Descent of dg cohesive modules for open covers on complex manifolds

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Abstract
We study the descent problem of cohesive modules on complex manifolds. For a complex manifold $X$ we can consider the Dolbeault dg-algebra $A(X)$ on it and Block in 2006 introduced a dg-category $P_{A(X)}$, called cohesive modules, associated with $A(X)$. The same construction works for any open subset $U \subset X$ and we obtain a dg-presheaf on $X$ given by $U \mapsto P_{A(U)}$. We prove that this dg-presheaf satisfies the descent property for any locally finite open cover of a complex manifold $X$. This generalizes part of the results of Ben-Bassat and Block in 2012, who studied the case that $X$ is covered by two open subsets.

Keywords Descent · Cohesive modules · Twisted complexes · Dg-categories

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1 Introduction
In [4], Block assigned a dg-category $P_A$, called cohesive modules, to a (curved) dg-algebra $A$. The dg-category of cohesive modules provides a way to enhance many well-known triangulated categories and has been studied in [5, 7, 15, 19].

In particular, for a compact complex manifold $X$ we consider the Dolbeault dg algebra $A^\bullet(X) = (A^{0,\bullet}(X), \bar{\partial})$. In this case a cohesive module consists of a complex of smooth vector bundles on $X$ with a $\bar{\partial}$-$\mathbb{Z}$-connection. Block proved in [4] that $P_{A(X)}$ gives a dg-enhancement of $D^b_{\text{coh}}(X)$, the bounded derived category of coherent sheaves on $X$. Based on this result, in [3] the authors proved the Grothendieck–Riemann–Roch theorem for coherent sheaves on compact complex manifolds. Moreover, [8]...
generalizes the result in [4] to the case that \( X \) is non-compact with a slightly more restricted definition of coherent sheaves.

The descent problem is also one of the original motivations of considering dg-enhancement of triangulated categories. In [2], Ben-Bassat and Block proved that for a compact complex manifold \( X \) and an open cover \( \{ U_1, U_2 \} \) of \( X \), the natural restriction functor

\[
P_A(X) \to P_A(U_1) \times P_A(U_1 \cap U_2) \to P_A(U_2)
\]

is a quasi-equivalence of dg-categories, where \( P_A(U_1) \times P_A(U_1 \cap U_2) \to P_A(U_2) \) denotes the homotopy fiber product. See [2, Theorem 6.7 and Theorem 7.4] for details.

**Remark 1.1** It is well known that for derived categories, the natural restriction functor

\[
D^b_{\text{coh}}(X) \to D^b_{\text{coh}}(U_1) \times D^b_{\text{coh}}(U_1 \cap U_2) \to D^b_{\text{coh}}(U_2)
\]

is not an equivalence of triangulated categories, see [16, Section 2.2].

In this paper we study the descent of \( P_A(X) \) for an arbitrary locally finite open cover \( U = \{ U_i \} \) of \( X \). In this case the homotopy fiber product on the right-hand side of (1) should be replaced by the more general homotopy limit, and we prove that the natural functor

\[
P_A(X) \to \text{Holim}_U P_A(U_i)
\]

is a quasi-equivalence of dg-categories.

According to [1, 6], the homotopy limit \( \text{Holim}_U P_A(U_i) \) can be realized concretely as the dg-category of twisted complexes, \( \text{Tw}(X, P_A, U_i) \) (See Sect. 3). Therefore the main result of this paper could be stated as:

**Theorem 5.5** Let \( X \) be a complex manifold and \( \{ U_i \} \) be a finite open cover of \( X \). Let \( A = (A^0, \cdot, \overline{\partial}, 0) \) be the Dolbeault dg-algebra on \( X \) and \( P_A \) be the dg-category of cohesive modules. Let \( \text{Tw}(X, P_A, U_i) \) be the dg-category of globally bounded twisted complexes on \( X \). Then the natural functor

\[
\mathcal{T} : P_A(X) \to \text{Tw}(X, P_A, U_i)
\]

is a dg-quasi-equivalence of dg-categories.

This result can be considered as a dg-categorification of the Čech–Dolbeault theory of cohomologies on complex manifolds.

Let us briefly mention the strategy of the proof. We want to construct a right adjoint functor of \( \mathcal{T} \), \( \mathcal{S} : \text{Tw}(X, P_A, U_i) \to P_A(X) \). However, we will see that the image of \( \mathcal{S} \) cannot be contained in \( P_A(X) \). Therefore we have to enlarge our dg-category to quasi-cohesive modules \( C_A(X) \) and get an adjoint pair

\[
\mathcal{T} : C_A(X) \rightleftarrows \text{Tw}(X, C_A, U_i) : \mathcal{S}.
\]

Then the proof of Theorem 5.5 consists of (i) a detailed study of the pair \( \mathcal{T} \rightleftarrows \mathcal{S} \) restricted to underlying complexes; and (ii) some general results on dg-categories and cohesive modules.
Remark 1.2 The strategy of the proof of Theorem 5.5 is similar to the proof of the main theorem in [2]. Nevertheless the way of gluing underlying complexes in this paper is very different from that in [2, Section 5]. See Sect. 4.

Remark 1.3 Actually we will prove Theorem 5.5 for locally finite open cover and the dg-category of \textit{globally bounded} twisted complexes. See Sect. 3.3 and Theorem 5.5. If the cover is finite, then any twisted complex is globally bounded.

Remark 1.4 In derived algebraic geometry, the dg-category of quasi-coherent sheaves on a space \( X \) is \textit{defined} to be the homotopy limit \( \text{Holim} \mathbb{U} \text{QCoh}(U_i) \) for suitable affine open cover \( \{ U_i \} \). See [10, Chapter 3] and [1, Section 1].

In contrast, in our case the dg-category \( \mathcal{P}_A(X) \) is defined by itself, and we do not require the cover \( \{ U_i \} \) to be affine. Therefore Theorem 5.5 is not merely a definition.

This paper is organized as follows: In Sect. 2 we review cohesive modules; in Sect. 3 we review twisted complexes. In particular we define the natural functor \( \mathcal{T} : \mathcal{P}_A(X) \to \text{Tw}(X, \mathcal{P}_A, U_i) \) as well as its right adjoint \( \mathcal{S} \). In Sect. 4 we temporarily ignore the \( \bar{\partial}-\mathbb{Z} \)-connection and focus on the gluing of underlying complexes. In Sect. 5 we take the \( \bar{\partial}-\mathbb{Z} \)-connection back and consider the descent of cohesive modules, where the main result of this paper is proved (Theorem 5.5).

2 A review of cohesive modules

2.1 Definition and basic facts

Let us first recall the definition of cohesive modules in [4].

Definition 2.1 ([4, Definition 2.4]) For a curved dg-algebra \( A = (A^\bullet, d_A, c) \), we define the dg-category \( \mathcal{P}_A \):

- An object \( \mathcal{E} = (E^\bullet, \mathbb{E}) \) in \( \mathcal{P}_A \), which we call a \textit{cohesive module}, is a \( \mathbb{Z} \)-graded (but bounded in both directions) right module \( E^\bullet \) over \( A^0 \) (\( A^0 \) is the zero degree part of \( A^\bullet \)) which is finitely generated and projective, together with a \( \mathbb{Z} \)-connection, which satisfies the usual Leibniz condition

\[
\mathbb{E}(e \cdot \omega) = \mathbb{E}(e) \cdot \omega + (-1)^{|e|} e \cdot d_A(\omega),
\]

\[
\mathbb{E} : E^\bullet \otimes_{A^0} A^\bullet \to E^\bullet \otimes_{A^0} A^\bullet
\]

that satisfies the integrability condition that the relative curvature vanishes

\[
F_E(e) = \mathbb{E} \circ \mathbb{E}(e) + e \cdot c = 0
\]

for all \( e \in E^\bullet \).

- The morphisms of degree \( k \), \( \mathcal{P}_A^k(\mathcal{E}_1, \mathcal{E}_2) \) between two cohesive modules \( \mathcal{E}_1 = (E_1^\bullet, \mathbb{E}_1) \) and \( \mathcal{E}_2 = (E_2^\bullet, \mathbb{E}_2) \) are

\[
\{ \phi : E_1^\bullet \otimes_{A^0} A^\bullet \to E_2^\bullet \otimes_{A^0} A^\bullet \text{ of degree } k \text{ and } \phi(ea) = \phi(e)a \forall a \in A^\bullet \}
\]
with differentials defined by
\[ d(\phi) (e) = E_2(\phi(e)) - (-1)^{|\phi|}. \]

The differential \( E \) decomposes into \( E = \sum E^i \) where
\[ E^i : E^\bullet \to E^{\bullet + 1 - i} \otimes_A A^i. \]

A similar decomposition applies to the morphism \( \phi \).

As in [4, Section 2.4] we could define the shift functor and the mapping cone in \( P_A \).

**Definition 2.2** For \( E = (E^\bullet, E) \), set \( E[1] = (E[1]^\bullet, E[1]) \), where \( E[1]^\bullet = E^{\bullet+1} \) and \( E[1] = -E \). Next for \( \phi : E \to F \) a degree zero closed morphism in \( P_A \), we define the mapping cone of \( \phi \), \( \text{Cone}(\phi) = (\text{Cone}(\phi)^\bullet, C_\phi) \) by
\[
\text{Cone}(\phi)^\bullet = \begin{pmatrix} F^\bullet \\ \oplus \\ E[1]^\bullet \end{pmatrix}
\]
and
\[
C_\phi = \begin{pmatrix} F & \phi \\ 0 & E[1] \end{pmatrix}.
\]

It is clear that \( P_A \) is a pre-triangulated dg-category hence its homotopy category \( \text{Ho}(P_A) \) is a triangulated category.

In this paper we also consider the degree zero part \( A^0 \) as a dg-algebra concentrated at degree zero with zero differential. In this case \( P_{A^0} \) is simply the dg-category of bounded complexes of finitely generated projective \( A^0 \)-modules. Let \( \text{For} : P_A \to P_{A^0} \) be the forgetful functor. It is clear that
\[ \text{For}(E^\bullet, E) = (E^\bullet, E^0). \]

It is also useful to consider the larger dg-category of quasi-cohesive modules \( C_A \): a quasi-cohesive module is a data \( Q = (Q^\bullet, Q) \) where everything is the same as in a cohesive module except that \( Q^\bullet \) is not required to be bounded, finitely generated and projective. Similarly we have \( C_{A^0} \).

We have the following definition of quasi-equivalence between quasi-cohesive modules.

**Definition 2.3** A degree zero closed morphism \( \phi \in C_A(E_1, E_2) \) is called a quasi-isomorphism if and only if \( \text{For}(\phi) : \text{For}(E_1) \to \text{For}(E_2) \) is a quasi-isomorphism of complexes of \( A^0 \)-modules.

The following result characterizes homotopy equivalences in \( P_A \).
Proposition 2.4 ([4, Proposition 2.9]) Let $E_1$ and $E_2$ be objects in $\mathcal{P}_A$. Then a degree zero closed morphism $\phi \in \mathcal{P}_A(E_1, E_2)$ is a homotopy equivalence if and only if $\phi$ is a quasi-isomorphism in the sense of Definition 2.3, i.e. For $(\phi): \text{For}(E_1) \rightarrow \text{For}(E_2)$ is a quasi-isomorphism of complexes of $A^0$-modules.

Proof See [4, Proposition 2.9].

Remark 2.5 The result in Proposition 2.4 is not true if one of $E_1$ and $E_2$ is not in $\mathcal{P}_A$.

For $\mathcal{P}_A$ the fully faithful Yoneda embedding $h: Z^0(\mathcal{P}_A) \rightarrow \mathcal{P}_A$ is given by $E \mapsto h_E = \mathcal{P}_A(\cdot, E)$. Similarly for $\mathcal{C}_A$ we have a fully faithful functor $\tilde{h}: Z^0(\mathcal{C}_A) \rightarrow \mathcal{P}_A$ given by $Q \mapsto \tilde{h}_Q = \mathcal{C}_A(\cdot, Q)$ considered as a module over $\mathcal{P}_A$. Here $Z^0(-)$ denotes the associated category with closed, degree zero morphisms.

We recall the following result, which justifies the name “quasi-isomorphism” in Definition 2.3.

Proposition 2.6 ([4, Proposition 3.9]) Let $\phi: Q_1 \rightarrow Q_2$ be a quasi-isomorphism in $\mathcal{C}_A$ as in Definition 2.3. Then the induced morphism $\tilde{h}_\phi: \tilde{h}_{Q_1} \rightarrow \tilde{h}_{Q_2}$ is a quasi-isomorphism in $\text{Mod-\mathcal{P}_A}$. The inverse is not true.

2.2 Pullback and pushforward

Next we consider the pullback and pushforward of (quasi-)cohesive modules. For simplicity we focus on dg-algebras instead of curved dg-algebras. Let $f: A \rightarrow B$ be a morphism between dg-algebras. We define a dg functor $f^*: \mathcal{P}_A \rightarrow \mathcal{P}_B$ as follows. Given $E = (E^\bullet, E)$ a cohesive module over $A$, define $f^*(E)$ to be $(E^\bullet \otimes_A B^0, E_B)$ where

$$E_B(e \otimes b) = E(e)b + (-1)^{|e|} e \otimes d_B(b).$$

We could check that $E_B$ is still a $\mathbb{Z}$-connection and satisfies $E_B^2 = 0$. The functor $f^*$ on morphisms is defined in the same way. We call $f^*$ the pullback functor.

We could define $f^*: \mathcal{C}_A \rightarrow \mathcal{C}_B$ in the same way. Moreover, given composable morphisms of dg-algebras $f$ and $g$, there is a natural equivalence $(f \circ g)^* \Rightarrow g^* f^*$ which satisfies the obvious coherence relation.

For $f: A \rightarrow B$ a morphism between dg-algebras, there is also a functor in the opposite direction $f_*: \text{Mod-\mathcal{P}_B} \rightarrow \text{Mod-\mathcal{P}_A}$ and $f_*: \text{Mod-\mathcal{C}_B} \rightarrow \text{Mod-\mathcal{C}_A}$ defined by composing with $f^*$. Now suppose that we are in the special case that the natural map

$$B^0 \otimes_A A^\bullet \rightarrow B^\bullet$$

is a quasi-isomorphism.
is an isomorphism. Then we will define a pushforward functor $f_* : \mathcal{C}_{\mathcal{B}} \to \mathcal{C}_{\mathcal{A}}$ as follows. Let $(Q^\bullet, \mathbb{Q})$ be a quasi-cohesive $\mathcal{B}$-module. We consider $Q^\bullet$ as a graded $\mathcal{A}^0$-module via $f$. By the assumption there is an isomorphism $Q^\bullet \otimes_{\mathcal{A}^0} \mathcal{A}^\bullet \cong Q^\bullet \otimes_{\mathcal{B}^0} \mathcal{B}^\bullet$.

Then we define $f_* (Q^\bullet, \mathbb{Q}) = (Q^\bullet, \mathbb{Q})$ where the right-hand side is the same as the left-hand side but considered as graded $\mathcal{A}^0$-modules and $\mathcal{A}$-module maps. We call $f_*$ the pushforward functor.

It is easy to check that $f_* : \mathcal{C}_{\mathcal{B}} \to \mathcal{C}_{\mathcal{A}}$ and $f_* : \text{Mod-} \mathcal{C}_{\mathcal{B}} \to \text{Mod-} \mathcal{C}_{\mathcal{A}}$ are compatible via the Yoneda embedding. Moreover, both $f^\star$ and $f_*$ are compatible with the forgetful functor $\text{For} : \mathcal{C}_{\mathcal{A}} \to \mathcal{C}_{\mathcal{A}^0}$.

Remark 2.7 In general, for $(E^\bullet, \mathbb{E})$ in $\mathcal{P}_{\mathcal{B}}$, its pushforward $f_*(E^\bullet, \mathbb{E})$ is not in $\mathcal{P}_{\mathcal{A}}$.

2.3 The Serre–Swan theorem

Let $X$ be a $C^\infty$-manifold which is not necessarily compact. Let $C^\infty (X)$ be the ring of complex-value $C^\infty$-functions on $X$. For a $C^\infty$ complex vector bundle $E$ on $X$, let $\Gamma (E)$ be the set of $C^\infty$-sections of $E$. It is clear that $\Gamma (E)$ is a $C^\infty (X)$-module.

We recall the following result.

**Theorem 2.8** (Serre–Swan theorem, [12, Theorems 12.29 and 12.32]) Let $X$, $C^\infty (X)$, and $\Gamma$ be as before. Then $\Gamma$ gives an equivalence of categories from the category of finite-dimensional $C^\infty$ complex vector bundles on $X$ to the category of finitely generated projective $C^\infty (X)$-modules.

In this paper we will use the terminologies finite-dimensional $C^\infty$ complex vector bundles and finitely generated projective $C^\infty (X)$-modules interchangeably.

2.4 Cohesive modules on complex manifolds

Let $X$ be a complex manifold, in this paper we consider the Dolbeault dg-algebra $A(X) = (A^{0, \bullet} (X), \bar{\partial}_X, 0)$

where $A^{0, \bullet} (X)$ is the set of $C^\infty$-$(0, \bullet)$-forms on $X$. We have the dg-category of cohesive modules $\mathcal{P}_{A(X)}$. Let $E = (E^\bullet, \mathbb{E})$ be an object in $\mathcal{P}_{A(X)}$ where $E^\bullet$ is a bounded graded finitely generated projective $A^{0, 0} (X) = C^\infty (X)$-module. By Theorem 2.8, $E^\bullet$ corresponds to a bounded graded finite dimensional $C^\infty$ vector bundle on $X$. In this viewpoint, $\mathbb{E}$ is a $\bar{\partial}Z$-connection on the graded vector bundle $E^\bullet$.

In the compact case we have the following theorem.

**Theorem 2.9** ([4, Theorem 4.3]) Let $X$ be a compact complex manifold and $A(X) = (A^{0, \bullet} (X), \bar{\partial}_X, 0)$ be the Dolbeault dg-algebra. Then the homotopy category $\text{Ho}(\mathcal{P}_{A(X)})$ is equivalent to $D_{\text{coh}}^b (X)$, the bounded derived category of complexes of coherent $\mathcal{O}_X$-modules, where $\mathcal{O}_X$ is the sheaf of holomorphic functions on $X$. 

\[\square\] Springer
Remark 2.10 It is well known that in this case $D_{\text{coh}}^b(X)$ is also equivalent to $D_{\text{perf}}(X)$, the derived category of perfect complexes of $\mathcal{O}_X$-modules.

Recall that a perfect complex of $\mathcal{O}_X$-modules is a complex $E^\bullet$ of $\mathcal{O}_X$-modules such that there exists an open covering $U_i$ of $X$ such that on each $U_i$ there exists a bounded complex of finitely generated locally free $\mathcal{O}_{U_i}$-modules $F^\bullet_i$ together with a quasi-isomorphism $F^\bullet_i \to E^\bullet|_{U_i}$.

When $X$ is non-compact, we need the following concept.

Definition 2.11 ([8, Definition 7.2]) Let $X$ be a complex manifold which is not necessarily compact. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on $X$. A complex of $\mathcal{O}_X$-modules $E^\bullet$ is called a globally bounded perfect complex if there exists an open covering $U_i$ of $X$ and integers $a < b$ and $N > 0$ such that on each $U_i$ there exists a bounded complex of finitely generated locally free $\mathcal{O}_{U_i}$-modules $F^\bullet_i$ which is concentrated in degrees $[a, b]$ and each $F^j_i$ has rank $\leq N$, together with a quasi-isomorphism $F^\bullet_i \to E^\bullet|_{U_i}$. We denote the derived category of globally bounded perfect complexes on $X$ by $D^B_{\text{perf}}(X)$.

It is clear that when $X$ is compact, any perfect complex on $X$ is globally bounded. However, this is no longer true for non-compact $X$. See [8, Remark 7.2].

Theorem 2.12 ([8, Theorem 8.3]) Let $X$ be a complex manifold and $A(X) = (A^0\bullet(X), \tilde{\partial}_X, 0)$ be the Dolbeault dg-algebra. Then the homotopy category $\text{Ho}(\mathcal{P}_A(X))$ is equivalent to $D^B_{\text{perf}}(X)$, the derived category of globally bounded perfect complexes on $X$.

2.5 The dg-presheaf $\mathcal{P}_A$ and the descent problem

Let $U \subset X$ be an open subset of $X$ and we define the dg-category of (quasi-)cohesive modules on $U$ as follows.

Definition 2.13 Let $U \subset X$ be an open subset of $X$. We define the dg-algebra $A(U)$ to be

$$A(U) = (A^0\bullet(U), \tilde{\partial}, 0).$$

Then we can define the dg-categories $\mathcal{P}_A(U)$, $\mathcal{P}_A^0(U)$, $\mathcal{C}_A(U)$, and $\mathcal{C}_A^0(U)$.

For an inclusion $U \subset V$ of open subsets we have the restriction map $r: A(V) \to A(U)$. Hence we get the pullback functor $r^*: \mathcal{P}_A(V) \to \mathcal{P}_A(U)$ as in Sect. 2.2. Therefore the assignment

$$U \mapsto \mathcal{P}_A(U)$$

gives a dg-presheaf on $X$ and we denote it by $\mathcal{P}_A$.

For an open cover $\mathcal{U} = \{U_i\}$ of $X$, its Čech nerve is a simplicial space

$$\cdots \coprod U_i \cap U_j \cap U_k \coprod U_i \cap U_j \coprod U_i.$$
and we consider the resulting cosimplicial diagram of dg-categories

\[ \prod P_A(U_i) \longrightarrow \prod P_A(U_i \cap U_j) \longrightarrow \prod P_A(U_i \cap U_j \cap U_k) \longrightarrow \cdots \]  

(2)

It is clear that the descent data of \( P_A \) with respect to the open cover \( \{U_i\} \) is given by the homotopy limit of Diagram (2) in \( \text{DgCat}_{\text{DK}} \), the category of all dg-categories with the Dwyer–Kan model structure. In Sect. 3 we will present this homotopy limit as the dg-category of twisted complexes. The main topic of this paper is to prove that \( P_A(X) \) is quasi-equivalent to the homotopy limit of Diagram (2).

We will also need the following result on pushforward.

**Proposition 2.14** For an inclusion \( U \subset V \) of open subsets let \( r : \mathcal{A}(V) \rightarrow \mathcal{A}(U) \) be the restriction map. Then we can define the pushforward functor

\[ r_* : C_c^0(U) \rightarrow C_c^0(V). \]

**Proof** Recall that \( \mathcal{A}^\bullet(U) \) is the set of \( C^\infty_\bullet \)-forms on \( U \). From the definition of differential forms it is clear that the natural homomorphism

\[ \mathcal{A}^0(U) \otimes \mathcal{A}^0(V) \rightarrow \mathcal{A}^\bullet(U) \]

is an isomorphism. The claim of the proposition follows from the construction in Sect. 2.2. \( \square \)

3 Twisted complexes

3.1 Definition and basic facts

Toledo and Tong [17] introduced twisted complexes in the 1970’s as a way to obtain global resolutions of perfect complexes of sheaves on a complex manifold. In 2015, Wei proved in [18] that the dg-category of twisted perfect complexes gives a dg-enhancement of the derived category of perfect complexes.

In this paper we give a slightly generalized definition of twisted complexes so that we could apply it in the descent problem of \( P_A \). For reference of twisted complexes see [13, Section 1] or [18, Section 2].

Let \( X \) be a paracompact topological space and \( \mathcal{F} \) be a dg-presheaf on \( X \). Let \( \mathcal{U} = \{U_i\} \) be a locally finite open cover of \( X \). Let \( U_{i_0 \ldots i_n} \) denote the intersection \( U_{i_0} \cap \cdots \cap U_{i_n} \).

Let \( \{E_i\} \) and \( \{F_i\} \) be two collections of objects in \( \mathcal{F}(U_i) \) for each \( U_i \). We can consider the map

\[ C^\bullet(\mathcal{U}, \text{Hom}^\bullet(E, F)) = \bigoplus_{p,q} C^p(\mathcal{U}, \text{Mor}^q_{\mathcal{F}}(E, F)). \]
An element \( u^{p,q} \) of \( C^p(\mathcal{U}, \text{Mor}_S^q(E, F)) \) gives an element \( u^{p,q}_{i_0\ldots i_p} \) of \( \text{Mor}_S^q(E_{i_p}, F_{i_0}) \) over each non-empty intersection \( U_{i_0\ldots i_n} \). Notice that we require \( u^{p,q} \) to be a morphism from the \( E \) on the last subscript of \( U_{i_0\ldots i_n} \) to the \( F \) on the first subscript of \( U_{i_0\ldots i_n} \).

We need to define the compositions of \( C^*(\mathcal{U}, \text{Mor}^*(E, F)) \). Let \( \{G_i\} \) be a third collection of objects. There is a composition map

\[
C^*(\mathcal{U}, \text{Mor}^*(F, G)) \times C^*(\mathcal{U}, \text{Mor}^*(E, F)) \to C^*(\mathcal{U}, \text{Mor}^*(E, G)).
\]

In fact, for \( u^{p,q} \in C^p(\mathcal{U}, \text{Mor}^q(F, G)) \) and \( v^{r,s} \in C^r(\mathcal{U}, \text{Mor}^s(E, F)) \), their composition \( (u \cdot v)^{p+r,q+s} \) is given by (see [13, Equation (1.1)])

\[
(u \cdot v)^{p+r,q+s}_{i_0\ldots i_{p+r}} = (-1)^{qr} u^{p,q}_{i_0\ldots i_p} v^{r,s}_{i_p\ldots i_{p+r}}
\]

where the right-hand side is the composition of sheaf maps.

In particular, \( C^*(\mathcal{U}, \text{Mor}^*(E, E)) \) becomes an associative algebra under this composition (it is easy but tedious to check the associativity).

There is also a Čech-style differential operator \( \delta \) on \( C^*(\mathcal{U}, \text{Mor}^*(E, F)) \) and of bidegree \((1,0)\) given by the formula

\[
(\delta u)^{p+1,q}_{i_0\ldots i_{p+1}} = \sum_{k=1}^{p} (-1)^{k} u^{p,q}_{i_0\ldots i_k} v^{r,s}_{i_k\ldots i_{p+1}}
\]

for \( u^{p,q} \in C^p(\mathcal{U}, \text{Mor}^q(E, F)) \). It is not difficult to check that the Čech differential satisfies the Leibniz rule.

Now we can introduce the notion of twisted complexes.

**Definition 3.1** Let \( X \) be a paracompact topological space and \( \mathfrak{F} \) be a dg-presheaf on \( X \). Let \( \mathcal{U} = \{U_i\} \) be a locally finite open cover of \( X \). A twisted complex consists of a collection objects \( E_i \) of \( \mathfrak{F}(U_i) \) together with a collection of morphisms

\[
a = \sum_{k \geq 0} a^{k,1-k}
\]

where \( a^{k,1-k} \in C^k(\mathcal{U}, \text{Mor}^{1-k}(E, E)) \) which satisfies the equation

\[
\delta a + a \cdot a = 0.
\]

More explicitly, for \( k \geq 0 \),

\[
\delta a^{k-1,2-k} + \sum_{i=0}^{k} a^{i,1-i} \cdot a^{k-i,1-k+i} = 0.
\]

We impose two additional requirements on \( a \):
(i) For any \( u_{p,q} \in \text{Mor}^{q}_{\mathcal{F}}(E_{i_{p}}, E_{i_{0}}) \), the assignment

\[
-1 \cdot a_{0_{1}} \cdot u_{p,q} - (-1)^{p+q} a_{i_{p} 0_{1}} \cdot u_{p,q} \in \text{Mor}^{q+1}_{\mathcal{F}}(E_{i_{p}}, E_{i_{0}})
\]

coincides with the differential in the dg-category \( \mathcal{F}(U_{i_{0} \ldots i_{p}}) \);

(ii) \( a_{i_{1} 0_{1}} \in \text{Mor}^{0}_{\mathcal{F}}(E_{i}, E_{i}) \) is invertible up to homotopy.

Twisted complexes on \((X, \mathcal{F}, \{U_{i}\})\) form a dg-category: the objects are twisted complexes \((E_{i}, a)\) and the morphisms from \( \mathcal{E} = (E_{i}, a) \) to \( \mathcal{F} = (F_{i}, b) \) are \( C^{\bullet}(\mathcal{U}, \text{Mor}^{\bullet}(E, F)) \) with the total degree. Moreover, the differential on a morphism \( \phi \) is given by

\[
d\phi = \delta \phi + b \cdot \phi - (-1)^{|\phi|} \phi \cdot a.
\]

We denote the dg-category of twisted complexes on \((X, \mathcal{F}, \{U_{i}\})\) by \( \text{Tw}(X, \mathcal{F}, U_{i}) \). If there is no danger of confusion we can simply denote it by \( \text{Tw}(X) \).

Here we list some special cases of twisted complexes for various dg-presheaves \( \mathcal{F} \):

(i) Let \((X, \mathcal{R})\) be a ringed space and \( \mathcal{F} = \text{Cpx} \) be the dg-presheaf which assigns to each open subspace \( U \) the dg-category of complexes of left \( \mathcal{R} \)-modules on \( U \), then the dg-category of twisted complexes \( \text{Tw}(X, \mathcal{F}, U_{i}) \) as in Definition 3.1 is exactly the dg-category of twisted complexes \( \text{Tw}(X, \mathcal{R}, U_{i}) \) as in [18, Definition 2.12].

(ii) Again let \((X, \mathcal{R})\) be a ringed space and \( \mathcal{F} = \text{Perf} \) be the dg-presheaf which assigns to each open subspace \( U \) the dg-category of bounded complexes of finitely generated locally free left \( \mathcal{R} \)-modules on \( U \), then the dg-category of twisted complexes \( \text{Tw}(X, \mathcal{F}, U_{i}) \) as in Definition 3.1 is exactly the dg-category of twisted perfect complexes \( \text{Tw}^{\text{perf}}(X, \mathcal{R}, U_{i}) \) as in [18, Definition 2.14], which is also called twisted cochain in [13].

(iii) Let \( X \) be a complex manifold and \( \mathcal{F} = \mathcal{P}_{\mathcal{A}} \). The dg-category of twisted complexes \( \text{Tw}(X, \mathcal{P}_{\mathcal{A}}, U_{i}) \) is the main subject of this paper.

(iv) Let \( X \) be a complex manifold and \( \mathcal{F} = \mathcal{P}_{\mathcal{A}_{0}} \). Then the dg-category of twisted complexes \( \text{Tw}(X, \mathcal{P}_{\mathcal{A}_{0}}, U_{i}) \) is the same as \( \text{Tw}^{\text{perf}}(X, \mathcal{A}_{0}, U_{i}) \) and we will further study it in Sect. 4.

(v) Let \( X \) be a complex manifold and \( \mathcal{F} = \mathcal{C}_{\mathcal{A}} \) or \( \mathcal{C}_{\mathcal{A}_{0}} \), the dg-presheaf of quasi-cohesive modules. The resulting dg-categories \( \text{Tw}(X, \mathcal{C}_{\mathcal{A}}, U_{i}) \) and \( \text{Tw}(X, \mathcal{C}_{\mathcal{A}_{0}}, U_{i}) \) play auxiliary roles in this paper.

The importance of twisted complexes in descent theory is illustrated by the following theorem.

**Theorem 3.2** ([1, 6]) *Let \( \mathcal{F} \) be a dg-presheaf. The dg-category \( \text{Tw}(X, \mathcal{F}, U_{i}) \) is quasi-equivalent to the homotopy limit of the cosimplicial diagram*

\[
\prod \mathcal{F}(U_{i}) \xrightarrow{\partial} \prod \mathcal{F}(U_{i} \cap U_{j}) \xrightarrow{\partial} \prod \mathcal{F}(U_{i} \cap U_{j} \cap U_{k}) \xrightarrow{\partial} \cdots
\]
It is clear that Theorem 3.2 applies to all above cases.

**Remark 3.3** Theorem 3.2 is proved in [6] under the condition that $\mathcal{F}$ sends finite coproducts to products. In [1] this condition is removed and the authors prove the result for arbitrary cosimplicial diagram of dg-categories.

In [18, Section 2.5] it has been shown that $\text{Tw}(X, \mathcal{F}, U_i)$ has a pre-triangulated structure for all above cases, hence $\text{HoTw}(X, \mathcal{F}, U_i)$ is a triangulated category. In more details we have the following definitions.

**Definition 3.4** *(Shift)* Let $E = (E_i^\bullet, a)$ be a twisted complex. We define its shift $E[1]$ to be $E_{i+1}^\bullet$ and $a_{i+1}^{k, k-1} = (-1)^{k-1} a_{i}^{k, k-1}$.

Moreover, let $\phi : E \rightarrow F$ be a morphism. We define its shift $\phi[1]$ as $\phi[1]^{p,q} = (-1)^q \phi^{p,q}$.

**Definition 3.5** *(Mapping cone)* Let $\phi^{\bullet, -\bullet}$ be a closed degree zero map between twisted perfect complexes $E = (E^\bullet, a^{\bullet, -\bullet})$ and $F = (F^\bullet, b^{\bullet, -\bullet})$, we define the mapping cone $G = (G^\bullet, c)$ of $\phi$ as follows (see [14, Section 1.1]):

$$G^i_n := E^{i+1}_n \oplus F^n_i$$

and

$$c_{i_0 \ldots i_k}^{k, 1-k} = \begin{pmatrix} (-1)^{k-1} a_{i_0 \ldots i_k}^{k, 1-k} & 0 \\ (-1)^{k} b_{i_0 \ldots i_k}^{k, 1-k} & \phi_{i_0 \ldots i_k}^{k, 1-k} \end{pmatrix}.$$

### 3.2 Quasi-isomorphisms between twisted complexes

For $\mathcal{F} = \mathcal{C}_A$ or $\mathcal{C}_A^0$, we have the following definition of a quasi-isomorphism between twisted complexes.

**Definition 3.6** Let $\phi : E \rightarrow F$ be a degree zero closed morphism in $\text{Tw}(X, \mathcal{C}_A^0, U_i)$. Then we call $\phi$ a *quasi-isomorphism* if and only if its $(0, 0)$-component

$$\phi^{0,0} : (E^\bullet_i, a^{0,1}_i) \rightarrow (F^\bullet_i, b^{0,1}_i)$$

is a quasi-isomorphism of complexes of $A^0(U_i)$-modules for each $i$.

Moreover, let $\phi : E \rightarrow F$ be a degree zero closed morphism in $\text{Tw}(X, \mathcal{C}_A, U_i)$. Then we call $\phi$ a *quasi-isomorphism* if and only if $\text{For}(\phi) : \text{For}(E) \rightarrow \text{For}(F)$ is a quasi-equivalence in $\text{Tw}(X, \mathcal{C}_A^0, U_i)$, where $\text{For}$ is the forgetful functor.

**Remark 3.7** A quasi-isomorphism is called a weak equivalence in [18, Definition 2.27].
**Lemma 3.8** Let $\mathcal{E}$ be an object in $\text{Tw}(X, \mathcal{P}_A, U_i)$ and $\mathcal{F}$ and $\mathcal{G}$ be two objects in $\text{Tw}(X, \mathcal{C}_A, U_i)$. Let $\phi: \mathcal{E} \rightarrow \mathcal{G}$ be a degree zero closed morphism in $\text{Tw}(X, \mathcal{C}_A, U_i)$ and $\psi: \mathcal{F} \rightarrow \mathcal{G}$ be a quasi-isomorphism in $\text{Tw}(X, \mathcal{C}_A, U_i)$. Then $\phi$ can be lifted to a closed degree zero morphism $\eta: \mathcal{E} \rightarrow \mathcal{F}$ up to homotopy, i.e. there exists an $\eta: \mathcal{E} \rightarrow \mathcal{F}$ such that $\psi \circ \eta = \phi$ up to homotopy. The same result holds if we replace $\text{Tw}(X, \mathcal{P}_A, U_i)$ and $\text{Tw}(X, \mathcal{C}_A, U_i)$ by $\text{Tw}(X, \mathcal{P}_{A^0}, U_i)$ and $\text{Tw}(X, \mathcal{C}_{A^0}, U_i)$, respectively.

**Proof** It is a standard spectral sequence argument. See [18, Lemma 2.30].

We have some further results on quasi-isomorphisms if both objects are in $\text{Tw}(X, \mathcal{P}_A, U_i)$.

**Proposition 3.9** Let $\mathcal{E}$ and $\mathcal{F}$ be objects in $\text{Tw}(X, \mathcal{P}_A, U_i)$. Then a degree zero closed morphism $\phi: \mathcal{E} \rightarrow \mathcal{F}$ is a quasi-isomorphism if and only if $\phi$ is a homotopy equivalence. The same result holds for $\text{Tw}(X, \mathcal{P}_{A^0}, U_i)$.

**Proof** By Definition 3.6, $\phi_i^{0,0}: \mathcal{E}_i \rightarrow \mathcal{F}_i$ is a quasi-isomorphism in $\mathcal{P}_{A(U_i)}$. Then by Proposition 2.4, we have its homotopy inverse $\psi_i: \mathcal{F}_i \rightarrow \mathcal{E}_i$ in $\mathcal{P}_{A(U_i)}$. By a simple spectral sequence argument which is the same as the proof of [4, Proposition 2.9], we can extend $\psi_i$ to a degree zero closed morphism in $\text{Tw}(X, \mathcal{P}_{A^0}, U_i)$. A similar argument works for $\text{Tw}(X, \mathcal{P}_{A^0}, U_i)$. See also [18, Proposition 2.31].

### 3.3 Globally bounded twisted complexes

When $\mathcal{F} = \text{Perf}, \mathcal{P}_A$, or $\mathcal{P}_{A^0}$, we have the following dg-subcategory of $\text{Tw}(X, \mathcal{F}, U_i)$.

**Definition 3.10** Let $\mathcal{F}$ be $\text{Perf}, \mathcal{P}_A$, or $\mathcal{P}_{A^0}$. A twisted complex $(E_i, a)$ in $\text{Tw}(X, \mathcal{F}, U_i)$ is called *globally bounded* if there exist integers $a < b$ and $N > 0$ such that on each $U_i$ the underline complex $E^*_i$ of $\mathcal{F}(U_i)$ is concentrated in degrees $[a, b]$ and each of the $E^k_i$’s has rank $\leq N$.

Globally bounded twisted complexes form a full dg-subcategory of $\text{Tw}(X, \mathcal{F}, U_i)$ and we denote it by $\text{Tw}^B(X, \mathcal{F}, U_i)$.

It is clear that $\text{Tw}^B(X, \mathcal{F}, U_i)$ inherits the pre-triangulated structure from $\text{Tw}(X, \mathcal{F}, U_i)$. Moreover, Lemma 3.8 and Proposition 3.9 apply to $\text{Tw}^B(X, \mathcal{P}_A, U_i)$ as well.

**Remark 3.11** If the open cover $\{U_i\}$ is finite, then $\text{Tw}^B(X, \mathcal{F}, U_i)$ coincides with $\text{Tw}(X, \mathcal{F}, U_i)$.

**Remark 3.12** In general, an analogue of the claim in Theorem 3.2 does not hold for $\text{Tw}^B$ unless the cover $\{U_i\}$ is finite.

**Remark 3.13** Using the same method as in [18], we can prove that the dg-category $\text{Tw}^B(X, \text{Perf}, U_i)$ gives a dg-enhancement of the derived category of globally bounded perfect complexes $D^B_{\text{perf}}(X)$. Nevertheless we do not need this result in this paper.
3.4 The twisting functor and the sheafification functor

For \( \mathcal{F} = \mathcal{C}_A \) or \( \mathcal{C}_{A^0} \), we define a pair of adjoint dg-functors

\[
\mathcal{T}: \mathcal{C}_A(X) \rightleftharpoons \text{Tw}(X, \mathcal{C}_A, U_i) : \mathcal{S}
\]

and study their properties in this subsection.

First we define the natural dg-functor \( \mathcal{T}: \mathcal{C}_A(X) \to \text{Tw}(X, \mathcal{C}_A, U_i) \).

**Definition 3.14** ([18, Definition 3.11]) Let \( Q = (Q^\bullet, Q) \) be an object in \( \mathcal{C}_A(X) \). We define its associated twisted complex \( \mathcal{T}(Q) \in \text{Tw}(X, \mathcal{C}_A, U_i) \) by restricting to the \( U_i \)'s. In more detail, we define \( (E^\bullet, a) = \mathcal{T}(Q) \) as

\[
E^n_i = Q^n|_{U_i}
\]

and

\[
a^0_{i0} = Q|_{U_i}, \quad a^1_{ij} = \text{id}_{Q^i|_{U_{ij}}} \quad \text{and} \quad a^k_{i1-k} = 0 \quad \text{for} \quad k \geq 2.
\]

The \( \mathcal{T} \) of morphisms is defined in a similar way. We call the dg-functor \( \mathcal{T}: \mathcal{C}_A(X) \to \text{Tw}(X, \mathcal{C}_A, U_i) \) the twisting functor. We can define \( \mathcal{T}: \mathcal{C}_{A^0}(X) \to \text{Tw}(X, \mathcal{C}_{A^0}, U_i) \) the same way.

The definition of \( \mathcal{S}: \text{Tw}(X, \mathcal{C}_A, U_i) \to \mathcal{C}_A(X) \) is more complicated. First we noticed that a twisted complex \( \mathcal{E} = (E^\bullet, a) \) is not a globally defined quasi-cohesive complex on \( X \). Nevertheless in this subsection we associate a global complex to each twisted complex.

Let \( E_i \) be an object in \( \mathcal{C}_A(U_i) \). By Proposition 2.14, we can use the pushforward \( r_* \) to treat \( E_i \) as an object in \( \mathcal{C}_A(X) \). Moreover, for \( E_{i0} \) in \( \mathcal{C}_A(U_{i0}) \), we can first restrict \( E_{i0} \) to \( \mathcal{C}_A(U_{i0}...i_k) \) and then pushforward to \( \mathcal{C}_A(X) \).

**Definition 3.15** ([18, Definition 3.1]) For a twisted complex of quasi-cohesive modules \( \mathcal{E} = (E^\bullet, a) \), we define the associated quasi-cohesive module \( \mathcal{S}(\mathcal{E}) \) on \( X \) as follows: for each \( n \), the degree \( n \) component \( \mathcal{S}^n(\mathcal{E}) \) is an \( A^0(X) \)-module

\[
\mathcal{S}^n(\mathcal{E}) := \prod_{p+q=n} \prod_{i_0...i_p} E^q_{i0}|_{U_{i0...i_p}}
\]

where the right-hand side is considered as an \( A^0(X) \)-module by pushforward.

The connection on \( \mathcal{S}^\bullet(\mathcal{E}) \) is defined to be \( \mathcal{S}(\mathcal{E}) = \delta + a \) considered as morphisms on \( X \).

It is obvious that \( (\mathcal{S}^\bullet(\mathcal{E}), \mathcal{S}(\mathcal{E})) \) is a quasi-cohesive module in \( \mathcal{C}_A(X) \). The functor \( \mathcal{S} \) on morphisms is defined the same way. We call \( \mathcal{S} \) the sheafification functor.

**Remark 3.16** The functor \( \mathcal{S} \) for \( \mathcal{P}_A \) is a generalization of the functor \( \tilde{\mathcal{A}} \) in [2, Definition 6.2], and \( \mathcal{S} \) for \( \mathcal{P}_{A^0} \) is a generalization of the functor \( \tilde{\mathcal{Y}} \) in [2, Equation (5.6)].
Remark 3.17 It is clear that $T$ restricts to a functor $T : \mathcal{P}_A(X) \to \text{Tw}(X, \mathcal{P}_A(U_i))$ as well as $T : \mathcal{P}_{A^0}(X) \to \text{Tw}(X, \mathcal{P}_{A^0}(U_i))$. On the other hand, the image of the functor $S : \text{Tw}(X, \mathcal{P}_A(U_i)) \to \mathcal{E}_A(X)$ is not contained in $\mathcal{P}_A(X)$.

Proposition 3.18

$$T : \mathcal{C}_A(X) \rightleftarrows \text{Tw}(X, \mathcal{C}_A(U_i)) : S$$

is a pair of adjoint functors. Moreover, the unit morphism of the adjunction $\epsilon(\mathcal{E}) : \mathcal{E} \to S \circ T(\mathcal{E})$ is a quasi-isomorphism (in the sense of Definition 2.3) for any object $\mathcal{E} \in \mathcal{E}_A(X)$. The same results applies to $T : \mathcal{C}_{A^0}(X) \rightleftarrows \text{Tw}(X, \mathcal{C}_{A^0}(U_i)) : S$.

Proof It is a routine check. \qed

Moreover, for a refinement $\{V_j\}$ of the open cover $\{U_i\}$, we can define the twisted functor and the sheafification functor the same way.

Proposition 3.19

$$T : \text{Tw}(X, \mathcal{C}_A(U_i)) \rightleftarrows \text{Tw}(X, \mathcal{C}_{A^0}(U_i)) : S$$

is a pair of adjoint functors. Moreover, the unit morphism of the adjunction $\epsilon(\mathcal{E}) : \mathcal{E} \to S \circ T(\mathcal{E})$ is a quasi-isomorphism (in the sense of Definition 3.6) for any object $\mathcal{E} \in \text{Tw}(X, \mathcal{C}_A(U_i))$. The same results applies to $T : \text{Tw}(X, \mathcal{C}_{A^0}(U_i)) \rightleftarrows \text{Tw}(X, \mathcal{C}_{A^0}(V_j)) : S$.

Proof It is a routine check. \qed

The goal of this paper is to prove that $T : \mathcal{P}_A(X) \to \text{Tw}(X, \mathcal{P}_A(U_i))$ is a quasi-equivalence of dg-categories.

4 The gluing of underlying complexes

In this section we study in more detail the adjunction

$$T : \mathcal{C}_{A^0}(X) \rightleftarrows \text{Tw}(X, \mathcal{C}_{A^0}(U_i)) : S.$$
Proof The proof is essentially the same as the proof of [8, Lemma 7.5].

Proposition 4.3 Let $X$ be a $C^\infty$-manifold with $A^0$ the sheaf of $C^\infty$-functions. Let $\{U_i\}$ be a locally finite open cover of $X$. Then for every globally bounded twisted complex $F = (F^\bullet, b) \in \text{Tw}^B(X, \mathcal{P}_{A^0}, U_i)$, there is an object $E = (E^\bullet, d) \in \mathcal{P}_{A^0(X)}$ together with a quasi-isomorphism $\phi : E \to S \mathcal{F}$ in $\mathcal{P}_{A^0(X)}$.

Moreover, the corresponding morphism $\eta_F \circ \mathcal{T}(\phi) : \mathcal{T}(E) \to \mathcal{F}$ is a homotopy equivalence in $\text{Tw}(X, \mathcal{P}_{A^0}, U_i)$, where $\eta_F : \mathcal{T} \circ S \mathcal{F} \to \mathcal{F}$ is the counit morphism.

Proof By Lemma 4.1, $S\mathcal{F}$ is a globally bounded perfect complex of $A^0$-modules on $X$. Then by Proposition 4.2, there exists a bounded complex of finitely generated locally free $A^0$-modules $E$ together with a quasi-isomorphism of complexes of $A^0$-modules $\phi : E \to S\mathcal{F}$. This proves the first part of the proposition.

We can then consider $\mathcal{T}(\phi) : \mathcal{T}(E) \to \mathcal{T}S\mathcal{F}$. By definition $\mathcal{T}(\phi)$ is a quasi-isomorphism as in Definition 3.6. By [18, Proposition 3.13], $\eta_F : \mathcal{T}S\mathcal{F} \to \mathcal{F}$ is also a quasi-isomorphism, so is the composition $\eta_F \circ \mathcal{T}(\phi) : \mathcal{T}(E) \to \mathcal{F}$. Since $\mathcal{T}(E)$ and $\mathcal{F}$ are both in $\mathcal{P}_{A^0(X)}$, by Proposition 3.9, $\eta_F \circ \mathcal{T}(\phi)$ is a homotopy equivalence.

5 The descent of cohesive modules

We first recall the following theorem on objects in $\mathcal{P}_A$.

Theorem 5.1 ([4, Theorem 3.13]) Let $A = (A^\bullet, d, c)$ be a curved dg-algebra and $Q = (Q^\bullet, Q)$ be a quasi-cohesive module over $A$. Then there is an object $E$ in $\mathcal{P}_A$ together with a quasi-isomorphism $\phi : E \to Q$ in $\mathcal{C}_A$, under either of the following two conditions:

(i) $Q$ is a quasi-finite quasi-cohesive module (see [4, Definition 3.12]);
(ii) $A$ is flat over $A^0$ and there is a bounded complex $(E^\bullet, E^0)$ of finitely generated projective right $A^0$-modules and an $A^0$-linear quasi-isomorphism $\Theta^0 : (E^\bullet, E^0) \to (Q^\bullet, Q^0)$.

Proof See [4, Theorem 3.13].

Basically Theorem 5.1 says that, under mild conditions, we can lift quasi-isomorphisms between underline cochain complexes to quasi-isomorphisms between cohesive modules. We will combine Theorem 5.1 and Proposition 4.2 in the descent problem of cohesive modules.

Proposition 5.2 Let $X$ be a complex manifold and $\{U_i\}$ be a locally finite open cover, then for every globally bounded twisted complex $F = (F^\bullet, b) \in \text{Tw}^B(X, \mathcal{P}_A, U_i)$, there is an object $E \in \mathcal{P}_{A(X)}$ together with a quasi-isomorphism $\phi : E \to S\mathcal{F}$ in $\mathcal{C}_{A(X)}$.

Moreover, the corresponding morphism $\eta_F \circ \mathcal{T}(\phi) : \mathcal{T}(E) \to \mathcal{F}$ is a homotopy equivalence in $\text{Tw}^B(X, \mathcal{P}_A, U_i)$.

Proof All functors are compatible with the forgetful functor $\text{For} : \mathcal{P}_A \to \mathcal{P}_{A^0}$. By Proposition 4.3, for every twisted complex $F = (F^\bullet, b) \in \text{Tw}^B(X, \mathcal{P}_A, U_i)$, there is an object $E_0 \in \mathcal{P}_{A^0(X)}$ together with a quasi-isomorphism $\phi_0 : E_0 \to S(\text{For}(F)) = \text{For}(S\mathcal{F})$. 
It is clear that $A^\bullet(X)$ is flat over $A^0(X)$. Therefore condition (ii) in Theorem 5.1 is satisfied, hence by Theorem 5.1 there exists an object $E \in \mathcal{P}_{A(X)}$ together with a quasi-isomorphism $\phi : E \to S(J)$ in $\mathcal{C}_{A(X)}$ such that $\text{For}(E) = E_0$ and $\text{For}(\phi) = \phi_0$.

The second half of the claim comes from Proposition 4.3 and the compatibility of the forgetful functor. 

**Proposition 5.3** Let $E$ be an object in $\mathcal{P}_{A(X)}$. Apply Proposition 5.2 to $T(E)$, we get $\tilde{E}$ in $\mathcal{P}_{A(X)}$ together with a quasi-isomorphism $\phi : \tilde{E} \to S(T(E))$ in $\mathcal{C}_{A(X)}$. Then $E$ and $\tilde{E}$ are homotopy equivalent in $\mathcal{P}_{A(X)}$.

**Proof** By Proposition 3.18, the unit morphism $\epsilon : E \to S(T(E))$ is a quasi-isomorphism. Then by Proposition 2.6, $\phi : \tilde{E} \to S(T(E))$ can be lifted to $\psi : \tilde{E} \to E$ so that $\epsilon \circ \psi \simeq \phi$ up to homotopy. Since both $\epsilon$ and $\phi$ are quasi-isomorphisms, so is $\psi$. Since both $\tilde{E}$ and $E$ are in $\mathcal{P}_{A(X)}$, by Proposition 2.4, $\psi$ is a homotopy equivalence. 

We will use the following lemma on dg-categories in the proof.

**Lemma 5.4** ([2, Lemma 2.3]) Suppose that $\mathcal{C}$ and $\mathcal{D}$ are dg-categories, which are full dg-subcategories of dg-categories $\mathcal{C}_{\text{big}}$ and $\mathcal{D}_{\text{big}}$, respectively. Let $F$ be a dg-functor from $\mathcal{C}_{\text{big}}$ to $\mathcal{D}_{\text{big}}$ which carries $\mathcal{C}$ into $\mathcal{D}$. Let $G$ be a dg-functor from $\mathcal{D}_{\text{big}}$ to $\mathcal{C}_{\text{big}}$ which is right adjoint to $F$. Suppose that $F$ and $G$ satisfy the following conditions:

(i) For each $d \in \mathcal{D}$ we have an object $c_d \in \mathcal{C}$ and a quasi-isomorphism $h_{c_d} \to \tilde{h}_{G(d)}$ in $\text{Mod-}\mathcal{C}$, where $h$ and $\tilde{h}$ are Yoneda embeddings;

(ii) For each $d \in \mathcal{D}$, let $c_d$ be as above and $\Theta_d \in \mathcal{C}_{\text{big}}(c_d, G(d))$ correspond to the identity of $c_d$ under $h_{c_d}(c_d) \to h_{G(d)}(c_d)$. Then the morphism

$$\Lambda_d := \eta_d \circ F(\Theta_d) : F(c_d) \to d$$

is a homotopy equivalence in the dg-category $\mathcal{D}$, where $\eta : F \circ G \to \text{id}_\mathcal{D}$ is the counit of the adjunction;

(iii) For each $c \in \mathcal{C}$, $c$ and $c_{F(c)}$ are homotopy equivalent in $\mathcal{C}$.

Then $F|_{\mathcal{C}}$ is a dg-quasi-equivalence from $\mathcal{C}$ to $\mathcal{D}$.

**Proof** See [2, Lemma 2.3]. 

Now we are ready to prove the main theorem of this paper.

**Theorem 5.5** Let $X$ be a complex manifold and $\{U_i\}$ be a locally finite open cover of $X$. Let $A = (A^0, \bullet, \overline{\partial}, 0)$ be the Dolbeault dg-algebra on $X$ and $\mathcal{P}_{A}$ be the dg-category of cohesive modules. Let $\text{Tw}^B(X, \mathcal{P}_{A}, U_i)$ be the dg-category of globally bounded twisted complexes on $X$. Then the twisting functor

$$\mathcal{T} : \mathcal{P}_{A(X)} \to \text{Tw}^B(X, \mathcal{P}_{A}, U_i)$$

is a dg-quasi-equivalence of dg-categories.

In particular, if $\{U_i\}$ be a finite open cover of $X$, then the twisting functor

$$\mathcal{T} : \mathcal{P}_{A(X)} \to \text{Tw}(X, \mathcal{P}_{A}, U_i)$$

is a dg-quasi-equivalence of dg-categories.
Proof We want to apply Lemma 5.4 to the adjunction

$$\mathcal{J}: \mathcal{C}_{A(X)} \rightleftarrows \text{Tw}^B(X, \mathcal{C}_A, U_i) : S.$$ 

In this case $\mathcal{C} = P_{A(X)}, \mathcal{C}_{\text{big}} = \mathcal{C}_{A(X)}, \mathcal{D} = \text{Tw}(X, P_{A}, U_i), \mathcal{D}_{\text{big}} = \text{Tw}(X, \mathcal{C}_A, U_i)$.

Condition (i) of Lemma 5.4 is obtained by the first assertion of Propositions 5.2 and 2.6. Condition (ii) is the second assertion of Proposition 5.2, and condition (iii) is given by Proposition 5.3. Therefore Lemma 5.4 tells us that

$$\mathcal{J}: P_{A(X)} \to \text{Tw}(X, P_{A}, U_i)$$

is a dg-quasi-equivalence of dg-categories. □

Remark 5.6 It is interesting to generalize the result in Theorem 5.5 to hypercovers as defined in [9], which is important because it has been shown that the suitable sheaf condition on dg-presheaves is that they satisfy descent for hypercovers instead of merely covers. See [11, Lemma 6]. However, the author believes that we should leave this problem to a future project.

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