Anti-van der Waerden numbers of 3-term arithmetic progressions.

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April 24, 2016

Abstract

The anti-van der Waerden number, denoted by \( aw([n], k) \), is the smallest \( r \) such that every exact \( r \)-coloring of \([n]\) contains a rainbow \( k \)-term arithmetic progression. Butler et. al. showed that \( \lceil \log_3 n \rceil + 2 \leq aw([n], 3) \leq \lceil \log_2 n \rceil + 1 \), and conjectured that there exists a constant \( C \) such that \( aw([n], 3) \leq \lceil \log_3 n \rceil + C \). In this paper, we show this conjecture is true by determining \( aw([n], 3) \) for all \( n \). We prove that for \( 7 \cdot 3^m - 2 + 1 \leq n \leq 21 \cdot 3^m - 2 \),

\[
aw([n], 3) = \begin{cases} 
m + 2, & \text{if } n = 3^m 
m + 3, & \text{otherwise.}
\end{cases}
\]

Keywords. arithmetic progression; rainbow coloring; unitary coloring; Behrend construction.

1 Introduction

Let \( n \) be a positive integer and let \( G \in \{[n], \mathbb{Z}_n\} \), where \([n] = \{1, \ldots, n\} \). A \textit{k-term arithmetic progression} (\( k \)-AP) of \( G \) is a sequence in \( G \) of the form

\[
a, a + d, a + 2d, \ldots, a + (k - 1)d,
\]

where \( d \geq 1 \). For the purposes of this paper, an arithmetic progression is referred to as a set of the form \( \{a, a + d, a + 2d, \ldots, a + (k - 1)d\} \). An \textit{r-coloring} of \( G \) is a function \( c : G \to [r] \), and such a coloring is called \textit{exact} if \( c \) is surjective. Given \( c : G \to [r] \), an arithmetic progression is called \textit{rainbow} (under \( c \)) if \( c(a + id) \neq c(a + jd) \) for all \( 0 \leq i < j \leq k - 1 \).

The \textit{anti-van der Waerden number}, denoted by \( aw(G, k) \), is the smallest \( r \) such that every exact \( r \)-coloring of \( G \) contains a rainbow \( k \)-AP. If \( G \) contains no \( k \)-AP, then \( aw(G, k) = |G| + 1 \); this is consistent with the property that there is a coloring of \( G \) with \( aw(G, k) - 1 \) colors that has no rainbow \( k \)-AP.

An \( r \)-coloring of \( G \) is \textit{unitary} if there is an element of \( G \) that is uniquely colored. The smallest \( r \) such that every exact unitary \( r \)-coloring of \( G \) contains a rainbow \( k \)-AP is denoted by \( aw_u(G, k) \). Similar to the anti-van der Waerden number, \( aw_u(G, k) = |G| + 1 \) if \( G \) has no \( k \)-AP.

Problems involving counting and the existence of rainbow arithmetic progressions have been well-studied. The main results of Axenovich and Fon-Der-Flaass [1] and Axenovich and Martin [2] deal with the existence of 3-APs in colorings that have uniformly sized color classes. Fox,

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Jungić, Mahdian, Nešetril, and Radoičić also studied anti-Ramsey results of arithmetic progressions in [6]. In particular, they showed that every 3-coloring of $[n]$ for which each color class has density more than $1/6$, contains a rainbow 3-AP. Fox et. al. also determined all values of $n$ for which $aw(Z_n, 3) = 3$.

The specific problem of determining anti-van der Waerden numbers for $[n]$ and $Z_n$ was studied by Butler et. al. in [4]. It is proved in [4] that for $k \geq 4$, $aw([n], k) = n^{1-o(1)}$ and $aw(Z_n, k) = n^{1-o(1)}$. These results are obtained using results of Behrend [3] and Gowers [5] on the size of a subset of $[n]$ with no $k$-AP. Butler et. al. also expand upon the results of [6] by determining $aw(Z_n, 3)$ for all values of $n$. These results were generalized to all finite abelian groups in [7]. Butler et. al. also provides bounds for $aw([n], 3)$, as well as many exact values (see Table 1).

$$
\begin{array}{c|cccccccccccc}
\text{n \backslash k} & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
3 & 3 & & & & & & & & & & & \\
4 & 4 & & & & & & & & & & & \\
5 & 4 & 5 & & & & & & & & & & \\
6 & 4 & 6 & & & & & & & & & & \\
7 & 4 & 6 & 7 & & & & & & & & & \\
8 & 5 & 6 & 8 & & & & & & & & & \\
9 & 4 & 7 & 8 & 9 & & & & & & & & \\
10 & 5 & 8 & 9 & 10 & 11 & & & & & & & \\
11 & 5 & 8 & 9 & 10 & 11 & 12 & & & & & & \\
12 & 5 & 8 & 10 & 11 & 12 & 13 & 14 & & & & & \\
13 & 5 & 8 & 11 & 11 & 12 & 13 & 14 & & & & & \\
14 & 5 & 8 & 11 & 12 & 13 & 14 & 14 & 15 & & & & \\
15 & 5 & 9 & 11 & 13 & 14 & 14 & 15 & & & & & \\
16 & 5 & 9 & 12 & 13 & 15 & 15 & 16 & 16 & 17 & & & \\
17 & 5 & 9 & 13 & 13 & 15 & 16 & 16 & 17 & & & & \\
18 & 5 & 10 & 14 & 14 & 16 & 17 & 17 & 18 & & & & \\
19 & 5 & 10 & 14 & 15 & 17 & 17 & 18 & 18 & 19 & & & \\
20 & 5 & 10 & 14 & 16 & 17 & 18 & 19 & 19 & 20 & 20 & 21 & \\
21 & 5 & 11 & 14 & 16 & 17 & 19 & 20 & 20 & 20 & 20 & 21 & \\
22 & 6 & 12 & 14 & 17 & 18 & 20 & 21 & 21 & 21 & 22 & & \\
23 & 6 & 12 & 14 & 17 & 19 & 20 & 21 & 22 & 22 & 22 & 23 & \\
24 & 6 & 12 & 15 & 18 & 20 & 20 & 22 & 23 & 23 & 23 & 24 & \\
25 & 6 & 12 & 15 & 19 & 21 & 21 & 23 & 23 & 24 & 24 & 24 & 25 \\
\end{array}
$$

Table 1: Values of $aw([n], k)$ for $3 \leq n \leq \frac{n+3}{2}$.

In this paper, we determine the exact value of $aw([n], 3)$, which answers questions posed in [4] and confirms the following conjecture:

**Conjecture 1.** [4] There exists a constant $C$ such that $aw([n], 3) \leq \lceil \log_3 n \rceil + C$, for all $n \geq 3$.

Our main result, Theorem 2, also determines $aw_u([n], 3)$ which shows the existence of extremal colorings of $[n]$ that are unitary.

**Theorem 2.** For all integers $n \geq 2$,

$$
aw_u([n], 3) = \begin{cases}
  m + 2, & \text{if } n = 3^m \\
  m + 3, & \text{if } n \neq 3^m \text{ and } 7 \cdot 3^{m-2} + 1 \leq n \leq 21 \cdot 3^{m-2}.
\end{cases}
$$

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In section 2, we provide lemmas that are useful in proving Theorem 2 and section 3 contains the proof of Theorem 2.

2 Lemmas

In [4, Theorem 1.6] it is shown that $3 \leq \text{aw}(Z_p, 3) \leq 4$ for every prime number $p$ and that if $\text{aw}(Z_p, 3) = 4$ then $p \geq 17$. Furthermore, it is shown that the value of $\text{aw}(Z_n, 3)$ is determined by the values of $\text{aw}(Z_p, 3)$ for the prime factors $p$ of $n$. We have included this theorem below with some notation change.

**Theorem 3.** [4] Let $n$ be a positive integer with prime decomposition $n = 2^{e_0}p_1^{e_1}p_2^{e_2} \cdots p_s^{e_s}$ for $e_i \geq 0$, $i = 0, \ldots, s$, where primes are ordered so that $\text{aw}(Z_p, 3) = 3$ for $1 \leq i \leq \ell$ and $\text{aw}(Z_{p_i}, 3) = 4$ for $\ell + 1 \leq i \leq s$. Then

$$\text{aw}(Z_n, 3) = \begin{cases} 
2 + \sum_{j=1}^{\ell} e_j + \sum_{j=\ell+1}^{s} 2e_j, & \text{if } n \text{ is odd} \\
3 + \sum_{j=1}^{\ell} e_j + \sum_{j=\ell+1}^{s} 2e_j, & \text{if } n \text{ is even.}
\end{cases}$$

We use Theorem 3 to prove the following lemma.

**Lemma 4.** Let $n \geq 3$, then $\text{aw}(Z_n, 3) \leq \lceil \log_3 n \rceil + 2$ with equality if and only if $n = 3^j$ or $2 \cdot 3^j$ for $j \geq 1$.

**Proof.** Suppose $n = 2^{e_0}p_1^{e_1}p_2^{e_2} \cdots p_s^{e_s}$ with $e_i \geq 0$ for $i = 0, \ldots, s$, where primes $p_1, p_2, \ldots, p_s$ are ordered so that $\text{aw}(Z_p, 3) = 3$ for $1 \leq i \leq \ell$ and $\text{aw}(Z_{p_i}, 3) = 4$ for $\ell + 1 \leq i \leq s$. We consider two cases depending on parity of $n$.

**Case 1.** Suppose $n$ is odd, that is $e_0 = 0$. Then $\text{aw}(Z_n, 3) = 2 + \sum_{j=1}^{\ell} e_j + \sum_{j=\ell+1}^{s} 2e_j$ by Theorem 3. Since $\text{aw}(Z_p, 3) = 3$ for odd primes $p \leq 13$, we have $p_i \geq 17$ for $i \geq \ell + 1$, and clearly $p_i \geq 3$ for $i \leq \ell$, therefore

$$3^{\text{aw}(Z_n, 3)} = 3^{2 + \sum_{j=1}^{\ell} e_j + \sum_{j=\ell+1}^{s} 2e_j} = 9 \cdot 3^{e_1} \cdot 3^{e_2} \cdots 3^{e_{\ell+1}} \cdot 9^{e_{\ell+1}} \cdots 9^{e_s} \leq 9 \cdot p_1^{e_1} \cdots p_s^{e_s} = 9n.$$

Note that the equality holds if and only if $n$ is a power of 3, that is $e_j = 0$ for $2 \leq j \leq s$. Therefore, $\text{aw}(Z_n, 3) \leq \lceil \log_3 n \rceil + 2$ for odd $n$, with equality if and only if $n = p_1^{e_1}$.

**Case 2.** Suppose $n$ is even, that is $e_0 \geq 1$. Then $\text{aw}(Z_n, 3) = 3 + \sum_{j=1}^{\ell} e_j + \sum_{j=\ell+1}^{s} 2e_j$ by Theorem 3. If $n = 2^{e_0} \cdot 3^j$ for $j \geq 1$, then by direct computation $\text{aw}(Z_n, 3) = 3 + j \leq 2 + \lceil \log_3 n \rceil$, with equality if and only if $e_0 = 1$. So suppose there is $i$ such that $p_i \neq 3$, and let $h = \frac{n}{2^{e_0}p_i}$. If $i \leq \ell$ then $p_i \geq 5$, and so $3 \cdot 3^{e_i} < 2^{e_0}p_i^{e_i}$ for all $e_0 \geq 1$ and $e_i \geq 1$. Therefore, since $h$ is odd, by the previous case

$$3^{\text{aw}(Z_n, 3)} = 3 \cdot 3^{e_i} \cdot 3^{\text{aw}(Z_h, 3)} \leq 3 \cdot 3^{e_i} \cdot 9h < 2^{e_0}p_i^{e_i} \cdot 9h = 9n.$$

If $i \geq \ell + 1$ then $p_i \geq 17$, and so $3 \cdot 9^{e_i} < 2^{e_0}p_i^{e_i}$ for all $e_0 \geq 1$ and $e_i \geq 1$. Then by the previous case

$$3^{\text{aw}(Z_n, 3)} = 3 \cdot 3^{e_i} \cdot 3^{\text{aw}(Z_h, 3)} \leq 3 \cdot 3^{e_i} \cdot 9h < 2^{e_0}p_i^{e_i} \cdot 9h = 9n.$$
Lemma 5. Let $N$ be an integer and $c$ be an exact $r$-coloring of $[N]$ with no rainbow $3$-AP, where $1$ and $N$ are colored uniquely. Then either the coloring $c$ is special or $|\{c(x) : x \equiv i \pmod{3} \text{ and } x \in [N]\}| \geq r - 1$ for $i = 1$ or $i = N$.

Proof. Observe that $N$ is even, otherwise $\{1, (N + 1)/2, N\}$ is a rainbow $3$-AP. We partition the interval $[N]$ into four subintervals $I_1 = \{1, \ldots, [N/4]\}$, $I_2 = \{[N/4] + 1, \ldots, N/2\}$, $I_3 = \{N/2 + 1, \ldots, 3[N/4]\}$, and $I_4 = \{3[N/4] + 1, \ldots, N\}$. Notice that every color other than $c(1)$ and $c(N)$ must be used in the subinterval $I_2$. To see this, assume $i$ is the missing color in $I_2$ distinct from $c(1)$ and $c(N)$. Let $x$ be the largest integer in $c_i(I_1)$. Since $N$ is even, we have $2x - 1 \leq 2[N/4] - 1 \leq N/2$, and so $2x - 1 \in I_2$ and $c(2x - 1) \neq i$. Therefore the $3$-AP $\{1, x, 2x - 1\}$ is a rainbow. If there is no such integer $x$ in $I_1$, then the integers colored with $i$ must be in the second half of the interval $[N]$, so we choose the smallest such integer $y$ in $c_i(I_3 \cup I_4)$. Then $\{2y - N, y, N\}$ is a rainbow $3$-AP since $c(2y - N) \neq i$, because $2y - N \in I_1 \cup I_2$. Similarly, every color other than $c(1)$ and $c(N)$ must be used in the subinterval $I_3$.

Throughout the proof we mostly drop (mod 3) and just say congruent even though we mean congruent modulo 3. We consider the following three cases.

Case 1: $N \equiv 0 \pmod{3}$. Assume $|\{c(x) : x \equiv i \pmod{3} \text{ and } x \in [N]\}| < r - 1$ for both $i = 1$ and $i = N$. So there are two colors, say $\text{red}$ and $\text{blue}$, such that no integer in $[N]$ colored with $\text{red}$ is congruent to 1, and no integer in $[N]$ colored with $\text{blue}$ is congruent to 0. We further partition the interval $I_2$ into subintervals $I_{2(i)}$ and $I_{2(ii)}$ so that $\ell(I_{2(i)}) \leq \ell(I_{2(ii)}) \leq \ell(I_{2(ii)}) + 1$, and partition the interval $I_3$ into subintervals $I_{3(i)}$ and $I_{3(ii)}$ so that $\ell(I_{3(i)}) \leq \ell(I_{3(ii)}) \leq \ell(I_{3(ii)}) + 1$. Then we have the following observations:

(i) $x \equiv 0$ for all $x \in c_{\text{red}}(I_3 \cup I_4)$ and $y \equiv 1$ for all $y \in c_{\text{blue}}(I_1 \cup I_2)$.

If there is an integer $r$ in $I_3 \cup I_4$ colored with $\text{red}$ and congruent to 2, then $2r - N \equiv 1$, and so $c(2r - N)$ is not $\text{red}$ by our assumption. Therefore the $3$-AP $\{2r - N, r, N\}$ is rainbow. Similarly, if there is an integer $b$ in $I_1 \cup I_2$ colored with $\text{blue}$ and congruent to 2, then $2b - 1 \equiv 0$, and so $c(2b) \neq \text{blue}$, forming a rainbow $3$-AP $\{1, b, 2b - 1\}$.

(ii) $x \equiv 2$ for all $x \in c_{\text{red}}(I_2)$ and $y \equiv 2$ for all $y \in c_{\text{blue}}(I_3)$.

If there is an integer $r$ in $c_{\text{red}}(I_2)$ congruent to 0, then $2r - 1 \equiv 2$ and $2r - 1 \in I_3 \cup I_4$ since $2r - 1 \geq N/2 + 1$. Therefore, $2r - 1$ is not colored with $\text{red}$ by the previous observation, and so the $3$-AP $\{1, r, 2r - 1\}$ is a rainbow. Similarly, if there is an integer $b$ in $c_{\text{blue}}(I_3)$ congruent to 1, then using $N$ we obtain the rainbow $3$-AP $\{2b - N, b, N\}$, because $2b - N \equiv 2$ and $2b - N \leq N/2$.

(iii) $c_{\text{red}}(I_{3(ii)}) = c_{\text{blue}}(I_{2(i)}) = \emptyset$.

If there is an integer $r$ in $I_{3(ii)}$ colored with $\text{red}$, then $2r - N \equiv 0$, by observation (i). Furthermore, $2r - N \leq N/2$ and $2r - N \geq 2(N/2 + \ell(I_{3(ii)}) + 1) - N \geq (2\ell(I_{3(ii)}) + 1) + 1 \geq [N/4] + 1$. So...
2r − N ∈ I_2 and hence it is not colored with red by observation (ii). Therefore, \{2r − N, r, N\} is a rainbow 3-AP. Similarly, if there is an integer b in I_2(i) colored with blue, then 2b−1 ∈ 1 and N/2 + 1 ≤ 2b−1 ≤ |3N/4|. So 2b−1 ∈ I_3 and hence it is not colored with blue by observation (ii). Therefore, \{1, b, 2b−1\} is a rainbow 3-AP.

(iv) \(c_{\text{red}}(I_2(\text{ii})) = c_{\text{blue}}(I_3(\text{ii})) = \emptyset\). Suppose there is an integer \(r\) in \(I_2(\text{ii})\) colored with red. Since the coloring of \(I_2\) contains both red and blue and there is no integer in \(I_2(\text{ii})\) colored with blue, by (iii), there must be an integer \(b\) in \(I_2(\text{ii})\) colored with blue. By (i) and (ii), \(b \equiv 1\) and \(r \equiv 2\). Wlog, suppose \(b > r\). Then \(2r−b \equiv 0\) and \(2r−b \in I_2\) since \(\ell(I_2(\text{ii})) \leq \ell(I_2(\text{ii})) + 1\). So \(2r−b\) is not colored red or blue and hence the 3-AP \(\{2r−b, r, b\}\) is rainbow. Therefore, there is no integer in \(I_2(\text{ii})\) that is colored with red. Similarly, there is no integer in \(I_3(\text{ii})\) that is colored with blue.

Recall that every color other than \(c(1)\) and \(c(N)\) is used in both intervals \(I_2\) and \(I_3\). Therefore, sets \(c_{\text{red}}(I_2(\text{ii}))\), \(c_{\text{blue}}(I_2(\text{ii}))\), \(c_{\text{red}}(I_3(\text{ii}))\), and \(c_{\text{blue}}(I_3(\text{ii}))\) are nonempty. Using above observations we next show that in fact these integers colored with blue and red in each subinterval are unique. Let \(B = \{b_1, \ldots, b_2\}\) be the shortest interval in \(I_2(\text{ii})\) which contains all integers colored with blue and let \(R = \{r_1, \ldots, r_2\}\) be the shortest interval in \(I_3(\text{ii})\) which contains all integers colored with red. Choose the largest integer \(x\) in \(c_{\text{red}}(I_2(\text{ii}))\) and consider two 3-APs \(\{x, b_1, 2b_1−x\}\) and \(\{x, b_2, 2b_2−x\}\). Since \(x\) is congruent to 2 and both \(b_1\) and \(b_2\) are congruent to 1, we have that both \(2b_1−x\) and \(2b_2−x\) are congruent to 0 and are contained in \(I_3\), otherwise the 3-APs are rainbow. Since all integers colored with blue in \(I_3\) are congruent to 2 by (ii), we have that \(2b_1−x\) and \(2b_2−x\) are both colored with red and so contained in \(R\). Therefore, \(2\ell(B)−1 ≤ \ell(R)\).

Now using the smallest integer in \(c_{\text{blue}}(I_3(\text{ii}))\), we similarly have that \(2\ell(R)−1 ≤ \ell(B)\). Since \(\ell(B) ≥ 1\) and \(\ell(R) ≥ 1\), we have that \(\ell(R) = \ell(B) = 1\), i.e. there are unique integers \(b\) in \(c_{\text{blue}}(I_2(\text{ii}))\) and \(r\) in \(c_{\text{red}}(I_3(\text{ii}))\).

Now for any integer \(\tilde{r}\) from \(c_{\text{red}}(I_2(\text{ii}))\) the integer \(2\tilde{r}−1\) must be colored with red, otherwise the 3-AP \(\{1, \tilde{r}, 2\tilde{r}−1\}\) is rainbow. Since \(2\tilde{r}−1 \in I_3\), it must be equal to the unique red colored integer \(r\) of \(I_3\). Therefore, there is exactly one such \(\tilde{r}\) in \(I_2(\text{ii})\), i.e. \(c_{\text{red}}(I_2(\text{ii})) = \{\tilde{r}\}\). Similarly, using \(N\) there is a unique integer \(\tilde{b}\) in \(I_3(\text{ii})\) colored with blue. Since \(\{1, \tilde{r}, r\}, \{\tilde{r}, b, r\}, \{b, r, \tilde{b}\}\), and \(\{b, \tilde{b}, N\}\) are all 3-APs, \(N = 7(\ell(\{b, \ldots, r\}))−1) + 1 = 7(r−b) + 1\).

Observe that if \(\tilde{r}\) is even, the integer \((\tilde{r} + N)/2\) in 3-AP \(\{\tilde{r}, (\tilde{r} + N)/2, N\}\) must be red and congruent to 1 since \(\tilde{r} \equiv 2\) by (ii), contradicting our assumption. So \(\tilde{r}\) is odd, and hence the integer \(r' = (\tilde{r} + 1)/2\) in \(I_1\) must be colored with red. Notice that there cannot be another integer \(x\) larger than \(r'\) in \(c_{\text{red}}(I_1)\), otherwise \(2x−1\) will be another integer colored with red in \(I_2\) distinct from \(\tilde{r}\). Now, since \(\ell(\{r', \ldots, \tilde{r}\}) = \ell(\{b, \ldots, r\})\) we have that \(\{r', r, N\}\) is a 3-AP, and so \(r'\) must be even. Suppose there are integers smaller than \(r'\) in \(c_{\text{red}}(I_1)\), and let \(z\) be the largest of them. Then \(2z−1\) is also in \(c_{\text{red}}(I_1)\) and must be equal to or larger than \(r'\) in \(I_1\). However, that is impossible because \(r'\) is even and there is no integer in \(c_{\text{red}}(I_1)\) larger than \(r'\). So \(r'\) is a unique integer in \(I_1\) colored with red. Similarly, there is a unique integer \(b'\) in \(I_4\) colored with blue. Therefore the 8-AP can be formed using integers \(1, r', \tilde{r}, b, r, \tilde{b}, b', N\) since \(\ell(\{b, \ldots, r\}) = \ell(\{\tilde{r}, \ldots, \tilde{r}\}) = \ell(\{b, \ldots, r\}) = \ell(\{b, \ldots, b\}) = \ell(\{\tilde{b}, \ldots, N\}) = \ell(\{b', \ldots, N\})\).

In order for this coloring to be special, it remains to show that \(c_{\text{blue}}(I_1) = c_{\text{red}}(I_4) = \emptyset\). If \(c_{\text{blue}}(I_1) \neq \emptyset\), then choose the largest integer \(y\) in it and consider the 3-AP \(\{1, y, 2y−1\}\). Since \(2y−1\) must be in \(c_{\text{blue}}(I_2)\) and the only integer in this set is \(b\), we have \(2y−1 = b\). However, we know that \(b\) is even because \(b = 2\tilde{b}−N\), a contradiction. Similarly, if \(c_{\text{red}}(I_4) \neq \emptyset\) choose the
smallest integer $x$ in it and consider the 3-AP \( \{2x - N, x, N\} \). Since \( 2x - N \) must be in \( c_{\text{red}}(I_3) \) and the only integer in this set is \( r \), we have \( 2x - N = r \). However, we know that \( r \) is odd because \( r = 2\tilde{r} - 1 \), a contradiction. This implies that \( c_{\text{red}}([N]) = \{r', \tilde{r}, r\} \) and \( c_{\text{blue}}([N]) = \{b, \tilde{b}, b'\} \), so the coloring is special.

**Case 2:** \( N \equiv 2 \ (\text{mod} \ 3) \). This case is analogous to Case 1.

**Case 3:** \( N \equiv 1 \ (\text{mod} \ 3) \). Assume \( |\{c(x) : x \equiv i \ (\text{mod} \ 3) \text{ and } x \in [N]\}| < r - 1 \) i.e. there are two colors, say red and blue, such that no integer in \( [N] \) colored with red or blue is congruent to 1. Recall that every color other than red and blue appears in \( I_2 \) and \( I_3 \). First, notice that all integers colored with red or blue in \( I_2 \) must be congruent modulo 3. Otherwise, choosing a red colored integer and a blue colored integer, we obtain a 3-AP whose third term is colored with red or blue and is congruent to 1 contradicting our assumption. Similarly, this is also the case for \( I_3 \). So suppose all integers in \( c_{\text{red}}(I_2) \cup c_{\text{blue}}(I_2) \) and \( c_{\text{red}}(I_3) \cup c_{\text{blue}}(I_3) \) are congruent modulo 3 to integers \( p \neq 1 \) and \( q \neq 1 \), respectively. Pick the largest integers from \( c_{\text{red}}(I_2) \) and \( c_{\text{blue}}(I_2) \) and form a 3-AP whose third term is in \( I_3 \). Then the third term is colored with red or blue and is congruent to \( p \). Therefore, \( p \equiv q \neq 1 \).

We further partition the interval \( I_2 \) into subintervals \( I_{2(i)} \) and \( I_{2(ii)} \), so that \( \ell(I_{2(i)}) \leq \ell(I_{2(ii)}) \leq \ell(I_{2(ii)}) + 1 \). If there exists \( x \in c_{\text{red}}(I_{2(ii)}) \cup c_{\text{blue}}(I_{2(ii)}) \), the integer \( 2x - 1 \) must be colored with \( c(x) \) and contained in \( I_3 \), so \( 2x - 1 \equiv p \) while \( x \equiv p \neq 1 \), a contradiction. So \( c_{\text{red}}(I_{2(ii)}) \cup c_{\text{blue}}(I_{2(ii)}) = \emptyset \). However, then the smallest integers of \( c_{\text{red}}(I_{2(ii)}) \) and \( c_{\text{blue}}(I_{2(ii)}) \) form a 3-AP whose first term is contained in \( I_{2(i)} \) and is colored with red or blue, a contradiction. This completes the proof of the lemma.

\[ \square \]

## 3 Proof of Theorem 2

Given a positive integer \( n \), define the function \( f \) as follows:

\[
  f(n) = \begin{cases} 
  m + 2, & \text{if } n = 3^m \\
  m + 3, & \text{if } n \neq 3^m \text{ and } 7 \cdot 3^{m-2} + 1 \leq n \leq 21 \cdot 3^{m-2}.
  \end{cases}
\]

In this section, we prove Theorem 2 by showing that \( aw([n], 3) = f(n) \) for all \( n \).

First, we show that \( f(n) \leq aw_u([n], 3) \) by inductively constructing a unitary coloring of \([n]\) with \( f(n) - 1 \) colors and no rainbow 3-AP. The result is true for \( n = 1, 2, 3 \), by inspection. Suppose \( n > 3 \) and that the result holds for all positive integers less than \( n \). Let \( n = 3h - s \), where \( s \in \{0, 1, 2\} \) and \( 2 \leq h < n \).

Let \( r = aw_u([h], 3) \). So there is an exact unitary \((r - 1)\)-coloring \( c \) of \([h]\) with no rainbow 3-AP. Let \( \text{red} \) be a color not used in \( c \). Define the coloring \( c_1 \) of \([n]\) such that if \( x \equiv 1 \ (\text{mod} \ 3) \), then \( c_1(x) = c((x + 2)/3) \), otherwise color \( x \) with \( \text{red} \). When \( s \neq 0 \), define the coloring \( c_2 \) of \([n]\) as follows: if \( x \not\equiv 0 \ (\text{mod} \ 3) \) then color \( x \) with \( \text{red} \); if \( x \equiv 0 \ (\text{mod} \ 3) \) then \( c_2(x) = c(x/3 + 1) \) when \( c(h) \) is the only unique color in \( c \) and \( c_2(x) = c(x/3) \) otherwise. Notice that \( c_2 \) is a unitary \( aw_u([h - 1], 3) \)-coloring when \( s \neq 0 \) and \( c_1 \) is a unitary \( r \)-coloring of \([n]\). Now consider a 3-AP \( \{a, b, 2b - a\} \) in \([n]\). If \( a \equiv b \neq 1 \), then \( a \) and \( b \) are colored with \( \text{red} \), and so the 3-AP is not a rainbow. If \( a \equiv b \equiv 1 \), then \( 2b - a \equiv 1 \), so this set corresponds to a 3-AP in \([h]\) with coloring \( c \), and hence the 3-AP is not rainbow. If \( a \not\equiv b \), then \( 2b - a \) is not congruent to \( a \) or \( b \), so two of the terms of the 3-AP are colored with \( \text{red} \), and hence the 3-AP is not rainbow under \( c_1 \). Similarly, this 3-AP is not rainbow under \( c_2 \). Therefore, \( c_1 \) and \( c_2 \) are unitary colorings of \([n]\) with no rainbow
3-AP. Also note that \(aw_u([n], 3) \geq aw_u([h], 3) + 1\) under \(c_1\) and \(aw_u([n], 3) \geq aw_u([h-1], 3) + 1\) under \(c_2\). We proceed with three cases determined by \(n/3\).

**Case 1.** First suppose \(7 \cdot 3^{m-2} + 1 \leq n \leq 3^m - 3\) or \(3^m \leq n \leq 21 \cdot 3^{m-2}\). By the induction hypothesis and using the coloring \(c_1\),

\[
aw_u([n], 3) \geq aw_u([h], 3) + 1 \geq f(h) + 1 = f(n).
\]

**Case 2.** Suppose \(n = 3^m - t\) where \(t \in \{1, 2\}\). Notice that \(h = 3^{m-1}\), so by induction and using coloring \(c_2\),

\[
aw_u([n], 3) \geq aw_u([h-1], 3) + 1 \geq f(h-1) + 1 = f(3^{m-1} - 1) + 1 = (m + 2) + 1 = f(n).
\]

The upper bound, \(aw([n], 3) \leq f(n)\), is also proved by induction on \(n\). For small \(n\), the result follows from Table 1. Assume the statement is true for all values less than \(n\), and let \(7 \cdot 3^{m-2} + 1 \leq n \leq 21 \cdot 3^{m-2}\) for some \(m\). Let \(aw([n]) = r + 1\), so there is an exact \(r\)-coloring \(\hat{c}\) of \([n]\) with no rainbow 3-AP. We need to show that \(r \leq f(n) - 1\). Let \([n_1, n_2, \ldots, n_N]\) be the shortest interval in \([n]\) containing all \(r\) colors under \(\hat{c}\). Define \(c\) to be an \(r\)-coloring of \([N]\) so that \(c(j) = \hat{c}(n_j)\) for \(j \in \{1, \ldots, N\}\). By minimality of \(N\) the colors of 1 and \(N\) are unique. If \([N]\) has at least \(r - 1\) colors congruent to 1 or \(N\), then \([n]\) has at least \(r - 1\) colors congruent to \(n_1\) or \(n_N\), respectively, so \(r \leq aw([n/3])\) and by induction \(r \leq f([n/3]) \leq f(n) - 1\). So suppose that is not the case, then by Lemma 5 we have that the coloring \(c\) is special.

Let \(N = 7q+1\) for some \(q \geq 1\), and let the 8-AP in this special coloring be \(\{1, r_1, r_2, b_1, r_3, b_2, b_3, N\}\), where \(r_1, r_2, r_3\) are the only integers colored red, \(b_1, b_2, b_3\) are the only integers colored blue and \(q = r_1 - 1\). If \(n \geq 9q\), then the 8-AP can be extended to a 9-AP in \(n\) by adding the 9th element to either the beginning or the ending. Wlog, suppose \(\{1, r_1, r_2, b_1, r_3, b_2, b_3, N, 2N-b_3\}\) correspond to a 9-AP in \([n]\). Since the coloring has no rainbow 3-AP, the color of \(2N-b_3\) is blue or \(c(N)\), so we have a 4-coloring of this 9-AP. However, \(aw([9], 3) = 4\) and hence there is a rainbow 3-AP in this 9-AP which is in turn a rainbow 3-AP in \([n]\). Therefore, \(n \leq 9q - 1\).

By uniqueness of red colored integer \(r_1\) in interval \(\{1, \ldots, r_2 - 1\}\), the colors of integers in interval \(\{r_1 + 1, \ldots, r_2 - 1\}\) is the same as the reversed colors of integers in \(\{2, \ldots, r_1 - 1\}\), i.e. \(c(r_1 + i) = c(r_1 - i)\) for \(i = 1, \ldots, q - 1\). Similarly, coloring of integers in interval \(\{r_2 + 1, \ldots, b_1 - 1\}\) is the reversed of the coloring of integers in interval \(\{r_1 + 1, \ldots, r_2 - 1\}\), and so on. This gives a rainbow 3-AP-free \((r - 2)\)-coloring of \(\mathbb{Z}_{2q}\). Therefore, \(r - 2 \leq aw(\mathbb{Z}_{2q}, 3) - 1\).

If \(q = 3^i\) for some \(i\), then \(n\) can not be a power of 3 because \(7 \cdot 3^i + 1 \leq n \leq 9 \cdot 3^i - 1\). Suppose \(n = 3^m\), then \(2q\) is not twice a power of 3 and clearly \(2q\) is not a power of 3. Therefore, by Lemma 4 we have

\[
r \leq aw(\mathbb{Z}_{2q}, 3) + 1 \leq \lceil \log_3(2q) \rceil + 2 \leq \lceil \log_3(2n/7) \rceil + 2 = \lceil \log_3(2 \cdot 3^m/7) \rceil + 2 = m + 1 \leq f(n) - 1.
\]

Suppose now that \(n \neq 3^m\). If \(q = 3^i\) for some \(i\) then \(i \leq m - 2\). Otherwise, if \(i \geq m - 1\) then \(q \geq 3^{m-1} \geq 1/7n\) which contradicts the fact that \(q < 1/7n\). Therefore, \(2q \leq 2 \cdot 3^{m-2} = 18 \cdot 3^{m-4}\) and so by induction and Lemma 4, \(r \leq aw(\mathbb{Z}_{2q}, 3) + 1 = aw([2q], 3) + 1 \leq m + 2 \leq f(n) - 1\). If \(q\) is not a power of 3, then again using Lemma 4, \(r \leq aw(\mathbb{Z}_{2q}, 3) + 1 \leq aw([2q], 3)\). Notice that \(6 \cdot 3^{m-3} + 2/7 \leq 2n/7 \leq 18 \cdot 3^{m-3}\), and so \(aw([2q], 3) \leq m + 2\) by induction. Therefore, \(r \leq m + 2 \leq f(n) - 1\).
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