Majorana fermions in topological superconductors with spin-orbit interaction

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Abstract. When a topological order is realized in a certain class of superconductors, it supports Majorana fermions on the edge or in the vortex core. Vortices with Majorana fermion modes are neither fermions nor bosons but non-Abelian anyons, obeying the non-Abelian statistics for which the exchange operations of particles are not commutative. Because of this remarkable feature, Majorana fermions play an important role for the realization of fault-tolerant topological quantum computation.

Here I would like to report our recent work on Majorana fermions and non-Abelian anyons in topological superconductors with spin-orbit interaction [1, 2, 3, 4, 5, 6]. Until recently, only spin-triplet superconducting states, such as such \( \nu = 5/2 \) fractional quantum Hall states or \( \text{Sr}_2\text{RuO}_4 \), had been known to exhibit Majorana fermions [7, 8, 9, 10, 11]. However, now it is revealed that spin-singlet superconductors also support Majorana fermions if we take into account the spin-orbit interaction. While a possible realization of non-Abelian anyon in an s-wave superconducting state was first discussed in the context of cosmology [1], by using a zero mode solution in a vortex by Jackiw and Rossi [12] (or its non-gauged version [13]), now there are many examples of such systems in condensed matter physics. In particular, a scheme using the spin-orbit interaction and Zeeman one, which was proposed in [2, 3, 4], is powerful to realize Majorana fermions in various systems. It is shown that Majorana fermions are realized in full-gapped s-wave superfluids/superconductors [3, 4, 14, 15]. Also, Majorana fermions is shown to possible even in nodal superconductors such as high-\( T_c \) cuprates [5].

We also show that a peculiar Majorana fermion is obtained in a certain non-centrosymmetric superconductor [6, 16, 17]. In sharp contrast to ordinary Majorana fermions, the novel Majorana fermion has a flat dispersion and preserves the time-reversal invariance.

1. What is Majorana fermion
First, I would like to explain briefly what a Majorana fermion is. The definition of a Majorana fermion is very simple. A Majorana fermion is a Dirac fermion which satisfies the Majorana condition. Its Hamiltonian is given by the Dirac Hamiltonian,

\[
\mathcal{H}(k) = \sigma \cdot k \quad (2D) \quad \text{or} \quad \mathcal{H}(k_x) = c k_x \quad (1D),
\]

and its wavefunction \( \Psi \) satisfies the Majorana condition

\[
\Psi = C \Psi^*,
\]

with a constant matrix \( C \). The latter equation implies that the particle \( \Psi \) is identified with its anti-particle \( \Psi^* \). In other words, the Majorana fermion is its own anti-particle. Originally, the
Majorana fermion was conceived as an elementary particle like a neutrino, but now it has been known that the Majorana fermion can be realized in topological superconductors [18].

A representative example of superconductors with Majorana fermion is a two-dimensional chiral $p$-wave superconductor [7]. The chiral superconductor is analogous to the quantum Hall state. Like a quantum Hall state, its bulk wave function has a non-zero Chern number and a one-dimensional Dirac fermion is realized as a topologically protected edge state. Furthermore, the Dirac fermion naturally satisfies the Majorana condition due to the superconductivity: In a superconductor, an electron is scattered into its anti-particle, or vice versa, by forming or destroying a Cooper pair. Thus, in the background of many Cooper pairs, a particle excitation cannot be distinguished from its anti-particle, thus it satisfies the Majorana condition naturally.

In a two-dimensional chiral $p$-wave superconductor, there also exist a Majorana fermion in a vortex. In two dimensions, a vortex can be considered as a small hole in the superconductor [7]. Thus as an edge state of the small hole, one obtains a topologically stable zero mode in a vortex. The zero mode $\gamma_0$ also satisfies the Majorana condition, which is given by

$$\gamma_0^\dagger = \gamma_0.$$  
(3)

At first sight, the Majorana condition (3) looks problematic. For an ordinary zero mode, $\gamma_0^\dagger$ is considered as the creation operator for the zero mode, but with the condition (3), this gives rise to contradiction, i.e., the creation operator $\gamma_0^\dagger$ becomes the annihilation operator $\gamma_0$ at the same time. Fortunately, the apparent contradiction can be avoided if we consider a pair of vortices [7, 8]. From the zero modes $\gamma_0^{(i)}$ ($i = 1, 2$) for a pair of vortices 1 and 2, one can construct a well-defined creation operator $\gamma^\dagger$, 

$$\gamma^\dagger = \frac{\gamma_0^{(1)} + i\gamma_0^{(2)}}{\sqrt{2}}.$$  
(4)

The creation operator $\gamma^\dagger$ is not the same as its conjugate annihilation operator $\gamma$ and it satisfies the standard anti-commutation relation,

$$\{\gamma^\dagger, \gamma\} = 1.$$  
(5)

Here one should note that spatially separate vortices are needed to define the creation operator. The nonlocal definition gives rise to a nonlocal quantum correlation between vortices, which changes the statistical properties of the vortices drastically. Actually the vortices obey a new kind of statistics called the non-Abelian anyon statistics [8]. Unlike ordinary fermions/bosons or Abelian anyons, one can create many degenerate states and manipulate them by exchanging non-Abelian anyons. The non-Abelian anyons are of particular interest since they can be considered as qubits for fault-tolerant quantum computers [19].

From the reason above, the two-dimensional chiral $p$-wave superconductor is very interesting. However, at the same time, the chiral $p$-wave superconductor is very special. 1) The chiral $p$-wave superconductor supports spin-triplet Cooper pairs. At present, however, only a few such as $^3$He superfluids or Sr$_2$RuO$_4$ have been widely accepted as spin-singlet superconductors. 2) The chiral $p$-wave superconductor is a full gapped but unconventional superconductor, while most unconventional superconductors including the high $T_c$ cuprate have nodes in their gap functions. 3) Finally, the chiral $p$-wave superconductor breaks the time-reversal symmetry. Thus a natural question arises: Which property is essential to realize the Majorana fermion? In the following, I will show that the answer is none of them: Majorana fermions are possible in spin-singlet superconductors. Furthermore, nodal superconductors also can support Majorana fermions. And even for time-reversal invariant system, we may have a peculiar time-reversal invariant Majorana fermion. Thus, Majorana fermions can be obtained without any special property of the chiral $p$-wave superconductivity. However, as I will show below, we find that the spin-orbit interaction is indispensable to realize Majorana fermions instead.
2. Majorana fermion in spin-singlet superconductors

Here I will show that Majorana fermion and non-Abelian anyon are possible even for spin-singlet s-wave superconducting states [1, 20, 3, 4, 14, 15]. This possibility was first considered in Ref.[1].

The basic idea of Ref.[1] is to consider a two dimensional Dirac fermion coupled with an s-wave condensate \( \Phi \),

\[
\mathcal{H} = \begin{pmatrix}
-i\sigma_i \partial_i & \Phi^* \\
\Phi & -i\sigma_i \partial_i
\end{pmatrix}
\]

where \( \sigma_i \) is the Pauli matrices. From the analysis of Jackiw and Rossi [12] (and its non-gauged version [13]), it has been known that there exist a zero mode in the vortex configuration given by \( \Phi = \Phi_0 f(r)e^{i\theta} \). Therefore, imposing the Majorana condition on the system, one has a Majorana zero mode in a vortex, which implies that the vortices are non-Abelian anyons [1].

When this idea was considered in 2003, no condensed matter system supporting a single Dirac fermion was known. Thus the above idea was applied to a cosmic string called axion string [1]. After the publication of Ref.[1], however, it was found that a single Dirac fermion is naturally realized on a surface of a topological insulator. Actually, Fu and Kane showed, independently of Ref.[1], that a similar Hamiltonian is realized in the interface between a topological insulator and an s-wave superconductor [20]. They also showed that the vortex obeys non-Abelian anyon statistics in the same reason mentioned above. Furthermore, the idea was generalized to unconventional superconductors [21]. Here one should note that the spin-orbit interaction is very important since the spin-orbit interaction is a key gradient to realize topological insulators.

In the above scheme of non-Abelian anyons, we need a special system which supports a two dimensional Dirac fermion. However, if one consider an two dimensional s-wave superconductor with Rashba spin-orbit interaction, one can realize a Majorana fermion from electrons with a usual parabolic dispersion [3, 4]. The model Hamiltonian considered in Refs.[3, 4] is

\[
\mathcal{H}(k) = \begin{pmatrix}
\varepsilon_k - h\sigma_z + g_k \cdot \sigma & i\Delta \sigma_y \\
-i\Delta \sigma_y & -\varepsilon_k + h\sigma_z + g_k \cdot \sigma^*\end{pmatrix}
\]

where \( k = (k_x, k_y) \) is the momentum in the xy plane, \( \varepsilon_k \) the electron energy measured from the Fermi energy, \( \Delta \) the s-wave gap function, and \( g_k \sim \lambda(k_y, -k_x) \) the Rashba spin-orbit interaction. We also consider the Zeeman magnetic field \( h = \mu B H_z \) in the z-direction. Contrary to the Hamiltonian (6), we consider here that \( \varepsilon_k \) is parabolic around \( k = 0 \) as a usual electron. To see the basic idea of Ref.[3], perform the following unitary transformation

\[
\mathcal{H}^D(k) = D\mathcal{H}(k)D^\dagger,
\]

where \( D = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i\sigma_y \\ i\sigma_y & 1 \end{pmatrix} \),

which leads to the dual Hamiltonian \( \mathcal{H}^D \),

\[
\mathcal{H}^D(k) = \begin{pmatrix}
\Delta - h\sigma_z & -i\varepsilon_k \sigma_y - i\lambda g_k \cdot \sigma \sigma_y \\
i\varepsilon_k \sigma_y + ig_k \sigma_y \cdot \sigma & -\Delta + h\sigma_z
\end{pmatrix}
\]

Interestingly, the spin-triplet gap function \(-i\lambda g_k \cdot \sigma \sigma_y\) is induced in the dual Hamiltonian, by the spin-orbit interaction of the original Hamiltonian. This implies that the original Hamiltonian \( \mathcal{H}(k) \) also has a property similar to the spin-triplet superconductor in the presence of the spin-orbit interaction. Actually, we can show that gapless Majorana edge state appears like a two dimensional chiral p-wave superconductor if the Zeeman magnetic field exceeds the critical value [3, 4],

\[
h > \sqrt{\Delta^2 + \mu^2}.
\]
One can also show that a vortex in the system supports a Majorana zero mode, so it becomes a non-Abelian anyon in this parameter region [3, 4, 14]. The bulk wave function has a non-zero Chern number in this parameter region [3, 4], thus the state is topologically equivalent to a chiral $p$-wave superconductor.

Here one might notice that a rather strong magnetic field, which is larger than the gap function $\Delta$, is needed to satisfy the condition (10). Under such a strong magnetic field, the orbital depairing effect may break the superconductivity, although the Pauli depairing effect is strongly suppressed due to the Rashba spin-orbit interaction [22, 23]. This difficulty, however, has been resolved by various systems, (i) $s$-wave superfluids in neutral ultracold fermionic atoms with larger generated Rashba spin-orbit interaction [3, 4], (ii) heterostructure semiconductor devices [14, 15], and (iii) heavy fermion systems [4].

Recently, a similar idea was applied to nanowire systems [9, 24], and it was found that the same condition (10) is required to obtain Majorana fermions at the boundary of the semiconductor nanowire on an $s$-wave superconductor [25, 26, 27].

3. Majorana fermion in nodal superconductors

Now we show that Majorana fermions are possible even in nodal superconductors [5]. The model we consider is a two-dimensional $d$-wave superconductor with the Rashba spin-orbit interaction and Zeeman term,

$$
\mathcal{H}(k) = \left( \begin{array}{cc}
\varepsilon_k - h \sigma_z + g_k \cdot \sigma & i \Delta(k) \sigma_y \\
- i \Delta(k) \sigma_y & -\varepsilon_k + h \sigma_z + g_k \cdot \sigma^*
\end{array} \right).
$$

The Hamiltonian is almost the same as Eq.(7), but instead of an $s$-wave gap function $\Delta$, we consider the $d$-wave gap functions,

$$
\Delta(k) \sim \Delta_0 k_x k_y, \quad \text{or} \quad \Delta(k) \sim \Delta_0 (k_x^2 - k_y^2).
$$

Gap-nodes exist in both $d$-wave pairings.

To understand what happens, we can use the dual transformation again. In a manner similar to (9), the spin-triplet gap function $-i g_k \cdot \sigma \sigma_z$ is induced in the dual Hamiltonian by the Rashba spin-orbit interaction. Therefore, one can expect that the original Hamiltonian is topologically similar to the spin-triplet superconductor. Indeed, although there are bulk nodes in the system, a gapless Majorana edge state appears like a chiral $p$-wave superconductor if the Zeeman magnetic field satisfies

$$
h > \mu.
$$

We can also show that there exist a single Majorana zero mode in a vortex in this parameter region, thus the vortex obeys non-Abelian anyon statistics [5]. On contrary to the $s$-wave case (see (10)), the condition (13) does not depend on the magnitude of the gap function $\Delta_0$. Therefore, by choosing a system with small $\mu$, the Majorana fermion can be realized in a weak magnetic field. The topological phase in nodal superconductors can be characterized by the parity of the Chern number [5].

4. Time-reversal invariant Majorana fermions

Finally, I would like to show the existence of Majorana fermion in two dimensional time-reversal invariant superconductors [6, 16, 17]. Usually, the time-reversal breaking is necessary to realize Majorana fermion in two dimensional superconductors. This is because the Majorana fermion on the edge has a linear dispersion

$$
E = c k_y,
$$
and it already breaks the time-reversal invariance. However, if the coefficient \( c \) vanishes, and the dispersion becomes flat,

\[
E = 0 \quad (15)
\]

then the Majorana fermion can be obtained even in time-reversal invariant systems. In Ref.[6], we have shown that such a Majorana fermion with flat dispersion is realized in a special kind of two dimensional nonecentrosymmetric superconductors.

We found that the spin-orbit interaction is indispensable to realize the dispersionless Majorana fermion in our model. Without the spin-orbit interaction, the single branch of the edge state with flat dispersion disappears, and no Majorana fermion is obtained [6, 16].

5. Summary
We show that Majorana fermions can be realized in various two dimensional superconductors other than chiral \( p \)-wave ones, if we take into account the spin-orbit interaction. In particular, \( s \)-wave superconductors or \( d \)-wave superconductors in high \( T_c \) cuprate can support Majorana fermions and non-Abelian anyons. From the stableness of \( s \)-wave pairing against disorder or a large superconducting gap of high \( T_c \) cuprate, the experimental detection of non-Abelian anyons in these schemes is expected to be feasible in comparison with that in chiral \( p \)-wave superconductors.

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