Minimal Volume Product of Three Dimensional Convex Bodies with Various Discrete Symmetries

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Abstract
We give the sharp lower bound of the volume product of three dimensional convex bodies which are invariant under a discrete subgroup of $O(3)$ in several cases. We also characterize the convex bodies with the minimal volume product in each case. In particular, this provides a new partial result of the non-symmetric version of Mahler’s conjecture in the three dimensional case.

Keywords Convex body · Volume product · Minimizing problem · Mahler conjecture

Mathematics Subject Classification 52A40 · 52A38

1 Introduction and Main Results
1.1 Mahler’s Conjecture and Its Generalization
Let $K$ be a convex body in $\mathbb{R}^n$, i.e., $K$ is a compact convex set in $\mathbb{R}^n$ with nonempty interior. Then $K \subset \mathbb{R}^n$ is said to be centrally symmetric if it satisfies that $K = -K$. Denote by $\mathcal{K}^n$ the set of all convex bodies in $\mathbb{R}^n$ equipped with the Hausdorff metric and by $\mathcal{K}^n_0$ the set of all $K \in \mathcal{K}^n$ which are centrally symmetric. The interior of $K \in \mathcal{K}^n$ is denoted by $\text{int } K$. For a point $z \in \text{int } K$, the polar body of $K$ with respect to $z$ is
defined by

\[ K^z = \{ y \in \mathbb{R}^n : (y - z) \cdot (x - z) \leq 1 \text{ for any } x \in K \}, \]

where \( \cdot \) denotes the standard inner product on \( \mathbb{R}^n \). Then the affine invariant quantity

\[ P(K) := \min_{z \in \text{int} K} \frac{|K| |K^z|}{|K^0|} \quad (1) \]

is called volume product of \( K \), where \( |K| \) denotes the \( n \)-dimensional volume of \( K \) in \( \mathbb{R}^n \). It is known that for each \( K \in \mathcal{K}_n^0 \) the minimum of (1) is attained at the unique point \( z \) on \( K \), which is called the Santaló point of \( K \) (see, e.g., [13]). For a centrally symmetric convex body \( K \in \mathcal{K}_n^0 \), the Santaló point of \( K \) is the origin \( o \). See [17, p.546] for further informations. In the following, the polar of \( K \) with respect to \( o \) is denoted by \( K^\circ \).

Mahler’s conjecture [10] states that for any \( K \in \mathcal{K}_n^0 \),

\[ P(K) \geq \frac{4^n}{n!} \quad (2) \]

should hold. He proved it in the case where \( n = 2 \) [9]. The three dimensional case was recently proved in [7] (see also [5] for a nice simple proof of an equipartition result used in [7]). Although the case that \( n \geq 4 \) is still open, \( n \)-cube or, more generally, Hanner polytopes satisfy the equality in (2) and are predicted as the minimizers of \( P \).

As for non-symmetric bodies there is another well-known conjecture for the lower bound of the volume product as follows.

**Conjecture** Any \( K \in \mathcal{K}_n \) satisfies that

\[ P(K) \geq \frac{(n + 1)^{n+1}}{(n!)^2} \]

with equality if and only if \( K \) is a simplex.

This was proved by Mahler for \( n = 2 \) [9] (see also [12]). Note that this conjecture remains open for \( n \geq 3 \), see e.g., [1,6,8].

On the other hand, Barthe and Fradelizi [1] obtained the sharp lower bound of \( P(K) \) when \( K \) has many reflection symmetries, in other words, when \( K \) is invariant under the action of a Coxeter group. It is worthwhile to estimate the volume product of the \( G \)-invariant convex body \( K \subset \mathbb{R}^n \) from below for more general discrete subgroups \( G \) of the orthogonal group \( O(n) \).

**Problem** Let \( G \) be a discrete subgroup of \( O(n) \). Denote by \( \mathcal{K}_n(G) \) the set of all convex bodies \( K \in \mathcal{K}_n \) which satisfy that \( K = g(K) \) for any \( g \in G \). Then, consider the minimizing problem

\[ \min_{K \in \mathcal{K}_n(G)} P(K) \]

and determine all the minimizers \( K \in \mathcal{K}_n(G) \).
Remark

• From this viewpoint, Mahler’s conjecture corresponds to the case $G = \{ E, -E \} \cong \mathbb{Z}_2$ ($E$ is the identity matrix) and the non-symmetric case corresponds to $G = \{ E \}$.

• Following [1] for a subset $A \subset \mathbb{R}^n$ we recall the subgroup of $O(n)$ defined by

$$O(A) = \{ g \in O(n) : g(A) = A \}.$$  

If $P \subset \mathbb{R}^n$ is a regular polytope with $o$ as the centroid, then $O(P)$ is a discrete subgroup of $O(n)$. The above problem was solved for $G = O(P)$ for every regular polytope $P \subset \mathbb{R}^n$ [1, Thm. 1 (i)].

In this paper, we shall focus on the three dimensional case of the above problem and consider $G$-invariant convex bodies $K \subset \mathbb{R}^3$ for a discrete subgroup $G$ of $O(3)$. Note that the classification of the discrete subgroups of $O(3)$ is well known. Before we explain the details, we recall the two dimensional case in which the problem has already been solved.

1.2 The Two Dimensional Results

For $\ell \in \mathbb{N}$, we put $\xi = 2\pi / \ell$ and

$$R_\ell := \begin{pmatrix} \cos \xi & -\sin \xi \\ \sin \xi & \cos \xi \end{pmatrix}, \quad V := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Up to conjugation, all the discrete subgroups of $O(2)$ are

$$C_\ell = \langle R_\ell \rangle \quad \text{and} \quad D_\ell = \langle R_\ell, V \rangle, \quad \ell \in \mathbb{N}.$$  

Note that $K^2(C_1) = K^2$ and $K^2(C_2) = K^2_0$. We denote by $\triangle_\ell$ the regular $\ell$-gon with $o$ as the centroid. For simplicity, we also denote $\triangle := \triangle_3$ and $\square := \triangle_4$.

The following is a summary of the known results.

Theorem 1.1 ([1,2,9])

(i) $\mathcal{P}(K) \geq \mathcal{P}(\triangle)$ holds for $K \in K^2(C_1)$ with equality if and only if $K$ is the image of $\triangle$ by a non-singular affine transformation of $\mathbb{R}^2$. The same inequality holds for $K \in K^2(D_1)$.

(ii) $\mathcal{P}(K) \geq \mathcal{P}(\square)$ holds for $K \in K^2(C_2)$ with equality if and only if $K$ is the image of $\square$ by a non-singular linear transformation of $\mathbb{R}^2$. The same inequality holds for $K \in K^2(D_2)$.

(iii) Assume that $\ell \geq 3$. Then $\mathcal{P}(K) \geq \mathcal{P}(\triangle_\ell)$ holds for $K \in K^2(C_\ell)$ with equality if and only if $K$ is a dilation or rotation of $\triangle_\ell$. The same inequality holds for $K \in K^2(D_\ell)$.  

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1.3 The Three Dimensional Results

We shall state known results and the main theorem in the case where \( n = 3 \). Let us first recall the list of all discrete subgroups of \( O(3) \) by using Schoenflies’ notation. We equip \( \mathbb{R}^3 \) with the usual orthogonal \( xyz \)-axes and put

\[
R_\ell := \begin{pmatrix}
\cos \xi & -\sin \xi & 0 \\
\sin \xi & \cos \xi & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad V := \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad H := \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix},
\]

where \( \ell \in \mathbb{N}, \xi := 2\pi/\ell \). Here \( R_\ell, V, \) and \( H \) mean the rotation through the angle \( 2\pi/\ell \) about the \( z \)-axis, the reflection with respect to the \( zx \)-plane, and the reflection with respect to the \( xy \)-plane, respectively. It is known that, up to conjugation, the discrete subgroups of \( O(3) \) are classified as the seven infinite families

\[
C_\ell := \langle R_\ell \rangle, \quad C_{th} := \langle R_\ell, H \rangle, \quad C_{tv} := \langle R_\ell, V \rangle, \quad S_2 \ell := \langle R_{2\ell}H \rangle, \\
D_\ell := \langle R_\ell, VH \rangle, \quad D_{td} := \langle R_{2\ell}H, V \rangle, \quad D_{th} := \langle R_\ell, V, H \rangle,
\]

and the following seven finite groups (see e.g., [3, Table 3.1]):

\[
T := \{ g \in SO(3) : g\Delta = \Delta \}, \quad T_d := \{ g \in O(3) : g\Delta = \Delta \}, \quad T_h := \{ \pm g : g \in T \}, \\
O := \{ g \in SO(3) : g\Diamond = \Diamond \}, \quad O_h := \{ g \in O(3) : g\Diamond = \Diamond \} = \{ \pm g : g \in O \}, \\
I := \{ g \in SO(3) : g\bigcirc = \bigcirc \}, \quad I_h := \{ g \in O(3) : g\bigcirc = \bigcirc \} = \{ \pm g : g \in I \},
\]

where \( \Delta = \Delta^3, \Diamond = \Diamond^3, \) and \( \bigcirc = \bigcirc^3 \) denote the regular tetrahedron (simplex), the regular octahedron, and the regular icosahedron with \( o \) as their centroids, respectively. Note that, for example, if we fix the configuration of \( \Delta \), then \( T_d \) is uniquely determined as the subgroup of \( O(3) \) without any ambiguity of conjugations. In other words, any discrete subgroup conjugated to \( T_d \) in \( O(3) \) is realized as \( \{ g \in O(3) : g\Delta’ = \Delta’ \} \), where \( \Delta’ = k\Delta \) for some \( k \in O(3) \). Using the notation of [1], \( T_d = O(\Delta) \cong O(\Delta’) \).

We also note that among the above classification \( C_\ell, D_\ell, T, O, I \) are indeed subgroups of \( SO(3) \), and that the Santaló point of \( K \in K^3(G) \) is the origin \( o \) except only for the two subgroups \( G = C_\ell \) or \( C_{tv}, \ell \in \mathbb{N} \).

Now we recall the known results concerning the three dimensional case. A convex body \( K \in K^3(D_{2h}) \) is 1-unconditional and \( D_{2h} \cong (\mathbb{Z}_2)^3 \) as groups.

**Theorem 1.2** ([11,15]) For \( K \in K^3(D_{2h}) \), we have

\[
\mathcal{P}(K) \geq \mathcal{P}(\Diamond) = \frac{32}{3}.
\]

The equality holds if and only if \( K \) is a three dimensional Hanner polytope [11,14], i.e., \( K \) is the image of the regular octahedron \( \Diamond \) or the cube \( \Diamond^o \) by a linear transformation of \( \mathbb{R}^3 \) by a diagonal matrix.
Theorem 1.3 ([1, Thm. 1])

(i) $P(K) \geq P(\triangle)$ holds for $K \in \mathcal{K}^3(T_d)$. The equality holds if and only if $K$ is a dilation of the regular tetrahedron $\triangle$ or $\triangle^\circ$.

(ii) $P(K) \geq P(\Diamond)$ holds for $K \in \mathcal{K}^3(O_h)$. The equality holds if and only if $K$ is a dilation of the regular octahedron $\Diamond$ or the cube $\Diamond^\circ$.

(iii) $P(K) \geq P(\bigcirc)$ holds for $K \in \mathcal{K}^3(I_h)$. The equality holds if and only if $K$ is a dilation of the regular icosahedron $\bigcirc$ or the regular dodecahedron $\bigcirc^\circ$.

(iv) Assume that $\ell \geq 3$. Then $P(K) \geq P(P_\ell)$ holds for $K \in \mathcal{K}^3(D_{\ell h})$, where $P_\ell$ is the $\ell$-regular right prism defined by

$$P_\ell := \text{conv}\left\{ \left( \begin{array}{c} \cos k\xi \\ \sin k\xi \\ 1 \end{array} \right), \left( \begin{array}{c} \cos k\xi \\ \sin k\xi \\ -1 \end{array} \right) : k = 0, \ldots, l - 1 \right\}.$$ 

Here, we denote by $\text{conv } S$ the convex hull of a set $S$.

Furthermore, the solution of Mahler’s conjecture for $n = 3$ [7, Thm. 1] corresponds to the case $G = S_2(\cong \mathbb{Z}_2)$, which immediately implies

Corollary 1.4 Assume that $G = C_{2h}, T_h, S_6, D_{3d},$ or $S_2$. For $K \in \mathcal{K}^3(G)$, we have

$$P(K) \geq P(\bigcirc) = \frac{32}{3}$$

with equality if and only if $K$ is a linear image of $\Diamond$ or $\Diamond^\circ$ which is invariant under the group $G$.

The following theorem is the main result of this paper.

Theorem 1.5 (i) $P(K) \geq P(\triangle)$ holds for $K \in \mathcal{K}^3(T)$. The equality holds if and only if $K$ is a dilation of the regular tetrahedron $\triangle$ or $\triangle^\circ$.

(ii) $P(K) \geq P(\Diamond)$ holds for $K \in \mathcal{K}^3(O)$. The equality holds if and only if $K$ is a dilation of the regular octahedron $\Diamond$ or the cube $\Diamond^\circ$.

(iii) $P(K) \geq P(\bigcirc)$ holds for $K \in \mathcal{K}^3(I)$. The equality holds if and only if $K$ is a dilation of the regular icosahedron $\bigcirc$ or the regular dodecahedron $\bigcirc^\circ$.

(iv) Assume that $\ell \geq 3$. Then $P(K) \geq P(P_\ell)$ holds for $K \in \mathcal{K}^3(C_{\ell h})$, where $P_\ell$ is the $\ell$-regular right prism defined in Theorem 1.3. The equality holds if and only if $K$ coincides with $P_\ell$ or $P_\ell^\circ$ up to a linear transformation in the abelian subgroup of $GL(3, \mathbb{R})$ defined by

$$G := \left\{ \left( \begin{array}{ccc} a \cos \theta & -a \sin \theta & 0 \\ a \sin \theta & a \cos \theta & 0 \\ 0 & 0 & b \end{array} \right) : a, b > 0 \right\}.$$ 

The same inequality holds for $K \in \mathcal{K}^3(D_{\ell})$. The equality holds if and only if $K$ coincides with $P_\ell$ or $P_\ell^\circ$ up to a linear transformation in
$G' := \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} : a, b > 0, \theta \in \mathbb{R} \right\}$.

In particular, Theorem 1.5 (i) provides a new partial result of the non-symmetric version of Mahler’s conjecture in the three dimensional case (cf. Theorem 1.3 (i)). The results from Theorems 1.2 to 1.5 are summarized as follows.

**Remark 1.6**  
- At present, the authors cannot treat the following cases successfully by the method of this paper:
  
  $C_{\ell}, \ell \in \mathbb{N}; \quad C_{\ell v}, \ell \in \mathbb{N}; \quad S_{2n}, n \neq 1, 3; \quad D_2; \quad D_{\ell d}, \ell \neq 3$.

- Except for $C_{\ell}$ and $C_{\ell v}, \ell \in \mathbb{N}$, each discrete subgroup of $O(3)$ acting on $\mathbb{R}^3$ fixes only one point (the origin $o$).

### 1.4 Method of the Proof and Organization of This Paper

The proof of this paper’s results is based on a simple inequality [7, Prop. 3.2] that gives rise to an effective method to estimate the volume product $P(K)$ from below, which is a natural extension of the volume estimate by Meyer. He gave in [11] an elegant proof of the result by Saint-Raymond [15] (see Theorem 1.2 above for $n = 3$) for 1-unconditional bodies $K \subset \mathbb{R}^n$ as follows. An 1-unconditional body $K$ is determined by the part $K_1$ in the first octant:

$$K_1 := K \cap \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \geq 0, \ldots, x_n \geq 0\}.$$ 

We denote the intersection of $\partial K$ with each positive part of coordinate axes by $P_1, \ldots, P_n$, respectively. For any point $P$ in $K_1$, we can make the polytope which is the convex hull of vertices $o, P_1, \ldots, P_n$, and $P$. The volume of this polytope is less than or equal to that of $K_1$. This inequality yields a test point contained in the dual (truncated) cone of $K_1$. Interchanging the role of $K_1$ and the dual cone, we get another test point in $K_1$. By paring these two test points, we obtain the sharp lower bound estimate of $P(K)$ for 1-unconditional bodies $K$.

Meyer’s estimate for the above $K_1$ can be generalized to the case of a convex truncated cone and it was used in [1]. Actually, $n$-dimensional convex bodies with many hyperplane symmetries were treated. In the argument in [1], the hyperplane symmetries (reflections) are essential. For instance, for an 1-unconditional body $K$, the symmetry yields that the polar body $K^\circ$ is also 1-unconditional.

In general, $K \in K^3(G)$ has a fundamental domain $\hat{K}$ with respect to the discrete group action by $G$. Then $\hat{K}$ is a convex truncated cone. If $G \subset O(n)$ is not a Coxeter group, then the corresponding region $\hat{K}^\circ$ for the polar $K^\circ$ of $K$ is not necessarily convex. Nevertheless, by using the inequality [7, Prop. 3.2], in some cases it is possible to obtain the sharp estimate of $P(K)$ even for convex bodies $K$ without enough hyperplane symmetries. In this paper we focus on the three dimensional case.
| $\ell$ | 1   | 2   | 3   | 4   | 5   | 6   | ... |
|-------|-----|-----|-----|-----|-----|-----|-----|
| $C_\ell$ | $\{E\}$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ | ... |
| $C_{\ell v}$ | $C_{1v}$ | $C_{2v}$ | $C_{3v}$ | $C_{4v}$ | $C_{5v}$ | $C_{6v}$ | ... |
| $C_{\ell h}$ | $C_{1v}$ | $\bullet C_{2h}$ | $\circ C_{3h}$ | $\circ C_{4h}$ | $\circ C_{5h}$ | $\circ C_{6h}$ | ... |
| $S_\ell$ | $C_{1v}$ | $\bullet S_2$ | $C_{3h}$ | $S_4$ | $C_{5h}$ | $\bullet S_6$ | ... |
| $D_\ell$ | $C_2$ | $D_2$ | $\circ D_3$ | $\circ D_4$ | $\circ D_5$ | $\circ D_6$ | ... |
| $D_{\ell v}$ | $C_{2v}$ | $\frac{1}{2} D_{2h}$ | $\dagger D_{3h}$ | $\dagger D_{4h}$ | $\dagger D_{5h}$ | $\dagger D_{6h}$ | ... |
| $D_{\ell d}$ | $C_{2h}$ | $D_{2d}$ | $\bullet D_{3d}$ | $D_{4d}$ | $D_{5d}$ | $D_{6d}$ | ... |
| $D_{\ell t}$ | $\circ T$ | $\frac{1}{2} T_d$ | $\bullet T_h$ | $\circ O$ | $\frac{1}{2} O_h$ | $\circ I$ | $\frac{1}{2} I_h$ |

Bold: duplicates; $\dagger$ The results in [1]; $\frac{1}{2}$ The result in [15]; $\bullet$ Results deduced from [7] (the case where $S_2 \subset G$ and $\diamond$ is a minimizer); $\circ$ New results (Theorem 1.5)
This paper is organized as follows. In Sect. 2, we review necessary facts about two dimensional bodies. Though all facts here may be well known, we give short proofs for completeness. In Sect. 3, we give a detailed exposition of the “signed volume estimate” for three dimensional \( G \)-invariant convex bodies. The principal estimate is the inequality in Lemma 3.10, which is frequently used in the arguments in Sect. 4. In Sect. 4, we prove the main result (Theorem 1.5). We apply the estimates obtained in Sect. 3 to the convex bodies in \( K^3(G) \) for each discrete subgroup \( G \subset O(3) \). By means of the inequality in Lemma 3.10, the estimates are reduced to the two dimensional result prepared in Sects. 2 and 3. We also characterize the equality condition for each case in Sect. 5.

2 The Two Dimensional Case

In this section, we prepare necessary facts about two dimensional bodies. Although they are essentially proved in [2], we give proofs of them for the sake of completeness. The estimate of the volume product \( P(K) \) of a three dimensional convex body \( K \) is eventually reduced to that case. In what follows, we denote by

\[
o, a, b, p, a^o, \text{ etc.}
\]

the position vectors of points \( o, A, B, P, A^o, \text{ etc.} \) in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \), respectively. And \( a \parallel b \) means that the two vectors \( a \) and \( b \) are parallel. We denote the positive hull generated by points \( A_1, \ldots, A_m \in \mathbb{R}^n \) by

\[
\text{pos} (A_1, \ldots, A_m) = \{ t_1 a_1 + \cdots + t_m a_m \in \mathbb{R}^n : t_1, \ldots, t_m \geq 0 \}.
\]

2.1 An Estimate for Two Dimensional Convex Truncated Cones

The following formula estimates the volume product of the intersection of a convex body \( K \subset \mathbb{R}^2 \) and a positive hull and was first proved in the remark after [2, Lem. 7].

**Lemma 2.1** Let \( K \subset \mathbb{R}^2 \) be a convex body containing the origin in its interior. Assume that \( A, B \in \partial K \) with \( a \parallel b \) and \( A^o, B^o \in \partial K^o \) with \( a^o \parallel b^o \) satisfying \( a \cdot a^o = b \cdot b^o = 1 \). We put \( L := K \cap \text{pos} (A, B) \) and \( L^o := K^o \cap \text{pos} (A^o, B^o) \). Then we have \( |L| \cdot |L^o| \geq (a - b) \cdot (a^o - b^o)/4 \).
Proof We put
\[
\begin{align*}
  a &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, &
  b &= \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, &
  a^\circ &= \begin{pmatrix} a_1^\circ \\ a_2^\circ \end{pmatrix}, &
  b^\circ &= \begin{pmatrix} b_1^\circ \\ b_2^\circ \end{pmatrix}.
\end{align*}
\]
Since \( a \cdot a^\circ = b \cdot b^\circ = 1 \), we have \( a_1 a_1^\circ + a_2 a_2^\circ = 1 \) and \( b_1 b_1^\circ + b_2 b_2^\circ = 1 \). For any point \( P(x, y) \) in \( K \), since the sum of the signed area of the triangle \( OAP \) and that of \( OPB \) is less than or equal to \( |L| \), we have
\[
\frac{1}{2} \left| \begin{matrix} a_1 & a_2 \\ x & y \end{matrix} \right| + \frac{1}{2} \left| \begin{matrix} b_1 & b_2 \\ x & y \end{matrix} \right| \leq |L| \quad \text{for any } \left( \begin{matrix} x \\ y \end{matrix} \right) \in K,
\]
which means that
\[
\frac{1}{2|L|} \left( -a_2 + b_2 \right) \in K^\circ.
\]
Similarly, we obtain
\[
\frac{1}{2|L^\circ|} \left( -a_2^\circ + b_2^\circ \right) \in K.
\]
These test points yield that
\[
4|L| |L^\circ| \geq (a_1 - b_1)(a_1^\circ - b_1^\circ) + (a_2 - b_2)(a_2^\circ - b_2^\circ) = (a - b) \cdot (a^\circ - b^\circ). \quad \square
\]

Remark 2.2 We can easily check that the above two test points are contained in \( L^\circ \) and \( L \), respectively. The proof of Lemma 2.1 is closely related to the second proof of [2, Lem. 7].

2.2 The Case of Cyclically Symmetric Bodies

The following lemma was first proved independently in [2, Cor. 4] and [1, Cor. 5]. It is useful for, especially, the case of cyclically symmetric bodies in \( \mathbb{R}^2 \).

Lemma 2.3 Under the same assumptions in Lemma 2.1, assume that \( B = R(A) \) and \( B^\circ = R(A^\circ) \), where \( R \) denotes the rotation of angle \( \xi \in (0, \pi) \) around \( o \). Then
\[
|L| |L^\circ| \geq \frac{1 - \cos \xi}{2}
\]
holds.

Proof We may assume that
\[
\begin{align*}
  a &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, &
  b &= \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \cos \xi \\ \sin \xi \end{pmatrix}, \\
  a^\circ &= \begin{pmatrix} a_1^\circ \\ a_2^\circ \end{pmatrix} = \begin{pmatrix} 1 \\ a_2^\circ \end{pmatrix}, &
  b^\circ &= \begin{pmatrix} b_1^\circ \\ b_2^\circ \end{pmatrix} = \begin{pmatrix} \cos \xi - a_2^\circ \sin \xi \\ \sin \xi + a_2^\circ \cos \xi \end{pmatrix}.
\end{align*}
\]
Then we obtain $4|L| |L^\circ| \geq (1 - \cos \xi)(1 - \cos \xi + a_2^\circ \sin \xi) - \sin \xi(a_2^\circ - \sin \xi - a_2^\circ \cos \xi) = 2(1 - \cos \xi)$.

The following lemma provides the same equality condition as in [1, Cor. 3] or [2, Lem. 7].

**Lemma 2.4** Under the same assumptions in Lemma 2.3, if $a \parallel a^\circ$ and the equality of (3) holds, then either (i) or (ii) below holds.

(i) $L = \text{conv}\{o, A, C, B\}$, $L^\circ = \text{conv}\{o, A^\circ, B^\circ\}$, where $c = (a + b)/(1 + \cos \xi)$.

(ii) $L = \text{conv}\{o, A, B\}$, $L^\circ = \text{conv}\{o, A^\circ, C^\circ, B^\circ\}$, where $c^\circ = (a^\circ + b^\circ)/(1 + \cos \xi)$.

**Proof** Under the setting of Lemma 2.3, the assumption $a \parallel a^\circ$ implies that $a_2^\circ = 0$. Hence, we can take the test points in the proof of Lemma 2.1 as

$$c^\circ := \frac{1}{2|L|} \left( \frac{-a_2 + b_2}{a_1 - b_1} \right) = \frac{1}{2|L|} \left( \sin \xi \begin{pmatrix} 1 - \cos \xi \\ 1 - \cos \xi \end{pmatrix} \right) \in L^\circ,$$

$$c := \frac{1}{2|L^\circ|} \left( \frac{-a_2^\circ + b_2^\circ}{a_1^\circ - b_1^\circ} \right) = \frac{1}{2|L^\circ|} \left( \sin \xi \begin{pmatrix} 1 - \cos \xi \\ 1 - \cos \xi \end{pmatrix} \right) \in L$$

(see Remark 2.2). Then we have

$$|L| \geq \text{the area of the quadrilateral } oACB = \frac{1 - \cos \xi}{2|L^\circ|},$$

$$|L^\circ| \geq \text{the area of the quadrilateral } oA^\circ C^\circ B^\circ = \frac{1 - \cos \xi}{2|L|}.$$

Hence, the equality holds if and only if $L$ and $L^\circ$ coincide with the quadrilaterals $oACB$ and $oA^\circ C^\circ B^\circ$, respectively. Here note that there exists a constant $\alpha$ such that

$$c = \frac{1}{2|L^\circ|} \left( \begin{pmatrix} \sin \xi \\ 1 - \cos \xi \end{pmatrix} \right) = \frac{\sin(\xi/2)}{|L^\circ|} \left( \begin{pmatrix} \cos(\xi/2) \\ \sin(\xi/2) \end{pmatrix} \right) = \frac{\alpha a + b}{2}.$$

Since the quadrilateral $oACB$ is convex, we have $\alpha \geq 1$. Similarly, we obtain

$$c^\circ = \alpha^\circ \frac{a^\circ + b^\circ}{2}, \quad \alpha^\circ \geq 1.$$

**Case $\alpha = 1$:** In this case, $L$ coincides with the triangle $oAB$ and the dual face of the edge $AB$ is the vertex $C^\circ$. That is, $a \cdot c^\circ = 1(= b \cdot c^\circ)$ holds and

$$\alpha^\circ = \frac{2}{1 + \cos \xi}.$$

Therefore, we obtain

$$c^\circ = \frac{2}{1 + \cos \xi} \cdot \frac{a^\circ + b^\circ}{2} = \frac{a^\circ + b^\circ}{1 + \cos \xi},$$

which means that the condition (ii) holds.
Case $\alpha > 1$: Since $C$ is a vertex of $L$, its dual face is an edge of $L^\circ$. The edge contains the segment $A^\circ C^o$ or $C^o B^o$. From $c \cdot c^o = 1$, we have $c \cdot a^o = 1$ or $c \cdot b^o = 1$. It follows that
\[
\alpha = \frac{2}{1 + \cos \xi}, \quad c = \frac{a + b}{1 + \cos \xi}.
\]
Then we have
\[
1 = c \cdot c^o = \frac{\alpha a^o}{2} (1 + \cos \xi) = \alpha^o.
\]
Hence, $L^\circ$ coincides with the triangle $oA^o B^o$, that is, the condition (i) holds. □

3 Preliminaries for the Three Dimensional Case

In this section, we recall the method of “signed volume estimate” introduced in [7]. The exposition here is simpler than that of [7] and applicable to various truncated cones. Although the method can be extended to the higher dimensional case, from now on, in this paper, we concentrate on the three dimensional case.

3.1 Signed Volume of the Cone of a Ruled Surface

Given a convex body $K \in K^3$ whose interior contains the origin $o$, for any $g \in O(3) \subset GL(3, \mathbb{R})$, we have
\[
(gK)^\circ = (g)^{-1}K^\circ = gK^\circ.
\]
Hence, $K \in K^3(G)$ implies that $K^\circ \in K^3(G)$ for each subgroup $G \subset O(3)$.

Let $C$ be an oriented piecewise $C^1$-curve in $\mathbb{R}^3$ and $r(t)$, $0 \leq t \leq 1$, a parametrization of $C$. Then we define a vector $\overline{C} \in \mathbb{R}^3$ by
\[
\overline{C} := \frac{1}{2} \int_C r \times dr = \frac{1}{2} \int_0^1 r(t) \times r'(t) dt,
\]
which is independent of the choice of a parametrization of $C$. If the curve $C$ is on a plane in $\mathbb{R}^3$ passing through the origin $o$, then $\overline{C}$ is a normal vector of the plane. Let us consider the ruled surface $o * C := \{\lambda u \in \mathbb{R}^3 : u \in C, \ 0 \leq \lambda \leq 1\}$. For any $x \in \mathbb{R}^3$, the signed volume of the solid
\[
x * (o * C) := \{(1 - \nu)x + \nu \xi \in \mathbb{R}^3 : \xi \in o * C, \ 0 \leq \nu \leq 1\}
\]
is defined by
\[
\frac{1}{3} x \cdot \overline{C} = \frac{1}{6} \int_0^1 \det (x, r(t), r'(t)) dt.
\]
Note that this quantity is essential for the signed volume estimate of truncated cones explained later (see Lemma 3.8).

Next, we examine the behavior of $\overline{C}$ under the action of $O(3)$. For any $g \in O(3)$, we obtain

$$g\overline{C} = \det(g)g\overline{C},$$

because

$$x \cdot g\overline{C} = \frac{1}{2} \int_0^1 \det(x, gr(t), gr'(t)) dt = \frac{1}{2} \det(g) \int_0^1 \det(g^{-1}x, r(t), r'(t)) dt = \det(g) g^{-1}x \cdot \overline{C} = x \cdot \det(g)g\overline{C}$$

holds for any $x \in \mathbb{R}^3$. In particular, we have

$$g\overline{C} = g\overline{C}, \quad g \in SO(3); \quad V\overline{C} = -V\overline{C}; \quad H\overline{C} = -H\overline{C};$$

where $V$ and $H$ are the elements of $O(3)$ defined in Sect. 1.3. We denote by $\overline{-C}$ the curve in $\mathbb{R}^3$ with the same image of the curve $C$ but with the opposite direction. (Note that the notation $\overline{-C}$ here is different from the one used in [7].) Then we obtain the formula

$$\overline{-C} = -\overline{C}.$$

### 3.2 Curves on $\partial K$ and Their Polars

Let $K \in K^3$ with $o \in \text{int } K$. For any two points $A, B \in \partial K$ with $a \parallel b$, let us introduce the oriented curve from $A$ to $B$ on the boundary $\partial K$ defined by

$$C(A, B) = C_K(A, B) := \{\rho_K((1-t)a + tb)((1-t)a + tb) : 0 \leq t \leq 1\},$$

where $\rho_K$ is the radial function of $K$ defined by $\rho_K(x) := \max \{\lambda \geq 0 : \lambda x \in K\}$ for $x \in \mathbb{R}^3 \setminus \{o\}$. For any points $A_1, \ldots, A_m \in \partial K$ with $a_i \parallel a_{i+1}$, we denote by $C_K(A_1, \ldots, A_m)$ the oriented curve on $\partial K$ consisting of the successive oriented curves $C(A_i, A_{i+1}), i = 1, \ldots, m - 1$;

$$C_K(A_1, \ldots, A_m) := C(A_1, A_2) \cup \ldots \cup C(A_{m-1}, A_m).$$

In particular, if $C_K(A_1, \ldots, A_m)$ is a simple closed curve, that is, $A_m = A_1$, then we denote by $S_K(A_1, \ldots, A_{m-1})$ the part of $\partial K$ enclosed by the curve such that the orientation of the part is compatible with that of the curve.

Let $K \in K^3$ with $o \in \text{int } K$ and denote by $\mu_K$ its Minkowski gauge. Note that $\rho_K(x) = 1/\mu_K(x)$. Assume that $\partial K$ is a $C^2$-hypersurface in $\mathbb{R}^3$. From now on, we
consider the following class of convex bodies:

$$\mathcal{K}^3 := \{ K \in \mathcal{K}^3 : o \in \text{int } K, \ K \text{ is strongly convex, } \partial K \text{ is of class } C^2 \},$$

where $K$ is said to be strongly convex if the Hessian matrix $D^2(\mu_K^2/2)(x)$ of $C^2$-function $\mu_K^2/2$ on $\mathbb{R}^3$ is positive definite for each $x \in \mathbb{R}^3$ with $|x| = 1$. For a convex body $K \in \mathcal{K}^3$, we define a $C^2$-map $\Lambda = \Lambda_K : \partial K \to \partial K^o$ by

$$\Lambda(x) = \nabla \mu_K(x), \quad x \in \partial K.$$ 

If $K$ is strongly convex with the boundary $\partial K$ of class $C^2$, then $K^o$ is strongly convex with $\partial K^o$ of class $C^1$, and the map $\Lambda : \partial K \to \partial K^o$ is a $C^1$-diffeomorphism satisfying that $x \cdot \Lambda(x) = 1$ (see [17, Sect. 1.7.2] and [7, Sect. 3.1]). Note that the curve $C(A, B)$ is in the plane passing through the three points $o$, $A$, and $B$, but its image $\Lambda(C(A, B))$ into $\partial K^o$ is not necessarily contained in a plane of $\mathbb{R}^3$.

**Lemma 3.1** Let $K \in \mathcal{K}^3$ be a convex body and $A, B \in \partial K$ be two points with $a \parallel b$. Let $H \subset \mathbb{R}^3$ be the plane passing through the three points $o, A, B$, and $\pi_H$ denotes the orthogonal projection onto $H$. Then $\pi_H \circ \Lambda_K = \Lambda_{K \cap H}$ holds on $\partial (K \cap H)$.

**Proof** It suffices to consider the case where the points $A, B$ are in the $xy$-plane, i.e., $H = \{ x = (x, y, z) \in \mathbb{R}^3 : z = 0 \}$. By definition, $\Lambda_K = \nabla \mu_K|_{\partial K}$ is a $C^1$-diffeomorphism from $\partial K$ to $\partial K^o$, where $\nabla \mu_K = (\partial_x \mu_K, \partial_y \mu_K, \partial_z \mu_K)$. Setting $L := K \cap H$, then $L \subset H$ is strongly convex with its boundary of class $C^2$, $o \in \text{int } L \subset H$, and $\mu_L = \mu_K|_{\partial L}$. Since $\nabla \mu_L = (\partial_x (\mu_K|_H), \partial_y (\mu_K|_H)) : H \to H$ and $\Lambda_L := \nabla \mu_L|_{\partial L} : \partial L \to \Lambda_L(\partial L)$, we have

$$\pi_H \circ \Lambda_K|_{\partial L} = (\partial_x \mu_K, \partial_y \mu_K)|_{\partial L} = \Lambda_L$$
on $\partial L$. \qed

We next turn to the case of $G$-invariant convex bodies in $\mathcal{K}^3$. For a subgroup $G$ of $O(3)$, let us introduce the class

$$\mathcal{K}^3(G) := \{ K \in \mathcal{K}^3 : gK = K \text{ for all } g \in G \}.$$

For a convex body $K$ in this class, the map $\Lambda_K$ behaves $G$-equivariantly as follows.

**Lemma 3.2** Let $K \in \mathcal{K}^3(G)$. For any $x \in \partial K$ and $g \in G$, we obtain

$$g \Lambda_K(x) = \Lambda_K(gx). \quad (4)$$

**Proof** Let $K \in \mathcal{K}^3$. We first suppose that $x \in \mathbb{R}^3 \setminus \{0\}$ and $g \in O(3)$. By the definition of $\mu_K$, we have $\mu_K(x) = \mu_{gK}(gx)$. By differentiating it,

$$\nabla \mu_K(x) = g \nabla \mu_{gK}(gx) = g^{-1} \nabla \mu_{gK}(gx)$$

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holds, because \( g \in O(3) \). Here, assume that \( K \in \tilde{K}^3(G) \), \( x \in \partial K \), and \( g \in G \). Then \( gK = K \) and \( gx \in \partial K \) hold. Since \( \Lambda_K = \nabla \mu_K \) on \( \partial K \), we obtain the equality (4).

Not every \( G \)-invariant convex body \( K \) is equipped with the map \( \Lambda_K \). However, owing to the following approximation result, which is a special case of [16, p. 438], it suffices to consider only the class \( \tilde{K}^3(G) \) for the purpose of this paper.

**Proposition 3.3** (Schneider) Let \( G \) be a discrete subgroup of \( O(3) \). Let \( K \in \tilde{K}^3(G) \) be a \( G \)-invariant convex body. Then, for any \( \varepsilon > 0 \) there exists a \( G \)-invariant convex body \( K_\varepsilon \in \tilde{K}^3(G) \) having the property that \( \delta(K, K_\varepsilon) < \varepsilon \), where \( \delta \) denotes the Hausdorff distance on \( K^3 \).

In Sect. 4, we frequently use Proposition 3.3 for various discrete subgroups of \( O(3) \).

**Remark 3.4** Note that Lemmas 3.1 and 3.2, and Proposition 3.3 hold for \( n \)-dimensional case by the same argument.

### 3.3 The Signed Area Estimate

Let \( K \in \tilde{K}^3 \). If we apply the method of signed volume estimate to \( K \) (see Sect. 3.4), then the lower bound estimate of the volume product \( \mathcal{P}(K) \) is reduced to the estimate of some two dimensional situation. Here we prepare such an estimate.

**Proposition 3.5** Let \( K \in \tilde{K}^3 \). For any two points \( A, B \in \partial K \) with \( a \parallel b \), it holds that

\[
\overline{C}(A, B) \cdot \Lambda(\overline{C}(A, B)) \geq (a - b) \cdot \frac{\Lambda(a) - \Lambda(b)}{4}.
\]

Moreover, if \( |a| = |b| \), \( a \parallel \Lambda(a) \), and \( b \parallel \Lambda(b) \) hold, then we obtain

\[
\overline{C}(A, B) \cdot \Lambda(\overline{C}(A, B)) \geq \frac{1 - \cos \xi}{2},
\]

where \( \xi \) is the angle between \( a \) and \( b \).

**Proof** Let \( H \subset \mathbb{R}^3 \) be the plane that contains the curve \( \overline{C}(A, B) \) and the origin \( o \). By Lemma 3.1, we have

\[
o \ast \pi_H(\Lambda_K(\overline{C}(A, B))) = o \ast \Lambda_{K \cap H}(\overline{C}(A, B)) = (K \cap H)^{oH} \cap \text{pos}(\Lambda_{K \cap H}(A), \Lambda_{K \cap H}(B)),
\]

where \( (K \cap H)^{oH} := \{ y \in H : y \cdot x \leq 1 \text{ for any } x \in K \cap H \} \). Putting \( L := (K \cap H) \cap \text{pos}(A, B) \), the convex set \( o \ast \pi_H(\Lambda_K(\overline{C}(A, B))) \) is the corresponding \( L^o \) in Lemma 2.1. Thus,

\[
|L| \leq \frac{\pi_H(\Lambda(a) - \Lambda(b))}{4} = (a - b) \cdot \frac{\Lambda(a) - \Lambda(b)}{4}
\]

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holds. Moreover, if $a \parallel \Lambda(a)$ and $b \parallel \Lambda(b)$ hold, then we further get $o \ast \pi_H(\Lambda_K(C(A, B))) = (K \cap H)^{oH} \cap \text{pos}(A, B)$. In this case, by means of Lemma 2.3, we can replace the right-hand side of the above inequality by $(1 - \cos \xi)/2$.

Finally, we have to check that $\overline{C(A, B)} \cdot \Lambda(C(A, B)) = |L| |L^o|$. Since $K$ has its boundary of class $C^2$, then the plane curve $C(A, B)$ on $\partial K$ is of class $C^2$ and, by definition, we can represent the vector $\overline{C(A, B)} \in \mathbb{R}^3$ as

$$\overline{C(A, B)} = |o \ast C(A, B)| n, \quad n := \frac{a \times b}{|a \times b|}. \quad (5)$$

Note that $o \ast C(A, B) = K \cap \text{pos}(A, B) = L$ and $n$ is the unit normal vector of the plane $H$. On the other hand, let $r(t)$ be a parametrization of the $C^1$-curve $\Lambda(C(A, B))$ on $\partial K^o$ with $r(0) = \Lambda(a)$ and $r(1) = \Lambda(b)$. Then we have

$$\overline{\Lambda(C(A, B))} \cdot n = \frac{1}{2} \int_0^1 (r(t) \times r'(t)) \cdot n \, dt.$$ 

This quantity is nothing but the area of the projection image of the surface $o \ast \Lambda(C(A, B))$ to $H$, which is a convex set in $H$. Consequently,

$$\overline{C(A, B)} \cdot \overline{\Lambda(C(A, B))} = |o \ast C(A, B)| n \cdot \overline{\Lambda(C(A, B))} = |L| |o \ast \pi_H(\Lambda(C(A, B)))| = |L| |L^o|. \quad \square$$

Furthermore, from Lemma 2.4 we obtain the following

**Proposition 3.6** Let $K \in \mathcal{K}^3$. Let $a, b \in \partial K$ with $a \parallel b$ and $|a| = |b|$. Assume that $a^o, b^o \in \partial K^o$ satisfies $a \cdot a^o = b \cdot b^o = 1, a \parallel a^o$, and $b \parallel b^o$. Let $H \subset \mathbb{R}^3$ be the plane passing through the three points $o, A, B$ and $\pi_H$ denotes the orthogonal projection onto $H$. Then, setting

$$L := K \cap \text{pos}(A, B), \quad L^o := \pi_H(K^o) \cap \text{pos}(A, B),$$

we have

$$|L| |L^o| \geq \frac{1 - \cos \xi}{2},$$

where $\xi$ is the angle between $a$ and $b$. The equality holds if and only if either the following (i) or (ii) is satisfied:

(i) $L = \text{conv} \{o, A, C, B\}, L^o = \text{conv} \{o, A^o, B^o\}$, where $c = (a + b)/(1 + \cos \xi)$;
(ii) $L = \text{conv} \{o, A, B\}, L^o = \text{conv} \{o, A^o, C^o, B^o\}$, where $c^o = (a^o + b^o)/(1 + \cos \xi)$.

**Remark 3.7** In the setting of Proposition 3.6, $\text{pos}(A, B) = \text{pos}(A^o, B^o)$ holds.
3.4 The Signed Volume Estimate

Now we are in position to state the “signed volume estimate”, which is a natural generalization of the standard volume estimate used in [11, I.2. Théorème] and [1, Lem. 11].

Lemma 3.8 ([7, Prop. 3.2]) Let $K \in \mathcal{K}^3$. Let $C$ be a piecewise $C^1$, oriented, simple closed curve on $\partial K$. Let $S_K(C) \subset \partial K$ be a piece of surface enclosed by the curve $C$ such that the orientation of the surface is compatible with that of $C$. Then for any point $x \in K$, the following inequality holds:

$$\frac{x \cdot C}{3} \leq |o \ast S_K(C)|.$$

Remark 3.9 Although in [7, Prop. 3.2] the boundary $\partial K$ is assumed to be $C^\infty$, the proof works without any changes under the assumptions that $K \in \mathcal{K}^3$ and the curve $C$ is piecewise $C^1$.

Lemma 3.10 Under the same assumptions as Lemma 3.8, the following inequality holds:

$$|o \ast S_K(C)|\frac{\Lambda(C)}{9} \geq \frac{\overline{C}}{3|o \ast S_K(C)|}.$$ 

Proof By Lemma 3.8, we have $x \cdot \overline{C}/3 \leq |o \ast S_K(C)|$ for any $x \in K$. Hence we get a test vector $\overline{C}/(3|o \ast S_K(C)|) \in K^\circ$. By the assumption, the $C^1$-map $\Lambda: \partial K \rightarrow \partial K^\circ$ can be defined. By applying Lemma 3.8 to a piece of surface $S_{K^\circ}(\Lambda(C)) \subset \partial K^\circ$, we obtain

$$\frac{\overline{C}}{3|o \ast S_K(C)|} \cdot \frac{\Lambda(C)}{3} \leq |o \ast S_{K^\circ}(\Lambda(C))|,$$

as claimed. \qed

In Sect. 3.1, we defined the vector $\overline{C} \in \mathbb{R}^3$ for a piecewise $C^1$-curve $C \subset \mathbb{R}^3$ by means of line integral. Let $K \in \mathcal{K}^3$ be a convex body. Let $A, B \in \partial K$ with $a \parallel b$. As we explained in the proof of Proposition 3.5, if $\partial K$ is of class $C^1$, then the vector $\overline{C}(A, B)$ is given by the formula (5). Note that the right-hand side of (5) is defined whenever the curve $C(A, B)$ is at least continuous. Hence, we define the vector $\overline{C}(A, B) \in \mathbb{R}^3$ by the formula (5) for such a non-smooth case.

Then we have the following fact, which is necessary for determining the equality conditions of Theorem 1.5 (see Sect. 5).

Lemma 3.11 (cf. [7, Lem. 6.2]) Let $K \in \mathcal{K}^3$ and $A_1, A_2, A_3 \in \partial K$. Assume that $C(A_1, A_2, A_3, A_1)$ is a simple closed curve on $\partial K$, that is, $S_K(A_1, A_2, A_3)$ is a “triangle” on $\partial K$. Then, for any $x \in K$ we have

$$\frac{x \cdot (\overline{C}(A_1, A_2) + \overline{C}(A_2, A_3) + \overline{C}(A_3, A_1))}{3} \leq |o \ast S_K(A_1, A_2, A_3)|.$$
with equality if and only if \( o \ast S_K(A_1, A_2, A_3) = x_0 \ast (o \ast C(A_1, A_2, A_3, A_1)) \) for some \( x_0 \in S_K(A_1, A_2, A_3) \).

**Proof** For the inequality part, see [7, Lem. 6.2]. Assume that the equality holds. By the construction, if \( x \notin o \ast S_K(A_1, A_2, A_3) \), then the equality does not hold. Hence, \( x \in o \ast S_K(A_1, A_2, A_3) \). Then the left-hand side is nothing but the volume of the cone over \( o \ast C(A_1, A_2, A_3, A_1) \) with the vertex \( x \). Since the cone is contained in \( o \ast S_K(A_1, A_2, A_3) \), the assumption implies that the two solids coincide and \( x \in S_K(A_1, A_2, A_3) \). The converse is obvious. \( \square \)

4 Proof of Theorem 1.5: Inequality

In this section, we prove the inequalities in Theorem 1.5 by case analysis. We start with the case \( G = T \).

4.1 The Case \( G = T \)

**Proposition 4.1** The inequality \( \mathcal{P}(K) \geq \mathcal{P}(\triangle) \) holds for any \( T \)-invariant convex body \( K \in K^3(T) \).

**Proof** We put
\[
a = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad b = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \quad c = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \quad d = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.
\]

Then the regular tetrahedron \( \triangle \) is represented as \( \triangle = \text{conv} \{A, B, C, D\} \), and we can easily compute that \( \mathcal{P}(\triangle) = 64/9 \). We denote by \( R_A, R_B, R_C, \) and \( R_D \) the rotations through the angle \( 2\pi/3 \) about the axes \( oA, oB, oC, \) and \( oD \), respectively. That is,
\[
R_A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad R_B = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad R_C = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad R_D = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.
\]

Under this setting, we see that these four elements generate the group \( T \):
\[
T = \langle R_A, R_B, R_C, R_D \rangle \subset SO(3).
\]

Let \( K \in K^3(T) \). Since the volume product \( \mathcal{P} \) is continuous with respect to the Hausdorff distance on \( K^3 \), by Proposition 3.3, it suffices to consider the case that \( K \in K^3 \).
\(\tilde{K}^3(T)\). By a dilation of \(K\), we may assume that \(A \in \partial K\). Then, by the \(T\)-symmetry, \(B, C, D \in \partial K\) also hold. Let us consider fundamental domains \(\tilde{K} := o \cdot S_K(A, B, C)\) and \(\tilde{K}^\circ := o \cdot \Lambda(S_K(A, B, C))\) of \(K\) and \(K^\circ\), respectively. Then, we have

\[|K| = 4|\tilde{K}|, \quad |K^\circ| = 4|\tilde{K}^\circ|\]

By applying Lemma 3.10 to the curve \(C = C_K(A, B, C, A)\), we obtain

\[
\frac{9}{16} |K| |K^\circ| = 9|\tilde{K}| |\tilde{K}^\circ| \geq (\overline{C}(A, B) + \overline{C}(B, C) + \overline{C}(C, A)) \cdot \left(\Lambda(C(A, B)) + \overline{\Lambda}(C(B, C)) + \overline{\Lambda}(C(C, A))\right).
\]

Next, let us compute the right-hand side of (6). Since \(\overline{C}(B, C) = R^2 \cdot (C(A, B))\), \(\overline{C}(B, C) = R \cdot (C(A, B))\), (7)

we have

\[
\overline{C}(A, B) + \overline{C}(B, C) + \overline{C}(C, A) = (E + R_D + R^2_D) \cdot \overline{C}(A, B).
\]

Note that \(\overline{C}(A, B) = \overline{C}(A, B)\) \(t\) \((0, 1/\sqrt{2}, -1/\sqrt{2})\) from (5). Hence,

\[
(E + R_D + R^2_D) \cdot \overline{C}(A, B) = \frac{\overline{C}(A, B)}{\sqrt{2}} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \sqrt{2} |\overline{C}(A, B)| \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}.
\]

On the other hand, by Lemma 3.2, we have \(\Lambda(C(B, C)) = R^2_D(\Lambda(C(A, B)))\) and \(\Lambda(C(C, A)) = R_D(\Lambda(C(A, B)))\) from (7), so that

\[
\overline{\Lambda}(C(A, B)) + \overline{\Lambda}(C(B, C)) + \overline{\Lambda}(C(C, A)) = (E + R_D + R^2_D) \cdot \overline{\Lambda}(C(A, B))
\]

holds. Putting \(t(x_1, x_2, x_3) := \overline{\Lambda}(C(A, B))\), we get

\[
\overline{C}(A, B) \cdot \overline{\Lambda}(C(A, B)) = \frac{x_2 - x_3}{\sqrt{2}} |\overline{C}(A, B)|.
\]

Since

\[
R^2_B \cdot R_C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad R^2_B \cdot R_C(C(A, B)) = C(B, A),
\]

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we obtain
\[
-\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \end{pmatrix} = \frac{1}{\Lambda_1(C(B,A))} = \frac{R^2_B R_C(\Lambda(C(A,B)))}{R^2_B R_C \Lambda(C(A,B))} = \begin{pmatrix} x_1 \\ -x_2 \\ -x_3 \\ \end{pmatrix},
\]
which implies that \(x_1 = 0\). Therefore,
\[
(E + R_D + R^2_D) \Lambda(C(A,B)) = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \\ \end{pmatrix} = (x_2 - x_3) \begin{pmatrix} 1 \\ 1 \\ -1 \\ \end{pmatrix}
\]
holds. Thus, the right-hand side of (6) is computed as
\[
\sqrt{2|C(A,B)|} \left( \begin{pmatrix} 1 \\ 1 \\ -1 \\ \end{pmatrix} \cdot (x_2 - x_3) \begin{pmatrix} 1 \\ 1 \\ -1 \\ \end{pmatrix} = 3 \sqrt{2(x_2 - x_3)|C(A,B)|} = 6 \frac{C(A,B)}{\Lambda(C(A,B))}.
\]
Finally, we estimate the right-hand side of the above equality from below. Putting \(t(y_1, y_2, y_3) := \Lambda(a)\), we have
\[
a \cdot \Lambda(a) = \frac{y_1 + y_2 + y_3}{\sqrt{3}} = 1, \quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \end{pmatrix} = \Lambda(a) = \Lambda(R_A a) = R_A \Lambda(a) = \begin{pmatrix} y_3 \\ y_1 \\ \end{pmatrix},
\]
which implies that \(\Lambda(a) = a\). Similarly, \(\Lambda(b) = b\) holds. By Proposition 3.5, we get
\[
\frac{C(A,B)}{\Lambda(C(A,B))} \geq \frac{a - b}{4} \cdot (\Lambda(a) - \Lambda(b)) = \frac{2}{3}.
\]
Consequently, we obtain
\[
|K| |K^o| \geq \frac{16}{9} \frac{C(A,B)}{\Lambda(C(A,B))} \geq \frac{64}{9},
\]
which completes the proof. \(\Box\)

### 4.2 The Case \(G = O\)

**Proposition 4.2** The inequality \(\mathcal{P}(K) \geq \mathcal{P}(\Diamond)\) holds for any \(O\)-invariant convex body \(K \in K^3(O)\).

**Proof** Let \(\Diamond\) be the regular octahedron with vertices \(\{\pm a, \pm b, \pm c\}\), where
\[
a := \begin{pmatrix} 1 \\ 0 \\ 0 \\ \end{pmatrix}, \quad b := \begin{pmatrix} 0 \\ 1 \\ 0 \\ \end{pmatrix}, \quad c := \begin{pmatrix} 0 \\ 0 \\ 1 \\ \end{pmatrix}.
\]
Then, all the elements of $O_h$ are described as
\[
\begin{pmatrix}
\pm 1 & 0 & 0 \\
0 & \pm 1 & 0 \\
0 & 0 & \pm 1
\end{pmatrix},
\begin{pmatrix}
\pm 1 & 0 & 0 \\
0 & 0 & \pm 1 \\
0 & \pm 1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & \pm 1 & 0 \\
0 & 0 & \pm 1 \\
\pm 1 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
\pm 1 & 0 & 0 \\
0 & 0 & \pm 1 \\
\pm 1 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & \pm 1 & 0 \\
0 & 0 & \pm 1 \\
\pm 1 & 0 & 0
\end{pmatrix}.
\]

In particular, $O = \{ g \in O_h : \det(g) = 1 \}$. And we see that $P(\Diamond) = 32/3$. Let $K \in K^3(O)$. In order to examine the volume product of $K$, by Proposition 3.3, we may assume that $K \in \hat{K}^3(O)$. Putting $\tilde{K} := o \ast S_K(A, B, C)$ and $\tilde{K}^o := o \ast \Lambda(S_K(A, B, C))$, by the $O$-symmetry, we have
\[
|K| |\tilde{K}^o| = 64|\tilde{K}| |\tilde{K}^o|.
\]

By Lemma 3.10, we obtain
\[
9|\tilde{K}| |\tilde{K}^o| \geq \left( \overline{C(A, B)} + \overline{C(B, C)} + \overline{C(C, A)} \right) \\
\cdot \left( \Lambda(\overline{C(A, B)}) + \Lambda(\overline{C(B, C)}) + \Lambda(\overline{C(C, A)}) \right).
\]

We put
\[
R := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad R_A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad R_B := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \in O,
\]

where $R$ is the rotation through the angle $2\pi/3$ about the axis passing through the origin $o$ and the point $(1, 1, 1)$, and $R_A$ and $R_B$ are rotations through the angle $\pi/2$ about the axes $oA$ and $oB$, respectively. Since
\[
\overline{C(B, C)} = R(\overline{C(A, B)}), \quad \overline{C(C, A)} = R^2(\overline{C(A, B)}),
\]
the right-hand side of (8) becomes
\[
(E + R + R^2) \overline{C(A, B)} \cdot (E + R + R^2) \Lambda(\overline{C(A, B)}).
\]

Note that $\overline{C(A, B)} = |\overline{C(A, B)}|^t(0, 0, 1)$. Here we put $t(x_1, x_2, x_3) := \Lambda(\overline{C(A, B)})$. Then, the right-hand side of (8) equals
\[
|\overline{C(A, B)}|^t \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 \end{pmatrix} = 3|\overline{C(A, B)}|(x_1 + x_2 + x_3).
On the other hand, since \( RR_A(C(A, B)) = R(C(A, C)) = C(B, A) \), we obtain
\[
\begin{pmatrix} x_2 \\ x_1 \\ -x_3 \end{pmatrix} = RR_A \Lambda(C(A, B)) = -\Lambda(C(A, B)) = - \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},
\]
hence \( x_2 = -x_1 \). Thus, the right-hand side of (8) equals
\[
3|C(A, B)|(x_1 + x_2 + x_3) = 3|C(A, B)|x_3 = 3C(A, B) \cdot \Lambda(C(A, B)).
\]
Since \( RAa = a \) and \( RBb = b \) hold, by a similar argument of the case \( G = T \), we obtain \( \Lambda(a) = a \) and \( \Lambda(b) = b \). Therefore, Proposition 3.5 yields that
\[
\frac{C(A, B) \cdot \Lambda(C(A, B))}{4} \geq \frac{a - b}{4} \cdot (\Lambda(a) - \Lambda(b)) = \frac{1}{2},
\]
so that we obtain \(|K| |K^\circ| \geq 32/3\), as claimed.

\[ \square \]

Remark 4.3 In this case, the minimum of \( \mathcal{P} \) is the same as the centrally symmetric case, that is, \( S_2 = \langle R_2 H \rangle = \langle -E \rangle \cong \mathbb{Z}_2 \). However, the generator \( -E \) of \( S_2 \) is not an element of \( O \). This means that Proposition 4.2 and [7, Thm. 1] are independent results.

4.3 The Case \( G = I \)

Proposition 4.4 The inequality \( \mathcal{P}(K) \geq \mathcal{P}(\odot) \) holds for any \( I \)-invariant convex body \( K \in \mathcal{K}^3(I) \).

Proof Let \( \odot \) be the regular icosahedron with the twelve vertices
\[
(0, \pm 1, \pm \phi), \quad (\pm \phi, 0, \pm 1), \quad (\pm 1, \pm \phi, 0),
\]
where \( \phi = (1 + \sqrt{5})/2 \) (see [4, pp. 52–53]). We put
\[
a := \begin{pmatrix} 0 \\ 1 \\ \phi \end{pmatrix}, \quad b := \begin{pmatrix} \phi \\ 0 \\ 1 \end{pmatrix}, \quad c := \begin{pmatrix} 1 \\ \phi \\ 0 \end{pmatrix}.
\]

Let \( R \) be the rotation through the angle \( 2\pi/3 \) about the axis \( \langle (1, 1, 1) \rangle \). Then, we have
\[
R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in I, \quad R(A) = B, \quad R(B) = C, \quad R(C) = A.
\]

By a simple calculation, we see that \(|\odot| = 10(3 + \sqrt{5})/3\) and \(|\odot^\circ| = 2(25 - 11\sqrt{5})\). Let \( K \in \mathcal{K}^3(I) \). By Proposition 3.3, we may assume that \( K \in \mathcal{K}^3(I) \). By a dilation,
we can also assume that \( A, B, C \in \partial K \). Putting \( \tilde{K} := o * S_K(A, B, C) \), \( \tilde{K}^\circ := o * \Lambda(S_K(A, B, C)) \), by the \( I \)-symmetry, we obtain

\[
|K| |K^\circ| = 400 |\tilde{K}| |\tilde{K}^\circ|.
\]

By Lemma 3.10, we get

\[
9 |\tilde{K}| |\tilde{K}^\circ| \geq (\tilde{C}(A, B) + \tilde{C}(B, C) + \tilde{C}(C, A)) \\
\cdot \left( \frac{\Lambda(\tilde{C}(A, B))}{\Lambda(C(A, B))} + \frac{\Lambda(\tilde{C}(B, C))}{\Lambda(C(B, C))} + \frac{\Lambda(\tilde{C}(C, A))}{\Lambda(C(C, A))} \right).
\]

Since

\[
\tilde{C}(B, C) = R(\tilde{C}(A, B)), \quad \tilde{C}(C, A) = R^2(\tilde{C}(A, B)) ,
\]

the right-hand side of (9) equals

\[
(E + R + R^2) \tilde{C}(A, B) \cdot (E + R + R^2) \frac{\Lambda(\tilde{C}(A, B))}{\Lambda(C(A, B))}.
\]

Here, by definition, we have \( \tilde{C}(A, B) \parallel \iota(1, \phi^2, -\phi) \). Let \( R_{AB} \) be the rotation through the angle \( \pi \) about the axis through \( o \) and the midpoint of the segment \( AB \). Then \( R_{AB} \in I \) and

\[
R_{AB} \Lambda(\tilde{C}(A, B)) = \Lambda(\tilde{C}(A, B)) = -\Lambda(\tilde{C}(B, A))
\]

hold. Note that the three vectors

\[
a \times b = \begin{pmatrix} 1 \\ \phi^2 \\ -\phi \end{pmatrix}, \quad a - b = \begin{pmatrix} -\phi \\ 1 \\ \phi - 1 \end{pmatrix}, \quad a + b = \begin{pmatrix} \phi \\ 1 \\ \phi + 1 \end{pmatrix}
\]

are orthogonal to each other. Putting \( \Lambda(\tilde{C}(A, B)) = y_1(a \times b) + y_2(a - b) + y_3(a + b), \) by (10), we obtain

\[-y_1(a \times b) - y_2(a - b) + y_3(a + b) = -y_1(a \times b) - y_2(a - b) - y_3(a + b),\]

so that \( y_3 = 0 \). We put \( \iota(1, \phi^2, -\phi) := \tilde{C}(A, B) \) for a nonzero real number \( x \). Then,

\[
\tilde{C}(A, B) \cdot \Lambda(\tilde{C}(A, B)) = x \begin{pmatrix} 1 \\ \phi^2 \\ -\phi \end{pmatrix} \cdot \{ y_1(a \times b) + y_2(a - b) \} = 4x y_1 \phi^2
\]
holds and the right-hand side of (9) becomes

\[ x(E + R + R^2) \left( \begin{array}{c} 1 \\ \phi^2 \\ -\phi \end{array} \right) \cdot (E + R + R^2) \left( \begin{array}{c} y_1 - \phi y_2 \\ \phi^2 y_1 + y_2 \\ -\phi y_1 + (\phi - 1)y_2 \end{array} \right) \]

\[ = 4xy_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 12xy_1 \]

\[ = \frac{3}{\phi^2} \overline{C}(A, B) \cdot \Lambda(\overline{C}(A, B)) = \frac{3(3 - \sqrt{5})}{2} \overline{C}(A, B) \cdot \Lambda(\overline{C}(A, B)). \]

We denote by \( R_A \) the rotation through the angle \( 2\pi/5 \) about the axis through \( o \) and \( A \). Then \( R_A \in I \) and \( R_Aa = a \) hold, so that we have \( \Lambda(a) \parallel a \). Similarly, \( \Lambda(b) \parallel b \) holds. Since \( |a| = |b| \), by Proposition 3.5, we obtain

\[ \overline{C}(A, B) \cdot \Lambda(\overline{C}(A, B)) \geq \frac{1}{2} \left( 1 - \frac{a \cdot b}{|a||b|} \right) = \frac{1}{2} \left( 1 - \frac{1}{\sqrt{5}} \right) . \]

Consequently,

\[ |K||K^\circ| \geq \frac{80}{3}(5 - 2\sqrt{5}). \]

Since the right-hand side equals the volume product of \( \circ \), it completes the proof. \( \square \)

### 4.4 The Case \( G = C_{\ell h} \)

**Proposition 4.5** Assume that \( \ell \geq 3 \). Then the inequality \( \mathcal{P}(K) \geq \mathcal{P}(P_\ell) \) holds for any \( C_{\ell h}\)-invariant convex body \( K \in \mathcal{K}^3(C_{\ell h}) \).

**Proof** Let \( K \in \mathcal{K}^3(C_{\ell h}) \) for \( \ell \geq 3 \). To prove the inequality for \( \mathcal{P} \), we may assume that \( K \in \mathcal{K}^3(C_{\ell h}) \) by Proposition 3.3. We put

\[ p := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad a := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad b := R_\ell a = \begin{pmatrix} \cos \xi \\ \sin \xi \\ 0 \end{pmatrix}, \]

where \( \xi = 2\pi/\ell \).
Recall that $G$ is the subgroup of $GL(3, \mathbb{R})$ defined in Theorem 1.5 (iv). Since $gh = hg$ holds for every $g \in \tilde{G}$ and every $h \in C_{\ell h}$, we see that $gK \in \tilde{K}(C_{\ell h})$ for any $g \in G$. Thus, we can assume that $P, A \in \partial K$. Setting $\tilde{K} := o * S_K(P, A, B)$ and $\tilde{K}^\circ := o * \Lambda(S_K(P, A, B))$, by the $C_{\ell h}$-symmetries of $K$ and $K^\circ$, we have

$$|K| = 2\ell|\tilde{K}|, \quad |K^\circ| = 2\ell|\tilde{K}^\circ|.$$ 

By Lemma 3.10, we obtain

$$9|\tilde{K}| |\tilde{K}^\circ| \geq (\tilde{C}(P, A) + \tilde{C}(A, B) + \tilde{C}(B, P))$$

$$\cdot \left( \Lambda(\tilde{C}(P, A)) + \Lambda(\tilde{C}(A, B)) + \Lambda(\tilde{C}(B, P)) \right).$$

(11)

Since $\tilde{C}(B, P) = R_\ell(\tilde{C}(A, P))$, we obtain

$$\tilde{C}(B, P) = R_\ell \tilde{C}(A, P) = -R_\ell \tilde{C}(P, A).$$

Since $\Lambda(\tilde{C}(B, P)) = \Lambda(R_\ell(\tilde{C}(A, P))) = R_\ell(\Lambda(\tilde{C}(A, P)))$ by Lemma 3.2, we get

$$\overline{\Lambda(\tilde{C}(B, P))} = R_\ell \overline{\Lambda(\tilde{C}(A, P))} = -R_\ell \overline{\Lambda(\tilde{C}(P, A))}.$$ 

It follows from the above equalities that the right-hand side of (11) equals

$$(E - R_\ell) \tilde{C}(P, A) \cdot (E - R_\ell) \overline{\Lambda(\tilde{C}(P, A))} + (E - R_\ell) \overline{\tilde{C}(P, A)} \cdot \overline{\Lambda(\tilde{C}(A, B))}$$

$$+ \tilde{C}(A, B) \cdot (E - R_\ell) \overline{\Lambda(\tilde{C}(P, A))} + \overline{\tilde{C}(A, B)} \cdot \overline{\Lambda(\tilde{C}(A, B))}.$$ 

Here we denote these four terms by (I), (II), (III), and (IV), respectively.

Let us start with the calculation of (III). For any $x = \iota(x_1, x_2, x_3) \in \mathbb{R}^3$, we have

$$(E - R_\ell)x = \begin{pmatrix} (1 - \cos \xi)x_1 + (\sin \xi)x_2 \\ -(\sin \xi)x_1 + (1 - \cos \xi)x_2 \\ 0 \end{pmatrix}.$$ 

On the other hand, by definition, $\overline{\tilde{C}(A, B)} \parallel \iota(0, 0, 1)$ holds, which immediately implies that (III) = 0. Next, since $\Lambda(\tilde{C}(A, B)) = \Lambda(H(\tilde{C}(A, B))) = H \Lambda(\tilde{C}(A, B))$, we obtain

$$\overline{\Lambda(\tilde{C}(A, B))} = H(\overline{\Lambda(\tilde{C}(A, B))}) = -H \overline{\Lambda(\tilde{C}(A, B))}.$$ 

Hence $\overline{\Lambda(\tilde{C}(A, B))} \parallel \iota(0, 0, 1)$ holds, which means that (II) = 0. Moreover, we can put $\iota(0, x_2, 0) := \tilde{C}(P, A)$ and $\iota(y_1, y_2, y_3) := \overline{\Lambda(\tilde{C}(P, A))}$ by their definitions. By
the above calculation, we get

(I) = \begin{pmatrix}
(\sin \xi) x_2 \\
(1 - \cos \xi) x_2 \\
0
\end{pmatrix} \cdot \begin{pmatrix}
(1 - \cos \xi) y_1 + (\sin \xi) y_2 \\
-(\sin \xi) y_1 + (1 - \cos \xi) y_2 \\
0
\end{pmatrix} = 2(1 - \cos \xi)x_2y_2

= 2(1 - \cos \xi)C(P, A) \cdot \Lambda(C(P, A)).

Consequently, we obtain

\[ |K| |K^o| \geq \frac{4\ell^2}{9} \left(2(1 - \cos \xi)C(P, A) \cdot \Lambda(C(P, A)) + \frac{C(A, B)}{\Lambda(C(A, B))}\right). \]  \hspace{1cm} (12)

Finally, we compute the right-hand side of (12). Since \( R_\ell p = p \) and \( p = t(0, 0, 1) \), we get \( R_\ell \Lambda(p) = \Lambda(p) \) and \( \Lambda(p) = p \). From \( Ha = a \) and \( b = R_\ell a \), we have \( H\Lambda(a) = \Lambda(a) \) and \( \Lambda(b) = R_\ell \Lambda(a) \). Thus, we can put

\[ \Lambda(a) = \begin{pmatrix}
z_1 \cos \xi - z_2 \sin \xi \\
z_1 \sin \xi + z_2 \cos \xi \\
0
\end{pmatrix}, \quad \Lambda(b) = \begin{pmatrix}
z_1 \cos \xi - z_2 \sin \xi \\
z_1 \sin \xi + z_2 \cos \xi \\
0
\end{pmatrix}, \]

for some \( z_1, z_2 \in \mathbb{R} \). Since \( z_1 = a \cdot \Lambda(a) = 1 \), Proposition 3.5 asserts that

\[ C(P, A) \cdot \Lambda(C(P, A)) \geq \frac{p - a}{4} \cdot (\Lambda(p) - \Lambda(a)) = \frac{1}{2}, \]

\[ C(A, B) \cdot \Lambda(C(A, B)) \geq \frac{a - b}{4} \cdot (\Lambda(a) - \Lambda(b)) = \frac{1 - \cos \xi}{2}, \]

so that

\[ |K| |K^o| \geq \frac{2\ell^2}{3}(1 - \cos \xi). \]

It is easy to check that the right-hand side equals the volume product of \( P_\ell \). \hfill \Box

4.5 The Case \( G = D_\ell \)

**Proposition 4.6** Assume that \( \ell \geq 3 \). Then the inequality \( \mathcal{P}(K) \geq \mathcal{P}(P_\ell) \) holds for any \( D_\ell \)-invariant convex body \( K \in \mathcal{K}^3(D_\ell) \).

**Proof** Let \( K \in \mathcal{K}^3(D_\ell) \) for \( \ell \geq 3 \). Recall that \( D_\ell = (R_\ell, VH) \). To prove the inequality, we may assume that \( K \in \mathcal{K}^3(D_\ell) \) by Proposition 3.3. We put

\[ p = \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}, \quad a = \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}, \quad q = \begin{pmatrix}
0 \\
0 \\
-1
\end{pmatrix}, \quad b = \begin{pmatrix}
\cos \xi \\
\sin \xi
\end{pmatrix}. \]
In a similar way as the proof of Proposition 4.5, by a linear transformation in $G'$, we can assume that $A, B, P, Q \in \partial K$. Let us consider the closed regions $\tilde{K} := o \ast S_K(P, A, B)$ and $\tilde{K}^\circ := o \ast \Lambda(S_K(P, A, B))$. Since $(V H) R_\ell^{-1}(S_K(P, A, B)) = S_K(Q, B, A)$ holds, we have

$$|K| = 2\ell |\tilde{K}|, \quad |K^\circ| = 2\ell |\tilde{K}^\circ|.$$  

Here, we show that

$$(E - R_\ell) \tilde{C}(P, A) \cdot \Lambda(C(A, B)) = 0. \quad (13)$$

For $g \in D_\ell$, we have $g(K) = K$ and $g(\partial K) = \partial K$, so that

$$g(C_K(X, Y)) = C_{gK}(g(X), g(Y)) = C_K(g(X), g(Y))$$

for any $x, y \in \partial K$ with $x \parallel y$. Since $R_\ell, VH \in D_\ell$, the above formula yields that

$$R_\ell VH(C(B, A)) = C(R_\ell VH(B), R_\ell VH(A)) = C(A, B).$$

Thus, by Lemma 3.2, we have

$$\Lambda(C(A, B)) = \Lambda(R_\ell VH(C(B, A))) = R_\ell VH(\Lambda(C(B, A))),$$

so that

$$\Lambda(C(A, B)) = R_\ell VH(\Lambda(C(B, A))) = R_\ell VH\Lambda(C(B, A)) = -R_\ell VH\Lambda(C(A, B)).$$

Putting $t(x_1, x_2, x_3) := \Lambda(C(A, B))$, we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \cos \xi & -\sin \xi & 0 \\ \sin \xi & \cos \xi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\cos \xi x_1 - \sin \xi x_2 \\ -\sin \xi x_1 + \cos \xi x_2 \\ x_3 \end{pmatrix}.$$  

Thus

$$\begin{pmatrix} 1 + \cos \xi & \sin \xi \\ \sin \xi & 1 - \cos \xi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (14)$$

holds. On the other hand, since

$$(E - R_\ell) \tilde{C}(P, A) = \begin{pmatrix} 1 - \cos \xi & \sin \xi & 0 \\ -\sin \xi & 1 - \cos \xi & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ |\tilde{C}(P, A)| \\ 0 \end{pmatrix}$$

$$= |\tilde{C}(P, A)| \begin{pmatrix} \sin \xi \\ 1 - \cos \xi \\ 0 \end{pmatrix}.$$  

we obtain

\[(E - R_\ell) \overline{C}(P, A) \cdot \Lambda(C(A, B)) = |\overline{C}(P, A)|((\sin \xi)x_1 + (1 - \cos \xi)x_2) = 0\]

by (14), so that (13) is verified.

Similarly as the case \(C_{\ell h}\), by the signed volume estimate, we obtain exactly the same inequality as (12). Since \(R_\ell p = p\) and \(p \cdot \Lambda(p) = 1\), we have \(\Lambda(p) = R_\ell \Lambda(p)\), so that \(\Lambda(p) = p\) holds. Similarly, \(V H a = a\) implies that \(\Lambda(a) = a\). Hence, \(\Lambda(b) = \Lambda(R_\ell a) = R_\ell \Lambda(a) = R_\ell a = b\) also holds. Thus, in the same way as in the case \(C_{\ell h}\), we can check that the right-hand side of (12) is greater than or equal to \(2\ell^2(1 - \cos \xi)/3\), and obtain the conclusion.

4.6 The Cases \(G = C_{2h}, T_h, S_6, D_{3d}\)

**Proof of Corollary 1.4** Recall that \(\Diamond\) is the regular octahedron, which is \(O_h\)-invariant. By [7, Thm. 1], \(\Diamond\) is a minimizer of \(\mathcal{P}\) on the set of \(S_2\)-invariant convex bodies. Under the matrix representations used in Sects. 1.3, 4.1, and 4.2, we have

\[
S_2 = \langle -E \rangle \subset C_{2h} = \langle R_2, H \rangle \subset T_h = \langle \pm R_A, \pm R_B, \pm R_C, \pm R_D \rangle \subset O_h, \\
S_2 \subset S_6 \subset D_{3d} \subset O'_h := gO_hg^{-1},
\]

where

\[
g := \begin{pmatrix}
\frac{\sqrt{2}}{\sqrt{3}} & -1/\sqrt{6} & -1/\sqrt{6} \\
0 & 1/\sqrt{2} & -1/\sqrt{2} \\
1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3}
\end{pmatrix} \in SO(3).
\]

It follows from \(\Diamond \in \mathcal{K}(O_h)\) that

\[
\mathcal{P}(\Diamond) = \min_{K \in \mathcal{K}(S_2)} \mathcal{P}(K) \leq \min_{K \in \mathcal{K}(C_{2h})} \mathcal{P}(K) \leq \min_{K \in \mathcal{K}(T_h)} \mathcal{P}(K) \leq \min_{K \in \mathcal{K}(O_h)} \mathcal{P}(K) \leq \mathcal{P}(\Diamond).
\]

Moreover, since \(\Diamond' := g\Diamond\) is also a minimizer of \(\mathcal{P}\) on \(\mathcal{K}(S_2)\) by [7, Thm. 1] and \(\Diamond' \in \mathcal{K}(O'_h)\), we obtain

\[
\mathcal{P}(\Diamond') = \min_{K \in \mathcal{K}(S_2)} \mathcal{P}(K) \leq \min_{K \in \mathcal{K}(S_6)} \mathcal{P}(K) \leq \min_{K \in \mathcal{K}(D_{3h})} \mathcal{P}(K) \leq \min_{K \in \mathcal{K}(O'_h)} \mathcal{P}(\Diamond') \leq \mathcal{P}(\Diamond').
\]

In addition, [7, Thm. 1] asserts that, if \(K\) is a minimizer of \(\mathcal{P}\) on the set of \(S_2\)-invariant convex bodies, then \(K\) or \(K^o\) is a parallelepiped. That is, \(K^o\) or \(K\) is a linear image of \(\Diamond\). Thus, we obtain the conclusion. \(\square\)
5 Proof of Theorem 1.5: Equality Conditions

5.1 The Case $G = T$

Proposition 5.1 Let $K \in \mathcal{K}^3(T)$. If $\mathcal{P}(K) = \mathcal{P}(\Delta)$ holds, then $K$ is a dilation of $\Delta$ or $\Delta^o$.

Proof Suppose that $K \in \mathcal{K}^3(T)$ satisfies $\mathcal{P}(K) = \mathcal{P}(\Delta)$, i.e., $K$ is a minimizer of $\mathcal{P}$. By a dilation of $K$, we may assume that $A, B, C, D \in \partial K$, where $A, B, C, D$ are the points used in Sect. 4.1. By the approximation result (Proposition 3.3), there is a sequence $\{K_m\}_{m \in \mathbb{N}} \subset \mathcal{K}^3(T)$ such that $K_m \to K$ as $m \to \infty$ in the Hausdorff distance. In addition, normalizing $K_m$ by their dilations, we can assume $A, B, C, D \in \partial K_m$. Notice that, even though this normalization is done, the convergence $K_m \to K$ as $m \to \infty$ still holds. We denote the curve $C_{K_m}(A, B)$ by $C_m(A, B)$ for short. For each convex body $K_m$, we can define the map $\Lambda_1^m: \partial K_m \to \partial K_m^o$. Let $H$ be the plane through $o, A,$ and $B$, and $\pi_H$ be the orthogonal projection onto $H$. We put

$$L_m := o \ast C_m(A, B) \subset H, \quad L_m^o := o \ast \pi_H(\Lambda_1^m(C_m(A, B))) \subset H.$$  

Then by the argument in the proof of Proposition 4.1, we have

$$\frac{9}{16} \mathcal{P}(K_m) \geq 6|L_m||L_m^o| \geq 4. \quad (15)$$

Now we examine the two dimensional subsets $L_m$ and $L_m^o$ of $H$. By the $T$-symmetry, we have $A^o := \Lambda_m(A) = A$ and $B^o := \Lambda_m(B) = B$, and

$$L_m = K_m \cap \text{pos}(A, B), \quad L_m^o = \pi_H(K_m^o) \cap \text{pos}(A, B)$$

hold. Putting

$$L := K \cap \text{pos}(A, B), \quad L^o := \pi_H(K^o) \cap \text{pos}(A, B),$$

then we obtain that $L_m \to L$ and $L_m^o \to L^o$ on $H$ in the Hausdorff distance. Since $K$ is a minimizer of $\mathcal{P}$, taking $m \to \infty$ in (15), we have

$$|L||L^o| = \frac{2}{3},$$

which means that Proposition 3.6 holds with equality. Thus, one of the following two cases occurs.

Case (i) $L = \text{conv} \{o, A, Z, B\}, L^o = \text{conv} \{o, A^o, B^o\}$, where $z = 3(a + b)/2$.
Case (ii) $L = \text{conv} \{o, A, B\}, L^o = \text{conv} \{o, A^o, Z^o, B^o\}$, where $z^o = 3(a^o + b^o)/2$.

First, we consider Case (ii).
Since $Z^o \in \pi_H(K^o)$, there exists $s \in \mathbb{R}$ such that $z^o + sa \times b \in K^o$. We note that $z^o \parallel (a^o + b^o)/2 = (a + b)/2$. Let $R$ be the rotation through the angle $\pi$ about the axis through $o$ and the midpoint of $A$ and $B$. Then, we see that $R \in T$ and

$$R(z^o + sa \times b) = z^o - sa \times b \in R(K^o) = K^o.$$ 

Thus, the convexity of $K^o$ implies that $Z^o \in K^o$. Moreover, since $o, A^o, B^o, Z^o \in \text{pos}(A, B)$, we get

$$L^o = \text{conv} \{o, A^o, Z^o, B^o \} \subset K^o \cap \text{pos}(A, B) \subset \pi_H(K^o) \cap \text{pos}(A, B) = L^o,$$

so that $L^o = K^o \cap \text{pos}(A, B)$. It follows from $A = A^o$ and $B = B^o$ that

$$L^o = o \ast C_{K^o}(A^o, B^o), \quad \overline{C_{K^o}(A^o, B^o)} = |L^o| \frac{a \times b}{|a \times b|}. \quad (16)$$

Putting $Y^o := R_D Z^o$ and $X^o := R_D^2 Z^o$, we get

$$o \ast C_{K^o}(C^o, A^o) = \text{conv} \{o, C^o, Y^o, A^o\},$$

$$o \ast C_{K^o}(B^o, C^o) = \text{conv} \{o, B^o, X^o, C^o\}.$$ 

As for $L$, since $L = \text{conv} \{o, A, B\} = K \cap \text{pos}(A, B)$, we have

$$L = o \ast C_K(A, B), \quad \overline{C_K(A, B)} = |L| \frac{a \times b}{|a \times b|}.$$ 

Now we consider the truncated convex cone $o \ast S_K(A, B, C)$. By the $T$-symmetry, we have

$$\overline{C_K(A, B)} + \overline{C_K(B, C)} + \overline{C_K(C, A)}$$

$$= (E + R_D + R_D^2) \overline{C_K(A, B)}$$

$$= |L|(E + R_D + R_D^2) \frac{a \times b}{|a \times b|} = \sqrt{2} |L| \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$ 

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Moreover, Lemma 3.11 asserts that

\[ \frac{\sqrt{2}}{3} |L| x \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \frac{x}{3} \left( C_K(A, B) + \overline{C}_K(B, C) + \overline{C}_K(C, A) \right) \leq |o \ast S_K(A, B, C)| = \frac{|K|}{4} \]

for any \( x \in K \). Hence,

\[ q^o := \frac{4 \sqrt{2}}{3} \frac{|L|}{|K|} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \in K^o. \]

We put \( \hat{K}^o := o \ast S_K^o(A^o, B^o, C^o) \), then it is easy to check that \( q^o \in \hat{K}^o \). By (16), we can calculate the volume of the solid \( Q^o \ast (o \ast C_K^o(A^o, B^o, C^o)) \subset \hat{K}^o \) as

\[ \frac{q^o}{3} \cdot (C_K^o(A^o, B^o) + \overline{C}_K^o(B^o, C^o) + \overline{C}_K^o(C^o, A^o)) \]

\[ = \frac{q^o}{3} \cdot (E + R_D + R_D^2)|L^o| \frac{a \times b}{|a \times b|} \]

\[ = \frac{4 \sqrt{2}}{9|K|} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \cdot \sqrt{2} |L^o| \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \frac{16}{9|K|} = \frac{|K^o|}{4}, \]

which is exactly the volume of \( \hat{K}^o \). Hence, we can apply Lemma 3.11 with equality to get \( \hat{K}^o = Q^o \ast (o \ast C_K^o(A^o, B^o, C^o, A^o)) \). By the convexity, the segment \( Q^o A^o \) is on the boundary of \( \hat{K}^o \) and that of \( K^o \). Here, the endpoints of the three vectors

\[ a^o = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ y^o = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}, \ z^o = \frac{1}{\sqrt{3}} \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \]

are on the plane defined by \( (x + y + z)/\sqrt{3} = 1 \). Thus, putting \( q^o = s^t(1, 1, -1) \), \( s > 0 \), we have \( a^o \cdot (q^o - a^o) \geq 0 \), i.e.,

\[ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot q^o = \frac{s}{\sqrt{3}} \geq 1, \]

so that \( s \geq \sqrt{3} \). On the other hand, \( a \cdot q^o = s/\sqrt{3} \leq 1 \) holds, which yields that \( s = \sqrt{3} \) and \( q^o = \sqrt{3}^t(1, 1, -1) \). Hence, \( q^o \cdot a = q^o \cdot b = q^o \cdot c = 1 \). By the definition of
the polar, we have $K \subset \{x \in \mathbb{R}^3 : x \cdot q^0 \leq 1\}$. Thus, we obtain

$$
\text{conv}\{o, A, B, C\} \subset K \cap \text{pos}(A, B, C)
$$

$$
\subset \{x : x \cdot q^0 \leq 1\} \cap \text{pos}(A, B, C) = \text{conv}\{o, A, B, C\},
$$

so that $K \cap \text{pos}(A, B, C) = \text{conv}\{o, A, B, C\}$. By the $T$-symmetry of $K$, we have $K = \text{conv}\{A, B, C, D\}$, which means that $K$ is a dilation of $\triangle$.

Finally, we consider Case (i). It is clear that $L = o \ast C_K(A, B)$. As for $L^\circ$, we have $L^\circ = \text{conv}\{o, A^\circ, B^\circ\}$ and $o, A^\circ(= A), B^\circ(= B) \in L^\circ$. It then follows from the definition of $L^\circ$ that

$$
L^\circ \subset K^\circ \cap \text{pos}(A, B) \subset \pi_H(K^\circ) \cap \text{pos}(A, B) = L^\circ,
$$

so that $K^\circ \cap \text{pos}(A, B) = L^\circ$, hence (16) holds. Then, similarly as Case (ii), Lemma 3.11 asserts that

$$
q := \frac{4\sqrt{2}|L^\circ|}{3|K^\circ|} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \in K.
$$

After that, by the same argument exchanging the roles of $K$ and $K^\circ$ in Case (ii), we get $K^\circ = \text{conv}\{A^\circ, B^\circ, C^\circ, D^\circ\}$. \hfill \Box

### 5.2 The Cases $G = O, I$

One can show the following similarly as Proposition 5.1, thus we omit their proofs.

**Proposition 5.2** (i) Let $K \in \mathcal{K}^3(O)$. If $\mathcal{P}(K) = \mathcal{P}(\odot)$ holds, then $K$ is a dilation of $\odot$ or $\odot^0$.

(ii) Let $K \in \mathcal{K}^3(I)$. If $\mathcal{P}(K) = \mathcal{P}(\odot)$ holds, then $K$ is a dilation of $\odot$ or $\odot^0$.

### 5.3 The Case $G = D_\ell$

**Proposition 5.3** Assume that $\ell \geq 3$. Let $K \in \mathcal{K}^3(D_\ell)$. If $\mathcal{P}(K) = \mathcal{P}(P_\ell)$ holds, then $K$ coincides with $P_\ell$ or $P^\circ_\ell$ up to a linear transformation in $\mathcal{G}'$.

**Proof** We use the symbols $p, a, q$, and $b$ defined in the proof of Proposition 4.6. Without loss of generality, we may assume that the minimizer $K$ satisfies $P, A, B \in \partial K$. By Proposition 3.3, there is a sequence $\{K_m\}_{m \in \mathbb{N}} \subset \mathcal{K}^3(D_\ell)$ such that $K_m \to K$ as $m \to \infty$. Normalizing $K_m$ by a linear transformation in $\mathcal{G}'$, we can assume that $P, A, B \in \partial K_m$ for every $m$. With this change, the convergence $K_m \to K$ as $m \to \infty$ still holds. Let $\pi_2$ and $\pi_3$ be the orthogonal projections to $zx$-plane and $xy$-plane, respectively. Let $H$ be the plane through $o, P, B$, and $\pi_H$ be the orthogonal projection onto $H$. For each $K_m$, by the same argument as in the proof of Proposition 5.1, we have $P^\circ := \Lambda_m(P) = P$, $A^\circ := \Lambda_m(A) = A$, and $B^\circ := \Lambda_m(B) = B$, where
\( \Lambda_m : \partial K_m \to \partial K^\circ \). Moreover, we put

\[
L_m := o * C_m(P, A) = K_m \cap \text{pos}(P, A),
\]

\[
L^\circ_m := o * \pi_2(\Lambda_m(C_m(P, A))) = \pi_2(K^\circ_m) \cap \text{pos}(P, A),
\]

\[
M_m := o * C_m(A, B) = K_m \cap \text{pos}(A, B),
\]

\[
M^\circ_m := o * \pi_3(\Lambda_m(C_m(A, B))) = \pi_3(K^\circ_m) \cap \text{pos}(A, B).
\]

As \( m \to \infty \), these two dimensional sets converge to

\[
L := K \cap \text{pos}(P, A), \quad L^\circ := \pi_2(K^\circ) \cap \text{pos}(P, A),
\]

\[
M := K \cap \text{pos}(A, B), \quad M^\circ := \pi_3(K^\circ) \cap \text{pos}(A, B),
\]

respectively, in the Hausdorff distance. Since the inequality (12) holds for each \( K_m \), by taking the limit \( m \to \infty \), we obtain

\[
|K| |K^\circ| \geq \frac{4\ell^2}{9} (2(1 - \cos \xi)|L| |L^\circ| + |M| |M^\circ|).
\]

Since \( K \) is a minimizer of \( \mathcal{P} \), by the inequality in Proposition 3.6, we get

\[
|L| |L^\circ| = \frac{1}{2}, \quad |M| |M^\circ| = \frac{1 - \cos \xi}{2},
\]

which implies that Proposition 3.6 holds with equality for \( L \) and also for \( M \). Hence, for the two dimensional subsets \( L \) and \( L^\circ \) one of the following two cases happens:

Case (A) \( L = \text{conv} \{o, P, C, A\} \) and \( L^\circ = \text{conv} \{o, P^\circ, A^\circ\} \), where \( c = p + a \);

Case (B) \( L = \text{conv} \{o, P, A\} \) and \( L^\circ = \text{conv} \{o, P^\circ, C^\circ, A^\circ\} \), where \( c^\circ = p^\circ + a^\circ \).

In addition, \( M \) and \( M^\circ \) are characterized as one of the following:

Case (a) \( M = \text{conv} \{o, A, D, B\} \) and \( M^\circ = \text{conv} \{o, A^\circ, B^\circ\} \), where \( d = (a + b)/(1 + \cos \xi) \);

Case (b) \( M = \text{conv} \{o, A, B\} \) and \( M^\circ = \text{conv} \{o, A^\circ, D^\circ, B^\circ\} \), where \( d^\circ = (a^\circ + b^\circ)/(1 + \cos \xi) \).

Thus, it suffices to consider only four cases. Before that, we give some remarks.

**Remark 5.4**

(i) In Case (B), since \( C^\circ \in L^\circ \subset \pi_2(K^\circ) \), there exists \( s \in \mathbb{R} \) such that \( c^\circ + t(0, s, 0) \in K^\circ \). In the following, we put \( c' := c^\circ + t(0, s, 0) \).

(ii) In Case (b), since \( D^\circ \in M^\circ \subset \pi_3(K^\circ) \), there exists \( t \in \mathbb{R} \) such that \( d^\circ + t(0, 0, t) \in K^\circ \). In addition, we have \( R_\ell VH(d^\circ + t(0, 0, t)) \in K^\circ \), so that

\[
\begin{pmatrix}
1 \\
\sin \xi/(1 + \cos \xi) \\
t
\end{pmatrix}, \begin{pmatrix}
1 \\
\sin \xi/(1 + \cos \xi) \\
-t
\end{pmatrix}, \begin{pmatrix}
1 \\
\sin \xi/(1 + \cos \xi) \\
0
\end{pmatrix} \in K^\circ.
\]
by the convexity of \( K^\circ \). Hence,

\[
M^\circ = \text{conv} \{ o, A^\circ, D^\circ, B^\circ \} \subset K^\circ \cap \text{pos}(A, B) \subset \pi_3(K^\circ) \cap \text{pos}(A, B) = M^\circ
\]

holds, so that \( M^\circ = \text{conv} \{ o, A^\circ, D^\circ, B^\circ \} = \text{conv}(A, B) \).

**Case (A) and (a)** Since \( P, A \in K^\circ \) and \( L^\circ = \text{conv} \{ o, P^\circ, A^\circ \} \), by the convexity of \( K^\circ \), we have

\[
L^\circ = \text{conv} \{ o, P^\circ, A^\circ \} \subset K^\circ \cap \text{pos}(P, A) \subset \pi_2(K^\circ) \cap \text{pos}(P, A) = L^\circ, \quad (17)
\]

that is, \( L^\circ = \pi_2(K^\circ) \cap \text{pos}(P, A) = K^\circ \cap \text{pos}(P, A) \). Similarly, \( M^\circ = K^\circ \cap \text{pos}(A, B) \). By exchanging the roles of \( K \) and \( K^\circ \), this case is reduced to Case (B) and (b) below.

**Case (A) and (b)** We put \( \tilde{K}^\circ := o \ast \mathcal{S}_{K^\circ}(P^\circ, A^\circ, B^\circ) = K^\circ \cap \text{pos}(P, A, B) \). Since \( L^\circ = K^\circ \cap \text{pos}(P, A) \) from (17) and \( M^\circ = \text{conv} \{ o, A^\circ, D^\circ, B^\circ \} \) from Remark 5.4(ii), \( \text{conv} \{ o, P^\circ, A^\circ, D^\circ, B^\circ \} \subset \tilde{K}^\circ \) holds. On the other hand, we have \( \pi_2(\tilde{K}^\circ) \subset L^\circ \) and \( \pi_3(\tilde{K}^\circ) \subset R_\ell(L^\circ) \) for \( \ell \geq 4 \), which mean that \( \tilde{K}^\circ \subset \text{conv} \{ o, P^\circ, A^\circ, D^\circ, B^\circ \} \) for \( \ell \geq 4 \). However, \( \pi_2(\tilde{K}^\circ) \subset L^\circ \) and \( \pi_3(\tilde{K}^\circ) \subset R_\ell(L^\circ) \) do not hold for \( \ell = 3 \). Therefore, we give another proof of \( \tilde{K}^\circ = \text{conv} \{ o, P^\circ, A^\circ, D^\circ, B^\circ \} \) which is applicable for all \( \ell \geq 3 \).

Let us also consider \( \hat{K} := o \ast \mathcal{S}_K(P, A, B) = K \cap \text{pos}(P, A, B) \). In the same way as \( \tilde{K}^\circ \), we have \( \text{conv} \{ o, P, C, A, B, R_\ell(C) \} \subset \hat{K} \). Then, it follows from the \( D_\ell \)-symmetries of \( K \) and \( K^\circ \) that

\[
2\ell \text{\mid conv} \{ o, P, C, A, B, R_\ell(C) \} = 2\ell \text{\mid conv} \{ o, P^\circ, A^\circ, D^\circ, B^\circ \} \leq |K| |K^\circ| = \frac{2\ell^2}{3}(1 - \cos \xi). \quad (18)
\]

On the other hand, by an easy calculation, we have \( \text{conv} \{ o, P, C, A, B, R_\ell(C) \} = (\sin \xi)/2 \) and \( \text{conv} \{ o, P^\circ, A^\circ, D^\circ, B^\circ \} = (\sin \xi)/3(1 + \cos \xi) \). Hence, the inequality (18) holds with equality. This means that \( K^\circ = \text{conv} \{ o, P^\circ, A^\circ, D^\circ, B^\circ \} \). Again, by the \( D_\ell \)-symmetry, \( K^\circ \) is nothing but a regular \( \ell \)-bipyramid such that the cross section of \( K^\circ \) with the \( xy \)-plane is the regular \( \ell \)-gon with vertices \( (R_\ell)^j(D^\circ), j = 0, \ldots, \ell - 1 \).

**Case (B) and (a)** We put \( \hat{K}^\circ := o \ast \mathcal{S}_{K^\circ}(P^\circ, A^\circ, B^\circ) = K^\circ \cap \text{pos}(P, A, B) \). Then, in a similar way as the above case, \( \text{conv} \{ o, A^\circ, B^\circ, P^\circ, C^\circ, R_\ell(C^\circ) \} \subset \hat{K}^\circ \) holds. Again, by an argument similar as in the above case, we obtain \( \hat{K}^\circ = \text{conv} \{ o, A^\circ, B^\circ, P^\circ, C^\circ, R_\ell(C^\circ) \} \). By the \( D_\ell \)-symmetry, \( K^\circ \) coincides with an \( \ell \)-regular right prism.
Case (B) and (b) We put \( \tilde{K}^o := o * S_{K^o}(P^o, C', A^o, D^o, B^o, R_\ell(C')) \). For each convex body \( K_m \in \tilde{K}^3(D_\ell) \), we have

\[
\frac{2\ell}{3|K_m|} \left( C_m(P, A) + C_m(A, B) + \overline{C}_m(B, P) \right)
\]

\[
= \frac{2\ell}{3|K_m|} \left( |L_m| \begin{bmatrix} \sin \xi \\ 1 - \cos \xi \\ 0 \end{bmatrix} + |M_m| \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \in K_m^o.
\]

As \( m \to \infty \), we obtain

\[
q^o := \frac{2\ell}{3|K|} \left( |L| \begin{bmatrix} \sin \xi \\ 1 - \cos \xi \\ 0 \end{bmatrix} + |M| \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \in K^o.
\]

Now, we calculate the (signed) volume of the solid

\[
\tilde{K}^o := Q^o \ast (o \ast C_{K^o}(P^o, C', A^o, D^o, B^o, R_\ell(C'), P^o)) \subset K^o.
\]

We put \( L_1^o := o \ast C_{K^o}(P^o, C') \) and \( L_2^o := o \ast C_{K^o}(C', A^o) \). Then we have

\[
\overline{C}_{K^o}(P^o, C') = \frac{|L_1^o|}{\sqrt{1 + s^2}} \begin{bmatrix} -s \\ 1 \\ 0 \end{bmatrix}, \quad \overline{C}_{K^o}(R_\ell(C'), P^o) = \frac{|L_1^o|}{\sqrt{1 + s^2}} \begin{bmatrix} s \cos \xi + \sin \xi \\ s \sin \xi - \cos \xi \\ 0 \end{bmatrix}.
\]

\[
\overline{C}_{K^o}(C', A^o) = \frac{|L_2^o|}{\sqrt{1 + s^2}} \begin{bmatrix} 0 \\ 1 \\ -s \end{bmatrix}, \quad \overline{C}_{K^o}(B^o, R_\ell(C')) = \frac{|L_2^o|}{\sqrt{1 + s^2}} \begin{bmatrix} \sin \xi \\ 0 \\ -\cos \xi \end{bmatrix}.
\]

\[
\overline{C}_{K^o}(A^o, B^o) = |M^o| \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad |L^o| = \frac{|L_1^o| + |L_2^o|}{\sqrt{1 + s^2}}.
\]

By a direct calculation, the signed volume of \( \tilde{K}^o \) equals

\[
\frac{q^o}{3} \cdot \left( |L^o| \begin{bmatrix} \sin \xi \\ 1 - \cos \xi \\ 0 \end{bmatrix} + |M^o| \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \frac{s|L_1^o|}{\sqrt{1 + s^2}} \begin{bmatrix} -(1 - \cos \xi) \\ \sin \xi \\ 0 \end{bmatrix} \right)
\]

\[
= \frac{2\ell}{9|K|} (2(1 - \cos \xi)|L^o| + |M^o| |M^o|) \frac{\ell}{3|K|} (1 - \cos \xi) = \frac{|K^o|}{2\ell}.
\]

On the other hand, by the \( D_\ell \)-symmetry, we have \( R_k^\ell(\tilde{K}^o) \subset K^o \), \( 0 \leq k \leq \ell - 1 \), and the 3-dimensional Lebesgue measure of \( R_k^\ell(\tilde{K}^o) \cap R_{k'}^\ell(\tilde{K}^o) \) is zero for \( 0 \leq k \neq k' \leq \ell - 1 \). These facts imply that the sum of the signed volume of the subsets \( \tilde{K}^o \), \( R_\ell(\tilde{K}^o), \ldots, R_{\ell-1}(\tilde{K}^o) \) of \( K^o \) is nothing but the volume of \( K^o \). Consequently, we obtain

\[
\tilde{K}^o \cup R_\ell(\tilde{K}^o) \cup \ldots \cup R_{\ell-1}(\tilde{K}^o) = K^o,
\]
which implies that $q^\circ \in \partial K^\circ$. Thus, the segment $Q^\circ A^\circ$ is on the boundary $\partial K^\circ$. Since the plane $\{x = 1\}$ contains the points $C^\circ, A^\circ, D^\circ$, the first coordinate of $q^\circ$ is greater than or equal to 1. On the other hand, since $a \cdot q^\circ \leq 1$, the first coordinate of $q^\circ$ equals 1. Since $|L| = 1/2, |M| = (\sin \xi)/2, |K| = (\ell \sin \xi)/3$, we obtain

$$q^\circ = \frac{\ell}{3|K|} \begin{pmatrix} \sin \xi \\ 1 - \cos \xi \\ \sin \xi \end{pmatrix} = \begin{pmatrix} 1 \\ 1 - \cos \xi/\sin \xi \\ 1 \end{pmatrix} = d^\circ + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$ 

By the $D_\ell$-symmetry, $K^\circ$ is the $\ell$-regular right prism with vertices $(R_\ell)^k(Q^\circ), (R_\ell)^k V H(Q^\circ), 1 \leq k \leq \ell$.}

5.4 The Case $G = C_{\ell h}$

**Proposition 5.5** Assume that $\ell \geq 3$. Let $K \in \mathcal{K}^3(C_{\ell h})$. If $P(K) = P(P_\ell)$, then $K$ coincides with $P_\ell$ or $P_\ell^\circ$ up to a linear transformation in $\mathcal{G}$.

**Proof** Similarly as in the proof of Proposition 5.3, there exists a sequence $\{K_m\}_{m \in \mathbb{N}} \subset \mathcal{K}^3(C_{\ell h})$ such that $K_m \to K$ as $m \to \infty$ with $P, A, B \in \partial K_m$. For each $K_m$, we have $P^\circ := \Lambda_m(P) = P$. However, in this case, we note that $A^\circ = A$ may not hold, where $A^\circ := \lim_{m \to \infty} \Lambda_m(A)$. Since $H a = a$, we have $H \Lambda_m(a) = \Lambda_m(a)$. By $a \cdot \Lambda_m(a) = 1$, we can represent $\Lambda_m(a)$ as

$$\Lambda_m(a) = \begin{pmatrix} 1 \\ a_m \\ 0 \end{pmatrix}, \quad a_m \in \mathbb{R}.$$ 

Here, let $\mathcal{R}_\theta$ be the rotation through the angle $\theta$ about the z-axis. Then, for each $m \in \mathbb{N}$ there exists $\theta_m \in [0, 2\pi)$ such that

$$\Lambda_{\mathcal{R}_{\theta_m} K_m}(a) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$ 

Taking a subsequence if necessary, we can assume $\theta_m \to \theta$ as $m \to \infty$ for some $\theta \in \mathbb{R}$. Using $\mathcal{R}_{\theta_m} K_m$ and $\mathcal{R}_\theta K$ instead of $K_m$ and $K$, respectively, we may assume that $\Lambda_m(a) = a$. Thus, we can get the conclusion in a similar way to the proof of Proposition 5.3. 

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