On relative weighted entropies with central moments weight functions

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Abstract

Following [1], the aim of this paper is to analyze the relative weighted entropy involving the central moments weight functions. We compare the standard relative entropy with the weighted case in two particular forms of Gaussian distributions. As an application, the weighted deviance information criterion is proposed.

Keywords: weighted entropy, conditional weighted entropy, relative weighted entropy, central moments weight functions, Gaussian distribution, weighted deviance information criterion, Bayesian analysis.

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1 Introduction: The weighted entropies

Let \( X \) be a real-valued random vector (RV) with a join probability density function (PDF) \( f \). The differential entropy (DE) of RV \( X \) is defined by

\[
H(X) = H(f) = \int_{\mathbb{R}^n} f(x) \log f(x) \, dx.
\]

(1)

The definition and a number of inequalities for a standard DE were illustrated in [9, 3, 7]. Furthermore, in [1, 6] the initial definition and results on weighted entropy was introduced. Following [10, 5, 8, 11], recently in [4, 2, 13, 14, 15], a similar method with standard DE drives to emerge certain properties and applications of information-theoretical weighted entropies with a number of determinant-related inequalities.

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Let \( x \in \mathbb{R}^n \mapsto \phi(x) \geq 0 \) be a given (measurable) function, called weighted function (WF). The weighted differential entropy (WDE) \( H^w_\phi(X) \) of a real-valued RV \( X \) with a PDF \( f \) is given by

\[
H^w_\phi(X) = H^w_\phi(f) := -\mathbb{E} \phi(X) \log f(X) = -\int_{\mathbb{R}^n} \phi(x) f(x) \log f(x) dx. \tag{2}
\]

Note that the WDE (2) is obtained for a given non-negative WF; when this function equals 1, the WDE coincides with the standard (Shannon) DE, (1).

We also assume the integrals in (1) and (2) absolutely converge, on the other hand the WDE and DE are finite. A standard agreement \( 0 \log 0 = 0 \).

We now give the definition of conditional DE and mutual DE for RVs, in view of the fact that these are the ones on which we shall focus in our analysis more.

**Definition 1.1** Let \( X_1 \in \mathbb{R}^{m_1} \) and \( X_2 \in \mathbb{R}^{m_2} \) be two RVs, with joint PDFs \( f(x_1, x_2) \) and marginal PDFs \( f_1(x_1) \) and \( f_2(x_2) \). The conditional DE of \( X_1 \) given \( X_2 \) is defined by

\[
H(X_1|X_2) = -\int_{\mathbb{R}^{m_1+m_2}} f(x_1, x_2) \log \frac{f(x_1, x_2)}{f_2(x_2)} dx_1 dx_2. \tag{3}
\]

Next for RV \( X = (X_1, X_2, \ldots, X_n) \), we use joint and marginal PDFs \( f_{X_1, \ldots, X_n} \) and \( f_{X_1}, f_{X_2}, \ldots, f_{X_n} \) to define the mutual DE by

\[
I(f_{X_1, \ldots, X_n}, f_{X_1} \ldots f_{X_n}) = \int_{\mathbb{R}^n} f(x) \log \frac{f(x)}{f_1(x_1) \ldots f_n(x_n)} dx. \tag{4}
\]

Note that motivated by continuity, we set \( 0 \log 0 = 0 \).

Here and below we use both notations \( f(x_1, \ldots, x_n) \) and \( f_{X_1, \ldots, X_n} \) for joint PDF allowing us to be flexible in shortening throughout the paper. In addition we employ both \( f_i(x_i) \) and \( f_{X_i} \) as marginal PDF of random variable \( X_i, i = 1, \ldots, n \).

The following theorem was proven in [3].

**Theorem 1.2** (Chain rule for the DE) Let \( X_1, \ldots, X_n \) be drawn according to joint density PDF \( f(x_1, \ldots, x_n) \), then

\[
H(X_1, X_2, \ldots, X_n) = \sum_{i=1}^n H(X_i|X_{i-1}, \ldots, X_1). \tag{5}
\]

One of the mutual information’s properties, the same as bivariate case, is which can be demonstrated also in terms of marginal entropy and conditional entropy. The proof comes directly if we rewrite (4) in Definition 1.1 and omitted.
Proposition 1.3 For RV $X \in \mathbb{R}^n$ with joint and marginal PDFs $f_{X_1, \ldots, X_n}$ and $f_{X_1}, f_{X_2}, \ldots, f_{X_n}$, we have
\[
I(f_{X_1, \ldots, X_n}, f_{X_1} \cdots f_{X_n}) = \sum_{i=1}^{n-1} [H(X_i) - H(X_i|X_{i+1}, \ldots, X_n)].
\] (6)

Remark: An alternative expression for mutual information, in terms of entropy and conditional entropy is derived as follows:
\[
I(f_{X_1, \ldots, X_n}, f_{X_1} \cdots f_{X_n}) = \sum_{i=1}^{n-1} E_{X_{i+1}, \ldots, X_n} [H(X_i) - H(X_i|x_{i+1}, \ldots, x_n)].
\] (7)

Here
\[
H(X_i|x_{i+1}, \ldots, x_n) = - \int_{\mathbb{R}} f(x_i|x_{i+1}, \ldots, x_n) \log f(x_i|x_{i+1}, \ldots, x_n) dx_i.
\] (8)

As the Definition 1.2 in [14]: Let $x \in \mathbb{R}^n \mapsto \phi(x) \geq 0$ be a WF. The conditional WE of RV $X_1 \in \mathbb{R}^{m_1}$ given $X_2 \in \mathbb{R}^{m_2}$ is defined by
\[
H_w^\phi(X_1|X_2) = - \int_{\mathbb{R}^{m_1+m_2}} \phi(x_1, x_2) f(x_1, x_2) \log \frac{f(x_1, x_2)}{\phi(x_1) f_2(x_2)} dx_1 dx_2,
\] (9)

and the mutual WE is given by
\[
I_w^\phi(f_{X_1, \ldots, X_n}, f_{X_1} \cdots f_{X_n}) = \int_{\mathbb{R}^n} \phi(x) f(x) \log \frac{f(x)}{f_1(x_1) \cdots f_n(x_n)} dx.
\] (10)

This concept is easily adapted to the weighted DE by using the quality of random variables, as explained in [1].

2 Relative weighted entropies

The contribution of our paper in this setting is thus twofold:

1. We briefly improve several theorems discovered in [17] and give alternative definitions in particular form of WF.

2. We reformulate these results for Gaussian distribution with two different covariance matrixes.
2.1 Central moments weight functions

As we said in this paper, basically this subsection, we deal with central moments WFs, of the form \( \phi(x) = \prod_{i=1}^{n} (x_i - a_i)^2 \) for constants \( a_1, \ldots, a_n \).

The naturalness of the definition of the WE and conditional WE is exhibited by the fact that the WE of a vector of random variables is the conditional WE of one plus the generalized conditional WE of the others. On the other hand the chain rule can thus be adapted to the WE; accordingly, we reformulate it as follows:

**Theorem 2.1** (Chain rule for the WE) Consider the RV \( \mathbf{X} = (X_1, \ldots, X_n) \) with joint PDF \( f(x_1, \ldots, x_n) \). Then for constants \( a_1, \ldots, a_n \) and given WF \( \phi(x) = \prod_{i=1}^{n} (x_i - a_i)^2 \)

\[
H_{\phi}^w(X_1, \ldots, X_n) = H_{\phi}^w(X_n|X_{n-1}, \ldots, X_1) + \sum_{i=1}^{n-1} H_{\psi_i}^w(X_i|X_{i-1}, \ldots, X_1).
\]

Here for constants \( a_1, \ldots, a_n \)

\[
\psi_i(x_1, \ldots, x_i) = \prod_{j=1}^{i} (x_j - a_j)^2 E \left( (X_{i+1} - a_{i+1})^2|X_i, \ldots, X_1 = (x_i, \ldots, x_1) \right)
\]

**Proof:** If \( (X_1, X_2) \) is a random pair, then in this particular case we have,

\[
H_{\phi}^w(X_1, X_2) = H_{\phi}^w(X_2|X_1) + H_{\psi_1}^w(X_1),
\]

Note that here \( \psi_1(x_1) = (x_1 - a_1)^2 E((X_2 - a_2)^2|X_1 = x_1) \). Now more generally assume triple random \( (X_1, X_2, X_3) \), similarly the WE is obtained by

\[
H_{\phi}^w(X_1, X_2, X_3) = H_{\phi}^w(X_3|X_2, X_1) + H_{\psi_2}^w(X_2|X_1) + H_{\psi_1}^w(X_1),
\]

The WFs \( \psi_1 \) and \( \psi_2 \) are given by using the form of \( \psi_i \) when \( i = 1, 2 \). Applying the same methodology and expanding the RV to \( n \) random variables, \( n > 3 \) we detect the given form by

\[
H_{\phi}^w(X_1, \ldots, X_n) = H_{\phi}^w(X_2, \ldots, X_n|X_1) + H_{\psi_1}^w(X_1)
\]

\[
= H_{\phi}^w(X_3, \ldots, X_n|X_2, X_1) + H_{\psi_2}^w(X_2|X_1) + H_{\psi_1}^w(X_1)
\]

\[
\cdots
\]

\[
= H_{\phi}^w(X_n|X_{n-1}, \ldots, X_1) + H_{\psi_{n-1}}^w(X_{n-1}|X_{n-2}, \ldots, X_1) + H_{\psi_n}^w(X_{n-1}|X_{n-2}, \ldots, X_1) + H_{\psi_2}^w(X_2|X_1) + H_{\psi_1}^w(X_1).
\]

This leads to the desired result. \( \square \)

In this stage, an immediate question crossed our mind which states shall we extend the similar conclusions due to the weighted entropies? In fact, among all equivalent expression for the mutual WE, as already observed in mutual information, the most applicable is represented by the WE and the conditional WE.
Theorem 2.2 Let us now consider the weighted mutual information, $I^w_{\phi}(f_{X_1, \ldots, X_n}, f_{X_1 \cdots f_{X_n}})$, then it can be written as follows:

$$I^w_{\phi}(f_{X_1, \ldots, X_n}, f_{X_1 \cdots f_{X_n}}) = \sum_{j=1}^{n-1} H^w_{\psi_j}(X_j) - H^w_{\phi}(X_1, \ldots, X_{n-1}|X_n),$$

(12)

where $\psi_j'(x_j) = (x_j - a_j)^2 E \left[ \prod_{i=1, i \neq j}^{n} (X_i - a_i)^2 | X_j = x_j \right]$.

**Proof:** By recalling (10), we observe that

$$I^w_{\psi}(f_{X_1, \ldots, X_n}, f_{X_1 \cdots f_{X_n}}) = \int_{\mathbb{R}^n} \prod_{i=1}^{n} (x_i - a_i)^2 f(x_1, \ldots, x_n) \log f(x_1, \ldots, x_{n-1}|x_n) dx$$

$$- \int_{\mathbb{R}^n} \prod_{i=1}^{n} (x_i - a_i)^2 f(x_1, \ldots, x_n) \sum_{j=1}^{n-1} \log f_j(x_j) dx$$

Consequently,

$$I^w_{\psi}(f_{X_1, \ldots, X_n}, f_{X_1 \cdots f_{X_n}}) = -H^w_{\phi}(X_1, \ldots, X_{n-1}|X_n)$$

$$- \int_{\mathbb{R}} \sum_{j=1}^{n-1} (x_j - a_j)^2 \int_{\mathbb{R}^{n-1}} \prod_{i=1, i \neq j}^{n} (x_i - a_i)^2 f(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n|x_j)f_j(x_j) \log f_j(x_j) dx$$

$$= -H^w_{\phi}(X_1, \ldots, X_{n-1}|X_n) - \sum_{j=1}^{n-1} \int_{\mathbb{R}} (x_j - a_j)^2 E \left[ \prod_{i=1, i \neq j}^{n} (X_i - a_i)^2 | X_j = x_j \right] f_j(x_j) \log f_j(x_j) dx_j.$$

Which is precisely the result that we are looking for. \qed

Considering real situation which there exist two dependent groups of components or on the other hand random vectors, in some experimental research we are looking for the discrimination between probability function while such vectors are independent and dependent, in fact using this methodology clarifies the effect of dependent random vectors. Indeed, it seems as much as the dependency between two groups of random data is stronger than the information among density functions should raise. This fact will be adopted specifically in Gaussian distribution in the next subsection throughout examples.

**Proposition 2.3** Suppose that $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, \ldots, Y_m)$ be RVs showing any real situation, with joint PDF $f(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and marginal multivariate PDFs $f_1(x_1, \ldots, x_n)$ and $f_2(y_1, \ldots, y_m)$ respectively. Then

$$D(f_{X|Y}|f_{X}) = H^w_{\phi_{X|Y}}(X) - H(X|Y),$$

(13)

here $\phi_{X|Y}(x) = f(x|y)/f_1(x)$ and $H(X|Y)$ is defined as [3].
In addition, let us here define the relative DE, known as the Kullback-Leibler divergence, for two given functions \( f \) and \( g \), \( D(f \| g) \) by

\[
D(f \| g) = \int_{\mathbb{R}^n} f(x) \log \frac{f(x)}{g(x)} \, dx.
\] (14)

Proof: The Proof is based on the equation (14) and straightforward. \( \square \)

Remark: We explicitly note that by taking expectation on \( D(f_{X|Y}, f_X) \) with respect to RV \( Y \), by virtue of (13), mutual DE can be yielded:

\[
I(f_{X,Y}, f_Xf_Y) = \mathbb{E}_Y\left[ D(f_{X|Y}, f_X) \right] = \mathbb{E}_Y\left[ H^w_{\phi_{X|Y}}(X) - H(X|y) \right].
\] (15)

Here, going back to weighted information measure we implicitly present the weighted information between \( f_{X|y} \) and \( f_X \). This probably makes reader even more interested, however we were also wondering whether the amount value of RVs associates the effects of dependency or not but let us to concentrate on this object in the next part of the paper.

Definition 2.4 For two functions \( x \in \mathbb{R}^n \mapsto f(x) \geq 0 \) and \( x \in \mathbb{R}^n \mapsto g(x) \geq 0 \), the relative \( WE \) (the weighted Kullback-Leibler divergence), for given \( WF \) \( \phi \) is defined by

\[
D^w_\phi(f \| g) = \int_{\mathbb{R}^n} \phi(x) f(x) \log \frac{f(x)}{g(x)} \, dx.
\] (16)

Theorem 2.5 With the same assumptions and analogue method as Proposition 2.3, for given \( WF \) \( \phi(x) = \prod_{i=1}^{n} (x_i - a_i)^2 \), constants \( a_1, \ldots, a_n \), one can obtain the following relationship:

\[
D^w_\phi(f_{X|Y}, f_X) = \int_{\mathbb{R}^n} \prod_{i=1}^{n} (x_i - a_i)^2 f(x|y) \log \frac{f(x|y)}{f(x)} \, dx
\]

\[
= \int_{\mathbb{R}^n} \prod_{i=1}^{n} (x_i - a_i)^2 f(x|y) \log f(x|y) \, dx
\]

\[
- \int_{\mathbb{R}^n} \prod_{i=1}^{n} (x_i - a_i)^2 \frac{f(x|y)}{f(x)} f(x) \log f(x) \, dx
\]

\[
= H^w_{\phi_{X|Y}}(X) - H^w_{\phi}(X|y).
\] (17)

where

\[
\phi'_{X|Y}(x) = \prod_{i=1}^{n} (x_i - a_i)^2 \left[ \frac{f(x|y)}{f_1(x)} \right]
\] (18)
Similar equations to (15) in terms of weighted case also can be seen,
\[ E_Y \left[ \prod_{j=1}^{m} (Y_j - a_j)^2 D_{\phi}^w(f_{X|y}, f_X) \right] \]
\[ = \int_{\mathbb{R}^m} \prod_{j=1}^{m} (y_j - a_j)^2 f_2(y) \int_{\mathbb{R}^n} \prod_{i=1}^{n} (x_i - a_i)^2 f(x|y) \log \frac{f(x|y)}{f_1(x)} \, dx \, dy \]
\[ = \int_{\mathbb{R}^{n+m}} \prod_{j=1}^{m} (y_j - a_j)^2 \prod_{i=1}^{n} (x_i - a_i)^2 f(x, y) \log \left[ \frac{f(x, y)}{f_1(x) f_2(y)} \right] \, dx \, dy, \]
hence at last piece of discussion in this subsection, we point out that the mutual WE can be implied by calculating \( m \)th order of moments for random vector \( Y \) while \( D_{\phi}^w(f_{X|y}, f_X) \cdot f(y) \) plays the rule of density function.

The WF \( \psi_X^w(f_{X|y}, f_X) \) applies the form as in (15).

### 2.2 Gaussian distribution

The Gaussian distribution is the most useful, and most studied, of the standard joint distributions in probability. A huge body of statistical theory depends on the properties of families of random variables whose joint distribution is at least approximately multivariate normal. As we know many fancy statistical procedures implicitly require bivariate (or multivariate, for more than two random variables) normality. Moreover, the hypothesis of dependency between random variables has been always the center of researcher’s attentions, hence in this subsection we focus on the dependent RVs with Gaussian distribution.

Throughout this part of our research we give two types of Gaussian examples with different covariance matrices. By using the same technique as before, general formulas for \( n = 3 \) are given. Furthermore we will observe the rule of coefficient correlation \( \rho \) in the relative measure for the weighted and standard forms.

Consider \( X \sim \mathcal{N} (\mu, \Sigma) \), one of the achievements in [3] explicitly shows that the entropy for this famous family does not depend on \( \mu \):
\[ H(X) = \frac{1}{2} \log \left[ (2\pi)^n |\Sigma| \right]. \quad (20) \]
where \( |\Sigma| \) is the determinant matrix \( \Sigma \).
However, in the Gaussian case, by virtue of (2) involving \( \phi(x) = \prod_{i=1}^{3} (x_i - \mu_i)^2 \), the WE admits a representation depending on mean \( \mu \), (see the Appendix):

\[
H_{\phi}(X) = \left[ \frac{1}{2} \log \left( (2\pi)^2 |\Sigma| \right) \right] \Xi + \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \Sigma_{ij}^{-1} \Lambda_{ij}.
\] (21)

Here

\[
\Lambda_{ij} = \left( \Sigma_{11} \Sigma_{22} + 2 (\Sigma_{12})^2 \right) \left( \Sigma_{33} \Sigma_{ij} + 2 \Sigma_{3i} \Sigma_{3j} \right),
\]

\[
\Xi = \Sigma_{11} \left( \Sigma_{22} \Sigma_{33} + 2 (\Sigma_{23})^2 \right) + 2 \Sigma_{12} \left( \Sigma_{12} \Sigma_{33} + 2 \Sigma_{13} \Sigma_{23} \right) + 2 \Sigma_{13} \left( 2 \Sigma_{12} \Sigma_{23} + \Sigma_{13} \Sigma_{22} \right).
\] (22)

Recall random pair \( X_1 = (X_1, X_2) \), having Gaussian distribution with mean \( \mu_1 = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \) and covariance matrix \( \Sigma_1 = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \). Hence the RV \( (X_1, X_2|X_3 = x_3) \sim \mathcal{N}(\overline{\mu}, \overline{\Sigma}) \), where

\[
\overline{\mu} = \begin{bmatrix} \mu_1 + \Sigma_{13} \Sigma_{33}^{-1} (x_3 - \mu_3) \\ \mu_2 + \Sigma_{23} \Sigma_{33}^{-1} (x_3 - \mu_3) \end{bmatrix},
\] (23)

\[
\overline{\Sigma} = \begin{bmatrix} \Sigma_{11} - \Sigma_{13} \Sigma_{33}^{-1} \Sigma_{31} & \Sigma_{12} - \Sigma_{13} \Sigma_{33}^{-1} \Sigma_{32} \\ \Sigma_{21} - \Sigma_{23} \Sigma_{33}^{-1} \Sigma_{31} & \Sigma_{22} - \Sigma_{23} \Sigma_{33}^{-1} \Sigma_{32} \end{bmatrix}.
\] (24)

Further, we represent the inclosed formula for \( D(f(x_1, x_2)|x_3, f(x_1, x_2)) \). Then we shall exploit later to compare it with the weighted one in order to catch a part of our main purpose in this work.

Going back to the Proposition 2.3 admits the representation

\[
D(f(x_1, x_2)|x_3, f(x_1, x_2)) = H_{\phi(x_1, x_2)|x_3}^{W}(X_1, X_2) - H((X_1, X_2)|x_3)
\]

\[
= \frac{1}{2} \log \left( (2\pi)^2 |\Sigma_1| \right) - \frac{1}{2} \log \left( (2\pi)^2 |\overline{\Sigma}| \right)
\]

\[
+ \frac{1}{2} \sum_{i,j=1,2} \Sigma_{ij}^{-1} \left\{ \Sigma_{ij} + \overline{\mu}_i \overline{\mu}_j - \mu_j \overline{\mu}_i - \mu_i \overline{\mu}_j + \mu_i \mu_j \right\}.
\] (25)

Note that here \( \phi(x_1, x_2)|x_3(x_1, x_2) = f(x_1, x_2|x_3) / f(x_1, x_2) \). For simplification and avoiding confusion we introduce \( \Sigma_{ij}^{-1} \) as the cells in the concentration matrix \( \Sigma_1^{-1} \).
The conditional DE $H((X_1, X_2)|x_3)$ follows directly from (20) and

$$(X_1, X_2|X_3 = x_3) \sim \mathcal{N}(\pi, \Sigma).$$

On the other hand, for $H^w_{\phi(X_1, X_2)|x_3}(X_1, X_2)$ we have the respective formula:

$$H^w_{\phi(X_1, X_2)|x_3}(X_1, X_2) = \log \left( (2\pi)|\Sigma|^\frac{1}{2} \right) + \frac{1}{2} \sum_{i,j=1,2} \Sigma_{ij}^{-1} \int_{\mathbb{R}^2} f(x_1, x_2|X_3)(x_i - \mu_i)(x_j - \mu_j)dx_1dx_2$$

$$= \log \left( (2\pi)|\Sigma|^\frac{1}{2} \right) + \frac{1}{2} \sum_{i,j=1,2} \Sigma_{ij}^{-1} \left\{ E[X_iX_j|X_3] - \mu_j E[X_i|X_3] - \mu_i E[X_j|X_3] + \mu_i\mu_j \right\}$$

$$= \log \left( (2\pi)|\Sigma|^\frac{1}{2} \right) + \frac{1}{2} \sum_{i,j=1,2} \Sigma_{ij}^{-1} \left\{ \Sigma_{ij} + \mu_i\mu_j - \mu_j\mu_i - \mu_i\mu_j + \mu_i\mu_j \right\}.$$

In addition, according to (17), we are entitled to give a comprehensive expression for $D^w_{\phi}(f(X_1, X_2)|x_3, f(X_1, X_2))$. Define

$$\Theta(x_3) = \mathbb{E} \left[ \prod_{i=1}^{2}(X_i - \mu_i)^2 | X_3 \right]$$

$$\Xi_{ij} = \mathbb{E} \left[ \prod_{k=1}^{2}(X_k - \mu_k)^2 (X_i - \mu_i)(X_j - \mu_j) | X_3 \right].$$

Then we get

$$H^w_{\phi}(X_1, X_2|X_3 = x_3) = \left[ \frac{1}{2} \log \left( (2\pi)^2|\Sigma| \right) \right] \Theta(x_3) + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \Sigma_{ij}^{-1} \Xi_{ij}.$$  (27)

We confine in the Appendix the calculations related to the precise value of $\Theta(x_3)$ and $\Xi_{ij}$.

Now for WF

$$\varphi'(X_1, X_2|x_3, x_2) = \prod_{i=1}^{2}(x_i - \mu_i)^2 \left[ \frac{f(x_1, x_2|X_3)}{f(x_1, x_2)} \right],$$

we draw the reader’s attention to the following assertion:

$$H^w_{\varphi(X_1, X_2)|x_3}(X_1, X_2) = \frac{1}{2} \log \left[ (2\pi)^2|\Sigma_1| \right] \Theta(x_3) + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \Sigma_{ij}^{-1} \Upsilon_{ij}.$$  (28)

Here $\Theta(x_3)$ is defined as in (26) and

$$\Upsilon_{ij} = \mathbb{E} \left[ \prod_{k=1}^{2}(X_k - \mu_k)^2 (X_i - \mu_i)(X_j - \mu_j)|X_3 \right].$$  (29)
As before the explicit expressions of $\Theta(x_3)$ and $\Upsilon_{ij}$ are given in Appendix.

Following Corollary 2.5, combine (27) and (28), and obtain the following quite long expression for the mutual WE:

$$D_w^\phi(f(X_1,X_2)|x_3, f(X_1,X_2)) = \frac{1}{2} \log \left[ \frac{|\Sigma_1|}{|\Sigma|} \right] \Theta(x_3) + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \Sigma_{ij}^{-1} \Upsilon_{ij} - \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \Sigma_{ij}^{-1} \Lambda_{ij}. \tag{30}$$

Consequently, it is now clear, both Kullbak-Leibler and weighted Kulback-Leibler information measures for Gaussian conditional pair $(X_1, X_2)$ precisely depend on mean and covariances between all random variables individually. Because of this fact, it is logical to wonder about the effect of correlations on them and whether this effect on kullback-Leibler information is completely analogous to the weighted one.

One obviously can imagine if the dependency between $(X_1, X_2)$ and $X_3$ is increasing then the density function of conditional vector $(X_1, X_2)|X_3 = x_3)$ becomes more far than joint density function $(X_1, X_2)$, on the other way we understand that knowing dependent random variable $X_3$ gives more information.

Indeed to prove this claim we shall present more evidences, so that two particular examples are considering in the following.

**Example 2.6** In Gaussian case assume $\mu = \Omega$ and $\Sigma = \begin{bmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & 0 \\ \rho^2 & 0 & 1 \end{bmatrix}$, then

$$(X_1, X_2|X_3 = x_3) \sim \mathcal{N}(\mu, \Sigma),$$

where

$$\mu = \begin{bmatrix} \rho^2 x_3 \\ 0 \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} 1 - \rho^4 & \rho \\ \rho & 1 \end{bmatrix}.$$

Following $\Sigma_1$ as before, i.e. the covariance matrix for pair $(X_1, X_2)$, one yields $|\Sigma_1| = 1 - \rho^2$, $|\Sigma| = 1 - \rho^2 - \rho^4$ and

$$\sum_{i,j=1,2} \Sigma_{ij}^{-1} \left\{ \Sigma_{ij} + \mu_i \mu_j - \mu_j \mu_i - \mu_i \mu_j + \mu_i \mu_j \right\} = \sum_{i,j=1,2} \Sigma_{ij}^{-1} \left\{ \Sigma_{ij} + \mu_i \mu_j \right\}
= \Sigma_{11}^{-1} \left( \Sigma_{11} + \mu_1^2 \right) + \Sigma_{12}^{-1} \Sigma_{12} + \Sigma_{21}^{-1} \Sigma_{21} + \Sigma_{22}^{-1} \Sigma_{22} = \frac{2 - 2\rho^2 + \rho^4(x_3^2 - 1)}{1 - \rho^2}. $$

Using the above expression in (25), we obtain

$$D(f(X_1,X_2)|x_3, f(X_1,X_2)) = \frac{1}{2} \log \left[ \frac{1 - \rho^2}{(1 - \rho^2 - \rho^4)} \right] + \frac{\rho^4}{2(1 - \rho^2)} (x_3^2 - 1) + 1. \tag{31}$$
Taking look into this equation, we are not explicitly able to realize if it is a monotonic function with respect to \( \rho \), but obviously it is an even function with respect to \( \rho \) and \( x_3 \), that is the relative DE does not depend on the sign of correlation coefficient \( \rho \). Although from (31) it is clear that the absolute value of \( x_3 \) effects on the information.

In Figure 2.2.1 we see that first the relative DE takes non-negative values (Gibbs inequality, see [3, 7]). Second the information raises when absolute value of \( \rho \) and \( x_3 \) is increasing which is completely coincide with our expected claim, observe Figures 2.2.1(B) and 2.2.1(C). On the other hand in this example the dependency between \( X_1 \) and \( X_3 \), \( \rho_2 \), and the discrimination between \( f(X_1, X_2 \mid x_3) \), \( f(X_1, X_2) \) change in the same direction. Although since we have concentrated on the information between distributions \( (X_1, X_2 \mid X_3 = x_3) \) and \( (X_1, X_2) \), the correlation between \( X_1 \) and \( X_2 \) is not in our attention.

Now, we shall switch to the relative WE in order to discover if we can extend the similar impression for the weighted one. It is worthwhile nothing that since in weighted information apart of probabilities we also add the amount values of RVs \( X_1, X_2 \), thus probably our perception changes.
Consider $\mu$ and $\Sigma$ as already given in Example 2.4, we get
\[
\Theta(x_3) = 1 + 2\rho^2 + \rho^4(x_3^2 - 1),
\]
where
\[
\Xi_{ij} = \alpha_{ij}(\rho) + \rho^4 x_3^2 \left( (1 - \rho^4)\Sigma_{ij} + 2\Sigma_{1i}\Sigma_{1j} \right).
\]
Then
\[
\int \alpha_{ij}(\alpha) = (1 - \rho^4) \left( \Sigma_{ij} + 2\Sigma_{2i}\Sigma_{2j} \right) + 2\rho \left( \rho\Sigma_{ij} + \Sigma_{1i}\Sigma_{2j} + \Sigma_{1j}\Sigma_{2i} \right) + \Sigma_{1i} \left( 2\rho\Sigma_{2j} + \Sigma_{1j} \right) + \Sigma_{1j} \left( 2\rho\Sigma_{2i} + \Sigma_{1i} \right).
\]
Set $\beta_{ij}(\rho) = (\overline{\mu}_i - \mu_i)(\overline{\mu}_j - \mu_j)$, we have
\[
\Upsilon_{ij} = \alpha_{ij}(\rho) + \beta_{ij}(\rho) \left[ 1 + 2\rho^2 + \rho^4(x_3^2 - 1) \right] + 2\rho^2 x_3(\overline{\mu}_j - \mu_j) \left( \Sigma_{1i} + 2\rho\Sigma_{2i} \right) + 2\rho^2 x_3(\overline{\mu}_i - \mu_i) \left( \Sigma_{1j} + 2\rho\Sigma_{2j} \right) + (\rho^4 x_3^2) \left( \Sigma_{ij} + 2\Sigma_{2i}\Sigma_{2j} \right).
\]
By virtue of (30), we obtain
\[
D^w_\phi(f_1(x_1,x_2)|x_3,f_2(x_1,x_2)) = \frac{1}{2} \log \left[ \frac{1 - \rho^2}{1 - \rho^2 - \rho^4} \right] \left( 1 + 2\rho^2 + \rho^4(x_3^2 - 1) \right) + \frac{1}{2} \left( 3(1 - \rho^4)^2 + 3(1 - \rho^4) + 6\rho^2 - 6\rho^4 - 6\rho^6 x_3^2 + 9\rho^4 x_3^2 - 6\rho^8 x_3^2 + \rho^8 x_3^2 \right) - \frac{1}{2} \frac{1}{1 - \rho^2 - \rho^4} \left( 6\rho^2 (1 - \rho^4) + 6(1 - \rho^4)^2 + 4\rho^4 (1 - \rho^4)x_3^2 - 12\rho^4 - 4\rho^6 (1 - \rho^4)x_3^2 \right).
\]
The following theorem was presented in [13]:

**Theorem 2.7** (The weighted Gibbs inequality) Given non-negative functions $f$, $g$ and $\phi$, assume the bound
\[
\int \phi(x)[f(x) - g(x)] dx \geq 0.
\]
Then
\[
D^w_\phi(f||g) \geq 0,
\]
with equality iff $g \equiv f$.

The condition (35) is re-written as
\[
\int \prod_{i=1}^2 (x_i - \mu_i)^2 \left[ f(x_1,x_2|x_3) - f(x_1,x_2) \right] dx_1 dx_2 = E \left[ \prod_{i=1}^2 (X_i - \mu_i)^2 | X_3 \right] - E \left[ \prod_{i=1}^2 (X_i - \mu_i)^2 \right] \geq 0.
\]
Correspondingly in this example we obtain:

$$\Theta(x_3) - E\left[\prod_{i=1}^{2} X_i^2\right] = \rho^4(x_3^2 - 1).$$

This states that for $x_3 \in (-1, 1)$ the condition (35) is violated and there is no guaranty that the relative WE takes non-negative values whereas $D^\rho_w(f(x_1, x_2)|x_3, \tilde{f}(x_1, x_2)) \geq 0$, see Figure 2.2.2(A).

Further, a similar pattern as the relative DE for the relative WE’s behavior is confirmed on Figure 2.2.2(B) and 2.2.2(C) with the same values of $\rho$ and $x_3$.

![Figure 2.2.2](image)

Our observations still are preliminary, and we think that further examples are needed here, to build a detailed picture. Hence let us now devote our efforts on another special case of Gaussian distribution which has been called from Example 3.4.1 page 39, [16]:

**Example 2.8** Let $X = (X_1, X_2, X_3)$ be distributed according to an $N(0, \Sigma)$ distribution, where $\Sigma_{ii} = 1, (i = 1, 2, 3)$ and $\Sigma_{12} = 1 - 2\rho, \Sigma_{13} = \Sigma_{23} = 1 - \rho, \ 0 < \rho < \frac{1}{2}$. For every fix
\( C = (C_1, C_2, C_3)^T \in \mathbb{R}^3. \) we can write,

\[
C^T \Sigma C = (1 - \rho)(C_1 + C_2 + C_3)^2 + \rho(C_1 + C_2)^2 + \rho C_3^2,
\]

Since \( C^T \Sigma C \geq 0 \) holds, and the equality holds if and only if \( C_1 = C_2 = C_3 = 0 \), so \( \Sigma \) is a positive define matrix. We have then,

\[
\overline{\mu} = \begin{bmatrix} (1 - \rho) x_3 \\ (1 - \rho) x_3 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \rho(2 - \rho) & -\rho^2 \\ -\rho^2 & \rho(2 - \rho) \end{bmatrix}.
\]

Owing to (25) we first calculate,

\[
\sum_{i,j=1,2}^{2} \Sigma_{ij}^{-1} \{ \overline{\Sigma}_{ij} + \overline{\mu}_i \overline{\mu}_j - \mu_i \overline{\mu}_j + \mu_i \mu_j \} = 1 + \rho + (1 - \rho)x_3^2.
\]
Therefore, for $\rho \in (0, \frac{1}{2})$, we drive

\[
D(f(x_1, x_2)|x_3, \bar{f}(x_1, x_2)) = \frac{1}{2} \log \left[ \frac{(2\pi)^2 4\rho (1 - \rho)}{\rho^2 (2 - \rho)^2 - \rho^4} \right] \cdot \Theta(x_3)
\]

\[
= \frac{1}{2} \log \left[ \frac{\rho^2 (2 - \rho)^2 - \rho^4}{(1 - \rho)^2 x_3^2} \right] + \frac{1}{2} \left[ \frac{1 + \rho + (1 - \rho)x_3^2}{\rho^2 (2 - \rho)^2 - \rho^4} \right] - 1.
\]

(36)

Next, Figure 2.2.3 shows a more interesting character of behavior. Function $D(f(x_1, x_2)|x_3, \bar{f}(x_1, x_2))$ takes non-negative values. However, the relative DE decreases in $\rho \in (0, \frac{1}{2})$. In other words, this is another example showing our perception holds true: when the dependency within RVs $X_1$ and $X_3$ raises the information increases as well.

Furthermore, the following expression for $\Theta(x_3)$ emerges:

\[
\Theta(x_3) = \rho^2 (2 - \rho)^2 + 4\rho^4 + 2\rho (2 - \rho) (1 - \rho)^2 x_3^2 - 4\rho^2 (1 - \rho)^2 x_3^2 + (1 - \rho)^4 x_3^4.
\]

(37)

Therefore owing to (30), one yields

\[
D_w^p(f(x_1, x_2)|x_3, \bar{f}(x_1, x_2)) = \frac{1}{2} \log \left[ \frac{4\rho (1 - \rho)}{\rho^2 (2 - \rho)^2 - \rho^4} \right] \cdot \Theta(x_3)
\]

\[
+ \frac{1}{8\rho (1 - \rho)} \left( (\Upsilon_{11} + \Upsilon_{22}) + (2\rho - 1)(\Upsilon_{12}) \right)
\]

\[- \frac{1}{2(\rho^2 (2 - \rho)^2 - \rho^4)} \left( \rho (2 - \rho)(\Lambda_{11} + \Lambda_{22}) + 2\rho^2(\Lambda_{12}) \right).
\]

(38)

Here $\Theta(x_3)$ is as (37) and we have written the open form of $\Lambda_{ij}$ and $\Upsilon_{ij}$ in Appendix.

Let us now check the statues of the condition (35):

\[
\Theta(x_3) - \left( \frac{1}{2} + (1 - 2\rho)^2 \right)
\]

\[
= \rho^2 (2 - \rho)^2 + 4\rho^4 + 2\rho (2 - \rho) (1 - \rho)^2 x_3^2 - 4\rho^2 (1 - \rho)^2 x_3^2 + (1 - \rho)^4 x_3^4 - 1 - (1 - 2\rho)^2.
\]

(39)

Analyzing (39), one can explore that the condition (35) doesn’t hold true for all values of $\rho$ and $x_3$ whereas as Figure 2.2.4 shows, the relative WE is non-negative (within the indicated range of $(\rho, x_3)$.

Finally, the plots given in Figure 2.2.4(B) and 2.2.4(C) give an impression that the behavior of $D_w^p(f(x_1, x_2)|x_3, \bar{f}(x_1, x_2))$ is more complicated. Other words, in this example the information doesn’t behave monotonically with respect to $\rho$ and $x_3$. Consequently, in spite of standard case, in weighted form one does not yield that the dependency between $X_1$ and $X_3$ effects directly on the information.
3 An application: Weighted deviance information criterion

Concluding this paper in this section, we briefly demonstrate an application of the relative DE and WE by exploiting Bayesian analysis in model selecting, cf. [18, 17].

Assume that $f(\tilde{y})$ and $g(\tilde{y})$ respectively represent the PDFs of the "true model" and the "approximating model" on the same measurable space. For given WF $\phi$, the relative WE or weighted Kullback-Leibler divergence is given by:

$$D_w^\phi(f \parallel g) = E_{\tilde{y}} \left[ \phi(\tilde{y}) \log f(\tilde{y}) \right] - E_{\tilde{y}} \left[ \phi(\tilde{y}) \log g(\tilde{y}) \right].$$

(40)

Note that such a quantity is not always non-negative. Namely the smaller the value of $D_w^\phi$, the closer we consider the model $g$ to be the true distribution. Hence in practice the first part of (40) is negligible in model comparison for given data $y = (y_1, \ldots, y_n)$ with weights $\phi(y)$. As $n$ increases to infinity, the following expression, weighted log-likelihood (say):

$$\frac{1}{n} L_w^\phi(\theta | y) := \frac{1}{n} \sum_{i=1}^{n} \log \left[ g(y_i | \theta) \phi(y_i) \right]$$
tends to $E[\phi(y) \log g(\tilde{y}|\theta)]$ by the law of large numbers. Here $\phi(y_i)$ is the weight for $y_i$ and $\tilde{y}$ is supposed to be an unknown but potentially observable quantity coming from the same distribution $f$ and independent of $y$.

Next in agreement with [12], we propose the weighted deviance information criterion (WDIC):

$$WDIC = DE^w_\phi(\hat{\theta}, y) + 2p^w_D, \quad (41)$$

as an adaptation of the Akaike information criterion for weighted case for Bayesian models. Consider the penalty of over-estimating $p^w_D$ by

$$p^w_D = E_{\theta|y} [DE^w_\phi(\theta, y) - DE^w_\phi(\hat{\theta}, y)] \quad (42)$$

in order to estimate the "effective number of parameters". Here

$$DE^w_\phi(\theta, y) = -2 \sum_{i=1}^{n} \log \left( g(y_i|\theta)^{\phi(y_i)} \right).$$

As far the full model specification of Bayesian statistics contains a prior function $\Pi(\theta)$ in addition to the likelihood, and the inference can be derived from the posterior distribution $\Pi(\theta|y) \propto L(\theta|y)\Pi(\theta)$, therefore $\hat{\theta}$ could be either posterior mean or mode. In practice the advantage of WDIC with respect to DIC is observed when the data has the utility (weight) non equal to one.

**Remark:** It would be interesting to investigate some simulation results as evidence of this fact by using Markov chain Monte Carlo (MCMC) method. This also is one of our intentions for future works.

### 4 APPENDIX

**Proof of (21):**

According to [2] for the Gaussian PDF and given WF $\phi(x) = \prod_{i=1}^{3} (x_i - \mu_i)^2$, one can write:

$$H^w_\phi(X) = \left[ \frac{1}{2} \log \left( (2\pi^3|\Sigma) \right) \right] E \left[ \prod_{k=1}^{3} (X_k - \mu_k)^2 \right]$$

$$+ \frac{1}{2} \int_{R^3} \prod_{k=1}^{3} (x_k - \mu_k)^2 f(x) \sum_{i=1}^{3} \sum_{j=1}^{3} (x_i - \mu_i) \Sigma_{ij}^{-1} (x_j - \mu_j) dx$$

$$= \left[ \frac{1}{2} \log \left( (2\pi^3|\Sigma) \right) \right] E \left[ \prod_{k=1}^{3} (X_k - \mu_k)^2 \right]$$

$$+ \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \Sigma_{ij}^{-1} E \left[ \prod_{k=1}^{3} (X_k - \mu_k)^2 (X_i - \mu_i)(X_j - \mu_j) \right]. \quad (43)$$
Set \( y_i = x_i - \mu_i \), then \( y_i \sim \mathcal{N}(0, \Sigma_{ij}) \) and moreover for odd \( M = r_1 + r_2 + \cdots + r_n \), \( E[\prod_{i=1}^n y_i^r] = 0 \). Now let us focus on the last expectation in (43) which takes the form:

\[
\Lambda_{ij} := E \left[ \prod_{k=1}^3 (X_k - \mu_k)^2(X_i - \mu_i)(X_j - \mu_j) \right] = E \left[ \prod_{k=1}^3 Y_k^2Y_iY_j \right]
\]

\[
= E[Y_1^2Y_2^2]E[Y_3^2]E[Y_iY_j] + E[Y_1^2Y_2^2]E[Y_3Y_i]E[Y_3Y_j] + E[Y_1^2Y_2^2]E[Y_3Y_i]E[Y_3Y_j] + E[Y_1^2Y_2^2]E[Y_1Y_3]E[Y_3Y_i]E[Y_3Y_j] + E[Y_1^2Y_2^2]E[Y_1Y_3]E[Y_3Y_i]E[Y_3Y_j] + E[Y_1^2Y_2^2]E[Y_1Y_3]E[Y_3Y_i]E[Y_3Y_j] + E[Y_1^2Y_2^2]E[Y_1Y_3]E[Y_3Y_i]E[Y_3Y_j] + E[Y_1^2Y_2^2]E[Y_1Y_3]E[Y_3Y_i]E[Y_3Y_j]
\]

\[
= \left( \Sigma_{11}\Sigma_{22} + 2(\Sigma_{12})^2 \right) \left( \Sigma_{33}\Sigma_{ij} + 2 \Sigma_{3i}\Sigma_{3j} \right).
\]

Next, in (43) we need to find one more expectation:

\[
\Xi := E \left[ \prod_{k=1}^3 (X_k - \mu_k)^2 \right] = E \left[ \prod_{k=1}^3 Y_k^2 \right] = E[Y_1^2] \left( E[Y_2^2]E[Y_3^2] + 2 \left( E[Y_2Y_3] \right)^2 \right)
\]

\[
+ 2 E[Y_1Y_2] \left( E[Y_1Y_3]E[Y_3^2] + 2 E[Y_1Y_3]E[Y_2Y_3] \right)
\]

\[
+ 2 E[Y_1Y_3] \left( 2 E[Y_1Y_2]E[Y_2Y_3] + E[Y_1Y_3]E[Y_2^2] \right)
\]

\[
= \Sigma_{11} \left( \Sigma_{22}\Sigma_{33} + 2(\Sigma_{23})^2 \right) + 2 \Sigma_{12} \left( \Sigma_{12}\Sigma_{33} + 2\Sigma_{13}\Sigma_{23} \right)
\]

\[
+ 2 \Sigma_{13} \left( 2\Sigma_{12}\Sigma_{23} + \Sigma_{13}\Sigma_{22} \right).
\]

By replacing the above expressions in (43) we can obtain

\[
H_{\phi}^X(X) = \left[ \frac{1}{2} \log \left( (2\pi)^3|\Sigma| \right) \right] \Xi + \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 \Sigma_{ij}^{-1} \Lambda_{ij},
\]

which is exactly what we are looking for. □

Proof of (27):

Recall the conditional WE:

\[
H_{\phi}^X(X_1, X_2|X_3 = x_3)
\]

\[
= - \int \phi(x_1, x_2)f(x_1, x_2|x_3) \log f(x_1, x_2|x_3) dx_1 \ dx_2
\]

\[
= \frac{1}{2} \log \left( (2\pi)^2|\Sigma_1| \right) E \left[ \prod_{i=1}^2 (X_i - \mu_i)^2|X_3 \right]
\]

\[
+ \frac{1}{2} \sum_{i,j=1,2} \Sigma_{ij}^{-1} \int \prod_{k=1}^2 (x_k - \mu_k)^2(x_i - \tilde{\mu}_i)(x_j - \tilde{\mu}_j) f(x_1, x_2|x_3) dx_1 \ dx_2.
\]

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Observe that
\[
E \left[ \prod_{i=1}^{2} (X_i - \mu_i)^2 | X_3 \right] = E \left[ \prod_{i=1}^{2} \left( (X_i - \overline{\mu}_i)^2 + 2(X_i - \overline{\mu}_i)(\overline{\mu}_i - \mu_i) + (\overline{\mu}_i - \mu_i)^2 \right) | X_3 \right]
\]
\[
= E [\prod_{i=1}^{2} (X_i - \overline{\mu}_i)^2] + E [\prod_{i=1}^{2} (\overline{\mu}_i - \mu_i)^2] + 4 E [\prod_{i=1}^{2} (\overline{\mu}_i - \mu_i)^2].
\]

where \((\bar{Y}_1, \bar{Y}_2) \sim N(\mathbf{0}, \Sigma), \overline{\mu} \) and \(\Sigma\) are as in (23) and (24). Therefore
\[
E \left[ \prod_{i=1}^{2} (X_i - \mu_i)^2 | X_3 \right] = \left( \Sigma_{11} \Sigma_{22} + 2(\Sigma_{12})^2 \right) + \Sigma_{11} \left( \Sigma_{23} \Sigma_{33}^{-1} (x_3 - \mu_3) \right)^2
\]
\[
+ \Sigma_{22} \left( \Sigma_{13} \Sigma_{33}^{-1} (x_3 - \mu_3) \right)^2 + 4 \Sigma_{12} \prod_{i=1}^{2} \left( \Sigma_{i3} \Sigma_{33}^{-1} (x_3 - \mu_3) \right)
\]
\[
+ \prod_{i=1}^{2} \left( \Sigma_{i3} \Sigma_{33}^{-1} (x_3 - \mu_3) \right)^2 := \Theta(x_3) \text{(say)}.
\]

Furthermore
\[
\overline{\alpha}_{ij} := E \left[ \prod_{k=1}^{2} (X_k - \mu_k)^2 (X_i - \overline{\mu}_i)(X_j - \overline{\mu}_j) | X_3 \right]
\]
\[
= E [\bar{Y}_1 \bar{Y}_2 \bar{Y}_i \bar{Y}_j] + \sum_{k=1}^{2} (\overline{\mu}_k - \mu_k)^2 E [\bar{Y}_k \bar{Y}_i \bar{Y}_j]
\]
\[
+ \prod_{k=1}^{2} (\overline{\mu}_k - \mu_k)^2 E [\bar{Y}_i \bar{Y}_j] + 4 \prod_{k=1}^{2} (\overline{\mu}_k - \mu_k) E [\bar{Y}_1 \bar{Y}_2 \bar{Y}_i \bar{Y}_j].
\]

Owing to \((\bar{Y}_1, \bar{Y}_2) \sim N(\mathbf{0}, \Sigma), \) we have the following list of assertions:
\[
E [\bar{Y}_1 \bar{Y}_2 \bar{Y}_i \bar{Y}_j] = \Sigma_{11} \left( \Sigma_{22} \Sigma_{ij} + 2 \Sigma_{21} \Sigma_{21} \right) + 2 \Sigma_{12} \left( \Sigma_{12} \Sigma_{ij} + \Sigma_{11} \Sigma_{21} + \Sigma_{1j} \Sigma_{21} \right)
\]
\[
+ \Sigma_{11} \left( 2 \Sigma_{12} \Sigma_{21} + \Sigma_{22} \Sigma_{11} \right) + \Sigma_{1j} \left( 2 \Sigma_{12} \Sigma_{21} + \Sigma_{22} \Sigma_{11} \right),
\]
and for \(i, j, k = 1, 2\)
\[
E [\bar{Y}_1 \bar{Y}_2 \bar{Y}_i \bar{Y}_j] = \Sigma_{12} \Sigma_{ij} + \Sigma_{11} \Sigma_{2j} + \Sigma_{1j} \Sigma_{21}, \quad E [\bar{Y}_1 \bar{Y}_2 \bar{Y}_i \bar{Y}_k] = \Sigma_{12} \Sigma_{jk} + 2 \Sigma_{ij} \Sigma_{21}, \quad E [\bar{Y}_1 \bar{Y}_2 \bar{Y}_i] = \Sigma_{ij}, \quad E [\bar{Y}_1 \bar{Y}_2] = \Sigma_{kk} \text{ and}
\]
\[
\overline{\mu}_k - \mu_k = \Sigma_{k3} \Sigma_{33}^{-1} (x_3 - \mu_3).
\]

which leads directly to the result.  □
Proof of (28):

By virtue of the definition of WE with given WF

$$\phi'_{(X_1,X_2)|x_3} = \prod_{i=1}^{2} (x_i - \mu_i)^2 \left[ \frac{f(X_1, X_2|x_3)}{f(X_1, X_2)} \right],$$

we can write:

$$H_{\phi'_{(X_1,X_2)|x_3}}^{\text{w}}(X_1, X_2)$$

$$= - \int_{\mathbb{R}^2} \prod_{i=1}^{2} (x_i - \mu_i)^2 f(x_1, x_2|x_3) \log f(x_1, x_2) dx_1 dx_2$$

$$= \frac{1}{2} \log [(2\pi)^2 |\Sigma_1|] E \left[ \prod_{i=1}^{2} (X_i - \mu_i)^2 |X_3 \right]$$

$$+ \frac{1}{2} \sum_{i,j=1,2} \Sigma_{ij}^{-1} E \left[ \prod_{k=1}^{2} (X_k - \mu_k)^2 (X_i - \mu_i)(X_j - \mu_j)|X_3 \right].$$

Here $\Theta(x_3)$ has been calculated already in (47). Moreover one yields

$$\gamma_{ij} := E \left[ \prod_{k=1}^{2} (X_k - \mu_k)^2 (X_i - \mu_i)(X_j - \mu_j)|X_3 \right]$$

$$= E \left[ \prod_{k=1}^{2} (Y_k^2 + 2Y_k(\mu_k - \mu_k) + (\mu_k - \mu)^2)(Y_i + (\mu_i - \mu))(Y_j + (\mu_j - \mu)) \right].$$

(50)

Applying RV $(Y_1, Y_2)$, (50) becomes

$$\gamma_{ij} = E [Y_1^2 Y_2^2 Y_i Y_j] + E [Y_1^2 Y_2^2] (\mu_i - \mu)(\mu_j - \mu)$$

$$+ 2 E [Y_1 Y_2 Y_i Y_j] (\mu_2 - \mu_2)(\mu_j - \mu_2) + 2 E [Y_1 Y_2 Y_i Y_j] (\mu_2 - \mu_2)(\mu_i - \mu_i)$$

$$+ E [Y_1^2 Y_1 Y_2] (\mu_2 - \mu_2)^2 + E [Y_1^2] (\mu_2 - \mu_2)^2(\mu_i - \mu_i)(\mu_j - \mu_j)$$

$$+ 2 E [Y_1 Y_2 Y_1^2] (\mu_1 - \mu_1)(\mu_j - \mu_j) + 2 E [Y_1 Y_2 Y_2^2] (\mu_1 - \mu_1)(\mu_i - \mu_i)$$

$$+ 4 E [Y_1 Y_2 Y_1 Y_2] (\mu_1 - \mu_1)(\mu_2 - \mu_2) + 4 E [Y_1 Y_2] \prod_{k=1}^{2} (\mu_k - \mu_k)(\mu_i - \mu_i)(\mu_j - \mu_j)$$

$$+ 2 E [Y_1 Y_2 Y_1] (\mu_j - \mu_j)(\mu_1 - \mu_1)(\mu_2 - \mu_2)^2 + 2 E [Y_1 Y_2 Y_1] (\mu_i - \mu_i)(\mu_1 - \mu_1)(\mu_2 - \mu_2)^2$$

$$+ E [Y_1 Y_2 Y_1] (\mu_1 - \mu_1)^2 + E [Y_1 Y_2] (\mu_1 - \mu_1)^2(\mu_i - \mu_i)(\mu_j - \mu_j)$$

$$+ 2 E [Y_2 Y_1 Y_1] (\mu_2 - \mu_2)(\mu_1 - \mu_1)^2(\mu_j - \mu_j) + 2 E [Y_2 Y_1 Y_2] (\mu_2 - \mu_2)(\mu_1 - \mu_1)^2(\mu_i - \mu_i)$$

$$+ E [Y_1 Y_2 Y_1] \prod_{k=1}^{2} (\mu_k - \mu_k)^2 + \prod_{k=1}^{2} (\mu_k - \mu_k)^2(\mu_i - \mu_i)(\mu_j - \mu_j).$$

(51)

Follow the expectations from (49). Hence the final relation is concluded.  \(\square\)
\( \bar{\lambda}_{ij} \) and \( \gamma_{ij} \) in Example 2.8

\[
\bar{\lambda}_{11} = 12\rho^5(2 - \rho) + 3\rho^3(2 - \rho)^3 + 4x_3^2\rho^2(2 - \rho)^2(1 - \rho)^2 \\
+ 2\rho^4x_3^2(1 - \rho)^2 + x_3^4\rho(2 - \rho)(1 - \rho)^4 - 12\rho^3x_3^2(2 - \rho)(1 - \rho)^2,
\]

\[
\bar{\lambda}_{12} = \bar{\lambda}_{21} = -9\rho^4(2 - \rho)^2 - 6\rho^6 - 6x_3^2(1 - \rho)^2\rho^3(2 - \rho) - \rho^2x_3^4(1 - \rho)^4 \\
+ 8\rho^4x_3^2(1 - \rho)^2 + 4\rho^2x_3^4(1 - \rho)^2(2 - \rho)^2,
\]

\[
\bar{\lambda}_{22} = 3\rho^4(2 - \rho)^3 + 12\rho^5(2 - \rho) + 4\rho^2x_3^2(2 - \rho)^2(1 - \rho)^2 + 2\rho^4x_3^4(1 - \rho)^2 \\
+ x_3^4\rho(2 - \rho)(1 - \rho)^4 - 12x_3^2\rho^3(2 - \rho)(1 - \rho)^2.
\]

Furthermore

\[
\gamma_{11} = 12\rho^5(2 - \rho) + 3\rho^3(2 - \rho)^3 + 2(1 - \rho)^2x_3^2(\rho^2(2 - \rho)^2 + 2\rho^4) \\
- 24(1 - \rho)^2x_3^2(2 - \rho) + 3(1 - \rho)^2x_3^2(2 - \rho)^2 + (1 - \rho)^6x_3^6 \\
+ 4(1 - \rho)^2x_3^2(\rho^2(2 - \rho)^2 - 2\rho^3(2 - \rho)) - 8\rho^2(1 - \rho)^4x_3^4 + 7\rho(2 - \rho)(1 - \rho)^4x_3^4.
\]

\[
\gamma_{12} = \gamma_{21} = -9\rho^4(2 - \rho)^2 - 6\rho^6 + 9(1 - \rho)^2x_3^2(\rho^2(2 - \rho)^2 + 2\rho^4) + (1 - \rho)^6x_3^6 \\
- 18(1 - \rho)^2x_3^2(2 - \rho) + 6\rho(2 - \rho)(1 - \rho)^4x_3^4 - 6\rho^2(1 - \rho)^4x_3^4.
\]

\[
\gamma_{22} = 3\rho^4(2 - \rho)^3 + 12\rho^5(2 - \rho) + 6(1 - \rho)^2x_3^2(\rho^2(2 - \rho)^2 + 2\rho^4) \\
- 24(1 - \rho)^2x_3^2(2 - \rho) + (1 - \rho)^6x_3^6 + 7\rho(2 - \rho)(1 - \rho)^4x_3^4 \\
- 8\rho^2(1 - \rho)^4x_3^4 + 3\rho^2(2 - \rho)^2(1 - \rho)^2x_3^2.
\]

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