Planck-scale effect through a new MDR

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Abstract

In order to the expected Planck-scale correction in the physical systems we have put forwarded a novel modified dispersion relation (MDR). It has a generalized structure. A specific choice of the function used in the construction of this MDR, it has Lorentz invariance. A toy model like relativistic harmonic oscillator has been studied to get the necessary Planck-scale correction. It has been found that each laves of harmonic oscillator acquires Planck-scale correction and the result agrees with negligible deviation with the result obtained for this system for the same purpose using generalized uncertainty relation (GUP). The relativistic Hydrogen atom problem has also been studied with this MDR and it is found that like harmonic oscillator each energy level of the Hydrogen atom too has got Planck-scale correction.

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I. INTRODUCTION

The study of probable scenario of different physical systems in the vicinity of Planck-scale is of huge interest since the strength of the gravitational interaction in that scale becomes comparable to the strength of the electromagnetic interaction. So, in the vicinity of Planck-scale, it necessities to take into account the effect of (quantum) gravity. But straightforward quantization of gravity and its direct insertion into the physical systems is still lying at far reaching stage. Therefore, the indirect way of including the quantum gravity effect has been receiving much attention over the years through the use of novel ideas like generalized Heisenberg uncertainty principle and modified dispersion relation (MDR), as these two important ideas have been playing remarkable role towards providing necessary Planck-scale corrections into the physical systems in indirect manner [1–9, 13]. This issue in the context of black body radiation was fond to be addressed in the articles [15–18]. To make statistical mechanics compatible to incorporate quantum gravity effect, formal development of it along with few applications in the thermodynamical systems have been explored in the articles [19–23]. Some experimental result also have been found to be reported to explain with the use of this type of generalization (modification) [24, 25].

In the articles [15–20, 22, 23, 26–30] the concept of GUP have been exercised extensively to incorporate Planck-scale effect into different physical systems and the concept of MDR has been extended for the same purpose in the articles [31–37]. Although these two novel concepts (GUP and MDR) have been exercised in independent manner to serve essentially the same purpose in different physical systems, it is not difficult to understand conceptually that MDR and GUP are complementary to each other, however it is fair to admit that there is no established direct one to one correspondence between these two in general. It depends on the choice of generalization of the uncertainty relation or on the choice of modification of the dispersion relation. We should mention at this stage that an attempt to establish the conceptual connection between GUP and MDR is carried out recently in the article [39]. There is indeed a special instance where a precise MDR has been proposed for a specific GUP [41].

Although these two substantial as well as potentially effective ideas got strong initial support from string theory [2, 4, 42, 43, 45–47], these two ideas were also supported immensely by the other quantum gravity candidates like loop quantum gravity [8, 44, 45] and space-time non-commutativity [38–41]. In these two modifications (though in principle should considered to be merged into one) Lorentz symmetry was needed to be ignored and that necessarily led to open up a new idea namely deformed special relativity (DSR) [52]. So a natural question may arise whether this modification can be made maintaining Lorentz symmetry. It is true that there are some experimental signature which was explained inviting the Lorentz violation in an essential way [53], but it would be nicer indeed if it could be
explained maintaining the Lorentz symmetry since violation of Lorentz symmetry invites unwanted problems to a
great extent because this symmetry is deeply rooted both in field theory as well as in the general theory of relativity.
The articles [10,12] also shows a possibility of framing MDR in a Lorentz symmetric manner where deformation of d’
Alemberian has been exercised. In this article, we, therefore, introduce a generalized modified dispersion relation in
such a way that it can be used in Lorentz invariant manner as well as with out maintaining that invariance. In this
context, we should mention that in the articles [53,54], relativistic quantum mechanical systems without quantum
gravity correction have been studied and at the same time the important article [55], is of worth mentionable where
possibility of incorporating invariant quantum gravity effect at the vicinity of Planck-scale has been discussed in
detail.

So it would be beneficial to designed a generalized MDR in such a fashion so that it can meet both the purpose:
Lorentz invariance as well as Lorentz non-invariance extension of any physical system towards incorporating the
Planck-scale effect of any physical system. With the MDR proposed here we have studied the toy model like relativistic
one dimensional harmonic oscillator and a three dimensional physical system like relativistic Hydrogen atom to include
the Planck-scale correction into their energy spectrum. Recently, these two systems has been studied in [56,57] for
the same purpose with the use of GUP.

The article is organized in the following manner. In Sec. I, we introduce a new generalized dispersion relation. Sec.
II is devoted to get the Planck-scale correction of relativistic harmonic oscillator using raising and lowering operator.
We have shown in Sec III that the same correction can be obtained if we use differential form of momentum operator.
Sec. IV contains a discussion of obtaining Planck-scale correction of the relativistic Hydrogen atom directly from
the Schr"{o}dinger’s equation. In Sec. V, the correction is computed using the average value of different powers of
momentum. Sec. VI contains a brief discussion and conclusion.

II. FORMULATION OF NEW DISPERSION RELATION TO GET THE PLANCK-SCALE
CORRECTION

The generalized MDR which we are going to formulate in order to incorporate quantum gravity effect can be casted
in Lorentz invariant as well as Lorentz non-invariant manner. So if the extension with this MDR is carried in a Lorentz
symmetric manner one need not be worried about the search of precise DSR corresponding to this MDR. The explicit
expression of this generalized MDR is

\[ P^\mu P_\mu = m_0^2 + \sum_k f_k (p^2 - E^2)^k. \]  (1)

Here \( f_k = \frac{\alpha_k}{(m_P)^k} g\left(\frac{E}{p}\right) \). In general, any Lorentz non-covariant structure of the function \( g\left(\frac{E}{p}\right) \) breaks the Lorentz
symmetry. However any \( p \) independent or Lorentz covariant structure of \( g\left(\frac{E}{p}\right) \) preserves Lorentz invariance. So this
new generalized MDR can be used in any physical system to incorporate Planck-scale correction in significant manner.

We have chosen the simplest possible Lorentz covariant structure of the function \( g\left(\frac{E}{p}\right) \), as \( g\left(\frac{E}{p}\right) = 1 \), which makes
\( f_k = \frac{\alpha_k}{(m_P)^k} \). So the MDR with which we will start our investigation is

\[ P^\mu P_\mu = m_0^2 + \sum_k \frac{\alpha_k}{(m_P)^k} (P^\mu P_\mu)^k. \]  (2)

Here \( m_0 \) is the rest mass of the particle considered for study, \( k \) is an integer that runs from \( k = 2 \) to any desired
order \( N \), \( P = (E, \vec{p}) \) and \( p = |\vec{p}| \). The parameter \( \alpha_k \) represents arbitrary \( N-2 \) number of free parameters which can be
fixed comparing the result obtained using this MDR with the experimental result or with the results obtained earlier through other reliable concept like GUP. The Planck-mass is characterized by the symbol \( m_P \) which is given
by \( m_P = \sqrt{\frac{\hbar G}{c^3}} \). Note that, reparametrization invariance in the action of a system can be maintained with this MDR
having manifestly lorentz covariance, since under \( p \rightarrow -p \) this MDR remains unchanged. Of course, we can define
in terms of Planck-length \( l_P \) in the following way

\[ P^\mu P_\mu = m_0^2 + \sum_k \eta_k (l_P)^k (P^\mu P_\mu)^k, \]  (3)

where \( \eta_k \) represents \( N - 2 \) number of parameters certainly different from \( \alpha_k \) and the Planck-length \( l_P = \sqrt{\frac{\hbar}{Gc^3}} \).

In general, there are infinite number of parameters and numerical computation is possible with an arbitrary large
numbers of such parameters, although analytical calculation will not always be possible with a desired numbers of
terms. In practice however we need not keep all these parameter to get the desired accuracy.
The symbol $c$ stands for velocity of light in vacuum in both the cases. We will choose natural unit $\hbar = 1$ and $c = 1$. To carry out our analytical investigations on relativistic harmonic oscillator and Hydrogen atom with this generalized MDR we will kept ourselves limited to $k = 2$. In due course, we will find that with the terms available for the the choice $k = 2$, the analytical computation towards Planck-scale correction for the said systems is tractable and it has good agreement with the result of the system studied earlier using the concept of GUP to incorporate the Planck-scale effect [9]. The MDR with $k = 2$ reads

$$E^2 = p^2 + m_0^2 + \frac{\alpha_2}{(m_P)^2}(p^2 - E^2)^2. \tag{4}$$

A careful look reveals that this MDR resembles the deformation of the $d'$ Alembertian ($\square = \partial_\mu \partial^\mu$) as it has been found in [10][12]. To be more precise: to include the dynamics of a physical system for any energy scale the deformed $d'$ Alembertian would be some desired function of the usual $d'$ Alembertian ($f(\square)$) having some constants that has to be fixed with the experimental result. So to get Planck-scale effect it is to be considered that the dynamics below the Plank scale will be governed with the usual $d'$ Alembertian and the dynamics in the vicinity will be governed by the usual $d'$ Alembertian along with deformation part of the $d'$ Alembertian in a judicious manner to get necessary agreement with the experimental result (if or when available). Since both the has bears the Lorentz symmetry the frame independence of the physical result will be protected. Of course velocity of light will remain as an invariant quantity.

The hitherto available literatures related MDR show that the MDRs contain the expression of $E$ as a function of $P$. This new MDR is not an exception to that, but it is true that the nature of the function is little generalized. The modification which is made here is only the enforcement of a physical symmetry which is none other than the celebrated Lorentz invariance which was not there in the MDR designed earlier. We have kept this point of view that in the vicinity of the Planck-scale violation of Lorentz invariance may occur but it can not be a basic criteria. On the other hand it would be admitted from all corners that Lorentz covariant structure is advantageous over Lorentz non-covariance in many respect because the basic foundation of quantum field theory and general theory of relativity is deeply rooted to the Poincar’e symmetry. The article [14] is an excellent example in this direction Therefore, this new MDR will certainly add new light in the formal development of MDR related theories. The novelty of this MDR is the welcome entry of the Lorentz symmetry and its capability to render Planck-scale correction in a frame independent manner which is lacking in the construction of MDR in the available literature dealing with Planck-scale correction with MDR.

Taking the square root of the above expression we get

$$E = \sqrt{p^2 + m_0^2 + \frac{\alpha_2}{(m_P)^2}(p^2 - E^2)^2}. \tag{5}$$

Equation (5) on binomial expansion results

$$E = \frac{p^2}{2m_0} + m_0 + \frac{\alpha_2 E_n^4}{2m_0(m_P)^2} - p^2\left(\frac{\alpha_2 E_n^4}{4m_0^3(m_P)^2} + \frac{\alpha_2 E_n^2}{m_0(m_P)^2}\right)$$

$$+ p^4\left(\frac{\alpha_2}{2m_0(m_P)^2} + \frac{\alpha_2 E_n^2}{2m_0^3(m_P)^2} - \frac{1}{8(m_0)^3} + \frac{3\alpha_2 E_n^2}{16m_0^3(m_P)^2}\right)$$

$$+ p^6\left(-\frac{\alpha_2}{4m_0^3(m_P)^2} + \frac{1}{16(m_0)^5} - \frac{3\alpha_2 E_n^2}{8m_0^5(m_P)^2} - \frac{5\alpha_2 E_n^4}{32m_0^5(m_P)^2}\right). \tag{6}$$

In the above expansion we have neglected terms containing higher order in $\alpha_2$ and retained the terms up to sixth order in $p$. Thus our modified Hamiltonian with this settings reads

$$H = \frac{p^2}{2m_0} + V + m_0 + \frac{\alpha E_n^4}{2m_0(m_P)^2} - p^2\left(\frac{\alpha E_n^4}{4m_0^3(m_P)^2} + \frac{\alpha E_n^2}{m_0(m_P)^2}\right)$$

$$+ p^4\left(\frac{\alpha}{2m_0(m_P)^2} + \frac{\alpha E_n^2}{2m_0^3(m_P)^2} - \frac{1}{8(m_0)^3} + \frac{3\alpha E_n^4}{16m_0^3(m_P)^2}\right)$$

$$+ p^6\left(-\frac{\alpha}{4m_0^3(m_P)^2} + \frac{1}{16(m_0)^5} - \frac{3\alpha E_n^2}{8m_0^5(m_P)^2} - \frac{5\alpha E_n^4}{32m_0^5(m_P)^2}\right). \tag{7}$$

In the above expression we have replaced $\alpha_2$ by $\alpha$ since there is no other $\alpha$’s in the body of the article. Here $V$ stands for the potential energy inserted by hand. To avoid confusion we would like to mention that $[q, p] = i\hbar$ and for modified position and momentum pair $(q_m, p_m)$ the canonical Poisson’s bracket will take a modified form and that
It is known that the raising and lowering operators are respectively given by

\[ \hat{a} = \frac{\alpha E_n}{\sqrt{2m_0}(m_0 p^2)} \frac{\alpha E_n}{\sqrt{2m_0}(m_0 p^2)} < p^2 > \left( \frac{\alpha E_n}{4m_0^2(m_0 p^2)} + \frac{\alpha E_n^2}{3m_0^2(m_0 p^2)} \right) \]

\[ + < p^4 > \left( \frac{\alpha}{2m_0(m_0 p^2)} + \frac{\alpha E_n^2}{2m_0^3(m_0 p^2)} \right) - \frac{1}{8(m_0)^3} + \frac{3\alpha E_n^4}{16m_0^5(m_0 p^2)} \]

\[ + < p^6 > \left( \frac{\alpha}{4m_0^2(m_0 p^2)} + \frac{1}{16(m_0)^5} - \frac{3\alpha E_n^2}{8m_0^4(m_0 p^2)} - \frac{5\alpha E_n^4}{32m_0^6(m_0 p^2)} \right), \]

where \( E_n^{(m)} \) and \( E_n \) are modified and unmodified eigenvalues respectively. This generalized expression shows that we have to evaluate the expectation values up to sixth power of momentum. To be precise we need the expressions of \( < p^2 >, < p^4 >, \) and \( < p^6 > \) of the system that would considered under investigation.

**III. RELATIVISTIC HARMONIC OSCILLATOR WITH MDR USING RAISING AND LOWERING REPRESENTATION OF MOMENTUM OPERATOR**

Let us first proceed to calculate the energy eigenvalues using raising and lowering representation of the momentum. It is known that the raising and lowering operators are respectively given by

\[ a = \frac{1}{\sqrt{2m_0\hbar}} (m_0\omega x + ip), \]

\[ a^\dagger = \frac{1}{\sqrt{2m_0\hbar}} (m_0\omega x - ip). \]

For the sake of convenience we will not set \( \hbar = 1 \) but \( c = 1 \) will be maintained from the starting point. However the final expression will be presented with \( \hbar = 1 \). If \( a \) and \( a^\dagger \) are operated separately on the \( n^{th} \) eigen state \( |n> \) we get

\[ a|n> = \sqrt{n}|n - 1 >, \]

\[ a^\dagger|n> = \sqrt{n+1}|n+1 >. \]

The momentum operator in terms of raising and lowering operator can be expressed as

\[ p = i\sqrt{\frac{m_0\omega\hbar}{2}} (a^\dagger - a) \]

Consecutive operation of \( p \) on the \( n^{th} \) eigen state for two, four and six times results the following

\[ p^2|n> = -\frac{m_0\omega\hbar}{2} \left[ \sqrt{n+1}\sqrt{n+2}|n + 2> - (2n + 1)|n> + \sqrt{n}\sqrt{n-1}|n - 2> \right], \]

\[ p^4|n> = \left( \frac{m_0\omega\hbar}{2} \right)^2 \left[ \sqrt{n+1}\sqrt{n+2}\sqrt{n+3}\sqrt{n+4}|n + 4> \right. \]

\[ - \sqrt{n+1}\sqrt{n+2}(4n+6)|n + 2> + (6n^2 + 6n + 3)|n> - \sqrt{n}\sqrt{n-1}(4n-2)|n - 2> + \sqrt{n}\sqrt{n-1}\sqrt{n-2}\sqrt{n-3}|n - 4> \right], \]

\[ p^6|n> = \left( \frac{m_0\omega\hbar}{2} \right)^3 \left[ \sqrt{n+1}\sqrt{n+2}\sqrt{n+3}\sqrt{n+4}\sqrt{n+5}\sqrt{n+6}|n + 6> \right. \]

\[ - \sqrt{n+1}\sqrt{n+2}\sqrt{n+3}(6n+15)|n + 4> + \sqrt{n+1}\sqrt{n+2}(15n^2 + 45n + 45)|n + 2> - (20n^3 + 30n^2 + 40n + 15)|n> \]

\[ + \sqrt{n}\sqrt{n-1}(15n^2 - 15n + 15)|n - 2> - \sqrt{n}\sqrt{n-1}\sqrt{n-2}\sqrt{n-3}(6n - 9)|n - 4> + \sqrt{n}\sqrt{n-1}\sqrt{n-2}\sqrt{n-3}\sqrt{n-4}\sqrt{n-5}|n - 6> \].
If we now take the inner product of the above expression with $|n>$ we will get the required expectation values of $p^2$, $p^4$ and $p^6$. The precise expression of the expectation values are

$$<p^2> = \frac{m_0\omega\hbar}{2}(2n+1),$$  \hspace{1cm} (17) \\
$$<p^4> = \left(\frac{m_0\omega\hbar}{2}\right)^2(6n^2 + 6n + 3),$$  \hspace{1cm} (18) \\
$$<p^6> = \left(\frac{m_0\omega\hbar}{2}\right)^3(20n^3 + 30n^2 + 40n + 15),$$  \hspace{1cm} (19)

which leads us to get the modified eigenvalues with this Lorentz symmetric modified dispersion relation

$$E_n^{(mho)} = E_n + m_0 + \frac{\alpha E_n^4}{2m_0(m_P)^2} - \frac{m_0\omega\hbar}{2}(2n+1)(\frac{\alpha E_n^4}{4m_0(m_P)^2} + \frac{\alpha E_n^2}{m_0(m_P)^2})$$

$$+ \left(\frac{m_0\omega\hbar}{2}\right)^2(6n^2 + 6n + 3)(\frac{\alpha}{2m_0(m_P)^2} + \frac{\alpha E_n^2}{2m_0^3(m_P)^2} - \frac{1}{8(m_0)^3} + \frac{3\alpha E_n^4}{16m_0^5(m_P)^2})$$

$$+ \left(\frac{m_0\omega\hbar}{2}\right)^3 \times (20n^3 + 30n^2 + 40n + 15)$$

$$+ \frac{\alpha}{4m_0^3(m_P)^2} + \frac{1}{16(m_0)^2} - \frac{3\alpha E_n^2}{8m_0^5(m_P)^2} - \frac{5\alpha E_n^4}{32m_0^7(m_P)^2}).$$  \hspace{1cm} (20)

Equation (20) reveals that each levels of hydrogen atom acquires Planck-scale correction which was also found in the articles [56, 57] where concept of GUP was employed to incorporate quantum gravity effect. This result is amenable to compare with the correction obtained in [9], using the concept of GUP for incorporation of quantum gravity correction. We will now turn to calculate the modified energy eigenvalues using the differential form of the momentum to get it confirmed whether these two agrees with each other.

**A. Calculation using differential form of momentum operator**

We know that momentum operator in differential form with the coordinate representation can is written down as

$$p = -ih\frac{\partial}{\partial x}, \quad i = \sqrt{-1},$$  \hspace{1cm} (21)

and the $n^{th}$ state eigenfunction of a harmonic oscillator is known to be

$$\psi_n = \sqrt{\frac{a}{\sqrt{\pi}2^n n!}}H_n(ax)e^{-\frac{x^2}{2}}, \quad a = \sqrt{\frac{m_0\omega}{\hbar}},$$  \hspace{1cm} (22)

where $H_n(ax)$, represents the Hermite polynomial of order n. Bringing the momentum operator in repeated action for two, four and six times and multiplying the obtained result by $\psi_n$ and hence integrating within the limit $-\infty$ to $\infty$ we get the expectation values of $p^2$, $p^4$ and $p^6$:

$$<p^2> = -\hbar^2 \frac{a}{\sqrt{\pi}2^n n!}\int_{-\infty}^{\infty} (a^4x^2 - a^2)e^{-a^2x^2}H_n(ax)H_n(ax)dx$$

$$- 4na^3 \int_{-\infty}^{\infty} xe^{a^2x^2}H_{n-1}(ax)H_n(ax)dx$$

$$+ 4n(n-1)a^2 \int_{-\infty}^{\infty} e^{a^2x^2}H_{n-2}(ax)H_n(ax)dx$$

$$= \frac{m_0\omega\hbar}{2}(2n+1),$$  \hspace{1cm} (24)
\[
< p^4 > = \frac{\alpha}{\sqrt{2\pi n!}} (16n(n-1)(n-2)(n-3)a^4 \int_{-\infty}^{\infty} e^{a^2 x^2} H_{n-4}(ax)H_n(ax)dx
- 32n(n-1)(n-2)a^5 \int_{-\infty}^{\infty} xe^{a^2 x^2} H_{n-3}(ax)H_n(ax)dx
+ 24n(n-1) \int_{-\infty}^{\infty} (a^6 x^2 - a^4) e^{a^2 x^2} H_{n-2}(ax)H_n(ax)dx
+ \int_{-\infty}^{\infty} (24an^5 x - 8an^7 x^3) e^{a^2 x^2} H_{n-1}(ax)H_n(ax)dx
+ \int_{-\infty}^{\infty} (3a^4 - 6a^6 x^2 + a^8 x^4) e^{a^2 x^2} H_n(ax)H_n(ax)dx] (25)
= \left( \frac{m_0 \omega \hbar}{2} \right)^2 (6n^2 + 6n + 3), (26)
\]

\[
< p^6 >= - \frac{\alpha}{\sqrt{2\pi n!}} (64n(n-1)(n-2)(n-3)(n-4)(n-5)a^6 \int_{-\infty}^{\infty} e^{a^2 x^2} H_{n-6}(ax)H_n(ax)dx
- 192n(n-1)(n-2)(n-3)(n-4)a^7 \int_{-\infty}^{\infty} xe^{a^2 x^2} H_{n-5}(ax)H_n(ax)dx
+ 240n(n-1)(n-2)(n-3) \int_{-\infty}^{\infty} (a^8 x^2 - a^6) e^{a^2 x^2} H_{n-4}(ax)H_n(ax)dx
+ n(n-1)(n-2) \int_{-\infty}^{\infty} (480a^7 x - 160a^9 x^3) e^{a^2 x^2} H_{n-3}(ax)H_n(ax)dx
+ n(n-1) \int_{-\infty}^{\infty} (180a^5 - 360a^8 x^2 + 60a^4 0x^4) e^{a^2 x^2} H_{n-2}(ax)H_n(ax)dx
+ n \int_{-\infty}^{\infty} (-180a^7 x + 120a^9 x^3 - 12a^1 1x^5) e^{a^2 x^2} H_{n-1}(ax)H_n(ax)dx
+ \int_{-\infty}^{\infty} (45a^8 x^2 - 15a^6 - 15a^3 0x^4 + a^1 2x^6) e^{a^2 x^2} H_n(ax)H_n(ax)dx (27)
= \left( \frac{m_0 \omega \hbar}{2} \right)^3 (20n^3 + 30n^2 + 40n + 15). (28)
\]

In the above computation we have used the following recurrence relations.
\[ H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x), \frac{dH_n(x)}{dx} = 2n H_{n-1}(x). (29) \]

The the standard integrals
\[
\int_{-\infty}^{\infty} e^{a^2 x^2} H_n(ax)H_n(x)dx = 2^n \sqrt{\pi n!} \delta_{mn}, (30)
\]
\[
\int_{-\infty}^{\infty} xe^{a^2 x^2} H_n(ax)H_n(x)dx = 2^{n-1} \sqrt{\pi n!} \delta_{m,n-1} + 2^n \sqrt{\pi} (n+1) \delta_{m,n+1}, (31)
\]
\[
\int_{-\infty}^{\infty} x^2 e^{a^2 x^2} H_n(ax)H_n(x)dx = 2^{n-2} \sqrt{\pi n!} \delta_{m,n-2} + (2n+1)2^{n-1} \sqrt{\pi} (n+1) \delta_{m,n+2}, (32)
\]
are also needed for the required computation. Using the above recurrence relations and the standard integrals the final expression of the modified eigenvalues come out to be
\[
E_n^{(mho)} = E_n + m_0 + \frac{\alpha E_n^4}{2m_0(m_P)^2} - \frac{m_0 \omega \hbar}{2} (2n+1)(\frac{\alpha E_n^4}{4m_0(m_P)^2} + \frac{\alpha E_n^2}{m_0(m_P)^2})
+ \frac{(m_0 \omega \hbar)^2 (6n^2 + 6n + 3)}{2m_0(m_P)^2} + \frac{\alpha E_n^2}{2m_0(m_P)^2} - \frac{1}{8(m_0)^3} + \frac{3\alpha E_n^4}{16m_0^3(m_P)^2})
+ \frac{(m_0 \omega \hbar)^3 (20n^3 + 30n^2 + 40n + 15)}{32m_0^4(m_P)^2}.
(33)
\]
Note that identical expression for eigenvalue $E^{(m)}_n$ has come out for both the cases: when momentum is expressed in terms of raising and lowering operator as well as when the differential form of the momentum operator is used for computation. This is of course, the expected scenario. The expression obtained in (20) and (33) do not have Lorentz symmetric structure although the the Lorentz covariant MDR has been used for computation of eigenvalues, because we have not considered the full relativistic theory. For full relativistic Hamiltonian the result ought to be Lorentz symmetric if at all the exact analytical extension is feasible in this situation.

Fig.1 shows a plot of the energy eigenvalues versus $n$ for $\alpha = 1.6 \times 10^{-39}$. Note that energy eigenvalues is plotted subtracting the rest mass energy from it. The spectrum agrees with spectrum obtained in the article [9] which was evaluated using a quadratic GUP. In our case modification is resulted with the introduction of a generalized MDR. It ravels once again that GUP and MDR essentially serve the same purpose which is indeed the obtaining the quantum gravity correction at the vicinity of Planck-scale. The agreement of our result with the result obtained in the article [9] is natural since the MDR and GUP basically serves the same purpose although it is fair to admit that there is no one to one correspondence between this generalized MDR used here and the MDR used in [9].

\[ \frac{p^2}{2m_0} \psi_{nlm} - \frac{e^2}{4\pi \varepsilon_0 r} \psi_{nlm} = E_n \psi_{nlm}, \]  
(34)

where

\[ \psi_{nlm} = R_{nl} Y_l^m(\theta, \phi) = u_{nl} r^l Y_l^m(\theta, \phi), \]  
(35)

IV. MODIFIED EIGENVALUES OF HYDROGEN ATOM USING SCHRODINGER'S EQUATION

Let us now proceed to obtain quantum gravity correction to the spectrum of Hydrogen atom. The Schrödinger’s equation for Hydrogen atom reads

\[ \frac{p^2}{2m_0} \psi_{nlm} - \frac{e^2}{4\pi \varepsilon_0 r} \psi_{nlm} = E_n \psi_{nlm}, \]  
(34)
This gives

\[ p^2 \psi_{nlm} = 2m_0[E_n + \frac{e^2}{4\pi\varepsilon_0 r}] \psi_{nlm}, \]  

and hence the expectation value of \( p^2 \) is given by

\[ < p^2 > = 2m_0[E_n + \frac{e^2}{4\pi\varepsilon_0}] < \frac{1}{r} >. \]  

If \( p^4 \) and \( p^6 \) on \( \psi_{nlm} \) get operated on (35) we ultimately have

\[ p^4(\psi_{nlm}) = 4m_0^2E_n[\psi_{nlm} + \frac{m_0e^2}{2\pi\varepsilon_0} \frac{2h^2 Y_l^m}{r^2} dR_{nl} + \frac{2m_0}{r} (E_n + \frac{e^2}{4\pi\varepsilon_0 r}) \psi_{nlm}], \]

\[ p^6(\psi_{nlm}) = 8m_0^3E_n^2[\psi_{nlm} + \frac{2m_0^2e^2E_n}{\pi\varepsilon_0} \frac{2h^2 Y_l^m}{r^2} dR_{nl} + (\frac{2m_0E_n}{r} + \frac{m_0e^2}{2\pi\varepsilon_0 r^2}) \psi_{nlm}] \]

\[ + \frac{m_0^2e^4}{4\pi^2\varepsilon_0} (m_0e^2 \frac{2m_0E_n}{r^2} \psi_{nlm} + \frac{4h^2 Y_l^m}{r^4} du_{nl} - \frac{6h^2}{r^4} \psi_{nlm}) \]

\[ + \frac{m_0^2e^2h^2}{\pi\varepsilon_0} \frac{l(l+1)h^2 Y_l^m}{r^4} dR_{nl} + \frac{12h^2}{r^5} \psi_{nlm} + \frac{5m_0e^2}{2\pi\varepsilon_0 r^3} - \frac{10m_0E_n}{r^3} \psi_{nlm} \]

These lead us to obtain the expectation value of \( p^4 \):

\[ < p^4 > = 4m_0^2E_n^2 + \frac{2m_0^2e^2E_n}{\pi\varepsilon_0} < \frac{1}{r} > + \frac{m_0^2e^4}{4\pi^2\varepsilon_0} < \frac{1}{r^2} >. \]

To get the desired result we have used the stand integral

\[ \int_0^\infty R_{nl} \frac{dR_{nl}}{dr} dr = 0. \]

The expectation value of \( p^6 \) is now found out in a straightforward manner:

\[ < p^6 > = 8m_0^3E_n^2 + \frac{6m_0^3e^2E_n^2}{\pi\varepsilon_0} < \frac{1}{r} > + \frac{3m_0^3e^4E_n}{2\pi^2\varepsilon_0} < \frac{1}{r^2} > + \left[ \frac{m_0^3e^6}{8\pi^3\varepsilon_0^3} - \frac{8m_0^2E_n e^2h^2}{\pi\varepsilon_0} \right] < \frac{1}{r^3} > \]

\[ - \frac{9m_0^2e^4h^2}{4\pi^2\varepsilon_0^3} < \frac{1}{r^4} > + \frac{m_0^2e^2h^2}{\pi\varepsilon_0} [6l(l+1)h^2 - 12h^2] < \frac{1}{r^5} >. \]

To this end, Feynmann-Hellman theorem helps a lot to get the expressions of \( < \frac{1}{r} > \) and \( < \frac{1}{r^2} > \): Ultimately, we see that

\[ < \frac{1}{r} > = \frac{1}{n^2a_0}, \]

\[ < \frac{1}{r^2} > = \frac{2}{n^3a_0^2(l+1)}, \]

where \( a_0 \) is the Bohr radius. To get the exact expression of \( < \frac{1}{r} >, < \frac{1}{r^2} > < \frac{1}{r^3} > \) Kramers’ relation [58] also has been employed here which is given by

\[ \frac{s+1}{n^2} < r^s > - (2s+1)a_0 < r^{s-1} > + \frac{s}{4} [(2l+1)^2 - s^2]a_0^2 < r^{s-2} >= 0. \]

If \( s = -1 \) is set in the Kramers’ relation it leads to obtain the expression of \( < \frac{1}{r^3} >:

\[ < \frac{1}{r^3} > = \frac{2}{a_0^3n^4(l+1)(2l+1)}. \]
Thus the modified eigenvalues for Hydrogen atom finally come out as

\[
< \frac{1}{r^4} > = \frac{12}{a_0^6n^3l(l+1)(2l+3)(4l^2-1)} - \frac{4}{a_0^6n^5(2l+3)(4l^2-1)}.
\]

(47)

Finally, putting \( s = -3 \) we get the expression of \( < \frac{1}{r^6} > \):

\[
< \frac{1}{r^6} > = \frac{20}{a_0^6n^3l(l^2-1)(l+2)(2l+3)(4l^2-1)} - \frac{3a_0^5n^5(l-1)(l+2)(2l+3)(4l^2-1)}{20} - \frac{4}{a_0^6n^5l(l^2-1)(l+2)(2l+1)}.
\]

(48)

Substituting these in the expression for \( < p^2 >, < p^4 > \) and \( < p^6 > \) we land on to the following.

\[
< p^2 > = 2m_0[E_n + \frac{\hbar^2}{m_0a_0^2}],
\]

(49)

\[
< p^4 > = \frac{8\hbar^4}{a_0^6n^3(2l+1)} + 4m_0^2E_n^2 + \frac{8m_0E_n\hbar^2}{n^2a_0^2},
\]

(50)

\[
< p^6 > = 8m_0^3E_n^3 + \frac{6m_0^3\epsilon^2E_n^2}{\pi\varepsilon_0} - \frac{8m_0^2E_n\epsilon^2\hbar^2}{\pi\varepsilon_0} \left[ \frac{a_0^6n^3l(l+1)(2l+3)(4l^2-1)}{a_0^6n^3l(l^2-1)(l+2)(2l+1)} \right]^2
\]

\[
+ \frac{3m_0^3\epsilon^4E_n}{2\pi\varepsilon_0^2} - \frac{8m_0^2E_n\epsilon^4\hbar^2}{\pi\varepsilon_0^2} \left[ \frac{a_0^6n^3l(l+1)(2l+3)(4l^2-1)}{a_0^6n^3l(l^2-1)(l+2)(2l+1)} \right]^2
\]

\[
+ \frac{9m_0^2\epsilon^6E_n^2}{4\pi\varepsilon_0^4} - \frac{8m_0^2E_n\epsilon^6\hbar^2}{\pi\varepsilon_0^4} \left[ \frac{a_0^6n^3l(l+1)(2l+3)(4l^2-1)}{a_0^6n^3l(l^2-1)(l+2)(2l+1)} \right]^2
\]

\[
+ \frac{m_0^4E_n^4}{8\pi\varepsilon_0^6} - \frac{8m_0^2E_n\epsilon^4\hbar^2}{\pi\varepsilon_0^6} \left[ \frac{a_0^6n^3l(l+1)(2l+3)(4l^2-1)}{a_0^6n^3l(l^2-1)(l+2)(2l+1)} \right]^2
\]

\[
- \frac{9m_0^2\epsilon^6\hbar^2}{4\pi\varepsilon_0^4} \left[ \frac{a_0^6n^3l(l+1)(2l+3)(4l^2-1)}{a_0^6n^3l(l^2-1)(l+2)(2l+1)} \right]^2
\]

\[
+ \frac{m_0^4E_n^4}{8\pi\varepsilon_0^6} - \frac{8m_0^2E_n\epsilon^4\hbar^2}{\pi\varepsilon_0^6} \left[ \frac{a_0^6n^3l(l+1)(2l+3)(4l^2-1)}{a_0^6n^3l(l^2-1)(l+2)(2l+1)} \right]^2
\]

Thus the modified eigenvalues for Hydrogen atom finally come out as

\[
E_n^{(aha)} = E_n + m_0 + \alpha E_n^4 \left[ \frac{1}{2m_0(m_p)^2} - \frac{2m_0[E_n + \frac{\hbar^2}{m_0a_0^2n^2}]^2}{8m_0E_n\hbar^2} \right]
\]

\[
+ \left[ \frac{8\hbar^4}{a_0^6n^3(2l+1)} + 4m_0^2E_n^2 + \frac{8m_0E_n\hbar^2}{n^2a_0^2} \right] \left[ \frac{a_0^6n^3l(l+1)(2l+3)(4l^2-1)}{a_0^6n^3l(l^2-1)(l+2)(2l+1)} \right]^2
\]

\[
+ \frac{3m_0^3\epsilon^4E_n}{2\pi\varepsilon_0^2} - \frac{8m_0^2E_n\epsilon^4\hbar^2}{\pi\varepsilon_0^2} \left[ \frac{a_0^6n^3l(l+1)(2l+3)(4l^2-1)}{a_0^6n^3l(l^2-1)(l+2)(2l+1)} \right]^2
\]

\[
+ \frac{9m_0^2\epsilon^6E_n^2}{4\pi\varepsilon_0^4} - \frac{8m_0^2E_n\epsilon^6\hbar^2}{\pi\varepsilon_0^4} \left[ \frac{a_0^6n^3l(l+1)(2l+3)(4l^2-1)}{a_0^6n^3l(l^2-1)(l+2)(2l+1)} \right]^2
\]

\[
+ \frac{m_0^4E_n^4}{8\pi\varepsilon_0^6} - \frac{8m_0^2E_n\epsilon^4\hbar^2}{\pi\varepsilon_0^6} \left[ \frac{a_0^6n^3l(l+1)(2l+3)(4l^2-1)}{a_0^6n^3l(l^2-1)(l+2)(2l+1)} \right]^2
\]

\[
- \frac{9m_0^2\epsilon^6\hbar^2}{4\pi\varepsilon_0^4} \left[ \frac{a_0^6n^3l(l+1)(2l+3)(4l^2-1)}{a_0^6n^3l(l^2-1)(l+2)(2l+1)} \right]^2
\]

\[
+ \frac{m_0^4E_n^4}{8\pi\varepsilon_0^6} - \frac{8m_0^2E_n\epsilon^4\hbar^2}{\pi\varepsilon_0^6} \left[ \frac{a_0^6n^3l(l+1)(2l+3)(4l^2-1)}{a_0^6n^3l(l^2-1)(l+2)(2l+1)} \right]^2
\]

\[
(\frac{-\alpha}{4m_0^3E_n^4} + \frac{1}{16(m_0)^2} - \frac{3\alpha E_n^2}{8m_0E_n(m_p)^2} - \frac{5\alpha E_n^4}{32m_0E_n(m_p)^2})^2.
\]

(52)
Like the harmonic oscillator this expression also does not have Lorentz invariance. The reason indeed is the same as we have maintained in earlier when we got the modified eigenvalues of the harmonic oscillator: the full relativistic theory in this case too is not possible to consider for obtaining analytical computation. What follows next is the computation of eigenvalues Hydrogen atom calculating the average values of different power of momentum as required.

A. Spectrum through computation of average values of different powers of momentum

The differential form of \( p^2 \) in polar coordinate is written down as

\[
p^2 = -\frac{\hbar^2}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) - \frac{\hbar^2}{r^2} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}.
\]

This can be rewritten as

\[
p^2 = p_r^2 + \frac{L^2}{r^2},
\]

where

\[
p_r^2 = -\frac{\hbar^2}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial}{\partial r})
\]

and

\[
L^2 = -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right].
\]

The eigenfunction for Hydrogen atom is given by

\[
\psi_{nlm} = \frac{u_{nl}(r)}{r} Y_{l}^{m}(\theta, \varphi),
\]

where \( Y_{l}^{m}(\theta, \varphi) \) represent the spherical harmonics. If we operate \( p^2 \) on \( \psi_{nlm} \) once, twice and thrice we get

\[
p^2(\psi_{nlm}) = -\frac{\hbar^2}{r} \frac{d^2 u_{nl}}{dr^2} + \frac{l(l+1)\hbar^2}{r^2} \psi_{nlm},
\]

\[
p^4(\psi_{nlm}) = -\frac{m_0 c^2 \hbar^2}{\pi \varepsilon_0 r^4} Y_l^m u_{nl} + \frac{m_0 c^2 \hbar^2}{\pi \varepsilon_0 r^3} Y_l^m \frac{du_{nl}}{dr},
\]

\[+ \frac{4m_0 \left( \frac{c^2}{\varepsilon_0} \right)^2}{r^3} Y_l^m u_{nl} + \frac{4m_0^2 E^2}{r} Y_l^m u_{nl},
\]

\[
+ \frac{2m_0^2 c^2}{\pi \varepsilon_0 r^2} Y_l^m u_{nl}
\]

\[
p^6(\psi_{nlm}) = 8m_0^3 E_n^2 \left[ E_n + \frac{c^2}{4 \pi \varepsilon_0 r} \right] \psi_{nlm}
\]

\[+ \frac{2m_0^2 c^2}{\pi \varepsilon_0} \frac{2h^2}{r^2} Y_l^m \frac{dR_{nl}}{dr} + \frac{2m_0}{r} \left( E_n + \frac{c^2}{4 \pi \varepsilon_0 r} \right) \psi_{nlm}
\]

\[+ \frac{m_0^2 c^4}{4 \pi \varepsilon_0} \frac{l(l+1)\hbar^2}{r^4} \psi_{nlm} - \frac{2m_0}{r^2} \left( \frac{l(l+1)\hbar^2}{2m_0 r^2} - \frac{c^2}{4 \pi \varepsilon_0 r} - E_n \right) \psi_{nlm}
\]

\[+ \frac{4h^2}{r^6} Y_l^m \frac{du_{nl}}{dr} - \frac{6h^2}{r^4} \psi_{nlm} + \frac{m_0 c^2 \hbar^2}{\pi \varepsilon_0} \frac{l(l+1)\hbar^2}{r^4} Y_l^m \frac{dR_{nl}}{dr}
\]

\[+ \frac{12h^2}{r^5} \psi_{nlm} + \frac{10m_0}{r^3} \left( \frac{l(l+1)\hbar^2}{2m_0 r^2} - \frac{c^2}{4 \pi \varepsilon_0 r} - E_n \right) \psi_{nlm}
\]

\[- \frac{\hbar^2}{r^3} Y_l^m \left( \frac{d^3 u_{nl}}{dr^3} + 12 \frac{du_{nl}}{r^3} \right) \psi_{nlm}.
\]

If we multiply the results by \( \psi_{nlm} \) and then integrate within the limit 0 to \( \infty \), 0 to \( \pi \) and 0 to \( 2\pi \) for \( r, \theta \) and \( \phi \) respectively we get the expectation value of \( p^2, p^4 \) and \( p^6 \) as

\[
\langle p^2 \rangle = 2m_0 \left[ E_n + \frac{c^2}{4 \pi \varepsilon_0 r} \right] \langle \frac{1}{r} \rangle
\]

\[
= 2m_0 \left[ E_n + \frac{\hbar^2}{m_0 a_0^2 n^2} \right],
\]

(61)
\[
\langle p^4 \rangle = 4m_0^2E_n^2 + \frac{2m_0^2\epsilon^2 E_n}{\pi \varepsilon_0} - \frac{1}{\rho^2} < \frac{m_0^2 \epsilon^4}{4\pi^2 \varepsilon_0^2} < \frac{1}{\rho^2} > \\
= \frac{8\hbar^4}{a_0^4 n^3(2l+1)} + 8m_0^2E_n^2 + \frac{8m_0 E_n \hbar^2}{n^2a_0^2}.
\]

(62)

\[
\langle p^6 \rangle = 8m_0^3E_n^3 + \frac{6m_0^3 \epsilon^2 E_n^2}{\pi \varepsilon_0} - \frac{1}{\rho^2} + \frac{3m_0^3 \epsilon^4 E_n}{2\pi^2 \varepsilon_0} - \frac{1}{\rho^2} + \frac{m_0^3 \epsilon^6}{8\pi^3 \varepsilon_0^3} - \frac{8m_0^2 E_n \hbar^2}{\pi \varepsilon_0} < \frac{1}{\rho^3} > \\
- \frac{9m_0^3 \epsilon^2 \hbar^2}{4\pi^2 \varepsilon_0^2} < \frac{1}{\rho^2} > + \frac{m_0^6 \epsilon^2 \hbar^2}{\pi \varepsilon_0} \left[ 6l(l+1)\hbar^2 - 12\hbar^2 \right] < \frac{1}{\rho^3} > \\
= 8m_0^3E_n^3 + \frac{6m_0^3 \epsilon^2 E_n^2}{\pi \varepsilon_0} - \frac{1}{\rho^2} + \frac{3m_0^3 \epsilon^6 E_n}{8\pi^3 \varepsilon_0^3} - \frac{8m_0^2 E_n \hbar^2}{\pi \varepsilon_0} < \frac{1}{\rho^3} > \\
- \frac{9m_0^3 \epsilon^2 \hbar^2}{4\pi^2 \varepsilon_0^2} < \frac{1}{\rho^2} > + \frac{m_0^6 \epsilon^2 \hbar^2}{\pi \varepsilon_0} \left[ 6l(l+1)\hbar^2 - 12\hbar^2 \right] < \frac{1}{\rho^3} > \\
+ \frac{20}{3a_0^4 n^3(l-1)(l+2)(2l+3)(4l^2-1)} - \frac{4}{3a_0^5 n^5(l^2-1)(l+2)(2l+3)(4l^2-1)}.
\]

(63)

Using he expectation value of \( p^2, p^4 \) and \( p^6 \) we finally land on to the required expression of the modified eigenvalues for Hydrogen atom:

\[
E_n^{(mha)} = E_n + m_o + \frac{\alpha E_n^4}{2m_0(m_p)^2} - 2m_0[E_n + \frac{\hbar^2}{m_0a_0^2}](\frac{\alpha E_n^4}{4m_0^2(m_p)^2} + \frac{\alpha E_n^2}{m_0^2(m_p)^2}) \\
+ \frac{8\hbar^4}{a_0^4 n^3(2l+1)} + 4m_0^2 E_n^2 + \frac{8m_0 E_n \hbar^2}{n^2a_0^2} \\
- \frac{9m_0^3 \epsilon^2 \hbar^2}{4\pi^2 \varepsilon_0^2} + \frac{m_0^6 \epsilon^2 \hbar^2}{\pi \varepsilon_0} \left[ 6l(l+1)\hbar^2 - 12\hbar^2 \right] \left[ \frac{20}{a_0^4 n^3(l^2-1)(l+2)(2l+3)(4l^2-1)} - \frac{4}{3a_0^5 n^5(l^2-1)(l+2)(2l+3)(4l^2-1)} \right].
\]

(64)

The expression of modified eigenvalues obtained in [64] is identical to the expression as we have already obtained in [52]. Thus the result that has been obtained using Schrodinger equation agrees with the result obtained using differential form of operator \( p^2 \). Like the harmonic oscillator each level of hydrogen atom too has acquired Planck-scale correction due to the strong the gravity background at that energy scale. In the article [54], too we fond that all the levels of Hydrogen atom got Planck-scale correction. Albeit the expression of corrected eigenvalues are different in these two cases it as expected to have agreement of the correction numerically with suitable settings of the parameters. It is true that the correction due to this generalized MDR is very small and it may be the case that it is beyond the scope of experimental verification even with the hitherto available advanced instrumental facilities. But from the theoretician point of view this correction cannot be ignored at the present state of time when interest towards
obtaining Planck-scale correction has been getting intensified. It is worth mentionable that the expression obtained in
\[ (52) \] and \[ (64) \] are not Lorentz symmetric because we have not considered the full relativistic theory since analytical
extension with full relativistic structure is not tractable with this framework.

V. DISCUSSION AND CONCLUSION

We have put forwarded a novel generalized modified dispersion relation which is expected to be equally useful in
order to incorporate the Planck-scale correction in any physical system. The interesting aspect of this novel MDR
is that one can use it maintaining the Lorentz covariance as well as without the maintenance of it. In fact, it is a
specific choice of the function \( g(E) \) which keeps the Lorentz symmetry intact, else it violates Lorentz symmetry.

When the function \( g(E) \) be Lorentz symmetric it can be considered as physics of modified \( d' \) Alambertian \( f(\square) \) as
introduced in [10]. Here modified part will render the Plank-scale effect within the system. For all energy scale this
MDR will be applicable: below Plank-scale the usual part will be useful to describe the dynamics, but in the vicinity
of the Plank-scale the modified dispersion will serve the purpose. A question of positive definite ness may arise, but it
will not create any problem here since the original contribution is dominating so far construction is concerned.

We have studied the toy model like relativistic harmonic oscillator with this MDR with a specific Lorentz symmetric
structure. To get first order correction one needs to compute the average values up to sixth power of momentum.

The spectrum is determined both by the use of raising and lowering operator and by the direct computation using the
coordinate representation of the momentum operator. The results comes out in agreement with each other that must
be the case indeed. This Planck-scale corrected spectrum of the harmonic oscillator would be useful in the study of
oscillating modes in the early stage of evolution of the Universe when gravity was expedited to be very strong. It
has already been mentioned that the modified spectrum of harmonic oscillator is in good agreement with spectrum
obtained in the article [9]. So our result also gives a message that MDR and GUP essentially serve the same purpose
towards having the information of the physics at the vicinity of Planck-scale in indirect manner.

The relativistic Hydrogen atom problem has also been studied with this MDR having Lorentz covariant structure.

Our investigation with this MDR reveals that like the harmonic oscillator each levels of Hydrogen atom too gets the
Planck-scale correction. Our endeavor suggests that the Hydrogen atom spectrum certainly had had the background
effect (Quantum Gravity) which was expected to play prominent role at the vicinity of Planck-scale at the time of
initial formation of it when the Universe was in infant stage. It may help us to have the information of the process
evolution of the Universe through the spectrum of the Hydrogen atom because at the time when Hydrogen atom
was formed initially it certainly encountered prominent quantum gravity effect. So our result may shade light on the
evolution history of the Universe from the study of Hydrogen or Hydrogen like atom.

This novel MDR may be useful to study the Planck-scale effect in any physical system. The gravity background
where ever was prominent (in the vicinity of Planck-scale), i.e when it play its role with the strength comparable
to electromagnetic background this novel MDR is expected to render its important service to provide the correction
required there due to the presence of the strong gravity background.

The Planck-scale corrected modified eigenvalues for both the harmonic oscillator and Hydrogen atom would have
Lorentz covariant shape for the specific MDR used here for computation of eigenvalues. However, it is not the case,
since the full relativistic Hamiltonian is not considered here. It is fair to admit that complete analytical solution for
eigenvalues with full relativistic Hamiltonian for the harmonic oscillator or Hydrogen atom with this MDR is not
tractable too. However if it would be possible to proceed with the full complicated structure that would certainly be
much involved and it would exhibit Lorentz symmetric expression.

To tell about the future prospect of this novel MDR we would like to add that the generalized uncertainty relation
depending to this MDR is indeed a matter of further investigation. It may be instructive to obtain Planck-scale
correction with the use of this MDR where ever that correction is needed. The extension of black-hole physics with
this MDR would be of interest.

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