Abstract

Theoretical proposals of scaling laws for the differential elastic scattering cross sections of protons are confronted with experimental data over a wide energy range. Different combinations of the transferred momentum and energy resulting from the solution of the definite partial differential equation are attempted as scaling variables. Reasonable scaling of the differential cross sections in the diffraction cone has been shown for one of these variables. The violation of the geometrical scaling is ascribed to the increase of the proton blackness with energy. The origin of high-t region violations of scaling laws is discussed.

The differential cross section for elastic scattering of particles \(d\sigma(s,t)/dt\) is the only measurable characteristics of this process. At any fixed energy \(s\), one presents a one-dimensional plot of its dependence on the transferred momentum \(t\). However, the possibility that the differential cross sections might be described as functions of a single scaling variable representing a definite combination of energy and transferred momentum has been discussed \([1, 2]\). No rigorous proof of this assumption has been proposed. Recently this property was obtained \([3]\) from the solution of the partial differential equation for the imaginary part \(\text{Im}A(s,t)\) of the elastic scattering amplitude. The equation has been derived by equating the two expressions for the ratio of the real to imaginary parts of the amplitude \(\rho(s,t)\). They were known from the local dispersion relations \([4, 5, 6]\) with the \(s\)-derivative and from the linear \(t\)-approximation \([1, 7]\) with the \(t\)-derivative. These expressions are, correspondingly,

\[
\rho(s,t) = \frac{\pi}{2} \left[ \frac{\partial \ln \text{Im}A(s,t)}{\partial \ln s} - 1 \right], \quad (1)
\]

and

\[
\rho(s,t) = \rho(s,0) \left[ 1 + \frac{\partial \ln \text{Im}A(s,t)}{\partial \ln |t|} \right]. \quad (2)
\]

Therefrom the following partial differential equation is valid

\[
p - f(x)q = 1 + f(x), \quad (3)
\]
where \( p = \partial u/\partial x; \quad q = \partial u/\partial y; \quad u = \ln \text{Im}A(s,t); \quad f(x) = 2\rho(s,0)/\pi \approx d\ln \sigma_t/dx; \quad x = \ln s; \quad y = \ln |t|; \quad \sigma_t \) is the total cross section. The variables \( s \) and \( |t| \) should be considered as scaled by the corresponding constant factors \( s_0^{-1} \) and \( |t_0|^{-1} \).

Eq. (3) can be rewritten in another way as

\[
\frac{\partial u}{\partial \ln \sigma_t} - \frac{\partial u}{\partial \ln t} = 1 + \frac{d\ln s}{d\ln \sigma_t}.
\]  

The general solution of Eq. (4) reveals the scaling law

\[
\frac{t}{s} \text{Im}A(s,t) = \phi(t\sigma_t).
\]  

For the differential cross section it looks like

\[
t^2d\sigma/dt = \phi^2(t\sigma_t),
\]  

if the real part of the amplitude is neglected compared to the imaginary part. Thus the scaling law is predicted not for the differential cross section itself but for its product to \( t^2 \). Let us note that the often used ratio (see, e.g., [8]) of \( d\sigma/dt \) to \( d\sigma/dt|_{t=0} \propto \sigma_t^2 \) is also a scaling function. However, expression (6) is more suitable for comparison with experiment.

The scaling law with the \( t\sigma_t \)-scale is known as ”geometrical scaling”. Different aspects of its violation are often discussed. Here we contribute another view of this problem.

The geometrical scaling violation is seen in Fig. 1 in the diffraction cone especially for the TOTEM data at 7 TeV. With the common approximation \( d\sigma/dt \propto \exp(Bt) \) in the diffraction cone one gets that the maximum of the function \( t^2d\sigma/dt \) displayed in Fig. 1 should be positioned at \( t_m\sigma_t = 2\sigma_t/B = 16\pi(1 - \exp(-\Omega(s))) \). It is important that it depends only on the opacity of protons \( \Omega(s) \) (see the Table 1 in the review paper [13]) but not on their radii. The shift of the maximum is completely determined by the energy increase of the opacity. Scaling is violated due to the stronger energy dependence of \( \sigma_t \) compared to the diffraction cone slope \( B \) observed in experiment. That shows the difference between the volume effect important for the total cross section and the surface effect responsible for the elastic processes and, consequently, for the diffraction cone slope. Their energy dependences coincide only if the opacity saturates.
Figure 1: (Color online) The values of $t^2 d\sigma/dt$ for pp-scattering at energies $\sqrt{s}$ from 4.4 GeV to 7 TeV as functions of $t \sigma_t$ with $\sigma_t$ provided by the corresponding experiment. The scale on the abscissa axis is defined by $t_0 = -1$ GeV$^2$ and $\sigma_t$ in GeV$^{-2}$. The data are from [9, 10, 11, 12].

Thus this simple geometrical scaling is not fulfilled at very high energies, even at low transferred momenta. We show below how this problem can be cured. Moreover, the scaling is much more strongly violated outside the diffraction region. It is also discussed in what follows.

In attempts to cure these problems we turn to the assumptions used in the derivation of the scaling law (6). The neglect by the real part of the amplitude in (6) is the most evident one. Its contribution is easily estimated using Eqs. (1), (2). One gets

$$t^2 d\sigma/dt = \phi^2(t \sigma_t)[1 + (d \ln \phi/d \ln(t \sigma_t))^2 \rho^2(s, t = 0)].$$

The second term violates scaling - albeit it is very small because of smallness of $\rho(s, t = 0)$ and does not pose any problem.

Another approximation is involved in the relation (1). It was guessed as the extension to non-zero transferred momenta of the first term in the series
expansion of the exact expression for $\rho(s,0)$ which looks like

$$
\rho(s,0) \approx \frac{1}{\sigma_t} \left[ \tan \left( \frac{\pi}{2} \frac{d}{d \ln s} \right) \right] \sigma_t = \frac{1}{\sigma_t} \left[ \frac{\pi}{2} \frac{d}{d \ln s} + \frac{1}{3} \left( \frac{\pi}{2} \right)^3 \frac{d^3}{d \ln s^3} + ... \right] \sigma_t.
$$

(8)

The terms with higher derivatives in $s$ were neglected. This assumption is quite reasonable because their contribution seems negligible for experimentally measured energy dependence of $\sigma_t$ and to any analytical fits.

More serious questions arise concerning Eq. (2). It looks as if only the first term in the $t$-expansion of $\rho(s,t)$ is taken into account in this relation. It could be satisfactory in the diffraction cone where $\text{Im} A(s,t) \propto \exp(Bt/2)$. Let us note here that according to (2) $\rho(s,t)$ should become ever smaller in the diffraction cone crossing zero at $t = t_m$ and be negative at larger $|t|$. Moreover, even dealing within a linear approximation, one gets negative values of $\rho$ in the region directly attached to the diffraction cone (known as the Orear region by the name of its discoverer) if $\rho(s,t)$ is approximated by its constant average value there [14].

The behavior of $\rho(s,t)$ may become there strongly non-linear in $t$ [14]. The solution of the unitarity equation for the imaginary part of the amplitude in the Orear region [15, 16] (see also the review paper [13]) $u \propto \rho(s,t) = 1 + \rho(s,0)\rho_t$ with $\rho_t$ denoting the average value of $\rho$ in this region. If $\rho_t$ is replaced by the non-averaged $\rho(s,t)$ and such $u$ inserted in Eq. (2), then the derivative of the imaginary part naturally produces the derivative of $\rho(t)$. The resulting differential equation for $\rho(t)$ was solved. The strongly non-linear $t$-dependence with large negative values of $\rho$ in the Orear region was obtained. The more rigorous approach was also attempted.

These indications suit quite well the results of the fit in the Orear region at 7 TeV [17] where the negative and quite large values of $\rho \approx -2.1$ had to be chosen in that region. No such tendency is provided directly by Eq. (2).

The violation of the simple geometrical scaling law (6) is clearly seen in Fig. 1. In the diffraction cone it is rather well satisfied at most energies except the highest one of 7 TeV. In the Orear region there is no scaling even at lower energies. We ascribe it to necessary modifications of Eq. (2). Until now we are unable to propose any admissible generalization of Eq. (2) at large $t$.

Nevertheless, we try to modify it at small $t$ in such a way to get better
Figure 2: (Color online) The values of $t^{2.4} d\sigma/dt$ for pp-scattering at energies $\sqrt{s}$ from 4.4 GeV to 7 TeV as functions of $t^{1.2}\sigma_t$. $t_0 = -1$ GeV$^2$ and $\sigma_t$ in GeV$^{-2}$.

scaling inside the diffraction cone even at 7 TeV compared to Fig. 1. This can be done by varying the coefficient in front of the linear term in Eq. (2) writing it now as

$$
\rho(s, t) = \rho(s, 0) \left[ 1 + \frac{1}{a} \frac{\partial \ln \text{Im} A(s, t)}{\partial \ln |t|} \right].
$$

(9)

Another way to interpret this modification is to say that all higher order terms in $t$-expansion sum to a constant. After the corresponding renormalization at $t = 0$ it is reduced to Eq. (9).

It can hardly be valid at large $t$ but may help at low $t$. The equation (3) is easily transformed and the final prediction is the following scaling law for the differential cross section

$$
t^{2a} d\sigma/dt = \omega(t^a \sigma_t).
$$

(10)
We have found that the cross section scales within the diffraction cone at all available energies in the best way at $a=1.2$. This is shown in Fig. 2. Thus the violation of the geometrical scaling in the diffraction cone has been cured with the help of a single parameter $a$. This parameter accounts for the energy increase of the protons blackness which leads to different energy behaviors of $\sigma_t$ and $B$ as mentioned above. The coincidence of maxima positions in Fig. 2 implies the relation $\sigma_t \propto B^{1.2}$. That shifts the TOTEM maximum to larger values of $t\sigma_t$ in Fig. 1 and allows it to fit other maxima at the scale $t^{1.2}\sigma_t$ in Fig. 2. We were able to get an approximate scaling in the diffraction cone by some modification of the linear in $t$ term only. The somewhat different height of the maxima may be ascribed to slight variations (oscillations?) of the differential cross section about a simple exponent near $t = t_m$ as has been claimed in some experimental studies [18]. The "effective" parameter $a$ accounts approximately for the opacity increase with energy. Some other relation (e.g., of the logarithmic shape) is possible but asks for more elaborated modifications of the Eq. (2).
The region outside the diffraction peak is still not described by this law. The scaling violation is noticeable, in particular, even in the shift of the positions of the minima closest to the diffraction cone (Its parametrization can be found in [8]) not to say about the tails. That asks for other terms to be added in Eq. (2). Non-linear in $t$ terms would give rise to non-linear modifications of Eq. (3).

The Figures 1 and 2 are redrawn as the log-log plots in Figs 3 and 4. Again, the fit with $a=1.2$ is better than with $a=1$.

We have obtained the similar plots both for $a=1$ and $a=1.2$ if $\sigma_t$ is approximated by the often used phenomenological dependences $\ln^2(s/s_0)$ and $s^\Delta$ with $\Delta=0.17$. Both approximations are not ideally suited for fits of the total cross section in the whole energy range and it reveals itself in the slight change of scaling shapes. The decline from the scaling shape, which we do not demonstrate here, is not strong and is determined by the accuracy of these approximations. Therefore the fit of Fig. 2 with experimentally known $\sigma_t$ is more reliable than those using the above approximations.

Figure 4: (Color online) The log-log plot of Fig. 2
To conclude, we have shown that simple geometrical scaling is not fulfilled even in the diffraction cone (Figs 1 and 3). The physics origin of its violation is that protons become more black with increasing energy. Nevertheless, the scaling law can be generalized to get the agreement in this region of transferred momenta so that the differential cross sections fill in a single curve at all available energies (Figs 2 and 4). The problem of different tails in the Orear region is related, in our opinion, to further extensions of the relation (2).

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