Simple algorithm for GCD of polynomials

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Based on the Bezout approach we propose a simple algorithm to determine the gcd of two polynomials which doesn’t need division, like the Euclidean algorithm, or determinant calculations, like the Sylvester matrix algorithm. The algorithm needs only \( n \) steps for polynomials of degree \( n \). Formal manipulations give the discriminant or the resultant for any degree without needing division nor determinant calculation.

I. INTRODUCTION

There exist different approach to determine the greatest common divisor (gcd) for two polynomials, most of them are based on Euclid algorithm [1] or matrix manipulation [4] [5] or subresultant technics [2]. All these methods requires long manipulations and calculations around \( O(m^2) \) for polynomials of degree \( m \). Bezout identity could be another approach. If \( P^{(m)}(x) \) is a polynomial of degree \( m \) and \( Q^{(m)}(x) \) is a polynomial of degree at least \( m \), the Bezout identity says that \( \text{gcd}(P^{(m)}(x), Q^{(m)}(x)) = s(x)P^{(m)}(x) + t(x)Q^{(m)}(x) \) where \( t(x) \) and \( s(x) \) are polynomials of degree less then \( m \). Finding \( s(x) \) and \( t(x) \) requires also \( O(m^2) \) manipulations. If we know that \( P^{(m)}(0) \neq 0 \) we propose here another approach which use only linear combination of \( P^{(m)}(x) \) and \( Q^{(m)}(x) \) and division by \( x \) to decrease the degree of both polynomials by 1.

II. ALGORITHM

Let’s take two polynomials \( P^{(m)}(x) \) and \( Q^{(m)}(x) \):

\[
P^{(m)}(x) = \sum_{k=0}^{m} p_k^{(m)} x^k ; \quad Q^{(m)}(x) = \sum_{k=0}^{m} q_k^{(m)} x^k
\]

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with $p_0^{(m)} \neq 0$ and $p_m^{(m)} \neq 0$. The corresponding list of coefficients are:

$$p^{(m)} = \{p_0^{(m)}, p_1^{(m)}, \ldots, p_{m-1}^{(m)}, p_m^{(m)}\} ; \quad q^{(m)} = \{q_0^{(m)}, q_1^{(m)}, \ldots, q_{m-1}^{(m)}, q_m^{(m)}\}$$

Let’s define $\Delta_m = q_m^{(m)} p_0^{(m)} - p_m^{(m)} q_0^{(m)}$. If $\Delta_m \neq 0$, we can build two new polynomials of degree $m - 1$ by cancelling the lowest degree term and the highest degree term:

$$\begin{cases} 
    P^{(m-1)}(x) = \frac{1}{x}(q_0^{(m)} P^{(m)}(x) - p_0^{(m)} Q^{(m)}(x)) \\
    Q^{(m-1)}(x) = q_m^{(m)} P^{(m)}(x) - p_m^{(m)} Q^{(m)}(x)
\end{cases} \tag{1}$$

or in matrix notation:

$$\begin{pmatrix} 
    x & P(x) \\
    Q(x)
\end{pmatrix}_{m-1} = 
\begin{pmatrix} 
    q_0^{(m)} & -p_0^{(m)} \\
    q_m^{(m)} & -p_m^{(m)}
\end{pmatrix} 
\begin{pmatrix} 
    P(x) \\
    Q(x)
\end{pmatrix}_m \tag{2}$$

and the reverse:

$$\begin{pmatrix} 
    P(x) \\
    Q(x)
\end{pmatrix}_m = \frac{1}{\Delta_m} 
\begin{pmatrix} 
    -p_m^{(m)} & p_0^{(m)} \\
    -q_m^{(m)} & q_0^{(m)}
\end{pmatrix} 
\begin{pmatrix} 
    x & P(x) \\
    Q(x)
\end{pmatrix}_{m-1} \tag{3}$$

If $\Delta_m = 0$ then we replace $Q^{(m)}(x)$ by $\tilde{Q}^{(m)}(x)$:

$$\begin{cases} 
    P^{(m)}(x) = P^{(m)}(x) \\
    \tilde{Q}^{(m)}(x) = x(p_0^{(m)} Q^{(m)}(x) - q_0^{(m)} P^{(m)}(x))
\end{cases} \tag{4}$$

This correspond to the manipulation on the list of coefficients:

$$\begin{cases} 
    p_k^{(m-1)} = q_0^{(m)} p_k^{(m)} - p_0^{(m)} q_k^{(m)} \\
    q_k^{(m-1)} = q_m^{(m)} p_k^{(m)} - p_m^{(m)} q_k^{(m)}
\end{cases} \quad k \in [0, m - 1] \tag{5}$$

or

$$\begin{cases} 
    q_k^{(m)} = p_0^{(m)} q_k^{(m)} - q_0^{(m)} p_k^{(m)} \\
    q_k^{(m)} = p_0^{(m)} q_k^{(m)} - q_0^{(m)} p_k^{(m)}
\end{cases} \quad k \in [1, m] \tag{6}$$

note that the new $\tilde{q}_1^{(m)} = 0$. Note also that $p_{m-1}^{(m-1)} = -q_0^{(m-1)} = -\Delta_m$ and this will remains true at all iteration ending with $p_0^{(0)} = -q_0^{(0)} = -\Delta_1$.

So we have the same Bezout argument, (we know that 0 is not a root of $P^{(m)}(x)$) the $\gcd(P^{(m)}(x), Q^{(m)}(x))$ must divide $P^{(m-1)}(x)$ and $Q^{(m-1)}(x)$ or $P^{(m)}(x)$ and $\tilde{Q}^{(m)}(x)$. Repeating the iteration, it must divide $P^{(m-2)}(x)$ and $Q^{(m-2)}(x)$. If we reach a constant :
\( P^{(0)}(x) = p_0^{(0)} \) and \( Q^{(0)}(x) = q_0^{(0)} = -p_0^{(0)} \) then \( \gcd(P^{(m)}(x), Q^{(m)}(x)) = 1. \) If we reach, at some stage \( j \) of iteration, \( P^{(m-j)}(x) = 0 \) or \( Q^{(m-j)}(x) = 0 \) then the previous stage \( j-1 \) contains the \( \gcd. \)

When dealing with numbers the recurrence could gives large numbers so we can normalise the polynomials by some constant

\[
P^{(m-1)}(x) = \frac{\alpha_{m-1}}{x}(q_0^{(m)} P^{(m)}(x) - p_0^{(m)} Q^{(m)}(x))
\]

\[
Q^{(m-1)}(x) = \beta_{m-1}(q_m^{(m)} P^{(m)}(x) - p_m^{(m)} Q^{(m)}(x))
\]

choosing for example \( \alpha \) and \( \beta \) such that the sum of absolute value of the coefficients of \( P^{(m-1)}(x) \) and \( Q^{(m-1)}(x) \) are 1: \( \alpha_{m-1} = \sum_{k=0}^{m-1} |p_k^{(m-1)}|, \beta_{m-1} = \sum_{k=0}^{m-1} |q_k^{(m-1)}|, \) or that the maximum of the coefficients is always 1: \( \alpha_{m-1} = \max(p_k^{(m-1)}), \beta_{m-1} = \max(q_k^{(m-1)}). \)

For example if \( P^{(8)} = x^8 + x^6 - 3x^4 - 3x^3 + 8x^2 + 2x - 5 \) and \( Q^{(8)} = 3x^6 + 5x^4 - 4x^2 - 9x + 21 \), after 8 iterations we have to deal with numbers of order \( 10^{15} \), while using the sum of absolute value or the maximum we obtain after 8 iterations the result which prove that the polynomials are co-prime:

\[
P^{(8)}(x) \left( \frac{699x^5 - 236877x^4 + 8107x^3 - 37558x^2 + 11607x + 158088}{130354x^8} \right) +
\]

\[
- Q^{(8)}(x) \left( \frac{233x^7 - 78959x^6 + 2547x^5 + 40120x^4 + 1938x^3 + 61457x^2 - 3839x - 37640}{130354x^8} \right) = 1
\]

\[
Q^{(8)}(x) \left( \frac{7528x^7 - 233x^6 + 86487x^5 - 2547x^4 - 62704x^3 - 24522x^2 - 1233x + 18895}{130354x^7} \right) +
\]

\[
- P^{(8)}(x) \left( \frac{22584x^5 - 699x^4 + 274517x^3 - 8107x^2 + 7446x - 79359}{130354x^7} \right) = 1
\]

In term of list manipulation we have:

\[
\text{if } \Delta_m \neq 0 \quad p^{(m-1)} = \text{Drop}[\text{First}[q^{(m)}], p^{(m)}] - \text{First}[p^{(m)}], q^{(m)}, 1]
\]

\[
q^{(m-1)} = \text{Drop}[\text{Last}[q^{(m)}], p^{(m)}] - \text{Last}[p^{(m)}], q^{(m)}, -1]
\]

where \( \text{First}[\text{list}] \) and \( \text{Last}[\text{list}] \) takes the first and the last element of the list respectively, while \( \text{Drop}[\text{list, 1}] \) and \( \text{Drop}[\text{list, -1}] \) drop the first and the last element of the list respectively. If \( \Delta_m = 0 \) then we know that \( p_0^{(m)} q_m^{(m)} - q_0^{(m)} p_m^{(m)} = 0 \) so the list \( p_0^{(m)} q^{(m)} - q_0^{(m)} p^{(m)} \) ends with 0 so the list manipulation is:

\[
\tilde{q}^{(m)} = \text{RotateRight}[\text{First}[p^{(m)}] q^{(m)} - \text{First}[q^{(m)}] p^{(m)}]
\]

where \( \text{RotateRight}[\text{list}] \) rotate the list to the right (\( \text{RotateRight}[[a,b,c]] = [c,a,b] \)).
Repeating these steps decrease the degree of polynomials. So or we reach a constant, and reversing the process enables us to find a combinations of $P^{(m)}$ and $Q^{(m)}$ which gives a monomial $x^k$ and the polynomials are co-prime, or we reach a 0-polynomial before reaching the constant and $P^{(m)}(x)$, $Q^{(m)}(x)$ have a non trivial \text{gcd}.

For example

\[
\begin{align*}
P^{(8)}(x) &= x^8 - 4x^6 + 4x^5 - 29x^4 + 20x^3 + 24x^2 + 16x + 48 \\
Q^{(8)}(x) &= x^8 + 3x^7 - 7x^4 - 21x^3 - 6x^2 - 18x \\
p^{(8)} &= \{48, 16, 24, 20, -29, 4, -4, 0, 1\} \\
q^{(8)} &= \{0, -18, -6, -21, -7, 0, 0, 3, 1\}
\end{align*}
\]

and let’s use the “max” normalisation. The first iteration says that \text{gcd} must divide $P^{(7)}(x)$ and $Q^{(7)}(x)$:

\[
\begin{align*}
P^{(7)} &= -\frac{1}{21x} Q^{(8)} \quad \text{and} \quad Q^{(7)} = \frac{1}{48}(P^{(8)} - Q^{(8)}) \\
P^{(7)}(x) &= -\frac{x^7}{21} - \frac{x^6}{7} + \frac{x^3}{3} + x^2 + \frac{2x}{7} + \frac{6}{7} \\
Q^{(7)}(x) &= -\frac{x^7}{16} - \frac{x^6}{12} + \frac{x^5}{24} - \frac{11x^4}{48} + \frac{5x^3}{8} + \frac{17x}{24} + 1
\end{align*}
\]

then \text{gcd} divide

\[
\begin{align*}
P^{(6)}(x) &= \frac{x^6}{78} - \frac{2x^5}{13} + \frac{2x^4}{13} + \frac{11x^3}{78} - \frac{67x^2}{13} + x - \frac{9}{13} \\
Q^{(6)}(x) &= \frac{x^6}{4} + \frac{x^5}{5} + \frac{11x^4}{10} + x^3 - \frac{33x^2}{20} + \frac{4x}{5} - \frac{3}{10}
\end{align*}
\]

then \text{gcd} divide

\[
\begin{align*}
P^{(5)}(x) &= \frac{22x^5}{57} + \frac{8x^4}{19} - \frac{31x^3}{19} + x^2 - \frac{115x}{57} + \frac{11}{19} \\
Q^{(5)}(x) &= -\frac{32x^5}{187} - \frac{19x^4}{187} + \frac{155x^3}{187} - \frac{151x^2}{187} + x - \frac{12}{17}
\end{align*}
\]

e etc. finally \text{gcd} divide

\[
\begin{align*}
P^{(3)}(x) &= x^3 + 3x^2 + x + 3 \\
Q^{(3)}(x) &= \frac{x^3}{3} + x^2 + \frac{x}{3} + 1
\end{align*}
\]
the next step will give \( Q^{(2)}(x) = 0 \) \((3Q^{(3)}(x) - P^{(3)}(x) = 0)\), with the last step:

\[
\begin{align*}
P^{(2)}(x) &= P^{(8)}(x) \left( \frac{88}{63x^4} + \frac{50}{63x^3} + \frac{229}{378x^2} + \frac{143}{378x} \right) + \\
-Q^{(8)}(x) \left( \frac{-704}{189x^5} - \frac{400}{189x^4} + \frac{164}{189x^3} + \frac{100}{189x^2} + \frac{143}{378x} \right) &= x^3 + 3x^2 + x + 3 \\
Q^{(2)}(x) &= P^{(8)}(x) \left( -\frac{6}{x^3} + x - \frac{1}{x} \right) - Q^{(8)}(x) \left( \frac{16}{x^4} - \frac{8}{x^2} + x + \frac{4}{x} - 3 \right) = 0
\end{align*}
\]

so we have \( \gcd(P^{(8)}(x), Q^{(8)}(x)) = x^3 + 3x^2 + x + 3 \)

Doing the algorithm on formal polynomials gives automatically the resultant or the discriminant of \( P^{(m)}(x) \) and \( Q^{(m)}(x) \).

For example for the \( \gcd \) of \( P^{(m)}(x) \) and \( P^{(m)}(x)' \) for formal polynomials (we always cancel the term \( x^{m-1} \) by translation) we have:

\[
P^{(3)}(x) = a \cdot x^3 + b \cdot x + c \quad Q^{(3)}(x) = 3a \cdot x^2 + b
\]
gives after 3 iterations the well known discriminant:

\[
(9abcx + 2b^3)P^{(3)}(x) - (3abcx^2 + (2b^3 + 9ac^2)x + 2b^2c)Q^{(3)}(x) = -a(4b^3 + 27ac^2)x^3
\]

\[
3b(2bx - 3c)P^{(3)}(x) + (9c^2 + 3bcx - 2b^2x^2)Q^{(3)}(x) = (4b^3 + 27ac^2)x^2
\]

For the general polynomial of degree 4:

\[
P^{(4)} = a \cdot x^4 + b \cdot x^2 + c \cdot x + d \quad Q^{(4)} = 4a \cdot x^3 + 2b \cdot x + c
\]
in 5 iterations we have, if \( 3c^2 - 8bd \neq 0 \) the discriminant is \([3]\)

\[
disc = 256a^2b^3 - 128ab^2d^2 + 144abc^2d - 27ac^4 + 16b^4d - 4b^3c^2
\]

and

\[
P^{(4)}((-4cx(16ad^2 + 12b^2d - 3bc^2) + 8x^2(-16abd^2 + 6ac^2d + 4b^3d - b^2c^2) + c^2(9c^2 - 32bd)) + \\
+ Q^{(4)}(cx^2(16ad^2 + 12b^2d - 3bc^2) - x(-64ad^3 + 16b^2d^2 - 38bc^2d + 9c^4) + \\
- 2x^3(-16abd^2 + 6ac^2d + 4b^3d - b^2c^2) - cd(9c^2 - 32bd)) = disc \cdot x^4
\]

\((9)\)

and

\[
P^{(4)}(2c(3bc^2 - 8b^2d + 32ad^2) - 8(-bc^2 + 4b^2d + 6ac^2d - 16abd^2)x + 4ac(9c^2 - 32bd)x^2) + \\
Q^{(4)}(2d(-3bc^2 + 8b^2d - 32ad^2) + 2c(-3bc^2 + 10b^2d - 8ad^2)x + \\
+ 2(-b^2c^2 + 4b^3d + 6ac^2d - 16abd^2)x^2 - ac(9c^2 - 32bd)x^3) = -disc \cdot x^3
\]

\((10)\)
if \(3c^2 - 8bd = 0\) the discriminant is
\[
\text{disc} = 27a^2c^4 + 18ab^3c^2 + 4b^6
\]
and
\[
P^{(4)}(-24ab^3cx^2 + 18abc^3 - 8b^5x) + \\
+ Q^{(4)}\left(\frac{3}{2}bcx(2b^3 - 3ac^2) + 6ab^3cx^3 - \frac{27ac^4}{4} + 2b^5x^2\right) = -\text{disc} x^3
\]
and
\[
P^{(4)}(4b^2(2b^3 + 9ac^2) + 24ab^3cx + 72a^2bc^2x^2) + \\
Q^{(4)}\left(-\frac{3}{2}bc(2b^3 + 9ac^2) - 2b^2(b^3 + 9ac^2)x - 6ab^3cx^2 - 18a^2bc^2x^3\right) = \text{disc} x^2
\]
A more formal case [3] is:
\[
P^{(m)}(x) = x^m + a x + b ; \quad P^{(m)}(x)' = m x^{m-1} + a
\]
so we have successively:
\[
k \in [0, m] : \quad p_k^{(m)} = b \delta_k^0 + a \delta_1^k + \delta_m^k \quad \text{and} \quad q_k^{(m)} = a \delta_0^k + m \delta_{m-1}^k
\]
so
\[
\begin{align*}
p_0^{(m)} &= b ; \quad p_m^{(m)} = 1 ; \quad q_0^{(m)} = a ; \quad q_m^{(m)} = 0 ; \quad \Delta_m = -a \\
k &\in [0, m-1] \quad \left\{
\begin{aligned}
p_k^{(m-1)} &= a^2 \delta_0^k - mb \delta_{m-2}^k + a \delta_m^k \\
q_k^{(m-1)} &= -a \delta_0^k - m \delta_{m-1}^k
\end{aligned}\right.
\end{align*}
\]
then
\[
\begin{align*}
p_0^{(m-1)} &= a^2 ; \quad p_{m-1}^{(m-1)} = a ; \quad q_0^{(m-1)} = -a ; \quad q_{m-1}^{(m-1)} = -m ; \quad \Delta_{m-1} = -(m-1)a^2 \\
k &\in [0, m-2] \quad \left\{
\begin{aligned}
p_k^{(m-2)} &= mab \delta_{m-3}^k + (m-1)a^2 \delta_{m-2}^k \\
q_k^{(m-2)} &= -(m-1)a^2 \delta_0^k + m^2b \delta_{m-2}^k
\end{aligned}\right.
\end{align*}
\]
this structure will repeat, indeed, if
\[
k \in [0, m-j] \quad \left\{
\begin{aligned}
p_k^{(m-j)} &= A_j \delta_{m-j-1}^k + B_j \delta_{m-j}^k \\
q_k^{(m-j)} &= -B_j \delta_0^k + C_j \delta_{m-j}^k
\end{aligned}\right.
\]
then \(p_0^{(m-j)} = 0, p_{m-j}^{(m-j)} = B_j, q_0^{(m-j)} = -B_j, q_{m-j}^{(m-j)} = C_j\), then \(\Delta_{m-j} = B_j^2\) and the next coefficients are:
\[
k \in [0, m-j-1] \quad \left\{
\begin{aligned}
p_k^{(m-j-1)} &= -B_j A_j \delta_{m-j-2}^k - B_j^2 \delta_{m-j-1}^k \\
q_k^{(m-j-1)} &= B_j^2 \delta_0^k + C_j A_j \delta_{m-j-1}^k
\end{aligned}\right.
\]
so we have the recurrence \( A_{j+1} = -B_jA_j \), \( B_{j+1} = -B_j^2 \) and \( C_{j+1} = C_jA_j \) from \( j = 2 \) (with \( A_2 = mab, B_2 = (m - 1)a^2, C_2 = m^2b \)) up to \( j = m - 2 \). At \( j = m - 1 \) we arrive then to:

\[
k \in [0, 1] \begin{cases} 
    p_k^{(1)} = A_{m-1} \delta_0^k + B_{m-1} \delta_1^k \\
    q_k^{(1)} = -B_{m-1} \delta_0^k + C_{m-1} \delta_1^k
\end{cases}
\]  

(18)

with \( p_0^{(1)} = A_{m-1}, p_1^{(1)} = B_{m-1}, q_1^{(1)} = C_{m-1} \) and \( q_0^{(1)} = -B_{m-1} \) so \( \Delta_1 = C_{m-1}A_{m-1} + B_{m-1}^2 \) and the last iteration gives the constant:

\[
\begin{cases} 
    p_0^{(0)} = -B_{m-1}^2 - A_{m-1}C_{m-1} \\
    q_0^{(0)} = C_{m-1}A_{m-1} + B_{m-1}^2
\end{cases}
\]  

(19)

the recurrence on \( B_j, A_j \) and \( C_j \) gives \( (j \geq 2) \)

\[
B_j = -(m-1)a^2)^{2^{j-2}}; \ A_j = m\ a\ b((m-1)a^2)^{-1+2^{j-2}}; \ C_j = (m-1)m^ja^jb^{-1}((m-1)a^2)^{2^{j-2}-j}
\]

so the final constant term is

\[
(m-1)^{-m+2^{m-2}+1}a^{2m-1-m}(m^mb^{m-1} + (m-1)^{m-1}p^m)
\]

we can factorise the constant and the discriminant is then [3]

\[
m^mb^{m-1} + (m-1)^{m-1}a^m
\]  

(20)

III. CONCLUSIONS

The algorithm developed here could be use for formal or numerical calculation of the \( \text{gcd} \) of two polynomials, or the discriminant and the resultant. It doesn’t use matrix manipulation nor determinant calculations and it takes \( O(n) \) steps to achieve the goal. It provide also the two polynomials needed for Bezout identity.

Appendix: 1

The Mathematica program for the algorithm is:

\[
\text{GCDList[\{list1_, list2_, P_, Q_\}] := \{ \\
\text{Drop[First[list] list1 - First[list] list2, 1]}, \\
\text{\}}}
\]
Drop[Last[list2] list1 - Last[list1] list2, -1],
(First[list2] P - First[list1] Q)/x,
Last[list2] P - Last[list1] Q
}

this routine doesn’t test the $\Delta_m$. The variable $P$ and $Q$ are there just for keeping track of the linear combination on $P$ and $Q$ which leads to the next step.

GCDListMax[{list1_, list2_, P_, Q_}] :=
Module[{p1, q1},
  If[Last[list2] First[list1] - Last[list1] First[list2] == 0,
    Return[{list1, RotateRight[First[list1] list2 - First[list2] list1],
      P, x (First[list1] Q - First[list2] P)}],
    p1 = Drop[First[list2] list1 - First[list1] list2, 1];
    q1 = Drop[Last[list2] list1 - Last[list1] list2, -1];
    Return[{p1/Max[p1], q1/Max[q1],
      1/Max[p1]/x (First[list2] P - First[list1] Q),
      1/Max[q1] (Last[list2] P - Last[list1] Q)}]]

this routine test the $\Delta_m$ and use the “max” to normalise the coefficients at each step.

[1] Knuth, D.E. The Art of Computer Programming, Vol. 2. Addison-Wesley, Reading, Mass., 1969

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[3] http://www2.math.uu.se/~svante/papers/sjN5.pdf

[4] Dario A. Bini and Paola Boito. 2007. Structured matrix-based methods for polynomial $\varepsilon$-gcd: analysis and comparisons. In Proceedings of the 2007 international symposium on Symbolic and algebraic computation (ISSAC ’07). Association for Computing Machinery, New York, NY, USA, 9-16. DOI:https://doi.org/10.1145/1277548.1277551

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