A RADIAL INTEGRABILITY RESULT CONCERNING BOUNDED
FUNCTIONS IN ANALYTIC BESOV SPACES WITH
APPLICATIONS

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Abstract. We prove that for every \( p \geq 1 \) there exists a bounded function in the
analytic Besov space \( B^p \) whose derivative is “badly integrable” along every radius.
We apply this result to study multipliers and weighted superposition operators acting
on the spaces \( B^p \).

1. Introduction

Let \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) denote the open unit disc in the complex plane \( \mathbb{C} \)
and let \( \mathcal{Hol}(\mathbb{D}) \) be the space of all analytic functions in \( \mathbb{D} \) endowed with the topology of
uniform convergence in compact subsets. Also, \( dA \) will denote the area measure on \( \mathbb{D} \),
normalized so that the area of \( \mathbb{D} \) is 1. Thus \( dA(z) = \frac{1}{\pi} dx \, dy = \frac{1}{\pi} r \, dr \, d\theta \).

For \( 0 \leq r < 1 \) and \( g \) analytic in \( \mathbb{D} \) we set

\[
M_p(r, g) = \left( \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^p \, d\theta \right)^{1/p}, \quad 0 < p < \infty,
\]

\[
M_\infty(r, g) = \max_{|z|=r} |g(z)|.
\]

For \( 0 < p \leq \infty \) the Hardy space \( H^p \) consists of those functions \( g \), analytic in \( \mathbb{D} \), for
which

\[
\|g\|_{H^p} \overset{\text{def}}{=} \sup_{0<r<1} M_p(r, g) < \infty.
\]

We refer to [14] for the theory of Hardy spaces.

For \( 0 < p < \infty \) and \( \alpha > -1 \) the weighted Bergman space \( A^p_\alpha \) consists of those
functions \( f \) in \( \mathcal{Hol}(\mathbb{D}) \) such that

\[
\|f\|_{A^p_\alpha} \overset{\text{def}}{=} \left( (\alpha + 1) \int_{\mathbb{D}} (1 - |z|^2)^\alpha |f(z)|^p \, dA(z) \right)^{1/p} < \infty.
\]

The unweighted Bergman space \( A^p_0 \) is simply denoted by \( A^p \). We refer to [15, 20, 26]
for the notation and results about Bergman spaces.

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FEDERJA-002).
For $1 < p < \infty$, the analytic Besov space $B^p$ is defined as the set of all functions $f$ analytic in $\mathbb{D}$ such that $f' \in A^p_{p-2}$. Thus a function $f \in \text{Hol}(\mathbb{D})$ belongs to $B^p$ if and only if $ho_p(f) < \infty$, where

$$
\rho_p(f) = \|f'\|_{A^p_{p-2}} = \left( (p-1) \int_{\mathbb{D}} (1 - |z|^2)^{p-2} |f'(z)|^p \, dA(z) \right)^{1/p}.
$$

All $B^p$ spaces ($1 < p < \infty$) are conformally invariant with respect to the semi-norm $\rho_p$. An important and well-studied case is the classical Dirichlet space $B^2$ (often denoted by $D$) of analytic functions whose image Riemann surface has a finite area.

We mention [4] as a fundamental reference for Besov spaces and Zhu’s monograph [26] as a very good place to find a lot of information about them.

The space $B^1$ requires a special definition: it is the space of all analytic functions $f$ in $\mathbb{D}$ for which $f'' \in A^1$. Although the corresponding semi-norm is not conformally invariant, the space itself is. Another possible definition (with a conformally invariant semi-norm) is given in [4], where $B^1$ was denoted by $M$.

It is well known that all the Besov spaces are contained in the space $BMOA$ (even more, in $VMOA$) and, hence, in the Bloch space $B$. We refer to [17] and [3] for the theory of these spaces. The inclusion $B^p \subset B^q$ yields

$$
(1.1) \quad B^1 \subset B^q \subset B^p \subset B, \quad 1 < q < p < \infty.
$$

Obtaining results about the integrability along radii of distinct classes of analytic functions in the unit disc has shown to be an important question in complex analysis which has attracted the attention of lots of authors over the years. One of the best known results in this line is due to Rudin [24] who showed the existence of an $H^\infty$-function $f$ for which the radial variation $V(f, \theta) = \int_0^1 |f'(re^{i\theta})| \, dr$ is infinite for every $\theta$ except possibly for those $\theta$ in a set of the first category and of measure zero. Bourgain [6] solved a question raised by Rudin by showing that, for $f \in H^\infty$, the set of those $\theta$ for which $V(f, \theta)$ is finite cannot be empty since it must have Hausdorff dimension 1.

In this paper we shall be concerned with radial integrability properties of $B^p$-functions. By the definition, it is clear that if $1 < p < \infty$ and $f \in \text{Hol}(\mathbb{D})$ then

$$
(1.2) \quad f \in B^p \iff \int_0^1 (1 - r)^{p-2} M_p(r, f')^p \, dr < \infty.
$$

Clearly, this implies that if $1 < p < \infty$ and $f \in B^p$ then

$$
(1.3) \quad \int_0^1 (1 - r)^{p-2} |f'(re^{i\theta})|^p \, dr < \infty, \quad \text{for almost every } \theta \in [0, 2\pi].
$$

In our first result we prove that (1.2) and (1.3) are sharp in a very strong sense connecting this with (1.1). Indeed, for $1 < q < p < \infty$ we prove the existence of a function $f \in B^p \cap H^\infty$ with $M_p(r, f')$ “as big as possible” and having “bad integrability properties of order $q$ along all the radii”. Before stating it, let us notice that throughout the paper we shall be using the convention that $C = C(p, \alpha, q, \beta, \ldots)$ will denote a positive constant which depends only upon the displayed parameters $p, \alpha, q, \beta, \ldots$ (which often will be omitted) but not necessarily the same at different occurrences.
Moreover, for two real-valued functions $E_1, E_2$ we write $E_1 \lesssim E_2$, or $E_1 \gtrsim E_2$, if there exists a positive constant $C$ independent of the arguments such that $E_1 \leq CE_2$, respectively $E_1 \geq CE_2$. If we have $E_1 \lesssim E_2$ and $E_1 \gtrsim E_2$ simultaneously then we say that $E_1$ and $E_2$ are equivalent and we write $E_1 \asymp E_2$.

**Theorem 1.** Suppose that $1 < q < p < \infty$ and let $\phi$ be a positive increasing function defined in $[0, 1)$ satisfying

$$\int_0^1 (1 - r)^{p-2} \phi(r)^p \, dr < \infty$$

and

$$\int_0^1 (1 - r)^{q-2} \phi(r)^q \, dr = \infty.$$

Then there exists a function $f \in B^p \cap H^\infty \setminus B^q$ with the following two properties:

$$M_p(r, f') \gtrsim \phi(r).$$

$$\int_0^1 (1 - r)^{q-2} |f'(re^{i\theta})|^q \, dr = \infty, \text{ for every } \theta \in [0, 2\pi].$$

A typical example of a function $\phi$ in the conditions of Theorem 1 is

$$\phi(r) = \frac{1}{(1 - r)^{\alpha}}, \quad 0 \leq r < 1,$$

$\alpha$ being a real number with $1 - \frac{1}{q} < \alpha < 1 - \frac{1}{p}$.

The substitute of Theorem 1 for $q = 1$ is the following.

**Theorem 2.** Suppose that $1 < p < \infty$ and let $\phi$ be a positive increasing function defined in $[0, 1)$ satisfying (1.4). Then there exists a function $f \in B^p \cap H^\infty \setminus B^1$ with the following two properties:

$$M_p(r, f') \gtrsim \phi(r).$$

$$\int_0^1 |f''(re^{i\theta})| \, dr = \infty, \text{ for every } \theta \in [0, 2\pi].$$

The proofs of Theorem 1 and Theorem 2 will be presented in Section 2. In Section 3 we shall apply these theorems to obtain results on weighted superposition operators and multipliers acting on the Besov spaces.
2. Proofs of the integrability results

Proofs of Theorem 1 and Theorem 2. Let \( \phi \) be as in Theorem 1. Set \( r_k = 1 - \frac{1}{2^k}, \quad k = 1, 2, \ldots. \)

Since \( \phi \) is increasing it easy to see that (1.4) implies that

\[
\sum_{k=1}^{\infty} \frac{1}{2^{k(p-1)}} \phi(r_k)^p < \infty.
\]

For \( k \geq 1 \), define \( a_k = \frac{1}{2^k} \phi(r_k) \) and set

\[
f(z) = \sum_{k=1}^{\infty} a_k z^{2^k}, \quad z \in \mathbb{D}.
\]

Then \( f \) is an analytic function in \( \mathbb{D} \) given by a power series with Hadamard gaps. Using (2.1) we see that

\[
\sum_{k=1}^{\infty} 2^k |a_k|^p = \sum_{k=1}^{\infty} \frac{1}{2^{k(p-1)}} \phi(r_k)^p < \infty.
\]

Then it follows that \( f \in B^p \) (see, e.g., [13, Theorem D]). Let \( q \) be the exponent conjugate to \( p \), that is, \( q = p/(p-1) \). Using Hölder’s inequality with the exponents \( p \) and \( q \) and (2.1), we obtain

\[
\sum_{k=1}^{\infty} 2^k/2^{kq/p} \phi(r_k) \leq \left( \sum_{k=1}^{\infty} \frac{1}{2^{kq/p}} \right)^{1/q} \left( \sum_{k=1}^{\infty} \frac{1}{2^{k(p-1)}} \phi(r_k)^p \right)^{1/p} < \infty.
\]

Then it follows that \( f \in H^\infty \) (even more, \( f \) belongs to the disc algebra).

Now, \( z f'(z) = \sum_{k=1}^{\infty} \phi(r_k) z^{2^k} \) (\( z \in \mathbb{D} \)) and then it follows that

\[
M_2(r, f')^2 \geq \sum_{k=1}^{\infty} \phi(r_k)^2 r^{2^k+1}, \quad 0 < r < 1.
\]

For a given \( r \in (0, 1) \) take \( k \in \mathbb{N} \) such that \( r_k \leq r < r_{k+1} \). Using (2.2), the facts that the functions \( M_2(\cdot, f') \) and \( \phi \) are increasing in \( (0, 1) \), and the simple estimate \( r_k^{2^k} \gtrsim 1 \), we obtain

\[
M_2(r, f')^2 \geq M_2(r_k, f')^2 \geq \phi(r_{k+1})^2 r_k^{2^{k+2}} \gtrsim \phi(r_{k+1})^2 \gtrsim \phi(r)^2.
\]

Now, since \( f \) is given by a power series with Hadamard gaps we have that \( M_2(r, f') \asymp M_2(r, f') \) (see, e.g., Theorem 8.20 in [27, Vol. I, p. 215]). Then (2.3) yields

\[
M_2(r, f') \gtrsim \phi(r).
\]

Actually, we have that \( M_\lambda(r, f') \asymp M_2(r, f') \) for all finite \( \lambda \) and, consequently, we can assert that

\[
M_\lambda(r, f') \gtrsim \phi(r), \quad 0 < \lambda < \infty.
\]

Using this with \( \lambda = q \) and (1.5) we obtain that \( f \notin B^q \). This and [13, Theorem D] imply \( \sum_{k=1}^{\infty} 2^k |a_k|^q = \infty \). Then (1.7) follows using a a result of Gnuschke [19, Theorem 1].

This finishes the proof of Theorem 1.
Since $B^1 \subset B^q$, $f \notin B^1$ and, hence, $\sum_{k=1}^{\infty} 2^k |a_k| = \infty$. Then the just mentioned result of Gnuschke also yields \cite{19}. Hence Theorem 2 is also proved.

3. Applications to weighted superposition operators and multipliers

Given an entire function $\varphi$, the superposition operator
\[ S_{\varphi} : \text{Hol}(\mathbb{D}) \longrightarrow \text{Hol}(\mathbb{D}) \]
is defined by $S_{\varphi}(f) = \varphi \circ f$.

The natural questions in this context are: If $X$ and $Y$ are two normed (metric) subspaces of $\text{Hol}(\mathbb{D})$, for which entire functions $\varphi$ does the operator $S_{\varphi}$ map $X$ into $Y$? When is $S_{\varphi}$ a bounded (or continuous) operator from $X$ to $Y$?

Let us remark that we are dealing with non-linear operators. Consequently, boundedness and continuity are not equivalent a priori. However, Boyd and Rueda \cite{8} have shown that for a large class of Banach spaces of analytic functions $X$ and $Y$, bounded superposition operators from $X$ to $Y$ are continuous.

These questions have been studied for different pair of spaces $(X, Y)$. See, for example, \cite{2,5,7,8,9,10,11,12,18,22,23} and the references therein.

Concerning Besov spaces a complete characterization of the entire functions $\varphi$ which map $B^p$ into $B^q$ ($1 \leq p < q < \infty$) is given in \cite{9}. In particular, the following result is taken from \cite{9}.

**Theorem A.** Suppose that $1 \leq q < p < \infty$ and let $\varphi$ be an entire functions. Then the superposition operator $S_{\varphi}$ maps $B^p$ into $B^q$ if and only if $\varphi$ is constant.

We shall use Theorem 1 and Theorem 2 to obtain a new proof of Theorem A. Actually, we shall extend this result to the setting of weighted superposition operators.

If $\varphi$ is an entire function and $w \in \text{Hol}(\mathbb{D})$, the weighted superposition operator
\[ S_{\varphi,w} : \text{Hol}(\mathbb{D}) \longrightarrow \text{Hol}(\mathbb{D}) \]
is defined by
\[ S_{\varphi,w}(f)(z) = w(z) \varphi(f(z)), \quad f \in \text{Hol}(\mathbb{D}), \quad z \in \mathbb{D}. \]
In other words, $S_{\varphi,w} = M_w \circ S_{\varphi}$, where $M_w$ is the multiplication operator defined by
\[ M_w(f)(z) = w(z)f(z), \quad f \in \text{Hol}(\mathbb{D}), \quad z \in \mathbb{D}. \]

Note that $S_{\varphi} = S_{\varphi,w}$ with $w(z) = 1$, for all $z \in \mathbb{D}$. The literature on weighted superposition operators is not as wide as the one concerning superposition operators. We mention \cite{12} and \cite{21} as recent papers dealing with them. We can prove the following result.

**Theorem 3.** Suppose that $1 \leq q < p < \infty$, $w \in \text{Hol}(\mathbb{D})$, $w \not\equiv 0$, and $\varphi$ is an entire function. Then the weighted superposition operator $S_{w,\varphi}$ maps $B^p$ into $B^q$ if and only if $w \in B^q$ and $\varphi$ is constant.
When \( w \equiv 1 \), Theorem 3 reduces to Theorem A.

**Proof of Theorem 3.** Suppose that \( 1 \leq q < p < \infty \).

It is trivial that if \( \phi \) is constant and \( w \in B^q \) then \( S_{\phi,w} \) maps \( B^p \) into \( B^q \).

It is also trivial that if \( \phi \) is constant and not identically zero, and \( S_{\phi,w} \) maps \( B^p \) into \( B^q \), then \( w \in B_q \).

It remains to prove that if \( \phi \) is not constant, and \( S_{\phi,w}(B^p) \subset B^q \), then \( w \equiv 0 \).

Take \( a \in \mathbb{C} \) such that \( \phi(a) \neq 0 \) and let \( h \) be the constant functions defined by \( h(z) = a \), for all \( z \in \mathbb{D} \). Since \( h \in B^p \), it follows that
\[
S_{\phi,w}(h) = \phi(a) \cdot w \in B^q.
\]

This implies that
\[
(3.1) \quad w \in B^q.
\]

We have to show that \( w \equiv 0 \). Since \( B^1 \subset B^s \) for all \( s > 1 \), it suffices to consider the case \( q > 1 \).

So, suppose that \( q > 1 \) and \( w \not\equiv 0 \). Let us use Theorem 1 to pick a function \( f \in B^p \cap H^\infty \) satisfying (1.7). Since \( S_{\phi,w}(B^p) \subset B^q \), we deduce that
\[
S_{\phi,w}(f) = w \cdot (\phi \circ f) \in B^q,
\]
that is,
\[
\int_\mathbb{D} (1 - |z|^2)^{q-2} |w'(z)\phi(f(z)) + w(z)\phi'(f(z))f'(z)|^q \, dA(z) < \infty.
\]

Since \( f \in H^\infty \), we also have that \( \phi \circ f \in H^\infty \). This and (3.1) imply that
\[
\int_\mathbb{D} (1 - |z|^2)^{q-2} |w(z)\phi'(f(z))f'(z)|^q \, dA(z) < \infty.
\]

Then it follows that
\[
\int_\mathbb{D} (1 - |z|^2)^{q-2} |w(z)\phi'(f(z))f'(z)|^q \, dA(z) < \infty
\]
and, hence,
\[
(3.2) \quad \int_0^1 (1 - r)^{q-2} |w(re^{i\theta})\phi'(f(re^{i\theta}))|^q |f'(re^{i\theta})|^q \, dr < \infty, \text{ for almost every } \theta \in [0,2\pi].
\]

Clearly, \( \phi' \circ f \not\equiv 0 \) and \( \phi' \circ f \in H^\infty \). These facts and the well known inclusion \( B^q \subset \text{BMOA} \) readily imply that
\[
(3.3) \quad w \cdot (\phi' \circ f) \not\equiv 0 \quad \text{and} \quad w \cdot (\phi' \circ f) \subset H^\lambda \text{ for all } \lambda \in (0,\infty).
\]

Using Fatou’s theorem and the Riesz uniqueness theorem [13, Chapter 2], we see that the function \( w \cdot (\phi' \circ f) \) has a finite and non-zero radial limit at almost every point \( e^{i\theta} \) of \( \partial \mathbb{D} \). This and (3.3) imply that
\[
\int_0^1 (1 - r)^{q-2} |f'(re^{i\theta})|^q \, dr < \infty, \text{ for almost every } \theta \in [0,2\pi].
\]

This is in contradiction with (1.7). □
Now we turn to apply Theorem\textsuperscript{11} to study multipliers acting on Besov spaces. For \( g \in \mathcal{H}ol(\mathbb{D}) \), the multiplication operator \( M_g \) is defined by
\[
M_g(f)(z) \overset{\text{def}}{=} g(z)f(z), \quad f \in \mathcal{H}ol(\mathbb{D}), \ z \in \mathbb{D}.
\]
If \( X \) and \( Y \) are two spaces of analytic function in \( \mathbb{D} \) (which will always be assumed to be Banach or \( F \)-spaces continuously embedded in \( \mathcal{H}ol(\mathbb{D}) \)) and \( g \in \mathcal{H}ol(\mathbb{D}) \) then \( g \) is said to be a \textbf{multiplier} from \( X \) to \( Y \) if \( M_g(X) \subseteq Y \). The space of all multipliers from \( X \) to \( Y \) will be denoted by \( M(X,Y) \) and \( M(X) \) will stand for \( M(X,X) \). Using the closed graph theorem we see that if \( g \in M(X,Y) \) then \( M_g \) is a bounded operator from \( X \) into \( Y \).

It is well known that if \( X \) is nontrivial then \( M(X) \subseteq H^\infty \) (see, e.g., \cite{1} Lemma 1.1 or \cite{25} Lemma 1.10). Clearly, this implies the following:
\[
\text{(3.4)} \quad \text{If } Y \text{ is nontrivial and } Y \subseteq X \text{ then } M(X,Y) \subseteq H^\infty.
\]

The spaces of multipliers \( M(B^p, B^q) \) have been studied in a good number of papers. The following result is part of \cite{16} Theorem 2].

**Theorem B.** \( \text{If } 1 \leq q < p < \infty \text{ then } M(B^p, B^q) = \{0\} \).

The proof of this result in \cite{16} uses, among other facts, a decomposition theorem for Besov spaces and Khinchine’s inequality. We will show next that Theorem\textsuperscript{B} can be obtained as a consequence of Theorem\textsuperscript{11}.

**Proof of Theorem\textsuperscript{B}** Since \( B^1 \subseteq B^s \) for all \( s > 1 \), it suffices to prove the result in the case \( 1 < q < p < \infty \). So, assume this and that \( M_g(B^p) \subseteq B^q \).

Suppose that \( g \neq 0 \).

Since the constant function 1 belong to \( B^p \), we see that \( g \in B^q \). Also, (3.4) and the inclusion \( B^q \subseteq B^s \) imply that \( g \in H^\infty \). Thus we have
\[
\text{(3.5)} \quad g \in B^q \cap H^\infty.
\]

Use Theorem\textsuperscript{11} to pick a function \( f \in B^p \cap H^\infty \) satisfying (1.7). Since \( M_g(B^p) \subseteq B^q \), we have that \( M_g(f) = g \cdot f \in B^q \), that is
\[
\text{(3.6)} \quad \int_{\mathbb{D}} (1 - |z|^2)^{q-2} |g'(z)f(z) + g(z)f'(z)|^q dA(z) < \infty.
\]

Since \( g \in B^q \) and \( f \in H^\infty \) we see that
\[
\int_{\mathbb{D}} (1 - |z|^2)^{q-2} |g'(z)f(z)|^q dA(z) < \infty.
\]

This and (3.6) imply that
\[
\int_{\mathbb{D}} (1 - |z|^2)^{q-2} |g(z)f'(z)|^q dA(z) < \infty.
\]

and, hence,
\[
\text{(3.7)} \quad \int_0^1 (1 - r)^{q-2} |g(re^{i\theta})f'(re^{i\theta})|^q d\theta < \infty, \quad \text{for almost every } \theta \in [0, 2\pi].
\]
Since $g \in H^\infty$ and $g \not\equiv 0$, $g$ has a finite non-zero radial limit at almost every point $e^{i\theta}$ of $\mathbb{D}$. This and (3.7) imply that
\[
\int_0^1 (1-r)^{q-2} |f'(re^{i\theta})|^q \, d\theta < \infty, \quad \text{for almost every } \theta \in [0, 2\pi].
\]
This is in contradiction with (1.7). □

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