On Perturbation theory for the Fokker–Planck equation

Francisco M. Fernández
INIFTA (UNLP, CCT La Plata–CONICET), División Química Teórica, Blvd. 113 S/N, Sucursal 4, Casilla de Correo 16, 1900 La Plata, Argentina
E-mail: fernande@quimica.unlp.edu.ar
Abstract. We discuss a recent application of the Modified Homotopy Perturbation Method to the Fokker–Planck equation and show that the selected examples do not have any connection with actual physical problems.

1. Introduction

In a recent paper Jafari and Aminataei [1] proposed the application of the homotopy perturbation method (HPM) to the Fokker–Planck equation. More precisely, they applied a modified HPM (MHPM) to several examples and concluded that “we infer that our new MHPM and its coincident HPM is a powerful tool for solving a Fokker–Planck equation.” The purpose of this paper is to discuss those results. In Sec. 2 we outline the problem. In Sec. 3 we briefly consider the perturbation method proposed by the authors. In Sec. 4 we examine each example and its result. In Sec. 5 we analyze the authors’ conclusions about the convergence of their method. Finally, in Sec. 6 we draw our own general conclusions.

2. The Fokker–Planck equation

Jafari and Aminataei [1] discussed Fokker–Planck equations of the form
\[
\frac{\partial u(x,t)}{\partial t} = - \left[ \sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i(x,t) + \sum_{i,j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j} B(x,t) \right] u(x,t), \quad u(x,0) = f(x) \tag{1}
\]
where \( x = (x_1, x_2, \ldots, x_N) \). The Fokker–Planck equation describes the time evolution of the probability density of the position of a particle or other observables. Therefore, \( u(x,t) \) should be an integrable function that satisfies the boundary conditions of the problem. In general it is not easy to solve Eq. (1) for a given plausible initial condition \( f(x) \).

Many textbooks discuss applications of the Fokker–Planck and diffusion equations; one of the simplest illustrative examples is given by [2]
\[
\frac{\partial C(x,t)}{\partial t} = D \frac{\partial^2 C(x,t)}{\partial x^2} \tag{2}
\]
On Perturbation theory for the Fokker–Planck equation

where $C(x, t)$ is the concentration of the species and $D$ is a constant diffusion coefficient. If we assume that $C(x, 0) = C_0\delta(x)$, where $\delta(x)$ is the Kronecker delta function, then

$$C(x, t) = \frac{C_0}{2(\pi D t)^{1/2}} \exp\left(-\frac{x^2}{4Dt}\right)$$  \hspace{1cm} (3)

3. The homotopy story

In order to solve the Fokker–Planck equation (1) Jafari and Aminataei [1] resorted to the HPM. Basically it consists of introducing a parameter $p$ into the differential equation and expanding the solution of the resulting equation $H(v, p) = 0$ in a Taylor series about $p = 0$: $v = v_0 + v_1p + \ldots$. In other words, HPM is just an ordinary perturbation approach with a fancy name. However, Jafari and Aminataei [1] went further and proposed a modified homotopy perturbation method (MHPM) that they described as follows: “We can extend $H(v, p)$ to $H(v, c, p)$. In the MHPM, we assume that the solution of $H(v, c, p) = 0$ can be written as a power series in $p$ and $c$; i.e. $v = c_0v_0 + c_1v_1p + \ldots$, where $c = [c_1, c_2, \ldots]$. Adomian decomposition and spectral methods are special cases of this homotopy (means MHPM).” We clearly appreciate two facts: first, there is no power series in $c$ and, second, if we redefine the perturbation corrections as $\tilde{v}_j = c_jv_j$ then we have the ordinary HPM because the perturbation corrections are determined by the boundary conditions of the problem.

4. The experiments

Jafari and Aminataei [1] conducted several experiments to show the performance of this new approach. In what follows we discussed them.

Experiment 1:

The authors chose the trivial nonlinear differential equation $\partial u/\partial x + u^2 = 0$, $u(0) = 1$ that admits the exact solution $u(x) = 1/(1 + x)$. Notice that this equation does not have any connection whatsoever with the Fokker–Planck one [1]. Therefore, it is unlikely that this example gives us any clue about the performance of the HPM on
On Perturbation theory for the Fokker–Planck equation

After solving the HPM equations they realized that the perturbation corrections $v_j = (-x)^j$ are merely the terms of the Taylor expansion of the solution about $x = 0$. Of course, if one substitutes the Taylor series $u(x) = 1 + u_1 x + u_2 x^2 + \ldots$ into the differential equation one easily obtains a recurrence relation for the coefficients:

$$u_{n+1} = -\frac{1}{n+1} \sum_{j=0}^{n} u_j u_{n-j}, \quad n = 0, 1, \ldots, u_0 = 1$$  (4)

It seems that the good old Taylor approach is more efficient than the HPM and provides a more compact expression for the solution.

**Experiment 2:**

The second example is given by

$$\frac{\partial u}{\partial t} = \hat{L} u, \quad \hat{L} = \frac{\partial}{\partial x} + \frac{\partial^2}{\partial x^2}, \quad u(x, 0) = x$$  (5)

where we have introduced the linear operator $\hat{L}$ in order to facilitate the following discussion. The first question that one may ask is: what is the physical meaning of that initial condition? Obviously, $u(x, 0)$ is unbounded and cannot be a good candidate for a probability density. It is not difficult to guess why the authors chose it: the solution follows straightforwardly from $u(x, t) = e^{t\hat{L}} u(x, 0) = x + t$. In other words, you cannot miss it because $\hat{L}^n u(x, 0) = 0$ for all $n > 1$. It is not surprising that the solution is also unbounded and does not resemble any physical probability density, like, for example, Eq. (3).

**Experiment 3:**

In this case the authors chose coefficients $A$ and $B$ that are rather complicated and arbitrary functions of $x$ and $t$. There is no physical justification for their choice or for the initial condition, except that the differential equation admits the exact solution $u(x, t) = \sinh(x)e^t$. In this case the authors managed to derive the Taylor series for $e^t$ about $t = 0$. Once again we are in front of a solution with no physical meaning whatsoever.

For brevity I will summarize the results of the remaining experiments. **Experiment 4:** Taylor series about $t = 0$ of $u(x, t) = xe^t$. **Experiment 5:** Taylor series about $t = 0$
of $x^2e^t$. **Experiment 6:** Taylor series about $t = 0$ of $(1 + x)e^t$. We appreciate that the authors merely chose tailor–made toy problems fabricated with the only purpose of obtaining the known answer. Besides, their HPM always yielded the Taylor series that one derives straightforwardly as illustrated in **Experiment 1**.

5. Convergence

The authors also tried to prove the convergence of their method. However, they simply copied the equations for the Taylor series shown in most textbooks on calculus and did not arrive to any useful conclusion. For example, they failed to tell the reader that the Taylor series for $1/(1 + x)$ converges for $|x| < 1$ or that the expansion for $e^t$ converges for all $t$.

6. Conclusions

I think that after the short discussion given above it is unnecessary to add that the paper of Jafari and Aminataei [1] should not have been published in the first place. It may sound politically incorrect but we think that is high time to leave euphemisms aside and discuss seriously what is happening in today’s science. There has recently been far too many such papers published elsewhere [3–16]. We will summarize the main results for the reader’s benefit. For example, Chowdhury and Hashim [4] applied the HPM to obtain the Taylor series about $x = 0$ of the functions $y(x) = e^{x^2}$, $y(x) = 1 - x^3/3!$, $y(x) = \sin(x)/x$, and $y(x) = x^2 + x^8/72$. By means of the same method Chowdhury et al [5] derived the Taylor series about $t = 0$ for the solutions of the simplest population models. Bataineh et al [3] went a step further and resorted to the even more powerful homotopy analysis method (HAM) and calculated the Taylor expansions about $x = 0$ of the following two–variable functions: $y(x, t) = e^{x^2+\sin t}$, $y(x, t) = x^2 + e^{x^2+t}$, $y(x, t) = x^3 + e^{x^2-t}$, $y(x, t) = t^2 + e^{x^3}$, $y(x, t) = -2 \ln(1 + tx^2)$ and $y(x, t) = e^{-tx^2}$. Zhang et al [15] derived the first two terms of the Taylor expansion about $t = 0$ of $u(x, t) = -2 \sec h^2[(x - 2t)/2]$, and $u(x, t) = -(15/8) \sec h^2[(x - 5t/2)]$. 
Bataineh et al. [9] modified HAM to produce MHAM and found the Taylor expansions of the functions \( u(t) = t^2 - t^3 \) and \( u(t) = 1 + t^2/16 \). Sadighi and Ganji [14] calculated the Taylor expansions about \( t = 0 \) of \( u(x, t) = 1 + \cosh(2x)e^{-4it} \) and \( u(x, t) = e^{3i(x+3t)} \) by means of HPM and Adomian decomposition method (ADM), and verified that the results were exactly the same. By means of HPM Rafiq et al. [13] also derived polynomial functions like \( y(x) = x^4 - x^3, \ y(x) = x^2 + x^3 \) and \( y(x) = x^2 + x^8/72 \). Özis and Agirseven [12] expanded \( u(x, t) = x^2e^t, \ u(x, y, t) = y^2 \cosh t + x^2 \sinh t \) (and other such functions) about \( t = 0 \) by means of the HPM. Bataineh et al. [8] used HAM to obtain expansions about \( x = 0 \) for \( w(x, t) = xe^{-t} + e^{-x}, \ w(x, t) = e^{x+t+t^2}, \ w(x, t) = e^{t+x^2} \) and \( w(x, t) = e^{t^2+x^2} \). They thus managed to reproduce earlier ADM results. Koçak and Yıldırım [16] applied HPM to a 3D Green’s function for the dynamic system of anisotropic elasticity.

There are many more articles where the authors applied HPM, HAM and ADM and produced results that anybody would easily obtain by means of a straightforward Taylor expansion of the model differential equations. In some cases the examples are tailor–made toy models as those discussed above, in others the Taylor series is unsuitable for an acceptable description of the physics of the system. In our opinion this unhappy situation is becoming rather preoccupying, to say the least. Notice that Frank [17] criticized a previous article on the Fokker–Planck equation in this same journal. Unfortunately, the production of such kind of papers is inexhaustible.

The reader may find the discussion of other articles elsewhere [18–29]. We recommend the most interesting case of the predator–prey model that predicts a negative number of rabbits [21].

[1] Jafari M A and Aminataei A 2009 Phys. Scr. 80 055001 (5 pp).
[2] McQuarrie D A 2000 Statistical Mechanics (University Science Books, Sausalito).
[3] Bataineh A S, Noorani M S M, and Hashim I 2007 Phys. Lett. A 371 72.
[4] Chowdhury M S H and Hashim I 2007 Phys. Lett. A 365 439.
[5] Chowdhury M S H, Hashim I, and Abdulaziz O 2007 Phys. Lett. A 368 251.
[6] Esmaeilpour M and Ganji D D 2007 Phys. Lett. A 372 33.
On Perturbation theory for the Fokker–Planck equation

[7] Rafei M, Daniali H, Ganji D D, and Pashaei H 2007 Appl. Math. Comput. 188 1419.
[8] Bataineh A S, Noorani M S M, and Hashim I 2008 Phys. Lett. A 372 613-618.
[9] Bataineh A S, Noorani M S M, and Hashim I 2008 Phys. Lett. A 372 4062.
[10] Chowdhury M S H and Hashim I 2008 Phys. Lett. A 372 1240.
[11] Inc M 2008 Phys. Lett. A 372 356-360.
[12] Öziş T and Agirseven D 2008 Phys. Lett. A 372 5944.
[13] Rafiq A, Ahmed M, and Hussain S 2008 Phys. Lett. A 372 4973.
[14] Sadighi A and Ganji D D 2008 Phys. Lett. A 372 465.
[15] Zhang B-G, Li S-Y, and Liu Z-R 2008 Phys. Lett. A 372 1867-1872.
[16] Kocak H and Yıldırım A 2009 Phys. Lett. A 373 3145.
[17] Frank T D 2008 Phys. Scr. 78 067001 (2pp).
[18] Fernández F M, Perturbation Theory for Population Dynamics, arXiv:0712.3376v1
[19] Fernández F M, On Some Perturbation Approaches to Population Dynamics, arXiv:0806.0263v1
[20] Fernández F M, On the application of homotopy-perturbation and Adomian decomposition
  methods to the linear and nonlinear Schrödinger equations, arXiv:0808.1515v1
[21] Fernández F M, On the application of the variational iteration method to a prey and predator
  model with variable coefficients, arXiv.0808.1875v2
[22] Fernández F M, On the application of homotopy perturbation method to differential equations,
  arXiv:0808.2078v2
[23] Fernández F M, Homotopy perturbation method: when infinity equals five, 0810.3318v1
[24] Fernández F M 2009 Phys. Scr. 79 055003 (2pp.).
[25] Fernández F M, Amazing variational approach to chemical reactions, arXiv:0906.0950v1
  [physics.chem-ph]
[26] Fernández F M, On the homotopy perturbation method for Boussinesq-like equations,
  arXiv:0907.4481v1 [math-ph]
[27] Fernández F M, Perturbation approaches and Taylor series, arXiv:0910.0149v1 [math-ph]
[28] Fernández F M, On a simple approach to nonlinear oscillators, arXiv:0910.0600v1 [math-ph]
[29] Fernández F M 2009 Appl. Math. Comput. 215 168.