CONVERGENCE OF EXPONENTIAL ATTRACTORS FOR A TIME SPLITTING APPROXIMATION OF THE CAGINALP PHASE-FIELD SYSTEM

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Abstract. We consider a time semi-discretization of the Caginalp phase-field model based on an operator splitting method. For every time-step parameter $\tau$, we build an exponential attractor $M_\tau$ of the discrete-in-time dynamical system. We prove that $M_\tau$ converges to an exponential attractor $M_0$ of the continuous-in-time dynamical system for the symmetric Hausdorff distance as $\tau$ tends to 0. We also provide an explicit estimate of this distance and we prove that the fractal dimension of $M_\tau$ is bounded by a constant independent of $\tau$.

1. Introduction. The Caginalp phase-field system, which has been proposed in [9] to model phase transition phenomena such as melting-solidification, has a great importance in materials science. Other models of phase transition can also be derived from it as singular limits (e.g. the Allen-Cahn, the Cahn-Hilliard and the Stefan problems). We refer the reader to [1, 3, 7, 10, 8, 16, 31, 40] and the references therein for more details.

In this paper, we will consider the Caginalp system where all physical parameters are set equal to one. It reads

\[ \frac{\partial \varphi}{\partial t} - \Delta \varphi + g(\varphi) = u, \tag{1} \]
\[ \frac{\partial u}{\partial t} - \Delta u = -\frac{\partial \varphi}{\partial t}, \tag{2} \]

where $\varphi$ is the order parameter, $u$ is the relative temperature, and $g$ is the derivative of a double-well potential $G$.

The asymptotic behaviour of this model has been extensively studied. Convergence to a stationary state, existence of global and exponential attractors has been proved for regular and singular potentials, and for various types of boundary conditions, see [4, 6, 12, 13, 14, 24, 26, 27, 29, 30, 43].

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In this paper, we consider a time semi-discretization of (1)-(2) based on an operator splitting approach. For every time-step parameter $\tau > 0$ small enough, the scheme defines a discrete dynamical system; the case $\tau = 0$ corresponds to the continuous dynamical system associated to (1)-(2). Our purpose is to build a family $\mathcal{M}_\tau$ of exponential attractors associated to these dynamical systems and which is continuous at $\tau = 0$ for the symmetric Hausdorff distance.

An exponential attractor is a compact positively invariant set which contains the global attractor, has finite fractal dimension and attracts exponentially the trajectories. In comparison with the global attractor, an exponential attractor is expected to be more robust to perturbations: global attractors are generally upper semi-continuous with respect to perturbations, but the lower semi-continuity can be proved only in some particular cases (see e.g. [2, 28, 33, 35, 37, 38, 39]). We can also note that an exponential attractor is not necessarily unique.

In the initial construction proposed by Eden et al. [17], based on a “squeezing property”, the continuity of exponential attractors was shown for classical Galerkin approximations, but only up to a time shift (see also [23, 25]). In [20], Efendiev, Miranville and Zelik proposed a construction of exponential attractors based on a “smoothing property” and an appropriate error estimate, where the continuity holds without time shift. Their construction has been adapted to many situations, including singular perturbations. We refer the reader to the review [28] and the references therein for more details.

In [32], the second author was able to use the construction in [20] in order to build a robust family of exponential attractors for a time semi-discretization of a generalized Allen-Cahn equation. An abstract result was first derived, and then applied to the backward Euler scheme. The robustness result in [20, 32] also includes an upper bound on the fractal dimension of the family of exponential attractors (and therefore of the global attractors). This bound is independent of the time step and, although crude, it is explicit in terms of the physical parameters of the model. In contrast, only a few authors have obtained upper bounds independent of the discretization parameters for the dimensions of attractors, by using more specific methods [41, 42, 44].

Our proof in this paper uses the abstract result from [32], but here, the time discretization is a first order operator splitting method, which consists in decoupling the equations (1) and (2): at every time iteration, a semi-linear elliptic equation is first solved, followed by a linear elliptic equation. In comparison, the backward Euler scheme would require the resolution of a semi-linear elliptic system at every time iteration.

We choose this particular scheme because operator splitting methods, also known as fractional step methods, are ubiquitous in the numerical resolution of evolutionary systems of partial differential equations. They allow a very efficient reduction of the computational complexity. The Chorin-Temam [15, 36] scheme for the resolution of the incompressible Navier-Stokes equation, the convex splitting of the energy for Allen-Cahn and Cahn-Hilliard type equations [21, 22], or the exponential formula for the sum of maximal monotone operators [5] are some famous (nonexhaustive) examples of these methods.

The paper is structured as follows. The estimates for the continuous problem are first given in Section 2, their discrete counterparts are derived in Section 3, and the error between the discrete solution and the continuous solution is estimated in Section 4. The main result, Theorem 5.3, is stated in the last section.
2. The continuous problem.

2.1. The continuous semi-group. We consider the Caginalp system (1)-(2) in 
\( \Omega \times (0, +\infty) \), where \( \Omega \) is a bounded domain of \( \mathbb{R}^I \) (\( I = 1, 2 \text{ or } 3 \)) with smooth boundary \( \partial \Omega \). The function \( g \) is a polynomial of odd degree with a positive leading coefficient,

\[
g(s) = \sum_{j=0}^{2p-1} b_j s^j, \quad b_{2p-1} > 0, \quad p \geq 2.
\]

For sake of simplicity, we assume that \( p = 2 \) when \( I = 3 \) (no condition on \( p \) is required if \( I = 1 \) or 2). This guarantees that the continuous imbeddings

\[
H^1(\Omega) \subset L^{4p-2}(\Omega) \subset L^{2p}(\Omega)
\]

hold true for \( I \in \{1, 2, 3\} \). A typical choice for the nonlinearity is \( g(s) = s^3 - s \).

The potential \( G \) is defined by

\[
G(s) = \sum_{j=0}^{2p-1} \frac{b_j}{j+1} s^{j+1}.
\]

Following [6], we will work with the unknowns

\[
(\varphi, v) := (\varphi, u + \varphi),
\]

and we consider Neumann boundary conditions. The problem ("problem (P)"") reads

\[
\begin{align*}
\frac{\partial \varphi}{\partial t} - \Delta \varphi + g(\varphi) + \varphi &= v \text{ in } \Omega \times (0, \infty), \\
\frac{\partial v}{\partial t} - \Delta v &= -\Delta \varphi \text{ in } \Omega \times (0, \infty), \\
\frac{\partial \varphi}{\partial \nu} &= 0 \text{ on } \partial \Omega \times (0, \infty), \\
\varphi(x, 0) &= \varphi_0(x), \quad v(x, 0) = v_0(x) \quad x \in \Omega.
\end{align*}
\]

Here, \( \nu \) is the outward unit normal to the boundary \( \partial \Omega \). We set

\[
H = (L^2(\Omega))^2 \quad \text{and} \quad V = (H^1(\Omega))^2,
\]

where the norm in \( L^2(\Omega) \) is denoted \( |\cdot|_0 \) and the scalar product \((\cdot, \cdot)_0\); the norm in \( H^1(\Omega) \) is \( \|\cdot\|_1 = |\cdot|_0^2 + |\nabla \cdot |_0^2 \), where \( |\cdot|_1 = |\nabla \cdot |_0 \) is the seminorm.

We note that any solution \( v \) of problem (P) satisfies

\[
\int_{\Omega} v(x, t) \, dx = \int_{\Omega} v_0(x) \, dx.
\]

It is therefore convenient to introduce the function spaces

\[
H_\beta = \{(\varphi, v) \in H; \frac{1}{|\Omega|} \int_{\Omega} v(x) \, dx = \beta\} \quad \text{and} \quad \mathcal{H}_\alpha = \bigcup_{|\beta| \leq \alpha} H_\beta,
\]

where \( \beta \in \mathbb{R} \) and \( \alpha > 0 \); \( H_\beta \) is an affine subspace of \( H \) whereas \( \mathcal{H}_\alpha \) is a closed convex (unbounded) subset of \( H \). We denote \( \mathbb{R}_+ \) the interval \([0, +\infty)\).

The following result is proved in [6]:

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Theorem 2.1. For any \((\varphi_0, v_0) \in H_\beta\), Problem (P) has a unique solution \((\varphi, v)\) which satisfies \((\varphi, v) \in C^1(\mathbb{R}_+; H_\beta)\) and \((\varphi, v) \in L^\infty(0, T; H_\beta) \cap L^2(0, T; V), \varphi \in L^{2p}(Q_T)\) for all \(T > 0\), where \(Q_T = \Omega \times (0, T).\) If, furthermore, \((\varphi, v) \in V \cap H_\beta,\) then \((\varphi, v) \in L^\infty(0, T; V) \cap L^2(0, T; (H^2(\Omega))^2)\) and \((\varphi_t, v_t) \in L^2(Q_T).\)

Moreover, the mapping 
\[ S_0(t) : (\varphi_0, v_0) \mapsto (\varphi(t), v(t)) \]
is Lipschitz continuous on \(H\) for all \(t \geq 0.\)

As a consequence, \(\{S_0(t) : t \geq 0\}\) is a (continuous-in-time) semi-group on \(H_\alpha\) (cf. Section 5.1).

2.2. Some useful inequalities. By Poincaré’s inequality (see e.g. [6]), there exists a constant \(c_P = c_P(\Omega)\) such that for all \(v \in H^1(\Omega),\)
\[ |v|^2_0 \leq |\Omega| \left( \frac{1}{|\Omega|} \int_\Omega v \, dx \right)^2 + c_P |\nabla v|^2_0. \] (10)

By considering the leading coefficients of the polynomials \(G\) and \(g,\) it is easily seen (see e.g. [37]) that
\[ \frac{b_{2p-1}}{4p} s^{2p} - c_1 \leq G(s) \leq \frac{b_{2p-1}}{p} s^{2p} + c'_1 \quad \forall s \in \mathbb{R}, \] (11)
and
\[ \frac{1}{2} b_{2p-1} s^{2p} - c_2 \leq g(s)s \leq \frac{3}{2} b_{2p-1} s^{2p} + c'_2 \quad \forall s \in \mathbb{R}. \] (12)
for some positive constants \(c_1, c'_1, c_2, c'_2.\) There is also a nonnegative constant \(c_g\) such that
\[ g'(s) \geq -c_g \quad \forall s \in \mathbb{R}. \] (13)

Thus, by the mean value theorem,
\[ |g(a) - g(b)|(a - b) \geq -c_g(a - b)^2 \quad \forall a, b \in \mathbb{R}. \] (14)

The following lemma, based on the Sobolev imbeddings and the restriction on \(p\) when \(I = 3,\) is standard (see e.g. [32]).

Lemma 2.2. Let \(w_1, w_2 \in H^1(\Omega)\) such that \(\|w_i\|_1 \leq R_1 (i = 1, 2),\) and \(w_3 \in L^2(\Omega).\) Then
\[ \int_\Omega |g(w_1) - g(w_2)||w_3| \, dx \leq h_1(R_1) \|w_1 - w_2\|_1\|w_3\|_0, \]
where \(h_1(R_1) = h_1(R_1, I, \Omega, p, g)\) is monotonic in \(R_1.\)

2.3. A priori estimates for the solution. In this section, we collect some results from [6].

Proposition 2.3 (Absorbing set in \(H_\alpha\)). For any \(\alpha > 0,\) there exist a constant \(R_0 = R_0(\alpha) > 0\) and a monotonic function \(T_0(\cdot)\) such that for all \((\varphi_0, v_0) \in H_\alpha,\)
\[ \|S(t)(\varphi_0, v_0)\|_H \leq R_0(\alpha) \quad \text{if} \ t \geq T_0(\|W(\varphi_0, v_0)\|_H). \]

Let \(r\) denote a positive constant. We have:

Proposition 2.4 (Absorbing set in \(H_\alpha \cap V\)). For any \(\alpha > 0,\) there exists a constant \(R_1 = R_1(\alpha) > 0\) such that for all \((\varphi_0, v_0) \in H_\alpha,\)
\[ \|S(t)(\varphi_0, v_0)\|_V \leq R_1(\alpha) \quad \text{if} \ t \geq T_0(\|W(\varphi_0, v_0)\|_H) + r. \]
These results imply the existence of a compact and connected global attractor $\mathcal{A}_\alpha$ for the semi-group $\{S_0(t) : t \geq 0\}$ on $\mathcal{H}_\alpha$ (see [6, Theorem 3.2]).

For the existence of an exponential attractor, we need a few additional estimates. The following result (Theorem 2.2 in [6]) will prove useful:

**Proposition 2.5.** There exists a positive constant $D$ depending only on $g$ such that

$$\|S_0(t)(\varphi_0, v_0)\|_H \leq e^{Dt}\|(\varphi_0, v_0)\|_H \quad \text{for all } t \geq 0.$$  

As a shortcut, $S_0(t)(\varphi_0, v_0)$ is denoted $(\varphi(t), v(t))$ in the remainder of this section. We have (see [6, (2.15)]):

**Proposition 2.6.** For any $R_1 > 0$ and $T > 0$, there exists a constant $C_1 = C_1(R_1, T)$ such that for all $(\varphi_0, v_0) \in V$ with $\|(\varphi_0, v_0)\|_V \leq R_1$, for all $t \in [0, T]$,

$$\|(\varphi(t), v(t))\|^2 + \int_0^t |\Delta \varphi_0|^2 + |\Delta v_0|^2 \, ds \leq C_1(R_1, T).$$

Next, we prove:

**Proposition 2.7.** For any $R_1 > 0$ and $T > 0$, there exists a constant $C_2 = C_2(R_1, T)$ such that for all $(\varphi_0, v_0) \in V$ with $\|(\varphi_0, v_0)\|_V \leq R_1$, for all $t \in [0, T]$,

$$\int_0^t \left| \frac{\partial \varphi}{\partial t}(s) \right|^2 \, ds \leq C_2(R_1, T).$$

In particular, for all $0 \leq t_1, t_2 \leq T$, we have

$$\|(\varphi(t_1), v(t_1)) - (\varphi(t_2), v(t_2))\|^2 \leq C_2(R_1, T)|t_1 - t_2|.$$

**Proof.** First, we multiply (3) by $\frac{\partial \varphi}{\partial t}$ and we integrate over $\Omega$. We find

$$\left| \frac{\partial \varphi}{\partial t} \right|^2 \, dt + \frac{1}{2} \frac{d}{ds} |\nabla \varphi_0|^2 + \frac{d}{ds} (G(\varphi), 1_0) + |\varphi|^2 \, dt = (v, \frac{\partial \varphi}{\partial t})_0 \leq \frac{1}{2} |v_0|^2 + \frac{1}{2} \left| \frac{\partial \varphi}{\partial t} \right|^2 |\varphi_0|_0.$$

We integrate this relation over $[0, t]$, we use (11), the Sobolev imbedding $H^1(\Omega) \subset L^{2p}(\Omega)$, and we find

$$\int_0^t \left| \frac{\partial \varphi}{\partial t}(s) \right|^2 \, ds \leq C(\|\varphi_0\|_1) + \int_0^t |v(s)|^2 \, ds,$$  

for some positive constant $C(\|\varphi_0\|_1) = C(\|\varphi_0\|_1, \Omega, p, c_1, c'_1)$.

Next, we multiply (4) by $\frac{\partial v}{\partial t}$ and we integrate over $\Omega$. After some standard computations, we obtain

$$\int_0^t \left| \frac{\partial v}{\partial t}(s) \right|^2 \, ds \leq \|v_0\|^2 + \int_0^t |\Delta \varphi(s)|_0^2 \, ds.$$  

We add (16) and (17) and we conclude, using Proposition 2.6, that (15) holds. The conclusion follows by a standard calculation. \qed

**2.4. Estimates for the difference of two solutions.** The purpose here is to obtain a “smoothing property” (cf. [28]), which is the key in our construction of the exponential attractor for the continuous dynamical system. In contrast, a “squeezing property” was used in [6] in the construction of an exponential attractor for the same problem. We note that in the Hilbert setting, the “smoothing property” implies the “squeezing property” (see [18, Remark (i)]).
In Section 2.4, \((\varphi_1,v_1)\) and \((\varphi_2,v_2)\) are two solutions to Problem (P) corresponding to different initial conditions \((\varphi_{1,0},v_{1,0})\) and \((\varphi_{2,0},v_{2,0})\), and we denote \((\psi,w) = (\varphi_1 - \varphi_2,v_1 - v_2)\) their difference. It satisfies
\[
\frac{\partial \psi}{\partial t} - \Delta \psi + [g(\varphi_1) - g(\varphi_2)] + \psi = w \text{ in } \Omega \times (0, \infty), \tag{18}
\]
\[
\frac{\partial w}{\partial t} - \Delta w + \Delta \psi = 0 \text{ in } \Omega \times (0, \infty), \tag{19}
\]
\[
\frac{\partial \psi}{\partial \nu} = 0 \text{ on } \partial \Omega \times (0, \infty), \tag{20}
\]
\[
\psi(x,0) = \psi_0(x), \quad w(x,0) = w_0(x) \quad x \in \Omega, \tag{21}
\]
where \(\psi_0 = \varphi_{1,0} - \varphi_{2,0}\) and \(w_0 = v_{1,0} - v_{2,0}\).

We have:

**Lemma 2.8.** For all \(t \geq 0\),
\[
|\psi(t)|_0^2 + |w(t)|_0^2 + \int_0^t (|\psi(s)|_1^2 + |w(s)|_1^2) ds \leq (|\psi_0|_0^2 + |w_0|_0^2)e^{D_g t},
\]
where \(D_g\) depends only on \(c_g\).

**Proof.** We first multiply (18) by \(\psi\) and integrate over \(\Omega\). We obtain
\[
\frac{1}{2} \frac{d}{dt} |\psi_0|^2 + |\nabla \psi_0|^2 + |\psi_0|^2 = \frac{1}{2} (|g(\varphi_1) - g(\varphi_2)|,\psi_0 + (w,\psi_0)
\]
\[
\leq c_g |\psi_0|^2 + \frac{1}{2} |w_0|^2 + \frac{1}{2} |\psi_0|^2,
\]
where we have used (14). Thus,
\[
\frac{d}{dt} |\psi_0|^2 + 2 |\nabla \psi_0|^2 + |\psi_0|^2 \leq 2c_g |\psi_0|^2 + |w_0|^2. \tag{22}
\]
Similarly, on multiplying (19) by \(w\), we obtain
\[
\frac{d}{dt} |w_0|^2 + |\nabla w_0|^2 \leq |\nabla \psi_0|^2. \tag{23}
\]
Now, we add (22) to (23) and we use Gronwall’s lemma. The conclusion follows with \(D_g = \max\{2c_g,1\}\).

The following result is an \(H-V\) “smoothing property”.

**Lemma 2.9.** Let \(R_1 > 0\) and \(T > 0\). There exists a constant \(C_3 = C_3(R_1,T)\) such that
\[
||(|\psi(T),w(T)||_V \leq C_3(R_1,T)||(|\psi_0,w_0)||_H
\]
if \(||(|\varphi_{i,0},v_{i,0}||_V \leq R_1 \quad (i = 1, 2)\).

We note that the constant \(C_3(R_1,T)\) blows up as \(T \to 0^+\).

**Proof.** By Proposition 2.6, we have
\[
||(|\varphi_i(t),v_i(t)||_V \leq C_1(R_1,T) \quad \forall t \in [0,T],
\]
for \(i = 1, 2\). We multiply (18) by \(\frac{\partial \psi}{\partial t}\) and integrate over \(\Omega\). We find
\[
\left|\frac{\partial \psi}{\partial t}\right|_0^2 + \frac{1}{2} \frac{d}{dt} |\nabla \psi_0|^2 + \frac{1}{2} \frac{d}{dt} |\psi_0|^2 = -\frac{1}{2} (|g(\varphi_1) - g(\varphi_2)|,\frac{\partial \psi}{\partial t}) + (w,\frac{\partial \psi}{\partial t})_0.
\]
Using Lemma 2.2 and Young’s inequality, we obtain
\[
\left| \frac{\partial \psi}{\partial t} \right|^2 + \frac{1}{2} \frac{d}{dt} |\nabla \psi|^2 + \frac{1}{2} \frac{d}{dt} |\psi|^2 \leq \left| \frac{\partial \psi}{\partial t} \right|^2 + \frac{1}{2} C'(R_1, T) \| \psi \|^2 + \frac{1}{2} |w|^2,
\]
for some constant $C'(R_1, T)$. Thus, for all $t \in [0, T]$,
\[
t \frac{d}{dt} \| \psi \|^2 \leq tC'(R_1, T) \| \psi \|^2 + t|w|^2.
\tag{24}
\]
Next, we multiply (19) by $\frac{\partial w}{\partial t}$. This yields
\[
\left| \frac{\partial w}{\partial t} \right|^2 + \frac{1}{2} \frac{d}{dt} |\nabla w|^2 = -(\Delta \psi, \frac{\partial w}{\partial t})_0.
\]
Using Young’s inequality and multiplying by $t$, we obtain that for all $t \geq 0$,
\[
t \frac{d}{dt} |\nabla w|^2 \leq t|\Delta \psi|^2.
\tag{25}
\]
On adding (24) and (25), we find
\[
t \frac{d}{dt} (\| \psi \|^2 + |w|^2) \leq C'(R_1, T)(t \| \psi \|^2) + t|w|^2 + t|\Delta \psi|^2,
\]
for all $t \in [0, T]$. Let $a(t) = \| \psi(t) \|^2 + |w(t)|^2$. Since $\frac{d}{dt}[t a(t)] = t \frac{d}{dt} a(t) + a(t)$, we deduce from the estimate above that
\[
\frac{d}{dt}[t a(t)] \leq C'(R_1, T)[t a(t)] + a(t) + t|w(t)|^2 + t|\Delta \psi(t)|^2,
\]
for all $t \in [0, T]$. Gronwall’s lemma implies that
\[
t a(t) \leq \exp(C'(R_1, T)t) \left( \int_0^t a(s)ds + T \int_0^t |w|^2 ds + \int_0^t s|\Delta \psi|^2 ds \right),
\tag{26}
\]
for all $t \in [0, T]$.

In order to estimate $|\Delta \psi|^2$, we multiply (18) by $-\Delta \psi$. We find
\[
\frac{1}{2} \frac{d}{dt} |\nabla \psi|^2 + |\Delta \psi|^2 + |\nabla \psi|^2 = ([g(\varphi_1) - g(\varphi_2)] \Delta \psi)_0 - (w, \Delta \psi)_0
\leq \frac{1}{2} |\Delta \psi|^2 + \frac{1}{2} C''(R_1, T) \| \psi \|^2 + |w|^2,
\]
where in the last line, we have used Lemma 2.2 and Young’s inequality. Thus, for all $t \in [0, T]$,
\[
\frac{d}{dt} |\nabla \psi|^2 + |\Delta \psi|^2 \leq C''(R_1, T) \| \psi \|^2 + 2|w|^2.
\]
On multiplying by $t$ and adding $|\nabla \psi(t)|^2$, we find
\[
\frac{d}{dt} [t|\nabla \psi(t)|^2] + t|\Delta \psi(t)|^2 \leq tC''(R_1, T) \| \psi(t) \|^2 + 2t|w(t)|^2 + |\nabla \psi(t)|^2.
\]
Integrating on $[0, t]$ yields
\[
t|\nabla \psi(t)|^2 + \int_0^t s|\Delta \psi(s)|^2 ds \leq [1 + tC''(R_1, T)] \int_0^t \| \psi \|^2 ds + 2T \int_0^t |w(s)|^2 ds,
\tag{27}
\]
for all $t \in [0, T]$. The conclusion follows from (26), (27) and Lemma 2.8.
3. The time semi-discrete problem. For the time semi-discretization, we apply a splitting scheme to the system (18)-(19) in the variables \((\varphi, v)\). Let \( \tau > 0 \) denote the time step. The scheme reads: let \((\varphi^0, v^0) \in H_\beta\) and for \(n = 0, 1, 2, \ldots\), let \((\varphi^{n+1}, v^{n+1}) \in V \cap H_\beta\) solve

\[
\frac{1}{\tau} (\varphi^{n+1} - \varphi^n) - \Delta \varphi^{n+1} + \varphi^{n+1} + g(\varphi^{n+1}) = v^n \text{ in } \Omega, \tag{28}
\]

\[
\frac{1}{\tau} (v^{n+1} - v^n) - \Delta v^{n+1} = -\Delta \varphi^{n+1} \text{ in } \Omega, \tag{29}
\]

\[
\frac{\partial \varphi^{n+1}}{\partial \nu} = \frac{\partial v^{n+1}}{\partial \nu} = 0 \text{ on } \partial \Omega. \tag{30}
\]

This scheme is nonlinear in the \(\varphi^{n+1}\) variable and linear in the \(v^{n+1}\) variable. By induction, we see that

\[
(v^n, 1) = (v^0, 1), \quad \forall n \geq 0. \tag{31}
\]

Thus, an important feature of the continuous problem (see (8)) is preserved on the discrete level.

3.1. The discrete semi-group. The following result shows that the discrete semi-group \(S_\tau(\varphi^0, v^0) = (\varphi^n, v^n)\) is well-defined.

**Theorem 3.1.** Assume that \( \tau \leq 1/(c_g + 1) \). Then for every \((\varphi, v) \in H_\beta\), there exists a unique \((\psi, w) \in V \cap H_\beta\) such that

\[
\frac{1}{\tau} (\psi - \varphi) - \Delta \psi + \psi + g(\psi) = v \text{ in } \Omega, \tag{32}
\]

\[
\frac{1}{\tau} (w - v) - \Delta w = -\Delta \psi \text{ in } \Omega, \tag{33}
\]

\[
\frac{\partial \psi}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial \Omega. \tag{34}
\]

Moreover, the mapping \(S_\tau : (\varphi, v) \mapsto (\psi, w)\) is Lipschitz continuous from \(H\) into \(V\), with

\[
\|S_\tau(\varphi, v) - S_\tau(\tilde{\varphi}, \tilde{v})\|_V \leq \frac{4}{\tau} \|(\varphi, v) - (\tilde{\varphi}, \tilde{v})\|_H, \quad \forall (\varphi, v, \tilde{\varphi}, \tilde{v}) \in H. \tag{35}
\]

We note that the Lipschitz constant blows up as \( \tau \to 0^+ \).

**Proof.** Let \((\varphi, v) \in H_\beta\). We first obtain a solution \(\psi\) of (32) by minimizing the functional

\[
\mathcal{G}(\zeta) = \frac{1}{2\tau} \|\zeta - \varphi\|^2_0 + \frac{1}{2} \|\nabla \zeta\|^2_0 + \frac{1}{2} \|\zeta\|^2_0 + (G(\zeta), 1)_0 - (v, \zeta)_0
\]

on \(H^1(\Omega)\). We may then obtain \(w \in H^1(\Omega)\) with \((w, 1)_0 = (v, 1)_0 = \beta|\Omega|\) which solves (33) by application of the Lax-Milgram theorem.

Now, we prove the Lipschitz continuity of \(S_\tau\) (this will also imply uniqueness of \((\psi, w)\)). Let \((\varphi, v) \in H_1\), and consider a corresponding solution \((\tilde{\psi}, \tilde{w})\) of (32)-(34). Then the difference \((\delta \psi, \delta w) = (\psi, w) - (\tilde{\psi}, \tilde{w})\) satisfies

\[
\frac{1}{\tau} \delta \psi - \Delta \delta \psi + \delta \psi + g(\psi) - g(\tilde{\psi}) = \delta v + \frac{1}{\tau} \delta \varphi, \tag{36}
\]

\[
\frac{1}{\tau} \delta w - \Delta \delta w = -\Delta \delta \psi + \frac{1}{\tau} \delta v, \tag{37}
\]
where $(\delta \varphi, \delta v) = (\varphi, v) - (\hat{\varphi}, \hat{v})$. On multiplying (36) by $\delta \psi$, using (14), $1/\tau \geq c_g$, and the Cauchy-Schwarz inequality, we find

$$|\nabla \delta \psi|^2 + |\delta \psi|^2 \leq |\delta v|^2 |\delta \psi| + \frac{1}{\tau} |\delta \varphi| |\delta \psi|.$$  

Next, we multiply (37) by $\delta w$ and we use Young’s inequality. We obtain

$$\frac{1}{\tau} |\delta w|^2 + \frac{1}{2} |\nabla \delta w|^2 \leq \frac{1}{2} (|\delta \varphi, \delta w|^2 + |\delta w|^2)^{1/2}.$$  

On adding these two estimates, using $1 \leq 1/\tau$ and the Cauchy-Schwarz inequality, we find

$$\frac{1}{2} \|\delta \psi\|^2 + \frac{1}{2} \|\delta w\|^2 \leq \frac{1}{2} (|\delta \varphi, \delta w|^2 + |\delta w|^2)^{1/2}.$$  

This implies the Lipschitz continuity (35) of $S_\tau$ as claimed. 

In the remainder of the paper, we assume that the time step $\tau$ of the discrete system satisfies at least $0 < \tau \leq 1/(1 + c_g)$. We also denote $\lfloor \cdot \rfloor$ the floor integer function.

3.2. A priori estimates for the solution, uniform in $\tau$. Throughout Section 3.2, $(\varphi^n, \psi^n)_n$ denotes a sequence in $\mathcal{H}_n$ which complies with (28)-(30). The following well-known identity will prove useful:

$$(a - b, a) = \frac{1}{2} (|a|^2 - |b|^2 + |a - b|^2) \quad \forall a, b \in L^2(\Omega).$$  

**Proposition 3.2** (Absorbing set in $\mathcal{H}_n$). For any $\alpha > 0$, there exist a constant $R_\alpha = R\alpha > 0$ independent of $\tau$ and a monotonic function $T_\alpha$ independent of $\tau$ such that for all $(\varphi^n, \psi^n) \in \mathcal{H}_n$.

$$\| (\varphi^n, \psi^n) \| \leq R_\alpha \quad \text{if } n \tau \geq T_\alpha(\| (\varphi^n, \psi^n) \|).$$

**Proof.** We multiply (28) by $\varphi^{n+1}$. We obtain

$$\frac{1}{\tau} (\varphi^{n+1} - \varphi^n, \varphi^{n+1})_0 + |\nabla \varphi^{n+1}|_0^2 + |\varphi^{n+1}|_0^2 + (g(\varphi^{n+1}), \varphi^{n+1})_0 = (v^n, \varphi^{n+1})_0.$$  

We use (12) and the inequality $|ab| \leq \varepsilon a^2 + \varepsilon b^2 + C_\varepsilon$ with $\varepsilon \leq b_{2p-1}/4$. We find

$$\frac{1}{\tau} (\varphi^{n+1} - \varphi^n, \varphi^{n+1})_0 + |\nabla \varphi^{n+1}|_0^2 + |\varphi^{n+1}|_0^2 + \frac{1}{4} b_{2p-1} (|\varphi^{n+1}|_0^{2p}, 1)_0 \leq \varepsilon |\varphi^n|^2 + C_\varepsilon',$$  

with $C'_\varepsilon = (C_\varepsilon + c_2)|\Omega|$. Next we multiply (29) by $v^{n+1}$, we use the Cauchy-Schwarz inequality and Young’s inequality. We obtain

$$\frac{1}{\tau} (v^{n+1} - v^n, v^{n+1})_0 + |\nabla v^{n+1}|_0^2 \leq \frac{1}{2} |\nabla \varphi^{n+1}|_0^2 + \frac{1}{2} |\nabla v^{n+1}|_0^2.$$  

Now we add (39) to (40) and we use (38). This yields

$$\frac{1}{2\tau} (|\varphi^{n+1}|_0^2 - |\varphi^n|^2_0 + |\varphi^{n+1} - \varphi^n|^2_0) + \frac{1}{2\tau} (|v^{n+1}|_0^2 - |v^n|^2_0 + |v^{n+1} - v^n|^2_0)$$  

$$\quad + \frac{1}{2} |\nabla \varphi^{n+1}|_0^2 + |\varphi^{n+1}|_0^2 + \frac{1}{2} |\nabla v^{n+1}|_0^2 \leq \varepsilon |\varphi^n|^2_0 + C'_\varepsilon \quad \forall n \geq 0.$$  

Recall that $(v^{n+1}, 1)_0 = (v^n, 1) = |\Omega|\beta$ (cf. (31)). Using the Poincaré inequality (10) and choosing $\varepsilon \leq 1/(4c_P)$, namely $\varepsilon = \min\{b_{2p-1}/4, 1/(4c_P)\}$, we find

$$\frac{1}{2\tau} (|\varphi^{n+1}|_0^2 - |\varphi^n|^2_0 + |\varphi^{n+1} - \varphi^n|^2_0) + \frac{1}{2\tau} (|v^{n+1}|_0^2 - |v^n|^2_0 + |v^{n+1} - v^n|^2_0)$$  

$$\quad + |\varphi^{n+1}|_0^2 + \frac{1}{2\tau} |\nabla v^{n+1}|_0^2 \leq \frac{1}{4c_P} |\varphi^n|^2_0 + \frac{1}{2} C(\beta, \Omega, \varepsilon) \quad \forall n \geq 0.$$
where \( \frac{1}{2}C(\beta, \Omega, \varepsilon) = C_\varepsilon + |\Omega|\beta^2/(2c_P) \). On multiplying by \( 2\tau \), we find
\[
(1 + 2\gamma\tau)a_{n+1} \leq (1 + \gamma\tau)a_n + \tau C \quad \forall n \geq 0,
\]
with \( a_n = \|\phi^n, v^n\|_{TV}^2, \gamma = \min\{1/2, 1/(2c_P)\} > 0 \) and \( C = C(\beta, \Omega, \varepsilon) \). By induction, this yields
\[
a_n \leq \tilde{\gamma}^n a_0 + (1 - \tilde{\gamma}^n) \frac{C}{\gamma} \quad \forall n \geq 0, \tag{42}
\]
with \( \tilde{\gamma} = (1 + \gamma\tau)/(1 + 2\gamma\tau) < 1 \). We note that \( \exp(s/2) \leq (1 + 2s)/(1 + s) \) for all \( s \in [0, 1/2] \). Choosing \( s = \gamma\tau \), which belongs to \([0, 1/2]\) since \( \tau \leq 1/(1 + c_g) \leq 1 \), we see that \( \tilde{\gamma} \leq \exp(-\gamma\tau/2) \), and we find
\[
a_n \leq \exp(-\gamma n\tau/2)a_0 + \frac{C}{\gamma} \quad \forall n \geq 0.
\]

The assertion of the theorem follows by taking
\[
\mathcal{R}_0 = (1 + C/\gamma)^{1/2} = \mathcal{R}_0'(\alpha, \Omega, b_{2p-1}, c_P, c_2, p)
\]
and \( \mathcal{T}_0'(R) = (2/\gamma) \ln(R^2) \).

We note that (42) implies \( a_n \leq a_0 + C/\gamma \) for all \( n \geq 0 \). Thus, we have:

**Proposition 3.3** (Bound on bounded sets). For all \( R > 0 \), there exists a constant \( C(R) \) independent of \( \tau \) such that \( \|\phi^n, v^n\|_H \leq R \) implies \( \|\phi^n, v^n\|_H \leq C(R) \) for all \( n \geq 0 \).

For the absorbing set in \( V \), we will need the following lemma from [34].

**Lemma 3.4** (Discrete uniform Gronwall lemma). Let \( n_0, N \in \mathbb{N}, a_1, a_2, a_3, \tau, r' > 0 \) and \((d^n), (g^n), (h^n)\) be three sequences of nonnegative real numbers which satisfy
\[
\frac{d^{n+1} - d^n}{\tau} \leq g^n d^n + h^n, \quad \forall n \geq n_0,
\]
and
\[
\tau \sum_{n=k_0}^{k_0+N} g^n \leq a_1, \quad \tau \sum_{n=k_0}^{k_0+N} h^n \leq a_2, \quad \tau \sum_{n=k_0}^{k_0+N} d^n \leq a_3,
\]
for all \( k_0 \geq n_0 \), with \( r' = \tau N > 0 \). Then
\[
d^n \leq \left(a_2 + \frac{a_3}{r'}\right) \exp(a_1), \quad \forall n \geq n_0 + N.
\]

In the next result, \( r \geq 2 \) is a fixed positive constant (cf. Proposition 2.4).

**Proposition 3.5** (Absorbing set in \( \mathcal{H}_\alpha \cap V \)). For any \( \alpha > 0 \), there exists a constant \( \mathcal{R}'_1 = \mathcal{R}'_1(\alpha) > 0 \) independent of \( \tau \) such that for all \((\phi^0, v^0) \in \mathcal{H}_\alpha \),
\[
\|\phi^n, v^n\|_V \leq \mathcal{R}'_1(\alpha) \quad \text{if } n\tau \geq \mathcal{T}'_0(\|\phi^0, v^0\|_H) + 1 + r.
\]

**Proof.** We multiply (28) by \(-\Delta \phi^{n+1}\) and (29) by \( v^{n+1} \), and we add the two resulting equations. Using
\[
\int_{\Omega} g(\phi^{n+1}) \Delta \phi^{n+1} dx = \int_{\Omega} g'(\phi)|\nabla \phi^{n+1}|^2 dx
\geq - c_g |\nabla \phi^{n+1}|_0^2 = c_g (\phi^{n+1}, \Delta \phi^{n+1})_0,
\]
Young’s inequality and (38), we find
\[
\frac{1}{2\tau^2}(|\varphi^{n+1}|^2 - |\varphi^n|^2 + |\varphi^{n+1} - \varphi^n|^2) + \frac{1}{2\tau^2}(|v^{n+1}|^2 - |v^n|^2 + |v^{n+1} - v^n|^2)
\]
\[
+ \frac{1}{4}(|\Delta \varphi^{n+1}|^2 + \frac{1}{2}(|\Delta v^{n+1}|^2) \leq 2c_2^2|\varphi^{n+1}|^2 + 2|v^n|^2 \quad \forall n \geq 0.
\]
(43)

Thus, by Proposition 3.5, for \(n\tau \geq \mathcal{T}_0((\varphi^0, v^0)_H)\), we have
\[
\frac{1}{\tau}(|\varphi^{n+1}|^2 - |\varphi^n|^2 + |v^{n+1}|^2 - |v^n|^2) \leq C_g \mathcal{R}_0^2(\alpha),
\]
where \(C_g = \max\{4c_2^2, 4\}\). On the other hand, estimate (41) shows that
\[
\frac{1}{\tau}(\|\varphi^{n+1}, v^{n+1}\|_H^2 - \|\varphi^n, v^n\|_H^2) + |\varphi^{n+1}|^2 + |v^{n+1}|^2 \leq 2\varepsilon \mathcal{R}_0^2(\alpha) + 2C_\varepsilon,
\]
for all \(n\tau \geq \mathcal{T}_0((\varphi^0, v^0)_H)\). We set \(d^n = |\varphi^n|^2 + |v^n|^2\). Let \(k_0, N \in \mathbb{N} \setminus \{0\}\) with \(k_0 \tau \geq \mathcal{T}_0((\varphi^0, v^0)_H) + \tau\) and \(N = \lfloor r/\tau \rfloor \geq 2\). Summing the estimate above from \(n = k_0 - 1\) to \(k_0 + N - 1\), we obtain
\[
\tau \sum_{n=k_0}^{k_0+N} d^n \leq (N + 1)\tau(2\varepsilon \mathcal{R}_0^2(\alpha) + 2C_\varepsilon) + \mathcal{R}_0^2(\alpha) \leq C(\alpha, r).
\]
We are in position to apply the discrete uniform Gronwall lemma (Lemma 3.4) with \(g^n = 0, a_1 = 0\), and \(h^n = C_g \mathcal{R}_0^2(\alpha)\) (a constant). We find
\[
d^n \leq (N + 1)\tau C_g \mathcal{R}_0^2(\alpha) + \frac{C(\alpha, r)}{r'} \quad \forall n \geq k_0 + N,
\]
with \(r' = N\tau \in [r - \tau, r]\). This shows the assertion. \(\square\)

Let us now derive a useful estimate.

**Proposition 3.6.** For all \(R_1 > 0\) and \(T > 0\), there exists a constant \(C_1'(R_1, T)\) independent of \(\tau\) such that \(\|((\varphi^0, v^0))_V \leq R_1\) and \(0 \leq n\tau \leq T\) imply
\[
\|((\varphi^n, v^n))_V^2 + \sum_{k=0}^{n-1} \|((\varphi^{k+1} - \varphi^k, v^{k+1} - v^k))_V^2 \leq C'_1(R_1, T).
\]

**Proof.** Let \(\varepsilon' = 1/(4c_2^2)\), and define the following norm on \(V\),
\[
\|\psi, w\|_{V, \varepsilon'}^2 = |\psi|^2 + |w|^2 + \varepsilon' |\psi|^2 + \varepsilon' |w|^2
\]
which is equivalent to the norm \(\|\cdot\|_V\). We multiply (43) by \(\varepsilon'\) and we add the result to (41). This yields
\[
\frac{1}{\tau}(\|\varphi^{n+1}, v^{n+1})_V^2 - \|\varphi^n, v^n\|_V^2 + \|\varphi^{n+1} - \varphi^n, v^{n+1} - v^n\|_{V, \varepsilon'}^2)
\]
\[
\leq (2\varepsilon + 2\varepsilon')|v^n|^2 + 2C_\varepsilon \quad \forall n \geq 0.
\]
In other words, we have
\[
b_{n+1} + c_n \leq (1 + \gamma \tau)b_n + 2\tau C_\varepsilon' \quad \forall n \geq 0,
\]
where \(b_n = \|((\varphi^n, v^n))_{V, \varepsilon'}^2\), \(c_n = \|((\varphi^{n+1} - \varphi^n, v^{n+1} - v^n))_{V, \varepsilon'}^2\), and \(\gamma = 2\varepsilon + 2\varepsilon'\). By induction, we find
\[
b_n + \sum_{k=0}^{n-1} c_k \leq (1 + \gamma \tau)b_0 + \frac{2}{\gamma} C_\varepsilon' (1 + \gamma \tau)^n - 1 \quad \forall n \geq 0.
\]
The result follows by noting that \((1 + \gamma \tau)^n \leq \exp(\gamma n \tau) \leq \exp(\gamma T)\) if \(0 \leq n \tau \leq T\).

### 3.3. Estimates for the difference of solutions, uniform in \(\tau\)

In this section, \((\varphi^n, v^n)\) and \((\hat{\varphi}^n, \hat{v}^n)\) are two sequences generated by the scheme (28)-(30). We denote \(\psi^n = \varphi^n - \hat{\varphi}^n\) and \(w^n = v^n - \hat{v}^n\) their difference, which satisfies

\[
\frac{1}{\tau}(\psi^{n+1} - \psi^n) - \Delta \psi^{n+1} + \psi^{n+1} + g(\varphi^{n+1}) - g(\hat{\varphi}^{n+1}) = w^n \quad \text{in } \Omega, \tag{44}
\]

\[
\frac{1}{\tau}(w^{n+1} - w^n) - \Delta w^{n+1} = -\Delta \psi^{n+1} \quad \text{in } \Omega, \tag{45}
\]

\[
\frac{\partial \psi^{n+1}}{\partial \nu} = \frac{\partial w^{n+1}}{\partial \nu} = 0 \quad \text{on } \Omega. \tag{46}
\]

We first have (compare with Lemma 2.8):

**Lemma 3.7.** Assume that \(\tau \leq 1/(4c_g)\). Then for all \(n \geq 0\) we have

\[
|\psi^n|_0^2 + |w^n|_0^2 + \tau \sum_{k=0}^{n-1} (|\psi^{k+1}|_1^2 + |w^{k+1}|_1^2) \leq \exp(D'_g n \tau) (|\psi^0|_0^2 + |w^0|_0^2),
\]

where \(D'_g\) is a constant which depends only on \(c_g\).

**Proof.** We multiply (44) by \(\psi^{n+1}\) and (45) by \(w^{n+1}\). We add the resulting equations, we use (13), (38) and Young’s inequality. This yields

\[
\frac{1}{\tau}(\psi^{n+1}^2|_0 - \psi^n|_0^2 + |w^{n+1}|_0^2 - |w^n|_0^2) + \|\psi^{n+1}\|_1^2 + |w^{n+1}|_1^2 \leq 2c_g (\psi^{n+1})_0^2 + \|w^n\|_1^2
\]

for all \(n \geq 0\). Thus, for all \(n \geq 0\), we have

\[
a_{n+1} + \tau b_n \leq \frac{1 + \tau}{1 - 2c_g \tau} a_n,
\]

where \(a_n = |\psi^n|_0^2 + |w^n|_0^2\) and \(b_n = \|\psi^{n+1}\|_1^2 + |w^{n+1}|_1^2\). Now we note that

\[
\frac{1 + s}{1 - 2c_g s} \leq 1 + D'_g s \leq \exp(D'_g s) \quad \forall s \in [0, 1/(4c_g)],
\]

for some constant \(D'_g\) which depends only on \(c_g\). By induction, we obtain

\[
a_n + \tau \sum_{k=0}^{n-1} b_k \leq \exp(D'_g n \tau) a_0 \quad \forall n \geq 0,
\]

and the claim is proved. \(\square\)

The smoothing property reads as follows (compare with Lemma 2.9).

**Proposition 3.8.** Let \(R_1 > 0\) and \(T > 0\). There exist two constants \(\tau = \tau(R_1, T) > 0\) and \(C_4(R_1, T) > 0\) such that for all \(\tau \in (0, \tau]\), for all \(0 \leq n \tau \leq T\), \(\|\varphi^0, v^0\|_V \leq R_1\) and \(\|\hat{\varphi}^0, \hat{v}^0\|_V \leq R_1\) imply

\[
n \tau \|\psi^n, w^n\|_V^2 \leq C_4(R_3, T) \|\psi^0, w^0\|_H^2.
\]

**Proof.** Let \(R_1 > 0\) and \(T > 0\). By Proposition 3.6, for all \(n\) such that \(0 \leq n \tau \leq T\), we have

\[
\|\varphi^n, v^n\|_V^2 \leq C_1'(R_1, T) \quad \text{and} \quad \|\hat{\varphi}^n, \hat{v}^n\|_V^2 \leq C_1'(R_1, T).
\]
We multiply (44) by \((\psi^{n+1} - \psi^n)/\tau\), we use (38), Lemma 2.2 and Young’s inequality. We find
\[
\frac{1}{\tau}(\|\psi^{n+1}\|_1^2 - \|\psi^n\|_1^2) \leq C(R_1, T)\|\psi^{n+1}\|_1^2 + |w^n|_0^2, \quad 0 \leq n\tau \leq T,
\] (47)
for some constant \(C(R_1, T)\). Next, we multiply (45) by \((w^{n+1} - w^n)/\tau\), we use (38) and Young’s inequality. This yields
\[
\frac{1}{\tau}(\|\psi^{n+1}\|_1^2 - |w^n|_1^2) \leq |\Delta \psi^{n+1}|_0^2, \quad n \geq 0.
\] (48)
We multiply (47) and (48) by \(n\tau\) and we add the resulting estimates. We find
\[
n(\|\psi^{n+1}\|_1^2 - \|\psi^n\|_1^2 + |w^{n+1}|_1^2 - |w^n|_1^2) \\
\leq C(R_1, T)n\tau\|\psi^{n+1}\|_1^2 + n\tau|w^n|_0^2 + n\tau|\Delta \psi^{n+1}|_0^2,
\]
for all \(0 \leq n\tau \leq T\). We set \(a_n = \|\psi^n\|_1^2 + |w^n|_1^2\). Using
\[
na_{n+1} - a_n = (n+1)a_{n+1} - na_n - a_{n+1},
\]
we see that
\[
(n+1)a_{n+1} - na_n \leq C(R_1, T)\tau(n+1)a_{n+1} + a_{n+1} + n\tau|w^n|_0^2 + n\tau|\Delta \psi^{n+1}|_0^2,
\]
for all \(0 \leq n\tau \leq T\). We let \(\tau_1 = \tau_1(R_1, T) := 1/(2C(R_1, T))\). If \(0 < \tau \leq \tau_1\), then \(C(R_1, T)\tau \leq 1/2\) and we have
\[
\frac{1}{1 - C(R_1, T)\tau} \leq 1 + 2C(R_1, T)\tau.
\]
The estimate above becomes
\[
b_{n+1} \leq (1 + 2C(R_1, T)\tau ) \left( b_n + a_{n+1} + T|w^n|_0^2 + n\tau|\Delta \psi^{n+1}|_0^2 \right),
\]
for all \(0 \leq n\tau \leq T\), where \(b_n = na_n\). Since \(b_0 = 0\), we obtain by induction that
\[
b_n \leq (1 + 2C(R_1, T)\tau )\sum_{k=0}^{n-1} (a_{k+1} + T|w^k|_0^2 + k\tau|\Delta \psi^{k+1}|_0^2),
\] (49)
for all \(0 \leq n\tau \leq T\).

Next, we estimate the term \(|\Delta \psi^{n+1}|_0\). We multiply (44) by \(-\Delta \psi^{n+1}\), we use (38), Lemma 2.2 and Young’s inequality. We find
\[
\frac{1}{\tau}(\|\psi^{n+1}\|_1^2 - |\psi^n\|_1^2) + |\Delta \psi^{n+1}|_0^2 \leq C'(R_1, T)\|\psi^{n+1}\|_1^2 + 2|w^n|_0^2,
\]
for all \(0 \leq n\tau \leq T\), for some constant \(C'(R_1, T)\). On multiplying by \(n\tau\) and adding \(|\psi^{n+1}|_1^2\), we find
\[
(n+1)|\psi^{n+1}|_1^2 - n|\psi^n|_1^2 + n\tau|\Delta \psi^{n+1}|_0^2 \leq C'(R_1, T)n\tau|\psi^{n+1}|_1^2 + 2n\tau|w^n|_0^2;
\]
for all \(0 \leq n\tau \leq T\). By induction, we obtain
\[
n|\psi^n|_1^2 + \sum_{k=0}^{n-1} k\tau|\Delta \psi^{k+1}|_0^2 \leq TC'(R_1, T)\sum_{k=0}^{n-1} \|\psi^{k+1}\|_1^2 + 2T\sum_{k=0}^{n-1} |w^k|_0^2,
\] (50)
for all \(0 \leq n\tau \leq T\). The conclusion follows from (49), (50) and Lemma 3.7, with
\[
\tau = \min \{ \tau_1(R_1, T), 1/(4c_g), 1/(1 + c_g) \}.
\] \(\square\)
4. Finite time uniform error estimate. For the error estimate on a finite time interval, we follow the (now standard) methodology described in [38].

We consider a sequence \((\varphi^n, v^n)\) generated by (28)-(30). To the sequence \((\varphi^n)_n\), we associate two functions \(\varphi_\tau, \overline{\varphi}_\tau : \mathbb{R}_+ \to L^2(\Omega)\), namely

\[
\varphi_\tau(t) := \varphi^n + \frac{t-n\tau}{\tau}(\varphi^{n+1} - \varphi^n), \quad t \in [n\tau, (n+1)\tau],
\]

and

\[
\overline{\varphi}_\tau(t) := \varphi^{n+1} + \frac{n\tau}{\tau}(\varphi^n - \varphi^{n+1}), \quad t \in [n\tau, (n+1)\tau).
\]

We associate two functions \(v_\tau, \overline{v}_\tau : \mathbb{R}_+ \to L^2(\Omega)\) to the sequence \((v^n)_n\) in a similar way. We also use the function \(\underline{v}_\tau : \mathbb{R}_+ \to L^2(\Omega)\) defined by

\[
\underline{v}_\tau(t) := v^n, \quad t \in [n\tau, (n+1)\tau).
\]

We assume that \((\varphi^0, v^0)\) belongs to \(V \cap H_\beta\). Then \((\varphi_\tau, v_\tau)\) belongs to \(C^0(\mathbb{R}_+; V \cap H_\beta)\) and \((\overline{\varphi}_\tau, \overline{v}_\tau) \in L^\infty(\mathbb{R}_+; V \cap H_\beta)\). The scheme (28)-(30) can be rewritten

\[
\frac{\partial \varphi_\tau}{\partial t} - \Delta \varphi_\tau + \varphi_\tau + g(\overline{\varphi}_\tau) = v_\tau, \quad \text{a.e. } t \geq 0, \quad (51)
\]

\[
\frac{\partial v_\tau}{\partial t} - \Delta v_\tau = -\Delta \overline{\varphi}_\tau \quad \text{in } [H^1(\Omega)]', \quad \text{a.e. } t \geq 0. \quad (52)
\]

Let \((\varphi, v)\) denote a solution of (3)-(6) with \((\varphi_0, v_0) \in V \cap H_\beta\), and set \(e_\tau = \varphi_\tau - \varphi, \theta_\tau = v_\tau - v\). We prove the following error estimate.

**Theorem 4.1.** For all \(R_1 > 0\) and \(T > 0\), there is a constant \(Q(R_1, T)\) independent of \(\tau\) such that \((\varphi^0, v^0) = (\varphi_0, v_0)\) and \(\|(\varphi^0, v^0)\|_V \leq R_1\) imply

\[
\sup_{t \in [0, N\tau]} \|e_\tau(t), \theta_\tau(t)\|_H \leq Q(R_1, T)\tau^{1/2},
\]

where \(N = \lfloor T/\tau \rfloor\).

**Proof.** Let \(R_1 > 0\), \(T > 0\) and \(N = \lfloor T/\tau \rfloor\). By Proposition 3.6, for all \(n\) such that \(0 \leq n\tau \leq T\), we have \(\|(\varphi^n, v^n)\|_V^2 \leq C^1(R_1, T)\). The functions \(\varphi_\tau, \overline{\varphi}_\tau, v_\tau, \overline{v}_\tau\) and \(\underline{v}_\tau\) are therefore uniformly bounded by \(\sqrt{C^1} = \sqrt{C^1(R_1, T)}\) for the V-norm on \([0, N\tau]\). Equation (51) reads

\[
\frac{\partial \varphi_\tau}{\partial t} - \Delta \varphi_\tau + \varphi_\tau + g(\overline{\varphi}_\tau) = v_\tau - \Delta[\varphi_\tau - \overline{\varphi}_\tau] + [\varphi_\tau - \overline{\varphi}_\tau] + [g(\varphi_\tau) - g(\overline{\varphi}_\tau)] - [v_\tau - \underline{v}_\tau], \quad (53)
\]

We subtract (3) from (53), and we multiply the resulting equation by \(e_\tau\). Using (13), we find

\[
\frac{1}{2} \frac{d}{dt} |e_\tau|^2 + \|e_\tau\|_1^2 \leq \varepsilon g |e_\tau|^2 + \|\theta_\tau\|_0 |e_\tau|_0 + \|\varphi_\tau - \overline{\varphi}_\tau\|_1 \|e_\tau\|_1 \\
+ \int_\Omega |g(\varphi_\tau) - g(\overline{\varphi}_\tau)| |e_\tau| \, dx + |v_\tau - \underline{v}_\tau|_0 |e_\tau|_0. \quad (54)
\]

Now, equation (52) reads

\[
\frac{\partial v_\tau}{\partial t} - \Delta v_\tau = -\Delta \varphi_\tau - \Delta[v_\tau - \overline{\varphi}_\tau] + \Delta[\varphi_\tau - \overline{\varphi}_\tau]. \quad (55)
\]

We subtract (4) from (55) and we multiply the resulting equation by \(\theta_\tau\). We find

\[
\frac{1}{2} \frac{d}{dt} |\theta_\tau|^2 + |\theta_\tau|_1^2 \leq |e_\tau|_1 |\theta_\tau|_1 + |v_\tau - \overline{\varphi}_\tau|_1 |\theta_\tau|_1 + |\varphi_\tau - \overline{\varphi}_\tau|_1 |\theta_\tau|_1. \quad (56)
\]
Adding (54) and (56), using Young’s inequality and Lemma 2.2, we find

\[
\frac{1}{2} \frac{d}{dt} \left( |e_{r}(t)|^2 + |\theta_{r}(t)|^2 \right) \leq \left( c_{g} + 1 \right) |e_{r}|^2 + |\theta_{r}|^2 + \left( \frac{3}{2} + h^2(\sqrt{C_{1}}) \right) \| \varphi_{r} - \bar{\varphi}_{r} \|^2 + |v_{r} - \bar{v}_{r}|^2 + |v_{r} - \bar{v}_{r}|^2,
\]

for all \( t \in [0, N\tau] \). Since \( e_{r}(0) = 0 \) and \( \theta_{r}(0) = 0 \), the classical Gronwall lemma yields

\[
|e_{r}(t)|^2 + |\theta_{r}(t)|^2 \leq C_{1}'' \exp(C_{g}T) \int_{0}^{N\tau} \left( \| \varphi_{r} - \bar{\varphi}_{r} \|^2 + |v_{r} - \bar{v}_{r}|^2 + |v_{r} - \bar{v}_{r}|^2 \right) ds
\]

for all \( t \in [0, N\tau] \), where \( C_{g} = 2(c_{g} + 1) \) and \( C_{1}'' = 3 + 2h^2(\sqrt{C_{1}}) = C'_{1}(R_{1}, T) \).

On the interval \([n\tau, (n + 1)\tau)\) we have \( \| \varphi_{r} - \bar{\varphi}_{r} \|_1 \leq \| \varphi^{n+1} - \varphi^n \|_1 \), \( |v_{r} - \bar{v}_{r}|_0 \leq |v^{n+1} - v^n|_0 \) and \( |v_{r} - \bar{v}_{r}|_1 \leq |v^{n+1} - v^n|_1 \), so

\[
\int_{0}^{N\tau} \| \varphi_{r} - \bar{\varphi}_{r} \|^2 + |v_{r} - \bar{v}_{r}|^2 + |v_{r} - \bar{v}_{r}|^2 ds \leq \tau \sum_{n=0}^{N-1} \| (\varphi^{n+1} - \varphi^n, v^{n+1} - v^n) \|^2
\]

This shows, using Proposition 3.6, that

\[
|e_{r}(t)|^2 + |\theta_{r}(t)|^2 \leq [C'_{1}''(R_{1}, T) \exp(C_{g}T)C'_{1}(R_{1}, T)] \tau, \quad \forall t \in [0, N\tau].
\]

The proof is complete.

\[\square\]

5. Convergence of exponential attractors.

5.1. Some (standard) definitions. Before stating our main result, we recall some definitions (see e.g. [20, 32, 37]). The Hilbert space \( H \) is defined as previously by (7) and \( K \) denotes a closed subset of \( H \). A continuous-in-time semi-group \( \{S(t), t \in \mathbb{R}_{+}\} \) on \( K \) is a family of (nonlinear) operators such that \( S(t) \) is a continuous operator from \( K \) into itself, for all \( t \in \mathbb{R}_{+} \), with \( S(0) = \text{Id} \) (identity in \( K \)) and

\[
S(t+s) = S(t) \circ S(s), \quad \forall s, t \in \mathbb{R}_{+}.
\]

A discrete-in-time semi-group \( \{S(t), t \in \mathbb{N}\} \) on \( K \) is a family of (nonlinear) operators which satisfy these properties with \( \mathbb{R}_{+} \) replaced by \( \mathbb{N} \). A discrete-in-time semi-group is usually denoted \( \{S^{n}, n \in \mathbb{N}\} \), where \( S(= S(1)) \) is a continuous (nonlinear) operator from \( K \) into itself.

A (continuous or discrete) semi-group \( \{S(t), t \geq 0\} \) defines a (continuous or discrete) dynamical system: if \( u_{0} \) is the state of the dynamical system at time 0, then \( S(t)u_{0} \) is the state at time \( t \geq 0 \). The term “dynamical system” will sometimes be used instead of “semi-group”.

**Definition 5.1** (Global attractor). Let \( \{S(t), t \geq 0\} \) be a continuous or discrete semi-group on \( K \). A set \( A \subset K \) is called the global attractor of the dynamical system if the following three conditions are satisfied:

1. \( A \) is compact in \( H \);
2. \( A \) is invariant, i.e. \( S(t)A = A \), for all \( t \geq 0 \);
3. \( A \) attracts all bounded sets in \( K \), i.e., for every bounded set \( B \) in \( K \),

\[
\lim_{t \to +\infty} \text{dist}_{H}(S(t)B, A) = 0.
\]
Here, $\text{dist}_H$ denotes the non-symmetric Hausdorff semi-distance in $H$ between two subsets, which is defined as

$$\text{dist}_H(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_H.$$ 

It is easy to see, thanks to the invariance and the attracting property, that the global attractor, when it exists, is unique [37].

Let $A \subset H$ be a (relatively compact) subset of $H$. For $\varepsilon > 0$, we denote $N_\varepsilon(A, H)$ the minimum number of balls of $H$ of radius $\varepsilon > 0$ which are necessary to cover $A$. The fractal dimension of $A$ (see e.g. [17, 37]) is the number

$$\text{dim}_F(A) = \limsup_{\varepsilon \to 0} \frac{\log(N_\varepsilon(A, H))}{\log(1/\varepsilon)} \in [0, +\infty].$$

**Definition 5.2** (Exponential attractor). Let $\{S(t), \ t \geq 0\}$ be a continuous or discrete semi-group on $K$. A set $M \subset K$ is an exponential attractor of the dynamical system if the following three conditions are satisfied:

1. $M$ is compact in $H$ and has finite fractal dimension;
2. $M$ is positively invariant, i.e. $S(t)M \subset M$, for all $t \geq 0$;
3. $M$ attracts exponentially the bounded subsets of $K$ in the following sense:

$$\forall B \subset K \text{ bounded}, \quad \text{dist}_H(S(t)B, M) \leq Q\|B\|_H e^{-\alpha t}, \quad t \geq 0,$$

where the positive constant $\alpha$ and the monotonic function $Q$ are independent of $B$. Here, $\|B\|_H = \sup_{b \in B} \|b\|_H$.

The exponential attractor, if it exists, contains the global attractor (actually, the existence of an exponential attractor yields the existence of the global attractor, see [2, 11]). We also note that if $K$ is bounded, then in the definition above, we may obviously replace (3) by:

$$(3\text{bis}) \quad M \text{ attracts } K \text{ exponentially}, \quad \text{i.e.}$$

$$\text{dist}_H(S(t)K, M) \leq Ce^{-\alpha t}, \quad t \geq 0,$$

for some positive constants $C$ and $\alpha$.

### 5.2. The main result

Recall that $\mathcal{H}_\alpha$ (see (9)) is a closed convex subset of $H$. We have seen (Theorem 2.1) that $\{S_0(t), \ t \in \mathbb{R}_+\}$ defines a continuous-in-time dynamical system on $\mathcal{H}_\alpha$, and that for every $\tau > 0$ small enough (Theorem 3.1), $\{S^n_\tau, \ n \in \mathbb{N}\}$ defines a discrete-in-time dynamical system on $\mathcal{H}_\alpha$. We have:

**Theorem 5.3.** Let $\tau_0 = 1/(1 + c_\theta)$. For every $\tau \in (0, \tau_0]$, the discrete dynamical system $\{S^n_\tau, \ n \in \mathbb{N}\}$ possesses an exponential attractor $\mathcal{M}_\tau$ on $\mathcal{H}_\alpha$, and the continuous dynamical system $\{S_0(t), \ t \in \mathbb{R}_+\}$ possesses an exponential attractor $\mathcal{M}_0$ on $\mathcal{H}_\alpha$ such that:

1. the fractal dimension of $\mathcal{M}_\tau$ is bounded, uniformly with respect to $\tau \in [0, \tau_0]$,

$$\text{dim}_F \mathcal{M}_\tau \leq c_3,$$

where $c_3$ is independent of $\tau$;

2. $\mathcal{M}_\tau$ attracts the bounded sets of $\mathcal{H}_\alpha$, uniformly with respect to $\tau \in (0, \tau_0]$, i.e.

$$\forall \tau \in (0, \tau_0], \ \forall B \subset \mathcal{H}_\alpha \text{ bounded}, \quad \text{dist}_H(S^n_\tau B, \mathcal{M}_\tau) \leq Q\|B\|_H e^{-c_4n\tau}, \ n \in \mathbb{N},$$

where the constant $c_4$ and the monotonic function $Q$ are independent of $\tau$;
3. the family \( \{ \mathcal{M}_\tau, \tau \in [0, \tau_0] \} \) is continuous at 0,
\[
dist_{\text{sym}}(\mathcal{M}_\tau, \mathcal{M}_0) \leq c_5 T^{c_6},
\]
where \( c_5 \) and \( c_6 \in (0,1) \) are independent of \( \tau \) and \( \dist_{\text{sym}} \) denotes the symmetric Hausdorff distance between sets, defined by
\[
dist_{\text{sym}}(A, B) := \max\{ \dist_H(A, B), \dist_H(B, A) \}.
\]

**Proof.** We apply Theorem 2.5 in [32] with the spaces \( H \) and \( V \) defined by (7), and the set
\[
\mathcal{B} = \{(\psi, w) \in \mathcal{H}_\alpha : \| (\psi, w) \|_V \leq \max\{ R_1(\alpha), R_1'(\alpha) \} \}.
\]
We note that \( V \) is compactly imbedded in \( H \), and that \( \mathcal{B} \) is absorbing in \( \mathcal{H}_\alpha \), uniformly with respect to \( \tau \in [0, \tau_0] \). The estimates of Sections 2-4 show that assumptions (H1)-(H9) of Theorem 2.5 in [32] are satisfied. Thus, the conclusions of Theorem 5.3 hold for \( \tau \in [0, \tau_0'] \), for some \( \tau_0' \in (0, \tau_0] \) small enough (we note that Theorem 2.5 is stated for a family of semi-groups which act on the whole space \( H \), but with a minor modification of its proof, it can be applied to our situation).

For \( \tau \in [\tau_0, \tau_0'] \), we set \( T = T'_0(\| \mathcal{B} \|_H) + 2 + r \) and \( \bar{T}_\tau = \lfloor T/\tau \rfloor \), so that \( \bar{T}_\tau \mathcal{B} \subset \mathcal{B} \) (Proposition 3.5). Estimate (35) shows that \( \bar{T}_\tau : \mathcal{B} \to \mathcal{B} \) satisfies a smoothing property with a constant \( \Lambda = \Lambda(T, \tau_0', \tau_0) \) independent of \( \tau \in [\tau_0, \tau_0'] \). By Proposition 1 in [19], the dynamical system generated by iterates of the map \( \bar{T}_\tau : \mathcal{B} \to \mathcal{B} \) has an exponential attractor \( \mathcal{M}_\tau^d \), i.e. a compact and positively invariant subset of \( \mathcal{B} \) which has finite fractal dimension bounded by a constant independent of \( \tau \in [\tau_0, \tau_0'] \), and which satisfies
\[
\dist_H(\bar{T}_\tau^n \mathcal{B}, \mathcal{M}_\tau^d) \leq 2\| \mathcal{B} \|_H 2^{-n}, \quad n \in \mathbb{N}.
\]
Next, we define
\[
\mathcal{M}_\tau = \bigcup_{0 \leq n \tau \leq T} \bar{T}_\tau^n \mathcal{M}_\tau^d.
\]
A standard argument (see e.g. [32]) shows that \( \mathcal{M}_\tau \) is an exponential attractor for \( \mathcal{S}_\tau \) on \( \mathcal{H}_\alpha \), with a fractal dimension bounded by a constant independent of \( \tau \in [\tau_0, \tau_0'] \), and which attracts the bounded sets of \( \mathcal{H}_\alpha \), uniformly with \( \tau \in [\tau_0, \tau_0'] \). This concludes the proof (the continuity holds only at \( \tau = 0 \)).

As in [32, Corollary 6.2], we have:

**Corollary 5.4.** For every \( \tau \in [0, \tau_0] \), the semi-group \( \{ \mathcal{S}_\tau(t), t \geq 0 \} \) possesses a global attractor \( \mathcal{A}_\tau \) in \( \mathcal{H}_\alpha \) which is bounded in \( V \), compact and connected in \( H \). Moreover, \( \dist_H(\mathcal{A}_\tau, \mathcal{A}_0) \to 0 \) as \( \tau \to 0^+ \), and the fractal dimension of \( \mathcal{A}_\tau \) is bounded by a constant independent of \( \tau \).

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