Huygens’ principle for the Dirac equation in spacetime of non-constant curvature

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Abstract

In this article we give sufficient and necessary conditions for the validity of the Huygens’ principle for the Dirac equation in the non-constant curvature spacetime of the Friedmann-Lemaître-Robertson-Walker models of cosmology. The Huygens’ principle discussed for the equation of a field with mass $m = 0$ as well as a massive spin-$\frac{1}{2}$ field undergoing a red shift of its wavelength as the universe expands.

Keywords  Dirac equation · Einstein & de Sitter model · FLRW models · Huygens’ principle ·

1 Introduction

In this article we give sufficient and necessary conditions for the validity of the Huygens’ principle for the Dirac equation in the non-constant curvature spacetime of the Friedmann-Lemaître-Robertson-Walker models of cosmology. We use the definition of the Huygens’ principle due to Hadamard [23] as the absence of tails. Thus, the field equations satisfy the Huygens’ principle if and only if the solution has no tail, that is, the solution depends on the source distributions on the past null cone of the field only and not on the sources inside the cone.

The Dirac equation and its quantization in curved spacetime are of great interest due to the role of spin-$\frac{1}{2}$ particles in astrophysics and cosmology. Recent observational confirmation of the expansion of the universe and the quantum field theory demand a detailed investigation of the solutions of the Dirac equation in curved spacetime (see, e.g., [4, 19] and bibliography therein). The standard models of Cosmology provide such backgrounds, which form a family of curved backgrounds of FLRW models. For the de Sitter spacetime in [25] a fundamental solution of the Dirac operator and an explicit formula for the solution of the Cauchy problem are obtained. In [20] an examination of these explicit formulas gave an answer to such an interesting question in the physics of fundamental particles as a validity of the Huygens’ principle in the de Sitter spacetime. It was proved in [24] that the Klein-Gordon equation in the de Sitter universe obeys the Huygens’ principle only for the particle with the mass $\sqrt{2}$ (in the system of units with $c = \hbar = H = 1$, where $c$ is the speed of light, $\hbar$ is the Planck’s constant, and $H$ is the Hubble constant), while for the zero mass fields the incomplete Huygens’ principle defined in [24] holds. In the case of a spatially curved spacetime the last was revealed in [13]. In fact, the fundamental solutions to the operators can provide information such as the Huygens’ principle, that is impossible to obtain by numerical solutions of differential equations.

For the Dirac equation in a curved four-dimensional spacetime, the Huygens’ principle is generally violated by its solutions, due to the mass term in the equation and the curvature of spacetime [9, 23]. The presence or absence of tails for waves has been established for some spacetime metrics, including constant curvature metrics [23]. In fact, the study of the Huygens’ principle has important applications to quantum field theory and cosmology, especially in the inflationary theories of the early universe. The fact that the support of the commutator or the anticommutator-distribution, respectively, lies on the null-cone if and only if the Huygens’ principle holds for the corresponding equation [15, 18] shows significance of the Huygens’ principle for quantum field theory. The Huygens’ principle has been studied in the context of cosmology for classical fields [7] and for gravitational waves in a curved background (see, e.g., [14]).
For the Dirac operator in the cosmological context, the Huygens’ principle is discussed in [2, 5, 12]. The violations of the Huygens’ principle [11] may have a consequence for relativistic quantum communication. For the fields obeying the Huygens’ principle communication through massless fields is confined to the light cone. The violation of the Huygens’ principle makes possible a leakage of information towards the inside of the light cone [11]. According to [5] the violation of the Huygens’ principle has unexpected consequences in the propagation of information from the early Universe to the current era [8]. In [5] are studied conformal and minimal couplings of a test massless scalar field in a cosmological background. The conclusion made in [5] is that the signals received today generically contains overlapped information about the past from both timelike and light connected events.

In the theory of partial differential equations the Huygens’ principle has been used in the estimates of the Bahouri-Gérard concentration compactness method and Strichartz estimates (see, e.g., [13, 20]) as well as in the study of global solvability of nonlinear hyperbolic equations (see, e.g., [1]).

Our consideration is based on the constructed in [27] the fundamental solution of the Dirac operator and the explicit formulas for the solution of the Cauchy problem for the Dirac equation in the FLRW spacetime with both accelerating and decelerating expansion or contraction. The spatially flat FLRW models considered in [27] had the metric tensor that, in Cartesian coordinates, is written as follows

\[
g_{\mu \nu} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -a^2(t) & 0 & 0 \\
0 & 0 & -a^2(t) & 0 \\
0 & 0 & 0 & -a^2(t)
\end{pmatrix}, \quad \mu, \nu = 0, 1, 2, 3,
\]

where the scale factor \(a(t) = a_0 t^\ell, \ell \in \mathbb{R}, t > 0, \) and \(x \in \mathbb{R}^3, x_0 = t.\) If \(\ell < 0\) the spacetime is contracting. In the case of \(\ell > 1\) the expansion is accelerating (with horizon), while for \(0 < \ell < 1\) the expansion is decelerating. In the case of the Milne spacetime [6, 10, 21] \(\ell = 1.\) The FLRW spacetimes with the scale factors \(a(t) = a_0 t^{2/3}\) and \(a(t) = a_0 t^{1/2}\) are modeling the matter dominated universe and the radiation dominated universe, respectively (see, e.g., [17]).

It was proved by Wünsch [23] that if the massive \((m \neq 0)\) Dirac equation obeys the Huygens’ principle, then spacetime has a constant curvature. Accordingly, it was admitted in [27] that the mass term of the field can be changing in time and vanishing at future infinity. The problem of the time variation of the spin-\(\frac{1}{2}\) massive particle in the early universe has been studied in physical literature (see [22] and the references therein). In the present paper, the decay assumption on the mass term has been made in the context of the expanding universe. More exactly, the model is determined by the Dirac operator

\[
\mathcal{D}(t, \partial_t, \partial_x) := i \gamma^0 \partial_t + i \frac{1}{a(t)} \gamma^1 \partial_{x_1} + i \frac{1}{a(t)} \gamma^2 \partial_{x_2} + i \frac{1}{a(t)} \gamma^3 \partial_{x_3} + \frac{3 \dot{a}(t)}{2a(t)} \gamma^0 - mt^{-1} \mathbb{I}_4,
\]

where \(m \in \mathbb{C}\) and where the contravariant gamma matrices are

\[
\gamma^0 = \begin{pmatrix}
\mathbb{I}_2 & \mathbb{O}_2 \\
\mathbb{O}_2 & -\mathbb{I}_2
\end{pmatrix}, \quad \gamma^k = \begin{pmatrix}
\mathbb{O}_2 & \sigma^k \\
-\sigma^k & \mathbb{O}_2
\end{pmatrix}, \quad k = 1, 2, 3.
\]

Here \(\sigma^k\) are the Pauli matrices

\[
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

and \(\mathbb{I}_n, \mathbb{O}_n\) denote the \(n \times n\) identity and zero matrices, respectively.

This model includes the equation of a neutrino with \(m = 0\) as well as a massive spin-\(\frac{1}{2}\) field undergoing a red shifting of its wavelength as the universe expands. Thus, in the present paper we consider the Dirac equation in the spacetime with the metric tensor \([1]\), that is,

\[
\mathcal{D}(t, \partial_t, \partial_x) \Psi = F.
\]

Recall that a retarded fundamental solution for the Dirac operator [2] is a four-dimensional matrix

\[
\mathcal{E}^{ret}(x, t; x_0, t_0; m) = \delta(x - x_0)\delta(t - t_0) \mathbb{I}_4, \quad (x, t), (x_0, t_0) \in \mathbb{R}^3 \times \mathbb{R}_+,
\]

in the study of global solvability of nonlinear hyperbolic equations (see, e.g., [1]).
and with the support in the \textit{chronological future} (causal future) \(D_+(x_0, t_0)\) of the point \((x_0, t_0) \in \mathbb{R}^3 \times \mathbb{R}_+\). The advanced fundamental solution (propagator) \(E^{adv} = E^{adv}(x, t; x_0, t_0; m)\) solves the equation (3) and has the support in the \textit{chronological past} (causal past) \(D_-(x_0, t_0)\). The forward and backward light cones are defined as the boundaries of

\[
D_\pm (x_0, t_0) := \{(x, t) \in \mathbb{R}^3 \times \mathbb{R}_+ ; \ |x - x_0| \leq \pm (\phi(t) - \phi(t_0)) \},
\]

where \(\phi(t) := \frac{1}{2} t^{1-\ell} \) if \(\ell \neq 1\). In fact, any intersection of \(D_-(x_0, t_0)\) with the hyperplane \(0 < t = const < t_0\) determines the so-called \textit{dependence domain} for the point \((x_0, t_0)\), while the intersection of \(D_+(x_0, t_0)\) with the hyperplane \(t = const > t_0 > 0\) is the so-called \textit{domain of influence} of the point \((x_0, t_0)\).

In [27] was defined the right co-factor

\[
\mathcal{D}^{co}(t, \partial_t, \partial_x) := it^{-\frac{3}{2}} \gamma^0 (t^{im} U - t^{im} L) \frac{\partial}{\partial \ell} + it^{-\frac{3}{2}} \sum_{k=1}^{3} \gamma^k (t^{im} U + t^{im} L) \frac{\partial}{\partial x_k}
\]

of the Dirac operator \(\mathcal{D}(t, \partial_t, \partial_x)\) of [2] such that the composition \(\mathcal{D}(t, \partial_t, \partial_x) \mathcal{D}^{co}(t, \partial_t, \partial_x)\) was a diagonal matrix of operators. Here, in order to distinguish upper and lower 2-spinors, the upper-left corner and lower-right corner matrices,

\[
\gamma^U = \left( \begin{array}{cc} \mathbb{I}_2 & 0_2 \\ 0_2 & 0_2 \end{array} \right) = \frac{1}{2} (I_4 + \gamma^0), \quad \gamma^L = \left( \begin{array}{cc} 0_2 & 0_2 \\ \mathbb{I}_2 & 0_2 \end{array} \right) = \frac{1}{2} (I_4 - \gamma^0),
\]

respectively, will be used.

Let \(\Delta\) be the Laplace operator in \(\mathbb{R}^3\). Denote by \(E^w(x, t)\) the distribution that is the fundamental solution to the Cauchy problem for the wave equation in the Minkowski spacetime

\[
E^w_t - \Delta E^w = 0, \quad E^w(x, 0) = \delta(x), \quad E^w_t(x, 0) = 0.
\]

According to Theorem 1.2 [27], for every positive \(\varepsilon > 0\) and \(t > \varepsilon\) the fundamental solution \(E_+(x, t; x_0; m; \varepsilon)\) to the Cauchy problem, that is, a distribution satisfying

\[
\left\{ \begin{array}{l}
\mathcal{D}(t, \partial_t, \partial_x) E_+(x, t; x_0; m; \varepsilon) = 0_4, \\
E_+(x, t; x_0; m; \varepsilon) = \delta(x - x_0) I_4,
\end{array} \right.
\]

is given as follows

\[
E_+(x, t; x_0; m; \varepsilon) = -i \varepsilon^1 t^{-\frac{3}{2}} \sum_{\ell=0}^{m} (1 - \ell)^{-1} \mathcal{D}^{co}(x, t, \partial_t, \partial_x)^{-1} \mathcal{D}(t, \partial_t, \partial_x) \gamma^0
\]

\[
\times \int_{0}^{\phi(t) - \phi(\varepsilon)} \left( K_1 (r, t; m; \varepsilon) \mathbb{I}_2 \quad K_1 (r, t; -m; \varepsilon) \mathbb{I}_2 \right) E^w(x - x_0, r) dr,
\]

where

\[
K_1 (r, t; m; \varepsilon) := 2^{2i \frac{t}{1-\ell}} \varepsilon^{2i \frac{m}{1-\ell} - 1} \left( (\phi(t) + \phi(\varepsilon))^2 - r^2 \right)^{\frac{m}{1-\ell}} F \left( i \frac{m}{1-\ell}, i \frac{m}{1-\ell}; 1 - \frac{m}{1-\ell}; 
\frac{(\phi(t) - \phi(\varepsilon))^2 - r^2}{(\phi(t) + \phi(\varepsilon))^2 - r^2} \right).
\]

Henceforth, \(F(\alpha, \beta; \gamma; z)\) is the hypergeometric function (see, e.g., [3]). In order to write the solution to the Cauchy problem in an explicit form, we use the integral operator

\[
K_1(x, t; D_x; m; \varepsilon) \varphi(x, t) := -i \varepsilon^1 t^{-\frac{3}{2}} \sum_{\ell=0}^{m} (1 - \ell)^{-1} \int_{0}^{\phi(t) - \phi(\varepsilon)} K_1 (r, t; m; \varepsilon) \int_{\mathbb{R}^3} E^w(x - y, r) \varphi(y) dy dr, \quad \varphi \in C^\infty_0 (\mathbb{R}^3).
\]

Theorem 1.3 [27] in the case of source free equation provides the representation of the solution to the Cauchy problem

\[
\left\{ \begin{array}{l}
\mathcal{D}(t, \partial_t, \partial_x) \Psi(x, t) = 0, \quad t > \varepsilon > 0, \\
\Psi(x, \varepsilon) = \Phi(x),
\end{array} \right.
\]

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with $m \in \mathbb{C}$, as follows
\[
\Psi(x,t) = \mathcal{O}(t, \partial_t, \partial_x) \gamma^0 \left( \frac{K_1(x,t,D_x;m;\varepsilon)}{\mathcal{O}_2} \right) \mathcal{O}_2 \mathcal{O}_2 \left[ \Phi(x,t), \quad t > \varepsilon > 0. \quad (9) \right]
\]

We say that the equation (8) obeys the Huygens’ principle if the solution $\Psi$ vanishes at all points which cannot be reached from the support of initial data $\Phi$ by a null geodesic. The main result of this paper is the following theorem.

**Theorem 1.1** The solution of the Dirac equation (8), with $\ell \in \mathbb{R}$, $\ell \neq 1$, and the mass $m \in \mathbb{C}$ obeys the Huygens’ principle if and only if the mass term takes the values $m = 0, \pm i(\ell - 1)$.

According to Theorem 1.1 for the spin-$\frac{1}{2}$ particle like neutrino, the radiation and matter dominated universes are to some extent opaque unless we accept that the mass of particle is either zero or decaying in time. Theorem 1.1 guarantees that the Huygens’ principle is fulfilled and, consequently, the tail does not contribute to the dissipation of the the propagating field. In that sense for the spin-$\frac{1}{2}$ massless or decaying in time imaginary massive fields with $m = \pm i(\ell - 1)$, the matter dominated universe is transparent.

A discussion of an imaginary mass parameter in the Dirac equation from the physical point of view is given in [26].

The rest of this paper is organized as follows. In Section 2, we prove the sufficiency part of Theorem 1.1 by the representation formulas obtained in [27] for the solution of the Dirac equation in the FLRW spacetime. In Section 3, the proof of the necessity part of Theorem 1.1 is reduced to the verification of the asymptotic (for large time) behavior of some integral. The asymptotic analysis is carried out separately for the cases $\ell > 1$ and $\ell < 1$ in Section 4 and Section 5, respectively.

## 2 Huygens’ principle. Sufficient Conditions.

The fundamental solution $E^w$ to the Cauchy problem for the wave equation in the Minkowski spacetime can be written as $E^w = \partial_t Y^w$, where
\[
Y^w_{tt} - \Delta Y^w = 0, \quad Y^w(x,0) = 0, \quad Y^w_t(x,0) = \delta(x).
\]

In order to write the solution to the Cauchy problem for (8) we use the operator $K_1(x,t,D_x;m;\varepsilon)$ defined in (7). According to Theorem 1.3 [27], in the case of source-free equation, the solution to the Cauchy problem (8) with $m \in \mathbb{C}$ is given by (9). For $m = 0$ we obtain $K_1(r,t;0;\varepsilon) = \phi(\varepsilon)^{-1}$ and, consequently,
\[
K_1(x,t,D_x;0;\varepsilon)[\varphi](x,t) := -i\varepsilon^\frac{1}{2}(1 - \ell)^{-1} \int_0^{\phi(t) - \phi(\varepsilon)} \int_{\mathbb{R}^3} E^w(x-y,r)\varphi(y) dy dr, \quad \varphi \in C_0^\infty(\mathbb{R}^n).
\]

Hence the solution
\[
K_1(x,t,D_x;0;\varepsilon)[\varphi](x,t) = -i\varepsilon^\frac{1}{2}(1 - \ell)^{-1} \int_{\mathbb{R}^3} Y^w(x-y,\phi(t) - \phi(\varepsilon))\varphi(y) dy, \quad \varphi \in C_0^\infty(\mathbb{R}^n),
\]
obeys the Huygens’ principle.

For the case of the values of mass $m = \pm i(1 - \ell)$ we have
\[
K_1(r,t;0;\varepsilon) = \frac{r^2}{2} \left( \frac{1}{2} \varepsilon^3 \ell^3 - 3 \varepsilon^2 \ell^2 - 2 \varepsilon \ell^2 + \frac{3}{2} \varepsilon - 3 \right) + \left( \frac{1}{2} \varepsilon^2 - 3 \ell^2 + 2 \varepsilon - 1 \right) \ell^2 - \frac{3}{2} \varepsilon - 3 \beta^3 - 2 \beta - 1 \varepsilon + \frac{3}{2} \varepsilon - 1 + \frac{3}{2} \varepsilon^2 \ell^2 - 2 \varepsilon - 1 + \frac{3}{2} \varepsilon - 1 + \frac{3}{2} \varepsilon - 1,
\]
\[
K_1(r,t;0;\varepsilon) = - (\ell - 1) \varepsilon^\ell.
\]

In the next theorem we consider the family of more general kernels $K_1(r,t;m;\varepsilon)$, which leads to the Huygens’ principle.
Theorem 2.1 If the kernels $K_1 (r, t; \pm m; \varepsilon)$ can be represented in the form

$$K_1 (r, t; \pm m; \varepsilon) = r^2 a_\pm (\pm m; \varepsilon) + b_\pm (t; \pm m; \varepsilon),$$

then the Dirac equation (8) obeys the Huygens’ principle.

Proof. First, we consider the case of plus. According to (7) we have

$$K_1 (x, t, D_x; m; \varepsilon) = 1 + i\varepsilon \int_0 (1 - \varepsilon)^{-1} a_+(m; \varepsilon) \int_0^\phi (t - \phi) \int_\mathbb{R} E^w (x - y, r) \varphi (y) \, dy \, dr$$

$$- i\varepsilon \int_0^\phi (t - \phi) \int_\mathbb{R} E^w (x - y, r) \varphi (y) \, dy \, dr,$$

where $\varphi \in C_0^\infty (\mathbb{R}^n)$. For the last term of the previous relation we have

$$\int_0^\phi (t - \phi) \int_\mathbb{R} E^w (x - y, r) \varphi (y) \, dy \, dr = \int_\mathbb{R} E^w (x - y, \phi (t) - \phi (\varepsilon)) \varphi (y) \, dy, \quad \varphi \in C_0^\infty (\mathbb{R}^n).$$

This term obeys the Huygens’ principle since $E^w$ does it. Next we consider the first term of that relation (10):}

$$\int_0^\phi (t - \phi) \int_\mathbb{R} E^w (x - y, r) \varphi (y) \, dy \, dr = \left( \int_\mathbb{R} E^w (x - y, r) \varphi (y) \, dy \right) \left( \int_\mathbb{R} E^w (x - y, r) \varphi (y) \, dy \right) \, dr.$$

In view of (9) and the structure of $D^\alpha (t, \partial_t, \partial_x)$, the following lemma completes the proof of Theorem 2.1.

Lemma 2.2 The function

$$\frac{\partial}{\partial x_j} \int_0^\phi (t - \phi) \int_\mathbb{R} E^w (x - y, r) \varphi (y) \, dy \, dr, \quad \varphi \in C_0^\infty (\mathbb{R}^n), \quad j = 1, 2, 3,$$

satisfies the Huygens’ principle.

Proof. Indeed, we apply the Kirchhoff’s formula and consider, for instance, the case of $j = 3$. Then up to an unimportant factor, the possible tail is:

$$\frac{\partial}{\partial x_j} \int_0^\phi (t - \phi) \int_\mathbb{R} E^w (x - y, r) \varphi (y) \, dy \, dr = \int \int \int_{|y| \leq \phi (t) - \phi (\varepsilon)} \frac{\partial}{\partial y_3} \varphi (x + y) \, dy_1 \, dy_2 \, dy_3$$

$$= \int \int \int_{y_1^2 + y_2^2 \leq \phi (t) - \phi (\varepsilon)} \varphi \left( x_1 + y_1, x_2 + y_2, x_3 + \sqrt{(\phi (t) - \phi (\varepsilon))^2 - y_1^2 - y_2^2} \right)$$

$$- \varphi \left( x_1 + y_1, x_2 + y_2, x_3 - \sqrt{(\phi (t) - \phi (\varepsilon))^2 - y_1^2 - y_2^2} \right) \, dy_1 \, dy_2.$$

For every $t > \varepsilon$ the points

$$(x_1 + y_1, x_2 + y_2, x_3 + \sqrt{(\phi (t) - \phi (\varepsilon))^2 - y_1^2 - y_2^2}) \in \mathbb{R}^3,$$

belong to the sphere of the radius $(\phi (t) - \phi (\varepsilon))$ in $\mathbb{R}^3$, that is, the domain of integration does not intersect the interior of the domain of dependence. Thus, the tail is empty and the theorem is proved. \qed
3 Huygens’ principle. The necessary conditions

The proof of the necessity part will be carried out in two steps. The first step is the choice of the special initial spinor that is a radial spinor with a support in small neighborhood of the origin. The second step is in the establishing asymptotic behavior of the solution for large time at the spatial origin. If the value of the solution at the spatial origin differs from zero for the large time, then the Huygens’ principle is violated.

According to Theorem 1.3 [27], the solution to the Cauchy problem (3) with \( m \in \mathbb{C} \), is given by (4), where the right co-factor \( \mathcal{P}^{co}(t, \partial_t, \partial_x) \) is given by (4), while \( \gamma^U \) and \( \gamma^L \) are defined in (5). The proof of the necessity part of Theorem 1.3 is based on the large time asymptotics of the tail of solution. The initial data \( \Phi(x) = (\Phi_0(x), \Phi_1(x), \Phi_2(x), \Phi_3(x))^T \) will be chosen radial having small support. Consider the solution of the Cauchy problem with the radial function \( \Phi(x) = \Phi(r) \), supp \( \Phi \subset \{ x \in \mathbb{R}^n ; |x| \leq \min\{1/3, \varepsilon/|\ell - 1|\} \} \), \( \varepsilon \in (0, 1) \):

\[
\Psi(x, t) = \mathcal{P}^{co}(t, \partial_t, \partial_x) \gamma^0 \begin{pmatrix}
K_1(x, t, D_x; m; \varepsilon)[\Phi_0(x)] \\
K_1(x, t, D_x; m; \varepsilon)[\Phi_1(x)] \\
K_1(x, t, D_x; -m; \varepsilon)[\Phi_2(x)] \\
K_1(x, t, D_x; -m; \varepsilon)[\Phi_3(x)]
\end{pmatrix} (x, t), \quad t > \varepsilon > 0.
\]

If we choose the initial data

\[
\Phi(x) = (\Phi_0(x), 0, 0, 0)^T,
\]
then the solution \( \Psi(x, t) = (\Psi_0(x, t), \Psi_1(x, t), \Psi_2(x, t), \Psi_3(x, t))^T \) is given by

\[
\Psi(x, t) = \begin{pmatrix} K_1(x, t, D_x; m; \varepsilon)[\Phi_0(x)] \\
0 \\
0 \\
0 \end{pmatrix} \begin{pmatrix} 1 + \frac{\gamma^L t}{1 - \ell} \int_0^t \int_{\mathbb{R}^n} \gamma^U \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} 0 \\
0 \\
0 \\
0 \end{pmatrix} \begin{pmatrix} (x, t) \end{pmatrix}, \quad t > \varepsilon > 0.
\]

The first component of \( \Psi(x, t) \) is

\[
\Psi_0(x, t)
= \begin{pmatrix} K_1(x, t, D_x; m; \varepsilon)[\Phi_0(x)] \\
0 \\
0 \\
0 \end{pmatrix} (x, t)
= \begin{pmatrix} K_1(x, t, D_x; m; \varepsilon)[\Phi_0(x)] \\
0 \\
0 \\
0 \end{pmatrix} \begin{pmatrix} 1 + \frac{\gamma^L t}{1 - \ell} \int_0^t \int_{\mathbb{R}^n} \gamma^U \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} 0 \\
0 \\
0 \\
0 \end{pmatrix} (x, t), \quad t > \varepsilon > 0.
\]

If we choose the initial data

\[
\Phi(x) = (0, 0, \Phi_2(x), 0)^T,
\]
then the solution \( \Psi(x, t) = (\Psi_0(x, t), \Psi_1(x, t), \Psi_2(x, t), \Psi_3(x, t))^T \) is given by

\[
\Psi(x, t) = \begin{pmatrix} K_1(x, t, D_x; m; \varepsilon)[\Phi_2(x)] \\
0 \\
0 \\
0 \end{pmatrix} \begin{pmatrix} 1 + \frac{\gamma^L t}{1 - \ell} \int_0^t \int_{\mathbb{R}^n} \gamma^U \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} 0 \\
0 \\
0 \\
0 \end{pmatrix} \begin{pmatrix} (x, t) \end{pmatrix}, \quad t > \varepsilon > 0.
\]

The third component of \( \Psi(x, t) \) is

\[
\Psi_3(x, t)
= \begin{pmatrix} K_1(x, t, D_x; m; \varepsilon)[\Phi_2(x)] \\
0 \\
0 \\
0 \end{pmatrix} (x, t)
= \begin{pmatrix} K_1(x, t, D_x; m; \varepsilon)[\Phi_2(x)] \\
0 \\
0 \\
0 \end{pmatrix} \begin{pmatrix} 1 + \frac{\gamma^L t}{1 - \ell} \int_0^t \int_{\mathbb{R}^n} \gamma^U \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} 0 \\
0 \\
0 \\
0 \end{pmatrix} \begin{pmatrix} (x, t) \end{pmatrix}, \quad t > \varepsilon > 0.
\]
Denote either \( \varphi(y) := \Phi_0(y) \) or \( \varphi(y) := \Phi_2(y) \). We need to find a large time asymptotics only for the integral

\[
\int_0^{\phi(t) - \phi(\varepsilon)} \left( \frac{\partial}{\partial t} K_1(r, t; \pm m; \varepsilon) \right) \int_{\mathbb{R}^3} \mathcal{E}^w(x - y, r) \varphi(y) \, dy \, dr,
\]

since the term obtained by differentiation of the upper limit,

\[
t^{-\ell} K_1(\phi(t) - \phi(\varepsilon), t; \pm m; \varepsilon) \int_{\mathbb{R}^3} \mathcal{E}^w(x - y, \phi(t) - \phi(\varepsilon)) \varphi(y) \, dy,
\]

obeys the Huygens’ principle. Hence, we consider

\[
\int_0^{\phi(t) - \phi(\varepsilon)} \left( \frac{\partial}{\partial t} K_1(r, t; \pm m; \varepsilon) \right) \partial_r \int_{\mathbb{R}^3} \mathcal{E}^w(x - y, \phi(t)) \varphi(y) \, dy \, dr.
\]

Denote

\[\Psi(x, t) := \int_0^{\phi(t) - \phi(\varepsilon)} \left( \frac{\partial}{\partial t} K_1(r, t; \pm m; \varepsilon) \right) \partial_r \mathcal{E}(x, \phi(t)) \, dr,\]

where the notation

\[\mathcal{E}(x, \phi(t)) := \int_{\mathbb{R}^3} \mathcal{E}^w(x - y, \phi(t)) \varphi(y) \, dy\]

has been used. Then

\[
\Psi(x, t) = \left( \frac{\partial}{\partial t} K_1(\phi(t) - \phi(\varepsilon), t; \pm m; \varepsilon) \right) \mathcal{E}(x, \phi(t) - \phi(\varepsilon)) \, dr
\]

\[\quad - \int_0^{\phi(t) - \phi(\varepsilon)} \left( \frac{\partial}{\partial r} \frac{\partial}{\partial t} K_1(r, t; \pm m; \varepsilon) \right) \mathcal{E}(x, r) \, dr.
\]

In particular, by the Kirchhoff’s formula we have \( \mathcal{E}(0, r) = r \varphi(r) \) and

\[\mathcal{E}(0, \phi(t) - \phi(\varepsilon)) = (\phi(t) - \phi(\varepsilon)) \varphi_0(\phi(t) - \phi(\varepsilon)) = 0\]

for sufficiently large \( t \), that is, if \( \phi(t) - \phi(\varepsilon) > \tilde{\varepsilon} \). Consequently, for large \( t \) we have

\[
\Psi(0, t) = -\int_0^{\phi(t) - \phi(\varepsilon)} \left( \frac{\partial}{\partial r} \frac{\partial}{\partial t} K_1(r, t; \pm m; \varepsilon) \right) r \varphi(r) \, dr
\]

\[= \int_0^{\tilde{\varepsilon}} \left( \frac{\partial}{\partial t} K_1(r, t; \pm m; \varepsilon) \right) \frac{\partial}{\partial r} (r \varphi(r)) \, dr.
\]

The outline of the remaining part of the proof is as follows. In the next sections we study the asymptotics of the function \( \frac{\partial}{\partial r} K_1(r, t; \pm m; \varepsilon) \) as \( t \to \infty \). The principal term of this asymptotics for the cases of \( m \neq 0, \pm i(\ell - 1) \) is the function of \( r \) that allows us to find the initial function \( \varphi(r) \) such that the scalar product \( \frac{\partial}{\partial r} (r \varphi(r)) \) with the principal term differs from zero. Thus, \( \Psi(0, t) \neq 0 \) for sufficiently large time, and the Huygens’ principle is violated for these values of \( m \).

It is easily seen that it suffices to study the asymptotic behavior of the function

\[
\frac{\partial}{\partial r} \left( \left( (\tau + 1)^2 - A^2 \right)^{-M} F \left( M, M; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right) \right)
\]

for \( M \in \mathbb{C}, M \neq 0, \pm 1, \) as \( \tau \to 0 \) if \( \ell > 1 \) or as \( \tau \to \infty \) if \( \ell < 1 \). For \( M = 0, \pm 1 \) the Huygens’ principle holds.

Here \( \tau = t^{1-\ell} \) while \( A^2 := (\ell - 1)^2 \tau^2 \) is sufficiently small, say \( A^2 \leq 1/9 \). It is important that

\[
0 < \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} < 1 \quad \text{when} \quad \tau \in (0, 1/2) \cup (2, \infty) \quad \text{and} \quad A^2 \in [0, 1/9],
\]
as well as
\[
\lim_{\tau \to 0} \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} = \lim_{\tau \to \infty} \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} = 1 \quad \text{uniformly on } A^2 \in [0, 1/9].
\]
Moreover, we will use the finite sum
\[
\sum_{k=N_1}^{N_2} \tau^{a_k} \left( h_k(A, M) + \tilde{h}_k(A, M)(\ln(\tau))^{b_k} \right)
\]
of the terms of the asymptotic (as \( t \to \infty \)) series
\[
\sum_{k=-\infty}^{\infty} \tau^{a_k} \left( h_k(A, M) + \tilde{h}_k(A, M)(\ln(\tau))^{b_k} \right)
\]
where \( a_k \in \mathbb{C}, k = 0, \pm 1, \pm 2, \ldots \), and the real parts \( \Re a_k, k = 0, \pm 1, \pm 2, \ldots \) are such that
\[
\ldots < \Re a_{k-1} < \Re a_k < \Re a_{k+1} < \ldots, \quad \lim_{k \to -\infty} \Re a_k = -\infty, \quad \lim_{k \to \infty} \Re a_k = \infty.
\]
It is crucial that if the coefficient \( h_k(A, M) \) or \( \tilde{h}_k(A, M) \) is independent of \( A \), then
\[
\int_{0}^{\frac{1}{\tau}} h_k(A, M) \tau^{a_k} \frac{\partial}{\partial r}(r\varphi(r)) \, dr = 0 \quad \text{or} \quad \int_{0}^{\frac{1}{\tau}} \tilde{h}_k(A, M) \tau^{a_k} \frac{\partial}{\partial r}(r\varphi(r)) \, dr = 0
\]
for every function \( \varphi \in C_0^\infty(0, \tau) \). On the other hand, it will be shown that all coefficients \( h_k(A, M), \tilde{h}_k(A, M) \) are polynomials in \( \sqrt{1-A^2} \) or \( \ln(1-A^2) \) (if \( \ell < 1 \)) or rational in \( 1-A^2 \) (if \( \ell > 1 \)) functions. Thus, it is enough to prove the existence of the depending on \( A \) coefficient \( h_k(A, M) \) or \( \tilde{h}_k(A, M) \). Then the existence of a function \( \varphi \in C_0^\infty(0, \tau) \) such that
\[
\Psi(0, t) = \int_{0}^{\frac{1}{\tau}} \left( \frac{\partial}{\partial t} K_1(r, t; \pm m; \tau) \right) \frac{\partial}{\partial r}(r\varphi(r)) \, dr \neq 0 \quad \text{for sufficiently large } t,
\]
is evident. This completes the proof of the necessity part of Theorem 1.1.

4 Asymptotics of \( \frac{\partial}{\partial t} K_1(r, t; \pm m; \tau) \) when \( \ell > 1 \)

In this section we show that if \( \ell > 1 \), then the principal term of the asymptotics of \( \frac{\partial}{\partial t} K_1(r, t; \pm m; \tau) \) for the case of \( M \neq 0, \pm 1 \) is a function of \( r \). Taking into account the scaling, henceforth we can suppose \( \tau = 1 \).

Denote
\[
M := \frac{im}{\ell - 1}, \quad A := (\ell - 1)r, \quad \tau := t^{1-\ell} \to 0 \quad \text{as} \quad t \to \infty,
\]
\[
z := \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} = 1 - \frac{4\tau}{1-A^2} + \frac{8\tau^2}{(1-A^2)^2} + \frac{4(A^2+3)\tau^3}{(1-A^2)^3} + O(\tau^4) \quad \text{as} \quad \tau \to 0.
\]
(12)

For the function defined by (6) we obtain
\[
\frac{\partial}{\partial t} K_1(r, t; m; 1) = C(m, \ell) t^{-\ell} \left( (1+\tau)^2 - A^2 \right)^{M-2} F(A, M; \tau), \quad C(m, \ell) \neq 0,
\]
(13)

where
\[
F(A, M; \tau) := 2M \left( 1 - A^2 - \tau^2 \right) F \left( 1 - M, 1 - M; 2; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right),
\]
(14)
For the case of minus, $M := -\frac{im}{\tau - 1}$, we have similar representation for the derivative $\frac{\partial}{\partial \tau} K_1 (r, t; \pm m; 1)$. We note, that for the values of $M = 0, \pm 1$, the function $\frac{\partial}{\partial \tau} K_1 (r, t; \pm m; 1)$ is independent of $r$:

$$\frac{\partial}{\partial \tau} \left( ((\tau + 1)^2 - A^2)^M F (-M, -M; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2}) \right) = 0 \text{ if } M = 0,$$

$$\frac{\partial}{\partial \tau} \left( ((\tau + 1)^2 - A^2)^M F (-M, -M; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2}) \right) = 4\tau \text{ if } M = 1,$$

$$\frac{\partial}{\partial \tau} \left( ((\tau + 1)^2 - A^2)^M F (-M, -M; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2}) \right) = -\frac{1}{4\tau^2} \text{ if } M = -1. \tag{17}$$

This indicates the fact that for these values of $M$ the Dirac equation obeys the Huygens’ principle.

### 4.1 The case of $\ell > 1$ and $M \in \mathbb{C}, \; M \neq \frac{1}{2} k, \; k = 0, \pm 1, \pm 2, \ldots$

We note here that

$$((\tau + 1)^2 - A^2)^M = (1 - A^2)^M + 2M\tau (1 - A^2)^{M-1} - M\tau^2 (1 - A^2)^{M-2} (A^2 - 2M + 1) + O (\tau^3) \tag{18}$$

as $\tau \searrow 0$, that together with (13) allows us to ignore all terms of the expansion of the function $F(A, M; \tau)$ except the first one.

**Proposition 4.1** Assume that $\ell > 1, \; M \in \mathbb{C}, \; M \neq \frac{1}{2} k, \; k = 0, \pm 1, \pm 2, \ldots$, and $A \in [0, 1/2]$. The function $F(A, M; \tau)$ (14) has the following expansion at $\tau = 0$:

$$F(A, M; \tau) = \frac{2M\Gamma(2M) \left((A^2(1 - 2M) + 1)\right)}{(1 - 2M)\Gamma(M + 1)^2} \tag{19}$$

$$+ \frac{\tau^{2M} 16^M \Gamma \left(\frac{1 - M}{2}\right) \Gamma \left(1 + M - 2M\right)}{I(1 - M)^2 (1 - A^2)^{-2M}} \left((A^2 - 1) + \tau 2(M - 2) + O (\tau^2) \right) \left(\tau^{2M} + 1\right).$$

Then the principal term in the series of the function $F(A, M; \tau)$ is

$$\frac{\tau^{2M} 16^M \Gamma \left(-\frac{2M}{2}\right) \Gamma \left(1 - A^2\right)^{1-2M}}{(1 - A^2)^{1-2M}} \text{ if } \Re(M) < \frac{1}{2},$$

$$\frac{2M\Gamma(2M) \left((A^2(1 - 2M) + 1)\right)}{(1 - 2M)\Gamma(M + 1)^2} \text{ if } \Re(M) > \frac{1}{2},$$

$$\frac{(1 + 2iB) \left((A^2 + i)\right)}{2\Gamma \left(\frac{1}{2} + iB\right)^2} \frac{\Gamma \left(1 + 2iB\right)}{\Gamma \left(\frac{1}{2} - iB\right)} (1 - A^2)^{-2iB} \text{ if } M = \frac{1}{2} + iB, \; B \in \mathbb{R} \setminus \{0\}. \tag{20}$$

**Proof.** In the function (14) we use the notation of $z$ from (12) and the expansion

$$1 - z = \frac{4\tau}{1 - A^2} - \frac{8\tau^2}{(1 - A^2)^2} + O(\tau^3).$$

The hypergeometric function is defined (see, e.g., [3] Sec.2.1)) by the series

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n \tag{20}$$

If $z$ is fixed and $|z| < 1$, then $F(a, b; c; z)$ is entire analytic function of the parameters $a$, $b$, and $c$ in the complex plane $\mathbb{C}$. (See, e.g., [3] Sec.2.1.6.) If $\Re(c - a - b) > 0$, then $F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$.

There is a formula (3 Sec.2.10) that ties together the values of $F(a, b; c; z)$ at the points $z = 0$ and $z = 1$:

$$F(a, b; c; z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a, b; a+b+c+1; 1-z) + (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F(c-a, c-b, c-a-b+1; 1-z),$$
where \(|\text{arg}(1 - z)| < \pi\), \(|1 - z| < 1\), and \(c - a - b \neq \pm 1, \pm 2, \ldots\). Hence,

\[
F(-M, -M; 1; z) = \frac{\Gamma(1 + 2M)}{\Gamma(1 + M)} F(-M, -M; -2M; 1 - z) + (1 - z)^{1 + 2M} \frac{\Gamma(-2M - 1)}{\Gamma(-M)^2} \times F(1 + M, 1 + M; 2 + 2M; 1 - z), \quad 1 + 2M \neq \pm 1, \pm 2, \ldots,
\]

(21)

\[
F(1 - M, 1 - M; 2; z) = \frac{\Gamma(2M)}{\Gamma(M)^2} F(1 - M, 1 - M; -2M + 1; 1 - z) + (1 - z)^{2M} \frac{\Gamma(-2M)}{\Gamma(1 - M)^2} F(1 + M, 1 + M; 2M + 1; 1 - z), \quad 2M \neq \pm 1, \pm 2, \ldots.
\]

(22)

It follows

\[
F\left(-M, -M; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2}\right) = \frac{\Gamma(2M + 1)}{\Gamma(M + 1)^2} \left(1 - \frac{2M}{1 - A^2}\right) - \left(1 - A^2\right)^{-2M - 1} \frac{\Gamma(-2M)}{2(M + 1)\Gamma(-M)^2} \left[\frac{1 - \frac{2M}{1 - A^2}}{1 - A^2}\right] + \tau^{2M} O\left(\tau^3\right) + O\left(\tau^2\right).
\]

Further,

\[
F\left(1 - M, 1 - M; 2; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2}\right) = -\tau \frac{4(M - 1)^2\Gamma(2M)}{(A^2 - 1)(1 - 2M)\Gamma(M + 1)^2} + \frac{\Gamma(2M)}{\Gamma(M + 1)^2} + O\left(\tau^2\right)
\]

\[
+ \left(1 - A^2\right)^{-2M} \left(\frac{4^{M + 1} \Gamma(1 - M)^2}{(2M + 1)2(A^2 - 1) M\Gamma(1 - M)^2}\right) - \frac{4^{M + 1} \Gamma(1 - M)^2}{2M\Gamma(1 - M)^2} + O\left(\tau^2\right).
\]

Consequently, \((19)\) holds. Proposition is proved. \(\square\)

Lemma 4.1 Assume that \(\ell > 1\), \(M \in \mathbb{C}, M \neq \frac{1}{2}, k = 0, \pm 1, \pm 2, \ldots\), and \(A \in [0, 1/2]\). Then the principal term in the asymptotic expansion of the function \(\frac{\partial}{\partial M} K_1(r; t; m; 1)\) is

\[
t^{2M(1 - \ell) - \ell} t^{16M^2M} \frac{\Gamma(-2M)}{\Gamma(1 - M)^2} \left(1 - A^2\right)^{-1 - M} \text{ if } \Re(M) < \frac{1}{2},
\]

\[
t^{1 - 2\ell} \frac{2M\Gamma(2M)}{(1 - 2M)\Gamma(M + 1)^2} \left(2A^2(1 - 2M) + 1\right) (1 - A^2)^M \text{ if } \Re(M) > \frac{1}{2},
\]

\[
t^{1 - 2\ell} \frac{2B(1 + 2iB)\Gamma(1 + 2iB)}{2B(2A^2B + i) (1 - A^2)^M} - t^{(1 + 2iB)(1 - \ell)} \frac{(2^2 + 4iB\Gamma(-2iB))}{\Gamma(\frac{1}{2} - iB)^2} (1 - A^2)^M \text{ if } M = \frac{1}{2} + iB, \quad B \in \mathbb{R} \setminus \{0\}.
\]

Proof. It follows from Proposition \([10]\) and \([13]\). \(\square\)

4.2 The case of \(\ell > 1\) and \(M = n, n = 1, 2, \ldots\)

In this case according to the definition \([6]\) we have

\[
\frac{\partial}{\partial t} K_1(r; t; m; \varepsilon) = C(n, \ell, \varepsilon)(1 - \ell) t^{1 - \ell} \frac{\partial}{\partial \tau} \left((\tau + 1)^2 - A^2\right)^n F\left(-n, -n; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2}\right), \quad C(n, \ell, \varepsilon) \neq 0.
\]
**Lemma 4.2** For \( n = 1, 2, \ldots \) the asymptotic expansion of the function \( \frac{\partial}{\partial t}K_1(r, t; m; \varepsilon) \) contains the term

\[
-C(n, \ell, \varepsilon)(1 - \ell)\tau^{-2\ell} \frac{4n\Gamma(2n)}{(2n - 1)\Gamma(n + 1)^2} (1 - A^2)^{n-2} (A^2(2n - 1) - 1).
\]

**Proof.** For \( n = 1, 2, \ldots \) the hypergeometric function \( F(-n, -n; 1; z) \) is the following polynomial

\[
F(-n, -n; 1; z) = 1 + \sum_{j=1}^{n} \left( \frac{\Gamma(n+1)}{\Gamma(j+1)\Gamma(n+1-j)} \right)^2 z^j.
\] (23)

We use (12) and obtain

\[
\frac{\partial}{\partial t}K_1(r, t; m; \varepsilon)
= C(n, \ell, \varepsilon)(1 - \ell)\tau^{-\ell} \left[ (\tau + 1)^2 - A^2 \right]
\times \left( 1 + \sum_{j=1}^{n} \left( 1 + \frac{4\tau}{A^2 - 1} + \frac{8\tau^2}{(A^2 - 1)^2} + \frac{4(A^2 + 3)\tau^3}{(A^2 - 1)^3} + O(\tau^4) \right) \frac{\Gamma(n+1)}{\Gamma(j+1)\Gamma(n+1-j)} \right)^2
\]

\[
= -C(n, \ell, \varepsilon)(1 - \ell)\tau^{-\ell} \left\{ \frac{4n^2(1 - A^2)^{n-2}(A^2(2n - 1) - 1)\Gamma(2n)}{(2n - 1)\Gamma(n+1)^2} + O(\tau^2) \right\}
\]

for \( n = 1, 2, \ldots \). The lemma is proved. \( \square \)

**4.3 The case of \( \ell > 1 \) and \( M = -n, n = 1, 2, \ldots \)**

If \( M = -n, n = 1, 2, \ldots \), then according to the definition (8) we have

\[
K_1(r, t; m; \varepsilon) := C(n, \ell, \varepsilon) \left( (\tau + 1)^2 - A^2 \right)^{-n} F \left( n, n; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right), \quad C(n, \ell, \varepsilon) \neq 0,
\]

and

\[
\frac{\partial}{\partial \tau} \left( (\tau + 1)^2 - A^2 \right)^{-n} F \left( n, n; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right)
= -2n(\tau + 1) \left( (\tau + 1)^2 - A^2 \right)^{-n-1} F \left( n, n; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right)
+ 4n^2(2A^2 + \tau^2 - 1) \left( (\tau + 1)^2 - A^2 \right)^{-n-2} F \left( n+1, n+1; 2; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right).
\] (24)

**Lemma 4.3** The principal term of the expansion of

\[
\frac{\partial}{\partial \tau} \left( (\tau + 1)^2 - A^2 \right)^{-n} F \left( n, n; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right)
\]

as \( \tau \to 0 \) is

\[
\begin{cases}
-\frac{1}{4} \tau^{-2}, & \text{if } n = 1, \\
-4^{1-2n}n^2(1 - A^2)^{n-1} \frac{\Gamma(2n)}{\Gamma(n + 1)^2} \tau^{-2n}, & \text{if } n = 2, 3, 4, \ldots
\end{cases}
\]

**Proof.** For \( n = 1 \) we use (20) and \( F(1, 1; 1; z) = \frac{1}{\tau z} \). Hence

\[
\frac{\partial}{\partial \tau} \left( (\tau + 1)^2 - A^2 \right)^{-1} F \left( 1, 1; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right) = -\frac{1}{4\tau^2}.
\]
For $n = 2$,
\[
F(2, 2; 1; z) = \sum_{n=0}^{\infty} \frac{\Gamma(2+n)\Gamma(2+n)}{\Gamma(1+n)} \frac{z^n}{n!} = \sum_{n=0}^{\infty} (1+n)^2 z^n = \frac{z + 1}{(1-z)^3}
\]
and, consequently,
\[
\frac{\partial}{\partial \tau} \left( (\tau + 1)^2 - A^2 \right)^{-2} F \left( 2, 2; 1; \frac{(\tau-1)^2 - A^2}{(\tau+1)^2 - A^2} \right) = -\frac{3}{32\tau^4} \left( 1 - A^2 \right) - \frac{1}{32\tau^2}.
\]  
(25)

Consider the case of $n = 3, 4, \ldots$. There is a typo in the first statement of Lemma 6.3 [20]. The correct version is
\[
\lim_{x \to 0} x^{2n-1} F(n, n; 1; 1-x) = F(1-n, 1-n; 1; 1) = \frac{\Gamma(2n-1)}{\Gamma(n)^2}, \quad n = 2, 3, \ldots,
\]
\[
\lim_{x \to 0} x^{2n-2} F(n, n; 2; 1-x) = F(2-n, 2-n; 2; 1) = \frac{\Gamma(2n-2)}{\Gamma(n)^2}, \quad n = 2, 3, \ldots.
\]

For the proof we appeal to [3] (14) Sec. 2.1] and to elementary relations, which can be verified, for instance, by [20] and the multiplication of the series in the left-hand sides of the relations
\[
(1-z)^{2n-1} F(n, n; 1; z) = F(1-n, 1-n; 1; z) \quad \text{for all} \quad 0 < z < 1, \quad n = 1, 2, 3, \ldots,
\]
\[
(1-z)^{2n-2} F(n, n; 2; z) = F(2-n, 2-n; 2; z) \quad \text{for all} \quad 0 < z < 1, \quad n = 1, 2, 3, \ldots
\]
or by [3] (14) Sec. 2.10]. Then with $a = b = n, c = 1$, it holds $a + b - c = 2n - 1 \geq 3$ and we have
\[
F(n, n, 1; z) = (1-z)^{-2n+1} \frac{\Gamma(2n-1)}{\Gamma(n)^2} \sum_{k=0}^{2n-2} \frac{((-n+1)k)^2}{(-2n+2)k!}(1-z)^k,
\]
while with $a = b = n, c = 2$, it holds $a + b - c = 2n - 2 \geq 3$ and we have
\[
F(n, n, 2; z) = (1-z)^{-2n+2} \frac{\Gamma(2n-2)}{\Gamma(n)^2} \sum_{k=0}^{2n-2} \frac{((-n+2)k)^2}{(-2n+3)k!}(1-z)^k,
\]
where $-\pi < \arg(1-z) < \pi$. According to [12] we can write
\[
1 - z = 1 - \frac{(\tau-1)^2 - A^2}{(\tau+1)^2 - A^2}; \quad \lim_{\tau \to 0} (1-z) = 0.
\]
Thus, from [24] we derive
\[
\frac{\partial}{\partial \tau} \left( (\tau + 1)^2 - A^2 \right)^{-n} F \left( n, n; 1; \frac{(\tau-1)^2 - A^2}{(\tau+1)^2 - A^2} \right)
= -2n(\tau+1) \left( (\tau+1)^2 - A^2 \right)^{-n-1} \left( \frac{\Gamma(2n-1)}{\Gamma(n)^2} + o(1) \right) \left( 1 - \frac{(\tau-1)^2 - A^2}{(\tau+1)^2 - A^2} \right)^{1-2n}
+ 4n^2 \left( A^2 + \tau^2 - 1 \right) \left( (\tau+1)^2 - A^2 \right)^{-n-2} \left( \frac{\Gamma(2n)}{\Gamma(n+1)^2} + o(1) \right) \left( 1 - \frac{(\tau-1)^2 - A^2}{(\tau+1)^2 - A^2} \right)^{-2n}.
\]

Furthermore, there are easily verified, for instance, by induction, the following expansions
\[
\left( 1 - \frac{(\tau-1)^2 - A^2}{(\tau+1)^2 - A^2} \right)^{-2n} = \left( \frac{\tau}{1-A^2} \right)^{-2n} \left( 4^{-2n} - \frac{4^{1-2n}n^2\tau}{A^2-1} + O(\tau^2) \right), \quad n > 1,
\]
\[
\left( 1 - \frac{(\tau-1)^2 - A^2}{(\tau+1)^2 - A^2} \right)^{1-2n} = \left( \frac{\tau}{1-A^2} \right)^{1-2n} \left( 4^{-2n} + \frac{2^{3-4n}(1-2n)\tau}{A^2-1} + O(\tau^2) \right), \quad n > 2,
\]
\[
\left( (\tau+1)^2 - A^2 \right)^{-n-1} = (1-A^2)^{-n-1} - 2\tau \left( n+1 \right) \left( 1-A^2 \right)^{-n-2} + O(\tau^2),
\]
\[
\left( A^2 + \tau^2 - 1 \right) \left( (\tau+1)^2 - A^2 \right)^{-n-2} = - (1-A^2)^{-n-1} + 2(n+2)\tau \left( 1-A^2 \right)^{-n-2} + O(\tau^2).
\]

Lemma 4.4

For the following lemma is decisive for the asymptotics of the function

Indeed,

According to Lemma 7.1 \([26]\), for \(z\) we get

Hence,

This completes the proof of the lemma.

4.4 The case of \(\ell > 1\) and \(M = \frac{1}{2} + n, n = 0, 1, 2, \ldots\)

The following lemma is decisive for the asymptotics of the function \(\frac{\partial}{\partial m} K_1(r, t; m; \varepsilon)\).

Lemma 4.4 For \(A \in [0, 1/2]\) and all \(n = 1, 2, 3, \ldots\)

Proof. Indeed,

According to Lemma 7.1 \([26]\), for \(z \leq 0\) the following formulas hold

Then

\[ F \left( -n - \frac{1}{2}, -n - \frac{1}{2} - \frac{4\tau}{(\tau + 1)^2 - A^2} \right) = \frac{\Gamma(2n + 2)}{\Gamma(\frac{3}{2} + n)^2} \left( n - \frac{1}{2} \right)^2 4\Gamma(2n + 1) \left( (\tau + 1)^2 - A^2 \right) \frac{\Gamma(n + \frac{3}{2})}{\Gamma(\frac{3}{2} + n)^2} O(\tau^2) \]
Hence,

\[(\tau + 1)((\tau + 1)^2 - A^2)^{n - \frac{7}{2}} F\left(-n - \frac{1}{2}, -n - \frac{1}{2}; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2}\right)\]

\[+ (2n + 1) (A^2 + \tau^2 - 1) ((\tau + 1)^2 - A^2)^{-\frac{7}{2}} F\left(\frac{1}{2} - n, \frac{1}{2} - n; 2; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2}\right)\]

\[= (\tau + 1)((\tau + 1)^2 - A^2)^{n - \frac{7}{2}} \left[\frac{\Gamma(2n + 2)}{\Gamma(n + 3/2)} - \frac{(n + 4)^2 4\Gamma(2n + 1)}{(\tau + 1)^2 - A^2} = O(\tau^2)\right] + (2n + 1) (A^2 + \tau^2 - 1) ((\tau + 1)^2 - A^2)^{-\frac{7}{2}} \left[\frac{\Gamma(2n + 1)}{(n + 3/2)^2} - \frac{(\frac{1}{2} - n)^2 4\Gamma(2n)}{(\tau + 1)^2 - A^2} = O(\tau^2)\right] + (2n + 1) (1 - A^2)^{-\frac{7}{2}} (2A^2 n - 1) \frac{\Gamma(2n)}{2^{n + 3/2}} = O(\tau^2) .\]

Lemma is proved.

\[\square\]

### 4.5 The case of \(\ell > 1\) and \(M = -\frac{1}{2} - n, n = 0, 1, 2, \ldots\)

For \(n = 1, 2, \ldots\), from \(\text{[13]}\) we obtain the following relation

\[
\frac{\partial}{\partial t} K_1 (r, t; m; 1) = C(n, \ell, \varepsilon) t^{-\ell} \left((1 + \tau^2 - A^2)^{-\frac{7}{2}} - n \right) F\left(A, -\frac{1}{2} - n; \tau\right), \quad C(n, \ell, \varepsilon) \neq 0 .
\]

**Lemma 4.5** There exist numbers \(C_0(\ell, \varepsilon) \neq 0\) and \(C_1(n, \ell, \varepsilon) \neq 0\) such that if \(n = 0\), then

\[
\frac{\partial}{\partial t} K_1 (r, t; m; 1) = C_0(\ell, \varepsilon)(1 - A^2)^{-\frac{7}{2}} t^{-1} + t^{-\ell} \ln(t) O(1) ,
\]

while

\[
\frac{\partial}{\partial t} K_1 (r, t; m; 1) = C_1(n, \ell, \varepsilon) (1 - A^2)^{-\frac{7}{2}} n - \ell (1 + o(1)) \quad \text{as} \quad t \to \infty \quad \text{if} \quad n = 1, 2, \ldots .
\]

**Proof.** First, we use \(\text{[15]}\) :

\[
((1 + \tau^2 - A^2)^{-\frac{7}{2}}) = (1 - A^2)^{-\frac{7}{2}} - (2n + 5)\tau (1 - A^2)^{-\frac{7}{2}} + \frac{1}{2}(2n + 5)\tau^2 (1 - A^2)^{-\frac{7}{2}} (A^2 + 2n + 6) + O(\tau^3) .
\]

If we plug \(M = -\frac{1}{2} - n, n = 0, 1, 2, \ldots\) in the definition \(\text{[13]}\) of \(F(A, M; \tau)\), then

\[
F\left(A, -\frac{1}{2} - n; \tau\right) = -(2n + 1) (1 - A^2 - \tau^2) F\left(n + \frac{3}{2}, n + \frac{3}{2}; 2; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2}\right) \]

\[- (1 + \tau) (1 + \tau^2 - A^2) F\left(n + \frac{1}{2}, n + \frac{1}{2}; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2}\right), \quad n = 0, 1, 2, \ldots .
\]

Denote

\[
z := \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \in (0, 1), \quad 1 - z = \frac{4\tau}{(\tau + 1)^2 - A^2} \to 0 \quad \text{as} \quad \tau \to 0 .
\]
For \( n = 1, 2, 3, \ldots \), the following asymptotics
\[
(1 + \tau) \left( (1 + \tau)^2 - A^2 \right) F \left( n + \frac{1}{2}, n + \frac{1}{2}; 1; \frac{(1 - \tau)^2 - A^2}{(1 + \tau)^2 - A^2} \right)
+ (2n + 1) (1 - \tau^2 - A^2) F \left( n + \frac{3}{2}, n + \frac{3}{2}; 2; \frac{(1 - \tau)^2 - A^2}{(1 + \tau)^2 - A^2} \right)
= (2n + 1) (1 - \tau^2 - A^2) (1 - z)^{-(2n+1)} \left\{ \frac{\Gamma(2n+1)}{[\Gamma(n + \frac{1}{2})]^2} + (1 - z)O(1) \right\}
\]
is proved in [26, subsection 7.1, p.30]. In particular,
\[
\lim_{\tau \to 0} (1 - z)^{(2n+1)} \left[ (1 + \tau) \left( (1 + \tau)^2 - A^2 \right) F \left( n + \frac{1}{2}, n + \frac{1}{2}; 1; z \right)
+ (2n + 1) (1 - \tau^2 - A^2) F \left( n + \frac{3}{2}, n + \frac{3}{2}; 2; z \right) \right]
= (2n + 1) \frac{\Gamma(2n+1)}{[\Gamma(n + \frac{1}{2})]^2} (1 - A^2), \quad n = 1, 2, 3, \ldots
\]
Together with
\[
\left( 1 - \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right)^{-2n-1} = (1 - A^2)^{2n+1} r^{2n-1} - 1 + O(1)
\]
the relation (27) proves the second statement of Lemma 4.5. For the case of \( n = 0 \) we apply [13], [14], and the result of Section 8 [26].

\[
(1 + \tau) \left( (1 + \tau)^2 - A^2 \right) F \left( \frac{1}{2}, \frac{1}{2}; 1; \frac{(1 - \tau)^2 - A^2}{(1 + \tau)^2 - A^2} \right) + (1 - \tau^2 - A^2) F \left( \frac{3}{2}, \frac{3}{2}; 2; \frac{(1 - \tau)^2 - A^2}{(1 + \tau)^2 - A^2} \right)
= (1 + \tau) \left( (1 + \tau)^2 - A^2 \right) \left( \frac{1}{\pi} \left[ 2\psi(1) - 2\psi \left( \frac{1}{2} \right) - \ln(z) \right] + z \ln(z)O(1) \right)
+ (1 - \tau^2 - A^2) \left( \frac{4z - 1}{\pi} - O(1) \ln(z) \right)
= (1 - A^2)^2 \frac{1}{\pi} \tau^{-1} - O(1) \ln(t).
\]
The lemma is proved.

4.6 The case of \( \ell > 1 \) and \( M = \frac{1}{2} \)

Lemma 4.6 For \( M = \frac{1}{2} \),
\[
\frac{\partial}{\partial \tau} \left( (\tau + 1)^2 - A^2 \right)^{\frac{3}{2}} F \left( -\frac{1}{2}, -\frac{1}{2}; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right)
= -4 \frac{1}{\pi} (1 - A^2)^{-3/2} \tau (A^2 - \ln(4 - 4A^2) + \ln(\tau) + 1) + O \left( \tau^2 \right),
\]
and the principal term of the expansion is \( -4 \frac{1}{\pi} (1 - A^2)^{-3/2} \tau \ln(\tau) + O \left( \tau \right) \).

Proof. We apply the results of subsection 7.3 [26] together with [13] and \( M - 2 - 3/2 = -4 \).
4.7 The case of $\ell > 1$ and $M = -\frac{1}{2}$

Lemma 4.7 The following asymptotic expansion holds:

$$\frac{\partial}{\partial \tau} \left((\ell + 1)^2 - A^2\right)^{-\frac{1}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(\ell - 1)^2 - A^2}{(\ell + 1)^2 - A^2}\right) = -\frac{1}{\pi \sqrt{1 - A^2}} + O(\tau) \quad \text{as} \quad \tau \to 0.$$

Proof. We have

$$\frac{\partial}{\partial \tau} \left((\ell + 1)^2 - A^2\right)^{-\frac{1}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(\ell - 1)^2 - A^2}{(\ell + 1)^2 - A^2}\right) = -((\ell + 1)(\ell + 1)^2 - A^2)^{-\frac{1}{2}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(\ell - 1)^2 - A^2}{(\ell + 1)^2 - A^2}\right) + (A^2 + \tau^2 - 1) \left((\ell + 1)^2 - A^2\right)^{-\frac{1}{2}} F\left(\frac{3}{2}, \frac{3}{2}; 2; \frac{(\ell - 1)^2 - A^2}{(\ell + 1)^2 - A^2}\right).$$

For $F\left(\frac{1}{2}, \frac{1}{2}; 1; z\right)$ we apply [3] (Sec. 2.10) with $a = b = \frac{1}{2}$, $c = 1$, $m = 0$, and obtain

$$F\left(\frac{1}{2}, \frac{1}{2}; 1; z\right) = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{[\frac{1}{2}]_n^2}{[n]^2} [\bar{h}_n - \ln(1 - z)] (1 - z)^n \quad \text{(29)}$$

$$= \frac{4 \ln (2) - \ln (1 - z)}{\pi} + \frac{2 - 4 \ln(2) + \ln(1 - z)}{4\pi} (z - 1) + \frac{3(12 \ln(2) - 7 - 3 \ln(1 - z))}{64\pi} (z - 1)^2 + O ((z - 1)^3).$$

If $c - a - b = -m$, then we apply [3] (14) Sec. 2.10):

$$F(a, b, a + b - m; z) = \frac{\Gamma(m)(1 - z)^{-m} \sum_{n=0}^{m-1} (a - m)_n (b - m)_n (1 - z)^n}{\Gamma(a) \Gamma(b) (1 - m)_n n!} \quad \text{and obtain}$$

$$\frac{\Gamma(m)(1 - z)^{-m} \sum_{n=0}^{m-1} (a - m)_n (b - m)_n (1 - z)^n}{\Gamma(a) \Gamma(b) (1 - m)_n n!} + \frac{(-1)^m}{\Gamma(a - m) \Gamma(b - m)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(n + m)_n n!} [\bar{h}_n - \ln(1 - z)] (1 - z)^n,$$

where $-\pi < \arg(1 - z) < \pi$, $a, b, \neq 0, -1, -2, \ldots$, and $\sum_{n=0}^{m-1}$ is set zero if $m = 0$.

For the function $F\left(\frac{3}{2}, \frac{3}{2}; 2; z\right)$, we set $a = b = \frac{3}{2}$, $c = 2$, $m = 1$, and obtain

$$F\left(\frac{3}{2}, \frac{3}{2}; 2; z\right) = \frac{4}{\pi} (1 - z)^{-1} - \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{[\frac{3}{2}]_n^2}{(n + 4)_n n!} [\bar{h}_n - \ln(1 - z)] (1 - z)^n \quad \text{(30)}$$

$$= \frac{4}{\pi} (1 - z)^{-1} + \frac{3 - 2 \ln(4) + \ln(1 - z)}{16\pi} + \frac{3(-17 + 12 \ln(4) - 6 \ln(1 - z))}{16\pi} (z - 1)$$

$$- \frac{15(-14 + 10 \ln(4) - 5 \ln(1 - z))}{64\pi} (z - 1)^2 + O ((z - 1)^3).$$

Taking into account [12] we write

$$F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{(\ell - 1)^2 - A^2}{(\ell + 1)^2 - A^2}\right) = \frac{\log (4 (1 - A^2)) - \log(\tau)}{\pi} + \frac{(- \log (4 (1 - A^2)) + \log(\tau) + 2)}{\pi (A^2 - 1)} \tau + O(\tau^2),$$

$$F\left(\frac{3}{2}, \frac{3}{2}; 2; \frac{(\ell - 1)^2 - A^2}{(\ell + 1)^2 - A^2}\right) = \frac{1 - A^2}{\pi \tau} + \frac{5 - \log (4 (1 - A^2)) + \log(\tau)}{\pi},$$

$$- \frac{(4 A^2 + 18 \log (1 - A^2) - 18 \log(\tau) - 47 + 36 \log(2))}{4\pi (1 - A^2)} \tau + O(\tau^2).$$

On the other hand,
We appeal to the function $K$ where asymptotic expansion

**Proposition 5.1**
Assume that

$$-(\tau + 1) (\tau + 1)^2 - A^2)^{-\frac{5}{2}} = -\frac{1}{(1 - A^2)^{3/2}} + \frac{(A^2 + 2)}{(1 - A^2)^{5/2}} \tau + O (\tau^2),$$

$$(A^2 + \tau^2 - 1) ((\tau + 1)^2 - A^2)^{-\frac{5}{2}} = -\frac{1}{(1 - A^2)^{3/2}} + \frac{5}{(1 - A^2)^{5/2}} \tau + O (\tau^2).$$

Then imply

$$\frac{\partial}{\partial \tau} \left( (\tau + 1)^2 - A^2 \right)^{-\frac{5}{2}} F \left( \frac{1}{2}, \frac{1}{2}; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right) = -\frac{1}{\pi \sqrt{1 - A^2} \tau} + O (\tau).$$

The lemma is proved. \hfill \Box

## 5 Asymptotics of $\frac{\partial}{\partial t} K_1 (r, t; \pm m; \varepsilon)$ when $\ell < 1$

We appeal to the function $K_1 (r, t; m; \varepsilon)$ of [8]. Taking into account the scaling, henceforth we can suppose $\varepsilon = 1$. Assume $\ell < 1$ and denote

$$M := \frac{im}{\ell - 1}, \quad A := (\ell - 1) r, \quad \tau := t^{1-\ell} \to \infty \quad \text{as} \quad t \to \infty, \quad z = \frac{(t^{1-\ell} - 1)^2 - A^2}{(t^{1-\ell} + 1)^2 - A^2} = \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2}.$$

Then, according to [8], we write

$$K_1 (r, t; m; 1) := C(m, \ell) \left( 1 + \tau \right)^2 - A^2 \right)^M F \left( -M, -M; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right), \quad C(m, \ell) \neq 0,$$

and with the constant $C_1 (m, \ell) = (1 - \ell) C(m, \ell)$ obtain

$$\frac{\partial}{\partial t} K_1 (r, t; m; 1) = C_1 (m, \ell) t^{-\ell} \frac{\partial}{\partial \tau} \left( (1 + \tau)^2 - A^2 \right)^M F \left( -M, -M; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right).$$

Further, we consider

$$\frac{\partial}{\partial \tau} \left( (1 + \tau)^2 - A^2 \right)^M F \left( -M, -M; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right) = (1 + \tau)^2 - A^2 \right)^M F \left( -M, -M; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right),$$

where

$$F(A, M; \tau) := 2M (1 - A^2 - \tau) F \left( 1 - M, 1 - M; 2; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right)$$

$$- (1 + \tau) \left( (1 + \tau)^2 - A^2 \right) F \left( -M, -M; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right),$$

similar to [13] and [14], respectively. We note that any term of the asymptotics that is independent of $r$ obeys the Huygens' principle and can be neglected. (Compare with [15], [16], and [17].)

### 5.1 The case of $\ell < 1$ and $M \in \mathbb{C}$, $2M \neq 0, \pm 1, \pm 2, \ldots$

**Proposition 5.1** Assume that $\ell < 1$ and $M \in \mathbb{C}$, $2M \neq 0, \pm 1, \pm 2, \ldots$, and $A \in [0, 1/3]$. Then the following asymptotic expansion

$$\frac{\partial}{\partial \tau} \left( (1 + \tau)^2 - A^2 \right)^M F \left( -M, -M; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right)$$

(31)
\[
\tau^2 M \left[ -\frac{2M \Gamma(1 - 2M) \Gamma(2M)}{\Gamma(-2M)[\Gamma(M + 1)]^2} \tau^{-1} + \frac{4(M - 1) (A^2(2M - 1) - 1) \Gamma(2M)}{(1 - 2M)[\Gamma(M)]^2} \tau^{-3} + O \left( \frac{1}{\tau^5} \right) \right] \\
- \frac{4^{2M+1} \Gamma(1 - 2M)}{2(2M + 1)M[\Gamma(-M)]^2} \tau^{-2} - \frac{3 \left[ 4^{2M+1}(M + 1) (A^2(2M + 3) + 1) \Gamma(1 - 2M) \right]}{2(2M + 3)(2M + 1)M[\Gamma(-M)]^2} \tau^{-4} + O \left( \frac{1}{\tau^5} \right)
\]
holds as \( \tau \to \infty \).

**Proof.** It is easily seen that

\[
1 - z = 1 - \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} = \frac{4}{\tau} - \frac{8}{\tau^2} + \frac{4(A^2 + 3)}{\tau^3} - \frac{16(A^2 + 1)}{\tau^4} + \frac{4(A^4 + 10A^2 + 5)}{\tau^5} + O \left( \frac{1}{\tau^5} \right)
\]

as \( \tau \to \infty \). We apply the previous expansion and the relations (21), (22) to obtain

\[
F \left( -M, -M; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right)
\]

\[
= \frac{2M \Gamma(2M)}{\Gamma(M + 1)^2} - \frac{4M^2 \Gamma(2M)}{\Gamma(M + 1)^2} \tau^{-1} - \frac{8M^2 \Gamma(2M)}{(1 - 2M)\Gamma(M)^2} \tau^{-2}
\]

\[
+ \frac{4M^2 (A^2(6M - 3) + 2M(2M(M + 1) + 1) - 1) \Gamma(2M)}{3(1 - 2M)\Gamma(M + 1)^2} \tau^{-3}
\]

\[
- \frac{16M^4 (A^2(6M - 9) + M(M^2 + M - 1) - 4) \Gamma(2M)}{3(2M - 3)(1 - 2M)\Gamma(M + 1)^2} \tau^{-4} + O \left( \frac{1}{\tau^5} \right)
\]

\[
+ \tau^{-2M} \left[ \frac{4^{2M+1} \Gamma(1 - 2M)}{2M(2M + 1)\Gamma(-M)^2} \tau^{-1} - \frac{4^{2M+2} \Gamma(1 - 2M)}{(2M + 1)\Gamma(-M)^2} \tau^{-2}
\]

\[
+ \frac{4^{2M+1} (A^2(4M(M + 2) + 3) + 4M(M + 1)^2 + 1) \Gamma(1 - 2M)}{2(2M + 3)M(2M + 1)\Gamma(-M)^2} \tau^{-3}
\]

\[
- \frac{16^{M+1}(M + 1) (A^2(6M + 9) + 2M(M + 2) + 3) \Gamma(1 - 2M)}{6(2M + 3)(2M + 1)\Gamma(-M)^2} \tau^{-4}
\]

\[
+ \frac{4^{2M+1}(M + 1)\Gamma(1 - 2M)}{6(2M + 3)(2M + 5)M(2M + 1)\Gamma(-M)^2}
\]

\[
\times \left[ 3A^2(2M + 1)(2M + 3)(2M + 5) + 6A^2(2M + 3)(2M + 5)(2M(M + 1) + 1)
\]

\[
+ 2M(2M(M + 3) + 3)(2M(M + 3) + 7) + 9 \right] \tau^{-5} + O \left( \frac{1}{\tau^6} \right)
\]

and

\[
F \left( 1 - M, 1 - M; 2; \frac{(1 - \tau)^2 - A^2}{(\tau + 1)^2 - A^2} \right)
\]

\[
= \frac{\Gamma(2M)}{\Gamma(M + 1)^2} + \frac{4(M - 1) \Gamma(2M)}{(1 - 2M)\Gamma(M + 1)^2} \tau^{-1} - \frac{4(M - 1)(M - 2M + 2) \Gamma(2M)}{(1 - 2M)\Gamma(M + 1)^2} \tau^{-2}
\]

\[
+ \frac{4(M - 1)(3A^2(M - 1)(2M - 3) + M(M(2M - 4)M + 17) - 45) + 27) \Gamma(2M)}{3(2M - 3)(1 - 2M)\Gamma(M + 1)^2} \tau^{-3}
\]

\[
- \frac{8(M - 1)\Gamma(2M)}{3((2M - 3)(1 - 2M)\Gamma(M + 1)^2)}
\]

\[
\times \left( 3A^2(2M - 3)((M - 2)M + 2) + M(M((M - 4)M + 11) - 23) + 30 \right) \tau^{-4}
\]

\[
+ O \left( \frac{1}{\tau^5} \right) + \tau^{-2M} \left[ \frac{16^{M+1}(1 - 2M)}{2M\Gamma(1 - M)^2} + \frac{4^{2M+1}(1 - 2M) \Gamma(1 - 2M)}{2M(2M + 1)\Gamma(1 - M)^2} \tau^{-1}
\]

\[
+ O \left( \frac{1}{\tau^5} \right) \right]
\]
Lemma 5.2

The function

\[
\Phi(x, y) = \frac{2^{4M} (M(A^2(2M + 1) + 2(M - 2)M + 1) + 4) \Gamma(1 - 2M)}{M(2M + 1) \Gamma(1 - M)^2} x^{-2}
\]

\[
\frac{4^{2M+1} \Gamma(1 - 2M)}{6(2M + 3)M(2M + 1) \Gamma(1 - M)^2}
\]

\[
\times (3A^2(2M + 1)(2M + 3)((M - 1)M - 1) + (M - 1)M (4M^3 - 4M + 15) - 27) \tau^{-3}
\]

\[
\frac{16M \Gamma(1 - 2M)}{6(2M + 3)M(2M + 1) \Gamma(1 - M)^2}
\]

\[
\times (3A^4M(2M + 1)^2(2M + 3) + 6A^2(2M + 3)(M(4M((M - 1)M - 1) + 11) + 8)
\]

\[
+ M(2M(2M((M - 1)M + 1) + 5) - 47) + 57) + 144 \tau^{-4} + O\left(\frac{1}{\tau^5}\right)
\]

as \( \tau \to \infty \). On the other hand,

\[
\left( (1 + \tau)^2 - A^2 \right)^{M-2} = \tau^{2M-4} \left[ 1 + \frac{2(M - 2)}{\tau} - \frac{(M - 2)(A^2 - 2M + 5)}{\tau^2}
\right]
\]

\[
- \frac{2}{3} \frac{(M - 3)(M - 2)(3A^2 - 2M + 5)}{3 \tau^3}
\]

\[
+ \frac{(M - 3)(M - 2)(3A^4 - 12(A^2 + 2)M + 42A^2 + 4M^2 + 35)}{6 \tau^4} + O\left(\frac{1}{\tau^5}\right)
\]

as \( \tau \to \infty \). Then we put the last expressions into (31) and obtain (51). The proposition is proved. \( \square \)

Corollary 5.1

If \( \Re(M) > -\frac{1}{2} \), then in the expansion (31) there is the term containing \( A \):

\[
\tau^{2M-4} \frac{4(M - 1)(A^2(2M - 1) - 1) \Gamma(2M)}{(1 - 2M) \Gamma(M)^2}
\]

If \( \Re(M) < -\frac{1}{2} \), then in the expansion (31) there is the term containing \( A \):

\[
- \frac{3}{2(2M + 3)(2M + 1)M \Gamma(1 - M)^2} \tau^{-4}
\]

If \( M = -\frac{1}{2} + iB \), \( B \in \mathbb{R} \setminus \{0\} \), then in the expansion (31) there are the terms containing \( A \):

\[
- \frac{1}{4 \tau^4} \left[ \frac{8(2B + 3i)(2A^2(B + i) + i)}{\Gamma(-\frac{1}{2} + iB)^2 \Gamma(3 - 2iB)} \tau^{2iB} + \frac{3i4^{2iB}(2B - i)(2A^2(B - i) - i)}{(B - i) \Gamma(\frac{1}{2} - iB)^2 \Gamma(1 + 2iB)} \right]
\]

5.2 The case of \( \ell < 1 \) and \( M = n, n = 2, 3, \ldots \)

Lemma 5.2

The function

\[
\frac{\partial}{\partial \tau} \left( (1 + \tau)^2 - A^2 \right)^n F\left(-n, -n; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2}\right)
\]

for \( n = 2, 3, \ldots \) is a polynomial in \( \tau \) and \( A^2 \), which varies in \( A^2 \) and contains at least one term \( \tau^a A^{2b} \) with \( a, b \neq 0 \).

Proof. To verify the first statement we just use (23). For \( M = n \), where \( n = 2, 3, \ldots \) we put \( z = \frac{\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \) in (23), and obtain for the function of (33) the polynomial

\[
\frac{\partial}{\partial \tau} \left( (1 + \tau)^2 - A^2 \right)^n \left[ 1 + \sum_{j=1}^{n} \frac{\Gamma(n + 1)}{\Gamma(j + 1) \Gamma(-j + n + 1)} \left( \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right)^j \right]
\]
For Lemma 5.3 \( a, b \neq 0 \), in fact, \( \tau \) as \( A = \tau - 1 \) the polynomial equals to \([2(n + 2)\tau + (2n - 2)](4\tau)^{n-1}\), while at \( A = \tau + 1 \) it takes value \((2n\tau - 2n)(4\tau)^{n-1}\). Thus, there is at least one nonvanishing term of the polynomial containing \( \tau^a A^b \) with \( a, b \neq 0 \). We can use such term with the greatest \( a \), in fact, \( a = 2n - 3 \). The lemma is proved. \( \Box \)

5.3 The case of \( \ell < 1 \) and \( M = -n, n = 1, 2, 3, \ldots \)

In the case of \( M = -n \), with \( n = 1, 2, 3, \ldots \), we appeal to (24) to prove the next lemma.

**Lemma 5.3** For \( n = 2, 3, 4, \ldots \), in the expansion of the function

\[
\frac{\partial}{\partial \tau} \left( ((\tau + 1)^2 - A^2)^{-n} F \left( n, n; 1; \frac{\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right) \right)
\]

as \( \tau \to \infty \) there exists the term

\[-\frac{2^{1-2n} (1 - A^2)^{n-1} \Gamma \left( n + \frac{1}{2} \right)}{\sqrt{\pi} \Gamma(n)} \frac{1}{\tau^{2n}}.\]

**Proof.** The case of \( M = -1, -2 \) is given by Lemma 4.3 and (25). For the case of \( n = 3, 4, \ldots \) we appeal to (14) section 2.1 and to elementary relation (20). Hence, by definition (20) we have

\[ F(n, n; 1; z) = (1 - z)^{1-2n} F(1 - n, 1 - n; 1; z) = (1 - z)^{1-2n} \sum_{k=0}^{n-1} \frac{[(1-n)k]^2}{[k!]^2} z^k. \]

Then

\[ \frac{\partial}{\partial \tau} \left( ((\tau + 1)^2 - A^2)^{-n} F \left( n, n; 1; \frac{\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right) \right) = 4^{1-2n} \tau^{-2n} P(A; n, \tau), \]

where

\[ P(A; n, \tau) := \tau \sum_{k=0}^{n-1} \frac{[(1-n)k]^2}{[k!]^2} \left( A^2 ((n - 1)(\tau + 1) - 2k) - (\tau^2 - 1)(2k + (n - 1)(\tau - 1)) \right) \]

\[ + 4^{1-2n} (1 - 2n) \sum_{k=0}^{n-1} \frac{[(1-n)k]^2}{[k!]^2} \left( ((\tau - 1)^2 - A^2)^k ((\tau + 1)^2 - A^2)^{-k+n-1} \right) \]

is polynomial in \( \tau \) and \( A^2 \). We set \( \tau = 0 \) and obtain

\[ P(A; n, 0) = -\frac{2^{1-2n} (1 - A^2)^{n-1} \Gamma \left( n + \frac{1}{2} \right)}{\sqrt{\pi} \Gamma(n)}. \]
Hence, the asymptotic expansion of the function \((\ref{34})\) contains the term
\[-\tau^{-2n} \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi} \Gamma(n)} \left(1 - A^2\right)^{-n-1}.\]

The lemma is proved. \(\Box\)

Thus, in the asymptotic expansion of \((\ref{34})\) we can choose the term with greatest degree in \(\tau\) among all containing variable \(A^2\).

5.4 The case of \(\ell < 1\) and \(M = \frac{1}{2} + n, n = 1, 2, \ldots\)

Lemma 5.4 For \(M = \frac{3}{2} (n = 1)\) we have
\[
\frac{\partial}{\partial \tau} \left( (\tau + 1)^{-\frac{3}{2}} A^{-\frac{3}{2}} F \left( \frac{3}{2}, \frac{3}{2}; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right) \right) = -\frac{1}{4\pi^2} + \frac{6A^2 - 6 \ln(\tau) + 5 - \ln(4)}{16\pi^4} + O \left( \frac{1}{\tau^5} \right).
\]

For \(n = 2, 3, 4, \ldots\), in the asymptotic expansion of the function
\[
\frac{\partial}{\partial \tau} \left( (\tau + 1)^{-\frac{3}{2}} A^{-\frac{3}{2}} F \left( n + \frac{1}{2}, n + \frac{1}{2}; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right) \right)
\]
as \(\tau \to \infty\) there exists the term with \(\tau^{-4}\) depending on \(A\), namely, this term is
\[
\frac{3\Gamma(2n) (4^{-2n-1} (2n - 1) (2A^2(n - 1) - 1) \Gamma(n + \frac{3}{2})^2 (n - 1))}{\Gamma(n + \frac{3}{2})^2 (n - 1)} \frac{1}{\tau^4}.
\]

Proof. For \(n = 1, 2, 3, 4, \ldots\), we have to discuss
\[
\frac{\partial}{\partial \tau} \left( (\tau + 1)^{-\frac{3}{2}} A^{-\frac{3}{2}} F \left( n + \frac{1}{2}, n + \frac{1}{2}; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right) \right).
\]
We make the change of variable \(\lambda = \tau^{-1}\) and consider the asymptotic series about \(\lambda = 0\):
\[
\frac{\partial}{\partial \lambda} \left( \left( \frac{1}{\lambda} + 1 \right)^{-\frac{3}{2}} A^{-\frac{3}{2}} F \left( n + \frac{1}{2}, n + \frac{1}{2}; 1; \frac{(\lambda - 1)^2 - A^2}{(\lambda + 1)^2 - A^2} \right) \right)
\]
\[
= (2n + 1)\lambda^{2n} (\lambda (-A^2\lambda + \lambda + 2) + 1)^{-n-\frac{1}{2}} F(A, n; \lambda),
\]
where has been used the notation
\[
F(A, n; \lambda) := (\lambda + 1) (\lambda (-A^2\lambda + \lambda + 2) + 1) F \left( n + \frac{1}{2}, n + \frac{1}{2}; 1; \frac{(\lambda - 1)^2 - A^2}{(\lambda + 1)^2 - A^2} \right)
- \lambda((2n + 1)) (A^2 - 1) \lambda^2 + 1) F \left( n + \frac{3}{2}, n + \frac{3}{2}; 2; \frac{(\lambda - 1)^2 - A^2}{(\lambda + 1)^2 - A^2} \right).
\]

Denote
\[
z := \frac{(\lambda - 1)^2 - A^2}{(\lambda + 1)^2 - A^2} \rightarrow 1 \quad \text{as} \quad \lambda \rightarrow 0,
\]
then we can use \((\ref{32})\) as \(\lambda = \tau^{-1} \rightarrow 0\). According to \((\ref{3})\) (Sec.\ 2.10) for \(n = 1, 2, \ldots\) with \(a = b = n + \frac{1}{2}\), \(c = 1, c - a - b = -2n\), and \(m = 2n\), we have:
\[
F \left( n + \frac{1}{2}, n + \frac{1}{2}; 1; z \right) = (1 - z)^{-2n} \frac{\Gamma(2n)}{\Gamma(n + \frac{3}{2})} \sum_{k=0}^{2n-1} \frac{[(\frac{1}{2} - n)k]^2}{(1 - 2n)k!} (1 - z)^k
+ \frac{1}{\Gamma(\frac{1}{2} - n)} \sum_{k=0}^{\infty} \frac{[(\frac{1}{2} - n)k]^2}{(k + 2n)k!} \left[ \ln(1 - z) \right] (1 - z)^k
= (1 - z)^{-2n} \frac{\Gamma(2n)}{\Gamma(n + \frac{3}{2})} \sum_{k=0}^{2n-1} \frac{[(\frac{1}{2} - n)k]^2}{(1 - 2n)k!} (1 - z)^k + |\ln(1 - z)|O(1),
\]

where \(-\pi < \arg(1-z) < \pi\), \(a, b \neq 0, -1, -2, \ldots\)

\[
\bar{h}_n = \psi(1+n) + \psi(1+n+m) - \psi(a+n) - \psi(b+n).
\]

(35)

The function \(\psi(z)\) is the logarithmic derivative of the gamma function: \(\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}\).

Similarly, since \(a = b = n + \frac{3}{2}\), \(c = 2\), \(c - a - b = -m\), \(m = 2n + 1\), we have

\[
F\left(n + \frac{3}{2}, n + \frac{3}{2}; z\right) = \frac{\Gamma(2n+1)(1-z)^{-2n-1}}{[\Gamma(n+\frac{3}{2})]^2} \sum_{k=0}^{2n} \frac{[\frac{1}{2} - n]_k^2}{(-2n)_kk!} (1-z)^k
\]

\[
+ \frac{(-1)^{n+1}}{\Gamma(\frac{1}{2} - n)} \sum_{k=0}^\infty \frac{[(n + \frac{3}{2})]_k^2}{(k + 2n + 1)_kk!} \left[h_k - \ln(1-z)\right] (1-z)^k
\]

\[
= (1-z)^{-2n-1} \frac{\Gamma(2n+1)}{[\Gamma(n+\frac{3}{2})]^2} \sum_{k=0}^{2n} \frac{[\frac{1}{2} - n]_k^2}{(-2n)_kk!} (1-z)^k + \ln(1-z)|O(1)\]

Hence,

\[
F(A, n; \lambda)
\]

\[
\begin{align*}
&= (1-z)^{-2n-1} \left[ (\lambda + 1) \left( -A^2 \lambda + \lambda + 2 \right) + 1 \right] \\
&\times \left\{ \frac{\Gamma(2n)}{[\Gamma(n+\frac{3}{2})]^2} \sum_{k=0}^{2n-1} \frac{[\frac{1}{2} - n]_k^2}{(1-2n)_kk!} (1-z)^{k+1} + (1-z)^{2n+1} \ln(1-z)|O(1) \right\} \\
&- \lambda((2n+1)) ((A^2 - 1) \lambda^2 + 1) \left\{ \frac{\Gamma(2n+1)}{[\Gamma(n+\frac{3}{2})]^2} \sum_{k=0}^{2n} \frac{[\frac{1}{2} - n]_k^2}{(-2n)_kk!} (1-z)^k + (1-z)^{2n+1} \ln(1-z)|O(1) \right\}
\end{align*}
\]

\[
= (1-z)^{-2n-1} \frac{\Gamma(2n)}{[\Gamma(n+\frac{3}{2})]^2} \\
\times \left[ (\lambda + 1) \left( -A^2 \lambda + \lambda + 2 \right) + 1 \right] \left\{ \sum_{k=0}^{2n-1} \frac{[\frac{1}{2} - n]_k^2}{(1-2n)_kk!} (1-z)^{k+1} + (1-z)^{2n+1} \ln(1-z)|O(1) \right\} \\
- \lambda((2n+1)) ((A^2 - 1) \lambda^2 + 1) \left\{ \frac{2n}{[\Gamma(n+\frac{3}{2})]^2} \sum_{k=0}^{2n} \frac{[\frac{1}{2} - n]_k^2}{(-2n)_kk!} (1-z)^k + (1-z)^{2n+1} \ln(1-z)|O(1) \right\}
\]

Since \(n \geq 2\), we can truncate the sums as follows:

\[
F(A, n; \lambda) = (1-z)^{-2n-1} \frac{\Gamma(2n)}{[\Gamma(n+\frac{3}{2})]^2} \\
\times \left[ (\lambda + 1) \left( -A^2 \lambda + \lambda + 2 \right) + 1 \right] \left\{ \sum_{k=0}^{2n} \frac{[\frac{1}{2} - n]_k^2}{(1-2n)_kk!} (1-z)^{k+1} + (1-z)^4|O(1) \right\} \\
- \lambda((2n+1)) ((A^2 - 1) \lambda^2 + 1) \left\{ \frac{2n}{[\Gamma(n+\frac{3}{2})]^2} \sum_{k=0}^{2n} \frac{[\frac{1}{2} - n]_k^2}{(-2n)_kk!} (1-z)^k + (1-z)^4O(1) \right\}
\]

It is evident that for \(n \geq 2\) the following asymptotic expansion holds

\[
(2n+1)\lambda^{2n} \left( -A^2 \lambda + \lambda + 2 \right) + 1)^{-n-\frac{3}{2}} \left( 1 - \left( \frac{1}{4} - 1 \right)^2 - A^2 \right)^{-2n-1}
\]

\[
= 4^{-2n-1} (2n+1) \frac{1}{\lambda} + 4^{-2n-1} (2n-3)(2n+1) + 2^{-4n-3}(2n-3)(2n+1) (-A^2 + 2n - 4) \lambda + O(\lambda^2)
\]
On the other hand
\[
(\lambda + 1) \left( \lambda \left( -A^2 \lambda + \lambda + 2 \right) + 1 \right) \left\{ \sum_{k=0}^{2} \frac{[(\frac{1}{2} - n)k]^2}{(1 - 2n)k!} (1 - z)^{k+1} + (1 - z)^4 O(1) \right\} \\
- \lambda(2n + 1) \left( (A^2 - 1) \lambda^2 + 1 \right) \left\{ \sum_{k=0}^{2n} \frac{[(\frac{1}{2} - n)k]^2}{(n + \frac{1}{2})^2} (1 - z)^{k} + (1 - z)^2 O(1) \right\}
\]
\[
= \frac{\lambda(\lambda + 1)}{(n - 1) (\lambda (A^2(-\lambda) + \lambda + 2) + 1)^2} \times \left[ \left( \lambda^2 \left( -8A^2(n - 1) + 8n^3 - 44n^2 + 78n - 41 \right) + 4 (A^2 - 1)^2 \lambda^4(n - 1) + 4 (A^2 - 1) \lambda^3(n - 1)(2n - 5) - 4\lambda(n - 1)(2n - 5) + 4(n - 1) \right) \right.
\]
\[
= \frac{\lambda n(2n + 1) \left( (A^2 - 1) \lambda^2 + 1 \right)}{12 \left( n + \frac{1}{2} \right)^2} \left\{ \frac{6\lambda^2(2n - 1)(3 - 2n)^2}{n (\lambda (A^2(-\lambda) + \lambda + 2) + 1)^2} \right\} + \frac{12\lambda(1 - 2n)^2}{n ((A^2 - 1) \lambda^2 - 2\lambda - 1)} + \frac{\lambda^3(2n - 1)(3 - 2n)^2(5 - 2n)^2 + 24}{(n - 1)n ((A^2 - 1) \lambda^2 - 2\lambda - 1)^3 + 24}\right]\]
\]
implies
\[
(2n + 1)\lambda^{2n} \left( \lambda \left( A^2(-\lambda) + \lambda + 2 \right) + 1 \right)^{-n - \frac{3}{2}} F(A, n; \lambda) = \frac{\Gamma(2n)}{|\Gamma(n + \frac{1}{2})|^2} \left( 4^{2n} - 3 \frac{4^{2n-1}(2n - 1)(2A^2(n - 1) - 1)}{n - 1} \lambda^2 + o(\lambda^2) \right).
\]
This completes the proof of the second statement of the lemma. The proof of the first statement of the lemma is straightforward. \(\square\)

### 5.5 The case of \(\ell < 1\) and \(M = -\frac{1}{2} - n, n = 1, 2, \ldots\)

Taking into account the scaling, we from now on can suppose \(\varepsilon = 1\). Assume \(\ell < 1\) and denote

\[
K_1(r, t; m; 1) := C(m, \ell) \left( (1 + \tau)^2 - A^2 \right)^{\frac{1}{2} + n} F \left( -\frac{1}{2} - n, -\frac{1}{2} - n; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right), \quad C(m, \ell) \neq 0.
\]

Consider the function

\[
\frac{\partial}{\partial \tau} K_1(r, t; m; 1) := C(m, \ell)(1 - \ell) t^{-\ell} \frac{\partial}{\partial \tau} \left( (1 + \tau)^2 - A^2 \right)^{\frac{1}{2} + n} F \left( -\frac{1}{2} - n, -\frac{1}{2} - n; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right).
\]

**Lemma 5.5** In the asymptotic expansion of the function

\[
\frac{\partial}{\partial \tau} \left( (1 + \tau)^2 - A^2 \right)^{\frac{1}{2} + n} F \left( -\frac{1}{2} - n, -\frac{1}{2} - n; 1; \frac{(\tau - 1)^2 - A^2}{(\tau + 1)^2 - A^2} \right)
\]
as \(\tau \to \infty\) there is a term containing \(A^2\):

\[
\tau^{2n-2} 2^{2n+1} \frac{(2A^2 n - 1) \Gamma(n)}{\sqrt{\pi} \Gamma \left( n - \frac{1}{2} \right)}.
\]

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Proof. We consider the series about $\lambda = 0$ after the change of variable $\lambda = \tau^{-1}$:

$$
\frac{\partial}{\partial \lambda} \left( \left( \frac{1}{\lambda + 1} \right)^{2} - A^2 \right)^{n + \frac{1}{2}} F \left( -n - \frac{1}{2}, -n - \frac{1}{2}; 1; \frac{(\lambda - 1)^2 - A^2}{(1 + \frac{1}{\lambda})^2 - A^2} \right).
$$

We remind that $\varphi \in C_0^\infty (0, 1)$ implies $A^2 = (1 - \ell)^2 r^2 > \text{constant} > 0$. Then,

$$
\frac{\partial}{\partial \lambda} \left( \left( \frac{1}{\lambda + 1} \right)^{2} - A^2 \right)^{n + \frac{1}{2}} F \left( -n - \frac{1}{2}, -n - \frac{1}{2}; 1; \frac{(\lambda - 1)^2 - A^2}{(1 + \frac{1}{\lambda})^2 - A^2} \right) = -(2n + 1) \left((\lambda + 1)^2 - A^2\lambda^2\right)^{n - \frac{1}{2}} \lambda^{-2n-2} \mathcal{F}(\lambda; n, A),
$$

where

$$
\mathcal{F}(\lambda; n, A) := (\lambda + 1) \left( \lambda (A^2(-\lambda) + \lambda + 2) + 1 \right)^2 F \left( -n - \frac{1}{2}, -n - \frac{1}{2}; 1; 1 - \frac{4\lambda}{(\lambda + 1)^2 - (A\lambda)^2} \right)
$$

$$
+ \lambda(2n + 1) \left( (A^2 - 1) \lambda^2 + 1 \right) (\lambda + 1)^2 - A^2\lambda^2 \right) F \left( \frac{1}{2} - n, \frac{1}{2} - n; 1; 1 - \frac{4\lambda}{(\lambda + 1)^2 - (A\lambda)^2} \right).
$$

We consider the function $\mathcal{F}(\lambda; n, A)$ around the point $\lambda = 0$. In order to substitute the following expansion

$$
z := 1 - \frac{4\lambda}{(\lambda + 1)^2 - (A\lambda)^2} = 1 - 4\lambda + 8\lambda^2 - 4 (A^2 + 3) \lambda^3 + O (\lambda^4)
$$

in the series for the function $\mathcal{F}(\lambda; n, A)$ we appeal to \cite{12} Sec. 2.10:

$$
F(a, b; a + b + m; z) = \frac{1}{\Gamma(a + b + m)} \sum_{n=0}^{m-1} \frac{(a)_n (b)_n}{(1 - m)_n n!} (1 - z)^n
$$

$$
= \frac{\Gamma(m)}{\Gamma(a + m) \Gamma(b + m)} \sum_{n=0}^{m-1} (a)_n (b)_n \sum_{n=0}^{\infty} \frac{(a + m)_n (b + m)_n}{(n + m)!} [h''_n - \ln(1 - z)] (1 - z)^n,
$$

where $-\pi < \arg(1 - z) < \pi$, $a, b \neq 0, -1, 2, \ldots$,

$$
h''_n = \psi(n + 1) + \psi(n + m + 1) - \psi(a + n + m) - \psi(b + n + m),
$$

and the term $\sum_{n=0}^{m-1}$ in the expression for $F(a, b; a + b + m; z)$ is to be interpreted as zero when $m = 0$. Hence with $m = 2n + 2$, $n = 1, 2, \ldots$, we obtain

$$
F \left( -n - \frac{1}{2}, -n - \frac{1}{2}; 1; z \right) = \frac{\Gamma(2n + 2)}{\Gamma(n + \frac{3}{2})^2} \sum_{k=0}^{2n+1} \frac{[(-n - \frac{1}{2})^2] k!^2 (1 - z)^k + (1 - z)^{2n+2} \ln(1 - z)O(1)},
$$

and with $m = 2n + 1$, $n = 1, 2, \ldots$, we derive

$$
F \left( \frac{1}{2} - n, \frac{1}{2} - n; 2; z \right) = \frac{\Gamma(2n + 1)}{\Gamma(n + \frac{3}{2})^2} \sum_{k=0}^{2n} \frac{[\frac{1}{2} - n]^2 k! (1 - z)^k + (1 - z)^{2n+1} \ln(1 - z)O(1)}.
$$

We keep the terms of order $(1 - z)^2$ and of order $(1 - z)$ in the finite sums

$$
F \left( -n - \frac{1}{2}, -n - \frac{1}{2}; 1; z \right) = (2n + 1) \frac{\Gamma(2n + 1)}{\Gamma(n + \frac{3}{2})^2} \sum_{k=0}^{2} \frac{[(-n - \frac{1}{2})^2] k! (1 - z)^k + (1 - z)^3 \ln(1 - z)O(1)},
$$

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and
\[ F\left( \frac{1}{2} - n, \frac{1}{2} - n; 2; z \right) = \frac{\Gamma(2n+1)}{|\Gamma(n+\frac{3}{2})|^2} \sum_{k=0}^{\infty} \frac{[(\frac{1}{2} - n)_k]^2}{(-2n)_k k!} (1 - z)^k (1 - z)^2 \ln(1 - z)|O(1), \]
respectively. On the other hand
\[
(\lambda + 1) \left( \lambda \left( A^2(-\lambda) + \lambda + 2 \right) + 1 \right)^2 = 1 + 5\lambda + (10 - 2A^2) \lambda^2 + (10 - 6A^2) \lambda^3 + O(\lambda^4), \\
(\lambda + 1)^2 - A^2 \lambda^2 = (2n + 1)\lambda + (4n + 2)\lambda^2 + O(\lambda^4),
\]
and
\[
((\lambda + 1)^2 - A^2 \lambda^2)^n - \frac{5}{4} = 1 + (2n - 5)\lambda - \frac{1}{2}(2n - 5)(A^2 - 2n + 6)\lambda^2 + O(\lambda^3).
\]
We substitute in $-(2n + 1)(\lambda + 1)^2 - A^2 \lambda^2)^n - \frac{1}{2} \mathcal{F}(\lambda; n, A)$ the relations \((37), (38), (39), (40)\) and obtain
\[-(2n + 1)(\lambda + 1)^2 - A^2 \lambda^2)^n - \frac{1}{2} \mathcal{F}(\lambda; n, A) = \frac{2n + 1}{\Gamma(n + \frac{3}{2})^2} + \frac{2n + 1}{\Gamma(n + \frac{3}{2})^2} (2A^2 n - 1) \Gamma(n) \lambda^2 + O(\lambda^3).
\]
The lemma is proved. \(\square\)

5.6 The case of \(\ell < 1 \) and \( M = \frac{1}{2} \)

Similar to what was done in the previous subsection, we replace \( \tau \) with \( \lambda = \tau^{-1} \rightarrow 0 \) as \( \tau \rightarrow \infty \).

**Lemma 5.6** In the expansion for small \( \lambda \searrow 0 \) of the function
\[
\frac{\partial}{\partial \lambda} \left( \left( \frac{1}{\lambda} + 1 \right)^2 - A^2 \right) \frac{-\frac{1}{2}}{F\left( \frac{1}{2} \frac{1}{2}; 1; \frac{1}{2} \frac{1}{2} - A^2 \right)} = ((1 - A^2) \lambda^2 + 2 \lambda + 1)^{-\frac{1}{2}} \mathcal{F}(A, \lambda),
\]
there is a term $-4\pi^{-1}(6A^2 + 3)\lambda^2 \ln(\lambda)$, which is depending on \( A \).

**Proof.** Indeed,
\[
\frac{\partial}{\partial \lambda} \left( \left( \frac{1}{\lambda} + 1 \right)^2 - A^2 \right) \frac{-\frac{1}{2}}{F\left( \frac{1}{2} \frac{1}{2}; 1; \frac{1}{2} \frac{1}{2} - A^2 \right)} = ((1 - A^2) \lambda^2 + 2 \lambda + 1)^{-\frac{1}{2}} \mathcal{F}(A, \lambda),
\]
where
\[
\mathcal{F}(A, \lambda) := \left\{ (-A^2 \lambda^3 - A^2 \lambda^2 + \lambda^3 + 3\lambda^2 + 3\lambda + 1)F\left( \frac{1}{2} \frac{1}{2}; 1; \frac{1}{2} \frac{1}{2} - A^2 \right) \right. \\
\left. -(A^2 \lambda^3 - \lambda^3 + \lambda)F\left( \frac{3}{2} \frac{3}{2}; 2; \frac{1}{2} \frac{1}{2} - A^2 \right) \right\}.
\]

Then it is easy to see the following asymptotic expansions
\[
((1 - A^2) \lambda^2 + 2 \lambda + 1)^{-\frac{1}{2}} = 1 - 5\lambda + \frac{5}{2}(A^2 + 6)\lambda^2 - \frac{35}{2}(A^2 + 2)\lambda^3 + O(\lambda^4), \\
1 - \frac{\frac{3}{2} - A^2}{(\frac{3}{2} + 1)^2 - A^2} = 4\lambda - 8\lambda^2 + 4(A^2 + 3)\lambda^3 + O(\lambda^4).\]
Hence, according to \(^{20}\) and \(^{30}\), we have

\[
F\left(\frac{1}{2} \cdot 1; 1; \frac{1}{2} \left(\frac{1}{\lambda} - 1\right)^2 - A^2 \right) = \frac{2 \ln(2) - \ln(\lambda)}{\pi} + \frac{2 \ln(2) - \ln(\lambda)}{\pi} \lambda + \frac{-4A^2 - 9 + 2 \ln(2) - \ln(\lambda)}{4\pi} \lambda^2 + O(\lambda^3)
\]

and

\[
F\left(\frac{3}{2} \cdot \frac{3}{2}; 1; \frac{1}{2} \left(\frac{1}{\lambda} - 1\right)^2 - A^2 \right) = \frac{1}{\pi} + \frac{5 - 2 \ln(2) + \ln(\lambda)}{\pi} + \frac{-4A^2 + 47 - 18 \ln(4) + 18 \ln(\lambda)}{4\pi} \lambda^2 + O(\lambda^3).
\]

It remains to substitute obtained expansions into \(^{11}\) to derive

\[
\mathcal{F}(A, \lambda) = \frac{-1 + \ln(4) - \ln(\lambda)}{\pi} + \frac{5(-1 + \ln(4) - \ln(\lambda))}{\pi} \lambda + \frac{-4A^2 - 4A^2 \ln(4) - 52 + 86 \ln(2) - 43 \ln(\lambda) + 4A^2 \ln(\lambda)}{4\pi} \lambda^2 + O(\lambda^3).
\]

Thus,

\[
\frac{\partial}{\partial \lambda} \left(\left(\frac{1}{\lambda} + 1\right)^2 - A^2\right)^{-\frac{1}{2}} F\left(\frac{1}{2} \cdot \frac{1}{2}; 1; \frac{1}{2} \left(\frac{1}{\lambda} - 1\right)^2 - A^2 \right) = \begin{cases} 1 - 5\lambda + \frac{5}{2} (A^2 + 6) \lambda^2 - \frac{35}{2} (A^2 + 2) \lambda^3 + O(\lambda^4) \\
\times \left\{ \frac{-1 + \ln(4) - \ln(\lambda)}{\pi} + \frac{5(-1 + \ln(4) - \ln(\lambda))}{\pi} \lambda + \frac{-4A^2 - 4A^2 \ln(4) - 52 + 86 \ln(2) - 43 \ln(\lambda) + 4A^2 \ln(\lambda)}{4\pi} \lambda^2 + O(\lambda^3) \right\} 
\end{cases}
\]

Finally, we obtain

\[
\frac{\partial}{\partial \lambda} \left(\left(\frac{1}{\lambda} + 1\right)^2 - A^2\right)^{-\frac{1}{2}} F\left(\frac{1}{2} \cdot \frac{1}{2}; 1; \frac{1}{2} \left(\frac{1}{\lambda} - 1\right)^2 - A^2 \right) = -\frac{- \ln(\lambda) - 1 + \ln(4)}{\pi} + \frac{-(6A^2 + 3) \ln(\lambda) - 14A^2 + 12A^2 \ln(2) - 12 + 6 \ln(2) A^2}{4\pi} \lambda^2 + O(\lambda^3).}
\]

The lemma is proved. \(\Box\)

5.7 The case of \(\ell < 1\) and \(M = -\frac{1}{2}\)

We appeal to the arguments which have been used in subsection \(^{5.5}\) and consider around the point \(\lambda = 0\) the function \((\lambda + 1)^2 - A^2 \lambda^2)^{-\frac{1}{2}} \mathcal{F}(\lambda; 0, A)\), where \(\mathcal{F}(\lambda; 0, A)\) is defined in \(^{26}\).

**Lemma 5.7** In the expansion of the function \(^{76}\) for small \(\lambda \searrow 0\) there is a term \((-2A^2 - 3 + 4 \ln(2))/\pi\) depending on \(A\).

**Proof.** Indeed, we have

\[
\frac{\partial}{\partial \lambda} \left(\left(\frac{1}{\lambda} + 1\right)^2 - A^2\right)^{1/2} F\left(-\frac{1}{2}; 1; 1; \frac{(1/\lambda - 1)^2 - A^2}{(1 + 1/\lambda)^2 - A^2}\right) = \frac{1}{\lambda^2 (A^2 + 2 + 2\lambda + 1)^{3/2}} \mathcal{F}(\lambda, A),
\]

26
Furthermore, according to subsection 5.5, we can write

\[ F(\frac{1}{2}, \frac{1}{2}; 1; z) = \frac{4}{\pi} + \frac{z - 1}{\pi} + \frac{-2 \ln(1-z) - 5 + 8 \ln(2)}{16\pi} (z-1)^2 \]

\[-\frac{3(-2 \ln(1-z) - 5 + 8 \ln(2))}{64\pi} (z-1)^3 + O((z-1)^4)\]

and

\[ F\left(\frac{1}{2}, \frac{1}{2}; 2; z\right) = \frac{4}{\pi} + \frac{4 \ln(2) - \ln(1-z) - 3}{\pi} (z-1) - \frac{3(-6 \ln(1-z) - 17 + 24 \ln(2))}{16\pi} (z-1)^2 \]

\[+ \frac{15(-5 \ln(1-z) - 14 + 20 \ln(2))}{64\pi} (z-1)^3 + O((z-1)^4)\]

We substitute the relation (12) in the expansions for \( F\left(\frac{1}{2}, \frac{1}{2}; 1; z\right) \) and \( F\left(\frac{1}{2}, \frac{1}{2}; 2; z\right) \) and derive

\[ F\left(-\frac{1}{2}, \frac{1}{2}; 1; \left(\frac{1}{\lambda} - 1\right)^2 - A^2\right) \]

\[= \frac{4}{\pi} - \frac{4}{\pi} \lambda + \frac{-2 \ln(\lambda) + 3 + 4 \ln(2)}{\pi} \lambda^2 + \frac{-4A^2 + 2 \ln(\lambda) - 3 - 4 \ln(2)}{\pi} \lambda^3 + O(\lambda^4)\]

\[ F\left(\frac{1}{2}, \frac{1}{2}; 2; \left(\frac{1}{\lambda} - 1\right)^2 - A^2\right) \]

\[= \frac{4}{\pi} - \frac{4(-\ln(\lambda) - 3 + 2 \ln(2))}{\pi} \lambda + \frac{10 \ln(\lambda) + 19 - 20 \ln(2)}{\pi} \lambda^2 \]

\[+ \frac{4A^2 \ln(\lambda) + 16A^2 - 8A^2 \ln(2) + 15 \ln(\lambda) + 26 - 30 \ln(2)}{\pi} \lambda^3 + O(\lambda^4)\]

On the other hand,

\[-\frac{1}{\lambda^2}(-A^2 \lambda^2 + \lambda^2 + 2 \lambda + 1)^{3/2} = -\frac{1}{\lambda^2} + \frac{3}{\lambda} - \frac{3}{2} (A^2 + 4) + \frac{5}{2} (3A^2 + 4) \lambda - \frac{15}{8} (A^4 + 12A^2 + 8) \lambda^2 \]

\[+ \frac{21}{8} (5 (A^2 + 4) A^2 + 8) \lambda^3 + O(\lambda^4)\]

Thus, we obtain

\[ \frac{\partial}{\partial \lambda} \left( \left(\frac{1}{\lambda} + 1\right)^2 - A^2\right)^{1/2} \ F\left(\frac{1}{2}, \frac{1}{2}; 1; \left(\frac{1}{\lambda} - 1\right)^2 - A^2\right) \]

\[= -\frac{4}{\pi \lambda^2} + \frac{-2A^2 - 3 + 4 \ln(2) - 2 \ln(\lambda)}{\pi} + O(\lambda^2)\]

The lemma is proved. \( \square \)

The Huygens' principle's is a local property, that is, it can be verified for the small time. On the other hand, the proof of the necessity part presented in this paper is based on the large time asymptotics. In fact, by taking into account the scaling invariance of the operator (2), the proof can be easily modified to the small time by the implementing a large auxiliary parameter. In this paper, we avoid that modification, which creates an unnecessary cumbersome, even though it could be a useful tool for the further generalizations.
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