ON HOROWITZ AND SHELAH’S BOREL MAXIMAL EVENTUALLY DIFFERENT FAMILY

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Abstract. We show there is a closed (in fact effectively closed, i.e., \( \Pi^0_1 \)) eventually different family (working in ZF or less).

1. Introduction

A. We call a set \( E \) an eventually different family (of functions from \( \mathbb{N} \) to \( \mathbb{N} \)) if and only if \( E \subseteq \mathbb{N}^\mathbb{N} \) and any two distinct \( f_0, f_1 \in E \) are eventually different, i.e., \( \{n \mid f_0(n) = f_1(n)\} \) is finite; such a family is called maximal if and only if it is maximal with respect to inclusion among eventually different families (we abbreviate maximal eventually different family by medf).

In [2] Horowitz and Shelah prove the following (working in ZF).

Theorem 1.1 ([2]). There is a \( \Delta^1_1 \) (i.e., effectively Borel) maximal eventually different family.

This was surprising as the analogous statement is false in many seemingly similar situations: e.g., infinite so-called mad families cannot be analytic [5] (see also [9]). In a more recent, related result [1] Horowitz and Shelah obtain a \( \Delta^1_1 \) maximal cofinitary group.

In this note we present a short and elementary proof of the following improvement of Theorem 1.1:

Theorem 1.2. There is a \( \Pi^0_1 \) (i.e., effectively closed) maximal eventually different family.

To prove this we first define an medf in a simpler manner than [2] (its defining formula will be \( \Sigma^0_3 \lor \Pi^0_3 \)). We then show that we can produce from any arithmetic medf a new medf whose definition contains one less existential quantifier. The main result follows.

Note: Theorem 1.2 was announced by Horowitz and Shelah without proof in [2]; the proof in the present paper was found by the author while studying their construction of a \( \Delta^1_1 \) medf in [2].

In a related paper [8] the present author presents a further simplification of the construction and positively answers the following question of Asger Törnquist [10]: Given \( F : \mathbb{N} \to \mathbb{N} \) such that \( \lim\inf_{n \to \infty} F(n) = \infty \), is there a Borel (or even compact) medf in the restricted space \( \mathcal{N}^*_F = \{g \in \mathbb{N}^\mathbb{N} \mid (\forall n \in \mathbb{N}) g(n) \leq F(n)\} \)?

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B. We fix some notation and terminology (generally, our reference for notation is [3]). ‘∃\infty’ means ‘there are infinitely many...’ ^N\mathbb{N} means the set of functions from \mathbb{N} to \mathbb{N} and ^s<\mathbb{N}\mathbb{N} means the set of finite sequences from \mathbb{N}; we write h(s) for the length of s when s \in ^s<\mathbb{N}\mathbb{N}. For s, t \in ^n\mathbb{N}, s \prec t is the concatenation of s and t, i.e., the unique u \in ^{\text{lh}(s)+\text{lh}(t)}\mathbb{N} such that s \subseteq u and (\forall k < \text{lh}(t)) u(\text{lh}(s) + k) = t(k).

We write f_0 = \infty f_1 to mean that f_0 and f_1 are not eventually different (they are infinitely equal). Two sets A, B \subseteq \mathbb{N} are called almost disjoint if and only if A \cap B is finite, and an almost disjoint family is a set A \subseteq \mathcal{P}(\mathbb{N}) any two elements of which are almost disjoint.

Qualifications like ‘... is recursive (i.e., computable) in...’ are applied to subsets of \mathbf{H}(\omega), the set of hereditarily finite sets. Consult [7, 4, 3] for more on the (effective) Borel and projective hierarchies, i.e., on \Pi_1^0, \Pi_1^0(F), \Delta_1^0, ... sets.

All results in this paper can be derived in ZF (or in fact, in a not so strong subsystem of second order arithmetic).

C. This note is organized as follows. In Section 2 we make some motivating observations, leading to Lemma 2.5 which gives an abstract recipe for creating maximal eventually different families. We take the opportunity to give a rough sketch of the proof of Theorem 1.1 as given by Horowitz and Shelah in [2].

We then give a simpler construction instantiating the recipe from Lemma 2.5 and yielding a medf which is \Sigma_3^0 \lor \Pi_3^0 in Section 3. Lastly, we show how to get rid of all existential quantifiers in Section 3. This requires mangling the family, but the new family is still maximal eventually different.

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2. The recipe

Definition 2.1. Fix a computable (i.e., \Delta_1^0) bijection n \mapsto s_n of \mathbb{N} with ^s<\mathbb{N}\mathbb{N} and write s \mapsto \# s for its inverse. Given f: \mathbb{N} \to \mathbb{N}, let e(f): \mathbb{N} \to \mathbb{N} be the function defined by

\[
e(f)(n) = \# f \downarrow n.
\]

Clearly \{e(f) \mid f \in ^N\mathbb{N}\} is an eventually different family. At first sight, it may seem a naive strategy to make it also maximal by varying the definition of e(f) so that it leaves f intact on some infinite set. But this is just how [2] succeeds.

Definition 2.2. Let f: \mathbb{N} \to \mathbb{N}.

A. Let B(f) = \{2n + 1 \mid s_n \subseteq f\}.

B. For a set B \subseteq \mathbb{N}, let \bar{e}(f, B): \mathbb{N} \to \mathbb{N} be the function defined by

\[
\bar{e}(f, B)(n) = \begin{cases} f(n) & \text{if } n \in B, \\ \# f \downarrow n & \text{if } n \notin B. \end{cases}
\]

Remark 2.3. Note for later that f is recursive in \bar{e}(f, B(f)) as \bar{e}(f, B(f)) \upharpoonright 2\mathbb{N} = e(f) \upharpoonright 2\mathbb{N}.

The family \mathcal{E}_0 = \{\bar{e}(f, B(f)) \mid f \in ^N\mathbb{N}\} is spanning, i.e., (\forall h \in ^N\mathbb{N})(\exists g \in \mathcal{F}) h = \infty g. Interestingly, \mathcal{E}_0 is also in some sense close to being eventually different: For if
\[ \hat{e}(f, B(f))(n) = \hat{e}(f', B(f'))(n) \]

for infinitely many \( n \), almost all of these \( n \) must lie in \( B(f) \cup B(f') \) and hence as \( \{ B(f) \mid f \in \mathbb{N}^{\mathbb{N}} \} \) is an almost disjoint family,

\[ (\exists \infty n \in B(f)) f(n) = e(f')(n) \]

or the same holds with \( f \) and \( f' \) switched.

The brilliant idea of Horowitz and Shelah is the following: Ensure maximality with respect to \( f \) which look like \( e(f') \) on an infinite set using \( e(f') \); restrict the use of \( \hat{e} \) to \( f \) which don’t look like they arise from \( e \) on some infinite subset of \( B(f) \) to avoid the situation described above. We make these ideas precise in the following definition and in Lemma 2.5 below.

**Definition 2.4.** Let a function \( f : \mathbb{N} \to \mathbb{N} \) and \( X \subseteq \mathbb{N} \) be given. We say \( f \) is \( \infty \)-coherent on \( X \) if and only if there is \( f' \in \mathbb{N}^{\mathbb{N}} \) and infinite \( X' \subseteq X \) such that \( f \upharpoonright X' = e(f') \upharpoonright X' \).

We can now give a general recipe for constructing a medf.

**Lemma 2.5.** Suppose that \( T \subseteq \mathbb{N}^{\mathbb{N}} \) and \( C : \mathbb{N}^{\mathbb{N}} \to \mathcal{P}(\mathbb{N}) \) is a function such that

(A) If \( f \notin T \), there is an infinite set \( X' \subseteq C(f) \) and \( f' \in \mathbb{N}^{\mathbb{N}} \) such that \( f \upharpoonright X' = e(f') \upharpoonright X' \); i.e., \( f \) is \( \infty \)-coherent on \( C(f) \).

(B) If \( f \in T \), for no \( f' \in \mathbb{N}^{\mathbb{N}} \) does \( f \) agree with \( e(f') \) on infinitely many points in \( C(f) \); i.e., \( f \) is not \( \infty \)-coherent on \( C(f) \).

(C) \( \{ C(f) \mid f \in T \} \) is an almost disjoint family.

Then

\[ E = \{ \hat{e}(f, C(f)) \mid f \in T \} \cup \{ e(f) \mid f \notin T \} \]

is a maximal eventually different family.

Of course the challenge here is to define \( C \) and \( T \) so that \( E \) is \( \Delta^1_1 \); before we discuss this aspect, we prove the lemma.

For the sake of this proof it will be convenient to define the map \( \hat{e} : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}} \) as follows: For \( f \in \mathbb{N}^{\mathbb{N}} \) let \( \hat{e}(f) \) be the function defined by

\[ \hat{e}(f) = \begin{cases} \hat{e}(f, C(f)) & \text{if } f \in T, \\ e(f) & \text{otherwise}. \end{cases} \] (1)

Clearly \( E = \{ \hat{e}(f) \mid f \in \mathbb{N}^{\mathbb{N}} \} \).

**Proof of Lemma 2.5.** To show \( E \) consists of pairwise eventually different functions, fix distinct \( g_0 \) and \( g_1 \) from \( E \) and suppose \( g_i = \hat{e}(f_i) \) for each \( i \in \{0, 1\} \). Clearly we can disregard the set

\[ N = \{ n \in \mathbb{N} \mid g_0(n) = e(f_0)(n) \text{ and } g_1(n) = e(f_1)(n) \} \]

as \( g_0 \) and \( g_1 \) can only agree on finitely many such \( n \).

If \( n \notin N \) then it must be the case that for some \( i \in \{0, 1\} \), \( f_i \in T \) and \( n \in C(f_i) \); suppose \( i = 0 \) for simplicity. By (C) we may restrict our attention to \( C(f_0) \setminus C(f_1) \) where \( g_0 \) agrees with \( f_0 \) and \( g_1 \) agrees with \( e(f_1) \). But \( f_0 \) and \( e(f_1) \) can’t agree on an infinite subset of \( C(f_0) \setminus C(f_1) \) by (B).

It remains to show maximality. So let \( f : \mathbb{N} \to \mathbb{N} \) be given. If \( f \in T \) we have \( \hat{e}(f) \upharpoonright C(f) = f \upharpoonright C(f) \) and \( \hat{e}(f) \in E \) by definition.

If on the other hand \( f \notin T \) there is \( f' \in \mathbb{N}^{\mathbb{N}} \) such that \( e(f') \) agrees with \( f \) on an infinite subset of \( C(f) \). As \( \hat{e}(f') \in E \) it suffices to show \( f = \infty \hat{e}(f') \).
If \( f' \notin T \) as well this is clear as \( \hat{e}(f') = e(f') \). If on the contrary \( f' \in T \), we have \( f \neq f' \) and so \( C(f) \cap C(f') \) is finite by (C). So \( \hat{e}(f') \) agrees with \( e(f') \) for all but finitely many points in \( C(f) \) and hence agrees with \( f \) on infinitely many points. \( \square \)

Note that letting \( T = \{ f \in \mathbb{N} \mid f \) is not \( \infty \)-coherent on \( B(f) \} \) and \( C(f) = B(f) \) the requirements of Lemma 2.5 are trivially satisfied; but the resulting \( \mathcal{E} \) will not be Borel (only \( \Pi^1_1 \vee \Sigma^1_1 \)). On the other hand if \( T \) is \( \Delta^1_1 \) and \( C: \mathbb{N} \to \mathcal{P}(\mathbb{N}) \) is \( \Sigma^1_1 \), then \( \mathcal{E} \) is \( \Delta^1_1 \), and in fact it follows that \( \mathcal{E} \) is \( \Delta^1_1 \) in this case because\(^1\) it is a medf and so

\[
h \notin \mathcal{E} \iff (\exists g \in \mathbb{N}) h \neq g \wedge h \Rightarrow g \wedge g \in \mathcal{E}.
\]

(Of course the function \( C: \mathbb{N} \to \mathcal{P}(\mathbb{N}) \) is also automatically \( \Delta^1_1 \).) We may view the task at hand to be: find a reasonably effective process producing from a function \( f \) either a subset of \( B(f) \) where \( f \) agrees with some \( e(f') \) or a set \( C(f) \subseteq B(f) \) on which \( f \) can be seen effectively to not be \( \infty \)-coherent.

From this we can sketch what is arguably the core of Horowitz and Shelah’s construction from [2]. The present author has not verified whether their construction yields an arithmetic family.

**Proof of Theorem 1.1.** Given \( f: \mathbb{N} \to \mathbb{N} \) define a coloring of unordered pairs from \( \mathbb{N} \) as follows (supposing without loss of generality that \( k < k' \)):

\[
c(\{k, k'\}) = \begin{cases} 
0 & \text{if } lh(s_f(k)) = k, \; lh(s_f(k')) = k', \; \text{and } s_f(k) \subseteq s_f(k'), \\
1 & \text{otherwise.}
\end{cases}
\]

Let \( T \) be such that for every \( f \in T \) there is an infinite set \( X \subseteq B(f) \) which is 1-homogeneous, i.e., \( c \) assigns the color 1 to every unordered pair from \( X \), and for every \( f \notin T \) there is an infinite 0-homogeneous \( X \subseteq B(f) \). Then (A) holds. For \( f \in T \) let \( C(f) \) be some infinite 1-homogeneous \( X \subseteq B(f) \); for \( f \notin T \) let \( C(f) = B(f) \). Then (B) and (C) hold by definition and by Lemma 2.5 \( \mathcal{E} \) is a medf.

By the proof of the Infinite Ramsey Theorem, the set \( T \) can be chosen to be \( \Delta^1_1 \) and the function \( C: \mathbb{N} \to \mathcal{P}(B(f)) \) can be chosen to be \( \Sigma^1_1 \). Thus \( \mathcal{E} \) as defined in Lemma 2.5 is \( \Delta^1_1 \). \( \square \)

In the next section, we essentially replace the appeal to the Infinite Ramsey Theorem by a simple instance of the law of excluded middle.

3. **A MAXIMALLY EVENTUALLY DIFFERENT FAMILY WITH A SIMPLE DEFINITION**

We now give a simpler construction of a family satisfying the requirements of Lemma 2.5.

**Definition 3.1 (The medf \( \mathcal{E} \)).**

A. Let \( f: \mathbb{N} \to \mathbb{N} \). Define a binary relation \( \prec_f \) on \( \mathbb{N} \) by

\[
m \prec_f m' \iff \left( \left( lh(s_f(m)) = m \right) \right) \wedge \left( \left( lh(s_f(m')) = m' \right) \wedge \left( \left( s_f(m) \not\subseteq s_f(m') \right) \right) \right)
\]

B. Let \( T \) be the set of \( f: \mathbb{N} \to \mathbb{N} \) such that

\[
(\forall m \in B(f))(\exists m \in B(f) \setminus m)(\forall m' \in B(f) \setminus m) \sim(m \prec_f m')
\]

We also say \( f \) is **tangled** to mean \( f \in T \).

\(^1\)In this context, the much more general Theorem 1.4.23 in [2] p. 15] deserves mention; compare also [3] 35.10, p. 285].
Lemma 3.2. The set \( f \) by (i) and (ii) above. For (A), suppose
\[
E = \{ f \mid f \in \mathbb{N}^\mathbb{N} \}
\]
where \( \hat{e}(f) \) is the function defined as in (I):
\[
\hat{e}(f) = \begin{cases} 
\bar{e}(f, C(f)) & \text{if } f \in \mathcal{T}, \\
e(f) & \text{otherwise.}
\end{cases}
\]

We want to call the following to the readers attention:
(i) \( \{ C(f) \mid f \in \mathbb{N}^\mathbb{N} \} \) is an almost disjoint family (as \( C(f) \subseteq B(f) \) by definition).

(ii) When \( f \) is tangled, \( C(f) \) is an infinite set by (2) and for no \( f' \in \mathbb{N}^\mathbb{N} \) does \( f \)
agree with \( f' \) on infinitely many (or in fact, just two) points in \( C(f) \) — i.e.,
\( f \) is not \( \infty \)-coherent on \( C(f) \).

Lemma 3.3. The set \( E \) is a maximal eventually different family.

Proof. We show that Lemma 2.3 can be applied. Requirements (C) and (B) hold by (I) and (II) above. For (A), suppose \( f \) is not tangled, i.e.,
\[
(\exists n \in B(f))(\forall m \in B(f) \setminus n)(\exists m' \in B(f) \setminus m) m \prec_{f} m'.
\]

Let \( m_0 \) be the least witness to the leading existential quantifier above; by recursion let \( m_{i+1} \) be the least \( m' \) in \( B(f) \) above \( m \) such that \( m_i \prec_{f} m' \). Letting \( f' = \bigcup \{ s_{f(m)} \mid i \in \mathbb{N} \} \) yields a well-defined function in \( \mathbb{N}^\mathbb{N} \) such that \( f = \infty e(f') \), i.e., \( f \)
is \( \infty \)-coherent on \( C(f) \).

It is obvious that \( E \) is \( \Delta^1_1 \). We now show a stronger result.

Lemma 3.4. The set \( E \) is in the Boolean algebra generated by the \( \Sigma^0_3 \) sets in \( \mathbb{N}^\mathbb{N} \).

Proof. By construction \( g \in E \) if and only if the following holds of \( g \) (see Remark 2.3):
(I) \( (\forall n \in \mathbb{N}) \) \( \text{lht}(s_g(2n)) = 2n \), and
(II) \( (\forall n \in \mathbb{N})(\forall m \leq n) s_g(2m) \subseteq s_g(2n) \), and letting \( f = \bigcup_{n \in 2\mathbb{N}} s_g(2n) \).
(III) either the following three requirements hold:
(a) \( f \) is tangled and
(b) \( (\forall n \in \mathbb{N}) n \in C(f) \Rightarrow g(n) = f(n) \) and
(c) \( (\forall n \in \mathbb{N}) n \notin C(f) \Rightarrow g(n) = e(f)(n) \);
(IV) or both of the following hold:
(a) \( f \) is not tangled and
(b) \( (\forall n \in \mathbb{N}) g(n) = e(f)(n) \).

As \( C(f) \) is \( \Pi^1_1(f) \) for \( f \in \mathcal{T} \) and \( \Pi^1_1 \) is \( \Pi^0_3(f) \), clearly \( \Pi^1_1 \) is \( \Pi^1_0(g, f) \). Likewise
(IV) is \( \Sigma^0_3(g, f) \). As \( f \) is recursive in \( g \), \( \Pi^1_1 \) can be expressed by a \( \Pi^0_3(g) \) formula and
(IV) can be expressed by a \( \Sigma^0_3(g) \) formula (substitute each expression of the form \( f(n) = m \) by \( s_g(2n+2)(n) = m \) and \( f \upharpoonright n \) by \( s_g(2n) \upharpoonright n \)).

4. MANGLING AWAY EXISTENTIAL QUANTIFIERS

We use the following lemma to reduce the complexity of the family \( E \).

Lemma 4.1. Let \( \xi < \omega_1 \). Suppose there is a \( \Pi^0_{\xi+2} \) maximal eventually different family. Then there is a \( \Pi^0_{\xi+1} \) maximal eventually different family.
Proof. Suppose
\[ f \in \mathcal{E} \iff (\forall n \in \mathbb{N})(\exists m \in \mathbb{N}) \Psi(n, m, f), \]
where \( \Psi(n, m, f) \) is \( \Pi^0_2 \). For each \( f \in \mathcal{E} \) let \( g_f : \mathbb{N} \to \mathbb{N} \) be the function such that for each \( n \in \mathbb{N} \), \( g_f(n) \) is the least \( m \) satisfying \( \Psi(n, m, f) \).

We construct a set \( \mathcal{E}^* \) of functions from \( \mathbb{N} \) to \( \mathbb{N} \) as follows. Given \( f \in \mathcal{E} \), let \( f^* : \mathbb{N} \to \mathbb{N} \) be the following function: for \( n \in \mathbb{N} \) and \( i \in \{0, 1\} \) let

\[ f^*(2n + i) = \begin{cases} f(n) & \text{for } i = 0; \\ \#(f \upharpoonright n + 1 \setminus (g_f \upharpoonright n + 1)) & \text{for } i = 1. \end{cases} \]

It is straightforward to check that \( \mathcal{E}^* \) is a medf as every function will agree with an element of \( \mathcal{E}^* \) on infinitely many even numbers.

Lastly, \( \mathcal{E}^* \) is \( \Pi^0_{\xi + 1} \): Let \( \Psi'(n, m, h) \) denote the formula obtained from \( \Psi(n, m, f) \) by replacing each occurrence of \( f(m) = n \) by \( h(2m) = n \). Clearly \( \Psi' \) is \( \Pi^0_1 \).

Let \( S_2 \) denote the recursive set \( \{m \in \mathbb{N} \mid (\forall n \in \mathbb{N}) s_m \in ^{\mathbb{N}} \mathbb{N} \} \) and given \( m \in S_2 \), write \( f_m \) for \( s_m \upharpoonright h(2m) \) and \( g_m \) for the function \( t : n \to \mathbb{N} \) given by \( k \mapsto s_m(n + k) \). In other words, if \( m = \#(f \upharpoonright n + 1 \setminus (g_f \upharpoonright n + 1)) \) as above in the definition of \( f^* \), then \( f_m = f \upharpoonright n + 1 \) and \( g_m = g_f \upharpoonright n + 1 \). Clearly \( m \mapsto f_m \) and \( m \mapsto g_m \) are both recursive on \( S_2 \).

It is straightforward to check that \( h \in \mathcal{E}^* \) if and only if for every \( n \in \mathbb{N} \) all of the following hold:

(i) \( h(2n + 1) \in S_2 \land h(2n) = f_{h(2n + 1)}(n) \)
(ii) \( \Psi'(n, g_{h(2n + 1)}(n), h) \)
(iii) \( (\forall m < g_{h(2n + 1)}(n)) \neg \Psi'(n, m, h) \).

Requirement (i) is \( \Delta^0_1(h) \); (ii) is \( \Pi^0_1(h) \) and (iii) is \( \Sigma^0_1(h) \). So \( \mathcal{E}^* \) is \( \Pi^0_{\xi + 1} \).

In fact (but we have no use for this) it is possible to carry out out a similar construction as the above for limit \( \xi \). This would give a second proof that there is a \( \Pi^0_3 \) medf based on the construction of any \( \Delta^1_1 \) medf regardless of its precise complexity, and a version of the above lemma.

Corollary 4.2. There is a \( \Pi^0_3 \) maximal eventually different family.

Proof. By Lemma [3] there is an arithmetic (in fact \( \Sigma^0_3 \lor \Pi^0_3 \)) medf so we obtain a \( \Pi^0_3 \) medf by the previous lemma. \( \square \)

References

[1] Haim Horowitz and Saharon Shelah, A Borel maximal cofinitary group, \( \text{arxiv:1610.01344[math.LO]} \) October 2016.
[2] \( \text{arxiv:1610.01344[math.LO]} \) May 2016.
[3] Alexander S. Kechris, Classical descriptive set theory, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR 1321597
[4] Richard Mansfield and Galen Weitkamp, Recursive aspects of descriptive set theory, Oxford Logic Guides, vol. 11, The Clarendon Press, Oxford University Press, New York, 1985, with a chapter by Stephen Simpson. MR 786122
[5] A. R. D. Mathias, Happy families, Ann. Math. Logic 12 (1977), no. 1, 59–111. MR 0409107
[6] Ben Miller, An introduction to classical descriptive set theory, See http://www.logic.univie.ac.at/millerb45/notes/dst.pdf 2015.
[7] Yiannis N. Moschovakis, Descriptive set theory, second ed., Mathematical Surveys and Monographs, vol. 155, American Mathematical Society, Providence, RI, 2009. MR 2526093
[8] David Schrittesser, Compactness of maximal eventually different families, arXiv:1704.04751[math.LO] April 2017.
[9] Asger Törnquist, Definability and almost disjoint families, arXiv:1503.07577[math.LO] March 2015.
[10] personal communication, February 2017.

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