LETTER

Hyperentanglement purification and concentration assisted by diamond NV centers inside photonic crystal cavities

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Abstract

Hyperentanglement has attracted much attention due to its fascinating applications in quantum communication. However, it is impossible to purify a pair of photon systems in a mixed hyperentangled state with errors in two degrees of freedom using linear optical elements only, far different from all the existing entanglement purification protocols in a degree of freedom (DOF) for quantum systems. Here, we investigate the possibility of purifying a spatial-polarization mixed hyperentangled Bell state with the errors in both the spatial-mode and polarization DOFs, resorting to the nonlinear optics of a nitrogen-vacancy (NV) center in a diamond embedded in a photonic crystal cavity coupled to a waveguide. We present the first hyperentanglement purification protocol for purifying a pair of two-photon systems in a mixed hyperentangled Bell state with the errors in two DOFs. We also propose an efficient hyperentanglement concentration protocol for a partially hyperentangled Bell pure state, which has the maximal success probability in principle. These two protocols are useful in long-distance quantum communication with hyperentanglement.

(Some figures may appear in colour only in the online journal)

1. Introduction

Entanglement, a unique phenomenon in quantum physics, is an essential resource in quantum information processing, such as quantum computing [1], quantum key distribution [2], quantum secure direct communication [3, 4], quantum teleportation [5], quantum dense coding [6, 7], quantum secret sharing [8, 9], and so on. In long-distance quantum communication, entangled photon pairs always act as a quantum channel and they are usually produced locally and inevitably polluted by the environment noise in the distribution process between the two legitimate users. With current technology, the photon signals can be transmitted no more than several hundreds of kilometers in an optical fiber or a free space, so quantum repeaters are required to connect the two neighboring nodes in a long-distance quantum communication network. Also, the entanglement of photon pairs will decrease in the storage process, which will decrease the fidelity and the security of long-distance quantum communication protocols. In quantum repeaters, entanglement purification and entanglement concentration are two necessary quantum techniques for improving the entanglement of the entangled systems.

Entanglement purification is used to obtain high-fidelity nonlocal entangled states from mixed states with less entanglement [10–19]. To date, many entanglement purification protocols (EPPs) have been proposed for polarization photon systems, resorting to nonlinear optics [10–13] or
linear optics [14–19]. Entanglement concentration is used to obtain the maximally entangled states from partially entangled pure states [20–27]. Some of the most important entanglement concentration protocols (ECPs) were proposed for the polarization entanglement of photon systems with the parameters of the partially entangled pure state either accurately known [20–24] or unknown [24–27] to the two remote users in quantum communication.

Hyperentanglement, defined as the entanglement of a quantum system in multiple degrees of freedom [28–30], has attracted much attention due to its fascinating applications in quantum information processing. It can be used to assist polarization entanglement purification [13–19], polarization Bell-state analysis [31–35] and polarization quantum repeater [36]. In 2008, Barreiro et al [37] exceeded the channel capacity limit of superdense coding with polarization-orbital-angular-momentum hyperentanglement using linear optics. Recently, some hyperentangled Bell-state analysis protocols with nonlinear optics have been proposed for increasing the channel capacity of long-distance quantum communication [38–40].

Although there are some interesting entanglement purification and entanglement concentration protocols, they are focused on only a degree of freedom (DOF) of quantum systems. There is no hyperentanglement purification protocol (hyper-EPP). Moreover, it is impossible to complete the hyperentanglement purification for a pair of quantum systems entangled in two or more DOFs with linear optical elements, far different from the existing EPPs [14–19]. Here, we investigate the possibility of achieving the hyperentanglement purification of two-photon systems entangled in two or more DOFs with linear optical elements, far different from the existing EPPs [14–19]. Entanglement concentration is used to concentrate partially hyperentangled Bell states with errors in both the polarization and spatial-mode DOFs. Also, we propose an efficient hyperentanglement protocol (hyper-ECP) which can be used to obtain the maximally entangled states from partially entangled pure state either accurately known [20–24] or unknown [24–27] to the two remote users in quantum communication.

![Figure 1](image1.png)

**Figure 1.** The optical transition of an NV center embedded in a photonic crystal cavity with circularly polarized photons. (a) One-side cavity-NV-center (cavity-waveguide-NV-center) system. (b) The spin preserving optical transition between the ground state and the excited state of an NV center.

spin triplet with a splitting of 2.88 GHz between the magnetic sublevels $|0\rangle$ ($m_s = 0$) and $|\pm 1\rangle$ ($m_s = \pm 1$), and its orbit state is $|E_0\rangle$. The specifically excited state $|A_2\rangle = \frac{1}{\sqrt{2}}(|E_-| + 1) + |E_+| - 1) [41]$, which is induced by spin–orbit and spin–spin interactions and C3v symmetry [42], is robust with stable symmetry. It decays with an equal probability to the two ground states $|-1\rangle$ and $|+1\rangle$ with radiation of right ($|R\rangle$, $\sigma_R = +1$) and left ($|L\rangle$, $\sigma_- = -1$) circularly polarized photons, respectively. Here, $|E_\pm\rangle$ ($|E_0\rangle$) are the orbit states with the angular momentum projections $\pm 1$ (0) along the NV axis.

The input–output property of the cavity-NV-center system shown in figure 1(a) can be described by the Heisenberg equations of motion for the cavity field operator $\hat{a}$ and dipole operator $\hat{\sigma}_-$ [43],

$$\frac{d\hat{a}}{dt} = -i(\omega_c - \omega) + \frac{\eta}{2}\hat{a} - g \hat{\sigma}_- - \sqrt{\eta} \hat{a}_{in},$$

$$\frac{d\hat{\sigma}_-}{dt} = -i(\omega_c - \omega) + \frac{\gamma}{2} \hat{\sigma}_- - g \hat{\sigma}_z \hat{a},$$

where $\omega_c$, $\omega$, and $\omega_k$ are the frequencies of the energy transition, the waveguide channel mode, and the cavity mode, respectively. $g$ is the coupling strength between the cavity and the NV center. $\gamma/2$, $\eta/2$, and $\kappa/2$ are the decay rates of the NV center, the cavity field, and the cavity side leakage mode, respectively. $\hat{a}_{in}$ and $\hat{a}_{out}$ are the input and output field operators of the waveguide, and they satisfy the relation $\hat{a}_{out} = \hat{a}_{in} + \sqrt{\eta} \hat{a}$. The reflection coefficient ($\hat{a}_{out}/\hat{a}_{in}$) of the cavity-NV-center system can be obtained in the weak excitation limit (the NV center is dominantly in the ground state and $\langle \sigma_z \rangle = -1$) [44, 45], that is,

$$r(\omega) = \frac{|1(\omega_c - \omega) + \frac{\eta}{2}|^2 |1(\omega_c - \omega) - \frac{\eta}{2} + \frac{1}{2}\sigma_z|^2 + g^2}{|1(\omega_c - \omega) + \frac{\eta}{2}|^2 |1(\omega_c - \omega) - \frac{\eta}{2} + \frac{1}{2}\sigma_z|^2 + g^2}.$$

In an ideal condition, the cavity side leakage can be neglected, and the NV center is resonant with the cavity ($\omega_c = \omega_k = \omega_0$). If the NV center is uncoupled to the cavity ($g = 0$), the reflection coefficient is $r_0(\omega) = -1$. If the NV center is coupled to the cavity ($4g^2 \gg \eta\gamma$), the reflection coefficient is $r(\omega) \rightarrow 1$. That is,

$$|R\rangle, (-1) \rightarrow |R\rangle, (-1), |R\rangle, (+1) \rightarrow |-R\rangle, (+1),$$

$$|L\rangle, (-1) \rightarrow |-L\rangle, (-1), |L\rangle, (+1) \rightarrow |L\rangle, (+1).$$

### 2. Hyper-EPP for a mixed hyperentangled Bell state with cavity-NV-center systems

#### 2.1. Parity-check QNDS for the polarization and spatial-mode DOFs of two-photon systems

The schematic diagram for an NV center in a diamond embedded in a photonic crystal cavity coupled to a waveguide is shown in figure 1. The negatively charged NV center consists of a substitutional nitrogen atom and an adjacent vacancy with six electrons from the nitrogen and three carbons surrounding the vacancy. The ground state is an electronic
This input–output property of a cavity-NV-center system can be used to construct parity-check QNDs for both the polarization (NV$_1$ in figure 2(a)) and the spatial-mode (NV$_2$ in figure 2(b)) DOFs of a two-photon system. Suppose that the initial states of NV$_1$ and NV$_2$ are $\frac{1}{\sqrt{2}}(|−1⟩ + |+1⟩)_e$ and $\frac{1}{\sqrt{2}}(|−1⟩ + |+1⟩)_c$, respectively.

(1) **Polarization parity-check QND.** The polarization parity-check QND (P-QND) is constructed with CPBSs and NV$_1$ shown in figure 2(a). If the polarization DOF of the two-photon system AC is in a Bell state, we first let the photon A pass through CPBS (CPBS$_1$ and CPBS$_2$), NV$_1$, and CPBS (CPBS$_3$ and CPBS$_4$), and then put the photon C into CPBS (CPBS$_5$ and CPBS$_6$), NV$_1$, and CPBS (CPBS$_7$ and CPBS$_8$). After interaction, the state of the system composed of NV$_1$ and two photons with polarizations becomes

$$|\phi^±_e⟩_{AC}∅|+⟩_e → |\phi^±_e⟩_{AC}∅|+⟩_e,$$

$$|\psi^±_c⟩_{AC}∅|+⟩_c → |\psi^±_c⟩_{AC}∅|−⟩_c.$$  (4)

Here, $|\phi^±_e⟩_{AC} = \frac{1}{\sqrt{2}}((|R⟩ ± |L⟩))_{AC}$, $|\psi^±_c⟩_{AC} = \frac{1}{\sqrt{2}}((|L⟩ ± |R⟩))_{AC}$, and $|±⟩_e = \frac{1}{\sqrt{2}}(|−1⟩ ± |+1⟩)_e$. The same polarization operations are performed on the two spatial modes $i_1$ and $i_2$ ($i = a, c$) without affecting the states of the photons in the spatial-mode DOF. By measuring the state of NV$_1$ in the orthogonal basis $|+⟩_e$, $|−⟩_e$, we can distinguish the even-parity polarization Bell states $|\phi^±_a⟩_{AC}$ from the odd-parity polarization Bell states $|\psi^±_a⟩_{AC}$. That is, the two-photon system is in an even-parity polarization Bell state if NV$_1$ is projected into the state $|+⟩_e$. Otherwise, the two-photon system is in an odd-parity polarization Bell state.

(2) **Spatial-mode parity-check QND.** The spatial-mode parity-check QND (S-QND) is constructed with NV$_2$ and HWPs as shown in figure 2(b). If we have a photon in the state $\alpha|R⟩ + \beta|L⟩$ put into NV$_2$ and HWP$_1$, after interaction, the state of the system composed of NV$_2$ and the photon is transformed into

$$\alpha|R⟩ + \beta|L⟩) ⊕ |+⟩_e → (\alpha|R⟩ + \beta|L⟩) ⊕ |−⟩_e.$$  (5)

By measuring the state of NV$_2$ in the orthogonal basis $|+⟩_e$, $|−⟩_e$, we can distinguish the case with an even number of photons $(|+⟩_e)$ from that with an odd number of photons $(|−⟩_e)$ which have interacted with NV$_2$. Therefore, we can distinguish the even-parity spatial-mode Bell states $|\phi^±_e⟩_{AC} = \frac{1}{\sqrt{2}}((|a⟩|c⟩ ± |a⟩|c⟩))_{AC}$ from the odd-parity spatial-mode Bell states $|\psi^±_e⟩_{AC} = \frac{1}{\sqrt{2}}((|a⟩|c⟩ ± |a⟩|c⟩))_{AC}$ by putting the two spatial-modes $a_2$ and $c_2$ into NV$_2$ and HWP (HWP$_1$ and HWP$_2$) in sequence, without influencing the states of the photons in the polarization DOF.

(3) **Measurement on an NV center system.** By applying a Hadamard operation on NV$_1$, the states $|+⟩_e$ and $|−⟩_e$ can be rotated to the states $|−⟩_e$ and $|+⟩_e$, respectively. If we have an auxiliary photon $p(|ϕ_p⟩ = \frac{1}{\sqrt{2}}(|R⟩ + |L⟩))$ put into CPBS$_1$, NV$_1$, and CPBS$_3$ as shown in figure 2(a), after interaction, the state of the system composed of NV$_1$ and the photon $p$ is changed as follows:

$$\frac{1}{\sqrt{2}}(|R⟩ + |L⟩)|−⟩_e → \frac{1}{\sqrt{2}}(|R⟩ + |L⟩)|−⟩_e,$$

$$\frac{1}{\sqrt{2}}(|R⟩ + |L⟩)|+⟩_e → \frac{1}{\sqrt{2}}(|R⟩ + |L⟩)|−⟩_e.$$  (6)

**Figure 2.** (a) Schematic diagram of the polarization parity-check QND (P-QND). (b) Schematic diagram of the spatial-mode parity-check QND (S-QND). (c) Schematic diagram of our hyper-EPP for a mixed hyperentangled Bell state with both spatial-mode bit-flip errors and polarization bit-flip errors, resorting to S-QNDS and P-QNDS. Bob performs the same operations as Alice by replacing the photons A and C with the photons B and D, respectively. NV$_1$ (NV$_2$) represents a one-side cavity-NV-center system which is used to perform polarization (spatial-mode) parity-check QND on the photon pair AC. CPBS$_i$ ($i = 1, 2, \ldots$) represents a polarizing beam splitter in the circular basis, which transmits the photon in the right-circular polarization $|R⟩$ and reflects the photon in the left-circular polarization $|L⟩$, respectively. R$_3$ represents a half wave plate which is used to perform a Hadamard operation on the polarization DOF. HWP$_1$ (HWP$_2$) represents a half wave plate which is used to perform a polarization phase-flip operation $σ_p^± = |R⟩⟨R| − |L⟩⟨L|$.

$D_j$ ($j = L_1, R_1, R_2, L_2$) represents a single-photon detector. BS represents a 50:50 beam splitter which is used to perform a Hadamard operation on the spatial-mode DOF of the photon. $k_1$ and $k_2$ represent the two spatial-modes of the photon $k$ ($k = a, c, b, d$).
We can read out the state of NV$_1$ by measuring the output state of the auxiliary photon $p$ with orthogonal linear polarization basis. If the auxiliary photon $p$ is in the state $\frac{1}{\sqrt{2}}(|R\rangle + |L\rangle)$, the state of the NV center is $|1\rangle_{e_1}$. Otherwise, the state of the NV center is $|1\rangle_{e_1}$. The state of NV$_2$ can be measured in the same way.

2.2. Hyper-EPP for a mixed hyperentangled Bell state with parity-check QNDs

We suppose that there are two identical two-photon systems in a mixed hyperentangled state as follows:

$$\rho_{AB} = \left[ F_1|\phi^+\rangle_p|\phi^+\rangle_p + (1 - F_1)|\psi^+\rangle_p|\psi^+\rangle_p \right]_{AB} \otimes \left[ F_2|\phi^+\rangle_S|\phi^+\rangle_S + (1 - F_2)|\psi^+\rangle_S|\psi^+\rangle_S \right]_{CD},$$

$$\rho_{CD} = \left[ F_1|\phi^+\rangle_p|\phi^+\rangle_p + (1 - F_1)|\psi^+\rangle_p|\psi^+\rangle_p \right]_{CD} \otimes \left[ F_2|\phi^+\rangle_S|\phi^+\rangle_S + (1 - F_2)|\psi^+\rangle_S|\psi^+\rangle_S \right]_{CD}.\quad (7)$$

Here the subscripts AB and CD represent two photon pairs shared by the two remote parties in quantum communication, say Alice and Bob, respectively. The two photons A and C belong to Alice, and the two photons B and D belong to Bob. $F_1$ and $F_2$ represent the probabilities of $|\phi^+\rangle_p$ and $|\phi^+\rangle_S$, respectively. The subscripts P and S represent the polarization and the spatial-mode DOFs of a two-photon system, respectively. The initial state of the four-photon system ABCD is $\rho_0 = \rho_{AB} \otimes \rho_{CD}$. It can be viewed as the mixture of 16 maximally hyperentangled pure states.

The principle of our hyper-EPP for a mixed hyperentangled Bell state with both polarization bit-flip errors and spatial-mode bit-flip errors is shown in figure 2(c). Both Alice and Bob perform the same S-QND and P-QND on the polarization and the spatial-mode DOFs of photon pairs AC and BD, and the entanglement of a pair of two-photon systems in the two DOFs can be purified independently. For describing the principle of our hyper-EPP explicitly, we describe the principle of our spatial-mode EPP in detail, and the principle of our polarization EPP is the same as for our spatial-mode EPP.

The spatial-mode state $|\phi^+\rangle_{AB} \otimes |\phi^+\rangle_{CD}$ is described as

$$|\phi^+\rangle_{AB} \otimes |\phi^+\rangle_{CD} = \frac{1}{2}\left( |a_1b_1c_1d_1\rangle + |a_2b_2c_2d_2\rangle \right) + |a_1b_2c_2d_1\rangle + |a_2b_1c_1d_2\rangle. \quad (8)$$

Alice and Bob first perform the spatial-mode S-QNDs on the two photon pairs AC and BD, respectively. If the two photon pairs are both in either an even-parity or odd-parity spatial mode, the spatial-mode state of the four-photon system ABCD is projected into $|\Phi_1\rangle = \frac{1}{\sqrt{2}}(|a_1b_1c_1d_1\rangle + |a_2b_2c_2d_2\rangle)$ or $|\Phi_2\rangle = \frac{1}{\sqrt{2}}(|a_1b_1c_2d_2\rangle + |a_2b_2c_1d_1\rangle)$, respectively. The state $|\Phi_1\rangle$ can be transformed into $|\Phi_1\rangle$ by performing spatial-mode bit-flip operations on the photons C and D. Then Alice and Bob perform Hadamard operations on the spatial-mode DOF of the two photons C and D, respectively (with 50:50 BSs), and the state $|\Phi_1\rangle$ is transformed into

$$|\Phi'_1\rangle = \frac{1}{2\sqrt{2}}\left( (|a_1b_1\rangle + |a_2b_2\rangle)(|c_1d_1\rangle + |c_2d_2\rangle) + (|a_1b_1\rangle - |a_2b_2\rangle)(|c_1d_1\rangle + |c_2d_2\rangle) \right).$$

(9)

If the two clicked photon detectors of photons C and D are in the even-parity spatial mode, the two-photon system AB is projected into the maximally entangled state $|\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|a_1b_1\rangle + |a_2b_2\rangle)_{AB}$. If the outcome of two clicked detectors is in the odd-parity spatial mode, a phase-flip operation $\sigma_z^2 = |b_1\rangle\langle b_1| - |b_2\rangle\langle b_2|$ performed on the photon B is required to obtain the state $|\phi^+\rangle_{AB}$. The spatial-mode states $|\phi^+\rangle_{AB} \otimes |\psi^+\rangle_{CD}$ and $|\phi^+\rangle_{AB} \otimes |\psi^+\rangle_{CD}$ are described as

$$|\phi^+\rangle_{AB} \otimes |\psi^+\rangle_{CD} = \frac{1}{2}\left( |a_1b_1c_1d_2\rangle + |a_2b_2c_2d_1\rangle + |a_1b_1c_2d_1\rangle + |a_2b_2c_1d_2\rangle \right),$$

(10)

If the spatial-mode states of the two photon pairs AC and BD are in the even-parity mode and the odd-parity mode, the state of the four-photon system ABCD is projected into $|\Phi_3\rangle = \frac{1}{\sqrt{2}}(|a_1b_1c_1d_2\rangle + |a_2b_2c_2d_1\rangle)$ or $|\Phi_4\rangle = \frac{1}{\sqrt{2}}(|a_1b_1c_2d_2\rangle + |a_2b_2c_1d_1\rangle)$. As Alice and Bob cannot identify which one of the photon pairs AB and CD has bit-flip errors, the two photon pairs have to be discarded in this case. The two photon pairs also have to be discarded if the spatial-mode states of the two photon pairs AC and BD are in the odd-parity mode and the even-parity mode, respectively.

The spatial-mode state $|\psi^+\rangle_{AB} \otimes |\psi^+\rangle_{CD}$ is described as

$$|\psi^+\rangle_{AB} \otimes |\psi^+\rangle_{CD} = \frac{1}{2}\left( |a_1b_2c_1d_2\rangle + |a_2b_1c_1d_2\rangle + |a_1b_2c_1d_1\rangle + |a_2b_1c_2d_1\rangle \right).$$

(11)

The spatial-mode states of the two photon pairs AC and BD are both in either the even-parity mode or the odd-parity mode, which is the same as $|\phi^+\rangle_{AB} \otimes |\phi^+\rangle_{CD}$. Therefore, Alice and Bob cannot distinguish the case in which both the two photon pairs have bit-flip errors from that without bit-flip errors.

The principle of our polarization EPP is the same as our spatial-mode one. That is, Alice and Bob pick up the cases in which the two photon pairs AC and BD are in the same polarization parity mode and discard the ones in different polarization parity modes.

After a round of our hyper-EPP process, the state of the photon pair AB kept is transformed into

$$\rho'_{AB} = \left[ F'_1|\phi^+\rangle_p|\phi^+\rangle_p + (1 - F'_1)|\psi^+\rangle_p|\psi^+\rangle_p \right]_{AB} \otimes \left[ F'_2|\phi^+\rangle_S|\phi^+\rangle_S + (1 - F'_2)|\psi^+\rangle_S|\psi^+\rangle_S \right]_{CD}. \quad (12)$$

Here $F'_1 = \frac{F_1}{F_1 + (1 - F_1)}, \quad F'_2 = \frac{F_2}{F_2 + (1 - F_2)},$ and $F_1 > 1/2$ ($i = 1, 2$). The fidelity of $|\phi^+\rangle_{AB} |\phi^+\rangle_{CD}$ in equation (12) is $F' = F'_1 \times F'_2$. With iteration of our hyper-EPP process several times, the fidelity of this two-photon hyperentangled Bell state can be improved (shown in figure 3 for the cases with $F_1 = F_2$).

Our hyper-EPP is constructed to purify a mixed hyperentangled Bell state with bit-flip errors in both the spatial-mode and the polarization DOFs. Usually, the two spatial modes of a photon are composed of two fibers.
are unknown to Alice and Bob, and they satisfy the relation α,β,γ
and C are kept by Alice, and the two photons B and D are
shared by Alice and Bob, respectively. The two photons A
Here, the subscripts AB and CD represent two photon pairs
polarization errors, such as a polarization bit-flip error in
in a nonlocal partially hyperentangled Bell state in both the
We suppose that there are two identical two-photon systems
3. Hyper-ECP for a partially hyperentangled Bell
state can also be purified with our hyper-EPP. 
However, the phases of two wavepackets for a photon are not
stable as they are influenced by thermal fluctuation, vibration,
and the imperfection of the fibers [46]. In principle, the
stable as they are influenced by thermal fluctuation, vibration,
difficult for a photon to penetrate from one fiber to the other.
Figure 3. The fidelity of the two-photon hyperentangled Bell state
in our hyper-EPP. F′ alters with the iteration number of
hyperentanglement purification processes n and the initial fidelity of
the mixed hyperentangled Bell state F1. Here, we assume F1 = F2.

(paths). In fact, the spatial-mode state of photon pairs is
far more robust than their polarization state in a practical
transmission over optical fiber channels [15–19]. In particular,
the probability that the bit-flip error takes place in the
spatial-mode DOF of a photon pair is negligible as it is
difficult for a photon to penetrate from one fiber to the other.
However, the phases of two wavepackets for a photon are not
stable as they are influenced by thermal fluctuation, vibration,
and the imperfection of the fibers [46]. In principle, the
phase-flip errors can be transformed into bit-flip errors with
local Hadamard operations in both the spatial-mode and the
phase-flip errors can be transformed into bit-flip errors with
local operation, and this mixed hyperentangled Bell state
has different

Figure 4. Schematic diagram of our hyper-ECP for a partially
hyperentangled Bell state with unknown parameters, resorting to
nonlinear optics. Bob performs the same operations as Alice
by replacing the photons AC and P-QND with the photons BD and
S-QND, respectively.

The principle of our hyper-ECP for a nonlocal partially
hyperentangled Bell-class state with unknown parameters is shown in figure 4. The initial state of the four-photon system
ABCD is |Ψ0⟩ = |ψ0⟩AB ⊗ |ψ0⟩CD. Alice can divide the states
of the two-photon system AC into two groups with the P-QND
whose principle is shown in figure 2(a), and Bob can divide the
states of the two-photon system BD into two groups with the
S-QND shown in figure 2(b). They pick up the odd-parity
terms of the polarization DOF and the odd-parity terms of the
spatial-mode DOF with the same parameters. The state of the four-photon system with the selected terms is

\[ |Ψ_1⟩ = \frac{1}{2} (|RRLL⟩ + |LLRR⟩)_{ABCD} \]

\[ \otimes (|a_2b_2c_1d_1⟩ + |a_1b_1c_2d_2⟩). \]

The probability for obtaining this state is \( p(1) = 4|αβγδ|^2 \).
Alice and Bob can obtain the photon pair AB in the maximally
hyperentangled Bell state |φ+⟩_{AB} with Hadamard
operations, detections, and local phase-flip operations on this
four-photon system, in the same way as that discussed for our
hyper-EPP.

If the state of the four-photon system ABCD is in
an even-parity polarization mode and an even-parity spatial
mode, the two-photon system AB is projected into |Ψ1⟩_{AB}
after the detections on two photons CD and local phase-flip
operations on B. Here

\[ |ψ_1⟩_{AB} = y(α^2|RR⟩ + β^2|LL⟩)_{AB} \otimes (γ^2|a_1b_1⟩ + δ^2|a_2b_2⟩). \]

and \( y = \frac{1}{\sqrt{(|α|^4 + |β|^4)(|γ|^4 + |δ|^4)}} \). This is a hyperentangled
state with less entanglement, which takes place with the probability of \( p'(1) = (|α|^4 + |β|^4)(|γ|^4 + |δ|^4) \), and it can be
distilled to the maximally hyperentangled Bell state with another round of our hyper-ECP. That is, another photon pair
A'B' is required, which has the identical nonlocal partially
hyperentangled Bell state with the photon pair AB. Alice and
Bob perform P-QND and S-QND on the photon pairs AA' and
BB', respectively, and they pick up the odd-parity terms of the
polarization DOF and the odd-parity terms of the spatial-mode
DOF with the same parameters, respectively. In this way, the

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state with cavity-NV-center systems

We suppose that there are two of identical two-photon systems
in a nonlocal partially hyperentangled Bell state in both the
polarization and spatial-mode DOFs, that is,

\[ |ψ_0⟩_{AB} = (α|RR⟩ + β|LL⟩)_{AB} \otimes (γ|a_1b_1⟩ + δ|a_2b_2⟩), \]
\[ |ψ_0⟩_{CD} = (α|RR⟩ + β|LL⟩)_{CD} \otimes (γ|c_1d_1⟩ + δ|c_2d_2⟩). \]

Here, the subscripts AB and CD represent two photon pairs
shared by Alice and Bob, respectively. The two photons A
and C are kept by Alice, and the two photons B and D are
kept by Bob. α, β, γ, and δ are four real parameters that
are unknown to Alice and Bob, and they satisfy the relation
\( |α|^2 + |β|^2 = |γ|^2 + |δ|^2 = 1 \).
maximally hyperentangled Bell state can be obtained with the probability of $p(21) = 4|\beta|^4(\alpha^4 + |\beta|^4)/[|\alpha|^4 + |\beta|^4]$ after Hadamard operations, detections, and local phase-flip operations. If the state of the four-photon system $ABAB'$ is not in an odd-parity polarization mode and an odd-parity spatial-mode, a third round of the hyper-ECP process is required.

In the other two cases, in which the outcomes of the two DOFs are in different parity modes, there is one DOF of the two-photon system $AB$ projected into the maximally entangled Bell state. That is,

$$|\psi_3\rangle_{AB} = y_1(|RR\rangle + |LL\rangle)_{AB} \otimes (|\alpha|^2|a_1b_1\rangle + |\beta|^2|a_2b_2\rangle),$$

$$|\psi_4\rangle_{AB} = y_2(|RR\rangle + |LL\rangle)_{AB} \otimes (|a_1b_1\rangle + |a_2b_2\rangle),$$

where $y_1 = \frac{1}{\sqrt{2(|\alpha|^2 + |\beta|^2)}}$ and $y_2 = \frac{1}{\sqrt{2(|\alpha|^2 + |\beta|^2)}}$. These two states are obtained with the probabilities of $p'(1) = 2|\beta|^2(|\gamma|^4 + |\delta|^4)$ and $p'(1) = 2|\gamma|^2(|\alpha|^4 + |\beta|^4)$, respectively. In a second round of hyper-ECP process, the maximally hyperentangled Bell state can be obtained with the probabilities of $p(2) = 4|\beta|^4|\alpha|^2(|\gamma|^4 + |\delta|^4)$ and $p(2) = 4|\alpha|^4|\beta|^2(|\alpha|^4 + |\beta|^4)$, respectively. If the state of the four-photon system $ABAB'$ is not in an odd-parity spatial-mode (similar to $|\psi_3\rangle_{AB}$) or an odd-parity polarization mode (similar to $|\psi_4\rangle_{AB}$), a third round of hyper-ECP process is required.

The success probability $P$ of our hyper-ECP becomes much higher by the iteration of our hyper-ECP process with the states preserved in the latter three cases, which have been discarded in the hyper-ECP with linear optics [24]. That is, the success probabilities of each round are $p(1), p(2) = p(21) + p(1) + p(22) + p(23), \ldots$. The total success probability of our hyper-ECP with $n$ rounds is $P = \sum p(n)$.

In the case $|\alpha| < |\beta|$, the relation between the success probability $P$ and the parameter $|\alpha|^2$ is shown in figure 5. If the parameter is $2|\alpha|^2 = 0.9$, the success probability of our hyper-ECP is $P = 80.8\%$ with the iteration number $n = 5$ (nearly equivalent to the maximal value $4|\alpha|^4 = 81\%$), while it is $P = 24.5\%$ for the hyper-ECP with linear optics [24]. That is, the success probability $P$ of our hyper-ECP can achieve the maximal success probability in principle.

4. Discussion and summary

An NV center in a diamond is a promising candidate for quantum information processing with its long electron-spin decoherence time even at room temperature [47, 48], and it has nanosecond manipulation time for an individual NV center [49]. The interaction between an NV center and a circularly polarized photon may be largely enhanced by coupling with the photonic crystal cavity [50]. Therefore, the fidelities of the P-QND and the S-QND in our hyper-EPP and hyper-ECP are mainly influenced by the coupling strength and cavity side leakage. With the definition $F = \langle (\psi|\psi) \rangle^2$, we can obtain the fidelities of the two QNDs (for even-parity states),

$$F_P = \frac{(|r_0|^2 + |r|^2 + 2(|r_0| + |r| + 2)^2}{16(|r_0|^4 + |r|^4 + 2)(|r_0|^2 + |r|^2 + 2)},$$

$$F_S = \frac{1}{8} \left( \frac{|r_0|^2 + |r|^2}{4(|r_0|^2 + |r|^2)} \right)^2 \frac{|r_0| + |r| + 2}{4(|r_0|^2 + |r|^2 + 2)}.$$

Here $|\psi\rangle$ is the ideal final state and $\langle \psi|\psi \rangle$ is the final state considering experimental factors. The spontaneous decay rate of an NV center is $\gamma = 2\pi \times 15 \text{ MHz}$ [41], and the cavity side leakage rate can reach $10\% = \gamma = 2\pi \times 10 \text{ GHz}$ [51] with a quality factor of $Q \approx 10^4$. If the coupling strength is $g = 0.1\eta$, the fidelities of these two QNDs are $F_P = 97.1\%$ and $F_S = 98.6\%$, respectively. That is, both the P-QND and the S-QND are feasible with current technology.

With linear optics, we can only produce one qubit error for a pair of two-photon systems in a nonmaximally hyperentangled state. That is, a pair of two-photon systems in a mixed hyperentangled Bell state with both polarization bit-flip errors and spatial-mode bit-flip errors cannot be purified with linear optics [24]. Therefore, nonlinear optics is required to implement the hyper-EPP for a mixed hyperentangled Bell state with two or more qubit errors in two DOFs. Our hyper-EPP is the first one for a mixed hyperentangled spatial-polarization Bell state with the errors in both the two DOFs with the P-QNDs and S-QNDs assisted by one-side cavity-NV-center systems. As the phase-flip errors in both the polarization and the spatial-mode DOFs can be transformed into the bit-flip errors in these two DOFs [12, 14], our hyper-EPP can be used to purify a mixed hyperentangled Bell state completely in principle. Also, we have proposed an efficient hyper-ECP for partially hyperentangled Bell states with unknown parameters, resorting to nonlinear optics. Our hyper-ECP can obtain the maximal success probability in principle by iteration of the hyper-ECP process with the states that have been discarded in the hyper-ECP with linear optics [24].

In summary, we have investigated the possibility of improving the entanglement of nonlocal nonmaximally...
hyperentangled Bell states with nonlinear optics. Both our hyper-EPP and hyper-ECP are implemented with the P-QNDs and S-QNDs that are constructed with nonlinear optics of cavity-NV-center systems. We have analyzed the experimental feasibility of the two QNDs in two DOFs and shown that they can be implemented with current technology. These two protocols are useful for increasing the channel capacity of long-distance quantum communication with hyperentanglement.

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