Abstract—When the scale of communication networks has been growing rapidly in the past decades, it becomes a critical challenge to extract fast and accurate estimation of key state parameters of network links, e.g., transmission delays and dropped packet rates, because such monitoring operations are usually time-consuming. Based on the sparse recovery technique reported in [Wang et al. (2015) IEEE Trans. Information Theory, 61(2): 1028–1044], which can infer link delays from a limited number of measurements using compressed sensing, we particularly extend to networks with dynamic changes including link insertion and deletion. Moreover, we propose a more efficient algorithm with a better theoretical upper bound. The experimental result also demonstrates that our algorithm outperforms the previous work in running time while maintaining similar recovery performance, which shows its capability to cope with large-scale networks.

Index Terms—Compressed sensing, link delay estimation, network tomography, sparse recovery

I. INTRODUCTION

With the increasing trend of online services, including video conferencing and live streaming, over the internet, the need of high QoS (Quality-of-Service) guarantees has become urgent in recent years. The service providers are thus required to keep maintaining network utilization and performance. More precisely, a network management system has to efficiently and repeatedly detect network congestions and estimate the status of link delays, if any exists. The related works which discussed this problem, i.e., network tomography introduced by Vardi [17], involved measuring origin-to-destination path delays from aggregated measurements of the individual links along such paths in a given network. That is, these past studies on (active) network tomography measured end-to-end characteristics by sending probe packets from sender vertices to receiver vertices, i.e., (actively) probing the network [8], [9], [12], [21].

Obviously, the computational load of visiting all link delays by measuring all pairs of origin-to-destination paths is heavy, especially for large-scale communication networks. A common way to resolving this problem is a divide-and-conquer approach which can reduce the size of a given network to smaller ones. Hence, an initial estimation of state parameters can be derived based on the principle of local information [21]. On the other hand, for its stochastic counterpart, where the link attributes follow a given probabilistic distribution, the EM (Expectation Maximization) algorithm was used to determine the unknown state parameters. Readers may refer to the survey [4] for related studies. Recently, Firooz and Roy [8], Mahyar et al. [14], [15], Ghalebi et al. [10], Wang et al. [18], and Xu et al. [19], [20] proposed efficient approaches to reducing the number of measurements by using the concept of compressed sensing. Compressed sensing (CS) [3], [6] has been widely studied in the last decade, which can be deemed as a combination of two phases: sensing and signal recovery. The first phase is usually applied to data acquisition and compression (or dimensionality reduction) simultaneously, while the second phase aims to recover sparse signals. Here we say a signal is k-sparse if the number of nonzero entries of the signal is at most k. Based on the fact that there are usually a small number of links with large delays, the past studies that applied compressed sensing to network tomography in that the parameters of link delays are represented by a sparse vector, and a measurement matrix by taking advantage of the network topology is designed.

Compared with [10], [14], [15], [20], our study and [8], [18] used a hub set to measure the other links, so that we can select the set of remaining links randomly and this kind of measurement matrix has been proved to satisfy the RIP condition [2]. Specifically, random matrices are known to satisfy δck < θ with high probability provided one chooses m = O(ck/θ2 log(N/K)) measurements. [7] shows that ℓ1-minimization recovers all k-sparse vectors provided the sensing matrix satisfies θ2k < 0.414. The above theoretical results indicate that a sparse signal can be recovered from incomplete measurements under certain conditions.

More precisely, Wang et al. [18] considered a model for recovering sparse signals when satisfying some graph constraints. This model finds applications to estimation of network parameters because the parameters such as link delays, as mentioned, are usually sparse. They adopted a line graph model to represent the topological constraints to ensure the network’s connectivity, where the constraints allow that a subset of vertices can be aggregated measured if and only if they induce a connected subgraph. Moreover, their two assumptions on the graph constraints are as follows:

H1. A vertex subset V’ of a graph G can be measured together in one measurement if and only if the subgraph induced by V’ is connected.

H2. The measurement is an additive sum of values at the corresponding vertices.
Here we follow the two assumptions and our goal is to reduce the computation cost of transforming the network tomography problem to a sparse recovery problem. In particular, we extend it to networks with dynamic changes, where link insertion and deletion are allowed. That is, we consider the problem model with dynamic graph constraints that can be measured without recomputing all the state parameters under dynamic link operations.

A. Main Contributions

The key results obtained in this study are summarized as follows:

1) We consider link delay estimation in a dynamic network tomography model, where dynamic link operations including insertion and deletion are allowed.

2) We propose a faster algorithm and prove that it has a better theoretical upper bound on running time, compared with Wang et al.’s work [18].

3) The experimental result confirms that our algorithm also outperforms [18] in running time, while maintaining similar recovery performance. Therefore, it can be particularly applied to a large-scale network environment.

II. Problem Model

We consider a communication network $G = (V, E)$, where $V$ denotes the set of vertices with cardinality $|V| = n$ and $E$ is the set of links with cardinality $|E| = m$. Let $V = \{v_1, v_2, \ldots, v_n\}$, let $d_i$ be the degree of $v_i$, and denote the average vertex degree as $d = \frac{1}{n} \sum_{i=1}^{n} d_i$. Wang et al.’s algorithm [18] adopted a line graph model $L_G = (V_L, E_L)$, where every vertex in $L_G$ corresponds to a link in $G$, i.e., $V_L = E$, and two vertices are adjacent in $L_G$ if and only if their corresponding links in $G$ are incident, i.e., sharing a common end vertex. Similarly, let $d'$ denote the average vertex degree in the line graph $L_G$.

In [18], every link in $E = \{e_1, \ldots, e_m\}$ is associated with a nonnegative delay $x_t$, $1 \leq t \leq m$, where $x = (x_1, \ldots, x_m)$ is the unknown signal to be recovered. The authors assumed that $x$ is a $k$-sparse vector; that is, the number of nonzero entries of $x$ is at most $k$. Based on that, they can take $N$ measurements, $N \ll m$ for sparse signal recovery, where we let $y$ denote the measurement vector of length $N$. In addition, let $A$ be the $N \times m$ binary measurement matrix, where each row of $A$ represents a path along which entry 1 (or 0) denotes a visited (or unvisited) link (i.e., $A_{i,j} = 1$ if link $j$ contributes to the $i$th measurement, and $A_{i,j} = 0$ otherwise). Hence, one can have the compact form $y = Ax$.

Conventionally, compressed sensing (CS) requires that each entry of $A$ be drawn from i.i.d random variable (randomness constraint) for reconstructing $x$ given $y$ and $A$. However, since $G$ is not a complete graph, it leads to the difficulty in the design of $A$. To solve this problem, Wang et al. [18] used the concept of a connected dominating set (CDS) on the line graph $L_G$ of a given graph $G$. Let $C$ denote a CDS on $L_G$. Due to the property of a CDS, every vertex $v \in V_L \setminus C$ can be connected to $C$. That is, we can keep connectivity by using the vertices in $C$ (as hub vertices) to connect other vertices. Moreover, we can use the hub vertices to recover others. The above procedure is performed repeatedly until all the delays are recovered.

For the implementation, a measurement is to sum up all delays along a close path that includes the hub vertices and the delays that we want to measure. Notice that we have to deduce the delay of the hub vertices to exactly get the delay we want to measure. More precisely, assume the total delay is $y_t$ and the hub delay is $y_h$, and let what we want to measure be $y_t - y_h = (A_t - A_h)x$. Therefore, $A_t - A_h$ is the measurement array that satisfies the randomness constraint [18]. The key idea of our algorithm uses a similar manner, i.e., the concept of CDS, to ensure the randomness constraint. The major difference between our algorithm and Wang et al.’s approach is the selection of hub vertices; that is, finding a CDS in the initial round.

We further extend the model to a dynamic scenario in which link insertion and deletion are allowed. Therefore, each input can be considered a sequence of instances, $I_1, I_2, \ldots, I_T$, where each instance $I_t$ is updated from $I_{t-1}$, $1 < t \leq T$, by inserting or deleting links in $G$. Precisely, every link $e_i$ is associated with a $T$-dimensional vector, $x_t$, where each entry $x_{t,i}$ is derived from $x_{t-1,i}$ by inserting or deleting edge $e_i$ at time $t$. That is, $x_t = (x_{t,1}, x_{t,2}, \ldots, x_{t,m})$ denotes the unknown signal to recover at time $t$, $1 \leq t \leq T$. The objective is to recover and estimate the link delays at each time slot without repeating the whole procedure of selecting the set of hub vertices.

III. Proposed Method

Here we use the following topological property of the original graph to achieve a better selection of hub vertices.

Property 1: In a graph, a maximal independent set $S$ is also a dominating set; that is, every vertex is either in $S$ or has at least one neighbor in $S$.

Lemma 1: A matching in a graph $G$ corresponds to an independent set in the line graph $L_G$ of $G$.

Proof: Please see the supplementary material for proof. □

Algorithm 1: Construction of measurements for a graph $G$

Input : $G = (V, E)$, $L_G = (V_L, E_L)$
Output : All the measurements
1: Find a maximum matching $M$ in $G$;
2: Let $C^* = \text{Connecting}(G, M)$ and measure the summation delay of $C^*$; where $C^*$ corresponds to a CDS on $L_G$;
3: Design $f(|T|, k) + 1$ measurements and the corresponding matrix to recover $x_T$, where $T = V_L \setminus C$;
4: Measure the vertex delays $x_C$ directly;

Subroutine 1: Connecting($G, M$)

Input : $G = (V, E)$, a maximum matching $M$
Output : A connected edge set $C^*$
1: Let $C^* = M$, and randomly select a vertex $u$
2: while $C^*$ is unconnected do
3: Find an edge $(u, v) \notin C^*$ and $(u, k), (v, r) \in M$;
4: $C^* := C^* + (u, v)$;
5: end while
Our algorithm is designed as follows and described in Algorithm 1. We first find a maximum matching in a given graph $G$, and connect all the matched vertices to construct a connected edge set $C^*$. (see Subroutine 1). Note that the edge set $C^*$ corresponds to a CDS $C$ in the line graph.

Based on [18], by letting every vertex in the CDS $C$ be a hub, we can design an appropriate number of measurements to recover $x_T$ and derive $\hat{x}_T$ in the line graph $L_G = (V_L, E_L)$, where $T = V_L \setminus C$ and $x_T$ is the subvector of $x$ with indices in $T$. Here we let the appropriate number of measurements be $f(|T|, k) + 1$, where $f(|T|, k)$ represents the number of measurements constructed to identify $k$-sparse vectors associated with a complete graph of $n$ vertices [18]. Then, we directly measure each of the remaining delays in $x_C$ and obtain $\hat{x}_C$, i.e. $x_v$ with $v \in C$ in $L_G$.

We will discuss the strategy of dealing with dynamic link operations later in Section IV.

A. Analysis and Time Complexity

Both of our algorithm and Wang et al.’s approach [18] recover link delays by using the concept of CDS (i.e., hub vertices). That is, in order to compare our algorithm with that reported in [18] from a theoretical perspective, we mainly consider the size of the CDS, i.e. the cardinality of the set of hub vertices $C$, as well as its construction time in the initial round. Recall that Wang et al. first calculated the radius in the line graph and then picked the central vertex to be the root vertex. Secondly, they used the breadth-first search (BFS) and treated the non-leaf vertices as the hub vertices. The whole process of their procedure to generate a set of hub vertices costs $O(|E|^2 \log |E| + |E||E_L|)$ time in total [18].

Given a network $G = (V, E)$, where $|V| = n$ and $|E| = m$, the cardinality of a maximum matching is at most $n/2$ so the number of edges connecting the maximum matching to form $C^*$ is at most $(n/2) - 1$, which implies that the cardinality of $C^*$ is bounded by $n$ in the initial round. By contrast, the next lemma shows that the number of the hub vertices, generated by the BFS method in [18], is $O(n \log n)$ in the worst case.

**Lemma 2:** The cardinality of the hub selected by Wang et al. [18] is at least $n \log d$, where $d$ is the average degree in a given network $G = (V, E)$.

**Proof:** Please see the supplementary material for proof. □

Consider the time complexity of Algorithm 1 the time cost mainly comes from the computation of the maximum matching of a graph. Since Subroutine 1 can be solved at most linear in the number of vertices, Micali and Vazirani [16] showed that the maximum matching problem can be solved in $O(|E||V|^{0.5})$ time, which is much faster than [18] that takes at least quadratic time proportional to the number of edges in a given network $G$. Hence, Algorithm 1 is more capable of coping with large-scale networks.

IV. Dynamic Strategy

In this section, we introduce our strategy to deal with dynamic link operations, including link insertion and deletion. First, assume the given network remains connected when link deletion is allowed. The dynamic procedure is described by Algorithm 2. For link insertion of edge $(i, j)$, we first find an augmenting path by Micali and Vazirani [16], if any exists, with respect to the original maximum matching $M$ in the graph $G \cup \{(i, j)\}$. Then, we derive the new maximum matching $M^*$ and connect all the vertices in $M^*$ to form a new set $C^*$ using Subroutine 1. On the other hand, for link deletion, we consider the following cases. If $(i, j) \in M$, we still use Subroutine 1 to obtain a new $C^*$. Otherwise, if $(i, j) \notin M$, that is, $M$ is still the maximum matching, and if $(i, j) \notin C^*$, we can use Subroutine 1 to derive a new $C^*$. Otherwise, if $(i, j) \in C^*$, the original $C^*$ is optimal and $C^*$ does not need to be updated. As shown in Algorithm 2, the time cost of tackling dynamic link operations, i.e., finding an augmenting path, is at most linear in the number of edges. In addition, Subroutine 1 takes at most linear time in the number of vertices. Thus, the time complexity $O(n^2)$ of our dynamic strategy is faster than the one $O(n^{2.5})$ of re-running our proposed algorithm, where “re-running” means that there are dynamic link operations but we execute Algorithm 1 without using the dynamic strategy.

V. Experimental Result

We conducted numerical experiments on the machine with Intel i5-6500(3.20GHz) of CPU with 16GB DDR3 of RAM, running Windows 10 x64. All programs were compiled by MATLAB R2015a and running time was measured using one single thread.

There are two major issues we consider for link delay estimation via sparse recovery: execution time and recovery performance. We evaluated the time cost of our algorithm from the following perspectives. We first considered the B-A
model [11, 18], where it is a scale-free network set to contain 500 and 1000 vertices under different average degrees \( d = 10 \) and 20 (see Table I), respectively. We compared the time costs of re-running our algorithm and adopting the proposed dynamic strategy with edge deletion randomly (see Table II).

Obviously, the experimental result demonstrates that our algorithm outperforms Wang et al.’s approach [18] in terms of running time. Precisely, our algorithm runs at least thousand times faster than Wang et al.’s approach. Moreover, we have the following observations from Table I and Table II: 1) As shown in Table II, the execution time of Wang et al.’s algorithm is more than ten thousand seconds. By contrast, when the size of input networks grows, there is only a slight increase in running time of our algorithm because our algorithm takes \( O(|E||V|^{0.5}) \) time only. 2) Table II depicts the effectiveness of our algorithm to deal with dynamic operations. It is observed that our algorithm with dynamic strategy runs ten times faster than its static version of re-running the algorithm without relying on the dynamic strategy. Moreover, when increasing the size of the scale-free network, the time cost of the dynamic strategy increases slightly, which is significantly smaller than the increase cost of the static version.

As for the recovery performance, we refer to the standard evaluation procedure for sparse recovery [5]. [11] and define a procedure to be success when all supports are found and \( \frac{\|x - x_0\|_2}{\|x_0\|_2} < 0.02 \). Consider the recovery performance in a given scale-free network with \( |V| = 500 \) and \( d = 10 \). We generated a vector \( x_0 \) with a random support set, where the sparsity rate, \( r := k/m \), increases by 0.05 from 0.05 to 0.35. Next, the values of entries in the support set were drawn from a uniform distribution with the range \([5(1-r), 5]\) and others entries were drawn from the one with range \([0, 0.001(1-r)]\).

We set the measurement ratio \( N/m \) to be 0.3, 0.4 and 0.5. As shown in Fig. 1, the recovery performance of our algorithm outperforms Wang et al.’s method under low to moderate sparsity rates. For example, when \( N/m = 0.5 \), our algorithm dominates their results until the sparsity is up to 25%. In addition, we also evaluated the recovery performance of our algorithm with two different network densities. As shown in Fig. 2, we verify our algorithm in two different scale-free networks with \( |V| = 500 \), and \( d = 10 \) and 20, respectively. It can be observed that, under the same sparsity rate, our algorithm exhibits better recovery performance in a more dense network.

More precisely, recall that \( N = f(|T|, k) + |C| \), where \( f(|T|, k) \) is the number of measurements to reconstruct \( x_T \) and \( C \) is the hub that is fixed and determined by the number of vertices of the input graph. When the density of the network is increased, the ratio of \( f(|T|, k)/N \) is increased correspondingly, leading to the advantage of more measurements for recovering \( x_T \). Therefore, our method has better recovery performance in a more dense network.

**VI. Conclusion**

This study has extended Wang et al.’s work [18] to link delay estimation with dynamic link operations. Moreover, our algorithm ensures superior performance against the previous approach in execution time while maintaining similar recovery performance. Since our algorithm has significantly reduced the time expense, it is well-expected to be implemented in large-scale networks.

There are several open discussions for this study. Note that both of our algorithm and Wang et al.’s approach [18] attempted to recover link delays by using the concept of connectivity. It would be worthwhile to develop another graph-theoretic approaches to effectively identify a hub. Moreover, in this study, our dynamic strategy allows deleting one edge in a time. Multiple edges deletion (or insertion) is an interesting issue for further study.
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In this section, we present the proofs of the following lemmas in Sec. III.

**Lemma 1.** A matching in a graph $G$ corresponds to an independent set in the line graph $L_G$ of $G$.

**Proof:** A matching $M$ is a set of edges that have no common end vertices. When we consider the line graph $L_G$, each vertex in $L_G$ corresponds to an edge in $G$. Because every two edges in $M$ are not incident in $G$, in the line graph $L_G$ any two vertices in $M$ are not adjacent to each other. $M$ is thus an independent set in $L_G$.

**Lemma 2.** The cardinality of the hub selected by Wang et al. [18] is at least $n \log d$, where $d$ is the average degree in a given network $G = (V, E)$.

**Proof:** Based on the analysis reported in [13], the number of non-leaf vertices in a BFS tree is close to $\frac{m \ln d}{d}$. Therefore, the cardinality of the hub approximates $m \ln d$, in the corresponding line graph model $L_G(V_L, E_L)$ of $G$, where $|V_L| = m$, $|E_L| = \frac{\sum_{i=1}^{n} d_i^2 - 2m}{2}$, and $d'$ is the average degree of $L_G$. Note that because $|E_L| = \frac{d'n}{m} = \frac{\sum_{i=1}^{n} d_i^2 - 2m}{2}$, it implies $d' = \frac{\sum_{i=1}^{n} d_i^2 - 2m}{m} = \frac{2 \sum_{i=1}^{n} d_i^2}{m} - 2$. By the Cauchy-Schwarz inequality, we can derive:

$$\left(\sum_{i=1}^{n} d_i^2\right)\left(\sum_{i=1}^{n} 1^2\right) \geq \left(\sum_{i=1}^{n} d_i\right)^2$$

$$\Rightarrow \sum_{i=1}^{n} d_i^2 \times n \geq (nd)^2,$$

which implies $d' \geq \frac{2nd^2}{nd} - 2 = 2d - 2$. Thus, the cardinality of the hub is close to $m \ln d' \leq \frac{n \ln(2d - 2)}{4d - 4}$, which approximates $n \log d$. Notice that the number of hub vertices selected by our approach is at most $n$. In contrast, the cardinality of the hub derived by [18] $n \log d$, depends on the value of $d$, which ranges from $n$ to $n \log n$. 