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TOEPLITZ OPERATORS AND SOLVABLE C*-ALGEBRAS
ON HERMITIAN SYMMETRIC SPACES

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Bounded symmetric domains (Cartan domains and exceptional domains) are higher-dimensional generalizations of the open unit disc. In this note we give a structure theory for the C*-algebra $\mathcal{T}$ generated by all Toeplitz operators $T_f(h) := P(fh)$ with continuous symbol function $f \in C(S)$ on the Shilov boundary $S$ of a bounded symmetric domain $D$ of arbitrary rank $r$. Here $h$ belongs to the Hardy space $H^2(S)$, and $P : L^2(S) \rightarrow H^2(S)$ is the Szegö projection. For domains of rank 1 and tube domains of rank 2, the structure of $\mathcal{T}$ has been determined in [1, 2]. In these cases Toeplitz operators are closely related to pseudodifferential operators. For the open unit disc, $\mathcal{T}$ is the C*-algebra generated by the unilateral shift.

The structure theory for the general case [12] is based on the fact that $D$ can be realized as the open unit ball of a unique Jordan triple system $Z$ [7, Theorem 4.1]. Denoting the Jordan triple product by $\{uv^*w\}$, a tripotent $e \in Z$ satisfies $\{ee^*e\} = e$. Tripotents generalize the partial isometries of matrix algebras and determine the boundary structure of $D \subset Z$ (cf. [7, Theorem 6.3]). Our principal result ([12]; cf. also [3, 4, 8]) is the following;

**THEOREM 1.** The Toeplitz C*-algebra $\mathcal{T}$ associated with a bounded symmetric domain $D \subset Z$ of rank $r$ is solvable of length $r$, i.e. there exists a chain

$$\{0\} = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_r \subset I_{r+1} = \mathcal{T}$$

of closed two-sided ideals $I_k$ such that for $0 \leq k \leq r$ there is a C*-algebra isomorphism ("$k$-symbol")

$$\sigma_k : I_{k+1}/I_k \rightarrow C(S_k) \otimes K(H_k),$$

where $S_k$ denotes the compact manifold of all tripotents $e \in Z$ of rank $k$ and $K(H_k)$ denotes the C*-algebra of all compact operators on a Hilbert space $H_k$. Further, $\dim(H_k) = \infty$ for $k < r$ and $\dim(H_r) = 1$. 

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COROLLARY. The spectrum of \( \mathcal{T} \) can be identified with the set of all tripotents of \( Z \). The ideal \( I_r \) is the closed commutator ideal of \( \mathcal{T} \) and \( \mathcal{T}/I_r \cong C(S) \), where \( S = S_r \) is the Shilov boundary. Further, \( I_1 = K(H^2(S)) \).

The proof of Theorem 1 is based on a detailed study of the harmonic analysis in \( H^2(S) \) [10] and of the fine structure of single Toeplitz operators [11]. Since the Toeplitz \( C^* \)-algebra \( \mathcal{T} \) associated with a reducible bounded symmetric domain \( D \) can be realized as a tensor product, we may assume that \( D \) is irreducible. Let \( \mathcal{P}(Z) \) denote the polynomial algebra on \( Z \) and let \( K \) be the largest connected group of biholomorphic automorphisms of \( D \) fixing the origin.

The next result [10], based on ideas from [6], applies to domains equivalent to a tube domain (generalized upper half-plane). In this case the Jordan triple system \( Z \) is actually a unital Jordan algebra.

**THEOREM 2.** Suppose the domain \( D \) is of tube type. Then
\[
\mathcal{P}(Z) \approx C[N] \otimes \mathcal{H}(Z),
\]
where \( N \) denotes the norm function ("generalized determinant") of the Jordan algebra \( Z \) and \( \mathcal{H}(Z) \) is the space of all harmonic polynomials (for the commutator subgroup of \( K \)).

In order to apply Theorem 2 to a general domain \( D \subset Z \), consider for \( 1 \leq k \leq r \) the Jordan algebra \( Z_k := \{z \in Z : \{ee^*z\} = z\} \) of rank \( k \) with unit element \( e := e_{r+1-k} + \cdots + e_r \), where \( \{e_1, \ldots, e_r\} \) denotes a frame of orthogonal minimal tripotents of the Jordan triple system \( Z \) [7, §5]. Denote by \( N_k \) the norm function of \( Z_k \), viewed as a polynomial on \( Z \). The Peter-Weyl decomposition of \( H^2(S) \), determined in [9] and described case by case in [5], can now be realized as follows [10]:

**THEOREM 3.** The irreducible \( K \)-module \( E_m \subset \mathcal{P}(Z) \) with signature \( m_1 \geq m_2 \geq \cdots \geq m_r \geq 0 \) is generated by the conical polynomial \( N_m = N_1^{l_1} N_2^{l_2} \cdots N_r^{l_r} \), where \( m_k = l_k + \cdots + l_r \) for all \( k \).

The \( K \)-invariant scalar product \( (u|v) \) on \( Z \) given by the generic trace [7, 4.15] induces a differential scalar product \( (p|q)_Z \) for polynomials \( p, q \in \mathcal{P}(Z) \) [6, III.1]. Let \( (\ |\ )_S \) be the integral scalar product in \( H^2(S) \). Using integral formulas for semisimple Lie groups, the relationship between these \( K \)-invariant scalar products can be computed explicitly [11]. Let \( (r,s,t) \) denote the type of \( D \), defined via the Peirce decomposition of \( Z \) [7, Theorem 3.14].

**THEOREM 4.** For every signature \( m = (m_1, \ldots, m_r) \) and all \( p, q \in E_m \), we have
\[
\frac{(p|q)_Z}{(p|q)_S} = \prod_{j=1}^r \frac{(m_j + \frac{1}{2}s(r-j) + t)!}{(\frac{1}{2}s(r-j) + t)!}.
\]

As a consequence of Theorem 4, the fine structure of "polynomial" Toeplitz operators (generating \( \mathcal{T} \)) can be related to polynomial differential operators \( h(z)(\partial/\partial z) \), where \( z \) denotes the "coordinate" of \( Z \) (cf. [11]):
THEOREM 5. Suppose \( l(z) = (z|^v) \) is a linear form. Then
\[
T^*_l(p) = \sum_{j=1}^{r} \left( m_j + s \left( \frac{r-j}{2} + t \right) \right)^{-1} \left( \left( v \frac{\partial}{\partial z} \right) p \right)_{m-\varepsilon_j},
\]
\[
T_l(p) = \sum_{j=1}^{r} \left( m_j - s \left( \frac{j-1}{2} \right) \right)^{-1} \left( \left( zv^* z \right) \frac{\partial}{\partial z} \right) p_{m+\varepsilon_j},
\]
for all \( p \in E_m \), the subscript denoting the Peter-Weyl component for signature
\[
m \pm \varepsilon_j = (m_1, \ldots, m_{j-1}, m_j \pm 1, m_{j+1}, \ldots, m_r).
\]

COROLLARY. The commutator \([T^*_l, T_l]\) is a "diagonal" operator respecting the Peter-Weyl decomposition of \( H^2(S) \).

Theorem 5 enables us to construct the irreducible representations of the Toeplitz \( C^* \)-algebra \( T \) [11]. For a tripotent \( e \in Z \), the Jordan triple system \( Z_e := \{ w \in Z : \{ e w e \w = 0 \} \} \) contains the bounded symmetric domain \( D \cap Z_e \) with Shilov boundary \( S_e \). For \( f \in C(S) \) define \( f_e \in C(S_e) \) by \( f_e(w) := f(e+w) \).

Consider the "peaking functions"
\[
h^i_e(z) := c_i(\exp(z|e))^i
\]
for \( i \geq 0 \), where \( c_i > 0 \) is a constant such that \( \|h^i_e\| = 1 \).

THEOREM 6. For each tripotent \( e \in Z \) there exists an irreducible representation \( \sigma_e (\"e-symbol\") \) of \( T \) on the Hardy space \( H^2(S_e) \), such that \( \sigma_e(T_f) = T^*_f e \) for all \( f \in C(S) \) and
\[
\lim_{i \to -\infty} \|A(h^i_e \cdot q) - h^n_e \sigma_e(A)q\| = 0
\]
for all \( q \in P(Z_e) \) and all operators \( A \) in a dense *-subalgebra of \( T \).

For \( 0 \leq k \leq r \) let \( I_k \subset \mathcal{T} \) be the joint kernel of all \( e \)-symbol homomorphisms \( \sigma_e \) for tripotents \( e \in S_k \) of rank \( k \). Theorem 1 now follows from the fact that \( I_1 \) consists of compact operators. More generally, the ideals \( I_k \) have an internal characterization [12]:

THEOREM 7. For \( 0 \leq k \leq r \) let \( P_k \) denote the orthogonal projection from \( H^2(S) \) onto the Hilbert sum of all \( K \)-modules \( E_m \) satisfying \( m_{k+1} = \cdots = m_r = 0 \). Then \( I_{k+1} \) is the \( C^* \)-algebra generated by all operators \( T^*_p P_k T_q \) for \( p, q \in P(Z) \).

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