EXACT SEQUENCES FOR LOCALLY CONVEX SUBALGEBRAS OF PIMSNER ALGEBRAS WITH AN APPLICATION TO QUANTUM HEISENBERG MANIFOLDS

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We prove six-term exact sequences of Pimsner-Voiculescu type for certain subalgebras of the Cuntz-Pimsner algebras. This sequence may, in particular, be applied to smooth subalgebras of the Quantum Heisenberg Manifolds in order to compute the generators of their cyclic cohomology. Further, our results include the known results for smooth crossed products. Our proof is based on a combination of arguments from the setting of Cuntz-Pimsner algebras and the Toeplitz proof of Bott-periodicity.

1 Introduction

In classical bivariant $K$-theory, the Pimsner-Voiculescu-six-term-exact-sequence (PMV-sequence) \cite{PV80} is one of the main results, allowing to calculate the $K$-theory of crossed products in many cases. While proved originally for monovariant $K$-theory, it can be proved more simply using $KK$-theory, see \cite{Bla98}. J. Cuntz (\cite{Cun84}) has given a proof that applies, at least partially, to all split exact, homotopy invariant stable functors on the category of $C^*$-algebras with values in the category of abelian groups. His proof is based on the notion of quasihomomorphism, a sort of linearized version of $*$-homomorphism. These techniques were later generalized to the setting of locally convex (or even bornological) algebras (\cite{CMR07}). The by now classical Toeplitz proof of Bott periodicity (\cite{Cun84}) may then be modified to yield a proof of the six-term exact sequence, and J. Cuntz gives in \cite{Cum05} an analogous sequence involving smooth crossed products in the locally convex setting.

Meanwhile, M. Pimsner presents in \cite{Pim97} a simultaneous generalisation of crossed products and Cuntz-Krieger algebras, and proves a PMV-type se-
sequence, using a certain universal algebra that is an extension of the algebra corresponding to the crossed products by a stabilisation of the base-algebra.

We will show that, under suitable conditions, one may construct such a sequence for locally convex subalgebras of the Pimsner algebras by considering certain subalgebras of the tensor products of the Toeplitz algebra already used for the construction of \( kk \)-theory in [Cun05]. One of the main results is then the identification of this extended algebra with the base algebra as formulated in Theorem 27. The proof is based on the techniques introduced by Cuntz in the Toeplitz proof of Bott periodicity in the smooth setting. The passage from crossed product to our setting naturally causes difficulties when considering the equivalence between the subalgebra generated by the canonical copy of the base algebra in its corresponding Toeplitz-algebra. It is here that we have to impose further size-restrictions on the algebras we start with.

Changing the stabilisations in the construction of \( kk \), or rather by passing to other versions of the smooth Toeplitz extension defined by using Schatten-classes, has some influence on the growth conditions we impose in section 7, but we will not go into this for the moment.

As an application, we construct six-term-exact-sequences for the Quantum Heisenberg Manifolds (QHM) in \( kk \) and cyclic theory, that allow to completely determine the cyclic invariants of the QHM. The \( C^* \)-QHM can be considered as Pimsner-algebras, and therefore the long exact sequences are available. When dealing with cyclic theory, one has to stay in the smooth category. We show thus that the smooth QHM satisfy our growth conditions, and our techniques thus apply. Consequently, we get Theorem 39 below as a concrete result.

We leave the application to an explicit calculation of the cyclic periodic cohomology of the QHM to a future article (see [Gab]).

2 Preliminaries

We denote the completed projective tensor product by \( \otimes_\pi \). By a locally convex algebra we will mean a complete locally convex vector space that is at the same time a topological algebra, i.e., the multiplication is continuous. Hence if \( A \) is a locally convex algebra, this means that for every continuous seminorm \( p \) on \( A \) there is a continuous seminorm \( q \) on \( A \) such that for all \( a, b \in A \)

\[
p(ab) \leq q(a)q(b).
\]

If \( A \) is a locally convex algebra, we denote by \( ZA \) the locally convex algebra of differentiable functions from \([0,1]\) all of whose derivatives vanish at the endpoints. We denote by \( ev_t^A: ZA \to A \) the evaluations in \( t \in I \).
Definition 1. Let $\varphi_0, \varphi_1 : A \to B$ be a homomorphism of locally convex algebras. A diffeotopy is by definition a homomorphism $\Phi : A \to ZB$ such that $ev_i^B \circ \Phi = \varphi_i$ for $i = 1, 2$.

Definition 2. The smooth compact operators are defined as those compact operators $A \in B(l^2\mathbb{N})$ such that, if $(a_{i,j})$ is the representation of $A$ with respect to the standard basis, then for all $m, n \in \mathbb{N}$:

$$\sup_{i,j \in \mathbb{N}} (1 + i)^n(1 + j)^m |a_{i,j}| \leq \infty.$$  

They are topologized by the increasing family of seminorms $||_m,n$ with

$$||A||_{m,n} := \sum_{i,j}(1 + i)^m(1 + j)^n|a_{i,j}|.$$  

Note that the elements of $K^\infty \otimes_\pi B$ are just the matrices with rapidly decreasing coefficients in $B$ (see [CMR07], chapter 2, 3.4).

Definition 3. A functor $H$ on the category of locally convex algebras with values in abelian groups is called

- $K^\infty$-stable if the natural inclusion $\theta : A \to K^\infty \otimes_\pi A$, $a \mapsto e \otimes a$ obtained by any minimal idempotent $e \in K^\infty$ induces an isomorphism

$$H(\theta) : H(A) \to H(K^\infty \otimes_\pi A),$$

- diffeotopy invariant, if $H(ev_t^A)$ is independent of $t \in [0, 1]$ for every locally convex algebra $A$,

- split exact if for every short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of locally convex algebras admitting a split, the sequence

$$0 \longrightarrow H(A) \longrightarrow H(B) \longrightarrow H(C) \longrightarrow 0$$

is exact.
3 Pimsner Algebras

In this section, we give a short introduction to the construction presented by M. Pimsner in \cite{Pim97}, and collect some necessary details. See also \cite{Kat03} for a generalized version (which we will only use later on). In the sequel, we will simply refer to a $U(1)$-action as a gauge-action.

For given right Hilbert module $E$ over a $C^*$-algebra $A$ we denote by $\mathcal{B}_A(E)$ the adjointable operators on $E$. Let $(E, \varphi)$ be a $C^*$-auto-correspondence of $A$. That is, $E$ is a Hilbert $A$-module, and $\varphi : A \to \mathcal{B}_A(E)$ a representation of $A$ on $E$ by adjointable operators. We assume throughout that $\varphi$ is injective and $E$ a full $A$-module, and we suppress $\varphi$ whenever there is no risk of confusion. The full Fock-space of $E$ is defined as vector space $\mathcal{F}_E := \bigoplus_{n=0}^{\infty} E \otimes_n$, where $E \otimes_n$ denotes the $n$-fold tensor product $E \otimes_A \ldots \otimes_A E$. There is a $U(1)$-action on $\mathcal{F}_E$ given by $\lambda.(\xi_1 \otimes \cdots \otimes \xi_n) := \lambda^n \xi_1 \otimes \cdots \otimes \xi_n$, which induces an $\mathbb{N}$-grading; the projection onto the degree-zero subspace with respect to this grading is denoted $Q$. The Pimsner-Toeplitz algebra $\mathcal{T}_E$ is defined as $C^*(T_{\xi} | \xi \in E) \subseteq \mathcal{B}(\mathcal{F}_E)$. Here $T_\xi$ is the operator

$$\mathcal{F}_E \ni \eta_1 \otimes \cdots \otimes \eta_n \rightarrow \xi \otimes \eta_1 \otimes \cdots \otimes \eta_n \in \mathcal{F}_E^{n+1}.$$ 

The gauge-action on $\mathcal{F}_E$ induces a gauge-action on $\mathcal{B}_A(E)$ by conjugation; $\mathcal{T}_E$ is invariant for this action. It thus inherits a $\mathbb{Z}$-grading, and we denote the subspace of degree $n$ by $T_{n}^E$.

Define the ideal $J(E) \subseteq \mathcal{B}_A(\mathcal{F}_E)$ as the sub-$C^*$-algebra generated by the subalgebras

$$\mathcal{B}_A \left( \bigoplus_{n=0}^{N} \mathcal{F}_E^n \right), \quad N \in \mathbb{N}.$$ 

The (Cuntz-)Pimsner algebra is by definition $\mathcal{O}_E := \mathcal{T}_E / J(E)$ (more precisely, the image of $\mathcal{T}$ in the quotient $\mathcal{B}(\mathcal{F}_E) / J(E)$); the $U(1)$-action preserves the quotient, hence passes to a Gauge action on $\mathcal{O}_E$ and induces again an $\mathbb{Z}$-grading. An element $x$ in $\mathcal{T}_E$ or $\mathcal{O}_E$ is of degree $n$ iff $\gamma_t(x) = e^{2\pi int}x$. Note also that the maps $\xi \mapsto T_\xi$ and $S \mapsto S_\xi$ are isometric and linear, where $S$ is $T$ followed by the quotient map.

We assume now that $E$ is an Hilbert $A$-bimodule with compatible scalar products $A(\cdot | \cdot)$ and $(\cdot | \cdot)_A$:

$$\forall \zeta, \xi, \eta \in E : \quad A(\zeta | \xi) \eta = \zeta (\xi | \eta)_A.$$ 

Then note that on vectors of degree greater zero

$$T_\xi T_\zeta (\eta_1 \otimes \cdots \otimes \eta_n) = \xi \otimes (\zeta | \eta_1)_A \eta_2 \otimes \cdots \otimes \eta_n = A(\xi | \eta) \eta_1 \otimes \cdots \otimes \eta_n,$$
and $T_ξT_ξ^* = 0$ on vectors of degree zero. Hence $T_ξT_ξ^* = (1 - Q)_A \langle ξ|η \rangle$. Thus in the Pimsner algebra, we have an honest equality $S_ξS_ξ^* = A \langle ξ|η \rangle$. Therefore, denoting the dual $E^*$ of $E$ by $E^\otimes^{-1}$, $O_E$ identifies to the Hilbert-module $\oplus_{n \in \mathbb{Z}} E^\otimes A n$, equipped with the obvious product.

4 Morita contexts and split exact functors

The following definition of Morita context is basically from [Cun05]. We add some isomorphisms to the definitions in [Cun05] in order to make the existence of a Morita-context a weaker condition than being isomorphic.

Definition 4. Let $A$ and $B$ be locally convex algebras. Then a Morita-context from $A$ to $B$ is given by data $(ϕ, D, ψ, ξ_i, η_i)$, where $D$ is a locally convex algebra, $ϕ : A \to D$, $ψ : B \to D$ are isomorphisms onto subalgebras of $D$, and sequences $η_i, ξ_i$ in $D$ such that

(i) $η_iϕ(A)ξ_j \in ψ(B)$ for all $i, j$

(ii) $(η_iϕ(a)ξ_j)_{ij} \in \mathbb{K}^\infty \otimes_π ψ(B)$

(iii) $\sum ξ_iη_iϕ(a) = ϕ(a)$ for all $a \in A$ (convergence in $ϕ(A)$).

With this definition:

- If $ϕ : A \to B$ is an isomorphism, then we get a Morita context $(ϕ^+, B^+, .^+)$ from $A$ to $B$, where $B^+$ is the unitization of $B$, $b \mapsto b^+$ the canonical embedding, and $ϕ^+ = ϕ \circ .^+$

- In particular, there is now a canonical Morita-context from $A$ to $A$, for any locally convex algebra $A$

- If $B \subseteq \mathbb{K}^\infty \otimes_π C$ is a subalgebra, and we call a corner in $B$ a subalgebra $A \subseteq B$ of the form $\sum_{i=0}^k e_{0i}B \sum_{i=0}^k e_{i0}$, then there is a trivial context from $A$ to $B$

- If $B$ is row and column-stable ($e_{0i}B, Be_{i0} \subseteq B$ for all $i$), then there is a context from $B$ to $A$, where $B$ is a subalgebra of $\mathbb{K}^\infty \otimes_π A$.

Definition 5. Let $H$ be a $\mathbb{K}^\infty$-stable functor, then we define $θ : A \to \mathbb{K}^\infty \otimes_π B$, $a \mapsto (ψ^{-1}(η_kϕ(a)ξ_j))_{ij}$ and $H(ϕ, D, ψ, ξ_i, η_i) := H(θ)$.

Note that for $a, a' \in A$ we have

$$(ψ^{-1}(η_kϕ(a)ξ_j))_{ij}(ψ^{-1}(η_kϕ(a')ξ_i))_{kl} = (ψ^{-1}(η_kϕ(a) \sum m ξ_mη_mϕ(a')ξ_i))_{ij}$$
which equals $\theta(aa')$ by [iii] in the definition of a Morita context. Hence $\theta$ is indeed a homomorphism.

**Definition 6.** A Morita-bicontext between locally convex algebras $A$ and $B$ is given by two Morita-contexts from $A$ to $B$ and $B$ to $A$ respectively, of the form $(\varphi, D, \psi, \xi^A_i, \eta^A_i)$ and $(\psi, D, \varphi, \xi^B_j, \eta^B_j)$ such that

1. $\varphi(A)\xi^A_i \xi^B_j \subseteq \varphi(A)$, $\eta^B_j \eta^A_i \varphi(A) \subseteq \varphi(A)$ (left compatibility)

2. $\psi(B)\xi^B_j \xi^A_i \subseteq \psi(B)$, $\eta^A_i \eta^B_j \psi(B) \subseteq \psi(B)$ (right compatibility).

**Theorem 7.** Given two Morita contexts as in the above definition, and a difftopy invariant, $\mathbb{K}^\infty$-stable functor $H$, we have

$$H(\varphi, D, \psi, \xi^A_i, \eta^A_i) \circ H(\varphi, D, \psi, \xi^B_j, \eta^B_j) = \text{id}_{H(A)}$$ if they are left compatible

$$H(\varphi, D, \psi, \xi^A_i, \eta^A_i) \circ H(\psi, D, \varphi, \xi^B_j, \eta^B_j) = \text{id}_{H(B)}$$ if they are right compatible

**Proof.** The proof is an adaptation of the one from [Cun05] Lemma 7.2. Denote the isomorphism $H(A) \rightarrow H(\mathbb{K}^\infty \otimes \pi A)$ given by $\mathbb{K}^\infty$-stability by $\varepsilon_A$. Then more precisely, we have to show that

$$H(\varphi, D, \psi, \xi^B_j, \eta^B_j) \circ \varepsilon_B^{-1} \circ H(\varphi, D, \psi, \xi^A_i, \eta^A_i)$$

is invertible. We suppose left compatibility; then denoting $\theta^A$ and $\theta^B$ the maps $A \rightarrow \mathbb{K}^\infty \otimes \pi B$ and $B \rightarrow \mathbb{K}^\infty \otimes \pi A$ determined by the two contexts, and multiplying by $\varepsilon_{\mathbb{K}^\infty \otimes \pi A}$ on the left, we see that it suffices to show that the composition $(\mathbb{K}^\infty \otimes \theta_B) \circ \theta_A$ induces an invertible map under $H$. Now this is the map

$$a \mapsto (\varphi^{-1}(\eta^B_i(\hat{\xi}^A_i(a)\xi^B_j)), i, j)$$

and it is difftopic to the stabilisation as follows. The $L \times L$ matrix with entries $\varphi^{-1}(\hat{\eta}_a(t)\varphi(a)\xi^A_\beta(t))$, $a, \beta \in \mathbb{N}^2 \cup \{0\}$ with

$$\hat{\xi}_0(t) = \cos(t)1 \quad \hat{\xi}_1(t) = \sin(t)\xi^B_j \xi^A_i$$

$$\hat{\eta}_0(t) = \cos(t)1 \quad \hat{\eta}_j = \sin(t)\eta^B_j \eta^A_i$$

yields a difftopy of the map in equation (1).
5 Quasihomomorphisms

We collect some basic properties of quasihomomorphisms as introduced first in [Cun83], and later adapted to the locally convex setting. See for example [CT06] and [Cun05] for details.

**Definition 8.** A quasihomomorphism between locally convex algebras $A$ and $B$ relative to $\hat{B}$, denoted $(\alpha, \bar{\alpha}) : A \rightrightarrows \hat{B} \triangleright B$, is given by a pair $(\alpha, \bar{\alpha})$ of homomorphisms into an algebra $\hat{B}$ that contains $B$ such that $(\alpha - \bar{\alpha})(A) \subseteq B$, $\alpha(A)B \subseteq B$ and $B\alpha(A) \subseteq B$.

Note that the definition is symmetric in $\alpha$ and $\bar{\alpha}$. Also, the definition can be generalized, basically by supposing that $\alpha, \bar{\alpha}$ have image in the multipliers of $B$, but we will not go into this. That quasihomomorphisms may be used as a tool to prove properties of $K$-theory (on the category of $C^*$-algebras) goes back to [Cun84].

**Lemma 9.** Let $(\alpha, \bar{\alpha}) : A \rightrightarrows \hat{B} \triangleright B$ be a quasihomomorphism, $H$ a split exact functor with values in the category of abelian groups. Then

(i) $(\alpha, \bar{\alpha})$ induces a morphism, denoted $H(\alpha, \bar{\alpha})$, from $H(A)$ to $H(B)$,

(ii) for every morphism $\varphi : D \rightarrow A$, $(\alpha \circ \varphi, \bar{\alpha} \circ \varphi)$ is a quasihomomorphism and:

$$H(\alpha, \bar{\alpha}) \circ H(\varphi) = H(\alpha \circ \varphi, \bar{\alpha} \circ \varphi)$$

(iii) if $\varphi : \hat{B} \rightarrow \hat{C}$ is a morphism and $C \leq \hat{C}$ an ideal such that $(\varphi \circ \alpha, \varphi \circ \bar{\alpha}) : A \rightrightarrows \hat{C} \triangleright C$ is a quasihomomorphism, then $H(\varphi \circ \alpha, \varphi \circ \bar{\alpha}) = H(\varphi) \circ H(\alpha, \bar{\alpha})$,

(iv) If $A$ is generated as a locally convex algebra by a set $X$, $B \leq \hat{B}$ an ideal and $\alpha, \bar{\alpha} : A \rightarrow \hat{B}$ are two homomorphisms such that $(\alpha - \bar{\alpha})(X) \subseteq B$, then $(\alpha, \bar{\alpha})$ is a quasihomomorphism $A \rightrightarrows \hat{B} \triangleright B$

(v) if $\alpha - \bar{\alpha}$ is a morphism and orthogonal to $\bar{\alpha}$, then $H(\alpha, \bar{\alpha}) = H(\alpha - \bar{\alpha})$

(vi) if $H$ is diffeotopy invariant, $(\alpha, \bar{\alpha}) : A \rightrightarrows Z\hat{B} \triangleright ZB$ a quasihomomorphism, then the homomorphisms $H(ev_t \circ \alpha, ev_t \circ \bar{\alpha})$ is independent of $t \in I$.

**Proof.** To prove (iv), we let $A$ be generated by $X$, and $x, x' \in X$; then

$$\alpha(xx') - \bar{\alpha}(xx') = (\alpha(x) - \bar{\alpha}(x'))\alpha(x') + \bar{\alpha}(x')(\alpha(x') - \bar{\alpha}(x')).$$

The proof of the other statements can be found in the references given above. □
For subsets $X$ and $Y$ in a locally convex algebra $B$, we say $Y$ is $X$-stable if $XY, YX \subseteq Y$.

**Definition 10.** Let $X,Y \subseteq B$ be subsets. Then $XYX$ is defined as the smallest closed $X$-stable, hence locally convex, subalgebra of $B$ containing $Y$.

$XYX$ is obviously independent of the size (but not the topology) of the ambient algebra: If $B \subseteq B'$ as a closed subalgebra and $X,Y \subseteq B$, then it doesn’t matter if we intersect $X$-stable subalgebras of $B$ or $B'$. We denote by $X^+$ the set $X$ with a unit adjoint.

**Lemma 11.** $XYX$ is the locally convex algebra generated by

$$\left\{ \sum_{finite} x_iy_ix_i' \big| x_i, x_i' \in X^+, y_i \in Y \right\}$$

and thus depends only on the locally convex algebras $LC(X)$ and $LC(Y)$ generated in $B$ by $Y$ and $X$, respectively:

$$XYX = XL(Y)X = LC(X)YL(X).$$

**Definition 12.** For a pair of morphisms $\alpha, \bar{\alpha} : A \rightarrow \hat{B}$ we call the quasihomomorphism $(\alpha, \bar{\alpha}) : A \rightrightarrows \hat{B} \triangleright B$ with $B := \alpha(A)(\alpha - \bar{\alpha})(A)\alpha(A)$ the associated quasihomomorphism. We call a quasihomomorphism minimal, if it is associated to $\alpha, \bar{\alpha} : A \rightarrow \hat{B}$.

By the above Lemma, if $X$ generates $A$, then $\alpha(X)(\alpha - \bar{\alpha})(X)\alpha(X) = \alpha(A)(\alpha - \bar{\alpha})(A)\alpha(A)$, using the notation of Lemma 11 above.

### 6 The Smooth Toeplitz algebra and Extension

We recall the definition and properties of the smooth Toeplitz algebra as introduced in [Cun97], compare Satz 6.1 therein:

**Definition 13.** The smooth Toeplitz algebra $T^\infty$ is defined as the direct sum $K^\infty \oplus C^\infty(S^1)$ of locally convex vector spaces with multiplication induced from the inclusion into the $C^*$-Toeplitz algebra.

It follows from [Cun97] that $T^\infty$ is a nuclear locally convex algebra. We denote $S$ and $\hat{S}$ the generators of $T^\infty$, set $e := 1 - SS$, and extract from the proof of Lemma 6.2 in [Cun97] the following fact:
Lemma 14. There exists a unital difftopy $\varphi_t$ such that

$$\varphi_t(S) = S(1 - e) \otimes 1 + f(t)(e \otimes S) + g(t)Se \otimes 1,$$

where $f, g$ are smooth functions such that $f(0) = 0$, $f(1) = 1$, $g(0) = 1$ and $g(1) = 0$ and all derivatives of $f$ and $g$ vanish in $0, 1$.

Thus $\varphi_0 : \mathcal{T}^{\infty} \rightarrow \mathcal{T}^{\infty} \otimes \pi \mathcal{T}^{\infty}$, $x \mapsto x \otimes 1$ is the canonical inclusion into the first variable, and $\varphi_1$ is the map determined by $S \mapsto \varphi_0(S^2S) + (1 - SS) \otimes S$.

To unclutter notation, we denote $\varphi_0(x) := \hat{x}$ and $e \otimes x =: \check{x}$ for all $x \in \mathcal{T}^{\infty}$.

We also recall that that the smooth Toeplitz algebra fits into an extension

$$0 \rightarrow \mathbb{K}^{\infty} \rightarrow \mathcal{T}^{\infty} \rightarrow \pi \mathcal{L}(U) \rightarrow 0$$

where $\mathcal{L}(U)$ denotes the algebra $\mathcal{C}^{\infty}(\mathbb{T})$ of smooth functions on the torus. The extension is linearly split by a continuous map $\rho : \mathcal{L}(U) \rightarrow \mathcal{T}^{\infty}$. The ideal is the algebra $\mathbb{K}^{\infty}$ of smooth compact operators, which is isomorphic (as a vector space) to $s \otimes s$, where $s$ is the space of rapidly decreasing sequences, and hence $\mathbb{K}^{\infty}$ is nuclear. We caution the reader that the smooth compacts are not isomorphism-invariant if we view them as represented inside some $\mathbb{K}$, where $\mathbb{K}$ denotes the compacts on a separable, infinite dimensional Hilbert space. They are rapidly decreasing with respect to a choice of base.

7 A Morita-Context for locally convex subalgebras of $\mathcal{O}_E$

We assume we are given a $C^*$-auto-correspondence $E$ over $A$, and a self-adjoint, locally convex subalgebra $D \subseteq \mathcal{O}_E$. Define $\mathcal{A} := D \cap A$ – considered as a locally convex algebra with the subspace topology. We define

$$\mathcal{E} := \{ \xi \in E | S\xi \in D \} = S^{-1}(D),$$

which is a linear subspace by linearity of $S$, a right module over the idealizer of $\mathcal{A}$ and has an $\mathcal{A}$-valued scalar product, as is easily checked. We assume throughout that $1_A \in \mathcal{A}$ and (for convenience), that $\mathcal{A}$ is complete.

Definition 15. We define $\mathcal{T}^{\infty}_D$, the Toeplitz-algebra of $D$, as the closed subalgebra of $D \otimes \pi \mathcal{T}^{\infty}$ (completed projective tensor product) generated by

$$\mathcal{A} \otimes 1, \ S(\mathcal{E}) \otimes S, \ \check{S}(\mathcal{E}) \otimes \check{S},$$

and by $\iota : \mathcal{A} \rightarrow \mathcal{T}^{\infty}_D$ the canonical inclusion. We denote $\mathcal{T}^0_D$ the algebra generated by the same generators, but without closure.
If we don’t assume that \( \mathcal{A} \) is complete, we have to deal later on with a functor on the category of noncomplete locally convex algebras. The tensor-products have to be replaced everywhere by noncompleted tensor products, and the subalgebras are then just the algebraically generated ones. Much of what we do still works in this case.

**Definition 16.** We define the algebra \( C \) as the closed subalgebra in \( D \otimes_{\pi} T^{\infty} \) generated by \( \mathcal{A} \otimes e, S(\mathcal{E}) \otimes Se \) and \( S(\mathcal{E}) \otimes eS \); denote \( \iota_{C} : \mathcal{A} \to C, a \mapsto a \otimes e \) the canonical inclusion. Define \( \alpha := \text{id}_{\mathcal{T}^{\infty}_{D}} \) and \( \bar{\alpha} := \text{Ad}(1 \otimes S)_{|T^{\infty}_{D}} \). Then \( (\alpha, \bar{\alpha}) : T^{\infty}_{D} \Rightarrow D \otimes_{\pi} T^{\infty} \supseteq C \) is a quasiisomorphism by design, in fact, the minimal quasiisomorphism associated to \( \alpha, \bar{\alpha} \).

Note that \( \bar{\alpha} \) is continuous, because multiplication in the locally convex algebra \( D \otimes_{\pi} T^{\infty} \) is continuous, hence \( \text{Ad}(1 \otimes S) \) is continuous when restricted to the subalgebra \( T^{\infty}_{D} \). Note that \( (\alpha, \bar{\alpha}) \) is minimal because \( C \) is \( \alpha(T^{\infty}_{D}) \)-stable.

We denote \( \pi_{D} : T^{\infty}_{D} \to D \otimes_{\pi} \text{LC}(U) \) the restriction of the projection (\( \otimes_{\pi} \) preserves surjectivity) defined as

\[
\text{id}_{D} \otimes_{\pi} : D \otimes_{\pi} T^{\infty} \to D \otimes_{\pi} \text{LC}(U).
\]

In order to simplify upcoming calculations, we extend the definition of \( S(\xi) = S_{\xi} \) as follows. If \( I \in \mathbb{N}^{n} \) is a multiindex, then we denote a tuple \( (\xi_{1}, \ldots, \xi_{n}) \) of elements of \( \mathcal{E} \) by \( \xi_{I} \) and define \( S_{\xi_{I}} := S_{\xi_{I_{1}}} \cdots S_{\xi_{I_{n}}} \); we extend \( \bar{S} \) in the same manner. We will also refer to \( \xi_{I} \) as an \( \mathcal{E} \)-multivector, and write \( |\xi_{I}| := |I| \).

Using the relations in \( D \) and Lemma 11, we see that \( C = T_{D}^{\infty}(1 \otimes e)T_{D}^{\infty} \) is the locally convex algebra generated by \( \sum x_{k,l} \otimes S_{k}eS_{l} \), where \( x_{k,l} \) is a finite sum of elements of the form \( S_{k}eS_{l} \) with \( |\xi_{I}| = k, |\xi_{J}| = l \) (and where we identify \( \mathcal{A} \) with elements of degree zero in the obvious way). Thus \( C \subseteq D \otimes_{\pi} \mathbb{K}^{\infty} \subseteq D \otimes_{\pi} T^{\infty} \).

We continue to fix a locally convex subalgebra \( D \subseteq \mathcal{O}_{E} \) such that \( D \cap \mathcal{A} \) is complete locally convex and unital. We want to analyse the exact sequence

\[
0 \longrightarrow \text{Ker}(\pi_{D}) \longrightarrow T^{\infty}_{D} \overset{\pi_{D}}{\longrightarrow} \text{Im}(\pi_{D}) \longrightarrow 0,
\]

and its homological invariants. To start out, we treat \( T^{\infty}_{D} \).

**Definition 17.** Let \( \xi_{I} \) be a sequence of \( \mathcal{E} \)-multivectors. Then we define

\[
\Xi_{i} := S_{\xi_{I}} \otimes eS_{|\xi_{I}|} \quad \text{and} \quad \bar{\Xi}_{i} := \bar{S}_{\xi_{I}} \otimes e\bar{S}_{|\xi_{I}|},
\]

the which are elements in \( D \otimes_{\pi} \mathbb{K}^{\infty} \).

**Lemma 18.** \( \bar{\Xi}_{i}C\Xi_{j} \subseteq \mathcal{A} \otimes e \) for all \( i, j \in \mathbb{N} \); for \( x = \sum_{k,l=1}^{\infty} x_{k,l} \otimes S_{k}eS_{l} \in C \)

\[
p \otimes q(\Xi_{i}x\Xi_{j}) = p(\bar{S}_{\xi_{I}}x_{|\xi_{I}|}S_{\xi_{J}})q(e)
\]

for every continuous seminorm \( p \) on \( \mathcal{A} \) and \( q \) on \( \mathbb{K}^{\infty} \).
Proof. Let $x \in C$, and assume first that $x = \sum_{k,l} x_{k,l} \otimes S^k e \bar{S}$ is a finite sum. Then
\[
\Xi_i x \Xi_j = \sum_{k,l} \tilde{S}_{\xi_i} x_{k,l} \bar{S}_{\xi_j} \otimes e S^{|\xi_i|} S^{|\xi_j|} e \bar{S}.
\]
\[
= \sum_{k,l} \tilde{S}_{\xi_i} x_{k,l} \bar{S}_{\xi_j} \otimes \delta_{|\xi_i|,|\xi_j|} e \bar{S}.
\]
\[
= \tilde{S}_{\xi_i} x_{|\xi_i|,|\xi_j|} \bar{S}_{\xi_j} \otimes e,
\]
and the latter is clearly an element of $A \otimes e$. By continuity, the result extends to $C$.

For $x$ a finite sum the above yields the equality, and for $x \in C$ with representation $x = \lim_n (x_n = \sum_{k,l} x_{n,k,l} \otimes S^k e \bar{S}^l)$ as a limit of finite sums $x_n$ we obtain
\[
p \otimes q(\Xi_i x \Xi_j) = \lim_n p \otimes q(\tilde{S}_{\xi_i} x_{n,|\xi_i|,|\xi_j|} \bar{S}_{\xi_j} \otimes e) = p \otimes q(\tilde{S}_{\xi_i} x_{|\xi_i|,|\xi_j|} \bar{S}_{\xi_j} \otimes e)
\]
from which the result follows.

For simplicity, we assume now that $i \mapsto |\xi_i|$ is of the form $(1, \ldots, 1, 2, \ldots, 2, 3, \ldots)$, where the same value is obtained on intervals of fixed length $l$, and that the products $S_{\xi_i}, \bar{S}_{\xi_i}$ of same length sum up to one. We will call such sequences $S_{\xi_i}, \bar{S}_{\xi_i}$ admissible.

**Lemma 19.** If $S_{\xi_i}, \bar{S}_{\xi_i}$ are admissible and $p(S_{\xi_i} \bar{S}_{\xi_i})$ of polynomial growth in $i$ for all continuous seminorms $p$, then $\left( \sum_{i=1}^N \Xi_i \Xi_i \right) N$ is an approximate unit for $D \otimes_{\pi} \mathbb{K}^\infty$.

Proof. Let $x \in D \otimes_{\pi} \mathbb{K}^\infty$, and present $x = \sum_{m,n=1}^\infty x_{m,n} \otimes S^m e \bar{S}^n$ with rapidly decreasing coefficients $x_{m,n} \in D$. Thus for fixed $N$
\[
\left( \sum_{i=0}^N \Xi_i \Xi_i \right) x = \sum_{m,n=0}^N S_{\xi_i} \bar{S}_{\xi_i} x_{m,n} \otimes S^{|\xi_i|} e \bar{S}^{|\xi_i|} S^m e \bar{S}^n
\]
\[
= \sum_{m,n=0}^N \sum_{i=0}^N (\delta_{|\xi_i|,m} S_{\xi_i} \bar{S}_{\xi_i}) x_{m,n} \otimes S^m e \bar{S}^n.
\]
Now let $\varepsilon > 0$, $p$ a continuous seminorm on $D$ and fix $s, t \in \mathbb{N}$. Choose a finite subset $F = \mathbb{N}_{\leq M} \times \mathbb{N}_{\leq M} \subseteq \mathbb{N}^2$ such that
\[
\sum_{(m,n) \in \mathbb{N}^2 \setminus F} p(x_{m,n}) \|S^m e \bar{S}^n\|_{s,t} \leq \varepsilon.
\]
We may use the hypothesis to ensure that $F$ has the property that for all $F \subseteq F'$
\[ \sum_{(m,n) \in \mathbb{N}^2 \setminus F, \hspace{1em} i \leq m} \text{sup}\{p(S_{\xi}, \bar{S}_{\xi})\} p(x_{m,n}) || S^m e \bar{S}^n ||_{s,t} \leq \varepsilon. \]

Let $N \geq IM$; then
\[ p \otimes ||_{s,t} \left( \sum_{i=0}^{N} S_{\xi_i} \bar{S}_{\xi_i} x - x \right) \]
\[ \leq \sum_{(m,n) \in F} p \otimes ||_{s,t} \left( \left( \sum_{i : i \leq N, |\xi_i| = m} S_{\xi_i} \bar{S}_{\xi_i} \right) - 1 \right) x_{m,n} \otimes S^m e \bar{S}^n \]
\[ + \sum_{(m,n) \in \mathbb{N}^2 \setminus F} p \otimes ||_{s,t} \left( \left( \sum_{i : i \leq N, |\xi_i| = m} S_{\xi_i} \bar{S}_{\xi_i} \right) - 1 \right) x_{m,n} \otimes S^m e \bar{S}^n \]
\[ \leq 0 + \sum_{(m,n) \in \mathbb{N}^2 \setminus F} p \otimes ||_{s,t} (x_{m,n} \otimes S^m e \bar{S}^n) \]
\[ + \sum_{(m,n) \in \mathbb{N}^2 \setminus F} p \otimes ||_{s,t} \left( \sum_{i : i \leq N, |\xi_i| = m} S_{\xi_i} \bar{S}_{\xi_i} x_{m,n} \otimes S^m e \bar{S}^n \right) \leq 2\varepsilon. \]

\[ \square \]

**Remark 20.** This result is certainly not optimal, but sufficient for our applications. One may suppose that the $S_{\xi_i} \bar{S}_{\xi_i}$ is just an approximate unit with an additional property (and thus not ordered by degree) on the appropriate sets, and a similar proof still goes through. Further, as the above condition is rather related to a lower bound for the size of the sequence $S_{\xi_i}$, it would be surprising if polynomial growth was strictly necessary at this point. After all, the $S_{\xi_i} \bar{S}_{\xi_i}$ are positive elements in the base-$C^*$-algebra that sum up to one.

**Definition 21.** We call $D$ tame if it admits admissible sequences $(S_{\xi_i})_i$, $(\bar{S}_{\xi_i})_i$ of polynomial growth.

**Theorem 22.** If $D$ is tame, then there is a Morita bi-context from $A$ to $C$.

**Proof.** This follows by combining Lemmas 18 and 19 above, and using as the inverse context the constant sequences $1 \otimes e$.

Hence by Theorem 7 we have the following

**Corollary 23.** If $H$ is a diffeotopy invariant $K^\infty$-stable functor on the category of locally convex algebras, then
\[ H(C) \sim H(A). \]
8 Equivalence of the Toeplitz and base algebra

We set \( \Phi : T_{D}^{\infty} \rightarrow Z(D \otimes T^{\infty} \otimes T^{\infty}) \) as the restriction of \( \text{id}_{D} \otimes (\varphi)_{t} \), where \( (\varphi)_{t} : T^{\infty} \rightarrow Z(T^{\infty} \otimes_{\pi} T^{\infty}) \) denotes the diffotopy from Lemma 11. Further define \( \Psi : T_{D}^{\infty} \rightarrow Z(D \otimes T^{\infty} \otimes_{\pi} T^{\infty}) \) as the restriction of the map which is constant \( \text{Ad}(1 \otimes \hat{S}) \), where \( \text{Ad}(1 \otimes \hat{S})(x) := (1 \otimes \hat{S})(x)(1 \otimes \hat{S}) \) for all \( x \in D \otimes_{\pi} T^{\infty} \otimes_{\pi} T^{\infty} \).

We let \( (\Phi, \Psi) : T_{D}^{\infty} \Rightarrow Z(D \otimes_{\pi} T^{\infty} \otimes_{\pi} T^{\infty}) \supset C' \) denote the minimal quasihomomorphism associated to \( \Phi, \Psi \), and \( C' := \text{ev}_{t}(C') \) the fibre over \( t \in [0, 1] \) of \( C' \). We have the following

Lemma 24. \( C' \) is the closed algebra generated by

\[
S_{\xi} \otimes (f(t)\hat{S} + g(t)\hat{S}\hat{e}), a \otimes \hat{e}, \hat{S}_{\xi} \otimes (f(t)\hat{S} + g(t)\hat{e}S), \quad \xi \in \mathcal{E}, a \in A.
\]

Proof. Using Lemma 11 it suffices to check that \( (\Phi - \Psi)(T_{D}^{\infty}) \) is \( \Psi(T_{D}^{\infty}) \)-stable, and it suffices to do so on the generators. We have got the product of sets:

\[
\{S_{\xi} \otimes (f(t)\hat{S} + g(t)\hat{S}\hat{e}), a \otimes \hat{e}\} \{S_{\xi}' \otimes \hat{S}^{2}\hat{S}, a' \otimes \hat{S}\hat{S}, \hat{S}_{\xi}' \otimes \hat{S}^{2}\}\} = 0
\]

because, in \( D \otimes_{\pi} T^{\infty} \otimes_{\pi} T^{\infty} \) we may factor the right hand set by a \( 1 \otimes \hat{S} \) on the left, which is orthogonal to the left factor. Further:

\[
\hat{S}_{\xi} \otimes (f(t)\hat{S} + g(t)\hat{e}S) \cdot S_{\xi} \otimes \hat{S}^{2}\hat{S} = 0
\]

by factoring out \( 1 \otimes \hat{S}^{2} \). But

(2) \[
(\hat{S}_{\xi} \otimes (f(t)\hat{S} + g(t)\hat{e}S)) \cdot (a' \otimes \hat{S}\hat{S}) = \hat{S}_{\xi}a' \otimes g(t)\hat{e}S.
\]

However,

\[(a \otimes \hat{e})(S_{\xi} \otimes (f(t)\hat{S} + g(t)\hat{S}\hat{e})) = aS_{\xi} \otimes f(t)\hat{S}\]

lies in the algebra, thus so does the result in equation (2) because \( A \) is unital. \( \Psi(T_{D}^{\infty}) \)-left stability follows similarly.

The fibre \( C_{t} \) is thus not constant. It’s size over intermediate points is larger than the endpoints. We denote \( \iota_{1} \) the inclusion of \( T_{D}^{\infty} \rightarrow C'_{t} \) induced by \( T^{\infty} \rightarrow T^{\infty} \otimes T^{\infty}, x \mapsto \hat{x} \).

Definition 25. We denote \( \hat{C} \) the closed subalgebra generated by the constants \( a \otimes k \otimes 1, S_{\xi} \otimes k \otimes S, \hat{S}_{\xi} \otimes k \otimes \hat{S} \) in \( Z(D \otimes_{\pi} T^{\infty} \otimes_{\pi} T^{\infty}) \).
Theorem 26. If $D$ is tame, then there is a Morita context $\Xi_i', \bar{\Xi}_i'$ from $C'$ to $\bar{C} \subseteq Z(D \otimes \pi T^\infty \otimes \pi T^\infty)$ that coincides with the context $\Xi, \bar{\Xi}$ in the fibre over zero.

Proof. We may use the constants $\Xi_i' := \Xi_i \otimes 1$ as a Morita context on $C'$. The conditions are checked as in the previous case for $\Xi_i$ and $\bar{\Xi}_i$.

We denote by $\sigma$ the resulting homomorphism

$$\sigma : C' \to Z(D \otimes \pi T^\infty \otimes \pi T^\infty) \otimes \pi K^\infty,$$

and by $\sigma_t$ its restriction to the fibre over $t$. Recall from Definition 15 that $\iota : A \to T_D^\infty$ is the inclusion, from Definition 16 that $\iota_C : A \to C$ denotes the canonical inclusion, and that $(\alpha, \bar{\alpha}) : T_D^\infty \to D \otimes T^\infty \cong C$ is the quasihomomorphism induced by the identity and $\text{Ad}(1 \otimes S)$.

Theorem 27. Let $H$ be a split exact functor from the category of locally convex algebras to the category of abelian groups. Set $x := H(\iota)$ and $y := H(\alpha, \bar{\alpha})$. Such that

(i) if $H(\iota_C) : H(A) \to H(C)$ is right invertible, $y$ is right invertible;

(ii) if $H$ is difftopy invariant and $K^\infty$-stable, $D$ tame, then $y$ is left invertible.

Hence if $D$ is tame and $H$ a split exact, difftopy invariant $K^\infty$-stable functor, $H(T_D^\infty) \sim H(A)$, and the isomorphism is implemented by $\iota$.

Proof. The first part is clear by Lemma 9. In fact:

$$H(\alpha, \bar{\alpha}) \circ H(\iota) = H(\alpha \circ \iota, \bar{\alpha} \circ \iota) = H(\iota_C).$$

Hence if $z$ is a right inverse for $H(\iota_C)$, then $H(\iota) \circ z$ is a right inverse of $y$.

In order to prove (ii) we show that there exists a morphism $z'$ such that $z' \circ y$ is left invertible. For $z'$, we take the inclusion of $D \otimes \pi T^\infty \to D \otimes \pi T^\infty \otimes \pi T^\infty$, in other words, we simply view $(\alpha, \bar{\alpha})$ as a quasihomomorphism into a larger algebra, namely as the quasihomomorphism $(\Phi_0, \Psi_0)$. Because $\varphi_\iota$ is unital, $\Phi_1(a \otimes 1) = a \otimes 1 \otimes 1 = (\text{Ad}(1 \otimes \hat{S})(a \otimes \hat{1})) + (a \otimes \hat{1})$,

$$\Phi_1(S_\xi \otimes S) = S_\xi \otimes (S^2 \hat{S} + \hat{S}) = (\text{Ad}(1 \otimes \hat{S})(S_\xi \otimes S)) + (S_\xi \otimes \hat{S}).$$

Furthermore $\text{Im(Ad}(1 \otimes \hat{S}))$ and $\text{Im}(\iota_1)$ are orthogonal because $\hat{S}e = 0 = eS$.

We now enlarge again the range of the quasihomomorphism by composing
with the stabilisation $\sigma$. We thus set $\bar{\Phi} := \sigma \circ \Phi$ and $\bar{\psi} := \sigma \circ \Psi$. This yields a quasihomomorphism

$$(\bar{\Phi}, \bar{\psi}) : T^\infty_D \to Z(D \otimes_{\pi} T^\infty \otimes_{\pi} T^\infty) \otimes K^\infty \supset \bar{C} \otimes_{\pi} K^\infty.$$ 

Note that $\bar{\Phi}_t = \sigma_t \circ \Phi_t$.

Hence we may apply Lemma 9 to deduce:

$$H(\bar{\Phi}_0, \bar{\Psi}_0) = H(\bar{\Phi}_1, \bar{\Psi}_1) = H(\sigma_1) \circ H(\text{Ad}(1 \otimes \hat{S}) \oplus \iota_1, \text{Ad}(1 \otimes \hat{S})) = H(\sigma_1) \circ H(\iota_1).$$

Now $\iota_1$ corresponds to the stabilisation map under the obvious isomorphism $C' \approx T^\infty_D \otimes K^\infty$, and therefore $y$ is left invertible.

Now if $D$ is tame, then $y$ has a right inverse because $H(\iota_C)$ has a right inverse – being invertible by Theorem 22. As we have shown that $y$ has also a left inverse above, $y$ is invertible (and $H(\iota_C) \circ H(\tau)$ is its inverse, where $\tau$ is the stabilisation coming from the Morita bicontext $C \to A$).

This proves the theorem. \hfill $\square$

The basic idea is to show invertibility by using a Morita context that identifies the noncanonical copy of $K(H) \otimes A$ sitting at 0 in $C'$ with a subalgebra of the stabilisation of $T_\alpha$, and extends from zero to a context on all fibres.

### 9 Determination of the Kernel and Quotient

We now determine the remaining terms in

$$0 \to \text{Ker}(\pi_D) \to T^\infty_D \to \text{Im}(\pi_D) \to 0.$$ 

We call the above sequence canonically split, if the split $\text{id}_D \otimes \rho : D \otimes_{\pi} C^\infty(S^1) \to D \otimes_{\pi} T^\infty$, where $\rho$ is the split of the smooth Toeplitz extension, restricts to a split of $\pi_D$. We suppose in all this section that the above sequence is canonically split.

**Proposition 28.** If $1 \otimes e$ is in $\text{Ker}(\pi_D)$, then

$$C = T^\infty_D(1 \otimes e)T^\infty_D = \text{Ker}(\pi_D).$$
Proof. $C = T_D^\infty(1 \otimes e)T_D^\infty$ is clear by Lemma 11. $T_D^\infty(1 \otimes e)T_D^\infty \subseteq \text{Ker}(\pi_D)$ is obvious by hypothesis. Because

$$0 \longrightarrow D \otimes \pi \mathbb{K}^\infty \longrightarrow D \otimes \pi T^\infty \overset{1 \otimes \pi}{\longrightarrow} D \otimes \pi LC(U) \longrightarrow 0$$

stays exact, $\text{Ker}(\pi_D) = \text{Ker}(1 \otimes \pi) \cap T_D^\infty$.

Now let $x \in \text{Ker}(\pi_D)$, $\varepsilon > 0$, and fix a continuous seminorm $p$ on $T_D^\infty$, choose a continuous seminorm $q$ such that there is $c_{\rho_D}$ with $p(\rho_D(y)) \leq c_{\rho_D}q(y)$, and choose a continuous seminorm $p'$ on $T_D^\infty$ such that $q(\pi_D(x')) \leq c_{\rho_D}p'(x')$ for all $x'$. As $x \in T_D^\infty$, we may choose a finite sum $x_0 = \sum_{I,J} S_{\xi_I} \tilde{S}_{\xi_J} \otimes S^{[I]} \tilde{S}^{[J]}$ with $p(x - x_0), p'(x - x_0) < \varepsilon$. Then $x_0 - \rho_D(\pi_D(x_0)) \in \text{Ker}(\pi_D)$ and

$$p(x - (x_0 - \rho_D(\pi_D(x_0)))) \leq p(x - x_0) + p(\rho_D(\pi_D(x - x_0))) \leq (1 + c_{\rho_D}c_{\pi_D})\varepsilon.$$

Now $x_0 - \rho_D(\pi_D(x))$ is easily seen to be an element in the ideal generated by $1 \otimes e$ in $T_D^\infty$, and thus the other inclusion is proved.

**Definition 29.** Let $D \subseteq \mathcal{O}_E$ be a locally convex algebra. Then we call $D$ gauge-smooth if the application

$$\mathcal{O}_E \rightarrow \mathcal{C}(\mathbb{T}) \otimes \pi \mathcal{O}_E, \ x \mapsto [t \mapsto \gamma_t(x)]$$

restricts to a continuous homomorphism

$$\tau : D \rightarrow \mathcal{C}^\infty(\mathbb{T}) \otimes \pi D.$$

We say $D$ is generated by $\mathcal{E}$ if the algebra generated by $S(\mathcal{E})$, $\tilde{S}(\mathcal{E})$ and $D \cap A$ is dense in $D$.

In the setting of metrizable algebras, it suffices that $D$ be invariant and its elements smooth for the gauge action.

**Proposition 30.** Let $D \subseteq \mathcal{O}_E$ be a locally convex gauge-smooth subalgebra generated by $\mathcal{E}$. Then $\text{Im}(\pi_D)$ is isomorphic to $D$.

**Proof.** The existence of the splitting implies that the image of $\pi_D$ is closed, hence generated by $S_{\xi} \otimes U$, $\tilde{S}_{\xi} \otimes U$ and $A \otimes 1$. The (co)restriction of

$$\text{id} \otimes \text{ev}_1 : \mathcal{O}_E \otimes \mathcal{C}(\mathbb{S}^1) \rightarrow \mathcal{O}_E \otimes \mathbb{C} \simeq \mathcal{O}_E,$$

to $\text{Im} \pi_D$ defines a continuous morphism of algebras $\text{Im} \pi_D \rightarrow D$. Checking on generators, one sees that $\tau$ is it’s inverse. $\square$
10 The long exact sequences

We now use the theory $kk$ to show that there are exact sequences in $kk$ and bivariant cyclic theory that are compatible with the Chern-Character and the boundary maps. In particular, we get, specialising the first algebra to $C$, such sequences in $K$-theory and cyclic theory.

However, to give our first result, we do not really need $kk$-theory, but we obtain a six-term-exact sequence for every half exact $K^\infty$-stable functor.

**Definition 31.** A functor $H$ on the category of locally convex algebras with values in abelian groups is called half-exact if for every linearly split short exact sequence

$$0 \to A \to B \to C \to 0$$

the sequence

$$H(A) \to H(B) \to H(C)$$

obtained by applying $H$ is exact.

**Definition 32.** If $D \subseteq \mathcal{O}_E$ is a unital locally convex subalgebra of a Pimsner algebra such that $D \cap A \subseteq D$ is closed, then we say that $D$ is a smooth Pimsner algebra if it is tame, gauge-smooth, generated by $E$ and canonically split.

Note that $1 \otimes e \in \ker(\pi_D)$ holds because there is a frame for $E$ inside $D$ by hypothesis.

**Theorem 33.** Let $D \subseteq \mathcal{O}_E$ be a smooth Pimsner algebra that. Then for every half-exact, difftimey invariant and $K^\infty$-stable functor there is a six-term exact sequence

$$H(A) \to H(A) \to H(D)$$

$$H(\mathcal{S}D) \gets H(\mathcal{S}A) \gets H(\mathcal{S}A)$$

where $\mathcal{S}$ denotes the smooth suspension.

**Proof.** First, extend the half exact-functor in the usual fashion to a homology theory $H_n$ – compare, e.g., Lemma 4.1.5 [CT06]. Observe that $H_n$ has automatically Bott-periodicity, either by carrying over the proof from [Cum05], or by universality of $kk$. Apply the long exact sequence for $H$ and Theorem 8.5 to the short exact sequence

$$0 \to \ker(\pi_D) \to T_D^\infty \to \pi_D \to \im(\pi_D) \to 0.$$
By Theorem 22, we may replace the kernel, using the Morita bi-context, with \(A\). Furthermore \(T_\infty^D\) may be replaced by \(A\) using Theorem 27 (half exactness implies split exactness). An application of Proposition 30 shows that \(\text{Im}(\pi_D)\) can be replaced with \(D\).

We now make use of the version of \(kk\) for \(m\)-algebras introduced in [Cun97].

**Theorem 34.** Let \(D\) be as above; suppose \(D\) is an \(m\)-algebra, and the Chern-Connes character is a complex isomorphism on \(A\) (eventually with coefficients), then it is also a complex isomorphism \(HP(D) \to K_0(D)\) (with coefficients). In particular, this holds with coefficients \(C\) for any \(D\) with commutative base \(A\).

**Proof.** This follows by an application of the five Lemma to

\[
\begin{align*}
&\xymatrix{
HP_*(B, A) \ar[rr] && HP_*(B, A) \\
& HP_*(B, D) \ar[dl] \ar[urr] & \ar[dl] \\
kk_*(B, A) & & kk_*(B, A) \\
& kk_*(B, D) & \ar[u] \\
}
\end{align*}
\]

where the maps between \(kk\) and cyclic theory denote the bivariant Chern-Connes character from [Cun97], the others come from the long exact sequence.

\[\square\]

**11 Application to Quantum Heisenberg Manifolds**

**Definition 35 (Quantum Heisenberg Manifolds).** Given two real numbers \(\mu, \nu\) and an integer \(c > 0\), we define the Quantum Heisenberg Manifolds (QHMs or \(D^c_{\mu, \nu}\)) as the \(C^*\)-completion of the algebra

\[
D_0 = \{ F \in C_c(\mathbb{Z} \to C_b(\mathbb{R} \times S^1)) | F(p, x + 1, y) = e(-cp(y - p\nu))F(p, x, y)\}
\]

where \(e(x) = e^{2\pi ix}\), equipped with the multiplication:

\[
(F_1 \cdot F_2)(x, y, p) = \sum_{q \in \mathbb{Z}} F_1(x, y, q)F_2(x - q2\mu, y - q2\nu, p - q),
\]

and where we have now switched to the notational standard, writing the \(\mathbb{Z}\)-variable \(p\) in the last component.
In the following, we will write $D$ instead of $D_{\mu,\nu}^c$ whenever there is no risk of confusion. One can prove that the involution is:

$$F^*(x, y, p) = F(x - 2p\mu, y - 2p\nu, p).$$

It has been proved in [AEE98] that $D$ carries an action of $S^1$ – the gauge action – that turns it into a generalised crossed product. Moreover, in [Kat03], Katsura proved that all generalised crossed products are (Katsura-)Pimsner algebras.

The grading of $D$ as a Pimsner algebra, is given by the spectral subspaces of the gauge action. Hence:

$$(4) \quad D^{(n)} = \{ F(p, x, y) = \delta_{n,p} f(x) | f(x + 1, y) = e(-cn(y - n\nu))f(x, y) \}$$

**Lemma 36.** Given $(c, \mu, \nu) \in \mathbb{N}^* \times \mathbb{R}^2$ and an integer $n \in \mathbb{Z}$, there is a frame of two elements $\xi^n_1, \xi^n_2$ in $D^{(n)}$. In fact

$$(\xi^n_1)^* \xi^n_1 + (\xi^n_2)^* \xi^n_2 = 1 \quad \xi^n_1(\xi^n_1)^* + \xi^n_2(\xi^n_2)^* = 1.$$ 

Furthermore, we can choose $\xi^n_1$ and $\xi^n_2$ to be smooth functions.

**Proof.** On $D^{(0)}$, we get a frame from $\xi_1 = 1$, $\xi_2 = 0$. For $n \neq 0$, let $U$, $V$ be small neighbourhoods of $[0, 1/2]$ and $[1/2, 1]$, respectively, and $f_1, f_2$ a partition of unity subordinate to $U$, $V$. Set $\chi_i = \sqrt{f_i}$; thus $\chi_1^2 + \chi_2^2 = 1$.

We define $\xi^n_1$ on $U \times S^1$ by setting $\xi^n_1(x, y) = \chi_1(x)$. Choosing $U$ and $V$ small enough $\xi_1$ and, $\xi_2$ may be assumed to vanish with all derivatives on the boundary of a fundamental domain, and using the equation (4), $\xi^n_1$ can then be extended to an element of $D^{(n)}$.

A similar process can be applied to $\chi_2$ to obtain $\xi^n_2$. An easy computation on a well chosen fundamental domain yields

$$(\xi^n_1)^* \xi^n_1 = \chi_1 \chi_i \quad \xi^n_1(\xi^n_1)^* = \chi_1 \chi_i,$$

and then $\chi_1^2 + \chi_2^2 = 1$ ensures that

$$(\xi^n_1)^* \xi^n_1 + (\xi^n_2)^* \xi^n_2 = 1 \quad \xi^n_1(\xi^n_1)^* + \xi^n_2(\xi^n_2)^* = 1.$$ 

Finally, notice that if $\chi_i$ is smooth, then so is $\xi^n_i$. \qed

Recall that the Heisenberg group $H_1$ is the subgroup of $GL_3(\mathbb{R})$ of the matrices

$$M(r, s, t) := \begin{pmatrix} 1 & s & t \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix} \quad \text{where } r, s, t \in \mathbb{R}$$
It has been proved (see [Rie89], section 5) that the Heisenberg group acts on \( D \). We use the following expression for the action:

\[
\alpha_{(r,s,t)}(F)(x, y, p) = e^\left(-p(t + cs(x - r - p\mu))\right) F(x - r, y - s, p).
\]

The infinitesimal generators \( \partial_i \), for \( i = 1, 2, 3 \) are

\[
\partial_1(F)(x, y, p) = -\frac{\partial F}{\partial x}(x, y, p) \quad \partial_3(F)(x, y, p) = -i2\pi p F(x, y, p)
\]

\[
\partial_2(F)(x, y, p) = -\frac{\partial F}{\partial y}(x, y, p) - i2\pi cp(x - p\mu) F(x, y, p).
\]

Note that \( \alpha_{(0,0,t)} \) is just the gauge action. As \( M(0,0,t) \) is in the center, the action of \( H_1 \) preserves the grading and \( \partial_1, \partial_2 \) commute with \( \partial_3 \). A short calculation shows:

\[
[\partial_1, \partial_2] = -c\partial_3 \quad [\partial_1, \partial_3] = 0 \quad [\partial_2, \partial_3] = 0
\]

**Definition 37.** We define the smooth Quantum Heisenberg Manifold \( D \) as the subalgebra of \( H_1 \)-smooth elements of \( D \).

Note that \( D \) inherits the \( \mathbb{N} \)-grading from \( D \), and that \( D^1 \) is actually a bimodule over \( D^0 \).

We recall for the reader’s convenience that the algebra of smooth elements for a Lie-group action on a \( C^\ast \)-algebra \( A \) form a holomorphically closed Fréchet subalgebra \( A \) of \( A \), equipped with the family of seminorms induced by the action of the universal enveloping algebra \( U(\mathfrak{g}) \) of the Lie algebra.

**Proposition 38.** The smooth Quantum Heisenberg Manifold is a Fréchet algebra and a smooth Pimsner algebra.

**Proof.** Unitality is clear by definition, and \( D \) is Fréchet as noted above. \( D \cap D^0 = C^\infty(\mathbb{T}^2) \) with it’s natural topology, and is thus closed. It is gauge smooth, because \( \alpha_{(0,0,t)} \) is the gauge action itself (compare the remark after Definition 29).

It remains to show that \( D \) is tame. By Lemma 36, there is a frame with two elements for every \( D^n \); denote \( \xi_n \) the sequence of frames ordered by degree. Then it clearly satisfies the conditions on the degree. Because scaling and adding do not change the equivalence-class of a family of seminorms, we may apply Poincaré-Birkoff-Witt (see [Hum78], 17.4) to see that it is enough to show that the seminorms of the form

\[
p_{n_3,n_2,n_1}(d) := ||\partial_3^{n_3}\partial_2^{n_2}\partial_1^{n_1}(d)||_\infty, \quad d \in D
\]
yield polynomially increasing sequences on $\xi_n$. We show: There is a constant $C$ that does not depend on $n$ such that

\[(7) \quad p_{n_3,n_2,n_1}(\xi_n) \leq C(1+n)^{n_3+2n_2}.\]

Trivializing over $U$ or $V$, we may assume that $\xi_n$ is $\chi_1$ or $\chi_2$ (compare Lemma 36) and calculate in the first case:

$$||p_{n_3,n_2,n_1}\xi_n|| = ||\partial_{n_3}^3\partial_{n_2}^2\partial_{n_1}^1\chi_1|| \leq Cn^{n_3}||\partial_{n_2}^2\chi_1^{(n_1)}|| \leq Cn^{n_3+n_2}||(x+|\mu|n)^{n_2}||$$

where $C$ is a universal constant, from whence the above inequality follows in this case. The case of $\chi_2$ is treated similarly.

Hence, applying Theorem 33 to $kk$ and $HP$, we get

**Theorem 39.** There is a commutative diagram

\[
\begin{array}{cccccc}
K_0(\mathcal{C}^\infty(T^2)) & \longrightarrow & K_0(\mathcal{C}^\infty(T^2)) & \longrightarrow & K_0(D) \\
\downarrow & & \downarrow & & \downarrow \\
HP_0(\mathcal{C}^\infty(T^2)) & \longrightarrow & HP_0(\mathcal{C}^\infty(T^2)) & \longrightarrow & HP_0(D) \\
\downarrow & & \downarrow & & \downarrow \\
HP_1(D) & \longrightarrow & HP_1(\mathcal{C}^\infty(T^2)) & \longrightarrow & HP_1(D) \\
\downarrow & & \downarrow & & \downarrow \\
K_0(D) & \longrightarrow & K_1(\mathcal{C}^\infty(T^2)) & \longrightarrow & K_1(\mathcal{C}^\infty(T^2))
\end{array}
\]

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