Inexact GMRES Policy Iteration
for Large-Scale Markov Decision Processes

Matilde Gargiani ∗ Dominic Liao-McPherson ∗
Andrea Zanelli ∗∗ John Lygeros ∗

∗ Automatic Control Laboratory (IfA), ETH Zürich, Switzerland
(e-mail: {gmatilde, dliaomc}@ethz.ch, jlygeros}@ethz.ch).
∗∗ Institute for Dynamic Systems and Control (IDSC), ETH, Zurich
(e-mail: zanellia@ethz.ch)

Abstract: Policy iteration enjoys a local quadratic rate of contraction, but its iterations are
computationally expensive for Markov decision processes (MDPs) with a large number of states.
In light of the connection between policy iteration and the semismooth Newton method and
taking inspiration from the inexact variants of the latter, we propose inexact policy iteration,
a new class of methods for large-scale finite MDPs with local contraction guarantees. We then
design an instance based on the deployment of GMRES for the approximate policy evaluation
step, which we call inexact GMRES policy iteration. Finally, we demonstrate the superior
practical performance of inexact GMRES policy iteration on an MDP with 10000 states, where
it achieves a ×5.8 and ×2.2 speedup with respect to policy iteration and optimistic policy
iteration, respectively.

Keywords: Optimal control; Dynamic programming; GMRES; Inexact semismooth Newton

1. INTRODUCTION

Stochastic optimal control problems arise in a variety
of applications across different fields (Bertsimas and Lo,
1998; Elamvazhuthi and Berman, 2015) and can be compactly expressed in mathematical terms via a recursive
functional equation known as the Bellman equation (Bell-
man, 1952). Dynamic programming (DP) comprises all
methods to solve the Bellman equation, such as value iter-
ation (VI), policy iteration (PI) and their variants (Bert-
sekas, 2012). Empirical evidence has shown that, among
the dynamic programming methods, PI tends to enjoy the
fastest rate of convergence. In addition, Gargiani et al.
(2022) have proved that for finite MDPs PI is an in-
case of the semismooth Newton method and therefore,
by exploiting the structural properties of the Bellman
equation, it is possible to conclude local quadratic rate
of convergence. Even though PI converges in very few
iterations, its time performance degrades rapidly with the
size of the state space. In fact, at each iteration PI requires
the exact solution of a system of linear equations with
dimension equal to the number of states. While the total
number of iterations is not dependent on the size of the
MDP, the computational complexity of the exact policy
evaluation step is strongly dependent on it, diminishing
the computational advantages of PI.

An intuitive way to improve the time complexity of PI
is to solve the system of linear equations inexactly. This
is the main idea behind optimistic policy iteration (OPI),
where the policy evaluation is solved approximately with

a finite number of VI steps (Bertsekas, 2012). Variants of
this method include, e.g., the deployment of the Gauss-
Seidel and mini-batch versions of VI (Gargiani et al.,
2021). In (Mrkaic, 2002) the author explores the practical
performance of variants of OPI where Krylov methods are
used for the approximate policy evaluation step instead
of VI. The benchmarks show significant performance im-
provements with respect to PI and OPI for finite MDPs
arising from the discretization of stochastic growth models.
Variants of OPI are also studied in (Huang et al., 2011,
2012) for financial pricing problems. Their numerical ex-
amples show that OPI-type methods are generally signif-
icantly faster in terms of CPU time compared to the full
PI scheme. Finally, Wingate and Seppi (2005) study the
performance of different OPI-type methods when used in
combination with prioritization, partitioning and reorder-
ing heuristics.

In light of the connection between policy iteration and the
semismooth Newton method and inspired by the inexact
variants of the latter, we propose inexact policy iteration,
a new class of dynamic programming methods (Section 3).
As in OPI, in inexact policy iteration methods the policy
evaluation step is carried out only approximately with an
iterative solver; however, the number of inner iterations is
not fixed a priori, but dictated by a stopping condition
which depends on the infinity-norm of the Bellman resi-
dual function. Unlike (Mrkaic, 2002), we provide a rigorous
analysis of the local contraction properties of the methods
in this class (Subsection 3.2). In Subsection 3.1 we design
an instance based on the deployment of GMRES (Saad
and Schultz, 1986), which we call inexact GMRES pol-
icy iteration (iGMRES-PI), and we also give theoretical
and empirical insights on the advantages of GMRES with

∗ This work was supported by the European Research Council under
the Horizon 2020 Advanced under Grant 787845 (OCAL).
respect to VI for the approximate solution of the policy evaluation step. Finally, in Section 4 we demonstrate the performance superiority of iGMRES-PI on a large-scale MDP with 10000 states versus the only 500 states MDP used in (Mrkaic, 2002). Section 2 is dedicated to the description of the problem setting and the necessary background material.

2. PROBLEM SETTING & BACKGROUND

We consider infinite horizon discounted cost problems for MDPs \( \{S, A, P, g, \gamma\} \) comprising a finite state space \( S = \{1, \ldots, n\} \), a finite action space \( A = \{1, \ldots, m\} \), a transition probability function \( P : S \times A \times S \to [0, 1] \) that defines the probability of ending in state \( s' \) when applying action \( a \) in state \( s \), a stage-cost function \( g : S \times A \to \mathbb{R} \) that associates to each state-action pair a bounded cost, and a discount factor \( \gamma \in (0, 1) \). Throughout the paper, with a slight abuse of notation we use \( A(s) \) to denote the nonempty subset of actions that are allowed at state \( s \), \( p_{ss'}(a) = P(s, a, s') \) for the probability of transitioning to state \( s' \) when the system is in state \( s \) and action \( a \) is selected with \( \sum_{s' \in S} p_{ss'}(a) = 1 \) for all \( s \in S \) and \( a \in A(s) \).

A deterministic stationary control policy \( \pi : S \to A \) is a function that maps states to actions, with \( \pi(s) \in A(s) \). We use \( \Pi \) to denote the set of all deterministic stationary control policies, from now on simply policies. At step \( t \) of the decision process under the policy \( \pi \in \Pi \), the system is in some state \( s_t \) and the action \( a_t = \pi(s_t) \) is applied. The discounted cost \( \gamma g(s_t, a_t) \) is accrued and the system transitions to a state \( s_{t+1} \) according to the probability distribution \( P(s_t, a_t, \cdot) \). This process is repeated leading to the following cumulative discounted cost

\[
V^\pi(s) = \lim_{T \to \infty} \mathbb{E} \left[ \sum_{t=0}^{T-1} \gamma^t g(s_t, \pi(s_t)) \right] \bigg| s_0 = s, \quad (1)
\]

where \( \{s_0, \pi(s_0), s_1, \pi(s_1), \ldots\) is the state-action sequence generated by the MDP under policy \( \pi \) with initial state \( s_0 \), and the expected value is taken with respect to the corresponding probability measure over the space of sequences. The transition probability distributions induced by policy \( \pi \) can be compactly represented by the rows of an \( n \times n \) row-stochastic matrix \( P^\pi_{ss'} = p_{ss'}(\pi(s)) \) for all \( s, s' \in S \) and the costs induced by policy \( \pi \) by the vector \( g^\pi = [g(1, \pi(1)), \ldots, g(n, \pi(n))]^T \in \mathbb{R}^n \). The optimal cost is defined as

\[
V^*(s) = \min_{\pi \in \Pi} V^\pi(s), \quad \forall s \in S. \quad (2)
\]

Any policy \( \pi^* \in \Pi \) that attains the optimal cost is called an optimal policy. Notice that in (2) we restrict our attention to stationary deterministic policies as in our setting there exists a policy in this class that attains \( V^* \) (Bertsekas, 2012, Section I.1.4).

Equations (1) and (2) admit recursive formulations which are known as the Bellman equations. In particular

\[
V^\pi(s) = g(s, \pi(s)) + \gamma \sum_{s' \in S} p_{ss'}(a) V^\pi(s') \quad \forall s \in S, \quad (3)
\]

is the Bellman equation associated with policy \( \pi \), and

\[
V^*(s) = \min_{\pi \in \Pi} \left\{ g(s, \pi(s)) + \gamma \sum_{s' \in S} p_{ss'}(a) V^*(s') \right\} \quad \forall s \in S, \quad (4)
\]

is the Bellman equation associated with the optimal cost. Given the cost \( V : S \to \mathbb{R}^n \), any policy which satisfies the following equation

\[
\pi(s) \in \arg \min \left\{ g(s, \pi(s)) + \gamma \sum_{s' \in S} p_{ss'}(a) V(s') \right\} \forall s \in S
\]

is called greedy with respect to the cost \( V \). We denote with \( \text{GreedyPolicy}(V) \) the operator which extracts a greedy policy associated with \( V \).

Starting from the Bellman equations we can define two mappings, \( T^\pi : \mathbb{R}^n \to \mathbb{R}^n \) and \( T : \mathbb{R}^n \to \mathbb{R}^n \), where \( T^\pi V = g^\pi + \gamma P^\pi V \) and \( TV = \min_{\pi \in \Pi} \{g^\pi + \gamma P^\pi V\} \). These mappings are known as the Bellman operators and allow one to rewrite (3) and (4) in a compact form, \( V^\pi = T^\pi V^\pi \) and \( V^* = TV^* \), respectively. The Bellman operators are \( \gamma \)-contractive, monotone and shift-invariant and, in our setting, have \( V^\pi \) and \( V^* \) as their unique fixed-points, respectively. We refer to (Bertsekas, 2012) for a detailed discussion on the properties of the Bellman operators.

2.1 Dynamic Programming

DP comprises the methods to solve (4) (Bertsekas, 2012). In this work we focus on variants of PI and VI. In particular, given an arbitrary initial cost vector \( V_0 \in \mathbb{R}^n \), VI is the fixed-point iteration

\[
V_{k+1} = T V_k, \quad (6)
\]

and enjoys global linear convergence to \( V^* \) and \( V^* \), respectively, with a \( \gamma \)-contraction rate. PI instead starts with an arbitrary policy \( \pi_0 \in \Pi \) and alternates two steps: policy evaluation

\[
V_{k+1}^\pi = (I - \gamma P^\pi)^{-1} g^\pi, \quad (8)
\]

and policy improvement

\[
\pi_{k+1} = \arg \min_{\pi \in \Pi} \{g^\pi + \gamma P^\pi V_{k+1}^\pi\}. \quad (9)
\]

Exact PI converges in a finite number of iterations, but the worst-case upper bound for large state spaces could be dramatic. Fortunately, PI enjoys global linear convergence to \( V^* \) with rate \( \gamma \). In addition, the convergence rate superiority of PI with respect to VI has been long suggested by extensive empirical evidence and only recently proved for this setting. In particular, Gargiani et al. (2022) show that the solution of the Bellman equation (4) can be expressed as the root of the so-called Bellman residual function

\[
r(V) = V - TV. \quad (10)
\]

Consequently, solving the Bellman equation corresponds to computing the root of the Bellman residual function. Applying a semismooth variant of Newton’s method to (10) yields the iteration

\[
V_{k+1} = V_k - J_k^{-1} r(V_k), \quad (11)
\]

where \( J_k \) is an element in Clarke’s generalized Jacobian of \( r \) at \( V_k \). The authors of (Gargiani et al., 2022) show that the PI iterate is an instance of (11) and that the policy evaluation step corresponds to the solution of the Newtonian linear system

\[
r(V_k) + J_k (V - V_k) = 0, \quad (12)
\]

where \( J_k = I - \gamma P^\pi_k \). This equivalence, together with the structural properties of the Bellman residual function, proves local quadratic convergence of PI.
Despite its fast convergence rate, PI iterations are computationally expensive for MDPs with a large number of states. In particular, in the scenarios where $n$ is large the policy evaluation step is not practical as it requires the exact solution of an $n$-dimensional system of linear equations. An alternative is OPI, where the policy evaluation step is carried out approximately with a fixed number $W$ of VI steps. This number is generally selected to trade-off computational complexity and convergence rate. Notice that when $W = 1$ we resort to VI and when $W \to \infty$ we resort to PI. We refer to (Bertsekas, 2012, Chapter 2) for a thorough analysis of VI, PI and OPI.

Even if the semismooth Newton method enjoys a fast rate of convergence, computing the exact solution of (12) using a direct method can be expensive if the number of unknowns is large. A more computationally efficient solution in the large-scale case consists in solving (12) only approximately with some iterative linear solver and using a certain stopping rule. These are the principles behind inexact semismooth Newton methods (Izmailov and Solodov, 2014; Martínez and Qi, 1995). In particular, $V_{k+1}$ is no longer required to exactly solve (12), but only to satisfy

$$\|r(V_k) + J_k (V_{k+1} - V_k)\| \leq \alpha_k \|r(V_k)\|$$

(13)

for some $\alpha_k \in [0, 1]$. The sequence $\{\alpha_k\}$ is called forcing sequence and it greatly affects both local convergence properties and robustness of the method (Izmailov and Solodov, 2014). Different iterative linear solvers can be used to approximately solve (12) (Hackschbusch, 1994). Often Krylov subspace methods, such as the generalized minimal residual method (GMRES) (Saad and Schultz, 1986), are deployed in large-scale scenarios.

### 2.2 GMRES

Consider a general system of linear equations

$$Ax = b,$$  

(14)

where $b \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ is a non-singular matrix. Starting from an initial guess $x_0 \in \mathbb{R}^n$ with residual $\Phi(x_0) = b - Ax_0$, GMRES (Saad and Schultz, 1986) generates a sequence $\{x_i\}$ of approximate solutions to (14) with

$$x_i = \arg \min_x \{\|b - Ax\|_2 : x \in x_0 + \mathcal{K}_i\},$$

(15)

where $\mathcal{K}_i = \text{span} \{\Phi_0, A\Phi_0, A^2\Phi_0, \ldots, A^{i-1}\Phi_0\}$ is known as the $i$-th Krylov subspace and $\Phi_0 = \Phi(x_0)$. In particular, at each iteration GMRES generates an orthonormal basis of $\mathcal{K}_i$ via the Arnoldi’s method (Saad and Schultz, 1986) and then deploys it to solve (15). Unlike the conjugate gradient method, the orthonormal bases can not be computed with a short recurrence. When $i$ increases the number of stored vectors increases like $i$ and the number of multiplication like $0.5i^2 n$. A practical variant of GMRES, denoted as GMRES(i), consists in restarting the algorithm after every $i$ iterations.

GMRES with exact arithmetic converges to the solution of (14) in at most $n$ steps. Its convergence rate though is greatly affected by the distribution of the eigenvalues of the coefficient matrix (Campbell et al., 1997). This is exemplified in Figure 1, where GMRES is used to solve the linear systems $A_1 x = b$ and $A_2 x = b$. In particular, $A_2$ is a matrix with non-clustered complex eigenvalues, while all the eigenvalues of $A_2 \subset \mathbb{R}^{100 \times 100}$ are contained in the circle of center $(1,0)$ and radius $0.9$ in the complex plane. In the first scenario the norm of the residual is significantly decreased only when $i = 100$, while in the second scenario we observe $R$-linear convergence with a fast rate starting from the first iteration.

We refer to (Saad and Schultz, 1986; Campbell et al., 1997) for a detailed description of GMRES and its convergence properties. See (Saad and Schultz, 1986, Algorithm 3) for a pseudocode description of GMRES.

### 3. INEXACT POLICY ITERATION METHODS

We define a novel variant of PI for large-scale scenarios, which we call inexact policy iteration. This class of methods is based on approximately solving the policy evaluation step with an iterative linear solver. The methods in this class start with an initial guess of the optimal cost $V_0 \in \mathbb{R}^n$ and then at every iteration extract a greedy policy associated with the current iterate $V_k \in \mathbb{R}^n$, which is used to compute an element in Clarke’s generalized Jacobian. The next iterate $V_{k+1} \in \mathbb{R}^n$ is selected as an approximate solution of the Newtonian linear system

$$(I - \gamma P^{x_k}) V = g^{x_k}$$

(16)

which verifies the stopping condition in (13) with the infinity-norm. Because of the specific structure of the Bellman residual function, (13) simplifies to

$$\|g^{x_k} - (I - \gamma P^{x_k}) V_{k+1}\| \leq \alpha_k \|g^{x_k} - (I - \gamma P^{x_k}) V_k\|.$$

In principle, any iterative solver for linear systems with non-singular coefficient matrices can be used to generate an approximate solution of (16), such as VI, its mini-batch version (Gargiani et al., 2021) and GMRES. Notice that, when VI is deployed as inner solver, we obtain a variant of OPI where the number of inner iterations is not selected a priori, but dictated by the stopping condition. See Algorithm 1 for a pseudocode description of a general inexact policy iteration method.

#### 3.1 Inexact GMRES Policy Iteration

We deploy the presented algorithmic framework to design a novel DP method for large-scale applications. The selection of the inner solver is important for the performance
of the overall scheme, as a more efficient solver will require less time to meet the stopping condition, leading to an overall faster method. Given the particular structure of the Newtonian linear system in (16), we propose to deploy GMRES as iterative linear solver in Step 7 of Algorithm 1. In particular, the coefficient matrices are non-singular (Gargiani et al., 2022, Proposition 3.3), but, unless stronger assumptions on the geometry of the underlying MDP hold, we cannot rely on symmetry. In addition, as discussed in Section 2.2, GMRES has a particularly favorable convergence behavior for the case of coefficient matrices with clustered eigenvalues. The following lemma demonstrates that this is exactly the scenario encountered in inexact policy iteration methods as the eigenvalues of the coefficient matrices of the Newtonian linear systems are clustered in a circle of radius less than 1.

Lemma 1. For any π ∈ Π, the eigenvalues of I − γPπ are contained in a circle centered at (1, 0) and with radius γ in the complex plane.

Proof. The result follows directly from the fact that the eigenvalues of Pπ are contained in a circle centered at (0, 0) and with radius 1 in the complex plane (Berman and Plemmons, 1994, Theorem 5.3 in Chapter 2).

Finally, as depicted in Figure 2 for policy evaluation, the convergence rate of GMRES appears to be more robust against the discount factor compared to that of VI. Algorithm 2 provides a pseudocode description of iGMRES-PI, where we have adopted the restarted version of GMRES to reduce the computational and storage complexity.

3.2 Theoretical Analysis

In this section we provide an analysis of the local convergence properties of inexact policy iteration methods for finite MDPs with discount factor γ ∈ (0, 1). We start by characterizing the Lipschitz constant of the Bellman residual function and deriving an upper bound on the infinity-norm of the inverse of the coefficient matrix of the Newtonian linear system in (16).

Lemma 2. Let r : R^n → R^n be the Bellman residual function as defined in (10). Then,

\[ \|r(V_1) - r(V_2)\|_\infty \leq (1 + \gamma)\|V_1 - V_2\|_\infty, \quad \forall V_1, V_2 \in \mathbb{R}^n. \]

Proof. As shown in (Gargiani et al., 2022), r is piecewise affine with selection functions rs(V) = V = TζV = (I − γPπ)V − gπ for all π ∈ Π. Piecewise affine functions are globally Lipschitz continuous and their Lipschitz constant is given by the maximum over the norms of the coefficient matrices of their selection functions (Facchinei and Pang, 2003, Proposition 4.2.2). Therefore, by exploiting the fact that for any π ∈ Π the matrices Pπ are row-stochastic, we obtain

\[ \max_{\pi \in \Pi} \|I - \gamma P\|_{\infty} = \max_{\pi \in \Pi} \max_{s \in S} \sum_{s' \in S \setminus \{s\}} | - \gamma p_{ss'}(\pi(s)) | + | 1 - \gamma p_{ss}(\pi(s)) | = \max_{\pi \in \Pi} \gamma (1 - p_s(\pi(s))) + 1 - \gamma p_{ss}(\pi(s)) \leq 1 + \gamma, \]

Algorithm 1 Inexact Policy Iteration

1: Initialization: V_0 ∈ R^n, α ∈ (0, 1), K > 0
2: for k = 0, 1, ..., K − 1 do
3: \quad π_k ← GreedyPolicy(V_k)
4: \quad J^{π_k} = (I - \gamma P^{π_k})
5: \quad V_{k+1} ← V_k
6: \quad while \|g^{π_k} - J^{π_k}V_{k+1}\|_\infty > \alpha \|g^{π_k} - J^{π_k}V_k\|_\infty do
7: \quad \quad V_{k+1} ← IterativeLinearSolver(J^{π_k}, g^{π_k}, V_{k+1})
8: \quad end while
9: end for

Algorithm 2 GMRES(W) Policy Iteration

1: Initialization: V_0 ∈ R^n, α ∈ (0, 1), K, W > 0
2: for k = 0, ..., K − 1 do
3: \quad π_k ← GreedyPolicy(V_k)
4: \quad J^{π_k} ← (I - \gamma P^{π_k})
5: \quad V_{k+1} ← V_k
6: \quad Φ_k ← g^{π_k} - J^{π_k}V_k
7: \quad Q ← 0_{n×(W+1)}
8: \quad H ← 0_{(W+1)×W}
9: \quad [Q]_1 \leftarrow Φ_k / \|Φ_k\|_2
10: i ← 1
11: while \|g^{π_k} - J^{π_k}V_{k+1}\|_\infty > \alpha \|Φ_k\|_\infty do
12: \quad q ← J^{π_k}[Q]_i
13: \quad for j = 1, ..., i do
14: \quad \quad [H]_{ji} ← [Q]_{ji}q
15: \quad \quad q \leftarrow q - [H]_{ji}[Q]_{j}q
16: \quad end for
17: \quad \|q\|_2 = 0 then
18: \quad \bar{y} ← \text{arg min}_y \|\Phi_2 \cdot e_1 - Hy\|_2
19: \quad V_{k+1} ← \bar{Q}\bar{y} + V_k
20: \quad go to line 35
21: \quad if \|q\|_2 = 0 then
22: \quad \bar{y} ← \text{arg min}_y \|\Phi_2 \cdot e_1 - Hy\|_2
23: \quad V_{k+1} ← \bar{Q}\bar{y} + V_k
24: \quad if i = W then
25: \quad \bar{Q} ← 0_{n×(W+1)}
26: \quad H ← 0_{(W+1)×W}
27: \quad [Q]_1 \leftarrow (g^{π_k} - J^{π_k}V_{k+1}) / \|g^{π_k} - J^{π_k}V_{k+1}\|_2
28: \quad i ← 1
29: \quad else
30: \quad i ← i + 1
31: \quad end if
32: \quad end if
33: \quad end if
34: end while
35: end for

Fig. 2. Convergence of GMRES (red) and VI (blue) for policy evaluation for different values of γ and n = 100.
which concludes the proof. □

Lemma 3. For any $\pi \in \Pi$ the following inequality holds
\[
\| (I - \gamma P^\pi)^{-1} \|_\infty \leq \frac{1}{1 - \gamma} .
\]

Proof. Since $\| \gamma P^\pi \|_\infty = \gamma < 1$ for any $\pi \in \Pi$, then $I - \gamma P^\pi$ is invertible and $(I - \gamma P^\pi)^{-1} = \sum_{k=0}^{\infty} (\gamma P^\pi)^k$ (Sheldon, 2020, Chapter 10). Therefore for any $\pi \in \Pi$
\[
\| (I - \gamma P^\pi)^{-1} \|_\infty = \sum_{k=0}^{\infty} \| \gamma P^\pi \|^k \|_\infty \leq \sum_{k=0}^{\infty} \| \gamma P^\pi \|^k = \frac{1}{1 - \gamma} ,
\]
where the first inequality follows from the properties of the infinity-norm and the last equality follows from the properties of the geometric series. □

The following theorem characterizes the local contraction of inexact policy iteration methods.

Theorem 4. (Local contraction). Consider a general inexact policy iteration method as given in Algorithm 1. Assume that $\pi_k$ in Step 3 is a non-spurious greedy policy (Gargiani et al., 2022, Definition 3.2) and let $\{\alpha_k\}$ be a sequence of positive numbers contained in $[0, \alpha[$, with $\alpha \in (0, \frac{1}{1-\gamma})$. Then there exists a neighborhood of $V^*$ such that, for any $V_0 \in \mathbb{R}^n$ in this neighborhood, the inexact policy iteration method is Q-linearly convergent to $V^*$ with rate $\frac{\alpha}{1+\gamma}$. If $\lim_{k \to \infty} \alpha_k = 0$, then the method enjoys local Q-superlinear convergence.

Proof. Since by assumption $\pi_k$ is a non-spurious greedy policy, then $I - \gamma P^\pi_k$ is an element in Clarke’s generalized Jacobian of $r$ at $V_k$ (Gargiani et al., 2022). In addition, since $r$ is globally CD-regular (Gargiani et al., 2022, Proposition 3.3), the sequence (11) is globally well-defined. In the following derivations we use $J_k = I - \gamma P^\pi_k$ and $\Delta V_k = V_{k+1} - V_k$
\[
\| V_{k+1} - V^* \|_\infty = \| V_k + \Delta V_k - V^* \|_\infty \leq \| V_k - J_k r(V_k) \|_\infty + \| J_k \Delta V_k \|_\infty \leq \| J_k r(V_k) \|_\infty + \| J_k \Delta V_k \|_\infty .
\]

(1) $\leq \frac{1}{1 - \gamma} \| r(V_k) - J_k r(V_k) \|_\infty + \| J_k \Delta V_k \|_\infty$
(2) $\leq \frac{1}{1 - \gamma} \| r(V_k) - J_k r(V_k) \|_\infty + \alpha_k \| r(V_k) \|_\infty$
(3) $\leq \frac{1}{1 - \gamma} \| r(V_k) - r(V^*) - J_k (V_k - V^*) \|_\infty + \alpha_k \| r(V_k) - r(V^*) \|_\infty$

where (a) follows from Lemma 3, (b) from the stopping condition (13) and (c) from Lemma 2.

Since $r$ is strongly semismooth at $V^*$, there exists a neighborhood $\mathcal{N}(V^*)$ such that, if $V_k \in \mathcal{N}(V^*)$, then
\[
\| V_{k+1} - V^* \|_\infty \leq \mathcal{O} \left( \| V_k - V^* \|_\infty \right) + \frac{1 + \gamma}{1 - \gamma} \| \alpha_k \| \| V_k - V^* \|_\infty ,
\]
from which we can conclude local Q-linear convergence with rate $\frac{1 + \gamma}{1 - \gamma} \alpha_k < 1$ if $\{\alpha_k\} \subseteq [0, \alpha[$ with $\alpha \in (0, \frac{1}{1-\gamma})$ and local Q-superlinear convergence if $\lim_{k \to \infty} \alpha_k = 0$. □

The results of Theorem 4 show that the local convergence properties of inexact policy iteration methods are strongly affected by the forcing sequence. In addition, because of the specific structure of the problem at hand, we can compute the range of $\alpha$-values for which local convergence is guaranteed as it solely depends on $\gamma$.

4. NUMERICAL EVALUATION

We evaluate the performance of iGMRES-PI against PI and OPI on an MDP with 10000 states, 40 actions and $\gamma = 0.95$. The methods are implemented in Python using NumPy (Charles et al., 2020) and the simulations are run on an Intel(R) Core(TM) i7-10750H CPU @ 2.60GHz architecture. We enforce single-core execution for all algorithms to ensure a fair comparison.

In Figure 3a we visualize the infinity-norm of the suboptimality gap versus the number of outer iterations. As shown in Figure 3a, PI enjoys the fastest rate of convergence, followed by iGMRES-PI and OPI. As expected, the convergence rate of OPI improves by increasing the number of inner iterations $W$. When considering time instead of outer iterations, the situation changes dramatically for PI. Its expensive iterations result in PI being the slowest converging method in terms of wall-clock time. As shown in Figure 3b, PI takes $\sim 87$ seconds to reach convergence. A better trade-off between convergence rate and computational complexity is offered by OPI with $W = 50$ and $W = 80$, which achieves convergence in $\sim 36$ and $\sim 33$ seconds, respectively. Our iGMRES-PI greatly outperforms both PI and OPI, achieving convergence in only $\sim 15$ seconds and therefore attaining a $\times 5.8$ and $\times 2.2$ speedup with respect to PI and OPI, respectively.

We then run the same benchmarks increasing the discount factor to 0.99. As depicted in Figures 3c and 3d, this increase has a dramatic effect on the performance of OPI, while PI and iGMRES-PI’s performance is essentially unaltered. These empirical results are in line with our observations in Figure 2.

5. CONCLUSIONS & FUTURE WORK

Taking inspiration from inexact semismooth Newton methods, we define a novel class of DP methods for large-scale applications which we call inexact policy iteration. We provide local contraction guarantees for the methods in this class and propose iGMRES-PI, an instance of inexact policy iteration based on the deployment of GMRES for the approximate policy evaluation step. We validate the performance superiority of iGMRES-PI against PI and OPI on a large-scale MDP.

Future work includes providing global convergence guarantees for inexact policy iteration, boosting the performance of GMRES via the design of an ad-hoc preconditioner for policy evaluation and the study of high-performance parallel and distributed variants of iGMRES-PI.
Fig. 3. Performance of iGMRES-PI (red), PI (black) and OPI with a different number of inner iterations $W$ (blue) on an MDP with $n = 10000$ and $m = 40$. 

REFERENCES

Bellman, R. (1952). On the theory of dynamic programming. *Proceedings of the National Academy of Sciences*, 38(8), 716–719.

Berman, A. and Plemmons, R.J. (1994). *Nonnegative Matrices in the Mathematical Sciences*. Society for Industrial and Applied Mathematics.

Bertsekas, D.P. (2012). *Dynamic Programming and Optimal Control*, volume 2. Athena Scientific, fourth edition.

Bertsimas, D. and Lo, A.W. (1998). Optimal control of execution costs. *Journal of Financial Markets*, 1(1), 1–50.

Campbell, S., Ipsen, I., Kelley, C., and Meyer, C. (1997). GMRES and the minimal polynomial. *BIT Numerical Mathematics*, 36.

Charles, R.H. et al. (2020). Array programming with NumPy. *Nature*, 585(7825), 357–362.

Elamvazhuthi, K. and Berman, S. (2015). Optimal control of stochastic coverage strategies for robotic swarms. *Proceedings - IEEE International Conference on Robotics and Automation*, 2015, 1822–1829.

Facchinei, F. and Pang, J. (2003). *Finite-Dimensional Variational Inequalities and Complementarity Problems*, volume I. Springer.

Gargiani, M., Martineili, A., Martinez, M., and Lygeros, J. (2021). Parallel and flexible dynamic programming via the randomized mini-batch operator. ArXiv:2110.02901.

Gargiani, M., Zanelli, A., Liao-McPherson, D., and Lygeros, J. (2022). Dynamic programming through the lens of semismooth Newton-type methods (extended version). ArXiv:2203.08678.

Hackbusch, W. (1994). *Iterative Solution of Large Sparse Systems of Equations*, volume 95 of *Applied Mathematical Sciences*. Springer, New York, first edition.

Huang, Y., Forsyth, P.A., and Labahn, G. (2011). Methods for pricing American options under regime switching. *SIAM Journal on Scientific Computing*, 33(5), 2144–2168.

Huang, Y., Forsyth, P.A., and Labahn, G. (2012). Combined fixed point and policy iteration for Hamilton–Jacobi–Bellman equations in finance. *SIAM Journal on Numerical Analysis*, 50(4), 1861–1882.

Izmailov, A. and Solodov, M. (2014). *Newton-Type Methods for Optimization and Variational Problems*. Springer.

Martinez, J. and Qi, L. (1995). Inexact Newton methods for solving nonsmooth equations. *Journal of Computational and Applied Mathematics*, 60(1), 127–145. Proceedings of the International Meeting on Linear/Nonlinear Iterative Methods and Verification of Solution.

Mrkaic, M. (2002). Policy iteration accelerated with Krylov methods. *Journal of Economic Dynamics and Control*, 26(4), 517–545.

Saad, Y. and Schultz, M.H. (1986). GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems. *Siam Journal on Scientific and Statistical Computing*, 7, 856–869.

Sheldon, A. (2020). *Measure, Integration & Real Analysis*. Springer International Publishing.

Wingate, D. and Seppi, K.D. (2005). Prioritization methods for accelerating MDP solvers. *Journal of Machine Learning Research*, 6, 851–881.