Classification of viscoelastic models with integer and fractional order derivatives

A I Krusser and M V Shitikova
Research Centre on Dynamics of Solids and Structures, Voronezh State Technical University, Voronezh 304006, Russia
an.krusser@yandex.ru

Abstract. In the world literature there exists a wide variety of papers devoted to linear viscoelastic models. This work was initiated by the absence of a single generally accepted classification of viscoelastic models. We focused on the basic mechanical models, namely, the Kelvin-Voigt, Maxwell, standard linear solid and Jeffreys models. All other models are different combinations of basic elements connected in series or in parallel. The classification also includes viscoelastic models with fractional derivatives and fractional operators.

1. Introduction
Recently, polymeric materials, possessing viscoelastic properties, have become widespread. In order to visualize the work of various viscoelastic materials, it is advisable to consider analogous mechanical models. They involve a combination of pure elastic and viscous elements connected in series or in parallel [1,2].

An ideal linear elastic element is represented by a spring (Hooke’s element), the elongation of which is directly proportional to the applied force [1]

\[ F = Ea, \]  

where \( F \) is the applied force, \( a \) is the elongation of a spring, \( E \) is the modulus of elasticity of the spring. The spring has the property of accumulating mechanical energy.

An ideal viscous element is represented by a dashpot (Newton's element), the rate of deformation of which is directly proportional to the applied force

\[ F = \eta Da, \]  

where \( \eta \) is the coefficient of viscosity of a dashpot, \( D \) denotes differentiation with respect to time. The dashpot has the property of dissipating the mechanical energy.

In the theory of linear viscoelasticity, the models involving the combination of springs and dashpots are considered.
2. Classic viscoelastic models with integer order derivatives

2.1. Two-element models

2.1.1. Maxwell model. The connection of elastic and viscous elements in series is the simplest scheme, called the Maxwell model, shown in Figure 1a. In this case, the force \( F \) is the same for the spring and the dashpot, and the elongation of a system \( a \) is equal to the sum of their elongations [1]:

\[
D_a = \frac{1}{E} D F + \frac{1}{\eta} F. \tag{3}
\]

2.1.2. Kelvin-Voigt model. The parallel connection of elastic and viscous elements is known as the Kelvin-Voigt model shown in Figure 1b. In this case, deformations of the spring and the dashpot \( a \) are equal, and the force \( F \) is equal to the sum of forces stretching the elements [1]:

\[
F = E a + \eta D a. \tag{4}
\]

![Figure 1. Schemes of two-element models: a) Maxwell model; b) Kelvin-Voigt model](image)

2.2. Three-element models

Three-element models consist of combinations of two springs and one dashpot or one spring and two dashpots. In the literature there is a wide variety of names for these models and there is no single structure for their classification.

2.2.1. Three-element elastic models or models of a standard linear solid. Three-element elastic models [1], also called the models of standard linear solid [3-9], or Zener model [3-5,8,9], and Poynting-Thomson model [3,7,10,11], are obtained by adjoining a spring in series to the Kelvin-Voigt element (Figure 2a), or in parallel to the Maxwell element (Figure 2b). Therefore, these models are also called the standard three-parameter Voigt and Maxwell models, respectively [7]. Applying the combination rules for series-parallel connection of the elements and performing the simplest transformations, a general equation could be obtained that determines the relationship between the force \( F \) and deformation \( a \) of the model [1]:

\[
DF + p_0 F = q_1 D a + q_0 a, \tag{5}
\]

where \( p_0 = \frac{E_i + E_2}{\eta} \); \( q_1 = E_i \); \( q_0 = \frac{E_i E_2}{\eta} \) for the model in Figure 2a,

\[
p_0 = \frac{E_1}{\eta}; \quad q_1 = E_1 + E_2; \quad q_0 = \frac{E_1 E_2}{\eta} \] for the model in Figure 2b.
2.2.2. Three-element viscous models, or models of the standard linear liquid. In the literature, three-element viscous models [1] are also called Jeffreys models [3,4,12], or models of standard linear liquid [3,4]. They are obtained by adjoining a dashpot in series to the Kelvin-Voigt element (Figure 3a), or in parallel to the Maxwell element (Figure 3b). Therefore, these models are also called nonstandard three-parameter Voigt and Maxwell models, respectively [7].

Applying the combination rules for the series-parallel connection of the elements and performing the simplest transformations, a general equation could be obtained which determines the relationship between the force $F$ and deformation $a$ of the model [1]:

$$DF + p_0 F = q_2 D^2 a + q_1 D a,$$

where $p_0 = \frac{E}{E_1 + \eta_2}$; \( q_2 = \frac{\eta_1 \eta_2}{E_1 + \eta_2} \); \( q_1 = \frac{E \eta_1}{E_1 + \eta_2} \) for the model in Figure 3a, and

$p_0 = \frac{E}{\eta_1}$; \( q_2 = \eta_2 \); \( q_1 = \frac{E(\eta_1 + \eta_2)}{\eta_1} \) for the model in Figure 3b.

The main relations of the models are expressed in terms of the force $F$ and elongation $a$, however, these relations can be written in terms of the stress and strain for an element with a cross-sectional area $A$ and length $L$ considering that $F = \sigma A$ and $a = \varepsilon L$. Since the rules for series-parallel connection of elements are the same in both cases, than each model that describes the relationship between the force and elongation will be the same in general as the model that relates the stress and strain. The constants of the last model are obtained from the constants of the first model by multiplying by the constant $A / L$ [1].
More complex models involving a larger number of elements represent various combinations of the basic models, among them four-element models, or Burgers models [3,4,7,9,12,13], five-element models [10], and also the systems consisting of a large number of elements known in the literature as the generalized Maxwell and Voigt models [1,2,9], or Wiechert model [6,9].

3. Models of viscoelasticity with fractional order derivatives

Linear viscoelastic models were generalized by introducing a fractional derivative of various orders, resulting in viscoelastic models based on fractional order operators. Scott-Blair and colleagues showed that the shear stress is proportional to the fractional derivative of the shear deformation with respect to time, but they could not give a definition of the fractional derivative that would satisfy mathematicians of that period of time [14].

The simplest fractional derivative element proposer by Scott-Blair was later named by Koeller as a «spring-pot» [15], and the model equation is written in the form

$$\sigma = Et_0^\gamma D^\gamma \varepsilon \quad (0 < \gamma < 1),$$

(7)

where $\sigma$ is the stress, $\varepsilon$ is the deformation, $\tau_\sigma = \eta / E$ is the retardation time, $D^\gamma$ is the Riemann-Liouville fractional derivative, and $\gamma$ is the order of the fractional derivative, which some time is called as the memory parameter.

Thus, Koeller showed that this equation more accurately describes two boundary cases, namely: at $\gamma = 0$ the «spring-pot» corresponds to a pure elastic material (elastic solid), and at $\gamma = 1$ the equation describes a material with the perfect memory (viscous fluid).

In the 60s of the last century, the simplest viscoelastic models involving fractional derivatives were proposed. They are based on the replacement of an integer order derivative in classical viscoelastic models by the fractional Riemann-Liouville derivative [16]:

$$^{rl} D^\gamma \sigma = \frac{d}{dt} \int_0^t \frac{\sigma(t')}{\Gamma(1-\gamma)(t-t')^\gamma} dt',$$

(8)

where $0 < \gamma < 1$ is the fractional parameter, and $\Gamma(1-\gamma)$ is the Gamma function.

3.1. Maxwell model with fractional derivatives

The Maxwell model with fractional derivatives shown in Figure 4a was first introduced by Meshkov in 1967 [17] and has the form:

$$\sigma + \tau_\varepsilon^\gamma D^\gamma \sigma = E_\varepsilon \tau_\varepsilon^\gamma D^\gamma \varepsilon,$$

(9)

where $E_\varepsilon$ is the non-relaxed (instantaneous) elastic modulus, and $\tau_\varepsilon$ is the relaxation time.

The Maxwell fractional derivative model was also independently introduced by Caputo [18].

The equation of the Maxwell model with fractional derivatives could be represented in the form of the Boltzmann-Volterra relations with a fractional exponential function with a weakly singular kernel of heredity [19]

$$\sigma = E_\varepsilon [\varepsilon - \int_0^t \gamma_\varepsilon (-t'/\tau_\varepsilon) \varepsilon(t-t')dt'],$$

(10)

or could be expressed in terms of the fractional operator [20]
\[ \sigma = E_\alpha [1 - \gamma^*_\alpha (t') \varepsilon(t)], \]  

(11)

where \( \gamma^*_\alpha (t') = \frac{1}{1 + \gamma^*_\alpha D^\gamma} \) is the dimensionless fractional operator \([21]\),

\[ \varepsilon^*_\gamma (t'/\tau_e) = \frac{t'^{-1}}{\tau_e} \sum_{n=0}^{\infty} \frac{(-1)^n (t'/\tau_e)^n}{\Gamma(n+1)} \] is Rabotnov’s fractional exponential function, which reduces to the ordinary exponential function at \( \gamma = 1 \).

Koeller \([22]\) presented a model consisting of a finite number of Maxwell elements with fractional derivatives connected in parallel (Figure 5a), and wrote down the relationship between the strain and the total stress of the model, which is the sum of the stresses in the individual Maxwell elements, which has the strict physical meaning only when it is matched with the Rabotnov model \([23, 24]\), written in terms of the Rabotnov’s dimensionless fractional operator:

\[ \sigma = \sum_{n=0}^{\infty} \sigma_n = E_\alpha [1 - \gamma^*_\alpha (t'_{\gamma_e}) \varepsilon(t)]. \]

(12)

3.2. Kelvin-Voigt model with fractional derivatives

The Kelvin-Voigt model with the fractional derivative (Figure 4b) was first introduced by Shermergor in 1966 \([19]\) and has the form:

\[ \sigma = E_0 \varepsilon + E_\alpha \gamma^*_\alpha D^\gamma \varepsilon, \]

(13)

where \( E_0 \) is the relaxed elastic modulus. At \( \gamma = 1 \) the model described by equation (13) turns into the classical Kelvin-Voigt model (4).

\[ a) \quad b) \quad c) \]

\[ \sigma \quad \sigma \quad \sigma \]

\[ E \quad E_1 \quad E_0 \]

\[ E \tau_f^\gamma \quad E \tau_f^\gamma \quad \]

\[ \sigma \quad \sigma \quad \sigma \]

\[ \sigma \quad \sigma \quad \]

\[ E_1 \quad E_1 \quad \]

\[ \sigma \quad \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]

\[ \]
The equation of the Kelvin-Voigt model with fractional derivatives could also be represented in the form of the Boltzmann-Volterra relations with the fractional exponential function as a weakly singular kernel of heredity [19]

\[ \varepsilon = J_0 \gamma^f_\nu (-t' / \tau_\nu) \sigma(t-t')dt', \]

or could be expressed in terms of the fractional operator [20]

\[ \varepsilon = J_0 \gamma^f_\nu (\tau_\nu^f \sigma(t)). \]

Koeller [22] presented a model consisting of a finite number of Kelvin-Voigt elements with fractional derivatives connected in series (Figure 5b), and wrote down the relationship between stress and the total deformation of the model, which has the strict physical meaning only when it is matched with the Rabotnov model [23,24] written in terms of the Rabotnov’s dimensionless fractional operator:

\[ \varepsilon = \sum_{n=0}^{N+1} E_n \gamma^f_n (\tau_n^f \sigma(t)). \]

where \( N+1 \) is the total number of the elements of the model.

3.3. Standard linear solid model with fractional derivatives

The model of the standard linear solid with fractional derivatives (Figure 4c) was for the first time introduced by Meshkov in 1967 [17] and has the form:

\[ \sigma + \gamma^f_\nu D^\gamma \varepsilon = E_0 (\varepsilon + \gamma^f_\sigma D^\gamma \varepsilon), \]

where \( \left( \frac{\tau_\nu}{\tau_\sigma} \right)^\gamma = \frac{E_0}{E_\infty} = \frac{J_0}{J_\infty}, J_\infty = E_\infty^{-1} \) is the is non-relaxed (instantaneous) compliance, and \( J_0 = E_0^{-1} \) is the relaxed (prolonged) compliance.

This model was also independently introduced by Caputo [28].

The fractional model of the standard linear solid (17) is known in the literature as a four-parameter fractional model, since in order to uniquely define the relationship between the stress and strain functions, it is necessary that four parameters (constants) are preassigned: \( E_0, E_\infty, \tau_\nu, \gamma \).

Bagley and Torvik [29] analyzed the four-parameter model and found it thermodynamically and mechanically stable under steady-state harmonic conditions. Pritz [30] theoretically analyzed a model of the standard linear solid with four parameters and showed that it is able to describe the change in the dynamic modulus and loss coefficient of real viscoelastic materials over a wide frequency range provided that there is one symmetric loss peak. Welch [31] investigated the quasistatic viscoelastic
behavior of polymer materials defined by a fractional four-parameter model. Huang et al. [32] used the standard linear solid model with fractional derivatives to describe the creep of concrete taking into account its age and duration of the load.

The equation of the fractional derivatives standard linear solid model could be represented also in the form of the Boltzmann-Volterra relations with a fractional exponential function as a weakly singular kernel of heredity [33]

\[ \sigma = E_\infty [e - \nu_\varepsilon \int_0^t \left( \frac{t'}{\tau} \right) e(t-t')dt'], \] (18)

or could be expressed in terms of a fractional operator [20]

\[ \sigma = E_\infty [1 - \nu_\varepsilon \mathcal{S}_\varepsilon(t')]e(t), \] (19)

where \( \nu_\varepsilon = \Delta E E_\infty^{-1} \), and \( \Delta E = E_\infty - E_0 \) is the defect of elastic modulus.

The applications of the simplest viscoelastic models have been discussed by many authors, the review of which could be found in [9, 34, 35, 38].

4. Conclusion
Due to the appearance of new class of polymers, the behavior of which is often described by linear viscoelastic models, many scientists have devoted their papers for study and comparison of such materials. Starting from the end of the 19th century and ending with modern literature reviews, there is a wide variety of names for the same mechanical models of viscoelastic materials presented in various sources. This paper presents a generalized classification of standard viscoelastic models consisting of springs and dashpots connected in series and in parallel.

It should be noted that linear viscoelastic models were generalized by introducing a fractional derivative of various fractional orders, resulting in the appearance of viscoelastic models based on fractional order operators. In addition to the simplest models of viscoelasticity with fractional derivatives, more complex models are known in the literature, among them: systems with several fractional parameters [23, 35], several relaxation (retardation) times [22, 24, 31, 36, 37], models with fractional elements [38-40], as well as nonlinear models of viscoelasticity [41-43]. The use of fractional models is largely due to the fact that less parameters are required to represent the viscoelastic behavior of materials than when using traditional integer-order models. Therefore, such models allow the rheological parameters to be changed over a wide range. In addition, there is better agreement between the experimental data for models with fractional derivatives than for models with integer derivatives [44].

Acknowledgements
The first author was supported by RFBR, project number 20-38-70143.

References
[1] Bland D R 1960 The theory of linear viscoelasticity (New York: Pergamon Press)
[2] Christensen R M 1971 Theory of viscoelasticity: An introduction (New York and London: Academic press)
[3] Kruzik M and Roubicek T 2019 Mathematical Methods in Continuum Mechanics of Solids (Switzerland AG: Springer Nature)
[4] Mainardi F and Spada G 2011 Eur. Phys. J. Special Topics 193 133–60
[5] Nonnenmacher T F and Glöckle W G 1991 Phil. Mag. Lett. 64 89-93
[6] Roylance D 2001 Dep. Mater. Sci. Eng. Inst. Technol. 2139 1–37
[7] Tschoegl N 1989 The Phenomenological Theory of Linear Viscoelastic Behavior: An Introduction (Berlin: Springer-Verlag)
[8] Ward I M 1983 Mechanical Properties of Solid Polymers (Chichester: Wiley)
[9] Zhou X Q, Yu D Y, Shao X Y, Zhang S Q and Wang S 2016 Compos. Struct. 136 460–80
[10] Bogomolov V 2016 Autom. Transport 38 117–25 [in Russian]
[11] Liang J and Guangli H 2019 Adv. Civ. Eng. 2019 1-8
[12] Reyner M 1960 Deformation, strain and flow: An elementary introduction to rheology (London: H. K. Lewis & Co. Ltd)
[13] Malkin A I and Isayev A I 2017 Rheology Concept, Methods and Applications (ChemTec Publishing)
[14] Scott-Blair G W 1944 J. Sci. Instrum. 21 80–4
[15] Koeller R C 1984 J. Appl. Mech. 51 299–307
[16] Samko S G, Kilbas A A and Marichev O I 1993 Fractional integrals and derivatives (theory and applications) (Switzerland: Gordon and Breach)
[17] Meshkov S I 1967 J. Appl. Mech. Tech. Phys. 8 100–2
[18] Caputo M and Mainardi F 1971 Riv. Nuovo Cimento 1 161–98
[19] Shermegor D T 1966 J. Appl. Mech. Tech. Phys. 7 85–7
[20] Rossikhin Yu A and Shitikova M V 2019 Encyclopedia of Continuum Mechanics Fractional Operator Models of Viscoelasticity (Germany: Springer-Verlag GmbH)
[21] Rossikhin Yu A and Shitikova M V 2014 Frac. Calculus Appl. Anal. 17 674–83
[22] Koeller R C 1986 Acta Mech. 58 251–64
[23] Rossikhin Yu A and Shitikova M V 2007 Frac. Calculus Appl. Anal. 10 111-121
[24] Rossikhin Yu A, Shitikova M V and Shcheglova T A 2008 Comp. Math. Appl. 59 1727-1744
[25] Eldred L B, Baker W P and Palazotto A N 1995 AIAA J. 33 547–50
[26] Caputo M 1967 Geophys. J. R. Astron. Soc. 13 529–39
[27] Smit W and de Vries H 1970 Rheol. Acta 6 525–34
[28] Caputo M and Mainardi F 1971 Pure Appl. Geophys. 91 134–47
[29] Bagley R L and Torvik P J 1986 J. Rheol. 30 133–55
[30] Pritz T 1996 J. Sound Vib. 195 103–15
[31] Welch S W J, Rorrer R A L and Duren R G 1999 Mech. Time-Depend. Mater. 3 279 - 303
[32] Huang Y, Xiao L, Bao T and Liu Y 2019 Mech. Time-Depend. Mater. 23 361–72
[33] Rabotnov Yu N 1948 Prikl. Matem. Mekh. 12 81–91 [in Russian]
[34] Rossikhin Yu A and Shitikova M V 1997 Appl. Mech. Rev. 50 15 - 67
[35] Rossikhin Yu A and Shitikova M V 2010 Appl. Mech. Rev. 63 010801
[36] Song D Y and Jiang T Q 1998 Rheol. Acta 37 512–17
[37] Arikoglu A 2014 Rheol. Acta 53 219–33
[38] Schiessel H, Metzler R, Blumen A and Nonnenmacher T F 1995 J. Wys. A: Math 28 6567-84
[39] Heymans N and Bauwens J-C 1994 Rheol. Acta 33 210–19
[40] Liu J G and Xu M Y 2006 Mech. Time-Depend. Mater. 10 263-79
[41] Adolfsson K. 2004 Nonlinear Dyn. 38 233–46
[42] Zopf C, Hoque S E and Kaliske M 2015 Comput. Mater. Sci. 98 287–96
[43] King A W 2019 Nonlinear Fractional Order Derivative Models of Components and Materials in Hearing Aids and Transducers (Denmark: Technical University of Denmark, Department of Electrical Engineering)
[44] Wollscheid D and Lion A 2014 Comput. Mech. 53 1015–31
[45] Bagley R L and Torvik P J 1983 J. of Rheology 27 201-10