The $N = 2$ supersymmetric Toda lattice hierarchy and matrix models

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Abstract

We propose a new integrable $N = 2$ supersymmetric Toda lattice hierarchy which may be relevant for constructing a supersymmetric one–matrix model. We define its first two Hamiltonian structures, the recursion operator and Lax–pair representation. We provide partial evidence for the existence of an infinite-dimensional $N = 2$ superalgebra of its flows. We study its bosonic limit and introduce new Lax–pair representations for the bosonic Toda lattice hierarchy. Finally we discuss the relevance this approach for constructing $N = 2$ supersymmetric generalized Toda lattice hierarchies.

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1 Introduction.

In this paper we construct the $N = 2$ version of the Toda lattice hierarchy. We write down the Hamiltonians, the recursion operator and the bosonic as well as the fermionic flows.

There are several reasons to study this problem. First, one would like to extend also to discrete integrable hierarchies the program of $N = 2$ supersymmetrization which has proven very successful for differential hierarchies. Secondly: it is well-known that from a bosonic integrable hierarchy of the Toda type one can construct a differential hierarchy; it is interesting to see whether this holds true also for the $N = 2$ supersymmetric extension. Finally, there have been a few attempts to find a supersymmetric generalization of the one-matrix model. It is arguable whether these attempts have been successful. We believe it is worth trying a different course, based on the remark that most information concerning exactly solvable matrix models is contained in the underlying integrable hierarchies. The idea is to find first the $N = 2$ supersymmetric extension of the integrable hierarchy that characterizes the one-matrix model, and then try to reconstruct the features of the supersymmetric model which is at the origin of it. Here we have completed the first step in this direction.

Behind these motivations is the basically unanswered question concerning the relevance of $N = 2$ supersymmetric extensions of integrable hierarchies particularly in connection with 2D cohomological field theories. It is well-known that such theories can be obtained as twisted 2D $N = 2$ supersymmetric theories. On the other hand many integrable hierarchies correspond to cohomological field theories (for example models corresponding to the A and D series). The same hierarchies are also likely to admit an $N = 2$ extension. Although we do not know enough yet about supersymmetric hierarchies, it is likely that the above is not an accidental coincidence. This paper adds some further evidence in this direction.

The original motivation for this paper was to find an $N = 2$ supersymmetric extension of the Toda hierarchy underlying hermitean one-matrix model. This is defined by the semi-infinite matrix

$$Q = \sum_j (E_{jj+1} + a_j E_{jj} + b_j E_{jj-1})$$

and by the flows

$$\frac{\partial Q}{\partial t_k} = [Q^k_+, Q],$$

where the subscript $+$ means the upper triangular part of a matrix including the main diagonal, and $E_{ij}$ is the matrix with entries $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. Actually our final result is more general than this. It also includes, for example, the $N = 2$ versions of finite lattice hierarchies.

It remains for us to point out that our construction of the $N = 2$ Toda lattice hierarchy presented here hinges upon some recent results on $N = 2$ hierarchies, [1, 2, 3], and to describe the content of this paper. In section 2 we introduce the first two Hamiltonian structures and the lowest lying (bosonic and fermionic) Hamiltonians. In section 3 we compute the recursion operator and, starting from it, in section 4 we write down recursion formulas for all the flows as well as explicit formulas for the first two sets of flows. Section 5 shows that our hierarchy actually possesses $N = 2$ supersymmetry. We conjecture that it is characterized by a more general non-abelian algebra. Then we the possible reductions of our general hierarchy: one leads to the lattice hierarchy characteristic of one-matrix model; the others are reductions over finite lattices. In section 6 we show how we arrived at the formulation of the present hierarchy.
starting from the $N = 2$ supersymmetric Nonlinear Schrödinger (NLS) hierarchy and the f–Toda equations. We also give indications of how it is possible to extend the construction of this paper to generalized $N = 2$ Toda lattice hierarchies. In section 7 we discuss Lax–pair formulations of our hierarchy and in section 8 their bosonic limits. An appendix is devoted to the Lagrangian formulation of the first flow.

2 Hamiltonian structure of the $N = 2$ super Toda lattice hierarchy.

The $N = 2$ supersymmetric Toda lattice hierarchy comprises four different classes of bi-Hamiltonian flows. The first is a hierarchy of commuting bosonic flows with the evolution times $t_l$ ($l \in \mathbb{N}$) and Hamiltonians $H_l$ which are in involution, and in the bosonic limit they reproduce the flows and Hamiltonians of the corresponding bosonic Toda lattice hierarchy. In addition to $H_l$, there exist one more bosonic and two fermionic series of Hamiltonians $U_l$, $S_l$ and $\overline{S}_l$, respectively, which are integrals of the $t_l$-flows generated by Hamiltonians $H_l$ and commute with them. In other words the flows generated by them represent symmetries of the $t_l$-flows. In general, all Hamiltonians as well as their flows form a nonabelian algebra. Altogether, these data may be considered as a general definition of the extended integrable hierarchy we are looking for.

In this section we describe the bi-Hamiltonian structure of the hierarchy; later on we will present the flows, their properties, algebra and different representations.

Let us introduce first some notations:

$$H_{a,l} \equiv \{U_l, S_l, \overline{S}_l, H_l\}, \quad \tau_{a,l} \equiv \{q_l, \theta_l, \overline{\theta}_l, t_l\}, \quad a, b = 1, ..., 4,$$

$$O_{A,i} \equiv \{b_i, a_i, \beta_i, \overline{\beta}_i, \alpha_i, \overline{\alpha}_i\}, \quad A, B = 1, ..., 6, \quad i \in \mathbb{Z},$$

where $\tau_{a,l}$ are evolution times corresponding to Hamiltonians $H_{a,l}$. More precisely, $\theta_l$ and $\overline{\theta}_l$ ($t_l$ and $q_l$) are Grassmann odd (even) variables; $b_i$ and $a_i$ ($\beta_i, \overline{\beta}_i, \alpha_i$ and $\overline{\alpha}_i$) are bosonic (fermionic) fields which depend on all evolution times.

Lattice Hamiltonians (1) are represented in the following general form:

$$H_{a,l} = \sum_{j=-\infty}^{\infty} h_{a,l,j}.$$  

As a consequence of our definition of the hierarchy, any Hamiltonian density $h_{a,l,j}$ is expected to satisfy an equation with respect to the evolution time $t_l$, which has the form of a lattice conservation law,

$$\frac{\partial}{\partial t_l} h_{a,l,j} = f_j - f_{j-1} \equiv (\Delta f)_j, \quad \sum_{j=-\infty}^{\infty} (\Delta f)_j = 0 \Rightarrow \frac{\partial}{\partial t_l} H_{a,l} = 0,$$

where $f_j$ is a polynomial of the lattice fields $O_{A,i}$ (2). In what follows we call the operator $\Delta$ lattice derivative and assume a suitable boundary conditions for the fields $O_{A,i}$ in order for the last equality in eqs. (3) to be satisfied.
The first nontrivial fermionic, \( S_l \) and \( \overline{S}_l \), and bosonic, \( U_l \) and \( H_l \), Hamiltonians are:

\[
U_0 = \sum_{j=-\infty}^{\infty} \ln b_j; \quad U_1 = \sum_{j=-\infty}^{\infty} \left( -\frac{\beta_j \overline{\beta}_j}{b_j} + \alpha_j \sum_{i=-\infty}^{j-1} \overline{\alpha}_i \right),
\]

\[
S_1 = \sum_{j=-\infty}^{\infty} \alpha_j, \quad S_2 = \sum_{j=-\infty}^{\infty} (\beta_j + \alpha_j \sum_{i=-\infty}^{j-1} \frac{\beta_i \overline{\beta}_i}{b_i}),
\]

\[
\overline{S}_1 = - \sum_{j=-\infty}^{\infty} \overline{\alpha}_j, \quad \overline{S}_2 = \sum_{j=-\infty}^{\infty} (\overline{\beta}_j - \alpha_j \sum_{i=j+1}^{\infty} \frac{\beta_i \overline{\beta}_i}{b_i}),
\]

\[
H_1 = \sum_{j=-\infty}^{\infty} (a_j + \frac{\beta_j \overline{\beta}_j}{b_j}), \quad H_2 = \sum_{j=-\infty}^{\infty} \left( \frac{1}{2} a_j^2 + b_j - \beta_j \overline{\alpha}_j + \alpha_j \overline{\beta}_j \right). \quad (5)
\]

Their dimensions in length can be chosen as \([S_l] = [\overline{S}_l] = -l + 1/2\) and \([H_l] = [U_l] = -l\). Then the dimensions of all the fields are completely fixed and are defined by: \([b_i] = -2\), \([a_i] = -1\), \([\beta_i] = [\overline{\beta}_i] = -3/2\) and \([\alpha_i] = [\overline{\alpha}_i] = -1/2\). Let us remark the local character of the densities corresponding to the Hamiltonians \( H_l \) \(\square\).

A bi–Hamiltonian system of equations can be represented in the following general form:

\[
\frac{\partial}{\partial t_{a,l}} O_{A,i} = \{H_{a,l+1}, O_{A,i}\}_1 = \{H_{a,l}, O_{A,i}\}_2 \\
\equiv \sum_{j=-\infty}^{\infty} \sum_{B=1}^{6} (J_1)_{AB,ij} \frac{\delta}{\delta O_{B,j}} H_{a,l+1} = \sum_{j=-\infty}^{\infty} \sum_{B=1}^{6} (J_2)_{AB,ij} \frac{\delta}{\delta O_{B,j}} H_{a,l}, \quad (6)
\]

where \( J_1 \) and \( J_2 \) are supermatrices of the first and second Hamiltonian structures

\[
(J_p)_{AB,ij} = -(-1)^{d_A d_B} (J_p)_{BA,ji} \equiv -(-1)^{d_A d_{H_A}} \{O_{A,i}, O_{B,j}\}_p, \quad p = 1, 2, \quad (7)
\]

and the brackets \( \{.,.\}_p \) are corresponding graded Poisson brackets with the properties

\[
\{O_{A,i}, O_{B,j}\}_p = -(-1)^{d_A d_B} \{O_{B,j}, O_{A,i}\}_p, \\
\{O_{A,i}, O_{B,j} O_{C,k}\}_p = \{O_{A,i}, O_{B,j}\}_p O_{C,k} + (-1)^{d_A d_B} O_{B,j} \{O_{A,i}, O_{C,k}\}_p, \quad (8)
\]

and satisfying the graded Jacobi identities

\[
(-1)^{d_A d_C} \{\{O_{A,i}, O_{B,j}\}_p, O_{C,k}\}_p + (-1)^{d_B d_A} \{O_{B,i}, O_{C,j}\}_p, O_{A,k}\}_p \\
+ (-1)^{d_C d_B} \{O_{C,i}, O_{A,j}\}_p, O_{B,k}\}_p = 0. \quad (9)
\]

Here, the \( d_A \) is the Grassmann parity of the fields \( O_{A,i} \), \( d_A = 0 \) (1) for bosonic (fermionic) fields, and the \( d_{H_A} \) is the Grassmann parity of the Hamiltonians \( H_{a,l} \) \(\square\).

The explicit form for the \( J_1 \) and \( J_2 \) can be obtained using their definitions \(\square\) and the following explicit expressions for the first,

\[
\{b_i, a_j\}_1 = b_i(-\delta_{i,j} + \delta_{i,j+1}), \\
\{a_i, \beta_j\}_1 = \beta_j \delta_{i,j}, \\
\{a_i, \overline{\beta}_j\}_1 = -\overline{\beta}_j \delta_{i,j-1}, \\
\{\beta_i, \beta_j\}_1 = -b_j \delta_{i,j}, \\
\{\alpha_i, \overline{\alpha}_j\}_1 = -\delta_{i,j} + \delta_{i,j+1}. \quad (10)
\]
and for the second,

\[
\begin{align*}
\{b_i, b_j\}_2 &= b_ib_j(\delta_{i,j+1} - \delta_{i,j-1}), \\
\{b_i, a_j\}_2 &= b_ia_j(-\delta_{i,j} + \delta_{i,j+1}), \\
\{b_i, \beta_j\}_2 &= b_i\beta_j\delta_{i,j+1}, \\
\{b_i, \overline{\beta}_j\}_2 &= -b_i\overline{\beta}_j\delta_{i,j-1}, \\
\{b_i, \alpha_j\}_2 &= b_i\alpha_j\delta_{i,j}, \\
\{b_i, \overline{\alpha}_j\}_2 &= -b_i\overline{\alpha}_j\delta_{i,j}, \\
\{a_i, a_j\}_2 &= b_i\delta_{i,j+1} - b_j\delta_{i,j-1}, \\
\{a_i, \beta_j\}_2 &= a_i\beta_j\delta_{i,j} - b_j\alpha_j\delta_{i,j-1}, \\
\{a_i, \overline{\beta}_j\}_2 &= -a_i\overline{\beta}_j\delta_{i,j-1} - b_i\overline{\alpha}_j\delta_{i,j}, \\
\{a_i, \alpha_j\}_2 &= \beta_j\delta_{i,j}, \\
\{a_i, \overline{\alpha}_j\}_2 &= -\overline{\beta}_j\delta_{i,j-1}, \\
\{\beta_i, \beta_j\}_2 &= -\beta_i\beta_j\delta_{i,j-1}, \\
\{\beta_i, \overline{\beta}_j\}_2 &= -\beta_i\overline{\beta}_j\delta_{i,j} + b_i\delta_{i,j+1}, \\
\{\overline{\beta}_i, \alpha_j\}_2 &= -\alpha_j\overline{\beta}_j\delta_{i,j} - b_i\delta_{i,j-1}, \\
\{\alpha_i, \overline{\alpha}_j\}_2 &= \frac{\beta_i\overline{\beta}_j}{b_i}\delta_{i,j} + a_j\delta_{i,j+1},
\end{align*}
\]

(11)

Poisson brackets structures, where only nonzero brackets are written down.

The algebra of the Poisson brackets (10) possesses the discrete inner automorphism \(\sigma_j\) defined by

\[
\begin{align*}
\sigma_j b_i \sigma_j^{-1} &= b_{j-i}, & \sigma_j a_i \sigma_j^{-1} &= -a_{j-i-1}, \\
\sigma_j \beta_i \sigma_j^{-1} &= \overline{\beta}_{j-i}, & \sigma_j \overline{\beta}_i \sigma_j^{-1} &= \beta_{j-i}, \\
\sigma_j \alpha_i \sigma_j^{-1} &= \overline{\alpha}_{j-i}, & \sigma_j \overline{\alpha}_i \sigma_j^{-1} &= \alpha_{j-i}.
\end{align*}
\]

(12)

Under the action of these transformations the overall signs of all Poisson brackets of the algebra (11) are reversed. Due to this as well as to the property \(\sigma_j^2 = 1\), one can conclude that the \(\sigma_j\)-transformations are involution transformations of the algebra (11).

Now taking into account the following transformation properties of the Hamiltonians (3)

\[
\sigma_j \{U_l, S_l, \overline{S}_l, H_l\} \sigma_j^{-1} = (-1)^l \{U_l, \overline{S}_l, S_l, H_l\}
\]

(13)

and defining the transformation properties of their evolution parameters (4) by the formula

\[
\sigma_j \{q_l, \theta_l, \overline{\theta}_l, t_l\} \sigma_j^{-1} = (-1)^{l-1} \{q_l, \overline{\theta}_l, \theta_l, t_l\},
\]

(14)

one easily recognizes that the bi-Hamiltonian system (3) also possesses the involution \(\sigma_j\) (12), (14).

A remark is in order: the appearance of \(b_j\) in the denominator both in the Poisson brackets (11) and Hamiltonians (3), is an artifact of the basis. One can avoid it if instead of the fields \(\beta_i, \overline{\beta}_i\) one uses the following ones: \(\beta_i \mapsto b_i\beta_i, \overline{\beta}_i \mapsto \overline{\beta}_i\) (or \(\beta_i \mapsto \beta_i, \overline{\beta}_i \mapsto b_i\overline{\beta}_i\)). However, we prefer to deal with the old fields \(\beta_i, \overline{\beta}_i\) as the transformation properties (12) with respect to the involution \(\sigma_j\) are simpler.
3 Recursion operator of the $N = 2$ super Toda lattice hierarchy.

One can check by direct, but rather tedious calculations that the graded Jacobi identities \cite{[3]} are satisfied for the algebras \((10)\)–\((11)\). Moreover, they are also satisfied for their sum $\mu_1 \{\} + \mu_2 \{\}$, where $\mu_1$ and $\mu_2$ are arbitrary parameters. Thus, the Hamiltonian structures $J_1$ and $J_2$ are compatible, i.e. they form a Hamiltonian pair which can be used to construct the hereditary recursion operator $R_{AB,ij}$ according to the following general rule:

$$R_{AB,ij} \equiv \sum_{k=-\infty}^{\infty} \sum_{C=1}^{6} (J_2)_{AC,ik}(J_1^{-1})_{CB,kj}, \quad (15)$$

where $(J_1^{-1})_{CB,kj}$ is the inverse matrix of the first Hamiltonian structure,

$$\sum_{k=-\infty}^{\infty} \sum_{C=1}^{6} (J_1)_{AC,ik}(J_1^{-1})_{CB,kj} = \sum_{k=-\infty}^{\infty} \sum_{C=1}^{6} (J_1^{-1})_{AC,ik}(J_1)_{CB,kj} = \delta_{A,B}\delta_{i,j},$$

$$(J_1^{-1})_{1,ij} = \frac{1}{b_i b_j} \beta_i \beta_j \sum_{k \geq 1} \delta_{i+k,j} - \frac{\beta_i \beta_j}{b_j} \sum_{k \geq 1} \delta_{i-k,j},$$

$$(J_1^{-1})_{12,ij} = -(J_1^{-1})_{21,ji} = \frac{1}{b_i} \sum_{k \geq 1} \delta_{i-k,j},$$

$$(-1)^d_{\alpha} (J_1^{-1})_{13,ij} = (J_1^{-1})_{31,ji} = -\frac{\beta_j}{b_i b_j} \sum_{k \geq 1} \delta_{i-k+1,j},$$

$$(-1)^d_{\alpha} (J_1^{-1})_{14,ij} = (J_1^{-1})_{41,ji} = \frac{\beta_j}{b_i b_j} \sum_{k \geq 1} \delta_{i-k,j},$$

$$(J_1^{-1})_{34,ij} = (J_1^{-1})_{43,ji} = (-1)^d_{\alpha} \frac{1}{b_i} \delta_{i,j},$$

$$(J_1^{-1})_{56,ji} = (J_1^{-1})_{65,ij} = (-1)^d_{\alpha} \sum_{k \geq 1} \delta_{i-k+1,j}.$$  \quad (16)

Here, only nonzero matrix elements are presented.

Another important consequence of \cite{[3]} and the compatibility of the Hamiltonian structures are the following involution properties

$${\mathcal H}_{a,l}, \{\mathcal H}_{a,m}\}_2 = \{\mathcal H}_{a,l}, \mathcal H_{a,m}\}_1 = 0 \quad (17)$$

of the Hamiltonians \cite{[14]}.

The recursion operator \cite{[13]} can be represented by $(6\infty) \times (6\infty)$ supermatrix with the following general structure:

$$R = \pmatrix{B_{(2\infty)\times(2\infty)}, \quad & (1)_{d_{\alpha} F_{(2\infty)\times(4\infty)}}, & B_{(4\infty)\times(4\infty)},}$$

where $B_{(n\infty)\times(n\infty)}$ $(F_{(n\infty)\times(m\infty)})$ is a boson- (fermion-) valued square (rectangular) matrix of dimension $(n\infty) \times (n\infty)$ $(m\infty) \times (m\infty)$ which is the same both for the case of bosonic and

\footnote{For details concerning integrable Hamiltonian systems, see, e.g., the review \cite{[3]} and references therein.}
fermionic Hamiltonians (1), (5). The dependence of the recursion operator on the Grassmann nature of the Hamiltonians appears in (13) only via the factors \((-1)^{d_{ia}}\). Substituting eqs. (11) and (16) into eq. (15), one can obtain the following explicit expressions for its entries:

\[
B_{11,ij} = (a_{i-1} - \frac{\beta_j \beta_i}{b_i}) \delta_{i,j} + \frac{b_i}{b_j} (a_{i-1} - a_i) \sum_{k \geq 1} \delta_{i,j-k}, \quad B_{12,ij} = b_i (\delta_{i,j} + \delta_{i,j+1}),
\]

\[
B_{21,ij} = \frac{1}{b_j} (b_i \sum_{k \geq 1} \delta_{i,j-k+1} - (\alpha_{i+1} \beta_i + \beta_i \alpha_i) \sum_{k \geq 1} \delta_{i,j-k} - b_{i+1} \sum_{k \geq 1} \delta_{i,j-k-1}), \quad B_{22,ij} = a_i \delta_{i,j},
\]

\[
F_{13,ij} = -\beta_j \delta_{i,j}, \quad F_{14,ij} = \beta_j \delta_{i,j}, \quad F_{15,ij} = b_i \alpha_i \sum_{k \geq 1} \delta_{i,j+k-1}, \quad F_{16,ij} = -b_i \alpha_i \sum_{k \geq 1} \delta_{i,j-k+1},
\]

\[
F_{23,ij} = \alpha_i \delta_{i,j}, \quad F_{24,ij} = a_j \delta_{i,j-1}, \quad F_{25,ij} = -\beta_i \sum_{k \geq 1} \delta_{i,j+k-1}, \quad F_{26,ij} = -\beta_i \sum_{k \geq 1} \delta_{i,j-k+1},
\]

and

\[
F_{31,ij} = \frac{1}{b_j} (b_i \alpha_i \sum_{k \geq 1} \delta_{i,j-k+1} - a_i \beta_i \sum_{k \geq 1} \delta_{i,j-k} - \beta_j \beta_i \sum_{k \geq 1} \delta_{i,j+k-1}),
\]

\[
F_{41,ij} = \frac{1}{b_j} (b_i \alpha_i \sum_{k \geq 1} \delta_{i,j-k} + a_i \beta_i \sum_{k \geq 1} \delta_{i,j-k+1} + \alpha_i \beta_j \delta_i \sum_{k \geq 1} \delta_{i,j+k}),
\]

\[
F_{32,ij} = \beta_i \sum_{k \geq 1} \delta_{i,j+k-1}, \quad F_{42,ij} = -\beta_i \sum_{k \geq 1} \delta_{i,j+k+1},
\]

\[
F_{51,ij} = \frac{1}{b_j} (\beta_{i-1} \sum_{k \geq 1} \delta_{i,j-k+1} - \beta_i \sum_{k \geq 1} \delta_{i,j-k} - \alpha_i \beta_j \beta_i \sum_{k \geq 1} \delta_{i,j+k}), \quad F_{52,ij} = \alpha_i \sum_{k \geq 1} \delta_{i,j+k},
\]

\[
F_{61,ij} = \frac{1}{b_j} (\beta_{i+1} \sum_{k \geq 1} \delta_{i,j-k} - \beta_i \sum_{k \geq 1} \delta_{i,j-k+1} + \alpha_i \beta_j \delta_i \sum_{k \geq 1} \delta_{i,j+k-1}), \quad F_{62,ij} = -\alpha_i \sum_{k \geq 1} \delta_{i,j+k},
\]

\[
B_{33,ij} = -\frac{\beta_j \beta_j}{b_j} \sum_{k \geq 1} \delta_{i,j+k-1}, \quad B_{34,ij} = \frac{\beta_i \beta_j}{b_j} \sum_{k \geq 1} \delta_{i,j+k-1},
\]

\[
B_{35,ij} = \beta_i \alpha_i \sum_{k \geq 1} \delta_{i,j+k-1} - b_i \sum_{k \geq 1} \delta_{i,j+k}, \quad B_{43,ij} = \frac{\beta_j \beta_i}{b_j} \sum_{k \geq 1} \delta_{i,j+k},
\]

\[
B_{44,ij} = \frac{\beta_j \beta_i}{b_j} \sum_{k \geq 1} \delta_{i,j+k}, \quad B_{46,ij} = \alpha_i \beta_i \sum_{k \geq 1} \delta_{i,j-k+1} + b_i \sum_{k \geq 1} \delta_{i,j-k},
\]

\[
B_{53,ij} = \delta_{i,j+1} - \alpha_i \beta_j \sum_{k \geq 1} \delta_{i,j+k}, \quad B_{54,ij} = \frac{\alpha_i \beta_j}{b_j} \sum_{k \geq 1} \delta_{i,j+k},
\]

\[
B_{55,ij} = -\frac{\beta_j \beta_j}{b_i} \sum_{k \geq 1} \delta_{i,j+k-1} - a_i \sum_{k \geq 1} \delta_{i,j+k},
\]

\[
B_{63,ij} = \frac{\alpha_i \beta_j}{b_j} \sum_{k \geq 1} \delta_{i,j+k-1}, \quad B_{64,ij} = -\delta_{i,j-1} + \frac{\beta_j \alpha_i}{b_j} \sum_{k \geq 1} \delta_{i,j+k-1},
\]

\[
B_{66,ij} = -\frac{\beta_j \beta_j}{b_i} \sum_{k \geq 1} \delta_{i,j-k+1} - a_i \sum_{k \geq 1} \delta_{i,j-k}, \quad B_{36,ij} = R_{45,ij} = B_{56,ij} = B_{65,ij} = 0.
\]
4 Flows of the $N = 2$ super Toda lattice hierarchy.

The hierarchy starts with the flows corresponding to the evolution times $q_0$, $\theta_1$, $\bar{\theta}_1$ and $t_1$, because the Hamiltonians $U_0$, $S_1$, $\bar{S}_1$ and $H_1$ (3) lie in the center of the first Hamiltonian structure, i.e.,

$$\{U_0, O_{A,i}\} = \{S_1, O_{A,i}\} = \{\bar{S}_1, O_{A,i}\} = \{H_1, O_{A,i}\} = 0.$$ (21)

Taking into account eqs. (3), the relations (21) obviously demonstrate that there are no flows with the evolution times $q_{-1}$, $\theta_0$, $\bar{\theta}_0$ and $t_0$. The first nontrivial fermionic and bosonic flows with the times $q_0$, $\theta_1$, $\bar{\theta}_1$ and $t_1$ can be derived using the Hamiltonians $U_0$, $S_1$, $\bar{S}_1$, $H_1$ (or $U_1$, $S_2$, $\bar{S}_2$, $H_2$) (4), respectively, and the second (or first) Hamiltonian structure (11) ((10)),

$$\frac{\partial}{\partial \theta_0} b_j = 0, \quad \frac{\partial}{\partial \theta_0} a_j = 0,$$

$$\frac{\partial}{\partial \theta_0} \beta_j = \beta_j, \quad \frac{\partial}{\partial \theta_0} \bar{\beta}_j = -\bar{\beta}_j,$$

$$\frac{\partial}{\partial \theta_0} \alpha_j = \alpha_j, \quad \frac{\partial}{\partial \theta_0} \bar{\alpha}_j = -\bar{\alpha}_j,$$ (22)

$$\frac{\partial}{\partial \theta_1} b_j = -b_j \alpha_j, \quad \frac{\partial}{\partial \theta_1} a_j = -\beta_j,$$

$$\frac{\partial}{\partial \theta_1} \beta_j = 0, \quad \frac{\partial}{\partial \theta_1} \bar{\beta}_j = -b_j - \alpha_j \bar{\beta}_j,$$

$$\frac{\partial}{\partial \theta_1} \alpha_j = 0, \quad \frac{\partial}{\partial \theta_1} \bar{\alpha}_j = a_j + \frac{\beta_j \bar{\beta}_j}{b_j},$$ (23)

$$\frac{\partial}{\partial \theta_1} b_j = -b_j \bar{\alpha}_j, \quad \frac{\partial}{\partial \theta_1} a_{j-1} = \bar{\beta}_j,$$

$$\frac{\partial}{\partial \theta_1} \beta_j = -b_j + \beta_j \bar{\alpha}_j, \quad \frac{\partial}{\partial \theta_1} \bar{\beta}_j = 0,$$

$$\frac{\partial}{\partial \theta_1} \alpha_j = -a_{j-1} - \frac{\beta_j \bar{\beta}_j}{b_j}, \quad \frac{\partial}{\partial \theta_1} \bar{\alpha}_j = 0,$$ (24)

$$\frac{\partial}{\partial t_1} b_j = b_j (a_j - a_{j-1}), \quad \frac{\partial}{\partial t_1} a_j = b_{j+1} - b_j + \beta_j \bar{\alpha}_j + \alpha_{j+1} \bar{\beta}_{j+1},$$

$$\frac{\partial}{\partial t_1} \beta_j = a_j \beta_j - b_j \alpha_j, \quad \frac{\partial}{\partial t_1} \bar{\beta}_j = -a_{j-1} \bar{\beta}_j - b_j \bar{\alpha}_j,$$

$$\frac{\partial}{\partial t_1} \alpha_j = \beta_j - \beta_{j-1}, \quad \frac{\partial}{\partial t_1} \bar{\alpha}_j = \bar{\beta}_j - \bar{\beta}_{j-1}.$$ (25)

The higher flows can be generated from flows (22)–(25) using the recurrence relations

$$\frac{\partial}{\partial s_{\alpha, l+1}} O_{A,i} = \sum_{j = -\infty}^{\infty} \sum_{B=1}^{6} R_{AB,ij} \frac{\partial}{\partial s_{\alpha, l}} O_{B,j} = (K_{a,l+1})_{A,i}.$$ (26)

Substituting eqs. (3) and (18)–(20) into (26), one can obtain the following explicit expressions for them:

$$\frac{\partial}{\partial s_{\alpha, l+1}} b_i = b_i \left[ \frac{\partial}{\partial s_{\alpha, b}} \left( a_i + a_{i-1} + \frac{\beta_j \bar{\beta}_j}{b_j} \right) \right] + a_{i-1} \frac{\partial}{\partial s_{\alpha, l}} \sum_{j=i}^{\infty} \ln b_j - a_i \frac{\partial}{\partial s_{\alpha, l}} \sum_{j=i+1}^{\infty} \ln b_j$$

$$+ (-1)^{d_{\alpha}} \bar{\alpha}_i \frac{\partial}{\partial s_{\alpha, l}} \sum_{j=-\infty}^{i} \alpha_j - (-1)^{d_{\alpha}} \bar{\alpha}_i \frac{\partial}{\partial s_{\alpha, l}} \sum_{j=i}^{\infty} \bar{\alpha}_j.$$
\[
\frac{\partial}{\partial \tau_{a,l+1}} a_i = \frac{\partial}{\partial \tau_{a,l}} (b_i + b_i + \frac{1}{2} a_i^2 - \beta_j \overline{b}_j + \alpha_{j+1} \overline{b}_{j+1}) \\
- (b_i - b_i + \beta_j \overline{b}_j + \alpha_{j+1} \overline{b}_{j+1}) \frac{\partial}{\partial \tau_{a,l}} \sum_{j=i+1}^{\infty} \ln b_j \\
- (-1)^{d_{\tau_a}} \overline{b}_{i+1} \frac{\partial}{\partial \tau_{a,l}} \sum_{j=-\infty}^{i} \alpha_j - (-1)^{d_{\tau_a}} \beta_i \frac{\partial}{\partial \tau_{a,l}} \sum_{j=-\infty}^{\infty} \overline{\alpha}_j,
\]

\[
\frac{\partial}{\partial \tau_{a,l+1}} \beta_i = \frac{\partial}{\partial \tau_{a,l}} (b_i \alpha_i) + (-1)^{d_{\tau_a}} \beta_i \frac{\partial}{\partial \tau_{a,l}} \sum_{j=-\infty}^{i} (a_j + \beta_j \overline{b}_j) \\
+ (-1)^{d_{\tau_a}} (b_i \alpha_i - a_i \beta_i) \frac{\partial}{\partial \tau_{a,l}} \sum_{j=1}^{\infty} \ln b_j - (b_i - \beta_i \overline{b}_i) \frac{\partial}{\partial \tau_{a,l}} \sum_{j=-\infty}^{i} \alpha_j,
\]

\[
\frac{\partial}{\partial \tau_{a,l+1}} \overline{b}_i = - \frac{\partial}{\partial \tau_{a,l}} (b_i \overline{b}_i) - (-1)^{d_{\tau_a}} \overline{b}_i \frac{\partial}{\partial \tau_{a,l}} \sum_{j=1}^{i-1} (a_j + \beta_j \overline{b}_j) \\
+ (-1)^{d_{\tau_a}} (b_i \overline{b}_i + a_i - \beta_i \overline{b}_i) \frac{\partial}{\partial \tau_{a,l}} \sum_{j=1}^{\infty} \ln b_j + (b_i + \alpha_i \overline{b}_i) \frac{\partial}{\partial \tau_{a,l}} \sum_{j=-\infty}^{\infty} \overline{\alpha}_j,
\]

\[
\frac{\partial}{\partial \tau_{a,l+1}} \alpha_i = \frac{\partial}{\partial \tau_{a,l}} (\beta_{i-1} + a_{i-1} \alpha_i) + (-1)^{d_{\tau_a}} \alpha_i \frac{\partial}{\partial \tau_{a,l}} \sum_{j=-\infty}^{i-1} (a_j + \beta_j \overline{b}_j) \\
- (a_{i-1} + \beta_j \overline{b}_j) \frac{\partial}{\partial \tau_{a,l}} \sum_{j=-\infty}^{i} \alpha_j - (-1)^{d_{\tau_a}} \beta_i \frac{\partial}{\partial \tau_{a,l}} \sum_{j=1}^{\infty} \ln b_j + (-1)^{d_{\tau_a}} \beta_{i-1} \frac{\partial}{\partial \tau_{a,l}} \sum_{j=-\infty}^{\infty} \ln b_j,
\]

\[
\frac{\partial}{\partial \tau_{a,l+1}} \overline{\alpha}_i = - \frac{\partial}{\partial \tau_{a,l}} (\overline{b}_{i+1} - a_i \overline{b}_i) - (-1)^{d_{\tau_a}} \overline{b}_i \frac{\partial}{\partial \tau_{a,l}} \sum_{j=1}^{i-1} (a_j + \beta_j \overline{b}_j) - (a_i + \beta_j \overline{b}_j) \frac{\partial}{\partial \tau_{a,l}} \sum_{j=-\infty}^{\infty} \overline{\alpha}_j \\
+ (-1)^{d_{\tau_a}} \overline{b}_{i+1} \frac{\partial}{\partial \tau_{a,l}} \sum_{j=1}^{\infty} \ln b_j - (-1)^{d_{\tau_a}} \overline{b}_i \frac{\partial}{\partial \tau_{a,l}} \sum_{j=-\infty}^{\infty} \ln b_j,
\]

(27)

where \(d_{\tau_a}\) is the Grassmann parity of the evolution time \(\tau_{a,l}\). Everywhere in these expressions only the lattice densities of the Hamiltonians \(U_0, S_1, \overline{S}_1\) and \(H_1\) \(^{[3]}\) appear under the sign of summation over the lattice points and under the action of time derivatives. Remembering that the \(t_l\)-derivative of any Hamiltonian density \(^{[3]}\) can be represented as the lattice derivative of some local functions of the fields \(O_{A,l}\) \(^{[2]}\), one can conclude that all \(t_l\)-flows and densities of their Hamiltonians \(H_l\) are local. This is, in general, not the case for other flows and Hamiltonians of the hierarchy under consideration.

Let us comment now about the effect of the involution \(\sigma_j\) on the flows. One can check by direct calculations that, if the flows with evolution times \(\tau_{a,l}\) possess the involution \(\sigma_j\) \(^{[2], [4]}\), then the bosonic flows with the times \(q_{l+1}\) and \(t_{l+1}\) generated via formulae \(^{[27]}\) also possess the involution \(\sigma_j\), but the fermionic flows with the times \(\theta_{l+1}\) and \(\overline{q}_{l+1}\) have the following transformation properties with respect to \(\sigma_j\):

\[
\sigma_j \frac{\partial}{\partial \theta_{l+1}} \sigma_j^{-1} = (-1)^l \left( \frac{\partial}{\partial q_{l+1}} + \left( \frac{\partial}{\partial \theta_{l+1}} U_0 \right) \frac{\partial}{\partial \theta_{l+1}} \right), \quad \sigma_j \frac{\partial}{\partial \overline{q}_{l+1}} \sigma_j^{-1} = (-1)^l \left( \frac{\partial}{\partial \overline{q}_{l+1}} + \left( \frac{\partial}{\partial \theta_{l+1}} U_0 \right) \frac{\partial}{\partial \overline{q}_{l+1}} \right),
\]

(28)

where we have used the following obvious relations: \(\frac{\partial}{\partial \tau_{a,l}} H_1 = \frac{\partial}{\partial q_0} U_0 = \frac{\partial}{\partial \theta_0} U_0 = 0\). The \(\sigma_j\)-transformed flows are also admissible flows of the hierarchy, because the additional second terms on the right hand sides of eqs. \(^{[28]}\) also commute with the \(t_l\)-flows\(^{[4]}\). So, the \(\sigma_j\)-transformation

\(^{[2]}\)The relations \(\frac{\partial}{\partial \theta_{l+1}} \frac{\partial}{\partial \tau_{a,l}} U_0 = \frac{\partial}{\partial \tau_{a,l}} \frac{\partial}{\partial \theta_{l+1}} U_0 = 0\) are satisfied according to the definition of the hierarchy given at
only makes a change of basis in the space of the fermionic flows. One can introduce a new basis for them, which is invariant with respect to the involution $\sigma_j$; one defines it as the basis containing all the flows calculated via recurrence relations (27), except for the $\bar{f}_j$-flows, which should be calculated via the formula:

$$
\frac{\partial}{\partial t_{n+1}} = (-1)^{j} \sigma_j \frac{\partial}{\partial t_{j+1}} \sigma_j^{-1}.
$$  (29)

In what follows we adopt such definition.

As an example, we present the second flows, i.e. those corresponding to the evolution times $q_1$, $\theta_2$ and $t_2$,

$$
\frac{\partial}{\partial q_1} b_j = -b_j (\alpha_j \sum_{i=-\infty}^{j-1} \bar{\alpha}_i + \alpha_j \sum_{i=j+1}^{\infty} \alpha_i), \quad \frac{\partial}{\partial q_1} a_j = -\beta_j \sum_{i=-\infty}^{j} \bar{\alpha}_i + \bar{\beta}_j \sum_{i=j+1}^{\infty} \alpha_i,
$$  (30)

$$
\frac{\partial}{\partial \theta_2} b_j = b_j \sum_{i=j}^{\infty} \alpha_i - \beta_j \bar{\alpha}_j \sum_{i=j+1}^{\infty} \alpha_i, \quad \frac{\partial}{\partial \theta_2} \bar{b}_j = b_j \sum_{i=-\infty}^{j} \bar{\alpha}_i + \alpha_j \bar{\beta}_j \sum_{i=j+1}^{\infty} \alpha_i,
$$

$$
\frac{\partial}{\partial q_1} \alpha_j = \beta_{j-1} + a_{j-1} \sum_{i=j}^{\infty} \alpha_i + \frac{\beta_j \bar{\beta}_j}{b_j} \sum_{i=j+1}^{\infty} \alpha_i, \quad \frac{\partial}{\partial q_1} \bar{\alpha}_j = \bar{\beta}_{j+1} - a_j \sum_{i=-\infty}^{j} \bar{\alpha}_i - \frac{\beta_j \bar{\beta}_j}{b_j} \sum_{i=-\infty}^{j-1} \bar{\alpha}_i,
$$

$$
\frac{\partial}{\partial \theta_2} b_j = b_j (-\beta_{j-1} - \alpha_j \sum_{i=-\infty}^{j} (a_i + \frac{\beta_{i-1} \bar{\beta}_{i-1}}{b_{i-1}}) + (a_j - a_{j-1}) \sum_{i=j}^{\infty} \alpha_i),
$$  (31)

$$
\frac{\partial}{\partial \theta_2} a_j = -b_j \alpha_j - \beta_j \sum_{i=-\infty}^{j} (a_i + \frac{\beta_j \bar{\beta}_j}{b_i}) + (b_{j+1} - b_j + \beta_j \bar{\alpha}_j + \alpha_{j+1} \bar{\beta}_{j+1}) \sum_{i=j+1}^{\infty} \alpha_i,
$$

$$
\frac{\partial}{\partial \theta_2} \beta_j = (b_j \alpha_j - a_j \beta_j) \sum_{i=j+1}^{\infty} \alpha_i,
$$

$$
\frac{\partial}{\partial \theta_2} \bar{\beta}_j = -b_{j-1} \bar{\beta}_j + (a_j - a_{j-1}) \alpha_j \bar{\beta}_j + (a_{j-1} \bar{\beta}_j - b_j \bar{\alpha}_j) \sum_{i=j+1}^{\infty} \alpha_i
$$

$$
- (b_j + \alpha_j \bar{\beta}_j) \sum_{i=-\infty}^{j} (a_i + \frac{\beta_j \bar{\beta}_j}{b_i}),
$$

$$
\frac{\partial}{\partial \theta_2} \alpha_j = b_{j-1} \sum_{i=j}^{\infty} \alpha_i - \beta_j \sum_{i=j+1}^{\infty} \alpha_i,
$$

$$
\frac{\partial}{\partial \theta_2} \bar{\alpha}_j = b_{j+1} - \beta_j \bar{\alpha}_j + \bar{\beta}_{j+1} \sum_{i=j+2}^{\infty} \alpha_i - \bar{\beta}_j \sum_{i=j+1}^{\infty} \alpha_i - \frac{a_j \beta_j \bar{\beta}_j}{b_j} + (a_j + \frac{\beta_j \bar{\beta}_j}{b_j}) \sum_{i=-\infty}^{j} (a_i + \frac{\beta_j \bar{\beta}_j}{b_i}),
$$

$$
\frac{\partial}{\partial \theta_2} b_j = b_j (a_j - a_{j-1} + b_{j+1} - b_{j-1} + \beta_{j-1} \alpha_{j-1} - \beta_j \bar{\alpha}_j + \alpha_{j+1} \bar{\beta}_{j+1} - \alpha_j \bar{\beta}_j),
$$  (32)

$$
\frac{\partial}{\partial \theta_2} a_j = b_{j+1} (a_{j+1} + a_j - a_j \bar{\alpha}_j + \alpha_{j+1} \bar{\beta}_{j+1}) - b_j (a_j + a_{j-1} - a_j \bar{\alpha}_j) + \beta_{j+1} \bar{\beta}_{j+1} - \beta_j \bar{\beta}_j
$$

$$
+ a_j (\beta_j \bar{\alpha}_j + \alpha_{j+1} \bar{\beta}_{j+1}),
$$

$$
\frac{\partial}{\partial \theta_2} \beta_j = a_j^2 \beta_j + b_{j+1} \beta_j - b_j \beta_{j-1} + \beta_j \alpha_{j+1} \bar{\beta}_{j+1} - a_{j-1} \beta_j a_j,
$$

$$
\frac{\partial}{\partial \theta_2} \bar{\beta}_j = -a_j^2 \bar{\beta}_j + b_{j+1} \beta_j - b_j \beta_{j-1} + \bar{\beta}_j \beta_{j+1} \bar{\beta}_{j+1} - a_{j-1} \beta_j \bar{\alpha}_j,
$$

$$
\frac{\partial}{\partial \theta_2} \alpha_j = b_j \alpha_j - b_j \alpha_{j-1} + a_j \beta_j - a_j \beta_{j-1},
$$

$$
\frac{\partial}{\partial \theta_2} \bar{\alpha}_j = b_{j+1} \bar{\alpha}_j + b_j \bar{\alpha}_j + a_j \beta_j - a_j \beta_{j+1}.
$$
The flow with the evolution time $\bar{\theta}_2$ can be easily derived from flow (31) via formula (24), and we do not write it down here. Let us remark the nonlocality of the $\frac{\partial}{\partial q_2}$ and $\frac{\partial}{\partial \bar{q}_2}$ flows.

Actually in the previous expressions (30), (31) and (32) calculated via recurrence relations (27) we have introduced an additional change in the space of the flows,

$$\frac{\partial}{\partial q_1} \to \frac{\partial}{\partial q_1} - S_1 \frac{\partial}{\partial q_1} + \bar{S}_1 \frac{\partial}{\partial \bar{q}_1}, \quad \frac{\partial}{\partial q_2} \to \frac{\partial}{\partial q_2} - H_1 \frac{\partial}{\partial q_1}, \quad \frac{\partial}{\partial \bar{q}_2} \to \frac{\partial}{\partial \bar{q}_2}. \quad (33)$$

The reason is that in this way the flows are Hamiltonian with respect to the second Hamiltonian structure (14) with Hamiltonians $U_1$, $S_2$ and $H_2$ (3), respectively.

At this point one can go on and construct the Hamiltonians $U_2$, $S_3$ and $H_3$ which generate the flows (30), (31) and (32) via the first Hamiltonian structure (10). As an example, we present here the expression for the Hamiltonian $H_3$,

$$H_3 = \sum_{j=-\infty}^{\infty} \left( \frac{1}{3} a_j^3 + b_j(a_j + a_{j-1} - \alpha_j \bar{\alpha}_j) + \beta_j(\bar{\beta}_j + \bar{\beta}_{j+1}) + a_j(\alpha_{j+1} \bar{\beta}_{j+1} - \beta_j \bar{\alpha}_j) \right), \quad (34)$$

which is characterized by a lattice–local Hamiltonian density.

## 5 Superalgebra of the flows and reductions.

The above constructed integrable hierarchy possesses $N = 2$ supersymmetry. Indeed, using the explicit expressions (22)–(25) for the first four flows, one can easily check that the structure relations of the $N = 2$ supersymmetry are satisfied:

$$\{ \frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_1} \} = \{ \frac{\partial}{\partial q_1}, \frac{\partial}{\partial \bar{q}_1} \} = \{ \frac{\partial}{\partial \bar{q}_1}, \frac{\partial}{\partial \bar{q}_1} \} = \frac{\partial}{\partial q_1}, \quad \frac{\partial}{\partial q_1}, \quad \frac{\partial}{\partial \bar{q}_1}, \quad \{ \frac{\partial}{\partial q_1}, \frac{\partial}{\partial \bar{q}_1} \} = 0, \quad \{ \frac{\partial}{\partial q_0}, \frac{\partial}{\partial q_1} \} = -\frac{\partial}{\partial q_1}, \quad \{ \frac{\partial}{\partial q_0}, \frac{\partial}{\partial \bar{q}_1} \} = -\frac{\partial}{\partial \bar{q}_1}, \quad \{ \frac{\partial}{\partial \bar{q}_0}, \frac{\partial}{\partial \bar{q}_1} \} = -\frac{\partial}{\partial \bar{q}_1}, \quad (35)$$

where the bracket $\{ , \}$ denotes the anticommutator. Using flows (22)–(25) and (30)–(32), one can also derive the following important (anti)commutation relations:

$$\{ \frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2} \} = \{ \frac{\partial}{\partial q_2}, \frac{\partial}{\partial \bar{q}_2} \} = 0, \quad \{ \frac{\partial}{\partial q_0}, \frac{\partial}{\partial q_2} \} = \frac{\partial}{\partial q_2}, \quad \{ \frac{\partial}{\partial q_0}, \frac{\partial}{\partial \bar{q}_2} \} = \frac{\partial}{\partial \bar{q}_2}, \quad \{ \frac{\partial}{\partial q_1}, \frac{\partial}{\partial \bar{q}_2} \} = -(\frac{\partial}{\partial \bar{q}_2} + H_1 \frac{\partial}{\partial \bar{q}_1}), \quad (36)$$

where $H_1$ is the Hamiltonian from the set (3). Let us remark that the appearance of the $H_1$ on the right hand side of eq. (36) is a basis artifact and can be excluded by introducing a new basis in the space of Hamiltonians (3). Indeed, after introducing the new Hamiltonians $\bar{U}_1$, $\bar{S}_2$ and $\bar{S}_2$,

$$\bar{U}_1 = U_1 + \frac{1}{2} S_1 \bar{S}_1, \quad \bar{S}_2 = S_2 - \frac{1}{2} H_1 S_1, \quad \bar{S}_2 = \bar{S}_2 - \frac{1}{2} H_1 \bar{S}_1, \quad (37)$$

the corresponding flows,

$$\frac{\partial}{\partial q_1} = \frac{\partial}{\partial q_1} + \frac{1}{2} S_1 \frac{\partial}{\partial q_1} - \frac{1}{2} \bar{S}_1 \frac{\partial}{\partial \bar{q}_1}, \quad \frac{\partial}{\partial q_2} = \frac{\partial}{\partial q_2} - \frac{1}{2} S_1 \frac{\partial}{\partial \bar{q}_1} - \frac{1}{2} H_1 \frac{\partial}{\partial \bar{q}_1}, \quad \frac{\partial}{\partial \bar{q}_2} = \frac{\partial}{\partial \bar{q}_2} - \frac{1}{2} \bar{S}_1 \frac{\partial}{\partial q_1} - \frac{1}{2} H_1 \frac{\partial}{\partial q_1}, \quad (38)$$

the graded commutators between the new and the remaining flows become the same as in eqs. (36), except that the term with $H_1$ in the last anticommutator disappears. The graded
comutators (35) and (36) of the flows can also be derived from the Poisson brackets (10)–(11) between the Hamiltonians (1), (5) if use is made of the homomorphism

\[ H_{a,l} = \{H_{b,m}, H_{c,n}\}_2 = \{H_{b,m+1}, H_{c,n}\}_1 \Rightarrow \frac{\partial}{\partial \tau_{a,l}} = [\frac{\partial}{\partial \tau_{b,m}}, \frac{\partial}{\partial \tau_{c,n}}], \] (39)

where the brackets \([A, B]\) between any operators \(A\) and \(B\) mean the graded commutator, \([A, B] \equiv AB - (-1)^{d_A d_B} BA\) and it is understood that the pair of the indices \((a, l)\) is determined by the two pairs \((b, m)\) and \((c, n)\). For example, using (39) and (17), one can derive the following (anti)commutators:

\[ [\frac{\partial}{\partial \tau_{a,l}}, \frac{\partial}{\partial \tau_{a,m}}] = 0. \] (40)

Considering the results obtained in this section, and taking into account that the superalgebra formed by the flows should be invariant with respect to the automorphism transformations (14), it is reasonable to conjecture the existence of a basis in the space of the flows where the algebra of the flows is:

\[ \{\frac{\partial}{\partial \theta_{l}}, \frac{\partial}{\partial \theta_{m}}\} = \{\frac{\partial}{\partial \theta_{l}}, \frac{\partial}{\partial t_{m}}\} = \{\frac{\partial}{\partial \theta_{l}}, \frac{\partial}{\partial \theta_{m}}\} = \{\frac{\partial}{\partial t_{l}}, \frac{\partial}{\partial \theta_{m}}\} = \{\frac{\partial}{\partial t_{l}}, \frac{\partial}{\partial t_{m}}\} = 0, \]

\[ \{\frac{\partial}{\partial q_{l}}, \frac{\partial}{\partial \theta_{m}}\} = \frac{\partial}{\partial \theta_{l+m}}, \quad \{\frac{\partial}{\partial q_{l}}, \frac{\partial}{\partial t_{m}}\} = -\frac{\partial}{\partial \theta_{l+m}}, \quad \{\frac{\partial}{\partial q_{l}}, \frac{\partial}{\partial \theta_{m}}\} = -\frac{\partial}{\partial t_{l+m}}. \] (41)

From these expressions one can see that the operator \(\frac{\partial}{\partial q_1}\) acts as recursion operator for the fermionic flows, while their knowledge allows to generate the bosonic flows corresponding to the Hamiltonians \(H_t\).

In the last part of this section we discuss the consistent reductions of the Toda lattice hierarchy. The three subsets of fields (4),

\[ O_1 \equiv \{b_i, a_i, \beta_i / \sqrt{b_i}, \alpha_{i+1}, \alpha_i\}, \quad m \leq i, \quad m \in \mathbb{N} \]
\[ O_2 \equiv \{b_i, a_i-1, a_{m-1}, \beta_i, \beta_i / \sqrt{b_i}, \alpha_i, \alpha_m, \alpha_{i-1}, \alpha_{m-1}\}, \quad 1 \leq i \leq m - 1, \]
\[ O_3 \equiv \{b_i, a_i-1, \beta_i / \sqrt{b_i}, \beta_i / \sqrt{b_i}, \alpha_i, \alpha_{i-1}\}, \quad i \leq 0, \] (42)

form subalgebras of the algebras (10) and (11). A simple inspection of flows (22)–(32) shows that modulo automorphism \(\sigma_j\) (12) they admit at least two different nonsingular reductions,

I) \(O_3 = 0, \quad \text{or} \quad \text{II) } O_1 = O_3 = 0. \) (43)

Algebras (10)–(11) as well as the Hamiltonians from the set (5) and (34), except \(U_0\), are nonsingular on the shell of the constraints I) or II) (13). The first reduction produces the semi-infinite lattice hierarchy where the lattice is bounded on the left (this is related to the bosonic lattice hierarchy of the one–matrix model, see Introduction). The second one generates finite lattice hierarchies (the bound is on the left and the right simultaneously).

\(^3\text{I.e., under the adjoint action of the operator }\frac{\partial}{\partial q_1}\text{ every fermionic flow transforms into the next fermionic flow.}\)
6 Origin of the $N = 2$ super Toda lattice hierarchy.

The $N = 2$ discrete Toda lattice hierarchy, as it has been presented so far, may appear to have come out of the blue. It is time to explain how we were lead to this construction by relating it to previously known hierarchies. Ancestors of the present paper can be considered refs. \cite{1, 2, 3} as well as \cite{4}. As one might suspect, there is a one–to–one correspondence between the lattice hierarchy defined above and the differential $N = 2$ supersymmetric NLS hierarchy. The relation between the two is however rather non–trivial.

To start with, let us introduce a new basis \{\{u, v, \psi, \bar{\psi}, \alpha, \bar{\alpha}\} in the space of the fields \{b, a, \beta, \bar{\beta}, \alpha, \bar{\alpha}\}, defined by the following transformation:

$$b_j = -u_j v_j, \quad a_j = (\ln v_j)' + \psi_j \bar{\psi}_j, \quad \beta_j = v_j \psi_j, \quad \bar{\beta}_j = u_j \bar{\psi}_j,$$

(44)

where the $'$ means the derivative with respect $t_1$, and in what follows we also use the notation $\frac{\partial}{\partial t_1} \equiv \partial$. This transformation is invertible, and the inverse transformation has the form:

$$u_j = -b_j \exp(-\partial^{-1}(a_j + \frac{\beta_j \bar{\beta}_j}{b_j})), \quad v_j = \exp \partial^{-1}(a_j + \frac{\beta_j \bar{\beta}_j}{b_j}),$$

$$\psi_j = \beta_j \exp(-\partial^{-1}(a_j + \frac{\beta_j \bar{\beta}_j}{b_j})), \quad \bar{\psi}_j = -\frac{\beta_j}{b_j} \exp \partial^{-1}(a_j + \frac{\beta_j \bar{\beta}_j}{b_j}).$$

(45)

Let us stress that the transformation (44) is nonholonomic, i.e., it is not reducible to a point transformation of the initial phase space.

In the new basis (44), the first flow (25) become

$$(\ln(u_{j+1} v_j))' = \psi_{j+1} \bar{\psi}_{j+1} - \psi_j \bar{\psi}_j, \quad (\ln v_j)'' = u_j v_j - u_{j+1} v_{j+1} + \psi_{j+1} \bar{\psi}_{j+1} - \psi_j \bar{\psi}_j,$$

$$\left(\frac{\psi_j}{v_j}\right)' = v_j \psi_j - v_{j-1} \psi_{j-1}, \quad \left(\frac{\bar{\psi}_j}{v_j}\right)' = u_j \bar{\psi}_j - u_{j+1} \bar{\psi}_{j+1}, \quad \psi_j' = u_j \alpha_j, \quad \bar{\psi}_j' = v_j \bar{\alpha}_j,$$

(46)

and coincides with the minimal $N = 2$ supersymmetric Toda chain equations, introduced in \cite{1, 3} and called f–Toda, extended by two additional fields $\alpha_j$ and $\bar{\alpha}_j$. They are compatible with the fermionic flows

$$\frac{\partial}{\partial \theta_j} u_j = -\psi_j', \quad \frac{\partial}{\partial \theta_j} v_j = 0, \quad \frac{\partial}{\partial \theta_j} \psi_j = 0, \quad \frac{\partial}{\partial \theta_j} \bar{\psi}_j = v_j, \quad \frac{\partial}{\partial \theta_j} \alpha_j = 0, \quad \frac{\partial}{\partial \theta_j} \bar{\alpha}_j = (\ln v_j)',$$

(47)

$$\frac{\partial}{\partial v_j} u_j = 0, \quad \frac{\partial}{\partial v_j} v_j = -\psi_j', \quad \frac{\partial}{\partial v_j} \psi_j = u_j, \quad \frac{\partial}{\partial v_j} \bar{\psi}_j = 0, \quad \frac{\partial}{\partial v_j} \alpha_j = (\ln u_j)', \quad \frac{\partial}{\partial v_j} \bar{\alpha}_j = 0.$$n

(48)

These correspond to vector fields that generate the transformations of the $N = 2$ supersymmetry.

In \cite{1} it was demonstrated that the flow equations belonging to the differential $N = 2$ NLS hierarchy, are invariant with respect to the f–Toda transformations in the sense that different solutions are related by f–Toda transformations. Let us consider an example involving the second flow equations of the $N = 2$ NLS hierarchy \cite{3, 4}. These equations are

$$\frac{\partial}{\partial \theta_2} u_j = -u_j'' + 2u_j^2 v_j + 2u_j (\psi_j \bar{\psi}_j' - \psi_j' \bar{\psi}_j) + 2u_j' \psi_j \bar{\psi}_j,$$

$$\frac{\partial}{\partial \theta_2} v_j = v_j'' - 2u_j v_j^2 - 2v_j (\psi_j \bar{\psi}_j' - \psi_j' \bar{\psi}_j) + 2v_j' \psi_j \bar{\psi}_j,$$

$$\frac{\partial}{\partial \theta_2} \psi_j = -\psi_j'' + 2u_j v_j \psi_j - 2\psi_j' \bar{\psi}_j \psi_j', \quad \frac{\partial}{\partial \theta_2} \bar{\psi}_j = \bar{\psi}_j'' - 2u_j v_j \bar{\psi}_j - 2v_j \bar{\psi}_j' \psi_j,$$

(49)
where here we have appended a label $j$ in order to mark different solutions. Solutions labeled by neighboring $j$’s are related by the f-Toda chain equations (14). In fact, using the transformations (14) and (15), one can rewrite eqs. (19) as well as eqs. (17) and (18) in terms of the fields $b_j, a_j, \beta_j, \bar{\beta}_j$. Thus, in the old basis they become

$$\frac{\partial}{\partial t}b_j = -b_j'' + 2(b_j a_j)' + 2b_j' \left( \frac{\beta_j \bar{\beta}_j}{b_j} \right)' ,$$

$$\frac{\partial}{\partial t}a_j = a_j'' + 2a_j a_j' + 2b_j 2b_j' + 2(\beta_j \bar{\beta}_j)' + a_j \frac{\beta_j \bar{\beta}_j}{b_j}' ,$$

$$\frac{\partial}{\partial t}\beta_j = -\beta_j'' + 2(a_j \beta_j)' , \quad \frac{\partial}{\partial t}\bar{\beta}_j = \bar{\beta}_j'' + 2(a_j \bar{\beta}_j)' - 2(\beta_j (\ln b_j)')' ,$$

$$\frac{\partial}{\partial \theta}b_j = \beta_j' - a_j \beta_j , \quad \frac{\partial}{\partial \theta}a_j = -\beta_j ,$$

$$\frac{\partial}{\partial \theta}\beta_j = 0 , \quad \frac{\partial}{\partial \theta}\bar{\beta}_j = -b_j - a_j \frac{\beta_j \bar{\beta}_j}{b_j} + \frac{\beta_j \beta_j'}{b_j} ,$$

$$\frac{\partial}{\partial \theta}a_j = 0 , \quad \frac{\partial}{\partial \theta}\bar{\alpha} = a_j + \frac{\beta_j \bar{\beta}_j}{b_j} ,$$

$$\frac{\partial}{\partial \theta}b_j = a_j \beta_j + b_j \frac{\beta_j \bar{\beta}_j}{b_j}' , \quad \frac{\partial}{\partial \theta}a_j = \beta_j + \left( \frac{\beta_j}{b_j} \right)'' + \left( \frac{a_j \beta_j}{b_j} \right)' ,$$

$$\frac{\partial}{\partial \theta}\beta_j = -b_j - \beta_j \frac{\beta_j \bar{\beta}_j}{b_j}' - a_j \frac{\beta_j \beta_j'}{b_j} , \quad \frac{\partial}{\partial \theta}\bar{\beta}_j = 0 ,$$

$$\frac{\partial}{\partial \theta}a_j = -a_j + (\ln b_j)' - \frac{\beta_j \beta_j'}{b_j} , \quad \frac{\partial}{\partial \theta}\bar{\alpha} = 0 ,$$

respectively. One can check easily by direct substitution of the flows (23), (24), (25) and (32) into eqs. (50), (51) and (52) that they are indeed identically satisfied. Of course, the same procedure can be applied to any flow belonging to the $N = 2$ supersymmetric NLS hierarchy. Thus, flows of the $N = 2$ supersymmetric Toda lattice hierarchy can be reconstructed using the set of the corresponding $N = 2$ supersymmetric NLS flows via the f-Toda chain equations.

As one can expect at this point, the recursion operators of the $N = 2$ supersymmetric Toda lattice and NLS hierarchies are related. Indeed, if we use the first flow equations (25) we can derive the following relations:

$$a_{j-1} = a_j - (\ln b_j)'' , \quad b_{j+1} + \alpha_{j+1} \bar{\beta}_{j+1} = a_j' + b_j - \beta_j \bar{\alpha}_j ,$$

$$\beta_{j-1} = \beta_j - \alpha_j' , \quad \bar{\beta}_{j+1} = \bar{\beta}_j - \bar{\alpha}_j' , \quad \sum_{j=-\infty}^{i} \alpha_j = \partial^{-1} \beta_i , \quad \sum_{j=i}^{\infty} \bar{\alpha}_j = \partial^{-1} \bar{\beta}_i ,$$

$$\sum_{j=i+1}^{\infty} \ln b_j = -\partial^{-1} a_i , \quad \sum_{j=-\infty}^{i-1} \left( a_j + \frac{\beta_j \bar{\beta}_j}{b_j} \right) = \partial^{-1} (b_i + \alpha_i \bar{\beta}_i) ,$$

which, upon substitution into the recurrence relations (27) transform them into relations involving only the fields $\{b_i, a_i, \beta_i, \bar{\beta}_i, \alpha_i, \bar{\alpha}_i\}$ defined at the same lattice point. After applying transformations (14)–(15) to the new basis $\{v_j, \psi_j, \bar{\psi}_j, \alpha_j = \psi_j' / u_j, \bar{\alpha}_j = \bar{\psi}_j' / v_j\}$, one can verify that the resulting relations reproduce the corresponding recursion relations for the $N = 2$
supersymmetric NLS hierarchy derived in [1] in terms of $N=2$ superfields. For the bosonic Toda lattice hierarchy a similar procedure was developed in [3].

We would also like to note that at least three sets of Hamiltonians ([1], $S_l$, $T_l$ and $H_l$, can be recovered using the connection between the $N=2$ supersymmetric Toda lattice and NLS hierarchies. Let us shortly explain the main steps of such procedure.

We take the sum of the f–Toda chain equations ([10]) over the lattice points and differentiate with respect to some evolution time $T_l$ of the $N=2$ super NLS hierarchy. We remark that the right–hand sides of the resulting expressions become identically equal to zero and the left–hand sides become the full derivatives with respect to the time $t_1$ corresponding to the first flow of the $N=2$ supersymmetric Toda lattice hierarchy.

\[
\frac{\partial}{\partial t_1} \left[ \sum_{j=-\infty}^{\infty} \frac{\partial}{\partial T_l} \left( \ln(u_{j+1}v_j) \right) \right] = 0, \quad \frac{\partial}{\partial t_1} \left[ \sum_{j=-\infty}^{\infty} \frac{\partial}{\partial T_l} \left( \frac{\psi_j^\prime}{u_j} \right) \right] = 0, \quad \frac{\partial}{\partial t_1} \left[ \sum_{j=-\infty}^{\infty} \frac{\partial}{\partial T_l} \left( \frac{\overline{\psi}_j'}{v_j} \right) \right] = 0, \quad (54)
\]

where we have used a very important property, the commutativity the $N=2$ super NLS and f–Toda flows, $[\frac{\partial}{\partial t_1},\frac{\partial}{\partial T_l}]=0$. The equations (54) have the form of conservation laws, and it is obvious that the expressions inside the square brackets are the Toda–lattice Hamiltonians. Thus, any flow of the $N=2$ super NLS hierarchy can generate some Hamiltonian for the $N=2$ supersymmetric Toda lattice hierarchy. Now we substitute the explicit expressions for the first three bosonic and fermionic flows of the $N=2$ super NLS hierarchy into the expressions inside the square brackets of eqs. (54); then we use eqs. (46) in order to express higher derivatives with respect to $t_1$ that appear in the calculations in terms of its first and zeroth derivatives; we make the transformations (44)–(45) and eliminate the operator $\partial^{-1}$ via formulae (53). We have checked that the expressions inside the square brackets of (54) indeed reproduce the integrals $S_1, S_2, T_1, S_2, H_1, H_2, H_3$ ([1], [4]).

The above illustrated connection with the $N=2$ NLS hierarchy via the f–Toda equations explains the origin of the definitions and properties of the previous sections. Let us end this section with two remarks.

First, we can extend the above correspondence to the $a=4$ $N=2$ super-KdV hierarchy. Using the mapping [4] which connects the $N=2$ super-NLS and the $a=4$ $N=2$ super-KdV hierarchies, as well as using transformations (44) and (45), one can derive the mapping

\[
s_j = -a_j, \quad r_j = b_j + \beta_j((\overline{\beta}_j^\prime)^{\prime} + a_i \overline{\beta}_j), \quad \xi_j = \frac{1}{2}\beta_j, \quad \overline{\xi}_j = -\frac{1}{2}\overline{\beta}_j - \frac{1}{2}((\overline{\beta}_j^\prime)^{\prime} + a_i \overline{\beta}_j)^{\prime},
\]

which connects our eqs. (50) and the second flow equations

\[
\frac{\partial}{\partial t_2} r_j = -r_j^\prime - 2(r_j s_j)^{\prime} - 8(\xi_j \overline{\xi}_j)^{\prime}, \quad \frac{\partial}{\partial t_2} s_j = s_j^\prime - 2 s_j^\prime s_j - 2 r_j^\prime, \quad \frac{\partial}{\partial t_2} \xi_j = -\xi_j^\prime - 2(s_j \xi_j)^{\prime}, \quad \frac{\partial}{\partial t_2} \overline{\xi}_j = \overline{\xi}_j^\prime - 2(s_j \overline{\xi}_j)^{\prime}
\]

belonging to the $a=4$ $N=2$ super-KdV hierarchy. Here $s_j$ and $r_j$ ($\xi_j$ and $\overline{\xi}_j$) are bosonic (fermionic) fields with the scaling dimensions 1 and 2 (3/2 and 3/2), respectively.

Finally we would like to point out the possibility to apply the approach developed in this section to the wide class of the $N=2$ supersymmetric generalized Toda lattice equations.

\[\text{In eqs. (44) only the relations generating nontrivial independent Hamiltonians are presented.}\]
constructed recently in \[2\]. They are related with the \(N = 2\) supersymmetric \((n, m)\) Generalized NLS hierarchies (GNLS) \[9\] in the same way as the above \(N = 2\) supersymmetric Toda lattice and NLS hierarchies are related. Recently the recursion operators for the \(N = 2\) super \((n, m)\)-GNLS hierarchies were constructed in \[10\]. Using them and the above method one can derive the recursion operators for the corresponding \(N = 2\) supersymmetric generalized Toda lattice hierarchies. The details will be given elsewhere.

7 Lax–pair representation of the \(N = 2\) super Toda lattice hierarchy.

The more compact way to formulate an integrable hierarchy is by means of a Lax–pair. In the present case we can do that by means of the recursion operator \(R_{AB,ij}\) \[18\]–\[20\]. In terms of it the Lax–pair representation for the bosonic flows of the \(N = 2\) supersymmetric Toda lattice hierarchy is:

\[
\frac{\partial}{\partial \tau_{a,l}} R_{AB,ij} = \sum_{k=-\infty}^{\infty} \sum_{C=1}^{6} ((K_{a,l}')_{AC,ik} R_{CB,kj} - R_{AC,ik} (K_{a,l}')_{CB,kj}), \quad a = 1, 4, \tag{57}
\]

where \((K_{a,l}')_{AC,ik}\) is the matrix of Fréchet derivatives of the function \((K_{a,l})_{A,i}\) defined in eq. \[26\].

For completeness we present also the operator representation for the fermionic flows,

\[
\epsilon \frac{\partial}{\partial \tau_{a,l}} R_{AB,ij} = \sum_{k=-\infty}^{\infty} \sum_{C=1}^{6} (\epsilon (K_{a,l}')_{AC,ik} R_{CB,kj} + R_{AC,ik} \epsilon (K_{a,l}')_{CB,kj}), \quad a = 2, 3, \tag{58}
\]

where we have introduced an additional Grassmann parameter \(\epsilon\) in order to derive a closed relation containing only two supermatrices, \(R_{AB,ij}\) and \((K_{a,l}')_{AC,ik}\) at \(a = 2, 3\).

The representation \[57\] can be treated as the integrability condition for the linear system

\[
\sum_{j=-\infty}^{\infty} \sum_{C=1}^{6} R_{AC,ij} N_{C,j} = \lambda N_{A,i},
\]

\[
\frac{\partial}{\partial \tau_{a,l}} N_{A,i} = \sum_{j=-\infty}^{\infty} \sum_{C=1}^{6} (K_{a,l}')_{AC,ij} N_{C,j}, \quad a = 1, 4, \tag{59}
\]

where \(N_{C,j}\) are its eigenfunctions and \(\lambda\) is the spectral parameter. The Hamiltonians which are in involution can be derived using the formula\[5\]

\[
H_l = str(R^l). \tag{60}
\]

Acting \(p\)-times with the hereditary recursion operator \[18\]–\[20\] on the first Hamiltonian structure \(J_1\) \[10\] one can derive the \((p + 1)\)-th Hamiltonian structure,

\[
J_{p+1} = R^p J_1, \tag{61}
\]

\[5\) Let us recall the definition of the supertrace for a supermatrix of the form \[18\]: \(str(R) = tr(B_{(2\infty)\times(2\infty)}) - tr(B_{(4\infty)\times(4\infty)})\).
which is compatible with \( J_1 \). Thus, almost all information about the whole hierarchy is encoded in the recursion operator \((18)-(21)\).

Let us remark the existence of another linear system which includes only bosonic wave functions. One can check by a direct computation that the following linear system
\[
(1 + \frac{\beta_{j+1}(aj\bar{\beta}_j + b_j\bar{\pi}_j + \alpha_{j+1}\bar{\pi}_j\bar{\beta}_j)}{b_jb_{j+1}})M_{j+1} + (b_j + \frac{(a_j(b_j-1)\beta_j + b_j^2\alpha_j)\bar{\beta}_{j-1}}{b_{j-1}b_j})M_{j-1} + (a_j + \frac{\beta_j\bar{\beta}_{j-1}}{b_{j-1}} + \alpha_{j+1}\bar{\pi}_j - \frac{a_j\alpha_{j+1}\bar{\pi}_j}{b_j} + \frac{a_{j+1}(b_{j+1}-b_j)\beta_{j+1}\bar{\beta}_{j+1}\alpha_{j+1}\bar{\pi}_j}{b_jb_{j+1}^2})M_j = 0, \tag{62}
\]
is consistent if the first flow equations \((25)\) are satisfied. Here, \( M_j \) are bosonic wave functions which in consequence of \((23)\) and \((62)-(63)\) satisfy the following second order differential equation
\[
\frac{\partial}{\partial t_j}M_j = -(b_j + \frac{(a_j(b_{j-1}-b_j)\beta_j + b_j^2\alpha_j)\bar{\beta}_{j-1}}{b_{j-1}b_j})M_{j-1} + \beta_j(\bar{\beta}_j - \frac{\bar{\beta}_{j-1}}{b_{j-1}})M_j, \tag{63}
\]
for any values of the index \( j \). However, the linear system \((62)-(63)\) provides a Lax–pair representation valid only in a week sense, i.e. only when the operator equation are restricted on the shell of the wave functions satisfying eq. \((62)\). Unfortunately this is an obstacle at least for a straightforward derivation of the higher Hamiltonians and flows.

### 8 Bosonic limit.

In the previous section we found two Lax pair representation of our hierarchy, although the second one is only a ‘conditional’ Lax pair. To give a closer look to such Lax pairs, it is convenient to examine their bosonic limit. In this limit (i.e., as \( \alpha_j = \bar{\pi}_j = \beta_j = \bar{\beta}_j = 0 \)) it is possible to introduce a spectral parameter into the linear system \((62)-(63)\) to derive the Lax–pair representation in a strong, operator form. Indeed, in the bosonic limit the first flow \((25)\) admit the one parameter group of invariance transformations \( a_j \rightarrow a_j - \gamma \), where \( \gamma \) is an arbitrary constant parameter. Applying this transformation to the bosonic limit of the linear system \((62)-(63)\), it becomes
\[
(L_0M)_i \equiv M_{j+1} + b_jM_{j-1} + a_jM_j = \gamma_M, \tag{65}
\]
and reproduces the well known form for the linear system corresponding to the bosonic Toda lattice (see, e.g., \([11]\) and references therein). This means that the Toda lattice equations are the integrability conditions of the Lax–pair representation,
\[
\frac{\partial}{\partial t_j}L_0 = [A_0, L_0], \tag{66}
\]
which is valid in a strong, operator sense. This Lax pair is to be compared with the one in the Introduction, which appears in the one–matrix model.
It is interesting to remark that in (63) the transformation parameter \( \gamma \) plays the role of the spectral parameter. The supersymmetric flow (25) does not admit such transformation, and, actually, it is not clear how to introduce a spectral parameter into (22) (which would be necessary in order to construct a Lax–pair representation in a strong, operator sense). Moreover, in the case under consideration, it is even not obvious that it is possible at all, because the linear system includes only bosonic wave functions.

Let us come now to a comparison between the bosonic limit of the Lax–pair representation (57), (59) with the representation (25). In the bosonic limit the recursion operator (18–20) has a block-diagonal structure and can be represented by a direct sum of the following two matrices:

\[
R_1 = \begin{pmatrix}
  a_{i-1} + \frac{b_i}{b_j} (a_i - a_{i-1}) \sum_{k \geq 1} \delta_{i,j-k}, & b_i (\delta_{i,j} + \delta_{i,j+1}) \\
  -\frac{1}{b_j} (b_{i+1} \sum_{k \geq 1} \delta_{i,j-k-1} - b_i \sum_{k \geq 1} \delta_{i,j-k+1}), & a_i \delta_{i,j}
\end{pmatrix},
\]

\[
R_2 = \begin{pmatrix}
  0, & 0, & -b_i \sum_{k \geq 1} \delta_{i,j+k}, & 0 \\
  0, & 0, & b_i \sum_{k \geq 1} \delta_{i,j-k}, & 0 \\
  0, & -\delta_{i,j-1}, & 0, & -a_i \sum_{k \geq 1} \delta_{i,j-k} \\
  \delta_{i,j+1}, & 0, & -a_{i-1} \sum_{k \geq 1} \delta_{i,j+k}, & 0
\end{pmatrix}.
\]

The matrix \( R_1 \) is the recursion operator of the bosonic Toda lattice hierarchy, while the interpretation of the matrix \( R_2 \) is a bit obscure. For it is not easy to interpret the following fact: the traces \( \text{tr}(R_2^l) \) are equal to zero at least for the first two values of the \( l = 1 \) and \( l = 2 \), and it is not possible to generate the Toda lattice Hamiltonians \( h_l \) using the standard prescription \( h_l = \text{tr}(R_2^l) \).

Calculating the matrix of Fréchet derivatives (\( K_1' \)) corresponding to the first flow (25) and finding its bosonic limit\(^6\), one can observe that it also splits into a direct sum of two matrices,

\[
K_1' = \begin{pmatrix}
  (a_i - a_{i-1}) \delta_{i,j}, & b_i (\delta_{i,j} - \delta_{i,j+1}) \\
  \delta_{i,j-1} - \delta_{i,j}, & 0
\end{pmatrix},
\]

\[
K_2' = \begin{pmatrix}
  a_i \delta_{i,j}, & 0, & -b_i \delta_{i,j}, & 0 \\
  0, & -a_{i-1} \delta_{i,j}, & 0, & -b_i \delta_{i,j} \\
  0, & \delta_{i,j+1}, & 0, & 0 \\
  \delta_{i,j}, & -\delta_{i,j-1}, & 0, & 0
\end{pmatrix},
\]

respectively. Comparison between (57)–(58) and (65) shows that in this way we produce two different Lax–pair representations with the matrices \( R_1, K_1' \) and \( R_2, K_2' \), respectively, which do not coincide with the corresponding matrices \( L_0, A_0 \) of the representation (25). Moreover, their algebraic origins are also different: the dimension of the Lax operator \( L_0 \) (63) corresponds to the fundamental representation of the \( sl(\infty) \) algebra, while the dimension of the recursion operator \( R_1 \) (57) corresponds to its adjoint representation. The former representation has a simpler matrix structure, and, in this sense, it is preferable to the latter. Moreover the representation (65) arises naturally in the context of the bosonic one-matrix model, (12). It is plausible to think that it is also relevant for the corresponding supermatrix model, if any. However the relationship between the two representations is not evident, and, for the time being, we can say that they are complementary representations.

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\(^{6}\)Let us recall that linear systems corresponding to supersymmetric integrable equations contain, as a rule, both bosonic and fermionic wave functions.

\(^{7}\)One should pay attention to the fact that these operations are not commutative, so, their order is crucial.
Let us comment now on the general structure of the supersymmetric counterpart of the first representation, i.e., supersymmetric Lax–pair representation whose bosonic limit contains the Lax operator $L_0$ (65). Keeping in mind the analogy with the bosonic case, it is reasonable to conjecture that a general structure of such supersymmetric Lax operator $L$ is similar to the general structure (18) of the recursion operator $R$, but their dimensions are different, and the relation between them is the same as in the bosonic case. Finally, we conjecture the following form of $L$ (compare (69) with (18)):

$$L = \left( \begin{array}{c}
B_{\infty} \times \infty, & F_{\infty} \times 2(\infty), & B_{2(\infty)} \times 2(\infty) \\
F_{2\infty} \times \infty, & B_{2\infty} \times 2(\infty), & | = L_0,
\end{array} \right)$$

(69)

where $|$ denotes the bosonic limit. The matrix (69) corresponds to the fundamental representation of the $sl(\infty|2\infty)$ superalgebra. As for the matrix $B_{2\infty} \times 2(\infty)|$, it either corresponds to some Lax operator of the bosonic Toda lattice or identically vanishes.

9 Conclusion.

In this paper we have presented in any detail a new $N = 2$ supersymmetric Toda lattice hierarchy and indicated the method to construct generalizations of it. The former can be thought of as the discrete version of the $N = 2$ differential NLS hierarchy. The latter would correspond to the discrete version of the differential $N = 2$ GNLS hierarchies, [9]. A few years ago $N = 2$ discrete Toda lattice hierarchies were proposed, [13]: it would be interesting to know whether they bear any relation to our hierarchies. The relation, if it exists, is rather non–trivial. There appear to be no straightforward restriction that may lead to our hierarchies, perhaps some more complicated coset construction is necessary.

Finally we would like to make some comments on relevant consequences of our result. The $N = 2$ hierarchy presented in this paper, as pointed out in the introduction, is the extension of the discrete integrable hierarchy that arises in the one–matrix random model. Starting from the Virasoro constraints of the latter and the fact that in any model these constraints must be consistent with the underlying hierarchy, we are very likely to already possess all the information we need in order to write down the complete set of super–Virasoro constraints consistent with the $N = 2$ Toda lattice hierarchy. This means that we can completely calculate the free energy and correlators of the would–be $N = 2$ one–matrix model (see, for example, the second reference of [5]) and, therefore, to completely determine it. Moreover, the correlators in the non-supersymmetric case are interpretable as correlators of a topological field theory. Therefore, once we know the correlators in the $N = 2$ case, it will be interesting to examine what kind of new information about topological field theories they may provide.

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Appendix. Canonical basis and Lagrangian formulation of the $t_1$-flow equations.

It is interesting that the first flow (25) of our $N = 2$ hierarchy admits a Lagrangian formulation. Let us introduce the new coordinates $\{x_j, p_j, \xi_j, \bar{\xi}_j, \eta_j, \bar{\eta}_j\}$ in the phase space (2)

$$\begin{align*}
    b_j &= e^{x_j-x_{j-1}}, & a_j &= -p_j, \\
    \beta_j &= -e^{x_j} \xi_j, & \bar{\beta}_j &= e^{-x_j} \bar{\xi}_j, \\
    \alpha_j &= \eta_{j-1} - \eta_j, & \bar{\alpha}_j &= \bar{\eta}_j
\end{align*}$$

(70)

for which the first Hamiltonian structure (11) becomes canonical,

$$\{x_i, p_j\}_1 = \delta_{i,j}, \quad \{\xi_i, \bar{\xi}_j\}_1 = \delta_{i,j}, \quad \{\eta_i, \bar{\eta}_j\}_1 = \delta_{i,j}. \quad (71)$$

In terms of these coordinates the Hamiltonian $H_2$ (3) is

$$H_2 = \sum_{j=-\infty}^{\infty} \left( \frac{1}{2} p_j^2 + e^{x_j-x_{j-1}} + e^{x_j} \xi_j \bar{\xi}_j + e^{-x_j} (\eta_{j-1} - \eta_j) \bar{\xi}_j \right) \quad (72)$$

It generates the following $t_1$-flow equations (23) via the first Hamiltonian structure (1).

$$\begin{align*}
    \frac{\partial}{\partial t_1} x_j &= p_j, & \frac{\partial}{\partial t_1} p_j &= e^{x_{j+1}-x_j} - e^{x_j-x_{j-1}} - e^{x_j} \xi_j \bar{\xi}_j + e^{-x_j} (\eta_{j-1} - \eta_j) \bar{\xi}_j, \\
    \frac{\partial}{\partial t_1} \xi_j &= e^{-x_j-1} (\eta_j - \eta_{j-1}), & \frac{\partial}{\partial t_1} \bar{\xi}_j &= e^{x_j} \bar{\xi}_j, \\
    \frac{\partial}{\partial t_1} \eta_j &= -e^{x_j} \xi_j, & \frac{\partial}{\partial t_1} \bar{\eta}_j &= e^{-x_j} \bar{\xi}_{j+1} - e^{-x_j-1} \bar{\xi}_j,
\end{align*}$$

(73)

admitting the automorphism $\sigma_j$ (12), which has the following realization in terms of the new coordinates (70):

$$\begin{align*}
    \sigma_j x_i \sigma_j^{-1} &= -x_{i-1}, & \sigma_j p_i \sigma_j^{-1} &= -p_{i-1}, \\
    \sigma_j \xi_i \sigma_j^{-1} &= -\xi_{i-1}, & \sigma_j \bar{\xi}_i \sigma_j^{-1} &= -\bar{\xi}_{i-1}, \\
    \sigma_j \eta_i \sigma_j^{-1} &= -\sum_{k=-i}^{\infty} \bar{\eta}_{j+k}, & \sigma_j \bar{\eta}_i \sigma_j^{-1} &= \eta_{j-i} - \eta_{j-1}.
\end{align*}$$

(74)

Let us mention that besides the coordinates $\eta_j$ and $\bar{\eta}_j$, there is one more set of the canonical coordinates $\bar{\eta}_j$ and $\bar{\bar{\eta}}_j$ with the lattice–local equations of motion, related with the former ones by the transformation

$$\bar{\eta}_j = \eta_{j-1} - \eta_j, \quad \bar{\bar{\eta}}_j = \sum_{k=1}^{\infty} \bar{\eta}_{j-k} \Leftrightarrow \bar{\eta}_j = -\bar{\bar{\eta}}_{j+1}, \quad \eta_j = -\sum_{k=0}^{\infty} \bar{\eta}_{j-k}, \quad (75)$$

which does not include the fields $\xi_j$ and $\bar{\xi}_j$. In addition to this, of course there exist other canonical transformations of the Poisson algebra (71) which mix all fields and also lead to local equations for them.

Following the standard procedure, one can derive the Lagrangian $\mathcal{L}$ and action $S$,

$$\begin{align*}
    S &= \int dt L \equiv \int dt \left[ \sum_{j=-\infty}^{\infty} \left( p_j \frac{\partial}{\partial t_1} x_j + \xi_j \frac{\partial}{\partial t_1} \bar{\xi}_j + \eta_j \frac{\partial}{\partial t_1} \bar{\eta}_j \right) - H_2 \right] \\
    &= \int dt \sum_{j=-\infty}^{\infty} \left[ \frac{1}{2} \left( \frac{\partial}{\partial t_1} x_j \right)^2 + \xi_j \frac{\partial}{\partial t_1} \bar{\xi}_j + \eta_j \frac{\partial}{\partial t_1} \bar{\eta}_j - e^{x_j-x_{j-1}} - e^{x_j} \xi_j \bar{\xi}_j - e^{-x_j} (\eta_{j-1} - \eta_j) \bar{\xi}_j \right].
\end{align*}$$

(76)

\footnote{Here, for convenience we have reversed the $t_1$-sign, $t_1 \rightarrow -t_1$.}
which can be important in connection with the quantization problem. As usual, the variation of the action $S$ with respect to the fields $\{x_j, \xi_j, \bar{\xi}_j, \eta_j, \bar{\eta}_j\}$ produces the equations of motion (72) for them, where the momenta $p_j$ are replaced by $\frac{\partial}{\partial t_1} x_j$. If, in addition to the momenta, the fields $\eta_j$ and $\bar{\eta}_j$ are also eliminated from eqs. (73) by means of corresponding equations expressing them in terms of the fields $\{x_j, \xi_j, \bar{\xi}_j\}$ and their $t_1$-derivatives, the remaining equations become

$$
\frac{\partial^2}{\partial t_1^2} x_j = e^{x_{j+1} - x_j} - e^{x_j - x_{j+1}} - \xi_j \frac{\partial}{\partial t_1} \xi_j + \bar{\xi}_{j+1} \frac{\partial}{\partial t_1} \xi_{j+1},
$$

$$
\frac{\partial}{\partial t_1} (e^{x_j - x_{j+1}} \frac{\partial}{\partial t_1} \xi_j) = e^{x_j - x_{j+1}} \xi_{j+1} - e^{x_j - x_{j-1}} \xi_j,
$$

and reproduce the set of the restricted f-Toda chain equations (3), which form a subset of the f-Toda chain equations (see section 6).

The canonical basis (70) is not important only for the Lagrangian formulation, there is also another reason to introduce it. This is the known fact that the basis (70) can be more convenient than the old one (9) to study the reductions (12)–(13) of some structures characterizing the infinite $N = 2$ supersymmetric Toda lattice hierarchy. Indeed, for the case of the bosonic Toda lattice hierarchy, in [14] it was argued that for the recursion operator such reduction is not admitted at all in the old basis $\{a_i, b_i\}$, whereas the reduction is possible in the canonical basis $\{x_i, p_i\}$ (in both bases the Hamiltonian structures can be reduced). One expects that similar arguments should be relevant also for the supersymmetric case. Having this in mind, we present the second Poisson structure of the N=2 super Toda lattice hierarchy in the canonical basis,
where we have introduced the antisymmetric lattice function $\varepsilon_{i,j}$,

$$\varepsilon_{i,j} = -\varepsilon_{j,i} \equiv 1, \quad i > j$$

(79)

with the evident properties:

$$\varepsilon_{-i,-j} = -\varepsilon_{i,j}, \quad \varepsilon_{i,j-1} - \varepsilon_{i,j} = \delta_{i,j-1} + \delta_{i,j},$$

(80)

and only nonzero brackets are written down. Together with the inverse matrix for the first Hamiltonian structure (71), which can be easily derived due to the very simple structure of those equations, the relations (78) define the recursion operator in the canonical basis (70). Its explicit expression can be easily obtained and will not be written down here.

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