Super-Additivity and Entanglement Assistance in Quantum Reading

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March 7, 2022

Abstract
Quantum information theory determines the maximum rates at which information can be transmitted through physical systems described by quantum mechanics. Here we consider the communication protocol known as quantum reading. Quantum reading is a protocol for retrieving the information stored in a digital memory by using a quantum probe, e.g., shining quantum states of light to read an optical memory. In a variety of situations using a quantum probe enhances the performances of the reading protocol in terms of fidelity, data density and energy dissipation. Here we review and characterize the quantum reading capacity of a memory model, defined as the maximum rate of reliable reading. We show that, like other quantities in quantum information theory, the quantum reading capacity is super-additive. Moreover, we determine conditions under which the use of an entangled ancilla improves the performance of quantum reading.

1 Introduction

The scope of quantum information theory is to determine how and how much information can be stored, processed, and transmitted through physical systems behaving according to the laws of quantum mechanics \cite{1}. In particular, one is interested in transmitting classical or quantum information, possibly in the presence of physical constraints (e.g., limited energy of bandwidth) or additional resources (e.g., quantum entanglement or feedback communication) \cite{2}.

In the most common setting, one is given a quantum communication channel, that is, a physical process that transforms quantum states at the input into quantum states at the output, as for example an optical fiber does, see e.g., \cite{3,4}. A channel of this kind is also called a quantum-quantum (QQ) channel. Another kind of channel is the so-called classical-quantum (CQ) channel, which maps classical states (that is, probability distribution over a set of symbols) into quantum states. To send classical information through a QQ channel, the sender (Alice) first encodes classical states into quantum states by applying a suitable CQ channel at the input of the QQ channel, as represented in Figure 1a). In this setting, the CQ channel plays the role of an encoding map. At the other end of the QQ channel, the receiver (BOB) collects the output and measures it to decode the classical information sent by Alice.

Quantum reading (QR) is a communication protocol that is based on a different rationale \cite{5,6}. Instead of having a given communication channel and encoding classical information by choosing the input states, in QR the sender encodes information by choosing an element from a collection of QQ channels. Then, to decode, the receiver probes the QQ channels with a quantum states, collects the output and measures it.

The prototypical example of QR is that of an optical memory, e.g. a CD or DVD, where information is encoded in a memory cell by means of the physical properties of the substrate, e.g., its reflectivity or phase. For this reason, QR has been mainly considered in the context of optical realizations \cite{7,8,9,10,11,12,13,14,15}. For example, a memory cell with low or high reflectivity may encode a logical “0” or “1”. To read this information the receiver must shine a laser beam on the memory cell, and the collect the reflected beam (see Figure 2). There are proven advantages in using quantum states of light to perform this task, for instance increased fidelity and data density, reduced energy consumption and dissipation \cite{5}. 

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From a more abstract point of view, QR can be represented as shown in Figure 1b). The encoding of a symbol \( x \) belonging to an alphabet \( \mathcal{X} \) can be modeled as a control-QQ channel (a generalization of a control-unitary channel [1]) where the value of \( x \) determines which of the QQ channel in a set \( \Phi = \{ \phi_x \}_{x \in \mathcal{X}} \) should be applied.

Following [6] we refer to the set \( \Phi = \{ \phi_x \}_{x \in \mathcal{X}} \) as a “memory cell”. One can define the quantum reading capacity of \( \Phi \) as the maximum rate (in bits per use of the memory cell) that can be reliably transmitted from the sender to the receiver using the encoding procedure specified by \( \Phi \). Indeed, in previous works several notions of capacity have been defined according to which constraints are assumed or additional resources are allowed [6]. In this paper we present further results concerning quantum reading capacities, in particular we show that QR capacity is super-additive, and discuss under which conditions the assistance of entanglement enhances the QR capacity.

The paper proceeds as follows. In Section 2 we review a few basic notions and definitions. In Section 3 we analyze the case of noiseless QR. The property of super-additivity of QR is discussed in Section 4 and the case of noisy QR is considered in Section 5. Finally, Section 6 is devoted to zero-error QR capacity, and Section 7 is for conclusions.

## 2 Quantum reading capacities

A QR protocol comprises an encoding and a decoding stage. During the encoding stage, which is essentially classical, the sender Alice encodes messages \( i = 1, 2, \ldots, M \) using codewords of length \( n \), \( x^n(i) = x_1(i)x_2(i) \cdots x_n(i) \), where \( x_k(i) \in \mathcal{X} \). Each codeword identifies a corresponding sequence of quantum channels from the memory cell \( \Phi = \{ \phi_x \}_{x \in \mathcal{X}} \), e.g., \( \phi^n_{x^n(i)} = \phi_{x_1(i)} \otimes \phi_{x_2(i)} \otimes \cdots \phi_{x_n(i)} \). During the decoding stage, the receiver Bob prepares a state \( \rho^n \), also called a transmitter, which is used to probe the sequence of quantum channels \( \phi^n_{x^n(i)} \). Finally, Bob collects and measures the output to retrieve the encoded message.

In analogy with other quantum communication protocol we introduce the following definitions.

**Definition 1 (Quantum Reading protocol)** A \((M, n, \epsilon)\)-QR protocol for a memory cell \( \Phi = \{ \phi_x \}_{x \in \mathcal{X}} \) is defined by an encoding map \( E \) from \( i = 1, \ldots, M \) to \( \mathcal{X}^{\otimes n} \), a transmitter state \( \rho^n \), and a measurement with POVM
have across pairs, triplets, etc., of different uses of the memory cell, that is, that the maximum is indeed obtained when the Holevo information, and information theory \cite{16, 17}, the latter can be expressed as:

\[
\rho = \sum_{x \in \mathcal{X}} \rho_x \overline{x}
\]

where the maximum is over all probability distribution over the alphabet \( \mathcal{X} \), say that the transmitter, we can restrict to the family of QR protocols for which the transmitter has the form, \( \rho^n = \rho^\otimes n \), that is, it is a separable state across different uses of the memory cell. The maximum QR rate that can be achieved under this constraint is defined as \( C^1(\Phi) \). We have

\[
C^1(\Phi) = \max_{\rho} C^1(\Phi|\rho),
\]

where \( C^1(\Phi|\rho) \) denotes the maximum QR rate achievable for a given \( \rho \). Applying known results of quantum information theory \cite{16, 17}, the latter can be expressed as:

\[
C^1(\Phi|\rho) = \max_{\{p_x, \phi_x(\rho), \} x \in \mathcal{X}} \chi(\{p_x, \phi_x(\rho), \} x \in \mathcal{X})
\]

where the maximum is over all probability distribution over the alphabet \( \mathcal{X} \).

\[
\chi(\{p_x, \phi_x(\rho), \} x \in \mathcal{X}) = S\left(\sum_{x \in \mathcal{X}} p_x \phi_x(\rho)\right) - \sum_{x \in \mathcal{X}} p_x S(\phi_x(\rho)),
\]

is the Holevo information, and \( S(\sigma) = -\text{Tr}(\sigma \log \sigma) \) denotes the von Neumann entropy. It can be easily shown that the maximum is indeed obtained when \( \rho \) is a pure state \cite{6}.

Similarly we can define the QR capacities \( C^k(\Phi) \), for \( k = 1, 2, \ldots \), where the transmitter state is separable across pairs, triplets, etc., of different uses of the memory cell, that is, \( \rho^n = \rho^\otimes (n/k) \) (for \( n \) multiple of \( k \)). We have

\[
C^k(\Phi|\rho^k) = \frac{1}{k} \max_{\{p_{x^k}, \phi_{x^k}(\rho^k), \} x^k \in \mathcal{X}^k} \chi(\{p_{x^k}, \phi_{x^k}(\rho^k), \} x^k \in \mathcal{X}^k)
\]

Clearly \( C(\Phi) \geq C^k(\Phi) \geq C^h(\Phi) \) for \( k > h \). If the inequality is strict, that is, \( C^k(\Phi) > C^1(\Phi) \) for some \( k \), we say that the \( C(\Phi) \) is super-additive.

More generally, the transmitter state can be chosen to be entangled with an ancilla, which Bob retains and measures jointly, see Figure\footnote{3} In this case we speak of entanglement-assisted QR. Notice that an entanglement-assisted QR protocol for a memory cell \( \Phi = \{\phi_x\} \) is equivalent to an unassisted protocol for the extended memory cell \( \Phi \otimes \text{id} = \{\phi_x \otimes \text{id}\} \), where \( \text{id} \) denotes the identity channel acting on the ancilla. The entanglement-assisted QR capacity is hence given by the expression

\[
C^1_{EA}(\Phi) = C(\Phi \otimes \text{id}).
\]

Similarly, we can define the assisted QR capacities \( C^k_{EA}(\Phi) \) by constraining the transmitter to be separable across groups of \( k \) uses of the memory cell. Clearly, we have \( C^1_{EA}(\Phi) \geq C(\Phi) \), and \( C^k_{EA}(\Phi) \geq C^h(\Phi) \). If, for some value of \( k \), this inequality is strict we say that the assistance of entanglement enhances the QR capacity \( C^k(\Phi) \) of the memory cell.
3 Noiseless quantum reading

We first consider a noiseless setting in which the QQ channels in the memory cell are unitary transformations, that is, $\phi_x(p) = U_x p U_x^\dagger$. For the sake of simplicity we consider the binary setting, $x \in \{0, 1\}$, with the unitaries acting in a finite-dimensional Hilbert space of dimension $d$.

Let us consider the Holevo information,

$$\chi\left(\{p_x, U_x |\psi\rangle\langle\psi|U_x^\dagger\}_{x=0,1}\right) = S\left[ p U_0 |\psi\rangle\langle\psi|U_0^\dagger + (1-p) U_1 |\psi\rangle\langle\psi|U_1^\dagger \right]$$

(6)

$$= S\left[ p |\psi\rangle\langle\psi| + (1-p) U |\psi\rangle\langle\psi|U^\dagger \right],$$

(7)

where $p = p_0$, and $U = U_0^\dagger U_1$. From Equation (6) we obtain $C^1$ by maximization of the Holevo information. This is equivalent to maximizing the von Neuman entropy of the state $\sigma = p |\psi\rangle\langle\psi| + (1-p) U |\psi\rangle\langle\psi|U^\dagger$. In order to do that, it is convenient to introduce an unit vector $|\psi_\perp\rangle$ such that $\langle\psi|\psi_\perp\rangle = 0$ and

$$U |\psi\rangle = \alpha |\psi\rangle + \sqrt{1-|\alpha|^2} |\psi_\perp\rangle.$$  

(8)

In the system of orthonormal vectors $|\psi\rangle, |\psi_\perp\rangle$, the state $\sigma$ is represented by the density matrix:

$$\tilde{\sigma} = \left( \begin{array}{cc} p + (1-p)|\alpha|^2 & (1-p)\alpha \sqrt{1-|\alpha|^2} \\ (1-p)\alpha^* \sqrt{1-|\alpha|^2} & (1-p)(1-|\alpha|^2) \end{array} \right).$$

(9)

The maximum von Neumann entropy of $\sigma$ is achieved in correspondence to the maximum determinant of the matrix $\tilde{\sigma}$. We have,

$$\det \tilde{\sigma} = p(1-p)(1-|\alpha|^2),$$

(10)

which is maximized for $p = 1/2$ and in correspondence of the minimum value of $|\alpha|^2 = |\langle\psi|U|\psi\rangle|^2$. Let us denote as $\{|j\rangle\}_{j=0,...,d-1}$ the eigenvectors of $U$, and as $e^{i\theta_j}$ the corresponding eigenvalues. We expand $|\psi\rangle$ in the basis of eigenvectors, $|\psi\rangle = \sum_j \psi_j |j\rangle$, which yields $\alpha = \sum_j |\psi_j|^2 e^{i\theta_j}$. The reading capacity is hence obtained by putting $\alpha = \alpha_{\text{min}}$, with

$$|\alpha_{\text{min}}|^2 = \min_{\{\psi_j\}|\sum_j |\psi_j|^2 = 1} \sum_{jj'} |\psi_j|^2 |\psi_{j'}|^2 e^{i(\theta_j - \theta_{j'})},$$

(11)

which finally yields

$$C^1 = h\left(1 - \frac{|\alpha_{\text{min}}|^2}{2}\right),$$

(12)

where $h(x) = -x \log x - (1-x) \log (1-x)$.

Let us now consider the entanglement-assisted QR capacity $C_{EA}^1$. To compute $C_{EA}^1$ we can repeat the reasoning of above with $U$ replaced by $U \otimes I$. Notice that $U \otimes I$ has the same eigenvalues of $U$ (but with higher multiplicity). We can then consider a system of eigenvectors of $U \otimes I$, denoted as $\{|jk\rangle\}$, where $|jk\rangle$ are the eigenvectors with shared eigenvalue $e^{i\theta_j}$. Expanding the transmitter state $|\psi\rangle$ in this basis we obtain $|\psi\rangle = \sum_{jk} \psi_{jk} |jk\rangle$, which yields $\alpha = \sum_{jk} |\psi_{jk}|^2 e^{i\theta_j} = \sum_j (\sum_k |\psi_{jk}|^2) e^{i\theta_j}$. We then obtain the same expression for $|\alpha_{\text{min}}|^2$ as in Equation (11) upon replacing $\sum_{jk} |\psi_{jk}|^2 \to |\psi_{j}|^2$. In conclusion, we have obtained that $C^1 = C_{EA}^1$, that is, the assistance of entanglement does not enhance the QR capacity $C^1$ in the noiseless setting.
As an example, let us consider the case of qubit unitaries \((d = 2)\). We have
\[
|\psi\rangle = |\psi_0\rangle + |\psi_1\rangle
\]
and
\[
|\alpha_{\text{min}}|^2 = \min_{\{\psi_0, \psi_1: |\psi_0|^2 + |\psi_1|^2 = 1\}} |\psi_0|^4 + |\psi_1|^4 + 2|\psi_0|^2|\psi_1|^2 \cos(\delta \theta),
\]
with \(\delta \theta = |\theta_1 - \theta_0|\). The minimum is hence obtained for
\[
|\psi_0|^2 = |\psi_1|^2 = 1/2
\]
and yields
\[
|\alpha_{\text{min}}| = \sqrt{\frac{1 + \cos(\delta \theta)}{2} = |\cos(\delta \theta/2)|}.
\]
Finally from (12) we obtain
\[
C^1 = h\left(\sin^2(\delta \theta/4)\right).
\]

4 Super-additivity

Let us consider the case of a binary memory cell composed of two qubit unitary transformations, \(U_0\) and \(U_1\). We now show that this cell exhibits the phenomenon of super-additivity.

For given \(k > 1\), let us consider the Holevo information
\[
\chi^k = \chi\left(\{p_{x^k}, \phi_{x^k}(|\psi\rangle\langle\psi|)\}\right)
\]
with a probability distribution such that \(p_{00...0} = p_{11...1} = 1/2\). That is, we are only considering, with equal probability, the unitary transformations \(U_0^\otimes k\) and \(U_1^\otimes k\). The Holevo information then reads
\[
\chi^k = S\left(\frac{1}{2} |\psi\rangle\langle\psi| + \frac{1}{2} U^\otimes k |\psi\rangle\langle\psi| U^\otimes k\right),
\]
with \(U = U_0^\dagger U_1\).

Let us denote the eigenvectors of \(U\) as \(|0\rangle\) and \(|1\rangle\), with corresponding eigenvalues \(e^{i\theta_0}\) and \(e^{i\theta_1}\). As a transmitter state we chose the entangled state \(|\psi\rangle = (|0\rangle^\otimes k + |1\rangle^\otimes k) / \sqrt{2}\). We then obtain
\[
C^k \geq \frac{1}{k} h\left(\sin^2(k\delta \theta/4)\right).
\]

We hence have found that for any \(k\) there exist values of \(\delta \theta\) such that \(C^k > C^1\), that is, QR is super-additive. We can also write a lower bound on the ultimate QR capacity, that is,
\[
C \geq \sup_k \frac{1}{k} h\left(\sin^2(k\delta \theta/4)\right).
\]

See Figure 4 for a comparison among different QR capacities.
5 Noisy quantum reading

Going beyond the case of noiseless quantum reading, we consider a simple yet physically motivated example of noisy binary quantum reading where the two encoding maps are of the form

$$\phi_x(\rho) = (1-q)U_x \rho U_x^\dagger + q \rho_0,$$

where $\rho_0$ is the maximally mixed state in $d$ dimensions, with $x = 0, 1$ and $q \in [0, 1]$.

Let us first consider the QR capacity $C^1$. Putting $U = U_0^0 U_1$, the Holevo information in Equation (2) reads

$$\chi = S \left( (1-q) \langle \rho_0 | \psi \rangle \langle \psi | + p_1 U | \psi \rangle \langle \psi | U^\dagger \rangle + q \rho_0 \right) - S \left( (1-q) U | \psi \rangle \langle \psi | U^\dagger + q \rho_0 \right)$$

$$= S \left( (1-q) \langle \rho_0 | \psi \rangle \langle \psi | + p_1 U | \psi \rangle \langle \psi | U^\dagger \rangle + q \rho_0 \right) - \eta(1-q + q/d) - (d-1)\eta(q/d),$$

where $\eta(y) = -y \log y$. The maximization of the Holevo information is thus reduced to the maximization of the von Neumann entropy of the state $\sigma = (1-q) \langle \rho_0 | \psi \rangle \langle \psi | + p_1 U | \psi \rangle \langle \psi | U^\dagger \rangle + q \rho_0$. It is convenient to expand this state in a basis defined by the vector $|\psi\rangle$, the vector $|\psi_\perp\rangle$ (such that $\langle \psi | \psi_\perp \rangle = 0$ and $U | \psi \rangle = \alpha | \psi \rangle + \sqrt{1-|\alpha|^2} | \psi_\perp \rangle$), and any other set of $d-2$ vectors. In this basis the state $\sigma$ is represented by the density matrix

$$\tilde{\sigma} = \left( \begin{array}{cc} (1-q)(1-p) & \frac{\alpha^* \sqrt{1-|\alpha|^2}}{d} \\ \frac{\alpha \sqrt{1-|\alpha|^2}}{d} & (1-q)(1-p)(1-|\alpha|^2) \end{array} \right).$$

The maximum von Neumann entropy of $\sigma$ corresponds to the maximum determinant of $\tilde{\sigma}$, where

$$\det \tilde{\sigma} = \left( p(1-p)(1-q)^2(1-|\alpha|^2) + \frac{q(1-q) - q^2|\alpha|^2}{d^2} \right)^{d-2}.$$ (23)

For any given $q$, the determinant is maximized for $p = (1-p) = 1/2$ and in correspondence of the minimum value of $|\alpha|^2$. We hence obtain

$$C^1 = \eta \left( \frac{q}{d} + \frac{1}{2} \frac{1-|\alpha_{\min}|}{d} \right) + \eta \left( \frac{q}{d} + \frac{1}{2} \frac{1-|\alpha_{\min}|}{d} \right) - \eta(1-q + q/d) - \eta(q/d),$$ (24)

where $|\alpha_{\min}|$ is given by Equation (11).

For example, in the case $d = 2$, using Equation (24), we obtain

$$C^1 = \eta \left( \frac{q}{2} + (1-q) \cos (\delta \theta/4)^2 \right) + \eta \left( \frac{q}{2} + (1-q) \sin (\delta \theta/4)^2 \right) - \eta(1-q/2) - \eta(q/2),$$ (25)

5.1 Entanglement assisted reading

Unlike the noiseless case, the assistance of entanglement can be beneficial in the noisy setting. We consider the entanglement-assisted QR capacity $C^1_{E_A}$, which can be computed by maximization of the Holevo information

$$\chi = S \left( (1-q) \langle \rho_0 | \psi \rangle \langle \psi | + p_1 (U \otimes \mathbb{I}) | \psi \rangle \langle \psi | (U^\dagger \otimes \mathbb{I}) \right) + q \rho_0 \otimes \rho_A$$

$$- S \left( (1-q) (U \otimes \mathbb{I}) | \psi \rangle \langle \psi | (U^\dagger \otimes \mathbb{I}) + q \rho_0 \otimes \rho_A \right),$$ (26)

where $| \psi \rangle$ is a joint pure state for the $BA$ system, comprising both Bob output and the ancillary system, and $\rho_A = \text{Tr}_B(| \psi \rangle \langle \psi |)$ denotes the reduced state of the ancilla. Without loss of generality we can assume that the dimension of the ancilla $A$ equals that of Bob system $B$. Moreover, as an example we take $| \psi \rangle$ to be a maximally entangled state in the $BA$ system, which implies $\rho_A = \rho_0 = \mathbb{I}/d$. We then obtain

$$\chi = S \left( (1-q) \langle \rho_0 | \psi \rangle \langle \psi | + p_1 (U \otimes \mathbb{I}) | \psi \rangle \langle \psi | (U^\dagger \otimes \mathbb{I}) \right) + q \mathbb{I}/d^2$$

$$- S \left( (1-q) (U \otimes \mathbb{I}) | \psi \rangle \langle \psi | (U^\dagger \otimes \mathbb{I}) + q \mathbb{I}/d^2 \right)$$

$$= S \left( (1-q) \langle \rho_0 | \psi \rangle \langle \psi | + p_1 (U \otimes \mathbb{I}) | \psi \rangle \langle \psi | (U^\dagger \otimes \mathbb{I}) \right) + q \mathbb{I}/d^2$$

$$- \eta(1-q - q/d^2) - (d^2 - 1)\eta(q/d^2),$$ (28)
The rate of the QR protocol is in close relation with the problem of discriminating between two unitary transformations. AQR is closely related to the problem of quantum channel discrimination [5, 6]. The particular case of noiseless 6 Zero-error capacity (extension to the higher dimension is straightforward). Given that the spectrum of the unitary zero-error QR rate. Coming back to the results of [18], let us consider the case of two-dimensional unitaries Φ = {ϕα}α∈X, measurement with POVM elements |ψ⟩ where
\[ C^{1}_{EA} \geq \eta \left( \frac{q}{d^2} + \frac{(1-q)(1+|\alpha|)}{2} \right) + \eta \left( \frac{q}{d^2} + \frac{(1-q)(1-|\alpha|)}{2} \right) - \eta(1 - q + q/d^2) - \eta(q/d^2), \] where |α|^2 = |⟨ϕ|U⊗|ψ⟩|^2 and |ψ⟩ is a maximally entangled state.

For the sake of simplicity, let us now consider the case d = 2 and consider a system of eigenvectors of U ⊗ I, denoted as {⟨jk⟩}, where |jk⟩ is an eigenvector with eigenvalue e^{iθ}. The maximally entangled state |ψ⟩ can be represented, without loss of generality, as |ψ⟩ = \sum_{j=0,1} 2^{-1/2}|jj⟩, which implies α = \frac{1}{2} \sum_{j=0,1} e^{iθ}, and in turn yields |α| = |cos(δθ/2)|. Substituting this value for α in Equation (32) with d = 2 we obtain
\[ C^{1}_{EA} \geq \eta \left( \frac{q}{4} + (1-q) \cos(\deltaθ/4)^2 \right) + \eta \left( \frac{q}{4} + (1-q) \sin(\deltaθ/4)^2 \right) - \eta(1 - 3q/4) - \eta(q/4). \] (33)

By comparison with Equation (25) it follows that, unlike the noiseless setting, the assistance of entanglement is beneficial (that is, C^{1}_{EA} > C^{1}) in the presence of noise (see Figure 5).

### 6 Zero-error capacity

QR is closely related to the problem of quantum channel discrimination [5, 6]. The particular case of noiseless QR is hence in close relation with the problem of discriminating between two unitary transformations. According to [18], two unitaries, U_0 and U_1, can always be perfectly discriminated, if enough copies of them are provided and using a suitable input state and, possibly, a collective measurement. To relate in a formal way this feature of the problem of unitary discrimination with QR, we need to consider the notion of zero-error QR capacity.

**Definition 3 (Zero-error quantum reading protocol)** A (M,n) zero-error QR protocol for a memory cell Φ = {ϕx}x∈X is defined by an encoding map E from i = 1, . . . , M to X⊗n, a transmitter state ρ^n, and a measurement with POVM elements {Λ(j)}j∈J, such that the average probability of error in decoding is zero. The rate of the QR protocol is R = \frac{1}{n} log M.

From the definition of zero-error QR protocol it follows that of zero-error QR capacity as the maximum zero-error QR rate. Coming back to the results of [18], let us consider the case of two-dimensional unitaries (extension to the higher dimension is straightforward). Given that the spectrum of the unitary U = U_0^1 U_1 is e^{iθ}, e^{iθ}, [18] proved that the unitaries U_0^n and U_1^n are perfectly distinguishable if n ≥ \frac{π}{δθ} (with δθ = |θ_0 − θ_1|). This result implies that the zero-error QR capacity of the binary unitary memory cell is
\[ C_0 = \frac{1}{|π/δθ|}. \] (34)

The zero-error capacity is plotted in Figure 4.
7 Conclusions

We have presented several new results concerning the properties of super-additivity and the use of entanglement as a resource to enhance the QR capacities. We have proven that the assistance of entanglement does not increase the QR capacity in the noiseless setting, where the memory cell consists of unitary transformations. On the contrary, we have shown with an example that the assistance of entanglement may enhance the QR capacity when the memory cell consists of noisy quantum channels. We have also shown that the QR capacity, like other quantities in quantum information theory [19, 20], exhibits the phenomenon of super-additivity.

As already pointed out in previous works [5, 6], the protocol of QR is closely related to the task of quantum channel discrimination. For example, the fact that the assistance of entanglement enhances the QR capacity for a noisy memory cell, mirrors the fact that the use of an entangled ancilla may improve the discrimination between quantum channels (see e.g. [21]). At a more formal level, the analogy between QR and quantum channel discrimination can be appreciated through the notion of zero-error QR capacity, as discussed in Section 6. QR is also closely related to the task of parameter estimation, see e.g. [22] (this connection was also discussed in [23]). For example, notice that the QR capacity is super-additive in the region of small values of δθ (see Figure 4). This is the regime in which discriminating between two unitaries is essentially equivalent to estimating a small variation of the value of a relative phase.

Acknowledgements

S.P. has been supported by the EPSRC (‘qDATA’, EP/L011298/1).

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