Conformal Matrix Models as an Alternative to Conventional Multi-Matrix Models

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Abstract

We introduce conformal multi-matrix models (CMM) as an alternative to conventional multi-matrix model description of two-dimensional gravity interacting with $c < 1$ matter. We define CMM as solutions to (discrete) extended Virasoro constraints. We argue that the so defined alternatives of multi-matrix models represent the same universality classes in continuum limit, while at the discrete level they provide explicit solutions to the multi-component KP hierarchy and by definition satisfy the discrete $W$-constraints. We prove that discrete CMM coincide with the $(p,q)$-series of 2d gravity models in a well-defined continuum limit, thus demonstrating that they provide a proper generalization of Hermitian one-matrix model.
1 Introduction

Matrix models are used nowadays to describe non-perturbative partition functions, which interpolate between various sets of two-dimensional conformal models coupled to 2d gravity, and thus serve to approach the old-standing goal of devising a universal partition function of entire string theory. Considerable progress so far is reached in unification of all tachyon-free string models, associated with $c < 1$ minimal conformal models, i.e. the models from $(p, q)$-series plus 2d gravity. Namely, in [2] it has been suggested that the non-perturbative partition function $Z_p[T_k]$, interpolating between all the models with given $p$, possesses the following properties:

(i) $Z_p[T_k]$ is a $\tau$-function of $p$-reduced KP hierarchy, which is completely independent of all $T_k$ with $k = 0 \mod p$;

(ii) $Z_p[T_k]$ satisfies an infinite set of differential equations of the form

$$W_{i}(a)Z_p = 0, \quad a = 2, \ldots, p; \quad i \geq 1 - a,$$

where $W_{i}(a)$ are (harmonics of) the generators of Zamolodchikov’s $W_p$-algebra [3, 4], (see sect.4 below for explicit expressions of $W_{i}(a)$ in terms of $T$ and $\partial/\partial T$). These two statements (and actually the second one alone) are enough to define $Z_p[T]$ completely. Further interpolation between different values of $p$ is described in terms of Generalized Kontsevich Model ([5, 6] and references therein).

Though the above formulated suggestion of [2] can be just used as a reasonable definition of non-perturbative partition functions (and these are the corollaries of this definition which should be studied to proceed further with the string program), there are still many interesting but unresolved problems, which concern the original motivation of that suggestion. The most direct motivation is usually supposed to come from the study of the multi-scaling continuum limit of discrete multi-matrix models (the number of matrices being $p - 1$). While being completely understood in the simplest case of $p = 2$ (one-matrix model) [7], such continuum limit was never honestly studied for any $p > 2$. This problem

1 Partition function of a given $(p, q)$ string model (i.e. the generating functional of all the correlation functions in the model) arises from $Z_p[T_k]$, if all the time-variables $T_k = 0$, except for $T_1$ and $T_{p+q}: Z_{p,q}[T] \leftrightarrow P[T_1, 0, \ldots, 0, T_{p+q}, 0, \ldots]$. Existing formalism does not respect explicitly the symmetry between $p$ and $q$. 
lem attracted certain attention, but remains unresolved because of considerable technical difficulties.

In the present paper we are going to argue that there is a way around all these difficulties. Moreover, we continue to advocate the central idea of [8] that one has to look for the origin of all essential features of continuum double-scaling limit of (multi-) matrix models in their discrete counterparts. From this point of view one accepts that the conventional multi-matrix models, as defined in [3, 10, 11] are not the simplest possible discrete representatives of the relevant classes of universality. This is more or less clear already from the fact that these do not satisfy any simple Ward identities at the discrete level (or at least, the structure of these identities — so called \( \bar{W} \)-constraints [12] — is rather different from the \( W \)-constraints ([1]). Moreover, the advantage of being solved by orthogonal polynomial technique is not enough to present them in the form which respects the whole integrable structure.

Below we shall confirm the suggestion of [1] that much better representatives of the same universality classes are provided by somewhat different “multi-matrix” models or just solutions to integrable hierarchies and string equations which can be written in the multiple-integral form essentially different from conventional multiple integrals of multi-matrix models (by “interaction” of the nearest neighbours as well as by specific choice of integration contours). We shall call them conformal multi-matrix models (CMM), because of their relation to the formalism of conformal field theory. These models are by definition constructed to satisfy \( W \)-constraints at discrete level. Moreover, some especially interesting solution to these discrete \( W \)-constraints immediately at the discrete level possess a rich integrable structure which is in some sense less rough than the integrable structure found in the case of ordinary multi-matrix models [8, 13]. More concretely, the partition function of the standard multi-matrix models is a \( \tau \)-function of Toda lattice hierarchy in the first and the last times, the other times describing only the parameterization of the point of the Grassmannian. Thus, this case corresponds to extremely non-economic usage of the whole variety of parameters (times) in the problem. The way to rule out this drawback is to involve all these parameters as new times of a more rich hierarchy, namely, some multi-component hierarchy. It turns out that our CMM are related to the multi-component KP hierarchy of the \( SL(p) \) type (or generalized AKNS type [14, 15]).
Thus, we can claim that the proper viewpoint on the one-matrix model is to consider it as a reduced 2-component KP hierarchy.

In what follows we shall discuss some generic (especially integrability) properties of CMM in more details than it was done in [1], and give a detailed description of the relevant continuum limit (analogous to the presentation of [7] for $p = 2$, i.e. including the definition of Kazakov variables $t_k \to T_k$, reduction, rescaling of partition function and the transformation of discrete $W$-constraints into the continuum ones [1], see also [16]). As a result, we argue that the continuum limit

$$\lim_{N \to \infty} \left\{ Z_{p,N}^{\text{CMM}, \text{red}}[t_k] \right\}^{1/p} = Z_p[T_k]$$

(2)

is exactly a solution to eqs. (1), i.e. the relevant partition function of the $c < 1$ string theory.

The plan of the paper is as follows. In the sect.2 we remind (and make more explicit) the construction of CMM from ref.[1] and discuss the form of (Zamolodchikov’s) $W$-constraints in these discrete models. In the sect.3 integrability properties of $Z_p^{\text{CMM}}[t]$ are discussed. The sect.4 is describes the $p$-reduction of CMM, the analogue of Kazakov variables and the transformation of discrete into continuum $W$-constraints mainly along lines of the ref.[16]. Concluding remarks are given in the sect.5.

2 Conformal multi-matrix models

In this section we shall remind the ideas of [1] and give the definition of CMM. First, we show that the simplest example of discrete Hermitian 1-matrix model can be easily reformulated in these terms.

Indeed, Hermitian one-matrix model ($p = 2$) can be defined as a solution to discrete Virasoro constraints:

$$L_n Z_{2,N}[t] = 0, \quad n \geq -1$$

$$L_n \equiv \sum_{k=0}^{\infty} k t_k \partial / \partial t_{k+n} + \sum_{a+b=n} \partial^2 / \partial t_a \partial t_b$$

$$\partial Z_{2,N} / \partial t_0 = -NZ_{2,N}$$

(3)

The Virasoro generators (3) have the well-known form of the Virasoro operators in the theory of one free scalar field. If we look for such solution in terms of holomorphic
components of the scalar field

\[ \phi(z) = \hat{q} + \hat{p} \log z + \sum_{k \neq 0} \frac{J_{-k}}{k} z^{-k} \]

\[ [J_n, J_m] = n \delta_{n+m,0}, \quad [\hat{q}, \hat{p}] = 1 \]

the procedure is as follows. Define vacuum states

\[ J_k |0 \rangle = 0, \quad \langle N | J_{-k} = 0, \quad k > 0 \]

\[ \hat{p} |0 \rangle = 0, \quad \langle N | \hat{p} = N \langle N | \]

the stress-tensor\[ T(z) = \frac{1}{2} [\partial \phi(z)]^2 = \sum T_n z^{-n-2}, \quad T_n = \frac{1}{2} \sum_{k>0} J_{-k} J_{k+n} + \frac{1}{2} \sum_{a+b=n, a,b \geq 0} J_a J_b, \]

\[ T_n |0 \rangle = 0, \quad n \geq -1 \]

and the Hamiltonian

\[ H(t) = \frac{1}{\sqrt{2}} \sum_{k>0} t_k J_k = \oint_{C_0} V(z) j(z) \]

\[ V(z) = \sum_{k>0} t_k z^k, \quad j(z) = \frac{1}{\sqrt{2}} \partial \phi(z). \]

Now one can easily construct a “conformal field theory” solution to (3) in two steps. First,

\[ L_n \langle N | e^{H(t)} \ldots = \langle N | e^{H(t)} T_n \ldots \]

can be checked explicitly. As an immediate consequence, any correlator of the form

\[ \langle N | e^{H(t)} G |0 \rangle \]

\((N \text{ counts the number of zero modes of } G)\) gives a solution to (3) provided

\[ [T_n, G] = 0, \quad n \geq -1 \]

Second, the conformal solution to (11) (and therefore to (3)) comes from the properties of 2d conformal algebra. Indeed, any solution to

\[ [T(z), G] = 0 \]

\[ ^{2}\text{For the sake of brevity, we omit the sign of normal ordering in the evident places, say, in the expression for } T \text{ and } W \text{ in terms of free fields.} \]

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is a solution to (11), and it is well-known that the solution to (12) is a function of screening charges

\[ Q_\pm = \oint J_\pm = \oint e^{\pm \sqrt{2} \phi}. \tag{13} \]

With a selection rule on zero mode it gives

\[ G = \exp \left( \frac{1}{N!} Q_+^N \right). \tag{14} \]

(Of course, the general case must be \( G \sim Q_+^N M Q_-^M \) but the special prescription for integration contours, proposed in [4], implies that the dependence of \( M \) can be irrelevant and one can just put \( M = 0 \).) In this case the solution

\[ Z_{2,N}[t] = \langle N | e^{H(t)} \exp Q_+ | 0 \rangle \tag{15} \]

after computation of the free theory correlator, analytic continuation of the integration contour (see more detailed discussion of this point below) gives well-known result

\[ Z_{2,N} = (N!)^{-1} \prod_{i=1}^N dz_i \exp \left( - \sum t_k z_i^k \right) \Delta_N^2(z) = \]

\[ = (N! \text{Vol } U(N))^{-1} \int DM \exp \left( - \sum t_k M^k \right) \tag{16} \]

\[ \Delta_N = \prod_{i<j} (z_i - z_j) \]

in the form of multiple integral over spectral parameter or integration over Hermitian matrices.

In the case of \( p = 2 \) (Virasoro) constraints this is just a useful reformulation of the Hermitian 1-matrix model. However, in what follows we are going to use this point of view as a constructive one. Indeed, instead of considering a special direct multi-matrix generalization of (16) [9, 10, 11] one can use powerful tools of conformal theories, where it is well known how to generalize almost all the steps of above construction: first, instead of looking for a solution to Virasoro constraints one can impose extended Virasoro or \( W \)-constraints on the partition function. In such case one would get Hamiltonians in terms of multi-scalar field theory, and the second step is generalized directly using screening charges for \( W \)-algebras. The general scheme looks as follows

(i) Consider Hamiltonian as a linear combination of the Cartan currents of a level one Kac-Moody algebra \( \mathcal{G} \)

\[ H(t^{(1)}, \ldots, t^{(\text{rank } \mathcal{G})}) = \sum_{\lambda,k>0} t_k^{(\lambda)} \mu_\lambda J_k, \tag{17} \]
where \( \{ \mu_i \} \) are basis vectors in Cartan hyperplane, which, say for \( SL(p) \) case are chosen to satisfy
\[
\mu_i \cdot \mu_j = \delta_{ij} - \frac{1}{p}, \quad \sum_{j=1}^{p} \mu_j = 0.
\]

(ii) The action of differential operators \( W^{(a)}_i \) with respect to times \( \{ t_k^{(\lambda)} \} \) can be now defined from the relation
\[
W^{(a)}_i \langle N | e^{H(t)} \rangle \ldots = \langle N | e^{H(t)} W^{(a)}_i \rangle \ldots, \quad a = 2, \ldots, p; \quad i \geq 1 - a,
\]
(18)
where
\[
W^{(a)}_i = \oint z^{a+i-1} W^{(a)}(z)
\]
\[
W^{(a)}(z) = \sum_{\lambda} [\mu_\lambda \partial \phi(z)]^a + \ldots
\]
(19)
are spin-\( a \) \( W \)-generators of \( W_p \)-algebra written in terms of rank \( G \)-component scalar fields [4].

(iii) The conformal solution to (1) arises in the form
\[
Z_{p,N}^{CM}\{ t \} = \langle N | e^{H(t)} G \{ Q^{(a)} \} | 0 \rangle
\]
(20)
where \( G \) is an exponential function of screenings of level one Kac-Moody algebra
\[
Q^{(\alpha)} = \oint J^{(\alpha)} = \oint e^{\alpha \phi}
\]
(21)
\( \{ \alpha \} \) being roots of finite-dimensional simply laced Lie algebra \( G \). (For the case of non-simply laced case see [17]. Below \( G = SL(p) \) if not stated otherwise.) The correlator (20) is still a free-field correlator and the computation gives it again in a multiple integral form
\[
Z_{p,N}^{CM} \{ t \} \sim \prod_{\alpha} \prod_{i=1}^{N_\alpha} dz_i^{(\alpha)} \exp \left( - \sum_{\lambda,k>0} t_k^{(\lambda)} (\mu_\lambda \alpha)(z_i^{(\alpha)})^k \right) \times
\]
\[
\times \prod_{(\alpha,\beta)} \prod_{i,j=1}^{N_\alpha N_\beta} (z_i^{(\alpha)} - z_j^{(\beta)})^{\alpha \beta}
\]
(22)
The expression (22) is what we shall study in this paper: namely the solution to discrete \( W \)-constraints (1) which can be written as multiple integral over spectral parameters \( \{ z_i^{(\alpha)} \} \) (this integral is sometimes called “eigenvalue model”). The difference with the one-matrix case (16) is that the expressions (22) have rather complicated representation
in terms of multi-matrix integrals. Namely, the only non-trivial (Van-der-Monde) factor can be rewritten in the (invariant) matrix form:

$$\prod_{i=1}^{N} \prod_{j=1}^{N} (z_i^{(\alpha)} - z_j^{(\beta)})^{\alpha \beta} = \left[ \det \{ M^{(\alpha)} \otimes I - I \otimes M^{(\beta)} \} \right]^{\alpha \beta},$$

(23)

where $I$ is the unit matrix. Still this is a model with a chain of matrices and with closest neighbour interactions only (in the case of $SL(p)$).

The purpose of this paper is to show that CMM, defined by (22) as a solution to the $W$-constraints has indeed a very rich integrable structure and possesses a natural continuum limit. To pay for these advantages one should accept a slightly less elegant matrix integral with the entries like (23).

The first non-trivial example (which we use as a demonstration making all our statements in the paper more clear) is the $p = 3$ associated with Zamolodchikov’s $W_3$-algebra and serves as alternative to 2-matrix model. In this particular case one obtains

$$H(t, \bar{t}) = \frac{1}{\sqrt{2}} \sum_{k \geq 0} (t_k J_k + \bar{t}_k \bar{J}_k)$$

(24)

$$W_2^{(n)} = L_n = \sum_{k=0}^{\infty} (kt_k \partial / \partial t_{k+n} + k\bar{t}_k \partial / \partial \bar{t}_{k+n}) +$$

$$+ \sum_{a+b=n} (\partial^2 / \partial t_a \partial t_b + \partial^2 / \partial \bar{t}_a \partial \bar{t}_b)$$

(25)

$$W_3^{(n)} = \sum_{k,l>0} (kt_k t_l \partial / \partial t_{k+n+l} - k\bar{t}_k \bar{t}_l \partial / \partial \bar{t}_{k+n+l} - 2kt_k \bar{t}_l \partial / \partial \bar{t}_{k+n+l}) +$$

$$+ 2 \sum_{k>0} \left[ \sum_{a+b=n+k} (kt_k \partial^2 / \partial t_a \partial t_b - kt_k \partial^2 / \partial \bar{t}_a \bar{t}_b - 2k\bar{t}_k \partial^2 / \partial t_a \partial \bar{t}_b) \right] +$$

$$+ \frac{4}{3} \sum_{a+b+c=n} (\partial^3 / \partial t_a \partial t_b \partial t_c - \partial^3 / \partial t_a \partial \bar{t}_b \partial \bar{t}_c),$$

(26)

where times $t_k$ and $\bar{t}_k$ correspond to the two orthogonal directions in $SL(3)$ Cartan plane. (We use the standard specification of the Cartan basis: $e = \alpha_1 / \sqrt{2}$, $\bar{e} = \sqrt{3} \nu_2 / \sqrt{2}$. This basis is convenient for discussing integrability properties of the model in the sect.3, for the continuum limit we will use another basis in Cartan plane connected with $t \pm \bar{t}$.) In this case one has six screening charges $Q^{(\pm \alpha_i)} (i = 1, 2, 3)$ which commute with

$$W^{(2)}(z) = T(z) = \frac{1}{2} [\partial \phi(z)]^2$$

(27)
and
\[ W^{(3)}(z) = \sum_{\lambda=1}^{3}(\mu_\lambda \partial \phi(z))^3, \] (28)

where \( \mu_\lambda \) are vectors of one of the fundamental representations (3 or \( \bar{3} \)) of \( SL(3) \).

The particular form of integral representation (22) depends on particular screening insertions to the correlator (20). Following ref. [1] we will concentrate on the solutions which have no denominators. One of the reasons of such choice is that these solutions possess the most simple integrable structure of the forced hierarchy type [18, 19] (see sect.3), though the other ones can still be analyzed in the same manner.

One of the possible ways to avoid the problem with other solutions is the following prescription with the integration contours. First, as in any conformal theory, one has to consider closed contours in the definition of screening charges (13), (21). Then, the denominators can be easily integrated out using the ordinary residue technique. However, in the final result, it is necessary to continue contours to their specific locations defining any concrete model. Say, in the simplest example of the Hermitian one-matrix model the integration runs along real line. This is not too simple procedure, because only after this the solution starts to be non-trivial. (In a sense, with properly defined contour \( Q^{(i)}|0\rangle = \int_\mathbb{R} J^{(i)}|0\rangle \neq |0\rangle \), what would be the case for closed contour). Thus, the prescription to get rid of denominators implies that one first integrates over some part of the spectral variables and then uses an analytic continuation. However, even besides this prescription the integrable structure of generic solutions (22) can be analyzed in the free-fermion formalism and we shall return to this problem in the sect.3.

The simplest solutions which have no denominators correspond to specific correlators
\[ Z_{p,N}^{\text{CMM}}[\{t\}] = \langle N|e^{H(t)}\prod_i \exp Q_{\alpha_i}|0\rangle \] (29)
when we take \( \alpha_i \) to be “neighbour” (not simple!) roots: \( \langle \alpha_i, \alpha_j \rangle = 1 \). In the case of \( SL(3) \) this corresponds, say, to insertions of only \( Q_{\alpha_1} \) and \( Q_{\alpha_2} \)
\[ Z_{3,M,N}[t, \bar{t}] \equiv Z_{M,N}[t, \bar{t}] = \frac{1}{N!M!}\langle N, M|e^{H(t,\bar{t})}(Q^{(\alpha_1)})^N(Q^{(\alpha_2)})^M|0\rangle = \frac{1}{N!M!}\int \prod dx_i dy_i \exp \left(-\sum [V(x_i) + \bar{V}(y_i)]\right) \Delta^2_N(x) \Delta^2_M(y) \prod_{i,j} (x_i - y_j) \] (30)

This expression can be examined for its integrable properties by methods developed in [18] (see sect.3).
Before we proceed to a detailed investigation of the integrable structure of CMM, let us make two remarks about possible generalizations. The first one concerns supersymmetric matrix models. This problem was first examined in [20] where the authors looked for a solution to the system of equations \( L_n Z = 0 \) and \( G_m Z = 0 \), with generators \( \{L_n, G_m\} \) forming the \( N = 1 \) superconformal algebra. The solution was found in the form of a multiple integral over even and odd (Grassmannian) parameters, in the form similar to that considered above. It is necessary to point out that in our language this is nothing but trivial generalization of the one-field case which has to be substituted by scalar superfield. Then the insertion of screenings of \( N = 1 \) superconformal algebra immediately leads to the result by [20]. From this point of view the real problem with supersymmetric generalization can arise only in the \( N = 2 \) case because of the lack of appropriate screening operators.

The second possible generalization is to the case of “deformed” Virasoro–\( W \) constraints via Feigin-Fuchs-Dotsenko-Fateev procedure. In such case formulas (13) and (20) are still valid, one has only change the definitions (6), (19) to

\[
T = \frac{1}{2} [\partial \phi(z)]^2 + \alpha_0 \partial^2 \phi
\]  

(31)

(and corresponding formulas for higher \( W \)-algebras with \( c < \text{rank} G \)) with corresponding deformations of generators (1) written in terms of differential operators. Deformed generators will now commute with new screening charges

\[
Q_{\pm} = \oint e^{\alpha_{\pm} \phi}, \quad \alpha_{\pm} = \alpha_0 \pm (\alpha_0 + 2)^{1/2}
\]

(32)
in the case of deformed \( p = 2 \) theory (minimal models). The only problem which might arise here is connected with the integrability of such system — expressions (32) have no natural representation via free fermions for generic \( \alpha_0 \). To avoid this problem one should consider a \( j \)-differential system [21]. After the Miwa transform [22] of the time-variables

\[
t_k = \frac{1}{k} \sum_j a_j \xi_j^{-k}
\]

(33)

our correlators acquire the form of

\[
\langle e^{H(t)} Q^N_+ Q^M_- \rangle = \frac{\langle \prod_i \Phi_i(\xi_i) Q^N_+ Q^M_- \rangle}{\prod_{i<j}(\xi_i - \xi_j)^{a_i a_j}}
\]

(34)
with

\[ \Phi_i(z) = \exp[a_i \phi(z)], \quad \Delta_i = a_i^2/2 - a_i \alpha_0 \]  

(35)

One can choose \( \{a_j\} \) in such a way that \( \{\Phi_j(z)\} \) become primaries of minimal models, in such case the correlators (34) will satisfy certain differential equations following from the null-vector conditions (in the particular case of 2-level degeneration they will be identical with the Virasoro constraints in Miwa variables). This can be also easily generalized to the \( W \)-case of several scalar fields and several sets of times. In this case one should introduce in (33) vectors \( t_k \) and \( \alpha_j \). This leads to independent differential equations which again correspond to the independent null-vector conditions. Such construction trivially reveals the origin of the correspondence between Virasoro constrained \( \tau \)-functions and correlators in the minimal models.

Unfortunately the construction is valid in its present form only for discrete models (solution to discrete constraints) and is rather difficult to generalize to continuum models. The reason is that in the latter case we have to impose the twisted boundary conditions on scalar fields and thus (i) there is nothing like selection rule (14) which in the discrete case is provided by a zero mode and (ii) eq.(7) is no longer satisfied. We are going to return to this problem in a separate publication (see also [24]).

3 Integrability of conformal multi-matrix models

In this section we are going to present a detailed investigation of integrable structure of CMM, based on the application of the general technique of [18]. In the subsect.3.1 we will derive a determinant representation for CMM partition function which is an indication to the fact that we deal with an integrable system and the partition function is a certain \( \tau \)-function. As a direct result of the determinant formula we obtain the Hirota bilinear relation satisfied by partition function of CMM, which is a generalization of that one for the Hermitian one-matrix model. In the subsect.3.2 we are going to apply the fermionic operator formalism for CMM which brings us to a conclusion that the partition function of CMM is a \( \tau \)-function of a multi-component Kadomtsev-Petviashvili hierarchy which
obeys the constraint
\[ \sum_{k=1}^{p} \partial_t \eta^{(k)}_n \tau^{(p)}(\{t\}) = 0, \quad n = 1, 2, \ldots, \] (36)
where \( p \) is the number of components. This \( \tau \)-function satisfies \( W \)-constraints (4) and is a discrete counterpart of the analogous statement in the continuum limit found in \[2\].

### 3.1 Determinant representation

Let us first remind briefly the results of [18] for the simplest example of \( p = 2 \). The partition function of \( p = 2 \) case [16] can be written in the form
\[ Z_{2,N}(t) = \det_{N \times N} [\partial_{ij} + \tau(t)] = \tau_N(t) \] (37)
with
\[ \partial_{ij} C(t) = \partial_{ij} \tau(t) \] (38)
which implies that it is a \( \tau \)-function of the Toda chain hierarchy (or a \( \tau \)-function of AKNS-reduction of 2-component KP hierarchy [25]; see ref. [18] and sect.3.2 below for details). Eq.(38) means that \( C(t) \) just has an integral representation
\[ C(t) = \int d\mu(z) \exp \sum t_k z_k, \] (39)
where \( d\mu \) is some measure; these are the Virasoro constraints which fix the concrete measure and the contour of integration in (39). The determinant form (37) is an explicit manifestation of the fact that the partition function does satisfy the Hirota bilinear relations, the simplest one of which in this particular case takes the form
\[ \tau_N(t) \frac{\partial^2 \tau_N(t)}{\partial t_1^2} - \left( \frac{\partial \tau_N(t)}{\partial t_1} \right)^2 = \tau_{N+1}(t)\tau_{N-1}(t) \] (40)
The fact that in eq.(37) we are dealing with a (forced) Toda chain reduction of generic Toda lattice (two-component KP hierarchy) is reflected in the specific feature of the matrix in (37): \( C_{ij} = C_{i+j} \) [14, 13, 19].

Now we are going to generalize (37) and (40). In the case of the \( p = 3 \) model (30) one has to introduce two functions instead of (39):
\[ C(t) = \int dz \exp[-V(z)], \quad \bar{C}(t) = \int dz \exp[-\bar{V}(z)] \] (41)
where
\[ V(z) = \sum_{k>0} t_k z^k, \quad \bar{V}(z) = \sum_{k>0} \bar{t}_k z^k \]
and
\[ \partial_n C(t) = \partial^n t_1 C(t), \quad \partial_n \bar{C}(\bar{t}) = \partial^n \bar{t}_1 \bar{C}(\bar{t}) \] (42)

The determinant representation has now the form \((\partial \equiv \partial/\partial t_1, \bar{\partial} \equiv \partial/\partial \bar{t}_1)\)
\[
Z_{N,M}(t, \bar{t}) = \det
\begin{bmatrix}
C & \partial C & \cdots & \partial^{N-1} C & \bar{C} & \bar{\partial} C & \cdots & \bar{\partial}^{M-1} \bar{C}
\partial C & \partial^2 C & \cdots & \partial^N C & \bar{\partial} C & \bar{\partial}^2 C & \cdots & \bar{\partial}^M \bar{C}
\partial^{N+M-1} C & \partial^{N+M} C & \cdots & \partial^{N+M-2} C & \bar{\partial}^{N+M-1} \bar{C} & \bar{\partial}^{N+M} \bar{C} & \cdots & \bar{\partial}^{2N+M-2} \bar{C}
\end{bmatrix}
\equiv \tau_{N,M}(t, \bar{t}) (43)
\]
which is exactly the double-Wronskian representation of a \(\tau\)-function [26].

We can prove the determinant representation (43) as follows. Instead of eq.(30) let us consider slightly more general expression by introducing the external sources \(\{\beta_i\}\) and \(\{\bar{\beta}_i\}\):
\[
Z_{3;M,N}^{\text{CMM}}[t, \bar{t} | \beta, \bar{\beta}] = \frac{1}{N!M!} \times \int \prod dx_i dy_i \exp \left( - \sum \left[ V(x_i) - \beta_i x_i + \bar{V}(y_i) - \bar{\beta}_i y_i \right] \right) \Delta_N(x) \Delta_M(y) \Delta(x, y), (44)
\]
where \(\Delta(x, y) \equiv \Delta_N(x) \Delta_M(y) \prod_{i,j} (x_i - y_j)\) is \((N+M) \times (N+M)\) Van-der-Monde determinant. Using the derivatives with respect to \(\beta_i\) and \(\bar{\beta}_i\) one can get rid of \(\Delta(x, y)\) in the integrand thus obtaining
\[
Z_{3;M,N}^{\text{CMM}}[t, \bar{t} | \beta, \bar{\beta}] = \Delta(\partial_\beta, \partial_{\bar{\beta}}) F(t, \beta) \bar{F}(\bar{t}, \bar{\beta}), (45)
\]
where \(F\) and \(\bar{F}\) have the specific form of Kontsevich-like integrals :
\[
F(t, \beta) = \int \prod dx_i \exp \left( - \sum \left[ V(x_i) - \beta_i x_i \right] \right) \Delta_N(x),
\]
\[
\bar{F}(\bar{t}, \bar{\beta}) = \int \prod dx_i \exp \left( - \sum \left[ \bar{V}(x_i) - \bar{\beta}_i x_i \right] \right) \Delta_N(x).
\]
Using the trick with the differentiation over \(\beta\) and \(\bar{\beta}\) again one can represent these expressions in the determinant form (compare with [3, 4]):
\[
F(t, \beta) = \det[\partial_\beta^{-1} C(t, \beta_j)] = \det[\partial_\beta^{-1} C(t, \beta_j)], \quad i, j = 1, \ldots, N, (46)
\]
\[ F(\bar{t}, \bar{\beta}) = \det[\partial_{\bar{t}_i}^{-1} \tilde{C}(\bar{t}, \bar{\beta}_j)] = \det[\partial_{\bar{t}_i}^{-1} \tilde{C}(\bar{t}, \bar{\beta}_j)], \quad i, j = 1, \ldots, M, \quad (47) \]

where

\[ C(t, \beta) \equiv \int dz \exp[-V(z) + \beta z], \quad \tilde{C}(\bar{t}, \bar{\beta}) \equiv \int dz \exp[-\bar{V}(z) + \bar{\beta} z]. \]

Substitution of eqs. (46) and (47) into eq. (45) leads to the expression which contains the sum of \(M!N!\) terms analogous to (43) (each term corresponds to a particular permutation of \(\{\beta_i\}\) and \(\{\bar{\beta}_i\}\)). Finally, in the limit of \(\beta_i = \bar{\beta}_i = 0\) it reproduces the eq. (43).

Now the generalization of (43) for \(p \geq 3\) is quite obvious: \(\tau\)-function has “\((p-1)\)-tuple” Wronskian form for \((\sum N_i) \times (\sum N_i)\) matrix (multiplied by the factor \(\prod N_i!\)) with corresponding \(C_i(t_i) = \int \exp[-V_i(z)]dz\) \((i = 1, \ldots, p-1)\).

From representation (43) it is easy to derive that

\[ \frac{\partial^2}{\partial t_1 \partial \bar{t}_1} \log \tau_{N,M}(t, \bar{t}) = \frac{\tau_{N+1,M-1}(t, \bar{t})\tau_{N-1,M+1}(t, \bar{t})}{\tau_{N,M}(t, \bar{t})}, \quad (48) \]

This expression can be also easily extended to the \((p-1)\)-matrix case:

\[ \frac{\partial^2}{\partial t_1^{(i)} \partial \bar{t}_1^{(j)}} \log \tau_{\{N_k\}}(t, \bar{t}) = \frac{\tau_{\{N_i+1,N_j-1,\ldots\}}(t, \bar{t})\tau_{\{N_i-1,N_j+1,\ldots\}}(t, \bar{t})}{\tau_{\{N_k\}}^2(t, \bar{t})}, \quad (49) \]

This equations of motion can be immediately derived from the corresponding determinant representation like the case of Wronskian solutions of KP- or Toda lattice hierarchies (see, for example, [26]) and are the first Hirota bilinear equations generalizing (40).

### 3.2 Fermionic representation

Now we shall proceed to the representation of the solutions to CMM in terms of free fermion correlation functions. Such a representation (invented for integrable hierarchies by Kyoto school [27]) allows one to establish some of the properties of the system under consideration in an elegant and “physical” way.

Again, first we are going to show that the solution to Hermitian one-matrix model is nothing but AKNS-reduction of 2-component KP hierarchy [8, 18]. Indeed, the \(\tau\)-function of 2-component KP hierarchy is by definition the correlator

\[ \tau_{N,M}^{(2)}(x, y) = \langle N, M | e^{H(x,y)}G| N + M, 0 \rangle \quad (50) \]
where

\[ H(x, y) = \sum_{k>0} (x_k J_k^{(1)} + y_k J_k^{(2)}) \]  

(51)

\[ J^{(i)}(z) = \sum J_k^{(i)} z^{-k-1} = \psi^{(i)}(z) \psi^{(i)*}(z) \]  

(52)

\[ \psi^{(i)}(z) \psi^{(i)*}(z') = \frac{\delta_{ij}}{z - z'} + \ldots \]  

(53)

Now we are going to demonstrate that (15) is equivalent to (50) for certain \( G \) for which (50) depends only on the differences \( x_k - y_k \). To do this we have to make use of the free-fermion representation of \( SL(2)_{k=1} \) Kac-Moody algebra:

\[ J_0 = \frac{1}{2} (\psi_1^{(1)} \psi_1^{(1)*} - \psi_2^{(2)} \psi_2^{(2)*}) = \frac{1}{2} (J^{(1)} - J^{(2)}) \]

\[ J_+ = \psi_2^{(2)} \psi_1^{(1)*} \quad J_- = \psi_1^{(1)} \psi_2^{(2)*} \]  

(54)

Now let us take \( G \) to be the following exponent of a quadratic form

\[ G \equiv \exp \left( \int \psi_2^{(2)} \psi_1^{(1)*} \right) \]  

(55)

The only term which contributes into the correlator (50) due to the charge conservation rule is:

\[ G_{N,M} \equiv G_{N,-N} \delta_{M,-N} = \frac{1}{N!} : \left( \int \psi_2^{(2)} \psi_1^{(1)*} \right)^N : \delta_{M,-N} \]  

(56)

Now we bosonize the fermions

\[ \psi^{(i)*} = e^{\phi_i}, \quad \psi^{(i)} = e^{-\phi_i} \]

\[ J^{(1)} = \partial \phi_1, \quad J^{(2)} = \partial \phi_2 \]  

(57)

and compute the correlator

\[ \tau^{(2)}(x, y) \equiv \tau^{(2)}_{N,-N}(x, y) = \frac{1}{N!} \langle N, -N | \exp \left( \sum_{k>0} (x_k J_k^{(1)} + y_k J_k^{(2)}) \right) \left( \int : \psi_2^{(2)} \psi_1^{(1)*} : \right)^N |0\rangle = \]

\[ = \frac{1}{N!} \langle N, -N | \exp \left( \phi \left[ X(z) J^{(1)}(z) + Y(z) J^{(2)}(z) \right] \right) \left( \int : \exp (\phi_1 - \phi_2) : \right)^N |0\rangle \]

Introducing the linear combinations \( \sqrt{2} \phi = \phi_1 - \phi_2, \sqrt{2} \bar{\phi} = \phi_1 + \phi_2 \) we finally get

\[ \tau^{(2)}(x, y) = \frac{1}{N!} \langle \exp \left( \frac{1}{\sqrt{2}} \int [X(z) + Y(z)] \partial \bar{\phi}(z) \right) \rangle \times \]

\[ \times \langle N | \exp \left( \frac{1}{\sqrt{2}} \int [X(z) - Y(z)] \partial \phi(z) \right) \left( \int : \exp \sqrt{2} \phi : \right)^N |0\rangle = \tau^{(2)}_N (x - y) \]  

(58)
since the first correlator is in fact independent of $x$ and $y$. Thus, we proved that the
$	au$-function (50) indeed depends only on the difference of two sets of times \( \{x_k - y_k\} \). So,
we obtained here a particular case of the 2-component KP hierarchy (50) and

(i) requiring the elements of Grassmannian to be of the form \( \text{Grassmannian} \) we actually performed
a reduction to the 1-component case\(^3\).

(ii) we proved in (58) that this is an AKNS-type reduction for the \( \tau \)-function (57)\[^{14, 25} \].

The above simple example already contains all the basic features of at least all the
\( A_p \) cases. Indeed, the reduction (58) is nothing but \( SL(2) \)-reduction of a generic \( GL(2) \) situation. In other
words, the diagonal \( U(1) \) \( GL(2) \)-current \( \tilde{J} = \frac{1}{2}(J^{(1)} + J^{(2)}) = \frac{1}{2\sqrt{2}} \partial \tilde{\phi} \) decouples. This is an invariant statement which can be easily
generalized to higher \( p \) cases.

In the case of \( SL(p) \) we have to deal with the \( p \)-component hierarchy and instead of
(50) for generic \( \tau \)-function one has

\[
\tau^{(p)}_N(x) = \langle N | e^{H(x)} G | 0 \rangle \tag{59}
\]

\[ N = \{N_1, \ldots, N_p\}, \quad x = \{x^{(1)}, \ldots, x^{(p)}\} \]

and now we have \( p \) sets of fermions \( \{\psi^{(i)*}, \psi^{(i)}\} i = 1, \ldots, p \). The Hamiltonian is given by

Cartan currents of \( GL(p) \)

\[
H(t) = \sum_{i=1}^{p} \sum_{k>0} x_k^{(i)} J_k^{(i)} \tag{60}
\]

\[
J^{(i)}(z) = \psi^{(i)} \psi^{(i)*}(z)
\]

and the element of the Grassmannian in the particular case of CMM is given by an
exponents of the other currents

\[
J^{(ij)} = \psi^{(i)} \psi^{(j)}, \quad J^{(ij)*} = \psi^{(i)*} \psi^{(j)*}, \quad i \neq j \tag{61}
\]

\(^3\) Note that the idea to preserve both indices in (58) leads immediately to additional insertions either
of \( \psi^{(1)*} \) or \( \psi^{(2)*} \) to the right vacuum \( |0\rangle \), so that it is no longer annihilated at least by the \( T_{-1} \) Virasoro
generator, or in other words this ruins the string equation. Thus only the particular reduction (53) seems
to be consistent with string equation. This choice of indices just corresponds to that considered originally
in \[23\].
i.e.

\[ G \equiv \prod \exp(Q^{(ij)}) \exp(\bar{Q}^{(ij)}) \exp(Q^{(ij)*}) \]  
\[ Q^{(ij)} = \oint J^{(ij)}, \quad \bar{Q}^{(ij)} = \oint \bar{J}^{(ij)}, \quad Q^{(ij)*} = \oint J^{(ij)*}, \quad i \neq j \]  

Since (61) are the $SL(p)_1$ Kac-Moody currents, (62) play the role of screening operators in the theory under consideration. It deserves mentioning that they are exactly the $SL(p)$ (not $GL(p)$) -screenings and thus the $\tau$-function (59) does not depend on $\{\sum_{i=1}^{p} x_k^{(i)}\}$, i.e. we obtain the constraint (36).

In the case of $SL(3)$ this looks as follows. The screenings are

\[ Q^{(\alpha)} = \oint J^{(\alpha)}, \]  

where $\{\alpha\}$ is the set of the six roots of $SL(3)$. In terms of fermions or bosons the screening currents look like

\[ J^{(\alpha_1)} = \psi^{(1)*} \psi^{(2)*} = \exp(\phi_1 + \phi_2) \]
\[ J^{(\alpha_2)} = \psi^{(2)*} \psi^{(3)*} = \exp(\phi_2 + \phi_3) \]
\[ J^{(\alpha_3)} = \psi^{(1)*} \psi^{(3)*} = \exp(\phi_3 - \phi_1) \]
\[ J^{(-\alpha_1)} = \psi^{(1)} \psi^{(2)} = \exp(-\phi_1 - \phi_2) \]
\[ J^{(-\alpha_2)} = \psi^{(2)} \psi^{(3)} = \exp(-\phi_2 - \phi_3) \]
\[ J^{(-\alpha_3)} = \psi^{(3)} \psi^{(1)*} = \exp(\phi_1 - \phi_3) \]  

The particular $\tau$-function is now described in terms of the correlator

\[ \tau_{N}^{(3)}(x) = \langle N | e^{H(x)} G | 0 \rangle \]  

The condition of Cartan neutrality is preserved by compensation of charges between the operator (64) and left vacuum $\langle N \rangle$ in (66). It is obvious that in such case due to the condition of Cartan neutrality of the correlator (like in Wess-Zumino models) the mode

\[ \bar{J} = \partial \tilde{\phi} = \frac{1}{\sqrt{p}} \sum_{i=1}^{p} \partial \phi_i \]  

decouples from the correlator, and

\[ \tau_{N}^{(3)}(x) = \langle N | e^{H(x)} G | 0 \rangle = \]
\[ \langle 0 | \exp \left( \sum_{k>0} J_k \sum_{i=1}^3 x^{(i)}_k \right) |0 \rangle \tilde{\phi} \langle N | e^{H(t, \bar{t})} G |0 \rangle = \tau^\text{red}_N(t, \bar{t}), \]  

(68)

where the first correlator in the second row is trivially equal to unity. For the specific choice of the operator \( G \) in (68)

\[ G = G_{1,2} = \exp \left( \int J^{(\alpha_1)} \right) \exp \left( \int J^{(\alpha_2)} \right) \]  

(70)

we reproduce the formula (43).

Thus, the general statement is that for \( A_p \)-series we obtain an “AKNS-type” \( SL(p) \)-reduction of \( p \)-component KP hierarchy. Generalization to other groups is easy. Indeed, starting from the corresponding \( W \)-algebra we automatically end up with the element of the Grassmannian with proper group properties. Thus, we have a tool to construct hierarchies of the matrix model type (i.e. satisfying a string equation) with a given symmetry group. All properties of these hierarchies are automatically dictated by the symmetry. Say, the defining property of (generalized) AKNS hierarchy is the independence of the corresponding \( p \)-component KP \( \tau \)-function of the sum of times [14]. On the other hand, it really corresponds to \( SL(p) \) reduction from theoretical group point of view [25].

It is even more interesting to remark again that the general AKNS system is defined for arbitrary set of zero times:

\[ \tau_{N,M} \sim \langle N | \ldots | M \rangle \]  

with the only restriction \( \sum N_i = \sum M_i \) due to the charge conservation law. But the condition that \( T_{-1} \) annihilates the right vacuum requires \( M \) to be zero (it is interesting to note that the original paper [28], devoted to multi-component KP hierarchy, was dealing with exactly this type of restriction). In the \( SL(2) \) case this gives rise to a one-parameter \( \tau \)-function \( \tau_{N,-N} \), what allows one to rewrite the system as a Toda (chain)-type hierarchy, since the Toda-type system should depend only on a single zero-time (in this case the determinant of the type (43) can be re-ordered to have a block form with the given symmetry properties between different blocks). In particular, the first equation from the 2-component hierarchy which is of the form (48) [29], in this case transforms into eq.(40) which is just a Toda chain equation.
As to the Toda-like representation of CMM, in the simplest $SL(2)$-case the result is indeed equivalent to the Toda chain hierarchy [8, 18, 19]. In the fermionic language this connection is established by the following substitution in the element of the Grassmannian

$$\psi^{(1)}(z) \rightarrow \psi(z), \quad \psi^{(2)}(z) \rightarrow \psi(\frac{1}{z})$$

(71)

and the same for $\psi^*$'s. This is a reflection of the fact that Toda system is described by the two marked points (say, 0 and $\infty$) and corresponds to two glued discs, so it can be also described by two different fermions. This might lead to a general phenomenon, when any multi-component solution to CMM is actually related to (some reduction) of a multi-component Toda lattice.

4 Double-scaling limit

In this section we consider the central issue of the connection between matrix models and 2d gravity theory – continuum limit. Even in the most investigated case of Hermitian one-matrix model this is a rather sophisticated procedure, especially if one wants to reproduce the whole continuum integrable structure of [2]. Moreover, in the case of conventional multi-matrix model this procedure is still unknown.

Below we demonstrate that the advantages of CMM make it possible to define honestly the double-scaling limit in these theories along the line of [7] (see also [16]). This means that the set of times of discrete model undergo a Kazakov-like transformation to the continuum times. Discrete $W$-constraints are transformed into constraints (1) of continuous model.

4.1 Results of [7] for the one matrix model

To begin with let us briefly remind the main points of [7].

It has been suggested in [2] that the square root of the partition function of the continuum limit of one-matrix model is subjected to the Virasoro constraints

$$L_n^{\text{cont}} \sqrt{Z} ds = 0, \quad n \geq -1,$$

(72)
where
\[ \mathcal{L}_{n}^{\text{cont}} = \sum_{k=0}^{n}\left(k + \frac{1}{2}\right)T_{2k+1} \frac{\partial}{\partial T_{2(k+n)+1}} + G \sum_{0 \leq k \leq n-1} \frac{\partial^2}{\partial T_{2k+1} \partial T_{2(n-k-1)+1}} + \]
\[ + \frac{\delta_{0,n}}{16} + \frac{\delta_{-1,n}T_{1}^{2}}{(16G)} \]  
are modes of the stress tensor
\[ \mathcal{T}(z) = \frac{1}{2} \partial \Phi^{2}(z) = \frac{1}{16z^2} = \sum \frac{\mathcal{L}_{n}}{z^{n+2}}. \]  

It was shown in [7] that these equations which reflect the \( W^{(2)} \)-invariance of the partition function of the continuum model can be deduced from analogous constraints in Hermitian one-matrix model by taking the double-scaling continuum limit. The procedure (generalized below to CMM) is as follows.

The partition function of Hermitian one-matrix model can be written in the form
\[ Z\{t_{k}\} = \int DM \exp Tr \sum_{k=0}^{n} t_{k} M^{k} \]  
and satisfies [30, 31] the discrete Virasoro constraints (3).

In order to obtain the above-mentioned relation between \( W \)-invariance of the discrete and continuum models one has to consider a reduction of model (75) to the pure even potential \( t_{2k+1} = 0 \).

Let us denote by the \( \tau_{N}^{\text{red}} \) the partition function of the reduced matrix model
\[ \tau_{N}^{\text{red}}\{t_{2k}\} = \int DM \exp Tr \sum_{k=0}^{n} t_{2k} M^{2k} \]  
and consider the following change of the time variables
\[ g_{m} = \sum_{n \geq m} \frac{(-)^{n-m}\Gamma\left(n + \frac{3}{2}\right)a^{-n-\frac{1}{2}}}{(n-m)!\Gamma\left(m + \frac{3}{2}\right)} T_{2n+1}, \]  
where \( g_{m} \equiv m t_{2m} \) and this expression can be used also for the zero discrete time \( g_{0} \equiv N \) that plays the role of the dimension of matrices in the one-matrix model. Derivatives with respect to \( t_{2k} \) transform as
\[ \frac{\partial}{\partial t_{2k}} = \sum_{n=0}^{k-1} \frac{\Gamma\left(k + \frac{1}{2}\right)a^{n+\frac{1}{2}}}{(k-n-1)!\Gamma\left(n + \frac{3}{2}\right)} \frac{\partial}{\partial T_{2n+1}}, \]  
(79)
where the auxiliary continuum times $\tilde{T}_{2n+1}$ are connected with “true” Kazakov continuum times $T_{2n+1}$ via
\begin{equation}
T_{2k+1} = \tilde{T}_{2k+1} + a \frac{k}{k + 1/2} \tilde{T}_{2(k-1)+1},
\end{equation}
and coincide with $T_{2n+1}$ in the double-scaling limit when $a \to 0$.

Let us rescale the partition function of the reduced one-matrix model by exponent of quadratic form of the auxiliary times $\tilde{T}_{2n+1}$
\begin{equation}
\tilde{\tau} = \exp \left( -\frac{1}{2} \sum_{m,n \geq 0} A_{mn} \tilde{T}_{2m+1} \tilde{T}_{2n+1} \right) \tau_N^{\text{red}}
\end{equation}
with
\begin{equation}
A_{nm} = \frac{\Gamma \left( n + \frac{3}{2} \right) \Gamma \left( m + \frac{3}{2} \right)}{2\Gamma^2 \left( \frac{3}{2} \right)} \frac{(-)^{n+m} a^{-n-m-1}}{n!m!(n+m+1)(n+m+2)}.
\end{equation}
Then a direct though tedious calculation [7] demonstrates that the relation
\begin{equation}
\frac{\hat{L}_n \tilde{\tau}}{\tilde{\tau}} = a^{-n} \sum_{p=0}^{n+1} C_{n+1}^p (-1)^{n+1-p} L_{2p}^{\text{red}} \tau_N^{\text{red}},
\end{equation}
is valid, where
\begin{equation}
L_{2n}^{\text{red}} \equiv \sum_{k=0}^{n} kt_{2k} \frac{\partial}{\partial t_{2(k+n)}} + \sum_{0 \leq k \leq n} \frac{\partial^2}{\partial t_{2k} \partial t_{2(n-k)}}
\end{equation}
and
\begin{align}
\hat{\mathcal{L}}_{-1} &= \sum_{k \geq 1} \left( k + \frac{1}{2} \right) T_{2k+1} \frac{\partial}{\partial T_{2(k-1)+1}} + \frac{T_i^2}{16G}, \\
\hat{\mathcal{L}}_0 &= \sum_{k \geq 0} \left( k + \frac{1}{2} \right) T_{2k+1} \frac{\partial}{\partial T_{2k+1}}, \\
\hat{\mathcal{L}}_n &= \sum_{k \geq 0} \left( k + \frac{1}{2} \right) T_{2k+1} \frac{\partial}{\partial T_{2(k+n)+1}} \\
&+ \sum_{0 \leq k \leq n-1} \frac{\partial}{\partial T_{2k+1}} \frac{\partial}{\partial T_{2(n-k-1)+1}} - \frac{(-)^n}{16a^n}, \quad n \geq 1.
\end{align}
Here $C_p^p = \frac{n!}{p!(n-p)!}$ are binomial coefficients.

These Virasoro generators differ from the Virasoro generators (73) [2, 32] by terms which are singular in the limit $a \to 0$. At the same time $L_{2p}^{\text{red}} \tau$ at the r.h.s. of (83) do not need to vanish, since
\begin{equation}
0 = L_{2p} \tau \bigg|_{t_{2k+1}=0} = L_{2p}^{\text{red}} \tau \bigg|_{t_{2k+1}=0} + \frac{\partial^2 \tau}{\partial t_{2k+1} \partial t_{2(n-i-1)+1}} \bigg|_{t_{2k+1}=0}.
\end{equation}
It was shown in [7] that these two origins of difference between (73) and (85) actually cancel each other, provided eq.(83) is rewritten in terms of the square root $\sqrt{\tilde{\tau}}$ rather than $\tilde{\tau}$ itself:

$$
\mathcal{L}_{n}^{\text{cont}} \sqrt{\tilde{\tau}} = a^{-n} \sum_{p=0}^{n+1} C_{n+1}^{p} (-1)^{n+1-p} \frac{L_{2p} \tau}{\tau} \bigg|_{\tilde{t}_{2k+1}=0} (1 + O(a)) .
$$

(87)

The proof of this cancelation, as given in [7], is not too much simple and makes use of integrable equations for $\tau$.

In our consideration of CMM below we will use a more economical way to define the change of the time-variables $t \rightarrow T$ (also proposed in [7]), implied by the scalar field formalism. The Kazakov change of the time variables (78,79) can be deduced from the following prescription. Let us consider the free scalar field with periodic boundary conditions ((98) for $p = 2$)

$$
\partial \varphi(u) = \sum_{k \geq 0} g_{k} u^{2k-1} + \sum_{k \geq 1} \frac{\partial}{\partial t_{2k}} u^{-2k-1},
$$

(88)

and analogous scalar field with antiperiodic boundary conditions:

$$
\partial \Phi(z) = \sum_{k \geq 0} \left( (k + \frac{1}{2}) T_{2k+1} z^{k-\frac{1}{2}} + \frac{\partial}{\partial T_{2k+1}} z^{-k-\frac{3}{2}} \right).
$$

(89)

Then the equation

$$
\frac{1}{\tau} \partial \Phi(z) = a \frac{1}{\tau_{\text{red}}} \partial \varphi(u) \tau_{\text{red}}, \quad u^{2} = 1 + az
$$

(90)

generates the correct transformation rules (78), (79) and gives rise to the expression for $A_{nm}$ (82). Taking the square of the both sides of the identity (90),

$$
\frac{1}{\tau} T(z) = \frac{1}{\tau_{\text{red}}} T(u) \tau_{\text{red}},
$$

(91)

one can obtain after simple calculations that the same relation (83) is valid.

### 4.2 On the proper basis for CMM

In this subsection we would like to discuss briefly the manifest expressions for constraint algebras of the sect.2 in terms of time-variables. Indeed, for convenience of taking the continuum limit, the time variables should be redefined (i.e. the integrable flows of the previous section are not suitable in the continuum limit). In other words, this is the
question what is the proper reduction, or what combinations of the “integrable” times should be eliminated.

To begin with, we consider the simplest non-trivial case of \( p = 3 \). Then introducing the scalar fields

\[
\partial \phi^{(1)}(z) = \sum_k k t_k^{(1)} z^{k-1} + \sum_k \frac{\partial}{\partial t_k^{(1)}} z^{-k-1}, \tag{92}
\]

\[
\partial \phi^{(2)}(z) = \sum_k k t_k^{(2)} z^{k-1} + \sum_k \frac{\partial}{\partial t_k^{(2)}} z^{-k-1}, \tag{93}
\]

with \( t_k^{(1)} = (i \tilde{t}_k + t_k)/2\sqrt{2} \), \( t_k^{(2)} = (i \tilde{t}_k - t_k)/2\sqrt{2} \), one obtains the expressions:

\[
W^{(2)}(z) = \frac{1}{2} \partial \phi^{(1)}(z) \partial \phi^{(2)}(z), \tag{94}
\]

\[
W^{(3)}(z) = \frac{1}{3\sqrt{3}} \sum_i (\partial \phi^{(i)}(z))^3. \tag{95}
\]

instead of (25) and (26).

This choice of basis in the Cartan plane is adequate to the continuum limit of the system under consideration, as the latter one is described by completely analogous expressions [2]. Now let us describe this basis in more invariant terms and find the generalization to arbitrary \( p \).

Comparing (95) with (28), we can conclude that \( \partial \phi^{(i)} \equiv \beta_i \partial \phi \) corresponds to the basis

\[
\beta_{1,2} = \frac{1}{2} (\sqrt{3} \mu_2 \pm i \alpha_2). \tag{96}
\]

This basis has the properties

\[
\beta_1 \cdot \beta_2 = 1, \quad \beta_1 \cdot \beta_1 = 0, \quad \beta_2 \cdot \beta_2 = 0. \tag{97}
\]

Now it is rather evident how this basis should look in the case of general \( p \). Due to [2] we can guess what is the choice of the proper scalar fields:

\[
\partial \phi^{(i)}(z) = \sum_k k t_k^{(i)} z^{k-1} + \sum_k \frac{\partial}{\partial t_k^{(p-1)}} z^{-k-1}. \tag{98}
\]

This choice certainly corresponds to the basis with defining property (it can be observed immediately from the relations (98) and (18)):

\[
\beta_i \cdot \beta_j = \delta_{p,i+j}, \tag{99}
\]
the proper choice of the Hamiltonians in (17) being

\[ H = \sum_{i,k} t^{(i)}_{k} \beta_i \cdot J_k, \]

what determines new times adequate to the continuum limit.

Let us construct the basis (99) in a manifest way. To begin with, we define a set of vectors \( \{ \mu_i \} \) with the property:

\[ \mu_i \cdot \mu_j = \delta_{ij} - \frac{1}{p}, \quad \sum_i \mu_i = 0. \]

The \( W^{(n)} \)-algebra can be written in this basis as follows [4]:

\[ W^{(n)} = (-)^{n+1} \sum_{1 \leq j_1 < \ldots < j_k \leq p} \prod_{m=1}^{n} (\mu_{j_m} \cdot \partial \phi), \quad n = 1, 2, \ldots, p. \]

Now the basis (99) can be constructed from (101) by diagonalization of the following cyclic permutation [2, 33]:

\[ \mu_i \rightarrow \mu_{i+1}, \quad \mu_p \rightarrow \mu_1 \quad i = 1, \ldots, p - 1. \]

This transformation has \( \{ \beta_i \} \) as its eigenvectors, their manifest expressions being of the form:

\[ \beta_k = \frac{1}{\sqrt{p}} \sum_{j=1}^{p} \exp \left\{ \frac{2\pi i}{p} jk \right\} \mu_j, \quad k = 1, 2, \ldots, p - 1. \]

It is trivial to check that the properties (99) are indeed satisfied. One can immediately rewrite the corresponding \( W \)-generators in the basis of \( \beta_i \)'s. After all, one obtain the expressions similar to the continuum \( W \)-generators [2, 33], but with the scalar fields defined as in (98) and without the “anomaly” corrections appearing in the continuum case due to the twisted boundary conditions. These corrections can be correctly reproduced by taking the \( p \)-th root of the partition function as well as simultaneously doing the reduction (see subsect. 4.3-4.5).

Thus, the proposed procedure allows one to take the continuum limit immediately transforming the scalar fields as elementary building blocks.

### 4.3 The two-matrix model example

In this section we consider the simplest example of the \( p = 3 \) CMM.
From \{t\}- to \{T\}-variables. To describe this change of variables we shall use the scalar-field formalism.

Consider the set of scalar fields (92), (93) and perform the reduction

\[ t_{3k+1}^{(i)} = t_{3k+2}^{(i)} = 0, \quad i = 1, 2, \quad k = 0, 1, \ldots. \]  
(105)

Then

\[ \partial \varphi^{(1)}(u) = \sum_{k \geq 0} g_k^{(1)} u^{3k-1} + \sum_{k \geq 1} \frac{\partial}{\partial t_{3k}^{(1)}} u^{-3k-1}, \]

\[ \partial \varphi^{(2)}(u) = \sum_{k \geq 0} g_k^{(2)} u^{3k-1} + \sum_{k \geq 1} \frac{\partial}{\partial t_{3k}^{(1)}} u^{-3k-1}, \]  
(106)

and \( g_k^{(i)} = k t_{3k}^{(i)}, \quad k = 1, 2, \ldots, \) \( g_0^{(i)} = N^{(i)} \) were introduced in the subsect. 4.2. \( N^{(1,2)} \) are “the dimensions of the matrices” used in our two-matrix model. Then we put

\[ \frac{1}{\tau} \partial \Phi^{(i)}(z) = au^{i-2} \frac{1}{\tau} \partial \varphi^{(i)}(u) \tau^{\text{red}}, \quad u^3 = 1 + az, \quad i = 1, 2, \]  
(107)

where \( \Phi^{(i)}(z) \) are the scalar fields of the continuum model 2

\[ \partial \Phi^{(1)}(z) = \sum_{k \geq 0} \left( k + \frac{1}{3} \right) T_{3k+1} z^{k-\frac{4}{3}} + \frac{\partial}{\partial T_{3k+1}} z^{-k-\frac{5}{3}} \right); \]

\[ \partial \Phi^{(2)}(z) = \sum_{k \geq 0} \left( k + \frac{2}{3} \right) T_{3k+2} z^{k-\frac{4}{3}} + \frac{\partial}{\partial T_{3k+1}} z^{-k-\frac{5}{3}} \right), \]  
(108)

and

\[ T_{3k+i} = \tilde{T}_{3k+i} + a \frac{k}{k+i/3} \tilde{T}_{3(k-1)+i}, \quad i = 1, 2. \]  
(109)

The relation (107) gives rise to the following Kazakov-like change of the time variables

\[ g_m^{(i)} = \sum_{n \geq m} \frac{(-)^{n-m} \Gamma(n+1+i/3) a^{-n-\frac{4}{3}}}{(n-m)! \Gamma(m+i/3)} T_{3n+i}, \quad i = 1, 2 \]  
(110)

and equations (110) can be also continued to the zero discrete times \( g_0^{(i)} \). It follows from (107) that the derivatives with respect to \( t_{3k}^{(i)} \), \( i = 1, 2 \) transform as

\[ \frac{\partial}{\partial t_{3k}^{(i)}} = \sum_{n=0}^{k-1} \frac{\Gamma(k+i/3)}{(k-n-1)! \Gamma(n+1+i/3)} \frac{\partial}{\partial T_{3n+i}}, \quad i = 1, 2 \]  
(111)

To cancel the terms like \( T_{3k+2} z^{-k-\frac{4}{3}} \) and \( T_{3k+1} z^{-k-\frac{5}{3}} \) with \( k \geq 0 \) at the left hand sides of (107) we have to rescale the partition function of the continuum model \( \tilde{\tau} \) by means of an
exponent of some quadratic form

$$\tilde{\tau} = \exp \left( - \sum_{m,n \geq 0} A_{mn} \tilde{T}_{3n+2} \tilde{T}_{3n+1} \right) \tau^{\text{red}},$$  \hspace{1cm} (112)

where $A_{nm}$ has the form

$$A_{nm} = \frac{\Gamma \left( n + \frac{5}{3} \right) \Gamma \left( m + \frac{4}{3} \right)}{\Gamma \left( \frac{2}{3} \right) \Gamma \left( \frac{1}{3} \right)} \frac{(-)^{n+m} a^{-n-m-1}}{n! m! (n + m + 1)(n + m + 2)}. \hspace{1cm} (113)$$

**Constraint algebras of the two-matrix model.** It was suggested in [2] that the continuum “two-matrix” model possesses the $W^{(3)}$ (including Virasoro) symmetry, where Virasoro generators $L_n$ and generators of the $W^{(3)}$ algebra are constructed from the scalar fields (108) in the following way

$$T(z) = \frac{1}{2} :\partial \Phi^{(1)}(z) \partial \Phi^{(2)}(z): - \frac{1}{9z^2} = \sum \frac{L_n}{z^{n+2}}, \hspace{1cm} (114)$$

$$W^{(3)}(z) = \frac{1}{3\sqrt{3}} \left( :\left( \partial \Phi^{(1)}(z) \right)^3 : + :\left( \partial \Phi^{(2)}(z) \right)^3 : \right) = \sum \frac{W_n^{(3)}}{z^{n+3}}, \hspace{1cm} (115)$$

One can easily derive a relation between the generators $\tilde{L}_n$ and the corresponding generators $L_{3n}^{\text{red}}$, associated with reduction (105):

$$\frac{1}{\tau} \tilde{L}_n \tilde{\tau} = a^{-n} \sum_{p=0}^{n+1} C_{p+1}^{n+1} (-)^{n+1-p} \frac{L_{3p}^{\text{red}}}{\tau^{\text{red}}}, \quad n \geq -1, \hspace{1cm} (116)$$

where Virasoro generators $L_{3n}^{\text{red}}$ are defined by the same formula (104), only with $\phi$ substituted by “reduced” fields $\varphi$:

$$T(u) = \frac{1}{2} :\partial \varphi^{(1)}(u) \partial \varphi^{(2)}(u): = \sum_n u^{-3n-2} L_{3n}^{\text{red}}, \hspace{1cm} (117)$$

and

$$\tilde{L}_{-1} = \sum_{k \geq 1} \left( \left( k + \frac{1}{3} \right) T_{3k+1} \frac{\partial}{\partial T_{3k-2}} + \left( k + \frac{2}{3} \right) T_{3k+2} \frac{\partial}{\partial T_{3k-1}} \right) + \frac{2}{9} T_1 T_2,$$

$$\tilde{L}_0 = \sum_{k \geq 0} \left( \left( k + \frac{1}{3} \right) T_{3k+1} \frac{\partial}{\partial T_{3k+1}} + \left( k + \frac{2}{3} \right) T_{3k+2} \frac{\partial}{\partial T_{3k+2}} \right),$$

$$\tilde{L}_n = \sum_{k-m=-n} \left( \left( k + \frac{1}{3} \right) T_{3k+1} \frac{\partial}{\partial T_{3m+1}} + \left( k + \frac{2}{3} \right) T_{3k+2} \frac{\partial}{\partial T_{3m+2}} \right)$$

$$+ \sum_{m+k=-n-1} \frac{\partial}{\partial T_{3k+2}} \frac{\partial}{\partial T_{3m+1}} \left( (-)^n \frac{9a^n}{9a^n} \right), \quad n \geq 1. \hspace{1cm} (118)$$
Eq. (116) is a direct consequence of the relation
\[ \frac{1}{\tau} T(z) \tilde{\tau} = \frac{1}{\tau} \partial \Phi^{(1)}(z) \partial \Phi^{(2)}(z) \tilde{\tau} = \frac{1}{\tau_{\text{red}}} \partial \varphi^{(1)}(u) \partial \varphi^{(u)}(u) \tau_{\text{red}} = \frac{1}{\tau_{\text{red}}} T(u) \tau_{\text{red}}, \]
\[ u^3 = 1 + az \]  \hspace{2cm} (119)

(compare with (107)). As in the 1-matrix case, the generators (118) differ from the continuum generators \(L_n\) of [2] by singular \(e\)-number terms. Instead, again \(L_{3n}^{\text{red}}\) do not exactly annihilate \(\tau_{\text{red}}\). We assume that again these two effects cancel each other, provided eq. (116) for \(\tilde{\tau}\) is rewritten in terms of the cubic root \(\sqrt[3]{\tilde{\tau}}\). In other words, doing accurately the reduction procedure in the Virasoro constraints of the discrete two-matrix model one can rewrite (116) in the form
\[ \frac{1}{\sqrt[3]{\tau}} L_{\text{cont}} \sqrt[3]{\tilde{\tau}} = a^n \sum_{p=0}^{n+1} C_{n+1}^p \left( -1 \right)^{n+1-p} \frac{L_{3p} \tau}{\tau} (1 + O(a)) \bigg|_{t_{3k+l}=0; \ i=1,2}, \quad n \geq -1, \]  \hspace{2cm} (120)

Thus, we conclude that the Virasoro constraints of the continuum matrix model are indeed implied by the corresponding Virasoro constraints of the discrete conformal two-matrix model.

It follows from (115) that generators of the \(W^{(3)}\) symmetry for the continuous model can be written in the form
\[ \mathcal{W}_n^{(3)} = 3 \sum_{k,m \geq 0} \left( k + \frac{1}{3} \right) \left( m + \frac{1}{3} \right) T_{3k+1} T_{3m+1} \frac{\partial}{\partial T_{3(k+m+n)+1}} + 3 \sum_{k,m \geq 0} \left( k + \frac{2}{3} \right) \left( m + \frac{2}{3} \right) T_{3k+2} T_{3m+2} \frac{\partial}{\partial T_{3(k+m+n-1)+2}} + 3 \sum_{m,p \geq 0} \left( m + p - n + \frac{4}{3} \right) T_{3(m+p-n+1)+1} \frac{\partial}{\partial T_{3m+2}} \frac{\partial}{\partial T_{3p+2}} + 3 \sum_{m,p \geq 0} \left( m + p - n + \frac{2}{3} \right) T_{3(m+p-n)+2} \frac{\partial}{\partial T_{3m+1}} \frac{\partial}{\partial T_{3p+1}} + \sum_{k,m \geq 0} \frac{\partial}{\partial T_{3k+2}} \frac{\partial}{\partial T_{3m+2}} \frac{\partial}{\partial T_{3(n-k-m)-4}} + \frac{\partial}{\partial T_{3k+1}} \frac{\partial}{\partial T_{3m+1}} \frac{\partial}{\partial T_{3(n-k-m)-2}}, \] \hspace{2cm} (121)

where \(n \geq -2\) and the terms with negative values of the indices should be omitted. The generators \(\mathcal{W}_{-2}^{(3)}\) and \(\mathcal{W}_{-1}^{(3)}\) (but not the \(\mathcal{W}_0^{(3)}\)) have additional terms, cubic in times:
\[ \mathcal{W}_{-1}^{(3)} = \frac{1}{27} T_1^3 + \cdots \quad \text{and} \quad \mathcal{W}_{-2}^{(3)} = \frac{8}{27} T_2^3 + \frac{4}{9} T_1^2 T_4 + \cdots. \]  \hspace{2cm} (122)
Similar to the Virasoro case one can show that the relation between the generators of the $W^{(3)}$ symmetry for the discrete model and the generators $\tilde{W}_{n}^{(3)}$ is

$$\frac{1}{\tau} \tilde{W}_{p}^{(3)} \tilde{\tau} = a^{-p} \sum_{n=0}^{p+2} C_{p+2}^{n} (-)^{p-n} \frac{W^{(3)\text{red}}_{3n} \tau^{\text{red}}_{\text{red}}}{\tau^{\text{red}}_{\text{red}}}, \quad p \geq 2$$  \quad (123)

and follows from the identity

$$\left( \frac{\partial \Phi^{(1)}(z)}{\tilde{\tau}_{\text{red}}} \right)^{3} + \left( \frac{\partial \Phi^{(2)}(z)}{\tilde{\tau}_{\text{red}}} \right)^{3} = a^{3} \left[ \frac{1}{u^{3}} \left( \frac{\partial \varphi^{(1)}(u)}{\tau^{\text{red}}_{\text{red}}} \right)^{3} + \left( \frac{\partial \varphi^{(2)}(u)}{\tau^{\text{red}}_{\text{red}}} \right)^{3} \right], \quad (124)$$

where the generators $W^{(3)\text{red}}_{n}$ of the discrete model are defined after the reduction \[105\] by the relation

$$\left[ \frac{1}{u^{3}} \left( \frac{\partial \varphi^{(1)}(u)}{} \right)^{3} + \left( \frac{\partial \varphi^{(2)}(u)}{} \right)^{3} \right] = \sum_{n} u^{-3n-3} W^{(3)\text{red}}_{3n}. \quad (125)$$

After reduction of the discrete model eq.\[123\] can be rewritten in the form

$$\frac{1}{\sqrt{\tau}} W_{p}^{(3)} \text{cont} \frac{3}{\tau} = a^{-p} \sum_{n=0}^{p+2} C_{p+2}^{n} (-)^{p-n} \frac{W^{(3)\text{cont}}_{3n} \tau}{\tau}, \quad p \geq 2, \quad \text{eq.\[123\]}$$  \quad (126)

where $W_{p}^{(3)\text{cont}}$ are the generators of the $W^{(3)}$-symmetry from the paper \[3\] (see \[121\] and \[122\]).

### 4.4 The general case

It is easy to generalize the example of $p = 3$ to general multi-matrix models using the formalism of scalar fields with $\mathbb{Z}_{p}$-twisted boundary conditions. Let us introduce $p-1$ sets of the discrete times $t_{k}^{(i)}, i = 1, 2, \ldots, p-1$ and $k = 0, 1, \ldots$ for the discrete $(p-1)$-matrix model and consider the reduction

$$t_{pk+j}^{(i)} = 0, \quad i, j = 1, 2, \ldots, p-1, \quad k = 0, 1, \ldots \quad (127)$$

Choose the discrete and continuum scalar fields in the form

$$\partial \varphi^{(i)}(u) = \sum_{k \geq 0} g_{k}^{(i)} u^{k-1} + \sum_{k \geq 1} \frac{\partial}{\partial t_{pk}^{(p-i)}} u^{-p-1}, \quad g_{k}^{(i)} = k t_{pk}^{(i)} \quad (128)$$

$$\partial \Phi^{(i)}(z) = \sum_{k \geq 0} \left\{ \left( k + \frac{i}{p} \right) T_{pk+i}^{k-\frac{i-1}{p}} + \frac{\partial}{\partial T_{pk+p-i}^{k-\frac{2i-1}{p}}} z^{-\frac{k-2i+1}{p}} \right\}, \quad (129)$$
\[ T_{pk+i} = \tilde{T}_{pk+i} + a \frac{k}{k + i/p} \tilde{T}_{p(k-1)+i}, \; i = 1, 2, \ldots, p - 1. \] (130)

Then the equations
\[ au^{i+p+1} \frac{1}{\tau_{\text{red}}} \frac{\partial \phi^{(i)}(u)\tau_{\text{red}}}{\partial} = \frac{1}{\tau} \frac{\partial \Phi^{(i)}(z)\bar{\tau}}{\partial}, \; u^p = 1 + az, \; i = 1, 2, \ldots, p - 1, \] (131)
generate the Kazakov-like change of the time variables
\[ g_m^{(i)} = \sum_{n \geq m} \frac{(-)^{n-m} \Gamma (n + 1 + \frac{i}{p}) a^{-n-\frac{i}{p}}}{(n-m)! \Gamma (m + \frac{i}{p})} T_{pm+i}, \; i = 1, 2, \ldots, p - 1 \] (132)
\[ \frac{\partial}{\partial t_{pk}^{(i)}} = \sum_{n=0}^{k-1} \frac{\Gamma (k + \frac{i}{p}) a^{n+i+\frac{i}{p}}}{(k - n - 1)! \Gamma (n + 1 + \frac{i}{p})} \frac{\partial}{\partial T_{pn+i}} T_{pm+i}, \; i = 1, 2, \ldots, p - 1. \] (133)

Using the eq. (131) and considerations similar to those used in the previous subsections one can show that there is relation between tilded continuum generators \( \tilde{W}_n^{(i)}, \; i = 2, \ldots, p \) of the \( \mathcal{W} \)-symmetry and reduced generators of the discrete \( \mathcal{W} \)-symmetry \( W_{pk}^{(i)\text{red}} \) of the form
\[ \frac{1}{\tau} \tilde{W}_n^{(i)} \bar{\tau} = a^{-n} \sum_{s=0}^{n+i-1} C_s^{n+i-1} (-)^{n+i-1-s} \frac{W_{ps}^{(i)\text{red}}\tau_{\text{red}}}{\tau_{\text{red}}}, \; n \geq -i + 1, \] (134)
where rescaled \( \tau \)-function is defined
\[ \bar{\tau} = \exp \left( -\frac{1}{2} \sum_{i=1}^{p-1} \sum_{m,n \geq 0} A^{(i)}_{mn} \tilde{T}_{pm+i} \tilde{T}_{pn+p-i} \right) \tau_{\text{red}} \{ t_{pk} \} \] (135)
and matrices \( A^{(i)}_{mn} \) are determined by
\[ A_{nm}^{(i)} = \frac{\Gamma (n + \frac{p+i}{p}) \Gamma (m + \frac{2p-i}{p})}{\Gamma \left( \frac{i}{p} \right) \Gamma \left( \frac{p-i}{p} \right)} \frac{(-)^{n+m} a^{-n-m-1}}{n! m! (n + m + 1) (n + m + 2)}, \; i = 1, 2, \ldots, p - 1. \] (136)

This relation again corresponds to the identity
\[ \frac{W(u)\tau_{\text{red}}}{\tau_{\text{red}}} = \mathcal{W}(z)\bar{\tau}, \] (137)
where \( u^p = 1 + az \).

Performing the proper reduction procedure (127), which eliminates all but the time-variables of the form \( t_{pk}^{(i)} \) (i.e. leaves the \( 1/p \) fraction of the entire quantity of variables) we can obtain the relation
\[ \frac{1}{\sqrt{\tau}} \tilde{W}_n^{(i)} \frac{\phi}{\tau} = a^{-n} \sum_{s=0}^{n+i-1} C_s^{n+i-1} (-)^{n+i-1-s} \frac{W_{ps}^{(i)\tau}}{\tau^{i/i_{pk} \neq 0}} (1 + O(a)), \; n \geq -i + 1, \] (138)
where $\mathcal{W}_n^{(i)}$ is the $\mathcal{W}$-generators of the paper \[2\. Thus, we proved the $W$-invariance of the partition function of the continuum $p - 1$-matrix model and found the explicit relation between its partition function and corresponding partition function of the discrete $(p - 1)$-matrix model.

4.5 On the reduction of the partition function

To conclude this section we would like to discuss the problem of reduction \[105\] and \[127\] of the partition function in detail, with accuracy up to (non-leading) $c$-number contributions and only after the continuum limit is taken. More precisely, we reformulate the condition of a proper reduction in the continuum limit in order to reduce it to more explicit formulas which can be immediately checked. As a by-product of our consideration we obtain some restrictions on the integration contour in the partition function \[22\].

To get some insight, let us consider the simplest case of the Virasoro constrained Hermitian one-matrix model \[7\]. Before the reduction the Virasoro operators read as in \[76\]. Then their action on $\log \tau$ can be rewritten as

$$\left[ \sum_k kt^k \frac{\partial \log \tau}{\partial t_{k+n}} + \sum_m \frac{\partial^2 \log \tau}{\partial t_m \partial t_{n-m}} \right] + \sum_m \left[ \frac{\partial \log \tau}{\partial t_m} \frac{\partial \log \tau}{\partial t_{n-m}} \right] = 0. \quad (139)$$

After the reduction, we obtain

$$\left[ \sum_k 2kt_k t^{2k} \frac{\partial \log \tau_{red}}{\partial t_{2k+2n}} + \sum_m \frac{\partial^2 \log \tau_{red}}{\partial t_m \partial t_{2n-2m}} + \sum_m \frac{\partial^2 \log \tau_{red}}{\partial t_{2m+1} \partial t_{2n-2m-1}} \right] + \sum_m \left[ \frac{\partial \log \tau_{red}}{\partial t_{2m}} \frac{\partial \log \tau_{red}}{\partial t_{2n-2m}} \right] = 0 \quad (140)$$

under the condition

$$\left. \frac{\partial \log \tau_{red}}{\partial t_{odd}} \right|_{t_{odd}=0} = 0. \quad (141)$$

The last formula is a direct consequence of the “Schwinger-Dyson” equation induced by the transformation of the reflection $M \to -M$ in \[75\]. Indeed, due to the invariance of the integration measure under this transformation one can conclude that the partition function \[73\] depends only on a quadratic form of odd times.

Thus, the second derivatives of $\log \tau$ over odd times do not vanish, and are conjectured to satisfy the relation

$$\sum_m \frac{\partial^2 \log \tau_{red}}{\partial t_{2m} \partial t_{2n-2m}} \sim \sum_m \frac{\partial^2 \log \tau_{red}}{\partial t_{2m+1} \partial t_{2n-2m-1}}. \quad (142)$$
where the sign \( \sim \) implies that this relation should be correct only after taking the continuum limit. In this case one obtains the final result (cf. (87))

\[
\left[ \sum_k k t_{2k} \frac{\partial \log \sqrt{\tau_{\text{red}}}}{\partial t_{2k+2n}} + \sum_m \frac{\partial^2 \log \sqrt{\tau_{\text{red}}}}{\partial t_{2m} \partial t_{2n-2m}} \right] = 0. \quad (143)
\]

Thus, it remains to check the correctness of the relation (142). To do this, one should use the manifest equations of integrable (Toda chain) hierarchy, and after direct but tedious calculations [7] one obtains the result different from the relation (142) by \( c \)-number terms which are singular in the limit \( a \to 0 \) and just cancell corresponding items in (85) (this is certainly correct only after taking the continuum limit).

All this (rather rough) consideration can be easily generalized to the \( p \)-matrix model case. In this case one should try to use all \( W_{\text{red}}^{(i)} \)-constraints with \( 2 \leq i \leq p \). Thus, the second derivatives should be replaced by higher order derivatives, and one obtain a series of equations like (139). It is the matter of trivial calculation to check that these equations really give rise to the proper constraints satisfied by \( \sqrt{\tau} \) (cf. (120), (34) and (138)) provided by the two sets of the relations like (141) and (142).

Namely, the analog of the relation (141) in the \( p \)-matrix model case is the cancellation of all derivatives with incorrect gradation, i.e. with the gradation non-equal to zero by modulo \( p \). The other relation (142) should be replaced now by the conditions of the equality (in the continuum limit) of all possible terms with the same correct gradation. In the simplest case of \( p = 3 \) these are

\[
\begin{align*}
\sum_m \frac{\partial^2 \log \tau_{\text{red}}}{\partial t_{3m+1} \partial t_{3n-3m-1}} & \sim \sum_m \frac{\partial^2 \log \tau_{\text{red}}}{\partial t_{3m} \partial t_{3n-3m}}, \\
\sum_{m,k} \frac{\partial^3 \log \tau_{\text{red}}}{\partial t_{3m+1} \partial t_{3n-3(m+k)+1} \partial t_{3k-2}} & \sim \sum_{m,k} \frac{\partial^3 \log \tau_{\text{red}}}{\partial t_{3m+2} \partial t_{3n-3(m+k)+2} \partial t_{3k-4}} \\
& \sim \sum_{m,k} \frac{\partial^3 \log \tau_{\text{red}}}{\partial t_{3m} \partial t_{3n-3(m+k)} \partial t_{3k}}.
\end{align*}
\]

Again, this second condition is correct modulo some singular in the limit of \( a \to 0 \) terms, which appear only in the case of even \( p \). Unfortunately, we do not know the way to prove this statement without using the integrable equations, what is very hard to proceed in the case of higher \( p \).

On the other hand, the cancellation of derivatives with incorrect gradation can be trivially derived from the “Schwinger-Dyson” equations given rise by the transformations...
$M \to \exp \left\{ \frac{2\pi ki}{p} \right\} M \ (0 < k < p)$ of the integration variable in the corresponding matrix integral, the integration measure being assumed to be invariant. In its turn, it implies that the integration contour, instead of real line, should be chosen as a set of rays beginning in the origin of the co-ordinate system with the angles between them being integer times $\frac{2\pi}{p}$. This rather fancy choice of the integration contour is certainly necessary to preserve $Z_p$-invariance of $p$-matrix model system.

5 Conclusion

In this paper we proposed a new point of view on the discrete matrix formulation of the unified theory of $2d$ gravity coupled to minimal series of $c < 1$ matter. As alternative to conventional discrete multi-matrix models which are extremely hard to investigate by imposing some differential equations (Ward identities etc.) on matrix path integral, we introduced another class of models which satisfy “conformal” constraints by definition and actually possess much richer integrable structure, essentially as rich as the one arising in continuum limit. This is also up to now the only case when continuum double scaling interpolating limit (of multi-matrix models) can be performed honestly, reproducing the results of [2].

If matrix models are considered as certain solutions to integrable theories, the conformal multi-matrix models satisfy the equations of multi-component KP hierarchy. More precisely, they correspond to particular reduction of multi-component KP hierarchy, the reduction of AKNS type, which is further constrained to be consistent with discrete $W$-constraints and/or discrete string equation. We found a determinant representation for a certain subclass of such solutions, which can be also considered as a generalization of those corresponding to orthogonal polynomials, leading to forced hierarchies. All this must shed light on the origin of the integrability in $2d$ gravity and the work only started in this direction.

So far there was no real progress in taking the continuum limit of conventional multi-matrix models, and most of the facts were rather introduced axiomatically in this case. In contrast to this, CMM do have a nice continuum limit, which can be described in the same terms as the continuum limit of 1-matrix case. This shows that the fact of discrete
$W$-invariance or, put differently, the proper choice of a particular representative within the universality class, is crucial to expose the origin of string equation of 2d gravity theory in a manifest form.

Certainly there must be a deep connection between CMM and unified description of the continuum theory via Generalized Kontsevich Model (GKM) [2, 3, 19, 34, 35]. We know [19] that there exists also a reformulation of a discrete Hermitian 1-matrix model in terms of GKM, the integrability being established using formulas very close to those of the sect.3. It is possible also to take the continuum limit of 1-matrix case in “internal GKM terms”, what is much easier than by technique developed in [7] and sect.4 above. All these facts implies that there should be a sort of reformulation of the CMM in GKM-like terms.

There is another still unresolved question on interpretation of conformal fields appearing in the above formalism and their relations to the conformal fields of Polyakov’s formulation of 2d gravity. Of course, there shouldn’t be any direct correspondence, but there can be a kind of duality between “world-sheet” and “spectral” Riemann surfaces. The key role of $W$-constraints found in the paper and the presence of multi-component scalar fields might also have a meaning in the framework of $W$-gravity theory [36, 37].

We hope to return to all these problems elsewhere.

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