Poisson approximation for large-contours in low-temperature Ising models

Pablo A. Ferrari \textsuperscript{a} Pierre Picco \textsuperscript{b}

\textsuperscript{a}Universidade de S\~{a}o Paulo, IME, Cx. Postal 66281, 05315-970, S\~{a}o Paulo, Brazil; email: pablo@ime.usp.br, http://www.ime.usp.br/~pablo

\textsuperscript{b}CNRS-CPT Luminy Case 907, 13288 Marseille, France

Abstract

We consider the contour representation of the infinite volume Ising model at any fixed inverse temperature $\beta > \beta^*$, the solution of $\sum_{\theta \in \Theta} e^{-\beta|\theta|} = 1$. Let $\mu$ be the infinite-volume “+” measure. Fix $V \subset \mathbb{Z}^d$, $\lambda > 0$ and a (large) $N$ such that calling $G_{N,V}$ the set of contours of length at least $N$ intersecting $V$, there are in average $\lambda$ contours in $G_{N,V}$ under $\mu$. We show that the total variation distance between the law of $(\gamma : \gamma \in G_{N,V})$ under $\mu$ and a Poisson process is bounded by a constant depending on $\beta$ and $\lambda$ times $e^{-(\beta - \beta^*)N}$. The proof builds on the Chen-Stein method as presented by Arratia, Goldstein and Gordon. The control of the correlations is obtained through the loss-network space-time representation of contours due to Fernández, Ferrari and Garcia.

Key words: Peierls contours; Animal models; Loss networks; Large contours; Ising model; Poisson approximation; Chen-Stein method

AMS Classification: Primary: 60K35 82B 82C 60F17 60F05

1 Introduction

The infinite volume Ising model is one of the most studied in Statistical Physics. One of the fundamental historical fact was the Peierls argument for the existence of more than one phase at low enough temperature —which we assume throughout this paper— in any dimension $d \geq 2$. This argument was set by Dobrushin as the existence of at least two extremal Gibbs states, $\nu^+$ and $\nu^-$ that are obtained as infinite volume limits of Gibbs measures with $+$ resp. $-$ boundary conditions. In the Physics literature, a configuration of spins which is typical with respect to $\nu^+$, is frequently described as a set of islands of $-$ within a sea of $+$; the contours being the boundaries of the islands. Various questions can be asked about typical configuration of contours
at low temperature. A very simple one that does not seem to have attracted attention is: what is the number of occurrences of contours larger than a fixed (big) $N$, that intersect a fixed volume $V = V(N)$ say centered at the origin. To describe the physical idea of rarity of the island of $-$, a Poisson type behavior naturally comes in mind. We prove indeed that the distribution of these occurrences, under the condition that the mean number of occurrences in the volume $V(N)$ is a fixed value $\lambda$, approaches sharply a Poisson process of mean $\lambda$. The paper is organized as follows: in section 2, we give the definitions, state the main result and the Arratia-Goldstein-Gordon formulation of the Chen-Stein Method used in the proof. In section 3 we recall the graphical construction of the loss-network space-time representation of contours of Fernández-Ferrari-Garcia which we apply to obtain the relevant bounds. In section 4 we prove the main result.

2 Definitions and the main result

Peierls introduced a map between typical configurations of $\nu^+$ or $\nu^-$ into an ensemble of objects —the contours— interacting only by perimeter-exclusion. See, for instance, Section 5B of Dobrushin [6], for a concise and rigorous account of this mapping. Contours are hyper surfaces formed by a finite number of $(d - 1)$-dimensional unit cubes —links for $d = 2$, plaquettes for higher dimensions— centered at points of $\mathbb{Z}^d$ and perpendicular to the edges of the dual lattice $\mathbb{Z}^d + (\frac{1}{2}, \cdots, \frac{1}{2})$. To formalize their definition, let us call two plaquettes adjacent if they share a $(d - 2)$-dimensional face. A set of plaquettes, $\gamma$, is connected if for any two plaquettes in $\gamma$ there exists a sequence of adjacent plaquettes in $\gamma$ joining them. The set $\gamma$ is closed if every $(d - 2)$-dimensional face is covered by an even number of plaquettes in $\gamma$. Contours are connected and closed sets of plaquettes. For example, in two dimensions contours are closed polygonals. Two contours $\gamma$ and $\theta$ are said compatible if no plaquette of $\gamma$ is adjacent to a plaquette of $\theta$. In this case we write $\gamma \sim \theta$. In two dimensions, therefore, contours are compatible if and only if they do not share the endpoint of a link. In three dimensions two compatible contours can share vertices, but not sides of plaquettes. Ising spin configurations in a bounded region with “$+$” (or “$-$”) boundary condition are in one-to-one correspondence with families of pairwise compatible contours.

The set of possible contours contained in $\Lambda \subset \mathbb{Z}^d$ will be denoted $G[\Lambda]$. The set of configurations of compatible contours is $\mathcal{X}[\Lambda] = \{\eta \in \{0, 1\}^{G[\Lambda]}; \eta(\gamma) \eta(\theta) = 0$ if $\gamma \not\sim \theta\}$. We denote $|\gamma|$ the number of plaquettes of the contour $\gamma$ and define the finite-volume Gibbs measure $\mu_\Lambda$ on $\mathcal{X}[\Lambda]$; for $\eta \in \mathcal{X}[\Lambda]$,

$$\mu_\Lambda(\eta) = \frac{\exp\left(-\beta \sum_{\gamma} |\gamma| \eta(\gamma)\right)}{Z_\Lambda} \quad (1)$$
We fix $\beta > \beta^*$, the solution of $\alpha_0(\beta) = 1$, where $\alpha_0(\beta) := \sum_{\theta, \gamma \geq 0} e^{-\beta |\theta|}$. Then there exists a unique weak limit $\mu = \lim_{\lambda \to \mathbb{Z}^d} \mu_\lambda$ [7,8]. Notice that $\beta^*$ is strictly bigger than $\beta_p$, the Peierls inverse-temperature, which is the infimum of the $\beta$ making the previous sum finite. Let $p_\gamma = \int \mu(d\eta) \eta(\gamma)$ be the probability of the presence of contour $\gamma$ under $\mu$. Fix $N > 0$, let $V = V(N) \subset \mathbb{Z}^d$ and $\lambda > 0$ be such that if we define $G = G(N, V, \lambda) := \{ \kappa \in G[\mathbb{Z}^d] : |\kappa| \geq N \} \cap V \neq \emptyset$ as the set of contours with length at least $N$ intersecting $V$, then $\sum_{\gamma \in G} p_\gamma = \lambda$. Hence $V$ is a set such that in average there are $\lambda$ contours of length at least $N$ intersecting $V$. Define the processes $X := (\eta(\gamma) : \gamma \in G)$, $Y := (Y(\gamma) : \gamma \in G)$ where $\eta$ has distribution $\mu$ and $Y(\gamma)$ are iid with Poisson distribution of mean $p_\gamma$. We denote $\mathcal{L}(X)$ the law of a process $X$ and $\|\mathcal{L}(X) - \mathcal{L}(Y)\|_{TV}$ the total variation distance between the laws of the processes $X$ and $Y$ [2].

**Theorem 1** Fix $\beta > \beta^*$, then for any $\beta' \in (\beta^*, \beta)$, if $M(\beta, \beta')$ is given in (26) below we have:

$$\|\mathcal{L}(X) - \mathcal{L}(Y)\|_{TV} \leq M(\beta, \beta') N^{d+1} \lambda e^{-(\beta - \beta') N}$$

(2)

The proof of this theorem is an application of the Chen–Stein method [10,5,2,3] as proposed in [2]. In order to describe it, let the distance between contours be $d(\gamma, \theta) := \min\{|x - y| : x \in \gamma, y \in \theta\}$. For $D = D(N) > 0$ to be fixed later we define $B_\gamma = B_\gamma(N, D) = \{ \theta \in G : d(\gamma, \theta) < D \}$, a “$D$-neighborhood” of $\gamma$. Let $p_{\gamma \theta} = \int \mu(d\eta) \eta(\gamma) \eta(\theta)$ and define $b_i = b_i(N, D)$ by

$$
b_1 := \sum_{\gamma \in G} \sum_{\theta \in B_\gamma} p_{\gamma \theta}, \quad b_2 := \sum_{\gamma \in G} \sum_{\gamma \neq \theta \in B_\gamma} p_{\gamma \theta}, \quad b_3 := \sum_{\gamma \in G} \mathbb{E} \mathbb{E}\{\eta(\gamma) - p_\gamma | \sigma(\theta) : \theta \in G \setminus B_\gamma\}\}
$$

Theorem 2 of [2] shows that the total variation distance between the Poisson process $Y$ and the process $X$ is dominated by $2(2b_1 + 2b_2 + b_3)$. To prove Theorem 1 we choose $D$ equal to $N$ times an appropriate constant and then find bounds on $b_i$. The bound on $b_1$ does not offer problems, but the bound on $b_2$ requires the inequality $p_{\gamma \theta} \leq p_\gamma e^{-\beta |\theta|} + p_\theta e^{-\beta |\gamma|}$ proved in Lemma 4 below. The Chen-Stein method has been applied mostly to examples where $b_3 = 0$ [2], while in our case $b_3 \neq 0$. To dominate $b_3$ we control the correlations between “far” contours in Lemma 7 below. The proof of those lemmas is based on the graphical representation of a loss network, a Markov process having $\mu$ as invariant measure [7,8]. There are many papers dealing with Poisson approximations; we limit ourselves to quote the books [1] and [3] —with a nice introduction explaining the Chen-Stein method—, the original paper of Chen [5] and a monograph of Stein [10]. There are a few results for random
fields. The number of "−" spins in the "+" measure of the Ising model at low temperature and/or high external "+" magnetic field converges to Poisson \([4,9]\). In this case the variables are positively associated and the Chen-Stein method works via cluster expansions giving explicit convergence rates \([9]\). In \([8]\) the approximation is established for the contours of the Ising model at low temperature and without magnetic field. The available bounds for the total variation distance are of the order of the inverse of the volume of the observed set \(V\) \([9,8]\).

3 Graphical construction

In this section we quickly review the construction and some results of \([7]\) relevant to our proof. To each contour \(\gamma \in \mathbb{G}(\mathbb{Z}^d)\) we associate an independent (of everything) marked Poisson process \(N_\gamma\) on \(\mathbb{R}\) with rate \(e^{-\beta|\gamma|}\). We call \(T_k(\gamma) \in \mathbb{R}, \gamma \in \mathbb{G}(\mathbb{Z}^d)\), the ordered time-events of \(N_\gamma\) with the convention that \(T_0(\gamma) < 0 < T_1(\gamma)\). For each occurrence time \(T_i(\gamma)\) of the process \(N_\gamma\) we choose an independent mark \(S_i(\gamma)\) exponentially distributed with mean 1. At the Poisson time-event \(T_1(\gamma)\) a contour \(\gamma\) appears and it lasts \(S_i(\gamma)\) time units. The resultant object is the random family \(C = \{((\gamma,T_i(\gamma),S_i(\gamma)) : i \in \mathbb{Z} : \gamma \in \mathbb{G}\}\). A marked point \((\gamma,T_k(\gamma),S_k(\gamma)) \in C\) is identified with \(\gamma \times [T_k(\gamma),T_k(\gamma) + S_k(\gamma)]\), the cylinder with basis \(\gamma\), birth-time \(T_k(\gamma)\) and lifetime \(S_k(\gamma)\). The life of the cylinder is the time interval \([T_k(\gamma),T_k(\gamma) + S_k(\gamma)]\). For a generic cylinder \(C = (\gamma,t,s)\), we use the notation \(\text{Basis}(C) = \gamma\), Birth \((C) = t\), Death \((C) = t + s\), Life \((C) = [t,t + s]\). In the sequel \(\mathbb{P}\) and \(\mathbb{E}\) are the probability and expectation in the space where \(C\) is defined.

For \(t \in \mathbb{R}\) we define \(\xi_t(\gamma,C) = \sum_{C \in \mathbb{C}} \text{1}\{\text{Basis}(C) = \gamma, \text{Life}(C) \geq t\}\). The above process, called the free network, is a product of independent stationary birth-and-death processes on \(\mathbb{N}^{\mathbb{G}(\mathbb{Z}^d)}\) with \(t \in \mathbb{R}\) whose generator is given by

\[
A^0 f(\xi) = \sum_{\gamma \in \mathbb{G}(\mathbb{Z}^d)} e^{-\beta|\gamma|} [f(\xi + \delta_\gamma) - f(\xi)] + \sum_{\gamma \in \mathbb{G}(\mathbb{Z}^d)} \xi(\gamma) [f(\xi - \delta_\gamma) - f(\xi)].
\]

The invariant (and reversible) measure for this process is the product measure \(\mu^0\) on \(\mathbb{N}^{\mathbb{G}(\mathbb{Z}^d)}\) with Poisson marginals: \(\mu^0\{\xi(\gamma) = k\} = \exp\left(e^{-\beta|\gamma|}\right)(e^{-\beta|\gamma|})^k/k!\).

In particular, for any \(t \in \mathbb{R}\), \(\mathbb{E} f(\xi_t) = \mu^0 f\).

We say that cylinders \(C\) and \(C'\) are incompatible and write \(C' \not\sim C\) if and only if \(\text{Basis}(C) \not\sim \text{Basis}(C')\) and \(\text{Life}(C) \cap \text{Life}(C') \neq \emptyset\). We say that two sets of cylinders \(\mathbf{A}\) and \(\mathbf{A}'\) are incompatible and write \(\mathbf{A} \not\sim \mathbf{A}'\) if there is a cylinder in \(\mathbf{A}\) incompatible with a cylinder in \(\mathbf{A}'\). Otherwise we use the sign \(\sim\) for compatibility. For any cylinder \(C\) define the set of ancestors of \(C\) as
the set of cylinders in $C$ born before $C$ that are incompatible with $C$: $A^C_1 = \{C' \in C; C' \not\subset C, \mathrm{Birth}(C') < \mathrm{Birth}(C)\}$. Recursively for $n \geq 2$, the $n$th generation of ancestors of $C$ is $A^C_n = \{C'' : C'' \in A^C_1\}$ for some $C'' \in A^C_{n-1}$.

Let the clan of $C$ be the union of its ancestors: $A^C = \bigcup_{n \geq 1} A^C_n$. Under the condition $\beta > \beta^*$ all cylinders in $C$ have a finite clan with probability one. This property called no backwards oriented percolation is essential to show that the loss network $\eta$ can be constructed in a stationary way for $t \in \mathbb{R}$. Assume that there is no backwards oriented percolation. The construction is as follows.

All cylinders in $C$ are classified as kept and erased. Since all clans are finite, we can write $C = \bigcup_{n \geq 0} C_n$, where $C_n := \{C \in C : A^C \neq \emptyset, A^C_{n+1} = \emptyset\}$. Inductively we set $K_0 = C_0, D_0 = \emptyset, K_n = \{C \in C_n \setminus \bigcup_{i=0}^{n-1} (D_i \cup K_i) : \{C\} \sim \bigcup_{i=0}^{n-1} K_i\}$ and $D_n = C_n \setminus [K_n \cup \bigcup_{i=0}^{n-1} (D_i \cup K_i)]$. Let the set of kept cylinders be $K = \bigcup_n K_n$ and the set of erased cylinders be $D = \bigcup_n D_n$. Clearly $K \cup D = C$. The event $\{C \in K\}$ is measurable with respect to the sigma field generated by $A^C$. In words, it is sufficient to know the (finite) clan of $C$ to know if $C$ is kept or erased. The stationary loss network is defined by $\eta(C) = \sum_{C \in K} 1\{\text{Basis}(C) = \gamma, \text{Life}(C) \ni t\}$. The process $\eta$ is Markovian with generator

$$Af(\eta) = \sum_{\gamma \in G} e^{-\beta|\gamma|} 1\{\eta + \delta, \in \chi\} [f(\eta + \delta_\gamma) - f(\eta)] + \sum_{\gamma \in G} \eta(\gamma)[f(\eta - \delta_\gamma) - f(\xi)].$$

The unique invariant (and reversible) measure for this process is the Gibbs measure $\mu$. In particular, for any $t \in \mathbb{R}$, $E\eta(t) = \mu f$. We study properties of $\mu$ by studying the law of $\eta_0$, the stationary loss network at time zero.

The presence/absence of contours intersecting a region $\Lambda$ at time $t$ depends only on the set $A^{\Lambda,t}$, the union of the clans of the cylinders of $C$ with basis intersecting $\Lambda$ and life containing time $t$, that is $A^{\Lambda,t} = \{C' \in A^C : \text{Basis}(C) \cap \Lambda \neq \emptyset, \text{Life}(C) \ni t\}$. In particular $\eta_0(\gamma)$ is a (deterministic) function of $A^{\Lambda,t}$ defined by $\eta_0(\gamma, C) = \eta_0(\gamma, A^{\Lambda,t})$. The reason is that in order to determine if $C \in K$ it suffices to look at $A^C$. When $t = 0$ we will use the notation $A^\Lambda$ instead of $A^{\Lambda,0}$.

Let $C_{\gamma,t,r} = (\gamma,-t,t+r)$ be a cylinder with Birth$(C_{\gamma,t,r}) = -t$, Basis$(C_{\gamma,t,r}) = \gamma$, Life$(C_{\gamma,t,r}) = t + r$. Conditioning to the birth time of the (kept) cylinder alive at time zero with basis $\gamma$,

$$p_\gamma = \int_{(\mathbb{R}^+)^2} \mathbb{P}(C_{\gamma,t,r} \in K \mid C_{\gamma,t,r} \in C) e^{-t} e^{-r} e^{-\beta|\gamma|} dt dr$$
**Coupling of clans.** For disjoint sets Λ and Υ of \( Z^d \) it is possible to construct \((A^Λ, A^Υ, \hat{A}^Λ, \hat{A}^Υ)\), a coupling between four sets of cylinders satisfying (a) \( A^Λ = \hat{A}^Λ \) and \( A^Υ = \hat{A}^Υ \), in distribution; (b) \( A^Λ \cup A^Υ = A^Λ \cup A^Υ \); (c) \( \hat{A}^Λ \) and \( \hat{A}^Υ \) are independent; (d) If \( \hat{A}^Λ \sim \hat{A}^Υ \), then the marginals coincide: \( \hat{A}^Λ = A^Λ \) and \( \hat{A}^Υ = A^Υ \). We write \( P \) and \( E \) for the probability and the expectation of the coupling. The following bound for the probability of incompatibility between clans follows as in the proof of (2.13) of [8]: for any \( \beta' \in (β^*, β) \),

\[
P(\hat{A}^Λ \not\sim \hat{A}^Υ) \leq 2 (1 - \alpha_0(\beta'))^{-2} \sum_{x \in Λ} \sum_{y \in Υ} |x - y| e^{-|x-y|}. \tag{5}
\]

4 Proof of the theorem

The proof of the theorem is based on a sequence of lemmata.

**Lemma 2** Denote \( ρ = \alpha_0(β) \). Then for \( γ \) such that \( |γ| \geq N \),

\[
\exp\{-|γ|\} \leq p_γ \leq \exp\{-β|γ|\} \tag{6}
\]

**PROOF.** Bounding the conditional probability inside the integral in (4) by one we get the upper bound. For the lower bound notice that the event \( \{C_{γ,t,r} \in K\} \) is measurable with respect to the sigma field generated by the cylinders born before \(-t\), the birth time of \( C_{γ,t,r} \). Hence

\[
P(C_{γ,t,r} \in K \mid C_{γ,t,r} \in C) = P\left( \bigcap_{θ:θ≠γ} \{η_θ(γ, A^θ, -t) = 0\} \right) \tag{7}
\]

\[
\geq P\left( \bigcap_{θ:θ≠γ} \{ξ_θ(γ, C) = 0\} \right) = \exp\left( \sum_{θ:θ≠γ} \exp\{-β|θ|\} \right) \geq e^{-ρ|γ|} \tag{8}
\]

The first inequality follows from \( η_t(γ) \leq ξ_t(γ) \); the second identity from the distribution of \( \xi_t \) described in Section 3. Inserting this inequality in (4) we get the left inequality in (6).

**Lemma 3** The following inequalities hold

\[
|V| \leq λ\left( \sum_{γ \in G_0} |γ|^{-1} p_γ \right)^{-1} \leq λN e^{(β+ρ)N} \tag{9}
\]

**PROOF.** The first inequality is immediate. Bounding below the sum in (9) by one term (choose any \( γ_0 \) such that \( |γ_0| = N \) we dominate the middle term
in (9) by \(|\gamma_0|/p_{\gamma_0}\) which using the left inequality in (6) is bounded by the rhs of (9).

**Lemma 4** We have \(p_{\gamma \theta} \leq p_\gamma e^{-\beta |\theta|} + p_\theta e^{-\beta |\gamma|}\)

**PROOF.** Notice that \(p_{\gamma \theta} = 0\) if \(\gamma \not\sim \theta\). Recall the notation used in (4). Consider compatible \(\gamma\) and \(\theta\) and condition to the birth times of the kept cylinders alive at time zero with bases \(\gamma\) and \(\theta\) to obtain as in (4),

\[
p_{\gamma \theta} = \int_{(R^+)^4} \mathbb{P}(C_{\gamma,t,r} \in K, C_{\theta,s,w} \in K \mid C_{\gamma,t,r} \in C, C_{\theta,s,w} \in C) \\
\times e^{-t} e^{-s} e^{-r} e^{-w} e^{-\beta |\gamma|} e^{-\beta |\theta|} \ ds \ dt \ dr \ dw \tag{10}
\]

For \(s < t\) the event \(\{C_{\gamma,t,r} \in K\}\) is measurable with respect to the sigma field generated by \(\{C: \text{Birth}(C) \leq -t\}\). Hence it is independent of \(\{C_{\theta,s,w} \in K\}\) and we have

\[
1\{s < t\} \mathbb{P}(C_{\gamma,t,r} \in K, C_{\theta,s,w} \in K \mid C_{\gamma,t,r} \in C, C_{\theta,s,w} \in C) \\
\leq 1\{s < t\} \mathbb{P}(C_{\gamma,t,r} \in K \mid C_{\gamma,t,r} \in C) \\
= 1\{s < t\} \mathbb{P}(C_{\gamma,t,r} \in K \mid C_{\gamma,t,r} \in C) \\
\leq \mathbb{P}(C_{\gamma,t,r} \in K \mid C_{\gamma,t,r} \in C) \tag{11}
\]

Analogously \(1\{s \geq t\} \mathbb{P}(C_{\gamma,t,r} \in K, C_{\theta,s,w} \in K \mid C_{\gamma,t,r} \in C, C_{\theta,s,w} \in C) \leq \mathbb{P}(C_{\theta,s,w} \in K \mid C_{\theta,s,w} \in C).\) This and (11) allow us to factorize the integrals in (10) and then use (4) to get the lemma.

**Lemma 5** The following bound holds for \(b_1\), for all \(\varepsilon \leq d(\beta + \rho)\)

\[
b_1 \leq \sum_{\gamma \in G} (D + |\gamma|)^d p_\gamma \sum_{\theta \in C_0} e^{-\beta |\theta|} \tag{12}
\]

\[
\leq 2 \lambda N \left( \frac{\beta + \rho}{\beta - \beta^*} \right) \left( \frac{D + 1}{\varepsilon} \right)^d e^{-(\beta - \beta^* - \varepsilon)N} \tag{13}
\]

**PROOF.** Inequality (12) follows immediately from (6). To prove (13) it is sufficient to show that for all \(\varepsilon < d(\beta + \rho)\),

\[
\sum_{\gamma \in G} (D + |\gamma|)^d p_\gamma \leq 2 \lambda N \left( \frac{\beta + \rho}{\beta - \beta^*} \right) \left( \frac{D + 1}{\varepsilon} \right)^d \exp(\varepsilon N). \tag{14}
\]
Lemma 6 The following bound holds for $b_2$

$$b_2 \leq \lambda \left( \frac{D+1}{\varepsilon} \right)^d \left( 2N \left( \frac{\beta + \rho}{\beta - \beta^*} \right) + d^d \right) e^{-(\beta - \beta^* - \varepsilon)N}$$  \hspace{1cm} (16)

PROOF. By Lemma 4, $b_2$ is bounded above by

$$\sum_{\gamma \in G} (D + |\gamma|)^d p_{\gamma} \sum_{\theta \in G_0} e^{-\beta|\theta|} + \sum_{\gamma \in G} (D + |\gamma|)^d e^{-\beta|\gamma|} \sum_{\theta \in G_0} p_{\theta}$$  \hspace{1cm} (17)

The first term was controlled in the previous lemma. The second one can be treated in a similar way.

Lemma 7 The following bound holds for $b_3$. For any $\beta' \in (\beta^*, \beta)$, letting $\rho' = \alpha_0(\beta')$

$$b_3 \leq 2\lambda e^{-(\beta - \beta')N} + Q(\beta, \beta') N \lambda D^d e^{A(\beta, \beta')N} e^{-(\beta - \beta')D}$$  \hspace{1cm} (18)

where $Q(\beta, \beta') = 4(1 - \rho')^{-2}(\beta - \beta')^{-2}(2d - 1)^2(\beta + \rho)$ and $A(\beta, \beta') = ((\beta + \rho)(\beta - \beta')^{-1} + 1)(\beta + \rho) \log(2d - 1)$.

PROOF. Since $\sigma(\eta_0(\theta, C) : \theta \in G \setminus B_\gamma) \subset \sigma(A^{\theta, 0} : \theta \in G \setminus B_\gamma))$

$$b_3 \leq \sum_{\gamma \in G} \mathbb{E} \left[ \mathbb{E} \left\{ \eta_0(\gamma, A^\gamma) - p_\gamma \right| \sigma(A^\theta : \theta \in G \setminus B_\gamma) \right\} \right]$$

$$= \sum_{\gamma \in G} \mathbb{E} \left[ \mathbb{E} \left\{ \eta_0(\gamma, A^\gamma) - \eta_0(\gamma, \hat{A}^\gamma) \right| \sigma(A^\theta : \theta \in G \setminus B_\gamma) \right\} \right]$$

$$\leq \sum_{\gamma \in G} \mathbb{E} \left\{ 1 \left\{ \hat{A}^\gamma \neq \bigcup_{\theta \in G \setminus B_\gamma} \hat{A}^\theta \right\} (\eta_0(\gamma, A^\gamma) + \eta_0(\gamma, \hat{A}^\gamma)) \right\}$$  \hspace{1cm} (19)

$$\leq \sum_{\gamma \in G} \mathbb{E} \left\{ 1 \left\{ \hat{A}^\gamma \neq \bigcup_{\theta \in G \setminus B_\gamma} \hat{A}^\theta \right\} (\eta_0(\gamma, A^\gamma) + \eta_0(\gamma, \hat{A}^\gamma)) \right\}$$  \hspace{1cm} (20)
where the identity comes from property (c) of the coupling defined in Section 2 and the second inequality from property (d). We divide the sum in (20) in two parts. The first one is over those \( \gamma \in G \) of length bigger than \( KN \), for some \( K \) to be chosen later. Dominating the indicator function by one, this sum is bounded by

\[
\sum_{\gamma \in G, |\gamma| \geq KN} 2p_\gamma \leq 2\lambda e^{(\beta + \rho)N} e^{-(\beta - \beta^*)KN} \tag{21}
\]

The second sum is over those \( \gamma \in G \) with length less than \( KN \). Dominating \( \eta_0(\cdot) \) by one, the summand is bounded by four times

\[
\mathbb{P}(\widehat{A}^\gamma \cap \bigcup_{\theta \in G \setminus B_\gamma(N)} \widehat{A}^\theta) \leq (1 - \rho')^{-2} \sum_{x \in \gamma} \sum_{y \in d(\gamma, \{y\}) > D} |x - y| e^{-(\beta - \beta')|x-y|} \leq \frac{|\gamma| D^d}{(1 - \rho')^2 (\beta - \beta')^2} e^{-(\beta - \beta')D} \tag{22}
\]

using (5) in the first inequality. Hence the second sum is bounded by

\[
\frac{4\lambda (KN) (2d - 1)^{NK+2} D^d}{(1 - \rho')^2 (\beta - \beta')^2} e^{-(\beta - \beta')D} e^{(\beta + \rho)N} \tag{23}
\]

using (9) to bound \(|V|\). The sum (20) is then bounded above by the sum of (21) and (23). Fixing \( K = \frac{\beta + \rho}{\beta - \beta'} + 1 \), we get (18).

**Proof of Theorem 1.** It remains to choose \( D \) in such a way that \( 2(2b_1 + 2b_2 + b_3) \) is bounded above by the right hand side of (2). We already know that \( b_1 \) and \( b_2 \) decay as \( e^{-(\beta - \beta^* - \varepsilon)N} \). If we choose \( D = \delta N \) in (18), where

\[
\delta = \frac{\beta + \rho}{\beta - \beta'} \left[ 1 + \log(2d - 1) \left( \frac{1}{\beta - \beta^*} + \frac{1}{\beta + \rho} \right) \right] + 1, \tag{24}
\]

then

\[
b_3 \leq \frac{4\lambda (\beta + \rho)^d N^{d+1}}{(1 - \rho')^2 (\beta - \beta')^{2d+2}} e^{-(\beta - \beta')N}. \tag{25}
\]

With this bound and the choice \( \varepsilon = \beta' - \beta^* \) in (13) and (16) we get (2). The constant \( M(\beta, \beta') \) is given by

\[
M(\beta, \beta') = 10 \frac{(\beta + \rho)^d}{(\beta - \beta')^{2d+2}} \left( \frac{d^d}{(\beta' - \beta^*)^d} + \frac{2(\log(2d - 1))^2}{(1 - \rho')^2} \right) \tag{26}
\]
Acknowledgements

We thank Leonardo Moledo for his contribution to Lemma 4 and Roberto Fernández for many discussions, specially about plaquettes.

This paper started when PAF was visiting CPT at Marseille. It was written when PAF was Chargé de Recherche associé at Université de Rouen. PAF thanks support of Fundação de Amparo à Pesquisa do Estado de São Paulo and FINEP. PP used financial support of the project Cofecub-USP 45/97.

References

[1] D. Aldous, Probability approximation via the Poisson clumping heuristic. *Lecture Notes in Math.* (Springer, Berlin 1987)

[2] R. Arratia, L. Goldstein and L. Gordon, Two moments suffice for Poisson approximations: The Chen-Stein method, *Ann. Probab.* 17, (1989) 9–25.

[3] A. D. Barbour, L. Holst and S. Janson, *Poisson Approximation* (Clarendon Press, Oxford, 1992).

[4] A. D. Barbour and P. E. Greenwood, Rates of Poisson approximation to finite range random fields, *Ann. Appl. Prob.* 3, 91 (1993).

[5] L. H. Y. Chen, Poisson approximation for dependent trials, *Ann. Probab.* 3 (1975) 534–545.

[6] R. L. Dobrushin. Perturbation methods of the theory of Gibbsian fields. In *Lectures on probability theory and statistics (Saint-Flour, 1994)*, pages 1–66. (Springer, Berlin, 1996).

[7] R. Fernández, P. A. Ferrari and N. L. Garcia, Measures on contour, polymer or animal models. A probabilistic approach. *Markov Processes and Related Fields.* 4 (1998) 479–497.

[8] R. Fernández, P. A. Ferrari and N. L. Garcia, Loss network representation of Peierls contours. Preprint. Los Alamos Archive math. PR/9806131, (1998).

[9] A. Ganesh, B. M. Hambly, N. O’Connell, D. Stark and P. J. Upton, Poissonian behavior of Ising spin systems in an external field. *J. Stat. Phys.* (2000).

[10] C. M. Stein, *Approximate Computations of Expectations* (IMS, Hayward, Calif. 1986).