The effect of fast movement in dissipative PDEs with the forcing term

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Abstract

This version of the paper should be considered as an announcement of results, it is incomplete and we present sketches of proofs.

We describe a topological method to study dynamics with rapidly oscillating vector fields. As an example we apply the technique to the Burgers equation with nonautonomous forcing and the periodic boundary conditions. We prove that for large initial condition integral the equation admits a globally attracting solution defined on the real line. We show that the technique applies to 2D Navier-Stokes equations.

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1 Introduction

Let us consider a dissipative PDE with periodic boundary condition in $\mathbb{R}^d$ (i.e. on the $d$-dimensional torus $T^d$) of the following form

$$
\frac{du}{dt} = \nu \Delta u + N(\nabla u, u) + f(t, x)
$$

where $\nu > 0$, $u \in \mathbb{R}^d$ and $f \in \mathbb{R}^d$ is the forcing term.

Additionally, we assume that (1) is invariant with respect the transformation $(t, x) \mapsto (t, x + at)$, where $a \in \mathbb{R}^d$ and we have the following conservation law

$$
\frac{d}{dt} \int_{T^d} u(t, x) = \int_{T^d} f(t, x).
$$

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If, we assume that
\[ \int_{T^d} f(t, x) = 0, \]  
then
\[ \frac{1}{(2\pi)^d} \int_{T^d} u(t, x) = u_0. \]  

For the problem for large \( \|u_0\| \) we are interested in the existence of bounded solutions of the form \( u_0 + O\left( \frac{\|f\|}{\|u_0\|} \right) \) defined for all \( t \in \mathbb{R} \), which attracts all other solutions. The important point is that we do not assume that \( \|f\| \) is small, we will just assume that \( f \) is a finite trigonometric polynomial in respect space variables with time dependent coefficients with bounded time derivatives.

We consider two models: one dimensional Burgers equation and the Navier-Stokes system in dimensional two and three. For the Burgers equation in dimension one we established the result described above. For Navier-Stokes equations in 2D, the result is true for a generic direction of \( u_0 \) and in 3D for generic direction \( u_0 \) we establish the existence of small locally attracting solution.

The basic idea in our approach can be described as follows: for generic choices of \( u_0 \) the transformation of (1) to a coordinate frame in which the average of \( u \) vanishes, leads to (1) with rapidly oscillating forcing term. We prove that in the transformed system this very rapid oscillation is effectively equivalent to small forcing term of the size \( O\left( \frac{\|f\|}{\|u_0\|} \right) \). As the result we obtain an absorbing set very close to 0 and then by topological reasoning we show that it contains an orbit bounded by \( O(\|f\|/\|u_0\|) \) defined for all \( t \in \mathbb{R} \). By reversing the initial coordinate change we obtain an attraction orbit of the form \( u_0 + O\left( \frac{\|f\|}{\|u_0\|} \right) \).

The underlining idea is that the rapid oscillating could be effectively ‘integrated out’ (or averaged) without assuming the smallness of oscillating term. This idea is known for some time in the numerical analysis, where it is used to obtain effective quadratures and ODE solvers for system with rapid oscillations (see [I] and the references cited there). To see how this idea can be exploited to obtain rigorous results about the dynamics, invariant sets etc see [ZSr].

Similar results to ours can be found in literature. In [JKM] for any \( \nu > 0 \) the authors established the existence of a globally attracting solution for 1D viscous Burgers equation with periodic boundary conditions, under assumption that forcing is periodic in time. Without the proof they state this solution scales as \( u_0 + O(1/\|u_0\|) \) (for fixed forcing and \( \nu \)). Our result is for more general forcing, moreover we are able to establish the exponential convergence rate to the attracting solution, while in [JKM] the authors clearly indicated that they cannot make such claim and they asked for the convergence rate in one of the stated problems [JKM, Problem 3(i)]. The method in [JKM] appears to be restricted to the scalar equation on one-dimensional domains, partially due to the use of the maximum principles.
1.1 Notation

Consider nonautonomous ODE

\[ x'(t) = f(t, x(t)), \]  

(5)

where \( x \in \mathbb{R}^n \) and \( f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) is regular enough to guarantee the uniqueness of the initial value problem \( x(t_0) = x_0 \). We set \( \varphi(t_0, t, x_0) = x(t_0 + t) \), where \( x(t) \) is a solution of (5) with initial condition \( x(t_0) = x_0 \). Obviously in each context it will be clearly stated what is the ordinary differential equation generating \( \varphi \). We will sometimes refer to \( \varphi \) as to the local process generated by (5).

For matrix \( U \) by \( U^t \) we will denote its transpose. For a square matrix \( U \) we will denote its spectrum by \( \text{Sp}(U) = \{ \lambda \in \mathbb{C} \mid \lambda \text{ is an eigenvalue of } U \} \). If \( A \in \mathbb{R}^{n \times n} \) by \( \mu(A) \) we will denote its logarithmic norm, which is defined by

\[ \mu(A) = \lim_{h \to 0^+} \frac{\|I + hA\| - 1}{h} . \]  

(6)

For the properties of logarithmic norm and its relation with the Lipschitz constant for the flow induced by ODEs see [KZ] and literature cited there. The logarithmic norm depends on the norm used. We will always assume that we are using the euclidean norm.

Quite of the following expression will appear \( h_l(t) = e^{lt} - 1 \), where \( l \) is a fixed parameter. For \( l = 0 \) we will understand \( h_0 \) as follows \( h_0(t) = \lim_{l \to 0} e^{lt} - 1 = t \).

For a function of several variables we will often use \( D_t f \) and \( D_z f \) to denote the partial derivatives. For example, \( D_z f(t, z) = \frac{\partial f}{\partial z}(t, z) \).

2 Basic estimates

In this section we rewrote some estimates from [ZSr].

2.1 Linear nonautonomous equations

Assume that \( A : \mathbb{R} \to \mathbb{R}^{n \times n} \) is continuous and for \( k = 1, \ldots, m \) \( u_k : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \) is \( C^1 \), \( g_k : \mathbb{R} \to \mathbb{R} \) is continuous.

Let us consider the following non-autonomous non-homogenous linear ODE

\[ x'(t) = A(t)x(t) + \sum_{k=1}^{m} g_k(\omega_k t)u_k(t), \quad x \in \mathbb{R}^n. \]  

(7)

Let \( G_k(t) \) be a primitive of \( g_k \), so

\[ G_k'(t) = g_k(t). \]  

(8)

Assume that \( \|G_k(t)\| \) are bounded.
Let $M(t, t_0)$ be a fundamental matrix of solutions of the homogenous version of (7)
\[ x'(t) = A(t)x(t). \] (9)
This means that for any $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$ the function $x(t) = M(t, t_0)x_0$ solves (9) with the initial condition $x(t_0) = x_0$.

It is well known that $M$ has the following properties.
\[
M(t_0, t) = I, \quad M(t_0, t)^{-1} = M(t, t_0), \quad \frac{\partial}{\partial t} M(t, t_0) = A(t)M(t, t_0), \quad \frac{\partial}{\partial t_0} M(t, t_0) = -M(t, t_0)A(t_0).
\] (10) (11) (12) (13)

The general solution of (7) is given by
\[
\varphi(t_0, t, x_0) = M(t_0 + t, t_0)x_0 + \int_0^t M(t_0 + t, t_0 + s) \sum_{k=1}^m g_k(\omega_k(t_0 + s))u_k(t_0 + s)ds
\] (14)

Let us estimate the integral in the above formula. Using the integration by parts and (13) we obtain for $k = 1, \ldots, m$
\[
D_k(t + t_0) := \int_0^t g_k(\omega_k(t_0 + s))M(t_0 + t, t_0 + s)u_k(t_0 + s)ds = \left\{ \begin{array}{l}
G(\omega_k(t_0 + s))M(t_0 + t, t_0 + s)u_k(t_0 + s) \bigg|^{s=t}_{s=0} + \\
-\frac{1}{\omega_k} \int_0^t G_k(\omega_k(t_0 + s)) \frac{\partial}{\partial s} (M(t_0 + t, t_0 + s)u_k(t_0 + s)) ds = \\
\frac{1}{\omega_k} \left( G_k(\omega_k(t_0 + t))u_k(t_0 + t) - G_k(\omega_k t_0)M(t_0 + t, t_0)u_k(t_0) \right) + \\
\frac{1}{\omega_k} \int_0^t G_k(\omega_k(t_0 + s))M(t_0 + t, t_0 + s)A(t_0 + s)u_k(t_0 + s)ds + \\
-\frac{1}{\omega_k} \int_0^t G_k(\omega_k(t_0 + s))M(t_0 + t, t_0 + s)u_k'(t_0 + s)ds
\end{array} \right. 
\] (15)

(For Galerkin projections of dissipative PDEs while $\|A\|$ will not have any uniform bound independent of the projection dimension, we expect $\|Au_k\|$ to be uniformly bounded.)

### 2.2 Estimates for a nonlinear problem

Consider problem
\[ x' = f(t, x) + \sum_{k=1}^m g_k(\omega_k t)u_k(t, x) \] (16)
and its oscillation free version

\[ y' = f(t, y). \]  

(17)

Let \( \varphi(t_0, t, x) \) be the local process induced by (16) and \( \varphi_a(t_0, t, x) \) is the local process induced by (17).

**Lemma 2.1.** Let \( x : [t_0, t_0 + h] \to \mathbb{R}^n \) and \( y : [t_0, t_0 + h] \to \mathbb{R}^n \) be solutions to (16) and (17), respectively, such that \( x(t_0) = y(t_0) \).

Let \( W \) be a compact set, such that for any \( t \in [0, h] \) the segment joining \( x(t_0 + t) \) and \( y(t_0 + t) \) is contained in \( W \).

Assume that for \( k = 1, \ldots, m \) \( C'_k(t) = g_k(t) \).

Assume that there exist constants \( l, C(\ldots) \) such that for all \( k = 1, \ldots, m \) holds

\[
\sup_{z \in W, s \in [t_0, t_0 + h]} \| f(s, z) \| = C(f),
\]

(18)

\[
\sup_{t \in \mathbb{R}} \| G_k(t) \| = C(G_k),
\]

(19)

\[
\sup_{t \in \mathbb{R}} \| g_k(t) \| = C(g_k),
\]

(20)

\[
\mu(D_z f(s, z)) = l
\]

(21)

\[
\sup_{z \in W, s \in [t_0, t_0 + h]} \| u_k(s, z) \| = C(u_k)
\]

(22)

\[
\sup_{z, z_1 \in W, s \in [t_0, t_0 + h]} \| D_z f(s, z) u_k(s, z_1) \| = C(D_z f u_k)
\]

(23)

Then for \( t \in [0, h] \) holds

\[
\| x(t_0 + t) - y(t_0 + t) \| \leq \sum_k \frac{1}{|\omega_k|} b_k(t)
\]

(24)

where the continuous nondecreasing function \( b_k : [0, h] \to \mathbb{R}_+ \) depends on constants \( C(f), l, C(g_i), C(G_i), C(u_i), C(D_z f u_i), C(\frac{\partial u_k}{\partial t}) \) and \( C(\frac{\partial u_k}{\partial s}) \).

We have

\[
b_k(t) \leq C(G_k) (C(u_k)(1 + e^{lt}) + C(D_z f u_k) + C(\frac{\partial u_k}{\partial t}) + C(\frac{\partial u_k}{\partial z}) (C(f) + \sum_i C(u_i) C(g_i)) (e^{lt} - 1) \frac{l}{t})
\]
Proof: Let \( z(t) = x(t) - y(t) \). We have
\[
z'(t) = f(x(t)) - f(y(t)) + \sum_{k=1}^{m} g_k(\omega_k t)u_k(t, x) = \\
\left( \int_0^1 D_x f(t, s(x(t) - y(t)) + y(t)) ds \right) \cdot z(t) + \sum_{k=1}^{m} g_k(\omega_k t)u_k(t, x(t)).
\]
Therefore
\[
z'(t) = A(t)z(t) + \sum_{k=1}^{m} g_k(\omega_k t)u_k(t, x(t)),
\]
where
\[
A(t) = \left( \int_0^1 D_x f(t, s(x(t) - y(t)) + y(t)) ds \right).
\]
Let \( M(t_1, t_0) \) is the fundamental matrix of solutions for \( x' = A(t)x \).
Since \( z(t_0) = 0 \), then from (14) and (15) it follows that
\[
z(t_0 + t) = \sum_k D_k(t + t_0)
\]
where
\[
D_k(t + t_0) = \frac{1}{\omega_k} (G_k(\omega_k(t_0 + t))u_k(t_0 + t, x(t + t_0)) - G_k(\omega_k t_0)M(t_0 + t, t_0)u_k(t_0, x(t_0)) + \\
\frac{1}{\omega_k} \int_0^t G_k(\omega_k (t_0 + s))M(t_0 + t, t_0 + s)A(t_0 + s)u_k(t_0 + s, x(t_0 + s)) ds + \\
- \frac{1}{\omega_k} \int_0^t G_k(\omega_k (t_0 + s))M(t_0 + t, t_0 + s) \left( \frac{d}{ds}u_k(t_0 + s, x(t_0 + s)) \right) ds
\]
From the standard estimate using the logarithmic norms (see also Lemma 4.1 in [KZ]) we know that for \( t \geq 0 \) holds
\[
\|M(t + t_0, t_0)\| \leq \exp(lt).
\]
We obtain the following estimate of \( D_k(t) \) for \( t \in [0, h] \)
\[
|\omega_k| \cdot \|D_k(t + t_0)\| \leq C(u_k)C(G_k)(1 + e^{lt}) + C(D_z f u_k) C(G_k) \int_0^t e^{l(t-s)} ds + \\
C(G_k) \left( C \left( \frac{\partial u_k}{\partial t} \right) + C \left( \frac{\partial u_k}{\partial z} \right) \sup_{s \in [0, h]} \|x'(t_0 + s)\| \right) \int_0^t e^{l(t-s)} ds \leq \\
C(u_k)C(G_k)(1 + e^{lt}) + C(D_z f u_k) C(G_k)(e^{lt} - 1)/l + \\
C(G_k) \left( C \left( \frac{\partial u_k}{\partial t} \right) + C \left( \frac{\partial u_k}{\partial z} \right) \left( C(f) + \sum_i C(u_i)C(g_i) \right) \right) (e^{lt} - 1)/l
\]
It appears to be reasonable to expect that $b_k(0) = 0$ and this is true for bounded range of $\omega_k$, however for $\omega_k$ unbounded this does not hold, as the following example shows.

Consider the equation
\[
x' = x + \exp(i\omega t), \quad x \in \mathbb{C}.
\]

The solution is
\[
x(t_0 + t) = x(t_0)e^t + e^{i\omega t_0} \frac{e^{i\omega t} - e^t}{1 + i\omega}.
\]

hence
\[
b(t) = \frac{|e^{i\omega t} - e^t|}{|i - 1/\omega|}.
\]

It is clear that for any $t > 0$, such that $e^t < 1$ holds
\[
b(t) \geq \limsup_{\omega \to \infty} |e^{i\omega t} - e^t| \geq \limsup_{\omega \to \infty} |\sin(\omega t)| - e^t > 1 - e^t > 0.
\]

### 3 Viscous Burgers equation with periodic boundary conditions on the line

The Burgers equation was proposed in [B] as a mathematical model of turbulence. There is a significant number of applications of the Burgers equation, see e.g. [Wh]. We consider the initial value problem for viscous Burgers equation on the real line with periodic boundary conditions and a non-autonomous forcing $F$, i.e.

\[
\begin{align*}
  u_t(t, x) + u(t, x) \cdot u_x(t, x) - \nu u_{xx}(t, x) &= f(t, x), & t \in [t_0, \infty), & x \in \mathbb{R}, \\
  u(t, x) &= u(t, x + 2\pi), & t \in [t_0, \infty), & x \in \mathbb{R}, \\
  f(t, x) &= f(t, x + 2\pi), & t \in \mathbb{R}, & x \in \mathbb{R}, \\
  u(t_0, x) &= u_0(x), & t_0 \in \mathbb{R}, & x \in \mathbb{R},
\end{align*}
\]

where $\nu > 0$.

We will use the Fourier series to study (33). Let
\[
u(t, x) = \sum_{k \in \mathbb{Z}} a_k(t) \exp(ikx).
\]

It is straightforward to write the problem (33) in the Fourier basis. We obtain the following infinite ladder of equations
\[
\frac{da_k}{dt} = -\frac{i}{2} \sum_{k_1 \in \mathbb{Z}} a_{k_1} \cdot a_{k-k_1} + \lambda_k a_k + f_k(t), \quad t \in [t_0, \infty), \quad k \in \mathbb{Z},
\]
where

\[ a_k(t_0) = \frac{1}{2\pi} \int_{0}^{2\pi} u_0(x)e^{-ikx} \, dx, \quad k \in \mathbb{Z}, \quad (36a) \]

\[ f_k = \frac{1}{2\pi} \int_{0}^{2\pi} f(t,x)e^{-ikx} \, dx, \quad k \in \mathbb{Z}, \quad (36b) \]

\[ \lambda_k = -\nu k^2. \quad (36c) \]

The reality of \( u \) and \( f \) implies that for \( k \in \mathbb{Z} \)

\[ a_k = \overline{a_{-k}}, \quad f_k = \overline{f_{-k}}, \quad \text{for } t \in \mathbb{R}. \quad (37) \]

In view of the above variables \( \{a_k\}_{k \in \mathbb{Z}} \) are not independent, this motivates the following definition.

**Definition 3.1.** In the space of sequences \( \{a_k\}_{k \in \mathbb{Z}} \), where \( a_k \in \mathbb{C} \), we will say that the sequence \( \{a_k\} \) satisfies the reality condition iff

\[ a_k = \overline{a_{-k}}, \quad k \in \mathbb{Z}. \quad (38) \]

We will denote the set of sequences satisfying (38) by \( R \). It is easy to see that \( R \) is a vector space over the field \( \mathbb{R} \).

We will assume that the initial condition for (35) satisfies

\[ \frac{1}{2\pi} \int_{0}^{2\pi} u_0(x) \, dx = \alpha, \quad \text{for a fixed } \alpha \in \mathbb{R}. \quad (39) \]

We will require additionally that \( f_0(t) = 0 \) for \( t \in \mathbb{R} \), and then (39) implies that \( a_0(t) \) is constant, namely

\[ a_0(t) = \alpha, \quad \forall t \geq t_0. \quad (40) \]

**Definition 3.2.** For any given number \( m > 0 \) the \( m \)-th Galerkin projection of (35) is

\[ \frac{da_k}{dt} = -i \frac{k}{2} \sum_{|k-k_1| \leq m} a_{k_1} \cdot a_{k-k_1} + \lambda_k a_k + f_k(t), \quad t \in [t_0, \infty), \quad |k| \leq m. \quad (41) \]

Note that the condition (40) holds also for all Galerkin projections (41) as long as \( f_0(t) = 0 \) for all \( t \in \mathbb{R} \). Also observe that the reality condition (38) is invariant under all Galerkin projections (41), i.e. if \( a_k(t_0) = \overline{a_{-k}(t_0)} \), then \( a_k(t) = \overline{a_{-k}(t)} \) for all \( t > t_0 \) if the solution of (41) exists up to that time.

**Definition 3.3.** Let \( \| \cdot \|: \mathbb{R} \to \mathbb{R} \) be given by

\[ \|a\| = \begin{cases} |a| & \text{if } a \neq 0, \\ 1 & \text{if } a = 0. \end{cases} \]
Definition 3.4. Let $H$ be the space $l_2(\mathbb{Z}, \mathbb{C})$, i.e. $u \in H$ is a sequence $u : \mathbb{Z} \to \mathbb{C}$ such that $\sum_{k \in \mathbb{Z}} |u_k|^2 < \infty$ over the coefficient field $\mathbb{R}$. The subspace $\tilde{H} \subset H$ is defined by
\[
\tilde{H} := \left\{ \{a_k\} \in H : \text{there exists } 0 \leq C < \infty \text{ such that } |a_k| \leq \frac{C}{|k|^4} \text{ for } k \in \mathbb{Z} \right\}.
\]

Definition 3.5. Let the space $H'$ be given by
\[
H' := \tilde{H} \cap \mathbb{R}.
\]

Let us comment on Definitions 3.4 and 3.5. Despite the fact that we are dealing with complex sequences we use as the coefficient field the set of real numbers, because the reality condition is not compatible with the complex multiplication.

The choice of the particular subspace $H'$ is motivated by the fact that the order of decay of coefficients $\{a_k\} \in H'$ is sufficient for the uniform convergence of $\sum a_k e^{ikx}$ and every term appearing in (33a).

3.1 The effect of the moving coordinate frame

Let us transform the Burgers equation to a coordinate frame, which is moving with the velocity $c$. Since the function $u(t, x)$ has the meaning of the velocity, hence this transformation on the function level works as follows. The function $u(t, x)$ is transformed into a function $v(t, x)$
\[
v(t, x) = u(t, x + ct) - c. \tag{42}
\]

We have the following easy lemma.

Lemma 3.6. Assume that $\int_0^{2\pi} f(t, x)dx = 0$ and $\frac{1}{2\pi} \int_0^{2\pi} u_0(x)dx = c$. Let $u(t, x)$ be a solution of (33). Then function
\[
v(t, x) = u(t, x + ct) - c. \tag{43}
\]
is a solution of (33) with the forcing term $\tilde{f}(t, x) = f(t, y + ct)$ and $\int_0^{2\pi} v(0, x)dx = 0$.

Proof: We have
\[
u(t, x) = v(t, x - ct) + c. \tag{44}
\]
Hence
\[
u_t(t, x) = v_t(t, x - ct) - cv_x(t, x - ct)
\]
\[
u_x(t, x) = v_x(t, x - ct)
\]
\[
u_{xx}(t, x) = v_{xx}(t, x - ct)
\]
Therefore from
\[
u_t(t, x) + u(t, x)u_x(t, x) = \nu u_{xx}(t, x) + f(t, x)
\]

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we obtain
\[ v_t(t, x - ct) - cv_x(t, x - ct) + (v(t, x - ct) + c)v_x(t, x - ct) = \nu v_{xx}(t, x - ct) + f(t, x) \]
\[ u_t(t, x - ct) + v(t, x - ct)v_x(t, x - ct) = \nu v_{xx}(t, x - ct) + f(t, x) \]
\[ v_t(t, x) + v(t, x)v_x(t, x) = \nu v_{xx}(t, x) + f(t, x, x + ct). \]

Observe that from the space periodicity of \( f \) it follows that for any \( t \) holds
\[ \int_0^{2\pi} f(t, x + ct)dx = \int_0^{2\pi} f(t, x)dx \quad (45) \]
We also have for all \( t \)
\[ \int_0^{2\pi} v(t, x)dx = \int_0^{2\pi} (u(t, x - ct) - c)dx = \int_0^{2\pi} u(t, x)dx - 2\pi c = 0. \quad (46) \]

### 3.2 The action of this transformation on the Fourier modes

Assume that
\[ f(t, x) = \sum_{k \in \mathbb{Z}} f_k(t) \exp(ikx) \]

Observe that the Fourier expansion for the new forcing term is given as follows
\[ \hat{f}(t, x) = f(t, x + ct) = \sum_{k \in \mathbb{Z}} f_k(t) \exp(ik(x + ct)) = \sum_{k \in \mathbb{Z}} f_k(t) \exp(ikct) \exp(ikx) \]

Hence
\[ \hat{f}_k(t) = f_k(t) \exp(ikct). \quad (47) \]

If \( u(t, x) = \sum_{k \in \mathbb{Z}} u_k(t) \exp(ikx) \) with \( u_0(t) = c \), then
\[ v(t, x) = u(t, x + ct) - c = \sum_{k \in \mathbb{Z} \setminus \{0\}} u_k(t) \exp(ik(x + ct)) = \sum_{k \in \mathbb{Z} \setminus \{0\}} (u_k(t) \exp(ikct)) \exp(ikx) \]

Hence
\[ v_0(t) = 0 \]
\[ v_k(t) = u_k(t) \exp(ikct), \quad k \neq 0. \]
4 Burgers equation with large average speed

The Burgers equation in the Fourier domain is

\[ a_k' = -\lambda_k a_k - ika_0 a_k + N_k(z) + f_k(t), \quad k \in \mathbb{Z} \setminus \{0\} \]  

(48)

where \( a_0 \in \mathbb{R} \) and

\[ N_k(z) = -\frac{ik}{2} \sum_{k_i \in \mathbb{Z} \setminus \{0, k\}} a_k a_{k-k_i}. \]

(49)

Let

\[ \omega_k = -ka_0. \]

(50)

Observe that the transformation (compare previous section)

\[ x_k = a_k \exp(-i\omega_k t) \]

(51)

preserves the reality condition, i.e. if \( \overline{a_k} = a_k \), then \( \overline{x_k} = x_k \) and

\[ N_k(x(a)) = N_k(a). \]

(52)

Therefore we obtain the system

\[ x_k' = -\lambda_k x_k + N_k(x) + f_k(t) \exp(ika_0 t) = F_k(t, x), \quad k \in \mathbb{Z} \setminus \{0\}. \]

(53)

for the referencing reason we will also write autonomous problem

\[ x_k' = -\lambda_k x_k + N_k(x) = \tilde{F}_k(x), \quad k \in \mathbb{Z} \setminus \{0\}, \]

(54)

and the whole forcing term

\[ g_k(t) = f_k(t) \exp(ika_0 t). \]

(55)

As before we assume that \( f_k = 0 \) for \( |k| > K \), otherwise we have to assume something about the decay rate of \( f_k \), which also should not be a problem.

We have

\[ (z|N(z)) = 0 \]

(56)

this gives us an absorbing forward invariant ball of the size independent form the dimension of the Galerkin projection, both for (53) and (54). Let us denote the radius \( R \).

Let us recall some results regarding existence of forward invariant absorbing sets for (53).

**Definition 4.1.** [Cy, Def. 4.6] Let \( N_0 \geq 0 \), \( \varphi_n \) be a local process induced by the \( n \)-th Galerkin projection of (53). A set \( A \subset H' \) is called the absorbing set for large Galerkin projections of (53), if for any pair \((t_0, u_0) \in \mathbb{R} \times H' \) there exists \( t_1(u_0) \geq 0 \) such that for all \( n > N_0 \) and all \( t_0 \in \mathbb{R} \), \( t \geq t(u_0) \) holds \( \varphi_n(t_0, t, P_n u_0) \in P_n A \). Moreover, \( P_n A \) is forward invariant for \( \varphi_n \).
Definition 4.2. [Cy, Def. 3.1] Energy of (35) is given by the formula

\[ E\{a_k\} = \sum_{k \in \mathbb{Z}} |a_k|^2. \] (57)

Energy of (35) with \(a_0\) excluded is given by the formula

\[ \mathcal{E}\{a_k\} = \sum_{k \in \mathbb{Z} \setminus \{0\}} |a_k|^2. \] (58)

Lemma 4.3. Let \(s_l \geq 4\). For the system (54) there exist a time independent forward invariant trapping isolating segment

\[ \mathcal{R} := \left\{ \{a_k\} \in H': E(\{a_k\}) \leq r_l \text{ and } |a_k| \leq \frac{C_l}{|k|^s_l} \right\}, \]

such that for any forcing \(f\), such that \(f_0 = 0\), \(f_k = 0\) for \(|k| > J\), and \(f_k(t) = \tilde{f}_k(t), |\tilde{f}_k(t)| \leq D\) for \(|k| \leq J, t \in \mathbb{R}\) the following inequality holds

\[ \mu(D_x F(\mathcal{R}, \mathcal{R})) < 0, \]

where \(\mu\) is the block-infinity logarithmic norm (the Euclidean norm on \(2 \times 2\) blocks, and the maximum of sum of rows), with all blocks two-dimensional, as introduced in [CyZ].

Proof Observe that the forcing term in (53) does not depend on \(x\), therefore we may write \(D_x F(\mathcal{R}, \mathcal{R}) = D\tilde{F}(\mathcal{R})\).

In order to show that \(\mu(D\tilde{F}(\mathcal{R})) < 0\) we have to show that the following values are strictly negative for all \((i) > 0\)

\[ \sup_{x \in \mathcal{R}} \mu_e \left( \frac{\partial \tilde{F}(i)}{\partial x(i)} \right) + \sup_{x \in \mathcal{R}} \sum_{j \neq (i)} \frac{\partial \tilde{F}(i)}{\partial x(j)} (x), \]

where \(\mu_e\) is the logarithmic norm of the block inducted by the Euclidean norm. Using the explicit formulas for the partial derivatives of (53) derived in [CyZ, Supplementary material] we obtain the following formulas

\[ \sup_{x \in \mathcal{R}} \mu_e \left( \frac{\partial \tilde{F}(i)}{\partial x(i)} \right) + 2ci \sup_{x \in \mathcal{R}} \sum_{j \neq (i)} \left| \Re(x_{i-j}) \right| + \left| \Re(x_{i+j}) \right| + \left| \Im(x_{i-j}) \right| + \left| \Im(x_{i+j}) \right| \leq \]

\[ \sup_{x \in \mathcal{R}} \mu_e \left( \frac{\partial \tilde{F}(i)}{\partial x(i)} \right) + 2ci \sup_{x \in \mathcal{R}} \sum_{j \neq 0} \left| \Re(x_j) \right| + \left| \Re(x_j) \right| + \left| \Im(x_j) \right| + \left| \Im(x_j) \right|. \]

For the logarithmic norm of the block inducted by the Euclidean norm we have

\[ \sup_{x \in \mathcal{R}} \mu_e \left( \frac{\partial \tilde{F}(i)}{\partial x(i)} \right) = -\nu i^2. \]
Hence if
\[
\nu > 2c \sup_{x \in R} \sum_{j \neq 0} |\Re(x_j)| + |\Re(x_j)| + |\Im(x_j)| + |\Im(x_j)|
\]
the inequality \(\mu(D_e F(\mathbb{R}, \mathbb{R})) < 0\) holds.

The theorem below is a main building block for the construction of the absorbing set.

**Theorem 4.4.** [CYZ, Thm. 2.8] Assume that \(f_k\) for \(t \in \mathbb{R}\) satisfies \(f_k(t) = f_{-k}(t), f_k(t) = 0\) for \(|k| > J\), and \(f_0(t) = 0\). Let \(\{a_k\}_{k \in \mathbb{Z}} \in H, s > 0.5, E_0 = \sup_{k \in \mathbb{Z}} E(t_k) < \infty, \tilde{E} > E_0, D = 2^{s-\frac{1}{2}} + \frac{2^{s-1}}{\sqrt{2s-1}}, C > \sqrt{\tilde{E}N^s}, N >\)

max \(\left\{ J, \left( \frac{\sqrt{\tilde{E}D}}{\nu} \right)^2 \right\} \). Then

\[
W(\tilde{E}, N, C, s) = \left\{ \{a_k\} \in R \mid \mathcal{E}(\{a_k\}) \leq \tilde{E}, |a_k| \leq \frac{C}{|k|^s} \right\}
\]
is a trapping region (i.e. is forward invariant) for each Galerkin projection of \(\tilde{E}\) restricted to the invariant subspace given by \(a_k = \frac{1}{a-k}\).

**Theorem 4.5.** [CYZ, Thm 2.9] Assume that \(f_k\) for \(t \in \mathbb{R}\) satisfies \(f_k(t) = f_{-k}(t), f_k(t) = 0\) for \(|k| > J\), and \(f_0(t) = 0\). Let \(\varepsilon > 0, E_0 = \sup_{k \in \mathbb{R}} E(t_k) < \infty, \tilde{E} > E_0, N \) is defined in Thm. 4.4. Put

\[
s_i = i/2 \text{ for } i \geq 2,
\]
\[
D_i = 2^{s_i-\frac{1}{2}} + \frac{2^{s_i-1}}{\sqrt{2s_i-1}} \text{ for } i \geq 2,
\]
\[
C_2 = \varepsilon + \left( \frac{1}{2\tilde{E}} + \sup_{0 < |k| \leq J} \frac{|g_k| + |\tilde{g_k}(t)|}{|k|} \right)/\nu,
\]
\[
C_i = \varepsilon + \left( C_{i-1} \sqrt{\tilde{E}D_{i-1}} + \sup_{0 < |k| \leq J} \frac{|k|^{s_i-2} (|g_k| + |\tilde{g_k}(t)|)}{\nu} \right) \text{ for } i > 2(60)
\]

Then for all \(i \geq 2\), and \(\tilde{C}_i > \sqrt{\tilde{E}N^{s_i}}\)

\[
H \supseteq W_i(\tilde{E}, C_i, \varepsilon) := \left\{ \{a_k\}_{k \in \mathbb{Z}} \in R \mid \mathcal{E}(\{a_k\}_{k \in \mathbb{Z}}) \leq \tilde{E}, |a_k| \leq \frac{C_i}{|k|^s} \right\} \bigcap W(\tilde{E}, N, \tilde{C}_i, s_i),
\]
is an absorbing set for large Galerkin projections of \(\tilde{E}\) restricted to the invariant subspace given by \(a_k = \frac{1}{a-k}\).

In the next lemma we show that for any forcing there exists a value \(M > 0\) such that \((I - P_M)\) projection of the global attractor is within \(\mathcal{R}\).
Lemma 4.6. Let $f$ be a forcing such that $f_0 = 0$, $f_k = 0$ for $|k| > J$, and $f_k(t) = \tilde{f}_k(t)$, $|f_k'(t)| \leq D$ for $|k| \leq J$, $t \in \mathbb{R}$. Let $\mathcal{R}$ be the trapping region such that $\mu(D_2 F(\mathbb{R}, \mathcal{R})) < 0$ proved to exist in Lemma 4.3, and $s_i \geq 4$ is the decay rate of coefficients within $\mathcal{R}$, let $\mathcal{W}_{2(s_i+1)}$ be the absorbing set proved to exist in Lemma 4.5.

Then there exist $M > 0$ such that $(I - P_M)\mathcal{W}_{2(s_i+1)} \subset (I - P_M)\mathcal{R}$.

Proof. This is consequence of the fact that the decay rate of coefficients in $\mathcal{W}_{2(s_i+1)}$ is larger than in $\mathcal{R}$. Observe that the following condition

$$\frac{C_i}{k^{|s_i|}} > \frac{\hat{C}_{2(s_i+1)}}{|s_i+1|},$$

is satisfied for $|k| > \frac{\hat{C}_{2(s_i+1)}}{C_i}$,

where $\hat{C}_{2(s_i+1)} = \min\{C_{2(s_i+1)}, \tilde{C}_{2(s_i+1)}\}$, $C_{2(s_i+1)}, \tilde{C}_{2(s_i+1)}$ are the constants appearing in the definition of $\mathcal{W}_{2(s_i+1)}$ in Lemma 4.3. $C_i$ is the constant appearing in the definition of $\mathcal{R}$ in Lemma 4.3.

Therefore it is enough to take $M = \frac{\hat{C}_{2(s_i+1)}}{C_i}$.

Theorem 4.7. Let $f$ be a forcing such that $f_0 = 0$, $f_k = 0$ for $|k| > J$, and $f_k(t) = \tilde{f}_k(t)$, $|f_k'(t)| \leq D$ for $|k| \leq J$, $t \in \mathbb{R}$. Let $a_0$ be the constant appearing in the oscillatory terms in (53), i.e. $\exp(ik_0a_0)$.

There exists $a_0 > 0$ such that for all $a_0 > a_0$

- there exists $\mathfrak{a} : \mathbb{R} \to H'$, which is a solution of (53),
- $\mathfrak{a}(t) = \mathcal{O}(\frac{1}{|a_0|})$ for all $t \in \mathbb{R}$,
- $\mathfrak{a}$ attracts all orbits in $H'$.

Sketch of a proof. From Lemma 4.6 it follows that the $(I - P_M)$ part of $\mathcal{R}$ is forward invariant (because the absorbing set is forward invariant, and the whole absorbing set is contained in $\mathcal{R}$). This is an important point, because from now we will just worry about the behavior of lower modes.

Therefore, let us restrict our attention to the $P_M$ part of the considered sets.

Let $\mathcal{R}_0$ be a scaled $\mathcal{R}$, e.g. $\mathcal{R}_0 = s \cdot \mathcal{R}$, where $s \in (0, 1)$. In sequel $r_l$ is the 'energy radius' parameter of $\mathcal{R}$.

Let $\tilde{a}_0$ be sufficiently large. Now we prove that there exists $h_0$ such that all solutions of (53) and (54) with an initial condition $z(t_0) \in \mathcal{R}_0$ satisfy $z(t_0 + t) \in \mathcal{R}$ for $t \in [0, h_0]$ for any $|a_0| > \tilde{a}_0$ and $t_0$. This can be accomplished by adjusting first $h_0$ to the scaling factor $s$ so that (it is enough to do it for finite number of directions, because on the tail everything is fine)

$$h_0 2 \|f\| r_l = r_l^2 - (sr_l)^2.$$  (61)
For solutions of (54) satisfying $y(t_0) \in \mathcal{R}_0$ and $t \in [0, h_0]$ due to (56) condition we have
\[ \frac{d\|z\|}{dt} \leq -L\|z\|, \]
and hence
\[ \|y(t_0 + t)\| \leq \exp(-Lt)\|y(t_0)\|, \]
where $L = \min_{k \in \mathbb{N}} \lambda_k = \nu$.

If $y : [t_0, t_0 + h_0] \to \mathbb{R}^n$ is solution of (54) and $z : [t_0, t_0 + h_0] \to \mathbb{R}^n$ is a solution of (53), such that $z(t_0) = y(t_0) \in \mathcal{R}_0$, then from Lemma 2.1 and the fact that we have fast decaying tails we have for $t \in [0, h_0]$
\[ \|y(t_0 + t) - z(t_0 + t)\| \leq \frac{b(t)}{a_0}, \]
for some increasing and continuous function $b(t)$.

Combining the above inequalities we have for $z(t_0) \in \mathcal{R}_0$ and $t \in (0, h_0]$
\[ \|z(t_0 + t)\| \leq \|z(t_0 + t) - y(t_0 + t)\| + \|y(t_0 + t)\| \leq \frac{b(t)}{\omega} + \exp(-Lt\|z(t_0)\|). \]

Now we show that there exist a sufficiently large value of $\tilde{a}_0$, such that $P_M \mathcal{R}_0$ is mapped into itself by the time shift by $h_0$. Namely we need the following inequality
\[ \frac{b(h_0)}{\tilde{a}_0} + \exp(-Lh_0)s r_l < r_{\min}, \]
where $r_{\min} = \frac{sC_l|\mathcal{M}| s l}{M}.$

Clearly, such value of $\tilde{a}_0$ that (64) is satisfied exists, and $P_M \mathcal{R}_0$ is mapped into itself for all $a_0 \geq \tilde{a}_0$.

Between the time steps $h_0$ the orbit cannot go further than ($b$ is a nondecreasing function – see Lemma 2.1).
\[ \frac{b(h_0)}{a_0} \]

Now, if $\tilde{a}_0$ is so large that the orbit does not escape $\mathcal{R}$ between the time steps, the orbit stays within the negative logarithmic norm regime for all times. Therefore by showing that $\mathcal{R}_0$ is mapped into itself by the $h_0$ time shift from [CyZ, Theorem 6.15] we get the existence of an orbit $\mathcal{T} : \mathbb{R} \to H'$, which is a solution of (54). Moreover, because this orbit is contained in $\mathcal{R}$ for all times from [CyZ, Theorem 6.15] it follows that this orbit attracts all other orbits within $\mathcal{R}$.

Now, we show that $P_M \mathcal{W}_{2(s_l+1)}$ is mapped into $P_M \mathcal{R}$ analogically, possibly adjusting the $a_0$ value (increase its value to be sufficiently large).

Let time $\delta$ be such that for the (54) problem $P_M \mathcal{W}_{2(s_l+1)}$ is mapped into $P_M \mathcal{R}_0$. We know that $z$ – the solution of (53) after this time satisfies
\[ \|z(t_0 + \delta)\| \leq \frac{b(\delta)}{a_0} + \exp(-L\delta)\|z(t_0)\|. \]
If necessary we increase the $a_0$ value so that the solution of (53) after time $\delta$ is within $P_M R$.

Hence the full orbit which attracts all orbits exists and if $f$ is $T$-periodic, then in fact it must have period $T$ for (53).

\section{Navier-Stokes equation}

\subsection{The Navier-Stokes equation}

We will use the following notation. For $z \in \mathbb{C}$, by $\overline{z}$ we denote the conjugate of $z$. For any two vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ from $\mathbb{C}^n$ or $\mathbb{C}^\infty$ we set (if it makes sense)

\begin{align*}
(u|v) &= \sum_i u_i \overline{v_i}, \\
(u \cdot v) &= \sum_i u_i v_i.
\end{align*}

The general $d$-dimensional Navier-Stokes system (NSS) is written for $d$ unknown functions $u(t, x) = (u_1(t, x), \ldots, u_d(t, x))$ of $d$ variables $x = (x_1, \ldots, x_d)$ and time $t$, and the pressure $p(t, x)$.

\begin{align*}
\frac{\partial u_i}{\partial t} + \sum_{k=1}^d u_k \frac{\partial u_i}{\partial x_k} &= \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f^{(i)}(t, x) \quad (66) \\
\text{div } u &= \sum_{i=1}^d \frac{\partial u_i}{\partial x_i} = 0 \quad (67)
\end{align*}

The functions $f^{(i)}(t, x)$ are the components of the external forcing, $\nu > 0$ is the viscosity.

We consider (66),(67) on the torus $\mathbb{T}^d = (\mathbb{R}/2\pi)^d$ (i.e. we consider the periodic boundary conditions).

An easy computation shows that

\begin{equation}
\frac{d}{dt} \int_{\mathbb{T}^d} u(t, x) dx = \int_{\mathbb{T}^d} f(t, x) dx. \quad (68)
\end{equation}

The periodic boundary conditions enable us to use the Fourier series. We write

\begin{align*}
u(t, x) &= \sum_{k \in \mathbb{Z}^d} u_k(t) e^{i(k, x)}, \quad p(t, x) = \sum_{k \in \mathbb{Z}^d} p_k(t) e^{i(k, x)} \quad (69)
\end{align*}

Observe that $u_k(t) \in \mathbb{C}^d$, i.e. they are $d$-dimensional vectors and $p_k(t) \in \mathbb{C}$. We will always assume that

\begin{equation}
f_0 = 0. \quad (70)
\end{equation}
From (68) and (70) it follows that
\[ u_0(t) = u_0(0) = c. \]  (71)

Observe that (67) is reduced to the requirement \( u_k \perp k \). Namely
\[
\text{div } u = \sum_{k \in \mathbb{Z}^d} i(u_k(t), k)e^{i(k,x_x)} = 0
\]
\[
(u_k, k) = 0 \quad k \in \mathbb{Z}^d
\]

We obtain the following infinite ladder of differential equations for \( u_k \) (see for example [ZNS])
\[
\frac{du_k}{dt} = -i \sum_{k_1 \neq 0} (u_{k_1}|k)u_{k-k_1} - \nu k^2 u_k - i p_k k + f_k, \quad k \in \mathbb{Z}^d. \]  (72)

Here \( f_k \) are components of the external forcing. Let \( \cap_k \) denote the operator of orthogonal projection onto the \((d-1)\)-dimensional plane orthogonal to \( k \). Observe that since \( (u_k, k) = 0 \), we have \( \cap_k u_{k} = u_k \). We apply the projection \( \cap_k \) to (72). The term \( p_k k \) disappears and we obtain
\[
\frac{du_k}{dt} = -i \sum_{k_1} (u_{k_1}|k) \cap_k u_{k-k_1} - \nu k^2 u_k + \cap_k f_k \]  (73)

The pressure is given by the following formula
\[
- i \sum_{k_1} (u_{k_1}|k)(I - \cap_k)u_{k-k_1} - i p_k k + (I - \cap_k)f_k = 0 \]  (74)

Observe that solutions of (73) satisfy incompressibility condition \( (u_k, k) = 0 \).

The subspace of real functions which can be defined by \( u_k = u_k \) for all \( k \in \mathbb{Z}^d \) is invariant under (73). In the sequel, we will investigate the equation (73) restricted to this subspace.

Observe that since \( u_0(t) = u_0 \) the system (73) becomes
\[
\frac{du_k}{dt} = -i \sum_{k_1 \neq 0} (u_{k_1}|k)\cap_k u_{k-k_1} - (\nu k^2 + i(u_0|k))u_k + \cap_k f_k, \quad k \in \mathbb{Z}^d \setminus \{0\}. \]  (75)

5.2 Effect of the introduction of uniformly moving coordinate system

As in the case of the Burgers equation we consider NSS in the moving coordinate frame. Under the assumption of \( f_0 \) and in coordinate frame moving with the velocity \( a \), such that
\[
a = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} u(0, x)dx \]  (76)
the function
\[ v(t, x) = u(t, x + at) - a \]  
will satisfy NSS equation with the forcing term \( \hat{f}(t, x) = f(t, x + at) \) and
\[ \int_{\mathbb{T}^d} v(0, t) = 0. \]  

On the Fourier series level we obtain
\[ \hat{f}_k(t) = f_k(t) e^{i(k,a)t}. \]

We see that we have the same phenomenon as for the Burgers equation.

5.3 Our result on NSS

**Theorem 5.1.** Let \( J \subset \mathbb{Z}^d \), be symmetric and finite. Consider NSS with periodic boundary conditions and forcing term \( f(t, x) = \sum_{k \in J} f_k(t) \exp(ikx) \), such that (70 holds, with a constraint
\[ \int_{\mathbb{T}^d} u(t, x) \, dx = a. \]

Assume that there exist constants \( C(f_k) \) and \( C \left( \frac{df_k}{dt} \right) \) for \( k \in J \), such that
\[ |f_k(t)| \geq C(f_k), \quad \left| \frac{df_k}{dt} (t) \right| \leq C \left( \frac{df_k}{dt} \right), \quad t \in \mathbb{R}, k \in J, \]
\[ (a, k) \neq 0, \quad k \in J. \]

Then there exists a full orbit of size \( O(1/\min(|k \cdot a|)) \) (we fix \( f \) and \( \nu \)). If \( d = 2 \), then it attracts all orbits.

**Sketch of the proof:** In dimension two, the proof follows the same pattern as for the Burgers equation, but the role of the energy is taken by the enstrophy. The existence of the trapping regions and the estimate for the logarithmic norms are given in [ZNS, ZNS2].

In dimension three the existence of small trapping regions is shown in [ZNS], so we obtain a locally attracting orbit, only.

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