FINITENESS OF CLASSES OF RELATIVE EQUILIBRIA FOR THE $n$-BODY PROBLEM

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Abstract. This paper has been withdrawn by the author. Due to a crucial error, being that Lemma 2.7 is incorrect, which makes the argument of the proof of the main theorem void, this paper has been withdrawn by the author.

1. Introduction

The $n$-body problem of celestial mechanics is the problem of deducing the dynamics of $n$ point masses with time dependent coordinates $q_1,..., q_n \in \mathbb{R}^2$ and respective masses $m_1,...,m_n$ (see Definition 2.1). A relative equilibrium is a solution to such a problem where the $q_1,..., q_n$ form a configuration of $n$ points rotating around one point (see [16] and Definition 2.2). Any two relative equilibrium solutions to an $n$-body problem are said to be equivalent if one can be transformed into the other through rotation, or scalar multiplication. We call a set of equivalent relative equilibria a class of relative equilibria.

In this paper, we will solve the sixth Smale problem, which can be formulated as the question of whether the number of classes of relative equilibria is finite, given any choice of positive real numbers $m_1,...,m_n$ as masses (see [11]). The earliest document in which this question is posed is, to the author’s best knowledge, ‘The Analytical Foundations of Celestial Mechanics’ by Wintner (see [16]).

Before stating our main theorem, some background information about the sixth Smale problem is in order:

In 1998, V.I. Arnold, on behalf of the International Mathematical Union, invited top mathematicians from all over the world to suggest problems that are to be the focus of the next century’s mathematical research. Arnold’s initiative was partly inspired by Hilbert’s famous list from 1900 (see for example [3] and [13] for Hilbert’s list).

One of the mathematicians contacted by Arnold was Field medalist Steve Smale, who consequently published a list of 18 open problems, entitled ‘Mathematical Problems for the next Century’ (see [11]), of which the sixth problem is the focus of this paper. Smale’s sixth problem has been solved for $n = 3$ by A. Wintner (see [16]), $n = 4$ by M. Hampton and R. Moeckel (see [12]) and for $n = 5$ by A. Albouy and V. Kaloshin (see [2]), but for $n > 5$ the problem has remained open, until now.

Results on the finiteness of subclasses of relative equilibria can be found in [5], [6], [14] and [7]. In [9], M. Shub showed that the set of all relative equilibria if the masses are given is compact and G. Roberts showed in [8] that for the five-body problem, if one of the masses is negative, a continuum of relative equilibria exists.

For further background information and a more detailed overview see [1], [12], [10] and [8]. In this paper, we will prove the following theorem:
Theorem 1.1. If the masses of the \( n \)-body problem of celestial mechanics are given, the number of classes of relative equilibria that solve the \( n \)-body problem is finite.

In order to prove this theorem, we will first formulate some definitions and needed results, which will be done in section 2. Then we will prove Theorem 1.1 in section 3. The proof of Theorem 1.1 is inspired by [4], [9] and [15].

2. Background Theory

The results in this section are not new and can either be found in works such as [16], or are statements that follow naturally from existing results, but have been included nonetheless to make this paper self-contained.

The first definition we will need is, of course, a precise definition of the \( n \)-body problem:

Definition 2.1. Let \( n \in \mathbb{N} \). Let \( q_1, \ldots, q_n \in \mathbb{R}^2 \) be the time dependent coordinates of \( n \) point masses with respective masses \( m_1, \ldots, m_n \). Assume that the functions \( q_i : \mathbb{R} \to \mathbb{R}^2 \), \( i \in \{1, \ldots, n\} \) are twice differentiable. By the \( n \)-body problem, we mean the problem of finding solutions to the equations of motion described by

\[
\ddot{q}_i = \sum_{j=1, j \neq i}^{n} \frac{m_j (q_j - q_i)}{\|q_j - q_i\|^3}, \quad n \geq 3 \tag{2.1}
\]

Furthermore, for notational purposes, we need the following notation:
Let \( \theta : \mathbb{R} \to \mathbb{R} \) be a function. Then

\[
T(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}
\]

Next, we will define relative equilibria, relative equilibria classes and take specific representative relative equilibria from these classes, which will then later on be used to prove that the number of relative equilibria classes is finite in section 3.

Definition 2.2. Let \( \theta : \mathbb{R} \to \mathbb{R} \) be a twice continuously differentiable function. Let \( Q_1, \ldots, Q_n \in \mathbb{R}^2 \) be constant. We then call any solution \( q_i = T(\theta) Q_i \), \( i = 1, \ldots, n \), of (2.1) a relative equilibrium.

In order to formulate a definition for relative equilibria classes, we need the following two lemmas:

Lemma 2.3. Let \( c \in \mathbb{R} \) be a constant and let \( \{q_i\}_{i=1}^{n} \) be a solution to (2.1). If \( \{cq_i\}_{i=1}^{n} \) is a solution as well, then \( c = 1 \).

Proof. If the set of functions \( \{cq_i\}_{i=1}^{n} \) is a solution of (2.1), then

\[
c\ddot{q}_i = \sum_{j=1, j \neq i}^{n} \frac{m_j (cq_j - cq_i)}{\|cq_j - cq_i\|^3} = \sum_{j=1, j \neq i}^{n} \frac{m_j (q_j - q_i)}{c^2 \|q_j - q_i\|^3}.
\]

Multiplying both sides of (2.2) by \( c^2 \) gives

\[
c^3 \ddot{q}_i = \sum_{j=1, j \neq i}^{n} \frac{m_j (q_j - q_i)}{\|q_j - q_i\|^3}.
\]

As \( \{q_i\}_{i=1}^{n} \) is a solution of (2.1), we get that

\[
c^3 \ddot{q}_i = \sum_{j=1, j \neq i}^{n} \frac{m_j (q_j - q_i)}{\|q_j - q_i\|^3} = \ddot{q}_i.
\]

(2.2)
So \(c^3 \bar{q}_i = \bar{q}_i\), which means that \(c = 1\).

**Lemma 2.4.** For any relative equilibrium solution \(\{T(\theta)Q_i\}_{i=1}^n\) of (2.1) as described in Definition 2.2, we have that \(\theta'\) is constant and

\[
(\theta')^2 Q_i = \sum_{j=1, j \neq i}^n \frac{m_j(Q_i - Q_j)}{||Q_i - Q_j||^3}.
\]

**Proof.** Let \(q_i = T(\theta)Q_i\), \(i \in \{1, \ldots, n\}\) be a relative equilibrium solution as described in Definition 2.2. Note that

\[
(T(\theta))' = \theta'T(\theta) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

and thus that

\[
(\theta')^2 Q_i = \sum_{j=1, j \neq i}^n \frac{m_j(Q_i - Q_j)}{||Q_i - Q_j||^3}.
\]

Inserting \(q_i = T(\theta)Q_i\), \(q_j = T(\theta)Q_j\) and (2.3) into (2.1) gives

\[
(\theta''T(\theta) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - (\theta')^2 T(\theta)Q_i = (\theta''T(\theta) - (\theta')^2 T(\theta)Q_i = T(\theta) \sum_{j=1, j \neq i}^n \frac{m_j(Q_j - Q_i)}{||T(\theta)(Q_j - Q_i)||^3}.
\]

As \(T(\theta)\) is a rotation matrix, \(||T(\theta)(Q_j - Q_i)|| = ||Q_j - Q_i||\). Multiplying both sides of (2.4) with \(T(-\theta)\) gives

\[
(\theta'' T(\theta) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - (\theta')^2 I) Q_i = \sum_{j=1, j \neq i}^n \frac{m_j(Q_j - Q_i)}{||Q_j - Q_i||^3}.
\]

where \(I\) is the \(2 \times 2\) identity matrix.

Note that for (2.5) to make sense as a differential equation, we need that either \(\theta'\) is constant, or \(Q_i = Q_j\) for \(i, j \in \{1, \ldots, n\}\). Thus \(\theta'\) is constant, \(\theta'' = 0\) and

\[
(\theta')^2 Q_i = \sum_{j=1, j \neq i}^n \frac{m_j(Q_i - Q_j)}{||Q_i - Q_j||^3}.
\]

This completes the proof. \(\Box\)

Because of Lemma 2.4, for any rotation matrix \(T(\theta)\) used to construct a relative equilibrium, \(\theta\) must be a linear function. Note that for any constant \(c \in \mathbb{R}_{\geq 0}\) this means that if \(q_1 = T(\theta)Q_1, \ldots, q_n = T(\theta)Q_n\) solve (2.1), where \(Q_1, \ldots, Q_n \in \mathbb{R}^2\) are constant, then \(T(c \theta) e^{-\frac{c}{2}} Q_1, ..., T(c \theta) e^{-\frac{c}{2}} Q_n\) solve (2.1) as well and if \(U\) is a \(2 \times 2\) unitary matrix and \(q_1(t), ..., q_n(t)\) is a solution of (2.1), then \(Uq_1(t), ..., Uq_n(t)\) is a solution as well. We formulate the following definition:

**Definition 2.5.** If \(U\) is a \(2 \times 2\) unitary matrix, \(c \in \mathbb{R}_{\geq 0}\) a constant, \((q_1', ..., q_n')\) and \((q_{1b}', ..., q_{nb}')\) are relative equilibrium solutions of (2.1), and \((q_{1a}', ..., q_{na}') = ((Uq_1a)', ..., (Uq_na)'), \(q_{1b}', ..., q_{nb}')\) are called equivalent and a class of such solutions is called a class of relative equilibria.

Note that by Lemma 2.3 if the rotation \(T(\theta)\) of a relative equilibrium is given, it is defined up to multiplication with a unitary matrix. Thus, from this point onward, we will assume \(\theta\) to be given.

Because the proof of our main theorem will heavily rely on compact sets of relative equilibria, we will need the following results:
Lemma 2.6. Let $p, n \in \mathbb{N}$. Let $\{x_{ik}\}_{k=1}^\infty \subset \mathbb{R}^p$, $i \in \{1, \ldots, n\}$ be $n$ limit sequences with limit $x_{1,0} \in \mathbb{R}^p$. Then we can construct a subsequence $\{(x_{1k_1}, \ldots, x_{nk_k})\}_{k=1}^\infty$ of $\{(x_{1k}, \ldots, x_{nk})\}_{k=1}^\infty$, reconsidering the sequences $\{x_{ik}\}_{k=1}^\infty$ in terms of $i$ if needed, such that for any $j \in \{1, \ldots, n-1\}$ either $\|x_{jk_1}\| > \|x_{j+1,k_1}\|$ for all $k_1 \in \mathbb{N}$, or $\|x_{jk_1}\| = \|x_{j+1,k_1}\|$ for all $k_1 \in \mathbb{N}$.

Proof. Define for $i \in \{1, \ldots, n\}$ the quantity $M_{ik} = \max\{\|x_{jk}\| | j \in \{i, \ldots, n\}\}$. Then there has to be a fixed $i_1 \in \{1, \ldots, n\}$ for which $M_{1k} = \|x_{i_1,k}\|$ for infinitely many values of $k$. Take a subsequence $\{(x_{1k_1}, \ldots, x_{nk_k})\}_{k=1}^\infty$ of $\{(x_{1k}, \ldots, x_{nk})\}_{k=1}^\infty$ such that $M_{1k_1} = \|x_{i_1,k_1}\|$ for all $k_1 \in \mathbb{N}$. Renumber the $x_{ik_1}$ such that $i_1 = 1$. Then there has to be an $i_2 \in \{2, \ldots, n\}$ such that $\|x_{i_2k_1}\| = M_{2k_1}$ for infinitely many values of $k_1$. Take a subsequence $\{(x_{1k_1}, \ldots, x_{nk_k})\}_{k=1}^\infty$ of $\{(x_{1k_1}, \ldots, x_{nk_k})\}_{k=1}^\infty$ such that $\|x_{i_2k_1}\| = M_{2k_1}$ for all $k_2 \in \mathbb{N}$ and renumber the $x_{ik_1}$ in such a way that $i_1 = 1, i_2 = 2$. By induction, we can construct a subsequence $\{(x_{1k_1}, \ldots, x_{nk_k})\}_{k=1}^\infty$ of $\{(x_{1k}, \ldots, x_{nk})\}_{k=1}^\infty$ such that

$$\|x_{1k_1}\| = M_{1k_1} \geq \ldots \geq \|x_{nk_k}\| = M_{nk_k}$$

for all $l_n \in \mathbb{N}$.

For each $j \in \{1, \ldots, n-1\}$, there has to be an infinite amount of values of $l_n$ for which $\|x_{jk_1}\| > \|x_{j+1,k_1}\|$, or there has to be an infinite amount of values of $l_n$ for which $\|x_{jk_1}\| = \|x_{j+1,k_1}\|$. Thus, we take a subsequence $\{(x_{1k_1}, \ldots, x_{nk_k})\}_{k=1}^\infty$ of $\{(x_{1k_1}, \ldots, x_{nk_k})\}_{k=1}^\infty$ such that for each $j \in \{1, \ldots, n-1\}$ either $\|x_{jk_1}\| > \|x_{j+1,k_1}\|$ for all $l \in \mathbb{N}$, or $\|x_{jk_1}\| = \|x_{j+1,k_1}\|$ for all $l \in \mathbb{N}$. □

Lemma 2.7. Let $n \in \mathbb{N}$ and $\{c_{1k}\}_{k=1}^\infty, \ldots, \{c_{nk}\}_{k=1}^\infty \subset \mathbb{R}$ be limit sequences that converge to 0 for which $|c_{1k}| > \ldots > |c_{nk}|$ for all $k \in \mathbb{N}$. Then

$$\frac{1}{1 + c_{1k}} + \cdots + \frac{1}{1 + c_{nk}}$$

are linearly independent in terms of $k$.

Proof. Let $a_1, \ldots, a_n \in \mathbb{R}$ be constants and assume that

$$0 = \sum_{j=1}^n \frac{a_j}{1 + c_{jk}}.$$

Then for $k$ large enough

$$0 = \sum_{j=1}^n \sum_{p=0}^\infty a_j(-1)^p c_{jk}^p = \sum_{p=0}^\infty \sum_{j=1}^n a_j(-1)^p c_{jk}^p$$

$$= \sum_{p=0}^\infty c_{1k}^p (-1)^p \left( a_1 + \sum_{j=2}^n a_j \left( \frac{c_{jk}}{c_{1k}} \right)^p \right)$$

By construction, the terms $c_{jk}^p \left( a_1 + \sum_{j=2}^n a_j \left( \frac{c_{jk}}{c_{1k}} \right)^p \right)$ are linearly independent, meaning that $a_1 + \sum_{j=2}^n a_j \left( \frac{c_{jk}}{c_{1k}} \right)^p = 0$ for all $p \in \{0, 1, 2, \ldots\}$ and as $1 > \frac{c_{1k}}{c_{1k}} > \ldots > \frac{c_{nk}}{c_{1k}}$ this means that the $\left( \frac{c_{1k}}{c_{1k}} \right)^p$ are linearly independent.
terms of \( p. \) which means that \( a_1 = ... = a_n = 0. \) So
\[
\frac{1}{1 + c_{1k}}, ..., \frac{1}{1 + c_{nk}}
\]
are linearly independent in terms of \( k. \) \( \Box \)

**Corollary 2.8.** Let \( n \in \mathbb{N} \) and \( \{c_{1k}\}_{k=1}^{\infty}, \{c_{nk}\}_{k=1}^{\infty} \subset \mathbb{R} \) be limit sequences that converge to 0 for which \( |c_{1k}| > ... > |c_{nk}| \) for all \( k \in \mathbb{N}. \)

Let \( n \in \mathbb{N}, l_1, ..., l_n \in \mathbb{N}, \) let \( \{\beta_{ijk}\}_{k=1}^{\infty}, i \in \{1, ..., n\}, j \in \{1, ..., l_i\} \) be real valued sequences for which \( \lim_{k \to \infty} \beta_{ijk} = 0 \) and let
\[
v_{ij} = \left( \begin{array}{c} x_{ij} \\ y_{ij} \end{array} \right) \in \mathbb{R} \times \mathbb{R}, \ i \in \{1, ..., n\}, \ j \in \{1, ..., l_i\}
\]
and all \( v_{ij}, i \in \{1, ..., n\}, j \in \{1, ..., l_i\} \) are nonzero and lie in the same half-plane. Then the vectors
\[
\left\{ \frac{1}{1 + c_{ik}} \sum_{j=1}^{l_i} T(\beta_{ijk})v_{ij} \right\}_{i=1}^{n}
\]
are linearly independent in terms of \( k. \)

**Proof.** Suppose \( a_1, ..., a_n \in \mathbb{R} \) are constants and
\[
0 = \sum_{i=1}^{n} a_i \frac{1}{1 + c_{ik}} \sum_{j=1}^{l_i} T(\beta_{ijk})v_{ij}
\]
Then using that \( \cos \beta_{ijk} = 1 + o(1) \) and \( \sin \beta_{ijk} = o(1) \) gives
\[
0 = \sum_{i=1}^{n} a_i \frac{1}{1 + c_{ik}} \sum_{j=1}^{l_i} \left( x_{ij} + o(1) \right) \left( y_{ij} + o(1) \right)
\]
and as the \( \frac{1}{1 + c_{ik}} \) are linearly independent by Lemma 2.7 we have that
\[
0 = \sum_{i=1}^{n} a_i \frac{1}{1 + c_{ik}} \sum_{j=1}^{l_i} \left( x_{ij} \right) \left( y_{ij} + o(1) \right)
\]
and thus that
\[
a_i \sum_{j=1}^{l_i} v_{ij} = 0 \text{ for all } i \in \{1, ..., n\}.
\]
As the \( v_{ij}, i \in \{1, ..., n\}, j \in \{1, ..., l_i\} \) all lie in the same half-plane and are nonzero, we have that
\[
\sum_{j=1}^{l_i} v_{ij} \neq 0 \text{ for all } i \in \{1, ..., n\},
\]
so \( a_1 = ... = a_n = 0. \) This completes the proof. \( \Box \)

**Lemma 2.9.** Let \( n \in \mathbb{N} \) and let \( \{\beta_{ik}\}_{k=1}^{\infty}, i \in \{1, ..., n\} \) be real valued sequences for which \( \lim_{k \to \infty} \beta_{ik} = 0 \) and \( |\beta_{1k}| > ... > |\beta_{nk}| \) for all \( k \in \mathbb{N}. \) Let \( v_1, ..., v_n \in \mathbb{R}^2 \) be nonzero, constant vectors. Then the vectors \( T(\beta_{1k})v_1, ..., T(\beta_{nk})v_n \) are linearly independent.
Proof. Suppose $a_1,...,a_n \in \mathbb{R}$ and

$$0 = \sum_{i=1}^{n} a_i T(\beta_{ik}) v_i.$$ 

Then writing

$$v_i = \begin{pmatrix} v_{i1} \\ v_{i2} \end{pmatrix}, \quad i \in \{1,...,n\}$$

gives

$$0 = \sum_{i=1}^{n} a_i \begin{pmatrix} \cos \beta_{ik} & -\sin \beta_{ik} \\ \sin \beta_{ik} & \cos \beta_{ik} \end{pmatrix} \begin{pmatrix} v_{i1} \\ v_{i2} \end{pmatrix}.$$ 

Expanding $\cos \beta_{ik}$ and $\sin \beta_{ik}$ as power series then gives

$$0 = \sum_{i=1}^{n} a_i \sum_{p=0}^{\infty} \left( (-1)^p \frac{\beta_{ik}^{2p}}{(2p)!} \begin{pmatrix} 0 & 0 \\ 0 & (-1)^p \frac{\beta_{ik}^{2p+1}}{(2p+1)!} \end{pmatrix} \right) \begin{pmatrix} v_{i1} \\ v_{i2} \end{pmatrix}$$

which means, by the linear independence of

$$\beta_{ik}^{2p+1} \sum_{i=1}^{n} a_i \left( \frac{\beta_{ik}}{\beta_{1k}} \right)^{2p+1} \begin{pmatrix} v_{i1} \\ v_{i2} \end{pmatrix}$$

and

$$\beta_{ik}^{2p} \sum_{i=1}^{n} a_i \left( \frac{\beta_{ik}}{\beta_{1k}} \right)^{2p} \begin{pmatrix} v_{i1} \\ v_{i2} \end{pmatrix}$$

in terms of $k$ and consequently the linear independence of $\left( \frac{\beta_{ik}}{\beta_{1k}} \right)^{2p}$ and $\left( \frac{\beta_{ik}}{\beta_{1k}} \right)^{2p+1}$ in terms of $p$, that

$$a_i \begin{pmatrix} v_{i1} \\ v_{i2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad a_i \begin{pmatrix} -v_{i2} \\ v_{i1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and thus, as the $v_i$ are nonzero, that $a_i = 0, i \in \{1,...,n\}$. This completes the proof. \hfill $\square$

**Corollary 2.10.** Let $n \in \mathbb{N}$ and let $\{\beta_{ik}\}_{k=1}^{\infty}, i \in \{1,...,n\}$ be real valued sequences for which $\lim_{k \to \infty} \beta_{ik} = 0$ and $|\beta_{ik}| > ... > |\beta_{nk}|$ for all $k \in \mathbb{N}$.

Let $l_1,...,l_n \in \mathbb{N}$ and let $\{r_{jk}\}_{k=1}^{\infty} \subset \mathbb{R}, j \in \{1,...,l_i\}, i \in \{1,...,n\}$ be limit sequences which converge to 1. Let $v_{i1},...,v_{il_i} \in \mathbb{R}^2, i \in \{1,...,n\}$ be nonzero, constant vectors that all lie in the same half-plane. Then the vectors

$$\left\{ T(\beta_{ik}) \sum_{j=1}^{l_i} r_{jk} v_{ij} \right\}_{i=1}^{n}$$

are linearly independent in terms of $k$. 
Proof. Suppose \( a_1, ..., a_n \in \mathbb{R} \) be constants and

\[
0 = \sum_{i=1}^{n} a_i T(\beta_{ik}) \sum_{j=1}^{l_k} r_{jk} v_{ij}.
\]

As \( r_{jk} = 1 + o(1) \) and the \( T(\beta_{ik}) \) are linearly independent in terms of \( k \) by Lemma 2.9, this means that

\[
0 = \sum_{i=1}^{n} a_i T(\beta_{ik}) \sum_{j=1}^{l_k} v_{ij}.
\]

Again by Lemma 2.9 as all the \( v_{ij} \) lie in the same half-plane and thus \( \sum_{j=1}^{l_k} v_{ij} \neq 0 \), this means that \( a_i = 0 \), \( i \in \{1, ..., n\} \), so the vectors

\[
\left\{ T(\beta_{ik}) \sum_{j=1}^{l_k} r_{jk} v_{ij} \right\}_{i=1}^{n}
\]

are linearly independent in terms of \( k \). \( \square \)

Lemma 2.11. Let \( n \in \mathbb{N} \), \( n \geq 3 \). Suppose that there exist limit sequences \( \{Q_{1k}\}_{k=1}^{\infty}, \{Q_{nk}\}_{k=1}^{\infty} \subset \mathbb{R}^2 \) with respective limits \( Q_{10}, ..., Q_{n0} \) and suppose that

\[
(\theta')^2 Q_{ik} = \sum_{j=1}^{n} \frac{m_j(Q_{ik} - Q_{jk})}{\|Q_{ik} - Q_{jk}\|^3}
\]

for all \( k \in \mathbb{N} \) and \( \theta' \) as in Lemma 2.4. Then at most one of the limits \( Q_{10}, ..., Q_{n0} \) is equal to zero.

Proof. Assume that the contrary is true. According to Lemma 2.6, we can take subsequences \( \{Q_{1k}\}_{l=1}^{\infty}, \{Q_{nk}\}_{l=1}^{\infty} \subset \mathbb{R}^2 \) and renumber the \( Q_{ik} \) in terms of \( i \) such that

\[
\|Q_{1k_l}\| \geq ... \geq \|Q_{nk_l}\|
\]

for all \( l \in \mathbb{N} \). Let \( j^* \in \{1, ..., n\} \) be the first value for which \( \lim_{l \to \infty} Q_{j^*k_l} = 0 \). Note that by (2.6),

\[
(\theta')^2 Q_{j^*k_l} = \sum_{j<j^*} \frac{m_j(Q_{j^*k_l} - Q_{j^*k_l})}{\|Q_{j^*k_l} - Q_{j^*k_l}\|^3} + \sum_{j>j^*} \frac{m_j(Q_{j^*k_l} - Q_{j^*k_l})}{\|Q_{j^*k_l} - Q_{j^*k_l}\|^3}
\]

As the vectors \( Q_{1k_l}, ..., Q_{nk_l} \) may be multiplied with constant rotation matrices, we may assume that

\[
Q_{j^*k_l} = \left( \begin{array}{c} \|Q_{j^*k_l}\| \\ 0 \end{array} \right)
\]

for all \( l \in \mathbb{N} \). If \( j^* < n \), then the terms in the summation \( \sum_{j>j^*} \frac{m_j(Q_{j^*k_l} - Q_{j^*k_l})}{\|Q_{j^*k_l} - Q_{j^*k_l}\|^3} \) all lie in the right half plane and thus do not cancel each other out in the summation \( \sum_{j>j^*} \frac{m_j(Q_{j^*k_l} - Q_{j^*k_l})}{\|Q_{j^*k_l} - Q_{j^*k_l}\|^3} \) and as \( \left\| \frac{m_j(Q_{j^*k_l} - Q_{j^*k_l})}{\|Q_{j^*k_l} - Q_{j^*k_l}\|^3} \right\| = \frac{m_j}{\|Q_{j^*k_l} - Q_{j^*k_l}\|^3} \), the outcome of that summation goes to infinity for \( l \) going to infinity. The summation \( \sum_{j<j^*} \frac{m_j(Q_{j^*k_l} - Q_{j^*k_l})}{\|Q_{j^*k_l} - Q_{j^*k_l}\|^3} \)
converges to a constant. As the left-hand side of (2.7) goes to zero, this gives a contradiction. Thus, there is at most one limit $Q_{i,0}$ equal to zero, $i \in \{1, \ldots, n\}$.

We now have all we need to prove Theorem 1.1

3. Proof of Theorem 1.1

Assume that the number of equivalence classes of relative equilibrium solutions is infinite. M. Shub (see [9]) has proven that the set of vectors $(Q_1^t, \ldots, Q_n^t) \in \mathbb{R}^{2n}$, $Q_1, \ldots, Q_n$ as described in Definition 2.2, is a compact set in $\mathbb{R}^{2n}$. Because of Bolzano-Weierstrass, we can therefore take a convergent subsequence $\{(Q_{1k}^t, \ldots, Q_{nk}^t)\}_{k=1}^\infty$ with limit $(Q_{1,0}^t, \ldots, Q_{n,0}^t) \in \mathbb{R}^{2n}$. Because of Lemma 2.4 this means that

$$
(b')^2 Q_{ik} = \sum_{j=1}^{n} \frac{m_j(Q_{ik} - Q_{jk})}{\|Q_{ik} - Q_{jk}\|^3}
$$

for all $k \in \mathbb{N}$.

Because of Lemma 2.6 we may renumber the $Q_{ik}$ in terms of $i$ such that

$$
\|Q_{ik}\| = \max\{\|Q_{ik}\| \mid i \in \{1, \ldots, n\}\}
$$

for all $k \in \mathbb{N}$, taking a subsequence if needed. As the vectors $\{Q_{ik}\}_{k=1}^\infty$ are unique up to multiplication with a unitary matrix that is independent of $i$, we may fix $Q_{1k}$ to be equal to

$$
\begin{pmatrix}
(r_{1k}\|Q_{1,0}\|) \\
0
\end{pmatrix}
$$

where $\lim_{k \to \infty} r_{1k} = 1$ by Lemma 2.11.

As $\lim_{k \to \infty} Q_{ik} = Q_{i,0}$, $i \in \{1, \ldots, n\}$, we can write

$$
Q_{ik} = r_{ik} T(\alpha_{ik}) Q_{i,0},
$$

where either $\lim_{k \to \infty} r_{ik} = 1$, or possibly $\lim_{k \to \infty} r_{ik} = 0$ for at most one $i \in \{1, \ldots, n\}$ because of Lemma 2.11 and $\lim_{k \to \infty} \alpha_{ik} = 0$.

By fixing the angle of $Q_{1k}$, there must exist $r_{ik}$ that are not equal to $r_{1k}$, or $T(\alpha_{ik})$ that are not equal to the identity matrix. Otherwise, we have that $Q_{ik} = r_{1k} Q_{i,0}$, $i \in \{1, \ldots, n\}$ and then the $r_{1k} Q_{i,0}$, $i \in \{1, \ldots, n\}$ have to fulfill Lemma 2.4. In that case, by Lemma 2.2, $r_{1k} = 1$ for all $k \in \mathbb{N}$, which would mean that our limit sequence is finite. So for our limit sequence to be infinite, there have to be $r_{ik}$ that are not equal to $r_{1k}$, or $T(\alpha_{ik})$ that are not equal to the identity matrix.

We will show that under this assumption, there exist linearly independent terms with respect to $k$ in

$$
(b')^2 \begin{pmatrix}
\|Q_{1,0}\| \\
0
\end{pmatrix} = \frac{1}{r_{1k}} \sum_{j=2}^{n} \frac{m_j(Q_{1k} - Q_{jk})}{\|Q_{1k} - Q_{jk}\|^3}
$$

which will lead to a contradiction.

Note that

$$
\frac{Q_{1k} - Q_{jk}}{\|Q_{1k} - Q_{jk}\|^3} = \frac{1}{\|Q_{1k} - Q_{jk}\|^2} \frac{Q_{1k} - Q_{jk}}{\|Q_{1k} - Q_{jk}\|}
$$

$$
= \frac{1}{\|Q_{1,0} - Q_{j,0}\|^2} \frac{1}{\|Q_{1,0} - Q_{j,0}\|^2} \frac{Q_{1k} - Q_{jk}}{\|Q_{1k} - Q_{jk}\|}
$$
Define
\[ c_{jk} = \frac{\|Q_{1k} - Q_{jk}\|^2 - \|Q_{1,0} - Q_{j,0}\|^2}{\|Q_{1,0} - Q_{j,0}\|^2} \]  
(3.5)
and
\[ \frac{Q_{1k} - Q_{jk}}{\|Q_{1k} - Q_{jk}\|} = T(b_{jk}) \frac{Q_{1,0} - Q_{j,0}}{\|Q_{1,0} - Q_{j,0}\|} \]  
(3.6)
for sequences \( \{b_{jk}\}_{k=1}^{\infty}, j \in \{2, \ldots, n\} \), \( \lim_{k \to \infty} b_{jk} = 0. \)

For notational convenience, we will define
\[ E_j = \frac{Q_{1,0} - Q_{j,0}}{\|Q_{1,0} - Q_{j,0}\|} \]  
(3.7)
and
\[ d_j = \|Q_{1,0} - Q_{j,0}\|. \]  
(3.8)
Thus, because of (3.4), (3.5), (3.6) and (3.7), we have that
\[ T \] is possible that there are no nonconstant rotations

Corollary 2.10, provided that there are nonconstant
are at least two sequences \( \{\} \)

Next, we will consider the case that there are
infinitely many values of \( c_{jk} \) are equal to zero. In that case, the
right hand side of (3.3) is a linear combination of vectors that all lie in the right
half-plane, where at least one of the \( b_{jk} \) is nonconstant.
Thus, by Lemma 2.6 and Corollary 2.10 we have a contradiction.

Next, we will consider the case that there are \( c_{jk} \) that are not equal to zero for
infinitely many values of \( k \).
By Lemma 2.6 and Corollary 2.8 taking a subsequence if necessary, either there
are at least two sequences \( \{c_{j_1k}\}_{k=1}^{\infty} \) \( \{c_{j_2k}\}_{k=1}^{\infty}, \) where \( j_1, j_2 \in \{2, \ldots, n\} \) and \( |c_{j_1k}| \geq |c_{j_2k}| \), or \( c_{j_1k} = -c_{j_2k}, \) which gives at least two linearly independent terms in the
right hand side of (3.3), which would mean a contradiction, or all the \( c_{jk} \) are
the same for all values of \( i \). In that case, we have a contradiction by Lemma 2.6 and
Corollary 2.10 provided that there are nonconstant \( T(b_{jk}) \) in terms of \( k \). However,
it is possible that there are no nonconstant rotations \( T(b_{jk}) \), in which case
\[ (\theta')^2 \left( \frac{\|Q_{1,0}\|}{0} \right) = \frac{1}{(1 + c_{jk})r_{1k}} \sum_{j=2}^{n} m_j d_j E_j, \]
which is a contradiction as long as \( (1 + c_{2k})r_{1k} \) is not constant. Thus, we suppose
that \( (1 + c_{2k})r_{1k} \) is constant. Recall from (3.5) that
\[ c_{jk} = \frac{\|Q_{1k} - Q_{jk}\|^2 - \|Q_{1,0} - Q_{j,0}\|^2}{\|Q_{1,0} - Q_{j,0}\|^2}, \]
so
\[ 1 + c_{jk} = \frac{\|Q_{1k} - Q_{jk}\|^2}{\|Q_{1,0} - Q_{j,0}\|^2}. \]
As \( \lim_{k \to \infty} c_{jk} = 0 \) and \( \lim_{k \to \infty} r_{1k} = 1 \), we must have that for \( (1 + c_{2k})r_{1k} \) to be constant,
\( (1 + c_{2k})r_{1k} = 1 \) and thus that
\[ \frac{\|Q_{1k} - Q_{jk}\|^2}{\|Q_{1,0} - Q_{j,0}\|^2} r_{1k} = 1 \text{ and } \|Q_{1k} - Q_{jk}\| = r_{1k}^{-\frac{1}{2}} \|Q_{1,0} - Q_{j,0}\|. \]
\[ (3.9) \]
Note that (3.3) can be rewritten as

\[-(\theta')^2 \left( \left\| Q_{1,0} \right\| \right) = - \sum_{j=2}^{n} \frac{m_j}{\left\| Q_{1k} - Q_{jk} \right\|^{3}} \left( \left\| Q_{1,0} \right\| \right) + \frac{1}{r_{1k}} \sum_{j=2}^{n} \frac{m_j Q_{jk}}{\left\| Q_{1k} - Q_{jk} \right\|^{3}},\]

which in turn, using (3.2) and (3.3), can be rewritten as (3.10)

\[-(\theta')^2 \left( \left\| Q_{1,0} \right\| \right) = - \frac{1}{r_{1k}} \sum_{j=2}^{n} \frac{m_j}{\left\| Q_{1,0} - Q_{j,0} \right\|^{3}} \left( \left\| Q_{1,0} \right\| \right) + \frac{1}{r_{1k}} \sum_{j=2}^{n} \frac{m_j Q_{jk}}{\left\| Q_{1,0} - Q_{j,0} \right\|^{3}},\]

Let \( \gamma_j \) be the angle between \( Q_{1k} \) and \( Q_{1k} - Q_{jk} \). Because the \( b_{jk} \) are by construction zero, \( \gamma_j \) is constant. Because of the cosine rule,

\[
\left\| Q_{jk} \right\|^2 = r_{1k}^2 \left\| Q_{1,0} \right\|^2 + \left\| Q_{1k} - Q_{jk} \right\|^2 - 2r_{1k} \left\| Q_{1,0} \right\| \left\| Q_{1k} - Q_{jk} \right\| \cos \gamma_j
\]

(3.11)

\[= r_{1k}^2 \left\| Q_{1,0} \right\|^2 - 2 \frac{1}{r_{1k}} \left\| Q_{1,0} \right\| \left\| Q_{1,0} - Q_{j,0} \right\| \cos \gamma_j + r_{1k}^{-1} \left\| Q_{1,0} - Q_{j,0} \right\|^2.\]

Inserting (3.11) into (3.10) and writing \( A_j = \frac{Q_{1,0} - Q_{j,0}}{\left\| Q_{1,0} \right\|} \) gives

\[-(\theta')^2 \left( \left\| Q_{1,0} \right\| \right) = - \frac{1}{r_{1k}} \sum_{j=2}^{n} \frac{m_j}{\left\| Q_{1,0} - Q_{j,0} \right\|^{3}} \left( \left\| Q_{1,0} \right\| \right) + \frac{1}{r_{1k}} \sum_{j=2}^{n} \frac{m_j Q_{jk}}{\left\| Q_{1,0} - Q_{j,0} \right\|^{3}},\]

(3.12)

Let \( \Delta_k = r_{1k}^{-2} \) and \( B_j = 1 - 2A_j \cos \gamma_j + A_j^2 \). Then (3.12) becomes

\[-(\theta')^2 \left( \left\| Q_{1,0} \right\| \right) = - \frac{1}{1 + \Delta_k} \sum_{j=2}^{n} \frac{m_j}{\left\| Q_{1,0} - Q_{j,0} \right\|^{3}} \left( \left\| Q_{1,0} \right\| \right) + \frac{1}{1 + \Delta_k} \sum_{j=2}^{n} \frac{m_j (B_j - 2\Delta_k A_j \cos \gamma_j + \Delta_k (2 + \Delta_k) A_j^2)^{1/2}}{\left\| Q_{1,0} - Q_{j,0} \right\|^{3}} T(\alpha_{jk}) Q_{j,0},\]

(3.13)

Note that the terms of

\[
\sum_{j=2}^{n} \frac{m_j (B_j - 2\Delta_k A_j \cos \gamma_j + \Delta_k (2 + \Delta_k) A_j^2)^{1/2}}{\left\| Q_{1,0} - Q_{j,0} \right\|^{3}} T(\alpha_{jk}) Q_{j,0},
\]

(3.14)

are linearly independent as long as long as the \( B_j \), the \( A_j \), or the \( \cos \gamma_j \) are distinct. So vectors \( Q_{j,0} \) for which the \( B_j \), the \( A_j \) and the \( \cos \gamma_j \) are the same are only \( Q_{j,0} \) and its reflection in the first coordinate axis, if that vector is in the summation of (3.14). Thus, there have to be \( m_j, j \in \{1, ..., n\} \), which are equal to zero. This is a contradiction. Thus, the number of classes of relative equilibria is finite if the masses \( m_1, ..., m_n \) are given. This completes the proof.

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