On compressing sinh-Gordon solutions

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This paper is concerned with a class of approximate non-linear transformations that compress solutions of the (generalized) sinh-Gordon equation into parametrically small regions in two-dimensional spacetime. Given the sinh-Gordon field near a time-slice, a long Nambu-Goto string can be constructed in three-dimensional anti-de Sitter space. The string is then approximated to arbitrary accuracy by a slightly smoothed piecewise linear string of \(N\) segments. The corresponding sinh-Gordon field has a comb-like structure and its size is controlled by the amount of smoothing applied to the segmented string. In a (singular) large-\(N\) limit, the transformation commutes with time evolution. As an example, a static cosh-Gordon solution is discussed in detail. The corresponding smooth and segmented string solutions are obtained and the compressed cosh-Gordon potential is investigated.

1. Introduction. Completely integrable models in two dimensions play an important role in physics. Since they possess sufficiently many conserved quantities they are exactly solvable thus providing a controlled laboratory in which various phenomena can be studied. The focus of this paper is on the hyperbolic version of the celebrated sine-Gordon equation: the classical (generalized) sinh-Gordon model. The equation of motion is given by

\[
\partial_+ \partial_- \alpha + e^\alpha - uw e^{-\alpha} = 0
\]

where \(\alpha(\sigma^+, \sigma^-)\) is the sinh-Gordon field, \(u(\sigma^-)\) and \(v(\sigma^+)\) are auxiliary fields, and \(\sigma^\pm = \frac{1}{2}(\tau \pm \sigma)\) are light-cone coordinates in two dimensions \((\partial_- \equiv \partial_{\sigma^-} \) and \(\partial_+ \equiv \partial_{\sigma^+})\). The equation describes constant mean curvature surfaces in \(\mathbb{R}^{2,1}\), extremal surfaces in AdS\(_3\) [1], and it governs the evolution of the mesonic mean field in the two-dimensional Gross-Neveu model [2, 3].

In regions where \(uv = 0\), eqn. [1] reduces to the Liouville equation. Outside these regions, after a coordinate transformation given by \(d\hat{\sigma}^- = \sqrt{|u(\sigma^-)|} d\sigma^-\), \(d\hat{\sigma}^+ = \sqrt{|v(\sigma^+)|} d\sigma^+\), the equation takes the standard sinh-Gordon or cosh-Gordon form

\[
\hat{\partial}_+ \hat{\partial}_- \hat{\alpha} + e^{\hat{\alpha}} \pm e^{-\hat{\alpha}} = 0
\]

where \(\hat{\alpha} = \alpha - \log \sqrt{|uv|}\). The variables \(\hat{\sigma}^\pm\) will be called balanced coordinates.

The non-linear sinh-Gordon equation is integrable [1] [4] and possesses singular soliton solutions. Multisoliton solutions can be constructed by the inverse scattering method. In this Letter the relationship between the sinh-Gordon equation and long Nambu-Goto strings (extremal surfaces) in AdS\(_3\) will be exploited to obtain scheme-dependent non-linear transformations which compress solution of the sinh-Gordon equation into parametrically small regions. The flowchart of the map is illustrated in FIG.1

2. Strings in AdS\(_3\). In the first step of the transformation one needs to obtain a smooth string embedding in AdS\(_3\). A unit size AdS\(_3\) can be immersed into an \(\mathbb{R}^{2,2}\) ambient space via the equation,

\[
Y \cdot Y = Y_{-2} - Y_0^2 + Y_1^2 + Y_2^2 = -1 \quad Y \in \mathbb{R}^{2,2}.
\]

A part of global AdS\(_3\) is covered by the Poincaré patch. The metric on the patch is given by

\[
ds^2 = -dt^2 + dx^2 + dy^2.
\]

The coordinates \(t, x, y\) are related to \(Y\) via the following transformation

\[
(t, x, y) = \left(\frac{Y_0}{Y_2 - Y_{-1}}, \frac{Y_1}{Y_2 - Y_{-1}}, \frac{1}{Y_2 - Y_{-1}}\right),
\]

whose inverse on the hyperboloid is

\[
Y = \left(\frac{-1 + t^2 - x^2 - y^2}{2y}, \frac{t}{y}, \frac{x}{y}, \frac{1 + t^2 - x^2 - y^2}{2y}\right).
\]

The boundary of AdS is located at \(y = 0\). In the ambient space the boundary is the set of points that satisfy \(Y^2 = 0\) with the identification \(Y \equiv aY\) (with \(a \in \mathbb{R}^+\)).

The string can be mapped into AdS\(_3\) by taking the target space to be \(\mathbb{R}^{2,2}\) and then forcing the string to
lie on the hyperboloid using a Lagrange multiplier. In
conformal gauge the action is given by

$$S = -\frac{T}{2} \int d\tau d\sigma (\partial_\tau Y^\mu \partial_\sigma Y_\mu - \partial_\tau Y^\mu \partial_\tau Y_\mu + \lambda(Y^2 + 1)),$$

where $T$ is the string tension and $Y(\tau, \sigma) \in \mathbb{R}^{2,2}$ is the
embedding function. The equation of motion in lightcone coordinates is

$$\partial_+ \partial_\tau Y - (\partial_\tau Y \cdot \partial_\tau Y) Y = 0.$$  

Due to the gauge choice, the equations are supplemented
by the Virasoro constraints

$$\partial_- Y \cdot \partial_+ Y = \partial_+ Y \cdot \partial_- Y = 0.$$  

From the string embedding the sinh-Gordon field can be
computed via

$$\alpha = \log |\partial_- Y \cdot \partial_+ Y|.$$  

If the unit normal vector $N$ is defined such that

$$N_a = e^{-\alpha} \epsilon_{abcd} Y^b \partial_- Y^c \partial_+ Y^d$$

then the auxiliary fields are given by

$$u = -N \cdot \partial_- \partial_+ Y = \partial_- N \cdot \partial_+ Y = -Y \cdot \partial_- \partial_+ N$$
$$v = N \cdot \partial_+ \partial_+ Y = \partial_+ N \cdot \partial_+ Y = Y \cdot \partial_+ \partial_+ N \quad (4)$$

The string equations of motion guarantee that $u = u(\sigma^-)$
and $v = v(\sigma^+)$ and that the quantities satisfy the
generalized sinh-Gordon equation $[3,4]$. Note that while
changing $N \rightarrow -N$ flips the signs of both $u$ and $v$, their
product which appears in $[1]$ remains invariant.

3. Auxiliary linear system. Given a solution of the
generalized sinh-Gordon equation, one would like to construct
the string embedding. In order to do this one has to solve an auxiliary scattering problem. In terms of $\alpha, u, v$ the $SL(2)$
Lax matrices are $[5]

$$B^-_L = \begin{pmatrix} \frac{1}{4} \partial_- \alpha & \frac{1}{\sqrt{2}} e^{\frac{\alpha}{2}} \\ \frac{1}{\sqrt{2}} e^{-\frac{\alpha}{2}} & -\frac{1}{4} \partial_+ \alpha \end{pmatrix} \quad B^+_L = \begin{pmatrix} -\frac{1}{4} \partial_- \alpha & -\frac{1}{\sqrt{2}} e^{\frac{\alpha}{2}} \\ \frac{1}{\sqrt{2}} e^{-\frac{\alpha}{2}} & \frac{1}{4} \partial_+ \alpha \end{pmatrix}$$
$$B^-_R = \begin{pmatrix} -\frac{1}{4} \partial_- \alpha & -\frac{1}{\sqrt{2}} e^{\frac{\alpha}{2}} \\ \frac{1}{\sqrt{2}} e^{-\frac{\alpha}{2}} & \frac{1}{4} \partial_+ \alpha \end{pmatrix} \quad B^+_R = \begin{pmatrix} \frac{1}{4} \partial_- \alpha & -\frac{1}{\sqrt{2}} e^{\frac{\alpha}{2}} \\ \frac{1}{\sqrt{2}} e^{-\frac{\alpha}{2}} & \frac{1}{4} \partial_+ \alpha \end{pmatrix}.$$  

It is easy to check that the flatness conditions

$$\partial_- B^+_L - \partial_+ B^-_L + [B^-_L, B^+_L] = 0$$
$$\partial_- B^+_R - \partial_+ B^-_R + [B^-_R, B^+_R] = 0$$

are in fact equivalent to $[1]$. Consider the left and right
auxiliary linear systems

$$\partial_- \psi^L_{\alpha} + (B^-_L)_{\alpha}^\beta \psi^L_{\beta} = 0, \quad \partial_+ \psi^L_{\alpha} + (B^+_L)_{\alpha}^\beta \psi^L_{\beta} = 0,$$
$$\partial_- \psi^R_{\alpha} + (B^-_R)_{\alpha}^\beta \psi^R_{\beta} = 0, \quad \partial_+ \psi^R_{\alpha} + (B^+_R)_{\alpha}^\beta \psi^R_{\beta} = 0. \quad (5)$$

where $\psi^L_{\alpha}$ and $\psi^R_{\alpha}$ are two-component spinors$^1$. Each of these systems have two linearly independent solutions, denoted by $\psi^L_{\alpha,a}$ and $\psi^R_{\alpha,a}$, (with $a, \alpha, \alpha = 1, 2$). These can be normalized so that

$$\epsilon^{\beta \gamma} \psi^L_{\alpha,a} \psi^L_{\beta,b} = \epsilon_{ab}, \quad \epsilon^{\beta \gamma} \psi^R_{\alpha,a} \psi^R_{\beta,b} = \epsilon_{ab} \quad (6)$$

where $\epsilon$ is the $2 \times 2$ Levi-Civita tensor. Finally, the string
embedding is given by$^2$

$$Y_{\alpha a} = \begin{pmatrix} Y_{\alpha 1} + Y_{\beta 2} & Y_{\alpha 1} - Y_{\beta 2} \\ Y_{\alpha 1} + Y_{\beta 2} & Y_{\alpha 1} - Y_{\beta 2} \end{pmatrix}. \quad \psi^L_{\alpha,a} \psi^R_{\beta,b} = \psi^L_{\alpha,a} M^{\alpha \beta} \psi^R_{\beta,b} \quad (7)$$

where $M = \text{diag}(1,1)$.

4. Segmented strings. Once the smooth$^3$ string embedding is obtained, one can proceed by approximating it with a segmented string. Segmented strings are AdS generalizations of piecewise linear strings in flat space $[7,8]$. They consist of elementary AdS$_3$ segments which themselves are linear subspaces in $\mathbb{R}^{2,2}$. Kinks between the segments move with the speed of light which ensures that the string remains segmented at all times. The scalar curvature of the induced metric is constant everywhere, except for points on the worldsheet where left- and right-moving kinks collide. At these locations the curvature diverges (similarly to the case of a polyhedron).

The segmentation procedure is depicted in FIG.2. The diagram shows the string worldsheet (yellow surface) embedded into AdS$_3$. A time-slice on the worldsheet is indicated by a black line. One can discretize this curve according to any preferred scheme. This will result in a set of points $\{P_i\}$.

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$^1$ The index $\alpha$ is not to be confused with the sinh-Gordon potential.

$^2$ For the scattering problem in an $SU(1,1)$ basis, see $[6]$.

$^3$ In general the embedding will only be smooth almost everywhere.

Namely, the string may contain cusps which correspond to singular solitons in the sinh-Gordon theory.
These points are indicated by black dots in the figure. For a full set of initial data, one needs to further specify another set of points denoted by \( \{ Q_i \} \) (red dots in FIG. 2) which are null-separated from their neighbors. This condition means that they must lie at the triple intersection of future lightcones and the smooth worldsheet as in the figure. The initial data for the segmented string will then be defined by the null zigzag \( P_1, Q_1, P_2, Q_2, \ldots \). If the points are taken to be in \( \mathbb{R}^{2,2} \), then any three consecutive points define a linear AdS \(_2\) subspace: a segment of the string\(^4\). On the Poincaré patch the segments are given by the equations

\[
(x - x_i)^2 + y^2 - (t - t_i)^2 = R_i^2 \tag{8}
\]

where \( i \) labels the segments. Each segment has three parameters \( \{ x_i, t_i, R_i \} \). The total number of segments will be denoted by \( N \).

**4. Smoothing.** One can compute the generalized sinh-Gordon fields corresponding to a segmented string. Inside the AdS\(_2\) patches \( u = v = 0 \), on the kink worldlines and at the kink collision points (vertices) they are ill-defined. In order for these fields to take finite values, the segmented string has to be smoothed. This can be achieved by smoothing the initial conditions, for instance by using Mikhailov’s solution \(12\). The embedding is given in terms of the position function of the string endpoint on the boundary,

\[
t(t_r, y) = t_r + \frac{y}{\sqrt{1 - x_0(t_r)^2}}
\]

\[
x(t_r, y) = x_0(t_r) + \frac{y x_0'(t_r)}{\sqrt{1 - x_0'(t_r)^2}}
\]

where \( x_0(t_r) \) specifies the endpoint of the string in terms of the retarded time \( t_r \). The resulting string will have non-linear waves moving in one direction.

Consider the motion given by

\[
x_0(t_r) = \theta(\sqrt{1 + t_r^2} - 1),
\]

where \( \theta \) is the Heaviside function. Mikhailov’s embedding gives a segmented string that consists of two pieces: a vertical segment at \( x = 0 \) attached to another one with radius \( R = 1 \). The relativistic acceleration of the string endpoint (the inverse radius of the segment) jumps as seen in FIG. 3. One can smooth the kink by choosing a new \( x_0(t_r) \) function which is equal to \( x_0(t_r) \) outside the region \( (-\frac{\pi}{2}, \frac{\pi}{2}) \) and whose acceleration is smooth (red dashed line in the figure). The average slope of the acceleration function will be of the order of \( \Delta a/\varepsilon \) where \( \Delta a \) is the jump in the acceleration. In order to have an estimate for the \( u \) field across this region, the smooth motion of the endpoint can be approximated by taking \( x_0(t_r) = \Delta a t_r^2/\varepsilon \) which will produce a linear acceleration function for small positive \( t_r \). After changing to lightcone coordinates in Mikhailov’s embedding, one gets \( u \approx 6 \Delta a/\varepsilon \). This means that in balanced coordinates the width of the smoothed kink is \( \Delta k \) (see FIG. 4)

\[
L \propto \sqrt{u \varepsilon} \propto \sqrt{\Delta a \varepsilon}.
\]

If there are \( N \) segments, then \( \Delta a \) scales as \( N^{-1} \) and \( \varepsilon \) can also be taken to scale in the same way. The balanced linear size on the worldsheet is then proportional to \( N^0 \sqrt{\varepsilon} \).

In the final step of the transformation, the sinh-Gordon fields \( \alpha, u, v \) are computed. As \( N \to \infty \), the segmented string approaches the original smooth string and thus their time evolution will be the same.

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\(^4\) The segmented string is an exact Nambu-Goto string solution. Time evolution can be obtained by reflecting the points on their two neighbors (e.g. \( P_2 \) on \( Q_1 \) and \( Q_2 \)) as described in \(^7\)\(^11\).
5. An example. Consider the function

\[ \alpha = -\log \left[ k \sn^2 \left( \frac{\sigma}{\sqrt{2k}} - k^2 \right) \right] \] (9)

where \( \sn(u/m) \) is the Jacobi elliptic function and \( k \geq 1 \) is a parameter. It solves (1) with \( u = -v = 1 \), i.e. it is a static cosh-Gordon solution.

In order to compress the function, the first step is constructing the corresponding string embedding. The spinor solutions of (5) are found to be

\[ \psi^1 = e^{-\omega_-} \left( \sqrt{\frac{2}{\sn}} + \frac{2(\cn \dn + \sn \sqrt{k^2 - 1})}{\sn + \sqrt{\sn^2}} \right) \]

\[ \psi^2 = e^{-\omega_-} \left( \sqrt{\frac{1 + \sn^2}{8(k^2 - 1)}} \right) \] \( -\frac{\sn \sqrt{k^2 - 1} - \cn \dn}{\sn + \sqrt{\sn^2}} \)

\[ \psi^3 = e^{\omega_+} \left( \sqrt{\frac{2}{\sn}} - \frac{2(\cn \dn + \sn \sqrt{k^2 - 1})}{\sn + \sqrt{\sn^2}} \right) \]

\[ \psi^4 = e^{-\omega_+} \left( -\frac{\sn \sqrt{k^2 - 1} + \cn \dn}{\sqrt{8(k^2 - 1)}} \right) \] \( -\frac{1 - \sn^2}{8(k^2 - 1) \sn} \)

where

\[ \Pi_{\pm} = \Pi \left( \pm \frac{\tau}{\sqrt{2k}} \right) \]

\[ \omega_{\pm} = \sqrt{k - k^{-1}} \left( -\sqrt{k} \Pi_{\pm} + \frac{\tau + \sigma}{2\sqrt{2}} \right) \]

and \( \sn, \cn \) and \( \dn \) denote Jacobi elliptic functions with arguments \( \left( \frac{\sigma}{\sqrt{2k}} - k^2 \right) \), and \( \am \) is the Jacobi amplitude function. If one eliminates \( \tau \) in favor of Poincaré time \( t \), then the embeddings will have the form \( x(\sigma, t) = t \hat{x}(\sigma) \) and \( y(\sigma, t) = t \hat{y}(\sigma) \). Thus they scale linearly with time while their shape is constant.

The shape \( \hat{x}(\hat{y}) \) of such an embedding satisfies the equation of motion

\[ \hat{y} \left( \hat{x}^2 + \hat{y}^2 - 1 \right) \hat{x}'' + (2 - 2\hat{y}^2) \hat{x}^3 + 4\hat{x}\hat{y}\hat{x}^2 + (2 - 2\hat{x}^2) \hat{x}' = 0 \]

which can be derived from the Nambu-Goto action. FIG. 6(left) shows the embedding on the Poincaré patch for various values of \( k \). As \( k \to 1^+ \), at \( t = 1 \) the string is mapped into a vanishingly small region near the origin and its size increases slowly as time evolves. By applying an \( \SO(2, 2) \) isometry transformation the size of the string can be kept finite as the limit is taken. Thus at \( k = 1 \) the string is static. In the following, let us concentrate on this special case. Its shape \( x(y) \) satisfies the equation

\[ 2\hat{x}' + 2(\hat{x}')^3 - y \hat{x}'' = 0 \]

which is indeed invariant under rescaling the coordinates by a constant factor. The explicit solution centered at \( x = 0 \) has two branches\(^5\)

\[ x(y) = \mp x_0 \pm \frac{y^3}{3y_0^2} \frac{2F_1 \left( \frac{1}{2}, \frac{3}{4}, \frac{7}{4}, \frac{y^4}{y_0^4} \right)}{2} \]

where \( x_0 = \frac{\sqrt{\pi(4)}}{3\Gamma(4)} \) \( y_0 \) and \( y_0 \) is a parameter. The string touches the boundary at \( x = \pm x_0 \). In FIG. 6(middle) the embedding is plotted (background gray curve).

One can switch to balanced coordinates via the coordinate transformation given by

\[ t = \frac{y_0 \tau}{\sqrt{2}}, \quad y = y_0 \sn \left( \frac{\sigma}{\sqrt{2}}, -1 \right) \]

\(^5\) The Ricci scalar of the induced metric is \( R = -2 \left( 1 + (y/y_0)^4 \right) \).

Since \( R < R_0 \) (where \( R_0 = -2 \) is the curvature of AdS\(_3\)) the solution in balanced coordinates satisfies the cosh-Gordon equation. Close to the boundary \( (y \to 0) \) the solution asymptotes to AdS\(_2\).
In these coordinates \( u = -v = 1 \) and it is easy to check that \( \alpha \) is indeed given by (9) with \( k = 1 \).

The next step is to segment the string. Since in an appropriate frame the string is static, one can apply the shooting technique described in Appendix B of [11]. The only parameter is the number \( N \) of string segments. The resulting segmented string is periodic in Poincaré time. An example with \( N = 13 \) is depicted in FIG. 6 (middle). If each segment is subdivided into \( n \) pieces then one has \( nN \) segments in total. After the subdivision, a smoothing filter can be applied on the points. For instance one can repeatedly apply the transformation

\[
P'_i \propto \lambda(P_{i-1} + \lambda P_{i+1}) + P_i \quad \text{such that} \quad (P'_i)^2 = -1,\]

for points in the refined segmented string, and the same for \( Q'_i \). This step smoothes the points similarly to a discretized heat equation. After an iteration \( Q'_i - P'_i \) and \( Q''_i - P''_i \) are generically not null, but one can define new \( Q''_i \) positions such that they lie in the AdS2 spanned by \( \{P'_i, Q'_i, P'_{i+1}\} \) and that \( (Q''_i - P''_i)^2 = (Q''_i - P''_{i+1})^2 = 0 \). The points \( P''_i = P'_i \) are not changed. In the next iteration one smoothes the initial data given by the null zigzag \( \{P''_i, Q''_i\} \). Let \( q \) denote the number of smoothing iterations applied on the string. For sufficiently large \( n \), the resulting string is a good approximation for a smooth string. An example is depicted in FIG. 6 (right).

Finally, the generalized sinh-Gordon fields \( \alpha, u, v \) can be computed using worldsheet points and normal vectors which all take values in \( \mathbb{R}^{2,2} \). For instance in FIG. 7 for the patch with normal vector \( N_1 \) one has

\[
e^\alpha = a^{-2}(V_{10} - V_{00})(V_{01} - V_{00}) \quad \text{(10)}
\]

where \( a \) is the lattice spacing. A discrete version of (4) is given by

\[
v \approx a^{-2}(N_1 - N_2)(V_{01} - V_{00}) \quad \text{(11)}
\]

and an approximate \( u \) function can similarly be computed. FIG. 8 (middle) shows an example for \( v \) for a slightly smoothed string with \( N = 30 \) segments. After switching to balanced coordinates, the cosh-Gordon field is seen to have peaks traveling with the speed of light, see FIG. 8 (bottom). Time-slices are shown in FIG. 9. For comparison, the original potential given in [2] with \( N = 10 \) is plotted in black. The difference between the two figures is the number of smoothing iterations applied on the subdivided segmented string. Without any smoothing the balanced width of the string would be zero and the spikes would diverge.

Finally, one can check that the balanced width \( L \) of the compressed function indeed scales as expected. Based on the properties of the heat equation, one expects that an appropriate smoothing parameter can be defined as \( \varepsilon = \sqrt{q/n} \) and that \( L \propto \varepsilon \). This is indeed born out by numerical calculations: for an example with \( N = 10 \) segments a numerical fit gave \( L \propto n^{\gamma} \) with \( \gamma = -0.495 \pm 0.01 \) and \( L \propto q^\delta \) with \( \delta = 0.246 \pm 0.01 \), confirming the expectations.
FIG. 9: Top: A static cosh-Gordon solution from eqn. (9) with $k = 1$ (black curve) and its compressed version (blue curve) on a time-slice. The compressed function is obtained from a smoothed segmented string with $N = 30$ segments. The arrow indicates the reduction of the balanced width. The snapshot is taken at the moment when left- and right-moving peaks collide. Bottom: A smaller amount of smoothing yields a more compressed (and more singular) potential.

6. Discussion. This paper presented a transformation of the generalized sinh-Gordon field that approximately commutes with time-evolution. The transformation has been defined by constructing from the sinh-Gordon field a long Nambu-Goto string in AdS$_3$. The (almost everywhere) smooth string is approximated by a segmented string. The segmented string is smoothed and the new sinh-Gordon field is computed from the embedding. The new function occupies a smaller region in balanced co-ordinates in which the equation has a standard sinh- or cosh-Gordon form. The transformed function has peaks (see FIG. 9) which travel with the speed of light.

For the transformation, one can use various segmentation and smoothing schemes. In a given scheme one is still left with two parameters: the number of string segments $N$, and the smoothing parameter $\varepsilon$. It is plausible that in an appropriate large-$N$ limit the scheme-dependence goes away. It would be interesting to see if one can define the transformation in a closed form similarly to an auto-Bäcklund transformation.

The equations of motion for a string in AdS$_3$ have an internal $SO(2,2)$ symmetry acting on the vectors \[ \{ Y, e^{-\frac{\varepsilon}{2} \partial_- Y}, e^{-\frac{\varepsilon}{2} \partial_+ Y}, N \} \] [5]. Note that this symmetry has been broken by both smoothing procedures discussed in the paper.

The results made use of the equivalence of the sinh-Gordon equation and the string equation of motion in AdS$_3$. Similar equivalences exist in higher dimensional AdS spacetimes which allows for the extension of the compressing transformation to other systems, such as the $B_2$ Toda theory which is related to strings moving in AdS$_4$ [13].

The transformed function contains $N$ peaks which superficially look similar to boosted sinh-Gordon solitons. Solitons can be thought of as particles whose motion is governed by the hyperbolic Ruijsenaars-Schneider model with a special coupling value [14]. Since the balanced size of the transformed domain vanishes as $\varepsilon \to 0$, it is plausible to think that this gives an interesting limit of the model. Perhaps this limit leads to a dual description of the string.

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