of the corresponding Grassmanian is equal to $(l - 1)(M - l + 1)$, we obtain the estimate

$$\text{codim}(Y_{1,l-1,l-1} \subset L_{1,l}) \geq l\left(\frac{M - l + 2}{2}\right) - (l - 1)(M - l + 1) =$$

$$= \frac{M - l + 1}{2}(lM - l^2 + 2).$$

It is easy to check that the minimum of the latter expression is attained at $l = M - 1$ and equal to $M + 1$. This proves the inequality (2) for $a = l - 1$.

Consider the case $a \leq l - 2$. Fixed a linear subspace $P \subset \mathbb{P}^{M-1}$ of codimension $a$ and set $Y_{1,l,a}^o(P) \subset Y_{1,l,a}$ to be the set of such tuples that there exists an irreducible component $W$ of the set $Z(p_1, \ldots, p_l)$ such that $\langle W \rangle = P$ and $p_l|_W \equiv 0$. Let $Y_{1,l,a}(P) \subset Y_{1,l,a}$ be the closure of the set $Y_{1,l,a}^o(P)$.

Let us estimate the codimension of the set $Y_{1,l,a}(P)$ in $L_{1,l}$. We use the method of the proof of Proposition 4 in [1]: there are $l - a - 1$ polynomials

$$p_{l-1}, \ldots, p_{l-a-1}$$

such that $W$ is an associated subvariety of the set $(p_1, \ldots, p_{l-a-1})$ (see [1, Definition 3]), so that

$$\text{codim}(Y_{1,l,a}(P) \subset L_{1,l}) \geq (a + 1)(2(M - a) - 1),$$

and thus

$$\text{codim}(Y_{1,l,a} \subset L_{1,l}) \geq (a + 1)(2(M - a) - 1) - a(M - a) =$$

$$= a(M - 1 - a) + 2(M - a) - 1.$$

The minimum of this function, quadratic in $a$, on the interval $[0, l - 2]$ is attained at one of its endpoints. For $a = 0$ we get $2M - 1$. For $a = l - 2$ we get

$$l(M - l + 1) + 1.$$

The minimum of the latter function on the interval $[2, M - 1]$ is attained at its endpoints and equal to $2M - 1$. This completes the proof of the first inequality.

Let us prove the second inequality. We follow the same scheme as above. Let us define the set $Y_{2,l,a}$ as the closure of the set $Y_{2,l,a}^o$ in the space of truncated tuples
\( L_{2,t} = \{(p_1, \ldots, p_t)\}, \ t \leq k \), which is determined by the following condition: there exists an irreducible component \( W \) of the set \( Z(p_1, \ldots, p_t) \), the linear span of which is of codimension \( a \geq 2 \). Obviously, it is sufficient to prove the inequality

\[
\text{codim}(Y_{2,t,a} \subset L_{2,t}) \geq M + 1
\]

for each \( a = 2, \ldots, l \). Consider first the case \( a = l \). Here \( W = \langle W \rangle \) is a linear space and we get the estimate

\[
\text{codim}(Y_{2,t,a}(P) \subset L_{2,t}) = l\frac{(M-l)(M-l+1)}{2} - l(M-l) =
\]

\[
= l(M-l)\frac{M-l-1}{2},
\]

which is by far stronger than we need.

Consider the case \( a \leq l-1 \). Fix a linear subspace \( P \subset \mathbb{P}^{M-1} \) of codimension \( a \) and construct the set \( Y_{2,t,a}(P) \) of such tuples \( (p_1, \ldots, p_t) \), that the closed set \( Z(p_1, \ldots, p_t) \) has an irreducible component \( W \) such that \( \langle W \rangle \) is of codimension \( a \geq 2 \). Among the polynomials \( p_1, \ldots, p_t \) we can find \((l-a)\) quadrics \( p_{i_1}, \ldots, p_{i_{l-a}} \), such that \( p_{i_1}|p_1, \ldots, p_{i_{l-a}}|P \) form a good sequence and \( W \) is its associated subvariety (see the proof of Proposition 4 in [1]). This implies the estimate

\[
\text{codim}(Y_{2,t,a}(P) \subset L_{2,t}) \geq a(2(M-a-1) + 1),
\]

so that

\[
\text{codim}(Y_{2,t,a} \subset L_{2,t}) \geq a(2(M-a) - 1) - a(M-a) = a(M-a-1).
\]

It is easy to check that the minimum of the right hand side on the interval \([2, l-1]\) is attained at \( a = 2 \) and equal to \( 2(M-3) \geq M + 1 \) for \( M \geq 7 \). This proves the second inequality of Proposition 1. Proof of Theorem 2 is complete.

3. Proof of birational superrigidity. Starting from this moment, we fix a complete intersection \( V(f_1, \ldots, f_k) \subset \mathbb{P}^{M+k} \), satisfying the regularity condition and the conditions of being correct in the quadratic terms at every point. Assume that the variety \( V \) is not birationally superrigid. This implies in an easy way (see [1-3]), that on \( V \) there exists a mobile linear system \( \Sigma \subset |nH| \) with a maximal singularity, that is, there are: a (non-singular) projective model \( V^\natural \), a birational morphism \( \varphi: V^\natural \to V \) and an exceptional divisor \( E^\natural \subset V^\natural \) such that the Noether-Fano inequality

\[
\text{ord}_{E^\natural} \varphi^*\Sigma > na(E^\natural, V)
\]

holds, that is, \( E^\natural \) realizes a non-canonical singularity of the pair \((V, \frac{1}{n}\Sigma)\).

The irreducible subvariety \( B = \varphi(E^\natural) \subset V \) is the centre of the maximal singularity \( E^\natural \). It is easy to check that \( \text{mult}_B \Sigma > n \).

There are three options for the codimension of the subvariety \( B \):
(i) $\text{codim } B = 2$,
(ii) $\text{codim } B = 3$,
(iii) $\text{codim } B \geq 4$,
and we must show that none of them takes place. If $\text{codim } B = 2$, then word for word the same arguments as in [1, Sec. 1.1] give a contradiction. Those arguments work for any variety $V$ (without any conditions of general position), satisfying the equality $A^2V = \mathbb{Z}H^2$, where $H$ is the positive generator of the Picard group, and $A^2$ is the numerical Chow group of codimension two cycles. (The case $\text{codim } B = 2$ can also be excluded by the argument used below in codimension three.)

If $\text{codim } B = 3$, then we come to a contradiction with the following fact.

**Proposition 2.** For any subvariety $W \subset V$ of dimension $\geq k$ the following inequality holds:

$$\text{mult}_W \Sigma \leq n.$$ 

**Proof.** This is Proposition 3.6 in [9]. (Recall that $\dim V = M \geq k + 3$ by assumption, so that if $\text{codim } B = 3$, then $\dim B \geq k$ and [9, Proposition 3.6] applies.)

Therefore, we may assume that the third case takes place: $\text{codim } B \geq 4$. Moreover, we may assume that the codimension of the subvariety $B$ is minimal among all centres of maximal singularities of the linear system $\Sigma$, in particular, $B$ is not contained in the centre $B'$ of another maximal singularity $E'$, if they do exist. Let $o \in B$ be a point of general position, $\lambda: \tilde{V} \to V$ its blow up, $E = \lambda^{-1}(o) \cong \mathbb{P}^{M-1}$ the exceptional divisor. By the symbol $\tilde{\Sigma}$ we denote the strict transform of the mobile system $\Sigma$ on $\tilde{V}$. Let

$$Z = (D_1 \circ D_2)$$

be the self-intersection of the system $\Sigma$, that is, the scheme-theoretic intersection of two generic divisors $D_1, D_2 \in \Sigma$. By the symbol $\tilde{Z}$ we denote the strict transform of the effective cycle $Z$ on $\tilde{V}$. The following fact is true.

**Proposition 3.** There exist a linear subspace $P \subset E$ of codimension two, satisfying the inequality

$$\text{mult}_o Z + \text{mult}_P \tilde{Z} > 8n^2.$$  

If $\text{mult}_o Z \leq 8n^2$, then the linear subspace $P$ is uniquely determined by the linear system $\Sigma$.

**Proof.** Let $X \ni o$ be a generic germ of a smooth subvariety of dimension $\text{codim } B \geq 4$. By the assumption on minimality of $\text{codim } B$, the pair $(X, \frac{1}{n} \Sigma_X)$ is canonical outside the point $o$, but not canonical at that point, where $\Sigma_X = \Sigma|_X$ is the restriction of the mobile system $\Sigma$ onto $X$. Now, applying [10, Proposition 4.1], we obtain the required claim.

Let us consider the tangent hyperplanes $T_o F_i$, $i = 1, \ldots, k$ (in the coordinates $z_s$ they are given by the equations $q_{i,1} = 0$). Let $T_i = V \cap T_o F_i$ be the corresponding tangent divisors, $\text{mult}_o T_i = 2$ (the equality holds because of the regularity condition), and let $\tilde{T}_i \subset \tilde{V}$ be their strict transforms on $\tilde{V}$. The equation $q_{i,2}|_\mathbb{T} = 0$ defines the projectivised tangent cone $T_i^E = (\tilde{T}_i \circ E)$, since $E = \mathbb{P}(\mathbb{T}) \cong \mathbb{P}^{M-1}$. Since none of
the irreducible components of the effective cycle \((T_1^E \circ \ldots \circ T_k^E)\) of codimension \(k\) is contained in a linear space of codimension two in \(E\), for a generic hyperplane \(\Lambda \ni P\) in \(E\) we get
\[
\text{codim}_E(T_1^E \cap \ldots \cap T_k^E \cap \Lambda) = k + 1. \tag{3}
\]

Let \(L \in |H|, L \ni o\) be a generic hyperplane section, such that \(\tilde{L} \cap E = \Lambda\). By genericity, none of the components of the cycle \(Z\) is contained in \(L\), so that \(Z_L = (Z \circ L)\) is an effective cycle of codimension three on \(V\), satisfying the estimate
\[
\text{mult}_o Z_L \geq \text{mult}_o Z + \text{mult}_P \tilde{Z} > 8n^2,
\]
and \(\text{deg} Z_L = \text{deg} Z = dn^2\). Therefore, there exists an irreducible subvariety \(C \subset V\) of codimension three (an irreducible component of the cycle \(Z_L\)), such that
\[
\frac{\text{mult}_o C}{\text{deg} C} > \frac{8}{d}.
\]

Obviously, the support of the projectivised tangent cone \(C_E = (\tilde{C} \circ E)\) is contained in the hyperplane \(\Lambda\). Let \(\Delta = \{(\lambda_1 q_{1.2} + \ldots + \lambda_k q_{k.2})| \tau = 0\}\) be the linear system of quadrics on \(E\), spanned by the quadrics \(T_i^E, \ldots, T_k^E\). By the equality \(\text{(3)}\), for \((k - 2)\) generic divisors in this system, for simplicity of notations let them be just \(T_i^E, \ldots, T_{k-2}^E\), such that
\[
\text{codim}_E(C_E \cap T_1^E \cap \ldots \cap T_{k-2}^E) = k + 1.
\]

Therefore, for the codimension in a neighborhood of the point \(o\) we also have
\[
\text{codim}_o(C \cap T_1 \cap \ldots \cap T_{k-2}) = k + 1.
\]

Now let us construct a sequence of irreducible subvarieties \(R_0, R_1, \ldots, R_{k-2}\) with the following properties:

(i) \(R_0 = C\), \(R_i \ni o\) for all \(i = 1, \ldots, k - 2\);

(ii) \(R_{i+1}\) is an irreducible component of the effective cycle \((R_i \circ T_{i+1})\), for which the ratio
\[
\frac{\text{mult}_o}{\text{deg}}
\]
takes the maximal value.

Obviously, if \(R_i \ni o\) and \(R_{i+1}\) satisfies the property (ii), then \(\text{mult}_o R_{i+1} > 0\), so that \(R_{i+1} \ni o\) as well. Moreover, \(\text{codim}_o(C \cap T_1 \cap \ldots \cap T_{i+1}) = i + 4\) and \(\text{codim} R_i = i + 3\), so that \(R_1 \not\subset T_{i+1}\) and the construction described above is possible. It is easy to see that for all \(i = 1, \ldots, k - 2\)
\[
\frac{\text{mult}_o}{\text{deg}} R_i \geq 2^i \cdot \frac{\text{mult}_o}{\text{deg}} C.
\]

Set \(R = R_{k-2}\). This is an irreducible subvariety of codimension \(k + 1\), \(o \in R\) and the inequality
\[
\frac{\text{mult}_o}{\text{deg}} R > \frac{2^{k+1}}{d}
\]
holds. Therefore, by [1, Corollary 1], we obtain the estimate

\[ \frac{2^{k+1}}{d} - 1 < \frac{\text{mult}_a R}{\deg R} \leq \frac{3d_k}{2d_k - 2} \cdot 2^b \cdot \left( \prod_{d_i \geq 3} d_i \right)^{-1}, \]

where \( b = k - \# \{ i \mid 1 \leq i \leq k, d_i = 2 \} \). The proof, given in [1, Sec. 2.2-2.3] does not make use of the condition \( M \geq 2k + 1 \) and so works for any complete intersection, satisfying the regularity condition. Since

\[ d = 2^{k-b} \left( \prod_{d_i \geq 3} d_i \right), \]

we obtain the inequality

\[ 1 < \frac{3d_k}{4d_k - 4}, \]

which is not true for \( d_k \geq 4 \). This completes the proof of Theorem 3, and so that of Theorem 1. Q.E.D.

**Remark 1.** The key difference of the arguments of the present paper from the proof given in [1] is that the combination of the \( 4n^2 \)-inequality with the Lefschetz theorem (\( A^i V = ZH^i \) for \( i > [M/2] \)) is replaced by the \( 8n^2 \)-inequality. This makes it possible to avoid the assumption that \( k < [M/2] \), which is needed to apply the Lefschetz theorem. The \( 4n^2 \)-inequality [2,3,11] goes back to the classical paper of V.A.Iskovskikh and Yu.I.Manin on the three-dimensional quartic [12]. The \( 8n^2 \)-inequality was known since 2000 (see [4]), however, its proof in [4,13] generated some doubts and, as it turned out, contained an essential gap indeed, which was corrected only very recently [10,14].

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