Symmetries and conservation laws of the one-dimensional shallow water magnetohydrodynamics equations in Lagrangian coordinates

E I Kaptsov\(^1\)\(\ast\), S V Meleshko\(^1\)\(\ast\) and V A Dorodnitsyn\(^2\)\(\ast\)

\(^1\) School of Mathematics, Institute of Science, Suranaree University of Technology, Nakhon Ratchasima 30000, Thailand
\(^2\) Keldysh Institute of Applied Mathematics, Russian Academy of Science, Miusskaya Pl. 4, Moscow 125047, Russia

E-mail: sergey@math.sut.ac.th

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Abstract
Symmetries of the one-dimensional shallow water magnetohydrodynamics equations (SMHD) in Gilman’s approximation are studied. The SMHD equations are considered in case of a plane and uneven bottom topography in Lagrangian and Eulerian coordinates. Symmetry classification separates out all bottom topographies which yields substantially different admitted symmetries. The SMHD equations in Lagrangian coordinates were reduced to a single second order PDE. The Lagrangian formalism and Noether’s theorem are used to construct conservation laws of the SMHD equations. Some new conservation laws for various bottom topographies are obtained. The results are also represented in Eulerian coordinates. Invariant and partially invariant solutions are constructed.

Keywords: shallow water, magnetohydrodynamics, Lagrangian coordinates, Lie symmetries, conservation laws, exact invariant solutions

1. Introduction

The shallow water approximation for the study of large-scale processes in plasma magnetohydrodynamics plays the same fundamental role as the analogous approximation in neutral fluid hydrodynamics: it is an alternative of solving the full magnetohydrodynamics system of heavy fluid with free boundary. Shallow water approximation in magnetic hydrodynamics is
used for the study of the Sun tachocline; mechanical processes in the production of aluminum by electrolysis [1, 2] (see also literature therein). When MHD is added in the form of a toroidal field to be perturbed by waves, the resulting MHD Rossby waves can behave quite differently than their hydrodynamic counterparts [3]. The nonlinear dynamics of such flows, described by the complete system of magnetohydrodynamics equations on all scales is a difficult task for analytical study.

The Gilman shallow water magnetohydrodynamics model (SMHD) [1] is applied for describing the global dynamics of the solar tachocline, a thin layer at the base of the solar convection zone. It was demonstrated that the tachocline can be regarded as two-dimensional shallow water in the presence of a flat magnetic field [1]. This result significantly increased the utility of the shallow water approximation for astrophysical problems and led to numerous publications on SMHD.

Properties of the SMHD equations as a nonlinear system of hyperbolic conservation laws were studied in [4], where the foundations laid for constructing accurate shock-capturing numerical schemes for the SMHD equations. In [5], a Hamiltonian formulation of the SMHD was used, and a dispersive system, based on the model [1] was constructed to avoid unphysical cusp-like singularities in finite amplitude magnetogravity waves. It was shown in [6] that the SMHD model may be systematically derived by vertical averaging of the full MHD equations for the rotating magnetofluid under the gravity influence. In [6] the author proposes a more formal approach that clarifies the main hypothesis underlying Gilman’s model and gives multi-layer generalizations allowing for incorporation of the baroclinic effects. A number of properties of the model [1] was investigated in [2], where self-similar discontinuous and continuous solutions were found. The local well-posedness in time of the one-dimensional model was proven in [7]. The structural stability of shock waves and current-vortex sheets in the SMHD was studied in [8].

Based on the model [1], a large number of numerical discretizations have been constructed [9–12], including discretizations for the case of non-flat bottom topography.

Although the model [1] is the very common, it is not the only SMHD model. While the most works on SMHD consider initially toroidal magnetic fields, in neutron star-related applications, however, it makes sense to consider initially vertical magnetic fields. The SMHD equations in the external vertical magnetic field were covered in [13–18]. The extended SMHD model in the presence of vorticity and magnetic currents was proposed in [19]. Motion of an ideal fluid flow under the influence of a constant external field (gravitational or magnetic) may also be modeled by means of modified shallow water equations [20] for which the authors have recently constructed conservative symmetry-preserving finite-difference schemes [21].

Despite the many publications on the SMHD the symmetries of the SMHD equations have not yet been studied. The symmetries of the equations express important physical properties of the model and they are closely related with its conservation laws [22, 23]. They also allow finding exact solutions reducing partial differential equations (PDEs) to ordinary differential equations.

The most of cited research papers are devoted to the SMHD equations in Eulerian coordinates. An alternative to the Eulerian description is the Lagrangian one, which has some advantages. Lagrangian approach for studying motions of continuous medium is essentially based on a description of the history of the motion of each specific particle of the medium and it is always implied in the formulation of physical laws [24, 25]. It is also worth noting that in numerical simulation of plasma physics and astrophysics it is often easier to set boundary conditions in mass Lagrangian coordinates [26]. For some models it is possible to construct a Lagrangian for the equations [27–32] (see also references therein), whereas there are no Lagrangians in Eulerian description. Knowing the Lagrangian and the symmetries
of the equation makes it possible to derive conservation laws by a simple algorithmic procedure [33].

In the present paper the authors fill this gap in the study of symmetries of the one-dimensional SMHD equations [1]. The SMHD equations are considered in case of uneven bottom topography in both Lagrangian and Eulerian coordinates. The presence of an arbitrary bottom needs a group classification, which consists of identifying bottom topographies for which the admitted Lie algebras are extended. The Lagrangian formalism and Noether’s theorem allow constructing conservation laws of the equations, including new conservation laws for various bottom topographies.

The paper is organized as follows. Gilman’s model in Eulerian coordinates, as well as its one-dimensional version with uneven bottom topography, are given in section 2. In section 3 the model is presented in mass Lagrangian coordinates, and it is shown that the system reduces to a single second-order PDE, while the remaining equations are integrated. Complete group classification of this equation with respect to the bottom topography is performed in section 4. In section 5 the Lagrangian and Hamiltonian for the SMHD equations are found. Using Noether’s theorem, conservation laws are obtained in mass Lagrangian coordinates, and their counterparts in Eulerian coordinates are also presented. Section 6 is devoted to exact solutions. Invariant and partially invariant solutions are obtained there. The results are summarized in section 7.

2. The SMHD equations in Eulerian coordinates

In general, the SMHD model proposed in [1], has the form

\[ h_t + \nabla' \cdot (hu) = 0, \]  
\[ u_t = \nabla \left( \frac{H \cdot H}{2} \right) - \nabla \left( \frac{u \cdot u}{2} \right) - (\hat{k} \times u) \hat{k} \cdot \nabla \times u + (\hat{k} \times H) \hat{k} \cdot \nabla \times H - g \nabla h = 0, \]  
\[ H_t = \nabla \times (u \times H) + (\nabla' \cdot u) H - (\nabla' \cdot H) u, \]  
\[ \nabla' \cdot (H \cdot u) = 0, \]  

where \( u = (u, v, 0) \) is the two-dimensional velocity vector, \( H = (H^x, H^y, 0) \) is the two-dimensional magnetic field vector, \( \hat{k} = (0, 0, 1) \) is the unit vector in the vertical direction, \( h \) characterizes a deviation of the free surface from the undisturbed level, \( \nabla' \cdot \) is the horizontal divergence operator, \( \hat{k} \cdot \nabla \times \) is the vertical component of the curl operator, and the constant \( g \neq 0 \) characterizes the gravitational acceleration.

Equation (1a) means that the free surface is a material surface and particles initially on this surface remain there. Equation (1b) describes the evolution of the horizontal velocity field, where it is taken into account that the usual hydrostatic equation in the presence of a magnetic field must be replaced by the condition

\[ \nabla p = g \nabla h - \nabla \left( \frac{H \cdot H}{2} \right). \]

Equation (1c) describes the evolution of the horizontal magnetic field. In contrast to the standard MGD equations, there is horizontal divergence of both velocity and magnetic field allowed for, both of which arise in this system as a result of the deformation of the free surface. Equation (1d) means that the magnetic field initially in the surface remains there and remains locally parallel to the free surface (a modified form of the divergence-free condition for magnetic fields).
Remark 1. For convenience, the reduced magnetic vector field $\mathbf{H}$ instead of the magnetic vector field $\mathbf{e}_H$ is used, which are related as

$$\mathbf{H} = \frac{\mathbf{e}_H}{\sqrt{4\pi \rho}},$$

where $\rho$ is the density of the fluid, assumed constant for shallow water.

By analogy with the standard shallow water equations, the function $b(x,y)$ characterizing topology of the bottom can be introduced [34]. In coordinate form (1a)–(1d) with uneven bottom becomes

\begin{align*}
h_t + uh_x + hu_x + hv_y + hv = 0, \\
u_t + uu_x + vv_y - H^x H^x_x - H^y H^y_y + gh_x = b_x, \\
v_t + uv_x + vv_y - H^x H^y_x - H^y H^y_y + gh_y = b_y, \\
H^x_t + uH^x_x + vH^y_y - u_x H^x - u_y H^y = 0, \\
H^y_t + uH^x_x + vH^y_y - v_x H^x - v_y H^y = 0, \\
h_x H^x + h_y H^y + h H^x_y = 0.
\end{align*}

Assuming that all dependent functions only depend on the single space variable $x$, the latter system brought to the form

\begin{align*}
h_t + uh_x + hu_x &= 0, \quad (2a) \\
u_t + uu_x - H^x H^x_x + gh_x &= b', \quad (2b) \\
v_t + uv_x - H^x H^y_x &= 0, \quad (2c) \\
H^x_t + uH^x_x - u_x H^x &= 0, \quad (2d) \\
H^y_t + uH^x_x - v_x H^y &= 0, \quad (2e) \\
h_x H^x + h H^x_y = 0. \quad (2f)
\end{align*}

Notice that by means of (2a) and (2f) equation (2d) can be rewritten as

$$h_x H^x + h H^x_y = (h H^x)_t = 0.$$

Hence, due to (2f) one derives that

$$h H^x = a,$$

where $a$ is constant.

3. The one-dimensional SMHD equations in Lagrangian coordinates

Similar to the gas dynamics equations [35] one can introduce mass Lagrangian coordinates $(s, t)$, where $x = \varphi(s,t)$, and

$$\varphi_t(s,t) = \tilde{u}(s,t), \quad \varphi(s,t) = \frac{1}{\tilde{h}(s,t)}.$$

4
Here \( \tilde{u}(s,t) = u(\varphi(s,t),t) \), \( \tilde{h}(s,t) = h(\varphi(s,t),t) \). The sign tilde ‘‘ is omitted in further formulas. System (2a)–(2f) in mass Lagrangian coordinates reduces to the equations

\[
\begin{align*}
\frac{1}{\tilde{h}}_s - u_t &= 0, \\
u_t - a^2 \left( \frac{1}{\tilde{h}}_s \right) - gh \tilde{h}_s &= b', \\
\phi_t - aH_t &= 0, \\
H_t - av_t &= 0,
\end{align*}
\]  

(5a)–(5d)

where equations (2d) and (2f) become irrelevant due to (2a) and (3).

In (5b), the bottom topography is described by the function \( b(x) \), where \( x = \varphi(s,t) \), and the following relation for the differentials \( dt, ds \) and \( dx \) holds \( [36] \)

\[
ds = h dx - hu dt,
\]

(6)

that means

\[
s_t = -hu, \\
s_x = h.
\]

The general solution of (5c) and (5d) follows from the d’Alembert formula for the wave equation and has the form

\[
v = f_1(s + at) + f_2(s - at), \\
H^t = f_1(s + at) - f_2(s - at),
\]

where \( f_1 \) and \( f_2 \) are arbitrary functions of their arguments.

In variables \( t, s, \varphi \) system (5a)–(5d) reduces to the only second-order PDE:

\[
\varphi_{tt} - \left( a^2 \varphi_s - \frac{g}{2\varphi_t^2} \right)_s = b'.
\]

(8)

Therefore, the study of (2a)–(2f) in mass Lagrangian coordinates is reduced to the analysis of a single equation (8).

Notice that if \( a = 0 \), then (8) corresponds to the classical one-dimensional shallow water equation, considered in mass Lagrangian coordinates. Symmetries and conservation laws of this case (\( a = 0 \)) have already been studied in [29] (see also [28, 37]). Hence, for further analysis it is assumed that \( a \neq 0 \).

Remark 2. In mass Lagrangian coordinates, Equations for inclined bottom \( b' = b_0 = \text{const} \) reduce to equations for a horizontal bottom by means of the following transformation \([2, 38]\)

\[
\varphi = \tilde{\varphi} + b_0 t^2.
\]

4. Lie group classification

For the sake of simplicity in the present section we use the notation \( c = b' \).

Calculations yield the following equivalence group:

\[
\begin{align*}
X_1 &= \frac{\partial}{\partial s}, \\
X_2 &= \frac{\partial}{\partial t}, \\
X_3 &= \frac{\partial}{\partial \varphi}, \\
X_4 &= 2\varphi \frac{\partial}{\partial \varphi} + s \frac{\partial}{\partial s} + t \frac{\partial}{\partial t} + 3g \frac{\partial}{\partial g}, \\
X_5 &= 2\varphi \frac{\partial}{\partial t} + 4s \frac{\partial}{\partial s} + t \frac{\partial}{\partial t} + 3a \frac{\partial}{\partial a}, \\
X_6 &= \varphi \frac{\partial}{\partial \varphi} + s \frac{\partial}{\partial s} + t \frac{\partial}{\partial t} - c \frac{\partial}{\partial c}.
\end{align*}
\]
We also use the following involutions:

(a) : \( \varphi \mapsto -\varphi, \ g \mapsto -g, \ c \mapsto -c, \)
(b) : \( t \mapsto -t, \ a \mapsto -a. \)

**Remark 3.** A shift of the bottom function \( b = \tilde{b} + k, \) where \( k \) is constant, does not change (2a)-(2f).

The equivalence transformations allow reducing certain constants in the results of symmetry classification. The group classification of (8) considered for the case \( ag \neq 0. \)

An admitted generator is sought in the form

\[ X = \xi^t \frac{\partial}{\partial t} + \xi^s \frac{\partial}{\partial s} + \zeta^\varphi \frac{\partial}{\partial \varphi}, \]

which is prolonged for derivatives by standard formulas of the group analysis [22]. Solving the determining equations

\[ X \left( \varphi_{tt} - \left( a^2 \varphi_s - \frac{a}{2\varphi_s^2} \right)_s - b' \right) \bigg|_{(8)} = 0, \]

one derives the classifying equation

\[ \zeta'' - \zeta c'' - 2k_1(\varphi c' + c) = 0, \]  

where

\[ \zeta^\varphi = 2k_1\varphi + \zeta(t), \ \xi^t = 2k_1s + k_2, \ \xi^s = 2k_1t + k_3, \]

and \( k_i, i = 1, 2, 3 \) are arbitrary constants.

For any bottom function \( b(x), \) (8) admits the generators

\[ X_1 = \frac{\partial}{\partial s}, \quad X_2 = \frac{\partial}{\partial t}. \]

The Lie algebra consisting of these generators is called the kernel of admitted Lie algebras of (8).

Notice that differentiating (9) with respect to \( t, \) one finds that

\[ \zeta' c''' = 0. \]  

Further analysis depends on a choice of the function \( c. \) According to (11), one needs to consider \( c'' \neq 0 \) and \( c''' = 0. \) Results of this analysis are presented in table 1, where the first column lists the function \( c \) (up to equivalence transformations), the second column gives the corresponding bottom topography, extensions of the kernel of admitted Lie algebras are given in the third column. The generators presented in table 1 are

\[ X_3 = \frac{\partial}{\partial \varphi}, \quad X_4 = t \frac{\partial}{\partial \varphi}, \quad X_5 = (\varphi + q) \frac{\partial}{\partial \varphi} + s \frac{\partial}{\partial s} + t \frac{\partial}{\partial t}, \]

\[ X_6 = \varphi \frac{\partial}{\partial \varphi} + s \frac{\partial}{\partial s} + t \frac{\partial}{\partial t}, \quad X_7 = e^{ikt} \frac{\partial}{\partial \varphi}, \quad X_8 = e^{-ikt} \frac{\partial}{\partial \varphi}, \]

\[ X_9 = \cos kt \frac{\partial}{\partial \varphi}, \quad X_{10} = \sin kt \frac{\partial}{\partial \varphi}. \]
Table 1. Group classification.

| c(φ)  | b(x)  | Extension   |
|-------|-------|-------------|
| 0     | q₁    | X₃, X₄, X₆  |
| φ⁻¹   | ln x  | X₅          |
| φ     | ½x² + qx | X₇, X₈     |
| −φ    | −½x² + qx | X₉, X₁₀    |

5. Lagrangian and Hamiltonian formalism of (8)

5.1. Lagrangian

Equation (8) can be represented as the Euler–Lagrange equation. For this purpose one has to find a Lagrangian of the form

$$ L = L(t, s, \varphi, \varphi_t, \varphi_s) $$

such that the equation

$$ \frac{\delta L}{\delta \varphi} = 0, \quad (12) $$

is equivalent to (8), where

$$ \frac{\delta}{\delta \varphi} = \frac{\partial}{\partial \varphi} - D_t \left( \frac{\partial}{\partial \varphi_t} \right) - D_s \left( \frac{\partial}{\partial \varphi_s} \right) $$

is the variational derivative, and $D_t, D_s$ are the total derivatives with respect to $t$ and $s$. This is called the Helmholtz problem [39].

Substituting $\varphi_t$ found from (8) into (12), and splitting it with respect to parametric derivatives, one derives an overdetermined system of differential equations in which $t, s, \varphi, \varphi_t$ and $\varphi_s$ are considered as independent variables. Solving the resulting system of equations, one finally arrives at the Lagrangian

$$ L = \frac{\varphi_t^2}{2} - \frac{a^2 \varphi_s^2}{2} - \frac{g}{2 \varphi_s} + b. \quad (13) $$

One of applications of the Lagrangian consists of deriving conservation laws using Noether’s identity, which explicitly shows that invariance of the Lagrangian on solutions of the Euler–Lagrange equation yields conservation laws for this equation (for example, see [40]):

$$ X L + L D_j \xi^j = (\eta - \xi^i \varphi) \frac{\delta L}{\delta \varphi_j} + D_j \left( \xi^i L + (\eta - \xi^i \varphi) \frac{\partial L}{\partial \varphi_j} \right), $$

where $j \in \{t, s\}$, and the invariance of Lagrangian means that the left hand side of the latter identity is divergent.

5.2. Hamiltonian

The Lagrangian (13) can be written in the form

$$ L(x, x_s, \dot{x}) = \frac{1}{2} \dot{x}^2 + F(x, x_s), \quad (14) $$

7
where \( F = -\left(\frac{\partial^2 x}{\partial t^2} + \frac{\partial b(x)}{\partial x}\right) \), dot \(^\prime\) means the derivative with respect to \( t \). Introducing the variable \( \xi = \dot{x} \), the Lagrangian \( \tilde{\mathcal{L}} \) becomes
\[
\tilde{\mathcal{L}}(\xi, x, x_s) = \frac{1}{2} \xi^2 + F(x, x_s).
\]
The Euler–Lagrange equation
\[
\ddot{x} = \frac{\delta F}{\delta x}
\]  
(15)
in the Euler–Lagrange form can be rewritten as
\[
\dot{\eta} = \frac{\delta \tilde{\mathcal{L}}}{\delta \xi}, \quad \dot{\xi} = \xi,
\]
where \( \xi \) is found from the equation
\[
\eta = \frac{\delta \tilde{\mathcal{L}}}{\delta \xi} = \dot{\xi}.
\]
As the Lagrangian (14) is nonsingular [41], then one can derive the Hamiltonian as follows [42].

Using the Legendre transformation
\[
\mathcal{H} = \dot{x} \mathcal{L}_i - \mathcal{L} = \frac{1}{2} \dot{x}^2 - F,
\]
the Hamiltonian becomes
\[
\mathcal{H}(\eta, x, x_s) = \frac{1}{2} \eta^2 - F(x, x_s).
\]  
(16)
The Hamiltonian equations are
\[
\dot{\xi} = \frac{\delta \mathcal{H}}{\delta \eta}, \quad \dot{\eta} = -\frac{\delta \mathcal{H}}{\delta \xi}.
\]  
(17)
Substituting the Hamiltonian (16) into (17), they are
\[
\dot{x} = \eta, \quad \dot{\eta} = -\frac{\delta F}{\delta x}.
\]
Hence, one notes that Hamiltonian equations (17) coincide with the Euler–Lagrange equations (15).

Recall that the canonical Hamiltonian equations (we denote \( x = q, \eta = p \))
\[
\dot{q} = \frac{\partial \mathcal{H}}{\partial p}, \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q},
\]
can be obtained by varying the action functional
\[
\delta \int_{t_1}^{t_2} \left( p\dot{q} - \mathcal{H}(t, q, p) \right) dt = 0
\]
in the phase space \( (q, p) \) (see, for example, [42, 43]).

Lie point symmetries have to be written in the variables \( (t, q, p) \) as
\[
X = \xi(t, q, p) \frac{\partial}{\partial t} + \eta(t, q, p) \frac{\partial}{\partial q} + \zeta(t, q, p) \frac{\partial}{\partial p}.
\]
The newly established Hamilton identity derived in [44]
\[
\zeta \dot{q} + p \dot{\eta} - \mathcal{H} \dot{\zeta} = \xi \left( D_t \mathcal{H} - \frac{\partial \mathcal{H}}{\partial t} \right) - \eta \left( \dot{p} + \frac{\partial \mathcal{H}}{\partial q} \right) + \zeta \left( \dot{q} - \frac{\partial \mathcal{H}}{\partial p} \right) + D_t [p \eta - \xi \mathcal{H}]
\]
shows the relation between invariance of the Hamiltonian (which means the left hand side of the identity is zero) and first integral on the solution of Hamilton equations. Notice that the first expression in the right hand side of the identity is zero on the solution of Hamilton equations. Further we use the Lagrangian approach to find conservation laws.

5.3. Conservation laws

In the present section, the conservation laws for equation (8) in mass Lagrangian coordinates and equations (5a), (5b) in Eulerian coordinates, found by applying Noether’s theorem, are presented.

The local conservation law of a studied system of equations has the form of a divergent expression that vanishes on solutions of the system,

\[(T^1)_t + (T^2)_s = 0,\]

where the conserved quantities \(T^1\) and \(T^2\) are usually called the density and the flux of the conservation law.

As in the previous section, here \(c = b'\). Each conservation law is preceded by the symmetry which was used in applying Noether’s theorem. The case \(c = q_1\) is not considered further, because of Remark 2, it reduces to the case \(c = 0\).

- Case \(c\) is arbitrary.
  
  \[X_1: \quad (\varphi_t \varphi_s)_t - \left( \frac{\varphi_t^2 + a^2 \varphi_s^2}{2} - \frac{g}{\varphi_s} + b \right)_s = 0, \quad (18)\]
  
  \[X_2: \quad \left( \frac{\varphi_t^2 + \varphi_s^2}{2} + \frac{g}{2\varphi_s} - b \right)_t + \left( \frac{g\varphi_t}{2\varphi_s^2} - \varphi_t \varphi_s \right)_s = 0. \quad (19)\]

- Case \(c = 0\).
  
  \[X_3: \quad (\varphi)_t + \left( \frac{g}{2\varphi_s^2} - a^2 \varphi_s \right)_s = 0, \quad (20)\]
  
  \[X_4: \quad (t\varphi_t - \varphi)_t + \left( \frac{tg}{2\varphi_s^2} - ta^2 \varphi_s \right)_s = 0. \quad (21)\]

- Case \(c = \varphi\).
  
  \[X_7: \quad ((\varphi - \varphi_t - \varphi_s) e^t)_t + \left( \left( a^2 \varphi_s + \varphi_t + \varphi - \frac{g}{2\varphi_s^2} \right) e_x^t \right)_s = 0, \quad (22)\]
  
  \[X_8: \quad ((\varphi_t + \varphi_s + \varphi) e^{-t})_t + \left( \left( \varphi - a^2 \varphi_s - \varphi_t + \frac{g}{2\varphi_s^2} \right) e^{-t} \right)_s = 0. \quad (23)\]

- Case \(c = -\varphi\).
  
  \[X_9: \quad (\varphi \sin t + \varphi_t \cos t)_t - \left( \left( a^2 \varphi_s - \frac{g}{2\varphi_s^2} \right) \cos t \right)_s = 0, \quad (24)\]
  
  \[X_{10}: \quad (\varphi \cos t - \varphi_s \sin t)_t + \left( \left( a^2 \varphi_s - \frac{g}{2\varphi_s^2} \right) \sin t \right)_s = 0. \quad (25)\]
Conserved quantities \((\epsilon T^i, \epsilon T^2)\) and \((\epsilon^c T^i, \epsilon^c T^2)\) in Lagrangian and Eulerian coordinates are related as
\[
\epsilon T^i = h^i T^i, \quad \epsilon T^2 = hu^i T^i + T^2.
\]
Thus, the following conservation laws for \((5a)\) and \((5b)\) in Eulerian coordinates are obtained. The operators of total differentiation with respect to Eulerian coordinates \(t\) and \(x\) are denoted as \(D_t\) and \(D_x\).

- **Case \(c\) is arbitrary.**

\[
\begin{align*}
X_1: & \quad D_t(2u) + D_x \left( u^2 - (H^2)^2 + 2gh - 2b \right) = 0, \\
X_2: & \quad D_t \left\{ (u^2 + (H^2)^2 - 2b)h + gh^2 \right\} + D_x \left\{ (u^2 - (H^2)^2 + 2gh - 2b)hu \right\} = 0.
\end{align*}
\]

- **Case \(c = 0.\)**

\[
\begin{align*}
X_3: & \quad D_t(uh) + D_x \left( h^2 - (H^2)^2 \right) (u + \frac{gh^2}{2}) = 0, \\
X_4: & \quad D_t \left( ((t - x)h) + D_x \left( \left( (a^2 - (H^2)^2) - xu \right) h + \frac{tg^2}{2} \right) \right) = 0.
\end{align*}
\]

- **Case \(c = x.\)**

\[
\begin{align*}
X_7: & \quad D_t [ ((x - u)h - 1) e^r ] + D_x \left[ \left( x + \left( (H^2)^2 - u^2 + xu \right) h - \frac{gh^2}{2} \right) e^r \right] = 0, \\
X_8: & \quad D_t [ ((x + u)h + 1) e^{-r} ] + D_x \left[ \left( x + \left( (H^2)^2 + xu \right) h + \frac{gh^2}{2} \right) e^{-r} \right] = 0.
\end{align*}
\]

- **Case \(c = -x.\)**

\[
\begin{align*}
X_9: & \quad D_t (h(u \cos t + x \sin t)) + D_x \left[ \left( u^2 - (H^2)^2 + \frac{gh}{2} \right) h \cos t + xhu \sin t \right] = 0, \\
X_{10}: & \quad D_t (h(u \sin t - x \cos t)) + D_x \left[ \left( u^2 - (H^2)^2 + \frac{gh}{2} \right) h \sin t - xhu \cos t \right] = 0.
\end{align*}
\]

**Remark 4.** The transition from conservation laws in coordinates \(t, s, \varphi\) to conservation laws in coordinates \(t, s, h, u\) for \((5a)\) and \((5b)\) can be carried out straightforward using formulas \((4)\) and therefore is not given here. The only difficulty is that the conservation laws whose densities or fluxes explicitly include the variable \(x = \varphi\) should be supplemented by \((4)\). This is true for the conservation laws \((21)–(25)\) and also for \((18)\) and \((19)\) in case \(b' \neq 0\). For example, the conservation law \((22)\) in terms of the variables \(t, s, h, u\) should be written as
\[
\begin{align*}
\left( \left( x - u - \frac{1}{h} \right) e^r \right)_t + \left( \left( x + u + \frac{a^2}{h} - \frac{gh^2}{2} \right) e^r \right)_s &= 0, \\
x_t = u, & \quad x_r = \frac{1}{h}.
\end{align*}
\]
6. Invariant and partially invariant solutions

The knowledge of admitted Lie group allows one to construct some invariant and partially invariant solutions of \((2a)–(2f)\). The procedure for obtaining substantially different invariant solutions is based on finding an optimal system of subalgebras of the admitted Lie algebra of the studied equations \([22, 45]\). Choosing a subalgebra, say \(L\), one has to find universal invariant of the subalgebra by solving the equation

\[
XJ = 0, \quad \forall X \in L,
\]

where \(J\) depends on all dependent and independent variables. Separating the universal invariant into two parts, one derives a representation of the invariant solution, which, after substitution it into the original equations, provides a reduced system of equations. The main advantage of the reduced system is that it contains fewer independent variables.

The kernel of admitted Lie algebras \((10)\) gives two one-dimensional subalgebras: \(\{X_1\}\) and \(\{X_2 + \mathcal{D}X_1\}\). The subalgebra \(\{X_1\}\) does not provide invariant solutions, because invariant solutions in this case have the form \(\varphi = \varphi(t)\), which is impossible due to the requirement \(\varphi_s \neq 0\) following from \((4)\). Solutions invariant with respect to the subalgebra \(\{X_2 + \mathcal{D}X_1\}\) are called travelling wave type solutions, which are discussed in the next section.

6.1. Travelling wave type solutions of \((8)\)

Here the general case of arbitrary bottom topography is studied, while the further sections consider specific invariant solutions based on the group classification results.

A solution of a travelling wave type for \((8)\) is defined by the assumption \(\varphi = \varphi(z)\), where \(z = s - \mathcal{D}t\). Substituting the representation of the solution into \((8)\), it becomes

\[
\left(\mathcal{D}^2 - a^2 - \frac{8}{\varphi^2}\right) \varphi'' = b'.
\]

Integrating it, due to the equivalence transformation corresponding to the generator \(\partial / \partial s\), one gets

\[
(\mathcal{D}^2 - a^2)(\varphi')^3 - 2b\varphi' + 2g = 0.
\]

The latter equation can be represented as \(\varphi' = F(\varphi)\). As \(u = -\mathcal{D}\varphi'\) and \(h = 1/\varphi'\), then in Eulerian coordinates \(u(x, t) = -\mathcal{D}F(x)\) and \(h(x, t) = F^{-1}(x)\), where the function \(F(x)\) satisfies the equation

\[
(\mathcal{D}^2 - a^2)F^3 - 2bF + 2g = 0.
\]

Thus, the travelling wave solution corresponds to a stationary solution for the functions \(u\) and \(h\) in Eulerian coordinates. Let \(\chi(x)\) be such that \(\chi'(x) = h(x)\). Substituting \(s = \mathcal{D}t + \chi(x)\) into \((7)\), one also obtains the functions \(v(x, t)\) and \(H'(x, t)\) in an explicit form in Eulerian coordinates.

6.2. Invariant solutions for extensions of the kernel

Because the case \(c = 0\) corresponds to the gas dynamics equations in Lagrangian coordinates \([46]\), our study in the present section is restricted by the cases of logarithmic and parabolic bottom topographies. In the case of two independent variables consideration of subalgebras of dimensions two and higher leads to non-trivial exact solutions. According to table \(1\), for logarithmic and parabolic bottom, \((8)\) admits three- and four-dimensional Lie algebras. For all such algebras, optimal systems of subalgebras are already known and are given in \([47]\).
6.2.1. Case $c = \varphi^{-1}$. Consider the Lie algebra $\{X_1, X_2, X_6\}$. Its optimal system of one-dimensional subalgebras consists of
$$\{X_1\}, \{X_6\}, \{X_2 + \mathcal{D}X_1\}.$$ 

- Consideration of the one-dimensional subalgebra $\{X_6\}$ leads to the solutions of the form
$$\varphi = tQ(z), \quad z = s/t.$$

Substituting into (8), one gets the ODE
$$\left(z^2 - a^2 - \frac{g}{Q^3}\right)Q'' - \frac{1}{Q} = 0.$$

- As studied earlier in section 6.1, the subalgebra $\{X_2 + \mathcal{D}X_1\}$ leads to the solutions of the form
$$\varphi = Q(s - \mathcal{D}t).$$

Substituting into (8), one gets the reduction
$$\left(\varphi^2 - a^2 - \frac{g}{Q^3}\right)Q'' - \frac{1}{Q} = 0.$$

Multiplying by $Q'$ and integrating, one finds
$$\frac{1}{2} \left(\varphi^2 - a^2\right)Q'^2 + \frac{g}{Q} - \ln Q = C_1,$$
where $C_1$ is an arbitrary constant, which can be cancelled by equivalence transformations. Further in the present section $C_1$ and $C_2$ denote constants of integration. The following particular solution can be found in case $\mathcal{D} = \pm a$, namely
$$\varphi = \frac{g(s \pm at + C_2)}{W_0(g(s \pm at + C_2) \exp(C_1g - 1))},$$
where $W_0$ is the principal branch of the Lambert W function [48].

6.2.2. Case $c = \varphi$. The optimal system of one-dimensional subalgebras of the Lie algebra $\{X_1, X_2, X_7, X_8\}$ is
$$\{X_1\}, \{X_2 + \mathcal{D}X_1\}, \{X_7 + \beta X_8 + \mu X_1\}.$$ 

- $\{X_2 + \mathcal{D}X_1\}$ leads to the reduction
$$\left(\varphi^2 - a^2 - \frac{g}{Q^3}\right)Q'' - Q = 0,$$
where $\varphi = Q(s - \mathcal{D}t)$. Multiplying by $Q'$ and integrating, one finds
$$\frac{1}{2} \left(\varphi^2 - a^2\right)Q'^2 + \frac{g}{Q} - \frac{Q^2}{2} = C_1.$$
In case $\mathcal{D} = \pm a$ one gets the particular solution
\[ \varphi = \zeta - \frac{2C_1}{\zeta}, \quad \zeta^3 = 3(s \pm at + C_2)g + \sqrt{9(s \pm at + C_2)^2g^2 + 8C_1^3}. \] (29)

- \{X_7 + \beta X_8 + \mu X_1\}, $\mu \neq 0$, leads to the reduction
\[ Q'' - Q = 0, \quad \varphi = \frac{s}{\mu}(e^t + \beta e^{-t}) + Q(t). \]

This gives the invariant solution
\[ \varphi = \left( C_1 + \frac{s}{\mu} \right) e^t + \left( C_2 + \frac{\beta s}{\mu} \right) e^{-t}. \] (30)

6.2.3. Case $c = -\varphi$.

The optimal system of one-dimensional subalgebras of the Lie algebra $\{X_1, X_2, X_9, X_{10}\}$ is
\[ \{X_1\}, \{X_2 + \mathcal{D}X_1\}, \{X_9 + \mu X_1\}, \{X_{10} + \mu X_1\}. \]

- For $\{X_2 + \mathcal{D}X_1\}$, similar to (28), one obtains
\[ \frac{1}{2} (\mathcal{D}^2 - a^2)Q^2 + \frac{g}{Q^2} + \frac{Q'^2}{2} = C_1. \]

In case $\mathcal{D} = \pm a$ there is the particular solution
\[ \varphi = \zeta + \frac{2C_1}{\zeta}, \quad \zeta^3 = -3(s \pm at + C_2)g + \sqrt{9(s \pm at + C_2)^2g^2 - 8C_1^3}. \] (31)

- \{X_9 + \mu X_1\}, $\mu \neq 0$, leads to the reduction
\[ Q'' + Q = 0, \quad \varphi = \frac{s}{\mu} \cos t + Q(t). \]

This gives the invariant solution
\[ \varphi = C_1 \sin t + \left( \frac{s}{\mu} + C_2 \right) \cos t. \] (32)

- Similar to the previous case, for the one-dimensional subalgebra \{X_{10} + \mu X_1\} one derives the invariant solution
\[ \varphi = C_1 \cos t + \left( \frac{s}{\mu} + C_2 \right) \sin t. \] (33)

The results are summarized in table 2. The particular solutions (27), (29) and (31) are not presented in the table because of their cumbersome forms. As mentioned above, the subalgebra $\{X_1\}$ gives no invariant solutions. It is also assumed $\mu \neq 0$ throughout the table, as for $\mu = 0$ no invariant solutions. Notice that the traveling wave type solutions corresponding to the subalgebra $\{X_2 + \mathcal{D}X_1\}$ are particular cases of the solutions studied in section 6.1.
6.2.4. Invariant solutions in Eulerian coordinates. Invariant solutions obtained in Lagrangian coordinates can be rewritten in Eulerian coordinates. Among invariant solutions only the case of the scaling group corresponding to $X_6$ requires special explanation. All other invariant solutions are given in table 3.

For representation of invariant solution corresponding to the logarithmic bottom and the subalgebra $\{X_6\}$ one solves (26) with respect to $s$, namely,

$$s = tR(z),$$  (34)

where $z = x/t$ and $R = Q^{-1}$.}

| Table 2. Invariant solutions in mass Lagrangian coordinates. |
|--------------------------------------------------------------|
| $c(\varphi)$ | Subalgebra | Invariant solution or reduction |
| $\varphi^{-1}$ | $X_6$ | $\varphi = tQ(z)$, $z = \frac{1}{t}$, $(z^2 - a^2 - \frac{D}{a}) Q'' - \frac{1}{a} = 0$ |
| $X_2 + \mathcal{D}X_1$ | $\varphi = Q(s - \mathcal{D}t)$, $(\mathcal{D}^2 - a^2) Q'' + \frac{2e}{s} - 2\ln Q = \text{const}$ |
| $\varphi$ | $X_2 + \mathcal{D}X_1$ | $\varphi = Q(s - \mathcal{D}t)$, $(\mathcal{D}^2 - a^2) Q'' + \frac{2e}{s} - Q^2 = \text{const}$ |
| $X_7 + \beta X_8 + \mu X_1$ | $\varphi = \left( C_1 + \frac{a}{\mu} \right)e^t + \left( C_2 + \frac{2\mu}{\eta} \right)e^{-t}$ |
| $-\varphi$ | $X_2 + \mathcal{D}X_1$ | $\varphi = Q(s - \mathcal{D}t)$, $(\mathcal{D}^2 - a^2) Q'' + \frac{2e}{s} + Q^2 = \text{const}$ |
| $X_9 + \mu X_1$ | $\varphi = C_1 \sin t + \left( \frac{a}{\mu} + C_2 \right) \cos t$ |
| $X_{10} + \mu X_1$ | $\varphi = C_1 \cos t + \left( \frac{a}{\mu} + C_2 \right) \sin t$ |

| Table 3. Some invariant solutions in Eulerian coordinates. |
|-------------------------------------------------------|
| $b(x)$ | Subalgebra | Invariant solution or reduction |
| arbitrary | $X_2 + \mathcal{D}X_1$ | See section 6.1 |
| $\ln x$ | $X_6$ | $h = R'$, $u = z + \frac{R}{s}$, $v = f_1(\zeta_+ + f_2(\zeta_-)$, $H^k = \frac{R}{s}$, $H^f = f_1(\zeta_+) - f_2(\zeta_-)$, $\zeta_+ = \left( \frac{R}{s} \right)^{\pm} \pm$ at, |
| $\frac{x^2}{2}$ | $X_7 + \beta X_8 + \mu X_1$ | $h = \frac{\mu}{x + \beta s}$, $u = \frac{1}{x + \beta s} \left( (e^2 - \beta) s + 2(C_1 \beta - C_2) e^t \right)$, $v = f_1(\zeta_+) + f_2(\zeta_-)$, $H^k = \frac{2\mu}{s}(e^t + \beta e^{-t})$, $H^f = f_1(\zeta_+) - f_2(\zeta_-)$, $\zeta_+ = \left( e^t - C_1 e^2 - C_2 \right) \pm \mu \pm$ at. |
| $\frac{\sqrt{2}}{2}$ | $X_9 + \mu X_1$ | $h = \frac{\mu}{x \tan t}$, $u = \frac{\zeta_+}{\cos t} - x \tan t$, $v = f_1(\zeta_+) + f_2(\zeta_-)$, $H^k = \frac{2\sin t}{\mu}$, $H^f = f_1(\zeta_+) - f_2(\zeta_-)$, $\zeta_+ = \left( \frac{x}{\sin t} = C_1 \tan t - C_2 \right) \mu \pm$ at. |
| $X_{10} + \mu X_1$ | $h = \frac{\mu}{x \cot t}$, $u = x \cot t - \frac{D}{a}$, $v = f_1(\zeta_+) + f_2(\zeta_-)$, $H^k = \frac{2\sin t}{\mu}$, $H^f = f_1(\zeta_+) - f_2(\zeta_-)$, $\zeta_+ = \left( \frac{1}{\sin t} = C_1 \cot t - C_2 \right) \mu \pm$ at. |
Differentiating (34) with respect to $t$ and $x$ and taking into account (6), one derives the following equations in Eulerian coordinates
\[ zR' - R = hu, \quad R' = h. \] (35)

Then, one can write the function $R$ explicitly as
\[ R = (z - u)h. \] (36)

Substituting (35) into (2a)–(2f), one also obtains the constraint
\[ gR'' + \frac{(a^2 - R^2)R'''}{R^{'}} - \frac{1}{z} = 0. \] (37)

The invariant solution is
\[ h = R', \quad u = z - \frac{R}{R'}, \quad v = f_1(\zeta_+) + f_2(\zeta_-), \]
\[ H^x = \frac{a}{R'}, \quad H^y = f_1(\zeta_+) - f_2(\zeta_-), \quad \zeta_{\pm} = \left( \frac{R}{R'} \pm a \right) t, \] (38)

where $z = x/t$, and the function $R(z)$ must satisfy (37).

The remaining solutions in Eulerian coordinates corresponding to (30), (32) and (33) are derived in a similar way.

In order to demonstrate how the magnetic field affects the velocity field $u$, we take a closer look at a couple of particular solutions here.

(a) First, we consider the newly obtained solution for the parabolic bottom $\frac{v^2}{2}$ corresponding to the generator $X_7 + \beta X_8 + \mu X_1$:
\[ h = e^{\frac{\mu}{v + \beta - \gamma}}, \quad u = \frac{1}{\beta + \gamma} \left( (e^{2t} - \beta)x + 2(C_1 + C_2)e^t \right), \]
\[ v = f_1(\zeta_+) + f_2(\zeta_-), \quad H^x = \frac{a}{\mu} (e^t + \beta e^{-t}), \quad H^y = f_1(\zeta_+) - f_2(\zeta_-), \]
\[ \zeta_{\pm} = (xe^t - C_1 e^{2t} - C_2) \frac{1}{\beta + \gamma} \pm at. \]

We choose the particular case
\[ \mu = 1, \quad \beta = 1, \quad C_1 = 2, \quad C_2 = 1, \]
\[ f_1(\zeta) = \frac{1}{2} \sin \zeta, \quad f_2(\zeta) = \frac{1}{4} \sin \zeta. \]

For magnetic free ($a = 0, \mathbf{H} = 0$) solution we get
\[ h = \frac{1}{e^t + e^{-t}}, \quad u = \frac{(e^{2t} - 1)x + 2e^t}{e^{2t} + 1}, \quad v = \frac{x e^t - 2e^{2t} - 1}{e^{2t} + 1}, \]
\[ H^x = 0, \quad H^y = 0. \]

For $a = 1, \mathbf{H} \neq 0$, the solution is
\[ h = \frac{1}{e^t + e^{-t}}, \quad u = \frac{(e^{2t} - 1)x + 2e^t}{e^{2t} + 1}, \quad v = \frac{(t - 2)e^{2t} + xe^t + t - 1}{e^{2t} + 1}, \]
\[ H^x = e^t + e^{-t}, \quad H^y = v. \]

In this case, only the component $v$ of the velocity changes under the influence of the magnetic field. In figure 1, the velocity field is shown in gray for $\mathbf{H} = 0$ and in black for $\mathbf{H} \neq 0$. 

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Figure 1. Evolution of $v$ in the cases $H = 0$ (grey) and $H \neq 0$ (black) in the region $x \in [0, 25]$ for $t \in [2, 5]$.

It can be seen from the figure that in the presence of a magnetic field in the selected region, the velocity component $v$ changes its direction.

(b) Consider the solution (38) for the logarithmic bottom corresponding to the generator $X_6$. We choose the following parameters and functions

$$
g = 1, \quad z_0 = 0.1, \quad R(z_0) = 0, \quad R'(z_0) = 0.25, \quad
f_1(\zeta) = \frac{1 + a}{2} \zeta, \quad f_2(\zeta) = \frac{1 - a}{2} \zeta.
$$

The numerical results for the velocity components $u$ and $v$ for $t \in [0.1, 1.1], x \in [1, 5]$ are given in figure 2. Here the velocity fields have a significant change in slope near the region where $zR' = R$.

6.3. Simple waves

According to the group analysis method, a simple wave is a partially invariant solution, where all unknown functions depend on a single function [22, 49, 50]. Riemann wave is a particular case of simple waves for equations which can be written in Riemann invariants\(^3\). Simple waves of (2a)–(2f) are discussed here.

Consider (8)

$$
\varphi_{tt} - \left( a^2 \varphi_x - \frac{g}{2 \varphi_x} \right)_x = b'.
$$

\(^3\) Sometimes a simple wave is also called by a Riemann wave.
In Eulerian coordinates this equation corresponds to the system of equations

\[ \begin{align*}
    &h_t + uh_x + hu_x = 0, \\
    &u_t + uu_x + \left( \frac{a^2}{h^2} + g \right) h_x = b'.
\end{align*} \tag{39} \]

Equation (39) coincide with the gas dynamics equations for isentropic flows, where the sound speed is

\[ \lambda = \sqrt{\frac{a^2}{h^2} + gh}. \]

Let \( \sigma(h) \) be a function such that \( \sigma'(h) = \frac{1}{\lambda} \). Using the Riemann invariants [35, 36]\(^4\)

\[ r = u + \sigma(h), \quad l = u - \sigma(h), \]

Equation (39) are rewritten in the form

\[ \begin{align*}
    &r_t + (u + \lambda)r_x = b', \\
    &l_t + (u - \lambda)l_x = b'.
\end{align*} \]

Consider case \( b' = 0 \). As in the classical gas dynamics for a Riemann wave one of the Riemann invariants is assumed to be constant. For example, assume that \( r = k \), where \( k \) is constant. In Lagrangian coordinates this Riemann wave is defined by the equation

\[ \varphi_t + \sigma(1/\varphi) = k. \tag{40} \]

Solutions of this equation can be found by the Cauchy method (method of characteristics). Comprehensive study of constructing solutions of equations even more general than (40) is given in [52] (v.4). In particular, using notations of [52], equation (40) has the form

\[ f(p, q) = p + \sigma(1/q) - k = 0, \]

\(^4\) Riemann waves of (39) are discussed in [2, 51].
where \( \frac{d\sigma(1/\alpha)}{dq} = -\sqrt{a^2 + \frac{g}{q}} \). The complete integral is

\[ \varphi = s\alpha + t(k - \sigma(1/\alpha)) + \beta, \]

where \( \alpha \) and \( \beta \) are constant. Using the envelope, one can construct singular and general integrals, as well as a solution of a Cauchy problem. Thus, one obtains the general solution of the Riemann wave in Lagrangian coordinates.

**Remark 5.** System of equations (2a)–(2f) only has trivial solutions of simple wave type: it is reducible to an invariant solution. One class of such invariant solutions is considered in [2, 51], where the authors studied solutions of (2a)–(2f) invariant with respect to the admitted generator \( t\partial_t + x\partial_x \).

Indeed, for a nontrivial simple wave type solution of (2a)–(2f) one has to assume that \( aH^\prime \neq \text{const} \). These solutions can be represented in the form

\[ u = U(h), \quad H^\prime = \frac{a}{h}, \quad v = V(h), \quad H^\prime = H(h), \]

where \( U(h), V(h) \) and \( H(h) \) are some functions. Substituting these functions into (39), one obtains

\[
\begin{align*}
    h_t + Uh_{x} + hU'h = 0, \\
    U'(h + Uh') + \left( \frac{a}{h} + g \right) h_t = 0, \\
    V'(h + Uh') - ah^{-1}H'h = 0, \\
    H'(h + Uh') - ah^{-1}V'h = 0.
\end{align*}
\]

As \( H^\prime \neq \text{const} \), then \( h_t \neq 0 \). Eliminating \( h_t \), found from the first equation of (41), substituting it into the other equations, and taking into account that \( h_t \neq 0 \), one derives

\[
\begin{align*}
    U'^2 &= \frac{a^2}{h^2} + \frac{g}{h}, \\
    V'hU' + ah^{-1}H^\prime = 0, \quad ah^{-1}V' + H'hU' = 0.
\end{align*}
\]

As (43) compose a homogeneous system of linear algebraic equations with respect to \( H^\prime \) and \( V' \), and due to \( H^\prime \neq 0 \), one derives that

\[ U'^2 = \frac{a^2}{h^2}. \]

Comparison of the latter relation with (42) gives the contradiction to the condition that \( g \neq 0 \).

**7. Conclusion**

The present paper is devoted to the Lie group analysis of the one-dimensional SMHD equations within Gilman’s model. The SMHD equations are considered in cases of a plane and uneven bottom topography.

We would like to outline three important results. Firstly, it is shown that the system of equations written in Eulerian coordinates reduces to a single second-order PDE in Lagrangian coordinates, while the rest of equations were integrated explicitly. Complete Lie group classification with respect to bottom topography of the single equation is performed.

Secondly, the Lagrangian formalism and Noether’s theorem are used to construct conservation laws of the equations. For all cases of a bottom topographies the conservation laws are constructed, including some new conservation laws. These results are also represented in Eulerian coordinates.
Thirdly, invariant and partially invariant solutions are constructed. The symmetry classification separated out all bottom topographies into four classes. For each of the classes an extension of the kernel of admitted Lie algebras is found and all invariant solutions are considered. The kernel of admitted Lie algebras consists of the shifts with respect to time and the mass Lagrangian space coordinate that allows constructing traveling wave solutions. It should be noted that such solutions in Eulerian coordinates correspond to stationary solutions for $h, u$ and $H_x$, whereas for $v$ and $H_y$ we obtained the general solution, which is expressed through $h$ in explicit form with two arbitrary functions. Almost all invariant solutions are found in explicit form both in mass Lagrangian coordinates and in Eulerian coordinates. Analysis of Riemann waves in mass Lagrangian coordinates and comparison of them with simple waves in Eulerian coordinates are presented in the paper.

Data availability statement

No new data were created or analysed in this study.

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ORCID iDs

E I Kaptsov https://orcid.org/0000-0001-7984-4238
S V Meleshko https://orcid.org/0000-0002-3205-5650
V A Dorodnitsyn https://orcid.org/0000-0003-0860-8806

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