On the relation between the complex Toda and Volterra lattices

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Abstract
We give an analytic, sufficient condition for the existence of the Bäcklund transformation between the semiinfinite Toda and Volterra lattices, in the complex case, extending previous results given for the real case.

1 Introduction
The semiinfinite Toda lattice

\[
\begin{align*}
\dot{\alpha}_n(t) &= \lambda_{n+1}^2(t) - \lambda_n^2(t) \\
\dot{\lambda}_{n+1}(t) &= \frac{\lambda_{n+1}(t)}{2}(\alpha_{n+1}(t) - \alpha_n(t))
\end{align*}
\]

(1)

(where dot means differentiation with respect to \( t \in \mathbb{R} \)) is a well-known differential system with a remarkable property: integrability (cf. [3]). Integrability reveals itself as a multitude of properties, like the existence of an infinite set of conservation laws, symmetries, solutions that can be written explicitly, inverse scattering solvability, etc. In this letter we center our attention in one of these features: the existence of a so-called Bäcklund transformation that, from a given solution of the system, produces another, different solution.

The case of the real Toda lattice, i.e. when the dependent variables \( \alpha_n \) and \( \lambda_n \) are real functions, has been thoroughly studied, being one of the cornerstones of modern integrability theory. The complex case, when \( \alpha_n \) and \( \lambda_n \) are complex functions, has been found (much later) to have physical applications,
like the description of the asymptotic behaviour of some N-soliton solutions of
the nonlinear Schrödinger equation [6]. The algebraic nature of both problems
is the same, in the sense that they admit the same Lax pair representation:

\[ \dot{J}(t) = [J(t), K(t)] \]

is equivalent to (1), where \([J(t), K(t)]\) is the commutator \(J(t)K(t) - K(t)J(t)\)
of the operators represented by the semiinfinite matrices
\[
J(t) = \begin{pmatrix}
\alpha_1(t) & \lambda_2(t) & 0 & 0 & \cdots \\
\lambda_2(t) & \alpha_2(t) & \lambda_3(t) & 0 & \cdots \\
\lambda_3(t) & \alpha_3(t) & \ddots & \ddots & \ddots \\
\cdots & \cdots & \cdots & \ddots & \ddots \\
\end{pmatrix}, \quad K(t) = \frac{1}{2} \begin{pmatrix}
0 & -\lambda_2(t) & 0 & 0 & \cdots \\
\lambda_2(t) & 0 & -\lambda_3(t) & 0 & \cdots \\
\lambda_3(t) & 0 & \ddots & \ddots & \ddots \\
\cdots & \cdots & \cdots & \ddots & \ddots \\
\end{pmatrix}.
\]

Nevertheless, questions like the prolongability of solutions in time are not easily
treated by the techniques used in the real case. In fact, the first reference to these
problems we know of is the recent preprint [4], where an algebraic treatment is
used.

As it has been said above, we are going to study here an special property of
the Toda lattice, the existence of a Bäcklund transformation. For the real Toda
lattice, this problem has been analysed in [5] and [7]. We extend this analysis
to the complex Toda lattice. We are going to provide an analytical criterion,
related with the spectral structure of the matrix \(J(t)\). We believe it is the
first criterion independent of special algebraic structures, for the complex lattice
case, and one of its good features is that, once the correct setting and structures
have been introduced (cf. [10]), the mathematical proofs are straightforward,
basically induction.

2 The existence of the complex Bäcklund transformation

The Bäcklund transformation appears through the intervention of a second sys-
tem, the semiinfinite Volterra or Langmuir lattice

\[ \dot{\gamma}_{n+1}(t) = \gamma_{n+1}(t) \left( \gamma_{n+2}(t) - \gamma_n(t) \right), \quad n \in \mathbb{N}, \quad (\gamma_1 \equiv 0). \]

We call solution of system (1) (resp. (2)) to any sequence of differentiable,
complex valued functions of a real variable \(\{\lambda_n(t), \alpha_n(t)\}\) (resp. \(\{\gamma_n(t)\}\)), \(n \in \mathbb{N}\),
satisfying the system.

Consider the tridiagonal matrix \(J(t)\). We use the representation

\[ J(t) = \Re J(t) + i \Im J(t), \]

where, if \(A_{r,s}\) is the element in the \(r\)-th row and \(s\)-th column of a matrix \(A\), we
have \((\Re J(t))_{r,s} = \Re J_{r,s}(t)\), \((\Im J(t))_{r,s} = \Im J_{r,s}(t)\) and \(\Re z\), \(\Im z\) denote the
real and imaginary part of \( z \). An important set of conditions will be

\[
\Re J(t) \text{ selfadjoint, } \| \Im J(t) \| < +\infty, \quad t \in \mathbb{R},
\]

where \( \| \Im J(t) \| = \sup_{\| x \|=1} \| \Im J(t)x \| . \)

**Theorem 1.** Let \( \{ \lambda_n(t) \}, \{ \alpha_n(t) \}, \{ \gamma_n(t) \}, n \in \mathbb{N} \), such that (5) are satisfied and the relations

\[
\begin{align*}
\lambda_1(t) & \equiv \gamma_1(t) \equiv 0, \\
\lambda_{n+1}^2(t) & = \gamma_{2n}(t)\gamma_{2n+1}(t), \quad \alpha_n(t) = \gamma_{2n-1}(t) + \gamma_{2n}(t) + C, \quad n \in \mathbb{N},
\end{align*}
\]

hold for some \( C \in \mathbb{C} \) with

\[
d(C, \text{Conv}(\sigma(\Re J(t)))) > \| \Im J(t) \| \quad \text{for any } t \in \mathbb{R}, \tag{7}
\]

where \( \text{Conv}(\cdot) \) denotes the convex hull and \( \sigma(\cdot) \) the spectrum. Then \( \{ \gamma_n(t) \}, n \in \mathbb{N} \), is a solution of the Volterra lattice (4) if and only if \( \{ \lambda_n(t), \alpha_n(t) \}, n \in \mathbb{N} \), is a solution of the Toda lattice (3).

If \( J(t) \) is selfadjoint, then \( \Im J(t) = 0 \) in (4). In this case the constant \( C \) can be substituted by any value not belonging to the convex hull of the spectrum of \( J(t) \), giving rise to the following relation between the Volterra and real Toda lattices.

**Corollary 1.** Let \( J(t) \) be selfadjoint for all \( t \in \mathbb{R} \). Consider sequences \( \{ \lambda_n(t) \}, \{ \alpha_n(t) \}, \{ \gamma_n(t) \}, n \in \mathbb{N} \) satisfying (4), with \( C \notin \text{Conv}(\sigma(J(t))) \) for each \( t \in \mathbb{R} \). Then \( \{ \gamma_n(t) \}, n \in \mathbb{N} \), is a solution of (3) if and only if \( \{ \lambda_n(t), \alpha_n(t) \}, n \in \mathbb{N} \), is a solution of (1).

In addition, the relations (6) between solutions of (1) and (3) yield a relation between two different solutions of (1), as stated in the following result, supplementary to Theorem 1.

**Theorem 2.** Let \( \{ \lambda_n(t), \alpha_n(t) \}, n \in \mathbb{N} \), be a solution of (1) satisfying (5), and let \( C \in \mathbb{C} \) such that (7) holds. Then there exists a solution \( \{ \tilde{\lambda}_n(t), \tilde{\alpha}_n(t) \}, n \in \mathbb{N} \), of (1) and a solution \( \{ \gamma_n(t) \}, n \in \mathbb{N} \), of (3), such that

\[
\begin{align*}
\lambda_{n+1}^2(t) & = \gamma_{2n}(t)\gamma_{2n+1}(t), \quad \alpha_n(t) = \gamma_{2n-1}(t) + \gamma_{2n}(t) + C, \quad n \in \mathbb{N}, \\
\tilde{\lambda}_{n+1}^2(t) & = \gamma_{2n+1}(t)\gamma_{2n+2}(t), \quad \tilde{\alpha}_n(t) = \gamma_{2n}(t) + \gamma_{2n+1}(t) + C, \quad n \in \mathbb{N}.
\end{align*}
\]

For each fixed \( C \in \mathbb{C} \) satisfying (7), the sequences \( \{ \tilde{\lambda}_n(t), \tilde{\alpha}_n(t) \}, \{ \gamma_n(t) \}, n \in \mathbb{N} \), verifying (8), (9) with \( \gamma_1(t) \equiv 0 \) are unique.

## 3 Proofs

Concerning Theorem 1 if \( \{ \gamma_n(t) \} \) is a solution of (4) and \( \{ \lambda_n(t), \alpha_n(t) \} \) is taken as in (4) with any \( C \in \mathbb{C} \), then this sequence of functions is immediately
a solution of (1). Consequently, it is enough to prove Theorem 2 to complete the proof of Theorem 1. This is the purpose of this section.

We consider that \( \{ \lambda_n(t), \alpha_n(t) \} \) is a solution of (1) such that (4) are satisfied. The use of orthogonal polynomials was proposed in \cite{7} as a tool for proving Theorem 2 in the case when the operators \( J(t), t \in \mathbb{R} \), are selfadjoint. Our results are an extension to complex Toda lattices. We use the sequence of polynomials (in \( z \)) given by the recurrence

\[
\begin{align*}
P_{n+1}(t, z) &= (z - \alpha_{n+1}(t))P_n(t, z) - \lambda_{n+1}^2(t)P_{n-1}(t, z), \quad n \geq 0 \\
P_{-1}(t, z) &\equiv 0, \quad P_0(t, z) \equiv 1
\end{align*}
\]

(10)

for each \( t \in \mathbb{R} \), which has the role of a parameter. The dependence of \( \{ P_n(t, z) \} \) on \( t \) is instrumental in order to establish the relations (8) and (9). The following auxiliary result, consequence of \cite{7} Lemma 2, p. 523], and whose proof is immediate by induction using (10), describes such dependence.

**Lemma 1.** If \( \{ \lambda_n(t), \alpha_n(t) \} \) is a solution of (1), then

\[
P_n(t, z) = -\lambda_{n+1}^2(t)P_{n-1}(t, z), \quad n \in \mathbb{N},
\]

where the derivative \( \dot{P}_n(t, z) \) is taken with respect to \( t \in \mathbb{R} \).

For each \( t \in \mathbb{R} \) the set of zeros of each polynomial \( P_n(t, z) \) is the spectrum of \( J_n(t) \), \( \sigma(J_n(t)) \), being \( J_n(t) \) the \( n \times n \) submatrix formed by the first \( n \) rows and columns of \( J(t) \). The relation between the spectra of \( J(t) \) and its real part was studied in \cite{11}, and we need the following result, derived there. We suppose that \( C \) satisfies \cite{7}. The expressions \( \Re J_n(t) \) and \( \Im J_n(t) \) denote the submatrices formed by the first \( n \) rows and columns of \( \Re J(t) \) and \( \Im J(t) \), respectively, and \( \| \cdot \| \) is the norm of each operator in the space where it is defined, i.e. either \( \ell^2 \) or \( \mathbb{C}^n \).

**Lemma 2.** (cf. \cite{11} lemmas 1 and 2) For the restrictions \cite{14}, for each \( n \in \mathbb{N} \) we have

1.- \( \| \Im J_n(t) \| \leq \| \Re J(t) \| \)

2.- If \( d(C, \sigma(\Re J_n(t))) > \| \Im J_n(t) \| \) then \( P_n(t, C) \neq 0 \)

For \( n \in \mathbb{N} \) fixed, from the well-known fact that \( \sigma(\Re J_n(t)) \subset \text{Conv}(\sigma(\Re J(t))) \) and Lemma 2 it follows that \( P_n(t, C) \neq 0 \).

As in \cite{7} Th. 1, consider the sequence of monic polynomials \( \{ Q_n^{(C)}(t, z) \} \) defined by

\[
Q_n^{(C)}(t, z) = \frac{P_{n+1}(t, z) - P_{n+1}(t, C)}{z - C} P_n(t, z), \quad n = 0, 1, \ldots, \quad t \in \mathbb{R}
\]

(11)

(cf. \cite{2} p. 35]). Let us prove that these polynomials satisfy a three-term recurrence relation. The fact, though, that the functions \( \lambda_n(t), \alpha_n(t), n \in \mathbb{N} \), are complex-valued, prevents the use of the standard proof using the corresponding moment functional, because now it is not positive-definite. Nevertheless, a direct proof is still possible.
\textbf{Lemma 3.} The sequence of polynomials \( \{Q_n^{(C)}(t, z)\} \) satisfies the three-term recurrence relation

\[ Q_n^{(C)}(t, z) = (z - \tilde{\alpha}_n(t))Q_{n-1}^{(C)}(t, z) - \tilde{\lambda}_n^2(t)Q_{n-2}^{(C)}(t, z), \quad n \in \mathbb{N} \] \hspace{1cm} (12)

\[ Q_0^{(C)} = 0, \quad Q_0^{(C)} = 1 \]

being

\[ \tilde{\alpha}_n(t) = \frac{P_{n+1}(t, C)}{P_n(t, C)} + \alpha_{n+1}(t) - \frac{P_n(t, C)}{P_{n-1}(t, C)} \]

\[ \tilde{\lambda}_n^2(t) = \lambda_n^2(t) \frac{P_{n-2}(t, C)P_n(t, C)}{P_{n-1}(t, C)} \] \hspace{1cm} (13)

Proof of Lemma \( \text{3} \) Formula \( \text{11} \) gives \( Q_0^{(C)}(t, z) = 1 \), and also gives the same expression than \( \text{12} \) for \( Q_1^{(C)}(t, z) \). The sequence of polynomials given by \( \text{12} \) is unique. Thus, it is enough to check that, for \( n \geq 2 \), the sequence of polynomials given by \( \text{11} \) satisfy \( \text{12} \). Substituting \( (z - C)Q_{n-1}^{(C)}(t, z) \) and \((z - C)Q_{n-2}^{(C)}(t, z) \) given by \( \text{11} \) in the right hand side of \( \text{12} \) yields (suppressing explicit \( t \)-dependence for brevity)

\[ (z - \tilde{\alpha}_n) \left( P_n(z) - \frac{P_n(C)}{P_{n-1}(C)}P_{n-1}(z) - \tilde{\lambda}_n^2 \left( P_{n-1}(z) - \frac{P_{n-1}(C)}{P_{n-2}(C)}P_{n-2}(z) \right) = \right. \]

\[ (z - \tilde{\alpha}_n)P_n(z) - \left[ (z - \tilde{\alpha}_n) \frac{P_n(C)}{P_{n-1}(C)} + \tilde{\lambda}_n^2 \right] P_{n-1}(z) + \tilde{\lambda}_n^2 \frac{P_{n-1}(C)}{P_{n-2}(C)}P_{n-2}(z). \] \hspace{1cm} (14)

From \( \text{13} \) we have \( \tilde{\lambda}_n^2 \frac{P_{n-1}(C)}{P_{n-2}(C)} = \lambda_n^2 \frac{P_{n-1}(C)}{P_{n-2}(C)} \), and taking into account also \( \text{10} \), \( z - \tilde{\alpha}_n = z - C + \lambda_n^2 \frac{P_{n-1}(C)}{P_{n-2}(C)} + \frac{P_n(C)}{P_{n-1}(C)} \), \( n \in \mathbb{N} \). Thus \( \text{14} \) becomes

\[ \left( z - \alpha_{n+1} - \frac{P_{n+1}(C)}{P_n(C)} + \frac{P_n(C)}{P_{n-1}(C)} \right) P_n(z) - \left( (z - C) \frac{P_n(C)}{P_{n-1}(C)} + \lambda_{n+1}^2 \right. \]

\[ + \frac{P_n^2(C)}{P_{n-1}^2(C)} + \lambda_n^2 \frac{P_{n-2}(C)}{P_{n-1}(C)} \right) P_{n-1}(z) + \lambda_n \frac{P_{n-1}(C)}{P_{n-2}(C)}P_{n-2}(z) \]

\[ = (z - \alpha_{n+1})P_n(z) - \lambda_n^2 P_{n-1}(z) - \frac{P_{n+1}(C)}{P_n(C)}P_n(z) - \frac{P_{n-1}(C)}{P_{n-2}(C)}P_{n-1}(z) \]

\[ - \frac{P_n(C)}{P_{n-1}(C)} \left( z - C + \frac{P_n(C)}{P_{n-1}(C)} + \lambda_n^2 \frac{P_{n-2}(C)}{P_{n-1}(C)} \right) P_{n-1}(z) + \lambda_n^2 \frac{P_{n-1}(C)}{P_{n-2}(C)}P_{n-2}(z). \]

Using \( \text{10} \) in the last expression, we get

\[ P_{n+1}(z) - \frac{P_{n+1}(C)}{P_n(C)}P_n(z) + \frac{P_{n+1}(C)}{P_{n-1}(C)}P_{n-1}(z) - \frac{P_{n+1}(C)}{P_{n-1}(C)}(z - \alpha_n)P_{n-1}(z) \]

\[ + \lambda_n^2 \frac{P_{n+1}(C)}{P_{n-1}(C)}P_{n-2}(z) = P_{n+1}(z) - \frac{P_{n}(C)}{P_{n-1}(C)}P_{n}(z), \]
which is \((z - C)Q_n^{(C)}(z)\).

Lemma 1 and 3 imply that the sequence \(\{\tilde{\lambda}_n(t), \tilde{\alpha}_n(t)\}, n \in \mathbb{N}\), is a solution of the Toda lattice 1. Define the sequence \(\{\gamma_n(t)\}, n \in \mathbb{N}\), as

\[
\gamma_1(t) = 0, \quad \gamma_{2n}(t) = -\frac{P_n(t, C)}{P_{n-1}(t, C)}, \quad \gamma_{2n+1}(t) = -\lambda_{n+1}^2 \frac{P_{n-1}(t, C)}{P_n(t, C)},
\]

(15)

Lemma 1 again implies that \(\{\gamma_n(t)\}\) is a solution of the Volterra lattice 3. Besides, the relations 8 and 9 are readily checked.

Uniqueness of the sequences obtained satisfying 8 and 9 can be checked directly. Suppose that there exists another sequence \(\{\tilde{\gamma}_n(t)\}\) with \(\tilde{\gamma}_1(t) \equiv 0\). Then 8 and 9 imply that

\[
\gamma_{2n-1}(t) + \gamma_{2n}(t) = \tilde{\gamma}_{2n-1}(t) + \tilde{\gamma}_{2n}(t), \quad n \in \mathbb{N},
\]

(16)

\[
\gamma_{2n}(t)\gamma_{2n+1}(t) = \tilde{\gamma}_{2n}(t)\tilde{\gamma}_{2n+1}(t), \quad n \in \mathbb{N}.
\]

(17)

For \(n = 1\), 16 means that \(\gamma_2(t) = \tilde{\gamma}_2(t)\) and then 17 leads to \(\gamma_3(t) = \tilde{\gamma}_3(t)\). In general, if \(\gamma_{2n-1}(t) = \tilde{\gamma}_{2n-1}(t)\), 16 implies that \(\gamma_{2n}(t) = \tilde{\gamma}_{2n}(t)\) and 17 that \(\gamma_{2n+1}(t) = \tilde{\gamma}_{2n+1}(t)\) (note that, because of 15, we have \(\gamma_{2n}(t) \neq 0\), \(n \in \mathbb{N}\)). So, both sequences coincide. The uniqueness of \(\{\tilde{\lambda}_n(t), \tilde{\alpha}_n(t)\}\) follows from the uniqueness of \(\{\gamma_n(t)\}\) and from 9. This proves Theorem 2.

4 Remarks and Conclusions

Theorem 2 provides a method for constructing families of solutions of 11 from a given solution, real or complex. Choosing \(C \notin \mathbb{R}\) provides new complex solutions of 11 and 3. On the contrary, if \(C \in \mathbb{R} \setminus \text{Conv}(\sigma(J(t)))\) for all \(t \in \mathbb{R}\), and the starting solution is real, the relations 13 and 15 show that the new solutions are also real, with \(\lambda_n(t), \gamma_n(t) > 0, t \in \mathbb{R}, n \in \mathbb{N}\).

On the other hand, in theorems 11 and 2 some restriction on the value of \(C\) is necessary. We can understand this fact by taking arbitrary sequences \(\{\lambda_n(t)\}, \{\alpha_n(t)\}, \{\gamma_n(t)\}\) with \(\lambda_n(t) \neq 0, n \geq 2, t \in \mathbb{R}\) (not necessarily solutions of 11 and 3), and \(C \in \mathbb{C}\) verifying 9. Then, if \(\{P_n(t, z)\}\) is the sequence of polynomials given by 10, we have \(P_n(t, C) \neq 0\) for all \(t \in \mathbb{R}\). In fact, these conditions imply that the \(\{\gamma_n(t)\}\) given in 11 is the only sequence verifying 9, as can be deduced from the proof of Theorem 2.

With the premises and the notation given in Theorem 2 define, for each \(t \in \mathbb{R}\), the polynomials

\[
S_n^{(C)}(t, z) = P_n(t, z^2 + C), \quad S_{2n+1}^{(C)}(t, z) = zQ_n(t, z^2 + C), \quad n \in \mathbb{N}.
\]

Using 10 and 11 it is easy to see that the sequence \(\{S_n^{(C)}(t, z)\}, n \in \mathbb{N}\), verify the recurrence relation

\[
\begin{align*}
S_n^{(C)}(t, z) &= zS_{n-1}^{(C)}(t, z) - \gamma_n(t)S_{n-2}^{(C)}(t, z), \quad n \in \mathbb{N} \\
S_{-1}^{(C)} &\equiv 0, \quad S_0^{(C)} \equiv 1
\end{align*}
\]
In other words, the relations between the sequences \( \{P_n\}, \{Q_n^{(C)}\}, \{S_n^{(C)}\} \) are analogous to those given in [2] Th. 9.1, p. 46, corresponding to polynomials independent of \( t \) associated with constants that satisfy [3] and [4].

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