Nonlinear Fourier transform and probability distributions

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Abstract. The paper describes some probabilistic and combinatorial aspects of nonlinear Fourier transform associated with the AKNS-ZS problems. In the first of the two main results, we show that a family of polytopes that appear in a power expansion of the nonlinear Fourier transform is distributed according to the beta probability distribution. We establish this result by studying an Euler type discretization of the nonlinear Fourier transform. This approach provides our second main result, discovering a novel discrete probability distribution that approximates the beta distribution. The numbers of alternating ordered partitions of an integer into distinct parts are distributed according to our new distribution. Using another discretization, we also find a formula for the values of alternating ordered partitions into non-distinct parts. We find a connection between this discretization and the multinomial distribution.

AMS classification scheme numbers: 37K15, 42A99, 60E05, 05A17
1. Introduction

The nonlinear Fourier transform is the central object of the inverse scattering transform theory used to solve and analyze the integrable nonlinear partial differential equations. A significant class of integrable equations are the equations of the AKNS-ZS type. This class contains, e.g. sine-Gordon and nonlinear Schroedinger equations. The pioneering work was done by Ablowitz, Kaup, Newel and Segur, [1], [2], and by Zakharov and Shabat, [15]. In this paper, we shall consider the nonlinear Fourier transform \( \mathcal{F} \) which appears in the study of the periodic AKNS-ZS problems. To every well-behaved function \( u(x) : [0, 1] \to \mathbb{C} \) it assigns the doubly infinite sequence \( \{ \mathcal{F}[u](n) \}_{n \in \mathbb{Z}} \) of \( SU(2) \) matrices, given by \( \mathcal{F}[u](n) = (-1)^n \Phi(x = 1, n) \), where \( \Phi(x, n) \) is the solution of the linear initial value problem

\[
\Phi_x(x, n) = L(x, n) \cdot \Phi(x, n), \quad \Phi(0, n) = I.
\]

The coefficient matrix \( L(x, n) \) is given by

\[
L(x, n) = \begin{pmatrix} \pi i n & u(x) \\ -u(x) & -\pi i n \end{pmatrix}.
\]

The transformation \( \mathcal{F} \) can be thought of as a non-linearisation of the usual Fourier transformation. Namely, we have

\[
\mathcal{F}[u](n) = I + \begin{pmatrix} 0 & F[u](n) \\ -F[\bar{u}](n) & 0 \end{pmatrix} + \sum_{d=2}^{\infty} A_d[u](n),
\]

where \( F \) is the linear Fourier transform and \( u \to A_d[u] \) are matrix-valued nonlinear operators. The amount of literature on various aspects of the inverse scattering method is vast, so we shall only mention a few works in which the Fourier analysis aspect is more pronounced. The foundational work was done by the originators mentioned above. Nonlinear Fourier transforms of functions, defined on \( \mathbb{R} \) and \( \mathbb{R}^+ \), were studied by I. Gelfand, A. Fokas and B. Pelloni in [4], [5], [7], and in their other works. A different, but closely related transformation is described by T. Tao and C. Thiele in [11]. Some aspects of the transformation, defined above, were studied in [9] and [10].

Below, we shall consider \( \mathcal{F} \), together with two of its discretizations. Many authors studied discretizations of transformations similar to \( \mathcal{F} \), but usually acting on the functions defined on \( \mathbb{R} \) or \( \mathbb{R}^+ \), see e.g. [13], [14], [12]. M. Ablowitz and J. Ladik discovered a discretization that preserves the integrability of the AKNS-ZS systems, see [3].

In this paper we shall describe some probabilistic and combinatorial aspects of \( \mathcal{F} \) which stem from its nonlinearity. Our first result concerns a certain set of polytopes. For every positive integer \( d \) and every \( l \in [0,1] \), the polytope \( \hat{D}_d(l) \) is given by

\[
\hat{D}_d(l) = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d; \; 1 \geq x_1 \geq x_2 \geq \ldots \geq x_d \geq 0; \; \sum_{j=1}^{d} (-1)^{j-1} x_j = l \}.
\]
Denote by $D_d(l)$ the orthogonal projection of $\hat{D}_d(l)$ on the hyperplane $\{(x_1, \ldots, x_{d-1}, 0) \in \mathbb{R}^d\}$. These polytopes appear in the power expansion of $\mathcal{F}[u_c](n)$ for the constant function $u_c(x) \equiv u$.

We shall see that

$$\mathcal{F}[u_c](n) = I + \sum_{d=1}^\infty u^d \int_0^1 \text{Vol}(D_d(l)) \left( \begin{array}{cc} 0 & e^{-2\pi i nl} \\ -e^{2\pi i nl} & 0 \end{array} \right)^d dl. \quad (1)$$

**Theorem 1** For every dimension $d$, the volumes of polytopes $D_d(l)$ are essentially distributed according to the beta distribution with the shape parameters $(\frac{d}{2}, \frac{d}{2} + 1)$, if $d$ is even, and $(\frac{d+1}{2}, \frac{d+1}{2})$, if $d$ is odd. More concretely, we have the following expression:

$$\text{Vol}(D_d(l)) = \frac{1}{d!} \begin{cases} 
\frac{1}{B(\frac{d}{2}, \frac{d}{2}+1)} l^{\frac{d}{2} - 1} (1 - l)^{\frac{d}{2}} = p_\beta(l; \frac{d}{2}, \frac{d}{2} + 1); & d \text{ even} \\
\frac{1}{B(\frac{d+1}{2}, \frac{d+1}{2})} l^{\frac{d+1}{2} - 1} (1 - l)^{\frac{d+1}{2}} = p_\beta(l; \frac{d+1}{2}, \frac{d+1}{2}); & d \text{ odd},
\end{cases} \quad (2)$$

where $p_\beta(l; a, b)$ denotes the probability density function of the distribution Beta$(a, b)$.

Expressions (1) and (2) point to the importance of the beta distribution for the nonlinear Fourier transform $\mathcal{F}$. In the proof of the above theorem, we shall use the relation between the polytopes and $\mathcal{F}$ in an essential way. The beta distribution is one of the oldest and most important probability distributions with a broad spectrum of applications in different areas of probability and statistics, particularly in Bayesian statistical inference. In recent times it is mentioned in virtually every book on machine learning and related topics. The beta distribution also appears connected with polytopes, although in a setting very different from ours. In [6] and in many other works, Kabluchko, Thale, and Zaporozhets, together with coworkers, describe exciting results concerning the relations between volumes and angles of random polytopes on the one hand, and beta distributions on the other.

To formulate and to prove theorem 1 we used an appropriate discretization of $\mathcal{F}$, namely the transformation $\mathcal{F}_N$ described in section 2. The study of this discretization leads to another result concerning beta distribution.

**Theorem 2** Let the discrete probability distribution Beta$_N(a, b)$ be given by the probability mass function

$$P_N(\lambda; a, b) = c(N) \left( \begin{array}{c} a + b + 1 \\ N \end{array} \right) \left( \begin{array}{c} N \lambda - 1 \\ a \end{array} \right) \left( \begin{array}{c} N - N\lambda \\ b \end{array} \right)$$

which is defined on the set of values $\lambda \in \left\{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}\right\}$. All the parameters, except for $\lambda$, are integers and $C(N)$ is the normalizing factor. This probability distribution is a discrete approximation of the continuous beta distribution with the probability density function

$$p_\beta(x; a, b) = \frac{1}{B(a + 1, b + 1)} x^a (1 - x)^b, \quad x \in [0, 1].$$
More concretely, let the sequence of integers \( \{l_N\}_{N \in \mathbb{N}} \) be such that \( \lambda_N < N \) and \( \lim_{N \to \infty} \frac{\lambda_N}{N} = \lambda \in [0, 1] \). Then we have
\[
\lim_{N \to \infty} P_N(l_N; a, b) = p_\beta(\lambda; a, b).
\]
We also have \( c(N) = 1 + O(\frac{1}{N}) \).

A different approach to discretize the beta distribution is introduced by A. Punzo in [8].

Theorem 2 stems from the study of the numbers \( AQ_N(l, d) \) which count the ordered alternating partitions of \( l \) into \( d \) distinct parts not greater than \( N - 1 \),
\[
AQ_N(l, d) = \#\{(l_1, l_2, \ldots, l_d); \text{ } N - 1 \geq l_1 > l_2 > \ldots > l_d \geq 0; \sum_{j=1}^d (-1)^{j-1}l_j = l\}. \tag{3}
\]
We shall prove that the numbers \( AQ_N(l, d) \) are essentially distributed according to the distribution \( P_N \), given in theorem 2. This fact is not surprising, since the numbers \( AQ_N(l, d) \) can be viewed as discretizations of the volumes \( \text{Vol}(D_d(l)) \). Just as the numbers \( \text{Vol}(D_d(j)) \) are closely related to \( \mathcal{F} \), so are the numbers \( AQ_N(j, d) \) closely related to \( \mathcal{F}_N \). In proposition 3 of section 3 we prove the following formula which gives the expression of \( AQ_N(l, d) \) in terms of the transformation \( \mathcal{F}_N \):
\[
AQ_N(l, d) = \frac{N^d}{d!} \left( \frac{d}{du} \right)_{u=0}^{N-1} \left( \begin{array}{cc} e^{2\pi i \frac{u}{N}} & 0 \\ 0 & e^{-2\pi i \frac{u}{N}} \end{array} \right) \cdot \mathcal{F}_N(\vec{u}_c)(n) \cdot \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)_{1,1}^{-d},
\]
where \( \vec{u}_c = (u, u, \ldots, u) \) is the constant discrete function.

Let now \( AP_N(l, d) \) denote the number of ordered alternating partitions of \( l \) into \( d \) non-distinct parts not greater than \( N - 1 \). It turns out that finding the values of \( AP_N(l, d) \) demands a different approach. In section 2 we introduce another discretization \( G_N \) of \( \mathcal{F} \) and this transform enables us to find formulae for \( AP_N(l, d) \). In section 4 we prove the following proposition.

**Proposition 1** The number \( AP_N(l, d) \) of ordered alternating partitions of number \( l \) into \( d \) non-distinct parts is given by
\[
AP_N(l, d) = \frac{N^d}{d!} D_d \left( \sum_{n=0}^{N-1} \left( \begin{array}{cc} e^{2\pi i \frac{n}{N}} & 0 \\ 0 & e^{-2\pi i \frac{n}{N}} \end{array} \right) D^d G_N(\vec{u})(n) \cdot \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)_{1,1}^{-d} \bigg|_{\vec{u}=0}. \right.
\]
Here \( \vec{u} = (u, u, \ldots, u_{N-1}) \in \mathbb{R}^d \) and the operators \( D_d \) and \( D^d \) are defined by
\[
D_d = \sum_{k \in [0, d]^N} \frac{\partial^{\sum_{j=0}^{N-1} k_j}}{\partial^{k_0} u_0 \partial^{k_1} u_{k_1} \ldots \partial^{k_{N-1}} u_{N-1}} \quad \text{and} \quad D^d(f(\vec{u})) = \left( \frac{d}{ds} \right)^d |_{s=0} f(s\vec{u}).
\]
The subscript \((1, 1)\) denotes the upper right term of the \(2 \times 2\) matrix.

A central object in the study of the numbers \( AQ_N(l, d) \) is the vector \( \vec{l} = (l_1, l_2, \ldots, l_d) \) together with its alternating sum \( l = \sum_{j=1}^d (-1)^{j-1}l_j \). This vector appears in the study of \( AP_N(l, d) \) in an implicit way. Let \( \vec{k} = (k_0, k_1, \ldots, k_{N-1}) \) be a vector of nonnegative integers. Then the analogue of \( \vec{l} \) is the vector of those indices \( (l_1, l_2, \ldots, l_d) \) from \( \{0, \ldots, N - 1\} \) for which the components \( k_j \) of \( \vec{k} \) are odd integers. The analogue
of the alternating sum is the function $\text{alt}$, given by $\text{alt}(\vec{k}) = \sum_{j=d}^{1}(-1)^{d-j}l_j$. In section \[ we give a longer but clearer description of $\text{alt}(\vec{k})$.

The discretization $G_N$ turns out to be related to the multinomial distribution. The connection between the two objects can be seen in a variety of ways. One of them is the following proposition which we prove in section \[.

**Proposition 2** Let $\vec{X} = (X_0, X_2, \ldots, X_{N-1})$ be a random vector with values in $(\mathbb{N} \cup \{0\})^N$, and let the probability of the event $\vec{X} = \vec{k}$ be given by the multinomial distribution

$$P(\vec{X} = \vec{k}) = P_{N,d}(\vec{u}, \vec{k}) = \left( \begin{array}{c} d \\ k_0, k_1 \ldots k_{N-1} \end{array} \right) u_0^{k_0} u_1^{k_1} \ldots u_{N-1}^{k_{N-1}},$$

where $\sum_{j=0}^{N-1} u_j = 1$. Then the probability $P_{\text{alt}}(l)$ of the event that $\vec{X}$ will assume a value $\vec{k}$ with $\text{alt}(\vec{k}) = l$ is equal to

$$P_{\text{alt}}(l) = \sum_{\text{alt}(\vec{k}) = l} P_{N,d}(\vec{u}, \vec{k}) = \left( N^d \sum_{n=0}^{N-1} \left( \begin{array}{c} e^{2\pi i n} \\ -e^{-2\pi i n} \end{array} \right) \right) \cdot D^d G_N[\vec{u}](n) \cdot \left( \begin{array}{c} 0 \\ -1 \\ 0 \end{array} \right)^{-d}_{1,1},$$

where $G_N$ is the discretization of $F$ appearing in proposition \[.

2. Discrete nonlinear Fourier transforms

We have defined the nonlinear Fourier transform of functions $u(x) : [0, 1] \to \mathbb{C}$ in the introduction. Definition in this form is usually given in the texts which study the integrable ANKS-ZS equations. We shall rather represent $F$ in a different gauge. Let $G(x, n) = \text{diag}(e^{-\pi inx}, e^{\pi inx})$ be the (diagonal) matrix of our gauge transformation. The transformed coefficient matrix is $L^G(x, n) = G_x \cdot G^{-1}(x, n) + G(x, n) \cdot L(x, n) \cdot G^{-1}(x, n)$. Its explicit expression is

$$L^G(x, n) = \left( \begin{array}{cc} 0 & e^{-2\pi i nx}u(x) \\ -e^{2\pi i nx}u(x) & 0 \end{array} \right).$$

In the new gauge $F[u](n)$ becomes $F^G[u](n) = \Phi^G(x = 1, n)$, where $\Phi^G(x, n)$ is the solution of the linear initial value problem

$$\Phi^G_x(x, n) = L^G(x, z) \cdot \Phi^G(x, n), \quad \Phi^G(0, n) = I.$$  

We have $\Phi^G(x, n) = G(x, n) \cdot \Phi(x, n)$ and, since $n \in \mathbb{Z}$, we have $F[u](n) = F^G[u](n)$.

The solution to the problem \[ can be given in the form of the Dyson series.

$$\Phi^G(x, n) = I + \sum_{d=1}^{\infty} \int_{\Delta_d(x)} L^G(x_1, n) \cdot L^G(x_2, n) \cdots L^G(x_d, n) \, d\vec{x},$$

where $\Delta_d(x)$ is the ordered simplex of dimension $d$ with the edge length equal to $x$,

$$\Delta_d(x) = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d; x \geq x_1 \geq x_2 \geq \ldots \geq x_d \geq 0\}.$$

Let us denote

$$E(x, n) = \left( \begin{array}{cc} e^{\pi i x n} & 0 \\ 0 & e^{-\pi i x n} \end{array} \right), \quad J = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right),$$

\[.
and let \( u(x) \) be real valued. Then we have \( L^G(x, n) = u(x) E(-2x, n) \cdot J \). Matrices \( E(x, n) \) and \( J \) do not commute. Instead, have the relation
\[
E(x, n) \cdot J = J \cdot E(-x, n).
\]

Using (8) in the Dyson series and evaluating at \( x = 1 \) gives
\[
\mathcal{F}[u](n) = I + \sum_{d=1}^{\infty} \int_{\Delta_d(1)} u(x_1) u(x_2) \cdots u(x_d) E\left(-2\left(\sum_{j=1}^{d} (-1)^{j-1} x_j\right), n\right) \cdot J^d \, d\bar{x}
\]
which, upon setting \( x_1 - x_2 + \cdots + (-1)^{d-1} x_d = l \), can be rewritten as
\[
\mathcal{F}[u](n) = I + \sum_{d=1}^{\infty} \int_{0}^{1} E(-2l, n) \left( \int_{D_d(l)} u(x_1) u(x_2) \cdots u(x_d) \, dx \right) \cdot J^d \frac{1}{\sqrt{d}} \, dl
\]
\[
= I + \sum_{d=1}^{\infty} \int_{0}^{1} E(-2l, n) \left( \int_{D_d(l)} U(x_1, x_2, \ldots, x_{d-1}; l) \, dx_1 \cdots dx_{d-1} \right) J^d \, dl,
\]
where the polytope \( \hat{D}_d(l) \) is given by
\[
\hat{D}_d(l) = \{ (x_1, x_2, \ldots, x_d) \in \Delta_d(1); \sum_{j=1}^{d} (-1)^{j-1} x_j = l \},
\]
and \( D_d(l) \) is its projection on the hyperplane \( x_d = 0 \). We denoted
\[
U(x_1, x_2, \ldots, x_{d-1}; l)) = u(x_1) \cdots u(x_{d-1})u((-1)^{d-1}(l - (x_1 - x_2 + \cdots + (-1)^{d-2}x_{d-1})).
\]
By means of some linear algebra one can show that the volume forms \( d_l \bar{x} \) and the Euclidean form \( dx_1 \cdots dx_{d-1} \) are related by \( d_l \bar{x} = \sqrt{d} \, dx_1 \cdots dx_{d-1} \).

In the case where \( u_c(x) \equiv u \) is a constant function, we get
\[
\mathcal{F}[u_c](n) = I + \sum_{d=1}^{\infty} u^d \int_{0}^{1} \text{Vol}(D_d(l)) \, E(-2l, n) \cdot J^d \, dl.
\]

**Discretization** \( \mathcal{F}_N \) We have obtained the nonlinear Fourier transform from an initial value problem for a particular first-order linear differential equation. An obvious approach to construct a discretization is to replace the differential equation with a suitable difference equation. Let \( \bar{u} = (u_0, u_1, \ldots, u_{N-1}) \in \mathbb{R}^N \) be a vector which plays a role of a function of a discrete variable. Let the \( L \)-matrix be given by
\[
L_N(k; n) = \begin{pmatrix}
0 & e^{-2\pi i \frac{kn}{N}} u_k \\
-e^{2\pi i \frac{kn}{N}} u_k & 0
\end{pmatrix}.
\]

**Definition 1** Let \( k, n \in \{0, 1, \ldots, N - 1\} \). Discrete nonlinear Fourier transform \( \mathcal{F}_N[\bar{u}] \) of \( \bar{u} \) is defined by \( \mathcal{F}_N[\bar{u}](n) = \Phi_N(k = N - 1, n) \), where \( \Phi_N \) is the solution of the difference initial value problem
\[
N \left( \Phi_N(k + 1, n) - \Phi_N(k, n) \right) = L_N(k; n) \cdot \Phi_N(k, n), \quad \Phi_N(0, n) = I.
\]
Solving the above initial value problem and evaluating at $k = N - 1$ gives

$$\mathcal{F}_N[\tilde{u}](n) = \prod_{k=N-1}^{0} \left( I + \frac{1}{N} L_N(k, n) \right),$$

and this can be expanded into

$$\mathcal{F}_N[\tilde{u}](n) = I + \sum_{d=1}^{N} \frac{1}{N^d} \sum_{N-1\geq l_1 > l_2 > \ldots > l_d \geq 0} L_N(l_1, n) \cdot L_N(l_2, n) \cdots L_N(l_d, n). \quad (11)$$

This expression is a discrete analogue of the Dyson expansion \ref{eq:Dyson}. Let us introduce the notation

$$E_\delta(l, n) = E(l, \frac{n}{N}), \quad l, n \in \{0, 1, \ldots, N-1\}$$

where $E$ is given by \ref{eq:coefficient}, and the subscript $\delta$ refers to the use in the discretized context. The coefficient matrix $L_N$ can be written in the form

$$L_N(l, n) = u_l \cdot E_\delta(-2l, n) \cdot J,$$

with $J$ also defined in \ref{eq:coefficient}. By means of relation \ref{eq:copies}, we can collect all the copies of $J$ in (11) on the right. Let $\tilde{u}_c = (u, \ldots, u)$ be a constant vector. We get

$$\mathcal{F}_N[\tilde{u}_c](n) = I + \sum_{d=1}^{N} \left( \frac{u}{N} \right)^d \sum_{N-1\geq l_1 > l_2 > \ldots > l_d \geq 0} E_\delta(-2l, n) \cdot J^d.$$  

If we denote $l = l_1 - l_2 + \ldots + (-1)^{d-1}l_d$, we can finally write

$$\mathcal{F}_N[\tilde{u}_c](n) = I + \sum_{d=1}^{N-1} \left( \frac{u}{N} \right)^d \sum_{l=0}^{N-1} E_\delta(-2l, n) \sum_{(l_1, \ldots, l_d) \in \hat{D}_d(\delta)} J^d, \quad (12)$$

where

$$\hat{D}_d(\delta) = \{ (l_1, l_2, \ldots, l_d); \ N-1 \geq l_1 > l_2 > \ldots > l_d \geq 0, \sum_{j=1}^{d} (-1)^{j-1}l_j = l \}.$$  

**Discretization $\mathcal{G}_N$**  
Another approach to discretize $\mathcal{F}$ is to start with $\mathcal{F}[u_s]$, where $u_s$ is a step function, $u_s(x) = \sum_{l=0}^{N-1} u_l \chi_{[l,l+(1/N)]}(x)$. This can be computed directly. We have

$$\mathcal{F}[u_s](n) = \text{Exp}(\frac{1}{N} L(N-1, n)) \cdot \text{Exp}(\frac{1}{N} L(N-2, n)) \cdots \text{Exp}(\frac{1}{N} L(0, n)).$$

This discretization has its merits, but we will simplify it by separating the spatial and spectral parameters:

$$\frac{1}{N} L(l, n) = \left( \frac{\pi n}{N} \begin{array}{cc} 0 & 0 \\ 0 & -\frac{\pi n}{N} \end{array} \right) + \left( 0 \begin{array}{cc} 0 \\ -\frac{u}{N} \end{array} \right).$$

The Baker-Campbell-Hausdorff formula gives

$$\text{Exp}(\frac{1}{N} L(l, n)) = \left( e^{\frac{\pi n}{N} l} 0 \right) \cdot \left( \cos \frac{u}{N} \sin \frac{u}{N} \right) + O\left( \frac{1}{N^2} \right)$$

$$= E_\delta(1, n) \cdot R\left( \frac{u}{N} \right) + O\left( \frac{1}{N^2} \right).$$
Definition 2 Let \( l, n, \in \{0, 1, \ldots, N - 1\} \). Discrete nonlinear transform \( G_N[\vec{u}] \) of \( \vec{u} \) is defined by
\[
G_N[\vec{u}](n) = E_\delta(1, n) \cdot R\left(\frac{u_{N-1}}{N}\right) \cdots E_\delta(1, n) \cdot R\left(\frac{u_1}{N}\right) \cdot E_\delta(1, n) \cdot R\left(\frac{u_0}{N}\right).
\]

Using the obvious identity \( E_\delta(1, n) = E_\delta(l + 1, n) \cdot E_\delta(-l, n) \) we can express \( G_N \) in the form
\[
G_N[\vec{u}](n) = \prod_{l=N-1}^{0} A_d E_\delta(-l, n) R\left(\frac{u_l}{N}\right) = \prod_{l=N-1}^{0} \left(\cos \frac{u_l}{N} I + \sin \frac{u_l}{N} E_\delta(-2l, n) \cdot J\right).
\]

It is easy to see that \( F_N \) and \( G_N \) are related by the formula
\[
G_N[\vec{u}](n) = C[\vec{u}] F_N[\tan (\vec{u}/N)](n),
\]
where \( C[\vec{u}] = \prod_{l=0}^{N-1} \cos (u_l/N) \), and \( \tan (\vec{u}/N) = (\tan (u_0/N), \ldots, \tan (u_{N-1}/N)) \). We see that for small \( \vec{u} \) the two discretizations differ only very little. We note that, unlike \( F_N \), the discretization \( G_N \) takes values in \( SU(2) \).

3. Ordered alternating partitions with distinct parts

This section will first describe the connection between the numbers of ordered alternating partitions with distinct parts \( AQ_N(l, d) \) and the transformation \( F_N \). The results will lead us to the unexpected connection between the transformation \( F \), volumes of polytopes \( D_d(l) \), and the beta distribution, described in theorem \( \text{[1]} \). We shall also provide a novel discretization of the beta distribution, given by theorem \( \text{[2]} \).

Let us recall definition \( \text{[3]} \)

\[ AQ_N(l, d) = \#\{(l_1, l_2, \ldots, l_d); \ N - 1 \geq l_1 > l_2 > \ldots > l_d \geq 0; \ \sum_{j=1}^{d} (-1)^{j-1} l_j = l\} \]

Proposition 3 The power series expansion of \( F_N[u_c] \) around \( u = 0 \) is given by
\[
F_N[u_c](n) = I + \sum_{d=1}^{N} \left(\frac{u}{N}\right)^d \sum_{l=0}^{N-1} AQ_N(l, d) E_\delta(-2l, n) \cdot J^d.
\]  

The number \( AQ_N(l, d) \) of alternating partitions of \( l \) into \( d \) distinct parts not greater than \( N - 1 \) is given by the equation
\[
AQ_N(l, d) = \frac{N^d}{d!} \left(\frac{d}{du}\right)^d |_{u=0} \sum_{n=0}^{N-1} E_\delta(2l, n) \cdot F_N[u_c](n) \cdot J^{-d}\right)_{1,1}.
\]  

Proof: The first formula of proposition follows immediately from equation \( \text{[12]} \). We only have to notice that \( AQ_N(l, d) \) is equal to the number of elements in \( \hat{D}_d^{disc}(l) \). To get \( \text{[15]} \), we multiply both sides of \( \text{[14]} \) by \( J^{-d} \) and then perform the inverse linear discrete Fourier transform on both sides. To isolate the term containing the \( d \)-th power of \( u \), we take the \( d \)-th derivative with respect to \( u \), evaluate at \( u = 0 \), and get formula \( \text{[15]} \).

There is an explicit formula for the function \( AQ_N(l, d) \). We have:
Proposition 4 For any \( N \in \mathbb{N} \), \( d \leq N \) and \( l \in \{0, \ldots, N-1\} \), we have
\[
AQ_N(l, d) = \begin{cases} 
\binom{l-1}{\lfloor \frac{d-1}{2} \rfloor} \binom{N-l}{\lfloor \frac{d}{2} \rfloor} & \text{if } d \text{ even} \\
\binom{l}{\lfloor \frac{d-1}{2} \rfloor} \binom{N-l-1}{\lfloor \frac{d}{2} \rfloor} & \text{if } d \text{ odd}
\end{cases}
\]
(16)

Above we use the definition of the binomial symbol for which \( \binom{a}{b} = 0 \) for negative \( a \).

Proof: Let us define \( \tilde{A}Q_N(l, d) = \#\{(l_1, \ldots, l_d); N \geq l_1 > \ldots > l_d \geq 1, \text{ and } \sum_{j=1}^{d} (-1)^{j-1} l_j = l\} \).

We claim that for \( \tilde{A}Q_N(l, d) \) we have
\[
\tilde{A}Q_N(l, d) = \binom{l-1}{\lfloor \frac{d-1}{2} \rfloor} \binom{N-l}{\lfloor \frac{d}{2} \rfloor}
\]
(17)

The formula can be proved by induction on \( N \). For \( N = 2 \), formula (17) can be checked by hand. It is an easy exercise to show that \( \tilde{A}Q_N(l, d) \) satisfies the recursion relation
\[
\tilde{A}Q_N(l, d) = \tilde{A}Q_{N-1}(l, d) + \tilde{A}Q_{N-1}(N-l, d-1).
\]

By the induction hypothesis, the above equation becomes
\[
\tilde{A}Q_N(l, d) = \binom{l-1}{\lfloor \frac{d-1}{2} \rfloor} \binom{N-l-1}{\lfloor \frac{d}{2} \rfloor} + \binom{N-l-1}{\lfloor \frac{d-2}{2} \rfloor} \binom{l-1}{\lfloor \frac{d-1}{2} \rfloor}
\]
\[
= \binom{l-1}{\lfloor \frac{d-1}{2} \rfloor} \binom{N-l}{\lfloor \frac{d}{2} \rfloor},
\]
and this proves (17). The second equality above comes from the recurrence relation of the Pascal triangle.

Finally, we observe that
\[
AQ_N(l, d) = \tilde{A}Q_N(l, d), \text{ for } d \text{ even, and } AQ_N(l, d) = \tilde{A}Q_N(l+1, d), \text{ for } d \text{ odd.}
\]

These relations, together with formula (17), prove the proposition.

\[\square\]

If we insert the result of the above proposition in proposition \( \Box \) we get the following corollary:

Corollary 1 The power series of \( \mathcal{F}_N[u_c](n) \) around \( u = 0 \) is given by
\[
\mathcal{F}_N[u_c](n) = I + \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} (-1)^k \left( \frac{u}{N} \right)^{2k} \sum_{l=0}^{N-1} \binom{l-1}{k-1} \binom{N-l}{k} \cdot E_\delta(-2l, n)
\]
\[
+ \sum_{k=0}^{\lfloor \frac{N+1}{2} \rfloor} (-1)^k \left( \frac{u}{N} \right)^{2k+1} \sum_{l=0}^{N-1} \binom{l}{k} \binom{N-l-1}{k} \cdot E_\delta(-2l, n) \cdot J.
\]
We shall now prepare the necessary tools for the proof of theorem \[\Box\]  First, we shall consider the appropriate limit of \(AQ_N(l,d)\) when \(N\) goes to infinity. The subset \(\hat{D}^{\text{disc}}_d(l)\) of the discrete ordered simplex

\[
\Delta^{\text{disc}}_d(N) = \{(l_1, l_2, \ldots, l_d) \in (N \cup \{0\})^d; N - 1 \geq l_1 > l_2 > \ldots > l_d \geq 0\}
\]

with the edge of size \(N\) is given by one equation. Its size \(AQ_N(l,d)\) is therefore of the order \(N^{d-1}\).

**Lemma 1** Let \(\lambda\) be a real number in \([0,1]\) and let \(\{l_N\}_{N \in \mathbb{N}}\) be a sequence of positive integers such that \(\lambda_N < N\) and \(\lim_{N \to \infty} \frac{l_N}{N} = \lambda\). Then we have

\[
\lim_{N \to \infty} \frac{d!}{N^{d-1}} AQ_N(l_N,d) = \begin{cases} \\
\frac{1}{B\left(\frac{d}{2}, \frac{d}{2}+1\right)} \lambda^{d-1} (1-\lambda)^{\frac{d}{2}} = p_\beta(\lambda; \frac{d}{2}, \frac{d}{2}+1); \quad d \text{ even} \\
\frac{1}{B\left(\frac{d+1}{2}, \frac{d+1}{2}\right)} \lambda^{\frac{d-1}{2}} (1-\lambda)^{\frac{d+1}{2}} = p_\beta(\lambda; \frac{d+1}{2}, \frac{d+1}{2}); \quad d \text{ odd} \end{cases} \quad (18)
\]

**Proof:** We shall prove the formula only for even \(d\). The proof for odd \(d\) is essentially the same. For \(d = 2m\), formula \((16)\) gives

\[
AQ_N(l_N,d) = \left(\frac{l_N - 1}{m - 1}\right) \left(\frac{N - l_N}{m}\right).
\]

This expression can be expanded into

\[
AQ_N(l_N,d) = \frac{1}{(m - 1)! m!} \prod_{k=0}^{m-2} ((l_N - 1) - k) \prod_{k=0}^{m-1} ((N - l_N) - k),
\]

which gives

\[
AQ_N(l_N,d) = \frac{1}{(m - 1)! m!} \left((N \frac{l_N}{N} - 1)^{m-1} + O((N\lambda)^{m-2})\right) \left((N - N \frac{l_N}{N})^m + O((N\lambda)^{m-1})\right);
\]

and, due to \(d - 1 = m + (m - 1)\) and \(\lim_{N \to \infty} \frac{l_N}{N} = \lambda\),

\[
\lim_{N \to \infty} \frac{1}{N^{d-1}} AQ_N(\lambda_N, d) = \frac{1}{(m - 1)! m!} \lambda^{m-1} (1 - \lambda)^m.
\]

The definition of the Euler beta function for positive integers gives \(\frac{d!}{(m - 1)! m!} = \frac{1}{B(m,m+1)}\), and this proves formula \((18)\) for even \(d\).

**Proof of theorem \[\Box\]** Recall the set \(\hat{D}^{\text{disc}}_d(l)\), given by formula \((13)\). Rescaling it by the factor \(1/N\) gives the set

\[
\hat{D}^{\text{disc}}_d\left(\frac{l}{N}\right) = \left\{\left(\frac{l_1}{N}, \frac{l_2}{N}, \ldots, \frac{l_d}{N}\right); \frac{N - 1}{N} \geq \frac{l_1}{N} > \frac{l_2}{N} > \ldots > \frac{l_d}{N} \geq 0, \sum_{j=1}^{d} (-1)^{j-1}l_j = l\right\}
\]

which contains the same number of points as \(\hat{D}^{\text{disc}}_d(l)\), but lies in the polytope \(\hat{D}_d(l)\). Let \(D^{\text{disc}}_d(l)\) denote the orthogonal projection of \(\hat{D}^{\text{disc}}_d(l)\) on the hyperplane \(\{x_1, \ldots, x_{d-1}, 0\} \subset \mathbb{R}^d\). The number \(\sharp \hat{D}^{\text{disc}}_d\left(\frac{l}{N}\right)\) of points in \(\hat{D}^{\text{disc}}_d\left(\frac{l}{N}\right)\) is clearly equal to the number of points \(\sharp D^{\text{disc}}_d\left(\frac{l}{N}\right)\) in the projection. So, on the one hand, the number \(\sharp D^{\text{disc}}_d\left(\frac{l}{N}\right)\) is equal to \(AQ_N(l; d)\), while on the other, the value \(\frac{1}{N^{d-1}} \sharp D^{\text{disc}}_d\left(\frac{l}{N}\right)\)
Nonlinear Fourier transform and probability distribution

is approximately equal to the volume \( \text{Vol}(D_d(l)) \) of the projection of \( \hat{D}_d(l) \) on the hyperplane \( x_d = 0 \) in \( \mathbb{R}^d \). Let now \( \{\lambda_N\}_{N \in \mathbb{N}} \) be a sequence of rationals \( l/N \) converging to \( \lambda \in [0, 1] \). We have

\[
\lim_{N \to \infty} \frac{1}{N^{d-1}} AQ_N(N\lambda_N, d) = \text{Vol}(D_d(\lambda)).
\]

The above expression, together with lemma 1, proves our theorem 1.

\[\square\]

Proof of theorem 2: The proof is an obvious adaptation of the proof of lemma 1. We only have to replace the particular values \( m + 1 \) and \( m \) of the shape parameters by an arbitrary pair \( a \) and \( b \) of integers. Then the same calculations as those performed in the proof of lemma 1 yield the proof.

The number \( c(N)^{-1} \) is the discrete integral of the function

\[
Q_N(\lambda_N, a, b) = \frac{(a + b + 1)!}{N^{a+b}} \binom{N\lambda_N - 1}{a} \binom{N - N\lambda_N}{b}
\]

over the discrete interval \( \lambda_N \in (0, \frac{1}{N}, \ldots, \frac{N-1}{N}) \) with the volume form \((1/N)\). Multiplying by \( c(N) \) normalizes \( Q_N \) to a probability mass function whose integral has to be equal to 1.

\[\square\]

4. General ordered alternating partitions

In this section, our goal is to express the numbers \( AP_N(l, d) \) of ordered alternating partitions of \( l \) into \( d \) parts not greater than \( N - 1 \). The parts in \( AP_N(l, d) \) need not be distinct. The numbers \( AP_N(l, d) \) will be expressed in terms of the transformation \( G_N \) rather than \( F_N \) which yielded the numbers \( AQ_N(l, d) \).

Let \( \vec{u} = (u_0, u_1, \ldots, u_{N-1}) \in \mathbb{R}^N \) be a discrete function. Recall that \( G_N[\vec{u}] \) is given by

\[
G_N[\vec{u}](n) = E_\delta(1, n) \cdot R\left(\frac{u_{N-1}}{N}\right) \cdot E_\delta(1, n) \cdot R\left(\frac{u_{N-2}}{N}\right) \cdots E_\delta(1, n) \cdot R\left(\frac{u_0}{N}\right).
\]

We have seen in section 2 that

\[
G_N[u](n) = \prod_{l=1}^{N-1} \left( \cos \left( \frac{u_l}{N} \right) \cdot I + \sin \left( \frac{u_l}{N} \right) E_\delta(-2l, n) \cdot J \right).
\]

Let us consider the derivative of order \( d \), namely \( D^dG_N[\vec{u}](n) = (\frac{d}{ds})^d |_{s=0} G_N[s\vec{u}](n) \). The generalized Leibniz rule gives

\[
D^dG_N[\vec{u}](n) = \sum_{\vec{k}} \binom{d}{k_0, k_1, \ldots, k_{N-1}} \prod_{l=1}^{N-1} \left( \cos \left( \frac{s u_l}{N} \right) + \sin \left( \frac{s u_l}{N} \right) E_\delta(-2l, n) \cdot J \right)^{(k_l)} |_{s=0}, \quad (19)
\]

where we sum over all \( \vec{k} = (k_0, k_1, \ldots, k_{N-1}) \) such that \( \|\vec{k}\|_1 = k_0 + k_1 + \ldots + k_{N-1} = d \). Let \( p(k) = k \mod 2 \) be the parity of \( k \), and let us define the operator

\[\text{alt}: (\mathbb{N} \cup \{0\})^N \to \mathbb{N} \cup \{0\}\]
Another important function in this section is \( \text{odd}(\vec{k}) \). By definition it is equal to the number of odd components in the integral vector \( \vec{k} = (k_0, k_1, \ldots, k_{N-1}) \in (\mathbb{N} \cup \{0\})^N \).

The following two examples should clarify the formula for \( \text{alt}(\vec{k}) \). Let first \( N = 6, d = 4 \), and \( \vec{k} = (1,1,1,1,0,0) \). Then, \( \text{alt}(\vec{k}) = 3 - 2 + 1 - 0 = 2 \). Here we get \( \text{alt}(\vec{k}) = 2 \) and \( \text{odd}(\vec{k}) = 4 \). Let now \( N = 6, d = 8 \), and \( \vec{k} = (0,1,2,2,3,0) \). Then, \( \text{alt}(\vec{k}) = 4 - 4 + 4 - 3 + 3 - 2 + 2 - 1 = 3 \), so \( \text{alt}(\vec{k}) = 3 \) and \( \text{odd}(\vec{k}) = 2 \). The values \( \|\vec{k}\|_1 = d \) and \( \text{odd}(\vec{k}) \) are equal only when all the components \( k_j \) of \( \vec{k} \) are equal either to 0 or to 1.

The following proposition follows directly from the definition of the function \( \text{alt} \).

**Proposition 5** The function \( \text{alt} \) has the following three properties:

(i) The even components \( k_j \) of \( \vec{k} \) do not contribute to \( \text{alt}(\vec{k}) \).

(ii) Replacing any odd component \( k_j \) of \( \vec{k} \) by 1 does not change \( \text{alt} \).

(iii) In the \( L_1 \)-sphere \( S^d_1 = \{ \vec{k}; \|\vec{k}\|_1 = d \} \) we have

\[
\sum_{\vec{k}; \|\vec{k}\|_1 = d} 1 = A P_N(l, d).
\]

The factors of (19) are equal to

\[
\left( \frac{d}{ds} \right)^{k_1}_{|s=0} \left( \cos \left( s \frac{u_1}{N} \right) I + \sin \left( s \frac{u_1}{N} \right) E_\delta(-2l, n) \cdot J \right) = \left( \frac{u_1}{N} \right)^{k_1} E_\delta(-2l, n)^{p(k_1)} J^{k_1}.
\]

The exponential factor \( E_\delta(-2l, n) \) appears if and only if \( k_1 \) is odd. Formula (19) can be rewritten as

\[
D^d G_N^l(n) = \sum_{m=0}^{d} \left( \frac{1}{N} \right)^d \left( \sum_{\|\vec{k}\|_1 = d \atop \text{odd}(\vec{k}) = m} \left( k_0, k_1, \ldots, k_{N-1} \right) u_0^{k_0} u_1^{k_1} \cdots u_{N-1}^{k_{N-1}} \prod_{l_m > \ldots > l_1} E_\delta(-2l, n)^{J^{k_j}} \right).
\]

The indices \( l_j \in \{l_1, \ldots, l_m\} \subset \{0, 1, \ldots, N-1\} \) in the product at the end of our formula are those for which the component \( k_{l_j} \) of the vector \( \vec{k} = (k_0, k_1, \ldots, k_{N-1}) \) is an odd integer. As in the previous section, we use the relation \( J \cdot E(l, n) = E(-l, n) \cdot J \) and get

\[
D^d G_N^l(n) = \left( \sum_{\|\vec{k}\|_1 = d} \left( \frac{1}{N} \right)^d \left( k_0, k_1, \ldots, k_{N-1} \right) u_0^{k_0} u_1^{k_1} \cdots u_{N-1}^{k_{N-1}} \cdot E_\delta(-2 \text{alt}(\vec{k}), n) \right) \cdot J^d \cdot (20)
\]

The first two properties of \( \text{alt} \) from proposition (5) allow us to replace the alternating sums

\[
l_m - l_{m-1} + l_{m-2} - \ldots + (-1)^{m-1} l_1, \quad m = 1, \ldots, d,
\]
where \( l_j \) are odd, by the values \( \text{alt}(\vec{k}) \) of vectors \( \vec{k} \) appearing in the sum in (20).

**Proof of proposition 2**: Let \( P_{N,d} \) denote the probability mass function of the binomial distribution,

\[
P_{N,d}(\vec{u}, \vec{k}) = \binom{d}{k_0, k_1, \ldots, k_{N-1}} u_0^{k_0} u_1^{k_1} \cdots u_{N-1}^{k_{N-1}}.
\]

We divide the set of non-negative integer valued functions \( \{\vec{k}\} \subset (\mathbb{N} \cup \{0\})^{N-1} \) into disjoint subsets with respect to the values of the function \( \text{alt}: \{\vec{k}\} \to \mathbb{N} \cup \{0\} \). This gives

\[
D^d G_N[\vec{u}](n) = \sum_{l=0}^{N-1} \left( \frac{1}{N} \right)^d \left( \sum_{\vec{k}: \text{alt}(\vec{k})=l} P_{N,d}(\vec{u}, \vec{k}) \right) \cdot E_\delta(-2l, n) \cdot J^d. \tag{21}
\]

We can now apply the discrete inverse linear Fourier transform and obtain

\[
\sum_{\vec{k}: \text{alt}(\vec{k})=l} P_{N,d}(\vec{u}, \vec{k}) \cdot I = N^d \left( \sum_{n=0}^{N-1} E_\delta(2l, n) \cdot D^d G_N[\vec{u}](n) \right) \cdot J^{-d}. \tag{22}
\]

This is the formula that we had to prove.

**Proof of proposition 1**: Recall the differential operator \( \mathcal{D}_d \), defined in proposition 1 of the introduction. First, we observe that

\[
\mathcal{D}_d \left( \sum_{\vec{k}: \text{alt}(\vec{k})=l} P_{N,d}(\vec{u}, \vec{k}) \right)|_{\vec{u}=0} = d! \cdot AP_N(l, d).
\]

This follows from the third part of proposition 5 and from the fact that for two vectors \( \vec{k}_a \) and \( \vec{k}_b \) we have the following possibilities:

\[
\partial_{\vec{k}_a} \left[ \binom{d}{k_0^a, k_1^a, \ldots, k_{N-1}^a} u_0^{k_0^a} u_1^{k_1^a} \cdots u_{N-1}^{k_{N-1}^a} \right] = \begin{cases} 
p(u_0, \ldots, u_{N-1}) ; & \|\vec{k}_a\|_1 < d \\
d! \delta_{a,b} ; & \|\vec{k}_a\|_1 = d \\
0 ; & \|\vec{k}_a\|_1 > d \end{cases},
\]

where \( p(u_0, \ldots, u_{N-1}) \) is a polynomial without the constant term and \( \delta_{a,b} \) is the Kronecker delta. From (22) we now get

\[
AP_N(l, d) = \frac{N^d}{d!} \mathcal{D}_d \left( \sum_{n=0}^{N-1} E_\delta(2l, n) \cdot D^d G_N[\vec{u}](n) \right)|_{\vec{u}=0} \cdot J^{-d},
\]

which proves the proposition.

\[\square\]

5. Acknowledgement

I am grateful to Matjaž Konvalinka for a very helpful discussion. The research for this paper was supported in part by the research programme Analysis and Geometry, P1-0291, and the research project Analysis, Equations, and Partial Differential Equations, J1-9104, funded by the Slovenian Research Agency.
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