On Bayesian Estimation via Divergences

Mohamed Cherfi

Laboratoire de Statistique Théorique et Appliquée (LSTA)
Equipe d’Accueil 3124
Université Pierre et Marie Curie – Paris 6
Tour 15-25, 2ème étage
4 place Jussieu
75252 Paris cedex 05

Abstract.
In this Note we introduce a new methodology for Bayesian inference through the use of \( \phi \)-divergences and the duality technique. The asymptotic laws of the estimates are established.

Keywords: Bayes; Divergences; Markov Chain Monte Carlo; Posterior distribution

1 Introduction

Bayesian techniques are particularly attractive since they can incorporate information other than the data into the model in the form of prior distributions. Another feature which make them increasingly attractive is that they can handle models that are difficult to estimate with classical methods by use of simulation techniques, see for instance Robert (2001).

The aim of this Note is to discuss the use of divergences as a basis for Bayesian inference. The use of divergence measures in a Bayesian context has been considered in Dey and Birmiwal (1994) and Peng and Dey (1995). Most of this work is concerned with the use of divergence measures to study Bayesian robustness when the priors are contaminated and to diagnose the effect of outliers.

In order to estimate model parameters and circumvent possible difficulties encountered with the likelihood function, we follow up common robustification ideas, see for instance Hanousek (1990, 1994), and propose to replace the likelihood in the formula of the posterior distribution by the dual form of the divergence that lead to estimators that are both robust and efficient and include the expected a posteriori estimator (EAP) as a benchmark. A major advantage of the method is that it does not require additional accessories such as

E-mail address: mohamed.cherfi@gmail.com
as kernel density estimation or other forms of nonparametric smoothing to produce non-
parametric density estimates of the true underlying density function in contrast with the
method proposed by Hooker and Vidyashankar (2011) which is based on the concept of a
minimum disparity procedure introduced by Lindsay (1994). The plug-in of the empirical
distribution function is sufficient for the purpose of estimating the divergence in the case
of i.i.d. data. The proposed estimators are based on integration rather than optimization.
Other reasons, which are commonly put forward to use the proposed approach is compu-
tational attractiveness through the use of MCMC and can easily handle a large number of
parameters.
The outline of the Note is as follows. Together with a brief review of definitions and
properties of divergences, Section 2 discusses the procedure to obtain the estimates. In
Section 3, we give the limit laws of the proposed estimators. Some final remarks conclude
the Note.

2 Estimation

2.1 Background on dual divergences inference

Keziou (2003) and Broniatowski and Keziou (2009) introduced the class of dual diver-
gences estimators for general parametric models. In the following, we shortly recall their
context and definition.
Recall that the \( \phi \)-divergence between a bounded signed measure \( Q \) and a probability \( P \)
on \( \mathcal{D} \), when \( Q \) is absolutely continuous with respect to \( P \), is defined by
\[
D_\phi(Q, P) := \int _\mathcal{D} \phi \left( \frac{dQ}{dP}(x) \right) dP(x),
\]
where \( \phi \) is a convex function from \( (-\infty, \infty) \) to \([0, \infty]\) with \( \phi(1) = 0 \).
Different choices for \( \phi \) have been proposed in the literature. For a good overview, see
Pardo (2006). Well-known class of divergences is the class of the so called “power diver-
genences” introduced in Cressie and Read (1984) (see also Liese and Vajda (1987) chapter
2); it contains the most known and used divergences. They are defined through the class
of convex functions
\[
x \in [0, +\infty[, \phi_\gamma(x) := \frac{x^\gamma - \gamma x + \gamma - 1}{\gamma (\gamma - 1)}
\]
if \( \gamma \in \mathbb{R} \setminus \{0, 1\} \), \( \phi_0(x) := -\log x + x - 1 \) and \( \phi_1(x) := x \log x - x + 1 \).
Let \( X_1, \ldots, X_n \) be an i.i.d. sample with p.m. \( P_{\theta_0} \). Consider the problem of estimating
the population parameters of interest \( \theta_0 \), when the underlying identifiable model is given
by \( \{P_{\theta} : \theta \in \Theta\} \) with \( \Theta \) a subset of \( \mathbb{R}^d \). Here the attention is restricted to the case where
the probability measures $\mathbb{P}_\theta$ are absolutely continuous with respect to the same $\sigma$-finite measure $\lambda$; correspondent densities are denoted $p_\theta$.

Let $\phi$ be a function of class $C^2$, strictly convex and satisfies:

\[
\int \left| \phi' \left( \frac{p_\theta(x)}{p_\alpha(x)} \right) \right| p_\theta(x) \, dx < \infty.
\]  

(2)

By Lemma 3.2 in Broniatowski and Keziou (2006), if the function $\phi$ satisfies: There exists $0 < \eta < 1$ such that for all $c$ in $[1 - \eta, 1 + \eta]$, we can find numbers $c_1, c_2, c_3$ such that

\[
\phi(cx) \leq c_1 \phi(x) + c_2 |x| + c_3,
\]

for all real $x$,

(3)

then the assumption (2) is satisfied whenever $D_\phi(\mathbb{P}_\theta, \mathbb{P}_\alpha)$ is finite. From now on, $\mathcal{U}$ will be the set of $\theta$ and $\alpha$ such that $D_\phi(\mathbb{P}_\theta, \mathbb{P}_\alpha) < \infty$. Note that all the real convex functions $\phi$, pertaining to the class of power divergences defined in (1) satisfy the condition (3).

Under (2), using Fenchel duality technique, the divergence $D_\phi(\theta, \theta_0)$ can be represented as resulting from an optimization procedure, this elegant result was proven in Keziou (2003), Liese and Vajda (2006) and Broniatowski and Keziou (2009). Broniatowski and Keziou (2006) called it the dual form of a divergence, due to its connection with convex analysis.

Under the above conditions, the $\phi$-divergence:

\[
D_\phi(\mathbb{P}_\theta, \mathbb{P}_{\theta_0}) = \int \phi \left( \frac{p_\theta(x)}{p_{\theta_0}(x)} \right) p_{\theta_0}(x) \, dx,
\]

can be represented as the following form:

\[
D_\phi(\mathbb{P}_\theta, \mathbb{P}_{\theta_0}) = \sup_{\alpha \in \mathcal{U}} \int h(\theta, \alpha) \, d\mathbb{P}_{\theta_0},
\]  

(4)

where $h(\theta, \alpha) : x \mapsto h(\theta, \alpha, x)$ and

\[
h(\theta, \alpha, x) := \int \phi' \left( \frac{p_\theta(x)}{p_\alpha(x)} \right) p_\theta - \left[ p_\theta(x) \phi' \left( \frac{p_\theta(x)}{p_\alpha(x)} \right) \left( \frac{p_\theta(x)}{p_\alpha(x)} \right) - \phi \left( \frac{p_\theta(x)}{p_\alpha(x)} \right) \right].
\]  

(5)

Since the supremum in (4) is unique and is attained in $\alpha = \theta_0$, independently upon the value of $\theta$, by replacing the hypothetical probability measure $\mathbb{P}_{\theta_0}$ by the empirical measure $\mathbb{P}_n$ define the class of estimators of $\theta_0$ by

\[
\hat{\alpha}_\phi(\theta) := \arg \sup_{\alpha \in \mathcal{U}} \int h(\theta, \alpha) d\mathbb{P}_n, \quad \theta \in \Theta,
\]  

(6)

where $h(\theta, \alpha)$ is the function defined in (5). This class is called “dual $\phi$-divergence estimators” ($D_\phi$DE’s), see for instance Keziou (2003) and Broniatowski and Keziou (2009).
Formula (6) defines a family of $M$-estimators indexed by the function $\phi$ specifying the divergence and by some instrumental value of the parameter $\theta$, called here escort parameter, see also Broniatowski and Vajda (2009).

Application of dual representation of $\phi$-divergences have been considered by many authors, we cite among others, Keziou and Leoni-Aubin (2008) for semi-parametric two-sample density ratio models, robust tests based on saddlepoint approximations in Toma and Leoni-Aubin (2010), Toma and Broniatowski (2010) have proved that this class contains robust and efficient estimators and proposed robust test statistics based on divergences estimators. Bootstrapped $\phi$-divergences estimates are considered in Bouzebda and Cherfi (2011), extension of dual $\phi$-divergences estimators to right censored data are introduced in Cherfi (2011a), for estimation and tests in copula models we refer to Bouzebda and Keziou (2010) and the references therein. Performances of dual $\phi$-divergence estimators for normal models are studied in Cherfi (2011b).

2.2 Estimation

Let us now turn to the estimation using divergences in our setting. For the parameter $\theta$ consider a prior density $\pi(\theta)$ on $\Theta$, and let $\rho(x, \theta) : \mathbb{R} \times \Theta$ be a suitable function. Then Hanousek (1990) considered the following Bayes-type or B-estimator of $\theta_0$, corresponding to the prior density $\pi(\theta)$ and generated by the function $\rho(x, \theta)$,

$$
\hat{\theta}^*_n = \frac{\int_{\Theta} \theta \exp \{- \sum_{i=1}^{n} \rho(X_i, \theta)\} \pi(\theta) \, d\theta}{\int_{\Theta} \exp \{- \sum_{i=1}^{n} \rho(X_i, \theta)\} \pi(\theta) \, d\theta}
$$

(7)

if both integrals exist. This type of estimators is often called Laplace type estimators see for instance Chernozhukov and Hong (2003).

The posterior $M$-estimator is defined as

$$
\hat{\theta}^+_n = \arg \max_{\theta \in \Theta} \left( - \sum_{i=1}^{n} \rho(X_i, \theta) + \ln \pi(\theta) \right).
$$

(8)

Hanousek (1990) showed that $\hat{\theta}^*_n$ is asymptotically equivalent to the $M$-estimator generated by $\rho$ for a large class of priors and under some conditions on $\rho$ and $P_{\theta_0}$. The asymptotic equivalence provides the access to the study of asymptotics for B-estimators via the $M$-estimators.

In the context of the Bayesian methods examined in this Note, instead of a likelihood function, our work will use a criterion function $\int h(\theta, \alpha) \, dP_n$. Inference is based on the $\phi$-posterior

$$
p_{\phi, n}(\alpha|X_1, \cdots, X_n) = \frac{\exp \{ nP_n h(\theta, \alpha) \} \pi(\alpha)}{\int_{\Theta} \exp \{ nP_n h(\theta, \alpha) \} \pi(\alpha) \, d\alpha}.
$$

(9)
A risk function is the expected loss or error in which the researcher incurs when choosing a certain value for the parameter estimate. Let $\mathcal{L}_n(u)$ be a loss function. The risk function takes the form

$$R_n(\tilde{\alpha}) = \int_{\mathcal{U}} \mathcal{L}_n(\alpha - \tilde{\alpha}) P_{\phi,n}(\alpha|X_1, \cdots, X_n) \, d\alpha,$$

(10)

where $P_{\phi,n}(\alpha|X_1, \cdots, X_n)$ is the $\phi$-posterior density, $\tilde{\alpha}$ is the selected value, and $\alpha$ is all other possible values we are integrating over. The loss function can penalize the selection of $\alpha$ asymmetrically, and is a function of the selected value and the rest of the possible values of the parameters in $\mathcal{U}$.

The dual $\phi$-divergence Bayes type estimator minimizes the expected loss for different forms of the loss function

$$\hat{\alpha}_\phi^*(\theta) = \arg \inf_{\tilde{\alpha} \in \mathcal{U}} R_n(\tilde{\alpha}).$$

(11)

Choosing different loss functions will change the objective function such that the estimators bear different interpretations. For instance, when the loss is squared error ($\mathcal{L}_n(u) = |\sqrt{n}u|^2$), for fixed $\theta$, the dual $\phi$-divergence Bayes type estimator is defined as

$$\hat{\alpha}_\phi^*(\theta) = \int_{\mathcal{U}} \alpha P_{\phi,n}(\alpha|X_1, \cdots, X_n) \, d\alpha := \frac{\int_{\mathcal{U}} \alpha \exp \{nP_n h(\theta, \alpha)\} \pi(\alpha) \, d\alpha}{\int_{\mathcal{U}} \exp \{nP_n h(\theta, \alpha)\} \pi(\alpha) \, d\alpha},$$

(12)

if both integrals exist, other familiar forms obtained for different loss functions are modes, medians and quantiles.

The posterior dual $\phi$-divergences estimator is defined as

$$\hat{\alpha}_\phi^+(\theta) = \arg \sup_{\alpha \in \mathcal{U}} (P_n h(\theta, \alpha) + \ln \pi(\alpha)).$$

(13)

It is obvious that posterior dual $\phi$-divergences estimates naturally inherit the properties of dual $\phi$-divergences estimates and hence we focus on dual $\phi$-divergences Bayes type estimators only.

Remark 1

1. The EAP estimator, which is the mean of the posterior distribution, belongs to the class of estimates (12). Indeed, it is obtained when $\phi(x) = -\log x + x - 1$, that is as the dual modified $KL_m$-divergence estimate. Observe that $\phi'(x) = -\frac{1}{x} + 1$ and $x\phi'(x) - \phi(x) = \log x$, hence

$$\int h(\theta, \alpha) dP_n = -\int \log \left(\frac{dP_n}{d\alpha} \right) dP_n.$$
Keeping in mind definitions (12), we get

$$\hat{\alpha}^{KL}_m(\theta) := \frac{\int U \alpha \prod_{i=1}^{n} p_{\alpha}(X_i) \pi(\alpha) \, d\alpha}{\int U \prod_{i=1}^{n} p_{\alpha}(X_i) \pi(\alpha) \, d\alpha},$$

independently upon $\theta$.

2. If new data $X_{n+1}, \ldots, X_N$ are obtained, the posterior for the combined data $X_1, \ldots, X_N$ can be obtained by using posterior after $n$ observations, $p_{\phi,n}(\alpha|X_1, \ldots, X_n)$ as a prior $\alpha$:

$$p_{\phi,n}(\alpha|X_1, \ldots, X_N) \propto p_{\phi,n}(\alpha|X_1, \ldots, X_n) \times p_{\phi,n}(X_{n+1}, \ldots, X_N|\alpha).$$

3 **Asymptotic properties**

In this section we state the asymptotic normality of the estimates based on the $\phi$-posterior and evaluate their limiting variance. The hypotheses handled here are similar to those used in Keziou (2003) and Broniatowski and Keziou (2009) in the frequentist case, these conditions are mild and can be satisfied in most of circumstances. From now on, $\overset{D}{\longrightarrow}$ denotes the convergence in distribution.

(R.1)

$$\sup_{\alpha \in \Theta} |\mathbb{P}_n h(\theta, \alpha) - \mathbb{P}_{\theta_0} h(\theta, \alpha)| \overset{a.s.}{\longrightarrow} 0.$$

(R.2) There exists a neighborhood $N(\theta_0)$ of $\theta_0$ such that the first and second order partial derivatives (w.r.t $\alpha$) of $\phi' \left( \frac{p_\theta(x)}{p_\alpha(x)} \right) p_\theta(x)$ are dominated on $N(\theta_0)$ by some integrable functions. The third order partial derivatives (w.r.t $\alpha$) of $h(\theta, \alpha, x)$ are dominated on $N(\theta_0)$ by some $\mathbb{P}_{\theta_0}$-integrable functions.

Let

$$S := -\mathbb{P}_{\theta_0} \frac{\partial^2}{\partial \alpha^2} h(\theta, \theta_0)$$

and

$$V := \mathbb{P}_{\theta_0} \frac{\partial}{\partial \alpha} h(\theta, \theta_0)^\top \frac{\partial}{\partial \alpha} h(\theta, \theta_0).$$

Observe that the matrix $S$ is symmetric and positive since the second derivative $\phi''$ is nonnegative by the convexity of $\phi$.

(R.3) The matrices $S$ and $V$ are non singular.
For \( \alpha \) in an open neighborhood of \( \theta_0 \), using (R.2) by a Taylor expansion

\[
P_n h(\theta, \alpha) - P_n h(\theta, \theta_0) = (\alpha - \theta_0)^\top U_n(\theta_0) - \frac{1}{2} (\alpha - \theta_0)^\top S(\alpha - \theta_0) + R_n(\alpha),
\] (14)

(R.4) Given any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that, the probability of the event

\[
\sup_{|\alpha - \theta_0| \leq \delta} |R_n(\alpha)| \geq \epsilon
\] (15)

 tends to zero as \( n \to \infty \).

**Remark 2**

1. Using Example 19.8 in *van der Vaart* (1998), it is clear that the class of functions \( \{\alpha \mapsto h(\theta, \alpha); \alpha \in \Theta\} \) is a Glivenko-Cantelli class of functions for all fixed \( \theta \), that (R.1) holds.

2. Conditions (R.2) and (R.3) are about usual regularity properties of the underlying model, they guarantee that we can interchange integration and differentiation and the existence of the variance-covariance matrices, they are similar to regularity conditions used in *Keziou* (2003) and *Broniatowski and Keziou* (2009) in the frequentist case.

3. Condition (R.4) easily holds when there is enough smoothness. It requires that the remainder term of the expansion can be controlled in a particular way over a neighborhood of \( \theta_0 \).

Define

\[
t := \sqrt{n} (\alpha - \Delta_n), \quad \Delta_n := \theta_0 + S^{-1} U_n(\theta_0),
\] (16)

and \( p_{\phi,n}^*(t) \) be the \( \phi \)-posterior density of \( t \).

The following theorem states that under some regularity conditions, for large \( n \), \( p_{\phi,n}^*(\cdot) \) is approximately a random normal density in the \( L_1 \) sense.

**Theorem 1** Let \( \pi(\theta) \) be any prior that is continuous and positive at \( \theta_0 \) with \( \int |\theta| \pi(\theta) \, d\theta \).

Then under Conditions (R.1-4)

\[
\int \left| p_{\phi,n}^*(t) - \left( \frac{\det S}{2\pi} \right)^{d/2} \exp \left\{ -\frac{1}{2} t^\top S t \right\} \right| \, dt \overset{P}{\to} 0.
\] (17)

We now state the principal result of this section. **Theorem 2** is concerned with the efficiency and asymptotic normality of the proposed estimates. See *Ibragimov and Has’minskii* (1981) and *Strasser* (1981) for more on the consistency and efficiency of Bayes estimators.
Theorem 2  Let $\pi(\theta)$ be any prior that is continuous and positive at $\theta_0$ with $\int |\theta| \pi(\theta) \, d\theta$. Assume that Conditions (R.1-4) hold, then as $n$ tends to infinity

$$V^{-1/2} S \sqrt{n} \left( \hat{\alpha}_\phi^* (\theta) - \theta_0 \right) \overset{d}{\to} \mathcal{N} (0, I).$$

Remark 3  If $\theta = \theta_0$, then $S^T V^{-1} S = I_{\theta_0}$ the information matrix, so that $\hat{\alpha}_\phi^* (\theta_0)$ is consistent and asymptotically efficient. The consequence is that the value of the escort parameter should be taken as a consistent estimator of $\theta_0$, see Cherfi (2011a,b) for relevant discussion on this subject.

4  Concluding remarks

We have introduced a new estimation procedure in parametric models that combine divergences method with Bayesian analysis, it generalizes the expected a posteriori estimate. The proposed estimators are based on integration rather than optimization. These estimators are often much easier to compute in practice than the $\arg \sup$ estimators (6), especially in the high-dimensional setting; see, for example, the discussion in Liu et al. (2008).

In order to compute these estimators, using Markov chain Monte Carlo methods, we can draw a Markov chain,

$$S = (\alpha^{(1)}; \alpha^{(2)}; \cdots ; \alpha^{(B)});$$  \hspace{1cm} (18)

whose marginal density is approximately given by $p_{\phi,n}(\cdot)$, the $\phi$-posterior distribution. Then the estimate $\hat{\alpha}_\phi^* (\theta)$ is computed as

$$\hat{\alpha}_\phi^* (\theta) = \frac{1}{B} \sum_{i=1}^{B} \alpha^{(i)}. \hspace{1cm} (19)$$

Consider the construction of confidence intervals for the quantity $f(\theta_0)$, for a given continuously differentiable function $f : \Theta \rightarrow \mathbb{R}$. Define

$$C_n (\epsilon) := \inf \left\{ x : \int_{f(\alpha) \leq x} \alpha p_{\phi,n}(\alpha) \, d\alpha \geq \epsilon \right\}. \hspace{1cm} (20)$$

Then the dual $\phi$-divergence Bayes type estimator confidence interval is given by $\left[ C_n \left( \frac{\epsilon}{2} \right); C_n \left( 1 - \frac{\epsilon}{2} \right) \right]$. These confidence intervals can be constructed simply by taking the $\frac{\epsilon}{2}$th and $\frac{\epsilon}{2}$th quantiles of the MCMC sequence

$$f(S) = \left( f(\alpha^{(1)}); f(\alpha^{(2)}); \cdots ; f(\alpha^{(B)}) \right), \hspace{1cm} (21)$$
and thus are quite simple in practice.
The very peculiar choice of the escort parameter defined through \( \theta = \theta_0 \) has same limit properties as the posterior mean. This result is of some relevance, since it leaves open the choice of the divergence, while keeping good asymptotic properties, we expect that it can also be used directly to provide robust inference, we leave this study for a subsequent paper.
The problem of the choice of the divergence remain an open question and need more investigation.

5 Proofs

Our arguments follow those presented by Lehmann and Casella (1998), the main difference is due to the non-likelihood setting. See also Chernozhukov and Hong (2003) for similar arguments. We often use \( M \) to denote a generic finite constant and \( I \) to denote the identity matrix. The smallest-eigenvalue of a matrix \( S \) is denoted as mineig(\( S \)).

5.1 Proof of Theorem 1

Define
\[
t := \sqrt{n} (\alpha - \Delta_n), \, \Delta_n := \theta_0 + S^{-1} U_n (\theta_0),
\]
then
\[
p^{*}_n (t) = \frac{1}{\sqrt{n}} \frac{p_{\phi,n} \left( \frac{t}{\sqrt{n}} + \Delta_n \right)}{\pi \left( \frac{t}{\sqrt{n}} + \Delta_n \right) \exp \left\{ n \pi_n h \left( \theta, \frac{t}{\sqrt{n}} + \Delta_n \right) \right\}}
\]
\[
= \frac{1}{c_n} \frac{\int \pi \left( \frac{u}{\sqrt{n}} + \Delta_n \right) \exp \left\{ n \pi_n h \left( \theta, \frac{u}{\sqrt{n}} + \Delta_n \right) \right\} \, du}{\pi \left( \frac{t}{\sqrt{n}} + \Delta_n \right) \exp \left\{ \omega (t) \right\}},
\]
where
\[
\omega (t) := n \pi_n h \left( \theta, \frac{t}{\sqrt{n}} + \Delta_n \right) - n \pi_n h (\theta, \theta_0) - \frac{n}{2} U_n (\theta_0) ^\top S^{-1} U_n (\theta_0),
\]
and
\[
c_n := \int \pi \left( \frac{u}{\sqrt{n}} + \Delta_n \right) \exp \left\{ \omega (u) \right\} \, du.
\]
Lemma 1 Let

\[ J_1 = \int \left| \pi \left( \frac{t}{\sqrt{n}} + \Delta_n \right) e^{\omega(t)} - \pi (\theta_0) e^{-\frac{1}{2} t^\top S t} \right| dt, \quad (25) \]

then if (R.1-4) hold, \( J_1 \xrightarrow{P} 0. \)

By Lemma 1, we have that

\[ c_n \xrightarrow{P} \int \pi (\theta_0) e^{-\frac{1}{2} t^\top S t} dt = \pi (\theta_0) \sqrt{\frac{(2\pi)^d}{|\det S|}}. \quad (26) \]

Observe that

\[ \int \left| p_{\phi,n}^*(t) - \left( \frac{\det S}{2\pi} \right)^{d/2} \exp \left\{ -\frac{1}{2} t^\top S t \right\} \right| dt = \frac{J}{c_n}, \]

where

\[ J := \int \left| \pi \left( \frac{t}{\sqrt{n}} + \Delta_n \right) e^{\omega(t)} - c_n \sqrt{\frac{|\det S|}{(2\pi)^d}} e^{-\frac{1}{2} t^\top S t} \right| dt \quad (27) \]

By (26), to show (17) it is enough to show that \( J \xrightarrow{P} 0. \) But, \( J \leq J_1 + J_2 \) where \( J_1 \) is given by (25) and

\[ J_2 = \int \left| c_n \sqrt{\frac{|\det S|}{(2\pi)^d}} e^{-\frac{1}{2} t^\top S t} - \pi (\theta_0) e^{-\frac{1}{2} t^\top S t} \right| dt. \]

Observe that

\[ J_2 = \left| c_n \sqrt{\frac{|\det S|}{(2\pi)^d}} - \pi (\theta_0) \right| \int e^{-\frac{1}{2} t^\top S t} dt \xrightarrow{P} 0. \quad (28) \]

By Lemma 1 and (28), \( J_1 \) and \( J_2 \) tend to zero in probability, and this completes the proof.

Proof of Lemma 1

Let

\[ U_n(\theta_0) := P_n \frac{\partial}{\partial \alpha} h(\theta, \theta_0). \]

Using (R.2) and (R.3) in connection with the central limit theorem (CLT), we can see that

\[ \sqrt{n} V^{-1/2} U_n(\theta_0) \xrightarrow{d} \mathcal{N}(0, I). \quad (29) \]
Write
\[ \omega(t) = -\frac{1}{2} t^\top St + R_n \left( \frac{t}{\sqrt{n}} + \Delta_n \right). \] (30)

To prove that the integral (25) tends to zero in probability, divide the range of integration into the three parts:

(i) \( |t| \leq M \),

(ii) \( |t| \geq \delta \sqrt{n} \),

(iii) \( M < |t| < \delta \sqrt{n} \),

and show that the integral over each of the three tends to zero in probability.

Part(i):
\[ \int_{|t| \leq M} \left| \pi \left( \frac{t}{\sqrt{n}} + \Delta_n \right) e^{\omega(t)} - \pi(\theta_0) e^{-\frac{1}{2} t^\top St} \right| \, dt \xrightarrow{P} 0. \]

To prove this result, we shall show that for every \( 0 < M < \infty \),
\[ \sup_{|t| \leq M} \left| \pi \left( \frac{t}{\sqrt{n}} + \Delta_n \right) e^{\omega(t)} - \pi(\theta_0) e^{-\frac{1}{2} t^\top St} \right| \xrightarrow{P} 0. \] (31)

Substituting the expression (30) for \( \omega(t) \), (31) is seen to follow from the following two facts

\[ \sup_{|t| \leq M} \left| \pi \left( \frac{t}{\sqrt{n}} + \Delta_n \right) - \pi(\theta_0) \right| \xrightarrow{P} 0 \] (32)

and
\[ \sup_{|t| \leq M} \left| R_n \left( \frac{t}{\sqrt{n}} + \Delta_n \right) \right| \xrightarrow{P} 0. \] (33)

The first fact is obvious from the continuity of \( \pi \) and because by Condition (R.3) and (29):
\[ \sqrt{n}S^{-1}U_n(\theta_0) = O_P(1), \] (34)

so that \( \Delta_n \xrightarrow{P} \theta_0 \).

Given (34), the second fact follows from Condition (R.2), and
\[ \sup_{|t| \leq M} \left| \Delta_n + \frac{t}{\sqrt{n}} - \theta_0 \right| = O_P\left( \frac{1}{\sqrt{n}} \right). \]

Part(ii):
\[ \int_{M < |t| < \delta \sqrt{n}} \left| \pi \left( \frac{t}{\sqrt{n}} + \Delta_n \right) e^{\omega(t)} - \pi(\theta_0) e^{-\frac{1}{2} t^\top St} \right| \, dt \xrightarrow{P} 0. \]
For the second part, since the integral of the second term is finite and can be made arbitrarily small by setting $M$ large, it suffices to show that for the integrand of the first term is bounded by an integrable function with probability $\geq 1 - \epsilon$. More precisely, we shall show that given $\epsilon > 0$, there exists $\delta > 0$ and $C < \infty$ such that for sufficiently large $n$,

$$P\left[ \pi \left( \frac{t}{\sqrt{n}} + \Delta_n \right) e^{\omega(t)} \leq C e^{-\frac{1}{2} t^T S t} \text{ for all } |t| < \delta \sqrt{n} \right] \geq 1 - \epsilon. \quad (35)$$

By the fact that $\Delta_n \xrightarrow{P} \theta_0$ and the continuity of $\pi$, we can drop the factor $\pi \left( \frac{t}{\sqrt{n}} + \Delta_n \right)$ from consideration, so that it remains to establish such a bound for $\exp\{\omega(t)\}$. By definition of $\omega(t)$ (30)

$$\exp\{\omega(t)\} \leq \exp\left\{ -\frac{1}{2} t^T S t + R_n \left( \frac{t}{\sqrt{n}} + \Delta_n \right) \right\}. \quad (36)$$

Since $|\Delta_n - \theta_0| = o_P(1)$, it follows that with probability arbitrarily close to 1, for $n$ sufficiently large,

$$\left| \Delta_n + \frac{t}{\sqrt{n}} - \theta_0 \right| < 2\delta' \text{ for all } |t| \leq \delta' \sqrt{n}. $$

Thus, by Condition (R.4), there exists some small $\delta'$ and large $M$ such that the latter inequality implies

$$P\left[ \sup_{M \leq |t| \leq \delta' \sqrt{n}} \left| R_n \left( \frac{t}{\sqrt{n}} + \Delta_n \right) \right| \leq \frac{1}{4} \min_{eig}(S) \right] \geq 1 - \epsilon.$$

Combining this fact with (34), we see that (36), for some $C > 0$, is

$$\exp\{\omega(t)\} \leq C \exp\left\{ -\frac{1}{2} t^T S t \right\}, \quad (37)$$

for all $t$ satisfying (ii), with probability arbitrarily close to 1, and this establishes (35).

Part(iii):

$$\int_{|t| \geq \delta \sqrt{n}} \left| \pi \left( \frac{t}{\sqrt{n}} + \Delta_n \right) e^{\omega(t)} - \pi (\theta_0) e^{-\frac{1}{2} t^T S t} \right| \ dt \xrightarrow{P} 0.$$

As in (ii), the second term in the integrand can be neglected. Therefore we only need to show

$$\int_{|t| \geq \delta \sqrt{n}} \pi \left( \frac{t}{\sqrt{n}} + \Delta_n \right) e^{\omega(t)} \ dt \xrightarrow{P} 0.$$

Recalling the definition of $t$, the term is bounded by

$$\int_{|\alpha - \Delta_n| \geq \delta} \pi (\alpha) \exp\left\{ n P_n \left( h(\theta, \alpha) - n P_n h(\theta, \theta_0) - \frac{n}{2} U_n(\theta_0)^T S^{-1} U_n(\theta_0) \right) \right\} \ d\alpha.$$
By (R.4), for any \( \delta > 0 \), there exists \( \epsilon > 0 \), such that
\[
\sup_{|\alpha - \theta_0| \geq \delta} \frac{1}{P_n h (\theta, \alpha) - P_n h (\theta, \theta_0)} \leq -\epsilon.
\] (38)

Since \( \Delta_n \overset{P}{\to} \theta_0 \), therefore with probability tending to 1, there exists \( \epsilon \) such that
\[
\sup_{|\alpha - \Delta_n| \geq \delta} \frac{1}{\exp \{ nP_n h (\theta, \alpha) - nP_n h (\theta, \theta_0) \}} \leq e^{-n\epsilon}.
\]

Since \( \exp \{ -\frac{n}{2} U_n(\theta_0) \top S^{-1} U_n(\theta_0) \} = O_P(1) \), the entire term is bounded by
\[
C \sqrt{n} e^{-n\epsilon} \int \pi (\alpha) \ d\alpha = o_P(1),
\]
with probability tending to 1.

The entire proof is now completed by combining all terms.

5.2 Proof of Theorem 2

We have
\[
V^{-1/2} \sqrt{n} (\tilde{\alpha}_\phi^* (\theta) - \theta_0) = V^{-1/2} \sqrt{n} (\tilde{\alpha}_\phi^* (\theta) - \Delta_n) + V^{-1/2} \sqrt{n} (\Delta_n - \theta_0).
\]

By the CLT, the second term has the limit distribution \( \mathcal{N} (0, I) \), so that it only remains to show that
\[
\sqrt{n} (\tilde{\alpha}_\phi^* (\theta) - \Delta_n) \overset{P}{\to} 0.
\] (39)

Observe that
\[
\tilde{\alpha}_\phi^* (\theta) = \int \alpha p_{\phi, n}(\alpha) \ d\alpha
\]
\[
= \int \left( \frac{t}{\sqrt{n}} + \Delta_n \right) p_{\phi, n}^*(t) \ dt
\]
\[
= \frac{1}{\sqrt{n}} \int t p_{\phi, n}^*(t) \ dt + \Delta_n,
\]
and hence
\[
\sqrt{n} (\tilde{\alpha}_\phi^* (\theta) - \Delta_n) = \int t p_{\phi, n}^*(t) \ dt.
\] (40)

Thus,
\[
\sqrt{n} |\tilde{\alpha}_\phi^* (\theta) - \Delta_n| = \left| \int t p_{\phi, n}^*(t) \ dt - \int t \left( \frac{\det S}{2\pi} \right)^{d/2} \exp \left\{ -\frac{1}{2} t^\top S t \right\} \ dt \right|
\]
\[
\leq \int |t| \left| p_{\phi, n}^*(t) \ dt - \left( \frac{\det S}{2\pi} \right)^{d/2} \exp \left\{ -\frac{1}{2} t^\top S t \right\} \right| \ dt,
\]
which tends to zero in probability by Theorem 1.
References

Bouzebda, S. and Cherfi, M. (2011). General bootstrap for dual $\phi$-divergences estimates. *Journal of Probability and Statistics*, In Press, Accepted Manuscript.

Bouzebda, S. and Keziou, A. (2010). Estimation and tests of independence in copula models via divergences. *Kybernetika*, 46(1), 178–201.

Broniatowski, M. and Keziou, A. (2006). Minimization of $\phi$-divergences on sets of signed measures. *Studia Sci. Math. Hungar.*, 43(4), 403–442.

Broniatowski, M. and Keziou, A. (2009). Parametric estimation and tests through divergences and the duality technique. *J. Multivariate Anal.*, 100(1), 16–36.

Broniatowski, M. and Vajda, I. (2009). Several applications of divergence criteria in continuous families. Technical Report 2257, Academy of Sciences of the Czech Republic, Institute of Information Theory and Automation.

Cherfi, M. (2011a). Dual divergences estimation for censored survival data. *Arxiv preprint arXiv:1106.2627*.

Cherfi, M. (2011b). Dual $\phi$-divergences estimation in normal models. *Arxiv preprint arXiv:1108.2999*.

Chernozhukov V. and Hong. H. (2003). An MCMC Approach to Classical Estimation. *Journal of Econometrics*, 115, 293–346.

Cressie, N. and Read, T. R. C. (1984). Multinomial goodness-of-fit tests. *J. Roy. Statist. Soc. Ser. B*, 46(3), 440–464.

Dey, D. K. and Birmiwal, L. R. (1994). Robust Bayesian analysis using divergence measures. *Statist. Probab. Lett.*, 20(4), 287–294.

Hanousek, J. (1990). Robust Bayesian type estimators and their asymptotic representation. *Statist. Decisions*, 8(1), 61–69.

Hanousek, J. (1994). Generalized Bayesian-type estimators. Robust and sensitivity analysis. *Kybernetika (Prague)*, 30(3), 271–278.

Hooker, G. and Vidyashankar, A. (2011). Bayesian Model Robustness via Disparities.

Ibragimov, I. A. and Has’minskii, R. Z. (1981). *Statistical Estimation — Asymptotic Theory*. Springer-Verlag, New York.
Keziou, A. (2003). Dual representation of $\phi$-divergences and applications. *C. R. Math. Acad. Sci. Paris*, **336**(10), 857–862.

Keziou, A. and Leoni-Aubin, S. (2008). On empirical likelihood for semiparametric two-sample density ratio models. *J. Statist. Plann. Inference*, **138**(4), 915–928.

Lehmann, E. L. and Casella, G. (1998). *Theory of point estimation*. Springer Texts in Statistics. Springer-Verlag, New York, second edition.

Liese, F. and Vajda, I. (1987). *Convex statistical distances*, volume 95 of *Teubner-Texte zur Mathematik [Teubner Texts in Mathematics]*. BSB B. G. Teubner Verlagsgesellschaft, Leipzig. With German, French and Russian summaries.

Liese, F. and Vajda, I. (2006). On divergences and informations in statistics and information theory. *IEEE Trans. Inform. Theory*, **52**(10), 4394–4412.

Lindsay, B. G. (1994). Efficiency versus robustness: the case for minimum Hellinger distance and related methods. *Ann. Statist.*, **22**(2), 1081–1114.

Liu, J. S. Tian, L. and Wei, L. J. (2008). Implementation of estimating-function based inference procedures with MCMC samplers. *Journal of American Statistical Association*.

Pardo, L. (2006). *Statistical inference based on divergence measures*, volume 185 of *Statistics: Textbooks and Monographs*. Chapman & Hall/CRC, Boca Raton, FL.

Peng, F. and Dey, D. (1995). Bayesian analysis of outlier problems using divergence measures. *Canadian Journal of Statistics*, **23**(2), 199–213.

Robert, C. (2001). *The Bayesian choice: from decision-theoretic foundations to computational implementations*, Springer, New York.

Strasser, H. (1981). Consistency of maximum likelihood and Bayes estimates. *Ann. Statist.*, **9**, 1107–1113.

Toma, A. and Broniatowski, M. (2010). Dual divergence estimators and tests: robustness results. *Journal of Multivariate Analysis*, **102**(1), 20–36.

Toma, A. and Leoni-Aubin, S. (2010). Robust tests based on dual divergence estimators and saddlepoint approximations. *Journal of Multivariate Analysis*, **101**(5), 1143–1155.

van der Vaart, A. W. (1998). *Asymptotic statistics*, volume 3 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge.