Low-Energy Behavior of Gluons and Gravitons from Gauge Invariance

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Workshop on “Supersymmetric Field Theories”
This talk is based on the work done together with Zvi Bern, Scott Davies and Josh Nohle

“Low-Energy Behavior of Gluons and Gravitons: from Gauge Invariance”,
arXiv:1406:6987 [hep-th].

See also the related work by J. Broedel, M. de Leeuw, J. Plefka and M. Rosso
“Constraining subleading soft gluon and graviton theorems”,
arXiv:1406.6574 [hep-th]
Plan of the talk

1. Introduction
2. Scattering of a photon and \( n \) scalar particles
3. Scattering of a graviton and \( n \) scalar particles
4. Soft limit of \( n \)-gluon amplitude
5. Soft limit of \( n \)-graviton amplitude
6. Comments on loop corrections: gauge theory
7. Comments on loop corrections: gravity
8. What about soft theorems in string theory?
9. Soft theorem for dilaton
10. Conclusions
11. Outlook
Three kinds of symmetries with different physical consequences.

Global unbroken symmetries as isotopic spin or $SU(3)_V$ in three-flavor QCD.

Unique vacuum annihilated by the symmetry generator: $Q_a |0\rangle = 0$

Particles are classified according to multiplets of this symmetry and all particles of a multiplet have the same mass.

If isotopic spin were an exact symmetry, the proton and the neutron would have the same mass.

This would have happened in QCD if the lowest two quarks would have had the same mass.

This is not the case because the mass matrix of the quarks breaks explicitly $SU(2)$ and even more $SU(3)$ flavor symmetry.
Then, we have the global spontaneously broken symmetries as $SU(3)_L \times SU(3)_R$ (broken to $SU(3)_V$) symmetry in QCD for zero mass quarks.

Degenerate vacua: $Q_a|0\rangle = |0'\rangle$.

Not realized in the spectrum, but it implies the presence of massless particles, called Goldstone bosons.

They are the pions in QCD with 2 flavors.

This is one physical consequence of the spontaneous breaking.

Another one is the existence of low-energy theorems.

The $\pi\pi$ scattering amplitude is fixed at low energy.

One gets the two scattering lengths:

$$a_0 = \frac{7m_\pi}{32\pi F_\pi^2}; \quad a_2 = -\frac{m_\pi}{16\pi F_\pi^2}$$

explicit breaking by a mass term.

Scattering amplitude is zero for massless pions at low energy because Goldstone bosons interact with derivative coupling implying a shift symmetry.
Finally, we have the local gauge symmetries for massless spin 1 and spin 2 particles.

Local gauge invariance is necessary to reconcile the theory of relativity with quantum mechanics.

It allows a fully relativistic description, but eliminating, at the same time, the presence of negative norm states in the spectrum of physical states.

Although described by $A_{\mu}$ and $G_{\mu\nu}$, both photons and gravitons have only two physical degrees of freedom in $d=4$,

and respectively

$$d - 2 \quad \text{and} \quad \frac{(d - 2)(d - 1)}{2} - 1$$

in $d$ space-time dimensions.

Another consequence of gauge invariance is charge conservation that, however, follows from the global part of the gauge group.

Yet another physical consequence of local gauge invariance is the existence of low-energy theorems for photons and gravitons:

[F. Low, 1958; S. Weinberg, 1964]
Let us consider Compton scattering on spinless particles.

The scattering amplitude $M_{\mu\nu}$ is gauge invariant:

$$k_{1}^{\mu} M_{\mu\nu} = k_{2}^{\nu} M_{\mu\nu} = 0$$

The previous conditions determine the scattering amplitude for zero frequency photons and one gets the Thompson cross-section:

$$\sigma_T = \frac{8\pi}{3} \left( \frac{e^2}{mc^2} \right)^2 = \frac{8\pi}{3} r_{cl}$$

where $r_{cl}$ is the classical radius of a point particle of mass $m$ and charge $e$. 
The interest on the soft theorems was recently revived by [Cachazo and Strominger, arXiv:1404.1491[hep-th]].

They study the behavior of the $n$-graviton amplitude when the momentum $q$ of one graviton becomes soft ($q \sim 0$).

They suggest a universal formula for the subleading term $O(q^0)$.

The leading term $O(q^{-1})$ was shown to be universal by Weinberg in the sixties.

In a previous paper Strominger et al derived the Weinberg universal behavior from the Ward identities of the BMS transformations.

They speculate that also the next to the leading term follows from the BMS transformations.

In the following we show that the first three leading terms are a direct consequence of gauge invariance.
The scattering amplitude $M_{\mu}(q; k_1 \ldots k_n)$, involving one photon and $n$ scalar particles, consists of two pieces:

\[
A_{\mu n}^{\mu}(q; k_1, \ldots, k_n) = \sum_{i=1}^{n} e_i \frac{k_i^{\mu}}{k_i \cdot q} T_n(k_1, \ldots, k_i + q, \ldots, k_n) + N_{\mu n}^{\mu}(q; k_1, \ldots, k_n).
\]

and must be gauge invariant for any value of $q$:

\[
q_{\mu} A_{\mu n}^{\mu} = \sum_{i=1}^{n} e_i T_n(k_1, \ldots, k_i + q, \ldots, k_n) + q_{\mu} N_{\mu n}^{\mu}(q; k_1, \ldots, k_n) = 0.
\]
Expanding around \( q = 0 \), we have

\[
0 = \sum_{i=1}^{n} e_i \left[ T_n(k_1, \ldots, k_i, \ldots, k_n) + q_\mu \frac{\partial}{\partial k_{i\mu}} T_n(k_1, \ldots, k_i, \ldots, k_n) \right]
\]

\[
+ q_\mu N_\mu^n(q = 0; k_1, \ldots, k_n) + \mathcal{O}(q^2).
\]

At leading order, this equation is

\[
\sum_{i=1}^{n} e_i = 0,
\]

which is simply a statement of charge conservation [Weinberg, 1964]

At the next order, we have

\[
q_\mu N_\mu^n(0; k_1, \ldots, k_n) = - \sum_{i=1}^{n} e_i q_\mu \frac{\partial}{\partial k_{i\mu}} T_n(k_1, \ldots, k_n).
\]
This equation tells us that $N_{n}^{\mu}(0; k_1, \ldots, k_n)$ is entirely determined in terms of $T_n$ up to potential pieces that are separately gauge invariant.

However, it is easy to see that the only expressions local in $q$ that vanish under the gauge-invariance condition $q_{\mu}E^{\mu} = 0$ are of the form,

$$E^{\mu} = (B_1 \cdot q)B_{2}^{\mu} - (B_2 \cdot q)B_{1}^{\mu},$$

where $B_{1}^{\mu}$ and $B_{2}^{\mu}$ are arbitrary vectors (local in $q$) constructed with the momenta of the scalar particles.

The explicit factor of the soft momentum $q$ in each term means that they are suppressed in the soft limit and do not contribute to $N_{n}^{\mu}(0; k_1, \ldots, k_n)$.

We can therefore remove the $q_{\mu}$ leaving

$$N_{n}^{\mu}(0; k_1, \ldots, k_n) = -\sum_{i=1}^{n} e_i \frac{\partial}{\partial k_{i\mu}} T_n(k_1, \ldots, k_n),$$

thereby determining $N_{n}^{\mu}(0; k_1, \ldots, k_n)$ as a function of the amplitude without the photon.
Inserting this into the original expression yields

\[ A_n^\mu(q; k_1, \ldots, k_n) = \sum_{i=1}^{n} \frac{e_i}{k_i \cdot q} \left[ k_i^\mu - iq_\nu J_i^{\mu\nu} \right] T_n(k_1, \ldots, k_n) + \mathcal{O}(q), \]

where

\[ J_i^{\mu\nu} \equiv i \left( k_i^\mu \frac{\partial}{\partial k_{i\nu}} - k_i^\nu \frac{\partial}{\partial k_{i\mu}} \right), \]

is the orbital angular-momentum operator and \( T_n(k_1, \ldots, k_n) \) is the scattering amplitude involving \( n \) scalar particles (and no photon).

The amplitude with a soft photon with momentum \( q \) is entirely determined in terms of the amplitude without the photon up to \( \mathcal{O}(q^0) \).

This goes under the name of F. Low’s low-energy theorem.
Low’s theorem is unchanged at loop level for the simple reason that even at loop level, all diagrams containing a pole in the soft momentum are of the form shown, with loops appearing only in the blob and not correcting the external vertex.

Can we get any further information at higher orders in the soft expansion?

One order further in the expansion, we find the extra condition,

\[
\frac{1}{2} \sum_{i=1}^{n} e_i q_\mu q_\nu \frac{\partial^2}{\partial k_{i\mu} \partial k_{i\nu}} T_n(k_1, \ldots, k_n) + q_\mu q_\nu \frac{\partial N^\mu_n}{\partial q_\nu} (0; k_1, \ldots, k_n) = 0.
\]

This implies

\[
\sum_{i=1}^{n} e_i \frac{\partial^2}{\partial k_{i\mu} \partial k_{i\nu}} T_n(k_1, \ldots, k_n) + \left[ \frac{\partial N^\mu_n}{\partial q_\nu} + \frac{\partial N^\nu_n}{\partial q_\mu} \right] (0; k_1, \ldots, k_n) = 0,
\]
Gauge invariance determines only the symmetric part of the quantity \( \frac{\partial N_{\nu}^\mu}{\partial q_\mu} (0; k_1, \ldots, k_n) \).

The antisymmetric part is not fixed by gauge invariance.

Indeed, this corresponds exactly to the gauge invariant terms considered above.

Then, up to this order, we have

\[
A_\mu^\mu (q; k_1, \ldots, k_n) = \sum_{i=1}^{n} \frac{e_i}{k_i \cdot q} \left[ k_i^\mu - iq_\nu J_i^{\mu\nu} \left( 1 + \frac{1}{2} q_\rho \frac{\partial}{\partial k_i^\rho} \right) \right] T_n(k_1, \ldots, k_n)
+ \frac{1}{2} q_\nu \left[ \frac{\partial N_{\nu}^\mu}{\partial q_\nu} - \frac{\partial N_{\mu}^\nu}{\partial q_\mu} \right] (0; k_1, \ldots, k_n) + O(q^2).
\]

It is straightforward to see that one gets zero by saturating the previous expression with \( q_\mu \).
In order to write our universal expression in terms of the amplitude, we contract $A_n^\mu(q; k_1, \ldots, k_n)$ with the photon polarization $\varepsilon_{q\mu}$.

Finally, we have the soft-photon limit of the single-photon, $n$-scalar amplitude:

$$A_n(q; k_1, \ldots, k_n) \rightarrow \left[ S^{(0)} + S^{(1)} \right] T_n(k_1, \ldots, k_n) + \mathcal{O}(q),$$

where

$$S^{(0)} \equiv \sum_{i=1}^{n} e_i \frac{k_i \cdot \varepsilon_{q}}{k_i \cdot q},$$

$$S^{(1)} \equiv -i \sum_{i=1}^{n} e_i \frac{\varepsilon_{q\mu} q_{\nu} J_{i}^{\mu\nu}}{k_i \cdot q},$$

where $J_{i}^{\mu\nu}$ is the angular momentum.
One graviton and n scalar particles

In the case of a graviton scattering on $n$ scalar particles, one can write

$$M^{\mu\nu}_n(q; k_1, \ldots, k_n) = \sum_{i=1}^{n} \frac{k_i^\mu k_i^\nu}{k_i \cdot q} T_n(k_1, \ldots, k_i + q, \ldots, k_n) + N^{\mu\nu}_n(q; k_1, \ldots, k_n),$$

$N^{\mu\nu}_n(q; k_1, \ldots, k_n)$ is symmetric under the exchange of $\mu$ and $\nu$.

For simplicity, we have set the gravitational coupling constant to unity.

On-shell gauge invariance implies

$$0 = q_\mu M^{\mu\nu}_n(q; k_1, \ldots, k_n)$$

$$= \sum_{i=1}^{n} k_i^\nu T_n(k_1, \ldots, k_i + q, \ldots, k_n) + q_\mu N^{\mu\nu}_n(q; k_1, \ldots, k_n).$$
At leading order in $q$, we then have

$$\sum_{i=1}^{n} k_i^\mu = 0,$$

It is satisfied due to momentum conservation.

If there had been different couplings to the different particles, it would have prevented this from vanishing in general.

This shows that gravitons have universal coupling [Weinberg, 1964]).

At first order in $q$, one gets

$$\sum_{i=1}^{n} k_i^\nu \frac{\partial}{\partial k_{i\mu}} T_n(k_1, \ldots, k_n) + N_{n}^{\mu\nu}(0; k_1, \ldots, k_n) = 0,$$

while at second order in $q$, it gives

$$\sum_{i=1}^{n} k_i^\nu \frac{\partial^2}{\partial k_{i\mu} \partial k_{i\rho}} T_n(k_1, \ldots, k_n) + \left[ \frac{\partial N_{n}^{\mu\nu}}{\partial q_\rho} + \frac{\partial N_{n}^{\rho\nu}}{\partial q_\mu} \right] (0; k_1, \ldots, k_n) = 0.$$
As for the photon, this is true up to gauge-invariant contributions to $N^\mu\nu_n$.

However, the requirement of locality prevents us from writing any expression that is local in $q$ and not sufficiently suppressed in $q$.

Using the previous equations, we write the expression for a soft graviton as

$$M^\mu\nu_n(q; k_1 \ldots k_n) = \sum_{i=1}^n \frac{k^\nu_i}{k_i \cdot q} \left[ k_i^\mu - iq_\rho J_i^\mu\rho \left( 1 + \frac{1}{2} q_\sigma \frac{\partial}{\partial k_i^\sigma} \right) \right] T_n(k_1, \ldots, k_n) + \frac{1}{2} q_\rho \left[ \frac{\partial N^\mu\nu_n}{\partial q_\rho} - \frac{\partial N^\rho\nu_n}{\partial q_\mu} \right] (0; k_1, \ldots, k_n) + O(q^2).$$

This is essentially the same as for the photon except that there is a second Lorentz index in the graviton case.

Unlike the case of the photon, the antisymmetric quantity in the second line of the previous equation can also be determined from the amplitude $T_n(k_1, \ldots, k_n)$ without the graviton.
Saturating the previous expression with $q^\mu$ we get of course zero.

If we instead saturate it with $q^\nu$, we get

$$q^\nu M_n^{\mu\nu}(q; k_1, \ldots, k_n)$$

$$= \frac{1}{2} q^\rho q^\sigma \left\{ \sum_{i=1}^{n} \left( k_i^\mu \frac{\partial}{\partial k_i^\rho} - k_i^\rho \frac{\partial}{\partial k_i^\mu} \right) \frac{\partial}{\partial k_i^\sigma} T_n(k_1, \ldots, k_n) + \left[ \frac{\partial N_n^{\mu\sigma}}{\partial q^\rho} - \frac{\partial N_n^{\rho\sigma}}{\partial q^\mu} \right] (0; k_1, \ldots, k_n) \right\} = 0 ,$$

The vanishing follows from the equation above (implied by gauge invariance), remembering that $N_n^{\mu\nu}$ is a symmetric matrix.

Therefore the amplitude is gauge invariant.
The same equation allows us to write the relation,

\[-i \sum_{i=1}^{n} J_{i}^{\mu \rho} \frac{\partial}{\partial k_{i \sigma}} T_{n}(k_{1}, \ldots, k_{n}) = \left[ \frac{\partial N_{n}^{\rho \sigma}}{\partial q_{\mu}} - \frac{\partial N_{n}^{\mu \sigma}}{\partial q_{\rho}} \right] (0; k_{1}, \ldots, k_{n}),\]

which fixes the antisymmetric part of the derivative of $N_{n}^{\mu \nu}$ in terms of the amplitude $T_{n}(k_{1}, \ldots, k_{n})$ without the graviton.
Using the previous equation, we can then rewrite the terms of $\mathcal{O}(q)$ as follows:

$$M_{n}^{\mu\nu}(q; k_1, \ldots, k_n)|_{\mathcal{O}(q)}$$

$$= -\frac{i}{2} \sum_{i=1}^{n} \frac{q_{\rho} q_{\sigma}}{k_{i} \cdot q} \left[ k_{i}^{\nu} J_{i}^{\mu\rho} \frac{\partial}{\partial k_{i\sigma}} - k_{i}^{\sigma} J_{i}^{\mu\rho} \frac{\partial}{\partial k_{i\nu}} \right] T_{n}(k_1, \ldots, k_n)$$

$$= -\frac{i}{2} \sum_{i=1}^{n} \frac{q_{\rho} q_{\sigma}}{k_{i} \cdot q} \left[ J_{i}^{\mu\rho} k_{i}^{\nu} \frac{\partial}{\partial k_{i\sigma}} - (J_{i}^{\mu\rho} k_{i}^{\sigma}) \frac{\partial}{\partial k_{i\nu}} \right] T_{n}(k_1, \ldots, k_n)$$

$$- J_{i}^{\mu\rho} k_{i}^{\sigma} \frac{\partial}{\partial k_{i\nu}} + (J_{i}^{\mu\rho} k_{i}^{\sigma}) \frac{\partial}{\partial k_{i\nu}} \right] T_{n}(k_1, \ldots, k_n)$$

$$= \frac{1}{2} \sum_{i=1}^{n} \frac{1}{k_{i} \cdot q} \left[ \left( (k_{i} \cdot q)(\eta^{\mu\nu} q^{\sigma} - q^{\mu} \eta^{\nu\sigma}) - k_{i}^{\mu} q^{\nu} q^{\sigma} \right) \frac{\partial}{\partial k_{i}^{\sigma}}$$

$$- q_{\rho} J_{i}^{\mu\rho} q_{\sigma} J_{i}^{\nu\sigma} \right] T_{n}(k_1, \ldots, k_n).$$
Finally, we wish to write our soft-limit expression in terms of the amplitude, so we contract with the physical polarization tensor of the soft graviton, $\varepsilon_{q\mu\nu}$.

We see that the physical-state conditions set to zero the terms that are proportional to $\eta^{\mu\nu}$, $q^\mu$ and $q^\nu$.

We are then left with the following expression for the graviton soft limit of a single-graviton, $n$-scalar amplitude:

$$M_n(q; k_1, \ldots, k_n) \to \left[ S^{(0)} + S^{(1)} + S^{(2)} \right] T_n(k_1, \ldots, k_n) + O(q^2),$$

where

$$S^{(0)} \equiv \sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} k_i^{\mu} k_i^{\nu}}{k_i \cdot q},$$

$$S^{(1)} \equiv -i \sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} k_i^{\mu} q_\rho J_i^{\nu\rho}}{k_i \cdot q},$$

$$S^{(2)} \equiv -\frac{1}{2} \sum_{i=1}^{n} \frac{\varepsilon_{\mu\nu} q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma}}{k_i \cdot q}.$$
These soft factors follow from gauge invariance and agree with those computed by Cachazo and Strominger.

We have also looked at higher-order terms and found that gauge invariance does not fully determine them in terms of derivatives acting on $T_n(k_1, \ldots, k_n)$. 
We consider a tree-level color-ordered amplitude where gluon $n$ becomes soft with $q \equiv k_n$.

Being the amplitude color-ordered, we have to consider only two poles.
\[ A_{n}^{\mu_1;\mu_1\cdots\mu_{n-1}}(q; k_1, \ldots, k_{n-1}) = \frac{\delta_{\rho}^{\mu_1} k_1^\mu + \eta^{\mu_1\rho} q_{\rho} - \delta_{\rho}^{\mu} q_{\mu 1}}{\sqrt{2}(k_1 \cdot q)} A_{n-1}^{\rho\mu_2\cdots\mu_{n-1}}(k_1 + q, k_2, \ldots, k_{n-1}) \]

\[ - \frac{\delta_{\rho}^{\mu_{n-1}} k_{n-1}^\mu + \eta^{\mu_{n-1}\mu} q_{\rho} - \delta_{\rho}^{\mu} q_{\mu n-1}}{\sqrt{2}(k_{n-1} \cdot q)} A_{n-1}^{\mu_1\cdots\mu_{n-2}\rho}(k_1, \ldots, k_{n-2}, k_{n-1} + q) \]

\[ + N_n^{\mu_1;\mu_1\cdots\mu_{n-1}}(q; k_1, \ldots, k_{n-1}). \]

We have dropped terms from the three-gluon vertex that vanish when saturated with the external-gluon polarization vectors in addition to using the current-conservation conditions,

\[ (k_1 + q)_\rho A_{n-1}^{\rho\mu_2\cdots\mu_{n-1}}(k_1 + q, k_2, \ldots, k_{n-1}) = 0, \]

\[ (k_{n-1} + q)_\rho A_{n-1}^{\mu_1\cdots\mu_{n-2}\rho}(k_1, \ldots, k_{n-2}, k_{n-1} + q) = 0, \]

which are valid once we contract with the polarization vectors carrying the \( \mu_j \) indices.
By introducing the spin-one angular-momentum operator,

\[ (\Sigma^\mu\sigma^i)_{\mu i \rho} \equiv i (\eta^\mu\mu_i \eta^{\rho\sigma} - \eta^{\mu\rho} \eta^\mu\sigma_i), \]

we can write the total amplitude as

\[
A^\mu;\mu_1\cdots\mu_{n-1}_n (q; k_1, \ldots, k_{n-1})
= \frac{\delta^\mu_1 k^\mu_1 - iq^\sigma (\Sigma^\mu\sigma^i)_{\mu i \rho}^1}{\sqrt{2(k_1 \cdot q)}} A^{\rho\mu_2\cdots\mu_{n-1}}_{n-1} (k_1 + q, k_2, \ldots, k_{n-1})
- \frac{\delta^\mu_{n-1} k^\mu_{n-1} - iq^\sigma (\Sigma_{n-1}^\mu\sigma^i)_{\mu i \rho}^{n-1}}{\sqrt{2(k_{n-1} \cdot q)}} A^{\mu_1\cdots\mu_{n-2}\rho}_{n-1} (k_1, \ldots, k_{n-2}, k_{n-1} + q)
+ \mathcal{N}^\mu;\mu_1\cdots\mu_{n-1}_n (q; k_1, \ldots, k_{n-1}).
\]

Notice that the spin-one terms independently vanish when contracted with \( q^\mu \).
On-shell gauge invariance requires

\[
0 = q_\mu A_{n}^{\mu;\mu_1\cdots\mu_{n-1}}(q; k_1, \ldots, k_{n-1})
\]

\[
= \frac{1}{\sqrt{2}} A_{n-1}^{\mu_1\mu_2\cdots\mu_{n-1}}(k_1 + q, k_2, \ldots, k_{n-1})
\]

\[
- \frac{1}{\sqrt{2}} A_{n-1}^{\mu_1\cdots\mu_{n-2}\mu_{n-1}}(k_1, \ldots, k_{n-2}, k_{n-1} + q)
\]

\[
+ q_\mu N_{n}^{\mu;\mu_1\cdots\mu_{n-1}}(q; k_1, \ldots, k_{n-1}).
\]

For \( q = 0 \), this is automatically satisfied.

At the next order in \( q \), we obtain

\[
- \frac{1}{\sqrt{2}} \left[ \frac{\partial}{\partial k_{1\mu}} - \frac{\partial}{\partial k_{n-1\mu}} \right] A_{n-1}^{\mu_1\cdots\mu_{n-1}}(k_1, k_2 \ldots k_{n-1})
\]

\[
= N_{n}^{\mu;\mu_1\cdots\mu_{n-1}}(0; k_1, \ldots, k_{n-1}).
\]

Similar to the photon case, we ignore local gauge-invariant terms in \( N_{n}^{\mu;\mu_1\cdots\mu_{n-1}} \) because they are necessarily of a higher order in \( q \).

Thus, \( N_{n}^{\mu;\mu_1\cdots\mu_{n-1}}(0; k_1, \ldots, k_{n-1}) \) is determined in terms of the amplitude without the soft gluon.
With this, the total expression becomes

\[
A_n^{\mu_1;\mu_1\cdots\mu_{n-1}}(q; k_1 \ldots k_{n-1}) \\
= \left( \frac{k_{\mu_1}^{\mu}}{\sqrt{2}(k_1 \cdot q)} - \frac{k_{\mu_{n-1}}^{\mu}}{\sqrt{2}(k_{n-1} \cdot q)} \right) A_{n-1}^{\mu_1\cdots\mu_{n-1}}(k_1, \ldots, k_{n-1}) \\
- \frac{iq_\sigma(J_1^{\mu\sigma})^{\mu_1}_{\mu}}{\sqrt{2}(k_1 \cdot q)} A_{n-1}^{\rho\mu_2\cdots\mu_{n-1}}(k_1, \ldots, k_{n-1}) \\
+ \frac{iq_\sigma(J_{n-1}^{\mu\sigma})^{\mu_{n-1}}_{\mu}}{\sqrt{2}(k_{n-1} \cdot q)} A_{n-1}^{\mu_1\cdots\mu_{n-2}\rho}(k_1, \ldots, k_{n-1}) + \mathcal{O}(q),
\]

where

\[
(J_i^{\mu\sigma})^{\mu_i\rho} \equiv L_i^{\mu\sigma} \eta^{\mu_i\rho} + (\Sigma_i^{\mu\sigma})^{\mu_i\rho},
\]

with

\[
L_i^{\mu\sigma} \equiv i \left( k_i^{\mu} \frac{\partial}{\partial k_{i\sigma}} - k_i^{\sigma} \frac{\partial}{\partial k_{i\mu}} \right) ;
\]

\[
(\Sigma_i^{\mu\sigma})^{\mu_i\rho} \equiv i (\eta^{\mu\mu_i} \eta^{\rho\sigma} - \eta^{\mu\rho} \eta^{\mu_i\sigma}).
\]
In order to write the final result in terms of full amplitudes, we contract with external polarization vectors.

We must pass polarization vectors $\varepsilon_{1\mu_1}$ and $\varepsilon_{n-1\mu_{n-1}}$ through the spin-one angular-momentum operator such that they will contract with the $\rho$ index of, respectively, $A^{\rho\mu_2\cdots\mu_{n-1}}(k_1, \ldots, k_{n-1})$ and $A^{\mu_1\cdots\mu_{n-2}\rho}_{n-1}(k_1, \ldots, k_{n-1})$.

It is convenient write the spin angular-momentum operator as

$$\varepsilon_{i\mu_i}(\Sigma_i^{\mu\sigma})^\mu_i \rho A^\rho = i \left( \varepsilon_i^\mu \frac{\partial}{\partial \varepsilon_i \sigma} - \varepsilon_i^\sigma \frac{\partial}{\partial \varepsilon_i \mu} \right) \varepsilon_{i\rho} A^\rho.$$
We may therefore write

$$A_n(q; k_1, \ldots, k_{n-1}) \to \left[ S_n^{(0)} + S_n^{(1)} \right] A_{n-1}(k_1, \ldots, k_{n-1}) + O(q),$$

where

$$S_n^{(0)} \equiv \frac{k_1 \cdot \varepsilon_n}{\sqrt{2 (k_1 \cdot q)}} - \frac{k_{n-1} \cdot \varepsilon_n}{\sqrt{2 (k_{n-1} \cdot q)}},$$

$$S_n^{(1)} \equiv -i \varepsilon_n \mu \sigma q_{\sigma} \left( \frac{J_{1}^{\mu \sigma}}{\sqrt{2 (k_1 \cdot q)}} - \frac{J_{n-1}^{\mu \sigma}}{\sqrt{2 (k_{n-1} \cdot q)}} \right).$$

Here

$$J_i^{\mu \sigma} \equiv L_i^{\mu \sigma} + \Sigma_i^{\mu \sigma},$$

where

$$L_i^{\mu \nu} \equiv i \left( k_i^{\mu} \frac{\partial}{\partial k_i^{\nu}} - k_i^{\nu} \frac{\partial}{\partial k_i^{\mu}} \right), \quad \Sigma_i^{\mu \sigma} \equiv i \left( \varepsilon_i^{\mu} \frac{\partial}{\partial \varepsilon_i^{\sigma}} - \varepsilon_i^{\sigma} \frac{\partial}{\partial \varepsilon_i^{\mu}} \right).$$
As before the amplitude is the sum of two pieces:

\[
M_n^{\mu \nu; \mu_1 \nu_1 \cdots \mu_{n-1} \nu_{n-1}} (q; k_1, \ldots, k_{n-1})
= \sum_{i=1}^{n-1} \frac{1}{k_i \cdot q} \left[ k_i^{\mu} \eta^{\mu_i \alpha} - iq_\rho (\Sigma_i^{\mu \rho})^{\mu_i \alpha} \right] \left[ k_i^{\nu} \eta^{\nu_i \beta} - iq_\sigma (\Sigma_i^{\mu \sigma})^{\nu_i \beta} \right]
\times M_{n-1}^{\mu_1 \nu_1 \cdots \mu_{n-1} \nu_{n-1}} (k_1, \ldots, k_i + q, \ldots, k_{n-1})
\]

\[
+ N_n^{\mu \nu; \mu_1 \nu_1 \cdots \mu_{n-1} \nu_{n-1}} (q; k_1, \ldots, k_{n-1}),
\]

where

\[
(\Sigma_i^{\mu \rho})^{\mu_i \alpha} \equiv i \left( \eta^{\mu \mu_i} \eta^{\alpha \rho} - \eta^{\mu \alpha} \eta^{\mu_i \rho} \right).
\]
On-shell gauge invariance implies

\[ 0 = q_\mu M_n^{\mu \nu ; \mu_1 \nu_1 \cdots \mu_{n-1} \nu_{n-1}}(q; k_1, \ldots, k_{n-1}) \]

\[ = \sum_{i=1}^{n-1} \left[ k_i^\nu \eta_i^{\nu_i \beta} - iq_\rho (\Sigma_i^{\nu \rho})^{\nu_i \beta} \right] M_{n-1}^{\mu_1 \nu_1 \cdots \mu_i \cdots \mu_{n-1} \nu_{n-1}}(k_1, \ldots, k_i + q, \ldots, k_{n-1}) \]

\[ + q_\mu N_n^{\mu \nu ; \mu_1 \nu_1 \cdots \mu_{n-1} \nu_{n-1}}(q; k_1, \ldots, k_{n-1}) . \]

Proceeding as before we end up getting

\[ M_n^{\mu \nu ; \mu_1 \nu_1 \cdots \mu_{n-1} \nu_{n-1}}(q; k_1, \ldots, k_{n-1}) \]

\[ = \sum_{i=1}^{n-1} \frac{1}{k_i \cdot q} \left\{ k_i^\mu k_i^\nu \eta_i^{\mu_i \alpha} \eta_i^{\nu_i \beta} \right\} \]

\[ - \frac{i}{2} q_\rho \left[ k_i^\mu \eta_i^{\mu_i \alpha} \left[ L_i^{\nu \rho} \eta_i^{\nu_i \beta} + 2(\Sigma_i^{\nu \rho})^{\nu_i \beta} \right] + k_i^\nu \eta_i^{\nu_i \beta} \left[ L_i^{\mu \rho} \eta_i^{\mu_i \alpha} + 2(\Sigma_i^{\mu \rho})^{\mu_i \alpha} \right] \right] \]

\[ - \frac{1}{2} q_\rho q_\sigma \left[ \left[ L_i^{\mu \rho} \eta_i^{\mu_i \alpha} + 2(\Sigma_i^{\mu \rho})^{\mu_i \alpha} \right] \left[ L_i^{\nu \sigma} \eta_i^{\nu_i \beta} + 2(\Sigma_i^{\nu \sigma})^{\nu_i \beta} \right] - 2(\Sigma_i^{\mu \rho})^{\mu_i \alpha}(\Sigma_i^{\nu \sigma})^{\nu_i \beta} \right] \]

\[ \times M_{n-1}^{\mu_1 \nu_1 \cdots \mu_{n-1} \nu_{n-1}}(k_1, \ldots, k_i, \ldots, k_{n-1}) + \mathcal{O}(q^2) . \]
In order to write our expression in terms of amplitudes, we saturate with graviton polarization tensors using $\varepsilon_{\mu\nu} \rightarrow \varepsilon_{\mu} \varepsilon_{\nu}$ where $\varepsilon_{\mu}$ are spin-one polarization vectors.

As we did for the case with gluons, we must pass the polarization vectors through the spin-one operators.

$$M_n(q; k_1, \ldots, k_{n-1}) = \left[ S_n^{(0)} + S_n^{(1)} + S_n^{(2)} \right] M_{n-1}(k_1, \ldots, k_{n-1}) + \mathcal{O}(q^2)$$

where

$$S_n^{(0)} \equiv \sum_{i=1}^{n-1} \frac{\varepsilon_{\mu\nu} k_i^\mu k_i^\nu}{k_i \cdot q},$$

$$S_n^{(1)} \equiv -i \sum_{i=1}^{n-1} \frac{\varepsilon_{\mu\nu} q^\rho J_i^{\nu\rho}}{k_i \cdot q},$$

$$S_n^{(2)} \equiv -\frac{1}{2} \sum_{i=1}^{n-1} \frac{\varepsilon_{\mu\nu} q^\rho J_i^{\mu\rho} q^\sigma J_i^{\nu\sigma}}{k_i \cdot q}.$$
Here

\[ J_i^{\mu\sigma} \equiv L_i^{\mu\sigma} + \Sigma_i^{\mu\sigma}, \]

with

\[ L_i^{\mu\sigma} \equiv i \left( k_i^\mu \frac{\partial}{\partial k_i^\sigma} - k_i^\sigma \frac{\partial}{\partial k_i^\mu} \right), \]

\[ \Sigma_i^{\mu\sigma} \equiv i \left( \varepsilon_i^\mu \frac{\partial}{\partial \varepsilon_i^\sigma} - \varepsilon_i^\sigma \frac{\partial}{\partial \varepsilon_i^\mu} \right). \]
Comments on loop corrections: gauge theory

- At one-loop the amplitude will have in general IR and UV divergences.
- We are not giving here a complete study of them.
- The one-loop contributions have been classified into the factorizing ones and the non-factorizing ones.
- We will concentrate here to the factorizing ones.
- They modify the vertex present in the pole term.
- For the gauge theory they are of the type shown in the figure.
They have been computed in QCD and are given by:

\[
D^{\mu, \text{fact}} = \frac{i}{\sqrt{2}} \frac{1}{3} \frac{1}{(4\pi)^2} \left( 1 - \frac{n_f}{N_c} + \frac{n_s}{N_c} \right) (q - k_a)^\mu \left[ (\varepsilon_n \cdot \varepsilon_a) - \frac{(q \cdot \varepsilon_a)(k_a \cdot \varepsilon_n)}{(k_a \cdot q)} \right]
\]

[Z. Bern, V. Del Duca, C.R. Schmidt, 1998]
[Z. Bern, V. Del Duca, W.B. Kilgore, C.R. Schmidt, 1999]

- It is both IR and UV finite and the limit \( \varepsilon \to 0 \) has been taken.
- It is non-local because of the pole in \((qk_a)\).
- It is gauge invariant under the substitution \( \varepsilon_q \to q \).
- It does not contribute to the leading soft behavior.
Attaching to it the rest of the amplitude

\[ D^\text{fact}_\mu \frac{-i}{2q \cdot k_a} J^\mu, \]

\( J^\mu \) is a conserved current:

\[ (q + k_a)_\mu J^\mu = 0, \]

assuming that all the remaining legs are contracted with on-shell polarizations.

We can trade \( k_a \) with \( q \) and we get immediately:

\[ D^\text{fact}_\mu \frac{-i}{2q \cdot k_a} J^\mu = O(q^0), \]

No leading \( O\left(\frac{1}{q}\right) \) correction from the factorizing contribution to the one-loop soft functions.
A similar calculation can be done for the gravity case.

We consider only the case in which scalar fields circulate in the loop.

The result of this calculation is:

\[
D_{\mu\nu, \text{fact,s}} = \frac{i}{(4\pi)^2} \left( \frac{\kappa}{2} \right)^3 \frac{1}{30} \left[ (\varepsilon_n \cdot \varepsilon_a) - \frac{(q \cdot \varepsilon_a)(k_a \cdot \varepsilon_n)}{(q \cdot k_a)} \right] \\
\times \left( (q \cdot \varepsilon_a)(k_a \cdot \varepsilon_n) - (\varepsilon_n \cdot \varepsilon_a)(q \cdot k_a) \right) k_{\alpha}^\mu k_{\alpha}^\nu + \mathcal{O}(q^2),
\]
As in the gauge-theory case, the diagrams $D_{\mu\nu, \text{fact}, s}$ contract into a conserved current:

$$(k_a + q)^\mu J_{\mu\nu} = f(k_i, \epsilon_i)(k_a + q)_\nu, \quad (k_a + q)^\nu J_{\mu\nu} = f(k_i, \epsilon_i)(k_a + q)_\mu.$$  

This means

$$k_a^\mu k_a^\nu J_{\mu\nu} = (k_a + q)^\mu (k_a + q)^\nu J_{\mu\nu} + O(q) = f(k_i, \epsilon_i)(k_a + q)^2 + O(q) = 2f(k_i, \epsilon_i)q \cdot k_a + O(q) = O(q).$$

We therefore have

$$D_{\mu\nu, \text{fact}, s} \frac{i}{2q \cdot k_a} J_{\mu\nu} = O(q).$$

No modification of the two first leading terms.

As in QCD, we expect that the contribution of other particles circulating in the loop will not modify this result.
What about soft theorems in string theory?

- In superstring the soft theorems have been investigated by B.U.W. Schwab, arXiv:1406.4172 and M. Bianchi, Song He, Yu-tin Huang and Congkao Wen, arXiv:1406.5155.

- Here we give just few examples in the bosonic string.

- One gluon and three tachyons:

\[
A_\mu(p_1, p_2, q, p_3) \sim \sqrt{2\alpha'} \frac{\Gamma(1 + 2\alpha' p_3 q)\Gamma(1 + 2\alpha' p_2 q)}{\Gamma(1 + 2\alpha'(p_2 + p_3)q)} \\
\times \left( \frac{p_{2\mu}}{2\alpha' p_3 q} - \frac{p_{3\mu}}{2\alpha' p_3 q} \right)
\]

- One graviton(dilaton) and three tachyons \((p_1 + p_2 + p_3 = -q)\):

\[
A_{\mu\nu}(p_1, p_2, p_3, q) \sim \left( \frac{p_{1\mu}p_{1\nu}}{p_1 q} + \frac{p_{2\mu}p_{2\nu}}{p_2 q} + \frac{p_{3\mu}p_{3\nu}}{p_3 q} \right) \\
\times \frac{\Gamma(1 + \frac{\alpha'}{2} p_1 q)\Gamma(1 + \frac{\alpha'}{2} p_2 q)\Gamma(1 + \frac{\alpha'}{2} p_3 q)}{\Gamma(1 - \frac{\alpha'}{2} p_1 q)\Gamma(1 - \frac{\alpha'}{2} p_2 q)\Gamma(1 - \frac{\alpha'}{2} p_3 q)}
\]
One gluon and 4 tachyons
With [R. Marotta]

\[ A_\mu(p_1, p_2, p_3, q, p_4) \sim \int_0^1 dz_3 (1 - z_3)^{2\alpha'} p_2 p_3 z_3^{2\alpha'} p_3 p_4 \]

\[ \times \int_0^{z_3} dz_4 (1 - z_4)^{2\alpha'} p_2 q (z_3 - z_4)^{2\alpha'} p_3 q z_4^{2\alpha'} p_4 q \]

\[ \times \left[ \frac{p_2\mu}{1 - z_4} + \frac{p_3\mu}{z_3 - z_4} - \frac{p_4\mu}{z_4} \right] \]

It is gauge invariant: \( q^\mu A_\mu = 0 \).

The last two lines are equal to \((z_4 = z_3 t)\)

\[ z_3^{2\alpha'} (p_3 + p_4) q \int_0^1 dt (1 - t)^{2\alpha'} p_3 q t^{2\alpha'} p_4 q (1 - z_3 t)^{2\alpha'} p_2 q \]

\[ \times \left[ \frac{z_3 p_2\mu}{1 - z_3 t} + \frac{p_3\mu}{1 - t} - \frac{p_4\mu}{t} \right] \]
They are equal to

\[
Z_3^{2\alpha'(p_3+p_4)q} \left[ \frac{\Gamma(1 + 2\alpha'p_4q)\Gamma(2\alpha'p_3q)}{\Gamma(2 + 2\alpha'(p_3 + p_4)q)} \right] z_3 \\
\times {}_2F_1(1 - 2\alpha'p_2q, 1 + 2\alpha'p_4q; 2 + 2\alpha'(p_3 + p_4)q; z_3) \\
+ \frac{\Gamma(2\alpha'p_4q + 1)\Gamma(1 + 2\alpha'p_3q)}{\Gamma(1 + 2\alpha'(p_3 + p_4)q)} \left( -\frac{p_4\mu}{2\alpha'p_4q} \right) \\
\times {}_2F_1(-2\alpha'p_2q, 2\alpha'p_4q; 1 + 2\alpha'(p_3 + p_4)q; z_3) \\
+ \frac{p_3\mu}{2\alpha'p_3q} (1 - z_3)^{2\alpha'p_2q} \\
\times {}_2F_1(-2\alpha'p_2q, 2\alpha'p_3q; 2\alpha'(p_3 + p_4)q + 1; -\frac{z_3}{1 - z_3})
\]

In the soft limit up to the order \(q^0\) we can forget the ratio of \(\Gamma\)-functions, we can approximate the last two \(_2F_1\) with 1 and the first one with: \(_2F_1(1, 1; 2; z_3)z_3 = -\log(1 - z_3)\).
In this way we get:

\[
\int_0^1 dz_3 (1 - z_3)^{2\alpha' p_2 p_3} z_3^{2\alpha' p_3 p_4} \left[ - \log(1 - z_3) p_{2\mu} + z_3^{2\alpha'} (p_3 + p_4) q \left( \frac{p_{3\mu}}{2\alpha' p_3 q} (1 - z_3)^{2\alpha' p_2 q} - \frac{p_{4\mu}}{2\alpha' p_4 q} \right) \right]
\]

It can be written as follows:

\[
\frac{1}{2\alpha'} \left[ \frac{p_{3\mu}}{p_3 q} - \frac{p_{4\mu}}{p_4 q} + \frac{q^\rho J^{(3)}_{\mu\rho}}{p_3 q} - \frac{q^\rho J^{(4)}_{\mu\rho}}{p_4 q} \right] \times \int_0^1 dz_3 (1 - z_3)^{\alpha' (p_2 + p_3)^2 - 2} z_3^{\alpha' (p_3 + p_4)^2 - 2}
\]

The last integral is the amplitude for four tachyons and

\[
J^{(3,4)}_{\mu\rho} = p_{(3,4)\mu} \frac{\partial}{\partial p_{(3,4)\rho}} - p_{(3,4)\rho} \frac{\partial}{\partial p_{(3,4)\mu}}
\]
Soft theorem for dilaton

- The soft dilaton behavior in string theory goes back to the 70s [Ademollo et al., 1975] and [Shapiro, 1975].
- The loop amplitudes in the bosonic string are divergent because of the dilaton tadpole, corresponding to a zero momentum dilaton disappearing in the vacuum.
- In the previous papers it was proposed how to get rid of these divergence renormalizing the slope of the Regge trajectory and the string coupling constant.
- The soft theorem for a dilaton can, in principle, be computed starting from the expression that we obtained for the graviton except that now we cannot neglect terms proportional to $\eta^{\mu\nu}$ as we did in the case of a graviton.
For the graviton we got:

\[ M_n^{\mu\nu}(q; k_1 \ldots k_n) = \sum_{i=1}^{n} \frac{k_i^\nu}{k_i \cdot q} \left[ k_i^\mu - iq_\rho J_i^{\mu\rho} \right] T_n(k_1, \ldots, k_n) \]

\[ + \frac{1}{2} \sum_{i=1}^{n} \frac{1}{k_i \cdot q} \left[ \left( (k_i \cdot q)(\eta^{\mu\nu} q^\sigma - q^\mu \eta^{\nu\sigma}) - k_i^\mu q^\nu q^\sigma \right) \frac{\partial}{\partial k_i^\sigma} \right] - q_\rho J_i^{\mu\rho} q_\sigma J_i^{\nu\sigma} \] \[ T_n(k_1, \ldots, k_n). \]

and we have neglected the terms in the third line because the graviton polarization satisfies the identities:

\[ q^\mu \epsilon_{\mu\nu} = q^\nu \epsilon_{\mu\nu} = \eta^{\mu\nu} \epsilon_{\mu\nu} = 0 \]

In other words, gauge invariance imposes:

\[ q_\mu M_n^{\mu\nu} = f(k_i)q^\nu \implies q_\mu (M_n^{\mu\nu} - f(k_i)\eta^{\mu\nu}) = 0 \]

The extra term with \( \eta^{\mu\nu} \) is irrelevant for the graviton, but not for the dilaton.
Let us forget for a moment this problem and, in the case of the dilaton, let us saturate $M_{\mu\nu}^n$ with the dilaton projector:

$$(\eta_{\mu\nu} - q_\mu \bar{q}_\nu - q_\nu \bar{q}_\mu) M_{\mu\nu}^n ; \quad q^2 = \bar{q}^2 = 0 ; \quad q\bar{q} = 1$$

We get

$$S^{(0)} + S^{(1)} + S^{(2)} = -\sum_{i=1}^{n} \frac{m_i^2}{k_i q} \left(1 + q^\rho \frac{\partial}{\partial k_{i\rho}} + \frac{1}{2} q^\rho q^\sigma \frac{\partial^2}{\partial k_{i\rho} \partial k_{i\sigma}}\right)$$

$$- \sum_{i=1}^{n} k_{i\mu} \frac{d}{dk_{i\mu}} + 2$$

$$- \sum_{i=1}^{n} k_{i\mu} q_\sigma \frac{\partial^2}{\partial k_{i\mu} \partial k_{i\sigma}} + \frac{1}{2} (k_i q) \frac{\partial^2}{\partial k_{i\mu} \partial k_{i\mu}}$$

We have checked the previous expression up to order $q^0$ computing the amplitude involving a dilaton and $n$ closed tachyons.
It is given by

\[ M_{\mu\nu}^n \sim \int \frac{\prod_{i=1}^n d^2 z_i}{dV_{abc}} \prod_{i<j} |z_i - z_j|^{\alpha' k_i k_j} \int d^2 z \prod_{i=1}^n |z - z_i|^{\alpha' k_i q} \]

\[ \times \alpha' \sum_{i=1}^n \frac{k_i^\mu}{z - z_i} \sum_{i=1}^n \frac{k_i^\nu}{\bar{z} - \bar{z}_i} \]

In the soft limit \((q \to 0)\) we can put directly \(q = 0\) in the non-diagonal terms, while we have to be more careful with the diagonal terms that provide the terms of order \(q^{-1}\).

We have checked that the amplitude for both the graviton and the dilaton satisfies the general low energy theorems derived above up to the order \(q^0\).

No extra term proportional to \(\eta^{\mu\nu}\) is needed to reproduce the previous amplitude and also the amplitude involving massless closed string states.

For amplitudes involving \(N\) massless open strings, one needs to add a term \(\eta^{\mu\nu} \frac{N-2}{4}\).
Conclusions

▶ We have extended Low’s proof of the universality of sub-leading behavior of photons to non-abelian gauge theory and to gravity.

▶ On-shell gauge invariance can be used to fully determine the first sub-leading soft-gluon behavior at tree level.

▶ In gravity the first two subleading terms in the soft expansion can also be fully determined from on-shell gauge invariance.

▶ We have considered the factorizing contribution to both gauge theories and gravity.

▶ In non-abelian gauge theories the leading term is not affected by it, but the next to the leading is affected.

▶ Similarly in gravity the first two leading terms are not affected by the factorizing contribution, but the next term is affected.

▶ For the dilaton gauge invariant terms may appear at the order $q^0$ and therefore they cannot be obtained using gauge invariance as for the graviton.
It would be nice to have under control, together with the factorizing contribution, also the ones involving both the IR and the UV divergences at one loop.

In gauge theory they are well established, but in gravity some more work has to be done.

It would be very nice to extract everything from string theory in the limit of $\alpha' \to 0$. 