Fuzzy Sphere and Hyperbolic Space from Deformation Quantization

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Abstract

We explicitly construct noncommutative * products on circularly symmetric two dimensional space by using the technique of Fedosov’s deformation quantization. Especially, on constant curvature spaces i.e., $S^2$ and $H^2$, we get $su(2)$ and $su(1,1)$ algebra respectively. These are candidates of * products applicable to noncommutative field theories or noncommutative gauge theories on spaces with nontrivial symplectic structure.
1 Introduction

Since the relation between string theory and noncommutative geometry was discussed in [1], noncommutative field theories and noncommutative gauge theories have been investigated enthusiastically from various viewpoints.

Many authors use the Moyal product[1] as noncommutative associative * product for explicit calculations. It corresponds to a constant NS-NS $B$-field background in flat space in the context of string theory. On the other hand, at least formally, more general * products which may correspond to string theory on nonconstant $B$-field background in curved space are defined by some authors[2]. However, explicit form of * products other than the Moyal product has been scarcely discussed in physical context[3].

In this paper, we use the technique of Fedosov’s deformation quantization[3] to get explicit forms of * products on nontrivial backgrounds. For simplicity, we investigate * products on circularly symmetric two dimensional spaces. Specifically, we focus on constant curvature spaces $S^2$, $H^2$ and $\mathbb{R}^2$, and explicitly construct * products which are different from the Moyal product. We also discuss some physical applications of our * products.

2 Construction of * product

Here we review the construction of Fedosov’s * product very briefly[3], and apply this procedure to circularly symmetric two dimensional spaces.

First, for a given symplectic manifold $(M, \Omega_0)$, we define the Weyl algebra bundle $W$ which has $\circ$ product of the Moyal type and its Abelian connection $D$ with some input parameter. For $\text{Ker} D \subset W$ (which is called flat section $W_D$), we get a one to one correspondence with $C^\infty (M)[[\hbar]]$, where $\hbar$ is the deformation parameter. We denote the map from $C^\infty (M)[[\hbar]]$ to $W_D$ as $Q$, and its inverse map as $\sigma$. Then Fedosov’s * product on $C^\infty (M)[[\hbar]]$ is defined by

$$a_0 * b_0 := \sigma(Q(a_0) \circ Q(b_0)), \ a_0, b_0 \in C^\infty (M)[[\hbar]].$$

(1)

This is a solution of the problem of deformation quantization, i.e.,

1. Here we call $*= \exp \left( \frac{i}{\hbar} \frac{\partial}{\partial x} \theta^{ij} \frac{\partial}{\partial y^j} \right)$ with constant $\theta^{ij} = -\theta^{ji}$ the Moyal product.
2. [2],[3], for example.
3. In [4], nonassociative star product which generalizes [2],[3] is discussed to describe D-brane in curved backgrounds.
4. See [3],[4] for details.
is associative and its commutator $[,]_*$ is expanded as

$$[ , ]_* = i\hbar\{ , \} + \mathcal{O}(\hbar^2) \quad (2)$$

where $\{ , \}$ is the Poisson bracket with respect to the symplectic form $\Omega_0$.

Now, we apply this procedure to a two dimensional space $M$ with metric

$$ds^2 = e^{\Phi(r)}(dr^2 + r^2 d\theta^2), \quad (3)$$

where $\Phi(r)$ is some function of $r$ only (i.e. circularly symmetric space) for simplicity. Its volume form is given by

$$\Omega_0 = e^{\Phi(r)}rdr \wedge d\theta, \quad (4)$$

and we identify it with symplectic form. Using Fedosov’s procedure with the input

$$\Omega_1 = 0, \quad \nabla = d, \quad \mu = \frac{1}{3}e^{-\Phi(r)}r^{-1}(y^1)^2y^2, \quad (5)$$

we get an Abelian connection $D$ as

$$Da = da - \delta a + \frac{i}{\hbar}(r \circ a - a \circ r), \quad a \in W,$nabla r = e^{-\Phi(r)}r^{-1}(y^1)^2y^2,$nabla r \circ r = 0.$

For this Abelian connection $D$, we solve the equation $Da = 0$ and get the map $Q : C^\infty(M[[\hbar]] \to W_D$ as

$$a = Q(a_0(r, \theta)) = a_0 \left( G(r, y^1), \theta + \frac{y^2}{r} \right), \quad (7)$$

where $G(r, y^1)$ is given by

$$\int_r^{G(r,y^1)} e^{\Phi(r')}r'dr' = y^1r. \quad (8)$$

Then we can define a $*$ product on $M$ by eq.(1).

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5 See 4, 6 for the meaning of $\nabla, \Omega_1, \mu, \delta$. Here we choose these parameters in such a way that the iteration formula (eq.(21) of 4) which gives an Abelian connection is satisfied trivially, i.e., $\nabla r + \frac{1}{\hbar}r \circ r = 0$. Then we get $r = \delta \mu + \delta^{-1}(d(\omega_{ij}y^j) - \Omega_1)$ for the input 4.
3 \textbf{S}^2 \textbf{case}

In this section we apply the result of §2 to the case \( M = S^2 \). We consider 2-sphere \( S^2 \) with radius \( R \), which is defined as two dimensional surface embedded in \( \mathbb{R}^3 \):

\[
(X^1)^2 + (X^2)^2 + (X^3)^2 = R^2.
\]  

We parametrize the coordinate \( X^i, i = 1, 2, 3 \) on \( S^2 \) as

\[
X^1 = \frac{2R^2 r}{r^2 + R^2} \cos \theta, \quad X^2 = \frac{2R^2 r}{r^2 + R^2} \sin \theta, \quad X^3 = \frac{R^2 - R^2}{r^2 + R^2},
\]

\( r \geq 0, \quad 0 \leq \theta \leq 2\pi \). 

(10)

Then the metric of \( S^2 \), \( ds^2 = (dX^1)^2 + (dX^2)^2 + (dX^3)^2 \), is given by

\[
ds^2 = \frac{4R^4}{(r^2 + R^2)^2}(dr^2 + r^2 d\theta^2),
\]

and the conformal factor \( e^\Phi \) of eq.(3) is identified as

\[
e^\Phi(r) = \frac{4R^4}{(r^2 + R^2)^2}.
\]

(12)

From eqs. (12), (7) and (1), we get the explicit form of our \( * \) product on \( S^2 \):

\[
a_0(r, \theta) * b_0(r, \theta)
= \left( a_0 \left( \sqrt{\frac{r^2 + y^1 r(r^2 + R^2)}{1 - y^1 r(r^2 + R^2)}}, \theta + \frac{y^2}{r} \right) \exp \left( -\frac{i\hbar}{2} \left( \frac{\partial}{\partial y^1} \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \theta} \frac{\partial}{\partial y^1} \right) \right) \right) 
\cdot b_0 \left( \sqrt{\frac{r^2 + y^1 r(r^2 + R^2)}{1 - y^1 r(r^2 + R^2)}}, \theta + \frac{y^3}{r} \right) \bigg|_{y^1=y^2=0}.
\]

(13)

By using this definition, we can calculate \( * \) product of the \( S^2 \) coordinate \( X^i \) (10). In particular, we have

\[
[X^i, X^j]_* = \frac{i\hbar}{R^2} \varepsilon^{ijk} X^k,
\]

(14)

\[
X^1 * X^1 + X^2 * X^2 + X^3 * X^3 = R^2 \left( 1 - \frac{\hbar^2}{4R^4} \right),
\]

(15)

where \( \varepsilon^{ijk} \) is the antisymmetric tensor with \( \varepsilon^{123} = +1 \). Eq.(14) means that the commutators of \( X^i \)'s form \( su(2) \) algebra which is known as fuzzy sphere algebra, and eq.(13) means that its radius is given by \( R\sqrt{1 - \frac{\hbar^2}{4R^4}} \) which is deformed by \( O(\hbar^2) \) from the original radius \( R \) of commutative \( S^2 \) (7). Namely, we have obtained a fuzzy sphere by deforming \( S^2 \) using the \( * \) product (13).
4 $H^2$ case

In this section we apply the result of §2 to the case $M = H^2$. Calculation is quite similar to the $S^2$ case (§3). We consider two dimensional hyperbolic space $H^2$ with radius $R$, which is defined as two dimensional surface embedded in $\mathbb{R}^{1,2}$:

$$-(Y^0)^2 + (Y^1)^2 + (Y^2)^2 = -R^2, \quad Y^0 > 0.$$  \hfill (16)

We parametrize the coordinates $Y^i, i = 0, 1, 2$ on $H^2$ as

$$Y^0 = R \frac{r^2}{R^2 - r^2}, \quad Y^1 = \frac{2R^2r}{R^2 - r^2} \cos \theta, \quad Y^2 = \frac{2R^2r}{R^2 - r^2} \sin \theta,$$

$$0 \leq r \leq R, \quad 0 \leq \theta \leq 2\pi.$$  \hfill (17)

Then, the metric of $H^2$, $ds^2 = -(dY^0)^2 + (dY^1)^2 + (dY^2)^2$, and the conformal factor are given respectively by

$$ds^2 = \frac{4R^4}{(R^2 - r^2)^2} (dr^2 + r^2 d\theta^2),$$

$$e^\Phi(r) = \frac{4R^4}{(R^2 - r^2)^2}.$$  \hfill (18)

(19)

From eqs. (19), (7) and (1), we get the explicit form of our $\ast$ product on $H^2$:

$$a_0(r, \theta) \ast b_0(r, \theta)$$

$$= \left( a_0 \left( \sqrt{\frac{r^2 + \frac{y^i}{2R^2} r(R^2 - r^2)}{1 + \frac{y^i}{2R^2} r(R^2 - r^2)}} , \theta + \frac{y^2}{r} \right) \exp \left( -\frac{i\hbar}{2} \left( \frac{\partial}{\partial y^1} \frac{\partial}{\partial \tilde{y}^1} - \frac{\partial}{\partial y^2} \frac{\partial}{\partial \tilde{y}^2} \right) \right) \right) \cdot b_0 \left( \sqrt{\frac{r^2 + \frac{y^i}{2R^2} r(R^2 - r^2)}{1 + \frac{y^i}{2R^2} r(R^2 - r^2)}} , \theta + \frac{y^2}{r} \right)_{y^1 = y^2 = 0}. \hfill (20)$$

By using this definition, we obtain the following $\ast$ products of the $H^2$ coordinate $Y^i$ (17):

$$[Y^0, Y^1]_\ast = \frac{i\hbar}{R} Y^2, \quad [Y^2, Y^0]_\ast = \frac{i\hbar}{R} Y^1, \quad [Y^1, Y^2]_\ast = -\frac{i\hbar}{R} Y^0,$$

$$-Y^0 \ast Y^0 + Y^1 \ast Y^1 + Y^2 \ast Y^2 = -R^2 \left( 1 - \frac{\hbar^2}{4R^4} \right).$$  \hfill (21)

(22)

Eq.(21) means that commutators of $Y^i$’s form $su(1, 1)$ algebra which corresponds to isometry of $H^2$, and eq.(22) means that its radius is given by $R \sqrt{1 - \frac{\hbar^2}{4R^4}}$ which is deformed by $O(\hbar^2)$ from the original radius $R$ of commutative $H^2$ (10). Namely, we get fuzzy hyperbolic space by deforming $H^2$ using the $\ast$ product (20).
5 Large $R$ limit and $\mathbb{R}^2$

Here we consider large radius limit of the results of §3 and §4. The sectional curvature of $S^2$ ($H^2$ ($\mathbb{R}^2$)) is $\frac{1}{R^2}$ ($-\frac{1}{R^2}$), which tends to $+0$ ($-0$) in the limit $R \to \infty$. Therefore they approach the flat space $\mathbb{R}^2$ in the large $R$ limit in the usual commutative picture. How about it from the noncommutative viewpoint?

For comparison, we construct a $\ast$ product on $\mathbb{R}^2$ following the method of §2. We adopt as its flat metric

$$ds^2 = 4(dr^2 + r^2 d\theta^2)$$

(23)

with its front factor 4 chosen so that (23) coincides with the large $R$ limit of (11) and (18).

With $e^\Phi = 4$, we get the explicit form of our $\ast$ product on $\mathbb{R}^2$:

$$a_0(r, \theta) \ast b_0(r, \theta) = \left( a_0 \left( \sqrt{r^2 + \frac{y^1 r}{2}}, \theta + \frac{y^2}{r} \right) \exp \left( -\frac{i\hbar}{2} \left( \frac{\partial}{\partial y^1} \frac{\partial}{\partial y^2} - \frac{\partial}{\partial y^2} \frac{\partial}{\partial y^1} \right) \right) \right) \cdot b_0 \left( \sqrt{r^2 + \frac{y^1 r}{2}}, \theta + \frac{y^2}{r} \right) .$$

(24)

Then, we can calculate the $\ast$ products of the complex coordinate $z := re^{i\theta}, \bar{z} := re^{-i\theta}$:

$$z \ast z = \sqrt{r^4 - \frac{\hbar^2}{16}}e^{2i\theta} = z \ast \bar{z}, \quad z \ast \bar{z} = r^2 - \frac{\hbar}{4}, \quad \bar{z} \ast z = r^2 + \frac{\hbar}{4},$$

(25)

The commutator $[z, \bar{z}]_\ast$ coincides with that of the usual Moyal product for Cartesian coordinates on $\mathbb{R}^2$, but $\ast$ product itself is different from the Moyal product. This difference comes from ambiguity of deformation quantization.

We can calculate the commutator $[z, \bar{z}]_\ast$ also in the $S^2$ and $H^2$ cases. For $S^2$, from eq.(13) we get

$$[z, \bar{z}]_\ast = -\frac{\hbar}{2} \left( \frac{1}{1 - \left( \frac{\hbar}{4R^2} \right) (r^2 + \bar{R}^2)} \right) = -\frac{\hbar}{2R^2} (R^2 + z \ast \bar{z})(R^2 + \bar{z} \ast z).$$

(26)

And for $H^2$, from eq.(20) we get

$$[z, \bar{z}]_\ast = -\frac{\hbar}{2R^2} (R^2 - r^2) \left( \frac{1}{1 - \left( \frac{\hbar}{4R^2} \right) (r^2 + \bar{R}^2)} \right) = -\frac{\hbar}{2R^2} (R^2 - z \ast \bar{z})(R^2 - \bar{z} \ast z).$$

(27)

Both eqs.(26) and (27) are reduced to $[z, \bar{z}]_\ast = -\frac{\hbar}{2}$ (25) as $R \to \infty$. In other words, the $\ast$ product which we obtained in §2 connects $su(2)$ algebra (or fuzzy $S^2$) with $su(1, 1)$ algebra (or fuzzy $H^2$) through $R = \infty$. 

5
6 An application

In the previous sections, we explicitly calculated * products by using Fedosov’s formulation. They are candidates of * product for defining noncommutative field theory or noncommutative gauge theory on fuzzy $S^2$, $H^2$ and $R^2$.

As an example, we discuss four dimensional noncommutative $U(1)$ gauge theory with one scalar field which is given by the action

$$ S = \text{Tr} \left( \frac{1}{4} G^{IJ} G^{KL} F_{IK} \ast F_{JL} + \frac{1}{2} G^{IJ} D_I \phi \ast D_J \phi \right). \tag{28} $$

We assume that only two dimensional space is noncommutative (1,2 direction), and use a general formulation of noncommutative gauge theory of [3]:

$$ G^{IJ} = \delta^{IJ}, \ I, J = 1, \cdots, 4, $$
$$ F_{IJ} = \partial_I A_J - \partial_J A_I - i[A_I, A_J], \ J_{12} = -J_{21} = 1, \text{others} = 0, $$
$$ \partial_I = \frac{i}{\hbar} [-J_{IJ} \tilde{\phi}^J, ]_* \ I = 1, 2, \ \partial_3 = \frac{\partial}{\partial x^3}, \ \partial_4 = \frac{\partial}{\partial x^4} $$
$$ D_I \phi = \partial_I \phi - i[A_I, \phi]. \tag{29} $$

Here, $\tilde{\phi}^I$ is the “canonical” noncommutative coordinate satisfying

$$ \frac{i}{\hbar} [\tilde{\phi}^1, \tilde{\phi}^2]_* = 1. \tag{30} $$

Its explicit form is

$$ \tilde{\phi}^1 = \frac{2Rr}{\sqrt{r^2 + R^2}} \cos \theta, \quad \tilde{\phi}^2 = \frac{2Rr}{\sqrt{r^2 + R^2}} \sin \theta \tag{31} $$

for fuzzy $S^2$ [13],

$$ \tilde{\phi}^1 = \frac{2Rr}{\sqrt{R^2 - r^2}} \cos \theta, \quad \tilde{\phi}^2 = \frac{2Rr}{\sqrt{R^2 - r^2}} \sin \theta \tag{32} $$

for fuzzy $H^2$ [20], and

$$ \tilde{\phi}^1 = 2r \cos \theta, \quad \tilde{\phi}^2 = 2r \sin \theta \tag{33} $$

for fuzzy $R^2$ [24]. The action (28) is invariant under noncommutative $U(1)$ gauge transformation:

$$ \delta_\lambda A_I = \partial_I \lambda - i[A_I, \lambda], \quad \delta_\lambda \phi = -i[\phi, \lambda]. \tag{34} $$

The symbol Tr is trace for the * product satisfying $\text{Tr} f \ast g = \text{Tr} g \ast f$ [3], but we can discuss equations of motion without using the explicit form of the trace.
The equations of motion of (28) are
\[ D^I F_{IJ} = -i[\phi, D_J \phi], \quad D^I D_I \phi = 0, \] (35)
and we obtain a solution by solving the \(U(1)\) noncommutative BPS equation:
\[ B_I = D_I \phi, \quad I = 1, 2, 3, \quad \partial_4 = 0, \quad A_4 = 0, \quad B_I := \frac{1}{2} \varepsilon^{IJK} \left( F_{JK} + \frac{J_{JK}}{\hbar} \right). \] (36)
Under the ansatz
\[ A_1 + i A_2 = i f_A(l, x^3) (\ddot{\phi}^1 + i \ddot{\phi}^2), \quad A_3 = 0, \]
\[ \phi = f(l, x^3), \quad l := \sqrt{(\ddot{\phi}^1)^2 + (\ddot{\phi}^2)^2 + (x^3)^2}, \] (37)
eq(36) can be rewritten as
\[ \partial_3 G^{(m)} - 4 \partial_L f^{(m)} = \sum_{2n+k=m, \ n \geq 1} \frac{4 \partial_{2n+1}^{(2n+1)} f^{(k)}}{(2n+1)!} + \sum_{2n+k+k' = m-1} \frac{4G^{(k')} \partial_{2n+1}^{(k')} f^{(k)}}{(2n+1)!}, \]
\[ \partial_3 f^{(m)} - \partial_L (L G^{(m)}) = \sum_{2n+k=m, \ n \geq 1} \frac{\partial_{2n+1}^{(2n+1)} (L G^{(k)})}{(2n+1)!} \] (38)
with
\[ L := (\ddot{\phi}^1)^2 + (\ddot{\phi}^2)^2, \quad f = \sum_{k=0}^{\infty} \hbar^k f^{(k)}, \quad \left( \frac{1}{\hbar} + f_A \right)^2 = \frac{1}{\hbar^2} + \frac{1}{\hbar} \sum_{k=0}^{\infty} \hbar^k G^{(k)}. \] (39)
We can solve (38) order by order in \(\hbar\), and we get
\[ f = \frac{g}{l} - \hbar g^2 \left( \frac{2x^3}{l^4} - \frac{1}{l^3} \right) + \hbar^2 \left( \frac{8g^3x^3}{l^6} - \frac{g}{4l^5} + \left( \frac{4g}{8} + 10g^3 \right) \frac{(x^3)^2}{l^7} \right) + O(\hbar^3), \]
\[ f_A = \frac{g}{l(l + x^3)} + \hbar g^2 \left( \frac{2}{l^4} - \frac{1}{l^3(l + x^3)} - \frac{1}{2l^2(l + x^3)^2} \right) \]
\[ + \hbar^2 \left( \frac{8g^3}{l^5} + \frac{4g^3}{l^5(l + x^3)^2} + \frac{g^3}{l^4(l + x^3)^2} + \frac{g^3}{2l^3(l + x^3)^3} - \left( \frac{5g}{8} + 10g^3 \right) \frac{x^3}{l^7} \right) + O(\hbar^3), \] (40)
as a solution such that it becomes the \(U(1)\) Dirac monopole in the commutative limit (i.e., \(\hbar \to 0\)). In the fuzzy \(\mathbb{R}^2\) case (33), the \(O(\hbar)\) terms coincide with those in (3) which solved the equations of motion with the usual Moyal product.
7 Conclusion and discussion

In this paper we have presented explicit construction of * products on two dimensional constant curvature spaces $S^2$, $H^2$ and $\mathbb{R}^2$. We have found that the algebras of the * products represent fuzzy $S^2$, $H^2$ and $\mathbb{R}^2$ because the commutators of the * product form $su(2)$, $su(1, 1)$ and Heisenberg algebra respectively. The commutators $[z, \bar{z}]_*$ for fuzzy $S^2$ and $H^2$ are reduced to that of fuzzy $\mathbb{R}^2$ in the large $R$ limit. In this sense, fuzzy $S^2$ and $H^2$ approach to fuzzy $\mathbb{R}^2$ as $R \to \infty$. This is consistent with usual commutative picture.

In §6 we applied explicit form of our * products to $U(1)$ noncommutative BPS equation (36), and obtained its solution to $O(h^2)$. In eq.(36) the * product appears only in the commutator $[\ , \ ]_*$. Therefore, eq.(36) is solved unifiedly for fuzzy $S^2$, $H^2$ and $\mathbb{R}^2$ by using “canonical” noncommutative coordinate $\tilde{\phi}^I$ (30). In other words, we can get a solution of eq.(36) even if the definition of * is different as long as we use “canonical” noncommutative coordinate $\tilde{\phi}^I$ for the * product.

To study the effects of the difference of * products themselves, we should consider noncommutative equations containing “bare” * products. Its typical example is $\phi * \phi = \phi$ which is essentially the equation for noncommutative soliton [7]. Even for the $\mathbb{R}^2$ case, the * product which we get here is different from the usual Moyal product, and hence $\phi \sim \exp(-r^2)$ is not a solution of $\phi * \phi = \phi$. It is a future problem to find an explicit solution of it and to investigate its meaning.

For fuzzy $S^2$, * product is usually defined by using representation matrix of $su(2)$ and spherical harmonic function, and depends on the size of matrix. On the other hand our * product depends on the deformation parameter $h$, so they are very different in appearance. It is also a future problem to study an explicit relation between them. If the relation becomes clear, our * product may give some suggestions to string theory in the literature [8] for example.

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7 In the case of the Moyal product, this is a solution.
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