Ageing, dynamical scaling and conformal invariance

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Abstract

Building on an analogy with conformal invariance, local scale transformations consistent with dynamical scaling are constructed. Two types of local scale invariance are found which act as dynamical space-time symmetries of certain non-local free field theories. The scaling form of two-point functions is completely fixed by the requirement of local scale invariance. These predictions are confirmed through tests in the 3D ANNNI model at its Lifshitz point and in ageing phenomena of simple ferromagnets, here studied through the kinetic Ising model with Glauber dynamics.

\(^1\)presented at the International Conference on Theoretical Physics TH2002, Paris 22–27 July 2002

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Scale invariance is a central notion of modern theories of critical and collective phenomena. We are interested in systems with strongly anisotropic or dynamical criticality. In these systems, two-point functions satisfy the scaling form

\[ G(t, r) = b^{2x} G(b^\theta t, b^r) = t^{-2x/\theta} \Phi \left( r t^{-1/\theta} \right) = r^{-2x} \Omega \left( t r^{-\theta} \right) \]

(1)

where \( t \) stands for ‘temporal’ and \( r \) for ‘spatial’ coordinates, \( x \) is a scaling dimension, \( \theta \) the anisotropy exponent (when \( t \) corresponds to physical time, \( \theta = z \) is called the dynamical exponent) and \( \Phi, \Omega \) are scaling functions. Physical realizations of this are numerous, see [1] and references therein. For isotropic critical systems, \( \theta = 1 \) and the ‘temporal’ variable \( t \) becomes just another coordinate. It is well-known that in this case, scale invariance with a constant rescaling factor \( b \) can be replaced by the larger group of conformal transformations \( b = b(t, r) \) such that angles are preserved. It turns out that in the case of one space and one time dimensions, conformal invariance becomes an important dynamical symmetry from which many physically relevant conclusions can be drawn [2].

Given the remarkable success of conformal invariance descriptions of equilibrium phase transitions, one may wonder whether similar extensions of scale invariance also exist when \( \theta \neq 1 \). Indeed, for \( \theta = 2 \) the analogue of the conformal group is known to be the Schrödinger group [3, 4] (and apparently already known to Lie). While applications of the Schrödinger group as dynamical space-time symmetry are known [5], we are interested here in the more general case when \( \theta \neq 1, 2 \). We shall first describe the construction of these local scale transformations, show that they act as a dynamical symmetry, then derive the functions \( \Phi, \Omega \) and finally comment upon some physical applications. For the sake of conciseness, we shall present the main results formally as propositions. For details we refer the reader to [6].

The defining axioms of our notion of local scale invariance from which our results will be derived, are as follows (for simplicity, in \( d = 1 \) space dimensions).

1. We seek space-time transformations with infinitesimal generators \( X_n \), such that time undergoes a Möbius transformation

\[ t \rightarrow t' = \frac{\alpha t + \beta}{\gamma t + \delta} \; ; \; \alpha \delta - \beta \gamma = 1 \]

(2)

and we require that even after the action on the space coordinates is included, the commutation relations \([X_n, X_m] = (n - m)X_{n+m}\) remain
valid. This is motivated from the fact that this condition is satisfied for both conformal and Schrödinger invariance.

2. The generator $X_0$ of scale transformations is $X_0 = -t\partial_t - \theta^{-1}r\partial_r - x/\theta$ with a scaling dimension $x$. Similarly, the generator of time translations is $X_{-1} = -\partial_t$.

3. Spatial translation invariance is required.

4. Since the Schrödinger group acts on wave functions through a projective representation, generalizations thereof should be expected to occur in the general case. Such extra terms will be called mass terms. Similarly, extra terms coming from the scaling dimensions should be present.

5. The generators when applied to a two-point function should yield a finite number of independent conditions, i.e. of the form $X_n G = 0$.

**Proposition 1**: Consider the generators

$$X_n = -t^{n+1}\partial_t - \sum_{k=0}^{n} \binom{n+1}{k+1} A_{k0} t^{n-k} \partial_r - \sum_{k=0}^{n} \binom{n+1}{k+1} B_{k0} t^{n-k}$$

where the coefficients $A_{k0}$ and $B_{k0}$ are given by the recurrences $A_{n+1,0} = \theta A_{n0} A_{10}$, $B_{n+1,0} = \frac{\theta}{n-1} (n B_{n0} A_{10} - A_{n0} B_{10})$ for $n \geq 2$ where $A_{00} = 1/\theta$, $B_{00} = x/\theta$ and in addition one of the following conditions holds: (a) $A_{20} = \theta A_{10}$ (b) $A_{10} = A_{20} = 0$ (c) $A_{20} = B_{20} = 0$ (d) $A_{10} = B_{10} = 0$. These are the most general linear first-order operators in $\partial_t$ and $\partial_r$ consistent with the above axioms 1. and 2. and which satisfy the commutation relations $[X_n, X_{n'}] = (n - n')X_{n+n'}$ for all $n, n' \in \mathbb{Z}$.

Closed but lengthy expressions of the $X_n$ for all $n \in \mathbb{Z}$ are known [6]. In order to include space translations, we set $\theta = 2/N$ and use the short-hand $X_n = -t^{n+1}\partial_t - a_n \partial_r - b_n$. We then define

$$Y_m = Y_{k-N/2} = -\frac{2}{N(k+1)} \left( \frac{\partial a_k(t,r)}{\partial r} \partial_r + \frac{\partial b_k(t,r)}{\partial r} \right)$$

where $m = -\frac{N}{2} + k$ and $k$ is an integer. Clearly, $Y_{-N/2} = -\partial_r$ generates space translations.
Proposition 2: The generators $X_n$ and $Y_m$ defined in eqs. (3,4) satisfy the commutation relations

$$
[X_n, X_{n'}] = (n - n')X_{n+n'} , \quad [X_n, Y_m] = \left( n \frac{N}{2} - m \right) Y_{n+m}
$$

in one of the following three cases: (i) $B_{10}$ arbitrary, $A_{10} = A_{20} = B_{20} = 0$ and $N$ arbitrary. (ii) $B_{10}$ and $B_{20}$ arbitrary, $A_{10} = A_{20} = 0$ and $N = 1$. (iii) $A_{10}$ and $B_{10}$ arbitrary, $A_{20} = A_{10}^2$, $B_{20} = \frac{3}{2}A_{10}B_{10}$ and $N = 2$.

In each case, the generators depend on two free parameters. The physical interpretation of the free constants $A_{10}, A_{20}, B_{10}, B_{20}$ is still open. In the cases (ii) and (iii), the generators close into a Lie algebra, see [6] for details. For case (i), a closed Lie algebra exists if $B_{10} = 0$.

Turning to the mass terms, we now restrict to the projective transformations in time, because we shall only need those in the applications later. It is enough to give merely the ‘special’ generator $X_1$ which reads for $B_{10} = 0$ as follows [6]

$$
X_1 = -t^2 \partial_t - N \text{tr} \partial_r - N x \tau - \alpha \tau^2 \partial_t^{N-1} - \beta \tau^2 \partial_r^{2(N-1)/N} - \gamma \partial_r^{2(N-1)/N} \tau^2
$$

where $\alpha, \beta, \gamma$ are free parameters (the cases (ii,iii) of Prop. 2 do not give anything new). Furthermore, it turns out that the relation $[X_1, Y_{N/2}] = 0$ for $N$ integer is only satisfied in one of the two cases (I) $\beta = \gamma = 0$ which we call Type I and (II) $\alpha = 0$ which we call Type II. In both cases, all generators can be obtained by repeated commutators of $X_{-1} = -\partial_t$, $Y_{-N/2} = -\partial_r$, and $X_1$, using [6]. Commutators between two generators $Y_m$ are non-trivial and in general only close on certain ‘physical’ states. One might call such a structure a weak Lie algebra. These results depend on the construction [6] of commuting fractional derivatives satisfying the rules $\partial^{a+b}_r = \partial^a_r \partial^b_r$ and $[\partial^a_r, r] = a \partial^{a-1}_r$ (the standard Riemann-Liouville fractional derivative is not commutative, see e.g. [7]).

For $N = 1$, the generators of both Type I and Type II reduce to those of the Schrödinger group. For $N = 2$, Type I reproduces the well-known generators of 2D conformal invariance (without central charge) and Type II gives another infinite-dimensional group whose Lie algebra is isomorphic to the one of 2D conformal invariance [6].

Dynamical symmetries can now be discussed as follows, by calculating the commutator of the ‘Schrödinger-operator’ $S$ with $X_1$. We take $d = 1$ and $B_{10} = 0$ for simplicity.
Proposition 3: The realization of Type I sends any solution $\psi(t,r)$ with scaling dimension $x = 1/2 - (N - 1)/N$ of the differential equation

$$S\psi(t,r) = \left(-\alpha \partial_t^N + \left(\frac{N}{2}\right)^2 \partial_r^2\right)\psi(t,r) = 0$$ (7)

into another solution of the same equation.

Proposition 4: The realization of Type II sends any solution $\psi(t,r)$ with scaling dimension $x = (\theta - 1)/2 + (2 - \theta)\gamma/(\beta + \gamma)$ of the differential equation

$$S\psi(t,r) = \left(-\beta + \gamma\right)\partial_t + \frac{1}{\theta^z}\partial_r^\beta\psi(t,r) = 0$$ (8)

into another solution of the same equation.

In both cases, $S$ is a Casimir operator of the ‘Galilei’-subalgebra generated from $X_{-1}, Y_{-N/2}$ and the generalized Galilei-transformation $Y_{-N/2+1}$. The equations (7,8) can be seen as equations of motion of certain free field theories, where $x$ is the scaling dimension of that free field $\psi$. These free field theories are non-local, unless $N$ or $\theta$ are integers, respectively.

From a physical point of view, these wave equations suggest that the applications of Types I and II are very different. Indeed, eq. (7) is typical for equilibrium systems with a scaling anisotropy introduced through competing uniaxial interactions. Paradigmatic cases of this are so-called Lifshitz points which occur for example in magnetic systems when an ordered ferromagnetic, a disordered paramagnetic and an incommensurate phase meet (see [8] for a recent review). On the other hand, eq. (8) is reminiscent of a Langevin equation which may describe the temporal evolution of a physical system, with a dynamical exponent $z = \theta$. In any case, causality requirements can only be met by an evolution equation of first order in $\partial_t$.

Next, we find the scaling functions $\Phi, \Omega$ in eq. (1) from the assumption that $G$ transforms covariantly under local scale transformations.

Proposition 5: Local scale invariance implies that for Type I, the function $\Omega(v)$ must satisfy

$$\left(\alpha \partial_v^{N-1} - v^2 \partial_v - N x\right)\Omega(v) = 0$$ (9)

together with the boundary conditions $\Omega(0) = \Omega_0$ and $\Omega(v) \sim \Omega_\infty v^{-Nx}$ for $v \to \infty$. For Type II, we have

$$\left(\partial_u + \theta(\beta + \gamma) u \partial_u^{\beta - z} + 2\theta(2 - z)\gamma \partial_u^{1 - z}\right)\Phi(u) = 0$$ (10)
with the boundary conditions \( \Phi(0) = \Phi_0 \) and \( \Phi(u) \sim \Phi_\infty u^{-2x} \) for \( u \to \infty \). Here \( \Omega_{0,\infty} \) and \( \Phi_{0,\infty} \) are constants. The ratio \( \beta/\gamma \) turns out to be universal and related to \( x \). From the linear differential equations (9,10) the scaling functions \( \Omega(v) \) and \( \Phi(u) \) can be found explicitly using standard methods [6].

![Scaling function \( \Psi(v) \) of the spin-spin correlator at the Lifshitz point in the 3D ANNNI model. The main plot the scaling of selected data for several values of \( r_\perp \) is shown. In the inset, the full set of numerical data (gray points) is compared to the solution of eq. (9). The data are from [9].](image)

Figure 1: Scaling function \( \Psi(v) \) of the spin-spin correlator at the Lifshitz point in the 3D ANNNI model. The main plot the scaling of selected data for several values of \( r_\perp \) is shown. In the inset, the full set of numerical data (gray points) is compared to the solution of eq. (9). The data are from [9].

Given these explicit results, the idea of local scale invariance can be tested in specific models. Indeed, the predictions for \( \Omega(v) \) coming from Type I with \( N = 4 \) nicely agree with cluster Monte Carlo data [9] for the spin-spin and energy-energy correlators of the 3D ANNNI model at its Lifshitz point. Extensive simulations led to a considerable improvement in the precision of the estimation of the location of the Lifshitz point and the Lifshitz point exponents \( \alpha_L = 0.18(2), \beta_L = 0.238(5), \gamma_L = 1.36(3) \), respectively, of the specific heat, the magnetization and the susceptibility were obtained [9]. These values are in good agreement with the results of a careful two-loop study of the ANNNI model within renormalized field theory [10]. In figure 1 we show the scaling function \( \Psi(v) := v^\zeta \Omega(v) \) of the spin-spin correlator. Here \( \zeta = (2/\theta)(d_\perp + \theta)/(2 + \gamma_L/\beta_L) \), where \( d_\perp = 2 \) is the number of transverse
dimensions without competing terms and $\theta \simeq 1/2$ within the numerical accuracy of the data (see [9, 10, 6] for a fuller discussion of this point). Clearly, the data agree with the prediction of local scale invariance.

On the other hand, the predictions of Type II have been tested extensively in the context of ageing ferromagnetic spin systems to which we turn now. Consider a ferromagnetic spin system (e.g. an Ising model) prepared in a high-temperature initial state and then quenched to some temperature $T$ at or below the critical temperature $T_c$. Then the system is left to evolve freely (for recent reviews, see [11, 12]). It turns out that clusters of a typical time-dependent size $L(t) \sim t^{1/z}$ form and grow, where $z$ is the dynamical exponent. Furthermore, two-time observables such as the response function $R(t,s;r-r') = \delta(\sigma_r(t))/\delta h_{r'}(s)$ depend on both $t$ and $s$, where $\sigma_r$ is a spin variable and $h_{r'}$ the conjugate magnetic field. This breaking of time-translation invariance is called ageing. We are mainly interested in the autoresponse function $R(t,s) = R(t,s;0)$. One finds a dynamic scaling behaviour $R(t,s) \sim s^{-1-a}f_R(t/s)$ with $f_R(x) \sim x^{-\lambda_R/z}$ for $x \gg 1$ and where $\lambda_R$ and $a$ are exponents to be determined.

In order to apply local scale invariance to this problem, we must take into account that time translation invariance does not hold. The simplest way to do this is to remark that the Type II-subalgebra spanned by $X_0, X_1$ and the $Y_m$ leaves the initial line $t = 0$ invariant, see (6). Therefore the autoresponse function $R(t,s)$ is fixed by the two covariance conditions $X_0R = X_1R = 0$.

**Proposition 6:** For a statistical non-equilibrium system which satisfies local scale invariance the autoresponse function $R(t,s)$ takes the form

$$R(t,s) = r_0 (t/s)^{1+a-\lambda_R/z} (t-s)^{1-a}, \quad t > s$$

where $a$ and $\lambda_R/z$ are non-equilibrium exponents and $r_0$ is a normalization constant.

The functional form of $R$ is completely fixed once the exponents $a$ and $\lambda_R/z$ are known. Similarly, $R(t,s;r) = R(t,s)\Phi(r(t-s)^{-1/z})$ gives the spatio-temporal response, with the scaling function $\Phi(u)$ determined by (10) [6].

The reader might wonder whether invariance under the special transformation generated by $X_1$ actually holds in spite of the absence of time translation invariance. An answer to this question can be given [13] for ageing systems below criticality, where $z = 2$ for a non-conserved order parameter. In this case, the fluctuations in the system merely come from the disordered initial conditions and the presence of these must be taken into account when


considering the transformation of the effective action \( S \) (in the context of Martin-Siggia-Rose theory, see e.g. [1] and refs therein).

**Proposition 7:** [13] Consider an ageing system described by a local action \( S \) and which is invariant under spatial translations, phase shifts, Galilei transformations and dilatations with \( z = 2 \). Then \( \delta S = 0 \) under the action of the special Schrödinger transformation \( X_1 \).

A quantitative test of the prediction (11) is best carried out through the consideration of an integrated response function, such as the thermoremanent magnetization \( M_{\text{TRM}}(t, s) = h \int_0^s du R(t, u) \), where the system is quenched in a small magnetic field \( h \) which is turned off after the waiting time \( s \) has elapsed. The magnetization is then measured at a later time \( t \). One has the following scaling form for \( s \) large [14]

\[
M_{\text{TRM}}(t, s)/h = r_0 s^{-a} f_M(t/s) + r_1 s^{-\lambda_R/z} g_M(t/s)
\]  

(12)

Local scale invariance furnishes, via (11), explicit predictions for the scaling functions \( f_M(x) \) and \( g_M(x) \) [14] where \( r_{0,1} \) are normalization constants. In concrete systems such as the 2D Glauber-Ising model where exponents \( a = 1/z = 1/2 \) and \( \lambda_R/z \sim 0.6 - 0.8 \), the two terms are often of the same order of magnitude and one might expect to find a strong cross-over behaviour in the scaling of \( M_{\text{TRM}} \).

We now illustrate this crossover explicitly. In the left and middle panel of figure 2 [14], we display \( M_{\text{TRM}}(t, s) \) as a function of the waiting time \( s \) for the the kinetic 3D spherical model with a non-conserved order parameter, for two values of the initial magnetization \( M_0 \). Here, the full curve gives the exact solution of the Langevin equation, the dash-dotted line gives the asymptotic scaling according to (12) and the dotted and dashed lines give the individual contributions. The cross-over between a contribution \( \sim s^{-\lambda_R/z} \) for \( s \) small and a contribution \( \sim s^{-a} \) for \( s \) large is clearly seen. In the right panel of figure 2 analogous data for the 2D Glauber-Ising model are shown. The full curves are the predictions of local scale invariance where the constants \( r_{0,1} \) were fixed using the data for \( x = 3 \). Then, for any different value of \( x \), a definite prediction without any free parameter at all is obtained. We compare this with data for \( x = 5 \) and \( x = 7 \) and find a nice agreement. We point out that our data disagree with a recent suggestion [16] that the exponent \( a = 1/4 \) in the 2D Glauber-Ising model at \( T < T_c \).

The prediction (11) for the autoresponse has been confirmed in several physically distinct systems undergoing ageing, see [12, 17, 18, 19, 15] and
Figure 2: Scaling of the thermoremanent magnetization $M_{TRM}(t, s)$ as a function of $s$. Data for the 3D spherical model with initial average magnetization $M_0 = 0.5$ (left) and $M_0 = 0$ (middle) are shown, with $x = t/s = 5$ and the exponents $a$ and $\lambda_R/z$ taken from [15]. At the right, data from the 2D Glauber-Ising model at $T = 0$ and with the exponents $a = 1/2$ and $\lambda_R = 1.54$ are displayed. The values of $x$ are 3 (circles), 5 (squares), 7 (diamonds). The data are from [14].

As an example, we show data from the 3D Glauber-Ising model, both below and at criticality, in figure 3. The expected scaling behaviour $M_{TRM}(t, s)/h = s^{-a}\rho(t/s)$ is well realized and the form of $\rho(x)$ agrees nicely with local scale invariance, where the values of $a$ and of $\lambda_R/z$ were taken from the literature, see [17, 6] and references therein. These confirmations (which do go beyond free field theory) suggest that $M_{TRM}(t, s)$ should hold independently of (i) the value of the dynamical exponent $z$ (ii) the spatial dimensionality $d > 1$ (iii) the numbers of components of the order parameter and the global symmetry group (iv) the spatial range of the interactions (v) the presence of spatially long-range initial correlations (vi) the value of the temperature $T$ (vii) the presence of weak disorder. Very recently, the space-time response functions $R(t, s; r)$ were calculated in the 2D and 3D Glauber-Ising model after a quench to below the critical point. Our results agree perfectly with the prediction of local scale invariance with $z = 2$ [20].

3A recent two-loop calculation [21] in the kinetic $O(N)$-model at criticality recovers [11], up to a small correction. A detailed discussion will be given elsewhere.
Summarizing, we have shown that local scale transformations exist for any value of $z$ or $\theta$, act as dynamical symmetries of certain non-local free field theories and appear to be realized as space-time symmetries in some strongly anisotropic critical systems of physical interest. We also showed how the theory of local scale invariance may be applied to resolve standing questions in ageing phenomena.

It is a pleasure to thank M. Pleimling for a yearlong fruitful collaboration, C. Godrèche, J.-M. Luck, M. Paeßens, A. Picone and J. Unterberger for their collaboration on the work described here and P. Calabrese, H.W. Diehl, A. Gambassi and M. Shpot for useful discussions or correspondence. This work was supported by CINES Montpellier (projet pmn2095) and the Bayerisch-Französisches Hochschulzentrum (BFHZ).
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