Noncommutative Thermofield Dynamics

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Abstract

The real-time operator formalism for thermal quantum field theories, thermofield dynamics, is formulated in terms of a path-integral approach in non-commutative spaces. As an application, the two-point function for a thermal non-commutative $\lambda\phi^4$ theory is derived at the one-loop level. The effect of temperature and the non-commutative parameter, competing with one another, is analyzed.

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1 Introduction

The quantum field theory formulated in a non-commutative space-time has attracted attention over the last decades due to a variety of motives. In particular, it has been associated with the behavior of non-abelian gauge fields and with phenomenological effects in condensed matter physics, including aspects of phase transitions \cite{1,2,3,4,5,6,7,8}. In addition, there are some results coming from string theory. In this case, Connes, Douglas e Schwarz \cite{9} showed that an M-theory can be equivalent to a supersymmetric Yang-Mills field in a non-commutative torus; a result explored and expanded by Seiberg e Witten \cite{10}.

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A key ingredient of non-commutative quantum field theories is the mixing of ultraviolet and infrared divergence (UV-IR mixing) arising simultaneously in perturbation schemes. Such a divergence is a signature of non-commutativity due to the appearance of non-planar diagrams; and a way to regularize the UV divergences is by requiring that the external momenta be away from the IR regime. The UV-IR mixing is relevant for applications of non-commutativity to condensed matter systems, including the finite temperature cases, as in the quantum hall effect [1]. For instance, a way for testing the non-commutativity in space coordinates is to consider the limit of high temperature [11]. These elements have motivated several studies to take into account temperature, and all them are based on the Matsubara formalism [12, 13, 14, 15, 16].

In thermal field theory one finds two approaches that are introduced by different choices of complex-time contours. The imaginary-time (Matsubara) formalism [17] takes the Boltzmann factor, \( \exp(-\beta H) \), where \( \beta \) is the inverse of temperature \( T \) (\( \beta = 1/T \)), under a Wick rotation in the time evolution, such that the time \( t \) is mapped into an imaginary time: \( t \to t = i\tau \), with \( 0 \leq \tau \leq \beta \). The other formalism is called real-time and, since there is no imaginary time, it can be applied to non-equilibrium problems. The first real-time approach is due to Schwinger [18] and Keldysh [19] including a functional development. Another method considering real-time is called thermofield dynamics (TFD) [20, 21], which is based on the notion of linear space and representation theory. Both formalisms, Keldysh-Schwinger and TFD, are essentially the same in terms of results for equilibrium thermodynamics [22].

TFD was proposed by Takahashi and Umezawa in order to overcome difficulties with the imaginary-time [23]. The basic elements they used were a doubling of the Hilbert space and a Bogoliubov transformation. The latter is a rotation, associated with the doubling, that leads to thermalization. This formalism has been developed for practical purposes. These include perturbative schemes and Feynman rules that follow in parallel with the zero-temperature quantum field theory [20, 24] and applications to superconductivity [25], magnetic systems like ferromagnets and paramagnets [26], quantum optics [27, 28, 29, 30, 31], transport phenomena [32], d-branes [33, 34] and particle physics [24].

Formally the real-time theory can be studied by using TFD in the context of \( \ast \)-algebras [35] and symmetry groups [21, 36, 37]. In this analysis the structure of the Tomita-Takesaki (standard) representation [38, 39] is used and, in particular, the physical meaning of the doubling is fully identified with the notion of thermo-Lie groups [40, 41]. Along these lines, there are several possibilities to explore thermal effects. As an example, the kinetic theory has been formulated by analyzing representations of symmetry groups [21] and elements of the q-group have been considered, where the effect of temperature is related to a deformation in the Weyl-Heisenberg algebra [42, 43].

One interesting aspect coming naturally from the algebraic structure of TFD is that the propagator is written in two pieces: one describes the \( T = 0 \) theory, while the other gives rise to the temperature effect. This is not the case of the imaginary time formalism, where the propagator reduces to that of a 3-dimensional Euclidian theory in the momentum space, involving an infinite
sum of the Matsubara frequencies: $w_n = 2\pi n/\beta$ in the case of bosons and $w_n = \pi(2n+1)/\beta$ for fermions. TFD should then be of interest when the effect of temperature is competing with another parameter, as in a non-commutative approach. In the present work, our main goal is just to analyze such a competition, by applying TFD to non-commutative theories.

The TFD procedure is often carried out in the canonical quantization scheme. Here we consider the thermal field quantization by using a doubled path-integral formalism \[41, 44, 45\], but extended to non-commutative spaces. Although we concentrate on the $\lambda \phi^4$ theory, our procedure can be generalized to other interacting non-commutative fields. As an application, we calculate the two-point function at the one-loop level, with real time, showing explicitly the effect of temperature and non-commutativity.

The presentation is organized as follows. In Section 2 we review some basic aspects of TFD for, in Section 3, developing the thermal path integral formalism for interacting fields. The generalization to non-commutative theories is given in Section 4. The analysis of the interplay between temperature and the non-commutative parameter is carried out for a $(1+1)$-dimensional $\lambda \phi^4$ in Section 5. Our final concluding remarks are given in Section 6.

## 2 The doubling in TFD

The duplication in TFD is defined by mapping each operator $A$ acting in the Hilbert space $\mathcal{H}$, into another operator $\tilde{A}$ acting in a Hilbert space $\tilde{\mathcal{H}}$. The tilde mapping $\tilde{}: A \rightarrow \tilde{A}$, called tilde conjugation rule, is introduced from general elements of symmetry and is basically the anti-unitary modular conjugation in $c^*$-algebras \[21\] given by

\[
(A_i A_j) = \tilde{A}_i \tilde{A}_j, \tag{2.1}
\]

\[
(c A_i + A_j) = c \tilde{A}_i + \tilde{A}_j, \tag{2.2}
\]

\[
(A_i^\dagger) = (\tilde{A}_i)^\dagger, \tag{2.3}
\]

\[
(\tilde{A}_i) = A_i, \tag{2.4}
\]

\[
[A_i, \tilde{A}_j] = 0. \tag{2.5}
\]

Representation in the Hilbert space $\mathcal{H}_T = \mathcal{H} \otimes \tilde{\mathcal{H}}$ is studied by considering the generator of symmetries as defined by

\[
\tilde{A} = A - \tilde{A}. \tag{2.6}
\]

In particular the generator of time translation is given by $\tilde{H} = H - \tilde{H}$, where $H$ is the Hamiltonian of the system.

In order for a quantum field theory in the doubled Hilbert space $\mathcal{H}_T$ to be constructed, we consider initially a free boson system, and apply the tilde conjugation rules to introduce, up to normalization factors, the following generating
where $\tilde{L} = L - \bar{L}$ is the doubled Lagrangian density, given by

$$\tilde{L} = \frac{1}{2} \phi (\Box + m) \phi - J \phi - \frac{1}{2} \bar{\phi} (\Box + m) \bar{\phi} + \bar{J} \phi \quad (2.8)$$

Such a functional can then be written as

$$Z_0 \simeq \exp \{ -\frac{i}{2} \int dxdy \left[ J(x)(\Box + m^2) - i\varepsilon \right]^{-1} J(y) - \bar{J}(x)(\Box + m^2 + i\varepsilon) \bar{J}(y) \} \quad (2.9)$$

From $\tilde{L}$, we write the equations of motion, such that, the Feynman propagator for the non-tilde variables is given as usual, $(\Box + m^2 + i\varepsilon) G_0(x) = -\delta(x)$. For the tilde variables we have, $(-1)(\Box + m^2 - i\varepsilon) \tilde{G}_0(x) = -\delta(x)$, resulting that

$$\tilde{G}_0(x) = -G_0^*(x). \quad (2.10)$$

With these results, we find the normalized functional

$$Z_0[J^T, J] = \exp \left\{ -\frac{i}{2} \int dxdy [J^T(x)G_0(x-y)J(y)] \right\}, \quad (2.11)$$

where

$$J(x) = \begin{pmatrix} J(x) \\ \bar{J}(x) \end{pmatrix}, \quad J^T(x) = (J(x), \bar{J}(x)), \quad (2.12)$$

and

$$G_0(x) = (G_0^{ab}(x)) = \begin{pmatrix} G_0(x) & 0 \\ 0 & -G_0^*(x) \end{pmatrix}. \quad (2.13)$$

We show then that

$$G_0(x-y) = \frac{i}{\delta J(y) \delta J^T(x)} \frac{\delta^2 Z[J^T, J]}{\delta J(y) \delta J^T(x)} |_{J = J^T = 0}, \quad (2.14)$$

where the short notation we have used is such that, for instance, $G_0^{ab}(x-y) = i\delta^2 Z[J^T, J]/\delta J_a(x) \delta J_b(y)|_{J = J^T = 0}$. The component $G_0^{11}(x)$ is the physical component. On the other hand, $G_0^{22}(x) = \tilde{G}_0(x) = -G_0^*(x)$ is, up to the complex conjugation and the minus sign, the physical component as well. The role played by the doubling is that we can explore linear mapping among the tilde and non-tilde components of the propagator; or in an equivalent way, among the components of the generating functional. This is the mechanism to introduce an extra parameter in the theory, which will be used as the temperature.
3 Bogoliubov transformation and thermal-path integral

We proceed by introducing rotations with the nature of a Bogoliubov transformation in the Fourier components of the propagator. We have first a mapping 

\[ B(\beta) : Z_0[\mathcal{J}^T, \mathcal{J}] \to Z_0[\mathcal{J}^T, \mathcal{J}; \beta], \]

such that

\[
G(x - y; \beta)^{ab} = \frac{1}{(2\pi)^4} \int d^4k G_0(k; \beta)^{ab} e^{-ik(x-y)} \tag{3.1}
\]

where \( G(k; \beta)^{ab} = \mathcal{B}^{-1}(k_0) G_0(k)^{ab} \mathcal{B}(k_0) \), with

\[
(G_0(k)^{ab}) = \begin{pmatrix}
\frac{1}{k^2-m^2+i\epsilon} & 0 \\
0 & \frac{1}{k^2-m^2-i\epsilon}
\end{pmatrix}, \tag{3.2}
\]

\[
\mathcal{B}(k; \beta) = \begin{pmatrix}
u(k; \beta) & -v(k; \beta) \\
v(k; \beta) & u(k; \beta)
\end{pmatrix}, \tag{3.3}
\]

such that \( u^2(k; \beta) - v^2(k; \beta) = 1 \) and

\[
u(k; \beta) = \frac{1}{[1 - e^{-\beta w_k}]}^{1/2}, \tag{3.4}
\]

\[
v(k; \beta) = \frac{1}{[e^{\beta w_k} - 1]}^{1/2}. \tag{3.5}
\]

Using the definition of \( \mathcal{B}(k; \beta) \), the components \( G(k; \beta)^{ab} \) read \([21]

\[
G_0(k; \beta)^{11} = G_0(k) - n(k_0; \beta) [G_0(k) - G_0^*(k)], \tag{3.6}
\]

\[
G_0(k; \beta)^{22} = -G_0^*(k) + n(k_0; \beta) [G_0(k) - G_0^*(k)], \tag{3.7}
\]

\[
G_0(k; \beta)^{12} = G_0^*(k; \beta) = -[n(k_0; \beta) + n(k_0)^2]^{1/2} [G_0(k) - G_0^*(k)], \tag{3.8}
\]

where \([3.8] \) and

\[
n(k_0; \beta) = \frac{1}{e^{\beta w_k} - 1}. \tag{3.9}
\]

The generating functional is then given by

\[
Z_0[\mathcal{J}^T, \mathcal{J}; \beta] = \exp\left\{-\frac{i}{2} \int dxdy[J^T(x)G_0(x - y; \beta)J(y)]\right\}. \tag{3.10}
\]

Therefore, it is simple to show that, the thermal two-point function of the free scalar field is

\[
\tau^{ab}_0(x - y; \beta) = -\frac{\delta^2 Z[J^T, \mathcal{J}, \beta]}{\delta J_a(x) \delta J_b(y)} \big|_{J^T = 0}. \tag{3.11}
\]

reproducing in a right way the results coming from the canonical formalism. In order to treat interaction, we consider

\[
\tilde{L} = \frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{m^2}{2} \phi^2(x) + L_{\text{int}}
\]

\[
-\frac{1}{2} \partial_\mu \phi(x) \partial^\mu \bar{\phi}(x) + \frac{m^2}{2} \bar{\phi}^2(x) - \bar{L}_{\text{int}}. \tag{3.12}
\]
In this case, as for the zero-temperature formalism and considering first the doubling structure, the functional $Z[J^T, J]$ fulfills the following equation

$$ (\Box + m^2) \frac{\delta Z[J^T, J]}{\delta J(x)} + \hat{L}_{\text{int}} \left( \frac{1}{i} \frac{\delta}{\delta J} \frac{1}{i} \frac{\delta}{\delta J^T} \right) Z[J^T, J] = J(x) Z[J^T, J], \quad (3.13) $$

with solution

$$ Z[J^T, J] = N \exp \left[ i \int dx \hat{L}_{\text{int}} \left( \frac{1}{i} \frac{\delta}{\delta J} \frac{1}{i} \frac{\delta}{\delta J^T} \right) \right] Z_0[J^T, J], \quad (3.14) $$

where $\hat{L}_{\text{int}} \left( \frac{1}{i} \frac{\delta}{\delta J} \frac{1}{i} \frac{\delta}{\delta J^T} \right) = L_{\text{int}} \left( \frac{1}{i} \frac{\delta}{\delta J} \frac{1}{i} \frac{\delta}{\delta J^T} \right) - L_{\text{int}} \left( \frac{1}{i} \frac{\delta}{\delta J^T} \right)$. In order to introduce a temperature-dependent generating functional, we map $B(\beta) : Z[J^T, J] \to Z[J^T, J; \beta]$, by mapping $B(\beta) : Z_0[J^T, J] \to Z_0[J^T, J; \beta]$, as before, resulting in

$$ Z[J^T, J; \beta] = \exp \left[ i \int dx \hat{L}_{\text{int}} \left( \frac{1}{i} \frac{\delta}{\delta J} \frac{1}{i} \frac{\delta}{\delta J^T} \right) \right] Z_0[J^T, J; \beta] / \exp \left[ i \int dx \hat{L}_{\text{int}} \left( \frac{1}{i} \frac{\delta}{\delta J^T} \right) \right] Z_0[J^T, J; \beta] |_{J = J^T = 0}. \quad (3.15) $$

When $\beta \to \infty$ ($T \to 0$), the usual results for zero temperature are recovered. Now we turn our attention to the non-commutative $\lambda \phi^4$ theory at finite temperature and real time, using the generating functional introduced above.

### 4 TFD in the non-commutative plane

We obtain a non-commutative field theory changing the usual product between fields by the Moyal $\star$-product [2] in the Lagrangian density, i.e.

$$ (\phi_1 \star \phi_2)(x) = e^{i \theta_{\mu \nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x'^\nu} } \phi_1(x + \xi) \phi_2(x + \zeta) |_{\xi = \zeta = 0}, \quad (4.1) $$

where the non-commutative parameter $\theta_{\mu \nu}$ is skew-symmetric ($\theta_{\mu \nu} = -\theta_{\nu \mu}$). This product reproduces a non-commutative space-time [46]

$$ [x^\mu, x'^\nu] = x^\mu \star x'^\nu - x'^\nu \star x^\mu = 2i \theta_{\mu \nu}. \quad (4.2) $$

Then action of a $\phi^4$-theory over this non-commutative space-time is [47]

$$ S = S_0 + S_{\text{int}} = \int d^4x \left[ \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{g}{4!} (\phi \star \phi \star \phi \star \phi)(x) \right]. \quad (4.3) $$

With the results in the last section, we introduce the generating functional for this non-commutative theory in the TFD formalism, i.e.

$$ Z[J^T, J; \beta] = \exp \left[ i \int dx \hat{L}_{\text{int}} \left( \frac{1}{i} \frac{\delta}{\delta J} \frac{1}{i} \frac{\delta}{\delta J^T} \right) \right] Z_0[J^T, J; \beta] / \exp \left[ i \int dx \hat{L}_{\text{int}} \left( \frac{1}{i} \frac{\delta}{\delta J^T} \right) \right] Z_0[J^T, J; \beta] |_{J = J^T = 0}. \quad (4.4) $$
Let us expand this functional, up to a normalizing factor,

\[
Z[J^T, J, \beta]^* = Z_0[J^T, J, \beta] + \frac{ig^2}{4!} \sum_{n=1}^{2} (-1)^{n+1} a_n
+ \frac{1}{2!} \left( \frac{ig^2}{4!} \right)^2 \sum_{n=1}^{2} \sum_{m=1}^{2} (-1)^{n+m+2} b_{nm} + \ldots,
\]

(4.5)

where

\[
a_n = \int dz \left( \frac{\delta}{\delta J_n(z)} \right)^4 \times Z_0[J^T, J, \beta],
\]

(4.6)

\[
b_{nm} = \int \int dz dw \left( \frac{\delta}{\delta J_n(z)} \right)^4 \left( \frac{\delta}{\delta J_m(w)} \right)^4 \times Z_0[J^T, J, \beta].
\]

(4.7)

In order to carry out the \( \times \) -product, the currents are written in momentum space

\[
J_a(x) = \int \frac{dk}{(2\pi)^4} e^{i(k^\mu x_\mu)} J_a(k),
\]

(4.8)

\[
\frac{\delta}{\delta J_a(x)} = \int \frac{dk}{2\pi} e^{i(k^\mu x_\mu)} \frac{\delta}{\delta J_a(k)}.
\]

(4.9)

The product in the interacting Lagrangian then gives rise to

\[
\left( \frac{\delta}{\delta J_n(z)} \right)^4 = \int \prod_{i=1}^{4} \frac{dk_i}{(2\pi)^4} e^{-\frac{i}{2}(k_i \theta k_2)} e^{-\frac{i}{2}(k_3 \theta k_4)} e^{-i\sum_{j=1}^{4} k_j^\mu z_\mu} \frac{\delta}{\delta J_n(k_i)},
\]

(4.10)

where \( k_i \theta k_j \equiv k_i^\mu \theta_{\mu \nu} k_j^\nu \). Therefore, we obtain

\[
a_n = \int \prod_{i=1}^{4} \frac{dk_i}{(2\pi)^{12}} e^{-\frac{i}{2} \sum_{i=1}^{4} (k_i \theta k_2)} \frac{\delta}{\delta J_n(k_i)} \times Z_0[J^T, J, \beta].
\]

(4.11)

The two-point function of this non-commutative field theory is

\[
\tau^{ab}(k_1, k_2; \beta)^* = -\frac{\delta^2 Z[J^T, J, \beta]^*}{\delta J_a(k_2) \delta J_b(k_1)} \bigg|_{J^T = J = 0};
\]

(4.12)

such that, up to the first order in the interacting term, we have

\[
\tau_{ab}^{11}(k_1, k_2; \beta)^* = -\frac{ig^2}{4!} \sum_{n=1}^{2} (-1)^{n+1} \frac{\delta a_n}{\delta J_a(k_2) \delta J_b(k_1)} \bigg|_{J^T = J = 0}.
\]

(4.13)

The only physical relevant part of this matrix two-point function is the component \( \tau_{11}^{11}(k_1, k_2; \beta)^* \). However, it contains all the components of the free matrix propagator and, therefore, the temperature effect, that is
\[ \tau_{11}^{11}(k_1, k_2; \beta)^* = -\frac{g^2}{6}\Lambda(k_1, k_2)G_0^{11}(k_1; \beta)G_0^{11}(k_2; \beta) \int \frac{dp}{(2\pi)^4} G_0^{11}(p; \beta) \]

\[ -\frac{g^2}{3}G_0^{11}(k_1; \beta)G_0^{11}(k_2; \beta) \int \frac{dp}{(2\pi)^4} \Lambda(k_1, p)\Lambda(k_2, p)G_0^{11}(p; \beta) \]

\[ +\frac{g^2}{6}\Lambda(k_1, k_2)G_0^{21}(k_1; \beta)G_0^{21}(k_2; \beta) \int \frac{dp}{(2\pi)^4} G_0^{22}(p; \beta) \]

\[ +\frac{g^2}{3}G_0^{21}(k_1; \beta)G_0^{21}(k_2; \beta) \int \frac{dp}{(2\pi)^4} \Lambda(k_1, p)\Lambda(k_2, p)G_0^{22}(p; \beta), \]

(4.14)

where

\[ \Lambda(k_i, k_j) \equiv e^{-\frac{i}{2\theta}k_i\theta k_j} \]

and \( G_0^{ij} \) are the components of the TFD propagator for the Klein-Gordon field.

These results are analyzed in the following section.

We note in the above expression for the two-point function \( \tau_{11}^{11}(k_1, k_2; \beta)^* \) that there are two kinds of integrals (diagrams). In the first and third lines the phase term \( \Lambda(k_i, k_j) \) is not in the integrand. This leads to planar diagrams, which are equivalent to the usual diagrams for the commutative theory, when \( \theta \to 0 \). In the second and fourth lines there are phase terms in the integrand. This leads to non-planar diagrams. Now, non-planar diagrams are typical of non-commutative theories and leads to the UV-IR mixing divergence. This divergence appears when one attempts to renormalize the theory in the UV regime. This is possible only if the external momenta do not go to zero, i.e., IR regime.

5 \( \beta, \theta \)-two point function in \((1+1)\)-dimensions

We find the structure of the function \( \tau_{11}^{11}(k_1, k_2; \beta)^* \), given in Eq. (4.14), by considering \( \theta \) a small quantity, such that \( \Lambda(k_i, k_j) \approx 1 - \frac{i}{2}k_i\theta k_j \). Since \( G_0^{11}(k_1; \beta) \) is given by Eq. (3.6), the \( T = 0 \) and \( \theta = 0 \) contribution is

\[ \tau_{11}^{11}(k_1, k_2) = -\frac{g^2}{2}G_0(k_1)G_0(k_2) \int \frac{dp}{(2\pi)^4} G_0(p), \]

(5.1)

recovering the usual result. The first line in Eq. (4.14) provides, for instance, the following \( \beta \) and \( \theta \)-dependent term:

\[ I(k_1, k_2; T, \theta) = \frac{g^2}{6}i\theta k_2 n(k_1; \beta)G_0(k_1)G_0(k_2) \int \frac{dp}{(2\pi)^4} G_0(p). \]

(5.2)

In this case, the non-commutativity, characterized by the parameter \( \theta \), is competing with the temperature, described by \( n(k_1; \beta) \). Notice that for high enough
temperature, considering \( \theta \) fixed, this terms are present in the two-point function; and so interfering in the measurable variables, calculated from \( \tau_{11}^{11} \), as the cross-section.

The interplay between temperature and the non-commutativity parameter to the non-planar part of \( \tau_{11}^{11}(k_1, k_2) \) is better observed in the case of 1+1 dimensions. In this limit we have

\[
\omega_k^2 = k_0^2 = k^2 + m^2 = k_1^2 + k_2^2 + m^2 \rightarrow k_1^2 + m^2, \quad (5.3)
\]

\[
k\theta p = \sum_{i,j=0}^{3} k_i \theta^{ij} p_j \rightarrow k_0 \theta^{01} p_1 + k_1 \theta^{10} p_0 = \theta(k_0 p_1 - k_1 p_0), \quad (5.4)
\]

and the non-planar part of the two-point function is

\[
\tau_{np}^{11}(k_1, k_2; \beta) \equiv \frac{g^2}{3} \left\{-G_0^{11}(k_1; \beta)G_0^{11}(k_2; \beta) \int \frac{dp}{(2\pi)^2} \Lambda(k_1, p)\Lambda(k_2, p)G_0^{11}(p; \beta) \right. \\
\left. +G_0^{21}(k_1; \beta)G_0^{21}(k_2; \beta) \int \frac{dp}{(2\pi)^2} \Lambda(k_1, p)\Lambda(k_2, p)G_0^{22}(p; \beta) \right\}. \quad (5.5)
\]

Now we write \( \omega_k^2 = k^2 + m^2 \), \( k\theta p = \theta(k_0 p_1 - k_1 p_0) \) and define two dimensionless parameters \( \alpha \equiv \beta \omega_k, \gamma \equiv k\theta p \). When the temperature is greater than the field mass (\( \beta m \ll 1 \)) we have \( \alpha \simeq \beta k_1 \) and \( \gamma \simeq \alpha \frac{2}{\pi}(p_1 - p_0) \). In this limit we have

\[
\Lambda(\gamma) = e^{-\frac{\pi}{2}\gamma} \simeq 1 - \frac{i}{2}\gamma, \quad (5.6)
\]

\[
n(\alpha) = \frac{1}{e^{\alpha} - 1} \simeq \frac{1}{\alpha}. \quad (5.7)
\]

The non-planar part of the two-point function is

\[
\tau_{np}^{11}(k_1, k_2; \beta) = \frac{g^2}{3} \left\{-A(k_1, k_2) + \frac{1}{\beta} B(k_1, k_2) + \theta C(k_1, k_2) + \frac{\theta}{\beta} D(k_1, k_2) \right\}, \quad (5.8)
\]

where

\[
A(k_1, k_2) = G_0(k_1) \left[ \int \frac{dp}{(2\pi)^2} G_0(p) \right] G_0(k_2), \quad (5.9)
\]

\[
B(k_1, k_2) = 2\pi i \left\{ \frac{\delta(k_1^2 - m^2)}{\omega_{k_1}} \left[ \int \frac{dp}{(2\pi)^2} G_0(p) \right] G_0(k_2) \\
+G_0(k_1) \left[ \int \frac{dp}{(2\pi)^2} G_0(p) \right] \frac{\delta(k_2^2 - m^2)}{\omega_{k_2}} \right. \\
\left. +G_0(k_1) \left[ \int \frac{dp}{(2\pi)^2} \frac{\delta(p^2 - m^2)}{\omega_{p}} \right] G_0(k_2) \right\}, \quad (5.10)
\]

\[
C(k_1, k_2) = \frac{i}{2} \left[ \omega_{k_1} + \omega_{k_2} \right] \left\{ G_0(k_1) \left[ \int \frac{dp}{(2\pi)^2}(p_1 - p_0) G_0(p) \right] G_0(k_2) \right\}, \quad (5.11)
\]
\[ D(k_1, k_2) = \pi [\omega_{k_1} + \omega_{k_2}] \frac{\delta(k_1^2 - m^2)}{\omega_{k_1}} \left[ \int \frac{dp}{(2\pi)^2} (p_1 - p_0) G_0(p) \right] G_0(k_2) \]
\[ + G_0(k_1) \left[ \int \frac{dp}{(2\pi)^2} (p_1 - p_0) \frac{\delta(k_2^2 - m^2)}{\omega_{k_2}} \right] G_0(k_2) \]
\[ + G_0(k_1) \left[ \int \frac{dp}{(2\pi)^2} \frac{(p_1 - p_0) \delta(p^2 - m^2)}{\omega_p} \right] G_0(k_2) \].

(5.12)

From the right-hand side of equation (5.8), we identify the dependence on \( \theta \) and \( \beta \) explicitly: in the first term, \( -A(k_1, k_2) \) is independent of \( \theta \) and \( \beta \) and, in addition, this term is the value of the two-point function in the case of \( T = 0 \) and \( \theta = 0 \); the second term depends on the temperature, only; the third on the commutative parameter; and the fourth term depends on the interplay of commutativity and temperature.

We note that Eq. (5.8) shows how is the leading behavior of these non-planar diagrams with respect to temperature and non-commutativity. Now, these diagrams are responsible for the existence of a singular behavior in the limit \( \theta \to 0 \), in the usual case of zero temperature \( \beta \to \infty \). It is clear from this expansion that even in the case of finite temperature, there will be no modification of the singular behavior of the limit \( \theta \to 0 \).

6 Concluding remarks

In short, in this paper we have developed a path integral formalism in the context of the thermofield dynamics (TFD) to calculate the two-point function up the one-loop level of the non-commutative \( \phi^4 \) theory. The main result is the nature of contributions arising from different terms in the propagator: there is a term with \( T = \theta = 0 \), where \( T \) accounts for the temperature and \( \theta \) for the non-commutativity; there are terms depending on \( \theta \) only; terms depending on \( T \) only; and (mixed) terms depending on both \( T \) and \( \theta \), as that one given in Eqs. (5.2) and (5.8). For a fixed \( \theta \) and high temperature, the contribution of mixed terms cannot be trivially discarded, and so it can contribute for measured quantities. A similar analysis is still valid for high order perturbative terms and that can include non-equilibrium effects, since we have considered a real (not imaginary) time approach. Beyond that, many important aspects remain to be studied, as the application to the problems of non-commutative gauge theory and renormalization, that have been addressed in different ways \cite{2, 17, 18}.

For instance, in the context of the real-time formalism, it would be interesting to investigate the renormalization proof for \( \phi^4 \) as carried out by Grosse and Wulkenhaar \cite{49, 50, 51}.

The results derived here emerge basically from the algebraic structure of TFD, the c*-algebra, where the tilde conjugation rules are identified with the modular conjugation in the standard representation. In this context, the Bogoliubov transformation, the other basic TFD ingredient, corresponds to a linear transformation involving the commutants of the von Neumann algebra. It is important to emphasize that several of these algebraic elements are also found
in non-commutative theories. A consequence is that the case \( T = \theta = 0 \) is easily identified since the temperature and the non-commutativity are implemented by mappings, in a von Neumann algebra, connected to identity. This observation points to a close connection of such formalisms that deserves more studies.

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