Stationary Spiral Structure and
Collective Motion of the Stars in a Spiral Galaxy

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Most fully developed galaxies have a vivid spiral structure, but the formation and evolution of the spiral structure are still an enigma in astrophysics. In this paper, according to the standard Newtonian gravitational theory and some observational facts, we derive an idealized model for spiral galaxy, and give a possible explanation to the spiral structure. We solve some analytic solutions to a spiral galaxy, and obtain manifest relations between mass density and speed. From the solution we get some interesting results: (I) The spiral pattern is a stationary or static structure of density wave, and the barred galaxy globally rotate around an axis at tiny angular speed. (II) All stars in the disc of a barred spiral galaxy move in almost circular orbits. (III) In the spiral arms, the speed of stars takes minimum and the stellar density takes maximum. (IV) The mass-energy density of the dark halo is compensatory for that of the disc, namely, it takes minimum in the spiral arms. This phenomenon might reflect the complicated stream lines of the dark halo.

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I. INTRODUCTION

Most fully developed galaxies have a vivid spiral structure, which has profound influences on the generation and evolution of stars. During the last 60 years many efforts have been made to reveal the nature of the spiral structure. Despite the progress made in understanding the amplification mechanisms of the spiral arms and the role of the spiral arms in the dynamics of galaxies, the formation and evolution of the spiral structure are still an enigma.
in astrophysics\cite{1, 2, 3}. We even do not know if the spirals are a long-lived phenomenon, or are short-lived and regenerated many times during the galactic evolution. C. C. Lin and F. H. Shu suggested that, the spiral structure should be a quasi-stationary and stable density waves, which are excited by over-reflection at co-rotation due to some nonlinear effects\cite{4, 5}. But some other opinions favor short-lived or recurrent spiral patterns which are developed in the galactic disks via swing-amplification in a rotating disk with shear\cite{6}, or by an external force\cite{7}. The recent observational results seem to favor a long-lived pattern. In contrast the nearby galaxies with the distant ones observed through the Hubble Space Telescope and from the ground\cite{8, 9}, the results show that there is only mild evolution in the relationship between radial size and stellar mass for galactic disks within the redshift region $0 < z < 1$.

In the noncentral potential, the orbit of a star is usually an unclosed and complicated spatial curve, which sensitively depends on the initial speed and direction\cite{1}. So the stellar system should be more reasonably treated as fluid rather than mass points. Although the dynamics for the two models is essentially the same Newtonian mechanics, but the initial and boundary conditions are different. In the case of fluid, the stream lines are consistent field, which are a natural result of the generation of stars from nebulae.

W. Dehnen and J. Binney fitted the mass distribution within the Milky Way to the observational data such as the rotation curve and Oort’s constants\cite{10}, but they found the fitting mass distribution is ill determined. In \cite{11, 12}, the authors made numerical simulations for 2-dimensional stellar hydrodynamics of a flat galactic disc embedded in dark matter halo, and the model solves the Boltzmann equations up to second order moment\cite{1}. A dynamical approach to explain the formation of both spirals and rings in barred galaxies was proposed in \cite{13, 14}. It is based on the orbital motion driven by the unstable equilibrium points of a given rotating bar potential, and then the spirals, rings and pseudo-rings are related to the invariant manifolds associated to the periodic orbits around these equilibrium points. By adjusting dynamical parameters of the host galaxy, we get the spiral and ring like structures.

If arms of a spiral galaxy were constructed by fixed material, the arms would become more and more tightly wound, since the matter nearer to the center of the galaxy rotates faster than the matter at the edge of the galaxy. The arms would become indistinguishable from the rest of the galaxy after only a few orbits. In order to avoid this winding problem, Lin and Shu proposed the density wave theory\cite{4, 5}. They suggested that the arms are not
constructed by stationary materials, but instead made up of areas of greater density similar to a traffic jam on a highway. More specifically, for some dynamical reasons, the luminous materials such as stars intend to move slowly and spend more time inside the spiral arms and move rapidly outside the spiral arms. Then an inhomogeneous and quasi-stationary distribution of mass density wave is developed, which forms the spiral arms. In the sense of morphology, this theory is quite successful in some aspects. However, the dynamical explanation for the formation of density wave is still unclear. The calculations of this paper attempt to give a clear dynamical explanation for the density wave.

According to the following observational facts, we find the problem can be simplified and described by an idealized fluid model, and then the nature of the spiral structure might be disclosed by dynamical approach.

1. By observation, except for the center part, all stars in a spiral galaxy are mainly distributed in a thin disc, so the spiral structure should be a visible display of collective motion of these stars, and the collective motion can be approximately described by the 2-dimensional hydrodynamics.

2. The collision among stars rarely occurs, so the fluid of star flow is zero-pressure and inviscid, and the stars move along geodesics.

3. All stars are driven by the average background gravity of total mass-energy distribution in the galaxy, and the background mass density and gravity can be constrained by some empirical data such as the flat rotation curve.

These are some basic assumptions of the following calculations. The results show that, the stationary solutions to such system indeed exist, and the distributions of mass density and speed do have stable spiral structure. So this idealized model and the solutions might shed some lights on the enigma of galactic structure.

II. HYDRODYNAMICS FOR STARS IN A SPIRAL GALAXY

As discussed below, a fully dynamical approach to the galactic structure in the context of general relativity strongly depends on the properties of the dark halo, so it is unrealistic at present due to the lack of such knowledge[15, 16]. Since the gravity in a galaxy is very
weak except for the region near the center, the Newtonian gravitational theory is accurate enough to simulate the spiral structure.

Denote the total effective mass-energy density of a galaxy by \( \rho \), the Newtonian gravitational potential by \( \Phi \), we have the dynamical equation for Newtonian gravity (see [17] or the appendix)

\[
\partial_\alpha \partial^\alpha \Phi = -4\pi G \rho,
\]

(2.1)

where \( \partial_\alpha \partial^\alpha = \partial_t^2 - \nabla^2 \) is the d’Alembert operator. For the stars moving in the gravity, the dynamical equation of the flow is given by

\[
(\partial_t + \vec{V} \cdot \nabla) \vec{V} = -\nabla \Phi.
\]

(2.2)

Theoretically, the complete dynamical equation system should include the continuity equation and the dynamics of the dark halo. But these equations strongly depend on the equation of state of dark matter and dark energy, which is not clearly discovered. However, as shown below, the lack of this knowledge can be compensated by other empirical data such as the flat rotation speed curve, and then some important information of galaxy such as the total mass density and the spiral structure, can be inversely derived. That is to say, we do the inverse treatments as done in [10, 13, 14].

For convenience we rewrite (2.1) and (2.2) in the spherical coordinate system \((t, r, \theta, \phi)\). By 3-dimensional tensor operation, we get the following equations

\[
c^{-2} \partial_t^2 \Phi - \left[ \partial_r^2 + \frac{2}{r} \partial_r + \frac{1}{r^2} \left( \partial_\theta^2 + \cot \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\phi^2 \right) \right] \Phi + 4\pi G \rho = 0,
\]

(2.3)

\[
(\partial_t + V_r \partial_r + V_\theta \partial_\theta + V_\phi \partial_\phi) V_r - r V_\theta^2 - r \sin^2 \theta V_\phi^2 + \partial_r \Phi = 0,
\]

(2.4)

\[
(\partial_t + V_r \partial_r + V_\theta \partial_\theta + V_\phi \partial_\phi) V_\theta + \frac{2}{r} V_r V_\theta - \sin \theta \cos \theta V_\phi^2 + \frac{\partial_\theta \Phi}{r^2} = 0,
\]

(2.5)

\[
(\partial_t + V_r \partial_r + V_\theta \partial_\theta + V_\phi \partial_\phi) V_\phi + \frac{2}{r} V_r V_\phi + 2 \cot \theta V_\theta V_\phi + \frac{\partial_\phi \Phi}{(r \sin \theta)^2} = 0,
\]

(2.6)

where the velocities are defined in the form of 3-dimensional contra-variant vectors

\[
V_r = \frac{dr}{dt}, \quad V_\theta = \frac{d\theta}{dt}, \quad V_\phi = \frac{d\phi}{dt}.
\]

(2.7)

For the stationary spiral structure globally precessing around \( z \)-axis at constant angular speed \( \Omega \) (namely the pattern speed), under the coordinate transformation \( \varphi = \phi - \Omega t \), the
solution will be static in coordinate system \((t, r, \theta, \phi)\), namely independent of \(t\). Then we get the following equations for such galaxy

\[
c^{-2}\Omega^2 \partial_t^2 \Phi - \left[ \partial_r^2 + \frac{2}{r} \partial_r + \frac{1}{r^2} \left( \partial_\theta^2 + \cot \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\phi^2 \right) \right] \Phi + 4\pi G \rho = 0, \tag{2.8}
\]

\[
(V_r \partial_r + V_\theta \partial_\theta + V_\phi \partial_\phi) V_r - r[V_\theta^2 + \sin^2 \theta (V_\phi + \Omega)^2] + \partial_\theta \Phi = 0, \tag{2.9}
\]

\[
(V_r \partial_r + V_\theta \partial_\theta + V_\phi \partial_\phi) V_\theta + \frac{2}{r} V_r V_\theta - \sin \theta \cos \theta (V_\phi + \Omega)^2 + \frac{\partial_\theta \Phi}{r^2} = 0, \tag{2.10}
\]

\[
(V_r \partial_r + V_\theta \partial_\theta + V_\phi \partial_\phi) V_\phi + \left( \frac{2}{r} V_r + 2 \cot \theta V_\theta \right) (V_\phi + \Omega) + \frac{\partial_\phi \Phi}{(r \sin \theta)^2} = 0. \tag{2.11}
\]

In (2.9)-(2.11), some terms have manifest physical meanings, \(r[V_\theta^2 + \sin^2 \theta (V_\phi + \Omega)^2]\) stands for centrifugal force, and \((\frac{2}{r} V_r + 2 \cot \theta V_\theta) \Omega\) the Coriolis force.

Considering an unwarped galaxy with two stationary spiral aims, the total mass-energy density and potential can be generally expanded by spherical harmonics \(Y_{lm}(\theta, \phi)\) with even \(m\). To second order terms, we equivalently have the following approximation

\[
\rho = \rho_0 + (\rho_1 + \rho_2 \cos 2\varphi + \rho_3 \sin 2\varphi) \sin^2 \theta, \tag{2.12}
\]

\[
\Phi = \Phi_0 + (\Phi_1 + \Phi_2 \cos 2\varphi + \Phi_3 \sin 2\varphi) \sin^2 \theta, \tag{2.13}
\]

where all \((\rho_n, \Phi_n)\) are functions of \(r\), and \((\rho_0 \geq 0, \rho_1 \geq 0)\). Accordingly, the velocity of the stars in the disc, also to second order terms, should be

\[
V_r = W_1 \cos 2\varphi + W_2 \sin 2\varphi, \quad V_\theta = 0, \tag{2.14}
\]

\[
V_\phi = \omega_0 + \omega_1 \cos 2\varphi + \omega_2 \sin 2\varphi, \tag{2.15}
\]

where all \((W_n, \omega_n)\) are functions of \(r\).

Substituting (2.12) and (2.13) into (2.8), we get the relations between \(\Phi_n\) and \(\rho_n\) as follows

\[
\partial_r^2 \Phi_0 + \frac{2}{r} \partial_r \Phi_0 + \frac{4}{r^2} \Phi_1 = 4\pi G \rho_0, \tag{2.16}
\]

\[
\partial_r^2 \Phi_1 + \frac{2}{r} \partial_r \Phi_1 - \frac{6}{r^2} \Phi_1 = 4\pi G \rho_1, \tag{2.17}
\]

\[
4c^{-2}\Omega^2 \Phi_k + \partial_r^2 \Phi_k + \frac{2}{r} \partial_r \Phi_k - \frac{6}{r^2} \Phi_k = 4\pi G \rho_k, \quad (k = 2, 3). \tag{2.18}
\]

Substituting (2.13)-(2.15) into (2.9) and (2.11), and constraining \(\theta = \frac{1}{2} \pi\), we get respectively

\[
0 = \frac{1}{4} \partial_r(W_1^2 + W_2^2) + \partial_r(\Phi_0 + \Phi_1) - \frac{1}{2} r(2\omega_0^2 + \omega_1^2 + \omega_2^2) + \omega_1 W_2 - \omega_2 W_1 +
\]

\[
2[(\omega_0 - \Omega) W_2 - r\omega_0 \omega_1 + \frac{1}{2} \partial_r \Phi_2] \cos 2\varphi +
\]

\[
2[(\Omega - \omega_0) W_1 - r\omega_0 \omega_2 + \frac{1}{2} \partial_r \Phi_3] \sin 2\varphi + \cdots, \tag{2.19}
\]
and

\[
0 = \frac{1}{2}(r\partial_r\omega_1 + 2\omega_1)W_1 + \frac{1}{2}(r\partial_r\omega_2 + 2\omega_2)W_2 + \\
[(r\partial_r\omega_0 + 2\omega_0)W_1 + 2(\omega_0 - \Omega)r\omega_2 + \frac{2}{r}\Phi_3]\cos 2\varphi + \\
[(r\partial_r\omega_0 + 2\omega_0)W_2 + 2(\Omega - \omega_0)r\omega_1 - \frac{2}{r}\Phi_2]\sin 2\varphi + \cdots .
\]

(2.20)

The equation (2.10) automatically holds. Theoretically, we can solve \((\Omega, W_j, \omega_j)\) and some relations among \((\rho_k, \Phi_k)\) by condition that the coefficients of (2.19) and (2.20) vanish.

According to the high-accurate observational data\([18, 19, 20]\), we find the rotation curves of most spiral galaxies are approximately flat, then we can assume

\[
\omega_0 = \frac{v}{r} - \Omega, \quad (r \in [R_0, R_1]),
\]

(2.21)

where \(v\) is a constant speed with typical value \(|v| = 200 \sim 400\text{km/s}\), \([R_0, R_1]\) is the effective region of (2.21), with their typical values as \(R_0 = 100 \sim 500\text{pc}\), and \(R_1 = 10 \sim 60\text{kpc}\) to be approximately the visible radius of the galaxy. Equivalently, we can assume \((\omega_0 > 0, v > 0, \Omega > 0)\) in calculation, because the dynamical equations (2.3)-(2.6) have reversal invariance under transformation \(\phi \rightarrow -\phi\). We use the empirical condition (2.21) to replace the dynamical equations for the background, then the problem is greatly simplified. In addition, for a given galaxy, instead of constant \(v\), we can use fitting function \(v = v(r)\) in (2.21) to get more accurate results and larger effective domain\([21, 22, 23]\). However in this case, we can usually get numerical results only.

Substituting (2.21) into (2.19) and (2.20), by setting the coefficients of \((\sin 2\varphi, \cos 2\varphi)\) terms to zero, we can solve

\[
\omega_1 = -K\left[\frac{1}{2}v\partial_r\Phi_3 + \frac{2}{r}(v - r\Omega)\Phi_2\right],
\]

(2.22)

\[
\omega_2 = -K\left[\frac{1}{2}v\partial_r\Phi_3 + \frac{2}{r}(v - r\Omega)\Phi_3\right],
\]

(2.23)

\[
W_1 = K[(v - r\Omega)r\partial_r\Phi_3 + 2v\Phi_3],
\]

(2.24)

\[
W_2 = K[(v - r\Omega)r\partial_r\Phi_2 + 2v\Phi_2],
\]

(2.25)

\[
K \equiv (2r^2\Omega^2 - 4rv\Omega + v^2)^{-1}.
\]

(2.26)

By the zeroth order terms in (2.19) and (2.20), we have

\[
\partial_r(\Phi_0 + \Phi_1) + \frac{1}{4}\partial_r(W_1^2 + W_2^2) = \frac{1}{2}r(2\omega_0^2 + \omega_1^2 + \omega_2^2) - \omega_1W_2 + \omega_2W_1,
\]

(2.27)
and
\[(r \partial_r \omega_1 + 2 \omega_1)W_1 + (r \partial_r \omega_2 + 2 \omega_2)W_2 = 0.\] (2.28)

Substituting (2.22)-(2.26) into (2.27) and (2.28), we get two constraints for \(\Phi_k(r)\) of the background potentials.

Obviously, the disc satisfies the mass conservation law independent of the dark halo, so we have the 2-dimensional continuity equation for all stars and baryonic particles moving in the disc as follows[3, 11]
\[0 = \partial_t \Sigma + \nabla \cdot (\vec{V} \Sigma) = (\partial_t + V_r \partial_r + V_\phi \partial_\phi) \Sigma + \left(\partial_r V_r + \partial_\phi V_\phi + \frac{1}{r} V_r\right) \Sigma,\] (2.29)
where \(\Sigma\) stands for the surface mass density of the stars and baryons in the disc. It should be mentioned that, (2.8)-(2.11) are expressed in spherical coordinate system due to the background gravity, but (2.29) is expressed in polar coordinate system. In the stationary case, (2.29) becomes
\[(V_r \partial_r + V_\phi \partial_\phi) \Sigma + (\partial_r V_r + \partial_\phi V_\phi + \frac{1}{r} V_r) \Sigma = \nabla \cdot (\vec{V} \Sigma) = 0.\] (2.30)
(2.29) or (2.30) is the equation to describe the mass density of the stars.

III. RESOLUTION TO THE EQUATIONS

A. Solution to a Barred Spiral Galaxy

The general solutions to the above underdetermined equation system are quite complicated and unnecessary. The most important case to understand the nature of spiral structure is the stable and terminal state of a galaxy, which is similar to the eigenstate of a micro particle. At first, we consider the case that all stars move in the orbits near circle, for which the analytic solutions can be solved. In this case the radial speed is a high order little term, then we have
\[W_1 = 0, \quad W_2 = 0.\] (3.1)

By (2.24) and (2.25), noticing the symmetry between \((\Phi_2, \Phi_3)\), we get the equivalent solution, except for an initial phase of \(\varphi\), as follows
\[\Phi_2 = -\frac{q}{r^2} (v - r\Omega)^2, \quad \Phi_3 = 0,\] (3.2)
where \( q \geq 0 \) is a constant. Substituting (3.1) and (3.2) into the above equations (2.16)-(2.18) and (2.22)-(2.28), we finally get

\[
\begin{align*}
\omega_1 &= \frac{q}{r^3}(v - r\Omega), \quad \omega_2 = 0, \quad \rho_3 = 0, \\
\rho_1 &= -\rho_0 + \frac{1}{4\pi G} \left( \frac{q^2v}{12r^6}(16r\Omega - 15v) - \frac{2v^2}{r^2} \ln \frac{r}{r_0} + \frac{2\Phi_0}{r^2} + \frac{v^2}{r^2} \right), \\
\rho_2 &= \frac{q}{2\pi Gr^4} \left( (3r^2\Omega^2 - 6rv\Omega + 2v^2) - \frac{2r^2\Omega^2}{c^2}(v - r\Omega)^2 \right), \\
0 &= \Phi_0 + \Phi_1 + \frac{q^2}{24r^4}(6r^2\Omega^2 - 8vr\Omega + 3v^2) - v^2 \ln \frac{r}{r_0},
\end{align*}
\]

where \( r_0 > 0 \) is a constant with length dimension.

Substituting the solutions into (2.9) and (2.11), we can check the truncation error of the equations reads

\[
\Delta(2.9) = -\frac{q^2}{2r^5}(v - r\Omega)^2 \cos 4\varphi, \quad \Delta(2.11) = -\frac{q^2}{r^6}(v - r\Omega)^2 \sin 4\varphi,
\]

which are higher order terms. So the above solution is a good approximation. If \( q = 0 \), the solutions are exact, which correspond to the galaxy without spirals in the disc. To get more accurate results, one should introduce higher order terms such as \((\cos 4\varphi, \sin 4\varphi)\) terms in (2.12)-(2.15).

By (2.21)-(3.2), we get the speed of the stellar flow

\[
V_r = 0, \quad V_\theta = 0, \quad V_\varphi = \frac{v}{r} + \frac{q}{r^3}(v - r\Omega) \cos 2\varphi.
\]

Substituting it into (2.30), we get the continuity equation as \( \partial_\varphi (V_\varphi \Sigma) = 0 \), which yields the mass density distribution of the stars as

\[
\Sigma(r, \varphi) = \frac{vr^2 \varrho(r)}{vr^2 + q(v - r\Omega) \cos 2\varphi},
\]

where \( \varrho(r) \) is a density function determined by boundary conditions. Apparently, the density distribution of the stars display a profile of barred spiral galaxy.

To determine \((\rho_0, \rho_1)\), we need another condition related to the dynamics of the black halo. In the case of \( \rho_1 \propto \rho_0 \), we can solve

\[
\begin{align*}
\rho_0 &\sim \frac{n^2 - n - 4}{4\pi G} \frac{q^2}{r^6} \left( \frac{9v^2}{2(n - 4)(n + 3)r^6} - \frac{\Omega^2}{(n - 2)(n + 1)r^4} \right) + \\
&\quad \frac{n^2 - n - 4}{4\pi G} \left( \frac{a}{r^{2+n}} + \frac{3v^2}{n(n-1)r^2} \right), \\
\rho_1 &\sim \frac{(3 - n)(n - 2)}{n^2 - n - 4} \rho_0,
\end{align*}
\]

(3.10)
in which \( a \) and \( \frac{1}{2}(\sqrt{17} + 1) \leq n \leq 3 \) are constants. (3.10) and (3.11) provide a heuristic mass density for the dark halo in barred spiral galaxy within the range \([R_0, R_1]\).

### B. Solution to a Spiral Galaxy

In an ordinary spiral galaxy, the potentials \( \Phi_2 \) and \( \Phi_3 \) should take the following form

\[
\Phi_2 = P^{-1} \cos(\xi r + \varphi_0), \quad \Phi_3 = P^{-1} \sin(\xi r + \varphi_0),
\]

where \( \xi \) is a constant, \( \varphi_0 \) the initial phase of \( \varphi \) and \( P(r) \) a function of \( r \). Obviously we can set \( \varphi_0 = 0 \) by a translation of \( \varphi \) similar to the barred spiral case. Substituting (3.12) into (2.28), we get a linear equation for \( P \)

\[
P'' - \frac{4v}{r(v - r\Omega)}P' + \left( \xi^2 + \frac{4\Omega^2}{rv} + \frac{8\Omega}{r^2} \right) P = 0.
\]

(3.13)

The equation (3.13) is independent of \( \cos(\xi r + \varphi_0) \) and \( \sin(\xi r + \varphi_0) \), which reflects (3.12) touches the nature of the spiral structure.

The solution to (3.13) can be expressed by the Hankel function with complex parameters, which is much complicated. Here we examine the simplest case of (3.13), the static case with \( \Omega = 0 \). The analytic solution to this case can be expressed in a clear form, and it is heuristic to reveal the nature of a spiral galaxy. In the general cases, the numerical simulation is more convenient and efficient.

Setting \( \Omega = 0 \), (3.13) becomes a Bessel-like equation

\[
P'' - \frac{4v}{r}P' + \left( \xi^2 + \frac{8}{r^2} \right) P = 0.
\]

(3.14)

The solution to (3.14) is given by

\[
P = \sqrt{r^5} \left[ C_1 J_\alpha(\xi r) + C_2 J_{-\alpha}(\xi r) \right], \quad \alpha = \frac{1}{2} \sqrt{7},
\]

(3.15)

where \((C_1, C_2)\) are constants, \((J_\alpha, J_{-\alpha})\) are Bessel functions with complex parameters, which are defined by

\[
J_\nu(x) = \left( \frac{x}{2} \right)^\nu \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(\nu + k + 1)} \left( \frac{x}{2} \right)^{2k}, \quad (\nu = \pm \alpha).
\]

(3.16)

Substituting (3.15) into (2.18) and (2.22)-(2.26), we can check \((\rho_2, \rho_3)\) and speed all have spiral structure like (3.12). Again by (2.30), we find \( \Sigma \) also has spiral structure. Substituting
the results into (2.27), we get a constraint for \((\Phi_0, \Phi_1)\) similar to (3.6). Since the concrete expressions are long and complicated, and can be derived by straightforward calculation, we do not display them here.

**IV. DISCUSSION AND CONCLUSION**

In the context of general relativity, the whole dynamical equation system for the galactic evolution should be the Einstein’s field equation

\[
G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -\kappa T^{\mu\nu}, \quad (\kappa \equiv \frac{8\pi G}{c^4}), \tag{4.1}
\]

combined with the energy-momentum conservation law and the equation of state of the gravitating source. For the spinors with interactions, the classical approximation gives

\[
T^{\mu\nu} = (\rho_{\text{tot}} + P)U^\mu U^\nu + (W - P)g^{\mu\nu}, \tag{4.2}
\]

where \(W\) is a potential corresponding to the interaction terms, which acts as negative pressure. According to the energy-momentum conservation law or Bianchi identity \(T^{\mu\nu}_{;\nu} = 0\), we can derive the continuity equation \(U_\mu T^{\mu\nu}_{;\nu} = 0\) and the equation of motion for the source as follows

\[
U^\mu \partial_\mu (\rho_{\text{tot}} + W) = -(\rho_{\text{tot}} + P)U^\mu_{;\mu}, \tag{4.3}
\]

\[
(\rho_{\text{tot}} + P)U^\mu U^\nu_{;\nu} = (g^{\mu\nu} - U^\mu U^\nu) \partial_\nu (P - W). \tag{4.4}
\]

For the nonlinear dark spinors, we have \(W \sim \rho_{\text{tot}} \gg P\). In this case, by (4.4) we find the stream lines of the spinors are quite different from the geodesics \(U^\nu U_{\mu;\nu} = 0\). So unless the nature of the dark matter is disclosed, a fully relativistic simulation for the dynamics of galaxy is impossible. In (2.1), the effects of \((P, W)\) are merged into one effective mass-energy density \(\rho\), so the treatment is much simplified.

The above procedure provides a method to research the structure of a galaxy via dynamical approach, which naturally connect the hydrodynamics with empirical data. The above solutions provide manifest analytic relations and functions for a spiral galaxy, which are much helpful to understand the structure and property of the galaxy. These solution verifies that the spiral arms are stationary or even static stellar density wave distribution.

In the case of barred spiral galaxy with \(\Omega \neq 0\), by (3.5), we find \(\rho_2(r) = 0\) has a root \(r = R \lesssim v/\Omega\), which means the bar vanishes at radius \(r = R \sim R_1\). By (3.3), we get
ω_1(R)\equiv 0, which implies the density wave vanishes near r = R, and then a ring of stars will form. The above solution is derived under the assumption (2.21), so the effective domain of the solution is also [R_0, R_1], the effective domain of (2.21). These conclusions can be used to estimate the global angular speed

\[ \Omega \approx \frac{v}{R_1} \sim 300\text{km/s/(30kpc)} = 10\text{km/s/kpc} \sim 0.002''/\text{year}, \]

which is less than the previously estimated pattern speed 30 \sim 60\text{km/s/kpc}[24].

By (3.7) and (3.8), we learn all stars in the disc move in the almost circular orbits. This conclusion is reasonable and coincident with facts. Through some mechanisms and long-term evolution, the baryons were shifted and concentrated into the disc, and their orbits of motion became harmonious in a regular galaxy. Otherwise violent collisions will occur frequently among the moving stars, and then the galaxy become a hell.

By (3.3) and (3.5), we find \( \omega_1 > 0 \) and \( \rho_2 > 0 \) hold simultaneously. This means where the mass density of the background is larger, where the potential is lower and the stellar speed is higher. On the other hand, by (3.9) we find that the higher the stellar speed, the lower the number density. This means the mass density of the disc should be compensatory for that of the dark halo. Namely, the density of dark halo \( \rho \) should take minimum in the spiral arms where \( \Sigma \) take maximum or vice versa. This is a strange but interesting phenomenon, which reflects the complicated stream lines of the dark halo. This phenomenon might also be one reason why the dark matter and dark energy can be hardly detected in the vicinity of the solar system.

The relations (3.1) and (3.4)-(3.6) as well as (3.12) provide some information for the distribution and property of the total mass density and potential. How to combine the results with the dynamics of the background is an interesting problem, which might be a shortcut to study the properties of the dark matter and dark energy.

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V. APPENDIX: THE DERIVATION OF THE EQUATIONS (2.1) AND (2.2)

In this appendix, we derive the dynamics (2.1) and (2.2) of the Newtonian gravitational system from the Einstein’s relativistic dynamics (4.1)-(4.4) by weak-field and low-speed approximation. Some fundamental contents can be found in [17], but here we make more systematic and detailed study for galactic dynamics. For convenience, we take $c = 1$ as unit of velocity. Noticing the facts that the collisions among stars rarely occur, and the trajectories of the ordinary matter such as electrons and baryons are almost geodesics, so for the stars, the following zero-pressure and inviscid energy-momentum tensor holds

$$T^\mu_\nu^s = \rho_s U^\mu U_\nu,$$  \hfill (5.1)

in which $\rho_s$ is the comoving mass density of the stars, and $U^\mu$ is the 4-vector speed of the stellar flow. Since the ordinary matter satisfies the mass-energy conservation law independent of the dark halo, we have $T^\mu_\nu^s = 0$. Expressing it in the form of equations of continuity and motion, we get the dynamical equations for the stars

$$U^\mu \partial_\mu \rho_s + \rho_s U^\mu _;^\mu = 0, \quad U^\nu U^\mu _;^\mu = 0.$$  \hfill (5.2)

The total energy-momentum tensor of the galaxy is still given by (4.2), and satisfies the
dynamical equations (4.3) and (4.4). By (4.1) and (4.2), we get

$$R = \kappa(\rho_{tot} + 4W - 3P),$$  \hspace{1cm} (5.3)

where $R = g_{\mu\nu}R^{\mu\nu}$ is the scalar curvature. Substituting (5.3) into (4.1), we get

$$R^{\mu\nu} = -\kappa(\rho_{tot} + P)U^\mu U^\nu + \frac{1}{2}\kappa(\rho_{tot} + 2W - P)g^{\mu\nu},$$  \hspace{1cm} (5.4)

where $U^\mu$ is the average 4-vector speed of all gravitating source.

In order to make weak-field approximation, we choose the harmonic coordinate system, which leads to usual Cartesian coordinate system when making linearization of metric. Then we have the de Donder coordinate condition

$$\Gamma^\mu \equiv g^{\alpha\beta}\Gamma^\mu_{\alpha\beta} = -\frac{1}{\sqrt{g}}\partial_\nu(\sqrt{g}g^{\mu\nu}) = 0,$$  \hspace{1cm} (5.5)

where $g = | \text{det}(g) |$. Denote the Minkowski metric by

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1).$$  \hspace{1cm} (5.6)

For weak-field approximation, we have the linearization for the metric

$$g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu}, \quad g^{\mu\nu} \equiv \eta^{\mu\nu} - h^{\mu\nu},$$  \hspace{1cm} (5.7)

$$h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}, \quad h = h^{\mu\mu} = \eta^{\mu\nu}h_{\mu\nu},$$  \hspace{1cm} (5.8)

$$g \doteq 1 + h, \quad \sqrt{g} \doteq 1 + \frac{1}{2}h.$$  \hspace{1cm} (5.9)

In what follows, we directly use $= \doteq$ to replace $\doteq$. By straightforward calculation, we get the linearization for other parameters

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}\eta^{\mu\nu}(\partial_\alpha h_{\nu\beta} + \partial_\beta h_{\alpha\nu} - \partial_\nu h_{\alpha\beta}),$$  \hspace{1cm} (5.10)

$$\Gamma^\mu = \partial_\nu(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h),$$  \hspace{1cm} (5.11)

$$R_{\mu\nu} = \frac{1}{2}\partial_\alpha \partial^\alpha h_{\mu\nu} - \frac{1}{2}(\eta_{\mu\lambda}\partial_\nu\Gamma^\lambda + \eta_{\nu\alpha}\partial_\mu\Gamma^\alpha),$$  \hspace{1cm} (5.12)

$$R^{\mu\nu} = \frac{1}{2}\partial_\alpha \partial^\alpha h^{\mu\nu} - \frac{1}{2}(\eta^{\mu\alpha}\partial_\nu\Gamma^\alpha + \eta^{\nu\alpha}\partial_\mu\Gamma^\alpha),$$  \hspace{1cm} (5.13)

$$R = \frac{1}{2}\partial_\alpha \partial^\alpha h - \partial_\alpha \Gamma^\alpha.$$  \hspace{1cm} (5.14)

In the harmonic coordinate system, we have

$$\Gamma^\mu = \partial_\nu(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h) = 0,$$  \hspace{1cm} (5.15)

$$R_{\mu\nu} = \frac{1}{2}\partial_\alpha \partial^\alpha h_{\mu\nu}, \quad R^{\mu\nu} = \frac{1}{2}\partial_\alpha \partial^\alpha h^{\mu\nu},$$  \hspace{1cm} (5.16)

$$R = \frac{1}{2}\partial_\alpha \partial^\alpha h, \quad G^{\mu\nu} = \frac{1}{2}\partial_\alpha \partial^\alpha (h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h).$$  \hspace{1cm} (5.17)
By (5.15) and (5.17) we find if $\Gamma^\mu = 0$ at any given time $t = t_0$, it will always hold due to the Bianchi identity $G^{\mu \nu} = 0$.

In order to compare with electromagnetism and to understand the physical meaning of the parameters, denote

$$\Phi = \frac{1}{2}h_{tt} = \frac{1}{2}h^{tt}, \quad \vec{A} = (h^{tx}, h^{ty}, h^{tz}) = -(h_{tx}, h_{ty}, h_{tz}), \quad (5.18)$$

$$H = (h_{ab}) = (h^{ab}), \quad \{a, b\} \in \{1, 2, 3\}, \quad \vec{B} = \nabla \times \vec{A}. \quad (5.19)$$

In the International System of Units, we have the order of magnitude for the components of metric

$$c^2|h_{ab}| \sim c|A_k| \sim |\Phi| \ll 1, \quad (a \neq b), \quad (5.20)$$

which means $|h_{ab}| \ll |A_k| \ll |\Phi| \ll 1$ if taking $c = 1$ as unit.

For the present purpose, we define the stellar speed $\vec{V}$ by

$$\vec{V} \equiv \frac{1}{U_0}(U^1, U^2, U^3), \quad (5.21)$$

which is approximately equivalent to the usual definition. For galaxies, we have the following order of magnitude

$$|\vec{V}| \sim 300 \text{km/s} = 10^{-3}c, \quad \vec{A} \sim \kappa \vec{V}, \quad h_{ab} \sim \kappa |\vec{V}|^2, \quad (a \neq b), \quad (5.22)$$

in which the coefficient $\kappa$ is also a number of little value. Then according to

$$1 = \sqrt{g_{\mu \nu}U^\mu U^\nu} = (1 + 2\Phi - 2\vec{A} \cdot \vec{V} + g_{ab}V^a V^b) \frac{1}{2}U^0, \quad (5.23)$$

by omitting $O(V^2)$ terms, the low-speed assumption gives

$$U^0 = 1 - \Phi + \vec{A} \cdot \vec{V}. \quad (5.24)$$

Substituting (5.21) and (5.24) into (5.2) and omitting the high order terms, we get the continuity equation and motion equation for stars

$$(\partial_t + \vec{V} \cdot \nabla) \rho_s = -\rho_s[\nabla \cdot \vec{V} + (\partial_t \Phi + \nabla \cdot \vec{A})], \quad (5.25)$$

$$(\partial_t + \vec{V} \cdot \nabla) \vec{V} = -\nabla \Phi + (-\partial_t \vec{A} + \vec{V} \partial_t \Phi) + \vec{V} \times \vec{B} + \vec{V} \cdot \partial_t H. \quad (5.26)$$

In (5.25), we used the de Donder condition $\Gamma^0 = 0$ in the form

$$\frac{1}{2} \partial_t (h_{xx} + h_{yy} + h_{zz}) = -(\partial_t \Phi + \nabla \cdot \vec{A}). \quad (5.27)$$
The equation of motion (5.26) has a similar structure to the electrodynamics. From it we learn that, $\Phi$ gives the Newtonian gravitational potential, and $\vec{A}$ leads to gravimagnetic field $\vec{B}$. By (5.22), the zeroth order approximations of (5.25) and (5.26) are just (2.29) and (2.2) respectively.

By (5.3) and (5.17), we have

$$\partial_\alpha \partial^\alpha h = 2\kappa (\rho_{\text{tot}} + 4W - 3P). \tag{5.28}$$

By (5.28), (5.4) and (5.16), we get the dynamical equations for $h^{\mu\nu}$

$$\partial_\alpha \partial^\alpha h^{\mu\nu} = -2\kappa (\rho_{\text{tot}} + P)U^\mu U^\nu + \kappa (\rho_{\text{tot}} + 2W - P)\eta^{\mu\nu}, \tag{5.29}$$

$$\partial_\alpha \partial^\alpha \chi^{\mu\nu} = -2\kappa [(\rho_{\text{tot}} + P)U^\mu U^\nu + (W - P)\eta^{\mu\nu}], \tag{5.30}$$

where $\chi^{\mu\nu} = h^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} h$. If the average speed of the dark halo is also small, omitting $O(\vec{U}^2)$ from (5.29) we get $h^{xx} = h^{yy} = h^{zz} \equiv 2\Phi$, $h^{ab} = 0, (a \neq b)$ and

$$\partial_\alpha \partial^\alpha \Phi = \frac{1}{2} \partial_\alpha \partial^\alpha h^{00} = -4\pi G \rho, \tag{5.31}$$

$$\partial_\alpha \partial^\alpha \Psi = \frac{1}{2} \partial_\alpha \partial^\alpha h^{kk} = -4\pi G \tilde{\rho}, \tag{5.32}$$

where $\rho$ and $\tilde{\rho}$ are the effective mass density. Their zeroth order approximation gives

$$\rho = \rho_{\text{tot}} [2(\mathcal{U}^0)^2 - 1] - 2W + P[2(\mathcal{U}^0)^2 + 1] \approx \rho_{\text{tot}} - 2W + 3P. \tag{5.33}$$

$$\tilde{\rho} \approx \rho_{\text{tot}} + 2W - P. \tag{5.34}$$

The equation (5.31) is just (2.1). For the dark matter or dark energy with large enough negative pressure, by (5.33) we may even have $\rho < 0$, which means the Newtonian gravity becomes repulsive in this case. So detecting the dynamical behavior of the galactic dark halo may be a shortcut to investigate the weird properties of dark matter or energy.