THE VIRIAL THEOREM FOR ACTION-GOVERNED THEORIES

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Abstract. We describe a simple derivation of virial relations, for arbitrary physical systems that are governed by an action. The virial theorem may be derived directly from the action, \( A \), with no need to go via the equations of motion, and is simply a statement of the stationarity of the action with respect to certain variations in the degrees of freedom. Only a sub-class of solutions that obey appropriate boundary conditions satisfy each virial relation. There is a set of basic virial relations (one for each degree of freedom) of the form

\[ \frac{\partial A^i}{\partial c} \bigg|_{c=1} = 0, \]

where \( A^i \) is the action in which one degree of freedom has been scaled by a constant factor \( c \). Linear combinations of these may be put in simple forms, taking advantage of homogeneity properties of the action, and of dimensional considerations. When some of the degrees of freedom are of the same type a tensor virial theorem presents itself; it may be obtained in a similar way by considering the variation of the action under linear transformations among these degrees of freedom. Further generalizations are discussed. Symmetries of the action may lead to identities involving the virial relations. Beside pointing to a unified provenance of the virial relations, and affording general systematics of them, our method is a simple prescription for deriving such relations. It is particularly useful for treating high-derivative and non-local theories. We bring several examples to show that indeed the usual virial relations are obtained by this procedure, and also to produce some new virial relations.

1. Introduction

In default of a general definition of the virial theorem we describe it, drawing from known examples (see, for some of many,[1-9], and, in particular,[10]): It consists of a set of global (integral) relations that are satisfied by a subclass of solutions of the equations of motion. This sub-class is defined by requiring that the solutions obey certain boundary requirements. The derivation of these relations proceeds as follows: One contracts the equations of motion with functions of the degrees of freedom and integrate over the variables on which the latter depend. One then integrates by parts so as to reduce the order of derivatives, as many times as possible–discarding boundary terms, as one proceeds, by imposing requirements on the boundary behavior of the solutions. One ends up with a virial theorem consisting of (1) the set of integral relations, and (2) the set of boundary conditions under which they apply.

The applications of this poor-man’s substitute for the equations of motion are many and varied (see e.g. [3][10][11-13], and the many uses described in [10] ). Its usefulness draws partly from the fact that it involves derivatives of the degrees of freedom of lower order than appear in the equations of motion. (We shall show, in fact, that the order of derivation appearing is the same as that in the action itself.) Thus, for example, the applications in astrophysics require knowledge of only particle velocities, which can be measured, and not of the accelerations, which cannot. This feature also makes the virial theorem a useful tool for checking numerical solutions of the equations of motion.

Here, we give a systematic treatment of the virial theorem, for arbitrary systems governed by equations that are derived from a variational principle, in a way that highlights its origin, and greatly facilitate its derivation in the case of higher order theories, and especially for non-local theories.

We explain how the virial relations emerge from the action in sections 2 and 3, showing

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that they are tantamount to stationarity of the action under certain increments in the degrees of freedom; when the variation of the action under such increments may be expressed simply, a convenient virial relation results. We demonstrate the procedure with various examples in section 4.

2. Derivation of the virial relations from the action

Virial relations may be derived for action-governed systems as follows: The system is described by $d$ degrees of freedom $f_i[\eta^{(i)}], \quad 1 \leq i \leq d$, where, for each $i$, $\eta^{(i)}$ is the set of variables, $\eta^{(i)}, ..., \eta^{(i)}$, on which $f_i$ depends (time, path length, space-time coordinates for fields, Fourier variables if the degree of freedom is described in some Fourier space, etc.). One assumes that there exits an action $A$: a functional of $f_i$, such that under a variation $\delta f_i$ in $f_i$

$$\delta A = \sum_i \int d\eta^{(i)} \, Q_i \, \delta f_i + \sum_i (F_i(\delta f_i))_{\text{ends}},$$  \hspace{1cm} (1)$$

where $Q_i(f_1, ..., f_d, \eta^{(i)})$ is a functional of the degrees of freedom and a function of $\eta^{(i)}$ only; $F_i$ is a linear operator acting on $\delta f_i$, and may depend as a functional on all the $f_i$s. The term $(\text{ends})$ is a boundary term evaluated at the boundary of the respective $\eta^{(i)}$ space. One then postulates that the physical states are described by $f_i$s for which $A$ is stationary under variations $\delta f_i$ that annihilate the ends term. These $f_i$s are then the solutions of the equations of motion

$$Q_i(f_1, ..., f_d, \eta^{(i)}) = 0 \quad 1 \leq i \leq d. \hspace{1cm} (2)$$

Consider now, as a specific change, a rescaling of a single degree of freedom $f_i$

$$f_i \rightarrow (1 + \epsilon)f_i = f_i + \delta f_i. \hspace{1cm} (3)$$

The change in the action is then, by eq. (1)

$$\delta A = \epsilon \int d\eta^{(i)} \, Q_i \, f_i + \epsilon (F_i(f_i))_{\text{ends}}. \hspace{1cm} (4)$$

On the other hand, we can write

$$\delta A = \epsilon \left. \frac{\partial A_i}{\partial c} \right|_{c=1}, \hspace{1cm} (5)$$

where

$$A_i(c_1, ..., f_i, ..., f_d) \equiv A(f_1, ..., cf_i, ..., f_d). \hspace{1cm} (6)$$

We thus have

$$\left. \frac{\partial A_i}{\partial c} \right|_{c=1} = \int d\eta^{(i)} \, Q_i \, f_i + (F_i(f_i))_{\text{ends}}. \hspace{1cm} (7)$$

Equation (7) is but an identity. If we now specialize to solutions of the equations of motion (2) we get

$$\left. \frac{\partial A_i}{\partial c} \right|_{c=1} = (F_i(f_i))_{\text{ends}}, \hspace{1cm} (8)$$

which is non-trivial (i.e. is not satisfied by generic $f_i[\eta^{(i)}]$ that are not solutions of the equations of motion ). It tells us that certain expressions derived from the action are pure boundary terms.

The virial relation results when we further restrict ourselves to solutions for which the ends terms may be discarded. This may happen because the particular $f_i$s satisfy boundary conditions in $\eta^{(i)}$ space that render $(F_i(f_i))_{\text{ends}}$ zero, or, more generally, because for a large enough $\eta^{(i)}$ volume the ends term remains finite while $A$ becomes infinite (in which case we normalize by the volume, and the ends term vanish in the limit). We then obtain as our desired virial relations

$$\left. \frac{\partial A_i}{\partial c} \right|_{c=1} = 0 \quad 1 \leq i \leq d, \hspace{1cm} (9)$$
each satisfied on its own sub-class of solutions for which the ends term can be discarded. Clearly, eq. (9) is just the relation we would get by taking the equation of motion derived by varying $A$ with respect to $f_l$, then multiplying by $f_l$, then integrating by parts as many times as needed to reduce the order of derivatives back to that in $A$. The present method is then but a way to shortcut this return journey to the equations of motion.

Linear combination of relations (9) with coefficients $\alpha$—which should hold under the appropriate boundary behavior—may be conveniently written as

$$\left. \frac{\partial A^{\alpha_1\ldots\alpha_d}}{\partial c} \right|_{c=1} = 0,$$

(10)

where

$$A^{\alpha_1\ldots\alpha_d}_c = A(c^{\alpha_1} f_1, ..., c^{\alpha_d} f_d).$$

(11)

Some of these linear combinations may be more useful than others. For example, some may be reduced to simpler forms by employing homogeneity properties of the action, or its parts. If there is a choice of $\alpha_i$ for which $A^{\alpha_1\ldots\alpha_d} = c^\beta A$, then, by eq. (10), $A = 0$ is one of the virial relations. (If $\beta = 0$ the $d$ relations (9) are dependent, but the theory is then degenerate.) This is the case for any system with a Newtonian kinetic action and harmonic forces, in which case $A = 0$ implies the equality of mean kinetic and potential energies. It is also the case for the Einstein-Hilbert action in vacuum, in which case the resulting relation is just the integral of the trace of the equation of motion.

More generally, the action may be split into terms

$$A = \sum_a A_a,$$

(12)

all homogeneous with the same set of $\alpha_i$s, but each with its own $\beta_a$. Then

$$\sum_a \beta_a A_a = 0$$

(13)

is one of the virial relations. This is the case for a system of particles with a Newtonian kinetic action, and forces that are homogeneous in the positions (e.g. power-law forces). Equation (13) then tell us that the ratio of mean kinetic and potential energies is fixed for the theory. It is also the case for General Relativity with matter in the form of charged particles, and electromagnetic fields (see sec. 6).

We may also apply dimensional analysis to obtain some useful combinations of the virial relations. Suppose the action depends on some constants $q_p$, $1 \leq p \leq n$. Consider then, for instance, how the different quantities change under a change in the units of length by a factor $c$. Depending on their respective dimensions, $f_l$ will change to $c^{\alpha_l} f_l$, $q_p$ will change to $c^{\gamma_p} q_p$, while $A$ will change to $c^\beta A$. (We assume that $\eta^{(i)}$ have no length dimension, as when they denote the time, or if they do we redefine them not to have such dimension, and then we have to add to the $q_p$ another constant of the dimensions of length.) Dimensional analysis then tells us that

$$A(c^{\alpha_1} f_1, ..., c^{\alpha_i} q_1, ...) = c^\beta A,$$

(14)

and so

$$A^{\alpha_1\ldots\alpha_d}_c = c^\beta A(f_l, ..., f_d, c^{\gamma_1} q_1, ..., c^{\gamma_n} q_n).$$

(15)

From this and eq. (10) we then obtain

$$\beta A - \sum_{p=1}^n \gamma_p q_p \frac{\partial A}{\partial q_p} = 0,$$

(16)

as one of the virial relations. This is particularly useful when $A$ depends only on a small number of constants.

3. Tensor virial theorems and further generalizations
Often, some of the degrees of freedom are of the same type—say \( f_i \) for \( 1 \leq i \leq K \), collectively designated \( \vec{f} \). Then, further virial relations suggest themselves. (Examples of degrees of freedom of the same type are the position coordinates of a particle, or indeed, of different particles, the components of the electromagnetic vector potential field, the elements of the metric tensor etc.)

We now consider the variation of the action under variations of the form

\[
\vec{f} \rightarrow \mathcal{M} \vec{f},
\]

(17)

where \( \mathcal{M} = 1 + \mathcal{E} \) (with \( \mathcal{E} \) infinitesimal) is a constant \( K \times K \) matrix. From eq. (1) we have

\[
\delta A = \sum_{ij} \mathcal{E}_{ij} \left[ \int d\eta \, Q_i \, f_j + [\mathcal{F}_i(f_j)]_{\text{ends}} \right].
\]

(18)

(All the \( Q_i \) and \( f_i \), \( i \leq K \), are defined on the same \( \eta \) space.)

On the other hand, defining

\[
\mathcal{A}_M(\vec{f}, f_{K+1}, \ldots, f_D) = \mathcal{A}(\mathcal{M} \vec{f}, f_{K+1}, \ldots, f_D),
\]

(19)

we see that

\[
\delta A = \sum_{ij} \mathcal{E}_{ij} V_{ij},
\]

(20)

where

\[
V_{ij} = \frac{\partial \mathcal{A}_M}{\partial \mathcal{M}_{ij}} \bigg|_{\mathcal{M}_{ij} = \delta_{ij}}.
\]

(21)

Thus

\[
V_{ij} = \int d\eta \, Q_i \, f_j + [\mathcal{F}_i(f_j)]_{\text{ends}}.
\]

(22)

As in section 2, we confine ourselves to solutions of the equations of motion for which the ends term in eq. (22) can be discarded and then

\[
V_{ij} = 0,
\]

(23)

which is the desired \( K \times K \)-tensor virial theorem (the diagonal elements of relation (23) are the same as eq. (9)). In addition we still have, of course, the relations of type (9) for all the degrees of freedom \( f_i \), \( i > K \).

Symmetries of the action under transformations of \( \vec{f} \) lead to constraints on the tensor virial relations. For example, say that \( \mathcal{A} \) is invariant under a rotations in \( \vec{f} \) space, i.e., under \( \vec{f} \rightarrow U \vec{f} \), for any orthogonal matrix (continuously connected to the unit matrix). Then, from eq. (20), \( \sum_{ij} \mathcal{E}_{ij} V_{ij} = 0 \) for any antisymmetric matrix \( \mathcal{E} \) (the generators of the orthogonal transformations). Thus, \( V \) must be a symmetric matrix: \( V_{ij} = V_{ji} \) is then an identity, and there are only \( K(K + 1)/2 \) independent relations. Note that symmetries of the action do not necessarily imply symmetries of the above type where only some of the degrees of freedom are transformed.

The information held by the equations of motion on their solutions is infinitely greater than that supplied by the finite number of discrete, global relations that we have discussed so far. It should be obvious then that one can derive an infinite number of further such relations. These emerge if we consider the variation of the action under general infinitesimal increments of the form

\[
f_i \rightarrow f_i + \xi_i(f_1, \ldots, f_d).
\]

(24)

Here \( \xi_i \) may be a function of the degrees of freedom and also a functional of them, but it must, of course, be “of the same type” as \( f_i \) itself (in the sense that the action is defined if we replace \( f_i \) by \( \xi_i \)). As before, if the ends term \([\mathcal{F}_i(\xi_i)]_{\text{ends}}\) may be discarded, the variation of \( \mathcal{A} \) under (24) vanishes on solutions of the equations of motion. When, for given \( \mathcal{A} \) and \( \xi \), \( \delta \mathcal{A} \) can be written in a simple form—as was the case for the linear increments we had discussed earlier—a useful relation may be obtained. The point is that because the increment (24) is defined in
terms of the degrees of freedom themselves it enables us to formulate the extremum condition on \( A \), and the required boundary conditions, in terms of the \( f_i \)'s themselves thus leading to a virial-like relation. When it is not necessary to have a relations written in terms of system properties alone, it may be useful to consider more general increments in which \( \xi_i \) is also an explicit function of \( \eta_i(t) \). Thus considering \( f_i \rightarrow \epsilon \eta_i \hat{f}_i \) will lead to higher-tensorial-order relations (for a system of Newtonian particles they correspond to relations connected with higher moments of the mass distribution—see section 5).

While these further relations are not as amenable to many of the uses as their predecessors, they may be useful, depending on the particular case. For example, in connection with checks on numerical solutions of the equations of motion the may provide checks of aspects not covered by the basic relations (see section 4).

We have discussed virial relations as they apply to the full sub-class of solutions that satisfy the appropriate boundary conditions. When one limits himself further to special solutions—such as ones with certain symmetry properties—the virial theorem may, in general, be simplified; we shall not deal with such further restrictions.

4. Examples

Now let us look at some examples, mainly to demonstrate the procedure and various relevant points discussed generally in the previous sections. We start with the archetype of all virial theorems: that for a system of \( N \) Newtonian particles interacting via a general \( N \)-body potential. The action for a finite time span \((t_1, t_2)\) is

\[
A = A_K + A_P, \tag{25}
\]

with the kinetic action

\[
A_K = T^{-1} \int_{t_1}^{t_2} \frac{1}{2} \sum_i m_i \left( \frac{d\vec{r}_i}{dt} \right)^2 dt, \tag{26}
\]

and the potential action

\[
A_P = -T^{-1} \int \Phi[\vec{r}_1(t), ..., \vec{r}_N(t)] dt. \tag{27}
\]

We have normalized the standard action by \( T \equiv t_2 - t_1 \), to make \( A \) finite for \( T \rightarrow \infty \).

The degrees of freedom are the \( 3N \) components \( r_i^\alpha \) of the positions. We get, straightforwardly, from eq. (9) the \( 3N \) virial relations

\[
\langle m_i (\dot{r}_i^\alpha)^2 \rangle - \langle \vec{r}_i^\alpha \partial \Phi / \partial r_i^\alpha \rangle = 0, \tag{28}
\]

where

\[
\langle Q \rangle \equiv T^{-1} \int_{t_1}^{t_2} Q \ dt. \tag{29}
\]

Usually this average is taken in the limit \( t_1 \rightarrow -\infty, \) and \( t_2 \rightarrow \infty, \) and it then holds for solutions in which all particles are bounded in a finite volume. The standard scalar virial theorem for this case:

\[
\langle \sum_i m_i v_i^2 \rangle - \langle \sum_i \vec{r}_i \cdot \vec{\nabla} \vec{r}_i \Phi \rangle = 0, \tag{30}
\]

is obtained by differentiating \( A[\vec{r}_1(t), ..., \vec{r}_N(t)] \) with respect to \( c \) (at \( c = 1 \)). There is a \( 3N \times 3N \) tensor virial theorem

\[
\langle m_i \dot{r}_i^\alpha \dot{r}_j^\beta \rangle - \langle r_i^\alpha \partial \Phi / \partial r_j^\beta \rangle = 0, \tag{31}
\]

which is obtained from eq. (23) , by considering \( r_i^\alpha \rightarrow r_i^\alpha + \epsilon r_j^\beta \), for fixed values of all indices. Neither term is symmetric when the masses are not all equal, because neither term in the action is symmetric under rotations in configuration space. The standard tensor virial theorem
is obtained by taking the trace over the particle indices, or by considering $r_i^a \to r_i^a + \epsilon r_i^a$ for all values of $i$ but fixed $\alpha, \beta$.

$$\langle \sum_i m_i \dot{r}_i^a \dot{r}_i^a - (\dot{r}_i^a \partial \Phi / \partial r_i^a) \rangle = 0. \quad (32)$$

The kinetic term is now identically symmetric because the kinetic action is symmetric under simultaneous rotations of all the $\vec{r}_i$. The antisymmetric part of the potential term is $\sum_i \vec{r}_i \times \vec{\nabla} \Phi$ which vanishes identically if $\Phi$ is rotationally invariant, in which case relation (32) is identically symmetric in $\alpha \beta$.

We now add a field degree of freedom by considering a system of $N$ gravitation particles of masses $m_i$ and we also want to solve for the (Newtonian) gravitational field $\varphi(\vec{r})$ which is thus also a degree of freedom. The action is

$$\mathcal{A} = T^{-1} \int dt \left\{ \sum_i \frac{1}{2} m_i \dot{v}_i^2 - \sum_i m_i \varphi(r_i) - \frac{1}{8\pi G} \int (\vec{\nabla} \varphi)^2 d^3 r \right\}. \quad (33)$$

It gives the standard equation of motion of the particles in the potential $\varphi$, and variation over $\varphi$ gives the Poisson equation for $\varphi$ with the density $\rho(\vec{r}) = \sum_i m_i \delta^3(\vec{r} - \vec{r}_i)$.

Beside the scalar and tensor virial relations we wrote in the previous example (with $\Phi(\vec{r}_1(t), ..., \vec{r}_N(t)) = \sum_i m_i \varphi(\vec{r}_i(t))$), we now have a relation resulting from $\varphi \to \epsilon \varphi$ which is read directly from the action—the three terms in $\mathcal{A}$ being homogeneous in $\varphi$, with powers 0, 2, and 1 respectively:

$$\langle \sum_i m_i \varphi(r_i) \rangle + \frac{1}{4\pi G} \langle \int (\vec{\nabla} \varphi)^2 d^3 r \rangle = 0. \quad (34)$$

To demonstrate the potential usefulness of relations resulting from the more general types of variations eq. (24), consider an infinitesimal increment

$$\varphi \to \epsilon J(\varphi). \quad (35)$$

Substituting in the action, and taking the first order in $\epsilon$ we get

$$\langle \sum_i m_i J(\varphi(r_i)) \rangle + \frac{1}{4\pi G} \langle \int J'(\varphi)(\vec{\nabla} \varphi)^2 d^3 r \rangle = 0, \quad (36)$$

with the boundary requirement being that $J(\varphi) = 0$. Clearly, this infinite set of relations captures more of the information in the equation of motion. By choosing $J$ properly we may accentuate different regions of space, for example regions where there are better measurements, or where better approximations can be made.

Consider now a system of particles governed by a general kinetic action, so the action is of the form

$$\mathcal{A} = \mathcal{A}_K(\vec{r}_1(t), ..., \vec{r}_N(t), q_1, ..., q_n) - T^{-1} \int \Phi(\vec{r}_1(t), ..., \vec{r}_N(t)) dt. \quad (37)$$

where $\mathcal{A}_K$ is a general functional of the particle trajectories $\vec{r}_i(t)$ (including higher-derivative or, indeed, non-local actions, in which case the full world lines of the particles are integrated on), and $q_0$ are constants. We apply eq. (16) noting that $\beta = 2$ (with our choice of normalization $\mathcal{A}$ has dimensions of energy), and that $\Phi$ must be of the form

$$\Phi = \Phi_0 \phi(\vec{r}_1/l, ..., \vec{r}_N/l). \quad (38)$$

Here, $\Phi_0$, and $l$ are constants with the dimensions of energy and length, respectively. Thus, we obtain in generalization of the scalar virial theorem resulting from $\vec{r}_i(t) \to (1 + \epsilon)\vec{r}_i(t)$ (see [13])

$$2\mathcal{A}_K - \sum_{p=1}^n g_p g_p \frac{\partial \mathcal{A}_K}{\partial q_p} - \langle \sum_{i=1}^N \dot{r}_i \cdot \vec{\nabla} \phi(\vec{r}_1) \Phi \rangle = 0. \quad (39)$$
As in eq. (16), \( \gamma_p \) are the powers of length in the dimensions of the constants \( q_p \).

When \( A_K \) is a Lagrangian action:

\[
A_K = T^{-1} \int L_K[\vec{r}_i, \vec{r}_i^{(1)}, \vec{r}_i^{(2)}, ..., q_p] \, dt, \tag{40}
\]

with the kinetic Lagrangian \( L_K \) depending on higher time derivatives \( \vec{r}_i^{(l)} \) of the trajectories, we can write, in generalization of the standard scalar virial theorem, either

\[
\left\langle \sum_{i=1}^N \sum_l r_i^{(l)} \frac{\partial L_K}{\partial \dot{r}_i^{(l)}} \right\rangle - \left\langle \sum_{i=1}^N \vec{r}_i \cdot \vec{\nabla} \vec{r}_i \Phi \right\rangle = 0, \tag{41}
\]

obtained from the variations of \( A \) under \( \vec{r}_i(t) \to c\vec{r}_i(t), \) \( 1 \leq i \leq N \), or, equivalently, from eq. (39),

\[
2(L_K) - \sum_{p=1}^n \gamma_p q_p \frac{\partial (L_K)}{\partial q_p} - \left\langle \sum_{i=1}^N \vec{r}_i \cdot \vec{\nabla} \vec{r}_i \Phi \right\rangle = 0. \tag{42}
\]

The latter is particularly useful when \( L_K \) depends only on a small number of constants.

The tensor virial theorem, also read directly from \( A \) is

\[
V_{\alpha\beta} \equiv \left\langle \sum_{i=1}^N \sum_l r_i^{(l)} \frac{\partial L_K}{\partial \dot{r}_i^{(l)}} \right\rangle - \left\langle \sum_{i=1}^N r_i^\alpha \frac{\partial \Phi}{\partial r_i^\beta} \right\rangle = 0 \tag{43}
\]

\((r_i^{(l)})^\alpha\) is the \( l \)th time derivative of the \( \alpha \) component of \( \vec{r}_i \). Rotational invariance of \( A_K \) implies that the kinetic part of \( V_{\alpha\beta} \) is symmetric, as an identity, although this is not manifest in eq. (43).

For a general relativistic system of particles of masses \( m_i \) and electric charges \( e_i \), the action is[14]

\[
A = A_G + A_p + A_{EM} + A_e, \tag{44}
\]

where

\[
A_G = -\frac{1}{16\pi G} \int R g^{1/2} d^4 x \tag{45}
\]

is the Einstein action for the metric;

\[
A_p = -\sum_i m_i \int d\lambda \left[ -g_{\mu
u}(x_i) \frac{dx_i^\mu}{d\lambda} \frac{dx_i^\nu}{d\lambda} \right]^{1/2} \tag{46}
\]

is the particle action;

\[
A_{EM} = -\frac{1}{4} \int (A_{\nu,\mu} - A_{\mu,\nu}) g^{\mu
u} g^{\alpha\beta} (A_{\beta,\alpha} - A_{\alpha,\beta}) g^{1/2} d^4 x \tag{47}
\]

is the action for the electromagnetic field; and

\[
A_e = \sum_i e_i \int d\lambda \frac{dx_i^\mu}{d\lambda} A_\mu(x_i) \tag{48}
\]

is the field-particle-interaction contribution. The degrees of freedom here are the particle space-time coordinates \( x_i^\mu(\lambda) \), the four electromagnetic-potential fields \( A_\mu(x) \), and the components, \( g_{\mu\nu}(x) \), of the metric tensor.

Under \( g_{\mu\nu} \to c g_{\mu\nu} \) the terms in the action are homogeneous: \( A_G \to c A_G, \ A_p \to c^{1/2} A_p, \) and \( A_{EM}, \) and \( A_e \) are invariant. Thus we get as one of the virial relations

\[
A_G + \frac{1}{2} A_p = 0. \tag{49}
\]
This can be shown to be just the space-time integral of the trace of the (Einstein) equation of motion. This result is easily seen to be general: The definition of the energy-momentum tensor, \(T_{\mu\nu}\), is such that under a change \(\delta g_{\mu\nu}\) we have \(\delta A \equiv (1/2) \int \delta g_{\mu\nu} T^{\mu\nu} g^{1/2} d^4x\) \[14\]; so, under \(g_{\mu\nu} \rightarrow (1 + \epsilon) g_{\mu\nu}, \delta A = (\epsilon/2) \int T g^{1/2} d^4x\), for whatever the source is. The boundary requirements for this type of relation is satisfied if the metric becomes flat at spatial infinity (so no gravitational waves escape to infinity), and if there are periodic boundary conditions in time (e.g., a stationary solution).

There is also homogeneity under \(A_\alpha \rightarrow c A_\alpha\), which gives a virial relation
\[
2A_{EM} + A_e = 0
\] (requiring that no EM waves escape to infinity).

There are tensor virial theorems for \(g_{\mu\nu}\) and for \(A_\alpha\); the latter, for example, gotten by considering \(A_\alpha \rightarrow A_\alpha + \epsilon A_\beta\) reads

\[
V_\beta^\alpha \equiv - \int A_{\beta,\rho} g^{\mu\gamma} g^{\alpha\nu}(A_{\nu,\gamma} - A_{\gamma,\nu}) g^{1/2} d^4x + \sum_i e_i \int d\lambda dx_i A_\beta(x_i) = 0.
\] (51)

All these relations may be easily derived from the equations of motion, as is always the case with low-order theories. We bring them here to show how they follow from our systematic procedure, by mere inspection of the action, and without having to derive the equations of motion first.

By employing a more general conformal increment
\[
g_{\mu\nu} \rightarrow g_{\mu\nu} + \epsilon J(g_{\mu\nu}, A_\alpha, ..)g_{\mu\nu},
\] eq. (49) generalizes to
\[
\int \left[-(8\pi G)^{-1} R + T \right] J g^{1/2} d^4x = 0,
\] (53)
and there is a similar generalization for eq. (50).

The action may have terms quadratic in the curvature tensor—as is the case for the first order semi-classical corrections to General Relativity obtained from string theory—such as
\[
\int R^2 g^{1/2} d^4x, \quad \int R_{\mu\nu} R^{\mu\nu} g^{1/2} d^4x, \quad \int R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} g^{1/2} d^4x.
\] (54)

These are invariant under \(g_{\mu\nu} \rightarrow c g_{\mu\nu}\), and do not modify relation (49). In order to capture them in a virial relation we may need to employ a tensor relation, or use one that is induced by increments of type (24).

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