Non-Hermitian interaction representation and its use in relativistic quantum mechanics

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Abstract

The textbook interaction-picture formulation of quantum mechanics is extended to cover the unitarily evolving systems in which the Hermiticity of the observables is guaranteed via an ad hoc amendment of the inner product in Hilbert space. These systems are sampled by the Klein-Gordon equation with a space- and time-dependent mass term.

Keywords
relativistic quantum mechanics; unitary evolution; non-Hermitian interaction picture; state vectors; Schrödinger-type equations; observables; Heisenberg-type equations; Klein-Gordon particles; instant- and position-dependent mass;
1 Introduction

Quantum mechanics formulated in the Dirac’s alias interaction picture offers a fairly universal representation of the unitarily evolving quantum systems as well as a useful methodical bridge between the most economical Schrödinger picture and its slightly older, more intuitive Heisenberg-picture alternative [1]. Unfortunately, in the eyes of mathematical physicists the overall image of the interaction picture approach is partially damaged by its frequent use in the areas prohibited by the Haag’s theorem [2]. The applicability of the recipe was rigorously disproved in multiple models of quantum fields [3]. Nevertheless, one may feel surprised that the review paper “Nine formulations of quantum mechanics” [4] did not mention the interaction picture even without moving into the quantum field theory. This makes an impression of a misunderstanding because the authors of the review admit, at the same time, the usefulness of having alternative approaches. They emphasized that from a purely technical point of view “no formulation produces a royal road to quantum mechanics” [4].

In the technical setting we are witnessing a remarkable new progress during the last few years. The conventional menu of pictures was enriched by the descriptions of the unitarily evolving quantum systems in which the optimal representations of certain observables appear non-Hermitian (cf., e.g., [5, 6]). In particular, multiple specific merits of the non-Hermitian evolution equations were found to follow from the use of the generalized stationary non-Hermitian Schrödinger picture [7, 8, 9]. In this amended framework people started using, with great success and impact, innovated stationary non-Hermitian Hamiltonians $H \neq H^\dagger$ with real spectra and exhibiting certain unusual features like parity-time symmetry, etc (see the recent review of history [10] and a few more detailed comments in section 2 below).

The scope of the early extensions of the non-Hermitian formalism to non-stationary dynamical scenarios [11, 12] was strongly limited by the traditional belief that every consistent form of the non-stationary generator of the evolution of wave functions (let us denote this generator by a dedicated symbol $G(t)$) must be observable. This belief proved unexpectedly deeply rooted (cf. also the early extensive discussion [13] or Theorem 2 in [8]). Once this belief was reclassified as artificial and redundant, the non-Hermitian formalism was immediately generalized to non-stationary systems [14, 15]. Its rather complicated technical nature inspired its further simplifications [16, 17] and, finally, a reduction to the special stationary non-Hermitian Heisenberg picture [18] with certain specific practical merits as discussed, e.g., in Refs. [19, 20].

The difficulty of a return to the full-fledged non-stationary formalism of the non-Hermitian interaction picture remained perceived as a challenge [21, 22, 23]. In our present
paper we shall return to the subject, with the emphasis on the building of a methodical bridge between the alternative non-Hermitian formulations of quantum mechanics. We shall summarize the recent developments in this direction, formulate a new consistent version of the non-Hermitian interaction picture and we shall illustrate its applicability via a realistic, relativistic quantum-mechanical model.

The material will be organized as follows. In sections 2 and 3 we shall describe the necessary methodical preliminaries. We shall recall the notation of review [15] and we shall explain that the phenomenological models with both stationary and non-stationary non-Hermitian generators of wave functions can be perceived as unitary and phenomenologically admissible. In sections 4, 5, 6 and 7 the formalism will be made complete while in section 8 the theory will be applied to the Klein-Gordon model, with the mass term being both space- and time-dependent. In the last two sections 9 and 10 we shall discuss some of the possible further innovative consequences of our approach in the broader context of physics.

2 Stationary quantum systems in the non-Hermitian Schrödinger picture

The existing nontrivial applications of the non-Hermitian representations of unitary quantum systems are almost exclusively constructed in the Schrödinger picture (SP) framework. The main reason lies in the enhanced importance of the mathematical economy of the non-Hermitian formalism, much more rarely achieved in the alternative Heisenberg picture (HP, [18, 20]).

2.1 Conventional Hermitian observables: Limits of applicability

An arbitrary quantum system $\mathcal{S}$ may be studied in the most common Schrödinger picture of textbooks, in principle at least. The system is represented, in pure state, by an element of a preselected Hilbert space $\mathcal{H}^{(\text{textbook})}$, i.e., by a ket-vector $|\psi^{(\text{SP})}(t)\rangle \in \mathcal{H}^{(\text{textbook})}$ which is time-dependent. After one prepares this ket vector at an initial time $t_i = 0$, the task is to predict the result of a measurement performed at a future time $t_f > 0$. This prediction is probabilistic. Whenever we are interested in the measurement of an observable represented by a stationary self-adjoint operator $q_{(\text{SP})}$ (or, in general, by the variable $q_{(\text{SP})(t)}$), the prediction is prescribed in terms of matrix element

$$\langle \psi^{(\text{SP})}(t_f)|q_{(\text{SP})}(t_f)|\psi^{(\text{SP})}(t_f)\rangle .$$

(1)
The experimentally relevant information about the unitary evolution of the system $S$ is all carried by the wave-ket solutions of Schrödinger equation
\[
\frac{i}{\hbar} \frac{\partial}{\partial t} |\psi(t)\rangle = \mathbf{h}_{(SP)} |\psi(t)\rangle, \quad |\psi(t)\rangle \in \mathcal{H}^{(\text{textbook})}.
\] (2)
The unitarity of the evolution is equivalent, due to the Stone’s theorem [24], to the self-adjoint nature of the system’s Hamiltonian, $\mathbf{h}_{(SP)} = \mathbf{h}^\dagger_{(SP)}$ in $\mathcal{H}^{(\text{textbook})}$.

For the stationary self-adjoint Hamiltonians $\mathbf{h}_{(SP)} \neq \mathbf{h}_{(SP)}(t)$ the solution of Eq. (2) may proceed via their diagonalization. In realistic systems $S$ even this diagonalization may happen to be a formidable, practically next-to-impossible task. One of its efficient simplifications is provided by a non-Hermitian modification of Schrödinger picture. This approach, usually attributed to Dyson [5], found a number of successful applications in nuclear physics [6]. Its key features will be described in the next paragraph.

2.2 The change of Hilbert space using stationary Dyson map

Dyson [5] revealed that in many-fermion quantum systems $S_{(Dyson)}$ one of the key technical obstacles lies in an extremely unfriendly nature of the underlying conventional fermionic Hilbert space $\mathcal{H}^{(\text{textbook})}$ as well as in an equally unfriendly nature of the eigenstates $|\psi_n\rangle$ of the system’s stationary self-adjoint Hamiltonians $\mathbf{h}_{(SP)}$. As long as at least a part of the difficulty lies in the fermion-fermion correlations, Dyson proposed that these correlations might be separated and simulated via an ad hoc operator,
\[
|\psi_n^{(correlated)}\rangle = \Omega_{(Dyson)} |\psi_n^{(simplified)}\rangle.
\] (3)
The best candidates for the correlation operators $\Omega_{(Dyson)}$ appeared to be non-unitary,
\[
\Omega_{(Dyson)} \neq \Omega_{(Dyson)}(t), \quad \Omega_{(Dyson)}^\dagger \Omega_{(Dyson)} = \Theta_{(stationary)} \neq I.
\] (4)
Dyson decided to make their “trial and error” choice $n-$independent. Thus, he replaced Eq. (3) by an overall ansatz
\[
|\psi^{(SP)}(t)\rangle = \Omega_{(Dyson)} |\psi^{(Dyson)}(t)\rangle, \quad |\psi^{(Dyson)}(t)\rangle \in \mathcal{H}^{(\text{friendlier})}.
\] (5)
in which the map $\Omega_{(Dyson)} \neq \Omega_{(Dyson)}(t)$ from the simpler Hilbert space remains stationary.

In the numerous applications of such an ansatz in nuclear physics [6] the mapping was reinterpreted as a replacement of the unfriendly initial fermionic Hilbert space $\mathcal{H}^{(\text{textbook})}$ by a more suitable bosonic Hilbert space $\mathcal{H}^{(\text{friendlier})}$. In the majority of these applications the mapping was kept stationary but, otherwise, flexible, non-unitary. The stationarity
assumption implied that the Schrödinger Eq. (2) in $\mathcal{H}^{(\text{textbook})}$ gets replaced by its “inter-
acting boson model” avatar

$$i\hbar \frac{\partial}{\partial t} |\psi^{(\text{Dyson})}(t)\rangle = H_{\text{Dyson}} |\psi^{(\text{Dyson})}(t)\rangle, \quad |\psi^{(\text{Dyson})}(t)\rangle \in \mathcal{H}^{(\text{friendlier})}. \quad (6)$$

It lives in the friendlier space and contains a simplified isospectral Hamiltonian,

$$H_{\text{Dyson}} = \Omega^{(-1)}_{\text{Dyson}} h_{(SP)} \Omega_{\text{Dyson}}. \quad (7)$$

Needless to add that the variational determination of spectra via $H_{\text{Dyson}}$ proved superior in a number of realistic calculations [6].

3 The Stone’s theorem revisited

In a brief conceptual detour let us now address one of the apparent paradoxes encountered after the use of non-unitary maps (4).

3.1 The Dyson-proposed return to Hermiticity

The replacement (7) of the known and realistic Hamiltonian $h_{(SP)}$ defined in $\mathcal{H}^{(\text{textbook})}$ by its simpler isospectral partner $H_{\text{Dyson}}$ acting in $\mathcal{H}^{(\text{friendlier})}$ is accompanied by the loss of the physical status of the Hilbert space caused by the non-unitarity of the map. As long as the product (4) called metric remains time-independent, its emergence introduces a comparatively marginal technical complication. It suffices to convert the auxiliary, unphysical Hilbert space $\mathcal{H}^{(\text{friendlier})}$ into its unitarily non-equivalent partner, i.e., into another Hilbert space $\mathcal{H}^{(\text{standard})}$. The latter space re-acquires the physical-state status because it becomes, by construction, unitarily equivalent to $\mathcal{H}^{(\text{textbook})}$.

The conversion is easy. One merely keeps the linear space $\mathcal{V}$ of the ket-vectors $|\psi\rangle$ unchanged and realizes the transition to the desired $\mathcal{H}^{(\text{standard})}$ by the mere $\text{}$ ad hoc amelioration of the bra vectors,

$$\langle \psi | \rightarrow \langle \psi |_{\Theta^{(\text{stationary})}} \equiv \langle \psi |_{\Theta} \quad \text{for} \quad \mathcal{H}^{(\text{friendlier})} \rightarrow \mathcal{H}^{(\text{standard})}. \quad (8)$$

The new bra-vectors are metric-dependent. In the terminology of functional analysis we just changed the definition of the dual alias bra-vector space of the linear functionals, $\mathcal{V}' \rightarrow \mathcal{V}'_{\Theta}$. Thus, we replaced the unphysical Dirac’s bra-ket inner product $\langle \psi | \chi \rangle$ by its physical standard-space amendment,

$$\langle \psi | \chi \rangle \rightarrow \langle \psi |_{\Theta} | \chi \rangle \equiv \langle \psi |_{\Theta} | \chi \rangle. \quad (9)$$
Schematically, the situation is depicted in Fig. 1

In Fig. 1 we emphasize the strict equivalence of the phenomenological predictions which are, in principle, available in both of the physical Hilbert spaces $\mathcal{H}^{(\text{textbook})}$ and $\mathcal{H}^{(\text{standard})}$. This explains why the theory does not contradict the Stone’s theorem \([21]\). Indeed, this theorem states that the unitarity of the evolution requires the self-adjointness of the generator. Obviously, by construction, such a condition is satisfied in both of the physical Hilbert spaces, viz., in $\mathcal{H}^{(\text{textbook})}$ (where we have $\mathfrak{h} = \mathfrak{h}^\dagger$, as we must have) as well as in $\mathcal{H}^{(\text{standard})}$ (where we only have to understand that the self-adjointness is usually represented by formula $H = H^\dagger := \Theta^{-1}H^\dagger\Theta$, i.e., by its representation in auxiliary $\mathcal{H}^{(\text{friendlier})}$).

### 3.2 The predictions of measurements revisited

In practice, we are always performing all calculations in the auxiliary space $\mathcal{H}^{(\text{friendlier})}$. Whenever needed, we only perform the easy transition to $\mathcal{H}^{(\text{standard})}$ by means of the introduction of the non-trivial ad hoc metric operator $\Theta \neq I$. The unitarity of the evolution is equally guaranteed in both of the representations of the quantum system in question, therefore.

The results of measurements find their correct interpretation in either one of the two, mutually unitarily equivalent physical Hilbert spaces $\mathcal{H}^{(\text{standard})}$ and $\mathcal{H}^{(\text{textbook})}$. The main advantage of the resulting triple-representation formalism is that the majority of the con-
structive considerations may be, in the auxiliary Hilbert space $\mathcal{H}^{(friendlier)}$, maximally simplified. Only the definitions of the observables $q_{(SP)}$ are usually deduced inside $\mathcal{H}^{(textbook)}$. Thus, one has to use an analogue of Eq. (7) and one has to pull the operators up to the auxiliary Hilbert space $\mathcal{H}^{(friendlier)}$. This yields their non-Hermitian representations in $\mathcal{H}^{(friendlier)}$,

$$Q_{(Dyson)} = \Omega_{(Dyson)}^{-1} q_{(SP)} \Omega_{(Dyson)} \neq Q_{(Dyson)}^\dagger .$$

(10)

In the case of the observable of position this transformation of operators other than Hamiltonians was sampled in [25]. In all of the similar cases the experiment-predicting formula (1) may be replaced by its – friendlier – alternative since

$$\langle \psi_{(Dyson)}(t_f) | Q_{(Dyson)}(t_f) | \psi_{(Dyson)}(t_f) \rangle = \langle \psi_{\Theta}(t_f) | Q_{(Dyson)}(t_f) | \psi_{\Theta}(t_f) \rangle .$$

(11)

The pragmatic appeal of Hamiltonians $H_{(Dyson)}$ and of the other non-Hermitian observables $Q_{(Dyson)}$ results from an implicit assumption that the use of the respective mappings (7) and (10) simplifies the evaluation of matrix elements (11). Otherwise, the use of the non-unitary map of Eq. (5) and of the non-Hermitian version of Schrödinger equation would not be sufficiently well motivated.

### 3.3 Best known special case: Stationary $\mathcal{PT}$–symmetric systems

In applications, the stationary maps $\Omega_{(Dyson)}$ are sought in a trial-and-error manner [5, 6]. Bender with Boettcher [26] innovated the strategy and proposed an inversion of the Dyson’s flowchart. This inspired a number of innovative studies of Eq. (6). All of them were based on an input choice of a tentative stationary candidate $H_{(Dyson)} \neq H_{(Dyson)}^\dagger$ for the Hamiltonian. These toy-model operators were mostly chosen Krein-space self-adjoint alias $\mathcal{PT}$–symmetric,

$$H_{(Dyson)} \mathcal{PT} = \mathcal{PT} H_{(Dyson)} ,$$

(12)

with $\mathcal{PT}$ meaning parity-time-reflection (see an extensive summary of these developments in deeply physics-oriented review paper [7]).

In the resulting narrower, $\mathcal{PT}$–symmetric quantum mechanics the main task lies again in the ultimate evaluation of experimental predictions (11). For this purpose people usually reconstruct the observables-Hermitizing Hilbert-space metric $\Theta = \Theta_{(stationary)}$ from the Hamiltonian, i.e., as a solution of the time-independent relation

$$H^\dagger \Theta = \Theta H , \quad \Theta = \Omega^\dagger \Omega$$

(13)

which follows from the self-adjointness constraint $\mathfrak{h} = \mathfrak{h}^\dagger$ and from definition (7).
People often introduce also another observable $Q$. The task of making the theory fully consistent is then more complicated \cite{27}. In the light of Eq. (10) the same metric must satisfy also the second equation

$$Q^{\dagger}_{(\text{Dyson})}\Theta = \Theta Q_{(\text{Dyson})}.$$  \hspace{1cm} (14)

It is worth adding that physicists often need the factor map $\Omega_{(\text{Dyson})}$ as well. This is another ambiguous task in general. For the sake of simplicity this ambiguity is usually ignored and the special, unique self-adjoint $\Omega_{(\text{special})} = \sqrt{\Theta}$ is used \cite{8}.

4 \hspace{1cm} Non-stationary quantum systems

In a way explained in the influential 2007 letter \cite{12} the possibilities of an innovative, time-dependent choice of the Hilbert-space mapping in ansatz (5) are rather restricted. The reasons may be found discussed in review paper \cite{8} where it has been proved that in an extended, non-stationary theory with time-dependent metric one has the choice between the loss of the unitarity of the evolution and the loss of the observability of the Hamiltonian operator. In our more or less immediate reaction \cite{28} to the latter no-go theorem we pointed out that the puzzle is purely terminological. The observability of the generator of the evolution of the state-ket-vectors in Schrödinger Eq. (6) is not necessary \cite{18, 23}. In the non-stationary non-Hermitian cases it is possible to admit that the generator $G(t)$ differs from the observable Hamiltonian $H_{(\text{Dyson})}(t)$ of Eq. (7) by a new, Coriolis-force component $\Sigma(t)$ of a purely kinematical origin (to be defined by Eq. (19) in the next paragraph). Thus, one can speak about a non-Hermitian interaction picture (NIP).

4.1 \hspace{1cm} Schrödinger equation for ket vectors

After one accepts the non-stationary quantum-evolution scenario, the theory becomes complicated. Multiple new open questions emerge. They concern not only the NIP theory itself (e.g., the perception of the adiabatic approximation \cite{29, 30} or of the so called geometric phases \cite{21, 22}) but also its practical applications, say, in the analysis of the observability-breaking processes \cite{17, 31} or in an efficient elimination of the time dependence from Schrödinger equations \cite{16, 32}.

In a brief reminder of the time-dependent version of quantum theory in its generalized, non-Hermitian and non-stationary triple-Hilbert-space formulation of Refs. \cite{14, 15} let us abbreviate or drop the subscripts and superscripts. The presentation of the theory may then start from a non-stationary version of ansatz (5),

$$|\psi(t)\rangle = \Omega(t)|\psi(t)\rangle \in \mathcal{H}^{(T)} , \quad |\psi(t)\rangle \in \mathcal{H}^{(F)}.$$  \hspace{1cm} (15)
The insertion in the original Schrödinger Eq. (2) does not lead to Eq. (6) but to its non-stationary update
\begin{equation}
\frac{i}{\partial t} |\psi(t)\rangle = G(t) |\psi(t)\rangle
\end{equation}
to be solved in $\mathcal{H}^{(F)}$, with the “unobservable Hamiltonian”\begin{equation}
G(t) = H(t) - \Sigma(t)
\end{equation}
containing the “observable Hamiltonian”\begin{equation}
H(t) = \Omega^{(-1)}(t)\hbar_{(sp)}(t)\Omega(t)
\end{equation}
and the “Coriolis-force Hamiltonian”\begin{equation}
\Sigma(t) = i\Omega^{-1}(t)\dot{\Omega}(t), \quad \dot{\Omega}(t) = \frac{d}{dt} \Omega(t).
\end{equation}
Operator $H(t)$ remains defined by Eq. (7). It keeps the quasi-Hermitian form,\begin{equation}
H^\dagger(t)\Theta(t) = \Theta(t)H(t), \quad \Theta(t) = \Omega^\dagger(t)\Omega(t),
\end{equation}
i.e., the status of an observable, viz., of an instantaneous energy.

### 4.2 Schrödinger equation for bra vectors

Using the same notation convention as above and abbreviating $\Theta(t)|\psi(t)\rangle \equiv |\psi_\Theta(t)\rangle$ we replace the ansatz of Eq. (15) by its dual-space alternative
\begin{equation}
|\psi(t)\rangle = [\Omega^\dagger(t)]^{-1} |\psi_\Theta(t)\rangle \in \mathcal{H}^{(T)}, \quad |\psi(t)\rangle \equiv \Theta(t)|\psi(t)\rangle \in \mathcal{H}^{(F)}.
\end{equation}
This leads to the complementary Schrödinger equation in $\mathcal{H}^{(F)}$,
\begin{equation}
\frac{i}{\partial t} |\psi_\Theta(t)\rangle = G^\dagger(t) |\psi_\Theta(t)\rangle
\end{equation}
(see [14]; a misprint is to be removed in [15]).

Whenever we decide to study the evolution of pure states, our pair of Schrödinger equations must be complemented by the choice of initial values of $|\psi(t)\rangle$ and $|\psi_\Theta(t)\rangle$, say, at $t = t_i = 0$. This choice has the well known physical meaning reflecting the experimental preparation of the quantum system in question. In the non-Hermitian dyadic notation the pure states of system $S$ may and should be treated as represented by the elementary projectors
\begin{equation}
\pi_{\psi,\Theta}(t) = |\psi(t)\rangle \frac{1}{\langle \psi_\Theta(t)|\psi(t)\rangle} \langle \psi_\Theta(t)|.
\end{equation}
From here, one could also very quickly move to the non-Hermitian statistical quantum mechanics where one prepares and works with the statistical mixtures of states characterized, conveniently, by the non-Hermitian density matrices of the form

$$\hat{\rho}(t) = \sum_k |\psi^{(k)}(t)\rangle p_k \langle \psi^{(k)}(t)| \Theta(t) \langle \psi^{(k)}(t)|,$$

(24)

For the time-independent preparation probabilities \(p_k \neq p_k(t)\) one would then simply get, as an immediate consequence of Eqs. (16) and (22), the evolution equation

$$i \partial_t \hat{\rho}(t) = G(t) \hat{\rho}(t) - \hat{\rho}(t) G(t),$$

(25)
i.e., the non-Hermitian version of the Liouvillean evolution picture.

### 4.3 Heisenberg equations for observables

Operator \(Q(t)\) of any observable must be compatible with the requirement

$$Q(t) \Theta(t) = \Theta(t) Q(t)$$

(26)

which guarantees its NIP observability inherited from the Hermiticity of \(q(t)\). Formula (26) represents just a transfer of the self-adjointness property \(q(t) = q(t)^\dagger\) from \(H(T)\) via the time-dependent generalization of the stationary definition by Eq. (10),

$$Q(t) = \Omega(-1)(t) q_{SP}(t) \Omega(t).$$

(27)

The differentiation of Eq. (27) with respect to time (denoted by overdot) now yields the Heisenberg-type evolution equation

$$i \frac{\partial}{\partial t} Q(t) = Q(t) \Sigma(t) - \Sigma(t) Q(t) + K(t), \quad K(t) = \Omega(-1)(t)i q_{SP}(t) \Omega(t).$$

(28)

We have two possibilities. In the general, non-stationary scenario characterized by the nonvanishing time-derivative \(q_{SP}(\dot{})(t)\) the evaluation of \(K(t)\) (i.e., of the necessary input information) would require the full knowledge of the nonstationary Dyson maps \(\Omega(t)\). Naturally, Heisenberg Eq. (28) itself would be then redundant because also the very operators \(Q(t)\) would be obtainable, by the same mapping, from their partners \(q_{SP}(t)\) (which, by themselves, could be evaluated from their derivatives, by ordinary integration).

In such a scenario one would not have any reason for leaving the \(T\)-space. Let us, therefore, restrict our attention to the more common, stationary models in which the partial derivatives \(q_{SP}(\dot{})(t)\) vanish. This will make the recipe of Eq. (28) sensible because the related operator \(K(t)\) would be vanishing as well. Alternatively, we may also admit the models in which a non-vanishing operator \(K(t)\) would be prescribed in advance. In both of these situations the solution of Heisenberg Eq. (28) would make sense.
5 The physics of unitary evolution

5.1 Broader context: Haag’s theorem

Schematically, the results of our preceding considerations may be interpreted as certain preparatory steps towards a general non-Hermitian interaction-picture recipe. In it one may relax all of the ad hoc simplification assumptions, having to work just with the three separate, independent unitary-evolution generators $G(t)$ (for kets), $G^\dagger(t)$ (for their duals in $\mathcal{H}^{(\text{standard})}$) and $\Sigma(t)$ (Coriolis, for the operators of observables).

![Diagram of Hilbert spaces and their respective roles in the general non-stationary case.]

The situation reflecting the general non-stationary quantum dynamics is summarized in Fig. 2. We may find its conventional, single-Hilbert-space quantum theory predecessor in the textbooks. Indeed, once we choose $G(t) = H(t)$ (i.e., once we have $\Sigma(t) = 0$ and the trivially evolving operators) we may speak about the non-Hermitian version of Schrödinger picture. Similarly, the alternative special choice of $G(t) = 0$ (yielding $\Sigma(t) = H(t)$ and the trivially evolving bras and kets) may be perceived as determining the non-Hermitian version of the Heisenberg picture (cf. Ref. [18]).

Obviously, the general formalism finds its textbook analogue in the so-called interaction picture. A brief comment may prove useful in this context. It is well known that the applicability of the interaction picture is severely restricted, in multiple quantum field theories, by the famous Haag’s theorem [2]. The theorem states, in essence, that due
to the complicated mathematical nature of these field theories, the above-mentioned one-to-one correspondence between the two physical Hilbert spaces of Fig. 2 may cease to exist, making the approach deeply mathematically inconsistent. In our present language this would mean that the Hilbert spaces $\mathcal{H}^{(\text{textbook})}$ and $\mathcal{H}^{(\text{standard})}$ would happen to be unitarily inequivalent. Fortunately, it is usually forgotten that the no-go nature of the Haag’s theorem does not extend to the ordinary quantum mechanics, i.e., to the theory which is considered in our present paper.

5.2 Initial values

The solution of the triplet of the NIP evolution Eqs. (16), (22) and (28) can only lead to the meaningful phenomenological predictions under the correct specification of the initial values. Thus, the two kets $|\psi(t_i)\rangle$ and $|\psi_\Theta(t_i)\rangle$ and the initial value of the observable of interest (say, $Q(t_i)$) must be given in advance. The problem of the specification of the initial operator value $Q(t_i)$ is subtle because we must require that this operator also satisfies the physical observability constraint (26) at the initial time $t = t_i = 0$.

From the point of view of mathematics the choice of the initial values is arbitrary. In contrast, this choice carries the most relevant information about physics. Thus, the non-Hermitian nature of the NIP representation makes the guarantee of the consistency of the initial values most important (cf., e.g., the numerical experiments with the changes of initial values in [29]).

5.3 The measured quantities

The most straightforward use of the NIP recipe would be based on the assumption that we know both the generators $G(t)$ and $\Sigma(t)$. The two Schrödinger-type equations may be then solved to define the two state vectors $|\psi(t)\rangle$ and $|\psi_\Theta(t)\rangle$ at all times $t > t_i = 0$. Similarly, the knowledge of $\Sigma(t)$ enables us to recall Heisenberg Eq. (28) and to reconstruct operators $Q(t)$ up to the time of measurement $t = t_f > t_i = 0$. In this manner we can evaluate, in principle at least, the ultimate experimental prediction

$$\langle \psi_\Theta(t_f)|Q(t_f)|\psi(t_f)\rangle.$$ (29)

In contrast to the stationary case of Eq. (11) we may notice that the explicit knowledge of the operator of metric becomes, within such a triple-equation recipe, redundant. Its role is played by the vector $|\psi_\Theta(t_f)\rangle$. The apparent paradox is easily explained via the trivial mathematical identity

$$i \frac{\partial}{\partial t} \Theta(t) = \Theta(t)\Sigma(t) - \Sigma^\dagger(t)\Theta(t).$$ (30)
This shows that the knowledge of the metric is formally equivalent to the knowledge of the Coriolis-force generator $\Sigma(t)$. It is worth adding that due to Eq. (20), the latter mathematical identity is equivalent to its alternative version

$$i \frac{\partial}{\partial t} \Theta(t) = G^\dagger(t)\Theta(t) - \Theta(t)G(t).$$

By several authors [21, 23, 29], the latter differential equation for operators was picked up as a tool of the construction of the metric. Naturally, such a highly non-economical recipe proved sufficiently efficient for the purposes of the study of certain next-to-trivial illustrative non-Hermitian non-stationary two-by-two matrix models in loc. cit.

6 Special cases: Pre-selected generators $G(t)$

Quantum models using any form of the “kinematical” input operator $G(t)$ will combine an enhanced flexibility with technical complications. At least some of its applications may prove both interesting and feasible. For example, in the light of Theorem 2 of review paper [8] and of the related comments in [16, 23], the non-Hermitian interaction picture seems to be the only formulation of quantum mechanics of unitary systems in which the non-Hermitian versions of Schrödinger equations would be allowed to contain time-dependent, albeit non-observable, Hamiltonian-like generators $G(t)$ and $G^\dagger(t)$.

6.1 Non-Hermitian Heisenberg picture ($G = 0$)

In Heisenberg picture (HP, cf. Ref. [18]) the state-vectors are, by definition, stationary. Their generator is trivial, $G_{(HP)}(t) \equiv 0$. Due to Eq. (31), the HP metric operator cannot be non-stationary, therefore. Vice versa, any description of dynamics during which the metric would be changing requires a replacement of the oversimplified HP formalism by its suitable (i.e., perhaps, perturbatively tractable) NIP generalization.

In the non-Hermitian Heisenberg picture of Ref. [18] with vanishing $G = 0$ both the bra and ket state vectors are constant. This indicates that the isolated choice of any other generator $G(t)$ has a more or less purely kinematical character and need not carry any information about the dynamics. Basically, the “missing” information about the dynamics will enter the picture via the exhaustive description of the quantum system $S$ at the initial instant $t = t_i$.

In the special non-Hermitian HP case the problem is slightly simplified. From Eq. (17) we may deduce that

$$\Sigma_{(HP)}(t) = H(t)$$
i.e., that the second generator becomes observable. Thus, we are free to accept any form of such an observable Hamiltonian $H(t)$ and/or any form of the other observable $Q(t)$ as an independent, additional input information about the dynamics of the quantum system in question.

In the non-Hermitian Heisenberg picture both of the NIP Schrödinger equations drop out and only the solution of the Heisenberg Eqs. (28) is asked for. Possibly, differential Eq. (19) defines mapping $\Omega = \Omega(t)$ from its pre-selected initial value at $t = t_i = 0$. Such a constructive version of the non-Hermitian Heisenberg picture already has its physical applications [20]. In the near future the non-Hermitian HP and NIP formalisms might also prove needed and applied in quantum cosmology [8, 31].

6.2 Extended non-Hermitian Heisenberg picture ($G \neq G(t)$)

The most straightforward generalization of the non-Hermitian HP recipe may be based on the use of a constant nontrivial generator of the evolution of wave functions. Such an “extended Heisenberg picture” (EHP) with $G_{(EHP)}(t) = G_{(EHP)}(0) \neq 0$ has certain unexpected mathematical simplicity features (cf. Eq. Nr. 21 in [16]). It proved also particularly suitable, in the words of the Abstract of Ref. [16], for the description of the “evolution of the manifestly time-dependent self-adjoint quantum Hamiltonians $h(t)$”.

Recently, both of these EHP applicability features were rediscovered and illustrated, via Rabi-type Hamiltonian $h_{(SP)}(t)$, in rapid communication [32]. The observation also renewed the interest in the the systems (first discussed by Bíla [29]) for which the operator $G(t)$ is given in advance. In the present setting, the key technical advantage of these quantum systems is represented by the above-mentioned replacement of the construction of the metric via Eqs. (30) or (31) by the mere solution of Schrödinger Eq. (22).

7 General non-Hermitian interaction picture

7.1 Reconstruction of the basis

One of the immediate consequences of the hypothetical knowledge of the non-observable Hamiltonian $G(t)$ is that any change of the initial ket-vector may be reflected by a new solution of Schrödinger Eq. (16) and, mutatis mutandis, any change of the initial bra-vector may be used as an initial value for the parallel repeated solution of our second Schrödinger Eq. (22). Once we decide to repeat the process $N$-times, the knowledge of initial $N$-plets

$$|\psi_1(0)\rangle, |\psi_2(0)\rangle, \ldots, |\psi_N(0)\rangle$$

(32)
and

\[ |\psi_1(0)\rangle, |\psi_2(0)\rangle, \ldots, |\psi_N(0)\rangle \]  

will generate, in principle as well as in practice, the related respective \(N\)-plets of state vectors

\[ |\psi_1(t)\rangle, |\psi_2(t)\rangle, \ldots, |\psi_N(t)\rangle \]

and

\[ |\psi_1(\Theta(t))\rangle, |\psi_2(\Theta(t))\rangle, \ldots, |\psi_N(\Theta(t))\rangle \]

at all times \(t\).

One should now contemplate an arbitrary given physical quantum system \(S\) which is prepared (i.e., known) at some initial time (say, \(t = 0\)) and which is to be analyzed in its NIP representation, i.e., using a pre-selected operator \(G(t)\). The preparation of this system in several (i.e., \(N\)) independent pure states is then equivalent to our knowledge of the respective initial pairs of vectors \(|\psi_j(t)\rangle\) and \(|\psi_j(\Theta(t))\rangle\). In an extreme case we may assume that the initial \(N\)-plet (with \(N \leq \infty\) in general) is, at \(t = 0\), bi-orthonormal,

\[ \langle \psi_m(\Theta(0)) | \psi_n(0) \rangle = \delta_{m,n}, \quad m, n = 1, 2, \ldots, N \]  

and complete,

\[ \sum_{n=1}^{N} |\psi_n(0)\rangle \langle \psi_n(\Theta(0))| = I. \]  

The assumption of our knowledge of \(G(t)\) at all \(t > 0\) then enables us to construct the pairs \(|\psi_j(t)\rangle\) and \(|\psi_j(\Theta(t))\rangle\) such that

\[ \sum_{n=1}^{N} |\psi_n(t)\rangle \langle \psi_n(\Theta(t))| = I, \quad \langle \psi_1(\Theta(t)) | \psi_n(t) \rangle = \delta_{m,n}, \quad m, n = 1, 2, \ldots, N. \]  

Their respective completeness and bi-orthonormality survive at all times.

### 7.2 Reconstruction of the metric

The hypothetical NIP representability of a given quantum system \(S\) must reflect the existence of its (by assumption, overcomplicated and technically inaccessible but existing) Hermitian, textbook representation using the self-adjoint operators of observables in \(\mathcal{H}^{(T)}\). The system \(S\) must have its non-Hermitian SP representation at \(t = 0\). The \(N\)-plets (32) and (33) of pure states may be expected to be prepared, at \(t = 0\), as the respective left and right eigenstates of a relevant observable. Typically, we work with a Hamiltonian and obtain its spectral representation, therefore,

\[ H(t) = \sum_{n=1}^{N} |\psi_n(t)\rangle E_n(t) \langle \psi_n(\Theta(t))| . \]
Such an instantaneous energy operator is, by definition, related to its isospectral Hamiltonian partner $\mathbf{h}$ which is self-adjoint in $\mathcal{H}^{(T)}$. The same parallelism will also apply to any other observable, i.e., in our notation, to any operator $Q(t)$.

In the light of paragraph 4.3 we will consider the “tractable” scenarios in which the SP energies themselves are conserved, i.e., $E_n(t) = E_n(0) = E_n$. The key benefit of this assumption is that we may now recall Ref. [33] and get an important mathematical result, viz., the solution of the time-dependent quasi-Hermiticity relation (31) at all times.

**Theorem 1** For a given generator $G(t)$ and for the two initial vector sets (32) and (33) with properties (34) and (35), the metric operator $\Theta(t)$ has the following formal representation in $\mathcal{H}^{(F)}$,

$$\Theta(t) = \sum_{n=1}^{N} |\psi_{n,\Theta}(t)\rangle \langle \psi_{n,\Theta}(t)|.$$ (38)

**Proof.** Along the lines outlined in Ref. [33], formula (38) follows from completeness (36) and from the definition of the physical bra vectors $\langle \psi_{n,\Theta}(t)|$ in $\mathcal{H}^{(S)}$ (the superscript abbreviates the “standard” physical Hilbert space, unitarily equivalent to $\mathcal{H}^{(T)}$). The representation is formal because in some pathological cases with $N = \infty$ its right-hand-side series need not converge. A less formal extension of validity of the theory to these cases would require a more rigorous mathematical specification of the properties of the biorthonormal bases [34].

We see that via the repeated solution of Schrödinger equations one can obtain the spectral-like formula for the metric “for free”. Formula (38) indicates that the initial choice of the biorthonormal basis determines the unique initial-value choice of the metric $\Theta(t)$ with $t_i = 0$ as its byproduct. *Vice versa*, by definition, the initial choice of $\Theta(0)$ would enable us to generate the set of initial bras (33) from a pre-selected set (32) of initial kets. The condition of the mutual compatibility (20) of $\mathcal{H}(0)$ with $\Theta(0)$ of Eqs. (37) and (38) follows from properties (36). The important remaining freedom is that the energies $E_n$ in the standard spectral representation (37) of the observable Hamiltonian are free parameters of the theory. Their numerical values remain unrestricted in the NIP theoretical framework, therefore.

### 7.3 Reconstruction of the Dyson mappings $\Omega(t)$

The operator of metric (38) must be self-adjoint and strictly positive in $\mathcal{H}^{(F)}$ so that it can be diagonalized, under suitable mathematical conditions, via a unitary operator $U(t)$,

$$\Theta(t) = U(t)\theta^2(t)U(t)^\dagger.$$ (39)
The matrix $\theta^2(t)$ of the real and positive eigenvalues of $\Theta(t)$ and its square root $\theta(t)$ are both diagonal, real, finite and non-vanishing. The diagonalization of the metric is purely numerical. In practice we either work with finite-dimensional Hilbert spaces, or we truncate them to a finite dimension $N < \infty$.

Up to another, independent unitary-matrix ambiguity $V(t)$, we may factorize the metric into the product of the entirely general class of non-stationary Dyson’s mappings,

$$\Omega(t) = V^†(t)\theta(t)U(t).$$

At this moment we are already able to construct the Coriolis force $\Sigma(t)$ so that we may construct a new, tilded-operator sum $\tilde{H}(t) = G(t) + \Sigma(t)$. It is instructive to verify that the latter operator is equal to the old untilded unobservable Hamiltonian of Eq. (37). One reveals that the demonstration of the identity $\tilde{H}(t) = H(t)$ is easy due to the validity of Theorem I.

8 Application in relativistic quantum mechanics

The authors of the textbooks on quantum theory are very well aware that a consequent incorporation of all of the effects of the relativistic kinematics requires a transition from quantum mechanics to the quantum field theories in which the number of particles is not conserved. The relativistic versions of the models in quantum mechanics are considered as certain sufficiently satisfactory approximations of the physical reality. The choice of the Dirac equation describing fermions is preferred. The reasons range from the applicability of Pauli principle (explaining antifermions as holes in the Dirac’s sea) up to an agreement of the theory with experiments, say, for hydrogen atom [35].

8.1 Stationary Klein-Gordon equation

Within the framework of the relativistic quantum mechanics the widespread preference of the study of the Dirac equation is, certainly, a paradox because its bosonic Klein-Gordon (KG) alternative is less mathematically complicated. In one of its simplest versions written in units $\hbar = c = 1$,

$$\left(\frac{\partial^2}{\partial t^2} + D\right)\psi^{(KG)}(\vec{x}, t) = 0, \quad D = -\Delta + m^2$$

one does not consider any external electromagnetic field so that the kinetic energy is represented by the elementary Laplacean $\Delta$. The dynamics is also reduced to a scalar external field simulated by a suitable position-dependence of the mass term, $m^2 = m^2(\vec{x})$. 

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The most common physical interpretation of Eq. (41) was proposed by Feshbach and Villars (FV, [36]). They changed the variables

$$\psi^{(KG)}(\vec{x}, t) \rightarrow \langle \vec{x} | \psi^{(FV)}(t) \rangle = \left( \begin{array}{c} i \partial_t \psi^{(KG)}(\vec{x}, t) \\ \psi^{(KG)}(\vec{x}, t) \end{array} \right).$$  (42)

(cf. also Ref. [37]) and they replaced the hyperbolic partial differential Eq. (41) by its parabolic partial differential mathematical equivalent

$$i \frac{\partial}{\partial t} |\psi^{(FV)}(t)\rangle = H^{(FV)} |\psi^{(FV)}(t)\rangle.$$  (43)

This is an evolution equation with stationary generator

$$H^{(FV)} = \begin{pmatrix} 0 & D \\ I & 0 \end{pmatrix} \neq H^{(FV)}(t).$$  (44)

In the most common Hilbert space of states $\mathcal{H}^{(FV)} = \mathcal{L}^2(\mathbb{R}^3) \bigoplus \mathcal{L}^2(\mathbb{R}^3)$ such a generator of the evolution of the FV wave functions cannot be interpreted as standard Hamiltonian because it is manifestly non-Hermitian, $H^{(FV)} \neq H^{(FV)}\dagger$.

A partial resolution of the puzzle was suggested by Pauli and Weisskopf [38]. They noticed that, translated to the modern language, the non-Hermitian operator $H^{(FV)}$ is $\mathcal{P}\mathcal{T}$—symmetric alias self-adjoint in an ad hoc Krein space. This means that this operator is tractable as self-adjoint with respect to a new, auxiliary, indefinite inner product,

$$\langle \psi_1 | \psi_2 \rangle \rightarrow \langle \psi_1, \psi_2 \rangle^{(Krein)} = \langle \psi_1 | \mathcal{P}^{(FV)} | \psi_2 \rangle.$$  (45)

On this background, after the standard introduction of an external electromagnetic field, the model can be perceived as a charge-conserving evolution of the physical KG field in which the number of particles is not conserved (for details see, e.g., chapter XII of [39]).

The alternative, more natural Mostafazadeh’s proposal (cf. [40, 41]) was based on the stationarity property (44). The idea consisted in the further change of the inner product,

$$\langle \psi_1, \psi_2 \rangle^{(Krein)} \rightarrow \langle \psi_1, \psi_2 \rangle^{(Mostafazadeh)} = \langle \psi_1 | \Theta^{(stationary)} | \psi_2 \rangle.$$  (46)

The physics-restoring operators $\Theta^{(stationary)} = \Theta^{(stationary)}\dagger$ were required positive definite (cf. also [6] and [8] in this context). The initial, manifestly unphysical Hilbert space $\mathcal{H}^{(FV)}$ (with inner products $\langle \psi_1 | \psi_2 \rangle$) was replaced by the Mostafazadeh’s Hilbert space $\mathcal{H}^{(M)}$ which differed from $\mathcal{H}^{(FV)}$ by the correct and physical inner product (46) defined in terms of the so called metric operator $\Theta^{(stationary)}$ [42]. Eq. (44) was reinterpreted as Hamiltonian living in physical Hilbert space $\mathcal{H}^{(M)}$ with inner product (46).
In opposite direction, any Hamiltonian \( H^*_F \) with property
\[
H^*_F V^\dagger \Theta_{\text{stationary}} = \Theta_{\text{stationary}} H_F
\] (47)
may be perceived as self-adjoint with respect to the ad hoc product (46). In the Klein-Gordon case, in particular, relation (47) may be satisfied by the Hilbert-space metrics of the closed block-matrix form
\[
\Theta_{\text{stationary}} = \left( \begin{array}{cc} 1/\sqrt{D} & 0 \\ 0 & \sqrt{D} \end{array} \right).
\] (48)
Formulae (46) and (48) yield the first-quantized interpretation of the Klein-Gordon Eq. (41) in which the probability density of finding the relativistic spinless massive particle is never negative.

8.2 Non-stationary scenario

From the point of view of relativistic kinematics the stationary position-dependence \( m^2 = m^2(\vec{x}) \) of the mass term violates the Lorentz covariance of the interaction. A privileged coordinate frame must be used so that a weakening \( m^2 = m^2(\vec{x}, t) \) of the assumption would be highly desirable.

8.2.1 The NIP generator \( G(t) \) of KG ket-vectors

The acceptance of the hypothesis \( m^2 = m^2(\vec{x}, t) \) leads to the replacement of Eq. (41) by its non-stationary form
\[
\left( \frac{\partial^2}{\partial t^2} + D(t) \right) \psi^{(KG)}(\vec{x}, t) = 0, \quad D(t) = -\Delta + m^2(\vec{x}, t).
\] (49)
The same change of variables as above defines the NIP ket-vector wave functions
\[
\langle \vec{x} | \psi^{(NIP)}(t) \rangle = \left( i \frac{\partial}{\partial t} \psi^{(KG)}(\vec{x}, t) \right). \] (50)
Its use converts Eq. (49) into the KG realization of the non-stationary Schrödinger Eq. (16),
\[
i \frac{\partial}{\partial t} |\psi^{(NIP)}(t)\rangle = \left( \begin{array}{cc} 0 & D(t) \\ I & 0 \end{array} \right) |\psi^{(NIP)}(t)\rangle
\] (51)
This equation contains the KG generator \( G_{(NIP)}(t) \). Its spectrum may, but need not, be real [15]. In the current literature it is called “generator” [14, 15], “Hamiltonian” [21, 29], “nonobservable Hamiltonian” [23] or “unobservable Hamiltonian” [32].
8.2.2 The NIP generator $G^i(t)$ of KG bra-vectors

Bíla’s preprint [29] deserves to be recalled as one of the first studies of the methodical scenario in which one starts the analysis of quantum dynamics from the knowledge of $G(t)$. Using an elementary toy-model generator $G(t)$ the author solved Eq. (51) and revealed that the qualitative features of the metric $\Theta(t)$ depend rather strongly on the choice of this metric operator at the initial time $t = t_i$. Later, similar constructive case studies were published in Refs. [16, 21, 23].

The weak point of the Bíla’s preprint lies in the complicated operator form of equation (31). This key to evolution was later called “time-dependent quasi-Hermiticity relation” [23]. In paragraph 4.2 above we recommended a more economical approach. We felt inspired by our recent studies [16, 18] in which we analyzed the consequences of the choice of a vanishing or constant $G(t)$, respectively. The role of the initial conditions proved more important than expected while the role of the generator itself appeared to be merely technical. Moreover, the use of the non-stationary metric $\Theta(t)$ was replaced by the solutions of the second Schrödinger equation.

We came to the conclusion that one may skip the solution of Eq. (31). In the implementation of the NIP recipe (based on the knowledge of metric) one is strongly recommended to replace the multiplicative definition of

$$\left| \psi_{\Theta}^{(\text{NIP})} \right\rangle = \Theta(t) \left| \psi^{(\text{NIP})} \right\rangle$$

by the KG form of the second Schrödinger Eq. (22),

$$\frac{i}{\hbar} \frac{\partial}{\partial t} \left| \psi_{\Theta}^{(\text{NIP})}(t) \right\rangle = \begin{pmatrix} 0 & I \\ D^*(t) & 0 \end{pmatrix} \left| \psi_{\Theta}^{(\text{NIP})}(t) \right\rangle .$$

(53)

It is worth noticing that one could even admit here the complex effective mass terms $m^2(\vec{x}, t) \notin \mathbb{R}$.

8.2.3 The Coriolis NIP generator $\Sigma(t)$

The replacement of the construction of operators by the construction of vectors will simplify the calculations. This will also render them feasible far beyond the most popular two-by-two matrix toy models. One can predict that in the nearest future such a form of the NIP formalism might also find multiple other applications, most of which could closely parallel the existing uses of Hermitian interaction picture. One can expect that in a way guided by the parallel an enhanced attention will be paid to the models with the dynamics dominated by the HP-resembling non-Hermitian generator $\Sigma(t)$. Naturally, after one pre-selects $G(t)$, the choice of $\Sigma(t)$ ceases to be unconstrained, mainly due to the deeply physical role played
by the experiment-related preparation of the initial-state vectors \(^{(32)}\) and \(^{(33)}\) and of the related initial-instant energy \(H^{(NIP)}(t_i)\) as prescribed by Eq. \((37)\). Moreover, we know that the evolution of the latter observable proceeds via the Heisenberg-type Eq. \((28)\), i.e.,

\[
i \frac{\partial}{\partial t} H^{(NIP)}(t) = H^{(NIP)}(t) \Sigma^{(NIP)}(t) - \Sigma^{(NIP)}(t) H^{(NIP)}(t) + K^{(NIP)}(t)
\]

or, equivalently,

\[
i \frac{\partial}{\partial t} H^{(NIP)}(t) = G^{(NIP)}(t) H^{(NIP)}(t) - H^{(NIP)}(t) G^{(NIP)}(t) + K^{(NIP)}(t)
\]

with some vanishing or pre-determined SP-variability term

\[K^{(NIP)}(t) = \Omega^{(-1)}(t) i \hbar^{(SP)}(t) \Omega(t).\]

As long as \(\Sigma^{(NIP)}(t) = H^{(NIP)}(t) - G^{(NIP)}(t)\) there remains no freedom in the choice of this operator. Subsequently, as long as we have, from definition,

\[
i \frac{\partial}{\partial t} \Omega^{(NIP)}(t) = \Omega^{(NIP)}(t) \Sigma^{(NIP)}(t), \tag{56}
\]

the only ambiguity of \(\Omega^{(NIP)}(t)\) is contained in its initial-value specification.

9 Discussion

9.1 Open problems

In the non-stationary NIP formalism our intuition need not work and our expectations may prove wrong. For this reason it is extremely fortunate that one can simulate many unusual features of quantum systems \(S\) using classical optics \([43]\). Moreover, the results of the experimental tests and classical optical simulations may also find an alternative, manifestly non-unitary-evolution interpretations in the effective-operator descriptions of open quantum systems \([44]\).

In all of these dynamical regimes one of the most interesting open questions is the problem of the domain of the survival of validity of the usual adiabatic hypothesis. This hypothesis may be violated, and violated in an unexpected manner \([30]\). Thus, even in the domain of the safely unitary quantum evolutions our understanding of the limits of the validity of the adiabatic approximation seems to be far from complete. The obstacles encountered in such a context were summarized in preprint \([29]\). They may be separated into three subcategories. In the first one one deals with the problems of incompleteness of the information about dynamics \([45]\). Even if we start from a trivial \(G(t)\), the whole
NIP formalism degenerates to the non-Hermitian Heisenberg picture \[18, 20\]. This implies that the answers to all of the questions about stability and/or instability remain strongly dependent on the number of additional assumptions about the dynamics.

In the second subcategory of problems one encounters an *ambiguity* related to the choice of the representation of the correct physical Hilbert space \(\mathcal{H}^{(S)}\). This form of flexibility already represented a serious methodical challenge in the traditional stationary models \[6\]. In the present non-stationary context it reflects the freedom of preparation of the system \(S\) in a pure state at \(t = 0\). It was again Bíla \[29\] who addressed this problem in 2009. He illustrated the relevance of the initial conditions via a two-by-two-matrix toy model. The resulting demonstration of the existence of deeply different evolution scenarios was sampled in Fig. Nr. 1 of *loc. cit.*. Recently, a similar approach and analogous analysis of another two-by-two matrix model were published in Ref. \[23\].

The third group of the currently open questions concerns the *economy* of the alternative construction strategies. In this sense we offered here a new approach, circumventing partially the main weakness of the above-cited constructions which were all based on the solution of the operator evolution Eq. (31). Even in its truncated, \(N\)–dimensional Hilbert space approximation this equation is a rather complicated coupled set of ordinary differential equations for as many as \(N^2\) unknown matrix elements of \(\Theta(t)\). Although we managed to weaken the difficulty, we had to postpone the practical numerical tests of the economy of the present approach to a separate study. After all, the difficulty of such a project is underlined by the freshmost study \[23\] in which the solution of the operator evolution Eq. (31) (without a help by an ansatz for \(\Omega(t)\)) was only performed at \(N = 2\).

Table 1: Sample of notation conventions used in the literature.

| symbol | meaning | Ref. \[8\] | Ref. \[23\] | Ref. \[6\] |
|--------|---------|------------|------------|------------|
| \(\Omega\) | Dyson’s map, see Eq. (15) | \(\rho\) | \(\eta\) | \(S\) |
| \(\Theta\) | \(= \Omega^\dagger \Omega\), the metric | \(\eta_+\) | \(\rho\) | \(\tilde{T}\) |
| \(\ket{\psi}\) | state vector, Eq. (16) | \(\ket{\psi}\) | \(\Psi\) | \(\ket{\Psi}\) |
| \(G\) | the generator of kets, Eq. (17) | – | \(H\) | – |
| \(\hbar\) | textbook Hamiltonian, Eq. (18) | \(\hbar\) | \(\hbar\) | – |
| \(\Sigma\) | Coriolis Hamiltonian, Eq. (19) | – | unabbr. | – |
| \(H\) | observable Hamiltonian, Eq. (20) | \(H\) | \(\tilde{H}\) | \(H\) |
| \(\ket{\psi_{\Theta}}\) | dual state vector, Eq. (21) | \(\ket{\phi}\) | \(\rho \Psi\) | \(\tilde{T} \ket{\Psi}\) |
9.2 Terminology: A Rosetta stone

One of the consequences of the novelty of the NIP picture in both of its stationary and non-stationary versions is the diversity of notation (see Table 1). This is partly due to the replacement of the traditional Hamiltonian in $H^{(T)}$ by the triplet of operators acting in $H^{(F)}$. All of them participate in the description of the unitary evolution of a given system $S$ so that all of them could be called “Hamiltonians”. In [15] our present convention was developed to distinguish, without subscripts or superscripts, between the observable $H(t)$ of Eq. (18) and the two not necessarily observable generators $G(t)$ (changing the state-vector kets via Schrödinger-type Eq. (17)) and $\Sigma(t)$ (prescribing the time-dependence of every operator of an observable via the respective Heisenberg-type Eq. (28)).

In the non-stationary NIP formalism it is rather counterintuitive that the generators $G(t)$ and $\Sigma(t)$ are, in general, not observable while the observable Hamiltonian $H(t)$ itself is just their sum. Some authors decided to treat $G(t)$ as “the” Hamiltonian [23, 29, 32]. No comments are usually made on the fact that its spectrum is, in general, complex even if the evolution of system $S$ itself is unitary.

Besides the single-observable scope of predictions [29] we are often interested in the knowledge of metric for some other reasons. A few of them may be found listed in Ref. [25]. In paragraph 4.1 we also emphasized the formal equivalence between the respective “time-dependent” and “time-independent” quasi-Hermiticity relations (31) and (20) for the metric. Now we have to add that this is certainly not an equivalence from the point of view of applications. The latter relation is algebraic while the former one is differential, much more complicated to solve. For this reason, even when one needs to know the metric, its reconstruction should proceed via Eq. (20). The preference of Eq. (31) by some of the above-cited authors looks rather naive.

A slightly discouraging aspect of Eq. (20) may be seen in the fact that one has to determine both of the “unknown” operator components $H(t)$ and $\Theta(t)$ of Eq. (20) at once. A hint to the resolution of the apparent paradox was provided in paragraph 4.3 above. We argued there that the observable Hamiltonian $H(t)$ must be perceived as carrying a necessary addition to the input information about dynamics. In other words, even after an exhaustive exploitation of the knowledge of $G(t)$ the definition of quantum system $S$ remains as incomplete as in the above-mentioned Heisenberg picture where $G_{(HP)}(t) = 0$.

10 Summary

Any quantum system $S$ can be described using any formulation alias picture, in principle at least [1]. The textbooks offer a more or less standard menu of pictures, each one
of which may prove technically most appropriate for a specific $S$. **Vice versa**, the development of new pictures of unitary evolution may help to extend the class of tractable quantum systems $S$. Our paper may be read as an illustration of the latter statement. We recalled the “three-Hilbert-space” picture of Ref. [15] and we demonstrated that its NIP reformulation may offer new insights in relativistic quantum mechanics.

Our specific task of a consistent and constructive interpretation of non-stationary Klein-Gordon quantum-mechanics systems had several aspects. The most important one may be seen in having us forced to work with Schrödinger equations containing unobservable generators $G(t)$ and/or $\Sigma(t)$. This is a methodical feature of the new theory which looks counterintuitive [13]. After one admits that the Dyson’s maps can be time-dependent, the acceptance of the unobservability becomes natural, understandable and well founded [15, 23, 28].

A few equally important phenomenological challenges connected with our present first-quantization approach to the non-stationary Klein-Gordon system were listed as open problems. We provided several answers to some of them. Incidentally, we improved also the computational economy of the evaluation of the experimental predictions (29). The feasibility of their computation may be perceived as a very core of the theory. Typically, the authors of Refs. [23, 29, 32] used these prediction formulae in combination with the metric-multiplication definition (8) where one needs to know the metric. Thus, these authors had to solve the operator evolution Eq. (31). In contrast we noticed that the relevant information about the metric is all carried by the single bra-vector $\langle \psi_\Theta(t) |$. Thus, we replaced the latter recipe by the direct solution of the vector-evolution Schrödinger-type Eq. (22). In this way we eliminated the complicated, Heisenberg-equation-resembling evolution rule (31) as redundant. We explained that such an operator differential-equation rule is just a combination of identity (30) with the much simpler, more natural and fully transparent algebraic condition of observability (20).

We may summarize that the key mathematical feature of our present recipe is that it prescribes a specific time-evolution of the physical inner product in the Hilbert space of states. This leads to the standard probabilistic interpretation of the theory while opening new perspectives. In the context of physics, one of the most concise characteristics of our present approach to the non-stationary Klein-Gordon problem may be seen in its parallelism to the conventional interaction picture. The main difference from the latter formulation of quantum dynamics (and from its non-Hermitian stationary SP formulations [8]) is that we admit that the generators $G(t)$ and $\Sigma(t)$ need not be observable in general. Still, the unitarity of the underlying physical system is preserved, guaranteed by the fact that the manifestly non-unitary evolution of $|\psi^{(\text{NIP})}(t)\rangle$ (controlled by a Schrödinger-type equation) is strictly compensated by the non-unitarity of the evolution of the operators of
observables (controlled, in parallel, by non-Hermitian Heisenberg-type equations).

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