Mirror Symmetry and Toric Degenerations of Partial Flag Manifolds

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Abstract

In this paper we propose and discuss a mirror construction for complete intersections in partial flag manifolds $F(n_1,\ldots,n_l,n)$. This construction includes our previous mirror construction for complete intersection in Grassmannians and the mirror construction of Givental for complete flag manifolds. The key idea of our construction is a degeneration of $F(n_1,\ldots,n_l,n)$ to a certain Gorenstein toric Fano variety $P(n_1,\ldots,n_l,n)$ which has been investigated by Gonciulea and Lakshmibai. We describe a natural small crepant desingularization of $P(n_1,\ldots,n_l,n)$ and prove a generalized version of a conjecture of Gonciulea and Lakshmibai on the singular locus of $P(n_1,\ldots,n_l,n)$.

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1 Introduction

Using combinatorial dualities for reflexive polyhedra and Gorenstein cones together with the theory of generalized GKZ-hypergeometric functions, one can extend the calculation of the number $n_d$ of rational curves of degree $d$ on the generic quintic threefold in $P^4$ by Candelas, de la Ossa, Green, and Parkes [10] to the case of Calabi-Yau complete intersections in toric varieties [3, 4, 5, 9].

Another class of examples which includes Calabi-Yau quintic 3-folds are Calabi-Yau complete intersections in homogeneous Fano varieties $G/P$ where $G$ is a semisimple Lie group and $P$ is its parabolic subgroup. It is a priori not clear how to find an appropriate mirror family for these varieties, because $G/P$ is not a toric variety in general. In [6], we described a mirror construction for complete interesections in the Grassmannian $G(k,n)$, which turned out to involve a degeneration of $G(k,n)$ to a certain singular toric Fano variety $P(k,n)$ introduced by Sturmfels in [27].

In this paper we consider the extension of our methods to the case of complete intersections in arbitrary partial flag manifolds and give complete proofs of statements from [6].

It turns out that the Plücker embedding of any such flag manifold $F := F(n_1, \ldots, n_l, n)$ admits a flat degeneration to a Gorenstein toric Fano variety $P(n_1, \ldots, n_l, n)$. This deformation has been studied recently by Gonciulea and Lakshmibai in [23, 17, 18]. The “mirror-dual” toric variety $P_{\Delta(n_1,\ldots,n_l,n)}$ associated with a reflexive polyhedron $\Delta(n_1,\ldots,n_l,n)$ has a nice combinatorial description in terms of a certain graph $\Gamma := \Gamma(n_1,\ldots,n_l,n)$ that was introduced by Givental for the case of the complete flag manifolds [15]. The idea of toric degenerations has been discussed in more general framework in [7].

Using residue formula, we compute explicitly a series $\Phi_F := \Phi_F(q_1, \ldots, q_l)$ associated with the graph $\Gamma$ and conjecture that $\Phi_F$ gives a solution to the quantum $D$-module associated with Gromov-Witten classes and quantum cohomology of the partial flag manifold $F$. We note that there is no essential difficulty in checking the conjecture in each particular case at hand, because it involves only calculations in the small quantum cohomology ring of $F$, for which explicit formulas are known [11]. Applying the “trick with factorials” (see [1], or §1.2 below) to a Calabi-Yau complete intersection in $F$, we obtain $\Phi_F$ as a specialization of the toric GKZ-hypergeometric series, from which
the instanton numbers (that is, the virtual numbers of rational curves on
the Calabi-Yau) can be computed via the standard procedure (see e.g. [5]).
As the validity of this trick was shown recently for general homogeneous
spaces [22], this implies that any instanton numbers computed via the usual
“mirror symmetry method” are automatically proven to be correct in all
cases for which our conjecture on $\Phi_F$ holds. The series $\Phi_F$ of complete flag
manifolds has been investigated by Schechtman [26].

The paper is organized as follows. In Section 2 we introduce main com-
binatorial notions used in the definition of a Gorenstein toric Fano variety
$P(n_1, \ldots, n_l, n)$ associated with a given partial flag manifold $F(n_1, \ldots, n_l, n)$.
In Section 3 we investigate singularities of $P(n_1, \ldots, n_l, n)$ and show that
these singularities can be smoothed by a flat deformation to the partial flag
manifold $F(n_1, \ldots, n_l, n)$. As a consequence of our results, we prove a genera-
ized version of a conjecture of Gonciulea and Lakshmibai about the singular
locus of $P(n_1, \ldots, n_l, n)$ [18]. In Section 4 we discuss quantum differential
systems following ideas of Givental [14, 15, 16]. Finally, in Section 5 we ex-
plain the mirror construction for Calabi-Yau complete intersections in partial
flag varieties $F$ and the computations of the corresponding hypergeometric
series $\Phi_F$.

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2 Toric varieties associated with partial flag manifolds

In this section we explain how to associate to an arbitrary partial flag manifold \( F(n_1, \ldots, n_l, n) \) certain combinatorial objects: a graph \( \Gamma(n_1, \ldots, n_l, n) \), a reflexive polytope \( \Delta(n_1, \ldots, n_l, n) \) and a Gorenstein toric Fano variety \( P(n_1, \ldots, n_l, n) \).

2.1 The graph \( \Gamma(n_1, \ldots, n_l, n) \)

Let \( k_1, k_2, \ldots, k_{l+1} \) be a fixed sequence of positive integers. We set \( n_0 = 0 \), \( n_i := k_1 + \ldots + k_i \) (\( i = 1, \ldots, l+1 \)), and \( n := n_{l+1} \). Denote by \( F(n_1, \ldots, n_l, n) \) the partial flag manifold parametrizing sequences of subspaces

\[
0 \subset V_1 \subset V_2 \subset \cdots \subset V_l \subset \mathbb{C}^n,
\]

with \( \dim V_i = n_i \) (\( i = 1, \ldots, l \)). Then

\[
\dim F(n_1, \ldots, n_l, n) = \sum_{i=1}^{l} (n_i - n_{i-1})(n - n_i).
\]

To simplify notations, we shall often write \( F \) instead of \( F(n_1, \ldots, n_l, n) \), if there is no confusion about the numbers \( n_1, \ldots, n_l, n \). By a classical result of Ehresmann ([12]), a natural basis for the integral cohomology of \( F \) is given by the Schubert classes. These are Poincaré dual to the fundamental classes of the closed Schubert cells \( C_w \subset F \), parametrized by permutations \( w \in S_n \) modulo the subgroup

\[
W(k_1, \ldots, k_{l+1}) := S_{k_1} \times \cdots \times S_{k_{l+1}} \subset S_n.
\]

In particular, the Picard group of \( F \), which is isomorphic to \( H^2(F, \mathbb{Z}) \), is generated by \( l \) divisors \( C_1, \ldots, C_l \), corresponding to the simple transpositions \( \tau_i \in S_n \) exchanging \( n_i \) and \( n_i + 1 \).

Definition 2.1.1 Denote by \( \Lambda := \Lambda(n_1, \ldots, n_l, n) \) the standard ladder diagram consisting of unit squares (the number of unit squares in \( \Lambda \) is equal to the dimension of \( F \)) corresponding to the Schubert cell of maximal dimension in the flag manifold \( F \). We place the ladder diagram \( \Lambda \) in the lower
left corner of a $n \times n$-square $Q$. The lower left corner of $\Lambda$ (or of $Q$) will be denoted by $O_0$. We denote by $O_i$ ($i \in \{1, \ldots, l\}$) the common vertex of the diagonal squares $Q_i$ of size $k_i \times k_i$ and $Q_{i+1}$ of size $k_{i+1} \times k_{i+1}$ (the figure below illustrates the case $l = 4$).

![Figure 1](image)

**Definition 2.1.2** Let $\Lambda = \Lambda(n_1, \ldots, n_l, n)$ be the above ladder diagram. We associate with $\Lambda$ the following:

- $D = D(n_1, \ldots, n_l, n)$, the set of centers of unit squares in $\Lambda$: we place a dot at the center of each unit square and call elements of $D$ **dots**.
- $S = S(n_1, \ldots, n_l, n)$, the set consisting of $(l + 1)$ **stars**: an element of $S$ is obtained by placing a star at the $(1/2, 1/2)$-shift of the lower left corner of each of the diagonal squares $Q_i$ ($i \in \{1, \ldots, l + 1\}$).
- $E = E(n_1, \ldots, n_l, n)$, the set of oriented horizontal and vertical segments connecting adjacent elements of $D \cup S$: the vertical segments
are oriented downwards, and the horizontal segments are oriented to the right.

**Definition 2.1.3** \( \Gamma := \Gamma(n_1, \ldots, n_l, n) \) is the oriented graph whose set of vertices is \( D \cup S \), and whose set of oriented edges is \( E \).

Such a graph \( \Gamma \) (without the orientation!) is shown in figure 2 below. The edges of \( \Gamma \) are drawn with solid lines.

**Figure 2**

**Definition 2.1.4** We denote by \( L(D) \cong \mathbb{Z}^{|D|}, L(S) \cong \mathbb{Z}^{|S|}, \) and \( L(E) \cong \mathbb{Z}^{|E|} \) the free abelian groups (or lattices) generated by the sets \( D, S \) and \( E \).

We remark that the lattices \( L(D) \oplus L(S) \) and \( L(E) \) can be viewed as the groups of 0-chains and 1-chains of the graph \( \Gamma \). Then the boundary map in the chain complex is

\[ \partial : L(E) \rightarrow L(D) \oplus L(S), \quad e \mapsto h(e) - t(e), \]
where \( h, t : E \to D \cup S \) are the maps that associate to an oriented edge \( e \in E \) its **head** and its **tail** respectively.

Definition 2.1.5 A **box** \( b \) in \( \Gamma \) is a subset of 4 edges \( \{e, f, g, h\} \subset E \) which form together with their end-points a connected subgraph \( \Gamma_b \subset \Gamma \), such that the topological space associated to \( \Gamma_b \) is homeomorphic to a circle. The set of boxes in \( \Gamma \) will be denoted by \( B \).

It is easy to see that

\[
H_0(\Gamma) = \text{Coker}(\partial) \cong \mathbb{Z} \quad H_1(\Gamma) = \text{Ker}(\partial) \cong \mathbb{Z}^{|B|}
\]

We also consider the projection \( \varrho : L(D) \oplus L(S) \to L(D) \) and the composed map

\[
\delta := \varrho \circ \partial : L(E) \to L(D).
\]

Since one can regard the groups \( L(E) \) and \( L(D) \) together with the homomorphism \( \delta \) as the relative chain complex of the topological pair \( (\Gamma, S) \), we have

\[
H_0(\Gamma, S) = \text{Coker}(\delta) = 0 \quad H_1(\Gamma, S) = \text{Ker}(\delta) \cong \mathbb{Z}^{|B|+I}
\]

Definition 2.1.6 A **roof** \( R_i, i \in \{1, 2, \ldots, l\} \) is the set of \( k_i + k_{i+1} \) edges of \( \Gamma \) forming the oriented path that runs along the upper right “boundary” of \( \Gamma \) between the \( i \)-th and the \( (i + 1) \)-st stars in \( S \).
**Definition 2.1.7** The **corner** $C_b$ of a box $b = \{e, f, g, h\} \in B$ is the pair of edges $\{e, f\} \subset b$ meeting at the lower left vertex of $\Gamma_b$. So a corner $C_b$ contains one vertical edge $e$ and one horizontal edge $f$ such that $h(e) = t(f)$.

The roofs and corners give a decomposition of the set $E$ of edges of the graph $\Gamma$ into a disjoint union of subsets:

$$E = \mathcal{R}_1 \cup \ldots \cup \mathcal{R}_t \cup \bigcup_{b \in B} \mathcal{C}_b.$$ 

This decomposition is shown in the figure below.

**Definition 2.1.8** The **opposite corner** $C_b^-$ of a box $b = \{e, f, g, h\} \in B$ is the pair of edges $\{g, h\} \subset b$ meeting at the upper right vertex of $\Gamma_b$. An opposite corner $C_b^-$ contains one vertical edge $h$ and one horizontal edge $g$, such that $h(g) = t(h)$.

By elementary arguments one obtains:

**Proposition 2.1.9** The elements

$$\rho_b = \sum_{e \in \mathcal{C}_b^-} e - \sum_{e \in \mathcal{C}_b} e,$$
where \( b \) runs over the set \( B \), form a natural \( \mathbf{Z} \)-basis of \( \text{Ker}(\partial) \subset L(E) \). Moreover, the elements
\[
\rho_i = \sum_{e \in \mathcal{R}_i} e \quad (i \in \{1, \ldots, l\})
\]
and
\[
\rho_b = \sum_{e \in \mathcal{C}_b} e - \sum_{e \in \mathcal{C}_b^-} e, \quad b \in B
\]
form a natural \( \mathbf{Z} \)-basis of \( \text{Ker}(\delta) \subset L(E) \).

\[
\blacksquare
\]

2.2 The toric variety \( P(n_1, \ldots, n_l, n) \)

We denote again by \( \delta \) the \( \mathbf{R} \)-scalar extension \( L(E) \otimes \mathbf{R} \rightarrow L(D) \otimes \mathbf{R} \) of the homomorphism \( \delta : L(E) \rightarrow L(D) \).

**Definition 2.2.1** The polyhedron \( \Delta := \Delta(n_1, \ldots, n_l, n) \) associated to \( F \) is the convex hull of the set \( \delta(E) \subset L(D) \otimes \mathbf{R} \), where the set \( E \) is identified with the standard basis of \( L(E) \otimes \mathbf{R} \cong \mathbf{R}^{\mid E\mid} \).

In order to describe the faces of the polyhedron \( \Delta \) we introduce some further combinatorial objects associated to the ladder diagram \( \Lambda \).

**Definition 2.2.2** (i) A **positive path** \( \pi \) in the diagram \( \Lambda \) is a path obtained by starting at one of the points \( O_i \) \( (i = 1, \ldots, l) \) and moving either downwards, or to the left along some \( n \) edges of \( \Lambda \), until the lower left corner \( O_0 \) is reached (see fig. 5). We denote by \( \Pi \) the set of positive paths, and by \( \Pi_i \) the set of positive paths connecting \( O_i \) and \( O_0 \), so that
\[
\Pi = \Pi_1 \cup \ldots \cup \Pi_l
\]
Note that the number of elements in \( \Pi_i \) is \( N_i = \binom{n}{n_i} \)

(ii) A **meander** is a collection of positive paths \( \{\pi_1, \ldots, \pi_l\} \) \( (\pi_i \in \Pi_i) \), with the property that the union
\[
\pi_1 \cup \cdots \cup \pi_l
\]
is a tree with endpoints \( O_0, O_1, \ldots, O_l \).

The set of all meanders is denoted by \( \mathcal{M} \).
Theorem 2.2.3. There is a natural bijection between the codimension 1 faces of $\Delta$ and the set $\mathcal{M}$ of meanders.

Proof: Since every face $\Theta$ of $\Delta$ is given by its supporting hyperplane, it follows from the exact sequence

$$0 \rightarrow \text{Ker}(\delta) \rightarrow L(E) \otimes \mathbb{R} \xrightarrow{\delta} L(D) \otimes \mathbb{R} \rightarrow 0$$

that this hyperplane can be described by a linear function

$$\lambda : L(E) \otimes \mathbb{R} \rightarrow \mathbb{R}$$

which vanishes on $\text{Ker}(\delta)$ and satisfies the conditions

$$\lambda(v) \leq 1, \text{ for all } v \in L(E) \otimes \mathbb{R} \text{ with } \delta(v) \in \Delta,$$

and

$$\delta(v) \in \Theta \text{ iff } \lambda(v) = 1 \text{ and } \delta(v) \in \Delta.$$
Let us show that every meander \( m = \{\pi_1, \ldots, \pi_l\} \in \mathcal{M} \) defines such a linear function \( \lambda_m \). We define the value of \( \lambda_m \) on \( e \in E \) by the formula:

\[
\lambda_m(e) := 1 - \sum_{i: \pi_i \cap e \neq \emptyset} |\mathcal{R}_i|.
\]

(1)

It follows that \( \lambda_m(e) = 1 \) if the meander \( m \) doesn’t intersect \( e \), and \( \lambda_m(e) \) is negative if \( m \) intersects \( e \). Now we show that the linear function \( \lambda_m \) satisfies the requirement \( \lambda_m|_{\text{Ker}(\delta)} = 0 \). By 2.1.9, it suffices to prove that

\[
\sum_{e \in \mathcal{R}_i} \lambda_m(e) = 0,
\]

(2)

for all \( i \in \{1, 2, \ldots, l\} \), and

\[
\sum_{e \in \mathcal{C}_b} \lambda_m(e) = \sum_{e \in \mathcal{C}_b^-} \lambda_m(e).
\]

(3)

for all \( b \in B \).

We remark first that every roof \( \mathcal{R}_i, i \in \{1, 2, \ldots, l\} \) contains exactly one edge \( e_i \in E \) intersecting the positive path \( \pi_i \in m \), for which

\[
\lambda_m(e_i) := 1 - |\mathcal{R}_i| < 0.
\]

On the other hand, \( \lambda(e) = 1 \), for each \( e \in \mathcal{R}_i, e \neq e_i \). It follows that

\[
\sum_{e \in \mathcal{R}_i} \lambda_m(e) = 0 \quad \forall i \in \{1, 2, \ldots, l\}.
\]

Now let \( b \in B \) be an arbitrary box. Since the positive paths of the meander \( m \) form a tree, only the following 3 cases can occur:

Case 1. The meander \( m \) doesn’t intersect edges in \( b \). Then \( \lambda_m(e) = 1 \) for all 4 edges of \( b \), hence (3) holds.

Case 2. The meander \( m \) intersects exactly two edges in \( b \). Then \( m \) intersects exactly one edge \( e' \in b \) belonging to \( \mathcal{C}_b \) and exactly one edge \( e'' \in b \) belonging to \( \mathcal{C}_b^- \). By the formula (1) for \( \lambda_m \), we have \( \lambda_m(e') = \lambda_m(e'') \). So again the relation (3) holds.

Case 3. The meander \( m \) intersects exactly three edges in \( b \). Then \( m \) intersects both edges \( e', e'' \in b \) belonging to \( \mathcal{C}_b^- \) and exactly one edge \( e''' \in b \) belonging to \( \mathcal{C}_b \). By (1)

\[
\lambda_m(e''') = \lambda_m(e') + \lambda_m(e'') - 1.
\]
Again the relation (3) holds.

Therefore, by Proposition 2.1.9, \( \lambda_m|_{\text{Ker}(\delta)} = 0 \).

Let \( \Theta_m \) be the face of \( \Delta \) defined by the supporting affine hyperplane \( \lambda_m(\cdot) = 1 \). We claim that \( \Theta_m \) has codimension 1. Since \( \Theta_m \) is the convex hull of the lattice points in \( \Delta \) corresponding to the edges \( e \in E \) on which \( \lambda_m \) takes the value 1, it is sufficient to show that any linear function \( \lambda' \) satisfying \( \lambda'|_{\text{Ker}(\delta)} = 0 \) and \( \lambda'(v) = 1 \) for all \( v \in L(E) \otimes \mathbb{R} \) with \( \delta(v) \in \Theta_m \) must coincide with \( \lambda_m \). Indeed, by 2.1.9, the value of such a linear function \( \lambda' \) is uniquely determined on each edge \( e \) of each roof \( R_i \) (1 \( \leq i \leq l \)):

\[
\lambda'(e) = \begin{cases} 
1- |R_i| & \text{if } \pi_i \cap e \neq \emptyset \\
1 & \text{otherwise}
\end{cases}
\]

Next we remark that if for some box \( b \in B \) we have shown that

\( \lambda'(e) = \lambda_m(e) \)

holds \( \forall e \in C_b^- \), then, by 2.1.9 and (3), we obtain

\[
\sum_{e \in C_b} \lambda'(e) = \sum_{e \in C_b} \lambda_m(e)
\]

and therefore

\( \lambda'(e) = \lambda_m(e) \ \forall e \in C_b \),

since only one edge \( e \in C_b \) can be intersected by \( m \) (see Cases 1-3). Since we have established the equality \( \lambda'(e) = \lambda_m(e) \) for all \( e \in R_1 \cup \cdots \cup R_l \), the above arguments imply the equality \( \lambda'(e) = \lambda_m(e) \) for all \( e \in E \).

Now we prove that any codimension-1 face \( \Theta \) of \( \Delta \) can be obtained from some meander \( m \in M \). For this purpose, it suffices to show that if a supporting linear function \( \lambda \) defines a face \( \Theta \subset \Delta \), then there exists a meander \( m \in M \) with \( \Theta \subset \Theta_m \). The latter is equivalent to the condition \( \lambda(e) < 1 \) for all edges \( e \in E \) such that \( e \cap m \neq \emptyset \).

First we remark that the linear function \( \lambda \) cannot attain the value 1 on all edges of the roof \( R_1 \), because \( \lambda \) vanishes on the element \( \rho_1 \in \text{Ker}(\delta) \) (see 2.1.9). Now start a positive path \( \pi_1 \) at \( O_1 \) whose first nonempty intersection with edges of the opposite corner \( C_b^- \) of some box \( b \in B \) occurs on an edge \( e_1 \in R_1 \) with \( \lambda(e_1) < 1 \). Since

\[
\sum_{e \in C_b} \lambda(e) = \sum_{e \in C_b^-} \lambda(e),
\]

13
the value of \( \lambda \) on at least one of the two edges of \( C_b \) has to be strictly less then 1. We prolong our path through that edge and enter a next box, where the same reasoning applies. Continuing this, we complete a positive path \( \pi_1 \) from \( O_1 \) to \( O_0 \) crossing only edges where \( \lambda \) is strictly less than 1. Now we repeat this construction for each of the \( O_i \) in subsequent order, starting at \( O_2 \) ... etc. If in the process of constructing a positive path \( \pi_i \) we collide with some already constructed positive path \( \pi_j \) \((j < i)\), we just follow from this point the path \( \pi_j \). In the end, we produce a meander with the required property.

We conclude that \( \Theta_m \) \((m \in \mathcal{M})\) are all the codimension-1 faces of \( \Delta \).

\begin{corollary}
\( \Delta(n_1, \ldots, n_l, n) \) is a reflexive polyhedron.
\end{corollary}

\begin{proof}
The statement follows immediately from Theorem 2.2.3 and from the integrality of the supporting linear function \( \lambda_m \) (see Definition 4.1.5 in [3]).
\end{proof}

\begin{definition}
The complete rational polyhedral fan \( \Sigma = \Sigma(n_1, \ldots, n_l, n) \) is the fan defined as the collection of cones over all faces of \( \Delta \). The toric variety \( P_\Sigma \) associated to the fan \( \Sigma \) will be denoted by \( P = P(n_1, \ldots, n_l, n) \)
\end{definition}

Using one of the equivalent characterizations of reflexive polyhedra (see Theorem 4.1.9 in [3]), we obtain from 2.2.4:

\begin{proposition}
\( P(n_1, \ldots, n_l, n) \) is a Gorenstein toric Fano variety.
\end{proposition}

\section{Further properties of \( P(n_1, \ldots, n_l, n) \)}

\subsection{Singular locus}

\begin{definition}
Define \( \hat{P} = \hat{P}(n_1, \ldots, n_l, n) \) to be the toric variety \( P_\hat{\Sigma} \) associated to the fan \( \hat{\Sigma} \), obtained by refining the fan \( \Sigma \) to a simplicial one, whose one-dimensional cones are the same as the ones of \( \Sigma \) (i.e., they are generated by the lattice vectors \( \{\delta(e), e \in E\} \subset L(D) \)) and whose combinatorial structure is given by the following \(|B| + l \) primitive collections:

\[ R_1, R_2, \ldots, R_l \text{ and } C_b, b \in B. \]
\end{definition}
In other words, the cones of maximal dimension of the fan \( \hat{\Sigma} \) are defined by taking all edges \( e \in E \) except one from each roof and from each corner.

**Proposition 3.1.2** The variety \( \hat{P} \) is a small toric desingularisation of \( P \).

**Proof:** We have to show that each cone of \( \hat{\Sigma} \) is contained in a cone of \( \Sigma \), and each cone of \( \hat{\Sigma} \) is generated by a part of a basis. It suffices to prove the above properties for cones of \( \hat{\Sigma} \) of maximal dimension.

Choose an edge \( e_i \) in each roof \( R_i \) (\( i = 1, \ldots, l \)) and an edge \( f_b \) in each corner \( C_b \), \( b \in B \). This choice determines a \(|D|\)-dimensional cone \( \sigma \) in \( \hat{\Sigma} \). For each \( i = 1, \ldots, l \) there exists a unique positive path from \( O_i \) to \( O_0 \) with the following two properties:

- \( \pi_i \) crosses the edge \( e_i \);
- if \( \pi_i \) enters a box \( b \), then it crosses the edge \( f_b \).

It is easy to see that the union \( \pi_1 \cup \ldots \cup \pi_l \) of these paths is a meander. Indeed, if a union of positive paths as above is not a tree, then there must exist a box \( b \in B \) with both edges of the corner \( C_b \) intersecting the union of positive paths. This contradicts the second of the above conditions. Therefore the set of edges \( \{e_i\}_i \cup \{f_b\}_{b} \) defines uniquely a meander \( m \in \mathcal{M} \), and the cone \( \sigma \) is contained in the cone over the face \( \Theta_m \subset \Delta \). On the other hand, the elements \( \{\rho_i\}_{i=1,\ldots,l} \) and \( \{\rho_b\}_{b} \) together with the set

\[
G_{\sigma} := E \setminus (\{e_i\}_i \cup \{f_b\}_{b})
\]

form a \( \mathbb{Z} \)-basis of \( L(E) \). By \[2.1.9\], the set of generators of \( \sigma \) (i.e., the \( \delta \)-image of \( G_{\sigma} \)) is a \( \mathbb{Z} \)-basis of \( L(D) \).

The desingularization morphism \( \hat{P} \rightarrow P \) induced by the refinement \( \hat{\Sigma} \) of \( \Sigma \) is small (i.e., contracts no divisor), because the sets of 1-dimensional cones in \( \hat{\Sigma} \) and \( \Sigma \) are the same. \( \square \)

There is another way to describe \( \hat{P} \), namely as an iterated toric fibration over \( \mathbb{P}^1 \): One starts with the product of projective spaces

\[
\mathbb{P}^{|R_l|-1} \times \cdots \times \mathbb{P}^{|R_l|-1}
\]
corresponding to the roofs. Then one chooses a corner $C_b$ of a box $b \in B$ whose opposite corner $C_b^-$ belongs to a roof. This choice allows to define a toric bundle over $\mathbb{P}^1$ with the fibre $\mathbb{P}^{[R_i]^{-1}} \times \cdots \times \mathbb{P}^{[R_l]^{-1}}$. Then one adds a new corner $C_{b'}$ of a box $b' \in B$ whose opposite corner $C_{b'}^-$ is contained in the union of roofs and $C_b$ ... etc. At each stage of this process one gets a toric fibre bundle over $\mathbb{P}^1$, with fibre the space constructed in the previous step. Using this description of $\hat{P}$, one obtains an alternative proof of the fact that the anti-canonical divisor on $P$ is Cartier and ample, i.e., that the polyhedron $\Delta$ is reflexive.

**Definition 3.1.3** Let $b \in B$ be an arbitrary box. Define $W_b \subset P$ to be the closure of the torus orbit in $P$ corresponding to the 3-dimensional cone $\sigma_b$ generated by the $\delta$-image of the 4-element set $b$.

**Theorem 3.1.4** The singular locus of $P$ consists of codimension 3 strata $W_b$, $b \in B$. These are conifold strata, i.e. transverse to a generic point of $W_b$ the variety $P$ has an ordinary double point.

**Proof:** Since the desingularization morphism $\varphi : \hat{P} \to P$ is small, $P$ is smooth in codimension 2. Moreover, the singular locus of $P$ is precisely the union of toric strata in $P$ over which the morphism $\varphi$ is not bijective. According to the main result of [25], the exceptional locus $Ex(\varphi) \subset \hat{P}$ (i.e., $\varphi^{-1}(Sing(P))$) is the union of toric strata covered by rational curves contracted by $\varphi$. On the other hand, since $\hat{P}$ is an iterated toric bundle, the Mori cone $NE(\hat{P})$ is a simplicial cone generated by the classes of the primitive relations

$$\sum_{e \in R_i} \delta(e) = 0 \quad i = 1, \ldots, l$$

and

$$\sum_{e \in C_b} \delta(e) = \sum_{e \in C_b^-} \delta(e), \quad b \in B$$

(see sections 2 and 4 in [3]). Since the morphism $\varphi$ is defined by the semiample anticanonical class of $\hat{P}$, it contracts exactly the extremal rays in $NE(\hat{P})$ defined by the primitive relations corresponding to the boxes $b \in B$. The rational curves representing each such class cover the codimension-2 strata $\hat{W}_b$, $b \in B$, corresponding to the 2-dimensional cones in $\hat{\Sigma}$ spanned by
the $\delta$-images of the edges forming the opposite corner $C_b^c$. These strata are contracted, with $\mathbb{P}^1$-fibres, to the codimension 3 strata $W_b$ in $P$ corresponding to the 3-dimensional cones $\sigma_b \in \Sigma$ over the quadrilateral faces $\Theta_b$ of $\Delta$ whose vertices are $\delta$-images of the edges in $b$ ($b \in B$). It follows that $\bigcup_{b \in B} W_b$ is exactly the singular locus of $P$. □

### 3.2 Canonical flat smoothing

Let $F = F(n_1, \ldots, n_l, n)$ be a partial flag manifold. The semiample line bundles $\mathcal{O}(C_1), \ldots, \mathcal{O}(C_l)$ associated to the Schubert divisors $C_1, \ldots, C_l$ define the Plücker embedding of $F$ into a product of projective spaces:

$$
\phi : F \hookrightarrow \mathbb{P}^{N_1-1} \times \cdots \times \mathbb{P}^{N_l-1},
$$

where $N_i = \binom{n}{n_i}$. We will always consider $F$ as a smooth projective variety together with this embedding.

We describe now an embedding of $P$ in the same product of projective spaces.

**Definition 3.2.1** For each $e \in E$, let $H_e$ be the toric Weil divisor on $P$ determined by the 1-dimensional cone of $\Sigma$ spanned by the vector $\delta(e)$.

For every edge $e \in \bigcup_{i=1}^l R_i$ which is part of a roof, denote by $U(e)$ the subset of $E$ consisting of the edge $e$, together with all edges $f \in E$ which are either directly below $e$ in the graph $\Gamma$, if $e$ is horizontal, or directly to the left of $e$, if $e$ is vertical.

Fix $1 \leq i \leq l$. For $e \in R_i$ consider the Weil divisor $\sum_{f \in U(e)} H_f$.

**Lemma 3.2.2** For each $e \in R_i$, the Weil divisor $\sum_{f \in U(e)} H_f$ is Cartier. Moreover, if $e' \in R_i$ is another edge in the same roof, then the associated divisor $\sum_{f' \in U(e')} H_{f'}$ is linearly equivalent to $\sum_{f \in U(e)} H_f$.

**Proof:** To each edge $e \in R_i$, and each positive path $\pi \in \Pi_i$, joining $O_i$ with $O_0$, we associate a linear function $\pi[e] : L(E) \rightarrow \mathbb{Z}$ defined by

$$
\pi[e](g) = \begin{cases} 
0, & \text{if } \pi \cap g = \emptyset \text{ and } g \notin U(e), \\
0, & \text{if } \pi \cap g \neq \emptyset \text{ and } g \in U(e), \\
-1, & \text{if } \pi \cap g = \emptyset \text{ and } g \in U(e), \\
1, & \text{if } \pi \cap g \neq \emptyset \text{ and } g \notin U(e),
\end{cases}
$$
It is an elementary exercise to check that $\pi[e]$ vanishes on the elements

$$\rho_j = \sum_{g \in \mathcal{R}_j} g \ (j \in \{1, \ldots, l\})$$

and

$$\rho_b = \sum_{g \in \mathcal{C}_b} g - \sum_{g \in \mathcal{C}_b} g, \ b \in B.$$  

It follows from Proposition 2.1.9 that $\pi[e]$ descends to a linear function on $L(D)$.

To show that $\sum_{f \in U(e)} H_f$ is Cartier, it suffices to construct for each maximal dimensional cone $\sigma$ in $\Sigma$ an integral linear function

$$\lambda_{\sigma} : L(E) \rightarrow \mathbb{Z},$$

which vanishes on $ker(\delta)$, and satisfies

- $\lambda_{\sigma}(g) = 0$, for all $g \in E$ such that $\delta(g) \in \sigma$ and $g \notin U(e)$.

- $\lambda_{\sigma}(g) = -1$, for all $g \in E$ such that $\delta(g) \in \sigma$ and $g \in U(e)$.

By Theorem 2.2.3, every maximal cone $\sigma$ is determined by a meander $m = (\pi_1, \pi_2, \ldots, \pi_l)$, and $\delta(g) \in \sigma$ if and only if the meander does not intersect the edge $g$ (cf. the proof of 2.2.3). It follows that

$$\lambda_{\sigma} := \pi_i[e]$$

satisfies the above conditions, where $\pi_i$ is the positive path in $m$ which joins $O_i$ with $O_0$. Hence $\sum_{f \in U(e)} H_f$ is Cartier. Note that the functional $\lambda_{\sigma}$ defined above does not depend on the positive paths $\pi_j$ ($j \neq i$) in $m$ that do not intersect the roof $\mathcal{R}_i$.

To prove the second part of the lemma, define an integral linear function

$$\mu : L(E) \rightarrow \mathbb{Z},$$

by

$$\mu(g) = \begin{cases} 
-1, & \text{if } g \in U(e), \\
1, & \text{if } g \in U(e'), \\
0, & \text{otherwise}.
\end{cases}$$

As above, one can easily check that $\mu$ vanishes on $ker(\delta)$, hence it descends to a linear function on $L(D)$. The descended linear function defines a rational function on $P$, whose divisor is $\sum_{f \in U(e)} H_f - \sum_{f' \in U(e')} H_{f'}$. This finishes the proof of the Lemma.  

\[ \square \]
Definition 3.2.3 For each \( i = 1, 2, \ldots, l \), the line bundle associated to the roof \( R_i \) is

\[
L_i := \mathcal{O}( \sum_{f \in U(e)} H_f ),
\]

for some edge \( e \in R_i \).

It follows from Lemma 3.2.2 that \( L_i \) does not depend on the choice of the edge \( e \in R_i \).

We note that for each maximal dimensional cone \( \sigma \) the linear function \( \lambda_{\sigma} \) defined in the proof of 3.2.2 satisfies \( \lambda_{\sigma}(g) \geq 0 \) for all \( g \in E \) such that \( g \notin \sigma \). This implies that the line bundle \( \mathcal{O}(\sum_{f \in U(e)} H_f) \) is generated by global sections (cf. [13], p.68). We will now identify the space of global sections.

The Cartier divisor \( \sum_{f \in U(e)} H_f \) determines a rational convex polyhedron \( \Delta[e] \) in the dual vector space \( L(D)^* \otimes \mathbb{R} \), given by

\[
\Delta[e] = \left\{ \lambda \in L(D)^* \otimes \mathbb{R} : \begin{array}{l}
\lambda(\delta(g)) \geq -1, \forall g \in U(e), \\
\lambda(\delta(g)) \geq 0, \forall g \in E \setminus U(e)
\end{array} \right\}.
\]

The space of global sections of the line bundle \( \mathcal{O}(\sum_{f \in U(e)} H_f) \) has a natural basis, indexed by the lattice points in \( \Delta[e] \). By its very definition, for each positive path \( \pi \in \Pi_i \), the linear function \( \pi[e] \) introduced in the proof of 3.2.2 gives such a lattice point.

Proposition 3.2.4 For each \( i = 1, 2, \ldots, l \), the space of global sections of \( L_i \) has a natural basis parametrized by the set \( \Pi_i \) of positive paths connecting \( O_i \) and \( O_0 \).

Proof: Choose an edge \( e \in R_i \). We have to show that the only lattice points in \( \Delta[e] \) are the ones given by \( \pi[e], \pi \in \Pi_i \). Let \( \lambda : L(E) \rightarrow \mathbb{Z} \) be any linear function vanishing on \( \ker(\delta) \), and such that the descended linear function is in \( \Delta[e] \).

Since on the one hand \( \lambda \) vanishes on every

\[
\rho_j = \sum_{g \in R_j} g \quad (j \in \{1, \ldots, l\}),
\]

and on the other hand \( \lambda \) can be negative only on edges in \( U(e) \), there are exactly two possibilities:
(I) $\lambda(g) = 0$ for all $g \in \cup_{j=1}^{l} R_j$;

(II) $\lambda(e) = -1$, there exists an edge $h$ in the $i$th roof $R_i$ with $\lambda(h) = 1$, and $\lambda(g) = 0$ for all $g \in \cup_{j \neq i} R_j \setminus \{e, f\}$.

If (I) holds, then we start a positive path $\pi$ at $O_i$ that intersects the roof $R_i$ at the edge $e$. Let $b$ be the box containing $e$ in its opposite corner and let $f$ be the other edge in $U(e)$ contained in this box. If $\lambda(f) = 0$, we prolong the path through the edge $f$, and enter a next box $b'$, where we have the same situation as before (i.e., there is another edge $f' \in U(e)$, and if $\lambda(f') = 0$, then we prolong the path through $f'$ etc.). So we may assume that $\lambda(f) = -1$. The edge $f$ is part of the corner $C_b$ of $b$. Let $f''$ be the other edge in $C_b$. Since $\lambda$ vanishes on all elements

$$\rho_b = \sum_{g \in C_b} g - \sum_{g \in C_b^-} g, \quad b \in B,$$

$\lambda(f'')$ must be strictly positive (hence at least 1). We prolong the path $\pi$ through the edge $f''$, and enter a next box $b''$, for which $f''$ is part of the opposite corner. Now $\lambda$ is nonnegative on all four edges of $b''$, and $\lambda(f'') \geq 1$. It follows that there must be an edge $f'''$ in the corner $C_{b''}$, with $\lambda(f''') \geq 1$. We prolong the path through this edge, and enter a next box, where the same reasoning applies. Continuing this, we complete eventually a positive path $\pi$.

Consider the linear function $\nu := \lambda - \pi[e]$ on $L(E)$. By construction, and the definition of $\pi[e]$, the functional $\nu$ is nonnegative on all edges $g \in E$. On the other hand, $\nu$ vanishes on the generators of $\ker(\delta)$ described in $\text{2.1.9}$, since both $\lambda$ and $\pi[e]$ do. We claim that $\nu$ is identically zero on $L(E) \otimes \mathbb{R}$. Indeed, since $\nu$ is nonnegative and $\nu(\sum_{g \in R_j} g) = 0$ for $j = 1, 2, \ldots, l$, it follows that $\nu$ takes the value zero on each edge in the union of all roofs. Similarly, if $\nu$ vanishes on each of the edges of the opposite corner $C_{b''}$ of some box, then it must vanish on each of the edges of the corner $C_b$ as well. From these two facts, one obtains inductively that $\nu$ takes the value zero on every $g \in E$. Hence $\lambda = \pi[e]$.

Assume now that (II) holds. In this case, we start a positive path $\pi$ at $O_i$ that intersects the roof $R_i$ at the edge $h$. A reasoning entirely similar to that in case (I) shows that the path $\pi$ can be completed such that the functional $\lambda - \pi[e]$ is nonnegative on every edge. Hence we obtain again $\lambda = \pi[e]$. ◼

**Definition 3.2.5** The $(|D| + l)$-dimensional cone $C = C(n_1, \ldots, n_l, n)$ associated to the flag manifold $F$ is the convex polyhedral cone in the space
$\text{Im}(\partial) \otimes \mathbb{R}$ spanned by the vectors
\[ \partial(e) \in \text{Im}(\partial) \otimes \mathbb{R} \cong \mathbb{R}^{1^{|D|}+|S|-1} \]
with $e \in E$. We denote by $C^*$ the dual cone in the dual space $\text{Im}(\partial)^* \otimes \mathbb{R}$.

**Definition 3.2.6** Let $\pi \in \Pi$ be any positive path $\pi \in \Pi$. We associate to $\pi$ a linear function
\[ \lambda_\pi : L(E) \rightarrow \mathbb{Z} \]
by setting $\lambda_\pi(e) = 1$ if the path $\pi$ crosses the edge $e$, and $\lambda_\pi(e) = 0$ if it doesn’t.

**Remark 3.2.7** If the path $\pi$ enters a box $b \in B$, then it does so by crossing an edge which is part of the opposite corner $C_b^-$, and it has to leave $b$ by crossing an edge which is part of the corner $C_b^-$. It follows that the corresponding functional $\lambda_\pi$ is zero on $\text{Ker}(\partial) = H_1(\Gamma)$, hence it descends to a functional on $L(E)/\text{Ker}(\partial) = \text{Im}(\partial)$, still denoted by $\lambda_\pi$. By definition, $\lambda_\pi$ is a lattice point in the dual cone $C^* \subset \text{Im}(\partial)^* \otimes \mathbb{R}$.

**Theorem 3.2.8** The semigroup of lattice points in $C^*$ is minimally generated by the set of all $\lambda_\pi$, where $\pi$ runs over the set $\Pi$ of positive paths.

**Proof:** Let $\lambda : L(E) \rightarrow \mathbb{Z}$ with $\lambda|_{\text{Ker}(\partial)} = 0$ and $\lambda(e) \geq 0$ for all $e \in E$. We define the weight of $\lambda$ to be
\[ w(\lambda) = \sum_{e \in E} \lambda(e). \]
It is clear that $w(\lambda) \geq 0$, and that $w(\lambda) = 0$ iff $\lambda = 0$. Note also that $w(\lambda_\pi) = n$ for all $\pi \in \Pi$.

The statement of theorem will be proved if we show that $w(\lambda) \geq n$ for all non-zero integral linear functions $\lambda : L(E) \rightarrow \mathbb{Z}$ with $\lambda|_{\text{Ker}(\partial)} = 0$ and $\lambda(e) \geq 0$ for all $e \in E$, and, moreover, any such $\lambda$ is a non-negative integral linear combination of $\lambda_\pi$ ($\pi \in \Pi$).

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By 2.1.9, the requirement $\lambda|_{\text{Ker}(\partial)} = 0$ is equivalent to $\lambda(\rho_b) = 0$ for all $b \in B$, or
\[
\sum_{e \in C_b} \lambda(e) = \sum_{e \in C_b^\ominus} \lambda(e) \quad \text{for all } b \in B.
\]

As in the proof of Proposition 3.2.4, the above condition implies that if $\lambda \neq 0$, then there exists a roof $R_i$ containing an edge $e$ on which $\lambda$ is nonzero (hence $\lambda(e) \geq 1$). We start to construct a positive path $\pi_i$ from $O_i$ by choosing its edges in such a way that $e$ is the first edge of the graph $\Gamma$ intersected by $\pi_i$. Let $b \in B$ be a box containing $e$ in its opposite corner (i.e., $e \in C_b^\ominus$). Since $\lambda(\rho_b) = 0$, there must be an edge $f \in C_b$ such that $\lambda(f) > 0$ (hence, $\lambda(f) \geq 1$). We prolong the path $\pi_i$ through $f$ and enter a next box $b'$, for which $f \in C_{b'}^\ominus$. Again there must exist an edge $g \in C_{b'}$ such that $\lambda(g) \geq 1$. ... etc. Continuing this process, we eventually obtain a positive path $\pi_i$, which only crosses edges $e$ of $\Gamma$ having the property $\lambda(e) \geq 1$. This shows that $\lambda' := \lambda - \lambda_{\pi_i}$ is again an integral non-negative linear functional on $C$. On the other hand,
\[
w(\lambda') = w(\lambda) - w(\lambda_{\pi_i}) = w(\lambda) - n.
\]
Since $w(\lambda') \geq 0$, this shows that $w(\lambda) \geq n$. By induction on $w(\lambda)$, we can assume that $\lambda'$ is already a non-negative integral linear combination of $\lambda_{\pi_i}$, hence so is $\lambda = \lambda' + \lambda_{\pi_i}$. \qed

**Definition 3.2.9** We define a partial ordering on the set $\Pi$ of positive paths by declaring $\pi \geq \pi'$ if the path $\pi$ runs above the path $\pi'$. Figure 6
Remark 3.2.10 It is easy to see that the set $\Pi$ of positive paths together with the above partial ordering is a distributive lattice. The maximum $\max(\pi, \pi')$ for any two paths $\pi$ and $\pi'$ is the path bounding the union of the regions under $\pi$ and $\pi'$; similarly, $\min(\pi, \pi')$ bounds the intersection of these regions.

Definition 3.2.11 Consider the partition of the set of independent variables $\{z_\pi\}_{\pi \in \Pi}$ into $l$ disjoint subsets

$$\{z_\pi\}_{\pi \in \Pi_i}, \quad i = 1, \ldots, l,$$

and define $X = X(n_1, \ldots, n_l, n)$ to be the subvariety of

$$\mathbb{P}^{N_1-1} \times \mathbb{P}^{N_2-1} \times \cdots \times \mathbb{P}^{N_l-1}$$

given by the $l$-homogeneous quadratic equations

$$z_\pi z_{\pi'} - z_{\min(\pi, \pi')} z_{\max(\pi, \pi')} = 0,$$

for all pairs of noncomparable elements $\pi, \pi' \in \Pi$.

The variety $X$ has been investigated by N. Gonciulea and V. Lakshmibai in the papers [17, 18], where the following result has been proved:
Theorem 3.2.12 (i) $X(n_1,\ldots,n_l,n)$ is a $|D|$-dimensional, irreducible, normal, toric variety.

(ii) There exists a flat deformation

$$\rho : X \longrightarrow \text{Spec}(\mathbb{C}[t])$$

such that $\rho^{-1}(0) = X(n_1,\ldots,n_l,n)$, and $\rho^{-1}(t) = F(n_1,\ldots,n_l,n)$ for all $t \neq 0$.

The next theorem describes an isomorphism

$$X(n_1,\ldots,n_l,n) \cong P(n_1,\ldots,n_l,n).$$

Theorem 3.2.13 Let $P = P(n_1,\ldots,n_l,n)$ be the toric variety associated with a partial flag manifold $F = F(n_1,\ldots,n_l,n)$. The line bundles $L_i$ ($i = 1,\ldots,l$) define an embedding

$$\psi : P \hookrightarrow \mathbb{P}^{N_1-1} \times \mathbb{P}^{N_2-1} \times \cdots \times \mathbb{P}^{N_l-1},$$

whose image coincides with the toric variety $X(n_1,\ldots,n_l,n)$.

Proof: We have $X = \text{Proj}(\mathbb{C}[z_\pi ; \pi \in \Pi]/\mathcal{I})$, with $\mathcal{I}$ the ideal generated by the quadratic polynomials in (4), and $\text{Proj}$ is taken with respect to the $\mathbb{Z}^l$-grading given by

$$\deg(z_\pi) = (0,\ldots,0,1,0,\ldots,0), \text{ if } \pi \in \Pi_i.$$ If we identify $\Pi$ with the set $\{ \lambda_\pi, \pi \in \Pi \} \subset \text{Im}(\partial)^*$, then $\mathcal{I}$ is the toric ideal (see the definition in [27], p. 31) associated to this set (this is a standard fact about the ideals associated to distributive lattices; see for example Thm. 4.3 in [17] for a proof). Let $Y$ be the affine toric variety $\text{Spec}(\mathbb{C}[z_\pi ; \pi \in \Pi]/\mathcal{I})$. By Theorem 3.2.8 and Prop. 13.5 in [27], $Y$ coincides with the affine toric variety defined by the cone $C \subset \text{Im}(\partial) \otimes \mathbb{R}$, i.e. $\mathbb{C}[z_\pi ; \pi \in \Pi]/\mathcal{I}$ can be identified with the ring $\mathbb{C}[S_C]$ determined by the semigroup $S_C$ of lattice points in the dual cone $C^*$. Pick an edge $e_i \in R_i$ for each $1 \leq i \leq l$, and identify the line bundle $L_i$ with $\mathcal{O}(\sum_{f \in U(e_i)} H_f)$. For each $1 \leq i \leq l$, let $\Delta[e_i] \subset L(D)^* \otimes \mathbb{R}$ be the supporting polyhedron for the global sections of the line bundle $L_i$ (cf.
Proposition 3.2.4; recall that the lattice points in $\Delta[e_i]$ are given by the linear functions $\pi[e_i]$ ($\pi \in \Pi_i$) defined in the proof of Lemma 3.2.2. Define now for each $i$ a linear function $v[e_i] : L(E) \rightarrow \mathbb{Z}$ by

$$v[e_i](f) = \begin{cases} 
1, & \text{if } f \in U(e), \\
0, & \text{otherwise}.
\end{cases}$$

It is clear that $v[e_i]$ descends to a functional on $Im(\partial)$, and that for every path $\pi \in \Pi_i$ the functional $\lambda_\pi$ in Definition 3.2.6 coincides with $\pi[e_i] + v[e_i]$. Then the Minkowski sum $\sigma := \sigma_1 + \cdots + \sigma_l$ of these cones coincides with the cone $C^*$, since both $\sigma$ and $C^*$ are generated by the vectors $\{\lambda_\pi, \pi \in \Pi\}$. It follows that $P \cong \text{Proj}(C[S_C])$, where $\text{Proj}$ is taken with respect to the natural $\mathbb{Z}^l$-grading induced by the decomposition of $C^*$ into the Minkowski sum of the $\sigma_i$’s.

For each $1 \leq i \leq l$, choose an ordering $\{\pi_{i,1}, \pi_{i,2}, \ldots, \pi_{i,N_i}\}$ of the set $\Pi_i$. Let $s_{i,j} \in H^0(P, \mathcal{L}_i)$ denote the section determined by $\pi_{i,j}$ ($i = 1, \ldots, l$, $j = 1, \ldots, N_i$). The line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_l$ define a morphism

$$\psi : P \rightarrow \mathbb{P}^{N_1-1} \times \mathbb{P}^{N_2-1} \times \cdots \times \mathbb{P}^{N_l-1},$$

$$x \mapsto \left([s_{\pi_{1,1}}(x) : \ldots : s_{\pi_{1,N_1}}(x)], \ldots, [s_{\pi_{l,1}}(x) : \ldots : s_{\pi_{l,N_l}}(x)]\right).$$

By the above arguments, $\psi$ is the isomorphism

$$\text{Proj}(C[S_C]) \rightarrow \text{Proj}(C[z_\pi; \pi \in \Pi]/I),$$

and the theorem is proved. □

From 3.2.12 and 3.2.13 we obtain

**Corollary 3.2.14** There exists a flat deformation

$$\rho : \mathcal{X} \rightarrow \text{Spec}(C[t])$$

such that $\rho^{-1}(0) = P(n_1, \ldots, n_l, n)$ and $\rho^{-1}(t) = F(n_1, \ldots, n_l, n)$ for all $t \neq 0$.

**Remark 3.2.15** A description of the singular locus of $P$ was conjectured by N. Gonciulea and V. Lakshmibai in the case $F$ a Grassmannian (see [18]). Our Theorem 3.1.4 proves this conjecture and its generalization for arbitrary partial flag manifolds $F$.
4 Quantum differential systems

4.1 Quantum $\mathcal{D}$-module

In order to explain our mirror construction, we give a short overview of the quantum cohomology $\mathcal{D}$-module. The reader is referred to [14, 22] for details.

Let $V$ be a smooth projective variety. Denote $\{T_a\}_{a}$ and $\{T^a\}_{a}$ two homogeneous bases of $H^*(V, \mathbb{Q})$, dual with respect to the Poincaré pairing, i.e. such that

$$<T_a, T^b> = \delta_{a,b}.$$ 

We will consider only the even degree part of $H^*(V, \mathbb{Q})$ and will assume that $H^2(V, \mathbb{Z})$ and $H_2(V, \mathbb{Z})$ are torsion-free. We denote by 1 the fundamental class of $V$.

To simplify the exposition, suppose there is a basis $\{p_i, i = 1, 2, \ldots, l\}$ of $H^2(V, \mathbb{Z})$ consisting of nef divisors. Let $NE(V)$ be the Mori cone of $V$.

Introduce formal parameters $q_i$, $i = 1, \ldots, l$ and let $\mathbb{Q}[[q_1, \ldots, q_l]]$ be the ring of formal power series. The small quantum cohomology ring of $V$ will be denoted by $QH^*(V)$. This is the free $\mathbb{Q}[[q_1, \ldots, q_l]]$-module $H^*(V, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[[q_1, \ldots, q_l]]$, together with a new multiplication given by

$$T_a \circ T_b = \sum_{\beta \in NE(V)} \prod_{i=1}^l q_i^{<p_i, \beta>} \left( \sum_c I^V_{3, \beta}(T_a T_b T_c) T^c \right),$$

with $I^V_{3, \beta}(T_a T_b T_c)$ the 3-point, genus 0, Gromov-Witten invariants of $V$.

Remark 4.1.1 For the case of a partial flag manifold, the small quantum cohomology ring is well-understood. A presentation of this ring is known ([1, 19, 20]), as well as explicit formulas for quantum multiplication ([11]).

The operators of quantum multiplication with the generators $p_i$ give the quantum differential system, a consistent first order partial differential system (see e.g. [14]):

$$\hbar \frac{\partial}{\partial t_i} \bar{S} = p_i \circ \bar{S}, \ i = 1, \ldots, l$$

$$\hbar \frac{\partial}{\partial t_0} \bar{S} = 1 \circ \bar{S},$$
where $\vec{S}$ is an $H^*(V, \mathbb{Q})$-valued function in formal variables $t_0$, and $t_i = \log q_i$, $i = 1, ..., l$. Here $\hbar$ is an additional parameter.

Remarkably, a complete set of solutions to this system can be written down explicitly in terms of the so-called gravitational descendants $[14]$: 

$$\vec{S}_a^V : = e^{t_0/\hbar} \left( e^{pt/\hbar} T_a + \sum_{\beta \in NE(V) - 0} q^{p,\beta} \sum_b T^b \int_{[\overline{M}_{0,2}(V, \beta)]} e^{(e^{pt/\hbar} T_a) - c} \cup e^*_2(T_b) \right).$$

Here $\overline{M}_{0,2}(V, \beta)$ is Kontsevich’s space of stable maps, with evaluation morphisms $e_1, e_2 : \overline{M}_{0,2}(V, \beta) \to V$ at the two marked points, $[\overline{M}_{0,2}(V, \beta)]$ is the virtual fundamental class $[8, 24]$, and $c$ is the first Chern class of the line bundle over $\overline{M}_{0,2}(V, \beta)$ given by the cotangent line at the first marked point. Finally, $pt$ and $q^{p,\beta}$ are shorthand notations for $\sum_i p_i t_i$ and $\prod_i q_i^{p,\beta}$ respectively.

The quantum $\mathcal{D}$-module of $V$ is the $\mathcal{D}$-module generated by the functions $<\vec{S}, 1>$ for all solutions $\vec{S}$ to the above differential system.

A general conjecture about the structure of quantum $\mathcal{D}$-modules is Givental’s version of the mirror conjecture $[15]$:

**Conjecture 4.1.2** There exists a family $(M_q, \mathcal{F}_q, \omega_q)$ of (possibly noncompact) complex manifolds $M_q$, having the same dimension as $V$, together with holomorphic functions $\mathcal{F}_q$, and holomorphic volume forms $\omega_q$ such that the $\mathcal{D}$-module generated by integrals

$$\int_{\gamma \subset M_q} e^{(\mathcal{F}_q + t_0)/\hbar} \omega_q,$$

where $\gamma$ are suitable Morse-theoretic middle dimension cycles of the function $Re(\mathcal{F}_q)$, is equivalent to the quantum $\mathcal{D}$-module of $V$.

### 4.2 Complete intersections

Now assume that $V$ is Fano. Let $X$ be the zero locus of a generic section of a decomposable rank $r$ vector bundle

$$\mathcal{E} = \bigoplus_{j=1}^r L_j,$$
such that each $L_j$ is generated by global sections. In such a situation one can also define a quantum ring $\mathcal{Q}H^*(\mathcal{E})$ over the coefficient ring $\mathbb{Q}[[q_1,\ldots,q_l]]$, which encodes some of the enumerative geometry of rational curves on the complete intersection $X$. This leads to a quantum differential system for $(V,\mathcal{E})$ (see [16, 22]). We define degrees of $q_i$’s by requiring that

$$c_1(TV) - c_1(\mathcal{E}) = \sum (\deg q_i) p_i.$$  

Furthermore, we suppose that all degrees of $q_i$ are nonnegative (this is equivalent to the condition that $-K_X$ is nef). One can write down a similar complete set of solutions to the quantum differential system for $(V,\mathcal{E})$ [16]:

$$\bar{S}^V_a := \frac{e^{t_0/h} (e^{pt/h} T_a + \sum_{\beta \in NE(V)-0} q^{<p,\beta>} \sum_b T_b \int_0^{[\overline{M}_{0,2}(V,\beta)]} \frac{e^*(e^{pt/h} T_a) \cup e^*_2(T_b) \cup E_\beta}{h-c} \right)$$

where $E_\beta$ is the Euler class of the vector bundle on $\overline{M}_{0,2}(V,\beta)$ whose fibre over a point $(C,\mu; x_1, x_2)$ is the subspace of $H^0(\mu^*\mathcal{E})$ consisting of sections vanishing at $x_2$, and the rest of notations are as above.

Consider the cohomology valued functions

$$S_V := \sum_a <\bar{S}^V_a, 1 > T^a$$

and

$$S_\mathcal{E} := \sum_a <\bar{S}^\mathcal{E}_a, c_\tau(\mathcal{E}) > T^a.$$

These functions are given explicitly by the expressions

$$S_V = e^{(t_0+pt)/h} \left( 1 + \sum_{\beta \in NE(V)-0} q^{<p,\beta>} (c_1(\tau))_* \left( \frac{1}{h-c} \right) \right)$$

and

$$S_\mathcal{E} = e^{(t_0+pt)/h} \left( c_\tau(\mathcal{E}) + \sum_{\beta \in NE(V)-0} q^{<p,\beta>} (c_1(\tau))_* \left( \frac{e^*(c_\tau(\mathcal{E})) \cdot E_\beta}{h-c} \right) \right),$$

where now $E_\beta'$ is the Euler class of the vector bundle on $\overline{M}_{0,2}(V,\beta)$ whose fibre over a point $(C,\mu; x_1, x_2)$ is the subspace of $H^0(\mu^*\mathcal{E})$ consisting of sections vanishing at $x_1$. 28
Remark 4.2.1 If we view $X$ as an abstract variety, the general theory in §4.1 gives a $H^\ast(X,\mathbb{Q})$-valued function $S_X$. The functions $S_X$ and $S_E$ are closely related. For example, if $i^\ast : H^2(V,\mathbb{Z}) \to H^2(X,\mathbb{Z})$, where $i : X \hookrightarrow V$ is the inclusion, then $i^\ast(S_X) = S_E$.

Now consider a new cohomology valued function

$$I_E = e^{(t_0+p_0)/\hbar} \left( c_r(E) + \sum_{\beta \in NE(V)-0} q^{<p,\beta>} \prod_j \prod_{m=0}^{<c_1(L_j),\beta>} (c_1(L_j) + m\hbar)(\epsilon_1)_\ast \left( \frac{1}{\hbar - c} \right) \right).$$

In general it is very hard to compute $S_X$ or $S_E$ explicitly. However, note that $I_E$ can be computed directly from the function $S_V$ associated to the ambient manifold, which in many cases turns out to be more tractable. It is therefore extremely useful to have a result relating $S_E$ and $I_E$. Extending ideas of Givental, B. Kim [22] has recently proved the following theorem, which applies to the cases considered in this paper:

**Theorem 4.2.2** If $V$ is a homogeneous space and $X \subset V$ is the zero locus of a generic section of a nonnegative decomposable vector bundle $E$, then $S_E$ and $I_E$ coincide up to a weighted homogeneous triangular change of variables:

$$t_0 \to t_0 + f_0(q)\hbar + f_{-1}(q), \quad \log q_i \to \log q_i + f_i(q), \quad i = 1, \ldots, l,$$

where $f_{-1}, f_0, f_1, \ldots, f_l$ are weighted homogeneous formal power series supported in $NE(V) - 0$, with $\deg f_{-1} = 1$, and $\deg f_i = 0, i = 0, 1, \ldots, l$.

In particular, this implies that the coefficient $\Phi_V$ of the cohomology class $1 \in H^\ast(V,\mathbb{Q})$ in $S_V$, and the coefficient $\Phi_X$ of $c_r(E)$ in $I_E$ (specialized to $\hbar = 1$, $t_0 = 0$) are related in a very simple way. Namely, if

$$\Phi_V = \sum_{\beta \in NE(V)-0} a_\beta q^{<p,\beta>},$$
$$\Phi_X = \sum_{\beta \in NE(V)-0} b_\beta q^{<p,\beta>},$$

then

$$b_\beta = a_\beta \prod_{i=1}^{r} (<c_1(L_i), \beta>). \quad (5)$$

We will refer to Theorem 4.2.2 as the *quantum hyperplane section theorem*. The relation (5) above was called the “trick with factorials” in [3].
5 The mirror construction

In this section we give a partially conjectural mirror construction for partial flag manifolds, and use it to obtain an explicit hypergeometric series as the power series expansion of the integral representation. The case of Calabi-Yau complete intersections is then discussed in some detail.

5.1 Hypergeometric solutions for partial flag manifolds

Let $F = F(n_1, \ldots, n_l, n)$ be a partial flag manifold. In the notations of Section 4, we introduce $l$ independent variables $q_i, i = 1, 2, \ldots, l$ (each $q_i$ corresponds to the roof $R_i$), $|B|$ independent variables $\tilde{q}_b, b \in B$, and $|E|$ independent variables $y_e, e \in E$. Consider the following set of algebraically independent polynomial equations:

- Roof equations: For $i = 1, 2, \ldots, l$
  \[ F_i := \prod_{e \in R_i} y_e - q_i = 0. \]  

- Box equations: For $b = \{e, f, g, h\} \in B$
  \[ G_b := y_e y_f - \tilde{q}_b y_g y_h = 0, \]  

where $\{e, f\} = C_b$.

This set of equations was discussed by Givental [15], and was used to give an integral representation for the solutions to the quantum cohomology differential equations for the special case of complete flag manifolds. The results in that paper were the starting point for our investigations. We describe below Givental's result and our (conjectural) generalization to a general partial flag manifold.

Let $A^{|E|}$ be the complex affine space with the coordinates $y_e (e \in E)$. For fixed parameters values of $(q, \tilde{q}) := (q_1, \ldots, q_l, \ldots, \tilde{q}_b, \ldots)$

we obtain an affine variety:

\[ M_{q, \tilde{q}} := \{ u \in A^{|E|} \mid F_i = 0, i = 1, \ldots, l, G_b = 0, b \in B \}. \]
If all components of \((q, \tilde{q})\) are non-zero, \(M_{q, \tilde{q}}\) is isomorphic to the torus \((\mathbb{C}^*)^{|D|}\).

One can define on \(M_{q, \tilde{q}}\) a holomorphic volume form

\[
\omega_{q, \tilde{q}} := \text{Res}_{M_{q, \tilde{q}}} \left( \left. \frac{\Omega}{\prod_{i=1}^{t} \mathcal{F}_i \prod_{b \in B} \mathcal{G}_b} \right) \right),
\]

where

\[
\Omega := \wedge_{e \in E} dy_e.
\]

Let \(F = \sum_{e \in E} y_e\). Consider the integral

\[
I_{\gamma}(q, \tilde{q}) := \int_{\gamma} e^F \omega_{q, \tilde{q}},
\]

where \(\gamma \in H_{|D|}(M_{q, \tilde{q}}, \text{Re}(F) = -\infty)\). We put

\[
\Phi_{\gamma}(q_1, \ldots, q_l) := I_{\gamma}(q_1, \ldots, q_l, 1, 1, \ldots, 1).
\]

We can now formulate a precise version of conjecture 4.1.2:

**Conjecture 5.1.1** Let \(\vec{S}\) be any solution to the quantum differential system for \(F\). Then the component \(<\vec{S}, 1>\) can be expressed as \(\Phi_{\gamma}(q)\) for some \(\gamma \subset M_{q, 1}\).

**Remark 5.1.2** This conjecture generalizes Givental’s mirror theorem for complete flag manifolds [15].

**Definition 5.1.3** Let \(W\) denote the set of edges in the diagram \(\Lambda\) that intersect \(\Gamma\). We orient the vertical edges in \(W\) upwards and the horizontal edges to the right. Let \(V := B \cup \{0, 1, 2, \ldots, l\}\). For \(w \in W\), the tail \(t(w)\) of \(w\), is defined to be the box \(b_1 \in B\) where \(w\) starts. Similarly, the head \(h(w)\) of \(w\) is the box \(b_2 \in B\) where \(w\) ends. If \(w\) crosses the roof \(\mathcal{R}_i\), so that its “head” is outside the graph \(\Gamma\), we put \(h(w) := i\), and if the “tail” of \(w\) is outside \(\Gamma\), we put \(t(w) = 0\). In the sense of duality of planar graphs, the graph with vertices \(V\), edges \(W\), and incidence given by \(h, t : W \rightarrow V\) is dual to the graph \(\Gamma\) with all stars collapsed to one point.
Definition 5.1.4 For each cone \( \sigma \in \hat{\Sigma} \) of maximal dimension we define a cycle \( \gamma = \gamma_{q, \tilde{q}}(\sigma) \) in \( M_{q, \tilde{q}} \) by

\[
\gamma := \{ y_e | e \in E \text{ with } \delta(e) \in \sigma \}.
\]

Note that the \( y_f \)'s with \( \delta(f) \notin \sigma \) are determined uniquely by the \( y_e \)'s with \( \delta(e) \in \sigma \) and the roof and box equations (6), (7).

The cycle \( \gamma \) is a real torus, of dimension equal to \( \dim \mathbb{C}(M_{q, \tilde{q}}) = \dim \mathbb{C}(F) \). Since it is defined over the entire family of \( M_{q, \tilde{q}} \)'s, it is invariant under monodromy. The integral over this special cycle will be denoted by \( I(q, \tilde{q}) \).

Definition 5.1.5 The specialization \( \Phi_F(q) := I(q_1, \ldots, q_l, 1, \ldots, 1) \) is called the hypergeometric series of the partial flag manifold \( F \).

It turns out that \( I(q, \tilde{q}) \) has a nice power series expansion.

Theorem 5.1.6 \( I(q, \tilde{q}) = \sum_{m_1, \ldots, m_l, \ldots, m_b, \ldots} A_{m_1, \ldots, m_l, \ldots, m_b, \ldots} q_1^{m_1} \cdots q_l^{m_l} \prod_{b \in B} \tilde{q}_b^{m_b}, \) with

\[
A_{m_1, \ldots, m_l, \ldots, m_b, \ldots} := \frac{1}{(m_1!)^{k_1+k_2}} \frac{1}{(m_2!)^{k_2+k_3}} \cdots \frac{1}{(m_l!)^{k_l+k_{l+1}}} B_{m_1, \ldots, m_l, \ldots, m_b, \ldots},
\]

\[B_{m_1, \ldots, m_l, \ldots, m_b, \ldots} := \prod_{w \in W} \left( \frac{m_b(w)}{m_t(w)} \right). \]

Proof: By Leray's theorem, the integral is equal to

\[
\int_{T(\gamma_{q, \tilde{q}}(\sigma))} e^F \frac{\Omega}{\prod_{i=1}^l F_i \prod_{b \in B} G_b},
\]

where \( T \) is the tube map. For \( |q| < 1, |\tilde{q}| < 1 \), the cycle \( T(\gamma_{q, \tilde{q}}(\sigma)) \) is homologous to the cycle

\[
T := \{ y \in \mathbb{A}^{[E]} | y_e | = 1, \text{ for all } e \in E \}
\]

in the complement of the hypersurfaces \( y_e = 0 \). We now expand all the terms in the integrand:

\[
e^F = \sum_{d=0}^{\infty} \frac{1}{d!} F^d = \sum_{d_e \geq 0} \frac{\prod_{e \in E} y_e^{d_e}}{\prod_{e \in E} d_e!}.
\]
\[
\frac{1}{F_i} = \frac{1}{\prod_{e \in \mathcal{R}_i} y_e} \sum_{m_i \geq 0} \left( \frac{q_i}{\prod_{e \in \mathcal{R}_i} y_e} \right)^{m_i},
\]
\[
\frac{1}{G_b} = \frac{1}{y_{e,f} y_f} \sum_{m_b \geq 0} \left( \frac{\tilde{q}_b y_g y_h y_{e,f}}{y_e y_f} \right)^{m_b},
\]

where \(\{e, f\}\) makes up the corner and \(\{g, h\}\) the opposite corner of the box \(b = \{e, f, g, h\}\). The integral picks up precisely the constant coefficient of the following power series in the \(y_e\)'s, with parameters the \(q\)'s and \(\tilde{q}\)'s:

\[
\sum \frac{\prod_{e \in E} y_e^{d_e}}{\prod_{e \in E} d_e!} \prod_{i=1}^l \left( \frac{q_i}{\prod_{e \in \mathcal{R}_i} y_e} \right)^{m_i} \prod_{b \in B} \left( \frac{\tilde{q}_b y_g y_h}{y_e y_f} \right)^{m_b}
\]

Now there are three types of edges.

**Type I:** \(e \in \mathcal{R}_i\) for some \(i = 1, \ldots, l\). Then \(e\) also is edge of the opposite corner of a unique box \(b\). Only the terms with

\[d_e = m_i - m_b\]

will give a contribution.

**Type II:** \(e \in b \cap b'\), for two boxes \(b\) and \(b'\). We can then assume that \(e\) is part of the corner of \(b\), and the opposite corner of \(b'\). Only the terms with

\[d_e = m_b - m_{b'}\]

will give a contribution.

**Type III:** \(e\) is contained in a unique \(b \in B\). In this case \(e\) is part of the corner of \(b\). Only the terms with

\[d_e = m_b\]

will give a contribution.

Hence we see that the integral is given by the series:

\[
\sum_{m_i, m_b \geq 0} \frac{1}{\prod_{e \in E} d_e!} \prod_{i=1}^l q_i^{m_i} \prod_{b \in B} \tilde{q}_b^{m_b},
\]
where for each edge the number \( d_e \) is determined by the \( m_i \) and \( m_b \) by the above equations. We can rewrite this coefficient nicely in terms of binomial coefficients as follows. Each edge \( w \in W \) of the diagram \( \Lambda \) intersects precisely one edge \( e \in E \) of type I or type II. The corresponding coefficient \( d_e \) is then given by

\[
d_e = m_h(w) - m_t(w)
\]

Trivially,

\[
\frac{1}{\prod_{e \in E} d_e!} = \frac{\prod_{w \in W} m_h(w)!}{\prod_{w \in W} m_h(w)! \prod_{e \in E} d_e!}
\]

The heads of arrows \( w \in W \) which are not tails are the heads of arrows intersecting the edges of type I. The tails of arrows \( w \in W \) which are not heads are in bijection to the edges of type III. Hence, when we pull out a factor \( \prod_{i=1}^{l} \prod_{e \in R_i} (m_e!) \) from the denominator of the left hand side of the above equality, the other terms in numerator and denominator can precisely be combined into the product

\[
\prod_{w \in W} \left( \frac{m_h(w)}{m_t(w)} \right).
\]

This proves the result. \( \square \)

**Remark 5.1.7** Note that \( I(q, \tilde{q}) \) is the generalized hypergeometric series for the smooth toric variety \( \hat{P} \) defined in Section 3. The parameters \( q_1, \ldots, q_l \) correspond to the generators of \( Pic(\hat{P}) \) coming from the singular variety \( P \) (the pull-backs of the line bundles \( \mathcal{L}_1, \ldots, \mathcal{L}_l \)), while \( \tilde{q}_b \) correspond to the additional generators of \( Pic(\hat{P}) \).

Theorem 5.1.6 shows that it is very easy to write down the power series expansion for \( I(q, \tilde{q}) \) directly from the diagram.

**Example 5.1.8** \( F(2, 5) \) (the Grassmannian of 2-planes in \( \mathbb{C}^5 \))
Hence we read off:

\[ I(q, \tilde{q}) = \sum_{m,r,s \geq 0} \frac{1}{(m!)^5} \binom{s}{r} \binom{m}{r} \binom{m}{s}^2 q^m \tilde{q}_1 \tilde{q}_2. \]

**Example 5.1.9** \( F(3, 6) \) (the Grassmannian of 3-planes in \( \mathbb{C}^6 \))

Hence we read off:

\[ I(q, \tilde{q}) = \sum_{m,r,s,u,v} \frac{1}{(m!)^6} \binom{r}{u} \binom{v}{u} \binom{s}{r} \binom{s}{v} \binom{m}{s}^2 \binom{m}{v} q^m \tilde{q}_1 \tilde{q}_2 \tilde{q}_3 \tilde{q}_4. \]

**Example 5.1.10** \( F(1, 2, 3, 4) \) (the variety of complete flags in \( \mathbb{C}^4 \))
Hence we read off:

\[ I(q, \bar{q}) = \sum_{m_1, m_2, m_3, r, s, t} A_{m_1, m_2, m_3, r, s, t} q_1^{m_1} q_2^{m_2} q_3^{m_3} \bar{q}_1^{r} \bar{q}_2^{s} \bar{q}_3^{t}, \]

with

\[ A_{m_1, m_2, m_3, r, s, t} = \frac{1}{(m_1!)^2(m_2!)^2(m_3!)^2} \binom{r}{t} \binom{s}{t} \binom{m_1}{r} \binom{m_2}{r} \binom{m_3}{s} \binom{m_3}{s}. \]

A weaker version of Conjecture \[5.1.1\] is

**Conjecture 5.1.11** The series \( \Phi_F := I(q, 1) \) is the coefficient of the cohomology class 1 in the \( H^*(F, \mathbb{Q}) \)-valued function \( S_F \) describing the quantum \( \mathcal{D} \)-module of \( F \), that is,

\[ \Phi_F = 1 + \sum_{m_1, \ldots, m_l \neq 0} \left( \int_{M_{0,2}(F, \mathbb{Q})} e_1^* (e^C t \Omega_F) \right) \frac{e_1^* (e^C t \Omega_F)}{1 - c} \cup e_2^* (1) q_1^{m_1} \cdots q_1^{m_l}, \]

where \( C_t \) stands for \( C_1 t_1 + \cdots + C_l t_l \), with \( \{C_1, \ldots, C_l\} \) the Schubert basis of \( H^2(F, \mathbb{Q}) \), and \( \Omega_F \) is the cohomology class of a point.

**Remark 5.1.12** (i) Besides the case of complete flag manifolds (cf. Remark \[5.1.2\]), there is another case for which the above conjecture agrees with previously known results. Consider the partial flag manifold \( F := F(1, n - 1, n) \)
of flags $V^1 \subset V^{n-1} \subset \mathbb{C}^n$. The Plücker embedding identifies $F$ with a $(1,1)$ hypersurface in $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$. The hypergeometric series for $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ is

$$
\sum_{m_1, m_2 \geq 0} \frac{1}{(m_1!)^n (m_2!)^n} q_1^{m_1} q_2^{m_2},
$$

(cf. [10]), and by the quantum hyperplane section theorem ([10], [22]) we obtain that the hypergeometric series for $F$ is

$$
\sum_{m_1, m_2 \geq 0} \frac{(m_1 + m_2)!}{(m_1!)^n (m_2!)^n} q_1^{m_1} q_2^{m_2}.
$$

(8)

On the other hand, the recipe of Theorem 5.1.6 gives the formula

$$
\sum_{m_1, m_2 \geq 0} \frac{\sum_s \binom{m_1}{s} \binom{m_2}{s}}{(m_1!)^{n-1} (m_2!)^{n-1}} q_1^{m_1} q_2^{m_2}
$$

(9)

for the hypergeometric series of $F$. The identity $\sum_s \binom{m_1}{s} \binom{m_2}{s} = \binom{m_1+m_2}{m_1}$ implies that the series (8) and (9) coincide.

(ii) The quantum Pieri formula [11] gives explicitly the quantum product of a special Schubert class with a general one, and in particular the quantum product of a Schubert divisor with any other Schubert class. Using this, one can write down in reasonably low dimensional cases the quantum differential system for $F$ and reduce this first order system to higher order differential equations satisfied by the components. In particular, one can write down the differential operators annihilating the component $< \vec{S}, 1 >$ of any solution $\vec{S}$, and check by direct computation that the hypergeometric series $\Phi_F(q)$ of Theorem 5.1.6 is annihilated by these operators. In [14] this is done for the Grassmannians containing complete intersection Calabi-Yau 3-folds. For the complete flag manifolds, the operators are known to be the operators for the quantum Toda lattice (see [21]).

5.2 Calabi-Yau complete intersections in $F(n_1, \ldots, n_l, n)$

Recall that $\text{Pic}(F)$ is generated by the line bundles $\mathcal{O}(C_i)$, $i = 1, \ldots, l$, which also generate the (closed) Kähler cone. Hence any line bundle $\mathcal{H}$ on $F$ which is globally generated is of the form $\mathcal{O}(\vec{d}) := \mathcal{O}(\sum_{i=1}^l d^{(i)} C_i)$, with $d^{(i)}$
nonnegative. The common zero locus of \( r \) general sections of line bundles \( O(d_1), \ldots, O(d_r) \) will be denoted by \( X := X_{d_1, \ldots, d_r} \).

Assuming Conjecture 5.1.11, it follows from the quantum hyperplane section theorem that the hypergeometric series \( \Phi_X \) has the expression

\[
\Phi_X = \sum_m \prod_j (\sum_i d_j^{(i)} m_i)! A_{m_1, \ldots, m_l} q_1^{m_1} \cdots q_l^{m_l},
\]

where \( A_{m_1, \ldots, m_l} \) are the coefficients of \( \Phi_F \) in Theorem 5.1.6.

From now on the complete intersection \( X_{d_1, \ldots, d_r} \) is assumed to be a Calabi-Yau manifold. The construction of mirrors described in [6] for the case when \( F \) is a Grassmannian can be extended to the case of a general \( F \) as follows:

\( X \) can be regarded as the intersection of \( F \subset P^{N_1 - 1} \times \cdots \times P^{N_l - 1} \) with \( r \) general hypersurfaces \( Z_j (j = 1, \ldots, r) \) in \( P^{N_1 - 1} \times \cdots \times P^{N_l - 1} \), with \( Z_j \) of multidegree \( (d_1^{(1)}, \ldots, d_l^{(l)}) \). Let \( Y \) be the Calabi-Yau complete intersection of the same hypersurfaces with the toric degeneration \( P \) of \( F \).

For each edge \( e \in \cup_{i=1}^l R_i \), which is part of a roof, define polynomials

\[
\varphi_e(y) := \sum_{f \in U(e)} c_f y_f,
\]

where \( c_f \) are generically chosen complex numbers (recall that we have defined \( U(e) \) as the set consisting of \( e \), together with all edges in the graph \( \Gamma \) which are either directly below \( e \), if \( e \) is horizontal, or directly to the left of \( e \), if \( e \) is vertical).

Partition each of the roofs \( R_i, i = 1, \ldots, l \) into \( r \) disjoint subsets

\[
R_i = R_{i,1} \cup \cdots \cup R_{i,r}
\]

such that \( |R_{i,j}| = d_j^{(j)} \). It follows from Definition 3.2.3 and Theorem 3.2.13 that the toric Weil divisor

\[
\sum_{i=1}^l \sum_{e \in R_{i,j}} H_e
\]

is Cartier, and

\[
O(\sum_{i=1}^l \sum_{e \in R_{i,j}} H_e) \cong \mathcal{L}_1^{d_1^{(1)}} \otimes \cdots \otimes \mathcal{L}_l^{d_l^{(l)}}.
\]
Consider the torus $T$ in the affine space $\mathbb{A}^{|E|}$ given by the following set of equations:

- **Roof equations:** For $i = 1, 2, \ldots, l$
  \[ \prod_{e \in R_i} y_e = 1. \]

- **Box equations:** For $b = \{e, f, g, h\} \in B$
  \[ y_e y_f - y_g y_h = 0, \]

where $\{e, f\}$ form the corner $C_b$ of $b$.

Introduce additional independent variables $x_d$, $d \in D$, one for each generator of the lattice $L(D)$. For every edge $e \in E$, set

\[ x^{\delta(e)} := x_{h(e)}(x_{t(e)})^{-1}, \]

where, as before, $h(e)$ (resp. $t(e)$) is the head (resp. tail) of $e$. The torus $T$ can be identified with $\text{Spec}(\mathbb{C}[x_d, x_d^{-1}; d \in D])$, with the embedding $T \hookrightarrow \mathbb{A}^{|E|}$ induced by the ring homomorphism

\[ \mathbb{C}[y_e; e \in E] \longrightarrow \mathbb{C}[x_d, x_d^{-1}; d \in D], \quad y_e \mapsto x^{\delta(e)}. \]

With this identification, we obtain Laurent polynomials

\[ \varphi_e(x) := \sum_{f \in U(e)} c_f x^{\delta(f)}. \]

For $j = 1, \ldots, r$, let $\nabla_j$ be the Newton polyhedron of the Laurent polynomial

\[ P_j := 1 - \sum_{i=1}^{l} \sum_{e \in R_{i,j}} \varphi_e(x). \]

The polyhedra $\nabla_j$, $j = 1, \ldots, r$ define a nef-partition of the anticanonical class of $P$ (see definitions in [4, 5]), and according to [6] and [7], the mirror family $Y^*$ of the Calabi-Yau complete intersection $Y \subset P$ consists of Calabi-Yau compactifications of the general complete intersections in $T$ defined by the equations

\[ 1 - \sum_{i=1}^{l} \sum_{e \in R_{i,j}} \varphi_e(x) = 0, \quad j = 1, \ldots, r. \] (11)
Conjecture 5.2.1 Let $Y^*_0$ be a Calabi-Yau compactification of a general complete intersection in $T$ defined by the equations (11), with the additional requirement that the coefficients satisfy the relation

$$c_{f_1}c_{f_2} = c_{f_3}c_{f_4}$$

whenever $\{f_1, f_2, f_3, f_4\}$ make up a box $b \in B$, with $\{f_1, f_2\}$ forming the corner $C_b$ of $b$. Then a minimal desingularization of $Y^*_0$ is a mirror of a generic complete intersection Calabi-Yau $X \subset F$.

The main period of the mirror $Y^*$ of $Y$ is given by

$$\Phi_Y = \int_\gamma \text{Res}_{M_q \tilde{q}} \left( \Omega \prod_{j=1}^r \mathcal{E}_j \prod_{i=1}^r \mathcal{F}_i \prod_{b \in B} \mathcal{G}_b \right),$$

where the extra factors $\mathcal{E}_j$ come from the nef-partition of the anti-canonical class of $P$ described above. Specifically,

$$\mathcal{E}_j := 1 - \sum_{i=1}^l \sum_{e \in R_{i,j}} \sum_{f \in U(e)} y_{f}, \quad j = 1, \ldots, r.$$  

By direct expansion of the integral defining $\Phi_Y$ (as in Theorem 5.1.6), followed by the specialization $\tilde{q}_b = 1$, $b \in B$, one gets exactly the hypergeometric series $\Phi_X$.

Finally, we discuss some applications to the case when $X \subset F$ is a Calabi-Yau 3-fold.

First, as discussed in [6], our construction can be interpreted via conifold transitions. Indeed, by Theorem 3.1.4, if $X$ is generic, then its degeneration $Y \subset P$ is a singular Calabi-Yau 3-fold, whose singular locus consists of finitely many nodes. The resolution of singularities $\hat{P} \rightarrow P$ induces a small resolution $\hat{Y} \rightarrow Y$. In other words the (nonsingular) Calabi-Yau’s $X$ and $\hat{Y}$ are related by a conifold transition, and Conjecture 5.2.1 essentially states that their mirrors are related in a similar fashion.

Second, it is well understood (see e.g. [5]) that the knowledge of the hypergeometric series $\Phi_X$ for a Calabi-Yau 3-fold gives the virtual numbers of rational curves on $X$ via a formal calculation. In [6] we have used the hypergeometric series (10) to compute these numbers for complete intersections in Grassmannians.
5.3 List of Calabi-Yau complete intersection 3-folds

Recall that if $F := F(n_1, \ldots, n_l, n)$ is a partial flag manifold, then

$$\dim (F) = \sum_{i=1}^{l} (n_i - n_{i-1})(n - n_i). \quad (12)$$

In the Schubert basis of the Picard group, the anticanonical bundle of $F$ is given by

$$\omega_F^{-1} = \mathcal{O}\left(\sum_{i=1}^{l} (n_{i+1} - n_{i-1})C_i\right). \quad (13)$$

A (general) complete intersection Calabi-Yau 3-fold in $F$ is the common zero locus of $r := \dim (F) - 3$ general sections $s_j \in H^0(F, \mathcal{O}(d_j))$, where $\mathcal{O}(d_j)$, $j = 1, 2, \ldots, r$ are line bundles with $\bigotimes_{j=1}^{r} \mathcal{O}(d_j) = \omega_F^{-1}$. Hence, if $F$ contains a complete intersection Calabi-Yau 3-fold, then necessarily

$$\dim (F) \leq 3 + \sum_{i=1}^{l} (n_{i+1} - n_{i-1}) = n + n_l - n_1 + 3. \quad (14)$$

**Proposition 5.3.1** If $F := F(n_1, \ldots, n_l, n)$ is a partial flag manifold containing a complete intersection Calabi-Yau 3-fold, and $F$ is not a projective space, or one of the manifolds $F(1, n_1, n)$, then $n \leq 7$.

**Proof:** Using (14), after some manipulation, one can rewrite the inequality (14) as

$$(n_1 - 1)(n - n_1 - 1) + (n_2 - n_1)(n - n_2 - 1) + \ldots + (n_l - n_{l-1})(n - n_l - 1) \leq 4. \quad (15)$$

There are two cases.

**Case 1:** $n_1 > 1$. Then it is easy to see that $(n_1 - 1)(n - n_1 - 1) > 4$ for $n \geq 8$, unless $n_1 = n - 1$, in which case $F$ is a projective space.

**Case 2:** $n_1 = 1$. If $l = 1$, then $F$ is a projective space, so we may assume $l \geq 2$. As above, $(n_2 - 1)(n - n_2 - 1) > 4$ for $n \geq 8$, unless $n_2 = n - 1$, in which case $F = F(1, n - 1, n)$. \hfill \Box

**Remark 5.3.2** The flag manifold $F(1, n - 1, n)$ sits as a $(1, 1)$ hypersurface in $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$. Hence these cases (as well as the case when $F$ is projective space) can be viewed as particular instances of complete intersection Calabi-Yau’s in toric varieties.
We list below all the partial flag manifolds (not excluded by proposition 5.3.1) for which the inequality (14) is satisfied. The anticanonical class of $F$, denoted by $-K_F$, is expressed in terms of the natural Schubert basis of the Picard group. The last column of the table below contains the possible splittings of the anticanonical class into $\dim(F) - 3$ nonnegative divisors.

In general, there is a natural duality isomorphism

$$F(n_1, \ldots, n_l, n) \cong F(n - n_l, \ldots, n - n_1, n).$$

(16)

This is taken into account by listing only one of the two isomorphic flag manifolds. It may also be that the flag manifold is self-dual, i.e. (16) is an automorphism, and two families of complete intersection Calabi-Yau 3-folds corresponding to different splittings of the anticanonical class are interchanged by the duality automorphism. Whenever this happens (e.g., when $F$ parametrizes complete flags), only one of the two splittings of $-K_F$ is listed.
| $n$ | $F$ | $\dim(F)$ | $-K_F$ | splitting of $-K_F$ |
|-----|-----|----------|-------|-------------------|
| 7   | $F(2,7)$ | 10 | 7 | 7(1) |
| 7   | $F(1,2,7)$ | 11 | (2,6) | 2(1,0) + 6(0,1) |
| 7   | $F(1,5,7)$ | 14 | (5,6) | 5(1,0) + 6(0,1) |
| 7   | $F(1,2,6,7)$ | 15 | (2,5,5) | 2(1,0,0) + 5(0,1,0) + 5(0,0,1) |
| 6   | $F(2,6)$ | 8 | 6 | (2) + 4(1) |
| 6   | $F(3,6)$ | 9 | 6 | 6(1) |
| 6   | $F(1,2,6)$ | 9 | (2,5) | (2,0) + 5(0,1), (1,0) + (1,1) + 4(0,1), 2(1,0) + (0,2) + 3(0,1) |
| 6   | $F(1,3,6)$ | 11 | (3,5) | 3(1,0) + 5(0,1) |
| 6   | $F(1,4,6)$ | 11 | (4,5) | (2,0) + 2(1,0) + 5(0,1), 3(1,0) + (1,1) + 4(0,1), 4(1,0) + (0,2) + 3(0,1) |
| 6   | $F(1,2,5,6)$ | 12 | (2,4,4) | (2,0,0) + 4(0,1,0) + 4(0,0,1), (1,0,0) + (1,1,0) + 3(0,1,0) + 4(0,0,1), (1,0,0) + (1,0,1) + 4(0,1,0) + 3(0,0,1), 2(1,0,0) + (0,2,0) + 2(0,1,0) + 4(0,0,1), 2(1,0,0) + (0,1,1) + 3(0,1,0) + 3(0,0,1), 2(1,0,0) + 4(0,1,0) + (0,0,2) + 2(0,0,1) |
| 6   | $F(1,3,5,6)$ | 13 | (3,4,3) | 3(1,0,0) + 4(0,1,0) + 3(0,0,1) |
| 5   | $F(2,5)$ | 6 | 5 | (3) + 2(1), 2(2) + (1) |
| 5   | $F(1,2,5)$ | 7 | (2,4) | (2,0) + (0,2) + 2(0,1), (1,0) + (1,1) + (0,2) + (0,1), 2(1,1) + 2(0,1), 2(1,0) + 2(0,2), (1,0) + (1,2) + 2(0,1), (2,1) + 3(0,1) |
| 5   | $F(2,3,5)$ | 8 | (3,3) | (1,0) + (2,0) + 3(0,1), 2(1,0) + (1,1) + 2(0,1) |
| 5   | $F(1,3,5)$ | 8 | (3,4) | (3,0) + 4(0,1), (1,0) + (2,1) + 3(0,1), (1,1) + (2,0) + 3(0,1), (1,0) + (2,1) + 2(0,1), (1,0) + (2,0) + (0,2) + 2(0,1), 3(1,0) + 2(0,2), 3(1,0) + (0,1) + (0,3), 2(1,0) + (1,2) + 2(0,1) |
| $n$ | $F$ | $\text{dim}(F)$ | $-K_F$ | splitting of $-K_F$ |
|-----|-----|-----------------|--------|------------------|
| 5   | $F(1, 2, 4, 5)$ | 9 | (2, 3, 3) | 2(1, 0, 0) + (0, 3, 0) + 3(0, 0, 1) |
|     |              |    |         | 2(1, 0, 0) + (0, 1, 0) + (0, 3, 3) |
|     |              |    |         | (2, 1, 0) + 2(0, 1, 0) + 3(0, 0, 1) |
|     |              |    |         | (2, 1, 0) + 3(0, 1, 0) + 2(0, 0, 1) |
|     |              |    |         | (1, 2, 0) + (1, 0, 0) + (0, 1, 0) + 3(0, 0, 1) |
|     |              |    |         | 2(1, 0, 0) + (0, 2, 1) + (0, 1, 0) + 2(0, 0, 1) |
|     |              |    |         | 2(1, 0, 0) + (0, 1, 2) + 2(0, 1, 0) + (0, 0, 1) |
|     |              |    |         | (1, 0, 0) + (1, 0, 2) + 2(0, 1, 0) + (0, 0, 1) |
|     |              |    |         | (1, 1, 0) + (1, 0, 0) + 2(0, 1, 0) + 2(0, 0, 1) |
|     |              |    |         | (2, 0, 0) + (0, 2, 0) + (0, 1, 0) + 3(0, 0, 1) |
|     |              |    |         | (2, 0, 0) + (0, 0, 2) + 3(0, 1, 0) + (0, 0, 1) |
|     |              |    |         | 2(1, 0, 0) + (0, 2, 0) + (0, 1, 0) + (0, 0, 2) + (0, 0, 1) |
|     |              |    |         | (1, 1, 0) + (0, 1, 0) + 3(0, 1, 0) |
|     |              |    |         | (1, 0, 0) + (1, 0, 1) + 2(0, 1, 0) + 2(0, 0, 1) |
|     |              |    |         | (1, 0, 0) + (1, 0, 1) + (0, 1, 1) + (0, 0, 1) |
|     |              |    |         | 2(1, 0, 0) + 2(0, 1, 0) + (0, 0, 1) |
|     |              |    |         | (1, 0, 1) + (0, 2, 0) + (1, 0, 1) + 2(0, 0, 1) |
|     |              |    |         | (1, 0, 1) + (1, 0, 1) + (0, 1, 0) + 3(0, 0, 1) |
|     |              |    |         | (1, 0, 1) + (0, 2, 0) + (1, 0, 1) + (0, 0, 1) |
|     |              |    |         | (1, 1, 0) + (1, 0, 0) + 2(0, 1, 0) + (0, 0, 2) + (0, 0, 1) |
|     |              |    |         | (1, 1, 0) + (0, 1, 0) + (0, 0, 2) + (0, 0, 2) |
|     |              |    |         | (1, 1, 0) + (0, 2, 0) + (0, 1, 0) + 2(0, 0, 1) |
|     |              |    |         | (1, 1, 0) + (0, 0, 2) + (0, 0, 1) + (0, 0, 1) |

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