Thomas Precession, Berry potential and the Meron.

R. Shankar\textsuperscript{1} and Harsh Mathur\textsuperscript{2}

\textsuperscript{1}Departments of Physics and Applied Physics, P.O. Box 6666, Yale University, New Haven, CT 06511

\textsuperscript{2}A.T.&T Bell Laboratories

600 Mountain Avenue

Murray Hill NJ 07974

(September 1, 2018)

Abstract

We begin with a prior observation by one of us that Thomas precession in the nonrelativistic limit of the Dirac equation may be attributed to a nonabelian Berry vector potential. We ask what object produces the nonabelian potential in parameter space, in the same sense that the abelian vector potential arising in the adiabatic transport of a nondegenerate level is produced by a monopole, (centered at the point where the level becomes degenerate with another), as shown by Berry. We find that it is a \textit{meron}, an object in four euclidean dimensions with instanton number $\frac{1}{2}$, centered at the point where the doubly degenerate positive and negative energy levels of the Dirac equation become fourfold degenerate.
Consider the adiabatic transport of a nondegenerate level of a hermitian hamiltonian around a closed loop in parameter space. If $|\chi(R)\rangle$ is the instantaneous eigenket of the hamiltonian at a point $R$ in parameter space, Berry showed that (i) the state vector picks up a phase factor due to a vector potential
\[
A_\mu = \langle \chi | \partial_\mu \chi \rangle
\]
in addition to the integral of the energy along the circuit and that (ii) The source of this potential is a quantized magnetic monopole located at a point where the level becomes degenerate with another. In other words, the field tensor $F_{\mu\nu}$ derived from $A_\mu$ is that of a monopole.

The proper mathematical framework for describing Berry’s work was established by Simon. It was then pointed out by Wilczek and Zee that if the level being transported has an $m$-fold degeneracy, the effect of a closed circuit will now be given by $U(m)$ matrix obtained by a path-ordered integral of a nonabelian vector potential. These authors provided many illustrations.

The references mentioned up to this point, along with many others and very useful commentary may be found in the collection edited by Shapere and Wilczek.

The problem of two-fold degenerate levels becoming four-fold degenerate was discussed in its generality by Avron et al., using spin systems as a model. The relation of our work to theirs will be discussed at the end.

Our problem here stems from the observation by one of us that Thomas precession of electron spin in the nonrelativistic limit could be understood as arising due to nonabelian SU(2) Berry potential. The idea is as follows. For the free Dirac equation, the four levels at each momentum break up into two (spin) degenerate pairs with equal and opposite energies. Since there is a large gap separating the positive and negative levels, they will not mix under adiabatic evolution. (In practice the coulomb potential of the nucleus causes the adiabatic evolution). It was then a straightforward
matter to compute the vector potential

\[ A_{\mu}^{ab} = \langle a | \partial_{\mu} b \rangle \]  \hspace{1cm} (2)

starting from the degenerate positive energy spinors labeled by \( a \) and \( b \) and to show that the effect of this potential was to produce the Thomas precession. (It is of course possible to do the same for the negative energy states. Throughout this paper we will focus on just the positive energy states.)

Here we ask and answer the following question: *What is the source of this non-abelian potential?* We find it is *meron*, an object with Pontrayagin index equal to \( \frac{1}{2} \).

It is not necessary to be familiar with merons to follow this paper since their relevant features will be described later on.

Let us begin with the Dirac hamiltonian

\[ H = \gamma \cdot p + \beta m \]  \hspace{1cm} (3)

where the three \( \gamma \)'s and \( \beta \) are \( 4 \times 4 \) matrices which anticommute and whose square is unity. Introduce 4 *euclidean* gamma matrices

\[ \gamma_{\mu} = (\gamma, \beta) \]  \hspace{1cm} (4)

obeying

\[ \{ \gamma_{\mu} \gamma_{\nu} \} = 2 \delta_{\mu\nu} \]  \hspace{1cm} (5)

and a four vector

\[ p_{\mu} = (p, m). \]  \hspace{1cm} (6)

*Note that the fourth component of \( p_{\mu} \) is \( m \) and not the energy.* Then

\[ H = \not{p} \equiv \gamma_{\mu} p_{\mu} \quad (\mu = 1, 2, 3, 0). \]  \hspace{1cm} (7)

We shall use the representation
where

\[ \sigma_\mu = (\sigma, iI) \]  

and \( \sigma \) are the Pauli matrices and \( I \) is the \( 2 \times 2 \) identity.

Since

\[ H^2 = \not p \not p = p_\mu p_\mu = p^2 = p^2 + m^2 \]  

it follows the eigenvalues of \( H \) are \( \pm \sqrt{p^2 + m^2} = \pm p \). Since there should be a total of four levels, it follows they come in doubly degenerate pairs. The degeneracy may be traced back to fact that

\[ H = \gamma_2 \gamma_0 H^* \gamma_0 \gamma_2 \]  

which is Kramers’ degeneracy. The fact that the levels come with equal and opposite energies is due to the fact that

\[ \gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \]  

anticommutes with \( H \) and is unitary.

Let us now introduce a unitary matrix

\[ U = \frac{\sigma \cdot \mathbf{p} - im}{p} \]  

in terms of which

\[ H = \begin{bmatrix} 0 & pU^+ \\ pU & 0 \end{bmatrix} \]  

and the positive energy eigenvectors are
\[ |a\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} \chi_a \\ U\chi_a \end{bmatrix} \]  

(15)

where \( \chi_a \) and \( \chi_b \) are any two orthonormal two-component spinors. We choose them to be the canonical basis

\[ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \]  

(16)

The nonabelian vector potential is now given by a \( 2 \times 2 \) matrix

\[ A_{\mu}^{ab} = \frac{1}{2} \chi_a^\dagger U^\dagger \partial_\mu U \chi_b. \]  

(17)

Since \( \chi \) is a canonical basis vector we can write in compact notation

\[ A_\mu = \frac{1}{2} U^\dagger \partial_\mu U. \]  

(18)

Note that due to the factor of \( 1/2 \), this is not a pure gauge.

Explicit computation shows that

\[ A_0 = -\frac{i}{2p^2} \sigma \cdot p \quad A = \frac{i}{2p^2} (m\sigma + \sigma \times p) \]  

(19)

The vector potential has no radial component:

\[ p_\mu A_\mu = 0. \]  

(20)

The next step is to compute the field strength

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \]  

(21)

Given eq.(18), it follows that

\[ F_{\mu\nu} = \frac{1}{4} \left[ \partial_\mu U^\dagger \partial_\nu U - \partial_\nu U^\dagger \partial_\mu U \right] \]  

(22)

Although this expression may be evaluated in a straightforward but tedious manner, it is instructive to employ some tricks analogous to the ones used by Berry in the abelian case.
Let us first introduce some conventions. The kets $|a\rangle, |b\rangle, |c\rangle$ will stand for positive energy eigenvectors while $|A\rangle, |B\rangle, |C\rangle$ will stand for negative energy eigenvectors. We need two lemmas:

**Lemma 1:**

\[ \langle \partial a|B \rangle = \frac{\langle a|\partial H|B \rangle}{E_a - E_B} \]  

proven by taking the derivative of $\langle a|H|B \rangle = 0$ and using $\langle \partial a|B \rangle = -\langle a|\partial B \rangle$.

**Lemma 2:**

\[ \langle a|\partial H|b \rangle = 0 \quad a \neq b \]  

proven by taking the derivative of $\langle a|H|b \rangle = 0$ for the case $a \neq b$.

Now we have

\[ F_{ab}^{\mu\nu} = \partial_\mu \langle a|\partial_\nu b \rangle - (\mu \rightarrow \nu) + \langle a|\partial_\nu c\rangle \langle c|\partial_\mu b \rangle - (\mu \rightarrow \nu) \]  

\[ = \langle \partial_\mu a|\partial_\nu b \rangle - (\mu \rightarrow \nu) - \langle \partial_\mu a|c\rangle \langle c|\partial_\nu b \rangle - (\mu \rightarrow \nu) \]  

\[ = \langle \partial_\mu a|A\rangle \langle A|\partial_\nu b \rangle - (\mu \rightarrow \nu) \]  

where in the last line we note that the intermediate states are restricted to those of negative energy. We now invoke Lemma 1 to write

\[ F_{\mu\nu}^{ab} = \frac{1}{4p^2} \langle a|[\partial_\mu H, \partial_\nu H]|b \rangle - (\mu \rightarrow \nu) \]  

We now argue that we can enlarge the set of intermediate states to include the positive energy states well. First let $a \neq b$. Then no matter which state $c$ we introduce, it will be unequal to either $a$ or $b$, and the term will vanish by Lemma 2. Finally if $a = b$, then the contribution from $c$ will vanish if $c \neq a$ thanks to Lemma 2 and by antisymmetry if $a = b = c$.

We can now use completeness to reach the nice result

\[ F_{\mu\nu}^{ab} = \frac{1}{4p^2} \langle a|[\partial_\mu H, \partial_\nu H]|b \rangle \]
\[ \gamma_{\mu \nu} = \begin{pmatrix} \sigma_{\mu} \sigma_{\nu}^\dagger & 0 \\ 0 & \sigma_{\mu}^\dagger \sigma_{\nu} \end{pmatrix} \]  \hspace{1cm} (32)

If \( \mu \) and \( \nu \) are spatial indices, \( \gamma_{\mu \nu} \) is self-dual (same in the upper and lower blocks which are eigenspaces of \( \gamma_5 \) with eigenvalue \( \pm 1 \)), and anti-self-dual (of opposite sign in the two blocks) if one of the indices is 0.

Note however that the \( F_{\mu \nu} \) for our problem is not self-dual or anti-self-dual since it is given by the projection of \( \gamma_{\mu \nu} \) into the positive energy subspace. Indeed since \( \gamma_5 \) anticommutes with \( H \), its eigenvectors cannot be also eigenvectors of \( H \).

Using the explicit formulae for the eigenvectors, given by eqs.\((15-16)\) we find that

\[ F_{\mu \nu} = \frac{1}{4p^2} \left[ \sigma_{\mu} \sigma_{\nu}^\dagger + U^\dagger \sigma_{\mu}^\dagger \sigma_{\nu} U \right]. \]  \hspace{1cm} (33)

In the problem of Thomas precession, the particle goes around in a tiny loop (in the nonrelativistic limit) in some plane, say the the \( x - y \) plane, at fixed \( m \). The relevant field is the “magnetic field” along the \( z \)-axis, given in this limit by

\[ F_{xy} = B_z = \frac{is_z}{2m^2} + \mathcal{O}(p) \]  \hspace{1cm} (34)

in agreement with Reference 6. (In that paper the vector potential was given in a different gauge due to a different choice of eigenvectors.)

But now that we had the field in all of parameter space (and not just the nonrelativistic region) and we decided to explore some of its properties. We found that

\[ p_{\mu} F_{\mu \nu} = 0 = p_{\mu} A_\mu \]  \hspace{1cm} (35)

\[ D_\mu F_{\mu \nu} = \partial_\mu F_{\mu \nu} + [A_\mu, F_{\mu \nu}] \equiv DF = 0 \]  \hspace{1cm} (36)

\[ Tr(\hat{F} \hat{F}) = -8\pi^2 \delta^{(4)}(p). \]  \hspace{1cm} (37)
The first equations has a counterpart in Berry’s abelian monopole: there the magnetic field is radial, \( R \times B = 0 \), which can be written as \( R_i F_{ij} = 0 \). It is possible to choose the vector potential so that it too has no radial component. Likewise in our problem, only the circulation in tangent planes is nonzero.

The second equation tells us that \( F \) solves the Euclidean Yang-Mills equations. It is however not self-dual or anti-self-dual, nor is it required to be, since the corresponding action (the integral of \( Tr F^2 \)) is logarithmically divergent.

The last equation, (which reflects the fact that \( Tr E \cdot B = 0 \) here) tells us that the instanton density is zero everywhere, except possibly at the origin where the field strength is singular. Indeed one can argue that there must be a delta-function singularity there as follows.

The instanton density \( Q \) may written as the divergence of a vector \( K_\mu \) as follows:

\[
Q = -\frac{1}{16\pi^2} Tr(F\tilde{F}) \quad (38)
\]
\[
= \partial_\mu K_\mu \quad (39)
\]
\[
K_\mu = -\frac{1}{8\pi^2} \epsilon_{\mu\nu\alpha\beta} Tr[A_\nu(\partial_\alpha A_\beta + \frac{2}{3} A_\alpha A_\beta)] \quad (40)
\]

Thus the total instanton number inside any volume can be found by doing the surface integral of \( K_\mu \). Since \( K_\mu \) must have the form \( cp_\mu/p^4 \), by dimensional analysis, we can find \( c \) by considering a point on the m-axis. It is readily seen that \( c = 1/4\pi^2 \) which fixes

\[
Tr(F\tilde{F}) = -8\pi^2 \delta^{(4)}(p) \quad (41)
\]

which in turn implies \( Q = 1/2 \) as per eq.(39).

At this point it became clear that the field was that of a meron. The meron was first discovered as a solution to Yang-Mills equations by De Alfaro et al \[8\] with instanton number 1/2 and invoked by Callan et al \[9\] as a configuration that could enter the functional integral for the Yang-Mills field and explain confinement by...
producing the area law decay for the Wilson loop. We paraphrase their description of the relation between the meron and the instanton, which we found very enlightening. We will however do it in terms of the two-dimensional instanton which arises in the O(3) sigma model since it is easier to visualize. Consider the instanton of unit size which is obtained by placing a sphere of unit diameter on top of the plane and doing a stereographic projection and assigning to each point on the plane the unit vector associated with its inverse image: thus the origin gets assigned the south pole, the unit vector slowly starts rotating upwards as we move out and the equator gets mapped into a unit circle. As we move along this circle, the field lies in the plane and is radial. As we move further out, the field tilts up even more and finally at infinity, points upwards. If we measure the instanton number enclosed within any circle, it will grow from zero to a half by the time we reach the unit circle, and reach unity by the time we integrate out to infinity. Now we deform the map as follows. First we scale down the region inside the unit circle, which carries the lower hemisphere, to a circle of radius \( r < 1 \); push the region outside the unit circle (which carries the upper hemisphere) to outside a radius \( R > 1 \) and assign to the entire annulus \( r < 1 < R \), the radial equatorial field. The instanton density reaches the value 1/2 at \( r \), stays fixed out to \( R \) (since the entire annulus maps back to just the equator) and starts growing to unity as we go past \( R \) to infinity. If we now let \( r \to 0 \) and \( R \to \infty \) we get the meron. It will look just like the vortex of the x-y model, have a logarithmically large action, \( (\ln R/r) \) and carry no instanton density except at the origin. Our meron is related to the four dimensional instanton in just the same way: one takes the instanton, squeezes the region with half the winding number into the origin, sends the other half out to infinity and fills all of space with a configuration that has zero \( Tr \; FF \).

We now relate our work to the general analysis of Avron et al of hamiltonians with Kramers degeneracy, brought to our notice as this work was nearing completion.
These authors showed that the generic hamiltonian with Kramers degeneracy lives in a five dimensional Clifford algebra. Let us call the coefficients \( p_\mu = (p, m, M) \).

This corresponds to our adding a term \( M\gamma_5 \) to our \( H \). Because of our interest in the physical problem of the Thomas precession of the Dirac electron, we sliced this space along the plane \( M = 0 \). The relation of our four dimensional analysis to the five dimensional one is best explained by a simpler analogy. Consider the monopole that arises in Berry’s treatment of the abelian problem. It lives in three dimensions. It can be described by a vector potential with no radial component and a field tensor which too has circulation only in the planes tangential to the position vector. By surrounding the monopole with a unit sphere \( S^2 \) and measuring the flux (in proper units) we can obtain a topological invariant of the map (first Chern number) from the space of states to \( S^2 \). On this sphere the field strength is uniform and finite.

Suppose one were now to slice this space along the plane \( z = 0 \). On this plane one will see a radial field of the x-y vortex, i.e., the meron. The unit sphere \( S^2 \) will be sliced along the unit circle \( S^1 \) enclosing the meron. Consider now the integral of the monopole vector potential \( A_\mu = \langle \chi | \partial_\mu \chi \rangle \) along this circle. By Stokes’ theorem this equals the flux enclosed in any surface bounded by it. If one uses as the surface the upper hemisphere of the \( S^2 \), one gets half the topological charge, the contribution being uniform on the hemisphere. The other choice for the surface is the interior of the circle in the x-y plane, i.e., the disc with \( S^1 \) as its boundary. In the latter case the flux is zero except at the origin where the contribution is \( 1/2 \) due to all the flux flowing upwards into the upper hemisphere. Thus the person restricting himself to the plane can reconstruct the topological index associated with the \( S^2 \) by doing an integral on the \( S^1 \) he sees, thanks to Stokes theorem. (He must then remember to double to the answer for the lower hemisphere in this symmetric problem, or more generally, take the difference with another oppositely oriented line integral of the vector potential that describes the lower hemisphere.)
Returning to our problem, we must begin with $H = p_\mu \gamma_\mu = \not{p}$ in five dimensions, find its eigenvectors and define $A_\mu = \langle \chi | \partial_\mu \chi \rangle$ and find the ten component field strength $F_{\mu\nu}$. The result will be just as in eq. (31) (thanks to the lemmas)

$$F_{\mu\nu} = \frac{1}{2p^2} \langle a | \gamma_\mu \gamma_\nu | b \rangle$$

(42)

with the obvious change in the definition of $p$ and the range of indices. The field is still singular at the origin in five dimensions, obeys $p_\mu F_{\mu\nu} = 0$ and $DF = 0$. The former implies that the field is tangential. (This is true in any gauge. Also one can pick a gauge, as we did, in which $A_\mu$ itself has no radial component.) Thus at each point $p_\mu$, $F$ has only six nonzero components with “transverse” indices. We shall refer to them as $F$. This configuration is thus a natural generalization of the Berry monopole configuration in three dimensions.

Avron et al show that if we surround the origin by a unit sphere $S^4$,

- The (suitably normalized) integral of $Tr \, F \tilde{F}$ on the sphere (where the dual is with respect to the epsilon symbol $\hat{p}_\lambda \varepsilon_{\lambda\mu\nu\alpha\beta}$) is $\pm 1$ and measures a topological invariant (second Chern number or Pontrayagin index) of the map from space of states to $S^4$.

- $F$ is self-dual or anti-self-dual depending on the sign of the energy.

Since all points on the sphere are equal by symmetry, let us go to one with just $M = 1$, rest equal to zero. Now $\hat{p}_\lambda \varepsilon_{\lambda\mu\nu\alpha\beta}$ becomes just $\varepsilon_{\mu\nu\alpha\beta}$, the six nonzero components of $F$ have their indices going from 0 to 3. Since $H = \mu \gamma_5$ here, the eigenstates of $H$ are indeed eigenstates of $\gamma_5$ and the field is self(anti)-self-dual as observed earlier with respect to $\varepsilon_{\mu\nu\alpha\beta}$.

By focusing on the Thomas precession of the physical Dirac equation, we sliced this space along $M = 0$. Our plane contains the singularity at the origin and the sphere $S^3$ which is the equator of the $S^4$. The integral on this sphere of $K_\mu$, defined
in eq. (10), gives, by Stokes theorem, either half the topological charge associated with
the $S^4$ (if we view the $S^3$ as the boundary of the hemisphere of $S^4$) or the meron
charge if we view it as the boundary of the disc (or ball) contained in the slice. (In
the language of forms, since $Tr \widetilde{F} F$ is closed on the $S^4$, it can be
written locally as the derivative of a three form, which is just the dual of $K_{\mu}$. The integral
of the topological density associated with the upper hemisphere can be written via
Stokes’ theorem as the integral over the equatorial $S^3$ of this three form. But by
Stokes theorem, this also equals the integral of $Tr \widetilde{F} F$ within the ball in the $M = 0$
slice bounded by the same $S^3$, which gives now the meron’s winding number of a
half.)

To summarize, we looked at the adiabatic evolution of the Dirac hamiltonian and
asked what singularity in parameter space produces the nonabelian vector potential
which is behind the Thomas precession and found that it was the meron. Although we
computed the connection for all possible transports in parameter space, only closed
paths in momentum at fixed $m$ arise in atomic physics since $m$ is invariant. On the
other hand, in cosmological models in which $m$ can vary (say with some Higgs field,
which gives the fermion its mass) other components of the field can produce observable
effects. The connection to merons is amusing and possibly has other implications.
Callan et al offer many arguments for the meron’s previleged role before proposing it
as a configuration that can make important contributions to the Yang-Mills functional
integral. That the meron appears naturally in the Berry analysis provides yet another
argument. As for the topological ideas discussed here, they are naturally subsumed
by the very general five dimensional analysis of Avron et al . However the four
dimensional slice produces singular configurations interesting in their own right, just
like the x-y model vortices that arise on slicing the monople. We hope readers will find
the concrete example of the Dirac problem (with its relation to Thomas precession)
and the relation between the $S^4$ instanton of Avron et al in five dimensions and the
meron that appears in our four dimensional slice, a useful addition to our knowledge of instantons, merons, Thomas precession and the Berry connection.

RS acknowledges some wonderful discussions with Greg Moore. This report was supported by an NSF Grant DMR 9120525.
REFERENCES

[1] M.Berry, Proc. Roy. Soc., London A 392, 45, (1984).

[2] B.Simon, Phys.Rev.Lett., 51, 2167, (1983).

[3] F.Wilczek and A.Zee, Phys.rev. Lett., 52, 2111, (1984).

[4] Geometric Phases in Physics, Editors A.Shapere and F. Wilczek , World Scientific, New Jersey, USA, 1989.

[5] J.E.Avron, L.Sadun, J.Segert and B.Simon, Phys.Rev.Lett., 61,1329, (1988). For more details Comm. Math. Phys., 24, 595, (1989).

[6] H. Mathur, Phys. Rev. Lett., 67 ,3325, (1991).

[7] Solitons and Instantons, R.Rajaraman, North Holland, (1982).

[8] V.De Alfaro, S.Fubini and G.Furlan, Phys. Lett. 65B, 163, 91976).

[9] C.G.Callan, R.F.Dashen and D.J.Gross, Phys. Rev. D17, 2717, (1978).