Canonical extensions of lattices are more than perfect

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In memory of Bjarni Jónsson.

Abstract. In a paper published in 2015, we introduced TiRS graphs and TiRS frames to create a new natural setting for duals of canonical extensions of lattices. Here, we firstly introduce morphisms of TiRS structures and put our correspondence between TiRS graphs and TiRS frames into a full categorical framework. We then answer Problem 2 from our 2015 paper by characterising the perfect lattices that are dual to TiRS frames (and hence TiRS graphs). We introduce a new subclass of perfect lattices called PTi lattices and show that the canonical extensions of lattices are PTi lattices, and so are ‘more’ than just perfect lattices. We illustrate the correspondences between classes of our newly-described PTi lattices and classes of TiRS graphs by examples. We conclude by outlining a direction for future research.

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1. Introduction

One of the main tools to study lattice-based algebras has, in recent decades, been the theory of canonical extensions, which originated in the 1951–52 papers of Jónsson and Tarski [13,14]. We refer to Gehrke and Vosmaer [10] for a survey of the theory of canonical extensions of lattice-based algebras, and to recent papers by Gehrke [8] and Goldblatt [11], in particular to the first section of [11] called “A biography of canonical extension”.

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The canonical extensions of general (bounded) lattices were first introduced by Gehrke and Harding [9] as the complete lattices of Galois-closed sets associated with a polarity between the filter lattice and the ideal lattice of the given lattice. A new construction of the canonical extension of a general lattice was provided in [3] (see also [4]) where it was based on a topological representation of lattices by Ploščica [17]. The Ploščica representation presented a well-known representation of general lattices due to Urquhart [19] in the spirit of the theory of natural dualities of Clark and Davey [1]. It used maximal partial maps into the two-element set to represent elements of the first and second duals of a given lattice.

Another construction of canonical extensions of general lattices was presented in [2] where Ploščica’s topological representation was used in tandem with Gehrke’s representation of perfect lattices via RS frames. (For the latter we refer to the papers by Dunn, Gehrke and Palmigiano [6] and Gehrke [7].) In [2] we also demonstrated a one-to-one correspondence between TiRS frames (a subclass of RS frames) and TiRS graphs, which we introduced as an abstraction of the duals of general lattices in the Ploščica representation. This led to a new dual representation of the class of all finite lattices via finite TiRS frames, or equivalently finite TiRS graphs, which generalises the well-known Birkhoff dual representation between finite distributive lattices and finite posets from the 1930s. (Here we remark that every poset is a TiRS graph.) We use the term TiRS structures to refer to both TiRS graphs and TiRS frames without distinguishing between the two classes.

This paper has two goals:

1. To describe the appropriate morphisms of TiRS structures and hence to extend the one-to-one correspondence between the TiRS structures from [2] into a full categorical framework.
2. To describe the additional properties that perfect lattices dual to TiRS structures possess. This was listed as “Problem 2” in [2].

We also show that the canonical extensions of lattices are PTi lattices, which follows from their construction in [2] using Ploščica’s and Gehrke’s representations in tandem. Hence the canonical extensions of lattices are ‘more’ than just perfect lattices. We present an example of a perfect but not PTi lattice together with its dual TiRS graph, and an example of a PTi lattice that is not the canonical extension of any lattice, together with its dual TiRS frame. We describe the correspondences between subclasses of PTi lattices and subclasses of TiRS graphs and illustrate them in Figure 6.

The task of putting our correspondences between the class of PTi lattices and the class of TiRS graphs into a full categorical framework seems to be harder than the one we complete here in Section 3—we view the TiRS graph and TiRS frame morphisms we introduced in this paper as the starting point for the task of finding the right morphisms between PTi lattices. We propose this task as a direction for future research and in our conclusion we formulate it as an open problem.
2. Preliminaries

For a bounded lattice $L$, a completion of $L$ is defined to be a pair $(e, C)$ where $C$ is a complete lattice and $e: L \hookrightarrow C$ is an embedding. By a filter element (ideal element) of a completion $(e, C)$ of a bounded lattice $L$ we mean an element of $C$ which is a meet (join) of elements from $e(L)$. By $F(C)$ and $I(C)$ are denoted the sets of all filter and ideal elements of $C$, respectively. (We remark that in the older literature the filter (ideal) elements had been called closed (open) elements.) A completion $(e, C)$ of a bounded lattice $L$ is called dense if every element of $C$ can be expressed as both a join of meets and a meet of joins of elements from $e(L)$. A completion $(e, C)$ of $L$ is called compact if, for any sets $A \subseteq F(C)$ and $B \subseteq I(C)$ with $\bigwedge A \leq \bigvee B$, there exist finite subsets $A' \subseteq A$ and $B' \subseteq B$ such that $\bigwedge A' \leq \bigvee B'$. (We remark that the sets $A, B$ in the definition of compactness above can alternatively be taken as arbitrary subsets of $L$.)

Gehrke and Harding [9] defined abstractly the canonical extension $L^\delta$ of a general bounded lattice $L$ as a dense and compact completion of $L$. They proved that every bounded lattice $L$ has a canonical extension and that it is unique up to an isomorphism that fixes the elements of $L$. Concretely, they constructed $L^\delta$ as the complete lattice of Galois-stable sets of the polarity $R$ between the filter lattice $\text{Filt}(L)$ and the ideal lattice $\text{Idl}(L)$ of $L$ where the polarity is given by $(F, I) \in R$ if $F \cap I \neq \emptyset$. The complete lattice associated with that polarity was first studied by Hartonas and Dunn [12].

A filter-ideal pair $(F, I)$ will be called maximal if $F$ and $I$ are maximal with respect to being disjoint from one another. We shall later on use the following result from [9]. We recall that in a complete lattice $C$, we denote the sets of the completely join-irreducible elements and of the completely meet-irreducible elements by $J^\infty(C)$ and $M^\infty(C)$, respectively.

**Lemma 2.1** ([9, Lemma 3.4]). Let $(e, C)$ be a canonical extension of $L$.

1. $x \in J^\infty(C)$ if and only if $x = \bigwedge e[F]$ for some maximal pair $(F, I)$ of $L$;
2. $x \in M^\infty(C)$ if and only if $x = \bigvee e[I]$ for some maximal pair $(F, I)$ of $L$.

Further, each element of $C$ is a join of completely join irreducibles and a meet of completely meet irreducibles.

Ploščica’s dual [17] of a bounded lattice $L$ is a graph with topology, $D(L) = (\mathcal{L}^{\text{mp}}(L, 2), E, T)$, where $\mathcal{L}^{\text{mp}}(L, 2)$ is the set of maximal partial homomorphisms (MPHs, for short) from $L$ into the two-element lattice $2$. The graph relation $E$ is defined by

$$(f, g) \in E \text{ if } (\forall a \in \text{dom } f \cap \text{dom } g) f(a) \leq g(a),$$

or equivalently,

$$(f, g) \in E \text{ if } f^{-1}(1) \cap g^{-1}(0) = \emptyset.$$ 

The topology $T$ has as a subbasis of closed sets the set $\{V_a, W_a \mid a \in L\}$, with $V_a = \{f \in \mathcal{L}^{\text{mp}}(L, 2) \mid f(a) = 0\}$ and $W_a = \{f \in \mathcal{L}^{\text{mp}}(L, 2) \mid f(a) = 1\}$. The canonical extension of $L$ constructed in [3] is then taken to be the lattice
lattice \( G \) was then shown in [2]. We recall here some facts of this correspondence that will be needed in the next section.

TiRS graphs were defined by the present authors in [2] as an abstraction of the graphs \( D^\vartriangle (L) = (L^{mp}(L, \vartriangle), E) \) obtained from Ploščica’s duals of bounded lattices by forgetting the topology.

For a graph \( X = (X, E) \) and \( x \in X \), the sets \( \{ y \in X \mid (x, y) \in E \} \) and \( \{ y \in X \mid (y, x) \in E \} \) were denoted in [2] by \( Ex \) and \( E \bar{x} \) respectively. We defined the conditions (S), (R) and (Ti) for any graph \( X = (X, E) \) as follows:

(S) for every \( x, y \in X \), if \( x \neq y \) then \( xE \neq yE \) or \( Ex \neq E \bar{y} \);
(R) (i) for all \( x, z \in X \), if \( zE \subseteq xE \) then \( (z, x) \notin E \);
(ii) for all \( y, z \in X \), if \( E \vartriangle \subseteq E \bar{y} \) then \( (y, z) \notin E \);
(Ti) for all \( x, y \in X \), if \( (x, y) \in E \), then there exists \( z \in X \) such that \( zE \subseteq xE \) and \( E \vartriangle \subseteq E \bar{y} \).

A TiRS graph was in [2] defined as a graph \( X = (X, E) \) with a reflexive relation \( E \) and satisfying the conditions (R), (S) and (Ti). For any bounded lattice \( L \), its dual graph \( X = D^\vartriangle (L) \) is a TiRS graph [2, Proposition 2.3].

We further recall that a frame is a structure \( (X_1, X_2, R) \), where \( X_1 \) and \( X_2 \) are non-empty sets and \( R \subseteq X_1 \times X_2 \). For \( x_1 \in X_1 \) and \( y_1 \in X_2 \) we let \( x_1R = \{ y \in X_2 \mid x_1Ry \} \) and \( Ry_1 = \{ x \in X_1 \mid xRy_1 \} \). For an arbitrary frame \( F = (X_1, X_2, R) \) the conditions (S) and (R) are defined as follows:

(S) for all \( x_1, x_2 \in X_1 \) and \( y_1, y_2 \in X_2 \),
(i) \( x_1 \neq x_2 \) implies \( x_1R \neq x_2R \);
(ii) \( y_1 \neq y_2 \) implies \( Ry_1 \neq Ry_2 \).

(R) (i) for every \( x \in X_1 \) there exists \( y \in X_2 \) such that \( \neg(xRy) \) and \( \forall w \in X_1 ((w \neq x \& xR \subseteq wR) \Rightarrow wRy) \);
(ii) for every \( y \in X_2 \) there exists \( x \in X_1 \) such that \( \neg(xRy) \) and \( \forall z \in X_2 ((z \neq y \& Ry \subseteq Rz) \Rightarrow xRz) \).

The frames that satisfy the conditions (R) and (S) are called reduced separated frames, or RS frames for short, and were introduced by Gehrke [7] as a two-sorted generalisation of Kripke frames to be used for relational semantics of substructural logics.

The (Ti) condition introduced in [2] for frames \( (X_1, X_2, R) \) was motivated by the (Ti) condition on graphs:

(Ti) for every \( x \in X_1 \) and for every \( y \in X_2 \), if \( \neg(xRy) \) then there exist \( w \in X_1 \) and \( z \in X_2 \) such that
(i) \( \neg(wRz) \);
(ii) \( xR \subseteq wR \) and \( Ry \subseteq Rz \);
(iii) for every \( u \in X_1 \), if \( u \neq w \) and \( wR \subseteq uR \) then \( uRz \);
(iv) for every \( v \in X_2 \), if \( v \neq z \) and \( Rz \subseteq Rv \) then \( wRv \).

A TiRS frame was in [2] defined as a frame \( (X_1, X_2, R) \) that satisfies conditions (R), (S) and (Ti), i.e. it is an RS frame that satisfies condition (Ti). A one-to-one correspondence between TiRS graphs and TiRS frames was then shown in [2]. We recall here some facts of this correspondence that will be needed in the next section.
Definition 2.2 ([2, Definition 2.5]). Let $X = (X, E)$ be a graph. The associated frame $\rho(X)$ is the frame $(X_1, X_2, R_{\rho(X)})$ where

(i) $X_1 = X/\sim_1$ for the equivalence relation $\sim_1$ on $X$ given by
$$x \sim_1 y \text{ if } xE = yE;$$

(ii) $X_2 = X/\sim_2$ for the equivalence relation $\sim_2$ on $X$ given by
$$x \sim_2 y \text{ if } Ex = Ey;$$

(iii) $R_{\rho(X)}$ is the relation given by
$$[x]_1R_{\rho(X)}[y]_2 \iff (x, y) \notin E,$$

where $[x]_1$ and $[y]_2$ are, respectively, the $\sim_1$-equivalence class of $x$ and the $\sim_2$-equivalence class of $y$.

We omit the subscript $\rho(X)$ in $R_{\rho(X)}$ whenever it is clear to which relation $R$ refers.

If $X = (X, E)$ is a TiRS graph, then the associated frame $\rho(X) = (X_1, X_2, R_{\rho(X)})$ is a TiRS frame [2, Proposition 2.6]. Then it follows that if $L$ is a bounded lattice, $X = D^h(L)$ is its dual TiRS graph and $\rho(D^h(L))$ is the associated frame, then $\rho(D^h(L))$ is a TiRS frame (cf. [2, Corollary 2.7]).

Definition 2.3 ([2, Definition 2.8]). Let $F = (X_1, X_2, R)$ be a TiRS frame. The associated graph $\text{gr}(F)$ is $(H_F, K_F)$ where the vertex set $H_F$ is the subset of $X_1 \times X_2$ of all pairs $(x, y)$ that satisfy the following conditions:

(i) $\neg(x Ry),$

(ii) for every $u \in X_1$, if $u \neq x$ and $xR \subseteq uR$ then $uRy,$

(iii) for every $v \in X_2$, if $v \neq y$ and $Ry \subseteq Rv$ then $xRv,$

and the edge set $K_F$ is formed by the pairs $((x, y), (w, z))$ such that $\neg(xRz)$.

We omit the subscript $F$ in $H_F$ and in $K_F$ whenever it is clear which vertex set and edge set we refer to.

In [2, Proposition 2.10] we showed that if $F = (X_1, X_2, R)$ is a TiRS frame, then its associated graph $\text{gr}(F)$ is a TiRS graph.

Definition 2.4 ([2, Definition 2.11]). Two graphs $X = (X, E_X)$ and $Y = (Y, E_Y)$ are isomorphic (denoted $X \simeq Y$) if there exists a bijective map $\alpha: X \rightarrow Y$ such that
$$\forall x_1, x_2 \in X \ (x_1, x_2) \in E_X \iff (\alpha(x_1), \alpha(x_2)) \in E_Y$$

and we refer to such a map as the graph-isomorphism $\alpha: X \rightarrow Y$.

Two frames $F = (X_1, X_2, R_F)$ and $G = (Y_1, Y_2, R_G)$ are isomorphic (denoted $F \simeq G$) if there exists a pair $(\beta_1, \beta_2)$ of bijective maps $\beta_i: X_i \rightarrow Y_i$ $(i = 1, 2)$ with
$$\forall x_1 \in X_1 \forall x_2 \in X_2 \ (x_1R_F x_2 \iff \beta_1(x_1)R_G \beta_2(x_2))$$

and we refer to such a pair as the frame-isomorphism $(\beta_1, \beta_2): F \rightarrow G$. 
Now for a TiRS graph $X = (X, E)$, a map $\alpha_X : X \to \text{gr}(\rho(X))$ is defined by $\alpha_X(x) = ([x_1], [x_2])$. For any TiRS frame $F = (X_1, X_2, R)$ a map $\beta_1 : X_1 \to H_F/\sim_1$ is defined by $\beta_1(x) = [(x, y)]_1$, where $y \in X_2$ satisfies $(x, y) \in H$ and a map $\beta_2 : X_2 \to H_F/\sim_2$ is defined by $\beta_2(y) = [(x, y)]_2$, where $x \in X_1$ satisfies $(x, y) \in H$. (See [2, Section 2] for details.) The map $\beta_F : F \to \rho(\text{gr}(F))$ is defined to be the pair $(\beta_1, \beta_2)$. In the proof of the next result it was shown that $\alpha_X$ resp. $\beta_F$ are graph resp. frame isomorphisms and that the correspondence between TiRS graphs and TiRS frames is one-to-one.

**Theorem 2.5** ([2, Theorem 2.13]). Let $X = (X, E)$ be a TiRS graph and $F = (X_1, X_2, R)$ be a TiRS frame. Then

(a) the graphs $X$ and $\text{gr}(\rho(X))$ are isomorphic;

(b) the frames $F$ and $\rho(\text{gr}(F))$ are isomorphic.

### 3. TiRS graph and TiRS frame morphisms

In this section we extend the one-to-one correspondence between TiRS graphs and TiRS frames from Theorem 2.5. We define morphisms for both classes of structures and extend the definition of $\rho$ and $\text{gr}$ so that they become functors. We view these TiRS graph and TiRS frame morphisms as the starting point for the task of finding the right morphisms between PTi lattices (see Definition 4.1). We discuss this future research direction in the last section.

In Definition 3.1 we require our graph morphisms to be edge-preserving and to preserve sets of the form $xE$ and $Ex$. When the TiRS graph is a poset, these sets correspond to ‘principal up-sets’ and ‘principal down-sets’. As is seen in Definition 2.2, when we assign a TiRS frame to a TiRS graph, the two equivalence relations used to create the two sorts of the frame from the graph are defined via these principal up-sets and down-sets. In the very first step of the proof of Theorem 3.3 we use the properties (ii) and (iii) of Definition 3.1 to show that the map $\rho(\varphi)$ is well-defined. Likewise we need to consider for our frame morphisms the preservation of the sets of the form $xR$ and $Rx$. The properties in the definitions below are used repeatedly in the proof of Theorem 3.3.

**Definition 3.1.** Let $X = (X, E_X)$ and $Y = (Y, E_Y)$ be TiRS graphs. A **TiRS graph morphism** is a map $\varphi : X \to Y$ that satisfies the following conditions:

(i) for $x_1, x_2 \in X$, if $(x_1, x_2) \in E_X$ then $(\varphi(x_1), \varphi(x_2)) \in E_Y$;

(ii) for $x_1, x_2 \in X$, if $x_1E_X \subseteq x_2E_X$ then $\varphi(x_1)E_Y \subseteq \varphi(x_2)E_Y$;

(iii) for $x_1, x_2 \in X$, if $E_Xx_1 \subseteq E_Xx_2$ then $E_Y\varphi(x_1) \subseteq E_Y\varphi(x_2)$.

We note that every graph isomorphism and its inverse are TiRS graph morphisms.

**Definition 3.2.** Let $F = (X_1, X_2, R_F)$ and $G = (Y_1, Y_2, R_G)$ be TiRS frames. A **TiRS frame morphism** $\psi : F \to G$ is a a pair $\psi = (\psi_1, \psi_2)$ of maps $\psi_1 : X_1 \to Y_1$ and $\psi_2 : X_2 \to Y_2$ that satisfies the following conditions:

(i) for $x \in X_1$ and $y \in X_2$, if $\psi_1(x)R_G\psi_2(y)$ then $xR_Fy$;
(ii) for \(x, w \in X_1\), if \(xR_F \subseteq wR_F\) then \(\psi_1(x)R_G \subseteq \psi_1(w)R_G\);

(iii) for \(y, z \in X_2\), if \(R_Fy \subseteq R_Fz\) then \(R_G\psi_2(y) \subseteq R_G\psi_2(z)\);

(iv) for \(x \in X_1\) and \(y \in X_2\), if \((x, y) \in H_F\) then \((\psi_1(x), \psi_2(y)) \in H_G\).

We note that a frame isomorphism is a TiRS morphism.

We compare our choices of TiRS frame morphisms above with the frame morphisms of Deiters and Erné from [5]. Their frame morphisms (they call them quasi-embeddings, see [5, p. 7]) are pairs of maps—as we have for frames above—such that “iff” holds regarding the preservation of the relation. In condition (i) of our two definitions above we could use “iff” but we try to be as general as we can and only one of the implications is needed in our proofs of the statements in Theorem 3.3.

Our main result in this section puts our one-to-one correspondence between TiRS graphs and TiRS frames into a full categorical framework. The last two statements, (3) and (4), are illustrated by the diagrams in Figure 1.

**Theorem 3.3.** Let \(X = (X, E_X)\) and \(Y = (Y, E_Y)\) be TiRS graphs and let \(F = (X_1, X_2, R_F)\) and \(G = (Y_1, Y_2, R_G)\) be TiRS frames.

1. If \(\varphi : X \to Y\) is a TiRS graph morphism, then \(\rho(\varphi) : \rho(X) \to \rho(Y)\) defined by \(\rho(\varphi)([x]_1, [y]_2) = ([\varphi(x)]_1, [\varphi(y)]_2)\), for all \(x, y \in X\), is a TiRS frame morphism.

2. If the pair \(\psi = (\psi_1, \psi_2) : F \to G\) is a TiRS frame morphism, then the map \(\text{gr}(\psi) : \text{gr}(F) \to \text{gr}(G)\) defined by \(\text{gr}(\psi)(x, y) = (\psi_1(x), \psi_2(y))\), for all \((x, y) \in H_F\), is a TiRS graph morphism.

3. If \(\varphi : X \to Y\) is a TiRS graph morphism, then \(\text{gr}(\rho(\varphi)) \circ \alpha_X = \alpha_Y \circ \varphi\).

4. If \(\psi : F \to G\) is a TiRS frame morphism, then \(\rho(\text{gr}(\psi)) \circ \beta_F = \beta_G \circ \psi\).

**Proof.** (1) Our aim is to show that the pair \(\rho(\varphi) = (\rho(\varphi)_1, \rho(\varphi)_2)\) is a TiRS frame morphism from \(\rho(X)\) to \(\rho(Y)\), where \(\rho(\varphi)_1 : X/\sim_1 \to Y/\sim_1\) and \(\rho(\varphi)_2 : X/\sim_2 \to Y/\sim_2\) are given by \(\rho(\varphi)_1([x]_1) = [\varphi(x)]_1\) resp. \(\rho(\varphi)_2([x]_2) = [\varphi(x)]_2\), for all \(x \in X\). First we show that the map \(\rho(\varphi)_1\) is well defined. Let \(x, y \in X\). If \([x]_1 = [y]_1\) then \(xE_X = yE_X\) which implies \(\varphi(x)E_Y = \varphi(y)E_Y\) and so \([\varphi(x)]_1 = [\varphi(y)]_1\), by the definition of a TiRS graph morphism. Similarly we prove that we have \([\varphi(x)]_2 = [\varphi(y)]_2\) whenever \([x]_2 = [y]_2\). Next we prove that conditions (i) to (iv) of the definition of a TiRS frame morphism are satisfied by \(\rho(\varphi)\). For (i), let \(x, y \in X\) and assume \(\rho(\varphi)_1([x]_1)R_{\rho(Y)} \rho(\varphi)_2([y]_2)\).

![Figure 1. TiRS graph morphisms and TiRS frame morphisms](image-url)
Now \((\varphi(x), \varphi(y)) \notin E_Y\), thus \((x,y) \notin E_X\), and so \([x]_1 R_{\rho(X)}[y]_2\). For (ii), let \(x,w \in X\). Then the following holds:

\[
[x]_1 R_{\rho(X)} \subseteq [w]_1 R_{\rho(X)} \iff wE_X \subseteq xE_X \Rightarrow \varphi(w)E_Y \subseteq \varphi(x)E_Y
\]

\[
\iff [\varphi(x)]_1 R_{\rho(Y)} \subseteq [\varphi(w)]_1 R_{\rho(Y)}.
\]

Hence (ii) is satisfied. Similarly we conclude that (iii) holds. Finally (iv) follows from \([2, \text{Lemma 3.9}]\). As \(\varphi\) is a TiRS morphism, we also have \(\varphi(x) \notin E_X\) if \((x,x) \notin E_X\), \(\varphi(w) \notin E_Y\) if \((w,w) \notin E_Y\), \(\varphi(x) \notin E_Y\) if \((x,x) \notin E_Y\), and \(\varphi(w) \notin E_X\) if \((w,w) \notin E_X\).

(2) First we note that condition (iv) of the definition of a TiRS frame morphism satisfied by \(\varphi\) guarantees that the map \(\text{gr}(\varphi)\) is well defined. Next we prove that conditions (i) to (iii) of the definition of a TiRS graph morphism are satisfied by \(\text{gr}(\varphi)\). Let \((x,y),(w,z) \in H_{F}\). If \(((x,y),(w,z)) \in K_{F}\) then \(\text{gr}(\varphi)(x,y)\) is well defined. Next we prove that conditions (i) to (iii) of the definition of a TiRS graph morphism are satisfied by \(\text{gr}(\varphi)\). Let \((x,y),(w,z) \in H_{F}\). If \(((x,y),(w,z)) \in K_{F}\) then

\[
\text{gr}(\varphi)(x,y) = ((\varphi_1(x),\varphi_2(y)), (\varphi_1(w),\varphi_2(z))) \in K_{G}.
\]

Hence \((x,y),(w,z)\) satisfies (i). For (ii), we observe that \((x,y)K_{F} \subseteq (w,z)K_{F} \iff wR_{F} \subseteq xR_{F}\)

and

\[
\text{gr}(\varphi)(x,y)K_{G} \subseteq \text{gr}(\varphi)(w,z)K_{G} \iff \psi(x)R_{G} \subseteq \psi(y)R_{G},
\]

which follows from (iii) of \([2, \text{Lemma 3.9}]\). As \(\varphi\) is a TiRS morphism, we also have

\[
wR_{F} \subseteq xR_{F} \Rightarrow \psi_1(x)R_{G} \subseteq \psi_1(y)R_{G}.
\]

Hence \((x,y),(w,z)\) satisfies (ii). Similarly we conclude that (iii) also holds.

(3) Let \(x \in X\). We have that

\[
\text{gr}(\rho(\varphi)) \circ \alpha_X(x) = \text{gr}(\rho(\varphi))([x]_1, [x]_2)
\]

\[
= (\rho(\varphi)_1([x]_1), \rho(\varphi)_2([x]_2))
\]

\[
= ([\varphi(x)]_1, [\varphi(x)]_2)
\]

\[
= (\alpha_Y \circ \varphi)(x).
\]

(4) Let \(x \in X_{1}\). There exists \(y \in X_{2}\) such that \((x,y) \in H_{F}\). We get

\[
(\rho(\text{gr}(\varphi)) \circ \beta_F)(x) = \rho(\text{gr}(\varphi)([x,y]))_1
\]

\[
= [\text{gr}(\varphi)(x,y)]_1
\]

\[
= [(\psi_1(x),\psi_2(y))]_1,
\]

where \(\psi_1 : X_1 \rightarrow Y_1\) and \(\psi_2 : X_2 \rightarrow Y_2\) satisfy \(\psi = (\psi_1, \psi_2)\). Since \((x,y) \in H_{F}\) and \(\psi\) is a TiRS morphism, we also have \((\psi_1(x),\psi_2(y)) \in H_{G}\) and so

\[
[(\psi_1(x),\psi_2(y))]_1 = \beta_G(\psi(x)) = (\beta_G \circ \psi)(x).
\]

\[\square\]

**Corollary 3.4.** The category of TiRS graphs with TiRS graph morphisms is equivalent to the category of TiRS frames with TiRS frame morphisms via the functors given by \(\rho\) and \(\text{gr}\) as described above.
As mentioned at the beginning of this section, Deiters and Erné [5] in their definition of morphisms between two frames (contexts) use a pair of maps as we do above. They define what they call a quasi-isomorphism when both maps in the pair are surjective and that is all they need to obtain the isomorphism between the concept lattices in their [5, Lemma 2.1]. They also have the analog of our frame isomorphism when both maps in the pair are bijections.

There are also other definitions of morphisms between two frames (contexts) \( F = (X_1, X_2, R_F) \) and \( G = (Y_1, Y_2, R_G) \) that are used in the literature. Gehrke [7, Section 3] uses a pair of relations \( (R, S) \) where \( R \subseteq X_2 \times Y_1 \) and \( S \subseteq X_1 \times Y_2 \). More recently, Moshier [16] (see also Jipsen [15]) defined a context morphism to be a single relation \( S \subseteq X_1 \times Y_2 \).

4. Perfect lattices dual to TiRS structures

Consider a complete lattice \( C \) and let \( F(C) = (J_\infty(C), M_\infty(C), \leq) \). We will refer to \( F(C) \) as the frame coming from \( C \). For the opposite direction, consider an RS frame \( F = (X, Y, R) \). For \( A \subseteq X \) and \( B \subseteq Y \), let \( R_a(A) = \{ y \in Y \mid (\forall a \in A)(aRy) \} \) and \( R_d(B) = \{ x \in X \mid (\forall b \in B)(xRb) \} \).

Now consider the complete lattice of Galois-closed sets (ordered by inclusion): \( G(F) = \{ A \subseteq X \mid A = (R_d \circ R_a)(A) \} \).

By results from Gehrke [7, Section 2] we know that the completely join-irreducible elements and completely meet-irreducible elements of \( G(F) \) are identified as follows:

\[
J_\infty(G(F)) = \{ (R_d \circ R_a)(\{ x \}) \mid x \in X \} \quad \text{and} \quad M_\infty(G(F)) = \{ Ry \mid y \in Y \}.
\]

Below we introduce a condition that refines the class of perfect lattices. At the end of this section we will conclude that every perfect lattice that is the canonical extension of some bounded lattice will have this property. Because, as we shall see, it is the translation of the condition \((Ti)\) from RS-frames into perfect lattices, we give this condition the name \((PTi)\) where \(P\) stands for “Perfect”.

**Definition 4.1.** We say a perfect lattice satisfies the condition \((PTi)\) if for all \( x \in J_\infty(C) \) and for all \( y \in M_\infty(C) \), if \( x \nleq y \) then there exist \( w \in J_\infty(C) \), \( z \in M_\infty(C) \) such that

1. \( w \leq x \) and \( y \leq z \);
2. \( w \nleq z \); 
3. \( (\forall u \in J_\infty(C))(u < w \Rightarrow u \leq z) \); 
4. \( (\forall v \in M_\infty(C))(z < v \Rightarrow w \leq v) \).

In Figure 2 we give a pictorial depiction of the \((PTi)\) condition. We have indicated the sets \( \uparrow x, \uparrow w, \downarrow y \) and \( \downarrow z \). We see that the \((PTi)\) condition for \( C \) essentially starts with an arbitrary disjoint filter-ideal pair \( (\uparrow x, \downarrow y) \) generated by elements \( x \in J_\infty(C) \) and \( y \in M_\infty(C) \). It says that every such disjoint filter-ideal pair is contained in a maximal disjoint filter-ideal pair \( (\uparrow w, \downarrow z) \) where
Lemma 4.2. Let $C$ be a perfect lattice. If $C$ satisfies (PTi) then the RS frame $F(C) = (J^\infty(C), M^\infty(C), \leq)$ satisfies (Ti).

Proof. First observe that when translating the condition (Ti) from a general RS frame to $F(C)$ we have that $xR = \uparrow x$ and $Ry = \downarrow y$. The fact that $F(C)$ satisfies (Ti) follows then from the fact that $u < w$ implies $u \neq w$ and $\uparrow w \not\subseteq \uparrow u$. \hfill \Box

We want to characterise the condition (PTi) on the Galois closed sets arising from an RS frame $F = (X, Y, R)$. The following lemma will assist us in this task.

Lemma 4.3. Consider the RS frame $F = (X, Y, R)$. Then

(i) $w \in (R_\cap \circ R_\lor)(\{x\})$ if and only if $xR \subseteq wR$;
(ii) $(R_\cap \circ R_\lor)(\{w\}) \subseteq (R_\cap \circ R_\lor)(\{x\})$ if and only if $xR \subseteq wR$;
(iii) $(R_\cap \circ R_\lor)(\{x\}) \subseteq Ry$ if and only if $xRy$.

Proof. For (i) we have

\[
  w \in (R_\cap \circ R_\lor)(\{x\}) \iff (\forall z \in R_\lor(\{x\}))(wRz) \\
  \iff (\forall z \in Y)(xRz \Rightarrow wRz) \\
  \iff xR \subseteq wR.
\]

To help us prove (ii), we note that $R_\lor(\{x\}) = xR$ and $R_\lor(\{w\}) = wR$. If we assume that $xR \subseteq wR$ then the fact that $R_\cap : \wp(Y) \rightarrow \wp(X)$ is order-reversing gives us that $(R_\cap \circ R_\lor)(\{w\}) \subseteq (R_\cap \circ R_\lor)(\{x\})$. For the converse, if $(R_\cap \circ R_\lor)(\{w\}) \subseteq (R_\cap \circ R_\lor)(\{x\})$ then since $R_\lor : \wp(X) \rightarrow \wp(Y)$ is order-reversing and since $(R_\lor \circ R_\cap \circ R_\lor)(\{w\}) = R_\lor(\{w\})$, we get $xR \subseteq wR$. The statement (iii) is exactly [7, Proposition 2.6]. \hfill \Box
We want to prove that when an RS frame $F = (X, Y, R)$ satisfies the (Ti) condition, the perfect lattice of Galois-closed sets $G(F)$ satisfies (PTi). In order to make the proof easier to follow, it will be useful to translate the condition (PTi) from the setting of a general perfect lattice to the setting of $G(F)$.

**Lemma 4.4.** Let $F = (X, Y, R)$ be an RS frame. Assume that the following condition is satisfied by $F$:

For all $x \in X$ and all $y \in Y$, if $\neg (xRy)$ then there exist $p \in X$, $q \in Y$ such that

(i) $xR \subseteq pR$ and $Ry \subseteq Rq$;
(ii) $\neg (pRq)$;
(iii) $(\forall u \in X)(pR \not\subseteq uR \Rightarrow uR \not\subseteq Rq)$;
(iv) $(\forall v \in Y)(Rq \not\subseteq Rv \Rightarrow pR \not\subseteq Rv)$.

Then the lattice $G(F)$ satisfies (PTi).

**Proof.** This follows using Lemma 4.3 to translate the (PTi) conditions to the complete lattice $G(F)$. □

**Lemma 4.5.** Let $F = (X, Y, R)$ be an RS frame. If $F$ satisfies (Ti) then $G(F)$ satisfies (PTi).

**Proof.** Let $F = (X, Y, R)$ be an RS frame satisfying (Ti) (i.e. a TiRS frame). Take arbitrary $x \in X$ and $y \in Y$ and assume that $\neg (xRy)$. In the perfect lattice $G(F)$ coming from $F$ consider the sets $A = (R \circ \circ \circ R)(\{x\})$ and $B = Ry$. Then $A \in J^\infty(G(F))$, $B \in M^\infty(G(F))$ and $A \not\subseteq B$ using Lemma 4.3(iii).

We have $(R \circ \circ \circ R)(\{x\}) \not\subseteq Ry$ implies $(\exists w \in X)(xR \subseteq wR \& \neg (wRy))$, which in turn implies

$$(\exists w \in X)[xR \subseteq wR \& (\exists p \in X)(\exists q \in Y)(\neg (pRq) \& wR \subseteq pR \& Ry \subseteq Rq \& (\forall u \in X)(pR \not\subseteq uR \Rightarrow uR \not\subseteq Rq) \& (\forall v \in Y)(Rq \not\subseteq Rv \Rightarrow pR \not\subseteq Rv)]$$

The only part of the (PTi) condition for $G(F)$ that is not now immediate is the fact that we need $xR \subseteq pR$. This follows from $xR \subseteq wR \subseteq pR$ and the transitivity of set containment. □

Now we are ready to show that the canonical extensions of lattices are PTi lattices and so they indeed are ‘more’ than just perfect lattices. For this we cite our final result from [2]:

**Proposition 4.6 ([2, Corollary 3.11]).** Let $L$ be a bounded lattice and $X = D^\flat(L)$ be its dual TiRS graph. Let $\rho(X)$ be the frame associated to $X$ and $G(\rho(X))$ be its corresponding perfect lattice of Galois-closed sets.

The lattice $G(\rho(X))$ is the canonical extension of $L$.

The result is illustrated by the diagram in Figure 3. The given bounded lattice $L$ is firstly assigned its Ploščica dual space $D(L) = (L^{mp}(L, 2), E, T)$, and then the Ploščica dual graph $X = D^\flat(L) = (L^{mp}(L, 2), E)$ is obtained by forgetting the topology. This is a TiRS graph and so the frame $\rho(X)$ associated to $X$ in our one-to-one correspondence developed in [2] between TiRS graphs
and TiRS frames is a TiRS frame. Hence by Lemma 4.5 above, the perfect lattice $G(\rho(X))$ of Galois-closed sets corresponding in Gehrke’s representation to the frame $\rho(X)$ is a PTi lattice. By Proposition 4.6, the lattice $G(\rho(X))$ is the canonical extension of the given lattice $L$.

Hence we have our final result of this section:

**Theorem 4.7.** The canonical extension of a bounded lattice is a PTi lattice.

Gerhke and Vosmaer [10] showed that the canonical extension of a lattice need not be meet-continuous, and hence need not always be algebraic. Theorem 4.7 gives us further information about the structure of canonical extensions of bounded lattices.

## 5. Examples

Our goal in this section is to illustrate that the PTi condition adds to the current description of the canonical extension of a bounded lattice. We focus on non-distributive examples. Canonical extensions of distributive lattices are known to be completely distributive complete lattices. To show that our new condition does indeed add to the current description, we give an example of a perfect lattice that is not PTi. Giving an example of a PTi lattice that is not the canonical extension of a lattice would be the same as giving an example of a TiRS graph that is not of the form $(L_{mp}(L, 2), E)$ for some bounded lattice $L$. Hence this is the same as the representable TiRS graph (representable poset) problem.

Our goals are:

1. Give an example of a complete non-distributive lattice which is a PTi lattice but is not the canonical extension of any bounded lattice.
2. Give an example of a perfect non-distributive lattice that is not a PTi lattice.

**Example 5.1.** Consider the complete lattice $A_L$ depicted on the right in Figure 4. We will denote by $m$ the middle element of the infinite chain $\omega \oplus 1 \oplus \omega^9$ and $y$ is above the bottom and below the top but incomparable with all other elements.

The TiRS graph dual to $A_L$ is $X = \{p_i \mid i \in \omega\} \cup \{q_j \mid j \in \omega\} \cup \{k\}$ with the relation $E$ given by
Figure 4. A TiRS graph that is not the graph of MPH’s of any bounded lattice (left) and its dual PTi lattice $A_L$ that is not a canonical extension (right). The double-headed arrows on the graph emphasize that transitivity holds amongst the vertical edges

$$\{ (p_i, q_j) \mid i \leq j \} \cup \{ (q_j, q_i) \mid j \geq i \} \cup \{ (p_i, q_j) \mid i, j \in \omega \} \cup \{ (k, p_i) \mid i \in \omega \} \cup \{ (k, q_j) \mid j \geq 1 \}$$

(it is depicted on the left in Figure 4). To be clear, the $p_i$’s and $q_j$’s form a poset (it is transitive) that is order-isomorphic to $\omega \oplus \omega^\partial$ while the element $k$ is related to everything except the top of the chain.

Recall that MPE’s are ordered by: $\varphi \leq \psi$ if and only if $\varphi^{-1}(1) \subseteq \psi^{-1}(1)$. The MPE’s $\varphi_1$ and $\varphi_0$ are defined by $\varphi_1(x) = 1$ and $\varphi_0(x) = 0$ for all $x \in X$. All but one of the other MPE’s have $k \mapsto 0$ and then they split the chain at some point. When $\varphi$ splits the chain by sending the $p_i$’s to 0 and the $q_j$’s to 1 then you get the limit point in the middle of $A_L$. The interesting MPE is the map that does the following for $a \in X$:

$$\varphi(a) = \begin{cases} 1 & \text{if } a = k, \\ 0 & \text{if } a = q_0, \\ - & \text{otherwise.} \end{cases}$$

This interesting MPE is the incomparable point that makes the lattice $A_L$ non-distributive.

It is quite easy to show that the lattice $A_L$ is a PTi lattice; we indicated on the right in Figure 4 what the elements $w \in J^\infty(A_L)$ and $z \in M^\infty(A_L)$ are for the chosen elements $x \in J^\infty(A_L)$ and $y \in M^\infty(A_L)$. The fact that the lattice $A_L$ is not the canonical extension of any bounded lattice is harder to show and it follows from Proposition 5.2 below.
Figure 5. An RS frame that is not Ti (left) and its dual perfect lattice that is not PTi (right)

Proposition 5.2. There is no bounded lattice $L$ with embedding $e: L \to A_L$ such that $(e, A_L)$ is the canonical extension of $L$.

Proof. Suppose there is a bounded lattice $L$ and an embedding $e: L \to A_L$ such that $(e, A_L)$ is the canonical extension of $L$. Clearly the top and bottom element of $A_L$ are, respectively $e(1)$ and $e(0)$ where 1 and 0 are the top and bottom element of $L$. Now consider the set of elements $(A_L \setminus \{e(0), e(1), m\}$. It is easy to see that each of these elements is completely join-irreducible in $A_L$ and hence we have, by Lemma 2.1, that each of these elements is the meet of the embedding of a filter of $L$. Hence each of the elements of $(A_L \setminus \{e(0), e(1), m\}$ is a filter element. Dually, it is easy to see that each element of $(A_L \setminus \{e(0), e(1), m\}$ is completely meet-irreducible and again by Lemma 2.1 they are all the join of the embedding of an ideal of $L$ and hence are all ideal elements. Thus every element of $(A_L \setminus \{e(0), e(1), m\}$ is both ideal and filter and hence must be of the form $e(a)$ for some $a \in L$. Now consider the element $m$. Since $m = \bigwedge \omega^\partial$, and since every element of $\omega^\partial$ is the image of an element of $L$ under $e$, we have that $m$ is a filter element of $A_L$. Also, $m = \bigvee \omega$ and every element of $\omega$ is the image of an element of $L$ under $e$. Therefore $m$ is also an ideal element of $A_L$. Hence $m$ must be of the form $e(b)$ for some $b \in L$. Thus we have that $L \cong A_L$ and that the embedding $e$ is a bijection.

Now we show that $(e, A_L)$ cannot be the canonical extension of $L$. Observe that since $m = \bigwedge \omega^\partial = \bigvee \omega$ we have that $\bigwedge \omega^\partial \leq \bigvee \omega$. However, for any finite subset $A' \subseteq \omega^\partial$ and any finite subset $B' \subseteq \omega$ we will have $\bigvee B' < \bigwedge A'$. Hence $(e, A_L)$ is not a compact completion of $L$. □

Example 5.3. We consider the complete lattice $M_L$ depicted on the right in Figure 5. The order is given by the poset $\mathbf{1} \oplus \omega^\partial$ with an additional element $y$ incomparable to all elements except the top and the bottom. It can easily be seen that $M_L$ is a perfect lattice ($J^\infty(M_L) = M^\infty(M_L) = \{y\} \cup \{x_i \mid i \geq 1\}$). It is not PTi since there are no $w$ and $z$ for the pair $x_j \not\leq y$ ($j \geq 1$).

The RS frame corresponding to it was already mentioned in [2, p. 128] as an example of an RS frame which is not TiRS (it is indicated on the left in Figure 5): Let $X_1 = \{a_i\}_{i \in \omega}, X_2 = \{b_i\}_{i \in \omega}$ and let

$$R = \{(a_1, b_0), (a_0, b_1)\} \cup \{(a_i, b_j) \mid 2 \leq i, 1 \leq j \leq i\}. $$
By considering $\neg (a_0 R b_0)$ it is rather straightforward to show that $(X_1, X_2, R)$ does not satisfy (Ti).

Our final picture, Figure 6, describes the correspondence between PTi lattices and TiRS graphs and the correspondence between their important subclasses: (i) the canonical extensions of bounded lattices inside the PTi lattices, and representable graphs (as dual graphs of bounded lattices) inside the TiRS graphs; (ii) the canonical extensions of bounded distributive lattices inside the canonical extensions of bounded lattices, and representable posets (as dual graphs of bounded distributive lattices) inside the representable graphs.

A natural question that we asked already in [2, p. 126–127] was which TiRS graphs arise as duals of bounded lattices. In the case of bounded distributive lattices (denoted as BDLs in Figure 6) this question reduces to the question of which posets are representable posets. This seems to be an extremely hard problem. Examples of non-representable posets are also examples of non-representable graphs as any poset is automatically a TiRS graph. We mention an example of a non-representable poset due to Tan [18] from the 1970s: $T := \omega \oplus \omega^\partial$. The perfect lattice corresponding to this TiRS graph is the PTi lattice $T_L := \omega \oplus 1 \oplus \omega^\partial$.

6. Conclusion

In this paper we introduced morphisms of TiRS structures and we put our correspondence between TiRS graphs and TiRS frames from [2] into a full categorical framework. We also developed a correspondence between our newly-described PTi lattices and TiRS graphs as presented in Figure 6, and illustrated it with some examples. To put the latter correspondence into a full categorical framework, one needs to find the right concept of homomorphisms between PTi lattices. These would necessarily be complete lattice homomorphisms with additional properties. We propose the description of such homomorphisms as a future research direction.

We view the TiRS graph and TiRS frame morphisms we introduced in Section 3 as the starting point for the task of finding the right morphisms.
between PTi lattices. That the task of describing the right PTi morphisms will not be an easy one was illustrated already in [3] where our linking of the presented canonical extension construction to duality in a functorial way came at the cost of working with greatly enlarged dual spaces. Hence we end this paper with an open problem:

**Problem.** Describe a categorical duality between PTi lattices and TiRS graphs by characterising the homomorphisms between PTi lattices that would be dual to TiRS graph morphisms.

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