Existence and uniqueness of fractional mixed type integro-differential equations with non-instantaneous impulses through sectorial operator

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Abstract
The key purpose of this paper is to examine the existence and uniqueness of $PC$-mild solution of fractional order mixed type integro-differential equations with non-instantaneous impulses in Banach space through sectorial operator. Based on the Banach contraction principle, we develop the main results. At the end, an example is offered to explain theoretical outcomes.

Keywords
Fractional differential equations, mild solution, non-instantaneous impulses, fixed point theorem.

AMS Subject Classification
34K30, 35R12, 26A33.

1. Introduction

There are many areas of study in which the question of fractional differential equations has recently arisen as a significant method for modeling real-world problems. It has great uses in a variety of sciences and diverse fields of study, such as mechanics, polymer rheology, frequent variance of thermodynamics, blood flow phenomenon, etc. For additional information, see [1, 2] and the references in it.

Motivated by [1, 3, 4], in this paper we consider a class of fractional order mixed type integro-differential systems with non-instantaneous impulses of the form

$$D^\alpha x(t) = Ax(t) + J^{1-\alpha} f (t,x(t),K_1 x(t),K_2 x(t)),$$

where $D^\alpha$ is the Caputo fractional derivative of order $0 < \alpha \leq 1$, $J^{1-\alpha}$ is Riemann-Liouville fractional integral operator and $J = [0,T]$ is operational interval. The map $A : D(A) \subset X \to X$ is a nonlinear function and the functions $K_1$ and $K_2$ are defined by

$$K_1 x(t) = \int_0^t u(t,s,x(s))ds \quad \text{and} \quad K_2 x(t) = \int_0^T \tilde{u}(t,s,x(s))ds,$$

where $x(0) = x_0$.

References

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with the existence results of problem (1.1). An example is given in Section 4 to illustrate the results.

### 2. Preliminaries

Let us set \( J = [0, T], I_0 = [0, t_1], J_1 = (t_1, t_2], \ldots, J_{m-1} = (t_{m-1}, t_m], J_m = (t_m, t_{m+1}] \) and introduce the space \( PC(J, X) := \{ u : J \to X | u \in C(J, X), k = 0, 1, 2, \ldots, m, \text{ and there exist } u(t^+_k) \text{ and } u(t^-_k), k = 1, 2, \ldots, m, \text{ with } u(t^+_k) = u(t^-_k) \}. \) It is clear that \( PC(J, X) \) is a Banach space with the norm \( \| u \|_{PC} = \sup \{ \| u(t) \| : t \in J \}. \)

Let us recall the following definition of mild solutions for fractional evolution equations involving the Caputo fractional derivative.

**Definition 2.1.** [5] Caputo’s derivative of order \( \alpha > 0 \) with lower limit \( a, \) for a function \( f : [a, \infty) \to \mathbb{R} \) is defined as

\[
C^\alpha_a f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds = aP^{-\alpha} f^{(n)}(t)
\]

where \( n \geq \alpha > 0, n \in \mathbb{N}. \) The Laplace transform of the Caputo derivative of order \( \alpha > 0 \) is given as

\[
L \{ C^\alpha_a f(t); \lambda \} = \lambda^\alpha \tilde{f}(\lambda) - \sum_{k=0}^{n-1} \frac{\lambda^{\alpha-k} f^{(k)}(0)}{\Gamma(k+1)} - \frac{\lambda^{\alpha-n} f^{(n)}(0)}{\Gamma(n-\alpha)}, \quad n-1 < \alpha \leq n.
\]

**Definition 2.2.** [3] A closed and linear operator \( A \) is said to be sectorial if there are constants \( \omega \in \mathbb{R}, \theta \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right], M_A > 0, \) such that the following two conditions are satisfied:

1. \( \sum_{\theta, \omega} = \{ \lambda \in C : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta \} \subset \rho(A) \)
2. \( \| R(\lambda, A) \|_{L(X)} \leq \frac{M_A}{|\lambda - \omega|}, \lambda \in \sum_{\theta, \omega}, \)

where \( X \) is the complex Banach space with norm denoted \( \| \cdot \|_X. \)

**Lemma 2.3.** [3] Let \( f \) satisfies the uniform Holder condition with exponent \( \beta \in (0, 1) \) and \( A \) is a sectorial operator. Consider the fractional equations of order \( 0 < \alpha < 1 \)

\[
C^\alpha_a x(t) = Ax(t) + J^{1-\alpha} f(t), \quad t \in J = [a, T].
\]

\( a \geq 0, x(a) = x_0. \) Then a function \( x(t) \in C([a, T], X) \) is the solution of the equation (2.1) if it satisfies the following integral equation

\[
x(t) = Q_\alpha(t-a)x_0 + \int_a^t Q_\alpha(t-s)f(s)ds\]

where \( Q_\alpha(t) \) is solution operator generated by \( A \) defined as

\[
Q_\alpha(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{\lambda t} \lambda^{\alpha-1} (\lambda I - A)^{-1} d\lambda
\]

\( \Gamma \) is a suitable path lying on \( \sum_{\theta, \omega}. \)

**Remark 2.4.** If \( A \in \mathcal{A}'^\alpha(\theta_0, \omega_0), \) then strongly continuous \( \| Q_\alpha(t) \| \leq M_A e^{\omega t}. \) Let \( M_Q := \sup_{0 \leq t \leq T} \| Q_\alpha(t) \|_{L(X)}. \) So we have \( \| Q_\alpha(t) \|_{L(X)} \leq M_Q. \)

Now, we recall the following important Lemma which is very useful to prove our main result.

**Definition 2.5.** A function \( x \in PC(J, X) \) is said to be a \( PC \) solution of problem (1.1) if it satisfies the following relation:

\[
x(t) = Q_\alpha(t) x_0 + \int_0^t Q_\alpha(t-s) f(s, x(s), K_1 x(s), K_2 x(s))ds,
\]

\( t \in J = [0, T], g_i(t, x(t)), \) for all, \( i = 1, 2, \ldots, m. \)

**3. Existence Results**

In this section, we present and prove the existence and uniqueness of the system (1.1) under Banach contraction principle fixed point theorem.

From Definition 2.3, we define an operator \( \Upsilon : PC(J, X) \to PC(J, X) \) as where

\[
(\Upsilon x)(t) = Q_\alpha(t) x_0 + \int_0^t Q_\alpha(t-s) f(s, x(s), K_1 x(s), K_2 x(s))ds,
\]

\( t \in (s, t_1]. \)

\[
\delta(t, x, y) = \| f(t, x, y) - f(t, y, \tilde{y}) \| \leq L_{f_1} \| x_1 - y_1 \| + L_{f_2} \| x_2 - y_2 \| + L_{f_3} \| x_3 - y_3 \|
\]

for all \( (x_1, x_2, x_3), (y_1, y_2, y_3) \in X \) and every \( t \in J. \)

**H(\alpha, \mu)** The functions \( u, \tilde{u} : \Delta \times X \to X \) are continuous and there exist constants \( L_u, L_{\tilde{u}} > 0 \) such that

\[
\| u(t, x, y) - u(t, s, \tilde{y}) \| \leq L_u \| x - y \|
\]

for all, \( x, y \in X; \) and

\[
\| \tilde{u}(t, x, y) - \tilde{u}(t, s, \tilde{y}) \| \leq L_{\tilde{u}} \| x - y \|
\]

for all, \( x, y \in X; \)
(H(g)) For \( i = 1, 2, \ldots, m \), the functions \( g_i \in C ([s_i, s_{i+1}] \times X; X) \) and there exists \( L_{g_i} \in C (J, \mathbb{R}^+) \) such that

\[
\| g_i(t, x) - g_i(t, y) \| \leq L_{g_i} \| x - y \|
\]

for all \( x, y \in X \) and \( t \in [s_i, s_{i+1}] \).

**Theorem 3.1.** Expect that the hypotheses \( H(f), H(u, \tilde{u}) \) and \( H(g) \) hold, then problem (1.1) has a unique PC-mild solution \( x \in X \) provided

\[
\Lambda = \max_{1 \leq s_i \leq m} \left[ \tilde{M}Q_1L_{g_1} + \tilde{M}Q(t_{i+1} - s_i)L_{f_1} + L_{f_2}L_{u_1} + L_{f_3}L_{u_1} \right] < 1.
\]

**Proof.** For any \( x, y \in PC(J, X) \), by (3.1) and \( t \in [0, t_1] \), we obtain

\[
\| Yx(t) - Yy(t) \| = \left\| \int_0^t Q_1(t-s)f(s, x(s), K_1x(s), K_2x(s))ds \right\| \leq \left\| \int_0^t Q_2(t-s)f(s, y(s), K_1y(s), K_2y(s))ds \right\|
\]

\[
\leq \int_0^t \| Q_2(t-s) \| \left\| f(s, x(s), K_1x(s), K_2x(s)) \right\| ds \leq \tilde{M}Q_2 \int_0^t \| L_{f_1} + L_{f_2}L_{u_1} + L_{f_3}L_{u_1} \| \| x - y \|_{PC} ds
\]

For \( t \in (s_i, s_{i+1}] \), we have

\[
\| Yx(t) - Yy(t) \| \leq L_{g_i} \| x - y \|_{PC}.
\]

For \( t \in [s_i, t_{i+1}] \), we get

\[
\| Yx(t) - Yy(t) \| \leq \left\| Q_1(t - s_i)g(s_i, x(s_i)) + \int_{s_i}^t Q_1(t-s)f(s, x(s), K_1x(s), K_2x(s))ds \right\| \leq \tilde{M}Q_1L_{g_1} \| x - y \|_{PC} + \tilde{M}Q_1 \int_0^t \| Q_2(t-s) \| \left\| f(s, x(s), K_1x(s), K_2x(s)) \right\| ds \leq \tilde{M}Q_1L_{g_1} \| x - y \|_{PC} + \tilde{M}Q_1 \int_0^t \| Q_2(t-s) \| \left\| f(s, y(s), K_1y(s), K_2y(s)) \right\| ds \leq \tilde{M}Q_1L_{g_1} \| x - y \|_{PC} + \tilde{M}Q_1 \int_0^t \| Q_2(t-s) \| \left\| f(s, y(s), K_1y(s), K_2y(s)) \right\| ds \]

Thus

\[
\| Yx - Yy \| \leq \Lambda \| x - y \|_{PC}.
\]

Therefore, the operator \( Y \) is a contraction on \( PC(J, X) \) and there exists a unique mild solution of the system (1.1).

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4. **Application**

In this section, we are going to present an example to validate the results established in Theorem 3.1. Consider the fractional partial integro-differential equation of the form

\[
D^\alpha x(t, z) = \frac{\partial^2 x}{\partial z^2} + J^{1-\alpha}P(t, x(t, z)),
\]

\[
\int_0^t u(t, s, x(s, z))ds, \quad \int_0^T \tilde{u}(t, s, x(s, z))ds
\]

a.e. \((t, z) \in \cup_{i=1}^m [s_i, t_{i+1}] \times [0, \pi] \)

(4.1)

\[
x(t, 0) = x(t, \pi) = 0, \quad t \in [0, T]
\]

(4.2)

\[
x(0, z) = x_0(z), \quad z \in [0, \pi]
\]

(4.3)

\[
x(t, z) = G_i(t, x(t, z)), \quad t \in (s_i, s_{i+1}], \quad z \in [0, \pi],
\]

(4.4)

where \( 0 = t_0 = s_0 = s_1 < s_2 < \ldots < s_m < t_{m+1} = T, P \in C ([0, T] \times \mathbb{R}^3, \mathbb{R}) \) and \( G_i \in C ([s_i, s_{i+1}] \times \mathbb{R}, \mathbb{R}) \) for all \( i = 1, 2, \ldots, m \).

Let \( X = L^2([0, \pi]) \). Define an operator \( A : D(A) \subseteq X \rightarrow X \) by \( Ax = x'' \) with \( D(A) = \{ x \in X : x(0) = x(\pi) = 0 \} \).

It is well known that \( A \) is the infinitesimal generator of an analytic semigroup \( \{ T(t) : t \geq 0 \} \) in \( X \). Moreover, the subordination principle of solution operator implies that \( A \) is the infinitesimal generator of a solution operator \( \{ Q(t) \}_{t \geq 0} \) such that \( \| Q(t) \|_{L(X)} \leq Q_\delta \) for \( t \in [0, 1] \).

Set

\[
x(t)z = x(t, z), \quad J^{1-\alpha}f(t, x(t), K_1x(t), K_2x(t))z = J^{1-\alpha}P(t, x(t, z)) \int_0^T u(t, s, x(s, z))ds, \quad \int_0^T \tilde{u}(t, s, x(s, z))ds
\]

Thus with this set-up, equations (4.1)-(4.4) can be written in the abstract form for the system (1.1).

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