Noetherian properties of Fargues-Fontaine curves

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Abstract

We establish that the extended Robba rings associated to a perfect nonarchimedean field of characteristic $p$, which arise in $p$-adic Hodge theory as certain completed localizations of the ring of Witt vectors, are strongly noetherian Banach rings; that is, the completed polynomial ring in any number of variables over such a Banach ring is noetherian. This enables Huber’s theory of adic spaces to be applied to such rings. We also establish that rational localizations of these rings are principal ideal domains and that étale covers of these rings (in the sense of Huber) are Dedekind domains.

1 Introduction

The field of $p$-adic Hodge theory has recently been transformed by a series of new geometric ideas. Central among these is the reformulation of the basic theory by Fargues and Fontaine [5] (see also [4], [6], [12]) in terms of vector bundles on certain noetherian schemes associated to perfect nonarchimedean fields of characteristic $p$. While these schemes are not of finite type over a field, they have certain formal properties characteristic of proper curves; for instance, their Picard groups surject canonically onto $\mathbb{Z}$.

The so-called Fargues-Fontaine curves also admit canonical analytifications; more precisely, to each Fargues-Fontaine curve, one can functorially associate an object in Huber’s category of adic spaces [8] which maps back to the original scheme in the category of locally ringed spaces. The pullback functor on vector bundles induced by this morphism is an equivalence of categories [12, §8]; this constitutes a version of the GAGA principle.

One expects a similar result for coherent sheaves, but in order to build a theory of coherent sheaves on adic spaces, one must restrict to spaces satisfying certain noetherian hypotheses. Some care is needed because there is no analogue of the general Hilbert basis theorem for noetherian Banach rings: if $A$ is such a ring, then Tate algebras over $A$ (completion of polynomial rings over $A$ for the Gauss norm) is not known to be noetherian. One must thus consider adic spaces which locally arise from Banach rings for which the Tate algebras are all noetherian (i.e., these rings are strongly noetherian).

In this paper, we establish the strongly noetherian property for the rings used to build the adic Fargues-Fontaine curves (Theorem 3.2, Theorem 4.10); this answers a question of
Fargues [4]. These rings, which are derived from the Witt vectors over a perfect field which is complete with respect to a multiplicative norm, appear frequently in $p$-adic Hodge theory as extended Robba rings (e.g., see [12]). We also establish some finer properties of these rings: any rational localization is a finite direct sum of principal ideal domains (Theorem 7.11), and any étale covering in the sense of Huber is a finite direct sum of Dedekind domains (Theorem 8.5).

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2 Euclidean division for Witt vectors

We begin by recalling the basic setup, fixing notations, and reviewing the Euclidean division ring for certain rings of Witt vectors.

Hypothesis 2.1. Throughout this paper, let $p$ be a fixed prime, let $q$ be a power of $p$, let $L$ be a perfect field containing $\mathbb{F}_q$ which is complete with respect to the multiplicative nonarchimedean norm $|\cdot|$, let $E$ be a complete discretely valued field whose residue field contains $\mathbb{F}_q$, and fix a uniformizer $\varpi \in E$.

Definition 2.2. Let $\mathfrak{o}_L, \mathfrak{o}_E$ denote the valuation subrings of $L, E$. Define the rings

$$A_{L,E} = W(\mathfrak{o}_L)[[\varpi]] : \varpi \in L] \otimes_{W(\mathbb{F}_q)} \mathfrak{o}_E, \quad B_{L,E} = A_{L,E} \otimes_{\mathfrak{o}_E} E.$$

Note that each element of $A_{L,E}$ (resp. $B_{L,E}$) can be written uniquely in the form $\sum_{n \in \mathbb{Z}} \varpi^n [\varpi_n]$ for some $\varpi_n \in L$ which are zero for $n < 0$ (resp. for $n$ sufficiently small) and bounded for $n$ large. For $t \in [0, +\infty)$, define the “Gauss norm” function $\lambda_t : B_{L,E} \to \mathbb{R}$ by the formula

$$\lambda_t \left( \sum_{n \in \mathbb{Z}} \varpi^n [\varpi_n] \right) = \max \{ p^{-n} |\varpi_n|^t \}, \quad (2.2.1)$$

interpreting $|\varpi_n|^t$ as 1 in the case $t = 0$, so that $\lambda_0$ is the $\varpi$-adic absolute value.

Lemma 2.3. For $t \in [0, +\infty)$, the function $\lambda_t$ defines a multiplicative norm on $B_{L,E}$.

Proof. This is a straightforward consequence of the homogeneity properties of Witt vector arithmetic. See for instance [11, §4]. \hfill \Box

For the remainder of §2, fix some $r > 0.$
Lemma 2.6. For degree of multiplicities of all slopes of $x$
Proof. This follows from the multiplicative property of the norms in $[9]$; we have corrected these in the arguments that follow. (See $[12]$, setup here. In addition, there are a number of minor but confusing errors in the presentation.

Lemma 2.8. For $x \in A_{L,E}^r$ nonzero, there exists $\epsilon \in (0, 1)$ with the following property: for any $y \in A_{L,E}^r$, we can write $y = zx + w$ for some $z, w \in A_{L,E}^r$ subject to the following conditions.

(a) We have $\lambda_r(w) \leq \lambda_r(y)$.
(b) If $\lambda_r(w) > \epsilon \lambda_r(y)$, then $\deg(w) < \deg(x)$.

Definition 2.4. Let $A_{L,E}^r$ be the completion of $A_{L,E}$ with respect to $\lambda_r$. Note that $A_{L,E}^r$, $B_{L,E}^r$, and $\tilde{A}_{L,E}^r$ are rings that coincide with the rings denoted $\mathcal{R}_L^{\text{int}, r}, \mathcal{R}_L^{\text{bd}, r}$ in $[12]$.

Definition 2.5. For $x = \sum_{n \in \mathbb{Z}} p^n |x_n| \in B_{L,E}^r$ nonzero, define the 

Newton polygon

of $x$ as the portion of the boundary of the convex hull of the set

$$\bigcup_{n \in \mathbb{Z}} \{(x, y) \in \mathbb{R}^2 : x \leq \log_p |x_n|, y \geq n\},$$

with slopes in the range $(0, r]$. For $t \in (0, r]$, the multiplicity of $t$ in (the Newton polygon of) $x$ is the height of the segment of the Newton polygon of $x$ lying on a line of slope $t$, or 0 if no such segment exists; note that this quantity is always a nonnegative integer.

For $x \in A_{L,E}^r$, we define the degree of $x$, denoted $\deg(x)$, to be the largest $n$ realizing $\lambda_r(x) = \max_n \{p^{-n} |x_n|\}$, or equivalently, the sum of the $p$-adic valuation of $x$ plus the multiplicities of all slopes of $x$. By convention, we also put $\deg(0) = -\infty$.

Lemma 2.6. For $x_1, x_2 \in A_{L,E}^r$ nonzero and $t \in (0, r]$, the multiplicity of $t$ in (resp. the degree of) $x_1x_2$ is the sum of the multiplicities of $t$ in (resp. the degrees of) $f_1$ and $f_2$.

Proof. This follows from the multiplicative property of the norms $\lambda_r$ together with convex duality. We omit further details. \hfill $\square$

Remark 2.7. Note that for $x, y \in A_{L,E}^r$ such that $\lambda_r(x - y) < \lambda_r(x)$, we have $\deg(x) = \deg(y)$. This observation indicates that if one is willing to neglect lower-order terms, then degrees in our sense behave like the degrees of ordinary polynomials.

The ring $A_{L,E}^r$ admits a Euclidean division algorithm as described in $[9]$ Lemma 2.6.3. However, we opt to give a self-contained proof for several reasons. The level of generality in $[9]$ is at once too high (there are intended applications in which one considers somewhat smaller rings) and too low (the field $E$ therein is forced to be of characteristic 0) to match our setup here. In addition, there are a number of minor but confusing errors in the presentation in $[9]$; we have corrected these in the arguments that follow. (See $[12]$ §4.2 for errata in the context of $[9]$.)
Proof. Put \( m = \deg(x) \) and write \( x = \sum_{n=0}^{\infty} \omega^n[\mathfrak{p}_n] \). We may then choose \( \epsilon \in (0, 1) \) such that \( \lambda_r(\omega^n[\mathfrak{p}_n]) \leq \epsilon \lambda_r(x) \) for \( n > m \). We prove the claim for this value of \( \epsilon \).

We define a sequence \( y_0, y_1, \ldots \) as follows: take \( y_0 = y \), and given \( y_l = \sum_{n=0}^{\infty} \omega^n[\mathfrak{p}_n] \)\], put \( z_l = \sum_{n=0}^{\infty} \omega^n[\mathfrak{p}_l,n+m/\mathfrak{p}_m] \) and \( y_{l+1} = y_l - zx \). To prove the desired result, it suffices to deduce a contradiction under the assumptions that \( \lambda_r(y_l) > \epsilon \lambda_r(y) \) and \( \deg(y_l) \geq m \) for all \( l \). To see this, put \( x' = \sum_{n=0}^{m-1} \omega^n[\mathfrak{p}_n] \) and write

\[
y_{l+1} = \sum_{n=0}^{m-1} \omega^n[\mathfrak{p}_n] - z_l x' + *
\]

where \( \lambda_r(*) \leq \epsilon \lambda_r(y) \). Since \( \lambda_r(y_l+1) > \epsilon \lambda_r(y) \), we must have \( \lambda_r(y_{l+1} - *) > \epsilon \lambda_r(y) \). We must then also have \( \lambda_r(z_l x') > \epsilon \lambda_r(y) \), as otherwise by Remark \( \ref{remark:2.7} \) we would have \( \deg(y_{l+1}) = \deg(\sum_{n=0}^{m-1} \omega^n[\mathfrak{p}_n]) < m \).

This means that \( \deg(y_{l+1}) = \deg(z_l x') \). By Lemma \( \ref{lemma:2.6} \) we have \( \deg(z_l x') = \deg(z_l) + \deg(x') \). Since \( \deg(y_l) \geq m \), we have \( \deg(z_l) = \deg(y_l) - m \); on the other hand, evidently \( \deg(x') < m \). Consequently, \( \deg(y_{l+1}) = \deg(z_l x') < \deg(y_l) \); since \( \deg(y_l) \geq 0 \) for all \( l \), this yields the desired contradiction. \( \square \)

**Proposition 2.9.** For \( x, y \in A^r_{L,E} \) with \( x \neq 0 \), we can write \( y = zx + w \) for some \( z, w \in A^r_{L,E} \) with \( \lambda_r(w) \leq \lambda_r(y) \) and \( \deg(w) < \deg(x) \).

**Proof.** Choose \( \epsilon \in (0, 1) \) as in Lemma \( \ref{lemma:2.8} \). We define sequences \( y_0, y_1, \ldots \) and \( z_0, z_1, \ldots \) as follows. Take \( y_0 = y \). Given \( y_l \), if \( \deg(y_l) < \deg(x) \), put \( z_l = 0 \), \( y_{l+1} = y_l \). Otherwise, apply Lemma \( \ref{lemma:2.8} \) to write \( y_l = z_l x + w_l \) with \( \lambda_r(w_l) \leq \lambda_r(y_l) \) and either \( \lambda_r(w_l) \leq \epsilon \lambda_r(y_l) \) or \( \deg(w_l) < \deg(x) \), and put \( y_{l+1} = w_l \).

From the construction, we have \( \lambda_r(y_l) \leq \lambda_r(y) \) for all \( l \). If \( z_l = 0 \) for some \( l \), then the sum \( z = \sum_{l=0}^{\infty} z_l \) is obviously finite and \( y = zx = y_l \), so \( \lambda_r(y - zx) = \lambda_r(y_l) \leq \lambda_r(y) \) and \( \deg(y - zx) = \deg(y_l) < \deg(x) \). Otherwise, we have \( \lambda_r(y_l) \leq \epsilon \lambda_r(y) \) for all \( l \), so

\[
\lambda_r(z_l) \leq \lambda_r(x)^{-1} \max\{\lambda_r(y_l), \lambda_r(y_{l+1})\} \to 0
\]

and so the sum \( z = \sum_{l=0}^{\infty} z_l \) converges. We then have \( y - zx = \lim_{l \to \infty} y_{l+1} = 0 \), so we may take \( w = 0 \). \( \square \)

**Corollary 2.10.** The ring \( A^r_{L,E} \) is a Euclidean domain for the function \( \deg \), and hence a principal ideal domain.

## 3 The strong noetherian property

We now prove an analogue of the Hilbert basis theorem for the ring \( A^r_{L,E} \).

**Definition 3.1.** For any commutative Banach ring \( A \) with norm \( |\cdot| \), any nonnegative integer \( n \), and any \( n \)-tuple \( \rho = (\rho_1, \ldots, \rho_n) \) of positive real numbers, let \( A\{T_1/\rho_1, \ldots, T_n/\rho_n\} \) be the
completion of the ordinary polynomial ring $A[T_1, \ldots, T_n]$ with respect to the weighted Gauss norm

$$
\left| \sum_{i_1, \ldots, i_n=0}^{\infty} c_{i_1, \ldots, i_n} T_1^{i_1} \cdots T_n^{i_n} \right|_{\rho} = \max \{ \left| c_{i_1, \ldots, i_n} \right| \rho_1^{i_1} \cdots \rho_n^{i_n} \}. \quad (3.1.1)
$$

We may view $A\{T_1/\rho_1, \ldots, T_n/\rho_n\}$ as the subring of $A[[T_1, \ldots, T_n]]$ consisting of those series $\sum_{i_1, \ldots, i_n=0}^{\infty} c_{i_1, \ldots, i_n} T_1^{i_1} \cdots T_n^{i_n}$ for which $|c_{i_1, \ldots, i_n}| \rho_1^{i_1} \cdots \rho_n^{i_n} \to 0$ as $i_1 + \cdots + i_n \to \infty$, with the norm again given by (3.1.1). Note that if $|\bullet|$ is multiplicative, then so is $|\bullet|_{\rho}$ (Gauss’s lemma).

One would like to know that $A\{T_1/\rho_1, \ldots, T_n/\rho_n\}$ is noetherian whenever $A$ is, but this is only known under somewhat restrictive hypotheses, e.g., when $A$ is a nonarchimedean field [10, Theorem 5.2.6/1]. Over the course of [3] we will prove the following theorem, which answers a question of Fargues [4] by proving that $A'_{L,E}$ is strongly noetherian in the sense of Huber. This means that Huber’s theory of adic spaces, as developed in [8], applies to this ring; however, we will not pursue this point here.

**Theorem 3.2.** For $r > 0$, view $A'_{L,E}$ as a Banach ring using the norm $\lambda_r$. Then for any nonnegative integer $n$ and any $\rho_1, \ldots, \rho_n > 0$, the ring $R = A'_{L,E}\{T_1/\rho_1, \ldots, T_n/\rho_n\}$ is noetherian.

Our approach to the proof relies on some standard ideas from the theory of Gröbner bases; indeed, it can be used to give an alternate proof of [2, Theorem 5.2.6/1]. We start with the underlying combinatorial construction.

**Hypothesis 3.3.** For the remainder of [3] retain notation as in Theorem 3.2, let $H$ be an ideal of $R$, and let $I = (i_1, \ldots, i_n)$ and $J = (j_1, \ldots, j_n)$ (and subscripted versions thereof, such as $I_k = (i_{k,1}, \ldots, i_{k,n})$) denote elements of the additive monoid $\mathbb{Z}_{\geq 0}^n$.

**Definition 3.4.** We equip $\mathbb{Z}_{\geq 0}^n$ with the componentwise partial order $\leq$, for which $I \leq J$ if and only if $I_k \leq J_k$ for $i = 1, \ldots, n$. This partial order is a well-quasi-ordering: any infinite sequence contains an infinite nondecreasing subsequence.

We also equip $\mathbb{Z}_{\geq 0}^n$ with the graded lexicographic total order $\preceq$, for which $I \preceq J$ if either $i_1 + \cdots + i_n < j_1 + \cdots + j_n$, or $i_1 + \cdots + i_n = j_1 + \cdots + j_n$ and there exists $k \in \{1, \ldots, n\}$ such that $i_l = j_l$ for $l < k$ and $i_k = j_k$. Since $\preceq$ is a refinement of $\leq$, it is a well-ordering.

**Remark 3.5.** In commutative algebra, the only critical properties of $\preceq$ are that it is a well-ordering and that it refines $\leq$. In some cases (such as ours), it is also important that for any $I$, there are only finitely many $J$ with $J \preceq I$. In any case, there are many options for $\preceq$ with similar properties, giving rise to many different term orderings which are relevant for practical applications. See for instance [3].

We next define a notion of *leading terms* for elements of $R$. Note that a similar construction appears already in [10].

**Definition 3.6.** For $x = \sum_I x_I T^I \in R$ nonzero, define the *leading index* of $x$ to be the index $I$ which is maximal under $\preceq$ for the property that $|x_I T^I|_{\rho} = |x_I|_{\rho}$, and define the *leading coefficient* of $x$ to be the corresponding value of $x_I$. 
We can now define an analogue of a Gröbner basis for the ideal \( H \).

**Definition 3.7.** For each \( I \), let \( d_I \) be the smallest possible degree of the leading coefficient of an element of \( R \) with leading index \( I \), or \(+\infty\) if no such element exists. Note that if \( I_1 \leq I_2 \), then \( d_{I_2} \leq d_{I_1} \).

Since \( \mathbb{Z}_{\geq 0}^n \) is well-quasi-ordered under \( \leq \), the set of \( I \) for which \( d_I < +\infty \) contains only finitely many minimal elements with respect to \( \leq \). Consequently, the set of possible finite values of \( d_I \) is bounded above, and hence is finite. For each nonnegative integer \( d \), let \( S_d \) be the set of \( I \) which are minimal with respect to \( \leq \) for the property that \( d_I = d \); then \( S_d \) is finite for all \( d \) and empty for all but finitely many \( d \). Let \( S \) be the union of the \( S_d \). For each \( I \in S \), choose \( x_I \in H \setminus \{0\} \) with leading index \( I \) and leading coefficient of degree \( d_I \).

We claim that the finite set \( \{x_I : I \in S\} \) generates the ideal \( H \). As in the proof of Proposition 2.9, we first establish a certain approximate version of this statement, using an iterative construction and a proof by contradiction based on well-ordering properties.

**Lemma 3.8.** There exists \( \epsilon \in (0, 1) \) with the following property: for each \( y \in H \), there exist \( a_I \in R \) for \( I \in S \) such that \(|a_I|_\rho |x_I|_\rho \leq |y|_\rho \) and \(|y - \sum_{I \in S} a_I x_I|_\rho \leq \epsilon |y|_\rho \).

*Proof.* Write \( x_I = \sum_J x_{I,J} T^J \) and let \( c_I = x_{I,I} \) be the leading coefficient of \( x_I \). Let \( \epsilon \) be the maximum of \(|x_{I,J} T^J|_\rho / |c_I T^I|_\rho \) over all \( J \) for which \( I \prec J \) (or any value in \((0, 1) \) in case no such \( J \) exists); by construction, we have \( \epsilon \in (0, 1) \). We prove the claim for this value of \( \epsilon \).

We define \( y_l = H, a_{I,l} \in R \) for \( l = 0, 1, \ldots \) and \( I \in S \) as follows. Put \( y_0 = y \). Given \( y_l = \sum_J y_{l,J} T^J \), if \(|y_l|_\rho \leq \epsilon |y|_\rho \); put \( a_{I,l} = 0 \) and \( y_{l+1} = y_l \). Otherwise, \( y_l \) is nonzero, so it has a leading index \( J_l \). By construction, we can find an index \( I_l \in S \) such that \( I_l \leq J_l \) and \( d_{I_l} = d_{J_l} \). Apply Proposition 2.9 to write \( y_{l,J_l} = z_l c_{I_l} + w_l \) for some \( z_l, w_l \in A_{L,E} \) with \(|w_l| \leq |y_{l,J_l}| \) and \( \deg(w_l) < \deg(c_{I_l}) = d_{I_l} \). Put

\[
a_{I,l} = \begin{cases} z_l T^{J_l-I_l} & (I = I_l) \\ 0 & (I \neq I_l), \end{cases}
\]

\[
y_{l+1} = y_l - a_{I,l} x_{I_l}.
\]

If \(|y_l|_\rho \leq \epsilon |y|_\rho \) for some \( l \), then the sums \( a_I = \sum_{l=0}^{\infty} a_{I,l} \) are finite and have the desired effect. It thus suffices to derive a contradiction under the assumption that \(|y_l|_\rho > \epsilon |y|_\rho \) for all \( l \).

Define the \( \epsilon \)-support of \( y_l \) to be the finite set \( E_l \) consisting of those \( J \) for which \(|y_{l,J} T^J|_\rho > \epsilon |y|_\rho \) in particular, \( J_l \in E_l \). By virtue of our choice of \( \epsilon \), \( E_l \) and \( E_{l+1} \) agree for all indices \( J \) for which \( J_l \prec J \). In particular, since \( E_0 \) is finite, we can choose \( J_+ \) for which \( J \leq J_+ \) for all \( J \in E_0 \), and then \( J \geq J_+ \) for \( J \in E_l \) for all \( l \).

The set \( \{J \in \mathbb{Z}_{\geq 0}^n : J \geq J_+\} \) is finite, so there are only finitely many indices which occur as \( J_l \) for infinitely many \( l \). Let \( J \) be the largest such value with respect to \( \prec \); then there must be some nonnegative integers \( l < l' \) such that \( J_l = J_{l'} = J \) and \( J_k \prec J \) for \( l < k < l' \). But we then have

\[
|y_{l+1,J} T^J - y_{l',J} T^{J'}|_\rho \leq \epsilon |y|_\rho < |y_{l',J} T^{J'}|_\rho
\]

and hence

\[
\deg(y_{l+1,J}) < d_J, \deg(y_{l',J}) \geq d_{J}, |y_{l+1,J} - y_{l',J}| < |y_{l',J}|,
\]

which gives a contradiction by Remark 2.7. \( \square \)
We now finish as in the proof of Proposition \[2.9\]

**Lemma 3.9.** The finite set \( \{ x_I : I \in S \} \) generates the ideal \( H \). Consequently, Theorem \[3.2\] holds.

**Proof.** Choose \( \epsilon \in (0,1) \) as in Lemma \[3.8\]. For \( y \in H \), define sequences \( y_0, y_1, \ldots \) and \( a_{0,I}, a_{1,I}, \ldots \) for \( I \in S \) as follows: put \( y_0 = y \), and given \( y_I \), apply Lemma \[3.8\] to construct \( a_{I,I} \in R \) for \( I \in S \) such that \( |a_{I,I}|_\rho |x_I|_\rho \leq |y_I|_\rho \) and \( |y_I - \sum_{I \in S} a_{I,I} x_I|_\rho \leq \epsilon |y_I|_\rho \); then put \( y_{I+1} = y_I - \sum_{I \in S} a_{I,I} x_I \). By construction, \( |y_I|_\rho \leq \epsilon^I |y_I| \), so the sequence \( \{ y_I \} \) converges to zero and the sums \( a_I = \sum_{I=0}^\infty a_{I,I} \) converge to limits satisfying \( y = \sum_I a_I x_I \). \( \square \)

4 Some additional rings

We next define the rings that appear directly in the study of the adic spaces associated to Fargues-Fontaine curves, and use Theorem \[3.2\] to extend the strong noetherian property to these rings.

**Hypothesis 4.1.** Throughout \[4\] let \( I = [s,r] \) be a closed subinterval of \((0, +\infty)\).

**Definition 4.2.** Define \( \lambda_I = \max\{ \lambda_s, \lambda_r \} \); by Lemma \[2.3\] this is a power-multiplicative norm on \( B_{L,E}^I \). Let \( B_{L,E}^I \) be the completion of \( B_{L,E} \) with respect to \( \lambda_I \). Let \( B^I_{L,E} \) denote the subring of \( x \in B^I_{L,E} \) for which \( \lambda_I(x) \leq 1 \).

**Remark 4.3.** In the case \( E = \mathbb{Q}_p \), the ring \( B^I_{L,E} \) coincides with the ring \( \tilde{R}_L^I \) of \[12\]; in general, it appears under the notation \( B_I \) in \[5\].

The following may be considered an analogue of the Hadamard three circles inequality (compare \[12\], Lemma 4.2.3).

**Lemma 4.4.** For \( t_1, t_2 \in I \) and \( c \in [0,1] \), put \( t = t_1 t_2^{1-c} \). Then for all \( x \in B^I_{L,E} \),

\[
\lambda_I(x) \leq \lambda_{t_1}(x)^c \lambda_{t_2}(x)^{1-c}.
\]

**Proof.** By continuity, it suffices to check the inequality for \( x \in B_{L,E} \). From the shape of the formula \[2.2.1\], we may further reduce to the case where \( x = \omega^n [\tau_n] \) for some \( n \in \mathbb{Z} \), \( \tau_n \in L \). But in this case, the desired inequality becomes an equality. \( \square \)

**Corollary 4.5.** We have \( \lambda_I = \sup\{ \lambda_t : t \in I \} \).

**Corollary 4.6.** For any closed subinterval \( I' \) of \( I \), the map \( B^I_{L,E} \to B^{I'}_{L,E} \) is injective.

**Definition 4.7.** For \( x \in B^I_{L,E} \) nonzero, define the Newton polygon of \( x \) by choosing some \( x' \in B_{L,E} \) with \( \lambda_I(x' - x) < \lambda_I(x) \) for all \( t \in I \), forming the Newton polygon of \( x' \), then discarding segments corresponding to slopes not in \( I \). Note that this construction does not depend on the choice of \( x' \), and inherits the multiplicativity properties from the corresponding definition for \( A_{L,E} \) (Lemma \[2.6\]). We define the multiplicity of slopes as before, and the degree of \( x \) as the sum of all multiplicities, which is again a nonnegative integer.
Lemma 4.8. A nonzero element \( x \in B_{L,E}^I \) is a unit if and only if its degree is 0.

Proof. If \( x \) is a unit, it must have degree 0 by the multiplicativity property of Newton polygons (Definition 4.7). Conversely, if \( x \) has degree 0, then for some \( n \in \mathbb{Z}, \overline{x}_n \in L^x \) we have \( \lambda_t(x - \overline{x}_n) < \lambda_t(\overline{x}_n) \) for all \( t \in I \), so we may compute an inverse of \( x \) using a convergent geometric series. \( \square \)

Lemma 4.9. Choose \( \overline{z} \in L \) with \( |\overline{z}| = c \in (0,1) \). Then for \( \rho \in (0,1) \), we have isomorphisms of Banach rings

\[
B_{L,E}^I(T/\rho)/(T - [\overline{z}]) \cong B_{L,E}^{I'}, \quad I' = I \cap [\log_c \rho, +\infty),
\]

\[
B_{L,E}^I(T/\rho^{-1})/[T - [\overline{z}^{-1}]) \cong B_{L,E}^{I''}, \quad I'' = I \cap (0, \log_c \rho],
\]

\[
A_{L,E}^I(T)/(pT - [\overline{z}]) \cong B_{L,E}^{I'''}, \quad I''' = [-n^{-1} \log_c p, r],
\]

interpreting \( B_{L,E}^\ast \) as 0 if \( \ast \) is empty. Moreover, in case \( \rho \in \mathbb{P}^2 \), the integral closures of the images of \( B_{L,E}^{I',+} \) in \( B_{L,E}^{I'',+} \), \( B_{L,E}^{I''',+} \) are respectively \( B_{L,E}^{I',+}, B_{L,E}^{I'',+} \).

Proof. We check the first case in detail, the other cases being similar. Put \( t_0 = \log_c \rho \). For \( I' = \emptyset \), then \( \rho - [\overline{z}] = [\overline{z}](1 - [\overline{z}^{-1}]T) \) and \( [\overline{z}^{-1}]T|_\rho < 1 \), so \( \rho - [\overline{z}] \) is a unit in \( B_{L,E}^I(T/\rho) \) and so both sides of the desired equality are zero. We may thus assume hereafter that \( I' \neq \emptyset \), so that there is a well-defined map \( B_{L,E}^I(T/\rho) \to B_{L,E}^{I'} \) taking \( T \) to \( \overline{z} \).

For \( x, y \in B_{L,E}^I(T/\rho) \) with \( y = (T - [\overline{z}])x \), applying the multiplicative property of Gauss norms and then taking suprema yields \( |y|_\rho \geq |x|_\rho \) (compare [12, Lemma 2.8.8]). This means that multiplication by \( T - [\overline{z}] \) is a strict injective endomorphism of \( B_{L,E}^I(T/\rho) \). In particular, the ideal \( (T - [\overline{z}]) \) is closed, so \( B_{L,E}^I(T/\rho)/(T - [\overline{z}]) \) is a Banach ring.

Next, suppose that \( y = \sum_{n=0}^\infty y_n T^n \in B_{L,E}^I(T/\rho) \) maps to zero in \( B_{L,E}^{I'}(T/\rho) \). Put

\[
x_n = -\sum_{i=0}^n y_i [\overline{z}]^{i-n-1},
\]

so that in \( B_{L,E}^I(T) \) we have \( y = (T - [\overline{z}])x \) for \( x = \sum_{n=0}^\infty x_n T^n \). For \( t \in I \) with \( t < t_0 \), we see as above that \( T - [\overline{z}] \) is invertible in \( B_{L,E}^{[t]}(T/\rho) \) and so \( \rho^n \lambda_t(x_n) \to 0 \). For \( t \in I \) with \( t \geq t_0 \), we have \( \lambda_t([\overline{z}]) \leq \rho \) and so \( B_{L,E}^{[t]}(T/\rho)/(T - [\overline{z}]) \cong B_{L,E}^{[t]} \); hence \( \rho^n \lambda_t(x_n) \to 0 \) again. Using Corollary 4.5 we conclude that \( x \in B_{L,E}^I(T/\rho) \), so the map \( B_{L,E}^I(T/\rho)/(T - [\overline{z}]) \to B_{L,E}^{I'} \) is injective.

Next, put \( x = \overline{\omega}^n [\overline{x}_n] \) for some \( n \in \mathbb{Z}, \overline{x}_n \in L \). Let \( j \) be the smallest nonnegative integer such that \( c^{-j} |\overline{x}_n| \geq 1 \) and take \( y = \overline{\omega}^n [\overline{x}_n]^{-j} \). For \( t \in I \) with \( t \leq t_0 \), we have \( \rho^j \lambda_t(y) \leq \rho^j \lambda_t(y) = \lambda_t(y) \). For \( t \in I \) with \( t > t_0 \), in case \( j = 0 \) we obviously have \( \lambda_t(y) = \lambda_t(x) \); otherwise, we have \( c^{-j+1} |\overline{x}_n| < 1 \) and so \( \rho^j \lambda_t(y) < c^{j-1} \lambda_t(x) \). For \( z = yT^j \in B_{L,E}^I(T/\rho) \), we therefore have

\[
|z|_\rho \leq c^{j-1} \lambda_T(x). \quad (4.9.1)
\]

By (2.2.1), we may then lift any \( x \in B_{L,E}^{I'} \) to \( z \in B_{L,E}^I(T/\rho) \) so that (4.9.1) remains true. This implies that \( B_{L,E}^I(T/\rho)/(T - [\overline{z}]) \cong B_{L,E}^{I'} \) is strict surjective.
To conclude, it is sufficient to check that if $\rho \in p^{Q}$, then for any $x = \varpi^{i}[\pi_{n}]$ with $\lambda_{I'}(x) = 1$, we can lift some power of $x$ to $z \in B_{L,E}^{I} \{T/\rho\}$ with $|z|_{\rho} = 1$. Set notation as above. If $j = 0$, then

$$\lambda_{I'}(x) = \lambda_{s}(x) = \lambda_{s}(y) = \lambda_{I}(y) = |z|_{\rho}.$$ 

If $j > 0$, then $\lambda_{I'}(x) = \lambda_{t_{0}}(x) = p^{-n}|\pi_{n}|^{t_{0}}$. Since $\lambda_{I'}(x) = 1$ and $\rho \in p^{Q}$, after raising $x$ to a suitable power we have $|\pi_{n}z^{-j}| = 1$, so

$$|z|_{\rho} = \rho^{j}p^{-n} = \lambda_{t_{0}}(x) = \lambda_{I'}(x).$$

This completes the proof. □

By combining Theorem 3.2, Remark 4.3, and Lemma 4.9 we obtain the following result.

**Theorem 4.10.** View $B_{L,E}^{I}$ as a Banach ring using the norm $\lambda_{I}$. Then for any nonnegative integer $n$ and any positive real numbers $\rho_{1}, \ldots, \rho_{n}$, the ring $B_{L,E}^{I} \{T_{1}/\rho_{1}, \ldots, T_{n}/\rho_{n}\}$ is noetherian.

**Remark 4.11.** The fact that the rings $B_{L,E}^{I} \{T_{1}/\rho_{1}, \ldots, T_{n}/\rho_{n}\}$ are noetherian for $\rho_{1} = \cdots = \rho_{n} = 1$ means that $B_{L,E}^{I}$ is strongly noetherian in the sense of Huber. However, we do not know how to deduce this directly from the restricted form of Theorem 3.2 in which one only allows $\rho_{1} = \cdots = \rho_{n} = 1$: we need to allow arbitrary $\rho$ in order to fix the left endpoint of the interval $I$ using Lemma 4.9. We also do not know how to give a direct proof of Theorem 4.10 in the style of the proof of Theorem 3.2 except in the case where $I = [r, r]$ consists of a single point, in which case $\lambda_{I} = \lambda_{r}$ is again multiplicative.

**Remark 4.12.** One can make sense of $B_{L,E}^{[0, r]}$ by identifying it with $A_{L,E}^{r}[p^{-1}]$. This gives a Banach ring for the norm $\max\{\lambda_{0}, \lambda_{r}\}$; note that $\lambda_{0}$ is the $\varpi$-adic absolute value. Of course $B_{L,E}^{[0, r]}$ is noetherian because $A_{L,E}^{r}$ is. However, due to the mismatch of topologies, we do not know how to prove that $B_{L,E}^{[0, r]}$ is strongly noetherian.

**Remark 4.13.** One can also make sense of $B_{L,E}^{[r, +\infty]}$ by rescaling the Gauss norms, e.g., by setting

$$\lambda_{t} \left( \sum_{n \in \mathbb{Z}} \varpi^{n}[\pi_{n}] \right) = \sup\{p^{-n/(1+t)}|\pi_{n}|^{t/(1+t)}\}$$

so that

$$\lambda_{\infty} \left( \sum_{n \in \mathbb{Z}} \varpi^{n}[\pi_{n}] \right) = \sup\{|\pi_{n}|\}.$$ 

However, the resulting ring can be shown to be nonnoetherian, by exploiting the existence of elements with infinitely many distinct slopes in their Newton polygons (or equivalently, the fact that the maxima have become suprema).
5 A descent construction

Before continuing, we record a descent argument which will allow us to freely enlarge the field $L$ in what follows.

**Convention 5.1.** We adopt conventions concerning Banach rings, adic Banach rings, Gel’fand spectra, and adic spectra as in \[12\]. In particular, we write $\mathcal{M}(R)$ for the Gel’fand spectrum of the Banach ring $R$ and $\text{Spa}(R, R^+)$ for the adic spectrum of the adic Banach ring $(R, R^+)$. 

**Hypothesis 5.2.** Throughout \[15\] let $L'$ be a perfect overfield of $L$ which is complete with respect to a multiplicative nonarchimedean norm extending the norm on $L$.

**Lemma 5.3.** (a) The tensor product seminorms on $L' \otimes_L L'$ and $L' \otimes_L L' \otimes_L L'$ are power-multiplicative.

(b) The simplicial exact sequence

$$0 \to L \to L' \otimes_L L' \to L' \otimes_L L' \otimes_L L'$$

is almost optimal; that is, the quotient and subspace norms at each point coincide. Consequently, we may complete the tensor product to obtain another almost optimal exact sequence.

*Proof.* See \[12\] Remark 3.1.6].

Using Lemma 5.3 we obtain a descent property for ideals in $W(\mathfrak{o}_L)$.

**Lemma 5.4.** Equip $R = L' \widehat{\otimes}_L L'$ with the tensor product seminorm. Let $z$ be an element of $W(\mathfrak{o}_{L'})$ with the property that for any $\beta \in \mathcal{M}(R)$, the two images of $z'$ in $W(\mathfrak{o}_{\mathcal{H}(\beta)})$ generate the same ideal. Then $\mathfrak{z}'$ factors as a unit times an element of $W(\mathfrak{o}_L)$.

*Proof.* Let $\kappa_L, \kappa_{L'}$ be the residue fields of $L, L'$. Let $\mathfrak{o}_R$ be the subring of $R$ consisting of elements of norm at most 1. Let $\mathfrak{m}_R$ be the ideal of $\mathfrak{o}_R$ consisting of elements of norm strictly less than 1. Put $\kappa_R = \mathfrak{o}_R/\mathfrak{m}_R$. By Lemma 5.3(a) and \[12\] Theorem 2.3.10], the tensor product seminorm on $R$ can be computed as the supremum over $\mathcal{M}(R)$. Consequently, we have a canonical isomorphism $\kappa_R \cong \kappa_{L'} \otimes_{\kappa_L} \kappa_{L'}$.

Let $\iota_1, \iota_2$ denote the two maps $L' \to R$ and the induced maps $W(L') \to W(R)$. Put $q_0 = \iota_1(z')/\iota_2(z') \in W(R)$. By considering Newton polygons in $W(\mathfrak{o}_{\mathcal{H}(\beta)})$ for each $\beta$, we see that in fact $q_0 \in W(\mathfrak{o}_R)^\times$.

The projection of $q_0$ along $W(\mathfrak{o}_R^\times) \to \kappa_R^\times$ defines a descent datum on a one-dimensional vector space over $\kappa_{L'}$. By faithfully flat descent for modules, this vector space descends to $\kappa_L$; in other words, there exists $u_0 \in W(\mathfrak{o}_{L'})^\times$ such that for $q_1 = \iota_1(z'/u_0)/\iota_2(z'/u_0)$, we have $q_1 - 1 \in \ker(W(\mathfrak{o}_R) \to \kappa_R)$.

Define the submultiplicative norm $\lambda_1$ on $W(\mathfrak{o}_R)$ using the formula (2.2.1). We can then choose $\epsilon \in (0, 1)$ such that $\lambda_1(q_1 - 1) \leq \epsilon^2$. We construct sequences $u_1, u_2, \ldots \in W(\mathfrak{o}_{L'})^\times$ and $q_2, q_3, \ldots \in W(\mathfrak{o}_R)^\times$ as follows: given $q_1$, apply Lemma 5.3(b) to construct $v_1 \in W(\mathfrak{o}_{L'})$ with $\lambda_1(v_1) \leq \epsilon^{-1}\lambda_1(q_1)$ and $\iota_1(v_1) - \iota_2(v_1) = q_1$, then put $u_{i+1} = u_i(1 + v_i)$ and $q_{i+1} = \iota_1(z'/u_{i+1})/\iota_2(z'/u_{i+1})$. We then have $\lambda_1(q_i - 1) \leq \epsilon^{i+1}$ and hence $\lambda_1(v_i) \leq \epsilon^i$, so the $q_i$ converge to 1 and the $u_i$ converge to a limit $u \in W(\mathfrak{o}_{L'})^\times$ for which $z'/u \in W(\mathfrak{o}_L)$. 


6 Primitive elements of degree 1

We next focus attention on those elements of $W(\mathfrak{a}_L)$ which behave like monic linear polynomials in the variable $p$. These elements control much of the algebra and geometry of the rings we are considering. In particular, they give rise to a deformation retraction on $\mathcal{M}(B_{L,E}^I)$ as described in [11].

Definition 6.1. We say that $z = \sum_{n=0}^{\infty} \varpi^n [\bar{z}_n] \in W(\mathfrak{a}_L)$ is primitive of degree 1 if $\bar{z}_0 \in \mathfrak{m}_L \setminus \{0\}$ and $\bar{z}_1 \in \mathfrak{a}_L^\times$. For example, $\varpi - [\bar{\pi}]$ is primitive of degree 1 for any $\bar{\pi} \in \mathfrak{m}_L \setminus \{0\}$. For $z$ primitive of degree 1 with slope $r$, the ring $W(\mathfrak{a}_L)[p^{-1}]/(z)$ is a perfectoid field under the quotient norm induced by $\lambda$, [12 Theorem 3.5.3].

Definition 6.2. For $z \in W(\mathfrak{a}_L)$ primitive of degree 1 with slope $r$, let $H(z, \rho)$ be the quotient norm on $B_{L,E}\{T/(p^{-1})\}/(T - z)$ for the Gauss extension of $\lambda_r$. As in [11 Theorem 5.11], this norm is multiplicative.

Definition 6.3. Let $I$ be a closed interval in $(0, +\infty)$. For any $\beta \in \mathcal{M}(B_{L,E}^I)$, there exist a perfect overfield $L'$ of $L$ complete with respect to a multiplicative nonarchimedean norm extending the one on $L$ and some $\bar{\pi} \in \mathfrak{m}_L \setminus \{0\}$ such that the restriction of $H(\varpi - [\bar{\pi}], 0)$ to $B_{L,E}^I$ equals $\beta$.

Proof. Let $F'$ be a completed algebraic closure of $\mathcal{H}(\beta)$. Let $L'$ be the perfect field corresponding to $F'$ under the perfectoid correspondence [12 Theorem 3.5.3]; recall that $L'$ may be identified set-theoretically with the the inverse limit of $F'$ under the $p$-power map. We may then take $\bar{\pi}$ to be a coherent sequence of roots of $p$ in $F'$.

Definition 6.4. Let $I$ be a closed interval in $(0, +\infty)$. For any $\beta \in \mathcal{M}(B_{L,E}^I)$ and $\rho \in [0, 1]$, we may define a point $H(\beta, \rho) \in \mathcal{M}(B_{L,E}^I)$ by choosing $L', \bar{\pi}$ as in Lemma 6.3 and taking $H(\beta, \rho)$ to be the restriction of $H(\varpi - [\bar{\pi}], \rho)$. As in [11 Theorem 7.8], this construction does not depend on $L', \bar{\pi}$ and defines a continuous map $H : \mathcal{M}(B_{L,E}^I) \times [0, 1] \to \mathcal{M}(B_{L,E}^I)$. Note that $H(\beta, 0) = \beta$ while $H(\beta, 1) = \lambda_r$ for $r$ equal to the slope of $\varpi - [\bar{\pi}]$. We define the radius of $\beta$ to be the largest $\rho \in [0, 1]$ for which $H(\beta, \rho) = \beta$.

7 Structure of rational localizations

We next convert our previous observations into some structural properties of the rings obtained from $B_{L,E}^I$ by the formation of rational localizations.

Hypothesis 7.1. Throughout [17] let $I = [s, r]$ be a closed interval in $(0, +\infty)$. Let $(B_{L,E}^I, B_{L,E}^{I, +}) \to (C, C^+)$ be a rational localization.

Convention 7.2. Throughout [17] we will write $L'$ for an unspecified perfect overfield of $L$ complete with respect to a multiplicative nonarchimedean norm extending the one on $L$. For such $L'$, let $(C_L, C_L^+)$ denote the base extension of $(C, C^+)$ along $(B_{L,E}^I, B_{L,E}^{I, +}) \to (B_{L',E}^I, B_{L',E}^{I, +})$. 11
Lemma 7.3. The Banach ring \( C \) is uniform. In particular, by \([12, \text{Theorem 2.3.10}]\), the supremum over \( \mathcal{M}(C) \) computes the spectral norm on \( C \).

Proof. The ring \( B^I_{L,E} \) is preperfectoid \([12, \text{Theorem 5.3.9}]\), and hence stably uniform \([12, \text{Theorem 3.7.4}]\).

Lemma 7.4. Let \( x \in C \) be an element which is not a unit. Then for some \( L' \), we can find \( u \in m_{L'} \setminus \{0\} \) such that \( \overline{u} - \overline{[u]} \) divides \( x \) in \( C' \).

Proof. By Theorem 4.10, the ring \( C \) is noetherian, so all of its ideals are closed \( [12, \text{Remark 2.2.11}] \). Consequently, \( C/(x) \) is a nonzero Banach ring, so \( \mathcal{M}(C/(x)) \) is nonempty \([1, \text{Theorem 1.2.1}]\). Choose a point \( \gamma \in \mathcal{M}(C/(x)) \) and restrict it to \( \beta \in \mathcal{M}(B^I_{L,E}) \), then apply Lemma 6.3.

Corollary 7.5. For every \( x \in B_{L,E} \) nonzero, we can choose \( L' \) such that \( x \) factors in \( B^I_{L',E} \) as a unit times a finite product of primitive elements of degree 1.

Proof. This follows by Lemma 7.4 and consideration of slopes.

Lemma 7.6. For any \( \beta \in \mathcal{M}(C) \), there exist finitely many values \( 0 < \rho_1 < \cdots < \rho_m < 1 \) with the following properties.

(a) For \( J \) equal to any of \((0, \rho_1], [\rho_1, \rho_2], \ldots, [\rho_{m-1}, \rho_m], [\rho_m, 1]\), either \( H(\beta, \rho) \in \mathcal{M}(C) \) for all \( \rho \in J \), or \( H(\beta, \rho) \notin \mathcal{M}(C) \) for all \( \rho \) in the interior of \( J \).

(b) For \( J \) as in (a) for which the first alternative holds, for any \( x \in C \) the function 

\[
t \mapsto \log H(\beta, e^{-t})(x)
\]

is convex and continuous on \( -\log J \).

Proof. There is no harm in enlarging \( L \), so we may apply Lemma 7.4 to reduce to the case where \( \beta = H(\overline{\omega} - \overline{[u]}, 0) \) for some \( u \in m_L \setminus \{0\} \). As in \([12, \text{Remark 2.4.7}]\), we can find \( f_1, \ldots, f_n \in B_{L,E} \) generating the unit ideal in \( B^I_{L,E} \) for which

\[
\text{Spa}(C, C^+) = \{ v \in \text{Spa}(B^I_{L,E}, B^{I, +}_{L,E}) : v(f_i) \leq v(g) \quad (i = 1, \ldots, n) \}.
\]

Apply Corollary 7.3 to each of \( f_1, \ldots, f_n, g \) and let \( \overline{\omega} - [\overline{u_1}], \ldots, \overline{\omega} - [\overline{u_l}] \) be the list of all factors obtained. For each \( i \in \{1, \ldots, l\} \), note that

\[
H(\beta, \rho)(\overline{\omega} - [\overline{u_i}]) = \max\{ p^{-1} \rho, \beta([\overline{u} - [\overline{u_i}]) \};
\]

from this formula (and the fact that it suffices to check (b) for \( x \) in the dense subring \( B_{L,E}[g^{-1}] \) of \( C \)), we may easily deduce (a) and (b).

Corollary 7.7. Suppose that \( C \) is connected. Then for any \( \beta \in \mathcal{M}(C) \) of positive radius and any nonzero \( x \in C \), we have \( \beta(x) \neq 0 \).
Proof. By [12, Proposition 2.6.4], the connectedness of $C$ implies the connectedness of $\mathcal{M}(C)$. Consequently, for $x \in C$, if there exists a single $\beta \in \mathcal{M}(C)$ of positive radius with $\beta(x) \neq 0$, using Lemma 7.6 and the construction from Definition 6.4 to move up and down, we may show that $\gamma(x) = 0$ for all $\gamma \in \mathcal{M}(C)$. By Lemma 7.3 this forces $x = 0$. \hfill \square

Corollary 7.8. If $C$ is connected, then it is an integral domain.

Lemma 7.9. For any $m \in \text{Maxspec}(B^I_{L,E})$ and any positive integer $n$, the map $B^I_{L,E}/m^n \to C/m^nC$ is an isomorphism.

Proof. Since $B^I_{L,E}$ and $C_L$ are noetherian by Theorem 4.10, the proof of [2, Proposition 7.2.2/1] carries over: in the commutative diagram

\[
\begin{array}{c}
B^I_{L,E} \longrightarrow B^I_{L,E}/m^n \longrightarrow C/m^nC \\
\downarrow \quad \downarrow \quad \downarrow \\
C
\end{array}
\]

the dashed arrow exists by the universal property of rational localizations. Since the vertical arrows are surjective, so are $C \to B^I_{L,E}/m^n$ and $B^I_{L,E}/m^n \to C/m^nC$. On the other hand, the kernel of $C \to B^I_{L,E}/m^n$ contains $m^n$ and hence also $m^nC$, so $B^I_{L,E}/m^n \to C/m^nC$ is also injective.

Lemma 7.10. Suppose that $C$ is connected. For $x \in C$ nonzero, there are only finitely many $\beta \in \mathcal{M}(C)$ for which $\beta(x) \neq 0$.

Proof. Since $C$ is noetherian by Theorem 4.10 we may reduce to the case where $C$ is connected, and hence an integral domain by Corollary 7.8. It suffices to check that for each $\beta \in \mathcal{M}(C)$ for which $\beta(x) = 0$, there exists a neighborhood $U$ of $\beta$ in $\mathcal{M}(C)$ such that $\gamma(x) \neq 0$ for $\gamma \in U \setminus \{\beta\}$. Note that $\beta$ must have radius 0 by Corollary 7.7.

By Lemma 7.4 we can choose some $L'$ and some $\pi \in m_{L'} \setminus \{0\}$ such that $\beta' = H(\pi - [\pi], 0)$ restricts to $\beta$. Let $m$ be the ideal of $B^I_{L,E}$ generated by $\pi - [\pi]$; then $B^I_{L,E}/m \cong \mathcal{H}(\beta')$, so in particular $m$ is maximal. By Krull’s intersection theorem [3, Corollary 5.4], if $x \in m^nC'$ for all $n$ then $x$ vanishes in the local ring $C''_m$, but this yields a contradiction against Corollary 7.7.

By Lemma 7.3 we can find some $n$ such that $(\pi - [\pi])^n$ divides $x$ in $C'$ and the quotient $y$ has nonzero image in $\mathcal{H}(\beta')$.

We may thus choose a neighborhood $U'$ of $\beta'$ in $\mathcal{M}(C')$ such that $\gamma(y) \neq 0$ for all $\gamma \in U'$, and hence $\gamma(x) \neq 0$ for all $\gamma \in U' \setminus \{\beta'\}$. Since $\beta$ has radius 0, $U'$ restricts to a neighborhood of $U$ in $\mathcal{M}(C)$ of the desired form. \hfill \square

Theorem 7.11. The ring $C$ has the following properties.

(a) The ring $C$ is a direct sum of finitely many noetherian integral domains $C_1, \ldots, C_n$.

(b) For $i = 1, \ldots, n$, every element of $C_i$ can be written as an element of $W(\mathfrak{o}_L)$ times a unit.
(c) For \( i = 1, \ldots, n \), \( C_i \) is a principal ideal domain.

The fact that \( B_{L,E}^I \) itself is a principal ideal domain was known previously; see [12 Proposition 2.6.8].

Proof. We have (a) thanks to Theorem 4.10 and Corollary 7.8. We may thus assume hereafter that \( C \) itself is a noetherian integral domain.

Choose any nonzero \( x \in C \). By Lemma 7.10, there are only finitely many \( \beta \in \mathcal{M}(C) \) for which \( \beta(x) = 0 \). If there are no such \( \beta \), then \( x \) is a unit by [12, Corollary 2.3.5]. Otherwise, by Lemma 7.4, each such \( \beta \) may be lifted to \( H(\varpi - [\varpi], 0) \) for some \( L' \) and some \( \varpi \in m_{L'} \setminus \{0\} \).

We make a single choice of \( L' \) and then let \( u_1, \ldots, u_l \) be the resulting values of \( \varpi \). We may then apply Lemma 5.4 to the product \( \prod_{i=1}^l (\varpi - [u_i]) \) to write it as a unit times some element \( y_0 \in W(\mathfrak{o}_L) \), which then must be a divisor of \( x \) in \( C \). We thus form a sequence \( x_0 = x, x_1, \ldots \) of elements of \( C \) in which for each \( i \geq 0 \), we have \( x_i = y_i x_{i+1} \) for some \( y_i \in W(\mathfrak{o}_L) \), which is not a unit in \( C \). Since \( C \) is noetherian, we cannot extend this sequence indefinitely; we then have that \( x \) is the product of the \( y_i \) times a unit. This proves (b), which implies (c) because \( A_{L,E}^I \) is a principal ideal domain by Corollary 2.10. \( \square \)

8 Structure of étale morphisms

To conclude, we extend the preceding results to étale morphisms.

Hypothesis 8.1. Throughout §8, let \( (B_{L,E}^I, B_{L,E}^{I+}) \rightarrow (C, C^+) \) be a morphism of adic Banach rings which is étale in the sense of Huber. In particular, \( C \) is a quotient of \( B_{L,E}^I\{T_1, \ldots, T_n\} \) for some \( n \), so it is again noetherian by Theorem 4.10.

Lemma 8.2. There exist finitely many rational localizations \( \{(C, C^+) \rightarrow (D_i, D_i^+)\}_i \) such that \( \cup_i \text{Spa}(D_i, D_i^+) = \text{Spa}(C, C^+) \) and for each \( i \), \( (B_{L,E}^I, B_{L,E}^{I+}) \rightarrow (D_i, D_i^+) \) factors as a connected rational localization \( (B_{L,E}^I, B_{L,E}^{I+}) \rightarrow (C_i, C_i^+) \) followed by a finite étale morphism \( (C_i, C_i^+) \rightarrow (D_i, D_i^+) \) with \( D_i \) also connected.

Proof. See [8] Lemma 2.2.8]. \( \square \)

Lemma 8.3. The ring \( C \) is uniform.

Proof. Since \( C \) is strongly noetherian, \( (C, C^+) \) is sheafy [7, Theorem 2.2], so the claim may be checked locally. By Lemma 8.2, we may assume that \( (B_{L,E}^I, B_{L,E}^{I+}) \rightarrow (C, C^+) \) factors as \( (B_{L,E}^I, B_{L,E}^{I+}) \rightarrow (C_0, C_0^+) \rightarrow (C, C^+) \) where the first map is a rational localization and the second is a finite étale morphism. By [12, Theorem 5.3.9], \( B_{L,E}^I \) is preperfectoid, as then is \( C_0 \) by [12, Theorem 3.7.4] and \( C \) by [12, Proposition 3.7.5]. In particular, \( C \) is uniform. \( \square \)

Lemma 8.4. If \( C \) is connected, then it is an integral domain. Moreover, for any rational localization \( (C, C^+) \rightarrow (D, D^+) \) the map \( C \rightarrow D \) is injective.
Proof. Set notation as in Lemma \ref{lem:8.2}. For each $i$, the norm of any nonzero $x \in D_i$ to $C_i$ is nonzero, so by Corollary \ref{cor:7.7} we cannot have $\beta(x) = 0$ if $\beta \in M(D_i)$ restricts to a point of $M(C_i)$ of positive radius. Both claims follow at once.

Theorem 8.5. The ring $C$ is a direct sum of finitely many Dedekind domains.

Proof. Since $C$ is noetherian by Theorem \ref{thm:4.10}, we may reduce to the case where $C$ is connected, and hence an integral domain by Lemma \ref{lem:8.4}. It thus remains to prove that for any maximal ideal $m$ of $C$, the local ring $C_m$ is principal. Since $C$ is noetherian, we may check this after completion; by the proof of Lemma \ref{lem:7.9} this completion remains unchanged after replacing $C$ with $D$ for any rational localization $(C, C^+) \to (D, D^+)$ such that $mD \neq D$. By Lemma \ref{lem:8.2} we may thus reduce to the case where $C$ is finite étale over a rational localization, and then the claim follows from Theorem \ref{thm:7.11}(c).

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