A General Stochastic Maximum Principle For Optimal Control Of Stochastic Systems Driven By Multidimensional Teugel’s Martingales

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Abstract

A necessary maximum principle is proved for optimal controls of stochastic systems driven by multidimensional Teugel’s martingales. The multidimensional Teugel’s martingales are constructed by orthogonalizing the multidimensional Lévy processes. The control domain need not be convex, and the control is allowed to enter into the terms of Teugel’s martingales.

Keywords: Stochastic optimal control, Maximum principle, Backward stochastic differential equation, Lévy processes, Teugel’s martingals.

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1. Introduction

The stochastic maximum principle is one of the central topics in the stochastic optimal control theory. In the past four decades, a variety of results have been obtained on optimal stochastic control problems.(cf. for example, [1], [3], [5], [12], [14]-[17], [26], [31]). Two major advances in these works are worth mentioning. One is the definition of the adjoint processes and its characterization by Itô-type equations. This was contributed by Kushner [17] and Bismut [5], and summarized by Bensoussan [3] via functional analysis methods. Another advance is the idea of second-order variation in calculating the variation of the cost functional caused by the spike variation of the given optimal control. This was motivated by the study of the nonconvex optimal stochastic control of diffusion processes with the control entering into the diffusion term, and was developed by Peng [26]. On nonconvex controls of diffusion processes, we refer the reader to Kushner [17], Haussmann [14], Bensoussan [3], Hu [15], Hu and Peng [16], Peng [26] and Yong and Zhou [35].

It is well known that jump-diffusion process is an important class of processes for describing financial data. The stochastic maximum principle of jump-diffusion processes, where the control is unallowed into the jump terms, was considered by Boel [6], Boel and Varaiya [7], Rishel [28], Davis and Elliott [9] and Situ [31]. The further profound problem, where the control enters into the diffusion and jump terms and also some state constrains are imposed, was completely solved by Tang and Li [34] by applying the idea of second-order variation. On the convex controls of jump-diffusion, we refer the reader to Cadenillas[8], Framstad, Okesendal and Sulem [13], Shi and Wu [30].

The Lévy process (refers to Bertoin [4], Sato [29]) is a more general class of discontinuous processes than jump-diffusion processes. Nualart and Schoutens [22] obtained some interesting results. They introduce the power jump processes and the related Teugel’s martingales. Furthermore, they give a chaotic and predictable representation for a one-dimensional Lévy process, in terms of these orthogonalized Teugels martingales. Thus the martingale representation theorem for Lévy process satisfying some exponential moment condition was a consequence of the chaotic representation. Nualart and Schoutens [23] established the existence and uniqueness of solutions for BSDE driven by a one-dimensional Lévy process of the kind considered in Nualart and Schoutens [22]. Further progresses on the
subject were subsequently given by Bahlali, Eddahbi and Essaky [2], Ren[27], Lin[20]. Based on these Results, a stochastic linear-quadratic problem with Lévy processes was considered by Mitsui and Tabata [24], Tang and Wu [32]. The stochastic maximum principle, where the control enters into the diffusion and jump terms and also control domain is convex, was given by Meng and Tang [21], Tang and Zhang [33].

Recently, A chaotic and predictable representation theorem associated with multidimensional Lévy processes was obtained by Lin [19]. This extends the setting in Nualart and Schoutens [22] into the multidimensional Lévy processes. Furthermore, The existence and uniqueness of solutions for BSDEs driven by multidimensional Teugel’s martingales, which are constructed by orthogonalizing the multidimensional Lévy processes, was proved by Lin [20]. According to these results and following the research line of the paper in Peng [26] and Tang and Li [34], this paper discusses the general stochastic maximum principle where the control systems are driven by the multidimensional Teugel’s martingales. It is worth emphasizing that there are three main differences in our setting compared with Mitsui and Tabata [24], Tang and Wu [32], Meng and Tang [21] and Tang and Zhang [33]. First, in our paper, the each component in stochastic system is driven by a Teugel’s martingale which is generated by the multidimensional Lévy processes, while the each component in stochastic system in [21], [24], [32] and [33] is driven by a Teugel’s martingale which is generated by one component of multidimensional Lévy processes. Secondly, in our paper, the control domain need not be convex, while that in Meng and Tang [21], Tang and Zhang [33] is convex and therefore the second-order variation technique is unnecessary. Finally, the terminal state in our case is constrained while is not in Meng and Tang [21], Tang and Zhang [33].

The paper is organized as follows. Section 2 contains an introduction on chaotic and predictable representation theorem associated with multidimensional Lévy processes and BSDEs driven by multidimensional Teugel’s martingales. In Section 3, we give the statement of the problem, our main assumptions and some preliminary lemmas about the first- and second-order variational equation and variational inequality which will be used in the sequel. In Section 4, we derive the first- and second-order adjoint equations, and finally prove the necessary maximum principle. The conclusions are drawn in Section 5.

2. BSDE driven by multidimensional Teugel’s martingales

A \( \mathbb{R}^n \)-valued stochastic process \( X = \{X(t) = (X_1(t), X_2(t), \ldots, X_n(t))', t \geq 0\} \) defined in complete probability space \((\Omega, \mathcal{F}, P)\) is called Lévy process if \( X \) has stationary and independent increments and \( X(0) = 0 \). A Lévy process possesses a c\'adl\'ag modification and we will always assume that we are using this c\'adl\'ag version. If we let \( \mathcal{F}_t = \mathcal{G}_t \vee \mathcal{N} \), where \( \mathcal{G}_t = \sigma(X(s), 0 \leq s \leq t) \) is the natural filtration of \( X \), and \( \mathcal{N} \) are the \( P \)-null sets of \( \mathcal{F} \), then \( \{\mathcal{F}_t, t \geq 0\} \) is a right continuous family of \( \sigma \)-fields. We assume that \( \mathcal{F} \) is generated by \( X \). For an up-to-date and comprehensive account of Lévy processes we refer the reader to Bertoin [4] and Sato [29].

Let \( X \) be a Lévy process and denote by

\[
X(t-) = \lim_{\delta \to 0^+} X(t), \quad t > 0,
\]

the left limit process and by \( \Delta X(t) = X(t) - X(t-) \) the jump size at time \( t \). It is known that the law of \( X(t) \) is infinitely divisible with characteristic function of the form

\[
E[exp(i\theta \cdot X(t))] = (\phi(\theta))^t, \quad \theta = (\theta_1, \theta_2, \ldots, \theta_n) \in \mathbb{R}^n
\]

where \( \phi(\theta) \) is the characteristic function of \( X(1) \). The function \( \psi(\theta) = \log \phi(\theta) \) is called the characteristic exponent and it satisfies the following famous Lévy-Khinchine formula (Bertoin, [4]):

\[
\psi(\theta) = \frac{1}{2} \Sigma \theta \cdot \theta + ia \cdot \theta + \int_{\mathbb{R}^n} (exp(i\theta \cdot x) - 1 - i\theta \cdot x1_{|x| \leq 1})\nu(dx).
\]

where \( a, x \in \mathbb{R}^n \), \( \Sigma \) is a symmetric nonnegative-definite \( n \times n \) matrix, and \( \nu \) is a measure on \( \mathbb{R}^n \backslash \{0\} \) with \( \int (|x|^2 \wedge 1)\nu(dx) < \infty \). The measure \( \nu \) is called the Lévy measure of \( X \).

Throughout this paper, we will use the standard multi-index notation. We denote by \( \mathbb{N}_0 \) the set of nonnegative integers. A multi-index is usually denoted by \( p, p = (p_1, p_2, \cdots, p_n) \in \mathbb{N}_0^n \). Whenever \( p \) appears with subscript or superscript, it means a multi-index. In this spirit, for example, for \( x = (x_1, \cdots, x_n) \), a monomial in variables \( x_1, \cdots, x_n \).
is denoted by $x^p = x_1^{p_1} \cdots x_n^{p_n}$. In addition, we also define $p! = p_1! \cdots p_n!$ and $|p| = p_1 + \cdots + p_n$; and if $p, q \in \mathbb{N}^n_0$, then we define $\delta_{p,q} = \delta_{p_1,q_1} \cdots \delta_{p_n,q_n}$. 

In the remaining of the paper, we will suppose that

**Assumption 2.1.** the Lévy measure satisfies for some $\varepsilon > 0$, and $\lambda > 0$,

$$ \int_{|x| \geq \varepsilon} \exp(\lambda ||x||) \nu(dx) < \infty. $$

This implies that

$$ \int x^p \nu(dx) < \infty, \quad |p| \geq 2 $$

and that the characteristic function $E[\exp(i\theta \cdot X(t))]$ is analytic in a neighborhood of origin $\theta$. As a consequence, $X(t)$ has moments of all orders and the polynomials are dense in $L^2(\mathbb{R}^n, \nu \circ X(t)^{-1})$ for all $t > 0$.

Fix a time interval $[0, T]$ and set $L^2_T = L^2(\mathcal{F}_T, \mathbb{P}).$ We will denote by $\mathcal{P}$ the predictable sub-$\sigma$-field of $\mathcal{F}_T \otimes \mathcal{F}_0$. First we introduce some notation:

- Let $H_T^2$ denote the space of square integrable and $\mathcal{F}_t$-progressively one-dimensional measurable processes $\phi(t)$, $t \in [0, T]$ such that

$$ ||\phi||^2 = \mathbb{E} \left[ \int_0^T ||\phi(t)||^2 dt \right] < \infty. $$

- $M_T^2$ will denote the subspace of $H_T^2$ formed by predictable processes.

- $(H_T^2, \mathcal{F}_T)^m$ and $(M_T^2, \mathcal{F}_T)^m$ are the corresponding spaces of $m$-dimensional $\mathcal{F}_T$-valued processes equipped with the norm

$$ ||\phi||^2_{p,c} = \mathbb{E} \left[ \int_0^T \sum_{d=1}^m \sum_{p \in \mathbb{N}_0^d} |\phi^p_t|^2 dt \right] \quad k = 1, 2, \ldots, m, $$

$$ ||\phi||^2_{p,m} = \sum_{k=1}^m ||\phi_k(t)||^2_{p,c}, $$

where $\phi = (\phi_1, \phi_2, \cdots, \phi_m)$, $\phi_k = (\phi^p : p \in \mathbb{N}_0^n)$, $k = 1, 2, \cdots, m$ and $\mathbb{N}_0^d \defeq \{ p \in \mathbb{N}_0^n : |p| = d \}$.

- Set $\mathcal{H}_T^2 = H_T^2 \times (M_T^2)^m$.

Following Lin [19] we introduce power jump monomial processes of the form

$$ X(t)^{(p_1, \cdots, p_m)} \defeq \sum_{0 \leq s \leq t} (\Delta X_1(s))^{p_1} \cdots (\Delta X_n(s))^{p_n}. $$

The number $|p|$ is called the total degree of $X(t)^p$. Furthermore define

$$ Y(t)^{(p_1, \cdots, p_m)} \defeq X(t)^{(p_1, \cdots, p_m)} - \mathbb{E}[X(t)^{(p_1, \cdots, p_m)}] = X(t)^{(p_1, \cdots, p_m)} - m_p t, $$

the compensated power jump process of multi-index $p = (p_1, p_2, \cdots, p_n)$ where $m_p = \int \prod_{i=1}^n x_i^{p_i} \nu(dx)$. Under hypothesis

1. $Y(t)^{(p_1, \cdots, p_m)}$ is a normal martingale, since for an integrable Lévy process $Z$, the process $\{Z \cdot E[Z], t \geq 0\}$ is a martingale. We call $Y(t)^{(p_1, \cdots, p_m)}$ the Teugels martingale monomial of multi-index $(p_1, \cdots, p_n)$. 


Lemma 2.1. Some details about the technique and theory of orthogonal polynomials of several variables refer to Dunkl and Xu [11].

We can apply the standard Gram-Schmidt process with the graded lexicographical order to generate a biorthogonal basis \( \{H^p, p \in \mathbb{N}^n\} \), such that each \( H^p(\{p\} = d) \) is a linear combination of the \( Y^q \), with \( |q| \leq |p| \) and the leading coefficient equal to 1. We set

\[
H^p = Y^p + \sum_{q < p, |q| = |p|} c_q Y^q + \sum_{k=1}^{\lfloor p-1 \rfloor} \sum_{|q| = k} c_q Y^q,
\]

where \( p = \{p_1, \ldots, p_n\} \), \( q = \{q_1, \ldots, q_n\} \) and \( \prec \) represent the relation of graded lexicographical order between two multi-indexes. Some details about the technique and theory of orthogonal polynomials of several variables refer to Dunkl and Xu [11].

Set

\[
p(x)^p = x^p + \sum_{q < p, |q| = |p|} c_q x^q + \sum_{k=1}^{\lfloor p-1 \rfloor} \sum_{|q| = k} c_q x^q,
\]

\[
\tilde{p}(x)^p = x^p + \sum_{q < p, |q| = |p|} c_q x^q + \sum_{k=2}^{\lfloor p-1 \rfloor} \sum_{|q| = k} c_q x^q.
\]

Set

\[
H^p(t) = \sum_{0 \leq s \leq T} \left( (\Delta X_1)^{p_1} \cdots (\Delta X_n)^{p_n} + \sum_{q < p, |q| = |p|} c_q (\Delta X_1)^{q_1} \cdots (\Delta X_n)^{q_n} + \sum_{k=1}^{\lfloor p-1 \rfloor} \sum_{|q| = k} c_q (\Delta X_1)^{q_1} \cdots (\Delta X_n)^{q_n} \right).
\]

\[
-\frac{\partial}{\partial t} \mathbb{E} \left[ X^p(1) + \sum_{q < p, |q| = |p|} c_q X^q(1) + \sum_{k=1}^{\lfloor p-1 \rfloor} \sum_{|q| = k} c_q X^q(1) \right]
\]

\[
= \left( c_1 X_1(1) + \cdots + c_n X_n(1) \right) + \sum_{0 \leq s \leq T} \tilde{p}(\Delta X(s)) - \frac{\partial}{\partial t} \left[ c_1 X_1(1) + \cdots + c_n X_n(1) \right].
\]

Specially we have

\[
H^1(t) = c_1(X_1(t) - \mathbb{E}(X_1(1))),
\]

\[
H^2(t) = c_2(X_2(t) - \mathbb{E}(X_2(1))) + c_1 X_1(t) - \mathbb{E}(X_1(1)),
\]

\[
\vdots
\]

\[
H^n(t) = c_n(X_n(t) - \mathbb{E}(X_n(1))) + c_{n-1}(X_{n-1}(t) - \mathbb{E}(X_{n-1}(1))) + \cdots + c_1 X_1(t) - \mathbb{E}(X_1(1)).
\]

The main tool in the theory of BSDEs is the martingale representation theorem (cf. Pardoux and Peng [25]). Nualart and Schoutens [22] had proved the representation theorem associated with one-dimensional Lévy process, furthermore Nualart and Schoutens [23] had established the existence and uniqueness of solutions for BSDE driven by a one-dimensional Teugel’s martingale generated by the Lévy process. The main results in Lin [19] is the Predictable Representation Property (PRP) associated multidimensional Lévy processes:

Lemma 2.1. Every random variable \( F \) in \( L^2(\Omega, \mathcal{F}) \) has a representation of the form

\[
F = \mathbb{E}(F) + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}^d} \int_0^T \Phi^p(s) dH^p(s)
\]
where $\Phi^p(s)$ is predictable. This result is an extended version for the corresponding Theorem in Nualart and Schouten [22].

Taking into account the results and notation presented in the previous section, it seems natural to consider the BSDEs with the following form

$$-dY(t) = f(t, Y(t-), Z(t))dt - \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_0^e} z^p(t)dH^p(s), \quad Y(T) = \xi,$$

(2.2)

where

- $Y(t) = (Y_1(t), Y_2(t), \cdots, Y_m(t))'$.
- $Z(t) = (z^p(t))_{p \in \mathbb{N}_0^e}$, each component $z^p(t) = (z_1^p, \cdots, z_m^p)'$ is a $m$-variables $\mathcal{F}_t$ predictable function;
- $f = (f_1, f_2, \cdots, f_m)^\prime : \Omega \times [0, T] \times \mathbb{R}^m \times (M^2_\mathcal{F}(\mathbb{P}))^m \to \mathbb{R}^m$ is a measurable $m$–dimensional vector function such that $f(\cdot, 0, 0) \in (H^2_\mathcal{F})^m$.
- $f$ is uniformly Lipschitz in the first two components, i.e., there exists $C_k > 0$, $k = 1, 2, \cdots, m$, such that $dt \otimes dp$ a.s., for all $(y_1, z_1)$ and $(y_2, z_2)$ in $\mathbb{R}^m \times (F)^m$

$$|f_k(t, y_1, z_1) - f_k(t, y_2, z_2)| \leq C_k \left(||y_1 - y_2||_2 + ||z_1 - z_2||_{\mathbb{P}}\right), \quad k = 1, 2, \cdots, m.$$

- $\xi \in L^2(\Omega, \mathbb{F})$.

If $(f, \xi)$ satisfies the above assumptions, the pair $(f, \xi)$ is said to be standard data for BSDE. A solution of the BSDE is a pair of processes, $(Y(t), Z(t), 0 \leq t \leq T) \in H^2_\mathcal{F} \times (M^2_\mathcal{F}(\mathbb{P}))^m$ such that the following relation holds for all $t \in [0, T]$

$$Y(t) = \xi + \int_t^T f(s, Y(s-), Z(s))ds - \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_0^e} \int_t^T z^p(s)dH^p(s),$$

(2.3)

A key-result concerning the existence uniqueness of solution of BSDEs (2.2) is given by Lin [20]:

**Lemma 2.2.** Given standard data $(f, \xi)$, there exists a unique solution $(Y, Z)$ which solves the BSDE (2.3)

### 3. Notations and preliminary lemmas

Consider the following stochastic control system:

$$dx(t) = g(x(t-), v(t))dt + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_0^e} \gamma^p(x(t-), v(t))dH^p(t),$$

$$x(0) = x_0.$$  

(3.1)

Here and hereafter

$$g(x, v) : \mathbb{R}^m \times \mathcal{U} \to \mathbb{R}^m,$$

$$\gamma^p(x, v) : \mathbb{R}^m \times \mathcal{U} \to \mathbb{R}^m, \forall p \in \mathbb{N}^e,$$

and $\mathcal{U}$ is a nonempty subset of $\mathbb{R}^m$ (control domain). An admissible control $v(\cdot)$ is a $\mathcal{F}_t$–predictable process with values in $\mathcal{U}$ such that

$$||v(\cdot)|| := \sup_{0 \leq t \leq T} \left[ E|v(t)|^\gamma \right]^\gamma < \infty$$

(3.2)
We denote the set of all admissible controls by $\mathcal{U}_{ad}$. When $\mathcal{U} = \mathbb{R}^m$, we write $L^{\infty, p}_{\mathcal{X}}([0,1]; \mathbb{R}^m)$ for $\mathcal{U}_{ad}$. The terminal constraint is

$$
\mathbb{E}G(x_0, X(T)) \in Q \subset \mathbb{R}^k,
$$

where $G(\cdot) = (G^1(\cdot), \ldots, G^k(\cdot), \cdot)$ and $G^i(\cdot) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^k$ for $i = 1, 2, \ldots, k$.

The optimal control problem is to find a pair $(y_0, u() \in \mathbb{R}^m \times \mathcal{U}_{ad}$ such that (3.1) and (3.3) are satisfied and (3.4) is minimized.

Throughout the paper, we make the following assumptions:

**Assumption 3.1.** The vector functions $g(x, v), G(y, x), \ell(x, v), h(y, x)$ and $\gamma^p(x, v)(p \in \mathbb{N}_0)$ are twice continuously differentiable with respect to $x$ and $G(y, x), h(y, x)$ are differentiable in $y$. They and their derivatives in $x$ or $y$ are continuous in $(x, v)$ and $(y, x)$. The vector functions $g(x, v), G_s(y, x), G_{ss}(y, x), \ell_s(x, v), h_s(y, x), h_{ss}(y, x), \gamma^p(x, v)$ and $\gamma^p(x, v)$ are bounded. For $i = 1, \ldots, n$, are bounded by $(1 + |x| + |y| + |v|)$. The vector functions $G(y, x), \ell(x, v), h(y, x)$ are bounded by $(1 + |x|^2 + |y|^2 + |v|^2), g_s(x, v), G_s(x, v), G_{ss}(x, v), \ell_s(x, v), h_s(y, x)$, and

$$(i = 1, \ldots, n)$$

are bounded. Here $x_i, y_i (i = 1, \ldots, n)$ stand for the $i$-th coordinates of $x$ and $y$, respectively.

**Assumption 3.2.** The set $Q$ is closed and convex.

Let $(y_0, y(), u())$ be an optimal triplet of the problem. For the given $(x_0, v()) \in \mathbb{R}^m \times \mathcal{U}_{ad}$, write $y(s; v(), x_0)$ for the solution of (3.1). For $v(), v_1(), v_2() \in \mathcal{U}_{ad}$, denote

$$
\begin{align*}
\Delta m(s; v_2, v_1) & \overset{\text{def}}{=} m(y(s-), v_2) - m(y(s-), v_1), \\
\Delta m(s; v) & \overset{\text{def}}{=} m(y(s-), v) - m(y(s-), u(s)), \\
m(s; v_1) & \overset{\text{def}}{=} m(y(s), v_1), \\
m(s) & \overset{\text{def}}{=} m(y(s), u(s)),
\end{align*}
$$

with $m$ standing for $g, \gamma, \ell$ and all their (up to second-) derivatives in $x$.

For $I_0 \subset [0,1]$, let $|I_0|$ denote the Lebesgue measure of the set $I_0$. Let $v(), v_1(), v_2() \in \mathcal{U}_{ad}$. Define

$$
\tilde{d}(v_1(), v_2()) \overset{\text{def}}{=} \|[t \in [0,1]; E[v_1()-v_2()]]^2 > 0\|.
$$

For $p \in \{0, T\}, I_p \subset [0, T]$ and $v() \in \mathcal{U}_{ad}$, it is classical to construct a perturbed admissible control in the following way (spike variation):

$$
\begin{align*}
w^p(s) & \overset{\text{def}}{=} u(s)\chi_{[0,1]}(s) + v(s)\chi_{I_p}(s), \\
\gamma^p_0 \overset{\text{def}}{=} y_0 + |I_p| \eta, \\
\gamma^p() & \overset{\text{def}}{=} y(); w^p(), \gamma^p_0),
\end{align*}
$$

where $\gamma^p_0$ is uniform in $p$. Let $\gamma^p_0 = \gamma^p_0(\eta, \xi)$.
Lemma 3.1. We can prove that
\[
\hat{d}(u_r(\cdot), u(\cdot)) = |I_r|.
\] (3.8)
We can prove that \( u^r(\cdot) \in \mathcal{U}_{ad} \).

**Lemma 3.1.** Let the Assumption 3.1 hold. Then for \( v(\cdot), u(\cdot), u^r(\cdot) \in \mathcal{U}_{ad} \)

\[
\sup_{r \in [0, T]} \mathbb{E} \left[ |y(t; v(\cdot), x_0)|^8 \right] = O(1 + \|v(\cdot)\|)^8,
\]

\[
\sup_{r \in [0, T]} \mathbb{E} \left[ |y(t; u(\cdot), y_0) - y(t; u^r(\cdot), x_0)|^8 \right] = O(\hat{d}^2(u^r(\cdot), u(\cdot))(1 + \|u(\cdot)\| + \|u^r(\cdot)\|)^4),
\]

\[
\sup_{r \in [0, T]} \mathbb{E} \left[ |y_1(t; u^r(\cdot), u(\cdot))|^8 \right] = O(\hat{d}^4(u^r(\cdot), u(\cdot))(1 + \|u(\cdot)\| + \|u^r(\cdot)\|)^5),
\]

\[
\sup_{r \in [0, T]} \mathbb{E} \left[ |y_2(t; u^r(\cdot), u(\cdot))|^4 \right] = O(\hat{d}^4(u^r(\cdot), u(\cdot))(1 + \|u(\cdot)\| + \|u^r(\cdot)\|)^5),
\]

\[
\sup_{r \in [0, T]} \mathbb{E} \left[ |y(t; u^r(\cdot), y_0 + \hat{d}(u_r, u)\eta) - y(t; u, y_0) - y_1(t; u^r, u) - y_2(t; u^r, u)|^2 \right] = o(\hat{d}^2(u^r(\cdot), u(\cdot))(1 + \|u(\cdot)\| + \|u^r(\cdot)\|)^5), \text{ as } \hat{d}(u^r(\cdot), u(\cdot)) \to 0.
\] (3.9)

where \( y_1(\cdot), y_2(\cdot) \) are the solutions of

\[
y_1(t) = \int_0^t g_x(y(s), u(s))y_1(s)ds + \sum_{d=1}^d \sum_{p \in \mathbb{N}^d} \int_0^t \left[ \gamma^p_d(y(s), u(s))y_1(s) + \Delta \gamma^p_d(s, u^r(s), u(s)) \right] dH^p(s) \] (3.10)

\[
y_2(t) = \hat{d}(u^r(\cdot), u(\cdot))\eta + \int_0^t \left[ g_x(y(s), u(s))y_2(s) + \Delta g(s, u^r(s), u(s)) + \frac{1}{2} g_{xx}(y(s), u(s))y_1(s) \right] ds \]

\[
\quad + \sum_{d=1}^d \sum_{p \in \mathbb{N}^d} \int_0^t \left[ \gamma^p_d(y(s), u(s))y_2(s) + \frac{1}{2} \gamma^p_{xx}(y(s), u(s))y_1(s) \right] dH^p(s) \]

\[
\quad + \sum_{d=1}^d \sum_{p \in \mathbb{N}^d} \int_0^t \Delta \gamma^p_d(s, u^r(s), u(s))y_1(s)dH^p(s)
\]

(3.11)

where \( f_{xxy} = \sum_{i,j=1}^m f_{x(y)} y^i y^j \) for \( f = g, \gamma^p \).

Proof. Without loss of generality, we assume \( \eta = 0 \). Define

\[
\int_{\Lambda} g_0(s)dH^p(s) = : \int \chi_{\Lambda}(s)g_0(s)dH^p(s). \quad \forall p \in \mathbb{N}^n
\]

We have the following inequalities for \( p > 1 \):

\[
\mathbb{E} \left[ \int_{\Lambda} f_0(s)ds \right]^p \leq C_p |I_r|^{p-1} \mathbb{E} \int_{\Lambda} |f_0(s)|^p ds,
\]

\[
\mathbb{E} \left[ \int_{\Lambda} g_0(s)dH^p(s) \right]^p \leq C_p |I_r|^{p-1} \mathbb{E} \int_{\Lambda} |g_0(s, \cdot)|^2 ds, \quad \forall p \in \mathbb{N}^n.
\] (3.12)

By virtue of the Assumption 3.1, we have

\[
\sup_{r \in [0, T]} \mathbb{E} \left[ |y(t)|^8 \right] = O((1 + \|v(\cdot)\| + \|u(\cdot)\|)^8),
\]

\[
\sup_{r \in [0, T]} \mathbb{E} \left[ |\Delta g(t, u^r(s))|^4 \right] = O((1 + \|u^r(\cdot)\| + \|u(\cdot)\|)^4),
\]

\[
\sup_{r \in [0, T]} \mathbb{E} \left[ |\Delta \gamma^p(t, u^r(s))|^8 \right] = O((1 + \|u^r(\cdot)\| + \|u(\cdot)\|)^5), \quad \forall p \in \mathbb{N}^n.
\] (3.13)
Then the first four estimates of (3.9) are easily proved by using the familiar elementary inequalities
and the well-known Gronwall’s inequality.

Then we can obtain the following inequalities by using (3.12):

\[ \mathbb{E} \left[ \int_0^T \triangle g(t, u^p(t)) \right] = O(t^4(1 + ||v|| + ||u||)^4), \]
\[ \mathbb{E} \left[ \int_0^T \triangle g^p(t; u^p(t)) \right] = O(t^4(1 + ||v|| + ||u||)^8), \quad \forall p \in \mathbb{N}. \]

Then the first four estimates of (3.9) are easily proved by using the familiar elementary inequalities

\[ (m_1 + m_2)^i \leq C(m_1^i + m_2^i), \quad i = 4, 8 \]

and the well-known Gronwall’s inequality.

The proof for the last estimate follows. Set \( y_3 = y_1 + y_2 \). We have

\[ \int_0^\infty g(y + y_3, u^\rho)ds + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_0^*} \int_0^\infty \gamma^p(y + y_3, u^\rho)d\mathbb{H}^p(s), \]
\[ = \int_0^\infty \left[ g(y, u^\rho) + g(x(y, u^\rho), y_3) + \int_0^1 \int_0^1 A \left( g(x(y, u^\rho), u^\rho) - g(x(y, u^\rho), y_3) \right) ds \right] d\mathbb{H}^p(s) \]
\[ + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_0^*} \int_0^\infty \gamma^p(y, u^\rho)d\mathbb{H}^p(s) + \sum_{d=1}^{\infty} \int_0^\infty \int_0^1 A \left( g(x(y, u^\rho), u^\rho) - g(x(y, u^\rho), y_3) \right) d\mathbb{H}^p(s). \]

\( \left. \right. \]

\[ = \int_0^\infty g(y, u)ds + \int_0^\infty g(x(y, u), y_3)ds + \int_0^\infty \triangle g(s, u^\rho(s), u(s))ds \]
\[ + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_0^*} \int_0^\infty \gamma^p(y, u)d\mathbb{H}^p(s) + \int_0^\infty \int_0^1 A \left( g(x(y, u), u^\rho) - g(x(y, u), y_3) \right) d\mathbb{H}^p(s) \]
\[ + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_0^*} \int_0^\infty \gamma^p(y, u)d\mathbb{H}^p(s) + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_0^*} \int_0^\infty \triangle \gamma^p(s, u^\rho(s), u(s))d\mathbb{H}^p(s) \]
\[ + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_0^*} \int_0^\infty \frac{1}{2} \gamma^p(x(y, u)y_3(s)y_3(s)d\mathbb{H}^p(s) \]
\[ + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_0^*} \int_0^\infty \gamma^p(s, u^\rho(s), u(s))d\mathbb{H}^p(s) + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_0^*} \int_0^\infty \triangle \gamma^p(s, u^\rho(s), u(s))y_3d\mathbb{H}^p(s) \]
\[ = y(t) + y_3(t) - y_0 + \int_0^T G^p(s)ds + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_0^*} \int_0^T \mathbb{E}^p(s)d\mathbb{H}^p(s), \]
where
\[
G^\rho(s) = \frac{1}{2} g_{xx}(y(s), u(s))(y_2(s)y_2(s) + 2y_1(s)y_2(s)) \\
+ \Delta g_{xx}(y(s), u^\rho(s), u(s)) y_2(s) \\
+ \int_0^s \int_0^1 \lambda [g_{xx}(y + \lambda y_2), u^\rho] - g_{xx}(y, u)] d\lambda dy_3(y_3(s)) \\
\Xi^\rho(s) = \frac{1}{2} \gamma^\rho_{xx}(y(s), u(s))(y_2(s)y_2(s) + 2y_1(s)y_2(s)) \\
+ \Delta \gamma^\rho_{xx}(y(s), u^\rho(s), u(s)) y_2(s) \\
+ \int_0^s \int_0^1 \lambda [\gamma^\rho_{xx}(y + \lambda y_2), u^\rho] - \gamma^\rho_{xx}(y, u)] d\lambda dy_3(y_3(s))
\]

Since
\[
y(t) + y_3(t) = y_0 + \int_0^t g(y + y_3, u^\rho) ds + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_0^*} \int_0^t \gamma^\rho(y + y_3, u^\rho) dH^p(s) \\
- \int_0^t G^\rho(s) ds - \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_0^*} \int_0^t \Xi^\rho(s) dH^p(s).
\]
and
\[
y^\rho(t) = y_0 + \int_0^t g(y^\rho(s), u^\rho(s)) ds + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_0^*} \int_0^t \gamma^\rho(y^\rho(s), u^\rho(s)) dH^p(s),
\]
we can derive that
\[
(y^\rho - y - y_3)(t) = \int_0^t A^\rho(s)(y^\rho - y - y_3)(s) ds \\
+ \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_0^*} \int_0^t F^{\rho^p}(s)(y^\rho - y - y_3)(s) dH^p(s) \\
+ \int_0^t G^\rho(s) ds + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_0^*} \int_0^t \Xi^\rho(s) dH^p(s).
\]
\[
|A^\rho(s, \omega)| + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_0^*} |F^{\rho^p}(s, \omega)| \leq C \quad \forall s, \forall \omega.
\]
and
\[
\sup_{0 \leq t \leq T} E \left( \left( \int_0^t G^\rho(s) ds \right)^2 + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_0^*} \left( \int_0^t \Xi^{\rho^p}(s) dH^p(s) \right)^2 \right) = o(1 + \|u^\rho\| + \|u(\cdot)\|).
\]

From these we can use Itô’s formula and Gronwall’s inequality to obtain the fifth estimate (3.9). The proof is completed. □

**Lemma 3.2.** Assume that \( \ell(\cdot) \) is a scalar-valued Lebesgue integrable function defined on \([0, T]\). Then for \( \rho \in (0, T] \), there exists a measurable subset \( I_\rho \subset [0, T] \), such that
\[
I_{\|\cdot\|} = \rho, \quad I_{\ell(\cdot)} ds = \rho \int_{(0, T]} I(\cdot) ds + o(\rho), \quad \rho \to 0.
\] (3.15)

The proof is quite elementary and the reader is referred to [18].
4. Adjoint equations and the maximum principle

The Hamiltonian is defined as

\[ H(x, v, A, p, J) = \lambda \ell (x, v) + (p, g(x, v)) + \sum_{i=1}^{\infty} \sum_{p \in \mathcal{N}_d} (J^p, \gamma^p(x, v)) \]

This is a map from \( \mathbb{R}^m \times \mathcal{U} \times \mathbb{R} \times \mathbb{R}^m \times (\mathcal{M}_{L}^2(F))^m \) into \( \mathbb{R} \). Here we have used \( \cdot, \cdot \) for the scalar product of Euclidean spaces.

From Lemma 2.2 and Assumption 3.1, we see for the given \( p(T) \in L^2(\Omega, \mathcal{F}_T; \mathbb{R}^m), P(T) \in L^2(\Omega, \mathcal{F}_T; \mathbb{R}^{mxm}) \) that the Itô-type adjoint equations

\[
- dp(t) = \left[ g^p(y(t), u(t))p(t) + \sum_{i=1}^{\infty} \sum_{p \in \mathcal{N}_d} \gamma^p_i(y(t), u(t)) \right] dt \\
+ \sum_{i=1}^{\infty} \sum_{p \in \mathcal{N}_d} J^p_i(t) \theta^p_i(t) \\
p(T) = h_s(y(T)).
\]

(4.1)

and

\[
- dP(t) = \left[ g^p(y(t), u(t))p(t) + \sum_{i=1}^{\infty} \sum_{p \in \mathcal{N}_d} \gamma^p_i(y(t), u(t)) \right] dt \\
+ \sum_{i=1}^{\infty} \sum_{p \in \mathcal{N}_d} R^p_i(y(t), u(t)) + H_{xy}(y(t), u(t), \lambda, p(t), J(t)) \\
- \sum_{i=1}^{\infty} \sum_{p \in \mathcal{N}_d} R^p_i(t) \theta^p_i(t) \\
P(T) = h_{xx}(y(T)).
\]

(4.2)

admit unique solutions \((p(\cdot), \{J^p(\cdot))_{p \in \mathcal{N}_d}\}\) and \((P(\cdot), \{R^p(\cdot))_{p \in \mathcal{N}_d}\}\), with \(p(\cdot)\) and \(P(\cdot)\) being cadlag processes.

Define the following function:

\[ \Phi(s, z; \varepsilon) \equiv \inf_{(t, \xi) \in (-(\infty, J_{(s, z; e)}) - e) \times Q} \sqrt{\lvert t - s \rvert^2 + \lvert \xi - z \rvert^2} \]

(4.3)

Tang and Li [34] had proved the following result.

**Lemma 4.1.** For given \( \varepsilon > 0 \), the function \( \Phi(s, z; \varepsilon) \) is continuously differentiable on the open set \( \hat{Q} \equiv \{(s, z) : \Phi(s, z; \varepsilon) > 0\} \). Moreover, when \( \Phi(s, z; \varepsilon) > 0 \), we have

\[ \langle \Phi_s(s, z; \varepsilon), \hat{z} - z \rangle \leq 0, \forall \hat{z} \in Q, \]

\[ \Phi_{zz}(s, z; \varepsilon) \geq 0, \]

\[ |\Phi_s(s, z; \varepsilon)|^2 + |\Phi_{zz}(s, z; \varepsilon)|^2 = 1. \]

(4.4)

They introduce the smooth function \( \alpha(\cdot) \) defined by

\[ \alpha(t, z) \equiv \begin{cases} C \exp(t^2 + |z|^2 - 1)^{-1}, & t^2 + |z|^2 < 1, \\
0, & t^2 + |z|^2 \geq 1. \end{cases} \]

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Choose the constant $C$ such that
\[ \int_{\mathbb{R} \times \mathbb{R}^3} \alpha(t, z) \, dt \, dz = 1. \]

Set
\[ \alpha_\delta(t, z) = \delta^{-1(k+1)} \alpha \left( \frac{t}{\delta}, \frac{z}{\delta} \right). \] (4.5)

They also define the smooth approximation $\Psi(\cdot, \cdot; \varepsilon, \delta)$ of $\Phi(\cdot, \cdot; \varepsilon)$ as follows:
\[ \Psi(s, z; \varepsilon, \delta) \overset{\text{def}}{=} \int_{\mathbb{R} \times \mathbb{R}^3} \Phi(s - \bar{s}, z - \bar{z}; \varepsilon) \alpha_\delta(\bar{s}, \bar{z}) \, d\bar{s} \, d\bar{z} = 1. \] (4.6)

Then it is easy to have
\[ 0 \leq \Psi(J(u(\cdot), y_0), EG(y_0, y(T)); \varepsilon, \delta) \leq \varepsilon + \sqrt{2} \delta \]

Moreover, Tang and Li [34] gave the following lemma.

**Lemma 4.2.** For $\hat{Q}$ defined in Lemma 4.1, we have for $(s, z) \in \hat{Q}$,
\[ \lim_{\delta \to 0^+} \Psi_1(s, z; \varepsilon, \delta) = \Phi_1(s, z; \varepsilon), \]
\[ \lim_{\delta \to 0^+} \Psi_2(s, z; \varepsilon, \delta) = \Phi_2(s, z; \varepsilon). \] (4.7)

Our main result in this paper is almost similar to that in Tang and Li [34] in many places:

**Theorem 4.1.** Assume Assumptions 3.1 and 3.2 hold. Let $(y_0, y(\cdot), u(\cdot))$ be an optimal triplet. Then there exist
\[ 0 \leq \lambda \in \mathbb{R}, \quad \mu \overset{\text{def}}{=} \{ \mu^i \}_i \in \mathbb{R}^k, \]
\[ (p(\cdot), J(\cdot)) \in L_2^p(0, T; \mathbb{R}^m) \times L_2^p(0, T; (M_2^2(\bar{f}))^m) \]
\[ (P(\cdot), R(\cdot)) \in L_2^p(0, T; \mathbb{R}^{m \times m}) \times L_2^p(0, T; (M_2^2(\bar{f}))^{m \times m}) \]
such that we have the following.

1) The nontrivial condition
\[ |\lambda|^2 + |\mu|^2 = 1, \] (4.8)

is satisfied.

2) The Itô-type adjoint equations (4.1),(4.2), as well as
\[
\begin{cases}
    p(T) &= \lambda h_x(y_0, y(T)) + \sum_{j=1}^{k} \mu^j G_{x}^j(y_0, y(T)), \\
    p(0) &= -\lambda h_x(y_0, y(T)) - \sum_{j=1}^{k} \mu^j E G_{x}^j(y_0, y(T))
\end{cases}
\] (4.9)

and
\[ p(T) = \lambda h_{xx}(y_0, y(T)) + \sum_{j=1}^{k} \mu^j G_{xx}^j(y_0, y(T)), \] (4.10)

are satisfied, with $p(\cdot)$ and $P(\cdot)$ being cadlag processes.
3) The following maximum condition holds:

\[ H(y(s), v, \lambda, p(s), J(s)) - H(y(s), u(s), \lambda, K(s), J(s)) \]
\[ + \frac{1}{\varepsilon} \mathcal{P}(s) \left[ \sum_{d=1}^{\infty} \sum_{p \in \mathcal{P}} \Delta \gamma^p(s; v) \sum_{d=1}^{\infty} \Delta \gamma^p(s; v) \right] \]
\[ \geq 0, \quad \forall v(\cdot) \in \mathcal{U}, \quad a.e.a.s. \]  

(4.11)

4) The following transversality condition holds:

\[ \langle \mu, z - EG(y_0, y(T)) \rangle \geq 0, \quad \forall z \in \mathcal{Q}. \]  

(4.12)

**Proof**

Step 1. Applying Ekeland’s variational principle. We first consider the case that the set \( \mathcal{U}_{ad} \) is bounded in \( L^{\infty, \delta}_{0, p}[0, T]; \mathbb{R}^m \); the unbounded case can be reduced to the bounded case. Assume that

\[ \mathcal{U}_{ad} \text{ is bounded in } L^{\infty, \delta}_{0, p}[0, T]; \mathbb{R}^m \]  

(4.13)

An application of Ekeland’s variational principle will lead to the reduction of a general end-constraint problem to a family of free end-constraint problems.

Define the following auxiliary function

\[ J(v(\cdot), x_0; \varepsilon, \delta) = \Psi(J(v(\cdot), x_0), EG(x_0, x(T)); \varepsilon, \delta) \]  

(4.14)

with \( \Psi(\cdot, \cdot; \varepsilon, \delta) \) being defined as in (4.6). Then consider the metric space \((\mathbb{R}^m \times \mathcal{U}_{ad}, d)\) with the distance \( d \) defined by

\[ d((x_1, v_1()), (x_2, v_2()) = \sqrt{|x_1 - x_2|^2 + \triangle v_1()^2}. \]  

(4.15)

Tang and Li [34] verify that \( \Psi(\cdot, \cdot; \varepsilon, \delta) \) is complete and \( J(v(\cdot), x_0; \varepsilon, \delta) \) is continuous and bounded. Also, we have for any given \( \varepsilon > 0 \),

\[ \Phi(J(v(\cdot), x_0), EG(x_0, x(T)); \varepsilon) > 0, \quad \forall (x_0, v(\cdot)) \in \mathbb{R}^m \times \mathcal{U}_{ad}; \]
\[ \Phi(J(v(\cdot), y_0), EG(y_0, y(T)); \varepsilon) = \varepsilon; \]
\[ J(v(\cdot), x_0; \varepsilon, \delta) > 0, \quad \forall (x_0, v(\cdot)) \in \mathbb{R}^m \times \mathcal{U}_{ad}, \]
\[ \text{for sufficiently small } \delta > 0; \]
\[ J(u(\cdot), y_0; \varepsilon, \delta) \leq \varepsilon + 2D + \inf_{(x_0, v(\cdot)) \in \mathbb{R}^m \times \mathcal{U}_{ad}} J(v(\cdot), x_0; \varepsilon, \delta) \]

Therefore we can apply Ekeland’s variational principle (cf.[10]) and conclude that there exist \( u^{\varepsilon, \delta} \in \mathcal{U}_{ad} \) and \( y_0^{\varepsilon, \delta} \in \mathbb{R}^m \) such that

1) \[ J(u^{\varepsilon, \delta}, y_0^{\varepsilon, \delta}; \varepsilon, \delta) \leq \varepsilon + 2\delta; \]
2) \[ d((y_0^{\varepsilon, \delta}, u^{\varepsilon, \delta}()), (y_0, u(\cdot))) \leq \sqrt{\varepsilon + 2\delta}; \]
3) \[ J(v(\cdot), x_0; \varepsilon, \delta) \\overset{\text{def}}{=} J(v(\cdot), x_0; \varepsilon, \delta) + \sqrt{\varepsilon + 2\delta d((x_0, v(\cdot)), (y_0^{\varepsilon, \delta}, u^{\varepsilon, \delta}()))} \]
\[ \geq J(u^{\varepsilon, \delta}, y_0^{\varepsilon, \delta}), \quad \forall (x_0, v(\cdot)) \in \mathbb{R}^m \times \mathcal{U}_{ad}. \]  

(4.17)

Set

\[ J^\varepsilon \overset{\text{def}}{=} \Psi(J(u^{\varepsilon, \delta}(\cdot), y_0^{\varepsilon, \delta}), EG(y_0^{\varepsilon, \delta}, y_0^{\varepsilon, \delta}(T)); \varepsilon, \delta), \]
\[ \mu^\varepsilon \overset{\text{def}}{=} \Psi(J(u^{\varepsilon, \delta}(\cdot), y_0^{\varepsilon, \delta}), EG(y_0^{\varepsilon, \delta}, y_0^{\varepsilon, \delta}(T)); \varepsilon, \delta). \]  

(4.18)

and

\[ x_0 \overset{\text{def}}{=} x_0^{\varepsilon, \delta}(\cdot), \quad u(\cdot) \overset{\text{def}}{=} u^{\varepsilon, \delta}(\cdot). \]
Tang and Li [34] showed that for each sufficiently small \( \varepsilon > 0 \), we can choose \( \delta(\varepsilon) > 0 \) such that \( A^\varepsilon \geq 0 \) and \( \mu^\varepsilon \in \mathbb{R}^k \) satisfy the following:

\[
\lim_{\varepsilon \to 0} (|A^\varepsilon|^2 + |\mu^\varepsilon|^2) = 1, \quad< \mu^\varepsilon, z - E(g(\eta^\varepsilon, y^\varepsilon(T))) \leq \delta(\varepsilon) \leq \varepsilon. \tag{4.19}
\]

**Step 2. Computing the first-order component of the cost variation.** In this and the next steps, we look for the necessary conditions for the minimization of \( J(\nu(\cdot), x_0; \varepsilon, \delta) \) at \((y_0^\varepsilon, u^\varepsilon(\cdot))\).

For given \((\eta, \nu(\cdot)) \in \mathbb{R}^m \times \mathcal{U}_{def}\), set

\[
\begin{align*}
&u^\varepsilon(\cdot) = u^\varepsilon(\cdot) \chi_{[0,1]}(\cdot) + \nu(\cdot) \chi_{\ell}(\cdot), \\
&y_0^\varepsilon = y_0^\varepsilon + \|I_\rho\| \eta, \\
&y^\varepsilon(\cdot) = y^\varepsilon(\cdot; u^\varepsilon(\cdot), y_0^\varepsilon).
\end{align*}
\]  

(4.20)

We introduce, as in (3.4), the following simplified notations:

\[
\begin{align*}
\Delta m^\varepsilon(s; v) &\overset{\text{def}}{=} m(y^\varepsilon(s), v) - m(y^\varepsilon(s), u^\varepsilon(s)), \\
m^\varepsilon(s) &\overset{\text{def}}{=} m(y^\varepsilon(s), u^\varepsilon(s)),
\end{align*}
\]

(4.21)

with \( m \) standing for \( g, \gamma, \ell \) and all their (up to second-) derivatives in \( x \).

Let \( y^\varepsilon(\cdot) \) be the solution of (3.1) corresponding to \((y_0^\varepsilon, u^\varepsilon(\cdot))\). We define, as in (3.9) and (3.10), the half- and first-order processes \( y_1^\varepsilon(\cdot), y_2^\varepsilon(\cdot) \), respectively, by

\[
y_1^\varepsilon(t) = \int_0^t g_\varepsilon(y^\varepsilon(s), u^\varepsilon(s)) y_1^\varepsilon(s) ds + \sum_{\ell = 1}^\infty \sum_{p \in \mathcal{N}_\varepsilon} \int_0^t \left[ g_{\varepsilon p}(y^\varepsilon(s), u^\varepsilon(s)) y_1^\varepsilon(s) + \Delta g_{\varepsilon p}(s; u^\varepsilon(s), y_0^\varepsilon) \right] dH_p(s)
\]

(4.22)

and

\[
y_2^\varepsilon(t) = \int_0^t g_\varepsilon(y^\varepsilon(s), u^\varepsilon(s)) y_2^\varepsilon(s) ds + \sum_{\ell = 1}^\infty \sum_{p \in \mathcal{N}_\varepsilon} \int_0^t \left[ g_{\varepsilon p}(y^\varepsilon(s), u^\varepsilon(s)) y_2^\varepsilon(s) + \Delta g_{\varepsilon p}(s; u^\varepsilon(s), y_0^\varepsilon) y_1^\varepsilon(s) \right] dH_p(s) + \int_0^t \Delta g_{\varepsilon p}(s; u^\varepsilon(s), y_0^\varepsilon) y_2^\varepsilon(t) dH_p(s) + |I_\rho| \eta
\]

(4.23)

From Lemma 3.1, we can have

\[
\begin{align*}
&\sup_{0 \leq t \leq T} E|y_1^\varepsilon(t)|^8 = O(|I_\rho|^4), \\
&\sup_{0 \leq t \leq T} E|y_2^\varepsilon(t)|^8 = O(|I_\rho|^4), \\
&\sup_{0 \leq t \leq T} E|y^\varepsilon(t) - y_1^\varepsilon(t) - y_2^\varepsilon(t)|^2 = o(|I_\rho|^4), \\
&\quad \text{as} \quad |I_\rho| \to 0.
\end{align*}
\]  

(4.24)

In this step, we are to calculate the first-order component of the cost variation.
From 3) in (4.17) of Step 1, we have
\[ -|I_{p}| \sqrt{s} + 2\delta \sqrt{1 + |\eta|^2} \]
\[ \leq J(u^{p}(\cdot), y^{0}(0), s) - J(u^{c}(\cdot), y^{0}_0, s) \]
\[ \leq \mathcal{A}[J(u^{p}(\cdot), y^{0}_0 + |I_{p}|\eta) - J(u^{c}(\cdot), y^{0}_0)] \]
\[ + \sum_{j=1}^{n} \mu^{c}(E G^{j}(y^{0}_0 + |I_{p}|\eta, y^{p}(T)) - E G^{j}(y^{0}_0, y^{c}(T))) \]
\[ + O(J(u^{p}(\cdot), y^{0}_0 + |I_{p}|\eta) - J(u^{c}(\cdot), y^{0}_0)^2) \]
\[ + \sum_{j=1}^{n} O(E G^{j}(y^{0}_0 + |I_{p}|\eta, y^{p}(T)) - E G^{j}(y^{0}_0, y^{c}(T))^2) \]

Using (4.24), we have
\[ J(u^{p}(\cdot), y^{0}_0 + |I_{p}|\eta) - J(u^{c}(\cdot), y^{0}_0) \]
\[ = |I_{p}| < E h_{c}(y^{0}_0, y^{c}(T)), \eta > + E < h_{s}(y^{0}_0, y^{c}(T)), y^{1}_T + y^{2}_T > \]
\[ + E \int_{0}^{T} \ell_{s}(s, u^{c}(s)) y^{1}_T(s) + y^{2}_T(s) ds + \frac{1}{2} E \int_{0}^{T} y^{1}_T(s) \ell_{s}(s, y^{c}(s), u^{c}(s)) y^{2}_T(s) ds \]
\[ + E \int_{0}^{T} \Delta \ell_{c}(s, u^{c}(s)) ds + o(|I_{p}|) \]

and similarly
\[ E G^{j}(y^{0}_0 + |I_{p}|\eta, y^{p}(T)) - E G^{j}(y^{0}_0, y^{c}(T)) \]
\[ = |I_{p}| < E G^{j}_{s}(y^{0}_0, y^{c}(T)), \eta > + E < G^{j}_{s}(y^{0}_0, y^{c}(T)), y^{1}_T + y^{2}_T > \]
\[ + \frac{1}{2} E y^{1}_T(T) G^{j}_{s}(y^{0}_0, y^{c}(T)) y^{2}_T(T) + o(|I_{p}|) \]

From Lemma 2.2, we see that
\[-d p^{c}(t) = \left[ g^{T}(y^{c}(t), u^{c}(t)) p^{c}(t) \right. \]
\[ + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}^{2}} \gamma_{d}^{p}(y^{c}(t), u^{c}(t))^{T} J^{P^{c}}(t) + \mathcal{A}^{T} \ell_{c}(y^{c}(t), u^{c}(t)) \right] dt \]
\[ - \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}^{2}} J^{P^{c}}(t) d H^{p}(t) \]
\[ p^{c}(T) = \mathcal{A}^{T} h_{s}(y^{0}_0, y^{c}(T)) + \sum_{j=1}^{k} \mu^{c} G^{j}_{s}(y^{0}_0, y^{c}(T)). \]

and
\[-d P^{c}(t) = \left[ g^{T}(y^{c}(t), u^{c}(t)) P^{c}(t) + P^{c}(t) g_{s}(y^{c}(t), u^{c}(t)) \right. \]
\[ + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}^{2}} \gamma_{d}^{p}(y^{c}(t), u^{c}(t))^{T} P^{c}(t) y^{c}(t) + u^{c}(t) + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}^{2}} \gamma_{d}^{p}(y^{c}(t), u^{c}(t))^{T} R^{P^{c}}(t) \]
\[ + \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}^{2}} R^{P^{c}}(t) y^{c}(t) + H_{s}(y^{c}(t), u^{c}(t), \mathcal{A}^{c}, p^{c}(t), J^{c}(t)) \right] dt \]
\[ - \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}^{2}} R^{P^{c}}(t) d H^{p}(t) \]
\[ P^{c}(T) = \mathcal{A}^{c} h_{s}(y^{0}_0, y^{c}(T)) + \sum_{j=1}^{k} \mu^{c} G^{j}_{s}(y^{0}_0, y^{c}(T)). \]
have unique solutions \((p^c(\cdot), [J^p(\cdot)]_{\mu \in \mathcal{V}})\) and \((P^c(\cdot), [R^p(\cdot)]_{\mu \in \mathcal{V}})\) respectively, with \(p^c(\cdot)\) and \(P^c(\cdot)\) being cadlag processes.

Using Itô’s formula, we have from (4.22), (4.28) and (4.29), that

\[
E < \lambda^c h_\epsilon(y_{0\epsilon}, y^c(T)) + \sum_{j=1}^{d} \mu^c_j G_j^\epsilon(y_{0\epsilon}, y^c(T)) + \int_0^T \lambda^c \xi^c(s, u^c(s), y^c_1(T) + y^c_2(T) >
\]

\[
= E < p^c(0), y^c_1(T) + y^c_2(T) >
\]

\[
= E < p^c(0), y^c_1(T) + y^c_2(T) > + \sum_{d=1}^{\infty} \sum_{\mu \in \mathcal{V}} E \int_0^T (J^p(s), \lambda^c y^c_2(s, u^\mu(s)))ds
\]

\[
+ \frac{1}{2} E \int_0^T (p(s), g_x(y_1(s), u^\epsilon(s)))y^c_2(s)ds
\]

\[
+ \frac{1}{2} \sum_{d=1}^{\infty} \sum_{\mu \in \mathcal{V}} E \int_0^T (J^p(s), \gamma^p y^c_2(s, u^\mu(s)))y^c_2(s)ds
\]

Applying Itô’s formula to the matrix-valued processes

\[
Y(s) = y_1(s)y^T_1(s) = \begin{pmatrix} y_1^1 & \cdots & y_1^m \\ \vdots & \ddots & \vdots \\ y_1^m & \cdots & y_1^m \end{pmatrix}
\]

we have

\[
dY(t) = \left[ Y(t)g_y(t) + g_x(t)Y(t) + \sum_{d=1}^{\infty} \sum_{\mu \in \mathcal{V}} \gamma^p(t)Y(t)\gamma^p_2(t)^T + \Phi^c(t) \right] dt
\]

\[
+ \sum_{d=1}^{\infty} \sum_{\mu \in \mathcal{V}} \left[ Y(t)\gamma^p_2(t)^T + \gamma^p(t)Y(t) + \Omega^p(t) \right] dH^p(t)
\]

where

\[
\Phi^c(t) = \sum_{d=1}^{\infty} \sum_{\mu \in \mathcal{V}} \gamma^p(t)y_1(t)\gamma^p(t)y_1^T + \sum_{d=1}^{\infty} \sum_{\mu \in \mathcal{V}} \gamma^p(t, u^\mu(t))y_1(t)\gamma^p(t)^T
\]

\[
+ \sum_{d=1}^{\infty} \sum_{\mu \in \mathcal{V}} \gamma^p(t, u^\mu(t))\gamma^p(t, u^\mu(t))^T
\]

\[
\Omega^p(t) = y_1(t)\gamma^p(t, u^\mu(t))^T + \gamma^p(t, u^\mu(t))y_1(t)^T
\]

\[
+ \sum_{d=1}^{\infty} \sum_{\mu \in \mathcal{V}} \gamma^p(t, u^\mu(t))\gamma^p(t, u^\mu(t))^T
\]
From (4.33)-(4.36), we conclude for given 
Next choose the above 
This implies that 
\[ E \| h_{ij}(\gamma_0, y^c(T)) \| + \sum_{j=1}^{k}\mu^{ij} h_{ij}(\gamma_0, y^c(T)) h_{ij}(\gamma_0, y^c(T)) \| \leq C \sqrt{3e}, \]
\[ E \int_{I_0}^{T} E^{(s)}(v(s), u^\rho(s)) ds \leq -\sqrt{e} + 20(e) \sqrt{1 + |\eta|^2}, \]
Step 4. Passing to the limit. Without loss of generality, we assume that \( \lambda^\varepsilon \rightarrow \lambda, \mu^\varepsilon \rightarrow \mu, \) as \( \varepsilon \rightarrow 0^+ . \) Let \( \varepsilon \rightarrow 0^+ \). Equation (4.19)_2 gives the following:

\[
E \int_0^T (H(y(s), \nu^\varepsilon(s), \lambda, p(s), J(s)) - H(y(s), u(s), \lambda, p(s), J(s)))\,ds \\
+ \frac{1}{2}E \int_0^T \left( \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_0^d} \Delta^d \gamma^\varepsilon P(s; u^{\varepsilon}(s)) \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_0^d} \Delta^d \gamma^\varepsilon P^T(s; u^{\varepsilon}(s)) \right)\,ds \\
+ \frac{1}{2}E \int_0^T \left( \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_0^d} R^\varepsilon P(s) \sum_{d=1}^{\infty} \sum_{p \in \mathbb{N}_0^d} \Delta^d \gamma^\varepsilon P^T(s; u^{\varepsilon}(s)) \right)\,ds \\
\geq 0, \quad \forall v(\cdot) \in U_{ad},
\]

this implies (4.11). Furthermore, (4.11) is obtained from (4.19), (4.9)_2 is obtained from (4.39)_1, and the rest of Theorem 4.1 is checked from (4.28) and (4.29).

Step 5. The unbounded case of \( U_{ad} \) in \( L^{\infty, \varepsilon}(0, T]; \mathbb{R}^m) \). The proof procedure is the same as the step 5 in Tang and Li [34].

The proof of Theorem 4.1 is complete. \( \square \)

5. Conclusions

In this paper, necessary maximum principle for optimal control of stochastic system driven by multidimensional Teugel’s martingales is proved, where the multidimensional Teugel’s martingales are constructed by orthogonalizing the multidimensional Lévy processes. The control variable is allowed to enter the coefficients of the Teugel’s martingales, and the control domain is nonconcave. The technique for proving the maximum principle and the obtained result are almost similar to Peng [26] and Tang and Li [34].

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