Abstract
This paper provides a unified approach for detecting sample selection in nonparametric conditional quantile and mean functions. Our testing strategy consists of a two-step procedure: the first test is an omitted predictor test with the propensity score as omitted variable. This test has power against $\sqrt{n}$–alternatives. While failure to reject the null implies no selection, we cannot, as any omnibus test, distinguish between rejection due to genuine selection or to misspecification. Since differentiation of the latter has implications for nonparametric (point) identification and estimation of the conditional quantile function, our second test is designed to detect misspecification. Using only individuals with propensity score close to one, this test relies on an ‘identification at infinity’ argument, but accommodates cases of irregular identification. Finally, our testing procedure does not require any parametric assumptions on the selection equation, and all our results in the quantile case hold uniformly across quantile ranks in a compact set. We apply our procedure to test for selection in log hourly wages using UK Family Expenditure Survey data.

Key-Words: Nonparametric Estimation, Conditional Quantiles, Conditional Mean, Irregular Identification, Wild bootstrap.

JEL Classification: C12, C14, C21.

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1 Introduction

Empirical studies using non-experimental data are often plagued by the presence of non-random sample selection: individuals typically self select themselves into employment, training programs etc. on the basis of characteristics which are believed to be non-random and unobservable to the researcher(s) (Gronau, 1974; Heckman, 1974). In fact, it is well known that ignoring selection in conditional mean models induces a bias in the estimation, which can be additive (see e.g. Heckman, 1979; Das et al., 2003) or multiplicative (Jochmans, 2015) depending on the functional form of the model. In both cases, one can deal with the selection bias adopting a control function approach. On the other hand, until recently very little was known about the identification and estimation of conditional quantile functions in the presence of endogenous selection, see the recent survey by Arellano and Bonhomme (2017b). A notable exception is the case of sample selection in location shift models, where it induces a parallel shift in the quantile function, and hence can be taken into account by simply correcting for the selection bias as in the mean case. In all other cases, including linear quantile regression models, however, the presence of endogenous selection causes a rotation of the quantile function (Arellano and Bonhomme, 2017a). Hence, quantiles can no longer be corrected in a simple manner, e.g. by ‘adding’ the selection bias term. Moreover, as shown by these authors, in the absence of parametric assumptions, conditional quantile functions are only point identified ‘at infinity’ or when the joint distribution of outcome and selection error is real analytic, a condition which is difficult to verify in practice. However, even in the real analytic case, consistent estimation requires additional parametric restrictions (e.g. on the copula of the two errors). Hence, the importance of testing for sample selection a priori.

This paper provides a unified approach for detecting sample selection in conditional mean and conditional quantile functions. We do so in a completely nonparametric fashion, imposing only a minimal set of functional form assumptions on both the outcome and the selection equation(s). In fact, the only additional assumption is that selection (if present) affects outcome through the propensity score, the probability to be in the selected sample. This is a standard assumption in the selection literature (e.g., Das et al., 2003). Our objective is then to develop a rule for deciding between selection and non-selection, and to control the overall classification error.

To understand the heuristics of our testing strategy, note that we have selection if the conditional quantile (mean) error depends on the propensity score when the latter is in the interior of the unit interval. Instead, it is independent of the propensity score when the latter is one. This is because individuals with a propensity score equal to one are selected into the sample almost surely. The hypothesis of selection will therefore consist of the intersection of two different hypotheses. By contrast, we do not have selection if either the conditional quantile (mean) error is independent of the propensity score when the latter is in the interior, or when it depends on the propensity score, regardless whether the latter is in the interior or close to one. Hence, the hypothesis of non-selection consists of the union of two different hypotheses.

We formalize the above heuristic argument in a decision rule, which we implement in a two-step testing procedure. In the first step, we propose a test for omitted predictors, where the omitted predictor is the (estimated) propensity score. Here, the null hypothesis is that the conditional quantile (mean) error does not depend on the propensity score, when the latter is in the interior of its support. For the conditional quantile function, a corresponding test can be based on the statistic suggested by
Volgushev et al. (2013), while for the conditional mean case we use the statistic suggested by Delgado and Gonzalez-Manteiga (2001). For both conditional mean and quantile tests, the statistic converges in distribution to a functional of a Gaussian process under the null hypothesis of no omitted propensity score. Importantly, these tests have power against generic $\sqrt{n}$ local alternatives, where $n$ denotes the sample size. In addition, a key difference of our quantile test relative to the test of Volgushev et al. (2013) consists in the fact that our results hold uniformly over all quantile ranks in a compact subset of $(0, 1)$.

If we fail to reject the null hypothesis in the first step, we stop and decide for non-selection. This is because, the null hypothesis in the first step is part of the union of hypotheses forming the null of no selection in our decision rule. By contrast, if we reject the null in the first step, we cannot, as with other omnibus tests, distinguish between a rejection due to genuine endogenous selection or to omission of relevant predictor variable(s), which affect outcome and are not independent of the propensity score. As pointed out above, this is particularly relevant when the interest lies in conditional quantile functions as selection leads to a general loss of point identification in the nonparametric case (cf. Arellano and Bonhomme, 2017a). Indeed, the complement of the null in the first test is part of the intersection of hypotheses forming the null of selection in our rule.

Under the maintained assumption of no selection, our second test is therefore designed to detect misspecification due to omitted variable(s) correlated with the propensity score. This is achieved by a localized version of the first test, based only on individuals having a propensity score close to one, so that selection bias does no longer ‘bite’. Importantly, while the second step relies on a so called ‘identification at infinity’ argument, meaning that the support of the propensity score has to comprise the boundary point one, our test does allow for a thin set of observations close to the boundary, thus accommodating cases of so called irregular identification (Khan and Tamer, 2010).

If we do not reject the second test, we decide in favor of selection. In this case, we can estimate the nonparametric conditional quantiles on a subset of observations with propensity close to one only. The rate of convergence of the second test depends on both the degree of irregularity of the marginal density of the propensity score as well as on the size of the set of covariate values for which identification at infinity holds. We therefore suggest a studentized version of the test statistic, which is rate adaptive and converges weakly even if numerator and denominator of the statistic diverge individually at the same rate.

For both the test statistics in the first and the second step, we establish the first order validity of wild bootstrap critical values. Given the outcome of the test(s), we decide in favor or against selection, and obtain a bound for classification errors associated with our decision. Finally, we apply our testing procedure to test for selection in log hourly wages of females and males in the UK using data from the UK Family Expenditure Survey from 1995 to 2000. The same data was recently also used by Arellano and Bonhomme (2017a) to analyze gender wage inequality in the UK. We run our testing procedure on two different sub-periods, namely 1995-1997 and 1998-2000. As a preview of the results, we cannot find evidence for selection among females for the 1995-1997 period, but only for the 1998-2000 period. By contrast, while we reject the null of the first test for males with data from 1995 to 1997, our second test strongly suggests that this rejection may actually be due to misspecification of the quantile function, a feature that might have remained undetected without our testing procedure.

1 Thus, to obtain power in this test we require that the omitted predictor(s) are correlated with the propensity score, even when the latter is close to one.
Tests for conditional mean selection bias in a local average treatment effects framework have already been suggested by Black et al. (2017). These tests are based on the regression of parametric residuals from the null model on those variables which are assumed to affect selection (but not the outcome). Importantly, however, these tests rely on the correct (semi-)parametric specification of the conditional mean function under the null, and functional misspecification may lead to a loss in power. More recently, Breunig (2017) has proposed a test for the missing at random (MAR) assumption, failure of which implies a form of endogenous selection. As our first test, his null hypothesis is formulated as a test for omitted variables. However, while his test detects correlation between the selection variable and the instrumental variable(s), our procedure detects correlation between the (actual) outcome and the selection variable and can thus be viewed as complementary. On the other hand, Huber and Melly (2015) device a test for an independence assumption in a selection quantile model where the outcome equation is a location shift model, and the selection equation satisfies a single index restriction. This assumption is a necessary condition for consistency of various estimators for semi-, non-, and parametric conditional mean selection models (cf. Heckman 1979; Ahn and Powell 1993; Das et al. 2003), but relies on the correct specification of outcome and selection equation, and in particular rules out heterogeneity. Finally, Arellano and Bonhomme (2017a, supplementary material) outline a test for no selection in the case of a linear quantile model and parametric specification of the copula. To the best of our knowledge, our procedure is the first one to deliver a rule for deciding between selection and non selection, in both conditional quantile and conditional mean models, without imposing specific parametric restrictions.

The rest of the paper is organized as follows. Section 2 outlines the set-up and the decision rule. Section 3 then establishes the limiting behavior of the first test for omitted variables, and the first order validity of inference based on wild bootstrap critical values. Section 4 on the other hand derives the same results for the second, localized test, while Section 5 shows that the classification errors of our testing procedure are asymptotically controlled at pre-specified levels. Finally, Section 6 provides the empirical illustration testing for selection in log hourly wages of females and males in the UK. Section 7 concludes. Supplementary material contains the corresponding theoretical results for the conditional mean case, and a small set of simulations which demonstrate that our tests control size in finite samples.

2 Testing Procedure

We begin by outlining the data generating process. As it is customary in the sample selection literature, we postulate that the continuous outcome variable of interest, $y_i$, is observed if and only if $s_i = 1$, where $s_i$ denotes a binary selection indicator. For every individual $i$, we observe covariate(s) $x_i$ and instrumental variable(s) $z_i$. Here, $z_i$ is assumed to affect the process of selection into the sample governed by $s_i$, but not $y_i$ directly, an assumption which is testable in the context of the sample selection model (Kitagawa 2010). Note also that the variables $x_i$ and $z_i$ need not be disjoint, although our testing procedure requires some of the continuous variables in $z_i$ to be excluded from $x_i$ (cf. Assumption A.1 below).

Throughout the paper, the maintained assumption is that non-random selection (if present) and the

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2 We only consider the case of continuous outcomes in this paper. For generic inference methods for conditional quantile functions with discrete outcome variables see Chernozhukov et al. (2018).
instrumental variable(s) enter the conditional quantile or mean function only through the propensity
score \( p_i \equiv \Pr (s_i = 1|z_i) \), the probability to be in the selected sample for a given \( z_i \). In other words, we assume that
\[
\Pr \left( y_i \leq q_\tau (x_i) \bigg| x_i, z_i, s_i = 1 \right) = \Pr \left( y_i \leq q_\tau (x_i) \bigg| x_i, p_i, s_i = 1 \right) = \Pr \left( y_i \leq q_\tau (x_i) \bigg| x_i, p_i \right),
\]
almost surely, where \( q_\tau (x_i) \) denotes the conditional \( \tau \)-quantile of \( y_i \) given \( x_i \) and selection \( s_i = 1 \), the probability limit of the conditional local quantile regression estimator defined in (14) below. Likewise, in the case of the conditional mean, we have that
\[
E \left[ y_i \bigg| x_i, z_i, s_i = 1 \right] = E \left[ y_i \bigg| x_i, p_i \right].
\]
This condition, which was in fact also used by Das et al. (2003) for the conditional mean, is implied by standard threshold crossing selection models where \( s_i = 1 \{ p(z_i) > v_i \} \) and the unobservable error terms from the selection and the outcome equation are jointly independent of \( x_i \) and \( z_i \) (see Remark 1 below). In particular, note that we will only require that the propensity score is a smooth, but not necessarily monotonic function of \( z_i \). In fact, the conditions set out in (1) and (2) are the only ‘structure’ we impose on the way in which selection enters the conditional mean or quantile function.

**Remark 1:** To see that the condition in (1) is implied by the set-up of e.g. Arellano and Bonhomme (2017a), assume that potential outcome \( y_i^* \) is given by \( y_i^* = q(u_i, x_i) \) and \( s_i = 1 \{ p(z_i) > v_i \} \), where \((u_i, v_i)\) are assumed to be jointly statistically independent of \( z_i \) given \( x_i \). Assume also that:
\[
y_i = y_i^* s_i \quad \text{iff} \quad s_i = 1.
\]
Then, if \((u_i, v_i)\) are absolutely continuous w.r.t. Lebesgue measure, have standard uniform marginal distributions, and as \( F_{y_i^*|x}(y_i^*|x_i) \) and its inverse are strictly increasing, we obtain that
\[
\Pr \left( y_i^* \leq q_\tau (x_i) \bigg| x_i, z_i, s_i = 1 \right) = \Pr \left( q(u_i, x_i) \leq q_\tau (x_i) \bigg| x_i, z_i, v_i < p(z_i) \right) \equiv \Pr \left( q(u_i, x_i) \leq q_\tau (x_i) \bigg| x_i, p_i \right) \equiv \Pr \left( u_i \leq \tau \bigg| x_i, p_i \right).
\]
Note that in fact \( q_\tau (x_i) \), the ‘observed’ \( \tau \) quantile of \( y_i \) given \( x_i \) and \( s_i = 1 \), coincides with the \( \tau \) quantile of \( y_i^* \) given \( x_i \) when selection is random, i.e. when \( F_{y|x,s=1}(y|x, s = 1) = F_{y^*|x}(y|x) \) almost surely. A similar argument can be made for the conditional mean function.

Given the existence of a valid (continuous) instrument and under Equations (1) or (2), our aim is now to develop a rule for deciding between selection and no selection, and to obtain bounds on the classification error. As outlined in the introduction, this decision rule is based on the outcome of a two step statistical testing procedure. Hence, in order to formally state the hypotheses of selection and non-selection, we need to first outline the two set of hypotheses tested in the first and the second step.

In the first step, we test the hypothesis that the propensity score is not an omitted predictor, against its negation. In what follows, let \( \mathcal{T} = [\underline{\tau}, \bar{\tau}] \), where \( 0 < \underline{\tau} \leq \bar{\tau} < 1 \). Also, we use \( \mathcal{X} \) to denote a compact subset of the interior of the support of \( x_i \), \( R_x \), and \( \mathcal{P} = [\underline{p}, \bar{p}] \subset (0, 1) \) to denote a compact
subset of the support of \( p(z_i) \). In the conditional quantile case, we want to test, conditional on the selected sub-population, that:

\[
H_{0,q}^{(1)} : \Pr (\Pr (y_i \leq q_\tau (x_i) | x_i = x, p_i = p) = \tau) = 1 \quad \text{for all } \tau \in \mathcal{T}, \ x \in \mathcal{X}, \ \text{and } p \in \mathcal{P}
\]  

versus

\[
H_{A,q}^{(1)} : \Pr (\Pr (y_i \leq q_\tau (x_i) | x_i = x, p_i = p) = \tau) < 1 \quad \text{for some } \tau \in \mathcal{T}, \ x \in \mathcal{X}, \ \text{and } p \in \mathcal{P}\]

The logic behind \( H_{0,q}^{(1)} \) vs. \( H_{A,q}^{(1)} \) is that, given \( \{i\} \), it holds that:

\[
\Pr (y_i \leq q_\tau (x_i) | x_i, p_i) = \Pr (y_i \leq q_\tau (x_i) | x_i, p_i, s_i = 1).
\]

The last expression equals \( \tau \) if and only if \( \Pr (y_i \leq q_\tau (x_i) | x_i, p_i, s_i = 1) = \Pr (y_i \leq q_\tau (x_i) | x_i, s_i = 1) \).

**Remark 2:** Given the set-up of Arellano and Bonhomme (2017a), we have that

\[
\Pr (y_i^* \leq q_\tau (x_i) | x_i, z_i, s_i = 1) = \Pr (u_i \leq \tau | x_i, p_i) \equiv G_{x_i} (\tau, p_i),
\]

where \( G_{x_i} (\tau, p_i) = C_{x_i (\tau, p_i)} / p_i \) with \( C_{x_i} (\cdot, \cdot) \) denoting the joint distribution function of \( u_i \) and \( v_i \) conditional on \( x_i \). Clearly, when \( u_i \) and \( v_i \) are independent (which implies \( H_{0,q}^{(1)} \)), it holds that:

\[
G_x (\tau, p) = \Pr (u_i \leq \tau | x_i = x, p_i = p, s_i = 1) = \Pr (u_i \leq \tau | x_i = x, s_i = 1) = \tau.
\]

for all \( x \in \mathcal{X}, \ p \in \mathcal{P}, \) and \( \tau \in \mathcal{T} \).

In the conditional mean case, let \( m(x_i) \equiv E[y_i | x_i, s_i = 1] \) be the conditional mean function given \( x_i \) and \( s_i = 1 \). The corresponding hypotheses for this case read as:

\[
H_{0,m}^{(1)} : \Pr (\Pr ((y_i - m(x_i)) | x_i = x, p_i = p) = 0) = 1
\]  

for all \( x \in \mathcal{X}, \) and \( p \in \mathcal{P}, \) versus

\[
H_{A,m}^{(1)} : \Pr (\Pr ((y_i - m(x_i)) | x_i = x, p_i = p) = 0) < 1
\]

for some \( x \in \mathcal{X}, \) and \( p \in \mathcal{P}. \)

Under (1) or (2) and assumptions outlined in the next section, failure to reject \( H_{0,q}^{(1)} \) or \( H_{0,m}^{(1)} \) rules out endogenous selection asymptotically, with probability approaching one. Therefore, if we fail to reject the null hypothesis, we stop the testing procedure and decide against selection. By contrast, rejection in this first test can occur either due to genuine selection or due to an omitted variable in the outcome equation, which happens to be correlated with the propensity score. This is so, since the omitted predictor test, as any omnibus test, does not possess directed power against specific alternatives. In fact, suppose there is an omitted relevant predictor \( \pi_i \), which we define as follows:

\footnote{From here onwards, we make the conditioning on values of \( x_i \) and \( p_i \) explicit whenever required for clarity.}
Definition 1: Let \( \tilde{q}_r(x_i, \pi_i) \) and \( q_r(x_i) \) denote probability limits of two local polynomial quantile regressions for the selected subsample of \( y_i \) on \( x_i \) and \( \pi_i \) as well as on \( x_i \) only, respectively. We say that \( \pi_i \) is a relevant predictor if for some \( \tau \in T \) and \( \pi \in R_\pi \), where \( R_\pi \) denotes the support of \( \pi_i \), \( \tilde{q}_r(x, \pi) \neq q_r(x) \) for at least all \( x \) in a subset of \( \mathcal{X} \) with non-zero Lebesgue measure.

Therefore, if \( \pi_i \) is a relevant, omitted predictor which is correlated with \( p_i \), we expect indeed that

\[
\Pr \left( y_i \leq q_r(x_i) \middle| x_i = x, p_i = p \right) \neq \tau
\]

with positive probability for some \( \tau \in T \), \( x \in \mathcal{X} \), and \( p \in \mathcal{P} \).

Remark 3: Consider again the set-up of Remark 1, but suppose that the true \( \tau \) conditional quantile of \( y^*_i \) is given by \( \tilde{q}(\tau, x_i, \pi_i) \). Hence, \( \pi_i \) is an omitted predictor, which is assumed to be correlated with \( p_i \). Given A.2 below, and letting \( y^*_i = \tilde{q}(\tilde{u}_i, x_i, \pi_i) \), we can write

\[
\Pr \left( y^*_i \leq q_r(x_i) \middle| x_i = x_i, p_i = 1 \right)
= \Pr \left( \tilde{u}_i \leq \tau \middle| x_i, z_i, s_i = 1 \right)
= \Pr \left( \tilde{u}_i \leq \tau \middle| x_i, p_i \right)
= \tilde{G}_{x_i} (\tau, p_i) \neq \tau,
\]

with positive probability, where the last equality follows since \( \tilde{u}_i \) is a function of \( \pi_i \) (and \( x_i \)), which is correlated with \( p_i \), even in the absence of non-random selection.

Hence, we want to disentangle selection from relevant omitted predictors correlated with the propensity score. In order to impose no-selection as maintained hypothesis, we require the existence of at least one value \( z \) in the support of \( z_i \) s.t. \( p(z) = 1 \). This type of condition is typically labelled ‘identification at infinity’ in the nonparametric identification literature (e.g. Chamberlain, 1986) and requires the existence of a continuous instrument exhibiting sufficient independent variation from \( x_i \).

Note, however, that in Section 4 we will address concerns that the marginal density of \( p_i \) may not be bounded away from zero at \( p = 1 \) (so called irregular identification) resulting in very few observations with (estimated) propensity score close to one.

In the second step, we test the null hypothesis that the propensity score is an omitted predictor when close to one. We test

\[
H^{(2)}_{0,q} = \Pr (\Pr (y_i \leq q_r(x_i) | x_i = x, p_i = 1) = \tau) = 1
\]

for all \( \tau \in T \), and \( x \in \mathcal{X} \) for which identification at infinity holds (a more precise notion will be given in Section 4), versus

\[
H^{(2)}_{A,q} = \Pr (\Pr (y_i \leq q_r(x_i) | x_i = x, p_i = 1) = \tau) < 1
\]

for some \( \tau \in T \), and some \( x \). Analogously, for the conditional mean case, we test

\[
H^{(2)}_{0,m} = \Pr (\Pr ((y_i - m(x_i)) | x_i = x, p_i = 1) = 0) = 1
\]

for all \( x \in \mathcal{X} \) for which identification at infinity holds, versus

\[
H^{(2)}_{A,m} = \Pr (\Pr ((y_i - m(x_i)) | x_i = x, p_i = 1) = 0) < 1.
\]
For $j = q, m$, let $H_{S,j}$ and $H_{NS,j}$ denote the hypothesis of selection and of no-selection respectively. We have:

$$H_{S,j} = H_{A,j}^{(1)} \cap H_{0,j}^{(2)}, \quad (12)$$

and

$$H_{NS,j} = H_{0,j}^{(1)} \cup \left( H_{A,j}^{(1)} \cap H_{A,j}^{(2)} \right). \quad (13)$$

Thus, the hypothesis of selection is the intersection of $H_{A,j}^{(1)}$, the complement of $H_{0,j}^{(1)}$, and of $H_{0,j}^{(2)}$. This means that we choose selection if we reject $H_{0,j}^{(1)}$ and do not reject $H_{0,j}^{(2)}$.

By contrast, the hypothesis of non-selection is the union of $H_{0,j}^{(1)}$ and $H_{A,j}^{(1)} \cap H_{A,j}^{(2)}$. This means that we choose non-selection if either we fail to reject $H_{0,j}^{(1)}$ or we reject both $H_{0,j}^{(1)}$ and $H_{0,j}^{(2)}$. Thus, since $H_{NS,j}$ is defined as a union, if we fail to reject $H_{0,j}^{(1)}$, we stop and decide for non-selection. Our final goal is to compute asymptotic bounds for the classification errors $Pr(\text{choose } H_{S,j} | H_{NS,j} \text{ is true})$ and $Pr(\text{choose } H_{NS,j} | H_{S,j} \text{ is true})$, which we will do in Section 5.

3 First Test

We now introduce a statistic for testing $H_{0,q}^{(1)}$ vs. $H_{1,q}^{(1)}$, as defined in (4) and (5). The corresponding results for the conditional mean and $H_{0,m}^{(1)}$ vs. $H_{A,m}^{(1)}$ can be found in the supplementary material. Moreover, for notational simplicity, from here onwards we assume that all components of $x_i$ and $z_i$ are continuous. The extension to discrete elements in both vectors is immediate at the cost of more complicated notation and more lengthy arguments in the proofs. Also, note that one would generally expect the convergence rate of our statistics to depend only on the number of continuous elements in $x_i$ (and $z_i$) (cf. Li and Racine [2008]).

To implement our test, we rely on a statistic which was also used by Volgushev et al. [2013]. This statistic has the advantage of requiring an estimate of the conditional quantile function only under the null hypothesis, i.e. where the conditional quantile is a function of $x_i$ only. To estimate the conditional quantile functions, we use an r-th order local polynomial estimator based on the standard ‘check type’ objective function:

$$l_{\tau}(v) = 2v(\tau - 1 \{v \leq 0\}) \quad (4)$$

The local polynomial estimator is then given by:

$$\hat{b}_h(\tau, x) = \arg \min_b \frac{1}{nh^2} \sum_{i=1}^n l_{\tau} \left( y_i - b_0 - \sum_{0 \leq |t| \leq r} b_{t}(x_i - x)^t s_i K \left( \frac{x_i - x}{h} \right) \right) \quad (14)$$

is an estimator of $b_{h}^{\dagger}(\tau, x)$ with

$$b_{h}(\tau, x) = \arg \min_b \frac{1}{nh^2} \sum_{i=1}^n \mathbb{E} \left[ l_{\tau} \left( y_i - b_0 - \sum_{0 \leq |t| \leq r} b_{t}(x_i - x)^t s_i K \left( \frac{x_i - x}{h} \right) \right) \right] \quad (15)$$

4 In the discrete case, we can set up a local constant estimator in the direction of the discrete elements.

5 We borrow notation from Masry [1996] letting $t = (t_1, \ldots, t_{d_x})$, $|t| = \sum_{j=1}^{d_x} t_j$, and $\sum_{0 \leq |t| \leq r} = \sum_{j=0}^r \sum_{t_1=0}^r \cdots \sum_{t_{d_x}=0}^r$. 

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Here, $K(\cdot)$ denotes a $d_x$ dimensional product kernel and $h_x$ a corresponding bandwidth sequence that satisfies $h_x \to 0$ as $n \to \infty$. We use $\hat{q}_i(x) = \hat{b}_{0,h}(\tau, x)$, the first element of $\hat{b}_h(\tau, x)$, and set $\eta(x) = b_{0,h}(\tau, x)$. Finally, define $\hat{u}_i(x) \equiv y_i - \hat{q}_i(x)$, $u_i(x) \equiv y_i - q_i(x)$, and let $\bar{x} = (\bar{x}^1,...,\bar{x}^d)$, and $\bar{x} \in X$. The test statistic is given by:

$$Z^q_{1,n} = \sup_{\tau \in \mathcal{T}, (\bar{x}, \bar{p}) \in \mathcal{P}} |Z^q_{1,n}(\tau, \bar{x}, \bar{p}, \bar{p})|,$$

where

$$Z^q_{1,n}(\tau, \bar{x}, \bar{p}, \bar{p}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} s_i (1\{\hat{u}_i(x_i) \leq 0\} - \tau) \Pi_{j=1}^{d_x} 1\{x_{j,i} < \bar{x}_{j}\} 1\{p < \hat{p}_i < \bar{p}\}.$$

The statistic $Z_{1,n}(\tau, \bar{x}, \bar{p}, \bar{p})$ is very similar to the statistic suggested by Volgushev et al. (2013), but for two aspects. First, the omitted regressor $p_i$ is not observable and thus replaced by an estimate, $\hat{p}_i$. Under regularity and bandwidth conditions outlined below, we show however that the estimation error arising from $\hat{p}_i$ is asymptotically negligible. This is a well known result for estimates affecting the statistic only through a weight function (cf. Escanciano et al. 2014). Second, and more importantly, our test statistic is constructed taking the supremum also w.r.t. $\tau$ (over $\mathcal{T}$). We therefore do not just test for selection at a specific quantile level $\tau$, but across (almost) the entire conditional distribution of $y_i$ given $x_i$ and $s_i = 1$.

Heuristically, this is achieved via the use of a local polynomial quantile estimator for which Guerre and Sabbah (2012) established a Bahadur representation uniform over compact sets $\mathcal{X}$ and $\mathcal{P}$ $\square$. On the contrary, Volgushev et al. (2013) use the conditional quantile estimator of Dette and Volgushev (2008), which converges uniformly over the support of the covariates, but only pointwise in $\tau$. In fact, in the simulations, we do not find the assumption of compactness of $\mathcal{X}$ to be of great importance in finite samples.

In the sequel, we make the following assumptions:

**A.1** $(y_i, x'_i, z'_i, s_i) \subset R_y \times R_x \times R_z \times \{0, 1\}$ are identically and independently distributed. Let $\mathcal{X} \equiv X_1 \times ... \times X_{d_x}$ denote a compact subset of the interior of $R_x$. $z_i$ contains at least one variable which is not contained in $x_i$ and which is not $x_i$-measurable. The distributions of $x_i$ and $z_i$ have a probability density function with respect to Lebesgue measure which is strictly positive and continuously differentiable (with bounded derivatives) over the interior of their respective supports. Also, assume that the joint density function of $y_i$, $x_i$ and $p_i$ is uniformly bounded everywhere, and that $\Pr(s_i = 1|x, p) = \Pr(s_i = 1|p) > 0$ for all $x \in \mathcal{X}$ and $p \in \mathcal{P}$.

**A.2** The distribution function $F_{y|x,s=1}(\cdot, \cdot, \cdot)$ of $y_i$ given $x_i$ and selection $s_i = 1$ has a continuous probability density function $f_{y|x,s=1}(y|x, s = 1)$ w.r.t. Lebesgue measure which is strictly positive and bounded for all $y \in R_y$, $x \in \mathcal{X}$. The partial derivative(s) $\nabla_x F_{y|x,s=1}(y|x, s = 1)$ are continuous on

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*Since inference for extreme (conditional) quantile functions is different from and more complicated than inference for other quantiles (Chernozhukov and Fernandez-Val 2011), we restrict ourselves to this subset and relegate inference for extreme quantiles to future research.

**Qu and Yoon (2015)** recently presented a uniform (in $x_i$) Bahadur representation for the conditional (re-arranged) quantile estimator on an unbounded set $\mathcal{X}$. While this feature is certainly appealing, their representation does not hold uniformly in $\tau$. We therefore rely on a representation derived by Guerre and Sabbah (2012) see below for details), which holds uniformly on compact sets $\mathcal{X}$ and $\mathcal{T}$.

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Moreover, there exists a positive constant $C_1$ such that:

$$|f_{y|x,s=1}(y|x, s = 1) - f_{y|x,s=1}(y'|x', s = 1)| \leq C_1 \| (y, x) - (y', x') \|$$

for all $(y, x), (y', x') \in R_y \times X$. Also assume that $q_x(x)$ is $r + 1$-th times continuously differentiable on $X$ for all $\tau \in T$ with $r > \frac{1}{2}d_x$.

**A.3** There exists an estimator $\hat{p}(z_i)$ such that $\sup_{z \in Z} |\hat{p}(z) - p(z)| = o_p(n^{-\frac{1}{4}})$ with $Z$ a compact subset of $R_z$, and that:

$$\Pr (\exists i : z_i \in R_z \setminus Z, p(z_i) \in \mathcal{P}) = o(n^{-\frac{1}{4}}).$$

**A.4** For some positive constant $C_2$, it holds that:

$$|F_{p|x,u,r,s=1}(p|x, 0, s = 1) - F_{p|x,u,r,s=1}(p'|x', 0, s = 1)| \leq C_2 \| (p, x) - (p', x') \|$$

for all $\tau \in T$, $(p, p') \in \mathcal{P}$, and $(x, x') \in X$.

**A.5** The non-negative kernel function $K(\cdot)$ is a bounded, continuously differentiable function with uniformly bounded derivative and compact support on $[-1, 1]$. It satisfies $\int K(v)dv = 1$ as well as $\int vK(v)dv = 0$.

Assumption **A.1** imposes the existence of at least one continuous instrumental variable, and ensures the existence of selected observations for all values in $X$ and $\mathcal{P}$. Assumptions **A.2** and **A.4** on the other hand are rather standard smoothness assumptions, while **A.3** ensures that $p(z_i)$ can be estimated at a specific rate uniformly over $Z$ so that estimation error in $\hat{p}(z_i)$ is asymptotically negligible. In fact, in the case where $\hat{p}(z_i)$ is a local constant kernel estimator, the use of a second order kernel imposes restrictions on the dimensionality of the number of continuous regressors, namely $d_z < 4^{8}$.

Note also that a sufficient condition for the second part of **A.3** is the existence of sufficient moments. Letting ‘$\Rightarrow$’ denote weak convergence, we establish the asymptotic behavior of $Z_{1,n}^{q}$.

**Theorem 1:** Let Assumption **A.1-A.5** and Equation (1) hold. If as $n \to \infty$, $(nh_x^{2d_z})/ \log n \to \infty$, $nh_x^{4r} \log n \to 0$, then

(i) under $H_{b,q}^{(1)}$,

$$Z_{1,n}^{q} \Rightarrow Z_{1}^{q},$$

where $Z_{1}^{q}$ is the supremum of a zero mean Gaussian process whose covariance kernel is defined in the proof of Theorem 1.

(ii) under $H_{A,q}^{(1)}$, there exists $\varepsilon > 0$, such that

$$\lim_{n \to \infty} \Pr \left( Z_{1,n}^{q} > \varepsilon \right) = 1.$$  

More specifically, if we estimate $\hat{p}$ using a local constant estimator, and select the bandwidth via cross validation, we may choose $h_x$, the bandwidth of this estimator, to be of order $h_x = O(n^{-\frac{1}{4d_z+1}})$. In this case, when $d_z < 4$, the bias is of order $n^{-\frac{1}{4d_z+1}} = o(n^{-1/4})$, while for the standard deviation we obtain $(\sqrt{\pi n h_x})^{-1} = o \left( n^{-1/4} \right)$. On the contrary, if $d_z \geq 4$, we instead require a local polynomial estimator of order greater than one.
The results of Theorem 1 rely on an appropriate choice of $h_x$. Since our rate conditions require ‘under-smoothing’ as most of the nonparametric testing and inference literature, but also standard bias-variance considerations do not apply to this test set-up, cross-validation is not directly applicable in our setting. However, to still pick $h_x$ in a data-driven manner ensuring minimal bias at the same time, one possibility to select $h_x$ in practice could be to choose $h_x$ on the basis of cross-validation for a local polynomial estimator of order smaller than the one assumed for the test. For instance, if the assumed polynomial order for the test was three or higher (say $r = 3$ as an example), $h_x$ could be chosen by cross-validation for a local linear estimator, i.e. $h_x = O\left(n^{-\frac{1}{r+1}}\right)$. This in turn implies that $nh_x^4 \log(n) \to 0$ as well as $nh_x^{2d_x}/\log(n) \to \infty$ whenever $d_x < 4$.

As mentioned earlier, a second point worth noting about Theorem 1 is that the statistic converges at rate $\sqrt{n}$, despite the fact that it is based on a nonparametric quantile estimator. Indeed, using a similar line of arguments as in Volgushev et al. (2013), it can be shown that the test has power against a similar line of arguments as in Volgushev et al. (2013), it can be shown that the test has power against $\sqrt{n}$-local alternatives, an appealing feature in practice which does also apply to the test for the conditional mean outlined in the supplement.

Since the limiting distribution $Z_{1,n}^q$ depends on features of the data generating process, we derive a bootstrap approximation for it. In particular, we follow He and Zhu (2003), and use the bootstrap statistic:

$$Z_{1,n}^{*,q} (\tau; \bar{x}; x, p, \bar{p}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} s_i (B_{i,\tau} - \tau) \Pi_{j=1}^{d_x} 1\{\bar{x}_j < x_{j,i} < \bar{x}_j\} \left( (1\{\hat{p}_i < \bar{p}\} - 1\{\hat{p}_i < \bar{p}\}) - \left( \hat{F}_{p|x,u_\tau,s=1} (p|x_i,0, s_i = 1) - \hat{F}_{p|x,u_\tau,s=1} (p|x_i,0, s_i = 1) \right) \right),$$

(16)

where $B_{i,\tau} = 1 \{U_i \leq \tau\}$ with $U_i \overset{i.i.d.}{\sim} U(0,1)$, and independent of the sample, and:

$$\hat{F}_{p|x,u_\tau,s=1} (p|x_i,0, s_i = 1) = \frac{1}{nh_F^{x+\tau}} \sum_{j=1}^{n} s_j 1\{\hat{p}_j \leq p\} \frac{K \left( \frac{x_j-x_i}{h_F} \right)}{\frac{1}{nh_F^{x+\tau}} \sum_{j=1}^{n} s_j K \left( \frac{x_j-x_i}{h_F} \right)},$$

with $h_F \to 0$ as $n \to \infty$. The bootstrap test statistic is then given by:

$$Z_{1,n}^{*,q} = \sup_{\tau \in \mathcal{T}, (\bar{x}, x) \in \mathcal{X}, p, \bar{p} \in \mathcal{P}} |Z_{1,n}^{*,q} (\tau; \bar{x}; x, p, \bar{p}) |.$$

Let $c_{(1-\alpha),n,R}^{*(1)}$ be the $(1-\alpha)$ percentile of the empirical distribution of $Z_{1,n}^{*,q,1}, ..., Z_{1,n}^{*,q,R}$,where $R$ is the number of bootstrap replications. The following Theorem establishes the first order validity of inference based on the bootstrap critical values, $c_{(1-\alpha),n,R}^{*(1)}$.

**Theorem 1**: Let Assumption A.1-A.5 and Equation (1) hold. If as $n \to \infty$, $(nh_x^{2d_x})/\log n \to \infty$, $nh_x^4 \log n \to 0$, $h_F \to 0$, $nh_F^{d_x+1} \to \infty$, and $R \to \infty$, then

(i) under $H_{0,q}^{(1)}$

$$\lim_{n,R \to \infty} \Pr \left( Z_{1,n}^q \geq c_{(1-\alpha),n,R}^{*(1)} \right) = \alpha$$

\(^{9}\)In fact, the order of the bandwidth selected by cross-validation is too large for $nh_x^4 \log(n) \to 0$. 

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(ii) under $H_{A,q}^{(1)}$

$$\lim_{n,R \to \infty} \Pr \left( Z_{1,n}^q \geq c_{(1(1)\alpha),n,R}^{(1)} \right) = 1.$$  

Volgushev et al. (2013) suggest a variant of $Z_{1,n}^q (\tau, x, \pi, p)$ in which $\tau$ is replaced by

$$\hat{\tau} = n^{-1} \sum_{i=1}^{n} 1 \{ y_i \leq \hat{q}_\tau(x_i) \}.$$  

In their set-up, in which convergence is uniform in $x$ (over an increasing support), but pointwise in $\tau$, it is true that $\hat{\tau} - \tau = o_p(n^{-1/2})$. That is, replacing $\tau$ with $\hat{\tau}$ does not affect the limiting distribution and may yield improvement in the finite sample performance of the test. Here, however, this is not the case as $X$ and $P$ are compact. That is:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} s_i (1 \{ \hat{u}_\tau(x_i) \leq 0 \} - \tau) \Pi_{j=1}^{x_i} 1 \{ x_{j,i} < x_{j,i} < \pi_j \} 1 \{ p < \hat{p}_i < p \}$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} s_i (1 \{ u_\tau(x_i) \leq 0 \} - \tau) \Pi_{j=1}^{x_i} 1 \{ x_{j,i} < x_{j,i} < \pi_j \}$$

$$\times \left( (1 \{ p_i < \overline{p} \} - 1 \{ p_i < \underline{p} \} ) - (F_{p|x,u,\tau,s=1}(\overline{p}|x_i,0,s_i = 1) - F_{p|x,u,\tau,s=1}(\underline{p}|x_i,0,s_i = 1) ) \right) + o_p(n^{-1/2}).$$

Thus, in our set-up it does not hold true that $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} s_i (1 \{ \hat{u}_\tau(x_i) \leq 0 \} - \tau) = o_p(n^{-1/2}).$

4 Second Test

If we fail to reject the null hypothesis $H_{0,q}^{(1)}$, we can conclude that the propensity score is not an omitted predictor and thus there is no endogenous selection. This is so because the Type II error approaches zero in probability. On the other hand, if we reject the null, this may be either due to genuine non-random selection or instead due to an omitted regressor which is correlated with the propensity score, as the first test does not have directed power against either of the alternatives.

We therefore rely on an ‘identification at infinity’ argument, which in turn allows us discriminate between both alternatives. More specifically, provided $p(z) = 1$ for some $z \in Z$, under correct specification (i.e., in the absence of omitted predictors), whenever $p \to 1$, the selection bias approaches zero, or in other words it holds that

$$\lim_{p \to 1} \Pr (y_i \leq q_\tau(x_i)|x_i,p) = \tau.$$  

By contrast, when $\pi_i$ is also a relevant predictor in the of Definition 1 of Section 2 for value(s) $x \in X$ with $p(z)$ close to one (see Assumption A.8 below), then $\lim_{p \to 1} \Pr (y_i \leq q_\tau(x_i)|x_i,p) \neq \tau$ with positive probability. This heuristic motivates the second statistic based on observations with (estimated) propensity score close to one for testing $H_{0,q}^{(2)}$ vs. $H_{A,q}^{(2)}$ as defined in (8) and (9).

A common concern in the context of ‘identification at infinity’ is so called irregular identification (Khan and Tamer, 2010), where, although conditional quantiles are point identified, they cannot be

\[10\] The conditional mean counterpart can again be found in the supplementary material.
estimated at a regular convergence rate as the marginal density of $p_i$ may not be bounded away from zero at the evaluation point $p(z) = 1$. That is, heuristically, even if ‘identification at infinity’ holds, and for some value $z \in R_x$, $p(z)$ can reach one, it may be possible that observations in the neighborhood of one are very sparse in practice (‘thin density set’), and so convergence occurs at an irregular rate [Khan and Tamer, 2010]. To address this issue, we only use observations from parts of the support where the density of $p_i$ is bounded away from zero. Formally, this is implemented by introducing a trimming sequence, converging to zero at a sufficiently slow rate so that irregular identification is no longer a concern. Thus, let $\delta = 1 - H$ with $H \to 0$ and $H/h_p \to \infty$ as $n \to \infty$, where $H$ governs the speed of the trimming sequence $\delta$, while $h_p$ defines the window width around $\delta$. Then, for some fixed set $(x, \bar{x})$, the second test is based on the statistic

$$Z_{dp}^2 (\tau, \bar{z}, \bar{x}, 1) = \sum_{i=1}^n s_i (1 \{ \tilde{u}_r (x_i) \leq 0 \} - \tau) \Pi_{j=i}^{d_p} 1 \{ \bar{x}_j < x_{j,i} < \bar{x} \} K \left( \frac{\tilde{p}_i - 1}{h_p} \right) \left( \int K^2 (v) dv \sum_{i=1}^n s_i (1 \{ \tilde{u}_r (x_i) \leq 0 \} - \tau) \Pi_{j=i}^{d_p} 1 \{ \bar{x}_j < x_{j,i} < \bar{x} \} K \left( \frac{\tilde{p}_i - 1}{h_p} \right) \right)^{1/2}. \quad (17)$$

This statistic only uses observations with (estimated) propensity score $\tilde{p}_i \in (1 - h_p - H, 1 + h_p - H)$, and thus overcomes the issue of possible irregular identification as long as a sufficient number of observations are assumed to exist in this set (see below). Note here that the convergence speed of $H$ is inherently pegged to the tail behavior of the density of $p_i$ in the neighborhood of $p = 1$, which is of course unknown in practice. That is, the thinner the density tail of $p_i$, the slower $H$ has to go to zero. We discuss this issue and a potential data-driven way to select $H$ and $h_p$ in given finite samples further below.

In what follows, let $\xi_{j,n}, j = \{1, \ldots, d_x\}$ be a deterministic sequence that, for each $j$, may converge to 0 or to some $\xi_j > 0$ as $n \to \infty$. Furthermore, let

$$C_{0,n} \equiv \otimes_{j=1}^{d_x} [x_{0,j} - \xi_{j,n}, x_{0,j} + \xi_{j,n}],$$

for some point $x_0 \in \mathcal{X}$ defined in A.6 below. Note that $C_{0,n} \subseteq \otimes_{j=1}^{d_x} [\bar{x}_j, \tau_j]$. Finally, recall that $G_{x_j}(\tau, 1 - H) = \Pr(y_i \leq q_0(x_i)|x_i, 1 - H)$, and note that under $H_{0,y_i}^{(2)}$ it holds that $\lim_{H \to 0} G_{x_j}(\tau, 1 - H) = \tau$ for every $\tau \in \mathcal{T}$ and $x_j \in \lim_{n \to \infty} C_{0,n}$, and that $\lim_{H \to 0} \Pr(s_i = 1|1 - H) = 1$. We make the following additional assumptions:

**A.6** Assume there exists at least one point $x_0 \in \mathcal{X}$, such that for at least one $z \in R_z$, it holds that $p(z) = 1$. Moreover, there exists a strictly positive, continuous, and integrable function $g_{y,x,p}(y, x, 1)$ and $g_{x,p}(x, 1)$ such that for all $x \in C_{0,n}$ and $y \in R_y$:

$$\sup_{x \in C_{0,n}, y(x) \in R_y} \left( \frac{f_{y,x,p}(y, x, 1 - H)}{g_{y,x,p}(y, x, 1) H^\eta} - 1 \right) \to 0 \quad \text{and} \quad \sup_{x \in C_{0,n}} \left( \frac{f_{x,p}(x, 1 - H)}{g_{x,p}(x, 1) H^\eta} - 1 \right) \to 0$$

as $n \to \infty$ for some $0 \leq \eta < \eta < 1$. Moreover, for all $x \notin C_{0,n}$, and $y \in R_y$, it holds that $f_{y,x,p}(y(x), x, 1 - H) = 0$ for every $n$.

**A.7** The distribution function $F_{y|x,p,s=1}(\cdot | \cdot, \cdot)$ of $y_i$ given $x_i$, $p_i$, and selection $s_i = 1$ has a continuous probability density function $f_{y|x,p,s=1}(y|x, p, s = 1)$ w.r.t. Lebesgue measure. The functions $f_{y|x,p,s=1}(y|x_i, p_i, s_i = 1)$, $f_{p|x,s=1}(p|x_i, s_i = 1)$, and $\Pr(s_i = 1|p_i)$, and $f_{x,p}(x, p_i)$ are continuously differentiable w.r.t. to $p_i$ on $(0, 1)$ for all $x_i \in \mathcal{X}$ and $z_i \in \mathcal{Z}$. Moreover, assume that for every $x \in \mathcal{X}$ and
\( y, f_{y,x,p}(y,x,\cdot), f_{x,p}(x,\cdot), \) and \( f_p(\cdot) \) are left-continuous at \( p = 1 \).

**A.8** The set of \( x \in \mathcal{X} \) for which \( \pi_i \) is a relevant predictor is a subset of \( C_{0,n} \) (see Assumption A.6).

**A.9** Assume that for all \( x \in \mathcal{X} \) and \( \tau \in \mathcal{T} \), there exist positive constants \( C(x) \) and \( C \) such that:

\[
|G_x(\tau,1-H) - G_x(\tau,1)| \leq C(x)H^{1-\eta}
\]

as well as

\[
|\Pr(s_i = 1|1-H) - 1| \leq CH^{1-\eta}.
\]

Moreover, the following partial derivatives are bounded:

\[
\sup_{y \in R_y, x \in \mathcal{X}, p \in (0,1)} |\nabla_p f_{y|x,p,s=1}(y|x,p,s=1)| < C < \infty,
\]

\[
\sup_{x \in \mathcal{X}, p \in (0,1)} |\nabla_p f_{p|x,s=1}(p|x,s=1)| < C < \infty,
\]

\[
\sup_{x \in \mathcal{X}, p \in (0,1)} |\nabla_p \Pr(s_i = 1|p)| < C < \infty,
\]

and

\[
\sup_{x \in \mathcal{X}, p \in (0,1)} |\nabla_p f_{x,p}(x,p)| < C < \infty,
\]

where \( C \) may differ in each expression.

Assumption A.6 requires identification at infinity, for at least some values of the covariates. In particular, we allow for both the case of \( \xi_{j,n} = \xi_j > 0 \) and \( \xi_{j,n} \to 0 \) as \( n \to \infty \). The case of \( \xi_{j,n} = \xi_j > 0 \) for all \( j \in \{1, \ldots, d_x\} \), corresponds to the case of strong support, as we require the propensity score to approach one for all \( x \) in a set of non-zero Lebesgue measure in a compact subset of \( \mathbb{R}^{d_x} \). In the case when instead \( \xi_{j,n} \to 0 \) for some but not all \( j \), we require identification at infinity over a subset of non-zero Lebesgue measure in a compact subset of \( \mathbb{R}^{d_x} \), with \( d'_x < d_x \). Finally, if \( \xi_{j,n} \to 0 \) as \( n \to \infty \) for all \( j \), we only search over an interval shrinking to a singleton in \( \mathcal{X} \). Furthermore, in all cases we allow for so called irregular support, in the sense that \( f_{x,p}(x,1) \) is not necessarily bounded away from zero at \( p = 1 \). In fact, when \( \eta = 0 \), \( \lim_{H \to 0} f_{x,p}(x,1-H) \) is bounded away from zero for all \( x \in C_{0,n} \), while \( \eta > 0 \) corresponds to the case of irregular support (with a larger value of \( \eta \) representing thinner tails). That is, if \( \eta > 0 \), we allow for a thin set of observations with a propensity score close to one. Similarly, when \( \eta = 0 \), the first part A.9 becomes a standard Lipschitz condition, while as \( \eta \) gets closer to one and the tails of the densities in A.6 become thinner, we allow \( G_x(\tau,1-H) \) and \( \Pr(s_i = 1|1-H) \) to approach \( G_x(\tau,1) \) and 1, respectively, at a slower rate.

\[\sqrt{nh_p \left( \prod_{j=1}^{d_x} \xi_j \right)} H^n, \] where both \( \eta \) and \( \xi_{j,n} \) are of course unknown in practice. While it is immediate to see that the fastest achievable rate is \( \sqrt{nh_p} \) here, we are generally ignorant about the actual rate of convergence. To address this problem, we use a studentized statistic, which allows the convergence rate to vary depending on both the measure of the set of \( x \) for which \( p(z) = 1 \), and on the sparsity of observations around \( p \) close to one. That is, as we cannot infer the appropriate scaling factor, it

\[11\text{Note that under } H_0^{(2)}, \text{ we have that } G_x(\tau,1) = \tau \text{ almost surely.}\]
is crucial that, regardless of the ‘strength’ of the support and the degree of thinness of the set of observations with propensity score close to one, $Z_{2,n}(\tau, x, \tau, 1)$ and $\sqrt{\text{var}(Z_{2,n}(\tau, x, \tau, 1))}$ diverge at the same rate, so that the ratio still converges in distribution.

In the sequel, we study the asymptotic behavior of $\frac{Z_{2,n}^q(\tau, x, \tau, 1)}{\sqrt{\text{var}(Z_{2,n}^q(\tau, x, \tau, 1))}}$ as an empirical process over $\tau \in \mathcal{T}$, but for fixed values of the interval extremes $x$ and $\tau$. The reason is that for the case of $\xi_n \to 0$, the statistic does not depend on $x, \tau$, as it vanishes outside a shrinking interval of $x_0$.

**Theorem 2:** Let Assumptions A.1, A.3, A.5, A.6, A.7, A.8, A.9, and Equation (1) hold. If as $n \to \infty$, $(nh_{2d})/\log n \to \infty$, $nh_{2r} \log n \to 0$, $H \to 0$, $H/h_p \to \infty$, $nh_p H^{2-\eta} \left( \prod_{j=1}^{d} \xi_{j,n} \right) \to 0$, and $nh_p \left( \prod_{j=1}^{d} \xi_{j,n} \right) H^\eta \to \infty$, then

(i) under $H_{0,q}^{(2)}$,

$$\sup_{\tau \in \mathcal{T}} \left| \frac{Z_{2,n}^q(\tau, x, \tau, 1)}{\sqrt{\text{var}(Z_{2,n}^q(\tau, x, \tau, 1))}} \right| \Rightarrow Z_2^q,$$

where $Z_2^q$ is the supremum of a zero mean Gaussian process with covariance kernel defined in the proof of Theorem 2.

(ii) under $H_{A,q}^{(2)}$, there exists $\varepsilon > 0$, such that

$$\lim_{n \to \infty} \Pr \left( \sup_{\tau \in \mathcal{T}} \left| \frac{Z_{2,n}^q(\tau, x, \tau, 1)}{\sqrt{\text{var}(Z_{2,n}^q(\tau, x, \tau, 1))}} \right| > \varepsilon \right) = 1.$$

Theorem 2 establishes the limiting distribution of the studentized statistic. As the theoretical results crucially hinge on the tuning parameters $H$ and $h_p$, whose rates depend in turn on the unknown $\xi_{j,n}$ and $\eta$, some comments on possible choice of these parameters in practice are in order: fixing $\eta$ such that $nh_p H^{2-\eta} \left( \prod_{j=1}^{d} \xi_{j,n} \right) \to 0$ and $nh_p H^\eta \left( \prod_{j=1}^{d} \xi_{j,n} \right) \to \infty$ also for all $\eta < \eta$. Then, choosing $H = h_p^{-\varepsilon}$ for some $\varepsilon > 0$ arbitrarily small, and

$$h_p = cn^{-\frac{1}{5-\eta-2\varepsilon + \eta}} \log(n)^{-1},$$

where we will set the constant $\varepsilon = 1$ for simplicity in the following, note also that it holds that $nh_p^{(1-\varepsilon)(2-\eta)+1} \left( \prod_{j=1}^{d} \xi_{j,n} \right) \to 0$ regardless of whether $\left( \prod_{j=1}^{d} \xi_{j,n} \right)$ is bounded above zero or shrinks to zero. On the contrary, to ensure that indeed $nh_p^{(1-\varepsilon)\eta+1} \left( \prod_{j=1}^{d} \xi_{j,n} \right) \to \infty$ at the same time, observe that

$$nh_p^{(1-\varepsilon)\eta+1} \left( \prod_{j=1}^{d} \xi_{j,n} \right) = nn^{-\frac{1+\eta-\frac{1}{1-\eta+\varepsilon+\eta}}{5-\frac{1}{1-\eta-2\varepsilon + \eta}} \log(n)^{\eta(1-\varepsilon)+1} \left( \prod_{j=1}^{d} \xi_{j,n} \right)}$$

$$= \frac{n^{2(1-\varepsilon-\frac{1}{1-\eta+\varepsilon+\eta})}}{\log(n)^{\eta(1-\varepsilon)+1}} \left( \prod_{j=1}^{d} \xi_{j,n} \right) \to \infty$$

provided $\left( \prod_{j=1}^{d} \xi_{j,n} \right)$ goes to zero slower than $n^{-\frac{2(1-\varepsilon-\frac{1}{1-\eta+\varepsilon+\eta})}{5-\frac{1}{1-\eta-2\varepsilon + \eta}}}. Recalling that $\varepsilon$ can be set arbitrarily small, for a generic $\eta$, let now $h_p(\eta) = cn^{-\frac{1}{5-\eta-2\varepsilon + \eta}} \log(n)^{-1}$ and $H(\eta) = h_p(\eta)^{1-\varepsilon}$. Thus, in order
Theorem 2: inference based on the bootstrap critical values, $c$, the number of bootstrap replications. The following Theorem establishes the first order validity of \( \sqrt{n} \) thus possesses power against whose rate is at most nonparametric. By contrast, our first test converges at a parametric rate, and not ‘thin’, we would expect to select $q$ testing idea since the convergence rate of this statistic would be driven by the estimator of $\hat{q}$ as $n \to \infty$, otherwise they diverge at the same rate.

Finally, note that an alternative test for selection against non-selection could in principle be based on the null $q_\tau(x, 1) = q_\tau(x, p)$ for all $\tau \in \mathcal{T}$ and for some $x$ for which identification at infinity holds. A statistic for this null could be constructed using the weighted difference of the two corresponding estimators of $q_\tau(x, p)$ and $q_\tau(x, 1)$, respectively. In this paper, however, we do not pursue this alternative testing idea since the convergence rate of this statistic would be driven by the estimator of $q_\tau(x, 1)$, whose rate is at most nonparametric. By contrast, our first test converges at a parametric rate, and thus possesses power against $\sqrt{n}$ local alternatives. Moreover, a rejection of the above equality does not necessarily imply the presence of selection as rejection may occur whenever the omission of a relevant predictor has a differential effect on the quantile depending on whether the propensity score takes a value in the interior or at the boundary of its support.

As outlined in the proof of Theorem 2, quantile estimation error vanishes. This is because the statistic converges at a nonparametric rate, while quantile estimation error approaches zero at a parametric rate. Hence, when constructing the wild bootstrap statistic we do not have to ‘subtract’ an estimator of the conditional distribution of $p_i$. On the other hand, as the rate of convergence depends on the ‘degree’ of irregular identification at $p$ close to 1 and on the set of covariates for which identification at infinity holds, we also need a appropriately studentized bootstrap statistic, i.e.

$$Z_{2,n}^{*q} = \sup_{\tau \in \mathcal{T}} \left| \frac{Z_{2,n}^{q}(\tau, \underline{x}, \bar{x}, 1)}{\sqrt{\text{var}^*(Z_{2,n}^{q}(\tau, \underline{x}, \bar{x}, 1))}} \right|,$$

where

$$Z_{2,n}^{q}(\tau, \underline{x}, \bar{x}, 1) = \sum_{i=1}^{n} s_i(B_{i,\tau} - \tau) \Pi_{j=1}^{d_{x}} 1\{x_j < x_j,i < \bar{x}_j\} K\left(\frac{\hat{p}_i - \delta}{h_p}\right)$$

with $B_{i,\tau} = 1 \{U_i \leq \tau\}$ with $U_i \overset{i.i.d.}{\sim} U(0, 1)$ and independent of the sample, and

$$\text{var}^*(Z_{2,n}^{q}(\tau, \underline{x}, \bar{x}, 1)) = \left( \frac{1}{n} \sum_{i=1}^{n} (B_{i,\tau} - \tau)^2 \right) \left( \int K^2(v)dv \right) \sum_{i=1}^{n} s_i \Pi_{j=1}^{d_{x}} 1\{x_j < x_j,i < \bar{x}_j\} K\left(\frac{\hat{p}_i - \delta}{h_p}\right).$$

By noting that $\frac{1}{n} \sum_{i=1}^{n} (B_{i,\tau} - \tau)^2 = \tau(1 - \tau) + o_p^*(1)$, given (19), we see that whenever identification at infinity holds at all $x \in \mathcal{X}$ and the number of observations with propensity score in the interval $(1 - h_p - H, 1 + h_p - H)$ grows at rate $nh_p$, then both numerator and denominator in (18) are bounded in probability, otherwise they diverge at the same rate.

Let $c_{(1-\gamma),n,R}^{(2)}$ be the $(1 - \gamma)$ percentile of the empirical distribution of $Z_{2,n}^{q,1}, ..., Z_{2,n}^{q,R}$, where $R$ is the number of bootstrap replications. The following Theorem establishes the first order validity of inference based on the bootstrap critical values, $c_{(1-\gamma),n,R}^{(2)}$.

**Theorem 2**: Let Assumption A.1, A.3, A.5, A.6, A.7, A.8, A.9, and Equation (1) hold. If as $n \to \infty$, $(nh_p^{d_{x}})/\log n \to \infty$, $nh_p^{d_{x}} \log n \to 0$, $H \to 0$, $H/h_p \to \infty$, $nh_pH^{-\eta}(\prod_{j=1}^{d_{x}} \xi_{j,n}) \to 0,$
\( \lim_{n,R \to \infty} \Pr \left( Z_{q,n}^2 \geq c_{(1-\gamma),n,R}^{(2)} \right) = \gamma \)

(ii) under \( H_{A,q}^{(2)} \)

\( \lim_{n,R \to \infty} \Pr \left( Z_{q,n}^2 \geq c_{(1-\gamma),n,R}^{(2)} \right) = 1. \)

Theorem 2* establishes the first order validity of inference based on wild bootstrap critical values. Under \( H_{0,q}^{(2)} \), the studentized statistic and its bootstrap counterpart have the same limiting distribution. Under \( H_{A,q}^{(2)} \), the statistic diverges, as the numerator is of larger probability order than the denominator, while the bootstrap statistic remains bounded in probability.

5 Decision Rule

We now formalize the rules for deciding between selection and non selection, i.e. for choosing between \( H_{S,j} \) and \( H_{NS,j} \) as defined in (12) and (13) with \( j = q,m \).

Let \( c^{(1)}_{(1-\alpha),n,R} \) and \( c^{(2)}_{(1-\gamma),n,R} \) be respectively the \((1 - \alpha)\) and \((1 - \gamma)\) bootstrap critical values for either the quantile case (as defined in Theorem 1* and in Theorem 2*) or the mean case (as defined in the supplement). Based on the outcome of first and second test, we device the following decision rule

**Rule RS\(_{j,n}\):**

1. If \( Z_{q,n}^1 \geq c^{(1)}_{(1-\alpha),n,R} \) and \( Z_{2,n}^q (\tau, x, \tau, 1) \leq c^{(2)}_{(1-\gamma),n,R} \), we decide that \( H_{S,q} \) is true. That is, we decide in favor of selection.

2. If \( Z_{q,n}^1 \leq c^{(1)}_{(1-\alpha),n,R} \) or \( Z_{q,n}^1 \geq c^{(1)}_{(1-\alpha),n,R} \) and \( Z_{2,n}^q (\tau, x, \tau, 1) \geq c^{(2)}_{(1-\gamma),n,R} \), we decide that \( H_{NS,q} \) is true. That is, we decide in favor of non-selection.

The Theorem below establishes the validity of our procedure by showing that the mis-classification probabilities \( \Pr (\text{choose } H_{S,q} | H_{NS,q} \text{ is true}) \) and \( \Pr (\text{choose } H_{NS,q} | H_{S,q} \text{ is true}) \) are (asymptotically) controlled by our decision rule at levels \( \alpha \) and \( \gamma \), respectively.

**Theorem RS** Let all the Assumptions and the rate conditions in Theorems 1,1*,2 and 2* hold. Then,

\[ \lim_{n \to \infty} \Pr (\text{choose } H_{S,q} | H_{NS,q} \text{ is true}) \leq \alpha \]

and

\[ \lim_{n \to \infty} \Pr (\text{choose } H_{NS,q} | H_{S,q} \text{ is true}) \leq \gamma \]

If we fail to reject the first test, we decide against endogeneous selection and rely on nonparametric conditional quantile estimators which make use of all selected individuals. On the other hand, if we reject the first test but fail to reject the second one, we estimate the conditional quantile using only individuals with propensity score close to one. Finally, if we reject both tests, then there is evidence of a relevant omitted predictor and neither the estimator using all selected individuals nor the one using only those with propensity score close to one will deliver estimates consistent for the conditional quantile (mean) function of interest.
Since the estimator we use depends on the outcome of a testing procedure, one may be concerned about the size problem arising when one fails to reject a Hausman test of endogeneity, and then conducts inference based on OLS estimators, as outlined in Guggenberger (2010a,b). In terms of (3) in Remark 1, suppose that the correlation between $u_i$ and $p_i$ is weak, say of order $n^{-1/2}$. In this case, we may fail to reject the null of no selection, and thus move to estimate conditional quantiles using all selected individuals. If inference is based on a nonparametric quantile estimator, this estimator converges to its limiting distribution at a rate slower than $n^{1/2}$ and thus pre-testing would not represent a problem. It is only in the case where we decide to make inference using an estimator for a parametric conditional quantile function that the issue of size distortion may arise as in the set-up of Guggenberger (2010a,b). Moreover, while in the Hausman pre-testing case, the cost of always using instrumental variables is only in terms of efficiency loss, in our context the cost lies predominantly in a much slower convergence rate.

Finally, our testing procedure cannot disentangle selection and omitted regressor which is uncorrelated with $p$ when $p$ is close to one. In this case, we still decide in favor of selection, and so we estimate the quantiles using only observations with propensity score close to one. However, even in this case we make the ‘right’ decision in the sense that for observations with propensity score close to one, omitted predictor bias is not present.

6 Empirical Illustration

Our illustration is based on a subsample of the UK wage data from the Family Expenditure Survey used by Arellano and Bonhomme (2017a). As pointed out by these authors, due to changes in employment rates over time, simply examining wage inequality for females and males at work over time may provide a distorted picture of market-level wage inequality. We will therefore run our selection testing procedure on two different subsets of the data, namely 1995 to 1997, a period of increasing gross domestic product (GDP) growth rates, and 1998 to 2000, a period of high, but stable GDP growth rates. Unlike Arellano and Bonhomme (2017a), however, our testing procedure for selection will not rely on a parametric specification of the conditional log-wage quantile functions, but remain completely nonparametric.

The covariates we include in $x_i$ are dummies for marital status, education (end of schooling at 17 or 18, and end of schooling after 18), location (eleven regional dummies), number of kids (split by six age categories), time (year dummies), as well as age in years. This set of covariates is identical to the one used by Arellano and Bonhomme (2017a), but for the fact that the latter used cohort dummies instead of age in years. The continuous instrumental variable is given by the measure of potential out-of-work (welfare) income, interacted with marital status. This variable, which was also used by Arellano and Bonhomme (2017a), builds on Blundell et al. (2003) and is constructed for each individual in the sample (employed and non-employed) using the Institute of Fiscal Studies (IFS) tax and welfare-benefit simulation model.

The final sample for the years 1995-1997 comprises 21,263 individuals, 11,647 of which are females.

---

It is also noteworthy that our testing procedure only requires a second stage when we are unsure about the correct specification of the outcome function and the correlation of the unobserved factors with the instrument(s): when the instrumental variable(s) are free of these concerns as they have been constructed e.g. on the basis of a randomized control trial, a second stage is not required.

For the exact construction of the sample see their paper and references therein.
and 9,616 of which are males, respectively. The number of working females (males) with a positive log hourly wage in that sample is 7,761 (7,623). By contrast, for the 1998-2000 period we obtain 16,350 observations, 8,904 females and 7,446 males. The number of working females (males) in that sample are 5,931 (6,058).

All estimates are constructed using routines from the np package of [Hayfield and Racine](2008). More specifically, we estimate the propensity score \( \Pr(s_i = 1 | z_i) \) fully nonparametrically using an Epanechnikov second order kernel function for the continuous instrument and kernel functions as proposed by [Li and Racine](2008) for the remaining discrete controls (see above). To determine the bandwidth in a data-driven manner, we use the method outlined in [Racine](1993), conducting cross-validation on random subsets of the data (size \( n = 450 \)), selecting the median values over 50 replications. The conditional quantile function \( q_\tau(x_i) \) is estimated as in Equation (19) of [Li and Racine](2008), while the conditional distribution function \( F_{p|x,u,s=1}(\cdot | \cdot , \cdot) \) is constructed as in Equation (4) of the same paper. The bandwidths are again determined as before.\(^{14}\)

The quantile grid is chosen to be \( \mathcal{T} = \{.1, .2, .3, .4, .5, .6, .7, .8, .9 \} \).

To provide the reader with a better illustration of the potential magnitudes of selection into work, we replicate the predictions from the (estimated) parametric conditional quantile functions from Figure 1 of [Arellano and Bonhomme](2017a, p.16) for the sub-periods 1995-1997 and 1998-2000, see Figures 1 and 2, respectively. In these pictures, solid lines represent estimated uncorrected (for selection) conditional log-wage quantile functions, while dashed lines are the ones corrected for sample selection.\(^{15}\) Throughout, female quantile lines lie below the male quantile lines. The figures, which display selection corrections based on a linear quantile regression model and parametric selection correction, show little difference between the original and the corrected lines for both males and females for the 1995-1997 period (except for males at lower percentile levels), but more pronounced differences for the subsequent 1998-2000 period. In terms of magnitude, the effect of correction appears to be generally bigger for males than for females in both subperiods.

Turning to the test results in Table 1, we see that while we cannot find any evidence for selection during the 1995-1997 period for females at conventional significance levels, there is some evidence for males at the 10% significance level. In fact, taking a closer look at the results in Table 1 we observe that rejection for males occurs on the basis of the 10th percentile, which is in line with the graphical evidence in Figure 1. Switching over to Table 2, however, we obtain a different picture: for females, \( H_{0,q}^{(1)} \) is rejected at any conventional level, and rejection is most pronounced at the 20th and the 30th percentile. On the other hand, we cannot reject \( H_{0,q}^{(1)} \) for males. This failure to reject \( H_{0,q}^{(1)} \) for males is in contrast to the graphical evidence in Figure 2 and highlights the importance of formal testing under a more flexible specification.

Following our testing procedure outlined in Section 2, we perform the second test for males in the 1995-1997 period, and for females in the 1998-2000 period (see Table 3). Turning to the results, we strongly reject the null of no misspecification of the conditional quantile function \( q_\tau(x_i) \) for males, but fail to reject that null for females at any conventional levels. Both results appear to be robust to different choices of \( \delta \) and \( h_p \). Thus, under the assumption that out-of-work income is a valid instrument and indeed selection enters outcome as postulated in Equation (1), our test results suggest that there

\(^{14}\)Recall that there are no continuous covariates contained in \( x_i \), and thus the theoretical rate conditions do not directly apply here.

\(^{15}\)For the exact specification used, see [Arellano and Bonhomme](2017a).
is evidence for selection among females for the 1998-2000, but not for males. In fact, what appears to be selection among males during the 1995-1997 period may actually be attributed to misspecification of the conditional quantile function. [16]

[16] In a related paper, Kitagawa (2010) tested for the validity of the same instrumental variable (but for the interaction with marital status) in a similar data set on the basis of the UK Family Expenditure Survey used by Blundell et al. (2007). Although his test results are not directly informative here as his test is run on a much coarser set of covariates $x_i$ not including e.g. regional, marital, or family information, his evidence suggested that the conditional independence of the instrument and outcome (given $x_i$ and selection $s_i = 1$) may indeed be violated for some sub-groups (in particular, younger males with moderate levels of education). Thus, rejection in the second test for males could also be related to this feature.
Figure 1: Corrected and Uncorrected Log Hourly Wage Quantiles by Gender 1995-1997 (Arellano and Bonhomme, 2017a)

Note: male quantiles are always at the top, female ones at the bottom (solid lines: uncorrected quantiles; dashed lines: selection corrected quantiles)
Figure 2: Corrected and Uncorrected Log Hourly Wage Quantiles by Gender 1998-2000 [Arellano and Bonhomme 2017a]

Note: male quantiles are always at the top, female ones at the bottom (solid lines: uncorrected quantiles; dashed lines: selection corrected quantiles)
|        | Test 1 |        |        |
|--------|--------|--------|--------|
|        | Males  | Females|        |
| Statistic | 0.050  | 0.041  |        |
| 10%    | 0.050  | 0.015  |        |
| 20%    | 0.037  | 0.030  |        |
| 30%    | 0.031  | 0.041  |        |
| 40%    | 0.030  | 0.032  |        |
| 50%    | 0.040  | 0.034  |        |
| 60%    | 0.034  | 0.036  |        |
| 70%    | 0.040  | 0.027  |        |
| 80%    | 0.032  | 0.025  |        |
| 90%    | 0.022  | 0.019  |        |
| 90%-CV | 0.050  | 0.049  |        |
| 95%-CV | 0.053  | 0.053  |        |
| P-Value| 0.10   | 0.32   |        |
| # obs  | 7623   | 7761   |        |

Table 1: Test Results 1995-1997 (*Note: Number of Bootstrap Replications is 400*)

|        | Test 1 |        |        |
|--------|--------|--------|--------|
|        | Males  | Females|        |
| Statistic | 0.045  | 0.049  |        |
| 10%    | 0.033  | 0.021  |        |
| 20%    | 0.039  | 0.042  |        |
| 30%    | 0.044  | 0.049  |        |
| 40%    | 0.041  | 0.031  |        |
| 50%    | 0.045  | 0.032  |        |
| 60%    | 0.039  | 0.047  |        |
| 70%    | 0.035  | 0.042  |        |
| 80%    | 0.033  | 0.026  |        |
| 90%    | 0.032  | 0.032  |        |
| 90%-CV | 0.048  | 0.048  |        |
| 95%-CV | 0.049  | 0.051  |        |
| P-Value| 0.17   | 0.06   |        |
| # obs  | 6058   | 5931   |        |

Table 2: Test Results 1998-2000 (*Note: Number of Bootstrap Replications is 400*)
### Table 3: Test Results 1998-2000 (Note: Number of Bootstrap Replications is 1,000)

7 Conclusion

This paper introduces a novel testing procedure to detect sample selection in conditional quantile and mean functions, without imposing parametric assumptions on either the outcome or the selection equation. Our objective is to develop a decision rule that allows us to discriminate between selection and non-selection, and to demonstrate that the ‘mis-classification’ errors are controlled asymptotically. This is accomplished via two tests, the first of which is an omitted predictor test, with the estimated propensity score as omitted predictor. As with any omnibus test, rejection in the first step can be due to either selection or to the omission of a predictor which is correlated with the estimated propensity score. Since selection and misspecification have very different implications for
the identification of nonparametric (conditional) quantile functions, we aim at disentangling the two if we reject in the first step. That is, after rejection in the first test we proceed to the second test, which is a localized version of the first test, using only observations with (estimated) propensity score close to one. Here, non-selection is the maintained hypothesis, and thus a rejection implies general misspecification. Importantly, the second test, although relying on ‘identification at infinity’, allows for irregular identification by using observations close, but not too close to one. We establish the first order validity of bootstrap critical values based on the wild bootstrap.

In our empirical illustration, we test for sample selection in log hourly wages of females and males in the UK using data from the UK Family Expenditure Survey. Using the periods 1995-1997 and 1998-2000 as examples, we find evidence for selection among females for the 1998-2000, but not for males. In fact, what appears to be selection among males during the 1995-1997 period may actually be attributed to misspecification of the conditional quantile function.

8 Appendix

Proof of Theorem 1:

(i) In the following, let \( E_{S_n}[\cdot] \) denote the expectation operator conditional on the actual sample realizations. Moreover, since \( 1\{p \leq \hat{p}_i \leq \hat{p}\} = 1\{\hat{p}_i \leq \hat{p}\} - 1\{\hat{p}_i \leq \hat{p}\}, \) we will ignore the part of the statistic which involves \( 1\{\hat{p}_i \leq \hat{p}\} \) in the sequel. Thus:

\[
Z_{1,n}(\tau, \underline{x}, \overline{x}, \underline{p}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} s_i(1\{\hat{u}_\tau(x_i) \leq 0\} - \tau) \Pi_{j=1}^{d_x} 1\{x_j < x_{ji} < \overline{x}_j\} 1\{\hat{p}_i \leq \hat{p}\} + \sqrt{n} E_{S_n} \left[ s_i(1\{\hat{u}_\tau(x_i) \leq 0\} - \tau) \right]
\]

\[
\times \Pi_{j=1}^{d_x} 1\{x_j < x_{ji} < \overline{x}_j\} 1\{\hat{p}_i \leq \hat{p}\} - s_i(1\{u_\tau(x_i) \leq 0\} - \tau) \Pi_{j=1}^{d_x} 1\{x_j < x_{ji} < \overline{x}_j\} 1\{p_i \leq \hat{p}\} - E_{S_n} \left[ s_i(1\{u_\tau(x_i) \leq 0\} - \tau) \right]
\]

\[
\times \Pi_{j=1}^{d_x} 1\{x_j < x_{ji} < \overline{x}_j\} 1\{p_i \leq \hat{p}\} \right]
\]

\[
+ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} s_i(1\{\hat{u}_\tau(x_i) \leq 0\} - \tau) \Pi_{j=1}^{d_x} 1\{x_j < x_{ji} < \overline{x}_j\} 1\{\hat{p}_i \leq \hat{p}\} - E_{S_n} \left[ s_i(1\{\hat{u}_\tau(x_i) \leq 0\} - \tau) \right]
\]

\[
\times \Pi_{j=1}^{d_x} 1\{x_j < x_{ji} < \overline{x}_j\} 1\{\hat{p}_i \leq \hat{p}\} \right]
\]

\[
= I_n + II_n + III_n.
\]
For $II_n$, it holds that:

$$II_n = \sqrt{n}E_{S_n} \left[ s_i(1\{\hat{u}_r(x_i) \leq 0\} - 1\{u_r(x_i) \leq 0\}) \Pi_{j=1}^{\tau} 1\{x_j < x_{j,i} < \tau_j\} 1\{p_i \leq \bar{p}\}\right] + \sqrt{n}E_{S_n} \left[ s_i(1\{u_r(x_i) \leq 0\} - \tau) \Pi_{j=1}^{\tau} 1\{x_j < x_{j,i} < \tau_j\} 1\{\hat{p}_i \leq \bar{p}\} - 1\{p_i \leq \bar{p}\}\right] + \sqrt{n}E_{S_n} \left[ s_i(1\{\hat{u}_r(x_i) \leq 0\} - 1\{u_r(x_i) \leq 0\}) \Pi_{j=1}^{\tau} 1\{x_j < x_{j,i} < \tau_j\} 1\{\hat{p}_i \leq \bar{p}\} - 1\{p_i \leq \bar{p}\}\right] = II_{1,n} + II_{2,n} + II_{3,n}
$$

We start with $II_{1,n}$. Using A.1 and A.2, note that:

$$II_{1,n} = \sqrt{n} \int \int \{F_{u_r|x,p,s=1}(\hat{q}_r(x_i) - q_r(x_i)|x_i, p_i, s_i = 1) - F_{u_r|x,p,s=1}(0|x_i, p_i, s_i = 1)\} \times \Pr(s_i = 1|x_i, p_i) \Pi_{j=1}^{\tau} 1\{x_j < x_{j,i} < \tau_j\} 1\{p_i \leq \bar{p}\} f_{x_i}(x_i, p_i) dx_i dp_i
$$

$$= \sqrt{n} \int \int \{F_{u_r|x,p,s=1}(0|x_i, p_i, s_i = 1)(\hat{q}_r(x_i) - q_r(x_i))\} \times \Pi_{j=1}^{\tau} 1\{x_j < x_{j,i} < \tau_j\} 1\{p_i \leq \bar{p}\} f_{x_i}(x_i, p_i) dx_i dp_i \times \Pi_{j=1}^{\tau} 1\{x_j < x_{j,i} < \tau_j\} 1\{p_i \leq \bar{p}\} f_{x_i}(x_i, p_i) dx_i dp_i. \quad (20)$$

As for the second term on the right hand side (RHS) of (20), say $II_{1,n}^b$, it is majorized by

$$II_{1,n}^b = \sup_{\tau \in \mathcal{T}, x \in \mathcal{X}} \sqrt{n}(\hat{q}_r(x) - q_r(x))^2 \times \int \int |\nabla f_{u_r|x,p,s=1}(\xi_{r,n}(x)|x_i, p_i, s_i = 1) - q_r(x)| f_{x_i}(x_i, p_i) dx_i dp_i,$$

Since $f_{u_r|x,p,s=1}(u_r(x_i)|x_i, p_i, s_i = 1) = f_{u_r|x,p,s=1}(u_r(x_i)|x_i, s_i = 1)$ under $H_{0,q}^{(1)}$ and $f_{g|x,p,s=1}(q_r(x)|x_i, s_i = 1) = f_{g|x,p,s=1}(0|x_i, s_i = 1)$ a.s., Assumptions A.1 and A.2 imply that the term above is $o_p(1)$ once we show that $\sup_{\tau \in \mathcal{T}, x \in \mathcal{X}} \sqrt{n}(\hat{q}_r(x) - q_r(x))^2 = o_p(1)$. Now, from Theorem 1 in Guerre and Sabbah (2012), for polynomial order $r$ odd,

$$\sup_{\tau \in \mathcal{T}, x \in \mathcal{X}} \sqrt{n}(\hat{q}_r(x) - q_r(x))^2 \leq \sup_{\tau \in \mathcal{T}, x \in \mathcal{X}} \sqrt{n}(\hat{q}_r(x) - q^\dagger_r(x))^2 + \sup_{\tau \in \mathcal{T}, x \in \mathcal{X}} \sqrt{n}(q^\dagger_r(x) - q_r(x))^2$$

$$= \sup_{\tau \in \mathcal{T}, x \in \mathcal{X}} \sqrt{n}(\hat{q}_r(x) - q^\dagger_r(x))^2 + O\left(\sqrt{n}h^{2r}\right),$$

where $q^\dagger_r(x) = E[\hat{q}_r(x)]$, and from Theorem 2 of Guerre and Sabbah (2012)

$$\sup_{\tau \in \mathcal{T}, x \in \mathcal{X}} \sqrt{n}(\hat{q}_r(x) - q^\dagger_r(x))^2 = \sqrt{n}O_p\left(\frac{\log n}{nh^{4r}}\right) + \sqrt{n}O_p\left(\frac{\log n}{nh^{2r}}\right)^{\frac{3}{2}},$$

26
where the last term captures the error in the Bahadur representation of Theorem 2 in Guerre and Sabbah (2012). Hence, if \( n^{1/2}h^{2r} \log n \to 0 \) and \( n^{1/2}h^{d_x} / \log n \to \infty \), \( I_{1,n}^p = o_p(1) \). As for the first line of \( I_{1,n}^p \):

\[
\sqrt{n} \int \int \left\{ f_{u,x,p,s=1}(0|x_i,p_i,s_i = 1)(\hat{q}_r(x_i) - q_r(x_i)) \right\} \Pr(s_i = 1|x_i,p_i)
\times \prod_{i=1}^{d_x} \{ x_j < x_{j,i} < x_j \} \{ p_i \leq \overline{p} \} f_{x_p}(x_i,p_i)dx_idp_i
= \sqrt{n} \int F_{p|x,u,s=1}(\overline{p}|x_i,0,s_i = 1)f_{u,x,s=1}(0|x_i,s_i = 1) \Pr(s_i = 1|x_i)f_x(x_i)(\hat{q}_r(x_i) - q_r(x_i))
\times \prod_{i=1}^{d_x} \{ x_j < x_{j,i} < x_j \} dx_i.
\]

Note that \( \hat{q}_r(x_i) - q_r(x_i) = \hat{q}_r(x_i) - q_r^\dagger(x_i) + q_r^\dagger(x_i) - q_r(x_i) \) and also \( q_r^\dagger(x_i) - q_r(x_i) = O(h^{2r}) \) uniformly in \( \tau \), \( p \) and \( (\overline{x},\overline{\tau}) \). Next, let

\[
U_h(x_j - x_i) = \sum_{v|v|\leq \tau} \frac{1}{v!} \left( \frac{x_j - x_i}{h_x} \right)^v
\]

and

\[
J_j(\tau,x_i) = 2f_{y|x,s=1}(q_r^\dagger(x_j)|x_j,s_j = 1)s_j U_h(x_j - x_i)U_h(x_j - x_i)K_h(x_j - x_i),
\]

where

\[
K_h(\cdot) = K(\cdot/h_x) \times \ldots \times K(\cdot/h_x)
\]

denotes the \( d_x \) dimensional product kernel. From the Bahadur representation in Theorem 2 in Guerre and Sabbah (2012), uniformly in \( \tau \), \( (\overline{x},\overline{\tau}) \), and \( p \), the last line yields:

\[
\sqrt{n} \int F_{p|x,u,s=1}(\overline{p}|x_i,0,s_i = 1)f_{u,x,s=1}(0|x_i,s_i = 1) \Pr(s_i = 1|x_i)f_x(x_i)(\hat{q}_r(x_i) - q_r(x_i))
\times \prod_{i=1}^{d_x} \{ x_j < x_{j,i} < x_j \} dx_i
= \frac{-2}{\sqrt{nh_x^{d_x}}} \sum_{j=1}^{n} \int F_{p|x,u,s=1}(\overline{p}|x_i,0,s_i = 1)f_{u,x,s=1}(0|x_i,s_i = 1) \Pr(s_i = 1|x_i)f_x(x_i)
\times e^\prime \left( \frac{1}{nh_x^{d_x}} \sum_{j=1}^{n} J_j(\tau,x_i) \right)^{-1} s_j(1\{ y_j \leq q_r^\dagger(x_j) \} - \tau)U_h(x_j - x_i)K_h(x_j - x_i) \prod_{i=1}^{d_x} \{ x_j < x_{j,i} < x_j \} dx_i
+ o_p(1)
\]

By Assumptions A.2, A.5, a uniform law of large numbers for triangular arrays yields that:

\[
\frac{1}{nh_x^{d_x}} \sum_{j=1}^{n} J_j(\tau,x_i) \xrightarrow{p} J(\tau,x_i),
\]

where

\[
J(\tau,x_i) = 2f_{y|x,s=1}(q_r(x_i)|x_i,s_i = 1) \Pr(s_i = 1|x_i)f_x(x_i) \int U(v)U(v)^t K(v)dv.
\]
We can therefore re-write the first term on the RHS of the last equality in (21), as:

\[
-\frac{2}{\sqrt{nh_x}} \sum_{j=1}^{n} \int F_{\theta|x,u,s=1}(\rho|x_i,0,s_i = 1) f_{u|x,s=1}(0|x_i, s_i = 1) \Pr(s_i = 1|x_i) f_x(x_i) \\
\times e'(J(\tau,x_i))^{-1} s_j(1\{y_j \leq q^1_j(x_j)\} - \tau) \mathbf{U}_h(x_j - x_i) \mathbf{K}_h(x_j - x_i) \Pi^d_{j=1} 1\{z_j < x_{j,i} < \overline{z}_i\} dx_i \\
- \frac{2}{\sqrt{nh_x}} \sum_{j=1}^{n} \int F_{\theta|x,u,s=1}(\rho|x_i,0,s_i = 1) f_{u|x,s=1}(0|x_i, s_i = 1) \Pr(s_i = 1|x_i) f_x(x_i) \\
\times e' \left( \frac{1}{nh_x} \sum_{l=1}^{n} J_l(\tau,x_i) - J(\tau,x_i) \right)^{-1} s_j(1\{y_j \leq q^1_j(x_j)\} - \tau) \mathbf{U}_h(x_j - x_i) \mathbf{K}_h(x_j - x_i) \\
\times \Pi^d_{l=1} 1\{z_j < x_{l,i} < \overline{z}_i\} dx_i
\]

(22)

Noting that \( f_{y|x,s=1}(q_r(x_i)|x_i, s_i = 1) = f_{u|x,s=1}(0|x_i, s_i = 1) \) a.s., the first term of the above expression can be re-written as

\[
-\frac{1}{\sqrt{nh_x}} \sum_{j=1}^{n} \int F_{\theta|x,u,s=1}(\rho|x_i,0,s_i = 1) e' \left( \int \mathbf{U}(v) \mathbf{U}(v)^{T} \mathbf{K}(v) dv \right)^{-1} \\
\times s_j(1\{u_r(x_j) \leq 0\} - \tau) \mathbf{U}_h(x_j - x_i) \mathbf{K}_h(x_j - x_i) \Pi^d_{j=1} 1\{z_j < x_{j,i} < \overline{z}_i\} dx_i + o_p(1)
\]

\[
= -\frac{1}{\sqrt{nh_x}} \sum_{j=1}^{n} \int F_{\theta|x,u,s=1}(\rho|x_i,0,s_i = 1) e' \left( \int \mathbf{U}(v) \mathbf{U}(v)^{T} \mathbf{K}(v) dv \right)^{-1} \\
\times s_j(1\{u_r(x_j) \leq 0\} - \tau) \mathbf{U}_h(x_j - x_i) \mathbf{K}_h(x_j - x_i) \Pi^d_{j=1} 1\{z_j < x_{j,i} < \overline{z}_i\} dx_i \\
+ \frac{1}{\sqrt{nh_x}} \sum_{j=1}^{n} \int F_{\theta|x,u,s=1}(\rho|x_i,0,s_i = 1) e' \left( \int \mathbf{U}(v) \mathbf{U}(v)^{T} \mathbf{K}(v) dv \right)^{-1} \\
\times s_j(1\{u_r(x_j) \leq 0\} - 1\{u_r(x_j) \leq 0\}) \mathbf{U}_h(x_j - x_i) \mathbf{K}_h(x_j - x_i) \Pi^d_{j=1} 1\{z_j < x_{j,i} < \overline{z}_i\} dx_i = A_{1,n} + A_{2,n}
\]

Let \( t = (t_1, ..., t_{d_z}) \), via change of variables,

\[
A_{1,n} = -\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \int F_{\theta|x,u,s=1}(\rho|x_j + h_x t,0,s_j = 1) e' \left( \int \mathbf{U}(v) \mathbf{U}(v)^{T} \mathbf{K}(v) dv \right)^{-1} \\
\times (s_j(1\{u_r(x_j) \leq 0\} - \tau)) \mathbf{U}(t) \mathbf{K}(t) \Pi^d_{l=1} 1\{z_l < x_{l,i} + t_l h_x < \overline{z}_i\} dt
\]

\[
= -\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \int F_{\theta|x,u,s=1}(\rho|x_j,0,s_j = 1) e' \left( \int \mathbf{U}(v) \mathbf{U}(v)^{T} \mathbf{K}(v) dv \right)^{-1} \\
\times \left( \int \mathbf{U}(v) \mathbf{K}(v) dv \right) (s_j(1\{u_r(x_j) \leq 0\} - \tau)) \Pi^d_{l=1} 1\{z_l < x_{l,i} < \overline{z}_i\} + o_p(1)
\]

\[
= -\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \int F_{\theta|x,u,s=1}(\rho|x_j,0,s_j = 1) (s_j(1\{u_r(x_j) \leq 0\} - \tau)) \Pi^d_{l=1} 1\{z_l < x_{l,i} < \overline{z}_i\} \\
+ o_p(1)
\]

(23)

where the \( o_p(1) \) term holds again uniformly in \( \tau, x_i, \) and \( p(z_i) \) given Assumption A.4, and the last equality follows from the fact that \( e' \left( \int \mathbf{U}(v) \mathbf{U}(v)^{T} \mathbf{K}(v) dv \right)^{-1} \left( \int \mathbf{U}(v) \mathbf{K}(v) dv \right) = 1 \) as \( e \) contains zeros everywhere but for the first element. Moving to \( A_{2,n} \), given Assumptions A.2 and A.4, we can write
this term as:

\[ \frac{1}{\sqrt{n}h_x^{d_x}} \sum_{j=1}^{n} \left\{ \int F_{pl,x,s=1}(\bar{p}|x_i,0,s_i=1) \Pr(s_j = 1|x_j) \left( 1\{u^*_j(x_j) \leq 0\} - 1\{u_r(x_j) \leq 0\} \right) \right. \]

\times e' \left( \int \mathbf{U}(v) \mathbf{U}(v)' \mathbf{K}(v) dv \right)^{-1} \mathbf{U}_h(x_j - x_i) \Theta_h(x_j - x_i) \Pi_{l=1}^{d_x} 1\{x_l < x_{l,i} < \bar{x}_l\} dx_i

\left. - \sqrt{n} \frac{1}{h_x^{d_x}} E \left[ \int F_{pl,x,s=1}(\bar{p}|x_i,0,s_i=1) \Pr(s_j = 1|x_j) \left( 1\{u^*_j(x_j) \leq 0\} - 1\{u_r(x_j) \leq 0\} \right) \right. \right.

\times e' \left( \int \mathbf{U}(v) \mathbf{U}(v)' \mathbf{K}(v) dv \right)^{-1} \mathbf{U}_h(x_j - x_i) \Theta_h(x_j - x_i) \Pi_{l=1}^{d_x} 1\{x_l < x_{l,i} < \bar{x}_l\} dx_i \left. \right] \right)

\times e' \left( \int \mathbf{U}(v) \mathbf{U}(v)' \mathbf{K}(v) dv \right)^{-1} \mathbf{U}_h(x_j - x_i) \Theta_h(x_j - x_i) \Pi_{l=1}^{d_x} 1\{x_l < x_{l,i} < \bar{x}_l\} dx_i

= \ A_{21,n} + A_{22,n}.

Starting with \( A_{22,n} \), we have:

\[ \sqrt{n} \frac{1}{h_x^{d_x}} \int \int \Pr(s_j = 1|x_j) \left( F_{u_r|x,s=1}(q^*_r(x_j) - q_r(x_j)|x_j, s_j = 1) \right. \]

\left. - F_{u_r|x,s=1}(0|x_j, s_j = 1) \right) e' \left( \int \mathbf{U}(v) \mathbf{U}(v)' \mathbf{K}(v) dv \right)^{-1} \mathbf{U}_h(x_j - x_i) \Theta_h(x_j - x_i)

\times F_{pl,x,s=1}(\bar{p}|x_i,0,s_i=1) \Pi_{l=1}^{d_x} 1\{x_l < x_{l,i} < \bar{x}_l\} f_x(x_j) dx_j dx_i

\left. \right) \Pr(s_j = 1|x_j) f_x(x_j)

\times dx_i f_x(x_j) dx_j + o_p(1)

\]

\[ = \ o_p(1) \]

provided \( nh_x^{d_x} \rightarrow 0 \). Using Markov’s inequality, a similar argument yields that \( A_{21,n} = o_p(1) \) uniformly in \( \tau, x_i \) and \( p(z_i) \) provided again that \( nh_x^{d_x} \rightarrow 0 \). It remains to analyze the second term of Equation 22. Denote

\[ R_n(\tau, \bar{p}, x_i) = F_{pl,x,s=1}(\bar{p}|x_i,0,s_i=1) f_{u_r|x,s=1}(0|x_i, s_i = 1) \Pr(s_i = 1|x_i) f_x(x_i) \]

\times e' \left( \frac{1}{nh_x^{d_x}} \sum_{l=1}^{n} \mathbf{J}_l(\tau, x_i) - \mathbf{J}(\tau, x_i) \right)^{-1}
and observe that, as before, we can rewrite

\[-2 \sqrt{\frac{1}{nh_d^2}} \sum_{j=1}^{2n} \int R_n(p, \tau, x_i) s_j(1\{y_j \leq q_{\tau}(x_j)\} - \tau) U_h(x_j - x_i) K_h(x_j - x_i) \Pi_{l=1}^{d_x} 1\{x_l < x_{l,i} < x_l\} dx_i\]

\[= \frac{-2}{\sqrt{nh_d^2}} \sum_{j=1}^{2n} \int R_n(p, \tau, x_i) s_j(1\{y_j \leq q_{\tau}(x_j)\} - \tau) U_h(x_j - x_i) K_h(x_j - x_i) \Pi_{l=1}^{d_x} 1\{x_l < x_{l,i} < x_l\} dx_i\]

\[= B_{1,n} + B_{2,n}.\]

Starting with \(B_{1,n}\), note that:

\[B_{1,n} = \frac{-2}{\sqrt{nh_d^2}} \sum_{j=1}^{2n} s_j(1\{y_j \leq q_{\tau}(x_j)\} - \tau) R_n(p, \tau, x_j + ht) \Pi_{l=1}^{d_x} 1\{x_l < x_{l,j} + ht < x_l\} \int U(t)K(t) dt\]

\[= \frac{-2}{\sqrt{n}} \sum_{j \neq i} s_j(1\{y_j \leq q_{\tau}(x_j)\} - \tau) R_n(p, \tau, x_j) \Pi_{l=1}^{d_x} 1\{x_l < x_{l,j} < x_l\} \int U(t)K(t) dt + o_p(1)\]

uniformly in \(x, p, \tau\) by A.2, A.4, and A.5, and so

\[E[B_{12,n}] = \frac{-2}{\sqrt{n}} \sum_{j=1}^{n} E \left[ R_n(p, \tau, x_j) s_j(1\{y_j \leq q_{\tau}(x_j)\} - \tau) \Pi_{l=1}^{d_x} 1\{x_l < x_{l,j} < x_l\} \right] \int U(t)K(t) dt\]

\[= \frac{-2}{\sqrt{n}} \sum_{j=1}^{n} E \left[ R_n(p, \tau, x_j) \Pi_{l=1}^{d_x} 1\{x_l < x_{l,j} < x_l\} E[s_j(1\{y_j \leq q_{\tau}(x_j)\} - \tau)|x_j]\right] \int U(t)K(t) dt = 0,\]

as under \(H_0\), \(E[s_j(1\{y_j \leq q_{\tau}(x_j)\} - \tau)|x_j]\) is independent of \(p_j\), and \(E[s_j(1\{y_j \leq q_{\tau}(x_j)\} - \tau)] = 0.\)

Finally,

\[E[|B_{11,n}|] \leq \frac{1}{\sqrt{nh_d^2}} C \sup_{\tau \in T, x \in \mathcal{X}} \left| e' \left( \frac{1}{nh_d^2} \sum_{k=1}^{n} J_k(\tau, x_j) - J(\tau, x) \right) \right|^{-2} \int U(t)^2K(t)^2 dt = o \left( \frac{1}{\sqrt{nh_d^2}} \right) = o(1).\]
given $n h^{2d_x} / \log n \to \infty$. Also

$$E[B_{1,n}^2] = E \left[ \int F_{p|x,u,s=\{1\}}(x_j + h_x t, 0, s_j = 1) f_{u|x,s=\{1\}}(0|x_j + th_x, s_i = 1) \Pr(s_i = 1|x_j + h_x t)^2 
\times f_x(x_j + th_x)^2 e' \left( \frac{1}{nh_{dx}^2} \sum_{k=1}^n J_k(\tau, x_j + th_x) - J(\tau, x_j + th_x) \right)^2 U(t)^2 K(t)^2 
\times (s_j(1\{u(x_j) \leq 0\} - \tau)^2) \Pi_{l=1}^{d_x} 1\{x_l < x_{l,j} + t_l h_x < \overline{x}_l\} dt \right] 
\leq C \sup_{\tau \in T, x \in \mathcal{X}} \left( e' \left( \frac{1}{nh_{dx}^2} \sum_{k=1}^n J_k(\tau, x_j) - J(\tau, x) \right)^2 \int U(t)^2 K(t)^2 dt \right) E \left[ s_j(1\{u(x_j) \leq 0\} - \tau)^2 \right] 
+ o_p(1) 
= C \sup_{\tau \in T, x \in \mathcal{X}} \left( e' \left( \frac{1}{nh_{dx}^2} \sum_{k=1}^n J_k(\tau, x_j) - J(\tau, x) \right)^2 \int U(t)^2 K(t)^2 dt \right) \tau(1 - \tau) + o_p(1)$$

where the $o_p(1)$ term holds uniformly in $\tau$, $x$, and $p$ and follows from Assumptions A.1, A.2, and A.4. Since for any two symmetric non-singular matrices it holds that $A_1^{-1} - A_2^{-1} = A_2^{-1}(A_2 - A_1)A_1^{-1}$, we can bound

$$\sup_{\tau \in T, x \in \mathcal{X}} \left( e' \left( \frac{1}{nh_{dx}^2} \sum_{k=1}^n J_k(\tau, x_j) - J(\tau, x) \right)^2 \left( \frac{1}{nh_{dx}^2} \sum_{l=1}^n J_l(\tau, x) \right)^2 \int U(v)^2 K(v)^2 dv \right) 
\leq e' \inf_{\tau \in T, x \in \mathcal{X}} \left( \left\| J(\tau, x) \right\|^{-2} \sup_{\tau \in T, x \in \mathcal{X}} \left( \left\| J(\tau, x) - \frac{1}{nh_{dx}^2} \sum_{l=1}^n J_l(\tau, x) \right\| \right)^2 \right) \inf_{\tau \in T, x \in \mathcal{X}} \left( \left\| \frac{1}{nh_{dx}^2} \sum_{l=1}^n J_l(\tau, x) \right\|^{-2} \right) 
\times \int U(v)^2 K(v)^2 dv. \]

Now, $\inf_{\tau \in T, x \in \mathcal{X}} \left( \left\| J(\tau, x) \right\|^{-2} \right) > 0$ by A.1, A.2, and A.5, and, by the same assumptions and a uniform law of large numbers for triangular arrays, we have that $\inf_{\tau \in T, x \in \mathcal{X}} \left( \left\| \frac{1}{nh_{dx}^2} \sum_{l=1}^n J_l(\tau, x) \right\|^{-2} \right) = O_p(1)$ and

$$\sup_{\tau \in T, x \in \mathcal{X}} \left( \left\| J(\tau, x) - \frac{1}{nh_{dx}^2} \sum_{l=1}^n J_l(\tau, x) \right\|^{-2} \right) = o_p(1),$$

so that $B_{1,n}$ is of order $o_p(1)$ uniformly in $p$, $\tau$, and $x$. Finally, combining arguments used for $B_{1,n}$ above with the ones for $A_{21,n}$ and $A_{22,n}$ and the fact that $nh_{dx}^{4r} \to 0$ allows to conclude that this term is of order $o_p(1)$ uniformly in $p$, $\tau$, and $x$. 

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Turning to $II_{2, n}$ and recalling that $\hat{p}_i = \hat{p}(z_i)$ as well as $p_i = p(z_i)$, note that:

$$\sqrt{n}E_{S_n}\left[s_i \left\{1\{u_r(x_i) \leq 0\} - \tau\right\} \Pi_{j=1}^{d_x}1\{\bar{x}_j < x_{i,j} < \bar{x}_j\} \left(1\{\hat{p}(z_i) \leq \bar{p}\} - 1\{p(z_i) \leq \bar{p}\}\right)\right]$$

$$= \sqrt{n} \int \int \left(F_{p|x, u_r, s=1}(\bar{p} - \hat{p}_i) | u_r(x_i), s_i = 1\right) \times (1\{u_r(x_i) \leq 0\} - \tau) f_{u_r|x, s=1}(u_r(x_i)|x_i, s_i = 1) Pr(s_i = 1|x_i) \Pi_{j=1}^{d_x}1\{\bar{x}_j < x_{i,j} < \bar{x}_j\} f_x(x_i) du_r dx_i$$

$$= \sqrt{n} \int \int (1\{u_r(x_i) \leq 0\} - \tau) f_{u_r|x, s=1}(u_r(x_i)|x_i, s_i = 1) Pr(s_i = 1|x_i) \Pi_{j=1}^{d_x}1\{\bar{x}_j < x_{i,j} < \bar{x}_j\} f_x(x_i) du_r dx_i$$

$$Under H_{(1)}^{1(q)} it holds that f_{u_r|x, p, s=1}(u_r(x_i)|x_i, p, s_i = 1) = f_{u_r|x, s=1}(u_r(x_i)|x_i, s_i = 1) \left(\text{this is not true under the alternative}\right), \text{ and thus:}$$

$$\sqrt{n}E_{S_n}\left[s_i \left\{1\{u_r(x_i) \leq 0\} - \tau\right\} \Pi_{j=1}^{d_x}1\{\bar{x}_j < x_{i,j} < \bar{x}_j\} \left(1\{\hat{p}(z_i) \leq \bar{p}\} - 1\{p(z_i) \leq \bar{p}\}\right)\right]$$

$$= \sqrt{n} \int \int (1\{u_r(x_i) \leq 0\} - \tau) f_{u_r|x, s=1}(u_r(x_i)|x_i, s_i = 1) Pr(s_i = 1|x_i) \Pi_{j=1}^{d_x}1\{\bar{x}_j < x_{i,j} < \bar{x}_j\} f_x(x_i) du_r dx_i$$

$$= \sqrt{n} \int \int (\tau - \tau) f_{p|x, s=1}(\bar{p}|x_i, s_i = 1) Pr(s_i = 1|x_i) \Pi_{j=1}^{d_x}1\{\bar{x}_j < x_{i,j} < \bar{x}_j\} f_x(x_i) dx_i$$

$$= 0$$

It remains to analyze $II_{3, n}$. Noting that $|1\{\hat{p}(z_i) \leq \bar{p}\} - 1\{p(z_i) \leq \bar{p}\}| \leq 1\{|\hat{p}(z_i) - p(z_i)| > 0\}$ and likewise $|1\{u_r(x_i) \leq 0\} - 1\{u_r(x_i) \leq 0\}| \leq 1\{|u_r(x_i) - u_r(x_i)| > 0\}$, we can bound this term, using A.3, as follows:

$$|II_{3, n}| \leq \sqrt{n}C sup\limits_{z} |\hat{p}(z) - p(z)| \int_{x} 1\{|\hat{q}_r(x_i) - q_r(x_i)| > 0\} f_x(x_i) dx_i + o_p(1)$$

$$= \sqrt{n}C sup\limits_{z} |\hat{p}(z) - p(z)| \int_{x} 1\{|\hat{q}_r(x_i) - q_r(x_i)| > 0\} f_x(x_i) dx_i + o_p(1)$$

where the last line follows from the bandwidth conditions and Assumption A.3.

Finally, we turn to $III_n$. First, we apply Lemma A.1 of Escanciano et al. (2014) to the function classes $F_1 \equiv \{f_1(s, \tau, x) = s(1\{u_r(x) \leq 0\} - \tau)\Pi_{j=1}^{d_x}1\{x_j < x_{j} < \bar{x}_j\} : \{\tau, x, \bar{p}\} \in T \times \mathcal{X}\}$ and $F_2 \equiv \{f_2(p(z)) = 1\{p(z) \leq \bar{p}\} : \{p\} \in P\}$ to conclude that the product class $F_1 \times F_2$ is Donsker. Then, by Lemma A.3 in Escanciano et al. (2014), it follows that $III_n = o_p(1)$ uniformly in $\tau$, $x_i$, and $p_i$. Hence, recalling (23),

$$Z_{1, n}(\tau, \bar{x}, \bar{p}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (1\{u_r(x_i) \leq 0\} - \tau) \Pi_{j=1}^{d_x}1\{x_j < x_{i,j} < \bar{x}_j\} s_i \left(1\{p_i \leq \bar{p}\} - F_{p|x, u_r, s=1}(\bar{p}|x_i, 0, s_i = 1\right) + o_p(1),$$

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with the $o_p(1)$ term holding uniformly in $\tau, p$, and $(x, \overline{x})$. Thus, the process $Z_{1,n}^q (\tau, \overline{x}, \overline{p}, \overline{p}^\prime)$ converges weakly in $l^\infty (T \times X \times P)$, the Banach space of real bounded functions on $T \times X \times P$, with covariance kernel

$$
cov \left( Z_{1,n}^q (\tau, \overline{x}, \overline{p}, \overline{p}^\prime), Z_{1,n}^q (\tau', \overline{x}', \overline{p}', \overline{p}'') \right) = E \left[ (1 \{ u_r(x_i) \leq 0 \} - \tau) \Pi_{j=1}^{d} \{ z_j < x_{j,i} < \overline{x}_j \} (s_i (1 \{ p_i \leq \overline{p} \} - 1 \{ p_i \leq p' \})) - (F_{p|x,u,s=1}(\overline{p}|x_i,0,s_i=1) - F_{p|x,u,s=1}(p'|x_i,0,s_i=1)) \Pr(s_i=1|x_i) \right]$$

As an immediate consequence, we also obtain the weak convergence of any continuous functional and so:

$$Z_{1,n}^q \Rightarrow Z_1^q.$$

(ii) Under $H_{A,q}^{(1)}$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (1 \{ u_r(x_i) \leq 0 \} - \tau) \Pi_{j=1}^{d} \{ z_j < x_{j,i} < \overline{x}_j \} s_i \left( 1 \{ p_i \leq \overline{p} \} - F_{p|x,u,s=1}(\overline{p}|x_i,0,s_i=1) \right)$$

diverges to infinity. □

Proof of Theorem 1*

(i) We first show that $Z_{1,n}^q$ has the same distribution as $Z_{1,n}^q$ under the null, conditional on the sample, for all almost samples, i.e. we show that under $H_{0,q}^{(1)}$,

$$\lim_{n \to \infty} P \left( \omega : \sup_{v \in \mathbb{R}^+} \left| P^* \left( Z_{1,n}^q \leq v \right) - P \left( Z_{1,n}^q \leq v \right) \right| \right) = 1,$$

where $P^*$ denotes the bootstrap probability law, conditional on the sample. Moreover, let $E^*$ and $var^*$ be the bootstrap mean and variance operator conditional on the sample. Recalling that $B_{i,\tau} = 1 \{ U_i \leq \tau \}$ with $U_i \overset{i.i.d.}{\sim} U(0,1)$ and $U_i$ independent of the sample, note that:

$$E^* \left[ Z_{1,n}^q (\tau, \overline{x}, \overline{p}, \overline{p}^\prime) \right] = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} s_i \Pi_{j=1}^{d} \{ z_j < x_{j,i} < \overline{x}_j \} \left( 1 \{ \hat{p}_i \leq \overline{p} \} - \hat{F}_{p|x,u,s=1}(\overline{p}|x_i,0,s_i=1) \right) E^*[B_{i,\tau} - \tau] = 0.$$
Now, re-write $Z_{1,n}^{nq}(\tau, \bar{x}, \bar{x}, \bar{p})$ as:

$$
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (B_{i,\tau} - \tau) E \left[ s_i \prod_{j=1}^{d_x} 1\{x_{j,i} < x_{j,i} < \bar{x}_{j}\} \left(1\{p_i \leq p\} - F_{p|x,u,,s=1}(p|x_i,0,s_i = 1)\right) \right]
$$

$$
+ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (B_{i,\tau} - \tau) \left\{ s_i \prod_{j=1}^{d_x} 1\{x_{j,i} < x_{j,i} < \bar{x}_{j}\} \left(1\{p_i \leq p\} - F_{p|x,u,,s=1}(p|x_i,0,s_i = 1)\right) \right\}
$$

$$
- E \left[ s_i \prod_{j=1}^{d_x} 1\{x_{j,i} < x_{j,i} < \bar{x}_{j}\} \left(1\{p_i \leq p\} - F_{p|x,u,,s=1}(p|x_i,0,s_i = 1)\right) \right]
$$

$$
+ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (B_{i,\tau} - \tau) s_i \prod_{j=1}^{d_x} 1\{x_{j,i} < x_{j,i} < \bar{x}_{j}\} \left(1\{\hat{p}_i \leq \bar{p}\} - 1\{p_i \leq \bar{p}\}\right)
$$

$$
+ F_{p|x,u,,s=1}(p|x_i,0,s_i = 1) - \hat{F}_{p|x,u,,s=1}(p|x_i,0,s_i = 1)
$$

(25)

Using the conditional multiplier central limit Theorem 2.9.7 of Van der Vaart and Wellner (1996), the second term converges weakly to a centered Gaussian process for almost all samples. For the first term, as $(B_{i,\tau} - \tau)$ is independent of the sample, it also converges weakly to:

$$
\psi_i E \left[ s_i \prod_{j=1}^{d_x} 1\{x_{j,i} < x_{j,i} < \bar{x}_{j}\} \left(1\{p_i \leq p\} - F_{p|x,u,,s=1}(p|x_i,0,s_i = 1)\right) \right],
$$

where $\psi_i$ is a standard normal random variable. This establishes the weak convergence of the first two terms on the RHS of Equation (25) for almost all samples. The variance of these two terms is in fact given by:

$$
\frac{1}{n} \sum_{i=1}^{n} s_i \prod_{j=1}^{d_x} 1\{x_{j,i} < x_{j,i} < \bar{x}_{j}\} \left(1\{p_i \leq p\} - F_{p|x,u,,s=1}(p|x_i,0,s_i = 1)\right)^2 \text{var}^*(B_{i,\tau} - \tau)
$$

$$
= \frac{1}{n} \sum_{i=1}^{n} s_i \prod_{j=1}^{d_x} 1\{x_{j,i} < x_{j,i} < \bar{x}_{j}\} \left(1\{p_i \leq p\} - F_{p|x,u,,s=1}(p|x_i,0,s_i = 1)\right)^2 \tau (1 - \tau)
$$

$$
= E \left[ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} s_i (1\{y_i - q_r(x_i) \leq 0\} - \tau) \prod_{j=1}^{d_x} 1\{x_{j,i} < x_{j,i} < \bar{x}_{j}\} \{p_i \leq \bar{p}\} \right] \tau (1 - \tau)
$$

$$
+ o_p(1)
$$

with the $o_p(1)$ term holding uniformly in $p$, $\tau$, and $\bar{x}, \bar{x}$. Also, note that the covariance of these terms is given by:

$$
\frac{1}{n} \sum_{i=1}^{n} s_i \prod_{j=1}^{d_x} 1\{x_{j,i} < x_{j,i} < \bar{x}_{j}\} \left(1\{p_i \leq p\} - F_{p|x,u,,s=1}(p|x_i,0,s_i = 1)\right)
$$

$$
\prod_{j=1}^{d_x} 1\{x_{j,i} < x_{j,i} < \bar{x}_{j}\} \left(1\{p_i \leq p\} - F_{p|x,u,,s=1}(p|x_i,0,s_i = 1)\right)
$$

$$
\left(\min\{\tau, \tau^*\} - \tau \tau^*\right) E \left[ s_i \prod_{j=1}^{d_x} 1\{x_{j,i} < x_{j,i} < \bar{x}_{j}\} \left(1\{p_i \leq p\} - F_{p|x,u,,s=1}(p|x_i,0,s_i = 1)\right)
$$

$$
+ s_i \prod_{j=1}^{d_x} 1\{x_{j,i} < x_{j,i} < \bar{x}_{j}\} \left(1\{p_i \leq p\} - F_{p|x,u,,s=1}(p|x_i,0,s_i = 1)\right)\right]
$$

$$
+ o_p(1)
$$

uniformly over $\mathcal{X}$, $\mathcal{P}$ and $\mathcal{T}$. Finally, for the last term on the RHS of Equation (25), note that,
conditional on the sample, this term has expectation zero and is bounded since $B_{t,x}$ is bounded (and independent of the sample). Therefore, evoking $nh^{d_x+1} \to \infty$ and again Lemma A.1 and A.3 of Escanciano et al. (2014) as in the proof of Theorem 1, it follows that the expression is of order $o_p(1)$ uniformly over $\mathcal{X}$, $\mathcal{P}$ and $\mathcal{T}$. Thus, the statement in (23) follows.

(ii) $Z_{1,n}^q(\tau, x, p, p)$ has the same limiting distribution under both $H_{0,q}^{(1)}$ and $H_{A,q}^{(1)}$, while $Z_{1,n}^q(\tau, x, P, P)$ diverges to infinity under $H_{A,q}^{(1)}$.

**Proof of Theorem 2:**

(i) Given Assumption A.6,

$$Z_{2,n}^q(\tau, x, \tau, 1) = \frac{ZN_{2,n}^q(\tau, x, \tau, 1)}{ \left( \int K^2(v)dv \right)^\frac{1}{2} (1 + o_p(1))}$$

where $\mathcal{C}_{0,n} \subseteq \otimes_{j=1}^{d_x} [x_j, \bar{x}_j]$ and

$$ZN_{2,n}^q(\tau, x, \tau, 1) = \frac{1}{nh_pH^q \left( \prod_{j=1}^{d_x} \xi_{j,n} \right)} \sum_{i=1}^{n} s_i (1\{\hat{u}_t(x_i) \leq 0\} - \tau) 1 \{x_i \in \mathcal{C}_{0,n}\} K \left( \frac{\hat{p}_i - \delta}{h_p} \right) K \left( \frac{p_i - \delta}{h_p} \right)$$

The $o_p(1)$ term, which holds uniformly in $\tau \in \mathcal{T}$, follows from Assumption A.6, given that for all $x$ in the complement of $\mathcal{C}_{0,n}$, the set of observations with propensity score close to one is thinner than for all $x \in \mathcal{C}_{0,n}$. Hereafter, for brevity, we ignore the $(1 + o_p(1))$ term. Now,

$$ZN_{2,n}^q(\tau, x, \tau, 1) = \frac{1}{nh_pH^q \left( \prod_{j=1}^{d_x} \xi_{j,n} \right)} \sum_{i=1}^{n} s_i (1\{u_t(x_i) \leq 0\} - \tau) 1 \{x_i \in \mathcal{C}_{0,n}\} K \left( \frac{p_i - \delta}{h_p} \right)$$

$$+ \frac{n}{nh_pH^q \left( \prod_{j=1}^{d_x} \xi_{j,n} \right)} E_{\mathcal{S}_n} \left[ s_i (1\{\hat{u}_t(x_i) \leq 0\} - \tau) 1 \{x_i \in \mathcal{C}_{0,n}\} K \left( \frac{\hat{p}_i - \delta}{h_p} \right) \right]$$

$$- E_{\mathcal{S}_n} \left[ s_i (1\{u_t(x_i) \leq 0\} - \tau) 1 \{x_i \in \mathcal{C}_{0,n}\} K \left( \frac{p_i - \delta}{h_p} \right) \right]$$

$$- \frac{1}{nh_pH^q \left( \prod_{j=1}^{d_x} \xi_{j,n} \right)} \sum_{i=1}^{n} \left( s_i (1\{u_t(x_i) \leq 0\} - \tau) 1 \{x_i \in \mathcal{C}_{0,n}\} K \left( \frac{\hat{p}_i - \delta}{h_p} \right) \right)$$

$$+ \frac{1}{nh_pH^q \left( \prod_{j=1}^{d_x} \xi_{j,n} \right)} \sum_{i=1}^{n} \left( s_i (1\{\hat{u}_t(x_i) \leq 0\} - \tau) 1 \{x_i \in \mathcal{C}_{0,n}\} K \left( \frac{\hat{p}_i - \delta}{h_p} \right) \right)$$

$$- E_{\mathcal{S}_n} \left[ s_i (1\{\hat{u}_t(x_i) \leq 0\} - \tau) 1 \{x_i \in \mathcal{C}_{0,n}\} K \left( \frac{\hat{p}_i - \delta}{h_p} \right) \right]$$

$$= I_n + II_n + III_n$$
We derive the limiting distribution of $I_n$, and show that both $II_n$ and $III_n$ are $o_p(1)$, uniformly in $\tau \in \mathcal{T}$.

We begin by studying $I_n$ recalling that $G_{x_i}(\tau, p_i) = \Pr(y_i \leq q_{r}(x_i)|x_i, p_i)$ for notational simplicity. Note that this term can be decomposed as:

$$I_n = \frac{1}{\sqrt{nh^p}} \sum_{i=1}^{n} s_i(1\{u_{\tau}(x_i) \leq 0\} - G_{x_i}(\tau, p_i)) 1\{x_i \in C_{0,n}\} K\left(\frac{p_i - \delta}{h_p}\right)$$

$$\quad + \frac{1}{\sqrt{nh^p}} \sum_{i=1}^{n} s_i(G_{x_i}(\tau, p_i) - \tau) 1\{x_i \in C_{0,n}\} K\left(\frac{p_i - \delta}{h_p}\right)$$

$$\quad = I_{1,n} + I_{2,n}$$

The first term will be shown to drive the distribution, while the second term can be thought of as bias since $G_{x}(\tau, p) \rightarrow \tau$ only as $p \rightarrow 1$. Now, starting with the first term,

$$\frac{1}{h_p^p} \left[ s_i(1\{u_{\tau}(x_i) \leq 0\} - G_{x_i}(\tau, p_i)) 1\{x_i \in C_{0,n}\} K\left(\frac{p_i - \delta}{h_p}\right) \right]^2$$

$$= \frac{1}{h_p^p} \int_{C_{0,n}} \int_{\mathcal{C}_{0,n}} (1\{u_{\tau}(x_i) \leq 0\} - G_{x_i}(\tau, p_i))^2 K^2\left(\frac{p_i - \delta}{h_p}\right) f_{u_{\tau}, x, p, s=1}(u_{\tau}(x_i)|p_i, x_i, s_i = 1) \Pr(s_i = 1|p_i) f_{x, p}(x_i, p_i) du_{\tau}, dp_i dx_i$$

$$= \int_{\mathcal{C}_{0,n}} \int_{\mathcal{C}_{0,n}} (1\{u_{\tau}(x_i) \leq 0\} - G_{x_i}(\tau, 1 - H + h_p v))^2 f_{u_{\tau}, x, p, s=1}(u_{\tau}(x_i)|p_i, x_i, 1 - H + h_p v, s_i = 1)$$

$$\times K^2(v) \Pr(s_i = 1|1 - H + h_p v) f_{x, p}(x_i, 1 - H + h_p v) du_{\tau}, dv dx_i.$$
uniformly over $T$. Now, under $H_{0,q}^{(2)}$, using Assumption A.1, A.6, and A.9, this expression can be re-written further as (uniformly over $T$)

$$
\left( \int_{-1}^{1} K^2(v) \mathrm{d}v \right) \int_{C_{0,n}} G_{x_i}(\tau, 1 - H) (1 - G_{x_i}(\tau, 1 - H)) \Pr(s_i = 1|1 - H) g_{x,p}(x_i, 1) H \mathrm{d}x_i + O \left( h_p \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) \right)
$$

$$
\Pr(s_i = 1|1 - H) g_{x,p}(x_i, 1) H \mathrm{d}x_i + O \left( h_p \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) \right)
$$

$$
\Pr(s_i = 1|1 - H) g_{x,p}(x_i, 1) H \mathrm{d}x_i + O \left( h_p \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) \right)
$$

$$
\Pr(s_i = 1|1 - H) g_{x,p}(x_i, 1) H \mathrm{d}x_i + O \left( h_p \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) \right)
$$

$$
\Pr(s_i = 1|1 - H) g_{x,p}(x_i, 1) H \mathrm{d}x_i + O \left( h_p \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) \right)
$$

where the term $o\left( h_p \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) \right)$ follows using A.6 and is zero when $\eta = 0$ and $\xi_{j,n} > 0$ for all $j$ and $n$. Therefore:

$$
\text{var}(I_{1,n}) = \left( \int_{-1}^{1} K^2(v) \mathrm{d}v \right) \tau(1 - \tau) \int_{C_{0,n}} g_{x,p}(x_i, 1) H \mathrm{d}x_i + O \left( h_p \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) \right) + O \left( h_p \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) \right) + O \left( h_p \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) \right)
$$

Hence,

$$
\sqrt{\frac{1}{nh_p H \left( \prod_{j=1}^{d_x} \xi_{j,n} \right)}} \sum_{i=1}^{n} s_i \{ u_x(x_i) \leq 0 \} - G_x(\tau, 1) \prod_{j=1}^{d_x} \{ \tau_j \leq x_{i,j} \leq \tau_j \} \frac{K\left( \frac{p_i - \delta}{h_p} \right)}{h_p}
$$

converges pointwise by a CLT for triangular arrays. We now show that under the null, $I_{2,n} = o_p(I_{1,n})$, uniformly in $\tau \in T$ and for all $x_i \in C_{0,n}$.

$$
\frac{1}{h_p} E \left[ s_i(G_{x_i}(\tau, p_i) - \tau) 1 \{ x_i \in C_{0,n} \} K \left( \frac{p_i - \delta}{h_p} \right) \right]
$$

$$
= \frac{1}{h_p} \int_{0}^{1} \int_{C_{0,n}} (G_{x_i}(\tau, p_i) - \tau) K \left( \frac{p_i - \delta}{h_p} \right) \Pr(s_i = 1|p_i) f_{x,p}(x_i, p_i) \mathrm{d}p_i \mathrm{d}x_i
$$

Change of variables and mean value expansions around $h_p = 0$ together with Assumptions A.6 and A.9 yield again

$$
\int_{-1}^{1} H \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) \frac{h_p}{h_p} \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) H \mathrm{d}x_i + O \left( h_p \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) H \right)
$$

$$
= 0 + O \left( h_p \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) H \right)
$$
uniformly over $\mathcal{T}$. Hence, as $nh_p H^{2-\eta} \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) \to 0$ and $nh_p^3 H^n \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) \to 0$, $I_n = I_{1,n}(1 + o_p(1))$. As for $II_n$,

\[
II_n = \sqrt{n \frac{h_p H^n}{n} \left( \prod_{j=1}^{d_x} \xi_{j,n} \right)} \mathbb{E}_{\Sigma_n} \left[ s_i(1\{\hat{\mu}_r(x_i) < 0\} - 1\{u_r(x_i) \leq 0\}) \right] \{x_i \in \mathcal{C}_{0,n}\} K \left( \frac{p_i - \delta}{h_p} \right) + \sqrt{n \frac{h_p H^n}{n} \left( \prod_{j=1}^{d_x} \xi_{j,n} \right)} \mathbb{E}_{\Sigma_n} \left[ s_i(1\{u_r(x_i) \leq 0\} - \tau) \right] \{x_i \in \mathcal{C}_{0,n}\} K \left( \frac{\hat{p}_i - \delta}{h_p} \right) - K \left( \frac{p_i - \delta}{h_p} \right)
\]

uniformly over $\mathcal{T}$. We start with $II_{1,n}$. A mean value expansion yields:

\[
II_{1,n} = \sqrt{n \frac{h_p H^n}{n} \left( \prod_{j=1}^{d_x} \xi_{j,n} \right)} \int_0^1 \{F_{u_r|x,p,s=1}(\hat{q}_r(x_i) - q_r(x_i)|x_i, p_i, s_i = 1) - F_{u_r|x,p,s=1}(0|x_i, p_i, s_i = 1)\} \times \mathbb{P}(s_i = 1|p_i) \{x_i \in \mathcal{C}_{0,n}\} K \left( \frac{p_i - \delta}{h_p} \right) f_{x,p}(x_i, p_i) dx_i dp_i = \left( \sqrt{n \frac{h_p H^n}{n} \left( \prod_{j=1}^{d_x} \xi_{j,n} \right)} \int_0^1 \{F_{u_r|x,p,s=1}(0|x_i, p_i, s_i = 1)(\hat{q}_r(x_i) - q_r(x_i))\} \mathbb{P}(s_i = 1|x_i, p_i) \right) \times 1 \{x_i \in \mathcal{C}_{0,n}\} K \left( \frac{p_i - \delta}{h_p} \right) f_{x,p}(x_i, p_i) dx_i dp_i (1 + o_p(1)),
\]

where the $o_p(1)$ term holds uniformly in $(\tau, \xi, \sigma, \varphi)$ and follows by similar arguments as in the proof of Theorem 1. As for the first term, again by change of variables with $v = (p_i - \delta)/h_p$, using the fact that $\delta = 1 - H$, mean value expansion arguments together with A.5 and A.8 yield that

\[
\sqrt{n h_p^3 H^n} \int_{h_p^{-1}}^H \int f_{u_r|x,p,s=1}(0|x_i, 1 - H + vh_p, s = 1)(\hat{q}_r(x_i) - q_r(x_i)) \mathbb{P}(s_i = 1|1 - H + vh_p) \times 1 \{x_i \in \mathcal{C}_{0,n}\} K(v) f_{x,p}(x_i, 1 - H + vh_p) dx_i dv = \sqrt{n h_p^3 H^n} \int_{-1}^1 \int f_{u_r|x,p,s=1}(0|x_i, 1 - H, s = 1)(\hat{q}_r(x_i) - q_r(x_i)) \mathbb{P}(s_i = 1|1 - H) \times 1 \{x_i \in \mathcal{C}_{0,n}\} K(v) f_{x,p}(x_i, 1 - H) dx_i dv (1 + O(h_p))
\]

where we used Assumption A.9 and the fact that $\lim_{H \to 0} \mathbb{P}(s_i = 1|1 - H) = 1$. Because of $H/h_p \to \infty$, A.3, A.4, A.5, A.6, and the bandwidth conditions, similar arguments to the ones in the proof of
Theorem 1 yield that:

$$II_{1,n} = \left(\frac{nh_p}{H^n(\prod_{j=1}^{d_x} \xi_{j,n})}\int f_{u_r|x,p,s=1}(0|x_i, 1 - H, s = 1) (\tilde{q}_r(x_i) - q_r(x_i)) 1\{x_i \in C_{0,n}\} f_{x,p}(x_i, 1 - H)dx_i \times (1 + O(h_p))\right) = O\left(\sqrt{h_pH^n(\prod_{j=1}^{d_x} \xi_{j,n})}\right).$$

As for $II_{2,n}$,

$$II_{2,n} = \sqrt{\frac{n}{h_pH^n(\prod_{j=1}^{d_x} \xi_{j,n})}} E_{S_n} \left[ s_i(1\{u_r(x_i) \leq 0\} - \tau) 1\{x_i \in C_{0,n}\} \left( K\left(\frac{\hat{p}_i - \delta}{h_p}\right) - K\left(\frac{p_i - \delta}{h_p}\right)\right)\right]$$

$$= \frac{n}{h_pH^n(\prod_{j=1}^{d_x} \xi_{j,n})} \int \int \left( F_{u_r|x,p,s=1}(0|x_i, p_i, s_i = 1) - \tau \right) 1\{x_i \in C_{0,n}\} \Pr(s_i = 1|p_i)$$

$$\left( K\left(\frac{\hat{p}_i - \delta}{h_p}\right) - K\left(\frac{p_i - \delta}{h_p}\right)\right) f_{x,p}(x_i, p_i)dx_i dp_i$$

$$= 0$$

under $H_{0,q}^{(2)}$ since $F_{u_r|x,p,s=1}(0|x_i, p_i, s_i = 1) - \tau = F_{y|x,s=1}(q_r(x_i)|x_i, s_i = 1) - \tau = 0$. About $II_{3,n}$, given A.3 and the bandwidth conditions, another change of variables together with integration by parts yields that:

$$II_{3,n} = \frac{n}{h_pH^n(\prod_{j=1}^{d_x} \xi_{j,n})} \int \int \{F_{u_r|x,p,s=1}(\tilde{q}_r(x_i) - q_r(x_i)|x_i, p_i, s_i = 1) - F_{u_r|x,p,s=1}(0|x_i, p_i, s_i = 1)\}$$

$$\times \Pr(s_i = 1|p_i) 1\{x_i \in C_{0,n}\} \frac{1}{h_p} \nabla K\left(\frac{\hat{p}_i - \delta}{h_p}\right) (\hat{p}_i - p_i) f_{x,p}(x_i, p_i)dx_i dp_i (1 + o_p(1))$$

$$= o_p\left(\sqrt{h_pH^n(\prod_{j=1}^{d_x} \xi_{j,n})}\right).$$

uniformly over $T$.

Finally, we turn to $III_n$. Noting that the function class $F_1 \equiv \{f_1(p) = K\left(\frac{p - \delta}{h_p}\right) : h_p \in (0, 1], \delta \in [0, 1]\}$ is bounded by Assumption A.5 and the arguments from Lemma B.3 in Escanciano et al. (2014), we can again apply Lemma A.1 of Escanciano et al. (2014) to the function classes $F_1$ and $F_2 \equiv \{f_2(s, x) = s(1\{u_r(x) \leq 0\} - \tau) \Pi_{j=1}^{d_x} \{\xi_j < x_j < \bar{x_j}\} : \{\tau, \underline{x}, \bar{x}\} \in T \times X\}$ to conclude that the product class $F_1 \times F_2$ is Donsker. Then, by Lemma A.3 in Escanciano et al. (2014), it follows that $III_n = o_p(1)$ uniformly in $\tau, x_i, p_i$, as its counterpart in the proof of Theorem 1. Hence,

$$ZN_{2,n}^{q}(\tau, \underline{x}, \bar{x}, 1) = \frac{1}{\sqrt{n h_p}} \sum_{i=1}^{n} s_i(1\{u_r(x_i) \leq 0\} - \tau) 1\{x_i \in C_{0,n}\} K\left(\frac{p_i - \delta}{h_p}\right) (1 + o_p(1))$$

with the $o_p(1)$ term holding uniformly over $\tau \in T$.

As shown above, $ZN_{2,n}^{q}(\tau, \underline{x}, \bar{x}, 1)$, has a zero mean standard normal limiting distribution pointwise.
Moreover, using similar arguments, we obtain that:

\[
\left( \int K^2(v) dv \right) \frac{1}{nh_p H^n} \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) \sum_{i=1}^{n} s_i \left( 1 \{ \hat{u}_\tau(x_i) \leq 0 \} - \tau \right)^2 1 \{ x_i \in C_{0,n} \} K \left( \frac{p_i - \delta}{h_p} \right)
\]

\[
= \left( \int_{-1}^{1} K(v)^2 dv \right) \tau (1 - \tau) \frac{f_{C_{0,n}} g(x,1)x}{\left( \prod_{j=1}^{d_x} \xi_{j,n} \right)} + o_p(1)
\]

uniformly over \( \mathcal{T} \). The covariance kernel of the statistic is therefore given by:

\[
\text{cov} \left( Z^q_{2,n} (\tau, x, \bar{x}, 1), Z^q_{2,n} (\tau', x, \bar{x}, 1) \right)
\]

\[
= \lim_{n \to \infty} E \left[ \frac{\sum_{i=1}^{n} s_i (1 \{ u_{\tau}(x_i) \leq 0 \} - \tau) 1 \{ x_i \in C_{0,n} \} K \left( \frac{p_i - \delta}{h_p} \right)}{\left( \int K^2(v) dv \right) \sum_{i=1}^{n} s_i (1 \{ u_{\tau}(x_i) \leq 0 \} - \tau)^2 1 \{ x_i \in C_{0,n} \} K \left( \frac{p_i - \delta}{h_p} \right)} \right].
\]

Finally, by Lemma A.1 and B.3 of Escanciano et al. (2014), we can also conclude that numerator and denominator of \( Z^q_{2,n} (\tau, x, \bar{x}, 1) \) are Donsker, and hence by Theorem 2.10.6 of Van der Vaart and Wellner (1996), that \( Z^q_{2,n} (\tau, x, \bar{x}, 1) \) is Donsker as well. Thus, it follows that

\[
Z^q_{2,n} (\tau, x, \bar{x}, 1) = \frac{ZN^q_{2,n} (\tau, x, \bar{x}, 1)}{\left( \int K^2(v) dv \right) \frac{1}{nh_p H^n \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) \sum_{i=1}^{n} s_i (1 \{ u_{\tau}(x_i) \leq 0 \} - \tau)^2 1 \{ x_i \in C_{0,n} \} K \left( \frac{p_i - \delta}{h_p} \right)} \}^{1/2}
\]

converges weakly in \( l^\infty(\mathcal{T}) \), and by continuous mapping, so does the functional

\[
\sup_{\tau \in \mathcal{T}} \left| Z^q_{2,n} (\tau, x, \bar{x}, 1) \right|
\]

as postulated in the statement of part (i).

(ii) Now, if there is an omitted relevant regressor, given Assumption A.7,

\[
\frac{1}{nh_p \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) H^n} \sum_{i=1}^{n} s_i (G_{x_i} (\tau, p_i) - \tau) \Pi_{j=1}^{d_x} 1 \{ x_{i,j} \leq \bar{x}_{j} \} K \left( \frac{p_i - \delta}{h_p} \right)
\]

\[
= \lim_{n \to \infty} \frac{1}{h_p \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) H^n} E \left[ s_i (G_{x_i} (\tau, p_i) - \tau) 1 \{ x_i \in C_{0,n} \} K \left( \frac{p_i - \delta}{h_p} \right) \right] + o_p(1)
\]
and

\[
\frac{1}{nh_p H^n (\prod_{j=1}^{d_x} \xi_{j,n})} \sum_{i=1}^{n} s_i (1 \{ \hat{u}_i(x_i) \leq 0 \} - \tau)^2 \prod_{j=1}^{d_x} 1 \{ \xi_j \leq x_{i,j} \leq \overline{x}_j \} K^2 \left( \frac{\hat{p}_i - \delta}{h_p} \right)
\]

\[
\overset{P}{\lim}_{n \to \infty} \frac{1}{nh_p (\prod_{j=1}^{d_x} \xi_{j,n})} E \left[ s_i (G_x(\tau, p_i) + \tau^2 - \tau G_x(\tau, p_i))^2 1 \{ x_i \in C_{0,n} \} K^2 \left( \frac{p_i - \delta}{h_p} \right) \right] > 0.
\]

Thus, the test has power against \( \sqrt{nh_p \left( \prod_{j=1}^{d_x} \xi_{j,n} \right)} H^n \) alternatives.

**Proof of Theorem 2**:  

(i) We first show that \( Z_{S_{2,n}}^q \) has the same distribution of \( Z_{2,n}^q \) under the null, conditional on the sample, for almost all samples. That is, we show that under \( H_{0,q} \),

\[
\lim_{n \to \infty} P \left( \sum_{i=1}^{n} s_i \Pi_{j=1}^{d_x} 1 \{ \xi_j \leq x_{j,i} \leq \overline{x}_j \} K \left( \frac{\hat{p}_i - \delta}{h_p} \right) = 0 \right).
\]

Moreover:

\[
E^* \left[ Z_{S_{2,n}}^q (\tau, \xi, \overline{x}, 1) \right] = E^* [B_{i,\tau} - \tau] \frac{1}{\sqrt{nh_p}} \sum_{i=1}^{n} s_i \Pi_{j=1}^{d_x} 1 \{ \xi_j \leq x_{j,i} \leq \overline{x}_j \} K \left( \frac{\hat{p}_i - \delta}{h_p} \right) = 0.
\]

where \( P^* \) denotes the bootstrap probability law, conditional on the sample. Let \( E^* \) and \( var^* \) again be the bootstrap mean and variance operator conditional on the sample. Recalling that \( B_{i,\tau} = 1 \{ U_i \leq \tau \} \) with \( U_i \) i.i.d. \( U(0, 1) \) and \( U_i \) independent of the sample, note that:

\[
E^* \left[ Z_{S_{2,n}}^q (\tau, \xi, \overline{x}, 1) \right] = E^* [B_{i,\tau} - \tau] \frac{1}{\sqrt{nh_p}} \sum_{i=1}^{n} s_i \Pi_{j=1}^{d_x} 1 \{ \xi_j \leq x_{j,i} \leq \overline{x}_j \} K \left( \frac{\hat{p}_i - \delta}{h_p} \right) = 0.
\]

Moreover:

\[
var^* \left( Z_{S_{2,n}}^q (\tau, \xi, \overline{x}, 1) \right) = \frac{1}{nh_p} \sum_{i=1}^{n} s_i \Pi_{j=1}^{d_x} 1 \{ \xi_j \leq x_{j,i} \leq \overline{x}_j \} K^2 \left( \frac{\hat{p}_i - \delta}{h_p} \right) var^* (B_{i,\tau} - \tau) \]

\[
= \frac{1}{nh_p} \sum_{i=1}^{n} s_i \Pi_{j=1}^{d_x} 1 \{ \xi_j \leq x_{j,i} \leq \overline{x}_j \} K^2 \left( \frac{\hat{p}_i - \delta}{h_p} \right) \tau (1 - \tau)
\]

uniformly in \( \mathcal{T} \) given the bandwidth conditions and Assumption A.5. The expression after the last equality gives:

\[
E \left[ \frac{1}{h_p} s_i \Pi_{j=1}^{d_x} 1 \{ \xi_j \leq x_{j,i} \leq \overline{x}_j \} K^2 \left( \frac{p_i - \delta}{h_p} \right) \right] \tau (1 - \tau)
\]

\[
= \tau (1 - \tau) \left( \int_{-1}^{1} K^2 (v) \, dv \right) \int_{C_{0,n}} g_{x,p}(x_i, 1) H^n \, dx_i
\]

\[
+ O \left( H \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) \right) + O \left( H^n h_p \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) \right)
\]

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\begin{align*}
\bar{\text{var}}^*\left(Z_{2,n}^q (\tau, \bar{x}, \bar{\tau}, 1)\right) &= 1/n_p \sum_{i=1}^n s_i \Pi_{j=1}^{d_x} 1\{\xi_j < x_{j,i} < \bar{\xi}_j\} K^2 \left(\frac{\bar{\pi}_i - \delta}{h_p}\right) \bar{\text{var}}^* (B_{i,\tau} - \tau) \\
&= 1/n_p \sum_{i=1}^n s_i \Pi_{j=1}^{d_x} 1\{\xi_j < x_{j,i} < \bar{\xi}_j\} K^2 \left(\frac{\bar{\pi}_i - \delta}{h_p}\right) \left(1/n \sum_{j=1}^n (B_{j,\tau} - \tau)^2\right)
\end{align*}

Then, pointwise in \(\tau\), 
\[
\frac{Z_{2,n}^q (\tau, \bar{x}, \bar{\tau}, 1)}{\sqrt{\text{var}^* (Z_{2,n}^q (\tau, \bar{x}, \bar{\tau}, 1))}} \overset{d^*}{\to} N(0, 1),
\]
where \(d^*\) denotes convergence in distribution conditional on the sample. Now,

\begin{align*}
cov^*\left(\frac{Z_{2,n}^q (\tau, \bar{x}, \bar{\tau}, 1)}{\sqrt{\text{var}^* (Z_{2,n}^q (\tau, \bar{x}, \bar{\tau}, 1))}}, \frac{Z_{2,n}^q (\tau', \bar{x}, \bar{\tau}, 1)}{\sqrt{\text{var}^* (Z_{2,n}^q (\tau', \bar{x}, \bar{\tau}, 1))}}\right) &= \frac{\text{cov}^* (Z_{2,n}^q (\tau, \bar{x}, \bar{\tau}, 1), Z_{2,n}^q (\tau', \bar{x}, \bar{\tau}, 1))}{\sqrt{\text{var}^* (Z_{2,n}^q (\tau, \bar{x}, \bar{\tau}, 1))} \sqrt{\text{var}^* (Z_{2,n}^q (\tau', \bar{x}, \bar{\tau}, 1))}} + o_p^*(1) \\
&= \frac{1/n_p \sum_{i=1}^n s_i \Pi_{j=1}^{d_x} 1\{\xi_j, \xi < x_{j,i} < \bar{\xi}_j\} K^2 \left(\frac{\bar{\pi}_i - \delta}{h_p}\right)}{\sqrt{\text{var}^* (Z_{2,n}^q (\tau, \bar{x}, \bar{\tau}, 1))} \sqrt{\text{var}^* (Z_{2,n}^q (\tau', \bar{x}, \bar{\tau}, 1))}} \\
&\times \text{cov}^* (B_{i,\tau} - \tau, (B_{i,\tau'} - \tau')) + o_p^*(1) \\
&= \frac{1/n_p \sum_{i=1}^n s_i \Pi_{j=1}^{d_x} 1\{\xi_j < x_{j,i} < \bar{\xi}_j\} K^2 \left(\frac{\bar{\pi}_i - \delta}{h_p}\right)}{\sqrt{\text{var}^* (Z_{2,n}^q (\tau, \bar{x}, \bar{\tau}, 1))} \sqrt{\text{var}^* (Z_{2,n}^q (\tau', \bar{x}, \bar{\tau}, 1))}} \\
&\times (\min \{\tau, \tau'\} - \tau') + o_p^*(1) \\
&= \frac{1/n_p \sum_{i=1}^n s_i \Pi_{j=1}^{d_x} 1\{\xi_j < x_{j,i} < \bar{\xi}_j\} K^2 \left(\frac{\bar{\pi}_i - \delta}{h_p}\right)}{\sqrt{\text{var}^* (Z_{2,n}^q (\tau, \bar{x}, \bar{\tau}, 1))} \sqrt{\text{var}^* (Z_{2,n}^q (\tau', \bar{x}, \bar{\tau}, 1))}} + o_p^*(1) + o_p(1)
\end{align*}

Finally, as \(Z_{2,n}^q (\tau, \bar{x}, \bar{\tau}, 1)\) is stochastically equicontinuous in \(\tau\) by the arguments in the proof of Theorem 1, the statement in (27) follows.

(ii) \(Z_{2,n}^q\) has the same limiting distribution under both \(H_{0,q}^{(2)}\) and \(H_{A,q}^{(2)}\), while \(Z_{2,n}^q\) diverges to infinity under \(H_{A,q}^{(1)}\). \(\square\)

**Proof of Theorem RS**
\[
\lim_{n \to \infty} \Pr (\text{choose } H_{S,q} | H_{NS} \text{ is true}) \\
= \lim_{n \to \infty} \Pr \left( \text{reject } H_{0,q}^{(1)} \text{ and do not reject } H_{0,q}^{(2)} | H_{0,q}^{(1)} \text{ is true or } H_{0,q}^{(2)} \text{ false} \right) \\
\leq \lim_{n \to \infty} \Pr \left( \text{reject } H_{0,q}^{(1)} | H_{0,q}^{(1)} \text{ is true or } H_{0,q}^{(2)} \text{ false} \right) \\
= \lim_{n \to \infty} \Pr \left( \text{reject } H_{0,q}^{(1)} | H_{0,q}^{(1)} \text{ is true} \right) = \alpha
\]

and

\[
\lim_{n \to \infty} \Pr (\text{choose } H_{NS,q} | H_{S} \text{ is true}) \\
= \lim_{n \to \infty} \Pr \left( \text{do not reject } H_{0,q}^{(1)} \text{ OR reject } H_{0,q}^{(1)} \text{ and } H_{0,q}^{(2)} | H_{0,q}^{(1)} \text{ is false or } H_{0,q}^{(2)} \text{ true} \right) \\
\leq \lim_{n \to \infty} \Pr \left( \text{do not reject } H_{0,q}^{(1)} | H_{0,q}^{(1)} \text{ is false or } H_{0,q}^{(2)} \text{ true} \right) \\
+ \lim_{n \to \infty} \Pr \left( \text{reject } H_{0,q}^{(1)} \text{ and } H_{0,q}^{(2)} | H_{0,q}^{(1)} \text{ is false or } H_{0,q}^{(2)} \text{ true} \right) \\
= \lim_{n \to \infty} \Pr \left( \text{do not reject } H_{0,q}^{(1)} | H_{0,q}^{(1)} \text{ is false} \right) \\
+ \lim_{n \to \infty} \Pr \left( \text{reject } H_{0,q}^{(1)} \text{ and } H_{0,q}^{(2)} | H_{0,q}^{(1)} \text{ is false or } H_{0,q}^{(2)} \text{ true} \right) \\
\leq \lim_{n \to \infty} \Pr \left( \text{do not reject } H_{0,q}^{(1)} | H_{0,q}^{(1)} \text{ is false} \right) + \lim_{n \to \infty} \Pr \left( \text{reject } H_{0,q}^{(2)} | H_{0,q}^{(1)} \text{ is false or } H_{0,q}^{(2)} \text{ true} \right) \\
\leq \lim_{n \to \infty} \Pr \left( \text{do not reject } H_{0,q}^{(1)} | H_{0,q}^{(1)} \text{ is false} \right) + \lim_{n \to \infty} \Pr \left( \text{reject } H_{0,q}^{(2)} | H_{0,q}^{(2)} \text{ true} \right) \\
= 0 + \gamma.
\]

\(\square\)

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This supplementary material contains the technical assumptions required for the conditional mean case, the corresponding test statistics and formal convergence and consistency statements. More specifically, we outline the conditions and formal results for the first test in Section S1 and the ones for the second test in Section S3. The corresponding proofs can be found in Section S3. Finally, in Section S4 we examine the properties of the test statistics for the conditional quantile and mean case in finite samples through a small Monte Carlo study.

S1 First Test

The test statistic for the conditional mean is given by:

\[ Z_{1,n}^m = \sup_{(x, x', p, p') \in \mathcal{X} \times \mathcal{P}} |Z_{1,n}^m (x, x', p, p')|, \]

where

\[ Z_{1,n}^m (x, x', p, p') = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} s_i (y_i - \hat{m}(x_i)) f_{x}(x_i) \prod_{j=1}^{d_x} 1\{x_j < x_{j,i} < x_j\} 1\{p \leq \hat{p}_i \leq p\}. \]

We require the following technical assumptions:

S.1 \((y_i, x'_i, z'_i, s_i) \subset R_y \times R_x \times R_z \times \{0, 1\}\) are identically and independently distributed. Let \(\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_d_x\) denote a compact subset of the interior of \(R_x\). \(z_i\) contains at least one variable which is not contained in \(x_i\) and which is not \(x_i\)-measurable. The distributions of \(x_i\) and \(z_i\) have a probability density function with respect to Lebesgue measure which is strictly positive and continuously differentiable (with bounded derivatives) over the interior of their respective supports. Assume that \(\Pr(s_i = 1 | x, p) = \Pr(s_i = 1 | p) > 0\) for all \(p \in \mathcal{P}\) and \(x \in \mathcal{X}\). Moreover, assume that \(E[|y_i|^{2+\delta}] < \infty\) for some \(\delta > 0\).

S.2 Assume that \(m(x_i)\) and \(f_{x}(x_i)\) are \(q\) times differentiable over \(\mathcal{X}\) with uniformly bounded derivatives and \(q > d_x/2\).

S.3 The non-negative kernel function \(K(\cdot)\) is a bounded, continuously differentiable function with uniformly bounded derivative and compact support on \([-1, 1]\). It satisfies \(\int K(v)dv = 1\), \(\int v^l K(v)dv = 0\) for \(l = 1, \ldots, q - 1\), and \(\int |v^q| K(v)dv < \infty\).

S.4 The conditional distribution function \(F_{p|x,s=1}(p|\cdot, s=1)\) is continuously differentiable in \(x\) and \(p\) uniformly over \(\mathcal{X}\) and \(\mathcal{P}\), and for some \(C > 0\), satisfies:

\[ |\nabla_p F_{p|x,s=1}(p|x, s=1) - \nabla_{p'} F_{p'|x,s=1}(p'|x', s=1)| \leq C \| (p, p') - (x, p') \|. \]

for every \((x, x') \in \mathcal{X}\) and \((p, p') \in \mathcal{P}\).
There exists an estimator $\hat{p}(z_i)$ such that $\sup_{z \in Z} |\hat{p}(z) - p(z)| = o_p(n^{-\frac{1}{2}})$ with $Z$ a compact subset of $R_z$, and that:

$$\Pr(\exists i : z_i \in R_z \setminus Z, p(z_i) \in P) = o(n^{-\frac{1}{2}}).$$

Note that primitive conditions for Assumption S.1 can for instance be found in Escanciano et al. (2014).

**Theorem S1:** Let Assumptions S.1-S.5 and Equation (2) from the paper hold. If, as $n \to \infty$, It holds that (i) $h_x \to 0$ (ii) $nh_x^d \to \infty$, (iii) $nh_x^{2q} \to 0$, then (i) under $H_{0,m}^{(1)}$,

$$Z_{1,n}^m \Rightarrow Z_{1}^m,$$

where $Z_{1}^m$ denotes the supremum of a zero mean Gaussian process with covariance kernel defined in the proof of Theorem S1.

(ii) and under $H_{A,m}^{(1)}$, there exists $\varepsilon > 0$, such that

$$\lim_{n \to \infty} \Pr(Z_{1,n}^m > \varepsilon) = 1.$$

In the proof of Theorem S1 we establish that:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} s_i(y_i - \hat{m}(x_i)\hat{f}_x(x_i)) \prod_{j=1}^{d_x} 1\{x_j < x_{j,i} < \overline{x}_j\} 1\{p \leq \hat{p}_i \leq \overline{p}\}$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_x(x_i)(y_i - m(x_i)) \prod_{j=1}^{d_x} 1\{x_j < x_{j,i} < \overline{x}_j\} 1\{p_i \leq \overline{p}\} - 1\{p_i \leq \overline{p}\}$$

$$- (F_{p|x,s=1}(p|x_i, s_i = 1) - F_{p|x,s=1}(p|x_i, s_i = 1)) \Pr(s_i = 1|x_i)) + o_p(1),$$

uniformly over $X$ and $P$. Thus, a natural bootstrap version of $Z_{1,n}^m(x, \overline{x}, p)$ would be:

$$\tilde{U}^*_0(x, \overline{x}, p, \overline{p})$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} v_i f_x(x_i)(y_i - m(x_i)) \prod_{j=1}^{d_x} 1\{x_j < x_{j,i} < \overline{x}_j\} 1\{p_i \leq \overline{p}\} - 1\{p_i \leq \overline{p}\}$$

$$- (F_{p|x,s=1}(p|x_i, s_i = 1) - F_{p|x,s=1}(p|x_i, s_i = 1)) \Pr(s_i = 1|x_i),$$

where $v_i$, $i = 1, \ldots, n$ are i.i.d. random variables satisfying $E[v_i] = 0$ and $E[v_i^2] = 1$, which are independent of $(y_i, x_i)$. Of course, $\tilde{U}^*_0(x, \overline{x}, p, \overline{p})$ is infeasible since $p_i$, $F_{p|x,s=1}(p|x_i, s_i = 1)$ as well as $\Pr(s_i = 1|x_i)$ need to be estimated. Instead, the feasible bootstrap version we use for computational reasons (cf. Delgado and Gonzalez-Manteiga (2001)) is given by:

$$Z_{1,n}^{bm}(x, \overline{x}, p) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (y_i^* - \hat{m}^*(x_i)) s_i \hat{f}_x(x_i) \prod_{j=1}^{d_x} 1\{x_j < x_{j,i} < \overline{x}_j\} 1\{p \leq \hat{p}_i \leq \overline{p}\}$$
with $y_i^* = \hat{m}(x_i) + \hat{\varepsilon}_i^*$ and $\hat{\varepsilon}_i^* = v_i \varepsilon_i = v_i(y_i - \hat{m}(x_i))$, and

$$
\hat{m}^*(x_i) = \frac{1}{nh_x^d f_x(x_i)} \sum_{j=1}^{n} y_j^* K_h(x_i - x_j).
$$

The additional assumption we impose is:

**S.6** There exists an estimator $\hat{m}(x)$ such that:

$$
\sup_{x \in \mathcal{X}} |\hat{m}(x) - m(x)| = o_p(n^{-\frac{1}{2}}),
$$

where $\mathcal{X}$ is a compact subset defined in A.1. Moreover, $f_x(x_i) > c > 0$ for all $x_i \in \mathcal{X}$.

Assumption S.6 is a high-level condition and rather standard in the non- and semiparametric literature. If the estimator is a kernel estimator with a kernel function as defined in S.3, this condition will be satisfied in our context when $d_x < 4$.

**Theorem S1**: Let Assumptions S.1-S.6 and Equation (2) from the paper hold. If, as $n \to \infty$, It holds that (i) $h_x \to 0$ (ii) $nh_x^d \to \infty$, (iii) $nh_x^{2q} \to 0$, (iv) $R \to \infty$, then (i) under $H_{0,q}^{(1)}$

$$
\lim_{n,R \to \infty} \Pr Z_{1,n}^m \geq c_{(1-\alpha),n,R}^{\ast(1)} = \alpha
$$

(ii) under $H_{A,q}^{(1)}$

$$
\lim_{n,R \to \infty} \Pr Z_{1,n}^m \geq c_{(1-\alpha),n,R}^{\ast(1)} = 1.
$$

**S2 Second Test**

The second test is based on the statistic

$$
Z_{2,n}^m (\bar{x}, \bar{y}, 1) = \frac{\sum_{i=1}^{n} s_i (y_i - \hat{m}(x_i))^2 \hat{f}_x(x_i) \prod_{j=1}^{d_x} 1\{x_j < x_{j,i} \leq \bar{x}_j\} K_{h} \left( \frac{\bar{x}_j - \bar{x}_j}{h} \right)}{\left( \int K^2(u) dv \sum_{i=1}^{n} s_i (y_i - \hat{m}(x_i))^2 \hat{f}_x(x_i)^2 \prod_{j=1}^{d_x} 1\{x_j < x_{j,i} \leq \bar{x}_j\} K_{h} \left( \frac{\bar{x}_j - \bar{x}_j}{h} \right) \right)^{1/2}},
$$

where again $\delta = 1 - H$ with $H \to 0$, $H/h_p \to \infty$ as $n \to \infty$. The assumptions we require for the second test are almost identical to the ones needed for the conditional quantile case (we repeat them for completeness):

**S.7** Assume there exists at least one point $x_0 \in \mathcal{X}$, such that for at least one $z \in R_z$, it holds that $p(z) = 1$. Moreover, assume that for all $y$ and $x \in C_{0,n}$, there exist $0 \leq \eta \leq \bar{\eta} < 1$ and there exists a strictly positive, continuous, and integrable function $g_{y,x,p}(y, x, 1)$ and $g_{x,p}(x, 1)$ such that for all $y$ and $x \in C_{0,n}$:

$$
\sup_{x \in C_{0,n}, y \in R_y} \left( \left| \frac{f_{y,x,p}(y, x, 1 - H)}{g_{y,x,p}(y, x, 1) H^q} - 1 \right| \right) \to 0 \quad \text{and} \quad \sup_{x \in C_{0,n}} \left( \left| \frac{f_{x,p}(x, 1 - H)}{g_{x,p}(x, 1) H^q} - 1 \right| \right) \to 0
$$

as $n \to \infty$ for some $0 \leq \eta \leq \bar{\eta} < 1$. For all $x \notin C_{0,n}$ and $y \in R_y$, it holds that $f_{y,x,p}(y, x, 1 - H) = 0$ for all $H$.

**S.8** The distribution function $F_{y|x,p,s=1}(\cdot, \cdot, \cdot)$ of $y_i$ given $x_i$, $p_i$, and selection $s_i = 1$ has a continuous probability density function $f_{y|x,p,s=1}(y|x, p, s = 1)$ w.r.t. Lebesgue measure. The functions
\[ f_{y|x,p,s=1}(y|x_i, p_i, s_i = 1), f_{p|x,s=1}(p_i|x_i, s_i = 1), \Pr(s_i = 1|p_i), \text{ and } f_{x,p}(x_i, p_i) \text{ are continuously differentiable w.r.t. to } p_i \text{ on } (0,1) \text{ for all } x_i \in \mathcal{X} \text{ and } z_i \in \mathcal{Z}. \] Moreover, assume that for every \( x \in \mathcal{X} \) and \( y, f_{y,x,p}(y, x, \cdot) \) is left-continuous at \( p = 1 \).

**S.9** The set of \( x \in \mathcal{X} \) for which \( \pi \) is a relevant predictor is a subset of \( \mathcal{C}_{0,n} \) (see Assumption S.6).

**S.10** Assume that there exists a positive constant \( C \) and \( C' \) such that for every \( x \in \mathcal{X} \):

\[ |m_x(1 - H) - m_x(1)| \leq CH^{1-\eta} \]

and

\[ |\Pr(s_i = 1|1 - H) - 1| \leq C'H^{1-\eta}, \]

where \( m_x(\cdot) \equiv E[\varepsilon|x, p = \cdot] \). Moreover, the following partial derivatives are bounded:

\[ \sup_{y \in R, x \in \mathcal{X}, p \in (0,1)} |\nabla_p f_{y|x,p,s=1}(y|x, p, s = 1)| < \infty, \]

\[ \sup_{p \in (0,1), x \in \mathcal{X}} |\nabla_p f_{p|x,s=1}(p|x, s = 1)| < \infty, \]

\[ \sup_{x \in \mathcal{X}, p \in (0,1)} |\nabla_p \Pr(s_i = 1|p)| < \infty, \]

and

\[ \sup_{x \in \mathcal{X}, p \in (0,1)} |\nabla_p f_{x,p}(x, p)| < \infty. \]

**Theorem S2:** Let Assumptions S.1 through S.5, S.7 through S.10 as well as Equation (2) from the main paper hold. If, as \( n \to \infty \) and \( H \to 0 \), it holds that (i) \( h_x \to 0 \) (ii) \( nh_x^{d_x} \to \infty \), (iii) \( nh_x^{2q} \to 0 \), (iv) \( H/h_p \to \infty \), (v) \( nh_pH^{2-\eta} \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) \to 0 \), and (vi) \( nh_p \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) H^\eta \to \infty \), then

(i) under \( H_{0,m}^{(2)} \),

\[ \frac{Z_{m,n}^{2}(x, 1)}{\sqrt{\text{var}(Z_{m,n}^{2}(x, 1))}} \Rightarrow N(0, 1). \]

(ii) Under \( H_{A,m}^{(2)} \), there exists \( \varepsilon > 0 \), such that

\[ \lim_{n \to \infty} \Pr\left( \left| \frac{Z_{m,n}^{2}(x, 1)}{\sqrt{\text{var}(Z_{m,n}^{2}(x, 1))}} > \varepsilon \right| \right) = 1. \]

As a final remark, observe that, unlike in the quantile case, critical values can straightforwardly be constructed from the standard normal distribution as the limiting distribution is Gaussian with mean zero and variance one.

**S3 Proofs**

**Proof of Theorem S1**
(i) We ignore again the part involving \(1\{\hat{p}_i \leq p\}\). Let \(E\) denote the expectation operator and \(E_{S_n}\) denote the expectation conditional on the sample \(\{y_i, x_i', s_i\}_{i=1}^n\), so that:
\[
E_{S_n}[\hat{m}(x_i)] = \int \hat{m}(x_i)f_x(x_i)dx_i.
\]
Since under \(H_0^{(1)}\):
\[
E \left[ s_i(y_i - m(x_i))f_x(x_i) \prod_{j=1}^{d_x} 1\{x_j < x_{j,i} < x_{j,i}^*\}1\{p_i \leq p\} \right] = 0
\]
by iterated expectations, observe that:
\[
\begin{align*}
Z_{1,n}^m(\bar{x}, \bar{x}, \bar{p}) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n (y_i - \hat{m}(x_i))\hat{f}_x(x_i) \prod_{j=1}^{d_x} 1\{x_j < x_{j,i} < x_{j,i}^*\}1\{\hat{p}_i \leq \bar{p}\} \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^n f(x_i)s_i(y_i - m(x_i))1\{x_j < x_{j,i} < x_{j,i}^*\}1\{p_i \leq p\} + E_{S_n} \left[ \frac{1}{\sqrt{n}} \sum_{i=1}^n (y_i - \hat{m}(x_i))\hat{f}_x(x_i) \prod_{j=1}^{d_x} 1\{x_j < x_{j,i} < x_{j,i}^*\}1\{\hat{p}_i \leq \bar{p}\} \right] \\
&\quad + \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^n f(x_i)s_i(y_i - m(x_i)) \prod_{j=1}^{d_x} 1\{x_j < x_{j,i} < x_{j,i}^*\}1\{\hat{p}_i \leq \bar{p}\} \right\} \\
&\quad - E_{S_n} \left[ \frac{1}{\sqrt{n}} \sum_{i=1}^n (y_i - \hat{m}(x_i)) \prod_{j=1}^{d_x} 1\{x_j < x_{j,i} < x_{j,i}^*\}1\{\hat{p}_i \leq \bar{p}\} \right] \\
&\quad - \left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^n f(x_i)s_i(y_i - m(x_i)) \prod_{j=1}^{d_x} 1\{x_j < x_{j,i} < x_{j,i}^*\}1\{p_i \leq p\} \right\} \\
&= I_n + II_n + III_n,
\end{align*}
\]

We start with \(III_n\). Using Theorem 2.11.9 of Van der Vaart and Wellner (1996), whose assumptions can be verified using the same arguments as in the proof of Theorem 3.1 of Escanciano et al. (2014) and Assumptions S.1-S.5 paired with the bandwidth conditions, we can conclude that \(\mathcal{F}_1 \equiv \{f_1(s, y, x) = s(y - m(x))f_x(x)\prod_{j=1}^{d_x} 1\{x_j < x_{j,i} < x_{j,i}^*\} : (m, \{x, \bar{x}\}) \in \mathcal{M} \times \mathcal{X} \} \) is Donsker. Thus, by Lemma A.1 in Escanciano et al. (2014) the product class \(\mathcal{F}_1 \times \mathcal{F}_2\) is Donsker as well, where \(\mathcal{F}_2 \equiv \{f_2(p(z)) = 1\{p(z) \leq \bar{p}\} : p \in \mathcal{P}\}\), and by Lemma A.3 of Escanciano et al. (2014), it follows that \(III_n = o_p(1)\) uniformly over \(\mathcal{X}\) and \(\mathcal{P}\). For \(II_n\), note that:
\[
II_n = \frac{1}{\sqrt{n}h_x^d} \sum_{j=1}^n \int (m(x_i) - m(x_j))K_h(x_i - x_j) \prod_{j=1}^{d_x} 1\{x_j < x_{j,i} < x_{j,i}^*\}1\{\hat{p}_i \leq \bar{p}\} |x_i, s_i = 1| dF_x(x_i)
\]
\[
- \frac{1}{\sqrt{n}h_x^d} \sum_{j=1}^n \int \varepsilon_jK_h(x_i - x_j) \prod_{j=1}^{d_x} 1\{x_j < x_{j,i} < x_{j,i}^*\}1\{\hat{p}_i \leq \bar{p}\} |x_i, s_i = 1| \Pr(s_i = 1|x_i)f_x(x_i)dx_i
\]
\[
= II_n^A + II_n^B,
\]
which follows since under $H_{0,m}^{(1)}$:

$$
\frac{1}{\sqrt{n}h_x} \sum_{j=1}^{n} \int \int s_j \xi_i f(x_i|p_i) d\xi_i K_h(x_i-x_j) \prod_{l=1}^{d_x} 1\{x_j < x_{j,i} < \varpi_j\} 1\{\hat{p}(z_i) \leq \varrho\} f_{x,p}(x_i, p(z_i)) dx_j dz_i = 0.
$$

Standard Taylor expansion arguments using Assumptions S.2 through S.4 yield:

$$
II_n^A = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left\{ \int (m(x_j + h_x \nu) - m(x_j)) K(\nu) \prod_{l=1}^{d_x} 1\{x_l < x_{l,j} + h_x \nu < \varpi_l\} \right\} 
\times F_{p|x,s=1}(\hat{p}(z_i) \leq \varrho|x_j + h_x \nu, s_i = 1) \Pr(s_i = 1|x_j + h_x \nu) f_x(x_j + h_x \nu) d\nu 
= \frac{h_x^q}{\sqrt{n}} \sum_{j=1}^{n} \nabla_{x}^{(q)} m(x_j) \int \nu^K(\nu) d\nu \prod_{l=1}^{d_x} 1\{x_l < x_{l,j} < \varpi_l\} \left\{ F_{p|x,s=1}(p(z_j) \leq \varrho|x_j, s_j = 1) 
+ \nabla^{(1)}_p F_{p|x,s=1}(p(z_j) \leq \varrho|x_j, s_j = 1)(\hat{p}(z_j) - p(z_j)) \right\} \Pr(s_j = 1|x_j) f_x(x_j) + o_p(1)
= o_p(n^{-\frac{1}{2}})
$$

provided $nh_x^{2q} \to 0$. For $II_n^B$:

$$
II_n^B = -\frac{1}{\sqrt{n}} \sum_{j=1}^{n} f_x(x_j)(y_j - m(x_j)) \prod_{l=1}^{d_x} 1\{x_l < x_{l,i} < \varpi_l\} F_{p|x,s=1}(\hat{p} \leq \varrho|x_j, s_j = 1) \Pr(s_j = 1|x_j) \int K(\nu) d\nu + o_p(1)
= -\frac{1}{\sqrt{n}} \sum_{j=1}^{n} f_x(x_j)(y_j - m(x_j)) \prod_{l=1}^{d_x} 1\{x_l < x_{l,j} < \varpi_l\} F_{p|x,s=1}(p(z_j) \leq \varrho|x_j, s_i = 1) \Pr(s_j = 1|x_j) + o_p(1),
$$

where the second equality follows by a mean value expansion of $F_{p|x,s=1}(\hat{p}(z_j) \leq \varrho|x_j, s_j = 1) = F_{p|x,s=1}(p(z_j) \leq \varrho + (p(z_j) - \hat{p}(z_j))|x_j, s_j = 1)$ around $p$ and S.5. It thus follows that:

$$
I_n + II_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_x(x_i)(y_i - m(x_i)) \prod_{j=1}^{d_x} 1\{x_j < x_{j,i} < \varpi_j\} s_i \left\{ f_{x,p}(x_i, p(z_i)) - F_{p|x,s=1}(p(z_i) \leq \varrho|x_i, s_i = 1) \Pr(s_i = 1|x_i) \right\} + o_p(1).
$$

Thus, under Assumptions S.1 to S.5, the process $Z_{1,n}(x, \varpi, \varrho)$ converges weakly in $l^\infty(\mathcal{X} \times \mathcal{P})$, the Banach space of real bounded functions on $\mathcal{X} \times \mathcal{P}$. The covariance kernel of this Gaussian kernel is
given by:

$$E \left[ \left( f_x(x_i)(y_i - m(x_i)) \prod_{j=1}^{d_x} 1\{x_j < x_{j,i} < \overline{x}_j\} s_i\{(1\{p_i \leq \overline{p}\} - 1\{p_i \leq \overline{p}'\}) \right. \right.

\left. \left. - (F_p|x,s=1(p(z_i) \leq \overline{p}|x_i, s_i = 1) - F_p|x,s=1(p(z_i) \leq \overline{p}'|x_i, s_i = 1) \Pr(s_i = 1|x_i) \right) \right] \prod_{j=1}^{d_x} 1\{x_j < x_{j,i} < \overline{x}_j\} s_i\{(1\{p_i \leq \overline{p}\} - 1\{p_i \leq \overline{p}'\}) \right. \right.$$

$$\left. \left. - (F_p|x,s=1(p(z_i) \leq \overline{p}'|x_i, s_i = 1) - F_p|x,s=1(p(z_i) \leq \overline{p}'|x_i, s_i = 1) \Pr(s_i = 1|x_i) \right) \right]\right].$$

Under $H_{0,m}^{(1)}$, the weak convergence of any continuous functional follows as an immediate consequence:

$$\sup_{(\underline{x}, \overline{x}) \in \mathcal{X}, p \in \mathcal{P}} |Z_{1,n}^m (\underline{x}, \overline{x}, \underline{p}, \overline{p})| \Rightarrow Z_{1,n}^m.$$

(ii) Under the alternative:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_x(x_i)(y_i - m(x_i)) \prod_{j=1}^{d_x} 1\{x_j < x_{j,i} < \overline{x}_j\} s_i\{(1\{p_i \leq \overline{p}\} - F_p|x,s=1(p(z_i) \leq \overline{p}|x_i, s_i = 1) \Pr(s_i = 1|x_i) \}
$$

diverges to infinity. □

**Proof of Theorem S1**

(i) Under the assumption that $f_x(x_i) > c > 0$ for all $x_i \in \mathcal{X}$ in S.6, following the arguments of Delgado and Gonzalez-Manteiga (2001), one can establish that:

$$\sup_{(\underline{x}, \overline{x}) \in \mathcal{X}, (\underline{p}, \overline{p}) \in \mathcal{P}} \left| Z_{1,n}^m (\underline{x}, \overline{x}, \underline{p}, \overline{p}) - \tilde{U}_0^* (\underline{x}, \overline{x}, \underline{p}, \overline{p}) \right| = o_p(n^{-\frac{1}{2}})$$

where

$$\tilde{Z}_{1,n}^m (\underline{x}, \overline{x}, \underline{p}, \overline{p}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (y_i - \hat{m}^*(x_i)) s_i \hat{f}_x(x_i) \prod_{j=1}^{d_x} 1\{x_j < x_{j,i} < \overline{x}_j\} s_i\{p \leq p_i \leq \overline{p}\}$$

is the actual bootstrap statistic from before, but with $\hat{p}_i$ replaced by $p_i$. Thus, we need to establish that:

$$\sup_{(\underline{x}, \overline{x}) \in \mathcal{X}, (\underline{p}, \overline{p}) \in \mathcal{P}} \left| Z_{1,n}^m (\underline{x}, \overline{x}, \underline{p}, \overline{p}) - \tilde{Z}_{1,n}^m (\underline{x}, \overline{x}, \underline{p}, \overline{p}) \right| = o_p(n^{-\frac{1}{2}}).$$

But this follows by repeating similar to the ones above noting that $u_i$ are i.i.d. with mean zero and
The first term can be addressed as in the proof of Theorem S1 to show that it is $o(\varepsilon)$ where we used the fact that $v_i$ is independent of the sample $S_n$. More specifically:

$$Z_{1,n}^{m}(x, \tau, \overline{p}) - \tilde{Z}_{1,n}^{m}(x, \tau, \overline{p})$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (y_i - \hat{m}(x_i)) p_i \sum_{j=1}^{d_x} 1 \{x_{j,i} \leq \tau_j\} \{1 \{\hat{p}_i \leq p\} - 1 \{p_i \leq p\}\}$$

$$= E_{S_n}\left[\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (y_i - \hat{m}(x_i)) p_i \sum_{j=1}^{d_x} 1 \{x_{j,i} \leq \tau_j\} \{1 \{\hat{p}_i \leq p\} - 1 \{p_i \leq p\}\}\right]$$

$$- E_{S_n}\left[\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (y_i - \hat{m}(x_i)) p_i \sum_{j=1}^{d_x} 1 \{x_{j,i} \leq \tau_j\} \{1 \{\hat{p}_i \leq p\} - 1 \{p_i \leq p\}\}\right].$$

The second term can be shown to be $o_p(1)$ using again Theorem 2.11.9 of [Van der Vaart and Wellner (1996)] and Lemma A.3 of [Escanciano et al. (2014)]. By contrast, using iterated expectations and noting that $v_i$ is independent of the sample, we have for the first term:

$$E_{S_n}\left[\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (y_i - \hat{m}(x_i)) p_i \sum_{j=1}^{d_x} 1 \{x_{j,i} \leq \tau_j\} \{1 \{\hat{p}_i \leq p\} - 1 \{p_i \leq p\}\}\right]$$

$$= E_{S_n}\left[\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (y_i - \hat{m}(x_i)) p_i \sum_{j=1}^{d_x} 1 \{x_{j,i} \leq \tau_j\} \left(F_{p|x,s=1}(\hat{p}_i \leq p|x_i, s_i = 1) - F_{p|x,s=1}(p_i \leq p|x_i, s_i = 1)\right) Pr(s_i = 1|x_i)\right]$$

$$= \frac{1}{\sqrt{n h^d_x}} \sum_{j=1}^{n} E_{S_n}\left[\left(\hat{m}(x_i) - \hat{m}(x_j)\right) K_h(x_i - x_j) \sum_{j=1}^{d_x} 1 \{x_{j,i} \leq \tau_j\} \left(F_{p|x,s=1}(\hat{p}_i \leq p|x_i, s_i = 1) - F_{p|x,s=1}(p_i \leq p|x_i, s_i = 1)\right) Pr(s_i = 1|x_i)\right]$$

$$+ \frac{1}{\sqrt{n h^d_x}} \sum_{j=1}^{n} E_{S_n}\left[\varepsilon_i K_h(x_i - x_j) \sum_{j=1}^{d_x} 1 \{x_{j,i} \leq \tau_j\} \left(F_{p|x,s=1}(\hat{p}_i \leq p|x_i, s_i = 1) - F_{p|x,s=1}(p_i \leq p|x_i, s_i = 1)\right) Pr(s_i = 1|x_i)\right].$$

where we used the fact that $v_i$ is independent of the sample and $E_{S_n}[v_i] = 0$ for the term involving $\varepsilon_i$. Starting with the first term on the right hand side of the last equality, we can re-write this as:

$$\frac{1}{\sqrt{n h^d_x}} \sum_{j=1}^{n} \int (m(x_j) - m(x_i)) K_h(x_i - x_j) \sum_{j=1}^{d_x} 1 \{x_{j,i} \leq \tau_j\} \nabla_p F_{p|x,s=1}(\overline{p}|x_i, s_i = 1)$$

$$\times \sum_{j=1}^{d_x} 1 \{x_{j,i} \leq \tau_j\} \nabla_p F_{p|x,s=1}(\overline{p}|x_i, s_i = 1)\right) Pr(s_i = 1|x_i) f_x(x_i)dx_i.$$
\(X\) and \(P\). For the second term, given Assumption S.5, note that we can bound this expression by:

\[
\sqrt{n} \left( \sup_{x \in X} |\hat{m}(x) - m(x)| \right) \left( \sup_{z \in Z} |\hat{p}(z) - p(z)| \right) \\
\times \left( \frac{C}{nh^2} \sum_{j=1}^{n} \left| \mathbf{K}_h(x_j - x_j) \prod_{l=1}^{d_x} 1 \{ x_j < x_{l,i} < x_l \} \nabla_P F_{p|x,s=1}(\hat{p}|x_i, s_i = 1) \right| f_x(x_i) dx_i \right) + o_P(1)
\]

\[
= O(\sqrt{n}) o_P \left( \frac{1}{\sqrt{n}} \right) O_P(1) = o_P(1)
\]

given Assumptions S.3, S.4, and S.6, where \(o_P(1)\) holds uniformly over \(X\) and \(P\). Turning to the second term of (2), after change of variables and we obtain:

\[
\frac{1}{\sqrt{nh^2}} \sum_{j=1}^{n} \int v_j(y_j - m(x_j)) \mathbf{K}_h(x_j - x,j) \prod_{l=1}^{d_x} 1 \{ x_j < x_{l,i} < \bar{x}_l \} \left( F_{p|x,s=1}(\hat{p}|x_i, s_i = 1) - F_{p|x,s=1}(p \leq \bar{p}|x_i, s_i = 1) \right) \Pr(s_i = 1|x_i)f_x(x_i) dx_i \\
= \frac{1}{\sqrt{nh^2}} \sum_{j=1}^{n} \int v_j(y_j - m(x_j)) \prod_{l=1}^{d_x} 1 \{ x_j < x_{l,i} < \bar{x}_l \} \left( F_{p|x,s=1}(\hat{p}|x_j, s_j = 1) - F_{p|x,s=1}(p \leq \bar{p}|x_j, s_j = 1) \right) \Pr(s_j = 1|x_j)f_x(x_j) + o_P(1)
\]

\[
\left( \hat{p}_j - p_j \right) \Pr(s_j = 1|x_j)f(x_j) + o_P(1)
\]

Noting that:

\[
E_{S_n} \left[ v_j \epsilon_j \prod_{l=1}^{d_x} 1 \{ x_j < x_{l,i} < \bar{x}_l \} \nabla_P F_{p|x,s=1}(\bar{p}|x_j, s_j = 1) \left( \hat{p}(z_j) - p(z_j) \right) \Pr(s_j = 1|x_j)f(x_j) \right] = 0,
\]

one can again use arguments from before evoking Theorem 2.11.9 of [Van der Vaart and Wellner (1996)] and Lemma A.3 of [Escanciano et al. (2014)] to show that this term is of order \(o_P(1)\) uniformly over \(X\) and \(P\).

(ii) Under \(H_{A,q}^{(1)}\), the statistic \(Z_{1,n}^{m}(x, \bar{x}, \bar{p}, \bar{p})\) diverges, while the bootstrap statistic still converges in distribution. \(\square\)

**Proof of Theorem S2**

(i) Given Assumption S.7,

\[
Z_{2,n}^{m} (x, \bar{x}, 1) = \frac{Z \mathcal{N}_{2,n}^{m}(x, \bar{x}, 1)}{\left( \int K^2(v)dv \right)^{1/2} \left( \frac{1}{nh^2} \sum_{j=1}^{n} s_i(y_i - \hat{m}(x_i))^2 \frac{h}{nh^2} \right)^{1/2} \left( 1 + o_P(1) \right)}
\]

\[
\left( \int K^2(v)dv \right)^{1/2} \left( \frac{1}{nh^2} \sum_{j=1}^{n} s_i(y_i - \hat{m}(x_i))^2 \frac{h}{nh^2} \right)^{1/2} \left( 1 + o_P(1) \right)
\]

\[9\]
where $C_{0,n} \preceq \otimes_{j=1}^{d_x} [x_j, \bar{x}_j]$ and

$$ZN_{2,n}^m (\bar{x}, \bar{x}, 1) \equiv \frac{1}{\sqrt{nh_p H_n \prod_{j=1}^{d_x} \xi_{j,n}}} \sum_{i=1}^{n} s_i(y_i - \hat{m}(x_i)) f_x(x_i) 1 \{ x_i \in C_{0,n} \} K \left( \frac{p_i - \delta}{h_p} \right).$$

As in the proof of Theorem 2, for brevity, we ignore the $(1 + o_p(1))$ term in the following. Now,

$$ZN_{2,n}^m (\bar{x}, \bar{x}, 1) = \frac{1}{\sqrt{nh_p H_n \prod_{j=1}^{d_x} \xi_{j,n}}} \sum_{i=1}^{n} s_i(y_i - m(x_i)) f_x(x_i) 1 \{ x_i \in C_{0,n} \} K \left( \frac{p_i - \delta}{h_p} \right)$$

$$+ \frac{n}{\sqrt{nh_p H_n \prod_{j=1}^{d_x} \xi_{j,n}}} E_{S_n} \left[ s_i(y_i - \hat{m}(x_i)) f_x(x_i) 1 \{ x_i \in C_{0,n} \} K \left( \frac{\hat{p}_i - \delta}{h_p} \right) \right]$$

$$- s_i(y_i - m(x_i)) f_x(x_i) 1 \{ x_i \in C_{0,n} \} K \left( \frac{p_i - \delta}{h_p} \right)$$

$$- \frac{1}{\sqrt{nh_p H_n \prod_{j=1}^{d_x} \xi_{j,n}}} \sum_{i=1}^{n} (y_i - m(x_i)) f_x(x_i) 1 \{ x_i \in C_{0,n} \} K \left( \frac{p_i - \delta}{h_p} \right)$$

$$- E_{S_n} \left[ s_i(y_i - \hat{m}(x_i)) f_x(x_i) 1 \{ x_i \in C_{0,n} \} K \left( \frac{\hat{p}_i - \delta}{h_p} \right) \right]$$

$$+ \frac{1}{\sqrt{nh_p H_n \prod_{j=1}^{d_x} \xi_{j,n}}} \sum_{i=1}^{n} s_i(y_i - \hat{m}(x_i)) f_x(x_i) 1 \{ x_i \in C_{0,n} \} K \left( \frac{\hat{p}_i - \delta}{h_p} \right)$$

$$- E_{S_n} \left[ s_i(y_i - \hat{m}(x_i)) f_x(x_i) 1 \{ x_i \in C_{0,n} \} K \left( \frac{\hat{p}_i - \delta}{h_p} \right) \right]$$

$$= I_n + II_n + III_n$$

$III_n$ is $o_p(1)$ by the same arguments as in the proof of Theorem S1 and Assumption S.3. $I_n$ on the other can be treated in a similar manner as the corresponding terms in the proof of Theorem 2. That is, let $\varepsilon_i = y_i - m(x_i)$ and $E[\varepsilon_i | x_i, p_i, s_i = 1]$ denote conditional expectation of $\varepsilon_i$ given $x_i, p_i$, and $s_i = 1$. Then:

$$I_n = \frac{1}{\sqrt{nh_p H_n \prod_{j=1}^{d_x} \xi_{j,n}}} \sum_{i=1}^{n} s_i(\varepsilon_i - E[\varepsilon_i | x_i, p_i, s_i = 1]) f_x(x_i) 1 \{ x_i \in C_{0,n} \} K \left( \frac{p_i - \delta}{h_p} \right)$$

$$+ \frac{1}{\sqrt{nh_p H_n \prod_{j=1}^{d_x} \xi_{j,n}}} \sum_{i=1}^{n} s_i E[\varepsilon_i | x_i, p_i, s_i = 1] f_x(x_i) 1 \{ x_i \in C_{0,n} \} K \left( \frac{p_i - \delta}{h_p} \right)$$

The first term has clearly mean zero under both $H_{0,m}^{(2)}$ and the alternative, while the second term will only be of order $o_p(1)$ under $H_{0,m}^{(2)}$. More specifically, the first term satisfies a CLT for triangular
arrays, where the variance is given by:

\[
\frac{1}{h_p} E \left[ s_i (\varepsilon_i - E[\varepsilon_i|x_i, p_i, s_i = 1]) f_x(x_i) 1 \{x_i \in C_{0,n}\} \right] K \left( \frac{p_i - \delta}{h_p} \right)^2
\]

\[
= \frac{1}{h_p} \int_{-1}^{1} \int_{C_{0,n}} (\varepsilon_i - E[\varepsilon_i|x_i, p_i, s_i = 1])^2 f_x(x_i)^2 K^2 \left( \frac{p_i - \delta}{h_p} \right) f_{\varepsilon|x,p,s=1}(\varepsilon_i|x_i, s_i = 1) \times \text{Pr}(s_i = 1|p_i) f_{x,p}(x_i, p_i) d\varepsilon_i dp_i dx_i
\]

\[
= \int_{-1}^{1} \int_{C_{0,n}} (\varepsilon_i - E[\varepsilon_i|x_i, p_i, s_i = 1])^2 f_x(x_i)^2 f_{\varepsilon|x,p,s=1}(\varepsilon_i|x_i, 1 - H + h_pv, s_i = 1) K^2 v \Pr(s_i = 1|1 - H + h_pv) f_{x,p}(x_i, 1 - H + h_pv) d\varepsilon_i dx_i.
\]

Expanding around \( h_p = 0 \) and given Assumptions S.7 and S.10, the last term on the RHS above reads as

\[
\left( \int_{-1}^{1} K^2 (v) dv \right) \int_{C_{0,n}} (\varepsilon_i - E[\varepsilon_i|x_i, p_i, s_i = 1])^2 f_x(x_i)^2 f_{\varepsilon|x,p,s=1}(\varepsilon_i|x_i, s_i = 1) \text{Pr}(s_i = 1|1 - H) f_{x,p}(x_i, 1 - H) d\varepsilon_i dx_i
\]

\[
+ O \left( H^q h_p \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) \right)
\]

Now, under \( H_{0,m}^{(2)} \), using Assumptions S.1, S.3, and S.10, this expression can be re-written as:

\[
\left( \int_{-1}^{1} K^2 (v) dv \right) \int_{C_{0,n}} (\varepsilon_i - E[\varepsilon_i|x_i, p_i, s_i = 1])^2 f_x(x_i)^2 f_{\varepsilon|x,p,s=1}(\varepsilon_i|x_i, s_i = 1) g_{x,p}(x_i, 1) H^q d\varepsilon_i dx_i
\]

\[
+ O \left( H \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) + O \left( H^q \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) \right) \right)
\]

and so the variance is given by by

\[
\text{var}(I_{1,n}) = \left( \int_{-1}^{1} K^2 (v) dv \right) \int_{C_{0,n}} (\varepsilon_i - E[\varepsilon_i|x_i, p_i, s_i = 1])^2 f_x(x_i)^2 f_{\varepsilon|x,p,s=1}(\varepsilon_i|x_i, s_i = 1) g_{x,p}(x_i, 1) H^q d\varepsilon_i dx_i
\]

\[
+ O \left( H \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) + O \left( H^q \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) \right) \right) + O \left( H^q h_p \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) \right).
\]

Hence,

\[
\frac{1}{\sqrt{nh_p H^q \left( \prod_{j=1}^{d_x} \xi_{j,n} \right)}} \sum_{i=1}^{n} s_i (y_i - E[\varepsilon_i|x_i, p_i, s_i = 1]) f_x(x_i) 1 \{x_i \in C_{0,n}\} K \left( \frac{p_i - \delta}{h_p} \right)
\]

converges in distribution by a CLT for triangular arrays. For the second term of \( I_n \), we can use similar arguments to the proof of Theorem 2. That is, note that under \( H_{0,m}^{(2)} \) it holds that \( m_{x,1}(t) \equiv \)
\[ E[\varepsilon_i | x_i, p_i = 1, s_i = 1] = 0 \] almost surely. Thus:

\[
\frac{1}{h_p} \int_0^1 \int_{c_{0,n}} E[\varepsilon_i | x_i, p_i, s_i = 1] f_x(x_i) K\left( \frac{p_i - \delta}{h_p} \right) \Pr(s_i = 1 | p_i) f_{xp}(x_i, p_i) dp_i dx_i
\]

\[
= \int_{c_{0,n}} m_{x_i}(1 - H) f_x(x_i) K(v) \Pr(s_i = 1 | 1 - H) f_{xp}(x_i, 1 - H) dv dx_i
\]

\[
+ O\left( h_p \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) H^q \right)
\]

\[
= 0 + O\left( H \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) \right) + O\left( h_p \left( \prod_{j=1}^{d_x} \xi_{j,n} \right) H^q \right)
\]

where the last line follows from Assumption S.10. For \( II_n \), we can write:

\[
II_n = \sqrt{\frac{n}{h_p H^q \left( \prod_{j=1}^{d_x} \xi_{j,n} \right)}} E_{S_n} \left[ s_i \hat{\varepsilon}_i f_x(x_i) - \varepsilon_i f_x(x_i) \right] 1 \{ x_i \in c_{0,n} \} K\left( \frac{p_i - \delta}{h_p} \right)
\]

\[
= \sqrt{\frac{n}{h_p H^q \left( \prod_{j=1}^{d_x} \xi_{j,n} \right)}} E_{S_n} \left[ s_i \hat{\varepsilon}_i f_x(x_i) 1 \{ x_i \in c_{0,n} \} \left( K\left( \frac{p_i - \delta}{h_p} \right) - K\left( \frac{p_i - \delta}{h_p} \right) \right) \right]
\]

\[
+ \sqrt{\frac{n}{h_p H^q \left( \prod_{j=1}^{d_x} \xi_{j,n} \right)}} E_{S_n} \left[ s_i \hat{\varepsilon}_i f_x(x_i) 1 \{ x_i \in c_{0,n} \} \left( K\left( \frac{p_i - \delta}{h_p} \right) - K\left( \frac{p_i - \delta}{h_p} \right) \right) \right]
\]

\[
= II_{1,n} + II_{2,n} + II_{3,n}
\]

\( II_{2,n} \) is zero, while for the first term we can decompose as in the proof of Theorem S1:

\[
II_{1,n} = \frac{1}{\sqrt{nh_p H^q \left( \prod_{j=1}^{d_x} \xi_{j,n} \right)}} \sum_{j=1}^{n} \left( m(x_i) - m(x_j) \right) K_h(x_i - x_j) 1 \{ x_i \in c_{0,n} \} K\left( \frac{p_i - \delta}{h_p} \right)
\]

\[
\times f_{p|x,s=1}(p_i | x_i, s_i = 1) \Pr(s_i = 1 | x_i) f_x(x_i) dp_i dx_i
\]

\[
- \frac{1}{\sqrt{nh_p H^q \left( \prod_{j=1}^{d_x} \xi_{j,n} \right)}} \sum_{j=1}^{n} \varepsilon_j K_h(x_i - x_j) 1 \{ x_i \in c_{0,n} \} K\left( \frac{p_i - \delta}{h_p} \right)
\]

\[
\times f_{p|x,s=1}(p_i | x_i, s_i = 1) \Pr(s_i = 1 | x_i) f_x(x_i) dp_i dx_i
\]

Using the same arguments as in the proof of Theorem S1 together with the bandwidth conditions it follows that \( II_{1,n} \) is \( o_p(1) \). Similarly, using S.5, it also follows that the first \( III_n = o_p(1) \). Thus, since the denominator of \( Z_{2,n}^n(z, \pi, 1) \) can be analyzed by similar arguments, part (i) is an immediate consequence of the continuous mapping theorem.

(ii) Under \( H^{(2)}_{A,m} \), if there is an omitted relevant regressor, under S.10 it holds for the second term of \( I_n \) that:

\[
\lim_{n \to \infty} \frac{1}{h_p} E \left[ s_i m_{x_i}(p_i) f_x(x_i) 1 \{ x_i \in c_{0,n} \} K\left( \frac{p_i - \delta}{h_p} \right) \right] \neq 0,
\]
\[
\frac{1}{\sqrt{n h_p H^n (\prod_{j=1}^{d_x} \xi_{j,n})}} \sum_{i=1}^{n} \left( \int_{-1}^{1} K^2(v) \, dv \right) s_i (y_i - \hat{m}(x_i))^2 \hat{f}_x(x_i)^2 1 \{ x_i \in C_{0,n} \} K \left( \frac{\hat{p}_i - \delta}{h_p} \right) \xrightarrow{p} C > 0.
\]

## S4 Monte Carlo Simulation

In this section we examine the finite sample properties of our tests for the conditional quantile \textit{and} the conditional mean case in a small Monte Carlo study. Given the computational burden of the nonparametric quantile set-up, we only consider a limited set of scenarios to demonstrate that, under a rather wide range of bandwidth choices \( h_x \), our tests control size, and that in fact the first test also exhibits power under \( H^{(1)}_{A,q} \) and \( H^{(1)}_{A,m} \), respectively.

More specifically, for the conditional quantile case, we choose an outcome equation:
\[
y_i = \left( x_i - \frac{1}{2} \right)^2 + \frac{1}{2} \left( x_i + \frac{1}{5} \right) \varepsilon_i,
\]
which was also used in Volgushev et al. (2013), while for the conditional mean, we follow Delgado and Gonzalez-Manteiga (2001) and model the outcome as:
\[
y_i = 1 + \sin(10 \cdot x_i) + \varepsilon_i.
\]
In both cases, \( x_i \sim \mathcal{U}(0,1) \), while the corresponding selection equation is given by:
\[
s_i = 1 \{ z_i > v_i \},
\]
with \( z_i \sim \mathcal{N}(0,1) \) and:
\[
\begin{pmatrix} \varepsilon_i \\ v_i \end{pmatrix} \sim \mathcal{N} \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right).
\]
Thus, \( \rho \) controls the degree of selection with \( \rho = 0 \) corresponding to the case of no selection.\textsuperscript{17}

We consider three sample sizes \( n = \{1,000; 2,000; 3,000\} \), which, given a selection probability of approximately .5, implies an effective sample size for the outcome equation of around 500 to 1,500 observations, respectively.

In what follows, we assess the performance of our tests under three different designs, namely \( \rho = 0 \) (no selection), \( \rho = .25 \) (moderate selection), and \( \rho = .5 \) (strong selection). To construct the statistics, we first estimate the propensity score using a standard local constant estimator with second order Epanechnikov kernel choosing the corresponding bandwidth according to a rule of thumb, i.e.
\[
h_z = \hat{\sigma}(z) \cdot 2.34 \cdot n^{-\frac{1}{4}} \] with \( \hat{\sigma}(z) \) denoting the estimated standard deviation of \( z_i \).\textsuperscript{18}

\textsuperscript{17}Note that ‘identification at infinity’ is satisfied in this scenario, although there is no \( z \) such that \( p(z) = 1 \). While we do formally not consider this case (cf. A.6 and S.7), we found the open support \( p(z) \) to be of no relevance in the context of our simulations.

\textsuperscript{18}To construct this estimator as well as the estimators for the conditional mean, we use routines from the \texttt{np package} of Hayfield and Racine (2008). For the local polynomial quantile estimator, we use a routine from the \texttt{quantreg package} (?).
ourselves to a compact subset $\mathcal{X}$, we trim the outer 2.5% observations of the selected sample. In the quantile case, we then estimate $q_\tau(x_i)$ using a third order local polynomial estimator in line with the conditions of Theorem 1 choosing $T = \{.3, .4, .5, .6, .7\}$, and selecting from a range of bandwidths $h_x = \{.05, .1, .2, .3\}$ for the quantile case. For the first test, we use the same set of bandwidths $h_x$ to construct $F_{p|x,u,s=1}(\cdot|\cdot, \cdot)$. Since Volgushev et al. (2013) found the empirical distribution function $\hat{F}_p(\cdot)$ to work better in simulations with moderate sample sizes, we also experiment with this function as a substitute for $\hat{F}_{p|x,u,s=1}(\cdot|\cdot, \cdot, \cdot)$.

To implement the second test, we use a second order Epanechnikov kernel function to localize the statistic, and choose $\delta = \{0.98, 0.99, 1\}$ as well as $h_p = \{.05, .03, .02\}$, where $\delta$ and $h_p$ vary according to the sample size $n = \{1,000; 2,000; 3,000\}$.

We only report this narrow range of tuning parameter combinations to demonstrate that our test can control size in those cases as we indeed found the second test to be somewhat sensitive to the choice of $h_p$ and $\delta$ (with over-sizing when $h_p$ is chosen too large or $\delta$ is chosen too small, respectively). For the conditional mean tests, we estimate $m(x_i)$ and $f_x(x_i)$ using again the aforementioned local constant estimators over the bandwidths $h_x = \{.015, .025, .05, .075\}$. The second test uses the same kernel function and combination of $\delta$ and $h_p$ as the corresponding quantile test. The nominal size level is $\alpha = 10\%$ throughout.

Turning to the results in Table 1 we see that the first quantile test over-sizes somewhat for all bandwidth choices $h_x$ when $n = 1,000$, but rejection rates become close to the nominal level $\alpha = 10\%$ as the sample size increases, and it is only for $h_x = 0.05$ that the actual size is still quite far from the nominal level at $n = 3,000$. By contrast, when $\rho = 0.25$ and $\rho = 0.5$, we can observe that sample selection is picked up quickly and the test exhibits good power. Moving on to the second test in the second half of Table 1 size is, as expected, not only controlled at $\rho = 0$, but also at $\rho = 0.25$. It is only when $\rho = 0.5$ that the second test appears to over-size somewhat even at $n = 3,000$.

A similar behavior can be observed for the conditional mean case (Table 2): the first test controls size well when $h_x = 0.015$ and $h_x = 0.025$, and exhibits good power when $\rho = 0.25$ and $\rho = 0.5$ throughout. For the second test, we also find that the test controls size at the nominal level rather well (but for the case where $\rho = 0.5$, which appears to be notoriously difficult to handle), although in contrast to the quantile case we observe a bit of under-sizing at $\rho = 0$ and less over-sizing when $\rho = 0.5$. 

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|                  | # Obs. | \( h_x = .05 \) | \( h_x = .1 \) | \( h_x = .2 \) | \( h_x = .3 \) |
|------------------|--------|----------------|----------------|----------------|----------------|
| \( \rho = 0 \)   | \( n = 1,000 \) | 0.393           | 0.206          | 0.155          | 0.168          |
|                  | \( n = 2,000 \) | 0.226           | 0.166          | 0.153          | 0.161          |
|                  | \( n = 3,000 \) | 0.192           | 0.148          | 0.139          | 0.155          |
| \( \rho = .25 \) | \( n = 1,000 \) | 0.815           | 0.662          | 0.604          | 0.574          |
|                  | \( n = 2,000 \) | 0.893           | 0.823          | 0.795          | 0.776          |
|                  | \( n = 3,000 \) | 0.947           | 0.919          | 0.893          | 0.887          |
| \( \rho = .5 \)  | \( n = 1,000 \) | 0.993           | 0.983          | 0.976          | 0.973          |
|                  | \( n = 2,000 \) | 1.000           | 1.000          | 1.000          | 1.000          |
|                  | \( n = 3,000 \) | 1.000           | 1.000          | 1.000          | 1.000          |

|                  | # Obs. | \( h_x = .05 \) | \( h_x = .1 \) | \( h_x = .2 \) | \( h_x = .3 \) |
|------------------|--------|----------------|----------------|----------------|----------------|
| \( \rho = 0 \)   | \( n = 1,000 \) | 0.103           | 0.105          | 0.097          | 0.098          |
|                  | \( n = 2,000 \) | 0.103           | 0.104          | 0.099          | 0.097          |
|                  | \( n = 3,000 \) | 0.095           | 0.087          | 0.091          | 0.089          |
| \( \rho = .25 \) | \( n = 1,000 \) | 0.106           | 0.115          | 0.117          | 0.121          |
|                  | \( n = 2,000 \) | 0.101           | 0.114          | 0.110          | 0.108          |
|                  | \( n = 3,000 \) | 0.103           | 0.105          | 0.113          | 0.120          |
| \( \rho = .5 \)  | \( n = 1,000 \) | 0.159           | 0.175          | 0.191          | 0.196          |
|                  | \( n = 2,000 \) | 0.184           | 0.191          | 0.205          | 0.199          |
|                  | \( n = 3,000 \) | 0.165           | 0.175          | 0.177          | 0.176          |

Table 1: Conditional Quantile Tests (Note: No. of Monte Carlo Replications is 1,500; No. of Bootstrap Replications is 100; \( \delta \) and \( h_p \) are chosen as follows: \( \{n = 1,000 : \delta = .98, h_p = .05\} \), \( \{n = 2,000 : \delta = .99, h_p = .03\} \), and \( \{n = 3,000 : \delta = 1, h_p = .02\} \))
|                  | # Obs. | $h_x = .015$ | $h_x = .025$ | $h_x = .05$ | $h_x = .075$ |
|------------------|--------|--------------|--------------|-------------|-------------|
| **First Test, $\alpha = 10\%$** |        |              |              |             |             |
| $\rho = 0$       | $n = 1,000$ | 0.119        | 0.110        | 0.119       | 0.188       |
|                  | $n = 2,000$ | 0.113        | 0.118        | 0.135       | 0.270       |
|                  | $n = 3,000$ | 0.116        | 0.123        | 0.140       | 0.395       |
| $\rho = .25$    | $n = 1,000$ | 0.583        | 0.553        | 0.511       | 0.563       |
|                  | $n = 2,000$ | 0.820        | 0.816        | 0.786       | 0.833       |
|                  | $n = 3,000$ | 0.947        | 0.921        | 0.899       | 0.933       |
| $\rho = .5$     | $n = 1,000$ | 0.983        | 0.984        | 0.984       | 0.979       |
|                  | $n = 2,000$ | 1.000        | 1.000        | 1.000       | 1.000       |
|                  | $n = 3,000$ | 1.000        | 1.000        | 1.000       | 1.000       |

|                  | # Obs. | $h_x = .015$ | $h_x = .025$ | $h_x = .05$ | $h_x = .075$ |
|------------------|--------|--------------|--------------|-------------|-------------|
| **Second Test, $\alpha = 10\%$** |        |              |              |             |             |
| $\rho = 0$       | $n = 1,000$ | 0.084        | 0.091        | 0.095       | 0.095       |
|                  | $n = 2,000$ | 0.088        | 0.089        | 0.088       | 0.087       |
|                  | $n = 3,000$ | 0.081        | 0.086        | 0.088       | 0.087       |
| $\rho = .25$    | $n = 1,000$ | 0.129        | 0.113        | 0.121       | 0.119       |
|                  | $n = 2,000$ | 0.127        | 0.144        | 0.150       | 0.148       |
|                  | $n = 3,000$ | 0.105        | 0.125        | 0.122       | 0.119       |
| $\rho = .5$     | $n = 1,000$ | 0.232        | 0.233        | 0.225       | 0.223       |
|                  | $n = 2,000$ | 0.270        | 0.280        | 0.280       | 0.275       |
|                  | $n = 3,000$ | 0.192        | 0.221        | 0.211       | 0.201       |

Table 2: Conditional Mean Tests (Note: No. of Monte Carlo Replications is 1,500; No. of Bootstrap Replications is 100; $\delta$ and $h_p$ are chosen as follows: $\{n = 1,000 : \delta = .98, h_p = .05\}$, $\{n = 2,000 : \delta = .99, h_p = .03\}$, and $\{n = 3,000 : \delta = 1, h_p = .02\}$)

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