Associative Triple Systems with Nondegenerate Bilinear Forms

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Abstract

This paper is bringing a better knowledge of associative triple systems and their related algebraic structures. We prove that any associative triple system is either a $T^*$-extension of an associative triple system or an ideal of codimension one of a $T^*$-extension of an associative triple system. Moreover, we give several information about the structure of symmetric associative triple systems.

Keywords: Associative triple systems; Lie triple systems; $T^*$-extension

Introduction

Associative algebras with $n$-ary compositions in general and associative triple systems in particular play important roles in Lie and Jordan theories, geometry, analysis. For instance, associative triple systems give rise to Jordan triple systems [1-3] and Jordan triple systems give rise to 3-graded Lie algebras through the Tits Kantor Koecher construction [4,5], from which most simple Lie algebras can be obtained. Jordan triple systems also give rise to Lie triple systems through the K. Meyberg construction [6,7]. On the other hand, Lie triple systems give rise to graded Lie algebras, which are exactly the kind of Lie algebras associated to symmetric spaces. In geometry and analysis, various types of Jordan triple systems are used in the classifications of different classes of symmetric spaces [8-10].

After some preparations, using the notion of $T^*$-extension, we will show, in the first section, that there exist a symmetric associative triple systems which is not semisimple. This notion was introduced by Bordemann, who proved that every Jordan algebra of even dimension, which contains an isotropic ideal of dimension $n/2$, is a $T^*$-extension of a Jordan algebra and every odd dimensional Jordan algebra is an ideal of codimension one of a $T^*$-extension. Lin et al. extended this notion to Lie triple systems. We will show that the same theorem holds for associative triple systems. The proof in this case is different from the Jordan algebra case. It relies the construction of Lie triple systems associated to an associative one.

Definition 1

An associative triple system is a vector space $A$ over a field $K$ with a trilinear multiplication $(a,b,c)$ satisfying

\[
\{[a,b,c,d,e]=[a,b,c,d,e]=[a,b,c,d,e],
\]

for any $a,b,c,d,e \in A$.

Definition 2

A derivation of an associative triple system $A$ is a linear transformation $D$ of $A$ into $A$ such that

\[
D([a,b,c])=[D(a),b,c]+[a,D(b),c]+[a,b,D(c)],\forall a,b,c \in V
\]

The set of all derivations of $V$ is denoted by $Der(A)$. The set $Der(A)$ of derivations of $A$ is a Lie algebra of linear transformations, we call it the derivation algebra of $A$. Further, if $a,b \in A$, the linear maps $L(a,b)$ (resp.$R(a,b)$) defined on $A$ by $L(a,b) = (a,b,c)$ (resp. $R(a,b) = (c,a,b)$), $\forall c \in A$, is a derivation of $A$. The linear maps $L(a,b)$ (resp.$R(a,b)$) are called the left (resp.right) multiplications of $A$. $L(T,T) = \{ \sum_{\alpha \in J} a_\alpha \in A \}$ is a subalgebra of $Der(A)$, we call it we call it the inner derivation algebra of $A$, its element is called an inner derivation of $A$.

Definition 3

(1) A symmetric bilinear form $B$ on an associative triple system $A$ is called to be invariant if it is right and left invariant. That is

\[
\omega(a,[d,c,b]) - \omega(b,[d,c,a]) + \omega(c,[b,a,d]) + \omega(d,[a,b,c]) = 0,
\]

It was proved [11] that if $B$ is a symmetric and right invariant bilinear form on $A$, then $B$ is left invariant. Therefore, a symmetric bilinear form $B$ is invariant if and only if it is right invariant.

(2) We say that $(A,B)$ is a symmetric associative triple system if $B$ is a non-degenerate symmetric invariant bilinear form on $A$. Here $B$ is called a symmetric structure on $A$. A symmetric associative triple system $(A,B)$ is said to be reducible (or $B$-reducible) if it admits an ideal $I$ such that the restriction of $B$ to $I \times I$ is non-degenerate. Otherwise, we will say $(T,B)$ is irreducible.

(3) We say that $(A,\omega)$ is a symplectic associative triple system if $\omega$ is a non-degenerate skew-symmetric bilinear form on $T$ such that the identity

\[
\omega(a,[d,c,b]) - \omega(b,[d,c,a]) + \omega(c,[b,a,d]) + \omega(d,[a,b,c]) = 0,
\]

holds for any $a,b,c,d \in A$. Here $B$ is called a symplectic structure on $A$.

Definition 4

An element $\mathbb{f} \in Hom(A,A)$ is called $B$-symmetric, (resp.$B$-antisymmetric) if $B(f(a),b)=B(a,f(b))$, (resp.$B(f(a),b)=-B(a,f(b))$, $\forall a,b \in A$. Denote by $Hom(B,A)$ (resp.$Hom(B,A)$ the subspace of $B$-symmetric (resp.$B$-antisymmetric) endomorphism of $A$.

Theorem 1: A symmetric associative triple system $(A,B)$ admits a symplectic form $\omega$, if and only if there exists a $B$-antisymmetric invertible derivation $\delta$ of $A$ such that $\omega(a,b)=B(\delta(a),b), \forall a,b \in A$.

Proof. Since $B$ and $\omega$ are two nondegenerate bilinear forms, there
exists \( \delta \in \text{Hom}(A,A) \), such that \( \omega(a,b) = B((a,b),a,b,a) \). Further, since \( \omega \) is symplectic, then:
\[
B(B((a,c,d),d)) + B(B((a,d,c),d)) + B(B((a,b,c),d),d), \quad \forall a,b,c,d \in A.
\]

The fact that \( B \) is nondegenerate implies that \( \omega \) is a derivation of \( A \). Conversely, if \( \delta \) is a \( B \)-antisymmetric invertible derivation of \( A \), then it is clear that the bilinear form \( \omega: A \otimes A \rightarrow K \) defined by \( \omega(a,b) = B((a,b),a,b,a) \), is a symplectic form of \( A \).

**Theorem 2:** Let \( A_{n\times n} \) be a associative triple system and \( \Theta: A \times A \times A \rightarrow A \) be a trilinear map. Let \( T_{n}(A) = A \otimes A \) on which it is defined a symmetric bilinear distributive triple product

\[
(a + f, b + g, c + h) = (a,b,c) + (a,g,c) + (a,b,h),
\]

Then 

\[
(f, c)(d) = f((a, b, c))(d) = (g, d)(c) = (b, a, d),
\]

is an associative triple system if and only if \( \Theta \) satisfies

\[
\Theta(a, b, (c, r, t)) + \Theta(a, (b, c), r, t) = \Theta((a, b), c, r, t) + \Theta(a, b, (c, r, t)),
\]

\forall a, b, c, r, t \in A.

Furthermore, the bilinear form \( B \) defined on \( T_{n}(A) \) by

\[
B(a + f, b + g, c + h) = g(a) + f(b), \forall a, b, c, d \in A, f, g \in D,
\]

is an invertible scalar product on \( T_{n}(A) \) if and only if \( \Theta \) satisfies

\[
\Theta(a, b, c)(d) = \Theta(b, a, d)(c), \forall a, b, c, d \in A.
\]

The constructed symmetric associative triple system \( T_{n}(A, B) \) is called the \( T_{n} \)-extension of \( A \) by means of \( \Theta \).

**Proof.** Computation [12,13].

**Theorem 3:** Let \( A \) be a associative triple system which admits an invertible derivation \( D \) and a trilinear map \( \Theta: A \times A \times A \rightarrow A \) satisfying (2) and (3). Then the \( T \)-extension \( T_{n}(A) \) of \( A \) admits a symplectic structure.

Let \( \hat{D} \in \text{Hom}(T_{n}(A), T_{n}(A)) \) defined by

\[
\hat{D}(a + f) := D(a + f) - D(a), \forall a \in A \text{ and } f \in A^{*}.
\]

Then, \( \hat{D} \) is an invertible B-symmetric derivation of \( T_{n}(A) \). We get the result by Theorem 0.1.

**Example 1:** Let \( A \) be an associatiff triple system and \( n \in \mathbb{N} \). Consider the non-unitary associative algebra \( F_{t} = k[K]/t^{r}K[t] \). Define the bracket on the vector space \( A_{n} = T \otimes F_{t} \) by

\[
[x \otimes r, y \otimes s, z \otimes r'^{s}] = \{x, y, z \} \otimes r^{s}r'^{s},
\]

where \( x, y, z \in A \) and \( p, q, r \in \mathbb{N} \). Then \( A_{n} \) is a nilpotent associative triple system. The endomorphism \( D_{n} \) of \( A_{n} \) defined by

\[
D(x \otimes r) = p(x \otimes r)
\]

for any \( x \in A \) and \( p \in \mathbb{Z} \) is an invertible derivation of \( A_{n} \). Define the product on the vector space \( T_{n} = T \otimes F_{t} \) by

\[
[x + f, y + g, z + h] = [x,y,z]_{A} + [f,y,z]_{A} + [x,g,z]_{A} + [x,y,h]_{A}.
\]

With

\[
[f,y,z]_{A} = f([a,z,y]) \quad \text{and} \quad [x,y,h]_{A} = h([x,y,a]),
\]

for all \( x, y, z, a \in A_{n} \).

Define the bilinear form on \( T_{n} \) by \( B(X + f, Y + g) = f(Y) + g(X) \). Then \( (T_{n}, B) \) is a symmetric associative triple system. Define \( D \) on \( T_{n} \) by

\[
\hat{D}(X + f) = D(X) - f. \quad \text{Then,} \quad \hat{D}
\]

is a symmetric derivation which is skew-symmetric with respect to \( B \). Hence, the symmetric associative triple system \( (T_{n}, B) \) admits a symplectic structure.

**Proposition 1:** Let \( V \) be an associative triple system and \( Z(V) = \{a \in V \mid [a,b,c] = 0, \forall b,c \in V \} \) be the center of \( V \). Then, \( (Z(V))^{2} = T(V, V, V) \). Conversely, \( (Z(V))^{2} = T(V, V, V) \) using the invariance of \( B \) we get, \( B(y, b, a) = 0, c \in V \). Thus, \( (y, b, a) = 0, a, b \in V \) because \( B \) is nondegenerate. Hence, \( y \in Z(V) \). Consequently, \( (Z(V))^{2} = T(V, V, V) \).

**Definition 5:** Let \( (V, \langle , , \rangle) \) be an associative triple system. We define the descending series \( \langle V \rangle_{n+1} \) by \( V = V \subseteq V^{(1)} = (V, V, V) \subseteq V \subseteq V^{(n+1)} \subseteq \cdots \subseteq V^{(k)} \subseteq \cdots \subseteq V^{(k)} \subseteq \cdots \subseteq V \). If there exists \( n \in N \) such that \( V^{(n)} = 0 \) (resp. \( V^{(n)} = 0 \)), then \( A \) is called soluble (resp. nilpotent).

**Definition 6:** Let \( V \) be an associative triple system and \( B \) be an invariant scalar product on \( V \).

1. An ideal \( U \) of \( V \) is a subspace of \( A \) which satisfies \( U \subseteq V \) and \( U^{*} \subseteq U \).

2. An ideal \( U \) of \( V \) is said to be:

   (a) Abelian if \( [U, U, U] = 0 \).

   (b) Solvable (resp. nilpotent) if it is soluble (resp. nilpotent) as a associative triple system.

3. The largest solvable ideal of \( V \) is called the radical of \( V \) and denoted \( \text{Rad}(V) \).

4. The associative triple system \( (V, B) \) is called.

   (a) Semi-simple if it has no non trivial soluble ideal. That is \( \text{Rad}(V) = 0 \).

   (b) B-irreducible, if \( V \) contains no non-trivial nondegenerate ideal. The following lemma is straightforward.

**Lemma 1:** Let \( (V, B) \) be an associative triple system and \( U \) be an ideal of \( V \). Then,

\[
U = \{x \in V \mid B(x,y) = 0, \forall y \in U\} \text{ is an ideal of } V.
\]

If \( U \) is nondegenerate, then \( A = U \cup U^{*} \) and \( U^{*} \) is also nondegenerate.

**Lemma 2:** Let \( (V, B) \) be an associative triple system. Then, \( V = \bigoplus_{r} V_{r} \) where \( r \in N \) and such that for \( i \in \{1, \ldots, r\} \), \( V_{i} \) is a nondegenerate ideal of \( V \). \( V_{i} \) is \( B \)-irreducible as a associative triple system. For \( i \neq j \) and \( (x, y) \in V_{r} \times V_{s} \), we have \( B(x, y) = 0 \).

We preceede by induction on \( n = \dim(V) \). If \( n = 1 \), then the assertion is true. Suppose that every associative triple system of dimension less than \( n \) satisfies the proposition. Let \( (V, B) \) be an associative triple system of dimension \( n + 1 \). If \( V \) does not contain any non trivial nondegenerate
ideal, then the assertion is true for r=1. If not, let I be a non trivial nondegenerate ideal of V. By the Lemma 1, V=I⊕I. The result follows by applying the induction to I and I⊥:

**Proposition 2**: Let (V,B) be a semi-simple associative triple system and consider the decomposition V= ⊕  of V as in the Lemma 2.

If I is a simple ideal of V, then there exists  ∈ {1,...,r} such that V. 

For i ∈ {1,...,r}, V is simple.

(i) Let I be a non-trivial simple ideal of V. Assume that for all i ∈ {1,...,r} we have I∩V=Ψ={0}. Then, V∩V=Ψ=0. Hence, there exists  ∈ {1,...,r} such that V∩V. Since V∩V is an ideal of I and I is simple, then V∩V=Ψ. So, V∩V. The fact that V is B-irreducible and I is nondegenerate, imples that V∩V. (ii) Suppose that there exists i ∈ {1,...,r} such that V is not simple. Then, without loss of generality, we may write V= ((V∩V)) for 1≤i≤s, V is simple and V is not simple for s+1 ≤ r. Since V is semi-simple, then we can consider the decomposition V= ⊕  of V into the direct sum of its simple ideals. The assertion (i) imps that s=r, W.

The previous Proposition shows that, in the case of semisimples triples systems, the decomposition into the direct sum of orthogonal nondegenerate ideals coincides with the decomposition into a direct sum of simple ideals.

The following theorem presents a process of construction of a symmetric associative triple systems.

**Remark 1**: It is clear that I is an abelian ideal of T∗. Thus, T∗, I is not semi-simple. Moreover, if I is not nilpotent, then T∗, I is not nilpotent too. Consequently, the family of a symmetric associative triple systems contains strictly the of semi-simple associative triple systems and the symmetric nilpotent associative triple systems.

**Theorem 4**: Let (I,B) be a semi symmetric associative triple system of dimension n. Then, (I,B) is isomorphic to an I-extension (Tn(I,B),I) if and only if n is even and J contains an isotropic ideal of dimension n/2.

Let I be an isotropic ideal of J of dimension n/2. Since B is invariant and nondegenerate, then (J,B) is abelian. Now, let us consider the one dimensional abelian associative triple system K endowed with the bilinear form B:K×K→K defined by B(c,e)=c. Let J=IK be the associative triple system endowed with the triple product given by:

\[ x + y = \{x,y,z\} = \{x,y,z\}, \forall x,y,z \in J, y, z \in K. \]

We define on J the bilinear form B1 by:

\[ B_{1}(c,c) = 1 \text{, and } B_{1}(c,z) = B_{1}(z,c) = 0. \]

It is clear that (J,Bn) is a pseudo Euclidean associative triple system. Besides, I is a nondegenerate ideal of dimension 1 of J, Let d=1 such that B(d,d)=−1 and consider I=IK, where c=αd+I. It is easy to see that I is an ideal of J. In addition,

\[ B(c,e) = B(d,d) + B(c,c) = 0, \text{ and } B(x,e) = B(x,d) + B(x,c) = 0, \forall x \in I. \]

So, I is isotropic and \( \dim(I) = \frac{n+1}{2} \). Since dimension of J is even, then the theorem 4 imps that \( J_{1} \) is isomorphic to a \( T^{n} \)-extension of the associative triple system \( J/I \).

Consequently, \( J_{1}/I \) is isomorphic to \( J/I \), W.

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