ON CERTAIN ARITHMETIC FUNCTIONS INVOLVING
EXponential DIVisors, II.

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Dedicated to the memory of Professor M. V. Subbarao

Annales Univ. Sci. Budapest., Sect. Comp. 27 (2007), 155-166

1. Introduction

Let $n > 1$ be an integer of canonical form $n = p_1^{a_1} \cdots p_r^{a_r}$. The integer $d$ is called an exponential divisor (e-divisor) of $n$ if $d = p_1^{b_1} \cdots p_r^{b_r}$, where $b_1 \mid a_1, \ldots, b_r \mid a_r$, notation: $d \mid_e n$. By convention $1 \mid_e 1$. The integer $n > 1$ is called exponentially squarefree (e-squarefree) if all the exponents $a_1, \ldots, a_r$ are squarefree. The integer 1 is also considered to be e-squarefree.

Observe that $|\mu^e(n)| = 1$ or 0, according as $n$ is e-squarefree or not. For properties and generalizations of the e-convolution see [3], [4].

Other arithmetic functions regarding e-divisors, for example the number and the sum of e-divisors of $n$ were investigated by several authors, see the references given in the first part [11] of the present paper. We show that the corresponding error term can further be improved on the assumption of the Riemann hypothesis (RH), see Theorem 3.

In Theorem 2 we give a formula for $\sum_{n \leq x} \mu^e(n)$ without and with assuming RH. As far as we know there is no such result in the literature. We show that the error terms depend on estimates for the number of squarefree integers $\leq x$.

Consider now the exponential squarefree exponential divisors (e-squarefree e-divisors) of $n$. Here $d = p_1^{b_1} \cdots p_r^{b_r}$ is an e-squarefree e-divisor of $n = p_1^{a_1} \cdots p_r^{a_r} > 1$, if $b_1 \mid a_1, \ldots, b_r \mid a_r$ and $b_1, \ldots, b_r$ are squarefree. Note that the integer 1 is e-squarefree and it is not an e-divisor of $n > 1$.

We introduce the functions $t^e$ and $\kappa^e$, where $t^e(n)$ and $\kappa^e(n)$ denote the number of e-squarefree e-divisors of $n$ and the maximal e-squarefree e-divisor of $n$, respectively. These are the
exponential analogues of the functions representing the number of squarefree divisors of \( n \) (i.e. \( \theta(n) = 2^{\omega(n)} \), where \( \omega(n) = r \)) and the maximal squarefree divisor of \( n \) (the squarefree kernel \( \kappa(n) = \prod_{p|n} p \)), respectively.

The functions \( f(n) \) and \( \kappa(n) \) are multiplicative and for \( n = p_1^{\alpha_1} \cdots p_r^{\alpha_r} > 1 \),
\[
 t(n) = 2^{\omega(\alpha_1)} \cdots 2^{\omega(\alpha_r)}, \\
 \kappa(n) = p_1^{\kappa(\alpha_1)} \cdots p_r^{\kappa(\alpha_r)}.
\]

Asymptotic properties of the functions \( t(n) \) and \( \kappa(n) \) are given in Theorems 4, 5 and 7.

2. Results

The function \( \mu(n) \) is multiplicative and \( \mu(p^a) = \mu(a) \) for every prime power \( p^a \). Hence \( \mu(n) \in \{-1, 0, 1\} \) for every \( n \geq 1 \) and for every prime \( p \), \( \mu(p) = 1 \), \( \mu(p^2) = -1 \), \( \mu(p^3) = -1 \), \( \mu(p^4) = 0 \), .

According to a well-known result of H. Delange, cf. [1], Ch. 6, the function \( \mu(n) \) has a non-zero mean value given by
\[
m(\mu(n)) = \prod_p \left(1 + \sum_{a=2}^{\infty} \mu(a) - \mu(a-1) \right).
\]

An asymptotic formula for \( \mu(n) \) can be obtained from the following general result, which may be known.

**Theorem 1.** Let \( f \) be a complex valued multiplicative function such that \( |f(n)| \leq 1 \) for every \( n \geq 1 \) and \( f(p) = 1 \) for every prime \( p \). Then
\[
\sum_{n \leq x} f(n) = m(f)x + O(x^{1/2} \log x),
\]
where
\[
m(f) = \prod_p \left(1 + \sum_{a=2}^{\infty} f(p^a) - f(p^{a-1}) \right).
\]
is the mean value of \( f \).

Theorem 1 applies also for the multiplicative functions \( f = \mu^{\star}(n) \) and \( f = F \), where \( \mu^{\star}(p^a) = \mu^{\star}(a) = (-1)^{\omega(a)} \) representing the unitary exponential Möbius function, cf. [3], and \( F(p^a) = \lambda(a) = (-1)^{\Omega(a)} \) the Liouville function, with \( \Omega(a) \) denoting the number of prime power divisors of \( a \).

We prove for \( \mu^{\star}(n) \) the following more precise result.

**Theorem 2.** (i) The Dirichlet series of \( \mu^{\star}(n) \) is of form
\[
\sum_{n=1}^{\infty} \frac{\mu^{\star}(n)}{n^s} = \frac{\zeta(s)}{\zeta^2(2s)} U(s), \quad \text{Re} s > 1,
\]
where \( U(s) := \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \) is absolutely convergent for \( \text{Re} s > 1/5 \).

(ii)
\[
\sum_{n \leq x} \mu^{\star}(n) = m(\mu^{\star})x + O(x^{1/2} \exp(-c(\log x)^{\Delta})),
\]
where \( \Delta < 9/25 = 0.36 \) and \( c > 0 \) are constants.

(iii) Assume RH. Let \( 1/4 < r < 1/3 \) be an exponent such that \( D(x) := \sum_{n \leq x} \mu^2(n) - x/\zeta(2) = O(x^{1+\varepsilon}) \) for every \( \varepsilon > 0 \). Then the error term in (ii) is \( O(x^{(2-r)/(12-4r)+\varepsilon}) \) for every \( \varepsilon > 0 \).

The best known value – to our knowledge – of \( r \) is \( r = 17/54 \approx 0, 314814 \), obtained in [2], therefore the error term in (ii), assuming RH, is \( O(x^{0.91/202+\varepsilon}) \) for every \( \varepsilon > 0 \), where 91/202 \approx 0, 450495.
Theorem 3. If RH is true, then
\[ \sum_{n \leq x} |\mu^{(c)}(n)| = \prod_{p} \left( 1 + \sum_{a=4}^{\infty} \frac{\mu^2(a) - \mu^2(a-1)}{p^a} \right) x + O(x^{1/5+\varepsilon}), \]
for every \( \varepsilon > 0 \).

The function \( t^{(r)} \) is multiplicative and \( t^{(r)}(p^a) = 2^{\omega(a)} \) for every prime \( p^a \). Here for every prime \( p \), \( t^{(r)}(p) = 1 \), \( t^{(r)}(p^2) = t^{(r)}(p^3) = t^{(r)}(p^5) = 2 \), \( t^{(r)}(p^7) = 4 \), ... .

Theorem 4. (i) The Dirichlet series of \( t^{(c)} \) is of form
\[ \sum_{n=1}^{\infty} \frac{t^{(c)}(n)}{n^s} = \zeta(s)\zeta(2s)V(s), \quad \text{Re } s > 1, \]
where \( V(s) = \sum_{n=1}^{\infty} \frac{\upsilon(n)}{n} \) is absolutely convergent for \( \text{Re } s > 1/4 \).

(ii) \( \sum_{n \leq x} t^{(c)}(n) = C_1 x + C_2 x^{1/2} + O(x^{1/4+\varepsilon}) \),
for every \( \varepsilon > 0 \), where \( C_1, C_2 \) are constants given by
\[ C_1 := \prod_{p} \left( 1 + \frac{1}{p^2} + \sum_{a=6}^{\infty} \frac{2^{\omega(a)} - 2^{\omega(a-1)}}{p^a} \right), \]
\[ C_2 := \zeta(1/2) \prod_{p} \left( 1 + \sum_{a=4}^{\infty} \frac{2^{\omega(a)} - 2^{\omega(a-1)} - 2^{\omega(a-2)} + 2^{\omega(a-3)}}{p^{a/2}} \right). \]

Theorem 5.
\[ \limsup_{n \to \infty} \frac{\log t^{(c)}(n) \log \log n}{\log n} = \frac{1}{2} \log 2. \]

The function \( \kappa^{(c)} \) is multiplicative and \( \kappa^{(c)}(p^a) = p^{\kappa(a)} \) for every prime power \( p^a \). Hence for every prime \( p \), \( \kappa^{(c)}(p) = p \), \( \kappa^{(c)}(p^2) = p^2 \), \( \kappa^{(c)}(p^3) = p^3 \), \( \kappa^{(c)}(p^4) = p^2 \), ... .

To obtain an asymptotic formula for \( \kappa^{(c)} \) we use the following general theorem, of which parts (i) and (ii) are a variant of a result given in [6] and cited in the first part [11] of this paper.

Theorem 6. Let \( k \geq 2 \) be a fixed integer and \( f \) be a complex valued multiplicative arithmetic function satisfying
\( (a) \ f(p) = f(p^2) = ... = f(p^{k-1}) = 1 \) for every prime \( p \),
\( (b) \ there \ exists \ K > 0 \ such \ that \ |f(p^a)| \leq K \ for \ every \ prime \ power \ p^a \ with \ a \geq k + 1 \),
\( (c) \ there \ exist \ M > 0 \ and \ \beta \geq 1/(k + 1) \ such \ that \ |f(p^k)| \leq Mp^{-\beta} \ for \ every \ prime \ p \).

Then
(i) \( \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \frac{\zeta(s)}{\zeta(k s)}W(s), \quad \text{Re } s > 1, \)
where the Dirichlet series \( W(s) := \sum_{n=1}^{\infty} \frac{\upsilon(n)}{n} \) is absolutely convergent for \( \text{Re } s > 1/(k + 1) \).

(ii) \( \sum_{n \leq x} f(n) = C_f x + O(x^{1/k}\delta(x)) \),
where
\[ C_f := \prod_{p} \left( 1 + \sum_{a=k}^{\infty} \frac{f(p^a) - f(p^{a-1})}{p^a} \right). \]
and

\[ \delta(x) = \delta_A(x) := \exp(-A(\log x)^{3/5}(\log \log x)^{-1/5}), \]

A being a positive constant.

(iii) If RH is true, then the error term is \( O(x^{1/(k+1)+\varepsilon}) \) for every \( \varepsilon > 0 \).

**Theorem 7.**

\[
\sum_{n \leq x} \kappa^{(e)}(n) = \frac{1}{2} \prod_p \left(1 + \sum_{a=4}^{\infty} \frac{p^{\kappa(a)} - p^{1+\kappa(a)-1}}{p^a}\right) x^2 + O(x^{5/4}\delta(x)).
\]

If RH is true, then the error term is \( O(x^{6/5+\varepsilon}) \) for every \( \varepsilon > 0 \).

3. Proofs

**Proof of Theorem 1.** Let \( g = f \ast \mu \) in terms of the Dirichlet convolution. Then \( g \) is multiplicative, \( g(p) = f(p) - 1 = 0 \), \( g(p^a) = f(p^a) - f(p^{a-1}) \) and \( |g(p^a)| \leq |f(p^a)| + |f(p^{a-1})| \leq 2 \) for every prime \( p \) and every \( a \geq 2 \). Therefore \( |g(n)| \leq \ell(n)2^{\omega(n)} \) for every \( n \geq 1 \), where \( \ell(n) \) is the characteristic function of the squarefull integers and we have

\[
\sum_{n \leq x} f(n) = \sum_{d \leq x} g(d) = \sum_{d \leq x} g(d) \left( \frac{x}{d} + O(1) \right) = x \sum_{d \leq x} \frac{g(d)}{d} + O \left( \sum_{d \leq x} |g(d)| \right) = x \sum_{d=1}^{\infty} \frac{g(d)}{d} + O \left( x \sum_{d>x} \ell(d)2^{\omega(d)} \right) + O \left( \sum_{d \leq x} \ell(d)2^{\omega(d)} \right).
\]

Here

\[ \ell(n)2^{\omega(n)} = \sum_{d^e=n} \tau(d)h(e), \]

where \( \tau \) is the divisor function and \( h \) is given by

\[
\sum_{n=1}^{\infty} \frac{h(n)}{n^s} = \prod_p \left(1 + \frac{2}{p^{3s}} - \frac{1}{p^{4s}} - \frac{2}{p^{6s}} \right),
\]

absolutely convergent for \( \Re s > 1/3 \), cf. [7]. We obtain

\[
\sum_{n \leq x} \ell(n)2^{\omega(n)} = \sum_{e \leq x} h(e) \sum_{d \leq (x/e)^{1/2}} \tau(d) = \sum_{e \leq x} h(e) O \left( (x/e)^{1/2} \log(x/e) \right) = O \left( x^{1/2} \log x \sum_{e \leq x} |h(e)e^{-1/2} | \right) = O \left( x^{1/2} \log x \right),
\]

and by partial summation,

\[
\sum_{n > x} \frac{\ell(n)2^{\omega(n)}}{n} = O \left( x^{-1/2} \log x \right),
\]

which finishes the proof.

**Proof of Theorem 2.** (i) Let \( \mu_2(n) = \mu(m) \) or 0, according as \( n = m^2 \) or not, and let \( E_2(n) = 1 \) or 0, according as \( n = m^2 \) or not. The given equality is verified for \( \mu^{(e)} = \mu^2 \ast \mu_2 \ast u \), equivalent to \( u = \mu^{(e)} \ast \lambda \ast E_2 \), in terms of the Dirichlet convolution, where \( \lambda \) is the Liouville function. It is easy to check that \( u(p) = u(p^2) = u(p^3) = u(p^4) = 0 \), \( |(\lambda \ast E_2)(p^a)| \leq a \) for every prime power \( p^a \) with \( a \geq 1 \), hence \( |u(p^b)| \leq 1 + \sum_{a=1}^{b} |(\lambda \ast E_2)(p^a)| < b^2 \) for every prime power \( p^b \) with \( b \geq 5 \). We obtain that the Dirichlet series of the function \( u \) is absolutely convergent for \( \Re s > 1/5 \).
(ii) According to (i), \( \sum_{n \leq x} \mu^{(e)}(n) = \sum_{n \leq x} u(n)S(x/n) \), where

\[
S(x) := \sum_{n \leq x} \mu^2(n)\mu(d).
\]

We first estimate the sum \( S(x) \). Let \( \varrho = \varrho(x) \) such that \( 0 < \varrho < 1 \) to be defined later. If \( nd^2 \leq x \), then both \( n > \varrho^{-2} \) and \( d > \varrho \sqrt{x} \) can not hold good in the same time, therefore

\[
S(x) = \sum_{nd^2 \leq x \atop d \leq \sqrt{\varrho x}} \mu^2(n)\mu(d) + \sum_{n \leq \varrho^{-2} \atop \mu^2(n)\mu(d) - \sum_{d \leq \varrho^{-2}} \mu^2(n)\mu(d) = S_1(x) + S_2(x) - S_3(x),
\]

say. We use the following estimates of A. Walfisz [13]:

\[
M(x) := \sum_{n \leq x} \mu(n) = O(x\delta(x)), \quad E(x) := \sum_{n \leq x} \mu^2(n) = \frac{x}{\zeta(2)} + O(x^{1/2}\delta(x)).
\]

Note that \( \delta(x) \), defined in Section 2, is decreasing and \( x^\varepsilon \delta(x) \) is increasing for every \( \varepsilon > 0 \). By partial summation,

\[
R(x) := \sum_{n > x} \frac{\mu(n)}{n^2} = O(x^{-1/2}\delta(x)).
\]

Here

\[
S_1(x) = \sum_{d \leq \sqrt{\varrho}x} \mu(d)E(x/d^2) = \frac{x}{\zeta(2)} \sum_{d \leq \varrho \sqrt{x}} \frac{\mu(d)}{d^2} + O \left( x^{1/2} \sum_{d \leq \varrho \sqrt{x}} \frac{\delta(x/d^2)}{d} \right) =
\]

\[
= \frac{x}{\zeta(2)} \left( \frac{1}{\zeta(2)} - R(\varrho \sqrt{x}) \right) + O \left( x^{1/2}\varrho^{-2} \sum_{d \leq \sqrt{\varrho}x} \frac{1}{d} \right) =
\]

\[
= \frac{x}{\zeta^2(2)} + O \left( \varrho^{-1}x^{1/2}\delta(\varrho \sqrt{x}) \right) + O \left( x^{1/2}\varrho^{-2}\log x \right),
\]

\[
S_2(x) = \sum_{n \leq \varrho^{-2}} \mu^2(n)M((x/n)^{1/2}) = O \left( \sum_{n \leq \varrho^{-2}} (x/n)^{1/2}\delta((x/n)^{1/2}) \right) = O \left( \delta(\varrho \sqrt{x})x^{1/2} \sum_{n \leq \varrho^{-2}} \frac{1}{\sqrt{n}} \right) =
\]

\[
= O \left( \varrho^{-1}x^{1/2}\delta(\varrho \sqrt{x}) \right),
\]

\[
S_3(x) = M(\varrho \sqrt{x})E(\varrho^{-2}) = O \left( \varrho^{-1}x^{1/2}\delta(\varrho \sqrt{x}) \right).
\]

We obtain that

\[
S(x) = \frac{x}{\zeta^2(2)} + O \left( \varrho^{-1}x^{1/2}\delta(\varrho \sqrt{x}) \right) + O \left( x^{1/2}\delta(1/\varrho^2)\log x \right).
\]

Take \( \varrho = \exp(-\log x)^\beta \), where \( 0 < \beta < 1 \). Then \( \varrho \sqrt{x} = \exp(\frac{1}{2}(\log x)(-\log x)^\beta) \geq \exp\left(\frac{1}{4}(\log x)\right) = x^{1/4} \) for sufficiently large \( x \). Hence \( \delta(\sqrt{x}) \leq \delta(x^{1/4}) \leq \delta_B(x) \) with a suitable constant \( B > 0 \). For \( \beta < 3/5 \) we obtain \( \varrho^{-1}\delta(\sqrt{x}) \ll \exp((-\log x)\beta - B(\log x)^{3/5}(\log x)^{-1/5}) \ll \delta_C(x) \) with a suitable constant \( C > 0 \).

If \( \eta < 3/5 \), then \( \delta_B(x) \ll \exp(-A(\log x)^\beta) \) and obtain that \( \delta(\varrho^{-2}) \ll \exp(-A(2(\log x)^\beta)^\eta) = \exp(-D(\log x)^{\beta\eta}) \) with a suitable \( D > 0 \), where \( \beta\eta < 9/25 \).

Therefore,

\[
S(x) = \frac{x}{\zeta^2(2)} + O \left( x^{1/2} \exp(-c(\log x)^\Delta) \right),
\]

\[5\]
where \( \Delta < 9/25 \) and \( c > 0 \) are constants. Now,

\[
\sum_{n \leq x} \mu(x) n S(x/n) = \sum_{n \leq x} u(n) \left( \frac{x}{\zeta^2(2)n} + O \left( \left( \frac{x}{n} \right)^{1/2} \exp(-c(\log(x/n))^{\Delta}) \right) \right) = \frac{x}{\zeta^2(2)} \sum_{n \leq x} u(n) + O \left( \frac{x^{1/2}}{n^{1/2}} \sum_{n \leq x} |u(n)| \exp(-c(\log(x/n))^{\Delta}) \right),
\]

where, using that \( x^\varepsilon \exp(-c(\log x)^{\Delta}) \) is increasing for any \( \varepsilon > 0 \), the \( O \)-term is

\[
O \left( x^{1/2} \sum_{n \leq x} \frac{|u(n)|}{n^{1/2}} \left( \frac{x}{n} \right)^{-\varepsilon} \exp(-c(\log(x/n))^{\Delta}) \right) = O \left( x^{1/2} \sum_{n \leq x} \frac{|u(n)|}{n^{1/2}} \right) = O \left( x^{1/2} \exp(-c(\log(x))^{\Delta}) \right),
\]

for \( 1/2 - \varepsilon > 1/5 \). Furthermore,

\[
\sum_{n \leq x} \frac{u(n)}{n} = U(1) + O \left( \sum_{n > x} \frac{|u(n)|}{n} \right),
\]

with \( U(1) = \zeta^{-2}(2)m(\mu(x)) \) and \( \sum_{n > x} \frac{|u(n)|}{n} = O \left( x^{-3/5} \sum_{n > x} \frac{|u(n)|}{n^{3/2}} \right) = O(x^{-3/5}) \), which finishes the proof of (ii).

(iii) Assume RH. We use that, see \( [10] \),

\[
M(x) := \sum_{n \leq x} \mu(n) = O \left( x^{1/2} \omega(x) \right),
\]

where \( \omega(x) := \exp(A(\log x)(\log \log x)^{-1}) \), \( A \) being a positive constant, which gives by partial summation,

\[
R(x) := \sum_{n \leq x} \frac{\mu(n)}{n^2} = O(x^{-3/2} \omega(x)).
\]

Suppose that \( D(x) := \sum_{n \leq x} \mu^2(n) - \frac{x}{\zeta(2)} = O(x^{r+\varepsilon}) \) for every \( \varepsilon > 0 \), where \( 1/4 < r < 1/3 \). Then we obtain by similar computations that

\[
S_1(x) = \frac{x}{\zeta^2(2)} + O \left( \frac{x^{3/2} x^{1/4} \omega(\sqrt{\theta} x)}{\theta^{1/2} \theta^{1-2(r+\varepsilon)}} \right), \quad S_2(x) = O \left( \frac{x^{-3/2} x^{1/4} \omega(\sqrt{\theta} x)}{\theta^{1/2} \theta^{1-2(r+\varepsilon)}} \right), \quad S_3(x) = O \left( \frac{x^{-3/2} x^{1/4} \omega(\sqrt{\theta} x)}{\theta^{1/2} \theta^{1-2(r+\varepsilon)}} \right),
\]

Therefore

\[
S(x) = \frac{x}{\zeta^2(2)} + O \left( \frac{x^{-3/2} x^{1/4} \omega(\sqrt{\theta} x)}{\theta^{1/2} \theta^{1-2(r+\varepsilon)}} \right) + O \left( x^{1/2} \theta^{1-2(r+\varepsilon)} \right).
\]

Choose \( \theta = x^{-t} \), \( t > 0 \). Then \( x^{-3/2} x^{1/4} = x^{(6t+1)/4} \theta \sqrt{\theta} x = x^{1/2-t} \theta \sqrt{\theta} < x \), hence \( \omega(\sqrt{\theta} x) \ll \omega(x) \ll x^\varepsilon \) for every \( \varepsilon > 0 \) and obtain

\[
S(x) = \frac{x}{\zeta^2(2)} + O \left( x^{(6t+1)/4} \right) + O \left( x^{1/2-t(1-2r)+\varepsilon} \right).
\]

Take \( (6t+1)/4 = 1/2 - t(1-2r) \), this gives \( t = 1/(10-8r) \) leading to the common value \( (2-r)/(5-4r) + \varepsilon \) of the exponents.

**Proof of Theorem 3.** Apply Theorem 6 for \( f(n) = |\mu(x)|, k = 4 \) on the assumption of RH.

**Proof of Theorem 4.** The proof is similar to the proof of Theorem 1 of \( [11] \), see also \( [12] \) for a more general result of this type.
(i) To obtain the given equality let \( f = \mu_2 * \mu \), where \( \mu_2 \) is defined in the Proof of Theorem 2, and let \( v = t^{(c)} * f \). Here both \( f \) and \( v \) are multiplicative and it is easy to check that \( f(p^a) = f(p^3) = -1, f(p^5) = 1, f(p^6) = 0 \) for each \( a \geq 4 \), and \( v(p) = v(p^2) = v(p^3) = 0, v(p^5) = 2\omega(a) - 2\omega(a-1) - 2\omega(a-2) + 2\omega(a-3) \) for \( a \geq 4 \).

(ii) According to (i), \( t^{(c)} = v * \tau(2, \cdot) \), where \( \tau(1, 2, n) = \sum_{ab^2=n} 1 \) for which

\[
\sum_{n \leq x} \tau(1, 2, n) = \zeta(2)x + \zeta(1/2)x^{1/2} + O(x^{1/4}),
\]

cf. [4], p. 196-199. Therefore,

\[
\sum_{n \leq x} t^{(c)}(n) = \sum_{d \leq x} v(d) \sum_{e \leq x/d} \tau(1, 2, e)
\]

and we obtain the above result by usual estimates.

**Proof of Theorem 5.** We use the following general result given in [9]: Let \( F \) be a multiplicative function with \( F(p^a) = f(a) \) for every prime power \( p^a \), where \( f \) is positive and satisfying \( f(n) = O(n^\beta) \) for some fixed \( \beta > 0 \). Then

\[
\limsup_{n \to \infty} \frac{\log F(n) \log \log n}{\log n} = \sup_m \frac{\log f(m)}{m}.
\]

Take \( F(n) = t^{(c)}(n), f(a) = 2\omega(a) \). Here \( \omega(a) \leq a/2 \) and \( \frac{\log f(2)}{2} = \frac{1}{2} \log 2 \), which proves the result.

**Proof of Theorem 6.** (i), (ii) Take \( f = q_k * w \), in terms of the Dirichlet convolution, where \( q_k \) denotes the characteristic function of the \( k \)-free integers and use the estimate of A. Walfisz [13],

\[
\sum_{n \leq x} q_k(n) = \frac{x}{\zeta(k)} + O(x^{1/k} \delta(x)).
\]

For details cf. [5], [13].

(iii) If RH is true, then the error term of above is \( O(x^{1/(k+1)+\varepsilon}) \), according to the result of H. L. Montgomery and R. C. Vaughan [3], and take into account that \( W(s) \) is absolutely convergent for \( \text{Re}\, s > 1/(k+1) \).

**Proof of Theorem 7.** Apply Theorem 6 for \( f(n) = \kappa^{(c)}(n)/n, k = 4, \beta = 2 \), where \( f(p^k) = 1/p^2 \). Then by partial summation we obtain the result.

**Acknowledgement.** The author is grateful to Professor Imre Kátai for valuable suggestions.

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