REDUCTION THEOREMS FOR NOETHER’S PROBLEM

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ABSTRACT. Let $K$ be any field, and $G$ be a finite group. Let $G$ act on the rational function field $K(x(g) : g \in G)$ by $K$-automorphisms and $h \cdot x(g) = x(hg)$. Denote by $K(G) = K(x(g) : g \in G)^G$ the fixed field. Noether’s problem asks whether $K(G)$ is rational (= purely transcendental) over $K$. We will give several reduction theorems for solving Noether’s problem. For example, let $\tilde{G} = G \times H$ be a direct product of finite groups. Then $K(\tilde{G})$ is rational over $K(G)$. In particular, if $K(G)$ is rational (resp. retract rational) over $K$, so is $K(\tilde{G})$ over $K$.

1. Introduction and statements

Let $K$ be any field, and $G$ be a finite group. Let $G$ act on the rational function field $K(x(g) : g \in G)$ by $K$-automorphisms and $h \cdot x(g) = x(hg)$. Denote by $K(G) = K(x(g) : g \in G)^G$ the fixed field. Noether’s problem asks whether $K(G)$ is rational (= purely transcendental) over $K$. For a survey of Noether’s problem, see Swan’s paper [10].

The purpose of this article is to prove several reduction theorems when we try to solve Noether’s problem for some group.

Our first result generalizes [6, Proposition 7] (the case $p = 2$).

Theorem 1.1. Let $K$ be a field with char $K = p > 0$ and $\tilde{G}$ be a group extension defined by $1 \to \mathbb{Z}/p\mathbb{Z} \to \tilde{G} \to G \to 1$, where $G$ is a finite group. Then $K(\tilde{G})$ is rational over $K(G)$.

The meaning of the above conclusion is that there is a $K$-embedding of $K(G)$ into $K(\tilde{G})$, i.e. an injective $K$-linear homomorphism of fields from $K(G)$ into $K(\tilde{G})$, so that $K(\tilde{G})$ is rational over $K(G)$.

Note that, for any field $K$, if $G$ and $\tilde{G}$ are finite groups so that $K(\tilde{G})$ is rational over $K(G)$, then $K(\tilde{G})$ is rational (resp. stably rational, retract rational) over $K$ provided that so is $K(G)$. (Recall that “rational” $\Rightarrow$ “stably rational” $\Rightarrow$ “retract rational”. Here, a field $L$ is stably rational over $K$ if there exists a field which is rational over both $L$ and $K$. For the definition of retract rationality, see [9, Definition 3.2].)

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Corollary 1.2. Let $K$ be a field with $\text{char } K = p > 0$ and $\tilde{G}$ be a group extension defined by $1 \to H \to \tilde{G} \to G \to 1$, where $H$ and $G$ are finite groups. If either

(i) $H$ is a cyclic $p$-group or
(ii) $H$ is an abelian $p$-group lying in the center of $\tilde{G}$ or
(iii) $H$ is a $p$-group and $\tilde{G} \cong H \times G$, then $K(\tilde{G})$ is rational over $K(G)$.

Proof. In each of these cases, there is a subgroup $C \cong \mathbb{Z}/p\mathbb{Z}$ of $H$ such that $C$ is normal in $\tilde{G}$ and the extension $1 \to H/C \to \tilde{G}/C \to G \to 1$ belongs again to the same type (i), (ii) or (iii). □

Case (iii) of this result may be thought of as a generalization of Kuniyoshi’s Theorem: If $K$ is a field with $\text{char } K = p > 0$ and $G$ is a finite $p$-group, then $K(G)$ is rational over $K$. Also, by Kuniyoshi’s Theorem, one can view case (iii) of Corollary 1.2 as a consequence of our next result.

Theorem 1.3. Let $K$ be any field, and let $H$ and $G$ be finite groups. If $K(H)$ is rational (resp. stably rational, retract rational) over $K$, so is $K(H \times G)$ over $K(G)$.

In particular, if both $K(H)$ and $K(G)$ are rational (resp. stably rational, retract rational) over $K$, so is $K(H \times G)$ over $K$.\[
\]

Remark 1.4. Saltman already proved in [8, Theorem 1.5] that if $K(H)$ and $K(G)$ are retract rational over $K$, then so is $K(H \times G)$ over $K$.

Corollary 1.5. Let $\tilde{G} = H \times G$ be a direct product of finite groups, and let $K$ be a field. Assume that: (i) $H$ is an abelian group with exponent $e$, i.e. $e = \max\{\text{ord}(h) : h \in H\}$; and (ii) the field $K$ contains the $e$-th roots of unity. Then $K(\tilde{G})$ is rational over $K(G)$.

In particular, $K((\mathbb{Z}/2\mathbb{Z}) \times G)$ is rational over $K(G)$ for every field $K$ and every finite group $G$.

Proof. First, by Corollary 1.2 (iii), we can assume that the characteristic of $K$ does not divide $e$. Then, in this case, the result follows from Theorem 1.3 and Fischer’s Theorem: $K(H)$ is rational over $K$ for every finite abelian group $H$ of exponent $e$ and every field $K$ containing a primitive $e$-th root of unity [10, Theorem 6.1]. □

Remark 1.6. We don’t know whether Theorem 1.3 (or even Corollary 1.5) is valid for $\tilde{G}$, which is a semi-direct product but not a direct product. In fact, we don’t know whether there exist distinct prime numbers $p$ and $q$ such that $\tilde{G} = \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/q\mathbb{Z}$ is a non-abelian semi-direct product and $\mathbb{C}(\tilde{G})$ is not rational over $\mathbb{C}$. Note that $\mathbb{C}(\tilde{G})$ is retract rational over $\mathbb{C}$ [8, Theorem 3.5].

However, consider the non-abelian group $\tilde{G} = \mathbb{Z}/17\mathbb{Z} \rtimes \mathbb{Z}/16\mathbb{Z}$, where $\mathbb{Z}/16\mathbb{Z}$ acts faithfully on $\mathbb{Z}/17\mathbb{Z}$. By [8] Theorems 3.1 and 5.11, $\mathbb{Q}(\tilde{G})$ is not even retract rational over $\mathbb{Q}$, while it is known that $\mathbb{C}(\tilde{G})$ is rational over $\mathbb{C}$ [2].

For wreath products, we will prove the following.

Theorem 1.7. Let $K$ be any field, and let $H \wr G$ be the wreath product of finite groups $H$ and $G$. If $K(H)$ is rational (resp. stably rational) over $K$, so is $K(H \wr G)$ over $K(G)$.
In particular, if both $K(H)$ and $K(G)$ are rational (resp. stably rational) over $K$, so is $K(H \wr G)$ over $K$.

Remark 1.8. Assuming that $K$ is an infinite field, Saltman showed in \[8\, \text{Theorem 3.3}\] that $K(H \wr G)$ is retract rational over $K$ provided that so are $K(H)$ and $K(G)$.

Obvious combinations of the above theorems produce new rationality results as, for example, the following one.

Corollary 1.9. Let $K$ be a field, $p$ a prime number, and $G$ a $p$-Sylow subgroup of some finite symmetric group. If $K(\mathbb{Z}/p\mathbb{Z})$ is rational over $K$, so is $K(G)$ over $K$.

In particular, if $K$ is a field containing the $p$-th roots of unity, then $K(G)$ is rational over $K$.

Proof. It is well known (e.g. \[7\, \text{p. 177}\]) that $G$ is a direct product of iterated wreath products $\mathbb{Z}/p\mathbb{Z} \cdots \mathbb{Z}/p\mathbb{Z}$. Apply Theorem 1.3 and Theorem 1.7. □

We also have an application to Noether’s problem for dihedral groups.

Corollary 1.10. Let $K$ be any field, $n$ be an odd integer, and $D_n$ be the dihedral group of order $2n$.

(a) $K(D_{2n})$ is rational over $K(D_n)$. In particular, if $K(D_n)$ is rational (resp. retract rational) over $K$, so is $K(D_{2n})$.

(b) If $K(\mathbb{Z}/n\mathbb{Z})$ is rational over $K$, then both $K(D_n)$ and $K(D_{2n})$ are stably rational over $K$.

Proof. (a) If $n$ is odd, then $D_{2n} = \langle \sigma, \tau : \sigma^{2n} = \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle$ is a direct product of the groups $\langle \sigma^2, \tau \rangle \cong D_n$ and $\langle \sigma^n \rangle \cong \mathbb{Z}/2\mathbb{Z}$. Apply Corollary 1.5.

(b) It is easy to check that, for odd $n$, the map $(\mathbb{Z}/n\mathbb{Z}) \times D_n \to (\mathbb{Z}/n\mathbb{Z}) \wr (\mathbb{Z}/2\mathbb{Z})$ given by $(a, \sigma^b\tau^c) \mapsto ((a + b, a - b), \epsilon)$ is well-defined and is an isomorphism (see Section 4 for the definition of the wreath product). Hence, the stable rationality of $K(D_n)$ over $K$ follows from Theorem 1.3 and Theorem 1.7. Then, by (a), also $K(D_{2n})$ is stably rational over $K$. □

Remark 1.11. Note that part (a) is implicit in \[4\].

If $n$ is an odd integer and $K(\mathbb{Z}/n\mathbb{Z})$ is rational over $K$, the first-named author is able to show that $K(D_n)$ is indeed rational over $K$, by using other methods.

We will prove Theorem 1.1, Theorem 1.3, and Theorem 1.7 in Section 2, Section 6 and Section 4, respectively.

Standing notation. If $G$ is a finite group, we will write $V = \bigoplus_{g \in G} K \cdot x(g)$ as the regular representation space of $G$, where $G$ acts on $V$ by $h \cdot x(g) = x(hg)$ for any $g, h \in G$. Recall the definition $K(G) := K(x(g) : g \in G)^G$ given at the beginning of this section.

2. Proof of Theorem 1.1

Before proving Theorem 1.1 we recall two basic facts.

Theorem 2.1 \[Hajja and Kang \[3\, \text{Theorem 1}\].\] Let $G$ be a finite group acting on $L(x_1, \ldots, x_n)$, the rational function field of $n$ variables over a field $L$. Suppose that

(i) for any $\sigma \in G$, $\sigma(L) \subseteq L$;

(ii) the restriction of the action of $G$ to $L$ is faithful;
(iii) for any $\sigma \in G$, 
\[
\begin{pmatrix}
\sigma(x_1) \\
\sigma(x_2) \\
\vdots \\
\sigma(x_n)
\end{pmatrix} = A(\sigma) \cdot 
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix} + B(\sigma),
\]
where $A(\sigma) \in GL_n(L)$ and $B(\sigma)$ is an $n \times 1$ matrix over $L$.

Then there exist $z_1, \ldots, z_n \in L(x_1, \ldots, x_n)$ so that $L(x_1, \ldots, x_n) = L(z_1, \ldots, z_n)$ with $\sigma(z_i) = z_i$ for any $\sigma \in G$, any $1 \leq i \leq n$.

**Theorem 2.2** (Ahmad, Hajja and Kang [1, Theorem 3.1]). Let $L$ be any field, $L(x)$ the rational function field of one variable over $L$, and $G$ a group acting on $L(x)$. Suppose that, for any $\sigma \in G$, $\sigma(L) \subseteq L$ and $\sigma(x) = a_\sigma \cdot x + b_\sigma$, where $a_\sigma, b_\sigma \in L$ and $a_\sigma \neq 0$. Then $L(x)^G = L^G$ or $L^G(f)$ for some polynomial $f \in L[x]$. In fact, if the integer $m := \min \{\deg g(x) : g(x) \in L[x]^G, g(x) \notin L\}$ does exist, then $L(x)^G = L^G(f(x))$ for any $f(x) \in L[x]^G$ satisfying $\deg f = m$.

**Proof of Theorem 2.2** In this section, $K$ is a field with char $K = p > 0$ and $1 \to \mathbb{Z}/p\mathbb{Z} \to \tilde{G} \to G \to 1$. Let $c$ be a generator of the normal subgroup $\mathbb{Z}/p\mathbb{Z}$ and $\pi : \tilde{G} \to G \to 1$ be the given epimorphism.

Step 1. Let $u : G \to \tilde{G}$ be a section of $\pi$.

As before let $\tilde{V} = \bigoplus_{\tilde{g} \in \tilde{G}} K \cdot x(\tilde{g})$ and $V = \bigoplus_{g \in G} \tilde{K} \cdot x(g)$ be the regular representation spaces of $\tilde{G}$ and $G$ respectively.

Step 2. For each $\tilde{g} \in \tilde{G}$, define
\[
y(\tilde{g}) = \sum_{0 \leq i \leq p-1} x(c^i u(g)) \in \tilde{V},
\]
\[
z(\tilde{g}) = \sum_{0 \leq i \leq p-1} ix(c^i u(g)) \in \tilde{V},
\]
\[z = \sum_{g \in G} z(\tilde{g}) \in \tilde{V},
\]
\[W = \bigoplus_{\tilde{g} \in \tilde{G}} \bigoplus_{g \in G} \tilde{K} \cdot y(\tilde{g}).
\]

Note that $c \cdot y(\tilde{g}) = y(\tilde{g})$. As $G$-spaces, $W$ and $V$ are $G$-equivariant. Hence $K(W)^G \simeq K(G)$.

Step 3. We will examine the action of $\tilde{G}$ on $z(\tilde{g})$ and $z$.

It is clear that $c \cdot z(\tilde{g}) = z(\tilde{g}) - y(\tilde{g})$.

For any $h, g \in G$, suppose that $u(h) \cdot u(g) = c^n \cdot u(hg)$ and $u(h) \cdot c \cdot u(h)^{-1} = c^n$.

Note that $m$ is an integer depending on $g$ and $h$, and $n$ is invertible in $K$. When the element $h$ is fixed, we may write $m = m(g)$ to emphasize the dependence of $m$ on $g$.

We find that $u(h) \cdot z(\tilde{g}) = \sum_{0 \leq i \leq p-1} ix(u(h)c^i u(g)) = \sum_{0 \leq i \leq p-1} ix(c^{i+m} u(h) u(g)) = \sum_{0 \leq i \leq p-1} ix(c^{i+m} u(hg)) = c^n \cdot (1/n) \sum_{0 \leq i \leq p-1} ix(c^i u(hg)) = (1/n) z(hg) - (m/n) y(hg)$.

It follows that $u(h) \cdot z = (1/n) z - \sum_{g \in G}(m(g)/n) y(hg)$, where $m(g)$ denotes the integer $m$ depending on $g$. 

Step 4. Define \( \tilde{W} = W \oplus K \cdot z \). Then \( \tilde{W} \) is a faithful \( \tilde{G} \)-subspace of \( \tilde{V} \). By Theorem 2.1, \( K(\tilde{G}) \) is rational over \( K(\tilde{W})^{\tilde{G}} \).

Consider the pair \( \tilde{W} \) and \( W \) and apply Theorem 2.2. We find that \( K(\tilde{W})^{\tilde{G}} \) is rational over \( K(W)^G \). Since \( K(W)^G \cong K(G) \), we are done.

3. Proof of Theorem 1.3

Without loss of generality, we may assume that neither \( H \) nor \( G \) is the trivial group.
Step 1. Write \( \tilde{G} = H \times G \).
Let \( U = \bigoplus_{h \in H} K \cdot x(h) \) and \( V = \bigoplus_{g \in G} K \cdot x(g) \) be the regular representation spaces of \( \tilde{G} \) restricted to the action of \( H \) and \( G \) respectively.

For any element \( \tilde{g} \in \tilde{G} \), any \( u \otimes v \in U \otimes_K V \), define \( \tilde{g} \cdot (u \otimes v) = (h \cdot u) \otimes (g \cdot v) \) if \( \tilde{g} = hg \), where \( h \in H \) and \( g \in G \). It is easy to see that \( U \otimes_K V \) is isomorphic to the regular representation space of \( \tilde{G} \).

Step 2. Define

\[
\begin{align*}
  u_0 &= \sum_{h \in H} x(h) \in U, \quad v_0 = \sum_{g \in G} x(g) \in V, \\
  \tilde{U} &= \sum_{u \in U} K \cdot u \otimes v_0 \subset U \otimes_K V, \quad \tilde{V} = \sum_{v \in V} K \cdot u_0 \otimes v \subset U \otimes_K V.
\end{align*}
\]

It is easy to see that \( \tilde{U} \oplus \tilde{V} \) is a faithful \( \tilde{G} \)-subspace of \( U \otimes_K V \). Moreover, when restricted to the action of \( H \), the space \( \tilde{U} \) is \( H \)-equivariant isomorphic to the space \( U \), and similarly for \( \tilde{V} \) and \( V \) as \( G \)-spaces.

Step 3. By Theorem 2.1, \( K(\tilde{G}) = K(U \otimes_K V)^\tilde{G} \) is rational over \( K(U \oplus \tilde{V})^{\tilde{G}} \).

On the other hand, \( K(U \oplus \tilde{V})^{\tilde{G}} = (K(U \oplus \tilde{V})^H)^\tilde{G} \), which is \( K \)-isomorphic to \( K(H) \cdot K(G) \). We conclude that \( K(\tilde{G}) \) is rational over \( K(H) \cdot K(G) \). (Note that the composite \( K(H) \cdot K(G) \) is a free composite; i.e. the transcendence degree of it is the sum of those of \( K(H) \) and \( K(G) \).

Step 4. If \( K(H) \) is rational (resp. stably rational) over \( K \), it is easy to see that so is \( K(H) \cdot K(G) \) over \( K(G) \). Thus \( K(\tilde{G}) \) is rational (resp. stably rational) over \( K(G) \).

As to the retract rationality, from the definition of retract rationality [9] Definition 3.2], it is not difficult to show that (i) if \( K(H) \) is retract rational over \( K \), then \( K(H) \cdot K(G) \) is retract rational over \( K(G) \); and (ii) if both \( K(H) \) and \( K(G) \) are retract rational, then \( K(H) \cdot K(G) \) is retract rational over \( K \). Hence the result.

4. Proof of Theorem 1.7

Step 1. Write \( \tilde{G} = H \wr G \).
Recall the definition of the wreath product \( H \wr G \).
Define \( N = \bigoplus_{g \in G} H_g \), where each \( H_g \) is a copy of \( H \). When we write an element \( x = (\cdots, x_g, \cdots) \in N \), it is understood that \( x_g \) is the component of \( x \) in \( H_g \).

We will define a left action of \( G \) on \( N \) as follows. If \( \sigma \in G \) and \( x = (\cdots, x_g, \cdots) \in N \), define \( \sigma x = y \) where \( y = (\cdots, y_g, \cdots) \in N \) with \( y_g = x_{\sigma^{-1} g} \).
The wreath product $H \wr G$ is the semi-direct product $N \times G$. More precisely, if $x, y \in N$ and $\sigma, \tau \in G$, then $(x, \sigma) \cdot (y, \tau) = (x \cdot (\tau y), \sigma \tau)$. Thus we have

$$(\sigma x)(\tau y) = (\sigma \tau)(\tau^{-1} \cdot x \cdot y),$$

where $\sigma, \tau \in G$ and $x, y \in N$.

We will fix our notation for the group $\tilde{G} = H \wr G$, which will be used in subsequent discussions. The groups $N$ and $G$ may be identified (in the usual way) with subgroups of $G$. As above, if $x \in N$ and $\sigma \in G$, then $(x, \sigma)$ or $x \sigma$ denotes an element (and the same element) in $\tilde{G}$. For any $g \in G$, let $H_g$ be the subgroup of $N$ consisting of elements $x = (\cdot \cdot \cdot, x_{g'}, \cdot \cdot \cdot)$ satisfying the condition that $x_{g'} = 1$ for any $g' \in G \setminus \{g\}$.

Define a group isomorphism $\phi_g : H \to H_g$ such that, for any $h \in H$, if $x = \phi_g(h)$ and $x = (\cdot \cdot \cdot, x_{g'}, \cdot \cdot \cdot) \in H_g$, then $x_{g'} = h$.

Define a subgroup $M = \sum_{g \in G \setminus \{1\}} H_g$. Note that the coset decomposition of $\tilde{G}$ with respect to $M$ is given as $\tilde{G} = \bigcup (\sigma \cdot \phi_1(h))M$ where $\sigma$ and $h$ run over all elements in $G$ and $H$ respectively.

Step 2. Let $V = \bigoplus_{g \in G} K \cdot v(g)$ and $W = \bigoplus_{x \in N} K \cdot v(x)$ be the regular representation spaces of $G$ and $N$ respectively.

Define an action of $\tilde{G}$ on $V \otimes_K W$ by $(gx) \cdot (u(g') \otimes v(y)) = u(gg') \otimes v(g'^{-1} \cdot x \cdot y)$ (following Equation (1)), where $g, g' \in G$ and $x, y \in N$.

It follows that $V \otimes_K W$ is isomorphic to the regular representation space of $\tilde{G}$.

Step 3. For each $g \in G$, let $W_g = \bigoplus_{h \in H} K \cdot v(\phi_g(h))$ be the regular representation space of $H_g$. For any $g \in G \setminus \{1\}$, define

$w_g = \sum_{h \in H} v(\phi_g(h)) \in W_g.$

As in the proof of Theorem [13] we may regard $\bigotimes_{g \in G \setminus \{1\}} W_g$ as the regular representation space of $M$, and regard $\bigotimes_{g \in G} W_g$ as the regular representation space of $N$, i.e. $W$. Define

$w' = \bigotimes_{g \in G \setminus \{1\}} w_g \in \bigotimes_{g \in G \setminus \{1\}} W_g.$

Define

$w_0 = v(1) \otimes w' \in W, \quad u_0 = u(1) \otimes w_0 \in V \otimes_K W.$

Note that $x \cdot u_0 = u_0$ for any $x \in M$.

Step 4. For any $g \in G, h \in H$, define

$u(g; h) = (g \cdot \phi_1(h)) \cdot u_0 = u(g) \otimes (v(\phi_1(h)) \otimes w') \in V \otimes_K W.$

Note that, for $g, g' \in G$ and $h, h' \in H$, we have $g \cdot u(g'; h) = u(gg'; h), \phi_g(h) \cdot u(g; h') = u(gg; h'),$ and $\phi_g(h) \cdot u(g'; h') = u(g'; h')$ if $g \neq g'$.

For each $g \in G$, define

$U_g = \bigoplus_{h \in H} K \cdot u(g; h) \subset V \otimes_K W,$

and define

$\hat{U} := \bigoplus_{g \in G} U_g \subset V \otimes_K W.$

It is not difficult to show that $\hat{U}$ is a faithful $\tilde{G}$-subspace of $V \otimes_K W$. Note that $G$ permutes the spaces $U_g \ (g \in G)$ regularly; $H_g$ acts regularly on $U_g$, while $H_g$ acts trivially on $U_{g'}$ if $g \neq g'$. 
Step 5. Apply Theorem 2.1. We find that $K(\tilde{G})$ is rational over $K(\tilde{U}G)$. It remains to show that $K(\tilde{G})$ is rational (resp. stably rational) over $K(G)$ provided that $K(H)$ is rational (resp. stably rational) over $K$.

We consider first the situation when $K(H)$ is rational over $K$. Since $G$ permutes the spaces $U_g \ (g \in G)$ regularly, we may choose a transcendence basis $\{z(g;i) : 1 \leq i \leq d\}$ for $K(U_g)^H$ (where d is the order of $H$); i.e., we may write $K(U_g)^H = K(z(g;i) : 1 \leq i \leq d)$, such that $g \cdot z(g';i) = z(gg';i)$ for $1 \leq i \leq d$.

Thus $K(\tilde{U}G) = (K(\tilde{U}G)^N)^G = K(z(g;i) : g \in G, 1 \leq i \leq d)^G$. Apply Theorem 2.1. It is easy to see that $K(z(g;i) : g \in G, 1 \leq i \leq d)^G$ is rational over $K(z(g;1) : g \in G)^G$, which is isomorphic to $K(G)$.

Step 6. Assume now that $K(H)$ is stably rational over $K$. More precisely, suppose that $K(H)(t_1, \ldots, t_m)$ is rational over $K$.

Let $\{t(g;j)\}$ denote new indeterminates and define a $\tilde{G}$-space $\tilde{V}$ by

$$\tilde{V} := \bigoplus_{g \in G, 1 \leq j \leq m} K \cdot t(g;j),$$

where $g \cdot t(g';j) = t(gg';j)$ and $x \cdot t(g;j) = t(g;j)$ for any $g, g' \in G$, any $x \in N$, any $1 \leq j \leq m$.

Note that $\tilde{U} \oplus \tilde{V}$ is a faithful $\tilde{G}$-subspace of $(V \otimes_K W) \oplus \tilde{V}$. By Theorem 2.1 we find that $K((V \otimes_K W) \oplus \tilde{V})^\tilde{G}$ is rational over $K(V \otimes_K W)^G = K(G)$. Again by Theorem 2.1, $K((V \otimes_K W) \oplus \tilde{V})^\tilde{G}$ is rational over $K(\tilde{U} \oplus \tilde{V})^\tilde{G}$.

Now $K(\tilde{U} \oplus \tilde{V})^N = \prod_{g \in G} K(U_g)^H_{\tilde{g}}(t(g;j) : 1 \leq j \leq m)$, where each $K(U_g)^H_{\tilde{g}}$ is $K$-isomorphic to $K(H)$ with $g \cdot K(U_g)^H_{\tilde{g}} = K(U_g^g)^H_{\tilde{g}^g}$ for any $g, g' \in G$. For each $g \in G$, the field $K(U_g)^H_{\tilde{g}}(t(g;j) : 1 \leq j \leq m)$ is rational over $K$. As in Step 5, we may choose a transcendence basis $\{z(g;i) : 1 \leq i \leq d + m\}$ for $K(U_g)^H_{\tilde{g}}(t(g;j) : 1 \leq j \leq m)$ so that $G$ acts regularly on each set $\{z(g;i) : g \in G\}$, for every $1 \leq i \leq d + m$. The remaining arguments are quite similar to Step 5 and are omitted.

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