Research Article

Effects of Differently Located Clearance on the Dynamic Responses of a Two-Degree-of-Freedom Vibration System

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The mechanical model of a two-degree-of-freedom forced harmonic vibration system with multiclearance rigid constraints is established, considering the location schemes of symmetrical both-sided clearance and asymmetrical multiple clearance. Existence domains, correlative distributions, and bifurcation scenarios of periodic vibrations are analyzed using multiparameter and multiperformance cosimulation. Pattern diversity, distribution, and occurrence mechanism of the subharmonic impact motion sequences in the tongue-shaped transition regions among the neighboring fundamental periodic motions of the vibration systems are investigated. The emergent behavior of sticking process of fundamental periodic vibration, the occurrence law of chattering-impact motion, and the interaction of different modes of sticking are discussed. According to the sampling ranges of parameters, three multiple heterogeneous constraint conditions are explored; the effects of differently clearance location and values on the dynamic responses and the transition region of fundamental periodic vibrations and subharmonic motions are particularly analyzed. Hence, the reasonable clearance arrangement scheme and numerical optimization combination are determined and the ideal parameter domain of the vibration system is obtained.

1. Introduction

Clearance is inevitable in the mechanical structure that may be naturally produced during processing and assembly or necessarily existed in the mechanism movement and lubrication. No matter what the reason is, the clearance will inevitably lead to impact vibration and noise in the parts of the mechanical device during operation and aggravate the fatigue wear of the components. In recent years, the dynamics of vibration system with nonsmooth mechanical factors, such as clearances, constraints, friction, and hysteresis, have become a common concern project in mechanical and vibration engineering fields. The importance of the research on the nonsmooth mechanical systems with clearances and constraints is to improve the performance and realize the effective control and utilization of vibration and noise of mechanical equipment. There are multiple clearances for the institutions used in engineering. For example, the planar crank mechanism has multiple joint clearances [1–4]; the stamping forming machine has punch-die clearance and blank holder gap [5–7]; the gear system has symmetrical tooth side clearances or multiple tooth side clearances due to different transmission modes [8–11]; the gear rotor system has both bearing radial clearance and tooth side clearance [12–15]. Therefore, modelling of the above-mentioned mechanical systems requires a key description of multiclearance.

The dynamic researches of vibration system with clearance are generally based on qualitative analysis and numerical simulation of impact Poincaré mapping. However, due to the grazing singularity of impact mapping, some conventional qualitative analysis methods of smooth dynamic systems are difficult to directly apply to the study in global bifurcation of the system. Therefore, the dynamic response of mechanical systems is still a research hotspot of scholars’ attention. Most of the studies have focused on the
grazing bifurcation of periodic motions [16, 17], and the complete chattering-sticking impact motions [18–20] in nonsmooth mechanical systems. Wagg [21] introduced the chatter and sticking region of the impact oscillators and proved that within a certain range of parameter values, nonsmooth events, such as impact, grazing, and sticking, can occur simultaneously. Ing et al. [22] verified different experimental bifurcations with excitation frequency as the model parameter considering the mapping solutions among smooth subspaces simulation. Hős and Champneys [23] found the complete and incomplete chattering motions by controlling operating parameter and pointed out that the possibility about the sequences of period-doubling bifurcations resulted in the transition to complete chatter. Nordmark and Piiriönen [24] decomposed the impact phase into relative slip and stick based on the Strong impact law; investigated and validated the problems about chatter; and sustained slip and stick by numerical analysis of the rigid-contact model. From the current theoretical researches and experimental verifications, we found that chattering impact leads to energy transfer and dissipation, causes the system vibration and noise, and reduces mechanical efficiency. Accordingly, many scholars have devoted to the study of chattering prediction and suppression, reducing or eliminating the harm of chattering-impact vibration to the system [25–27].

In terms of dynamic experimental verification of the vibration system with clearance, Costa et al. and Liao et al. [28, 29] set up the impact oscillator experimental devices, respectively, and obtained the bifurcation diagrams and phase diagrams of the vibration system models by controlling the frequency, stiffness, clearance, and other parameters. Liu et al. [30–32] considered two-parameter bifurcation analysis, successively used experimental and numerical research methods to explore the dynamic behaviors of the vibration system with clearance, and observed chaotic motion, periodic motion, and period-doubling bifurcations of limit cycles. Wang and Luo [33] designed an equivalent electronic circuit of harmonically forced vibration system and realized dynamic simulation and experiment by adjusting the clearance, frequency, and other parameters. The experimental output result verified the various periodic-impact motions of the system. Muvengei et al. [34–37] conducted dynamic experiments on mechanical systems, such as slider-crank mechanism, gear transition, and rotor-bearing system, to further reveal the matching rules of dynamic parameters and nonlinear response characteristics.

The research objects mentioned above are still limited to vibration systems with single or double-sided clearance constraints. The dynamic parameter-matching research carried out is mostly about frequency, stiffness, damping, etc.; the clearance is more concerned about its value. The dynamics of the mechanical system with nonsmooth mechanical features and multiclearance are rarely reported. The models of a two-degree-of-freedom forced harmonic vibration system with symmetrical both-sided clearances and asymmetrical multiclearance are established. We focus on studying the dynamic characteristics of vibration system with multiclearance, revealing different pattern types, internal characteristics, and transition domains distributions of periodic motion groups using numerical simulation method. The second objective of the present article is to compare the characteristics of periodic motions and chattering-impact motion under the influence of different clearance positioning methods and therefore determine the clearance location schemes and reasonable parameter-matching ranges to achieve the coordinated optimization of the system dynamics.

2. Mechanical Models and Dynamic Equations

We consider a mechanical model of a two-degree-of-freedom forced harmonic vibration and impact system with clearances represented by rigid constraints as shown in Figure 1, which is similar to a stamping forming device [38]. The clearance location considered in the model includes the unilateral clearance between the punch and the die, and the installation clearance between the mould and the machine. \( M_i \) (\( i = 1, 2 \)) expresses the masses of two mass blocks, \( M_j \) (\( j = 0, 1, 2 \)) is the stiffness coefficient of Hookean spring, and \( C_j \) (\( j = 0, 1, 2 \)) is the damping coefficient of linear damper. The studied mechanical model is as follows: two mass blocks \( M_1 \) and \( M_2 \) are joined by a Hookean spring \( K_0 \) and a linear damper \( C_0 \), and the mass \( M_1 \) is attached to the right supporting plane by a Hookean spring \( K_1 \) and a linear damper \( C_1 \). The mass \( M_2 \) is attached to the left supporting plane by a Hookean spring \( K_2 \) and a linear damper \( C_2 \). There is a harmonic force \( F_i \sin(\Omega t + \tau) \) \((i = 1, 2)\) acting on the mass \( M_i \) \((i = 1, 2)\), where the amplitude of the harmonic force is \( F_i \) \((i = 1, 2)\), \( M_1 \) and \( M_2 \) have the same forcing frequency \( \Omega \) and phase angle \( \tau \). The mass blocks \( M_1 \) and \( M_2 \) reciprocate with the displacement \( X_i \) \((i = 1, 2)\) in horizontal direction, and the relative displacement of the two masses is represented by \( X_1 - X_2 \). For small harmonic forcing amplitude, the mass blocks exhibit linear forced vibration. The mass block \( M_1 \) impacts mutually with the mass block \( M_2 \) at the rigid displacement limit \( A_1 \) or \( -A_1 \) when \( X_1 - X_2 \) equals to the design clearance \( \Delta_1 \) or \( -\Delta_1 \), viz. \(|X_1 - X_2| = \Delta_1\). In addition, the mass block \( M_2 \) eventually hits the rigid displacement limit \( A_2 \), when the displacement \( X_2 \) equals to the design clearance \( \Delta_2 \). The velocities of \( M_1 \) and \( M_2 \) change suddenly, which are caused by constraint limits and impact collision and countermove with a new initial value until next mutual impacts. The energy loss during the collision is expressed by the coefficient of restitution \( R \). The system changes from two-degree-of-freedom linear impaction to two-degree-of-freedom nonlinear impaction.

When \(|X_1| < \Delta_2\) and \(|X_1 - X_2| < \Delta_1\), the differential equations of system motion between adjacent impacts are

\[
\begin{align*}
M_1 \ddot{X}_1 & = F_1 \sin(\Omega t + \tau) - K_0 (X_1 - X_2) - C_0 (\dot{X}_1 - \dot{X}_2) \\
M_2 \ddot{X}_2 & = F_2 \sin(\Omega t + \tau) - K_0 (X_1 - X_2) - C_0 (\dot{X}_1 - \dot{X}_2)
\end{align*}
\]
are as follows: 
\[
M_1\ddot{X}_1 + C_1\ddot{X}_1 + C_0(\dot{X}_1 - \dot{X}_2) + K_1X_1 + K_0(X_1 - X_2) = F_1 \sin(\Omega T + \tau), \\
M_2\ddot{X}_2 + C_2\ddot{X}_2 + C_0(\dot{X}_2 - \dot{X}_1) + K_2X_2 + K_0(X_2 - X_1) = F_2 \sin(\Omega T + \tau). 
\]

The introduced dimensionless variables and parameters are as follows:
\[
\mu_m = \frac{M_2}{M_1 + M_2}, \\
\mu_c = \frac{C_2}{C_1 + C_2}, \\
\mu_k = \frac{K_2}{K_1 + K_2}, \\
\mu_{c0} = \frac{C_0}{C_1 + C_0}, \\
\mu_{k0} = \frac{K_0}{K_1 + K_0}, \\
\omega_n = \sqrt{\frac{K_1}{M_1}}.
\]

\[
t = \omega_n T, \\
\omega = \frac{\Omega}{\omega_n}, \\
x_i = \frac{X_iK_1}{F_1 + F_2} (i = 1, 2), \\
\zeta = \frac{C_1}{2\sqrt{K_1M_1}}, \\
f_{20} = \frac{F_2}{F_1 + F_2}, \\
\delta_2 = \frac{\Delta_1K_1}{F_1 + F_2}, \\
\delta_1 = \frac{\Delta_1K_1}{F_1 + F_2}.
\]

The definition ranges of some dimensionless parameters are as follows: \(\mu_m \in (0, 1), \mu_c \in (0, 1),\) \(\mu_k \in (0, 1), \mu_{c0} \in (0, 1), \mu_{k0} \in (0, 1), f_{20} \in [0, 1],\) and \(R \in (0, 1).\)

Therefore, the nondimensional equations of the system are given as

\[
\ddot{x}_1 + 2\zeta_1 \dot{x}_1 + \frac{\omega_{c0}}{1 - \mu_{c0}}(\dot{x}_1 - \dot{x}_2) + x_1 + \frac{\omega_{k0}}{1 - \mu_{k0}}(x_1 - x_2) = (1 - f_{20})\sin(\omega t + \tau), \\
\mu_m \ddot{x}_2 + 2\zeta \dot{x}_2 + \frac{\omega_{c0}}{1 - \mu_{c}} \dot{x}_2 + 2\zeta \frac{\omega_{c0}}{1 - \mu_{c0}}(\dot{x}_2 - \dot{x}_1) + \frac{\mu_k}{1 - \mu_k}x_2 + \frac{\mu_{k0}}{1 - \mu_{k0}}(x_2 - x_1) = f_{20}\sin(\omega t + \tau).
\]

The impact equations at the rigid constraints \(A_1, \vec{A}_1\) with \(|x_1 - x_2| = \delta_1\) are described by

\[
\begin{aligned}
1 - \frac{\mu_m}{\mu_m} \dot{x}_{1+} + \dot{x}_{2+} &= 1 - \frac{\mu_m}{\mu_m} \dot{x}_{1-} + \dot{x}_{2-}, \\
R &= \frac{\dot{x}_{2+} - \dot{x}_{1+}}{\dot{x}_{1-} - \dot{x}_{2-}}.
\end{aligned}
\]

The impact equation at the rigid constraint \(A_2\) with \(x_2 = \delta_2\) is described by

\[
\dot{x}_{2+} = -R \dot{x}_{2-}.
\]
The multiparameter and multiobjective cosimulation analysis of vibration system with clearance can intuitively provides more information about pattern type, distribution condition, transition law, and bifurcation scenarios of the periodic vibrations in related parameter areas, reveals the relevance between dynamic characteristics and parameters from the system level, and provides reference for the reasonable matching range of parameters. Three Poincaré sections are defined to describe the impact characteristics of the system.

\[ \sigma_n = \left\{ (x_1, \dot{x}_1, x_2, \dot{x}_2, t) \in R^4 \times T \vert (x_1 - x_2) = (x_1 - x_2)_{\min}, \text{mod} \left( t = \frac{2\pi}{\omega} \right) \right\}. \]  

Impact mapping section is as follows:

\[ \sigma_{p1} = \left\{ (x_1, \dot{x}_1, x_2, \dot{x}_2, t) \in R^4 \times T \vert x_1 - x_2 = \delta_1, \dot{x}_1 - \dot{x}_2 = \dot{x}_{1+} - \dot{x}_{2+} \right\}, \]
\[ \sigma_{p2} = \left\{ (x_1, \dot{x}_1, x_2, \dot{x}_2, t) \in R^4 \times T \vert x_2 = \delta_2, \dot{x}_2 = \dot{x}_{2+} \right\}. \]

From the periodic mapping section \( \sigma_n \), we can determine the motion period number in a forcing period \( T_n = \frac{2\pi}{\omega} \) (\( n = 1, 2, 3, \ldots \)). The number of left impact \( p_1 \) (\( p_1 = 0, 1, 2, 3, \ldots \)) at rigid constraint \( A_1 \) and the number of right impact \( q_1 \) (\( q_1 = 0, 1, 2, 3, \ldots \)) at rigid constraint \( A_1 \) can be identified from the impact mapping section \( \sigma_{p1} \). The number of impact \( p_2 \) (\( p_2 = 0, 1, 2, 3, \ldots \)) at rigid constraint \( A_2 \) can be identified from the impact mapping section \( \sigma_{p2} \). Combining three Poincaré sections, the pattern type \( n-p_1-q_1 \) of the system at rigid constraints \( A_1 \) and \( A_2 \) and the pattern type \( n-p_2 \) at rigid constraint \( A_2 \) under the certain parameter conditions can be identified. We can establish the impact Poincaré mapping section by

\[ X^{(i+1)} = f(X^{(i)},\mu), \]

where \( X^{(i)} = (x_1^{(i)}, x_2^{(i)}, \dot{x}_1^{(i)}, \dot{x}_2^{(i)})^T, X^{(i+1)} = (x_1^{(i+1)}, x_2^{(i+1)}, \dot{x}_1^{(i+1)}, \dot{x}_2^{(i+1)})^T, X \in R^4, \mu \) is the system parameter, and \( \mu \in R^6 \).

Clearance constraint is a key parameter in dynamic performance matching design of the impact systems [39, 40]; therefore, it is necessary to carry out pertinent research in the influences of located clearances on the dynamics of the vibration system. In order to compare the influences of different clearance arrangements and values on the dynamics of the system, according to the value ranges of model parameters, we take the dimensionless parameters: \( \mu_m = 0.5, \mu_c = 0.5, \mu_k = 0.5, \mu_{\alpha} = 0.5, \mu_{\delta_0} = 0.5, \zeta = 0.1, R = 0.8, f_{\text{at}} = 1.0 \), as a set of unified parameters. It should be noted that when the mechanical model is applied to specific equipment, the parameters can be selected according to actual conditions or standards. In this study, we consider different clearance arrangements of vibration system shown in Figure 1 and focus on dynamics and correlation with structural parameters using multiparameter collaborative simulation method in the two-parameter \( (\omega, \delta) \) (\( i = 1, 2 \)) planes. All figures clearly show parameter domains and distribution laws of various periodic vibrations by different colors and symbols, such as complete chattering vibration with sticking is identified as 1-CIS (i.e., 1 - p - p or 1 - p motion), the not marked gray areas are long-period multi-impact vibrations and chaos.

3. Dynamics of the Two-Degree-of-Freedom Vibration System

3.1. Dynamics of the System with Symmetrical Both-Sided Clearances in the \((\omega, \delta_1)\) Parameter Plane. Suppose that the mass block \( M_z \) is connected with the left supporting plane without clearance, which means the clearance \( \delta_z \) is set to zero, the vibration system corresponding to Figure 1 became one with symmetrical both-sided clearances. Let the dimensionless clearance \( \delta_1 \) between the symmetrical fixed constraints \( A_1 \) and \( \overline{A}_1 \) equal gap changes; combine the Poincaré mapping \( \sigma_n \) and \( \sigma_{p1} \), and use two-parameter cosimulation method, so the transition domains distributions of periodic motions in the \((\omega, \delta_1)\) parameter plane with the unified parameters can be depicted in Figure 2(a). Figure 2(b) describes the relative impact velocity of mass blocks \( M_1 \) and \( M_2 \) at rigid constraints \( A_1 \) and \( \overline{A}_1 \) corresponding to Figure 2(a). As seen in Figure 2, the system mainly presents 1-0-0 and symmetrical 1-1-1 motions, except for the 3-1-1 subharmonic motion and \( n-0-1(n-1-0n \geq 1) \) one-sided impact motion in large clearance range. In high-frequency range, 1-0-0 motion undergoes real-grazing bifurcation with the clearance \( \delta_1 \) decreases and passes through \( G_{1-0-0} \) bifurcation line into symmetrical 1-1-1S motion; 1-1-1S motion occurs with pitchfork bifurcation and transitions to asymmetric 1-1-1AS motion. In low-frequency range, 1-0-0 impactless motion either goes through unstable 1-1-1 motion by bare-grazing bifurcation and embeds within the tongue-shaped domain or enters symmetrical 1-1-1S motion from real-grazing bifurcation. As the clearance \( \delta_1 \) is decreased, the 1-0-0 impactless motion parameter domain is gradually narrowing, and correspondingly, the 1-1-1S motion parameter domain is gradually increasing. When \( \delta_1 \) decreases to around 0.334 and continues to decrease, the symmetrical 1-1-1S parameter domain begins to decrease, the pattern types
of periodic vibrations increase, and the dynamic behavior becomes diverse. On the bottom left of the \((\omega, \delta_{1})\) parameter plane, the system mainly exhibits symmetric fundamental periodic vibration groups \(1-p-p\) \((p \geq 2)\) (the motion type whose periodic impact number is different, and whose motion period is one forcing period), complete chattering impact, incomplete chattering impact, and a small amount of sub-harmonic impact motions. When the excitation frequency \(\omega\) further decreases, symmetric \(1-p-p\) vibration groups experience series of real-grazing bifurcation lines \(G_{1} \rightarrow \cdots \rightarrow G_{p-p}\). At present, the system maintains original \(n=1\) periodic vibration. The each impact number of two masses \(M_{1}\) and \(M_{2}\) on the symmetrical rigid constraints \(A_{1}\) and \(A_{1}\) increases from \(p\) to \((p+1)\), and next time, the relative impact velocity between two mass blocks is \(\dot{x}_{1} \sim \dot{x}_{2} = 0\), which induces stable symmetrical \(1-(p+1)-(p+1)\) motion. When the impact number \(p\) increases to large enough, the system exhibits \(1-\overline{p} - \overline{p}\) incomplete chattering vibration. As \(\omega\) further decreases, the incomplete chattering vibration transitions to complete chattering vibration \(1-\overline{p} - \overline{p}\) through sliding bifurcation. Theoretically, the dynamic characteristics of complete chattering vibration presents as the number of impact sequences in the vibration period tends to be infinite and the impact velocity decays successively, which eventually causes the mass block sticks at the rigid constraint until the interaction constraint force between two masses \(M_{1}\) and \(M_{2}\) is reduced to zero, the masses deviate from the rigid constraint with the initial velocity being zero, and the sticking state ends. With increase in \(\omega\), the main transition from \(1-1-1\) periodic vibration to \(1-\overline{p} - \overline{p}\) complete chattering vibration can be summarized as follows:

\[
\begin{align*}
\omega \rightarrow 1 - \overline{p} - \overline{p} & \rightarrow \cdots \rightarrow 1 - \overline{p} - \overline{p} \rightarrow \cdots \rightarrow 1 - (p + 1) - (p + 1) \rightarrow 1 - p - p \rightarrow \cdots \rightarrow a \\
1 - 4 & \rightarrow 1 - 3 \rightarrow 1 - 2 \rightarrow 1 - 1 = 1,
\end{align*}
\]

where the symbol \(1 - \overline{p} - \overline{p}\) represents complete chattering vibration with sticking; \(1 - \overline{p} - \overline{p}\) represents incomplete chattering vibration without sticking; \(S\) \(\cdot\) \(Bif\) represents sliding bifurcation; and \(G\) \(\cdot\) \(Bif\) represents grazing bifurcation.

The impact mass blocks go through three motion states in one forcing period \(T_{n} = 2\pi n / \omega\): free motion, chattering sequence, and chattering sequence with sticking. The chattering impact is originated from unsteady regenerative effect of self-excited essentially. In a forcing period, the masses collide with the constraints infinitely, resulting in a chattering sequence. \(F_{i} (i = 1, 2)\) represents the resultant force of exciting force, spring restoring force, and damping force acting on the vibration mass block \(M_{i}\). \(N_{2}\) indicates the constraint force between mass block \(M_{2}\) and the fixed supporting base on the left. \(N_{12}\) indicates the interaction constraint force between mass blocks \(M_{1}\) and \(M_{2}\), \(N_{12}\) meets the condition: \((F_{1} - N_{12}/M_{1}) = (F_{2} - N_{12}/M_{2})\). The chattering time and sticking time of complete chattering vibration \(1-\overline{p} - \overline{p}\) are related to the values of frequency \(\omega\) and clearance \(\delta_{1}\). Generally, the smaller the \(\omega\) is, the shorter the chattering time will be, and the longer the corresponding sticking time \(t_{s}\) will be. As shown in Figures 3 and 4, the impact forces of the masses are increased with the smaller \(\delta_{1}\), the chattering time of the complete chattering vibration in one vibration period is decreased, and the corresponding sticking time \(t_{s}\) is increased. From Figure 4(c), we can see that when \(\delta_{1} = 0.25\), the sticking period of impact disappears.
and the system presents $1 - p - p$ incomplete chattering vibration.

The complexity of low-frequency periodic motions originates from the irreversibility mutual transition between neighboring fundamental periodic vibrations 1-$p$-$p$ and 1-$(p + 1)$-$(p + 1)$ [41]. Figure 5(a) shows the local refinement about the periodic vibrations of the symmetrical clearances vibration system in the low-frequency and small clearance range, it displays that the fundamental periodic vibration sequences 1-$p$-$p$ are arch-shaped distributed with obvious fractal characteristics. With decrease in clearance $\delta_1$ or frequency $\omega$, the window of 1-$p$-$p$ impact motion is gradually narrowing. Moreover, the transition process between the neighboring fundamental periodic vibrations is continuous and irreversible. This irreversibility leads to a series of singular points along with two types of transition regions: hysteresis region and tongue-shaped region [42, 43]. The reversibility of the mutual transitions between 1-$p$-$p$ and 1-$(p + 1)$-$(p + 1)$ neighboring fundamental periodic vibrations only occur at singular points. The hysteresis domain reflects the coexistence of two attractors of 1-$p$-$p$ and 1-$(p + 1)$-$(p + 1)$ motion, and both periodic motions are stable. The impact number between the two masses and any one of the constraints may be $p$ times or $(p + 1)$ times in an excitation forcing period because of the difference of the initial value conditions or the direction of parameter changing. The neighboring fundamental periodic vibrations are irreversible at the boundary of the hysteresis domains. As the excitation frequency or the clearance increases, 1-$p$-$p$ symmetric complete chattering vibration passes through a rising bifurcation $R_{1-p-p}$ and transits to 1-$p$-$p$ symmetric incomplete chattering vibration. Inversely, as the parameter $\omega$ or $\delta_1$ decreases, 1-$p$-$p$ symmetric incomplete chattering vibration passes through a sliding bifurcation $S_{1-p-p}$ and transits to 1-$p$-$p$ symmetric incomplete chattering vibration. In the symmetrical 1-$(p+1)$-$(p+1)$ motion parameter domain, with increase in $\omega$ or $\delta_1$, 1-$(p+1)$-$(p+1)$ motion develops into 1-$p$-$p$ motion by passing through a saddle-node bifurcation at the demarcation line $SN_{1-p-p}^{(p+1)}$, or transits into 1-$p$-$(p+1)$, 1-$(p+1)$-p, chaotic motion through a saddle-node bifurcation at $SN_{1-p-p}^{(p+1)}$. With decrease in $\omega$ or $\delta_1$, 1-$p$-p motion transits into stable 1-$(p(k+1)+1)$(p+1) motion by passing through a real-grazing bifurcation line $G_{1-p-p}^{(p+1)}$, or passes through a bare-grazing bifurcation line $G_{1-p-p}^{(p+1)}$ to unstable 1-$(p+1)$-$(p+1)$ motion, then embeds into subharmonic

![Figure 3](image1.png)  
![Figure 4](image2.png)
vibrations and chaotic motion. The upper bound of the hysteresis region is formed by the saddle-node bifurcation line $\text{SN}_{1,p}^{1,(p+1)-(p+1)}$ of $1-(p+1)-(p+1)$ motion, and the lower bound is formed by the real-grazing bifurcation line $G^b_{1,p}$ of $1-p$ motion. The tongue-shaped region is limited by the bare-grazing bifurcation line $G^b_{1,p}$ of $1-p$ motion and the saddle-node bifurcation line $\text{SN}_{1,p}^{1-(p+1)-(p+1)}$ of $1-(p+1)-(p+1)$ motion. Groups of subharmonic vibrations such as $2-(2p+1)-2p$, $2-(2p+2)-2p$, $3-(3p+1)-(3p+1)$ regularly appear in the tongue-shaped regions (Figure 5(a)).

The simulation results of Figure 5 describe in detail the transition between $1-p-p$ and $1-(p+1)-(p+1)$ neighboring fundamental periodic motions and the dynamics of parts of the tongue-shaped regions. Taking the mutual transitions between the neighboring $1-3-3$ and $1-4-4$ fundamental periodic vibration as an example, the singular points are marked as $X_1$ and $X_2$, the saddle-node bifurcation lines of $1-4-4$ periodic motion are marked as $\text{SN}_{1-4-4}$ and $\text{SN}_{1-4-4}$ in Figure 5(b), the real-grazing bifurcation line of $1-3-3$ periodic motion is marked as $G_{1,3,3}$ and the bare-grazing bifurcation line of $1-3-3$ periodic motion is marked as $G_{1,3,3}$ in Figure 5(c). Figure 5(d) are superposed of Figures 5(b) and 5(c). We can clearly observe that the hysteresis region $\text{HR}_4$ is surrounded by the saddle-node bifurcation line $\text{SN}_{1,4,4}$ of $1-4-4$ periodic motion and the real-grazing bifurcation line $G_{1,3,3}$ of $1-3-3$ periodic motion, and the tongue-shaped region is surrounded by the bare-grazing bifurcation line $G^b_{1,3,3}$ of $1-3-3$ periodic motion and the saddle-node bifurcation line $\text{SN}_{1,3,3}$ of $1-4-4$ periodic motion, which contains subharmonic impact motions, e.g., $2-7-6$, $2-6-8$, and $3-9-9$.

### 3.2 Dynamics of the System with Asymmetrical Multiclearance

in the $(\omega, \delta_1)$ Parameter Plane. When the clearance value $\delta_2$ between the mass $M_2$ and the left supporting plane is non-zero, due to the increase in the number of clearances, the rigid constraints cause the vibration system to present more complex non-smooth dynamics by comparison. Meanwhile the described dynamic model is an asymmetrical multiclearance vibration system. We compare the dynamic response of asymmetric multiclearance system with symmetric both-sided clearances system under the unified parameters, associated with the dimensionless clearance values $\delta_i$ ($i=1,2$) dynamic equivalent change simultaneously. The distribution domains and bifurcation diagram of various periodic vibrations of the asymmetrical multiclearance system corresponding to Figure 1 in the $(\omega, \delta_1)$ parameter plane are
shown in Figures 6(a) and 6(b). From the comparison with Figure 2(a) of the symmetrical clearance vibration system, it can be seen that under the same parameter conditions, the dynamics of the same constraint position of the system are quite different. The symmetrical periodic-impact motion groups are reduced, and the periodic motions of the multiclearance vibration system present an obvious asymmetrical form, which are affected by the clearance $A_2$ between the mass $M_2$ and the left supporting plane. The dynamic performance changes can be summarized as follows:

1. As shown in the $(\omega, \delta_1)$ parameter plane, although the constraints $A_1$ and $\overline{A_1}$ are symmetrical to each other. However, the number of mutual impact $q$ on the right constraint $A_1$ is always greater than the mutual impact number $p$ on the left constraint $\overline{A_1}$ or the same with each other.

2. It can be observed from Figure 6(a) that in high-frequency and large clearance range, two masses $M_1$ and $M_2$ mainly present impact-less 1-0-0 motion and symmetric 1-1-1S motion, which is similar to the symmetrical impact vibration system shown in Figure 2(a), except that the window of symmetric 1-1-1S motion decreases, and the window of asymmetric 1-1-1AS motion increases accordingly.

3. The fundamental periodic vibration groups existing on the bottom left of the $(\omega, \delta_1)$ parameter plane, in which symmetrical 1-1-p motion basically disappears, and is replaced by multiple groups of 1-0-q, 1-1-q, 1-2-q, etc., asymmetric fundamental periodic vibrations. As the clearance $\delta_1$ is decreased, the system presents 1-0-q single-impact motion under low-frequency vibration, mass blocks $M_1$ and $M_2$ only impact with the right side of the middle constraint $A_1$. 1-0-q single-impact motion groups still have arch-shaped distribution, but 1-1-q motion groups show in a transposition multilayer way and 1-p-q ($p \geq 2$) motion groups have kink-shaped distribution.

4. The fundamental periodic vibration sequences of 1-p-q are separated by a number of tongue-shaped domains containing subharmonic impact motions and chaotic motion, whose parameter domains are discontinuous. 1-p-q motion transits into asymmetric complete chattering vibration $1-\overline{p}-q$ with decrease in the parameter $\omega$ or $\delta_1$.

The simulation results which are shown in Figure 7 describe in detail the transition phenomena of neighboring fundamental periodic vibrations and tongue-shaped regions under the condition of small clearance and low frequency in the $(\omega, \delta_1)$ parameter plane. As seen in Figure 7(a), the parameter domains of 1-p-q fundamental periodic-impact motion groups is increased with the impact number $p$ at the left constraint $\overline{A_1}$, that gradually become narrow from left to right. With increase in $\omega$, 1-0-q single-impact motion goes through one-sided grazing bifurcation and transforms into fundamental periodic vibration groups of 1-1-q, 1-2-q and 1-3-q, 1-1-q motion either transits to 1-1-(q + 1) motion through a one-sided real-grazing bifurcation, or passes through the tongue-shaped region by a one-sided bare-grazing bifurcation into unstable 1-1-(q + 1) motion distribution domain. Moreover, through this unstable 1-1-(q + 1) motion, the system embeds chaotic motion or subharmonic impact motion. Figures 7(b) and 7(c) describe the pattern types, distribution law and transition characteristics of subharmonic motions within the tongue-shaped regions of neighboring 1-1-1, 1-1-2, 1-1-3 periodic-impact motions. Figure 7(b) shows that when clearance $\delta_1$ decreases or frequency $\omega$ is increased, there is a smooth and clear 1-2-q subharmonic impact motion parameter domain at the one-sided grazing bifurcation boundary of the neighboring 1-1-2 to 1-1-3 periodic motion. And, the tongue-shaped region contains 2-2-6, 3-3-8, 4-4-10, . . . , $n-p-2(p + 1)$ subharmonic motions and chaotic motion. Figure 7(c) further details the distribution of periodic vibrations within the tongue-shaped region from 1-1-1 to 1-1-2 motion with increase in $\omega$. In the two-dimensional parameter plane, the tongue-shaped region contains: 2-2-3, 3-3-5, 4-4-7, . . . , $n-p-(2p - 1)$ and 2-2-4, 3-3-6, 4-4-8, . . . , $n-p-2p$ periodic motion parameter islands. The positions of these parameter islands are distributed from bottom to top or from left to right, and their existence domains are decreased sequentially. There is not stable single-period multi-impact motion within the tongue-shaped region, so the above-mentioned subharmonic impact motions and chaotic motions occupy a dominant position within the tongue-shaped domain.

The dynamic responses of the asymmetric multiclearance vibration system in small clearance and high-frequency range can be obtained and verified by one-parameter simulation as depicted in Figure 8. When horizontally passing through the $(\omega, \delta_1)$ parameter plane at $\delta_1 = \delta_2 = 0.15$, the relative displacement of the two masses $M_1$ and $M_2$, the relative impact velocity of $M_1$ and $M_2$, and the impact velocity of mass $M_2$ with change in parameter $\omega$ can be numerically calculated. Figure 8(a) corresponds to the Poincaré mapping $\sigma_p$, Figure 8(b) corresponds to the Poincaré mapping $\sigma_{p1}$, and Figure 8(c) corresponds to the Poincaré mapping $\sigma_{p2}$. In order to further verify the stability and chaotic motion of the system, Figures 8(g) and 8(h) show the largest Lyapunov exponent of the system. When the system exhibits periodic motion, the largest Lyapunov exponent is less than zero; when bifurcation happens, the largest Lyapunov exponent is equal to zero; when the system exhibits chaotic motion, the largest Lyapunov exponent is greater than zero [44]. From the mutual correspondence between the bifurcation diagrams and the largest Lyapunov exponent diagram, it can be seen that the attractor characteristics of the system are consistent with the largest Lyapunov exponent.

Figures 8(d)–8(f) are the details of Figures 8(a)–8(c) in the high-frequency range. It can be observed that as the frequency $\omega$ is increased, a pitchfork bifurcation leads to the morphology change of symmetrical 1-1-1S motion. 1-1-1S motion transits from asymmetric 1-1-1AS motion and then embeds in the chaotic motion parameter domain. The symmetrical 1-1-1S motion represents that the dimensionless time $t$ is set to zero immediately when the
masses $M_1$ and $M_2$ impact at one of the rigid constraints $A_1$ or $\overline{A}_1$, next time, the impact of the two masses occurs at the same constraint position is $t = 2\pi/\omega$, and the dimensionless time between the two masses go out and home at the symmetrical constraint position $A_1$ or $\overline{A}_1$ is exactly $\pi/\omega$. The transition trend of 1-1-1S motion into 1-1-1AS can be verified from comparing the motion phase diagrams of Figures 9(a) and 9(b) in different excitation frequency periods. There are possibilities that the asymmetric 1-1-1AS periodic-impact motion shows an anti-symmetric morphology at the same time, due to the differences in constrained symmetry and initial conditions, as expressed in Figure 9(b) by different colors. The phase diagram of asymmetric 1-1-1AS motion shown in Figure 9(c) indicates that when $\omega$ continues to increase and crosses 2.048804, the two masses impact at the constraints showing Neimark–Sacker bifurcation (which is marked by the symbol N-S Bif in Figure 8(b)), resulting in multiple impacts long-period motion or chaos (Figure 9(d)).

3.3. Dynamics of the System with Asymmetrical Multiclearance in the $(\omega, \delta_1)$ Parameter Plane. Based on the Poincaré mapping $\sigma_{n}$ and $\sigma_{p_2}$, the parameter domains of periodic-impact motions in the $(\omega, \delta_2)$—two parameters—plane for the system with asymmetrical multiclearance are shown in Figure 10(a). Figure 10(b) shows the bifurcation diagram of impact velocity $x_{1-} - x_{2-}$ versus the clearance $\delta_2$ corresponding to Figure 10(a). As seen in Figure 10(a), the dynamic behavior of the system shows simple under large clearance condition. It mainly exhibits 1-0 and 1-1 motions. The critical lines of the parameter domains are the amplitude-frequency response curve of vibration system. With decrease in $\delta_2$, the window of 1-1 motion becomes narrow and transits into 1-2 motion by grazing bifurcation, which means that the impact number between the mass block $M_2$ and the rigid constraint $A_2$ is increased by one in a forcing period. There are narrow parameter islands of subharmonic motions, such as 2-4 and 3-6, contained in the parameter domain window of the 1-2 motion; the demarcation lines of 1-2, 2-4, and 3-6 parameter regions are the period-doubling

**Figure 6:** Two-parameter bifurcation diagrams showing the pattern distributions and transition law of periodic-impact motions for the asymmetrical multiclearance vibration system in the $(\omega, \delta_1)$ parameter plane. (a) Distribution domains of periodic motions. (b) Bifurcation diagram of relative impact velocity $\dot{x}_{1-} - \dot{x}_{2-}$.

**Figure 7:** The partial enlargement and details of Figure 6(a). (a) Distribution domains of 1-p-q motions. (b) Existence regions of subharmonic vibrations within the tongue-shaped regions. (c) Local details of Figure 7(b) ($\delta_1 \in [0.34, 0.376]; \omega \in [0.58, 0.66]$).
Figure 8: One-parameter bifurcation and largest Lyapunov exponent diagrams when $\delta_1 = \delta_2 = 0.15$. (a) Bifurcation diagram with $x_1-x_2$. (b) Bifurcation diagram with $\dot{x}_1 - \dot{x}_2$. (c) Bifurcation diagram with $\dot{x}_2$. (d–f) The partial enlargement of Figures 8(a)–8(c). (g, h) The largest Lyapunov exponent diagram of Figures 8(b) and 8(c).

Figure 9: Phase plane portraits of mass blocks $(M)_1$ and $(M)_2$. $\delta_1 = \delta_2 = 0.15$. (a) 1-1-1$_S$ motion, $\omega = 1.97$. (b) 1-1-1$_A$S motion, $\omega = 2.02$. (c) 1-1-1$_A$S motion with grazing contact, $\omega = 2.048803$. (d) Chaotic motion, $\omega = 2.048804$.

Figure 10: Two-parameter bifurcation diagrams showing the pattern distributions and transition law of periodic-impact motions for the asymmetrical multiclearance vibration system in the $(\omega, \delta_2)$ parameter plane. (a) Distribution domains of periodic-impact motions. (b) Bifurcation diagram of impact velocity $\dot{x}_2$. 
bifurcation lines. For small clearance $\delta_2$ and low-forcing frequency $\omega$, the transition phenomena of periodic vibrations are similar to the system with symmetrical clearances. The difference is that 1-$p$ fundamental periodic vibration groups lose their stability and continuity in the small clearance region, which are separated by chattering impact, subharmonic impact, and chaotic motions.

Figures 11(a)–11(c) are detailed descriptions of Figure 10(a) in small clearance region and illustrate the mutual transfer between fundamental periodic vibrations 1-$p$ and 1-$(p + 1)$. It can be observed from Figure 11(a) that the system has complete 1-$p$ periodic motion sequences in the parameter sampling range of $\delta_2 \in [0.32, 0.72]$ and $\omega \in [0.05, 1.25]$. When $\omega$ is decreased, 1-$p$ motion crosses the curve $G_{1-p}$ via a real-grazing bifurcation and transfers to 1-$(p + 1)$ motion. Continuous real-grazing bifurcations appear with gradually decreasing $\omega$, which lead to the impact number $p$ occurring at the constraint increases one by one in a forcing period, until $p$ is sufficiently large finite and 1-$p$ motion exhibits incomplete chattering vibration 1-$\bar{p}$. The impact velocity of the mass block $M_2$ gradually decreases with the increase of chattering sequences. With a further decrease in $\omega$, 1-$\bar{p}$ incomplete chattering impact transits into 1-$\bar{p}$ complete chattering impact through a sliding bifurcation. The mass block $M_2$ sticks to the constraint $A_2$ until the resultant force on $M_2$ is decreased to zero. $M_2$ leaves the constraint $A_2$ that ends the sticky state. With a constant clearance value, the excitation frequency interval of 1-$p$ motion turns small and moves to the left with increasing $p$. Therefore, the transition from 1-1 motion to 1-$\bar{p}$ complete chattering vibration in the $(\omega, \delta_2)$ parameter plane can be summarized as follows:

$$\omega: 1 - \bar{p} \rightarrow \cdots \rightarrow 1 - p \rightarrow \cdots \rightarrow 1 - (p + 1) \rightarrow 1 - p \rightarrow \cdots \rightarrow 1 - 4 \rightarrow 1 - 3 \rightarrow 1 - 2 \rightarrow 1 - 1. \tag{10}$$

It can be seen from Figures 11(b) and 11(c) that at the demarcation line $L: R_{1-p} \cap R_{1-(p + 1)}$ between the parameter domains of neighboring fundamental periodic vibration 1-$p$ and 1-$(p + 1)$, there are an array of tongue-shaped regions, including subharmonic motions and chaos. The upper demarcation lines of the tongue-shaped regions are limited from the bare-grazing bifurcation lines $G_{1-p}$ to the lower demarcation lines of the saddle-node bifurcation lines $SN_{1-(p+1)}$. The tongue-shaped regions are occupied by regular subharmonic vibrations and chaotic motion. Figure 11(b) depicts a series of subharmonic motion islands, such as 2-$\delta_2 + 1$, 3-$\delta_2 + 1$, ..., $n-(n + \delta_2 + 1)$ ($p \geq 4, n \geq 2$), which exist in the tongue-shaped regions. Figure 11(c) shows the existence of 2-$\delta_2 + 1$, 3-$\delta_2 + 1$, ..., $n-(n + \delta_2 + 1)$ ($p = 2, n = 2-5$) subharmonic vibrations and narrow 2-$\delta_2 + 1$, 2-$\delta_2 + 1$, 2-$\delta_2 + 1$, 2-$\delta_2 + 1$, ..., $n-(n + \delta_2 + 1)$ ($p = 2, n = 2-6$) subharmonic vibration windows within the tongue-shaped transition domain clamped by the neighboring 1-2 and 1-3 fundamental periodic vibrations. It also expresses the transition process of such subharmonic motions to chaos by period-doubling bifurcation sequences with increase in $\delta_2$.

When one of the following conditions is met, the vibration system with asymmetrical multiclearance will undergo complete chattering vibration. If the mass $M_2$ contacts the rigid constraint $A_2$ at an instantaneous velocity of zero, and the direction of the interaction force $N_{12}$ between the two masses points to the right, viz. $C_2$: $x_2 = \delta_2, x_{2c} = 0$ and $N_{12} \geq 0$. The two masses are sticking together at $A_2$, until $N_{12}$ changes its direction. $M_1$ and $M_2$ leave the constraint $A_2$ at the zero relative velocity. If the mass blocks $M_1$ and $M_2$ contact the left side of middle rigid constraint $\bar{A}_1$ at an instantaneous relative velocity of zero, the direction of the interaction force $N_{12}$ between the two masses points to the left, viz. $C_1$: $x_2 = -\delta_2, x_{2c} = 0$ and $N_{12} \leq 0$. The two masses are sticking at $\bar{A}_1$ until $N_{12}$ changes its direction. $M_1$ and $M_2$ leave the constraint $\bar{A}_1$ with the zero relative velocity. In summary, combining chattering-impact distribution domain both in $(\omega, \delta_1)$ and $(\omega, \delta_2)$ parameter planes, the sticking states of the system can be explained as follows:

1. When one of the above three conditions $C_1, C_2$, or $C_3$ is met, which can be corresponded into two situations: one is that only the condition $C_1$: $x_2 = \delta_2, x_{2c} = 0$ and $N_{12} \leq 0$ is met, the sticking only occurs at the rigid constraint $A_2$, the mass $M_2$ is at rest on the left supporting plane. The system is transformed into a single-degree-of-freedom impact oscillator with only mass $M_1$. Figures 12(a) and 12(b) confirm this phenomenon when $\delta_1 = \delta_2 = 0.35$ and $\omega = 0.40$. The other is that when the condition $C_2$ or $C_3$ is met, the sticking only occurs at the rigid constraint $A_1$ or $\bar{A}_1$. As shown in Figures 12(c) and 12(d), when $\delta_1 = \delta_2 = 0.1, \omega = 0.62$, and $\omega = 0.695$, the two masses are sticking together for synchronous motion temporarily. The system is transformed into a single-degree-of-freedom impact oscillator with two masses $M_1$ and $M_2$ sticking as a whole.

2. When two or all of the above three conditions $C_1, C_2$, and $C_3$ are met, the sticking occurs at two or all of the rigid constraints $A_2, A_1$, and $\bar{A}_1$. Figures 13(a) and
13(b) indicate that when $\delta_1 = \delta_2 = 0.1$, $\omega = 0.68$, the conditions $C_2$ and $C_3$ are met at the same time. The masses $M_1$ and $M_2$ are sticking as a whole at both constraints $A_1$ and $\overline{A}_1$. As seen in Figures 13(c) and 13(d), at small clearance $\delta_1 = \delta_2 = 0.06$, when $\omega = 0.645$, three conditions $C_1$, $C_2$, and $C_3$ are met simultaneously. The complete chattering vibration emerges at the rigid constraints $A_2$, $A_1$, and $\overline{A}_1$. The commonality of these situations is that the system degenerates to completely stationary state, and the two masses will not move temporarily until the sticky state ends.

The complete chattering-impact motion parameter distribution of the system with asymmetrical multiclearance does not have much regularity. The superposition effect between multiclearance makes the project of chattering prediction and suppression more complicated. However, the simulation results show that the completely stationary state domain where the vibration system satisfies more than two sticking conditions at the same time is extremely small. When designing and matching the system parameters, it is necessary to avoid such a static state domain, which can make the system dynamic behavior invalid.
4. Clearance Location Schemes of the Asymmetric Multiclearance System

The main mission of this section is to find the reasonable matching range of clearance parameter of the asymmetric multiclearance vibration system. Based on the analysis above mentioned, the asymmetric multiple clearances vibration system has high-dimensional and strong nonlinearity. The interaction of nonsmooth mechanical features will make the dynamic characteristics essential change. The mechanical model established in Figure 1 has certain universality, whose clearance \( \delta_1 \) can be seen as the assembly clearance between the two vibration devices, and clearance \( \delta_2 \) can be seen as the fixed clearance between the mechanical system and the installation benchmark. Subsequently, three installation conditions of the vibration system are discussed, that is, let the clearances at the constraints to be unequal, set one of the clearances fixed to study the influence of the other clearance changing on the dynamic responses.

4.1. Dynamics in the \((\omega, \delta_1)\) Parameter Plane When Clearance \( \delta_2 \) Is Being Fixed. In Figure 14, a clearance positioning scheme when clearance \( \delta_2 \) is being fixed with different selected values is considered to study the various periodic motion parameter domain of the system. The clearance \( \delta_2 \) between mass block \( M_2 \) and the supporting datum plane is defined as a fixed value, whose value range is discrete at equal intervals as \( \delta_2 = 0.25, 0.5, 0.75, \) and \( 1.0 \). It can be learnt from comparing the results when clearances equal changed as shown in Figure 6; when the clearance \( \delta_2 \) is fixed, the pattern types of periodic motions and their occurrence domains have been changed greatly. In the low-frequency and small clearance range, when the clearance \( \delta_2 = 0.25 \) as shown in Figure 14(a), the fundamental periodic vibration groups \( 1-0 \)-\( q, 1-1 \)-\( q, 1-2 \)-\( q, 1-3 \)-\( q, \ldots \) are distributed in a kink shape at the bottom left of the \((\omega, \delta_1)\) plane. As shown in Figure 14(b), when \( \delta_2 = 0.5, 1-0 \)-\( q \) one-sided impact fundamental periodic vibrations are no longer existed, whose parameter domains are replaced by \( 1-1 \)-\( q, 1-2 \)-\( q, 1-3 \)-\( q, \ldots \), periodic motions. The similarity between the Figures 14(a) and 14(b) is that, the motions in the low-frequency range are still dominated by asymmetric impact motion, the impact number \( q \) at the right constraint \( A_1 \) is always bigger than the impact number \( p \) at the constraint \( A_1 \). When the clearance \( \delta_2 \) is increased to 0.75, the system appears symmetrical fundamental periodic vibration \( 1-0 \)-\( p \) in the low-frequency domain (Figure 14(c)) The window of the symmetric fundamental periodic vibration parameter domain increases with \( \delta_2 \) increasing, and the corresponding asymmetric fundamental periodic vibration parameter domain decreases, as seen in Figure 14(d). The symmetrical fundamental periodic vibration groups are distributed in bands. As \( \omega \) is increased, the grazing bifurcation between neighboring fundamental periodic vibrations is also both side symmetrical. In the high-frequency range, when the fixed clearance \( \delta_1 \) is small, period-doubling impact-less motions such as \( 2-0 \)-\( 0, 4-0 \)-\( 0, \) and subharmonic motions are found in large clearance \( \delta_1 \) area, in addition to the \( 1-0 \)-\( 0 \) motion.

The effect of the fixed clearance \( \delta_2 \) on the fundamental periodic vibration group \( 1-p-p \) and the complete chattering vibration of the system can also be described by the time trajectories diagrams. Taking the assembly clearance value \( \delta_1 \) between two masses \( M_1 \) and \( M_2 \) as 0.25, the frequency \( \omega = 0.2 \), the analysis describes the dynamic performance of chattering-impact motion under the condition of different values of fixed clearance \( \delta_2 \). Figure 15 shows the variation law between relative displacement of masses \( M_1 \) and \( M_2 \) and dimensionless time. From Figure 15(a), it can be observed that when \( \delta_2 = 0.25 \), the mass blocks \( M_1 \) and \( M_2 \) shows sticking at the right constraint \( A_1 \) and the same time impactless at the left constraint \( A_1 \). With increase in \( \delta_2 \), periodic vibration of the masses \( M_1 \) and \( M_2 \) at the constraint \( A_1 \) becomes incomplete chattering; moreover, transits into complete chattering vibration with sticking via sliding bifurcation, and the sticking time increases as the clearance \( \delta_2 \) is increased. Figure 16 shows the variation law between relative velocity of \( M_1 \) and \( M_2 \) and dimensionless time. The complete chattering vibrations and sticking time of the two masses are the same as those reflected in Figure 15. In addition, it shows that the amplitude of the relative impact velocity of masses \( M_1 \) and \( M_2 \) decreases as the fixed clearance \( \delta_2 \) is increased. Accordingly, when the fixed clearance \( \delta_2 \) is larger, the vibration of the masses \( M_1 \) and \( M_2 \) is weaker under the same excitation condition, and the sticking time at the rigid constraints \( A_1 \) or \( A_1 \) is longer.

4.2. Dynamics in the \((\omega, \delta_2)\) Parameter Plane When Clearance \( \delta_1 \) Is Being Fixed. Figure 17 shows pattern distributions of periodic vibrations in the \((\omega, \delta_2)\) parameter plane, when the symmetrical clearance \( \delta_1 \) between the two mass blocks changing at the same time, \( \delta_1 \) is uniformly selected the value of 0.25, 0.5, 0.75, and 1.0 in its value range \((0, 1)\). The effect about different assembly clearance value \( \delta_1 \) on the distribution law of periodic motion parameter domain can be observed from comparing with Figures 17 and 10. Figure 17(a) illustrates that when the fixed assembly clearance \( \delta_1 \) is sufficiently small, \( \delta_1 \) has greater impact on the dynamic behavior of mass \( M_2 \) at the rigid constraint \( A_2 \). Especially, in the low-frequency interval, the fundamental periodic vibration groups \( 1-p \) are constantly disturbed by complete chattering vibration, chaotic motion, and irregular impact motion, which lead to the grazing bifurcation lines lose their stability. The grazing bifurcation line \( G_{11} \) between \( 1-1 \) and \( 1-2 \) motion is significantly longer, resulting in a relatively smaller parameter domain of the \( 1 \)-\( 1 \)motion; \( 1 \)-\( 1 \) motion is replaced by the window of \( 1-2 \) periodic motion mostly. Figures 17(b)–17(d) show that when the fixed assembly clearance \( \delta_1 \) is large, it has little effect on the distribution regular pattern of the periodic motion parameter domain in the low-frequency interval and greater effect in the high-frequency interval. \( 1-1 \) motion parameter domain increases with increase in \( \delta_1 \). Among that the period-doubling bifurcation lines in the fundamental periodic vibration parameter domains are the longest when \( \delta_1 = 0.5 \), as shown in Figure 17(b). It can be clearly observed that 2-2, 3-3, 4-4, 5-5 and other parameter islands are formed by period-
Figure 14: Periodic motions for the asymmetric multiclearance system with fixed clearance $\delta_2$ in the $(\omega, \delta_1)$ parameter plane. (a) $\delta_2 = 0.25$. (b) $\delta_2 = 0.5$. (c) $\delta_2 = 0.75$. (d) $\delta_2 = 1.0$.

Figure 15: Time trajectories for relative displacement amplitude, $\delta_1 = 0.25$ and $\omega = 0.2$. (a) $1 - 0 - \overline{q}$ motion, $\delta_2 = 0.25$. (b) $1 - \overline{p} - \overline{q}$ motion, $\delta_2 = 0.5$. (c) $1 - \overline{p} - \overline{q}$ motion, $\delta_2 = 0.75$. (d) $1 - \overline{p} - \overline{q}$ motion, $\delta_2 = 1.0$.

Figure 16: Time trajectories for relative impact velocity, $\delta_1 = 0.25$ and $\omega = 0.2$. (a) $1 - 0 - \overline{q}$ motion, $\delta_2 = 0.25$. (b) $1 - \overline{p} - \overline{q}$ motion, $\delta_2 = 0.5$. (c) $1 - \overline{p} - \overline{q}$ motion, $\delta_2 = 0.75$. (d) $1 - \overline{p} - \overline{q}$ motion, $\delta_2 = 1.0$.

Figure 17: Periodic motions for the asymmetric multiclearance system with fixed clearance $\delta_1$ in the $(\omega, \delta_2)$ parameter plane. (a) $\delta_1 = 0.25$. (b) $\delta_1 = 0.5$. (c) $\delta_1 = 0.75$. (d) $\delta_1 = 1.0$. 
4.3. Dynamics in the \((\omega, \delta)\) Parameter Plane When Clearance \(\delta_{1r}\) Is Being Fixed. Only fix the right assembly clearance \(\delta_{1r}\) between the mass blocks \(M_1\) and \(M_2\), then, the left clearance corresponding to it is represented by \(\delta_{1l}\), the distributions and transitions of periodic vibrations of the asymmetric multiclearance system in the \((\omega, \delta_{1l})\)- and \((\omega, \delta_{1r})\) parameter planes are simulation calculated under different values of \(\delta_{1l}\).

With the fixed clearance \(\delta_{1r} = 0.25\), the parameter domain distribution characteristics of periodic vibrations depicted in Figure 18(a) vary greatly. In the entire low-frequency range of \((\omega, \delta_{1l})\) parameter plane, the two masses exhibit complete chattering vibration, and \(1-p-q\) fundamental periodic vibration groups present a band-shaped distribution. In the large clearance domain represented by \(\delta_{1l}\), the pattern type of periodic motion is mainly occupied by \(1-0-q\) one-sided impact motion. With decrease in \(\omega\), \(1-0-q\) motion passes through \(G_{1-0-q}\) one-sided real-grazing bifurcation into \(1-(q+1)\) motion regularly. From the comprehensive observation of the \((\omega, \delta_{1l})\)- parameter plane, it can be known that \(n-p-q\) periodic motion parameter domain of the system always presents dynamic characteristics with the impact number \(p \leq q\). In the \((\omega, \delta_{1r})\) parameter plane presented in Figure 18(b), the parameter domains of \(1-p\) fundamental periodic vibration groups are kink shaped, and there are parameter islands containing various subharmonic motions and chaotic motion besides clear and stable grazing bifurcation lines \(G_{1-p}\) between \(1-p\) and \(1-(p+1)\) neighboring periodic motions. The fixed clearance \(\delta_{1r}\) is taken 0.5 in Figure 18(c), \(n-p-q\) periodic motion appears the impact number \(p \leq q\) characteristic above the \(1-1-1_{AS}\) parameter domain; on the contrary, it exhibits \(p \geq q\) characteristic below the \(1-1-1_{AS}\) parameter domain. The periodic motion domains are basically symmetrically distributed about \(\delta_{1l} = \delta_{1r} = 0.5\). In Figure 18(d), the fractal feature of the periodic vibration parameter domains is obvious, and the areas of 2-4, 3-6, 4-8 period-doubling bifurcation sequences included in the 1-2 motion existence domain are increased.

One-parameter bifurcation diagram, crossing the changing clearance \(\delta_{1l} = \delta_{1r} = 0.15\), can be obtained by a horizontal scan on the two-parameter bifurcation diagrams shown in Figures 18(c) and 18(d). Figure 19(a) shows the variation of the relative displacement of masses \(M_1\) and \(M_2\) with frequency \(\omega\), which corresponds to periodic Poincaré mapping \(\sigma_{n}\) following, Figure 19(b) corresponds to impact Poincaré mapping \(\sigma_{p}\), and Figure 19(c) corresponds to impact Poincaré mapping \(\sigma_{p1}\). The mass blocks \(M_1\) and \(M_2\) do not impact each other at the rigid constraint \(A_1\) or \(A_2\) when the excitation frequency \(\omega < 0.4135\) and keep 1-0-0 impactless motion (Figure 19(b)). After the frequency \(\omega\) is increased, the two masses first impact mutually at the left constraint \(A_1\) to obtain 1-1-0 periodic motion and then generate 1-2-0 periodic motion through a one-sided grazing bifurcation. Further increase in \(\omega\) up to \(\omega = 0.8525\), the masses impact each other at the constraint \(A_2\) for the first time. With increasing frequency, asymmetric \(1-1-1_{AS}\) motion occurs after a narrow window of 2-period motion. When \(\omega\) is increased into the high-frequency domain, the two masses successively show 1-1-0, 2-2-0, 2-1-0, one-sided impact periodic motions in the interval of \(\omega \in [1.905, 2.2755], [2.2755, 2.3605]\) and \([2.564, 2.966]\), with narrow multi-impact motion windows in them. Figure 19(c) illustrates the periodic vibration characteristics of \(M_2\) at the rigid constraint \(A_2\). In the low-frequency range of \(\omega < 0.556\), \(M_2\) always sticks to the constraint \(A_2\) until the restraining reaction force of the mass reduces to zero, \(M_2\) deviates from \(A_2\) with zero initial velocity, resulting in \(1-p\) fundamental periodic vibration, the sticking ends. With increase in \(\omega\), \(1-p\) periodic motion undergoes several grazing bifurcations, the impact number of the mass \(M_2\) at the rigid constraint \(A_2\) decreases one by one. With a further increase in the frequency, the periodic motion of \(M_2\) goes through inverse period-doubling bifurcation, 2-4 motion transfers to 1-2 motion. When \(\omega\) is up to 1.484, 1-1 motion is generated by a grazing bifurcation from 1-2 periodic motion. 1-0 impactless motion emerges from a narrow interval of period-doubling bifurcation and inverse period-doubling bifurcation at \(\omega = 1.905\), which corresponds to the 1-1-0 motion window of the two mass blocks shown in Figure 19(b). After \(\omega\) is increased more, the mass \(M_2\) exhibits long-periodic vibration and narrow chaotic motion.

The periodic motion domains of two mass blocks in the \((\omega, \delta_{1r})\) parameter plane are occupied with \(1-0-1, 1-1-0, 2-1-0, 2-0-1, \) etc, one-sided impact motions, as seen in Figure 18(e). The left-sided impact motion parameter domain \(n-p\)-0 gradually extends into the larger clearance domain, which forces \(1-0-1, 2-0-1, \) and other \(n-0-q\) right-sided impact motion parameter domain to be decreased. In the frequency range of \(\omega \in [1.22, 1.9325]\), 1-1-0 motion undergoes one-sided grazing bifurcation transition to 1-1-1(AS) periodic motion. In the higher \(\delta_{1l}\) clearance region, 1-1-1(AS) periodic motion parameter domain transits into 2-1-2 motion through one-sided period-doubling bifurcation as the clearance is increased. Oppositely, 1-1-1(AS) motion transits into 2-2-1 motion with decrease in \(\delta_{1l}\) and the two types of periodic motion parameter domains are basically symmetrically distributed. When the clearance value \(\delta_{1l}\) continues to increase to 1.0, the periodic vibrations in the \((\omega, \delta_{1l})\) parameter plane exhibit a large-area \(n-p-0\) left-sided impact motion, which means the mass blocks \(M_1\) and \(M_2\) no longer impact each other at the constraint \(A_1\) (Figure 18(g)). The periodic vibration distributions of the two mass blocks shown in Figures 18(f) and 18(h) have not changed much. Even in extreme cases \((\delta_{1r} = 1.0)\), the system does not induce other more complex irregular periodic motions. Except that only the period-doubling bifurcation sequences windows in the 1-1, 1-2 motion parameter domains are decreased with increase in the fixed clearance \(\delta_{1r}\). The areas of tongue-shaped regions nearby the demarcation of neighboring fundamental periodic vibrations become correspondingly smaller.
5. Conclusions

In this article, we studied the dynamics of a nonsmooth vibration system with multiclearance periodic excitation. It mainly presents fundamental periodic impact motion, incomplete chattering vibration, complete chattering vibration, and subharmonic impact motion in the global distribution. The correlative distributions and bifurcation scenarios of the periodic motion of the system are analyzed on the multiparameter plane, the relationship between the dynamic performance, and the clearance structure parameter are revealed to find the matching rule:

1. The location and the mutual influence between the clearances cause the multiclearance rigid constraints vibration system to exhibit complex and diverse dynamic characteristics under the same parameter conditions. The system mainly displays nonimpact and stable one-period motion in the large clearance domain. Because of the increase in the number of clearances and their interaction, the stable one-period motion domain gradually decreases. In the small clearance domain, the vibration system usually shows fundamental periodic motion sequence in the low-frequency range, and transitions to incomplete
and complete chattering-impact motion as the frequency decreases. Compared with the both-sided constraint vibration system, the constraint vibration system with more than two clearances reduces the fractal regularity of chattering motion due to the interaction of nonsmooth mechanical features, and even a completely static state domain appears.

(2) The local area nearby grazing bifurcation points on the boundary of periodic motion shows complex dynamics. The irreversibility of transfer mechanism between neighbouring fundamental periodic vibrations is revealed. This study has found the existence of singular points, hysteresis regions, and tongue-shaped regions nearby the boundaries of the parameter domain of neighboring fundamental periodic impact motions; among that, neighboring fundamental periodic impact motions coexist in the hysteresis region, and the transition and existence domains of series of atypical periodic motions within the tongue-shaped regions possess certain regular pattern. The increase in the clearance number causes the system to produce more subharmonic impact motion groups.

(3) Three types of multiple heterogeneous constraints are constructed. The simulation results of different clearance constraint conditions have discussed the key dynamic parameter, the clearance δ and its effect on the stability, and bifurcation of periodic vibration for the multiclearance vibration system. The fundamental periodic vibration sequences in the low-frequency range are highly sensitive to clearance parameter. When one of the clearances is fixed and the value is low, the multiple grazing bifurcations of the fundamental periodic vibrations accumulated. This results in a significant expansion of the chattering-sticking impact domain toward the direction of increasing frequency and severely squeezing the existence domains of fundamental periodic impact motion.

The restriction and mutual influence of multiclearance have superimposed effects on the periodic motion of the vibration system, which makes the dynamics of the system more complicated. When dynamically designing this type of multiclearance mechanical equipment, the clearance arrangement can be selected through periodic motion occurrence domains and bifurcation characteristics, and their values can be optimized to make the system work in one-period motion steadily and efficiently.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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