On the Homotopy Decomposition for the Quotient of a Moment–Angle Complex and Its Applications

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Abstract—We prove that the quotient of any real or complex moment–angle complex by any closed subgroup in the naturally acting compact torus on it is equivariantly homotopy equivalent to the homotopy colimit of a certain toric diagram. For any quotient we prove an equivariant homeomorphism generalizing the well-known Davis–Januszkiewicz construction for quasitoric manifolds and small covers. We deduce the formality of the corresponding Borel construction space under the natural assumption on the group action in the complex case, which leads to a new description of the equivariant cohomology for the quotients by any coordinate subgroups. We prove the weak toral rank conjecture for the partial quotient of a moment–angle complex by the diagonal circle action. We also give an explicit construction of partial quotients by circle actions to show that their integral cohomology may have arbitrary torsion.

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1. INTRODUCTION

The geometry and topology of moment–angle complexes and manifolds, as well as of quasitoric manifolds and small covers introduced in the seminal paper [14], are in the focus of study in toric topology. A wide family of topological spaces arising in toric topology is given by the class of partial quotients. It includes both all moment–angle complexes and all quasitoric manifolds and small covers. The term “partial quotient” was introduced in [11] for the quotient space of the complex moment–angle complex $\mathcal{Z}_K = (D^2, S^1)^K$ by any freely acting subtorus in $T^m = (S^1)^m$, where $K$ is a simplicial complex on the vertex set $[m] = \{1, 2, \ldots, m\}$. Note that in [21] the term “partial quotient” was used for the quotient of the complex moment–angle complex $\mathcal{Z}_K$ by an arbitrary closed subgroup (that is, a quasitorus) in $T^m$ acting freely on $\mathcal{Z}_K$. In this paper by a partial quotient of the (real or complex) moment–angle complex $(D^d, S^{d-1})^K$, $d = 1, 2$, we mean the corresponding quotient by the action of any freely acting closed subgroup $H_d$ in $(G_d)^m$, where $G_1 := \mathbb{Z}/2\mathbb{Z}$ and $G_2 := S^1$ (see [14]).

The notion of a polyhedral product introduced in [1] is an instance of a colimit for a certain diagram of topological spaces over a small category $\text{cat}$. The categorical approach to polyhedral products [39] includes the homotopy equivalence between any moment–angle complex [37] as well as any quasitoric manifold [45] and the homotopy colimits of toric diagrams in the terminology of [45], as well as of general partial quotients [20]. This elegant approach has several applications. For example, it implies that any quasitoric manifold is a rationally formal space [39] and that any complex Davis–Januszkiewicz space is formal [37].

Moment–angle complexes and quasitoric manifolds have already found numerous valuable applications in homotopy theory [4, 9, 15, 24, 25, 28–30], cobordism theory [13, 31, 33], hyperbolic geometry [8], and combinatorial commutative algebra [3, 32]. Unlike these two particularly important families of partial quotients, the geometry and topology of general partial quotients are

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still far from being well understood. Several authors have attacked the problem of describing the cohomology rings of general partial quotients [10, 19]. However, a complete and rigorous argument giving the multiplicative structure in the cohomology ring $H^*(Z_K/H; \mathbb{Z})$ of any partial quotient was given only recently in [21]. In addition, in [22, 43] another two classes of quotients of moment–angle complexes were introduced.

In this paper we introduce a new class of pairs $(K, H_d)$ where $K$ is any simplicial complex on $m$ vertices and $H_d$ is any closed subgroup in $G^m_d$ such that the following condition holds (see Condition 4.7): For any $I \subseteq J \in \text{cat } K$ the subgroup $H_d \cap G^I_d$ is mapped to the subgroup $H_d \cap G^J_d$ under the natural projection $G^I_d \to G^J_d$. Equivalently, in the diagram

$$
\begin{aligned}
G^J_d \cap H_d & \longrightarrow G^I_d \cap H_d \\
\downarrow & \downarrow \\
G^J_d & \longrightarrow G^I_d
\end{aligned}
$$

where the lower horizontal arrow is the natural projection, there exists an upper horizontal arrow making the diagram commutative.

The proposed class contains all partial quotients and in addition has some quotients by non-free actions on a moment–angle complex. The first main result of this paper is as follows (for precise definitions, see Section 2).

**Theorem** (Theorem 4.14). Suppose that $K$ and $H_d$ satisfy the above condition. Then one has

$$
\tilde{H}^i(\text{colim } D) = \lim \tilde{H}^i(D) \quad \text{and} \quad \lim \tilde{H}^j(\text{colim } D) = 0, \quad j > 0,
$$

where $D = S_d, BS_d$. In particular, $\tilde{H}^{\text{odd}}(\text{colim } BS_2; \mathbb{Z}) = 0$.

Notice that Theorem 4.14 generalizes the description of the Stanley–Reisner ring $\mathbb{Z}[K]$ as a limit. Theorem 4.14 computes the equivariant cohomology ring of a quotient (see Theorem 4.3 and the definition of $BS_d$ below). The second main result of this paper pertains to complex moment–angle complexes (that is, $d = 2$, and the corresponding index is omitted in the statement).

**Theorem** (Theorem 4.22). Suppose that the above condition holds for the pair $(K, H)$. Then the Eilenberg–Moore spectral sequence for the fiber inclusion to the Borel construction of the $L$-action on $Z_K/H$ is isomorphic to

$$
\text{Tor}_{H^*(BL)}^{i,j}(\text{colim } H^*(BS_2); \mathbb{Z}) \Rightarrow H^{i+j}(Z_K/H),
$$

where $L := T^m/H$. It collapses at the second page. In particular, the associated graded algebra of $H^*(Z_K/H)$ is isomorphic to $\text{Tor}_{H^*(BL)}^{i,j}(\text{colim } H^*(BS_2); \mathbb{Z})$.

The proof of Theorem 4.22 generalizes the known proof in the case of partial quotients (see, for example, [21]). To prove Theorems 4.14 and 4.22, we use the following structure theorems, which are interesting on their own.

**Theorem** (Theorem 3.3). For any $d = 1, 2$, any closed subgroup $H_d$ in $G^m_d$, and any simplicial complex $K$ on $[m]$, there is an $L_d$-equivariant homotopy equivalence of spaces

$$(D^d, S^{d-1})^K/H_d \simeq \text{hocolim } G^m_d/(G^I_d \cdot H_d),$$

where $L_d := G^m_d/H_d$ and $G^I_d \cdot H_d$ is the subgroup generated by $G^I_d := \prod_{i \in I} G_d$ and $H_d$ in $G^m_d$.

**Theorem** (Theorem 4.3). There is a homotopy equivalence

$$EL_d \times L_d (D^d, S^{d-1})^K/H_d \simeq \text{colim } B(G^I_d/(G^I_d \cap H_d))$$

for the Borel construction of the $L_d$-action on the quotient $(D^d, S^{d-1})^K/H_d$ of the moment–angle complex.
The proof of Theorem 3.3 is given in Sections 2 and 3. Theorem 3.3 was proved in a series of particular cases in [20, 40, 45]. In the case of a partial quotient the homotopy equivalence from Theorem 3.3 gives rise to an equivariant homeomorphism, where the standard realization of a homotopy colimit is used (Corollary 2.20). This result leads to the explicit $L_d$-CW-approximation for quotients of moment–angle complexes (Proposition 3.2). The last result (Corollary 2.20) generalizes the well-known Davis–Januszkiewicz construction [14] to the case of arbitrary quotients. We also mention the closely related general constructions for partial quotients from [2, 20].

In order to prove Theorem 4.3, we construct the homotopy equivalence between the diagram for the Borel construction of the $L_d$-action on $(D^d, S^{d-1})K/H_d$ and $B(G^I_d/(G^I_d \cap H_d))$. For $d = 2$ we prove the formality of the respective Borel construction (Theorem 4.21). Our proof uses an argument similar to [37].

The third and last main result of this paper is as follows.

**Theorem** (Theorem 5.7). Let $G$ be any finitely generated abelian group. Then there exists a simple polytope $P \subseteq \mathbb{R}^n$ with $m$ facets and a one-dimensional subtorus (circle) $H \subseteq T^m$ such that $H$ acts freely on the moment–angle manifold $Z_P$ and $H^*(Z_P/H)$ contains $G$ as a direct summand.

We prove Theorem 5.7 by using the Hochster-type formula from [43]. Furthermore, we show that the weak toral rank conjecture holds for the class of partial quotients of moment–angle complexes by the action of the diagonal circle action. We conclude the paper by a list of some related open problems.

2. HOMOTOPY DECOMPOSITION FOR QUOTIENTS OF MOMENT–ANGLE COMPLEXES

Unless explicitly stated otherwise, in this paper the cohomology groups of a topological space are the singular cohomology groups with integral coefficients. Given a simplicial complex $K$ on the vertex set $[m] := \{1, 2, \ldots, m\}$, the objects of $K$ together with the initial object (empty set) form a small category $\text{cat}K$ with the arrows induced by the natural inclusions of subsets of $[m]$. The formula $I \in \text{cat}K$ means that $I$ is an object of $\text{cat}K$, that is, either $I = \emptyset$ or $I \in K$.

2.1. Algebraic preliminaries and definitions of some diagrams. The proof of the following lemma is straightforward.

**Lemma 2.1.** Consider a commutative diagram of abelian group homomorphisms

\[
\begin{array}{ccc}
A' & \xrightarrow{a'} & B' \\
A & \downarrow{a} & B \\
C' & \xrightarrow{c'} & D' \\
C & \xrightarrow{c} & D
\end{array}
\]

where $a, a', c,$ and $c'$ are monomorphisms. Then the following diagram of group homomorphisms commutes, and each row in it is exact:

\[
\begin{array}{cccccc}
1 & \longrightarrow & A' & \xrightarrow{a'} & B' & \longrightarrow & B'/A' & \longrightarrow & 1 \\
1 & \longrightarrow & A & \xrightarrow{a} & B & \longrightarrow & B/A & \longrightarrow & 1 \\
1 & \longrightarrow & C' & \xrightarrow{c'} & D' & \longrightarrow & D'/C' & \longrightarrow & 1 \\
1 & \longrightarrow & C & \longrightarrow & D & \longrightarrow & D/C & \longrightarrow & 1
\end{array}
\]
Following [14], we make use of the notation
\[
G_d := \begin{cases}
\mathbb{Z}/2\mathbb{Z}, & d = 1, \\
S^1, & d = 2,
\end{cases}
\quad F_d := \begin{cases}
\mathbb{R}, & d = 1, \\
\mathbb{C}, & d = 2,
\end{cases}
\quad R_d := \begin{cases}
\mathbb{Z}/2\mathbb{Z}, & d = 1, \\
\mathbb{Z}, & d = 2.
\end{cases}
\]

We call any subgroup isomorphic to \(G_d^m\) for some \(m \geq 0\) a real torus for \(d = 1\) and a complex torus for \(d = 2\). Both real and complex tori will be referred to as tori.

Throughout the paper we use the standard formalism of polyhedral products (see [12]). Let \(H_d\) be any closed subgroup in \(G_d^m\). With a slight abuse of notation we identify the group \(G_I d := \prod_{i \in I} G_d\) with the isomorphic coordinate subgroup \((G_d, 1) \times \prod_{j \in [m] \setminus I} 1_j\) in \(G_d^m\), \(I \subseteq [m]\). In particular, by definition one has \(G_{\emptyset}^0 := \{1\}\). Denote by \(\varphi_I : G_d \to G_d^m / G_I d\) the natural quotient epimorphism of groups for any \(I \subseteq [m]\).

**Proposition 2.2.** For any \(I \subseteq J \in \text{cat } K\) the diagram
\[
\begin{array}{ccccccc}
1 & \to & G_I d / (G_I d \cap H_d) & \to & G_d^m / H_d & \to & G_d^m / (H_d \cdot G_I d) & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
1 & \to & G_J d / (G_J d \cap H_d) & \to & G_d^m / H_d & \to & G_d^m / (H_d \cdot G_J d) & \to & 1
\end{array}
\]

(2.1)

commutes and has exact rows, where \(H_d \cdot G_I d\) denotes the subgroup in \(G_d^m\) generated by \(H_d\) and \(G_I d\).

**Proof.** There is a short exact sequence of groups
\[
1 \to G_I d \cap H_d \to H_d \xrightarrow{\varphi_I | H_d} \varphi_I(H_d) \to 1,
\]
so that \(\varphi_I(H_d) \subseteq G_d^m / G_I d\). Clearly, (2.2) is functorial with respect to \(I \in \text{cat } K\) (that is, the upper face of the diagram (2.3) below is commutative). One has an isomorphism
\[
(G_d^m / G_I d) / \varphi_I(H_d) \cong G_d^m / (H_d \cdot G_I d).
\]

Then one constructs three out of four cubes of the following commutative diagram with exact rows and columns by applying Lemma 2.1 to the upper left cube:
There are two different ways to define the lower right cube of this diagram by applying Lemma 2.1. A simple check shows that these two cubes coincide. So the diagram (2.3) is well defined and commutative. The necessary diagram (2.1) is then given by restricting (2.3) to the bottom face.

Definition 2.3. Define $S_d, \kappa(G^m_d/H_d)$, and $Q_d$ to be the (cat $K$)-diagrams of topological spaces such that the arrow corresponding to $I \subseteq J \in \text{cat } K$ is given by the left, central, and right columns of the diagram (2.1), respectively, where $\kappa(G^m_d/H_d)$ is the constant (cat $K$)-diagram corresponding to $G^m_d/H_d$.

In what follows we write $Q_d = Q_d(K, H)$ to indicate the dependence of the diagram $Q_d$ on $K$ and $H_d$ and use similar notation for $S_d$ and $\kappa(G^m_d/H_d)$ when necessary.

Remark. Let $D \in \text{Top}^C$ be any diagram over a small category $C$ with values in the category Top of topological spaces. Suppose that any object of $D$ is a torus and any of its arrows is a group homomorphism. Then $D$ is called a toric diagram [45]. The diagrams $S_d, \kappa(G^m_d/H_d)$, and $Q_d$ are toric diagrams.

Corollary 2.4. There is a sequence of (cat $K$)-diagram morphisms

$$\kappa(1) \to S_d \to \kappa(G^m_d/H_d) \to Q_d \to \kappa(1)$$

given by (2.1). Objectwise, (2.4) is a short exact sequence of groups. The diagram $S_d$ is cofibrant.

Proof. The homomorphism $S_d(I \to J) : G^I_d/(G^I_d \cap H_d) \to G^J_d/(G^J_d \cap H_d)$ has a trivial kernel for any $I \subseteq J \subseteq K$, so $S_d$ is cofibrant by [45, Lemma 4.10]. The remaining claims are clear.

We will make use of the following basic properties of tori.

Proposition 2.5. (i) Any closed subgroup $H_d$ of the torus $G^m_d$ is a quasitorus; that is, it is isomorphic to the direct product of a finite abelian group and a complex torus.

(ii) For any closed subgroup $H_d$ of the torus $G^m_d$, the natural exact sequence of groups

$$1 \to H_d \to G^m_d \to G^m_d/H_d \to 1$$

splits if and only if $d = 1$ or $H_d$ is connected.

(iii) The quotient of the torus $G^m_d$ by any of its closed subgroups is isomorphic to a torus.

Proof. Assertion (i) is given in [38, Ch. 3, § 2, item 3v].

If $d = 1$, then the exact sequence from assertion (ii) splits as a sequence of finite-dimensional linear spaces over $\mathbb{Z}/2\mathbb{Z}$. If $d = 2$ and the group $H_d$ is connected, then there exists a subtorus $T$ in $G^m_d$ such that the equality $G^m_d = H_d \times T$ holds. The natural projection to the second factor coincides with the quotient homomorphism $G^m_d \to G^m_d/H_d$. A non-canonical section $T \to G^m_d$ of this projection gives a splitting of the exact sequence from (ii). If $d = 2$ and the group $H_d$ is not connected, then the connected group $G^m_d$ cannot be represented as a direct product of a disconnected group $H_d$ and the group $G^m_d/H_d$. This proves assertion (iii).

Recall that the image of a closed abelian Lie group with respect to an epimorphism is connected and abelian. This implies assertion (iii) directly, because the quotient homomorphism is epimorphic and closed.

2.2. Homotopy decomposition for $(D^d, S^{d-1})K/H_d$ and subgroup arrangements. In what follows we use the standard formalism of homotopy colimits for diagrams with values in (pointed) topological spaces. We refer to [7, 16, 45] for the foundations of the corresponding theory. Unless explicitly stated otherwise, throughout the paper we consider limits and colimits (as well as the corresponding homotopy analogs) over small categories cat$^{op}$ $K$ and cat $K$, respectively, with values in the category of compactly generated Hausdorff topological spaces Top only. Often in the text below we refer to a (cat $K$)- or (cat$^{op}$ $K$)-diagram $D$ by indicating its objects $D(I)$ (as a function of $I$) for brevity.
Let $K$ be a simplicial complex on $[m]$. Recall that there is a (cat $K$)-diagram $(D^d, S^{d-1})^I$ of topological spaces [12] given by the maps

$$(D^d, S^{d-1})^I \to (D^d, S^{d-1})^J, \quad (D^d, S^{d-1})^I := \prod_{i \in I} D^d_i \times \prod_{j \in [m]\setminus I} S^{d-1}_j \subseteq \prod_{i=1}^m D^d_i,$$

induced by the identity map $\text{Id}: D^d \to D^d$ and by the embedding of the boundary $S^{d-1} = \partial D^d \to D^d$, where $I \subseteq J \subseteq \text{cat } K$. The respective colimits

$$\mathbb{R}Z_K = (D^1, S^0)^K := \text{colim}(D^1, S^0)^I \quad \text{and} \quad Z_K = (D^2, S^1)^K := \text{colim}(D^2, S^1)^I$$

are called the real and complex moment–angle complexes, respectively [12]. Furthermore, the natural quotient homomorphisms

$$G^m_d/G^I_d \to G^m_d/G^I_d,$$

for any $I \subseteq J \subseteq \text{cat } K$, form another (cat $K$)-diagram $G^m_d/G^I_d$ of spaces [12].

**Proposition 2.6** [40; 12, Proposition 8.1.5]. The diagram $(D^d, S^{d-1})^I$ is cofibrant. One has the following homotopy equivalence:

$$(D^d, S^{d-1})^K \simeq \text{hocolim } G^m_d/G^I_d.$$

**Example 2.7.** Let $m = 2$, $d = 2$, and $K = \{\{1\}, \{2\}\}$. Then $Z_K = S^3$. In this case, the homotopy colimit of $G^m_d/G^I_d$ over cat $K$ is obtained by gluing the boundary components of the cylinder $I^1 \times T^2$ to two disjoint copies of $S^1$ along two different coordinate projections from $T^2 := (S^1)^2$ to $S^1$.

Let $H_d$ be any closed subgroup of the torus $G^m_d$. The subset $(D^d, S^{d-1})^I$ of $(D^d)^m$ is $G^m_d$-invariant with respect to the natural $G^m_d$-action on $(D^d)^m$. Hence, the subset $(D^d, S^{d-1})^I$ is $H_d$-invariant in $(D^d)^m$ for any $I \in \text{cat } K$. Then there are induced embeddings of the orbit spaces

$$(D^d, S^{d-1})^I/H_d \to (D^d, S^{d-1})^J/H_d, \quad (2.5)$$

where $I \subseteq J \subseteq K$, which form the (cat $K$)-diagram $(D^d, S^{d-1})^I/H_d$ of spaces.

**Proposition 2.8.** The (cat $K$)-diagram $(D^d, S^{d-1})^I/H_d$ is cofibrant. One has

$$(D^d, S^{d-1})^K/H_d = \text{colim}(D^d, S^{d-1})^I/H_d.$$

**Proof.** Any morphism in the diagram $(D^d, S^{d-1})^I/H_d$ is a closed immersion. Hence, the first claim follows by [45, Lemma 4.10]. The second claim follows from the commutation of colimits and the representation of the quotient by the $H_d$-action as the respective colimit. □

**Proposition 2.9.** One has the homotopy equivalence

$$\text{hocolim } S_d \simeq \bigcup_{I \in K} G^I_d/(G^I_d \cap H_d) \subseteq G^m_d/H.$$

**Proof.** The claim follows trivially since $S_d$ is cofibrant by Corollary 2.4, due to the projection lemma from [45]. □

Denote by

$$\pi_I: (D^d, S^{d-1})^I \to G^m_d/G^I_d$$

the map induced by the identity map $\text{Id}: S^{d-1} \to S^{d-1}$ and the projection $D^d \to 1$. Notice that the equality

$$\pi_I(gx) = \varphi_I(g)\pi_I(x)$$
holds for any $g \in G^m_d$ and any $x \in (D^d, S^{d-1})^l$. Hence, $\pi_I$ is an equivariant map with respect to the respective $H_d^r$- and $\varphi_I(H_d)$-actions. Therefore, the map $\pi_I$ induces the map of the orbit spaces

$$\tilde{\pi}_I: (D^d, S^{d-1})^l / H_d \to G^m_d / (G^l_d \cdot H_d).$$

**Proposition 2.10.** (i) The restriction

$$G^m_d / H_d \to G^m_d / (G^l_d \cdot H_d)$$

of $\tilde{\pi}_I$ to $G^m_d / H_d$ is the trivial principal fiber bundle with the fiber $G^l_d / (H_d \cap G^l_d)$.

(ii) The map $\tilde{\pi}_I$ is the trivial fiber bundle associated with the principal fiber bundle from (i) under the natural action of $G^l_d / (H_d \cap G^l_d)$ on $(D^d)^l / (H_d \cap G^l_d)$, where $(D^d)^l := (\prod_{i \in I} D^d_i)$. The fiber of $\tilde{\pi}_I$ is equal to $(D^d)^l / (H_d \cap G^l_d)$.

**Proof.** By Proposition 2.2, the sequence of groups

$$1 \to G^l_d / (H_d \cap G^l_d) \to G^m_d / H_d \to G^m_d / (G^l_d \cdot H_d) \to 1$$

(2.6)

is exact. By Propositions 2.5(ii) and 2.5(iii), each group in the sequence (2.6) is a torus (in particular, a Lie group), and the short exact sequence (2.6) of groups splits. Hence, there exists a section of the short exact sequence (2.6). Therefore, (2.6) defines a trivial fiber bundle. This proves assertion (i).

Let

$$G := G^l_d / (H_d \cap G^l_d), \quad X := G^m_d / H_d, \quad Y := (D^d)^l / (H_d \cap G^l_d).$$

The group $G$ acts on $X$ by left translations as a subgroup (see (2.6)). The natural embedding $G^l_d \to (D^d)^l$ is $G^l_d$-equivariant. Hence, this embedding induces the $G$-action on $Y$. Recall that the standard action of $G$ on $X \times Y$ given by the formula $g(x, y) := (gx, gg^{-1})$ for $x \in X$, $y \in Y$, and $g \in G$ has the orbit space $X \times_G Y := (X \times Y)/G$.

Denote by $d \times t'$ and $t \times t'$ arbitrary elements of $(D^d, S^{d-1})^l$ and $G^m_d$, respectively, where $d \in (D^d)^l$, $t \in G^l_d$, and $t' \in G^m_d$. Then, for instance, the natural $G^m_d$-action on $(D^d, S^{d-1})^l$ is given by the formula

$$g \cdot (d \times t') = ((\varphi_{[m]_l}I(g) \cdot d) \times (\varphi_I(g) \cdot t')), \quad g \in G^m_d.$$  

Consider the map

$$\Psi: X \times_G Y \to (D^d, S^{d-1})^l / H_d, \quad \left[([t \times t']_H_d, [d]_{H_d \cap G^l_d}]_G\right] \mapsto \left([(t \cdot d) \times t'_H_d\right]_H_d, \quad (2.7)$$

where, for instance, $[t \times t']_H_d$ denotes the $H_d$-orbit in $X$ represented by $t \times t'$. Any element from $G^m_d \times (D^d)^l$ representing the same $G$-orbit in $X \times_G Y$ as on the left-hand side of (2.7) has the form $(gh \cdot (t \times t'), r \cdot d \cdot g^{-1})$, where $g \in G^l_d$, $h \in H_d$, and $r \in H_d \cap G^l_d$. We compute the value of $\Psi$ on this element as follows:

$$\Psi\left([[gh \cdot (t \times t')]_H_d, [r \cdot d \cdot g^{-1}]_{H_d \cap G^l_d}]_G\right) = \left([\varphi_{[m]_l}I(gh)t \cdot \varphi_{[m]_l}I(g)^{-1} \cdot r \cdot d] \times \varphi_I(gh)t'_H_d\right]_{H_d}$$

$$= \left([\varphi_{[m]_l}I(h)t \cdot r \cdot d] \times \varphi_I(h)t'_H_d\right]_{H_d} = \left[gh \cdot ((t \cdot d) \times t'_H_d\right]_{H_d} = \left [(t \cdot d) \times t'_H_d\right]_{H_d}. \quad (2.8)$$

In the second equality we use the identity $\varphi_I(g) = 1$, which follows easily from the definition of $\varphi_I$. Hence, the map $\Psi$ is well defined.

Consider the map

$$\Phi: (D^d, S^{d-1})^l / H_d \to X \times_G Y, \quad [d \times t']_H_d \mapsto \left[[[1 \times t']_H_d, [d]_{H_d \cap G^l_d}]_G\right]. \quad (2.9)$$
Any element from \((D^d, S^{d-1})^I\) representing the same \(H_d\)-orbit in \((D^d, S^{d-1})^I/H_d\) as on the left-hand side of (2.9) has the form \(h \cdot (d \times t')\), where \(d \in (D^d)^I\), \(h \in H_d\), and \(t' \in G_{d}^{[m]\setminus J}\). We compute the value of \(\Phi\) on this element as follows:

\[
\Phi[h \cdot (d \times t')]_{H_d} = \left[([1 \times \varphi_I(h) \cdot t']_{H_d}, [\varphi_{[m]\setminus I}(h) \cdot d]_{H_d \cap G'_d}] \right]_G
= \left[([\varphi_{[m]\setminus I}(h) \cdot 1 \times \varphi_I(h) \cdot t']_{H_d}, [d]_{H_d \cap G'_d}] \right]_G = \left([([1 \times t']_{H_d}, [d]_{H_d \cap G'_d}] \right]_G.
\]

(2.10)

Hence, the map \(\Phi\) is well defined.

We prove that \(\Psi\) and \(\Phi\) are mutually inverse maps as follows:

\[
\Phi\Psi\left(([t \times t']_{H_d}, [d]_{H_d \cap G'_d}] \right]_G = \Phi[t \cdot d \times t']_{H_d} = \left([([1 \times t']_{H_d}, [t \cdot d]_{H_d \cap G'_d}] \right]_G = \left([([t \times t']_{H_d}, [d]_{H_d \cap G'_d}] \right]_G,
\]

\[
\Psi\Phi[d \times t']_{H_d} = \Psi\left([([1 \times t']_{H_d}, [d]_{H_d \cap G'_d}] \right]_G = [d \times t']_{H_d}.
\]

Notice that there is a diagram

\[
\begin{array}{ccc}
X \times G & Y & \xrightarrow{\Psi} & X/G \\
\downarrow & & & \downarrow \\
(D^d, S^{d-1})^I/H_d & \xrightarrow{G_{d}^{m}/(G_{d}^{I} \cdot H_d)} & G_{d}^{m}/(G_{d}^{I} \cdot H_d)
\end{array}
\]

(2.11)

where the right vertical arrow is the group isomorphism given by taking the respective quotient in (2.6), and the horizontal arrows are the respective projections. The diagram (2.11) is commutative because both compositions in (2.11) map the element on the left-hand side of (2.7) to \([t']_{H_d}\). This implies the claim about the fiber bundle from assertion (ii). The claim about triviality of the associated fiber bundle from assertion (ii) then follows directly from assertion (i).

**Corollary 2.11.** There is a homeomorphism of spaces

\[
(D^d, S^{d-1})_I^I/H_d \cong \left(G_{d}^{m}/(G_{d}^{I} \cdot H_d) \right) \times \left((D^d)^I/(H_d \cap G'_d)\right).
\]

**Proposition 2.12.** For any closed subgroup \(H_d\) of \(G_{d}^{m}\), the orbit space \((D^d)^m/H_d\) is contractible.

**Proof.** The homotopy given by mapping \(x\) to \((1-t)x\), where \(x \in (D^d)^I\) and \(t \in [0, 1]\), is \(H_d\)-equivariant due to the natural embedding of groups \(H_d \subseteq G_{d}^{m} \subseteq O(dm)\). Hence, it induces the deformation retraction of \((D^d)^m/H_d\) to a point.

**Proposition 2.13.** The diagram

\[
\begin{array}{ccc}
(D^d, S^{d-1})^I/H_d & \xrightarrow{f_1} & (D^d, S^{d-1})^I/H_d \\
\downarrow \pi_J & & \downarrow \pi_J \\
G_{d}^{m}/(G_{d}^{I} \cdot H_d) & \xrightarrow{f_2} & G_{d}^{m}/(G_{d}^{I} \cdot H_d)
\end{array}
\]

(2.12)

is commutative for any \(I \subseteq J \in \text{cat} K\), where \(f_1\) and \(f_2\) are the arrows from the respective diagrams.

**Proof.** Let \((d, t', t'') \in (D^d, S^{d-1})^I\) and \((d, d', t'') \in (D^d, S^{d-1})^J\), where \(d \in (D^d)^I\), \(t' \in (S^{d-1})^I\setminus J\), and \(t'' \in (S^{d-1})^{[m]\setminus J}\), and \(d' \in (D^d)^J\setminus J\). We check the commutativity of the diagram (2.12) as follows:

\[
f_2 \circ \pi_J([d, t', t'']_{H_d}) = f_2([([1, t', t'']_{G_{d}^{I} \cdot H_d}) = [([1, t, t'']_{G_{d}^{I} \cdot H_d}) = [([1, 1, t''])_{G_{d}^{I} \cdot H_d}.
\]

\[
\pi_J \circ f_1([d, t', t'']_{H_d}) = \pi_J([([d, t', t''])_{H_d}) = [([1, 1, t''])_{G_{d}^{I} \cdot H_d}.
\]

Hence, the diagram (2.12) is commutative.
Corollary 2.14. The maps \( \tilde{\eta}_I \), where \( I \) runs through \( \text{cat} \, K \), constitute a well-defined morphism \((D^d, S^{d-1})^I/ H_d \to Q_d \) of \( (\text{cat} \, K) \)-diagrams.

Recall that a partial quotient (see the Introduction) is a quotient of the moment–angle complex \((D^d, S^{d-1})^K \) by any closed freely acting subgroup \( H_d \) in \( G^m_d \). The following theorem has been known in the case of partial quotients (see [20, Corollary 2.3] and [21, Proposition 4.1]).

Theorem 2.15. For any \( d = 1, 2 \), any closed subgroup \( H_d \) in \( G^m_d \), and any simplicial complex \( K \) on \([m]\), there is a homotopy equivalence of spaces

\[
(D^d, S^{d-1})^K/ H_d \simeq \text{hocolim} \, G^m_d/(G^I_d \cdot H_d),
\]

where \( G^I_d \cdot H_d \) is the subgroup generated by \( G^I_d \) and \( H_d \) in \( G^m_d \).

Proof. Any arrow from the morphism of diagrams in Corollary 2.14 is a fiber bundle projection with contractible fiber onto the respective base by Proposition 2.12. By Proposition 2.8, the homotopy and projection lemmas [45] imply that

\[
(D^d, S^{d-1})^K/ H_d = \text{colim}(D^d, S^{d-1})^I/ H_d \simeq \text{hocolim} \, Q_d = \text{hocolim} \, G^m_d/(H_d \cdot G^I_d),
\]

which completes the proof. \( \square \)

Example 2.16. For \( H_d = 1 \), Theorem 2.15 gives the well-known homotopy colimit description for the moment–angle complex (see Proposition 2.6).

Example 2.17. Let \( d = 2, m = 2, K = \{\{1\}, \{2\}\} \), and let \( H_d = \mathbb{Z}/2\mathbb{Z} \) be a group acting on \( T^2 = (S^1)^2 \) by the formula \( h(x, y) = (-x, -y) \), where \( h \) is the generator of \( H_d \). Notice that \( T^2/ H_d \) is homeomorphic to \( T^2 \). Then one has \( Z_K = S^3 \) and \( Z_K/ H_d = \mathbb{R}P^3 \). The natural projection map \( p_i : T^2 = S^1 \times S^1 \to S^1 \)

to the \( i \)th factor is equivariant with respect to the \( H_d \)-action on \( T^2 \) and the action of the group \( \mathbb{Z}/2\mathbb{Z} \) on \( S^1 \) by the involution, where \( i = 1, 2 \). Hence, the map of the orbit spaces

\( q_i : T^2/ H_d \to S^1/(\mathbb{Z}/2\mathbb{Z}) \)

is well defined. We identify \( T^2/ H_d \) with \( T^2 \) and \( S^1/(\mathbb{Z}/2\mathbb{Z}) \) with \( S^1 \) by Proposition 2.5(iii). Then the homotopy colimit \( \text{hocolim} \, G^m_d/(H_d \cdot G^I_d) \) in the standard realization is equal to

\[
S^1 \sqcup_{0 \times q_1} I^1 \times T^2 \sqcup_{0 \times q_1} S^1.
\]  \( \quad \text{(2.13)} \)

The projection \( q_i \) is uniquely defined by the embedding of the corresponding character lattices. We describe this embedding. Choose the standard basis \( e_1, e_2 \) of the lattice \( \mathbb{Z}^2 \cong \text{Hom}(T^2, S^1) \). Next, choose the basis

\[ f_1 := e_1, \quad f_2 := e_1 + e_2 \]

in \( \mathbb{Z}^2 \). The latter basis agrees with the splitting of the torus \( T^2 \) into the direct product of the diagonal circle and the first coordinate circle. It is straightforward to deduce that the matrices

\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}
\]

give the homomorphisms \( p^*_1, p^*_2 \), and \( p^* \) of lattices in the basis \( f_1, f_2 \) which are induced by the torus homomorphisms \( p_1 \) and \( p_2 \) and by the natural projection \( p : T^2 \to T^2/(\mathbb{Z}/2\mathbb{Z}) \), respectively. There
is a commutative diagram of torus homomorphisms and the induced commutative diagram of lattice embeddings:

\[
\begin{array}{ccc}
T^2 & \xrightarrow{p_i} & S^1 \\
\downarrow p & & \downarrow \\
T^2/(\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{q_i} & S^1/(\mathbb{Z}/2\mathbb{Z})
\end{array} \quad \text{and} \quad
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{p^*} & \mathbb{Z}^2 \\
\mathbb{Z} & \xrightarrow{q^*} & \mathbb{Z}^2
\end{array}
\]

From this one deduces that the matrices

\[
\begin{pmatrix} 0 \\ 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -2 \end{pmatrix}
\]

give the homomorphisms \( q^*_1 \) and \( q^*_2 \) in the basis \( f_1, f_2 \).

**Construction 2.18.** Let \( H_d = G_d^0 \) for some \( I_0 \subseteq [m] \). Let \( K \) be any simplicial complex on \([m]\). The \((\text{cat} \ K)\)-diagram \( D_1 := Q(K, H_d) \) consists of objects \( G_{d}^1 I_0 / G_{d}^1 I_0 \). Let \( D_2 \) be the \((\text{cat} \ K/I_0)\)-diagram \( Q(K/I_0, 1) \), where the simplicial complex

\[
K/I_0 := \{ I \setminus I_0 | I \in K \}
\]

on \([m] \setminus I_0 \) is the excision of the simplicial complex \( K \) along \( I_0 \) (see [12]). By definition, one has \( D_1 = \alpha^* D_2 \), where

\[
\alpha : \text{cat} \ K \to \text{cat} \ (K/I_0), \quad I \mapsto I \setminus I_0,
\]

is the poset morphism and \( \alpha^* D_2 \) is the pullback of \( D_2 \) along \( \alpha \). In general, the homotopy colimits of \( D_1 \) and \( D_2 \) are not homotopy equivalent, as shown by the example of the cone \( K = \text{cone}_m \tilde{K} \) with apex at \( m \in [m] \) over \( \tilde{K} \) on \([m-1]\), and \( H_d = G_d^1 [m] \). Indeed, the join \( (D^d, S^{d-1})\tilde{K} \ast \text{pt} \simeq \text{hocolim} \ D_1 \) is contractible, whereas \( \text{hocolim} \ D_2 \simeq (D^d, S^{d-1})\tilde{K} \) is not contractible in general.

The following lemma is straightforward to prove.

**Lemma 2.19.** Let \( D \) be a \((\text{cat} \ K)\)-diagram of closed subspaces in a topological space \( X \), and let \( G(I) \) be a \((\text{cat} \ K)\)-diagram of closed subgroups in a group \( G \). Then one has the equality

\[
\text{colim}((D(I) \times G/G(I)) = (\text{colim}D(I)) \times G) / \sim,
\]

\[
(x, g) \sim (x', g') \iff x = x' \in D(I) \land g^{-1}g' \in G(I).
\]

The following corollary has been known in the case of partial quotients (see [20, Sect. 2] and [21, Sect. 4]). (For another variant of the corresponding generalization, see [2].)

**Corollary 2.20.** The quotient \( (D^d, S^{d-1})\tilde{K}/H_d \) is homeomorphic to the quotient

\[
((\text{colim}(D^d I)/(H_d \cap G_d^I)) \times (G_d^m / H_d)) / \sim,
\]

\[
(x, g) \sim (x', g') \iff x = x' \in (D^d I)/(H_d \cap G_d^I) \land g^{-1}g' \in G_d^I /(H_d \cap G_d^I).
\]

**Proof.** The claim follows directly from Corollary 2.11 and Lemma 2.19. \( \square \)

Let \( K = \partial P^m \) be the dual simplicial sphere to a simple polytope \( P^m \subset \mathbb{R}^n \). Suppose that the action of \( H_d \) on \((D^d, S^{d-1})\tilde{K}\) is free and that \( H_d \cong G_d^{m-n} \). The quotient \((D^d, S^{d-1})\tilde{K}/H_d \) is called a small cover for \( d = 1 \) and a quasitoric manifold for \( d = 2 \) (see [14]). Let \( G_d^m = G_d^m / H_d \) be the real or complex torus for \( d = 1 \) or \( d = 2 \), respectively. Consider the \((\text{cat} \ K)\)-diagram \( G_d^m / p(G^d_d) \) (see Proposition 3.2 below for the precise definition), where \( p : G_d^m \to G_d^m \) is the natural quotient homomorphism.

The following theorem was first proved for toric varieties in [45] and then generalized to quasitoric manifolds in [39]. We deduce it from Theorem 2.15.

**Theorem 2.21.**
Theorem 2.21 [39, 45]. Let \((K, H_d)\) be as above. Then there is a homotopy equivalence of spaces

\[
(D^d, S^{d-1})^K / H_d \simeq \text{hocolim} G^m_d / p(G^I_d).
\]

Proof. By the condition on \(H_d\), one has \(G^I_d \cap H_d = 1\) for any \(I \in K\). Then the commutative diagram (2.1) takes the form

\[
\begin{array}{cccccccc}
1 & \longrightarrow & G^I_d & \longrightarrow & G^m_d / H_d & \longrightarrow & G^m_d / (G^I_d \cdot H_d) & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & G^J_d & \longrightarrow & G^m_d / H_d & \longrightarrow & G^m_d / (G^J_d \cdot H_d) & \longrightarrow & 1
\end{array}
\]

for \(I \subseteq J \in \text{cat} K\). Hence, the diagram (2.14) defines the isomorphism \(Q_d \rightarrow G^m_d / p(G^I_d)\) of \((\text{cat} K)\)-diagrams. Now the desired homotopy equivalence follows by Theorem 2.15. \(\square\)

Remark. In the case of a small cover \(d = 1\), the homeomorphism from Corollary 2.20 is similar to the Davis–Januszkiewicz construction [14], because \((D^1, \text{pt})^K\) coincides with the cubical subdivision of the simple polytope \(P \subset \mathbb{R}^n\), where \(K = \partial P^*\). By relaxing the condition on \(H_d \cong G^m_d\) to the condition of only finite stabilizers of the respective action on \((D^d, S^{d-1})^K\) one obtains a homeomorphism similar to the one that was proved for quasitoric orbifolds in [41]. For general partial quotients a similar homeomorphism was proved in [20, 21] (see also [2]). Notice that the homeomorphism from Corollary 2.20 is in general different from the Davis–Januszkiewicz construction [14]. For a partial quotient the natural projection of orbit spaces

\[
(D^d, S^{d-1})^I / H_d \rightarrow (D^d, S^{d-1})^I / G^m_d = (I^1, 1)^I, \quad I \in \text{cat} K,
\]

is a trivial \((G^m_d / (H_d \cdot G^I_d))\)-bundle, where \(I^1 = [0, 1] \subset \mathbb{R}\). This observation allows us to recover the genuine homeomorphism from the Davis–Januszkiewicz construction.

3. EQUIVARIANT HOMOTOPY COLIMITS AND G-CW-COMPLEXES

In this section we prove the strengthening of Theorem 2.15 in the equivariant setting leading to \(G\)-CW-approximation for quotients of moment–angle complexes.

Let \(G\) be a topological group.

Definition 3.1 [44]. The equivariant union \(X = \text{colim}_{n \in \mathbb{Z}_{\geq 0}} X_n\) of \(G\)-spaces \(X_n\) is called a \(G\)-complex if there is a pushout

\[
X_{n+1} = X_n \cup_{\phi_n} \left( \bigsqcup_{\alpha \in A_n} D^{n_\alpha} \times G / H_\alpha \right)
\]

of \(G\)-spaces with the natural left \(G\)-action (left \(G\)-action on \(G / H_\alpha\) and trivial action on \(D^{n_\alpha}\)), where

\[
\phi_n : \bigsqcup_{\alpha \in A_n} S^{n_\alpha - 1} \times G / H_\alpha \rightarrow X_n
\]

is \(G\)-equivariant, \(D^{n_\alpha}\) is an \(n_\alpha\)-dimensional disk, and \(\{H_\alpha\}_{\alpha \in A_n}\) is a collection of closed subgroups in \(G\). If \(n_\alpha = n\) for any \(\alpha \in A_n\), then \(X\) is called a \(G\)-CW-complex.

The category \(G\text{-Top}\) of \(G\)-spaces (objects) and \(G\)-equivariant maps between these spaces (morphisms) has the Quillen model structure given by \(G\)-equivariant weak equivalences, \(G\)-equivariant Serre fibrations, and \(G\)-equivariant retracts of \(G\)-CW-complexes [7]. Since \(\text{cat} K\) is a Reedy category (see [16]), the category \(G\text{-Top}^{\text{cat} K}\) has the Reedy model structure given by objectwise
weak equivalences and fibrations, and cofibrations are given by morphisms \( D \to E \) such that
\[ D(I) \sqcup_{L_D(I)} L_E(I) \to E(I) \]
is a cofibration for any \( I \in K \), where
\[ L_D(I) := \hocolim_{(\text{cat } K)_{<I}} D \to D(I) \]
is the natural map for \( D \) (see [12, 16]).

The following proposition is straightforward to prove.

**Proposition 3.2.** Let \( G_I \) be a collection of closed subgroups in \( G \) such that \( G_I \subseteq G_J \) for any \( I \subseteq J \in \text{cat } K \). Define the (cat \( K \))-diagram \( D \) by
\[ D(I) := G/G_I, \quad D(I \to J) : G/G_I \to G/G_J, \]
where \( D(I \to J) \) is the natural projection. Then \( D \) is fibrant in \( G\text{-Top}^{\text{cat } K} \) and its homotopy colimit in \( G\text{-Top} \) is given by
\[ \hocolim D = \left( \bigsqcup_{I \in \text{cat } K} D(I) \times |(\text{cat } K)_{\geq I}| \right)/\sim, \quad (3.1) \]
where by definition \((d, \text{Inc}_{I \to J}(I')) \sim (D(I, J)(d), I') \) and \( \text{Inc}_{I \to J} : |(\text{cat } K)_{\geq J}| \to |(\text{cat } K)_{\geq I}| \) is the natural embedding. Furthermore, the decomposition (3.1) endows \( \hocolim D \) with the structure of a \( G\text{-CW}\)-complex.

Let \( G = G^m_d/H_d \). The natural \( G \)-action on \((D^d, S^{d-1})^K/H_d \) allows us to consider the (cat \( K \))-diagram \((D^d, S^{d-1})^I/H_d \) as a (cat \( K \))-diagram in \( G\text{-Top} \).

**Theorem 3.3.** For any closed subgroup \( H_d \) in \( G^m_d \) and any simplicial complex \( K \) on \([m]\), there is a \((G^m_d/H_d)\)-equivariant homotopy equivalence
\[ (D^d, S^{d-1})^K/H_d \simeq \hocolim G^m_d/(G^I_d \cdot H_d). \]

**Proof.** This follows from Theorem 2.15 by the standard properties of a homotopy colimit and from the \( G \)-equivariance of all arrows in (2.12). \(\square\)

**Remark.** Theorem 3.3 gives an explicit \( G\text{-CW} \)-approximation of the quotient \((D^d, S^{d-1})^K/H_d \) with cells
\[ \Delta^s(I_0 \supset \cdots \supset I_s) \times G^m_d/(H_d \cdot G^I_d), \quad I_0 \in \text{cat } K, \quad s \geq 0. \]
In the case of partial quotients this decomposition has been known (see [20, Sect. 2]). The homeomorphism from Corollary 2.20 is easily shown to be \( G \)-equivariant.

## 4. EQUIVARIANT COHOMOLOGY OF QUOTIENTS

### FOR MOMENT–ANGLE COMPLEXES

In this section we study the formality problem for the Borel construction of the natural \( L_d \)-action on the quotient \((D^d, S^{d-1})^K/H_d \) of the moment–angle complex for any closed subgroup \( H_d \) satisfying Condition 4.7 (see below). This condition is satisfied for any subgroup \( H_d \) that acts freely on the corresponding moment–angle complex. Notice that the formality of the corresponding Borel space for any freely acting subgroup (that is, for any partial quotient) was proved in [37]. We follow the ideas of [37] in our proof of formality for a wider class of quotients.

### 4.1. On the Borel construction.

The classifying space functor \( B : \text{TGrp} \to \text{Top} \) (see, for example, [39]) induces the functor \( \text{TGrp}^{\text{cat } K} \to \text{Top}^{\text{cat } K} \), which we denote by \( B \) with a slight abuse of notation. Thus, there are well-defined (cat \( K \))-diagrams \( BS_d, \kappa(BL_d), \) and \( BQ_d \). Consider the (cat \( K \))-diagram \( EL_d \times_{L_d} Q_d \) given by applying the (functorial) Borel construction to the diagram \( Q_d \) of \( L_d \)-spaces.
Proposition 4.1. There is an $L_d$-equivariant homotopy equivalence of $(\text{cat } K)$-diagrams

$$EL_d \times_{L_d} Q_d \rightarrow BS_d.$$ 

Proof. This follows directly from the fact that $L_d$ acts on the space $Q_d(I) = G^m_d/(H_d \cdot G^I_d)$ transitively with the kernel $S_d = G^m_d/(G^I_d \cap H_d)$ by (2.1). \hfill \Box

Thus, the Borel construction for the natural $L_d$-action on $(D^d, S^{d-1})^K/H_d$ takes the following form.

Corollary 4.2. For $G = L_d$, the following fibration in $G$-Top$^{\text{cat } K}$ holds:

$$Q_d \rightarrow BS_d \rightarrow \kappa(BL_d).$$ \hfill (4.1)

Theorem 4.3. There is a homotopy equivalence

$$EL_d \times_{L_d} (D^d, S^{d-1})^K/H_d \simeq \text{colim} B(G^I_d/(G^I_d \cap H_d))$$

for the Borel construction of the $L_d$-action on the quotient $(D^d, S^{d-1})^K/H_d$ of the moment–angle complex.

Proof. This follows directly from Proposition 4.1 by the homotopy lemma of [45]. \hfill \Box

Example 4.4. Suppose that the $H_d$-action on $(D^d, S^{d-1})^K$ is free. Then it follows from the standard properties of equivariant cohomology that the cohomology ring isomorphism (with $\mathbb{Z}$-coefficients)

$$H^*_G((D^d, S^{d-1})^K/H_d) \simeq H^*_G((D^d, S^{d-1})^K)$$

takes place. On the other hand, the freeness of the action implies that $H_d \cap G^I_d$ is a trivial group for any $I \in \text{cat } K$. Hence, the colimit of $BS_d$ is the Davis–Januszkiewicz space $DJ(K) = (\mathbb{F}_d \mathbb{P}^\infty, pt)^K$, whose cohomology ring with $R_d$-coefficients is isomorphic to the Stanley–Reisner ring $R_d[K]$ (see [14]). Thus Theorem 4.3 gives a correct answer in this case by [5] (for $d = 1$ we take the reduction of integral cohomology coefficients modulo 2).

As an application of Corollary 4.2 and Theorem 4.3, we describe the Borel construction for the quotient by any coordinate subgroup in $G^m_d$ (with not necessarily free action).

Corollary 4.5. Let $H_d = G^I_{d_0}$ for a fixed $I_0 \subseteq [m]$. Then for any complex $K$ the Borel construction of $L_d$-action on $(D^d, S^{d-1})^K/H_d$ is homotopy equivalent to the real or complex Davis–Januszkiewicz space $\mathbb{R}DJ(K/I_0)$ or $DJ(K/I_0)$ for $d = 1$ or 2, respectively. Furthermore, one has the ring isomorphism

$$H^*_L((D^d, S^{d-1})^K/H_d; R_d) \simeq R_d[K/I_0].$$

Proof. Notice that the natural group isomorphism

$$G^I_d/(G^I_d \cap G^I_{d_0}) \cong G^I_d\setminus I_0$$ \hfill (4.2)

holds for any $I \in \text{cat } K$. Hence, the diagram

$$\begin{array}{ccc}
G^I_d/(G^I_d \cap G^I_{d_0}) & \longrightarrow & G^I_d\setminus I_0 \\
S(I \rightarrow J) \downarrow & & \downarrow \\
G^J_d/(G^J_d \cap G^J_{d_0}) & \longrightarrow & G^J_d\setminus I_0
\end{array}$$

where both horizontal arrows are given by (4.2) and the right vertical arrow is the standard embedding, commutes for any $I \subseteq J \in \text{cat } K$. This diagram yields the isomorphism of $(\text{cat } K)$-diagrams $BS_d$ and $BG^I_d\setminus I_0$. Hence, one has

$$\text{colim } BS_d \cong \text{colim } BG^I_d\setminus I_0 = \text{colim } BG^I_{d_0}.$$ \hfill (4.3)
The last equality holds because the \((\text{cat } K)\)-diagram \(BG^m_d\) is cofibrant and has a singleton (point) as the object corresponding to any \(I \subseteq I_0\) such that \(I \in K\). The last expression in (4.3) is by definition the real or complex Davis–Januszkiewicz space for \(d = 1\) or \(2\), respectively. This proves the first claim. The second claim then follows from the first one by the standard computation for moment–angle complexes (see Example 4.4).

Example 4.6. The Borel construction of \(L_d\)-action on \((D^d, S^{d-1})^K / H_d\) for \(H_d = \{1\}\) and \(H_d = G^m_d\) is, up to homotopy equivalence, the respective Davis–Januszkiewicz space \((\mathbb{F}_d \mathbb{P}^\infty, pt)^K\) and the point, respectively, according to Corollary 4.5.

4.2. On a certain class of quotients for moment–angle complexes. Let \(K\) be any simplicial complex on \([m]\), and let \(H_d\) be any closed subgroup in \(G^m_d\). We introduce the following condition on the pair \((K, H_d)\).

Condition 4.7. For any \(I \subseteq J \in \text{cat } K\) the subgroup \(H_d \cap G^I_d\) is mapped to the subgroup \(H_d \cap G^J_d\) under the natural projection \(G^J_d \rightarrow G^I_d\). Equivalently, in the diagram

\[
\begin{array}{ccc}
G^I_d \cap H_d & \rightarrow & G^J_d \cap H_d \\
\downarrow & & \downarrow \\
G^I_d & \rightarrow & G^J_d
\end{array}
\] (4.4)

where the lower horizontal arrow is the natural projection, there exists an upper horizontal arrow making the diagram commutative.

Example 4.8. For any \(K\) and any \(H_d\) such that \(H_d\) acts freely on \((D^d, S^{d-1})^K\), both groups in the upper row of (4.4) are trivial. Hence, Condition 4.7 holds for any free action of \(H_d\) on \((D^d, S^{d-1})^K\).

Example 4.9. Let \(H_d = G^I_0_d\) for any fixed \(I_0 \subseteq [m]\), and let \(K\) be any simplicial complex. Notice that \(G^J_d \cap H_d = G^I_0_d\) for any \(I \subseteq [m]\). The natural projection \(G^J_d \rightarrow G^I_d\) sends the \(i\)th coordinate subgroup \((G_d)_i\) to 1 if \(i \notin I\) and acts as the identity if \(i \in I\). Hence, the image of \(G^J_d \cap H_d\) under this projection coincides with \(G^I_0_d \cap H_d\). We conclude that Condition 4.7 holds for the action of a coordinate subgroup \(H_d = G^I_0_d\) on \((D^d, S^{d-1})^K\). Notice that this action is not free in general.

Example 4.10. In [43] a certain class of closed subgroups \(H_d\) in \(G^m_d\) acting on \((D^d, S^{d-1})^K\) was introduced. Let \(d = 2\), \(m = 2\), and \(K = \Delta^1\). Consider the subgroup \(H_d = S^1\) of the two-dimensional torus corresponding to the kernel of the homomorphism given by the matrix \((1\ 1)\). It is easy to check that the action of the circle \(H_d = S^1\) on \((D^d, S^{d-1})^K\) belongs to this class and does not satisfy Condition 4.7 for \(I = \{1\}\) and \(J = \{1, 2\}\).

4.3. Twin diagrams and equivariant cohomology. Let \(D\) and \(D^\vee\) be \((\text{cat } K)\)- and \((\text{cat } \text{op } K)\)-diagrams with values in the category Top, respectively. Suppose that \(D(I) = D^\vee(I)\) for any \(I \in \text{cat } K\). Recall the following definition.

Definition 4.11 [37]. The diagrams \(D\) and \(D^\vee\) are called twin diagrams if the identity

\[
D^\vee(J \rightarrow I') \circ D(I \rightarrow J) = D(I \cap I' \rightarrow I') \circ D^\vee(I \rightarrow I \cap I')
\]

holds for any \(I, I' \subseteq J\) in \(\text{cat } K\).

Recall that any \((\text{cat } K)\)-diagram \(D\) of pointed topological spaces gives rise to the Bousfield–Kan type cohomological (with integral coefficients) spectral sequence \((E_D)_r^i\) (see [37]),

\[
(E_D)^{i,j}_2 = \lim^i \tilde{H}^j(D) \Rightarrow \tilde{H}^{i+j}(\text{hocolim } D).
\]
Theorem 4.12 [37, Lemma 3.8, Theorem 3.10]. Suppose that a (cat $K$)-diagram $D$ is cofibrant and has a twin. Then the second page of the Bousfield–Kan spectral sequence $(E^{r}_D)^{s,t}_2$ of $D$ is concentrated at $s = 0$. In particular, $(E^{r}_D)^{s,t}_2$ collapses at the second page $r = 2$.

Corollary 4.13 [37, Corollary 3.12]. If $D$ is cofibrant and has a twin, then one has
$$
\tilde{H}^i(\operatorname{colim} D) = \lim \tilde{H}^i(D) \quad \text{and} \quad \lim_i \tilde{H}^i(D) = 0, \quad j > 0.
$$

Theorem 4.14. Suppose that $K$ and $H_d$ satisfy Condition 4.7. Then one has
$$
\tilde{H}^i(\operatorname{colim} D) = \lim \tilde{H}^i(D) \quad \text{and} \quad \lim_i \tilde{H}^i(D) = 0, \quad j > 0,
$$
where $D = S_d, B S_d$. In particular, $\tilde{H}^{\text{odd}}(\operatorname{colim} B S_2; \mathbb{Z}) = 0$.

Proof. By hypothesis, the subgroup $H_d \cap G_d^J$ is mapped to the subgroup $H_d \cap G_d^I$ under the natural projection $G_d^J \to G_d^I$. Hence, there is a well-defined (cat$^{\text{op}}$ $K$)-diagram $S_d^J$, where
$$
S_d^J(J \to I) : G_d^J/(H_d \cap G_d^J) \to G_d^I/(H_d \cap G_d^I)
$$
is induced by the natural projection $G_d^J \to G_d^I$. Define $(B S_d)^{\vee} := B(S_d^J)$.

We check that the pairs $(S_d, S_d^J)$ and $(B S_d, (B S_d)^{\vee})$ are pairs of twin diagrams. Let $I, I' \subseteq J \in K$ be arbitrary. Consider the following diagram:

$$
\begin{array}{cccccccc}
1 & \rightarrow & H_d \cap G_d^{I \cap I'} & \rightarrow & G_d^{I \cap I'} & \rightarrow & S_d(I \cap I') & \rightarrow & 1 \\
1 & \rightarrow & H_d \cap G_d^I & \rightarrow & G_d^I & \rightarrow & S_d(I) & \rightarrow & 1 \\
1 & \rightarrow & H_d \cap G_d^{I'} & \rightarrow & G_d^{I'} & \rightarrow & S_d(I') & \rightarrow & 1 \\
1 & \rightarrow & H_d \cap G_d^I & \rightarrow & G_d^I & \rightarrow & S_d(J) & \rightarrow & 1
\end{array}
$$

(4.5)

In this diagram the horizontal short sequences are exact and are given by the embedding of the corresponding subgroup and by the quotient by its image. The remaining horizontal and vertical arrows are given by the corresponding projections and embeddings, respectively. Consider the left cube of this diagram. Its front and back faces are commutative by definition. Next, the upper and lower faces of the cube are commutative by Condition 4.7. A straightforward check shows that the right face of this cube is commutative. This implies that the left face of the left cube in (4.5) is commutative due to the monomorphic property of arrows from the horizontal short exact sequences in (4.5). Therefore the left cube in (4.5) is commutative. By Lemma 2.1 this implies that the right cube in (4.5) is well defined and commutative. Hence, the right square of the diagram (4.5) is commutative. It remains to notice that this square coincides with the square from the condition in Definition 4.11 (see (2.3)). (Each arrow of this square is labeled by the corresponding diagram.) It follows that the pair $(S_d, S_d^J)$ is a pair of twin diagrams. The proof of the claim that $(B S_d, (B S_d)^{\vee})$ is a pair of twin diagrams is obtained from this fact by applying the classifying space functor $B$ to the diagram (4.5). Clearly, the diagrams $S_d$ and $B S_d$ are cofibrant. We conclude that the necessary claim follows from Corollary 4.13. 

Corollary 4.15. Suppose that $K$ and $H_d$ satisfy Condition 4.7. Then one has the following ring isomorphism:
$$
\tilde{H}^*_{L_2}(D^d, S^{d-1})^K / H_d \cong \lim \tilde{H}^*(B(G_d^I/(G_d^I \cap H_d))).
$$
In particular, $\tilde{H}^{\text{odd}}_{L_2}(Z_K/H_2; \mathbb{Z}) = 0$.

Example 4.16. Let $H_d$ be a closed subgroup in $G^m_d$ that acts freely on $(D^d, S^{d-1})^K$ (for example, $H = \{1\}$). In this case the corresponding Borel construction is homotopy equivalent to
the Davis–Januszkiewicz space $(\mathbb{F}d^\infty, \text{pt})^K$. The cohomology ring of the latter space is isomorphic to the Stanley–Reisner ring (see [5, 14]). Hence, there is a ring isomorphism [12]

$$\tilde{H}_{d}((d^d, S^{d-1})^K/H_d; R_d) \cong R_d[K].$$

It is well known [12] that

$$R_d[K] \cong \lim_{I=(i_1, \ldots, i_q) \in \text{cat} \; K} R_d[v_{i_1}, \ldots, v_{i_q}]$$

for the Stanley–Reisner ring, where the arrows are the obvious monomorphisms to the polynomial ring $R_d[v_1, \ldots, v_m]$ with $\deg v_j := d_i$. Therefore, the group

$$\lim \tilde{H}(BG_d^I; R_d) \cong \lim_{I=(i_1, \ldots, i_q) \in \text{cat} \; K} (R_d[v_{i_1}, \ldots, v_{i_q}])_i$$

agrees with the respective component $(R_d[K])_i$ of the Stanley–Reisner ring (for $d = 1$ we take the reduction of coefficients modulo 2).

4.4. Formality of the Borel construction for the class of quotients for moment–angle complexes, and the Eilenberg–Moore spectral sequence. In this subsection we study only quotients of complex moment–angle complexes (that is, $d = 2$) and consider only cohomology with integral coefficients because of the use of the Eilenberg–Moore spectral sequences. For brevity we omit the subscript $d$ and replace $G_2$ with $T = S^1$ everywhere below. It is crucial that everywhere in Subsection 4.4 we assume that Condition 4.7 holds for the pair $(K, H)$.

The proof of the following proposition is similar to the proofs of [37, Lemma 4.7, Corollary 3.12] (notice that an analog of [37, Corollary 3.12] is given in Subsection 4.3).

**Proposition 4.17** [37]. Suppose that Condition 4.7 holds for the pair $(K, H)$. Then the natural homomorphism $g: C^*(\text{colim } BS) \rightarrow \text{lim } C^*(BS)$ is a quasi-isomorphism in $\text{dga}_\mathbb{Z}$, and the edge homomorphism (see [37]) $h: H^*(\text{colim } BS) \rightarrow \text{lim } H^*(BS)$ is an isomorphism in $\text{dga}_\mathbb{Z}$, where $C^*(\text{colim } BS)$ is the normalized singular cochain complex of $\text{colim } BS$.

In the following we need to describe the induced morphisms of chain and cochain complexes for tori under corresponding torus homomorphisms. Recall that for any complex compact torus $T = (S^1)^r$ there are simplicial sets $B( BN)$ and $BN$ given by

$$B( BN)_n := \{[b_{n-1}, \ldots, b_0] \mid b_i \in BN_1\}, \quad BN_n := \{[a_0, \ldots, a_{n-1}] \mid a_0, \ldots, a_{n-1} \in N\}, \quad n > 0,$$

and $B( BN)_0 = BN_0 := \{[\ ]\}$, where $N = \pi_1(T, e) \cong \mathbb{Z}^r$ and the explicit formulas for faces and degenerations of $B( BN)$ are given in [35, p. 87]. Any homomorphism of tori $f: T \rightarrow T'$ of ranks $r$ and $r'$, respectively, is given by the formula

$$(e^{2\pi i \varphi_1}, \ldots, e^{2\pi i \varphi_r}) \mapsto (e^{2\pi i \sum_q a_{1,q} \varphi_q}, \ldots, e^{2\pi i \sum_q a_{r',q} \varphi_q}), \quad \varphi_1, \ldots, \varphi_r \in [0, 1),$$

for some integer matrix $A = (a_{i,j}) \in \text{Mat}_{r', r}(\mathbb{Z})$, which we denote by $A = A(f)$, and vice versa. The bar construction of the simplicial set $B( BN)$ gives complexes whose homology and cohomology rings are isomorphic to the integral homology and cohomology rings of $BT$, respectively [35]. We denote the corresponding simplicial set and chain and cochain complexes by $\overline{W}(BT) = B(N)$, $\overline{W}_*(BT)$, and $\overline{W}^*(BT)$, respectively. (Notice that $\overline{W}(BT)$ is denoted by $\overline{W}(T)$ in [35].) Explicitly, the morphism $f_*: \overline{W}(BT) \rightarrow \overline{W}^*(BT')$ induced by $f$ is given by

$$f_*([b_{n-1}, \ldots, b_0]) = [f_*(b_{n-1}), \ldots, f_*(b_0)], \quad f_*([a_0, \ldots, a_{n-1}]) = [f_*(a_0), \ldots, f_*(a_{n-1})],$$

where $f_*(a) = Aa$ is the lattice morphism given by the matrix $A$. 


Lemma 4.18. There is a commutative diagram of simplicial sets

\[ \begin{array}{ccc}
\mathcal{W}(BT) & \longrightarrow & S(BT) \\
\downarrow f_* & & \downarrow f_* \\
\mathcal{W}(BT') & \longrightarrow & S(BT')
\end{array} \]

where any horizontal arrow is a homotopy equivalence and \( S(BT) \) is the simplicial set of singular simplices in \( BT \).

Proof. Notice that the simplicial set \( \mathcal{W}(BT) \) coincides with the bar construction \( B(\ast, BN, \ast) \) of the simplicial group \( BN \) (see [35, § 21]). By [21, Lemma 3.3], there exists a homotopy equivalence of simplicial sets

\[ F: BN \rightarrow S(T), \]

which is natural with respect to morphisms of the torus \( T \). The map \( F \) induces the homotopy equivalence

\[ F_*: \mathcal{W}(BT) = B(\ast, BN, \ast) \rightarrow B(\ast, S(T), \ast) \tag{4.6} \]

by naturality of the bar construction, which is again functorial with respect to morphisms of the group \( T \). By [34, Sect. 13], there is a homotopy equivalence

\[ B(\ast, S(T), \ast) \rightarrow S(BT), \tag{4.7} \]

which is also natural with respect to morphisms of the torus \( T \). One obtains the desired commutative diagram by taking the composition of (4.6) and (4.7) and using the aforementioned naturality properties. \( \square \)

In the following we consider a simplicial variant of the argument from [37]. Choose a generator \( v \) of the ring \( H^*(BS^1) \cong \mathbb{Z}[v] \) for the Eilenberg–MacLane space \( BS^1 = \mathbb{C}P^\infty \). Choose a cocycle \( \psi_v \in \mathcal{W}^r(BS^1) \) that represents \( v \) in the sense of homotopy equivalence from Lemma 4.18. Let a map \( \psi: H^*(BS^1) \rightarrow \mathcal{W}^*(BS^1) \) be given by \( v^q \mapsto (\psi_v)^q \), where the product on the right is with respect to the \( \cup_1 \)-product. For a complex compact torus \( T = (S^1)^r \), let

\[ \kappa: H^*(BT) \xrightarrow{\cong} H^*(BS^1)^{\otimes r} \xrightarrow{\otimes \psi} \mathcal{W}^*(BS^1)^{\otimes r} \]

be given by the composition of the Künneth isomorphism and \( \otimes \psi \). There is a zig-zag of quasi-isomorphisms

\[ \text{Hom}(\mathcal{W}_s(BS^1); \mathbb{Z})^{\otimes r} \rightarrow \text{Hom}(\mathcal{W}_s(BS^1)^{\otimes r}; \mathbb{Z}) \xleftarrow{\varepsilon^*} \mathcal{W}^*(BT), \]

where \( \varepsilon^* \) is the dual to the Eilenberg–Zilber map. For a group homomorphism \( f: T \rightarrow T' \) with the corresponding matrix \( A = (a_{i,j}) \in \text{Mat}_{r',r}(\mathbb{Z}) \), let

\[ A_*: \mathcal{W}_s(BS^1)^{\otimes r} \rightarrow \mathcal{W}_s(BS^1)^{\otimes r'} \quad \text{and} \quad A^*: \mathcal{W}^*(BS^1)^{\otimes r'} \rightarrow \mathcal{W}^*(BS^1)^{\otimes r} \]

be maps induced on degree 2 by the lattice morphisms with matrices \( A \) and \( A^t \), respectively.

Lemma 4.19. Let \( f: T \rightarrow T' \) be a homomorphism of compact complex tori of dimension \( r \) and \( r' \), respectively. Then there is a commutative diagram

\[ \begin{array}{ccc}
H^*(BT') & \xrightarrow{\kappa} & \mathcal{W}^*(BS^1)^{\otimes r'} & \longrightarrow & \text{Hom}(\mathcal{W}_s(BS^1)^{\otimes r'}; \mathbb{Z}) & \xleftarrow{\varepsilon^*} & \mathcal{W}^*(BT') \\
\downarrow f^* & & \downarrow A^* & & \downarrow \text{Hom}(A_*, Id) & & \downarrow f^* \\
H^*(BT) & \xrightarrow{\kappa} & \mathcal{W}^*(BS^1)^{\otimes r} & \longrightarrow & \text{Hom}(\mathcal{W}_s(BS^1)^{\otimes r}; \mathbb{Z}) & \xleftarrow{\varepsilon^*} & \mathcal{W}^*(BT) 
\end{array} \tag{4.8} \]

where each horizontal arrow is a quasi-isomorphism.
Proof. Let $H^*(BT) \cong \mathbb{Z}[v_1, \ldots, v_r]$ and $H^*(BT') \cong \mathbb{Z}[v'_1, \ldots, v'_r]$ be the isomorphisms given by the Künneth isomorphism. By the formulas

$$A^*(\psi_{v'_i}) = \sum_q a_{i,q} \psi_{v_q} \quad \text{and} \quad f^*(v'_i) = \sum_q a_{i,q} v_q \quad (4.9)$$

(and taking products) we get the two left vertical arrows in (4.8). This proves the commutativity of the left square in (4.8). Choose a generator $u \in H_2(BS^1)$ and let $\varphi_u \in W_2(BS^1)$ be a cycle representing $u$ in the sense of homotopy equivalence from Lemma 4.18. Then $W^*(BS^1) \rightarrow \text{Hom}(W_*(BS^1); \mathbb{Z})$ is given by $\psi_v \mapsto (\varphi_u)^*$, where $(\varphi_u)^*$ is the character dual to $\varphi_u$. Let $H_*(BT) \cong \mathbb{Z}[u_1, \ldots, u_r]$ and $H_*(BT') \cong \mathbb{Z}[u'_1, \ldots, u'_r]$ be the isomorphisms given by the Künneth isomorphism. The formulas

$$f_*(u_i) = \sum_q a_{q,i} u'_q \quad \text{and} \quad \text{Hom}(A_*, \text{Id})(((\varphi_u)^*)^*) = \sum_q a_{q,i}(\varphi_u'^*)_q \quad (4.10)$$

determine $f_*$ and $\text{Hom}(f_*, \text{Id})$, respectively. Then the commutativity of the middle square in (4.8) follows directly from (4.9) and (4.10). Notice that there is a diagram

$$\begin{array}{ccc}
W_*(BS^1)^{\otimes r} & \xrightarrow{e_z} & W_*(BT) \\
\downarrow A_* & & \downarrow f_* \\
W_*(BS^1)^{\otimes r'} & \xrightarrow{e_z} & W_*(BT')
\end{array} \quad (4.11)$$

where $e_z$ is the Eilenberg–Zilber map. The explicit formula for the Eilenberg–Zilber map (see [35, §29]) and (4.10) imply that the diagram (4.11) is commutative. This implies the commutativity of the right square in (4.8). □

Corollary 4.20. Suppose that Condition 4.7 holds for the pair $(K, H)$. Then there is a zig-zag of quasi-isomorphisms

$$\lim H^*(BS) \rightarrow D_1 \rightarrow \lim D_2 \leftarrow \lim W^*(BS) \leftarrow \lim C^*(BS), \quad (4.12)$$

where

$$D_1(I) := \overline{W}^*(BS^1)^{\otimes \text{rk} S(I)}, \quad D_1(I \rightarrow J) := A(S(I \rightarrow J))^*, \quad D_2(I) := \text{Hom}(\overline{W}(BS^1)^{\otimes \text{rk} S(I); \mathbb{Z}}), \quad D_2(I \rightarrow J) := \text{Hom}(A(S(I \rightarrow J))_*, \text{Id}).$$

Proof. Notice that $S(I \rightarrow J)$ is a monomorphic homomorphism of tori, and so $A(S(I \rightarrow J))$ is a monomorphic lattice homomorphism. This implies that each arrow in each diagram in (4.12) is an epimorphism. Therefore, any such diagram is fibrant in the model category $\text{dga}_{\mathbb{Z}}$ (see [12, Appendix C.1]). Hence, any limit in (4.12) is quasi-isomorphic to the respective homotopy limit (see [12, Appendix C.1]). It remains to use Lemma 4.19, because any quasi-isomorphism of diagrams in $[\text{cat}^{op} K, \text{dga}_{\mathbb{Z}}]$ induces a quasi-isomorphism of the respective homotopy limits (see [12, Appendix C.1]). □

Theorem 4.21. Suppose that Condition 4.7 holds for the pair $(K, H)$. Then the differential graded algebra $C^*(\text{colim} BS)$ is formal in $\text{dga}_{\mathbb{Z}}$.

Proof. Combining Proposition 4.17 and Corollary 4.20 yields the zig-zag

$$H^*(\text{colim} BS) \xrightarrow{h} \lim H^*(BS) \rightarrow \lim D_1 \rightarrow \lim D_2 \leftarrow \lim \overline{W}^*(BS) \leftarrow \lim C^*(BS) \xleftarrow{g} C^*(\text{colim} BS),$$

which completes the proof. □
For a Serre fibration \( p: E \to B \) with a connected fiber \( F \), the Eilenberg–Moore spectral sequence \((E^*,d)\) of the fiber inclusion has the second page
\[
E_2^{n,s} = \text{Tor}^{n,s}_{H^*(B)}(H^*(E),\mathbb{Z})
\]
(see [36, p. 233]), where the first grading is cohomological and the second is inner. If \( B \) is simply connected, then \((E^*,d)\) converges strongly to \( H^*(F) \) (see [36, p. 233]).

**Theorem 4.22.** Suppose that Condition 4.7 holds for the pair \((K,H)\). Then the Eilenberg–Moore spectral sequence for the fiber inclusion to the Borel construction of the \(L\)-action on \( Z_K/H \) is isomorphic to
\[
\text{Tor}_{H^*(BL)}^{i,j}(\lim H^*(BS);\mathbb{Z}) \Rightarrow H^{i+j}(Z_K/H).
\]
It collapses at the second page. In particular, the associated graded algebra of \( H^*(Z_K/H) \) is isomorphic to \( \text{Tor}_{H^*(BL)}^{i,j}(\lim H^*(BS);\mathbb{Z}) \).

**Proof.** The Eilenberg–Moore spectral sequence under consideration has the second page \( \text{Tor}_{H^*(BL)}^{i,j}(H^*(\text{colim} BS);\mathbb{Z}) \) and converges to \( \text{Tor}_{C^*(BL)}^{i,j}(C^*(\text{colim} BS);\mathbb{Z}) \). However, by the formality of \( BL \) and \( \text{lim} BS \) (see Theorem 4.21), these pages coincide. Hence, by Proposition 4.17 this spectral sequence collapses at the second page, which proves the first claim. The second claim then follows trivially from Theorem 14.14. □

**Remark.** Recall that the quotient \((D^d,S^{d-1})/H_d\) is called a partial quotient [12] if the corresponding \(H_d\)-action is free. In the case of partial quotients the claim of Theorem 4.22 has been known (see Example 4.16). We refer to [21] for the necessary bibliographical links and recent historical overview on the results about the cohomology groups and rings of partial quotients. Theorem 4.22 is a new generalization of previously known results on the above Eilenberg–Moore spectral sequence for partial quotients to the case of any (not necessarily freely acting) closed subgroup \( H \) (in \( T^m \)) satisfying Condition 4.7.

**Remark.** Puppe’s lemma [17] and the cofibrancy of \( BS_d \) imply that there is an \( L_d \)-equivariant Serre fibration
\[
hocolim Q_d \to \text{colim} BS_d \to BL_d.
\]
(4.13)
For \( d = 2 \), one can deduce that the Serre spectral sequence of the diagram of fibrations (4.1) evaluated at \( I \in \text{cat} K \), as well as of the fibration (4.13), collapses in the term \( E_3 \) (cf. [11, Proposition 7.36]).

### 5. COHOMOLOGY OF PARTIAL QUOTIENTS: CONCLUDING REMARKS AND OPEN PROBLEMS

In this section we are going to discuss the toral rank conjecture and torsion in the integral cohomology of partial quotients. The following conjecture was formulated by Halperin in [27].

**Conjecture 5.1.** Let \( X \) be a finite-dimensional CW complex. Then
\[
\text{hrk}(X) := \sum_{i \geq 0} \dim H^i(X;\mathbb{Q}) \geq 2^{\text{trk}(X)},
\]
where \( \text{trk}(X) \) denotes the maximal rank of a torus acting almost freely on \( X \).

In what follows we restrict our attention to (almost) free actions of toric subgroups in \( T^m \) on \( Z_K \) and on its partial quotients \((m = f_0(K)), \) and we denote by \( S^1_D \) the diagonal circle in \( T^m \). In this section we mainly study the family of spaces of the type \( Z_K/S^1_D \).

Recall that the *Buchstaber number* \( s(K) \) of a simplicial complex \( K \) is the maximal rank of a complex torus acting freely on \( Z_K \). Following the notation of [22], we denote by \( \Delta_m^k \) the \( k \)-skeleton.
of the \((m - 1)\)-dimensional simplex for \(m \geq 2\) and \(0 \leq k \leq m - 2\). The Buchstaber numbers for the spaces in this class were computed in [23].

**Theorem 5.2.** For any \(m \geq 2\) and \(0 \leq k \leq m - 2\), the partial quotient \(\mathcal{Z}_{\Delta_k} / S^1_D\) is a rationally formal space with torsion-free integral cohomology. Moreover, the following inequality holds:

\[
\text{hrk}(\mathcal{Z}_{\Delta_k} / S^1_D) \geq 2^{m-k-2}(k+2).
\]

**Proof.** Due to [22, Theorem 4.5.10], for any \(0 \leq k \leq m - 2\) one has

\[
\mathcal{Z}_{\Delta_k} / S^1_D \simeq \mathbb{C}P^{k+1} \lor \mathcal{Z}_{\Delta_{m-1}}^k \lor \left( \bigvee_{i=1}^k S^{2i-1} \ast \mathcal{Z}_{\Delta_{m-1-i}}^k \right) \lor (S^{2k+1} \ast T^{m-k-2}).
\]

Furthermore, by [26, Corollary 9.5], we obtain

\[
\mathcal{Z}_{\Delta_k} \simeq \bigvee_{j=k+2}^m (S^{k+j+1})^{\Sigma(n)}(i_{+1}).
\]

It follows immediately that the partial quotient \(\mathcal{Z}_{\Delta_k} / S^1_D\) is a rationally formal space with torsion-free integral cohomology.

One has the homotopy equivalence \(\Sigma T^m \simeq S^2 \lor \Sigma T^{n-1} \lor \Sigma^2 T^{n-1}\) for each \(n \geq 2\). It implies that \(\Sigma T^m\) is a homotopy wedge of spheres. Therefore,

\[
\text{hrk}(\mathcal{Z}_{\Delta_k} / S^1_D) = 1 + (\text{hrk}(\mathbb{C}P^{k+1}) - 1) + (\text{hrk}(\mathcal{Z}_{\Delta_{m-1}}) - 1)
\]

\[
+ \sum_{i=1}^k (\text{hrk}(\mathcal{Z}_{\Delta_{m-1-i}}) - 1) + (\text{hrk}(\Sigma T^{m-k-2}) - 1).
\]

Note that the formulas \(\text{hrk}(\mathbb{C}P^{k+1}) = k + 2\) and \(\text{hrk}(\Sigma T^{m-k-2}) = 2^{m-k-2}\) hold. Moreover, \(\text{hrk}(\mathcal{Z}_{\Delta_{m-1}}) \geq 2^{(m-1)-(k+1)} = 2^{m-k-2}\) due to [42, Theorem 10]. Hence, it follows that

\[
\text{hrk}(\mathcal{Z}_{\Delta_k} / S^1_D) \geq (k + 2) - 1 + 2^{m-k-2} + k(2^{m-k-2} - 1) + (2^{m-k-2} - 1) = 2^{m-k-2}(k + 2),
\]

which completes the proof. \(\square\)

The number \(s(P) = s(\partial P^+\ast)\) is called the Buchstaber number of a simple polytope \(P\).

**Problem 5.3.** 1. Does there exist a simplicial complex \(K\) on the vertex set \([m]\) and a toric subgroup \(H \subseteq T^m\) of rank \(r\), \(1 \leq r \leq s(K)\), acting freely on \(\mathcal{Z}_K\) such that the partial quotient \(\mathcal{Z}_K / H\) is not formal?

2. Does there exist a simple polytope \(P\) with \(m\) facets and a toric subgroup \(H \subseteq T^m\) of rank \(r\), \(1 \leq r \leq s(P)\), acting freely on \(\mathcal{Z}_P\) such that the partial quotient \(\mathcal{Z}_P / H\) is not formal?

If \(f_0(K) = m\) and \(\dim K = n - 1\), then the maximal rank of a toric subgroup in \(T^m\) acting almost freely on \(\mathcal{Z}_K\) equals \(m - n\) by [42, Lemma 8] and [14, Sect. 7.1].

**Lemma 5.4.** In the above notation, the maximal rank of a toric subgroup in \(T^m\) acting almost freely on \(\mathcal{Z}_K / S^1_D\) is less than or equal to \(m - n - 1\).

**Proof.** Obviously, the stabilizer of a point \(x \in (D^2, S^1)^I \subseteq \mathcal{Z}_K\) with respect to the \(T^m\)-action is equal to \(T^I\), where \(I \subseteq K\). Therefore, the stabilizer of a point \(x \in (D^2, S^1)^I / H \subseteq \mathcal{Z}_K\) with respect to the \(T^m\)-action is the subgroup \(H(I)\) generated by \(T^I\) and \(S^1 \subseteq T^m\). The subgroup \(H(I)\) does not depend on the choice of the point \(x \in (D^2, S^1)^I\).

A toric subgroup \(T^r \subseteq T^m\) acts almost freely on \(\mathcal{Z}_K / S^1_D\) if and only if the intersection \(T^r \cap H(I)\) is finite for all \(I \subseteq K\). In this case we have

\[
\text{rk} T^r + \text{rk} H(I) \leq \text{rk} T^m.
\]
The intersection $T^I \cap S^1_D$ is trivial by the condition on the almost free action. Hence, the group $H(I)$ is the inner direct product of the subgroups $T^I$ and $S^1_D$ in $T^m$. Therefore, one has $\text{rk} H(I) = n + 1$ for any $\dim I = n - 1$. It follows that

$$r = \text{rk} T^r \leq \text{rk} T^m - \text{rk} H(I) = m - n - 1,$$

as required. □

As already mentioned above, here we consider only (almost) free actions of toric subgroups in $T^m$ on $\mathbb{Z}_K$ and on its partial quotients. A weaker version of Halperin's conjecture for moment–angle complexes was proved in [42]. We show that a similar statement holds for partial quotients of moment–angle complexes by the diagonal circle action.

**Theorem 5.5.** For any simplicial complex $K$ of dimension $n - 1$ on $m$ vertices, one has the inequality

$$\text{hrk}(X) \geq 2^{\text{atr}(X)},$$

where $X = \mathbb{Z}_K / S^1_D$ and $\text{atr}(X)$ denotes the maximal rank of a toric subgroup in $T^m$ acting almost freely on $X$.

**Proof.** Consider the principal $S^1$-bundle $S^1_D \to \mathbb{Z}_K \to X$. The space $X$ is simply connected, which follows from the homotopy exact sequence of this bundle, because the moment–angle complex $\mathbb{Z}_K$ is 2-connected.

The inequality $	ext{hrk}(\mathbb{Z}_K) \leq \text{hrk}(X) \cdot \text{hrk}(S^1_D) = 2 \text{hrk}(X)$ follows directly from the Serre spectral sequence of this bundle. Hence, $\text{hrk}(X) \geq \text{hrk}(\mathbb{Z}_K)/2 \geq 2^{m-n-1}$, where the last inequality holds by [42, Theorem 10].

On the other hand, one has the inequality $\text{atr}(X) \leq m - n - 1$ by Lemma 5.4. □

We are grateful to Anton Ayzenberg for drawing our attention to the fact that the original version of Conjecture 5.1 remains open for both moment–angle complexes and more general partial quotients.

**Problem 5.6.** Prove Conjecture 5.1 for all partial quotients of moment–angle complexes, or find a counterexample in this class of spaces.

In what follows we discuss torsion in the integral cohomology of partial quotients.

**Theorem 5.7.** Let $G$ be a finitely generated abelian group. Then there exists a simple polytope $P \subseteq \mathbb{R}^n$ with $m$ facets and a toric subgroup of rank 1 (circle) $H \subseteq T^m$ such that $H$ acts freely on the moment–angle manifold $\mathbb{Z}_P$ and $H^*(\mathbb{Z}_P/H)$ contains the group $G$ as a direct summand.

**Proof.** Consider the Moore space $X$ for $G$, that is, $H^p(X) \cong G$ and $\tilde{H}^i(X) = 0$ for all $i \neq p$, given a certain $p \geq 1$. Take an arbitrary finite triangulation $K$ of $X$, and let $K'$ be obtained from $K$ by a stellar subdivision in a maximal simplex of $K$. Then there exist two distinct vertices $i$ and $j$ in the vertex set $[m]$ of $K'$ that are not linked by an edge.

Following [6], consider the full simplex $\Delta_{[m]}$ on the vertex set $[m]$ and let us cut off its faces, one by one, corresponding to the minimal non-faces of the complex $K'$. Then the nerve complex $\tilde{K}$ of the resulting simple polytope $P$ will be a polytopal sphere of dimension $m - 2$ with $M := m + |\text{MF}(K')|$ vertices, where $\text{MF}(K')$ denotes the set of minimal non-faces of the complex $K'$. Moreover, there still exists no edge connecting the vertices $i$ and $j$ of $\tilde{K}$, because $\{i, j\} \not\in \text{MF}(K') \subseteq \text{MF}(\tilde{K})$.

Now, consider the partition $\alpha$ of the set $[M]$ into $M - 1$ classes $\alpha_1, \ldots, \alpha_{M-1}$ with $\{i, j\}$ the only class consisting of two elements. Following the notation of [43], one has the map $\lambda_\alpha : [M] \to \mathbb{Z}^{M-1}$ that sends a vertex from the class $\alpha_k$ to $\bar{e}_k$, where $\{\bar{e}_1, \ldots, \bar{e}_{M-1}\}$ is the basis of the lattice $\mathbb{Z}^{M-1}$. Due to [43], we observe that the map $\lambda_\alpha$ gives rise to the toric subgroup $H_{\lambda_\alpha} \subset T^M$ which acts freely on the moment–angle manifold $\mathbb{Z}_P$. 

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Therefore, the space $X(\tilde{K}, \lambda_{\alpha}) \cong \mathbb{Z}_p/H$ will be the corresponding partial quotient. By our construction, it will be a smooth compact simply connected manifold of dimension $\dim \mathbb{Z}_p - 1 = 2m + |\text{MF}(K')| - 2$, and the equality $\tilde{K}_{\alpha[m-1]} = K'$ holds. Hence, by [43, Theorem 1.2], $H^q(\mathbb{Z}_p/H)$ contains $G$ as a direct summand when $q = p + m$. \qed

**Remark.** Let $H^*(\mathbb{Z}_K)$ be torsion-free. Then the integral cohomology rings of the partial quotients are also torsion-free for the moment–angle complexes $\mathbb{Z}_K$ from the class introduced in [43]: this follows immediately from [43, Theorem 1.2].

**Example 5.8.** Let $G = \mathbb{Z}/2\mathbb{Z}$. Take $X$ to be $\mathbb{RP}^2$. Consider its minimal triangulation on six vertices: $K = \mathbb{RP}^2_6$. It is well known that the set of the minimal non-faces $\text{MF}(K)$ consists of ten three-element sets. Hence, the two-dimensional simplicial complex $K'$ has $m = 7$ vertices, and the five-dimensional polytopal sphere $\tilde{K}$ has $M = 21$ vertices. Therefore, $\dim(\mathbb{Z}_p/H) = 26$ and its integral cohomology contains 2-torsion in degree $q = 9$.

Note that in general the orbit space $\mathbb{Z}_p/T^r$ has torsion in integral homology provided that the $T^r$-action is not free (see [18, Example 2.4]).

**Problem 5.9.** 1. Does there exist a simple polytope $P$ such that $H^*(\mathbb{Z}_p; \mathbb{Z})$ has torsion and the groups $H^*(\mathbb{Z}_p/T^r; \mathbb{Z})$ are free for any freely acting toric subgroup $T^r \subset T^m$, $1 \leq r \leq s(P)$? 2. Does there exist a simplicial complex $K$ such that $H^*(\mathbb{Z}_K; \mathbb{Z})$ has torsion and the groups $H^*(\mathbb{Z}_K/T^r; \mathbb{Z})$ are free for any freely acting toric subgroup $T^r \subset T^m$, $1 \leq r \leq s(K)$?

**Problem 5.10.** 1. Does there exist a simple polytope $P$ such that $H^*(\mathbb{Z}_p; \mathbb{Z})$ is a free group and $H^*(\mathbb{Z}_p/T^r; \mathbb{Z})$ has torsion for a certain freely acting toric subgroup $T^r \subset T^m$, $1 \leq r \leq s(P)$? 2. Does there exist a simplicial complex $K$ such that $H^*(\mathbb{Z}_K; \mathbb{Z})$ is a free group and $H^*(\mathbb{Z}_K/T^r; \mathbb{Z})$ has torsion for a certain freely acting toric subgroup $T^r \subset T^m$, $1 \leq r \leq s(K)$?

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