Singular Energy Distributions in Driven and Undriven Granular Media

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Abstract We study the kinetic theory of driven and undriven granular gases, taking into account both translational and rotational degrees of freedom. We obtain the high-energy tail of the stationary bivariate energy distribution, depending on the total energy $E$ and the ratio $x = \sqrt{E_W/E}$ of rotational energy $E_W$ to total energy. Extremely energetic particles have a unique and well-defined distribution $f(x)$ which has several remarkable features: $x$ is not uniformly distributed as in molecular gases; $f(x)$ is not smooth but has multiple singularities. The latter behavior is sensitive to material properties such as the collision parameters, the moment of inertia and the collision rate. Interestingly, there are preferred ratios of rotational-to-total energy. In general, $f(x)$ is strongly correlated with energy and the deviations from a uniform distribution grow with energy. We also solve for the energy distribution of freely cooling Maxwell Molecules and find qualitatively similar behavior.

Keywords Granular materials · Kinetic theory · Nonequilibrium statistical physics · Energy distribution · Rotation

1 Introduction

Energy dissipation has profound consequences in driven and undriven granular materials, especially in dilute gases, where the dynamics are controlled by collisions [1–3]. Dissipation is responsible for many interesting collective phenomena including clustering [4–8], formation of shocks [9–13], and hydrodynamic instabilities [14, 15]. Another consequence is the anomalous statistical physics that includes the non-Maxwellian velocity distributions [16–21] and the breakdown of energy equipartition in mixtures [22, 23].
For an elastic gas in equilibrium, the temperature, defined as the average kinetic energy, characterizes the entire distribution function including all of the moments, the bulk of the distribution, as well as the tail of the distribution. Outside of equilibrium, the temperature is not sufficient to characterize the energy distribution. Granular gases are inherently out of equilibrium and a complete characterization must therefore include the behavior of typical particles, the behavior of energetic particles, as well as the moments of the distribution. For example, the energy distribution may have power-law tails with divergent high-order moments [24–26] and consequently, the moments exhibit multiscaling [27]. Generally, non-equilibrium effects are pronounced in the absence of energy input to balance the dissipation but can be suppressed by injection of energy where the deviation from a Maxwellian distribution affects only extremely energetic particles [17, 28–30].

While there is substantial understanding of the energy distribution of frictionless granular gases, much less is known theoretically [31–38] and experimentally [39–41] when the rotational degrees of freedom are taken into account. It is difficult to measure the rotational motion experimentally, and the few available measurements are restricted to two-dimensions. Surface roughness and friction have important consequences and the hydrodynamic theory [42–45] must be modified, if the particles have spin [46]. Equipartition does not hold for the average rotational and translational temperature—neither in the free cooling case [33–36] nor for a driven system [37]. In general, rotational and linear degrees of freedom are correlated in direction [47].

In this paper, we investigate the nature of the full energy distribution, that is, the bivariate distribution of rotational and translational energy. Motivated by the fact that on average the total energy is not partitioned equally between rotational and translational degrees of freedom, we focus on the bivariate distribution \( P(E, x) \) of total energy \( E \) and the modified ratio \( x = \sqrt{E_w/E} \) of rotational to total energy. We thereby generalize the understanding of frictionless granular matter in terms of the energy distribution to rough grains.

Our starting point is the nonlinear Boltzmann equation with a collision rule that accounts for the coupling of translational and rotational motion due to tangential restitution. We study stationary solutions of the inelastic Boltzmann equation that describe steady states achieved through a balance between energy injections that are powerful but rare and energy dissipation through inelastic collisions. For high-energy particles we derive a linear equation for the bivariate energy distribution. The latter can be shown to factorize—\( P(E, x) = p(E)f(x) \)—into a product of the distribution of the total energy, \( p(E) \), and the distribution of the fraction of energy stored in the rotational degrees of freedom, \( f(x) \). The former distribution decays algebraically with energy: \( p(E) \sim E^{-\nu} \). The fraction of energy stored in rotational motion is universal for energetic particles in the sense that \( f(x) \) approaches a limiting distribution independent of energy. Furthermore, this quantity has a number of interesting features. First, the distribution is not uniform, as it would be, if equipartition were to hold. Second, the distribution is not analytic but has singularities at special energy ratios. Third, the distribution and in particular its singularities depend sensitively on the moment of inertia and the collision parameters. Only for energetic particles is this distribution well defined. In general, the partition of energy into rotational and translational motion depends on the magnitude of the energy. This paper specifically addresses two-dimensions, although the theoretical approach and the reported qualitative behavior are generic.

We also develop a general framework for describing high-energy collisions and we use this framework to study freely cooling Maxwell Molecules where the moments of the energy distribution can be found in a closed form. For example, the two granular temperatures corresponding to the rotational and translational motions are coupled and generally, they are not equal. The high-energy behavior found for driven steady-states extends to freely cooling gases.