Functional Calculus for Semi-Bounded Operators

Narinder S Claire

Abstract

We build on the work by Davies, extending the Helffer-Sjöstrand Functional Calculus domain for semi-bounded operators on Banach spaces given a priori controlled growth of the resolvents. We employ Seeley’s Extension Theorem to extend smooth functions on the half line to the whole line and thus indirectly define functions of these operators.

Introduction

Helffer and Sjöstrand \cite{HelfferSjostrand} introduced a new formula into the field of spectral theory

\[ f(H) := -\frac{1}{\pi} \oint\oint_{\gamma} \frac{\partial f}{\partial z} (z - H)^{-1} \, dx \, dy \]  

(0.1)

for self-adjoint operators. It was later shown by Davies \cite{Davies} that the Helffer-Sjöstrand formula had greater implications in spectral theory than it was originally intended for. He constructed a functional calculus for a much wider class of operators on spaces other than Hilbert under certain assumptions on the norms of the resolvents. He in addition showed that in the special case of self-adjoint operators this coincided with the classical functional calculus.

We show that we can widen the class of functions when we consider operators with spectrums which are bounded below, since positive operators take special treatment in the analysis of partial differential equations. The results enable explicit tractable definitions of certain functions of partial differential operators in particular the Heat Semigroup.

Our hypothesis in the pending analysis is

\( H \) is a closed densely defined operator on a Banach space \( \mathcal{B} \), with spectrum \( \sigma(H) \subset \mathbb{R} \). It has resolvent operators \( (z - H)^{-1} \) defined and bounded for all
$z \in \mathbb{C}$ satisfying:

$$\| (z-H)^{-1} \| \leq c|\text{Im}z|^{-1} \left( \frac{\langle z \rangle}{|\text{Im}z|} \right)^\alpha$$  \hspace{1cm} (0.2)

for some $\alpha \geq 0$ and all $z \notin \mathbb{R}$, where $\langle z \rangle := (1 + |z|^2)^{\frac{1}{2}}$

1 Preliminaries

We introduce our main concepts.

1.1 Algebra of Slow Decreasing functions

Definition 1.1 $\langle z \rangle := (1 + |z|^2)^{\frac{1}{2}}$ for all complex $z$

Definition 1.2 For $\beta \in \mathbb{R}$ let $S^\beta$ to be the set of all complex-valued smooth functions defined on $\mathbb{R}$ such that for every $n$ there is a positive constant $c_n$ where

$$\left| \frac{d^n f}{dx^n} \right| \leq c_n \langle x \rangle^{\beta-n}$$

Definition 1.3 We define the Algebra $\mathcal{A}$ as,

$$\mathcal{A} := \bigcup_{\beta < 0} S^\beta$$ \hspace{1cm} (1.3)

Remark 1.4 The strict inequality $\beta < 0$ is of importance and we conjecture that it cannot be extended to $\beta \leq 0$.

Lemma 1.5 (Davies \cite{2}) $\mathcal{A}$ is an algebra under pointwise multiplication. If $f \in \mathcal{A}$ then the expression

$$\|f\|_n := \sum_{r=0}^{n} \int_{-\infty}^{\infty} \left| \frac{d^r f}{dx^r} \right| \langle x \rangle^{r-1} dx$$ \hspace{1cm} (1.4)

defines a norm on $\mathcal{A}$ for each $n$ moreover $C^\infty_c (\mathbb{R})$ is dense in $\mathcal{A}$ with this norm. The completion $\mathcal{A}_n$ is a Banach space.

Lemma 1.6 The function $\langle x \rangle^{\beta}$ is in $\mathcal{A}$ for each $\beta < 0$
Proof

The statement follows from the observation that if $\beta < 0$ and $m \geq n$ then

$$x^n \langle x \rangle^{\beta - m} \leq \langle x \rangle^\beta$$

and

$$\frac{d (x^n \langle x \rangle^{\beta - m})}{dx} = nx^{n-1} \langle x \rangle^{\beta - m} + 2 (\beta - m) x^{n+1} \langle x \rangle^{\beta - m - 1}$$

Lemma 1.7 If $f \in \mathcal{A}$ and $\phi \in S^0$ then $\phi f \in \mathcal{A}$

Proof

The statement follows from the inequality

$$| \frac{d^r (\phi (x) f (x))}{dx^r} | = \left| \sum_{m=0}^r c_m \frac{d^{r-m} (\phi (x))}{dx^{r-m}} \frac{d^m (f (x))}{dx^m} \right|$$

$$\leq c_r \sum_{m=0}^r \frac{d^{r-m} (\phi)}{dx^{r-m}} \left| \frac{d (f (x))}{dx^m} \right|$$

$$\leq c_{r,\phi} \sum_{m=0}^r \langle x \rangle^{\beta - r}$$

$$\leq c_{r,\phi} \langle x \rangle^{\beta - r}$$

1.2 The Helffer Sjöstrand formula

We introduce the concept of almost analytic extensions due to Hörmander [4].

Definition 1.8 Let $\psi (s)$ be a smooth function of compact support on $\mathbb{R}$ such that

$$\psi (s) := \begin{cases} 1 & \text{if } |s| \leq 1 \\ 0 & \text{if } |s| \geq 2 \end{cases}$$

then we define

$$\sigma (x, y) := \psi \left( \frac{y}{\langle x \rangle} \right)$$

(1.5)

Definition 1.9 Given $f \in \mathcal{A}$ we define an almost analytic extension $\tilde{f}$ to the complex plane

$$\tilde{f} (x, y) := \left( \sum_{r=0}^n \frac{d^r f (x) (iy)^r}{dx^r r!} \right) \sigma (x, y)$$

(1.6)
and define
\[
\frac{\partial \tilde{f}}{\partial \tilde{z}} := \frac{1}{2} \left( \frac{\partial \tilde{f}}{\partial x} + i \frac{\partial \tilde{f}}{\partial y} \right)
\]  
(1.7)

**Definition 1.10** Given \( f \in \mathcal{A} \) and \( H \) satisfying our initial hypothesis we define
\[
f(\hat{H}) := -\frac{1}{\pi} \iint_{C} \frac{\partial \tilde{f}}{\partial \tilde{z}} (z - H)^{-1} \, dx \, dy
\]  
(1.8)

We recall some important results from [1] showing that we do indeed have a functional calculus.

**Lemma 1.11** (Davies [1])

i. If \( n > \alpha \) then subject to (0.2) the integral (1.8) is norm convergent for all \( f \) in \( \mathcal{A} \) and
\[
\|f(\hat{H})\| \leq c\|f\|_{n+1}
\]

ii. The operator \( f(\hat{H}) \) is independent of \( n \) and the cut-off function \( \sigma \), subject to \( n > \alpha \)

iii. If \( f \) is a smooth function of compact support disjoint from the spectrum of \( H \) then \( f(\hat{H}) = 0 \)

iv. If \( f \) and \( g \) are in \( \mathcal{A} \) then
\[
(fg)(\hat{H}) = f(\hat{H})g(\hat{H})
\]

v. If \( z \not\in \mathbb{R} \) and \( g_z(x) := (z - x)^{-1} \) for real \( x \) then \( g_z \in \mathcal{A} \) and
\[
g_z(\hat{H}) = (z - H)^{-1}
\]

**2 Semi-bounded Operators**

We modify our main hypothesis by assuming the spectrum of \( H \) is bounded below and without loss of generality \( \sigma(\hat{H}) \subseteq [0, \infty) \).

We introduce a new ring of functions \( \mathcal{A}^+ \).
Definition 2.1 $S_+^\beta$ is the set of smooth functions on $\mathbb{R}^+ \cup \{0\}$ with the same decaying property as $S^\beta$ that is for every $n$ there is positive constant $c_n$ such that
\[
|\frac{d^n f}{dx^n}| \leq c_n \langle x \rangle^{\beta-n}
\]
Then $\mathcal{A}^+$ is defined appropriately and similarly we define the Banach space $\mathcal{A}_n^+$ with norm
\[
\|f\|_{\mathcal{A}_n^+} := \sum_{r=0}^{n} \int_0^\infty \left| \frac{d^r f}{dx^r} \right| \langle x \rangle^{r-1} dx
\] (2.9)

2.1 Seeley’s Extension Theorem

We present a theorem due to Seeley [5] which gives a linear extension operator for smooth functions from the half space to the whole space. This extension operator is continuous for many topologies including uniform convergence of each derivative. We demonstrate a brief proof as it contributes to the proof of continuity for our topology $\|\|_{\mathcal{A}_n^+}$ for each $n$.

Definition 2.2 Given $f \in \mathcal{A}^+$, $\phi \in \mathcal{A}$ and real $a$ we define two operators on $\mathcal{A}^+$,
\[
(T_a f)(x) = f(ax)
\]
\[
(S_\phi f)(x) = \phi(x) f(x)
\]

Theorem 2.3 Seeley’s Extension Theorem.
There is a linear extension operator
\[
\mathcal{E} : C^\infty[0, \infty) \longrightarrow C^\infty(\mathbb{R})
\]
such that for all $x > 0$
\[
(\mathcal{E} f)(x) = f(x)
\]
The proof of the theorem centres on the following lemma

Lemma 2.4 There are sequences $\{a_k\}$, $\{b_k\}$ such that

i. $b_k < 0$

ii. $\sum_{k=0}^{\infty} |a_k| |b_k|^n < \infty$ for all non-negative integers $n$

iii. $\sum_{k=0}^{\infty} a_k (b_k)^n = 1$ for all non-negative integers $n$
iv. $b_k \to -\infty$

**Proof**

See [5]. We recall from the proof $b_k = -(2^k)$ and $|a_k| < e^4 2^{-k^2/2}$

**Proof of theorem**

Let $\phi \in C^{(\infty)}_c(\mathbb{R})$ such that

$$
\phi(x) = \begin{cases} 
1 & x \in [0, 1] \\
0 & x \geq 2 \\
0 & x \leq -1
\end{cases}
$$

we construct $\mathcal{E}$

$$(\mathcal{E} f)(x) := \begin{cases} 
\sum_{k=0}^{\infty} a_k (T_{b_k} S_\phi f)(x) & x < 0 \\
f(x) & x \geq 0
\end{cases}$$

The series is convergent since for all negative $x$ the sum has only finite non-zero terms. It is evident that $(\mathcal{E} f)(0) = f(0)$ and when $N \in \mathbb{N}$ then for all $x > -\frac{1}{2N}$

$$
\phi^{(n)}(x) = 0
$$

for all positive $n$ hence

$$(\mathcal{E} f)^{(n)}(x) = \sum_{k=0}^{\infty} a_k b_k^n \phi(b_k x) f^{(n)}(b_k x)$$

and we deduce that for all $n$ $\lim_{x \to 0^-} \frac{d^n (\mathcal{E} f)}{dx^n} (x) = \lim_{x \to 0^+} \frac{d^n (\mathcal{E} f)}{dx^n} (x)$ to complete the proof.

**Lemma 2.5** If $a > 1$ then $\|T_a\|_{A^n_+^{\infty}} \leq a^n$

**Proof**

$$
\|T_a f\|_{A^n_+^{\infty}} = \sum_{r=1}^{n} \int_0^{\infty} \frac{d^r f(ax)}{dx^r} |(ax)^{r-1} dx
$$

$$
\leq \sum_{r=1}^{n} a^r \int_0^{\infty} \frac{d^r f(ax)}{d(ax)^r} |(ax)^{r-1} d(ax)
$$

$$
= \sum_{r=1}^{n} a^r \int_0^{\infty} \frac{d^r f(x)}{dx^r} |(x)^{r-1} dx
$$
and the inequality follows.

**Lemma 2.6**  If $\phi \in A$ then $S_\phi$ is a bounded operator with respect to each norm $\| \cdot \|_{A^+_n}$

**Proof**

A simple application of Leibnitz gives

$$\frac{d^r (\phi(x)f(x))}{dx^r} = \sum_{m=0}^{r} c_r \frac{d^{r-m} (\phi(x))}{dx^{r-m}} \frac{d^m (f(x))}{dx^m}$$

then

$$\left| \frac{d^r (\phi(x)f(x))}{dx^r} \right| \leq c_r \sum_{m=0}^{r} d_{r-m,\phi} \langle x \rangle^{\beta -(r-m)} \frac{d^m (f(x))}{dx^m}$$

$$\leq c_{r,\phi} \sum_{m=0}^{r} \langle x \rangle^{m-r} \frac{d^m (f(x))}{dx^m}$$

and so we integrate to give

$$\int_0^{\infty} \left| \frac{d^r (\phi(x)f(x))}{dx^r} \right| \langle x \rangle^{r-1} dx \leq c_{r,\phi} \sum_{m=0}^{r} \int_0^{\infty} \frac{d^m (f(x))}{dx^m} \langle x \rangle^{m-1} dx$$

$$= c_{r,\phi} \|f\|_{A^+_n}$$

and hence we have our estimate

$$\|S_\phi f\|_n = \sum_{r=0}^{n} \int_0^{\infty} \left| \frac{d^r (\phi(x)f(x))}{dx^r} \right| \langle x \rangle^{r-1} dx$$

$$\leq c_{n,\phi} \sum_{r=0}^{n} \|f\|_{A^+_n}$$

$$\leq c_{n,\phi} \|f\|_{A^+_n}$$

**Theorem 2.7**  Seeley’s Extension Operator is a bounded operator on each of the normed vector spaces $A^+_n$

**Proof**
\[ \| \mathcal{E} f \|_{A_n} = \sum_{r=0}^{n} \int_{-\infty}^{\infty} \left| \frac{d^r}{dx^r} (\mathcal{E} f) \right| \langle x \rangle^{r-1} dx \]
\[ = \sum_{r=0}^{n} \int_{0}^{\infty} \left| \frac{d^r}{dx^r} (f(x)) \right| \langle x \rangle^{r-1} dx + \sum_{r=0}^{n} \int_{-\infty}^{0} \left| \sum_{k=0}^{\infty} a_k \frac{d^r}{dx^r} (\phi(b_k x) f(b_k x)) \right| \langle x \rangle^{r-1} dx \]
\[ = \| f \|_{A_n^+} + \| \sum_{k=0}^{\infty} a_k T_{-b_k} \phi f \|_{A_n^+} \]
\[ \leq \| f \|_{A_n^+} + \sum_{k=0}^{\infty} |a_k| \| \phi \| \| T_{-b_k} \| \| f \|_{A_n^+} \]
\[ \leq \| f \|_{A_n^+} + \left( \sum_{k=0}^{\infty} |a_k| |b_k| n \right) c_{n,\phi} \| f \|_{A_n^+} \]

and hence the extension operator is continuous.

### 2.2 The Functional Calculus

**Remark 2.8** If \( f \) and \( g \) are elements of \( A \) such that \( f|_{[0,\infty]} = g|_{[0,\infty]} \) and the spectrum of \( H \) is \([0,\infty)\) then it is not necessary that \( \text{supp}(f-g) \cap \sigma(H) \) is empty since \( \text{supp}(f-g) \cap \sigma(H) = \{0\} \) is possible and lemma 1.11 cannot be applied. This renders our problem non-trivial and justifies the technical detour.

**Theorem 2.9** If \( f \) is a smooth function on \( \mathbb{R} \) of compact support such that
\[ \text{supp}(f) = [-a,0] \]

and \( H \) is an operator satisfying our modified hypothesis with \( \sigma(H) \subseteq [0,\infty] \) then
\[ f(H) = 0 \]

**Proof**

Let \( \epsilon \in (0,1) \) and define
\[ f_\epsilon(x) := f(x + \epsilon) \]
so that \( \text{supp}(f_\epsilon) = [-a+\epsilon,-\epsilon] \).

By Lemma 1.11 \( (f_\epsilon(H)) = 0 \). For all \( n \) there are constants \( c_n \geq 0 \) such that
\[ \| \frac{d^n f}{dx^n} - \frac{d^n f_\epsilon}{dx^n} \|_{\infty} \leq c_n \epsilon \]
then

\[ \| f (H) \| = \| f (H) - f_\epsilon (H) \| \]

\[ \leq \sum_{r=0}^{n} \int_{-a+1}^{0} \left( \frac{d^r f (x)}{dx^r} - \frac{d^r f_\epsilon (x)}{dx^r} \right) \langle x \rangle^{r-1} dx \]

\[ \leq \sum_{r=0}^{n} \epsilon c_r \int_{-a+1}^{0} \langle x \rangle^{r-1} dx \]

\[ = \epsilon k_{n,f} \]

hence our result.

**Corollary 2.10** If \( f \) and \( g \) are in \( A \) such that \( f|_{[0, \infty]} = g|_{[0, \infty]} \) and \( \sigma (H) \subseteq [0, \infty] \) then \( f (H) - g (H) = 0 \)

**Theorem 2.11** If \( H \) is a closed densely defined operator on a Banach space \( B \), with spectrum \( \sigma (H) \subset [0, \infty) \), with resolvent operators \( (z - H)^{-1} \) defined and bounded for all \( z \in C \) satisfying:

\[ \| (z - H)^{-1} \| \leq c |Imz|^{-1} \left( \frac{\langle z \rangle}{|Imz|} \right)^{\alpha} \]

(2.10)

for some \( \alpha \geq 0 \) and all \( z \notin R \)
then there is a functional calculus \( \gamma_H : A^+ \rightarrow B (B) \) such that for all \( f \in A^+ \cap A \)

\[ \gamma_H (f) = -\frac{1}{\pi} \iint_{C} \frac{\partial \tilde{f}}{\partial z} (z - H)^{-1} dxdy \]

**Proof**
Let \( f^+ \in A^+ \), then by Seeley’s Extension Theorem there exists an extension \( f \in A \). We define \( \gamma_H (f^+) := f (H) \). This definition is independent of the particular extension by corollary 2.10. The functional analytic properties are inherited from the extension.

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Department of Mathematics
Strand
London WC2R 2LS
King’s College
England

e-mail: nclaire@mth.kcl.ac.uk