ON THE RATE OF EQUIDISTRIBUTION OF EXPANDING HOROSPHERES IN FINITE-VOLUME QUOTIENTS OF SL(2, C).

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ABSTRACT. Let Γ be a lattice in $G = \text{SL}(2, \mathbb{C})$. We give an effective equidistribution result with precise error terms for expanding translates of pieces of horospherical orbits in $\Gamma \backslash G$. Our method of proof relies on the theory of unitary representations.

1. Introduction

Let $G$ be a connected Lie group, and Γ a lattice in $G$. Various properties of orbits of horospherical subgroups of $G$ in the homogeneous space $\Gamma \backslash G$ have been studied by a multitude of authors. In particular, the effective equidistribution of expanding translates of such orbits has been established, cf. e.g. Kleinbock and Margulis [10].

For $G = \text{SL}(2, \mathbb{R})$, precise results relating the rate of equidistribution of orbits of the horocycle flow with the spectral theory of $\Gamma \backslash G$ have been obtained by a number of authors, e.g. Burger [4], Flaminio and Forni [6], Hejhal [9], Sarnak [17], Selberg (unpublished), Strömbergsson [19, 20], and Zagier [23]. One of the key parts of [3] is an integral formula [3, Lemma 1 (A)] for certain operators in irreducible unitary representations of $\text{SL}(2, \mathbb{R})$. More specifically, the formula relates horospherical averages with the action of a corresponding diagonal subgroup in an arbitrary irreducible unitary representation. One of the results in [3] obtained by use of the formula is an explicit rate of equidistribution for averages along horocycle orbits when $\Gamma$ is cocompact. In [19], use of the same formula is extended to prove explicit rates of equidistribution for non-cocompact lattices. In an ongoing project, we aim to generalize this method to obtain rates of equidistribution for other Lie groups. This note is a first report on this project, discussing only the case of $\text{SL}(2, \mathbb{C})$. This case has the benefits of allowing us to be completely explicit, and permitting comparisons with similar previously known results for hyperbolic 3-orbifolds. As we shall we see, a number of complications arise compared with $\text{SL}(2, \mathbb{R})$, both in expressing an integral formula corresponding to that of Burger (which we do in Proposition 4), as well as in the application of it to the problem of obtaining explicit rates of equidistribution.

From now on let $G = \text{SL}(2, \mathbb{C})$, and Γ a lattice in $G$. We denote by $\mu_G$ the unique Haar measure on $G$ such that the pushforward measure $\mu$ (under the map $g \mapsto \Gamma g$) on $\Gamma \backslash G$ is a probability measure. The group $G$ acts on $\Gamma \backslash G$ by right translation, and the action of the following subgroups of $G$ will be of particular interest to us:

$$A = \left\{ a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \mathbb{R} \right\},$$

and

$$N = \left\{ n_z = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \right\}.$$

Note that $N$ is a horospherical subgroup of $G$, relative to $A$, i.e.

$$N = \left\{ g \in G : \lim_{t \to \infty} a_{-t} ga_t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Let $\mu_N$ denote the Haar measure on $N$, chosen so that $\mu_N$ is the pushforward measure of the Lebesgue measure on $\mathbb{C}$ under the map $z \mapsto n_z$. The maximal compact subgroup $\text{SU}(2)$ of $G$ is denoted by $K$. The symmetric space $G/K$ can be identified with the three-dimensional hyperbolic upper half-space, and $M := \Gamma \backslash G/K$ is a finite-volume hyperbolic 3-orbifold ($M$...
is a manifold if $\Gamma$ is torsion-free). We use $\Delta$ to denote the Laplace-Beltrami operator on $\mathcal{M}$. Let $\lambda_1$ be the smallest positive eigenvalue for $-\Delta$ acting on $L^2(\mathcal{M})$, and define $s_1 \in \{1, 2\}$ by
\[
s_1 = \begin{cases} 
1 + \sqrt{1 - \lambda_1} & \text{if } \lambda_1 \in (0, 1) \\
1 & \text{otherwise}. 
\end{cases}
\]
In order to state our main result, we must introduce a Sobolev norm $\| \cdot \|_{W^m}$ on functions in $L^2(\Gamma \backslash G)$; it is discussed in greater detail in Section 5.1. We also let $\mathcal{Y}_\Gamma$ denote the invariant height function on $\Gamma \backslash G$. A stringent definition of $\mathcal{Y}_\Gamma$ is given in Section 3 for non-cocompact $\Gamma$, and in Section 5.2 for cocompact $\Gamma$. For now it suffices to view this function as a measure of how far into a cusp of $\Gamma \backslash G$ a point $p$ lies; for a fixed $x_0 \in \mathcal{M}$, let $\text{dist}(p)$ denote the distance between $x_0$ and the image of $p$ in $\mathcal{M}$. Then $\mathcal{Y}_\Gamma(p)$ is comparable to $e^{\text{dist}(p)}$.

**Theorem 1.** Let $B'$ be a connected compact subset of positive Lebesgue measure in $\mathcal{C}$ and assume that there exists a piecewise smooth, bi-Lipschitz mapping of the circle to $\partial B'$. If $B = \{ n \in N : z \in B' \}$, then there exists a constant $C(\Gamma, B') > 0$ such that for all $T \geq 0$, all $p \in \Gamma \backslash G$, and all $f \in L^2(\Gamma \backslash G)$ with $\|f\|_{W^7} < \infty$,
\[
\left| \frac{1}{\mu_N(B)} \int_B \mu_N(n) - \int_{\Gamma \backslash G} f \, d\mu \right| \leq C(\Gamma, B') \|f\|_{W^7} \left\{ \left( e^{-T} \mathcal{Y}_\Gamma(p) \right)^{2-s_1} + e^{-T} T^4 \right. \\
\left. + e^{-T} (1 + T^3) \mathcal{Y}_\Gamma(p) \right\}.
\]
We also prove a strengthened version of the above, Theorem 13 (cf. p. 24), where a bound on $C(\Gamma, B')$’s dependency on the set $B'$ is given.

By using the rate of exponential mixing (cf. [15, 14 Proposition 5.3]) and the so-called “Margulis’ mixing trick”, one may get a similar result (cf. [10 Proposition 2.4.8]). The rate one obtains in this manner, however, is worse than that which is achieved in Theorem 1. This is discussed in greater detail in the introduction to [21]. In [21], Södergren proves related results regarding the rate of effective equidistribution of pieces of closed horospheres in hyperbolic $n$-manifolds. Our Theorem 1 thus generalizes certain special cases of these results when $n = 3$, proving that this type of equidistribution even holds for translates of pieces of non-closed horospheres, and for test functions on the frame bundle of the manifold. We give an explicit statement of one such generalisation in Corollary 14 (cf. pg. 25).

Since the bounds in Theorem 1 are uniform in $p$, we may study the equidistribution of horospherical orbits by considering the point $pa_T$, giving

**Corollary 2.** Let $B$ satisfy the conditions of Theorem 1 and define $B_T := a_T Ba_{-T}$. Then for all $T \geq 0$, all $p \in \Gamma \backslash G$, and all $f \in L^2(\Gamma \backslash G)$ such that $\|f\|_{W^7} < \infty$,
\[
\left| \frac{1}{\mu_N(B_T)} \int_{B_T} f(pm) \, d\mu_N(n) - \int_{\Gamma \backslash G} f \, d\mu \right| \leq C(\Gamma, B') \|f\|_{W^7} \left\{ \left( e^{-T} \mathcal{Y}_\Gamma(pa_T) \right)^{2-s_1} + e^{-T} T^4 \right. \\
\left. + e^{-T} (1 + T^3) \mathcal{Y}_\Gamma(pa_T) \right\}.
\]

The corresponding result for $\text{SL}(2, \mathbb{R})$ is proved and stated as Proposition 3.1 in [19], the proof of which is the inspiration for the proof of Theorem 1. The equidistribution properties of orbits of horospherical subgroups are well-known; in fact, the celebrated results of Ratner give a precise understanding of the asymptotic behaviour of orbits of arbitrary unipotent subgroups.

It is well-known that for a given point $p$, the horospherical orbit $pN$ is either closed or dense. Furthermore, $pN$ is dense precisely when $pa_T$ is recurrent; there are therefore no closed horospheres in $\Gamma \backslash G$ when $\Gamma$ is cocompact. Noting that for cocompact $\Gamma$, $\mathcal{Y}_\Gamma(p) \ll 1$, Corollary 2 then provides an explicit rate of equidistribution for averaging sequences $B_T$ for every horosphere in $\Gamma \backslash G$. When $\Gamma$ is non-cocompact, however, there are many $p$ for which $pN$ is a closed horosphere. Moreover, there exist points $p$ for which $pa_T$ is recurrent, but
\[ \lim_{T \to \infty} e^{-T} \gamma_T(p \at T) \neq 0 \] (thus Corollary 2 does not by itself give effective equidistribution of every non-closed horosphere). Letting \( E \) be the exceptional set of such \( p \), and \( Q \) be the parabolic subgroup of upper triangular matrices in \( G \), we observe that \( EQ = E \); whether a point \( p \) is in \( E \) or not is therefore completely determined by \( p \)'s image in \( \Gamma \backslash G/Q \). We also note that, in a certain sense, \( E \) is small; by [16, Theorem 1], the image of \( E \) in \( G/Q \) has Hausdorff dimension zero. Note that the quotient \( G/Q \) may be identified with the “boundary sphere” of hyperbolic 3-space.

In [19], effective equidistribution of every non-closed horocycle is achieved ([19 Theorem 1]) by carefully splitting the horocycle into a number of pieces and using [19 Proposition 3.1] on all but one exceptional piece. Strömbergsson imposes an additional weighted supremum norm on the functions that are considered; it is this norm which is used to control the contribution from the exceptional piece of the horocycle. Moreover, it is shown in [19 Proposition 4.1] that one may not replace this supremum norm by a Sobolev norm of any order. We believe that by a similar type of argument, one should be able to obtain an effective equidistribution result for all non-closed horospheres in \( \Gamma \backslash G \). We do not attempt this here, however.

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2. Preliminaries

We recall some facts regarding the structure of \( G \), its Lie algebra, invariant measures and representations. The main references for this section are [11, Chapters 2, 5, 16], and [7, Chapter 7.4].

2.1. Invariant Measures and Iwasawa Decomposition. The Iwasawa decomposition of \( G \) is given by \( G = NAK \), where \( N \) and \( A \) are as previously defined, and \( K \) is \( SU(2) \), a maximal compact subgroup of \( G \). Each \( g \) in \( G \) has a unique decomposition \( g = n_za_k \), with \( n_\in N \), \( a_\in A \) and \( k \in K \) respectively. This decomposition gives rise to a corresponding decomposition of the Haar measure on \( G \); if \( g = n_za_k \), then \( d\mu_G(g) \) is a constant multiple of \( e^{-2t} dm(z) dt dk \), where \( dm \) is the Lebesgue measure on \( \mathbb{C} \) (i.e. \( dm(x + iy) = dx dy \)), and \( dk \) is the Haar measure of \( K \), normalized to be a probability measure. We choose the normalization of the Haar measure \( \mu_N \) on \( N \) such that if \( B' \subset \mathbb{C} \), and \( B = \{ n_\in N \} \), then \( \mu_N(B) = m(B') \). It will also be of use to define volumes of quotients other than \( \Gamma \backslash G \); let \( H \) be some group with Haar measure \( \mu_H \), and let \( \Xi \) be a discrete subgroup of \( H \). Then we define \( \mu_H(H/\Xi) \) (resp. \( \mu_H(\Xi \backslash H) \)) to be \( \mu_H(\mathcal{F}_\Xi) \), where \( \mathcal{F}_\Xi \) is a fundamental domain for \( H/\Xi \) (resp. \( \Xi \backslash H \)) in \( H \).

2.2. Lie Algebra. We denote by \( \mathfrak{g}_0 \) the Lie algebra of \( G \). It is a 6-dimensional real semisimple Lie algebra with a basis given by

\[
H = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_- = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]

\[
J = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad K_+ = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \quad K_- = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}.
\]

Note that

\[ \exp(tH) = a_t, \]

and

\[ \exp(xE_+ + yK_+) = n_{x+iy}. \]

The complexification of \( \mathfrak{g}_0 \) is denoted by \( \mathfrak{g} \), which has the universal enveloping algebra \( \mathcal{U}(\mathfrak{g}) \). We use \( \mathcal{U}^{\text{ad}}(\mathfrak{g}) \) to denote terms in the canonical filtration of \( \mathcal{U}(\mathfrak{g}) \). The center of \( \mathcal{U}(\mathfrak{g}) \), \( \mathcal{Z}(\mathfrak{g}) \),

\[ \frac{\mathbb{C}[G]}{\mathbb{C}[G]^\text{ad}}. \]
contains the following two elements:

\[ \Omega_1 = H^2 - J^2 - 2H + E_+E_- - K_+K_- , \]
\[ \Omega_2 = 2HJ - 2J + E_+K_- + K_+E_- . \]

The following identity will play an important role for us:

\[ H^4 - 4H^3 + (5 - \Omega_1)H^2 + 2(\Omega_1 - 1)H - (\Omega_1 + \frac{1}{2}\Omega_2^2) = E_+U_1 - K_+U_2 , \]

where \( U_1 \) and \( U_2 \) are the following elements in \( U^3(\mathfrak{g}) \):

\[ U_1 = \frac{1}{2}HE_+ + \frac{1}{2}JK_- - \frac{1}{2}E_+K_- - \frac{1}{2}K_+E_- - H^2E_- - HJK_- , \]
\[ U_2 = \frac{1}{2}HK_- - \frac{1}{2}JE_- + \frac{1}{2}K_+E_+ - H^2K_- - HJE_- . \]

2.3. Representation Theory. We now recall some basic facts from the theory of unitary representations. Let \((\pi, \mathcal{H})\) be a unitary representation of \(G\); i.e. \(\mathcal{H}\) is a separable Hilbert space, and \(\pi\) is a group homomorphism from \(G\) to the group of unitary operators on \(\mathcal{H}\) such that the map from \(G \times \mathcal{H}\) to \(\mathcal{H}\) given by

\[ (g, v) \mapsto \pi(g)v \]

is continuous. The representation \((\pi, \mathcal{H})\) is said to be irreducible if \(\mathcal{H}\) has no non-trivial proper closed subspace \(V\) such that \(\pi(G)V \subset V\). Each unitary representation \((\pi, \mathcal{H})\) of \(G\) has a direct integral decomposition

\[ (\pi, \mathcal{H}) \cong \left( \int_Z \| \pi_\zeta \| dv(\zeta), \int_Z \| \mathcal{H}_\zeta \| dv(\zeta) \right) , \]

where \(Z\) is a locally compact Hausdorff space, \(v\) is a positive Radon measure on \(Z\), and for \(v\)-a.e. \(\zeta\), \((\pi_\zeta, \mathcal{H}_\zeta)\) is an irreducible unitary representation of \(G\) (cf. eg. [11, Theorem 7.36]). The irreducible unitary representations of \(G\) are relatively easy to describe: if \((\pi, \mathcal{H})\) is an irreducible unitary representation of \(G\), then \((\pi, \mathcal{H})\) is isomorphic to either the trivial representation \((\pi_{triv}, \mathbb{C})\), a principal series representation \(\mathcal{P}^{(n, \nu)}\), where \((n, \nu) \in \{0\} \times i\mathbb{R}_{\geq 0} \cup \mathbb{N}_{>0} \times i\mathbb{R}\), or a complementary series representation \(\mathcal{P}^{(0, \nu)}\), where \(\nu \in (0, 2)\) (see [11, Theorem 16.2]).

The spherical unitary dual (the representations with a \(K\)-invariant vector) consists of the trivial representation and the representations \(\mathcal{P}^{(0, \nu)}\), where \(\nu \in i\mathbb{R}_{\geq 0} \cup (0, 2)\), and the tempered unitary dual consists of the principal series representations.

We let \(\mathcal{H}^\infty\) denote the space of smooth vectors for \((\pi, \mathcal{H})\); these are the vectors \(v\) for which the map from \(G\) to \(\mathcal{H}\) given by \(g \mapsto \pi(g)v\) is a \(C^\infty\) function on \(G\). It is well-known that \(\mathcal{H}^\infty\) is dense in \(\mathcal{H}\). For \(X \in \mathfrak{g}_0\), we define the operator \(d\pi(X)\) on \(\mathcal{H}^\infty\) as

\[ d\pi(X)v := \frac{d}{dt} \bigg|_{t=0} \pi(\exp(tX))v , \quad v \in \mathcal{H}^\infty . \]

We can extend this as a Lie algebra representation of \(\mathfrak{g}\) in the obvious way (i.e. for \(X \in \mathfrak{g}_0\), let \(d\pi(iX)v := id\pi(X)v\) and then to \(U^m(\mathfrak{g})\) by composition. We can now define Sobolev norms for the representation \((\pi, \mathcal{H})\) in the following manner: fix a basis \(X_1, \ldots, X_6\) for \(\mathfrak{g}_0\). For \(v \in \mathcal{H}^\infty\), define

\[ \|v\|_{W^m(\mathcal{H})}^2 := \sum_U \|d\pi(U)v\|^2_{\mathcal{H}} , \]

where the sum runs over all \(U\) that are monomials in \(X_1, \ldots, X_6\) of degree less than or equal to \(m\), including the term “1” of order zero (i.e. \(\|v\|_{\mathcal{H}}^2\) is one of the summands in the right-hand side of (5)). It is easy to check that for any \(m \geq 0\), there is a continuous function \(C : G \to \mathbb{R}_{>0}\) (independent of \((\pi, \mathcal{H})\)) such that

\[ \|\pi(g)v\|_{W^m(\mathcal{H})} \leq C(g)\|v\|_{W^m(\mathcal{H})} \quad \forall g \in G, \ v \in \mathcal{H}^\infty . \]
We note that given the direct integral decomposition of \((\pi, \mathcal{H})\) into irreducibles \([3]\), for \(v \in \mathcal{H}\) we have
\[
\|v\|_{\mathcal{H}}^2 = \int_{\mathcal{Z}} \|v_\zeta\|_{\mathcal{H}_\zeta}^2 \, dv(\zeta),
\]
and
\[
\pi(g)v = \int_{\mathcal{Z}} \pi_\zeta(g)v_\zeta \, dv(\zeta).
\]
Also, for \(v \in \mathcal{H}^\infty\),
\[
\|v\|_{W^m(\mathcal{H})}^2 = \int_{\mathcal{Z}} \|v_\zeta\|_{W^m(\mathcal{H}_\zeta)}^2 \, dv(\zeta).
\]
The direct integral decomposition of \((\pi, \mathcal{H})\) allows the construction of intertwining operators in the following manner: let \(f\) be a bounded, continuous function from \(\mathcal{Z}\) into \(\mathbb{C}\). We can then form the following operator: for \(v \in \mathcal{H}\), define
\[
T_f v := \int_{\mathcal{Z}} f(\zeta)v_\zeta \, dv(\zeta).
\]
Then for all \(g \in G\), \(v \in \mathcal{H}\);
\[
T_f \pi(g)v = \int_{\mathcal{Z}} \pi_\zeta(g)f(\zeta)v_\zeta \, dv(\zeta) = \pi(g)T_f v.
\]
We will also need intertwining operators of this kind where the scalar function is not necessarily uniformly bounded. By dropping the requirement that the function \(f\) in \((7)\) is uniformly bounded, we get operators that need not be defined on all of \(\mathcal{H}\), but may be bounded operators on \(\mathcal{H}^\infty\) with respect to various Sobolev norms \(\|\cdot\|_{W^m(\mathcal{H})}\).

Finally, we recall that if \((\pi, \mathcal{H})\) is irreducible, then by Schur’s lemma, elements of \(\mathcal{Z}(g)\) act as scalars on \(\mathcal{H}^\infty\). If \((\pi, \mathcal{H})\) is isomorphic to \(\mathcal{P}^{(\nu, \nu)}\), then the scalars for \(d\pi(\Omega_1)\) and \(d\pi(\Omega_2)\) are
\[
d\pi(\Omega_1) = \frac{n^2 + \nu^2}{4} - 1,
\]
and
\[
d\pi(\Omega_2) = \frac{\nu}{2}.
\]

3. Integral Formulas

We now prove the integral formulas for irreducible unitary representations of \(G\) that will be used in Section \([3]\). In this entire section we let \((\pi, \mathcal{H})\) be an irreducible unitary representation of \(G\). We also fix a compact subset \(B’\) of \(\mathbb{C}\), such that \(m(B’) > 0\) and the boundary \(\partial B’\) is a piecewise smooth simple closed curve. As before, we set
\[
B := \{n_z : z \in B’\} \subset N.
\]
For each vector \(v \in \mathcal{H}\) we define a function \(\psi_v : G \to \mathcal{H}\) by
\[
\psi_v(g) := \frac{1}{\mu_N(B)} \int_B \pi(n g) v \, d\mu_N(n).
\]
We note that we have
\[
\psi_v(g) = \frac{1}{m(B’)} \int_{B’} \pi(n_z g) v \, dm(z) = \frac{1}{m(B’)} \int_{B’} \pi(n_{z+iy} g) v \, dx \, dy
\]
(for the second equality we identify \(\mathbb{C}\) with \(\mathbb{R}^2\)). Note that \(\psi_v(g)\) depends linearly on \(v\). We apply this to \((11)\), giving that for \(v \in \mathcal{H}^\infty\),
\[
\psi_{d\pi(E^+U_1^+)}(g) - 4\psi_{d\pi(E^H)}(g) + (5 - \lambda_1)\psi_{d\pi(H^2)}(g) + 2(\lambda_1 - 1)\psi_{d\pi(H^1)}(g) - (\lambda_1 + \frac{1}{2}\lambda_2^2)\psi_v(g)
\]
\[
= \psi_{d\pi(E^+_+U_2^+)}(g) - \psi_{d\pi(K^+_+U_2^+)}v(g),
\]
where \( \lambda_1 = d\pi(\Omega_1) \) and \( \lambda_2 = d\pi(\Omega_2) \) are the scalars given at the end of Section 2.3 corresponding to \((\pi, \mathcal{H})\). We now restrict ourselves to studying the behaviour of \( \psi_v \) on \( A \); we define the following function from \( \mathbb{R} \) to \( \mathcal{H} \):

\[
\varphi_v(t) := \psi_v(a_t).
\]

We compute the various terms of (11) for \( \varphi_v \):

\[
\begin{align*}
 f_{\varphi(H)v}(t) &= \frac{1}{\mu_N(B)} \int_B \pi(na_t)d\pi(H)v\,d\mu_N(n) = \frac{1}{\mu_N(B)} \int_B \pi(na_t) \frac{d}{dr} \bigg|_{r=0} \pi(a_r)v\,d\mu_N(n) \\
 &= \frac{d}{dr} \bigg|_{r=0} \frac{1}{\mu_N(B)} \int_B \pi(na_t+r)v\,d\mu_N(n) = \frac{d}{dr} \bigg|_{r=0} \frac{1}{\mu_N(B)} \int_B \pi(na_t)v\,d\mu_N(n) = f_v'(t).
\end{align*}
\]

Hence

\[
(12) \quad f_{\varphi(H^m)v}(t) = f_v^{(m)}(t).
\]

We also have

\[
(13) \quad f_{\varphi(E+)v}(t) = \frac{1}{m(B')} \int_{B'} \pi(nx_+iy_t)v\,dx\,dy = \frac{1}{m(B')} \int_{B'} \pi(nx_+iy_t) \frac{d}{dt} \bigg|_{t=0} \pi(n_r)v\,dx\,dy \\
= \frac{1}{m(B')} \int_{B'} \frac{d}{dt} \bigg|_{t=0} \pi(nx_+re_+iy_t)v\,dx\,dy = \frac{e^t}{m(B')} \int_{B'} \partial_x \pi(nx_+iy_t)v\,dx\,dy.
\]

Likewise,

\[
(14) \quad f_{\varphi(K+)v}(t) = \frac{e^t}{m(B')} \int_{B'} \partial_y \pi(nx_+iy_t)v\,dx\,dy.
\]

Combining (11), (12), (13) and (14) gives

\[
\varphi_v^{(4)}(t) - 4\varphi_v^{(3)}(t) + (5 - \lambda_1)\varphi_v^{(2)}(t) + 2(\lambda_1 - 1)\varphi_v^{(1)}(t) - (\lambda_1 + \frac{1}{2}\lambda_2^2)\varphi_v(t) = \frac{e^t}{m(B')} \int_{B'} \left( \partial_x \pi(nx_+iy_t)v - \partial_y \pi(nx_+iy_t)d\pi(U_1)v - \partial_y \pi(nx_+iy_t)d\pi(U_2)v \right) dx\,dy.
\]

By Green’s Theorem, we then have

\[
(15) \quad \varphi_v^{(4)}(t) - 4\varphi_v^{(3)}(t) + (5 - \lambda_1)\varphi_v^{(2)}(t) + 2(\lambda_1 - 1)\varphi_v^{(1)}(t) - (\lambda_1 + \frac{1}{2}\lambda_2^2)\varphi_v(t) = \frac{e^t}{m(B')} \int_{B'} \pi(nx_+iy_t)d\pi(U_2)v\,dx\,dy + \pi(nx_+iy_t)d\pi(U_1)v\,dx\,dy.
\]

In Proposition 4, we present an integral representation of the solution to this (Hilbert space-valued) ODE that will prove to be useful in obtaining asymptotics for \( f_v(t) \) as \( t \) tends towards \(-\infty\). We first note, however, that if \((\pi, \mathcal{H})\) is one of the irreducible representations listed in Section 2.3 such that \( \lambda_2 = 0 \), we do not need to solve a fourth order differential equation. Indeed, in this case we may use the following identity

\[
(16) \quad H^3 - 3H^2 + (2 - \Omega_1)H + \Omega_1 = \frac{1}{2}\Omega_2 J + E_+V_1 + K_+V_2,
\]

where \( V_1, V_2 \) are the following elements of \( \mathcal{U}^2(g) \):

\[
(17) \quad V_1 = E_- - E_+H - \frac{1}{2}K_-J, \quad V_2 = -K_- + K_-H - \frac{1}{2}E_-J.
\]
In the same way that (11) implies (15), (16) implies, when \( d\pi(\Omega_2) = \lambda_2 = 0 \):

\[
(18) \quad f_v^{(3)}(t) - 3f_v^{(2)}(t) + (2 - \lambda_1)f_v^{(1)}(t) + \lambda_1f_v(t)
= \frac{\epsilon^t}{m(B')} \int_{\partial B'} \pi(n_{x+iy}a_t) d\pi(V_2)v \, dx + \pi(n_{x+iy}a_t) d\pi(V_1)v \, dy.
\]

For notational purposes we introduce the following function: for \( X, Y \in U(g) \), \( t \in \mathbb{R} \) and \( v \in H^\infty \), define

\[
I_v(X, Y, t) := \frac{\epsilon^t}{m(B')} \int_{\partial B'} \pi(n_{x+iy}a_t) d\pi(Y)v \, dx + \pi(n_{x+iy}a_t) d\pi(X)v \, dy.
\]

By using the values of \( \lambda_1 \) and \( \lambda_2 \) given in (8) and (9), we may rewrite (15) as

\[
(20) \quad (d\pi - (1 - \frac{r}{2})) (d\pi - (1 - \frac{r}{2})) (d\pi - (1 + \frac{r}{2})) f_v(t) = I_v(U_1, U_2, t).
\]

**Lemma 3.** Assume \( (\pi, \mathcal{H}) \cong \mathcal{P}^{(n,r)} \), where \( n > 0 \), and \( v \in H^\infty \). Let \( g_v(t) \) be defined by

\[
(21) \quad e^{(1-n/2)t} g_v(t) = (d\pi - (1 - \frac{r}{2})) (d\pi - (1 + \frac{r}{2})) (d\pi - (1 + \frac{r}{2})) f_v(t).
\]

Then

\[
(22) \quad g_v(t) = \int_{-\infty}^t e^{(n/2-1)s} I_v(U_1, U_2, s) \, ds.
\]

**Proof.** By (20) and (21),

\[
(d\pi - (1 - \frac{r}{2})) (e^{(1-n/2)t} g_v(t)) = I_v(U_1, U_2, t),
\]

so

\[
\frac{d}{dt} g_v(t) = e^{(n/2-1)t} I_v(U_1, U_2, t).
\]

The fundamental theorem of calculus then gives

\[
g_v(t) - g_v(r) = \int_r^t e^{(n/2-1)s} I_v(U_1, U_2, s) \, ds.
\]

By using the triangle inequality for integrals in (19), we get that

\[
\|I_v(U_1, U_2, s)\| \leq B \|v\|_{W^3(\mathcal{H})},
\]

so

\[
\|g_v(t) - g_v(r)\| < B \|v\| \int_r^t e^\frac{n}{2} ds = \frac{2}{n} \left( e^\frac{n}{2r} - e^\frac{n}{2} \right).
\]

From this uniform bound, we see that \( g_v(r) \) converges to some vector \( v_\infty \) as \( r \to -\infty \), and

\[
g_v(t) = v_\infty + \int_{-\infty}^t e^{(n/2-1)s} I_v(U_1, U_2, s) \, ds.
\]

It remains to prove that \( v_\infty = 0 \). We let \( w = (d\pi(H) - (1 - \frac{r}{2}))(d\pi(H) - (1 + \frac{r}{2}))(d\pi(H) - (1 + \frac{r}{2}))v \), and note that from the definition of \( g_v(r) \),

\[
g_v(r) = e^{(n/2-1)r} f_w(r).
\]

For \( n \geq 3 \), have

\[
\|g_v(r)\| = \|e^{(n/2-1)r} f_w(r)\| \leq e^{(n/2-1)r} \|w\| \leq e^{r/2} \|w\|,
\]

so \( v_\infty = 0 \). For \( n = 1 \) or \( n = 2 \), we use quantitative decay of matrix coefficients; let \( u \) be any vector in \( H^\infty \). We then have

\[
\left\langle e^{(n/2-1)r} f_w(r), u \right\rangle = \frac{1}{\mu_N(B)} \int_B e^{(n/2-1)r} \langle \pi(l) \pi(a_r) w, u \rangle \, d\mu_N(l)
= \frac{1}{\mu_N(B)} \int_B e^{(n/2-1)r} \langle \pi(a_r) w, \pi(l^{-1}) u \rangle \, d\mu_N(l).
\]
By [13] Proposition 5.3 (cf. [11] Propositions 7.14, 7.15 (c)), there exist \( \eta > 1/2 \) and \( C_\eta > 0 \), not depending on \( r \), such that
\[
|\langle \pi(a_r) w, \pi(t^{-1}) u \rangle| \leq C_\eta e^{\eta r} \|w\|_{W^2(\mathcal{H})} \|\pi(t^{-1}) u\|_{W^2(\mathcal{H})},
\]
giving
\[
|\langle e^{(n/2-1)r} f_w(r), u \rangle| \leq C_\eta e^{(\eta-1/2)r} \|w\|_{W^2(\mathcal{H})} \frac{1}{\mu_N(B)} \int_B \|\pi(t^{-1}) u\|_{W^2(\mathcal{H})} \, d\mu_N(l).
\]
Here the integral in the right-hand side is finite (cf. [11]); hence \( \langle e^{(n/2-1)r} f_w(r), u \rangle \to 0 \) as \( r \to -\infty \), and thus \( \langle v_\infty, u \rangle = 0 \). Since \( \mathcal{H}^\infty \) is dense in \( \mathcal{H} \), \( v_\infty = 0 \). \( \square \)

We are now able to prove the main result of this section:

**Proposition 4.** Given an irreducible unitary representation \((\pi, \mathcal{H})\) of \( G \), there exist \( \mathbb{C} \)-valued functions \( F, F_0, F_1, F_2 \), and elements \( X_1, X_2 \) of \( \mathcal{U}^2(\mathfrak{g}) \), all of which depend only on the isomorphism class of \((\pi, \mathcal{H})\), such that for any \( T \geq 0 \) and \( \nu \in \mathcal{H}^\infty \),
\[
(24) \quad f_\nu(-T) = \int_{-\infty}^0 F(T, t) I_\nu(X_1, X_2, t) \, dt + \sum_{m=0}^2 F_m(T) f_{d\pi(H^m)\nu}(0),
\]
and the following bounds hold, with all implied constants absolute:

(i) If \((\pi, \mathcal{H}) \cong \mathcal{P}^{(n, \nu)}\), where \( n > 0 \), then \(|F_2(T)| \ll \frac{(1+|\nu|)(1+T)e^{-T}}{n} \), for \( i = 0, 1 \), \(|F_i(T)| \ll \frac{(1+|\nu|)(1+T)e^{-T}}{(1+T^2)e^{-T}} \), for \( i = 1, 2 \), and
\[
|F(T, t)| \ll \frac{1}{n^2} \begin{cases} e^{\left(\frac{n}{2}-1\right)(T+t)} & \text{if } t \leq -T, \\ (1+t+T)e^{-(T+t)} & \text{if } t \geq -T. \end{cases}
\]

(ii) If \((\pi, \mathcal{H}) \cong \mathcal{P}^{(0, \nu)}\), where \( \nu \in i\mathbb{R}_{\geq 0} \), then \(|F_0(T)| \ll \frac{(1+|\nu|)(1+T^2)e^{-T}}{(1+T^2)e^{-T}} \), for \( i = 1, 2 \), and
\[
|F(T, t)| \ll \begin{cases} 0 & \text{if } t \leq -T, \\ (T+t)^2e^{-(T+t)} & \text{if } t \geq -T. \end{cases}
\]

(iii) If \((\pi, \mathcal{H}) \cong \mathcal{P}^{(0, \nu)}\), where \( \nu \in (0, 2) \), then \(|F_i(T)| \ll \nu^{-2} e^{\left(\frac{n}{2}-1\right)T} \), for \( i = 0, 1, 2 \), and
\[
|F(T, t)| \ll \frac{1}{\nu^2} \begin{cases} 0 & \text{if } t \leq -T, \\ e^{\left(\frac{n}{2}-1\right)(T+t)} & \text{if } t \geq -T. \end{cases}
\]

Furthermore, in cases (ii) and (iii), \( X_1 \) and \( X_2 \) may be taken as elements of \( \mathcal{U}^2(\mathfrak{g}) \).

In the proof below we obtain completely explicit formulas for \( F, F_0, F_1, F_2 \); and we see that we can take \( X_1 = U_1, X_2 = U_2 \) in case (i) (cf. [17]), and \( X_1 = V_1, X_2 = V_2 \) in cases (ii) and (iii) (cf. [17]). In case (iii), we have allowed the bounds to blow up as \( \nu \to 0 \) only to allow a simple statement; in fact the stronger bounds \(|F_i(T)| \ll \min\{1+T^2, \nu^{-2}\} e^{\left(\frac{n}{2}-1\right)T} \) and \(|F(T, t)| \ll \min\{T+t, \nu^{-2}\} e^{\left(\frac{n}{2}-1\right)(T+t)} \) can be deduced from the explicit formula.

**Proof.** Let
\[
\alpha_1 = 1 - \frac{n}{2}, \quad \alpha_2 = 1 - \frac{\nu}{2}, \quad \alpha_3 = 1 + \frac{\nu}{2}, \quad \alpha_4 = 1 + \frac{n}{2}.
\]
We first assume that \( n > 0 \). As in the proof of Lemma \[\text{[20]}\] we use \( \left( \prod_{i=1}^4 \left( \frac{d}{dr} - \alpha_i \right) \right) f_\nu(t) = I_\nu(U_1, U_2, t) \).

We now define the functions \( g_1(t), g_2(t), g_2(t) \) and \( g_1(t) \) to be such that
\[
f_\nu(t) = e^{\alpha_1 t} g_1(t),
\]
and for \( 3 \leq i \geq 1 \),
\[
\frac{d}{dt} g_{i+1}(t) = e^{(\alpha_i - \alpha_{i+1})t} g_i(t).
\]
From these definitions, we see that

\[ \frac{d}{dt}g_1(t) = e^{-\alpha_1 t} I_v(U_1, U_2, t). \]

Iterated integration of (25) gives

\[ f_v(-T) = g_4(0)e^{-\alpha_4 T} - g_3(0)e^{-\alpha_4 T} \int_{-T}^{0} e^{(\alpha_3 - \alpha_4)t_4} dt_4 \]
\[ + g_2(0)e^{-\alpha_4 T} \int_{-T}^{0} e^{(\alpha_3 - \alpha_4) t_4} \int_{t_4}^{0} e^{(\alpha_2 - \alpha_3) t_3} dt_3 dt_4 \]
\[ - e^{-\alpha_4 T} \int_{-T}^{0} e^{(\alpha_3 - \alpha_4) t_4} \int_{t_4}^{0} e^{(\alpha_2 - \alpha_3) t_3} \int_{t_3}^{0} e^{(\alpha_1 - \alpha_2) t_2} g_1(t_2) dt_2 dt_3 dt_4. \]

We use Lemma 3 and change the order of integration to get

\[ f_v(-T) = g_4(0)e^{-\alpha_4 T} - g_3(0)e^{-\alpha_4 T} \int_{-T}^{0} e^{(\alpha_3 - \alpha_4)t_4} dt_4 \]
\[ + g_2(0)e^{-\alpha_4 T} \int_{-T}^{0} e^{(\alpha_3 - \alpha_4)t_4} \int_{t_4}^{0} e^{(\alpha_2 - \alpha_3)t_3} dt_3 dt_4 \]
\[ + \int_{-\infty}^{0} I_v(U_1, U_2, t) F(T, t) dt, \]

where

\[ F(T, t) = -e^{-\alpha_1 t - \alpha_4 T} \int_{\max(t, -T)}^{t_2} \int_{-T}^{t_3} e^{\sum_{j=1}^{3} (\alpha_j - \alpha_{j+1}) t_{j+1}} dt_4 dt_3 dt_2. \]

From the definitions of the \( g_i \), we have that \( g_4(0) = f_v(0) \), \( g_3(0) = f_{d\nu(H)v}(0) - \alpha_4 f_v(0) \), and \( g_2(0) = f_{d\nu(H^2)v}(0) - (\alpha_3 + \alpha_4) f_{d\nu(H)v}(0) + \alpha_4 \alpha_3 f_v(0) \). By entering these into (26), and collecting terms, we obtain (24).

Turning our attention to \( (\pi, \mathcal{H}) \cong \mathcal{P}^{(0,v)} \), from (9) we see that (18) holds. We then rewrite this as

\[ \left( \prod_{i=1}^{3} \left( \frac{d}{dt} - \alpha_i \right) \right) f_v(t) = I_v(V_1, V_2, t). \]

We then solve this equation in the same manner as when \( n > 0 \), the main difference is that now we integrate \( I_v(V_1, V_2, t_1) \) from \( t_2 \), and not from \(-\infty\). This gives

\[ f_v(-T) = \int_{-\infty}^{0} F(T, t) I_v(V_1, V_2, t) dt + \sum_{m=0}^{2} F_m(T) f_{d\nu(H^m)v}(0), \]

where

\[ F(T, t) = \begin{cases} 
0 & \text{if } t < -T \\
- e^{-\alpha_1 t - \alpha_3 T} \int_{-T}^{t} \int_{-T}^{t_2} e^{(\alpha_1 - \alpha_2)t_2 + (\alpha_2 - \alpha_3)t_3} dt_3 dt_2 & \text{otherwise }
\end{cases} \]
\[ F_2(T) = e^{-\alpha_2 T} \int_{-T}^{0} e^{(\alpha_2 - \alpha_3)t_3} \int_{t_3}^{0} e^{(\alpha_1 - \alpha_2)t_2} dt_2 dt_3, \]
\[ F_1(T) = -e^{-\alpha_3 T} \int_{-T}^{0} e^{(\alpha_2 - \alpha_3)t_3} dt_3 \]
\[ - (\alpha_2 + \alpha_3) e^{-\alpha_3 T} \int_{-T}^{0} e^{(\alpha_2 - \alpha_3)t_3} \int_{t_3}^{0} e^{(\alpha_1 - \alpha_2)t_2} dt_2 dt_3, \]
and

$$F_0(T) = e^{-\alpha_3 T} + \alpha_3 e^{-\alpha_3 T} \int_{-T}^{0} e^{(\alpha_2 - \alpha_3) t_3} \, dt_3 + \alpha_2 \alpha_3 e^{-\alpha_3 T} \int_{-T}^{0} e^{(\alpha_2 - \alpha_3) t_3} \int_{t_3}^{0} e^{(\alpha_1 - \alpha_2) t_2} \, dt_2 \, dt_3.$$

Entering the numerical values of the $\alpha_i$s and repeated use of the triangle inequality now give the stated bounds. \qed

4. The Invariant Height Function and Geometry of $\Gamma \backslash \mathbb{H}^3$

In this section we define and establish certain properties of the in\textit{variant height function}, which can be seen as measuring how far into a cusp a point in $\Gamma \backslash G$ is; for this reason we assume throughout this entire section that $\Gamma$ is non-cocompact. The invariant height function will be needed for the pointwise Sobolev-type bounds of the next section. We also prove a bound on the average of the invariant height function along a translate of the boundary of $B$. This bound is stated in Proposition 6 and will be required when we apply Proposition 4 in the proof of Theorem 1 (see Section 6).

4.1. The Invariant Height Function. We start by recalling some of the main facts (the main reference of these are [5, Chapters 1, 2]) regarding the action of $G$ on the hyperbolic upper half-space $\mathbb{H}^3 = \{(z, r) : z \in \mathbb{C}, r \in \mathbb{R}_+\}$. For $g = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in G$, $(z, r) \in \mathbb{H}^3$, define

$$g \cdot (z, r) := \left( \frac{(az + b)(\overline{c}z + \overline{d}) + a\overline{c}r^2}{|cz + d|^2 + |c|^2 r^2}, \frac{r}{|cz + d|^2 + |c|^2 r^2} \right).$$

We also recall that this action extends uniquely to the boundary $\partial_{\infty} \mathbb{H}^3 = \{ \infty \} \cup \{(z, 0) : z \in \mathbb{C} \}$. It will be convenient to view $\mathbb{H}^3$ as the subset $\{z + rj : z \in \mathbb{C}, r \in \mathbb{R}_+\}$ of the quaternions. Letting $P = z + rj$, we can then write (27) in the more concise form

$$g \cdot P = \frac{aP + b}{cP + d}.$$

We note that

$$(n_z a_k) \cdot j = z + e^j;$$

this gives the standard identification of $G/K$ with $\mathbb{H}^3$. It is also useful to define

$$\text{ht}(z + rj) := r.$$

For $\eta \in \partial_{\infty} \mathbb{H}^3 \setminus \{ \infty \}$ and $\delta \in \mathbb{R}_+$, define

$$\mathcal{H}(\eta, \delta) := \{z + rj : |z - \eta|^2 + |r - \delta/2|^2 < (\delta/2)^2\};$$

note that this a Euclidean ball tangent to $\partial_{\infty} \mathbb{H}^3$ in the upper half-space model. Define also

$$\mathcal{H}(\infty, \delta) := \{ P \in \mathbb{H}^3 : \text{ht}(P) > \delta \}.$$

The sets $\mathcal{H}(\eta, \delta)$ are called horoballs, their boundaries in $\mathbb{H}^3$ are called horospheres. Since $\Gamma$ is a non-cocompact lattice, $\Gamma \backslash \mathbb{H}^3$ is a hyperbolic 3-orbifold with at least one cusp. We shall now define the in\textit{variant height function} $\mathcal{H}_\Gamma$ as a $\Gamma$-left and $K$-right invariant function on $G$. We may then also view $\mathcal{H}_\Gamma$ as a function on $\Gamma \backslash G$, as well as a function on $\mathbb{H}^3$—we shall abuse notation and also write $\mathcal{H}_\Gamma(p)$ for $p \in \Gamma \backslash G$, and $\mathcal{H}_\Gamma(P)$ for $P \in \mathbb{H}^3$. Recall that the cusps of $\Gamma$ (w.r.t. the action on $\mathbb{H}^3$) are the parabolic fixed points of $\Gamma$ on $\partial_{\infty} \mathbb{H}^3$. Let $\eta \in \partial_{\infty} \mathbb{H}^3$ be a cusp of $\Gamma$, and define the following subset of $G$:

$$\mathbb{N}_{\eta}^{(\Gamma)} := \{ h \in G : h \cdot \eta = \infty, \mu_N(N/(N \cap h\Gamma h^{-1})) = 1 \}.$$

Note that if $h_1, h_2 \in \mathbb{N}_{\eta}^{(\Gamma)}$, then for any $g \in G$,

$$\text{ht}(h_1 g \cdot j) = \text{ht}(h_2 g \cdot j).$$
We may therefore define
\[
ht_{\eta}(g \cdot j) := \text{ht}(hg \cdot j), \quad h \in N_\eta^G.
\]
We choose a maximal set \(\eta_1, \eta_2, \ldots, \eta_\kappa\) of \(\Gamma\)-inequivalent cusps (this is a finite set, due to \(\Gamma\) being a lattice), and define, for \(g \in G\):
\[
(28) \quad \mathcal{Y}_\Gamma(g) := \max_{i=1, \ldots, \kappa} \max_{\gamma \in \Gamma} \text{ht}_{\eta_i}(\gamma g \cdot j).
\]
Note that given \(g \in G, p \in \Gamma G,\) and \(P \in \mathbb{H}^3\) such that \(p = \Gamma g\) and \(P = g \cdot j\), we have \(\mathcal{Y}_\Gamma(p) = \mathcal{Y}_\Gamma(g) = \mathcal{Y}_\Gamma(P)\). In the proof of Lemma \(21\)a we will see that \(\mathcal{Y}_\Gamma\) does not depend on the choice of cusps, that is to say: given another maximal set of \(\Gamma\)-inequivalent cusps \(\eta'_1, \eta'_2, \ldots, \eta'_\kappa\), and letting \(\mathcal{Y}'_\Gamma\) be defined as in \((28)\), but with respect to this new choice of cusps, then \(\mathcal{Y}_\Gamma = \mathcal{Y}'_\Gamma\). As a function on \(\mathbb{H}^3\), \(\mathcal{Y}_\Gamma\) is comparable to the invariant height function defined in \([21]\) Section 2.3. We collect some properties of \(\mathcal{Y}_\Gamma\) which will be needed in the following lemma:

**Lemma 5.**

a) \(\forall \eta, \eta_0 \in G, \mathcal{Y}_\Gamma(\eta g_0) = \mathcal{Y}_{\eta_0}^{-1}(\eta g_0)\).

b) \(\forall s \in \mathbb{R}, \forall g \in G: \mathcal{Y}_\Gamma(ga_s) \leq \max\{e^s, e^{-s}\} \mathcal{Y}_\Gamma(g)\).

c) \(\forall c \in \mathbb{C}, \forall g \in G: \mathcal{Y}_\Gamma(gzn_c) \leq (1 + |z|^2) \mathcal{Y}_\Gamma(g)\).

d) Let \(C_0 = \sqrt{2/\sqrt{3}}\). For any two \(\Gamma\)-cusps \(\eta \neq \eta'\) and \(h \in N_\eta^G, h' \in N_{\eta'}^G:\)
\[
h^{-1} \cdot \mathcal{H}(\infty, C_0) \cap h^{-1} \cdot \mathcal{H}(\infty, C_0) = \emptyset.
\]
Consequently, for \(C \geq C_0\), the set \(\{P \in \mathbb{H}^3 : \mathcal{Y}_\Gamma(P) > C\}\) is a disjoint union of horoballs.

e) Let \(C\) be a fixed compact subset of \(G\). Then for all \(g \in G\),
\[
(29) \quad \sup_{h \in G} |\Gamma h \cap gC| \ll_{\Gamma, C} \mathcal{Y}_\Gamma(g)^2.
\]

**Proof.** Starting with a), let \(\eta\) be a cusp for \(\Gamma\). Then for any \(g_0 \in G, g_0^{-1} \cdot \eta\) is a cusp of \(g_0^{-1}\Gamma g_0,\) and \(N_\eta^G g_0 = N(g_0^{-1}\Gamma g_0)^\eta\). It follows from applying this with \(g_0 = \gamma_0 \in \Gamma\) that \(\max_{\gamma \in \Gamma} \text{ht}_{\eta}(\gamma g \cdot j)\) is invariant under replacing \(\eta\) by \(\gamma_0^{-1} \cdot \eta\), for any \(\gamma_0 \in \Gamma\). This shows that \(\mathcal{Y}_\Gamma\) is indeed independent of the choice of representatives \(\eta_1, \eta_2, \ldots, \eta_\kappa\). Now note that \(g_0^{-1} \cdot \eta_1, g_0^{-1} \cdot \eta_2, \ldots, g_0^{-1} \cdot \eta_\kappa\) is a maximal set of inequivalent cusps for \(g_0^{-1}\Gamma g_0\), and \(\max_{\gamma \in \Gamma} \text{ht}_{\eta_\kappa}(\gamma g_0 g \cdot j) = \max_{\gamma \in g_0\Gamma g_0} \text{ht}_{\eta_\kappa}(\gamma g \cdot j)\) for each \(i \in \{1, 2, \ldots, \kappa\}\). Hence \(\mathcal{Y}_\Gamma(\eta g_0) = \mathcal{Y}_{g_0^{-1}\Gamma g_0}(g),\) as claimed.

Part b) of the lemma follows from the fact that \((a b / c d) a_s = (c e^{s/2} / d e^{s/2})\), and \(|e^s - e^{-s}|^2 + |d|^2 e^{-s})^2 \leq \max\{e^s, e^{-s}\}((c)^2 + |d|^2)^{-1}.

To prove c), it suffices to prove that \(\text{ht}(gnz \cdot j) \leq \text{ht}(g \cdot j)(1 + |z|^2)^{s_2},\) i.e. that
\[
11 \leq (1 + |z|^2)^2 = |cz + d|^2 + |c|^2 |cz + d|^2.
\]
or, equivalently, \(|c|^2 \leq (1 + |z|^2)(1 + |z|^2)^2(1 + |z|^2)^2 - |d|^2 \leq 1 + |z|^2 |c|^2 - |c|^2 |c|^2 = |c|^2\).

For d), we may, after possibly conjugating \(\Gamma\), assume that \(\eta' = \infty\) and \(h' = (0, 0, 1)\). We then need to prove that
\[
(30) \quad \{h^{-1} \cdot P : \text{ht}(P) > C_0\} \cap \{P : \text{ht}(P) > C_0\} = \emptyset.
\]
Since \(h^{-1} \cdot \infty = \eta \neq \infty\), we may write \(h\) as \((a b / c d)\) with \(c \neq 0\). Assume now that \(P = z + e^j j\). Then
\[
\text{ht}(h^{-1} \cdot P) = \text{ht}(h^{-1} \cdot (z + e^j j)) = e^j \frac{e^j}{|cz + a|} \leq \frac{1}{|cz|^2 e^j}.
\]
Since \( e' = \text{ht}(P) \), we get that

\[
\{ h^{-1} \cdot P : \text{ht}(P) > C_0 \} \subset \left\{ P : \text{ht}(P) \leq (|c|^2 C_0)^{-1} \right\}.
\]

We now see that if \( (|c|^2 C_0)^{-1} \leq C_0 \), then the two sets in \( (30) \) will be disjoint. We will therefore prove that \( |c| \geq C_0^{-1} \). Since \( h \in N^{(1)}_h \), we have \( \mu_N(N/(N \cap hGh^{-1})) = 1 \). By the identification of \( \mu_N \) with the Lebesgue measure on \( C \) given in Section 2.1, we see that the set \( \{ z \in C : h^{-1}n_z h \in \Gamma \} \) is a unimodular lattice in \( C \); there therefore exists \( z_1 \in C \), \( 0 < |z_1| \leq C_0 \), such that \( h^{-1}n_z h \in \Gamma \) (cf. e.g. [4] Chapter 1). The same holds for \( \eta' = \infty \), \( h' = (0 \ 1) \), i.e. we can find \( z_2 \in C \), \( 0 < |z_2| \leq C_0 \), such that \( n_{z_2} \in \Gamma \). Let \( \Gamma' \) be the group generated by \( h^{-1}n_z h \) and \( n_{z_2} \). We have \( \Gamma' \subset \Gamma \), so \( \Gamma' \) is a discrete subgroup of \( G \), and

\[
h^{-1}n_z h = \begin{pmatrix} * & \ast \\ -c_{z_1} & * \end{pmatrix} \in \Gamma', \quad \begin{pmatrix} 1 & z_2 \\ 0 & 1 \end{pmatrix} \in \Gamma'.
\]

Shimizu’s Lemma ([5] Theorem 3.1) now applies, giving \(| -c^2 z_1 z_2 | \geq 1 \). Thus

\[
|c| \geq \frac{1}{\sqrt{|z_1 z_2|}} \geq C_0^{-1},
\]

as desired.

Finally, to prove e), we start by defining \( W(g) := \sup_{h \in G} |\Gamma h \cap gC| \). By using the fact that for all \( \gamma \in \Gamma \), \( |\Gamma h \cap gC| = |\Gamma(\gamma h) \cap gC| \), we get

\[
W(g) = \sup_{h \in gC} |\Gamma h \cap gC| \leq |\Gamma \cap gC|^{-1}|g|^{-1}.
\]

From this bound we see that \( W \) is uniformly bounded on any compact subset of \( G \). We also note that \( W \), like \( \mathcal{Y}_\Gamma \), is left \( \Gamma \)-invariant. These two observations reduce the problem to proving that \( (29) \) holds when \( \Gamma g \) lies far out in a cusp of \( \Gamma \backslash G \), say \( \mathcal{Y}_\Gamma(g) > C_0 e^{\Delta c_1} \), where \( C_0 \) is as in d), and

\[
\Delta c_1 = \sup_{g \in C_1} \text{dist}(g_1 \cdot j, j) \quad C_1 := CC^{-1},
\]

dist(\cdot, \cdot) being the hyperbolic distance in \( \mathbb{H}^3 \). We may now assume, by making a \( \Gamma \)-shift if necessary, that \( \mathcal{Y}_\Gamma(g) = \text{ht}_{\eta_i}(g \cdot j) \), where \( i \in \{1, 2, \ldots, k\} \).

For any \( \gamma \in \Gamma \cap gC_1 g^{-1} \) (cf. (31)), let \( g_\gamma = g^{-1} \gamma g \in C_1 \). Note that \( \gamma g = gg_\gamma \), and since \( N^{(1)}_{h_\gamma} \gamma^{-1} = N^{(1)}_{h_{g_\gamma}} \),

\[
\text{ht}_{\eta_i}(gg_\gamma \cdot j) = \text{ht}_{\eta_i}(\gamma g \cdot j) = \text{ht}_{\eta_i}(g \cdot j) > C_0 e^{\Delta c_1},
\]

\[
\text{ht}_{\eta_i}(gg_\gamma \cdot j) \geq \text{ht}_{\eta_i}(g \cdot j) e^{-\text{dist}(g_\gamma \cdot j, j)} \geq \text{ht}_{\eta_i}(g \cdot j) e^{-\Delta c_1} > C_0.
\]

From these inequalities we see that \( gg_\gamma \) is in the intersection of the sets \( \{ g' \in G : \text{ht}_{\eta_i}(g' \cdot j) > C_0 \} \) and \( \{ g' \in G : \text{ht}_{\eta_i}(g' \cdot j) > C_0 \} \). But by d), this intersection is empty unless \( \gamma \cdot \eta_i = \eta_i \). Hence \( W(g) \leq |\Gamma \cap gC_1 g^{-1}| = |\Gamma_{\eta_i} \cap gC_1 g^{-1}| \). Letting \( M = \{ (\epsilon, -1) : \epsilon \in C, |\epsilon| = 1 \} \), we note that by [3] Corollary 2.1.9], for \( h_i \in N^{(1)}_{h_\gamma} \), \( h_i \Gamma_{\eta_i} h_i^{-1} \subset MN \). Since \( \eta_i \) is a cusp of \( \Gamma \), \( h_i \Gamma_{\eta_i} h_i^{-1} \cap N = \{ n_z : z \in \Lambda \} \), where \( \Lambda \) is some lattice in \( C \). By the compactness of \( M \), there exist elements \( \gamma_1, \ldots, \gamma_r \in h_i \Gamma_{\eta_i} h_i^{-1} \) such that \( h_i \Gamma_{\eta_i} h_i^{-1} = \bigsqcup_{i=1}^r \gamma_i \{ n_z : z \in \Lambda \} \), giving

\[
|\Gamma_{\eta_i} \cap gC_1 g^{-1}| = \sum_{i=1}^r |h_i^{-1} \gamma_i \{ n_z : z \in \Lambda \} h_i \cap gC_1 g^{-1}|
\]

\[
= \sum_{i=1}^r \left| \left\{ z \in \Lambda : (h_ig^{-1})^{-1} \gamma_i n_z h_ig \in C_1 \right\} \right|.
\]
Lemma 7. Writing \( h_i g \cdot j = z_q + \gamma_T(g) j \), and \( \gamma_i^{-1} h_i g \cdot j = w_l + \gamma_T(g) j \), we see that for \( z \in \Lambda \) such that \( (h_i g)^{-1} \gamma n_z h_i g \in \mathcal{C}_1 \),
\[
\cosh(\Delta_{\mathcal{C}_1}) \geq \cosh \left( \text{dist} \left( (h_i g)^{-1} \gamma n_z h_i g \cdot j, j \right) \right) \\
= \cosh \left( \text{dist} \left( z + z_q + \gamma_T(g) j, w_l + \gamma_T(g) j \right) \right) \\
= 1 + \frac{|(w_l - z_q) - z|^2}{2 \gamma_T(g)^2},
\]
which implies \(|(w_l - z_q) - z| \ll \gamma_T(g)\). Since \( \Lambda \) is a lattice in \( \mathbb{C} \), the number of such \( z \) is \( \ll \gamma_T(g)^2 \), so summing over the \( k \) gives \( W(g) \ll \gamma_T(g)^2 \). \( \square \)

4.2. Boundary Integral of the Invariant Height Function. Using the previous lemma, we now state and prove the needed result on averages of \( \gamma_T \) along the boundary of translates of \( B \). We identify \( S^1 \) with \( \mathbb{R}/\mathbb{Z} \), and for \( x \in S^1 \), we write \( |x| \) for the distance to the point 0; in other words \( |x| = \min_{m \in \mathbb{Z}} |\bar{x} - m| \), where \( \bar{x} \) is any lift of \( x \) to \( \mathbb{R} \). The main result of this section can now be stated:

**Proposition 6.** Let \( \Gamma' \) be any lattice in \( G \), and let \( B' \) be a connected compact subset of \( \mathbb{C} \) such that \( 0 \in B' \) and there exists a piecewise smooth parametrization \( \gamma : S^1 \to \mathbb{C} \cap \partial B' \) with the following properties: i) for all \( t \in S^1 \) where \( \gamma(t) \) exists, \( |\gamma'(t)| = L \), the arc length of \( \partial B' \), and ii) there exists \( c > 0 \) s.t. \( \forall t_1, t_2 \in S^1 \), \( |\gamma(t_1) - \gamma(t_2)| \geq c|t_1 - t_2| \). Then for \( s \geq 0 \),
\[
\int_{\partial B'} \gamma_T(n_z a_{-s})|dz| \ll L(1 + R^2)(1 + Y) + \frac{L^2}{c} (1 + \log ((1 + R)(1 + Y))) + s,
\]
where \( R = \text{diam}(B') \), \( Y = \gamma_T(\left( \frac{1}{0} \right)) \), and the implied constant is absolute (in particular, the implied constant does not depend on \( \Gamma' \)).

We split the proof of the proposition into a series of lemmas. The notation and assumptions on \( B' \) from Proposition 6 will be used throughout the remainder of this section. Recall that by Lemma 3(d), the set \( \{ P \in \mathbb{H}^3 : \gamma_T(P) > 2 \} \) is a disjoint union of horoballs; we let \( H_{\Gamma'} \) be the family of these horoballs.

**Lemma 7.** Let \( \vartheta = 100 \max\{2, Y\} \). For fixed \( s \geq 0 \), define the following subset of \( S^1 \):
\[
\mathcal{I} := \{ t \in S^1 : \gamma_T(\left( \gamma(t) + e^{-s}j \right) \geq \vartheta \}. \]
Let \( \mathcal{H}(\eta_1, \delta_1), \mathcal{H}(\eta_2, \delta_2), \ldots, \mathcal{H}(\eta_N, \delta_N) \) be the distinct horoballs in \( H_{\Gamma'} \) that have non-empty intersection with \( \gamma(\mathcal{I}) + e^{-s}j \). Then
\[
\int_{\partial B'} \gamma_T(n_z a_{-s})|dz| \ll L \vartheta + \frac{L}{c} \sum_{k=1}^{N} \delta_k.
\]

**Proof.** We partition \( \mathcal{I} \) by the following sets:
\[
\mathcal{I}_k = \{ t \in \mathcal{I} : \gamma(t) + e^{-s}j \in \mathcal{H}(\eta_k, \delta_k) \} \quad 1 \leq k \leq N.
\]
Using the assumptions on \( \gamma \) and the definition of \( \mathcal{I} \), we have
\begin{equation}
\int_{\partial B'} \gamma_T(n_z a_{-s})|dz| = \int_{\partial B'} \gamma_T(z + e^{-s}j)|dz| = \int_{S^1} \gamma_T(\gamma(t) + e^{-s}j)|\gamma'(t)| dt
\leq L \vartheta + L \int_{\mathcal{I}} \gamma_T(\gamma(t) + e^{-s}j) dt = L \vartheta + L \sum_{k=1}^{N} \int_{\mathcal{I}_k} \gamma_T(\gamma(t) + e^{-s}j) dt.
\end{equation}
We now bound the contribution from the integral over one of the \( \mathcal{I}_k \)'s. Since \( \mathcal{I}_k \subset \mathcal{I} \), for \( t \in \mathcal{I}_k \) we have
\begin{equation}
\gamma_T(\gamma(t) + e^{-s}j) = \frac{2\delta_k e^{-s}}{|\gamma(t) - \eta_k|^2 + e^{-2s}} \geq \vartheta,
\end{equation}
which implies
\[ \frac{2\delta_k}{\vartheta} \geq e^{-s}. \]

Also, for all \( t \in I_k \),
\[ |\gamma(t) - \eta_k|^2 \leq \frac{2\delta_k e^{-s}}{\vartheta} - e^{-2s} =: \sigma_k^2. \]

Now, let \( \zeta_k(t) = \gamma(t) - \eta_k \), and define
\[ I_{k,m} = \left\{ t \in I_k : |\zeta_k(t)| \in \left( 2^{-(m+1)}\sigma_k, 2^{-m}\sigma_k \right) \right\}. \]

Then
\[
\int_{I_k} \mathcal{V}_{t}^{\varphi}(\gamma(t) + e^{-s}\mathbf{j}) \, dt = \int_{I_k} \frac{2\delta_k e^{-s}}{\zeta_k(t)^2 + e^{-2s}} \, dt = \sum_{m=0}^{\infty} \int_{I_{k,m}} \frac{2\delta_k e^{-s}}{\zeta_k(t)^2 + e^{-2s}} \, dt \\
\leq \sum_{m=0}^{\infty} \left( \frac{2\delta_k e^{-s}}{2^{-2(m+1)}\sigma_k^2 + e^{-2s}} \right) \int_{I_{k,m}} dt.
\]

Now, if \( t_1 \) and \( t_2 \) are both in \( I_{k,m} \), then
\[ |\zeta_k(t_1) - \zeta_k(t_2)| \leq 2^{1-m}\sigma_k. \]

Using \( |\zeta_k(t_1) - \zeta_k(t_2)| \geq c|t_1 - t_2| \) gives
\[ |t_1 - t_2| \leq c^{-1}2^{1-m}\sigma_k, \]

thus
\[ \int_{I_{k,m}} dt \leq c^{-1}2^{2-m}\sigma_k. \]

This now gives
\[
\int_{I_k} \mathcal{V}_{t}^{\varphi}(\gamma(t) + e^{-s}\mathbf{j}) \, dt \leq \sum_{m=0}^{\infty} \left( \frac{2\delta_k e^{-s}}{2^{-2(m+1)}\sigma_k^2 + e^{-2s}} \right) c^{-1}2^{2-m}\sigma_k \\
= \frac{8\delta_k (\epsilon^s\sigma_k)}{c} \sum_{m=0}^{\infty} \frac{2^{-m}}{2^{-2(m+1)}(\epsilon^s\sigma_k)^2 + 1}.
\]

For \( \epsilon^s\sigma_k \leq 1 \), this is clearly \( \ll \frac{\delta_k}{c} \). If \( \epsilon^s\sigma_k > 1 \), we choose \( M \) so that \( 2^M = \epsilon^s\sigma_k \), giving
\[
\ll \frac{8\delta_k \epsilon^s\sigma_k}{c} \left( \sum_{0 \leq m \leq M} \frac{2^{-m}}{2^{-2(m+1)}(\epsilon^s\sigma_k)^2 + 1} + \sum_{M < m} 2^{-m} \right) \\
\ll \frac{\delta_k \epsilon^s\sigma_k}{c}. \]

We have thus obtained the bound
\[ \int_{I_k} \mathcal{V}_{t}^{\varphi}(\gamma(t) + e^{-s}\mathbf{j}) \, dt \ll \frac{\delta_k}{c}, \]
which, when entered into (32), proves the lemma. \( \square \)

We now wish to bound the sum \( \sum_{k=1}^{N} \delta_k \) in Lemma 7. In order to do this, we first prove a lemma which provides a degree of separation between points of \( \partial B^d + e^{-s}\mathbf{j} \) that lie deep inside distinct horoballs. This will be used to bound the number of horoballs of a given diameter that \( \gamma(I) + e^{-s}\mathbf{j} \) has non-empty intersection with.
Lemma 8. Using the notation of Lemma 7 suppose $z_k + e^{-s} j \in \mathcal{H}(\eta_k, \delta_k) \cap \{ P \in \mathbb{H}^3 : \mathcal{V}_P(P) \geq \vartheta \}$, and $z_l + e^{-s} j \in \mathcal{H}(\eta_l, \delta_l) \cap \{ P \in \mathbb{H}^3 : \mathcal{V}_P(P) \geq \vartheta \}$, where $k \neq l$. Then

$$|z_k - z_l| \geq \left( 1 - \frac{4}{\vartheta} \right) \sqrt{\delta_k \delta_l}.$$  

Proof. Since $\mathcal{H}(\eta_k, \delta_k)$ and $\mathcal{H}(\eta_l, \delta_l)$ are disjoint, by Pythagoras’ theorem we have

$$|\eta_k - \eta_l|^2 \geq \frac{\delta_k^2}{4} + \frac{\delta_l^2}{4} - \left( \frac{\delta_k}{2} - \frac{\delta_l}{2} \right)^2 = \delta_k \delta_l.$$  

Hence, by the triangle inequality,

$$|z_k - z_l| \geq |\eta_k - \eta_l| - |z_k - \eta_k| - |z_l - \eta_l| \geq \sqrt{\delta_k \delta_l} - \frac{2\delta_k e^{-s}}{\vartheta} - \frac{2\delta_l e^{-s}}{\vartheta},$$

where we used (35). By (34), $e^{-s} \leq 2\vartheta^{-1} \min \{\delta_k, \delta_l\}$, hence

$$|z_k - z_l| \geq \sqrt{\delta_k \delta_l} - \frac{2}{\vartheta} \sqrt{\min \{\delta_k, \delta_l\}} \left( \sqrt{\delta_k} + \sqrt{\delta_l} \right) \geq \sqrt{\delta_k \delta_l} - \frac{4}{\vartheta} \sqrt{\delta_k \delta_l} = \left( 1 - \frac{4}{\vartheta} \right) \sqrt{\delta_k \delta_l}.$$  

\[\square\]

Lemma 9. Retaining the notation used in the previous lemmas, let $\delta_{\text{Max}} = \max_{1 \leq k \leq N} \delta_k$. Then

$$\sum_{k=1}^{N} \delta_k \ll \delta_{\text{Max}} + L (1 + \log \delta_{\text{Max}} + s).$$

Note that while $\log \delta_{\text{Max}}$ may be negative, $\log \delta_{\text{Max}} + s$ is always positive; in fact, from (34), $\log \delta_k + s > 0$ for all $k$.

Proof. For $m \in \mathbb{N}$, we define the following sets

$$S_m := \{ k \in \{1, 2, \ldots, N\} : \delta_k \in (2^{-(m+1)} \delta_{\text{Max}}, 2^{-m} \delta_{\text{Max}}) \}.$$  

Let $M$ be such that $2^M = \frac{2\delta_{\text{Max}}}{\vartheta}$. For $m > M$ and $k \in S_m$, we have

$$\delta_k \leq \delta_{\text{Max}} 2^{-m} < \delta_{\text{Max}} 2^{-M} = \frac{\vartheta e^{-s}}{2}.$$  

This gives

$$\frac{2\delta_k}{\vartheta} < e^{-s},$$

contradicting (34)! Thus for all $m > M$, $S_m = \emptyset$.

We now bound the sum of the diameters by

$$\sum_{k=1}^{N} \delta_k = \sum_{0 \leq m \leq M} \sum_{l \in S_m} \delta_l \leq \sum_{0 \leq m \leq M} 2^{-m} \delta_{\text{Max}} |S_m|.$$  

For each $l \in S_m$ we choose an element $z_l \in \gamma(I)$ such that $z_l + e^{-s} j \in \mathcal{H}(\eta_l, \delta_l)$. We may assume that $S_m = \{ l_1, l_2, \ldots, l_{N_m} \}$, with the numbering chosen so that the points $z_{l_1}, z_{l_2}, \ldots, z_{l_{N_m}}$ lie in this order along the curve $\partial B'$. Assuming $N_m \geq 2$, and using Lemma 8 to get a lower bound on $|z_{l_p} - z_{l_{p+1}}|$, gives

$$L \geq |z_{l_{N_m}} - z_{l_1}| + \sum_{p=1}^{N_m-1} |z_{l_p} - z_{l_{p+1}}| \geq \left( 1 - \frac{4}{\vartheta} \right) \sqrt{\delta_{l_{N_m}} \delta_{l_1}} + \sum_{p=1}^{N_m-1} \left( 1 - \frac{4}{\vartheta} \right) \sqrt{\delta_{l_p} \delta_{l_{p+1}} \geq \left( 1 - \frac{4}{\vartheta} \right) N_m 2^{-(m+1)} \delta_{\text{Max}}.$$
Thus
\[ |\mathcal{S}_m| = N_m \leq \max \left\{ 1, 2^{m+1} \left( 1 - \frac{4}{\vartheta} \right)^{-1} \frac{L}{\delta_{\text{Max}}} \right\} \]
\[ \leq 1 + 2^{m+1} \left( 1 - \frac{4}{\vartheta} \right)^{-1} \frac{L}{\delta_{\text{Max}}}, \]
giving
\[ \sum_{k=1}^{N} \delta_k \leq \sum_{0 \leq m \leq M} 2^{-m} \delta_{\text{Max}} |\mathcal{S}_m| \leq \delta_{\text{Max}} + 2 \left( 1 - \frac{4}{\vartheta} \right)^{-1} L (1 + M). \]

Using \( \vartheta \geq 200 \) and the definition of \( M \) gives
\[ \sum_{k=1}^{N} \delta_k \leq \delta_{\text{Max}} + 100 \frac{L}{49 \log 2} (2 \log 2 + \log \delta_{\text{Max}} + s). \]

\[ \square \]

**Proof of Proposition 6.** By applying Lemmas \([\text{?}]\) and \([\text{?}]\) we get

(36) \[ \int_{\partial B'} \mathcal{Y}(n_z a_{-s}) \, dz \ll L(\vartheta + \delta_{\text{Max}}) + \frac{L^2}{c} (1 + \log \delta_{\text{Max}} + s), \]
which reduces the problem to bounding \( \delta_{\text{Max}} \). Let \( \eta \) be one of the \( \eta_k \)s associated to a horoball with diameter \( \delta_{\text{Max}} \). We consider two cases:

**Case 1:** \( \exists z \in B' \) s.t. \( z + j \notin \mathcal{H}(\eta, \delta_{\text{Max}}) \).

Let \( \tilde{z} \in \partial B' \) be such that \( \tilde{z} + e^{-s} j \in \mathcal{H}(\eta, \delta_{\text{Max}}) \) and \( \mathcal{Y}(\tilde{z} + e^{-s} j) \geq \vartheta \). Then \( |z - \eta|^2 + 1 \geq \delta_{\text{Max}} \) and \( |\tilde{z} - \eta|^2 + e^{-2s} < \frac{2\delta_{\text{Max}}}{\delta_{\text{Max}}} e^{-s} \). Also, \( |z - \tilde{z}| \leq R \). Hence
\[ \delta_{\text{Max}} \leq 1 + |z - \eta|^2 \leq 1 + (|z - \tilde{z}| + |\tilde{z} - \eta|)^2 \]
\[ \leq 1 + \left( R + \left( \frac{2\delta_{\text{Max}}}{\vartheta} \right)^{1/2} \right)^2 \leq 1 + 2R^2 + \frac{4\delta_{\text{Max}}}{\vartheta} \leq 1 + 2R^2 + \frac{\delta_{\text{Max}}}{50}, \]
forcing
\[ \delta_{\text{Max}} \leq \frac{50}{49} (1 + 2R^2). \]

**Case 2:** \( B' + j \subset \mathcal{H}(\eta, \delta_{\text{Max}}) \).

In this case, \( \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \cdot j = 0 + j \in \mathcal{H}(\eta, \delta_{\text{Max}}) \), so

(37) \[ Y = \frac{2\delta_{\text{Max}}}{|\eta|^2 + 1} \geq 2, \]
and \( \vartheta = 100 Y \). We let \( \tilde{z} \) be as in Case 1, so, as before, \( |\tilde{z} - \eta|^2 \leq \frac{2\delta_{\text{Max}}}{\vartheta} \). Now,

(38) \[ |\eta|^2 \leq (|\tilde{z}| + |\tilde{z} - \eta|)^2 \leq (R + \sqrt{\frac{2\delta_{\text{Max}}}{\vartheta}})^2 \leq 2R^2 + \frac{\delta_{\text{Max}}}{25} Y. \]
Hence, by (37) and (38):
\[ Y \geq \frac{\delta_{\text{Max}}}{1 + 2R^2 + \frac{\delta_{\text{Max}}}{25} Y}, \]
giving
\[ \delta_{\text{Max}} \leq \frac{25}{24} (1 + 2R^2) Y. \]

We then have that in both Case 1 and Case 2,

(39) \[ \delta_{\text{Max}} \ll (1 + 2R^2) (1 + Y). \]
Entering (39) into (36) and using $\vartheta \ll 1 + Y$ gives
\[
\int_{\partial B'} \mathcal{Y}_{\Gamma}(n_{z-a_{-s}})|dz| \ll L(1 + R^2)(1 + Y) + \frac{L^2}{c} \left(1 + \log \left((1 + 2R^2)(1 + Y)\right) + s\right),
\]
which proves the proposition.

Proposition 6 will be applied for lattices $\Gamma' = g^{-1}\Gamma g$. The Sobolev inequalities of the next section will also require us to consider integrals along $\partial B'$ of $\mathcal{Y}_{\Gamma}$ raised to the power of some number between zero and one. We deal with both of these issues in the following corollary:

**Corollary 10.** For all $g \in G$ and $\alpha \in [0, 1]$, we have
\[
\int_{\partial B'} \mathcal{Y}_{\Gamma}(gn_{z-a_{-s}})^\alpha|dz| \ll_{\Gamma,\alpha} \frac{L^2(1 + R^2)}{c} (s + Y^\alpha),
\]
where now $Y = \mathcal{Y}_{\Gamma}(g)$.

**Proof.** We use the bound on $\gamma'$, Lemma 5(a), and Jensen’s inequality to get
\[
\int_{\partial B'} \mathcal{Y}_{\Gamma}(gn_{z-a_{-s}})^\alpha|dz| \leq L \int_{S^1} \mathcal{Y}_{g^{-1}\Gamma g}(\gamma(t) + e^{-s}j)^\alpha dt \leq L \left(\int_{S^1} \mathcal{Y}_{g^{-1}\Gamma g}(\gamma(t) + e^{-s}j) dt\right)^\alpha.
\]
By studying the proof of Proposition 6 we see that
\[
\left(\int_{S^1} \mathcal{Y}_{g^{-1}\Gamma g}(\gamma(t) + e^{-s}j) dt\right)^\alpha \ll \left((1 + R^2)(1 + Y') + \frac{L}{c} \left(1 + \log \left((1 + R)(1 + Y')\right) + s\right)\right)^\alpha,
\]
where $Y' = \mathcal{Y}_{g^{-1}\Gamma g}(\frac{1}{Y})$. Again by Lemma 5(a), $Y' = Y$. We now use the fact that $1 \ll_{\Gamma} Y$, and $L \geq 2|\gamma(\frac{1}{2}) - \gamma(0)| \geq c$, to get
\[
\ll_{\Gamma,\alpha} \frac{L^2}{c} (1 + R^2)Y^\alpha + \frac{L^2}{c} \left(1 + \log \left((1 + R)(1 + Y)\right) + s\right).
\]
This, after noting that $\frac{L^2}{c} (1 + \log \left((1 + R)(1 + Y)\right)) \ll_{\Gamma,\alpha} \frac{L^2}{c} (1 + R^2)Y^\alpha$, concludes the proof.

**Remark 1.** Note that any bi-Lipschitz mapping of $S^1$ to the boundary of some set $B' \subset \mathbb{C}$ may be reparametrized to satisfy the conditions of Proposition 6; this allows us to use Corollary 10 in the proof of Theorem 5.

5. **Decomposition of $L^2(\Gamma \setminus G)$ and Sobolev inequalities**

In this section we turn our attention to $L^2(\Gamma \setminus G)$; the main goal is to prove pointwise bounds for functions in an appropriate Sobolev space $W^m(\Gamma \setminus G)$. Unlike the previous section, we allow for $\Gamma$ to be cocompact (as well as non-cocompact). If $\Gamma$ is a non-cocompact lattice in $G$, functions in $W^m(\Gamma \setminus G)$ will generally not be bounded; they can grow in the cusps of $\Gamma \setminus G$. The rate of growth will be expressed in terms of the invariant height function of Section 4. This rate will also depend on spectral properties of the given function. For this reason we start with a discussion of the decomposition of $L^2(\Gamma \setminus G)$ into irreducible representations.

5.1. **Decomposition of $L^2(\Gamma \setminus G)$.** We now study the right-regular representation $(\rho, L^2(\Gamma \setminus G))$ of $G$ on $L^2(\Gamma \setminus G)$. Recall that $\rho$ is the right-translation operator: for all $f \in L^2(\Gamma \setminus G)$, $g \in G$ and $p \in \Gamma \setminus G$, $(\rho(g)f)(p) = f(pg)$.

We also recall that the Lie algebra acts on $\mathbb{C}$-valued functions on $\Gamma \setminus G$ as differential operators; for $p \in \Gamma \setminus G$, $X \in \mathfrak{g}_0$, and $f : \Gamma \setminus G \to \mathbb{C}$, let
\[
X f(p) = \frac{d}{dt} |_{t=0} f(p\exp(tX)).
\]
By allowing compositions and linear combinations over $\mathbb{C}$, we get an action of $U(\mathfrak{g})$. Define $W^m(\Gamma \setminus G)$ to be the space of functions in $C^m(\Gamma \setminus G)$ such that $Uf \in L^2(\Gamma \setminus G)$ for all $U \in U^m(\mathfrak{g})$. Analogously to (5), for a fixed basis of $\mathfrak{g}$, we define a norm $\| \cdot \|_{W^m}$ on $W^m(\Gamma \setminus G)$...
by \(\|f\|^2 = \sum_U \|Uf\|^2\), the sum being over all \(U\) that are monomials in the basis elements of degree not greater than \(m\). We note that for \(f \in L^2(\Gamma \backslash G)^\infty\) and \(U \in U^m(\mathfrak{g})\), we have \(Uf = dp(U)f\), so for such \(f\) we have \(\|f\|_{W^m} = \|f\|_{W^m(\mathcal{L}(\Gamma \backslash G))}\) (\(\|\cdot\|_{W^m(\mathcal{L}(\Gamma \backslash G))}\) denotes the norm on \(L^2(\Gamma \backslash G)^\infty\) defined in (5)). Moreover, \(L^2(\Gamma \backslash G)^\infty\) is dense in \(W^m(\Gamma \backslash G)\) with respect to \(\|\cdot\|_{W^m}\).

Recall that we may decompose \(L^2(\Gamma \backslash G)\) as

\[
L^2(\Gamma \backslash G) = L^2(\Gamma \backslash G)_{\text{disc}} \oplus L^2(\Gamma \backslash G)_{\text{cont}},
\]

where \(L^2(\Gamma \backslash G)_{\text{disc}}\) decomposes as a direct sum of irreducible unitary representations, and \(L^2(\Gamma \backslash G)_{\text{cont}}\) decomposes as a direct integral of irreducible unitary representations, with the measure associated to the integral having no atoms. We now let \(L^2(\Gamma \backslash G)_{\text{cusp}}\) denote the closed \(G\)-invariant subspace of \(L^2(\Gamma \backslash G)\) consisting of the cuspidal functions. A well-known result of Gelfand and Piatetski-Shapiro (see, for example, [8, Theorem 2]) gives that \(L^2(\Gamma \backslash G)_{\text{cusp}}\) decomposes into a direct sum of irreducible unitary representations of \(G\), each with finite multiplicity; \(L^2(\Gamma \backslash G)_{\text{cusp}}\) is therefore contained within \(L^2(\Gamma \backslash G)_{\text{disc}}\), and we write \(L^2(\Gamma \backslash G)_{\text{disc}}\) as the orthogonal sum

\[
L^2(\Gamma \backslash G)_{\text{disc}} = L^2(\Gamma \backslash G)_{\text{cusp}} \oplus L^2(\Gamma \backslash G)_{\text{res}},
\]

i.e. we let \(L^2(\Gamma \backslash G)_{\text{res}}\) be the orthogonal complement of \(L^2(\Gamma \backslash G)_{\text{cusp}}\) in \(L^2(\Gamma \backslash G)_{\text{disc}}\).

We will need some well-known facts regarding the connection between the spectral theory of the Laplace-Beltrami operator \(\Delta\) on the hyperbolic 3-orbifold \(\mathcal{M} = \Gamma \backslash G/K\) and the decomposition of \(L^2(\Gamma \backslash G)\). Let \(0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_M < 1\) be the eigenvalues of \(-\Delta\) (acting on \(L^2(\mathcal{M})\)) in \((0,1)\), counted with multiplicity (cf. eg. [5, Chapter 6]). These small eigenvalues may be parametrized thus: for \(\lambda_m \in (0,1)\), let \(s_m(2 - s_m)\), where \(s_m \in (1,2)\). For each \(s_m\), \(m \in \{1,2, \ldots M\}\), there exists a subrepresentation \(C^{s_m}\) of \((\rho, L^2(\Gamma \backslash G))\), and \(C^{s_m}\) is isomorphic to \(\mathcal{P}(0,2s_m-2)\). Moreover, any subrepresentation of \((\rho, L^2(\Gamma \backslash G))\) that is isomorphic to a complementary series representation is contained in \(\bigoplus_{m=1}^{M} C^{s_m}\). This is seen by noting that a \(K\)-invariant vector in a subrepresentation of \((\rho, L^2(\Gamma \backslash G))\) that is isomorphic to a complementary series representation \(\mathcal{P}(0,\nu)\) may be viewed as an eigenfunction to \(-\Delta\) in \(L^2(\mathcal{M})\), with eigenvalue \(1 - \frac{\nu^2}{4}\) (since \(\Omega_1\) acts as \(\Delta\) on the \(K\)-invariant vectors in \(L^2(\Gamma \backslash G)^\infty\)).

Finally, we recall that if \(\Gamma\) is cocompact, then \(L^2(\Gamma \backslash G)_{\text{cont}}\) is zero. On the other hand, if \(\Gamma\) is non-cocompact, then the irreducible unitary representations occurring in the direct integral decomposition of \(L^2(\Gamma \backslash G)_{\text{cont}}\) are all tempered. This may be seen by the previous identification of \(K\)-invariant vectors of \(L^2(\Gamma \backslash G)\) with elements of \(L^2(\mathcal{M})\), and the fact that the continuous spectrum of \(-\Delta\) in \(L^2(\mathcal{M})\) is the interval \([1, \infty)\) (cf. [5, Chapter 6]).

5.2. Sobolev Estimates. We now prove pointwise bounds for functions in some of the subspaces discussed in the previous section. For notational convenience, we first make the following definition:

\textbf{Definition 5.1.} For cocompact \(\Gamma\), let

\[
\mathcal{Y}_\Gamma(p) = 1 \quad \forall p \in \Gamma \backslash G.
\]

Note that by this definition, Lemma [5] and Corollary [10] trivially hold even for cocompact \(\Gamma\). By combining Lemma [5 e) and [1] Prop. B.2], we get:

\textbf{Lemma 11.} Let \(p \in \Gamma \backslash G\) and \(f \in W^{4}(\Gamma \backslash G)\). Then

\[
|f(p)| \ll_{\Gamma} \|f\|_{W^{4}} \mathcal{Y}_\Gamma(p).
\]

For \(f \in W^{5}(\Gamma \backslash G) \cap L^2(\Gamma \backslash G)_{\text{cusp}}\), we have the following uniform bound:
Lemma 12. For cuspidal functions in $W^5(\Gamma\backslash G)$, we have

$$|f(p)| \ll \|f\|_{W^5}.$$ 

This follows from [11 Lemma B.3]; we write out the proof as a preparation for Lemma [13] below.

Proof. Assume that $f \in W^5(\Gamma\backslash G) \cap L^2(\Gamma\backslash G)_{\text{cusp}}$ is $\mathbb{R}$-valued (for $\mathbb{C}$-valued functions, we may carry out the same arguments for the real and imaginary parts). Without loss of generality, we may assume that $\Gamma$ has a cusp at infinity, normalized so that the lattice $\Lambda = \{ z \in \mathbb{C} : n_z \in \Gamma \}$ is unimodular, and $p = \Gamma g$, where $g = na_0k_0$, is in the cuspidal region around $\infty$, i.e. $\mathcal{Y}_\Gamma(p) = e^t > 2$ (also, $\forall n' \in N$, $\mathcal{Y}_\Gamma(n'g) = \mathcal{Y}_\Gamma(g)$). Since $f$ is cuspidal, and in $C^5(\Gamma\backslash G)$, we have

$$\int_{(\Gamma \cap N) \backslash N} f(n g) n \mu_N(n) = 0.$$ 

Since $\Lambda$ is unimodular, there exists a fundamental parallelogram $F \subset \mathbb{C}$, $m(F) = 1$, for $\Lambda$ such that

$$\int_F f(n g) \, dm(z) = 0.$$ 

Now, $f$ is continuous and the above integral is zero, so there exists a $z_0 \in F$ s.t. $f(n_{z_0}g) = 0$. Using $n_{z_0}g = g k_0^{-1} n_{e^{-t}z_0k_0} = g \exp(e^{-t}X_{z_0,k_0})$, where

$$X_{z,k} := \text{Re}(z) \text{Ad}_{k^-1}(E_+) + \text{Im}(z) \text{Ad}_{k^-1}(K_+) \quad z \in \mathbb{C}, \ k \in K,$$

we have

$$f(g) = f(g) - f(n_{z_0}g) = - \int_0^{e^{-t}} \frac{d}{dr} f(g \exp(rX_{z_0,k_0})) \, dr$$

$$= - \int_0^{e^{-t}} (X_{z_0,k_0} f)(n_{e^t z_0}g) \, dr.$$ 

Hence, using Lemma [11] and $\mathcal{Y}_\Gamma(n_{e^t z_0}g) \equiv \mathcal{Y}_\Gamma(g) = e^t$,

$$|f(g)| \ll \sup_{z \in F, k \in K} \|X_{z,k} f\|_{W^4} \ll \|f\|_{W^5},$$

were the last bound holds since $F$ and $K$ are compact. \hfill $\square$

Remark 2. By using higher order derivatives, one may improve the bound in Lemma [12] to $\mathcal{Y}_\Gamma(p)^{-\alpha}$ for any $\alpha \geq 0$, provided one can use a Sobolev norm of sufficiently high order (cf. eg. [8 Lemma 10]); however we won’t need this.

For non-cuspidal elements of the $C^{s_m}$, we need a stronger pointwise bound than Lemma [11] provides. By studying the $K$-type decomposition of $C^{s_m}$, we are able to get the following bound:

Lemma 13. Let $s \in (1,2)$. For functions in $C^s \cap W^5(\Gamma\backslash G)$, we have

$$|f(p)| \ll_{\Gamma,s} \mathcal{Y}_\Gamma(p)^{2-s} \|f\|_{W^5}.$$ 

Proof. As in [22] Sections 4.1-4.4 and [14] Sections 2.1-2.3, we can decompose $C^s$ into irreducible $K$ (= SU(2)) representations $V_l$, each with multiplicity one in $C^s$. For each $l$ ($l$ is a non-negative integer), $V_l$ is a $2l + 1$-dimensional $K$-invariant vector space. There then exists an orthonormal basis of $C^s$ consisting of smooth functions $\varphi_{l,j}$, $-l \leq j \leq l$, aligned with the $K$-type decomposition (i.e. $\varphi_{l,j} \in V_l$) satisfying

$$J\varphi_{l,j} = ij\varphi_{l,j},$$

and

$$\Omega_K \varphi_{l,j} = l(l+1)\varphi_{l,j},$$
where $\Omega_K = -J^2 - \frac{1}{4}((E_+ - E_-)^2 + (K_+ + K_-)^2)$ (cf. Section 4.4). For $k \in K$, let $A_l(k)$ be the unitary $(2l + 1) \times (2l + 1)$ matrix defined by

$$A_l(k) = (a_{l,m,n}(k))_{-l \leq m,n \leq l} \quad \text{with} \quad a_{l,m,n}(k) = \langle \pi(k) \varphi_{l,j}, \varphi_{l,m} \rangle.$$

Note that

$$\varphi_{l,j}(pk) = \sum_{m=-l}^{l} a_{l,j,m}(k) \varphi_{l,m}(p).$$

As in the proof of Lemma 12, we may assume $\Gamma$ has a cusp at infinity, normalized so that $\Lambda = \{ z \in \mathbb{C} : n_z \in \Gamma \}$ is a unimodular lattice in $\mathbb{C}$, and restrict our attention to points lying in the cuspidal region at infinity (that is to say points $p = \Gamma g$, with $g = na_l k$ and $\mathcal{V}_g = e^t$).

Let $F$ be a fundamental parallelogram in $\mathbb{C}$ for $\Lambda$, and define

$$\phi_{l,j}(g) = \int_F \psi_{l,j}(n z g) dm(z).$$

By abusing notation slightly, let $\phi_{l,j}(t) := \phi_{l,j}(a_t)$. We note that $\phi_{l,j}$ is left $N$-invariant, so

$$\Omega_1 \phi_{l,j}(t) = (H^2 - J^2 - 2H + E_+ E_- - K_+ K_-) \phi_{l,j}(t) = (H^2 - J^2 - 2H) \phi_{l,j}(t),$$

and

$$\Omega_2 \phi_{l,j}(t) = (2HJ - 2J + E_+ K_- + K_+ E_-) \phi_{l,j}(t) = (2HJ - 2J) \phi_{l,j}(t).$$

In particular, since $\Omega_1 \phi_{l,j} = s(s - 2) \phi_{l,j}$ and $\Omega_2 \phi_{l,j} = 0$, we get that

$$\phi''_{l,j}(t) + j^2 \phi_{l,j}(t) - 2 \phi_{l,j}(t) = s(s - 2) \phi_{l,j}(t)$$

and

$$2ij \phi'_{l,j}(t) = 2ij \phi_{l,j}(t) = 0.$$  

Assuming $j \neq 0$, solving (13) gives $\phi_{l,j}(t) = A_{l,j} e^{t}$, for some $A_{l,j} \in \mathbb{C}$. Substituting this into (12) and noting that $j^2 - (s - 1)^2 \neq 0$ (since $j^2 \geq 1$, and $(s - 1)^2 \in (0, 1)$) gives $A_{l,j} = 0$ (i.e. for $j \neq 0$, $\phi_{l,j}(t)$ is identically zero). When $j = 0$, we solve (12) (13) gives no information), giving

$$\phi_{l,0}(t) = A_{l,0} e^{(2-s)t} + B_{l,0} e^{st},$$

where $A_{l,0}, B_{l,0} \in \mathbb{C}$. We now wish to prove that $B_{l,0} = 0$. Note that

$$2l + 1 = \sum_{j=-l}^{l} \| \varphi_{l,j} \|^2 = \sum_{j=-l}^{l} \int_{G \setminus G} |\varphi_{l,j}(g)|^2 d\mu_G(g)$$

$$\geq \sum_{j=-l}^{l} \int_F \int_{2}^{\infty} \int_{K} \left| \varphi_{l,j}(n z a_l k) \right|^2 e^{-2t} dk dt dm(z)$$

$$\geq \sum_{j=-l}^{l} \int_2^{\infty} e^{-2t} \left( \int_K \left| \varphi_{l,j}(n z a_l k) \right|^2 dm(z) \right) dk dt$$

$$= \sum_{j=-l}^{l} \int_2^{\infty} e^{-2t} \left( \sum_{m=-l}^{l} a_{l,j,m}(k) \phi_{l,m}(t) \right)^2 dk dt$$

$$= \int_2^{\infty} |\phi_{l,0}(t)|^2 e^{-2t} \left( \sum_{j=-l}^{l} |a_{l,j,0}(k)|^2 \right) dk dt$$

$$= \int_2^{\infty} |\phi_{l,0}(t)|^2 e^{-2t} dt.$$
We now apply the operator \( L \) decomposes as a direct integral over the tempered unitary dual of \( G \), giving

\[
\int_{F} f(n_{z}g) \, dm(z) = f(n_{z_{0}} g).
\]

We then have

\[
\sum_{l \geq 0} \sum_{|j| \leq l} d_{l,j} a_{l,j,0}(k) A_{l,0} \mathcal{X}_{T}(g)^{2-s} = f(g) + \int_{0}^{e^{-2t}} X_{w_{0},k} f(n_{e^{2t} w_{0}} g) \, dr,
\]

with \( X_{w_{0},k} \) defined as in the proof of Lemma 12. As before, we can bound the integral in the right-hand side of (44) by \( \|f\|_{W^{5}} \), giving

\[
|f(g)| \ll \|f\|_{W^{5}} + \mathcal{X}_{T}(g)^{2-s} \left| \sum_{l \geq 0} \sum_{|j| \leq l} d_{l,j} a_{l,j,0}(k) A_{l,0} \right|.
\]

Now, since \( f \) is sufficiently smooth, we may apply \( \Omega_{K}^{2} \) termwise in its \( \varphi_{l,j} \) decomposition, hence

\[
\|f\|_{W^{5}}^{2} \geq \|\Omega_{K}^{2} f\|^{2} = \sum_{l \geq 0} \sum_{|j| \leq l} |d_{l,j}|^{2} (l(l+1))^{4}.
\]

After recalling that \( |A_{l,0}| \ll \sqrt{2l+1} \), we may use the Cauchy-Schwartz inequality to bound the sum in the right-hand side of (45) by \( \|f\|_{W^{4}} \), giving

\[
|f(g)| \ll (1 + \mathcal{X}_{T}(g)^{2-s}) \|f\|_{W^{5}} \ll \mathcal{X}_{T}(g)^{2-s} \|f\|_{W^{5}}.
\]

\[\square\]

6. PROOF OF THEOREM 1

6.1. Proof of Theorem 1 Since \( \|\cdot\|_{W^{2}} \) is a stronger norm than both \( \|\cdot\|_{W^{4}} \) and \( \|\cdot\|_{L^{2}(\Gamma \backslash G)} \) on \( W^{2}(\Gamma \backslash G) \), Lemma 11 and the Cauchy-Schwartz inequality imply that for fixed \( p \) and \( T \), \( f \mapsto \frac{1}{\mu_{N}(B)} \int_{B} f(p n_{a-T} \sigma) \, d\mu_{N}(n) \) is a bounded linear functional on \( W^{2}(\Gamma \backslash G) \). By the density of \( L^{2}(\Gamma \backslash G)^{\infty} \) in \( W^{2}(\Gamma \backslash G) \), it is therefore sufficient to consider \( f \in L^{2}(\Gamma \backslash G)^{\infty} \). From the discussion in Section 5.1, we decompose \( (\rho, L^{2}(\Gamma \backslash G)) \) as the following orthogonal sum of representations

\[
(\rho, L^{2}(\Gamma \backslash G)) = (\rho, C \varphi_{0}) \oplus C^{s_{1}} \oplus C^{s_{2}} \oplus \ldots C^{s_{M}} \oplus (\rho, L^{2}(\Gamma \backslash G)_{\text{temp}}),
\]

where \( \varphi_{0} \equiv 1 \), the \( C^{s_{m}}, 1 \leq m \leq M \), are defined as in Section 5.1 and \( (\rho, L^{2}(\Gamma \backslash G)_{\text{temp}}) \) decomposes as a direct integral over the tempered unitary dual of \( G \). At the level of elements of \( L^{2}(\Gamma \backslash G) \), we write this decomposition as

\[
f = \int_{\Gamma \backslash G} f \, d\mu + \sum_{m=1}^{M} f_{m} + f_{\text{temp}}.
\]

We now apply the operator \( \frac{1}{\mu_{N}(B)} \int_{B} \rho \left( n_{a-T} \sigma \right) \, d\mu_{N}(n) \) to the previous equation, giving

\[
\frac{1}{\mu_{N}(B)} \int_{B} \left( n_{a-T} \sigma \right) f \, d\mu_{N}(n) = \int_{\Gamma \backslash G} f \, d\mu + \sum_{m=1}^{M} \frac{1}{\mu_{N}(B)} \int_{B} \rho \left( n_{a-T} \sigma \right) f_{m} \, d\mu_{N}(n)
\]

\[
+ \frac{1}{\mu_{N}(B)} \int_{B} \rho \left( n_{a-T} \sigma \right) f_{\text{temp}} \, d\mu_{N}(n).
\]
We will use Proposition 4 on the various summands of (46) (recall that we have assumed that \( \|f\|_{W^7} < \infty \)). Since \( \mathcal{C}^{s_m} \cong \mathcal{P}^{(0,2s_m-2)} \), we may apply the proposition directly to each \( f_m \), giving

\[
(47) \quad \frac{1}{\mu_N(B)} \int_B \rho(na_{-T}) f_m d\mu_N(n) = \int_{-T}^0 F(T, t) I_{f_m}(V_1, V_2, t) \, dt \\
+ \sum_{i=0}^2 \frac{F_i(T)}{\mu_N(B)} \int_B \rho(n)(H^i f_m) d\mu_N(n).
\]

By combining Lemma 13 and Corollary 10 (also applying Lemma 5 c) to accommodate for the fact that we require \( 0 \in B' \) in Proposition 8 but not in Theorem 5 with the definition of \( I_{f_m} \), (49), we get, for \( t \leq 0 \),

\[
|I_{f_m}(V_1, V_2, t)(p)| \ll \|f_m\|_{W^7} e^t \left( |t| + \mathcal{Y}_T(p)^{2-s_m} \right).
\]

Similarly, by combining Lemma 13 with Lemma 5 c, we get, for \( 0 \leq i \leq 2, \forall n \in B \),

\[
|(H^i f_m)(pn)| \ll \|f_m\|_{W^7} \mathcal{Y}_T(p)^{2-s_m}.
\]

These two bounds, combined with the bounds in Proposition 4(iii) and Proposition 13, allow the evaluation of both sides of (47) at \( p \), and give

\[
\left| \frac{1}{\mu_N(B)} \int_B f_m (pma_{-T}) d\mu_N(n) \right| \ll \|f_m\|_{W^7} \left( 2s_m - 2 \right)^{-2} \left\{ e^{(s_m-2)T} \mathcal{Y}_T(p)^{2-s_m} + \int_{-T}^0 e^{(s_m-2)(T+t)} e^t \left( |t| + \mathcal{Y}_T(p)^{2-s_m} \right) \, dt \right\},
\]

so

\[
(48) \quad \sum_{m=1}^M \left| \frac{1}{\mu_N(B)} \int_B f_m (pma_{-T}) d\mu_N(n) \right| \ll \|f\|_{W^7} \sum_{m=1}^M e^{(s_m-2)T} \mathcal{Y}_T(p)^{2-s_m}.
\]

We now wish to use the same method for \( f_{\text{temp}} \). The fact that \( f_{\text{temp}} \) is not necessarily contained in a single irreducible representation complicates matters, and we will need to use the intertwining operators discussed in Section 2.3. Assume that we have the direct integral decomposition

\[
(\rho, L^2(\Gamma \backslash G)_{\text{temp}}) \cong \left( \int_Z^{\oplus} \pi_\zeta \, dv(\zeta), \int_Z^{\oplus} H_\zeta \, dv(\zeta) \right),
\]

where each \((\pi_\zeta, H_\zeta)\) is isomorphic to an element of the tempered unitary dual, and write \( f_{\text{temp}} = \int_Z f_\zeta \, dv(\zeta) \). We partition \( Z \) into two parts: \( Z_0 = \{ \zeta \in Z : (\pi_\zeta, H_\zeta) \cong \mathcal{P}^{(0,\nu)}, \nu \in i\mathbb{R} \} \), and \( Z_1 = \{ \zeta \in Z : (\pi_\zeta, H_\zeta) \cong \mathcal{P}^{(n,\nu)}, \zeta > 0 \} \), and let \( f_{\text{temp,0}} = \int_{Z_0} f_\zeta \, dv(\zeta), f_{\text{temp,1}} = \int_{Z_1} f_\zeta \, dv(\zeta) \). We then have

\[
\left( \frac{1}{\mu_N(B)} \int_B \rho(na_{-T}) f_{\text{temp}} d\mu_N(n) \right) = \int_{Z_0} \left( \frac{1}{\mu_N(B)} \int_B \rho(na_{-T}) f_\zeta d\mu_N(n) \, dv(\zeta) \right) \\
+ \int_{Z_1} \left( \frac{1}{\mu_N(B)} \int_B \rho(na_{-T}) f_\zeta d\mu_N(n) \, dv(\zeta) \right)
\]

We now use Proposition 4, firstly on \( f_{\text{temp,0}} \), which gives

\[
\int_{Z_0} \left( \frac{1}{\mu_N(B)} \int_B \rho(na_{-T}) f_\zeta d\mu_N(n) \, dv(\zeta) \right) = \int_{Z_0} \int_{-T}^0 F_\zeta(T, t) I_{f_\zeta}(V_1, V_2, t) \, dt \, dv(\zeta) \\
+ \sum_{i=0}^2 \int_{Z_0} \frac{E_\zeta(T)}{\mu_N(B)} \int_B \rho(n)(H^i f_\zeta) d\mu_N(n) \, dv(\zeta),
\]
where we use the notation \( F^{\zeta}, F^{\zeta}_0, F^{\zeta}_1 \) and \( F^{\zeta}_2 \) to keep track of which \((\pi_\zeta, \mathcal{H}_\zeta)\) the scalar functions come from. These functions then define, for each \( T, t, \) intertwining operators \( Q(T, t), Q_0(T), Q_1(T), \) and \( Q_2(T) \) as in (49). The bounds in Proposition 3 (ii) give
\[
\|Q_0(T) f_{\text{temp}, 0}\| \ll e^{-T(t+1)}(T + t)^2\|f_{\text{temp}, 0}\|, \quad \|Q_i(T) f_{\text{temp}, 0}\| \ll e^{-T}(1 + T^2)\|f_{\text{temp}, 0}\|, \quad i = 1, 2,
\]
and similarly for any Sobolev norm \( \| \cdot \|_{W^m} \). For \( Q_0(T) \) we have, after letting \( \nu_\zeta \) denote the \( \nu \) parameter of the principal series representation isomorphic to \((\pi_\zeta, \mathcal{H}_\zeta)\),
\[
\|Q_0(T) f_{\text{temp}, 0}\| = \left( \int_{Z_0} |F^{\zeta}_0(T)|^2 \|f_\zeta\|_{\mathcal{H}_\zeta}^2 \, d\nu(\zeta) \right)^{1/2} \leq e^{-T(1 + T^2)} \left( \int_{Z_0} (1 + |\nu_\zeta|^2)^2 \|f_\zeta\|_{\mathcal{H}_\zeta}^2 \, d\nu(\zeta) \right)^{1/2}
\]
These operators intertwine with the action of \( G \), giving
\[
\frac{1}{\mu_N(B)} \int_B \rho(na_{-T}) f_{\text{temp}, 0} \, d\mu_N(n) = \int_{-T}^0 I_{Q(T, t) f_{\text{temp}, 0}}(V_1, V_2, t) \, dt + 2 \sum_{i=0}^2 \frac{1}{\mu_N(B)} \int_B \rho(n)(Q_i(T) H^i f_{\text{temp}, 0}) \, d\mu_N(n).
\]
The previously discussed bounds, combined with Lemma 11, Corollary 10 and Lemma 5 c) again allow evaluation at \( p \), and give
\[
|I_{Q(T, t) f_{\text{temp}, 0}}(V_1, V_2, t)(p)| \ll \|f_{\text{temp}, 0}\|_{W^6} e^{-(T+t)}(T + t)^2 e^t (|t| + \mathcal{Y}_1(p)),
\]
and \( \forall n \in B \),
\[
\|Q_i(T) H^i f_{\text{temp}, 0}(pmn)\| \ll \|f_{\text{temp}, 0}\|_{W^6} e^{-T}(1 + T^2)\mathcal{Y}_1(p) \quad i = 0, 1, 2.
\]
Combining these bounds gives
\[
\left| \frac{1}{\mu_N(B)} \int_B f_{\text{temp}, 0} \, (pma_{-T}) \, d\mu_N(n) \right| \ll \|f_{\text{temp}, 0}\|_{W^6} \left\{ e^{-T(1 + T^2)}\mathcal{Y}_1(p) + e^{-T}\int_{-T}^0 (T + t)^2 (|t| + \mathcal{Y}_1(p)) \, dt \right\},
\]
so
\[
\|Q_i(T) H^i f_{\text{temp}, 0}(pmn)\| \ll \|f_{\text{temp}, 0}\|_{W^6} \left\{ e^{-T(1 + T^3)}\mathcal{Y}_1(p) + e^{-T} T^4 \right\}.
\]
We proceed in the same manner for \( f_{\text{temp}, 1} \); Proposition 4 (i) is used, and we define intertwining operators \( Q, Q_0, Q_1 \) and \( Q_2 \) w.r.t. the functions \( F, F_0, F_1, F_2 \) to get
\[
\frac{1}{\mu_N(B)} \int_B \rho(na_{-T}) f_{\text{temp}, 1} \, d\mu_N(n) = \int_{-\infty}^0 I_{Q(T, t) f_{\text{temp}, 1}}(U_1, U_2, t) \, dt + 2 \sum_{i=0}^2 \frac{1}{\mu_N(B)} \int_B \rho(n)(Q_i(T) H^i f_{\text{temp}, 1}) \, d\mu_N(n).
\]
Here \( \|Q_2(T) H^2 f_{\text{temp}, 1}\| \ll e^{-T}(1 + T^2)\|f_{\text{temp}, 1}\|_{W^2} \), and, in a similar manner to (49), we get
\[
\|Q_i(T) H^i f_{\text{temp}, 1}\| \ll e^{-T}(1 + T)\|f_{\text{temp}, 1}\|_{W^{2+i}}, \quad \forall n \in B
\]
for \( i = 0, 1, 2 \). As before, we now use Lemmas 11 and 5 b) to get, for \( i = 0, 1, 2, \forall n \in B \),
\[
\|Q_i(T) H^i f_{\text{temp}, 1}(pm)\| \ll \|f_{\text{temp}, 1}\|_{W^{7+i}} e^{-T}(1 + T)\mathcal{Y}_1(p).
\]
Lemma 11, Corollary 10, Lemma 5 c), and Proposition 4 (i) give

\[ |I_{Q(T,t)f_{temp,1}}(U_1,U_2,t)(p)| \ll \|f_{temp,1}\|_{W^7} e^t (|t| + \mathcal{Y}_T(p)) \begin{cases} e^{-\frac{1}{2}(T+t)} & \text{if } t \leq -T \\ e^{-(T+t)}(1 + t + T) & \text{if } t \geq -T \end{cases} \]

So

\[
\left| \frac{1}{\mu_N(B)} \int_B f_{temp,1}(p) \, d\mu_N(n) \right| \ll \left( \frac{1}{\mu_N(B)} \right) \int_B f_{temp,1}(p) \, d\mu_N(n) \]

\[ \ll \|f\|_{W^7} \{ e^{-T}(1 + T^2)\mathcal{Y}_T(p) + e^{-T}T^3 \}. \]

Evaluating both sides of (46) at \( p \), and using the bounds (48), (50), and (51) gives

\[
\left| \frac{1}{\mu_N(B)} \int_B f(p) \, d\mu_N(n) - \int_{\Gamma \setminus G} f \, d\mu \right| \ll \|f\|_{W^7} \{ e^{-T}g_1(p)^2 + e^{-T}(1 + T^3)\mathcal{Y}_T(p) + e^{-T}T^4 \}. \]

\[ \square \]

6.2. The dependency on \( B' \). We now make explicit \( C(\Gamma, B') \)'s dependency on the set \( B' \) in Theorem 1 which gives our most exact result:

**Theorem 1'**. Let \( B' \) be a connected compact subset of \( \mathbb{C} \) such that there exists a piecewise smooth parametrization \( \gamma : S^1 \to \mathbb{C} \) of \( \partial B' \) with the following properties: i) for all \( t \in S^1 \) where \( \gamma'(t) \) exists, \( |\gamma'(t)| = L \), the arc length of \( \partial B' \), and ii) there exists \( c > 0 \) s.t. \( \forall t_1, t_2 \in S^1, |\gamma(t_1) - \gamma(t_2)| \geq c|t_1 - t_2| \). Assume that \( \text{diam}(B') = R \), and let \( z_0 \) be a point in \( B' \) with minimal distance to zero. Then the conclusion of Theorem 1 holds with

\[ C(\Gamma, B') \ll_L \frac{L^2(1 + R^2)(1 + |z_0|^2)(1 + e^{-1})}{m(B')} \]

**Proof**. Going through the proof of Theorem [1] we find that there are two types of bounds which implicitly depend on \( B' \). The first of these are bounds of the type: for \( \varphi \in W^m(\Gamma \setminus G) \),

\[
\left| \frac{1}{\mu_N(B)} \int_B \varphi(p) \, d\mu \right| \ll \|\varphi\|_{W^m} \mathcal{Y}_T(p)^\alpha,
\]

where \( \alpha \in (0, 1] \). By Lemma 5 c), and either Lemma 11 or Lemma 13 we have

\[ \left| \frac{1}{\mu_N(B)} \int_B \varphi(p) \, d\mu \right| \ll \left( 1 + (|z_0| + R)^2 \right) \|\varphi\|_{W^m} \mathcal{Y}_T(p)^\alpha. \]

The second type of bound where we previously neglected to explicitly write out the dependency on \( B' \) are bounds of the form

\[ |I_{\varphi}(X,Y,t)(p)| \ll \|\varphi\|_{W^m} e^{t} (|t| + \mathcal{Y}_T(p)^\alpha). \]

From (14), we have

\[ |I_{\varphi}(X,Y,t)(p)| \leq \frac{e^t}{m(B')} \int_{\partial B'} \int_{\partial B'} (Y\varphi)(p) \, dx + (X\varphi)(p) \, dy \]

\[ \ll \frac{e^t}{m(B')} \int_{\partial B'} \mathcal{Y}_T(p) \, dz, \]
where \( m \) and \( \alpha \) are chosen according to either Lemma 11 or Lemma 13. We now apply Corollary 10 giving

\[
|I_\varphi(X,Y,t)(p)| \ll_{\Gamma_0} \frac{e^t\|\varphi\|_{W^m}}{m(B')} \cdot \frac{L^2(1 + R^2)}{c} (|t| + \mathcal{Y}(gn_{z_0}))^\alpha.
\]

Once again, we use Lemma 5 c) to get

\[
I_\varphi(X,Y,t)(p)| \ll_{\Gamma_0} \frac{L^2(1 + R^2)(1 + |z_0|^2)}{m(B')c} \|\varphi\|_{W^m} e^t (|t| + \mathcal{Y}(p))^{\alpha}.
\]

Combining (52) and (53) gives the stated bound on \( C \).

We conclude by proving a generalisation of a result stated (though not proved) on [21, pg. 228] on the equidistribution of translates of rectangular pieces of horospheres. Note that while [21] requires the horosphere in question to be closed, this assumption is not needed in the following:

**Corollary 14.** Let \( \omega_1, \omega_2 \) be a basis for \( \mathbb{C} \) over \( \mathbb{R} \) such that \( |\omega_1| = |\omega_2| = 1 \). For \( e^{-T} \leq \delta_1 \leq \delta_2 \),

\[
\left\| \frac{1}{\delta_1 \delta_2} \int_0^\delta_2 \int_0^{\delta_1} f(pm_{\omega_1x + \omega_2y}a_{-T}) \, dx \, dy - \int_{\Gamma \setminus G} f \, d\mu \right\|_{W^7} \leq \mathcal{Y}(p)e^{-T}(1 + \delta_1^2)(1 + \delta_2^{-2}).
\]

**Proof.** Letting \( q = \lceil \frac{eT}{\delta_1} \rceil - 1 \geq 0 \), decomposing the integral as \( \int_{j=0}^{\delta_2} \int_{j=0}^{\delta_1} f(pm_{\omega_1x + \omega_2y}a_{-T}) \, dx \, dy - \int_{\Gamma \setminus G} f \, d\mu \), and defining \( p_j = pm_{\delta_1\omega_1} \) gives

\[
\left| \frac{1}{\delta_1 \delta_2} \int_0^\delta_2 \int_0^{\delta_1} f(pm_{\omega_1x + \omega_2y}a_{-T}) \, dx \, dy - \int_{\Gamma \setminus G} f \, d\mu \right| \leq \left| \frac{\delta_1 \delta_2 - q\delta_1}{\delta_1 \delta_2} \right| + \left| \frac{\delta_1 \delta_2 - q\delta_1}{\delta_1 \delta_2} \right| + \left| \frac{\delta_1 \delta_2 - q\delta_1}{\delta_1 \delta_2} \right| + \left| \frac{\delta_1 \delta_2 - q\delta_1}{\delta_1 \delta_2} \right|.
\]

Note here that \( f(pm_{\omega_1x + \omega_2y}a_{-T}) = p_j'm_{\omega_1x + \omega_2y}a_{-(T+\log \delta_1)} \), where \( p_j' := p_j m_{\delta_1} \). Set \( C_T = \max_{(u,v) \in [1/2]} C_{\Gamma, \mathcal{B}_{u,v}} \), where \( \mathcal{B}_{u,v} \) is the parallelogram spanned by \( u\omega_1 \) and \( v\omega_2 \). By Theorem 7, \( C_T \) is finite, and the above expression is

\[
\leq C_T \mathcal{Y}(p) e^{-T} \left\{ \frac{e^{-T}}{\delta_1} \left( (1 - q\delta_1^2)\mathcal{Y}(p'_j) + \frac{\delta_1 \delta_2 - q\delta_1}{\delta_1 \delta_2} \mathcal{Y}(p'_j) + \frac{\delta_1 \delta_2 - q\delta_1}{\delta_1 \delta_2} \right) \right\}.
\]
By Lemma 5 b) and c), $\mathcal{Y}(p_j') \leq \mathcal{Y}(p)(1 + j^2 \delta_1^2) \max\{\delta_1, \delta_1^{-1}\}$, so for $\alpha \in [0, 1]$:  
\[
\left(\frac{e^{-T}}{\delta_1}\right)^{\alpha} \left((1-q\delta_1\delta_2^{-1})\mathcal{Y}(p_j')^{\alpha} + \frac{\delta_1}{\delta_2} \sum_{j=0}^{q-1} \mathcal{Y}(p_j')^{\alpha}\right) \leq \left(\frac{\mathcal{Y}(p)e^{-T}(1 + \delta_2^2)}{\delta_1 \min\{\delta_1, \delta_1^{-1}\}}\right)^{\alpha},
\]
which gives the desired result. \hfill \Box

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