A solution of the bosonic and algebraic Hamiltonians by using an AIM

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Abstract
We apply the notion of an asymptotic iteration method (AIM) to determine the
eigenvalues of the bosonic Hamiltonians that include a wide class of quantum
optical models. We consider the solutions of the Hamiltonians, which are even
the polynomials of the fourth order with respect to the Boson operators. We
also demonstrate the applicability of the method for obtaining the eigenvalues
of the simple Lie algebraic structures. The eigenvalues of the multi-boson
Hamiltonians have been obtained by transforming them into the form of the
single boson Hamiltonian in the framework of the AIM.

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1. Introduction
The study of the same problems from different points of view leads to the progress of
science and includes a lot of mathematical taste. An iteration technique [1–7] has recently
been suggested to obtain the eigenvalues of the Schrödinger equation which improves both
analytical and numerical determination of the eigenvalues and has been developed for some
matrix Hamiltonians, arising from the development of fast computers [8, 9]. An asymptotic
iteration method (AIM) is very efficient to establish the eigenvalues of various quantum
mechanical systems because of their simplicity and low round-off error. This method has
widely been applied for the determination of the eigenvalues of the Schrödinger-type equations.
Encouraged by its satisfactory performance through comparisons with other methods, we feel
tempted to develop the AIM to obtain the eigenvalues of algebraic Hamiltonians. In contrast
to the solution of the Schrödinger equation by using the AIM including Coulomb, Morse,
harmonic oscillator, etc, type potentials, the study of the algebraic Hamiltonians [10–12] has
not attracted much attention in the literature. Such Hamiltonians have been found to be useful
in the study of the electronic properties of semiconductors, quantum dots and quantum wells.
It is evident that the formalism can also be developed for solving algebraic equations.
The algebraic techniques have been proven to be useful in the description of the physical problems in a variety of fields [11–18]. In recent years there has been a great deal of interest in quantum optical models which reveal new physical phenomena described by the Hamiltonians expressed as the nonlinear functions of Lie algebra generators or boson and/or fermion operators [19–23]. Such systems have often been analyzed by using numerical methods because the implementation of the Lie algebraic techniques to solve those problems is not very efficient and most of the other analytical techniques do not yield simple analytical expressions. They require tedious calculations. In principle, if a Hamiltonian is expressed by boson operators, one could rely directly on the known formulae of the action of boson operators on a state with a defined number of particles without solving differential equations. Apart from the mentioned method, sometimes the Hamiltonians can be put in a simple form by using the transformation properties of the bosons.

In this paper, the AIM is suggested and adapted to solve the bosonic Hamiltonians. We note that this has never been done before. As a particular case our model includes the solutions of the Hamiltonian of the multiphoton interactions and the Hamiltonian of the systems of photons and bosons expressed in a single-mode form. We briefly discuss the bosonic construction of the various Hamiltonians. These Hamiltonians are not only mathematically interesting but they also have potential interest in physics.

The paper is organized as follows. In section 2, we briefly review the properties of boson and its differential realization. The procedure for solving a bosonic Hamiltonian in the framework of the AIM is presented in this section. Section 3 is devoted to illustrating the determination of the eigenvalues of a bosonic Hamiltonian in the framework of the AIM. The bosonization of the physical Hamiltonians whose original forms are given as differential operators is discussed. As a practical example we illustrate the solution of the anharmonic oscillator and multiphoton interaction problems. In section 4, we introduce a technique to obtain the eigenvalues of the two-mode bosonic Hamiltonians by using the AIM. We present the application of the AIM in order to obtain eigenvalues for a class of models describing two-mode multiphoton processes. Finally, we comment on the validity of our method and remark on the possible use of our method in the different fields of physics.

2. Basic formalism and the solution of single boson Hamiltonian

In this section, we illustrate the solution of the single boson Hamiltonians by modifying the AIM. The usual differential realization of the annihilation operator $a$ and that of the creation operator $a^*$ are given by

$$a^* = \frac{1}{\sqrt{2}} \left( -\frac{\partial}{\partial x} + x \right), \quad a = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x} + x \right),$$

and they act on the state $|n\rangle$:

$$a^*|n\rangle = \sqrt{n} + 1|n + 1\rangle; \quad a|n\rangle = \sqrt{n} |n - 1\rangle,$$

with the commutation relation

$$[a, a^*] = 1.$$

A single boson Hamiltonian describing a physical system can be expressed as

$$H = \sum_i \gamma_{i,i} a^i (a^+)^i + \sum_{i,j(i \neq j)} \gamma_{i,j} a^i (a^+)^j,$$

where $\gamma_{i,j}$ is a constant. It is obvious that the first part of the Hamiltonian $H$ is diagonal and exactly solvable. The second part of the Hamiltonian $H$ includes non-diagonal terms and it
is usually solved by using various perturbation techniques. Our task is now to develop an AIM to obtain the eigenvalues of $H$. We assume that the action of $H$ on the state $|n\rangle$ produces the following three-term recurrence relation (or reduced to the three-term recurrence relation) such that

$$|n + 2\rangle = r_n |n + 1\rangle + s_n |n\rangle,$$

(4)

where $r_n$ and $s_n = E - s'_n$ are the functions of $n$. From the analogy of the AIM [1] it follows that (4) can be put in a more suitable form in order to obtain the eigenvalues $E$ of $H$. The reformulation of (4) provides the following equations:

$$n = 0; \quad |2\rangle = r_0 |1\rangle + s_0 |0\rangle = p_0 |1\rangle + q_0 |0\rangle$$
$$n = 1; \quad |3\rangle = r_1 |2\rangle + s_1 |1\rangle = p_1 |1\rangle + q_1 |0\rangle$$
$$\ldots$$
$$n = m; \quad |m + 2\rangle = r_m |m + 1\rangle + s_m |m\rangle = p_m |1\rangle + q_m |0\rangle,$$

(5)

where $p_m$ and $q_m$ are given by

$$p_m = r_m p_{m-1} + s_m p_{m-2}$$
$$q_m = r_m q_{m-1} + s_m q_{m-2},$$

(6)

with the initial conditions

$$p_{-1} = q_{-2} = 1 \quad \text{and} \quad p_{-2} = q_{-1} = 0.$$

To this end, we assume that $m$ is large enough and the states reach their asymptotic values. Thus, we can write

$$|m + 2\rangle = p_m |1\rangle + q_m |0\rangle$$
$$|m + 3\rangle = p_{m+1} |1\rangle + q_{m+1} |0\rangle.$$

(7)

After all we can concisely write that

$$\frac{p_m}{q_m} = \frac{p_{m+1}}{q_{m+1}} \quad \text{or} \quad q_m p_{m+1} - q_{m+1} p_m = 0.$$

(8)

The last equation can be solved for the eigenvalues $E$; then the last approximation leads to the determination of the eigenvalues of the Hamiltonian $H$. Before going further, we note that the eigenvalues of the associated problem can be obtained by using the following MATHEMATICA program code. Let us define $|n\rangle = f[n]$; then

$$k = 20; \quad \text{Do[f[n+2] = Simplify[r_f[n+1]+s_f[n]], \{n, 0, k\}]}$$

(*where k is the number of iterations*)

$$\text{NSolve[Coefficient[f[k+2], f[0]]*Coefficient[f[k], f[2]}$$

$$-\text{Coefficient[f[k+2], f[2]]*Coefficient[f[k], f[0]] == 0, E1}$$

(*E1 are the eigenvalues of H*)

In the following sections, we illustrate our task on an explicit example.
3. Eigenstate of the single boson Hamiltonians

In this section we study the determination of the single- and multi-boson Hamiltonians in the framework of the AIM.

3.1. Anharmonic oscillator.

The solution of the Schrödinger equation including anharmonic potential has attracted a lot of attention, arising its considerable impact on various branches of physics as well as biology and chemistry. Besides its importance in physics, biology and chemistry, in practice the anharmonic oscillator problem is always used to test the accuracy and efficiency of the unperturbative methods. In this section we take a new look at the solution of the anharmonic oscillator problem through the modified AIM. The equation is described by the Hamiltonian

\[ H = -\frac{d^2}{dx^2} + x^2 + \alpha x^4, \]  

where \(\alpha\) is a constant. Our task is now to demonstrate that the Hamiltonian (9) can be expressed in terms of the bosons. One way to express the Hamiltonian \(H\) with boson operators is to use an appropriate differential realization of bosons. Using the realization (1), the Hamiltonian (9) can be written as

\[ H = a^+ a + aa^+ + \frac{\alpha}{4} (a + a^+)^4. \]  

When the Hamiltonian (10) acts on the state \(|n\rangle\), the eigenvalue equation \(H|n\rangle = E|n\rangle\) can be transformed to the following recurrence relation:

\[ (H - E)|n\rangle = (2n + 1 - E)|n\rangle + \frac{3\alpha}{2} \left( n + n^2 + \frac{1}{2} \right)|n\rangle \]  

\[ + \alpha \sqrt{n(n+1)(n+2)} \left( n + \frac{3}{2} \right)|n+2\rangle + \alpha \sqrt{n(n-1)} \left( n - \frac{1}{2} \right)|n-2\rangle \]  

\[ + \frac{\alpha}{4} \sqrt{(n+1)(n+2)(n+3)(n+4)}|n+4\rangle \]  

\[ + \frac{\alpha}{4} \sqrt{n(n-1)(n-2)(n-3)}|n-4\rangle = 0. \]  

Here, the skill is to express the \(n\)th even state in terms of the \(|0\rangle\) and \(|2\rangle\) states and the \(n\)th odd state in terms of the \(|1\rangle\) and \(|3\rangle\) states. Applying the technique given in the previous section, we can obtain the following expressions:

\[ n = 0; \quad |4\rangle = p_0|0\rangle + q_0|2\rangle \]  

\[ n = 2; \quad |6\rangle = p_2|0\rangle + q_2|2\rangle \]  

\[ \ldots \]  

\[ n = m; \quad |m+4\rangle = p_m|0\rangle + q_m|2\rangle \]  

\[ n = m+2; \quad |m+6\rangle = p_{m+2}|0\rangle + q_{m+2}|2\rangle. \]  

The truncation of the state for the large values of \(m\) leads to the following relations:

\[ q_m p_{m+2} - p_m q_{m+2} = 0. \]  

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Table 1. The comparison of the eigenvalues of the anharmonic oscillator computed by the AIM 
[1], direct numerical integration method [28, 29] and by the present work, ATEM when $\alpha = 0.1$.

| $n$ | $E_{\text{present}}$ | $E[1]$ | $E[28, 29]$ |
|-----|----------------------|--------|-------------|
| 0   | 1.065 286            | 1.065 286 | 1.065 286 |
| 1   | 3.306 872            | 3.306 871 | 3.306 872 |
| 2   | 5.747 959            | 5.747 960 | 5.747 959 |
| 3   | 8.352 678            | 8.352 642 | 8.352 678 |
| 4   | 11.098 60            | 11.098 35 | 11.098 60 |
| 5   | 13.969 93            | 13.966 95 | 13.969 93 |

Here $p_i$ and $q_i$ can be calculated by using the following MATHEMATICA program code (again 
we define $|n\rangle = f[n]$):

\[
\begin{align*}
s1 &= \text{Collect}[\text{Simplify}[\text{Solve}[(H-E)f[n] \Rightarrow 0, f[n+4]], \{f[n+1]\}]] \\
&\quad (*f[n+4] \text{ is obtained from } (11)*)
\end{align*}
\]

\[
\begin{align*}
k &= 20; \text{Do}[f[n+4] &= \text{Simplify}[s1[[1,1,2]], \{n, 0, k\}] \\
&\quad (*\text{where } k \text{ is the number of iterations}*)
\end{align*}
\]

\[
\begin{align*}
\text{Solve}[\text{Coefficient}[f[k+4], f[0]]*\text{Coefficient}[f[k+2], f[2]] \\
- \text{Coefficient}[f[k+4], f[2]]*\text{Coefficient}[f[k+2], f[0]] \Rightarrow 0, E1]/. \alpha \rightarrow 0.1
\end{align*}
\]

(*gives the eigenvalues of the even state*)

\[
\begin{align*}
\text{Solve}[(\text{Coefficient}[f[k+3], f[1]]*\text{Coefficient}[f[k+1], f[3]]) \\
- \text{Coefficient}[f[k+3], f[3]]*\text{Coefficient}[f[k+1], f[1]]] \Rightarrow 0, E1]/. \alpha \rightarrow 0.1
\end{align*}
\]

(*gives the eigenvalues of the odd states*).

It is obvious that the program can easily be adapted for similar problems. The method introduced here gives accurate results for the bosonic Hamiltonian (10). The results are given in table 1. As shown in table 1, our data confirm some previous results. Note that the results are obtained after 20 iterations.

In the following subsections, it is shown that this asymptotic approach opens the way to the treatment of single boson quantum optical systems.

3.2. A simple multiphoton interaction Hamiltonian.

The Hamiltonian of the single-mode coherent light with an optically bistable two-photon medium is given by [24–26]

\[ H = \omega a^+a + \kappa(a^{+2} - a^2) + \Omega a^{+2}a^2, \] (14)

where $\omega$ is the frequency, and $\kappa$ and $\Omega$ are the real constants. The time development of the Hamiltonian (14) was studied by [24]. Here we study the determination of the eigenstate of the equation $H|n\rangle = E|n\rangle$. The action of the Hamiltonian on the state $|n\rangle$ can be written as

\[ (\omega n + \Omega(n - 1) - E)|n\rangle + \kappa(\sqrt{n(n - 1)}|n - 2\rangle - \sqrt{(n + 1)(n + 2)}|n + 2\rangle) = 0. \] (15)
Table 2. Eigenvalues \(E\) of the Hamiltonian (14), for \(\omega = 1\) and various values of \(\kappa\) and \(\Omega\).

| \(n\) | \(\kappa = \sqrt{3}/2; \Omega = 0\) | \(\kappa = 0.1; \Omega = 0.1\) | \(\kappa = 0.1; \Omega = 0.5\) | \(\kappa = 0.5; \Omega = 0.1\) |
|-------|----------------------------------|--------------------------------|--------------------------------|--------------------------------|
| 0     | 0.009 033 68                    | 0.006 654 83                  | 0.198 280 876 52              | 1.526 446 774 04               |
| 1     | 1.022 986 33                    | 1.011 994 512                 | 2.939 741 839                 | 4.477 321 501 23              |
| 2     | 2.230 860 41                    | 3.010 484 295                 | 5.167 778 494                 | 6.167 778 494                 |
| 3     | 3.635 725 96                    | 6.010 224 859                 | 9.029 296 095                 | 10.167 778 494               |
| 4     | 5.238 941 89                    | 10.010 131 290                | 15.010 086 374                | 16.167 778 494               |
| 5     | 7.041 178 81                    | 15.010 086 374                | 23.010 131 290                | 24.167 778 494               |

Our task is now to express the \(n\)th state in terms of the \(|0\rangle\) and \(|1\rangle\) states:

\[
\begin{align*}
  n = 0; & |2\rangle = p_0 |0\rangle \\
  n = 1; & |3\rangle = p_1 |1\rangle \\
  n = 2; & |4\rangle = p_2 |0\rangle \\
  \ldots
  n = m; & |m + 2\rangle = p_m |0\rangle \\
  n = m + 1; & |m + 3\rangle = p_{m+1} |0\rangle.
\end{align*}
\] (16)

It is obvious that the eigenvalues of (14) can be obtained for the even/odd eigenstate setting \(p_m = 0/p_{m+1} = 0\). In this case we have used the MATHEMATICA program code given in section 2. The results are given in table 2. We have checked that the Hamiltonian (14) can exactly be solved when \(\Omega = 0\). In this case for \(\kappa = \sqrt{3}/2\), eigenvalues \(E = 2n + \frac{1}{2}\), and we have obtained the same result by using the procedure given here.

Consequently, we have shown that the AIM can be applied to the determination of the eigenstate of the single boson system.

4. Eigenstate of multiboson Hamiltonians

In this section, we present the application of the AIM in order to obtain eigenvalues for a class of models describing two-mode multiphoton processes. In addition to the annihilation operator \(a\) and the creation operator \(a^+\), we introduce the operators \(b\) and \(b^+\) in the Hilbert space as

\[
b^+ = \frac{1}{\sqrt{2}} \left( -\frac{\partial}{\partial y} + y \right); \quad b = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial y} + y \right).
\] (17a)

The two-boson operators \(a\) and \(b\) obey the usual commutation relations

\[
[a, b] = [a, b^+] = [b, a^+] = [a^+, b^+] = 0, \quad [a, a^+] = [b, b^+] = 1.
\] (18)

Following a similar method which has been developed in the previous section, we try to determine the eigenvalues for a general class of two-mode multiphoton models. The Hamiltonian of such a system is given by

\[
H = r a^+ a + s b^+ b + \kappa (a^+ b^+ + b^+ a^+),
\] (19)

where \(r\) and \(k\) are the positive integers.
In this formalism when \( r = s \) the Hamiltonian (19) satisfies the \( SU(2) \) symmetry with the generators \([12, 27]\)

\[
J_+ = a^* b, \quad J_- = b^* a, \quad J_0 = \frac{i}{2} (a^* a - b^* b).
\]

(20)

These are the Schwinger representation of the \( su(2) \) algebra and they satisfy the commutation relations

\[
[J_+, J_-] = 2J_0, \quad [J_0, J_\pm] = \pm J_\pm.
\]

(21)

The fourth generator is the total boson number operator

\[
N = (a^* a + b^* b),
\]

(22)

which commutes with the \( su(2) \) generators. The Casimir operator of this structure is given by

\[
J = J_- J_+ + J_0 (J_0 + 1) = \frac{1}{2} N (N + 2).
\]

(23)

If we denote the eigenvalues of the operator \( J \) by

\[
J = j (j + 1),
\]

(24)

it is obvious that the irreducible representations of \( su(2) \) can be characterized by the total boson number \( N = 2j \). The application of the realization (20) on a set of \( 2j + 1 \) states leads to the \( (2j + 1) \)-dimensional unitary irreducible representation for each \( j = 0, 1/2, 1, \ldots \). If the basis states are \(|j, m\> (m = j, j - 1, \ldots, -j)\), then the action of the operators on the basis states is given by

\[
\begin{align*}
J_0 |j, m\> &= m |j, m\>

J_- |j, m\> &= \sqrt{(j + m)(j - m + 1)} |j, m + 1\>

C |j, m\> &= j (j + 1) |j, m\>.
\end{align*}
\]

(25)

An immediate practical consequence of these representations of \( su(2) \) algebra is that the Hamiltonian (19) can easily be expressed as

\[
H = \omega_s N + \kappa (J_+^2 + J_-^2).
\]

(26)

The eigenvalue equation \( H |j, m\> = E |j, m\> \) can be written as

\[
(2\omega s j - E) |j, m\> + \kappa \sqrt{\frac{(-1)^s (m + s - j - 1)! (m + s + j)!}{(m - j - 1)! (m + j)!}} |j, m + s\>

+ \kappa \sqrt{\frac{(-1)^s (-m + s - j - 1)! (-m + s + j)!}{(-m - j - 1)! (-m + j)!}} |j, m - s\> = 0,
\]

(27)

where \( N = 0, 1, 2, \ldots \). In this case the state \(|j, m + s\> \) can be expressed as follows:

\[
\begin{align*}
m &= -j; \quad |j, -j + s\> &= p_{-j} |j, -j\> + q_{-j} |j, -j - s\>
m &= -j + 1; \quad |j, -j + s + 1\> &= p_{-j+1} |j, -j + 1\> + q_{-j+1} |j, -j - s + 1\>
& \ldots
\end{align*}
\]

(28)

\[
\begin{align*}
m &= j - 1; \quad |j, j + s - 1\> &= p_{j-1} |j, j - 1\> + q_{j-1} |j, j - s - 1\>
m &= j; \quad |j, j + s\> &= p_{j-1} |j, j\> + q_{j-1} |j, j - s\>
\end{align*}
\]

with the boundary condition \(|j, -j - s\> = 0\). The Hamiltonian (26) is exactly solvable when \( s = 1 \) and after some straightforward treatment we can show that \( E = 2j + 2(n - j)\kappa \). For the values \( s = 2 \) and \( j = 3 \), the results are given in table 3.
Table 3. The eigenvalues of the Hamiltonian (26), for $\omega = 1$, $s = 2$ and $j = 3$.

| $m$ | $\kappa = \frac{1}{10}$ | $\kappa = \frac{1}{5}$ | $\kappa = \frac{1}{2}$ |
|-----|-------------------------|-------------------------|-------------------------|
| 0   | 12                      | 12                      | 12                      |
| $\pm 1$ | $\frac{1}{5}(57 \pm 2\sqrt{6})$ | $\frac{2}{5}(27 \pm 2\sqrt{6})$ | $(9 \pm 2\sqrt{6})$ |
| $\pm 2$ | $\frac{1}{5}(60 \pm 2\sqrt{15})$ | $\frac{2}{5}(30 \pm 2\sqrt{15})$ | $(12 \pm 2\sqrt{15})$ |
| $\pm 3$ | $\frac{1}{5}(63 \pm 2\sqrt{6})$ | $\frac{2}{5}(33 \pm 2\sqrt{6})$ | $(15 \pm 2\sqrt{6})$ |

5. Conclusion

The basic feature of our approach is to reformulate the AIM for obtaining the eigenvalues of the bosonic Hamiltonians. Furthermore, the technique given here has been used to determine the eigenvalues of the anharmonic oscillator, the multiphoton interaction problem and a class of models describing two-mode multiphoton processes. We have shown that the AIM gives accurate results for the eigenvalue of bosonic Hamiltonians.

As a further work the method presented here can be developed in various directions. A complete spectrum of the quasi-exactly solvable problems can be obtained in the framework of the method presented here. Since most of the quasi-exactly solvable problems can be expressed in terms of the generators of $su(1, 1)$ or $su(2)$ Lie algebra, the resulting recurrence relation can easily be solved by using the procedure given in this paper. The suggested approach can also be extended for solving boson–fermion systems. Before ending this work a remark is in order. This extension leads to the determination of the eigenvalues of various Hamiltonians: Jahn–Teller Hamiltonians [19], Rabi Hamiltonian [20], Hamiltonians of the Bose–Einstein condensation problems.

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