SHARP WELL-POSEDNESS RESULTS OF THE BENJAMIN-ONO EQUATION IN $H^s(\mathbb{T}, \mathbb{R})$ AND QUALITATIVE PROPERTIES OF ITS SOLUTION

PATRICK GÉRARD, THOMAS KAPPELER, AND PETAR TOPALOV

Abstract. We prove that the Benjamin–Ono equation on the torus is globally in time well-posed in the Sobolev space $H^s(\mathbb{T}, \mathbb{R})$ for any $s > -1/2$ and ill-posed for $s \leq -1/2$. Hence the critical Sobolev exponent $s_c = -1/2$ of the Benjamin–Ono equation is the threshold for well-posedness on the torus. The obtained solutions are almost periodic in time. Furthermore, we prove that the traveling wave solutions of the Benjamin–Ono equation on the torus are orbitally stable in $H^s(\mathbb{T}, \mathbb{R})$ for any $s > -1/2$. Novel conservation laws and a nonlinear Fourier transform on $H^s(\mathbb{T}, \mathbb{R})$ with $s > -1/2$ are key ingredients into the proofs of these results.

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1. Introduction

In this paper we consider the Benjamin-Ono (BO) equation on the torus,

\[ \partial_t v = H \partial_x^2 v - \partial_x (v^2), \quad x \in \mathbb{T} := \mathbb{R}/2\pi \mathbb{Z}, \quad t \in \mathbb{R}, \]

where \( v \equiv v(t,x) \) is real valued and \( H \) denotes the Hilbert transform, defined for \( f = \sum_{n \in \mathbb{Z}} \hat{f}(n)e^{inx}, \hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{-inx}dx, \) by

\[ Hf(x) := \sum_{n \in \mathbb{Z}} -i \text{sign}(n) \hat{f}(n) e^{inx} \]

with \( \text{sign}(\pm n) := \pm 1 \) for any \( n \geq 1 \), whereas \( \text{sign}(0) := 0 \). This pseudo-differential equation (ΨDE) in one space dimension has been introduced by Benjamin [7] and Ono [31] to model long, uni-directional internal gravity waves in a two-layer fluid. It has been extensively studied, both on the real line \( \mathbb{R} \) and on the torus \( \mathbb{T} \). For an excellent survey, including the derivation of (1), we refer to the recent article by Saut [33].

Our aim is to study low regularity solutions of the BO equation on \( \mathbb{T} \). To state our results, we first need to review some classical results on the well-posedness problem of (1). Based on work of Saut [32], Abdelouhab, Bona, Felland, and Saut proved in [1] that for any \( s \geq 3/2 \), equation (1) is globally in time well-posed on the Sobolev space \( H^s_r \equiv H^s(\mathbb{T}, \mathbb{R}) \) (endowed with the standard norm \( \| \cdot \|_s \), defined by (3) below), meaning the following:

(S1) Existence and uniqueness of classical solutions: For any initial data \( v_0 \in H^s_r \), there exists a unique curve \( v : \mathbb{R} \to H^s_r \) in \( C(\mathbb{R}, H^s_r) \cap C^1(\mathbb{R}, H^{s-2}_r) \) so that \( v(0) = v_0 \) and for any \( t \in \mathbb{R} \), equation (1) is satisfied in \( H^{s-2}_r \). (Since \( H^s_r \) is an algebra, one has \( \partial_x v(t)^2 \in H^{s-1}_r \) for any time \( t \in \mathbb{R} \).)

(S2) Continuity of solution map: The solution map \( S : H^s_r \to C(\mathbb{R}, H^s_r) \) is continuous, meaning that for any \( v_0 \in H^s_r, T > 0, \) and \( \varepsilon > 0 \) there exists \( \delta > 0 \), so that for any \( w_0 \in H^s_r \) with \( \|w_0 - v_0\|_s < \delta \), the solutions \( w(t) = S(t, w_0) \) and \( v(t) = S(t, v_0) \) of (1) with initial data \( w(0) = w_0 \) and, respectively, \( v(0) = v_0 \) satisfy \( \sup_{|t| \leq T} \|w(t) - v(t)\|_s \leq \varepsilon \).

In a straightforward way one verifies that

\[ H^{(-1)}(v) := \langle v|1 \rangle, \quad H^{(0)}(v) := \frac{1}{2} \langle v|v \rangle \]

are integrals of the above solutions of (1). Here \( \langle \cdot | \cdot \rangle \) denotes the \( L^2 \)-inner product,

\[ \langle f|g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f\overline{g}dx. \]
In particular it follows that for any $c \in \mathbb{R}$ and any $s \geq 3/2$, the affine space $H_{r,c}^s$ is left invariant by $\mathcal{S}$ where for any $\sigma \in \mathbb{R}$

\begin{equation}
H_{r,c}^\sigma := \{ w \in H_r^\sigma : \langle w \rangle_1 = c \}.
\end{equation}

In the sequel, further progress has been made on the well-posedness of (1) on Sobolev spaces of low regularity. The best results so far in this direction were obtained by Molinet by using the gauge transformation introduced by Tao [35]. Molinet’s results in [27] (cf. also [29]) imply that the solution map $\mathcal{S}$, introduced in (S2) above, continuously extends to any Sobolev space $H_r^s$ with $0 \leq s \leq 3/2$. More precisely, for any such $s$, $\mathcal{S} : H_r^s \to C(\mathbb{R}, H_r^s)$ is continuous and for any $v_0 \in H_r^s$, $\mathcal{S}(t, v_0)$ satisfies equation (1) in $H_r^{s-2}$. The fact that $\mathcal{S}$ continuously extends to $L_r^2 = H_r^0$, $\mathcal{S} : L_r^2 \to C(\mathbb{R}, L_r^2)$, can also be deduced by methods recently developed in [15]. Furthermore, one infers from [15] that any solution $\mathcal{S}(t, v_0)$ with initial data $v_0 \in L_r^2$ can be approximated in $C(\mathbb{R}, L_r^2)$ by solutions of (1) which are rational functions of $\cos x, \sin x$. We refer to these solutions as rational solutions.

In this paper we show that the BO equation is well-posed in the Sobolev space $H_r^{-s}$ for any $0 < s < 1/2$ and that this result is sharp. Since the nonlinear term $\partial_x v^2$ in equation (1) is not well-defined for elements in $H_r^{-s}$, we first need to define what we mean by a solution of (1) in such a space.

**Definition 1.** Let $s \geq 0$. A continuous curve $\gamma : \mathbb{R} \to H_r^{-s}$ with $\gamma(0) = v_0$ for a given $v_0 \in H_r^{-s}$, is called a global in time solution of the BO equation in $H_r^{-s}$ with initial data $v_0$ if for any sequence $(v_0^{(k)})_{k \geq 1}$ in $H_r^s$ with $\sigma > 3/2$, which converges to $v_0$ in $H_r^{-s}$, the corresponding sequence of classical solutions $\mathcal{S}(\cdot, v_0^{(k)})$ converges to $\gamma$ in $C(\mathbb{R}, H_r^{-s})$. The solution $\gamma$ is denoted by $\mathcal{S}(\cdot, v_0)$.

We remark that for any $v_0 \in L_r^2$, the solution $\mathcal{S}(\cdot, v_0)$ in the sense of Definition 1 coincides with the solution obtained by Molinet in [27].

**Definition 2.** Let $s \geq 0$. Equation (1) is said to be globally $C^0$—well-posed in $H_r^{-s}$ if the following holds:

(i) For any $v_0 \in H_r^{-s}$, there exists a global in time solution of (1) with initial data $v_0$ in the sense of Definition 1.

(ii) The solution map $\mathcal{S} : H_r^{-s} \to C(\mathbb{R}, H_r^{-s})$ is continuous, i.e. satisfies (S2).

Our main results are the following ones:

**Theorem 1.** For any $0 \leq s < 1/2$, the Benjamin-Ono equation is globally $C^0$—well-posed on $H_r^{-s}$ in the sense of Definition 2. For any $c \in \mathbb{R}$, $t \in \mathbb{R}$, the flow map $S^t = \mathcal{S}(t, \cdot)$ leaves the affine space $H_{r,c}^{-s}$, introduced in (4), invariant.

Furthermore, there exists a conservation law $I_{-s} : H_r^{-s} \to \mathbb{R}_{\geq 0}$ of (1)
satisfying
\[ \|v\|_{-s} \leq I_{-s}(v), \quad \forall v \in H^{-s}_r. \]

In particular, one has
\[ \sup_{t \in \mathbb{R}} \|S(t,v_0)\|_{-s} \leq I_{-s}(v_0), \quad \forall v_0 \in H^{-s}_r. \]

**Remark 1.** (i) Theorem 1 continues to hold on $H^s_r$ for any $s > 0$. See Corollary 8 in Appendix A.

(ii) Since by (2), the $L^2$-norm is an integral of (1), $I_{-s}$ in the case $s = 0$ can be chosen as $I_0(v) := \|v\|_0^2$. The definition of $I_{-s}$ for $0 < s < 1/2$ can be found in Remark 7 in Section 2. These novel integrals are one of the key ingredients for the proof of global $C^0-$well-posedness of (1) in $H^{-s}_r$ for $0 < s < 1/2$.

(iii) Note that global $C^0-$well-posedness implies the group property $S^{t_1} \circ S^{t_2} = S^{t_1+t_2}$. Consequently, $S^t$ is a homeomorphism of $H^{-s}_r$.

(iv) By Rellich’s compactness theorem, $S^t$ is also weakly sequentially continuous on $H^{-s}_r$, for any $0 \leq s < 1/2$ and $c \in \mathbb{R}$, hence in particular on $L^2_{r,0}$. Note that this contradicts a result stated in [28, Theorem 1.1]. Very recently, however, an error in the proof of the latter theorem has been found, leading to the withdrawal of the paper (cf. arXiv:0811.0505). A proof of this weak continuity property was indeed the starting point of the present paper.

The next result says that the well-posedness result of Theorem 1 is sharp.

**Theorem 2.** For any $c \in \mathbb{R}$, the Benjamin-Ono equation is ill-posed on $H^{-1/2}_{r,c}$. More precisely, there exists a sequence $(u^{(k)})_{k \geq 1}$ in $\bigcap_{n \geq 1} H^n_{r,0}$, converging strongly to 0 in $H^{-1/2}_{r,0}$, so that for any $c \in \mathbb{R}$, the solutions $S(t,u^{(k)} + c)$ of (1) of average $c$ have the property that the sequence of functions $t \mapsto |S(t,u^{(k)} + c)| e^{iz}$ does not converge pointwise to 0 on any given time interval of positive length.

**Remark 2.** It was observed in [5] that the solution map $S$ does not continuously extend to $H^{-s}_r$ with $s > 1/2$. More precisely, for any $c \in \mathbb{R}$, the authors of [5] construct a sequence $(v^{(k)}_0)_{k \geq 1}$ in $\bigcap_{n \geq 0} H^n_{r,c}$ of initial data so that for any $s > 1/2$ it converges to an element $v_0$ in $H^{-s}_{r,c}$ whereas for any $t \neq 0$, $(S(t,v^{(k)}_0))_{k \geq 1}$ diverges even in the sense of distributions. However, the divergence of $S(t,v^{(k)}_0)$ can be removed by renormalizing the flow by a translation of the space variable, $x \mapsto x + \eta_k t$. In the case $c = 0$, $\eta_k$ is given by $\|v^{(k)}_0\|_0^2$. We refer to [10] for a similar renormalization in the context of the nonlinear Schrödinger equation. In Appendix B, we construct a sequence of initial data in $\bigcap_{n \geq 0} H^n_{r,c}$ with the above convergence/divergence properties, but where such a renormalization is not possible.
Comments on Theorem 1 and Theorem 2. (i) A straightforward computation shows that $s_c = -1/2$ is the critical Sobolev exponent of the Benjamin-Ono equation. Hence Theorem 1 and Theorem 2 say that the threshold of well-posedness of (1) is given by the critical Sobolev exponent $s_c$.

(ii) In a recent, very interesting paper [34], Talbut proved by the method of perturbation determinants, developed for the KdV and the NLS equations by Killip, Visan, and Zhang in [24], that for any $0 < s < 1/2$, there exists a constant $C_s > 0$, only depending on $s$, so that any sufficiently smooth solution $t \to v(t)$ of (1) satisfies the estimate
\[
\sup_{t \in \mathbb{R}} \|v(t)\|_{-s} \leq C_s \left(1 + \|v(0)\|_{-s}^{1/2s}\right)^s \|v(0)\|_{-s}.
\]

We note that the integrals $I_{-s}$ of Theorem 1(iv) are of a different nature. Let us explain this in more detail. Our method for proving that the solution map $S$ of (1) continuously extends to $H_{r,s}^s$ for any $0 < s < 1/2$ consists in constructing a globally defined nonlinear Fourier transform $\Phi$, also referred to as Birkhoff map (cf. Section 2). It means that (1) can be solved by quadrature, when expressed in the coordinates defined by $\Phi$, which we refer to as Birkhoff coordinates. The integrals $I_{-s}$ of Theorem 1(iv) are tailored to show that $\Phi : H_{r,0}^{-s} \to h^{1/2-s}_+$ is onto (cf. Theorem 6). Actually, the map $\Phi$ is a key ingredient not only in the proof of Theorem 1, but in the proof of all results, stated in Section 1. In particular, with regard to Theorem 2, we note that the standard norm inflation argument, pioneered by [11] (cf. also [17, Appendix A] and references therein), does not apply for proving ill-posedness of (1) in $H_{r,c}^{-1/2}$ since the mean $\langle u | 1 \rangle$ is an integral of (1). Our proof of Theorem 2 is based in a fundamental way on the map $\Phi$ and its properties (cf. Section 7).

(iii) Using a probabilistic approach developed by Tzvetkov and Visciglia [37], Y. Deng [13] proved well-posedness result for the BO equation on the torus for almost every data with respect to a measure which is supported by $\bigcap_{\varepsilon>0} H_{r,\varepsilon}$ and for which $L^2_{r,\varepsilon}$ is of measure 0. Our result provides a deterministic framework for these solutions.

One of the key ingredients of our proof of Theorem 1 are explicit formulas for the frequencies of the Benjamin-Ono equation, defined by (9) below, which describe the time evolution of solutions of (1) when expressed in Birkhoff coordinates. They are not only used to prove the global well-posedness results for (1), but at the same time allow to obtain the following qualitative properties of solutions of (1).

**Theorem 3.** For any $v_0 \in H_{r,c}^{-s}$ with $0 < s < 1/2$ and $c \in \mathbb{R}$, the solution $S(t,v_0)$ has the following properties:

(i) The orbit $\{S(t,v_0) : t \in \mathbb{R}\}$ is relatively compact in $H_{r,c}^{-s}$.

(ii) The solution $t \mapsto S(t,v_0)$ is almost periodic in $H_{r,c}^{-s}$. 
Remark 3. Theorem 3 continues to hold for any initial data in $H_{r,c}^s$ with $s > 0$ arbitrary. See Corollary 8 in Appendix A. For $s = 0$, results corresponding to the ones of Theorem 3 have been obtained in [15].

In [3], Amick and Toland characterized the traveling wave solutions of (1), originally found by Benjamin [7]. It was shown in [15, Appendix B] that they coincide with the so called one gap solutions, described explicitly in [15]. Note that one gap potentials are rational solutions of (1) and evolve in $\bigcap_{n \geq 1} H_{r,0}^n$. In [4, Section 5.1] Angulo Pava and Natali proved that every travelling wave solution of (1) is orbitally stable in $H_{r}^{1/2}$. Our newly developed methods allow to complement their result as follows:

Theorem 4. Every traveling wave solution of the BO equation is orbitally stable in $H_{r}^{-s}$ for any $0 \leq s < 1/2$.

Remark 4. Theorem 4 continues to hold on $H_{r}^s$ for any $s > 0$. See Corollary 8 in Appendix A.

Method of proof. Let us explain our method for studying low regularity solutions of integrable PDEs / ΨDEs such as the Benjamin-Ono equation, in an abstract, informal way. Consider an integrable evolution equation (E) of the form $\partial_t u = X_H(u)$ where $X_H(u)$ denotes the Hamiltonian vector field, corresponding to the Hamiltonian $H$. In a first step we disregard the equation (E) and choose instead a family of Poisson commuting Hamiltonians $H_\lambda$, parametrized by $\lambda \in \Lambda$, with the property that the Hamiltonian $H$ is in the Poisson algebra, generated by the family $(H_\lambda)_{\lambda \in \Lambda}$, i.e., $\{H, H_\lambda\} = 0$ for any $\lambda \in \Lambda$. To study low regularity solutions of (E), the choice of $H_\lambda$, $\lambda \in \Lambda$, has to be made judiciously. Typically, the so called hierarchies, often associated with integrable PDEs/ΨDEs are not well suited families. Our strategy is to choose such a family with the help of a Lax pair formulation of (E), $\partial_t L = [B, L]$ where $L \equiv L_u$ and $B \equiv B_u$ are typically differential or pseudo-differential operators acting on Hilbert spaces of functions, with symbols depending on $u$, and where $[B, L]$ denotes the commutator of $B$ and $L$. At least formally, the spectrum of the operator $L$ is conserved by the flow of (E). The goal is to find a Lax pair $(L, B)$ for (E) with the property that the operator $L$ is well defined for $u$ of low regularity and then choose functions $H_\lambda$, encoding the spectrum of $L$, such as the (appropriately regularized) determinant of $L - \lambda$ or a perturbation determinant. We refer to such a function as a generating function. The key properties of $H_\lambda$ to be established are the following ones: (i) the flows of the Hamiltonian vector fields $X_{H_\lambda}$ are well defined for $u$ of low regularity and can be integrated globally in time; (ii) for $u$ sufficiently regular, $H$ can be expressed in terms of the generating function; (iii) the generating function can be used to construct Birkhoff coordinates so that the Hamiltonian vector field $X_H$, when expressed
in these coordinates, extends to spaces of $u$ of low regularity.

In the case of the Benjamin-Ono equation, this method is implemented as follows. In a first important step we prove that the operator $L_u$ (cf. (14)) of the Lax pair for the Benjamin-Ono equation, found by Nakamura [30], has the property that it is well defined for $u$ in the Sobolev spaces $H_{r,s}^{-s}$, $0 < s < 1/2$. See the paragraph Ideas of the proof of Theorem 5 in Section 2 for more details. By (27) in Section 3, our choice of the generating function is $H_\lambda(u) = (L_u + \lambda)^{-1}(1)$ and the Hamiltonian $\mathcal{H}$ of the Benjamin-Ono equation, when expressed in Birkhoff coordinates, is given by (11). The novel conservation laws of the Benjamin-Ono equation of Theorem 1, $I_{-s}: H_{r,s}^{-s} \to \mathbb{R}_{\geq 0}$, together with the results on the Lax operator $L_u$ for $u$ in $H_{r,0}^{-s}$ are the key ingredients to construct Birkhoff coordinates on $H_{r,s}^{-s}$ for any $0 < s < 1/2$. When expressed in these coordinates, equation (1) can be solved by quadrature.

Related work. Results on global well-posedness of the type stated in Theorem 1 have been obtained for other integrable PDEs such as the KdV, the KdV2, the mKdV, and the defocusing NLS equations. A detailed analysis of the frequencies of these equations allowed to prove in addition to the well-posedness results qualitative properties of solutions of these equations, among them properties corresponding to the ones stated in Theorem 3 – see e.g. [21],[22], [19], [20]. Very recently, sharp global well-posedness results for the cubic NLS, the mKdV equation, the KdV equation, and the fifth-order KdV equation on the real line were obtained in [17],[23], and, respectively, [9]. They are based on novel integrals constructed in [24] (cf. also [25]). By the same method, Killip and Visan provide in [23] alternative proofs of the global well-posedness results for the KdV equation on the torus obtained in [21]. However, to the best of our knowledge, their method does not allow to deduce qualitative properties of solutions of the KdV equation on $\mathbb{T}$ such as almost periodicity nor to obtain coordinates which can be used to study perturbations of the KdV equation by KAM type methods.

Subsequent work. One of the main novel features of the Benjamin–Ono equation, when compared from the point of view of integrable PDEs with the KdV equation or the cubic NLS equation, is that the Lax operator $L_u$ (cf. (14)), appearing in the Lax pair formulation of (1), is nonlocal. One of the consequences of $L_u$ being nonlocal is that the study of the regularity of the Birkhoff map and of its restrictions to the scale of Sobolev spaces $H_{r,0}^{s}$, $s \geq 0$, is quite involved. Further results on the Birkhoff map of the Benjamin-Ono equation in this direction will be reported on in subsequent work.

Organisation. In Section 2, we state our results on the extension of the Birkhoff map $\Phi$ (cf. Theorem 6) and discuss first applications. All
these results are proved in Section 3 and Section 4. In Section 5, we study the solution map \( S_B \) corresponding to the system of equations, obtained when expressing (1) in Birkhoff coordinates. These results are then used to study the solution map \( S \) of (1). In the same section we also introduce the solution map \( S_c \) (cf. (53)), defined in terms of the solution map of the equation (1) in the affine space \( H^s_{r,c}, c \in \mathbb{R} \), and study the solution map \( S_{c,B} \), obtained by expressing \( S_c \) in Birkhoff coordinates. With all these preparations done, we prove Theorem 1, Theorem 3, and Theorem 4, in Section 6. The proof of Theorem 2 is presented in Section 7. Finally, in Appendix A we study the restriction of the Birkhoff map to the Sobolev spaces \( H^s_{r,0} \) with \( s > 0 \) and discuss applications to the Benjamin-Ono equation, while in Appendix B we discuss results on ill-posedness of the Benjamin–Ono equation in \( H^{-s}_r \) with \( s > \frac{1}{2} \).

**Notation.** By and large, we will use the notation established in [15]. In particular, the \( H^s \)-norm of an element \( v \) in the Sobolev space \( H^s \equiv H^s(T, \mathbb{C}), s \in \mathbb{R} \), will be denoted by \( \|v\|_s \). It is defined by

\[
\|v\|_s = \left( \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} |\hat{v}(n)|^2 \right)^{1/2}, \quad \langle n \rangle = \max\{1, |n|\}.
\]

For \( \|v\|_0 \), we usually write \( \|v\| \). By \( \langle \cdot | \cdot \rangle \), we will also denote the extension of the \( L^2 \)-inner product, introduced in (3), to \( H^{-s} \times H^s \), \( s \in \mathbb{R} \), by duality. By \( H^s_+ \) we denote the Hardy space, consisting of elements \( f \in L^2(T, \mathbb{C}) \equiv H^0 \) with the property that \( \hat{f}(n) = 0 \) for any \( n < 0 \). More generally, for any \( s \in \mathbb{R} \), \( H^s_+ \) denotes the subspace of \( H^s \), consisting of elements \( f \in H^s \) with the property that \( \hat{f}(n) = 0 \) for any \( n < 0 \).

**Previous versions.** A first version of this paper appeared on arXiv in September 2019 and a second one with additional results in December 2019 – see [16]. In the current version, Section 7 (proof of ill-posedness of (1) in \( H^{-1/2}_r \)), Appendix A (restriction of the Birkhoff map to \( H^s_{r,0} \) and applications), and Appendix B (ill-posedness of (1) in \( H^{-s}_r \) for \( s > 1/2 \)) have been added and the introduction has been extended. To reflect better the content of the current version, the title of the paper has been changed.

2. The Birkhoff map \( \Phi \)

In this section we present our results on Birkhoff coordinates which will be a key ingredient of the proofs of Theorem 1 – Theorem 4. We begin by reviewing the results on Birkhoff coordinates proved in [15]. Recall that on appropriate Sobolev spaces, (1) can be written in
Hamiltonian form

$$\partial_t u = \partial_x(\nabla H(u)),$$

$$H(u) := \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{1}{2}(|\partial_x u|^2 - u^2) \right) dx$$

where $|\partial_x|^2$ is the square root of the Fourier multiplier operator $|\partial_x|$ given by

$$|\partial_x|f(x) = \sum_{n \in \mathbb{Z}} |n| \hat{f}(n)e^{inx}.$$

Note that the $L^2$-gradient $\nabla H$ of $H$ can be computed to be $|\partial_x|u - u^2$ and that $\partial_x \nabla H$ is the Hamiltonian vector field corresponding to the Gardner bracket, defined for any two functionals $F, G : \mathbb{H}_r^0 \to \mathbb{R}$ with sufficiently regular $L^2$-gradients by

$$\{F, G\} := \frac{1}{2\pi} \int_0^{2\pi} (\partial_x F) \nabla G dx.$$

In [15], it is shown that (1) admits global Birkhoff coordinates and hence is an integrable ΨDE in the strongest possible sense. To state this result in more detail, we first introduce some notation. For any subset $J \subset \mathbb{N}_0 := \mathbb{Z}_{\geq 0}$ and any $s \in \mathbb{R}$, $h^s(J) \equiv h^s(J, \mathbb{C})$ denotes the weighted $\ell^2$-sequence space

$$h^s(J) = \{(z_n)_{n \in J} \subset \mathbb{C} : \|(z_n)_{n \in J}\|_s < \infty\}$$

where

$$\|(z_n)_{n \in J}\|_s := \left( \sum_{n \in J} \langle n \rangle^{2s} |z_n|^2 \right)^{1/2}, \quad \langle n \rangle := \max\{1, |n|\}.$$

By $h^s(J, \mathbb{R})$, we denote the real subspace of $h^s(J, \mathbb{C})$, consisting of real sequences $(z_n)_{n \in J}$. In case where $J = \mathbb{N} := \{n \in \mathbb{Z} : n \geq 1\}$ we write $h^s_+ \text{ instead of } h^s(\mathbb{N})$. If $s = 0$, we also write $\ell^2$ instead of $h^0_+$ and $\ell^2_+$ instead of $h^0_+$. In the sequel, we view $h^s_+$ as the $\mathbb{R}$–Hilbert space $h^s(\mathbb{N}, \mathbb{R}) \oplus h^s(\mathbb{N}, \mathbb{R})$ by identifying a sequence $(z_n)_{n \in \mathbb{N}} \in h^s_+$ with the pair of sequences $(\text{Re} z_n)_{n \in \mathbb{N}}, (\text{Im} z_n)_{n \in \mathbb{N}} \in h^s(\mathbb{N}, \mathbb{R}) \oplus h^s(\mathbb{N}, \mathbb{R})$. We recall that $L^2_\mathbb{R} = H^0_\mathbb{R}$ and $L^2_{r,0} = H^0_{r,0}$. The following result was proved in [15]:

**Theorem 5.** ([15, Theorem 1]) There exists a homeomorphism

$$\Phi : L^2_{r,0} \to h^{1/2}_+, \quad u \mapsto (\zeta_n(u))_{n \geq 1}$$

so that the following holds:

(B1) For any $n \geq 1$, $\zeta_n : L^2_{r,0} \to \mathbb{C}$ is real analytic.

(B2) The Poisson brackets between the coordinate functions $\zeta_n$ are well-defined and for any $n, k \geq 1$,

$$\{\zeta_n, \zeta_k\} = -i\delta_{nk}, \quad \{\zeta_n, \zeta_k\} = 0.$$
It implies that the functionals $|\zeta_n|^2$, $n \geq 1$, pairwise Poisson commute,
\[
\{ |\zeta_n|^2, |\zeta_k|^2 \} = 0, \quad \forall n,k \geq 1.
\]

(B3) On its domain of definition, $H \circ \Phi^{-1}$ is a (real analytic) function, which only depends on the actions $|\zeta_n|^2$, $n \geq 1$. As a consequence, for any $n \geq 1$, $|\zeta_n|^2$ is an integral of $H \circ \Phi^{-1}$, $\{ H \circ \Phi^{-1}, |\zeta_n|^2 \} = 0$.

The coordinates $\zeta_n$, $n \geq 1$, are referred to as complex Birkhoff coordinates and the functionals $|\zeta_n|^2$, $n \geq 1$, as action variables.

**Remark 5.** (i) When restricted to submanifolds of finite gap potentials (cf. [15, Definition 2.2]), the map $\Phi$ is a canonical, real analytic diffeomorphism onto corresponding Euclidean spaces – see [15, Theorem 3] for details.

(ii) For any bounded subset $B$ of $L^2_{r,0}$, the image $\Phi(B)$ by $\Phi$ is bounded in $h^{1/2}_+$. This is a direct consequence of the trace formula, saying that for any $u \in L^2_{r,0}$ (cf. [15, Proposition 3.1]),
\[
\|u\|^2 = 2 \sum_{n=1}^{\infty} n |\zeta_n|^2.
\]

Theorem 5 together with Remark 5(i) can be used to solve the initial value problem of (1) in $L^2_{r,0}$. Indeed, by approximating a given initial data in $L^2_{r,0}$ by finite gap potentials (cf. [15, Definition 2.2]), one concludes from [15, Theorem 3] and Theorem 5 that equation (1), when expressed in the Birkhoff coordinates $\zeta = (\zeta_n)_{n \geq 1}$, reads
\[
\partial_t \zeta_n = \{ H \circ \Phi^{-1}, \zeta_n \} = i \omega_n \zeta_n, \quad \forall n \geq 1,
\]
where $\omega_n$, $n \geq 1$, are the BO frequencies,
\[
\omega_n = \partial_{|\zeta_n|^2} H \circ \Phi^{-1}.
\]

Since the frequencies only depend on the actions $|\zeta_k|^2$, $k \geq 1$, they are conserved and hence (8) can be solved by quadrature,
\[
\zeta_n(t) = \zeta_n(0) e^{i \omega_n (\zeta(0)) t}, \quad t \in \mathbb{R}, \quad n \geq 1.
\]

By [15, Proposition 8.1]), $H_B := H \circ \Phi^{-1}$ can be computed as
\[
H_B(\zeta) := \sum_{k=1}^{\infty} k^2 |\zeta_k|^2 - \sum_{k=1}^{\infty} \left( \sum_{p=k}^{\infty} |\zeta_p|^2 \right)^2,
\]
implying that the frequencies, defined by (9), are given by
\[
\omega_n(\zeta) = n^2 - 2 \sum_{k=1}^{\infty} \min(n,k) |\zeta_k|^2, \quad \forall n \geq 1.
\]

Remarkably, for any $n \geq 1$, $\omega_n$ depends linearly on the actions $|\zeta_k|^2$, $k \geq 1$. Furthermore, while the Hamiltonian $H_B$ is defined on $h^{1}_+$, the
frequencies $\omega_n$, $n \geq 1$, given by (12) for $\zeta \in h^1_+$, extend to bounded functionals on $\ell^2_+$,
\begin{equation}
\omega_n : \ell^2_+ \to \mathbb{R}, \; \zeta = (\zeta_k)_{k \geq 1} \mapsto \omega_n(\zeta).
\end{equation}
We will prove that the restriction $\mathcal{S}_0$ of the solution map of (1) to $L^2_{r,0}$, when expressed in Birkhoff coordinates,
$$\mathcal{S}_B : h^{1/2}_+ \to C(\mathbb{R}, h^{1/2}_+) \; ; \; (\zeta(0) \mapsto (\xi_n(0) e^{i \omega_n(\zeta(0)) t})_{n \geq 1}$$
is continuous – see Proposition 3 in Section 5. By Theorem 5, $\Phi : L^2_{r,0} \to h^{1/2}_+$ and its inverse $\Phi^{-1} : h^{1/2}_+ \to L^2_{r,0}$ are continuous. Since
$$\mathcal{S}_0 = \Phi^{-1}\mathcal{S}_B \Phi : L^2_{r,0} \to C(\mathbb{R}, L^2_{r,0}) \; ; \; u(0) \mapsto \Phi^{-1}\mathcal{S}_B(t, \Phi(u(0)))$$
it follows that $\mathcal{S}_0 : L^2_{r,0} \to C(\mathbb{R}, L^2_{r,0})$ is continuous as well. We remark that for any $u(0) \in L^2_{r,0}$, the solution $t \mapsto \mathcal{S}(t, u(0))$ can be approximated in $L^2_{r,0}$ by classical solutions of equation (1) (cf. Remark 5(i)) and thus coincides with the solution, obtained by Molinet in [27] (cf. also [29]).

Starting point of the proof of Theorem 1 is formula (56) in Subsection 5. We will show that it extends to the Sobolev spaces $H^{-s}_{r,0}$ for any $0 < s < 1/2$. A key ingredient to prove Theorem 1 is therefore the following result on the extension of the Birkhoff map $\Phi$ to $H^{-s}_{r,0}$ for any $0 < s < 1/2$:

**Theorem 6.** (Extension of $\Phi$.) For any $0 < s < 1/2$, the map $\Phi$ of Theorem 5 admits an extension, also denoted by $\Phi$,
$$\Phi : H^{-s}_{r,0} \to h^{1/2-s}_+, \; u \mapsto \Phi(u) := (\xi_n(u))_{n \geq 1},$$
so that the following holds:
(i) $\Phi$ is a homeomorphism.
(ii) There exists an increasing function $F_s : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ so that
$$\|u\|_{-s} \leq F_s(\|\Phi(u)\|_{1/2-s}) \quad \forall u \in H^{-s}_{r,0}.$$ 
(iii) $\Phi$ and its inverse map bounded subsets to bounded subsets.

**Remark 6.** (i) The Birkhoff map does not continuously extend to $H^{-1/2}_{r,0}$ – see Corollary 7 at the end of Section 7.
(ii) Results, developed in the course of the proof of Theorem 6 allow to study the restriction of the Birkhoff map to $H^s_r$ for any $s > 0$. See Proposition 5 in Appendix A for details.
(iii) Items (i) and (iii), combined with the Rellich compactness theorem, imply that for $0 \leq s < \frac{1}{2}$, the map $\Phi : H^{-s}_{r,0} \to h^{1/2-s}_+$ and its inverse $\Phi^{-1} : h^{1/2-s}_+ \to H^{-s}_{r,0}$ are weakly sequentially continuous on $H^{-s}_{r,0}$.

**Remark 7.** The above a priori bound for $\|u\|_{-s}$ can be extended to the space $H^{-s}_r$ as follows
$$\|v\|_{-s} \leq F_s(\|\Phi(v - [v])\|_{1/2-s}) + \|[v]\|, \quad [v] = \langle v|1\rangle, \quad \forall v \in H^{-s}_r.$$
For any $0 < s < 1/2$, the integral $I_{-s}$ in Theorem 1(iv) is defined as

$$I_{-s}(v) := F_s(\|\Phi(v - [v])\|_{1/2-s}) + [v].$$

Ideas of the proof of Theorem 6. At the heart of the proof of Theorem 1 in [15] is the Lax operator $L_u$, appearing in the Lax pair formulation in [30] (cf. also [8], [12], [14])

$$\partial_t L_u = [B_u, L_u]$$

of (1) – see [15, Appendix A] for a review. For any given $u \in L^2_r$, the operator $L_u$ is the first order operator acting on the Hardy space $H_+$,

$$L_u := -i\partial_x - T_u , \quad T_u(\cdot) := \Pi(u \cdot)$$

where $\Pi$ is the orthogonal projector of $L^2$ onto $H_+$ and $T_u$ is the Toeplitz operator with symbol $u$,

$$H_+ := \{ f \in L^2 : \hat{f}(n) = 0 \ \forall n < 0 \}.$$

The operator $L_u$ is self-adjoint with domain $H^1_+ := H^1 \cap H_+$, bounded from below, and has a compact resolvent. Its spectrum consists of real eigenvalues which are bounded from below. When listed in increasing order they form a sequence, satisfying

$$\lambda_0 \leq \lambda_1 \leq \cdots , \quad \lim_{n \to \infty} \lambda_n = \infty.$$

For our purposes, the most important properties of the spectrum of $L_u$ are that the eigenvalues are conserved along the flow of (1) and that they are all simple. More precisely, one has

$$\gamma_n := \lambda_n - \lambda_{n-1} - 1 \geq 0 , \quad \forall n \geq 1 .$$

The nonnegative number $\gamma_n$ is referred to as the $n$th gap of the spectrum $\text{spec}(L_u)$ of $L_u$, see [15, Appendix C] for an explanation of this terminology. For any $n \geq 1$, the complex Birkhoff coordinate $\zeta_n$ of Theorem 5 is related to $\gamma_n$ by $|\zeta_n|^2 = \gamma_n$ whereas its phase is defined in terms of an appropriately normalized eigenfunction $f_n$ of $L_u$, corresponding to the eigenvalue $\lambda_n$.

A key step for the proof of Theorem 6 is to show that for any $u \in H^{-s}_r$ with $0 < s < 1/2$, the Lax operator $L_u$ can be defined as a self-adjoint operator with domain included in $H^{1-s}_+$ and that its spectrum has properties similar to the ones described above in the case where $u \in L^2_r$. In particular, the inequality (15) continues to hold. Since the proof of Theorem 6 requires several steps, it is split up into two parts, corresponding to Section 3 and Section 4.

A straightforward application of Theorem 6 is the following result on isospectral potentials. To state it, we need to introduce some additional notation. For any $\zeta \in h^{1/2-s}_+$, define

$$\text{Tor}(\zeta) := \{ z \in h^{1/2-s}_+ : |z_n| = |\zeta_n| \ \forall n \geq 1 \}.$$
Note that $\mathrm{Tor}(\zeta)$ is an infinite product of (possibly degenerate) circles and a compact subset of $h_{1/2-s}^+$. Furthermore, for any $u \in H_{r,0}^{-s}$, let

$$\text{Iso}(u) := \{ v \in H_{r,0}^{-s} : \text{spec}(L_v) = \text{spec}(L_u) \}.$$ 

where as above, $\text{spec}(L_u)$ denotes the spectrum of the Lax operator $L_u := -i\partial_x - T_u$. The spectrum of $L_u$ continues to be characterized in terms of its gaps $\gamma_n$, $n \geq 1$, (cf. (15)) and the extended Birkhoff coordinates continue to satisfy $|\zeta_n|^2 = \gamma_n$, $n \geq 1$. An immediate consequence of Theorem 6 then is that [15, Corollary 8.1] extends as follows:

**Corollary 1.** For any $u \in H_{r,0}^{-s}$ with $0 < s < 1/2$,

$$\Phi(\text{Iso}(u)) = \text{Tor}(\Phi(u)).$$

Hence by the continuity of $\Phi^{-1}$, $\text{Iso}(u)$ is a compact, connected subset of $H_{r,0}^{-s}$.

### 3. Extension of $\Phi$. Part 1

In this section we prove the first part of Theorem 6, which we state as a separate result:

**Proposition 1.** (Extension of $\Phi$. Part 1) For any $0 < s < 1/2$, the following holds:

(i) For any $n \geq 1$, the formula in [15, (4.1)] of the Birkhoff coordinate $\zeta_n : L_{r,0}^2 \to \mathbb{C}$ extends to $H_{r,0}^{-s}$ and for any $u \in H_{r,0}^{-s}$, $(\zeta_n(u))_{n \geq 1}$ is in $h_{1/2-s}^-$. The extension of the map $\Phi$ of Theorem 5, also denoted by $\Phi$,

$$\Phi : H_{r,0}^{-s} \to h_{1/2-s}^+, \ u \mapsto \Phi(u) := (\zeta_n(u))_{n \geq 1},$$

maps bounded subsets of $H_{r,0}^{-s}$ to bounded subsets of $h_{1/2-s}^+$.

(ii) $\Phi$ is sequentially weakly continuous and one-to-one.

First we need to establish some auxiliary results related to the Lax operator $L_u$.

**Lemma 1.** Let $u \in H_{r,0}^{-s}$ with $0 \leq s < 1/2$. Then for any $f, g \in H_{1/2}^+$, the following estimates hold:

(i) There exists a constant $C_{1,s} > 0$ only depending on $s$, so that

$$\|fg\|_s \leq C_{1,s}^2 \|f\|_\sigma \|g\|_\sigma, \quad \sigma := (1/2 + s)/2.$$  

(ii) The expression $\langle u | f \rangle$ is well defined and satisfies the estimate

$$|\langle u | f \rangle| \leq \frac{1}{2} \|f\|_{1/2}^2 + \eta_s(\|u\|_{-s}) \|f\|^2$$

where

$$\eta_s(\|u\|_{-s}) := \|u\|_{-s}(2(1 + \|u\|_{-s}))^n C_{2,s}^2,$$

and $C_{2,s} > 0$ is a constant, only depending on $s$. 

Proof. (i) Estimate (17) is obtained from standard estimates of para-
multiplication (cf. e.g. [2, Exercise II.A.5], [6, Theorem 2.82, Theorem
2.85]). (ii) By item (i), $\langle u | f \rangle$ is well defined by duality and satisfies
\[
|\langle u | f \rangle| \leq \|u\|_{-s} \|f\|_s \leq \|u\|_{-s} C^2_{1,s} \|f\|^2_s.
\]
In order to estimate $\|f\|^2_s$, note that by interpolation one has $\|f\|_\sigma \leq \|f\|^{1+\varepsilon}_s \|f\|^{1-\varepsilon}_s$ and hence
\[
C_{1,s} \|f\|_s \leq \|f\|^{1+\varepsilon}_s (C_{2,s} \|f\|)^{1-\varepsilon}_s
\]
for some constant $C_{2,s} > 0$. Young’s inequality then yields for any
$\varepsilon > 0$
\[
(C_{1,s} \|f\|_s)^2 \leq \varepsilon \|f\|^2_{1/2} + \varepsilon^{-\alpha} (C_{2,s} \|f\|)^2,
\]
where $\alpha = \frac{1+2s}{1-2s}$. Estimate (18) then follows from (21) by choosing $\varepsilon = (2(1+\|u\|_{-s}))^{-1}$. □

Note that estimate (18) implies that the sesquilinear form $\langle T_u f | g \rangle$ on
$H^{1/2}_+$, obtained from the Toeplitz operator $T_u f := \Pi(u f)$ with symbol $u \in L^2_{r_0}$, can be defined for any $u \in H^{-s}_{r_0}$ with $0 \leq s < 1/2$ by setting
$\langle T_u f | g \rangle := \langle u | g f \rangle$ and that it is bounded. For any $u \in H^{-s}_{r_0}$, we then
define the sesquilinear form $Q^+_u$ on $H^{1/2}_+$ as follows
\[
Q^+_u(f,g) := \langle -i \partial_x f | g \rangle - \langle T_u f | g \rangle + (1 + \eta_a(\|u\|_{-s})) \langle f | g \rangle
\]
where $\eta_a(\|u\|_{-s})$ is given by (19). The following lemma says that the
quadratic form $Q^+_u(f,f)$ is equivalent to $\|f\|_{1/2}^2$. More precisely, the
following holds.

**Lemma 2.** For any $u \in H^{-s}_{r_0}$ with $0 \leq s < 1/2$, $Q^+_u$ is a positive,

sesquilinear form, satisfying
\[
\frac{1}{2} \|f\|_{1/2}^2 \leq Q^+_u(f,f) \leq (3 + 2\eta_a(\|u\|_{-s})) \|f\|_{1/2}^2,
\]
for all $f \in H^{1/2}_+$. Proof. (i) Using that $u$ is real valued, one verifies that $Q^+_u$ is sesquilin-
eral. The claimed estimates are obtained from (18) as follows: since
$\langle n \rangle \leq 1 + |n|$ one has $\|f\|_{1/2}^2 \leq \langle -i \partial_x f | f \rangle + \|f\|^2$, and hence by (18),
\[
|\langle T_u f | f \rangle| \leq \frac{1}{2} \langle -i \partial_x f | f \rangle + \left(\frac{1}{2} + \eta_a(\|u\|_{-s})\right) \|f\|^2.
\]
By the definition (22), the claimed estimates then follow. In particular,
the lower bound for $Q^+_u(f,f)$ shows that $Q^+_u$ is positive. □

Denote by $\langle f | g \rangle_{1/2} \equiv \langle f | g \rangle_{H^{1/2}_+}$ the inner product, corresponding to
the norm $\|f\|_{1/2}$. It is given by
\[
\langle f | g \rangle_{1/2} = \sum_{n \geq 0} \langle n | \hat{f(n)} \hat{g(n)} \rangle, \quad \forall f, g \in H^{1/2}_+.
\]
Furthermore, denote by $D : H^1_+ \to H^{-1}_+$ and $\langle D \rangle : H^1_+ \to H^{-1}_+$, $t \in \mathbb{R}$, the Fourier multipliers, defined for $f \in H^1_+$ with Fourier series $f = \sum_{n=0}^{\infty} \hat{f}(n)e^{inx}$ by

$$Df := -i\partial_x f = \sum_{n=0}^{\infty} n\hat{f}(n)e^{inx}, \quad \langle D \rangle f := \sum_{n=0}^{\infty} \langle n \hat{f}(n) e^{inx} \rangle.$$

**Lemma 3.** For any $u \in H^s_{r,0}$ with $0 \leq s < 1/2$, there exists a bounded linear isomorphism $A_u : H^1_+ \to H^1_+$ so that

$$\langle A_u f | g \rangle_{1/2} = Q^+_u(f,g), \quad \forall f, g \in H^1_+.$$

The operator $A_u$ has the following properties:

(i) $A_u$ and its inverse $A_u^{-1}$ are symmetric, i.e., for any $f, g \in H^1_+$,

$$\langle A_u f | g \rangle_{1/2} = \langle f | A_u g \rangle_{1/2}, \quad \langle A_u^{-1} f | g \rangle_{1/2} = \langle f | A_u^{-1} g \rangle_{1/2}.$$

(ii) The linear isomorphism $B_u$, given by the composition

$$B_u := \langle D \rangle A_u : H^1_+ \to H^1_+$$

satisfies

$$Q^+_u(f,g) = \langle B_u f | g \rangle, \quad \forall f, g \in H^1_+.$$

The operator norm of $B_u$ and the one of its inverse can be bounded uniformly on bounded subsets of elements $u$ in $H^s_{r,0}$.

**Proof.** By Lemma 2, the sesquilinear form $Q^+_u$ is an inner product on $H^1_+$, equivalent to the inner product $\langle \cdot | \cdot \rangle_{1/2}$. Hence by the theorem of Fréchet-Riesz, for any $g \in H^1_+$, there exists a unique element in $H^1_+$, which we denote by $A_u g$, so that

$$\langle A_u g | f \rangle_{1/2} = Q^+_u(g, f), \quad \forall f \in H^1_+.$$

Invitation for special issue in honor of Tony Bloch in Journal of Geometric Mechanics Then $A_u : H^1_+ \to H^1_+$ is a linear, injective operator, which by Lemma 2 is bounded, i.e., for any $f, g \in H^1_+$,

$$| \langle A_u g | f \rangle_{1/2} | = | Q^+_u(g, f) | \leq Q^+_u(g, g)^{1/2} Q^+_u(f, f)^{1/2} \leq (3 + 2\eta_u(\| u \|_{-1})) \| g \|_{1/2} \| f \|_{1/2},$$

implying that $\| A_u g \|_{1/2} \leq (3 + 2\eta_u(\| u \|_{-1})) \| g \|_{1/2}$.

Similarly, by the theorem of Fréchet-Riesz, for any $h \in H^1_+$, there exists a unique element in $H^1_+$, which we denote by $E_u h$, so that

$$\langle h | f \rangle_{1/2} = Q^+_u(E_u h, f), \quad \forall f \in H^1_+.$$

Then $E_u : H^1_+ \to H^1_+$ is a linear, injective operator, which by Lemma 2 is bounded, i.e.,

$$\frac{1}{2} \| E_u h \|_{1/2}^2 \leq Q^+_u(E_u h, E_u h) = \langle h | E_u h \rangle_{1/2} \leq \| h \|_{1/2} \| E_u h \|_{1/2}.$$
implying that \( \|E_u h\|_{1/2} \leq 2\|h\|_{1/2} \). Note that \( A_u(E_u h) = h \) and hence \( E_u \) is the inverse of \( A_u \). Therefore, \( A_u : H^{1/2}_+ \rightarrow H^{1/2}_+ \) is a bounded linear isomorphism. Next we show item (i). For any \( f, g \in H^{1/2}_+ \),
\[
\langle g | A_u f \rangle_{1/2} = \langle (A_u f) | g \rangle_{1/2} = Q^+_u(f, g) = Q^+_u(g, f) = \langle A_u g | f \rangle_{1/2}.
\]
The symmetry of \( A_u^{-1} \) is proved in the same way. Towards item (ii), note that for any \( f, g \in H^{1/2}_+ \), \( \langle f | g \rangle_{1/2} = \langle (D) f | g \rangle \) and therefore
\[
\langle A_u g | f \rangle_{1/2} = \langle (D) A_u g | f \rangle,
\]
implying that the operator \( B_u = (D) A_u : H^{1/2}_+ \rightarrow H^{-1/2}_+ \) is a bounded linear isomorphism and that
\[
\langle B_u g | f \rangle = Q^+_u(g, f), \quad \forall g, f \in H^{1/2}_+.
\]
The last statement of (ii) follows from Lemma 2.

We denote by \( L^+_u \) the restriction of \( B_u \) to \( \text{dom}(L^+_u) \), defined as
\[
\text{dom}(L^+_u) := \{ g \in H^{1/2}_+ : B_u g \in H_+ \}.
\]
We view \( L^+_u \) as an unbounded linear operator on \( H_+ \) and write \( L^+_u : \text{dom}(L^+_u) \rightarrow H_+ \).

**Lemma 4.** For any \( u \in H^{s}_+ \) with \( 0 \leq s < 1/2 \), the following holds:

(i) \( \text{dom}(L^+_u) \) is a dense subspace of \( H^{1/2}_+ \) and hence of \( H_+ \).

(ii) \( L^+_u : \text{dom}(L^+_u) \rightarrow H_+ \) is bijective and the right inverse of \( L^+_u \), \( (L^+_u)^{-1} : H_+ \rightarrow H_+ \), is compact. Hence \( L^+_u \) has discrete spectrum.

(iii) \( (L^+_u)^{-1} \) is symmetric and \( L^+_u \) is self-adjoint and positive.

**Proof.** (i) Since \( H_+ \) is a dense subspace of \( H^{1/2}_+ \) and \( B_u^{-1} : H^{1/2}_+ \rightarrow H^{1/2}_+ \) is a linear isomorphism, \( \text{dom}(L^+_u) = B_u^{-1}(H_+) \) is a dense subspace of \( H^{1/2}_+ \), and hence also of \( H_+ \).

(ii) Since \( L^+_u \) is the restriction of the linear isomorphism \( B_u \), it is one-to-one. By the definition of \( L^+_u \), it is onto. The right inverse of \( L^+_u \), denoted by \( (L^+_u)^{-1} \), is given by the composition \( \iota \circ B_u^{-1} | H_+ \), where \( \iota : H^{1/2}_+ \rightarrow H_+ \) is the standard embedding which by Sobolev’s embedding theorem is compact. It then follows that \( (L^+_u)^{-1} : H_+ \rightarrow H_+ \) is compact as well.

(iii) For any \( f, g \in H_+ \)
\[
\langle (L^+_u)^{-1} f | g \rangle = \langle A_u^{-1} | D \rangle^{-1} f | g \rangle = \langle A_u^{-1} | D \rangle^{-1} f | (D) g \rangle_{1/2}.
\]
By Lemma 3, \( A_u^{-1} \) is symmetric with respect to the \( H^{1/2}_+ \)-inner product. Hence
\[
\langle (L^+_u)^{-1} f | g \rangle = \langle (D) f | A_u^{-1} | D \rangle^{-1} g \rangle_{1/2} = \langle f | (L^+_u)^{-1} g \rangle,
\]
showing that \( (L^+_u)^{-1} \) is symmetric. Since in addition, \( (L^+_u)^{-1} \) is bounded it is also self-adjoint. By Lemma 2 it then follows that
\[
\langle L^+_u f | f \rangle = \langle (D) A_u f | f \rangle_{1/2} = \langle A_u f | f \rangle_{1/2} = Q^+_u(f, f) \geq \frac{1}{2} \|f\|_{1/2}^2,
\]
implying that $L_u^+$ is a positive operator.

We now define for any $u \in H^{−s}_{r,0}$ with $0 \leq s < 1/2$, the operator $L_u$ as a linear operator with domain $\text{dom}(L_u) := \text{dom}(L_u^+)$ by setting

$$L_u := L_u^+ - (1 + \eta_s(\|u\|_s)) : \text{dom}(L_u) \rightarrow H_+.$$  

Lemma 4 yields the following

**Corollary 2.** For any $u \in H^{−s}_{r,0}$ with $0 \leq s < 1/2$, the operator $L_u : \text{dom}(L_u) \rightarrow H_+$ is densely defined, self-adjoint, bounded from below, and has discrete spectrum. It thus admits an $L^2$–normalized basis of eigenfunctions, contained in $\text{dom}(L_u)$ and hence in $H^{1/2}_+$. 

**Remark 8.** Let $u \in H^{−s}_{r,0}$ with $0 \leq s < 1/2$ be given. Since $\text{dom}(L_u^+)$ is dense in $H^{1/2}_+$ and $L_u^+$ is the restriction of $B_u : H^{1/2}_+ \rightarrow H^{1/2}_+$ to $\text{dom}(L_u^+)$, the symmetry

$$\langle L_u^+ f | g \rangle = \langle f | L_u^+ g \rangle, \quad \forall f, g \in \text{dom}(L_u^+)$$

can be extended by a straightforward density argument as follows

$$\langle B_u f | g \rangle = \langle f | B_u g \rangle, \quad \forall f, g \in H^{1/2}_+.$$ 

Note that for any $f, g \in H^{1/2}_+$, $\langle B_u f | g \rangle = \langle (D)A_u f | g \rangle = \langle A_u f | g \rangle_{1/2}$ and hence by (22),

$$\langle B_u f | g \rangle = Q_u^+(f, g) = \langle Df - T_u f + (1 + \eta_s(\|u\|_{−s}))f | g \rangle,$$

yielding the following identity in $H^{−1/2}_+$,

$$B_u f = Df - T_u f + (1 + \eta_s(\|u\|_{−s}))f, \quad \forall f \in H^{1/2}_+.$$ 

Given $u \in H^{−s}_{r,0}$ with $0 \leq s < 1/2$, let us consider the restriction of $B_u$ to $H^{−s}_+$. 

**Lemma 5.** Let $u \in H^{−s}_{r,0}$ with $0 \leq s < 1/2$. Then $B_u(H^{1−s}_+) = H^{−s}_+$ and the restriction $B_{u|H^{−s}} := B_u|H^{−s}_+ : H^{1−s}_+ \rightarrow H^{−s}_+$ is a linear isomorphism. The operator norm of $B_{u|H^{−s}}$ and the one of its inverse are bounded uniformly on bounded subsets of elements $u \in H^{−s}_{r,0}$.

**Proof.** Since $1 − s > 1/2$, $H^{1−s}$ acts by multiplication on itself and on $L^2$, hence by interpolation on $H^r$ for $0 \leq r < 1 − s$. By duality, it also acts on $H^{−r}$, in particular with $r = s$. This implies that $B_{u|H^{1−s}} : H^{1−s}_+ \rightarrow H^{−s}_+$ is bounded. Being the restriction of an injective operator, it is injective as well. Let us prove that $B_{u|H^{1−s}}$ has $H^{−s}_+$ as its image.

To this end consider an arbitrary element $h \in H^{−s}_+$. We need to show that the solution $f \in H^{1/2}_+$ of $B_u f = h$ is actually in $H^{1−s}_+$. Write

$$Df = h + (1 + \eta_s(\|u\|_{−s}))f + T_u f.$$
Note that $h + (1 + \eta(s)(\|u\|_{\sigma})/f$ is in $H_+^{-s}$ and it remains to study $T_u f$. By Lemma 1(i) one infers that for any $g \in H_+^s$, with $\sigma = (1/2 + s)/2$, $$\langle T_u f | g \rangle = \langle u | g \overline{f} \rangle \leq \|u\|_{-s} \|g\|_s \leq C^2 \|u\|_{-s} \|g\|_s \|f\|_\sigma,$$

implying that $T_u f \in H_+^{-\sigma}$ and hence by (24), $f \in H_+^{1-\sigma}$. Since $1 - \sigma > 1/2$, we argue as at the beginning of the proof to infer that $T_u f \in H_+^{-s}$. Thus applying (24) once more we conclude that $f \in H_+^{-s}$. This shows that $B_u|_{H_+^{-s}} : H_+^{-s} \to H_+^{-s}$ is onto. Going through the arguments of the proof one verifies that the operator norm of $B_u|_{H_+^{-s}}$ and the one of its inverse are bounded uniformly on bounded subsets of elements $u \in H_0^{-s}$. This completes the proof of the lemma.

Lemma 5 has the following important

Corollary 3. For any $u \in H_0^{-s}$ with $0 \leq s < 1/2$, $\text{dom}(L_u^+ \subset H_+^{1-s}$. In particular, any eigenfunction of $L_u^+$ (and hence of $L_u$) is in $H_+^{1-s}$.

Proof. Since $H_+ \subset H_+^s$, one has $B_u^{-1}(H_+) \subset B_u^{-1}(H_+^s)$ and hence by Lemma 5, $\text{dom}(L_u^+) = B_u^{-1}(H_+) \subset H_+^{1-s}$.

With the results obtained so far, it is straightforward to verify that many of the results of [15] extend to the case where $u \in H_0^{-s}$. More precisely, let $u \in H_0^{-s}$ with $0 \leq s < 1/2$. We already know that the spectrum of $L_u$ is discrete, bounded from below, and real. When listed in increasing order and with their multiplicities, the eigenvalues of $L_u$ satisfy $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$. Arguing as in the proof of [15, Proposition 2.1], one verifies that $\lambda_n \geq \lambda_{n-1} + 1$, $n \geq 1$, and following [15, (2.10)] we define

$$\gamma_n(u) := \lambda_n - \lambda_{n-1} - 1 \geq 0.$$  

It then follows that for any $n \geq 1$, 

$$\lambda_n = n + \lambda_0 + \sum_{k=1}^n \gamma_k \geq n + \lambda_0.$$  

Since [15, Lemma 2.1, Lemma 2.2] continue to hold for $u \in H_0^{-s}$, we can introduce eigenfunctions $f_n(x, u)$ of $L_u$ corresponding to the eigenvalues $\lambda_n$, which are normalized as in [15, Definition 2.1]. The identities [15, (2.13)] continue to hold,

$$\lambda_n \langle 1 | f_n \rangle = -\langle u | f_n \rangle$$  

as does [15, Lemma 2.4], stating that for any $n \geq 1$

$$\gamma_n = 0 \quad \text{if and only if} \quad \langle 1 | f_n \rangle = 0.$$  

Furthermore, the definition [15, (3.1)] of the generating function $\mathcal{H}_\lambda(u)$ extends to the case where $u \in H_0^{-s}$ with $0 < s < 1/2$,

$$\mathcal{H}_\lambda : H_0^{-s} \to \mathbb{C}, \ u \mapsto \langle (L_u + \lambda)^{-1} | 1 \rangle.$$
and so do the identity [15, (3.2)], the product representation of \( H_\lambda(u) \), stated in [15, Proposition 3.1(i)],

\[
H_\lambda(u) = \frac{1}{\lambda_0 + \lambda} \prod_{n=1}^{\infty} \left(1 - \frac{\gamma_n}{\lambda_n + \lambda}\right),
\]

and the one for \(|\langle 1|f_n\rangle|^2\), \(n \geq 1\), given in [15, Corollary 3.1],

\[
|\langle 1|f_n\rangle|^2 = \gamma_n \kappa_n, \quad \kappa_n = \frac{1}{\lambda_n - \lambda_0} \prod_{p \neq n} \left(1 - \frac{\gamma_p}{\lambda_p - \lambda_n}\right).
\]

The product representation (28) then yields the identity (cf. [15, Proposition 3.1(ii)] and its proof),

\[
-\lambda_0(u) = \sum_{n=1}^{\infty} \gamma_n(u).
\]

Since \(\gamma_n(u) \geq 0\) for any \(n \geq 1\), one infers that for any \(u \in H_{r,0}^s\) with \(0 \leq s < 1/2\), the sequence \((\gamma_n(u))_{n \geq 1}\) is in \(\ell_1^1 \equiv \ell^1(\mathbb{N}, \mathbb{R})\) and

\[
\lambda_n(u) = n - \sum_{k=n+1}^{\infty} \gamma_k(u) \leq n.
\]

By (22), Lemma 2, and (23), we infer that \(-\lambda_0 \leq \frac{1}{2} + \eta_s(\|u\|_s),\) yielding, when combined with (30) and (31), the estimate

\[
n - \frac{1}{2} - \eta_s(\|u\|_s) \leq \lambda_n(u) \leq n, \quad \forall n \geq 0.
\]

In a next step we want consider the linear isomorphism

\[
B_{u;1-s} = B_u|_{H_{r}^{1-s}} : H_{r}^{1-s} \rightarrow H_{r}^{-s}
\]

on the scale of Sobolev spaces. By duality, \(B_{u;1-s}\) extends as a bounded linear isomorphism, \(B_{u,s} : H_{r}^{1} \rightarrow H_{r}^{1+s}\) and hence by complex interpolation, for any \(s \leq t \leq 1 - s\), the restriction of \(B_{u,s}\) to \(H_{r}^{1}\) gives also rise to a bounded linear isomorphism, \(B_{u;t} : H_{r}^{1} \rightarrow H_{r}^{1-t}\). All these operators satisfy the same bound as \(B_{u;1-s}\) (cf. Lemma 5). To state our next result, it is convenient to introduce the notation \(N_0 := \mathbb{Z}_{\geq 0}\). Recall that \(h^t(N_0) = h^t(N_0, \mathbb{C}), t \in \mathbb{R}\), and that we write \(\ell^2(N_0)\) instead of \(h^0(N_0)\).

**Lemma 6.** Let \(u \in H_{r,0}^s\) with \(0 \leq s < 1/2\) and let \((f_n)_{n \geq 0}\) be the basis of \(L_{r,0}^2\), consisting of eigenfunctions of \(L_u\) with \(f_n, n \geq 0\), corresponding to the eigenvalue \(\lambda_n\) and normalized as in [15, Definition 2.1]. Then for any \(-1 + s \leq t \leq 1 - s,\)

\[
K_{u;t} : H_{r}^{1} \rightarrow h^{t}(N_0), \quad f \mapsto (\langle f|f_n\rangle)_{n \geq 0}
\]

is a linear isomorphism. In particular, for \(f = \Pi u \in H_{r}^{-s}\), one obtains that \((\Pi u|f_n\rangle)_{n \geq 0} \in h^{-s}(N_0)\). The operator norm of \(K_{u;t}\) and the one
of its inverse can be uniformly bounded for $-1 + s \leq t \leq 1 - s$ and for $u$ in a bounded subset of $H_{r,0}^{-s}$.

**Proof.** We claim that the sequence $(\tilde{f}_n)_{n \geq 0}$, defined by

$$\tilde{f}_n = \frac{f_n}{(\lambda_n + 1 + \eta_u(||u||_{-s}))^{1/2}},$$

is an orthonormal basis of the Hilbert space $H_{+}^{1/2}$, endowed with the inner product $Q^+_u$. Indeed, for any $n \geq 0$ and any $g \in H_{+}^{1/2}$, one has

$$Q^+_u(\tilde{f}_n, g) = \langle L^+_u \tilde{f}_n | g \rangle = (\lambda_n + 1 + \eta_u(||u||_{-s}))^{1/2}(f_n | g \rangle .$$

As a consequence, for any $n, m \geq 0$, $Q^+_u(\tilde{f}_n, \tilde{f}_m) = \delta_{nm}$ and the orthogonal complement of the subspace of $H_{+}^{1/2}$, spanned by $(\tilde{f}_n)_{n \geq 0}$, is the trivial vector space $\{0\}$, showing that $(\tilde{f}_n)_{n \geq 0}$ is an orthonormal basis of $H_{+}^{1/2}$. In view of (32), we then conclude that

$$K_{u,t} : H_{+}^{1/2} \to h^{1/2}(N_0), f \mapsto (\langle f | f_n \rangle)_{n \geq 0}$$

is a linear isomorphism. Its inverse is given by

$$K_{u,t}^{-1} : h^{1/2}(N_0) \to H_{+}^{1/2}, (z_n)_{n \geq 0} \mapsto f := \sum_{n=0}^{\infty} z_n f_n.$$

By interpolation we infer that for any $0 \leq t \leq 1/2$, $K_{u,t} : H_{+}^{t} \to h^{t}(N_0)$ is a linear isomorphism. Taking the transpose of $K_{u,t}$ it then follows that for any $0 \leq t \leq 1/2$,

$$K_{u,-t} : H_{+}^{-t} \to h^{-t}(N_0), f \mapsto (\langle f | f_n \rangle)_{n \geq 0},$$

is also a linear isomorphism. It remains to discuss the remaining range of $t$, stated in the lemma. By Lemma 5, the restriction of $B_{u}^{-1}$ to $H_{+}^{-s}$ gives rise to a linear isomorphism $B_{u,1-s}^{-1} : H_{+}^{-s} \to H_{+}^{1-s}$. For any $f \in H_{+}^{-s}$, one then has

$$B_{u,1-s}^{-1}f = \sum_{n=0}^{\infty} \frac{\langle f | f_n \rangle}{(\lambda_n + 1 + \eta_u(||u||_{-s}))^{1/2}} f_n.$$

Since by our considerations above, $(\langle f | f_n \rangle)_{n \geq 0} \in h^{-s}(N_0)$ one concludes that the sequence $(\frac{\langle f | f_n \rangle}{(\lambda_n + 1 + \eta_u(||u||_{-s}))^{1/2}})_{n \geq 0}$ is in $h^{1-s}(N_0)$. Conversely, assume that $(z_n)_{n \geq 0} \in h^{1-s}(N_0)$. Then $(\langle \lambda_n + 1 + \eta_u(||u||_{-s}) z_n \rangle)_{n \geq 0}$ is in $h^{-s}(N_0)$. Hence by the considerations above on $K_{u,-s}$, there exists $g \in H_{+}^{-s}$ so that

$$\langle g | f_n \rangle = (\lambda_n + 1 + \eta_u(||u||_{-s})) z_n, \quad \forall n \geq 0.$$

Hence

$$g = \sum_{n=0}^{\infty} z_n (\lambda_n + 1 + \eta_u(||u||_{-s})) f_n = \sum_{n=0}^{\infty} z_n B_{u} f_n.$$
and \( f := B^{-1}g \) is in \( H^{1-s}_+ \) and satisfies \( f = \sum_{n=0}^{\infty} z_n f_n \). Altogether we have thus proved that
\[
K_{u;1-s} : H^{1-s}_+ \rightarrow h^{1-s}(\mathbb{N}_0), \quad f \mapsto (\langle f | f_n \rangle)_{n \geq 0},
\]
is a linear isomorphism. Interpolating between \( K_{u;-s} \) and \( K_{u;1-s} \) and between the adjoints of their inverses shows that for any \(-1 + s \leq t \leq 1 - s\),
\[
K_{u;t} : H^{t} \rightarrow h^{t}(\mathbb{N}_0), \quad f \mapsto (\langle f | f_n \rangle)_{n \geq 0}
\]
is a linear isomorphism. Going through the arguments of the proof one verifies that the operator norm of \( K_{u;t} \) and the one of its inverse can be uniformly bounded for \(-1 + s \leq t \leq 1 - s\) and for bounded subsets of elements \( u \in H^{-s}_r \).

With these preparations done, we can now prove Proposition 1(i).

**Proof of Proposition 1(i).** Let \( u \in H^{-s}_r \) with \( 0 \leq s < 1/2 \). By (29), one has for any \( n \geq 1 \),
\[
|\langle 1 | f_n \rangle|^2 = \gamma_n \kappa_n, \quad \kappa_n = \frac{1}{\lambda_n - \lambda_0} \prod_{p \neq n} (1 - \frac{\gamma_p}{\lambda_p - \lambda_n}).
\]
Note that the infinite product is absolutely convergent since the sequence \((\gamma_n(u))_{n \geq 1}\) is in \( \ell^{1+}_+ \) (cf. (30)). Furthermore, since
\[
1 - \frac{\gamma_p}{\lambda_p - \lambda_n} = \frac{\lambda_{p-1} + 1 - \lambda_n}{\lambda_p - \lambda_n} > 0, \quad \forall p \neq n
\]
it follows that \( \kappa_n > 0 \) for any \( n \geq 1 \). Hence, the formula \([15, (4.1)]\) of the Birkhoff coordinates \( \zeta_n(u) \), \( n \geq 1 \), defined for \( u \in L^2_{r,0} \),
\[
(33) \quad \zeta_n(u) = \frac{1}{\sqrt{\kappa_n(u)}} \langle 1 | f_n (\cdot, u) \rangle,
\]
extends to \( H^{-s}_r \). By (25) one has (cf. also \([15, (2.13)]\))
\[
\lambda_n (1 | f_n) = -\langle u | f_n \rangle = -\langle \Pi u | f_n \rangle.
\]
Since by Lemma 6, \((\langle \Pi u | f_n \rangle)_{n \geq 0} \in h^{-s}(\mathbb{N}_0)\) and by (32)
\[
n - \frac{1}{2} - \eta_n(\|u\|_{-s}) \leq \lambda_n(u) \leq n, \quad \forall n \geq 0,
\]
one concludes that
\[
(\langle 1 | f_n \rangle)_{n \geq 1} \in h^{1-s}_+, \quad \kappa_n^{-1/2} = \sqrt{n} + o(1)
\]
and hence \((\zeta_n(u))_{n \geq 1} \in h^{1/2-s}_+\). In summary, we have proved that for any \( 0 < s < 1/2 \), the Birkhoff map \( \Phi : L^2_{r,0} \rightarrow h^{1/2}_+ \) of Theorem 5 extends to a map
\[
H^{-s}_r \rightarrow h^{1/2-s}_+, \quad u \mapsto (\zeta_n(u))_{n \geq 1},
\]
which we again denote by $\Phi$. Going through the arguments of the proof one verifies that $\Phi$ maps bounded subsets of $H_{r,0}^{-s}$ into bounded subsets of $h_+^{1/2-s}$.

To show Proposition 1(ii) we first need to prove some additional auxiliary results. By (27), the generating function is defined as
\[ \mathcal{H}_\lambda : H_{r,0}^{-s} \to \mathbb{C}, \ u \mapsto ((L_u + \lambda)^{-1})^{-1}(1) \, . \]
For any given $u \in H_{r,0}^{-s}$, $\mathcal{H}_\lambda(u)$ is a meromorphic function in $\lambda \in \mathbb{C}$ with possible poles at the eigenvalues of $L_u$ and satisfies (cf. (28))
\[ \mathcal{H}_\lambda(u) = \frac{1}{\lambda_0 + \lambda} \prod_{n=1}^{\infty} \left( 1 - \frac{\gamma_n}{\lambda_n + \lambda} \right) \, . \]

**Lemma 7.** For any $0 \leq s < 1/2$, the following holds:

(i) For any $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $\mathcal{H}_\lambda : H_{r,0}^{-s} \to \mathbb{C}$ is sequentially weakly continuous.

(ii) $(\sqrt[n]{n})_{n \geq 1} : H_{r,0}^{-s} \to h_+^{1/2-s}$ is sequentially weakly continuous. In particular, for any $n \geq 0$, $\lambda_n : H_{r,0}^{-s} \to \mathbb{R}$ is sequentially weakly continuous.

**Proof.** (i) Let $(u^{(k)})_{k \geq 1}$ be a sequence in $H_{r,0}^{-s}$ with $u^{(k)} \to u$ weakly in $H_{r,0}^{-s}$ as $k \to \infty$. By the definition of $\zeta_n(u)$ (cf. (29) – (33)) one has $|\zeta_n(u)|^2 = \gamma_n(u)$. Since by Proposition 1(i), $\Phi$ maps bounded subsets of $H_{r,0}^{-s}$ to bounded subsets of $h_+^{1/2-s}$, there exists $M > 0$ so that for any $k \geq 1$
\[ \|u\|, \|u^{(k)}\| \leq M, \ \sum_{n=1}^{\infty} n^{1-2s} \gamma_n(u), \sum_{n=1}^{\infty} n^{1-2s} \gamma_n(u^{(k)}) \leq M \, . \]

By passing to a subsequence, if needed, we may assume that
\[ (\gamma_n(u^{(k)})^{1/2})_{n \geq 1} \to (\rho_n^{1/2})_{n \geq 1} \]
weakly in $h^{1/2-s}(\mathbb{N}, \mathbb{R})$ where $\rho_n \geq 0$ for any $n \geq 1$. It then follows that $(\gamma_n(u^{(k)}))_{n \geq 1} \to (\rho_n)_{n \geq 1}$ strongly in $\ell^1(\mathbb{N}, \mathbb{R})$. Define
\[ \nu_n := n - \sum_{p=0}^{\infty} \rho_p, \ \forall n \geq 0 \, . \]

Then for any $n \geq 1$, $\nu_n = \nu_{n-1} + 1 + \rho_n$ and $\lambda_n(u^{(k)}) \to \nu_n$ uniformly in $n \geq 0$. Since $L_{u^{(k)}} \geq \lambda_0(u^{(k)})$ we infer that there exists $c > | - \nu_0 + 1|$ so that for any $k \geq 1$ and $\lambda \geq c$,
\[ L_{u^{(k)}} + \lambda : H_+^{1-s} \to H_+^{-s} \]
is a linear isomorphism whose inverse is bounded uniformly in $k$. Therefore
\[ w_{\lambda}^{(k)} := (L_{u^{(k)}} + \lambda)^{-1}[1], \ \forall k \geq 1 \, , \]
is a well-defined, bounded sequence in $H_+^{1-s}$. Let us choose an arbitrary countable subset $\Lambda \subset [c, \infty)$ with one cluster point. By a diagonal procedure, we extract a subsequence of $(w^{(k)}_\lambda)_{k \geq 1}$, again denoted by $(w^{(k)}_\lambda)_{k \geq 1}$, so that for every $\lambda \in \Lambda$, the sequence $(w^{(k)}_\lambda)$ converges weakly in $H_+^{1-s}$ to some element $v_\lambda \in H_+^{1-s}$. By Rellich’s theorem

$$
(L_{u^{(k)}} + \lambda)w^{(k)}_\lambda \rightharpoonup (L_u + \lambda)v_\lambda
$$

weakly in $H_+^{1-s}$ as $k \to \infty$. Since by definition, $(L_{u^{(k)}} + \lambda)w^{(k)}_\lambda = 1$ for any $k \geq 1$, it follows that for any $\lambda \in \Lambda$, $(L_u + \lambda)v_\lambda = 1$ and thus by the definition of the generating function

$$
\mathcal{H}_\lambda(u^{(k)}) = \langle w^{(k)}_\lambda|1 \rangle \to \langle v_\lambda|1 \rangle = \mathcal{H}_\lambda(u), \quad \forall \lambda \in \Lambda.
$$

Since $\mathcal{H}_\lambda(u^{(k)})$ and $\mathcal{H}_\lambda(u)$ are meromorphic functions whose poles are on the real axis, it follows that the convergence holds for any $\lambda \in \mathbb{C}\setminus\mathbb{R}$. This proves item (i).

(ii) We apply item (i) (and its proof) as follows. As mentioned above, $\lambda_n(u^{(k)}) \to \rho_n$, uniformly in $n \geq 0$. By the proof of item (i) one has for any $c \leq \lambda < \infty$,

$$
\mathcal{H}_\lambda(u^{(k)}) \to \frac{1}{\nu_0 + \lambda} \prod_{n=1}^{\infty} \left(1 - \frac{\rho_n}{\nu_n + \lambda}\right)
$$

and we conclude that for any $\lambda \in \Lambda$

$$
\frac{1}{\lambda_0(u) + \lambda} \prod_{n=1}^{\infty} \left(1 - \frac{\gamma_n(u)}{\lambda_n(u) + \lambda}\right) = \mathcal{H}_\lambda(u) = \frac{1}{\nu_0 + \lambda} \prod_{n=1}^{\infty} \left(1 - \frac{\rho_n}{\nu_n + \lambda}\right).
$$

Since $\mathcal{H}_\lambda(u)$ and infinite product are meromorphic functions in $\lambda$, the functions are equal. In particular, they have the same zeroes and the same poles. Since the sequences $(\lambda_n(u))_{n \geq 0}$ and $(\nu_n(u))_{n \geq 0}$ are both listed in increasing order it follows that $\lambda_n(u) = \nu_n$ for any $n \geq 0$, implying that for any $n \geq 1$,

$$
\gamma_n(u) = \lambda_n(u) - \lambda_{n-1}(u) - 1 = \nu_n - \nu_{n-1} - 1 = \rho_n.
$$

By (35) we then conclude that

$$
(\gamma_n(u^{(k)})^{1/2})_{n \geq 1} \rightharpoonup (\gamma_n(u)^{1/2})_{n \geq 1}
$$

weakly in $H^{1/2-s}(\mathbb{N}, \mathbb{R})$. \hfill \square

**Corollary 4.** For any $0 \leq s < 1/2$ and $n \geq 1$, the functional $\kappa_n : H^{-s}_{r,0} \to \mathbb{R}$, introduced in (29), is sequentially weakly continuous.

**Proof.** Let $(u^{(k)})_{k \geq 1}$ be a sequence in $H^{-s}_{r,0}$ with $u^{(k)} \rightharpoonup u$ weakly in $H^{-s}_{r,0}$ as $k \to \infty$. By (31), one has for any $p < n$,

$$
\lambda_p(u^{(k)}) - \lambda_n(u^{(k)}) = p - n - \sum_{j=p+1}^{n} \gamma_j(u^{(k)})
$$
whereas for \( p > n \)

\[
\lambda_p(u^{(k)}) - \lambda_n(u^{(k)}) = p - n + \sum_{j=n+1}^{p} \gamma_j(u^{(k)}).
\]

By Lemma 7, one then concludes that

\[
\lim_{k \to \infty} (\lambda_p(u^{(k)}) - \lambda_n(u^{(k)})) - (\lambda_p(u) - \lambda_n(u)) = 0
\]

uniformly in \( p, n \geq 0 \). By the product formula (29) for \( \kappa_n \), it then follows that for any \( n \geq 1 \), one has \( \lim_{k \to \infty} \kappa_n(u^{(k)}) = \kappa_n(u) \). \( \square \)

Furthermore, we need to prove the following lemma concerning the eigenfunctions \( f_n(\cdot, u) \), \( n \geq 0 \), of \( L_u \).

**Lemma 8.** Given \( 0 \leq s < 1/2 \), \( M > 0 \) and \( n \geq 0 \), there exists a constant \( C_{s,M,n} \geq 1 \) so that for any \( u \in H^{-s}_r \) with \( \|u\|_{-s} \leq M \) and any \( n \geq 0 \)

\[
\|f_n(\cdot, u)\|_{1-s} \leq C_{s,M,n}.
\]

**Proof.** By the normalisation of \( f_n \), \( \|f_n\| = 1 \). Since \( f_n \) is an eigenfunction, corresponding to the eigenvalue \( \lambda_n \), one has

\[
-i\partial_x f_n = L_u f_n + T_u f_n = \lambda_n f_n + T_u f_n,
\]

implying that

\[
\|\partial_x f_n\|_{-s} \leq |\lambda_n| + \|T_u f_n\|_{-s}.
\]

Note that by the estimates (32),

\[
|\lambda_n| \leq \max\{n, |\lambda_0|\} \leq n + |\lambda_0|, \quad |\lambda_0| \leq 1 + \eta_s(\|u\|_{-s})
\]

where \( \eta_s(\|u\|_{-s}) \) is given by (19). Furthermore, since \( \sigma = (1/2 + s)/2 \) (cf. (17)) one has \( 1 - s > 1 - \sigma > 1/2 \), implying that \( H_+^{1-\sigma} \) acts on \( H^t_+ \) for every \( t \) in the open interval \( -(1-\sigma), 1-\sigma \). Hence

\[
\|T_u f_n\|_{-s} \leq C_s \|u\|_{-s} \|f_n\|_{1-\sigma}.
\]

Using interpolation and Young’s inequality (cf. (20), (21)), (39) yields an estimate, which together with (37) and (38) leads to the claimed estimate (36). \( \square \)

With these preparations done, we can now prove Proposition 1(ii).

**Proof of Proposition 1(ii).** First we prove that for any \( 0 \leq s < 1/2 \), \( \Phi: H^{-s}_{r,0} \to h^{1/2-s}_+ \) is sequentially weakly continuous: assume that \( (u^{(k)})_{k \geq 1} \) is a sequence in \( H^{-s}_{r,0} \) with \( u^{(k)} \rightharpoonup u \) weakly in \( H^{-s}_{r,0} \) as \( k \to \infty \). Let \( \zeta^{(k)} := \Phi(u^{(k)}) \) and \( \zeta := \Phi(u) \). Since \( (u^{(k)})_{k \geq 1} \) is bounded in \( H^{-s}_{r,0} \) and \( \Phi \) maps bounded subsets of \( H^{-s}_{r,0} \) to bounded subsets of \( h^{1/2-s}_+ \), the sequence \( (\zeta^{(k)})_{k \geq 1} \) is bounded in \( h^{1/2-s}_+ \). To show that \( \zeta^{(k)} \rightharpoonup \zeta \) weakly in \( h^{1/2}_+ \), it then suffices to prove that for any \( n \geq 1 \), \( \lim_{k \to \infty} \kappa_n(\zeta^{(k)}) = \kappa_n \). By the definition of the Birkhoff coordinates (33),
\[ \zeta_n^{(k)} = \langle 1 | f_n^{(k)} \rangle / (\kappa_n^{(k)})^{1/2} \] where \( \kappa_n^{(k)} := \kappa_n(u^{(k)}) \) and \( f_n^{(k)} := f_n(\cdot, u^{(k)}) \).

By Corollary 4, \( \lim_{k \to \infty} \kappa_n^{(k)} = \kappa_n \) and by Lemma 8, saying that for any \( n \geq 0 \), \( \|f_n\|_{1-s} \) is uniformly bounded on bounded subsets of \( H_{r,0}^{-s} \), \( \lim_{k \to \infty} \langle 1 | f_n^{(k)} \rangle = \langle 1 | f_n \rangle \) where \( \kappa_n := \kappa_n(u) \) and \( f_n := f_n(\cdot, u) \). This implies that \( \lim_{k \to \infty} \zeta_n^{(k)} = \zeta_n \) for any \( n \geq 1 \).

It remains to show that for any \( 0 < s < 1/2 \), \( \Phi : H_{r,0}^{-s} \to h_+^{1/2-s} \) is one-to-one. In the case where \( u \in L_{r,0}^2 \), it was verified in the proof of [15, Proposition 4.2] that the Fourier coefficients \( \hat{u}(k), \) \( k \geq 1 \), of \( u \) can be explicitly expressed in terms of the components \( \zeta_n(u) \) of the sequence \( \zeta(u) = \Phi(u) \). These formulas continue to hold for \( u \in H_{r,0}^{-s} \). This completes the proof of Proposition 1(ii). \( \square \)

4. Extension of \( \Phi \). Part 2

In this section we prove the second part of Theorem 6, which we again state as a separate proposition.

Proposition 2. (Extension of \( \Phi \). Part 2) For any \( 0 < s < 1/2 \), the map \( \Phi : H_{r,0}^{-s} \to h_+^{1/2-s} \) has the following additional properties:

(i) The inverse image of \( \Phi \) of any bounded subset of \( h_+^{1/2-s} \) is a bounded subset in \( H_{r,0}^{-s} \).

(ii) \( \Phi \) is onto and the inverse map \( \Phi^{-1} : h_+^{1/2-s} \to H_{r,0}^{-s} \) is sequentially weakly continuous.

(iii) For any \( 0 < s < 1/2 \), the Birkhoff map \( \Phi : H_{r,0}^{-s} \to h_+^{1/2-s} \) and its inverse \( \Phi^{-1} : h_+^{1/2-s} \to H_{r,0}^{-s} \) are continuous.

Remark 9. As mentioned in Remark 6, the map \( \Phi : L_{r,0}^2 \to h_+^{1/2} \) and its inverse \( \Phi^{-1} : h_+^{1/2} \to L_{r,0}^2 \) are sequentially weakly continuous.

Proof of Proposition 2(i). Let \( 0 < s < 1/2 \) and \( u \in H_{r,0}^{-s} \). Recall that by Corollary 2, \( L_u \) is a self-adjoint operator with domain \( \text{dom}(L_u) \subset H_+ \), has discrete spectrum and is bounded from below. Thus \( L_u - \lambda_0(u) + 1 \geq 1 \) where \( \lambda_0(u) \) denotes the smallest eigenvalue of \( L_u \). By the considerations in Section 3 (cf. Lemma 5), \( L_u \) extends to a bounded operator \( L_u : H_+^{1/2} \to H_+^{-1/2} \) and satisfies

\[ \langle L_u f | f \rangle = \langle Df | f \rangle - \langle u | f \bar{f} \rangle, \quad \forall f \in H_+^{1/2}. \]

By Lemma 1(i) one has \( |\langle u | f \bar{f} \rangle| \leq C_{1,s}^2 \|u\|_{-s} \|f\|_{1/2}^2 \) for any \( f \in H_+^{1/2} \) and hence

\[ \|f\|^2 \leq \langle (L_u - \lambda_0(u) + 1) f | f \rangle \]

\[ \leq \langle Df | f \rangle + C_{1,s}^2 \|u\|_{-s} \|f\|_{1/2}^2 + (-\lambda_0(u) + 1) \|f\|^2, \]

yielding the estimate

\[ \|f\|^2 \leq \langle (L_u - \lambda_0(u) + 1) f | f \rangle \leq M_u \|f\|_{1/2}^2 \]
where

\[(40)\quad M_u := C^2_{1,s}\|u\|_{-s} + (2 - \lambda_0(u)).\]

To shorten notation, we will for the remainder of the proof no longer indicate the dependence of spectral quantities such as \(\lambda_n\) or \(\gamma_n\) on \(u\) whenever appropriate. The square root of the operator \(L_u - \lambda_0 + 1\),

\[R_u := (L_u - \lambda_0 + 1)^{1/2} : H^1_+ \rightarrow H_+,
\]
can then be defined in terms of the basis \(f_n \equiv f_n(\cdot, u)\), \(n \geq 0\), of eigenfunctions of \(L_u\) in a standard way as follows: By Lemma 6, any \(f \in H^1_+\) has an expansion of the form \(f = \sum_{n=0}^{\infty} \langle f | f_n \rangle f_n\) where \((\langle f | f_n \rangle)_{n \geq 0}\) is a sequence in \(h^{1/2}(N_0)\). \(R_u f\) is then defined as

\[R_u f := \sum_{n=0}^{\infty} (\lambda_n - \lambda_0 + 1)^{1/2} \langle f | f_n \rangle f_n\]

Since \((\lambda_n - \lambda_0 + 1)^{1/2} \sim \sqrt{n}\) (cf. (32)) one has

\[\left(\langle \lambda_n - \lambda_0 + 1 \rangle^{1/2} \langle f | f_n \rangle \right)_{n \geq 0} \in \ell^2(N_0)
\]

implying that \(R_u f \in H_+\) (cf. Lemma 6). Note that

\[\| f \|^2 \leq \langle R_u f | R_u f \rangle = \langle R^2_u f | f \rangle \leq M_u \| f \|_{1/2}^2, \quad \forall f \in H^1_+
\]

and that \(R_u\) is a positive self-adjoint operator when viewed as an operator with domain \(H^1_+\), acting on \(H_+\). By complex interpolation (cf. e.g. [36, Section 1.4]) one then concludes that for any \(0 \leq \theta \leq 1\)

\[R^\theta_u : H^{\theta/2}_+ \rightarrow H_+, \quad \| R^\theta_u f \|^2 \leq M^\theta_u \| f \|_{\theta/2}^2, \quad \forall f \in H^1_+.
\]

Since by duality,

\[R^\theta_u : H_+ \rightarrow H^{-\theta/2}_+, \quad \| R^\theta_u g \|_{-\theta/2} \leq M^\theta_u \| g \|_2^2, \quad \forall g \in H_+,
\]

one infers, using that \(R^\theta_u : H_+ \rightarrow H^{-\theta/2}_+\) is boundedly invertible, that for any \(f \in H^{-\theta/2}_+\),

\[\| f \|_{-\theta/2}^2 \leq M^\theta_u \| R^{-\theta}_u f \|_2^2, \quad R^{-\theta}_u := (R^\theta_u)^{-1}.
\]

Applying the latter inequality to \(f = \Pi u\) and \(\theta = 2s\) and using that \(\Pi u = \sum_{n=1}^{\infty} (\Pi u | f_n) f_n\) and \(\langle \Pi u | f_n \rangle = -\lambda_n \langle 1 | f_n \rangle\) one sees that

\[(41)\quad \frac{1}{2} \| u \|_{-s}^2 = \| \Pi u \|_{-s}^2 \leq M_u^{2s} \Sigma
\]

where

\[\Sigma := \sum_{n=1}^{\infty} \lambda_n^2 (\lambda_n - \lambda_0 + 1)^{-2s} |\langle 1 | f_n \rangle|^2.
\]

We would like to deduce from (41) an estimate of \(\| u \|_{-s}\) in terms of the \(\gamma_n\)'s. Let us first consider \(M_u^{2s}\). By (40) one has

\[M_u^{2s} = 2^{2s} \max \{ (C^2_{1,s} \| u \|_{-s})^{2s}, (2 - \lambda_0(u))^{2s} \},
\]
yielding
\[(42)\quad M^2_s \leq (\|u\|_{-s}^2 (2C^2_{1,s})^{2s} + (2(2 - \lambda_0(u)))^{2s})\]

Applying Young’s inequality with \(1/p = s, 1/q = 1 - s\) one obtains
\[(43)\quad (\|u\|_{-s}^2 (2C^2_{1,s})^{2s} \Sigma \leq \frac{1}{4} \|u\|_{-s}^2 + ((4C^2_{1,s})^{2s} \Sigma)^{1/(1-s)},\]

which when combined with (41) and (42), leads to
\[\frac{1}{4} \|u\|_{-s}^2 \leq (4C^2_{1,s})^{2s} \Sigma)^{1/(1-s)} + (2(2 - \lambda_0(u)))^{2s} \Sigma.\]

The latter estimate is of the form
\[(44)\quad \|u\|_{-s}^2 \leq C_{3,s} \Sigma^{1/(1-s)} + C_{4,s} (2 - \lambda_0(u))^{2s} \Sigma,\]

where \(C_{3,s}, C_{4,s} > 0\) are constants, only depending on \(s\). Next let us turn to \(\Sigma = \sum_{n=1}^{\infty} \lambda^2_n (\lambda_n - \lambda_0 + 1)^{-2s} \|\langle 1|f_n \rangle\|^2.\) Since
\[\lambda_n = n - \sum_{k=n+1}^{\infty} \gamma_k, \quad \|\langle 1|f_n \rangle\|^2 = \gamma_n \kappa_n.\]

and
\[(45)\quad \kappa_n = \frac{1}{\lambda_n - \lambda_0} \prod_{p \neq n} (1 - \frac{\gamma_p}{\lambda_p - \lambda_n}).\]

the series \(\Sigma\) can be expressed in terms of the \(\gamma_n\)’s. To obtain a bound for \(\Sigma\) it remains to estimate the \(\kappa_n\)’s. Note that
\[\prod_{p \neq n} (1 - \frac{\gamma_p}{\lambda_p - \lambda_n}) \leq \prod_{p < n} (1 + \frac{\gamma_p}{\lambda_n - \lambda_p}) \leq e^{\sum_{p=1}^{n} \gamma_p} \leq e^{-\lambda_0}.\]

Since \((\lambda_n - \lambda_0)^{-1} = (n + \sum_{k=1}^{n} \gamma_k)^{-1} \leq n^{-1}\), it then follows that
\[0 < \kappa_n \leq \frac{e^{-\lambda_0}}{n}, \quad \forall n \geq 1.\]

Combining the estimates above we get
\[\Sigma \leq e^{-\lambda_0} \sum_{n=1}^{\infty} \lambda^2_n n^{-2s-1} \gamma_n.\]

By splitting the sum \(\Sigma\) into two parts, \(\Sigma = \sum_{n=-\lambda_0}^{n} + \sum_{n>\lambda_0}\) and taking into account that \(0 \leq \lambda_n \leq n\) for any \(n \geq -\lambda_0\) and \(|\lambda_n| \leq -\lambda_0\) for any \(1 \leq n < -\lambda_0\), one has
\[\Sigma \leq (1 - \lambda_0)^2 e^{-\lambda_0} \sum_{n=1}^{\infty} n^{1-2s} \gamma_n.\]

Together with the estimate (44) this shows that the inverse image by \(\Phi\) of any bounded subset of sequences in \(H^{1/2-s}_{r,0}\) is bounded in \(H^{-s}_{r,0}\).
Proof of Proposition 2(ii). First we prove that for any $0 < s < 1/2$, $\Phi : H_{r,0}^{-s} \to h_{+}^{1/2-s}$ is onto. Given $z = (z_{n})_{n \geq 1}$ in $h_{+}^{1/2-s}$, consider the sequence $\zeta^{(k)} = (\zeta^{(k)}_{n})_{n \geq 1}$, defined for any $k \geq 1$ by

$$
\zeta^{(k)}_{n} = z_{n} \quad \forall 1 \leq n \leq k, \quad \zeta^{(k)}_{n} = 0 \quad \forall n > k.
$$

Clearly $\zeta^{(k)} \to z$ strongly in $h_{+}^{1/2-s}$. Since for any $k \geq 1$, $\zeta^{(k)} \in h_{+}^{1/2}$, Theorem 5 implies that there exists a unique element $u^{(k)} \in L_{r,0}^{2}$ with $\Phi(u^{(k)}) = \zeta^{(k)}$. By Proposition 2(i), $\sup_{k \geq 1} \|u^{(k)}\|_{-s} < \infty$. Choose a weakly convergent subsequence $(u^{(k_{j})})_{j \geq 1}$ of $(u^{(k)})_{k \geq 1}$ and denote its weak limit in $H_{r,0}^{-s}$ by $u$. Since by Proposition 1, $\Phi : H_{r,0}^{-s} \to h_{+}^{1/2-s}$ is sequentially weakly continuous, $\Phi(u^{(k_{j})}) \to \Phi(u)$ weakly in $h_{+}^{1/2-s}$. On the other hand, $\Phi(u^{(k_{j})}) = \zeta^{(k_{j})} \to z$ strongly in $h_{+}^{1/2-s}$, implying that $\Phi(u) = z$. This shows that $\Phi$ is onto.

It remains to prove that for any $0 \leq s < 1/2$, $\Phi^{-1}$ is sequentially weakly continuous. Assume that $(\zeta^{(k)})_{k \geq 1}$ is a sequence in $h_{+}^{1/2-s}$, weakly converging to $\zeta \in h_{+}^{1/2-s}$. Let $u^{(k)} := \Phi^{-1}(\zeta^{(k)})$. By Proposition 2(i) (in the case $0 < s < 1/2$) and Remark 5(ii) (in the case $s = 0$), $(u^{(k)})_{k \geq 1}$ is a bounded sequence in $H_{r,0}^{-s}$ and thus admits a weakly convergent subsequence $(u^{(k_{j})})_{j \geq 1}$. Denote its limit in $H_{r,0}^{-s}$ by $u$. Since by Proposition 1, $\Phi$ is sequentially weakly continuous, $\Phi(u^{(k_{j})}) \to \Phi(u)$ weakly in $h_{+}^{1/2-s}$. On the other hand, by assumption, $\Phi(u^{(k_{j})}) = \zeta^{(k_{j})} \to \zeta$ and hence $u = \Phi^{-1}(\zeta)$ and $u$ is independent of the chosen subsequence $(u^{(k_{j})})_{j \geq 1}$. This shows that $\Phi^{-1}(\zeta^{(k)}) \to \Phi^{-1}(\zeta)$ weakly in $H_{r,0}^{-s}$.

Proof of Proposition 2(iii). By Proposition 1, $\Phi : H_{r,0}^{-s} \to h_{+}^{1/2-s}$ is sequentially weakly continuous for any $0 \leq s < 1/2$. To show that this map is continuous it then suffices to prove that the image $\Phi(A)$ of any relatively compact subset $A$ of $H_{r,0}^{-s}$ is relatively compact in $h_{+}^{1/2-s}$. For any given $\varepsilon > 0$, choose $N = N_{\varepsilon} \geq 1$ and $R = R_{\varepsilon} > 0$ as in Lemma 9, stated below. Decompose $u \in A$ as $u = u_{N} + u_{\perp}$ where

$$
u_{N} := \sum_{0 \leq |n| \leq N_{\varepsilon}} \hat{u}(n)e^{inx}, \quad u_{\perp} := \sum_{|n| > N_{\varepsilon}} \hat{u}(n)e^{inx}.
$$

By Lemma 9, $\|u_{N}\| < R_{\varepsilon}$ and $\|u_{\perp}\|_{-s} < \varepsilon$. By Lemma 6, applied with $\theta = -s$, one has

$$
K_{u_{\perp}}(\Pi u) = K_{u_{\perp}}(\Pi u_{N}) + K_{u_{\perp}}(\Pi u_{\perp}) \in h^{-s}(N_{0})
$$

where $K_{u_{\perp}}(\Pi u_{N}) = K_{u_{0}}(\Pi u_{N})$ since $\Pi u_{N} \in H_{+}$. Lemma 6 then implies that there exists $C_{A} > 0$, independent of $u \in A$, so that

$$
\|K_{u_{0}}(\Pi u_{N})\| \leq C_{A}R_{\varepsilon}, \quad \|K_{u_{\perp}}(\Pi u_{\perp})\|_{-s} \leq C_{A}\varepsilon.
$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, it then follows by Lemma 9 that $K_{u_{\perp}}(\Pi(A))$ is relatively compact in $h^{-s}(N_{0})$. Since by definition

$$
K_{u_{\perp}}(\Pi u) = \langle \Pi u | f_{n}(\cdot, u) \rangle, \quad \forall n \geq 0,
$$

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and since by (25),
\[ \zeta_n(u) \simeq \frac{1}{\sqrt{n}} \langle \Pi u | f_n(\cdot, u) \rangle \quad \text{as } n \to \infty \]
uniformly with respect to \( u \in A \), it follows that \( \Phi(A) \) is relatively compact in \( h_{+}^{1/2-s} \).

Now let us turn to \( \Phi^{-1} \). By Proposition 2(ii), \( \Phi^{-1} : h_{+}^{1/2-s} \to H_{r,0}^{-s} \) is sequentially weakly continuous. To show that this map is continuous it then suffices to prove that the image \( \Phi^{-1}(B) \) of any relatively compact subset \( B \) of \( h_{+}^{1/2-s} \) is relatively compact in \( H_{r,0}^{-s} \). By the same arguments as above one sees that \( \Phi^{-1} : h_{+}^{1/2-s} \to H_{r,0}^{-s} \) is also continuous. \( \square \)

It remains to state Lemma 9, used in the proof of Proposition 2(iii). It concerns the well known characterization of relatively compact subsets of \( H_{s}^{-r} \) in terms of the Fourier expansion
\[ u(x) = \sum_{n \neq 0} \hat{u}(n)e^{inx} \]
of an element \( u \) in \( H_{s}^{-r} \).

Lemma 9. Let \( 0 < s < 1/2 \) and \( A \subset H_{s}^{+} \). Then \( A \) is relatively compact in \( H_{s}^{+} \) if and only if for any \( \varepsilon > 0 \), there exist \( N_{\varepsilon} \geq 1 \) and \( R_{\varepsilon} > 0 \) so that for any \( f \in A \), the sequence \( \xi_{n} := \hat{f}(n), n \geq 0 \) satisfies
\[ \left( \sum_{n > N_{\varepsilon}} |n|^{-2s} |\xi_n|^2 \right)^{1/2} < \varepsilon, \quad \left( \sum_{0<n\leq N_{\varepsilon}} |\xi_n|^2 \right)^{1/2} < R_{\varepsilon}. \]

The latter conditions on \( (\xi_n)_{n \geq 0} \) characterize relatively compact subsets of \( h^{-s}(\mathbb{N}_0) \).

Proof of Theorem 6. The claimed statements follow from Proposition 1 and Proposition 2. In particular, item (ii) of Theorem 6 follows from Proposition 2(i) by setting
\[ F_{s}(R) := \sup_{\|\xi\|_{\frac{1}{2-s}} \leq R} \|\Phi^{-1}(\xi)\|_{-s}. \]

\( \square \)

5. Solution maps \( S_B \), \( S_B \) and \( S_c, S_{c,B} \)

In this section we provide results related to the solution map of (1), which will be used to prove Theorem 1 in the subsequent section.

Solution map \( S_B \) and its extension. First we study the map \( S_B \), defined in Section 2 on \( h_{+}^{1/2} \). Recall that by (12) – (13), the nth frequency of (1) is a real valued map defined on \( \ell_{+}^{2} \) by
\[ \omega_n(\zeta) := n^2 - 2 \sum_{k=1}^{n} k|\zeta_k|^2 - 2n \sum_{k=n+1}^{\infty} |\zeta_k|^2. \]
For any $0 < s \leq 1/2$, the map $S_B$ naturally extends to $h_+^{1/2-s}$, mapping initial data $\zeta(0) \in h_+^{1/2-s}$ to the curve

$$(46) \quad S_B(\cdot, \zeta(0)) : \mathbb{R} \to h_+^{1/2-s}, \ t \mapsto S_B(t, \zeta(0)) := (\zeta_n(0)e^{it\omega_n(\zeta)})_{n \geq 1}.$$ 

We first record the following properties of the frequencies.

**Lemma 10.** (i) For any $n \geq 1$, $\omega_n : \ell^2_+ \to \mathbb{R}$ is continuous and

$$|\omega_n(\zeta) - n^2| \leq 2n\|\zeta\|_0^2, \ \forall \zeta \in \ell^2_+; \quad |\omega_n(\zeta) - n^2| \leq 2\|\zeta\|_{1/2}^2, \ \forall \zeta \in h_+^{1/2}.$$ 

(ii) For any $0 \leq s < 1/2$, $\omega_n : h_+^{1/2-s} \to \mathbb{R}$ is sequentially weakly continuous.

*Proof.* Item (i) follows in a straightforward way from the formula (12) of $\omega_n$. Since for any $0 \leq s < 1/2$, $h_+^{1/2-s}$ compactly embeds into $\ell^2_+$, item (ii) follows from (i). \hfill \Box

From Lemma 10 one infers the following properties of $S_B$. We leave the easy proof to the reader.

**Proposition 3.** For any $0 \leq s \leq 1/2$, the following holds:

(i) For any initial data $\zeta(0) \in h_+^{1/2-s}$,

$$\mathbb{R} \to h_+^{1/2-s}, \ t \mapsto S_B(t, \zeta(0))$$

is continuous.

(ii) For any $T > 0$,

$$S_B : h_+^{1/2-s} \to C([-T, T], h_+^{1/2-s}), \ \zeta(0) \mapsto S_B(\cdot, \zeta(0)),$$

is continuous and for any $t \in \mathbb{R}$,

$$S_B^t : h_+^{1/2-s} \to h_+^{1/2-s}, \ \zeta(0) \mapsto S_B(t, \zeta(0)),$$

is a homeomorphism.

**Solution map $S_0$ and its extension.** Recall that in Section 2 we introduced the solution map $S_0$ of (1) on the subspace $L_{r, 0}^2$ of $L_r^2$, consisting of elements in $L_r^2$ with average 0, in terms of the Birkhoff map $\Phi$,

$$(47) \quad S_0 = \Phi^{-1}S_B\Phi : L_{r, 0}^2 \to C(\mathbb{R}, L_{r, 0}^2).$$

Theorem 6 will now be applied to prove the following result about the extension of $S_0$ to the Sobolev space $H_{r, 0}^{-s}$ with $0 < s < 1/2$, consisting of elements in $H_r^{-s}$ with average zero. It will be used in Section 6 to prove Theorem 1.

**Proposition 4.** For any $0 \leq s < 1/2$, the following holds:

(i) The Benjamin-Ono equation is globally $C^0$-well-posed on $H_{r, 0}^{-s}$.

(ii) There exists an increasing function $F_s : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ so that

$$\|u\|_{-s} \leq F_s(\|\Phi(u)\|_{1/2-s}), \quad \forall u \in H_{r, 0}^{-s}.$$
In particular, for any initial data \( u(0) \in H_{r,0}^s \),
\[
\sup_{t \in \mathbb{R}} \| S_t^0(u(0)) \|_{-s} \leq F_s(\| \Phi(u(0)) \|_{1/2-s}) .
\]

**Remark 10.** (i) By the trace formula (7), for any \( u(0) \in L_{r,0}^2 \), estimate (48) can be improved as follows,
\[
\| u(t) \| = \sqrt{2} \| \Phi(u(0)) \|_{1/2} = \| u(0) \| , \quad \forall t \in \mathbb{R} .
\]

(ii) We refer to the comments of Theorem 1 in Section 1 for a discussion of the recent results of Talbut [34], related to (48).

**Proof.** Statement (i) follows from the corresponding statements for \( S_B \)
in Proposition 3 and the continuity properties of \( \Phi \) and \( \Phi^{-1} \) stated in Theorem 6.

(ii) By Theorem 6 there exists an increasing function \( F_s : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \)
so that for any \( u \in H_{r,0}^{-s} \), \( \| u \|_{-s} \leq F_s(\| \Phi(u) \|_{1/2-s}) \). Since the norm of \( h^{1/2-s} \) is left invariant by the flow \( S_{\| \cdot \|_{-s}}^0 \), it follows that for any initial data \( u(0) \in H_{r,0}^s \), one has \( \sup_{t \in \mathbb{R}} \| S_t^0(u(0)) \|_{-s} \leq F_s(\| \Phi(u(0)) \|_{1/2-s}) \).

**Solution map \( S_c \).** Next we introduce the solution map \( S_c \) where \( c \) is a real parameter. Let \( v(t,x) \) be a solution of (1) with initial data \( v(0) \in H_{r,0}^s \) and \( s > 3/2 \), satisfying the properties (S1) and (S2) stated in Section 1. By the uniqueness property in (S1), it then follows that
\[
v(t,x) = u(t,x - 2ct) + c, \quad c = \langle v(0) | 1 \rangle
\]
where \( u \in C(\mathbb{R}, H_{r,0}^s) \cap C^1(\mathbb{R}, H_{r,0}^{s-2}) \) is the solution of the initial value problem
\[
\partial_t u = H \partial_x^2 u - \partial_x (u^2) , \quad u(0) = v(0) - \langle v(0) | 1 \rangle ,
\]
satisfying (S1) and (S2). It then follows that \( w(t,x) := u(t,x - 2ct) \) satisfies \( w(0) = u(0) \) and
\[
\partial_t w = H \partial_x^2 w - \partial_x (w^2) + 2c \partial_x w .
\]
By (49), the solution map of (51), denoted by \( S_c \), is related to the solution map \( S \) of (1) (cf. property (S2) stated in Section 1) by
\[
S(t,v(0)) = S_{\langle v(0) | 1 \rangle}(t, v(0) - \langle v(0) | 1 \rangle + \langle v(0) | 1 \rangle) , \quad \langle v(0) | 1 \rangle := \langle v(0) | 1 \rangle .
\]
In particular, for any \( s > 3/2 \),
\[
S_c : H_{r,0}^s \to C(\mathbb{R}, H_{r,0}^s) , \quad w(0) \mapsto S_c(\cdot, w(0))
\]
is well defined and continuous. Molinet’s results in [27] (cf. also [29]) imply that the solution map \( S_c \) continuously extends to any Sobolev space \( H_{r,0}^s \) with \( 0 \leq s \leq 3/2 \). More precisely, for any such \( s \), \( S_c : H_{r,0}^s \to C(\mathbb{R}, H_{r,0}^s) \) is continuous and for any \( v_0 \in H_{r,0}^s \), \( S_c(t,w_0) \) satisfies equation (1) in \( H_{r,0}^{s-2} \).

**Solution map \( S_{c,B} \) and its extension.** Arguing as in Section 2, we use Theorem 5, to express the solution map \( S_{c,B} \), corresponding to the
equation (51) in Birkhoff coordinates. Note that (51) is Hamiltonian, \(\partial_t w = \partial_{\xi} \nabla \mathcal{H}_c\), with Hamiltonian

\[
\mathcal{H}_c : H^s_{r,0} \to \mathbb{R}, \quad \mathcal{H}_c(w) = \mathcal{H}(w) + 2c\mathcal{H}^{(0)}(w)
\]

where by (2), \(\mathcal{H}^{(0)}(w) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} w^2 \, dx\). Since by Parseval’s formula, derived in [15, Proposition 3.1],

\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} w^2 \, dx = \sum_{n=1}^{\infty} n|\zeta_n|^2
\]

one has

\[
\mathcal{H}_{c,B}(\zeta) := \mathcal{H}_c(\Phi^{-1}(\zeta)) = \mathcal{H}_B(\zeta) + 2c \sum_{n=1}^{\infty} n|\zeta_n|^2,
\]

implying that the corresponding frequencies \(\omega_{c,n}, n \geq 1\), are given by

\[
(54) \quad \omega_{c,n}(\zeta) = \partial_{|\zeta_n|^2} \mathcal{H}_{c,B}(\zeta) = \omega_n(\zeta) + 2cn.
\]

For any \(c \in \mathbb{R}\), denote by \(S_{c,B}\) the solution map of (51) when expressed in Birkhoff coordinates,

\[
(55) \quad S_{c,B} : h^1_{r,0} \to C(\mathbb{R}, h^1_{r,0}), \zeta(0) \mapsto [t \mapsto (\zeta_n(0)e^{it\omega_n(\zeta(0))})_{n \geq 1}].
\]

Note that \(\omega_{0,n} = \omega_n\) and hence \(S_{0,B} = S_B\). Using the same arguments as in the proof of Proposition 3 one obtains the following

**Corollary 5.** The statements of Proposition 3 continue to hold for \(S_{c,B}\) with \(c \in \mathbb{R}\) arbitrary.

**Extension of the solution map \(S_c\).** Above, we introduced the solution map \(S_c\) on the subspace space \(L^2_{r,0}\). One infers from (52) that

\[
(56) \quad S_c = \Phi^{-1}S_{c,B}\Phi : L^2_{r,0} \to C(\mathbb{R}, L^2_{r,0}).
\]

Using the same arguments as in the proof of Proposition 4 one infers from Corollary 5 the following results, concerning the extension of \(S_c\) to the Sobolev space \(H^s_{r,0}\) with \(0 < s < 1/2\).

**Corollary 6.** The statements of Proposition 4 continue to hold for \(S_{c,B}\) with \(c \in \mathbb{R}\) arbitrary.

6. **Proofs Theorem 1, Theorem 3, and Theorem 4**

**Proof of Theorem 1.** Theorem 1 is a straightforward consequence of Proposition 4 and Corollary 6. \(\square\)

**Proof of Theorem 3.** We argue similarly as in the proof of [15, Theorem 2]. Since the case \(c \neq 0\) is proved be the same arguments we only consider the case \(c = 0\). Let \(u_0 \in H^s_{r,0}\) with \(0 \leq s < 1/2\) and let \(u(t) := S_0(t, u_0)\). By formula (46), \(\zeta(t) := S_B(t, \Phi(u_0))\) evolves on the torus \(\text{Tor}(\Phi(u_0))\), defined by (16).

(i) Since \(\text{Tor}(\Phi(u_0))\) is compact in \(h^1_{r,0}\) and \(\Phi^{-1} : h^1_{r,0} \to H^s_{r,0}\) is continuous, \(\{u(t) : t \in \mathbb{R}\}\) is relatively compact in \(H^s_{r,0}\).

(ii) In order to prove that \(t \mapsto u(t)\) is almost periodic, we appeal to Bochner’s characterization of such functions (cf. e.g. [26]) : a bounded continuous function \(f : \mathbb{R} \to X\) with values in a Banach space \(X\) is
Proof of Theorem 4. Since the general case can be proved by the same arguments we consider only the case \(c = 0\). By [15, Proposition B.1], the traveling wave solutions of the BO equation on \(\mathbb{T}\) coincide with the one gap solutions. Without further reference, we use notations and results from [15, Appendix B], where one gap potentials have been analyzed. Let \(u_0\) be an arbitrary one gap potential. Then \(u_0\) is \(C^\infty\)-smooth and there exists \(N \geq 1\) so that \(\gamma_N(u_0) > 0\) and \(\gamma_n(u_0) = 0\) for any \(n \neq N\). Furthermore, the orbit of the corresponding one gap solution is given by \(\{u_0(\cdot + \tau) : \tau \in \mathbb{R}\}\). Let \(0 \leq s < 1/2\). It is to prove that for any \(\varepsilon > 0\) there exists \(\delta > 0\) so that for any \(v(0) \in H^s_{r,0}\) with \(\|v(0) - u_0\|_{-s} < \delta\) one has

\[
\sup_{t \in \mathbb{R}} \inf_{\tau \in \mathbb{R}} \|v(t) - u_0(\cdot + \tau)\|_{-s} < \varepsilon.
\]

(57)

To prove the latter statement, we argue by contradiction. Assume that there exists \(\varepsilon > 0\), a sequence \((v^{(k)}(0))_{k \geq 1}\) in \(H^s_{r,0}\), and a sequence \((t_k)_{k \geq 1}\) so that

\[
\inf_{\tau \in \mathbb{R}} \|v^{(k)}(t_k) - u_0(\cdot + \tau)\|_{-s} \geq \varepsilon, \quad \forall k \geq 1, \quad \lim_{k \to \infty} \|v^{(k)}(0) - u_0\|_{-s} = 0.
\]

Since \(A := \{v^{(k)}(0) \mid k \geq 1\} \cup \{u_0\}\) is compact in \(H^s_{r,0}\) and \(\Phi\) is continuous, \(\Phi(A)\) is compact in \(h^{-1/2-s}_{+}\) and

\[
\lim_{k \to \infty} \|\Phi(v^{(k)}(0)) - \Phi(u_0)\|_{1/2-s} = 0.
\]

It means that

\[
\lim_{k \to \infty} \sum_{n=1}^{\infty} \eta_n^{1-2s} |\zeta_n(v^{(k)}(0)) - \zeta_n(u_0)|^2 = 0.
\]

Note that for any \(k \geq 1\),

\[
\zeta_n(v^{(k)}(t_k)) = \zeta_n(v^{(k)}(0)) e^{it_k \omega_n(v^{(k)}(0))}, \quad \forall n \geq 1
\]
and \( \zeta_n(u(t_k)) = \zeta_n(u_0) = 0 \) for any \( n \neq N \). Hence
\[
\lim_{k \to \infty} \sum_{n \neq N} n^{1-2s} |\zeta_n(v(k)(t_k))|^2 = 0 ,
\]
and since \( |\zeta_N(v(k)(t_k))| = |\zeta_N(v(k)(0))| \) one has
\[
\lim_{k \to \infty} \|\zeta_N(v(k)(t_k)) - \zeta_N(u_0)\| = 0 ,
\]
implying that \( \sup_{k \geq 1} |\zeta_N(v(k)(t_k))| < \infty \). It thus follows that the subset \( \{ \Phi(v(k)(t_k)) : k \geq 1 \} \) is relatively compact in \( h^{1/2-s} \) and hence \( \{ v(k)(t_k) : k \geq 1 \} \) relatively compact in \( H^{-s}_r \). Choose a subsequence \( (v(k_j)(t_{k_j}))_{j \geq 1} \) which converges in \( H^{-s}_r \) and denote its limit by \( w \in H^{-s}_r \).

By (58)–(59) one infers that there exists \( \theta \in \mathbb{R} \) so that
\[
\zeta_n(w) = 0 , \quad \forall n \neq N , \quad \zeta_N(w) = \zeta_N(u_0)e^{i\theta} .
\]
As a consequence, \( w(x) = u_0(x + \theta/N) \), contradicting the assumption that \( \inf_{r \in \mathbb{R}} \|v(k)(t_k) - u_0(\cdot + \tau)\|_{-s} \geq \varepsilon \) for any \( k \geq 1 \).

### 7. Proof of Theorem 2

In this section we prove Theorem 2. First we need to to make some preparations. We consider potentials of the form \( u(x) = v(e^{ix}) + \overline{v(e^{ix})} \), where \( v \) is a Hardy function, defined in the unit disc by
\[
v(z) = \frac{\varepsilon qz}{1 - qz} , \quad 0 < \varepsilon < q < 1 , \quad |z| < 1 .
\]
Note that
\[
\|u\|_{-1/2}^2 = 2\varepsilon^2 \sum_{n=1}^{\infty} n^{-1} q^{2n} = -2\varepsilon^2 \log(1 - q^2) .
\]
We want to investigate properties of the Birkhoff coordinates of \( u \). To this end we consider the Lax operator \( L_u = D - T_u \) for \( u \) of the above form. Since for any \( f \in H^1_+ \) and \( z \in \mathbb{C} \) in the unit disc
\[
T_u f(z) = \Pi((v + \overline{v})f)(z) = v(z)f(z) + \varepsilon q \frac{f(z) - f(q)}{z - q} ,
\]
the eigenvalue equation \( L_u f - \lambda f = 0 , \lambda \in \mathbb{R} \), reads
\[
z f'(z) - \left( \frac{\varepsilon qz}{1 - qz} + \frac{\varepsilon q}{z - q} - \mu \right) f(z) = f(q) \frac{\varepsilon q}{q - z} , \quad |z| < 1 ,
\]
where we have set \( \mu := -\lambda \). Note that if \( -\mu \) is an eigenvalue, then the eigenfunction \( f(z) \) is holomorphic in \( |z| < 1 \). Evaluating (61) in such a case at \( z = 0 \), one infers
\[
f(q) = \frac{\varepsilon + \mu}{\varepsilon} f(0) .
\]
Define for $|z| < q^{-1}$, $z \notin [0, q^{-1})$,
\[ \psi(z) := \frac{z^{\varepsilon+\mu}(1-qz)^\varepsilon}{(q-z)^\varepsilon}, \]
with the branches of the fractional powers chosen as
\[ \arg(z) \in (0, 2\pi), \quad \arg(q-z) \in (-\pi, \pi), \quad \arg(1-qz) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \]
Then (61) reads
\[ \frac{d}{dz}(\psi(z)f(z)) = f(q) \frac{\varepsilon q \psi(z)}{z(q-z)}. \]
As a consequence, $f(q) \neq 0$ since otherwise $\psi(z)f(z)$ would be constant and hence $\psi(z)^{-1}$ holomorphic on the whole unit disc, which is impossible since
\[ \frac{\psi(t+i0)}{\psi(t-i0)} = \frac{(t+i0)^{\varepsilon+\mu}(q-t+i0)^{\varepsilon}}{(q-t-i0)^{\varepsilon}(t-i0)^{\varepsilon+\mu}} = \begin{cases} e^{-2\pi i(\varepsilon+\mu)}, & t \in (0, q) \\ e^{-2\pi i \mu}, & t \in (q, q^{-1}) \end{cases} \]
and $e^{2\pi i \varepsilon} \neq 1$ for any $0 < \varepsilon < 1$. For what follows it is convenient to normalize the eigenfunction $f$ by $f(q) = 1$. Then the eigenvalue equation reads
\[ \frac{d}{dz}(\psi(z)f(z)) = g(z), \quad g(z) := \frac{\varepsilon q \psi(z)}{z(q-z)}, \quad |z| < 1. \]
Our goal is to prove that for an appropriate choice of the parameters $\varepsilon$ and $q$, $\mu = -\lambda_0(u)$ becomes arbitrarily large. So from now on we assume that $\mu > 0$. Note that by (64) one has
\[ f(z) = \frac{1}{\psi(z)} \int_0^z g(\zeta) \, d\zeta, \quad \{|z| < q^{-1}\} \setminus [0, q^{-1}) \].
Hence $-\mu$ will be an eigenvalue of $L_\mu$, if the right hand side of the latter expression extends to a holomorphic function in the disc $\{|z| < q^{-1}\}$. It means that $f$ can be continuously extended to $z = 0$ and $z = q$ and that
\[ f(t+i0) = f(t-i0), \quad \forall t \in (0, q) \cup (q, q^{-1}). \]
It is straightforward to check that in the case $\mu > 0$, $f$ extends continuously to $z = 0$. Furthermore, the identity (65) is verified for any $t \in (0, q)$ since
\[ \frac{g(s+i0)}{g(s-i0)} = e^{-2\pi i(\varepsilon+\mu)} = \frac{\psi(t+i0)}{\psi(t-i0)}, \quad \forall 0 < s < t. \]
It is then straightforward to verify that $f$ extends continuously to $z = q$ and that $f(q) = 1$. Next, let us examine the condition $f(t+i0) =$
f(t - i0) for t ∈ (q, 1). Notice that
\[ g(z) = \frac{d}{dz} \left( \frac{z^{\varepsilon}}{(q - z)^{\varepsilon}} \right) z^{\mu} (1 - qz)^{\varepsilon}, \]
so that with \( h(z) := \frac{z^{\varepsilon}}{(q - z)^{\varepsilon}} \frac{d}{dz} (z^{\mu} (1 - qz)^{\varepsilon}) \) one has, integrating by parts,
\[ \int_{0}^{\hat{z}} g(\zeta) \, d\zeta = \frac{z^{\varepsilon}}{(q - z)^{\varepsilon}} z^{\mu} (1 - qz)^{\varepsilon} - \int_{0}^{\hat{z}} h(\zeta) \, d\zeta = \psi(z) - \int_{0}^{\hat{z}} h(\zeta) \, d\zeta. \]

As a consequence, the condition \( f(t + i0) = f(t - i0) \), \( t \in (q, q^{-1}) \), reads
\[ \int_{0}^{t} h(\zeta + i0) \, d\zeta = \psi(t + i0) \int_{0}^{t} h(\zeta - i0) \, d\zeta, \quad \forall q < t < q^{-1}. \]

Since by (63), \( \psi(\zeta + i0) = e^{-2\pi i t} \psi(\zeta - i0) \) and \( h(\zeta + i0) = e^{-2\pi i t} h(\zeta - i0) \) for any \( q < \zeta < q^{-1} \), we conclude that the following condition
\[ \int_{0}^{q} h(\zeta + i0) \, d\zeta = e^{-2\pi i t} \int_{0}^{q} h(\zeta - i0) \, d\zeta \]
is necessary and sufficient for \(-\mu < 0\) to be an eigenvalue of \( L_\mu \). After simplification and using again that \( e^{2\pi i \varepsilon} \neq 1 \), this condition reads

(66) \[ F(\mu, \varepsilon, q) := \int_{0}^{q} t^{\varepsilon} \frac{d}{dt} \left( t^{\mu} (1 - qt)^{\varepsilon} \right) dt = 0 \]
or

(67) \[ F(\mu, \varepsilon, q) = \int_{0}^{q} \frac{t^{\mu + 1} (1 - qt)^{\varepsilon}}{(q - t)^{\varepsilon}} \left( \frac{\mu}{t} - \frac{\varepsilon q}{1 - qt} \right) dt = 0. \]

Notice that if \( \mu \geq \frac{\varepsilon q^2}{1 - q^2} \), then the latter integrand is strictly positive for any \( 0 < t < q \). In particular, one has

(68) \[ F\left( \frac{\varepsilon q^2}{1 - q^2}, \varepsilon, q \right) > 0. \]

On the other hand, let us fix \( \mu > 0 \) and study the limit of \( F(\mu, \varepsilon, q) \) as \( (\varepsilon, q) \to (0, 1) \). Clearly, one has

(69) \[ \lim_{(\varepsilon, q) \to (0, 1)} \int_{0}^{q} \frac{t^{\mu + 1} (1 - qt)^{\varepsilon}}{(q - t)^{\varepsilon}} \frac{\mu}{t} dt = \int_{0}^{1} \mu t^{\mu - 1} dt = 1. \]

To compute the limit of the remaining part of \( F(\mu, \varepsilon, q) \) is more involved. For any given fixed positive parameter \( 1 - q^2 < \theta < 1 \), split
the integral

\[ I(\mu, \varepsilon, q) := \int_0^q \frac{t^{\varepsilon+\mu}(1-qt)^\varepsilon}{(q-t)^\varepsilon} \frac{\varepsilon q}{1-qt} dt \]

into three parts,

\[ I_1(\mu, \varepsilon, q; \theta) + I_2(\mu, \varepsilon, q; \theta) + I_3(\mu, \varepsilon, q) := \int_{0}^{q(1-\theta)} + \int_{q(1-\theta)}^{q^3} + \int_{q^3}^{q}. \]

It is easy to check that

\[ 0 \leq I_1(\mu, \varepsilon, q; \theta) \leq C_1(\theta) \varepsilon \]

and that with the change of variable \( t := q - q(1 - q^2)y \) in \( I_3 \), one has

\[ 0 \leq I_3(\mu, \varepsilon, q) = \varepsilon q^{\mu+2} \int_{0}^{1} (1 - (1 - q^2)y)^{\varepsilon+\mu} (1 + q^2y)^{\varepsilon-\mu} dy \leq C_3 \varepsilon. \]

Using the same change of variable in \( I_2 \), we obtain

\[ I_2(\mu, \varepsilon, q; \theta) = \varepsilon q^{\mu+2} \int_{1}^{\theta/(1-q^2)} (1 - (1 - q^2)y)^{\varepsilon+\mu} y^{\varepsilon} (1 + q^2y)^{1-\varepsilon} dy. \]

Note that for \( 1 \leq y \leq \theta/(1-q^2) \), one has \( 1 - \theta \leq 1 - (1 - q^2)y \leq 1 \), and hence

\[ (1 - \theta)^{\mu+1} \leq (1 - (1 - q^2)y)^{\mu+\varepsilon} \leq 1. \]

Since

\[ \frac{1+y}{2} \leq y \leq 1+y, \quad q^2(1+y) \leq 1+q^2y \leq 1+y, \]

we then infer that

\[ \varepsilon q^{\mu+2}(1-\theta)^{\mu+1} \int_{1}^{\theta/(1-q^2)} \frac{dy}{1+y} \leq I_2 \leq \varepsilon q^{2\varepsilon+\mu+2} \int_{1}^{\theta/(1-q^2)} \frac{dy}{1+y}. \]

Using that as \( q \to 1 \),

\[ \frac{1}{-\log(1-q^2)} \int_{1}^{\theta/(1-q^2)} \frac{dy}{1+y} = \frac{1}{\log(1-q^2)} \log((1-q^2+\theta)/2) \to 1 \]

we then obtain

\[ (1-\theta)^{\mu+1} \leq \liminf_{(\varepsilon, q) \to (0, 1)} \frac{I_2(\mu, \varepsilon, q; \theta)}{-\varepsilon \log(1-q^2)} \leq \limsup_{(\varepsilon, q) \to (0, 1)} \frac{I_2(\mu, \varepsilon, q; \theta)}{-\varepsilon \log(1-q^2)} \leq 1. \]

Summarizing, we have proved that for any \( 0 < \theta < 1 \),

\[ (1-\theta)^{\mu+1} \leq \liminf_{(\varepsilon, q) \to (0, 1)} \frac{I(\mu, \varepsilon, q)}{-\varepsilon \log(1-q^2)} \leq \limsup_{(\varepsilon, q) \to (0, 1)} \frac{I(\mu, \varepsilon, q)}{-\varepsilon \log(1-q^2)} \leq 1. \]
Letting \( \theta \to 0 \), we conclude that
\[
I(\mu, \varepsilon, q) = -\varepsilon \log(1 - q^2)(1 + o(1)) \to +\infty
\]
for \((\varepsilon, q)\) satisfying
\[
\varepsilon \log(1 - q^2) \to -\infty .
\]
Therefore, if (71) holds, then by (67), (69), and (70)
\[
F(\mu, \varepsilon, q) \to -\infty , \quad \forall \mu > 0 .
\]
For any \( k \geq 1 \), let
\[
q_k^2 := 1 - e^{-\varepsilon_k^{-3/2}}
\]
with \( 0 < \varepsilon_k < q_k \) so small that
\[
F(k, \varepsilon_k, q_k) > 0 , \quad \frac{\varepsilon_k q_k^2}{1 - q_k^2} = \varepsilon_k e^{\varepsilon_k^{-3/2}}(1 - e^{-\varepsilon_k^{-3/2}}) > k .
\]
Set
\[
u^{(k)}(x) := 2\Re\left( \frac{\varepsilon_k q_k e^{ix}}{1 - q_k e^{ix}} \right) .
\]

**Lemma 11.** For any \( k \geq 1 \), \( u^{(k)} \in \bigcap_{n \geq 1} H^2_{r,0} \), and \( \lim_{k \to \infty} \|u^{(k)}\|_{-1/2} = 0 \) as well as \( \lim_{k \to \infty} \lambda_0(u^{(k)}) = -\infty \).

**Proof.** Expanding \( u^{(k)} \), \( k \geq 1 \), in its Fourier series, it is straightforward to check that \( u^{(k)} \in \bigcap_{n \geq 1} H^2_{r,0} \). By (60) we have \( \|u^{(k)}\|_{H^{-1/2}}^2 = \sqrt{\varepsilon_k} \to 0 \) and by (68), (72), \( \lambda_0(u^{(k)}) < -k \) and hence \( \lim_{k \to \infty} \lambda_0(u^{(k)}) = -\infty \). \( \square \)

In a next step we prove that for \( u \) of the form \( u = 2\Re\left( \frac{q e^{it\varepsilon}}{1 - q e^{it\varepsilon}} \right) \), \( L_u \) has only one negative eigenvalue. More precisely the following holds.

**Lemma 12.** For any \( 0 < \varepsilon < q < 1 \), \( F(\cdot, \varepsilon, q) \) has precisely one zero in \( \mathbb{R}_{>0} \). It means that \( \lambda_0(u) \) is the only negative eigenvalue of \( L_u \) and thus \( \lambda_1(u) \geq 0 \). Furthermore, \( \lambda_1(u) = 1 - \sum_{k \geq 2} \gamma_k(u) \leq 1 \) and
\[
0 \leq \sum_{n=2}^{\infty} \gamma_n(u) \leq 1 \quad \text{and} \quad \gamma_n(u) > 0 , \quad \forall n \geq 2 .
\]

**Proof.** The proof relies on an alternative formula for \( F(\mu, \varepsilon, q) \), obtained from (66) by integrating by parts. Choosing \( q \mu(1 - q^2)\varepsilon - t\mu(1 - qt)\varepsilon \) as antiderivative of \( \frac{d}{dt}(t\mu(1 - qt)\varepsilon) \) one gets
\[
F(\mu, \varepsilon, q) = \varepsilon q \int_0^q \frac{t \varepsilon}{(q - t)^\varepsilon} \frac{q \mu(1 - q^2)\varepsilon - t\mu(1 - qt)\varepsilon}{t(q - t)} \, dt .
\]
Consequently,
\[
\partial_\mu F(\mu, \varepsilon, q) = \varepsilon q \int_0^q \frac{t \varepsilon}{(q - t)^\varepsilon} \frac{q \mu(1 - q^2)\varepsilon \log q - t\mu(1 - qt)\varepsilon \log t}{t(q - t)} \, dt .
\]
Assume that $F(\mu, \varepsilon, q) = 0$ for some $\mu > 0$. Subtracting $F(\mu, \varepsilon, q) \log q$ from the above expression for $\partial_\mu F(\mu, \varepsilon, q)$, we infer from (75) that

$$\frac{\partial F}{\partial \mu}(\mu, \varepsilon, q) = \varepsilon q \int_0^q \frac{t^\varepsilon}{(q-t)^\varepsilon} \frac{t^\mu (1-qt)^\varepsilon}{t(q-t)} \log \left( \frac{q}{t} \right) \, dt > 0.$$ 

This implies that $F(\cdot, \varepsilon, q) = 0$ cannot have more than one zero in $\mathbb{R}_{>0}$. It means that $\lambda_0(u)$ is the only negative eigenvalue of $L_u$ and hence $\lambda_1(u) \geq 0$. Since by (31), $\lambda_n(u) = n - \sum_{k \geq n+1} \gamma_k(u)$ for any $n \geq 0$, it follows that $0 \leq \sum_{k \geq 2} \gamma_k(u) \leq 1$. For any $n \geq 0$, denote by $\hat{f}_n(\cdot, u)$ the eigenfunction of $L_u$, corresponding to $\lambda_n(u)$, normalized by $\hat{f}_n(q, u) = 1$. By (62), it then follows that for any $n \geq 1$, $\langle \hat{f}_n(\cdot, u)|1 \rangle = \hat{f}_n(0, u) \neq 0$ and hence by (26) that $\gamma_n(u) > 0$.

In a next step, given an arbitrary potential $u \in \bigcap_{n \geq 1} H^{{n}}_{r,0}$, we want to express the Fourier coefficient $\hat{u}(1) = \langle u|e^{ix} \rangle$ in terms of the Birkhoff coordinates $\zeta_n(u)$, $n \geq 1$, of $u$. Denote by $(f_p)_{p \geq 0}$ the orthonormal basis of eigenfunctions of $L_u$ with our standard normalization

$$\langle f_0|1 \rangle > 0, \quad \langle f_{n+1}|Sf_n \rangle > 0, \quad \forall n \geq 0.$$ 

Then we get

(76) \quad \hat{u}(1) = \sum_{p=0}^{\infty} \langle u|f_p \rangle \langle f_p|e^{ix} \rangle = \sum_{p=0}^{\infty} -\lambda_p \langle 1|f_p \rangle \sum_{n=0}^{\infty} \langle S^* f_p|f_n \rangle \langle f_n|1 \rangle .

The following lemma provides a formula for $\hat{u}(1)$ in terms of the Birkhoff coordinates $\zeta_n(u)$, $n \geq 1$. To keep the exposition as simple as possible we restrict to the case where $\gamma_n(u) > 0$ for any $n \geq 1$.

**Lemma 13.** For any $u \in \bigcap_{n \geq 1} H^{{n}}_{r,0}$ with $\gamma_n(u) > 0$ for any $n \geq 1$,

(77) \quad \hat{u}(1) = -\sum_{n=0}^{\infty} \sqrt{\frac{\mu_{n+1}(u) \kappa_n(u)}{\kappa_{n+1}(u)}} \zeta_{n+1}(u) \zeta_n(u) .

**Proof.** Recall that in the case $\gamma_{n+1}(u) \neq 0$, the matrix coefficients $M_{np} := \langle S^* f_p|f_n \rangle$ are given by [15, formulas (4.7), (4.9)]. Using that $|\langle f_{n+1}|1 \rangle|^2 = \kappa_{n+1} \gamma_{n+1}$ and $\zeta_{n+1} = \langle 1|f_{n+1} \rangle / \sqrt{\kappa_{n+1}}$, one then gets

$$M_{np} = \sqrt{\frac{\mu_{n+1}}{\kappa_{n+1}}} \frac{\langle f_p|1 \rangle}{\lambda_p - \lambda_{n+1}} \zeta_{n+1} .$$

Substituting this formula into the expression (76) for $\hat{u}(1)$ yields

$$\hat{u}(1) = -\sum_{n=0}^{\infty} \sqrt{\frac{\mu_{n+1} \kappa_n}{\kappa_{n+1}}} \zeta_{n+1} \zeta_n \sum_{p=0}^{\infty} \lambda_p |\langle 1|f_p \rangle|^2 .$$
where for convenience, $\zeta_0 := 1$. Since
\[
\sum_{p=0}^{\infty} \frac{\lambda_p |\langle 1 | f_p \rangle|^2}{\lambda_p - \lambda_n - 1} = \sum_{p=0}^{\infty} |\langle 1 | f_p \rangle|^2 + (\lambda_n + 1) H_{-\lambda_n - 1},
\]
and $H_{-\lambda_n - 1} = 0$ due to $\gamma_{n+1} > 0$, formula (77) follows.

Let us now consider the sequence $S_0(t, u^{(k)})$, $k \geq 1$, with $u^{(k)}$ given by (73). Since $\gamma_1(u^{(k)}) > k$ (Lemma 11), $\gamma_n(u^{(k)}) > 0$, $n \geq 2$, (Lemma 12), and since $\gamma_n$, $n \geq 1$, are conserved quantities of (1), it follows that formula (77) is valid for $S_0(t, u^{(k)})$ for any $t \in \mathbb{R}$ and $k \geq 1$. Hence
\[
\xi_k(t) := \langle S_0(t, u^{(k)}) | e^{ix} \rangle
\]
is given by
\[
\xi_k(t) = -\sum_{n=0}^{\infty} \sqrt{\frac{\mu_{n+1}(u^{(k)}) \kappa_n(u^{(k)})}{\kappa_{n+1}(u^{(k)})}} \zeta_{n+1}(S_0(t, u^{(k)})) \zeta_{n}(S_0(t, u^{(k)})).
\]
For any potential $u \in L^2_{r,0}$ and any $n \geq 1$, one has by (10), (12)
\[
\zeta_n(S_0(t, u)) = \zeta_n(u) e^{it\omega_n(u)}, \quad \omega_n(u) := n^2 - 2 \sum_{k=1}^{\infty} \min(k, n) \gamma_k(u),
\]
implying that $\omega_1(u) = 1 + 2\lambda_0(u)$ and for any $n \geq 1$,
\[
\omega_{n+1}(u) - \omega_n(u) = 2n + 1 - 2 \sum_{k=n+1}^{\infty} \gamma_k(u) = 1 + 2\lambda_n(u)
\]
so that, for any $n \geq 0$,
\[
\zeta_{n+1}(S_0(t, u)) \zeta_{n}(S_0(t, u)) = \zeta_{n+1}(u) \zeta_{n}(u) e^{it(1+2\lambda_n(u))}.
\]
Therefore, we have
\[
\xi_k(t) = -\sum_{n=0}^{\infty} \sqrt{\frac{\mu_{n+1}(u^{(k)}) \kappa_n(u^{(k)})}{\kappa_{n+1}(u^{(k)})}} \zeta_{n+1}(u^{(k)}) \zeta_{n}(u^{(k)}) e^{it(1+2\lambda_n(u^{(k)}))}. \tag{79}
\]

Proof of Theorem 2. First we consider the case $c = 0$. Our starting point is formula (79). Recall the product formulas of [15, Corollary 3.4]
\[
\kappa_0 = \prod_{p \geq 1} \left( 1 - \frac{\gamma_p}{\lambda_p - \lambda_0} \right), \quad \kappa_n = \frac{1}{\lambda_n - \lambda_0} \prod_{1 \leq p \neq n} \left( 1 - \frac{\gamma_p}{\lambda_p - \lambda_n} \right), \quad \forall n \geq 1
\]
and (cf. [15, formula (4.9)])
\[
\frac{\mu_{n+1}}{\kappa_{n+1}} = \frac{\lambda_n + 1 - \lambda_0}{\prod_{1 \leq p \neq n+1} \left( 1 - \frac{\gamma_p}{\lambda_p - \lambda_{n+1}} \right)}.
\]

---

1We are grateful to Louise Gassot for drawing our attention to this cancellation.
They yield
\[ \frac{\mu_{1+k_0}}{\kappa_1} = \prod_{p \geq 1} \frac{1 - \frac{\gamma_p}{\lambda_p - \lambda_0}}{1 - \frac{\gamma_{p+1}}{\lambda_{p+1} - \lambda_0}} = \prod_{p \geq 1} \left(1 - \left(\frac{\gamma_p}{\lambda_p - \lambda_0}\right)^2\right), \]
\[ \frac{\mu_{2+k_1}}{\kappa_2} = \left(1 + \frac{1}{\lambda_1 - \lambda_0}\right)(1 + \gamma_1)^{-1} \prod_{p \geq 2} \frac{1 - \frac{\gamma_p}{\lambda_p - \lambda_1}}{1 - \frac{\gamma_{p+1}}{\lambda_{p+1} - \lambda_1}}, \]
and for any \( n \geq 2, \)
\[ \frac{\mu_{n+k_n}}{\kappa_{n+1}} = \left(1 + \frac{1}{\lambda_n - \lambda_0}\right) \frac{1 + \frac{\gamma_n}{\lambda_n - \lambda_1}}{1 + \frac{\gamma_{n+1}}{\lambda_{n+1} - \lambda_1}} \prod_{2 \leq p \neq n} \left(1 - \frac{\gamma_p}{\lambda_p - \lambda_n}\right) \prod_{1 \leq p \neq n} \left(1 - \frac{\gamma_{p+1}}{\lambda_{p+1} - \lambda_n}\right). \]

Since by the definition of \( \gamma_1, \lambda_1(u^{(k)}) = \lambda_0(u^{(k)}) + 1 + \gamma_1(u^{(k)}) \) and since by Lemma 12, \( 0 \leq \lambda_1(u^{(k)}) \leq 1, \) it follows from Lemma 11 that
\[ \gamma_1(u^{(k)}) = \lambda_1(u^{(k)}) - (\lambda_0(u^{(k)}) + 1) \sim -\lambda_0(u^{(k)}) \to +\infty \]
and
\[ 1 - \left(\frac{\gamma_1}{\lambda_1 - \lambda_0}\right)^2 (u^{(k)}) = \left(1 - \frac{\gamma_1}{\lambda_1 - \lambda_0}\right)(1 + \frac{\gamma_1}{\lambda_1 - \lambda_0}) \sim \frac{2}{\gamma_1(u^{(k)})}. \]
Furthermore by (74), \( \sup_{k \geq 1} \sum_{n \geq 2} \gamma_n(u^{(k)}) \leq 1. \) One then concludes that
\[ \frac{\mu_{1+k_0}}{\kappa_1}(u^{(k)}) \sim \frac{2}{\gamma_1(u^{(k)})}, \quad \frac{\mu_{2+k_1}}{\kappa_2}(u^{(k)}) = O\left(\frac{1}{\gamma_1(u^{(k)})}\right), \]
and
\[ \frac{\mu_{n+k_n}}{\kappa_{n+1}}(u^{(k)}) = O(1), \quad \forall n \geq 2. \]
From the above estimates, we infer that the sequence of functions \((\xi_k(\cdot))_{k \geq 1},\) given by (79), is uniformly bounded for \( t \in \mathbb{R}. \) Assume that it converges pointwise to 0 on an interval \( I \) of positive length. Then by the dominated convergence theorem,
\[ \lim_{k \to \infty} \int_I \xi_k(t)e^{-it(1+2\lambda_0(u^{(k)}))} \, dt = 0. \]
On the other hand, since
\[ \xi_k(t)e^{-it(1+2\lambda_0(u^{(k)}))} = -\sqrt{\frac{\mu_1(u^{(k)})_{\kappa_1}(u^{(k)})}{\kappa_1(u^{(k)})}} \xi_1(u^{(k)}) \]
\[ -\sum_{n \geq 1} \sqrt{\frac{\mu_{n+1}(u^{(k)})_{\kappa_{n+1}}(u^{(k)})}{\kappa_{n+1}(u^{(k)})}} \xi_{n+1}(u^{(k)}) \zeta_n(u^{(k)})e^{it(\lambda_n(u^{(k)}) - \lambda_0(u^{(k))}}), \]
and \( \lambda_n(u^{(k)}) - \lambda_0(u^{(k)}) \geq |\lambda_0(u^{(k)})| \) for any \( n \geq 1 \), the above estimates yield
\[
\int_I \xi_k(t)e^{-it(1+2\lambda_0(u^{(k)}))} \, dt = -\frac{\sqrt{2}\xi_1(u^{(k)})}{\sqrt{\gamma_1(u^{(k)})}} |I| + O\left(\frac{1}{|\lambda_0(u^{(k)})|}\right)
\]
which does not converge to 0 as \( k \to \infty \), since \( |\xi_1(u^{(k)})| = \sqrt{\gamma_1(u^{(k)})} \) and \( |I| > 0 \). Hence the assumption that \( (\xi_k(\cdot))_{k \geq 1} \) converges pointwise to 0 on an interval \( I \) of positive length leads to a contradiction and thus cannot be true.

Finally, let us treat the case where \( c \in \mathbb{R} \) is arbitrary. Consider the sequence \( (u^{(k)} + c)_{k \geq 1} \) with \( (u^{(k)})_{k \geq 1} \) given by (73). By Lemma 11, \( \lim_{k \to \infty} (u^{(k)} + c) = c \) in \( H_{r,c}^{1/2} \). Furthermore, by (52) - (53) one has \( S(t, u^{(k)} + c) = S_c(t, u^{(k)}) + c \) and by (54) - (55),
\[
\zeta_n(S_c(t, u^{(k)})) = \zeta_n(u^{(k)}) e^{it(2cn + \omega_n(u^{(k)}))}.
\]
Hence with \( \xi_k(t) \) given by (78), one has
\[
\xi_{c,k}(t) := \langle S(t, u^{(k)} + c) | e^{ix} \rangle = e^{2ct} \xi_k(t), \quad \forall k \geq 1 .
\]
Since
\[
\int_I \xi_{c,k}(t)e^{-i2ct}e^{-it(1+2\lambda_0(u^{(k)}))} \, dt = \int_I \xi_k(t)e^{-it(1+2\lambda_0(u^{(k)}))} \, dt ,
\]
Theorem 2 follows from the proof of the case \( c = 0 \) treated above. \( \square \)

An immediate consequence of Theorem 2 is the following

**Corollary 7.** The Birkhoff map \( \Phi \) does not continuously extend to a map \( H_{r,0}^{-1/2} \to h_0^0 \).

**APPENDIX A. RESTRICTION OF \( \Phi \) TO \( H_{r,0}^s, s > 0 \)**

The purpose of this appendix is to study the restriction of the Birkhoff map \( \Phi \) to \( H_{r,0}^s \) with \( s > 0 \) and to discuss applications to the flow map of the Benjamin–Ono equation.

**Proposition 5.** For any \( s \geq 0 \), the restriction of the Birkhoff map \( \Phi \) to \( H_{r,0}^s \) is a homeomorphism from \( H_{r,0}^s \) onto \( h_+^{s+1/2} \). Furthermore, \( \Phi : H_{r,0}^s \to h_+^{s+1/2} \) and its inverse \( \Phi^{-1} : h_+^{s+1/2} \to H_{r,0}^s \) map bounded subsets to bounded subsets.

**Proof.** The case \( s = 0 \) is proved in [15]. We first treat the case \( s \in ]0,1] \). Assume that \( A \) is a bounded subset of \( L_{r,0}^2 \). Given \( u \in A \), let \( K_{u,s}, 0 \leq s \leq 1 \), be the linear isomorphism of Lemma 6,
\[
f \in H_+^s \to \langle (f | f_n) \rangle_{n \geq 0} \in h_+^s(N_0) , \quad f_n \equiv f_n(\cdot , u) .
\]
By Lemma 6, the operator $K_{u,s}$ is uniformly bounded with respect to $u \in A$. Since $L_{u,1} = -\Pi u$, one has

$$K_{u,s}(\Pi u) = (\langle \Pi u | f_n \rangle)_{n \geq 0} = (- \lambda_n(u) \langle 1 | f_n \rangle)_{n \geq 0}$$

$$(80) \quad = (- \lambda_n(u) \sqrt{\kappa_n(u)} \zeta_n(u))_{n \geq 0}$$

where for convenience we have set $\zeta_0(u) := 1$. Furthermore, by the proof of Proposition 1(i) one has that

$$(81) \quad \lambda_n(u) = n + o\left(\frac{1}{n}\right), \quad \sqrt{\kappa_n(u)} = \frac{1}{\sqrt{n}} \left(1 + o(1)\right),$$

uniformly with respect to $u \in A$. Therefore, for any $s \in [0,1]$, an element $u \in L^2_{r,0}$ belongs to $H^s_{r,0}$ if and only if $K_{u,0}(\Pi u) \in h^s(\mathbb{N})$ or, equivalently, $(\sqrt{n} \zeta_n(u))_{n \geq 1} \in h^s_+$, implying that $\Phi(u) \in h^{s+1/2}$. Furthermore, if $A$ is a bounded subset of $H^s_{r,0}$, then $\Phi(A)$ is a bounded subset of $h^{s+1/2}$. Conversely, if $B$ is a bounded subset of $h^{s+1/2}$, then $\Phi^{-1}(B)$ is bounded in $L^2_{r,0}$. As a consequence, the norm of $K_{u,s}^{-1} : h^s(\mathbb{N}) \to H^s_+$ is uniformly bounded with respect to $u \in \Phi^{-1}(B)$ and hence $\Phi^{-1}(B)$ is bounded in $H^s_{r,0}$.

Next we prove that for any $0 < s \leq 1$, $\Phi : H^s_{r,0} \to h^{s+1/2}$ and its inverse are continuous. Since $\Phi : L^2_{r,0} \to h^{1/2}$ is a homeomorphism, we infer from Rellich’s theorem and the boundedness properties of $\Phi$ and its inverse, derived above, that for any $s \in (0,1]$, $\Phi : H^s_{r,0} \to h^{s+1/2}$ and $\Phi^{-1} : h^{s+1/2} \to H^s_{r,0}$ are sequentially weakly continuous. As in the proof of Proposition 2(iii), to prove that $\Phi : H^s_{r,0} \to h^{s+1/2}$ is continuous, it then suffices to show that $\Phi$ maps any relatively compact subset $A$ of $H_{r,0}$ to a relatively compact subset of $h^{s+1/2}$. For $s = 1$, this is straightforward. Indeed, given any bounded subset $A$ of $H^1_{r,0}$, $A$ is relatively compact in $H^1_{r,0}$ if and only if $\{\Pi u \mid u \in A\}$ is a relatively compact in $L^2_+$. As $\{T_n \Pi u \mid u \in A\}$ is bounded in $H^1_+$, this amounts to say that $\{L_u(\Pi u) \mid u \in A\}$ is relatively compact subset in $L^2_+$. Since

$$\|L_u(\Pi u)\|^2 = \sum_{n \geq 0} \lambda_n(u)^2 |\langle 1 | f_n(\cdot, u)\rangle|^2,$$

we infer from (81) that $\|L_u(\Pi u)\|^2 = \|\Phi(u)\|^2_{\tilde{L}^{3/2}} + R(u)$, where $R$ is a weakly continuous functional on $H^1_{r,0}$. As a consequence, $\{L_u(\Pi u) \mid u \in A\}$ is relatively compact in $L^2_+$ if and only if $\Phi(A)$ is relatively compact in $h^{3/2}$. This shows that $\Phi : H^s_{r,0} \to h^{s+1/2}$ is continuous. In a similar way one proves that $\Phi^{-1} : h^{3/2} \to H^1_{r,0}$ is continuous. This completes the proof for $s = 1$.

To treat the case $s \in (0,1)$, we will use the following standard variant of Lemma 9.
Lemma 14. Let $s \in (0,1)$. A bounded subset $A_+ \subset H^s_+$ is relatively compact in $H^s_+$ if and only if for any $\varepsilon > 0$, there exist $N_\varepsilon \geq 1$ and $R_\varepsilon > 0$ so that for any $f \in A_+$, the sequence $\xi_n := \hat{f}(n), n \geq 0$ satisfies

$$\left( \sum_{n > N_\varepsilon} n^{2s} |\xi_n|^2 \right)^{1/2} < \varepsilon, \quad \left( \sum_{0 \leq n \leq N_\varepsilon} n^{2s} |\xi_n|^2 \right)^{1/2} < R_\varepsilon.$$ 

The latter conditions on $(\xi_n)_{n \geq 0}$ characterize relatively compact subsets of $h^s(\mathbb{N}_0)$.

We then argue as in the proof of Proposition 2(iii), using the operators $K_{u:s}$ and $K_{u:s}^{-1}$, to complete the proof of Proposition 5 in the case $s \in [0,1]$.

In order to deal with the case $s > 1$, we need the following lemma.

Lemma 15. Let $k$ be a nonnegative integer and assume that $A$ is a bounded subset of $H^k_{r,0}$. Then, for any $u \in A$ and $s \in [k, k+1]$, an element $f \in H^k_+$ is in $H^s_+$ if and only if for any $j = 0, \ldots, k$, $L_j f \in H^{s-k}_+$, with bounds which are uniform with respect to $u \in A$. Furthermore, $A$ is compact in $H^s_{r,0}$ if and only if for any $j = 0, \ldots, k$, $\{L^*_j \Pi u \mid u \in A\}$ is compact in $H^{s-k}_+$.

Proof of Lemma 15. The statement is trivial for $k = 0$. Let us first prove it for $k = 1$. Assume that $A$ is a bounded subset of $H^1_{r,0}$ and write $s = t + 1 + t \in [0,1]$. Then for any $f \in H^1_+$, one has $f \in H^s_+$ if and only if $f, Df$ belongs to $H^t_+$ with uniform bounds and with correspondence of compact subsets. For any $u \in H^1_{r,0}$, the operator $T_u$ maps bounded subsets of $H^t_+$ to bounded subsets of $H^1_+$. Hence for any $f \in H^1_+$, $f, Df \in H^t_+$ if and only if $f, L_u f \in H^1_+$ with bounds which are uniform with respect to $u \in A$ and with correspondence of compact subsets. This completes the proof for $k = 1$. For $k \geq 2$, we argue by induction. Assume $k \geq 2$ is such that the statement is true for $k - 1$. Let $A$ be a bounded subset of $H^k_{r,0}$, and let $s \in [k, k+1]$. Then for any $f \in H^k_+$, one has $f \in H^s_+$ if and only if $f, Df \in H^{s-1}_+$ with uniform bounds and correspondence of compact subsets. Since $s - 1 \geq k - 1 \geq 1$, $H^{s-1}_+$ is an algebra and hence $T_u$ maps bounded subsets of $H^{s-1}_+$ to bounded subsets of $H^{s-1}_+$. Consequently, $f, Df \in H^{s-1}_+$ if and only if $f, L_u f \in H^{s-1}_+$, with bounds which are uniform with respect to $u \in A$ and with correspondence of compact subsets. By the induction hypothesis, this is equivalent to $L^*_j f \in H^{s-k}_+$, for $j = 0, \ldots, k$ with bounds which are uniform with respect to $u \in A$ and with correspondence of compact subsets. □

Let us now come back to the proof of Proposition 5. We start with the case where $s = k$ is a positive integer and assume that $A$ is a bounded subset of $H^k_{r,0}$. Applying Lemma 15, one easily shows by induction on $k$ that for any $u \in A$, $L^*_j \Pi u \in L^k_+$ for $j = 0, \ldots, k$ with
bounds which are uniform with respect to $u \in A$ and with correspondence of compact subsets. In other words, for any $u \in A$, $\Pi u$ belongs to the domain of $L_u^k$, with bounds which are uniform with respect to $u \in A$ and with correspondence of compact subsets. On the other hand, if $u$ belongs to a bounded subset of $L_{r,0}^2$, $K_{u,0}$ maps the domain of $L_u^k$ into the space of sequences $(\xi_n)_{n \geq 0}$ such that $(\lambda_n(u)^{\ell} \xi_n)_{n \geq 0} \in h^0(\mathbb{N}_0)$ for every $\ell = 0, \ldots, k$, hence into $h^k(\mathbb{N}_0)$, with bounds which are uniform with respect to $u \in A$ and with correspondence of compact subsets. Hence for any $u \in A$, $\Phi(u) \in h_{+}^{k+1/2}$ with bounds, which are uniform with respect to $u \in A$ and with correspondence of compact subsets.

Finally, let $s \in (k, k + 1)$ and assume that $A$ is a bounded subset of $H_{r,0}^k$. It then follows by Lemma 15 that for any $u \in A$, $L_u^j u \in H_{+}^{s-k}$ for $j = 0, \ldots, k$ with bounds which are uniform with respect to $u \in A$ and with correspondence of compact subsets. Applying $K_{u,s-k}$ and Lemma 6, this means that $K_{u,s-k}(L_u^j u) \in h_{+}^{s-k}(\mathbb{N}_0)$ for $j = 0, \ldots, k$, with bounds which are uniform with respect to $u \in A$ and with correspondence of compact subsets. Since

$$K_{u,0}(L_u^j f) = (\lambda_n(u)^{\ell} \langle f | f_n \rangle)_{n \geq 0}$$

this implies $K_{u,0}(\Pi u) \in h_{s}(\mathbb{N}_0)$, and hence $\Phi(u) \in h_{+}^{s+1/2}$ with bounds which are uniform with respect to $u \in A$. The proof of the converse is similar, taking into account that by the case $s = k$ treated above, for any bounded subset $B$ of $h_{+}^{k+1/2}$, the set $\Phi^{-1}(B)$ is bounded in $H_{r,0}^k$. The correspondence of compact subsets is established similarly. □

Arguing as in the proofs of Theorems 1, 3 and 4, one deduces from Proposition 5 the following

**Corollary 8.** Let $s > 0$ and $c \in \mathbb{R}$. Then for any $t \in \mathbb{R}$, the flow map $S^t = S(t, \cdot)$ of the Benjamin–Ono equation leaves the affine space $H_{r,c}^s$, introduced in (4), invariant. Furthermore, there exists an integral $I_s : H_{r}^s \to \mathbb{R}_{\geq 0}$ of (1), satisfying

$$\|v\|_s \leq I_s(v), \quad \forall v \in H_{r}^s.$$

In particular, one has

$$\sup_{t \in \mathbb{R}} \|S(t, v_0)\|_s \leq I_s(v_0), \quad \forall v_0 \in H_{r}^s.$$

In addition, for any $v_0 \in H_{r,c}^s$, the solution $t \mapsto S(t, v_0)$ is almost periodic in $H_{r,c}^s$, and the traveling wave solutions are orbitally stable in $H_{r,c}^s$.

**Remark 11.** Corollary 8 significantly improves [18, Theorem 1.3], which provides polynomial (in $t$) bounds of solutions of (1) in $H_{r}^s$ for $1/2 < s \leq 1$. 
APPENDIX B. ILL-POSEDNESS IN $H^s_r$ FOR $s < -1/2$

The goal of this appendix is to construct for any $c \in \mathbb{R}$ a sequence $(v^{(k)}(t, x))_{k \geq 1}$ of smooth solutions of (1) so that

(i) $(v^{(k)}(0, \cdot))_{k \geq 1}$ has a limit in $H^{-s}_r$ for any $s > \frac{1}{2}$ and

(ii) for any $t \neq 0$, $(v^{(k)}(t, \cdot))_{k \geq 1}$ diverges in the sense of distributions, even after renormalizing the flow by a translation in the spatial variable.

Since the same arguments work for any $c \in \mathbb{R}$, we consider only the case $c = 0$. In a first step, let us review the results from [5], using the setup developed in this paper: the authors of [5] construct a sequence of one gap potentials $v^{(k)}(t, \cdot)$, $k \geq 1$, of the Benjamin-Ono equation so that $v^{(k)}(0, \cdot)$ converges in $H^{-s}_r$ for any $s > 1/2$, whereas for any $t \neq 0$, $(v^{(k)}(t, \cdot))_{k \geq 1}$ diverges in the sense of distributions. Without further reference, we use notations and results from [15, Appendix B], where one gap potentials have been analyzed. Consider the following family of one gap potentials of average zero,

$$u_{0,q}(x) = 2\text{Re}(q e^{ix} / (1 - q e^{ix})),$$

$0 < q < 1$.

The gaps $\gamma_n(u_{0,q})$, $n \geq 1$, of $u_{0,q}$ can be computed as

$$\gamma_{1,q} := \gamma_1(u_{0,q}) = q^2 / (1 - q^2), \quad \gamma_n(u_{0,q}) = 0, \quad \forall n \geq 2.$$

The frequency $\omega_{1,q} := \omega_1(u_{0,q})$ is given by (cf. (12))

$$\omega_{1,q} = 1 - 2\gamma_{1,q} = \frac{1 - 3q^2}{1 - q^2}.$$

The one gap solution, also referred to as traveling wave solution, of the BO equation with initial data $u_{0,q}$ is then given by

$$u_q(t, x) = u_{0,q}(x + \omega_{1,q}t), \quad \forall t \in \mathbb{R}.$$

Note that for any $s > 1/2$,

$$\lim_{q \to 1} u_{0,q} = 2\text{Re} \left( \sum_{k=1}^{\infty} e^{ikx} \right) = \delta_0 - 1$$

strongly in $H^{-s}_{r,0}$ where $\delta_0$ denotes the periodic Dirac $\delta$-distribution, centered at 0. Since $\omega_{1,q} \to -\infty$ as $q \to 1$, it follows that for any $t \neq 0$, $u_q(t, \cdot)$ diverges in the sense of distributions as $q \to 1$.

Note that by the trace formula (cf. (7)) $\|u_{0,q}\|^2 = 2\gamma_{1,q}$ and hence $\omega_{1,q} + \|u_{0,q}\|^2 = 1$, implying that a renormalization of the spatial variable by $\eta_q t$ with $\eta_q = \|u_{0,q}\|^2 = 1 - \omega_{1,q}$ removes the divergence, i.e., for any $s > 1/2$, $u_{0,q}(x + t)$ converges in $H^{-s}_{r,0}$ for any $t$ as $q \to 1$. In order to construct examples where no such renormalization is possible, we consider two-gap solutions. To keep the exposition as simple as possible, let $u$ be a two-gap potential with $\gamma_1 > 0, \gamma_2 > 0$ and hence $\gamma_n = 0$.
for any $n \geq 3$. From [15, Section 7] we know that $\Pi u(z) = -z \frac{Q'(z)}{Q(z)}$ where

\begin{equation}
Q(z) = \det(I - zM) , \quad \forall z \in \mathbb{C}, \quad |z| < 1 ,
\end{equation}

and $M := (M_{np})_{0 \leq n,p \leq 1}$ is a $2 \times 2$ matrix with $M_{np} = \sqrt{\frac{\nu_{n+1}^{\kappa_p} + \zeta_{n+1}^{\kappa_p}}{\nu_{n+1}^{\kappa_p} - \nu_{n+1}^{\kappa_p}}} \frac{\zeta_{n+1}^{\kappa_p}}{\nu_{n+1}^{\kappa_p}}$
or
$$M_{np} = \sqrt{\prod_{1 \leq q \neq n+1} (1 - \frac{\gamma_{n}}{\lambda_{q} - \lambda_{n}}} \frac{\zeta_{n+1}^{\kappa_p}}{\nu_{n+1}^{\kappa_p}} \frac{\gamma_{n}}{\nu_{n+1}^{\kappa_p}} .$$

Here $\zeta_{0} := 1$ and $\zeta_{n} = \sqrt{\gamma_{n}} e^{i\varphi_{n}} (1 \leq n \leq 2)$ whereas $\zeta_{n} = 0$ for any $n \geq 3$. Moreover, along the flow of the Benjamin-Ono equation, we have $\dot{\varphi}_{n} = \omega_{n}$, $1 \leq n \leq 2$, with
$$\omega_{1} = 1 - 2\gamma_{1} - 2\gamma_{2} , \quad \omega_{2} = 4 - 2\gamma_{1} - 4\gamma_{2} .$$

Let us express the entries of $M$ in terms of $\gamma_{1}, \gamma_{2}$ and $\varphi_{1}, \varphi_{2}$. First, we remark that

$$\kappa_{0} = \left( 1 - \frac{\gamma_{1}}{\lambda_{1} - \lambda_{0}} \right) \left( 1 - \frac{\gamma_{2}}{\lambda_{2} - \lambda_{0}} \right) = \frac{2 + \gamma_{1}}{(1 + \gamma_{1})(1 + \gamma_{2})} ,$$
$$\kappa_{1} = \frac{1}{\lambda_{1} - \lambda_{0}} \left( 1 - \frac{\gamma_{2}}{\lambda_{2} - \lambda_{1}} \right) = \frac{1}{(1 + \gamma_{1})(1 + \gamma_{2})} .$$

Then the above formula for $M_{np}$ yields

$$M_{00} = -\sqrt{\frac{\gamma_{1}(2 + \gamma_{1})(1 + \gamma_{1} + \gamma_{2})}{2 + \gamma_{1} + \gamma_{2}}} e^{i\varphi_{1}} , \quad M_{01} = \sqrt{\frac{1 + \gamma_{1} + \gamma_{2}}{1 + \gamma_{1}} \frac{1}{1 + \gamma_{2}}} ,$$
$$M_{10} = -\sqrt{\frac{\gamma_{2}}{2 + \gamma_{1} + \gamma_{2}}} e^{i\varphi_{2}} , \quad M_{11} = -\sqrt{\frac{\gamma_{1}\gamma_{2}(2 + \gamma_{1})}{1 + \gamma_{2}}} e^{i(\varphi_{2} - \varphi_{1})} .$$

Consequently, $Q(z) = 1 + \alpha z + \beta z^{2}$ with $\alpha = -\text{Tr}(M)$ and $\beta = \det(M)$ or

$$\alpha = \sqrt{\frac{\gamma_{1}(2 + \gamma_{1})}{1 + \gamma_{1}}} \left( \sqrt{\frac{1 + \gamma_{1} + \gamma_{2}}{2 + \gamma_{1} + \gamma_{2}}} e^{i\varphi_{1}} + \sqrt{\frac{\gamma_{2}}{1 + \gamma_{2}}} e^{i(\varphi_{2} - \varphi_{1})} \right) ,$$
$$\beta = \sqrt{\frac{(1 + \gamma_{1} + \gamma_{2})\gamma_{2}}{(1 + \gamma_{2})(1 + \gamma_{1} + \gamma_{2})}} e^{i\varphi_{2}} .$$

Taking the limit $\gamma_{1} \to \infty, \gamma_{2} \to \infty$, one sees that
$$\alpha = e^{i\varphi_{1}} + e^{i(\varphi_{2} - \varphi_{1})} + o(1) , \quad \beta = e^{i\varphi_{2}} + o(1) ,$$
which means that $Q(z) = (1 + q_{1}z)(1 + q_{2}z)$ where
$$q_{1} = e^{i\varphi_{1}} + o(1) , \quad q_{2} = e^{i(\varphi_{2} - \varphi_{1})} + o(1) .$$

Since $|q_{1}| < 1 , |q_{2}| < 1$, the family $u$ is bounded in $H_{r,\theta}^{-s}$ for any $s > \frac{1}{2}$.
We define our sequence \((v^{(k)}_0)_{k \geq 1}\) of initial data as the above two-gap potential \(u\) with \(\varphi_1(0) = \varphi_2(0) = 0\), and \(\gamma_1 = \gamma_2 := \gamma^{(k)}\) where \(\gamma^{(k)} \to \infty\). At \(t = 0\), we infer that for any \(z\) in the unit disc
\[
\lim_{k \to \infty} \Pi v^{(k)}_0(z) = \frac{2z}{1 + z}
\]
and hence \(v^{(k)}_0 \to 2(1 - \delta_x)\) in \(H^s_{r,0}\) for any \(s > \frac{1}{2}\). For \(t \neq 0\), we have
\[
\varphi_1(t) = t(1 - 4\gamma^{(k)}), \quad \varphi_2(t) - \varphi_1(t) = t(3 - 2\gamma^{(k)})
\]
and therefore
\[
\Pi v^{(k)}(z, t) = \frac{ze^{i\varphi_1(t)}}{1 + ze^{i\varphi_1(t)}} + \frac{ze^{i(\varphi_2 - \varphi_1)(t)}}{1 + ze^{i(\varphi_2 - \varphi_1)(t)}} + o(1).
\]
We thus conclude that \((\Pi v^{(k)}(z, t))_{k \geq 1}\) does not have a limit if \(t \neq 0\) for any \(z\) with \(0 < |z| < 1\), even after renormalizing \(z\) by a phase factor \(e^{i\eta_k(t)}\) with \(\eta_k(t)\) a function of our choice. This proves that for \(t \neq 0\), \((v^{(k)}(t, \cdot))_{k \geq 1}\) has no limit in \(D'(\mathbb{T})\), even after renormalizing it by a \(t\)-dependent translation. In particular, the renormalization of the spatial variable by \(\|v^{(k)}_0\|^2 t = (4 - \omega^{(k)}_2) t\) does not make \((v^{(k)}(t, \cdot))_{k \geq 1}\) convergent even in the sense of distributions.

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Laboratoire de Mathématiques d’Orsay, CNRS, Université Paris–Saclay, 91405 Orsay, France
E-mail address: patrick.gerard@math.u-psud.fr

Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, 8057 Zurich, Switzerland
E-mail address: thomas.kappeler@math.uzh.ch

Department of Mathematics, Northeastern University, 567 LA (Lake Hall), Boston, MA 0215, USA
E-mail address: p.topalov@northeastern.edu