‘Majorana mass’ fermions as untrue Majorana particles, rather endowed with pseudoscalar-type charges than genuinely neutral

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Abstract

The idea of a ‘Majorana mass’ to make a chiral neutrino really neutral is here reconsidered. It is pointed out that such an approach, unlike Majorana’s (non-chiral) old one, does not strictly lead, in general, to a true self-conjugate particle. This can be seen on directly using the basic definition (or fundamental representation) of charge conjugation $C$ in Quantum Field Theory, as an operation just acting on annihilation and creation operators and just expressing particle–antiparticle interchange. It is found, indeed, that the ‘active’ and ‘sterile’ whole fields which can be obtained from mixing the chiral components of two mutually charge-conjugate Dirac fields are themselves ‘charge conjugate’ to each other (rather than individually self-conjugate). These fields, taken as mass eigenfields (as in the ‘Majorana mass’ case), are shown to describe particles carrying pseudoscalar-type charges and being neutral relative to scalar-type charges only. For them, ‘$CP$ symmetry’ would be nothing but pure mirror symmetry, and $C$ violation (already implied in their respective ‘active’ and ‘sterile’ behaviors) should then involve time-reversal violation as well. The new (no longer strictly chargeless) ‘Majorana mass’ neutrino model still proves, however, neither to affect the usual expectation for a neutrinoless double $\beta$-decay, nor to prevent ‘active’ and ‘sterile’ neutrino varieties from generally taking different mass values. One has, on the other hand, that any fermion being just a genuine (i.e. really self-conjugate) Majorana particle cannot truly exist in two distinct – ‘active’ and ‘sterile’ – versions, and it can further bear only a unified mass kind which may at once be said to be either a ‘Majorana-like’ or a ‘Dirac-like’ mass kind.

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1 Introduction

The ‘nature’ of the neutrino mass (if any) has been for years one of the most intriguing puzzles of elementary-particle physics. The point still at issue is whether real neutrinos may even be self-conjugate (or self charge-conjugate) like Majorana particles [1], and further endowed with so-called ‘Majorana masses’ [2,3], or there actually exist mere ‘Dirac mass’ neutrinos looking like standard fermions (different from their own antiparticles). In this regard, the most general neutrino model available for each lepton family is believed to include an overall Lagrangian mass term with both ‘Dirac’ and ‘Majorana’ contributions, and with the latter contribution being the sum of two distinct mass terms for wholly ‘active’ and ‘sterile’ neutrino types [4]. Such a model, if suitably conceived with a ‘Majorana’ sector having just one nonzero term, relevant to a super-heavy ‘sterile’ neutrino type, can in particular be seen to account — via the well-known See–Saw Mechanism [5] — for the very small size to which the actual neutrino masses seem to be confined. Renewed interest in Majorana’s conjecture has been recently aroused also by the experimental discovery of Majorana bound states (or quasiparticles) in superconductors [6–12].

This paper deals with some subtle, and not yet thoroughly investigated, basic theoretical aspects concerning the idea itself of a ‘Majorana’ Lagrangian mass term as a way to get either an ‘active’ or a ‘sterile’ up-to-date version of the original (manifestly self-conjugate) Majorana neutrino field. It should, first of all, be reminded that the ‘Majorana mass’ construct cannot be really traced back to Majorana himself, nor can it be said essential for a self-conjugate fermion. Despite this, that a ‘Majorana mass’ fermion should just be a Majorana particle — i.e. a really neutral fermion — is normally regarded as quite an obvious conclusion, which automatically follows from an extended use of the well-known formula — Eq. (2) — defining the ‘charge conjugate’ of a standard Dirac field. According to common views, the full legitimacy of such a procedure is in particular believed to be unquestionable. Doing like that, however, one is not directly applying charge conjugation (or particle–antiparticle conjugation) as it is primarily defined within Quantum Field Theory (QFT): namely, an operation, $C$, truly acting on annihilation and creation operators and merely consisting in turning them into their own ‘charge conjugates’ (with no changes in either four-momenta or helicities) [13]. As already pointed out by Dvornikov in his canonical quantization of a massive Weyl field [14], this should not be taken as a negligible detail. One may guess it even better on going over to the zero-mass case. It is sufficient, for example, to take account of the straightforward hints here given in Sec. 2, on how to interpret the two couples of fermionic and antifermionic Weyl solutions in order to make sure that $C$ may really have no effects on helicities. These hints show that an approach like the usual one in defining the ‘charge conjugate’ of a Weyl field does not seem at all to lead to the appropriate choice. They also suggest the need for a more general check on the real consistency of a procedure that does nothing but borrow the standard definition of a ‘charge conjugate’ Dirac field. The simplest way to do so is just to make direct use of the above-mentioned fundamental representation of $C$. 

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Following this way, a basic new outcome is here obtained which overturns the current reading. It is found, indeed, that an active ‘Majorana mass’ fermion field and its sterile counterpart prove rather to be mutually charge-conjugate than individually self-conjugate, and so, at most, they may give rise to one and the same (really neutral) Majorana field if they are further imposed to coincide.

The formal aspects relevant to the whole question (including an explicit representation of the effective ‘new’ action of C on single chiral fields) are widely discussed in Sec. 3. In the subsequent section, moreover, it is shown that, regardless of whether the ‘Majorana mass’ or ‘Dirac mass’ case is considered, there generally exists a fully symmetrical link connecting both an active spin-$\frac{1}{2}$ field and its charge conjugate sterile counterpart with the corresponding pair of charge conjugate Dirac fields whence they have been constructed. This link is given by a unitary transformation being as well the inverse of itself, and it suitably allows an extended $(8 \times 8)$ matrix representation for C. In the light of the new formalism, on the other hand, the conclusion may also be drawn that a true (really self-conjugate) Majorana field can no longer turn out to be of two different – ‘active’ and ‘sterile’ – types, and furthermore (in strict accordance with its self-conjugate nature) it can be assigned only a unified mass kind which may at once be viewed as either a ‘Majorana-like’ or a ‘Dirac-like’ mass kind.

In Secs. 5, 6 and 7, a full insight is gained into the general variety of ‘charges’ that should now characterize a genuine ‘Majorana mass’ fermion and tell it from a genuine ‘Dirac mass’ fermion. The former particle, unlike the latter, should actually be endowed with pseudoscalar-type (or axial-type) charges and be ‘neutral’ as regards scalar-type charges only [15]. The ‘neutrality’ of it, in other words, is now to be meant no longer under C, but rather under a more restrictive ‘charge conjugation’ operation which leaves pseudoscalar-type charges unvaried and merely corresponds to a ‘scalar-charge conjugation’ operation. One such charged spin-$\frac{1}{2}$ particle would in turn amount to a ‘fermion’ or an ‘antifermion’ depending on either chirality involved. Thus, for instance, an active ‘Majorana mass’ neutrino is to be now referred to as a ‘lepton’ (having positive lepton number) or an ‘antilepton’ (having negative lepton number) according to whether being a left-handed or a right-handed particle, whereas the exact converse (with ‘lepton’ and ‘antilepton’ interchanged) should hold for the ‘charge conjugate’ sterile counterpart of it. In close connection with this, active and sterile ‘Majorana mass’ neutrinos may now be regarded as truly obeying ordinary mirror symmetry as just the analogue of ‘CP symmetry’ for Dirac neutrinos. A manifest (maximum) C violation is to be instead recognized in their (maximally) asymmetrical dynamical behaviors, and this should actually imply, in the light of the CPT theorem, a (maximum) time reversal violation as well (just counterbalancing the ‘recovered’ P symmetry). The new reading can also be seen, in particular, neither to influence the usual expectation for a neutrinoless double $\beta$-decay, nor to rule out the possibility — still compatible with CPT symmetry — of different mass values for the two (active and sterile) ‘Majorana mass’ neutrino versions.

In Sec. 8, finally, it is pointed out that a pair of charge conjugate spin-$\frac{1}{2}$ fields with identical masses, whether it may be a ‘Dirac mass’ or a ‘Majorana
mass’ field pair, can always be expressed as a linear combination of a couple of 
true Majorana fields with opposite CP intrinsic parities (and identical masses).

2 A ‘Majorana mass’ neutrino as not exactly a
genuine (really neutral) Majorana particle

According to Majorana’s early approach [1], a self-conjugate neutrino is a re-
ally neutral spin-$\frac{1}{2}$ particle which may be formally assigned, say, a Dirac field
solution of the special type

$$\psi_M(x) = \frac{1}{\sqrt{2}}[\psi(x) + \psi^c(x)]$$

(1)

($x \equiv x^\mu; \mu = 0, 1, 2, 3$), where $\psi^c(x)$, defined as

$$\psi^c(x) \equiv C\psi(x)C^{-1} = U_C\psi^T(x),$$

(2)

is the charge conjugate of a standard Dirac field solution $\psi(x)$, such that $\psi(x) \neq \psi^c(x)$. Here $U_C$ denotes the usual charge-conjugation (or $C$) matrix, and $\psi^T$ is the transpose of the adjoint solution $\psi^\dagger$. The field given by Eq. 1 has a manifest self-conjugate form:

$$\psi^c_M(x) = \frac{1}{\sqrt{2}}[\psi^c(x) + \psi(x)] = \psi_M(x),$$

(3)

and it can thus be automatically expanded in terms of net annihilation and cre-
ation operators coinciding with their own charge conjugates. This field, in other
words, is self-conjugate by definition; and the associated fermion, usually known
as a Majorana particle, is such that it cannot possibly be distinguished from its antiparticle. If we in particular split $\psi_M$ into a left-handed chiral component,
$\frac{1}{2}(1 - \gamma^5)\psi_M$, plus a right-handed one, $\frac{1}{2}(1 + \gamma^5)\psi_M$, where $\gamma^5 (\equiv i\gamma^0\gamma^1\gamma^2\gamma^3)$ is just denoting the chirality matrix, we then have that the former (latter) component taken alone would indifferently be able to describe a left-handed neutrino (right-handed antineutrino) as well as a left-handed antineutrino (right-handed neutrino):

$$\frac{1}{2}(1 \mp \gamma^5)\psi_M = \frac{1}{2}(1 \mp \gamma^5)\psi^c_M.$$  

(4)

In this regard, it is worth pointing out that $\psi_M$, also expressible in the form

$$\psi_M = \frac{1}{\sqrt{2}}\left[\frac{1}{2}(1 - \gamma^5)\psi + \frac{1}{2}(1 + \gamma^5)\psi^c\right] + \frac{1}{\sqrt{2}}\left[\frac{1}{2}(1 + \gamma^5)\psi + \frac{1}{2}(1 - \gamma^5)\psi^c\right],$$

(5)

can by no means be assigned any special sort of ‘handedness’ marking it as either an ‘active’ or a ‘sterile’ field. As shown by [1], one strictly has that ‘active’ and ‘sterile’ contributions are always equally present in $\psi_M$! This, indeed, implies that a neutrino just described by a field like $\psi_M$ would not be compatible with
the Standard Model (SM) [16–18], as it could give only half of the required contribution to the square modulus of the matrix element.

The equation obeyed by $\psi_M$ (still the Dirac one) may be derived, as usual, from a free spin-$\frac{1}{2}$ quantum field Lagrangian with a mass term proportional to

$$\bar{\psi}_M \psi_M,$$  \hspace{1cm} (6)

where $\bar{\psi}_M = \psi^\dagger_M \gamma^0$. This term looks just like the one for a standard Dirac particle; so, it does tell us nothing about the actual (self-conjugate) character of $\psi_M$, which can only be inferred from definition (1).

The current formal way of introducing a ‘self-conjugate’ neutrino is different from Majorana’s way — for a general review, see e.g. Ref. 19 or 20 — and is essentially based on a ‘reformulation’ of the Majorana neutrino theory in the light of parity-violating phenomenology [21, 22]. It leads to a neutrino type really compatible with the SM, being inspired by the idea of regarding a massive neutrino as merely an extension of a massless one. The new approach, in terms of basic chiral neutrino fields, relies just upon one peculiar requirement naturally fulfilled by the original Majorana field $\psi_M$: its being made up of two chiral components, $\frac{1}{2}(1 \mp \gamma^5)\psi_M$, that are subjected to the mutual link

$$\frac{1}{2}(1 \pm \gamma^5)\psi_M = U_C \left[ \frac{1}{2}(1 \mp \gamma^5)\psi_M \right]^T,$$  \hspace{1cm} (7)

as a result of the condition $\psi_M = \psi_M^c$ (recall that $U_C\gamma^5U_C^T = -\gamma^5U_C$ and $U_C^T = U_C^{-1}$). For a better insight into this point, let us start off by considering the purely left-handed (i.e. negative-chirality) neutrinos and purely right-handed (i.e. positive-chirality) antineutrinos as known from experience. As far as they are assumed to be massless, the question of the existence of opposite-chirality ‘complements’ of their own fields may somehow be ignored. This, strictly speaking, can no longer be the case in the presence of (not exactly zero) neutrino masses. If one is in particular thinking of real neutrinos and antineutrinos as standard massive elementary fermions and antifermions, one should be able to supply their (originally massless) Lagrangians — by adding e.g. suitable Higgs couplings — with mass terms proportional to

$$\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L = \bar{\psi}\psi$$  \hspace{1cm} (8)

and

$$\bar{\psi}_L^c\psi_R^c + \bar{\psi}_R^c\psi_L^c = \bar{\psi}^c\psi^c,$$  \hspace{1cm} (9)

respectively ($\bar{\psi}_L = \psi^\dagger_L \gamma^0$, and so on), where

$$\psi_L \equiv \frac{1}{2}(1 - \gamma^5)\psi, \quad \psi_R \equiv \frac{1}{2}(1 + \gamma^5)\psi$$  \hspace{1cm} (10)

and

$$\psi_L^c \equiv \frac{1}{2}(1 - \gamma^5)\psi^c, \quad \psi_R^c \equiv \frac{1}{2}(1 + \gamma^5)\psi^c.$$  \hspace{1cm} (11)
For clarity’s sake, it is worth noting that here symbols $\psi_{L,R}$ and $\psi^c_{L,R}$ are used quite symmetrically to denote the left- and right-handed chiral components of $\psi$ and those of $\psi^c$; so, one in particular has $\psi^c_{L,R} = (\psi^c)_{L,R}$, and not $\psi^c_{L,R} = (\psi^c)_{R,L}$ (as often encountered in the literature). There appears to be, on the other hand, an alternative formal way of constructing a congruous mass term for a neutrino; it consists in directly mixing the two (left-handed) neutrino and (right-handed) antineutrino fields themselves [2,3]. Doing like this, one is enabled to get, for instance, a scalar of the type

$$\bar{\psi}_L \psi^c_R + \bar{\psi}^c_R \psi_L = \bar{\psi}' \psi',$$

with the new (wholly ‘active’) field

$$\psi'(x) = \psi_L(x) + \psi_R(x)$$

replacing the original field $\psi(x) = \psi_L(x) + \psi_R(x)$. Note that (12) is different from (5) or (9) provided $\psi^c_R \neq \psi_R$, $\psi_L \neq \psi^c_L$. Such an approach to a massive neutrino does not necessarily need a complementary right-handed neutrino field, which could only enter into another (independent) mass term proportional to

$$\bar{\psi}_R \psi^c_L + \bar{\psi}^c_L \psi_R.$$

Of course, it is worth similarly stressing that this extra (wholly ‘sterile’) massive neutrino type could by no means be conjectured if it were $\psi^c_L = \psi_L$, $\psi_R = \psi^c_R$. From (8) or (9), it is therefore evident, in particular, that a nontrivial definition of the new field variable given by (13) (such that $\psi^c_R \neq \psi_R$, $\psi_L \neq \psi^c_L$) does indeed prevent it from being formally mistaken just for a $\psi_M$ field variable. Yet, since

$$U_C \psi^c_{L}^{\dagger T}(x) = \psi^c_R(x), \quad U_C \psi^c_{R}^{\dagger T}(x) = \psi_L(x),$$

one actually gets, in full analogy with (7),

$$\psi_R(x) = U_C \psi^c_{L}^{\dagger T}(x), \quad \psi_L(x) = U_C \psi^c_{R}^{\dagger T}(x),$$

with $\psi_R(x) = \frac{1}{2}(1 + \gamma^5)\psi'(x) = \psi^c_R(x)$ and $\psi_L(x) = \frac{1}{2}(1 - \gamma^5)\psi'(x) = \psi_L(x)$. Hence it is commonly argued that Eq. (16) is just defining a self-conjugate neutrino field variable, associated with a new kind of mass term — proportional to (12) — which is conventionally known as a ‘Majorana mass’ term. Such a conclusion relies on the fact that, by use of either (15) or (16), one globally obtains

$$\psi'(x) = U_C \psi^c_{L}^{\dagger T}(x).$$

Of course, in interpreting (16) as really a sufficient (besides necessary) condition to state that $\psi'(x)$ is self-conjugate, one is implicitly taking for granted that $U_C \psi^c_{L}^{\dagger T}(x)$ does correspond to the ‘charge conjugate’ of $\psi'(x)$, or that $U_C \psi^c_{L}^{\dagger T}(x) = U_C \psi^c_{L}^{\dagger T}(x)$ does correspond to the ‘charge conjugate’ of $\psi_L(x) = \psi^c_L(x)$. If so, it should then result

$$\psi''(x) \equiv C \psi'(x) C^{-1} = U_C \psi^c_{L}^{\dagger T}(x),$$

with

$$\bar{\psi}_L \psi^c_R + \bar{\psi}^c_R \psi_L = \bar{\psi}' \psi'',$$

for

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with

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$$\bar{\psi}_R \psi^c_L + \bar{\psi}^c_L \psi_R.$$
with the Majorana condition $\psi'(x) = \psi'^c(x)$ being automatically fulfilled. At first sight, since Eq. (18) is the exact analogue of Eq. (2), there seem to be no reasons for questioning it. Despite this, consider e.g. the two Weyl equations, into which the Dirac equation can be split up on going over to the zero-mass limit. As is well-known [12], the solutions of one Weyl equation amount to the couple of (massless) Dirac solutions

$$\psi_L(x), \psi_R^c(x) = U_C \psi_L^T(x),$$

whereas those of the other Weyl equation amount to the remaining couple of (massless) Dirac solutions

$$\psi_R(x), \psi_L^c(x) = U_C \psi_R^T(x).$$

Referring merely to the positive-energy contributions in the field expansions, we may in particular associate with the left-handed solutions, $\psi_L$ and $\psi_L^c$, a negative helicity and with the right-handed ones, $\psi_R$ and $\psi_R^c$, a positive helicity. Parity violation clearly occurs whenever (19) and (20) enter asymmetrically, and it becomes maximal just when only one couple of solutions is really involved. In the same way, as $C$ does not change helicities, also $C$ violation is expected to occur, with a maximal degree just when, for example, the only couple (19) appears to be available. Nevertheless, if we rewrite (19) as

$$\psi_L(x), \psi_R^c(x) = C\psi_R(x)C^{-1},$$

and (20), accordingly, as

$$\psi_R(x), \psi_L^c(x) = C\psi_L(x)C^{-1},$$

then, even on admitting that solutions (22) are suppressed, we cannot truly say that $C$ is violated, and we are in fact faced with a helicity inverting charge-conjugation operation! On the contrary, if we choose to interpret (19) as

$$\psi_L(x), \psi_R^c(x) = C\psi_L(x)C^{-1},$$

and (20), accordingly, as

$$\psi_R(x), \psi_L^c(x) = C\psi_R(x)C^{-1},$$

we immediately see that any asymmetry occurring between (23) and (24) really imply $C$ violation, with $C$ now being so defined as to really leave helicity unchanged.

In the light of these remarks on how to get an appropriate $C$ definition (not affecting helicity) for zero-mass spin-$\frac{1}{2}$ particles, it appears quite reasonable to try to check more carefully whether Eq. (18) is really consistent or not in the general framework of standard QFT. For this purpose, in order to form a clear idea on how to proceed, it may be useful to look first at Eq. (2). As is well-known, primarily putting $\psi^c(x) = C\psi(x)C^{-1}$ does indeed mean making
reference to the basic definition of charge conjugation \( C \), just coinciding with the fundamental representation of \( C \) (in the fermion–antifermion Fock space). Take e.g. the standard normal mode expansion of \( \psi(x) \) in terms of single ‘particle’ annihilation operators \( a^{(h)}(p) \) and ‘antiparticle’ creation operators \( b^{(h)}(p) \) obeying the usual anticommutation rules and being relevant to simultaneous eigenstates of momentum \( p \) and helicity \( h \). It looks like

\[
\psi(x) = \int \frac{d^4p}{(2\pi)^4 2p^0} \sum_h \left[ a^{(h)}(p)u^{(h)}(p)e^{-ip\cdot x} + b^{\dagger(h)}(p)v^{(h)}(p)e^{ip\cdot x} \right]
\]

(\( h = c = 1 \)), where \( u^{(h)}(p) \) and \( v^{(h)}(p) \) are four-spinor coefficients depending on four-momentum \( p \equiv (p^0, \mathbf{p}) \) \((p^0 > 0)\). We thus have, according to standard QFT, that \( C\psi(x)C^{-1} \) is the net field obtained from (25) as a result of the transformation

\[
Ca^{(h)}(p)C^{-1} = b^{(h)}(p), \quad Cb^{\dagger(h)}(p)C^{-1} = a^{\dagger(h)}(p).
\]

We also know that the subsequent equality, \( C\psi(x)C^{-1} = U_C\psi^{\dagger T}(x) \), does instead tell us how the field \( \psi^c(x) \equiv C\psi(x)C^{-1} \) as defined by means of (20) can equivalently be obtained via a suitable mapping, \( \psi(x) \rightarrow \psi^c(x) = U_C\psi^{\dagger T}(x) \), in the four-spinor space (such a mapping is indeed allowed by the fact that \( \psi^{\dagger T} \) and \( \psi^c \) share all four freedom degrees corresponding to the actual ‘particle’ and ‘antiparticle’ helicity eigenstates). Of course, Eq. (2) applies as well to a field, \( \psi_M(x) \), having the manifestly self-conjugate form (1), whose expansion can be obtained from (25) by making the substitutions \( a^{(h)}(p) \rightarrow a^{(h)}_M(p) \), \( b^{(h)}(p) \rightarrow a^{\dagger(h)}_M(p) \), where \( a^{(h)}_M(p) = \frac{1}{\sqrt{2}} [a^{(h)}(p) + b^{(h)}(p)] \) and \( a^{\dagger(h)}_M(p) = \frac{1}{\sqrt{2}} [b^{\dagger(h)}(p) + a^{\dagger(h)}(p)] \). Let us pass now to analyse Eq. (18), addressed to a field, \( \psi'(x) \), being such that

\[
\psi'(x) = \frac{1}{2}(1 - \gamma^5)\psi(x) + \frac{1}{2}(1 + \gamma^5)\psi^c(x),
\]

with \( \psi(x) \neq \psi^c(x) \). We shall have, quite similarly, that writing \( \psi^{lc}(x) \equiv C\psi'(x)C^{-1} \) does mean defining \( \psi^{lc}(x) \) as just that field which is obtained from \( \psi'(x) \) by merely turning every annihilation and creation operator into their respective charge-conjugates. As such operators can only be found within the \( \psi(x) \) and \( \psi^c(x) \) expansions, we may also write, due to the linearity property of \( C \) in the Fock space,

\[
\psi^{lc}(x) \equiv C\psi'(x)C^{-1} = \frac{1}{2}(1 - \gamma^5)\psi(x)C^{-1} + \frac{1}{2}(1 + \gamma^5)C\psi^c(x)C^{-1}.
\]

The actual check to be given to Eq. (18) should therefore concern \( C\psi'(x)C^{-1} = U_C\psi^{\dagger T}(x) \). In other words: Is the mapping \( \psi'(x) \rightarrow \psi^{lc}(x) = U_C\psi^{\dagger T}(x) \) consistently providing, as claimed, an equivalent way to get \( \psi^{lc}(x) \equiv C\psi'(x)C^{-1} \) from \( \psi'(x) \)? To answer this question, one has to do nothing else than directly applying prescription (25). Doing like this, one obtains

\[
\psi^{lc}(x) = \frac{1}{2}(1 - \gamma^5)\psi^c(x) + \frac{1}{2}(1 + \gamma^5)\psi(x),
\]
and hence, by use again of the compact notations \(10\) and \(11\),

\[
\psi'(x) = \psi'_L(x) + \psi'_R(x) \neq \psi'(x) = \psi_L(x) + \psi_R(x).
\] (30)

The key to \(30\) is just the \textit{linear} behavior of the fundamental representation of \(C\), as particularly regards its action inside the \textit{single} Dirac-field chiral components \(\psi_L\) and \(\psi'_R\). Such an outcome, if carefully examined, should not seem so surprising, especially in view of what we already know from the \(V - A\) theory \[23-25\]. Looking at Eq. \(27\), we can easily realize that purely replacing every annihilation and creation operator with their own charge-conjugates does amount to globally making the \textit{net} interchange \(\psi \leftrightarrow \psi^c\) relative to \(\frac{1}{2}(1 \mp \gamma^5)\); and this, indeed, fully agrees with the general fact that, if one takes \(\bar{\psi}_2 \gamma^\mu \psi_1\) and \(\bar{\psi}_2 \gamma^\mu (-\gamma^5) \psi_1\) as Dirac-field \textit{covariants} (as in the \(V - A\) theory), then one gets \(\bar{\psi}_2 \gamma^\mu (1 - \gamma^5) \psi_1 \leq \bar{\psi}_2 \gamma^\mu (1 - \gamma^5) \psi_1^T\) \[26\]. A comparison of Eqs. \(17\) and \(30\) gives, despite appearances,

\[
C \psi'(x) C^{-1} \neq U_C \psi'^T(x),
\] (31)

and then

\[
C \psi_L(x) C^{-1} \neq U_C \psi^T_L(x);
\] (32)

which ultimately means that standard QFT does \textit{not} really allow Eq. \(2\) to be extended to \(\psi'(x)\). By the way, the \textit{explicit} ‘new’ formal representations for \(C \psi_L(x) C^{-1}\) and \(C \psi'(x) C^{-1}\) will be discussed in Secs. 3 and 4: see Eqs. \(45\) and \(51\), respectively. It is obvious that, to avoid the inequalities \(30\), \(31\), and \(32\), one could just assume \(\psi = \psi^c = \psi = \psi^c\), but this would also \textit{cancel} the distinction between an ‘active’ \((\psi')\) and a ‘sterile’ \((\psi'^c)\) fermion field, as well as the distinction itself between a ‘Majorana’ and a ‘Dirac’ mass term! So, if the primary \(C\)-definition \(28\) is truly relied upon, one is indeed led to conclude that an ‘active’ fermion field of the type \(\psi_L(x) + (\psi^c)_R(x)\) and its ‘sterile’ counterpart \((\psi^c)_L(x) + \psi_R(x)\) are themselves a pair of mutually ‘charge conjugate’ (rather than \textit{individually} self-conjugate) fields. It will be shown in Sec. 6 that the apparent (conventional) \textit{full} neutrality of such fields is to be actually interpreted as a mere ‘neutrality’ restricted to \textit{scalar-type charges}. With the help of \(30\), it can be checked that

\[
\psi'(x) + \psi'^c(x) = \psi(x) + \psi^c(x).
\] (33)

This formally enables one to define the genuine Majorana field \(1\) \textit{also as}

\[
\psi_M(x) = \frac{1}{\sqrt{2}} [\psi'(x) + \psi'^c(x)],
\] (34)

where \(\psi'\) and \(\psi'^c\) are exactly identical to the field components within square brackets in \(35\).

The point at issue can also be faced in a reversed manner. We may begin, instead, by \textit{assuming} field \(\psi'\) to be truly self-conjugate, so that we are allowed to put \(\psi' = \psi^c\) \((\propto \psi_M)\). Since either \(\psi'\) or \(\psi'^c\) is to be still meant as in Eq. \(30\),
we shall then have as well $\psi = \psi^c$ and, after all, $\psi' = \psi$. This automatically eliminates the inequalities in Eqs. (30) and (31), but the chargeless fermion model obtained is not the same as the ‘Majorana mass’ conventional one. First, it is evident that setting $\psi' = \psi^c$ (in place of $\psi' \neq \psi^c$) does cause fields $\psi'$ and $\psi^c$ to lose their original ‘active’ and ‘sterile’ distinctive characters. Hence, in line with Eq. (5) or Eq. (34), but not in line with what is usually claimed, the general conclusion is to be drawn that there cannot exist two different − ‘active’ and ‘sterile’ − kinds of genuine (truly self-conjugate) Majorana fermions. Moreover, we have to consider that the net equality $\psi' = \psi$ does unavoidably make a ‘Majorana mass’ term (proportional to $\bar{\psi}' \psi'$) indistinguishable from a ‘Dirac mass’ term (proportional to $\bar{\psi} \psi$). So, it must be concluded as well that a strictly neutral spin-$\frac{1}{2}$ fermion which is supposed to bear a ‘Majorana mass’ can likewise be said to bear a ‘Dirac mass’ (or vice versa). In other words, there would be simply a unified mass kind for such a fermion, which may be equally taken for a ‘Majorana’ as for a ‘Dirac’ mass kind! This fully corresponds to the fact that we cannot manage to build more than one mass term from chiral fields like those in (4).

3 On the truly orthodox way of applying particle–antiparticle conjugation to single chiral fields within standard QFT

Owing to its subtlety, the whole question raised above deserves an even more detailed analysis. As shown by (32), and as already implied in the opening discussion on Weyl solutions, the core of the problem is just how to define the ‘charge conjugates’ of the unpaired chiral projections, $\psi_L$ and $\psi_R^c$, entering into (13). To start with, it is worth emphasizing that the standard Dirac-field ‘prescription’ (2) does not automatically extend to the individual chiral components of $\psi$. In principle, we may write

$$U_C \psi^T = U_C \left[ \frac{1}{2} (1 - \gamma^5) \psi \right]^T + U_C \left[ \frac{1}{2} (1 + \gamma^5) \psi \right]^T$$

(35)

as well as

$$U_C \psi^T = \frac{1}{2} (1 - \gamma^5) U_C \psi^T + \frac{1}{2} (1 + \gamma^5) U_C \psi^T,$$

(36)

where $\frac{1}{2} (1 - \gamma^5) U_C \psi^T = U_C \left[ \frac{1}{2} (1 - \gamma^5) \psi \right]^T$. From a strict formal viewpoint, we are thus faced with two possible alternative ways of defining the ‘charge conjugates’ of $\psi_L$ and $\psi_R$: we may put either

$$
\begin{cases}
C \psi_L(x) C^{-1} = U_C \psi_L^T(x) = (\psi^c)_R(x) \\
C \psi_R(x) C^{-1} = U_C \psi_R^T(x) = (\psi^c)_L(x)
\end{cases}
$$

(37)
or
\[
\begin{align*}
C\psi_L(x)C^{-1} & \equiv U_C\psi_R^\dagger_T(x) = (\psi^c)_L(x) \\
C\psi_R(x)C^{-1} & \equiv U_C\psi_L^\dagger_T(x) = (\psi^c)_R(x)
\end{align*}
\]  \hspace{1cm} (38)

and both these assumptions lead to the same overall result

\[
\psi^c = (\psi_L + \psi_R)^c = U_C(\psi_L^\dagger_T + \psi_R^\dagger_T) = U_C(\psi_R^\dagger_T + \psi_L^\dagger_T) = U_C\psi^\dagger_T.
\]  \hspace{1cm} (39)

Actually, that there may be some ‘freedom’ in defining a $C$ operation is not a novelty in the literature; see e.g. the quite similar reasonings made in Ref. [30], or those made in Ref. [27] by use of the ‘chiral helicity’ special construct, and see as well Ref. [14], where a new interpretation of the Majorana condition is proposed. In comparison with Refs. [27] and [14], the major distinctive feature can here be drawn from Eq. (39), which shows that the helicity of a massive spin-$\frac{1}{2}$ fermion is always preserved by $C$ no matter whether (37) or (38) is being adopted. Also, note that both in (37) and in (38) the matrix $U_C$ is regularly connecting field components with opposite chiralities. It is worth remarking, however, that in the zero-mass limit a ‘$C$ definition’ like (37) leads to the couples of solutions (21) and (22), whereas a ‘$C$ definition’ like (38) leads to the alternative couples of solutions (23) and (24). To tell which of the two options (37) and (38) is to be taken as the truly orthodox one according to standard QFT, it is enough to consider that only (38) is strictly consistent with the genuine (primary) $C$ definition (26). The remaining option, even though it is just as well allowed by (39) and may all the same reproduce Eq. (2), would rather correspond to a $C$ operation whose fundamental representation does no longer appear to be fully defined inside the (conventional) Fock space, as it would now consist of (26) supplemented by one ‘spurious’ basic prescription, $\frac{1}{2}(1 \mp \gamma^5) \rightarrow \frac{1}{2}(1 \pm \gamma^5)$, outside the Fock space. To this it should be added that the choice of either (37) or (38) – apparently irrelevant if we neglect (26) and we restrict ourselves to Eq. (2) – is made by no means irrelevant if the validity of Eq. (18) is also invoked, with $\psi'(x)$ being defined as in (13). The point is that Eq. (18) may really be claimed to hold only if the ‘wrong’ option (37) – just defining $\psi_L$ and $(\psi^c)_R$ as the ‘charge conjugates’ of each other – is adopted. So, after all, we may even choose (37) (as usually done) to extend Eq. (2) to a field like $\psi'(x)$, but in this way we are not keeping to the strict QFT prescription (26) and we are actually introducing a new ‘charge conjugation’ operation, say, $C'$, which is such that $C'\psi'(x)C'^{-1} = U_C\psi'^\dagger_T$ and $C'\psi_{L,R}(x)C'^{-1} = U_C\psi_{L,R}^\dagger_T(x)$, and which should not be confused with the one, $C$ itself, rigorously acting as ‘particle–antiparticle conjugation’ according to (26). It will be shown in Sec. 6 that $C'$ (normally mistaken for $C$) does amount, more precisely, to a mere ‘scalar-charge conjugation’ operation.

The difference between two such ways of defining a ‘charge conjugation’ operation is clearly brought to its extreme consequences (which affect helicities themselves) on passing to the zero-mass case. This has already been mentioned in the previous section; yet, for completeness’ sake, it may be worth trying to deal with that case in more detail, too. The zero-mass limit leads to a
special situation in which the matrix $U_C$ can ‘manifestly’ be seen to connect the two chiralities. This is due to the well-known fact that positive- and negative-energy massless eigenspinors with discordant (concordant) helicities are themselves chiral spinors with concordant (discordant) chiralities. So, if $\psi_L(x)$ and $\psi_R(x)$ are now taken as zero-mass fields, their respective normal mode expansions will become

$$\psi_L(x) = \int \frac{d^3p}{(2\pi)^32p^0} \left[ a^{(-)}(p)u^{(-)}(p)e^{-ip\cdot x} + b^{(+)}(p)v^{(+)}(p)e^{ip\cdot x} \right]$$  \hspace{1cm} (40)

and

$$\psi_R(x) = \int \frac{d^3p}{(2\pi)^32p^0} \left[ a^{(+)}(p)u^{(+)}(p)e^{-ip\cdot x} + b^{(-)}(p)v^{(-)}(p)e^{ip\cdot x} \right],$$  \hspace{1cm} (41)

where the superscripts $\mp$ still denote the eigenvalues of the helicity variable $\h$. Hence, recalling that particle–antiparticle conjugation $C$ leaves (by definition) helicity unvaried, we find e.g. that the ‘charge conjugate’ of the (massless) field $\psi_L(x)$ is to be strictly defined as

$$C\psi_L(x)C^{-1} \equiv U_C\psi_{R}^{\dagger T}(x) = (\psi^c)_L(x),$$  \hspace{1cm} (42)

with $U_C\psi_{R}^{\dagger T}(x)$ including every required positive-energy spinor $U_Cv^{(-)}(p)$, of negative helicity, as well as every required negative-energy spinor $U_Cu^{(+)}(p)$, of positive helicity, and not as

$$C\psi_L(x)C^{-1} \equiv U_C\psi_{L}^{\dagger T}(x) = (\psi^c)_R(x),$$  \hspace{1cm} (43)

with $U_C\psi_{L}^{\dagger T}(x)$ containing the corresponding unwanted spinors having interchanged helicities. On the other hand, this fully agrees with what has been inferred in Sec. 2 from comparing (23),(24) with (21),(22): it is definition (42), and not definition (43), that strictly requires also the presence of a (right-handed) field solution $\psi_R(x)$ (quite missing in neutrino phenomenology) thus leading us to conclude that $C$ itself is (maximally) violated by neutrino physics! Of course, we may come to (42) even without allowing for the specific normal mode expansions (40) and (41): it is sufficient simply to take account of the basic $C$-definition (26) to get

$$C\psi_L(x)C^{-1} = \frac{1}{2}(1 - \gamma^5)C\psi(x)C^{-1} = \frac{1}{2}(1 - \gamma^5)U_C\psi_{R}^{\dagger T}(x),$$  \hspace{1cm} (44)

where the former equality (leaving chirality unaffected) is just due to the linear behavior of $C$ in the Fock space. Hence it also follows, as already implied in (42),

$$C\psi_L(x)C^{-1} = U_C \left[ \frac{1}{2}(1 + \gamma^5)\psi(x) \right]^{\dagger T} = U_C\psi_{R}^{\dagger T}(x),$$  \hspace{1cm} (45)

and it may be concluded that a $C$-matrix identical with the conventional one is still available, provided that complex conjugation is supplemented by ‘L $\rightarrow$ R’
exchange. This, of course, does not affect the ‘whole’ Dirac result \( \psi^c = U_C \psi^{T\dagger} \) (with \( \psi = \psi_L + \psi_R \)) — see Eq. (39) — and it is just the way to maintain helicity invariance under \( C \). Note, on the other hand, that no \( 4 \times 4 \) matrix \( U'_C \) can further be found being such that \( C \psi_L(x) C^{-1} = U'_C \psi^{T\dagger}_L(x) \): to realize it, one may simply consider that positive-energy antifermions (fermions) annihilated (created) by a massless field like \( U'_C \psi_L(x) \), with

\[
\psi^{T\dagger}_L(x) = \int \frac{d^3p}{(2\pi)^3 2\hbar^0} \left[ a^{I(+)}(p) u^{I(+)}(p) e^{-ip \cdot x} + a^{I(-)}(p) u^{I(-)}(p) e^{ip \cdot x} \right],
\]

would rather have inverted helicities with respect to positive-energy fermions (antifermions) annihilated (created) by \( \psi_L(x) \). On the grounds of either Eq. (42) or Eq. (44), it can thus be argued, after all, that even for zero mass there is no net chirality flip actually induced by particle–antiparticle conjugation (26) (despite the unquestionable fact that \( U_C \) itself is manifestly connecting fields with opposite chiralities!).

4 A double variety of mutually ‘charge conjugate’ spin-1/2 fermion fields

Take Eq. (33), and consider the two (both admissible) formal ways, either (1) or (34), of defining a (manifestly self-conjugate) Majorana field \( \psi_M(x) \) in terms of mutually ‘charge conjugate’ spin-\( \frac{1}{2} \) fermion fields, either \( \psi(x), \psi^c(x) \) or \( \psi'(x), \psi'^c(x) \). These field pairs (no matter which of them is taken as a mass-eigenfield pair) can be formally put on an equal footing via the unitary transformation

\[
\begin{aligned}
\psi'(x) &= X_L \psi(x) + X_R \psi^c(x) \\
\psi'^c(x) &= X_R \psi(x) + X_L \psi^c(x)
\end{aligned}
\]

(47)

or the inverse one

\[
\begin{aligned}
\psi(x) &= X_L \psi'(x) + X_R \psi'^c(x) \\
\psi^c(x) &= X_R \psi'(x) + X_L \psi'^c(x)
\end{aligned}
\]

(48)

where

\[
X_L = \frac{1}{2} (1 - \gamma^5), \quad X_R = \frac{1}{2} (1 + \gamma^5)
\]

(49)

(and where, of course, \( X_L X_R = X_R X_L = 0, X_L^2 = X_R^2 = X_{L,R}, X_L X_R + X_R X_L = \gamma^5 \). Transformations (47) and (48), which can be suitably rewritten in the matrix forms

\[
\begin{pmatrix}
\psi'(x) \\
\psi'^c(x)
\end{pmatrix} = \begin{pmatrix} X_L & X_R \\ X_R & X_L \end{pmatrix} \begin{pmatrix} \psi(x) \\
\psi^c(x) \end{pmatrix}, \quad \begin{pmatrix}
\psi(x) \\
\psi^c(x)
\end{pmatrix} = \begin{pmatrix} X_L & X_R \\ X_R & X_L \end{pmatrix} \begin{pmatrix} \psi'(x) \\
\psi'^c(x) \end{pmatrix},
\]

(50)
make it possible to introduce a generalized matrix representation for $C$. Such an extension, clearly superfluous on dealing merely with the field pair $\psi(x), \psi'^c(x)$, turns out to be strictly needed if the field pair $\psi'(x), \psi'^c(x)$ is also allowed for. This is because the use of Eq. (45) along with $C\psi'^c_R(x)C^{-1} = U_C\psi'^c_L(x)$ — can only lead to the trivial overall result

$$C\psi'(x)C^{-1} \equiv \psi'^c(x) = U_C\psi'^c_U(x),$$

where $U_C\psi'^c_U(x)$ is merely an identical way to write $\psi'^c(x)$, and not an actual prescription to obtain $\psi'^c(x)$ from $\psi'(x)$! So, unlike what happens for $\psi(x) \overset{C}{\rightarrow} \psi'^c(x)$, with $\psi'^c(x) \equiv C\psi(x)C^{-1} = U_C\psi'^T(x)$, there seems to be no $4 \times 4$ matrix $U'_C$ effectively representing $\psi'(x) \overset{C}{\rightarrow} \psi'^c(x)$ and being such that $\psi'^c(x) \equiv C\psi'(x)C^{-1} = U'_C\psi'^T(x)$ (the problem does not apply, of course, to a genuine Majorana field $\psi_M$, since $\psi'_M = \psi_M = U_C\psi'^T_M = U_C\psi^T_M$). A deep motivation for this lack can be found when the normal mode expansions of $\psi'(x)$ and $\psi'^c(x)$ (taken as mass eigenfields) are derived — see Eqs. (64) and (65) below — and it is realized that (quite differently from the ‘Dirac mass’ case) there are indeed neither any ‘particle’ nor any ‘antiparticle’ helicity freedom degrees shared by these expansions. More precisely, one still has that the freedom degrees in question are four in all (as in the ‘Dirac mass’ case) but one also has that only two of them are included in the $\psi'(x)$ expansion and the other two in the $\psi'^c(x)$ expansion. It thus follows that $\psi'(x)$, as given by Eq. (64), and $\psi'^c(x)$, as given by Eq. (65), are actually belonging to two orthogonal field spaces; and this clearly makes quite inadmissible any mutual link of the type $\psi'^c(x) = U'_C\psi'^T(x)$ (which would instead require one and the same four-spinor field space for them both!). That said, let us begin by defining the ‘charge conjugate’ of the column matrix

$$\Psi(x) \equiv \begin{pmatrix} \psi(x) \\ \psi'^c(x) \end{pmatrix}. \quad (52)$$

We clearly have

$$C\Psi(x)C^{-1} = \begin{pmatrix} \psi^c(x) \\ \psi(x) \end{pmatrix} = \begin{pmatrix} U_C & 0 \\ 0 & U_C \end{pmatrix} \begin{pmatrix} \psi'^T(x) \\ \psi'^c_T(x) \end{pmatrix}, \quad (53)$$

where $\begin{pmatrix} U_C & 0 \\ 0 & U_C \end{pmatrix}$ is an $8 \times 8$ matrix (still made up, as expected, of $4 \times 4$ diagonal blocks). To obtain, likewise, the ‘charge conjugate’ of

$$\Psi'(x) \equiv \begin{pmatrix} \psi'(x) \\ \psi'^c(x) \end{pmatrix}, \quad (54)$$

it must be borne in mind that $C$ fundamentally acts just on the annihilation and creation operators included in the fields, without affecting the transformation matrix in either (47) or (48). The result is

$$C\Psi'(x)C^{-1} = \begin{pmatrix} \psi'^c(x) \\ \psi'(x) \end{pmatrix} = \begin{pmatrix} X_L & X_R \\ X_R & X_L \end{pmatrix} \begin{pmatrix} \psi^c(x) \\ \psi(x) \end{pmatrix}. \quad (55)$$
or ultimately, in full agreement with [51],

\[ C\Psi'(x)C^{-1} = \begin{pmatrix} \psi'^c(x) \\ \psi'(x) \end{pmatrix} = \begin{pmatrix} 0 & U_C \\ U_C & 0 \end{pmatrix} \begin{pmatrix} \psi'^cT(x) \\ \psi'^{cT}(x) \end{pmatrix}, \]

where we have substituted both [53] and

\[
\begin{pmatrix} \psi'^T(x) \\ \psi'^{cT}(x) \end{pmatrix} = \begin{pmatrix} X_L^T & X_R^T \\ X_R^T & X_L^T \end{pmatrix} \begin{pmatrix} \psi'^T(x) \\ \psi'^{cT}(x) \end{pmatrix}
\]

(57)

(and we have taken into account that \( U_CX_{L,R}^T = X_{R,L}UC \)). It thus turns out that the transformed 8 \times 8 matrix

\[
\begin{pmatrix} X_L & X_R \\ X_R & X_L \end{pmatrix} \begin{pmatrix} 0 & U_C \\ U_C & 0 \end{pmatrix} \begin{pmatrix} X_L^T & X_R^T \\ X_R^T & X_L^T \end{pmatrix} = \begin{pmatrix} 0 & U_C \\ U_C & 0 \end{pmatrix}
\]

(58)
is no more trivially made up of diagonal blocks. This generalized matrix representation of \( C \), given by Eqs. [53] and [54], has been built without necessarily specifying which of the two field pairs \( \psi(x), \psi'^c(x) \) and \( \psi'(x), \psi'^c(x) \) should be also a mass-eigenfield pair: what has been only assumed is the validity of the formal link [2] connecting \( \psi(x) \) and \( \psi'^c(x) \). So, after all, Eqs. [53] and [54] may be referred to as just the basic peculiar features generally distinguishing \( \psi(x), \psi'^c(x) \) and \( \psi'(x), \psi'^c(x) \).

By use of either [47] or [48] (and with the help of the \( X_{L,R} \) properties) it can further be checked that

\[ \tilde{\psi}'\gamma^\mu \psi - \tilde{\psi}'\gamma^\mu \psi'^c = \tilde{\psi}'\gamma^\mu (-\gamma^5)\psi' - \tilde{\psi}'\gamma^\mu (-\gamma^5)\psi'^c, \]

(59)

where the individual currents

\[ \tilde{\psi}'\gamma^\mu (-\gamma^5)\psi' = \tilde{\psi}'\gamma^\mu X_L\psi' - \tilde{\psi}'\gamma^\mu X_R\psi' = \tilde{\psi}'\gamma^\mu X_L\psi - \tilde{\psi}'\gamma^\mu X_R\psi'^c \]

(60)

and

\[ \tilde{\psi}'\gamma^\mu (-\gamma^5)\psi'^c = \tilde{\psi}'\gamma^\mu X_L\psi'^c - \tilde{\psi}'\gamma^\mu X_R\psi'^c = \tilde{\psi}'\gamma^\mu X_L\psi - \tilde{\psi}'\gamma^\mu X_R\psi \]

(61)

(just like the ordinary ones \( \tilde{\psi}\gamma^\mu \psi \) and \( \tilde{\psi}'\gamma^\mu \psi'^c \) are generally nonvanishing [14], in spite of the fact that \( \psi'(x) = U_C\psi'^cT(x) \) and \( \psi'^c(x) = U_C\psi'^cT(x) \).

Let us look first at the ‘Dirac mass’ special case, i.e. when, as usual, the field pair \( \psi(x), \psi'^c(x) \) is also a pair of mass eigenfields, the one defined by the expansion [25] and the other by the charge conjugate of [25]. In this case, the field \( \psi(x) \) and \( \psi'^c(x) \) — taken as free fields — are single solutions of the Dirac equation (with a given mass parameter \( m \)), the same cannot be said for \( \psi'(x) \) and \( \psi'^c(x) \), as it will clearly result

\[ i\gamma^\mu \partial_\mu \psi'(x) = m\psi'^c(x), \quad i\gamma^\mu \partial_\mu \psi'^c(x) = m\psi'(x) \]

(62)

(\( h = c = 1 \)). Yet, on the basis of [59], [60], and [61], it may be argued that fields \( \psi'(x) \) and \( \psi'^c(x) \) themselves seem in particular to enter into maximally-\( P \)-violating weak couplings like real ‘dynamical eigenfields’ – the former wholly
‘active’ and the latter wholly ‘sterile’ — thus giving, furthermore, a direct evidence of the maximum C-violation also implied in those couplings.

Let us pass now to the ‘Majorana mass’ special case, i.e. when, on the contrary, it is just the field pair $\psi'(x), \psi'^c(x)$ that stands for an actual pair of mass eigenfields. To make an explicit derivation of the normal mode expansions defining $\psi'(x)$ and $\psi'^c(x)$ in such a case, we may exploit the fact that these expansions should clearly have forms which are also available for the zero-mass limit. Splitting up both $\psi'$ and $\psi'^c$ into chiral components, such that

$$
\begin{cases}
\frac{1}{2}(1 - \gamma^5)\psi' = \frac{1}{2}(1 - \gamma^5)\psi, & \frac{1}{2}(1 + \gamma^5)\psi' = \frac{1}{2}(1 + \gamma^5)\psi^c \\
\frac{1}{2}(1 - \gamma^5)\psi'^c = \frac{1}{2}(1 - \gamma^5)\psi^c, & \frac{1}{2}(1 + \gamma^5)\psi^c = \frac{1}{2}(1 + \gamma^5)\psi,
\end{cases}
$$

(63)

we may, thus, simply substitute the zero-mass normal mode expansions of the single chiral-field couples $\frac{1}{2}(1 - \gamma^5)\psi, \frac{1}{2}(1 + \gamma^5)\psi^c$ and $\frac{1}{2}(1 - \gamma^5)\psi^c, \frac{1}{2}(1 + \gamma^5)\psi$ to obtain (for the nonzero-mass case at issue)

$$
\psi'(x) = \int \frac{d^3p}{(2\pi)^32p^0} \left\{ \left[ a(-)(p)u^(-)(p) + b^+(p)u^+(p) \right] e^{-ip \cdot x} \\
+ \left[ b^+(p)v^+(p) + a^-(p)v^-(p) \right] e^{ip \cdot x} \right\} \text{ ('Major. mass')} (64)
$$

and

$$
\psi'^c(x) = \int \frac{d^3p}{(2\pi)^32p^0} \left\{ \left[ b(-)(p)u^-(p) + a^+(p)u^+(p) \right] e^{-ip \cdot x} \\
+ \left[ a^+(p)v^+(p) + b^-(p)v^-(p) \right] e^{ip \cdot x} \right\} \text{ ('Major. mass')}, (65)
$$

with the superscripts ($\mp$) still denoting (negative and positive) helicities. Of course, as now the other two fields, $\psi(x)$ and $\psi^c(x)$, are no longer mass eigenfields, it is not surprising that their corresponding expansions obtained from (64) and (65) by use of (48) may not be the same as the (usual) ones for the ‘Dirac mass’ case. A glance at (64) and (65) shows that only operators such as $a^-(p)$ and $b^+(p)$ (plus their adjoints) enter into (64), and similarly, only operators such as $b^-(p)$ and $a^+(p)$ (plus their adjoints) enter into (65). This, at first sight, might lead one to mistake either $\psi'(x)$ or $\psi'^c(x)$ alone for a truly neutral spin $\frac{1}{2}$ field, merely endowed with two freedom degrees like $a^-(p) \equiv a^-(p), a^+(p) \equiv b^+(p)$ (as regards $\psi'$) or $b^-(p) \equiv b^-(p), b^+(p) \equiv a^+(p)$ (as regards $\psi'^c$). From comparing (64) and (65), it appears evident, however, that such pairs of annihilation operators, as well as the whole fields $\psi'(x)$ and $\psi'^c(x)$ themselves, are actually interchanged by particle–antiparticle conjugation (56), despite the fact that (64) and (65) also provide clear evidence for the individual formal constraints $\psi'(x) = U_C \psi'^c(x)$ and $\psi'^c(x) = U_C \psi'(x)$. To this it is worth adding that (as it should be expected) the two expansions (64) and (65) are fully consistent with Eq. (56).

Therefore, to come to a genuine self-conjugate field, it would be strictly necessary to impose the extra constraint $a^{(\mp)}(p) = b^{(\mp)}(p), b^{(\mp)}(p) = a^{(\mp)}(p)$;
which, indeed, would mean nothing else than trivially reducing \( \psi' \) and \( \psi'^c \) to one and the same field by means of the Majorana condition \( \psi' = \psi'^c \). On the other hand, if we truly admit that (64) and (65) are just defining ‘charged’ fields, we have also to admit, of course, that standard (i.e. scalar-type) charges are to be ruled out for them. It is thus left to see, after all, what new kind of ‘charges’ may ever characterize a pair of mutually ‘charge conjugate’ fermion fields each having only two (rather than four) freedom degrees, and what should be the new meaning to be assigned, accordingly, to each single relationship 

\[
\psi'(x) = U_C \psi'^T(x) \text{ and } \psi'^c(x) = U_C \psi'^cT(x).
\]

5 A spin-1/2 fermion with mass of the ‘Majorana’ (rather than ‘Dirac’) kind as a particle correspondingly endowed with pseudoscalar-type (rather than scalar-type) charges

Let us look again at either (47) or (48). If we still assume the two fermion fields \( \psi(x), \psi^c(x) \) – characterized by Eq. (2) – to be also mass eigenfields (as in the ‘Dirac mass’ case), we are clearly left with the usual ‘charged’ spin-\( \frac{1}{2} \) particles. If we instead suppose the field pair \( \psi'(x), \psi'^c(x) \) to be just an alternative pair of mass-eigenfields (as in the ‘Majorana mass’ case), we can no longer expect the associated fermions to be ‘really neutral’ (as commonly believed), the reason being because \( \psi'(x) \) and \( \psi'^c(x) \) are themselves two ‘mutually charge-conjugate’ (rather than individually self-conjugate) fields. In this case, as we have already pointed out, we should be actually faced with a new type of ‘charged’ spin-\( \frac{1}{2} \) particles.

To see what basic peculiar features may truly distinguish the latter ‘charged’ fermion type from the former one, it appears crucial to compare how, in the two cases, the single ‘particle’ and ‘antiparticle’ annihilation (creation) operators do transform under space reflection. In either case, each field that happens to be a mass eigenfield turns out to be also a solution of a Dirac-type equation. Thus, if Dirac invariance with respect to space reflection is always invoked, one gets e.g.

\[
P\psi(x^R)P^{-1} = i\gamma^0 \psi(x), \quad P\psi^c(x^R)P^{-1} = i\gamma^0 \psi^c(x)
\]

for the ‘Dirac mass’ case, and

\[
P'\psi'(x^R)P'^{-1} = i\gamma^0 \psi'(x), \quad P'\psi'^c(x^R)P'^{-1} = i\gamma^0 \psi'^c(x)
\]

for the ‘Majorana mass’ case, where \( x^R \equiv (t, -r) \) (and where the phase choice is such that it allows either \( P \) or \( P' \) to commute with \( C \) and to imply, as required, a fermion–antifermion relative intrinsic parity \( \sigma^2 = -1 \)). It can be argued that the two parity operators \( P \) and \( P' \) – the former defined by (66) and the latter by (67) – provide two noncoinciding representations of \( x \to x^R \) (just relevant to the two cases under consideration). That \( P \) and \( P' \) do not overlap can be easily shown by direct use of (47) or (48). Bearing in mind that either \( P \) or \( P' \)
is (in itself) nothing but a linear operator acting on annihilation and creation operators, one should expect, for example, that $P'$ applied to (48) still gives

$$\begin{align*}
P'\psi'(x)P'^{-1} &= X_L P'\psi'(x)P'^{-1} + X_R P'\psi^{ec}(x)P'^{-1}, \\
P'\psi^{ec}(x)P'^{-1} &= X_R P'\psi'(x)P'^{-1} + X_L P'\psi^{ec}(x)P'^{-1}.
\end{align*}$$

(68)

Hence, substituting (67) (and recalling that $X_L, R \gamma^0 = \gamma^0 X_R, L$), one is actually led to a result which is not the same as (66):

$$\begin{align*}
P'\psi'(x)P'^{-1} &= i\gamma^0 \psi^{ec}(x), \\
P'\psi^{ec}(x)P'^{-1} &= i\gamma^0 \psi'(x).
\end{align*}$$

(69)

If $\psi(x)$ and $\psi^{ec}(x)$ are still mass eigenfields (with usual normal mode expansions), then

$$P a^{(\mp)}(p) P^{-1} = i a^{(\pm)}(-p),$$

and so on (‘Dirac mass’); (70)

which means, strictly speaking, that standard Dirac particles — or ‘Dirac mass’ fermions — are bound to carry charges behaving just like ordinary scalars. This cannot be true for ‘Majorana mass’ fermions, i.e. for the new kind of ‘charged’ spin-$\frac{1}{2}$ particles associated with the alternative pair $\psi'(x), \psi^{ec}(x)$ of mass eigenfields. The fact is that, even though Eq. (67) is quite analogous to Eq. (60), the expansions strictly defining $\psi'(x)$ and $\psi^{ec}(x)$ as mass eigenfields are the nonstandard ones (64) and (65). So, space reflection now implies

$$P a^{(\mp)}(p) P^{-1} = i b^{(\pm)}(-p),$$

and so on (‘Majorana mass’), (71)

thus acting on ‘Majorana mass’ fermions in the same way as the whole CP operation acts on ‘Dirac mass’ fermions. This makes sense, of course, if and only if ‘Majorana mass’ fermions are assumed to carry nonzero charges behaving just like pseudoscalars. Due to such charges, one and the same particle of this kind is indeed predicted to behave like either a ‘fermion’ or an ‘antifermion’ according to the given chirality involved, so that, in the ultrarelativistic limit, it would naturally approach an exact two-component particle model. Associated with it, there should also be a nonvanishing (though not generally conserved) current, proportional e.g. to

$$\bar{\psi}'\gamma^\mu(-\gamma^5)\psi' = \bar{\psi}' L \gamma^\mu \psi'_L - \bar{\psi}' R \gamma^\mu \psi'_R$$

($\mu = 0, 1, 2, 3$) or to the current operator ‘charge conjugate’ to (72). For speed zero (or at least negligible compared with $c$) such a particle should be equally able to look like a ‘fermion’ (with negative chirality) as like an ‘antifermion’ (with positive chirality), while for ultrarelativistic speeds it should tend, depending on its helicity, to behave either in the former or in the latter manner only. Similarly, its individual helicity eigenstates could never turn out to be sharp ‘charge eigenstates’ but could only tend to become so in the ultrarelativistic limit (or in the limit of zero mass [29]). A particle like this cannot be strictly said to be ‘chargeless’ (as it would be for a true Majorana particle). The general fact is left, anyhow, that whatever spin-$\frac{1}{2}$ particle endowed with ‘Majorana mass’ would still be ‘neutral’ with respect to scalar-type charges.
A comparison of Eqs. (69), (71) with Eqs. (66), (70) shows that actually,

\[ P' = CP = PC \]  \hspace{1cm} (73)

Such a relationship, just equating parity \( P' \) for ‘Majorana mass’ fermions with \( CP \) for ‘Dirac mass’ fermions, may in particular lead one to speculate that, if all particles of the latter type were replaced by particles of the former type, the usual \( CP \) mirror symmetry of weak processes would then become nothing but a \textit{genuine} (ordinary) mirror symmetry! In view of Eq. (73), one may also put, for convenience,

\[ P = P'_{\text{ex}} \implies P' = CP'_{\text{ex}} = P'_{\text{ex}}C \]  \hspace{1cm} (74)

where \( P'_{\text{ex}} \) stands just for an ‘external’ parity operator (identical with \( P' \) except for not involving pseudoscalar-charge reversal). This shows that, on passing from the ‘Dirac mass’ case (when \( C \) may specifically be said to invert scalar-type charges) to the ‘Majorana mass’ case (when \( C \) may specifically be said to invert pseudoscalar-type charges), the standard effect of (maximum) \( P \) violation would indeed be reduced to a mere effect of (maximum) \( P'_{\text{ex}} \) violation.

6 ‘Scalar-charge conjugation’ and ‘pseudoscalar-charge conjugation’ operations

It follows from the foregoing that \( C \) does in principle reverse \textit{scalar-type} as well as \textit{pseudoscalar-type} charges, giving rise always to a full particle \( \leftrightarrow \) antiparticle interchange. If so, how can one \textit{separately} think of a ‘scalar-charge conjugation’ and a ‘pseudoscalar-charge conjugation’ operation? Let two such individual operations be denoted by \( C_s \) and \( C_{ps} \), respectively (with \( C_s^2 = C_{ps}^2 = 1 \)). Actually, it is only the \textit{product} of them both, i.e. \( C \) itself, that is strictly demanded to result in a \textit{pure} operation acting on annihilation and creation operators as in (26). It appears legitimate, therefore, to attempt to define these single charge-conjugation operations so that, regardless of the mass kind involved, they may simply fulfil the general requirements

\[ C_s \Psi(x) C_{s}^{-1} = C \Psi(x) C_{s}^{-1}, \quad C_s \Psi'(x) C_{s}^{-1} = \Psi'(x) \]  \hspace{1cm} (75)

and

\[ C_{ps} \Psi(x) C_{ps}^{-1} = \Psi(x), \quad C_{ps} \Psi'(x) C_{ps}^{-1} = C \Psi'(x) C_{s}^{-1}, \]  \hspace{1cm} (76)

where \( \Psi(x) \) and \( \Psi'(x) \) stand for the column matrices in Eqs. (52) and (54), and where \( C = C_{ps} C_s \) (\( = C_s C_{ps} \)). Definitions (75) and (76) naturally embody, in particular, the new achievement that ‘Majorana mass’ eigenfields are themselves non-neutral fields associated with pseudoscalar-type (rather than scalar-type) charges. Making use of transformation (47) or its inverse (48), we can see that neither \( C_s \) nor \( C_{ps} \) as given above may be fundamentally represented as \textit{purely} acting (like \( C \)) on annihilation and creation operators: for instance, it can be
easily checked that, for the two requirements in (75) to hold simultaneously, the 
\(C_s\) operation must be understood to be further such that

\[
\begin{align*}
C_s \left[ \frac{1}{2} (1 \mp \gamma^5) \psi \right] C_s^{-1} &= \frac{1}{2} (1 \mp \gamma^5) C \psi C^{-1} = U_C \left[ \frac{1}{2} (1 \mp \gamma^5) \psi \right]^T, \\
C_s \left[ \psi \right] C_s^{-1} &= \frac{1}{2} (1 \mp \gamma^5) C \psi C^{-1} = U_C \left[ \psi \right]^T. 
\end{align*}
\]

(77)

Still focusing on (75), let us compare \(C_s\) with \(C\) as represented in Eqs. (53) and (56). Due to (77), we shall now have not only

\[
C_s \Psi(x) C_s^{-1} = \left( U_C 0 \atop 0 U_C \right) \left( \psi^{\dagger T}(x) \atop \psi^{\dagger cT}(x) \right), 
\]

(78)

but also

\[
C_s \Psi'(x) C_s^{-1} = \left( U_C 0 \atop 0 U_C \right) \left( \psi'^{\dagger T}(x) \atop \psi'^{\dagger cT}(x) \right). 
\]

(79)

This leads us, in particular, to the conclusion that the conventional identity

\[
C \psi'(x) C^{-1} = U_C \psi'^{\dagger T}(x) \]

(80)

In the light of (77) and (80), it is immediate to see that \(C_s\) is exactly coinciding with the special ‘charge conjugation’ operation \(C'\) (distinct from \(C\)) mentioned in Sec. 3. As shown by (80), it is thus \(C_s\) that is normally mistaken for \(C\) on dealing with ‘Majorana mass’ fermions! This is a fundamental outcome which enables us to shed light, at last, on the new meaning to be assigned to the formal relationship \(\psi' = U_C \psi'^{\dagger T}\) (or its ‘charge conjugate’ \(\psi'^c = U_C \psi'^{c\dagger T}\)).

Strictly speaking, the constraint \(\psi' = U_C \psi'^{\dagger T}\) (\(\psi'^c = U_C \psi'^{c\dagger T}\)) does only express neutrality of \(\psi' (\psi'^c)\) with respect to scalar-type charges, and not real neutrality of \(\psi' (\psi'^c)\).

Bearing in mind transformation (47), let us then consider an active ‘Majorana mass’ neutrino, associated with a field of the \(\psi'\)-type, and a sterile one, associated with the partner field of the \(\psi'^c\)-type. Herein, two such ‘Majorana mass’ neutrino versions are charge conjugate to each other (and no longer individually self-conjugate). This, however, does not exactly mean that they can now be said to represent just a ‘lepton’ and just an ‘antilepton’ (as it normally happens for a ‘Dirac mass’ neutrino and the corresponding antineutrino). The point is that, unlike any standard neutrino–antineutrino pair, they would share a pseudoscalar (rather than scalar) ‘lepton number’ variety (proportional to chirality). We thus have that one and the same active ‘Majorana mass’ neutrino — associated with a current of the type (72) — could in turn be either a left-handed ‘lepton’ (with positive lepton number) or a right-handed ‘antilepton’ (with negative lepton number), and the exact converse (with ‘lepton’ and ‘antilepton’ interchanged) would in principle apply to the (‘charge conjugate’) sterile version of it. Note, on the other hand, that due to the presence of mass (which
breaks chirality conservation) the lepton number at issue may be conserved only in magnitude, and not in sign. Hence, with the help of (64), it is in particular easy to realize that such a neutrino (taken in its active version) could invariably give rise to a net neutrinoless double $\beta$-decay even without being a really neutral particle. It is worth pointing out, moreover, that a given left-handed (right-handed) neutrino would remain coupled to a given left-handed ‘charged lepton’ (right-handed ‘charged antilepton’), so that we should always be able, after all, to recognize single lepton families marked by their own ‘lepton-number conserving’ weak currents. This should be related to the general fact – already emphasized in Sec. 4 – that ‘Dirac mass’ fermions themselves, when involved in weak processes, are apparently described by ‘dynamical eigenfields’ structured just like $\psi'(x)$ and $\psi'^c(x)$ (as if they were as well carrying a pseudoscalar charge variety which is normally kept ‘hidden’ in their strict Dirac behaviors and may indeed be revealed once weak interaction is turned on [30–32]).

7 Pseudoscalar-type charges and the $CPT$ theorem

As is well-known, one familiar example of a pseudoscalar-type charge is just provided by magnetic charge [33–36]. The fact that the field $H$ generated by a magnetic dipole is an axial-vector (invariant under parity) clearly means that space reflection does also interchange the signs of the two poles (besides interchanging the spatial locations of them). The fact, on the other hand, that $H$ is inverted by time reversal (instead of being left unvaried like an electric field) shows that each pole does undergo once again a change in sign if space reflection is followed by time reversal. We therefore have that the overall effect of both space and time inversions on magnetic charge would be still the same as on electric charge. This holds as well for the whole $CPT$ operation, which would regularly turn a magnetic monopole with a given four-momentum into an opposite monopole with identical four-momentum. Such a conclusion (obviously extensible to whatever pseudoscalar-type charges) is also supported by the fact that the ‘proper’ relativistic transformation of strong reflection – essentially equivalent to $CPT$ [37–40] – acts identically on vector as on pseudovector currents.

That said, let us go back to the revised ‘Majorana mass’ fermion model herein proposed. The basic peculiar feature of it is obviously given by (71). In short, we have that the parity operator $P'$ (relevant to the ‘Majorana mass’ case) does naturally exert a ‘charge conjugating’ internal extra action, and this may clearly happen only for a particle just endowed with pseudoscalar-type charges. As already stressed in Sec. 4, such a result tells us that $P'$ is exactly the same as $CP$ for standard fermions. We also have, therefore, that the only symmetry under $CP$ ($= P'$) does not enable us to distinguish between scalar-type and pseudoscalar-type charges, and so it may equally allow, in principle, either the conjecture of a ‘Dirac mass’ neutrino (endowed with a scalar lepton number)
or the conjecture of a ‘Majorana mass’ neutrino (endowed with a \textit{pseudoscalar}
lepton number).

To enlarge the discussion to \textit{CPT} symmetry, let us just denote by \( T \) and
\( T' \) the two (antiunitary) operators representing time reversal in the pure ‘Dirac
mass’ and ‘Majorana mass’ cases, respectively. In the light of the above remarks on
the behaviors of pseudoscalar-type charges, such operators cannot be expected to
coincide, the reason being because \( T' \) is also demanded (just like \( P' \)) to include a ‘charge conjugating’ \textit{internal effect}. This can be properly
obtained in terms of ‘particle’ and ‘antiparticle’ annihilation (creation) operators
such as \( a^{(h)}(p) \) \( (a^{(h)\dagger}(p)) \) and \( b^{(h)}(p) \) \( (b^{(h)\dagger}(p)) \).

In other words, considering that (except for phase factors)
\[
T a^{(h)}(p) T^{-1} = a^{(h)}(-p), \quad \text{and so on} \quad \text{('Dirac mass')},
\]
(81)
it should correspondingly result
\[
T' a^{(h)}(p) T'^{-1} = b^{(h)}(-p), \quad \text{and so on} \quad \text{('Majorana mass')}. \quad \text{(82)}
\]

We thus have that \( T' \) can be formally related to \( T \) as follows:
\[
T' = C T,
\]
(83)
where, in analogy with (74), it may be set
\[
T = T'_\text{ex} \implies T' = C T'_\text{ex},
\]
(84)
with \( T'_\text{ex} \) standing for a mere ‘external’ time-reversal operator (identical with
\( T \) except for not involving pseudoscalar-charge reversal). From (73) and (83),
on the other hand, it can be seen that the overall \( PT \) and \( P'T' \) operators do
indeed \textit{coincide}. Hence,
\[
CPT = C P'T' = C P'_\text{ex} T'_\text{ex},
\]
(85)
and this is in full accordance with the above-mentioned fact that the whole
(equivalent) symmetry operation of \textit{strong reflection} does not make any distinction
between scalar-type and pseudoscalar-type charges.

It may therefore be concluded that the \textit{CPT} theorem is just as well available
for the special new kind of \textit{charged} particles to which ‘Majorana mass’ fermions
should strictly correspond. This, however, does not really mean that an active
field \( \psi' \) \( (= X_L \psi + X_R \psi_c) \) and its sterile counterpart \( \psi'^c \) \( (= X_L \psi^c + X_R \psi) \)
are bound to have \textit{identical} masses as a result of their being also a pair of
‘mutually charge-conjugate’ fields. If \( \psi' \) is coupled, say, to a mass \( m_L \), and
\( \psi'^c \), say, to a mass \( m_R \), we may, in other words, generally assume \( m_L \neq m_R \)
as in the conventional approach. The reason is just because, for fields having
such expansions as (64) and (65), the whole symmetry operation (85) is only
able (like \( P' \) and unlike \( C \)) to connect annihilation or creation operators always
included in \textit{one and the same} expansion. As an immediate consequence, one has
e.g. that the See–Saw Mechanism may still apply to neutrino masses without
spoiling \textit{CPT} symmetry.
Yet, there seems to be another intriguing feature to be pointed out. According to the new approach, fermions with ‘Majorana masses’ would indeed obey ordinary mirror symmetry (as just the analogue of CP symmetry for standard fermions) and they would also experience a manifest (maximum) C violation (due to the fact that an active ‘Majorana mass’ fermion and its sterile counterpart are now ‘charge conjugate to each other’). Hence, neglecting the extreme conjecture of CPT breakdown [41–43], we see that such fermions would as well obey symmetry under CT′ (= T′ ex), even though at the price of also experiencing a (maximum) time reversal (i.e. T′) violation, which would just counterbalance the ‘recovered’ ordinary mirror symmetry. In close connection with this, it is worth noting that merely releasing the constraint mL = mR would already break C and T′ individual symmetries, with no need to consider weak dynamics. We thus have, for example, that active and sterile ‘Majorana mass’ neutrino versions which are supposed to have masses mL ≠ mR should anynow be taken as particles intrinsically violating either C or time reversal.

8 Single ‘Dirac mass’ or ‘Majorana mass’ charged fermion fields as superpositions of pure Majorana neutral fields

We know from Ref. [1] — see also Refs. [10,20] — not only that a single Majorana neutral field ψM(x) can be obtained, via Eq. (1), as a superposition of two distinct (and mutually charge-conjugate) ‘Dirac mass’ charged fermion fields, but even that a single ‘Dirac mass’ charged fermion field ψ(x) may be seen, conversely, as a superposition of two distinct Majorana neutral fields (with opposite CP intrinsic parities). One has, more precisely,

\[ \psi(x) = \frac{1}{\sqrt{2}} \left[ \psi_M^+(x) + i\psi_M^-(x) \right] \quad ('\text{Dirac mass}') , \]

(86)

where ψM^+(x) = ψM(x), and where ψM^-(x) is a new (still manifestly self-conjugate) Majorana field defined as

\[ \psi_M^-(x) = -\frac{i}{\sqrt{2}} [\psi(x) - \psi^c(x)] . \]

(87)

The superscripts (±) have here been used to denote the CP intrinsic parities distinguishing two such neutral fields. A similar result holds for the ‘Dirac mass’ field being the charge conjugate of ψ(x), which may likewise be expressed in the form

\[ \psi^c(x) = \frac{1}{\sqrt{2}} \left[ \psi_M^+(x) - i\psi_M^-(x) \right] \quad ('\text{Dirac mass}') . \]

(88)

Whether we are considering (86) or (88), fields ψ_M^±(x) are understood to be mass eigenfields with identical eigenvalues, and their masses (the same as those carried by ψ and ψ^c) may be said to display ‘Dirac-like’ characters.
In the light of transformation (48) — which turns Eq. (1) into Eq. (34) — we may now, on the other hand, also think of a Majorana neutral field being a superposition of two mutually charge-conjugate ‘Majorana mass’ charged fermion fields, \( \psi'(x) \) and \( \psi'^c(x) \), with identical masses. This, indeed, is in line with the general fact that, due to condition (4), the mass term relevant to a true Majorana field may be claimed to be equally reminiscent of a ‘Dirac’ as of a ‘Majorana’ mass term. Starting from \( \psi'(x) \) and \( \psi'^c(x) \) (with masses \( m_L \) and \( m_R \) being in particular such that \( m_L = m_R \)), we can, thus, again construct two independent (manifestly self-conjugate) Majorana neutral fields as above. They read

\[
\psi_M^{(+)}(x) = \frac{1}{\sqrt{2}} [\psi'(x) + \psi'^c(x)], \quad \psi_M^{(-)}(x) = \frac{-i}{\sqrt{2}} [\psi'(x) - \psi'^c(x)]
\] (89)

and still have opposite \( CP \) intrinsic parities. Hence we can see that, in full analogy with the ‘Dirac mass’ case, fields \( \psi'(x) \) and \( \psi'^c(x) \) themselves (taken with identical masses) may be split, conversely, as follows:

\[
\begin{align*}
\psi'(x) &= \frac{1}{\sqrt{2}} \left[ \psi_M^{(+)}(x) + i\psi_M^{(-)}(x) \right] \\
\psi'^c(x) &= \frac{1}{\sqrt{2}} \left[ \psi_M^{(+)}(x) - i\psi_M^{(-)}(x) \right]
\end{align*}
\] (Majorana mass; \( m_L = m_R \)). (90)

Of course, the two neutral fields \( \psi_M^{(\pm)}(x) \) in (90) may correspondingly be said to have masses displaying ‘Majorana-like’ characters.

9 Concluding remarks

There are mainly two motivations underlying this paper. The former one is the purpose of throwing light upon some basic theoretical inconsistencies which appear to be present in the usual approach to Majorana fermions and ‘Majorana mass’ fermions. The latter one is the need of working out accordingly — with no departure from standard QFT — a formalism being really free of such inconsistencies and being further able to lead, after all, to a new insight into the whole subject.

As a starting point, a brief discussion has been made on how to interpret the two couples of massless Dirac field solutions (19) and (20) in order to avoid that charge conjugation \( C \) may happen to invert helicities. It has been argued that the appropriate reading is provided by (23) and (24), and not by (21) and (22), even though the latter choice seems just to come from a natural extension of the well-known Dirac ‘prescription’ (2). Opting for (23) and (24) has also been shown to be the only choice that correctly implies \( C \) violation as a result of any asymmetry occurring between (19) and (20). The fact is left, however, that the alternative reading is also the one which normally allows a ‘Majorana mass’ neutrino to be recognized as a self-conjugate particle! What, then, about the real nature of such a neutrino? To shed full light on the matter, a truly direct (and thus unambiguous) check has been tried, based on the primary QFT
definition itself of charge conjugation, i.e. its fundamental representation \( (26) \) (in the Fock space). This procedure has led us to conclude that a ‘Majorana mass’ neutrino, unlike the original neutrino by Majorana himself, cannot be really claimed to be genuinely self-conjugate. The point may be generally set as follows. Take the wholly active (sterile) fermion field which can be obtained from suitably mixing the chiral components of two mutually charge-conjugate Dirac fields. If \( C \) is exactly applied as in \((26)\), the net outcome is that such a field is correspondingly turned into its sterile (active) counterpart, and not into itself! This also shows that the standard formula \((2)\) − just suitable for defining the charge conjugate of a ‘Dirac mass’ fermion field − cannot be extended to ‘Majorana mass’ fermion fields (and thus be used as ‘proof’ of their real neutrality) without coming into conflict with the true \( C \) definition \((26)\).

If so, how can we explain the individual constraints, \( \psi'(x) = U_C \psi'^\dagger T(x) \) and \( \psi'^c(x) = U_C \psi'^c\dagger T(x) \), naturally applying to an active ‘Majorana mass’ fermion field \( \psi'(x) \) and its (‘charge conjugate’) sterile version \( \psi'^c(x) \)? The answer to this crucial question can actually be found in the new formalism which has been herein developed to remodel a fermion with mass of the ‘Majorana’ (rather than ‘Dirac’) type. The basic novelty is that replacing a ‘Dirac mass’ with a ‘Majorana mass’ does now mean passing from a standard fermion, endowed with scalar-type charges, to a fermion which is not really neutral but is endowed with pseudoscalar-type charges. A ‘charged’ spin-\( \frac{1}{2} \) particle like this should essentially be thought of as a chiral object in turn behaving like a ‘fermion’ or an ‘antifermion’ according to either sign of the associated chirality (where ‘fermion’ and ‘antifermion’ should clearly appear interchanged for the particle ‘charge conjugate’ to it). In such a framework, charge conjugation \( C \) proves to act as a true ‘particle–antiparticle conjugation’ operation which may generally be split into the product of a mere ‘scalar-charge conjugation’ and a mere ‘pseudoscalar-charge conjugation’ operation. Hence it can indeed be seen that the above constraints peculiar to \( \psi'(x) \) and \( \psi'^c(x) \) are just expressing neutrality of ‘Majorana mass’ fermions with respect to scalar-type charges.

This, however, does not mean that a genuinely neutral Majorana fermion cannot exist in nature. Such a particle − strictly described by a manifestly self-conjugate field like the one given in Eq. \((1) \) or \((34) \) − would regularly possess no charges at all (i.e. neither scalar-type nor pseudoscalar-type charges). Herein a ‘new’ model of it has been obtained, where some inconsistent features unavoidably affecting the conventional theory appear to be automatically removed. Firstly, in full compliance with Eq. \((4) \), one now has that a genuine Majorana fermion can no longer be assigned any special sort of ‘handedness’ marking it as just an ‘active’ or ‘sterile’ fermion: there may always be only a fifty-fifty probability for it to look like the former or the latter particle. This implies, for example, that a true Majorana neutrino cannot really be claimed to be quite compatible with the SM (contrary to what may clearly be said for a pure active ‘Majorana mass’ neutrino). Secondly, as rigorously demanded by the natural constraint \((4) \), the mass of a genuine Majorana fermion is now bound to be of a single kind, equally reminiscent of a ‘Majorana’ as of a ‘Dirac’ mass kind.
The (no longer chargeless) ‘Majorana mass’ fermion model here proposed does actually introduce a new sort of spin-$\frac{1}{2}$ particle which is somehow halfway between the Dirac one and the genuine Majorana one. The point is that a ‘Majorana mass’ fermion is now a particle endowed with pseudoscalar-type charges and still devoid of scalar-type charges: thus, it is ‘charged’ (like a Dirac particle) but it also retains only two freedom degrees (like a true Majorana particle). This model applies, in principle, to any wholly active or wholly sterile massive fermions (including SUSY ones) which are usually (and improperly) referred to as ‘Majorana fermions’ tout court. It in particular deals with the active and sterile versions of a ‘Majorana mass’ neutrino as a pair of ‘mutually charge-conjugate’ (rather than individually self-conjugate) particles which may always have two distinct mass values all the same (though now at the price of intrinsically violating $C$). The latter feature (still permitting a mass-generating mechanism like the See–Saw one) is just due to the following reason: unlike what happens for standard fermions (endowed with scalar-type charges), such (active and sterile) charge-conjugate neutrinos would not be really interchanged under $CPT$ (were it not so, on the other hand, $CPT$ itself would then be maximally violated!). Similarly, although a ‘Majorana mass’ neutrino is now predicted to bear a nonzero lepton number, we have that the conventional expectation for a neutrinoless double $\beta$-decay is left unaffected. This is simply because a pseudoscalar lepton number is as well a quantity that changes sign along with chirality. Yet, supposing that real neutrinos should truly turn out to be ‘Majorana mass’ (and not ‘Dirac mass’) fermions, we have also that their well-known phenomenology should then be reread in a way opposite to the usual one: the actual behaviors of them under space reflection and time inversion would indeed appear to have reversed meanings! The fact is that real neutrinos themselves should be admitted, accordingly, to be particles carrying pseudoscalar-type (rather than scalar-type) charges; so they would paradoxically seem to obey pure mirror symmetry (as just the analogue of ‘$CP$ symmetry’ for standard fermions) and to violate instead (to a maximal degree) either time reversal or particle–antiparticle conjugation $C$ (with possible far-reaching effects on the yet unsolved ‘time arrow’ fundamental question).

The last comment to be made goes beyond neutrino physics and is more generally addressed to the correspondence that has been herein set up between ‘Majorana masses’ and pseudoscalar-type charges. Since one has that an active ‘Majorana mass’ neutrino, whether viewed with no lepton number or with a pseudoscalar nonzero lepton number, is identically able to induce a net neutrinoless double $\beta$-decay, one could get the general idea that remodelling (active and sterile) ‘Majorana mass’ fermions as particles endowed with pseudoscalar-type charges (rather than genuinely neutral) should truly have no direct repercussions in experimental reality. Such an idea, however, would leave out of account the fact that pseudoscalar-type charges could themselves be at the origin of new interactions. A particularly significant example may come from considering magnetic charge, whose pseudoscalar nature (opposed to the scalar nature of electric charge) is already well-known. This indeed suggests that ‘Majorana mass’ fermions (just opposed to ‘Dirac mass’ ones) might now be seriously ex-
pected to be even the natural candidates for magnetic monopoles.

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