CP violations in $\pi^\pm$ Meson Decay

Gorazd Cvetič\textsuperscript{1}, C. S. Kim\textsuperscript{2}\textsuperscript{[1]} and Jilberto Zamora-Saá\textsuperscript{1}

\textsuperscript{1}Department of Physics, Universidad Técnica Federico Santa María, Valparaíso, Chile
\textsuperscript{2}Department of Physics and IPAP, Yonsei University, Seoul 120-749, Korea

(Dated: April 1, 2014)

We study the pion decays with intermediate on-shell neutrinos $N$ into two electrons and a muon, $\pi^\pm \to e^\pm N \to e^\pm e^\mp \mu^\pm \nu$. We investigate the branching ratios $Br_\pm = [\Gamma(\pi^- \to e^- e^\mp \mu^\pm \nu) \pm \Gamma(\pi^+ \to e^+ e^\mp \mu^\pm \nu)]/\Gamma(\pi^\pm \to e^\pm e^\mp \mu^\pm \nu)$ and the CP asymmetry ratio $A_{CP} = Br_-/Br_+$ for such decays, in the scenario with two different on-shell neutrinos. If $N$ is Dirac, only the lepton number conserving (LC) decays contribute (LC: $\nu_1 = \nu_\chi$ or $\bar{\nu}_\chi$); if $N$ is Majorana, both LC and lepton number violating (LV) decays contribute (LV: $\nu = \bar{\nu}_\mu$ or $\nu = {\nu}_\mu$). The results show that the CP asymmetry $A_{CP}$ is in general very small, but increases and becomes $\sim 1$ when the masses of the two intermediate neutrinos get closer to each other, i.e., when their mass difference becomes comparable with their decay width, $\Delta M_N \gg \Gamma_N$. The observation of CP violation in pion decays would be consistent with the existence of the well-motivated $\nu$MSM model with two almost degenerate heavy neutrinos.

PACS numbers: 14.60St, 11.30Er, 13.20Cz

I. INTRODUCTION

One of the outstanding issues in neutrino physics today is to clarify the Dirac or Majorana character of neutrino masses. If neutrinos are Dirac particles, they must have right-handed electroweak singlet components in addition to the known left-handed modes; in such case lepton number remains as a conserved quantity. Alternatively, if they are Majorana particles, they are indistinguishable from their antiparticles, and the lepton number in the reactions involving them may be violated. The nature of the neutrinos can be discerned via detection of neutrinoless double beta decays ($0\nu\beta\beta$) in nuclei\textsuperscript{[1]}, by considering specific scattering processes\textsuperscript{[2]}, or by studying rare meson decays\textsuperscript{[3]}\textsuperscript{[4]}. The experimental results to date are unable to distinguish between these two alternatives.

Among the principal tasks in neutrino physics are the ascertainment of the nature of the neutrino mass (Dirac or Majorana) and the CP violation in the neutrino sector. The measurement of neutrino oscillations\textsuperscript{[5]}\textsuperscript{[8]} suggests that the first three neutrinos are not massless but very light particles, with masses less than 1 eV. If these light masses are produced via a seesaw\textsuperscript{[9]} or related mechanism, then the existence of significantly heavier neutrinos is expected. Furthermore, there is a possibility of CP violation in the neutrino sector, both if neutrinos are Dirac or Majorana particles. In the Majorana case, though, the number of possible CP-violating phases in the PMNS matrix is larger.

If $n$ is the number of neutrino generations, the number of CP-violating phases is $(n-1)(n-2)/2$ in the Dirac case, and $n(n-1)/2$ in the Majorana case, cf. Ref.\textsuperscript{[10]}.

In this work, we investigate the possibility of measuring the CP asymmetry in the rare pion decays: The CP violation in the neutrino sector can be measured by neutrino oscillations\textsuperscript{[11]}. However, here we consider a scenario in which CP violation of the neutrino sector can be measured by investigating rare meson decays. We consider a scenario of two additional, sterile, almost degenerate neutrinos $N_j$ ($j = 1, 2$) with masses $M_N \sim 10^2$ MeV. Such neutrinos are not typically predicted by seesaw scenarios; nonetheless, there are models which predict such neutrinos\textsuperscript{[12]}\textsuperscript{[13]}, and they are not ruled out by experiments\textsuperscript{[14]}\textsuperscript{[15]}.

We note that the model, $\nu$MSM\textsuperscript{[12]}\textsuperscript{[13]}, proposes two almost degenerate Majorana neutrinos with mass between 100 MeV and a few GeV, in addition to a light Majorana neutrino of mass $\sim 10^1$ keV. The existence of such neutrinos is strongly motivated, because it can explain simultaneously the baryon asymmetry of the Universe, the pattern of light neutrino masses and oscillations, and can provide a dark matter candidate – cf.\textsuperscript{[16]} for a review, and\textsuperscript{[17]} for the allowed range of the sterile neutrinos in $\nu$MSM.\textsuperscript{[1]} The requirement that the lightest sterile neutrino be the dark matter candidate reduces the parameters of the model in such a way as to make the two heavier neutrinos nearly degenerate in mass.

Recently CERN-SPS has proposed a search of such heavy neutrinos, Ref.\textsuperscript{[19]}, in the decays of $D$, $D_s$ mesons. We

\textsuperscript{1} Electronic address: cskim@yonsei.ac.kr

\textsuperscript{[1]} The tentative evidence of a dark matter line, recently discussed in\textsuperscript{[18]}, is well within the regime predicted in $\nu$MSM in\textsuperscript{[17]}. We thank Marco Drewes for bringing this point to our attention.
are interested in the question whether in such models the CP violation in rare pion decays can be appreciable to cover the parameter space favored by theoretical models.

We investigate the rare decays of charged pions into three charged leptons and a light neutrino, with the two intermediate neutrinos $N_j$ in the decay being on-shell, and we look for a possibility of detection of CP asymmetries in such decays. The relevant processes are the lepton number conserving (LC) processes $\pi^\pm \rightarrow e^\pm N_j \rightarrow e^\pm e^\pm \mu^\mp\nu$ where $\nu = \nu_e$ for $\pi^+$ and $\nu = \bar{\nu}_e$ for $\pi^-$; and the lepton number violating (LV) processes, where $\nu = \bar{\nu}_\mu$ for $\pi^+$ and $\nu = \nu_\mu$ for $\pi^-$. If the $N_j$ neutrinos are Dirac, only LC decays contribute. If they are Majorana, both LC and LV decays contribute. In our previous work [4], we demonstrated that the decay branching ratios for these processes are very small but can be appreciable and could be measured in the future $\pi$ factories where huge numbers of pions will be produced, if the heavy-light neutrino mixing parameters are sufficiently large but still below the present upper bounds. Moreover, we showed that the consideration of the muon spectrum of these decays may allow us to distinguish whether the intermediate neutrinos are Dirac or Majorana.

We will investigate the branching ratios $Br_{X} \equiv \Gamma(\pi^X \rightarrow e^\pm e^\pm \mu^\mp\nu)/\Gamma(\pi \rightarrow \text{all})$ and the CP asymmetry ratio $A_{CP} \equiv Br_\mp/Br_+$ of the mentioned rare processes in the scenario of two intermediate on-shell neutrinos. We demonstrate that there exist scenarios where this CP asymmetry can be detected. In Sec. II we outline the formalism for the calculation of the various decay widths and branching ratios. The details of the calculation are given in Appendix A. In Sec. III we derive the expressions for the branching ratios $Br_{X}$ and for the CP asymmetry ratio $A_{CP}$, and present the numerical results. Additional details are given in Appendix B. In Sec. IV we present the conclusions.

II. THE PROCESSES AND FORMALISM FOR THE RARE PION DECAYS

We consider the lepton number violating (LV) process, Fig. 1, and the lepton number conserving (LC) process, Fig. 2. We note that if the intermediate neutrinos $N_j$ ($j = 1, 2$) are Majorana, both processes (LV and LC) take place; and if $N_j$ are Dirac, only the LC process takes place.

We will denote the mixing coefficient between the standard flavor neutrino $\nu_\ell$ ($\ell = e, \mu, \tau$) and the heavy mass eigenstate $N_j$ as $B_{\ell N_j}$ ($j = 1, 2$), i.e., this mixing element appears in the relation

$$\nu_\ell = \sum_{k=1}^{3} B_{\ell N_k} \nu_k + (B_{\ell N_1} N_1 + B_{\ell N_2} N_2) ,$$

where $\nu_k$ ($k = 1, 2, 3$) are the light mass eigenstates. We adopt the phase conventions of the book Ref. [10], i.e., all the CP-violating phases are incorporated in the PMNS matrix of mixing elements. The decay widths and asymmetries of these processes may become appreciable only if the two intermediate neutrinos $N_j$ are on-shell, i.e., if

$$(M_\mu + M_e) < M_{N_j} < (M_\mu - M_e) ,$$

i.e., when the masses $M_{N_j}$ are within the interval (106.2 MeV, 139 MeV).

From now on, unless otherwise stated, we will use the simplified notations for the decay widths of these rare processes:

$$\Gamma^{(X)}(\pi^\pm) \equiv \Gamma^{(X)}(\pi^\pm \rightarrow e^\pm e^\pm \mu^\mp\nu) , \quad (X = \text{LV, LC}) .$$
The decay widths $\Gamma^{(X)}(\pi^{\pm})$ can be written in the form

$$\Gamma^{(X)}(\pi^{\pm}) = \frac{1}{2} \frac{1}{2M_\pi} \frac{1}{(2\pi)^8} \int d_4 |T^{(X)}(\pi^{\pm})|^2 ,$$  

(4)

where $1/2!$ is the symmetry factor due to two final state electrons, and $d_4$ denotes the integration over the 4-particle final phase space

$$d_4 = \left( \prod_{j=1}^2 \frac{d^3\vec{p}_j}{2E_{e_j}(\vec{p}_j)} \right) \left( \prod_{j=1}^2 \frac{d^3\vec{p}_\mu}{2E_{\mu_j}(\vec{p}_\mu_j)} \right) \delta^{(4)}(p_\pi - p_1 - p_2 - p_\mu - p_\nu) ,$$  

(5)

and we denoted by $p_1$ and $p_2$ the momenta of $e^+$ from the left and the right vertex of the direct channels, respectively (and for the crossed channels just the opposite). The squared matrix element $|T^{(X)}(\pi^{\pm})|^2$ in Eq. (4) is a combination of contributions from $N_1$ and $N_2$ and from the two channels $D$ (direct) and $C$ (crossed), and is given explicitly in Eq. (A1) in Appendix A.

Combining Eqs. (4) and (A1), we obtain

$$\Gamma^{(X)}(\pi^{\pm}) = \sum_{i,j=1}^2 k_{i,j}^{(X)} \Gamma^{(X)}(DD^*)_{ij} + \Gamma^{(X)}(CC^*)_{ij} + \Gamma^{(X)}(DC^*)_{ij} + \Gamma^{(X)}(CD^*)_{ij} ,$$  

(6)

where $X = \text{LV, LC}$; the indices $(i, j)$ indicate contributions from $N_i$ and $N_j$ neutrino exchange amplitudes; and $k_{i,j}^{(X)}$ are the corresponding heavy-light mixing factors

$$k_{i,j}^{(LV)} = B_{eN_i}^2 , \quad k_{i,j}^{(LC)} = B_{eN_i} B_{N_j}^* , \quad k_{i,j}^{(X)} = \left( k_{i,j}^{(X)} \right)^* .$$  

(7)

In Eq. (6) we denoted by $\Gamma^{(X)}(YZ^*)_{ij}$ $(i, j = 1, 2)$ the elements of the normalized (i.e., without mixings) decay width matrices $\Gamma^{(X)}(YZ^*)$ $(X = \text{LV, LC}; Y, Z = D, C)$

$$\Gamma^{(X)}(XY^*)_{ij} = K^2 \frac{1}{2!} \frac{1}{2M_\pi} \frac{1}{(2\pi)^8} \int d_4 P_j^{(X)}(Y) P_j^{(X)}(Z)^* T^{(X)}_{ij}(Y^*) ,$$  

(8)

where the expressions for $T^{(X)}(Y^*)$ (with $X = \text{LV, LC}$) for the direct $(YZ^* = DD^*)$, crossed $(YZ^* = CC^*)$ and direct-crossed interference $(Y^* = DC^*, CD^*)$ appearing in Eq. (8) are given in Appendix A Eqs. (A2)-(A4). We note that $T^{(X)}(DD^*) = T^{(X)}_{-}(DD^*)$ and $T^{(X)}(CC^*) = T^{(X)}_{+}(CC^*)$, so that the terms $\Gamma^{(X)}(DD^*)_{ij}$ and $\Gamma^{(X)}(CC^*)_{ij}$ in Eq. (6) have no subscripts $\pm$. In Eq. (8), $P_j^{(X)}(Y)$ $(X = \text{LV, LC})$ represent the $N_j$ propagator functions of the direct and crossed channels $(Y = D, C)$

$$P_j^{(LC)}(D) = \frac{1}{(p_\pi - p_1)^2 - M_{N_j}^2 + i\Gamma_{N_j} M_{N_j}} , \quad P_j^{(LV)}(D) = M_{N_j} P_j^{(LC)}(D) ,$$  

(9a)

$$P_j^{(LC)}(C) = \frac{1}{(p_\pi - p_2)^2 - M_{N_j}^2 + i\Gamma_{N_j} M_{N_j}} , \quad P_j^{(LV)}(C) = M_{N_j} P_j^{(LC)}(C) ,$$  

(9b)

and $K^2$ constant is

$$K^2 = G_F^2 f_\pi^2 |V_{ud}|^2 \approx 2.983 \times 10^{-22} \text{ GeV}^{-6} .$$  

(10)

Several symmetry relations are valid between the normalized matrices (8), cf. Eqs. (A5) in Appendix A; the most important is that $\Gamma^{(X)}(DD^*) = \Gamma^{(X)}(CC^*)$ and that this $(2 \times 2)$ matrix is self-adjoint. Later we will see that the direct-crossed interference contributions $\Gamma^{(X)}(DC^*), \Gamma^{(X)}(CD^*)$ are suppressed by several orders of magnitude in comparison to $\Gamma^{(X)}(DD^*)$.

The branching ratios are obtained by dividing the calculated decay widths $\Gamma^{(X)}(\pi^{\pm})$, Eqs. (5)-(4) and (6), by the total decay width of the charged pion $\Gamma(\pi^{\pm} \rightarrow \text{all})$

$$\Gamma(\pi^{\pm} \rightarrow \text{all}) = 2.529 \times 10^{-17} \text{ GeV} \approx \frac{1}{8\pi} G_F^2 f_\pi^2 M_\pi^2 |V_{ud}|^2 \left( 1 - \frac{M_{N}^2}{M_{\pi}^2} \right)^2 .$$  

(11)
Another important quantity in the evaluations of $\Gamma^{(X)}(YZ)$ (and $\text{Br}^{(X)}(YZ)$) is the total decay width $\Gamma_{N_j}$ of the intermediate on-shell neutrinos, which for the mass range of interest [Eq. (2)] can be approximated in the following way:

$$\Gamma_{N_j} \approx C \bar{K}_j \Gamma(M_{N_j}),$$  

where

$$\Gamma(M_{N_j}) \equiv \frac{G_F^2 M_{N_j}^5}{192\pi^3},$$  

and $C = 2$ if $N_j$ is Majorana neutrino, and $C = 1$ if $N_j$ is Dirac neutrino. The factor $\bar{K}_j$ includes the heavy-light mixing factors dependence, from the charged channels and the neutral interaction channels mediated by Z. Using the results of Appendix C of Ref. [15], the factor $\bar{K}_j$ can be obtained

$$\bar{K}_j \approx 1.6 |B_{\ell N_j}|^2 + 1.1 (|B_{\mu N_j}|^2 + |B_{\tau N_j}|^2), \quad (j = 1, 2).$$  

The charged and neutral channel contributions produce only (light) neutrinos and $e^+e^-$; decays with muon in the final state are suppressed by a kinematical factor $f(M_{\mu}^2/M_{N_j}^2) < 10^{-2}$ and are neglected in the formula (14).²

### III. THE BRANCHING RATIOS AND THE CP ASYMMETRY FOR THE RARE DECAYS

In this Section we use the results of the previous Section to obtain the results for the branching ratios $\text{Br}_{\pm}^{(X)}$ and the CP asymmetry ratios $A_{CP}^{(X)} (X = LP, LC)$ of the discussed rare processes

$$\text{Br}_{\pm}^{(X)} = \frac{S_{\pm}^{(X)}(\pi)}{\Gamma^{(\pi^+ \rightarrow all)}(\pi) \pm \Gamma^{(\pi^- \rightarrow all)}(\pi)} = \frac{\Gamma^{(X)}(\pi^-) \pm \Gamma^{(X)}(\pi^+)}{\Gamma^{(\pi^- \rightarrow all)}(\pi) \pm \Gamma^{(\pi^\pm \rightarrow all)}(\pi)},$$  

$$A_{CP}^{(X)} = \frac{\text{Br}_{\pm}^{(X)} - \text{Br}_{\mp}^{(X)}}{\text{Br}_{\pm}^{(X)} + \text{Br}_{\mp}^{(X)}} = \frac{\Gamma^{(X)}(\pi^-) - \Gamma^{(X)}(\pi^+)}{\Gamma^{(X)}(\pi^-) + \Gamma^{(X)}(\pi^+)},$$  

where we recall the use of notations [3]. The total branching ratios are $\text{Br}_{\pm} = \text{Br}_{\pm}^{(LV)} + \text{Br}_{\pm}^{(LC)}$ when $N_j$ are Majorana neutrinos, and $\text{Br}_{\pm} = \text{Br}_{\pm}^{(LC)}$ when $N_j$ are Dirac neutrinos. It is useful to introduce the following notations related with the heavy-light neutrino mixing elements $B_{\ell N_j}$ and $B_{\mu N_j}$, where we adopt the convention $M_{N_2} > M_{N_1}$:

$$\kappa_\ell = \frac{|B_{\ell N_2}|}{|B_{\ell N_1}|}, \quad \kappa_\mu = \frac{|B_{\mu N_2}|}{|B_{\mu N_1}|}, \quad (17a)$$

$$B_{e N_j} = |B_{e N_j}|e^{i\theta_{e_j}}, \quad B_{\mu N_j} = |B_{\mu N_j}|e^{i\theta_{\mu_j}}, \quad (17b)$$

$$\theta^{(LV)} = 2(\theta_{e_2} - \theta_{e_1}), \quad \theta^{(LC)} = (\theta_{e_2} - \theta_{e_1}) - (\theta_{\mu_2} - \theta_{\mu_1}).$$  

It turns out (see later) that in our cases of interest the D-C interference contributions are negligible, and the resulting sums $S_{\pm}^{(X)}(\pi)$ of the decay widths are

$$S_{\pm}^{(LV)}(\pi) = \left[ \Gamma^{(LV)}(\pi^-) + \Gamma^{(LV)}(\pi^+) \right]$$

$$= 4|B_{e N_1}|^2 \left[ \Gamma^{(LV)}(DD^*)_{11} \left[ 1 + \kappa_\ell^2 \Gamma^{(LV)}(DD^*)_{22} + 2\kappa_\ell \cos \theta^{(LV)} \right] \delta_{1}^{(LV)} \right],$$  

$$S_{\pm}^{(LC)}(\pi) = \left[ \Gamma^{(LC)}(\pi^-) + \Gamma^{(LC)}(\pi^+) \right]$$

$$= 4|B_{e N_1}|^2 |B_{\mu N_1}|^2 \left[ \Gamma^{(LC)}(DD^*)_{11} \left[ 1 + \kappa_\ell^2 \kappa_\mu^2 \Gamma^{(LC)}(DD^*)_{22} + 2\kappa_\ell \kappa_\mu \cos \theta^{(LC)} \right] \delta_{1}^{(LC)} \right],$$  

² Eq. (14) is obtained by using Eqs. (C.6)-(C.9) of Ref. [15], for the channels $N_j \rightarrow e^+e^-\nu_\ell$, $\nu_\ell\bar{\nu}_\mu\nu_\mu$. The coefficients in the corresponding formula (2.3) of Ref. [2] are not correct.
where $\delta_j^{(X)}$ in the above quantities represent the (relative) contribution of the $N_1N_2$ interference channel

$$\delta_j^{(X)} \equiv \frac{\text{Re} \Gamma_j^{(X)}(DD^*)_{12}}{\Gamma_j^{(X)}(DD^*)_{jj}}, \quad (X = LV, LC; \; j = 1, 2) . \quad (19)$$

On the other hand, the difference $S_{-}^{(X)}(\pi)$ of the $\pi^-$ and $\pi^+$ rare decays is (where the $D-C$ interference terms are neglected)

$$S_{-}^{(LV)}(\pi) \equiv \left( \Gamma^{(LV)}(\pi^+) - \Gamma^{(LV)}(\pi^-) \right) = 8|B_{eN_1}|^2 \kappa^2 e^{-\sin \theta^{(LV)}} \text{Im} \chi^{(LV)}(DD^*)_{12} , \quad (20a)$$

$$S_{-}^{(LC)}(\pi) \equiv \left( \Gamma^{(LC)}(\pi^+) - \Gamma^{(LC)}(\pi^-) \right) = 8|B_{eN_1}|^2 |B_{\mu N_1}|^2 \kappa_e \kappa_\mu \sin \theta^{(LC)} \text{Im} \chi^{(LC)}(DD^*)_{12} . \quad (20b)$$

In these expressions we can recognize (a posteriori) the difference of the CP-odd phases as contained in the imaginary part of the product of propagators, $\text{Im} \chi^{(X)}(DD^*)_{12} \propto \text{Im} P_1^{(X)}(D)P_2^{(X)}(D^*)$, cf. Eqs. (17b)-(17c); while (sinus of) the difference of the CP-even phases is from the PMNS mixing matrix elements, cf. Eqs. (17b)-(17c); while (sinus of) the difference of the CP-even phases is contained in the imaginary part of the product of propagators, $\text{Im} \chi^{(X)}(DD^*)_{12} \propto \text{Im} P_1^{(X)}(D)P_2^{(X)}(D^*)$, cf. Eqs. (26) later.

In the limit of $\Gamma_{N_1} \rightarrow +0$, i.e., $\Gamma_{N_1} \ll M_{N_1}$, the expression for the “diagonal” decay width $\Gamma_j^{(X)}(DD^*)_{11}$ [and thus also for $\Gamma_j^{(X)}((DD^*)_{22})$] can be calculated analytically. The differential decay width $d\Gamma_j^{(X)}(DD^*)_{11}$ with respect to the muon energy $E_\mu$, in the $N_1$ rest frame, was obtained in Ref. [3], and the result of explicit integration of it over $E_\mu$, for the general case of not neglected electron mass ($M_e \neq 0$), is

$$\Gamma_j^{(X)}(DD^*)_{jj} \equiv \Gamma_j(DD^*)_{jj} = \frac{K^2}{192(2\pi)^4} \frac{M_{N_1}^{11}}{\Gamma_j} \lambda^{1/2}(\lambda - 1, x_{ej}) \left[ x_{ej} - 1 + x_{ej}(x_{ej} + 2 - x_{ej}) \right] \mathcal{F}(x_j, x_{ej}) , \quad (21)$$

where we use the notations

$$\lambda(y_1, y_2, y_3) = y_1^2 + y_2^2 + y_3^2 - 2y_1y_2 - 2y_2y_3 - 2y_3y_1 , \quad (22a)$$

$$x_{\pi j} = \frac{M_\pi^2}{M_{N_1}^2} , \quad x_{ej} = \frac{M_e^2}{M_{N_1}^2} , \quad x_j = \frac{M_j^2}{M_{N_1}^2} , \quad (j = 1, 2) , \quad (22b)$$

and the function $\mathcal{F}(x_j, x_{ej})$ is given in Appendix [B] Eq. [B2] where the derivation of this expression [21] is given. When $M_e = 0$, the results acquires a simpler form

$$\lim_{M_e \rightarrow 0} \frac{\Gamma_j^{(X)}(DD^*)_{jj}}{\Gamma_j} = \frac{K^2}{192(2\pi)^4} \frac{M_{N_1}^{11}}{\Gamma_j} (x_{ej} - 1)^2 f(x_j) , \quad (23)$$

where the function $f(x_j) = \mathcal{F}(x_j, 0)$ is

$$f(x_j) = 1 - 8x_j + 8x_j^3 - x_j^4 - 12x_j^2 \ln x_j . \quad (24)$$

We note that the expression [21] is the same for $X = LV$ and $X = LC$. In the range of the masses 0.117 GeV $< M_{N_1} < 0.136$ GeV the expression [23] differs from the exact expression [21] with Eq. [B2] by less than one per cent. However, for 0.106 GeV $< M_{N_1} < 0.117$ GeV and for 0.136 GeV $< M_{N_1} < 0.139$ GeV the deviation is more than one per cent. For values of $M_{N_1}$ close to the lower on-shell bound $M_{N_1} \approx 0.1062$ GeV the deviation is very large and the expression [21] with Eq. [B2] must be used instead of Eq. [23] for $\Gamma_j^{(X)}(DD^*)_{jj}$. We will use the general expression [21] unless otherwise stated.

Furthermore, we can also calculate analogously as $\Gamma_j^{(X)}(DD^*)_{jj}$ the analytic expression for the asymmetric difference $S_{-}^{(X)}$ in the limit $\Gamma_{N_1} \rightarrow +0$ ($\Gamma_{N_1} \ll M_{N_2} - M_{N_1}$). In order to explain this analogy, we note that in the limit $\Gamma_{N_1} \rightarrow +0$ it was crucial to use in the analytic calculation of $\Gamma_j^{(X)}(DD^*)_{jj}$ the identity

$$|P_j^{(LC)}|^2 = \left| \frac{-(p_\pi - p_1)^2 - M_{N_1}^2}{(p_\pi - p_1)^2 - M_{N_1}^2 + i \Gamma_{N_1} M_{N_1}} \right|^2 \approx \frac{\pi}{M_{N_1} \Gamma_{N_1}} \delta((p_\pi - p_1)^2 - M_{N_1}^2) ; \quad (j = 1, 2; \; \Gamma_{N_1} \ll M_{N_1}) . \quad (25)$$
On the other hand, in the difference $S_1^{(X)} \propto \text{Im} \Pi^{(X)}(DD^*)_{12}$ we have in the integrand of $\text{Im} \Pi^{(X)}(DD^*)_{12}$ as a factor the following combination of propagators:

$$\text{Im} P_1^{(LC)}(D) P_2^{(LC)}(D)^* = \frac{(p_N^2 - M_{N_2}^2) \Gamma_{N_2} M_{N_2} - \Gamma_{N_1} M_{N_1} (p_N^2 - M_{N_1}^2)}{(p_N^2 - M_{N_1}^2)^2 + \Gamma_{N_1} M_{N_1}^2}$$  

$$\approx P \left( \frac{1}{p_N^2 - M_{N_1}^2} \right) \pi \delta(p_N^2 - M_{N_2}^2) - \pi \delta(p_N^2 - M_{N_1}^2) P \left( \frac{1}{p_N^2 - M_{N_2}^2} \right)$$  

$$= \frac{\pi}{M_{N_2}^2 - M_{N_1}^2} \left[ \delta(p_N^2 - M_{N_2}^2) + \delta(p_N^2 - M_{N_1}^2) \right],$$

where $p_N = (p_2 - p_1)$ in the direct channel, and we assumed that $\Gamma_{N_j} \ll |\Delta M_N| = M_{N_j} - M_{N_i}$ in Eqs. (26b)-(26c). When $X = LV$, the corresponding combination of propagators is the same as in Eq. (26c) but with the additional factor $M_{N_i} M_{N_j}$. The expression (26c) has formally the same structure as the expression (25), except for the factors in front of the delta(s). Therefore, the integration over the final phase space can be performed formally in the same way. This then results in the expressions

$$\text{Im} \Pi^{(LV)}(DD^*)_{12} = \eta^{(LV)} K^2 \frac{1}{192(2\pi)^4} \frac{M_{N_2} M_{N_1}}{M_{N_2}^2 + M_{N_1}^2 (M_{N_2} - M_{N_1})}$$

$$\times \sum_{j=1}^2 M_{N_j}^{10} \lambda_{j/2} (x_{\pi j}, 1, x_{e j}) [x_{\pi j} - 1 + x_{e j} (x_{\pi j} + 2 - x_{e j})] \mathcal{F}(x_{\pi j}, x_{e j}),$$

$$\text{Im} \Pi^{(LC)}(DD^*)_{12} = \eta^{(LC)} K^2 \frac{1}{192(2\pi)^4} \frac{1}{M_{N_2}^2 + M_{N_1}^2 (M_{N_2} - M_{N_1})}$$

$$\times \sum_{j=1}^2 M_{N_j}^{12} \lambda_{j/2} (x_{\pi j}, 1, x_{e j}) [x_{\pi j} - 1 + x_{e j} (x_{\pi j} + 2 - x_{e j})] \mathcal{F}(x_{\pi j}, x_{e j}),$$

where the overall factor $\eta^{(X)}$ is equal to unity ($\eta^{(X)} = 1$) when $\Gamma_{N_j} \ll |\Delta M_N|$, i.e., when the identity (26c) can be applied. Nonetheless, when $\Gamma_{N_j} \ll |\Delta M_N|$, we have in general corrections to these formulas, in the form of $\eta < 1$, and the exact expression (26a) has to be used instead of the approximation (26c).

All these quantities can be evaluated also via numerical integrations over the final phase space, with finite widths $\Gamma_{N_j}$ in the propagators. The scalings $\Gamma^{(X)}(DD^*)_{jj} \propto \Gamma_{N_j}$, $\text{Im} \Pi^{(X)}(DD^*)_{12} \propto 1/|\Delta M_N|$, as suggested by Eqs. (21) and (27), are confirmed numerically (when $\Gamma_{N_j} \ll M_{N_j}$, and $\Gamma_{N_j} \ll |\Delta M_N|$, respectively). Furthermore, the numerical evaluations indicate clearly that the direct-crossed ($DC^*$ and $CD^*$) interference contributions to $S_1^{(X)}(\pi)$ are negligible in all considered cases, in comparison with the corresponding direct ($DD^*$) and crossed channel ($CC^*$) contributions. Namely, in the sum $S_1^{(X)}(\pi)$, the interference contributions $\text{Re} \Pi^{(X)}(DC^*)_{ij} \sim 10^{-37}$ GeV are approximately independent of $\Gamma_{N_j}$. On the other hand, $\Gamma^{(X)}(DD^*)_{jj} = \Gamma^{(X)}(CC^*)_{jj}$ is at $\Gamma_{N_j} = 10^{-4}$ GeV about two orders of magnitude larger than $\text{Re} \Pi^{(X)}(DC^*)_{ij}$. $\Gamma^{(X)}(DD^*)_{jj}$ grows at decreasing $\Gamma_{N_j}$ as 1/192 [Eq. (21)], while $\text{Re} \Pi^{(X)}(DC^*)_{ij}$ does not increase and becomes thus at $\Gamma_{N_j} < 10^{-4}$ GeV relatively negligible.

In the difference (asymmetry) $S_1^{(X)}(\pi)$, the $DC^*$ interference contribution $\text{Im} \Pi^{(X)}(DC^*)_{12} \sim 10^{-38}$ GeV is approximately independent of $\Delta M_N$. On the other hand, $\text{Im} \Pi^{(X)}(DD^*)_{12} = \text{Im} \Pi^{(X)}(CC^*)_{12}$ is at $\Delta M_N = 10^{-3}$ GeV about two orders of magnitude larger than $\text{Im} \Pi^{(X)}(DC^*)_{12}$. $\text{Im} \Pi^{(X)}(DD^*)_{12}$ grows at decreasing $\Delta M_N$ as 1/|$\Delta M_N$| [Eq. (27)], while $\text{Im} \Pi^{(X)}(DC^*)_{12}$ does not increase and becomes thus at $\Delta M_N < 10^{-3}$ GeV relatively negligible.

On the other hand, the numerical evaluations with $\Gamma_{N_j} \ll \Delta M_N$ give us the values of the $\delta_{ij}^{(X)}$ [cf. Eqs. (19) and (18)] and $\eta^{(X)}$ correction terms, due to non-negligible overlap of the $N_1$ with $N_2$ resonance. It turns out that these functions are independent of $X (= LV, LC)$, and that $\eta$ and $\delta \equiv (1/2)(\delta_1 + \delta_2)$ are effectively functions of only one

---

3 We note that there is no such overall correction factor in the expression (21) for $\Pi^{(X)}(DD^*)_{jj}$, because in our considered cases $\Gamma_{N_j} \ll M_{N_j}$ always and Eq. (21) is the correct expression then.
On the other hand, comparing the expressions (27) relevant for the CP asymmetries of the four-particle finite phase space, and are tabulated in Table I (with their estimated uncertainties due to numerical integrations).

The values of $\delta (= \delta^{(X)})$ and $\eta (= \eta^{(X)})$ as functions of $\Delta M_N/\Gamma_N$ can be obtained by numerical integrations over the four-particle finite phase space, and are tabulated in Table I (with their estimated uncertainties due to numerical integrations).

We note that the rare process decay widths $S^{(X)}_+(\pi)$, Eq. (18), are formally quartic in the heavy-light mixing elements $|B_{\ell N}|$, i.e., very small. Nonetheless, they are proportional to the expressions $\Gamma(DD^*)_{jj}$, Eq. (21), which in turn is proportional to $1/\Gamma_N$ due to the on-shellness of the intermediate $N_j$'s. This $1/\Gamma_N$ is proportional to $1/\tilde{\kappa}_j \sim 1/|B_{\ell N_j}|^2$ according to Eqs. (12-14). Therefore, the on-shellness of $N_j$'s makes the rare process decay widths significantly less suppressed by the mixings:

\begin{align}
\Gamma(DD^*)_{jj} &\propto 1/\Gamma_N \propto 1/\tilde{\kappa}_j \propto 1/|B_{\ell N_j}|^2, \\
S^{(X)}_+(\pi) &\propto |B_{\ell N_j}|^2.
\end{align}

On the other hand, comparing the expressions (27) relevant for the CP asymmetries $S^{(X)}_+(\pi)$ (20), with the expression (21) relevant for the decay widths $S^{(X)}_+(\pi)$ (18), we see that the asymmetries $S^{(X)}_-(\pi)$ are suppressed by mixings as $\sim |B_{\ell N}|^4$, making them in general much smaller than the decay widths $S^{(X)}_+(\pi) \propto |B_{\ell N}|^2$. However, the asymmetries are proportional to $1/\Delta M_N$ (where $\Delta M_N = M_{N_2} - M_{N_1} > 0$), cf. Eqs. (27). In general, $\Delta M_N \gg \Gamma_N$. Nonetheless, in a scenario where $\Delta M_N$ becomes very small and (almost) comparable with $\Gamma_N$, the asymmetries $S^{(X)}_-(\pi)$ can become comparable with the decay widths $S^{(X)}_+(\pi)$. A model with two almost degenerate neutrinos $N_j$ in the mass range of $\sim 10^2$ eV has been constructed and investigated in Ref. [13].

In particular, in this limit of two almost degenerate neutrinos $N_j$, where now $M_{N_1} \approx M_{N_2} \equiv M_N$, the formulas (21), (19) and (27) get simplified. In this case, it is convenient to introduce a “normalized” branching ratio $\bar{B}_\ell$:

\begin{equation}
\bar{B}_\ell(M_N) = \frac{1}{4\pi G_F^2 \Gamma(\pi^+ \rightarrow all)} \frac{1}{x_\pi^2} \lambda^{1/2}(x_\pi^-, 1, x_e) [x_\pi - 1 + x_e(x_\pi + 2) - x_e] \mathcal{F}(x, x_e),
\end{equation}

where we use the notations

\begin{equation}
x_\pi = \frac{M_\pi^2}{M_N^2}, \quad x_e = \frac{M_e^2}{M_N^2}, \quad x = \frac{M_e^2}{M_N^2}.
\end{equation}

In terms of this branching ratio $\bar{B}_\ell$, the formulas (21), (19) and (27) can be rewritten, in the mentioned almost degenerate scenario, as

\begin{align}
\frac{\Gamma(DD^*)_{jj}}{\Gamma(\pi^+ \rightarrow all)} &= \frac{1}{4CK_j} \bar{B}_\ell, \\
\Re \frac{\Gamma(DD^*)_{12}}{\Gamma(\pi^+ \rightarrow all)} &= \frac{\delta(y)}{2C(\tilde{\kappa}_1^2 + \tilde{\kappa}_2^2)} \bar{B}_\ell, \\
\Im \frac{\Gamma(DD^*)_{12}}{\Gamma(\pi^+ \rightarrow all)} &= \frac{\eta(y)/y}{2C(\tilde{\kappa}_1^2 + \tilde{\kappa}_2^2)} \bar{B}_\ell.
\end{align}

\begin{table}[h]
\centering
\caption{Values of $\delta(y)$ correction terms and $\eta(y)/y$ correction factors for various values of $y \equiv \Delta M_N/\Gamma_N$.}
\begin{tabular}{cccc}
$y \equiv \frac{\Delta M_N}{\Gamma_N}$ & $\log_{10} \frac{\delta_{1,2}}{y}$ & $\delta_{1,2}/y$ & $\eta(y)/y$
\hline
10.0 & 1.000 & 0.0100 ± 0.0005 & 0.984 ± 0.003 0.998 ± 3 × 10$^{-3}$
5.00 & 0.699 & 0.038 ± 0.002 & 0.957 ± 0.003 0.191 ± 0.001
2.50 & 0.398 & 0.137 ± 0.006 & 0.854 ± 0.003 0.342 ± 0.001
1.67 & 0.222 & 0.265 ± 0.005 & 0.730 ± 0.005 0.438 ± 0.003
1.25 & 0.097 & 0.392 ± 0.006 & 0.610 ± 0.007 0.488 ± 0.006
1.00 & 0.000 & 0.505 ± 0.010 & 0.498 ± 0.005 0.498 ± 0.005
\end{tabular}
\end{table}
where \( y \equiv \Delta M_N/\Gamma_N \). Similarly, after some algebra, we can rewrite in this scenario \((M_{N1} \approx M_{N2} \equiv M_N)\) the obtained branching ratios \( \text{Br}_+ \) and CP asymmetry ratios \( \mathcal{A}_{CP} \) for the considered rare decays, in terms of \( \text{Br}_+ \) and of the heavy-light mixing parameters. Below we present the results for the case when the neutrinos \( N_j \) are Dirac (Di), and when they are Majorana (Ma) neutrinos. The branching ratio \( \text{Br}_+ \) for the considered rare processes is

\[
\text{Br}_+^{(\text{Di})} = \frac{S_+^{(\text{LC})}(\pi)}{\Gamma(\pi^+ \rightarrow \text{all})} = \frac{2}{\Gamma(\pi^+ \rightarrow \text{all})} \sum_{j=1}^{2} \frac{|B_{\ell N_j}|^2|B_{\mu N_j}|^2}{k_j} + 4\delta(y) \frac{|B_{\ell N_1}||B_{\ell N_2}||B_{\mu N_1}||B_{\mu N_2}|}{(k_1 + k_2)} \cos \theta^{(\text{LC})} \text{Br}(M_N) \quad (33a)
\]

\[
\text{Br}_+^{(\text{Ma})} = \frac{S_+^{(\text{LV})}(\pi) + S_+^{(\text{LC})}(\pi)}{\Gamma(\pi^+ \rightarrow \text{all})} = \frac{2}{\Gamma(\pi^+ \rightarrow \text{all})} \sum_{j=1}^{2} \frac{|B_{\ell N_j}|^2|B_{\mu N_j}|^2}{k_j} + 2\delta(y) \frac{|B_{\ell N_1}||B_{\ell N_2}||B_{\mu N_1}||B_{\mu N_2}|}{(k_1 + k_2)} \cos \theta^{(\text{LC})} \text{Br}(M_N) \quad (33b)
\]

Here we took into account that in the Dirac case only the LC process contributes, while in the Majorana case both the lepton number violating (LV) and conserving (LC) processes contribute. The mixing parameters \( k_j \) (\( \sim |B_{\ell N_j}|^2 \)) are given in Eq. (14), and we took into account that in Eq. (12) for \( \Gamma_N \) the factor \( C \) is one in the Dirac case and is two in the Majorana case. The contributions of the \( N_1-N_2 \) overlap effects give the relative corrections of \( O(\delta) \) and are negligible when \( \Delta M_N > 10\Gamma_N \), cf. Table I.

The (CP asymmetry) branching ratio \( \text{Br}_- \) for the considered rare processes is

\[
\text{Br}_-^{(\text{Di})} = \frac{S_-^{(\text{LC})}(\pi)}{\Gamma(\pi^- \rightarrow \text{all})} = \frac{\Gamma^{(\text{LC})}(\pi^-) - \Gamma^{(\text{LC})}(\pi^+)}{\Gamma(\pi^+ \rightarrow \text{all})} = \frac{4|B_{\ell N_1}||B_{\ell N_2}||B_{\mu N_1}||B_{\mu N_2}|}{(k_1 + k_2)} \sin \theta^{(\text{LC})} \frac{\eta(y)}{y} \text{Br}(M_N) \quad (34a)
\]

\[
\text{Br}_-^{(\text{Ma})} = \frac{S_-^{(\text{LV})(\pi)} + S_-^{(\text{LC})(\pi)}}{\Gamma(\pi^- \rightarrow \text{all})} = \frac{\Gamma^{(\text{LV})(\pi^-) + \Gamma^{(\text{LC})(\pi^-) - \Gamma^{(\text{LC})(\pi^+}}}{\Gamma(\pi^+ \rightarrow \text{all})} = \frac{2|B_{\ell N_1}||B_{\ell N_2}|}{(k_1 + k_2)} \left( |B_{\ell N_1}||B_{\ell N_2}| \sin \theta^{(\text{LV})} + |B_{\mu N_1}||B_{\mu N_2}| \sin \theta^{(\text{LC})} \right) \frac{\eta(y)}{y} \text{Br}(M_N) \quad (34b)
\]

\[
\text{Br}_+^{(\text{Ma})} = \frac{2k_1|B_{\ell N_1}|^2}{(k_1 + k_2)} \left( \kappa_\ell|B_{\ell N_1}|^2 \sin \theta^{(\text{LV})} + \kappa_\mu|B_{\mu N_1}|^2 \sin \theta^{(\text{LC})} \right) \frac{\eta(y)}{y} \text{Br}(M_N). \quad (34c)
\]
Consequently, the usual CP asymmetry ratios $A_{CP}^{X}$ are obtained from Eqs. (33)-(34)

$$A_{CP}^{(Di)} \equiv \frac{Br_{-}^{(Di)}}{Br_{+}^{(Di)}} = \frac{\Gamma^{(LC)}(\pi^{-}) - \Gamma^{(LC)}(\pi^{+})}{\Gamma^{(LC)}(\pi^{-}) + \Gamma^{(LC)}(\pi^{+})}$$

$$= \frac{\sin \theta^{(LC)}}{y} \left[ \frac{1}{4} \frac{Br_{N_{1}}}{Br_{eN_{2}}} \left( 1 + \frac{\kappa_{2}}{\kappa_{1}} \right) + \frac{1}{4} \frac{Br_{\mu N_{2}}}{Br_{\mu N_{1}}} \left( 1 + \frac{\kappa_{2}}{\kappa_{1}} \right) + \delta(y) \cos \theta^{(LC)} \right] \eta(y)$$

$$A_{CP}^{(Ma)} \equiv \frac{Br_{-}^{(Ma)}}{Br_{+}^{(Ma)}} = \frac{\Gamma^{(LV)}(\pi^{-}) + \Gamma^{(LC)}(\pi^{-}) - \Gamma^{(LV)}(\pi^{+}) - \Gamma^{(LC)}(\pi^{+})}{\Gamma^{(LV)}(\pi^{-}) + \Gamma^{(LC)}(\pi^{-}) + \Gamma^{(LV)}(\pi^{+}) + \Gamma^{(LC)}(\pi^{+})}$$

$$= \frac{\sin \theta^{(LV)} + \frac{Br_{\mu N_{2}}}{Br_{\mu N_{1}}} \sin \theta^{(LC)}}{y} \left[ \frac{1}{4} \frac{Br_{eN_{1}}^{2} + Br_{\mu N_{1}}^{2}}{Br_{eN_{2}}^{2}} \left( 1 + \frac{\kappa_{2}}{\kappa_{1}} \right) + \frac{1}{4} \left( \frac{Br_{\mu N_{2}}^{2} + Br_{\mu N_{1}}^{2}}{Br_{eN_{1}}^{2}} \right) \left( 1 + \frac{\kappa_{2}}{\kappa_{1}} \right) + \delta(y) \left( \cos \theta^{(LV)} + \frac{Br_{\mu N_{1}}}{Br_{\mu N_{2}}} \cos \theta^{(LC)} \right) \right] \eta(y)$$

When $y \equiv \Delta M_N/\Gamma_N$ becomes large ($y > 10$), i.e., when $\Delta M_N > 10\Gamma_N$, Table 1 implies that the CP asymmetries become suppressed by the $\eta(y)/y$ factor. On the other hand, when $y < 10$ (i.e., $\Delta M_N < 10\Gamma_N$) and $|\theta^{(X)}| \sim 1$, the factor $\eta(y)/y$ is $\sim 1$ and the CP asymmetry ratios $A_{CP}^{X}$ become $\sim 1$, while all $Br_{\pm}$ become $\sim |Br_{eN_{1}}|^{2} Br(M_{N})$ ($\ell = e, \mu$).

FIG. 3: The normalized branching ratio $Br$, Eq. (30), as a function of the mass $M_{N_{1}} \approx M_{N_{2}} \equiv M_{N}$. The full formula was used (with $M_{e} = 0.511 \times 10^{-3}$ GeV). The formula for $M_{e} = 0$ case gives a line which is in this Figure indistinguishable from the depicted line.

---

4 If we also assume that $|Br_{eN_{1}}| \approx |Br_{eN_{1}}|$ (for $\ell = e, \mu, \tau$), then also $\tilde{K}_{1} \approx \tilde{K}_{2} \equiv \tilde{K}$, and the expressions for $A_{CP}$ become particularly simple

$$A_{CP}^{(Di)} = \frac{\sin \theta^{(LC)}}{1 + \delta(y) \cos \theta^{(LC)}} \frac{\eta(y)}{y} = \sin \theta^{(LC)} \frac{\eta(y)}{y} (1 + O(\delta))$$

$$A_{CP}^{(Ma)} = \left( \frac{|Br_{eN_{1}}^{2} \sin \theta^{(LV)} + |Br_{\mu N_{1}}^{2} \sin \theta^{(LC)}|}{|Br_{eN_{1}}^{2} + |Br_{\mu N_{1}}^{2}|} \right) \frac{\eta(y)}{y} (1 + O(\delta))$$. 

---
We present in Fig. 3 the normalized quantity $\overline{Br}$ as a function of $M_N$ in the on-shell kinematic interval (2); and in Figs. 4 the same curve near the lower end point $M_N \approx M_\mu + M_e$ ($= 0.1062$ GeV), where the effects of $M_e \neq 0$ are relatively appreciable. Further, in Fig. 5 we present the curves of the overlap suppression factors $\eta(y)/y$ and $\delta(y)$, as a function of the $N_1-N_2$ overlap parameter $y \equiv \Delta M_N/\Gamma_N$. On the other hand, the (CP asymmetry) branching ratio $Br_-$ in the case of mixing one and maximal CP phases (i.e., when $B_{\ell N_j} = 1$ for all $\ell$, and $\sin \theta^{(X)} = 1$; $Br^{(Di)} = Br^{(Ma)} \equiv Br_-$ then), as a function of $\Delta M_N$, is presented in Fig. 6. In that Figure, no overlap effects appear at the values of $\Delta M_N$ presented, i.e., $\eta = 1$.

Therefore, when $y \equiv \Delta M_N/\Gamma_N < 5$, i.e., in the almost degenerate case of two on-shell neutrinos $N_j$, we can expect in general the CP asymmetry ratio $A_{CP}$ of the considered rare process to be $\sim 1$. The branching ratio for this process, in the case of one $N$ neutrino, was considered in Ref. [4], and all the conclusions about the measurability of this branching ratio $Br \approx (1/2)Br_+$ can be translated into the conclusions about the measurability of the (CP asymmetry) branching ratio $Br_-$ in the described almost degenerate scenario, provided that $|\theta^{(LC)}|, |\theta^{(LV)}| \sim 1$.

This means that the CP asymmetries could be measured in the future pion factories in the described scenarios, provided that the heavy-light mixing parameters $|B_{\ell N_j}|^2$ ($\ell = e, \mu$) are not many orders of magnitude below the present experimental bounds. The present experimental bounds of the mixing parameters $|B_{\ell N_j}|^2$ ($\ell = e, \mu, \tau$) in the considered mass range (2), are: $|B_{e N_j}|^2 \lesssim 10^{-8}$ [20]; $|B_{\mu N_j}|^2 \lesssim 10^{-6}$ [21]; $|B_{\tau N_j}|^2 \lesssim 10^{-4}$ [22]; cf. also

It was considered in the $M_e = 0$ limit, but the general conclusions remain unchanged with respect to the $M_e = 0.511$ MeV case.
In these relations, we took into account that the LC process dominates over the LV process in the considered case, where resonances appear here \((\eta = 1)\). If in this case, in addition, \(B_{\ell N_j} \sim 10^2\), no suppression effects from the overlap of the \(N_1\) and \(N_2\) processes could be expected per year. The probability of (on-shell) neutrino \(N\) to decay inside a detector of length \(L = 10^1\) m in such pion factories is

\[
P_N \sim \frac{L}{\sqrt{\pi} \tau_N} = \frac{L \Gamma_N}{\sqrt{\pi}} \sim 10^{-2} \frac{\Gamma_N}{\sqrt{\pi}} \sim 10^{-3} \frac{\Gamma}{\sqrt{\pi}},
\]

where \(\Gamma \sim K_j \propto |B_{\ell N_j}|^2\). We should multiply the obtained branching ratios \(Br_\pm\) by such acceptance factors \(P_N\) to obtain the effective branching ratios \(Br_\pm^{(\text{eff})}\).

If the largest among the mixing elements \(|B_{\ell N_j}|^2 (\ell = e, \mu)\) are \(|B_{\mu N_j}|^2 (\sim |B_{\mu N}|^2) (j = 1, 2)\), i.e., if we have \(|B_{\mu N}|^2 \gg |B_{\ell N}|^2 (\sim |B_{\ell N}|^2)\), the formulas (36) with (33) and (34) give

\[
P_N Br_+^{(\text{D IDisposable\text{Ma}})} \sim 10^{-3}|B_{e N}|^2|B_{\mu N}|^2 \overline{\Gamma}(M_N) \sim |B_{e N}|^2|B_{\mu N}|^2 10^{-7},
\]

\[
P_N Br_-^{(\text{D IDisposable\text{Ma}})} \sim 10^{-3}|B_{e N}|^2|B_{\mu N}|^2 \sin \theta(X) \overline{\Gamma}(M_N) \sim |B_{e N}|^2|B_{\mu N}|^2 \sin \theta(LC) 10^{-7}.
\]

In these relations, we took into account that the LC process dominates over the LV process in the considered case, and that \(\overline{\Gamma} \sim 10^{-4}\) in most of the on-shell interval for the masses \(M_N \approx M_{N_2} \equiv M_N\), cf. Fig. 3. If in this case, in addition, \(|B_{\ell N_j}|^2 (\ell = e, \mu)\) are close to their present upper bounds, \(|B_{e N}|^2 \sim 10^{-8}\) and \(|B_{\mu N}|^2 \sim 10^{-6}\), this implies that \(P_N Br_+ \sim 10^{-21}\) and \(P_N Br_- \sim 10^{-21}\) (the latter provided \(\sin \theta(X) \approx 1\), implying that \(\sim 10^8\) events can be detected per year, with the difference between \(\pi^-\) and \(\pi^+\) decays also of the order \(\sim 10^6\). This number decreases in proportionality with the factor \(|B_{e N}|^2|B_{\mu N}|^2\) when this factor decreases. In this scenario there is almost no difference between the case when \(N_j\) are Dirac and the case when \(N_j\) are Majorana.

On the other hand, if the largest among the mixing elements \(|B_{\ell N_j}|^2 (\ell = e, \mu)\) are \(|B_{e N}|^2 (\sim |B_{e N}|^2) (j = 1, 2)\), i.e., if we have \(|B_{e N}|^2 \gg |B_{\mu N}|^2 (\sim |B_{\mu N}|^2)\), the formulas (36) with (33) and (34) give

\[
P_N Br_+^{(\text{D IDisposable\text{Ma}})} \sim 10^{-3}|B_{e N}|^2|B_{\mu N}|^2 \overline{\Gamma}(M_N) \sim |B_{e N}|^2|B_{\mu N}|^2 10^{-7},
\]

\[
P_N Br_-^{(\text{D IDisposable\text{Ma}})} \sim 10^{-3}|B_{e N}|^2|B_{\mu N}|^2 \sin \theta(LC) \overline{\Gamma}(M_N) \sim |B_{e N}|^2|B_{\mu N}|^2 \sin \theta(LC) 10^{-7}.\]

In this considered case, the LV process dominates over the LC process. If in this case, in addition, \(|B_{e N}|^2\) are close to their present upper bounds, \(|B_{e N}|^2 \sim 10^{-8}\) (and \(|B_{\mu N}|^2 \ll |B_{e N}|^2\), this implies that \(P_N Br_+^{(\text{Ma})} \sim 10^{-23}\).
neutrinos (and less events if \( N \) can be detected per year, with the difference between \( \pi^- \) and \( \pi^+ \) decays also of the order \( \sim 10^6 \), if \( N_j \) are Majorana neutrinos (and less events if \( N_j \) are Dirac neutrinos). This number decreases in proportionality with the factor \( |B_{eN}|^4 \) when this factor decreases. In this scenario there is a clear difference between the case when \( N_j \) are Dirac and the case when \( N_j \) are Majorana.

The mentioned present experimental upper bounds on the mixings (\( |B_{eN}|^2 \lessapprox 10^{-8}; |B_{\mu N}|^2 \lessapprox 10^{-6} \)) suggest that the first of the mentioned two scenarios is more probable, i.e., that the LC processes dominate over the LV processes.

The measurement of the CP asymmetries alone cannot distinguish between the Dirac and the Majorana character of intermediate neutrinos \( N_j \)'s. However, as argued in Ref. [4], the neutrino character could be determined from the measured differential decay rates of these processes with respect to the muon energy \( E_\mu \) in the \( N_j \) rest frame, \( d\Gamma/dE_\mu \), if the heavy-light mixing elements satisfy the relation \( |B_{eN}| \gtrsim |B_{\mu N}| \) (if \( |B_{eN}| \ll |B_{\mu N}| \), the LC process dominates).

IV. SUMMARY

We investigated the rare decays of charged pions, \( \pi^\pm \to e^\pm N_j \to e^{\pm}\mu^\mp \nu \), in scenarios with two heavy sterile neutrinos \( N_j \) (\( j = 1, 2 \)). Such scenarios allow the mentioned decays to proceed with exchange of on-shell intermediate neutrinos at the tree level, but are suppressed by the heavy-light neutrino mixing elements of the PMNS matrix. The mentioned decays can be of the lepton-number-conserving (LC) type (\( \nu = \nu_e, \bar{\nu}_e \)), or of the lepton-number-violating (LV) type (\( \nu = \bar{\nu}_\mu, \nu_\mu \)). If the \( N_j \) neutrinos are of Dirac nature, only LC decays take place; if they are of Majorana nature, both LC and LV decays take place. In Ref. [4] such processes were studied with a view to ascertain the nature of the intermediate neutrino \( N_j \), and it was shown there that it may be possible to do this in the future pion factories where the number of produced charged pions will be exceedingly high. In the present work, on the other hand, we investigated the possibility to ascertain the CP violation in such processes. Such a CP violation originates from the interference between the \( N_1 \) and \( N_2 \) exchange processes and the existence of possible CP-violating phases in the PMNS mixing matrix. We showed that such signals of CP violation could be detected in the future pion factories if we have (at least) two sterile neutrinos in the mentioned mass interval and such that their masses are almost degenerate, i.e., when the mass difference \( \Delta M_N \) between them is not many orders of magnitude larger than their decay width \( \Gamma_N \).

Therefore, our calculation suggests that the observation of CP violation in pion decays would be consistent with the existence of \( \nu \)MSM model [12, 13, 16], with the two almost degenerate heavy neutrinos in the lower mass range of the model. The Majorana nature of the neutrinos offers more possibilities of CP violation because there are more CP-violating phases in the PMNS matrix than in the case when the neutrinos have Dirac nature. On the other hand, the present experimental bounds on the heavy-light mixings allow higher rates and more appreciable CP-violating effects in these processes in the LC channels than in the LV channels, i.e., in the scenarios where the Majorana nature of the neutrinos is difficult to discern.

Acknowledgments

This work was supported in part by (Chile) FONDECYT Grant No. 1130599 (G.C. and C.S.K.), by CONICYT Fellowship “Beca de Doctorado Nacional” and Proyecto PIIC 2013 (J.Z.S.). The work of C.S.K. was supported by the NRF grant funded by the Korean Government of the MEST (No. 2011-0017430) and (No. 2011-0020333).

Appendix A: Explicit formulas

The squared matrix element \( |T^{(X)}(\pi^\pm)|^2 \) in Eq. (4), where \( X = \text{LV, LC} \), is a combination of contributions from \( N_1 \) and \( N_2 \) and from the two channels \( D \) (direct) and \( C \) (crossed)

\[
|T^{(X)}(\pi^\pm)|^2 = K^2 \sum_{i=1}^{2} \sum_{j=1}^{2} k_{i,\pm}^{(X)} k_{j,\pm}^{(X)} \\
\times \left[ P_i^{(X)}(D) P_j^{(X)}(D)^* T_{\pm}^{(X)}(DD^*) + P_i^{(X)}(C) P_j^{(X)}(C)^* T_{\pm}^{(X)}(CC^*) \right] + \left( P_i^{(X)}(D) P_j^{(X)}(C) T_{\pm}^{(X)}(DC^*) + P_i^{(X)}(C) P_j^{(X)}(D)^* T_{\pm}^{(X)}(CD^*) \right),
\]

(A1)
where the constant $K^2$ is given in Eq. (10), the mixing factors $k_{j,i}^{(X)}$ in Eq. (7), and $F_j^{(X)}(Y)$ represent the $N_j$ propagator functions Eq. (9) of the direct and crossed channels ($Y = D, C$).

Here we write down the explicit formulas for the direct $(D^*)$, crossed $(C^*)$ and direct-crossed interference $(DC^*)$ elements $[T_{\pm}^{(X)}(D^*), \; T_{\pm}^{(X)}(C^*), \; T_{\pm}^{(X)}(DC^*)]$ appearing in Eqs. (A1). For the lepton number violating (LV) process of Fig. 1, these are

\[
T_{\pm}^{(LV)}(DD^*) = 256(p_2 \cdot p_\nu) [\frac{1}{2} \cdot \frac{1}{2} (M_\pi^2 + M_\tau^2) + 2(p_1 \cdot p_\tau)(p_\mu \cdot p_\tau)] \equiv T_{\pm}^{(LV)}(D^*),
\]

\[
T_{\pm}^{(LV)}(CC^*) = 256(p_1 \cdot p_\nu) [\frac{1}{2} \cdot \frac{1}{2} (M_\pi^2 + M_\tau^2) + 2(p_2 \cdot p_\tau)(p_\mu \cdot p_\tau)] \equiv T_{\pm}^{(LV)}(C^*),
\]

\[
T_{\pm}^{(LV)}(DC^*) = 256(p_1 \cdot p_\nu) \left[ M_\pi^2 (p_2 \cdot p_\mu) - 2(p_2 \cdot p_\tau)(p_\mu \cdot p_\tau) \right] \equiv T_{\pm}^{(LV)}(DC^*),
\]

where we denoted

\[
\epsilon(q_1, q_2, q_3, q_4) = \epsilon^{\eta_1 \eta_2 \eta_3 \eta_4} (q_1 \eta_1 q_2 \eta_2 q_3 \eta_3 q_4 \eta_4),
\]

and $\epsilon^{\eta_1 \eta_2 \eta_3 \eta_4}$ is the totally antisymmetric Levi-Civita tensor with the sign convention $\epsilon^{0123} = +1$.

For the lepton number conserving (LC) process of Fig. 2, the corresponding expressions are

\[
T_{\pm}^{(LC)}(DD^*) = 256(p_2 \cdot p_\nu) \left[ (p_1 \cdot p_2) \left( M_\pi^2 - M_\tau^2 M_\pi^2 - 4M_\tau^2(p_1 \cdot p_\tau) + 4(p_1 \cdot p_\tau)^2 \right) + 2M_\tau^2 (p_2 \cdot p_\tau)(M_\pi^2 - p_1 \cdot p_\tau) \right] \equiv T_{\pm}^{(LC)}(D^*),
\]

\[
T_{\pm}^{(LC)}(CC^*) = 256(p_1 \cdot p_\nu) \left[ (p_1 \cdot p_2) \left( M_\pi^2 - M_\tau^2 M_\pi^2 - 4M_\tau^2(p_2 \cdot p_\tau) + 4(p_2 \cdot p_\tau)^2 \right) + 2M_\tau^2 (p_1 \cdot p_\tau)(M_\pi^2 - p_2 \cdot p_\tau) \right] \equiv T_{\pm}^{(LC)}(C^*),
\]

\[
T_{\pm}^{(LC)}(DC^*) = 256(p_1 \cdot p_\nu) \left[ (p_1 \cdot p_2)(M_\pi^2 - 2p_1 \cdot p_\tau) + M_\tau^2 (p_2 \cdot p_\tau)^2 \right] \equiv T_{\pm}^{(LC)}(DC^*),
\]

\[
T^{(LC)}(CD^*) = \left( T^{(LC)}(DC^*) \right)^*,
\]

where

\[
\epsilon(q_1, q_2, q_3, q_4) = \epsilon^{\eta_1 \eta_2 \eta_3 \eta_4} (q_1 \eta_1 q_2 \eta_2 q_3 \eta_3 q_4 \eta_4),
\]

and $\epsilon^{\eta_1 \eta_2 \eta_3 \eta_4}$ is the totally antisymmetric Levi-Civita tensor with the sign convention $\epsilon^{0123} = +1$. 

On the basis of these expressions, and the corresponding definitions of the normalized (i.e., without mixings) decay width matrices $\Gamma_{\pm}^{(X)}(Y \; Z^*)$ ($X = \text{LV, LC}; Y, Z = D, C$) in Eq. (5), and using the symmetry of the $d_4$ integration under the exchange $p_1 \leftrightarrow p_2$ (this because: $M_1 = M_2 = M_3$ in our considered case), the following symmetry relations hold between the elements of the decay width matrices of Eq. (5):

\[
\Gamma_{\pm}^{(X)}(DD^*)_{ij} = \Gamma_{\pm}^{(X)}(CC^*)_{ij}, \quad \Gamma_{\pm}^{(X)}(DD^*)_{ji} = \Gamma_{\pm}^{(X)}(DD^*)_{ij}^*,
\]

\[
\Gamma_{\pm}^{(X)}(CD^*)_{ij} = \Gamma_{\pm}^{(X)}(DC^*)_{ij}, \quad \Gamma_{\pm}^{(X)}(CD^*)_{ji} = \Gamma_{\pm}^{(X)}(CD^*)_{ij}^*,
\]
Appendix B: Explicit formula for $\Gamma^{(X)}$ when $M_e \neq 0$

The formula \[21\] is obtained by performing the integration of the differential decay width $d\Gamma^{(LV)}/dE_\mu$ over the muon energy $E_\mu$, in the rest frame of the $N_j$ neutrino. The expression for $d\Gamma^{(LV)}/dE_\mu$ is written explicitly, e.g., in Appendix A of Ref. [4]. This gives

$$\Gamma^{(LV)}(DD^*)_{jj} = K^2 \frac{1}{2!} \frac{1}{2M_\pi(2\pi)^6} \int d\lambda |P_j^{(LV)}(D)|^2 T^{(LV)}(DD^*)$$

(B1a)

$$= K^2 \frac{1}{2!} \frac{1}{2M_\pi(2\pi)^6} \int d\lambda \frac{\pi}{M_{N_j} \Gamma_{N_j}} \delta \left((p_\pi-p_1)^2 - M_{N_j}^2\right) M_{N_j}^2 M_{\mu}^\lambda T^{(LV)}(DD^*) = \cdots$$

(B1b)

$$= K^2 \frac{1}{(2\pi)^4 \Gamma_{N_j} \mu \pi} \lambda^{1/2}(M_{\mu}^2, M_{N_j}^2, M_{\ell}^2) \times \frac{1}{2M_{N_j}} \left[ M_{\mu}^2 (M_{N_j}^2 + M_{\ell}^2) - (M_{N_j}^2 - M_{\ell}^2)^2 \right]$$

$$\times \left( (M_{N_j}^2 + M_{\mu}^2 - M_{\ell}^2)/(2M_{\mu}) \right) dE_\mu E_\mu \sqrt{E_\mu^2 - M_{\mu}^2}$$

(B1c)

The ellipses in Eq. (B1b) indicate the analytic integrations over the four-particle final phase space of the process of Fig. 1(a) with the exception of $E_\mu$ (in the rest frame of $N_j$), performed in Ref. [4]. Eq. (B1c) then uses the differential decay width $d\Gamma^{(LV)}(DD^*)/dE_\mu$ obtained in Ref. [4]. The integration in Eq. (B1c) can be performed explicitly (in Ref. [4] it was performed only in the limit $M_e = 0$), and the result is Eq. (21) with notations (22) and the function $F(x_j, x_{e_j})$ given explicitly here

$$F(x_j, x_{e_j}) = \left\{ \begin{array}{l}
\lambda^{1/2}(1, x_j, x_{e_j})[(1 + x_j)(1 + x_{e_j}) - x_{e_j}(7 - 12x_j + 7x_j^2)] \\
-7x_{e_j}^2(1 + x_j) + 2x_{e_j}^3 - 24(1 - x_{e_j}^2)x_j^2 \ln 2 \\
+12 \left[ - x_j^2(1 - x_{e_j}^2) \ln x_j + (2x_j^2 - x_{e_j}^2)(1 + x_{e_j}) \right] \ln(1 + x_j + \lambda^{1/2}(1, x_j, x_{e_j}) - x_{e_j}) \\
+ x_{e_j}^2(1 - x_{j}^2) \ln \left( (1 - x_j)^2 + (1 - x_j)\lambda^{1/2}(1, x_j, x_{e_j}) - x_{e_j}(1 + x_j) \right) \right\},$$

(B2)

It turns out that the integration over the differential decay width of the lepton number conserving case, $d\Gamma^{(LC)}/dE_\mu$,

$$\Gamma^{(LC)}(DD^*)_{jj} = K^2 \frac{1}{(2\pi)^4 \Gamma_{N_j} \mu \pi} \lambda^{1/2}(M_{\mu}^2, M_{N_j}^2, M_{\ell}^2) \times \frac{1}{96M_{N_j}^2} \times$$

$$\times \int_{M_{\mu}}^{(M_{\mu}^2 + M_{\ell}^2 - M_{N_j}^2)/(2M_{\mu})} dE_\mu \left[ \frac{1}{M_{\mu}^2 + M_{N_j}(-2E_\mu + M_{N_j})} \right]^3$$

$$\times \left\{ 8 \sqrt{(E_\mu^2 - M_{\mu}^2)M_{N_j}^2 (2E_\mu - M_{N_j})M_{N_j} - M_{\mu}^2 + M_{\ell}^2)^2} \left[ M_{\mu}^2 M_{N_j}^2 - M_{\mu}^4 + M_{\ell}^4 (M_{\mu}^2 + 2M_{N_j}^2) - M_{\ell}^2 \right] \\
-2M_{\mu}^2 M_{N_j}^2 (M_{\mu}^2 + M_{N_j}^2 + 2M_{\ell}^2) - 2E_\mu^2 M_{N_j} \left( 5(M_{\mu}^2 + M_{N_j}^2) + M_{\ell}^2 \right) \\
+ E_\mu \left( 3M_{\mu}^2 + 10M_{\mu}^2 M_{N_j}^2 + 3M_{\mu}^2 + 3M_{\ell}^2 (M_{\mu}^2 + M_{N_j}^2) \right) \right\}$$

(B3)

6 Eq. (A.7) of that reference, with the corresponding replacements: $m_M \rightarrow M_\mu$, $m_N \rightarrow M_{N_j}$, $m_\ell \rightarrow M_\mu$, $m_1 = m_2 \rightarrow M_\mu$. 
gives the same result as the $X = LV$ case, i.e., Eqs. (21) with (B2). In Eq. (B3) we inserted the differential decay width $d\Gamma^{(LC)}_{jj}(DD^*)/dE_\mu$ as obtained in Eq. (A.14) of Ref. [4].

[1] G. Racah, Nuovo Cim. 14, 322 (1937); W. H. Furry, Phys. Rev. 56, 1184 (1939); H. Primakoff and S. P. Rosen, Rep. Prog. Phys. 22, 121 (1959); ibid. Phys. Rev. 184, 1925 (1969); ibid. Ann. Rev. Nucl. Part. Sci. 31, 145 (1981); J. Schechter and J. W. F. Valle, Phys. Rev. D 25, 2951 (1982); M. Doi, T. Kotani and E. Takasugi, Prog. Theor. Phys. Suppl. 83, 1 (1985); S. R. Elliott and J. Engel, J. Phys. G G 30, R183 (2004) [hep-ph/0405078]; V. A. Rodin, A. Faessler, F. Simkovic and P. Vogel, Nucl. Phys. A 766, 107 (2000) [Erratum-ibid. A 793, 213 (2007)] [arXiv:0706.4304 [nucl-th]].

[2] Y. -Y. Keung and G. Senjanović, Phys. Rev. Lett. 50, 1427 (1983); V. Tello, M. Nemevšek, F. Nesti, G. Senjanović and F. Vissani, Phys. Rev. Lett. 106, 151801 (2011) [arXiv:1011.3522 [hep-ph]]; M. Nemevšek, F. Nesti, G. Senjanović and V. Tello, arXiv:1112.3061 [hep-ph]; G. Senjanović, Riv. Nuovo Cim. 034, 1 (2011).

[3] L. S. Littenberg and R. E. Shrock, Phys. Rev. Lett. 68, 443 (1992); ibid., Phys. Lett. B 491, 285 (2000) [hep-ph/0005258]; C. Dib, V. Gribanov, S. Kovalenko and I. Schmidt, Phys. Lett. B 493, 82 (2000) [hep-ph/0006277]; A. Ali, A. V. Borisov and N. B. Zamorin, Eur. Phys. J. C 21, 123 (2001) [hep-ph/0104123]; M. A. Ivanov and S. G. Kovalenko, Phys. Rev. D 71, 053004 (2005) [hep-ph/0412198]. A. de Gouvea and J. Jenkins, Phys. Rev. D 77, 013008 (2008) [arXiv:0708.1344 [hep-ph]]

[4] A. Boyarsky, O. Ruchayskiy, D. Iakubovskyi and J. Franse, arXiv:1402.4119 [astro-ph.CO].

[5] L. Canetti, M. Drewes and M. Shaposhnikov, Phys. Rev. D 82, 053010 (2010) [arXiv:1005.4282 [hep-ph]]; J. C. Helo, S. Kovalenko and I. Schmidt, Nucl. Phys. B 853, 80 (2011) [arXiv:1005.1607 [hep-ph]].

[6] Y. Fukuda et al. [Super-Kamiokande Collaboration], Phys. Rev. Lett. 81, 1562 (1998) [hep-ex/9807003].

[7] Q. R. Ahmad et al. [SNO Collaboration], Phys. Rev. Lett. 89, 011301 (2002) [nucl-ex/0204008]; P. Lipari, Phys. Rev. D 64, 033002 (2001) [hep-ph/0102046]; Z. Rahman, A. Dasgupta and R. Adhikari, arXiv:1210.2603 [hep-ph], arXiv:1210.4801 [hep-ph].

[8] K. Eguchi et al. [KamLAND Collaboration], Phys. Rev. Lett. 90, 021802 (2003) [hep-ex/0212021].

[9] P. Minkowski, Phys. Lett. B 67, 421 (1977); M. Gell-Mann, P. Ramond and R. Slansky, in Sanibel Conference, “The Family Group in Grand Unified Theories,” Febr. 1979, CALT-68-700, reprinted in hep-ph/9804459 "Complex Spinors and Unified Theories,” Print 80-0576, published in: D. Freedman et al. (Eds.), “Supergravity”, North-Holland, Amsterdam, 1979; T. Yanagida, Conf. Proc. C 7902131, 95 (1979); S. L. Glashow, in: M. Levy et al. (Eds.), “Quarks and Leptons,” Cargese, Plenum, New York, 1980, p. 707; R. N. Mohapatra and G. Senjanović, Phys. Rev. Lett. 44, 912 (1980).

[10] S. Bilenky, Introduction to the Physics of Massive and Mixed Neutrinos, Lecture Notes in Physics 817, Springer Verlag, Berlin, Heidelberg, 2010.

[11] N. Cabibbo, Phys. Lett. B 72, 333 (1978).

[12] T. Asaka, S. Blanchet and M. Shaposhnikov, Phys. Lett. B 631, 151 (2005) [hep-ph/0503065]; T. Asaka and M. Shaposhnikov, Phys. Lett. B 620, 17 (2005) [hep-ph/0505013].

[13] G. Gorbunov and M. Shaposhnikov, JHEP 0710, 015 (2007) [arXiv:0705.1729 [hep-ph]].

[14] J. Beringer et al. [Particle Data Group Collaboration], Phys. Rev. D 86, 010001 (2012).

[15] A. Atre, T. Han, S. Pascoli and B. Zhang, JHEP 0905, 030 (2009) [arXiv:0901.0359 [hep-ph]], and references therein.

[16] A. Boyarsky, O. Ruchayskiy and M. Shaposhnikov, Ann. Rev. Nucl. Part. Sci. 59, 191 (2009) [arXiv:0901.0011 [hep-ph]].

[17] L. Canetti, M. Drewes and M. Shaposhnikov, Phys. Rev. Lett. 110, no. 6, 061801 (2013) [arXiv:1204.3902 [hep-ph]]; L. Canetti, M. Drewes, T. Frossard and M. Shaposhnikov, Phys. Rev. D 87, 093006 (2013) [arXiv:1208.4607 [hep-ph]].

[18] E. Bulbul, M. Markevitch, A. Foster, R. K. Smith, M. Loewenstein and S. W. Randall, arXiv:1402.2301 [astro-ph.CO]; A. Boyarsky, O. Ruchayskiy, D. Iakubovskyi and J. Franse, arXiv:1402.4119 [astro-ph.CO].

[19] W. Bombenbo et al., CERN-SPSC-2013-024, CERN-EOI-010, arXiv:1310.1762 [hep-ex]; R. Jacobson, “Search for heavy neutral neutrinos at the SPS,” presented at “Search for Heavy Neutral Neutrinos at the SPS,” presented at High Energy Physics in the LHC Era., UTFSM, Valparaiso, Chile, December 16-20, 2014, https://indico.cern.ch/contributionDisplay.py?contribId=215&confId=258287.

[20] M. Aoki et al. [PIENU Collaboration], Phys. Rev. D 84, 052002 (2011) [arXiv:1106.4055 [hep-ex]].

[21] T. Yanazaki, T. Ishikawa, Y. Akiba, M. Iwasaki, K. H. Tanaka, S. Ohtake, H. Tamura and M. Nakajima et al., Conf. Proc. C 840719, 262 (1984); S. R. Hayano, T. Taniguchi, T. Yamanaka, T. Tanimori, R. Enomoto, A. Ishibashi, T. Ishikawa and S. Sato et al., Phys. Rev. Lett. 49, 1305 (1982); A. Kusenko, S. Pascoli and D. Semikoz, JHEP 0511, 028 (2005) [hep-ph/0405198].

[22] J. Orloff, A. N. Rozanov and C. Santoni, Phys. Lett. B 550, 8 (2002) [hep-ph/0208075].

[23] O. Ruchayskiy and A. Ivashko, JHEP 1206, 100 (2012) [arXiv:1112.3319 [hep-ph]].

[24] Project X and the Science of the Intensity Frontier, white paper based on the Project X Physics Workshop, Fermilab, USA, 9-10 November 2009 (http://projectx.fnal.gov/pdfs/ProjectXwhitepaperJan.v2.pdf).

7 There is a typo in Eq. (A.16) of Ref. [4], i.e., in the expression for $d\Gamma^{(LC)}_{jj}(DD^*)/dE_\mu$ in the $M_\nu = 0$ limit: in the second line of that equation, $E^2_\ell$ must be replaced by $2E^2_\ell$. 
[25] S. Geer, private communication, 2012.