Effective Electric and Magnetic Local Actions for
$U_e(1) \times U_g(1)$ Electromagnetism:
Hodge Duality and Zero-Field Equation

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November 11, 2018

Abstract

Electromagnetism, Dynamical Symmetry Breaking, pseudo-photons, Zero Field Equation; packs: 03.50.De,11.15,-q,40. In this work it is considered a mechanism of dynamical symmetry breaking for extended $U_e(1) \times U_g(1)$ containing, one vector gauge field $A$ (photon) and one pseudo-vector gauge field $C$ (pseudo-photon). By choosing a particular solution of the equations of motion we obtain a functional description of either field in terms of the other one. In this way we obtain non-trivial configurations $C = C(A)$ or $A = A(C)$ such that, in the effective broken theories containing only one gauge field, the usual field discontinuities in the presence of both electric and magnetic monopoles (Dirac string or Wu-Yang non-trivial fiber-bundle) are absent. These fields regularity is achieved through corrections, inherited from the unbroken theory, to the standard definitions of electromagnetic fields and four-currents. Based in these results we also demonstrate non-triviality of the unbroken theory, in the presence of both electric and magnetic four-currents, there are no trivial solutions compatible with the equations of motion for both fields. Moreover we demonstrate that, although in complete agreement with the Maxwell equations, extended $U_e(1) \times U_g(1)$ electromagnetism is not equivalent to the standard variational $U(1)$ Maxwell electromagnetism. We further show that the unbroken theory is invariant under a combination of Hodge dualities for the gauge connections, $F \to \hat{\epsilon} \ast G$ and $G \to -\hat{\epsilon} \ast F$ ($\hat{\epsilon} = \pm 1$), that has as self-dual point, the field configuration corresponding to the zero-field equation $\tilde{C} = -\hat{\epsilon} A$. In addition this condition has the particularity of being the only configuration compatible with the mechanism of dynamical symmetry breaking proposed here that is also gauge invariant in the unbroken theory. These characteristics justify the zero-field equation as being a preferred configuration.
1 Introduction

The inclusion of magnetic charge in the Maxwell equations is justified by being the only theoretical explanation for quantization of electric charge due to the Dirac quantization condition $eg = n$ (Dirac). However the introduction of magnetic charge in single $U(1)$ photon theories implies the existence of non-physical extended singularities known as the Dirac string (Dirac) or the Wu-Yang fiber bundle (Wu & Yang 1975). These singularities are due to the violation of the Bianchi identities for the gauge field along a line (string) or a plane (the gluing of the bundles). A possible approach that eliminates extended singularities is to consider one extra auxiliary gauge field $C$. This framework has been originally put forward by Cabibbo and Ferrari (Cabibbo & Ferrari 1962) and further developed by Schwinger (Schwinger 1966). More recently, the extension to a theory with gauge group $U_c(1) \times U_g(1)$, containing both a photon $A$ and a pseudo-photon $C$, have been studied in (Singleton 1995, Cardoso de Mello & al. 1996, Berkovits 1996, Carneiro 1997, Castelo Ferreira 2005) and hold regular field solutions in the presence of both electric and magnetic monopoles. Independently of the existence of magnetic monopoles the inclusion of a pseudo-photon is also justified in the presence of external non-trivial field configurations that violate the gauge fields Bianchi identities which, at variational level, cannot be accounted by the Maxwell action for a single photon (Castelo Ferreira 2006). These studies indicate that the pseudo-photon may be a truly physical field, instead of a mathematical artifact, or a real physical particle that have so far not been directly detected. If the pseudo-photon is a real physical field it remains the pertinent question weather such theory has some sort of phase transition and which physical systems are in an unbroken phase, where both $A$ and $C$ fields are present, or in a broken phase of the theory were only the standard photon or pseudo-photon are present. Already available theoretical results, indicate that exist both physical systems which exhibit an unbroken phase (electromagnetism in the presence of non-regular external electromagnetic fields (Castelo Ferreira 2006)), physical systems which exhibit a broken electric phase (Anderson-Schwinger mechanism (Schwinger 1962, Anderson 1963, Proca 1988) for plasmon mass generation (Castelo Ferreira & Mendonça 2006, Mendonça & Castelo Ferreira 2006)) and systems that exhibit a broken magnetic phase (fractional Hall effect (Tsui & Anderson 1963, Proca 1988) for plasmon mass generation (Castelo Ferreira & Mendonça 2006, Mendonça & Castelo Ferreira 2006)).

The action with gauge symmetry $U_c(1) \times U_g(1)$ for electromagnetism, with one gauge vector field $A$ and one gauge pseudo-vector field $C$, coupled to electric and magnetic sources is given by $S = S_0 + S_{\text{Sources}}$ (Cardoso de Mello & al. 1996, Castelo Ferreira 2005),

$$S_0 = - \int_M \left[ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} - \frac{1}{4} \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} G_{\lambda\rho} \right],$$

where the gauge connections are $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $G_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$, and

$$S_{\text{Sources}} = - \int_M \left[ (A_\mu - \dot{\tilde{C}}_\mu) J^{\mu}_e - (\dot{\epsilon} \tilde{C}_\mu + \dot{\bar{A}}_\mu) J^{\mu}_g \right].$$

$\dot{A}$ and $\dot{C}$ do not constitute independent fields and are defined by the differential equations (Castelo Ferreira 2006)

$$\dot{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}, \quad \dot{G}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} G_{\lambda\rho},$$

where the dual gauge connections are $\dot{F}_{\mu\nu} = \partial_\nu \tilde{A}_\mu - \partial_\mu \tilde{A}_\nu$ and $\dot{G}_{\mu\nu} = \partial_\nu \tilde{C}_\mu - \partial_\mu \tilde{C}_\nu$. The electromagnetic physical fields in the unbroken theory corresponding to the above action are defined as (Singleton 1995, Castelo Ferreira 2005)

$$E^i = F^{0i} - \frac{1}{2} \epsilon^{0ijk} G_{jk}, \quad B^i = \dot{\epsilon} G^{0i} + \frac{1}{2} \epsilon^{0ijk} F_{jk}.$$

In the original works (Cabibbo & Ferrarri 1962, Schwinger 1966) the pseudo-photon was considered to be a non-physical auxiliary field, which through an appropriate constraint, would be effectively excluded from the
theory. The specific constraint considered in these original works, as well as in subsequent works (Cardoso de Mello et al. 1996, Berkovits 1996, Carneiro 1997) is the zero-field condition

\[ G^{\mu\nu} = \frac{\hat{\epsilon}}{2} \epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho} \quad \Leftrightarrow \quad \tilde{C}_\mu = -\hat{\epsilon} A_\mu . \]  

(5)

Alternative approaches were further developed in (Zwanzinger 1968, Brandt et al. 1979), with the drawback of not preserving either space isotropy or Lorentz invariance, as well as to consider a very massive pseudo-photons (Singleton 1995). In this work our aim is to investigate if effective theories with one single gauge field can be obtained from a theory with photons and pseudo-photons through some mechanism of dynamical symmetry breaking. In particular if the zero-field equation (5) is applicable in this framework and what are the consequences of such mechanism at the level of the effective broken theories. We are going to work mainly at variational level, having as starting point the above action

\[ S = S_0 + S_{\text{Sources}} \]  

given in (1) and (2), and using the equations of motion to find appropriate field configurations.

2 Effective Theories in the Absence of Sources

In this section we derive the equations of motion for action (1) and show that the generic non-trivial solutions kill half of the degrees of freedom of the full theory rendering either the standard Maxwell action, with only one gauge field, or its magnetic counterpart. As we will show both effective actions differ by a minus sign such that the pseudo-vector gauge field is a ghost field, as already shown in (Castelo Ferreira 2006).

For non-regular gauge fields the equations of motion obtained by varying the action (1) with respect to \( C_\nu \) and \( A_\nu \) are

\[ \partial_\mu G^{\mu\nu} = -\frac{\hat{\epsilon}}{2} \epsilon^{\mu\nu\lambda\rho} \partial_\mu F_{\lambda\rho} , \quad \partial_\mu F^{\mu\nu} = \frac{\hat{\epsilon}}{2} \epsilon^{\mu\nu\lambda\rho} \partial_\mu G_{\lambda\rho} . \]  

(6)

We note that these two equations are linear dependent, in particular are Hodge conjugated to each other, hence they constitute only 4 independent equations corresponding to \( \nu = 0, 1, 2, 3 \). Also a direct conclusion from the above equations is that the Bianchi identities for each field are related to the equations of motion for the other field. It is this fact that in the presence of external non-regular electromagnetic fields allow the induced electromagnetic fields to be expressed in terms of regular gauge fields only. External fields have been studied in detail in (Castelo Ferreira 2006), in the present study we consider only external electric and magnetic four-currents. We will first address the case of non-regular fields and later address the case of regular fields.

2.1 Electric Solutions for Non-Regular Fields

The non-trivial generic solutions for the equations of motion (6) are obtained by direct integration of the equations and are defined up to a closed 2-form, i.e. an antisymmetric tensor \( f_{\mu\nu} \), defined in terms of a regular field \( a_\mu \),

\[ G^{\mu\nu} = -\frac{\hat{\epsilon}}{2} \epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho} + \frac{\alpha_\epsilon}{2} \epsilon^{\mu\nu\lambda\rho} f_{\lambda\rho} , \quad f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu , \quad \epsilon^{\mu\nu\lambda\rho} \partial_\mu f_{\lambda\rho} = 0 . \]  

(7)

In these solutions we have considered, for convenience, a dimensionless constant that, up to rescaling of the fields \( a_\mu \), can be set to unity \( \alpha_\epsilon = \pm 1 \). By replacing the solution (7) in the original action (1), we obtain for each of the terms constituting the action, the following expressions

\[ + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} , \quad -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\alpha_\epsilon}{4} f_{\mu\nu} f^{\mu\nu} - \frac{2}{4} \hat{\epsilon} \alpha_\epsilon F_{\mu\nu} f^{\mu\nu} , \quad \frac{\hat{\epsilon}}{4} \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} G_{\lambda\rho} = -\frac{2}{4} F_{\mu\nu} F^{\mu\nu} + \frac{2}{4} \alpha_\epsilon F_{\mu\nu} f^{\mu\nu} . \]  

(8)

Here we used the identity

\[ \epsilon^{\mu\nu\delta\rho} X_{\delta\rho} \epsilon_{\mu\nu} \tilde{\gamma}_{\mu\nu} \tilde{Y}_{\mu\nu} = -4 X_{\mu\nu} Y^{\mu\nu} , \]  

(9)
valid for antisymmetric rank two tensors $X$ and $Y$ in a flat $3 + 1$-dimensional Minkowski space. Replacing the expressions (5) in the unbroken action (1) and electromagnetic field definitions (4) we obtain the effective action and electromagnetic field definitions

$$S_{\text{Electric}} = - \int_M \frac{1}{4} f_{\mu\nu} f^{\mu\nu} , \quad E^i = \hat{c} \alpha e f^{0i} , \quad B^i = \hat{c} \alpha e \epsilon^{0ijk} f_{jk} .$$

(10)

This action is recognized as the standard Maxwell action for the gauge field $\alpha \mu$ with gauge symmetry $U(1)$ and, for $\hat{c} \alpha e = +1$, the field definitions correspond to the standard ones of electromagnetism. The equations of motion for the broken theory are now given by

$$\partial_{\mu} f^{\mu\nu} = 0 .$$

(11)

These equations can consistently be obtained by direct replacement of the solution (7) in the equation of motion (6), or by a variation of the effective broken action (10) with respect to the field $\alpha \nu$. The equations of motion for $\alpha \nu$ (11), together with the respective Bianchi identities (7), constitute the standard Maxwell equations.

### 2.2 Magnetic Solutions for Non-Regular Fields

To obtain the effective magnetic solution we use the same approach. The non-trivial solution of (6) for $F_{\mu\nu}$ is, up to a generic antisymmetric tensor $g_{\mu\nu}$ defined in terms of a regular gauge field $c_{\mu}$, given by

$$F_{\mu\nu} = \frac{\hat{c}}{2} \epsilon^{\mu\nu\lambda\rho} g_{\lambda\rho} + \frac{\alpha_g}{2} \epsilon^{\mu\nu\lambda\rho} g_{\lambda\rho} , \quad g_{\mu\nu} = \partial_{\mu} c_{\nu} - \partial_{\nu} c_{\mu} , \quad \epsilon^{\mu\nu\lambda\rho} \partial_{\nu} g_{\lambda\rho} = 0 .$$

(12)

Again, for convenience, we considered a constant $\alpha_g = \pm 1$. Replacing this solution in the original unbroken action (1), we obtain the following expressions for each of the terms

$$- \frac{1}{4} G_{\mu\nu} G^{\mu\nu} = - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} ,$$

$$+ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} - \frac{\alpha_g}{2} g_{\mu\nu} g^{\mu\nu} - \frac{\hat{c}}{4} \alpha_g \epsilon G_{\mu\nu} g^{\mu\nu} ,$$

$$- \frac{\hat{c}}{4} \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} G_{\lambda\rho} = + \frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \frac{\alpha_g}{4} G_{\mu\nu} g^{\mu\nu} .$$

(13)

where again we used the identity (9). Hence from (11) we obtain the effective action, and from (4) the electromagnetic fields definitions

$$S_{\text{Magnetic}} = + \int_M \frac{1}{4} g_{\mu\nu} g^{\mu\nu} , \quad E^i = \frac{\alpha_g}{2} \epsilon^{0ijk} g_{jk} , \quad B^i = - \alpha_g g^{0i} .$$

(14)

This action has the opposite sign than the usual Maxwell action. At classical level this is not relevant, however upon quantization it renders negative energy eigenstates, thus the field $c_{\mu}$ is interpreted as a ghost field (Castelo Ferreira 2006) with gauge symmetry $U(1)$. By comparing these definitions with the original field definitions (4) for the unbroken theory, we conclude that for $\alpha_g = - \hat{c}$ are obtained the standard field definitions. The equations of motion are given by

$$\partial_{\mu} g^{\mu\nu} = 0 ,$$

(15)

and are obtained consistently, either by direct replacement of the solution (12) in the equation of motion (6), or by a variation of the effective broken action (14) with respect to the field $c_{\nu}$. The equations of motions for $c_{\nu}$ (15), together with the respective Bianchi identities (12), constitute the magnetic counterpart of the Maxwell equations.

### 2.3 Solutions for Regular Fields

For regular gauge fields the Hopf term in the action (11) is a total derivative and does not contribute to the equations of motion. Therefore, upon variation of the action (11) with respect to the gauge fields $C_{\nu}$ and $A_{\nu}$, we obtain the equations

$$\partial_{\mu} G^{\mu\nu} = 0 , \quad \partial_{\mu} F^{\mu\nu} = 0 .$$

(16)
Due to the fields being regular these equations are supplemented by the respective Bianchi identities

\[ \epsilon^{\mu \nu \lambda \rho} \partial_{\mu} G_{\lambda \rho} = 0 \, , \quad \epsilon^{\mu \nu \lambda \rho} \partial_{\mu} F_{\lambda \rho} = 0 \, . \]  

(17)

The most generic solution for both equations of motion (16) are, respectively,

\[ G^{\mu \nu} = \epsilon^{\mu \nu \delta \rho} h^{\delta \rho} \, , \quad F^{\mu \nu} = \epsilon^{\mu \nu \delta \rho} h^{\delta \rho} \, . \]  

(18)

for generic rank two tensors obeying \[ \epsilon^{\mu \nu \delta \rho} \partial_{\nu} h^{\delta \rho} = \epsilon^{\mu \nu \delta \rho} \partial_{\nu} h^{\delta \rho} = 0 \, . \]  

We are considering regular gauge fields only, hence without loss of generality, we can consider the field redefinitions

\[ h^{\beta \rho} = + \hat{\epsilon} g^{\beta \rho} \, , \quad h^{\beta \rho} = - \hat{\epsilon} f^{\beta \rho} \, . \]  

(19)

with \( f^{\beta \rho} \) and \( g^{\beta \rho} \) defined as in (7) and (12). In this way we retrieve the same expressions for the solutions that we have obtained for non-regular gauge fields in (7) and (12), respectively. The remaining of the proof follows in the same manner as for non-regular gauge fields, such that we obtain the respective effective electric action (10) and effective magnetic action (14). However it is important to stress that here, the solutions (19) for each gauge fields \( C \) and \( A \), are regular and the coupling of both sectors can only be fully justified by demanding consistence of the theory with the existence of non-regular gauge fields (Castelo Ferreira 2006) (or/and magnetic monopoles (Castelo Ferreira 2006)). To finalize our discussion we recall that the standard definitions of the electromagnetic fields and actions in electromagnetism correspond to setting the integration constants \( \alpha_e \) and \( \alpha_g \) to be

\[ \alpha_e = \hat{\epsilon} \, , \quad \alpha_g = - \hat{\epsilon} \, . \]  

(20)

Here \( \hat{\epsilon} \) stands for the relative terms of the Hopf term in (1).

2.4 On Trivial Solutions

So far we have not addressed the trivial solutions of the equations of motion. In addition to the non-trivial solutions (7) and (12) we can consider the cases \( \alpha_e = 0 \) and \( \alpha_g = 0 \) which correspond to

\[ \tilde{C}_{\mu} = + \hat{\epsilon} A_{\mu} \, . \]  

(21)

This solution holds a trivially null effective action \( S_{\text{eff}} = 0 \), which implies killing all the dynamics. Moreover, without any external sources or external fields, this solution is actually expected. The theory does not have any dynamics and the action is null. Hence the non-trivial solutions (7) and (12), although formally obeying the equations of motion (being an extrema of the action), cannot be physically justified as preferred in relation to the trivial solutions (21). In the next section we include external source terms which, as we will show, justify the non-trivial solutions as the only allowed ones, the trivial solution will no-longer be extremum of the action.

3 Inclusion of Source Terms

We proceed now to compute the solutions of the equations of motion in the presence of both electric and magnetic four-current densities. For non-regular gauge fields, upon variation of the full action \( S = S_0 + S_{\text{Sources}} \) given by (11) and (2) with respect to \( C_{\nu} \) and \( A_{\nu} \), we obtain the equations

\[ \partial_{\mu} G^{\mu \nu} = \hat{\epsilon} J_{\nu}^{\rho} - \frac{\hat{\epsilon}}{2} \epsilon^{\mu \nu \lambda \rho} \partial_{\mu} F_{\lambda \rho} \, , \]  

(22)

\[ \partial_{\mu} F^{\mu \nu} = J_{\nu}^{\rho} + \frac{\hat{\epsilon}}{2} \epsilon^{\mu \nu \lambda \rho} \partial_{\mu} G_{\lambda \rho} \, . \]  

(23)

We note that as opposed to the equations of motion (6), in the absence of sources, these equations are no longer Hodge conjugated to each other. Then, in order to obtain effective electric and magnetic theories, one has to choose which one to solve. In order to integrate (hence lower the order) of these equation we are
re-writing the action in terms of gauge invariant quantities (Schwinger 1951). So let us consider an Hodge decomposition for the currents that obey the continuity equations $\partial_\mu J^\mu_c = \partial_\mu J^\mu_g = 0$ required for gauge invariance of the full action,

\[
J^\mu_c = \frac{1}{2} \epsilon^{\mu\nu\delta\rho} \partial_\nu \phi^\delta_{\rho} + c^\mu_c, \quad \partial_\mu \phi^c_{\mu\nu} = 0, \\
J^\mu_g = \frac{1}{2} \epsilon^{\mu\nu\delta\rho} \partial_\nu \phi^\delta_{\rho} + c^\mu_g, \quad \partial_\mu \phi^g_{\mu\nu} = 0.
\]

(24)

Here $\phi_c$ and $\phi_g$ are close 2-forms and $c_c$ and $c_g$ are constant background currents. In the following we address local current densities only, hence we set $c_c = c_g = 0$. For constant backgrounds due either to topological non-trivial charge configurations, large gauge transformations or Wilson lines we generally have $c_c \neq 0$ and $c_g \neq 0$. See, for example (Schwinger 1962, Anderson 1963, Proca 1988, Castelo Ferreira & Mendonça 2006, Mendonça & Castelo Ferreira 2006), where a Proca mass for the surviving field is generated in presence of background currents. Using the above Hodge decompositions (24) we can rewrite the action terms containing the current densities in (22) as

\[
S_{\text{Sources}} = - \int_M \left[ \frac{1}{4} \epsilon^{\mu\nu\delta\rho} F_{\mu\nu} \phi^\delta_{\rho} + \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho} + \frac{\alpha_c}{2} \epsilon^{\mu\nu\lambda\rho} f_{\lambda\rho} \right] - \left( \frac{1}{4} \epsilon^{\mu\nu\delta\rho} G_{\mu\nu} \phi^\delta_{\rho} \right)
\]

(25)

In deriving this expression we have performed an integration by parts and discarded boundary terms. This action is explicitly gauge invariant, it only depends on the gauge connections, therefore the equations of motion can be written in terms of the gauge connections instead of its derivatives. For completeness in our discussion, it is also interesting to note that at classical level one can consider the degrees of freedom to be gauge connections instead of its derivatives. For completeness in our discussion, it is also interesting to note that at classical level one can consider the degrees of freedom to be gauge connections instead of the gauge fields $A$ and $C$.

3.1 Electrical Solutions for Non-Regular Fields

Using the Hodge decomposition (24) for the currents, we can integrate equation (22) obtaining

\[
G^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} G_{\lambda\rho} = \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho} + \frac{\alpha_c}{2} \epsilon^{\mu\nu\lambda\rho} f_{\lambda\rho}.
\]

(26)

Here $f$ is defined in terms of a regular field $a$, as given in equation (1) and, again for convenience, we consider a constant $\alpha_c = \pm 1$. Considering solution (26) and the Hodge decomposition (24) for the currents, we obtain for each of the terms constituting the action $S = S_0 + S_{\text{Sources}}$ given by (1) and (25), the following expressions

\[
\begin{align*}
+ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} &= + \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \\
- \frac{1}{4} G_{\mu\nu} G^{\mu\nu} &= + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\alpha_c}{4} f_{\mu\nu} f^{\mu\nu} - \frac{2 \hat{c} e}{4} F_{\mu\nu} f^{\mu\nu}, \\
- \frac{\hat{c}}{4} \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} G_{\lambda\rho} &= - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{2 \hat{c} e}{4} F_{\mu\nu} f^{\mu\nu}, \\
- \frac{\hat{c}}{4} \epsilon^{\mu\nu\lambda\rho} G_{\mu\nu} G_{\lambda\rho} &= - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{2 \hat{c} e}{4} F_{\mu\nu} f^{\mu\nu} + \frac{2}{4} f_{\mu\nu} f^{\mu\nu}, \\
- \frac{\hat{c}}{4} \epsilon^{\mu\nu\lambda\rho} G_{\mu\nu} G_{\lambda\rho} &= - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{2 \hat{c} e}{4} F_{\mu\nu} f^{\mu\nu}, \\
+ \frac{1}{4} \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} \phi^\lambda_{\rho} &= + \frac{1}{4} \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} \phi^\lambda_{\rho} - \frac{1}{4} \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} \phi^\lambda_{\rho} + \frac{\hat{c} e}{4} \epsilon^{\mu\nu\lambda\rho} f_{\mu\nu} \phi^\lambda_{\rho}, \\
- \frac{\hat{c}}{4} \epsilon^{\mu\nu\lambda\rho} G_{\mu\nu} \phi^\lambda_{\rho} &= - \frac{1}{4} \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} \phi^\lambda_{\rho} + \frac{\hat{c} e}{4} \epsilon^{\mu\nu\lambda\rho} f_{\mu\nu} \phi^\lambda_{\rho}, \\
+ \frac{\hat{c}}{4} \epsilon^{\mu\nu\lambda\rho} G_{\mu\nu} \phi^\lambda_{\rho} &= + \frac{1}{4} \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} \phi^\lambda_{\rho} - \frac{1}{4} \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} \phi^\lambda_{\rho}, \\
+ \frac{\hat{c}}{4} \epsilon^{\mu\nu\lambda\rho} G_{\mu\nu} \phi^\lambda_{\rho} &= + \frac{1}{4} \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} \phi^\lambda_{\rho}.
\end{align*}
\]

(27)
Where, again, we used identity (9). Hence the effective electric action is

\[
S_{\text{Electric}} = - \int_M \left[ \frac{1}{4} f_{\mu\nu} f^{\mu\nu} + \frac{\hat{\epsilon} \alpha_e}{4} \epsilon^{\mu\nu\rho\delta} f_{\mu\nu} \phi_{\delta\rho} + \frac{\hat{\epsilon} \alpha_e}{4} f^{\mu\nu} \phi_{\delta\rho} 
+ \frac{3}{4} \phi^{\mu\nu} \phi_{\delta\rho} + \frac{1}{4} \epsilon^{\mu\nu\rho\delta} \phi_{\delta\rho} \phi_{\delta\rho} \right].
\]  
(28)

The last two terms contribute to the vacuum energy and, at classical level, as long as we are not dealing with gravity, are irrelevant. For \( \hat{\epsilon} \alpha_e = +1 \), the first three terms, are recognized as the Maxwell action in the presence of local electric and magnetic charges (Schwinger 1951). Also it is important to stress that for regular \( \alpha \) and the currents obeying the Hodge decomposition (24), the third term is a total derivative. Then we obtain the effective action (up to a vacuum energy shift) and the effective electromagnetic fields definitions

\[
S_{\text{Electric}} = - \int_M \left[ \frac{1}{4} f_{\mu\nu} f^{\mu\nu} + a_\mu J_\mu \right],
E^i = \hat{\epsilon} \alpha_e f^{0i} + \phi^0 0i, \quad B^i = \hat{\epsilon} \alpha_e \epsilon^{0ijk} f_{jk} + \frac{1}{2} \epsilon^{0ijk} \phi^j_0.
\]  
(29)

In deriving these results we have integrated by parts the second and third terms of (28) and considered the identity \( \epsilon^{0ijk} 0j_k 0i \phi^0_0 = -2 \phi^0_0 \). This is the standard Maxwell action in the presence of electric sources and the field expressions are obtained by the direct replacement of the solution (30) for \( G \) in the fields definitions (1) of the unbroken theory. For \( \hat{\epsilon} \alpha_e = +1 \) these expressions correspond to the standard definitions in electromagnetism plus a contribution from the magnetic sources. This means that the magnetic current effects are still present in the effective theory, not at the level of the action but at the level of the electromagnetic fields definitions which are inherit from the unbroken theory.

### 3.2 Magnetic Solutions for Non-Regular Fields

Using the Hodge decomposition (24) for the currents we obtain that the generic solution for the equations of motion (29) is

\[
F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} \phi_\lambda^\rho + \frac{\hat{\epsilon} \alpha_e}{2} \epsilon^{\mu\nu\lambda\rho} G_{\lambda\rho} + \frac{\alpha_g}{4} g^{\mu\nu} g_{\lambda\rho}.
\]  
(30)

Again the field \( g \) is defined in terms of a regular field \( c \), as given in equation (12), and once more for convenience, we consider a constant \( \alpha_g = \pm 1 \). Replacing the solution (30) in the action \( S = S_0 + S_{\text{Sources}} \) given by (1) and (29) we obtain for each of the terms constituting the action the following expressions

\[
- \frac{1}{4} G_{\mu\nu} G^{\mu\nu} = - \frac{1}{4} G_{\mu\nu} G^{\mu\nu},
+ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} - \frac{\alpha_g}{4} g_{\mu\nu} g^{\mu\nu} - \frac{2 \hat{\epsilon} \alpha_g}{4} G_{\mu\nu} g^{\mu\nu},
- \hat{\epsilon} \frac{1}{4} \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} G_{\lambda\rho} = + \frac{2}{4} G_{\mu\nu} G^{\mu\nu} + \frac{\hat{\epsilon} \alpha_g}{4} g_{\mu\nu} g^{\mu\nu} + \frac{2 \hat{\epsilon}}{4} G_{\mu\nu} \phi_\rho^\mu \phi^\rho_\nu,
+ \frac{1}{4} \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} \phi^{\rho_\lambda}_{\rho_\nu} = - \frac{1}{2} \phi_\lambda^\rho \phi_\rho^\lambda - \frac{1}{4} \epsilon^{\mu\nu\lambda\rho} G_{\mu\nu} \phi_\rho^\lambda,
\]  
(31)

\[
- \frac{\hat{\epsilon}}{4} \epsilon^{\mu\nu\lambda\rho} G_{\mu\nu} \phi_\rho^\lambda = - \frac{\hat{\epsilon}}{4} \epsilon^{\mu\nu\lambda\rho} G_{\mu\nu} \phi_\rho^\lambda,
+ \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} \phi_\rho^\lambda = + \frac{\hat{\epsilon}}{4} \epsilon^{\mu\nu\lambda\rho} G_{\mu\nu} \phi_\rho^\lambda + \frac{\alpha_g}{4} \epsilon^{\mu\nu\lambda\rho} g_{\mu\nu} \phi_\rho^\lambda.
\]

Where again we used the identity (9). Hence we obtain the effective magnetic action

\[
S_{\text{Magnetic}} = - \int_M \left[ - \frac{1}{4} g_{\mu\nu} g^{\mu\nu} + \frac{\alpha_g}{4} \epsilon^{\mu\nu\lambda\rho} g_{\mu\nu} \phi_\rho^\lambda - \alpha_g g^{\mu\nu} \phi_\rho^\lambda 
- \frac{3}{4} \phi_\rho^\lambda \phi_\rho^\mu + \frac{1}{4} \epsilon^{\mu\nu\lambda\rho} \phi_\rho^\lambda \phi_\rho^\mu \right].
\]  
(32)
As in the electric case, the last two terms contribute to the vacuum energy. For \( \alpha_g = -\hat{\epsilon} \), the first three terms are recognized as the magnetic counterpart of the Maxwell action in the presence of local electric and magnetic charges (Castelo Ferreira 2006). For \( c \) regular and the currents obeying the Hodge decomposition (21) the third term is a total derivative. Then we obtain the effective action and the electromagnetic field definitions

\[
S_{\text{Magnetic}} = \int_M \left[ \frac{1}{4} g_{\mu\nu} e^{\mu\nu} + \hat{\epsilon} \phi_\mu J_\mu \right],
\]

for regular gauge fields the equations of motion for the action \( S = S_0 + S_{\text{Sources}} \) given by (1) and (2) are obtained by a variation with respect to \( C \) and \( A \), holding the equations

\[
\partial_\mu G^{\mu\nu} = \hat{\epsilon} J_\nu^g, \quad \partial_\mu F^{\mu\nu} = J_\nu^e.
\]

Using the current decompositions (24) the generic solutions for these equations are, respectively

\[
G^{\mu\nu} = -\hat{\epsilon} e^{\mu\nu\rho\sigma} \phi_\rho^\sigma, \quad F^{\mu\nu} = \frac{1}{2} e^{\mu\nu\rho\sigma} \phi_\rho^\sigma,
\]

where \( H_g \) and \( H_e \) are closed 2-forms such that \( \partial_\mu H_g^{\mu\nu} = \partial_\mu H_e^{\mu\nu} = 0 \). As we did in section 2 we can consider a field redefinition of the form

\[
H_g^{\mu\nu} = -\hat{\epsilon} e^{\mu\nu\rho\sigma} \phi_\rho^\sigma f_\rho^\sigma, \quad H_e^{\mu\nu} = \frac{1}{2} e^{\mu\nu\rho\sigma} \phi_\rho^\sigma g_\rho^\sigma,
\]

such that we obtain the same expressions for the solutions of non-regular gauge fields, as given in (26) and (30). The remaining of the proof follows in the same manner as for non-regular gauge fields, such that we obtain the same effective actions and electromagnetic field redefinitions given in (29) and (33). Again we stress that the coupling between both sectors is only fully justified by considering compatibility with the existence of non-regular field configurations.

3.4 Maxwell Equations and Current Densities Definitions

Let us address what are the consequences of the above results at the level of the Maxwell equations, in particular of the physical fields definitions (29) in the effective theory that, as we have seen, are inherit from the physical field definitions (4) of the original unbroken theory. In the electric broken theory the equations of motion and the Bianchi identities are given, respectively, by

\[
\partial_\mu f^{\mu\nu} = J_\nu^e, \quad \epsilon^{\mu\nu\lambda\rho} \partial_\mu f_{\lambda\rho} = 0.
\]

The equations of motion are obtained consistently either by direct replacement of the solution (26) in the equation of motion (1) or by a variation of the effective action (29) with respect to the field \( a_\nu \) while the Bianchi identities are due to the field \( a_\nu \) being regular by construction as given in (25) and (30). Equations (37) correspond to the 8 Maxwell equations \( (\nu = 0, 1, 2, 3) \), in order to rewrite these equations in terms of the electric and magnetic fields definitions (29) let us define the tensor \( F \)

\[
F^{\mu\nu} = E^{\mu\nu}, \quad F^{ij} = -\frac{1}{2} e^{ijk} B^k, \quad f^{\mu\nu} = F^{\mu\nu} - \phi_\mu^g \phi_\mu^g.
\]

Hence the Maxwell equations are written as

\[
\partial_\mu F^{\mu\nu} = J_\nu^e, \quad \epsilon^{\mu\nu\lambda\rho} \partial_\mu F_{\lambda\rho} = J_\nu^g.
\]
which are straightforwardly recognized as the generalized standard Maxwell equations in the presence of magnetic currents (Jackson 1975). In the first equation the term dependent on the magnetic current is null according to the Hodge decomposition \((24)\), i.e. \(\partial_q f_{\mu\nu} = 0\). In the second equation we have used the Hodge decomposition \((24)\) in order to rewrite the equation in terms of the magnetic current \(J^\mu_g = \epsilon^{\mu\nu\lambda\rho} \partial_q f_{\nu\rho}/2\). So we manage to recover the effects of magnetic currents in the broken theory at the level of the Maxwell equations. The gauge fields are regular and the re-appearance of magnetic currents is due to the electric and magnetic field definitions \((29)\). Although this construction may seem to consist in a simple redefinition of the gauge field \(A_\mu\), it is not so. In order to see it explicitly, let us note that \(\phi^g_{\mu\nu}\) could be written in terms of a four-vector field \(\phi^g_i\), obeying the following equations

\[
\phi^g_{\mu\nu} = \partial_\mu \phi^g_\nu - \partial_\nu \phi^g_\mu, \quad \epsilon^{\mu\nu\lambda\rho} \partial_q f_{\lambda\rho}/2 = J^\mu_g, \quad \partial_q \phi^g_\mu = \partial_\nu \phi^g_\mu.
\]

(40)

The second and third equations are obtained directly from the Hodge decomposition \((24)\) and are due to requiring compatible with the continuity equation \(\partial_\mu J^\mu_g\) (hence gauge invariance). We readily conclude that indeed, the dependence on \(\phi^g_{\mu\nu}\) of the field definitions \((29)\), cannot possible be eliminated by a redefinition of the gauge field \(a_\mu\), or equivalently of the gauge connection \(f_{\mu\nu}\). By construction \(a_\mu\) is a regular field obeying the Bianchi identities \((\epsilon^{\mu\nu\lambda\rho} \partial_q f_{\lambda\rho} = 0)\), while the field \(\phi^g_i\) is non-regular, its Bianchi identity is violated as given by the second equation in \((40)\). Given these results, and for completeness, we also note that \(f_{\mu\nu}\) and \(\phi^g_{\mu\nu}\) correspond, generally, to the two components of an Hodge decomposition of the tensor \(F_{\mu\nu}\) defined in equation \((28)\), that can also be defined in terms of a general field \(A_\mu = a_\mu + \phi^g_i\) such that its regular and non-regular parts are respectively \(a_\mu\) and \(\phi^g_i\). If one considers this field to be the physical gauge field we would simply obtain the original singularities in the gauge fields (the Dirac string and Wu-Yang fiber-bundle). Let us stress the crucial differences between our approach and the approaches which consider the gauge field \(A\) as the physical field: In the present construction we simply are not allowed to redefine the physical field \(a_\mu \rightarrow a_\mu + \phi^g_i\), by construction this field is regular. There is also another important point concerning our last comment, the full unbroken theory contains both photons and pseudo-photons degrees of freedom, while in any of the effective broken theories half of the degrees of freedom are truncated, nevertheless the physical content in the broken theories is inherited from the unbroken theory and we cannot freely choose what our fields are, they are constraint by the field configurations used in the dynamical symmetry breaking. Moreover, by the end of this section, we will have prove non-triviality of the full unbroken theory in the presence of both electric and magnetic monopoles which implies that it is not possible to describe the theory by the standard \(U(1)\) variational Maxwell theory. These crucial remarks clearly distinguish our broken effective theories from standard variational electromagnetism in the presence of magnetic monopoles. By construction the singularities are not included, and cannot be included, in the gauge fields.

In the effective theory the electric and magnetic charge \((Q_e\) and \(Q_g\)) and the electric and magnetic 3-current fluxes \((j^e_{i\phi}\) and \(j_g^{i\phi}\)) are defined as

\[
Q_e = \int_M dx^3 \partial_i f^{0i} + \frac{1}{2} \int_M dx^3 \partial_q f^{0q} \phi^g_{0i} = \int_M dx^2 \partial_j f^{0j} n_i = \int_M dx^3 J^0_e, \\
\rho^i_{e\phi} = \int_M dx^3 \partial_j f^{ij} + \frac{1}{2} \int_M dx^3 \partial_q f^{ij} \phi^g_{0j} = \int_M dx^2 \partial_k f^{ij} n_i = \int_M dx^3 J^i_e/2, \\
Q_g = \int_M dx^3 \epsilon^{ijk} \partial_j f_{0k} + \frac{1}{2} \int_M dx^3 \epsilon^{ijk} \partial_q f^{0j} \phi^g_{0k} = \frac{1}{2} \int_M dx^2 \epsilon^{ijk} \phi^g_{jk} n_i = \int_M dx^3 J^0_g, \\
\rho^i_{g\phi} = \int_M dx^3 \left[ \epsilon^{ijk} \partial_0 f_{jk} - 2 \epsilon^{ijk} \partial_j f_{0k} \right] + \frac{1}{2} \int_M dx^3 \left[ \epsilon^{ijk} \partial_0 \phi^g_{jk} - 2 \epsilon^{ijk} \partial_j \phi^g_{0k} \right] \\
= \frac{1}{2} \int_M dx^3 \epsilon^{ijk} \phi^g_{jk} - \int_0 \partial_M dx^2 \epsilon^{ijk} \phi^g_{0k} n_j = \int_M dx^3 J^i_g/2.
\]

(41)

This result is consistent with the previous results. Electric charge corresponds, as usual, to the \(U(1)\) group charge, however we must stress that magnetic charge definition is inherit from the unbroken theory and cannot, in the broken theory, be interpreted as group topological charge. Nevertheless the canonical variables of the theory are going to be sensitive to these currents. The canonical momenta of the unbroken theory and the respective effective canonical momenta for the broken electric theory are

\[
\pi^i_A = F^{0i} - \frac{\epsilon}{2} \epsilon^{0ijk} G_{jk} = E^i, \quad \Rightarrow \quad \pi^i_A = e \alpha_e f^{0i} + \phi^g_{0i} = E^i.
\]

(42)
The first expression is obtained by varying the action \( S \) with respect to \( \delta \partial_0 A_i \) (Castelo Ferreira 2006) and the second expression is obtained by replacing the solution \( (20) \) for the \( C \) field in the first expression. This is the same procedure we employed to obtain the effective definitions \( (29) \) of electromagnetic fields. For the case of of canonical momenta we can further derive this result directly from the broken theory. By noting that we have integrated by parts the third term in action \( (28) \) we obtain a time-boundary term of the form

\[
S_{\text{bound}} = \dot{\epsilon} \alpha_{e} \int d^{3}x A_i \phi^{\mu} \phi^{\nu} A_i A^{
u} \left| t_{2} - t_{1} \right|.
\]

(43)

This is actually not a standard boundary term, in more exact terms a are initial conditions for the equations of motion that ensure global conservation of magnetic charge. Nevertheless its effect holds the correction \( (42) \), as long as \( \dot{\epsilon} \alpha_{e} = +1 \), which coincides with the choice corresponding to the standard definition of electromagnetic fields in \( (29) \). Similar results hold for the magnetic effective theory.

As a very simple example let us consider the effective electric theory of a point-like static electric charge \( q_{e} \) located at \( \mathbf{r}_{e} = (x_{e}^{1}, x_{e}^{2}, x_{e}^{3}) \) and a point-like magnetic charge \( q_{g} \) located at \( \mathbf{r}_{g} = (x_{g}^{1}, x_{g}^{2}, x_{g}^{3}) \). The solution for the \( a \) field at a given point \( \mathbf{r} = (x^{1}, x^{2}, x^{3}) \) is the standard one, while the solution for \( \phi_{\mu \nu} \) can be inferred from the solution for \( G_{\mu \nu} \) corresponding to a magnetic monopole in the unbroken theory, i.e.

\[
a_{0}(\mathbf{r}) = \frac{q_{e}}{|\mathbf{r} - \mathbf{r}_{e}|}, \quad a_{i}(\mathbf{r}) = 0, \quad \phi^{\mu}_{0i} = 0, \quad \phi^{\mu}_{ij} = \frac{q_{e} \epsilon^{i j k} (x^{k} - x^{k}_{e})}{|\mathbf{r} - \mathbf{r}_{e}|^{3}}.
\]

(44)

As usual, due to considering point-like charges, we have singularities at \( \mathbf{r} = \mathbf{r}_{e} \) and \( \mathbf{r} = \mathbf{r}_{g} \), however as expected we have no extended singularities in the gauge field \( a \) of \( \phi \). The electromagnetic fields definitions \( (29) \) and the space-time equations of motion \( \) (Lorentz force) for a particle of electric charge \( q \) and mass \( m \) are

\[
E^{i}(\mathbf{r}) = \frac{q_{e}(x^{i} - x^{i}_{e})}{|\mathbf{r} - \mathbf{r}_{e}|^{3}}, \quad B^{i}(\mathbf{r}) = \frac{q_{g}(x^{i} - x^{i}_{g})}{|\mathbf{r} - \mathbf{r}_{g}|^{3}},
\]

\[
m \dot{x}^{i} = q_{e} \frac{(x^{i} - x^{i}_{e})}{|\mathbf{r} - \mathbf{r}_{e}|^{3}} + q_{g} \epsilon^{i j k} \dot{x}^{j} (x^{k} - x^{k}_{g}) |\mathbf{r} - \mathbf{r}_{g}|^{3}.
\]

(45)

Accordingly to the charge definitions \( (41) \) the total electric charge is \( Q_{e} = q_{e} \) and the total magnetic charge is \( Q_{g} = q_{g} \).

### 3.5 Again on Trivial Solutions

The trivial solutions \( (21) \) are no longer valid, we note that by including the source terms the equations of motion for \( G_{\mu \nu} \) and \( F_{\mu \nu} \) are no longer Hodge conjugate to each other, hence are linear independent (as opposed to \( (6) \) in the absence of sources). Also, in the presence of generic four-currents, considering the trivial solutions corresponding to solutions \( (20) \) and \( (40) \) with \( \alpha_{e} = 0 \) and \( \alpha_{g} = 0 \) is not possible. Simply they are not compatible with each other. Let us exemplify it, take the case of the electric trivial solution

\[
G^{\mu \nu} = \frac{\dot{\epsilon}}{2} \epsilon^{\mu \nu \lambda \rho} \phi^{\rho}_{\lambda} = \frac{\dot{\epsilon}}{2} \epsilon^{\mu \nu \lambda \rho} F_{\lambda \rho} \Rightarrow \begin{cases} J_{\mu}^{e} = \dot{\epsilon} \partial_{\mu} \phi^{e \mu} = 0, \\ \alpha_{g} = 0, \end{cases}
\]

(46)

where the first expression corresponds to solution \( (26) \) with \( \alpha_{e} = 0 \) (which is a solution of the equation of motion \( (22) \)) and the second ones to the remaining equation of motion \( (23) \) corresponding to this particular solution. Due to \( \partial_{\mu} \phi \), being regular by construction, this equation decouples in its Hodge components which must, independently of each other, obey the equation. In the presence of electric and magnetic charge these are not obeyed, hence the trivial solutions are not extrema of the action and the only allowed solutions are actually of the form \( (20) \) and \( (40) \) with \( \alpha_{e} \neq 0 \) and \( \alpha_{g} \neq 0 \). The respective broken theories have, for these non-trivial solutions, consistent equations of motion, as given in \( (43) \) for the electric case. It is interesting to note that this result also shows non-triviality of the unbroken theory. The same arguments are valid if, instead of magnetic monopoles, one considers external non-regular fields as in (Castelo Ferreira 2006).

In addition these results also explain why the broken effective theories are not equivalent to standard electromagnetism, as already mentioned before, the \( \text{broken} \) physical gauge fields cannot be freely chosen,
they must be compatible with the unbroken theory, in particular are regular. Hence we have conclude our
prove of non-equivalence between extended $U_c(1) \times U_g(1)$ electromagnetism and standard $U(1)$ variational
Maxwell theory. However we stress that our construction is in complete agreement with the Maxwell
equations which, we recall, have been inferred phenomenologically from experiments.

4 Singularities, Gauge Invariance and Zero-Field Equation

4.1 Singularities and Gauge Symmetry Breaking

Let us resume the construction developed in the previous sections. By solving the equations of motion
of the original theory containing two gauge fields we have managed to obtain one broken electric and one
broken magnetic effective theories with only one gauge field. We achieve these by choosing to replace in the
original action either $G$ or $F$ such that, at functional level, we are expressing each gauge field as a functional
of the other one, respectively $C = C(A)$ and $A = A(C)$. In our construction the effective theories given
by actions (29) and (33) are defined in terms of only regular fields $a$ and $c$. Hence the usual singularities
and/or discontinuities of the fields (Dirac,Wu & Yang 1975) are encoded in the non-trivial solutions for the
effectively excluded fields. This is the main novelty in our construction and we note that a similar feature
is already considered in (Cardoso de Mello & al. 1996, Berkovits 1996, Carneiro 1997). Concerning the
interpretation in terms of the degrees of freedom of the theory we note the dynamical symmetry breaking
conditions consist of 4 independent equations that constraint 8 field components $A_\mu$ and $C_\mu$ ($\mu = 0, 1, 2, 3$).
Therefore as expected we obtain 4 independent gauge fields encoded in $a_\mu$ or $c_\mu$, we start with 6 degrees
of freedom (4 physical) corresponding to one massless photon and one massless pseudo-photon and end up
with only 3 degrees of freedom (2 physical) corresponding to either the massless photon or the massless
pseudo-photon. As for gauge invariance there is a subtlety in the gauge symmetry breaking mechanism.
We start by having a gauge symmetry $U_c(1) \times U_g(1)$ and broke it down to one single $U(1)$ (either $U_c(1)$
or $U_g(1)$), however we did not properly address what is the exact relation between the surviving symmetry
and the original gauge symmetry. In order to see it explicitly take for example solution (7), as it stands
This solution is defined up to a closed 1-form uncharged under both groups, although it can be offset by a
gauge transformation it does not change any of the arguments that follow. Let us consider a
gauge transformation with regular gauge parameters $\Lambda_e$ and $\Lambda_g$ for some fields $\tilde{A}_\mu$, $\tilde{C}_\mu$ and $\tilde{a}_\mu$

$$A_\mu = \tilde{A}_\mu + \partial_\mu \Lambda_e - \hat{e} \partial_\mu \tilde{A}_g, \quad C_\mu = \tilde{C}_\mu + \partial_\mu \Lambda_g + \hat{e} \partial_\mu \tilde{A}_e.$$  

(48)

Each field is charged under both $U_c(1)$ and $U_g(1)$ as can explicitly be seen from the current couplings (2),
however we note that each of the fields is charged under one of the group currents and is topological charged
with respect to the other group (Castelo Ferreira 2006). Considering these gauge transformations (48) in
the solution for the field C (17) we obtain

$$\tilde{C}_\mu = \hat{e} \tilde{A}_\mu - \alpha_e a_\mu - 2 \hat{e} \partial_\mu \tilde{A}_g + 2 \hat{e} \partial_\mu \Lambda_e = \hat{e} \tilde{A}_\mu - \alpha_e a_\mu + 2 \partial_\mu \Lambda_e.$$  

(49)

As long as the parameter $\Lambda_g$ is regular $\tilde{A}_g = 0$, as expressed by the dual fields definition (3). Then the
equivalent parameter of the gauge transformation corresponding to the field $a$ that reproduces the gauge transformation in (19) is

$$\Lambda_e = -\frac{2 \hat{e}}{\alpha_e}.$$  

(50)

This equation constitutes a map between the original $U_c(1)$ and $U_g(1)$ such that $U_e(1) \cong U_c(1)$. For
the standard integration constants choice (20) we have $\alpha_e = \hat{e}$ and this map corresponds both, to a
sign inversion, and a rescaling of the gauge parameter. Therefore we conclude that the gauge symmetry
corresponding to the surviving field $a$, in the effective electric theory, is $U_c(1)$ corresponding to the symmetry
breaking $U_c(1) \times U_g(1) \rightarrow U_c(1)$. For the magnetic effective theory a similar result holds and we obtain
the map $\Lambda_g = -2 \hat{e} \Lambda_g/\alpha_g$ corresponding to the gauge symmetry $U_c(1) \times U_g(1) \rightarrow U_g(1)$. 

10
4.2 Hodge Duality as a Symmetry and Zero-Field Equation

Next we discuss a duality that constitutes an explicit symmetry of the action and its relation with the zero-field equation (5). This condition was considered in the original studies on magnetic monopoles (Dirac, 1948, Cabibbo & Ferrari 1962) and our main motivation is to investigate if it has some special meaning in relation to our mechanism of dynamical symmetry breaking. Let us start by noting that, relating each of the gauge connection (or equivalently each gauge field) through Hodge duality, leaves the action invariant. Considering the duality transformations

\[
\begin{align*}
F_{\mu\nu} &\rightarrow -\frac{\epsilon}{2}\epsilon^{\mu\nu\lambda\rho}G_{\lambda\rho} \\
G_{\mu\nu} &\rightarrow +\frac{\epsilon}{2}\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}
\end{align*}
\]

we obtain that the several terms in the actions (1) and (2) transform as

\[
\begin{align*}
\frac{1}{4} F_{\mu\nu} F^{\mu\nu} &\rightarrow \frac{1}{4} G_{\mu\nu} G^{\mu\nu} \\
-\frac{1}{4} G_{\mu\nu} G^{\mu\nu} &\rightarrow +\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\
-\frac{\epsilon}{4} \epsilon^{\mu\nu\lambda\rho} G_{\lambda\rho} F_{\mu\nu} &\rightarrow -\frac{\epsilon}{4} \epsilon^{\mu\nu\lambda\rho} G_{\lambda\rho} F_{\mu\nu} \\
+(A_{\mu} - \epsilon C_{\mu}) J_{\mu} &\rightarrow +(A_{\mu} - \epsilon C_{\mu}) J_{\mu} \\
-(\epsilon C + \hat{A}_{\mu}) J_{\mu} &\rightarrow -(\epsilon C + \hat{A}_{\mu}) J_{\mu},
\end{align*}
\]

hence the duality (51) leaves the Hodge terms invariant and transform the Maxwell terms into each other, being an exact global symmetry of the action, i.e. \( S_0 + S_{\text{Sources}} \rightarrow S_0 + S_{\text{Sources}} \). Also it is straight forward to show that both the electromagnetic fields (4) are invariant under this duality. The most interesting fact is that the zero-field equation (5) corresponds to the field self-dual condition \( \tilde{C}_\mu = -\epsilon A_\mu \) corresponding to the following relation between the field \( a_\mu \) and \( A_\mu \),

\[
a_\mu = \frac{2\epsilon}{\alpha_e} A^{\text{reg}}_\mu \Rightarrow \tilde{C}_\mu = \epsilon A^{\text{reg}}_\mu - \epsilon A^{\text{reg}}_\mu.
\]

We have explicitly decomposed the \( A = A^{\text{reg}} + A^{\text{non-reg}} \) field into a regular part \( A^{\text{reg}} \) and an non-regular part \( A^{\text{non-reg}} \). This is a generic approach and is due to the field \( a \) being regular by construction, such that the regular parts and the non-regular parts of the equations must be treated independently. Relating this result with the above gauge symmetry breaking we note that it corresponds to the case for which the map (50) does not holds a scaling of the gauge parameter. Moreover we note that condition (53) is gauge invariant under any regular gauge transformation of the unbroken theory. In this sense the zero-field equation is indeed a preferred solution, it is the only field configuration allowing dynamical symmetry breaking that is gauge invariant as well as duality invariant. It is missing to describe how the dynamical symmetry breaking is achieved for this particular condition. The only difference in relation to the previous treatment is that we have to account separately for the regular and non-regular components of the fields. With out loss of generality let us define the regular field \( \tilde{C}^{(0)}_\mu \) by subtracting the non-regular part of \( A \) from the \( \tilde{C} \) field

\[
\tilde{C}^{(0)}_\mu = \tilde{C}_\mu - \epsilon A^{\text{reg}}_\mu \Rightarrow \tilde{C}^{(0)}_\mu = -\epsilon A^{\text{reg}}_\mu.
\]

The second expression is directly obtained from the non-regular zero-field equation (3), hence it constitutes its regular version, being a particular case of our construction. The respective effective electric action and electromagnetic field definitions are

\[
S^{(0)}_{\text{Electric}} = -\int_M F^{\text{reg}}_{\mu\nu} F^{\text{reg}}_{\mu\nu}, \quad E^i = 2 F^{\text{reg}}_{0i}, \quad B^i = \epsilon^{0ijk} F^{\text{reg}}_{jk},
\]

given only in terms of the regular fields \( A^{\text{reg}} \). We note that the usual normalization is obtained by considering a rescaling of the fields by a factor of 1/2. In the presence of sources we can generalize this construction by including the non-regular field \( \varphi^\alpha_\mu \) as introduced in equation (40). Then the generalization of (54) is

\[
\tilde{C}^{(0,\varphi)}_\mu = \tilde{C}_\mu - \epsilon A^{\text{reg}}_\mu + \epsilon \varphi^\alpha_\mu \Rightarrow \tilde{C}^{(0,\varphi)}_\mu = -\epsilon A^{\text{reg}}_\mu.
\]
As already discussed the field $\varphi^g$ is non-regular as expressed by $\text{(1)}$, hence cannot be included either in the regular part of $A$ or in the field $a$ which is also regular by construction. We have shown that duality and gauge invariance of the zero-field equation differentiate it as a preferred configuration, however this choice is not mandatory. In principle the relation between the $a$ field and the $A$ field describes as the original degrees of freedom encoded in the $A$ field are mapped to the surviving degrees of freedom encoded in the $a$ field. If the mechanisms presented in this work exist in real world, it must be experiments to test this theoretical preference. Also it is important to stress that in this discussion we are assuming that there is a direct mapping between $a$ and $A$, our generic construction does not require any map between both fields $a$ and $A$, it simply kills the degrees of freedom encoded in the $C$ field and exchanges the degrees of freedom encoded in the $A$ field of the $U_e(1) \times U_g(1)$ theory, by the degrees of freedom encoded in the $a$ field of the effective $U_{e}(1)$ theory. Similar results hold for the magnetic case.

5 Conclusions

In this work we have presented a dynamical symmetry breaking mechanism for extended $U_e(1) \times U_g(1)$ electromagnetism containing both photons $A$ and pseudo-photons $C$. This mechanism renders broken effective $U_e(1)$ and $U_g(1)$ theories with only one gauge field incorporating both electric and magnetic four-currents. The most remarkable feature of our construction is that the extended singularities (Dirac string (Dirac) or Wu-Yang fiber bundle (Wu & Yang 1975)) characteristic of theories with one single $U(1)$ gauge field containing both electric and magnetic four-currents, are absent. Both four-currents are still present in the effective theories but are described through corrections to the standard definitions of the physical electromagnetic fields inherit from the unbroken theory. In the same manner, also the canonical momenta definitions gain corrections inherited from the unbroken theory. At variational level in the effective broken theories these corrections to the momenta are justified by properly considering the boundary contributions induced by the dynamical symmetry breaking mechanism. We have also shown that, in the presence of both electric and magnetic four-currents, trivial field configurations are not compatible with the equations of motion for both the gauge fields $A$ and $C$. This proves non-triviality of the theory. Moreover, although extended $U_e(1) \times U_g(1)$ electromagnetism is in complete agreement with the phenomenological Maxwell equations, these results imply that it is not equivalence to standard variational Maxwell theory.

In this work we have addressed only local current densities which can be coupled, at variational level, directly to the gauge connections. This construction have previously been considered in (Schwinger 1951) holding that the action is explicitly gauge invariant. However it excludes non-local currents such as constant background currents, non-trivial topological charge configurations, large gauge transformations, Wilson line effects and other topological effects. When such backgrounds are considered, depending on the specific framework, can be generated a Proca mass for the surviving gauge field (Schwinger 1962, Anderson 1963, Proca 1988). This is studied in detail and applied to unmagnetized plasmas in (Castelo Ferreira & Mendonça 2006, Mendonça & Castelo Ferreira 2006). We have also shown that there is a preferential field configuration that corresponds to the zero-field equation (Cabibbo & Ferrari 1962, Schwinger 1966). This configuration corresponds to the self-dual point of a duality constituted by a combination of Hodge dualities that is a global symmetry of the unbroken action. In addition the zero-field condition is also the only gauge field configuration that, when written explicitly as a non-differential relation between the gauge fields (instead of a condition written in terms of the connections), is gauge invariant. These characteristics justify the zero-field equation, from a theoretical point of view, as a preferred field configuration.

In the construction presented in this work the gauge fields regularity in the effective broken theories is accomplished by a correction to the physical fields and charge definitions inherit from the original unbroken theory instead of being achieved at effective action level. Although being a correct framework, a more desirable framework, would be to accomplished this program explicitly at the level of the action by some sort of mechanism. Although not clear to the author how to achieve this construction, possible approaches may be to consider extra fields. For example scalar fields (Witten 1979), in which case the corrections to the electric and magnetic field corrections may be justified as topological charge contributions due to the cross Hopf-term maintaining simultaneously $P$ and $T$ invariance. Also other possible mechanism could include a boundary conformal theory that offsets the boundary contributions of gauge transformations, similarly to Abelian NWZW-models (Wess & Zumino 1971, Witten 1983). This assumption is justified by
noting that the inherent quantum momenta can be described in terms of boundary contributions as we have shown. These topics deserve a detailed study somewhere else.

For last we note that our construction is assuming that monopoles are treated in the same footing as electrons do, being fundamental particles, i.e. Dirac monopoles (Dirac). Other possible approach to magnetic monopoles is to consider them as gauge configurations of non-Abelian gauge groups, i.e. ’t Hooft-Polyakov monopoles (’t Hooft 1974, Polyakov 1974). These can still be treated as particles in effective theories below the non-Abelian gauge symmetry breaking energy and our results could still be related to this framework. However for the particular case of BPS dyons (Julia & Zee 1975, Prasad & Sommerfield 1975, Bogomolny 1976) it was shown by Seiberg and Witten (Seiberg & Witten 1994, see also Witten 1995) that a duality between both electric and magnetic vacuum exists, which is distinct from the duality presented in our work. Furthermore as opposed to our construction Seiberg-Witten vacuum, besides being $P$ and $T$ violating, is chiral. Although not completely clear to the author at this stage, this fact may render both approaches non-equivalent, or at least mean that both approaches are describing different physical systems (at least concerning chiral symmetry breaking which is usually associated with the inclusion of fermionic effects). Concerning our approach and pseudo-photons, it is important to stress that the physical degrees of freedom are encoded in a gauge vector field ( photon) and gauge pseudo-vector field (pseudo-photon), as opposed to other approaches where two gauge vector fields are considered to correspond to one physical photon and one auxiliary non-physical photon which is integrated out of the theory. Also for pseudo-photon theories, exist already available theoretical results, which indicate that may exist in low-energy regimes, both physical systems which exhibit an unbroken phase (Castelo Ferreira 2006), physical systems which exhibit a broken electric phase (Castelo Ferreira & Mendonça 2006, Mendonça & Castelo Ferreira 2006) and systems that exhibit a broken magnetic phase (Castelo Ferreira 2007). Also, these works indicate that, the pseudo-photon may be a truly physical field, instead of a mathematical auxiliary field.

Acknowledgements This work was supported by SFRH/BPD/17683/2004.

References

[Anderson 1963] P. W. Anderson, Phys. Rev. 130 (1963) 439-442.

[Berkovits 1996] N. Berkovits, Phys. Lett. B395 (1997) 28-35, hep-th/9610134.

[Bogomolny 1976] E.B. Bogomolny, Sov. J. Nucl. Phys. 24 (1976) 449; Yad. Fiz. 24 (1976) 861.

[Brandt & al. 1979] R. A. Brandt, F. Neri and D. Zwanzinger, Phys. Rev. D19 (1979) 1153-1167.

[Cabibbo & Ferrari 1962] N. Cabibbo and E. Ferrari, Il Nuovo Cimento XXIII (1962) 1147-1154.

[Cardoso de Mello & al. 1996] P. C. R. Cardoso de Mello, S. Carneiro e M. C. Nemes, Phys. Lett. B384 (1996) 197-200, hep-th/9609218.

[Carneiro 1997] S. Carneiro, JHEP 9807 (1998) 022, hep-th/9702036.

[Castelo Ferreira 2005] P. Castelo Ferreira, J. Math. Phys. 47 (2006) 072902, hep-th/0510063.

[Castelo Ferreira & Mendonça 2006] P. Castelo Ferreira and J. T. Mendonça, hep-th/0601171.

[Castelo Ferreira 2006] P. Castelo Ferreira, accepted for publication in Phys. Rev. B, hep-ph/0609239.

[Castelo Ferreira 2007] P. Castelo Ferreira, hep-th/0703193; hep-th/0703194.

[Dirac] P. A. M. Dirac, Proc. Roy. Soc. A133 (1931) 60; Phys. Rev. 74 (1948) 817.

[Girvin & MacDonald 1987] S. M. Girvin and A. H. MacDonald, Phys. Rev. Lett. 58 (1987) 1252.

[Jackson 1975] J. D. Jackson, Classical Electrodynamics, 2nd Edition, John Wiley & Sons, 1975.

[Jain 1989] J. K. Jain, Phys. Rev. Lett. 63 (1989) 199.

[Julia & Zee 1975] B. Julia and A. Zee, Phys. Rev. D11 (1975) 2227.
[Laughlin 1982] R. B. Laughlin, Phys. Rev. Lett. 50 (1982) 1395.

[Mendonça & Castelo Ferreira 2006] J. T. Mendonça and P. Castelo Ferreira, Europhys. Lett. 75 (2006) 189, hep-th/0601166.

[Polyakov 1974] A. M. Polyakov, JETP Lett. 20 (1974) 194; Pisma Zh. Eksp. Teor. Fiz. 20 (1974) 430.

[Prasad & Sommerfield 1975] M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. 35 (1975) 760.

[Praca 1988] Alexandre Proca (1897-1955): Scientific Publications, G. A. Proca, 1988.

[Schwinger 1951] J. Schwinger, Phys. Rev. 82 (1951) 664-679.

[Schwinger 1962] J. Schwinger, Phys. Rev. 125 (1962) 397-398.

[Schwinger 1966] J. Schwinger, Phys. Rev. 144 (1966) 1087.

[Seiberg & Witten 1994] N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19, Erratum-ibid. B430 (1994) 485-486, hep-th/9407087; Nucl. Phys. B431 (1994) 484, hep-th/9408099.

[Singleton 1995] D. Singleton, Int. J. Theor. Phys. 35 (1996) 2419-2426, hep-th/9509157; Int. J. Theor. Phys. 34 (1995) 2453.

[T'Hooft 1974] G. 't Hooft, Nucl. Phys. B426 (1974) 276.

[Tsui & al. 1982] D. C. Tsui, H. L. Stormer and A. C. Gossard, Phys. Rev. Lett. 48 (1982) 1559.

[Wess & Zumino 1971] J. Wess, B. Zumino, Phys. Lett. B37 (1971) 95.

[Witten 1979] E. Witten, Phys. Lett. B86 (1979) 283-287.

[Witten 1983] E. Witten, Nucl. Phys. B223 (1983) 422-432.

[Witten 1995] E. Witten, Selecta Math. 1 (1995) 383, hep-th/9505186.

[Wu & Yang 1975] T. T. Wu and C. N. Yang, Phys. Rev. D12 (1975) 3845; Phys. Rev. D14 (1976) 437 hep-th/9701040.

[Zhang & al. 1989] S.C. Zhang, T. H. Hansson and S. Kivelson, Phys. Rev. Lett. 62 (1989) 82.

[Zwanzinger 1968] D. Zwanzinger, Phys. Rev. 176 (1968) 1489-1495; Phys. Rev. D3 (1971) 880-891.