Gluing pseudofunctors via \( n \)-fold categories

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Abstract

Gluing of two pseudofunctors has been studied by Deligne, Ayoub, and others in the construction of extraordinary direct image functors in étale cohomology, stable homotopy, and mixed motives of schemes. In this article, we study more generally the gluing of finitely many pseudofunctors. With the help of \( n \)-fold categories, we organize gluing data for \( n \) pseudofunctors into 2-categories and establish general criteria for the equivalence of such 2-categories. Results of this article are used in \cite{20} to construct extraordinary direct image functors in étale cohomology of Deligne-Mumford stacks.

Introduction

The extraordinary direct image functor \( Rf_{!} \), one of Grothendieck’s six operations, between derived categories of étale sheaves, was constructed in SGA 4 XVII \cite{6}. Here \( f \) is a morphism of schemes, compactifiable in the sense that it can be decomposed (noncanonically) into \( pj \), where \( j \) is an open immersion and \( p \) is a proper morphism. There are obvious candidates for \( Rp_{!} \) and \( Rp_{!} \), and the constructions \( j \mapsto Rf_{!} j \) and \( p \mapsto Rp_{!} p \) give rise to pseudofunctors to the 2-category of categories. To construct \( f \mapsto Rf_{!} \), Deligne developed a theory for gluing two such pseudofunctors \cite{6} Section 3 (Deligne’s original formulation uses the equivalent language of cofibered categories \cite{13} Section 8). Deligne’s theory extends to the gluing of two pseudofunctors with the same target, and has been applied to other contexts, such as the construction of \( Rf_{!} \) between triangulated categories of mixed motives \cite{5} Proposition 2.2.7. Ayoub developed a variant of Deligne’s theory \cite{1} Théorème 1.3.1 in order to construct \( Rf_{!} \) relative to stable homotopical pseudofunctors.

In this article, we study more generally the gluing of finitely many pseudofunctors with the same target. The case of three pseudofunctors is necessary for the construction of \( Rf_{!} \) and base change morphisms in \cite{20} for a morphism \( f \) of Deligne-Mumford stacks, because compactification of coarse spaces only allows us to decompose a morphism into a sequence of three morphisms.

Let \( C \) be a \((2,1)\)-category, namely a 2-category whose 2-cells are all invertible. Let \( D \) be a 2-category. For arrowy (i.e. defined by restricting morphisms, see Definition \ref{def:arrowy}) 2-subcategories \( A_{1}, \ldots, A_{n} \), we define a 2-category \( \text{GD}_{A_{1}, \ldots, A_{n}}(C, D) \), whose objects are collections \((F_{i}: A_{i} \to D)_{1 \leq i \leq n}\) of pseudofunctors endowed with some extra compatibility data between the pseudofunctors (see Remark \ref{rem:extra_compatibility} for an explicit description in the case \( n = 2 \) and Remarks \ref{rem:extra_compatibility_2} \ref{rem:extra_compatibility_3} for explicit descriptions in the general case). For \( n = 1 \), \( \text{GD}_{A_{1}}(C, D) = \text{PsFun}(A_{1}, D) \) is simply the 2-category of pseudofunctors from \( A_{1} \) to \( D \). Our main result is the following.

**Theorem.** Let \( B \) be an arrowy 2-subcategory of \( C \) containing \( A_{1} \) and \( A_{2} \). Under suitable assumptions on \( A_{1}, \ldots, A_{n}, B, C \), for every 2-category \( D \), the natural 2-functor (Remark \ref{rem:natural_2-functor} Construction \ref{const:natural_2-functor})

\[
Q_{D}: \text{GD}_{B, A_{3}, \ldots, A_{n}}(C, D) \to \text{GD}_{A_{1}, A_{2}}(C, D)
\]

is a 2-equivalence.

We refer the reader to Theorems \ref{thm:main_theorem} and \ref{thm:main_theorem_2} for more precise statements. The above theorem reduces the gluing of \( n \) pseudofunctors to the gluing of \( n - 1 \) pseudofunctors and can be applied recursively.

For \( n = 2 \) and \( B = C \), we get a 2-equivalence \( \text{PsFun}(C, D) \to \text{GD}_{A_{1}, A_{2}}(C, D) \), which is a common generalization of the results of Deligne and Ayoub. Even under their (more restrictive) hypotheses, our result has the advantage of being more precise in the sense that we establish a 2-equivalence of 2-categories, whereas previous results only dealt with objects of \( \text{GD}_{A_{1}, A_{2}}(C, D) \). This precision allows us to

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construct pseudonatural transformations such as the base change equivalence in \[20\]. We refer the reader to Remark 4.10 for more details on this comparison of results.

We study the gluing of pseudofunctors in the framework of \(n\)-fold categories. An \(n\)-fold category structure on a set of objects is an extended categorical structure consisting of \(n\) sets of morphisms (which could be considered as \(n\) directions), each endowed with its own composition law, and cells up to dimension \(n\) connecting the morphisms, usually visualized as hypercubes. This often encodes more information than a higher category structure of the same dimension, such as an \(n\)-category. There is, however, a rich interplay between extended categories and higher categories, of which the most relevant part to our study is the relation between \(n\)-fold categories and 2-categories. We construct an \(n\)-fold category \(Q_{A_1,...,A_n}\) associated to a 2-category \(C\), and a \((2,1)\)-categories \(\mathcal{Q}_{\mathcal{T}C}\) associated to an \(n\)-fold category \(C\). These constructions allow us to reduce the study of \(GD_{A_1,...,A_n}(C,D)\) for \(n \geq 2\) to the study of the \((2,1)\)-category \(\mathcal{Q}_{\mathcal{T}Q_{A_1,...,A_n}C}\).

The article is organized as follows. In Section 1, we fix some conventions and prove some preliminary results on 2-categories. In Section 2, after recalling the definition of an \(n\)-fold category, we investigate the relation between 2-categories and \(n\)-fold categories. In particular, we construct \(Q_{A_1,...,A_n}C\) and \(TC\). In Section 3, we study functorial properties of these constructions. In Sections 4 and 5, we apply these constructions to study the gluing of pseudofunctors and prove the main theorem. There are some technical differences between the two cases \(n = 2\) and \(n \geq 3\). We deal with them separately in the two sections. In Sections 6 through 8, we develop several tools for the application of the main theorem. In Sections 6 and 7, we introduce an alternative set of gluing data that only makes use of \(2\)-Cartesian squares instead of 2-commutative squares. This alternative set is easier to construct in applications. In Section 8, we check the axioms for gluing data in the case when the data are constructed by adjunction. For completeness, we give the proof of a preliminary result in Section 9.

In joint work with Yifeng Liu [17], we establish analogues of some results of this article in the \(\infty\)-categorical setting, which are used in [16] to construct Grothendieck’s six operations on Artin stacks. We remark that specific features of 2-categories have been exploited in this article and the full generality of our results cannot be deduced from [17].

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1 Preliminaries on 2-categories

In this section, we fix some conventions and notation on 2-categories and record some preliminary results. Throughout the article, we reserve the symbol \(\simeq\) for isomorphisms. The symbol will not be used for equivalences or 2-equivalences, which are instead stated verbally.

The notion of 2-categories was introduced by Ehresmann [8, note bibliographique, p. 324] and Bénabou [3, p. 3824]. We generally adopt the conventions of [4, Chapter 7] for basic concepts about 2-categories. In particular, a 2-functor is assumed to be strict and a pseudofunctor is not assumed to be strictly unital or strictly compatible with composition. A 2-equivalence is a pseudonatural functor \(F: C \rightarrow D\) such that there exist a pseudonatural functor \(G: D \rightarrow C\) and pseudonatural equivalences \(id_C \rightarrow GF\) and \(FG \rightarrow id_D\). In this case we say that \(F\) and \(G\) are 2-quasi-inverses of each other. We will use the term morphism for 1-cell of a 2-category. As in [11, Definition 1.4.26], 2-fiber products are taken in the weak sense.

Convention 1.1. Unless otherwise stated, all categories and 2-categories are assumed to be small. Big categories and big 2-categories are occasionally used as a linguistic tool to simplify the narrative. We let \(\text{Cat}\) denote the big category of categories and functors and \(\mathcal{2Cat}\) denote the big category of 2-categories and 2-functors.

Definition 1.2 and Notation 1.3 below are standard.

Definition 1.2. A \((2,1)\)-category is a 2-category whose 2-cells are invertible.
Notation 1.3. Let $C$ be a 2-category. We denote by $C^{coop}$ (resp. $C^{coop}$) the 2-category obtained from $C$ by reversing the morphisms and 2-cells (resp. 2-cells only). In other words $\text{Ob}(C^{coop}) = \text{Ob}(C)$ and, for any pair of objects $X$ and $Y$ of $C$, $C^{coop}(Y,X) = C^{op}(X,Y) = C(X,Y)^{op}$.

If $C$ is a $(2,1)$-category, inversion of the 2-cells defines an isomorphism $C \simeq C^{coop}$.

Notation 1.4. Let $C$ and $D$ be 2-categories. We denote by $2\text{Fun}(C,D)$ the 2-category of 2-functors $C \to D$. We denote by $\text{UPsFun}(C,D)$ (resp. $\text{PsFun}(C,D)$) the 2-category of strictly unital pseudofunctors (resp. pseudofunctors) $C \to D$. Morphisms of $2\text{Fun}(C,D)$, $\text{UPsFun}(C,D)$ and $\text{PsFun}(C,D)$ are pseudonatural transformations and 2-cells are modifications.

We will not use the more restrictive notion of 2-natural transformations between 2-functors.

We will need to work over a base 2-category as follows.

Definition 1.5. Let $D$ be a 2-category. A $D$-category is a pair $(C, F)$ consisting of a 2-category $C$ and a 2-functor $F : C \to D$. If $(B, E)$ and $(C, F)$ are $D$-categories, we define the 2-category of $D$-functors $D\text{Fun}((B, E), (C, F))$ to be the strict fiber at $E$ of the 2-functor $2\text{Fun}(B, C) \to 2\text{Fun}(B, D)$ induced by $F$. Objects, morphisms, 2-cells of this 2-category are called $D$-functors, $D$-natural transformations, $D$-modifications, respectively. Thus a $D$-functor $(B, E) \to (C, F)$ is a 2-functor $G : B \to C$ such that $E = FG$. If $G, H : (B, E) \to (C, F)$ are $D$-functors, a $D$-natural transformation is a pseudonatural transformation $\alpha : G \Rightarrow H$ such that $F \ast \alpha : FG \Rightarrow FH$ is id$_B$. A $D$-natural equivalence is a $D$-natural transformation $G \Rightarrow H$ that is an equivalence in $D\text{Fun}((B, E), (C, F))$. We say that a $D$-functor $G : (B, E) \to (C, F)$ is a $D$-equivalence if there exist a $D$-functor $H : (C, F) \to (B, E)$ and $D$-natural equivalences id$_C \Rightarrow GH$ and $HG \Rightarrow$ id$_B$. In this case we say that $G$ and $H$ are $D$-quasi-inverses of each other.

A $D$-equivalence $B \to C$ induces a 2-equivalence between the strict fiber categories $B_X \to C_X$ for every object $X$ of $D$.

Let us introduce a few terminologies on faithfulness.

Definition 1.6. Let $C$ and $D$ be 2-categories and let $F : C \to D$ be a pseudofunctor. We say that $F$ is 1-truncated if for every pair of objects $X$ and $Y$ of $C$, the functor

$$F_{XY} : C(X, Y) \to D(FX, FY)$$

is faithful. We say that $F$ is pseudofaithful (resp. pseudofull) if $F_{XY}$ is fully faithful (resp. essentially surjective) for all $X$ and $Y$. We say that $F$ is 2-faithful (resp. 2-fully faithful) if $F_{XY}$ is fully faithful and injective on objects (resp. an isomorphism of categories).

We say that a 2-subcategory of a 2-category is 2-faithful (resp. 2-fully faithful) if the inclusion 2-functor is 2-faithful (resp. 2-fully faithful). We say a 2-subcategory $C$ of a 2-category $D$ is arrowy if it is 2-faithful and $\text{Ob}(C) = \text{Ob}(D)$. An arrowy 2-subcategory of $D$ is thus determined by its set of morphisms.

Strict fibers of a 1-truncated 2-functor are 2-equivalent to categories.

Example 1.7. The inclusions $2\text{Fun}(C, D) \subseteq \text{UPsFun}(C, D) \subseteq \text{PsFun}(C, D)$ are 2-fully faithful.

Example 1.8. Let $F : C \to D$ be a 1-truncated 2-functor. For every 2-category $B$, the 2-functor $2\text{Fun}(B, C) \to 2\text{Fun}(B, D)$ induced by $F$ is 1-truncated. In particular, for every $D$-category $(B, E)$, the 2-category $D\text{Fun}((B, E), (C, F))$ is 2-equivalent to a category.

A 2-fully faithful $D$-functor $G : B \to C$ is a $D$-equivalence if for every object $Y$ of $C$, there exists an object $X$ of $B$ and an isomorphism $GX \to Y$ in $C$ whose image in $D$ is an identity.

Notation 1.9.

(1) Let $C$ be a set and let $D$ be a 2-category. We view $C$ as a discrete 2-category and denote by $D^{C}$ the 2-category of 2-functors $C \to D$. An object of $D^{C}$ is a map $C \to \text{Ob}(D)$. A morphism $\alpha : F \to F'$ of $D^{C}$ is a family

$$(\alpha(X) : FX \to F'X)_{X \in C}$$

of morphisms of $D$. A 2-cell $\Xi : \alpha \Rightarrow \beta$ of $D^{C}$ is a family

$$(\Xi(X) : \alpha(X) \Rightarrow \beta(X))_{X \in C}$$

of 2-cells of $D$.  

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(2) Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. We view $\text{UPsFun}(\mathcal{C}, \mathcal{D})$ and $\text{PsFun}(\mathcal{C}, \mathcal{D})$ as $\mathcal{D}^{\text{Ob}(\mathcal{C})}$-categories via the 1-truncated forgetful 2-functor $[-] : \text{PsFun}(\mathcal{C}, \mathcal{D}) \to \mathcal{D}^{\text{Ob}(\mathcal{C})}$.

The following is an immediate consequence of the definitions.

**Lemma 1.10.** Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories, let $F : \mathcal{C} \to \mathcal{D}$ be a pseudofunctor, let $H$ be an object of $\mathcal{D}^{\text{Ob}(\mathcal{C})}$, and let $\eta : |F| \Rightarrow H$ be an equivalence in $\mathcal{D}^{\text{Ob}(\mathcal{C})}$. Assume that to every morphism $f : X \to Y$ of $\mathcal{C}$ is associated a square in $\mathcal{D}$

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ \eta(X) \downarrow & \alpha_f & \downarrow \eta(Y) \\ HX & \xrightarrow{g_f} & HY \end{array}$$

where $\alpha_f$ is an invertible 2-cell. Then there exists a unique pair $(G, \epsilon)$, where $G : \mathcal{C} \to \mathcal{D}$ is a pseudofunctor and $\epsilon : F \Rightarrow G$ is a pseudonatural equivalence, such that $|G| = H$, $|\epsilon| = \eta$, $G(f) = g_f$ and $\epsilon(f) = \alpha_f$ for every morphism $f$ of $\mathcal{C}$.

Applying the lemma to the unital constraints of $F$, we obtain the following.

**Proposition 1.11.** Let $\mathcal{C}$ and $\mathcal{D}$ be 2-categories. Then the inclusion $\text{UPsFun}(\mathcal{C}, \mathcal{D}) \subseteq \text{PsFun}(\mathcal{C}, \mathcal{D})$ is a $\mathcal{D}^{\text{Ob}(\mathcal{C})}$-equivalence.

The proof of the following proposition is straightforward. For completeness we give the proof in Section 9.

**Proposition 1.12.** Let $F : \mathcal{C} \to \mathcal{D}$ be a pseudofunctor such that $|F| : \text{Ob}(\mathcal{C}) \to \text{Ob}(\mathcal{D})$ is a bijection. For every 2-category $\mathcal{E}$, we consider the $\mathcal{E}^{\text{Ob}(\mathcal{C})}$-functor

$$\Phi_{\mathcal{E}} : \text{PsFun}(\mathcal{D}, \mathcal{E}) \to \text{PsFun}(\mathcal{C}, \mathcal{E})$$

induced by $F$.

(1) If $F$ is pseudofull, then $\Phi_{\mathcal{E}}$ is 2-faithful for every 2-category $\mathcal{E}$.

(2) The following conditions are equivalent:

(a) $F$ is a 2-equivalence.

(b) There exist a pseudofunctor $G : \mathcal{D} \to \mathcal{C}$ such that $|G|$ is the inverse of $|F|$ and pseudonatural transformations $\eta : \text{id}_\mathcal{C} \Rightarrow GF$ and $\epsilon : FG \Rightarrow \text{id}_\mathcal{D}$ such that $\eta(X) = \text{id}_X$ and $\epsilon(Y) = \text{id}_Y$ for all objects $X$ of $\mathcal{C}$ and $Y$ of $\mathcal{D}$.

(c) $\Phi_{\mathcal{E}}$ is an $\mathcal{E}^{\text{Ob}(\mathcal{C})}$-equivalence for every 2-category $\mathcal{E}$.

**Construction 1.13.** For a 2-category $\mathcal{C}$, we define a category $\mathcal{OC}$ by $\text{Ob}(\mathcal{OC}) = \text{Ob}(\mathcal{C})$ and $(\mathcal{OC})(X, Y) = \pi_0(\mathcal{C}(X, Y))$ for objects $X$ and $Y$ of $\mathcal{C}$. Here $\pi_0$ is the set of connected components. For a pseudofunctor $F : \mathcal{C} \to \mathcal{D}$, there is an obvious functor $\mathcal{OC} : \mathcal{OC} \to \mathcal{OD}$. In particular, we obtain a functor $\mathcal{O} : \text{2Cat} \to \text{Cat}$. The obvious 2-functor $\mathcal{C} \to \mathcal{OC}$ exhibits $\mathcal{O}$ as a left adjoint to the inclusion functor $\text{Cat} \to \text{2Cat}$. The latter also admits a right adjoint $\text{2Cat} \to \text{Cat}$ sending a 2-category to its underlying category and a 2-functor to its underlying functor.

**Construction 1.14.** Let $\mathcal{C}$ be a 2-category. We define two $(2,1)$-categories, $\mathcal{LC}$ and $\mathcal{RC}$, such that $\text{Ob}(\mathcal{LC}) = \text{Ob}(\mathcal{RC}) = \text{Ob}(\mathcal{C})$, as follows. For objects $X$ and $Y$ of $\mathcal{C}$, $(\mathcal{RC})(X, Y)$ is the greatest subgroupoid of $\mathcal{C}(X, Y)$ and $(\mathcal{LC})(X, Y)$ the category obtained from $\mathcal{C}(X, Y)$ by inverting all morphisms of $\mathcal{C}(X, Y)$. (Section 1.1) there is no set-theoretic issue by Convention 111. If we let $(2,1)$Cat denote the category of $(2,1)$-categories and 2-functors, we obtain functors $\mathcal{L}, \mathcal{R} : \text{2Cat} \to (2,1)\text{Cat}$. The obvious 2-functor $\mathcal{C} \to \mathcal{LC}$ and the inclusion 2-functor $\mathcal{RC} \to \mathcal{C}$ exhibit $\mathcal{L}$ and $\mathcal{R}$ as left and right adjoints of the inclusion 2-functor $\mathcal{C} \to (2,1)\text{Cat} \to \text{2Cat}$, respectively.

Moreover, the obvious 2-functor $\mathcal{C} \to \mathcal{LC}$ and the inclusion 2-functor $\mathcal{RC} \to \mathcal{C}$ induce isomorphisms of 2-categories

$$\text{PsFun}(\mathcal{LC}, \mathcal{D}) \to \text{PsFun}(\mathcal{C}, \mathcal{D}), \quad \text{PsFun}(\mathcal{D}, \mathcal{RC}) \to \text{PsFun}(\mathcal{D}, \mathcal{C}),$$

for every $(2,1)$-category $\mathcal{D}$. For any 2-category $\mathcal{D}$, the 2-functor $\text{PsFun}(\mathcal{LC}, \mathcal{D}) \to \text{PsFun}(\mathcal{C}, \mathcal{D})$ identifies $\text{PsFun}(\mathcal{LC}, \mathcal{D})$ with the 2-fully faithful subcategory of $\text{PsFun}(\mathcal{C}, \mathcal{D})$ spanned by pseudofunctors that factor through $\mathcal{RD}$. 

4
2 2-categories and n-fold categories

In this section, after recalling the definitions of n-fold categories and n-fold functors in Definition 2.2, we investigate the relation between 2-categories and n-fold categories. The obvious functor \( \rho^* \) (Example 2.5 [2]) from the big category of categories to the big category of n-fold categories admits a left adjoint \( T \) (Remark 2.11). The goal of this section is to establish an analogue of this adjunction for 2-categories and pseudofunctors (Proposition 2.25). To formulate such an analogue, we construct an analogue \( Q \) (Definition 2.10) of \( \rho^* \), an analogue \( T \) (Definition 2.12) of \( T \), and the 2-category \( \text{PsFun}(\mathcal{C}, \mathcal{D}) \) (Definition 2.21) of pseudofunctors from an n-fold category \( \mathcal{C} \) to a 2-category \( \mathcal{D} \). These constructions will play an essential role in the study of 2-categories of gluing data in Sections 4 and 5.

The notion of n-fold categories was introduced by Ehresmann [11, Définition 15, p. 396]. His original definition proceeds by induction on \( n \). For our purpose, it is more convenient to adopt a direct combinatorial definition: an n-fold category is a collection of \( (\mathcal{C}_I)_I \) of sets, where \( I \) runs through subsets of \( \{1, \ldots, n\} \) and \( \mathcal{C}_I \) can be visualized as a set of \( I \)-hypercubes, endowed with various sources, targets, units, and compositions. To specify the compatibility between these data, it is convenient to introduce the following notation. Note that the map carrying \([0]\) to \([1]\) is a bijection from the set of subsets of \([1, \ldots, n]\) onto \([0, 1]^n\).

**Notation 2.1.** For \( m \geq 0 \), we let \([m]\) denote the totally ordered set \([0, 1, \ldots, m]\). We let \( \mathcal{I} \) denote the category whose objects are \([0]\) and \([1]\) and whose morphisms are increasing maps. In other words, the morphisms of \( \mathcal{I} \) are the identity maps, the face maps \( d_0, d_1 : [0] \to [1] \) and the degeneracy map \( s : [1] \to [0] \). We denote by \( \text{Ob}(\mathcal{C}) = \mathcal{C}([0]) \) and \( \text{Mor}(\mathcal{C}) = \mathcal{C}([1]) \) endowed with a composition map

\[
\mathcal{C}([1]) \times_{\mathcal{C}(d_1), \mathcal{C}([0]), \mathcal{C}(d_0)} \mathcal{C}([1]) \to \mathcal{C}([1]).
\]

We omit the maps \( \mathcal{C}(d_0) \) and \( \mathcal{C}(d_1) \) when no confusion arises.

For \( n \geq 0 \), we identify objects of \( \mathcal{I}^n \) with elements of \( \{0, 1\}^n \) and let \( \epsilon_i \) denote the element such that \( \epsilon_i(i) = 1 \) and \( \epsilon_i(j) = 0 \) for \( j \neq i \).

The following definition is similar to [10] Definition 2.2 [1].

**Definition 2.2.** An n-fold category is a functor \( \mathcal{C} : (\mathcal{I}^n)^{\text{op}} \to \text{Set} \) endowed with a composition map

\[
\circ^i : \mathcal{C}(\alpha') \times_{\mathcal{C}(\alpha)} \mathcal{C}(\alpha') \to \mathcal{C}(\alpha')
\]

for every \( 1 \leq i \leq n \) and every pair \((\alpha, \alpha')\) of objects of \( \mathcal{I}^n \) satisfying \( \alpha' = \alpha + \epsilon_i \), such that the following axioms hold:

1. (1) For all \( i \) and \((\alpha, \alpha')\) as above, the composite \( \mathcal{I}^{\text{op}} \xrightarrow{\mathcal{I}^i} (\mathcal{I}^n)^{\text{op}} \xrightarrow{C} \text{Set} \), where \( \iota \) is the obvious functor carrying \([0]\) to \( \alpha \) and 1 to \( \alpha' \), is a category.

2. (functoriality) For every \( 1 \leq i \leq n \) and every morphism \( \beta \to \alpha \) in \( \mathcal{I}^n \) satisfying \( \alpha_i = \beta_i = 0 \), the diagram

\[
\begin{array}{ccc}
\mathcal{C}(\alpha') & \xrightarrow{\circ^i} & \mathcal{C}(\alpha') \\
\downarrow & & \downarrow \\
\mathcal{C}(\beta') & \xrightarrow{\circ^i} & \mathcal{C}(\beta'),
\end{array}
\]

where \( \alpha' = \alpha + \epsilon_i \), \( \beta' = \beta + \epsilon_i \), commutes.

3. (interchange law) For all \( 1 \leq i < j \leq n \) and every object \( \alpha \) of \( \mathcal{I}^n \) satisfying \( \alpha_i = \alpha_j = 0 \), the diagram

\[
\begin{array}{ccc}
\mathcal{C}(\alpha') & \xrightarrow{\circ^i \times \circ^j} & \mathcal{C}(\alpha') \\
\downarrow & & \downarrow \circ^i \\
\mathcal{C}(\alpha') & \xrightarrow{\circ^i} & \mathcal{C}(\alpha')
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{C}(\alpha') & \xrightarrow{\circ^j \times \circ^i} & \mathcal{C}(\alpha') \\
\downarrow & & \downarrow \circ^j \\
\mathcal{C}(\alpha') & \xrightarrow{\circ^j} & \mathcal{C}(\alpha')
\end{array}
\]

\footnote{Our axiom [2] appears to be missing in [10] Definition 2.2.}
commutes. Here $\alpha' = \alpha + \epsilon_i, \bar{\alpha} = \alpha + \epsilon_j, \bar{\alpha}' = \alpha + \epsilon_i + \epsilon_j$, and $X$ is the limit of the diagram

\[
\begin{array}{ccc}
\mathbb{C}(\bar{\alpha}') & \longrightarrow & \mathbb{C}(\bar{\alpha}) \\
\downarrow & & \downarrow \\
\mathbb{C}(\bar{\alpha}) & \longrightarrow & \mathbb{C}(\alpha') \\
\end{array}
\]

Elements of the set $\text{Ob}(\mathbb{C}) = \mathbb{C}(\mathbf{0})$, where $\mathbf{0} = (0, \ldots, 0)$, are called objects of $\mathbb{C}$. Elements of $\mathbb{C}(\chi_I)$ are sometimes called $I$-morphisms of $\mathbb{C}$.

An $n$-fold functor $\mathbb{C} \to \mathbb{D}$ between $n$-fold categories $\mathbb{C}$ and $\mathbb{D}$ is a natural transformation compatible with the composition maps. We let $n\text{FoldFun}(\mathbb{C}, \mathbb{D})$ denote the set of $n$-fold functors from $\mathbb{C}$ to $\mathbb{D}$.

We let $n\text{FoldCat}$ denote the category of $n$-fold categories and $n$-fold functors.

**Example 2.3.** A 2-fold category is called a double category [71, Définition 10, p. 389]. A double category $\mathbb{C}$ consists of a set $\text{Ob}(\mathbb{C}) = \mathbb{C}(0, 0)$ of objects, a set $\text{Hor}(\mathbb{C}) = \mathbb{C}(1, 0)$ of horizontal morphisms, a set $\text{Ver}(\mathbb{C}) = \mathbb{C}(0, 1)$ of vertical morphisms, a set $\text{Sq}(\mathbb{C}) = \mathbb{C}(1, 1)$ of squares and is equipped with various sources, targets, and associative and unital compositions. We will sometimes use the notation $h = 1$ and $v = 2$.

**Remark 2.4.** Fiore and Paoli defined the $n$-fold nerve functor, which is a fully faithful functor from the category $n\text{FoldCat}$ to the big category of $n$-simplicial sets [10, Definition 2.14, Proposition 2.17]. As in the case $n = 1$ (see for example [18, Proposition 1.1.2.2]; see also [14, Proposition VI.2.2.3]), the essential image of the functor consists of $n$-simplicial sets satisfying the unique right lifting property with respect to the inclusions

$$\Lambda^m_k \boxtimes \boxtimes_{1 \leq j \leq n} \Delta^{m_j} \subseteq \Delta^m \boxtimes \boxtimes \Delta^{m_n},$$

$1 \leq i \leq n, 0 < k < m_i, m_1, \ldots, m_n \geq 0$. Here $\Lambda^m_k$ denotes the $k$-th horn in the $m_i$-simplex. Multisimplicial sets satisfying suitable lifting properties play an essential role in the theory of gluing functors between $\infty$-categories developed in [17].

**Notation 2.5.** Let $\mathcal{C}_1, \ldots, \mathcal{C}_n$ be categories. As in [10, Definition 2.9], we let $\mathcal{C}_1 \boxtimes \cdots \boxtimes \mathcal{C}_n$ denote the external product, which is the $n$-fold category $\mathbb{C}$ given by $\mathbb{C}(\alpha) = \prod_{1 \leq i \leq n} \mathcal{C}_i(\alpha_i)$ with $\phi^i$ given by the composition in $\mathcal{C}_i$. An object of $\mathcal{C}_1 \boxtimes \cdots \boxtimes \mathcal{C}_n$ is a collection $(X_1, \ldots, X_n)$ of objects $X_i$ of $\mathcal{C}_i$.

**Remark 2.6.** Let $\mathbb{C}, \mathbb{D}$ be $n$-fold categories. The $n$-fold functors $\mathbb{C} \to \mathbb{D}$ can be organized into an $n$-fold category $\mathbb{F}$ as follows. For $\alpha \in \mathbb{F}^i$, $\mathbb{F}(\alpha)$ is the set of collections of maps

$$(F_\beta : \mathbb{C}(\beta) \to n\text{FoldFun}((\alpha_1 \times \beta_1) \boxtimes \cdots \boxtimes (\alpha_n \times \beta_n), \mathbb{D}))_{\beta \in \mathbb{F}^n},$$

functorial in $\beta$ and compatible with compositions. Since $\mathbb{D}(\alpha) \simeq n\text{FoldFun}(\alpha_1 \boxtimes \cdots \boxtimes \alpha_n, \mathbb{D})$, $n\text{FoldFun}(\mathbb{C}, \mathbb{D}) \simeq \mathbb{F}(0, \ldots, 0)$. In the sequel we will not use this structure of $n$-fold category on the set $n\text{FoldFun}(\mathbb{C}, \mathbb{D})$.

**Definition 2.7.** Let $\phi : \{1, \ldots, m\} \to \{1, \ldots, n\}$ be a map and let $\mathbb{C}$ be an $n$-fold category. We define an $m$-fold category $\phi^* \mathbb{C}$ satisfying $(\phi^* \mathbb{C})(\epsilon_i) = \mathbb{C}(\phi(i))$ by $(\phi^* \mathbb{C})(\alpha) = n\text{FoldFun}(\boxtimes_{j=1}^n \prod_{\phi(i)=j} \alpha_i, \mathbb{C})$ for all $\alpha \in \mathbb{I}^m$. For $1 \leq i \leq m$, $\phi^i$ in $\phi^* \mathbb{C}$ is given by $\phi^i \in \mathbb{C}$. More formally, for $\alpha, \alpha' \in \mathbb{I}^m$ satisfying $\alpha' = \alpha + \epsilon_i$, $\phi^i$ is given by the map

$$(\phi^* \mathbb{C})(\alpha') \times_{(\phi^* \mathbb{C})(\alpha)} (\phi^* \mathbb{C})(\alpha') \simeq n\text{FoldFun} \left( \prod_{j=1}^n \tilde{\alpha}_k, \mathbb{C} \right) \to (\phi^* \mathbb{C})(\alpha')$$

induced by $d^2_\alpha$, where

$$\tilde{\alpha}_k = \begin{cases} 2 & \text{if } k = i, \\ \alpha_k & \text{if } k \neq i. \end{cases}$$

We obtain a functor $\phi^* : n\text{FoldCat} \to m\text{FoldCat}$.

**Example 2.8.**
(1) Let \( i_* : \{1\} \to \{1, \ldots, n\} \) be the map of image \( \{i\} \). For an \( n \)-fold category \( C \), \( \text{Ob}(i_*^* C) \simeq C(0, \ldots, 0) \) and \( \text{Mor}(i_*^* C) \simeq C(\epsilon_i) \). More generally, the functor \( \phi^* \) for \( \phi \) strictly increasing has been studied by Fiore and Paoli [10, Notation 2.12].

(2) Let \( \rho : \{1, \ldots, n\} \to \{1\} \). For a category \( C \), \( (\rho^* C)(\alpha) = \text{Fun}(\prod_{i=1}^n \alpha_i, C) \). In other words, \( I \)-morphisms of \( \rho^* C \) are commutative \( I \)-hypercubes in \( C \).

(3) Let \( t : \{1, 2\} \to \{1, 2\} \) be the map swapping 1 and 2. Then \( C^t = t^* C \) is the transpose of \( C \) in the sense that \( \text{Ob}(C^t) = \text{Ob}(C), \text{Hor}(C^t) = \text{Ver}(C), \text{Ver}(C^t) = \text{Hor}(C) \), and \( \text{Sq}(C^t) \) is obtained from \( \text{Sq}(C) \) by transposing the squares.

The functor \( i_*^* : \text{nFoldCat} \to \text{Cat} \) has the following 2-categorical refinement.

**Definition 2.9.** To any double category \( C \), one associates the underlying horizontal 2-category \( \text{HC} \) and the underlying vertical 2-category \( \text{VC} \). The underlying category of \( \text{HC} \) is \( i_*^* C = (\text{Ob}(C), \text{Hor}(C)) \) and the underlying category of \( \text{VC} \) is \( i_*^* C = (\text{Ob}(C), \text{Ver}(C)) \). A 2-cell \( \alpha : f \Rightarrow g \) in \( \text{HC} \) is a square in \( C \) of the form

\[
\begin{array}{ccc}
X & \overset{g}{\rightarrow} & Y \\
\downarrow & & \downarrow \\
X & \underset{f}{\rightarrow} & Y
\end{array}
\]

A 2-cell \( \alpha : f \Rightarrow g \) in \( \text{VC} \) is a square in \( C \) of the form

\[
\begin{array}{ccc}
X & \overset{f}{\rightarrow} & X \\
\downarrow & & \downarrow \\
Y & \underset{g}{\rightarrow} & Y
\end{array}
\]

We have isomorphisms \( \text{H}(C^t) \simeq (\text{VC})^\circ, \text{V}(C^t) \simeq (\text{HC})^\circ \) (Notation [13]).

More generally, to an \( n \)-fold category \( C \), and \( 1 \leq i, j \leq n, i \neq j \), one associates the 2-category

\[
\text{H}_{i,j} C = \begin{cases} 
\text{H}(i_*^* \epsilon_j C) & \text{if } i < j, \\
\text{V}(i_*^* \epsilon_i C) & \text{if } i > j,
\end{cases}
\]

where \( i_* : \{1, 2\} \to \{1, \ldots, n\} \) is the map sending 1 to \( i \) and 2 to \( j \). We obtain a functor \( \text{H}_{i,j} : \text{nFoldCat} \to 2\text{Cat} \).

**Definition 2.10.** Let \( C \) be an \( n \)-fold category. We define the category generated by \( C \), \( \mathcal{T} C \), satisfying \( \text{Ob}(\mathcal{T} C) = C(0, \ldots, 0) \), as follows. A morphism \( X \to Y \) of \( \mathcal{T} C \) is an equivalence class of paths

\[
X = X_0 \xrightarrow{f_1} X_1 \to \cdots \to X_{m-1} \xrightarrow{f_m} X_m = Y, \quad m \geq 0
\]

where \( f_i \in \coprod_{1 \leq k \leq n} C(\epsilon_k), 1 \leq i \leq m \) under the equivalence relation \( \sim \) generated by the conditions

(1) \( g * f \sim g * \text{id}_Y * f \), where \( X \xrightarrow{f} Y \xrightarrow{g} Z \) is a sequence of paths, \( \text{id}_Y \in C(\epsilon_k) \) is identity morphism of \( Y \).

(2) \( g * h' * h * f \sim g * (h' h) * f \), where \( X \xrightarrow{f} Y \xrightarrow{h} Y' \xrightarrow{h'} Y'' \xrightarrow{g} Z \) is a sequence of paths, \( h \) and \( h' \) belong to the same \( C(\epsilon_k) \) and \( h' h \) is their composition in \( C \).

(3) \( g * i * q * f \sim g * p * j * f \), where \( i, j \in C(\epsilon_k), p, q \in C(\epsilon_{k'}), k < k', D \in C(\epsilon_k + \epsilon_{k'}) \) is of the form

\[
\begin{array}{ccc}
X & \overset{j}{\rightarrow} & Y \\
\downarrow & & \downarrow \\
q & \underset{D}{\rightarrow} & p
\end{array}
\]

\[
\begin{array}{ccc}
Z & \overset{i}{\rightarrow} & W,
\end{array}
\]

g : W \to W' and \( f : X' \to X \) are paths.
Here * denote concatenation of paths. The identity morphism \( \text{id}_X : X \to X \) in \( \mathcal{T} \mathcal{C} \) is given by the path of length 0. Composition of morphisms is given by concatenation of paths.

We obtain a functor \( \mathcal{T} : \text{nFoldCat} \to \text{Cat} \).

**Remark 2.11.** Let \( \rho^* : \text{Cat} \to \text{nFoldCat} \) be as in Example 2.8 [2]. For every category \( \mathcal{D} \), we have an isomorphism of categories \( \mathcal{T} \rho^* \mathcal{D} \cong \mathcal{D} \) sending \( X \) to \( X \) and the morphism represented by a path \( f_m \cdot \cdots \cdot f_1 \), where \( f_i \in \prod_{1 \leq k \leq n} (\rho^* \mathcal{D})(\epsilon_k) \) to the composite \( f_m \cdots f_1 \). This isomorphism induces a bijection

\[
\text{Fun}(\mathcal{T} \mathcal{C}, \mathcal{D}) \cong \text{nFoldFun}(\mathcal{C}, \rho^* \mathcal{D}),
\]

functorial in \( \mathcal{C} \) and \( \mathcal{D} \), which exhibits \( \mathcal{T} \) as a left adjoint of \( \rho^* \).

We now proceed to give a 2-categorical analogue of the preceding remark. We start by a 2-categorical refinement of \( \mathcal{T} \).

**Definition 2.12.** Let \( \mathcal{C} \) be an \( n \)-fold category. We define the 2-category generated by \( \mathcal{C} \), \( \mathcal{T} \mathcal{C} \), satisfying \( \text{Ob}(\mathcal{T} \mathcal{C}) = \mathbb{C}(0, \ldots, 0) \), as follows. A morphism \( X \to Y \) of \( \mathcal{T} \mathcal{C} \) is a path

\[
X = X_0 \xrightarrow{f_1} X_1 \to \cdots \to X_{m-1} \xrightarrow{f_m} X_m = Y, \quad m \geq 0
\]

where \( f_i \in \prod_{1 \leq k \leq n} \mathcal{C}(\epsilon_k) \), 1 \( \leq i \leq m \). The identity morphism \( \text{id}_X : X \to X \) in \( \mathcal{T} \mathcal{C} \) is the path of length 0. Composition of morphisms is given by concatenation of paths and is denoted by \( * \). An atomic transformation between two paths sharing a source and a target is one of the following

1. \( \iota^k_{g,j} : g * \circ f \Rightarrow g * \circ \text{id}_Y \circ f \Rightarrow g * f \), where \( X \xrightarrow{f} Y \xrightarrow{g} Z \) is a sequence of paths, \( \text{id}_Y \) is identity morphism of \( Y \).

2. (composition) \( \gamma_{g,h,f} : g * h * \circ f \Rightarrow g * (h' \circ h) * \circ f \) or (decomposition) \( \delta_{g,h',h,f} : g * (h' \circ h) * \circ f \Rightarrow g * h' * \circ h \circ \circ f \), where \( X \xrightarrow{f} Y \xrightarrow{h} Y' \xrightarrow{h'} Y'' \xrightarrow{g} Z \) is a sequence of paths, \( h \) and \( h' \) belong to the same \( \mathcal{C}(\epsilon_k) \) and \( h' \circ h \) is their composition in \( \mathcal{C} \).

3. (square) \( \sigma_{p,q,f,j} : g * p * \circ j \Rightarrow g * p * \circ \circ j \circ \circ f \), where \( i,j \in \mathcal{C}(\epsilon_k) \), \( p,q \in \mathcal{C}(\epsilon_{k'}) \), \( k < k' \), \( D \in \mathcal{C}(\epsilon_k + \epsilon_{k'}) \) is of the form (2.10.1), \( g : W \to W' \) and \( f : X' \to X \) are paths.

A transformation \( f \Rightarrow g \) is a sequence of atomic transformations \( f = f_0 \Rightarrow f_1 \Rightarrow \cdots \Rightarrow f_{n-1} \Rightarrow f_n = g \). Composition of transformations is denoted by \( \circ \). If \( f,g : X \to Y, h : W \to X, h' : Y \to Z \) are paths and \( \alpha : f \Rightarrow g \) is a transformation, then one has the concatenated transformation \( h' \circ \alpha \circ h : h' * f \circ h \Rightarrow h' * g \circ h \).

Consider systems which associate to every pair of paths \((f,g)\) sharing a source and a target, an equivalence relation on the set of transformations \( f \Rightarrow g \). We say that one system \( \sim \) is finer than another system \( \sim' \) if \( \alpha \sim \beta \) implies \( \alpha \sim' \beta \). There is a finest system \( \sim \) satisfying the following conditions

1. (Stability under composition and concatenation) If \( f,g : X \to Y, h : W \to X, h' : Y \to Z \) are paths, \( \alpha : f \Rightarrow g, \eta : f' \Rightarrow f, \eta' : g \Rightarrow g' \) are transformations, then \( \eta' \circ \alpha \circ \eta \sim \eta' \circ \beta \circ \eta \) and \( h' * \alpha * h \sim h' * \beta * h \).

2. (Compatibility with concatenation) If \( f,f' : X \to Y, g,g' : Y \to Z \) are paths, \( \alpha : f \Rightarrow f', \beta : g \Rightarrow g' \) are transformations, then \((f' * \beta) \circ (\alpha * g) \sim (\alpha * g') \circ (f * \beta)\).

3. For \( 1 \leq k \leq n \) and every object \( X \) of \( \mathcal{C} \),

\[
\theta_{\text{id}_X,\text{id}_X}^k \circ \iota_{\text{id}_X,\text{id}_X}^k \sim \text{id}_{\text{id}_X}, \quad \iota_{\text{id}_X,\text{id}_X}^k \circ \theta_{\text{id}_X,\text{id}_X}^k \sim \text{id}_{\text{id}_X}.
\]

4. For \( 1 \leq k \leq n \) and \( f : X \to Y \) belonging to \( \mathcal{C}(\epsilon_k) \),

\[
\iota_{\text{id}_Y,\text{id}_Y}^k \sim \delta_{\text{id}_Y,\text{id}_Y}^k, \quad \iota_{\text{id}_Y}^k \sim \delta_{\text{id}_Y,\text{id}_Y}^k, \quad \iota_{\text{id}_Y}^k \sim \delta_{\text{id}_Y,\text{id}_Y}^k.
\]

5. For \( 1 \leq k \leq n \) and every sequence \( X \xrightarrow{f} Y \xrightarrow{g} Z \) of morphisms in \( \mathcal{C}(\epsilon_k) \),

\[
\delta_{\text{id}_Z,\text{id}_Z}^k \circ \gamma_{\text{id}_Z,\text{id}_Z}^k \sim \text{id}_{g * f}, \quad \gamma_{\text{id}_Z,\text{id}_Z}^k \circ \delta_{\text{id}_Z,\text{id}_Z}^k \sim \text{id}_{g * f}.
\]
For $1 \leq k \leq n$ and every sequence $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ of morphisms in $C(\epsilon_k)$,

$$\gamma_{\text{id}_W,h,g,f,\text{id}_X} \circ \gamma_{h,g,f,\text{id}_X} \sim \gamma_{\text{id}_W,h,g,f,\text{id}_X} \circ \gamma_{\text{id}_W,h,g,f,\text{id}_X}.$$  

For $1 \leq k, k' \leq n$ (neither $k$ nor $k'$ is $k'$) and $f: X \to Y$ belonging to $C(\epsilon_k)$, we have

$$\begin{cases}
\sigma_{\text{id}_Y,D,\text{id}_X} \circ f' \circ f \sim f' \circ f & \text{if } k < k', \\
\sigma_{\text{id}_X,D,\text{id}_X} \circ f' \circ f \sim f' \circ f & \text{if } k' < k,
\end{cases}$$

where $D = s(f) \in C(\epsilon_k + \epsilon_{k'})$ is the identity square

(2.12.1)

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y. \\
\end{array}
\]

For $1 \leq k, k' \leq n$ (neither $k$ nor $k'$ is $k'$) and $D'' = D' \circ k D$ in $C(\epsilon_k + \epsilon_{k'})$ of the form

(2.12.2)

\[
\begin{array}{ccc}
X_1 & \xrightarrow{j} & X_2 & \xrightarrow{j'} & X_3 \\
\downarrow & & D & & \downarrow \\
Y_1 & \xrightarrow{i} & Y_2 & \xrightarrow{i'} & Y_3,
\end{array}
\]

we have

$$\begin{cases}
\gamma_{p_3,j',j,\text{id}_X_1} \circ \sigma_{\text{id}_Y,D',j} \circ \sigma_{\text{id}_V,D,\text{id}_X} \sim \sigma_{\text{id}_Y,D',\text{id}_X_1} \circ \gamma_{\text{id}_Y,j',i,p_1} & \text{if } k < k', \\
\gamma_{\text{id}_Y,j',i,p_1} \circ \sigma_{\text{id}_V,D,\text{id}_X} \circ \sigma_{\text{id}_Y,D',j} \sim \sigma_{\text{id}_Y,D',\text{id}_X_1} \circ \gamma_{\text{id}_Y,j',j',\text{id}_X_1} & \text{if } k' < k.
\end{cases}$$

For $1 \leq k < k' < k'' \leq n$ and $C \in C(\epsilon_k + \epsilon_{k'} + \epsilon_{k''})$ of the form

\[
\begin{array}{ccc}
X' & \xrightarrow{b'} & Y' \\
\downarrow & & \downarrow \\
X & \xrightarrow{b} & Z, \\
\downarrow & & \downarrow \\
Z & \xrightarrow{a} & W \\
\downarrow & & \downarrow \\
W & \xrightarrow{w} & W',
\end{array}
\]

where $a, b, a', b' \in C(\epsilon_k), p, q, p', q' \in C(\epsilon_{k'})$, $x, y, z, w \in C(\epsilon_{k''})$, we have

$$\sigma_{\text{id}_W,f',b'} \circ \sigma_{p,j',\text{id}_X} \circ \sigma_{\text{id}_W,K,a} \sim \sigma_{w,K',\text{id}_X} \circ \sigma_{\text{id}_W,f,q} \circ \sigma_{a,l',\text{id}_X},$$

where $I, I', J, J', K, K'$ are respectively the right, left, front, back, bottom, top faces of the cube.

A 2-cell $f \Rightarrow g$ of $\mathcal{T}$ is an equivalence class of transformations under this system of equivalence relations. Composition of 2-cells is given by composition of transformations.

We obtain a functor $\mathbf{T}: \mathbf{nFoldCat} \to \mathbf{2Cat}$.

**Remark 2.13.** The functors $\mathbf{T}$ and $\mathcal{T}$ are related by the isomorphism $\mathcal{T} \simeq \mathcal{O}\mathcal{T}$, where $\mathcal{O}: \mathbf{2Cat} \to \mathbf{Cat}$ is the functor in Construction 1.13.

**Remark 2.14.** Let $\mathcal{C}$ be a category, considered as a 1-fold category. There is an obvious 2-functor $F: \mathbf{TC} \to \mathcal{C}$ sending a path $f_m \star \cdots \star f_1$ to its composition $f_m \cdots f_1$ in $\mathcal{C}$ and all 2-cells to identities. There is an obvious pseudofunctor $G: \mathcal{C} \to \mathbf{TC}$ sending $f$ to $f$ with coherence constraint given by $\iota$ and $\gamma$. We have $FG = \text{id}_C$ and there is an obvious pseudonatural equivalence $\text{id}_{\mathbf{TC}} \to GF$ given by $\gamma$. In particular, $F$ and $G$ are 2-equivalences.
Next we extend the functor $\rho^* : \text{Cat} \to n\text{FoldCat}$ (Example 2.8) to a functor $Q^n : 2\text{Cat} \to n\text{FoldCat}$. In the case $n = 2$, for a 2-category $C$, $Q^2C$ is the double category of up-squares (also known as quintets) of $C$ [2, 2.C.1, p. 272]. Objects of $Q^2C$ are objects of $C$, horizontal morphisms are morphisms of $C$, vertical morphisms are morphisms of $C$, and squares are up-squares in $C$, namely, diagrams in $C$ of the form

$$(2.14.1) \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ q \downarrow & \alpha \notag \downarrow & \downarrow p \\ Z & \xrightarrow{g} & W. \end{array}$$

In general, $I$-morphisms of $Q^nC$ are $I$-hypercubes in $C$ with 2-cells in suitable directions. Note that paths from the initial vertex to the final vertex of an $I$-hypercube correspond bijectively to total orders on $I$.

To give a precise definition of $Q^nC$, we introduce the following 2-category, parameterizing an $I$-hypercube with 2-cells (for $\alpha = \chi I$).

**Notation 2.15.** Let $\alpha$ be an object of $I^n$. We define an 2-category $\gamma_\alpha$ with $\text{Ob}(\gamma_\alpha) = \prod_{i=1}^n \alpha_i$ as follows. For objects $a, b$ of $\gamma_\alpha$, the category of morphisms $\gamma_\alpha(a, b)$ is given by a partially ordered set

$$\begin{cases} \text{set of total orders on } J_{a,b} = \{1 \leq i \leq n \mid a_i \neq b_i\} & \text{if } a \leq b, \\ \{\emptyset\} & \text{otherwise}. \end{cases}$$

For total orders $f$ and $g$ on $J_{a,b}$, there exists a 2-cell $f \Rightarrow g$ if and only if $\text{Inv}(f) \subseteq \text{Inv}(g)$, where $\text{Inv}(f) = \{(i, j) \mid i < j, \text{but } i \neq f j\} \subseteq J_{a,b}$ is the set of inversions. For $f : a \to b$ and $g : b \to c$, the composition $g \circ f$ is the unique order on $J_{a,c} = J_{a,b} \bigsqcup J_{b,c}$ extending $f$ and $g$ such that $i \lessdot gaf j$ for all $i \in J_{b,c}$ and $j \in J_{a,b}$.

Note that $\gamma_\alpha$ is the arrowy 2-subcategory of $T \boxtimes_{i=1}^n \alpha_i$ spanned by morphisms that do not contain $\text{id}^k$.

**Definition 2.16.** Let $C$ be a 2-category. We define an $n$-fold category $Q^nC$, isomorphic to $\rho^*C$ when $C$ is a 1-category, by $(Q^nC)(\alpha) = 2\text{Fun}(\gamma_\alpha, C)$. For $\alpha, \alpha' \in I^n$ satisfying $\alpha' = \alpha + \epsilon_i$, $\alpha'$ is given by composition in direction $i$, or more formally, by the map

$$((Q^nC)(\alpha'), ((Q^nC)(\alpha) \cong 2\text{Fun}(A, C) \to (Q^nC)(\alpha')$$

induced by $d^2_\alpha$, where $A \subseteq T \boxtimes_{j=1}^n \alpha_j$ (as in Definition 2.17) is the arrowy 2-subcategory spanned by morphisms that do not contain $\text{id}^k$ or segments of the form $a \to a + 2 \epsilon_i$.

We obtain a functor $Q^n : 2\text{Cat} \to n\text{FoldCat}$.

Let $A_1, \ldots, A_n$ be arrowy 2-subcategories of $C$. We denote by $Q(A_1, \ldots, A_n)C$ the greatest $n$-fold subcategory of $Q^nC$ such that $(Q(A_1, \ldots, A_n)C)(\epsilon) = 2\text{Fun}(\gamma_\epsilon, A_i)$. In other words, $I$-morphisms of $Q(A_1, \ldots, A_n)C$ are $I$-hypercubes in $C$ with 2-cells such that every edge in direction $i$ is in $A_i$.

By definition, $Q^nC = Q(A_1, \ldots, A_n)C$. The category $Q(A_1, \ldots, A_n)C$ is isomorphic to the underlying category of $A_i$. Moreover, for $i \neq j$, we have an isomorphism of 2-categories $H_{i,j}(Q(A_1, \ldots, A_n)C) \cong A_i$.

**Remark 2.17.** Let $S$ be the set of edges of the hypercube $\gamma_\alpha$, namely the set of morphisms $f : a \to b$ of $\gamma_\alpha$ such that $b = a + \epsilon_i$ for some $i$. By Lemma 2.14, for any pseudofunctor $F : \gamma_\alpha \to C$, there exists a unique pair $(G, \epsilon)$, where $G : \gamma_\alpha \to C$ is a 2-functor satisfying $G(f) = F(f)$ for all $f \in S$ and $\epsilon : F \to G$ is a pseudonatural equivalence such that $\epsilon(f)$ is the identity for $f \in S$ and $\epsilon(f)$ is induced by the coherence constraint of $F$ for every morphism $f$ of $\gamma_\alpha$.

**Example 2.18.** For $n = 2$, for arrowy 2-subcategories $A$ and $B$ of $C$, $Q(A, B)C$ is the double category whose objects are objects of $C$, horizontal morphisms are morphisms of $A$, vertical morphisms are morphisms of $B$, and squares are $(A, B)$-squares in $C$, namely, diagrams in $C$ of the form $\begin{array}{ccc} \alpha \to & & \beta \\ \beta \downarrow & \quad & \downarrow \gamma \\ \gamma \to & & \delta \end{array}$ where the horizontal morphisms $i, j$ are in $A$ and the vertical morphisms $p, q$ are in $B$. We have isomorphisms $H(Q(A, B)C) \cong A$ and $V(Q(A, B)C) \cong B$.

**Construction 2.19.** Let $C$ be an $n$-fold category. There is an obvious $n$-fold functor

$$(2.19.1) \quad C \to Q^nTC$$
sending $X \in \Ob(C)$ to $X$, $f \in C(\epsilon_i)$ to $f$, and $D \in C(\epsilon_k + \epsilon'_k)$ of the form \((2.10.1)\) to the square induced by $\sigma_{id_Y, D, id_Z}$. The image of a hypercube $C \in C(\alpha)$ of dimension $\geq 3$ under \((2.19.1)\) is uniquely determined by its vertices, edges, and faces (of dimension 2).

For $1 \leq i, j \leq n$, $i \neq j$, we define a pseudofunctor

\[(2.19.2)\]

$$H_{i,j}: C \to TC$$

as follows. To an object $X$ of $C$, we associate $X$. To $f \in C(\epsilon_i)$, we associate $f$. The coherence constraint is given by $\epsilon_{id_Y, D, id_Z}$ and $\alpha_{id_Z, g, id_X}$, where $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a sequence of morphisms in $C(\epsilon_i)$. If $i < j$ (resp. $i > j$), to any 2-cell $\alpha: f \Rightarrow g$ in $H_{i,j}C$, where $f, g: X \to Y$, we associate the composition

$$f \xrightarrow{\epsilon_{id_Y, D, id_Z}} f * id_Y \xrightarrow{\sigma_{id_Y, D, id_X}} id_Y * g \xrightarrow{\alpha_{id_Y, g, id_X}} g$$

(resp. $f \xrightarrow{\epsilon_{id_Y, D, id_Z}} f * id_Y \xrightarrow{\sigma_{id_Y, D, id_X}} id_Y * g \xrightarrow{\alpha_{id_Y, g, id_X}} g$),

where $D \in C(\epsilon_i + \epsilon_j)$ is the square induced by $\alpha$ of the form \((2.11.1)\) (resp. \((2.11.2)\)).

Let $\mathcal{D}$ be a 2-category. We define a 2-functor

\[(2.19.3)\]

$$F: \mathcal{TQ}^n \mathcal{D} \to \mathcal{D}$$

as follows. To an object $X \in \Ob(\mathcal{TQ}^n \mathcal{D}) = \Ob(\mathcal{D})$, we associate $X$. To a path $f_m \cdots f_1$, where $f_k \in \prod_{1 \leq k \leq n} (\mathcal{Q}^n \mathcal{D})(\epsilon_k)$, we associate the composite $f_m \cdots f_1$. We take $F(\sigma_{\alpha, D, f})$ to be the 2-cell induced by $D$. To other atomic transformations between paths, we associate identity 2-cells. Moreover, for $1 \leq i, j \leq n$, $i \neq j$, \[(2.19.2)\] induces a pseudofunctor

\[(2.19.4)\]

$$G_{i,j}: \mathcal{D} \simeq H_{i,j} \mathcal{Q}^n \mathcal{D} \to \mathcal{TQ}^n \mathcal{D}.$$

Note that $G_{i,j}$ does not depend on the choice of $j$.

**Proposition 2.20.** Let $\mathcal{D}$ be a 2-category. For $n \geq 2$, the 2-functor $F: \mathcal{TQ}^n \mathcal{D} \to \mathcal{D}$ is a 2-equivalence. More generally, if $A_1, \ldots, A_n$ are arrowy 2-subcategories of $\mathcal{D}$ and $A_i = \mathcal{D}$ for some $1 \leq i \leq n$, then the 2-functor $\mathcal{TQ}_{A_1, \ldots, A_n} \mathcal{D} \to \mathcal{D}$ induced by $F$ is a 2-equivalence.

**Proof.** We still use $F$ and $G_{i,j}$ to denote the induced functors between $\mathcal{TQ}_{A_1, \ldots, A_n} \mathcal{D}$ and $\mathcal{D}$. We have $FG_i = id_{\mathcal{D}}$. We define a pseudonatural equivalence $\epsilon: G_i F = id_{\mathcal{TQ}_{A_1, \ldots, A_n} \mathcal{D}}$ as follows. For any object $X$ of $\mathcal{D}$, we take $\epsilon(X) = id_X$. For any morphism $f: X \to Y$ of $\mathcal{D}$, we denote by $f^k \in (\mathcal{Q}^n \mathcal{D})(\epsilon_k)$ the image of $f$. We take $\epsilon(f^k) = id_{f^k}$, and take $\epsilon(f^k)$, $i < k$ (resp. $i > k$) to be the composition

$$f^i \xrightarrow{id_Y} f^i * id_Y \xrightarrow{\sigma_{id_Y, D, id_Z}} f^i * id_Y \xrightarrow{\alpha_{id_Y, g, id_X}} f^k \xrightarrow{id_Y} f^k$$

(resp. $f^i \xrightarrow{id_Y} f^i * id_Y \xrightarrow{\sigma_{id_Y, D, id_X}} f^i * id_Y \xrightarrow{\alpha_{id_Y, g, id_X}} f^k \xrightarrow{id_Y} f^k$),

where $D, E \in (\mathcal{Q}^n \mathcal{D})(\epsilon_i + \epsilon_k)$ are the squares

\[
\begin{array}{c}
X \\
\downarrow D \\
X \\
\downarrow F \\
Y \\
\end{array}
\quad \quad \quad
\begin{array}{c}
X \\
\downarrow F \\
Y \\
\end{array}
\]

induced by $id_f$. It is straightforward to check that this definition is compatible with 2-cells of $\mathcal{TQ}_{A_1, \ldots, A_n} \mathcal{D}$. \hfill $\square$

The functors $T: n\text{FoldCat} \to 2\text{Cat}$ and $\mathcal{Q}^n : 2\text{Cat} \to n\text{FoldCat}$, together with the natural transformations $\mathcal{T} \to \mathcal{Q}^n$ and $\mathcal{Q}^n \to \mathcal{T}$, do not form an adjunction per se. However, an analogue of \((2.11.1)\) does exist for pseudofunctors, with the analogue of the right-hand side of \((2.11.1)\) given as follows.

**Definition 2.21.** Let $C$ be an $n$-fold category and let $\mathcal{D}$ be a 2-category. We define the 2-category of pseudofunctors $\mathcal{PsFun}(C, \mathcal{D})$ as follows. Objects of $\mathcal{PsFun}(C, \mathcal{D})$ are pairs $(F, (F_i)_{1 \leq i \leq n})$ consisting of a natural transformation $F: C \to \mathcal{Q}^n \mathcal{D}$, where $C$ and $\mathcal{Q}^n \mathcal{D}$ are viewed as functors $(\mathcal{T}^n)^n \to \mathcal{Set}$, and pseudofunctors $F_i: \epsilon_i C \to \mathcal{D}$ extending $F_0$ and $F_{\epsilon_i}$. Here $\epsilon_i$ is the subcategory of $\mathcal{I}$ (Notation \ref{2.19}) spanned by the strictly order-preserving maps. For all $1 \leq k, k' \leq n$ ($k \neq k'$) and $D \in C(\epsilon_k + \epsilon'_k)$, we denote by $G_D$ the 2-cell in $F_{\epsilon_k + \epsilon_k}(D)$. The pair $(F, (F_i))$ is subject to the following conditions:
(a) For \( f: X \to Y \) belonging to \( \mathbb{C}(\epsilon_k) \), the following triangle (resp. with the horizontal arrow reversed)

\[
\begin{array}{c}
F_k(f) \\
\downarrow \\
F_k(f)F_k'(\text{id}_X^\epsilon) \xrightarrow{G_D} F_k'(\text{id}_X^\epsilon)F_k(f)
\end{array}
\]

commutes if \( k < k' \) (resp. \( k' < k \)), where \( D = s(f) \in \mathbb{C}(\epsilon_k + \epsilon_{k'}) \) is the identity square (2.1.11).

(b) For \( D'' = D' \circ D \) in \( \mathbb{C}(\epsilon_k + \epsilon_{k'}) \) of the form (2.1.22), the following pentagon (resp. with the horizontal arrows reversed)

\[
\begin{array}{c}
F_k(i'')F_k(i)F_k'(p_1) \xrightarrow{G_D} F_k(i'')F_k'(p_2)F_k(j) \xrightarrow{G_{D'}} F_k'(p_3)F_k(j')F_k(j) \\
\downarrow \\
F_k(i')F_k'(p_1) \xrightarrow{G_{D''}} F_k'(p_3)F_k(j')F_k(j)
\end{array}
\]

commutes if \( k < k' \) (resp. \( k' < k \)).

A morphism \( \mu: F \to F' \) of \( \text{PsFun}(\mathbb{C}, D) \) is a collection \( (\mu_i)_{1 \leq i \leq n} \) of morphisms (i.e. pseudonatural transformations) \( \mu_i: F_i \to F'_i \) of \( \text{PsFun}(\epsilon_i \mathbb{C}, D) \) satisfying \( |\mu_1| = \cdots = |\mu_n| \) (Notation 1.9 (2)) and such that for every square (2.1.11) in \( \mathbb{C} \), the cube

\[
\begin{array}{c}
F_X \xrightarrow{\alpha_X} F_Y \\
\downarrow F_k(j) \downarrow \alpha_Y \downarrow F_k'(p) \\
F_Z \xrightarrow{\alpha_Z} F_W \\
\downarrow F_k'(j) \downarrow \alpha_W \downarrow F_k'(p')
\end{array}
\]

is 2-commutative. Here the top, bottom, front, back, right, and left faces are respectively given by

\[
\begin{array}{c}
F_{k + \epsilon_{i'}}(D), F'_{k + \epsilon_{i'}}(D), \mu_k(i), \mu_k(j), \mu_k'(p), \text{ and } \mu_k'(q).
\end{array}
\]

A 2-cell \( \Xi: \mu \Rightarrow \nu \) of \( \text{PsFun}(\mathbb{C}, D) \) consists of a collection \( (\Xi_i)_{1 \leq i \leq n} \) of 2-cells (i.e. modifications) \( \Xi_i: \mu_i \Rightarrow \nu_i \) of \( \text{PsFun}(\epsilon_i \mathbb{C}, D) \) such that \( |\Xi_1| = \cdots = |\Xi_n| \). In other words, a 2-cell \( \mu \Rightarrow \nu \) of \( \text{PsFun}(\mathbb{C}, D) \) is a function from \( \text{Ob}(\mathbb{C}) \) to the set of 2-cells of \( D \) which is a modification \( \mu_i \Rightarrow \nu_i \) for all \( 1 \leq i \leq n \) at the same time.

Composition in \( \text{PsFun}(\mathbb{C}, D) \) is given by composition in \( D \).

We denote by \( \text{PsFun}^F(\mathbb{C}, D) \) the 2-fully faithful subcategory of \( \text{PsFun}(\mathbb{C}, D) \) spanned by pseudofunctors for which all the \( G_D \) are invertible 2-cells.

We view \( \text{PsFun}(\mathbb{C}, D) \) as a \( D^{\text{Ob}(\mathbb{C})} \)-category via the obvious forgetful 2-functor \( \text{PsFun}(\mathbb{C}, D) \to D^{\text{Ob}(\mathbb{C})} \).

We obtain a functor \( \text{PsFun}: n\text{FoldCat}^{\text{op}} \times 2\text{Cat} \to 2\text{Cat} \).

**Remark 2.22.** If \( \mathbb{C} \) is an ordinary category viewed as a 1-fold category, \( \text{PsFun}(\mathbb{C}, D) \) can be identified with the 2-category \( \text{PsFun}(\mathbb{C}, D) \).

For \( n = 2 \), the objects of \( \text{PsFun}(\mathbb{C}, D) \) can be identified with the double pseudofunctors \( \mathbb{C} \to \mathbb{Q}^2 \mathbb{D} \) in the sense of Shulman [19] Definition 6.1]. Recall that a double pseudofunctor from \( \mathbb{C} \) to another doubled category \( \mathbb{D} \) is weakly compatible with compositions in both directions. The notion of double pseudofunctors is more general than the notion of pseudo double functors by Fiore [2] Definition 6.4], which are weakly compatible with horizontal composition and strictly compatible with vertical composition.

**Remark 2.23.** The inclusion 2-functor \( \mathcal{R}D \to D \) induces an isomorphism of 2-categories \( \text{PsFun}(\mathbb{C}, \mathcal{R}D) \simeq \mathcal{R}\text{PsFun}^F(\mathbb{C}, D) \).
Construction 2.24. For 2-categories $C$ and $D$, we define a $\mathcal{D}_{\text{Ob}(C)}$-functor

\[(2.24.1) \quad Q : \text{PsFun}(C, D) \to \text{PsFun}(Q^n C, D)\]
as follows. To a pseudofunctor $F : C \to D$, we associate $(Q^n F, F, \ldots, F)$, where $Q^n F : Q^n C \to Q^n D$ is the natural transformation carrying a hypercube in $C$ with 2-cells to the hypercube in $D$ with 2-cells induced by $F$. More formally, $Q^n F$ carries $\lambda \in (Q^n C)(\alpha) = 2\text{Fun}(\gamma_{\alpha}, C)$ to the 2-functor associated to the pseudofunctor $F \circ \lambda$ in Remark 2.14. Here we have identified $\epsilon^* Q^n C$ with the underlying category of $C$. To a pseudonatural transformation $\alpha$ we associate $(\alpha, \ldots, \alpha)$. To a modification $\Xi$ we associate $(\Xi, \ldots, \Xi)$. We consider the composite $\mathcal{D}_{\text{Ob}(C)}$-functor

$$\Psi : \text{PsFun}(TC, D) \xrightarrow{Q} \text{PsFun}(Q^n TC, D) \to \text{PsFun}(C, D),$$

where the second 2-functor is induced by $\Phi$. For an $n$-fold category $C$, we define a $\mathcal{D}_{\text{Ob}(C)}$-functor

\[(2.24.2) \quad T : \text{PsFun}(C, D) \to \text{PsFun}(TC, TQ^n D)\]
as follows. To a pseudofunctor $(F, (F_k))$, we associate the pseudofunctor $TF : TC \to TQ^n D$ such that $(TF)(X) = FX$ for $X \in \text{Ob}(C)$, $(TF)(f) = Ff$ for $f \in C(\epsilon_k)$, the coherence constraint of $TF$ is trivial, $(TF)(\sigma_{g,f,j}) = \sigma_{Fg,Ff,j}$, and $T$ sends $\iota^*_k$ to the 2-cells induced by $\iota_{g,f,j}$, $\theta_{Fg,Ff,j}$, $\gamma_{g,h',h,f}$, $\delta_{g,h',h,f}$ to the 2-cells induced by $\iota_{g,f,j}$, $\theta_{g,f,j}$, $\gamma_{g,h',h,f}$, $\delta_{g,h',h,f}$ and the coherence constraint of $F_k$. For any morphism $\alpha = (\alpha_k) : F \to F'$ of $\text{PsFun}(C, D)$, $T\alpha : TF \to TF'$ is the pseudonatural transformation defined by $(T\alpha)(X) = \alpha X$ for $X \in \text{Ob}(C)$, $(T\alpha)(f) = (\alpha_k)(f)$ for $f \in C(\epsilon_k)$. For any 2-cell $\Xi : \alpha \Rightarrow \beta$ of $\text{PsFun}(C, D)$, $T\Xi : T\alpha \Rightarrow T\beta$ is the modification defined by $(T\Xi)(X) = \Xi X$ for $X \in \text{Ob}(C)$. We consider the composite $\mathcal{D}_{\text{Ob}(C)}$-functor

\[(2.24.3) \quad \Phi : \text{PsFun}(C, D) \xrightarrow{T} \text{PsFun}(TC, TQ^n D) \to \text{PsFun}(TC, D),\]

where the second 2-functor is induced by $\Phi$. We have $\Psi \Phi = \text{id}$.

Proposition 2.25. The $\mathcal{D}_{\text{Ob}(C)}$-functors $\Phi$ and $\Psi$ are $\mathcal{D}_{\text{Ob}(C)}$-quasi-inverses of each other.

Proof. There is a unique $\mathcal{D}_{\text{Ob}(C)}$-natural equivalence $\epsilon : \Phi \Psi \to \text{id}$ such that for any pseudofunctor $H : TC \to D$, $\epsilon H : \Phi \Psi H \to H$ is the pseudonatural equivalence defined by $(\epsilon H)(X) = \text{id}_{HX}$ for $X \in \text{Ob}(C)$ and $(\epsilon H)(f) = \text{id}_{Hf}$ for $f \in \coprod_{1 \leq k \leq n} C(\epsilon_k)$.

Remark 2.26. By construction $\Psi$ sends $\text{PsFun}(\mathcal{L}C, D)$ to $\text{PsFun}^\mathcal{L}(C, D)$ and $\Phi$ sends $\text{PsFun}^\mathcal{L}(C, D)$ to $\text{PsFun}(\mathcal{L}C, D)$. Thus $\Psi$ and $\Phi$ induce $\mathcal{D}_{\text{Ob}(C)}$-equivalences between $\text{PsFun}(\mathcal{L}C, D)$ and $\text{PsFun}^\mathcal{L}(C, D)$, $\mathcal{D}_{\text{Ob}(C)}$-quasi-inverses of each other.

Corollary 2.27. Let $C$ and $D$ be 2-categories. Then $Q : \text{PsFun}(C, D) \to \text{PsFun}(Q^n C, D)$ is a $\mathcal{D}_{\text{Ob}(C)}$-equivalence.

Proof. By construction, the composite $\mathcal{D}_{\text{Ob}(C)}$-functor

$$\text{PsFun}(C, D) \xrightarrow{Q} \text{PsFun}(Q^n C, D) \xrightarrow{\Phi} \text{PsFun}(TQ^n C, D)$$

is given by $F : TQ^n C \to C$ and consequently is a $\mathcal{D}_{\text{Ob}(C)}$-equivalence by Propositions 2.20 and 1.12 (2). The corollary then follows from Proposition 2.25.

3 Functionality with respect to the index set

In this section, we record some functorial properties of the operations $T$, $Q$ and $\text{PsFun}$ defined in the last section, with respect to the index set $\{1, \ldots, n\}$. Such properties will play an essential role in Section 5.

Construction 3.1. Let $C$ be an $n$-fold category and let $\phi : \{1, \ldots, m\} \to \{1, \ldots, n\}$ be a strictly increasing map. We define a 2-functor

\[(3.1.1) \quad T\phi^* C \to TC\]
as follows. To an object $X$, we associate $X$. To a morphism $f^k$, where $f \in (\phi^* C)(\epsilon_k) = C(\phi(\epsilon_k))$, we associate $f^{\phi(\epsilon_k)}$. To an atomic transformation in $T\phi^* C$, we associate the corresponding atomic transformation in $T C$.

For a sequence of strictly increasing maps

$$\{1, \ldots, l\} \xrightarrow{\phi'} \{1, \ldots, m\} \xrightarrow{\phi} \{1, \ldots, n\},$$

the composition

$$T(\phi\phi')^* C \simeq T\phi'^* \phi^* C \rightarrow T\phi^* C \rightarrow TC$$

equals the 2-functor induced by $\phi\phi'$.

**Construction 3.2.** Let $C$ be a 2-category, let $A_1, \ldots, A_n$ be arrowy 2-subcategories of $C$, and let $\phi: \{1, \ldots, m\} \rightarrow \{1, \ldots, n\}$ be a strictly increasing map. There is an obvious isomorphism of $m$-fold categories

$$(3.2.1)$$

$$\phi^* Q_{A_1, \ldots, A_n} C \simto Q_{A_{\phi(1)}, \ldots, A_{\phi(m)}} C$$

given by the bijection

$$(\phi^* Q^m C)(\alpha) = nFoldFun(\mathbb{Z}^m_{\alpha \neq 1} \prod_{\phi(i) = j} \alpha, Q^n C) \simeq (Q^m C)(\phi\alpha) = 2Fun(\gamma^\alpha, \gamma^\phi, C)$$

$$\simto 2Fun(\gamma^\alpha, C) = (Q^m C)(\alpha),$$

induced by the isomorphism of 2-categories $\gamma^\alpha \simto \gamma^\phi\alpha$ given by $\sigma_1$. Here $\alpha \in \mathbb{Z}^m$, $\gamma^\alpha$ is as in Notation 3.4, and $\gamma^\phi\alpha \in \mathbb{Z}^m$ is defined by $(\gamma^\phi\alpha)_i = \alpha_i$ for $1 \leq i \leq m$ and $(\gamma^\phi\alpha)_j = 0$ for $j$ not in the image of $\phi$.

For a sequence of strictly increasing maps (3.1.2), the composition

$$(\phi\phi')^* Q^m C \simeq \phi'^* \phi^* Q^m C \simto \phi'^* Q^m C \simto Q^m C$$

equals the isomorphism induced by $\phi\phi'$. The same equality holds for $dQ^m_{A_1, \ldots, A_n} C$.

**Construction 3.3.** Let $C$ be an $n$-fold category, let $D$ be a 2-category, and let $\phi: \{1, \ldots, m\} \rightarrow \{1, \ldots, n\}$ be a strictly increasing map. We define a $D^{Ob(C)}$-functor

$${\mathcal{P}}Fun(C, D) \rightarrow {\mathcal{P}}Fun(\phi^* C, D)$$

as follows. To an object $(F, (F_i)_{1 \leq i \leq n})$, we associate $(\epsilon \circ (\phi^* F), (F_{\phi(i)})_{1 \leq i \leq m})$, where $\epsilon: \phi^* Q^n D \rightarrow Q^n D$ is (3.2.1). To a morphism $(\mu_i)$, we associate $(\mu_{\phi(i)})$. To a 2-cell $(\Xi)$, we associate $(\Xi_{\phi(i)})$.

For a sequence of strictly increasing maps (3.1.2), the composition

$${\mathcal{P}}Fun(C, D) \rightarrow {\mathcal{P}}Fun(\phi^* C, D) \rightarrow {\mathcal{P}}Fun(\phi'^* \phi^* C, D) \simeq {\mathcal{P}}Fun((\phi\phi')^* C, D)$$

equals the $D^{Ob(C)}$-functor induced by $\phi\phi'$.

**Notation 3.4.** For every 2-category $C$ and every strictly increasing map $\phi: \{1, \ldots, m\} \rightarrow \{1, \ldots, n\}$, we let $F^\phi$ denote the composite 2-functor

$$TQ^m C \simeq T\phi^* Q^n C \rightarrow TQ^n C$$

induced by (3.1.1) and the inverse of (3.2.1).

**Construction 3.5.** Let $\phi: \{1, \ldots, m\} \rightarrow \{1, \ldots, n\}$ be an increasing map, not necessarily injective. The definition of the 2-functor (3.1.1) does not extend to this case, as there is no natural candidate for the image of $\sigma_{g,D,f}$, where $D \in (\phi^* C)(\epsilon_k + \epsilon_j)$ with $\phi(i) = \phi(j)$. However, for $n \geq 2$, we can extend the definition of $F_*$ (Notation 3.4) as follows. To an object $X$, we associate $X$. To a morphism $f_1 \cdots f_1$, where $f_i \in \prod_{1 \leq k \leq m}(Q^m C)(\epsilon_k)$ for $1 \leq i \leq a$, we associate $f_1 \cdots f_1$. To the 2-cell $\sigma_{g,D,f}$, where $D \in Q^m C(\epsilon_k + \epsilon_{k'})$ with $\phi(k) = \phi(k') = 1$, $k < k'$, we associate the composition

$$g * i * q * f \xrightarrow{\gamma_{\phi(1), \phi(1)}^\phi} g * i q * f \xrightarrow{\gamma_{\phi(2), \phi(2)}^\phi} g * p j * f \xrightarrow{\delta_{\phi(1), \phi(1)}^\phi} g * p * j * f,$$
where \( \alpha: i q \to pj \) is the image of the 2-cell in \( D \) under the pseudofunctor \( G_1: C \to TQ^oC \). To other atomic 2-cells of \( TQ^mC \), we associate the corresponding atomic transformation in \( TQ^oC \).

If \( A_1, \ldots, A_m, B_1, \ldots, B_n \) are arrowy 2-subcategory of \( C \) such that \( A_i \subseteq B_{\phi(i)} \) for all \( 1 \leq i \leq m \), then \( F_\phi \) restricts to a 2-functor
\[
TQ_{A_1, \ldots, A_m}C \to TQ_{B_1, \ldots, B_n}C.
\]

For a sequence of increasing maps \( \{1, \ldots, l\} \overset{\phi}{\to} \{1, \ldots, m\} \overset{\phi}{\to} \{1, \ldots, n\} \) with \( m, n \geq 2 \), the composition
\[
TQ^oC \xrightarrow{F_\phi} TQ^mC \xrightarrow{F_\phi} TQ^oC
\]
equals \( F_{\phi\phi} \).

**Proposition 3.6.** Let \( \phi: \{1, \ldots, m\} \to \{1, \ldots, n\} \) be an increasing map with \( n \geq 2 \) and let \( \phi' \) be a section of \( \phi \). Let \( A_1, \ldots, A_m \) be arrowy 2-subcategories of \( C \) such that \( A_i \subseteq A_{\phi(i)} \) for all \( 1 \leq i \leq m \). Then
\[
F_\phi: TQ_{A_1, \ldots, A_m}C \to TQ_{A_{\phi(1)}, \ldots, A_{\phi(m)}}C, \quad F_{\phi'}: TQ_{A_{\phi(1)}, \ldots, A_{\phi(m)}}C \to TQ_{A_1, \ldots, A_m}C
\]
are 2-quasi-inverses of each other.

**Proof.** We have \( F_\phi F_{\phi'} = F_{\phi\phi'} = id \). Furthermore, one can construct a pseudonatural equivalence \( F_{\phi'} F_\phi \Rightarrow id \) as in the proof of Proposition 2.20.

**Remark 3.7.** Let \( t: \{1, \ldots, n\} \to \{1, \ldots, n\} \) be the map sending \( i \) to \( n+1-i \). For an \( n \)-fold category \( C \) and arrowy \( 2 \)-categories \( C \) and \( D \), we have isomorphisms
\[
T(t^*C) \simeq (TC)^{\circ},
\]
\[
t^*(Q_{A_1, \ldots, A_n}C) \simeq Q_{A_{\circ 1}, \ldots, A_{\circ n}}C,\quad PsFun(C, D^{\circ}) \simeq PsFun(t^*C, D)^{\circ}.
\]

**Construction 3.8.** Let \( \phi: \{1, \ldots, m\} \to \{1, \ldots, n\} \) be an injective map, not necessarily increasing. We have the following analogues of Constructions 3.1, 3.2, 3.3. For any \( n \)-fold category \( C \), we have a 2-functor
\[
LT\phi^*: C \to LT\phi^*C
\]
(3.8.1)

sending \( \sigma_{g, D, f} \), with \( D \in C(\epsilon_i + \epsilon_j), \ i < j, \phi(i) > \phi(j) \), to \( \sigma_{g, D, f}^{-1} \). For any \( (2, 1) \)-category \( C \) and arrowy \( 2 \)-subcategories \( A_1, \ldots, A_n \), we have an isomorphism of \( m \)-fold categories
\[
\phi^*Q_{A_1, \ldots, A_n}C \simeq Q_{A_{\phi(1)}, \ldots, A_{\phi(n)}}C
\]
(3.8.2)

sending every square in \( (Q_{A_1, \ldots, A_n}C)(\epsilon_{\phi(i)} + \epsilon_{\phi(j)}), \ 1 \leq i < j \leq n, \phi(i) > \phi(j) \) to its transpose obtained by inverting the 2-cell. For any \( n \)-fold category \( C \) and any \( 2 \)-category \( D \), we have a \( D^{Ob(C)} \)-functor
\[
PsFun^C(C, D) \simeq PsFun^C(\phi^*C, D)
\]
(3.8.3)

Sending \( (F, (F_i)_{1 \leq i \leq n}) \) to \( (e \circ (\phi^*F), (F_{\phi(i)})_{1 \leq i \leq n}) \), where \( e: \phi^*Q^nRD \to Q^mRD \) is (3.8.2). Here we have used Remark 2.3.23.

Combining (3.8.1) and (3.8.2), we get a 2-functor
\[
E_\phi: LTQ^mC \to LTQ^oC.
\]

If \( \phi \) is a bijection, then (3.8.1) and (3.8.3) are isomorphisms.

**Construction 3.9.** Let \( \phi: \{1, \ldots, m\} \to \{1, \ldots, n\} \) be an arbitrary map with \( n \geq 2 \) and let \( C \) be a \( (2, 1) \)-category. Similarly to Construction 3.1, we have a 2-functor
\[
E_\phi: LTQ^mC \to LTQ^oC
\]
(3.9.1)

sending \( \sigma_{g, D, f} \) to \( \sigma_{g, D, f}^{-1} \), where \( D \in (Q^mC)(\epsilon_i + \epsilon_j), \ i < j, \phi(i) > \phi(j) \), and \( D^* \) is the transpose of \( D \) obtained by inverting the 2-cell.

We have the following analogue of Proposition 3.6.

**Proposition 3.10.** Let \( \phi: \{1, \ldots, m\} \to \{1, \ldots, n\} \) be a map with \( n \geq 2 \) and let \( \phi' \) be a section of \( \phi \). Let \( C \) be a \( (2, 1) \)-category and \( A_1, \ldots, A_m \) be arrowy 2-subcategories of \( C \) such that \( A_i \subseteq A_{\phi(i)} \) for all \( 1 \leq i \leq m \). Then
\[
E_\phi: LTQ_{A_1, \ldots, A_m}C \to LTQ_{A_{\phi(1)}, \ldots, A_{\phi(m)}}C,\quad E_{\phi'}: LTQ_{A_{\phi(1)}, \ldots, A_{\phi(m)}}C \to LTQ_{A_1, \ldots, A_m}C
\]
are 2-quasi-inverses of each other.
4 Gluing two pseudofunctors

In this section, we study the gluing of two pseudofunctors. The main result of this section is Theorem 4.9, which is the case \( n = 2 \) of the main theorem of this article. Although the theorem can be stated without reference to Section 2 (see Remarks 4.3 and 4.4), the constructions of Section 2 allow us to give a more conceptual interpretation, which is an essential ingredient of the proof. Indeed, we deduce Theorem 4.9 from a criterion involving fundamental groups in the 2-category \( \mathcal{LT}_{Q,A,B}C \) constructed in Section 2 (Theorem 4.13).

Throughout this section, \( C \) is a \((2,1)\)-category, and \( A \) and \( B \) are two arrowy (Definition 1.6) 2-subcategories of \( C \). In particular, \( \text{Ob}(A) = \text{Ob}(B) = \text{Ob}(C) \). The 2-functor \( T^Q_C \rightarrow C \) (2.19.3) induces a 2-functor

\[
E: \mathcal{LT}_{Q,A,B}C \rightarrow C.
\]

For any 2-category \( D \), the composite \( \text{Ob}(C) \)-functor

\[
\text{PsFun}(C, D) \xrightarrow{Q_D} \text{PsFun}^C(Q_{A,B}C, D) \xrightarrow{\Phi} \text{PsFun}(\mathcal{LT}_{Q,A,B}C, D),
\]

where \( Q_D \) is given by (2.24.1) and \( \Phi \) is a \( \text{Ob}(C) \)-equivalence by Remark 2.26, is induced by \( E \). Thus Proposition 1.12 applied to \( E \) gives the following.

**Proposition 4.1.**

(1) If \( E \) is pseudofull, then \( Q_D \) is 2-faithful for every 2-category \( D \).

(2) \( E \) is a 2-equivalence if and only if \( Q_D \) is a \( \text{Ob}(C) \)-equivalence for every 2-category \( D \).

**Definition 4.2.** We call \( \text{GD}_{A,B}(C, D) = \text{PsFun}^C(Q_{A,B}C, D) \) the 2-category of gluing data from \( C \) to \( D \) relative to \( A \) and \( B \).

**Remark 4.3.** Let us explicitly describe the \( \text{Ob}(C) \)-category \( \text{GD}_{A,B}(C, D) \). An object of it is a triple \((F_A, F_B, (G_D))\) consisting of an object \( F_A \) of \( \text{PsFun}(A, D) \), an object \( F_B \) of \( \text{PsFun}(B, D) \) satisfying \( |F_A| = |F_B| \), and a family of invertible 2-cells of \( D \)

\[
G_D : F_A(i)F_B(q) \Rightarrow F_B(p)F_A(j),
\]

\( D \) running over \((A, B)\)-squares in \( C \) of the form (2.14.1). The triple is subject to the following conditions:

(a) For any square \( D \) of the form

\[
\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
\cap & \alpha & \cap \\
X & \xrightarrow{j} & Y,
\end{array}
\]

the following square commutes

\[
\begin{array}{ccc}
F_A(i) & \xrightarrow{F_A(\alpha)} & F_A(j) \\
\cap & \cap & \cap \\
F_A(i)F_B(\text{id}_X) & \xrightarrow{G_D} & F_B(\text{id}_Y)F_A(j).
\end{array}
\]

(a’) For any square \( D \) of the form

\[
\begin{array}{ccc}
X & \xrightarrow{q} & Y \\
\cap & \alpha & \cap \\
X & \xrightarrow{p} & Y,
\end{array}
\]

the following square commutes

\[
\begin{array}{ccc}
F_B(q) & \xrightarrow{F_B(\alpha)} & F_B(p) \\
\cap & \cap & \cap \\
F_A(\text{id}_Y)F_B(q) & \xrightarrow{G_D} & F_B(p)F_A(\text{id}_X).
\end{array}
\]
Remark 4.4. Let us explicitly describe the $\mathcal{D}^{\text{Ob}(\mathcal{C})}$-functor

$$Q_D : \text{PsFun}(\mathcal{C}, \mathcal{D}) \to \text{GD}_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D}).$$

(b) If $D, D', D''$ are respectively the upper, lower and outer squares of the diagram

\[
\begin{array}{c}
X_1 \xrightarrow{i_1} Y_1 \\
\downarrow q \quad \quad \quad \quad \downarrow p \\
X_2 \xrightarrow{i_2} Y_2 \\
\downarrow q' \quad \quad \quad \quad \downarrow p' \\
X_3 \xrightarrow{i_3} Y_3,
\end{array}
\]

then the following pentagon commutes

\[
F_A(i_3)F_B(q')F_B(q) \xrightarrow{G_{D'}} F_B(p')F_A(i_2)F_B(q) \xrightarrow{G_D} F_B(p')F_B(p)F_A(i_1).
\]

(b') If $D, D', D''$ are respectively the left, right and outer squares of the diagram

\[
\begin{array}{c}
X_1 \xrightarrow{j} X_2 \xrightarrow{j'} X_3 \\
\downarrow p_1 \quad \quad \quad \quad \downarrow p_2 \quad \quad \quad \quad \downarrow p_3 \\
Y_1 \xrightarrow{i} Y_2 \xrightarrow{i'} Y_3,
\end{array}
\]

then the following pentagon commutes

\[
F_A(i')F_A(i)p_1 \xrightarrow{G_D} F_A(i')F_A(p_2)F_A(j) \xrightarrow{G_{D'}} F_A(p_3)F_A(j')F_A(j).
\]

A morphism $(F_A, F_B, G) \to (F'_A, F'_B, G')$ of $\text{GD}_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$ is a pair $(\alpha_A, \alpha_B)$ consisting of a morphism $\alpha_A : F_A \to F'_A$ of $\text{PsFun}(\mathcal{A}, \mathcal{D})$ and a morphism $\alpha_B : F_B \to F'_B$ of $\text{PsFun}(\mathcal{B}, \mathcal{D})$ such that $|\alpha_A| = |\alpha_B|$ and satisfying the following condition

(m) For any $(\mathcal{A}, \mathcal{B})$-square $D$ (2.14.1), the following hexagon commutes

\[
\begin{array}{c}
\alpha_0(W)F_A(i)F_B(q) \xrightarrow{\alpha_A(i)} F_A(i)\alpha_0(Z)F_B(q) \xrightarrow{\alpha_B(q)} F_A(j)F_B(p)\alpha_0(\alpha_0(X)) \xrightarrow{G_D} F_B(p)F_A(j)\alpha_0(\alpha_0(X)).
\end{array}
\]

Here $\alpha_0 = |\alpha_A| = |\alpha_B|$.

A 2-cell of $\text{GD}_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$ is a pair $(\Xi_A, \Xi_B) : (\alpha_A, \alpha_B) \Rightarrow (\alpha'_A, \alpha'_B)$ consisting of a 2-cell $\Xi_A : \alpha_A \Rightarrow \alpha'_A$ of $\text{PsFun}(\mathcal{A}, \mathcal{D})$ and a 2-cell $\Xi_B : \alpha_B \Rightarrow \alpha'_B$ of $\text{PsFun}(\mathcal{B}, \mathcal{D})$ such that $|\Xi_A| = |\Xi_B|$. The $\mathcal{D}^{\text{Ob}(\mathcal{C})}$-category structure of $\text{GD}_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$ is given by the 2-functor defined by

\[
(F_A, F_B, G) \mapsto |F_A| = |F_B|, \quad (\alpha_A, \alpha_B) \mapsto |\alpha_A| = |\alpha_B|, \quad (\Xi_A, \Xi_B) \mapsto |\Xi_A| = |\Xi_B|.
\]

In an object $(F_A, F_B, G)$ of $\text{GD}_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$, $G$ expresses compatibility between the pseudofunctors $F_A$ and $F_B$. The necessity of such compatibility for gluing $F_A$ and $F_B$ into a pseudofunctor $\mathcal{C} \to \mathcal{D}$ is also clear from the following remark.
For an object $F$ of $\text{PsFun}(\mathcal{C}, \mathcal{D})$, we have $Q_D(F) = (F|\mathcal{A}, F|\mathcal{B}, G)$. Here, for an $(\mathcal{A}, \mathcal{B})$-square $D$, $G_D$ is the composition

$$F(i)F(q) \Longrightarrow F(iq) \xrightarrow{F(\alpha)} F(pj) \Longrightarrow F(p)F(j).$$

For a morphism $\alpha: F \to F'$ of $\text{PsFun}(\mathcal{C}, \mathcal{D})$, $Q_D(\alpha)$ is

$$(\alpha|\mathcal{A}, \alpha|\mathcal{B}): (F|\mathcal{A}, F|\mathcal{B}, G) \to (F'|\mathcal{A}, F'|\mathcal{B}, G').$$

For a 2-cell $\Xi: \alpha \Rightarrow \alpha'$ of $\text{PsFun}(\mathcal{A}, \mathcal{B})$, $Q_D(\Xi)$ is

$$(\Xi|\mathcal{A}, \Xi|\mathcal{B}): (\alpha|\mathcal{A}, \alpha|\mathcal{B}) \Rightarrow (\alpha'|\mathcal{A}, \alpha'|\mathcal{B}).$$

**Remark 4.5.** The isomorphisms (3.8.2) and (3.8.3) induce an isomorphism of $O(\mathcal{C})$-categories

$$\Omega: GD_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D}) \cong GD_{\mathcal{B}, \mathcal{A}}(\mathcal{C}, \mathcal{D}),$$

which can be described as follows. To an object $(F, F', G)$ of $GD_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$, $\Omega$ associates $(F', F, G^*)$. Here, for every $(\mathcal{A}, \mathcal{B})$-square $D$, $G_D^* = G_D^{-1}$, where $D^*$ is the square obtained from $D$ by inverting $\alpha$. To a morphism $(\alpha_A, \alpha_B): (F_A, F_B, G) \to (F'_A, F'_B, G')$ of $GD_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$, $\Omega$ associates

$$(\alpha_B, \alpha_A): (F_B, F_A, G^*) \to (F'_B, F'_A, G'^*).$$

To a 2-cell $(\Xi_A, \Xi_B): (\alpha_A, \alpha_B) \Rightarrow (\alpha'_A, \alpha'_B)$ of $GD_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$, $\Omega$ associates

$$(\Xi_B, \Xi_A): (\alpha_B, \alpha_A) \Rightarrow (\alpha'_B, \alpha'_A).$$

**Definition 4.6.** We say that $\mathcal{A}$ and $\mathcal{B}$ generate $\mathcal{C}$ if for every morphism $f$ of $\mathcal{C}$, there exist morphisms $i_1, \ldots, i_n$ of $\mathcal{A}$ and $p_1, \ldots, p_n$ of $\mathcal{B}$ and a 2-cell $\eta: i_1 \cdots i_n \Rightarrow f$ of $\mathcal{C}$.

**Remark 4.7.** We have $\mathcal{A}$ and $\mathcal{B}$ generate $\mathcal{C}$ if and only if $\mathcal{E}: \mathcal{C} \to \text{QAx}_{\mathcal{A}, \mathcal{B}}\mathcal{C}$ is pseudofull. In particular, if $\mathcal{A}$ and $\mathcal{B}$ generate $\mathcal{C}$, then $Q_D$ is 2-faithful by Proposition 1.1.1.

**Definition 4.8.** We say that $(\mathcal{A}, \mathcal{B})$ is squarable in $\mathcal{C}$ if every pair of morphisms $i: Z \to W$ in $\mathcal{A}$ and $p: Y \to W$ in $\mathcal{B}$ with the same target can be completed into an $(\mathcal{A}, \mathcal{B})$-square 2-Cartesian in $\mathcal{C}$.

The main result of this section is the following.

**Theorem 4.9.** Let $\mathcal{C}$ be a $(2, 1)$-category and let $\mathcal{A}$, $\mathcal{B}$ be arrowy 2-subcategories of $\mathcal{C}$. Assume the following:

1. For every morphism $f: X \to Y$ of $\mathcal{C}$, there exist a morphism $i: X \to Z$ of $\mathcal{A}$, a morphism $p: Z \to Y$ of $\mathcal{B}$, and a 2-cell $\alpha: pi \Rightarrow f$ of $\mathcal{C}$.

2. $(\mathcal{B}, \mathcal{B})$ is squarable in $\mathcal{C}$.

Then

$$Q_D: \text{PsFun}(\mathcal{C}, \mathcal{D}) \to GD_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$$

is a $O(\mathcal{C})$-equivalence for every 2-category $\mathcal{D}$.

**Remark 4.10.** Let us recall the results of Deligne [6] Proposition 3.3.2 and Ayoub [1] Theorem 1.3.1. Both assume $\mathcal{C}$ to be an (ordinary) category. Consider the conditions [1] and [2] of Theorem 1.9 as well as the following conditions:

- (2a) Fiber products exist in $\mathcal{B}$ and are fiber products in $\mathcal{C}$.
- (2b) $\mathcal{C}$ admits fiber products and morphisms of $\mathcal{A}$ and $\mathcal{B}$ are stable under base change by morphisms of $\mathcal{C}$. In particular, all isomorphisms in $\mathcal{C}$ are in $\mathcal{A} \cap \mathcal{B}$. Moreover, the diagonal of every morphism in $\mathcal{C}$ is in $\mathcal{A}$.

Deligne assumes [1] and (2a), while Ayoub assumes [1] and (2b). In our language, their conclusions can be stated as saying that $Q_D$ induces a bijection between equivalence classes of $\text{PsFun}(\mathcal{C}, \mathcal{D})$ and $GD_{\mathcal{A}, \mathcal{B}}(\mathcal{C}, \mathcal{D})$. There are no implications between (2a) and (2b). Moreover, both are stronger than [2]. Thus Theorem 1.10 is a common generalization of the results of Deligne and Ayoub. Even under the assumptions of Deligne and Ayoub, the conclusion of Theorem 1.9 is more precise in the sense that it establishes a 2-equivalence of 2-categories. As we have remarked in the Introduction, this precision is useful in the construction of pseudonatural transformations.
By Proposition 4.11(2), the conclusion of Theorem 4.9 is equivalent to saying that $E: \mathcal{L}T\mathcal{Q}_{A,B}\mathcal{C} \to \mathcal{C}$ is a 2-equivalence. We will construct a 2-quasi-inverse $F$ as follows. The quadruple $(Z, i, p, \alpha)$ in Theorem 4.9(1) corresponds to the diagram

$$
\begin{array}{c}
X \xrightarrow{i} Z \\
\downarrow f \quad \downarrow \beta \\
Y
\end{array}
$$

is a called a compactification of $f$ (relative to $A$, $B$). Given a choice of compactification, $p * i$ is a candidate for $Ff$. To express the dependence on choices, we need to organize the set $\text{Comp}_{A,B}(f)$ of compactifications into 2-categories.

**Definition 4.11.** Let $f: X \to Y$ be a morphism of $\mathcal{C}$. Let $\mathcal{P}$ be an arrowy 2-subcategory of $\mathcal{C}$. We define the 2-category $\text{Comp}_{A,B}^\mathcal{P}(f)$ of compactifications of $f$ (relative to $A$, $B$, and $\mathcal{P}$) as follows. The objects are compactifications of $f$ (relative to $A$, $B$, independent of $\mathcal{P}$). A morphism $(Z, i, p, \alpha) \to (W, j, q, \beta)$ is a triple $(r, \gamma, \delta)$ consisting of a morphism $r: Z \to W$ of $\mathcal{P}$, and 2-cells $\gamma: ri \Rightarrow j$ and $\delta: qr \Rightarrow p$ of $\mathcal{C}$, fitting in the diagram

$$
\begin{array}{c}
X \xrightarrow{i} Z \\
\downarrow f \quad \downarrow W \\
Y
\end{array}
$$

where the outer triangle is $\alpha$. A 2-cell $(r, \gamma, \delta) \Rightarrow (r', \gamma', \delta')$ is a 2-cell $\epsilon: r \Rightarrow r'$ of $\mathcal{P}$ such that $\gamma = \gamma' \circ (\epsilon i)$ and $\delta = \delta' \circ (qe)$. We omit $A$ and $B$ from the notation when no confusion arises.

A compactification $(Z, i, p, \alpha)$ of $f$ gives a morphism $p * i: X \to Y$ of $\mathcal{T}\mathcal{Q}_{A,B}\mathcal{C}$. A morphism $(r, \gamma, \delta)$ of $\text{Comp}_{A,B}^\mathcal{P}(f)$ as above gives a 2-cell

$$
q * j \xrightarrow{q \ast j \ast \text{id}_X} q \ast j \ast \text{id}_X \xrightarrow{\text{id}_X \ast \delta \ast \text{id}_X} q \ast r \ast \text{id}_X \xrightarrow{\text{id}_X \ast \delta \ast \text{id}_X} qr \ast i \Rightarrow p * i
$$

of $\mathcal{T}\mathcal{Q}_{A,B}\mathcal{C}$, where the last 2-cell is given by the image of $\delta$ under the pseudofunctor $B \to \mathcal{T}\mathcal{Q}_{A,B}\mathcal{C}$ induced by $G_{\mathcal{V}}(2.19.3)$, and $D$ is the square

$$
\begin{array}{c}
X \xrightarrow{i} Z \\
\downarrow j \quad \downarrow r \\
X \xrightarrow{j} W
\end{array}
$$

given by $\gamma^{-1}$. This defines a 2-functor

$$
\text{Comp}_{A,B}^\mathcal{P}(f) \to (\mathcal{T}\mathcal{Q}_{A,B}\mathcal{C})(X,Y)^{op}.
$$

Composing with the functor inverting 2-cells in $\mathcal{E} = \mathcal{L}T\mathcal{Q}_{A,B}\mathcal{C}$, we get a 2-functor

(4.11.1) $$
G_f: \text{Comp}_{A,B}^\mathcal{P}(f) \to \mathcal{E}(X,Y),
$$

corresponding to a functor $O\text{Comp}_{A,B}^\mathcal{P}(f) \to \mathcal{E}(X,Y)$.

To investigate the compatibility of $F$ with composition, we also need to consider compactifications of sequences of morphisms. Recall that, for $n \geq 0$, $[n]$ denotes the totally ordered set $\{0, \ldots, n\}$. We define the 2-category of $n$-simplices of $\mathcal{C}$ to be the 2-category $\text{UPsFun}([n], \mathcal{C})$ of strictly unital pseudofunctors $[n] \to \mathcal{C}$ (Notation 1.4). A sequence of morphisms $X_0 \xrightarrow{f_1} \ldots \xrightarrow{f_n} X_n$ defines a 2-functor $[n] \to \mathcal{C}$, which we denote by $(f_n, \ldots, f_1)$. Let $\Delta_n$ be the partially ordered set

$$\{(k, l) \in [n] \times [n] \mid k \geq l\}.$$

The diagonal embedding $[n] \to \Delta_n$ induces a 2-functor $\text{UPsFun}(\Delta_n, \mathcal{C}) \to \text{UPsFun}([n], \mathcal{C})$. The following generalization of Definition 4.11 slightly generalizes [6, Définition 3.2.5] and [1, Section 1.3.1].
**Definition 4.12.** Let $\sigma$ be an $n$-simplex of $\mathcal{C}$ and let $\mathcal{P}$ be an arrowy 2-subcategory of $\mathcal{C}$. The 2-category of compactifications of $\sigma$ (relative to $A$, $B$ and $\mathcal{P}$), $\text{Comp}^\mathcal{P}(\sigma) = \text{Comp}^\mathcal{P}_{A,B}(\sigma)$, is the 2-faithful 2-subcategory of the strict fiber product

$$\text{UPsFun}(\Delta_n, \mathcal{C}) \times_{\text{UPsFun}([n], \mathcal{C})} \{\sigma\}$$

spanned by pseudofunctors $F$ such that $F(k \to k', l)$ is a morphism of $A$ and $F(k', l \to l')$ is a morphism of $B$, for all elements $(k, l)$ and $(k', l')$ of $\Delta_n$, and pseudonatural transformations $\alpha$ such that $\alpha(k, l)$ is a morphism in $\mathcal{P}$ for every element $(k, l)$ of $\Delta_n$. We denote the set of objects by $\text{Comp}^\mathcal{P}_{A,B}(\sigma)$, which does not depend on $\mathcal{P}$.

In particular, a compactification of $(f_n, \ldots, f_1)$ can be represented by a diagram of the form

with horizontal arrows in $A$, vertical arrows in $B$, and suitable 2-cells.

Let $d: [m] \to [n]$ be an increasing map and let $\sigma$ be an $n$-simplex. We have a 2-functor

$$(4.12.1) \quad \text{Comp}^\mathcal{P}(\sigma) \to \text{Comp}^\mathcal{P}(\sigma \circ d)$$

sending $F: \Delta_n \to \mathcal{C}$ to the composition $dF: \Delta_m \xrightarrow{d \times d} \Delta_n \xrightarrow{E} \mathcal{C}$.

We will deduce Theorem 4.9 from the following criterion.

**Theorem 4.13.** Let $\mathcal{C}$ be a $(2,1)$-category and let $A$, $B$ be arrowy 2-subcategories of $\mathcal{C}$. Let $E = \mathcal{L}\mathcal{T}\mathcal{Q}_{A,B}\mathcal{C}$. Assume the following:

1. For any morphism $f: X \to Y$ of $\mathcal{C}$, there exists a compactification $\kappa = (Z, i, p, \alpha)$ of $f$ relative to $A$, $B$ such that the homomorphism of fundamental groups $\pi_1(\mathcal{O}\text{Comp}^B(f), \kappa) \to \pi_1(\mathcal{E}(X, Y), p \ast i)$ induced by $G_f$ has trivial image.

2. For any sequence of two morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, $\text{Comp}^B(g, f)$ is connected.

Then the 2-functor $E: \mathcal{E} \to \mathcal{C}$ is a 2-equivalence.

Note that [2] implies that $\text{Comp}^B(f)$ is connected, so that [1] holds for every compactification of $f$.

**Remark 4.14.** The proof of Theorem 4.13 makes use of the following generalization of (4.11.1). There exists a unique way to associate, for every $n \geq 0$ and every $n$-simplex $\sigma$ of $\mathcal{C}$, a 2-functor

$$(4.14.1) \quad G_\sigma: \text{Comp}^B(\sigma) \to \text{UPsFun}([n], \mathcal{E})$$

satisfying the following conditions:

0. For every 0-simplex (i.e. object) $X$ of $\mathcal{C}$, the image of $G_X$ is $X$.

1. For every 1-simplex (i.e. morphism) $f$ of $\mathcal{C}$, $G_f$ is (4.11.1).
(2) For every 2-simplex \( \sigma \) of \( \mathcal{C} \), \( G_\sigma \) sends a compactification as partly shown by the diagram

\[
\begin{array}{c}
\xymatrix{ X \ar[r]^{q} & X' \ar[r]^{q} & X'' \ar[r]^{q} & Y \ar[r]^{r} & Y' \ar[r]^{r} & Y'' \ar[r]^{r} & Z }
\end{array}
\]

composes with the diagonal embedding \( \Delta_{n,j-1} \) is in \( B \), \( F(k', \ell \to \ell') \) is in \( B \). The case \( j = 0 \) is trivial as the diagonal embedding \([n] \to \Delta_{n,0}\) is an isomorphism. For \( j \geq 1 \), assume that \( F: \Delta_{n,j} \to \mathcal{C} \) has been constructed. Applying the hypothesis, we get a decomposition \( F((k+j-1, k) \to (k+j, k+1)) \simeq p_i \) for \( 0 \leq k \leq n-j \). We take \( F(k+j-1 \to k+j, k) = i \) and \( F(k+j, k \to k+1) = p \). In general, for \( P \leq Q \) in \( \Delta_{n,j} \) with \( P \neq Q \), we take

\[
F(P \to Q) = F(Q' \to Q) F(P' \to Q') F(P \to P')
\]

where \( P', Q' \in \Delta_{n,j-1} \) are given by

\[
P' = \begin{cases} P & \text{if } P \in \Delta_{n,j-1}, \\ P + (0,1) & \text{if } P \not\in \Delta_{n,j-1} \end{cases} \quad Q' = \begin{cases} Q & \text{if } Q \in \Delta_{n,j-1}, \\ Q - (1,0) & \text{if } Q \not\in \Delta_{n,j-1}. \end{cases}
\]

The coherence constraint of \( F \) is given by the obvious 2-cells. For \( j = n, \Delta_{n,n} = \Delta_n \) so that \( \text{Comp}(\sigma) \) is nonempty.

**Proof of Theorem 4.13.** For any set \( S \), we denote by \( S^\sim \) the category such that \( \text{Ob}(S^\sim) = S \) and for all elements \( X \) and \( Y \) of \( S \), there exists a unique morphism in \( S^\sim \) from \( X \) to \( Y \). By assumption, for any morphism \( f: X \to Y \) of \( \mathcal{C} \), \( G_f \) factorizes through a functor \( G_f^\sim: \text{Ob}(\text{Comp}^B(f)^\sim) \to \mathcal{E}(X,Y) \).
For every $f$, choose an object $\kappa_f$ of $\text{Comp}^B(f)$ such that $\kappa_{id_X} = ([X, id_X, id_X, id_{id_X}]$ for every object $X$ of $\mathcal{C}$. We denote by $d_i^n : [n-1] \to [n]$ and $s_i^n : [n+1] \to [n]$, $0 \leq i \leq n$, the face and degeneracy maps, respectively.

We construct a pseudofunctor $F : \mathcal{C} \to \mathcal{E}$ as follows. For any object $X$ of $\mathcal{C}$, we take $FX = X$. For any morphism $f$ of $\mathcal{C}$, we take $Ff = Gf(\kappa_f)$. A 2-cell $\alpha : f \Rightarrow g$ of $\mathcal{C}$ induces an isomorphism of 2-categories $H_{\alpha} : \text{Comp}^B(f) \to \text{Comp}^B(g)$ such that $G_HH_{\alpha} = G\alpha$. We take $F\alpha = G\alpha(H(\kappa_f) \Rightarrow \kappa_g)$.

We construct the coherence constraint of $F$ as follows. Let $X \xrightarrow{\lambda} Y \xrightarrow{\beta} Z$ in $\mathcal{C}$ be a sequence of morphisms of $\mathcal{C}$. We consider the 2-simplex $(g, f)$ of $\mathcal{C}$ and the 2-functor $G_{\alpha} : (g, f) \to \text{Comp}^B(g)$, $G_{\alpha}(\lambda)$ is a 2-simplex of $\mathcal{E}$ with edges $Gf(d_1^2\lambda)$, $Gf(d_2^2\lambda)$, $Gf(d_3^2\lambda)$. Applying Lemma 2.10 to $G_{\alpha}(\lambda)$, we get a unique pair $(F_x, \phi_x)$, where $F_x$ is a 2-simplex of $\mathcal{E}$ with edges $Ff$, $F(gf)$, $(g\alpha)F$, and $\phi_x : G_{\alpha}(\lambda) \to Fx$ is a morphism of $\text{UPSFun}(2, \mathcal{E}) \times_{\mathcal{E}^{\text{Comp}(2)}} \{(g, f)\}$ satisfying

$$\phi_x(0 \to 1) = Gf(d_1^2\lambda \Rightarrow \kappa_f), \quad \phi_x(0 \to 2) = Gf(d_2^2\lambda \Rightarrow \kappa_f), \quad \phi_x(1 \to 2) = Gf(d_3^2\lambda \Rightarrow \kappa_f).$$

By the uniqueness of the pair, for any morphism $\psi : \lambda \Rightarrow \lambda'$ of $\text{Comp}^B(g, f)$, we have $F\psi = F\alpha'$ and $\phi_\psi = \phi_\alpha \circ G_{\alpha}(\psi)$. It then follows from the connectedness of $\text{Comp}^B(g, f)$ that $F\alpha$ does not depend on the choice of $\lambda$ and we denote the corresponding 2-cell of $\mathcal{E}$ by $F_{\alpha} : Ff \Rightarrow F(gf)$.

A 2-cell $\alpha : f \Rightarrow f'$ of $\mathcal{C}$ induces an isomorphism of 2-categories $H_{\alpha} : \text{Comp}^B(g, f) \to \text{Comp}^B(g, f')$, compatible with $H_{\alpha}$ and $H_{\alpha'}$ and such that $G_{\alpha}H_{\alpha} = G_{\alpha'}$. Thus $F_{\alpha}f$ is functorial in $f$. Similarly $F_{\alpha}$ is functorial in $g$. For any object $X$ of $\mathcal{C}$, we denote by $u_X : id_X \Rightarrow \phi_{1X} \circ id_XF = F(id_X)$ the 2-cell in $\mathcal{E}$ induced by $id_X$ and $\phi_1$.

Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ be a sequence of morphisms of $\mathcal{C}$. We consider the 3-simplex $\sigma = (h, g, f)$ of $\mathcal{C}$ and the 2-functor $G_{\sigma}$. For any object $\lambda$ of $\text{Comp}^B(\sigma)$, $G_{\sigma}(\lambda)$ is a 3-simplex of $\mathcal{E}$ with faces $G_{\alpha}(\kappa_f)$, $G_{\alpha}(\kappa_g)$, $G_{\alpha}(\kappa_h)$, $G_{\alpha}(\kappa_{gf})$, and $G_{\alpha}(\kappa_{hf})$, and $\phi_{\alpha} : G_{\sigma}(\lambda) \to F\lambda$ is a morphism of $\text{UPSFun}(3, \mathcal{E}) \times_{\mathcal{E}^{\text{Comp}(3)}} \{(h, g, f)\}$ such that

$$\phi_{\alpha}(e) = G_{\sigma}(d_1^3\lambda \Rightarrow \kappa_{\sigma(e)})$$

for all edges $e$ of $[3]$. Here $d_i : [1] \to [3]$ denotes the map determined by $e$. By construction, $d_i^3F\lambda = Fd_i^3\lambda$ for $0 \leq i \leq 3$. Thus $F\lambda$ implies that the diagram

$$F(h)F(f)F(g) \xrightarrow{F_{\alpha}} F(h)F(\alpha)F(\alpha) \xrightarrow{F_{\alpha}} F(hg)F(f)$$

commutes, which proves the composition axiom.

Let $f : X \to Y$ be a morphism of $\mathcal{C}$. By construction, $F_{id_X,f}$ is given by $G_{id_X}(s_1^3\kappa_f)$ and $F_{f, id_Y}$ is given by $G_{f, id_Y}(s_1^3\kappa_f)$. Thus $F_{id_X,f}$ and $F_{id_Y}$ are induced by $\theta^h$ and $\theta^v$. In other words, the diagram

$$F(\alpha) \xrightarrow{u_X} F(\alpha F(id_X)) \xrightarrow{F_{f, id_Y}} F(f)F(id_X)$$

commutes, which proves the unit axiom. This finishes the construction of $F$.

We define a pseudonatural equivalence $\epsilon : EF \to id_X$ sending $X$ to $id_X$ as follows. To every morphism $f$ of $\mathcal{C}$, we associate the 2-cell $\pi : f \Rightarrow id_X$ in $\kappa_f = (Z, i, p, \alpha)$. We define a pseudonatural equivalence $\eta : id_E \to FE$ sending $X$ to $id_X$ as follows. To every morphism $f$ of $\mathcal{A}$ (resp. $B$), we associate the composition

$$f \xrightarrow{id_X \circ \sigma} F_{id_X,f} \xrightarrow{p \circ \alpha} \eta \xrightarrow{id_Y \circ \sigma} F(\alpha)F(id_X) \xrightarrow{p \circ \beta} F(id_X) \xrightarrow{p \circ \gamma} F(id_X).$$

(4.15.1)
where $D$ (resp. $D'$) is the square
\[
\begin{array}{ccc}
X & \rightarrow & Z \\
\downarrow^D & & \downarrow^p \\
X & \rightarrow & Y
\end{array}
\quad \text{(resp. )}
\begin{array}{ccc}
X & \rightarrow & Z \\
\downarrow^f & & \downarrow^p \\
Y & \rightarrow & Y
\end{array}
\]
induced by $\kappa_f$.

**Remark 4.16.** Similarly to (4.11.1), we have a 2-functor
\[(4.16.1) \quad G'_f : \text{Comp}^A(f) \to (\mathcal{T}\mathcal{Q}_{A,B}\mathcal{C})(X,Y) \to \mathcal{E}(X,Y).\]

It sends every morphism $(r,\gamma,\delta) : (Z,i,p,\alpha) \to (W,j,q,\beta)$ of $\text{Comp}^A(f)$ to the composition
\[
p * i \xrightarrow{r} \text{id}_{X,Y} * p * i \xrightarrow{\sigma_{\text{id}_{X,Y},p,i}} q * r * i \xrightarrow{\gamma_{q,r,i,\text{id}_{X,Y}}} q * ri \xrightarrow{\gamma} q * j,
\]
where $D$ is the square
\[
\begin{array}{ccc}
Z & \rightarrow & W \\
\downarrow^p & & \downarrow^q \\
Y & \rightarrow & Y
\end{array}
\]
given by $\delta^{-1}$. The analogue of Theorem 4.13 holds with $\text{Comp}^B$ replaced by $\text{Comp}^A$.

Moreover, (4.11.1) and (4.16.1) induce an object of $\mathbb{P}\mathbb{F}\mathbb{u}\mathbb{n}(\text{Comp}(f), \mathcal{E}(X,Y))$, where
\[
\text{Comp}(f) = \mathcal{Q}_{\text{Comp}^A(f),\text{Comp}^B(f)} \text{Comp}(f).
\]

Applying (2.24.3), we obtain a 2-functor
\[(4.16.2) \quad \mathcal{T}\text{Comp}(f) \rightarrow \mathcal{E}(X,Y)\]
extending (4.11.1) and (4.16.1), which will be used in the proof of Theorem 4.9.

By Proposition 4.1 (2), Theorem 4.9 follows from the following.

**Proposition 4.17.** Under the assumptions of Theorem 4.9, for every $n$-simplex $\sigma$ of $C$, $\text{Comp}^B(\sigma)^{\text{coop}}$ is directed. Moreover, the conditions of Theorem 4.13 is satisfied.

Here we say that a 2-category $D$ is **directed** if its underlying category is directed, namely, if it is nonempty and if for every pair of objects $X$ and $Y$ of $D$, there exists an object $Z$ of $D$ and morphisms $X \rightarrow Z$ and $Y \rightarrow Z$ of $D$. A directed 2-category is connected.

**Proof.** By Lemma 4.15 assumption (1) of Theorem 4.1 implies that $\text{Comp}^B(\sigma)$ is nonempty. Next we show the following:

(*) Every morphism of $n$-simplices $\sigma \rightarrow \tau$ in $C$, where $\tau$ is an $n$-simplex of $B$, is isomorphic to the composition $\sigma \xrightarrow{p} \tau' \xrightarrow{\rho} \tau$, where $p$ is a morphism of $n$-simplices of $B$ and $i(j)$ is a morphism of $A$ for all $0 \leq j \leq n$.

We proceed by induction on $n$. The case $n = 0$ is assumption (1) of Theorem 4.1. For $n \geq 0$, induction hypothesis provides $p \mid \{1,\ldots,n\}$ and $i \mid \{1,\ldots,n\}$. Applying assumption (2) of Theorem 4.9 we obtain a 2-commutative diagram in $C$
\[
\begin{array}{ccc}
\sigma(0) & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
\sigma(1) & \xrightarrow{i(1)} & \tau'(1) \\
& & \downarrow \\
& & \tau(1)
\end{array}
\]
where the square on the right is in $B$. Applying assumption (1) of Theorem 4.10 to $f$, we get $f \simeq qh$. Replacing $f$ by $h$, we may assume $f$ is a morphism of $A$. Then it suffices to take $\tau'(0) = X$. 

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Now let $F$ and $F'$ be two objects of $\text{Comp}^B_A(B(\sigma))$. Applying assumption \([2]\) of Theorem \([1.9]\) we obtain morphisms $F'' \to F$ and $F'' \to F'$ of $\text{Comp}^B_C(B(\sigma))$, where $F''(k,l)$ is a 2-fiber product $F(k,l) \times_{\sigma(k)} F'(k,l)$ in $C$. To show that $\text{Comp}^B_A(B(\sigma))$ is directed, it then suffices to show that, for every object $F$ of $\text{Comp}^B_C(B(\sigma))$, there exists a morphism $F' \to F$ of $\text{Comp}^B_A(B(\sigma))$ such that $F''$ is an object of $\text{Comp}^B_A(B(\sigma))$. We will construct a sequence

$$F' = F_n \to \cdots \to F_1 \to F_0 = F$$

of morphisms of $\text{Comp}^B_C(B(\sigma))$ such that $F_j | \Delta_j$ is an object of $\text{Comp}^B_A(B(\sigma)[j])$, $0 \leq j \leq n$. For $j \geq 1$, assume $F_j = F_{j-1}$ constructed. Applying \((*)\), we get a decomposition $F_j | \{j - 1\} \times \{j - 1\} \to \tau \to F_{j-1} | \{j\} \times \{j\}$. We take $F_j | \Delta_j = F_{j-1} | \Delta_{j-1}, F_j | \{j\} \times \{j\} = \tau, F_j(j,j) = F_{j-1}(j,j), F_j(\Delta_n - \Delta_j) = F_{j-1}(\Delta_n - \Delta_j)$. This finishes the proof of the fact that $\text{Comp}^B_A(B(\sigma))^{\text{comp}}$ is directed. In particular, we have verified condition \([2]\) of Theorem \([1.9]\).

The verification of condition \([1]\) of Theorem \([1.9]\) is reminiscent of the proof of \([1]\), Lemme 1.3.8. Let $f: X \to Y$ be a morphism of $C$. Endow $S = \text{Ob}(\text{Comp}^B(f))$ with the following preorder: $\kappa \leq \lambda$ if and only if there exists a morphism $\lambda \to \kappa$. Since $S$ is directed, it is simply connected. Thus it suffices to show that for every pair of morphisms $(r_1, r_2) = (W,j,q,\tau) \Rightarrow (Z,i,p,\sigma)$ of $\text{Comp}^B(f)$, $G_f(r_1) = G_f(r_2)$. Applying assumption \([2]\) of Theorem \([1.9]\) we obtain a diagram in $B$

\[
\begin{array}{ccc}
W \times_Y W & \longrightarrow & Z \times_Y W \\
\downarrow & & \downarrow \\
W \times_Y Z & \longrightarrow & Z \times_Y Z
\end{array}
\]

where all the squares are 2-Cartesian in $C$. Let $q_\alpha: W \times_Y W \to W, \alpha = 1,2$ be the projections. Applying \((*)\), we obtain a decomposition

\[
\begin{array}{ccc}
X & \longrightarrow & W' \\
\downarrow & \uparrow u & \downarrow \uparrow r_1 \times_Y r_2 \\
X & \longrightarrow & Z' \longrightarrow Z \times_Y Z,
\end{array}
\]

which induces 2-commutative squares

\[
\begin{array}{ccc}
W' & \longrightarrow & W \\
\downarrow & \uparrow q_\alpha^{-1} & \downarrow \uparrow r_\alpha \\
Z' & \longrightarrow & Z
\end{array}
\]

in $\text{Comp}^B(f)$. Since $\mathcal{E}(X,Y)$ is a groupoid, it suffices to show $G_f(p_1 s) = G_f(p_2 s)$ (and $G_f(q_1 t) = G_f(q_2 t)$). Applying \((*)\) again, we obtain a decomposition

\[
(4.17.1)
\]

where the composition of the second line is isomorphic to the diagonal. The diagram \([4.17.1]\) induces 2-commutative squares

\[
\begin{array}{ccc}
Z'' & \longrightarrow & Z' \\
\downarrow & \uparrow v & \downarrow \uparrow s \\
Z & \longrightarrow & Z \times_Y Z,
\end{array}
\]

where the composition of the second line is isomorphic to the diagonal. The diagram \([4.17.1]\) induces 2-commutative squares

\[
\begin{array}{ccc}
Z'' & \longrightarrow & Z' \\
\downarrow & \uparrow v' & \downarrow \uparrow s' \\
Z' & \longrightarrow & Z
\end{array}
\]

in $\text{Comp}^B(f)$. Thus it suffices to show $G_f(p_1 s') = G_f(p_2 s')$. Consider the 2-functor \([1.16.2]\). Note that $h$ induces a morphism in $\text{Comp}^A(f)$. Since $p_0 s' \ast h \simeq 1_{(Z,1,p,s')}$ in $\mathcal{T}\text{Comp}(f)$, the image of $p_0 s' \ast h$ under \([1.16.2]\) is the identity, so that $G_f(p_1 s') = G_f(p_2 s')$. One can also check more directly that
In this section, we study the gluing of finitely many pseudofunctors in general. The main result of this section is Theorem 5.9, which is the case \( n \geq 3 \) of the main theorem of this article. Although the constructions of Sections 2 and 3 can be avoided in the statement of the theorem (see Remarks 5.2, 5.3 and Construction 5.4), they allow us to give a more conceptual interpretation and is used in the proof (see for example the criterion of Theorem 5.10).

Throughout this section, \( C \) is a \( (2,1) \)-category, \( A_1, \ldots, A_n \), \( n \geq 1 \), are arrowy (Definition 1.6) \( 2 \)-subcategories of \( C \). Let \( D \) be a 2-category. By Remark 2.26, we have a \( D^{\text{Ob}(C)} \)-equivalence

\[
\Phi : \text{PsFun}^C(Q_{A_1}, \ldots, A_n, C, D) \to \text{PsFun}(\mathcal{C} \mathcal{T} Q_{A_1}, \ldots, A_n, C, D).
\]

We generalize Definition 4.2 as follows.

**Definition 5.1.** For \( n \geq 2 \), we call \( \text{GD}_{A_1, \ldots, A_n}(C, D) = \text{PsFun}^C(Q_{A_1}, \ldots, A_n, C, D) \) the 2-category of gluing data from \( C \) to \( D \), relative to \( A_1, \ldots, A_n \). For \( n = 1 \), we put \( \text{GD}_{A_1}(C, D) = \text{PsFun}(A_1, D) \).

**Remark 5.2.** Let us explicitly describe the \( D^{\text{Ob}(C)} \)-category \( \text{GD}_{A_1, \ldots, A_n}(C, D) \). An object of it is a pair

\[
((F_i)_{1 \leq i \leq n}, (G_{ij})_{1 \leq i < j \leq n}),
\]

where \( F_i : A_i \to D \) is an object of \( \text{PsFun}(A_i, D) \), and \( (F_i, F_j, G_{ij}) \) is an object of \( \text{GD}_{A_i A_j}(C, D) \), as described in Remark 4.8 satisfying the following condition:

**D** For \( 1 \leq i < j < k \leq n \) and any 2-commutative cube of the form

\[
(5.2.1)
\]

where \( a, b, a', b' \) are morphisms of \( A_i \), \( p, q, p', q' \) are morphisms of \( A_j \) and \( x, y, z, w \) are morphisms of \( A_k \), the following hexagon commutes

\[
\begin{align*}
F_i(a)F_j(b)F_k(x) & \xrightarrow{G_{ijk}} F_i(a)F_k(z)F_j(q') \xrightarrow{G_{ikj}} F_k(w)F_i(a')F_j(q') \\
\xrightarrow{G_{ijK}} F_j(p)F_i(b)F_k(x) & \xrightarrow{G_{ijp}} F_j(p)F_k(y)F_i(b') \xrightarrow{G_{ijl}} F_k(w)F_j(p')F_i(b')
\end{align*}
\]

where \( I, I', J, J', K, K' \) are respectively the right, left, front, back, bottom, top faces of the cube.
A morphism $((F_i),(G_{ij})) \to ((F'_i),(G'_{ij}))$ of GD_{A_1,\ldots,A_n}(\mathcal{C}, D)$ is a collection $(\alpha_i)_{1 \leq i \leq n}$ of morphisms $\alpha_i : F_i \to F'_i$ of PsFun(\mathcal{A}_i, D)$, such that for $1 \leq i < j \leq n$,

$$(\alpha_i, \alpha_j) : (F_i, F_j, G_{ij}) \to (F'_i, F'_j, G'_{ij})$$

is a morphism of GD_{A_i,A_j}(\mathcal{C}, D)$.

A 2-cell of GD_{A_1,\ldots,A_n}(\mathcal{C}, D)$ is a collection $(\Xi)_{1 \leq i \leq j \leq n}$: $(\alpha_i)_{1 \leq i \leq n} \Rightarrow (\alpha'_i)_{1 \leq i \leq n}$ of 2-cells $\Xi_i : \alpha_i \Rightarrow \alpha'_i$ of PsFun(\mathcal{A}_i, D)$ such that $|\Xi_i| = \cdots = |\Xi_n|$. The $\mathcal{D}^{\text{Ob}(\mathcal{C})}$-category structure of GD_{A_1,\ldots,A_n}(\mathcal{C}, D)$ is given by the 2-functor defined by

$$(\alpha,F,G) \mapsto |\Xi_0| = \cdots = |\Xi_n|,
\quad (\alpha_i) \mapsto |\Xi_i| = \cdots = |\Xi_n|,
\quad (\Xi) \mapsto |\Xi| = \cdots = |\Xi_n|.$$

**Remark 5.3.** Let us give an alternative description of the objects and morphisms of GD_{A_1,\ldots,A_n}(\mathcal{C}, D)$.

An object of it is a pair $((F_i)_{1 \leq i \leq n}, (G_{ij})_{1 \leq i,j \leq n})$ (here we do not assume $i < j$), where $F_i : \mathcal{A}_i \to D$ is an object of PsFun(\mathcal{A}_i, D)$ such that $|F_i| = \cdots = |F_n|$, and

$$G_{ij} : F_i(a)F_j(q) \Rightarrow F_j(p)F_i(b)$$

is an invertible 2-cell of $D$, $D$ running over $(\mathcal{A}_i, \mathcal{A}_j)$-squares in $\mathcal{C}$ (Example 2.13) of the form

$$\begin{array}{ccc}
X & \xrightarrow{b} & Y \\
\downarrow{q} & \searrow{\alpha} & \downarrow{p} \\
Z & \xrightarrow{a} & W,
\end{array}$$

satisfying condition (D) of Remark 5.2 for all $1 \leq i, j, k \leq n$ (again here we do not assume $i < j < k$) and the following conditions:

**(A)** For $1 \leq i,j \leq n$ and any square $D$ of the form

$$\begin{array}{ccc}
X & \xrightarrow{b} & Y \\
\downarrow{q} & \searrow{\alpha} & \downarrow{p} \\
X & \xrightarrow{a} & Y
\end{array}$$

where $a$ and $b$ are morphisms of $\mathcal{A}_i$, the following square commutes

$$F_i(a) \xrightarrow{F_i(\alpha)} F_i(b) \xrightarrow{G_{ij}D} F_i(\alpha)F_j(\text{id}_X) \xrightarrow{G_{ij}D} F_j(\text{id}_Y)F_i(\text{id}_Y).$$

**(O)** For any square $D$ \(\boxempty 3.1\) with $i = j$, the following square commutes

$$\begin{array}{ccc}
F_i(a) & \xrightarrow{F_i(\alpha)} & F_i(b) \\
\downarrow{G_{ii}D} & \searrow{G_{ii}D} & \downarrow{G_{ii}D} \\
F_i(a) & \xrightarrow{F_i(\alpha)} & F_i(b)
\end{array}$$

\(\text{In fact, given } (G_{ij})_{1 \leq i,j \leq n}, \text{ it suffices to take } (O) \text{ as a definition of } G_{ii}, \text{ and to put } G_{ji} = G_{ij}^\ast \text{ (Remark } 4.2) \text{ for } 1 \leq i < j \leq n. \)

A morphism $((F_i),(G_{ij})) \to ((F'_i),(G'_{ij}))$ of GD_{A_1,\ldots,A_n}(\mathcal{C}, D)$ is a collection $(\alpha_i)_{1 \leq i \leq n}$ of morphisms $\alpha_i : F_i \to F'_i$ of PsFun(\mathcal{A}_i, D)$ such that for every $(\mathcal{A}_i, \mathcal{A}_j)$-square $D$ \(\boxempty 5.31\), the following hexagon commutes

$$\begin{array}{ccc}
\alpha_0(W)F_i(a)F_j(q) & \xrightarrow{\alpha_j(a)} & F'_i(a)\alpha_0(Z)F_j(q) \\
\downarrow{G_{ij}D} & \xrightarrow{G_{ij}D} & \downarrow{G_{ij}D} \\
\alpha_0(W)F_j(p)F_i(b) & \xrightarrow{\alpha_i(p)} & F'_j(p)\alpha_0(Y)F_i(b) \\
\downarrow{G_{ij}D} & \xrightarrow{G_{ij}D} & \downarrow{G_{ij}D} \\
\alpha_0(W) & \xrightarrow{\alpha_i(b)} & \alpha_0(Y)F_j(p)F_i(b) \alpha_0(X)
\end{array}$$

Here $\alpha_0 = |\alpha_1| = \cdots = |\alpha_n|$. 

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Construction 5.4. Let $\phi: \{1, \ldots, n\} \to \{1, \ldots, m\}$ be a map and let $A_1, \ldots, A_n, B_1, \ldots, B_m$ be arrowy 2-subcategories of $C$ such that $A_i \subseteq B_{\phi(i)}$ for $1 \leq i \leq n$. The description in Remark 5.3 allows us to define a $D^{\text{Ob}(C)}$-functor $Q^\phi = Q_D^\phi: \text{GD}_{B_1, \ldots, B_m}(C, D) \to \text{GD}_{A_1, \ldots, A_n}(C, D)$ by

$$((F_i), (G_{ij})) \mapsto ((F_{\phi(i)}), (G_{\phi(i)\phi(j)})), \quad (\alpha_i) \mapsto (\alpha_{\phi(i)}), \quad (\Xi_i) \mapsto (\Xi_{\phi(i)}).$$

If $\phi': \{1, \ldots, m\} \to \{1, \ldots, l\}$ is a map, and $B_1', \ldots, B_l'$ are arrowy 2-subcategories of $C$ such that $B_j \subseteq B_{\phi'(j)}$ for $1 \leq j \leq l$, then we have $Q^{\phi'} = Q^\phi Q^{\phi'}$.

If $\phi$ is bijective and $A_i = B_{\phi(i)}$ for $1 \leq i \leq n$, then $Q^\phi$ is an isomorphism of $D^{\text{Ob}(C)}$-categories.

If $n \geq 2$, $m = 1$, then the diagram

$$
\begin{array}{ccc}
\text{PsFun}(C, D) & \xrightarrow{\phi} & \text{PsFun}(L \text{T}_{Q A_1, \ldots, A_n} C, D) \\
Q^\phi \downarrow & & \downarrow \text{PsFun}(E_{\phi}, D) \\
\text{GD}_{A_1, \ldots, A_n}(C, D) & \xrightarrow{\phi} & \text{PsFun}(L \text{T}_{Q A_1, \ldots, A_n} C, D).
\end{array}
$$

commutes. Here $E_{\phi}: L \text{T}_{Q A_1, \ldots, A_n} C \to C$ is the 2-functor induced by $2.19.3$.

If $n, m \geq 2$, then the diagram

$$(5.4.1) \quad \begin{array}{ccc}
\text{GD}_{B_1, \ldots, B_m}(C, D) & \xrightarrow{\phi} & \text{PsFun}(L \text{T}_{Q B_1, \ldots, B_m} C, D) \\
Q^\phi \downarrow & & \downarrow \text{PsFun}(E_{\phi}, D) \\
\text{GD}_{A_1, \ldots, A_n}(C, D) & \xrightarrow{\phi} & \text{PsFun}(L \text{T}_{Q A_1, \ldots, A_n} C, D)
\end{array}$$

commutes. Here $E_{\phi}: L \text{T}_{Q A_1, \ldots, A_n} C \to L \text{T}_{Q B_1, \ldots, B_m} C$ is the 2-functor induced by $3.9.1$. If moreover $\phi$ is injective, then $Q^\phi$ coincides with the $D^{\text{Ob}(C)}$-functor given by $3.8.2$ and $3.8.3$.

Applying Proposition 4.14 to $E_{\phi}$, we get the following generalization of Proposition 4.1.

Proposition 5.5. Assume $n \geq 2$.

1. If $E_{\phi}$ is pseudofull, then $Q^\phi_D$ is 2-faithful for every 2-category $D$.

2. $E_{\phi}$ is a 2-equivalence if and only if $Q^\phi_D$ is a $D^{\text{Ob}(C)}$-equivalence for every 2-category $D$.

Proposition 5.6. Let $\phi: \{1, \ldots, n\} \to \{1, \ldots, m\}$ be a map with a section $\phi'$. Assume $A_i \subseteq A_{\phi'\phi(i)}$ for all $1 \leq i \leq n$. Then

$$Q^\phi: \text{GD}_{A_{\phi'(1)}, \ldots, A_{\phi'(m)}}(C, D) \to \text{GD}_{A_1, \ldots, A_n}(C, D)$$

and $Q^{\phi'}$ are $D^{\text{Ob}(C)}$-quasi-inverses to each other.

Proof. We have $Q^{\phi'} Q^\phi = \text{id}$. For $m = 1$ we conclude by Propositions 5.5 and 2.20. For $m > 1$, we conclude by Propositions 5.5 and 3.10.

We can also construct a $D^{\text{Ob}(C)}$-natural equivalence $\epsilon: Q^\phi Q^{\phi'} \Rightarrow \text{id}$ more directly as follows. For any object $((F_i), (G_{ij}))$ of $\text{GD}_{A_1, \ldots, A_n} C$, we associate the morphism $Q^\phi Q^{\phi'} ((F_i), (G_{ij})) \to ((F_i), (G_{ij}))$ given by $\rho_{i, \phi'\phi(i)}: F_{\phi'\phi(i)} | A_i \to F_i$. See Remark 7.1 for the definition of $\rho_{i, \phi'\phi(i)}$.

In the remainder of this section, we will concentrate on the case $n \geq 3$, $m = n - 1$, $\phi(i) = \max\{0, i - 1\}$ for $1 \leq i \leq n$, $B_i = A_{i+1}$ for $2 \leq i \leq m$. We put $B = B_1$ and consider

$$(5.6.1) \quad \begin{array}{cc}
E = E_{\phi}: L \text{T}_{E A_1, \ldots, A_n} C \to L \text{T}_{Q B A_1, \ldots, A_n} C, \\
Q_D = Q^\phi_D: \text{GD}_{B, A_2, \ldots, A_n}(C, D) \to \text{GD}_{A_1, \ldots, A_n}(C, D).
\end{array}$$
Remark 5.7. If $A_1$ and $A_2$ generate $B$ (Definition 4.6), then $E$ is pseudofull, so that $Q_D$ is 2-faithful by Proposition 3.3 (1).

To state the main result of this section, we need to introduce some terminology.

Definition 5.8. Let $C$ be a (2,1)-category and let $A,B,A_i,A_j,A_k$ be arrowy 2-subcategories of $C$.

1. We say that $(A,B)$ is squaring in $C$ if every $(A,B)$-square (2.4.1) can be decomposed as

$$
\begin{array}{c}
\begin{array}{c}
X \ar@{-->}[rr]^{b} \ar@{-->}[rru]^{f} \ar@{-->}[rrd]_{g} & & Y' \ar@{-->}[uurr]^{k} \ar@{-->}[urr]^{r} \ar@{-->}[uurrdd]_{p} \\
\end{array} \\
\begin{array}{c}
Z \ar@{=}^{i} \ar@{=}^{j} \ar@{=}^{h} & & W \\
\end{array}
\end{array}
$$

where $k$ is a morphism of $A$, $r$ is a morphism of $B$, $f$ is a morphism of $A \cap B$, and the inner square is 2-Cartesian in $C$.

2. We say that $(A_i,A_j)$ is $A_k$-squaring in $C$ if every cube (5.2.1) can be decomposed as

$$
\begin{array}{c}
\begin{array}{c}
X' \ar@{=}[rr]^{a} \ar@{=}[rru]^{f} \ar@{=}[rrd]_{g} & & Y' \ar@{=}[uurr]^{b} \ar@{=}^{c} \ar@{=}^{d} \ar@{=}^{e} \\
\end{array} \\
\begin{array}{c}
Z' \ar@{=}^{\gamma} \ar@{=}^{\delta} \ar@{=}^{\epsilon} \ar@{=}^{\zeta} & & W' \\
\end{array}
\end{array}
$$

where $c,c'$ are morphisms of $A_i$, $r,r'$ are morphisms of $A_j$, $f,f'$ are morphisms of $A_i \cap A_j$, $v$ is a morphism of $A_k$, the bottom face $L$ and the top face $L'$ of the inner cube are 2-Cartesian in $C$.

Note that if the right face $I$ (resp. the front face $J$) of the inner cube is 2-Cartesian in $C$, so is the left face $I''$ (resp. the back face $J''$) by [15, Corollary 3.11].

If 2-fiber products exist in $B$ and are 2-fiber products in $C$ (cf. condition (2a) of Remark 4.10), then $(B,B)$ is squaring in $C$. One sufficient condition for $(A,B)$ to be squaring in $C$ is that 2-fiber products exist in $C$, and $A$ and $B$ are stable under 2-base change in $C$ and taking diagonals in $C$. One sufficient condition for $(A_i,A_j)$ to be $A_k$-squaring in $C$ is that $C$ admits 2-fiber products, and $A_i,A_j,A_k$ are stable under 2-base change in $C$ and taking diagonals in $C$.

The following is the main result of this section.

Theorem 5.9. Let $C$ be a (2,1)-category. Let $n \geq 3$ and let $A_1,\ldots,A_n,B$ be arrowy 2-subcategories of $C$. Assume the following:

1. For $3 \leq i \leq n$ and every morphism $f$ of $B$ (resp. $B \cap A_i$), there exist a morphism $a$ of $A_1$ (resp. $A_1 \cap A_i$), a morphism $p$ of $A_2$ (resp. $A_2 \cap A_j$), and a 2-cell $pa = f$ of $C$.

2. For $3 \leq i < j \leq n$, $A_1 \cap A_i \cap A_j$ and $A_2 \cap A_i \cap A_j$ generate $B \cap A_i \cap A_j$ (Definition 4.6).

3. $(A_2, A_3)$ is squarable in $B$ (Definition 4.6) and every 2-Cartesian $(A_2, A_3)$-square in $B$ is 2-Cartesian in $C$. For $3 \leq i \leq n$, $(A_2 \cap A_i, A_3 \cap A_i)$ is squarable in $B \cap A_i$.

4. For $3 \leq i, j \leq n$, $A_i$ and $A_i \cap A_j$ are stable under 2-base change in $C$ by morphisms of $A_i$ whenever such 2-base change exists. $(A_2, A_i)$ and $(A_2 \cap A_i, A_i)$ are squarable in $C$, $(B, A_i)$ and $(B \cap A_i, A_i)$ are squarable in $C$.
(5) For $3 \leq i \leq n$, $A_i \cap A_i$ and $A_2 \cap A_i$ are stable under 2-base change in $C$ by morphisms of $A_1$ whenever such 2-base change exists.

(6) For $3 \leq i, j \leq n$, $i \neq j$, $(B, A_j)$ is $A_i$-squaring.

Then

$$Q_D : GD_{S,A_3,...,A_n}(C,D) \rightarrow GD_{A_1,...,A_n}(C,D)$$

is a $D^{O(b)}_C$-equivalence for every 2-category $D$.

Note that assumptions [3][4][5] and [6] of Theorem 5.9 are all satisfied if $C$ admits 2-fiber products, $A_1, \ldots, A_n, B$ are stable under 2-base change in $C$, and $A_3, \ldots, A_n, B$ are stable under taking diagonals in $C$.

We will deduce Theorem 5.9 from an analogue of Theorem 4.13. To state it, we need to introduce some notation. Let $S, S', T$ be arrowy 2-subcategories of $C$. We denote by $Ar(S; T) \subseteq UPSFunct([1], C)$ the 2-faithful 2-subcategory spanned by strictly unital pseudofunctors that factor through $S$ and pseudonatural transformations $\alpha$ such that $a_0$ and $a_1$ are both morphisms of $T$. We define by $Tr(S; T) \subseteq UPSFunct([2], C)$ the 2-faithful 2-subcategory spanned by strictly unital pseudofunctors that factor through $S$ and pseudonatural transformations $\alpha$ such that $a_0, a_1, a_2$ are morphisms of $T$. We denote by $Sq(S, S'; T) \subseteq UPSFunct([1] \times [1], C)$ the 2-faithful 2-subcategory spanned by strictly unital pseudofunctors $F$ such that $F(0 \rightarrow 1, 0)$ and $F(0 \rightarrow 1, 1)$ are morphisms of $S$ and $F(0, 0 \rightarrow 1)$ and $F(1, 0 \rightarrow 1)$ are morphisms of $S'$ and by pseudonatural transformations $\alpha$ such that $\alpha(i, j), 0 \leq i, j \leq 1$ are all morphisms of $T$. Note that if $(A_i, A_j)$ is $A_k$-squaring in $C$, then $(A_i, A_j)$ is squaring in $C$ and $(Ar(A_k; A_i), Ar(A_k; A_j))$ is squaring in $Ar(A_k; C)$.

For $3 \leq i \leq n$, we consider the 2-functor

$$G_i : G_i := LTQ_{Ar(A_i; A_1), Ar(A_i; A_2)} Ar(A_i; B) \rightarrow Ar(A_i; B)$$

induced by $\\{1,1\\}$. For a morphism $f : x \rightarrow y$ of $Ar(A_i; B)$, we let $G_i^{-1}(f)$ denote the category of pairs $(g, \pi)$, where $g : x \rightarrow y$ is a morphism of $G_i$ and $\pi : G_i(y) \Rightarrow f$ is a 2-cell of $Ar(A_i; B)$.

**Theorem 5.10.** Let $C$ be a $(2, 1)$-category. Let $n \geq 3$ and let $A_1, \ldots, A_n, B$ be arrowy 2-subcategories of $C$ such that $A_1, A_2 \subseteq B$. Assume the following:

1. The 2-functor $G : G = LTQ_{A_1, A_2} B \rightarrow B$ is a 2-equivalence.
2. For every morphism $f : x \rightarrow y$ of $Ar(A_i; B)$, $3 \leq i \leq n$, the category $G_i^{-1}(f)$ is connected.
3. For $3 \leq i \leq n$, $Tr(A_i; A_1)$ and $Tr(A_i; A_2)$ generate $Tr(A_i; B)$.
4. For $3 \leq i, j \leq n$, $Sq(A_i, A_j; A_1)$ and $Sq(A_i, A_j; A_2)$ generate $Sq(A_i, A_j; B)$.

Then the 2-functor $E : LTQ_{A_1, \ldots, A_n} C \rightarrow LTQ_{B, A_3, \ldots, A_n} C$ is a 2-equivalence.

For $3 \leq i \leq n$, [3] implies that $Ar(A_i; A_1)$ and $Ar(A_i; A_2)$ generate $Ar(A_i; B)$, so that the category $G_i^{-1}(f)$ in [2] is nonempty. Note that for $n = 3$, [4] is a tautology.

**Proof.** For $3 \leq i \leq n$, let $B_i = Ar(A_i; B)$. We have source and target 2-functors $\tau_0, \tau_1 : B_i \rightarrow B$, sending an object $f : X \rightarrow Y$ of $B_i$ to $X$ and $Y$, respectively. These 2-functors induce 2-functors $G_i \rightarrow G$, which we still denote by $\tau_0$ and $\tau_1$. We have $\tau_0 G_i = G \tau_0$ and $\tau_1 G_i = G \tau_1$.

By Proposition 1.12 as $G$ is a pseudofunctor, there exists a pseudofunctor $H : B \Rightarrow G$ and pseudonatural transformations $\eta : id_B \Rightarrow HG$ and $\epsilon : GH \Rightarrow id_B$ such that $\eta(X) = id_X$ and $H(Y) = Y$, $\epsilon(Y) = id_Y$ for all objects $X$ and $Y$ of $C$.

Let $E = LTQ_{A_1, \ldots, A_n} C$, $D = LTQ_{B, A_3, \ldots, A_n} C$. We construct a 2-functor $F : D \rightarrow E$ as follows. For an object $X$ of $D$, we take $FX = X$. For a morphism of $D$ in length 1, we take $F(k^F) = f^{k+1}$ for $2 \leq k \leq n-1$, $f$ in $A_{k+1}$ and $F(k^1) = H(f)$ for $f$ in $B$. We thus obtain a functor from the underlying category of $D$ to the underlying category of $C$.

Next we define the effects of $F$ on the transformations giving rise to the 2-cells of $LTQ_{B, A_3, \ldots, A_n} C$ (Definition 2.12). We take $F(k^F) = F(k^F)$ for $2 \leq k \leq n-1$ and take $F(F(k^1) = H(f))$ to be the 2-cell induced by the coherence constraint $id_Y \Rightarrow H(id_Y)$ of $H$. In both cases, we take $F(k^F) = F(k^F)^{\sim}$ for $2 \leq k \leq n-1$, $f$ in $A_{k+1}$ and $F(F(k^1) = H(f))$ for $2 \leq k \leq n-1$ and $h$, $h'$ in $A_{k+1}$ and $f : A_{k+1} \rightarrow A_{k}$.

In both cases, we take $F(h^1) = F(h^1) = H(h)$ for $2 \leq k \leq n-1$ and $h$, $h'$ in $A_{k+1}$ and $F(F(h^1) = H(f))$ for $f$ in $B$. In both

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cases, we take $F(\delta_{g,h,k,f}) = F(\gamma_{g,h,k,f})^{-1}$. Now let $D \in (\mathcal{Q}\mathcal{E}_{A_0}, \ldots, A_n, \mathcal{C})(\epsilon_k + \epsilon_{k'})$, $k < k'$. For $k \geq 2$, we take $F(\sigma_{g,D,f}) = F(\tau_{g,D,f})$. For $k = 1$, we view $D$ as a morphism $x \to y$ of $B_{k+1} = \text{Ar}(A_{k+1}; B)$. An object $\rho = (D', \pi)$ of $G_{k+1}^{-1}(D)$, where $D' : x \to y$ is a morphism of $G_{k+1}$ and $\pi : G_{k+1}(D') \Rightarrow D$ is a 2-cell of $B_{k+1}$, we consider the 2-cell

$$\alpha_\rho : H\tau_1 D \ast x \xrightarrow{\pi^{-1}} H\tau_1 D' \ast x \xrightarrow{\alpha_{\pi}} \tau_1 D' \ast x \xrightarrow{\text{sym}, \rho^{-1}} y \ast \tau_0 D' \xrightarrow{\tau_0} y \ast \tau_0 D \xrightarrow{\text{sym}} y \ast H\tau_0 D.$$

It is straightforward to check that for a morphism $\rho \to \rho'$ of $G_{k+1}^{-1}(D)$, $\alpha_\rho = \alpha_{\rho'}$. Thus by the connectedness of $G_{k+1}^{-1}(D)$, $\alpha_\rho$ does not depend on the choice of $\rho$, and we take $F(\sigma_{g,D,f})$ to be the 2-cell induced by $\alpha_\rho$. We have thus defined a 2-functor $F : \mathcal{D} \to \mathcal{E}$, where $\mathcal{D}$ is the 2-category obtained from $\mathcal{TQ}_{\mathcal{E}_{A_0}} \ldots, A_n, \mathcal{C}$ by replacing the 2-cells by transformations.

To show that $F$ factors through a 2-functor $\mathcal{D} \to \mathcal{E}$, it suffices to check that the equivalence system defined by $\alpha \sim \beta$ if and only if $F(\alpha) = F(\beta)$ satisfies conditions (1) through (9) of Definition 2.12. Conditions (1) (2) (3), and (5) of Definition 2.12 follow immediately from the construction. For the other conditions, that only concerns the directions $A_1, \ldots, A_n$ are also trivial. Let us check the nontrivial cases. The case $k = 1$ of conditions (4) and (6) of Definition 2.12 follows from the unit and composition axioms of the coherence constraint of $H$, respectively. For the case $k' = 1$ of condition (7) of Definition 2.12 it suffices to take $\rho$ to be the object of $G_{k+1}^{-1}(D)$ given by the identity square $2.12.1$. For the case $k = 1$, condition (8) of Definition 2.12 it suffices to take $\rho$ to be the object of $G_{k+1}^{-1}(D)$ given by the identity on an object of the category $G^{-1}$ of pairs $(g, \pi)$, where $g : X \to Y$ is a morphism of $\mathcal{G}$ and $\pi : g \Rightarrow f$ is a 2-cell of $\mathcal{B}$. For the case $k' = 1$ of condition (8) of Definition 2.12 it suffices to take, for objects $\rho$ of $G_{k+1}^{-1}(D)$ and $\rho'$ of $G_{k+1}^{-1}(D')$, their composite $\rho' \circ \rho$ in $G_{k+1}^{-1}(D')$. For the case $k = 1$, condition (8) of Definition 2.12 it suffices to take objects of $G_{k+1}^{-1}(K), G_{k+1}^{-1}(K'), G_{k+1}^{-1}(J), G_{k+1}^{-1}(J')$ given by assumption (3) of Theorem 5.10. For the case $k = 1$, condition (9) of Definition 2.12 it suffices to take objects of $G_{k+1}^{-1}(K), G_{k+1}^{-1}(K'), G_{k+1}^{-1}(J), G_{k+1}^{-1}(J')$ given by assumption (4) of Theorem 5.10. This finishes the construction of the 2-functor $F : \mathcal{D} \to \mathcal{E}$.

We define a pseudonatural equivalence $\eta : \text{id}_\mathcal{E} \to FE$ sending $X$ to $\text{id}_X$ as follows. To every morphism $f$ of $A_i$, $3 \leq i \leq n$, we associate $\text{id}_f$. To every morphism $f$ of $A_1$ (resp. $A_2$), we associate $\eta(f)$.

We define a pseudonatural equivalence $\epsilon : EF \to \text{id}_\mathcal{E}$ sending $X$ to $\text{id}_X$ as follows. To every morphism $f$ of $A_i$, $3 \leq i \leq n$, we associate $\text{id}_f$. To every morphism $f$ of $B$ such that $H(f) = f_k \ast \cdots \ast f_1$, we associate the composition

$$f_k \ast \cdots \ast f_1 \xrightarrow{\eta(f)} f_k \circ \cdots \circ f_1 \xrightarrow{\text{sym}} f,$$

where the first 2-cell is given by $\gamma$.

**Proof of Theorem 5.7.** By Proposition 5.3 (2) it suffices to show that the assumptions of Theorem 5.9 imply the conditions of Theorem 5.10.

To check condition (1) of Theorem 5.10 it suffices to apply Theorem 4.10 to the arrow 2-subcategories $A_1$ and $A_2$ of $B$.

Next we check condition (2) of Theorem 5.10. To simplify the notation, we put $B_i = \text{Ar}(A_i ; B)$, $A_{k,i} = \text{Ar}(A_1; A_{k-1})$, $k = 1, 2$. We let $\tau_1 : B_i \to B$ denote the source and target 2-functors. We let $B'_i \subset \text{Ar}(A_1 ; B_i)$ (resp. $B''_i$) denote the arrow 2-subcategory of $B_i$ spanned by the morphisms $f$ of $B_i$ such that $\tau_0 f$ is in $A_i$ and $\tau_1 f$ is an identity (resp. morphisms of $B_i$ corresponding to 2-Cartesian squares in $C$). We put $A'_{k,i} = B'_i \cap A_{k,i}$, $A''_{k,i} = B''_i \cap A_{k,i}$. By Theorem 4.10, the 2-functors

$$G'_i : \mathcal{LQ}_{A_{k-1}, A_{k-2}, B'_i} \to B'_i, \quad G''_i : \mathcal{LQ}_{A_{k-1}, \mathcal{C}, B_{k-2}, B''_i} \to B''_i$$

are 2-equivalences. Let $f$ be a morphism of $B_i$. There exist $f'$ in $B'_i$, $f''$ in $B''_i$, and a 2-cell $f'' \Rightarrow f$ of $B_i$. These data induce a functor

$$F : G''_{k-1}(f') \times G''_{k-1}(f'') \to G^{-1}_{k-1}(f),$$

where $G''_{k-1}(f')$ and $G''_{k-1}(f'')$ are categories defined similarly to $G^{-1}_{k-1}(f)$ and are equivalent to the category $\{\ast\}$. Thus it suffices to show that every object $\rho = (f_m \ast \cdots \ast f_1, \pi)$ of $G''_{k-1}(f')$ is isomorphic to the image under $F$ of an object of $G''_{k-1}(f') \times G''_{k-1}(f'')$. Here each $f_s$, $1 \leq s \leq m$ is either a morphism of $A_1$, or a morphism of $A_2$. Note that, for every morphism $g$ of $A_{k,i}$, there exist $g'_i$ in $A'_{k,i}$, $g''_i$ in $A''_{k,i}$, and a 2-cell $g''_i \Rightarrow g$ in $A_{k,i}$. Thus, up to replacing $m$ by $2m$, we may assume that each $f_s$ is in $A_{k-1}$, or $A_{k-1}'$, or $A_{k-2}$, or $A_{k-2}'$. Moreover, if $g' : x \to y$ is a morphism of $A_{k-1}$, or $A_{k-1}'$, or $A_{k-2}$, or $A_{k-2}'$, then there exists a 2-cell $g' \Rightarrow h' \ast h'^*$ in $B_i$ with $h' : x \to v$ in
A^i_1 (resp. h'_i: x \to u in A^i_1, h'_v: u \to v in A^i_2), h'' : v \to z in A^i_1 (resp. A^i_2). Thus we may assume that 
\rho = \{ f_1'' \ast \cdots \ast f_n'' \ast f_1' \ast \cdots \ast f_n' \pi \}, where each f'_i is in A^i_1 or A^i_2, and each f''_i is in A^i_1 or A^i_2. In this case 
\rho is in the essential image of F.

Condition (3) of Theorem 5.10 follows from the fact that for every morphism \( f \) of Tr(A; B), there exists a 2-cell \( f \Rightarrow f_1'' f_1' f_2'' f_2' f_1 f_2 \), where \( f_k, f'_k, f''_k \) are in Tr(A; A), \( f''_k \) is given by a 2-Cartesian squares in \( C \), \( \tau_{0(0,1)} f''_k \) is given by a 2-Cartesian square in \( C \) and \( \tau_{2(0,1)} f''_k \) is an identity. Here \( \tau_{0(0,1)}, \tau_{1(1,2)} : Tr(A; B) \to B, \tau_0, \tau_2 : Tr(A; B) \to B \) are the obvious restriction 2-functors.

Condition (4) of Theorem 5.10 follows from the fact that for every morphism \( f \) of Sq(A; A; B), there exists a 2-cell \( f \Rightarrow f_1'' f_1' f_2'' f_2' f_1 f_2 \), where \( f_k, f'_k, f''_k \) are in Sq(A; A; A), \( f''_k \) is given by a 2-Cartesian square in \( C \), \( \tau_{0(0,1)} f''_k \) is in \( A \cap A \) and \( \tau_{1(1,2)} f''_k \) is an identity. Here \( \tau_{0(0,1)}, \tau_{1(1,2)} : Sq(A; A; B) \to B, \tau_0, \tau_1 : Sq(A; A; B) \to B, \) and \( \tau_{0,0} : Sq(A; A; B) \to B \) are the obvious restriction 2-functors.

6 Cartesian gluing data for two pseudofunctors

Let \( C \) be a \((2,1)\)-category, let \( A \) and \( B \) be two arrowy \( 2 \)-subcategorys of \( C \), and let \( D \) be a \( 2 \)-category. We studied the \( 2 \)-category \( GD_{A,B}(C, D) \) of gluing data in Section 4. One way to construct such data is by taking adjoints in base change isomorphisms (see Section 8). In many applications, these isomorphisms only exist for 2-Cartesian squares. In this section, we introduce a variant \( GD \) of \( GD_{A,B}(C, D) \), whose objects only make use of \( G_B \) for 2-Cartesian squares \( D \). The main result of this section is a criterion for the equivalence of \( GD_{A,B}(C, D) \) and \( GD \) (Theorem 6.5). This is used in the construction of \( RF \) for Deligne-Mumford stacks in \( 2D \) to produce the desired gluing data.

The idea of using Cartesian squares as an intermediary step to construct gluing data was already used by Deligne [6 5.1.5] and Ayoub [4 Section 1.6.5]. In [6 Section 5.1] it is possible to avoid this intermediary step by taking \( A \) to be spanned by dominant open immersions and \( B \) by proper morphisms, so that every \((A, B)\)-square is Cartesian. However, the intermediary step is necessary in other applications.

Construction 6.1. Let \( (F_A, F_B, G) \) be an object of \( GD_{A,B}(C, D) \), \( F_B = |F_A| = |F_B| \). Define an invertible morphism \( \rho : F_B|A \cap B \to F_A|A \cap B \) of PsFun(A \cap B, D) with \(|\rho| = \text{id}_{F_A} \) as follows. For any morphism \( f : X \to Y \) of \( A \cap B \), let \( \rho(f) \) be the composition

\[
F_B(f) \xrightarrow{G} F_A(\text{id}_Y)F_B(f) \xrightarrow{G} F_B(\text{id}_Y)F_A(f) \xrightarrow{f} F_A(f),
\]

where \( D \) is the left square in the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\| & \| & \| \\
X & \xrightarrow{f} & Y \\
\end{array}
\]

Denote the right square by \( D' \). Applying axiom (a) of Remark 4.3 to the outer square and axiom (b') of Remark 4.3 to the above diagram, one sees that \( \rho(f) \) is the inverse of the composition

\[
F_A(f) \xrightarrow{G} F_A(f)F_B(\text{id}_X) \xrightarrow{G} F_B(f)F_A(\text{id}_X) \xrightarrow{f} F_B(f).
\]

For any object \( X \) of \( C \), applying axiom (a) of Remark 4.3 to the constant square \([1] \times [1] \to C \) of value \( X \), one finds that the following diagram commutes

\[
\begin{array}{ccc}
\text{id}_{F_AX} & \xrightarrow{\rho(\text{id}_X)} & F_A(\text{id}_X) \\
\downarrow & & \downarrow \\
F_B(\text{id}_X) & \xrightarrow{G(\text{id}_X)} & F_A(\text{id}_X).
\end{array}
\]
For any sequence of morphisms \( X \xrightarrow{f} Y \xrightarrow{g} Z \), applying axioms (a), (b), (b') of Remark 4.3 to
\[
\begin{array}{c}
X \xrightarrow{f} Y \xrightarrow{g} Z \\
\downarrow \quad \downarrow \\
Y \xrightarrow{g} Z
\end{array}
\]
one finds that the following diagram commutes
\[
\begin{array}{ccc}
F_B(g)F_B(f) & \xrightarrow{\rho(g)\rho(f)} & F_A(g)F_A(f) \\
\downarrow & & \downarrow \\
F_B(gf) & \xrightarrow{\rho(gf)} & F_A(gf).
\end{array}
\]

Therefore, \( \rho \) is a pseudonatural equivalence.

**Remark 6.2.** The pseudonatural equivalence \( \rho \) has the following properties:

(c) If \( D \) is an \((A,B)\)-square (2.14.1) such that \( p, q \) are morphisms of \( A \cap B \), then the following hexagon commutes
\[
\begin{array}{ccc}
F_A(i)F_B(q) & \xrightarrow{\rho(q)} & F_A(i)F_A(q) & \xrightarrow{\rho(\alpha)} & F_A(iq) \\
G_D & & & & \downarrow \\
F_B(p)F_A(j) & \xrightarrow{\rho(p)} & F_A(p)F_A(j) & \xrightarrow{\rho(\alpha)} & F_A(pj).
\end{array}
\]

(c') If \( D \) is an \((A,B)\)-square (2.14.1) such that \( i, j \) are morphisms of \( A \cap B \), then the following hexagon commutes
\[
\begin{array}{ccc}
F_A(i)F_B(q) & \xrightarrow{\rho(j)^{-1}} & F_B(i)F_B(q) & \xrightarrow{\rho(\alpha)} & F_B(iq) \\
G_D & & & & \downarrow \\
F_B(p)F_A(j) & \xrightarrow{\rho(j)^{-1}} & F_B(p)F_B(j) & \xrightarrow{\rho(\alpha)} & F_B(pj).
\end{array}
\]

In fact, any square \( D \) in (c) can be decomposed as
\[
\begin{array}{ccc}
X \xrightarrow{j} Y & \xrightarrow{p} & Y \\
\downarrow \quad \downarrow \\
X \xrightarrow{j} Y & \xrightarrow{p} & W \\
\downarrow \quad \downarrow \\
X \xrightarrow{q} Z & \xrightarrow{i} & W \\
\downarrow \quad \downarrow \\
Z & \xrightarrow{i} & W
\end{array}
\]

Denote the upper left, upper right, middle, lower left, lower right squares by \( D_1, D_2, D_3, D_4 \) and \( D_5 \), respectively. Then \( G_{D_1} \) can be identified with \( \rho(p)^{-1} \) and \( G_{D_4} \) can be identified with \( \rho(q) \). By axiom (a) of Remark 4.3 \( G_{D_1} \) and \( G_{D_4} \) can be identified with identities and \( G_{D_3} \) can be identified with \( F_A(\alpha) \). Hence axioms (b) and (b') of Remark 4.3 imply that the hexagon in (c) commutes. Similarly, axioms (a'), (b) and (b') of Remark 4.3 imply (c').

**Definition 6.3.** Define a category \( \text{GD}^{\text{Cart}}_{A,B}(\mathcal{C}, \mathcal{D}) \) as follows. An object of this category is a quadruple \((F_A, F_B, (G_D), \rho)\) consisting of an object \( F_A \) of \( \text{PsFun}(A, \mathcal{D}) \), an object \( F_B \) of \( \text{PsFun}(B, \mathcal{D}) \), a family of invertible 2-cells of \( \mathcal{D} \)
\[
G_D : F_A(i)F_B(q) \Rightarrow F_B(p)F_A(j),
\]

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$D$ running over 2-Cartesian $(A, B)$-squares in $C$ of the form (2.14.1), and an invertible morphism $\rho: F_B \mid A \cap B \rightarrow F_A \mid A \cap B$ of $\text{PsFun}(A \cap B, D)$, such that $|\rho| = \text{id}_{F_B}$, where $F_B = |F_A| = |F_B|$, and satisfying conditions (b), (b') of Remark 4.3 and conditions (c), (c') of Remark 6.2 for 2-Cartesian $(A, B)$-squares.

A morphism $(F_A, F_B, G, \rho) \rightarrow (F_A', F_B', G', \rho')$ of $\text{GD}_{A,B}^\text{Cart}(C, D)$ is a pair $(\alpha_A, \alpha_B)$ consisting of a morphism $\alpha_A: F_A \rightarrow F_A'$ of $\text{PsFun}(A, D)$ and a morphism $\alpha_B: F_B \rightarrow F_B'$ of $\text{PsFun}(B, D)$, such that $|\alpha_A| = |\alpha_B|$, satisfying condition (m) of Remark 4.3 for 2-Cartesian $(A, B)$-squares, and the following condition

(n) The following square commutes

$$
\begin{array}{ccc}
F_B \mid A \cap B & \xrightarrow{\rho} & F_A \mid A \cap B \\
\alpha_B \mid A \cap B & & \alpha_A \mid A \cap B \\
F_B' \mid A \cap B & \xrightarrow{\rho'} & F_A' \mid A \cap B.
\end{array}
$$

A 2-cell of $\text{GD}_{A,B}^\text{Cart}(C, D)$ is a pair $(\Xi_A, \Xi_B): (\alpha_A, \alpha_B) \Rightarrow (\alpha'_A, \alpha'_B)$ consisting of a 2-cell $\alpha_A \Rightarrow \alpha'_A$ of $\text{PsFun}(A, D)$ and a 2-cell $\alpha_B \Rightarrow \alpha'_B$ of $\text{PsFun}(B, D)$ such that $|\alpha_A| = |\alpha_B|$. We view $\text{GD}_{A,B}^\text{Cart}(C, D)$ as a $\mathcal{D}^{\text{Ob}(C)}$-category via the 2-functor given by

$$
(F_A, F_B, G, \rho) \mapsto |F_A| = |F_B|, \quad (\alpha_A, \alpha_B) \mapsto |\alpha_A| = |\alpha_B|, \quad (\Xi_A, \Xi_B) \mapsto |\Xi_A| = |\Xi_B|.
$$

Construction (6.1) defines a $\mathcal{D}^{\text{Ob}(C)}$-functor

(6.3.1) $\text{GD}_{A,B}(C, D) \rightarrow \text{GD}_{A,B}^\text{Cart}(C, D)$,

which is clearly 2-faithful.

**Remark 6.4.** If $(A, B)$ is squaring in $C$ (Definition 5.8), then (6.3.1) is 2-faithful. In fact, for objects $(F_A, F_B, G)$ of $\text{GD}_{A,B}(C, D)$ and any morphism

$$(\alpha_A, \alpha_B): (F_A, F_B, G, \rho) \rightarrow (F'_A, F'_B, G', \rho')$$

of $\text{GD}_{A,B}^\text{Cart}(C, D)$ whose source and target are respectively the images of $(F_A, F_B, G)$ and $(F'_A, F'_B, G')$ under (6.3.1), $(\alpha_A, \alpha_B): (F_A, F_B, G) \rightarrow (F'_A, F'_B, G')$ is a morphism of $\text{GD}_{A,B}(C, D)$. Indeed, for any $(A, B)$-square $D$ (2.14.1), decomposing it as (5.8.1), we see that the following diagram commutes

$$
\begin{array}{ccc}
\alpha_0(W)F_A(i)F_B(q) & \xrightarrow{\alpha_0(W)F_A(i)F_B(r)F_B(f)} & \alpha_0(W)F_B(p)F_A(k)F_A(j) \\
\alpha_A(i) & \xrightarrow{\alpha_B(p)} & \alpha_A(k) \\
F_A'(i)F_B'(q) & \xrightarrow{F_A'(i)F_B'(r)\alpha_0(X)'F_B(f)} & F_B'(p)F_A'(k)F_A'(j) \\
\alpha_B(q) & \xrightarrow{\alpha_A(f)} & \alpha_A(j) \\
F_A'(i)F_B'(q)\alpha_0(X)' & \xrightarrow{F_A'(i)F_B'(r)\alpha_0(X)F_B(f)} & F_B'(p)F_A'(k)\alpha_0(X)F_A'(j) \\
\alpha_B(q) & \xrightarrow{\alpha_A(f)} & \alpha_A(j)
\end{array}
$$

Here $\alpha_0 = |\alpha_A| = |\alpha_B|$, and $D'$ is the inner square of (6.8.1).

**Theorem 6.5.** Let $C$ be a (2, 1)-category, let $A$ and $B$ be arrowy 2-subcategories of $C$, and let $D$ be a 2-category. Assume that every equivalence in $C$ is contained in $A \cap B$, and $(A, B), (A, A \cap B), (B, A \cap B)$ are squaring in $C$ (Definition 5.8). Then (6.3.1) is an isomorphism of $\mathcal{D}^{\text{Ob}(C)}$-categories.

**Proof.** We construct the inverse of (6.3.1) as follows. Let $(F_A, F_B, G, \rho)$ be an object of $\text{GD}_{A,B}^\text{Cart}(C, D)$. For any square $D$ (2.14.1), decompose it as (5.8.1), and denote the inner square by $D'$. Let $G_D$ be the composition

$$
\begin{array}{ccc}
F_A(i)F_B(q) & \xrightarrow{F_A(i)F_B(r)F_B(f)} & F_B(p)F_A(k)F_A(f) \\
\xrightarrow{G_D^\rho(f)} & & \xrightarrow{F_B(p)F_A'(k)F_A'(f)} \\
F_A'(i)F_B'(r) & \xrightarrow{F_A'(i)F_B'(r)\alpha_0(X)F_B(f)} & F_B'(p)F_A'(k)\alpha_0(X)F_A'(j).
\end{array}
$$
This does not depend on the choice of the decomposition. In fact, if

![Diagram](attachment:image.png)

is another decomposition with \( l \) in \( A \), \( s \) in \( B \), \( g \) in \( A \cap B \), \( D'' \) 2-Cartesian in \( C \), then they can be combined into

![Diagram](attachment:image.png)

where \( h \) is an equivalence. Applying the axioms of Remarks 4.3 and 6.2 to the following decomposition of \( D'' \)

![Diagram](attachment:image.png)

we obtain the following commutative diagram

\[
\begin{align*}
F_A(i)F_B(s) & \xrightarrow{G_{D''}} F_B(p)F_A(l) \\
F_B(\phi) & \xrightarrow{(b),(c)} F_A(\psi) \\
F_A(i)F_B(r)F_B(h) & \xrightarrow{F_A(i)F_B(r)F_A(h)} F_B(p)F_A(k)F_A(h).
\end{align*}
\]
Hence the following diagram commutes

\[
\begin{array}{c}
F_A(i)F_B(q) \\
\downarrow F_B(e) \\
F_A(i)F_B(s)F_B(g) = F_A(i)F_B(r)F_B(h)F_B(g) \\
\downarrow \rho(g) \\
F_A(i)F_B(s)F_A(g) \\
\downarrow \rho(h) \\
F_A(i)F_B(r)F_B(h)F_A(g) \\
\downarrow \rho(f) \\
F_A(i)F_B(r)F_A(g) \\
\downarrow \rho(f) \\
F_A(i)F_B(r)F_A(g) \\
\downarrow \rho(f) \\
F_A(i)F_B(r)F_A(g) \\
\downarrow \rho(f) \\
F_B(p)F_A(l)F_A(g) = F_B(p)F_A(k)F_A(h)F_A(g) = F_B(p)F_A(k)F_A(f). \\
\downarrow F_A(\omega) \\
F_B(p)F_A(l)F_A(j) \\
\downarrow F_A(\delta) \\
F_B(p)F_A(l)F_A(j) \\
\end{array}
\]

Next we show that \((F_A, F_B, \tilde{G})\) is an object of \(GD_{A,B}(C, D)\). Axioms (a) and (a') of Remark 6.3 for \(\tilde{G}\) follow from axioms (c) and (c') of Remark 6.2. Let \(D, D'\) and \(D''\) be squares as in axiom \((b')\) of Remark 6.4 for \(\tilde{G}\). Decompose it as

\[
\begin{array}{c}
X_1 \\
\downarrow f \\
W \\
\downarrow g \\
Z_1 \\
\downarrow k \\
Y_1 \\
\end{array}
\quad
\begin{array}{c}
X_2 \\
\downarrow j \\
X_3 \\
\downarrow b \\
Z_2 \\
\downarrow h \\
Y_2 \\
\end{array}
\]

where horizontal arrows are morphisms of \(A\), vertical arrows are morphisms of \(B\), oblique arrows are morphisms of \(A \cap B\), the squares \(E, E'\) and the square \(H\) containing \(\eta\) are 2-Cartesian in \(C\). Let \(E'' = E' \circ E, I = H \circ E\). Since \(I\) is the outer square of the diagram

\[
\begin{array}{c}
W \\
\downarrow q_1 \\
Y_1 \\
\end{array}
\quad
\begin{array}{c}
X_2 \\
\downarrow j \\
X_2 \\
\downarrow q_2h \\
Y_2 \\
\end{array}
\]

axiom \((b')\) of Remark 6.3 and axiom \((c')\) of Remark 6.2 imply the commutativity of the following triangle

\[
\begin{array}{c}
F_A(i)F_B(q) \\
\downarrow G_j \\
F_B(p_2)F_A(l) = F_B(q_2h)F_A(l). \\
\end{array}
\]
It follows that the following diagram commutes.
One establishes axiom (b) of Remark[4.3] for $\bar{G}$ in a similar way.

Let $(\alpha_A, \alpha_B) : (F_A, F_B, G, \rho) \to (F'_A, F'_B, G', \rho')$ be a morphism of $\text{GD}^\text{Cart}(\mathcal{C}, \mathcal{D})$. Then $(\alpha_A, \alpha_B) : (F_A, F_B, \bar{G}) \to (F'_A, F'_B, \bar{G}')$ is a morphism of $\text{GD}_{A,B}(\mathcal{C}, \mathcal{D})$ by Remark [6.3]. Let

$$(\Xi_A, \Xi_B) : (\alpha_A, \alpha_B) \Rightarrow (\alpha'_A, \alpha'_B)$$

be a 2-cell of $\text{GD}^\text{Cart}(\mathcal{C}, \mathcal{D})$. Then $(\Xi_A, \Xi_B)$ is a 2-cell of $\text{GD}_{A,B}(\mathcal{C}, \mathcal{D})$.

The 2-functor defined in this way is clearly the inverse of (6.3.1).

7 Cartesian gluing data for finitely many pseudofunctors

In this section, we generalize the definitions and results of the previous section to the case of finitely many pseudofunctors. The main result is a general criterion for the equivalence of $\text{GD}_{A,B}(\mathcal{C}, \mathcal{D})$ (Theorem 7.3).

Let $\mathcal{C}$ be a (2,1)-category, let $A_1, \ldots, A_n$ be arrowy 2-subcategories of $\mathcal{C}$, and let $\mathcal{D}$ be a 2-category.

Remark 7.1. Let $((F_i), G)$ be an object of $\text{GD}_{A_1,\ldots,A_n}(\mathcal{C}, \mathcal{D})$. For $1 \leq i,j \leq n$, $(F_i, F_j, G_{ij})$ is an object of $\text{GD}_{A_i,A_j}(\mathcal{C}, \mathcal{D})$. Let

$$\rho_{ij} : F_j | A_i \cap A_j \to F_i | A_i \cap A_j$$

be the morphism of $\text{PsFun}(A_i \cap A_j, \mathcal{D})$ associated to it by (6.3.1). Then $\rho_{ii} = \text{id}_F$ and $\rho_{ji} = \rho_{ij}^{-1}$. We claim that $\rho_{ij}$ has the following properties:

(E) For $1 \leq i,j,k \leq n$ and any $(A_i, A_j \cap A_k)$-square $D$ (6.3.1), the following square commutes

$$\begin{array}{ccc}
F_i(a) & \xrightarrow{\rho_{ik}(q)} & F_k(a) \\
G_{ik,D} & \downarrow & G_{ij,D} \\
F_k(p) & \xrightarrow{\rho_{jk}(p)} & F_j(p)
\end{array}$$

(F) (cocycle condition) For $1 \leq i,j,k \leq n$, the following triangle commutes

$$\begin{array}{ccc}
F_k | A_{ijk} & \xrightarrow{\rho_{ijk}} & F_j | A_{ijk} \\
\rho_{jk} | A_{ijk} & \downarrow & \rho_{ij} | A_{ijk} \\
F_i | A_{ijk}
\end{array}$$

Here $A_{ijk} = A_i \cap A_j \cap A_k$.

In fact, (E) follows from axiom (D) of Remark 5.2 applied to the cube

whose top and back faces are $D$ and whose other faces have identity 2-cells. Condition (F) follows from (E) applied to the square

for every morphism $f$ of $A_{ijk}$.
Definition 7.2. We define a 2-category \( GD_{A_1,\ldots,A_n}^{\text{Cart}}(C, D) \) as follows. An object of this 2-category is a triple \((F_i)_{1 \leq i \leq n}, (G_{ij})_{1 \leq i < j \leq n}, (\rho_{ij})_{1 \leq i < j \leq n}\), where \( F_i : A_i \rightarrow D \) is an object of \( \text{PsFun}(A_i, D) \) for \( 1 \leq i \leq n \), and \((F_i, F_j, G_{ij}, \rho_{ij})_{1 \leq i < j \leq n} \) satisfying condition (D) of Remark 5.2 for \( 1 \leq i < j < k \leq n \), and cubes with 2-Cartesian faces, condition (E) of Remark 7.1 for \( 1 \leq i \leq n \), where \( i \neq j, k \) and 2-Cartesian squares (here we put \( G_{ij} = G_{ji} \)), and condition (F) of Remark 7.1 for \( 1 \leq i < j < k \leq n \).

A morphism \((F_i, G, \rho) \rightarrow ((F_i', G', \rho')_{1 \leq i \leq n}) \) of \( GD_{A_1,\ldots,A_n}^{\text{Cart}}(C, D) \) is a collection \((\alpha_i)_{1 \leq i \leq n}\) of morphisms \( \alpha_i : F_i \rightarrow F_i' \) of \( \text{PsFun}(A_i, D) \) such that for \( 1 \leq i < j \leq n \), \((\alpha_i, \alpha_j) : (F_i, F_j, G_{ij}, \rho_{ij}) \rightarrow (F_i', F_j', G'_{ij}, \rho'_{ij}) \) is a morphism of \( GD_{A_1,\ldots,A_n}^{\text{Cart}}(C, D) \).

A 2-cell of \( GD_{A_1,\ldots,A_n}^{\text{Cart}}(C, D) \) is a collection \((\Xi_i)_{1 \leq i \leq n} : (\alpha_i)_{1 \leq i \leq n} \Rightarrow (\alpha'_i)_{1 \leq i \leq n} \) of 2-cells \( \alpha_i \Rightarrow \alpha'_i \) of \( \text{PsFun}(A_i, D) \) such that \( |\Xi_i| = \cdots = |\Xi_n| \).

We view \( GD_{A_1,\ldots,A_n}^{\text{Cart}}(C, D) \) as a \( D^\text{Ob(C)} \)-category via the 2-functor given by

\[
(F_i, G, \rho) \mapsto (|F_i| = \cdots = |F_n|, (\alpha_i) \mapsto |\alpha_i| = \cdots = |\alpha_n|, (\Xi_i) \mapsto |\Xi_i| = \cdots = |\Xi_n|).
\]

This \( D^\text{Ob(C)} \)-category coincides with the one defined in Construction 6.1 if \( n = 2 \).

Remark 7.2 defines a \( D^\text{Ob(C)} \)-functor

\[
GD_{A_1,\ldots,A_n}(C, D) \rightarrow GD_{A_1,\ldots,A_n}^{\text{Cart}}(C, D),
\]

which is clearly 2-faithful. If, for all \( 1 \leq i < j \leq n \), \( A_i \) and \( A_j \) are squaring in \( C \) (Definition 5.8), then \( 7.2.1 \) is 2-faithfully by Remark 6.4.

The following generalizes Theorem 6.5.

Theorem 7.3. Let \( C \) be a \((2,1)\)-category, let \( A_1, \ldots, A_n \) be arrow 2-subcategories of \( C \), and let \( D \) be a 2-category. Assume that every morphism of \( C \) that is an equivalence is contained in \( A_1 \cap \cdots \cap A_n \), and for \( 1 \leq i, j, k \leq n \), \( i \neq j \), \((A_i, A_j) \) and \((A_i, A_i \cap A_j) \) are squaring in \( C \) (Definition 5.8). Assume moreover that for pairwise distinct numbers \( 1 \leq i, j, k \leq n \), \((A_i, A_j) \) is \( A_k \)-squaring and \((A_i \cap A_j, A_k) \) is squaring. Then \( 7.2.1 \) is an isomorphism of \( D^\text{Ob(C)} \)-categories.

One sufficient condition for the assumptions of the Theorem 7.3 is that \( C \) admits 2-fiber products, and \( A_i \) is stable under 2-base change in \( C \) and taking diagonals in \( C \) for all \( 1 \leq i \leq n \).

Proof. We construct the inverse of \( 7.2.1 \) as follows. Let \((F_i, G, \rho) \) be an object of \( GD_{A_1,\ldots,A_n}(C, D) \). For \( 1 \leq i, j \leq n \), \( i \neq j \), let \((F_i, F_j, G_{ij}) \) be the image of \((F_i, F_j, G_{ij}, \rho_{ij}) \) under the inverse of \( 6.3.1 \). To show that \((F_i, G) \) is an object of \( GD_{A_1,\ldots,A_n}(C, D) \), it suffices to check axiom (D) of Remark 5.2 for \( 1 \leq i < j < k \leq n \).

First note that for pairwise distinct numbers \( 1 \leq i, j, k \leq n \), \( G \) satisfies axiom (D) of Remark 5.2 for cubes with 2-Cartesian faces, \( G \) and \( \rho \) satisfy axiom (E) of Remark 7.1 for 2-Cartesian squares, and \( \rho \) satisfies axiom (F) of Remark 7.1. Here for \( 1 \leq i < j \leq n \) we put \( G_{ij} = G_{ji} \) and \( \rho_{ij} = \rho_{ji}^{-1} \). Next we show that \( G \) satisfies axiom (E) of Remark 7.1 for pairwise distinct numbers \( 1 \leq i, j, k \leq n \) and all \((A_i, A_j \cap A_k)\)-squares \( D \) (5.3.1). Decompose \( D \) as

\[
\begin{array}{c}
X \\
\downarrow f \quad \downarrow \beta \\
X' \\
\downarrow \gamma \\
Y \\
\downarrow \chi \\
Z \\
\downarrow \alpha \\
W
\end{array}
\]

where \( c \) is a morphism of \( A_i \), \( r \) is a morphism of \( A_j \cap A_k \), \( f \) is a morphism of \( A_i \cap A_j \cap A_k \), and the inner square \( D' \) is 2-Cartesian. Then the following diagram commutes
For pairwise distinct numbers $1 \leq i, j, k \leq n$, we show axiom (D) of Remark 5.2 for $\bar{G}$ by descending induction on the number $m$ of pairs of 2-Cartesian opposite faces in the cube $\alpha_{ij}$. If $m = 3$, all the faces of the cube are 2-Cartesian, so the assertion is identical to axiom (D) for $G$. If $m < 3$, by symmetry, we may assume that either the bottom face $K$ or the top face $K'$ is not 2-Cartesian. Decompose the cube as $\bar{G}$. The inner cube has more than $m$ pairs of 2-Cartesian opposite faces, hence axiom (D) holds for the inner cube by induction hypothesis. Therefore, the following diagram commutes

Here $M$ is the square $X'X'YX$.

Any morphism $(\alpha_i): ((F_i), G, \rho) \to ((F'_i), G', \rho')$ of $\text{GD}_{A_1, \ldots, A_n}^{\text{Cart}}(C, D)$ is a morphism $((F_i), \bar{G}) \to ((F'_i), G')$ of $\text{GD}_{A_1, \ldots, A_n}(\bar{C}, D)$. Any 2-cell $(\xi_i): (\alpha_i) \Rightarrow (\alpha'_i)$ of $\text{GD}_{A_1, \ldots, A_n}^{\text{Cart}}(C, D)$ is a 2-cell of $\text{GD}_{A_1, \ldots, A_n}(C, D)$.

The $\mathcal{D}^{\text{Ob}(\mathcal{C})}$-functor defined in this way is clearly the inverse of $\text{GD}_{A_1, \ldots, A_n}$.

**Remark 7.4.** We can consider yet another 2-category of gluing data

$$\text{GD}_{A_1, \ldots, A_n}^{\text{Cart}}(C, D)' = \mathcal{P}_{\text{Fun}}^{\mathcal{C}}(\mathcal{Q}_{A_1, \ldots, A_n}^{\text{Cart}}(C, D))$$

by dropping $(\rho_{ij})$ from the definition of $\text{GD}_{A_1, \ldots, A_n}^{\text{Cart}}$, where $\mathcal{Q}_{A_1, \ldots, A_n}^{\text{Cart}}(C) \subseteq \mathcal{Q}_{A_1, \ldots, A_n}(C)$ is the $n$-fold subcategory spanned by 2-Cartesian squares. An $\infty$-categorical variant of $\mathcal{Q}_{A_1, \ldots, A_n}^{\text{Cart}}(C)$ is studied in [17] Section 5.

## 8 Gluing data and adjunction

In this section we show how to produce (Cartesian) gluing data from base change maps by taking adjoints. We deal with the axioms individually and refer the reader to Remark 8.3 for a synthesis. A somewhat more systematic treatment is possible in the $\infty$-categorical setting (for the gluing data mentioned in Remark 7.3) [16] Section 1.4]. Throughout this section, we fix a 2-category $\mathcal{D}$.

**Definition 8.1.** We define a 2-category $\mathcal{D}^{\text{adj}}$ satisfying $\text{Ob}(\mathcal{D}^{\text{adj}}) = \text{Ob}(\mathcal{D})$ by taking $\mathcal{D}^{\text{adj}}(X, Y)$ to be the category of adjoint pairs from $X$ to $Y$ for every pair $(X, Y)$ of objects of $\mathcal{D}$. More explicitly, a morphism $X \to Y$ of $\mathcal{D}^{\text{adj}}$ is a quadruple $(f, g, \eta, \epsilon)$ consisting of morphisms $f: X \to Y$, $g: Y \to X$ and 2-cells $\eta: \text{id}_Y \Rightarrow gf$, $\epsilon: gf \Rightarrow \text{id}_X$ of $\mathcal{D}$ such that the following triangles commute

\[
\begin{array}{ccc}
  f & \xRightarrow{\eta} & gf \\
  \downarrow \text{id}_f & & \downarrow \text{id}_g \\
  f & \equiv & fg \\
\end{array}
\quad
\begin{array}{ccc}
  g & \xRightarrow{\eta} & gf \\
  \downarrow \text{id}_g & & \downarrow \text{id}_f \\
  g & \equiv & gf \\
\end{array}
\]

The composition of $(f_1, g_1, \eta_1, \epsilon_1): X \to Y$ and $(f_2, g_2, \eta_2, \epsilon_2): Y \to Z$ is

$$(f_2f_1, g_1g_2, \eta_1\eta_2, \epsilon_1\epsilon_2): X \to Z,$$

where $\eta_1\eta_2$ is the composition

$$\text{id}_Z \xRightarrow{\eta_2} f_2g_2 \xRightarrow{\eta_1} f_2f_1g_1g_2$$

and $\epsilon_1\epsilon_2$ is the composition

$$g_1g_2f_1f_2 \xRightarrow{\epsilon_1} g_1f_1 \xRightarrow{\epsilon_2} \text{id}_X.$$
The identity morphism of an object $X$ is $(\text{id}_X, \text{id}_X, \text{id}_{\text{id}_X}, \text{id}_{\text{id}_X})$. A 2-cell $(f, g, \eta, \epsilon) \Rightarrow (f', g', \eta', \epsilon')$ of $\mathcal{D}^{\text{adj}}$ is a pair $(\alpha, \beta)$ of 2-cells $\alpha : f \Rightarrow f'$ and $\beta : g' \Rightarrow g$ of $\mathcal{D}$ such that the following squares commute

$$
definition{8.3}$$

The projection 2-functors $P_1 : \mathcal{D}^{\text{adj}} \rightarrow \mathcal{D}$ and $P_2 : \mathcal{D}^{\text{adj}} \rightarrow \mathcal{D}^{\text{coop}}$, sending $(f, g, \eta, \epsilon)$ to $f$ and $g$ respectively, are pseudofaithful (Definition 1.6).

**Remark 8.2.** Let $\mathcal{C}$ be a 2-category. Then $P_1$ and $P_2$ induce pseudofaithful 2-functors

$$
P_1 : \text{PsFun}(\mathcal{C}, \mathcal{D}^{\text{adj}}) \rightarrow \text{PsFun}(\mathcal{C}, \mathcal{D}), \quad P_2 : \text{PsFun}(\mathcal{C}, \mathcal{D}^{\text{adj}}) \rightarrow \text{PsFun}(\mathcal{C}, \mathcal{D}^{\text{coop}}).
$$

An object $F$ of $\text{PsFun}(\mathcal{C}, \mathcal{D})$ (resp. $\text{PsFun}(\mathcal{C}, \mathcal{D}^{\text{coop}})$) is in the image of $P_1$ (resp. $P_2$) if and only if for every morphism $a$ of $\mathcal{C}$, $F(a)$ can be completed into an adjoint pair $(F(a), g, \eta, \epsilon)$ (resp. $(f, F(a), \eta, \epsilon)$).

In the rest of this section, we fix a 2-category $\mathcal{C}$, a pseudofunctor $F : \mathcal{C} \rightarrow \mathcal{D}^{\text{coop}}$, a 2-faithful subcategory $\mathcal{B}$ of $\mathcal{C}$, and a pseudofunctor $B : \mathcal{B} \rightarrow \mathcal{D}^{\text{adj}}$ such that $P_2(B) = F | \mathcal{B}$. We do not assume that $\mathcal{C}$ is a $(2,1)$-category. We denote $F$ by

$$
f \mapsto f^*, \quad \alpha \mapsto \alpha^*,
$$

and the pseudofunctor $R = P_1(B) : \mathcal{B} \rightarrow \mathcal{D}$ by

$$
p \mapsto p_+, \quad \alpha \mapsto \alpha_+.
$$

**Construction 8.3.** In this section, it is convenient to use down-squares in $\mathcal{C}$

(8.3.1)

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
q \downarrow & \Downarrow_{\gamma} & p \\
Z & \xrightarrow{\gamma} & W.
\end{array}
$$

Let $D$ be such a square with $p$ and $q$ in $\mathcal{B}$. The base change map $B_D$ is by definition the following 2-cell of $\mathcal{D}$

$$
i^*p_+ \xrightarrow{n_+} i^*q^*i^*p_+ \Rightarrow q_*(q\eta(p))p_+ \xrightarrow{\alpha^*} q_*(p\eta)q^+p_+ \Rightarrow q_*(p\eta)q^+p_+=q_*(p\eta)q^+p_+.
$$

If $i$ and $j$ are also morphisms of $\mathcal{B}$, then $B_D$ is also the composition

$$
i^*(i^+(j)*j^*) \xrightarrow{\alpha^*} i^*(i^+(j)*j^*) \xrightarrow{\alpha^*} i^*(i^+(j)*j^*) \xrightarrow{\alpha^*} i^*(i^+(j)*j^*) \xrightarrow{\alpha^*} i^*(i^+(j)*j^*) \xrightarrow{\alpha^*} i^*(i^+(j)*j^*) \xrightarrow{\alpha^*} i^*(i^+(j)*j^*) \xrightarrow{\alpha^*} i^*(i^+(j)*j^*) \xrightarrow{\alpha^*} i^*(i^+(j)*j^*) \xrightarrow{\alpha^*} i^*(i^+(j)*j^*) \xrightarrow{\alpha^*} i^*(i^+(j)*j^*) \xrightarrow{\alpha^*} i^*(i^+(j)*j^*) \xrightarrow{\alpha^*} i^*(i^+(j)*j^*) \xrightarrow{\alpha^*} i^*(i^+(j)*j^*) \xrightarrow{\alpha^*} i^*(i^+(j)*j^*) \xrightarrow{\alpha^*} i^*(i^+(j)*j^*) \xrightarrow{\alpha^*} i^*(i^+(j)*j^*) \xrightarrow{\alpha^*} i^*(i^+(j)*j^*) \xrightarrow{\alpha^*} i^*(i^+(j)*j^*) \xrightarrow{\alpha^*} i^*(i^+(j)*j^*) \xrightarrow{\alpha^*} i^*(i^+(j)*j^*) \xrightarrow{\alpha^*} i^*(i^+(j)*j^*) \xrightarrow{\alpha^*} i^*(i^+(j)*j^*) \xrightarrow{\alpha^*} $
(1) Let $D$, $D'$, $D''$ be respectively the upper, lower and outer squares of the diagram in $C$

\[
\begin{array}{c}
X_1 \xrightarrow{i_1} Y_1 \\
q \downarrow \alpha \downarrow p
\end{array}
\quad
\begin{array}{c}
X_2 \xrightarrow{i_2} Y_2 \\
q' \downarrow \alpha' \downarrow p'
\end{array}
\quad
\begin{array}{c}
X_3 \xrightarrow{i_3} Y_3 \\
\end{array}
\]

where the vertical arrows are morphisms of $B$. Then the following diagram commutes

\[
i_3^* p_3^* \overset{B_{p'}}{\longrightarrow} q'^* i_2^* p_* \overset{B_D}{\longrightarrow} q'^* i_2^* \overset{B_{p'}}{\longrightarrow} q'^* i_2^* p_* \overset{B_{p'}}{\longrightarrow} (q'q)_i.
\]

(2) Let $D$, $D'$, $D''$ be respectively the left, right and outer squares of the diagram in $C$

\[
\begin{array}{c}
X_1 \xrightarrow{j} X_2 \xrightarrow{j'} X_3 \\
p_1 \downarrow \alpha \downarrow p_2 \downarrow \alpha' \downarrow p_3
\end{array}
\quad
\begin{array}{c}
Y_1 \xrightarrow{i} Y_2 \xrightarrow{i'} Y_3 \\
\end{array}
\]

where the vertical arrows are morphisms of $B$. Then the following diagram commutes

\[
i^* i'^* p_{3*} \overset{B_{p'}}{\longrightarrow} i^* j^* j'^* \overset{B_D}{\longrightarrow} p_{1*} j^* j'^* \overset{B_{p'}}{\longrightarrow} p_{1*} (j^* j')^*.
\]

Proof. This is equivalent to Propositions 1.1.11, 1.1.12. We provide a proof for the sake of completeness.

(1) The following diagram commutes

(2) Similar to (1).

In the rest of this section, we further fix a 2-faithful subcategory $\mathcal{A}$ of $C$ and a pseudofunctor $A: \mathcal{A} \to (D^{\text{coop}})^{\text{adj}}$ with $P_1(A) = F$. We denote the pseudofunctor $L = P_2(A): \mathcal{A} \to (D^{\text{coop}})^{\text{coop}} = D$ by

\[
i \mapsto i_1, \quad \alpha \mapsto \alpha_1.
\]
Construction 8.5. Let $D$ be a square \((\mathbf{s}, \mathbf{s})\) in $\mathcal{C}$ where $i$ and $j$ are morphisms of $\mathcal{A}$. The base change map $A_D$ is by definition the following 2-cell in $\mathcal{D}$

\[
\begin{array}{c}
\xymatrix{\mathcal{D} & \mathcal{D} \\
\mathcal{D} & \mathcal{D}}
\end{array}
\]

If $p$ and $q$ are also morphisms of $\mathcal{A}$, then $A_D$ is also the composition

\[
\begin{array}{c}
\xymatrix{\mathcal{D} & \mathcal{D} \\
\mathcal{D} & \mathcal{D}}
\end{array}
\]

We have an analogue of Proposition 8.4 for $A_D$.

Construction 8.6. Let $D$ be a square \((\mathbf{s}, \mathbf{s})\) in $\mathcal{C}$ where $i$ and $j$ are morphisms of $\mathcal{A}$, $p$ and $q$ are morphisms of $\mathcal{B}$. Then

\[
\begin{array}{c}
\xymatrix{\mathcal{D} & \mathcal{D} \\
\mathcal{D} & \mathcal{D}}
\end{array}
\]

is a 2-cell of $\mathcal{D}^{\text{ad}}$. In fact, the following diagrams commute

\[
\begin{array}{c}
\xymatrix{\mathcal{D} & \mathcal{D} \\
\mathcal{D} & \mathcal{D}}
\end{array}
\]

It follows that $B_D$ is invertible if and only if $A_D$ is. In this case, the diagram

\[
\begin{array}{c}
\xymatrix{\mathcal{D} & \mathcal{D} \\
\mathcal{D} & \mathcal{D}}
\end{array}
\]

commutes and we define $G_D : \ast q_* \Rightarrow p_* j !$ to be the composition. In fact, the following diagram commutes

\[
\begin{array}{c}
\xymatrix{\mathcal{D} & \mathcal{D} \\
\mathcal{D} & \mathcal{D}}
\end{array}
\]
where the hexagon $H$ commutes because the following diagram commutes

![Hexagon Diagram]

**Construction 8.7.** Let $f : X \to Y$ be a morphism of $\mathcal{A} \cap \mathcal{B}$. Then

$$(e_f^B, \eta_f^A) : (f^* f, f^* f, \eta_f^B, \eta_f^A, e_f^B, e_f^A) \Rightarrow (\text{id}_X, \text{id}_X, \text{id}_X, \text{id}_X)$$

is a 2-cell of $\mathcal{D}^{ad}$. It follows that $e_f^B$ is invertible if and only if $\eta_f^A$ is. In this case, the following diagram commutes

![Diagram](Diagram1)

and we define $\rho_f : f_! \Rightarrow f_*$ to be the composition. In fact, the following diagram commutes

![Diagram](Diagram2)

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a pair of composable morphisms of $\mathcal{A} \cap \mathcal{B}$ with $e_f^B$ and $e_g^B$ invertible. Then the following diagram commutes

![Diagram](Diagram3)

The following properties of $G_D$ and $\rho_f$ are similar to axioms (b), (b$'$) of Remark 4.3 and axioms (c), (c$'$) of Remark 6.3.

**Proposition 8.8.**

1. In the situation of Proposition 8.8 assume that the horizontal arrows are morphisms of $\mathcal{A}$, the vertical arrows are morphisms of $\mathcal{B}$, and $B_D$ and $B_{D'}$ are invertible. Then the following diagram commutes

![Diagram](Diagram4)

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(2) In the situation of Proposition 8.3.2 assume that the horizontal arrows are morphisms of \( A \), the vertical arrows are morphisms of \( B \), and \( B_D \) and \( B_{D'} \) are invertible. Then the following diagram commutes

\[
\begin{array}{ccc}
G_D & \rightarrow & G_{D'} \\
\downarrow & & \downarrow \\
p_1 \ast j & \rightarrow & p_3 \ast j'
\end{array}
\]

(3) Let \( D \) be a square (8.3.1) in \( C \) where \( i \) and \( j \) are morphisms of \( A \), \( p \) and \( q \) are morphisms of \( A \cap B \), \( \varepsilon_b^i \), \( \varepsilon_q^i \) and \( B_D \) are invertible. Then the following diagram commutes

\[
\begin{array}{ccc}
i \ast q & \leftarrow & (iq)_* \\
G_D & \downarrow & \downarrow \\
p \ast j & \leftarrow & (pj)_* \\
\end{array}
\]

(4) Let \( D \) be a square (8.3.1) in \( C \) where \( i \) and \( j \) are morphisms of \( A \cap B \), \( p \) and \( q \) are morphisms of \( B \), \( \varepsilon_i^b \), \( \varepsilon_j^b \) and \( B_D \) are invertible. Then the following diagram commutes

\[
\begin{array}{ccc}
i \ast q & \leftarrow & (iq)_* \\
G_D & \downarrow & \downarrow \\
p \ast j & \leftarrow & (pj)_*
\end{array}
\]

Proof. [1] Similar to [2]

[2] The following diagram commutes

\[
\begin{array}{ccc}
i \ast p_1 & \leftarrow & i \ast p_2 \ast j' \ast j \\
G_D & \downarrow & \downarrow \\
p_1 \ast j & \leftarrow & p_3 \ast j'
\end{array}
\]

[3] Similar to [4]
The following diagram commutes

where the pentagon commutes because the following diagram commutes

**Remark 8.9.** If $C$ is a $(2,1)$-category, $B_D$ and $G_D$ are invertible for every 2-Cartesian $(A,B)$-square $D$, and $\epsilon^B_f$ and $\rho_f$ are invertible for every morphism $f$ of $A \cap B$, then Proposition 8.8 shows that $(L, R, (G_D), \rho)$ is an object of $\text{GD}^{\text{Cart}}_A (C, D)$ (Definition 6.3). Here we have used the correspondence via inverting 2-cells between down-squares (8.3.1) used in this section and up-squares (2.14.1) used in earlier sections.

We conclude this section by a couple of criteria for the axioms for morphisms of (Cartesian) gluing data.

The following property is similar to condition (n) of Definition 6.3.

**Proposition 8.10.** Let $D$ be a square (8.3.1) in $C$ where $i$ and $j$ are morphisms of $A \cap B$, $\epsilon^B_i$, $\epsilon^B_j$ and $\alpha$ are invertible, and let $D'$ be the square obtained by inverting $\alpha$. Then the following diagram commutes

\[
\begin{array}{ccc}
  j \omega^* & \overset{\rho_{j \omega}}{\longrightarrow} & j \lambda^* \\
  A_D & \downarrow & B_{D'} \\
  p^* \iota & \overset{\rho_{p^* \iota}}{\longrightarrow} & p^* \iota.
\end{array}
\]
Proof. The following diagram commutes

The following property is similar to condition (m) of Remark 4.3.

Proposition 8.11. Let

\[(8.11.1)\]

be a cube in \(\mathcal{C}\), where \(i, j, i', j'\) are morphisms of \(A\), \(p, q, p', q'\) are morphisms of \(B\), and the 2-cells of the right, left, front, back, bottom, top faces, \(I, I', J, J', K, K'\), are respectively

\(py \Rightarrow wp', \; qx \Rightarrow zq', \; \beta : wa' \Rightarrow iz, \; \beta' : yj' \Rightarrow jx, \; pj \Rightarrow iq, \; p'j' \Rightarrow i'q'.\)

Assume that \(B_K\) and \(B_{K'}\) are invertible. Then the following diagram commutes

\[
\begin{array}{c}
\begin{array}{cccc}
\alpha & \beta & G & \beta' \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\alpha' & \beta' & G_K & \beta'' \\
\end{array}
\end{array}
\]

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Proof. The following diagram commutes

where the decagon is the outline of the following commutative diagram

where the octagon commutes by Proposition 8.4.

9 Proof of Proposition 1.12

Proof of Proposition 1.12 Let $G$ and $H$ be pseudo-functors from $\mathcal{D}$ to $\mathcal{E}$. We need to show that the functor

$$\text{PsNat}(G,H) \to \text{PsNat}(GF,HF) \quad \alpha \mapsto \alpha F$$

between categories of pseudonatural transformations is fully faithful and injective on objects. Let $\alpha$ and $\beta$ be pseudonatural transformations from $G$ to $H$.

We identify the set of modifications $\alpha \Rightarrow \beta$ and the set of modifications $\alpha F \Rightarrow \beta F$ with subsets of the set of 2-cells $|\alpha| \Rightarrow |\beta|$ in $\mathcal{E}^{\text{Ob}(\mathcal{C})}$. Let $\Xi: \alpha F \Rightarrow \beta F$ be a modification. Let $g: X \to Y$ be a morphism of $\mathcal{D}$. By the pseudofullness of $F$, there exists an invertible 2-cell $\delta: g \Rightarrow h = Ff$ in $\mathcal{D}$, where $f$ is a
morphism of \( C \). The front, back, left, right, and bottom squares of the cube

\[
\begin{array}{c}
\alpha(Y)G(g) \quad \Xi \quad H(g)\alpha(X) \\
G(\delta) \quad \beta(Y)G(g) \quad \Xi \quad H(\delta) \\
\alpha(Y)G(h) \quad \Xi \quad H(h)\alpha(X) \\
\beta(Y)G(h) \quad \Xi \quad H(h)\beta(X)
\end{array}
\]

commute. It follows that the top square commutes as well. Therefore, \( \Xi \) is a modification \( \alpha \Rightarrow \beta \).

Assume \( \alpha F = \beta F \). Let \( g \) and \( \delta \) be as above. The square

\[
\begin{array}{c}
\alpha(Y)G(g) \quad \alpha(g) \quad H(g)\alpha(X) \\
\alpha(Y)G(h) \quad \alpha(h) \quad H(h)\alpha(X)
\end{array}
\]

commutes. The same holds for \( \beta \). Since \( \alpha(h) = (\alpha F)(f) = (\beta F)(f) = \beta(h) \), we have \( \alpha(g) = \beta(g) \). Therefore, \( \alpha = \beta \).

Proof of Proposition \([1.10] \). Under assumption \([1.10] \), \( \eta \) and \( \epsilon \) induce \( \mathcal{E}^{\textbf{Ob}(C)} \)-natural equivalences \( \text{id} \rightarrow \Phi_\xi \Psi_\xi \) and \( \Psi_\xi \Phi_\xi \rightarrow \text{id} \), where \( \Psi_\xi : \text{PsFun}(\mathcal{C}, \mathcal{E}) \rightarrow \text{PsFun}(\mathcal{D}, \mathcal{E}) \) is induced by \( G \). Thus \([1.10] \) implies \([1.11] \).

To prove that \([1.11] \) implies \([1.12] \), let \( G : \mathcal{D} \rightarrow \mathcal{C} \) be a pseudofunctor endowed with pseudonatural equivalences \( \eta : \text{id}_\mathcal{C} \rightarrow GF \) and \( FG \rightarrow \text{id}_\mathcal{D} \). For every object \( Y \) of \( \mathcal{D} \), let \( G'Y = |F|^{-1}Y \). For every morphism \( g : Y \rightarrow Y' \) of \( \mathcal{D} \), choose a morphism \( G'g : G'Y \rightarrow G'Y' \) and an invertible 2-cell \( \psi_g \) in \( \mathcal{C} \):

\[
\begin{array}{c}
G'Y \quad \psi_g \quad G'Y' \\
\eta(G'Y) \quad \psi_g \quad \eta(G'Y')
\end{array}
\]

By Lemma \([1.12] \), this determines a pseudofunctor \( G' : \mathcal{D} \rightarrow \mathcal{C} \) such that \( |G'| : \text{Ob}(\mathcal{D}) \rightarrow \text{Ob}(\mathcal{C}) \) is the inverse of \( |F| \) and a pseudonatural equivalence \( \psi : G' \rightarrow G \) such that \( \psi(Y) = \eta(G'Y) \) for every object \( Y \) of \( \mathcal{D} \). For any morphism \( f : X \rightarrow X' \) of \( \mathcal{C} \), \( \eta \) induces the following square in \( \mathcal{C} \):

\[
\begin{array}{c}
X \quad \eta(X) \quad \eta(X') \\
GFX \quad GFX' \quad \eta(X')
\end{array}
\]
Since $\eta(X')$ is an equivalence in $\mathcal{C}$, $\psi \psi_f \eta_f^{-1}$ induces an invertible 2-cell $\eta'_f: f \Rightarrow G'Ff$. This defines a pseudonatural transformation $\eta': \text{id}_\mathcal{C} \rightarrow G'F$ satisfying $\eta'(X) = \text{id}_X$ for every object $X$ of $\mathcal{C}$. For every morphism $g: Y \rightarrow Y'$ in $\mathcal{D}$, choose a morphism $f: G'Y \rightarrow G'Y'$ and an invertible 2-cell $\alpha: g \Rightarrow Ff$. The composition

$$\epsilon_g: FG'g \overset{FG'\alpha}{\Rightarrow} FG'f \overset{(\eta'_f)^{-1}}{\Rightarrow} Ff \overset{\eta'_f}{\Rightarrow} g$$

does not depend on the choice of $(f, \alpha)$. This defines a pseudonatural transformation $\epsilon: FG' \rightarrow \text{id}_\mathcal{D}$ such that $\epsilon(Y) = \text{id}_Y$ for every object $Y$ of $\mathcal{D}$.

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