Dennis trace map for certain $K$-groups of categories with cofibrations

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Abstract

Let $C$ be a small category with cofibrations. In this paper, we define the $K$-theory and Hochschild homology groups of $C$ of order $Y$, where $Y$ is an ordered finite simplicial set with basepoint. Further, we construct the Dennis trace map between these groups.

Résumé

Soit $C$ une petite catégorie avec cofibrations. Dans cet article, nous définissons ses groupes de $K$-théorie et homologie de Hochschild d’ordre $Y$, où $Y$ est un ensemble simplicial ordonné et fini avec point de base. De plus, nous construisons le morphisme de Dennis entre ces groupes.

1 Introduction

In [3], Waldhausen introduced categories with cofibrations and defined their $K$-theory groups by means of the “$S$-construction”. For any $n \geq 0$, Waldhausen’s $S$-construction associates to a small category $C$ with cofibrations a category $S_nC$. An object of $S_nC$ is a chain of $n$ composable cofibrations in $C$ starting with the zero object (see (2.1)). Then, the objects of the simplicial category $\mathcal{S} = \{S_nC\}_{n \geq 0}$ determine a simplicial set $S = \{\text{obj}(S_nC)\}_{n \geq 0}$. Then, the $K$-theory groups of the category $C$ are defined to be the homotopy groups of the loop space $\Omega |SC|$ of the geometric realization of the simplicial set $SC$.

Let $\text{Ord}_*$ denote the category of finite, totally ordered sets with basepoint. Let $\text{SmCat}_0$ denote the category of small categories with zero objects. The starting point for this article is the fact that the association $n \mapsto S_nC$ extends naturally to a functor $S(C)$ from $\text{Ord}_*$ to $\text{SmCat}_0$. Then, if we take a simplicial object $Y : \Delta^{op} \rightarrow \text{Ord}_*$, i.e., $Y$ is an ordered finite simplicial set with basepoint, we consider the composition (see (2.4))

$$
\begin{align*}
S^Y(C) : \Delta^{op} & \rightarrow \text{Ord}_* \xrightarrow{S(C)} \text{SmCat}_0 \xrightarrow{\text{obj}} \text{Sets}_* \\
& \quad (1.1)
\end{align*}
$$

The purpose of this paper is to study the $K$-groups $K_p^Y(C)$, $p \geq 0$ of $C$ of order $Y$ which we define to be the homotopy groups of the loop space of the geometric realization of the simplicial set $S^Y(C)$ (see (2.5)). If $Y, Y' : \Delta^{op} \rightarrow \text{Ord}_*$ are simplicially homotopy equivalent as simplicial objects of $\text{Ord}_*$, we show that $K_p^Y(C) \cong K_p^{Y'}(C)$. Further, we describe a product structure $K_p^Y(C) \times K_q^{Y'}(D) \rightarrow K_{p+q}(E)$ for a bi-exact functor $F : \text{C} \times \text{D} \rightarrow \text{E}$ (see (2.7)).

In the second part of the paper, we want to define Dennis trace maps from $K_p^Y(C)$ to appropriate Hochschild homology groups. For this, we consider the geometric realization $|\text{CN}(S^Y(C))|$ of the bisimplicial set given by the cyclic nerve $\text{CN}(S^Y(C))$ of the simplicial category $S(C) \circ Y$. We define the Hochschild homology groups
$HH_p^Y(C)$ of $C$ of order $Y$ in terms of the singular homology of $|CN(S^Y(C))|$ with coefficients in a given field $k$ (see [3.3]). Again, if $Y, Y' : \Delta^{op} \rightarrow \text{Ord}_*$ are simplicially homotopy equivalent as simplicial objects of $\text{Ord}_*$, we show that $HH_p^Y(C) \cong HH_p^{Y'}(C)$. We show that the Hochschild homology groups also carry a product $HH_p^Y(C) \otimes HH_p^Z(D) \rightarrow HH_{p+q}^Y(E)$ for a bi-exact functor $F : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ (see [3,6]). Finally, for any $p \geq 0$, we construct a Dennis trace map $D_p^Y : K_p^Y(C) \rightarrow HH_p^Y(C)$.

2 The $K$-groups $K_p^Y(C)$ for a category with cofibrations

In this section and throughout this paper, we let $\mathcal{C}$ be a small category with cofibrations in the sense of Waldhausen [3]. In other words, $\mathcal{C}$ is a category with a zero object together with a subcategory $\text{coC}$ satisfying the axioms (Cof1) and (Cof2) below. The morphisms in $\text{coC}$ will be referred to as cofibrations and denoted by feathered arrows “$\rightarrow$”.

(Cof1) Every isomorphism in $\mathcal{C}$ is a cofibration. For any object $A$ in $\mathcal{C}$, the canonical morphism $0 \rightarrow A$ is a cofibration.

(Cof2) Given a cofibration $A \rightarrow B$, its pushout $C \bigsqcup_A B$ along any other morphism $A \rightarrow C$ exists in $\mathcal{C}$ and the canonical morphism $C \rightarrow C \bigsqcup_A B$ is a cofibration.

Given a cofibration $A \rightarrow B$ in $\mathcal{C}$, its pushout $0 \bigsqcup_A B$ along the morphism $A \rightarrow 0$ will be denoted by $B/A$. The canonical morphism from $B$ to the pushout $B/A = 0 \bigsqcup_A B$ is referred to as a quotient map and denoted by $B \rightarrow B/A$. The sequence $A \rightarrow B \rightarrow B/A$ is referred to as a cofibration sequence. A functor between categories with cofibrations is said to be exact if it takes 0 to 0, preserves cofibrations as well as the pushout diagrams arising from axiom (Cof2).

Given $\mathcal{C}$ as above, we let $S_n\mathcal{C}$ be the simplicial category associated to $\mathcal{C}$ by Waldhausen’s $S$-construction (see [3, 1.3]). More explicitly, for any $n \geq 0$, an object of $S_n\mathcal{C}$ is a sequence $(a_0, a_1, ..., a_{n-1})$ of composable cofibrations:

$$0 = A_0 \xrightarrow{a_0} A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} A_n$$

together with a choice of quotients $A_{ij} = A_i/A_j$. Further, for any $i \leq j \leq k$, $A_{ij} \rightarrow A_{ik} \rightarrow A_{jk}$ is a cofibration sequence.

We now let $\text{Ord}_*$ denote the category of finite totally ordered sets $(Z, *) = \{* < z_1 < z_2 < \ldots < z_k\}$ with basepoint $*$. A morphism $\phi : (Z, *) \rightarrow (Z', *)$ in $\text{Ord}_*$ satisfies $\phi(*) = *$ and $\phi(x) \leq \phi(y) \in Z'$ for any $x \leq y$ in $Z$. Let $\Gamma_*$ denote the subcategory of $\text{Ord}_*$ consisting of the objects $[n] = \{0 < 1 < 2 < \ldots < n\}$ (with basepoint 0) for any $n \geq 0$. Then, the category $\mathcal{C}$ determines a functor:

$$S(\mathcal{C}) : \Gamma_* \rightarrow \text{SmCat}_0 \quad [n] \mapsto S_n\mathcal{C}$$

(2.2)

where $\text{SmCat}_0$ denotes the category of small categories with zero objects. The morphisms in $\text{SmCat}_0$ are functors that preserve zero objects. Given a morphism $\phi : [n] \rightarrow [m]$ in $\Gamma_*$, we have an induced functor:

$$S(\phi) : S_n\mathcal{C} \rightarrow S_m\mathcal{C} \quad (a_0, ..., a_{n-1}) \mapsto (b_0, ..., b_{m-1}) \quad b_j := \prod_{i \leq j} a_i$$

(2.3)

where in (2.3), for any $0 \leq j \leq m - 1$, the cofibration $b_j = \prod_{i \leq \phi^{-1}(j), \leq n} a_i$ is the composition of the cofibrations $a_i$ where $i$ lies in the ordered set $\phi^{-1}(j)$ and $i < n$. In (2.3) it is understood that when $\phi^{-1}(j) \cap \{0, 1, 2, ..., n - 1\}$ is empty, we set $b_j = 1$. It is clear that the functor $S(\mathcal{C}) : \Gamma_* \rightarrow \text{SmCat}_0$ in (2.2) extends to a functor from $\text{Ord}_*$ to $\text{SmCat}_0$ that we continue to denote by $S(\mathcal{C}) : \text{Ord}_* \rightarrow \text{SmCat}_0$. We are now ready to define the $K$-groups of $\mathcal{C}$ with respect to an ordered finite simplicial set $Y$ with basepoint.
Definition 2.1. Let $\mathcal{C}$ be a category as above and let $Y: \Delta^{op} \to \text{Ord}_*$ be an ordered finite simplicial set with basepoint. Let $\text{Sets}_*$ denote the category of pointed sets. We consider the following composition of functors:

$$S^Y(\mathcal{C}) : \Delta^{op} \overset{Y}{\to} \text{Ord}_* \overset{S(\mathcal{C})}{\to} \text{SmCat}_0 \overset{\text{obj}}{\to} \text{Sets}_*$$

where $\text{obj} : \text{SmCat}_0 \to \text{Sets}_*$ is the functor that associates a category in $\text{SmCat}_0$ to its set of objects (with the zero object going to the basepoint). We consider the geometric realization $|S^Y(\mathcal{C})|$ of the pointed simplicial set $S^Y(\mathcal{C})$ in [2.3] and its loop space $\Omega|S^Y(\mathcal{C})|$. Then, we define $K$-theory groups $K^Y_p(\mathcal{C})$ of the category $\mathcal{C}$ of order $Y$ to be the homotopy groups:

$$K^Y_p(\mathcal{C}) := \pi_p(\Omega|S^Y(\mathcal{C})|) \quad \forall \ p \geq 0$$

We now show that homotopic maps of ordered simplicial sets determine identical morphisms on the $K$-groups defined above.

Proposition 2.2. Let $\mathcal{C}$ be a small category with cofibrations. Let $Y, Y' : \Delta^{op} \to \text{Ord}_*$ be ordered finite simplicial sets with basepoint. Let $f, g : Y \to Y'$ be morphisms of simplicial objects of $\text{Ord}_*$ that are simplicially homotopic. Then, $f$ and $g$ induce identical morphisms $K(f)_p = K(g)_p : K^Y_p(\mathcal{C}) \to K^{Y'}_p(\mathcal{C}) \ \forall \ p \geq 0$. In particular, if $Y$ and $Y'$ are simplicially homotopy equivalent as simplicial objects of $\text{Ord}_*$, $K^Y_p(\mathcal{C}) \cong K^{Y'}_p(\mathcal{C})$.

Proof. Let $f_n, g_n : Y_n \to Y'_n$ be the morphisms corresponding to $f$ and $g$ respectively at each level $n$. We are given that $f$ and $g$ are simplicially homotopic morphisms between simplicial objects of $\text{Ord}_*$; it follows that (see [4 § 8.3.1]) there are morphisms $h_{i,n} : Y_n \to Y'_{n+1}$, $0 \leq i \leq n$, $n \geq 0$ in $\text{Ord}_*$ such that $d^Y_{0,n+1}h_{0,n} = f_n$ and $d^Y_{n+1,n+1}h_{n,n} = g_n$ and

$$d^Y_{i+1,n}h_{j,n} = \begin{cases} h_{j-1,n-1}d^Y_{i,n} & \text{if } i < j \\ d^Y_{i+1,n}h_{i-1,n} & \text{if } i = j + 1 \\ h_{j,n-1}d^Y_{i,n} & \text{if } i \geq j + 1 \\ s^Y_{i+1,n}h_{j,n} = \begin{cases} h_{j,n+1} + s^Y_{i,n} & \text{if } i < j \\ h_{j,n+1}s^Y_{i,n} & \text{if } i \geq j \end{cases} \end{cases}$$

Here $d^Y_{i,n} : Y_n \to Y_{n-1}$ (resp. $d^Y_{i+1,n} : Y'_n \to Y'_{n-1}$ and $s^Y_{i,n} : Y_n \to Y_{n+1}$ (resp. $s^Y_{i+1,n} : Y'_n \to Y'_{n+1}$) for $0 \leq i \leq n$ are respectively the face and degeneracy maps of the simplicial object $Y$ (resp. $Y'$) of $\text{Ord}_*$. We now consider the simplicial sets $S^Y(\mathcal{C}) = \{S^Y(\mathcal{C})_n\}_{n \geq 0}$ and $S^{Y'}(\mathcal{C}) = \{S^{Y'}(\mathcal{C})_n\}_{n \geq 0}$ as defined in [2.3]. By definition, for any $n \geq 0$, $S^Y(\mathcal{C})_n = \text{obj}(S(\mathcal{C})_n)$ and $S^{Y'}(\mathcal{C})_n = \text{obj}(S(\mathcal{C})_n')$ along with induced maps $\text{obj}(S(\mathcal{C})(f_n)), \text{obj}(S(\mathcal{C})(g_n)) : S^Y(\mathcal{C})_n \to S^{Y'}(\mathcal{C})_n$. Then, the induced maps $\text{obj}(S(\mathcal{C})(h_{i,n})), \text{obj}(S(\mathcal{C})(s_{i,n})): S^Y(\mathcal{C})_n \to S^{Y'}(\mathcal{C})_{n+1}$ define a simplicial homotopy between the two maps $S^Y(\mathcal{C}) = \{\text{obj}(S(\mathcal{C})(f_n))\}_{n \geq 0}$ and $S^{Y'}(\mathcal{C}) = \{\text{obj}(S(\mathcal{C})(g_n))\}_{n \geq 0} : S^Y(\mathcal{C}) \to S^{Y'}(\mathcal{C})$ of simplicial sets.

It now follows from the definitions in [2.5] that the induced morphisms $K(f)_p := \pi_p(\Omega|S^Y(\mathcal{C})|), K(g)_p := \pi_p(\Omega|S^{Y'}(\mathcal{C})|) : K^Y_p(\mathcal{C}) = \pi_p(\Omega|S^Y(\mathcal{C})|) \to \pi_p(\Omega|S^{Y'}(\mathcal{C})|) = K^{Y'}_p(\mathcal{C})$ on the homotopy groups are identical.

Given small categories with cofibrations $\mathcal{C}, \mathcal{D}$, we consider the category $\mathcal{C} \times \mathcal{D}$. An object of $\mathcal{C} \times \mathcal{D}$ is a pair $(C, D)$ where $C \in \text{obj}(\mathcal{C})$ and $D \in \text{obj}(\mathcal{D})$. For $(C, D), (C', D') \in \text{obj}(\mathcal{C} \times \mathcal{D})$, the collection of morphisms from $(C, D)$ to $(C', D')$ in $\mathcal{C} \times \mathcal{D}$ is given by $\text{Hom}_{\mathcal{C}}(C, C') \times \text{Hom}_{\mathcal{D}}(D, D')$. Given a small category with cofibrations $\mathcal{E}$, we now recall that a functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is said to be bi-exact if it satisfies the following two conditions (see, for instance, [2 Definition 4.2.1]):

1. For any $C \in \text{obj}(\mathcal{C})$ (resp. $D \in \text{obj}(\mathcal{D})$), the functor $F(C, -) : \mathcal{D} \to \mathcal{E}$ (resp. $F(-, D) : \mathcal{C} \to \mathcal{E}$) is exact.
2. Given cofibrations $C \to C'$ and $D \to D'$ in the categories $\mathcal{C}$ and $\mathcal{D}$ respectively, the canonical morphism from $F(C, D) \amalg_{F(C, D)} F(C', D')$ to $F(C', D')$ is a cofibration in $\mathcal{E}$.
Proposition 2.3. Let $Y : \Delta^{op} \to \text{Ord}_*$ be an ordered finite simplicial set with basepoint. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be small categories with cofibrations and let $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ be a bi-exact functor. Then, there exists a product structure:

$$K^Y_p(\mathcal{C}) \times K^Y_q(\mathcal{D}) \to K^Y_{p+q}(\mathcal{E}) \quad \forall \ p, q \geq 0$$  \hspace{1cm} (2.7)

Proof. Given a bi-exact functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$, we consider the induced functors $F_n : S_n \mathcal{C} \times S_n \mathcal{D} \to S_n \mathcal{E}$, $n \geq 0$ defined as follows:

$$(0 = A_0 \to A_1 \to \cdots \to A_n) \times (0 = B_0 \to B_1 \to \cdots \to B_n) \in S_n \mathcal{C} \times S_n \mathcal{D}$$

$$
\begin{array}{c}
(0 = F(A_0, B_0) \to F(A_1, B_1) \to \cdots \to F(A_n, B_n)) \in S_n \mathcal{E}
\end{array}
$$ \hspace{1cm} (2.8)

We note that since $F$ is bi-exact, the morphisms $F(A_k, B_k) \to F(A_{k+1}, B_k)$ and $F(A_k, B_{k+1}) \to F(A_{k+1}, B_{k+1})$ are cofibrations in $\mathcal{E}$ for each $k \geq 0$. Hence, each morphism $F(A_k, B_k) \to F(A_{k+1}, B_{k+1})$ in (2.8) obtained by composing $F(A_k, B_k) \to F(A_{k+1}, B_k)$ and $F(A_{k+1}, B_k) \to F(A_{k+1}, B_{k+1})$ is a cofibration. Using the fact that $F$ is bi-exact, we also see that $F(A_k, 0) = F(0, B_k) = 0$. Hence, it follows that (2.8) induces a morphism $\text{obj}(S_n \mathcal{C}) \times \text{obj}(S_n \mathcal{D}) \to \text{obj}(S_n \mathcal{E})$ of pointed sets. Then, if we consider the ordered finite simplicial set $Y = \{Y_n\}_{n \geq 0}$, we have morphisms:

$$F^Y_n : \text{obj}(S(C)(Y_n)) \times \text{obj}(S(D)(Y_n)) \to \text{obj}(S(E)(Y_n)) \quad \forall \ n \geq 0$$  \hspace{1cm} (2.9)

From (2.9), it follows that we have a morphism $F^Y : S^Y(C) \times S^Y(D) \to S^Y(E)$ of pointed simplicial sets. Passing to geometric realizations and taking loop spaces, we have an induced map $\Omega(S^Y(C)) \wedge \Omega(S^Y(D)) \to \Omega(S^Y(E))$. The result is now clear from the definitions in (2.7).

\hspace{1cm} $\square$

3 Hochschild homology and the Dennis trace map

We recall that a cyclic set is a contravariant functor from Connes’ cyclic category $\Delta C$ to the category $\text{Sets}$ of sets (for details see, for instance, [1, § 6.1.2.1]). Given a small category $\mathcal{A}$, we can associate to it the cyclic set $CN(\mathcal{A}) = \{CN_n(\mathcal{A})\}_{n \geq 0}$ given by its cyclic nerve; in other words, for any $n \geq 0$, we set:

$$CN_n(\mathcal{A}) := \bigsqcap_{(A_0, \ldots, A_n) \in \text{obj}(\mathcal{A})^{n+1}} \text{Hom}_{\mathcal{A}}(A_1, A_0) \times \text{Hom}_{\mathcal{A}}(A_2, A_1) \times \cdots \times \text{Hom}_{\mathcal{A}}(A_0, A_n)$$ \hspace{1cm} (3.1)

By abuse of notation, given a small category $\mathcal{A}$, we will also let $CN(\mathcal{A}) = \{CN_n(\mathcal{A})\}_{n \geq 0}$ denote the underlying simplicial set of the cyclic set $CN(\mathcal{A})$.

Definition 3.1. Let $\mathcal{C}$ be a small category with cofibrations and let $Y : \Delta^{op} \to \text{Ord}_*$ be an ordered finite simplicial set with basepoint. Let $k$ be a given field. We consider the composition of functors:

$$CN(S^Y(\mathcal{C})) : \Delta^{op} \xrightarrow{Y} \text{Ord}_* \xrightarrow{S(\mathcal{C})} \text{SmCat}_0 \xrightarrow{CN} \text{SSets}$$ \hspace{1cm} (3.2)

where $\text{SSets}$ denotes the category of simplicial sets. Let $|CN(S^Y(\mathcal{C}))|$ denote the geometric realization of the bisimplicial set $CN(S^Y(\mathcal{C}))$. Then, we define the Hochschild homologies $HH^Y_p(\mathcal{C})$ of the category $\mathcal{C}$ of order $Y$ over the field $k$ to be the homology groups:

$$HH^Y_p(\mathcal{C}) := H_{p+1}(|CN(S^Y(\mathcal{C}))|, k) \quad \forall \ p \geq 0$$ \hspace{1cm} (3.3)
As noted before, the cyclic nerve of a small category is a cyclic set. As such, the bisimplicial set $CN(S^Y(C))$ in \[8.2\] is actually a “cyclic $\times$ simplicial set” (i.e., a cyclic set in one coordinate and a simplicial set in the other; see, for instance, \[2\] Appendix A.6). Taking the geometric realization first in the simplicial direction, we obtain a cyclic space whose geometric realization carries the structure of an $S^1$-space (see \[16\] § 7.1.4). Then, we can consider the cyclic geometric realization $|CN(S^Y(C))|^g_Y$ of $CN(S^Y(C))$ which is given by the Borel space $|CN(S^Y(C))|^g := ES^1 \times_{S^1} |CN(S^Y(C))|$ (see \[16\] § 7.2.2). Here $ES^1$ is any contractible space on which the topological group $S^1$ has a free action. We can define the cyclic homologies $HC_p(Y)$ of the category $C$ of order $Y$ to be the homology groups $HC_p(Y) := H_{p+1}(|CN(S^Y(C))|^g_Y, k) \forall p \geq 0$.

**Proposition 3.2.** Let $C$ be a small category with cofibrations and let $Y : \Delta^{op} \to Ord_*$ be an ordered finite simplicial set with basepoint. Then, the Hochschild and cyclic homologies of $C$ of order $Y$ fit into a long exact sequence:

$$\ldots \to HH_p(Y) \to HC_p(Y) \to HC_{p-2}(Y) \to HH_{p-1}(Y) \to \ldots \quad (3.4)$$

*Proof.* From \[16\] § 7.2.7, it follows that there exists a homotopy fibration $|CN(S^Y(C))| \to |CN(S^{Y'}(C))|^g_Y \to BS^1$, where $BS^1$ is the classifying space of the topological group $S^1$. Hence, it follows from \[16\] § 7.2.10 that the long exact sequence given by the homology spectral sequence corresponding to this fibration gives us the long exact sequence in \(3.4\). 

**Proposition 3.3.** Let $C$ be a small category with cofibrations. Let $Y, Y' : \Delta^{op} \to Ord_*$ be ordered finite simplicial sets with basepoint. Let $f, g : Y \to Y'$ be morphisms of simplicial objects of $Ord_*$ that are simplicially homotopic. Then, $f$ and $g$ induce identical morphisms $H(f)_p = H(g)_p : HH_p(Y) \to HH_p(Y'), \forall p \geq 0$. In particular, if $Y, Y'$ are simplicially homotopy equivalent as simplicial objects of $Ord_*$, $HH_p(Y) \cong HH_p(Y')$.

*Proof.* For any fixed $n \geq 0$, we consider the simplicial set $CN_n(S^Y(C))$ given by the composition:

$$CN_n(S^Y(C)) : \Delta^{op} \xrightarrow{Y} Ord_* \xrightarrow{S(C)} SmCat_0 \xrightarrow{CN_n} Sets \quad (3.5)$$

Then, as in the proof of Proposition 2.2, it follows that the maps of simplicial sets $CN_n(S^Y(C)), CN_n(S^{Y'}(C)) : CN_n(S^Y(C)) \to CN_n(S^{Y'}(C))$ induced by $f$ and $g$ respectively are simplicially homotopic.

If we consider the geometric realization of the cyclic $\times$ simplicial set $CN(S^Y(C))$ (resp. $CN(S^{Y'}(C))$) in the simplicial direction, we obtain the cyclic space $[CN_n(S^Y(C))]_{n \geq 0}$ (resp. $[CN_n(S^{Y'}(C))]_{n \geq 0}$). Here, the space $[CN_n(S^Y(C))]$ (resp. $[CN_n(S^{Y'}(C))]$) is given by the geometric realization of the simplicial set $CN_n(S^Y(C)) = \{CN_n(S(Y_m))\}_{m \geq 0}$ (resp. $CN_n(S^{Y'}(C)) = \{CN_n(S(Y'_m))\}_{m \geq 0}$). From the above, it follows that the morphisms between the cyclic spaces $[CN_n(S^Y(C))]_{n \geq 0}$ and $[CN_n(S^{Y'}(C))]_{n \geq 0}$ induced respectively by $f$ and $g$ are homotopic in each degree. Hence, the morphisms between the geometric realizations of the cyclic $\times$ simplicial sets $CN(S^Y(C))$ and $CN(S^{Y'}(C))$ induced by $f$ and $g$ respectively are homotopic. It follows from \(5\) that the induced maps $H(f)_p, H(g)_p : HH_p(Y) \to HH_p(Y')$ on the Hochschild homologies are identical. 

From the proof of Proposition 3.3 and the definition $HC_p(Y) := H_{p+1}(|CN(S^Y(C))|^g_Y, k)$, it is clear that given simplicially homotopic maps $f, g : Y \to Y'$ as above, an analogous result holds for induced maps on cyclic homologies.

**Proposition 3.4.** Let $Y : \Delta^{op} \to Ord_*$ be an ordered finite simplicial set with basepoint. Let $C, D$ and $E$ be small categories with cofibrations and let $F : C \times D \to E$ be a bi-exact functor. Then, there exists a product:

$$HH_p(Y) \otimes HH_q(D) \to HH_{p+q}(E) \quad \forall p, q \geq 0 \quad (3.6)$$
Proof. As in the proof of Proposition 3.5, for any \( n \geq 0 \), we have a functor \( F_n : S_n \mathcal{C} \times S_n \mathcal{D} \to S_n \mathcal{E} \) induced by \( F : \mathcal{C} \times \mathcal{D} \to \mathcal{E} \). Given \( Y = \{ Y_n \}_{n \geq 0} : \Delta^{op} \to \text{Ord} \), it is clear that the functors \( F_n, n \geq 0 \) induce \( F^n : S(\mathcal{C})(Y_n) \times S(\mathcal{D})(Y_n) \to S(\mathcal{E})(Y_n) \). Then, for any \( m \geq 0 \), we have a map:

\[
CN_m(F^n)_C : CN_m(S(\mathcal{C})(Y_n)) \times CN_m(S(\mathcal{D})(Y_n)) \to CN_m(S(\mathcal{E})(Y_n))
\]

This gives us a morphism \( S \) of geometric realizations. Combining with the definitions in (3.3), we obtain the map in (3.6).

The map of bisimplicial sets in (3.7) induces a morphism \( |CN(F^V)| : |CN(S^V(\mathcal{C}))| \times |CN(S^V(\mathcal{D}))| \to |CN(S^V(\mathcal{E}))| \) of geometric realizations. Finally, this gives us a product on homologies \( (\forall p, q \geq 0) \):

\[
H_{p+1}(|CN(S^V(\mathcal{C})), k) \otimes H_{q+1}(|CN(S^V(\mathcal{D})), k) \to H_{p+q+2}(|CN(S^V(\mathcal{E})), k)
\]

Comparing (3.8) with the definitions in (3.3), we obtain the map in (3.9).

\[\square\]

Proposition 3.5. Let \( \mathcal{C} \) be a small category with cofibrations and let \( Y : \Delta^{op} \to \text{Ord} \) be an ordered finite simplicial set with basepoint. Then, for each \( p \geq 0 \), there is a morphism \( \Delta^Y_p : K^Y_p(\mathcal{C}) \to \text{HH}^Y_p(\mathcal{C}) \) from the \( K \)-groups of \( \mathcal{C} \) of order \( Y \) to its Hochschild homology groups.

Proof. Given a small category \( \mathcal{A} \) in \( \text{SmCat}_0 \), for any \( n \geq 0 \), we have a map \( CN_0(\mathcal{A}) \to CN_n(\mathcal{A}) \) of sets that takes any \( A \in \text{obj}(\mathcal{A}) = CN_0(\mathcal{A}) \) to \( (A \xrightarrow{1} A \xrightarrow{1} A \xrightarrow{1} \ldots \xrightarrow{1} A) \in CN_n(\mathcal{A}) \) (map \( A \xrightarrow{1} A \) repeated \( n \) times). This gives us a morphism \( CN_0(\mathcal{A}) \to CN(\mathcal{A}) \) of simplicial sets (where \( CN_0(\mathcal{A}) \) is treated as a constant simplicial set) and hence a morphism \( CN_0 \to CN \) of functors from \( \text{SmCat}_0 \) to \( \text{SSets} \). Composing with \( S(\mathcal{C}) \circ Y : \Delta^{op} \to \text{SmCat}_0 \), we have a morphism \( \Delta^Y : S^Y(\mathcal{C}) \to CN(S^Y(\mathcal{C})) \) of functors from \( \Delta^{op} \to \text{SSets} \). The latter induces a map \( |\Delta^Y| : |S^Y(\mathcal{C})| \to |CN(S^Y(\mathcal{C}))| \) of geometric realizations. Combing with the definitions in (3.5) and (3.3), we have a morphism \( \Delta^Y_p : K^Y_p(\mathcal{C}) \to \text{HH}^Y_p(\mathcal{C}) \) given by the composition:

\[
K^Y_p(\mathcal{C}) = \pi_p(\Omega[S^Y(\mathcal{C})]) = \pi_{p+1}(|S^Y(\mathcal{C})|) \xrightarrow{\pi_{p+1}(|\Delta^Y|)} \pi_{p+1}(|CN(S^Y(\mathcal{C}))|) \to H_{p+1}(|CN(S^Y(\mathcal{C})), k) = \text{HH}^Y_p(\mathcal{C})
\]

\[\square\]

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