Notes on solution maps of abstract FDEs

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Let $\tau$ be a positive real number, $X$ be a Banach space, and $C := C([-\tau,0],X)$. For any $\phi \in C$, define $\|\phi\| = \sup_{-\tau \leq \theta \leq 0} \|\phi(\theta)\|_X$. Then $(C,\|\cdot\|)$ is a Banach space. Let $A$ be the infinitesimal generator of a $C_0$-semigroup \{$T(t)$\}_{t \geq 0} on $X$. Assume that $T(t)$ is compact for each $t > 0$, and there exists $M > 0$ such that $\|T(t)\| \leq M$ for all $t \geq 0$.

We consider the abstract functional differential equation

$$\begin{align*}
\frac{du(t)}{dt} &= Au(t) + F(t,u_t), \quad t > 0, \\
u_0 &= \phi \in C.
\end{align*}$$

(0.1)

Here $F : [0,\infty) \times C \to X$ is continuous and maps bounded sets into bounded sets, and $u_t \in C$ is defined by $u_t(\theta) = u(t + \theta)$, $\forall \theta \in [-\tau,0]$.

**Theorem A.** Assume that for each $\phi \in C$, equation (0.1) has a unique solution $u(t,\phi)$ on $[0,\infty)$, and solutions of (0.1) are uniformly bounded in the sense that for any bounded subset $B_0$ of $C$, there exists a bounded subset

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$B_1 = B_1(B_0)$ of $C$ such that $u_t(\phi) \in B_1$ for all $\phi \in B_0$ and $t \geq 0$. Then for any given $r > 0$, there exists an equivalent norm $\| \cdot \|_r^*$ on $C$ such that the solution maps $Q(t) := u_t$ of equation (0.1) satisfy $\alpha(Q(t)B) \leq e^{-rt}\alpha(B)$ for any bounded subset $B$ of $C$ and $t \geq 0$, where $\alpha$ is the Kuratowski measure of noncompactness in $(C, \| \cdot \|_r^*)$.

Proof. Define $\| x \|_r^* = \sup_{t \geq 0} \| T(t)x \|$, $\forall x \in X$. Then $\| x \| \leq \| x \|_r^* \leq M \| x \|$, and hence, $\| x \|_r^*$ is an equivalent norm on $X$. It is easy to see that

$$\| T(t)x \|_r^* = \sup_{s \geq 0} \| T(s)T(t)x \| = \sup_{s \geq 0} \| T(s + t)x \| \leq \| x \|_r^*, \quad \forall x \in X, \ t \geq 0,$$

which implies that $\| T(t) \|_r^* \leq 1$ for all $t \geq 0$. Thus, without loss of generality, we assume that $M = 1$.

Let $r > 0$ be given. Note that for each $\phi \in C$, the solution $u(t, \phi)$ of (0.1) satisfies the following integral equation

$$u(t) = \hat{T}(t)\phi(0) + \int_0^t \hat{T}(t-s)\hat{F}(s,u_s)ds, \quad t \geq 0,$$

$$u_0 = \phi \in C,$$

where $\hat{T}(t) = e^{-rt}T(t)$ and $\hat{F}(t,\varphi) = r\varphi(0) + F(t,\varphi)$, $\forall t \geq 0$, $\varphi \in C$. Then $\hat{T}(t)$ is also a $C_0$-semigroup on $X$ and $\| \hat{T}(t) \| \leq e^{-rt}$, $\forall t \geq 0$. Let $h(\theta) = e^{-r\theta}, \forall \theta \in [-\tau, 0]$, and define

$$\| \phi \|_r^* = \sup_{-\tau \leq \theta \leq 0} \frac{\| \phi(\theta) \|_X}{h(\theta)}, \quad \forall \phi \in C.$$

Then $\frac{1}{h(-\tau)}\| \phi \|_C \leq \| \phi \|_r^* \leq \| \phi \|_C$, and hence $\| \cdot \|_r^*$ is equivalent to $\| \cdot \|_C$. Clearly, $\| \phi(0) \|_X \leq \| \phi \|_r^*$, $\forall \phi \in C$. Define

$$(L(t)\phi)(\theta) = \begin{cases} \hat{T}(t+\theta)\phi(0), & t + \theta > 0, \\ \phi(t+\theta), & t + \theta \leq 0, \end{cases}$$

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and
\[
(Q(t)\phi)(\theta) = \begin{cases} 
\int_{0}^{t+\theta} \hat{T}(t+\theta-s)\hat{F}(s, u_s(\phi))ds, & t + \theta > 0, \\
0, & t + \theta \leq 0.
\end{cases}
\]

Thus, \( Q(t)\phi = L(t)\phi + \bar{Q}(t)\phi, \forall t \geq 0, \phi \in C, \) that is, \( Q(t) = L(t) + \bar{Q}(t), \forall t \geq 0. \)

We first show that \( L(t) \) is an \( \alpha \)-contraction on \((C, \| \cdot \|_r)\) for each \( t > 0 \). It is easy to see that \( L(t) \) is compact for each \( t > \tau \). Without loss of generality, we may assume that \( t \in (0, \tau] \) is fixed. For any \( \phi \in C \), we have
\[
\| L(t)\phi \|_r^* = \sup_{-\tau \leq \theta \leq 0} \frac{\| (L(t)\phi)(\theta) \|_X}{h(\theta)}
\leq \max \left\{ \sup_{-\tau \leq \theta \leq -t} \frac{\| \phi(t+\theta) \|_X}{h(t+\theta)} \frac{h(t + \theta)}{h(\theta)}, \sup_{-t \leq \theta \leq 0} \frac{\| \hat{T}(t+\theta)\phi(0) \|_X}{h(\theta)} \right\}
\leq \max \left\{ e^{-rt}\| \phi \|_r^*, \sup_{-\tau \leq \theta \leq 0} \frac{e^{-r(t+\theta)}\| \phi(0) \|_X}{h(\theta)} \right\}
= \max \left\{ e^{-rt}\| \phi \|_r^*, e^{-rt}\| \phi(0) \|_X \right\} \leq e^{-rt}\| \phi \|_r^*,
\]
which implies that \( \alpha(L(t)B) \leq e^{-rt}\alpha(B) \) for any bounded subset \( B \) of \( C \).

Thus, this contraction property holds true for all \( t > 0. \)

Next we prove that \( \bar{Q}(t) : C \to C \) is compact for each \( t > 0. \) Let \( t > 0 \) and the bounded subset \( B \) of \( C \) be given. By the uniform boundedness of solutions, there exists a real number \( K > 0 \) such that \( \| \hat{F}(s, u_s(\phi)) \|_X \leq K, \forall s \in [0, t], \phi \in B. \) It then follows that \( \bar{Q}(t)B \) is bounded in \( C. \) We only need to show that \( \bar{Q}(t)B \) is precompact in \( C. \) In view of the Arzela–Ascoli theorem for the space \( C := C([-\tau, 0], X), \) it suffices to prove that (i) for each \( \theta \in [-\tau, 0], \) the set \( \{(\bar{Q}(t)\phi)(\theta) : \phi \in B\} \) is precompact in \( X; \) and (ii) the set \( \bar{Q}(t)B \) is equi-continuous in \( \theta \in [-\tau, 0]. \) Clearly, statement (i) holds true if \( t + \theta \leq 0. \) In the case where \( t + \theta > 0, \) for any given \( \epsilon \in (0, t + \theta), \) we
have

\[
(\tilde{Q}(t)\phi)(\theta) = \int_0^{t+\theta-\epsilon} \hat{T}(t+\theta-s)\hat{F}(s,u_s(\phi))ds + \int_{t+\theta-\epsilon}^{t+\theta} \hat{T}(t+\theta-s)\hat{F}(s,u_s(\phi))ds
\]

\[
= \hat{T}(\epsilon) \int_0^{t+\theta-\epsilon} \hat{T}(t+\theta-\epsilon-s)\hat{F}(s,u_s(\phi))ds + \int_{t+\theta-\epsilon}^{t+\theta} \hat{T}(t+\theta-s)\hat{F}(s,u_s(\phi))ds.
\]

Define

\[
S_1 := \left\{ \hat{T}(\epsilon) \int_0^{t+\theta-\epsilon} \hat{T}(t+\theta-\epsilon-s)\hat{F}(s,u_s(\phi))ds : \phi \in B \right\}
\]

and

\[
S_2 := \left\{ \int_{t+\theta-\epsilon}^{t+\theta} \hat{T}(t+\theta-s)\hat{F}(s,u_s(\phi))ds : \phi \in B \right\}.
\]

Let \( \hat{\alpha} \) be the Kuratowski measure of noncompactness in \( X \). Since \( \hat{T}(\epsilon) \) is compact, it follows that

\[
\hat{\alpha}(\{(\tilde{Q}(t)\phi)(\theta) : \phi \in B\}) \leq \hat{\alpha}(S_1) + \hat{\alpha}(S_2) \leq 0 + 2K\epsilon = 2K\epsilon.
\]

Letting \( \epsilon \to 0^+ \), we obtain \( \hat{\alpha}(\{(Q(t)\phi)(\theta) : \phi \in B\}) = 0 \), which implies that the set \( \{(Q(t)\phi)(\theta) : \phi \in B\} \) is precompact in \( X \). It remains to verify statement (ii). Since \( \hat{T}(s) \) is compact for each \( s > 0 \), \( \hat{T}(s) \) is continuous in the uniform operator topology for \( s > 0 \) (see [2, Theorem 2.3.2]). It then follows that for any \( \epsilon \in (0, t) \), there exists a \( \delta = \delta(\epsilon) < \epsilon \) such that

\[
\|\hat{T}(s_1) - \hat{T}(s_2)\| < \epsilon, \quad \forall s_1, s_2 \in [\epsilon, t] \text{ with } |s_1 - s_2| < \delta.
\] (0.3)

We first consider the case where \( t \in (0, \tau] \). It is easy to see that

\[
\|(\tilde{Q}(t)\phi)(\theta)\|_X \leq K(t+\theta) \leq K\epsilon, \quad \forall \theta \in [-t,-t+\epsilon], \phi \in B.
\] (0.4)
For any $\phi \in B$ and $\theta_1, \theta_2 \in [-t + \epsilon, 0]$ with $0 < \theta_2 - \theta_1 < \delta$, it follows from (0.3) that

\[
\| (\tilde{Q}(t)\phi)(\theta_2) - (\tilde{Q}(t)\phi)(\theta_1) \|_X
= \left\| \int_0^{t - \epsilon + \theta_1} \left( \tilde{T}(t + \theta_2 - s) - \tilde{T}(t + \theta_1 - s) \right) \hat{F}(s, u_s(\phi)) ds \right\|_X
+ \left\| \int_{t - \epsilon + \theta_1}^{t + \theta_1} \tilde{T}(t + \theta_2 - s) \hat{F}(s, u_s(\phi)) ds \right\|_X
+ \left\| - \int_{t - \epsilon + \theta_1}^{t + \theta_1} \tilde{T}(t + \theta_1 - s) \hat{F}(s, u_s(\phi)) ds \right\|_X
\leq \epsilon K t + K(\theta_2 - \theta_1 + \epsilon) + K \epsilon
< (t + 3)K \epsilon.
\]  

(0.5)

Combining (0.4) and (0.5), we then obtain

\[
\| (\tilde{Q}(t)\phi)(\theta_2) - (\tilde{Q}(t)\phi)(\theta_1) \|_X < 2K \epsilon + (t + 3)K \epsilon = (t + 5)K \epsilon,
\]

for all $\theta_1, \theta_2 \in [-t, 0]$ with $0 \leq \theta_2 - \theta_1 < \delta$. Since $(\tilde{Q}(t)\phi)(\theta) = 0, \forall \theta \in [-\tau, -t]$, it follows that $\tilde{Q}(t)B$ is equi-continuous in $\theta \in [-\tau, 0]$. In the case where $t > \tau$, for any $\epsilon \in (0, t - \tau)$, the estimate in (0.5) implies that

\[
\| (\tilde{Q}(t)\phi)(\theta_2) - (\tilde{Q}(t)\phi)(\theta_1) \|_X < (t + 3)K \epsilon,
\]

for all $\theta_1, \theta_2 \in [-\tau, 0]$ with $0 \leq \theta_2 - \theta_1 < \delta$, and $\phi \in B$. Thus, $\tilde{Q}(t)B$ is equi-continuous in $\theta \in [-\tau, 0]$. It then follows that $\tilde{Q}(t) : C \to C$ is compact for each $t > 0$.

Consequently, for any $t > 0$ and any bounded subset $B$ of $C$, we have

\[
\alpha(\tilde{Q}(t)B) \leq \alpha(L(t)B) + \alpha(\tilde{Q}(t)B) \leq e^{-rt} \alpha(B).
\]

This completes the proof.
As an application example, we consider the following \( \omega \)-periodic reaction-diffusion system

\[
\begin{align*}
\frac{\partial u}{\partial t} &= D \triangle u + f(t, u_t), \quad x \in \Omega, \ t > 0, \\
\frac{\partial u}{\partial \nu} &= 0, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad x \in \partial \Omega, \ t > 0,
\end{align*}
\]

where \( D = \text{diag}(d_1, \ldots, d_m) \) with each \( d_i > 0 \), \( \Omega \subset \mathbb{R}^n \) is a bounded domain with the smooth boundary \( \partial \Omega \), and \( f(t, \phi) \) is \( \omega \)-periodic in \( t \in [0, \infty) \) for some \( \omega > 0 \). Let \( Y := C(\bar{\Omega}, \mathbb{R}^m) \) and assume that \( f \) is continuous and maps bounded subsets of \([0, \infty) \times C([-\tau, 0], Y)\) into bounded subsets of \( Y \). Let \( T(t) \) be the semigroup on \( Y \) generated by \( \frac{\partial u(t, x)}{\partial t} = D \triangle u(t, x) \) subject to the boundary condition \( \frac{\partial u}{\partial \nu} = 0 \). It is easy to see that \( \|T(t)\| \leq 1, \forall t \geq 0 \). By Theorem A, we then have the following result.

**Theorem B.** Assume that solutions of system (0.6) exist uniquely on \([0, \infty)\) for any initial data in \( C := C([-\tau, 0], Y) \) and are uniformly bounded. Then for each \( r > 0 \), there exists an equivalent norm \( \| \cdot \|_r \) on \( C \) such that for each \( t > 0 \), the solution map \( Q(t) = u_t \) of system (0.6) is an \( \alpha \)-contraction on \( (C, \| \cdot \|_r \) with the contraction constant being \( e^{-rt} \).

**Remark.** By using the theory of evolution operators (see, e.g., [1, Section II.11] and [2, Section 5.6]), one may extend Theorem A to the abstract functional differential equation \( \frac{du(t)}{dt} = A(t)u(t) + F(t, u_t) \) with \( u_0 = \phi \in C \) under appropriate assumptions.

**References**

[1] P. Hess, *Periodic-Parabolic Boundary Value Problems and Positivity*, Pitman Research Notes in Math., Series 247, Longman Scientific and
Technical, 1991.

[2] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.