STABILITY OF THE STOKES IMMERSED BOUNDARY PROBLEM WITH BENDING AND STRETCHING ENERGY

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ABSTRACT. We study the motion of a 1-D closed elastic string with bending and stretching energy immersed in a 2-D Stokes flow. In this paper we introduce the curves tangent angle function and the stretching function to describe the different deformations of the elastic string. These two functions are defined on the arc-length coordinate and the material coordinate respectively. With the help of the fundamental solution of the Stokes equation, we reformulate the problem into a parabolic system which is called the contour dynamic system. Under the non-self-intersecting and well-stretched assumptions on initial configurations, we establish the local well-posedness of the free boundary problem in Sobolev space. When the initial configurations are sufficiently close to the equilibrium state (i.e. an evenly parametrized circle), we prove that the solutions can be extended globally and the global solutions will converge to the equilibrium state exponentially as $t \to +\infty$.

1. Introduction

1.1. Presentation of the Problem. This paper is concerned with the hydrodynamics on the moving surface of a bilayer membrane immersed in a 2-D Stokes flow. Bilayer membranes are the outer layer of living cells whose thickness is much smaller than the length scale of the cell. The membranes undergo two different elastic deformations: bending and stretching. In general, we ignore the inertia of the membrane, and regard the membrane as a mathematical surface.

In 1973, based on similarities between lipid bilayers and nematic liquid crystals, Helfrich [15] proposes an elasticity model for bilayer membranes. Helfrich ignores the stretching deformation, because he finds that the shapes of non-spherical membranes is only governed by curvature. In his model, membranes are bend-resistant and represent minima of the following energy:

$$e_H = \int_{\Gamma} \left( \frac{c_1}{2} (H - B)^2 + c_2 K \right) d\sigma.$$  

Here $\Gamma$ is the surface representing the membrane, $H$ and $K$ are the mean curvature and the Gaussian curvature respectively, $B$ is the spontaneous curvature that reflects the initial or intrinsic curvature of the membrane, $c_1$ and $c_2$ are the elastic coefficients, and $d\sigma$ is the area form of the surface. This energy is known as the Helfrich energy. This model has been proved to be successful in explaining the shapes of cell membranes [10, 28, 29].

From a mechanical point of view, membrane tension is intimately related to membrane stretching. In a recent work, Lipowsky [23] reconsiders the tension within
membranes by minimizing the combined stretching and bending energy. The stretching energy is

$$e_s = \int_{\Gamma_{op}} \frac{c_3}{2} \left( \frac{\Delta a}{a} \right)^2 d\sigma,$$

where $\Gamma_{op}$ is the optimal surface which is evenly parametrized in material coordinate, $c_3$ is the elastic modulus of stretching, $\frac{\Delta a}{a}$ is the relative change per unit area.

The purpose of this paper is to study the motion of a membrane in 2-D Stokes flow, so we take both bending and stretching deformation into account. The bilayer membrane equips with the following free energy:

$$E = e_H + e_s + \int_{\Gamma} \lambda d\sigma.$$

Here $\lambda$ denotes the surface tension which is the Lagrange multiplier for area inextensibility of the membrane. We regard the bilayer membrane as a 1-D elastic string $\Gamma$, which is a Jordan curve parametrized by $X(s, t)$, where $s \in \mathbb{T}$ is the material coordinate (or the Lagrangian coordinate), $\mathbb{T} \overset{def}{=} \mathbb{R}/2\pi\mathbb{Z}$ is the 1-D torus, and $t \geq 0$ is the time variable. We also introduce $z(\alpha, t)$ to parametrize $\Gamma$, where $\alpha \in \mathbb{T}$ is the arc-length coordinate. Indeed, $z(\alpha, t)$ satisfies

$$z(\alpha(s, t), t) = X(s, t), \quad \alpha(s, t) = s + y(s, t), \quad \forall s \in \mathbb{T};$$

$$|z_\alpha(\alpha, t)| \overset{def}{=} s(t), \quad \forall \alpha \in \mathbb{T}.$$  

Here $s + y(s, t)$ is the transfer function between these two coordinates, $s$ stands for $\frac{1}{2\pi}$ of the perimeter of $\Gamma$. We call $y_s(s, t) = \partial_s y(s, t)$ the stretching function which quantifies the stretching deformation of the elastic string. Thus the free energy can be rewritten as

$$E = \frac{c_1 s}{2} \int_{\mathbb{T}} (\kappa - B)^2 d\alpha + \frac{c_3}{2} \int_{\mathbb{T}} |X_s|^2 ds + 2\pi \lambda s.$$

Here we use $\kappa$ to denote the curvature of the string.

In this paper, we only consider the case with $B \equiv 0$. We choose $c_1 = c_3 = 1$ for simplicity, and assume $\lambda \geq 0$ to be a constant. Therefore, the force applied on the string has the following formulation:

$$(1.1) \quad \bar{F}(s, t) = (4\kappa - \frac{1}{5} \partial_s)^2 \kappa - \frac{1}{2} \kappa^3) n + \frac{1}{s(1 + y_s)} \partial_s^2 X \overset{def}{=} \frac{1}{s(1 + y_s)} F(s, t),$$

where $n$ is the inward unit normal vector.

We now introduce the precise mathematical statement of the problem we are interested in. Let $\Omega_\alpha = \mathbb{R}^2 / \Gamma$, the flow field $u$ and pressure $p$ satisfy the following system:

$$\begin{cases}
\Delta u(x, t) = \nabla p(x, t), \\
\nabla \cdot u(x, t) = 0, \\
[-p(X(s, t), t) I + \tau(X(s, t), t)] \cdot n = \bar{F}(s, t), \\
[u](X(s, t), t) = 0, \\
u(X(s, t), t) = X(s, t), \\
|u|, |p| \to 0, \quad (x, t) \in \Omega_\alpha \times \mathbb{R}_+, \\
(x, t) \in \Omega_\alpha \times \mathbb{R}_+, \\
(s, t) \in \mathbb{T} \times \mathbb{R}_+, \\
(s, t) \in \mathbb{T} \times \mathbb{R}_+, \\
(s, t) \in \mathbb{T} \times \mathbb{R}_+, \\
(x) \to \infty,
\end{cases}$$

as $|x| \to \infty$.
where $\tau = \nabla u + \nabla u^\top$ is the stress of the bulk fluid, $\mathbf{F}$ is given in (1.1), $\mathbf{u}$ denotes the jump of $\mathbf{u}$ across the free boundary. The kinematic equation of the string $\mathbf{u}(\mathbf{X}(s, t), t) = \mathbf{X}_s(s, t) = 0$, $(s, t) \in \mathbb{T} \times \mathbb{R}_+$, means that the string moves along the flow. This system can be rewritten in the immersed boundary formulation:

\[
\begin{align*}
-\Delta \mathbf{u}(x, t) + \nabla p(x, t) &= \mathbf{f}(x, t), & (x, t) \in \mathbb{R}^2 \times \mathbb{R}_+, \\
\text{div } \mathbf{u}(x, t) &= 0, & (x, t) \in \mathbb{R}^2 \times \mathbb{R}_+, \\
\mathbf{u}(\mathbf{X}(s, t), t) &= \mathbf{X}_s(s, t), & (s, t) \in \mathbb{T} \times \mathbb{R}_+, \\
|\mathbf{u}|, |p| &\to 0, & \text{as } |x| \to \infty,
\end{align*}
\]

where

\[
\mathbf{f}(x, t) = \int_{\mathbb{T}} \mathbf{F}(s, t) \delta(x - \mathbf{X}(s, t)) ds,
\]

and $\delta$ is the 2-D delta measure. The first equation of (1.3) holds in the sense of distribution, and the expression of $\mathbf{f}$ shows that the force is only applied on the string. The immersed boundary formulation was initially introduced by Peskin [30]. It is easy to verify that (1.3) is equivalent to (1.2) if both $\mathbf{z}$ and $\mathbf{X}$ are sufficiently smooth [19].

In 2-D Stokes flow, the velocity field $\mathbf{u}$ and pressure $p$ can be solved from the force $\mathbf{f}$ by using boundary integral. It holds that

\[
\mathbf{u}(x, t) = \int_{\mathbb{R}^2} G(x - y) \mathbf{f}(y, t) dy, \quad p(x, t) = \int_{\mathbb{R}^2} Q(x - y) \mathbf{f}(y, t) dy,
\]

where

\[
G(x) = \frac{1}{4\pi}(-\ln |x| Id + \frac{x \otimes x}{|x|^2}), \quad Q(x) = \frac{x}{2\pi |x|^2},
\]

are the fundamental solutions [31], and $Id$ is the $2 \times 2$ identity matrix. The above formula shows that $\mathbf{u}$ and $p$ are determined by the configuration of the string. Since $G(x)$ is a single-layer potential, $\mathbf{u}$ is continuous on $\mathbb{R}^2$. It follows that

\[
\mathbf{u}(\mathbf{X}(s, t), t) = \int_{\mathbb{T}} G(\mathbf{X}(s, t) - \mathbf{X}(s', t)) \cdot \mathbf{F}(s', t) ds'.
\]

On the other hand, it holds that $\mathbf{X}_s(s, t) = \mathbf{u}(\mathbf{X}(s, t), t)$, and the fluid velocity on the string also determines the evolution of the membrane’s configuration. As a result, system (1.3) is equivalent to the following equation:

\[
\mathbf{X}_s(s, t) = \int_{\mathbb{T}} G(\mathbf{X}(s, t) - \mathbf{X}(s', t)) \cdot \mathbf{F}(s', t) ds'.
\]

1.2. Related Results. During the past several decades, several models have been developed to research the behaviors of bilayer membranes with or without surrounding fluid [5, 7, 18, 32, 35, 39]. There are also many analytic studies on the membrane dynamic problems. Without surrounding fluid, Hu-Song-Zhang [17] and Wang-Zhang-Zhang [38] analyze the dynamics of a membrane in 2-D and 3-D space respectively. They regard the membrane as a coupled system comprising a moving elastic surface and an incompressible membrane fluid. For the coupled fluid-structure interaction models, Cheng-Coutand-Shkoller [8, 9] obtain the local well-posedness of moving boundary problems which model the motion of a viscous incompressible fluid inside
of a bend-resistant elastic bio-membrane. In these papers, they study the membranes
with and without inertia, but they do not consider the fluid outside the membranes
in their models, which is different to the immersed boundary problem. The elastic
membrane with inertia is called the Koiter shell, more results about this model can
be found in [6, 14, 20, 27].

For other kinds of immersed boundary problems, Lin-Tong [22] study the coupled
motion of a 1-D closed elastic string immersed in a 2-D Stokes flow. The string they
considered behaves like a Hookean spring. That is to say, it only equips the following
stretching energy

\[ E = \frac{1}{2} \int_{\mathbb{T}} |X_s|^2 ds. \]

They prove the local-wellposedness of this model with an arbitrary initial configura-
tion in \( H^{3/2}(\mathbb{T}) \). Moreover, when the initial string configuration is sufficiently close to
an evenly parametrized circular configuration, they also prove that a global-in-time
solution uniquely exists, and will converge to the equilibrium configuration expo-
nentially as \( t \to +\infty \). The framework they developed is useful in treating immersed
boundary problems. The method we used in this paper is inspired from their work.
In a parallel work, Mori-Rodenberg-Spirn [26] study the same model and establish
well-posedness results in low-regularity Hölder spaces. They prove nonlinear sta-
bility of equilibrium states with explicit exponential decay estimates, and verify the
optimality of which numerically. In a recent paper [37], Tong studies the regularized
problem of this model, and derives error estimates under various norms.

When the elastic membrane is surrounded by inviscid fluids, the model become
to the one called the hydroelastic wave. Ambrose-Siegel [2] get the local well-
posedness of 2-D hydroelastic waves. In that model, the external force applied on
the surface is generated from the Helfrich energy. Liu-Ambrose [24] get similar results
for hydroelastic waves with mass.

1.3. Main Result. We first introduce some notations used throughout this paper.
We denote by \( \| \cdot \|_{L^p(\mathbb{T})}, \| \cdot \|_{H^s(\mathbb{T})}, \| \cdot \|_{H^s(\mathbb{T})} \)
the Lebesgue norm, the ordinary Sobolev norm and the homogeneous Sobolev norm on \( \mathbb{T} \)
for the arc-length coordinate \( \alpha \), and
\( \| \cdot \|_{L^p(\mathbb{T})}, \| \cdot \|_{H^s(\mathbb{T})}, \| \cdot \|_{H^s(\mathbb{T})} \)
for the material coordinate \( s \). When no confusion can arise, we will write

\[
\| \cdot \|_{L^p(\mathbb{T})} \to \| \cdot \|_{L^p(\mathbb{T})}, \quad \| \cdot \|_{H^s(\mathbb{T})} \to \| \cdot \|_{H^s(\mathbb{T})},
\]

for simplicity of notation. We define the following tangent angle function

\[ \theta \overset{\text{def}}{=} \arctan \left( \frac{z_\alpha^{(2)}}{z_\alpha^{(1)}} \right), \]

which is the angle between the strings tangent direction and the horizontal axis. Us-
ing \( \theta \), we can describe the shape of the string with the perimeter function \( s \). This idea
goes back at least as far as [16]. Though arctan is a multivalued function, we require
\( \theta \) to be continuous on \([-\pi, \pi]\). We emphasize that \( \theta \) is not continuous on \( \mathbb{T} \) due to
\( \theta(\pi) - \theta(-\pi) = 2\pi \). We denote by \( \theta_0 \) the tangent angle function from a initial configuration \( \mathbf{X}_0 \). In this paper, we always assume \((\theta, y_s, s)\) to be tangent angle function, stretching function and perimeter function corresponding to \( \mathbf{X} \). In the next section, we will show that one can reconstruct \( \mathbf{X} \) from \((\theta, y_s, s)\).

Given \( \beta_1, \beta_2 > 0 \), we introduce the non-self-intersecting assumption

\[
(1.6) \quad \frac{1}{|\alpha_1 - \alpha_2|} \int_{\alpha_2}^{\alpha_1} \left| (\cos(\theta), \sin(\theta))d\alpha' \right| \geq \beta_1, \quad \forall \alpha_1, \alpha_2 \in \mathbb{T},
\]

and the well-stretched assumption

\[
(1.7) \quad 1 + y_s(s, t) \geq \beta_2, \quad \forall s \in \mathbb{T},
\]

where \( |\alpha_1 - \alpha_2| \) is the distance between \( \alpha_1 \) and \( \alpha_2 \) on \( \mathbb{T} \). Assumption (1.6) ensures the string is a Jordan curve. If (1.7) holds, \( \alpha(s, t) \) is an invertible function, and we use \( s(\alpha, t) \) to denote its inverse.

For membrane dynamic problems involving only bending deformation or stretching deformation, one can study the evolution equations of the free surface in the arc-length coordinate or in the material coordinate respectively. However, such method is no longer suitable for membranes in which both bending and stretching deformation occur. If only the arc-length coordinate is used, the information of stretching deformation will be lost, while if only the material coordinated is used, the stabilizing effect of bending deformation is hard to reflect. To overcome this difficulty, we introduce two independent functions, the tangent angle function \( \theta \) and the stretching function \( y_s \), embodying bending and stretching deformation of the membrane respectively. \( \theta \) is defined in the arc-length coordinate, \( y_s \) is defined in the material coordinate, and we observe that the evolution equations of these two functions have favorable structures in their respective coordinates. Based on this idea, we got the following results.

**Theorem 1.1.** (Existence and uniqueness of local-in-time solution) Suppose \( \mathbf{X}_0 \) is a closed string which satisfies

\[
\theta_0 - \alpha \in H^{5/2}(\mathbb{T}), \; y_{0s} \in h^{3/2}(\mathbb{T}), \; s_0 \geq c > 0.
\]

Furthermore, we assume that (1.6)-(1.7) hold for some constants \( \beta_1, \beta_2 > 0 \). Then there exists \( T > 0 \) such that the immersed boundary problem (1.5) admits a unique solution \( \mathbf{X}(s, t) \) satisfying

\[
(1.8) \quad ||\theta||_{L^2_T H^{5/2} \cap L^2_T H^3} + ||y_s||_{L^2_T H^{3/2} \cap L^2_T H^3} \leq (3 + 4\sqrt{2}s_0^3/2)||\theta_0||_{H^{5/2}} + 5||y_{0s}||_{h^{3/2}},
\]

\[
(1.9) \quad ||\partial_t \theta||_{L^2_T H^1} + ||\partial_s y_s||_{L^2_T H^3} \leq 2(||\theta_0||_{H^{5/2}} + ||y_{0s}||_{h^{3/2}}),
\]

and

\[
(1.10) \quad \frac{1}{|\alpha_1 - \alpha_2|} \int_{\alpha_2}^{\alpha_1} (\cos(\theta(\alpha', t)), \sin(\theta(\alpha', t)))d\alpha' \geq \frac{1}{2}\beta_1, \quad \forall \alpha_1, \alpha_2 \in \mathbb{T}, \; t \in [0, T],
\]

\[
(1.11) \quad 1 + y_s(s, t) \geq \frac{1}{2}\beta_2, \quad \forall s \in \mathbb{T}, \; t \in [0, T].
\]
Remark 1.1. In Theorem [1.2], we only consider the function spaces related to variables \( \theta \) and \( y \). In fact, (1.8) implies that
\[
\begin{align*}
\mathbf{z}(\alpha, t) & \in L^\infty([0, T]; H^7/2(\mathbb{T})) \cap L^2([0, T]; H^5(\mathbb{T})), \\
\mathbf{X}(s, t) & \in L^\infty([0, T]; h^{5/2}(\mathbb{T})) \cap L^2([0, T]; h^3(\mathbb{T})).
\end{align*}
\]
One can see that the string has different regularities in different coordinates. This is the reason we introduce both the arc-length coordinate and the material coordinate.

Theorem 1.2. (Existence and uniqueness of global-in-time solution near equilibria.) There exists a constant \( \varepsilon > 0 \) such that, if \( \mathbf{X}_0 \) is a closed string and satisfies
\[
(1.12) \quad \|\theta_0 - \alpha\|_{H^{5/2}} + \|y_{0s}\|_{H^{3/2}} \leq \varepsilon,
\]
then there is a unique solution \( \mathbf{X} \in C([0, +\infty); h^{5/2}(\mathbb{T})) \cap L^2_{\text{loc}}([0, +\infty); h^3(\mathbb{T})) \) of the system (1.5) with initial data \( \mathbf{X}_0 \). The solution satisfies
\[
\begin{align*}
\theta_t(\alpha, t) & \in L^2_{\text{loc}}([0, +\infty); H^1(\mathbb{T})), \quad y_{s}(s, t) \in L^2_{\text{loc}}([0, +\infty); h^1(\mathbb{T})), \\
\|\theta - \alpha\|_{L^\infty([0, +\infty); H^{3/2})} + \|y_s\|_{L^\infty([0, +\infty); h^{1/2})} & \leq C\varepsilon,
\end{align*}
\]
\[
1 + y_s(s, t) \geq \frac{3}{4}, \quad \forall s \in \mathbb{T}, \ t \in [0, +\infty),
\]
where \( C \) is a constant.

Theorem 1.3. (Exponential convergence to the equilibria.) Let \( \mathbf{X}_0 \) be a closed string satisfying all the assumptions in Theorem [1.2] and let \( \mathbf{X} \) be the global solution obtained in Theorem [1.2] starting from \( \mathbf{X}_0 \). There exist universal constants \( \varepsilon, \gamma, C > 0 \) such that if in addition
\[
\|\theta_0 - \alpha\|_{H^{5/2}} + \|y_{0s}\|_{H^{3/2}} \leq \varepsilon,
\]
then it holds that
\[
\|\theta - \alpha\|_{H^{5/2}(t)} + \|y_s\|_{h^{3/2}(t)} \leq C e^{-\gamma t} \varepsilon.
\]
Furthermore, \( \mathbf{X} \) converges to an equilibrium configuration
\[
\mathbf{X}_\infty(s) \overset{\text{def}}{=} \sqrt{\frac{a}{\pi}} (\sin(s + \theta_\infty), \cos(s + \theta_\infty)) + x_\infty, \quad s \in \mathbb{T},
\]
and satisfies
\[
\|\mathbf{X} - \mathbf{X}_\infty\|_{h^{3/2}} \leq C \sqrt{\frac{a}{\pi}} e^{-\gamma t} \varepsilon,
\]
where \( a \) is the area enclosed by \( \mathbf{X}_0 \).

Remark 1.2. For other kinds of elastic membranes, such as one or two of \((c_1, c_2, \lambda)\) equals zero, similar results can be obtained by using the method developed herein.
The rest of this paper is organized as follows. In Section 2, we reformulate the problem to the contour dynamic system, and give an energy identity of this problem. In Section 3, we introduce a modified contour dynamic system and give a priori estimates of this system. Section 4 provides the local well-posedness of the modified system, and shows that the modified system is equivalent to the original system when the string is closed initially. In Section 5, by using the energy identity, we get the global-in-time existence of solutions to (1.5) provided that the initial data is sufficiently close to an equilibrium configuration. In Section 6, we observe that when z is closed to a circle, the first Fourier modes of $\theta - \alpha$ are extremely small compared to $||\theta - \alpha||_{L^2}$. Based on this observation, we prove the global solution gotten in Section 5 converges to an equilibrium configuration exponentially as $t \to +\infty$. Section 7 shows that the method developed in this paper can be applied to other kinds of immersed boundary problems. In the appendices, we state some auxiliary results and give a new proof to Fuglede’s isoperimetric inequality for the 2-dimensional case.

2. Reformulation of The Problem

In this section, we reformulate the immersed boundary problem to the contour dynamic system, and give an energy identity of this problem. The contour dynamic system is the combination of evolution equations for the tangent angle function, the material density function and the perimeter function, in which we can see the stabilization mechanism of the elastic force explicitly.

2.1. The arc-length coordinate and the material coordinate. From the definition of arc-length, it holds that

$$|z_{\alpha}(\alpha, t)| = s(t),$$

where $s$ is $\frac{1}{2\pi}$ of the perimeter of the string as we defined in the previous section. Let $n$ and $t$ denote the inward unit normal vector and the unit tangent vector of the free boundary, it holds that

$$n = \frac{z_{\alpha}}{s}, \quad t = \frac{z_{\alpha}}{s}.$$ 

Here $v^\perp = (-v^{(2)}, v^{(1)})$ for each two dimensional vector $v = (v^{(1)}, v^{(2)})$. Applying $\alpha$-derivation on $n$ and $t$, it follows that

$$t_\alpha = \kappa n, \quad n_\alpha = -\kappa s t.$$ 

Here we use the fact that

$$\kappa n = \frac{z_{\alpha\alpha}}{s^2} = \frac{z_{\alpha\alpha} \cdot z_{\alpha}}{s^3}.$$ 

Recalling the definition of the tangent angle function

$$\theta = \arctan\left(\frac{v^{(2)}}{v^{(1)}}\right),$$

one see immediately that

$$\kappa = \frac{\theta_\alpha}{s}, \quad n = (-\sin(\theta), \cos(\theta)) , \quad t = (\cos(\theta), \sin(\theta)).$$
As what we have mentioned above, the relation between \( X \) and \( z \) is
\[
(2.4) \quad X(s, t) = z(\alpha(s, t), t), \quad \alpha(s, t) = s + y(s, t),
\]
and it follows that
\[
(2.5) \quad X_i(s, t) = z_i(s + y(s, t), t) + y_i(s, t)\alpha_t(s + y(s, t), t).
\]
As \( z_\alpha = s \mathbf{t} \), we only have \( z_i(\alpha, t) \cdot \mathbf{n} = u(\mathbf{z}(\alpha, t), t) \cdot \mathbf{n} \). That is to say, \( z_i \) is not the real velocity of the string, and \( z \) is an abstract curve. We decompose \( z_i \) into the normal and tangent direction
\[
z_i = (z_i \cdot \mathbf{t}) \mathbf{t} + (z_i \cdot \mathbf{n}) \mathbf{n} \overset{\text{def}}{=} \mathbf{T} t + U n.
\]
Here \( U(\alpha, t) = u(\mathbf{z}(\alpha, t), t) \cdot \mathbf{n}(\alpha, t) \). Differentiating (2.1) and (2.2) in time, we get the evolution equations for \( \theta \) and \( s \):
\[
(2.6) \quad s_\alpha = \frac{z_{\alpha t} \cdot z_\alpha}{s} = \mathbf{T}_\alpha - \theta_\alpha U,
\]
\[
(2.7) \quad \theta_i = \frac{z_{\alpha t} \cdot z^\perp_\alpha}{s^2} = \frac{(z_i \cdot \mathbf{n})_\alpha}{s} - \frac{z_i \cdot z^\perp_\alpha}{s^2} = \frac{U_\alpha}{s} + \frac{\mathbf{T}}{s} \theta_\alpha.
\]
As \( \mathbf{T} \) is continuous on \( \mathbb{T} \), it holds that
\[
(2.8) \quad 2\pi s_\alpha = \int_{-\pi}^{\pi} \mathbf{T}_\alpha \, d\alpha - \int_{-\pi}^{\pi} \theta_\alpha U \, d\alpha = -\int_{-\pi}^{\pi} \theta_\alpha U \, d\alpha.
\]
Integrating (2.6) from \( -\pi \) to \( \alpha \), we have
\[
(2.9) \quad \mathbf{T}(\alpha, t) = \int_{-\pi}^{\alpha} \theta_\alpha(\alpha')U(\alpha') \, d\alpha' = \frac{\alpha + \pi}{2\pi} \int_{-\pi}^{\pi} \theta_\alpha U \, d\alpha + \mathbf{T}(t).
\]
Here \( \mathbf{T}(t) \) is a scalar function to be determined later. In the material coordinate, we can freely choose the starting point of the arc-length coordinate \( \alpha(-\pi, t) \). It follows from (2.5) that
\[
(2.10) \quad y_i(s, t) = \frac{1}{s}(X_i(s, t) \cdot \mathbf{t} - z_i(s + y(s, t), t) \cdot \mathbf{t}).
\]
We choose
\[
(2.11) \quad \mathbf{T}(t) = X_i(-\pi, t) \cdot \mathbf{t}(\alpha(-\pi, t), t),
\]
then we have
\[
(2.12) \quad y_i(-\pi, t) = \frac{1}{s}(X_i(-\pi, t) \cdot \mathbf{t} - \mathbf{T}(t)) = 0.
\]
In this paper, we always assume that \( y(-\pi, 0) = 0 \), so that \( y(-\pi, t) \equiv 0 \) and \( y(s, t) = \int_{-\pi}^{s} y_i(s', t) \, ds' \). We also have \( \alpha(-\pi, t) \equiv -\pi \), which means that at each time, the arc-length coordinate \( \alpha \) started at the same point of the string.

**Remark 2.1.** To simplify notation, when no confusion can arise, we will write
\[
y_i(s(\alpha, t), t) \rightarrow y_i(\alpha, t), \quad \theta(\alpha(s, t), t) \rightarrow \theta(s, t),
\]
and
\[
\int_T y_s(s(\alpha, t), t) + \theta(\alpha, t) d\alpha \rightarrow \int_T y_s + \theta d\alpha,
\]
\[
\int_T y_s(s, t) + \theta(\alpha(s, t), t) ds \rightarrow \int_T y_s + \theta ds.
\]

2.2. Velocity of the string. With the help of the fundamental solution \(G\), velocity fields on the string have the following expression

\[
(2.13) \quad u(z(\alpha, t), t) = \int_T G(z(\alpha, t) - X(s', t)) \cdot F(s', t) ds'.
\]

In the rest of paper, we use \(u(\alpha, t)\) and \(u(s, t)\) to denote \(u(z(\alpha, t), t)\) and \(u(X(s, t), t)\) respectively. From (2.4), we have

\[
(2.14) \quad X_s(s, t) = (1 + y_s(s, t))z_a(s + y(s, t), t),
\]

\[
(2.15) \quad X_{ss}(s, t) = (1 + y_s(s, t))^2 z_{aa}(s + y(s, t), t) + y_{ss}(s, t)z_a(s + y(s, t), t).
\]

Than, we rewrite (2.13) as follows:

\[
(2.16) \quad u(\alpha, t) = \int_T G(z(\alpha, t) - z(\alpha', t)) \cdot \left( (\lambda \theta_a n - \frac{\theta_{aaa}}{s^2} n - \frac{\theta^3}{s^2} n) (\alpha', t) + s \left( \frac{y_{ss}(s(\alpha', t), t) t}{1 + y_s(s(\alpha', t), t)} + (1 + y_s(s(\alpha', t), t)) \theta_a (\alpha', t) n \right) \right) d\alpha';
\]

\[
(2.17) \quad u(s, t) = \int_T G(X(s, t) - X(s', t)) \cdot \left( (\lambda \theta_a n - \frac{\theta_{aaa}}{s^2} n - \frac{\theta^3}{s^2} n) (\alpha(s', t), t) + (1 + y_s)(\lambda \theta_a n - \frac{\theta_{aaa}}{s^2} n - \frac{\theta^3}{s^2} n) (\alpha(s', t), t) \right) ds'.
\]

An easy computation shows that

\[
(2.18) \quad s(\lambda \frac{\theta_a n}{s} - \frac{\theta_{aaa}}{s^3} n - \frac{1}{2} \left( \frac{\theta^3}{s} \right) n) = \partial_s (\lambda t - \frac{\theta_{aaa}}{s^2} n - \frac{1}{2} \frac{\theta^3}{s^2} t),
\]

\[
(2.19) \quad (y_s t + (1 + y_s)^2 \theta_a n) = \partial_s ((1 + y_s) t).
\]

Therefore, it also holds that

\[
(2.20) \quad u = \text{p.v.} \int_T -\frac{\partial}{\partial \alpha'} G(X(s, t) - X(s', t)) \cdot \left( (1 + y_s) t \right) ds' + \text{p.v.} \int_T -\frac{\partial}{\partial \alpha'} G(z(\alpha, t) - z(\alpha', t)) \cdot (\lambda t - \frac{\theta_{aaa}}{s^2} n - \frac{1}{2} \frac{\theta^3}{s^2} t) d\alpha'.
\]

These three formulations of \(u\) will be used in different situations.

From the definitions of \(z\) and \(X\), one can see that

\[
(2.21) \quad z(\alpha, t) - z(\alpha', t) = s(t) \int_{\alpha'}^{\alpha} (\cos(\theta(\alpha'', t), t), \sin(\theta(\alpha'', t), t)) d\alpha'',
\]

\[
(2.22) \quad X(s, t) - X(s', t) = s(t) \int_{s'}^{s} \int_{\alpha'}^{\alpha} y_s(s'', t) ds'' (\cos(\theta(\alpha'', t), t), \sin(\theta(\alpha'', t), t)) d\alpha''.
\]
This indicates that $u$ is determined by $(\theta, y, s)$ and doesn’t depend on the exact position of $X$.

2.3. **Contour Dynamic System.** Now, we are in a position to introduce the following contour dynamic system.

**Proposition 2.1.** Assume that $X(s, t)$ is a closed string which satisfies (1.6)-(1.7) for some constants $\beta_1, \beta_2 > 0$, and $s > 0$, $\theta \in H^1(\mathbb{T})$, $y_s \in h^1(\mathbb{T})$, for $\forall t \in [0, T]$. The evolution equation of $X(s, t)$ in the 2-D Stokes immersed boundary problem (1.3) is equivalently given by

\begin{align*}
\theta_t(\alpha, t) &= \mathcal{L}(\theta)(\alpha, t) + g_\theta(\alpha, t), \quad \theta(\alpha, 0) = \theta_0(\alpha), \\
y_{ss}(s, t) &= \mathcal{Q}(y_s)(s, t) + g_y(s, t), \quad y_s(s, 0) = y_{0,s}(s, 0), \\
s_t(t) &= -\frac{1}{2\pi} \int_{-\pi}^{\pi} \theta_t u \cdot nd\alpha, \quad s(0) = s_0.
\end{align*}

Here

$$\mathcal{L}(\theta)(\alpha, t) = \frac{1}{4s^3} H(\theta_{aaa})(\alpha, t), \quad \mathcal{Q}(y_s)(s, t) = -\frac{1}{4} b(y_{ss})(s, t),$$

are two negative operators, $g_\theta$ and $g_y$ are the error terms with the following expressions:

\begin{align*}
g_\theta(\alpha, t) &= \frac{1}{4s^3} n \cdot [\mathcal{H}, n](\theta_{aaa})(\alpha, t) - \frac{1}{4} n \cdot [\mathcal{H}, t]\left(\frac{y_{ss}}{1 + y_s}\right)(\alpha, t) \\
&\quad - \frac{1}{4} n \cdot \mathcal{H}(1 + y_s)\theta_\alpha n - \frac{\lambda}{4s} n \cdot \mathcal{H}(\theta_\alpha n)(\alpha, t) + \frac{1}{8s^3} n \cdot \mathcal{H}((\theta_\alpha)^3 n)(\alpha, t) \\
&\quad + n \cdot \int_{\mathbb{T}} \left( \frac{\partial}{\partial \alpha} G(z(\alpha, t) - z(\alpha', t)) + \frac{1}{8\pi \tan(\frac{\alpha - \alpha'}{2})} Id\right) \\
&\quad \cdot \left( \frac{y_{ss}}{1 + y_s} t + (1 + y_s)\theta_\alpha n + \frac{\lambda \theta_\alpha}{s} n - \frac{\theta_{aaa}}{s^3} n - \frac{1}{2} \frac{(\theta_\alpha)^3}{s^3} n \right) d\alpha' \\
&\quad + \left( \int_{-\pi}^{\alpha} \theta_\alpha u \cdot nd\alpha' - \frac{\alpha + \pi}{2\pi} \int_{-\pi}^{\pi} \theta_\alpha u \cdot nd\alpha' - u \cdot t(\alpha, t) + \overline{\mathcal{T}(t)} \right) \frac{\theta_\alpha(\alpha, t)}{s},
\end{align*}

\begin{align*}
g_y(s, t) &= - (1 + y_s(s, t)) \frac{s}{s} - \frac{1}{4} t \cdot [b, t](y_{ss})(s, t) - \frac{1}{4} t \cdot b((1 + y_s)^2 \theta_\alpha n)(s, t) \\
&\quad + \frac{1}{4s^3} t \cdot [b, n](\partial_s \theta_\alpha)(s, t) - \frac{\lambda}{4s} t \cdot b((1 + y_s)\theta_\alpha n)(s, t) + \frac{1}{8s^3} t \cdot b((1 + y_s)(\theta_\alpha)^3 n)(s, t) \\
&\quad + t \cdot \int_{\mathbb{T}} \left( \frac{\partial}{\partial s} G(X(s, t) - X(s', t)) + \frac{1}{4 \pi \tan(\frac{s - s'}{2})} Id\right) \\
&\quad \cdot \left( y_s t + (1 + y_s)^2 \theta_\alpha n + (1 + y_s)(\frac{\lambda}{s} \theta_\alpha n - \frac{\theta_{aaa}}{s^3} n - \frac{1}{2} \frac{(\theta_\alpha)^3}{s^3} n) \right) ds',
\end{align*}
and \((H, b)\) are the Hilbert transform operators on \(T\) in the arc-length coordinate and in the material coordinate.

**Proof.** From (2.7) we know that

\[
\theta_t(\alpha, t) = \frac{1}{s}(u_\alpha(\alpha, t) \cdot n + (T(\alpha, t) - u(\alpha, t) \cdot t) \theta_\alpha(\alpha, t)).
\]

Here \(u_\alpha(\alpha, t) \cdot n\) is the most important term. By (2.13), it holds that

\[
u = n \cdot \text{p.v.} \int_T \left( \frac{z(\alpha) - z'(\alpha')}{4\pi|z(\alpha) - z(\alpha')|^2} \right) \text{Id} - \frac{1}{4\pi|z(\alpha) - z(\alpha')|^4} - \frac{1}{4\pi|z(\alpha) - z(\alpha')|^2} \cdot \left( \frac{s(\frac{ys}{1 + ys}, t)}{1 + ys} + (1 + ys)\theta_\alpha(\alpha', t)n + (\lambda\theta_\alpha n - \frac{\theta_{aaa} n}{s^2} - \frac{\theta^3 n}{s^2}) \right) \text{Id}.
\]

When \(|\alpha' - \alpha|\) is small, we formally find

\[
\frac{\partial}{\partial \alpha} G(z(\alpha, t) - z(\alpha', t)) \sim \frac{1}{4\pi\alpha' - \alpha} \sim \frac{1}{4\pi \tan(\frac{\alpha - \alpha'}{2})}.
\]

The Hilbert transform on \(T\) is defined as

\[
HY(\alpha) = \text{p.v.} \int_T \frac{Y(\alpha')}{2\pi \tan(\frac{\alpha - \alpha'}{2})} d\alpha'.
\]

Therefore, it follows that

\[
(2.26) \quad u_\alpha(\alpha, t) \cdot n
\]

\[
= \frac{1}{4\pi s^2} H(\theta_{aaa}) + \frac{1}{4\pi s^2} n \cdot [H, n](\theta_{aaa})(\alpha, t) - \frac{s}{4} n \cdot \frac{1}{8\pi s^2} H(\frac{ys}{1 + ys})(\alpha, t)
\]

\[
- \frac{s}{4} n \cdot H((1 + ys)\theta_\alpha n)(\alpha, t) - \frac{s}{4} n \cdot H(\theta_\alpha n)(\alpha, t) + \frac{1}{8\pi s^2} n \cdot H((\theta_\alpha^3)n)(\alpha, t)
\]

\[
+ n \cdot \int_T \left( \frac{\partial}{\partial \alpha} G(z(\alpha, t) - z(\alpha', t)) + \frac{1}{8\pi \tan(\frac{\alpha - \alpha'}{2})} \text{Id} \right)
\]

\[
\cdot \left( \frac{\frac{ys}{1 + ys} t + s(1 + ys)\theta_\alpha n + \lambda\theta_\alpha n - \frac{\theta_{aaa} n}{s^2} - \frac{1}{2} \frac{\theta^3 n}{s^2} \right) d\alpha'.
\]

Here \([H, n](\theta_{aaa}) = H(\theta_{aaa} n) - n H(\theta_{aaa})\) is a commutator.
Differentiating (2.10) in $s$, from (2.9) we have

\begin{equation}
(2.27) \quad y_{ss}(s, t) = \frac{1}{s} \frac{\partial}{\partial s} \left( X(s, t) \cdot t - z(s + y(s, t), t) \cdot t \right)
\end{equation}

\begin{equation}
\frac{1}{s} \left( u_s(s, t) \cdot t + (1 + y_s) u(s, t) \cdot \theta_{\alpha} n - (1 + y_s) \theta_{\alpha} u(s, t) \cdot n - (1 + y_s)^2 s \right)
\end{equation}

Similar to (2.26), it holds that

\begin{equation}
u_s(s, t) \cdot t \end{equation}

\begin{equation}
= -\frac{5}{4} b(y_{ss}) - \frac{s}{4} t \cdot [b, t](y_{ss}) - \frac{s}{4} t \cdot b((1 + y_s)^2 \theta_{\alpha} n) + \frac{1}{4 s^2} t \cdot b(1 + y_s) \theta_{\alpha} n
\end{equation}

\begin{equation}
+ t \cdot \int_{\mathbb{T}} \left( \frac{\partial}{\partial s} G(X(s, t) - X(s', t)) \right) + \frac{1}{4} \frac{1}{2 \pi \tan(\frac{s'}{2})} \frac{1}{2 \pi} \left( \frac{\partial}{\partial s} \right) y_{ss} t + \frac{s}{4} (1 + y_s)^2 \theta_{\alpha} n + (1 + y_s)(\lambda \theta_{\alpha} n - \frac{\theta_{\alpha \alpha}}{s^2} n - \frac{1}{2} (\theta_{\alpha}^2 n)) ds',
\end{equation}

where $\frac{s}{4} b Y(s) = p.v. \int_{\mathbb{T}} \frac{Y(s')}{2 \pi \tan(\frac{s'}{2})} ds'$

is the Hilbert transform on $\mathbb{T}$ in material coordinate. Then, one can deduce \textbf{(2.23)}-\textbf{(2.25)} immediately.

On the other hand, one can reconstruct $X$ from $(x, y, z)$. Indeed, recalling the definition of arc-length coordinate, we have

\begin{equation}
(2.28) \quad z(\alpha, t) = z(-\pi, t) + s \int_{-\pi}^{\alpha} (\cos(\theta(\alpha', t)), \sin(\theta(\alpha', t))) d\alpha'.
\end{equation}

From \textbf{(2.5)} and \textbf{(2.12)}, it is clear that

\begin{equation}
(2.29) \quad z(-\pi, t) = z(-\pi, 0) + \int_{0}^{t} z_{\alpha}(-\pi, t') dt' = X(-\pi, 0) + \int_{0}^{t} u(X(-\pi, t'), t') dt'.
\end{equation}

Consequently, it holds that

\begin{equation}
(2.30) \quad X(s, t) = X(-\pi, 0) + \int_{0}^{t} u(X(-\pi, t'), t') dt'
\end{equation}

\begin{equation}
+ s \int_{-\pi}^{s} \int_{s'}^{t} y(x(s', t'), t') ds'dt',
\end{equation}

This completes the proof. \hfill \Box

In the derivation of \textbf{(2.23)} and \textbf{(2.24)}, we extract the linear principal parts $\mathcal{L}(\theta)$ and $\mathcal{L}(y_s)$. This approach is known as small-scale decomposition which is introduced by
Beale, Hou, Lowengrub and Shelley in [4,16]. In what follows, we shall analyze the properties of the contour dynamic system.

2.4. Energy Dissipation. Solutions to the immersed boundary problem satisfy the following energy identity.

**Lemma 2.1.** Assume \( X \) is a solution to (1.5) with 
\[
s(t) > 0, \quad \theta \in L^2([0, T]; H^3(\mathbb{T})), \quad y_s \in L^2([0, T]; \dot{H}^2(\mathbb{T}))
\]
satisfying (1.6)–(1.7) for some constants \( \beta_1, \beta_2 > 0 \). It holds that 
\[
(2.31)
\frac{d}{dt} \left( \frac{1}{2s(t)} \int_\mathbb{T} \theta_\alpha^2(\alpha, t)d\alpha + 2\pi \lambda s(t) + \frac{s^2(t)}{2} \int_\mathbb{T} (1 + y_s)^2 ds \right) = -\int_{\mathbb{R}^2} |\nabla u(x, t)|^2 dx,
\]
where \( u \) is the velocity field.

**Proof.** From our assumption and the properties of the Green function \( G \), \( u \) is continuous on \( \mathbb{R}^2 \) and tends to 0 as \( |x| \to +\infty \). Since the first equation of (1.3) holding in the sense of distribution, we choose the test function to be \( u \), which implies
\[
\int_{\mathbb{R}^2} |\nabla u|^2 dx = \int_{\mathbb{R}^2} u(x, t) \cdot f(x, t) dx
\]
\[
= \int_{\mathbb{R}^2} \int_\mathbb{T} u(x, t) \cdot F(s, t)\delta(x - X(s, t)) ds dx
\]
\[
= \int_\mathbb{T} u(\alpha, t) \cdot (A\theta_\alpha n - \frac{\theta_{s\alpha\alpha} n}{s^2} - \frac{1}{2}\frac{\partial^3 n}{s^2}) d\alpha + \int_\mathbb{T} u(s, t) \cdot X_{ss}(s, t) ds.
\]
By (2.14), it holds that
\[
-\frac{d}{dt} \left( \frac{s^2}{2} \int_\mathbb{T} (1 + y_s)^2 ds \right) = -\frac{d}{dt} \frac{1}{2} \int_\mathbb{T} |(1 + y_s)z_s(\alpha(s, t), t)|^2 ds
\]
\[
= -\frac{d}{dt} \frac{1}{2} \int_\mathbb{T} |X_s(s, t)|^2 ds = -\int_\mathbb{T} X_s(s, t) \cdot X_{ss}(s, t) ds
\]
\[
= \int_\mathbb{T} u(X(s, t), t) \cdot X_{ss}(s, t) ds = \int_\mathbb{T} u(s, t) \cdot X_{ss}(s, t) ds.
\]
From (2.8), (2.7) and (2.25), we have
\[
-\frac{d}{dt} \left( \frac{1}{2s^2} \int_\mathbb{T} \theta_\alpha^2 d\alpha + 2\pi \lambda s \right)
\]
\[
= -\frac{1}{s} \int_\mathbb{T} \theta_\alpha \theta_{s\alpha} d\alpha + \frac{s^2}{2s^2} \int_\mathbb{T} \theta_\alpha^2 d\alpha - 2\pi \lambda s,\]
\[
= \frac{1}{s} \int_\mathbb{T} \theta_{s\alpha} \left( \frac{u \cdot n}{s} \right)_s + \frac{T}{s} \theta_\alpha d\alpha + \frac{s^2}{2s^2} \int_\mathbb{T} \theta_\alpha^2 d\alpha - 2\pi \lambda s,
\]
\[
= -\frac{1}{s^2} \int_\mathbb{T} \theta_{s\alpha\alpha} u \cdot n d\alpha - \frac{1}{2s^2} \int_\mathbb{T} \theta_\alpha^2 T_{s\alpha} d\alpha + \frac{s^2}{2s^2} \int_\mathbb{T} \theta_\alpha^2 d\alpha - 2\pi \lambda s,\]
\[
= -\frac{1}{s^2} \int_\mathbb{T} \theta_{s\alpha\alpha} u \cdot n d\alpha - \frac{1}{2s^2} \int_\mathbb{T} \theta_\alpha^2 (s_t + \theta_\alpha u \cdot n) d\alpha + \frac{s^2}{2s^2} \int_\mathbb{T} \theta_\alpha^2 d\alpha - 2\pi \lambda s.
\[
= \int_T \mathbf{u}(\alpha, t) \cdot (\lambda \theta_\alpha \mathbf{n} - \frac{\partial_{aaa} \mathbf{n}}{s^2} - \frac{1}{2} \frac{\theta_\alpha^3 \mathbf{n}}{s^2}) d\alpha.
\]

This proves the lemma.

**Remark 2.2.** From (2.18) and (2.19), it is immediately that \( \int_T F(s, t) ds = 0 \). Thanks to this fact, we do not suffer from the Stokes paradox of logarithmic growth of the velocity field \( \mathbf{u} \) at infinity.

3. **A Priori Estimates**

In this section, we derive the evolution equation to the oscillation part of the tangent angle function. Based on this equation, we introduce an a priori estimate of the dynamic system and give a priori estimates of this system.

3.1. **The oscillation part of the tangent angle function.** The tangent angle function can be split into its mean and its oscillation, i.e.,

\[
\theta(\alpha, t) = \bar{\theta}(\alpha, t) + \tilde{\theta}(t),
\]

where \( \bar{\theta}(t) = \frac{1}{2\pi} \int_{\alpha'} \theta(\alpha', t) d\alpha' \), and \( \tilde{\theta}(\alpha, t) = \theta(\alpha, t) - \bar{\theta}(t) \) is a zero-mean function.

From (2.7) and (2.9), one can see that

\[
\tilde{\theta}_t(t) = \frac{1}{2\pi} \int_{\alpha'} \tilde{\theta}(\alpha', t) d\alpha' = \frac{1}{2\pi s} \int_{\alpha'} \mathcal{T}(\alpha', t) \theta_\alpha(\alpha', t) d\alpha'.
\]

It follows that

\[
\tilde{\theta}_t(\alpha, t) = \frac{1}{s} (\mathbf{u}_s \cdot \mathbf{n} - \mathbf{u} \cdot \mathbf{t} \theta_\alpha + \mathcal{T} \theta_\alpha)(\alpha, t) - \frac{1}{2\pi s} \int_{\alpha'} \mathcal{T}(\alpha', t) \theta_\alpha(\alpha', t) d\alpha'.
\]

Next, we will show that all the information of deformation is contained in \( (\tilde{\theta}, y_s, s) \). That is to say, if we regard \( \tilde{\theta} \) as an independent variable, the following result holds.

**Lemma 3.1.** Given \( (\tilde{\theta}(\alpha, s), y_s(s), s) \), let \( \mathbf{z}, \mathbf{n} \) and \( \mathbf{t} \) be the functions defined in (2.16), (2.28) and (2.3), it holds that

\[
\frac{d}{d\tilde{\theta}} (\mathbf{u} \cdot \mathbf{n})(\tilde{\theta}, \alpha) = \frac{d}{d\tilde{\theta}} (\mathbf{u} \cdot \mathbf{t})(\tilde{\theta}, \alpha) = 0.
\]

The proof of this lemma can be found in Appendix C. As

\[
\partial_\alpha (\mathbf{u}(\mathbf{z}(\tilde{\theta}, \alpha)) \cdot \mathbf{n}(\tilde{\theta}, \alpha)) = \partial_\alpha (\mathbf{u}(\mathbf{z}(\tilde{\theta}, \alpha)) \cdot \mathbf{n}(\tilde{\theta}, \alpha)) + \mathbf{u}(\mathbf{z}(\tilde{\theta}, \alpha)) \cdot \theta_\alpha t(\tilde{\theta}, \alpha).
\]

Lemma 3.1 implies that \( \frac{d}{d\tilde{\theta}} (\partial_\alpha (\mathbf{u}(\mathbf{z}(\tilde{\theta}, \alpha)) \cdot \mathbf{n}(\tilde{\theta}, \alpha)) = 0. \)

We introduce

\[
\mathbf{n} = (-\sin(\tilde{\theta}), \cos(\tilde{\theta})), \quad \mathbf{t} = (\cos(\tilde{\theta}), \sin(\tilde{\theta})), \quad \mathbf{z}(\alpha, t) = s \int_{\alpha'} \mathcal{T}(\alpha', t, \tilde{\theta}) \, d\alpha',
\]

\[
\mathbf{u}(\alpha, t) = \int_T G(\mathbf{z}(\alpha, t) - \mathbf{z}(\alpha', t)) \cdot ((\lambda \bar{\theta}_\alpha \mathbf{n} + \frac{\theta_\alpha^3 \mathbf{n}}{s^2} - \frac{1}{2} \frac{\partial_{aaa} \mathbf{n}}{s^2} - \frac{\partial_{a} \mathbf{n}}{s^2} \mathbf{n}) (\alpha', t) + s \frac{y_s(\mathbf{z}(\alpha', t), t)}{1 + y_s(\mathbf{z}(\alpha', t), t)} (1 + y_s(t) \mathbf{z}(\alpha', t)) \mathbf{n} d\alpha'.
\]
From Lemma[3.1] we know that
\[ \dot{u}(\alpha, t) \cdot \dot{n}(\alpha, t) = u(\alpha, t) \cdot n(\alpha, t), \quad \ddot{u}(\alpha, t) \cdot \dot{t}(\alpha, t) = u(\alpha, t) \cdot \dot{t}(\alpha, t), \]
\[ \partial_{\alpha} (\dot{u}(\alpha, t)) \cdot \dot{n}(\alpha, t) = \partial_{\alpha} (u(\alpha, t)) \cdot n(\alpha, t). \]

Therefore, the evolution equation of \( \tilde{t} \) can be rewritten as
\[ \tilde{t}(\alpha, t) = \frac{1}{\tilde{s}} (\tilde{u} \cdot \dot{\tilde{n}} - \tilde{u} \cdot \tilde{t} \partial_{\alpha} \tilde{t}) - \frac{1}{2\pi \tilde{s}} \int_{-\pi}^{\pi} \tilde{T}(\alpha', t) \tilde{t}(\alpha', t) d\alpha'. \]

Here
\[ \tilde{T}(\alpha, t) = \int_{-\pi}^{\alpha} \partial_{\alpha}(\alpha') \tilde{u} \cdot \dot{\tilde{n}} d\alpha' - \frac{\alpha + \pi}{2\pi} \int_{-\pi}^{\pi} \tilde{t} \tilde{u} \cdot \dot{\tilde{n}} d\alpha' + \tilde{u}(\tilde{z}(-\pi, t), t) \cdot \tilde{t}(-\pi, t). \]

This shows that the evolution equation of \( \tilde{t}(\alpha, t) \) is independent of \( \tilde{t}(t) \), and \( \tilde{t}(t) \) is determined by \( (\tilde{t}, y_s, \tilde{s}) \).

3.2. Modified contour dynamic system. Based on the evolution equations of \( (\tilde{t}, y_s, \tilde{s}) \), we introduce the modified contour dynamic system. Giving function \( (\tilde{t}, \tilde{y}_s, \tilde{s}) \) which satisfies
\( (\tilde{t}(\alpha, t) - \alpha) \in H^3(\mathbb{T}), \quad \tilde{y}_s(s, t) \in h^1(\mathbb{T}), \quad \int_{-\pi}^{\pi} \tilde{t} d\alpha = 0, \quad \int_{-\pi}^{\pi} \tilde{y}_s d\tilde{s} = 0, \quad \tilde{s}(t) > 0, \quad \forall t \in [0, T], \)
we define the modified direction vectors by
\[ \tilde{n}(\alpha, t) = (-\sin(\tilde{t}(\alpha, t)), \cos(\tilde{t}(\alpha, t))), \quad \tilde{t}(\alpha, t) = (\cos(\tilde{t}(\alpha, t)), \sin(\tilde{t}(\alpha, t)), \tilde{t}(\alpha, t)), \]
and the modified velocity by
\[ \tilde{u}(\alpha, t) = \int_T G(\tilde{z}(\alpha, t) - \tilde{z}(\alpha', t)) \cdot (\lambda \tilde{\phi}_{\alpha} \tilde{n} - \tilde{\phi}_{\alpha \alpha} \tilde{n} - \tilde{\phi}_{\alpha \alpha} \tilde{n}) \tilde{u}(\alpha', t) d\alpha' + \int_T G(\tilde{z}(\alpha, t) - \tilde{X}(s', t)) \cdot \tilde{\phi}_s \tilde{t}(s') \tilde{t} + (1 + \tilde{y}_s)^2 \tilde{t}(\alpha(s', t)) \tilde{n}(s) \tilde{n}(s', t) d\alpha'. \]

\[ \tilde{T}(\alpha, t) = \int_{-\pi}^{\alpha} \tilde{t} \tilde{u}(\alpha', t) \tilde{n}(\alpha', t) d\alpha' - \frac{\alpha + \pi}{2\pi} \int_{-\pi}^{\pi} \tilde{t} \tilde{u}(\alpha', t) \tilde{n}(\alpha', t) d\alpha' + \tilde{T}(t). \]

Here
\[ \tilde{z}(\alpha, t) = \tilde{z}(t) \int_{-\pi}^{\alpha} (\cos(\tilde{t}(\alpha', t)), \sin(\tilde{t}(\alpha', t))) d\alpha' - \tilde{z}(t) \alpha \int_{-\pi}^{\alpha} (\cos(\tilde{t}(\alpha', t)), \sin(\tilde{t}(\alpha', t))) d\alpha' , \]
\[ \tilde{X}(s, t) = \tilde{z}(\alpha(s, t), t), \quad \alpha(s, t) = s + \int_{-\pi}^{s} \tilde{y}_s(s') ds', \]
\[ \tilde{T}(t) = \tilde{u}(\tilde{z}(-\pi, t), t) \cdot \pi(-\pi, t). \]
Then we introduce the following modified contour dynamic system:

\[
\tilde{\phi}_i(\alpha, t) = \frac{1}{\tilde{\xi}(t)} (\tilde{\mathbf{u}}_\alpha \cdot \tilde{\mathbf{n}} + (\tilde{T} - \tilde{\mathbf{u}} \cdot \tilde{\mathbf{i}})\tilde{\phi}_\alpha) - \frac{1}{2\pi \tilde{\xi}(t)} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \tilde{T}(t)\tilde{\phi}_\alpha d\alpha'' d\alpha' \\
= \frac{1}{4\tilde{\xi}(t)} \mathcal{H}(\tilde{\phi}_{\alpha\alpha})(\alpha, t) + \tilde{g}_\theta(\alpha, t),
\]

\[
\tilde{\theta}_i(t) = -\frac{1}{2\pi \tilde{\xi}(t)} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \tilde{T}(t)\tilde{\phi}_\alpha d\alpha',
\]

\[
\tilde{y}_{si}(s, t) = \frac{1}{\tilde{\xi}(t)} \partial_s (\tilde{\mathbf{u}} \cdot \tilde{\mathbf{i}}(\alpha(s, t), t) - \tilde{T}(\alpha(s, t), t)) = \frac{1}{\tilde{\xi}(t)} \tilde{\mathbf{u}}_s(s, t) \cdot \tilde{\mathbf{i}} - (1 + \tilde{y}_s)\tilde{\xi} \\
= -\frac{1}{4} b(\tilde{y}_{ss})(s, t) + \tilde{g}_s(s, t),
\]

\[
\tilde{s}_i(t) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\phi}_s(\alpha, t)\tilde{\mathbf{u}}(\alpha, t) \cdot \tilde{\mathbf{n}}(\alpha, t) d\alpha,
\]

with

\[
\tilde{\theta}(\alpha, 0) = \tilde{\theta}_0(\alpha), \quad \tilde{\theta}(0) = \tilde{\theta}_0, \quad \tilde{y}_s(s, 0) = y_{0s}(s, 0), \quad \tilde{s}(0) = s_0.
\]

Here \(\tilde{g}_\theta\) and \(\tilde{g}_s\) are the modified error terms which have similar expressions to \(g_\theta\) and \(g_s\). From above definitions, it is easy to verify that \(\int_T \tilde{\theta}_0 d\alpha = \int_T \tilde{g}_\theta ds = 0\). Therefore,

\[
\int_T \tilde{\theta} d\alpha = \int_T \tilde{\theta} d\alpha = \int_T \tilde{y}_{si} ds = \int_T \tilde{y}_s ds = 0.
\]

The reason to introduce such modified system is that, not every \(\tilde{\theta}\) satisfying \((\tilde{\theta} - \alpha) \in C(T)\) can reconstruct a closed string, in other word, \(\tilde{\theta}\) may not satisfy the following closed-string condition

\[
\int_{-\pi}^{\pi} (\cos(\tilde{\theta}), \sin(\tilde{\theta})) d\alpha = (0, 0).
\]

In the proof of Theorem 1.1, we need to use the Schauder fixed point theorem, which is valid only in a convex space. However, the set of \(\theta\) satisfying \((3.10)\) is not convex. To overcome such difficulty, we introduce \(\tilde{z}\) which is continuous on \(T\) \([24]\), then \((3.1)\) is well defined. We call \((\theta_0, y_{0s}, s_0)\) the closed-string initial data if \(\theta_0\) satisfies \((3.10)\). To solve this new system, we introduce the modified non-self-intersecting assumption and the modified well-stretched assumption:

\[
\frac{1}{|\alpha_1 - \alpha_2|} \left| \int_{\alpha_2}^{\alpha_1} (\cos(\tilde{\theta}), \sin(\tilde{\theta})) d\alpha' \right| - \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} (\cos(\tilde{\theta}), \sin(\tilde{\theta})) d\alpha \right| \geq \beta_1 > 0,
\]

\[
1 + \tilde{y}_s(s, t) \geq \beta_2 > 0.
\]

From above inequalities, one can see that

\[
\beta_1 \leq 1, \quad \beta_2 \leq \min_{\alpha \in T} (1 + \tilde{y}_s) \leq \frac{1}{2\pi} \int_T (1 + \tilde{y}_s) ds = 1.
\]
Under these two assumptions, for $\forall s_1, s_2 \in \mathbb{T}$, it holds that
\[
|\tilde{X}(s_1, t) - \tilde{X}(s_2, t)| \\
\geq \tilde{s} \left| \int_{\alpha(s_1, t)}^{\alpha(s_2, t)} (\cos(\tilde{\theta}), \sin(\tilde{\theta}))d\alpha' \right| - \tilde{s} \left| \frac{\alpha(s_1, t) - \alpha(s_2, t)}{2\pi} \right| \int_{-\pi}^{\pi} (\cos(\tilde{\theta}), \sin(\tilde{\theta}))d\alpha \\
\geq \beta_1 \tilde{s} |\alpha(s_1, t) - \alpha(s_2, t)| = \beta_1 \tilde{s} \int_{s_2}^{s_1} 1 + \tilde{y}_s(s', t)ds' \\
\geq \beta_1 \beta_2 \tilde{s} |s_1 - s_2|,
\]
where $|s_1 - s_2|$ is the distance between $s_1$ and $s_2$ on $\mathbb{T}$.

In the next section, we will show that this modified system is well-posed, and the solutions to (3.5)-(3.8) with closed-string initial data always satisfy (3.10). Then all these modified functions we defined above are actually the same to the functions defined in Section 2. Therefore, $(\tilde{\theta} + \tilde{\theta}, \tilde{y}, \tilde{s}, \tilde{z}, \tilde{u}, \tilde{X}, \tilde{T}, \tilde{n}, \tilde{i})$ is a solution to (2.23)-(2.25). In the rest of this paper, we omit the tilde on $(\tilde{\theta}, \tilde{\theta}, \tilde{y}, \tilde{s}, \tilde{z}, \tilde{u}, \tilde{X}, \tilde{T}, \tilde{n}, \tilde{i})$ for convenience.

### 3.3. Preliminaries

First, we introduce some fundamental lemmas.

**Lemma 3.2.** $\square$ Let $s \geq 1$ and assume $(\theta(\alpha) - \alpha) \in H^{s-1}(\mathbb{T})$, we have
\[
\|z\|_{H^s} \leq s(1 + \|\theta - \alpha\|_{H^{s-1}}).
\]

**Proof.** Recalling (3.3), we deduce that
\[
\tilde{z}_\alpha(\alpha, t) = \tilde{s}(t)(\cos(\theta(\alpha, t)), \sin(\theta(\alpha, t))) - \frac{\tilde{s}(t)}{2\pi} \int_{-\pi}^{\pi} (\cos(\theta(\alpha', t)), \sin(\theta(\alpha', t)))d\alpha'.
\]

The conclusion follows immediately. $\square$

**Lemma 3.3.** $\square$ For $\forall \psi \in H^i(\mathbb{T})$, the operator $[\mathcal{H}, \psi]$ is bounded from $H^0(\mathbb{T})$ to $H^{s-1}(\mathbb{T})$, it is also bounded from $H^{-1}(\mathbb{T})$ to $H^{s-2}(\mathbb{T})$. Thus, for $i = 0, -1$, we have
\[
\|[\mathcal{H}, \psi]f\|_{H^{s-1+i}} \leq \|f\|_{H^s}\|\psi\|_{H^i}.
\]

**Proof.** We write $[\mathcal{H}, \psi]$ as an integral operator:
\[
[H, \psi]f(\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(\alpha') (\psi(\alpha') - \psi(\alpha)) \frac{1}{\tan \left(\frac{\alpha' - \alpha}{2}\right)} d\alpha' \\
= \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(\alpha') \frac{\psi(\alpha') - \psi(\alpha)}{\alpha' - \alpha} \frac{\alpha' - \alpha}{\tan \left(\frac{\alpha' - \alpha}{2}\right)} d\alpha'.
\]

Here $\frac{\psi(\alpha') - \psi(\alpha)}{\alpha' - \alpha}$ is a divided difference, and $\frac{\alpha' - \alpha}{\tan \left(\frac{\alpha' - \alpha}{2}\right)}$ is an analytic function, the conclusion follows immediately. See $\square$ for more details.

**Lemma 3.4.** Suppose that $f \in H^1(\mathbb{T})$ and $g \in H^{1/2}(\mathbb{T})$, there exists a constant $C$ such that
\[
\|fg\|_{H^{1/2}} \leq C\|f\|_{H^1}\|g\|_{H^{1/2}}.
\]
**Proof.** Consider the operator $\mathcal{N} : H^k(\mathbb{T}) \to H^k(\mathbb{T})$ given by $\mathcal{N}(g) = fg$ for $k = 0, 1$. It is a bounded operator for $k = 0, 1$. Indeed

\[
\|fg\|_{L^2} \leq \|f\|_{L^\infty}\|g\|_{L^2} \leq \|f\|_{H^1}\|g\|_{L^2}; \\
\|fg\|_{H^1} \leq \|f_hg\|_{L^2} + \|f_{g_{h}}\|_{L^2} \leq \|f\|_{H^1}\|g\|_{H^1}.
\]

Then the interpolation theory implies that $\mathcal{N}$ is bounded from $H^{1/2}$ to itself [36]. □

The conclusions of the above two lemmas still hold in the material coordinate.

We introduce some notations that will be heavily used in the rest of this paper. For $\alpha, \alpha' \in \mathbb{T}$, let

\[
\tau(\alpha, \alpha') = \begin{cases}
\alpha' - \alpha + 2\pi, & \alpha' - \alpha < -\pi, \\
\alpha' - \alpha, & -\pi \leq \alpha' - \alpha < \pi, \\
\alpha' - \alpha - 2\pi, & \pi \leq \alpha' - \alpha,
\end{cases}
\]

which means that $\tau(\alpha, \alpha') \in [-\pi, \pi)$. We define

\[
L(\alpha, \alpha') = \frac{z(\alpha') - z(\alpha)}{\tau(\alpha, \alpha')}, \quad M(\alpha, \alpha') = \frac{z(\alpha') - z(\alpha)}{\tau(\alpha, \alpha')}, \quad N(\alpha, \alpha') = \frac{L(\alpha, \alpha') - z_0'(\alpha)}{\tau(\alpha, \alpha')},
\]

for $\alpha' \neq \alpha$, and

\[
L(\alpha, \alpha) = z_0(\alpha), \quad M(\alpha, \alpha) = z_0(\alpha), \quad N(\alpha, \alpha) = \frac{z_0(\alpha)}{2}.
\]

Note that $\tau(\alpha, \alpha')$ is not continuous at $|\alpha' - \alpha| = \pi$. For the functions involving $\tau$, i.e. $L(\alpha, \alpha')$, we write $\partial_\alpha L(\alpha, \alpha')$ in the sense of left derivative, and $\partial_{\alpha'} L(\alpha, \alpha')$ in the sense of right derivative. Therefore, it is easy to verify that

\[
\partial_\alpha L = N, \quad \partial_{\alpha'} L = \frac{z_0(\alpha') - L}{\tau}, \quad z_0(\alpha') = L + \tau(M - N).
\]

Similarly, for $s, s' \in \mathbb{T}$, let $l(s, s') = \tau(s, s')$. We define

\[
l(s, s') = \frac{X(s') - X(s)}{l(s, s')}, \quad m(s, s') = \frac{X_s(s') - X_s(s)}{l(s, s')}, \quad n(s, s') = \frac{l(s, s') - X_s(s)}{l(s, s')},
\]

for $s' \neq s$, and

\[
l(s, s) = s(1 + y_s)\mathbf{t}, \quad m(s, s) = s((1 + y_s)^2\sigma_1 \mathbf{n} + y_{ss} \mathbf{t}) , \quad n(s, s) = \frac{s}{2}((1 + y_s)^2\sigma_1 \mathbf{n} + y_{ss} \mathbf{t}).
\]

**Lemma 3.5.** Suppose $f(\alpha) \in H^{2/5}(\mathbb{T})$, $g(s) \in h^{2/5}(\mathbb{T})$, let

\[
\hat{f}(\alpha, \alpha') = \frac{f(\alpha') - f(\alpha)}{\tau(\alpha, \alpha')}, \quad g(s, s') = \frac{g(s') - g(s)}{l(s, s')},
\]

we have the following estimates:

1. For $\forall 1 \leq p \leq \infty$, it holds that

\[
\|f(\alpha, \cdot)\|_{L^p} \leq C\|\partial_\alpha f\|_{L^p}, \quad \|g(s, \cdot)\|_{L^p} \leq C\|\partial_\alpha g\|_{L^p}.
\]

2. There exists a universal constant $C$ such that

\[
\|f\|_{L^2} \leq C\|f\|_{H^{1/2}}, \quad \|\partial_\alpha f\|_{L^2} \leq C\|f\|_{H^{3/2}}, \quad \|\partial_{\alpha'}^2 f\|_{L^2} \leq C\|f\|_{H^{5/2}},
\]

\[
\|g\|_{L^p} \leq C\|g\|_{l^{1/2}}, \quad \|\partial_\alpha g\|_{L^p} \leq C\|g\|_{l^{3/2}}, \quad \|\partial_{\alpha'}^2 g\|_{L^p} \leq C\|g\|_{l^{5/2}}.
\]
Especially, it holds that
\begin{equation}
\|\partial_x n\|_{L^2} \leq C\|X\|_{H^{5/2}} \leq C\|1 + \|y_s\|_{L^\infty}\|1 + \|\theta\|_{H^2}\|(1 + \|y_s\|_{H^{3/2}} + \|\theta\|_{H^2}).
\end{equation}

(3) Let $M$ be the Hardy-Littlewood maximal operator on $\mathbb{T}$. Then for $\forall \alpha, \alpha' \in \mathbb{T}$, $\forall s, s' \in \mathbb{T}$, we have
\begin{equation}
|f(\alpha, \alpha')| \leq 2Mf_{\alpha}(\alpha), \quad |g(s, s')| \leq 2Mg_{s}(s).
\end{equation}

**Proof.** For the proofs of (3.13) and (3.15), we refer the readers to [22]. We only give the proof of (3.14). The idea is that homogeneous Sobolev norms can be described in terms of finite differences [3].

From the definitions, we have $\partial_s n = \partial^2_n$, therefore
\begin{align*}
\|\partial_s n\|_{L^2}^2 &= \int_T \int_T \frac{2(\mathbf{X}(s') - \mathbf{X}(s)) - 2(s' - s)\mathbf{X}_s(s) - (s' - s)^2 \mathbf{X}_{ss}(s)}{(s' - s)^3} \, ds' \, ds \\
&= \int_T \int_T \frac{2(\mathbf{X}(s' + s) - \mathbf{X}(s)) - 2s'\mathbf{X}_s(s) - s'^2 \mathbf{X}_{ss}(s)}{s'^3} \, ds' \, ds \\
&= \int_T \frac{1}{|s'^6|} \sum_{k \in \mathbb{Z}} |2e^{i\theta^2 k} - 2 - 2i\theta k + s'^2 k^2| |\mathbf{X}(k)|^2 \, ds'.
\end{align*}

We define
\begin{equation*}
\mathcal{F}(k) = \int_T \frac{2e^{i\theta k} - 2 - 2i\theta k + s'^2 k^2}{|s'|^6} \, ds'.
\end{equation*}

By the Taylor expansion, one can see that $\mathcal{F}$ is well defined. It is easily checked that $\mathcal{F}$ is a radial and homogeneous function of degree 5. This implies that the function $\mathcal{F}(k)$ is proportional to $|k|^5$. As a result, we have
\begin{equation*}
\|\partial_s n\|_{L^2}^2 = C\|X\|_{H^{5/2}}^2.
\end{equation*}

It follows from (3.4) that
\begin{equation*}
\|X\|_{H^{5/2}} \leq \|y_{ss} \mathbf{z}_s\|_{H^{3/2}} + s\|(1 + \|y_s\|_{L^\infty})^2 \theta_a \mathbf{n}\|_{H^{3/2}}.
\end{equation*}

Applying Lemma [3.4] we deduce that
\begin{align*}
\|y_{ss} \mathbf{z}_s\|_{H^{3/2}} &\leq s\|y_s\|_{H^{3/2}}(1 + \|y_s\|_{L^\infty})\|\theta\|_{H^2} \\
\|(1 + \|y_s\|_{L^\infty})^2 \theta_a \mathbf{n}\|_{H^3} &\leq (1 + \|y_s\|_{L^\infty})^5(1 + \|\theta\|_{H^2})(1 + \|y_s\|_{H^{3/2}} + \|\theta\|_{H^2}).
\end{align*}

Then, we conclude that
\begin{equation*}
\|X\|_{H^{5/2}} \leq s(1 + \|y_s\|_{L^\infty})^5(1 + \|\theta\|_{H^2})(1 + \|y_s\|_{H^{3/2}} + \|\theta\|_{H^2}).
\end{equation*}
3.4. Estimate of $\tilde{g}_\theta$ and $\tilde{g}_\alpha$. We start from the estimates of a special term.

**Lemma 3.6.** Suppose that $(\theta, y_\alpha, s)$ satisfies

\[ \theta(\alpha, t) - \alpha \in H^1(\mathbb{T}), \ y_\alpha(t, s) \in h^2(\mathbb{T}), \int_{-\pi}^{\pi} \theta d\alpha = \int_{-\pi}^{\pi} y_\alpha ds = 0, \ s(t) > 0, \ \forall t \in [0, T], \]

and (3.11)-(3.12) for some constants $\beta_1, \beta_2 > 0$. Then we have

\[
\left\| \int_T G(z(\cdot, t) - z(\alpha', t)) \cdot \theta_\alpha n(\alpha', t) d\alpha' \right\|_{L^2} \leq C \frac{1}{\beta_1} \| \theta - \alpha \|_{H^1},
\]

\[
\left\| \int_T G(X(\cdot, t) - X(s', t)) \cdot (1 + y_\alpha(s'))(\theta_\alpha n)(\alpha(s', t), t) ds' \right\|_{L^2} \leq C \frac{1}{\beta_1 \beta_2} \| \theta - \alpha \|_{H^1},
\]

where $C > 0$ is a constant.

**Proof.** From the fact that

\[
\int_T G(X(s, t) - X(s', t)) \cdot (1 + y_\alpha(s'))(\theta_\alpha n)(\alpha(s', t), t) ds'
\]

we only give the proof of the first inequality.

As $G$ is the Green function of the incompressible Stokes equation, and $\frac{z_\alpha}{|z_\alpha|}$ is the unit normal vector of the modified string $z$, it holds that (3.11)

\[
\int_T G(z(\alpha, t) - z(\alpha', t)) \cdot z_\alpha^-(\alpha', t) d\alpha' = 0.
\]

Therefore

\[
\int_T G(z(\alpha, t) - z(\alpha', t)) \cdot \theta_\alpha n(\alpha', t) d\alpha'
\]

\[
= \int_T G(z(\alpha, t) - z(\alpha', t)) \cdot (\theta_\alpha n - \frac{z_\alpha}{s})(\alpha', t) d\alpha'
\]

\[
= \int_T -\frac{\partial}{\partial \alpha'} G(z(\alpha, t) - z(\alpha', t)) \cdot (t(\alpha') - t(\alpha) - \hat{\theta}(\alpha') + \hat{\theta}(\alpha)) d\alpha'
\]

\[
= \int_T \left( \frac{L \cdot z_\alpha(\alpha')}{|L|^2} I_d + \frac{2L \cdot z_\alpha(\alpha') L \otimes L}{|L|^4} - \frac{z_\alpha(\alpha') \otimes L + L \otimes z_\alpha(\alpha')}{|L|^2} \right) 
\]

\[
\cdot \frac{1}{4\pi t} (t(\alpha') - t(\alpha) - \hat{\theta}(\alpha') + \hat{\theta}(\alpha)) d\alpha',
\]

where

\[
\hat{\theta}(\alpha, t) \overset{\text{def}}{=} \int_{-\pi}^{\alpha} (\sin(\theta(\alpha, t)), \cos(\theta(\alpha, t))) d\alpha' - \frac{\alpha}{2\pi} \int_{-\pi}^{\alpha} (\sin(\theta(\alpha, t)), \cos(\theta(\alpha, t))) d\alpha'.
\]

Consequently, by using Lemma 3.5 we have

\[
\left\| \int_T G(z(\alpha, t) - z(\alpha', t)) \cdot \theta_\alpha n(\alpha', t) d\alpha' \right\|_{L^2} \leq C \frac{1}{\beta_1} \| t - \hat{\theta} \|_{H^1}.
\]
As $\theta - \alpha \in C(\mathbb{T})$, it holds that $\theta(\pi) - \theta(-\pi) = 2\pi$. Therefore, it is easy to see that
\[
\int_{-\pi}^{\pi} (\sin(\theta(\alpha')), \cos(\theta(\alpha')))\theta_\alpha(\alpha')d\alpha' = 0.
\]
Finally we have
\[
\|t - t\|_{H^1} \leq \|\sin(\theta)(\theta_\alpha - 1)\|_{L^2} + \|\sin(\theta)(\theta_\alpha - 1)\|_{L^2} + \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} \sin(\theta)(\theta_\alpha - 1) d\alpha' \right|_{L^2} + \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} \cos(\theta)(\theta_\alpha - 1) d\alpha' \right|_{L^2} 
\leq C\|\theta - \alpha\|_{H^1}.
\]
\[\square\]

We emphasize that, by using the Poincaré inequality, we have $\|\theta - \alpha\|_{H^1} \leq \|\theta\|_{H^1}$ for $\gamma > 1$. However, as $\frac{1}{2\pi} \int_{-\pi}^{\pi} \theta_\alpha d\alpha = 1$, it only holds that $\|\theta\|_{H^1} \leq C(1 + \|\theta\|_{H^1})$. This is the reason we introduce the above lemma. Next, we give the estimates of $u$.

**Lemma 3.7.** Under the assumptions of Lemma 3.6 it holds that
\[
\|u\|_{L^\infty} \leq C(\lambda) \frac{1 + s^4}{\beta_1 \beta_2 s^2} (1 + \|\theta\|_{H^2})^2 (1 + \|y_s\|_{L^\infty})^2 (\|\theta\|_{H^2}^{1/2} + \|y_s\|_{H^2}^{1/2} + \|y_s\|_{H^2}),
\]
\[
\|u\|_{L^2} \leq C(\lambda) \frac{1 + s^4}{\beta_1 \beta_2 s^2} (1 + \|\theta\|_{H^2})^2 (1 + \|y_s\|_{L^\infty})^2 (\|\theta\|_{H^2} + \|y_s\|_{H^2}),
\]
where $C(\lambda) > 0$ is a constant only depends on $\lambda$.

**Proof.** Recalling (5.1), we deduce that
\[
\begin{align*}
 u(\alpha, t) &= \int_T G(\alpha, t) \cdot (\lambda \theta_\alpha n - \theta_\alpha n) + (\alpha, t) d\alpha' \\
&+ \int_T G(\alpha, t) \cdot s(y_s, s') \cdot \alpha(\alpha', s') + (1 + y_s^2) \theta_\alpha(\alpha', s') t d\gamma \\
&= \text{p.v.} \int_T \frac{\partial}{\partial s'} G(\alpha, t) \cdot (\lambda \theta_\alpha n - \theta_\alpha n) + \frac{\theta_\alpha^2}{s^2} t d\gamma \\
&+ \text{p.v.} \int_T \frac{\partial}{\partial s'} G(\alpha, t) \cdot (\alpha, t) t ds' \\
&= \frac{1}{4s^2} H(\theta_\alpha n) + \frac{1}{4s^2} (\theta_\alpha n) + \frac{s}{4} \delta(y_s) t - \frac{s}{4} \beta(y_s) + \frac{1}{4s^2} t ds'
\end{align*}
\]
\[
= \frac{1}{4s^2} \frac{L \cdot (M - N)}{|L|^2} Id + \frac{L \cdot (M - N) L \otimes L}{|L|^4} d\alpha' + \frac{2L \cdot (M - N) L \otimes L}{|L|^4} \cdot \theta_\alpha \alpha(\alpha', t) n d\alpha' \\
+ \frac{1}{4s^2} \frac{L \cdot (m - n)}{|l|^2} Id + \frac{L \cdot (m - n) l \otimes l}{|l|^4} d\alpha' + \frac{2L \cdot (m - n) L \otimes L}{|l|^4} \cdot \delta y_s(s', t) t ds' \\
= \frac{1}{4s^2} \frac{L \cdot (M - N)}{|L|^2} Id + \frac{L \cdot (M - N) L \otimes L}{|L|^4} d\alpha' + \frac{2L \cdot (M - N) L \otimes L}{|L|^4} \cdot \theta_\alpha \alpha(\alpha', t) n d\alpha' \\
+ \frac{1}{4s^2} \frac{L \cdot (m - n)}{|l|^2} Id + \frac{L \cdot (m - n) l \otimes l}{|l|^4} d\alpha' + \frac{2L \cdot (m - n) L \otimes L}{|l|^4} \cdot \delta y_s(s', t) t ds'.
\]
\[ + \int_T \left( \frac{L \cdot z_\alpha(\alpha')}{|L|^2} - 2L \cdot z_\alpha(\alpha')L \otimes L + \frac{z_\alpha(\alpha')}{|L|^2} \right) \theta (\alpha', \tau) \, d\alpha' + \frac{1}{4\pi} \left( \frac{1}{2} + 2\theta(\alpha') \right) d\alpha' \]

\[ + \int_T \left[ G(z(\alpha, t) - z(\alpha', t)) \cdot (\lambda - \frac{1}{2s^2}) \theta n(\alpha', t) \right] d\alpha' \]

\[ + \int_T [G(z(\alpha, t) - X(s', t)) \cdot s(1 + y_s(s')) (\theta n(\alpha(s'), t)) d\alpha' \]

Here we use the fact that \( z_\alpha(\alpha') = L + \tau(M - N) \) and \( X_s(s') = \lambda + \frac{4}{12} \). Note that

\[ 2 \tan\left( \frac{\alpha'}{2} \right) = -\frac{\pi}{2} \]

Using Lemma 3.2, Lemma 3.5, Lemma 3.6 and the property of Hardy-Littlewood maximal operator, we have

\[ |u| \leq \frac{1}{s^2} \left| \mathcal{H}(\theta n) \right| + \left| \mathcal{H}, n \right| (\theta n) + \frac{\lambda s^2 + \frac{1}{2} + s^3}{\beta_1} \| \theta - \alpha \|_{H^1} + \frac{1}{\beta_1} \left( 1 + \left| \theta n \right|_{L^2} \right) \| \theta - \alpha \|_{H^2} \]

\[ + s \left( |b(y_s)| + |b_t(y_s)| + \frac{1}{\beta_1 \beta_2} \left( (1 + \|y_s\|_{L^2})^2 \| \theta n \|_{L^2}^2 + \|y_s\|_{L^2} \|z\|_{L^2} \right) \right) \]

Therefore, one can achieve the results by using Lemma 3.3 and the Gagliardo-Nirenberg interpolation inequality.

Now, we give the estimates for \( \tilde{g}_\theta \) and \( \tilde{g}_y \).

**Lemma 3.8.** Suppose that \( (\theta(\alpha, t), y_s(s, t), s(t)) \) satisfies all the assumptions in Lemma 3.6 then for \( \forall \delta \in (0, \pi) \), it holds that

\[ (3.16) \quad \| \tilde{g}_y \|_{H^1} \leq C(\lambda) \left( \frac{1 + \frac{s^3}{\beta_1 \beta_2}}{s^3} \right) (1 + \|y_s\|_{H^{3/2}}) \left( 1 + \|\theta\|_{H^{3/2}} \right)^4 \left( \|\theta\|_{H^1} \right) + \frac{1}{\delta^{1/2}} \|\theta\|_{H^{5/2}} + \|y_s\|_{H^{3/2}} \]

where \( \beta_1, \beta_2 \) are defined in (3.11)-(3.12) and \( C(\lambda) > 0 \) is a constant that only depends on \( \lambda \).

**Proof.** Using the same technique in Lemma 3.6 we deduce that

\[ \tilde{g}_y(s, t) = -(1 + y_s)s \frac{\lambda}{s} - \frac{1}{4} \theta n(\alpha(s), t) \]

\[ + \int_T \left( \frac{\partial}{\partial s} G(X(s, t) - X(s', t)) + \frac{1}{4\pi} \right) d\alpha' \]

\[ \cdot \left( y_s + (1 + y_s)^2 \theta n + (1 + y_s)^2 \left( \frac{\theta n}{s} - \frac{1}{2s^3} \right) n - (1 + y_s) \frac{s^3 + \lambda s^2 - 1}{2s^4} \right) \]

\[ \def= I_1 + I_2. \]
Estimate of $I_1$. Recalling (3.8), we have

$$
\|(1 + y_s)^{\frac{\delta_t}{s}}\|_{H^1} \leq \|\theta\|_{H^1}\|y_s\|_{H^1}\|u\|_{L^2}.
$$

With the help of Lemma 3.3, it is easy to see that

$$
\|\mathbf{t} \cdot [\mathbf{b}, \mathbf{t}] (y_{ss})\|_{H^1} \leq \|\partial_s (\mathbf{t})\|_{L^2} \|y_s\|_{H^1} + \|\mathbf{b} \cdot \mathbf{t}\|_{H^1} \|y_s\|_{H^1} + \|\partial_s \mathbf{t}\|_{H^1} \|y_s\|_{H^1}
$$

$$
\leq C(1 + \|y_s\|_{H^1})^2(1 + \|\theta\|_{H^1})^2\|y_s\|_{H^1},
$$

$$
\|\mathbf{t} \cdot b((1 + y_s)^2 \theta_a \mathbf{n} - (1 + y_s)\frac{z_0}{s})\|_{H^1} \leq C(1 + \|y_s\|_{H^1})^3(1 + \|\theta\|_{H^2})^2\|\theta\|_{H^1},
$$

$$
\|\mathbf{t} \cdot [\mathbf{b}, \mathbf{n}] (\partial_s \theta_{ao})\|_{H^1} \leq C(1 + \|y_s\|_{H^1})^3(1 + \|\theta\|_{H^2})^2\|\theta\|_{H^1}
$$

$$
\leq C(1 + \|y_s\|_{H^1})^3(1 + \|\theta\|_{H^2})^2(\delta\|\theta\|_{H^1} + \frac{1}{\delta_{1/2}}\|\theta\|_{H^{3/2}}).
$$

In a similar way, we conclude that

$$
\|\partial_s I_1\|_{L^2} \leq C(\lambda) \frac{1 + s^3}{\beta_1 \beta_2 s^3}(1 + \|y_s\|_{H^1})^3(1 + \|\theta\|_{H^2})^2(\|y_s\|_{H^1} + \delta\|\theta\|_{H^1} + \frac{1}{\delta_{1/2}}\|\theta\|_{H^{3/2}}).
$$

Estimate of $I_2$. By direct computation, we deduce that

$$
\partial_s I_2 = \mathbf{t} \cdot \partial_s \int \left( \frac{\partial}{\partial s} G(\mathbf{X}(s, t) - \mathbf{X}(s', t)) + \frac{1}{4} \frac{1}{2\pi \tan(\frac{s}{2})} Id \right) \cdot \left( y_s \mathbf{t} + (1 + y_s)^2 \theta_a \mathbf{n} 
$$

$$
+ (1 + y_s) \left( \frac{\lambda}{s^2} - \frac{\theta_{ao}}{s^3} - \frac{1}{2} \frac{(\theta_a)^3}{s^3} \right) \mathbf{n} - (1 + y_s) \frac{s^3 + \lambda s^2 - \frac{1}{2} s^2}{2s^4} \mathbf{z}_0 \right) ds'
$$

$$
\partial_s \mathbf{t} \cdot \left( \frac{\partial}{\partial s} G(\mathbf{X}(s, t) - \mathbf{X}(s', t)) + \frac{1}{4} \frac{1}{2\pi \tan(\frac{s}{2})} Id \right) \cdot \left( y_s \mathbf{t} + (1 + y_s)^2 \theta_a \mathbf{n} 
$$

$$
+ (1 + y_s) \left( \frac{\lambda}{s^2} - \frac{\theta_{ao}}{s^3} - \frac{1}{2} \frac{(\theta_a)^3}{s^3} \right) \mathbf{n} - (1 + y_s) \frac{s^3 + \lambda s^2 - \frac{1}{2} s^2}{2s^4} \mathbf{z}_0 \right) ds'
$$

$$
\overset{\text{def.}}{=} I_2 + r_2,
$$

where $r_2$ is the most trouble term. From the definitions, we have $\mathbf{X}_s(s) = l - \mathbf{n}$. It follows that

$$
(3.17) \quad 4\pi \frac{\partial}{\partial s} G(\mathbf{X}(s, t) - \mathbf{X}(s', t))
$$

$$
= \frac{l \cdot \mathbf{X}_s(s)}{|l|^2 I_d} + \frac{2l \cdot \mathbf{X}_s(s) \otimes l}{|l|^4 I_d} - \frac{\mathbf{X}_s(s) \otimes l + l \otimes \mathbf{X}_s(s)}{|l|^2 I_d}
$$

$$
= \frac{1}{l} I_d - \frac{l \cdot \mathbf{n}}{|l|^2 I_d} - \frac{2l \cdot \mathbf{n} \otimes l}{|l|^4 I_d} + \frac{n \otimes l + l \otimes n}{|l|^2 I_d}.
$$

Note that $\frac{1}{|l(s, s')|}$ is not continuous at $(s, s')\neq (s, s)$ or $|s - s'| =\pi$. However, $\frac{1}{|l(s, s')|}$ and $\frac{\partial}{\partial s} G(\mathbf{X}(s, t) - \mathbf{X}(s', t))$ are continuous functions on $\mathbb{T} \times \mathbb{T}$. Therefore, it
holds that
\[
\partial \left( \frac{\partial}{\partial s} G(X(s,t) - X(s', t)) + \frac{1}{4\pi \tan(\frac{\pi s'}{2})} \right) \\
= \frac{\sin^2(\frac{\pi s'}{2}) - \frac{1}{4\pi^2} l^2}{\sin^2(\frac{\pi s'}{2})} \frac{1}{4\pi l^2} \frac{1}{4\pi l^4} \frac{1}{4\pi l^6} \\
= \frac{\sin^2(\frac{\pi s'}{2}) - \frac{1}{4\pi^2} l^2}{\sin^2(\frac{\pi s'}{2})} \frac{1}{4\pi l^2} \frac{1}{4\pi l^4} \frac{1}{4\pi l^6} \\
+ \frac{8l \cdot n \cdot nl \otimes l + 2l \cdot \partial_n \otimes l + 2l \cdot n(n \otimes l + l \otimes n) + 2l \cdot n(n \otimes l + l \otimes n)}{4\pi l^4}.
\]

Then, we rewrite \(i_2\) as follows:
\[
i_2 = t \cdot \int_T \left( \frac{\sin^2(\frac{\pi s'}{2}) - \frac{1}{4\pi^2} l^2}{\sin^2(\frac{\pi s'}{2})} \frac{1}{4\pi l^2} \frac{1}{4\pi l^4} \frac{1}{4\pi l^6} \right) \\
- \frac{2n \cdot nl \otimes l + 2l \cdot \partial_n \otimes l + 2l \cdot n(n \otimes l + l \otimes n) + 2l \cdot n(n \otimes l + l \otimes n)}{4\pi l^4} \\
+ \frac{\partial_n \otimes l + l \otimes \partial_n + 2n \otimes n}{4\pi l^2} \right) \cdot \left( y_{s\alpha} t + (1 + y_s)^2 \theta_a n \\
+ (1 + y_s)^3 \frac{\theta_{s\alpha s}}{s^3} - \frac{1}{2} \frac{\theta_{\alpha s}^3}{s^6} n - (1 + y_s)^3 \frac{s^3 + \lambda^2 - \theta_{s\alpha s}^4}{2s^4} \frac{\theta_{\alpha s}}{s^3} \right) ds'.
\]

By using Lemma 3.5, we have
\[
\| \int_T \frac{l \cdot \partial_n \otimes l}{|l|^4} \cdot (1 + y_s)^2 \theta_{\alpha s} n(s') ds' \|_{\mathcal{L}} \\
\leq \frac{1}{\beta_1 \beta_2} (1 + \|y_s\|_{\mathcal{L}^1})^{1/2} \|\partial_n \otimes l\|_{\mathcal{L}^2} \|\theta\|_{H^3} \\
\leq \frac{1}{\beta_1 \beta_2} (1 + \|y_s\|_{\mathcal{L}^1})^2 (1 + \|\theta\|_{H^3})^2 (\delta \|\theta\|_{H^4} + \frac{1}{\delta^2} \|\theta\|_{H^5}).
\]

In the same way, we deduce that
\[
\|i_2\|_{\mathcal{L}} \leq C(\lambda) \frac{1 + s^3}{\beta_1 \beta_2 s^3} (1 + \|y_s\|_{\mathcal{L}^1})^5 (1 + \|\theta\|_{H^2})^4 (\delta \|\theta\|_{H^4} + \frac{1}{\delta^2} \|\theta\|_{H^5} + \|y_s\|_{\mathcal{L}^1}),
\]
\[
\|r_2\|_{\mathcal{L}} \leq C(\lambda) \frac{1 + s^3}{\beta_1 \beta_2 s^3} (1 + \|y_s\|_{\mathcal{L}^1})^2 (1 + \|\theta\|_{H^2})^2 (\delta \|\theta\|_{H^4} + \frac{1}{\delta^2} \|\theta\|_{H^5} + \|y_s\|_{\mathcal{L}^1}).
\]

Finally, we conclude that
\[
\|\tilde{g}_s\|_{H^4} \leq C(\lambda) \frac{1 + s^3}{\beta_1 \beta_2 s^3} (1 + \|y_s\|_{\mathcal{L}^1})^5 (1 + \|\theta\|_{H^2})^4 (\delta \|\theta\|_{H^4} + \frac{1}{\delta^2} \|\theta\|_{H^5} + \|y_s\|_{\mathcal{L}^1}).
\]

\[
\|\tilde{g}_s\|_{H^4} \leq C(\lambda) \frac{1 + s^3}{\beta_1 \beta_2 s^3} (1 + \|\theta\|_{H^2})^4 (1 + \|y_s\|_{\mathcal{L}^1})^5 (\delta \|\theta\|_{H^4} + \frac{1}{\delta^2} \|\theta\|_{H^5} + \|y_s\|_{\mathcal{L}^1}).
\]

\]

\[
\textbf{Lemma 3.9.} \text{Suppose that } (\theta(\alpha, t), y_s(s, t), s(t)) \text{ satisfies all the assumptions in Lemma 3.6 then for } \forall \delta \in (0, \pi), \text{ it holds that}
\]
\[
(3.18) \|\tilde{g}_s\|_{H^4} \leq C(\lambda) \frac{1 + s^3}{\beta_1 \beta_2 s^3} (1 + \|\theta\|_{H^2})^4 (1 + \|y_s\|_{\mathcal{L}^1})^5 (\delta \|\theta\|_{H^4} + \frac{1}{\delta^2} \|\theta\|_{H^5} + \|y_s\|_{\mathcal{L}^1}).
\]
where $\beta_1, \beta_2$ are defined in (3.11)-(3.12) and $C(\lambda) > 0$ is a universal constant that only depends on $\lambda$.

The proof is similar to Lemma 3.8 and we omit the details.

4. Existence and Uniqueness of the Local-in-Time Solution

In this section, we will prove the local well-posedness of the modified contour dynamic system (3.5)-(3.8), and give the proof of Theorem 1.1.

4.1. Existence of solutions of the contour dynamic system. Let us introduce some notations before showing the existence of solutions of (3.5)-(3.8). For $T > 0$, we define

$$\Omega_T \overset{\text{def}}{=} \{ (\theta(s, t), y_s(s, t)) : \theta(s, t) - \alpha \in C(\mathbb{T}), \forall t \in [0, T],$$

$$\theta(s, t) \in L^\infty([0, T]; H^{1/2}(\mathbb{T})) \cap L^2([0, T]; \mathbb{H}^1(\mathbb{T})), \theta_t \in L^2([0, T]; \mathbb{H}^1(\mathbb{T})),$$

$$y_s \in L^\infty([0, T]; h^{1/2}(\mathbb{T})) \cap L^2([0, T]; h^2(\mathbb{T})), y_{st} \in L^2([0, T]; h^1(\mathbb{T})).$$

Given $(\theta_0, y_{0s}, s_0)$ which satisfies (3.10) and $\int_T \theta_0 \, d\alpha = \int_T y_0 \, ds = 0$, we also define

$$\Omega_{0, T}(\theta_0, y_{0s}, s_0) \overset{\text{def}}{=} \{ (\theta(s, t), y_s(s, t)) : \theta(s, t) - \alpha \in C(\mathbb{T}), \forall t \in [0, T],$$

$$\theta(s, t) \in L^\infty([0, T]; H^{1/2}(\mathbb{T})) \cap L^2([0, T]; \mathbb{H}^1(\mathbb{T})), \theta_t \in L^2([0, T]; \mathbb{H}^1(\mathbb{T})),$$

$$y_s \in L^\infty([0, T]; h^{1/2}(\mathbb{T})) \cap L^2([0, T]; h^2(\mathbb{T})), y_{st} \in L^2([0, T]; h^1(\mathbb{T})).$$

Here $\mathcal{L}_0(\theta) = \frac{1}{4\gamma_0} \mathcal{H}(\theta_{xax}).$ Be careful that not any $\theta \in \Omega_{0, T}$ can be a tangent angle function of a closed string. By Lemma 3.2 and Lemma 3.3 ($e^{t\mathcal{L}_0(\theta_0)} e^{t\mathcal{Y}_0(s_0)} \in \Omega_{0, T}(\theta_0, y_{0s}, s_0)$, thus $\Omega_{0, T}(\theta_0, y_{0s}, s_0)$ is nonempty. Furthermore, $\Omega_{0, T}(\theta_0)$ is convex and closed in $\Omega_T$. Then we are going to state the result.

**Proposition 4.1.** Assume $s_0 > 0$, $(\theta_0 - \alpha) \in H^{1/2}(\mathbb{T})$, $y_{0s} \in h^{1/2}(\mathbb{T})$ satisfying (3.10), $\int_T \theta_0 \, d\alpha = \int_T y_0 \, ds = 0$ and

$$\frac{1}{|\alpha_1 - \alpha_2|} \int_{\alpha_2}^{\alpha_1} (\cos(\theta(s), \sin(\theta(s))) \, da) \geq \bar{\beta}_1, \quad \forall \alpha_1, \alpha_2 \in \mathbb{T},$$

$$1 + y_0(s) \geq \bar{\beta}_2, \quad \forall s \in \mathbb{T},$$

for some constants $\bar{\beta}_1, \bar{\beta}_2 > 0$. Then there exists $T_0 = T_0(\bar{\beta}_1, \bar{\beta}_2, s_0, ||\theta_0||_{H^{1/2}(\mathbb{T})}, ||y_{0s}||_{h^{1/2}}) \in [0, \infty)$ and a solution with $(\theta(s, t), y_s(s, t)) \in \Omega_{T_0}$ and $s(t) \in C^1[0, T]$ of the modified system (3.5)-(3.8) satisfying

$$||\theta||_{L_0^\infty H^{1/2} L_0^{1/2} H^1} \leq (2 + 4 \sqrt{2} s_0^{3/2}) ||\theta_0||_{H^{1/2}} + ||y_{0s}||_{h^{1/2}}, \quad ||\partial_t \theta||_{L_0^{1/2} H^1} \leq ||\theta_0||_{H^{1/2}} + ||y_{0s}||_{h^{1/2}},$$

$$||y_s||_{L_0^\infty h^{1/2} L_0^{1/2} h} \leq 4 ||y_{0s}||_{h^{1/2}} + ||\theta_0||_{H^{1/2}}, \quad ||\partial_s y_s||_{L_0^{1/2} h} \leq ||\theta_0||_{H^{1/2}} + ||y_{0s}||_{h^{1/2}}.$$
Furthermore, for \( t \in [0, T_0] \), it holds that
\[
\frac{1}{|\alpha_1 - \alpha_2|} \left| \int_{\alpha_2}^{\alpha_1} (\cos(\theta), \sin(\theta))d\alpha' \right| - \left| \int_{-\pi}^{\pi} (\cos(\theta), \sin(\theta))d\alpha' \right| \geq \frac{1}{3}\tilde{\beta}_1, \quad \forall \alpha_1, \alpha_2 \in \mathbb{T}
\]
\[
1 + y_s(s, t) \geq \frac{1}{3}\tilde{\beta}_2, \quad \forall s \in \mathbb{T}.
\]

To prove this proposition, we need the following lemma.

**Lemma 4.1.** Assume \((\theta, y_s) \in \Omega_T, \theta(\alpha, 0) = \theta_0\) which satisfies (3.10). Furthermore, we assume that \(s_0 > 0\), \(\int_T \theta d\alpha = \int_T y_s ds \equiv 0\) and
\[
\frac{1}{|\alpha_1 - \alpha_2|} \left| \int_{\alpha_2}^{\alpha_1} (\cos(\theta_0), \sin(\theta_0))d\alpha' \right| \geq \tilde{\beta}_1,
\]
\[
1 + y_0(s) \geq \frac{1}{3}\tilde{\beta}_2.
\]

Then there exists a constant \( T = T(||\theta||_{L^\infty_T H^1}, ||y_s||_{L^\infty_T H^{1/2}}, \tilde{\beta}_1, \tilde{\beta}_2) \) such that for \( \forall t \in [0, T] \), it holds that
\[
(4.1) \quad \left| \int_{\alpha_2}^{\alpha_1} (\cos(\theta(\cdot, t)), \sin(\theta(\cdot, t)))d\alpha' \right| \geq \frac{2}{3}\tilde{\beta}|\alpha_1 - \alpha_2|,
\]
\[
(4.2) \quad \frac{1}{3}\tilde{\beta} \geq \left| \int_{-\pi}^{\pi} (\cos(\theta(\cdot, t)), \sin(\theta(\cdot, t)))d\alpha' \right|,
\]
and
\[
(4.3) \quad 1 + y_s(s, t) \geq \frac{1}{3}\tilde{\beta}_2.
\]

Moreover, (3.8) admits a unique solution \(s_{\theta, y_s} \) on \([0, T]\) satisfying
\[
(4.4) \quad \frac{1}{2}s_0 \leq s_{\theta, y_s}(t) \leq \frac{3}{2}s_0.
\]

**Proof.** From the assumption, we see that
\[
\left| \int_{\alpha_2}^{\alpha_1} (\cos(\theta(\cdot, t)), \sin(\theta(\cdot, t)))d\alpha' \right| - \left| \int_{\alpha_2}^{\alpha_1} (\cos(\theta_0(\cdot)), \sin(\theta_0(\cdot)))d\alpha' \right| \leq C|\alpha_1 - \alpha_2|T^{1/2}||\theta_0||_{L^2_T H^1}.
\]

For the same reason, we have
\[
\left| \int_{-\pi}^{\pi} (\cos(\theta(\cdot, t)), \sin(\theta(\cdot, t)))d\alpha' \right| - \left| \int_{-\pi}^{\pi} (\cos(\theta_0(\cdot)), \sin(\theta_0(\cdot)))d\alpha' \right| \leq CT^{1/2}||\theta_0||_{L^2_T H^1}.
\]
and
\[
|1 + y_s(s, t) - 1 - y_0(s)| \leq CT^{1/2}||y_s||_{L^2_T H^1}.
\]
Taking $T$ small enough, for all $t \in [0, T]$ we have
\[
\left| \int_{\Omega}^{\alpha_{1}} (\cos(h, t)), \sin(h, t))da'\right| \geq \frac{2}{3} \tilde{\beta}_{1} |\alpha_{1} - \alpha_{2}|, \\
\frac{1}{3} \tilde{\beta}_{1} \geq \left| \int_{-\pi}^{\pi} (\cos(h, t)), \sin(h, t))da'\right|, \\
1 + y_{s}(s, t) \geq \frac{1}{3} \tilde{\beta}_{2}.
\]
Recall that
\[
\partial_{t}s_{\theta,Y_{s}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta_{n} \cdot \left( \int_{T} G(z(\alpha, t) - z(\alpha', t)) \cdot (A\theta_{n} - \frac{\theta_{n}}{z_{\theta,Y_{s}}} n - \frac{\theta_{\alpha}}{z_{\theta,Y_{s}}} n)(\alpha', t)da' \right) \\
+ \int_{T} G(z(\alpha, t) - X(s', t)) \cdot s_{\theta,Y_{s}}(y_{s}(s', t) + (1 + y_{s})^{2} \theta_{n})(s', t)ds' \right) da,
\]
where $z$ and $X$ are constructed from $(\theta, y_{s}, s_{\theta,Y_{s}})$ in (3.3) and (3.4). From (4.1), (4.2) and Lemma [3.7] we know that
\[
(4.5) \quad \partial_{t}s_{\theta,Y_{s}} \leq \frac{1 + s_{\theta,Y_{s}}^{3}}{\tilde{\beta}_{1} \tilde{\beta}_{2} s_{\theta,Y_{s}}} (1 + \|\theta\|_{L_{T}^{\infty}H^{2}})^{2} (1 + \|y_{s}\|_{L_{T}^{\infty}H^{1}})(\|\theta\|_{L_{T}^{\infty}H^{2}} + \|y_{s}\|_{L_{T}^{\infty}H^{1}}).
\]
One can verify that (3.8) satisfies Lipschitz condition. Therefore, by the Cauchy-Lipschitz theorem, there exists a unique solution $s_{\theta,Y_{s}}$ to (3.8) with $s_{\theta,Y_{s}}(0) = s_{0}$. With the help of Gronwall’s inequality, we find a constant $T$ such that for $\forall t \in [0, T]$
\[
\frac{1}{2} s_{0} \leq s_{\theta,Y_{s}}(t) \leq \frac{3}{2} s_{0}.
\]
\[
\square
\]
Now, we are able to prove Proposition 4.1

**Proof of Proposition 4.1** By Lemma A.2 and Lemma A.3 for $\forall(\Theta, Y_{s}) \in \Omega_{0,T}(\theta_{0}, y_{0s}, s_{0})$, we have
\[
\|\Theta\|_{L_{T}^{2}H^{3/2} ; L_{T}^{2}H^{4}} \leq (2 + 2 s_{0}^{3/2})\|\theta_{0}\|_{H^{3/2}} + \|y_{0s}\|_{H^{3/2}}, \quad \|Y_{s}\|_{L_{T}^{2}H^{3/2} ; L_{T}^{2}H^{2}} \leq \|\theta_{0}\|_{H^{3/2}} + 4\|y_{0s}\|_{H^{3/2}}.
\]
From Lemma [4.1] there exists constant $\tilde{T}$ such that $(\Theta, Y_{s}, s_{\theta,Y_{s}})$ satisfies (4.1)-(4.4) on $[0, \tilde{T}]$. Here $s_{\theta,Y_{s}}$ is the solution to (3.8) with $s_{\theta,Y_{s}}(0) = s_{0}$. Note that $\tilde{T}$ only depends on $(\theta_{0}, y_{0s}, s_{0})$ and $\tilde{\beta}_{1}, \tilde{\beta}_{2}$. We define a map $V : \Omega_{0,T}(\theta_{0}, y_{0s}, s_{0}) \rightarrow \Omega_{0,T}(\theta_{0}, y_{0s}, s_{0})$ as follows. Given $(\Theta, Y_{s}) \in \Omega_{0,T}(\theta_{0}, S_{0})$, let $(\Phi, \mathcal{Y}_{s}) = V(\Theta, Y_{s})$ be the solution to
\[
(4.6) \quad \partial_{t}\Phi(\alpha, t) = L_{\theta,Y_{s}}(\Phi)(\alpha, t) + \tilde{g}_{\theta}(\alpha, t), \quad \Phi(\alpha, 0) = \theta_{0}(\alpha), \\
(4.7) \quad \partial_{t}\mathcal{Y}_{s}(s, t) = \mathcal{U}(Y_{s})(s, t) + \tilde{g}_{Y}(s, t), \quad \mathcal{Y}_{s}(s, 0) = y_{0s}(s),
\]
where $L_{\theta,Y_{s}} = \frac{1}{4\theta_{0,Y_{s}}} \mathcal{U}(\Phi_{a\alpha})$. To show that $V$ is well-defined, we first claim that $(\Phi, \mathcal{Y}_{s}) \in \Omega_{\tilde{T}}$. In fact, for $(\Theta, Y_{s}) \in \Omega_{0,T}(\theta_{0}, y_{0s}, s_{0})$, by Lemma [3.8] and Lemma [3.9]
we have
\[ \| \tilde{g}_0 \|_{L_t^{2,q}H^1} \leq C(1 + \| \Theta \|_{L_t^{2,q}H^1})^4 (1 + \| Y_s \|_{L_t^{2,q}H^1})^3 (\delta + \frac{\tilde{T}^{1/2}}{\delta^{1/2}})(\| \theta_0 \|_{H^{3/2}} + \| y_0 \|_{H^1}), \]
\[ \| \tilde{g}_Y \|_{L_t^{2,q}H^1} \leq C(1 + \| \Theta \|_{L_t^{2,q}H^1})^4 (1 + \| Y_s \|_{L_t^{2,q}H^1})^3 (\delta + \frac{\tilde{T}^{1/2}}{\delta^{1/2}})(\| \theta_0 \|_{H^{3/2}} + \| y_0 \|_{H^1}), \]
\[ \| \bar{\theta} \|_{L_t^{2,q}H^1} \leq C(1 + \| \theta_0 \|_{H^{3/2}} + \| y_0 \|_{H^1})^7 (\delta + \frac{\tilde{T}^{1/2}}{\delta^{1/2}})(\| \theta_0 \|_{H^{3/2}} + \| y_0 \|_{H^1}), \]
\[ \| \bar{\theta}_s \|_{L_t^{2,q}H^1} \leq C(1 + \| \theta_0 \|_{H^{3/2}} + \| y_0 \|_{H^1})^7 (\delta + \frac{\tilde{T}^{1/2}}{\delta^{1/2}})(\| \theta_0 \|_{H^{3/2}} + \| y_0 \|_{H^1}). \]

where \( C = (\lambda, s_{\theta,Y}, \tilde{\beta}_1, \tilde{\beta}_2) \) is a constant. Taking \( \bar{T} > 0 \) small enough, using Lemma A.2, Lemma A.3 and (4.5), we get the existence and uniqueness of the solution \((\Phi, \mathbf{Y}_s) \in \Omega_T\) to (4.6)-(4.7) which satisfies

\[ \| \partial_t \Phi \|_{L_t^{2,q}H^1} \leq C(1 + \| \theta_0 \|_{H^{3/2}} + \| y_0 \|_{H^1})^7 (\delta + \frac{\tilde{T}^{1/2}}{\delta^{1/2}})(\| \theta_0 \|_{H^{3/2}} + \| y_0 \|_{H^1}), \]
\[ \| \partial_s \mathbf{Y}_s \|_{L_t^{2,q}H^1} \leq C(1 + \| \theta_0 \|_{H^{3/2}} + \| y_0 \|_{H^1})^7 (\delta + \frac{\tilde{T}^{1/2}}{\delta^{1/2}})(\| \theta_0 \|_{H^{3/2}} + \| y_0 \|_{H^1}). \]

\( \Phi \) and \( \mathbf{Y}_s \) obviously have mean zero on \( T \) for \( t \in [0, \bar{T}] \).

Now, consider \( \mathbf{W} = \Phi - e^{\mathcal{L}t} \theta_0 \) and \( w_0 = \mathbf{Y}_s - e^{\mathcal{L}t} y_0 \) which solve

\[ \partial_t \mathbf{W}(\alpha, t) = \mathcal{L}_{\Theta,Y} \mathbf{W}(\alpha, t) + \tilde{g}_0(\alpha, t) + (\mathcal{L}_{\Theta,Y} - \mathcal{L}_0) e^{\mathcal{L}t} \theta_0(\alpha, t), \quad \mathbf{W}(\alpha, 0) = 0 \]
\[ \partial_t w_0(s, t) = \mathcal{L} w_0(s, t) + \tilde{g}_Y(s, t), \quad w_0(s, 0) = 0. \]

It follows that

\[ \| \mathbf{W} \|_{L_t^{2,q}H^{3/2} \cap L_t^{2,q}H^1} \leq C\| \mathbf{W} \|_{L_t^{2,q}H^1} + C\| \mathcal{L} \|_{\Theta,Y} - \mathcal{L}_0 \| e^{\mathcal{L}t} \theta_0 \|_{L_t^{2,q}H^1} \]
\[ \leq C(1 + \| \theta_0 \|_{H^{3/2}} + \| y_0 \|_{H^1})^7 (\delta + \frac{\tilde{T}^{1/2}}{\delta^{1/2}} + \tilde{T})(\| \theta_0 \|_{H^{3/2}} + \| y_0 \|_{H^1}), \]

and

\[ \| w_0 \|_{L_t^{2,q}H^{3/2} \cap L_t^{2,q}H^1} \leq C(1 + \| \theta_0 \|_{H^{3/2}} + \| y_0 \|_{H^1})^7 (\delta + \frac{\tilde{T}^{1/2}}{\delta^{1/2}} + \tilde{T})(\| \theta_0 \|_{H^{3/2}} + \| y_0 \|_{H^1}). \]

Here we use the fact that

\[ \| \mathcal{L}_0 - \mathcal{L}_{\Theta,Y} \|_{e^{\mathcal{L}t} \theta_0 \|_{L_t^{2,q}H^1}} \leq C\| \mathcal{L}_0 - \mathcal{L}_{\Theta,Y} \|_{\theta_0 \|_{H^{3/2}}} \leq C \int_0^\bar{T} |\partial_s s_{\Theta,Y}| dt \| \theta_0 \|_{H^{3/2}} \]
\[ \leq C \bar{T} (1 + \| \theta_0 \|_{H^{3/2}} + \| y_0 \|_{H^1})^7 \| \theta_0 \|_{H^{3/2}}. \]

We take \( \delta \) small enough and then take \( \bar{T} \) small enough to obtain

\[ \| \Phi - e^{\mathcal{L}t} \theta_0 \|_{L_t^{2,q}H^{3/2} \cap L_t^{2,q}H^1} \leq \| \theta_0 \|_{H^{3/2}} + \| y_0 \|_{H^1}, \quad \| \partial_s \Phi \|_{L_t^{2,q}H^1} \leq \| \theta_0 \|_{H^{3/2}} + \| y_0 \|_{H^1}, \]
\[ \| \mathbf{Y}_s - e^{\mathcal{L}t} y_0 \|_{L_t^{2,q}H^{3/2} \cap L_t^{2,q}H^1} \leq \| \theta_0 \|_{H^{3/2}} + \| y_0 \|_{H^1}, \quad \| \partial_s \mathbf{Y}_s \|_{L_t^{2,q}H^1} \leq \| \theta_0 \|_{H^{3/2}} + \| y_0 \|_{H^1}. \]
By the Aubin-Lions lemma, \( V(\Omega_0, T((\theta_0, y_{0s}), s_0)) \) is compact in \( C([0, T]; H^2(\mathbb{T})) \times C([0, T]; h^1(\mathbb{T})) \). As \( \Omega_0, T((\theta_0, y_{0s}), s_0) \) is convex, from the Schauder fixed point theorem, there exists a fixed point of map \( V \) in \( V(\Omega_0, T((\theta_0, y_{0s}), s_0)) \subset \Omega_0, T((\theta_0, y_{0s}), s_0) \). We denote this fixed point by \( (\theta, y, s) \in \Omega_T \), which is a solution to (3.5)-(3.8) satisfying

\[
\|\theta\|_{L^p_T H^{3/2}; L^2} \leq (2 + 4 \sqrt{2 s_0^{3/2}})\|\theta\|_{H^{3/2}} + \|y_{0s}\|_{h^{3/2}}, \|\partial_s \theta\|_{L^2 T} \leq \|\theta_0\|_{H^{3/2}} + \|y_{0s}\|_{h^{3/2}}, \|y_s\|_{L^2_T h^{3/2}} \leq 4\|y_{0s}\|_{h^{3/2}} + \|\theta_0\|_{H^{3/2}} + \|y_{0s}\|_{h^{3/2}}.
\]

Consequently, using Lemma 4.1, for \( \forall t \in [0, T] \), it holds that

\[
\frac{1}{|\alpha_1 - \alpha_2|} \int_{\alpha_2}^{\alpha_1} (\cos(\theta), \sin(\theta)) d\alpha' - \int_{-\pi}^{\pi} (\cos(\theta), \sin(\theta)) d\alpha' \geq \frac{1}{3} \tilde{B}_1, \forall \alpha_1, \alpha_2 \in \mathbb{T}
\]

\[
1 + y_s(s, t) \geq \frac{1}{3} \tilde{B}_2, \forall s \in \mathbb{T}
\]

This is our assertion. \( \square \)

In above proof we consider the modified error term \( \tilde{g}_\theta \) and \( \tilde{g}_y \). When \( (\theta, y, s) \) is a solution to (3.5)-(3.8) with closed-string initial data, \( \theta \) always satisfies (3.10). Indeed, we have the following result.

**Lemma 4.2.** Assume \( (\theta, y, s) \) is a solution to (3.5)-(3.8) with initial data satisfying

\[
\int_{-\pi}^{\pi} (\cos(\theta_0(\alpha)), \sin(\theta_0(\alpha))) d\alpha = (0, 0),
\]

then we have

\[
\int_{-\pi}^{\pi} (\cos(\theta(\alpha, t)), \sin(\theta(\alpha, t))) d\alpha = (0, 0).
\]

**Proof.** Recalling (3.1), (3.2), (3.5), one has

\[
\frac{d}{dt} \int_{-\pi}^{\pi} \sin(\theta) d\alpha = \int_{-\pi}^{\pi} \cos(\theta) d\alpha
\]

\[
= \int_{-\pi}^{\pi} \cos(\theta) \frac{1}{s}(u \cdot n) d\alpha + \int_{-\pi}^{\pi} \cos(\theta) \frac{T}{s} \theta_0 d\alpha - \frac{1}{2s} \int_{-\pi}^{\pi} \theta_0 d\alpha \int_{-\pi}^{\pi} \cos(\theta) d\alpha
\]

\[
= \int_{-\pi}^{\pi} \sin(\theta) \theta_0 \frac{1}{s} \frac{u \cdot n d\alpha}{s} - \int_{-\pi}^{\pi} \frac{1}{s} \sin(\theta) \frac{T}{s} \theta_0 d\alpha - \frac{1}{2s} \int_{-\pi}^{\pi} \theta_0 d\alpha \int_{-\pi}^{\pi} \cos(\theta) d\alpha
\]

\[
= - \frac{s_t}{s} \int_{-\pi}^{\pi} \sin(\theta) d\alpha - \frac{1}{2s} \int_{-\pi}^{\pi} \theta_0 d\alpha \int_{-\pi}^{\pi} \cos(\theta) d\alpha,
\]

where we use the fact that

\[
s_t = \theta_0 \frac{u \cdot n}{s}.
\]

Similarly, it holds that

\[
\frac{d}{dt} \int_{-\pi}^{\pi} \cos(\theta) d\alpha = - \frac{s_t}{s} \int_{-\pi}^{\pi} \cos(\theta) d\alpha + \frac{1}{2s} \int_{-\pi}^{\pi} \theta_0 d\alpha \int_{-\pi}^{\pi} \sin(\theta) d\alpha.
\]
Therefore, we have
\[
\frac{d}{dt} \left( (s \int_{-\pi}^{\pi} \sin(\theta) d\alpha)^2 + (s \int_{-\pi}^{\pi} \cos(\theta) d\alpha)^2 \right) = 2s \frac{\partial}{\partial s} \left( (s \int_{-\pi}^{\pi} \sin(\theta) d\alpha)^2 + (s \int_{-\pi}^{\pi} \cos(\theta) d\alpha)^2 \right) \\
+ 2s^2 \int_{-\pi}^{\pi} \sin(\theta) d\alpha \left( -\frac{\partial}{\partial s} \int_{-\pi}^{\pi} \sin(\theta) d\alpha - \frac{1}{2\pi s} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \theta_1 d\alpha \int_{-\pi}^{\pi} \cos(\theta) d\alpha \right) \\
+ 2s^2 \int_{-\pi}^{\pi} \cos(\theta) d\alpha \left( -\frac{\partial}{\partial s} \int_{-\pi}^{\pi} \cos(\theta) d\alpha + \frac{1}{2\pi s} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \theta_1 d\alpha \int_{-\pi}^{\pi} \sin(\theta) d\alpha \right) = 0.
\]
This completes the proof.

4.2. **Uniqueness of solutions of the contour dynamic system.** In this subsection we will prove the uniqueness of solutions to (3.5)-(3.8).

**Proposition 4.2.** Suppose \((\theta_0, y_0, z_0)\) the dynamic system \((3.5)-(3.8)\) admits at most one solution \((\theta, y, z) \in (\Omega_T, C^1[0, T])\) with initial data \((\theta_0, y_0, z_0)\).

**Proof.** Suppose \((\theta_1, y_{1s}, z_1), (\theta_2, y_{2s}, z_2) \in (\Omega_T, C^1[0, T])\) are two solutions to \((3.5)-(3.8)\) with initial data \((\theta_0, y_0, z_0)\). Let

\[
R = 1 + \|\theta_1\|_{L^r_T H^{\alpha/3} \cap L^{2}_T H^4} + \|\theta_2\|_{L^r_T H^{\alpha/3} \cap L^{2}_T H^4} + \|y_{1s}\|_{L^r_T H^{\alpha/3} \cap L^{2}_T H^4} + \|y_{2s}\|_{L^r_T H^{\alpha/3} \cap L^{2}_T H^4},
\]

\[
\Theta(\alpha, t) = \theta_1(\alpha, t) - \theta_2(\alpha, t), \quad Y_s(s, t) = y_{1s}(s, t) - y_{2s}(s, t).
\]

It holds that

\[
\partial_s \Theta(\alpha, t) = -\frac{1}{4s^3} \mathcal{H}(\Theta_{\alpha\alpha})(\alpha, t) + \frac{s_1^3 - s_2^3}{4s_1^3 s_2^3} \mathcal{H}(\theta_{2\alpha\alpha})(\alpha, t) + \tilde{g}_\alpha(\alpha, t) - \tilde{g}_\alpha(\alpha, t), \quad \Theta(\alpha, 0) = 0,
\]

\[
\partial_s Y_s(s, t) = -\frac{1}{4} \tilde{b}(Y_s)(s, t) + \tilde{g}_{y_1}(s, t) - \tilde{g}_{y_2}(s, t), \quad Y_s(0, s) = 0.
\]

Using Lemma 4.1 one can see that there exists \(T^*\) such that

\[
\|\frac{s_1^3 - s_2^3}{4s_1^3 s_2^3} \mathcal{H}(\theta_{2\alpha\alpha})\|_{L^2_T L^2} \leq C(s_0) \left( \int_0^T |s_1 - s_2|^2(t)\|\theta_1\|_{H^2(t)}^2 dt \right)^{1/2} \leq C(s_0) \left( \int_0^T |s_1 - s_2|^2(t)\|\theta_1\|_{H^2(t)}^2 dt \right)^{1/2} \leq C(s_0)\|\theta_2\|_{L^r_T H^{1/3}}^{1/3}\|\theta_2\|_{L^r_T H^4}^{1/3} \left( \int_0^T |s_1 - s_2|^3 dt \right)^{1/3} \leq C(R, s_0) T^{1/3}\|\Theta\|_{L^r_T L^2}.
\]
The last inequality follows from Lemma 4.2 and Lemma B.2. We claim that the following estimates hold for \( \forall \delta \in (0, 1) \):

\[
(4.8) \quad \|\tilde{g}_1 - \tilde{g}_2\|_{L^2} + \|\tilde{g}_1 - \tilde{g}_2\|_{L^2} \leq C(\delta)\|\Theta\|_{L^2_h^pH^3} + \frac{\tilde{T}_{1/2}}{\delta}\|\Theta\|_{L^2_h^pH^{1/2}} + \frac{\tilde{T}_{1/2}}{\delta}\|Y_s\|_{L^2_h^pH^{1/2}},
\]

where \( C = C(R, s_0, \lambda, \tilde{\beta}_1, \tilde{\beta}_2) \). One need to be careful that there are two transfer functions

\[
(4.9) \quad \alpha_1(s) = s + y_1(s), \quad \alpha_2(s) = s + y_2(s), \quad y_1(s) = \int_0^s y_1(s')ds', \quad y_2(s) = \int_0^s y_2(s')ds',
\]

and two inverse functions

\[
s_1(\alpha) = \alpha - y_1(s_1(\alpha)), \quad s_2(\alpha) = \alpha - y_2(s_2(\alpha)).
\]

For the difference terms defined on different coordinates, we have

\[
\theta_{1\alpha\alpha}(\alpha_1(s)) - \theta_{2\alpha\alpha}(\alpha_2(s)) = \int_{\alpha_2(s)}^{\alpha_1(s)} \theta_{1\alpha\alpha}d\alpha + \theta_{1\alpha\alpha}(\alpha_2(s)) - \theta_{2\alpha\alpha}(\alpha_2(s)) \leq \|\theta_{1\alpha\alpha}\|_{L^1} |y_1(s) - y_2(s)| + |\theta_{1\alpha\alpha}(\alpha_2(s)) - \theta_{2\alpha\alpha}(\alpha_2(s))|
\]

and

\[
y_{2s}(s_1(\alpha)) - y_{2s}(s_2(\alpha)) = \int_{s_2(\alpha)}^{s_1(\alpha)} y_{1s}d\alpha + y_{1s}(s_2(\alpha)) - y_{2s}(s_2(\alpha)) \leq \|y_{1s}\|_{L^1} |s_1(\alpha) - s_2(\alpha)| + |y_{1s}(s_2(\alpha)) - y_{2s}(s_2(\alpha))|.
\]

Next, we give the estimate for \( |s_1(\alpha) - s_2(\alpha)| \). From (4.9), it holds that

\[
s_1(\alpha) + y_1(s_1(\alpha)) = s_2(\alpha) + y_2(s_2(\alpha)).
\]

Without loss of generality, for fixed \( \alpha \), we assume \( s_1 > s_2 \), accordingly

\[
(4.10) \quad y_2(s_2(\alpha)) - y_1(s_2(\alpha)) = \int_{s_2(\alpha)}^{s_1(\alpha)} y_{2s}(s')ds' + s_1(\alpha) - s_2(\alpha) \geq \frac{1}{3}\tilde{\beta}_2(s_1(\alpha) - s_2(\alpha)),
\]

here we use the fact that \( y_{2s} \geq -1 + \frac{1}{3}\tilde{\beta}_2 \). It follows that

\[
|s_1(\alpha) - s_2(\alpha)| \leq 3\tilde{\beta}_2|y_2(s_2(\alpha)) - y_1(s_2(\alpha))|.
\]

One can complete the proof of (4.8) in a similar way to Lemma 3.8 we omit the details.

Choosing appropriate \( \delta \) and \( T \), and using Lemma A.2 and Lemma A.3, we conclude that

\[
\|\Theta\|_{L_t^\infty L_x^pH^5_0 \cap L_x^2H^3} + \|Y_s\|_{L_t^\infty H^{1/2} \cap L_x^2H^{1/2}} \leq \frac{1}{2}(\|\Theta\|_{L_t^\infty H^5_0 \cap L_x^2H^3} + \|Y_s\|_{L_t^\infty H^{1/2} \cap L_x^2H^{1/2}}),
\]

which implies \( \Theta \equiv 0 \) and \( Y_s \equiv 0 \). Hence \( s_1 \equiv s_2 \). \( \Box \)
4.3. Local well-posedness of the immersed boundary problem. Now, we are in the position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let \((\theta_0, y_0, s_0)\) be tangent angle function, the stretching function and the perimeter function of \(X_0\). Using Proposition 4.1 and Proposition 4.2, we can get \((\tilde{\theta}, y, s)\) which is the unique solution to (3.5)-(3.8). Then one can find \(\tilde{\theta}(t)\) and \(u(X(-\pi, t), t)\) by (3.6) and (2.17). Therefore, \(X(s, t)\) constructed from (2.22) is a solution to (2.23)-(2.25) satisfying (1.8)-(1.11). The uniqueness of \((\tilde{\theta}, y, s)\) implies that of \(\tilde{\theta}(t), u(X(-\pi, t), t)\) and \(X(s, t)\). This proves the theorem. \(\square\)

5. Existence and Uniqueness of Global-in-Time Solutions near Equilibrium Configurations

In this section, we prove the existence of a global solutions to (1.5) provided that the initial string configuration is sufficiently close to equilibrium. We introduce the following equilibrium configuration

\[
X_*(s) = (\sin(s), \cos(s)), \quad s \in \mathbb{T},
\]

which is a evenly parametrized unit circle, and it has the same expression in the are-length coordinate. The corresponding equilibrium state of tangent angle function, stretching function and perimeter function is

\[
(\theta_*(\alpha) = \alpha, \; y_*(s) = 0, \; s_* = 1).
\]

As the area enclosed by the string is unchanging, we choose the initial data \(X_0\) which satisfies

\[
\frac{1}{2} \int_{\mathbb{T}} X \cdot X_0^* ds = \pi. \tag{5.2}
\]

This means that the area enclosed by the string is \(\pi\). Next, we introduce a lemma which establishes the equivalence of the Sobolev norm distance and the energy difference between \((\theta, y, s)\) and the equilibrium configuration \((\theta_*, y_*, s_*)\). Our motivation is to transform the global coercive bound on the difference of energy into a more convenient quantity.

**Lemma 5.1.** Let \(X(s) \in h^2(\mathbb{T})\) be a closed convex string which satisfies (5.2), then it holds that

\[
\frac{1}{C} \left( \int_{\mathbb{T}} \theta_*^2 d\alpha + \frac{s_*^2}{2} \int_{\mathbb{T}} (1 + y_*)^2 ds + 2\pi \lambda s - 2\pi - 2\pi \lambda \right) \leq \frac{1}{2s} \int_{\mathbb{T}} (\theta - 1)^2 d\alpha + \frac{s^2}{2} \int_{\mathbb{T}} y^2 ds + \pi(s - 1) + 2\pi \lambda(s - 1) \leq C \left( \int_{\mathbb{T}} \theta_*^2 d\alpha + \frac{s_*^2}{2} \int_{\mathbb{T}} (1 + y_*)^2 ds + 2\pi \lambda s - 2\pi - 2\pi \lambda \right),
\]

where \(C > 0\) is a constant that only depends on \(s\).
Therefore, from Lemma 2.1 and Lemma 5.1, we get the following global estimate

\[ \frac{1}{2s} \int_\mathcal{T} (\theta_\alpha - 1)^2 d\alpha = \frac{1}{2s} \int_\mathcal{T} \theta_\alpha^2 + 1 - 2\theta_\alpha d\alpha = \frac{1}{2s} \int_\mathcal{T} \theta_\alpha^2 d\alpha - \frac{\pi}{s} \]

\[ = \frac{1}{2s} \int_\mathcal{T} \theta_\alpha^2 d\alpha - \pi + \frac{s-1}{s} \pi \geq \frac{1}{2s} \int_\mathcal{T} \theta_\alpha^2 d\alpha - \pi, \]

where the last inequality follows from the classical isoperimetric inequality. As the string is convex, using Lemma B.1 (Gages isoperimetric inequality), we have

\[ \frac{1}{2s} \int_\mathcal{T} \theta_\alpha^2 d\alpha = \frac{s}{2} \int_\mathcal{T} \kappa^2 d\alpha \geq s\pi. \]

Therefore, it holds that

\[ 2\left(\frac{1}{2s} \int_\mathcal{T} \theta_\alpha^2 d\alpha - \pi\right) = \frac{1}{2s} \int_\mathcal{T} (\theta_\alpha - 1)^2 d\alpha - \frac{s-1}{s} \pi + \frac{1}{2s} \int_\mathcal{T} \theta_\alpha^2 d\alpha - \pi \]

\[ \geq \frac{1}{2s} \int_\mathcal{T} (\theta_\alpha - 1)^2 d\alpha + \frac{(s-1)^2}{s} \pi \]

\[ \geq \frac{1}{2s} \int_\mathcal{T} (\theta_\alpha - 1)^2 d\alpha. \]

Similarly, one can deduce that

\[ \frac{1}{s} + \frac{1}{2} \left( \frac{s^2}{2} \int_\mathcal{T} (1 + y_\alpha)^2 ds - \pi \right) \leq \frac{s^2}{2} \int_\mathcal{T} y_\alpha^2 ds + \pi(s - 1) \]

\[ \leq (s + 1) \left( \frac{s^2}{2} \int_\mathcal{T} (1 + y_\alpha)^2 ds - \pi \right). \]

This completes the proof. \( \square \)

**Remark 5.1.** The assumption of the above lemma is reasonable. In the rest of this paper, we only consider the string closed to the equilibrium configuration. When \( \|\theta - \alpha\|_{H^2} \) is small enough, one has \( \theta_\alpha(\alpha) > 0, \forall \alpha \in \mathcal{T} \), which means that \( \mathbf{X} \) is a convex string. Furthermore, from the above proof, we also have

\[ (s - 1) \leq \frac{1}{4\pi} \|\theta_\alpha - 1\|_{L^2}. \]

Therefore, from Lemma 2.7 and Lemma 5.7, we get the following global estimate

\[ (\|\theta - \alpha\|_{H^1} + \|y_\alpha\|_{L^2})(t) \leq C(\|\theta_0 - \alpha\|_{H^1} + \|y_0\|_{L^2} + \lambda(s_0 - 1)). \]

As \( (\theta_\alpha, y_\alpha, s) = (\alpha, y_\alpha, s) = (0, s_0 = 1) \), we actually give the Sobolev norm distance estimates between \( (\theta, y_\alpha, s) \) and the equilibrium configuration in Proposition 4.1. Next, we will show that \( (\|\theta - \alpha\|_{H^1} + \|y_\alpha\|_{L^2})(t) \) cannot be big when \( (\|\theta - \alpha\|_{H^1} + \|y_\alpha\|_{L^2})(t) \) is small.

**Lemma 5.2.** Giving \( T > 0, R \leq \frac{1}{A} \), let \( \mathbf{X}(s, t) \) be a (local) solution to (1.5) satisfying (1.6) and (1.7) for some \( \beta_1, \beta_2 > 0 \) on [0, T] with initial data \( \mathbf{X}_0 \) satisfying (5.2). We also assume \( (\theta_\alpha, y_\alpha) \in \Omega_T \) such that

\[ \|\theta\|_{L_T^\infty H^{3/2} \cap L_2^4} + \|y_\alpha\|_{L_T^\infty H^{1/2} \cap L_2^4} \leq R. \]
If in addition
(5.5) \[ ||\theta - \alpha||^2_{H^{3/2}} + ||y_s||^2_{H^{3/2}} \geq c_s(||\theta - \alpha||_{H^1} + ||y_s||^{2/3}_{H^{3/2}}||y_s||^{1/3}_{L^2})^2, \quad t \in [0, T], \]
for some constant \( c_s = c_s(R, \lambda, \beta_1, \beta_2) > 0 \), one has
(5.6) \[ (||\theta - \alpha||_{H^{5/2}} + ||y_s||_{H^{3/2}})(t) \leq e^{-\delta}(||\theta_0 - \alpha||_{H^{5/2}} + ||y_0||_{H^{3/2}}). \]

Proof. Recalling (2.23)-(2.25), it holds that
\[
\begin{align*}
\partial_t(\theta(\alpha, t) - \alpha) &= \mathcal{L}(\theta)(\alpha, t) + g_\theta(\alpha, t), \\
\partial_s y_s(s, t) &= \mathcal{Q}(y_s)(s, t) + g_y(s, t).
\end{align*}
\]

From (5.3) we know that \( 1 \leq s \leq \frac{5}{4} \). Using Lemma 3.9 and Lemma 3.8 one can see that
(5.7) \[
\frac{d}{dt}(||\theta - \alpha||^2_{H^{3/2}} + ||y_s||^2_{H^{3/2}})(t) \leq -\frac{1}{8}(||\theta - \alpha||^2_{H^1} + ||y_s||^2_{H^1}) + C(||g_\theta||^2_{H^1} + ||g_y||^2_{H^1})(t).
\]

Noting that \( \partial_s \theta = 0 \), from Lemma 3.9 and Lemma 3.8 for \( \forall \delta \in (0, 1) \), we have
\[ ||g_\theta||_{H^1} + ||g_y||_{H^1} \leq C(R, \lambda, \beta_1, \beta_2)(\delta||\theta||_{H^1} + \frac{1}{\delta^{1/2}}||\theta||_{H^{5/2}} + ||y_s||_{H^{3/2}}). \]

We choose \( \delta \) small enough which is determined by \( (R, \lambda, \beta_1, \beta_2) \) such that
(5.8) \[
-\frac{1}{8}(||\theta - \alpha||^2_{H^1} + ||y_s||^2_{H^1}) + C(||g_\theta||^2_{H^1} + ||g_y||^2_{H^1})(t) \leq -\frac{1}{10}(||\theta - \alpha||^2_{H^1} + ||y_s||^2_{H^1}) + C_1(||\theta - \alpha||^2_{H^{5/2}} + ||y_s||^2_{H^{3/2}})(t).
\]

Here \( C_1 = C_1(R, \lambda, \beta_1, \beta_2) \) is a constant.

It follows from the Gagliardo-Nirenberg inequality that
\[
||\theta - \alpha||^2_{H^{5/2}} \leq C||\theta - \alpha||_{H^1}||\theta - \alpha||_{H^1}, \\
||y_s||^2_{H^{3/2}} \leq C||y_s||^{2/3}_{H^{3/2}}||y_s||^{1/3}_{L^2}||y_s||_{H^2}.
\]

Therefore, for
\[ c(t) = \frac{||\theta - \alpha||^2_{H^{5/2}} + ||y_s||^2_{H^{3/2}}}{(||\theta - \alpha||_{H^1} + ||y_s||^{2/3}_{H^{3/2}}||y_s||^{1/3}_{L^2})^2}, \]
it holds that
\[ c(t)(||\theta - \alpha||^2_{H^{5/2}} + ||y_s||^2_{H^{3/2}})(t) \leq C_2(||\theta - \alpha||^2_{H^1} + ||y_s||^2_{H^1})(t), \]
where \( C_2 > 1 \) is a universal constant. We see from (5.8) that
\[
\frac{d}{dt}(||\theta - \alpha||^2_{H^{5/2}} + ||y_s||^2_{H^{3/2}})(t) \leq (-\frac{c(t)}{10C_2} + C_1)(||\theta - \alpha||^2_{H^{5/2}} + ||y_s||^2_{H^{3/2}})(t).
\]
Choosing \( c_s = 10(1 + C_1)C_2 \), we arrive at (5.6) with the help of the Gronwall inequality.

Remark 5.2. As the equilibrium energy is not 0, and thus the nonlinear terms \( g_\theta \) and \( g_y \) are not high order small terms, one cannot get exponential decay estimates of \( (||\theta - \alpha||^2_{H^{5/2}} + ||y_s||^2_{H^{3/2}})(t) \) directly from (5.7).
Now, we give the proof of Theorem 1.2 in the case area $a = \pi$. The general case can be treated in the same way.

**Proof of Theorem 1.2**

Define

$$Q_\varepsilon = \{X \mid \|\theta - \alpha\|_{H^{1/2}} + \|y_s\|_{H^{1/2}} \leq \varepsilon\}.$$  

We claim that a universal constant $\varepsilon_\ast$ exists which will be clear below, for $\forall X \in Q_\varepsilon$, it holds that

$$\theta_\alpha(\alpha, t) \geq \frac{1}{2}, \forall \alpha \in \mathbb{T},$$

$$\frac{1}{|\alpha_1 - \alpha_2|} \int_{\alpha_2}^{\alpha_1} (\cos(\theta), \sin(\theta))d\alpha' \geq \frac{1}{\pi}, \forall \alpha_1, \alpha_2 \in \mathbb{T},$$

$$1 + y_s(s, t) \geq \frac{3}{4}, \forall s \in \mathbb{T}.$$

In fact, we have

$$\left| \int_{\alpha_2}^{\alpha_1} (\cos(\theta), \sin(\theta))d\alpha' \right|$$

$$\geq \left| \int_{\alpha_2}^{\alpha_1} (\cos(\alpha'), \sin(\alpha'))d\alpha' \right| - \left| \int_{\alpha_2}^{\alpha_1} (\cos(\theta) - \cos(\alpha'), \sin(\theta) - \sin(\alpha'))d\alpha' \right|$$

$$\geq \left( \frac{2}{\pi} - C_\ast \varepsilon_\ast \right) |\alpha_1 - \alpha_2|,$$

and

$$\theta_\alpha(\alpha, t) \geq 1 - C_\ast \varepsilon_\ast,$$

$$1 + y_s(s, t) \geq 1 - C_\ast \varepsilon_\ast,$$

where $C_\ast > 0$ is a universal constant coming from the Sobolev inequality. Hence, it suffices to take $\varepsilon_\ast = \min\{(3\pi C_\ast)^{-1}, 1\}$.

Combing above uniform estimates and Theorem 1.1 we know that there exists a universal constant $T_0 \in (0, 1)$ such that for $\forall X_0 \in Q_\varepsilon_\ast$, (1.5) admits a unique solution $X$ stating from $X_0$ and satisfying

$$\|\theta\|_{L_t^\infty H^{1/2} \cap L_t^2 H^1} + \|y_s\|_{L_t^\infty H^{1/2} \cap L_t^2 H^2} \leq 16(\|\theta_0\|_{H^{1/2}} + \|y_{0s}\|_{H^{1/2}}) \leq 16\varepsilon_\ast.$$

Moreover, for $\forall t \in [0, T_0], (5.10)-(5.12)$ holds. Therefore, taking $\varepsilon_\ast = \min\{(48\pi C_\ast)^{-1}, \frac{1}{64}\}$, we know that $X$ satisfies the assumption of Lemma 5.2 with $T = T_0$, $R = 16\varepsilon_\ast$, $\beta_1 = \frac{1}{2\pi}$ and $\beta_2 = \frac{1}{2}$, which are all universal constants.

From the energy dissipation equation (2.31), we have

$$\frac{d}{dt} \left( \frac{1}{2s(t)} \int_T \theta_\alpha^2(\alpha, t)d\alpha + \frac{s^2(t)}{2} \int_T (1 + y_s)^2 ds + 2\pi t s(t) \right) = - \int_{\mathbb{R}^2} |\nabla u|^2 dx,$$
which implies that

\[
\frac{1}{2s} \int_T \theta^2 \, d\alpha + \frac{s^2}{2} \int_T (1 + y_s)^2 \, ds + 2\pi \lambda s - 2\pi - 2\pi \lambda,
\]

\[
\leq \frac{1}{2s_0} \int_T \theta_{0\alpha}^2 \, d\alpha + \frac{s_0^2}{2} \int_T (1 + y_{0s})^2 \, ds + 2\pi \lambda s_0 - 2\pi - 2\pi \lambda.
\]

With the help of Lemma \(5.1\) and \(5.3\), one can see that

\[
\|\theta - \alpha\|_{L^\infty_t H^1} + \|y_s\|_{L^\infty_s L^2} \leq C_2(\|\theta_0 - \alpha\|_{H^1} + \|y_{0s}\|_{L^2}),
\]

where \(C_2\) is a constant only depends on \(\lambda\). Then, we choose

\[
\varepsilon = (8C_2c_\varepsilon^{3/2})^{-1} \varepsilon_*,
\]

which is the desired constant. Here \(c_* = c_\varepsilon(16\varepsilon_*, \lambda, \beta_1 = \frac{1}{2\pi}, \beta_2 = \frac{1}{2})\) is defined in Lemma \(5.2\). We claim that \(5.5\) holds if \(\|\theta - \alpha\|_{H^5/2} + \|y_s\|_{H^3/2}^2 \geq \frac{\varepsilon^2}{4}\). Indeed, as

\[
\frac{\|\theta - \alpha\|_{H^5/2}^2 + \|y_s\|_{H^3/2}^2}{(\|\theta - \alpha\|_{H^1} + \|y_s\|_{H^3/2})^{1/2} \|y_s\|_{H^3/2}} \geq \frac{\varepsilon^2}{8C_2^2\varepsilon^4/3 + 2^{5/3}C_2^{2/3} \varepsilon^{2/3}} \geq c_\varepsilon.
\]

Next, we show

\[
(\|\theta - \alpha\|_{H^5/2} + \|y_s\|_{H^3/2})(t) \leq \varepsilon_*, \quad \text{for } \forall t \in [0, T_0].
\]

As \(\|\theta - \alpha\|_{H^5/2} + \|y_s\|_{H^3/2}(t)\) is continuous on \([0, T_0]\), if \(\|\theta - \alpha\|_{H^5/2} + \|y_s\|_{H^3/2}(t_1) > \varepsilon_*\) for some \(t_1 \in [0, T_0]\), there must be \(t_0 < t_1\) such that

\[
(\|\theta - \alpha\|_{H^5/2} + \|y_s\|_{H^3/2})(t_0) = \varepsilon_*,
\]

\[
\varepsilon_* \leq (\|\theta - \alpha\|_{H^5/2} + \|y_s\|_{H^3/2})(t) \leq 16\varepsilon_*, \quad \forall t \in [t_0, t_1].
\]

Using Lemma \(5.2\) we have \(\|\theta - \alpha\|_{H^5/2} + \|y_s\|_{H^3/2}(t_1) \leq e^{-[t_1 - t_0]}\varepsilon_*\), which contradicts the assumption. Therefore, \(5.15\) holds for all \(t \in [0, T_0]\) and \(X(T_0) \in Q_\varepsilon\). We repeat the same steps like above and extend \(X\) to \([0, 2T_0]\), then existence of a global solution is established, and a universal estimate follows that

\[
\|\theta - \alpha\|_{L^\infty_t H^5/2} + \|y_s\|_{L^\infty_t H^3/2} \leq 8C_2c_\varepsilon^{3/2} \varepsilon_*,
\]

Accordingly, we arrive at the conclusion of this theorem. \(\Box\)

6. Exponential Convergence to Equilibrium Configurations

In this section, we prove that the global-in-time solution near equilibriums obtained in Theorem \(1.2\) converges exponentially in the Sobolev norms to an equilibrium configuration as \(t \to +\infty\). In what follows, we assume the area enclosed by the string is \(\pi\).
6.1. A lower bound of the rate of energy dissipation. A key step to prove the exponential convergence is to establish a lower bound of the rate of energy dissipation \( \int_{\mathbb{R}^2} |\nabla u|^2 \, dx \) in terms of \((||\theta - \alpha||_{H^2} + ||y_s||_{L^s} + s - 1)\).

Let \( Q_{e^*} \) be defined as in (5.9), \( e^* > 0 \) is to be determined. Let \( \Omega_X \subset \mathbb{R}^2 \) denote the bounded open domain enclosed by \( X(\mathbb{T}) \) where \( X \in Q_{e^*} \). Define the collection of all such domains to be

\[
\mathcal{M}_{e^*} = \{ \Omega_X \subset \mathbb{R}^2 : \partial \Omega_X = X(\mathbb{T}), X \in Q_{e^*} \}.
\]

We assume that \( e^* \) is sufficiently small such that domains in \( \mathcal{M}_{e^*} \) satisfy the uniform \( C^1 \) regularity condition with uniform constants. Indeed, as \( z \in H^{3/2}(\mathbb{T}) \), this is achievable due to the implicit function theorem and the Sobolev embedding \( H^3(\mathbb{T}) \hookrightarrow C^2(\mathbb{T}) \).

For the velocity field \( u \) determined by \( X \), we define

\[
(u)_{\Omega_X} = |\Omega_X|^{-1} \int_{\Omega_X} u \, dx, \quad (u)_{\partial \Omega_X} = |\partial \Omega_X|^{-1} \int_{\partial \Omega_X} u \, ds,
\]

where \( s \) is the arc-length parameter of \( \partial \Omega_X \). Then by the boundary trace theorem, it follows that

\[
\int_{\mathbb{R}^2} |\nabla u|^2 \, dx \geq \int_{\Omega_X} |\nabla u|^2 \, dx \geq C \int_{\partial \Omega_X} |u - (u)_{\Omega_X}|^2 \, ds
\]

\[
\geq C \int_{\partial \Omega_X} |u - (u)_{\partial \Omega_X}|^2 \, ds = C \int_T |u - (u)_{\partial \Omega_X}|^2 \, d\alpha.
\]

Before showing the above value is bounded from below by \((||\theta - \alpha||_{H^2} + ||y_s||_{L^s} + s - 1)\), we do some preparations.

Recalling that \( X_*(\alpha) = (\cos(\alpha), \sin(\alpha)) \), a direct computation shows that

\[
-\frac{\partial}{\partial \alpha'} G(X_*(\alpha, t) - X_*(\alpha', t)) = \frac{1}{8\pi} \begin{pmatrix} -\sin(\alpha + \alpha') - \frac{1}{\tan(\frac{\alpha}{2})} & \cos(\alpha + \alpha') \\ \cos(\alpha + \alpha') & \sin(\alpha + \alpha') - \frac{1}{\tan(\frac{\alpha}{2})} \end{pmatrix}.
\]

We use \( u \) to denote the velocity filed generated from \( X_* \), and it is obvious that \( u = 0 \). Recall that \( t(\alpha, t) = \int_{-\pi}^\alpha n(\alpha', t) \, d\alpha' \). By using the same technical in Lemma 3.6 we rewrite (2.20) to

\[
(6.1) \quad u(\alpha, t)
\]

\[
= p.v. \int_T \frac{\partial}{\partial \alpha'} G(z(\alpha, t) - z(\alpha', t)) \cdot (\lambda t - \lambda' + \frac{\theta_{aa} \cdot n}{\sin^2} - \frac{\theta^2 \cdot t}{2s^2} + \frac{\theta_{\alpha} \cdot n}{s} + \frac{t}{s} + s(1 + y_s) t - st) (\alpha', t) \, d\alpha'
\]

\[
= p.v. \int_T \left( \frac{\partial}{\partial \alpha'} G(z(\alpha, t) - z(\alpha', t)) + \frac{\partial}{\partial \alpha'} G(X_*(\alpha, t) - X_*(\alpha', t)) \right) \cdot (\lambda t - \lambda' + \frac{\theta_{aa} \cdot n}{\sin^2} - \frac{\theta^2 \cdot t}{2s^2} + \frac{\theta_{\alpha} \cdot n}{s} + \frac{t}{s} + s(1 + y_s) t - st) (\alpha', t) \, d\alpha'
\]

\[
+ p.v. \int_T \frac{\partial}{\partial \alpha'} G(X_*(\alpha, t) - X_*(\alpha', t)) \cdot (s(y_s - y_o) t(\alpha', t) \, d\alpha'
\]
\[ + \text{p.v.} \int_{\mathcal{T}} \frac{\partial}{\partial \alpha} G(\mathbf{X}_s(\alpha, t) - \mathbf{X}_s(\alpha', t)) \cdot (\lambda t - \lambda t - \frac{\hat{\theta}_{\alpha \alpha} \eta}{s^2} - \frac{\hat{\theta}_{\alpha \alpha} \eta}{s^2} + \frac{t}{2s^2} + s(1 + y), t - s t)(\alpha', t) d\alpha' \]

\[ \overset{\text{def}}{=} \mathcal{R}_1 + \mathcal{R}_2 + v. \]

Here \( y_s(\alpha, t) \overset{\text{def}}{=} y_s(s, t) \big|_{s=\alpha} \) is a function defined on arc-length coordinate. One can regard it as \( y_s(s, t) \) which have changed the notation of variable from \( s \) to \( \alpha \). We must be careful that \( y_s(\alpha, t) \) is different to \( y_s(s(\alpha, t), t) \). We introduce such special notation to emphasize this point. When \( \|y_s\|_{L^2} \) is small, there is little difference between \( \alpha \) and \( s(\alpha, t) \), so do \( y_s(\alpha, t) \) and \( y_s(s(\alpha, t), t) \). We also note that \( \|y_s\|_{L^2} = \|y_s\|_{L^2} \).

Next, we introduce some auxiliary functions

\[ D = \theta(\alpha, t) - \alpha, \quad \theta(\alpha, t) = \alpha + \eta D(\alpha, t), \quad y(\eta, \alpha, t) = \eta y(\alpha, t), \quad \eta \in [0, 1]; \]

\[ n_s(\alpha, t) = (\sin(\theta(\alpha, t)), \cos(\theta(\alpha, t)), \mathbf{t}(\alpha, t) = (\cos(\theta(\alpha, t)), \sin(\theta(\alpha, t))); \]

\[ t_s(\alpha, t) = \int_0^x (\sin(\theta(\alpha)), \cos(\theta(\alpha))) d\alpha - \frac{\alpha}{2\pi} \int_0^\pi (\sin(\theta(\alpha)), \cos(\theta(\alpha))) d\alpha; \]

\[ Dn_s(\alpha, t) = \int_0^\pi D(\alpha)(\sin(\alpha(\alpha)), \cos(\alpha(\alpha)), \alpha(\alpha)) d\alpha. \]

and an auxiliary velocity,

\[ v_s(\alpha, t) = \int_{\mathcal{T}} \frac{\partial}{\partial \alpha} G(\mathbf{X}_s(\alpha, t) - \mathbf{X}_s(\alpha, t)) \cdot (\mathbf{t}(-\alpha, t) - \mathbf{t}(-\alpha, t) - \frac{\hat{\theta}_{\alpha \alpha} \eta}{s^2} - \frac{\hat{\theta}_{\alpha \alpha} \eta}{s^2} + \frac{t}{2s^2} + s(1 + y), t - s t)(\alpha', t) d\alpha' \]

\[ \overset{\text{def}}{=} \int_{\mathcal{T}} h(\theta(\alpha, t), (\alpha')) d\alpha'. \]

In the following lemma, we linearize \( u \) and extract the principal part. The main idea is to apply Taylor expansion to the trigonometric functions.

**Lemma 6.1.** Assume \( X \in Q_\alpha \), for a sufficiently small constant \( \varepsilon \), it holds that

\[ u(\alpha, t) = \hat{\theta} \big|_{\eta=0} v(\alpha) + \mathcal{R}(\alpha), \]

where

\[ \|\mathcal{R}\|_{L^2} \leq \varepsilon \|\theta - \alpha\|_{H^2} + \|y_s\|_{L^2}, \]

with some universal constant \( C \).
Proof. Recalling (6.1), it is easy to see that $v_{\eta\eta=0} = 0$, $v_{\eta}\eta=1 = v$. By the mean value theorem respect to $\eta$, for each fixed $\alpha$ there exists an $\eta^* = \eta^*(\alpha) \in [0, 1]$ such that

$$v(\alpha) = v_{\eta\eta=1} - v_{\eta\eta=0} = \frac{\partial}{\partial \eta} \bigg|_{\eta=\eta^*(\alpha)} v_{\eta}(\alpha)$$

$$= \frac{\partial}{\partial \eta} \bigg|_{\eta=0} v_{\eta}(\alpha) + \frac{\partial}{\partial \eta} \bigg|_{\eta=\eta^*(\alpha)} v_{\eta}(\alpha) - \frac{\partial}{\partial \eta} \bigg|_{\eta=0} v_{\eta}(\alpha) = \frac{\partial}{\partial \eta} \bigg|_{\eta=0} v_{\eta}(\alpha) + R_3(\alpha).$$

It remains to verify (6.3) for

$$R = R_1 + R_2 + R_3.$$

We see at once that $||R_1||_{L^2} \leq \varepsilon_s(||\theta - \alpha||_{H^2} + ||y_s||_{\bar{c}})$. In the same way to Lemma 3.7, we have

$$||R_2||_{L^2} \leq C||\mathcal{H}||_{L^2} (||y_s(\alpha, t) - y_s(\alpha, t)||_{L^2} + C||y_s(\alpha, t) - y_s(\alpha, t)||_{L^2} \leq C\varepsilon_s||y_s||_{\bar{c}}.$$

By using the dominated convergence theorem, we have

$$\partial_{\eta} v_{\eta}(\alpha) = \int_{T} \partial_{\eta} h_{\eta}(\alpha, \alpha') d\alpha'$$

$$= \text{p.v.} \int_{T} - \frac{\partial}{\partial \alpha'} G(X_{\eta}(\alpha, t) - X_{\eta}(\alpha', t))$$

$$\cdot \left((\lambda D_{\eta} \eta - \frac{D_{\alpha} D_{\eta}}{\varepsilon^2} \eta - \theta_{\eta\alpha} D_{\eta} \eta - \frac{2D_{\alpha} \eta + \theta_{\eta\alpha}^2 D_{\eta} \eta}{\varepsilon^2} + \frac{2y_{\alpha} t_{\eta} + (1 + y_{\eta\alpha}) D_{\eta} \eta}{\varepsilon^2}\right) + (\lambda + s - \frac{1}{2\varepsilon^2})(\int_{-\pi}^{\alpha'} D_{\eta}(\alpha') d\alpha' - \frac{\alpha'}{2\pi} \int_{-\pi}^{\alpha'} D_{\eta}(\alpha') d\alpha').$$

It follows that,

$$R_3(\alpha) = \frac{\partial}{\partial \eta} \bigg|_{\eta=\eta^*(\alpha)} v_{\eta}(\alpha) - \frac{\partial}{\partial \eta} \bigg|_{\eta=0} v_{\eta}(\alpha)$$

$$= \text{p.v.} \int_{T} - \frac{\partial}{\partial \alpha'} G(X_{\eta}(\alpha, t) - X_{\eta}(\alpha', t)) \cdot \left(\eta^* \theta_{\eta\alpha} D_{\eta} \eta - \eta^* D_{\eta} \eta + \eta^* D_{\eta} \eta - \theta_{\eta\alpha} D_{\eta} \eta - \frac{2D_{\alpha} \eta + \theta_{\eta\alpha}^2 D_{\eta} \eta}{\varepsilon^2}\right) + (\lambda + s - \frac{1}{2\varepsilon^2})(\int_{-\pi}^{\alpha'} D_{\eta}(\alpha') d\alpha' - \frac{\alpha'}{2\pi} \int_{-\pi}^{\alpha'} D_{\eta}(\alpha') d\alpha').$$

It is easy to check that

$$||n_{\eta} - n_{s}||_{H^1} + ||t_{\eta} - t_{s}||_{H^1} \leq C||\theta - \alpha||_{H^1}.$$

Accordingly, similar to (6.4), we conclude that

$$||R_3||_{L^2} \leq C\varepsilon_s(||\theta - \alpha||_{H^2} + ||y_s||_{\bar{c}}).$$

This completes the proof.\qed

The following lemma gives the leading term of $u(\alpha)$. 

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Lemma 6.2. Under the assumptions of Lemma 6.1 it holds that

\[ \partial_{\eta|_{\eta=0}} v_\eta(\alpha) = \frac{1}{16\pi} \sum_{k \in \mathbb{Z} / 0} (N_k, M_k), \]

where

\[
N_k = \left[ \frac{1}{2s^2} k - \frac{1}{2s^2} \frac{k-1}{|k|} - \frac{1}{2s^2} \frac{k(k-1)^2}{|k|} + s \frac{k}{|k|} b_{k-1}(y_\alpha) \right. \\
+ \left( \frac{1}{2s^2} k + \frac{1}{2s^2} \frac{k(k+1)^2}{|k|} + \frac{1}{2s^2} \frac{k(k+1)}{|k|} \right) a_{k+1}(D) + \frac{s}{|k|} b_{k+1}(y_\alpha) \right. \\
+ i \left( \frac{1}{2s^2} k - \frac{1}{2s^2} \frac{k(k+1)}{|k|} - \frac{1}{2s^2} \frac{k(k+1)}{|k|} \right) b_{k-1}(D) + i s a_{k-1}(y_\alpha) \\
+ i \left( \frac{1}{2s^2} k + \frac{1}{2s^2} \frac{k(k+1)}{|k|} - \frac{1}{2s^2} \frac{k(k+1)}{|k|} \right) b_{k+1}(D) + i s b_{k+1}(y_\alpha) \]

\[
M_k = \left[ \frac{1}{2s^2} k - \frac{1}{2s^2} \frac{k-1}{|k|} + \frac{1}{2s^2} \frac{k(k-1)^2}{|k|} - s \frac{k}{|k|} a_{k-1}(y_\alpha) \right. \\
+ \left( \frac{1}{2s^2} k + \frac{1}{2s^2} \frac{k(k+1)}{|k|} - \frac{1}{2s^2} \frac{k(k+1)}{|k|} \right) b_{k-1}(D) - \frac{s}{|k|} a_{k+1}(y_\alpha) \\
+ i \left( \frac{1}{2s^2} k - \frac{1}{2s^2} \frac{k(k+1)}{|k|} - \frac{1}{2s^2} \frac{k(k+1)}{|k|} \right) a_{k-1}(D) - i s \frac{k}{|k|} b_{k-1}(y_\alpha) \\
+ i \left( \frac{1}{2s^2} k + \frac{1}{2s^2} \frac{k(k+1)}{|k|} + \frac{1}{2s^2} \frac{k(k+1)}{|k|} \right) a_{k+1}(D) + i s \frac{k}{|k|} b_{k+1}(y_\alpha) \right] e^{ik\alpha}.
\]

Here we use \( a_k \) and \( b_k \) to denote the Fourier coefficients such that

\[ a_k(f) = \int_{-\pi}^{\pi} \cos(k\alpha) f(\alpha) d\alpha, \quad b_k(f) = \int_{-\pi}^{\pi} \sin(k\alpha) f(\alpha) d\alpha. \]

From the above expression, it is obvious that

\[ \int_T \partial_{\eta|_{\eta=0}} v_\eta(\alpha) d\alpha = 0. \]

Proof. From the definitions, we have

\[
\partial_{\eta|_{\eta=0}} v_\eta(\alpha) = \text{p.v.} \int_T \frac{1}{2\pi} \begin{pmatrix}
- \sin(\alpha + \alpha') - \frac{1}{\tan(\frac{\alpha + \alpha'}{2})} \\
\cos(\alpha + \alpha') \sin(\alpha + \alpha') - \frac{1}{\tan(\frac{\alpha + \alpha'}{2})}
\end{pmatrix}
\cdot \left( \lambda + \frac{1}{2s^2} (Dn_* - Dn_* - \frac{D_{\alpha\alpha} n_* + D_{\alpha} t_*}{s^2} + s_y a_t) \right) (\alpha') d\alpha'.
\]
Direct calculations show that
\[
\int_{-\pi}^{\pi} \begin{pmatrix} -\sin(\alpha + \alpha') & \cos(\alpha + \alpha') \\ \cos(\alpha + \alpha') & \sin(\alpha + \alpha') \end{pmatrix} \cdot Dn_\alpha(d\alpha') = 0.
\]

For the same reason, one can deduce that
\[
\int_{-\pi}^{\pi} \begin{pmatrix} -\sin(\alpha + \alpha') & \cos(\alpha + \alpha') \\ \cos(\alpha + \alpha') & \sin(\alpha + \alpha') \end{pmatrix} \cdot \left( (\lambda + s - \frac{1}{2s^2})(Dn_\alpha - Dn_\alpha) - \frac{D_{\alpha\alpha}n_\alpha + D_{\alpha}t_\alpha}{s^2} + s_\alpha t_\alpha \right)(\alpha')d\alpha' = 0.
\]

Therefore,
\[
\partial_\eta|_{\alpha=0}v_\eta(\alpha) = -\frac{1}{4}H\left( (\lambda + s - \frac{1}{2s^2})(Dn_\alpha - Dn_\alpha) - \frac{D_{\alpha\alpha}n_\alpha + D_{\alpha}t_\alpha}{s^2} + s_\alpha t_\alpha \right)(\alpha).
\]

For \(k \neq 0\), we have
\[
a_k(Dn_\alpha) = \int_{-\pi}^{\pi} \cos(k\alpha) \begin{pmatrix} -D(\alpha)\sin(\alpha) \\ D(\alpha)\cos(\alpha) \end{pmatrix} d\alpha = \frac{1}{2} \left( b_{k-1}(D) - b_{k+1}(D) \right),
\]
\[
b_k(Dn_\alpha) = \frac{1}{2} \left( -a_{k-1}(D) + a_{k+1}(D) \right),
\]
and
\[
H(Dn_\alpha)(\alpha) = \sum_{k \in \mathbb{Z}/0} \left( -itd_{\alpha}a_k(Dn_\alpha) - \frac{\text{sgn}k}{2\pi} b_k(Dn_\alpha) \right)e^{ik\alpha} = \sum_{k \in \mathbb{Z}/0} \frac{1}{4\pi} \begin{pmatrix} \frac{k}{|k|}a_{k-1}(D) - \frac{k}{|k|}a_{k+1}(D) - \frac{k}{|k|}b_{k-1}(D) - \frac{k}{|k|}b_{k+1}(D) \\ \frac{k}{|k|}b_{k-1}(D) - \frac{k}{|k|}b_{k+1}(D) - \frac{k}{|k|}a_{k-1}(D) - \frac{k}{|k|}a_{k+1}(D) \end{pmatrix}.
\]

In this way, one can achieve the conclusion. \(\square\)
Next, we state an important observation which reflect a special property of the tangent angle functions.

**Lemma 6.3.** Let \( \gamma \geq 0 \), for \( X \in Q_{\varepsilon_*} \) with \( \varepsilon_* \) small enough, there exists a universal constant \( C \) such that

\[
\|D\|_{H^1}^2 \leq \frac{1}{\pi (1 - C \varepsilon_*^2)} \sum_{k=2}^{+\infty} (a_k^2(D) k^{2\gamma} + b_k^2(D) k^{2\gamma}),
\]

where \( \bar{D} \) is the mean value of \( D \).

**Proof.** From the assumption, \( \theta \) satisfies

\[
\int_{-\pi}^{\pi} \cos(\theta(\alpha))d\alpha = 0 \quad \text{and} \quad \theta(\pi) - \theta(-\pi) = 2\pi.
\]

It also holds that

\[
\int_{-\pi}^{\pi} \theta(\alpha) \cos(\theta(\alpha) - \bar{\theta})d\alpha = \int_{\theta(-\pi)}^{\theta(\pi)} \cos(\theta - \bar{\theta}) d\theta = 0.
\]

Then we have

\[
\int_{-\pi}^{\pi} (\theta(\alpha) - 1) \cos(\theta(\alpha) - \bar{\theta})d\alpha = 0.
\]

Therefore, one can deduce that

\[
a_1(D) = \int_{-\pi}^{\pi} (\theta - \alpha) \sin(\alpha)d\alpha = \int_{-\pi}^{\pi} (\theta - 1) \cos(\alpha)d\alpha
\]

\[
= \int_{-\pi}^{\pi} (\theta - 1)(\cos(\alpha) - \cos(\theta - \bar{\theta}))d\alpha
\]

\[
= 2 \int_{-\pi}^{\pi} (\theta - 1) \sin(\frac{\alpha + \theta - \bar{\theta}}{2}) \sin(\frac{\theta - \bar{\theta} - \alpha}{2})d\alpha
\]

\[
\leq C \int_{-\pi}^{\pi} (\theta - 1)^2 d\alpha \frac{1}{2} \int_{-\pi}^{\pi} \sin^2(\frac{\theta - \bar{\theta} - \alpha}{2})d\alpha\frac{1}{2}
\]

\[
\leq C \|\theta - 1\|_{L^2} \|\theta - \bar{\theta} - a\|_{L^2} \leq C \|D\|_{H^1} \|D - \bar{D}\|_{L^2}.
\]

Similarly, one has

\[
b_1(D) \leq C \|D\|_{H^1} \|D - \bar{D}\|_{L^2}.
\]

For any integrable real-valued function \( f \), it holds that

\[
a_k(f) = a_{-k}(f), \quad b_k(f) = -b_{-k}(f).
\]

From the above results, we have

\[
(1 - C \varepsilon_*^2) \|D\|_{H^1}^2 \leq \frac{1}{\pi} \sum_{k=2}^{+\infty} (a_k^2(D) k^{2\gamma} + b_k^2(D) k^{2\gamma}),
\]

which is the desired conclusion. \( \Box \)

Then, we can give the lower bound of the rate of energy dissipation.
Lemma 6.4. There exists a universal \( \varepsilon > 0 \) such that

\[
(6.5) \quad ||u - u_{crit}||_{L^2(\Omega)} \geq C(||\theta - \alpha||_{H^2} + ||\gamma||_E + \lambda^2(s-1)), \quad \forall \mathbf{X} \in Q_{\varepsilon},
\]

where \( Q_{\varepsilon} \) is defined in (5.9) and \( C \) is a universal constant. Since \( s \geq 1 \), (6.5) implies that

\[
\int_{\mathbb{R}^2} |\nabla u|^2 \, dx \geq C(||\theta - \alpha||_{H^2} + ||\gamma||_E + \lambda^2(s-1)), \quad \forall \mathbf{X} \in Q_{\varepsilon}.
\]

Proof. From Lemma 6.2, it holds that

\[
\int_{\mathbb{R}^2} |\partial_{\gamma} b_{\gamma} v_{\gamma}(\alpha)|^2 \, d\alpha
\]

\[
= \sum_{k=1}^{+\infty} \frac{1}{32\pi} \left[ \left( \frac{1}{2} + \frac{1}{s^2} k \right)^2 - \frac{1}{s^2} \left( \frac{1}{2} + \frac{1}{s^2} k \right)^2 \right] a_{k-1}(D) + \frac{k}{|k|} b_{k-1}(y_\alpha) \right]^2
\]

\[
+ \left( \frac{1}{2} + \frac{1}{s^2} k \right)^2 \left( \frac{1}{2} + \frac{1}{s^2} k \right)^2 \right] a_{k+1}(D) + \frac{k}{|k|} b_{k+1}(y_\alpha) \right]^2
\]

\[
+ \left( \frac{1}{2} + \frac{1}{s^2} k \right)^2 \left( \frac{1}{2} + \frac{1}{s^2} k \right)^2 \right] b_{k-1}(D) + \frac{k}{|k|} a_{k-1}(y_\alpha) \right]^2
\]

\[
+ \left( \frac{1}{2} + \frac{1}{s^2} k \right)^2 \left( \frac{1}{2} + \frac{1}{s^2} k \right)^2 \right] b_{k+1}(D) - \frac{k}{|k|} a_{k+1}(y_\alpha) \right]^2
\]

\[
\geq \frac{s^2}{64\pi} \left( a_{k-1}(y_\alpha) + b_{k-1}(y_\alpha) \right) - \frac{1}{8\pi} (\lambda + s - \frac{1}{2s^2}) \left( a_{k+1}(D) + b_{k+1}(D) \right)
\]

\[
+ \frac{s^2}{32\pi} \left( a_{k-1}(y_\alpha) + b_{k-1}(y_\alpha) \right) + \frac{(\lambda + s)^2}{12\pi} \left( a_{k-1}(D) + b_{k-1}(D) \right)
\]

\[
+ \sum_{k=3}^{+\infty} \frac{1}{32\pi} \left[ \left( \frac{k}{k-1} + \frac{2k^2 - 3k}{k-1} \right)^2 \right] \left( a_{k+1}(D) + b_{k+1}(D) \right)
\]

\[
+ s^2 \left( a_{k-1}(y_\alpha) + b_{k-1}(y_\alpha) \right) + s \left( \lambda + s \right) \left( \frac{k}{k-1} + \frac{2k^2 - 3k}{k-1} \right) \left( a_{k-1}(D) + b_{k-1}(D) \right)
\]

\[
\geq \frac{s^2}{64\pi} \sum_{k=1}^{+\infty} \left( a_{k-1}(y_\alpha) + b_{k-1}(y_\alpha) \right) - \frac{1}{8\pi} (\lambda + s - \frac{1}{2s^2}) \left( a_{k+1}(D) + b_{k+1}(D) \right)
\]
\[
\frac{1}{32\pi} \sum_{k=2}^{+\infty} \left( (\lambda + s)^2 + \frac{k^4}{s^4} \right) \left( a_k^2(D) + b_k^2(D) \right)
\geq C (s^2 \|y_s\|_{L^2}^2 + \frac{1}{s^2} \|\theta - \alpha\|_{H^2}^2 + (\lambda + s) \|\theta - \tilde{\theta}\|_{L^2}^2),
\]
the last inequality follows from Lemma 6.3. By using Lemma 6.1 and Lemma B.3, we have
\[
\|u - u_{\partial\mathcal{X}}\|_{L^2}^2 \geq C \|\partial_{\eta_{\eta=0}}v_{\eta}\|_{L^2}^2 - C \|R - \overline{R}\|_{L^2}^2 \geq C (\|y_s\|_{L^2}^2 + \|\theta - \alpha\|_{H^2}^2 + \lambda^2 (s - 1)),
\]
which actually ensures the conclusion. \(\square\)

Now, we are in a position to Theorem 1.3.

**Proof of Theorem 1.3.** From (2.31) and Lemma 6.4, taking \(\epsilon\) sufficiently small, we deduce that
\[
\frac{d}{dt} \left( \frac{1}{2\pi} \int_{\Omega} \theta^2 \, d\tilde{\alpha} + \frac{s^2}{2} \int_{T} (1 + y_s)^2 \, ds + 2\pi s^2 - 2\pi - 2\pi \lambda \right)
= -\int_{\mathbb{R}^2} |\nabla u|^2 \, dx 
\leq -C \left( \int_{T} (\theta - 1)^2 \, d\alpha + \int_{T} y_s^2 \, ds + \lambda (s - 1) \right),
\]
where \(C > 0\) is a universal constant. From Lemma 5.1, one can see that there exists a constant \(c_\gamma\), such that
\[
\|\theta - \alpha\|_{H^1} + \|y_s\|_{L^2} + \lambda |s| - 1| \leq Ce^{-\gamma t} (\|\theta_0 - \alpha\|_{H^1} + \|y_0\|_{L^2} + \lambda |s_0| - 1).
\]
In the proof of Theorem 1.2, we know that
\[
\|\theta - \alpha\|_{H^{1/2}}^2 + \|y_s\|_{H^{1/2}}^2 \leq c_* (\|\theta - \alpha\|_{H^1} + \|y_s\|_{H^{1/2}}^2 \|y_s\|_{L^2}^{1/2})^2,
\]
for some constant \(c_*\), therefore
\[
\|\theta - \alpha\|_{H^{1/2}}(t) + \|y_s\|_{H^{1/2}}(t) \leq Ce^{-\gamma_* t}.
\]
Recalling (3.6) and (2.17), one can see that there exists
\[
\lim_{t \to +\infty} \bar{\theta}(t) \to \theta_\infty, \quad \lim_{t \to +\infty} \int_0^t u(X(-\pi, t'), t') \, dt' \to x_\infty.
\]
This complete the proof. \(\square\)

7. Conclusion and further discussion

In this paper, we study the Stokes immersed boundary problem with bending and stretching energy and establish the global well-posedness of the problem in Sobolev space. We give a proof for the case with elastic coefficients \(c_1 = c_3 = 1\), and surface tension \(\lambda \geq 0\). From the proof, one can see that the results is independent of the choice of these coefficients. When \(c_1 = 0\) or \(c_3 = 0\), our framework still works.
For the case without stretching energy \((c_3 = 0)\), the force applied on the membranes only depend on the shape of the string. We only need to use \(z(\alpha)\) to parametrize the curve. However, \(z_t(\alpha, t) \cdot n = u(z(\alpha, t), t) \cdot n\), the velocity field of the fluid on the string determines the evolution of the membrane’s shape, so the free boundary problem \((1.2)\) is equivalent to the evolution equation of \(z\):

\[
z_t(\alpha, t) \cdot n = n \cdot \int_T G(z(\alpha, t) - z(\alpha', t)) \cdot F(\alpha', t)|z_\alpha(\alpha', t)|d\alpha',
\]

where

\[
F = \lambda n - \left( \frac{1}{\alpha^2} \partial_\gamma^2 \kappa + \frac{1}{2} \kappa^3 \right) n.
\]

Next, one can derive the contour dynamic system base on the evolution equations of \(\theta\) and \(s\). For simplicity, one can choose

\[
\overline{T}(t) = \frac{1}{4\pi} \int_{-\pi}^{\pi} u \cdot n_{\theta} d\alpha \int_{-\pi}^{\pi} \alpha \theta_\alpha d\alpha - \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} u(\alpha') \cdot n(\alpha') \theta_\alpha(\alpha') d\alpha' d\alpha.
\]

Thus we always have \(\bar{\theta}(t) = 0\).

For the case without bending energy and surface tension \((c_1 = 0, \lambda = 0)\), we go back to the model studied by Lin, Tong in [22]. In this case, the elastic energy do not contain any norm of \(\theta\). However, Lemma [B.3] (Fuglede’s isoperimetric inequality) shows that \(|\theta - \bar{\theta} - \alpha|_{L^2} \leq C(s - 1)\). Then we can use a similar method to get the global well-posedness of these free boundary problems. Furthermore, our method also works for the case \((c_1 = 0, c_3 > 0, \lambda > 0)\).

For the case with none but surface tension \((c_1 = 0, c_3 = 0, \lambda > 0)\), there are a lot of work about this problem [33, 34, 25]. The method establish in this paper give a new explanation on the stabilizing effect of surface tension for the Stokes flow.

**Appendix A. A Priori Estimates Involving \(L\) and \(\Omega\)**

Recalling that

\[
L(\theta)(\alpha, t) = \frac{1}{4s^3(t)} \mathcal{H}(\theta_{aaa})(\alpha, t), \quad \mathcal{L}(y)(s, t) = -\frac{1}{4} b(y_{ss})(s, t),
\]

we have following estimates under the assumption that \(c \geq s(t) \geq c_0 > 0\) for \(t \in [0, T]\).

**Lemma A.1.** For \(\forall \phi \in H^\gamma(\mathbb{T}), w \in h^\gamma(\mathbb{T})\) with arbitrary \(\gamma \in \mathbb{R}_+\), it holds that

1. \(\|e^{tL} \phi\|_{H^\gamma} \leq e^{-t(4c_1)} \|\phi\|_{H^\gamma}; \quad \|e^{tL} w\|_{h^\gamma} \leq e^{-t/4} \|w\|_{h^\gamma};\)
2. \(e^{tL} \phi \in C([0, +\infty); H^\gamma(\mathbb{T})); \quad e^{tL} w \in C([0, +\infty); h^\gamma(\mathbb{T}));\)
3. \(e^{tL} \phi \rightarrow \phi \text{ in } H^\gamma(\mathbb{T}) \text{ as } t \rightarrow 0^+, \quad e^{tL} w \rightarrow w \text{ in } h^\gamma(\mathbb{T}) \text{ as } t \rightarrow 0^+.\)

**Lemma A.2.** Assume \(T > 0\), let \(g \in L^2_T H^\gamma(\mathbb{T})\). The model equation

\[
(\theta(\alpha, t) - \alpha) = L(\theta)(\alpha, t) + g(\alpha, t), \quad \theta(\alpha, 0) = \theta_0(\alpha), \quad \alpha \in \mathbb{T}, t \geq 0,
\]

= \int_T G(z(\alpha, t) - z(\alpha', t)) \cdot F(\alpha', t)|z_\alpha(\alpha', t)|d\alpha',
\]

\[
F = \lambda n - \left( \frac{1}{\alpha^2} \partial_\gamma^2 \kappa + \frac{1}{2} \kappa^3 \right) n.
\]
Lemma B.2. Given two closed string $\kappa$ and $\alpha \in L_T^\infty H^{\gamma + \frac{1}{2}}(\mathbb{T}) \cap L_T^2 H^{\gamma + 3}(\mathbb{T})$ with $\theta_t \in L_T^2 H^\gamma(\mathbb{T})$. Furthermore, this solution satisfies
\[
\|\theta - \alpha\|_{L_T^2 H^{\gamma + 3}}^2(t) \leq e^{-(1/4e^3)}\|\theta_0 - \alpha\|_{L_T^2 H^{\gamma + 3}}^2 + 4e^3\|g\|_{L_T^2 H^\gamma}^2, \forall t \in [0, T],
\]
\[
\frac{d}{dt}\|\theta - \alpha\|_{L_T^2 H^{\gamma + 3}}^2(t) \leq -\frac{1}{4e^3}\|\theta - \alpha\|_{L_T^2 H^{\gamma + 3}}^2(t) + 4e^3\|g\|_{L_T^2 H^\gamma}^2(t), \forall t \in [0, T].
\]

Hence,
\[
\|\theta - \alpha\|_{L_T^2 H^{\gamma + 3}} \leq (1 + 2e^{3/2})\|\theta_0 - \alpha\|_{L_T^2 H^{\gamma + 3}} + C\|g\|_{L_T^2 H^\gamma}.
\]

It also holds that
\[
\|e^{t\theta_0} - \alpha\|_{L_T^\infty H^{\gamma + 3/2} \cap L_T^2 H^{\gamma + 1}} \leq (1 + 2e^{3/2})\|\theta_0 - \alpha\|_{L_T^2 H^{\gamma + 3/2}},
\]
\[
\|\partial_t \theta\|_{L_T^2 H^\gamma} \leq (1 + 2e^{3/2})\|\theta_0 - \alpha\|_{L_T^2 H^{\gamma + 3/2}} + C\|g\|_{L_T^2 H^\gamma}.
\]

Lemma A.3. Assume $T > 0$, let $g \in L_T^2 H^\gamma(\mathbb{T})$. The model equation
\[
y_{st} = \partial_t(y_s(s, t) + g(s, t)), \quad y_s(s, 0) = y_{0s}(s), \quad s \in \mathbb{T}, \quad t \geq 0,
\]

admits a unique solution $y_s \in L_T^\infty H^{\gamma + \frac{1}{2}}(\mathbb{T}) \cap L_T^2 H^{\gamma + 1}(\mathbb{T})$ with $y_{st} \in L_T^2 H^\gamma(\mathbb{T})$. Furthermore, this solution satisfies
\[
\|y_s\|_{L_T^\infty H^{\gamma + 1}}^2(t) \leq e^{-t/4}\|y_{0s}\|_{L_T^\infty H^{\gamma + 1}}^2 + 4\|g\|_{L_T^2 H^\gamma}^2, \forall t \in [0, T],
\]
\[
\frac{d}{dt}\|y_s\|_{L_T^\infty H^{\gamma + 1}}^2(t) \leq -\frac{1}{4}\|y_s\|_{L_T^\infty H^{\gamma + 1}}^2(t) + 4\|g\|_{L_T^2 H^\gamma}^2(t), \forall t \in [0, T].
\]

It also holds that,
\[
\|y_s\|_{L_T^\infty H^{\gamma + 1/2} \cap L_T^2 H^{\gamma + 1}} \leq \frac{3}{3}\|y_{0s}\|_{H^{\gamma + 3/2}} + 6\|g\|_{L_T^2 H^\gamma},
\]
\[
\|e^{t\theta_0}\|_{L_T^\infty H^{\gamma + 1/2} \cap L_T^2 H^{\gamma + 1}} \leq \frac{3}{3}\|y_{0s}\|_{H^{\gamma + 3/2}} + 6\|g\|_{L_T^2 H^\gamma}.
\]

Appendix B. Quantitative Isoperimetric Inequalities

Lemma B.1. (*Gage’s isoperimetric inequality [13]*) If $z$ is a closed, convex, $C^2$ string which satisfies (5.2), it holds that
\[
\frac{1}{2} \int_\mathbb{T} \kappa d\alpha \geq \pi,
\]
where $\kappa$ is the curvature of the string.

Lemma B.2. Given two closed string
\[
z_1(\alpha) = s_1 \int_{-\pi}^\alpha (\cos(\theta_1(\alpha')), \sin(\theta_1(\alpha'))) d\alpha', \quad z_2(\alpha) = s_2 \int_{-\pi}^\alpha (\cos(\theta_2(\alpha')), \sin(\theta_2(\alpha'))) d\alpha',
\]
if the areas enclosed by $z_1$ and $z_2$ are both $a$, it holds that
\[
\frac{s_1^2 - s_2^2}{s_1^2 s_2^2} \leq C\|\theta_1 - \theta_2\|_{L^2},
\]
where $C$ is a constant only depends on $a$. 
Proof. From the assumption we have
\[
a = -\frac{1}{2} \int_T z_1 \cdot z_{1\alpha} d\alpha = -\frac{1}{2} \int_T z_2 \cdot z_{2\alpha} d\alpha
\]
\[
= -\frac{s_1^2}{2} \int_{-\pi}^\pi (- \int_{-\pi}^\alpha \cos(\theta_1(\alpha')) d\alpha' \sin(\theta_1(\alpha)) + \int_{-\pi}^\alpha \sin(\theta_1(\alpha')) d\alpha' \cos(\theta_1(\alpha))) d\alpha
\]
\[
= -\frac{s_2^2}{2} \int_{-\pi}^\pi (- \int_{-\pi}^\alpha \cos(\theta_2(\alpha')) d\alpha' \sin(\theta_2(\alpha)) + \int_{-\pi}^\alpha \sin(\theta_2(\alpha')) d\alpha' \cos(\theta_2(\alpha))) d\alpha
\]
\[
= \frac{s_1^2 - s_2^2}{2} a = \int_{-\pi}^\pi \int_{-\pi}^\alpha (\cos(\theta_2(\alpha')) - \cos(\theta_1(\alpha'))) d\alpha' \sin(\theta_2(\alpha)) d\alpha
\]
\[
\quad \quad \quad \quad + \int_{-\pi}^\pi \int_{-\pi}^\alpha \cos(\theta_1(\alpha')) d\alpha' (\sin(\theta_2(\alpha)) - \sin(\theta_1(\alpha))) d\alpha.
\]
It follows that
\[
\frac{s_1^2 - s_2^2}{s_1^2} = \int_{-\pi}^\pi \int_{-\pi}^\alpha (\cos(\theta_2(\alpha')) - \cos(\theta_1(\alpha'))) d\alpha' \sin(\theta_2(\alpha)) d\alpha
\]
\[
\quad \quad \quad \quad + \int_{-\pi}^\pi \int_{-\pi}^\alpha \cos(\theta_1(\alpha')) d\alpha' (\sin(\theta_2(\alpha)) - \sin(\theta_1(\alpha))) d\alpha.
\]
Then, it is easy to show the result of this lemma. \( \square \)

Lemma B.3. (Fuglede isoperimetric inequality) Let \((\theta, s)\) be the tangent angle function and perimeter of a closed string which satisfies (5.2), then there exists \(\varepsilon > 0\) such that if in addition \(||\theta - \alpha||_{\bar{L}^1} \leq \varepsilon\), it holds that
\[
\frac{1}{C} (s - 1) \leq ||\theta - \bar{\theta} - \alpha||_{\bar{L}^2} \leq C (s - 1),
\]
where \(C\) is a constant.

Proof. Without loss of generality, we only consider the case \(\bar{\theta} = 0\). From Lemma [B.2] we know that
\[
\frac{s_1^2 - 1}{s_1^2} = \int_{-\pi}^\pi \int_{-\pi}^\alpha (\cos(\alpha') - \cos(\theta(\alpha'))) d\alpha' \sin(\alpha) d\alpha
\]
\[
\quad \quad \quad \quad + \int_{-\pi}^\pi \int_{-\pi}^\alpha \cos(\theta(\alpha')) d\alpha' (\sin(\alpha) - \sin(\theta(\alpha))) d\alpha.
\]
Using Taylor expansion, we have
\[
\sin(\theta) = \sin(\alpha) + D \cos(\alpha) - \frac{D^2}{2} \sin(\alpha) + O(D^3),
\]
\[
\cos(\theta) = \cos(\alpha) - D \sin(\alpha) - \frac{D^2}{2} \cos(\alpha) + O(D^3),
\]
where \(D\) stands for \((\theta(\alpha') - \alpha)\). Therefore, one can deduce that
\[
\frac{s_1^2 - 1}{s_1^2 = \frac{1}{2} \int_{-\pi}^\pi D^2(\alpha) d\alpha + \int_{-\pi}^\pi \int_{-\pi}^\pi D(\alpha') \sin(\alpha') d\alpha' D(\alpha) \cos(\alpha) d\alpha + R.
\]
Here \(R\) is the higher order error term and satisfies \(R \leq C\varepsilon ||D||_{\bar{L}^2}^2\).
Using Fourier expansion, we have
\[
\int_{-\pi}^{\pi} D(\alpha') \sin(\alpha')d\alpha' - \frac{\alpha}{2\pi} \int_{-\pi}^{\pi} D(\alpha') \sin(\alpha')d\alpha' = \sum_{k\in\mathbb{Z}/0} \mathbb{E}^{ik\alpha} \left( a_{k+1}(D) - a_{k-1}(D) - ib_{k+1}(D) + ib_{k-1}(D) \right) + \frac{1}{2\pi} \int_{-\pi}^{\pi} D \sin(\alpha) - \alpha D \sin(\alpha)d\alpha,
\]
\[
D(\alpha) \cos(\alpha) = \sum_{k\in\mathbb{Z}/0} \mathbb{E}^{ik\alpha} \left( a_{k+1}(D) + a_{k-1}(D) - ib_{k+1}(D) - ib_{k-1}(D) \right) + \frac{1}{2\pi} \int_{-\pi}^{\pi} D \cos(\alpha)d\alpha.
\]
Therefore, with the help of Lemma 6.3, it holds that
\[
\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} D(\alpha') \sin(\alpha')d\alpha' D(\alpha) \cos(\alpha)d\alpha
\]
\[
= \int_{-\pi}^{\pi} \left( \int_{-\pi}^{\pi} D(\alpha') \sin(\alpha')d\alpha' - \frac{\alpha}{2\pi} \int_{-\pi}^{\pi} D(\alpha') \sin(\alpha')d\alpha' \right) D(\alpha) \cos(\alpha)d\alpha
\]
\[
+ \frac{1}{2\pi} \int_{-\pi}^{\pi} D(\alpha') \sin(\alpha')d\alpha' \int_{-\pi}^{\pi} \alpha D(\alpha) \cos(\alpha)d\alpha
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} D \sin(\alpha) - \alpha D \sin(\alpha)d\alpha \int_{-\pi}^{\pi} D \cos(\alpha)d\alpha + \frac{1}{2\pi} \int_{-\pi}^{\pi} D \sin(\alpha)d\alpha \int_{-\pi}^{\pi} \alpha D \cos(\alpha)d\alpha
\]
\[
\leq C \|D\|_{L^2}(\|a_1(D)\| + \|b_1(D)\|) \leq C \varepsilon \|D\|_{L^2}^2.
\]
This actually finishes the proof. \(\square\)

There are similar results for the higher dimensional case \(11, 12\).

**Appendix C. Auxiliary Calculations**

**Lemma C.1.** Given \((\theta, \alpha), y, s, t, u, z, n\) and \(t\) be the functions defined in (2.16), (2.28) and (2.3), it holds that
\[
\frac{d}{d\theta} \left( (u \cdot n)\bar{\theta}(\alpha) \right) = \frac{d}{d\theta} \left( (u \cdot t)\bar{\theta}(\alpha) \right) = 0.
\]

**Proof.** By the definition, one has
\[
\cos(\theta) = \cos(\bar{\theta} + \bar{\theta}), \quad \sin(\theta) = \sin(\bar{\theta} + \bar{\theta}), \quad \frac{d}{d\theta} \cos(\theta) = -\sin(\theta), \quad \frac{d}{d\theta} \sin(\theta) = \cos(\theta).
\]

It follows that
\[
\frac{d}{d\theta} |z(\bar{\theta}, \alpha) - z(\bar{\theta}, \alpha')|^2
\]
\[
= \frac{2}{\pi^2} \int_{\alpha'}^{\alpha} \left( \cos(\theta(\alpha'')) \right) d\alpha'' \cdot \int_{\alpha'}^{\alpha} \left( -\sin(\theta(\alpha'')) \right) \cos(\theta(\alpha'')) d\alpha''
\]
\[
= 0.
\]
Therefore, we deduce that
\[
4\pi \frac{d}{d\theta} G(\mathbf{z}(\tilde{\theta}, \alpha) - \mathbf{z}(\tilde{\theta}, \alpha'))
+ \int_{\alpha'}^{\alpha} (- \sin(\theta(\alpha'')), \cos(\theta(\alpha'')))d\alpha'' \otimes \int_{\alpha'}^{\alpha} (\cos(\theta(\alpha'')), \sin(\theta(\alpha'')))d\alpha''
= \left| \int_{\alpha'}^{\alpha} (\cos(\theta), \sin(\theta))d\alpha'' \right|^2
\]
\[
+ \int_{\alpha'}^{\alpha} (\cos(\theta(\alpha'')), \sin(\theta(\alpha'')))d\alpha'' \otimes \int_{\alpha'}^{\alpha} (- \sin(\theta(\alpha''))), \cos(\theta(\alpha'')))d\alpha''
= \left| \int_{\alpha'}^{\alpha} (\cos(\theta), \sin(\theta))d\alpha'' \right|^2.
\]
For any scalar function \(f(\alpha) \in C(\mathbb{T})\) which is independent of \(\tilde{\theta}\), one can see
\[
4\pi \frac{d}{d\theta} \left( \mathbf{n}(\alpha) \cdot \int_{\mathbb{T}} G(\mathbf{z}(\tilde{\theta}, \alpha) - \mathbf{z}(\tilde{\theta}, \alpha')) f \cdot \mathbf{n}(\tilde{\theta}, \alpha')d\alpha' \right)
= -4\pi \int_{\mathbb{T}} f(\alpha') \left( \mathbf{t}(\tilde{\theta}, \alpha) \cdot G(\mathbf{z}(\tilde{\theta}, \alpha) - \mathbf{z}(\tilde{\theta}, \alpha')) \cdot \mathbf{n}(\tilde{\theta}, \alpha') \right)
\]
\[
+ \mathbf{n}(\tilde{\theta}, \alpha) \cdot G(\mathbf{z}(\tilde{\theta}, \alpha) - \mathbf{z}(\tilde{\theta}, \alpha')) \cdot t(\tilde{\theta}, \alpha') \right) d\alpha'
\]
\[
+ \mathbf{n}(\tilde{\theta}, \alpha) \cdot \int_{\mathbb{T}} \left( (- \sin(\theta), \cos(\theta))d\alpha'' \otimes \int_{\alpha'}^{\alpha} (\cos(\theta), \sin(\theta))d\alpha'' \right) f(\alpha') \cdot \mathbf{n}(\tilde{\theta}, \alpha')d\alpha'
\]
\[
+ \mathbf{n}(\tilde{\theta}, \alpha) \cdot \int_{\mathbb{T}} \left( (\cos(\theta), \sin(\theta))d\alpha'' \otimes \int_{\alpha'}^{\alpha} (- \sin(\theta), \cos(\theta))d\alpha'' \right) f(\alpha') \cdot \mathbf{n}(\tilde{\theta}, \alpha')d\alpha'
\]
\[
= \int_{\mathbb{T}} f(\alpha') \left| \mathbf{z}(\tilde{\theta}, \alpha) - \mathbf{z}(\tilde{\theta}, \alpha') \right| (- \cos(\theta(\alpha)) \sin(\theta(\alpha')) + \sin(\theta(\alpha)) \cos(\theta(\alpha'))
\]
\[
- \sin(\theta(\alpha)) \cos(\theta(\alpha')) + \cos(\theta(\alpha)) \sin(\theta(\alpha'))))d\alpha'
\]
\[
+ \int_{\mathbb{T}} f(\alpha') \left( (\int_{\alpha'}^{\alpha} \cos(\theta)d\alpha'' \right)^2 - (\int_{\alpha'}^{\alpha} \sin(\theta)d\alpha'' \right)^2 \sin(\theta(\alpha) + \theta(\alpha'))
\]
\[
+ \int_{\mathbb{T}} f(\alpha') \left( (\int_{\alpha'}^{\alpha} \cos(\theta)d\alpha'' \right)^2 - (\int_{\alpha'}^{\alpha} \sin(\theta)d\alpha'' \right)^2 \sin(\theta(\alpha) + \theta(\alpha'))
\]
\[
+ \int_{\mathbb{T}} f(\alpha') \left( (\int_{\alpha'}^{\alpha} \cos(\theta)d\alpha'' \right)^2 - (\int_{\alpha'}^{\alpha} \sin(\theta)d\alpha'' \right)^2 \sin(\theta(\alpha) + \theta(\alpha'))
\]
\[
= 0.
\]
For the same reason, it holds that
\[
\frac{d}{d\theta} (\mathbf{t}(\tilde{\theta}, \alpha) \cdot \int_{\mathbb{T}} G(\mathbf{z}(\tilde{\theta}, \alpha) - \mathbf{z}(\tilde{\theta}, \alpha')) f \cdot \mathbf{n}(\tilde{\theta}, \alpha')d\alpha') = 0,
\]
\[
\frac{d}{d\theta} (\mathbf{n}(\tilde{\theta}, \alpha) \cdot \int_{\mathbb{T}} G(\mathbf{z}(\tilde{\theta}, \alpha) - \mathbf{z}(\tilde{\theta}, \alpha')) f \cdot \mathbf{t}(\tilde{\theta}, \alpha')d\alpha') = 0.
\]
\[ \frac{d}{d\theta}(\mathbf{t}(\bar{\theta}, \alpha) \cdot \int_T G(z(\bar{\theta}, \alpha) - z(\bar{\theta}, \alpha')) f \cdot \mathbf{t}(\bar{\theta}, \alpha') d\alpha') = 0. \]

Thus, by (2.16) we arrive at
\[ \frac{d}{d\theta}(\mathbf{u}(z(\bar{\theta}, \alpha)) \cdot \mathbf{n}(\bar{\theta}, \alpha)) = 0, \quad \frac{d}{d\theta}(\mathbf{u}(z(\bar{\theta}, \alpha)) \cdot \mathbf{t}(\bar{\theta}, \alpha)) = 0. \]

\[\Box\]

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