Blow-up phenomena for a viscoelastic wave equation with strong damping and logarithmic nonlinearity

Tae Gab Ha¹ and Sun-Hye Park ²*

*Correspondence: sh-park@pusan.ac.kr
²Office for Education Accreditation, Pusan National University, Busan, South Korea
Full list of author information is available at the end of the article

Abstract

In this paper we consider the initial boundary value problem for a viscoelastic wave equation with strong damping and logarithmic nonlinearity of the form

\[ \begin{align*}
  u_{tt}(x, t) - \Delta u(x, t) + \int_0^t g(t - s) \Delta u(x, s) \, ds - \Delta u_t(x, t) &= |u(x, t)|^{p-2} u(x, t) \ln |u(x, t)| \\
  \text{in } \Omega \times (0, T),
\end{align*} \]

in a bounded domain \( \Omega \subset \mathbb{R}^n \), where \( g \) is a nonincreasing positive function. Firstly, we prove the existence and uniqueness of local weak solutions by using Faedo–Galerkin’s method and contraction mapping principle. Then we establish a finite time blow-up result for the solution with positive initial energy as well as nonpositive initial energy.

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1 Introduction

In this paper, we are concerned with the following viscoelastic wave equation with strong damping and logarithmic nonlinearity source:

\[ \begin{align*}
  u_{tt} - \Delta u + \int_0^t g(t - s) \Delta u(s) \, ds - \Delta u_t &= |u|^{p-2} u \ln |u| \quad \text{in } \Omega \times (0, T), \\
  u &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
  u(0) = u_0, \quad u_t(0) = u_1 \quad \text{on } \Omega,
\end{align*} \]

where \( \Omega \subset \mathbb{R}^n \), \( n \geq 1 \), is a bounded domain with smooth boundary \( \partial \Omega \). This type of equation is related to viscoelastic mechanics, quantum mechanics theory, nuclear physics, optics, geophysics and so on. For instance, the logarithmic nonlinearity arises in the inflation cosmology and super-symmetric fields in the quantum field theory. In the case \( n = 1, 2 \), Eq. (1.1) describes the transversal vibrations of a homogeneous viscous string and the
longitudinal vibrations of a homogeneous bar, respectively. For the physical point of view, we refer to [1–3] and the references therein.

During the past decades, the strongly damped wave equations with source effect

\[ u_{tt} - \Delta u - \omega \Delta u_t + \mu u_t = f(u) \]  

have been studied extensively on existence, nonexistence, stability, and blow-up of solutions. In the case of power nonlinearity \( f(u) = |u|^{p-2}u \), Sattinger [4] firstly considered the existence of local as well as global solutions for equation (1.4) with \( \omega = \mu = 0 \) by introducing the concepts of stable and unstable sets. Since then, the potential well method has become an important theory to the study of the existence and nonexistence of solutions [5–15]. Ikehata [8] gave properties of decay estimates and blow-up of solutions to (1.4) with linear damping (\( \omega = 0 \) and \( \mu > 0 \)). Gazzola and Squassina [6] proved the global existence and finite time blow-up of solutions to problem (1.4) with weak and strong damping (\( \omega > 0 \)). Liu [11] considered a viscoelastic version of (1.4). He investigated decay estimates for global solutions when the initial data enter the stable set and showed finite blow-up results when the initial data enter the unstable set.

In the case of logarithmic nonlinearity \( f(u) = u \ln |u| \), Ma and Fang [16] proved the existence of global solutions and infinite time blow-up to problem (1.4) with \( \omega = 1, \mu = 0 \), and \( k = 2 \). Lian and Xu [17] investigated global existence, energy decay and infinite time blow-up when \( \omega \geq 0 \) and \( \mu > -\omega \lambda_1 \), where \( \lambda_1 \) is the first eigenvalue of the operator \(-\Delta\) under homogeneous Dirichlet boundary conditions. The results of [16, 17] were obtained by use of the potential well method and the logarithmic Sobolev inequality.

By the way, there is not much literature for strongly damped wave equations with the logarithmic nonlinear source \( |u|^{p-2}u \ln |u| \). Recently, Di et al. [18] considered problem (1.1)–(1.3) when the kernel function \( g = 0 \). The presence of the Laplacian operator \(-\Delta u\) and the logarithmic nonlinearity \( |u|^{p-2}u \ln |u| \) causes some difficulty so that one cannot apply the logarithmic Sobolev inequality [19]. Thus, they discussed the global existence, uniqueness, energy decay estimates and finite time blow-up of solutions by modifying the potential well method. We also refer to [20, 21] and the references therein for problems with logarithmic nonlinearity.

Motivated by these results, we study the existence and finite time blow-up of weak solutions for problem (1.1)–(1.3) in the present work by applying the ideas in [11, 18]. To the best our knowledge, this is the first work in the literature that takes into account a viscoelastic wave equation with strong damping and logarithmic nonlinearity in a bounded domain \( \Omega \subset \mathbb{R}^n \).

The outline of this paper is as follows. In Sect. 2, we give materials needed for our work. In Sect. 3, we prove the local existence of solutions for problem (1.1)–(1.3) using Faedo–Galerkin’s method and contraction mapping principle. In Sect. 4, we establish a finite time blow-up result.

## 2 Preliminaries

In this section we give notations, hypotheses, and some lemmas needed in our main results.

For a Banach space \( X \), \( \| \cdot \|_X \) denotes the norm of \( X \). As usual, \( (\cdot, \cdot) \) and \( \langle \cdot, \cdot \rangle \) denote the inner product in the space \( L^2(\Omega) \) and the duality pairing between \( H^{-1}(\Omega) \) and \( H_0^1(\Omega) \),
\[ \|v\|_q \leq c_q \|\nabla v\|^2 \quad \text{for } v \in H^1_0(\Omega), \]

where

\[ 2 \leq q < \infty, \quad \text{if } n = 1, 2; \quad 2 \leq q \leq \frac{2n}{n-2}, \quad \text{if } n \geq 3. \]

We need the following lemma.

**Lemma 2.1** For each \( q > 0 \),

\[ \left| s^q \ln s \right| \leq \frac{1}{eq} \quad \text{for } 0 < s < 1 \quad \text{and} \quad 0 \leq s^{-q} \ln s \leq \frac{1}{eq} \quad \text{for } s \geq 1. \]

**Proof** We can easily show this from simple calculation. So, we omit it here. \( \square \)

**Lemma 2.2** ([9]) Let \( L(t) \) be a positive, twice differentiable function satisfying the inequality

\[ L(t) L''(t) - (1 + \delta)(L'(t))^2 \geq 0 \quad \text{for } t > 0, \]

with some \( \delta > 0 \). If \( L(0) > 0 \) and \( L'(0) > 0 \), then there exists a time \( T_* \leq \frac{L(0)}{\delta L'(0)} \) such that

\[ \lim_{t \to T_*} L(t) = +\infty. \]

With regard to problem (1.1)–(1.3), we impose the following assumptions:

\( (H_1) \) Hypotheses on \( p \).

The exponent \( p \) satisfies

\[ 2 < p < \infty, \quad \text{if } n = 1, 2; \quad 2 < p < \frac{2(n-1)}{n-2}, \quad \text{if } n \geq 3. \quad (2.1) \]

\( (H_2) \) Hypotheses on \( g \).

The kernel function \( g : [0, \infty) \to [0, \infty) \) is a nonincreasing and differentiable function satisfying

\[ 1 - \int_0^\infty g(s) \, ds := l > 0. \quad (2.2) \]

**Definition 2.1** Let \( T > 0 \). We say that a function \( u \) is a weak solution of problem (1.1)–(1.3) if

\[ u \in C([0, T]; H^1_0(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \cap C^2([0, T]; H^{-1}(\Omega)), \]
leads to

\[
\langle u_{tt}(t), w \rangle + (\nabla u(t), \nabla w) - \int_0^t g(t-s)(\nabla u(s), \nabla w) \, ds + (\nabla u_t(t), \nabla w)
\]

\[
= \int_\Omega |u(x,t)|^{p-2} u(x,t) \ln |u(x,t)| \, w \, dx
\]  

(2.3)

for any \( w \in H^1_0(\Omega) \) and \( t \in (0, T) \), and

\[
u(0) = u_0 \quad \text{in } H^1_0(\Omega), \quad u_t(0) = u_1 \quad \text{in } L^2(\Omega).
\]

3 Local existence of solutions

In this section we prove the local existence of solutions making use of the Faedo–Galerkin method and the contraction mapping principle. For a fixed \( T > 0 \), we consider the space

\[ \mathcal{H} = C([0, T]; H^1_0(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \]

with the norm

\[ \|v\|^2 = \max_{0 \leq t \leq T} (\|v(t)\|^2 + t \|\nabla v(t)\|^2). \]

To show the existence and uniqueness of local solution to problem (1.1)–(1.3), we firstly establish the following result.

Lemma 3.1 Assume that \((H_1)\) and \((H_2)\) hold. Then, for every \( u_0 \in H^1_0(\Omega) \), \( u_1 \in L^2(\Omega) \), \( v \in \mathcal{H} \), there exists a unique

\[ u \in C([0, T]; H^1_0(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \cap C^2([0, T]; H^{-1}(\Omega)) \]

such that \( u_t \in L^2([0, T]; H^1_0(\Omega)) \) and

\[
u_{tt} - \Delta u + \int_0^t g(t-s) \Delta u(s) \, ds - \Delta u_t = |v|^{p-2} v \ln |v| \quad \text{in } \Omega \times (0, T),
\]

\[
u = 0 \quad \text{on } \partial \Omega \times (0, T),
\]

\[
u(0) = u_0, \quad u_t(0) = u_1 \quad \text{on } \Omega.
\]

Proof Existence. Let \( \{w_j\}_{j \in \mathbb{N}} \) be an orthogonal basis of \( H^1_0(\Omega) \) which is orthonormal in \( L^2(\Omega) \) and \( W_m = \text{span}\{w_1, w_2, \ldots, w_m\} \), then there exist subsequences \( u_{00}^m \in W_m \) and \( u_{11}^m \in W_m \) such that \( u_{00}^m \rightharpoonup u_0 \) in \( H^1_0(\Omega) \) and \( u_{11}^m \rightharpoonup u_1 \) in \( L^2(\Omega) \), respectively. We will seek an approximate solution

\[ u^m(x, t) = \sum_{j=1}^m h_j^m(t)w_j(x) \]
satisfying
\[
(u''_m(t), w) + (\nabla u''_m(t), \nabla w) - \int_0^t g(t - s) (\nabla u''_m(s), \nabla w) \, ds + (\nabla u'_m(t), \nabla w)
\]
\[
= \int_\Omega |v(x, t)|^{p-2} v(x, t) \ln |v(x, t)| w(x) \, dx \quad \text{for } w \in W_m
\]
(3.4)
and the initial conditions
\[
uu_{m}(0) = u''_m, \quad uu'_m(0) = u'_m. \tag{3.5}\]

Since (3.4)–(3.5) is a normal system of ordinary differential equations, there exists a solution \( u''_m \) on the interval \([0, t_m] \subset [0, T] \). We obtain an a priori estimate for the solution \( u''_m \) so that it can be extended to the whole interval \([0, T] \) according to the extension theorem.

**Step 1. A priori estimate.** Replacing \( w \) by \( u''_m(t) \) in (3.4) and using the relation
\[
\int_0^t g(t - s) (\nabla u''_m(s), \nabla u'_m(t)) \, ds = -\frac{g(t)}{2} \| \nabla u''_m(t) \|^2 + \frac{1}{2} (g' \circ \nabla u''_m)(t)
\]
\[
- \frac{1}{2} \frac{d}{dt} \left( g \circ \nabla u''_m \right)(t) - \int_0^t g(s) \, ds \| \nabla u''_m(t) \|^2,
\]
where
\[
(g \circ \phi)(t) = \int_0^t g(t - s) \| \phi(t) - \phi(s) \|^2 \, ds,
\]
we have
\[
\frac{d}{dt} \left( \| u''_m \|^2 + \| \nabla u''_m \|^2 \right) + (g \circ \nabla u''_m)(t) + 2 \| \nabla u'_m \|^2
\]
\[
= (g' \circ \nabla u''_m)(t) - g(t) \| \nabla u''_m(t) \|^2 + 2 \int_\Omega |v(x, t)|^{p-2} v(x, t) \ln |v(x, t)| u''_m(x, t) \, dx.
\]

Integrating this over \((0, t)\) and making use of \((H_2)\),
\[
\| u''_m \|^2 + \| \nabla u''_m \|^2 + (g \circ \nabla u''_m)(t) + 2 \int_0^t \| \nabla u'_m(s) \|^2 \, ds
\]
\[
\leq \| u''_1 \|^2 + \| \nabla u''_m \|^2 + 2 \int_0^t \| v(s) \|^{p-2} v(s) \ln |v(s)| \| \nabla u''_m(s) \|_p \, ds. \tag{3.6}
\]

In order to estimate the last term in the right hand side of (3.6), we let
\[
\Omega_1 = \{ x \in \Omega : |u''_m(x, t)| < 1 \} \quad \text{and} \quad \Omega_2 = \{ x \in \Omega : |u''_m(x, t)| \geq 1 \}.
\]

Since \(2 < p < \frac{2n}{n-2}\), we can take \( \mu > 0 \) such that \(2 < p + \frac{\mu p}{p-1} < \frac{2n}{n-2}\). Applying Lemma 2.1, we infer that
\[
\| v^{p-2} v \ln |v| \|_{\frac{p}{p-1}}^\frac{p}{p-1} = \int_{\Omega_1} (|v|^{p-1} \ln |v|)^{\frac{p}{p-1}} \, dx + \int_{\Omega_2} (|v|^{-\mu + (p-1) \mu} \ln |v|)^{\frac{p}{p-1}} \, dx
\]
we used the fact that \( v \in \mathcal{H} \) in the last inequality. Here and in the sequel, \( C \) denotes a generic positive constant independent of \( m \) and \( t \) and different from line to line or even in the same line.

From (3.7), we see that

\[
2 \int_0^t \|v|^{p-2} v \ln |v| \| u_m \|_p \| u_t^m \|_p \, ds \leq C \int_0^t \| u_t^m(s) \|_p \, ds
\]

\[
\leq C \int_0^t \| \nabla u_t^m(s) \| \, ds \leq C T + \int_0^t \| \nabla u_t^m(s) \|^2 \, ds.
\]

Adapting this to (3.6), we get

\[
\| u_t^m(t) \|^2 + I \| \nabla u^m(t) \|^2 + (g \circ \nabla u^m)(t) + \int_0^t \| \nabla u_t^m(s) \|^2 \, ds \leq \| u_t^m \|^2 + \| \nabla u_t^m \|^2 + C T
\]

\[
\leq C.
\]

**Step 2. Passage to the limit.** So, there exists a subsequence of \( \{ u^m \} \), which we still denote by \( \{ u^m \} \), such that

\[
u^m \rightharpoonup u \quad \text{weakly star in } L^\infty(0, T; H_0^1(\Omega)),
\]

(3.8)

\[u_m^m \rightharpoonup u_t \quad \text{weakly star in } L^\infty(0, T; L^2(\Omega)),
\]

(3.9)

\[u_m^m \rightharpoonup u_t \quad \text{weakly in } L^2(0, T; H_0^1(\Omega)).
\]

(3.10)

Now, we integrate (3.4) over \((0, t)\) to get

\[
(u_t^m(t), w) - (u_t^m, w) + \int_0^t (\nabla u^m(s), \nabla w) \, ds - \int_0^t \int_0^\tau g(\tau - s)(\nabla u^m(s), \nabla w) \, ds \, d\tau
\]

\[
+ (\nabla u^m(t), \nabla w) - (\nabla u^m_0, \nabla w) = \int_0^t \int_\Omega |v(x, s)|^{p-2} v(x, s) \ln |v(x, s)| w(x) \, dx \, ds.
\]

Taking the limit \( m \to \infty \) in this, we have from (3.8) and (3.9) that

\[
(u_t(t), w) - (u_t, w) + \int_0^t (\nabla u(s), \nabla w) \, ds - \int_0^t \int_0^\tau g(\tau - s)(\nabla u(s), \nabla w) \, ds \, d\tau
\]

\[
+ (\nabla u(t), \nabla w) - (\nabla u_0, \nabla w) = \int_0^t \int_\Omega |v(x, s)|^{p-2} v(x, s) \ln |v(x, s)| w(x) \, dx \, ds.
\]

(3.11)
This remains valid for all \( w \in H^1_0(\Omega) \). Differentiating (3.11) with respect to \( t \), we have
\[
\langle u_t(t), w \rangle + (\nabla u(t), \nabla w) - \int_0^t g(t-s)(\nabla u(s), \nabla w) \, ds + (\nabla u(t), \nabla w)
= \int_\Omega |v(x,t)|^{p-2}v(x,t) \ln|v(x,t)| |w(x)| \, dx \quad \text{for} \ w \in H^1_0(\Omega).
\]
(3.12)

Now, we are left with verifying that the limit function \( u \) satisfies the initial conditions, that is,
\[
u(0) = u_0 \quad \text{in} \ H^1_0(\Omega), \quad u_t(0) = u_1 \quad \text{in} \ L^2(\Omega).
\]

From (3.8), (3.9), and Lion’s lemma [22], we get
\[
u^m \to u \quad \text{in} \ C([0,T];L^2(\Omega)).
\]
(3.13)

Thus, \( \nu^m(0) \to u(0) \) in \( L^2(\Omega) \). Since \( \nu^m(0) = \nu^m_0 \to u_0 \) in \( H^1_0(\Omega) \), we observe that
\[
u(0) = u_0 \quad \text{in} \ H^1_0(\Omega).
\]
(3.14)

Next, multiplying (3.4) by \( \phi \in C_0^\infty((0,T)) \) and integrating it over \((0,T)\), we find
\[
- \int_0^T (u_t^m(t), \phi'(t)) \, dt + \int_0^T (\nabla u^m(t), \nabla \phi(t)) \, dt
= \int_0^T \int_0^t g(t-s)(\nabla u^m(\tau), \nabla \phi(t)) \, d\tau \, dt
- \int_0^T (\nabla u^m(t), \nabla \phi(t)) \, dt = \int_0^T (|v(t)|^{p-2}v(t) \ln|v(t)|, \nabla \phi(t)) \, dt \quad \text{for} \ w \in W^m.
\]

Letting \( m \to \infty, \) we get
\[
- \int_0^T (u_t(t), \phi'(t)) \, dt + \int_0^T (\nabla u(t), \nabla \phi(t)) \, dt
= \int_0^T \int_0^t g(t-s)(\nabla u(\tau), \nabla \phi(t)) \, d\tau \, dt
- \int_0^T (\nabla u(t), \nabla \phi(t)) \, dt = \int_0^T (|v(t)|^{p-2}v(t) \ln|v(t)|, \nabla \phi(t)) \, dt \quad \text{for} \ w \in H^1_0(\Omega).
\]

This yields \( u_t \in L^2(0,T;H^{-1}(\Omega)) \). This and the fact that \( u_t \in L^2(0,T;H^1_0(\Omega)) \) imply that
\[
u_t \in C([0,T];H^{-1}(\Omega)).
\]

Thus, \( u_t^m(0) \to u_t(0) \) in \( H^{-1}(\Omega) \). Owing to \( u_t^m(0) = u_1^m \to u_1 \) in \( L^2(\Omega) \), we conclude
\[
u_t(0) = u_1 \quad \text{in} \ L^2(\Omega).
\]
Uniqueness. Let \( u \) and \( \tilde{u} \) be the solutions of the linearized problem (3.1)–(3.3) and \( w = u - \tilde{u} \). Then \( w \) satisfies

\[
    w_{tt} - \Delta w + \int_0^t g(t-s) \Delta w(s) \, ds - \Delta w_t = 0 \quad \text{in} \ \Omega \times (0, T),
\]

\[
    w = 0 \quad \text{on} \ \partial \Omega \times (0, T),
\]

\[
    w(0) = 0, \quad w_t(0) = 0 \quad \text{on} \ \Omega.
\]

By the same arguments of (3.6), we observe

\[
    \| w_t(t) \|^2 + l \| \nabla w(t) \|^2 + (g \circ \nabla w)(t) + 2 \int_0^t \| \nabla w_s(s) \|^2 \, ds \leq 0,
\]

and hence \( w \equiv 0 \). This completes the proof. \( \square \)

Now, we are ready to prove the local existence of problem (1.1)–(1.3).

**Theorem 3.1** Assume that (H₁) and (H₂) hold. Then, for the initial data \( u_0 \in H^1_0(\Omega) \), \( u_1 \in L^2(\Omega) \), there exists a unique solution \( u \) of problem (1.1)–(1.3).

**Proof** Existence. For \( M > 0 \) large enough and \( T > 0 \), we let

\[
    \mathcal{M}_T = \{ u \in \mathcal{H} : \| u \|_\mathcal{H} \leq M \}.
\]

For a given \( v \in \mathcal{H} \), there exists a unique solution \( u \) of problem (3.1)–(3.3). So, we can define a map \( S : \mathcal{M}_T \to \mathcal{H} \) by \( S(v) = u \). We will show that the map \( S \) is a contraction mapping on \( \mathcal{M}_T \). By a similar computation to that of (3.6), we find

\[
    \| u_t(t) \|^2 + l \| \nabla u(t) \|^2 + (g \circ \nabla u)(t) + 2 \int_0^t \| \nabla u_s(s) \|^2 \, ds
\]

\[
    \leq \| u_1 \|^2 + \| \nabla u_0 \|^2 + 2 \int_0^t \| \nabla u_s(s) \|^2 \, ds
\]

\[
    \leq \| u_1 \|^2 + \| \nabla u_0 \|^2 + 2 \int_0^t \| \nabla u_s(s) \|^2 \, ds
\]

\[
    + 2 \int_0^t \left\{ \left( \frac{1}{e(p-1)} \right)^{\frac{p}{p-1}} |\Omega_1| + c \left( \frac{p(p-1)}{p+1} \right)^{\frac{p}{p-1}} \left( \frac{1}{\varrho \mu} \right)^{\frac{p}{p-1}} \| \nabla v(s) \|^{\frac{p(p-1)}{p+1}} \right\} \frac{2^{(p-1)}}{p} \, ds
\]

\[
    \leq \| u_1 \|^2 + \| \nabla u_0 \|^2 + 2 \int_0^t \| \nabla u_s(s) \|^2 \, ds + CT(1 + M^{2(p-1)})
\]

we used \( v \in \mathcal{M}_T \) in the last inequality. Thus, we see

\[
    \| u_t(t) \|^2 + l \| \nabla u(t) \|^2 + (g \circ \nabla u)(t) \leq \| u_1 \|^2 + \| \nabla u_0 \|^2 + CT(1 + M^{2(p-1)}) \quad (\text{3.15})
\]

We take \( M > 0 \) large enough so that

\[
    \| u_1 \|^2 + \| \nabla u_0 \|^2 \leq \frac{M^2}{2}.
\]
then we choose $T > 0$ sufficiently small so that

$$CT \left( 1 + M^{2(p-1+\mu)} \right) \leq \frac{M^2}{2}.$$ 

From (3.15), we have $\|u\|_{H} \leq M$, that is,

$$S(M_T) \subset M_T.$$ 

It remains to show that $S$ is a contraction mapping. Let $v_1, v_2 \in M_T$, $u = S(v_1)$, $w = S(v_2)$ and $z = u - w$. Then $z$ satisfies

$$z_{tt} - \Delta z + \int_{0}^{t} g(t-s) \Delta z(s) \, ds - \Delta z_{t}$$

$$= |v_{1}|^{p-2}v_{1} \ln |v_{1}| - |v_{2}|^{p-2}v_{2} \ln |v_{2}| \quad \text{in } \Omega \times (0, T), \quad (3.16)$$

$$z = 0 \quad \text{on } \partial \Omega \times (0, T), \quad (3.17)$$

$$z(0) = 0, \quad z_{t}(0) = 0 \quad \text{on } \Omega. \quad (3.18)$$

Multiplying $z_{t}$ in (3.16) and integrating it over $(0, t)$,

$$\|z_{t}\|^2 + l\|\nabla z\|^2 + (g \circ \nabla z) + 2 \int_{0}^{t} \|\nabla z_{t}(s)\|^2 \, ds$$

$$\leq 2 \int_{0}^{t} \int_{\Omega} \left( |v_{1}|^{p-2}v_{1} \ln |v_{1}| - |v_{2}|^{p-2}v_{2} \ln |v_{2}| \right) z_{t} \, dx \, ds$$

$$= 2 \int_{0}^{t} \int_{\Omega} (p-1)|\xi|^{p-2} \ln |\xi| + |\xi|^{p-2}(v_{1} - v_{2})z_{t} \, dx \, ds$$

$$= 2 \int_{0}^{t} \int_{\Omega} |\xi|^{p-2}(v_{1} - v_{2})z_{t} \, dx \, ds + 2(p-1) \int_{0}^{t} \int_{\Omega} |\xi|^{p-2} \ln |\xi|(v_{1} - v_{2})z_{t} \, dx \, ds$$

$$:= \mathcal{E}_1 + \mathcal{E}_2, \quad (3.19)$$

where $\xi = \theta v_{1} + (1-\theta)v_{2}$, here $0 < \theta < 1$. Young’s inequality yields

$$\mathcal{E}_1 \leq 2 \int_{0}^{t} \|\xi\|_{L^{p}(\Omega)}^{p-2} \|v_{1} - v_{2}\|_{L^{2}(\Omega)} \|z_{t}\| \, ds$$

$$\leq 2c_{2}c_{L^{2}}^{p-2} \int_{0}^{t} \|\nabla \xi\|^{p-2} \|\nabla v_{1} - \nabla v_{2}\| \|z_{t}\| \, ds$$

$$\leq C \int_{0}^{t} \|\nabla \xi\|^{2(p-2)} \|\nabla v_{1} - \nabla v_{2}\|^{2} \, ds + \int_{0}^{t} \|\nabla z_{t}\|^{2} \, ds$$

$$\leq CM^{2(p-2)} T \|v_{1} - v_{2}\|_{L^{2}(\Omega)}^{2} + \int_{0}^{t} \|\nabla z_{t}\|^{2} \, ds \quad (3.20)$$

and

$$\mathcal{E}_2 \leq 2(p-1) \int_{0}^{t} \|\xi^{p-2} \ln |\xi|\|_{L^{p}(\Omega)} \|v_{1} - v_{2}\|_{L^{2}(\Omega)} \|z_{t}\| \, ds$$

$$\leq 2(p-1)c_{2}c_{L^{2}} \int_{0}^{t} \|\xi^{p-2} \ln |\xi|\|_{L^{p}(\Omega)} \|\nabla v_{1} - \nabla v_{2}\| \|z_{t}\| \, ds.$$ \quad (3.21)
Since $p - 2 < \frac{2}{n-2}$, there exists $\eta > 0$ such that $n(p - 2 + \eta) < \frac{2n}{n-2}$. By similar arguments to (3.7), we deduce

$$\| \xi \|_n^{p-2} \ln \| \xi \|_n \leq \left( \frac{1}{e(p-1)} \right)^n \| \Omega_1 \| + \int_{\Omega} (|\xi|^{-\eta} \ln |\xi|)^n |\xi|^{n(p-2+\eta)} \, dx$$

$$\leq \left( \frac{1}{e(p-1)} \right)^n \| \Omega_1 \| + e^{n(p-2+\eta)} \left( \frac{1}{e\eta} \right)^n \| \nabla \xi \|^{n(p-2+\eta)}$$

$$\leq C(1 + M^{n(p-2+\eta)}).$$

Applying this to (3.21), we get

$$\Xi_2 \leq C(1 + M^{p-2+\eta}) \int_0^t \| v_1 - v_2 \|_H \| \nabla z_t \| \, ds$$

$$\leq CT(1 + M^{p-2+\eta}) \| v_1 - v_2 \|_H^2 + \int_0^t \| \nabla z_t \|^2 \, ds. \quad (3.22)$$

Collecting (3.19), (3.20), (3.22), we arrive at

$$\| z_t(t) \|^2 + I \| \nabla z(t) \|^2 + (g \circ \nabla z)(t) \leq CT(1 + M^{2(p-2)} + M^{p-2+\eta}) \| v_1 - v_2 \|_H^2. \quad (3.23)$$

Taking $T > 0$ sufficiently small so that $CT(1 + M^{2(p-2)} + M^{p-2+\eta}) < 1$, we conclude

$$\| S(v_1) - S(v_2) \|_H < \| v_1 - v_2 \|_H.$$

Thus, the contraction mapping principle ensures the existence of weak solutions.

**Uniqueness.** Let $w$ and $z$ be the solutions of problem (1.1)–(1.3) and $U = w - z$. Then $U$ satisfies

$$U_{tt} - \Delta U + \int_0^t g(t-s) \Delta U(s) \, ds - \Delta U_t = |w|^{p-2} w \ln |w| - |z|^{p-2} z \ln |z| \quad \text{in } \Omega \times (0, T),$$

$$U = 0 \quad \text{on } \partial \Omega \times (0, T),$$

$$U(0) = 0, \quad U_t(0) = 0 \quad \text{on } \Omega.$$

By the same arguments as of (3.19), (3.20) and (3.21), we observe

$$\| U_t(t) \|^2 + \| \nabla U(t) \|^2 \leq C \int_0^t (\| U_t(s) \|^2 + \| \nabla U(s) \|^2) \, ds.$$

Gronwall’s inequality gives $U \equiv 0$. This completes the proof. \qed

## 4 Finite time blow-up of solutions

In this section we establish the blow-up of the weak solution for problem (1.1)–(1.3). For this purpose, we set the following functionals:

$$J(v) = \frac{1}{2} \left(1 - \int_0^\infty g(s) \, ds\right) \| \nabla v \|^2 - \frac{1}{p} \int_\Omega |v(x)|^p \ln |v(x)| \, dx + \frac{1}{p^2} \| v \|_p^p. \quad (4.1)$$
\[ I(v) = \left( 1 - \int_0^\infty g(s) \, ds \right) \| \nabla v \|^2 - \int_\Omega |v(x)|^p \ln |v(x)| \, dx, \quad (4.2) \]

then

\[ J(v) = \left( \frac{1}{2} - \frac{1}{p} \right) \left( 1 - \int_0^\infty g(s) \, ds \right) \| \nabla v \|^2 + \frac{1}{p} I(v) + \frac{1}{p^2} \| v \|^p_p. \quad (4.3) \]

Define the potential depth as

\[ d = \inf_{v \in H_0^1(\Omega) \setminus \{0\}} \sup_{\lambda > 0} f(\lambda v), \quad (4.4) \]

then, see e.g. [23–25],

\[ 0 < d = \inf_{v \in \mathcal{N}} J(v), \quad (4.5) \]

where \( \mathcal{N} \) is the well-known Nehari manifold given by

\[ \mathcal{N} = \left\{ v \in H_0^1(\Omega) \setminus \{0\} \mid I(v) = 0 \right\}. \]

**Lemma 4.1** For any \( v \in H_0^1(\Omega) \setminus \{0\} \), there exists a unique \( \lambda_* > 0 \) such that

\[ I(\lambda v) = \lambda \frac{\partial I(\lambda v)}{\partial \lambda} \begin{cases} > 0, & 0 < \lambda < \lambda_*, \\ = 0, & \lambda = \lambda_*, \\ < 0, & \lambda > \lambda_. \end{cases} \quad (4.6) \]

**Proof** For \( \lambda > 0 \), we have

\[ \frac{\partial}{\partial \lambda} f(\lambda v) = \lambda \left\{ 1 - \int_0^\infty g(s) \, ds \right\} \| \nabla v \|^2 - \lambda^{p-2} \int_\Omega |v(x)|^p \ln |v(x)| \, dx - \lambda^{p-2} \ln \lambda \| v \|^p_p \right\} \]

\[ := \lambda \mathcal{K}(\lambda v). \quad (4.7) \]

By simple calculation, we also get

\[ \frac{\partial}{\partial \lambda} \mathcal{K}(\lambda v) = -\lambda^{p-3} \left\{ (p-2) \int_\Omega |v(x)|^p \ln |v(x)| \, dx + (p-2) \ln \lambda \| v \|^p_p + \| v \|^p_p \right\} \]

\[ \begin{cases} > 0, & 0 < \lambda < \lambda_1, \\ = 0, & \lambda = \lambda_1, \\ < 0, & \lambda > \lambda_1, \end{cases} \]

where

\[ \lambda_1 = \exp \left( \frac{(p-2) \int_\Omega |v(x)|^p \ln |v(x)| \, dx + \| v \|^p_p}{(2-p)\| v \|^p_p} \right) < 1. \quad (4.8) \]
Since \( \lim_{\lambda \to 0^+} K(\lambda, v) = (1 - \int_{0}^{\infty} g(s) \, ds) \| \nabla v \| ^2 \geq 0 \) and \( \lim_{\lambda \to +\infty} K(\lambda, v) = -\infty \), there exists a unique \( \lambda_+ > \lambda_1 \) such that \( K(\lambda_+ v) = 0 \). From this and (4.7), we have

\[
\frac{\partial J(\lambda v)}{\partial \lambda} \begin{cases} > 0, & 0 < \lambda < \lambda_+, \\ = 0, & \lambda = \lambda_+, \\ < 0, & \lambda > \lambda_+. \end{cases}
\]

Noting that \( I(\lambda v) = \lambda \frac{dJ(\lambda v)}{d\lambda} \), which is verified by a direct computation, we complete the proof.

Now, we define the energy for problem (1.1)–(1.3) by

\[
E(t) = \frac{1}{2} \| u_t(t) \|^2 + \frac{1}{2} \left( 1 - \int_{0}^{t} g(s) \, ds \right) \| \nabla u(t) \|^2 + \frac{1}{2} (g \circ \nabla u)(t) - \frac{1}{p} \int_{\Omega} |u(x,t)|^p \ln |u(x,t)| \, dx + \frac{1}{p} \| u(t) \|^p, \tag{4.9}
\]

then

\[
E(t) \geq J(u(t)) + \frac{1}{2} \| u_t(t) \|^2 + \frac{1}{2} (g \circ \nabla u)(t) \geq J(u(t)). \tag{4.10}
\]

Replacing \( w \) in (2.3) by \( u_t(t) \) and using \((H_2)\), one sees

\[
\frac{d}{dt} E(t) + \| \nabla u_t(t) \|^2 = \frac{1}{2} \left( g' \circ \nabla u(t) \right)(t) - \frac{g(t)}{2} \| \nabla u(t) \|^2 \leq 0
\]

and hence

\[
E(t) + \int_{0}^{t} \| \nabla u_t(s) \|^2 \, ds \leq E(0) \quad \text{for} \quad 0 \leq t < T_{\text{max}}, \tag{4.11}
\]

where \( T_{\text{max}} \) is the maximal existence time of the solution \( u \) of problem (1.1)–(1.3).

**Lemma 4.2** Let \((H_1)\) and \((H_2)\) hold. If \( I(u_0) < 0 \) and \( E(0) < d \), then the solution \( u \) of problem (1.1)–(1.3) satisfies

\[
I(u(t)) < 0 \quad \text{and} \quad E(t) < d \quad \text{for} \quad t \in [0, T_{\text{max}}). \tag{4.12}
\]

**Proof** From (4.11), it is clear that \( E(t) < d \). Since \( I(u_0) < 0 \) and \( u \) is continuous on \([0, T_{\text{max}}), \)

\[
I(u(t)) < 0 \quad \text{for some interval} \quad [0, t_1) \subset [0, T_{\text{max}}). \tag{4.13}
\]

Let \( t_0 \) be the maximal time satisfying (4.13). Suppose \( t_0 < T_{\text{max}} \), then \( I(u(t_0)) = 0 \), that is,

\[
u(t_0) \in N.
\]

Thus, we have from (4.5)

\[
J(u(t_0)) \geq \inf_{v \in N} J(v) = d.
\]
But this is a contradiction for
\[ J(u(t_0)) \leq E(t_0) \leq E(0) < d. \]

**Theorem 4.1** Let the conditions \((H_1)\) and \((H_2)\) hold. Assume that \(I(u_0) < 0, E(0) = \alpha d, \)
where \(\alpha < 1,\) and the kernel function \(g\) satisfies
\[
\int_0^\infty g(s) ds \leq \frac{p - 2}{p - 2 + \frac{1}{(1 - \alpha)2p + 2\alpha(1 - \alpha)}},
\]
(4.14)
where \(\hat{\alpha} = \max\{0, \alpha\}.\) Moreover, suppose that \((u_0, u_1) > 0\) when \(E(0) = 0.\) Then the solution \(u\) of problem \((1.1)-(1.3)\) blows up in finite time.

**Proof** By contradiction, suppose that the solution \(u\) is global. For any \(T > 0,\) we consider \(L : [0, T] \to \mathbb{R}^+\) defined by
\[
L(t) = \|u(t)\|^2 + \int_0^t \|\nabla u(s)\|^2 ds + (T - t)\|\nabla u_0\|^2 + b(t + T_0)^2,
\]
(4.15)
where \(T_0 > 0\) and \(b \geq 0,\) which are specified later. Then
\[
L(t) > 0 \quad \text{for } t \in [0, T],
\]
(4.16)
\[
L'(t) = 2(u(t), u_t(t)) + \|\nabla u(t)\|^2 - \|\nabla u_0\|^2 + 2b(t + T_0)
\]
\[
= 2(u(t), u_t(t)) + 2 \int_0^t (\nabla u(s), \nabla u_t(s)) ds + 2b(t + T_0),
\]
(4.17)
and, from (1.1),
\[
L''(t) = 2\|u_t(t)\|^2 - 2\|\nabla u(t)\|^2 + 2 \int_0^t g(t - s)(\nabla u(t), \nabla u(s)) ds
\]
\[
+ 2 \int_\Omega |u(x, t)|^p \ln |u(x, t)| dx + 2b
\]
\[
= 2\|u_t(t)\|^2 - 2 \left(1 - \int_0^t g(s) ds\right)\|\nabla u(t)\|^2
\]
\[
- 2 \int_0^t g(t - s)(\nabla u(t), \nabla u(t) - \nabla u(s)) ds + 2 \int_\Omega |u(x, t)|^p \ln |u(x, t)| dx
\]
\[
+ 2b,
\]
(4.18)
for almost every \(t \in [0, T].\) By the Cauchy–Schwartz inequality and (4.15), we see that
\[
\frac{(L'(t))^2}{4} = \left((u(t), u_t(t)) + \int_0^t (\nabla u(s), \nabla u_t(s)) ds + b(t + T_0)\right)^2
\]
\[
\leq \left(\|u(t)\|^2 + \int_0^t \|\nabla u(s)\|^2 ds + b(t + T_0)^2\right) \left(\|u_t(t)\|^2 + \int_0^t \|\nabla u_t(s)\|^2 ds + b\right)
\]
\[
= (L(t) - (T - t)\|\nabla u_0\|^2) \left(\|u_t(t)\|^2 + \int_0^t \|\nabla u_t(s)\|^2 ds + b\right)
\]
\[
\leq L(t) \left(\|u_t(t)\|^2 + \int_0^t \|\nabla u_t(s)\|^2 ds + b\right).
\]
(4.19)
Thus, we have from (4.18) and (4.19) that

\[ L(t)L''(t) - \frac{p + 2}{4}(L'(t))^2 \geq L(t)F(t), \]

(4.20)

where

\[
F(t) = -p\|u_t(t)\|^2 - 2\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2 + 2\int_\Omega |u(x, t)|^p \ln|u(x, t)|\,dx
- 2\int_0^t g(t-s)(\nabla u(t), \nabla u(t) - \nabla u(s))\,ds - (p + 2)\int_0^t \|\nabla u_t(s)\|^2\,ds - pb. \tag{4.21}
\]

Applying (4.9) to this and using (4.11) and Young’s inequality, we get

\[
F(t) = -2pE(t) + (p - 2)\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2 + p(g \circ \nabla u)(t) + \frac{2}{p}\|u(t)\|^p_p
- (p + 2)\int_0^t \|\nabla u_t(s)\|^2\,ds - 2\int_0^t g(t-s)(\nabla u(t), \nabla u(t) - \nabla u(s))\,ds - pb
\geq -2pE(0) + (p - 2)\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|^2 + p(g \circ \nabla u)(t) + \frac{2}{p}\|u(t)\|^p_p
+ (p - 2)\int_0^t \|\nabla u_t(s)\|^2\,ds - 2\int_0^t g(t-s)(\nabla u(t), \nabla u(t) - \nabla u(s))\,ds - pb
\geq -2pE(0) + \left(p - 2\right) - \left(p - 2 + \frac{1}{\epsilon}\right)\int_0^t g(s)\,ds\|\nabla u(t)\|^2 + (p - \epsilon)(g \circ \nabla u)(t)
+ \frac{2}{p}\|u(t)\|^p_p + (p - 2)\int_0^t \|\nabla u_t(s)\|^2\,ds - pb,
\]

(4.22)

where \(\epsilon > 0\). We now consider the initial energy \(E(0)\) divided into three cases: \(E(0) < 0\), \(E(0) = 0\), and \(0 < E(0) < d\).

**Case 1:** \(\alpha < 0\), i.e. \(E(0) < 0\).

Taking \(\epsilon = p\) in (4.22) and choosing \(0 < b \leq -2E(0)\), we have from (4.14)

\[
F(t) \geq p(-2E(0) - b) + \left(p - 2\right) - \left(p - 2 + \frac{1}{p}\right)\int_0^t g(s)\,ds\|\nabla u(t)\|^2
+ \frac{2}{p}\|u(t)\|^p_p + (p - 2)\int_0^t \|\nabla u_t(s)\|^2\,ds \geq 0.
\]

(4.23)

**Case 2:** \(\alpha = 0\), i.e. \(E(0) = 0\).

Taking \(\epsilon = p\) in (4.22) and \(b = 0\), we see from (4.14) that

\[
F(t) \geq \left(p - 2\right) - \left(p - 2 + \frac{1}{p}\right)\int_0^t g(s)\,ds\|\nabla u(t)\|^2
+ \frac{2}{p}\|u(t)\|^p_p + (p - 2)\int_0^t \|\nabla u_t(s)\|^2\,ds \geq 0.
\]

(4.24)

**Case 3:** \(0 < \alpha < 1\), i.e. \(0 < E(0) < d\).
Taking $\epsilon = (1 - \alpha) p + 2 \alpha$ in (4.22), we find

$$F(t) \geq -2pE(0) + \left( (p - 2) - \left( p - 2 + \frac{1}{(1 - \alpha)p + 2\alpha} \right) \int_0^t g(s) \, ds \right) \| \nabla u(t) \|^2$$

$$+ \alpha(p - 2)(g \circ \nabla u)(t) + \frac{2}{p} \| u(t) \|^p + (p - 2) \int_0^t \| \nabla u(t) \|^2 \, ds - pb. \quad (4.25)$$

Due to the condition (4.14), it follows that

$$(p - 2) - \left( p - 2 + \frac{1}{(1 - \alpha)p + 2\alpha} \right) \int_0^t g(s) \, ds \geq \alpha(p - 2) \left( 1 - \int_0^\infty g(s) \, ds \right), \quad (4.26)$$

and hence

$$F(t) \geq -2pE(0) + \alpha(p - 2) \left\{ \left( 1 - \int_0^\infty g(s) \, ds \right) \| \nabla u(t) \|^2 + (g \circ \nabla u)(t) \right\}$$

$$+ \frac{2}{p} \| u(t) \|^p + (p - 2) \int_0^t \| \nabla u(t) \|^2 \, ds - pb$$

$$\geq -2pE(0) + \alpha(p - 2) \left( 1 - \int_0^\infty g(s) \, ds \right) \| \nabla u(t) \|^2 + \frac{2\alpha}{p} \| u(t) \|^p - pb. \quad (4.27)$$

On the other hand, it is noted that $I(u(t)) < 0$ for all $t \in [0, T]$ from Lemma 4.2. So, Lemma 4.1 ensures that the existence of $\lambda_* \in (0, 1)$ satisfying $I(\lambda_* u(t)) = 0$. Hence, from (4.3) and (4.5)

$$d \leq I(\lambda_* u(t)) = \left( \frac{1}{2} - \frac{1}{p} \right) \left( 1 - \int_0^\infty g(s) \, ds \right) \lambda_*^2 \| \nabla u(t) \|^2 + \frac{\lambda_*^p}{p^2} \| u(t) \|^p$$

$$< \frac{p - 2}{2p} \left( 1 - \int_0^\infty g(s) \, ds \right) \| \nabla u(t) \|^2 + \frac{1}{p^2} \| u(t) \|^p. \quad (4.28)$$

Since $u$ is continuous on $[0, T]$, there exists $\kappa > 0$ such that

$$d + \kappa < \frac{p - 2}{2p} \left( 1 - \int_0^\infty g(s) \, ds \right) \| \nabla u(t) \|^2 + \frac{1}{p^2} \| u(t) \|^p \quad \text{for all } t \in [0, T].$$

From this and (4.27), we get

$$F(t) \geq -2p\kappa d + 2ap \left( \frac{p - 2}{2p} \left( 1 - \int_0^\infty g(s) \, ds \right) \| \nabla u(t) \|^2 + \frac{1}{p^2} \| u(t) \|^p \right) - pb$$

$$> 2ap\kappa - pb. \quad (4.29)$$

Choosing $b > 0$ sufficiently small so that $2ap\kappa - pb > 0$, we obtain

$$F(t) \geq 0. \quad (4.30)$$

Adapting (4.23), (4.24), (4.30) to (4.20), we infer

$$L(t)L''(t) - \frac{p + 2}{4} (L'(t))^2 \geq 0. \quad (4.31)$$
Now it remains to show $L'(0) > 0$. In the case of $E(0) = 0$, the condition $(u_0, u_1) > 0$ gives

$$L'(0) = 2(u_0, u_1) > 0.$$ \hfill (5.1)

For the cases of $E(0) < 0$ and $0 < E(0) < d$, we choose $T_0$ large enough so that

$$L'(0) = 2(u_0, u_1) + 2bT_0 > 0.$$ \hfill (5.2)

Thus, we conclude from Lemma 2.2 that

$$\lim_{t \to T^-_*} L(t) = +\infty$$ \hfill (4.32)

for

$$T_* \leq \frac{4L(0)}{(p-2)L'(0)} = \frac{2\|u_0\|^2 + 2T\|\nabla u_0\|^2 + 2bT_0^2}{(p-2)((u_0, u_1) + bT_0)}.$$ \hfill (4.33)

Thus, we deduce that

$$T_* \leq \frac{2\|u_0\|^2 + 2bT_0^2}{(p-2)((u_0, u_1) + (p-2)bT_0 - 2\|\nabla u_0\|^2).}$$ \hfill (4.33)

From (4.15), (4.32) and (4.33), we have

$$\lim_{t \to T_*} \left(\|u(t)\| + \int_0^T \|\nabla u(s)\|^2 ds\right) = +\infty.$$ \hfill (4.34)

This contradicts our assumption that the weak solution is global. Thus, we conclude that the weak solution $u$ to problem (1.1)–(1.3) blows up in finite time. \hfill \Box

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Author details
1Department of Mathematics, and Institute of Pure and Applied Mathematics, Jeonbuk National University, Jeonju, South Korea. 2Office for Education Accreditation, Pusan National University, Busan, South Korea.

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