On a possible dynamical scenario leading to a generalised Gamma distribution

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In this report I present a possible scenario which can lead to the emergence of a generalised Gamma distribution first presented by C Tsallis et al. as the distribution of traded volumes of stocks in financial markets. This propose is related with superstatics and the notion of moving average commonly used in econometrics.

I. THE Γ-DISTRIBUTION

The Γ-distribution is a general distribution that is verified in processes where the waiting times between variables that follow a Poisson distribution are significant. It involves two free parameters, usually labeled by $\alpha$ and $\theta$ and defined as [1],

$$p_{\alpha,\theta}(x) = x^{\alpha-1} \exp\left(-\frac{x}{\theta}\right) \frac{\Gamma[\alpha]}{\theta^\alpha}. \quad (1)$$

A special case of Γ-distribution is to consider $\alpha = r/2$ and $\theta = 2$. In this case the distribution is called $\chi^2$-distribution and represents the probability of get a value $\zeta$ of a variable that is obtained by the summation of $r$ independent squared variables $\xi_i$ associated with the Gaussian distribution with null mean and unitary variance [1],

$$\zeta = \sum_{i=1}^{r} \xi_i^2. \quad (2)$$

The same form presented in Eq. (1) can be obtained as the stationary solution of the following differential stochastic equation

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\[ dx_t = -\gamma (x_t - \theta) \, dt + k\sqrt{x}dW_t. \] (3)

Considering the Itô convention for stochastic differentials I am able to write the Fokker-Planck equation [2],

\[ \frac{\partial p(x,t)}{\partial t} = \frac{\partial}{\partial x} \left[ \gamma (x_t - \theta) \, p(x,t) \right] + \frac{1}{2} \frac{\partial^2}{\partial x^2} \left[ k^2 x \, p(x,t) \right], \] (4)

whose stationary solution is

\[ p_{\alpha,\theta}(x) = \frac{\alpha^\alpha}{\Gamma[\alpha]} \frac{x^{\alpha-1}}{\theta^\alpha} \exp \left( -\frac{\alpha}{\theta} x \right) \] (5)

with \( \alpha = 2\gamma\theta k^{-2} \). Performing a simple variable change \( x \rightarrow \frac{x}{\alpha} \), it is possible to transform Eq. (5) into Eq. (1) and the Itô-Langevin equation (3)

\[ dx = -\gamma (x - \alpha \theta) \, dt + \sqrt{2\gamma \theta} \sqrt{x} dW_t. \] (6)

For a question of simplicity let me represent \( \theta \) as \( \beta^{-1} \). So Eq. (1) will be written as,

\[ p_{\alpha,\beta}(x) = \frac{\beta^\alpha}{\Gamma[\alpha]} x^{\alpha-1} \exp (-\beta x). \] (7)

In figure 1 are depicted some examples of \( \Gamma(x) \) distributions.

![Gamma distributions](image)

**FIG. 1.** Representation of \( \Gamma(x) \) vs \( x \) for some values of \( \alpha \) and \( \theta \). (I) - \( \alpha = 2 \) and \( \theta = 2 \); (II) - \( \alpha = 4 \) and \( \theta = 2 \); (III) - \( \alpha = 2 \) and \( \theta = 4 \). The inset presents the same curves, but in a log–log scale.
II. INTRODUCING THE GENERALISED $\Gamma$-DISTRIBUTION

Let one now suppose that parameter $\theta$ in Eq. (3) is in fact a stochastic variable on time scale larger than the characteristic time scale $\gamma^{-1}$. This means that $p_{\alpha,\beta}(x)$ is, for this case, a conditional probability density function $p_{\alpha}(x \mid \beta)$. If the random process for $\beta$ is associated with a SPDF, $\Pi(\beta)$, then the SPDF for $x$ variable, $P(x)$, is simply given by

$$P(x) = \int p_{\alpha}(x \mid \beta) \, \Pi(\beta) \, d\beta. \quad (8)$$

Among the various distributions for non-negative variables let one consider that $\beta$ is associated, itself, with a $\Gamma$-distribution,

$$P(x) = \int p_{\alpha}(x \mid \beta) \, \Pi(\beta) \, d\beta. \quad (9)$$

which can be associated to a microscopic equation similar to Eq. (6).

Calculating the integral presented in equation (8) one gets,

$$P(x) = \frac{\Gamma[\alpha + \lambda]}{\Gamma[\alpha] \Gamma[\lambda]} \omega^\alpha x^{\alpha-1} (1 + \omega x)^{-(\alpha + \lambda)}. \quad (10)$$

Defining $\bar{\theta} = \frac{1}{\omega(\alpha + \lambda)}$ and $q = 1 + \frac{1}{\alpha + \lambda}$, Eq. (10) can be rewritten as,

$$P(x) = \frac{\Gamma\left[\frac{1}{q-1}\right]}{\Gamma[\alpha] \Gamma\left[\frac{1}{q-1} - \alpha\right]} x^{\alpha-1} \left\{1 - (1 - q) \frac{x}{\bar{\theta}}\right\}^{\frac{1}{1-q}}, \quad (11)$$

$$P(x) \equiv \frac{\Gamma\left[\frac{1}{q-1}\right]}{\Gamma[\alpha] \Gamma\left[\frac{1}{q-1} - \alpha\right]} x^{\alpha-1} \exp_q \left[-\frac{x}{\bar{\theta}}\right], \quad (12)$$

which I will call the $q\Gamma$-distribution. This kind of distribution was already verified, at least, for the distribution of traded volumes of stocks in financial markets [3]. For the limit $q \to 1$, the usual $\Gamma$-distribution is recovered, which corresponds to $\Pi(\beta) = \delta(\beta - \frac{1}{\theta})$. Some examples of $q\Gamma$-distribution are presented in Fig. 2.
FIG. 2. Representation of $\Gamma(x)$ vs $x$ for some values of $\alpha$ and $\theta$. (I) - $q = 1.2$, $\alpha = 2$ and $\theta = 2$; (II) - $q = 1.1$, $\alpha = 2$ and $\theta = 2$; (III) - $q = 1.1$, $\alpha = 2$ and $\theta = 4$; (IV) - $q = 1.1$, $\alpha = 4$ and $\theta = 2$.

The inset presents the same curves, but in a log-log scale.

The same form is presented in Figs. 7 and 8 of Ref. [3].

This problem of fluctuations in some intensive parameter of the dynamical equation(s) that describe(s) the evolution of a system [4] was recently studied by C. Beck in the context of Langevin equation with fluctuating temperature [5] and extended together with Eddie G.D. Cohen [6] who defined it as superstatistics (a statistic of statistics). This superstatistics presents a close relation to the non-extensive statistical mechanics framework based on the entropic form [7,8],

$$S_q = \frac{1 - \int [p(x)]^q dx}{q - 1}. \quad (13)$$

For the problem of the distribution of traded volume of stocks in financial markets the presence of fluctuations in $\theta$, or the mean value of the scaled variable $\alpha x$, it is similar to the problem of the moving average in the analysis of the volatility useful in the reproduction of some empirical facts like the autocorrelation function and the so-called leverage effect, see e.g. Ref. [9].

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