Maximum nullity of Cayley graph

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Abstract One of the most interesting problems on maximum nullity (minimum rank) is to characterize $M(G)$ ($mr(G)$) for a graph $G$. In this regard, many researchers have been trying to find an upper or lower bound for the maximum nullity. For more results on this topic, see [6], [7], [11] and [12]. In this paper, by using a result of Babai [5], which presents the spectrum of a Cayley graph in terms of irreducible characters of the underlying group, and using representation and character of groups, we give a lower bound for the maximum nullity of Cayley graph, $X_S(G)$, where $G = \langle a \rangle$ is a cyclic group, or $G = G_1 \times \cdots \times G_t$ such that $G_1 = \langle a \rangle$ is a cyclic group and $G_i$ is an arbitrary finite group, for some $2 \leq i \leq t$, with determine the spectrum of Cayley graphs.

Keywords Cayley graphs · Spectra of graphs · Maximum nullity

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1 Introduction

For a positive integer $n$, let $S_n(\mathbb{R})$ be the set of all symmetric matrices of order $n$ over the real number. Suppose that $A \in S_n(\mathbb{R})$. Then the graph of $A$ which is denoted by $H(A)$ is a graph with the vertex set $\{u_1, \ldots, u_n\}$ and the edge set $\{u_i \sim u_j : a_{ij} \neq 0, 0 \leq i < j \leq n\}$. It should be noted that the diagonal of $A$ has no role in the determining of $H(A)$.
The set of symmetric matrices of a graph \( G \) is the set \( S(G) = \{ A \in S_n(\mathbb{R}) : H(A) = G \} \). The minimum rank of a graph \( G \) of order \( n \) is defined to be the minimum cardinality between the rank of symmetric matrices in \( S(G) \) and denoted by \( mr(G) \). Similarly, the maximum nullity of \( G \) is defined to be the maximum cardinality between the nullity of symmetric matrices in \( S(G) \); and is denoted by \( M(G) \). Clearly, \( mr(G) + M(G) = n \).

One of the most interesting problems on minimum rank is to characterize \( mr(G) \) for graphs. In this regard, many researchers have been trying to find an upper or lower bound for the minimum rank. For more results on this topic, see [6], [7], [11] and [12].

The adjacency matrix of a graph \( G \) is the matrix \( A_G \) whose the entry \( a_{ij} = 1 \) if and only if vertices \( u_i \) and \( u_j \) are adjacent, and \( a_{ij} = 0 \) otherwise. The eigenvalues of \( G \) are the eigenvalues of \( A_G \), and the spectrum of \( G \) is the collection of its eigenvalues together with multiplicities. If \( \lambda_1, \ldots, \lambda_t \) are distinct eigenvalues of a graph \( G \) with respective multiplicity \( n_1, \ldots, n_k \), then we denote the spectrum of \( G \) by

\[
\text{spec}(G) = \left[ \lambda_1^{n_1}, \ldots, \lambda_t^{n_t} \right].
\]  

Let \( G \) be a group, and let \( S \) be a subset of \( G \) that is closed under taking inverse and does not contain the identity, \( e \). Then the Cayley graph, \( X_S(G) \), is the graph with vertex set \( G \) and edge set

\[
E = \{ g_1 \sim g_2 : g_1g_2^{-1} \in S \}.
\]  

Since \( S \) is inverse-closed and does not contain the identity, it is a simple fact that \( X(G,S) \) is undirected and has no loop.

In 1979, Babai [5] presented the spectrum of a Cayley graph in terms of irreducible characters of the underlying group \( G \). The following important theorem was the result of this paper.

**Theorem 11** [5] *Let \( G \) be a finite group of order \( n \) whose irreducible characters (over \( \mathbb{C} \)) are \( \chi_1, \ldots, \chi_h \) with respective degree \( n_1, \ldots, n_h \). Then the spectrum of the Cayley graph \( X_S(G) \) can be arranged as \( \Lambda = \{ \lambda_{ijk} : i = 1, \ldots, h; j, k = 1, \ldots, n_i \} \) such that \( \lambda_{i1} = \ldots = \lambda_{ijn_i} \) (this common value will be denoted by \( \lambda_{ij} \)), and

\[
\lambda_{i1} + \ldots + \lambda_{ijn_i} = \sum_{s_1, \ldots, s_t \in S} \chi_i \left( \prod_{l=1}^{t} s_l \right),
\]

for any natural number \( t \).*

In this paper, by using a result of Babai, we give a lower bound for the maximum nullity of Cayley graph, \( X_S(G) \), where \( G = \langle a \rangle \) is a cyclic group, or \( G = G_1 \times \cdots \times G_t \) such that \( G_1 = \langle a \rangle \) is a cyclic group and \( G_i \) is an arbitrary finite group, for some \( 2 \leq i \leq t \), with determine the spectrum of Cayley graphs.
2 Preliminaries

For any positive integer \( n \), define \( \text{Möbius} \) number, \( \mu(n) \), as the sum of the primitive \( n \)-th roots of unity. It has values in \( \{-1, 0, 1\} \) depending on the factorization of \( n \) into prime factors.

1. \( \mu(1) = 1 \),
2. \( \mu(n) = 0 \), if \( n \) has a squared factor,
3. \( \mu(n) = (-1)^k \), if \( n \) is a square-free with \( k \) number of prime factors.

Suppose that \( k \) is a positive integer. The number of solutions of \( y_1 + \cdots + y_r \equiv t \pmod{k} \), where \( y_1, \ldots, y_r \) and \( t \) are belonged to the least non-negative residue system modulo \( k \), is obtained in terms of the von Sterneck function, \( \Phi(n,k) \). In particular von Sterneck studied the case where the polynomial resulting from the expansion is reduced modulo a positive integer. This function is used in several equivalent forms; and in the form used by Hölder [13],

\[
\Phi(k,n) = \frac{\phi(n)}{\phi(n/(n,k))} \mu(n/(n,k)),
\]

(2.1)

where \( k \) and \( n \) are positive integers, \( (n,k) \) is the greatest common divisor of \( k \) and \( n \), \( \phi(n) \) is the \text{Euler totient}, and \( \mu(n) \) is the \text{Möbius} number. In the sequel, the following fundamental result is obtained by Hölder.

\[
\Phi(r,n) = \sum_{(r,n)=1} \exp(2\pi i r k / n).
\]

(2.2)

This properties was also studied by Von Sterneck in 1902 [10], Nicol and Vandiver in 1954 [16], and Tom M. Apostol in 1972 [4].

Suppose that \( B(k,n) = \{ t \in \mathbb{N} : t \leq n, (t,n) = k \} \), and let \( \omega = \exp(2\pi i / n) \). Then the following function is called Ramanujan sum, and is denoted by \( C(r,n) \).

\[
\sum_{k \in B(1,n)} \omega^{kr}, 0 \leq r \leq n - 1,
\]

(2.3)

In [17], it was obtained that \( C(r,n) \) have only integral values, for some positive integers \( r \) and \( n \). Also, (2.2) and (2.3) state that \( \Phi(r,n) = C(r,n) \).

Lemma 21 Suppose that \( n > 1 \) and \( d > 1 \) are two positive integers such that \( d \mid n \). Also, let \( B(d,n) = \{ t \in \mathbb{N} : t \leq n, (t,n) = d \} \). Then

1. If \( t \in B(d,n) \), then \( C(t,n) = C(d,n) \),
2. \( |B(d,n)| = \phi(n/d) \).

Proof. The proof is straightforward.

Theorem 22 [1] For Euler totient \( \phi \) and positive integer \( n \), we have \( \sum_{d \mid n} \phi(d) = n \).

Lemma 23 [14] The irreducible character of \( G \times H \) is \( \chi \times \psi \) such that \( \chi \) and \( \psi \) are the irreducible characters of \( G \) and \( H \) respectively. the value of \( \chi \times \psi \) for any \( g \in G \) and \( h \in H \) is \( (\chi \times \psi)(g,h) = \chi(g)\psi(h) \).

Lemma 24 [14] Let \( G = \langle a \rangle \) be a cyclic group of order \( n \). Then irreducible characters of \( G \) are \( \rho_j(a^k) = \omega^{jk} \), where \( j, k = 0, 1, \ldots, n-1 \).
3 Main theorems

In the following theorem, we determine the spectrum of Cayley graph $X_S(G)$ whose $G$ is a cyclic group of order $n$. Here, we define $F(n_i) = (-1)^{k_i} \phi(n)/\phi(n_i)$, where $k_i$ is the number of prime factors in the decomposition of $n_i$.

**Theorem 31** Let $n$ be a positive integer and $D$ be its divisors set. Also, let $G = \langle a \rangle$ be a cyclic group of order $n$ and $S = \{a^i : i \in B(1,n)\}$. Then

$$\text{spec}(X_S(G)) = \left[\phi(n)^1, 0^{\sum_{x \in X} \phi(x)}, F(d_1)^{\phi(d_1)}, ..., F(d_t)^{\phi(d_t)}\right],$$

where $X = \{d \in D : p^2 \mid d\}$, for a prime $p$; and $d_i \in D \setminus X$, for some $1 \leq i \leq t$.

**Proof.** First, suppose that $n$ is a prime number. Thus $X_S(G)$ is isomorphic to the complete graph $K_n$, and so

$$\text{spec}(X_S(G)) = \left[(p - 1)^1, -1^{(p-1)}\right]. \quad (3.1)$$

Now, consider the case in that $n$ is not prime. Let $\lambda_{d_i}$ be the eigenvalue of $X_S(G)$ corresponding to character of $\chi_{d_i}$, for some $d_i \in D$. By Lemma 11 $\lambda_{d_i} = C(n, d_i)$, and by the form used by Hölder in (2.1), we have

$$\lambda_{d_i} = \frac{\phi(n)}{\phi(d_i)} \mu(d_i). \quad (3.2)$$

On the other hand, lemma 21 implies that the multiplicity of $\lambda_{d_i}$ is equal to $\phi(d_i)$. If $d_i = 1$, then $\lambda_n = \phi(n)$ with multiplicity 1. Also, if $p^2 \mid d_i$, then definition of Möbius number implies that $\lambda_{d_i} = 0$. For other cases, $\lambda_{d_i} = F(d_i)$. □

The following theorem which is proven by S. Akbari et al. [3], help us to make a connection between the multiplicity of the eigenvalues of a graph $G$ and its maximum nullity $M(G)$.

**Theorem 32** [3] Let $G$ be a graph of order $n$, and let $\lambda_i$ be its eigenvalue with respective multiplicity $\nu_i$. Then $M(G) \geq \nu_i$.

As a result, Theorems 31 and 32 state the following corollary.

**Corollary 33** Let $n$ be a positive integer and $D$ be its divisors set. Also, let $G = \langle a \rangle$ be a cyclic group of order $n$, and let $S = \{a^i : i \in B(1,n)\}$. For some prime $p$ and $d_i \in D$, the followings are established.

1. If $n$ has a squared factor, then $M(X_S(G)) \geq \max \left\{ \sum_{p^2 \mid d_i} \phi(d_i), \phi(d_i) \right\}$.
2. If $n$ is a square-free, then $M(X_S(G)) \geq \phi(d_i)$. 
**Definition 34** Let $G$ be a group, and let $S$ be a subset of $G$. Also, let $\Lambda = \{\chi_1, \ldots, \chi_k\}$ be the set of irreducible characters with degree 1 of $G$. A character $\chi_i \in \Lambda$ is defined to be an $\ell$-index character of $G$, if has the same value $\ell$ on all letters in $S$; in other word, $\chi_i \in \Lambda$ is an $\ell$-index character of $G$ if $\chi(s_i) = \ell$, for all $s_i \in S$. In the sequel, An $\ell$-index number of $G$ is defined to be the number of $\ell$-index characters of $G$, and is denoted by $N_G(\ell)$.

**Theorem 35** Let $n$ be a positive integer whose divisors set is denoted by $D$. Also, let $G_1 = \langle a \rangle$ be a cyclic group of order $n$, and let $S' = \{a^i : i \in B(1, n)\}$. Suppose that $G_2, \ldots, G_t$ are some arbitrary finite groups, and let $S_k$ be a subset of $G_k$, for some $2 \leq k \leq t$. If $S = \{(a^i, \alpha_1, \ldots, \alpha_t) : a^i \in S', \alpha_k \in S_k\}$, then for some prime $p$ and $d_i \in D$, the followings are established.

1. If $n$ has a square factor, then
   \[
   M_X(G_1 \times \cdots \times G_t) = \max \left\{ \prod_{i=2}^t \left( N_{G_i}(\ell_i) | S_i \right) \left( \sum_{p|d_i} \phi(d_i) \right) : \prod_{i=2}^t \left( N_{G_i}(\ell_i) | S_i \right) \left( \phi(d_i) \right) \right\}.
   \]

2. If $n$ is a square-free, then
   \[
   M_X(G_1 \times \cdots \times G_t) \geq \left( \prod_{i=2}^t \left( N_{G_i}(\ell_i) | S_i \right) \left( \phi(d_i) \right) \right).
   \]

**Proof.** For some $2 \leq k \leq t$, suppose that $\rho_{j_k}$ are the $\ell_k$-index irreducible characters with degree 1 of $G_k$, and let $\chi_{s_k}$ be an irreducible character of $G_1$. Let $\lambda_{j_1 \cdots j_t}$ and $\lambda_{s_k}$ be the eigenvalues of $X_S(G_1 \times \cdots \times G_t)$ and $X_{S'}(G_1)$ corresponding to irreducible characters of $\chi_{s_k} \chi_{j_1} \chi_{j_2} \cdots \chi_{j_t}$, respectively. Lemma 23 implies that

\[
\lambda_{j_1 \cdots j_t} = \sum_{(g_1, \ldots, g_t) \in S} \left( \chi_{s_k} \times \rho_{j_2} \times \cdots \times \rho_{j_t} \right)(g_1, \ldots, g_t)
= \sum_{(g_1, \ldots, g_t) \in S} \left( \chi_{s_k} \rho_{j_2}(g_2) \times \cdots \times \rho_{j_t}(g_t) \right)
= \left( \prod_{k=2}^t \left( N_{G_k}(\ell_k) | S_k \right) \right) \sum_{s' \in S'} \left( \chi_{s_k}(s') \right).
\]

We have,

\[
\lambda_{j_1 \cdots j_t} = \left( \prod_{k=2}^t \left( N_{G_k}(\ell_k) | S_k \right) \right) (\lambda_{s_k}).
\]

Hence, by Theorem 24 if $n$ is a square-free, then $\lambda_{j_1 \cdots j_t}$ is an eigenvalue of $X_S(G_1 \times \cdots \times G_t)$ with multiplicity

\[
\left( \prod_{k=2}^t \left( N_{G_k}(\ell_k) | S_k \right) \right) (\phi(d_i)).
\]
and if \( n \) is divided by a prime number, then \( \lambda_{\frac{n}{p}} \) is an eigenvalue of \( X_{S}(G_1 \times \cdots \times G_t) \) with multiplicity
\[
\left( \prod_{k=2}^{t} \left( \frac{N_{G_k}(\ell_k)}{|S_k|} \right) \right) (\phi(d_i)),
\]
where \( \lambda_{\frac{n}{p}} \neq 0 \), or with multiplicity
\[
\left( \prod_{k=2}^{t} \left( \frac{N_{G_k}(\ell_k)}{|S_k|} \right) \right) \left( \sum_{p^i \mid d_i} \phi(d_i) \right),
\]
where \( \lambda_{\frac{n}{p}} \neq 0 \). Therefore, theorem 32 completed the proof. \( \square \)

We apply the theorem to the dihedral groups \( D_n \), with presentation
\[
\langle a, b : a^n = b^2 = e, (ab)^2 = e \rangle.
\]
For simplicity, we treat the case of odd \( n \) only. For \( n = 2m + 1 \) there are \( m \) irreducible character of degree 2 and 2 of degree 1. Recall their character tables [14, chap.18], where \( \omega \) denotes a primitive \( n \)th root of unity.

| \( \chi_j \) | \( a^j \) | \( \omega^{jk} + \omega^{-jk} \) | \( a^j b \) | \( j = 1, \ldots, m \) |
|---------|--------|-----------------|--------|-----------------|
| \( \chi_{m+1} \) | 1 | -1 | \( j = 1, \ldots, m \) |
| \( \chi_{m+2} \) | 1 | 1 | \( j = 1, \ldots, m \) |

Table 1 Character table of dihedral groups \( D_n \), where \( n = 2m + 1 \).

**Example 36** Let \( G = \langle a \rangle \) be a group of order an odd \( n \), and let \( S = \{ (a^i, a^j) : a^i \in G, a^j \in D_n, i, j \in B(1, n) \} \). Obviously, \( \chi_{m+1} \) and \( \chi_{m+2} \) are two 1-index irreducible characters of \( D_n \), and so \( N_{D_n}(1) = 2 \). Hence, by Theorem 33 we have

1. If \( n \) has a square factor, then
\[
M(Cay(G \times D_n : S)) \geq \max \{ 2\phi(n) \left( \sum_{p^i \mid d_i} \phi(d_i) \right), 2\phi(n) (\phi(d_i)) \}.
\]
2. If \( n \) is square-free, then
\[
M(Cay(G \times D_n : S)) \geq 2\phi(n) (\phi(d_i)).
\]

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