Boundary Orbits in Static Spacetimes

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The study of circular orbits in spacetime is of astrophysical importance. The identification and classification of circular orbits in both static spherically (and axially) symmetric and also stationary spacetimes remains an active area of interest. Even in the simplest static spherically symmetric case, it is well known that the introduction of a cosmological constant in vacuum leads to the study of quartic polynomials in order to locate boundary orbits, those that straddle between stable and unstable orbits. These orbits are often referred to as “marginally stable orbits” or “indifferently stable orbits”. A comprehensive study of texts offers little clarification as to the stability or instability of these boundary orbits. Here we argue that the direct use of second order perturbation theory immediately shows that these boundary orbits are unstable in the perturbative sense.

I. INTRODUCTION

The study of circular timelike geodesic orbits is a fundamental aspect of relativistic astrophysics. Determining the inner most stable circular orbits (ISCOs) in the fields of black holes and neutron stars provides information on the accretion disk region of these objects (e.g. [1]), as the inner radius of the accretion disk is typically assumed to occur at the ISCO. During the accretion process the gas particles’ gravitational potential energy is depleted and the gas is heated up. The amount of radiation energy in this process is equal to the gravitational binding energy of the ISCO [2]. The efficiency can be determined by dividing the gravitational binding energy of the ISCO by the rest mass energy of the particle [3]. ISCOs also play a crucial role in the field of gravitational waves as they mark the transition of the compact binaries from the slow inspiralling phase to the fast plunge phase [4]. The gravitational waves which are generated by these sources can be classified according to the frequency at which the phase transition occurs (see [5, 6] and below).

With the introduction of charge and/or a cosmological constant, it is not difficult to see that outer most stable circular orbits (OSCOs) will also arise (e.g. [7]). We refer to boundary orbits as those that straddle between stable and unstable circular orbits and so include both ISCOs and OSCOs. (We avoid the use of the term “marginally stable orbits”. See [8], [9] and [10]. The orbits are also referred to as “indifferently stable”. For example [11].) With the existence of both ISCOs and OSCOs the search for boundary orbits leads directly to the study of polynomials. For example, recently Sturm’s theorem has been used to study boundary orbits in the Kottler (Schwarzschild-de Sitter) spacetime [12].

Studying the existence and stability of circular orbits is, of course, a fundamental problem in dynamical systems and there are a number of approaches that can be used. Indeed, there are a number of types of stability [13]. For example, in addition to the use of the elementary “effective potential” approach, it is now common to see the use of Lyapunov exponents (e. g. [14]). However, as we show, the direct use of second order perturbation theory immediately shows that boundary orbits are unstable in the perturbative sense and it is this type of instability we are interested in here. This instability is, at first sight, of no apparent deep physical significance. However, it is certainly of pedagogical significance (think of the name, ISCO). Without a clear statement about the stability of boundary orbits, one has to skate awkwardly about the issue. For example, in one of the most influential texts of all time [15], the boundary orbits in Schwarzschild are described both as stable and unstable. In another influential text [16] the boundary orbits are described as stable in a figure but (and without justification) unstable in the text. A careful treatment is, for example, given in [17], but the boundary orbits are not explicitly addressed. Generally speaking, texts consider only ISCOs, and they are considered “stable”. The same can be said of many, but not all, papers.

II. THE STATIC SPHERICALLY SYMMETRIC CASE

A. Background Calculations

First we construct stable circular timelike geodesic orbits in a static spherically symmetric field. We follow the notation of [19]. The field takes the form

$$ds^2 = \frac{dr^2}{1 - \frac{2m(r)}{r}} + r^2 d\Omega^2 - e^{2\Phi(r)} dt^2,$$  \hspace{1cm} (1)

where $d\Omega^2$ is the metric of a unit sphere. Throughout we refer to the function $\Phi(r)$ as the “potential” and $m(r)$ as the effective gravitational “mass”. It is immediately clear from [11] that all geodesic orbits are stably planar.

1 This is actually no longer true, given the observation of gravitational waves. “Bumpy” spacetimes can be tuned arbitrarily close to the Kerr metric. However, orbits approaching an ISCO can be qualitatively different depending on whether the ISCO is determined by the onset of instability in the radial or vertical direction. See [18]. Here we are concerned with a direct treatment of the much simpler static case.
(say $\theta = \pi/2$) and have two constants of motion, the “energy” $\gamma = e^{2\Phi(r)} r^2 \dot{\phi}$ (\`$\dot{\phi}$ being the proper time derivative) and “angular momentum” $l = r^2 \dot{\phi}$. In the timelike case then
\begin{equation}
\dot{r}^2 f(r) + \mathcal{V}(r) = \gamma^2, \tag{2}
\end{equation}
where
\begin{equation}
f(r) = \frac{e^{2\Phi(r)}}{1 - 2m(r)/r} > 0 \tag{3}
\end{equation}
and
\begin{equation}
\mathcal{V}(r) = e^{2\Phi(r)}(1 + \frac{l^2}{r^2}). \tag{4}
\end{equation}
Setting $\dot{r} = \ddot{r} = 0$, $r > 0$ it follows from the timelike geodesic equations that
\begin{equation}
\gamma^2 = \frac{e^{2\Phi}}{1 - r\Phi'} \tag{5}
\end{equation}
and
\begin{equation}
l^2 = \frac{r^3 \Phi'}{1 - r\Phi'} \tag{6}
\end{equation}
where $' = d/dr$. The existence of these circular orbits requires
\begin{equation}
0 < r\Phi' < 1. \tag{7}
\end{equation}
Note that from (4) and (6) on a circular orbit ($r = r_0$)
\begin{equation}
\mathcal{V}'(r_0) = 0 \tag{8}
\end{equation}
and
\begin{equation}
\mathcal{V}''(r_0) = 2e^{2\Phi} \left( \frac{3\Phi' + r\Phi'' - 2r(\Phi')^2}{r(1 - r\Phi')}. \tag{9}
\right)
\end{equation}

B. Radial Perturbations

Next, we require that the timelike circular geodesics be stable against radial perturbations. Other perturbations (in energy and angular momentum) are of no interest here as explained below. Let $r_0$ be a circular orbit and consider $r = r_0 + \delta$ where $\delta << r_0$. Taking expansions\(^3\) of $\mathcal{V}(r)$ and $f(r)$ about $r = r_0$ it follows from (2) that to order $\delta$
\begin{equation}
\delta + \frac{\mathcal{V}''(r_0)}{2f(r_0)} \delta = 0, \tag{10}
\end{equation}
so that $\mathcal{V}''(r_0) > 0$ for stability. From (9) then
\begin{equation}
3\Phi' + r\Phi'' > 2r(\Phi')^2 \tag{11}
\end{equation}
for stable circular orbits. Clearly $\mathcal{V}''(r_0) < 0$ leads to instability. The case $\mathcal{V}''(r_0) = 0$ defines the boundary orbits. To examine stability in this case we must go to order $\delta^2$. We then find
\begin{equation}
\delta + \frac{\mathcal{V}''(r_0)}{2f(r_0)} \delta^2 = 0, \tag{12}
\end{equation}
and so the solutions for $\delta$ are Weierstrass $\wp$ functions (e.g. (20)). We therefore interpret these orbits as unstable and we refer to the boundary orbits as $r_*$ where $\mathcal{V}''(r_*) = 0$.

C. Newtonian Comparison and a glance at More Involved Boundary Orbits

The stability condition (11) can be considered a refinement of the Newtonian condition
\begin{equation}
3\Phi' + r\Phi'' > 0 \tag{13}
\end{equation}
for conservative central fields. In Table 1 we compare conditions (10) and (13) in the simplest cases, a point mass in Newton’s theory and the Schwarzschild solution\(^4\)

| Theory | $\Phi$ | (10) | (13) |
|--------|-------|-----|-----|
| Newton | $-m/r$ | $r > 2m$ | $m > 0$ |
| Einstein | $\frac{1}{2}\ln(1 - \frac{2m}{r})$ | $r > 6m$ | $r > 4m$ |

These simple results quickly give way to more involved relations. For example, an inclusion of the cosmological constant (here we take $\Lambda \geq 0$) gives rise to the Newtonian potential $\Phi(r) = -m/r - \Lambda r^2/6$ and the Kottler (Schwarzschild - de Sitter) solution $\Phi(r) = \frac{1}{2}\ln(1 - 2m/r - \Lambda r^2/3)$. In the Newtonian case the boundary orbits satisfy $3\Phi'(r_*) + r\Phi''(r_*) = 0$ and so
\begin{equation}
r_* = \sqrt[3]{\frac{3m}{4\Lambda}}. \tag{14}
\end{equation}
The Newtonian circular orbits are stable for $r < r_*$, with $r_*$ given by (14).

\(^3\) Strictly speaking, of course, only (13) is appropriate in Newtonian theory and (11) in general relativity. However, we include this table to show, for example, that Newton’s theory somehow “knows” that with the correct stability criterion, $r > 2m$ for stable circular orbits.

\(^4\) The expansions do not involve reference to the metric components and so (10) and (12) apply to the axial case discussed below.
In the Kottler case the boundary orbits \( \mathcal{V}'(r_*) = 0 \) give

\[
4\Lambda r^4 - 15m\Lambda r^3 + 3mr + 18m^2 = 0, \tag{15}
\]
so that \( r_* \) has up to 2 distinct values. As mentioned previously, equation (15) has been examined recently via Sturm’s theorem by Ono et al. \( ^{12} \). The inclusion of charge (the Reissner Nordström de Sitter solution with \( \Phi(r) = \frac{1}{2} \ln(1-2m/r-\Lambda r^2/3+e^2/r^2) \)) requires numerical methods to determine the boundary orbits. These are given by

\[
4\Lambda r^6 - 15m\Lambda r^5 + 12\Lambda e^2r^4 - 3mr^3 + 18m^2r^2 - 27me^2r_* + 12e^4 = 0 \tag{16}
\]
so that \( r_* \) has up to 6 distinct values.

D. Changes in Energy and Angular Momentum

Given \( r_* \), \( \Phi'_* \) and \( \Phi_* \), \( \gamma_* \) and \( l_* \) are given uniquely by (5) and (6) respectively. The energy and angular momentum of boundary orbits is therefore predetermined. Note also that \( \gamma_* = l_* = 0 \) but the nature of the associated critical points depends on \( \Phi''_* \). The point here is that when considering perturbations about the boundary orbits \( r_* \), changes in \( \gamma \) and \( l \) can not be considered.

III. THE STATIC AXIALLY SYMMETRIC CASE

A. A Metric of Sufficient Generality

We start with the axially symmetric metric

\[
ds^2 = 2H dt d\phi + B^2 dr^2 + C^2 d\theta^2 + F^2 d\phi^2 - e^{2\Phi} dt^2 \tag{17}
\]
where \( H, B, C, F \) and \( \Phi \) are functions of \( r \) and \( \theta \). The Killing vector \( \xi^\alpha = \partial_\alpha \) satisfies

\[
\xi_\alpha \nabla^\beta \xi_\delta = 0 \tag{18}
\]
if and only if \( H = 0 \) which is our static condition.

B. Equatorial Orbits in Reflection Symmetric Spacetimes

By equatorial orbits we mean orbits in the subspace \( \theta = \pi/2 \) and by reflection symmetric we mean

\[
\left. \frac{\partial X}{\partial \theta} \right|_{\theta = \pi/2} = 0 \tag{19}
\]
where \( X \in (B, C, F, \Phi) \). Under these conditions, there is relatively little change from the spherical case. Whereas \( ^2 \) remains unchanged, equations (3) through (6) become

\[
f = B^2e^{2\Phi(r)} > 0, \tag{20}
\]

\[
\mathcal{V} = e^{2\Phi}(1 + \frac{l^2}{F^2}), \tag{21}
\]

\[
\gamma^2 = \frac{e^{2\Phi} F'}{F' - F\Phi}, \tag{22}
\]

and

\[
l^2 = \frac{F^3 F'}{F - F'} \tag{23}
\]
respectively where \( \Phi' \equiv \partial / \partial r |_{\theta = \pi/2} \). Moreover, whereas \( \mathcal{V} \) remains unchanged, \( \Phi \) becomes

\[
\mathcal{V}' = 2e^{2\Phi} \left( \frac{(3(F')^2 - FF''\Phi' + FF'(\Phi'' - 2(\Phi')^2)}{F(F' - F\Phi')} \right). \tag{24}
\]
The condition for the stability of circular orbits remains \( \mathcal{V}'(r_0) > 0 \) and boundary orbits are once again given by \( \mathcal{V}'(r_*) = 0 \).

C. Vacuum

The static axially symmetric vacuum solutions of the Einstein equations are the Weyl metrics\(^{21} \)

\[
ds^2 = -e^{2U} dt^2 + e^{-2U} (e^{2\gamma}(dr^2 + dz^2) + r^2 d\phi^2), \tag{25}
\]
where \( U \) and \( \gamma \) are functions of \( r \) and \( z \). The simplest case is the Zipoy \( ^{22} \) - Voorhees \( ^{23} \) solution which has been very widely studied (see, for example, \( ^{11} \)).

The next most widely studied case is the Curzon \( ^{24} \) - Chazy \( ^{25} \) solution. (For a recent study of the geometrical properties of this solution see \( ^{26} \). For a recent study of timelike geodesic orbits in this solution see \( ^{27} \).)

In Stachel coordinates \( ^{28} \) the solution is given by

\[
ds^2 = e^{-\frac{2}{\sqrt{3}}(dr^2 + r^2 d\theta^2)} + r^2 \sin(\theta)^2 d\phi^2 - e^{-\frac{2}{\sqrt{3}}} dt^2. \tag{26}
\]

From (24) and (26) it now follows that

\[
\gamma'' = \frac{2(x^2 - 6x + 4)}{x^4 e^{2/\sqrt{3}}(x - 2)} \tag{27}
\]
where \( x \equiv r/m \). Even though the metric (26) remains regular at \( r = 2m \) \( ^{26} \), the spacetime is divided into two regions as regards circular timelike orbits in the equatorial plane. For \( r > 2m \), \( x_* = 3 + \sqrt{5} \) and the timelike circular orbits are stable for \( x > x_* \) and unstable for \( x \leq x_* \). The minimum value of \( l^2 \) is at \( l^2 \) which is > 0. For \( r = 2m \) neither \( \gamma \) nor \( l \) exist. For \( r < 2m \), \( l \) does

\(^4\) Weyl metrics do not allow for \( \Lambda \) as \( G_{\rho\phi} = -G_{z\phi} \) which necessitates \( \Lambda = 0 \).
not exist. There is then only one boundary orbit \(29\), \(r_\ast = (3 + \sqrt{5})m \approx 5.24m\). This is all remarkably similar to the Schwarzschild vacuum, with \(2m\) replaced by \(3m\) and \((3 + \sqrt{5})m\) replaced by \(6m\). Since the boundary orbits in the one-particle Curzon-Chazy solution is so simple, one might expect the same in the two-particle case \(30\). Unfortunately, this turns out not to be the case.

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