General parametrization of black holes: the only parameters that matter

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The general parametrization of a black-hole spacetime in arbitrary metric theories of gravity includes an infinite set of parameters. It is natural to suppose that essential astrophysically observable quantities, such as quasinormal modes, parameters of shadow, electromagnetic radiation and accretion matter in the vicinity of a black hole, must depend only on a few of these parameters. Starting from the parametrization for spherically symmetric configurations in the form of infinite continued fraction, we suggest a compact representation of the asymptotically flat spherically symmetric and slowly rotating black holes in terms of only three and four parameters respectively. This approximate representation of a black-hole metric should allow one to describe physical observables in the region of strong gravity.

Introduction. Recent development of observations of black holes (BHs) in gravitational and electromagnetic spectra promises to determine the near-horizon geometry of BHs in the future and, thereby, to test the Einstein theory and its alternatives in the strong gravity regime \cite{1}. Therefore, it is important to have a general parametrized description of a black-hole spacetime in arbitrary metric theories of gravity, similar in the spirit to the parametrized post-Newtonian (PPN) formalism, and valid not only far from the BH, but in the whole space outside the event horizon (EH). Indeed, such a unified description allows one to consider various phenomena not in each theory of gravity, case by case, but using the general representation, so that constraining of the parameters there would show which theory of gravity is closer to the experimental data. For spherically symmetric BHs this parametrization was suggested in the form of the infinite continued fraction expansion in terms of the compact radial coordinate \cite{2}. It was further extended to the case of axially symmetric BHs \cite{3} and used for finding a number of analytical BH metrics \cite{4} approximating numerical solutions \cite{5-11}. Various phenomena in the background of these parametrized black-hole metrics, such as quasinormal modes (QNMs) \cite{12}, particle motion, Hawking radiation \cite{13} and others, were studied in \cite{14-18}.

In the general case the parametrization includes an infinite number of parameters. However, it is natural to expect that physical quantities, which are potentially observable in astrophysical phenomena around BHs, must depend only on a few of these parameters. In addition, one would not believe that in the true theory of gravity these observable quantities deviate from their Einsteinian values by orders, rather than by, at most, tens of percents. Otherwise, so strong deviations would be visible in the weak-field regime as well. The exception would occur supposing that the BH metric has the Kerr form in the whole space, except a very small region near its surface, where the deviation is strong. Then, such a geometry would be almost indistinguishable from the Kerr one, leaving a weak imprint only in the form of gravitational echoes \cite{19}.

When considering a parametrized approximate representation of some exact BH solution one should formulate the criterion of sufficient accuracy of the approximation. The physical "effect" which must be tested in the course of experiments is the deviation of one or another physical quantity (such as QNMs, parameters of the shadow, etc.) from their Einsteinian values. Therefore it is natural to require that this effect must be at least one order larger than the relative error of the approximation due to the truncation of the infinite series.

In the present paper we consider a great number of examples of BH metrics and show that a spherically symmetric asymptotically BH can be very well approximated by the following line element

\begin{equation}
\begin{aligned}
ds^2 &= -N^2(r)dt^2 + B^2(r)N^{-2}(r)dr^2 + r^2 d\Omega^2, \\
N^2(r) &= 1 - r_0(\epsilon + 1)/r + r_0^3(\epsilon + a_1)/r^3 - r_0^4a_1/r^4, \\
B^2(r) &= (1 + r_0^2b_1/r^2)^2, \\
\end{aligned}
\end{equation}

Here \( r_0 \) is the EH, so that \( N(r_0) = 0; \epsilon, a_1 \) and \( b_1 \) are some parameters, such that when they all are equal to zero, the Schwarzschild limit is reproduced. Within the approximation \cite{1} the deviation of observable quantities are at least one order larger than the relative error. For more accurate approximation, such that the error is two orders smaller than the "effect", one can use a straightforward procedure to introduce additional coefficients, \( a_2 \) and \( b_2 \), in the metric functions. Further we show that this representation can be easily generalized to the case of slowly rotating BHs and mention some approaches on how to extend this description to the arbitrary rotation.

The continued fraction expansion. Following \cite{2}, we use the dimensionless variable \( x \equiv 1 - (r_0/r) \), so that
$x = 0$ corresponds to the EH, while $x = 1$ corresponds to spatial infinity. In addition, we rewrite the metric function $N$ as $N^2 = x A(x)$, where $A(x) > 0$ for $0 \leq x \leq 1$. Using the new parameters $\epsilon$, $a_0$, and $b_0$, the functions $\tilde{A}$ and $\tilde{B}$ can be written as

$$A(x) = 1 - \epsilon (1 - x) + (a_0 - \epsilon) (1 - x)^2 + \tilde{A}(x) (1 - x)^3,$$

$$B(x) = 1 + b_0 (1 - x) + \tilde{B}(x) (1 - x)^2.$$  

Here the coefficient $\epsilon$ measures the deviation of $r_0$ from $2M$, $\epsilon = -1 + (2M/r_0)$. The coefficients $a_0$ and $b_0$ can be seen as combinations of the PPN parameters: $a_0 = (\beta - \gamma)(1 + \epsilon)^2/2$, $b_0 = (\gamma - 1)(1 + \epsilon)/2$. Current observational constraints on the PPN parameters imply $a_0 \sim b_0 \sim 10^{-4}$.

The functions $\tilde{A}$ and $\tilde{B}$ are introduced through infinite continued fraction in order to describe the metric near the horizon (i.e., for $x \simeq 0$),

$$\tilde{A}(x) = \frac{a_1}{1 + \frac{a_2 x}{1 + \ldots}} , \quad \tilde{B}(x) = \frac{b_1}{1 + \frac{b_2 x}{1 + \ldots}},$$  

where $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ are dimensionless constants to be constrained from observations of phenomena near the EH. At the horizon only the first two terms of the expansions survive, $\tilde{A}(0) = a_1$, $\tilde{B}(0) = b_1$, which implies that near the horizon only the lower order terms of the expansions are important.

**Observable quantities.** Conditionally, we could divide physical effects characterizing BHs in the regime of strong gravity into two categories. The first type of physical processes are almost completely determined by the *near-horizon* zone, e.g., thermodynamic properties, Hawking radiation or gravitational echoes at very late times, which appear due to a strong modification of a BH metric in a small region near the horizon. Whatever important and intriguing, none of these effects is likely to be observed for astrophysical BHs in the near future. The second type of physical processes are related to ongoing observations in the electromagnetic and gravitational spectra. Their characteristics are determined by the BH geometry in the region around the peak of the effective potential. For example, the position of the innermost stable circular orbit (ISCO) of the Schwarzschild BH ($x = 2/3$) defines the region, essential for accretion, while the peak of the function $P(x) \equiv (1 - x)^2 x A(x)$ ($x = 1/3$ for the Schwarzschild BH) stipulates the region which is essential for the photon sphere and the position of the shadow cast by a BH as well as for the values of QN frequencies radiated by the BH. This region, located at some distance from the BH, but not much farther than ISCO, we shall call the *radiation region*. In the extreme cases, in which we do not consider here, e.g., for an extremely rotating BH, these two regions may be approaching each other. Here, we construct a compact and simple representation for the BH metric with the help of only a few parameters which would be effective when describing the second class of processes related to plausible astrophysical observations. We shall further call such metrics *moderate* and discuss conditions for the BH to have a moderate metric.

For this purpose we consider the general parametrization [2] and see at how many orders of the continued fraction expansion this parametrization can be truncated in order to describe the above astrophysical processes for a great number of BH metrics. We shall measure all BH parameters in units of the EH. Their particular values in the metrics under consideration are chosen in order to achieve considerable deviation of observables from the Schwarzschild BH. However, this is not always possible, as in some cases, e.g., in Gauss-Bonnet theories, the BH geometry reaches its extremal state already under a very small deviation from the Schwarzschild limit. Our bunch of metrics are given in table 1 in abbreviated forms and includes: two particular examples of BHs in the Einstein-ether theory (Eq. (51) of [20] with $\epsilon_{13}^{(4)} = 0.9$ (Ether1) and Eq. (58) of [20] with $r_0 = 0.9$ (Ether2), the quantum corrected BH obtained by Kazarov and Solodukhin [21] for $a = 0.86$ (KS), a number of regular BH metrics obtained in the context of nonlinear electrodynamics or within other approaches (in the Heinebe-Euler electrodynamics for $Q = 0.1$ and $a = 10^{4}$ [22] (HE), Hayward metric [23] for $q = 0.85$, Bronnikov metric [24] for $M = 0.95$, Bardeen spacetime [25] for $a = 0.5$). Various BHs with a scalar field and higher curvature corrections are considered: Einstein-dilaton-Maxwell BH [26] ($\phi_0 = 0$) for $Q = 1$ (EdM), the BH with a coupling $f(\phi) = \exp(5\phi^2)$ between Maxwell and scalar fields [6, 27] for $P_0 = 0.55$ (EsM), BHs in theories with higher curvature corrections, such as Einstein-Weyl gravity [8, 28] for $p = 1.1$ (E-Weyl), Einstein-dilaton-Gauss-Bonnet [9, 10] for $p = 0.6$ (EdGB) and its generalization to other couplings to the scalar field, Einstein-scalar-Gauss-Bonnet [11, 12] with the coupling $f(\phi) = 1/(4\phi)$ for $p = 0.6$ (EsGB1) and the coupling $f(\phi) = \ln(\phi)/4$ for $p = 0.99$ (EsGB2). In addition, several other examples were studied: Casadio-Fabbri-Mazzacurati BH [29, 30] for $r_0 = 0$ (CFM1) and $r_0/(2M) = 0.99$ (CFM2) in the context of the braneworld model, and Johannsen-Psaltis ad hoc metrics [31] for the nonzero values of $\epsilon_2 = 5$ and $\epsilon_3 = 0.5$ (JP1) and for the only nonzero $\epsilon_{10} = 4 \cdot 10^{-3}$ (JP2).

The simple and illustrative characteristics, which we consider here, are: the radius of the BH shadow $R_\sigma$, which can be found if one calculates the maximum of the function $P(x)$, $r_0/R_\sigma^2 = \max P(x) = P(x_m)$, and the Lyapunov exponent $\lambda$, which depends on the second derivative in the same point $x_m$, $2\lambda^2 r_0^2 = -P(x_m) P''(x_m)/B^2(x_m)$. Eikonal regime of QNMs [32] can be represented through the frequency of the null geodesics and the Lyapunov exponent $\omega = R_\star^{-1}(\ell + 1/2) - i\lambda(n + 1/2) + O(\ell^{-1}).$ (4)

The basic quantity characterizing the effectiveness of the truncation of the continued fraction at a given order is the ratio of the “effect”, that is of the deviation of the...
observable quantity from its Schwarzschild limit for the exact metric, to the relative error due to truncation of its analytic approximation.

The top part of table I contains examples of BHs with moderate metrics. The parameters for each configuration are chosen in order to achieve considerable deviation from the Schwarzschild BH. From the data shown in table I we see that the simplest Äther1 and Äther2 metrics are reproduced exactly already at the first order of approximation, representing a trivial example. For other examples the truncation of the continued fraction at the first order provides the relative error, which is at least one order less than the effect. Second-order approximation further increases the accuracy (usually by one order).

The bottom part of table I is devoted to the examples, for which the error of the first-order approximation is comparable with the effect. We notice that even for near-extremal cases the effect for shadows and Lyapunov exponents (hence for QNMs) are relatively small and hardly exceed 10%. For the EdGB theory the BH geometry deviates only slightly from the Einstein limit, so that even the near extremal BHs allow for effect of only a few percents. This is in agreement with our intuitive definition of such geometries, which deviate from the Schwarzschild BH only near the horizon.

In order to estimate the accurateness of the approximation in the radiation region we also calculate location of the innermost stable circular orbit (ISCO) of a massive particle. In table II we present the orbital frequency at ISCO, $\Omega_{ISCO}$, which is an observable quantity, and the relative errors. Again we see that for the moderate metrics the error is considerably smaller than the effect and the second-order approximation significantly improves the accuracy. The only exception is the E-Weyl BH, which requires further increasing of the approximation orders for the chosen parameter $p = 1.1$. However, such a BH has very small effective mass, being, thereby, not an appropriate object for a viable accretion process.

It is important to understand that relatively large errors for the examples from the bottom of table II is not for the whole range of physical parameters within the corresponding theories. It usually occurs to near extremal BH states, while far from it our approximation still shows a good convergence.

The first-order approximation provides small relative error if the functions $\tilde{A}(x)$ and $\tilde{B}(x)$ are well approximated by a constant, namely, their value at the horizon, $\tilde{A}(0) = a_1$ and $\tilde{B}(x) = b_1$. The approximation by constant is good only if the functions $A(x)$ and $B(x)$ do not change strongly between the horizon and the radiation region. Indeed, for the moderate metrics we observe the relatively slow change of the metric function $A(x)$ and $B(x)$, starting from the EH and until the radiation zone.

We present an illustration on Fig. 1 where we compare moderate metrics, JP1 and Bardeen, with rather an artificial example JP2 with vanishing all the lower $\epsilon_i$ ($i < 10$), except for an extremely high $\epsilon_{10}$. This example is designed to understand the cases which cannot be effectively described by our approach. The JP1 and Bardeen metrics are very well approximated already at the first order (see table II). Thus, only three parameters $a_1$, $b_1$ and $\epsilon$ are sufficient to describe the observable quantities. JP2 geometry, however, has the metric functions which change strongly near the EH and reaches its asymptotic regime at a relatively short distance from the BH. This is appropriate to spacetimes representing, for
FIG. 1. Plots of \( A(x) \) (top panel) and \( P(x) \equiv (1 - x)^2 \pi \alpha A(x) \) (bottom panel) for JP2 BH (red, top), JP1 BH (blue), Schwarzschild BH (black), and Bardeen BH (green, bottom).

example, various modifications of the BH geometry in the near-horizon region only due to quantum corrections or other new physics near the surface of the compact object. Such spacetimes are not likely to be distinguished from the Einsteinian BHs in the near future, because their geometries can be tested either through direct observation of Hawking radiation, or, still elusive, echoes at very late times. Although such a behavior, with a sudden change only in the near-horizon region, cannot be excluded, it should be considered as much less probable, and, what is more important, such cases will not essentially influence the basic astrophysically observable quantities.

Thus, we conclude that for the astrophysically relevant observational phenomena (shadows, QNMs, accretion) the BH with moderate errors are the most important targets. Although for a stress test of our approach we considered here general PN behavior, one can practically neglect the BHs charge and choose \( a_0 = b_0 = 0 \). The line element of such BHs can be well approximated by \( (4) \).

In this case the problem of computation of \( x_m \) is reduced to the solution of the quadratic equation, hence the shadow radius \( R_s \) can be found in a closed but cumbersome form. It depends almost linearly on \( a_1 \) and decreases as \( a_1 \) grows,

\[
\frac{R_s^2}{R_s^2} = \frac{(1 - x_0)^2 x_0^2 (2 x_0 - 3)}{5 x_0^2 - 10 x_0 + 2} + (1 - x_0)^3 x_0 a_1 + (1 - 6 x_0)^2 (1 - x_0)^8 (5 x_0^2 - 10 x_0 + 2) x_0^2 + 12 (5 x_0^2 - 10 x_0 + 5) x_0 - 1 \equiv O(a_2^2),
\]

where \( x_0 \) is the compact coordinate for the photon circular orbit, satisfying the cubic equation, \( 1 - 2 e - 3 \{1 - (4/3) e \} x_0 - 15 e x_0^2 + 5 e x_0^3 = 0 \), and monotonically increases with \( e \). For small \( e \) we find \( x_0 = (1/3) + (14/81) e + (154/729) e^2 + (3122/19683) e^3 + O(e^4) \). Similarly, for the Lyapunov exponent \( \lambda \) we obtain

\[
\lambda^2 \equiv \frac{3(5 x_0^3 - 10 x_0^2 + 5 x_0 - 1)}{(5 x_0^3 - 10 x_0^2 + 2)^2 (1 + b_1(1 - x_0)^2)^2} (1 + x_0)^2 (120 x_0^3 - 255 x_0^2 + 145 x_0^2 - 47 x_0 + 7) - \frac{2 (5 x_0^3 - 10 x_0^2 + 5 x_0 - 1) (1 + b_1(1 - x_0)^2)^2}{(5 x_0^3 - 10 x_0^2 + 5 x_0 - 1) (1 + b_1(1 - x_0)^2)^2} b_1 a_1 + O(a_2^2).
\]

Since both quantities depend almost linearly on \( a_1 \), one can expect that the error due to the approximation remains one order smaller than the effect as long as the metric stays moderate. If one needs to achieve the approximation in which the error would be two orders less than the effect, the second order can be used via consideration of non-zero \( a_2 \) and \( b_2 \) in \( (3) \).

The approximation \( (4) \) can be extended to the small rotation regime as \( ds^2 = -dt^2 - (M \sin^2 \theta / r) d\Phi d\theta \), which implies that corrections owing to the modification of gravity must be much larger than those due to rotation, i.e., \( a/M << a_1, b_1 \) but also that the second order corrections given by \( a_2 \) and \( b_2 \) are negligible. Thus, in the hierarchy of corrections, the above \( \sim d\Phi d\theta \)-term is between the first- and second-order corrections in the radial direction.

In the case of generic rotation, we see no way to make the approximation as simple as in the spherical case. However, one can notice that a simpler parametrization is possible \( (3) \) for the subclass of Carter’s metrics \( (4) \), which can be obtained through the Newman-Janis algorithm \( (4) \). In order to find a similar simple parametrization for highly rotating BHs we need many more examples of axially-symmetric BH solutions in modified theories of gravity, which are currently lacking.

Conclusions. We have shown that, in order to estimate astrophysical observable quantities of spherically symmetric and slowly rotating BHs, it is sufficient to parameterize the spacetime of the BH with only three parameters. This compact form can considerably simplify further modelling of astrophysical phenomena in the background of a generic BH and should help to constrain the BH geometry in the future. Compact objects which are characterized by a sudden change of the metric functions in the near-horizon zone only, as, for example, various BH mimickers similar to Damour-Solodukhin wormholes \( (4) \) or BHs with quantum corrections owing to the cloud of quantized fields in the Plank scale region near the EH, cannot be well approximated in this way, and, at the same time, are not likely to affect processes of radiation in the gravitational and electromagnetic spectra (such as shadows, gravitational waves and accretion) which are observed nowadays.

Acknowledgments. R. A. K. thanks 9-03950S GACR grant and “RUDN University Program 5-100”. A. Z. was supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq).
