HIGHLIGHTS FROM “THE RAMANUJAN PROPERTY FOR SIMPLICIAL COMPLEXES”

URIYA A. FIRST

Abstract. This paper brings the main definitions and results from “The Ramanujan Property for Simplicial Complexes”. No proofs are given.

Given a simplicial complex $X$ and a group $G$ acting on $X$, we define Ramanujan quotients $X$. For $G$ and $X$ suitably chosen this recovers Ramanujan $k$-regular graphs and Ramanujan complexes in the sense of Lubotzky, Samuels and Vishne. Deep results in automorphic representations are used to give new examples of Ramanujan quotients when $X$ is the affine building of an inner form of $\text{GL}_n$ over a local field of positive characteristic.

This is an overview of the paper “The Ramanujan Property for Simplicial Complexes” [11]. We bring all main results and definitions (simplified at times), but give no proofs. Rather, precise references to the relevant places in [11] are provided.

Introduction

Let $X$ be a connected $k$-regular graph. Denote by $\lambda(X)$ the maximal absolute value of an eigenvalue of the adjacency matrix of $X$, excluding $k$ and $-k$. Graphs for which $\lambda(X)$ is bounded away from $k$ are called expander graphs. They enjoy many good combinatorial properties; see [27] for a survey.

The graph $X$ is called Ramanujan if $\lambda(X) \leq 2\sqrt{k-1}$. This definition is motivated by the Alon–Boppana Theorem [37], stating that for any $\varepsilon > 0$, only finitely many non-isomorphic $k$-regular graphs satisfy the tighter bound $\lambda(X) \leq 2\sqrt{k-1} - \varepsilon$. Furthermore, the interval $[-2\sqrt{k-1}, 2\sqrt{k-1}]$ is the spectrum of the $k$-regular tree ([35, p. 252, Apx. 3], [22, Th. 3]), which is the universal cover of any $k$-regular graph. Ramanujan graphs can therefore be thought of as having the smallest possible spectrum one can expect of an infinite family of graphs, or as finite approximations of the infinite $k$-regular tree.

The spectral properties of Ramanujan graphs manifest in some desired combinatorial properties. For example, Ramanujan graphs are supreme expanders, and have a large chromatic number if not bipartite. Some known constructions have large girth as well. Constructing infinite families of non-isomorphic $k$-regular Ramanujan graphs is considered difficult. The first such families were introduced by Lubotzky, Phillips and Sarnak [26] and independently by Margulis [34], assuming $k - 1$ is prime. Morgenstern [36] has extended this to the case $k - 1$ is a prime power. These works rely on deep results of Delinge [7] and Drinfeld [8] concerning the Ramanujan–Petersson conjecture for $\text{GL}_2$. The existence of infinitely many
The Ramanujan property for simplicial complexes

$k$-regular bipartite Ramanujan graphs for arbitrary $k$ was later shown by Marcus, Spielman and Srivastava [33] using different methods; the non-bipartite case remains open.

A high-dimensional generalization of Ramanujan graphs, called Ramanujan complexes, was suggested by Cartwright, Solé and Žuk [6], and later refined by Lubotzky, Samuels and Vishne [31] (see [20] for another generalization of Ramanujan graphs). These complexes are quotients of the affine Bruhat–Tits building of $\mathrm{PGL}_d(F)$, denoted $\mathcal{B}_d(F)$, where $F$ is a non-archimedean local field. The building $\mathcal{B}_d(F)$ is a contractible simplicial complex of dimension $d - 1$; its construction is recalled in [6A] below. The spectrum of a quotient of $\mathcal{B}_d(F)$, i.e. a simplicial complex whose universal cover is $\mathcal{B}_d(F)$, consists of the common spectrum of a certain family of $d - 1$ linear operators associated with the quotient, called the Hecke operators. According to Lubotzky, Samuels and Vishne [31], a quotient of $\mathcal{B}_d(F)$ is Ramanujan if its spectrum, which is a subset of $\mathbb{C}^{d-1}$, is contained in the spectrum of the universal cover $\mathcal{B}_d(F)$ together with a certain family of $d$ points in $\mathbb{C}^{d-1}$, called the trivial spectrum.

Li [25, Thm. 4.1] proved a theorem in the spirit of the Alon–Boppana Theorem for quotients of $\mathcal{B}_d(F)$: If $\{X_n\}_{n \in \mathbb{N}}$ is a family of such quotients satisfying a mild assumption, then the closure of the union of the spectra of $\{X_n\}_{n \in \mathbb{N}}$ (in $\mathbb{C}^{d-1}$) contains the spectrum of the universal cover $\mathcal{B}_d(F)$. Ramanujan complexes can therefore be thought of as having the smallest possible spectrum that can be expected of an infinite family, or as spectral approximations of the universal cover $\mathcal{B}_d(F)$, similarly to Ramanujan graphs. When $d = 2$, the complex $\mathcal{B}_d(F)$ is a regular tree, and Ramanujan complexes are just Ramanujan graphs in the previous sense.

The existence of infinite families of Ramanujan complexes was shown by Lubotzky, Samuels and Vishne in [31] (see also [30]), using Lafforgue’s proof of the Ramanujan–Petersson conjecture for $\mathrm{GL}_d$ in positive characteristic [23]. Li [25] has independently obtained very similar results using a special case of the conjecture established by Laumon, Rapoport and Stuhler [24, Th. 14.12]. As in the case of graphs, Ramanujan complexes enjoy various good combinatorial properties: They have high chromatic number [10, §6], good mixing properties [10, §4], they satisfy Gromov’s geometric expansion property [13] (see also [16, 21]), and the constructions of [31] have high girth in addition [28].

The Ramanujan property of quotients of $\mathcal{B}_d(F)$ is measured with respect to the spectrum of the Hecke operators. In a certain sense, to be made precise below, these operators capture all spectral information in dimension 0. Therefore, we regard the spectrum of Lubotzky, Samuels and Vishne as the 0-dimensional spectrum. However, one can associate other operators with a simplicial complex such that their spectrum affects combinatorial properties. For example, this is the case for the high-dimensional Laplacians; see for instance [41, 11, 15, 40]. Other examples are adjacency operators between various types of facets. These operators are high-dimensional in nature and so their spectrum is a priori not determined by the spectrum of the Hecke operators. The work [11], which is summarized in this manuscript, treats these and other high-dimensional operators, and constructs examples of complexes which are Ramanujan relative to such operators.

---

1 The proof in [31] assumed the global Jacquet–Langlands correspondence for $\mathrm{GL}_n$ in positive characteristic that was established later in [3].

2 The notion of Ramanujan complexes used in [25] is slightly weaker than the one used in [31], but the constructions of [25] are in fact Ramanujan in the sense of [31].
In more detail, let $X$ be a simplicial complex and let $G$ be a group of automorphisms of $X$ satisfying certain mild assumptions. For example, one can take $X$ to be a $k$-regular tree $\mathcal{T}_k$ and $G = \text{Aut}(\mathcal{T}_k)$, or $X = \mathcal{B}_d(F)$ and $G = \text{PGL}_d(F)$.

Even more generally, $X$ can be an affine Bruhat-Tits building (see [1]), and $G$ can be a group of automorphisms acting on $X$ in a sufficiently transitive manner. We consider quotients of $X$ by subgroups of $G$, called $G$-quotients for brevity, and associate several types of spectra with each of them. Among these spectra is the (non-oriented) $i$-dimensional spectrum. When $X = \mathcal{T}_k$ and $G = \text{Aut}(\mathcal{T}_k)$, or when $X = \mathcal{B}_d(F)$ and $G = \text{PGL}_d(F)$, our 0-dimensional spectrum coincides with the spectra of quotients of regular graphs and quotients of $\mathcal{B}_d(F)$ discussed earlier.

The main results of [11] are as follows:

1. If $\{X_n\}_{n \in \mathbb{N}}$ is a family of $G$-quotients of $X$ satisfying a mild assumption, then the closure of $\bigcup_{n \in \mathbb{N}} \text{Spec}(X_n)$ contains $\text{Spec}(X)$ (Theorem 4.1).

This generalizes Li’s aforementioned theorem, and leads to a notion of Ramanujan $G$-quotients of $X$:

- A $G$-quotient of $X$ is Ramanujan (relative to a particular type of spectrum) if its spectrum is contained in the union of the spectrum of $X$ with the trivial spectrum (Definition 4.4).

In analogy with Ramanujan graphs and Ramanujan complexes, Ramanujan $G$-quotients have the smallest possible spectrum one can expect of an infinite family of $G$-quotients of $X$, or alternatively, they can be regarded as spectral approximations of the covering complex $X$. When $X$ is a $k$-regular tree (resp. $\mathcal{B}_d(F)$), the quotients of $X$ which are Ramanujan in dimension 0 are precisely the Ramanujan graphs (resp. Ramanujan complexes in the sense of [11]).

Next, we give a representation-theoretic criterion for being Ramanujan (Theorem 5.2): Let $\Gamma \leq G$ be a subgroup such that $\Gamma \backslash X$ is a finite simplicial complex and $X \rightarrow \Gamma \backslash X$ is a cover map.

2. Let $x_1, \ldots, x_t$ be representatives for the orbits of the action of $G$ on the $i$-dimensional cells in $X$, and write $K_i = \text{Stab}_G(x_n)$. Then $X$ is Ramanujan in dimension $i$ if and only if every irreducible unitary $G$-subrepresentation of $L^2(\Gamma \backslash G)$ with $V^{K_1} + \cdots + V^{K_t} \neq 0$ is tempered or finite-dimensional.

2'. $\Gamma \backslash X$ is completely Ramanujan if and only if every irreducible unitary $G$-subrepresentation of $L^2(\Gamma \backslash G)$ is tempered or finite-dimensional.

The completely Ramanujan condition means that the complex is Ramanujan with respect to “any type” of spectrum. In particular, it is Ramanujan in all dimensions and Ramanujan relative to all high-dimensional Laplacians. However, it is also Ramanujan relative to other operators such as the adjacency operator of the graph obtained from $\Gamma \backslash X$ by taking triples of vertices $(u_1, u_2, u_3)$ with $d(u_1, u_2) = d(u_2, u_3) = 1$ and saying that $(u_1, u_2, u_3)$ is adjacent to $(v_1, v_2, v_3)$ if $u_2 = v_1$ and $v_3 = v_2$. The result (2) can be refined into a one-to-one correspondence between the $i$-dimensional spectrum of $\Gamma \backslash X$ and a certain class of $G$-subrepresentations of $L^2(\Gamma \backslash G)$ (Theorem 5.3).

In case $X$ is the affine Bruhat-Tits building of a simple algebraic group $G$ over the global field of $F$, $G = G(F)$, and $\Gamma$ is an arithmetic cocompact lattice in $G$, our criterion can be restated in terms of automorphic representations of $G$ (Theorem 6.3). The latter is used together with deep results about automorphic representations (particularly [23] and [3]) to show:

\[ \text{It should be pointed out that this does not mean that the regular graph obtained in this manner is Ramanujan. Rather, the spectrum of its adjacency operator is contained in the union of the spectrum of the adjacency operator of the graph obtained from $X$ and the trivial spectrum.} \]
(3) Let $F$ be a non-archimedean local field with $\text{char } F > 0$, let $D$ be a central division algebra over $F$, let $G = \text{PGL}_d(D) := \text{GL}_d(D)/F^\times$, and let $B_d(D)$ be the affine Bruhat–Tits building of $G$ (cf. 6A). Then there exist infinite families of $G$-quotients of $B_d(D)$ which are completely Ramanujan (Theorem 6.3).

Particular $G$-quotients of $B_d(D)$ which are completely Ramanujan are constructed in 6B. When $D = F$, our Ramanujan $G$-quotients are the Ramanujan complexes constructed by Lubotzky, Samuels and Vishne [31]. Thus, the Ramanujan complexes of [31], which are Ramanujan in dimension 0 according to our setting, are in fact completely Ramanujan. When $d = 2$, our construction gives rise to Ramanujan graphs, which seem to be new when $D \neq F$.

The paper is organized as follows: Section 1 is a brief introduction to the spectral theory of $\ast$-algebras. It brings some auxiliary definitions and results used later in the text. Section 2 recalls simplicial complexes and certain facts about $\ell$-groups acting on them. In Section 3, we introduce our notion of spectrum together with examples and supplementary results. In Section 4, we present a generalization of Li’s Theorem (reminiscent of the Alon–Boppana Theorem) and introduce the trivial spectrum, which leads to the definition of Ramanujan quotients. Section 5 gives a representation-theoretic criterion for a quotient of $X$ to be Ramanujan. Some consequence are discussed. Finally, in Section 6, we recall the construction of the affine Bruhat–Tits building of $\text{PGL}_d(D)$ and describe an infinite family of completely Ramanujan quotients of it, provided $\text{char } F > 0$. We also explain how the problem is translated into a statement about automorphic representations.

Notation

Throughout, all vector spaces are over $\mathbb{C}$. An algebra means a unital $\mathbb{C}$-algebra. Modules are assumed to be unital Subalgebras are not required to have the same unity as the ambient algebra. For an algebra $A$, a left $A$-module $V$ and $a \in A$, denote by $a|_V$ the linear operator $v \mapsto av \in \text{End}_\mathbb{C}(V)$.

If $V$ is a hilbert space, then $S^1(V)$ denotes the unit sphere of $V$. If $X$ is a set, then we write $Y \subseteq_f X$ to denote that $Y$ is a finite subset of $X$.

For a set $X$, we let $\tilde{\ell}^2(X)$ denote the set of functions $\varphi : X \to \mathbb{C}$ with finite support. We endow $\tilde{\ell}^2(X)$ with the inner product $\langle \varphi, \psi \rangle = \sum_{x \in X} \varphi(x) \overline{\psi(x)}$. This makes $\tilde{\ell}^2(X)$ into a pre-Hilbert space. Its completion is the Hilbert space of square-summable functions on $X$, denoted $\ell^2(X)$. The vector space $\tilde{\ell}^2(X)$ admits a standard basis $\{e_x\}_{x \in X}$ defined by

$$e_x(y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}.$$  

If $Y$ is another set and $f : X \to Y$ is any function, then we define $f_* : \tilde{\ell}^2(X) \to \tilde{\ell}^2(Y)$ by $(f_* \varphi)(y) = \sum_{x \in f^{-1}(y)} \varphi(x)$ for all $\varphi \in \tilde{\ell}^2(Y)$, $y \in Y$. In particular, we have

$$f_* e_x = e_{f(x)} \quad \forall x \in X.$$  

Recall that an $\ell$-group is a locally compact totally disconnected Hausdorff topological group. Such groups admit a basis of neighborhoods at the identity consisting of compact open subgroups.

---

4In [11], algebras and modules are not assumed to be unital.
1. *-Algebras

In the basis of our definition of spectrum of simplicial complexes lies the spectral theory of idempotented *-algebra, as developed in [11, Ch. 2]. We shall restrict here to unital *-algebra for the sake of simplicity. General *-algebras are discussed in [39], for instance.

1A. Unitary Representations. Let $A$ be an algebra. An involution on $A$ is a map $\dagger : A \to A$ such that $a^{\dagger \dagger} = a$, $(a+b)^{\dagger} = a^{\dagger} + b^{\dagger}$, $(ab)^{\dagger} = b^{\dagger}a^{\dagger}$ and $(\alpha a)^{\dagger} = \overline{\alpha}a^{\dagger}$ for all $a, b \in A$ and $\alpha \in \mathbb{C}$. A *-algebra is an algebra $A$ equipped with an involution, which is always denoted by $\dagger$.

**Example 1.1.** The commutative algebra $A = \mathbb{C}[X_1, \ldots, X_t]$ is a *-algebra with respect to the unique involution $*$ satisfying $X_i^* = X_i$ for all $i$.

A **unitary representation** of $A$ is a Hilbert space $V$ equipped with a left $A$-module

such that

(1U) $\langle au, v \rangle = \langle u, a^*v \rangle$ for all $a \in A$ and $u, v \in V$,

(2U) for all $a \in A$, the operator $a|_V : V \to V$ is bounded.

We say that $V$ is **irreducible** if it does not have a proper nonzero closed $A$-submodule. Let $\text{Rep}^u(A)$ denote the category whose objects are unitary representations of $A$ and whose morphisms are continuous $A$-module homomorphisms. Morphisms preserving the inner product are called unitary. We let $\text{Irr}^u(A)$ denote the class of irreducible unitary representations of $A$.

Let $\{V_i\}_{i \in I} \subseteq \text{Rep}^u(A)$. The direct sum $\bigoplus_i V_i$ admits an obvious inner-product making it into a pre-Hilbert space. The completion of $\bigoplus_i V_i$ is denoted $\bigoplus_i \overline{V_i}$. If $\sup_i ||a|_{V_i}| < \infty$ for all $a \in A$, then the diagonal action of $A$ on $\bigoplus_i \overline{V_i}$ extends to $\bigoplus_i V_i$ and we may regard $\bigoplus_i V_i$ as a unitary representation of $A$. We denote this by writing $\bigoplus_i V_i \in \text{Rep}^u(A)$. When $I$ is finite, we always have $\bigoplus_i V_i = \bigoplus_i V_i \in \text{Rep}^u(A)$.

We write $V_1 \leq V_2$ if there is a unitary injective $A$-homomorphism from $V_1$ to $V_2$.

We now recall several well-known facts about unitary representations.

**Theorem 1.2** (Schur’s Lemma; [11, Th. 2.6]). Let $V \in \text{Irr}^u(A)$. Then the continuous $A$-endomorphisms of $V$ are $\mathbb{C}\text{id}_V$.

**Corollary 1.3** ([11, Cor. 2.7]). If $A$ is commutative, then all irreducible unitary representations of $A$ are 1-dimensional.

**Proposition 1.4** ([11, Prp. 2.8]). Let $V, V' \in \text{Rep}^u(A)$. If there exists a continuous $A$-module isomorphism $f : V \to V'$, then there exists a unitary $A$-module isomorphism $g : V \to V'$.

1B. Spectrum and The Unitary Dual. Let $A$ be a *-algebra. The isomorphism class of $V \in \text{Rep}^u(A)$ is denoted $[V]$. The set of isomorphism classes of irreducible unitary representations of $A$ is called the **unitary dual** of $A$ and denoted

$$\hat{A} = \{ [V] : V \in \text{Irr}^u(A) \}.$$  

A subset $S$ of $\hat{A}$ is called **bounded** if $\bigoplus_{[V] \in S} V \in \text{Rep}^u(A)$, or equivalently, if $\sup_{[V] \in S} ||a|_{V}| < \infty$ for all $a \in A$.

---

5 Some texts use the term *topologically irreducible*. 
Example 1.5. Consider $A = \mathbb{C}[X_1, \ldots, X_t]$ with the unique involution satisfying $X_i^* = X_i$. By Corollary [11, 3.3] irreducible unitary representations of $A$ are 1-dimensional. In particular, they are irreducible $A$-modules. For $\lambda = (\lambda_1, \ldots, \lambda_t) \in \mathbb{C}$, let $V_\lambda = A/(X_1 - \lambda_1, \ldots, X_t - \lambda_t)$. It is well-known that $\{V_\lambda | \lambda \in \mathbb{C}^t\}$ form a complete set irreducible $A$-modules up to isomorphism. However, only those $V_\lambda$ for which $\lambda \in \mathbb{R}^t$ can be made into unitary representations of $A$. The unitary dual of $A$ is therefore in one-to-one correspondence with $\mathbb{R}^n$. Explicitly, the isomorphism $\widehat{A} \to \mathbb{R}^n$ is given by $[V] \mapsto (X_1|V, \ldots, X_t|V)$.

More generally, if $A$ is any (commutative) $*$-algebra generated as a $*$-algebra by $a_1, \ldots, a_t$, then the map

$$[V] \mapsto (a_1|V, \ldots, a_t|V) \in \mathbb{C}^t$$

gives an embedding of $\widehat{A}$ in $\mathbb{C}^t$. See [11, Rm. 2.23] for details about its image.

We make $\widehat{A}$ into a topological space as follows: Let $V \in \text{Irr}^u(A)$, $v \in S^1(V)$, $\varepsilon > 0$ and $F \subseteq_f A$. We define

$$N_{V,v,\varepsilon,F} \subseteq \widehat{A}$$

to be the set of all isomorphism classes $[U] \in \widehat{A}$ for which there is $u \in U$ such that $|\langle av, v \rangle - \langle au, u \rangle| < \varepsilon$ for all $a \in F$.

Note that $u$ is not required to be a unit vector. The possible sets $N_{V,v,\varepsilon,F}$ form a subbasis for a topology on $\widehat{A}$.

Let $[V] \in \widehat{A}$ and $V' \in \text{Rep}^u(A)$. We say that $V$ is weakly contained in $V'$ and write $V \prec V'$ if the following equivalent conditions hold (cf. [11, Lm. 2.15]):

(a) For all $v \in S^1(V)$, $\varepsilon > 0$ and $F \subseteq_f A$, there exists $v' \in S^1(V')$ such that $|\langle av, v \rangle - \langle av', v' \rangle| < \varepsilon$ for all $a \in F$.

(b) There exists $v \in S^1(V)$ such that for all $\varepsilon > 0$ and $F \subseteq_f A$, there is $v' \in V'$ such that $|\langle av, v \rangle - \langle av', v' \rangle| < \varepsilon$ for all $a \in F$.

For example, if $V \leq V'$, then $V \prec V'$. The converse is false in general. The $A$-spectrum of $V'$ is a subset of $\widehat{A}$ defined as

$$\text{Spec}_A(V') = \{[V] \in \widehat{A} : V \prec V'\}.$$ 

The following proposition shows that when $A$ is commutative and generated as a $*$-algebra by $a_1, \ldots, a_t$, there is a topological embedding of $\widehat{A}$ in $\mathbb{C}^t$ such that for every $V' \in \text{Rep}^u(A)$, the set $\text{Spec}_A(V')$ corresponds to the common (continuous) spectrum of $(a_1, \ldots, a_t)$ on $V$, denoted $\text{Spec}(a_1|V', \ldots, a_t|V')$. The $A$-spectrum is therefore essentially equivalent to the common spectrum of $(a_1, \ldots, a_t)$.

Proposition 1.6 ([11, Pr. 2.22]). Assume $A$ is commutative and generated as a $*$-algebra by $a_1, \ldots, a_t$. For $V \in \text{Irr}^u(A)$, denote by $\lambda_V \in \mathbb{C}^t$ the unique common eigenvalue of $(a_1, \ldots, a_t)$ on $V$ (cf. Corollary [11, 3.3]). Then the map

$$[V] \mapsto \lambda_V : \widehat{A} \to \mathbb{C}^t$$

is a topological embedding. In addition, for all $V' \in \text{Rep}^u(A)$, we have

$$\text{Spec}(a_1|V', \ldots, a_t|V') = \{\lambda_V | [V] \in \text{Spec}_A(V')\}.$$ 

Finally, a subset $S \subseteq \widehat{A}$ is bounded if and only if its image in $\mathbb{C}^t$ is bounded.

The unitary dual of finitely generated non-commutative algebras is not Hausdorff in general [11, Ex. 2.24].

We mention here several useful results about the $A$-spectrum.
Proposition 1.7 ([11] Pr. 2.28]). Let $V' \in \text{Rep}^u(A)$. Then $\text{Spec}_A(V')$ is bounded and closed in $\hat{A}$.

Proposition 1.8 ([11] Cor. 2.30]). Let $0 \neq V' \in \text{Rep}^u(A)$ and let $a \in A$. There is $[V] \in \text{Spec}_A(V')$ such that $\|a[V]\| = \|a[V']\|$. In particular, $\text{Spec}_A(V') \neq \emptyset$.

Theorem 1.9 ([11] Th. 2.31]). Let $\{V_i\}_{i \in I}$ be a family of unitary representations of $A$ such that $\bigoplus_{i \in I} V_i \in \text{Rep}^u(A)$. Then $\text{Spec}_A(\bigoplus_{i \in I} V_i) = \bigcup_{V_i \in \text{Spec}_A(V)} \text{Spec}_A(V_i)$.

1C. Subalgebras. Let $A$ be a $*$-algebra. A $*$-subalgebra of $A$ is a subalgebra $B$ such that $B^* = B$. If $V \in \text{Rep}^u(A)$, then $BV$ is a unitary representation of $B$ (notice that $BV$ is closed in $V$ since $B$ has a unity). When the unities of $B$ and $A$ are the same, we have $BV = V$. The following theorem shows that the $A$-spectrum of $V$ determines the $B$-spectrum of $BV$.

Theorem 1.10 ([11] Th. 2.35]). For all $V' \in \text{Rep}^u(A)$, we have
\[ \text{Spec}_B(BV') = \{ [U] \in \hat{B} : \text{there is } [V] \in \hat{A} \text{ with } U \leq BV \}. \]

Corollary 1.11 ([11] Cor. 2.37]). Let $V' \in \text{Rep}^u(A)$, and let $a_1, \ldots, a_t \in A$ be elements generating a commutative $*$-subalgebra of $A$. Then
\[ \text{Spec}(a_1[V'], \ldots, a_t[V]) = \bigcup_{[V] \in \text{Spec}_A(V')} \text{Spec}(a_1[V], \ldots, a_t[V]). \]

Theorem 1.12 ([11] §2[I]). Let $e \in A$ be an idempotent with $e^* = e$ and let $f = 1 - e$. Then $eV \in \text{Irr}^u(eAe)$ for all $V \in \text{Irr}^u(A)$ with $eV \neq 0$, and for all $V' \in \text{Rep}^u(A)$, we have
\[ \text{Spec}_{eAe}(eV') = \{ [eV] \mid [V] \in \text{Spec}_A(V'), eV \neq 0 \}, \]
\[ \text{Spec}_{fAf}(fV') = \{ [fV] \mid [V] \in \text{Spec}_A(V' \setminus fV) \}, \]
\[ \text{Spec}_A(V') = \{ [V] \in \hat{A} : [eV] \in eAe \text{ or } [fV] \in fAf \}. \]

In particular, $\text{Spec}_A(V')$ determines $\text{Spec}_{eAe}(eV')$ and $\text{Spec}_{fAf}(fV')$, and vice versa.

In general, it is not true that for every $U \in \text{Irr}^u(eAe)$ there is $V \in \text{Irr}^u(A)$ with $U \cong eV$ [11] Rm. 2.43).

1D. Pre-Unitary Representations. Let $A$ be a $*$-algebra. A pre-unitary representation of $A$ is a pre-Hilbert space $V$ endowed with a left $A$-module structure satisfying conditions (U1) and (U2) of [1A]

If $V$ is a pre-unitary representation, then the action of $A$ extends to the completion $\overline{V}$, which then becomes a unitary representation. The $A$-spectrum of $V$ is defined to be the $A$-spectrum of $\overline{V}$.

2. Simplicial Complexes

2A. Simplicial Complexes. Recall that a simplicial complex consists of a non-empty set of finite sets $X$ such that subsets of sets in $X$ are also in $X$. A partially ordered set $(Y, \leq)$ that is isomorphic to $(X, \subseteq)$ for some simplicial complex $X$ will also be called a simplicial complex.

The elements of a simplicial complex $X$ are called cells. It is locally finite if every cell in $X$ is contained in finitely many cells. We let $X^{(i)}$ denote the sets in $X$ of cardinality $i + 1$. Elements of $X^{(i)}$ are called $i$-dimensional cells, or just $i$-cells. The vertex set of $X$ is $X_{\text{vert}} := \bigcup_{x \in X} x$. By abuse of notation, we sometimes refer to elements of $X^{(0)}$ as vertices. The dimension of $X$ is the maximal $i$ such that $X^{(i)} \neq \emptyset$.
If \( x \) and \( x' \) are distinct cells in \( X \), then the combinatorial distance of \( x \) from \( x' \), denoted \( d(x, x') \), is the minimal \( t \in \mathbb{N} \) such that there exists a sequence of cells \( y_1, \ldots, y_t \) with \( x \leq y_1, x' \leq y_t \) and \( y_i \cap y_{i+1} \neq \emptyset \) for all \( 1 \leq i < t \). We further set \( d(x, x) = 0 \). (This agrees with the combinatorial distance in graphs.) The ball of radius \( n \) around \( x \), \( B_X(x, n) \), consists of the cells in \( X \) of distance \( n \) or less from \( x \). When \( X \) is locally finite, all the balls \( B_X(x, n) \) \((x \neq \emptyset)\) are finite. We say that \( X \) is connected if \( X = \bigcup_{n \geq 0} B_X(x, n) \) for some (and hence all) \( x \in X - \{\emptyset\} \), or equivalently, if \( d(x, x') < \infty \) for all \( x, x' \in X \).

A morphism of simplicial complexes \( f : X \to Y \) consists of a function \( f : X_{\text{vert}} \to Y_{\text{vert}} \) such that for all \( i \) and \( x \in X^{(i)} \), we have \( f(x) := \{ f(v) \mid v \in x \} \in Y^{(i)} \). The induced maps \( X^{(i)} \to Y^{(i)} \) and \( X \to Y \) are also denoted \( f \). A morphism \( f : X \to Y \) is a cover map if it is a cover map when \( X \) and \( Y \) are realized as topological spaces in the obvious way. This is equivalent to saying that \( f : X_{\text{vert}} \to Y_{\text{vert}} \) is surjective and the induced map \( f : \{ x \in X : v \in x \} \to \{ y \in Y : f(v) \in y \} \) is bijective for all \( v \in X_{\text{vert}} \). In this case, the deck transformations of \( f : X \to Y \) are the automorphisms \( h \) of \( X \) satisfying \( f \circ h = f \). We let \( \text{Cov} \)

denote the category of locally finite connected simplicial complexes with cover maps as morphisms.

**2B. Orientation.** Let \( X \) be a simplicial complex. An ordered cell in \( X \) consists of a pair \((x, \leq)\) where \( x \in X \) and \( \leq \) is a full ordering of the vertices of \( x \). Two orders on \( x \) are equivalent if one can be obtained from the other by an even permutation. We denote by \( [x, \leq] \) the equivalence class of \((x, \leq)\) and call it an oriented cell. When \( |x| > 1 \), we also write \([x, \leq]^{\text{op}}\) to denote \( x \) endowed with orientation different from the one induced by \( \leq \). For \( \{v_0, \ldots, v_t\} \in X^{(i)} \), let
\[
[v_0 v_1 \ldots v_t] = \left[ \{v_0, \ldots, v_t\}, \; v_0 < v_1 < \cdots < v_t \right].
\]
The collection of oriented \( i \)-dimensional cells is denoted \( X^{(i)}_{\text{ori}} \).

For every \( i > 0 \), define
\[
\Omega^{\pm}_i(X) := \ell^2(X^{(i)}_{\text{ori}}) \\
\Omega^-_i(X) := \{ \varphi \in \Omega^+_i(X) : \varphi(x^{\text{op}}) = -\varphi(x) \text{ for all } x \in X^{(i)}_{\text{ori}} \} \\
\Omega^+_i(X) := \{ \varphi \in \Omega^+_i(X) : \varphi(x^{\text{op}}) = \varphi(x) \text{ for all } x \in X^{(i)}_{\text{ori}} \}
\]
The inner product on \( \ell^2(X^{(i)}_{\text{ori}}) \) makes all three spaces into pre-Hilbert spaces. We further write
\[
\Omega^+_0(X) = \Omega^-_0(X) = \Omega^+_i(X) := \ell^2(X^{(i)}_{\text{ori}})
\]
and endow them with the inner product \( \langle \varphi, \psi \rangle_{\Omega^+_i(X)} = 2 \langle \varphi, \psi \rangle_{\ell^2(X^{(i)}_{\text{ori}})} \). The space \( \Omega^-_i(X) \) (resp. \( \Omega^+_i(X) \)) is the space of \( i \)-dimensional forms (resp. anti-forms) on \( X \).

Observe that \( \Omega^+_i \) is a covariant functor from \( \text{Cov} \) to \( \text{pHil} \), the category of pre-Hilbert spaces with (non-continuous) linear maps as morphisms; a morphism \( f : X \to Y \) in \( \text{Cov} \) is mapped to the linear map \( f_* : \ell^2(X^{(i)}_{\text{ori}}) \to \ell^2(Y^{(i)}_{\text{ori}}) \) determined by \( f_{|x} = e_{fx} \) for all \( x \in X^{(i)}_{\text{ori}} \) (\( fx \) is defined in the obvious way; cf. the notation section). Likewise, \( \Omega^-_i \) and \( \Omega^+_i \) are subfunctors of \( \Omega^+_i \). Notice that \( \Omega^+_i(X) \) is naturally isomorphic to \( \ell^2(X^{(i)}_{\text{ori}}) \) (as pre-Hilbert spaces). We will therefore occasionally identify \( \Omega^+_i(X) \) with \( \ell^2(X^{(i)}_{\text{ori}}) \).
Recall that the boundary map \( \partial_{i+1, X} : \Omega^i_{i+1}(X) \to \Omega^i_i(X) \) and the coboundary map \( \delta_{i, X} : \Omega^i_i(X) \to \Omega^i_{i-1}(X) \) are defined by
\[
(\partial_{i+1, X}\psi)[v_0 \ldots v_i] = \sum_{v \in X_{\text{vert}} \atop \{v, v_0, \ldots, v_i\} \in X^{(i+1)}} \psi[v v_0 \ldots v_i] \]
\[
(\delta_{i, X}\psi)[v_0 \ldots v_{i+1}] = \sum_{j=0}^{i+1} (-1)^j \psi[v_0 \ldots \hat{v}_j \ldots v_{i+1}] \]
(\(\hat{v}_j\) means omitting the \(j\)-th entry). It is easy to check that \(\delta^*_{i, X} = \partial_{i+1, X}\). The upper, lower and total \(i\)-dimensional Laplacians are defined by
\[
\Delta^+_{i, X} = \partial_{i+1, X}\delta_{i, X}, \quad \Delta^-_{i, X} = \delta_{i-1, X}\partial_{i, X}, \quad \Delta_i = \Delta^+_i + \Delta^-_i, \]
respectively (with the convention that \(\Delta^-_0 = 0\)). It is easy to check that \(\partial_{i+1} = \{\partial_{i+1, X}\}_{X \in \text{cov}} \) is a natural transformation from \(\Omega^i_{i+1} \) to \(\Omega^i_i\), and \(\delta_i = \{\delta_{i, X}\}_{X \in \text{cov}} \) is a natural transformation from \(\Omega^i_i\) to \(\Omega^i_{i-1}\). Likewise, \(\Delta^+_i, \Delta^-_i\) and \(\Delta_i\) are natural transformations from \(\Omega^i_i\) into itself.

2C. Group Actions. Let \(G\) be an \(\ell\)-group (see the notation section). By a \(G\)-complex we mean a (locally finite, connected) simplicial complex \(\mathcal{X}\) on which \(G\) acts faithfully via automorphisms and such that for all nonempty \(x \in \mathcal{X}\), the stabilizer \(\text{Stab}_G(x)\) is a compact open subgroup of \(G\). The \(\ell\)-groups for which a given simplicial complex \(\mathcal{X}\) is a \(G\)-complex are characterized in the following proposition.

**Proposition 2.1** ([1] Pr. 3.3). Let \(\mathcal{X}\) be a simplicial complex. Give \(\mathcal{X}\) the discrete topology and \(\text{Aut}(\mathcal{X})\) the topology of pointwise convergence. Then:

(i) \(\text{Aut}(\mathcal{X})\) is an \(\ell\)-group and \(\mathcal{X}\) is a \(\text{Aut}(\mathcal{X})\)-complex. Consequently, \(\mathcal{X}\) is a \(G\)-complex for any closed subgroup \(G\) of \(\text{Aut}(\mathcal{X})\).

(ii) If \(\mathcal{X}\) is a \(G\)-complex for an \(\ell\)-group \(G\), then the action map \(G \to \text{Aut}(\mathcal{X})\) is a closed embedding.

A \(G\)-complex \(\mathcal{X}\) is called **almost transitive** if \(G \setminus \mathcal{X}\) is finite.

**Example 2.2** ([1] Ex. 3.4). Let \(G\) be an almost simple algebraic group over a non-archimedean local field \(F\), let \(Z\) be the center of \(G\), and let \(G = G(F)/Z(F)\). Let \(B\) be the **affine Bruhat–Tits building** of \(G\) (see [17], [5]; a more elementary treatment in the case \(G\) is classical can be found in [2]). The building \(B\) is a simplicial complex carrying a faithful \(G\)-action making it into an almost transitive \(G\)-complex.

Let \(\mathcal{X}\) be a \(G\)-complex and let \(\Gamma\) be a subgroup of \(G\). The partial order on \(\mathcal{X}\) induces a partial order on \(\Gamma \setminus \mathcal{X}\) given by \(\Gamma \leq \Gamma y \iff \gamma x \subseteq y\) for some \(\gamma \in \Gamma\). However, \(\Gamma \setminus \mathcal{X}\) is not a simplicial complex in general ([1] Ex. 3.5(ii)), and even when this is the case, the projection map \(x \mapsto \Gamma x : \mathcal{X} \to \Gamma \setminus \mathcal{X}\) may not be a cover map ([1] Ex. 3.6). When both conditions hold, we call \(\Gamma \setminus \mathcal{X}\) a **\(G\)-quotient** of \(\mathcal{X}\) and write

\[
\Gamma \leq \mathcal{X} \leq G. \]

In this case, \(\Gamma\) coincides with the group of deck transformations of \(\mathcal{X} \to \Gamma \setminus \mathcal{X}\) ([1] Pr. 3.9).

**Proposition 2.3** ([1] Pr. 3.7, Pr. 3.9). Let \(\Gamma \leq G\). Then \(\Gamma \setminus \mathcal{X}\) is a simplicial complex if and only if

\(\text{(C1)}\) \(\{v_1, \ldots, v_n\} = \{\Gamma v_1, \ldots, \Gamma v_n\}\) implies \(\Gamma \{u_1, \ldots, u_n\} = \{\Gamma v_1, \ldots, v_n\}\) for all \(\{u_1, \ldots, u_n\}, \{v_1, \ldots, v_n\} \subseteq X_{\text{vert}}\).

\(\text{In general, } G(F)/Z(F)\) is not the same as \((G/Z)(F)\). More precisely, one has an exact sequence
\[
1 \to Z(F) \to G(F) \to (G/Z)(F) \to H^1_{\text{fpp}}(F, Z).\]
In this case, \( \dim(\Gamma \backslash X) = \dim X \) and the map \( x \mapsto \Gamma x : X \to \Gamma \backslash X \) is a morphism of simplicial complexes. The latter map is a cover map if and only if

\[ (C2) \quad \Gamma \cap \text{Stab}_G(v) = \{1_G\} \quad \text{for all } v \in X_{\text{crit}}. \]

In this case, \( \Gamma \) acts freely on \( X - \{\emptyset\} \) and \( \Gamma \) is discrete in \( G \).

The proposition can be used to prove the following elegant criterion for when \( \Gamma \leq_X G \).

**Corollary 2.4** ([11 Cor. 3.10]). \( \Gamma \leq_X G \iff d(v, \gamma v) > 2 \) for all \( 1 \neq \gamma \in \Gamma \) and \( v \in X_{\text{crit}} \).

**Corollary 2.5** ([11 Cor. 3.11]). If \( \Gamma \leq_X G \), then \( \Gamma' \leq_X G \) for all \( \Gamma' \leq \Gamma \), and \( g^{-1} \Gamma g \leq_X G \) for all \( g \in G \).

We finish with noting that if \( \Gamma \leq_X G \), then \( \Gamma \backslash X \) is finite if and only if \( \Gamma \) is cocompact in \( G \), and in this case \( G \) is unimodular [11 Pr. 3.13].

### 3. Spectrum in Simplicial Complexes

In this section, we introduce our notion of spectrum of simplicial complexes. Our definition is general and we shall demonstrate how it specializes to other notions of spectrum considered in the literature, e.g. the spectrum of \( k \)-regular graphs. The idea is to choose a \( \ast \)-algebra of operators whose spectrum we wish to investigate and apply the spectral theory of Section [11]

3A. Associated Operators. Let \( \mathcal{C} \) be a subcategory of \( \text{Cov} \), the category of locally finite connected simplicial complexes with cover maps as morphisms ([2A]), and let \( F \) be a (covariant) functor from \( \mathcal{C} \) to \( \text{pHi}1 \), the category of pre-Hilbert spaces with (non-continuous) linear maps as morphisms.

**Definition 3.1** ([11 Def. 4.5, §4C]). An associated operator of \( (\mathcal{C}, F) \), or just a \( (\mathcal{C}, F)\)-operator, is a natural transformation \( a = \{a_X\}_{X \in \mathcal{C}} : F \to F \) that admits a dual, i.e. a natural transformation \( a^\ast = \{a_X^\ast\}_{X \in \mathcal{C}} : F \to F \) such that for all \( X \in \mathcal{C} \), the operator \( a_X^\ast \) is the dual of \( a_X : FX \to FX \) relative to the inner product of \( FX \).

The collection of all \( (\mathcal{C}, F)\)-operators is denoted

\[ A(\mathcal{C}, F) \].

An algebra of \( (\mathcal{C}, F)\)-operators is a subset of \( A(\mathcal{C}, F) \) that is closed under addition, composition, scaling by elements of \( \mathbb{C} \), and taking the dual transformation.

The definition still works if one replaces \( \text{Cov} \) with the category of locally finite connected simplicial complexes with arbitrary morphisms.

**Example 3.2** ([11 Ex. 4.3, §4B]). Take \( \mathcal{C} = \text{Cov} \). We observed in [2B] that \( \Omega^+_i : \text{Cov} \to \text{pHi}1 \) is a functor and that the \( i \)-dimensional Laplacian \( \Delta_i : \Omega^+_i \to \Omega^+_i \) is a self-dual natural transformation. Thus, \( \Delta_i \) is a \( (\text{Cov}, \Omega^+_i) \)-operator, and \( \mathbb{C}[\Delta_i] \), the \( \mathbb{C} \)-algebra spanned by \( \Delta_i \) in \( A(\text{Cov}, \Omega^+_i) \), is an algebra of \( (\text{Cov}, \Omega^+_i) \)-operators.

In the next examples, we identify \( \Omega^+_i(X) \) with \( \tilde{L^2}(X^{(i)}) \) (cf. [2B]).

**Example 3.3** ([11 Ex. 4.1, §4B]). Let \( \mathcal{C} \subseteq \text{Cov} \) be the full subcategory of \( k \)-regular graphs. For \( X \in \mathcal{C} \), the vertex adjacency operator \( a_{0,X} : \Omega^+_0(X) \to \Omega^+_0(X) \) is defined by

\[ (a_{0,X} \varphi)u = \sum_v \varphi u \quad \forall \varphi \in \Omega^+_0(X), \ u \in X^{(0)}, \]
where the sum is taken over all \( v \in X^{(0)} \) connected by an edge to \( u \). Then \( a_0 := \{a_0,v \} \in \mathcal{C} \) is an associated operator of \((\mathcal{C}, \Omega_+^0)\) and \( \mathbb{C}[a_0] \) is an algebra of \((\mathcal{C}, \Omega_+^0)\)-operators. In fact, \( \mathbb{C}[a_0] = A(\mathcal{C}, \Omega_+^0) \) \( \text{[11] Ex. 4.20} \) (this is false for a general subcategory \( \mathcal{C} \)).

**Example 3.4** \( \text{[11] Ex. 4.2} \). Let \( X \) be a simplicial complex and assume \( i, j \) satisfy \( 0 \leq i < j \leq 2i + 1 \). Define \( a_{i,j,X} : \Omega_i^+(X) \to \Omega_j^+(X) \) by

\[
(a_{i,j,X} \phi) x = \sum_{y \in X^{(i)}, x \in y} \phi y \quad \forall \phi \in \Omega_j^+(X), \; x \in X^{(i)}.
\]

That is, the evaluation of \( a_{i,j,X} \phi \) at an \( i \)-cell \( x \) adds the values of \( \phi \) on \( i \)-cells \( y \) whose union with \( x \) is a \( j \)-cell. (In the notation of Example \( \text{[5] Ex. 4.3} \) we have \( a_{0,1,X} = a_{1,1,X} \).) Then \( a_{i,j} \) is a \((\text{Cov}, \Omega_i^+)\)-operator and the algebra spanned by \( a_{i,j}, a_{i+1,j+1}, \ldots, a_{2i+1,2j+1} \) is an algebra of \((\text{Cov}, \Omega_i^+)\)-operators. It is not commutative when \( i > 0 \).

3B. Spectrum. Let \( A \) be algebra of \((\mathcal{C}, F)\)-operators and let \( X \in \mathcal{C} \). Then \( A \) acts on \( FX \) via \( a \cdot v = axv \) (\( a = \{a,v \} \in \mathcal{A}, \; v \in FX \); the action is not necessarily unital). If the elements of \( A \) act continuously on \( FX \), then \( AFX \) is a pre-unitary representation of \( A \), and we define the \( A \)-spectrum of \( X \) to be

\[
\text{Spec}_A(X) = \text{Spec}_{A(FX)}(X)
\]

(if \( A \) contains \( \text{id}_F : F \to F \), we have \( AFX = FX \)). When \( A = A(\mathcal{C}, F) \) we also write \( \text{Spec}_{\mathcal{C}, F}(X) \) or \( \text{Spec}_F(X) \) instead of \( \text{Spec}_A(X) \).

When \( \mathcal{C} \) is understood from the context, we call \( \text{Spec}_{\Omega_i^+}(X) \) (resp. \( \text{Spec}_{\Omega_i^-}(X) \), \( \text{Spec}_{\Omega_i^0}(X) \)) the non-oriented (resp. oriented, full) \( i \)-dimensional spectrum, and denote it by \( \text{Spec}_i(X) \) (resp. \( \text{Spec}_{-i}(X) \), \( \text{Spec}_{\pm i}(X) \)). There is no ambiguity for \( i = 0 \) since \( \Omega_0^+ = \Omega_0^- = \Omega_0^0 \).

**Example 3.5** \( \text{[11] Ex. 4.8} \). Let \( \mathcal{C} \) be the category of \( k \)-regular graphs, and write \( A_0 = A(\mathcal{C}, \Omega_0^+) \). It can be checked that

\[
A_0 = \mathbb{C}[a_0]
\]

where \( a_0 \) is the vertex adjacency operator of Example \( \text{[5] Ex. 3} \). By Proposition \( \text{[11] Ex. 4.10} \) we can identify \( \hat{A}_0 \) with a subset of \( \mathbb{C} \) (this set is \( \mathbb{R} \), in fact) such that for all \( V \in \text{Rep}_0^0(A_0) \), the set \( \text{Spec}_{A_0}(V) \) corresponds to \( \text{Spec}(a_0|U) \). Therefore, for all \( X \in \mathcal{C} \), the datum of \( \text{Spec}_{A_0}(X) \) is equivalent to the spectrum of the vertex adjacency operator of \( X \). Otherwise stated, the (non-oriented) 0-dimensional spectrum is essentially the same as the usual spectrum of \( k \)-regular graphs.

Likewise, the non-oriented 1-dimensional spectrum turns out to be equivalent to the spectrum of the edge adjacency operator (for our particular choice of \( \mathcal{C} \)).

**Example 3.6.** Let \( F \) be a local non-archimedean field, let \( G = \text{PGL}_d(F) \) and let \( B_d(F) \) be the affine Bruhat-Tits building of \( G \) (see \( \text{[31] \S 2} \) or \( \text{[6A] below} \)). Let \( \mathcal{C} \) be the collection \( \{\Gamma \in \text{Aut}(B_d(F)) \mid \Gamma \leq B_d(F), \; G\} \). Then \( \mathcal{C} \) can be made into a category (cf. Definition \( \text{[3.7] below} \)) such that the (non-oriented) 0-dimensional spectrum of complexes in \( \mathcal{C} \) is equivalent to the spectrum of quotients of \( B_d(F) \) as defined by Lubotzky, Samuels and Vishne in \( \text{[31]} \). We give more details about this in \( \text{[6A]} \).

If \( A \) is an algebra of \((\mathcal{C}, F)\)-operators containing an algebra \( B \) of \((\mathcal{C}, F)\)-operators, then the \( A \)-spectrum (when defined) determines the \( B \)-spectrum, thanks to Theorem \( \text{[1.10] below} \). In addition, by Corollary \( \text{[1.11] below} \) for all \( a_1, \ldots, a_t \in A \) such that \( a_1, \ldots, a_t, a_1', \ldots, a_t' \) commute, the \( A \)-spectrum of \( X \in \mathcal{C} \) (when defined) determines the common spectrum of \((a_1, X), \ldots, (a_t, X)\).
3C. Elementary Functors. We now restrict our attention to special families of subcategories \( \mathcal{C} \subseteq \text{Cov} \) and functors \( F : \mathcal{C} \to \text{pHil} \) arising from almost transitive \( G \)-complexes. Henceforth, \( G \) is an \( \ell \)-group and \( X \) is an almost transitive \( G \)-complex.

**Definition 3.7 (\cite{11} Def. 4.6).** We define the subcategory
\[
\mathcal{C} = \mathcal{C}(G, X) \subseteq \text{Cov}
\]
as follows: The objects of \( \mathcal{C} \) are \( \{ \Gamma \setminus X \mid \Gamma \leq X \ G \} \) (see \cite{22}), where \( 1 \setminus X \) is identified with \( X \). The morphisms of \( \mathcal{C} \) are given as follows:

- For all \( \Gamma \leq X \ G \), set \( \text{Hom}_X(\mathcal{C}, \Gamma \setminus X) = \{ p \gamma \circ g \mid g \in G \} \), where \( p \gamma \) is the quotient map \( x : X \to \Gamma \setminus X \).
- For all \( 1 \neq \Gamma' \leq \Gamma \leq X \ G \), set \( \text{Hom}_X(\Gamma' \setminus X, \Gamma \setminus X) = \{ p \gamma' \gamma \} \) where \( p \gamma' \gamma \) is the quotient map \( \Gamma' \setminus X \to \Gamma \setminus X \).
- All other Hom-sets are empty.

In particular, \( \text{End}_X(\mathcal{C}, X) = G \).

We call \( \mathcal{C}(G, X) \) the category of \( G \)-quotients of \( X \).

**Definition 3.8.** A functor \( F : \mathcal{C}(G, X) \to \text{pHil} \) is elementary if there exists a covariant functor \( S : \mathcal{C}(G, X) \to \text{Set} \) such that

(1) There is a unitary natural isomorphism \( \bar{F} \circ S \cong F \).

(2) For all \( x \in SX \), the group \( \text{Stab}_S(x) \) is compact open in \( G \) and contained in the stabilizer of a nonempty cell in \( X \) (the action of \( G \) on \( SX \) is via \( S \)).

(3) For all \( \Gamma \leq X \ G \), the map \( \Gamma \setminus SX \to S(\Gamma \setminus X) \) given \( \Gamma x \mapsto (S\gamma)X \) (notation as in Definition 3.7) is an isomorphism.

(4) \( G \setminus SX \) is finite.

The functor \( F \) is called semi-elementary if there is another functor \( F' : \mathcal{C}(X, G) \to \text{pHil} \) such that \( F \circ F' \) is elementary.

**Example 3.9 (\cite{11} Ex. 4.12).** The functors \( \Omega^+ \) and \( \Omega^- \) are elementary. Indeed, take \( S \) to be \( X \mapsto X^{(i)} \) and \( X \mapsto X^{(i)} \), respectively (to verify (3), use \cite{11} Pr. 3.16]). The functor \( \Omega_0 \) is also elementary.

For \( i > 0 \), the functor \( \Omega^+ \) is semi-elementary since \( \Omega^\pm = \Omega^+ \oplus \Omega^- \).

For semi-elementary \( F : \mathcal{C}(G, X) \to \text{pHil} \), it is possible to give an alternative description of \( A(\mathcal{C}(G, X), F) \). In addition, the \( F \)-spectrum is always defined, i.e. \( A(\mathcal{C}(G, X), F) \) acts continuously on \( FX \) for all \( X \in \mathcal{C}(G, X) \).

Notice that \( FX \) admits an obvious left \( G \)-action since \( \text{End}_X(\mathcal{C}(G, X), X) = G \).

**Theorem 3.10 (\cite{11} Th. 4.15).** Let \( \mathcal{C} = \mathcal{C}(G, X) \), let \( F : \mathcal{C} \to \text{pHil} \) be semi-elementary, and write \( A = A(\mathcal{C}, F) \). Then:

(i) The map \( \{ aX \}_{X \in \mathcal{C}} \mapsto aX : A \to \text{End}_G(FX) \) is an isomorphism of \( * \)-algebras, where the involution on \( \text{End}_G(FX) \) is given by taking the dual with respect to the inner product on \( FX \).

(ii) For every \( a \in A \), there is \( M = M(a) \in \mathbb{R}_{\geq 0} \) such that \( ||a||_{FX} \leq M \) for all \( X \in \mathcal{C} \). In particular, \( \text{Spec}_A(X) \) is defined for all \( X \in \mathcal{C} \).

Let \( \mathcal{C} = \mathcal{C}(G, X) \), let \( F : \mathcal{C} \to \text{pHil} \) be elementary, and write \( F = \bar{F} \circ S \) as in Definition 3.7. Theorem 3.10 allows us to identify \( \text{End}_G(FX) \) with \( A := A(\mathcal{C}, F) \). For \( a \in \text{End}_G(FX) \) and \( \Gamma \setminus X \in \mathcal{C} \), the action of \( a \) on \( F(\Gamma \setminus X) \) can be described as follows: Let \( x \in SX \) and write \( ax = \sum_{y \in SY} \alpha_y e_y \) with \( \{ \alpha_y \} \subseteq \mathbb{C} \) (see the notation section). Then, upon identifying \( S(\Gamma \setminus X) \) with \( \Gamma \setminus SX \) as in (3), we have
\[
ae_{\Gamma x} = \sum_{y \in SY} \alpha_y e_{\Gamma y}
\]
Some examples demonstrating how Theorem 3.10 can be used to determine \(A(\mathcal{C}(G,X), F)\) can be found in [11 §4D]. It also yields the following result.

**Proposition 3.11.** Let \(\Gamma' \leq \Gamma \leq_X G\) with \([\Gamma : \Gamma'] < \infty\), let \(F : \mathcal{C}(G,X) \to \text{pHi1}\) be semi-elementary, and let \(A\) be an algebra of \((\mathcal{C}(G,X), F)\)-operators. Then \(\text{Spec}_A(\Gamma' \setminus X) \subseteq \text{Spec}_A(\Gamma \setminus X)\).

**3D. Further Remarks.** Let \(\mathcal{C}\) be a subcategory of \(\text{Cov}\) and let \(F, F' : \mathcal{C} \to \text{pHi1}\) be functors. It sometimes happen that there are dependencies between the \(F\)-spectrum and the \(F'\)-spectrum. For example, the full \(i\)-dimensional spectrum determines the non-oriented and oriented \(i\)-dimensional spectra, and vice versa. See [11 §4E, §4F] for an extensive discussion about this.

We also note that if \(X, X' \in \mathcal{C}\) are isomorphic simplicial complexes, then \(\text{Spec}_A(X)\) and \(\text{Spec}_A(X')\) may still differ, because \(X\) and \(X'\) may not be isomorphic in \(\mathcal{C}\). In some cases, this problem is only ostensible [11 Pr. 4.32]. However, this issue does occur for the spectrum of \(\text{PGL}_d(F)\)-quotients of the affine Bruhat–Tits building of \(\text{PGL}_d(F)\) as defined in [26] (cf. Example 3.4); see [11 Rm. 4.33] for more details.

### 4. Optimal Spectrum

For this section, let \(X\) be an almost transitive \(G\)-complex, let \(\mathcal{C} = \mathcal{C}(G,X)\), let \(F : \mathcal{C} \to \text{pHi1}\) be semi-elementary, and let \(A\) be an algebra of \((\mathcal{C}, F)\)-operators. A theorem in the spirit of the Alon–Boppana Theorem applies to the \(A\)-spectrum of \(\text{finite } G\)-quotients of \(X\).

**Theorem 4.1 ([11 Th. 5.1]).** Let \(\{\Gamma_\alpha\}_{\alpha \in I}\) be a family of subgroups of \(G\) such that \(\Gamma_\alpha \leq_X G\) for all \(\alpha \in I\), and one of the following conditions, which are equivalent, is satisfied:

1. For every compact \(C \subseteq G\) with \(1 \in C\), there exist \(\alpha \in I\) and \(g \in G\) such that \(C \cap g^{-1} \Gamma_\alpha g = 1\).
2. For every \(n \in \mathbb{N}\), there exists \(\alpha \in I\) and \(v \in X_{\text{vert}}\) such that quotient map \(X \to \Gamma_\alpha \setminus X\) is injective on the ball \(B_X(v,n)\).

Then

\[
\bigcup_\alpha \text{Spec}_A(\Gamma_\alpha \setminus X) \supseteq \text{Spec}_A(X)
\]

**Example 4.2 ([11 Ex. 5.3]).** Let \(\Gamma_1 \supseteq \Gamma_2 \supseteq \Gamma_3 \supseteq \ldots\) be subgroups of \(G\) satisfying \(\bigcap_{n} \Gamma_n = 1\) and \(\Gamma_1 \leq_X G\). Then Theorem 4.1 applies to \(\{\Gamma_n\}_{n \in \mathbb{N}}\).

Together with Proposition 1.6 the theorem implies:

**Corollary 4.3 ([11 Cor. 5.4]).** Let \(\{\Gamma_\alpha\}_{\alpha \in I}\) be as in Theorem 4.1 and suppose \(a_1, \ldots, a_t \in A(\mathcal{C}, F)\) satisfy \(a_i a_j = a_j a_i\) and \(a_i a_j^* = a_j^* a_i\) for all \(i, j\). Write \(V_\alpha = F(\Gamma_\alpha \setminus X)\) and \(V = FX\). Then

\[
\bigcup_\alpha \text{Spec}(a_1|_{V_\alpha}, \ldots, a_t|_{V_\alpha}) \supseteq \text{Spec}(a_1|_V, \ldots, a_t|_V)
\]

When \(G = \text{PGL}_d(F)\) for a local non-archimedean field \(F\), \(X\) is the affine Bruhat–Tits building of \(G\), \(F = \Omega_0^+\), and \(a_1, \ldots, a_{d-1}\) are Hecke operators (the definitions are recalled in [6A] below), Corollary 4.3 is a result of Li [25 Th. 4.1].

We proceed with describing the **trivial \(A\)**-spectrum. These are special points in the unitary dual \(\hat{A}\) that occur in the \(A\)-spectrum of finite \(G\)-quotients of \(X\).
Let $N$ be an open finite index subgroup of $G$. Then $F\mathcal{X}$ is a left $N$-module and we can form the space of $N$-coinvariants

$$(F\mathcal{X})_N = F\mathcal{X}/\text{span}\{v - gv \mid v \in F\mathcal{X}, \, g \in N\}.$$ 

Since $A$ acts on $F\mathcal{X}$ via $N$-equivariant linear maps, the action of $A$ descends to $(F\mathcal{X})_N$. It turns out that $A(F\mathcal{X})_N$ can be endowed with an inner product making it into a unitary representation of $A$, and the unitary isomorphism class of $A(F\mathcal{X})_N$ is independent of the inner product \cite[Lm. 5.6]{11}. We write

$$\mathfrak{T}_{A,N} = \text{Spec}_A(A(F\mathcal{X})_N)$$

and

$$\mathfrak{T}_A = \bigcup_N \mathfrak{T}_{A,N}$$

where $N$ ranges over the open finite index subgroups of $G$.

The set $\mathfrak{T}_A$ is called the trivial $A$-spectrum. The name is justified by the following two results, stating that the question of which points of $\mathfrak{T}_A$ occur in $\text{Spec}_A(G/\mathcal{X})$ is often a matter of which open finite index subgroups of $G$ contain $\Gamma$.

**Proposition 4.4** \cite[Pr. 5.7]{11}. Let $\mathcal{X}$ be a finite $G$-quotient of $\mathcal{X}$ such that $\mathcal{X} \leq N$. Then $\mathfrak{T}_{A,N} \subseteq \text{Spec}_A(G/\mathcal{X})$. Furthermore, if $N' \leq N$ is open of finite index, then $\mathfrak{T}_{A,N'} \subseteq \mathfrak{T}_{A,N}$.

**Proposition 4.5** \cite[Pr. 6.34]{11}. Assume $F$ is elementary, $A = A(\mathcal{E}, F)$, and write $F = \mathcal{O} \circ \mathcal{S}$ as in Definition 4.3. Let $x_1, \ldots, x_t$ be representatives for the $G$-orbits in $\mathcal{X}$ and write $\mathcal{K}_n = \text{Stab}_G(x_n)$ ($1 \leq n \leq t$). Let $\Gamma \leq \mathcal{X} G$ be such that $\Gamma \mathcal{X}$ is finite, and let $N$ be a normal finite-index subgroup of $G$ such that $G/N$ is abelian. Then:

(i) $\mathfrak{T}_{A,N} = \bigcup_{n=1}^t \mathfrak{T}_{A,NK_n}$.

(ii) If $N$ contains $\mathcal{K}_n$ for some $n$, then $\Gamma \subseteq N \iff \mathfrak{T}_{A,N} \subseteq \text{Spec}_A(G/\mathcal{X})$.

We note that in many important cases, $G$ contains a minimal open finite index subgroup $N$ such that $G/N$ is abelian, in which case Proposition 4.5 applies. For example, this holds when $\mathcal{X}$ is a $k$-regular tree and $G = \text{Aut}(\mathcal{X})$ \cite{46} or \cite{35}, or when $\mathcal{X}$ is the affine Bruhat–Tits building of an almost simple algebraic group $G$ over a local field $F$ and $G = \text{im}(G(F) \to \text{Aut}(\mathcal{X}))$.

Recall from the introduction that it is of interest to construct infinite families of finite $G$-quotients of $\mathcal{X}$ such that their $A$-spectrum is “as small as possible”, since this is likely to manifest in good combinatorial properties. The previous discussion suggests the following definition.

**Definition 4.6** \cite[Def. 5.14]{11}. A $G$-quotient $\Gamma \mathcal{X}$ is $A$-Ramamnujan if

$$\text{Spec}_A(G/\mathcal{X}) \subseteq \mathfrak{T}_A \cup \text{Spec}_A(\mathcal{X})$$

The $A$-Ramamnujan $G$-quotients of $\mathcal{X}$ can be regarded as those quotients whose spectrum is as small as one might expect of a decent infinite family of $G$-quotients. Alternatively, when finite, they can be regarded as spectral approximations of $\mathcal{X}$.

Here are several possible specializations of Definition 4.6:

- $\Gamma \mathcal{X}$ is $F$-Ramamnujan if it is $A(F, \mathcal{E})$-Ramamnujan.
- $\Gamma \mathcal{X}$ is Ramamnujan in dimension $i$ if it is $\Omega_i^+$-Ramamnujan.
- $\Gamma \mathcal{X}$ is completely Ramamnujan if it is $F$-Ramamnujan for any semi-elementary functor $F : \mathcal{E} \to \text{hil}$. (By Proposition 4.8 below, it is enough to check this when $F$ is elementary.)
Example 4.7 ([11] Ex. 5.15]). (i) Take $\mathcal{X}$ to be a $k$-regular tree, $G = \text{Aut}(\mathcal{X})$, $F = \Omega^1_0$ and $A = A(\mathcal{C}, F)$. As explained in Example 5.3 the $F$-spectrum can be canonically identified with the spectrum of the vertex adjacency operator. Careful analysis (see [11] Ex. 5.12) shows that under this identification $\mathcal{X}$ is same as the usual trivial spectrum of $k$-regular graphs — and that $\text{Spec}_A(\mathcal{X}) = \{-2\sqrt{k} - 1, 2\sqrt{k} - 1\}$ ([13] p. 252, Apx. 3, for instance). Thus, a $k$-regular graph is Ramanujan in dimension 0 (or $\Omega^0$-Ramanujan) as a $G$-quotient of $\mathcal{X}$ if and only if it is Ramanujan in the classical sense. In fact, we will see later that $k$-regular graphs are Ramanujan in dimension 0 if and only if they are completely Ramanujan.

(ii) Let $F$ be a non-archimedean local field, let $G = \text{PGL}_d(F)$ and let $\mathcal{X}$ be the affine Bruhat–Tits building of $G$. It can be checked that a $G$-quotient of $\mathcal{X}$ is Ramanujan in dimension 0 if and only if it is Ramanujan in the sense of Lubotzky, Samuels and Vishne [31]; see [11] Ex. 5.13, Ex. 5.15(ii)].

The $A$-Ramanujan property behaves well as $A$ and $\Gamma$ vary.

Proposition 4.8 ([11] Pr. 5.16]). Let $F, F' : \mathcal{C} \to \text{phil}$ be semi-elementary functors, let $A$ be an algebra of $(\mathcal{C}, F)$-operators, and let $\Gamma' \leq \Gamma \leq \mathcal{X} G$. Then:

(i) $\Gamma \backslash \mathcal{X}$ is $F \oplus F'$-Ramanujan if and only if $\Gamma \backslash \mathcal{X}$ is $F$-Ramanujan and $F'$-Ramanujan.

(ii) If $\Gamma \backslash \mathcal{X}$ is $A$-Ramanujan, then $\Gamma \backslash \mathcal{X}$ is $B$-Ramanujan for any $*$-subalgebra $B \subseteq A$.

(iii) If $[\Gamma : \Gamma'] < \infty$ and $\Gamma' \backslash \mathcal{X}$ is $A$-Ramanujan, then $\Gamma' \backslash \mathcal{X}$ is $A$-Ramanujan.

Existence of $A$-Ramanujan $G$-quotients cannot be guaranteed in general. This follows implicitly from [29]. Rather, it is more reasonable to hope that under certain assumptions, finite $G$-quotients of $\mathcal{X}$ would admit $A$-Ramanujan covers; see [11] Rm. 5.17 for further discussion.

5. Representation Theory

Unless otherwise indicated, $G$ is a unimodular $\ell$-group, $\mu_G$ is a fixed Haar measure of $G$, $\mathcal{X}$ is an almost transitive $G$-complex, and $F : \mathcal{C}(G, \mathcal{X}) \to \text{phil}$ is an elementary functor. We shall give a criterion for when $\Gamma \backslash \mathcal{X}$ is $F$-Ramanujan which is phrased in terms of certain unitary representation of $G$.

5A. Unitary Representations. As usual, a unitary representation of $G$ is a Hilbert space $V$ carrying a $G$-module structure such that the action $G \times V \to V$ is continuous and $\langle gu, gv \rangle = \langle u, v \rangle$ for all $u, v \in V, g \in G$. The representation $V$ is irreducible if $V$ does not contain closed $G$-submodules other than 0 and $V$. The category of unitary representations of $G$ with continuous $G$-homomorphisms is denoted $\text{Rep}^u(G)$, and the class of irreducible representations is denoted $\text{Ir}^u(G)$. The unitary dual of $G$, denoted $\hat{G}$, is the collection of unitary isomorphism classes in $\text{Ir}^u(G)$.

For every $K \leq G$, we write $V^K = \{ v \in V : kv = v \text{ for all } k \in K \}$.

Example 5.1 ([11] Ex. 6.1, Ex. 6.2)]. Let $\Gamma$ be a discrete subgroup of $G$. Since $G$ is unimodular, the left coset space $\Gamma \backslash G$ admits a right $G$-invariant measure $\mu_{\Gamma \backslash G}$, unique up to scaling. Then $L^2(\Gamma \backslash G)$ is a unitary representation of $G$ with respect to the left $G$-action given by

$$(g \varphi)x = \varphi(xg) \quad \forall g, x \in G, \varphi \in L^2(\Gamma \backslash G).$$

When $\Gamma = 1$, we have $\mu_{\Gamma \backslash G} = \mu_G$ (up to scaling), and $L^2(\Gamma \backslash G) = L^2(1 \backslash G)$ is the right regular representation of $G$. 
Let $V \in \operatorname{Irr}^u(G)$ and $V' \in \operatorname{Rep}^u(G)$. Recall that $V$ is weakly contained in $V'$, denoted $V \prec V'$, if for all $v \in \mathcal{S}(V)$, $\varepsilon > 0$ and compact $C \subseteq G$, there exists $v' \in \mathcal{S}(V')$ such that 

$$
|\langle gv, v \rangle - \langle gv', v' \rangle| < \varepsilon \quad \forall g \in G.
$$

We further write 

$$
\text{Spec}_G(V') = \{ [V] \in \hat{G} : V \prec V' \}.
$$

The representation $V$ is called tempered if it weakly contained in $L^2(1 \backslash G)$, the right regular representation of $G$. (See [38] §2.4 for equivalent definitions of temperedness when $G$ is the group of points of a reductive algebraic group over a local field.)

Finally, an irreducible representation $V \in \operatorname{Irr}^u(G)$ is said to have finite action if one of the following conditions, which are equivalent [11 Lm. 6.19], hold:

(a) The image of $G$ in the unitary group of $V$ is finite.
(b) There is an open subgroup of finite index $N \leq G$ such that $V = V^N$.

Representations with finite action are finite-dimensional, but the converse is false in general. However, if $\Gamma$ is a cocompact lattice in $G$, then irreducible representations that are weakly contained in $L^2(\Gamma \backslash G)$ have finite action if and only if they are finite dimensional [11 Lm. 6.20].

### 5B. A Criterion for Being Ramanujan.

**Theorem 5.2** ([11] Th. 6.22]). Write $F = \hat{\ell}^2 \circ S$ where $S$ is as in Definition 3.8. Let $x_1, x_2, \ldots, x_t$ be representatives of the $G$-orbits in $S\mathcal{X}$, and let $K_n = \text{Stab}_G(x_n)$ $(1 \leq n \leq t)$. Then:

(i) $\Gamma \backslash \mathcal{X}$ is $F$-Ramanujan if and only if every irreducible unitary representation $V \prec L^2(\Gamma \backslash G)$ satisfying $V^{K_1} + \cdots + V^{K_t} \neq 0$ is tempered (i.e. $V \prec L^2(1 \backslash G)$) or has finite action.

(ii) $\Gamma \backslash \mathcal{X}$ is completely Ramanujan if and only if every irreducible unitary representation $V \prec L^2(\Gamma \backslash G)$ is tempered or has finite action.

When $\Gamma \backslash \mathcal{X}$ is finite, one can replace “finite action” with “finite dimension” and “$V \prec L^2(\Gamma \backslash G)$” with “$V \prec L^2(\Gamma \backslash G)$”.

The criterion is a consequence of the following theorem, which gives more information about how the $F$-spectrum of $\Gamma \backslash \mathcal{X}$ is related to the irreducible subrepresentations of $L^2(\Gamma \backslash \mathcal{X})$.

**Theorem 5.3** ([11] Th. 6.21]). Keep the notation of Theorem 5.2. For any subset $T$ of the unitary dual $\hat{G}$, define 

$$
T^{(K_1, \ldots, K_t)} = \{ [V] \in T : V^{K_1} + \cdots + V^{K_t} \neq 0 \}.
$$

Then there exists an additive functor 

$$
\mathcal{F} : \operatorname{Rep}^u(G) \to \operatorname{Rep}^u(A(\mathcal{E}, F))
$$

with the following properties:

(i) $\mathcal{F}$ induces an embedding $[V] \mapsto [\mathcal{F}V] : \hat{G}^{(K_1, \ldots, K_t)} \to \hat{A}(\mathcal{E}, F)$, denoted $\hat{F}$.

(ii) For all $[V] \in \hat{G}^{(K_1, \ldots, K_t)}$ and $V' \in \operatorname{Rep}^u(G)$, we have $V \prec V' \iff \mathcal{F}V \prec \mathcal{F}V'$.

(iii) $\mathcal{F}(L^2(\Gamma \backslash G)) = F(\Gamma \backslash \mathcal{X})$ for all $\Gamma \leq X G$.

(iv) Let $[V] \in \hat{G}^{(K_1, \ldots, K_t)}$. Then $\hat{F}[V]$ is in $\Sigma_{A(\mathcal{E}, F)}$ (the trivial $A(\mathcal{E}, F)$-spectrum) if and only if $V$ has finite action. More precisely, we have 

$$
\hat{F}(\text{Spec}_G(L^2(\Gamma \backslash G))^{(K_1, \ldots, K_t)}) = \Sigma_{A(\mathcal{E}, F), N} \text{ for any open finite-index subgroup } N \leq G.
$$
It is also possible to describe $A(\mathcal{C}, F)$ using the Hecke algebra of $G$. See [11, Th. 6.10] for details.

We now give some consequences of Theorems 5.2 and 5.3.

Marcus, Spielman and Srivastava [33] proved that every bipartite $k$-regular graph $X$ has a Ramanujan double cover $X' \to X$. That is, all eigenvalues of the vertex adjacency operator of $X'$ not arising from $X$ are in the interval $[-2\sqrt{k-1}, 2\sqrt{k-1}]$. This was extended to covers of any prescribed degree by Hall, Puder and Sawin [17]. Using Theorem 5.2, one can show the following result, which is reminiscent of the Ramanujan–Petersson conjecture.

**Corollary 5.4 (11 Cor. 6.28, Pr. 6.30).** Let $X$ be a $k$-regular tree, let $G = \text{Aut}(X)$ and let $H$ be the index-$2$ subgroup of $G$ consisting of automorphisms preserving the canonical 2-coloring of $X^{(0)}$. Then for any cocompact $\Gamma' \leq H$ and $r \in \mathbb{N}$, there exists a sublattice $\Gamma' \trianglelefteq \Gamma$ of index $r$ such that every irreducible unitary subrepresentation $V$ of the orthogonal complement of $L^2(\Gamma \backslash G)$ in $L^2(\Gamma' \backslash G)$ is tempered.

The following proposition follows from Theorem 5.2 and representation-theoretic properties of the relevant group $G$.

**Proposition 5.5 (11 Pr. 6.30, Pr. 6.31).** (i) Let $X$ be a $k$-regular tree, and let $G = \text{Aut}(X)$. Then a $k$-regular graph, viewed as a $G$-quotient of $X$, is Ramanujan in dimension 0 (i.e. Ramanujan in the classical sense) if and only if it is completely Ramanujan.

(ii) Let $F$ be a local non-archimedean field, let $G = \text{PGL}_d(F)$ with $d \in \{2, 3\}$, let $X$ be the affine Bruhat–Tits building of $G$, and let $\Gamma \leq X \, G$. Then $\Gamma \backslash X$ is Ramanujan in dimension 0 if and only if $\Gamma \backslash X$ is Ramanujan in all dimensions.

Let $H$ be an open finite index subgroup of $G$. Then $X$ is an almost transitive $H$-complex. For $\Gamma \leq X \, H$, we can consider $\Gamma \backslash X$ both as a $G$-quotient and as an $H$-quotient of $X$. However, using Theorem 5.2(ii), one can show that this does not affect the completely Ramanujan property.

**Theorem 5.6 (11 Th. 6.36).** Let $\Gamma \leq X \, H$. Then $\Gamma \backslash X$ is completely Ramanujan as an $H$-quotient of $X$ if and only if $\Gamma \backslash X$ is completely Ramanujan as a $G$-quotient of $X$.

6. Ramanujan Complexes

Let $F$ be a local non-archimedean field of positive characteristic, let $D$ be a central division $F$-algebra, let $d \geq 2$, let $G = \text{PGL}_d(D) := \text{GL}_d(D)/F^\times$, and let $\mathcal{B}_d(D)$ be the affine Bruhat–Tits building of $G$ (we recall its construction below).

Theorem 5.2 can be applied together with deep results about automorphic representations to show that $\mathcal{B}_d(D)$ has infinitely many non-isomorphic $G$-quotients which are completely Ramanujan. This was shown in the case $D = F$ by Lubotzky, Samuels and Vishne [31], with the difference that they proved that the $G$-quotients were Ramanujan in dimension 0.

6A. The Building of $\text{PGL}_d(D)$. Recall that $r := [D : F]^{1/2}$ is an integer called the degree of $D$. If $\overline{F}$ is an algebraic closure of $F$, then $M_d(D) \otimes_F \overline{F} \cong M_d(\overline{F})$ (as $\overline{F}$-algebras). The restriction of the determinant map $\det : M_d(\overline{F}) \to \overline{F}^\times$ to $M_d(D)$ is called the reduced norm and denoted $\text{Nrd}_{M_d(D)/F}$. It well-defined and takes values in $F$.

Let $\eta : F \to \mathbb{Z} \cup \{\infty\}$ denote the additive valuation of $F$. By [43, §12], $\eta$ extends uniquely to an additive valuation $\eta_D : D \to \mathbb{R} \cup \{\infty\}$ given by:

$$\eta_D(x) = r^{-1} \eta(\text{Nrd}_{D/F}(x)) .$$
Since the cardinality $q$ of the residue field of $(F,\eta)$ is finite, $\text{im}(\eta_D) = \frac{1}{r}\mathbb{Z}$ and the residue division ring of $(D,\eta_D)$ is the Galois field of cardinality $q^r$. [31] Th. 14.3.

Fix an element $\pi_D \in D$ with $\eta_D(\pi_D) = \frac{1}{r}$ and write

$$\mathcal{O}_D = \{ x \in D : \eta_D(x) \geq 0 \}.$$

We make $\text{GL}_d(D)$ into a topological group by giving it the subspace topology induced from the inclusion $\text{GL}_d(D) \subseteq M_d(D) \cong F^{r \times d^2}$, and give $G = \text{PGL}_d(D) := \text{GL}_d(D)/F^{r \times d^2}$ the quotient topology. Then $G$ is an $\ell$-group. The quotient map $\text{GL}_d(D) \to \text{PGL}_d(D)$ is denoted by $g \mapsto \overline{g}$.

The affine Bruhat–Tits building of $G$, denoted $\mathcal{B}_d(D)$, is a simplicial complex constructed as follows: Let $K$ be the subgroup of $G$ generated by the images of $\text{GL}_d(\mathcal{O}_D)$ and

$$\begin{bmatrix}
\pi_D \\
\vdots \\
\pi_D 
\end{bmatrix}$$

in $G$. The vertices of $\mathcal{B}_d(D)$ are $G/K$. To define the edges of $\mathcal{B}_d(F)$, let

$$g_1 = \begin{bmatrix}
\pi_D & 1 \\
1 & \ddots \\
& \ddots & 1
\end{bmatrix}, \ldots, g_{d-1} = \begin{bmatrix}
\pi_D \\
& \ddots \\
& \ddots & 1
\end{bmatrix} \in \text{GL}_d(F).$$

Two vertices $gK, g'K \in G/K$ are adjacent if

$$g^{-1}g' \in K \cup K\overline{\pi}K \cup K\overline{\pi}K \cup \ldots K\overline{\pi}^{d-1}K,$$

and the $i$-dimensional cells of $\mathcal{B}_d(D)$ are the $(i+1)$-cliques, namely, they are sets $\{h_0K, \ldots, h_{i+1}K\} \subseteq G/K$ consisting of pairwise adjacent vertices. The resulting complex is indeed a pure $(r-1)$-dimensional contractible simplicial complex, which carries additional structure making it into an affine building; see [1] §6.9 or [2] for further details.

There is an obvious left action of $G$ on $\mathcal{B}_d(D)$, making the latter into an almost transitive $G$-complex.

**Example 6.1.** The complex $\mathcal{B}_d(D)$ is a $q^r + 1$ regular tree. Its $G$-quotients are therefore $(q^r + 1)$-regular graphs.

We now explain why the 0-dimensional spectrum of $G$-quotients of $\mathcal{B}_d(F)$ in the sense of [32] is equivalent to the spectrum defined by Lubotzky, Samuels and Vishne in [31] (basing on Cartwright, Solé and Žuk [6]).

The spectrum of Lubotzky, Samuels and Vishne is defined as follows: Consider the map $c : \text{GL}_d(D) \to \mathbb{Z}/d\mathbb{Z}$ given by

$$c(g) = \eta(\text{Nrd}_{M_d(D)/F}(g)) + d\mathbb{Z}.$$ 

It induces a $(d-1)$-coloring of the directed edges of $\mathcal{B}_d(D)$ given by

$$C_1(\overline{g}K, g'K) := c(g^{-1}g') \in \mathbb{Z}/d\mathbb{Z}, \quad \forall g, g' \in \text{GL}_d(D).$$

This is a $(d-1)$-coloring because $\text{im}(C_1) = \mathbb{Z}/d\mathbb{Z} - \{0\}$.

The coloring $C_1$ descends to the $G$-quotients of $\mathcal{B}_d(F)$. If $X$ is a such a quotient, we define linear operators

$$a_{1,X}, \ldots, a_{d-1,X} : \Omega^+_0(X) \to \Omega^+_0(X)$$

by

$$(a_{i,X} \varphi)x = \sum_{y \in X_{\text{vert}} \atop C_1(x,y) = i} \varphi y \quad \forall \varphi \in L^2(X_{\text{vert}}), \ x \in X_{\text{vert}}.$$
The operators $a_{1,X}, \ldots, a_{d-1,X}$ are called the colored adjacency operators or Hecke operators of $X$. It turns out that they commute with each other and that $a_{i,X}^* = a_{d-i,X}$ for all $i$. According to Lubotzky, Samuels and Vishne [31], when $D = F$, the spectrum of a $G$-quotient $X$ of $B_d(D)$ is
\[ \text{Spec}(a_{1,X}, \ldots, a_{d-1,X}) \subseteq \mathbb{C}^{d-1}. \]
This definition also makes sense when $D \neq F$.

Now, write $\mathcal{E} = \mathcal{E}(G, B_d(D))$ and $a_i = \{a_{i,X}\}_{X \in \mathcal{E}}$. Then $a_1, \ldots, a_{d-1}$ are operators associated with $(\mathcal{E}, \Omega^0_{\mathcal{E}})$ and they span a commutative $*$-subalgebra of $A(\mathcal{E}, \Omega^0_{\mathcal{E}})$. When $D = F$, it is known that $\mathbb{C}[a_1, \ldots, a_{d-1}] = A(\mathcal{E}, \Omega^0_{\mathcal{E}})$ ([32 Ch. V]; see also [11 Ex. 6.14]). Thus, by virtue of Proposition 1.6, the spectrum of Lubotzky, Samuels and Vishne [31] is equivalent to the 0-dimensional spectrum of $G$-quotients of $B_d(F)$.

Lubotzky, Samuels and Vishne [31] also defined Ramanujan $G$-quotients of $B_d(F)$ as those quotients whose spectrum is contained in union of the spectrum of $B_d(F)$ (which they determined in [31 Th. 2.11]) with a certain set of points in $\mathbb{C}^{d-1}$ called the trivial spectrum (see [31 §2.3, Def. 1.1]). The $G$-quotients which are Ramanujan in this sense are precisely the $G$-quotients which are Ramanujan in dimension 0 according to our definition [11 Ex. 5.15(ii)].

**Remark 6.2.** The assertion $\mathbb{C}[a_1, \ldots, a_{d-1}] = A(\mathcal{E}, \Omega^0_{\mathcal{E}})$ seems to be correct for general $D$. We were unable to find a source, however.

6B. Ramanujan Quotients. Keep the notation of [6A]. We now state our main result, which gives particular infinite families of completely Ramanujan $G$-quotients of $B_d(D)$. We remind the reader that we assume char $F > 0$.

We introduce additional notation: There is a global field $k$ with a place $\eta$ such that the completion of $k$ at $\eta$, denoted $k_\eta$, is $F$. Let $\mathcal{V}$ be the set of places of $k$. The additive valuation corresponding to $\nu \in \mathcal{V}$ is also denoted $\nu$. We further write
\[ O_\nu = \{\alpha \in k_\nu : \nu(\alpha) \geq 0\}, \]
\[ R = \{\alpha \in k : \nu(\alpha) > 0 \text{ for all } \eta \neq \nu \in \mathcal{V}\}. \]

Let $E$ be a central division $k$-algebra of dimension $(rd)^2$. Then for every $\nu \in \mathcal{V}$, there is a central division $k_\nu$-algebra $D_\nu$ and $m_\nu \in \mathbb{N}$ such that $E_\nu := E \otimes_k k_\nu \cong M_{m_\nu}(D_\nu)$. Suppose that $E$ is chosen such that
- $E_\eta \cong M_d(D)$, i.e. $D_\eta \cong D$ and $m_\eta = d$.
- There is $\theta \in \mathcal{V}$ such that $E_\theta$ is a division ring, i.e. $m_\theta = 1$.

Existence of a suitable $E$ for any prescribed $D$ and $d$ follows from the Albert–Brauer–Hasse–Noether Theorem ([33 Rem. 32.12(ii)], for instance).

The functor $A \mapsto \text{Aut}_{A,\text{alg}}(E \otimes_k A)$ from commutative $k$-algebras to groups is representable by an affine group scheme over $k$, denoted $\text{PGL}_{1,E}$. We write $H = \text{PGL}_{1,E}$ for brevity. By the Skolem–Noether Theorem, $H(L) = (E \otimes_k L)\times/L\times$ for every field extension $L/k$. Fix a closed embedding $j : H \to \text{GL}_n$. For any integral domain $S$ whose fraction field $L$ contains $k$, and for any $I \subseteq S$, we write
\[ H(S) = j(H(L)) \cap \text{GL}_n(S) \]
\[ H(S,I) = \ker(H(S) \to \text{GL}_n(S) \to \text{GL}_n(S/I)) \]
We assume that
- $H(O_\theta) = \text{im}(O_{E_\theta}^\times \to E_\theta^\times/k_\theta^\times = H(k_\theta))$
where $O_{E_\theta}$ is defined as in [6A]. The existence of an embedding $j : H \to \text{GL}_n$ with this property is shown in [31 §5].
Finally, recall that the ideals of $R$ correspond to functions $\mathring{n} : \mathcal{V} - \{\eta\} \to \mathbb{N} \cup \{0\}$ of finite support. The ideal corresponding to $\mathring{n}$ is
\[ I(\mathring{n}) = \{ \alpha \in R : \nu(\alpha) \geq \mathring{n}(\alpha) \text{ for all } \nu \in \mathcal{V} - \{\eta\} \} . \]

We write
\[ \Gamma(\mathring{n}) = H(R, I(\mathring{n})) . \]

Since $H$ is $k$-anisotropic, $\Gamma(\mathring{n})$ is a cocompact lattice in $H(k_\mathring{n}) = \text{PGL}_d(D)$ (see [11 §7F]). For every $\nu \in \mathcal{V} - \{\eta\}$, there is $n_0 \in \mathbb{N}$ such that $\Gamma(\mathring{n}) \leq \text{B}_d(D) \text{PGL}_d(D)$ whenever $\mathring{n}(\nu) \geq n_0$ [11 Rm. 7.26(ii)].

**Theorem 6.3** ([11 Th. 7.22]). Let $\mathring{n} : \mathcal{V} - \{\eta\} \to \mathbb{N} \cup \{0\}$ be a function of finite support such that $\Gamma(\mathring{n}) \leq \text{B}_d(D) \text{PGL}_d(D)$. Assume that either
1. $\mathring{n}(\emptyset) = 0$, or
2. $D = F$ and $d$ is prime.

Then $\Gamma(\mathring{n}) \backslash \text{B}_d(D)$ is completely Ramanujan.

When $D = F$, Theorem 6.3 is just Theorem 1.2 of [31] with the difference that we show complete Ramanujan-ness whereas [31] shows Ramanujan-ness in dimension 0 (cf. [23]).

**Example 6.4** ([11 Ex. 7.23]). For every $\Gamma = \Gamma(\mathring{n})$ as in Theorem 6.3, the spectrum of the $i$-dimensional Laplacian $\Delta_i$ of $\Gamma \backslash \text{B}_d(D)$ is contained in the union of the spectrum of the $i$-dimensional Laplacian of $\text{B}_d(D)$ with the trivial spectrum of $\Delta_i$ (which is $\mathbb{T}[\Delta_i]$, where $\mathbb{T}[\Delta_i]$ is the subalgebra of $A(\mathcal{C}[\Delta_i], \Omega_+^i)$ spanned by $\Delta_i$; use Proposition 1.6 and Proposition 4.3). This also holds for the adjacency operators considered in Example 6.4. Therefore, when $d = 2$, the quotient $\Gamma \backslash \text{B}_2(D)$ is a Ramanujan $(q^2 + 1)$-regular graph.

Theorem 6.3 is proved by using the criterion of Theorem 5.2(ii) together with deep results about automorphic representations, particularly the proof of the Ramanujan–Petersson conjecture for $\text{GL}_n$ in positive characteristic due to Lafforgue [23], and the global Jacquet–Langlands correspondence for $\text{GL}_n$ in positive characteristic, established in [3].

In the next subsection, we give brief details about how to translate the statement of Theorem 6.3 into a statement about automorphic representations, which can then be proved using results from [23], [3] and related works. The details of the latter can be found in [11 §7F–§7I].

6C. Automorphic Representations. Keeping all previous notation, let $S \subseteq \mathcal{V}$ be a finite subset and let $\mathbb{A}^S$ denote the $k$-ades away from $S$, namely
\[ \mathbb{A}^S = \prod_{\nu \in \mathcal{V} - S} k_\nu := \left\{ (a_\nu)_\nu \in \prod_{\nu \in \mathcal{V} - S} k_\nu : a_\nu \in \mathcal{O}_\nu \text{ for almost all } \nu \right\} . \]

We give $\prod_{\nu \in \mathcal{V} - S} \mathcal{O}_\nu$ the product topology, and topologize $\mathbb{A}^S$ by viewing it as a disjoint union of (additive) cosets of $\prod_{\nu \in \mathcal{V}} \mathcal{O}_\nu$. If $G$ is an algebraic group over $k$, then we topologize $G(\mathbb{A}^S)$ by choosing a closed embedding $G \hookrightarrow \text{SL}_n$ and giving $G(\mathbb{A}^S)$ the topology induced from $\text{SL}_n(\mathbb{A}) \subseteq M_n(\mathbb{A}) \cong \mathbb{A}^n$. This makes $G(\mathbb{A}^S)$ into an $\ell$-groupootnote{When $\text{char } k = 0$ and $S$ does not contain all archimedean places, $G(\mathbb{A}^S)$ is not an $\ell$-group but rather a locally compact group. Nevertheless, the discussion to follow applies when $\text{char } k = 0$ after some modifications.}, its topology is independent of the embedding.

As usual, $k$ is embedded diagonally in $\mathbb{A}^S$, and we write $\mathbb{A} := \mathbb{A}^0$. We shall occasionally view $\mathbb{A}$ as $k_\emptyset \times \mathbb{A}^{\{\emptyset\}}$. 


Let $G$ be a semisimple algebraic group over $k$ with center $Z$. Recall that an automorphic representation of $G$ is an irreducible representation $V$ of $G(\mathbb{A})$ that is weakly contained in $L^2(G(\mathfrak{a}) \backslash G(\mathbb{A}))$ (cf. Example 5.1). Every such representation can be written as a restricted tensor product $V = \bigotimes_{\nu} V_{\nu}$ with $V_{\nu} \in \operatorname{Irr}(G(k_\nu))$; see [12] or [11, §7C]. The factor $V_{\nu}$ is called the $\nu$-local factor of $V$.

Assume $G$ is $k$-anisotropic, and choose a compact open subgroup $K^\nu \leq G(\mathbb{A}^{(\nu)})$. Our assumption implies that $G(k) \backslash G(\mathbb{A})$ is compact and discrete, hence finite. Let $(1, g_1), \ldots, (1, g_t) \in G(k_\nu) \times G(\mathbb{A}^{(\nu)})$ be representatives for the double cosets. For each $1 \leq i \leq t$, define

$$\Gamma_i = G(k) \cap (G(k_\nu) \times g_i K^\nu g_i^{-1})$$

and view $\Gamma_i$ as a subgroup of $G(k_\nu)$. It is a standard fact that there is an isomorphism of topological (right) $G(k_\nu)$-spaces

$$(6.1) \quad \bigsqcup_{i=1}^t \Gamma_i \backslash G(k_\nu) \to G(k) \backslash G(\mathbb{A})/(1 \times K^\nu)$$

given by sending $\Gamma_i g$ to $G(k)(g, g_i)(1 \times K^\nu)$. In particular, $\Gamma_i$ is a cocompact lattice in $G(k_\nu)$ for all $1 \leq i \leq t$. Assume further that $G$ is almost simple, let $G = G(k_\nu)/Z(k_\nu)$ and write

$$\mathfrak{T}_i = \operatorname{im}(\Gamma_i \to G).$$

Then $G$ acts faithfully on the affine Bruhat–Tits building $B$ of $G(k_\nu)$, making it into an almost transitive $G$-complex (Example 2.2). Using (6.1) and Theorem 5.2 one can show:

**Theorem 6.5** ([11, Th. 7.4]). Let $F : \mathcal{C}(G, B) \to \mathfrak{phil}$ be an elementary functor (e.g. $\Omega^+_{\mathcal{L}}$ or $\Omega^+_{\mathcal{D}}$), write $F \cong \mathfrak{p} \circ S$ as in Definition 5.5, let $x_1, \ldots, x_s$ be representatives for the $G$-orbits in $SB$, and let $L_j = \operatorname{Stab}_{G(k_\nu)}(x_j) \times K^\nu$ ($1 \leq j \leq s$). Assume that for any automorphic representation $V = \bigotimes_{\nu} V_{\nu}$ of $G$ with $V^{L_1} \times \ldots \times V^{L_s} \neq 0$ (resp. $V^{Z(k_\nu)\times K^\nu} \neq 0$), the local factor $V_{\nu}$ is tempered or finite-dimensional. Then $\mathfrak{T}_i \backslash B$ is $F$-Ramanujan (resp. completely Ramanujan) for every $1 \leq i \leq t$ such that $\mathfrak{T}_i \leq_B G$. The converse holds when $\mathfrak{T}_i \leq_B G$ for all $1 \leq i \leq t$.

Applying Theorem 6.5 with $G = H$ (cf. [13]), a suitable $K^\nu$ and $g_1 = 1$ allows one to translate the statement of Theorem 6.5 into a statement about the automorphic representations of $H$, which can be proved using powerful results about the latter (see [11, §7F–§7I]).

We mention here several places where such ideas were applied in the literature, sometimes implicitly or in an equivalent formulation:

- Lubotzky, Phillips and Sarnak [26], and independently Margulis [34], constructed infinite families of Ramanujan ($p + 1$)-regular graphs for every prime $p$ using results of Eichler [9] and Igusa [19] about modular forms. (See also Deligne’s proof of the Ramanujan–Petersson conjecture for modular forms [7].) In the previous setting, this corresponds to taking $k = Q$ and $G$ to be an inner form of $\text{PGL}_2$ which splits over $k_\nu$.

- Morgenstern [36] used Drinfeld’s proof of the Ramanujan–Petersson conjecture for $\text{GL}_2$ when $\operatorname{char} k > 0$ [8] to construct infinite families of Ramanujan $(q + 1)$-regular graphs for every prime power $q$. Again, the corresponding group $G$ is an inner form of $\text{PGL}_2$. 


• Lubotzky, Samuels and Vishne [31] applied Lafforgue’s proof of the Ramanujan–Petersson conjecture for GL_d when char \( k > 0 \) [23] to construct infinite families of quotients of \( B_{d}(F) \) which are Ramanujan in dimension 0. The corresponding group \( G \) is an inner form of \( \text{PGL}_n \) which splits over \( k_\eta \).

• Li [25] independently gave similar constructions of Ramanujan complexes, using results of Laumon, Rapoport and Stuhler, who proved a special case of the Ramanujan–Petersson conjecture for anisotropic inner forms of \( \text{GL}_n \) [24, Th. 14.12].

• Ballantine and Ciubotaru [4] constructed infinite families of Ramanujan \( (q+1, q^3+1) \)-biregular graphs for every prime power \( q \). The corresponding group \( G \) is an inner form of \( \text{SU}(3) \), and they use the classification of the automorphic spectrum of \( G \) due to Rogawski [44].

We hope our work will facilitate further results of this kind.

References

[1] Peter Abramenko and Kenneth S. Brown. Buildings: Theory and applications, volume 248 of Graduate Texts in Mathematics. Springer, New York, 2008.
[2] Peter Abramenko and Gabriele Nebe. Lattice chain models for affine buildings of classical type. Math. Ann., 322(3):537–562, 2002.
[3] A. I. Badulescu and Ph. Roche. Global Jacquet-Langlands correspondence for division algebras in characteristic \( p \). 2013. Preprint (currently available at http://arxiv.org/abs/1302.5289).
[4] Cristina Ballantine and Dan Ciubotaru. Ramanujan bigraphs associated with \( \text{SU}(3) \) over a \( p \)-adic field. Proc. Amer. Math. Soc., 139(6):1939–1953, 2011.
[5] François Bruhat and Jacques Tits. Groupes réductifs sur un corps local. Inst. Hautes Études Sci. Publ. Math., (41):5–251, 1972.
[6] Donald I. Cartwright, Patrick Solé, and Andrzej Łuk. Ramanujan geometries of type \( \tilde{A}_n \). Discrete Math., 269(1-3):35–43, 2003.
[7] Pierre Deligne. La conjecture de Weil. I. Inst. Hautes Études Sci. Publ. Math., (43):273–307, 1974.
[8] V. G. Drinfel’d. Proof of the Petersson conjecture for GL(2) over a global field of characteristic \( p \). Funktsional. Anal. i Prilozhen., 22(1):34–54, 96, 1988.
[9] Martin Eichler. Quaternäre quadratische Formen und die Riemannsche Vermutung für die Kongruenzzetafunktion. Arch. Math., 5:355–366, 1954.
[10] Shai Evra, Konstantin Golubev, and Alexander Lubotzky. Mixing properties and the chromatic number of Ramanujan complexes. Int. Math. Res. Not. IMRN, (22):11520–11548, 2015.
[11] Uriya First. The Ramanujan property for simplicial complexes. 2016. Preprint (currently available at https://arxiv.org/abs/1605.02664).
[12] D. Flath. Decomposition of representations into tensor products. In Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, pages 179–183. Amer. Math. Soc., Providence, R.I., 1979.
[13] Jacob Fox, Mikhail Gromov, Vincent Lafforgue, Assaf Naor, and János Pach. Overlap properties of geometric expanders. J. Reine Angew. Math., 671:49–83, 2012.
[14] Konstantin Golubev. On the chromatic number of a simplicial complex. 2016. Preprint (currently available at http://arxiv.org/abs/1306.4818).
[15] Konstantin Golubev and Ori Parzanchevski. Spectrum and combinatorics of ramanujan triangle complexes. 2014. Preprint (currently available at http://arxiv.org/abs/1406.6666).
[16] Mikhail Gromov. Singularities, expanders and topology of maps. Part 2: From combinatorics to topology via algebraic isoperimetry. Geom. Funct. Anal., 20(2):416–526, 2010.
[17] Chris Hall, Doron Puder, and William F. Sawin. Ramanujan coverings of graphs. 2016. Preprint (currently available at http://arxiv.org/abs/1506.02335).
[18] G. Harder. Minkowskie Reduktionstheorie über Funktionenkörpern. Invent. Math., 7:33–54, 1969.
[19] Jun-ichi Igusa. Fibre systems of Jacobian varieties. III. Fibre systems of elliptic curves. Amer. J. Math., 81:453–476, 1959.
[20] Bruce W. Jordan and Ron Livné. The Ramanujan property for regular cubical complexes. Duke Math. J., 105(1):85–103, 2000.
[21] Tali Kaufman, David Kazhdan, and Alexander Lubotzky. Ramanujan complexes and bounded degree topological expanders. In 55th Annual IEEE Symposium on Foundations of Computer Science—FOCS 2014, pages 484–493. IEEE Computer Soc., Los Alamitos, CA, 2014.

[22] Harry Kesten. Symmetric random walks on groups. Trans. Amer. Math. Soc., 92:336–354, 1959.

[23] Laurent Lafforgue. Chtoucas de Drinfeld et correspondance de Langlands. Invent. Math., 147(1):1–241, 2002.

[24] G. Laumon, M. Rapoport, and U. Stuhler. Dr-elliptic sheaves and the Langlands correspondence. Invent. Math., 113(2):217–338, 1993.

[25] W.-C. W. Li. Ramanujan hypergraphs. Geom. Funct. Anal., 14(2):380–399, 2004.

[26] A. Lubotzky, R. Phillips, and P. Sarnak. Ramanujan graphs. Combinatorica, 8(3):261–277, 1988.

[27] Alexander Lubotzky. Expander graphs in pure and applied mathematics. Bull. Amer. Math. Soc. (N.S.), 49(1):113–162, 2012.

[28] Alexander Lubotzky and Roy Meshulam. A Moore bound for simplicial complexes. Bull. Lond. Math. Soc., 39(3):353–358, 2007.

[29] Alexander Lubotzky and Tatiana Nagnibeda. Not every uniform tree covers Ramanujan graphs. J. Combin. Theory Ser. B, 74(2):202–212, 1998.

[30] Alexander Lubotzky, Beth Samuels, and Uzi Vishne. Explicit constructions of Ramanujan complexes of type \( \tilde{A}_d \). Israel J. Math., 149:267–299, 2005.

[31] A. Nilli. On the second eigenvalue of a graph. Discrete Math., 91(2):207–210, 1991.

[32] Hee Oh. Uniform pointwise bounds for matrix coefficients of unitary representations and applications to Kazhdan constants. Duke Math. J., 113(1):133–192, 2002.

[33] Theodore W. Palmer. Banach algebras and the general theory of \( \ast \)-algebras. Vol. 2, volume 79 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2001.

[34] Ori Parzanchevski, Ron Rosenthal, and Ran J. Tessler. Isoperimetric inequalities in simplicial complexes. Combinatorica, pages 1–33, 2015.

[35] Vladimir Platonov and Andrei Rapinchuk. Algebraic groups and number theory, volume 139 of Pure and Applied Mathematics. Academic Press, Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen.

[36] Jacques Tits. Reductive groups over local fields. In Automorphic forms, representations and \( L \)-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, pages 29–69. Amer. Math. Soc., Providence, R.I., 1979.