Dynamics of Subcritical Bubbles in First Order Phase Transition

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We derive the Langevin and the Fokker-Planck equations for the radius of $O(3)$-symmetric subcritical bubbles as a phenomenological model to treat thermal fluctuation. The effect of thermal noise on subcritical bubbles is examined. We find that the fluctuation-dissipation relation holds and that in the high temperature phase the system settles down rapidly to the thermal equilibrium state even if it was in a nonequilibrium state initially. We then estimate the typical size of subcritical bubbles as well as the amplitude of fluctuations on that scale. We also discuss their implication to the electroweak phase transition.

§ 1. Introduction

The dynamics of the electroweak phase transition is important in the light of electroweak baryogenesis.1) For the scenario to be successful the phase transition must be of first order with supercooling in order to attain a nonequilibrium state necessary for baryogenesis.2) Though the effective potential of the Higgs field obtained by perturbation possesses a barrier between false and true vacua at the critical temperature,3) it has not been certain if the phase transition accompanies supercooling. The purpose of this paper is to negatively confirm this. If the thermal fluctuation around the symmetric vacuum is too large, the phase transition proceeds without supercooling and no baryogenesis is expected.

For the last five years, some attempts have been made by several different methods.3)–9) For example, Gleiser and Kolb4)–7) have discussed that the conventional picture of the first-order phase transition through nucleation of critical bubbles is applicable in the minimal standard model only in the case of relatively light Higgs mass, $m_H \leq 70$ GeV, and that otherwise subcritical fluctuations play a dominant role to realize emulsion of false and true vacua even above the critical temperature. In their analysis, however, it has been assumed that the typical scale of subcritical bubbles is given by the correlation length of the Higgs field. Meanwhile, as a way to understand the dynamics of quantum fields in a finite-temperature but nonequilibrium situation, Gleiser and Ramos7) have derived the Langevin equation for a scalar field extending Morikawa's method.10) The equation has also been numerically analyzed assigning white noise on a lattice, and it has been concluded that the sufficient phase mixing happens in any experimentally-allowed range of Higgs mass, where again the lattice spacing has been taken to be comparable to the correlation length.8) Because the amplitudes of thermal fluctuation changes drastically depending on the length scale,
it is important to determine physically the typical length scale of fluctuations which dominate the dynamics of the phase transition. As a first step along this line, in a previous paper we obtained the typical scale of $O(3)$-symmetric subcritical bubbles by constructing the effective Hamiltonian for their radius and taking a thermal average. We thereby concluded that the electroweak phase transition is dominated by subcritical bubbles with any experimentally-allowed value of Higgs mass. Unfortunately, however, we cannot deny the fact that our approach was also too phenomenological to be viable from fundamental points of view.

In the present paper, applying nonequilibrium statistical field theory we derive the Langevin and the Fokker-Planck equations for the Higgs field to clarify the detailed structure of the electroweak phase transition. Starting with the effective action we first derive these equations for generic field configurations. We then adopt a Gaussian ansatz for field configuration to model a subcritical bubble and yield the effective Langevin and Fokker-Planck equations for its radius. This is a kind of variational approach which is adopted because, unlike the critical bubble, these subcritical bubbles do not constitute a solution of field equations but should merely be regarded as a model of field configurations. As a result our previous approach will be justified under some conditions. Since we construct the Langevin equation in the perturbation theory, the effect from the environment is automatically taken into account. We need to calculate some loop corrections to obtain the friction term. The friction term will be estimated in the quasi-adiabatic approximation, while the noise term comes from the imaginary part of the effective action. The Fokker-Planck equation can be derived directly from the Langevin equation and then we find that its static solution is of the form which we assumed as a probability amplitude in the previous paper.

The rest of the present paper is organized as follows. In § 2, we review the non-equilibrium quantum field theory and discuss the statistical aspects of the theory. In § 3, we derive the Langevin and Fokker-Planck equations for the radius of subcritical bubbles. We examine the effect of dissipation and thermal noise on the electroweak phase transition. Section 4 is devoted to a summary and discussion. Throughout the paper we use the units $c=\hbar=1$.

§ 2. Effective Lagrangian of Higgs fields in the thermal bath

2.1. Basics of the non-equilibrium quantum field theory

In order to derive the Langevin equation and the Fokker-Planck equation for the subcritical bubbles, we use the non-equilibrium quantum field theory. This subsection is a brief review of its basic technique. The ordinary quantum field theory, which mainly deals with transition amplitudes in particle reactions, is not useful when we study statistical dynamics of macroscopic objects like bubbles. This is because we need the temporal evolution of some kind of classical order parameters with definite initial condition and not simply the transition amplitude of particle reactions with fixed initial and final conditions. The most appropriate extension of the field theory to deal with these issues is to generalize the time contour of integration to the closed form. More precisely, the
time integration contour is generalized so that it runs from minus infinity to plus infinity and then back to the minus infinity again. This formalism, often called in-in formalism of quantum field theory, yields various quantum averages of operators evaluated in the in-state without specifying out-state. On the other hand the ordinary quantum field theory, often called in-out formalism of quantum field theory, yields quantum averages of operators evaluated with an in-state at one end and an out-state at the other.

The partition function in the in-in formalism for a real scalar field is defined to be

$$Z[J] \equiv \text{Tr}\left[ T\left( \exp\left[i \int_c J \varphi \right]\right) \rho \right]$$

$$= \text{Tr}\left[ T_{+}\left( \exp\left[i \int_{+} J \varphi_{+} \right]\right) T_{-}\left( \exp\left[-i \int_{-} J \varphi_{-} \right]\right) \rho \right],$$

(2.1)

where the suffix $c$ in the integral means that the time integration contour runs from minus infinity to plus infinity and then back to the minus infinity again. All the field quantity is defined on this closed time contour. In the above, $X_{+}$ represents a field component $X$ which is restricted on the forward branch ($-\infty$ to $+\infty$) of the time contour and $X_{-}$ stands for that restricted on the backward branch ($+\infty$ to $-\infty$). In the rest of this paper we often use the following notation: $X_{\pm} = X_{+} - X_{-}$ and $X_{c} = (X_{+} + X_{-})/2$. The symbol $T_{+}$ designates the operator ordering with respect to this closed time contour, and accordingly $T_{-}$ designates the ordinary time ordering and $T_{-}$ the anti-time ordering. Here $J$ means the external field. Although $J_{+}$ and $J_{-}$ are the same actually, we regard that they are different from each other for technical reasons and we only set $J_{+} = J_{-}$ at the end of calculations. The symbol $\rho$ represents the initial density matrix, and the field $\varphi(x)$ is in the Heisenberg picture.

In the interaction picture, this partition function becomes

$$Z[J] = \text{Tr}\left[ T\left( \exp\left[i \int_c J \varphi + i \int_c V[\varphi] \right] \right) \right]$$

$$= \exp\left(-i \int_c V \left[ \frac{\delta}{i \delta J} \right] \right) \text{Tr}\left[ T\left( \exp\left(i \int_c J \varphi \right) \right) \right]$$

$$= \exp\left(-i \int_c V \left[ \frac{\delta}{i \delta J} \right] \right) \exp\left[-\frac{i}{2} \int_c \int_c J(x) G_0(x, y) J(y) \right],$$

(2.2)

where we have used the Wick theorem which holds not only in the vacuum state but also in the thermal state with $\rho = \exp(-H/T)$. In the latter the above propagator is, in the momentum representation,
\[
G_0(p) = \frac{G_F(p) G_+(p)}{G_-(p) G_F(p)}
= \begin{pmatrix}
\frac{1}{p^2 - m^2 + i\epsilon} & -2\pi i\delta(p^2 - m^2)
-2\pi i(\theta(p_0) + n(p))\delta(p^2 - m^2)
-2\pi i(\theta(-p_0) + n(p))\delta(p^2 - m^2)
\frac{1}{p^2 - m^2 - i\epsilon} & -2\pi i\delta(p^2 - m^2)
\end{pmatrix},
\]

where \( n(p) \) is the thermal distribution function: \( n(p) = \frac{\exp(\omega(p)/T) - 1}{\omega} \) and \( \omega = \sqrt{p^2 + m^2} \).

By the Legendre transformation of the partition function, we obtain the generalized effective action, or generating functional of the vertex functions:

\[
\phi(x) = \frac{\delta \ln Z}{\delta f(x)},
\]

\[
\Gamma[\phi] = -i \ln Z[J] - \int f \phi.
\]

The equality

\[
\frac{\delta \Gamma[\phi]}{\delta \phi(x)} = -J(x)
\]

immediately follows as in the in-out formalism and this form is often used as a generalized classical equation of motion for the variable \( \phi(x) \). However, this effective action has an imaginary part. For example, if we parameterize the kernel of two-point part of \( \Gamma \) as

\[
\hat{\Gamma}^{(2)}(x, y) = \begin{pmatrix}
D + iB & i(B - A)

i(B + A) & -D + iB
\end{pmatrix},
\]

the imaginary part of \( \Gamma^{(2)} \) is

\[
\text{Im} \Gamma^{(2)}[\phi_c, \phi_s] = \frac{1}{2} \int \int \phi_s(x) B(x - x') \phi_s(x').
\]

We can rewrite this expression by introducing a real auxiliary field \( \xi(x) \):

\[
\exp(i\Gamma[\phi]) = \int [d\xi] P[\xi] \exp \left[ i \text{Re} \Gamma + \int i \xi \phi_s \right],
\]

where

\[
P[\xi] = \exp \left[ -\frac{1}{2} \int \int B^{-1} \xi \right]
\]

is a normalizable positive definite statistical weight for the fields \( \xi(x) \). If we apply the variational principle for the exponent of the integrand of Eq. (2·8), we obtain an equation of motion for \( \varphi_c(x) \) as
This is a Langevin type stochastic differential equation with a nonlocal kernel \( A(x - x') \). If the time scale of change in \( \phi_c(x) \) is small compared with that of radiative corrections and \( \phi_c(x) \) is nearly homogeneous in space, this term reduces to the familiar friction term:

\[
\square \phi_c(x) + V'(\phi_c(x)) + \eta \phi_c(x) = \xi(x) \tag{2.11}
\]

with

\[
\eta = -i \lim_{k \to 0} \frac{\partial A(k)}{\partial k^0}, \tag{2.12}
\]

where \( V_{\text{eff}} \) is the effective potential and \( A(k) \) is the Fourier transform of \( A \).

If we define the statistical average as

\[
\langle \cdots \rangle_t = \int [d\xi] P[\xi] \cdots , \tag{2.13}
\]

we then obtain

\[
\langle \xi(x) \xi(x') \rangle_t = B(x - x'). \tag{2.14}
\]

Thus we can construct consistent statistical field theory in the in-in formalism of quantum field theory.

In the actual application of this formalism to the subcritical bubbles, we need to calculate the dissipative and diffusive kernels \( A \) and \( B \) at finite temperature. This is the subject of the next subsection.

2.2. The fluctuation-dissipation relation and the stationary distribution for Higgs fields

For simplicity, we consider the following Lagrangian of a singlet Higgs field \( \phi \):

\[
\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} \lambda \phi^4 + i \bar{\psi} \gamma^\mu \partial_\mu \phi - f \phi \bar{\psi} \phi . \tag{2.15}
\]

Strictly, we must include interactions with \( Z \), \( W \)-bosons and all quarks for the detail of the electroweak phase transition. In particular, contributions of \( Z \) and \( W \) boson are crucial to yield the cubic term in the effective potential with one-loop corrections. However, the essence of non-equilibrium phase transition can be fully obtained by the above simple model as we will see soon. According to the calculation by Morikawa\(^{10}\) and Gleiser et al.\(^{7}\), the effective action becomes, up to one-loop corrections,

\[
\Gamma[\phi_c, \phi_d] = \int d^4x \left\{ \phi_d(x)[ - \square - V(t)] \phi_c(x) - \frac{\lambda}{4!} (4 \phi_d(x) \phi_c^3(x) + \phi_c^3(x) \phi_c(x)) \right\}
\]

\[
- 2 \int d^4x d^4x' A_1(x - x') \phi_d(x) \phi_c(x') - \int d^4x d^4x' A_2(x - x')
\]
\[
\times \left[ \phi_d(x) \phi_c(x) \phi_c^2(x') + \frac{1}{4} \phi_d(x) \phi_c(x) \phi_\sigma^2(x') \right] \\
+ \frac{i}{2} \int d^4x d^4x' \left[ B_1(x-x') \phi_d(x) \phi_d(x') + B_2(x-x') \phi_d(x) \phi_c(x) \phi_c(x') \right]
\]

(2.16)

where

\[
V(t) = m^2 + \frac{\lambda}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1+2n(\omega)}{2\omega(\mathbf{k})},
\]

(2.17)

\[
A_1(x-x') = f^2 \text{Re} \left[ -i S_{\alpha\beta}^F(x-x') S_{\gamma\eta}^F(x'-x) \right] \theta(t-t'),
\]

(2.18)

\[
A_3(x-x') = \frac{\lambda^2}{2} \int \frac{d^3k}{(2\pi)^3} e^{ik \cdot (x-x')} \int \frac{d^3q}{(2\pi)^3} \text{Im} \left[ G_\sigma(q, t-t') G_\sigma(q-k, t-t') \right] \theta(t-t'),
\]

(2.19)

\[
B_1(x-x') = f^2 \text{Im} \left[ -i S_{\alpha\beta}^F(x-x') S_{\gamma\eta}^F(x'-x) \right],
\]

(2.20)

\[
B_2(x-x') = \frac{\lambda^2}{2} \int \frac{d^3k}{(2\pi)^3} e^{ik \cdot (x-x')} \text{Re} \int \frac{d^3q}{(2\pi)^3} \left[ G_\sigma(q, t-t') G_\sigma(q-k, t-t') \right].
\]

(2.21)

Here \( S_{\alpha\beta}^F(x-x') \) is the thermal Green's function of the fermion defined by

\[
S_{\sigma}(x-x') = -i \text{Tr} \left[ T_\mu(\psi(x) \bar{\psi}(x')) \rho \right]
= \int \frac{d^4p}{(2\pi)^4} \left[ \frac{1}{\gamma^\mu p_\mu + i\epsilon} + 2\pi i\gamma^\mu p_\mu f(p) \delta(p^2) \right] e^{-ip(x-x')},
\]

(2.22)

where \( f(p) = (\exp(|p|/T) + 1)^{-1} \).

We can rewrite this expression by introducing two real auxiliary fields \( \xi_1(x) \) and \( \xi_2(x) \):

\[
\exp(i\Gamma[\varphi]) = \int [d\xi_1][d\xi_2] P_1[\xi_1] P_2[\xi_2] \exp \left[ i\text{Re} \Gamma + i \int (\xi_1 \phi_d + \xi_2 \phi_c \phi_d) \right],
\]

(2.23)

where

\[
P_i[\xi_i] = \exp \left[ -\frac{1}{2} \int d^4x' \xi_i B_i^{-1} \xi_i \right].
\]

(2.24)

Applying the variational principle to Eq. (2.23) as in the previous subsection, we obtain the Langevin equation:

\[
\Box \phi_c(x) + V_{\text{ren}}(\phi_c(x)) + 2 \int d^4x' \int_0^t dt' \phi_c(x') A_1(x-x')
\]

Here we use a massless propagator as an approximation. In fact once the phase transition starts, the top quark has a space-time dependent mass due to the variation of the Higgs field. The maximum value of the top mass, however, remains as small as \( f_\sigma \sim 50 \text{ GeV} \) at the critical temperature \( T_c \approx 93 \text{ GeV} \). We may therefore conclude that the result would not change significantly even if we used the more complicated propagator with nontrivial space-time dependence or simply a massive propagator.
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Assuming that $\phi(x)$ is nearly homogeneous and changes slowly in time, we obtain

$$
\Box \phi_c(x) + V_{\text{eff}}(\phi_c(x)) + \eta_1 \dot{\phi_c}(x) + \eta_2 \phi_c^2(x) \phi_c(x) = \xi_1(x) + \phi_c(x) \xi_2(x)
$$

with

$$
\eta_1 = 2 \int_0^\infty dt \int d^3 x A_1(x, t) \xi_1,
$$

$$
\eta_2 = 2 \int_0^\infty dt \int d^3 x A_2(x, t) \xi_2
$$

and

$$
\langle \xi_1(x) \xi_1(x') \rangle = B_1(x - x').
$$

For the contribution from only the self-coupling, Gleiser and Ramos\(^7\) have obtained the result $\eta_2 = (96/\pi T) \ln(T/\mu_T)$ for the friction coefficient and

$$
\langle \xi_2(x) \xi_2(x') \rangle = 2 T \eta_2 \delta^3(x - x') \delta(t - t'),
$$

in the high temperature limit. However, for $\lambda \sim 10^{-2}$ (Higgs mass $m_H \sim 60$ GeV) the correlation time of the noise is $\Delta_{\text{noise}} \sim (\text{Decay width})^{-1} \sim 1536 \pi (\lambda^2 T)^{-1} \sim 10^7 T^{-1}$ which is much larger than the typical scale $\sim T^{-1}$. Hence, the above approximation is not guaranteed in Ref. 7).

In our present analysis, on the other hand, the most dominant contribution comes from the Yukawa interaction with $1 \gg f \gg \lambda$, so that terms proportional to $\lambda^2$ are negligible and it suffices to consider the following Langevin equation:

$$
\Box \phi_c + V_{\text{eff}}(\phi_c) + \eta_1 \dot{\phi_c} = \xi_1.
$$

To obtain the friction term we must prepare the full propagator of fermion. Up to one-loop order, the renormalized Green’s function becomes

$$
S_F(p, t) = \frac{e^{-\Gamma_f |t|}}{2 \omega_p} \left[ (\gamma^0 \omega_p \epsilon(t) + \gamma \cdot p) f(-\omega_p + i \Gamma_f) e^{-i \omega_p |t|} 
\right.
\left. - (\gamma^0 \omega_p \epsilon(t) + \gamma \cdot p) f(\omega_p + i \Gamma_f) e^{i \omega_p |t|} \right],
$$

where $\Gamma_f$ is the decay width given by

$$
\Gamma_f(\omega_p) = - \gamma^0 \text{Im}(\Sigma) \big|_{\rho^s=0} \sim \frac{f^2}{8 \pi} T
$$

in the high temperature limit. The friction coefficient then becomes

$$
\eta_1 = 2 f^2 \text{Re} \left[ i \int_0^\infty dt \int \frac{d^3 k}{(2\pi)^3} S_F(\mathbf{k}, t) S_F^\dagger(\mathbf{k}, -t) \right]
\left. = 2 f^2 \sin(\beta \Gamma_f) \int_0^\infty dt e^{-2 \Gamma_f t} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\cosh(\beta \omega_H) + \cosh(\beta \Gamma_f)} \left[ \sinh(\beta \omega_H) \sin(2 \omega_H t) 
\right.
\left. + \sin(\beta \Gamma_f) \cos(2 \omega_H t) \right] \right]
$$
Here we use the approximation that the external momentum is zero. In the case of Yukawa interaction with top quark, $\beta \Gamma_\gamma \approx 0.04$. Thus, we can expand the last expression by $\beta \Gamma_\gamma$ and to the lowest order we find

$$\eta \approx \frac{f^2}{4 \pi^2 \Gamma_\gamma^2} \int_0^\infty \frac{d\omega \omega_k^2}{\cosh(\beta \omega_k) + \cos(\beta \Gamma_\gamma)}.$$  \hspace{1cm} (2.34)

Taking the same approximation used in the derivation of Eq. (2.34), the autocorrelation of noise becomes

$$\langle \xi_i(x) \xi_i(x') \rangle = B_i(x - x')$$

$$= - f^2 \text{Im} \left[ i \int \frac{d^3 k}{(2\pi)^3} S^I_{\phi \phi}(\mathbf{k}, t - t') S^I_{\phi \phi}(\mathbf{k}, t' - t) \right] \delta^3(x - x')$$

$$= f^2 \int \frac{d^3 k}{(2\pi)^3} \frac{e^{-2\gamma \beta |t - t'|}}{[\cosh(\beta \omega_k) + \cos(\beta \Gamma_\gamma)]^2} \left\{ -2 \cosh(\beta \omega_k) \cosh(\beta \omega_k) + \cos(\beta \Gamma_\gamma) \cos(2 \omega_k |t - t'|) + 2 \sin(\beta \omega_k) \sin(\beta \Gamma_\gamma) \sin(2 \omega_k |t - t'|) \right\} \delta^3(x - x').$$  \hspace{1cm} (2.36)

In the limit $\Gamma_\gamma \gg 1$ (strong coupling limit $f \gg 1$),

$$\langle \xi_i(x) \xi_i(x') \rangle \approx \frac{f^2}{\Gamma_\gamma} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{[\cosh(\beta \omega_k) + \cos(\beta \Gamma_\gamma)]^2} \left\{ -2 \cosh(\beta \omega_k) \cosh(\beta \omega_k) + \cos(\beta \Gamma_\gamma) \cos(2 \omega_k |t - t'|) + 2 \sin(\beta \omega_k) \sin(\beta \Gamma_\gamma) \sin(2 \omega_k |t - t'|) \right\} \delta(t - t') \delta^3(x - x')$$

$$\approx 2 \eta \tau \delta(t - t') \delta^3(x - x').$$  \hspace{1cm} (2.37)

Properly speaking, one cannot take the above limit in the minimal standard model. But, we expect that the approximation does not give drastic changes on the final result as long as we concentrate on a time scale larger than $\Gamma_\gamma^{-1}$.

Since the ordinary fluctuation-dissipation relation holds as above, we expect that the static solution is the canonical distribution. Introducing a new variable

$$\frac{d\phi_c}{dt} = p_{\phi_c},$$  \hspace{1cm} (2.38)

we rewrite Eq. (2.31) as

$$\frac{dp_{\phi_c}}{dt} = \Delta \phi - V_{\phi_c}(\phi_c) - \eta \tau p_{\phi_c} + \xi_1.$$  \hspace{1cm} (2.39)

The Fokker-Planck equation for the above Langevin equation becomes

$$\frac{\partial W[\phi_c(x), \phi_c(x), t]}{\partial t} = \int d^3 x \left\{ - \frac{\delta}{\delta \phi_c(x)} \left( \frac{\delta H}{\delta \phi_c(x)} W \right) + \frac{\delta}{\delta \phi_c(x)} W \right\}$$
where
\[ \eta(x) = \begin{pmatrix} \phi_c(x) \\ \rho_{\phi_c}(x) \end{pmatrix}, \]
and \( J \) is the probability current density with the expression
\[ J = \begin{pmatrix} 0 & 1 \\ -1 & -\eta_1 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \phi_c} W + T \frac{\delta W}{\delta \phi_c} \\ \frac{\delta H}{\delta \rho_{\phi_c}} W + T \frac{\delta W}{\delta \rho_{\phi_c}} \end{pmatrix}. \]

Here \( H \) is the Hamiltonian given by
\[ H[\phi_c, \rho_{\phi_c}] = \int dx^3 \left[ \frac{1}{2 \mu} \left( \frac{d\phi_c}{dt} \right)^2 + \frac{1}{2} \left( \nabla \phi_c \right)^2 + V_{\text{en}}(\phi_c) \right]. \]

From the expression of the probability current density we easily find a static solution of the system as
\[ W_{\text{st}} \propto \exp \left[ -\frac{H[\phi_c, \rho_{\phi_c}]}{T} \right]. \]

On the other hand, normalizable dynamical solutions may be expressed in the form\(^{13}\)
\[ W[\phi_c(x), \rho_{\phi_c}(x); t] = \sum_n \Psi_n[\phi_c(x), \rho_{\phi_c}(x)] e^{-(\mathcal{H}/2T)} e^{-\lambda_n t}, \]
where the eigenfunction \( \Psi_n \) satisfies
\[ -\lambda_n \Psi_n = \int d^3 x \left[ -\frac{\delta \mathcal{H}}{\delta \rho_{\phi_c}} \frac{\delta \Psi_n}{\delta \phi_c} + \frac{\delta \mathcal{H}}{\delta \phi_c} \frac{\delta \Psi_n}{\delta \rho_{\phi_c}} + \eta_1 \frac{\delta^2 \mathcal{H}}{\delta \rho_{\phi_c}^2} \Psi_n - \eta_1 \frac{1}{4T} \left( \frac{\delta \mathcal{H}}{\delta \rho_{\phi_c}} \right)^2 \Psi_n \\
+ \eta_1 T \frac{\delta^2 \Psi_n}{\delta \rho_{\phi_c}^2} \right]. \]

The lowest eigenvalue \( \lambda_0 \) is of course zero, while the next lowest eigenvalue gives the time scale of relaxation to the thermal equilibrium. For its estimation we only have to consider the Hermitian part of (2.44),\(^{13}\)
\[ \lambda_n^{(H)} \Psi_n^{(H)} = \int d^3 x 2\eta_1 T \left[ -\frac{1}{2} \frac{\delta \Psi_n^{(H)}}{\delta \rho_{\phi_c}^{(H)}}(x) + \frac{1}{2} \left( \frac{p_{\phi_c}(x)}{2T} \right)^2 \Psi_n^{(H)} - \frac{\delta^{(0)}(0)}{4T} \Psi_n^{(H)} \right]. \]

The above equation is an infinite collection of harmonic oscillators with their ground-state energy subtracted. We thus find formally \( \lambda_1^{(H)} = \eta_1 \), that is, the time scale of relaxation is in general given by the inverse of the friction coefficient in the original Langevin equation. The relaxation time reads \( \Delta t_{\text{relax}} = \text{Re}(\lambda_1)^{-1} \geq \lambda_1^{(H)} = 3\pi^{-1} T^{-1} \).
\( \sim m_w^2 \sim 10^{-2}\text{GeV}^{-1} \sim 10^{-27}\text{s} \) for the electroweak Higgs fields. On the other hand the expansion time scale of the universe in this epoch is

\[
\Delta t_{\text{expand}} = \frac{1}{H} \sim 10^{-12}\text{s}.
\]  

(2.46)

As \( \Delta t_{\text{cl}x} \ll \Delta t_{\text{expand}} \), the system is almost always in the thermal equilibrium.

By integrating over \( \rho_{\phi_c} \) in (2.42) we obtain the probability distribution function \( P_{\text{st}}[\phi_c] \) for \( \phi_c \):

\[
P_{\text{st}} \propto \exp \left[ -\frac{{\mathcal{F}}[\phi_c]}{T} \right],
\]

(2.47)

where

\[
{\mathcal{F}}[\phi_c] := \int d^3x \left[ \frac{1}{2}(\nabla \phi_c)^2 + V_{\text{eff}}(\phi_c) \right],
\]

(2.48)

where \( V_{\text{eff}}(\phi_c) \) is the effective potential with loop corrections. Thus, we find the ordinary expression for the probability distribution function.

We now obtained the basic tools for understanding the dynamics of the electroweak phase transition. In the next section, we estimate the amplitude of the thermal fluctuation by adopting the Gaussian ansatz for the subcritical bubbles.

\section{3. The dynamics of subcritical bubbles}

\subsection{3.1. Derivation of the Langevin equation for the radius}

Having derived the equilibrium probability distribution function (2.42), we can calculate the characteristic spatial scale of thermal fluctuation by adopting, say, a Gaussian ansatz for the Higgs field as was done in the previous paper.\(^9\) However, since we are interested in the kinematics of subcritical bubbles, we adopt a variational principle approach to analyze the effective action directly. We choose a trial functional of the form

\[
\phi_\perp = \phi_0 \exp \left( -\frac{r^2}{R_\perp^2(t)} \right), \quad r := |x|,
\]

(3.1)

to derive the effective Langevin equation for the radius \( R(t) \). The above ansatz is reasonable because its \( O(3) \) symmetry helps to minimize the free energy and we are treating bubble with a thick wall at or above the critical temperature. Here \( \phi_0 \) should be identified with a local minimum of the effective potential which would become the global minimum below the critical temperature and is the most expected value apart from \( \phi = 0 \). We shall investigate the kinematics of subcritical bubbles with amplitude \( \phi_0 \). The detailed discussion for the above ansatz has been done in the previous paper.\(^9\)

Thus the degree of freedom has reduced to one and we then insert Eq. (3.1) into the effective action (2.16). First we calculate the real part. The result is
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\[ \text{Re}[\Gamma(R_+, R_-)] = 2\pi^{3/2} \phi_0^2 \int dt \left[ \frac{15}{32\sqrt{2}} (R_+ R_+ - R_- R_- - \frac{3}{8\sqrt{2}} (R_+ - R_-) \right. \\
\left. \left. - \left( \frac{1}{8\sqrt{2}} m^2 + \frac{\lambda^2}{768 \phi_0^2} \right) (R_+^2 - R_-^2) \right) \right. \\
\left. - 2 \int dt dt' \int \left[ \frac{d^3 k}{(2\pi)^3} \right] (R_+^3(t)e^{-\frac{1}{2}k R_+ R_+^2(t)} - R_-^3(t)e^{-\frac{1}{2}k R_- R_-^2(t)}) \right. \\
\left. \times (R_+^3(t')e^{-\frac{1}{2}k R_+ R_+^2(t')} - R_-^3(t')e^{-\frac{1}{2}k R_- R_-^2(t')}) \right) \mathcal{A}(\mathbf{k}, t - t'), \quad (3\cdot2) \]

where

\[ \mathcal{A}(\mathbf{k}, t - t') = \frac{f^2 \pi^3}{2} \phi_0^2 \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \text{Re}[i S_{\phi}(\mathbf{p}, t - t') S_{\phi}(\mathbf{p} - \mathbf{k}, t' - t)] \theta(t - t'). \quad (3\cdot3) \]

The imaginary part which generates the noise term becomes

\[ \text{Im}[\Gamma(R_+, R_-)] = \frac{1}{2} \int dt dt' \int \frac{d^3 k}{(2\pi)^3} (R_+^3(t)e^{-\frac{1}{2}k R_+ R_+^2(t)} - R_-^3(t)e^{-\frac{1}{2}k R_- R_-^2(t)}) \right. \\
\left. \times (R_+^3(t')e^{-\frac{1}{2}k R_+ R_+^2(t')} - R_-^3(t')e^{-\frac{1}{2}k R_- R_-^2(t')}) \mathcal{B}(\mathbf{k}, t - t'), \quad (3\cdot4) \]

where

\[ \mathcal{B}(\mathbf{k}, t - t') = \frac{f^2 \pi^3}{2} \phi_0^2 \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \text{Im}[i S_{\phi}(\mathbf{p}, t - t') S_{\phi}(\mathbf{p} - \mathbf{k}, t' - t)]. \quad (3\cdot5) \]

This can be rewritten with an auxiliary fields \( \xi(\mathbf{k}, t) \):

\[ \exp[i \times i \text{Im}[\Gamma(R_+, R_-)]] = \int [d\xi] P[\xi] \exp \left[ i \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2} \int dt \xi(\mathbf{k}, t) \right. \\
\left. \times (R_+^3(t)e^{-\frac{1}{2}k R_+ R_+^2(t)} - R_-^3(t)e^{-\frac{1}{2}k R_- R_-^2(t)}) \mathcal{B}(\mathbf{k}, t - t') \delta^3(\mathbf{k} - \mathbf{k}') \xi(\mathbf{k}', t') \right]. \quad (3\cdot6) \]

where

\[ P[\xi] = N \exp \left[ -\frac{1}{2} \int dt dt' \int \frac{d^3 k d^3 k'}{(2\pi)^3} \xi(\mathbf{k}, t) \mathcal{B}^{-1}(\mathbf{k}, t - t') \delta^3(\mathbf{k} - \mathbf{k}') \xi(\mathbf{k}', t') \right]. \quad (3\cdot7) \]

Thus the effective action for \( R_+(t) \) becomes

\[ S_{\text{eff}}(R_+, R_-) = \text{Re}[\Gamma(R_+, R_-)] + \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2} \int dt \xi(\mathbf{k}, t) \right. \\
\left. \times (R_+^3(t)e^{-\frac{1}{2}k R_+ R_+^2(t)} - R_-^3(t)e^{-\frac{1}{2}k R_- R_-^2(t)}) \right]. \quad (3\cdot8) \]

From \( \delta S_{\text{eff}}/\delta \beta = 0 \), we obtain the effective equation for \( R_+(t) \):

\[
\frac{d^2 R_+}{dt^2} + \frac{1}{2R_+} \left( \frac{dR_+}{dt} \right)^2 + \frac{2}{5} \frac{1}{R_+} + \left( \frac{2}{5} m^2 + \frac{\lambda}{60\sqrt{2}} \phi_0^2 + \cdots \right) R_+ = 0.
\]
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Assuming that the time-dependence of $R(t)$ is weak, the last term on the left-hand side of this equation is expanded as

$$
\frac{8\sqrt{2}}{15\pi^{3/2}\phi_0^2} \int_0^t \int_0^\infty \frac{d^3 k}{(2\pi)^3} \tilde{A}(k, t-t') (3R_c(t) - k^2R_c^3(t)) e^{-\frac{1}{2}k^2R_c^2(t')}. $$

The first and second terms in the last expression give the loop correction on the effective potential and friction term $F_f$, respectively. Consequently $F_f \sim \eta R_c$, where $\eta \sim \eta_0/4\sqrt{2}$.

Next we turn to the noise term, which we denote as

$$
\tilde{\xi}(t) := \frac{8\sqrt{2}}{15\pi^{3/2}\phi_0^2} \int_0^t \int_0^\infty \frac{d^3 k}{(2\pi)^3} (3R_c(t) - k^2R_c^3(t)) e^{-\frac{1}{2}k^2R_c^2(t')}. $$

By using the definition $\tilde{\xi}(k, t)$ the self-correlation under the limit $T \gg 1$ is given by

$$
\angle(\tilde{\xi}(k, t)\tilde{\xi}(k', t')) = \mathcal{D}(k, t-t') \delta(k - k')
$$

and then the correlation of $\tilde{\xi}(t)$ becomes

$$
\angle(\tilde{\xi}(t)\tilde{\xi}(t')) = \frac{8}{225\pi^3\phi_0^2R_c^3} \int d^3 k \int d^3 x \angle(\xi(x)\xi(x')) (3R_c^2(t) - k^2R_c^4(t))
\times e^{-\frac{1}{2}k^2R_c^2(t')(3R_c^2(t') - k^2R_c^4(t')) e^{-\frac{1}{2}k^2R_c^2(t')}}
\approx \frac{4\eta T}{15\pi^{3/2}\phi_0^2R_c^2} \delta(t-t') = : \tilde{\eta} \delta(t-t').
$$

In the above result one might suspect the $R_c$-dependence of the diffusion coefficient, but one shall understand by the later discussion that it is a quite reasonable result. Consequently the Langevin equation becomes

$$
\frac{d^2R_c}{dt^2} + \frac{1}{2R_c} \left( \frac{dR_c}{dt} \right)^2 + \frac{2}{5} \frac{1}{R_c} + \alpha(T)R_c = -\tilde{\eta} \frac{dR_c}{dt} + \tilde{\xi}(t),
$$
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where \( \eta := (\pi / 12\sqrt{2}) T \) and

\[
\alpha(T) = \frac{2}{5} m^2 + \frac{\lambda}{60\sqrt{2}} \phi_0^2 + \cdots.
\]

(3.15)

3.2. The Fokker-Planck equation for the radius and its static solution

In order to find the equilibrium state we transform the Langevin equation (3.14) to the Fokker-Planck equation. The Langevin equation can be rewritten as

\[
\frac{dP_c}{dt} = \frac{P_c^2}{2R_c M} - \frac{2}{5} \frac{M}{R} + M \alpha(T) R_c - \eta P_c + M \xi(t),
\]

(3.16)

where \( P_c \) is the canonical momentum defined by

\[
P_c := \frac{\partial L_{\text{eff}}}{\partial \dot{R}_c} = \frac{15\pi^{3/2}}{8\sqrt{2}} \phi_0^2 R_c M = M \dot{R}_c.
\]

(3.17)

with \( M(T, R_c) = (15\pi^{3/2} \phi_0^2 R_c / 8\sqrt{2}) \) and \( L_{\text{eff}} \) is the Lagrangian of the deterministic part of the Langevin equation:

\[
L_{\text{eff}}(R_c, V_c) = \frac{1}{2} M V_c^2 - \frac{2}{5} M - \frac{1}{3} M \alpha(T) R_c^2.
\]

(3.18)

Now we introduce the probability distribution function \( W(R_c, P_c; t) \) of \( R_c \) and \( P_c \). Then we obtain the following Fokker-Planck equation from the above Langevin equation:

\[
\frac{\partial}{\partial t} W(R_c, P_c; t) = \frac{\partial}{\partial R_c} \left( -\frac{P_c}{M} W \right) + \frac{\partial}{\partial P_c} \left[ -\frac{P_c^2}{2R_c M} + \frac{2}{5} \frac{M}{R_c} + M \alpha(T) R_c + \eta P_c \right] W
\]

\[
+ \frac{1}{2} M^2 \eta \frac{\partial^2}{\partial P_c^2} W.
\]

(3.19)

The above equation can be expressed by using the effective Hamiltonian

\[
H_{\text{eff}}(R_c, P_c) := P_c \dot{R}_c - L_{\text{eff}} = \frac{1}{2} M P_c^2 + \frac{2}{5} M + \frac{1}{3} M \alpha(T) R_c^2
\]

(3.20)

as

\[
\frac{\partial}{\partial t} W(R_c, P_c; t) = \frac{\partial}{\partial R_c} \left( -\frac{\partial H_{\text{eff}}}{\partial P_c} W \right) + \frac{\partial}{\partial P_c} \left[ \left( \frac{\partial H_{\text{eff}}}{\partial R_c} + M \eta \frac{\partial H_{\text{eff}}}{\partial P_c} \right) W \right]
\]

\[
+ \frac{1}{2} M^2 \eta \frac{\partial^2}{\partial P_c^2} W.
\]

(3.21)

The probability current density becomes
\[ J = \begin{pmatrix} 0 & 1 \\ -1 & -M\eta \end{pmatrix} \left( \begin{pmatrix} \frac{\partial H_{\text{eff}}}{\partial R_c} W + T \frac{\partial W}{\partial R_c} \\ \frac{\partial H_{\text{eff}}}{\partial P_c} W + T \frac{\partial W}{\partial P_c} \end{pmatrix} \right), \]

and Eq. (3·21) reduces to \( \partial_t W + \nabla \cdot J = 0 \), where \( \nabla = (\partial_{R_c}, \partial_{P_c}) \). Here we have used the relation between the diffusion and friction coefficients:

\[ \frac{1}{2} M^2 \tilde{\eta} = TM\eta. \]  (3·22)

Then the static solution becomes

\[ W_{\text{st}}(R, P) \propto \exp \left( -\frac{H_{\text{eff}}}{T} \right). \]  (3·23)

As in the case of Higgs fields the relaxation time scale is equal to \( \eta^{-1} \) and is again much shorter than that of the expansion of the universe. Thus the distribution is always in the stationary state.

Finally, we note the fluctuation-dissipation relation exactly holds. Let us recall the case of the Brownian motion of a massive particle. In such a system the relation between the diffusion and friction coefficients is

\[ (\text{Diffusion coefficient}) = (\text{Friction coefficient}) \times (\text{Temperature}) \times (\text{Mass}). \]  (3·24)

As the mass term corresponds to \( M \) in the present case, the above relation also holds. The \( R_c \)-dependence of \( W_{\text{st}} \) is a direct consequence of the three-dimensional volume effect. Further this dependence is physically reasonable. When one compares large bubbles and small bubbles, the destruction due to the thermal noise is not effective for large one.

3.3. The size of subcritical bubbles and the amplitude of the thermal fluctuation

Now we can estimate the amplitude of the thermal fluctuation. We first calculate the averaged radius of the subcritical bubble. Note that previously all authors have assumed the correlation length for the typical spatial scale of the thermal fluctuation. As we will see soon, the scale is a calculable quantity in high temperature phase. In this subsection, we extend the above analysis to the electroweak phase transition. For this purpose we start from the following effective Hamiltonian corresponding to Eq. (3·20):

\[ H_{\text{eff}}(R_c, P_c) = \frac{1}{2M} P_c^2 + \frac{2}{5} M + \frac{1}{3} M a_{\text{ew}}(T) R_c^2, \]  (3·25)

where

\[ a_{\text{ew}}(T) = \frac{4}{5} D(T^2 - T_2^2) - \frac{8\sqrt{2}}{15\sqrt{3}} ET\phi_0 + \frac{1}{10\sqrt{2}} \phi_0^2 \lambda_T . \]  (3·26)

and
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\[ \phi_0 = \frac{3ET}{2\lambda_T} \left[ 1 + \sqrt{1 - \frac{8\lambda_T D}{9E^2 T^2}} \left( T^2 - T_0^2 \right) \right]. \]  

(3.27)

In the above expression, some coefficients can be fixed by electroweak particles:

\[ D = \frac{1}{24} \left[ 6 \left( \frac{m_w}{\sigma} \right)^2 + 3 \left( \frac{m_Z}{\sigma} \right)^2 + 6 \left( \frac{m_t}{\sigma} \right)^2 \right] \sim 0.169, \]  

(3.28)

\[ E = \frac{1}{12\pi} \left[ 6 \left( \frac{m_w}{\sigma} \right) + 3 \left( \frac{m_Z}{\sigma} \right)^2 \right] \sim 0.00965, \]  

(3.29)

\[ \lambda_T = \frac{1}{16\pi^2} \left[ \sum_b g_b \left( \frac{m_b}{\sigma} \right)^4 \ln \left( \frac{m_b^2}{c_b T^2} \right) - \sum_f g_F \left( \frac{m_F}{\sigma} \right)^4 \ln \left( \frac{m_F^2}{c_F T^2} \right) \right] \sim 0.0350, \]  

(3.30)

\[ T_2 = \sqrt{\left( m_H^2 - 8\alpha^2 \right)/4D} \]  

(3.31)

and

\[ B = \frac{1}{64\pi^2} \left[ 6 \left( \frac{m_w}{\sigma} \right)^2 + 3 \left( \frac{m_Z}{\sigma} \right)^4 - 12 \left( \frac{m_t}{\sigma} \right)^2 \right] \sim -0.00456, \]  

(3.32)

where we have used \( m_w = 80.6 \) GeV, \( m_Z = 91.2 \) GeV, \( m_t = 174 \) GeV and \( \sigma = 246 \) GeV.\(^{14}\)

Further we have assumed \( m_H = 60 \) GeV which is the minimum value experimentally allowed.

Let us sketch the potential change with temperature. As the temperature decreases, the non-symmetric vacuum appears at \( T = T_1 = T_2 / \sqrt{1 - 9E^2 / 8\lambda_T D} \sim 93.52 \) GeV and then the hight of the two vacua coincides with each other at the critical temperature \( T = T_c = T_2 / \sqrt{1 - E^2 / \lambda_T D} \sim 93.43 \) GeV. Finally, the barrier between two vacua disappears at \( T = T_2 \).

As there exists the interaction with other particles in the present case, the expression of the friction and diffusion term obtained in the previous section should be modified accordingly. However, as long as the fluctuation dissipation relation holds, which has been confirmed in the presence of bosonic interaction in Ref. 7) and fermionic interaction in the present paper, the distribution of Higgs fields is given by

\[ W_{st} \propto \exp \left( -\frac{H_{\text{eff}}}{T} \right). \]  

(3.33)

Because the system is always in the stationary state, the typical scale can be calculated by the ordinary canonical averaging. In the high temperature phase the thermal average of \( R \) is given by

\[ \langle R_c \rangle = \frac{\int dP_c dR_c e^{\frac{H_{\text{eff}}}{T}}}{\int dP_c e^{\frac{H_{\text{eff}}}{T}}} \]  

\[ = \int_0^\infty dR_c R_c^{3/2} \exp \left[ -\frac{2M}{5T} \frac{M_{\text{dew}}}{3T} R_c^2 \right] \]  

\[ = \int_0^\infty dR_c R_c^{1/2} \exp \left[ -\frac{2M}{5T} \frac{M_{\text{dew}}}{3T} R_c^2 \right]. \]
In the low temperature phase, $\alpha_{ew}(T)$ is negative and accordingly the averaged radius diverges. However, this is not relevant. We are interested in whether the system has supercooling or not in the high temperature phase only.

Having seen that $\langle R \rangle$ is the important scale of subcritical bubbles near the critical temperature, we shall now estimate the amplitude of the fluctuation of $\phi$ around $\phi=0$ on this particular scale, adopting the trial function

$$\phi = \phi_0 \exp \left( - \frac{r^2}{\langle R \rangle^2} \right).$$

Then the free energy becomes

$$F(\phi_0, T) = \left[ \frac{3\pi^{3/2}}{4\sqrt{2}} \langle R \rangle + \frac{\pi^{3/2}}{2\sqrt{2}} D(T^2 - T_\ast^2) \langle R \rangle^3 \right] \phi_0^2 - \frac{\pi^{3/2}}{3\sqrt{3}} E T \phi_0^3 \langle R \rangle^3 + \frac{\pi^{3/2}}{32} \lambda T \phi_0^4 \langle R \rangle^3. \quad (3.36)$$

Thus the RMS amplitude of $\phi$ at the symmetric vacuum is

$$\sqrt{\phi^2} = \sqrt{\frac{\int d\phi \phi^2 e^{-F(\phi_0, T)/T}}{\int d\phi e^{-F(\phi_0, T)/T}}} \sim \frac{\phi_0}{\sqrt{3 + \frac{16 D(T^2 - T_\ast^2) T^2}{\pi^3 \phi_0^4}}}.$$ \quad (3.37)

Its temperature dependence is depicted in Fig. 1. In the same way as in the case of $\langle R \rangle$, the above approximation result gives the lower bound for $\sqrt{\phi^2}$. At $T = T_c$ and $T = T_1$, one obtains numerically $\sqrt{\phi^2}(T = T_c) = 36.2$ GeV and $\sqrt{\phi^2}(T = T_1) = 28.2$ GeV, respectively. This exceeds the first reflection point $\phi_* = (ET/\lambda T)$

$$\sqrt{(E^2 T^2 / \lambda T^2) - (2D(T^2 - T_\ast^2)/3\lambda T)},$$

which implies that the perturbation theory breaks down and that one can no longer conclude that electroweak phase transition is of first order.

The above argument implies that the phase transition is not accompanied by supercooling. The transition rate between two vacua is much larger than the expansion rate of the universe. In fact, it is given by

Fig. 1. The RMS of thermal fluctuations.

The unit of the vertical axis is GeV.
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\[ \Gamma \sim m_h \exp \left( -\frac{F(\phi_0, T)}{T} \right), \quad (3.38) \]

where \( F \) is the free energy of subcritical bubble configuration. As \( F \sim T \) during the high temperature phase, \( \Gamma \sim 10^5 \text{ GeV} (\gg H \sim 10^{-13} \text{ GeV}) \). Hence the fraction of two vacua is determined by the height of the potential and therefore the fraction is almost the same at \( T = T_c \). This means that any supercooling does not happen and critical bubbles cannot be borne.

§ 4. Summary and discussion

In the present paper we have examined the relaxation of the system from non-equilibrium states and found that the system settles down rapidly to the stationary distribution, which is simply the ordinary thermal distribution. For the Higgs fields it is

\[ P_{\text{eq}}[\phi] \propto \exp \left( -\frac{\mathcal{F}[\phi]}{T} \right), \quad (4.1) \]

where \( \mathcal{F}[\phi] \) is the free energy. For the radius of subcritical bubbles,

\[ W_{\text{eq}}(R, P) \propto \exp \left( -\frac{H_{\text{eff}}(R, P)}{T} \right), \quad (4.2) \]

where \( H_{\text{eff}}(R, P) \) is the effective Hamiltonian. First we estimated by using \( W_{\text{eq}}(R, P) \) the mean radius of subcritical bubbles which determines the spatial scale of the thermal fluctuation. Next we estimated the field fluctuation by using \( P_{\text{eq}}[\phi] \). We conclude that the electroweak phase transition is quite weakly of first order and therefore the standard baryogenesis does not work in minimal standard model, apart from the too small magnitude of \( CP \) violation.

Writing down directly the Langevin equation for the radius we have a glimpse of the kinematics of subcritical bubbles. In the fluctuation-dissipation relation the dependence on radius is a natural result from the three volume effect. For larger subcritical bubble the effect of diffusion is smaller.

Finally we discuss validity of some approximations employed here. First in the derivation of the Langevin equation (2.31) we have assumed the approximate homogeneity on the Higgs fields. In the derivation of the Langevin equation (3.14) for the radius we have used a similar approximation, which is correct if \( T > \langle R_c \rangle^{-1} \). In fact \( \langle R_c \rangle^{-1}/T \sim (E/\lambda T)^2 \sim 0.1 \). Thus the approximation is justified. Second, we have taken the strong coupling limit to obtain the white noise. In practice it is colored and in general the diffusion term is written by an integral expression, which would be too complicated to treat analytically. We expect, however, as long as we focus on a time scale larger than \( \Gamma^{-1} \) the more elaborate analysis of the colored noise would yield the same result and that in this sense our simplified analysis suffices. We hope to consider this subject further in a future publication. Third, we did not work out the calculation of the loop correction by gauge bosons on the friction and diffusion coefficients. Although it is essential to obtain the cubic term in the effective poten-
tial, their contribution to the friction and diffusion coefficients is smaller than top quark.

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