Fast fault-tolerant filtering of quantum codewords

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The stabilization of a quantum computer by repeated error correction can be reduced almost entirely to repeated preparation of blocks of qubits in quantum codeword states. These are multiparticle entangled states with a high degree of symmetry. The required accuracy can be achieved by measuring parity checks, using imperfect apparatus, and rejecting states which fail them. This filtering process is considered for \( t \)-error-correcting codes with \( t > 1 \). It is shown how to exploit the structure of the codeword and the check matrix, so that the filter is reduced to a minimal form where each parity check need only be measured once, not \( t \) times by the (noisy) verification apparatus. This both raises the noise threshold and also reduces the physical size of the computer. A method based on latin rectangles is proposed, which enables the most parallel version of a logic gate network to be found, for a class of networks including those used in verification. These insights allowed the noise threshold to be increased by an order of magnitude.

The concept of quantum computing has given fundamental insights into the laws of Nature and promises powerful new computing capability, beyond the range of any other type of computer, if it can be realized in practice \[1,2,3\]. A central consideration in both these aspects is the intrinsic sensitivity of quantum processes to the inevitable imperfections in physical devices. There exist protocols based on quantum error correction (QEC) \[4,5,6,7\] which allow successful quantum computing in the presence of low-level noise throughout the computer at all times—this is called fault-tolerance. A set of ideas which allow QEC to be fault-tolerant were put forward by Shor \[8,9,15\]. There followed further insights which generalized or improved the speed and space-efficiency of the methods \[10,11,12,13,14\].

The noise level which can be tolerated in QEC has been estimated by analysis of the networks of operations involved \[15,16,17\]. In \[18,19\] the present author analysed QEC networks which included several further improvements in efficiency, and found that the tolerated noise could be considerably higher than was previously possible. This letter presents the basic insights which generated these efficiency improvements.

This work is significant not only to the practical task of building a quantum computer, but also to two other areas. First, it contributes to our understanding of the thermodynamics of controlled entangled systems. This subject has only been explored a little up till now, but has revealed some striking behaviour; for example fault-tolerant (FT) methods lead to a phase transition in a set of qubits stabilized by QEC, in which the order parameter is related to the size of clusters of qubits whose entanglement is preserved at finite temperature \[10,21\]. The present work extends the range of possible transition temperatures. Secondly, basic insights into quantum network constructions improve our understanding of quantum processing methods in general, extending their use both in quantum algorithms and in realizing physical effects which exploit entanglement for other purposes.

The main result of FT QEC is to achieve a logical error rate of \( O(\epsilon^{t+1}) \) per logical operation followed by recovery, where \( \epsilon \) is the imprecision or noise of the elementary operations on physical qubits or per time-step for resting qubits, when a \( t \)-error correcting quantum code is employed. In order to do this, a combination of well-chosen code structure, network construction, and repetition is employed. In this letter I exploit the first two ingredients so as to avoid the third. I also show how to arrange the relevant networks using a method based on latin rectangles so that they are as parallel as possible and hence require a minimal number of time-steps.

This study assumes the most efficient way to achieve FT QEC known to the author, as follows. Ancillary qubits are prepared in quantum codeword states (e.g. the encoded logical zero state \( |0\rangle_L \)), then they are coupled them to the ‘data’ qubits which store the (encoded) logical information, then the ancilla are measured and the classical information obtained is decoded by classical processing in order to deduce the corrective operation to be applied to the data \[11\]. This method works for all CSS codes; these are important because they include good codes and allow relatively simple, and therefore robust, FT gate constructions \[8,12,13,14\]. The method relies on the following concepts. First, the properties of the en-
coding lead to the correct movement of error information from the data qubits to the measurement apparatus applied to the ancilla, without extracting any of the quantum information stored in the data. Secondly, the ancilla needs only to be checked for one type of error (either $\sigma_x \equiv X$ or $\sigma_z \equiv Z$; I assume $X$ here), because only one type propagates from ancilla to data when the two are coupled (when the desired propagation of $Z$ errors from data to ancilla takes place, then $X$ errors propagate in the other direction, and vice versa). The ancilla errors which are not detectable by this verification remain in the ancilla and render the deduced information concerning data errors (i.e. the syndrome) unreliable; the third ingredient is to use several independently prepared ancillas to extract the same syndrome, and take a majority vote. These methods are typically analysed under the assumption that different gates in the network, and qubits at different positions or times, fail independently with probability $\epsilon$. The degree to which this assumption can be relaxed without significantly affecting the results is an area of active investigation [16, 18, 21, 22].

Let the processes which cause imperfection be called ‘failures’ and the resulting imperfections in the state of the qubits be called ‘errors’. Each error $e$ is a tensor product of Pauli operators. Let $P_e(\epsilon)$ be the probability that the ancilla’s state differs from the desired state by $e$. We define the error to be uncorrelated if $P_e$ satisfies

$$P_e(\epsilon) = a_w \epsilon^s \mathrel{|} s \geq w_c \text{ for } w_c \leq t; s > t \text{ for } w_c > t,$$

and we define $e$ to be correlated otherwise. Here $w_c$ is the weight of $e$ (the number of qubits it affects). The coefficients $a_w$ (which depend on the code and the networks) should not be unreasonably large.

The central resource in the QEC protocol is an ancilla prepared in an approximation to $|0\rangle_L$ having uncorrelated $X$ errors. If this is available, then after coupling it to the data (in order to obtain the syndrome of the data) the probability of finding uncorrectable errors in the data scales in the right way, and the main result of FT QEC theory applies.

Ancilla verification is achieved by measuring all those observables $M$ in the stabilizer of $|0\rangle_L$ which anticommute with $X$ errors (and therefore whose measured eigenvalues reveal the presence of $X$ errors). The problem is that a failure of $w$ of these measurements might allow an $X$ error of weight $w$ to go undetected (see figure 1a). This was avoided in Shor’s and subsequent work by repeating the measurements $t+1$ or more times. This solution is costly in noise tolerance or computer size or both because for a large computation $t \gg 1$.

I will now show how to avoid the repetition in the case of an arbitrary CSS code.

The ancilla is to be placed in the codeword state $|0\rangle_L = \sum_{u \in C} |u \rangle$ where $C$ is the classical code on which the CSS code is based. When an approximation to $|0\rangle_L$ is prepared by any means, there are typically many locations in the preparation where a single failure results in a high-weight error in the output state; I assume the worst case that the preparation leaves an error of any non-zero weight with probability proportional to $\epsilon$. The subsequent measurements of observables $M$ in the $Z$ part of the stabilizer (equivalently, of parity checks which $|0\rangle_L$ ought to satisfy) must satisfy two conditions: (1) no correlated $X$ errors are introduced and (2) all correlated $X$ errors are detected with high enough probability: the probability that an error of weight $w$ goes undetected must scale as $\epsilon^{w^2(w-1)}$. Condition (1) is guaranteed by the fact that $X$ and $Z$ errors propagate differently, so that a network of controlled-gates to store $X$ parity information into one verification bit will cause $Z$- but not $X$-error-propagation around the ancilla bits. I will prove that a (properly constructed) noisy network also satisfies (2), where the network requires only a single measurement of each check, and a single physical qubit to accumulate each logical check bit, therefore it is minimal and QEC is rapid.

We require that failures of low weight in the verification network only cause ancilla errors of low weight to be ‘missed’. The non-trivial situation is when at least one failure occurs in the preparation and at least one in the verification; the case $t = 1$ (single-error-correcting code) is trivial since this is already a second-order process; this is why the question has not arisen in discussions of the $[[7,1,3]]$ code and its concatenations.

Consider the syndrome given by a complete set of all the checks in $H$, where each check is measured by preparing a single verification qubit in $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, operating $CZ$ gates from verification bit to ancilla at locations given by a row of $H$, then measuring the verification bit in the $|+\rangle , |−\rangle$ basis. A sufficient condition for CSS codes to achieve the desired behaviour is that every syndrome of weight $w_s \leq t$ can be produced by an error of weight $w_c \leq w_s$. If this holds then failure of $w = w_s$ gates in the verifier, allowing the error to pass unnoticed, will result in an ancilla error of weight at most $w$. This is true whether the error was produced in the preparation stage or the veri-
We have now arranged that each parity check need only be measured once when an ancilla is verified. This shows that even a minimal QEC network, in the sense of one performing minimal filtering of ancillas, can be fault-tolerant. This reduction from \( t + 1 \) repetitions to 1 permits both a large reduction in the size of the computer and also an increase in the tolerated memory noise, since with repeated verification \( O(t + 1) \) more ancillas would have to be prepared in parallel in order to keep the overall recovery rate up, and each ancilla would also have to survive \( O(t + 1) \) times longer before it can be used. A large computation requires \( t \) in the range 7 (e.g. \([127, 29, 15] \) BCH code) to 15 (e.g. \([7, 1, 3] \) code concatenated twice).

Repetition is still used to extract several copies of the syndrome, to guard against correcting the computer on the basis of a wrong syndrome. Next I reduce this also. Suppose the data has an error \( e \) whose true syndrome is \( s \). If the syndrome were extracted bit by bit, as in \([3, 1, 1] \), then a single failure can result in a single error \( g \) in the syndrome. If \( s + g \) is accepted the ‘correction’ of the data will leave the error \( e + f \) in the data, where \( f \) is the coset leader of the erroneous syndrome \( s + g \). \( f \) is in general unrelated to \( e \), therefore the final state of the computer is liable to contain an error of weight \( \text{wt}(e) + \text{wt}(f) \) with probability \( \sim e^{\text{wt}(e)+1} \); the QEC soon breaks down since \( \text{wt}(f) \) can be greater than 1. However, in the method under discussion, the syndrome is extracted indirectly by allowing the error \( e \) to propagate to the ancilla and then measuring the ancilla. A failure which gives a single error \( g \) in the ancilla merely changes the error in the ancilla to \( e + g \). Assuming this is correctable, the interpretation of the ancilla measurement outcome identifies \( e + g \) as the error to be corrected in the data; the final situation is then to leave the data with error \( g \) with probability \( \sim e^{\text{wt}(e)+1} \), which is merely a small addition to the probability of single errors in the data, so is harmless. There remains a small need for syndrome extraction repetition to guard against the comparatively few failure locations that give larger weight \( Z \) errors in the ancilla.

Next I will show how to compress the time required by the preparation and verification (\( H \)) networks. I already noted that, in order to be fault-tolerant, the \( H \) network is constructed for \( H \) in the standard form \((A, I)\). This has the additional advantage of almost minimizing the number of 1’s in the matrix and hence the number of gates in the verification network. The network of \( CZ \) gates between verification bits and ancilla bits can be parallelized to the degree that any group of \( CZ \) gates involving different pairs of bits can take place simultane-

![FIG. 1: An example of the verification construction. Both networks show a complete set of parity check measurements to detect bit-errors in the codeword \(|00000\rangle + |11111\rangle\). (a) poorly constructed network; in several places a single failure would allow a two-bit error to go undetected, therefore repetition is needed. (b) well-constructed network: for any input ancilla state, at the output the ancilla has uncorrelated \( X \) errors (and may have correlated \( Z \) errors) whenever the check measurements all give zero.](image)
ously. The problem of minimizing the number of time-steps is equivalent to the problem of forming a latin rectangle the size of \( A \), that is \( r \times (n - r) \), using an alphabet of minimal size, where the places where \( A \) has a zero need not be filled. Suppose we form such a rectangle using integers from 1 to \( N \), then each integer gives the time-step in which the ancilla qubit of that column is coupled to the verification bit for that row: the fact that no symbol appears twice on a column guarantees that no ancilla bit is involved in more than one gate at once; the fact that no symbol appears twice on a row guarantees that no verifier bit is involved in more than one gate at once. An example is given in figure 2. It is clear that \( N \geq w_{\text{max}} \) where \( w_{\text{max}} \) is the largest weight of a row or column of \( A \), and it can be shown from Hall’s theorem in combinatorics that a latin rectangle exists for this smallest possible \( N \) [24]. The verification network is completed by a single \( CZ \) from the verification bits to the last \( r \) ancilla bits; these can be simultaneous, so the total number of time-steps for verification is \( N + 1 + T_m \) where \( T_m \) is the time required for measurement of a set of qubits.

![Latin Rectangle Example](image)

FIG. 2: Construction of the \( G \) and \( H \) networks, illustrated for the example of a \([21,3,5]\) code. The \( A \) matrix is shown, with blanks where zeros appear, and the other locations numbered in a latin rectangle. The numbers indicate the time step in which each controlled-gate is applied.

The method to prepare the \( |0\rangle_L \) state need not involve networks of gates, but if a network is used then the same analysis shows that the \( G \) network can be accomplished in \( N \) time-steps using \( CX \) gates (on bits prepared in \( |0\rangle \) or \(|+\rangle \)).

To conclude, all CSS codes allow a network to prepare verified ancillas which is both fault-tolerant and minimal; such networks can also be compressed in time by a general procedure based on latin squares. The results described have the common theme of using structure in the design of the QEC network to serve to enhance its ability to extract entropy from the computer. In information theoretic terms, the structure is a form of negative entropy in the ancillas, which allows them to absorb more entropy from the data. The practical result is that fewer checking operations and timesteps are needed to run the QEC protocol, so that both a saving in computer size and an increase in memory noise tolerated is obtained. The saving is by a factor of the order of \( t \), the number of errors correctable by the code, which is in the range approximately 7 to 15 for a large quantum computation.

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[23] This is an example of degeneracy in quantum coding.
[24] I am indebted to a referee for pointing this out.