Ultraviolet asymptotics of glueball propagators

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Abstract: We point out that perturbation theory in conjunction with the renormalization group (RG) puts a severe constraint on the structure of the large-$N$ non-perturbative glueball propagators in $SU(N)$ pure YM, in QCD and in $\mathcal{N} = 1$ SUSY QCD with massless quarks, or in any confining asymptotically-free gauge theory massless in perturbation theory. For the scalar and pseudoscalar glueball propagators in pure YM and QCD with massless quarks we check in detail the RG-improved estimate to the order of the leading and next-to-leading logarithms by means of a remarkable three-loop computation by Chetyrkin et al. We investigate as to whether the aforementioned constraint is satisfied by any of the scalar or pseudoscalar glueball propagators computed in the framework of the AdS String/ large-$N$ Gauge Theory correspondence and of a recent proposal based on a Topological Field Theory underlying the large-$N$ limit of YM. We find that none of the proposals for the scalar or the pseudoscalar glueball propagators based on the AdS String/ large-$N$ Gauge Theory correspondence satisfies the constraint, actually as expected, since the gravity side of the correspondence is in fact strongly coupled in the ultraviolet. On the contrary, the Topological Field Theory satisfies the constraint that follows by the asymptotic freedom.
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1 Introduction and Conclusions

1.1 Introduction

In the last years several proposals for the non-perturbative glueball propagators of QCD-like confining asymptotically-free gauge theories have been advanced, based on the AdS String/ large-N Gauge Theory correspondence \cite{1} and more recently on a Topological Field Theory (TFT) underlying the large-N limit of the pure Yang-Mills (YM) theory \cite{2} \cite{3} \cite{4} \cite{5} \cite{6}.

Above all these proposal aim to elucidate, at least in the large-N limit, the most fundamental feature of the infrared of large-N QCD-like confining asymptotically-free gauge theories, i.e. the existence of a mass gap in the pure glue sector, as opposed to the massless spectrum of gluons in perturbation theory.

However, these proposals predict a variety of spectra for large-N QCD different among themselves, asymptotically quadratic for large masses \cite{1} \cite{7} \cite{8} \cite{9} \cite{10} or exactly linear \cite{11} \cite{3} \cite{4} \cite{5} \cite{1} in the square of the glueball masses and in general do not agree about the qualitative and quantitative details of the low-energy spectrum but for the existence of the mass gap.

In view of the importance of the problem that these proposals aim to answer and in order to discriminate between the various proposals it is worth investigating whether there is any constraint that we know by the fundamental principles of any confining asymptotically-free gauge theory that any supposed answer for the non-perturbative glueball propagators has to satisfy.

In fact, we do know with certainty the implications of the asymptotic freedom for the large-momentum asymptotic behavior of any gauge invariant correlation function.

In this paper we do not discuss at all the theoretical justification of the various proposal that we examine, leaving it to the original papers. We limit ourselves to check whether or not the constraint that follows by the asymptotic freedom and by the renormalization group in the ultraviolet (UV) is satisfied by any given proposal. Indeed, the importance of this constraint has been pointed out since the early days of large-N QCD \cite{12}, see also \cite{13}.

In fact the purpose of this paper is threefold.

1.2 Implications of the renormalization group and of the asymptotic freedom

Firstly, in sect.(2) we point out that perturbation theory in conjunction with the renormalization group (RG) severely constraints the asymptotic behavior of glueball propagators in pure SU(N) Yang-Mills, in QCD and in $\mathcal{N} = 1$ SUSY QCD with massless quarks, or in any confining asymptotically-free gauge theory massless to every order of perturbation theory.

Indeed, we show in this paper, on the basis of RG estimates, that the most fundamental object involved in the problem of the mass gap \footnote{Exact linearity in the TFT refers to the joint large-N spectrum of scalar and pseudoscalar glueballs. The TFT in its present formulation does not contain information about higher spin glueballs.}, the scalar (S) glueball propagator in any (confining) asymptotically-free gauge theory with no perturbative physical mass scale, up to

\footnote{The lightest glueball is believed to be a scalar in pure YM and in the ’t Hooft large-N limit of QCD.}
unphysical contact terms, i.e. distributions supported at coinciding points, has the following
universal, i.e. renormalization-scheme independent, large-momentum asymptotic behavior:
\[
\int \langle \frac{\beta(g)}{g_N} tr \left( \sum_{\alpha\beta} F_{\alpha\beta}^2(x) \right) \frac{\beta(g)}{g_N} tr \left( \sum_{\alpha\beta} F_{\alpha\beta}^2(0) \right) \rangle_{\text{conn}} e^{ip \cdot x} d^4 x
\]
\[= C_{Sp^4} \left[ \frac{1}{\beta_0} \log \frac{p^2}{\Lambda_{\text{MS}}^2} \left( 1 - \frac{\beta_1}{\beta_0} \log \log \frac{p^2}{\Lambda_{\text{MS}}^2} \right) + O \left( \frac{1}{\log^2 \frac{p^2}{\Lambda_{\text{MS}}^2}} \right) \right] \tag{1.1}
\]
Analogously for the pseudoscalar (PS) propagator:
\[
\int \langle \frac{g^2}{N} tr \left( \sum_{\alpha\beta} F_{\alpha\beta} \tilde{F}_{\alpha\beta}(x) \right) \frac{g^2}{N} tr \left( \sum_{\alpha\beta} F_{\alpha\beta} \tilde{F}_{\alpha\beta}(0) \right) \rangle_{\text{conn}} e^{ip \cdot x} d^4 x
\]
\[= C_{PS} p^4 \left[ \frac{1}{\beta_0} \log \frac{p^2}{\Lambda_{\text{MS}}^2} \left( 1 - \frac{\beta_1}{\beta_0} \log \log \frac{p^2}{\Lambda_{\text{MS}}^2} \right) + O \left( \frac{1}{\log^2 \frac{p^2}{\Lambda_{\text{MS}}^2}} \right) \right] \tag{1.2}
\]
and for a certain linear combination as well, the anti-seldual (ASD) propagator:
\[
\frac{1}{2} \int \langle \frac{g^2}{N} tr \left( \sum_{\alpha\beta} F_{\alpha\beta}^{-2} (x) \right) \frac{g^2}{N} tr \left( \sum_{\alpha\beta} F_{\alpha\beta}^{-2}(0) \right) \rangle_{\text{conn}} e^{ip \cdot x} d^4 x
\]
\[= C_{ASD^p} p^4 \left[ \frac{1}{\beta_0} \log \frac{p^2}{\Lambda_{\text{MS}}^2} \left( 1 - \frac{\beta_1}{\beta_0} \log \log \frac{p^2}{\Lambda_{\text{MS}}^2} \right) + O \left( \frac{1}{\log^2 \frac{p^2}{\Lambda_{\text{MS}}^2}} \right) \right] \tag{1.3}
\]
where \(F_{\alpha\beta} = F_{\alpha\beta} - \tilde{F}_{\alpha\beta}\) and \(\tilde{F}_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} F_{\gamma\delta}\).

The explicit dependence on the particular \(\Lambda_{\text{MS}}\) scale in Eq.(1.1)-Eq.(1.3) is illusory.
A change of scheme affects only the \(O \left( \frac{1}{\log^2 \frac{p^2}{\Lambda_{\text{MS}}^2}} \right)\) terms. The coincidence of the asymptotic behavior, up to the overall normalization constants that are computed in sect.(3), \(C_{S}, C_{PS}, C_{ASD}\), is due to the coincidence of the naive dimension in energy, 4, and of the one-loop anomalous dimension, \(\gamma(g) = -2\beta_0 g^2 + \cdots\), of these operators deprived of the factors of \(\beta(g)\) or of \(g^2\). Euclidean signature is always understood in this paper unless otherwise specified.

1.3 Perturbative check of the RG estimates

Secondly, in sect.(3) we check the correctness of our RG estimate on the basis of an explicit very remarkable three-loop computation \(3\) performed by Chetyrkin et al.[14] [15] in pure \(SU(N)\) \(YM\) and in \(SU(3)\) \(QCD\) with \(n_f\) massless Dirac fermions in the fundamental representation. For example, we show that in pure \(SU(N)\) \(YM\) Chetyrkin et al. result [14]
\[3\]The earlier two-loop computation was performed in [16].
[15] can be rewritten by elementary methods as:

\[
\frac{1}{2} \int \left( \frac{g^2}{N} \text{tr} \left( \sum_{\alpha \beta} F^{-2}_{\alpha \beta}(x) \right) \frac{g^2}{N} \text{tr} \left( \sum_{\alpha \beta} F^{-2}_{\alpha \beta}(0) \right) \right)_{\text{conn}} e^{-ip \cdot x} d^4 x
\]

\[
= (1 - \frac{1}{N^2}) \frac{p^4}{2\pi^2 \beta_0} (2g_{\overline{MS}}^2(\frac{p^2}{\Lambda_{\overline{MS}}^2}) - 2g_{\overline{MS}}^2(\frac{\mu^2}{\Lambda_{\overline{MS}}^2}))
\]

\[
+ (a + \tilde{a} - \frac{\beta_1}{\beta_0}) g_{\overline{MS}}^4(\frac{p^2}{\Lambda_{\overline{MS}}^2}) - (a + \tilde{a} - \frac{\beta_1}{\beta_0}) g_{\overline{MS}}^4(\frac{\mu^2}{\Lambda_{\overline{MS}}^2}) + O(g^6)
\]

where \(a\) and \(\tilde{a}\) are two scheme-dependent constants that are defined in sect.(3.5) and \(g_{\overline{MS}}\) is the 't Hooft coupling constant in the \(\overline{MS}\) scheme. In Eq.(1.4) the terms that depend on \(g(\frac{\mu^2}{\Lambda_{\overline{MS}}^2})\) correspond in the coordinate representation to distributions supported at coincident points (contact terms), and therefore they have no physical meaning. Remarkably, the correlator without the contact terms does not in fact depend on the arbitrary scale \(\mu\) (within \(O(g^6)\) accuracy) as it should be. The running coupling constant \(g_{\overline{MS}}^2(\frac{p^2}{\Lambda_{\overline{MS}}^2})\) occurs in Eq.(1.4) with two-loop accuracy and it is given by:

\[
g_{\overline{MS}}^2(\frac{p^2}{\Lambda_{\overline{MS}}^2}) = g_{\overline{MS}}^2(\frac{\mu^2}{\Lambda_{\overline{MS}}^2})(1 - \beta_0 g_{\overline{MS}}^2(\frac{\mu^2}{\Lambda_{\overline{MS}}^2}) \log \frac{p^2}{\mu^2})
\]

\[
- \beta_1 g_{\overline{MS}}^4(\frac{\mu^2}{\Lambda_{\overline{MS}}^2}) \log \frac{p^2}{\mu^2} + \beta_2 g_{\overline{MS}}^4(\frac{\mu^2}{\Lambda_{\overline{MS}}^2}) \log^2 \frac{p^2}{\mu^2} + \cdots
\]

(1.5)

Therefore, the perturbative computation furnishes an expansion of the correlator in powers of \(g_{\overline{MS}}^2(\mu)\) and of logarithms. This expansion has been rearranged by elementary methods in terms of the two-loop running coupling \(g_{\overline{MS}}^2(\frac{p^2}{\Lambda_{\overline{MS}}^2})\) in Eq.(1.4).

At this point our basic strategy to check the RG estimates of sect.(2) consists in substituting in Eq.(1.4) instead of Eq.(1.5) the RG-improved expression for \(g_{\overline{MS}}^2(\frac{p^2}{\Lambda_{\overline{MS}}^2})\) given by:

\[
g_{\overline{MS}}^2(\frac{\mu^2}{\Lambda_{\overline{MS}}^2}) = \frac{1}{\beta_0 \log \frac{p^2}{\Lambda_{\overline{MS}}^2}} \left[ 1 - \frac{\beta_1}{\beta_0} \log \frac{p^2}{\log \frac{p^2}{\Lambda_{\overline{MS}}^2}} \right] + O\left( \frac{\log^3 \frac{p^2}{\Lambda_{\overline{MS}}^2}}{\log \frac{p^2}{\Lambda_{\overline{MS}}^2}} \right)
\]

(1.6)

The \(\overline{MS}\) scheme is indeed defined [17] in such a way to cancel the term of order of \(\frac{1}{\log^2 \frac{p^2}{\Lambda_{\overline{MS}}^2}}\) that would occur in Eq.(1.6) in other schemes. By subtracting the unphysical contact terms and by substituting the RG-improved two-loop asymptotic expression for \(g_{\overline{MS}}^2(\frac{p^2}{\Lambda_{\overline{MS}}^2})\) it follows the actual large-momentum scheme-independent asymptotic behavior of Eq.(1.4):

\[
\frac{1}{2} \int \left( \frac{g^2}{N} \text{tr} \left( \sum_{\alpha \beta} F^{-2}_{\alpha \beta}(x) \right) \frac{g^2}{N} \text{tr} \left( \sum_{\alpha \beta} F^{-2}_{\alpha \beta}(0) \right) \right)_{\text{conn}} e^{-ip \cdot x} d^4 x
\]

\[
= (1 - \frac{1}{N^2}) \frac{p^4}{2\pi^2 \beta_0} \left[ \frac{1}{\beta_0 \log \frac{p^2}{\Lambda_{\overline{MS}}^2}} \left( 1 - \frac{\beta_1}{\beta_0} \log \frac{p^2}{\log \frac{p^2}{\Lambda_{\overline{MS}}^2}} \right) + O\left( \frac{1}{\log^2 \frac{p^2}{\Lambda_{\overline{MS}}^2}} \right) \right]
\]

(1.7)

as opposed to the perturbative behavior that would follow by Eq.(1.5). The asymptotic result in the other cases is checked similarly.
1.4 AdS/Large-N Gauge Theory correspondence and disagreement with the RG estimates

Thirdly, in this subsection and in the next one, we inquire whether the large-$N$ non-perturbative scalar or pseudoscalar propagators actually computed in the literature agree or disagree with the RG estimate.

We find, to the best of our knowledge, that all the scalar propagators presently computed in the literature in the framework of the AdS String/ large-$N$ Gauge Theory correspondence disagree with the universal asymptotic behavior.

We should mention that the comparison of the asymptotics of the scalar glueball propagators in the AdS approach with YM or with QCD at the lowest non-trivial order of perturbation theory has been already performed in [18] [19] [20] [21], but with somehow different conclusions. The reasons is that in [18] [19] [20] [21] the comparison has been performed only with the one-loop result for the scalar glueball propagator, i.e. only with the first term in Eq.(3.8), that is conformal in the UV. No higher order of perturbation theory and no RG improvement has been taken into account in the comparison, as instead we do in this paper.

Here we enumerate the models based on the AdS/Gauge Theory correspondence for which we could find explicit computations of the scalar glueball propagator in the literature.

In the Hard Wall model (Polchinski-Strassler background [9] in the so called bottom-up approach):

\[ \int \langle \text{tr} F^2(x) \text{tr} F^2(0) \rangle e^{-ip \cdot x} d^4 x \sim p^4 \left[ \frac{K_1(\frac{p}{\mu})}{I_1(\frac{p}{\mu})} - \log \frac{p^2}{\mu^2} \right] \]

where $K_1, I_1$ are the modified Bessel functions [18]. The asymptotic behavior [18] is conformal in the UV:

\[ p^4 \left[ \frac{K_1(\frac{p}{\mu})}{I_1(\frac{p}{\mu})} - \log \frac{p^2}{\mu^2} \right] \sim -p^4 \left[ \log \frac{p^2}{\mu^2} + O(e^{-2\mu/p}) \right] \]

with $p = \sqrt{p^2}$. Indeed, as recalled in appendix A, in the coordinate representation:

\[ -\int p^4 \log \frac{p^2}{\mu^2} e^{ip \cdot x} \frac{d^4 p}{(2\pi)^4} \sim \frac{1}{x^8} \]

and, as observed in [18] [19] [20] [21], it matches the one-loop large-$N$ QCD result for the perturbative glueball propagator displayed in the first term of Eq.(3.8). Nevertheless, it disagrees by a factor of $(\log p)^2$ with the correct asymptotic behavior in Eq.(1.1).

The Soft Wall model (bottom-up approach) [11] implies the same leading conformal asymptotic behavior [18] [19] [20] [21] in the UV for the scalar glueball propagator:

\[ \int \langle \text{tr} F^2(x) \text{tr} F^2(0) \rangle e^{-ip \cdot x} d^4 x \sim -p^4 \left[ \log \frac{p^2}{\mu^2} + O(\frac{\mu^2}{p^2}) \right] \]

that therefore disagrees in the UV by the same factor of $(\log p)^2$.

A more interesting example of the AdS string / large-$N$ Gauge Theory correspondence from the point of view of first principles applies to the cascading $\mathcal{N} = 1$ SUSY YM
theory (top-down approach) [22] [23], because in this case the correct asymptotically-free $β$ function of the cascading theory is exactly reproduced in the supergravity approximation in the Klebanov-Strassler background [22] [23]. Nevertheless, the asymptotic behavior of the scalar correlator is [24] [25]:

$$\langle \text{tr} F^2(x) \text{tr} F^2(0) \rangle \propto e^{-i p \cdot x} d^4 x \sim p^4 \log^3 \frac{p^2}{\mu^2}$$

that disagrees by a factor of $(\log p)^4$ with the correct asymptotic behavior in Eq.(1.1).

### 1.5 Topological Field Theory and agreement with the RG estimates

Finally, in sect.(4) we prove that in the large-$N$ limit of pure $SU(N) YM$ the ASD glueball propagator computed in [5][4][3] [4]:

$$\frac{1}{2} \sum_{\alpha\beta} F^{-2}_{\alpha\beta}(x) \frac{g^2}{N} \text{tr} \left( \sum_{\alpha\beta} F^{-2}_{\alpha\beta}(0) \right) = \frac{1}{\pi^2} \sum_{k=1}^{\infty} \frac{g_k^4 \Lambda^2_W}{p^2 + k \Lambda^2_W}$$

agrees with the universal RG estimate in Eq.(1.7).

Since the proposal for the TFT underlying large-$N$ YM is recent and not widely known we add here a few explanations, but for the purposes of this paper the reader can consider Eq.(1.8) just as a phenomenological model factorizing the ASD glueball propagator on a spectrum linear in the masses squared with certain residues.

Yet, to say it in a nutshell, the rationale behind Eq.(1.8) is as follows. In [2] [4] [6] it is shown that there is a TFT trivial [4] [6] at $N = \infty$ underlying the large-$N$ limit of $YM$. At $N = \infty$ the TFT is localized on critical points [3] [4]. However, at the first non-trivial $1/N$ order the ASD propagator of the TFT arises computing non-trivial fluctuations around the critical points of the TFT [3] [5].

In Eq.(1.8) $F^{-2}_{\alpha\beta}$ is the anti-selfdual part of the curvature $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + i \frac{g}{\sqrt{N}} [A_\alpha, A_\beta]$ with the canonical normalization defined in Eq.(2.8), $\Lambda^2_W$ is the renormalization group invariant scale in the scheme in which it coincides with the mass gap and $g_k = g(\frac{\Lambda^2}{\Lambda^2_W}) = k$ is the ’t Hooft running coupling constant at the scale of the pole (in Minkowski space-time) in the scheme defined in [2], that is recalled in sect.(4). In fact, the analysis of the UV behavior of Eq.(1.8) has already been performed at the order of the leading logarithm occurring in Eq.(1.3) in [5]. Here we go one step further comparing Eq.(1.8) with Eq.(1.3) at the order of the next-to-leading logarithm. Our basic strategy to obtain the large momentum asymptotics of Eq.(1.8) is as follows. We write the RHS of Eq.(1.8) as a sum of physical terms and contact terms according to [5]:

$$\frac{1}{\pi^2} \sum_{k=1}^{\infty} \frac{g_k^4 \Lambda^2_W}{p^2 + k \Lambda^2_W} = \frac{1}{\pi^2} \sum_{k=1}^{\infty} \frac{g_k^4 \Lambda^2_W}{p^2 + k \Lambda^2_W} + \frac{1}{\pi^2} \sum_{k=1}^{\infty} g_k^4 \Lambda^2_W (k \Lambda^2_W - \Lambda^2_W)$$

The first sum contains the physical terms that in Minkowski space-time carry the pole singularities, while the second sum contains the contact terms, that we ignore in the following. We now consider only the physical terms and to find the leading UV behavior we use the Euler-McLaurin formula according to the technique first introduced by Migdal [26] and

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4We use here a manifestly covariant notation as opposed to the one of the TFT employed in [5][4][3].

5We understand that Migdal technique has been known to him for decades.
employed in \cite{5}:

\[ \sum_{k=k_1}^{\infty} G_k(p) = \int_{k_1}^{\infty} G_k(p) \, dk - \sum_{j=1}^{\infty} B_j \left[ \partial_{k_1} G_k(p) \right]_{k=k_1} \quad (1.10) \]

where \( B_j \) are the Bernoulli numbers. In our case the terms proportional to the Bernoulli numbers involve negative powers of \( p \) and they are therefore suppressed with respect to the first term which behaves as the inverse of a logarithm, so that we ignore them as well. We have:

\[
\begin{align*}
\frac{1}{\pi^2} \sum_{k=1}^{\infty} \frac{g_k^4 \Lambda^2 W^2}{p^2 + k \Lambda^2 W^2} & \sim \frac{1}{\pi^2} p^4 \int_{1}^{\infty} \frac{1}{\beta_0 \log^2 \frac{k}{c}} \left( 1 - \frac{2\beta_1 \log \log \frac{k}{c}}{\beta_2 \log \frac{k}{c}} \right) \frac{dk}{k + \frac{p^2 W}{c^2}} \\
& \sim \frac{1}{\pi^2} p^4 \int_{1}^{\infty} \frac{1}{\beta_0 \log^2 \frac{k}{c}} \left( 1 - \frac{2\beta_1 \log \log \frac{k}{c}}{\beta_2 \log \frac{k}{c}} \right) \frac{dk}{k + \frac{p^2 W}{c^2}} 
\end{align*}
\]  

\[ \quad (1.11) \]

where we have used the \( \text{RG} \)-improved asymptotic behavior for large \( k \) of the running coupling constant \( g_k \) at the scale of the \( k \)-th pole, i.e. on shell (in Minkowski space-time):

\[ g_k^2 \sim \frac{1}{\beta_0 \log \frac{k}{c}} \left( 1 - \frac{\beta_1 \log \log \frac{k}{c}}{\beta_2 \log \frac{k}{c}} \right) \quad (1.12) \]

The constant \( c \) is related to the scheme that occurs in the non-perturbative calculation \cite{2}\cite{3}\cite{4}\cite{5}. The actual value of \( c \) is not relevant in this paper since we study only the universal asymptotic behavior. In sect.(4) we compute the universal leading and next-to-leading behavior of the integral in Eq.(1.11) and the result is:

\[
\begin{align*}
\frac{1}{\pi^2} p^4 \int_{1}^{\infty} \frac{1}{\beta_0 \log^2 \frac{k}{c}} \left( 1 - \frac{2\beta_1 \log \log \frac{k}{c}}{\beta_2 \log \frac{k}{c}} \right) \frac{dk}{k + \frac{p^2 W}{c^2}} \\
& = \frac{1}{\pi^2} \beta_0 p^4 \left[ \frac{1}{\beta_0 \log \left( \frac{1}{c} + \frac{p^2 W}{c^2} \right)} - \frac{\beta_1}{\beta_0} \frac{\log \log \left( \frac{1}{c} + \frac{p^2 W}{c^2} \right)}{\log \left( \frac{1}{c} + \frac{p^2 W}{c^2} \right)} \right] + O \left( \frac{1}{\log^2 \frac{p^2 W}{c^2}} \right) \\
& = \frac{1}{\pi^2} \beta_0 p^4 \left[ \frac{1}{\beta_0 \log \left( \frac{p^2 W}{c^2} \right)} \left( 1 - \frac{\beta_1}{\beta_0} \frac{\log \log \left( \frac{p^2 W}{c^2} \right)}{\log \left( \frac{p^2 W}{c^2} \right)} \right) \right] + O \left( \frac{1}{\log^2 \frac{p^2 W}{c^2}} \right) 
\end{align*}
\]  

\[ \quad (1.13) \]

1.6 Conclusions

The preceding result, for the \textit{ASD} glueball propagator computed in the \textit{TFT} underlying large-\( N \) pure \textit{YM}, agrees perfectly in the large-\( N \) limit with the universal part of the renormalization group improved expression of the perturbative result Eq.(1.7).

The agreement is due to the conspiracy between the residues of the poles, that are proportional to the fourth power of the coupling constant renormalized on shell times the fourth power of the glueball mass at the pole, and the exact linearity of the joint scalar and pseudoscalar spectrum of the square of the mass of the glueballs in the \textit{ASD} correlator of the \textit{TFT}.
To the best of our knowledge this is the only non-perturbative result for the scalar or pseudoscalar glueball propagator proposed in the literature that agrees with large-\(N\) \(YM\) perturbation theory and the renormalization group.

While this agreement is not by itself a guarantee of correctness of Eq.(1.8) it deserves further investigations, both at theoretical level and of further checks.

Besides, our analysis shows that the \(AdS/\text{Large-}N\) Gauge Theory correspondence in any of its present strong coupling incarnations, the bottom-up or the top-down approach, for which scalar glueball propagators are available in the literature, does not capture, not even approximatively, the fundamental ultraviolet feature of \(YM\) or of \(QCD\) or of any large-\(N\) confining asymptotically-free gauge theory in the pure glue sector.

While this conclusion is certainly known to some experts (see for just one example [23]), we think that it is not widely recognized that constructing theories that are conformal in the ultraviolet, as the Hard or the Soft Wall models, or even with the correct beta function but in the strong coupling phase, as the Klebanov-Strassler supergravity background, is not at all a good approximation for the correct result in the ultraviolet. In this paper, for the first time with leading and next-to-leading logarithmic accuracy, we have computed quantitatively the measure of the disagreement.

Finally, given the disagreement between the propagators of the \(TFT\) and the propagators of the \(AdS/\text{Large-}N\) Gauge Theory correspondence in the infrared for the first few lower-mass glueballs, a careful critical analysis of the two approaches at level of numerical lattice data is needed, and also at theoretical level of further constraints arising by the \(OPE\) and by the low-energy theorems of Shifman-Vainshtein-Zakharov (\(SVZ\)).

2 Renormalization group estimates on the universal behavior of correlators

2.1 Definitions

The \(SU(N)\) pure \(YM\) theory is defined by the partition function:

\[
Z = \int DA e^{-\frac{1}{2g_{YM}^2} \int tr F^2(x) d^4x}
\]

where we use the simplified notation \(tr F^2(x) = \sum_{\alpha\beta} tr (F_{\alpha\beta}^2)\). Introducing the ’t Hooft coupling constant \(g\) [27]:

\[
g^2 = g_{YM}^2 N
\]

the partition function reads:

\[
Z = \int DA e^{-\frac{N}{2g^2} \int tr F^2(x) d^4x}
\]

According to ’t Hooft [27] the large-\(N\) limit is defined with \(g\) fixed when \(N \rightarrow \infty\).

For the structure of large-\(N\) glueball propagators see [12] and for reviews of the large-\(N\) limit see [13] and [28]. The normalization of the action in Eq.(2.1) corresponds to choosing
the gauge field $A_\alpha$ in the fundamental representation of the Lie algebra, with generators normalized as:

$$tr (t^a t^b) = \frac{1}{2} \delta^{ab}$$

(2.4)

In Eq.(2.1) $F_{\alpha\beta}$ is defined by:

$$F_{\alpha\beta}(x) = \partial_\alpha A_\beta - \partial_\beta A_\alpha + i[A_\alpha, A_\beta]$$

(2.5)

We refer to the normalization of the action in Eq.(2.1) as the Wilsonian normalization. Perturbation theory is formulated with the canonical normalization, obtained rescaling the field $A_\alpha$ in Eq.(2.1) by the coupling constant $g_{YM} = \frac{2}{\sqrt{N}}$:

$$A_\alpha \rightarrow g_{YM} A_\alpha^{can}$$

(2.6)

in such a way that in the action the kinetic term becomes independent on $g$:

$$\frac{1}{2} \int tr(F^2(A^{can})) dx$$

(2.7)

where:

$$F_{\alpha\beta} = \partial_\beta A_\alpha^{can} - \partial_\alpha A_\beta^{can} + ig_{YM}[A_\alpha^{can}, A_\beta^{can}]$$

(2.8)

From now on we will simply write $F_{\alpha\beta}$ for the curvature as a function of the canonical field, without displaying the superscript $can$.

2.2 A short summary of perturbation theory and of the renormalization group

We recall the relation between bare and renormalized two-point connected correlators of a multiplicatively renormalizable gauge-invariant scalar operator $O$ of naive dimension in energy $D$:

$$G^{(2)}(p, \mu, g(\mu)) = Z^2_0(\frac{\Lambda}{\mu}, g(\Lambda))G_0^{(2)}(p, \Lambda, g(\Lambda))$$

(2.9)

where $G_0^{(2)}$ is the bare connected correlator in momentum space, computed in some regularization scheme with cutoff $\Lambda$, and $\mu$ is the renormalization scale:

$$G_0^{(2)}(p, \Lambda, g(\Lambda)) = \int \langle O(x) O(0) \rangle_{conn} e^{ip \cdot x} d^4 x \equiv \langle O(p) O(-p) \rangle_{conn}$$

(2.10)

Since $YM$ or $QCD$ with massless quarks or $N = 1$ SUSY $YM$ with massless quarks is massless to every order of perturbation theory and since $O$ has naive dimension $D$ we can write:

$$G^{(2)}(p, \mu, g(\mu)) = p^{2D-4} G^{(2)}_{DL}(\frac{p}{\mu}, g(\mu))$$

(2.11)

where $G^{(2)}_{DL}$ is a dimensionless function. The Callan-Symanzik equation for the dimensionless two-point renormalized correlator expresses the independence of the bare two-point correlator with respect to the subtraction point $\mu$:

$$\frac{dG_0^{(2)}}{d \log \mu} \bigg|_{\Lambda, g(\Lambda)} = 0$$

(2.12)
\[
\left( \frac{\partial}{\partial \log \mu} + \beta(g) \frac{\partial}{\partial g(\mu)} + 2\gamma(g) \right) G^{(2)}_{DL}(\frac{p}{\mu}, g(\mu)) = 0
\] (2.13)

where we have defined the beta function with respect to the renormalized coupling \( g(\mu) \):
\[
\beta(g) = \left. \frac{\partial g}{\partial \log \mu} \right|_{\Lambda, g(\Lambda)}
\] (2.14)

and the anomalous dimension:
\[
\gamma(g) = -\left. \frac{\partial \log Z_\gamma}{\partial \log \mu} \right|_{\Lambda, g(\Lambda)}
\] (2.15)

Eq. (2.13) can be rewritten taking into account the dependence of \( G^{(2)}_{DL} \) on the momentum \( p = \sqrt{p^2} \):
\[
\left( \frac{\partial}{\partial \log p} - \beta(g) \frac{\partial}{\partial g} - 2\gamma(g) \right) G^{(2)}_{DL}(\frac{p}{\mu}, g(\mu)) = 0
\] (2.16)

The general solution of Eq. (2.16) is:
\[
G^{(2)}_{DL}(\frac{p}{\mu}, g(\mu)) = \mathcal{G}(g(\frac{p}{\mu}, g(\mu))) e^{2 \int_{g(\mu)}^{g(p)} \frac{\gamma(g)}{\beta(g)} \, dg} \equiv \mathcal{G}_O(g(p)) Z^2_O(\frac{p}{\mu}, g(\mu))
\] (2.17)

The running coupling \( g(\frac{p}{\mu}, g(\mu)) \), that we briefly denote by \( g(p) \), solves:
\[
\frac{\partial g(p)}{\partial \log p} = \beta(g(p))
\] (2.18)

with the initial condition \( g(1, g(\mu)) = g(\mu) \).

The multiplicative renormalized factor \( Z_\gamma(\frac{p}{\mu}, g(\mu)) \) satisfies:
\[
\gamma(g) = -\frac{\partial \log Z_\gamma}{\partial \log \mu}
\] (2.19)

and from now on it is thought as a finite dimensionless function of \( g(\mu) \) and \( g(p) \) only.

\( Z_\gamma(g(p), g(\mu)) \):
\[
Z_\gamma = e^{\int_{g(\mu)}^{g(p)} \frac{\gamma(g)}{\beta(g)} \, dg}
\] (2.20)

Eq. (2.17) expresses the solution of the RG equation as a product of a RG invariant (RGI) function \( \mathcal{G}_O \) of \( g(p) \) only and of a multiplicative factor \( Z^2_O \) that is determined by the anomalous dimension \( \gamma(g) \) and by the beta function \( \beta(g) \). \( \mathcal{G}_O \) and \( Z_\gamma \) can be computed order by order in renormalized perturbation theory.

From Eq. (2.18), that represents the coupling constant flow as a function of the momentum, we obtain the well-known behavior of the RG-improved ’t Hooft running coupling constant with one- and two-loop accuracy, starting from the one- and two-loop perturbative beta function:
\[
\beta(g) = -\beta_0 g^4 - \beta_1 g^6 + \cdots
\] (2.21)

where for pure YM:
\[
\beta_0 = \frac{11}{3} \frac{1}{(4\pi)^2}
\]
\[
\beta_1 = \frac{34}{3} \frac{1}{(4\pi)^4}
\] (2.22)
With two-loop accuracy we get:

\[
\frac{dg}{d\log p} = -\beta_0 g^3 - \beta_1 g^5
\]

\[
\Rightarrow \int_{g(\mu)}^{g(p)} \frac{1}{\beta_0 g^2} (1 - \frac{\beta_1}{\beta_0} g^2) dg = -\log \frac{p}{\mu}
\]

\[
\Rightarrow \frac{1}{\beta_0} \left( \frac{1}{2g(\mu)^2} - \frac{1}{2g(p)^2} \right) - \frac{\beta_1}{\beta_0^2} \log \frac{g(p)}{g(\mu)} = -\log \frac{p}{\mu}
\]

\[
g^2(p) = \frac{g^2(\mu)}{1 + 2\beta_0 g^2(\mu) \log \frac{p}{\mu} - 2\frac{\beta_1}{\beta_0} g^2(\mu) \log \frac{g(p)}{g(\mu)}}
\]

\[
\sim \frac{1}{2\beta_0 \log \frac{p}{\mu}} \left( 1 + \frac{\beta_1}{\beta_0^2} \log \frac{g(p)}{g(\mu)} \right) \sim \frac{1}{2\beta_0 \log \frac{p}{\mu}} \left( 1 - \frac{\beta_1}{2\beta_0^2} \log \frac{p}{\mu} \right)
\]

\[
= \frac{1}{\beta_0 \log \frac{p}{\mu}} \left( 1 - \frac{\beta_1}{\beta_0^2} \log \frac{p}{\mu} \right) + O \left( \frac{1}{\log^2 \frac{p}{\mu}} \right)
\]

(2.23)

This is the well known actual UV asymptotic behavior of the running coupling constant. The function $G_0$ in Eq.(2.17) is not known from general principles but can be computed in perturbation theory as a function of $g(\mu)$ and then expressed in terms of $g(p)$, since $G_0$ is RG1. Similarly, we can evaluate $Z_0$ using again the one-loop or two-loop perturbative expressions for $\beta(g)$ and $\gamma_0(g)$:

\[
\gamma_0(g) = -\gamma_0^2 - \gamma_1^4 + \cdots
\]

(2.24)

With one-loop accuracy:

\[
Z_0^2 \sim \left( \frac{g^2(p)}{g^2(\mu)} \right)^{\gamma_0(\mu)} \sim \left( \log \frac{P}{\mu} \right)^{-\gamma_0(\mu)}
\]

(2.25)

and with two-loop accuracy we have:

\[
Z_0^2 \sim \left[ \frac{1}{2\beta_0 \log \frac{p}{\mu}} \left( 1 - \frac{\beta_1}{\beta_0^2} \log \frac{p}{\mu} \right) \right]^{\gamma_0(\mu)} \sim \left( \frac{1}{2\beta_0 \log \frac{p}{\mu}} \left( 1 - \frac{\beta_1}{\beta_0^2} \log \frac{p}{\mu} \right) \right) + O \left( \frac{1}{\log^2 \frac{p}{\mu}} \right)
\]

(2.26)
In evaluating the last two expressions we have used the two-loop RG-improved expression for \( g(p) \) given by Eq. (2.23). From the two-loop RG-improved expression in Eq. (2.26) it follows that the leading and next-to-leading logarithms for \( Z_2 \) are determined only by \( \beta_0 \), \( \beta_1 \) and by \( \gamma_0(\mathcal{O}) \), that are in fact universal, i.e. scheme independent. Indeed, the two-loop coefficient of the anomalous dimension \( \gamma_1(\mathcal{O}) \) does not occur in the first \( \log \log \frac{\mu}{\mu'} \) term, but only in terms that have a subleading behavior as powers of logarithms. Keeping only up to the next-to-leading term in \( Z_2^2 \), we obtain for the universal logarithmic behavior of the dimensionless two-point correlator:

\[
G^{(2)}_{\alpha F^2}(p, \mu) \sim \left[ \left( \frac{1}{2\beta_0 \log \frac{\mu}{\mu'}} \right) \left( 1 - \frac{\beta_1 \log \log \frac{\mu}{\mu'}}{2\beta_0^2 \log \frac{\mu}{\mu'}} \right) \right]^{\gamma_0(\mathcal{O}) \beta_0} \mathcal{G}(g(p))
\] (2.27)

Thus our aim, in order to get asymptotic estimates, is to determine the one-loop coefficient of the anomalous dimension \( \gamma_0(\mathcal{O}) \) and the RGI function \( \mathcal{G}(g) \) for our operators \( \mathcal{O} \).

### 2.3 Anomalous dimension of \( trF^2 \) and of \( tr\tilde{F} \)

The operator \( \frac{\beta(g)}{g} trF^2 \) is proportional to the conformal anomaly, that is the functional derivative with respect to a conformal rescaling of the metric of the renormalized effective action that must be RGI. Therefore \( \frac{\beta(g)}{g} trF^2 \) is RGI as well. Hence its anomalous dimension vanishes and, using the notation of the previous section, the form of its correlator, ignoring possible contact terms that will be taken into account in sect. (3), is:

\[
G^{(2)}_{\beta F^2}(p, \mu, g(\mu)) = p^4 \mathcal{G}(\frac{\beta(g)}{g} F^2)(g(p))
\] (2.28)

On the other hand \( trF^2 \) is not RGI and therefore its correlator is:

\[
G^{(2)}_F(p, \mu, g(\mu)) = p^4 \mathcal{G}_F(g(p)) Z^2_F(\frac{p}{\mu}, g(\mu))
\] (2.29)

Since the relation between \( G^{(2)}_{\beta F^2}(p, \mu, g(\mu)) \) and \( G^{(2)}_F(p, \mu, g(\mu)) \) is:

\[
G^{(2)}_{\beta F^2}(p, \mu, g(\mu)) = \left( \frac{\beta(g)}{g} \right)^2 G^{(2)}_F(p, \mu, g(\mu))
\] (2.30)

it follows that \( \left( \frac{\beta(g)}{g} \right)^2 \) and \( Z^2_F(\frac{p}{\mu}, g(\mu)) \) must combine in such a way to obtain a function of \( g(p) \) only:

\[
\left( \frac{\beta(g(\mu))}{g(\mu)} \right)^2 Z^2_F(\frac{p}{\mu}, g(\mu)) \mathcal{G}_F(g(p)) = \mathcal{G}(\frac{\beta(g)}{g} F^2)(g(p))
\] (2.31)

To find the anomalous dimension \( \gamma_{F^2} \) of \( trF^2 \) we exploit once again the property of \( \frac{\beta(g)}{g} trF^2 \) being RGI. Its two-point correlator must indeed satisfy the equation:

\[
\left( \frac{\partial}{\partial \log p} - \beta(g) \frac{\partial}{\partial g} - 4 \right) G^{(2)}_{\beta F^2}(p, \mu, g(\mu)) = 0
\] (2.32)
where the last term occurs because we are considering the complete correlator and not its dimensionless part. Using Eq. (2.30) we find the anomalous dimension of $tr F^2$:

$$\left(\frac{\partial}{\partial \log p} - \beta(g) \frac{\partial}{\partial g} - 4\right) \left[ \left(\frac{\beta(g)}{g}\right)^2 G_{F^2}^{(2)}(p, \mu, g(\mu)) \right] = 0$$

$$\Rightarrow \left[ \left(\frac{\beta(g)}{g}\right)^2 \frac{\partial}{\partial \log p} - \left(\frac{\beta(g)}{g}\right) \frac{\partial}{\partial g} - 4 \left(\frac{\beta(g)}{g}\right)^2 \right] G_{F^2}^{(2)}(p, \mu, g(\mu)) = 0 \quad (2.33)$$

From this equation it follows:

$$\gamma_{F^2}(g) = g \frac{\partial}{\partial g} \left(\frac{\beta(g)}{g}\right) \quad (2.34)$$

With two-loop accuracy this expression reads:

$$\gamma_{tr F^2}(g) = -2 \beta_0 g^2 - 4 \beta_1 g^4 + \cdots \quad (2.35)$$

Keeping only the first term, we can derive the expression for $Z^2(\frac{p}{\mu}, g(\mu))$ with one-loop accuracy:

$$Z^2(\frac{p}{\mu}, g(\mu)) \sim \frac{g^4(p)}{g^4(\mu)} \quad (2.36)$$

Finally, the correlator of $\frac{\beta(g)}{g} tr F^2$, with one-loop accuracy, is:

$$G_{\frac{\beta(g)}{g} F^2}^{(2)}(p, \mu, g(\mu)) = p^4 \beta_0^2 g^4(\mu) \frac{g^4(p)}{g^4(\mu)} G_{F^2}(g(p)) = p^4 \beta_0^2 g^4(p) G_{F^2}(g(p)) \quad (2.37)$$

We can repeat similar calculations for the operator $tr F \tilde{F}$ in order to compute its anomalous dimension, using the property of $g^2 tr F \tilde{F}$ being RGI. Indeed $g^2 tr F \tilde{F}$ is the density of the second Chern class or topological charge. The Callan-Symanzik equation is:

$$\left(\frac{\partial}{\partial \log p} - \beta(g) \frac{\partial}{\partial g} - 4\right) G_{g^2 F \tilde{F}}^{(2)}(p, \mu, g(\mu)) = \left[ g^4 \frac{\partial}{\partial g} g^4(\mu) \right] G_{F \tilde{F}}^{(2)}(g(p)) = 0 \quad (2.38)$$

from which we obtain the anomalous dimension of $tr F \tilde{F}$:

$$\gamma_{F \tilde{F}}(g) = 2 \frac{\beta(g)}{g} = -2 \beta_0 g^2 - 2 \beta_1 g^4 + \cdots \quad (2.39)$$

We notice that while the one-loop anomalous dimensions of $tr F^2$ and of $tr F \tilde{F}$ coincide, the two-loop anomalous dimensions are different. This means that the operator $tr F^{-2}$ has a well defined anomalous dimension only at one loop, in agreement with the fact that it belongs to the large-$N$ one-loop integrable sector of Ferretti-Heise-Zarembo [29]. Therefore, only the universal part of its correlator, that is determined by the one-loop anomalous dimension and by the two-loop $\beta$ function, can be meaningfully compared with the non-perturbative computation in Eq.(4.1).
2.4 Universal behavior of correlators

Knowing the naive dimension $D$ and the anomalous dimension of a (scalar) operator $O_D$, we can write the asymptotic form for $p >> \mu$ of its correlator obtained by the RG theory. Indeed, as we recalled in sect.(2.2), assuming multiplicative renormalizability, the RG-improved form of the Fourier transform of the correlator is given by:

$$G^{(2)}(p^2) = \int \langle O_D(x)O_D(0) \rangle_{conn} e^{ip \cdot x} d^4 x = p^{2D-4} G_{O_D}(g(p)) Z_{O_D}^2 \left( \frac{p}{\mu}, g(\mu) \right)$$  \hspace{1cm} (2.40)$$

where the power of $p$ is implied by dimensional analysis, $G_{O_D}$ is a dimensionless function that depends only on the running coupling $g(p)$, and $Z_{O_D}^2$ is the contribution from the anomalous dimension. But in fact in general the correlator of $O_D$ is not even multiplicatively renormalizable because of the presence of contact terms. These terms would affect the UV asymptotic behavior, but they are non-physical and therefore they must be subtracted. In fact, they spoil the positivity of the correlator in Euclidean space in the momentum representation, that is required by the Kallen-Lehmann representation (see the comment below Eq.(3.21)).

In the coordinate representation of the correlator, for $x \neq 0$, contact terms do not occur. Therefore, a strategy to avoid that contact terms interfere with the RG improvement is to pass to the coordinate scheme [30], where the correlator is multiplicatively renormalizable, to compute its RG-improved expression, to go back to the momentum representation, and eventually to subtract the contact terms.

In the coordinate representation for $x \neq 0$ the solution of the Callan-Symanzik equation reads:

$$G^{(2)}_{O_D}(x) = \langle O_D(x)O_D(0) \rangle_{conn} = \left( \frac{1}{x^2} \right)^D G_{O_D}(g(x)) Z_{O_D}^2 (x\mu, g(\mu))$$  \hspace{1cm} (2.41)$$

with $x = \sqrt{x^2}$, where we have denoted by $g(x)$ the running coupling in the coordinate scheme [30] and by an abuse of notation we have used the same names $G$ and $Z$ for the RGI function and renormalization factor in the coordinate and momentum representation.

The function $G(g(p))$ can be guessed at the lowest non-trivial order, since the correlator must be conformal at the lowest non-trivial order in perturbation theory, that implies $G(g(x)) \sim const$. Hence:

$$G(p) \sim const \log \frac{p}{\mu}$$  \hspace{1cm} (2.42)$$

Indeed, in appendix A we show that $\int p^{2D-4} \log \frac{p}{\mu} e^{ip \cdot x} d^4 p = const \left( \frac{1}{x^2} \right)^D$ that is conformal in the coordinate representation. The explicit dependence on $\mu$, that contradicts RG invariance in the momentum representation, is due to the fact that the correlator in the momentum representation, as opposed to the coordinate representation, is not really multiplicatively renormalizable because (scale dependent) contact terms arise. This can be understood observing that in the coordinate representation for $x \neq 0$ the lowest-order correlator is independent on the scale $\mu$ but it is not an integrable function, in such a way that its Fourier transform needs a regularization, that introduces the arbitrary scale $\mu$. 

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Naively, we can already derive the leading UV asymptotic behavior:

\[ C_{OD}^{(2)}(p^2) \sim p^{2D-4} \log \frac{p^2}{\mu^2} \left( \frac{g^2(p)}{g^2(\mu)} \right)^{\frac{\gamma_0(O_D)}{\beta_0}} \sim p^{2D-4}(g^2(p))^{\frac{\gamma_0(O_D)}{\beta_0}} - 1 \]  

(2.43)

where we have used the fact that \( g^2(p) \sim \frac{1}{\log(p^2)} \). It easy to check that for \( D = 4 \) and \( \gamma_0(O_D) = 2 \beta_0 \) this estimate coincides with Eq.(1.1)-Eq(1.3).

However, this estimate assumes multiplicative renormalizability in the momentum representation and it does not take into account the occurrence of contact terms in the momentum representation of the correlators.

Nevertheless, in the next section we confirm by direct computation that after subtracting the contact terms the actual behavior of the scalar and of the pseudoscalar correlator agrees with the estimate in Eq.(2.43).

3 Perturbative check of the universal behavior of correlators

In this section we obtain the explicit form of the three-loop correlators of \( trF^2 \) and of \( tr\tilde{F}F \) starting from their imaginary parts that have been computed in \([14][15]\) in the \( \overline{\text{MS}} \) scheme. The \( \overline{\text{MS}} \) scheme can be defined as the scheme in which the two-loop RG-improved running coupling does not contain \( \frac{1}{\log^2 \frac{p^2}{\Lambda^2}} \) contributions \([17]\). More precisely, we consider the equation for the running coupling constant that follows from the two-loop \( \beta \) function:

\[
\log \frac{p}{\Lambda_s} = \int_{g(\Lambda_s)}^{g(p)} \frac{dq}{\beta(g)} = \frac{1}{2\beta_0 g^2(p)} + \frac{\beta_1}{\beta_0^2} \log(g(p)) + C + \cdots
\]

(3.1)

where \( C \) is an arbitrary integration constant and \( \Lambda_s \) is the RGI scale in a generic scheme \( s \). The value of \( C \) in the \( \overline{\text{MS}} \) scheme is chosen in such a way to cancel the \( \frac{1}{\log^2 \frac{p^2}{\Lambda^2}} \) term in the solution:

\[
g^2_s(p) = \frac{1}{\beta_0 \log \frac{p^2}{\Lambda_s^2}} \left[ 1 - \frac{\beta_1}{\beta_0^2} \frac{\log(\beta_0 \log \frac{p^2}{\Lambda_s^2})}{\log \frac{p^2}{\Lambda_s^2}} + \frac{C}{\log \frac{p^2}{\Lambda_s^2}} \right]
\]

(3.2)

\[ \Rightarrow C = \beta_1 \beta_0 \log(\beta_0) \]

The result reported in \([14]\), for \( trF^2 \) in the \( SU(3) \) YM theory, is:

\[
\text{Im} \langle trF^2(p)trF^2(-p) \rangle_{\text{conn}} = \frac{8}{4\pi} p^4 \left\{ 1 + \frac{\alpha_{\overline{\text{MS}}}(\mu)}{\pi} \left[ \frac{73}{4} - \frac{11}{2} \log \frac{p^2}{\mu^2} \right] \right. \\
+ \left( \frac{\alpha_{\overline{\text{MS}}}(\mu)}{\pi} \right)^2 \left[ \frac{37631}{96} - \frac{363}{8} \zeta(2) - \frac{495}{8} \zeta(3) \\
- \frac{2817}{16} \log \frac{p^2}{\mu^2} + \frac{363}{16} \log^2 \frac{p^2}{\mu^2} \right] \right\}
\]

(3.3)

where \( \alpha_s = \frac{\alpha_{\overline{\text{MS}}}}{4\pi} \) and \( \alpha_{\overline{\text{MS}}} \) is \( \alpha_s \) in the \( \overline{\text{MS}} \) scheme. Firstly, we want to find from Eq.(3.3) the result for the \( SU(N) \) YM theory and we want to express the result in terms of the
't Hooft coupling in the $\overline{\text{MS}}$ scheme $g_{\overline{\text{MS}}}$. In fact, this operation is quite easy since it is known, and it can be checked in [15], that at this order of perturbation theory the rank of the gauge group enters the result only through the Casimir factor $C_A = N$. Therefore, to obtain the general result it is sufficient to divide by 3 and to multiply by $N$ the coefficient of $\alpha^2_{\overline{\text{MS}}}$ and to divide by 9 and to multiply by $N^2$ the coefficient of $\alpha^2_{\overline{\text{MS}}}$. The factors of $N$ and of $N^2$ are then absorbed in the definition of the 't Hooft coupling constant. We obtain:

$$\text{Im} \langle tr F^2(p) tr F^2(-p) \rangle_{\text{conn}} = \frac{N^2 - 1}{4\pi} p^4 \left\{ 1 + g^2_{\overline{\text{MS}}} (\mu) \left( \frac{73}{3(4\pi)^2} - 2 \frac{11}{3(4\pi)^2} \log \frac{p^2}{\mu^2} \right) 
+ g^4_{\overline{\text{MS}}} (\mu) \left[ \frac{37631}{54(4\pi)^4} - \frac{242}{3(4\pi)^2} \zeta(2) - \frac{110}{(4\pi)^3} \zeta(3) 
- \frac{313}{(4\pi)^4} \log \frac{p^2}{\mu^2} + \frac{121}{3(4\pi)^2} \log^2 \frac{p^2}{\mu^2} \right] \right\}$$

(3.4)

From Eq.(3.4) we derive the complete expression of the correlator, assuming the correlator in the form:

$$\langle tr F^2(p) tr F^2(-p) \rangle_{\text{conn}} = -\frac{N^2 - 1}{4\pi} p^4 \log \frac{p^2}{\mu^2} \left[ 1 + g^2_{\overline{\text{MS}}} (\mu) \left( f_0 - \beta_0 \log \frac{p^2}{\mu^2} \right) 
+ g^4_{\overline{\text{MS}}} (\mu) \left[ f_1 + f_2 \log \frac{p^2}{\mu^2} + f_3 \log^2 \frac{p^2}{\mu^2} \right] \right]$$

(3.5)

We extract the imaginary part of Eq.(3.5) that arises from the imaginary part of the logarithm in Minkowski signature, $\log(-\frac{p^2}{\mu^2}) = \log \frac{p^2}{\mu^2} - i\pi$. We obtain:

$$\text{Im} \langle tr F^2(p) tr F^2(-p) \rangle_{\text{conn}} = \frac{(N^2 - 1)}{4\pi} p^4 \left\{ 1 + f_0 g^2 (\mu) + (f_1 - f_3 \pi^2) g^4 (\mu) 
- 2 \beta_0 g^2 (\mu) \log \frac{p^2}{\mu^2} + 2 f_2 g^4 (\mu) \log \frac{p^2}{\mu^2} + 3 f_3 g^4 (\mu) \log^2 \frac{p^2}{\mu^2} \right\}$$

(3.6)

Finally, comparing Eq.(3.4) with Eq.(3.6) we determine the values of the coefficients $f_i$:

$$f_0 = \frac{73}{3(4\pi)^2}$$

$$f_1 - f_3 \pi^2 = \left( \frac{37631}{54} - \frac{242}{3} \zeta(2) - 110 \zeta(3) \right) \frac{1}{(4\pi)^4}$$

$$-2 \beta_0 = -2 \frac{11}{3(4\pi)^2}$$

$$2 f_2 = -\frac{313}{(4\pi)^4} \Rightarrow f_2 = -\frac{313}{2(4\pi)^4}$$

$$3 f_3 = \frac{121}{3(4\pi)^2} \Rightarrow f_3 = \frac{121}{9(4\pi)^4} \Rightarrow f_3 = \beta_0^2$$

$$\Rightarrow f_1 = \left( \frac{37631}{54} - 110 \zeta(3) \right) \frac{1}{(4\pi)^4}$$

(3.7)
Therefore, the correlator is:

\[
\langle trF^2(p)trF^2(-p)\rangle_{\text{conn}} = -\frac{(N^2 - 1)}{4\pi^2} p^4 \log \frac{p^2}{\mu^2} \left[ 1 + g^2(\mu) \left( f_0 - \beta_0 \log \frac{p^2}{\mu^2} \right) \right. \\
\left. + g^4(\mu) \left( f_1 + f_2 \log \frac{p^2}{\mu^2} + \beta_0^2 \log^2 \frac{p^2}{\mu^2} \right) \right] \tag{3.8}
\]

Similarly, the imaginary part of the correlator of \(trF \tilde{F}\), already written in [15] for the gauge group \(SU(N)\), is:

\[
\text{Im} \langle trF \tilde{F}(p)trF \tilde{F}(-p)\rangle_{\text{conn}} = \frac{N^2 - 1}{4\pi} p^4 \left\{ 1 + \frac{\alpha_{\text{MS}}(\mu)}{\pi} \left[ N \left( \frac{97}{12} - \frac{11}{6} \log \frac{p^2}{\mu^2} \right) \right] \\
+ \frac{\alpha_{\text{MS}}(\mu)^2}{\pi^2} \left[ N^2 \left( \frac{51959}{864} - \frac{121}{24} \zeta(2) - \frac{55}{8} \zeta(3) \right) \\
- \frac{1135}{48} \log \frac{p^2}{\mu^2} + \frac{121}{48} \log^2 \frac{p^2}{\mu^2} \right] \right\} \tag{3.9}
\]

We obtain:

\[
\langle trF \tilde{F}(p)trF \tilde{F}(-p)\rangle_{\text{conn}} = -\frac{(N^2 - 1)}{4\pi^2} p^4 \log \frac{p^2}{\mu^2} \left[ 1 + g^2_{\text{MS}}(\mu) \left( \tilde{f}_0 - \beta_0 \log \frac{p^2}{\mu^2} \right) \right. \\
\left. + g^4_{\text{MS}}(\mu) \left( \tilde{f}_1 + \tilde{f}_2 \log \frac{p^2}{\mu^2} + \beta_0^2 \log^2 \frac{p^2}{\mu^2} \right) \right] \tag{3.10}
\]

where:

\[
\tilde{f}_0 = \frac{97}{3(4\pi)^2} \\
\tilde{f}_1 = \frac{51959}{54} - 110\zeta(3) \frac{1}{(4\pi)^4} \\
-2\beta_0 = -2 \frac{11}{3(4\pi)^2} \\
2\tilde{f}_2 = -\frac{1135}{3(4\pi)^4} \Rightarrow \tilde{f}_2 = -\frac{1135}{6(4\pi)^4} \tag{3.11}
\]

### 3.1 Correlator of \(\frac{\beta(g)}{g_N} trF^2\) in \(SU(N)\) YM (two loops)

We now determine the \(UV\) asymptotic behavior for the correlators by employing their \(RG\)-improved expression. Firstly, we recall that in every generic scheme labelled by \(a\) the relation between the coupling constant at two different scales is, with one-loop accuracy:

\[
\frac{1}{g_a^2(\mu)} = \frac{1}{g_a^2(p)} - \beta_0 \log \frac{p^2}{\mu^2} \tag{3.12}
\]

This relation is necessary to express the correlators in their \(RG\)-improved form. As a starting simplified example we consider the two-loop expression of the correlator of \(\beta_0 \frac{g^2}{N} trF^2\)
We obtain for the correlator:

\[
\langle \beta_0 \frac{g^2}{N} tr F^2(p) \beta_0 \frac{g^2}{N} tr F^2(-p) \rangle_{\text{conn}} \\
\sim - \beta_0^2 p^4 g_{\text{MS}}^2 \frac{\mu^2}{g_{\text{MS}}^2} \log \frac{p^2}{\mu^2} \left[ 1 + g_{\text{MS}}^2 \frac{\mu^2}{g_{\text{MS}}^2} \left( f_0 - \beta_0 g_{\text{MS}}^2 \right) \right] 
\]

(3.13)

This expression is renormalization group invariant with one-loop accuracy, since the factor \( \left( \frac{\beta_0 g^2}{g} \right)^2 \) is \( \beta_0^2 g^4 \) if we employ the one-loop \( \beta \) function. The finite term \( f_0 g_{\text{MS}}^2(\mu) \) can be absorbed in a change of scheme. Indeed, defining:

\[
g_a^2(\mu) = g_{\text{MS}}^2(\mu)(1 + a g_{\text{MS}}^2(\mu))
\]

(3.14)

it follows:

\[
g_a^4(\mu) = g_{\text{MS}}^4(1 + 2a g_{\text{MS}}^2(\mu) + a^2 g_{\text{MS}}^4(\mu)) + O(g^{10})
\]

\[
g_{\text{MS}}^2(\mu) = g_a^2(\mu)(1 - a g_a^2(\mu)) + O(g^8)
\]

\[
= g_a^2(\mu)(1 - a g_a^2(\mu) + 2a^2 g_a^4(\mu)) + O(g^8)
\]

\[
g_{\text{MS}}^4(\mu) = g_a^4(\mu)(1 - 2a g_a^2(\mu) + 5a^2 g_a^4(\mu)) + O(g^{10})
\]

(3.15)

We obtain for the correlator:

\[
\langle \beta_0 \frac{g^2}{N} tr F^2(p) \beta_0 \frac{g^2}{N} tr F^2(-p) \rangle_{\text{conn}} \\
\sim - \beta_0^2 p^4 \log \frac{p^2}{\mu^2} g_a^4(\mu) \left[ 1 + (f_0 - 2a) g_a^2(\mu) - \beta_0 g_a^2(\mu) \log \frac{p^2}{\mu^2} + O(g^4 \log \frac{p^2}{\mu^2}) \right] 
\]

(3.16)

To cancel the finite term it is sufficient to put:

\[
a = \frac{f_0}{2}
\]

(3.17)

Hence we obtain:

\[
\langle \beta_0 \frac{g^2}{N} tr F^2(p) \beta_0 \frac{g^2}{N} tr F^2(-p) \rangle_{\text{conn}} \\
\sim - \beta_0^2 p^4 \log \frac{p^2}{\mu^2} g_a^4(\mu) \left[ 1 - \beta_0 g_a^2(\mu) \log \frac{p^2}{\mu^2} + O(g^4 \log \frac{p^2}{\mu^2}) \right] 
\]

(3.18)

At this order the term in square brackets is precisely the renormalization factor necessary to renormalize two powers of \( g_a(\mu) \). We obtain:

\[
\langle \beta_0 \frac{g^2}{N} tr F^2(p) \beta_0 \frac{g^2}{N} tr F^2(-p) \rangle_{\text{conn}} \\
\sim - \beta_0^2 p^4 g_a^2(\mu) g_a^2(p) \log \frac{p^2}{\mu^2} \left( 1 + O(g^4 \log \frac{p^2}{\mu^2}) \right)
\]

(3.19)

From Eq. (3.12) we express the logarithm in terms of the coupling constant:

\[
\beta_0 \log \frac{p^2}{\mu^2} = \frac{1}{g_a^2(p)} - \frac{1}{g_a^2(\mu)}
\]

(3.20)
The correlator becomes:

\[
\langle \beta_0 \frac{g^2}{N} tr F^2(p) \beta_0 \frac{g^2}{N} tr F^2(-p) \rangle_{\text{conn}}
\]

\[
\sim -\beta_0 p^4 g_2^2(\mu) g_2^2(p) \left( \frac{1}{g_2^2(p)} - \frac{1}{g_2^2(\mu)} \right) \left( 1 + O(g^4 \log \frac{p^2}{\mu^2}) \right)
\]

\[
= \beta_0 p^4 \left( g_2^2(p) - g_2^2(\mu) \right) \left( 1 + O(g^4 \log \frac{p^2}{\mu^2}) \right)
\]

(3.21)

The second term in the last line is in fact a contact term that has no physical meaning, therefore it may depend on the arbitrary scale \( \mu \) since it must be subtracted anyway. The physical term is positive, despite the correlator that we started with was negative. This is an important feature, since a negative physical term would have been in contrast with the Kallen-Lehmann representation, that requires a positive spectral function.

### 3.2 Correlator of \( \frac{\beta(g)}{gN} tr F^2 \) in \( SU(N) \) YM (three loops)

We now consider the three-loop result Eq.(3.8), this time including also the correct normalization factors:

\[
\langle \frac{g^2}{N} tr F^2(p) \frac{g^2}{N} tr F^2(-p) \rangle_{\text{conn}}
\]

\[
= -\left( 1 - \frac{1}{N^2} \right) g_{\text{MS}}^4(\mu) \frac{p^4}{4\pi^2} \log \frac{p^2}{\mu^2} \left[ 1 + g_{\text{MS}}^2(\mu) \left( f_0 - \beta_0 \log \frac{p^2}{\mu^2} \right) \right]
\]

\[
+ g_{\text{MS}}^4(\mu) \left( f_1 + f_2 \log \frac{p^2}{\mu^2} + \beta_0^2 \log^2 \frac{p^2}{\mu^2} \right)
\]

(3.22)

This correlator is not supposed to be RGI, because the factor of \( (\frac{\beta(g)}{g})^2 = g^4 (1 + \frac{g_\beta}{\beta_0} g^2)^2 \) is missing. We can eliminate the finite terms in the correlator by a redefinition of the scheme:

\[
g_{ab}^2(\mu) = g_{\text{MS}}^2(\mu) \left( 1 + a g_{\text{MS}}^2(\mu) + b g_{\text{MS}}^4(\mu) \right)
\]

\[
\Rightarrow g_{\text{MS}}^4(\mu) = g_{ab}^4(\mu) \left( 1 - 2a g_{ab}^2(\mu) + (2b + 5a^2) g_{ab}^4(\mu) \right) + O(g^{10})
\]

(3.23)
Substituting we obtain:

\[
\frac{g^2}{N} tr F^2(p) \frac{g^2}{N} tr F^2(-p) \mid_{\text{conn}} = -\left(1 - \frac{1}{N^2}\right) g_{ab}(\mu) (1 - 2 a g_{ab}(\mu) + (2b + 5a^2) g_{ab}(\mu)) \frac{p^4}{4\pi^2} \log \frac{p^2}{\mu^2}
\]

\[
\times \left[ 1 + f_0 g_{ab}(\mu) (1 - a g_{ab}(\mu)) - \beta_0 g_{ab}(\mu) (1 - a g_{ab}(\mu)) \log \frac{p^2}{\mu^2} + f_1 g_{ab}(\mu) + f_2 g_{ab}(\mu) \log \frac{p^2}{\mu^2} + \beta_0^2 g_{ab}(\mu) \log^2 \frac{p^2}{\mu^2} \right]
\]

\[
= -\left(1 - \frac{1}{N^2}\right) g_{ab}(\mu) \frac{p^4}{4\pi^2} \log \frac{p^2}{\mu^2}
\]

\[
\times \left[ 1 + f_0 g_{ab}(\mu) (1 - a g_{ab}(\mu)) - \beta_0 g_{ab}(\mu) (1 - a g_{ab}(\mu)) \log \frac{p^2}{\mu^2} + f_1 g_{ab}(\mu) + f_2 g_{ab}(\mu) \log \frac{p^2}{\mu^2} + \beta_0^2 g_{ab}(\mu) \log^2 \frac{p^2}{\mu^2} \right]
\]

(3.24)

We eliminate the two finite terms choosing:

\[
a = \frac{f_0}{2}
\]

\[
f_1 + 5\left(\frac{f_0}{2}\right)^2 + 2b - \frac{f_0^2}{2} = 0
\]

\[
\Rightarrow b = \frac{3}{8} f_0 - \frac{f_1}{2}
\]

(3.25)

With this choice of \(a\) the coefficient of the \(g^4 \log \frac{p^2}{\mu^2}\) term becomes:

\[
f_2 + 3 \beta_0 a = f_2 + \frac{3}{2} f_0 \beta_0 = -\frac{68}{3(4\pi)^4} = -2 \beta_1
\]

(3.26)

Therefore, the correlator reads:

\[
\frac{g^2}{N} tr F^2(p) \frac{g^2}{N} tr F^2(-p) \mid_{\text{conn}} = -\left(1 - \frac{1}{N^2}\right) g_{ab}(\mu) \frac{p^4}{4\pi^2} \log \frac{p^2}{\mu^2}
\]

\[
\times \left[ 1 - \beta_0 g_{ab}(\mu) \log \frac{p^2}{\mu^2} - 2 \beta_1 g_{ab}(\mu) \log \frac{p^2}{\mu^2} + \beta_0^2 g_{ab}(\mu) \log^2 \frac{p^2}{\mu^2} \right]
\]

(3.27)

We notice that the expression in square brackets is the two-loop \(Z\) factor determined by the anomalous dimension of \(tr F^2\) according to Eq.(2.35). The coefficient \(2\beta_1\) should become \(\beta_1\) if we multiply the correlator in Eq.(3.27) by the factor of \(1 + \frac{\beta_1}{\beta_0} g_{ab}(\mu)^2\), in order to
make the correlator $RGI$:

\[
\frac{\langle \beta(g_{ab}) \rangle}{N_{g_{ab}}} \frac{tr F^2(p)}{tr F^2(-p)} \bigg| \text{conn} \nonumber
\]

\[
= -\left(1 - \frac{1}{N^2}\right) \beta_0 g_{ab}(\mu) \frac{p^4}{4\pi^2} \left(1 + \frac{\beta_1}{\beta_0} g_{ab}(\mu)\right)^2 \log \frac{p^2}{\mu^2} \nonumber
\]

\[
\times \left[1 - \beta_0 g_{ab}(\mu) \log \frac{p^2}{\mu^2} - 2\beta_1 g_{ab}(\mu) \log \frac{p^2}{\mu^2} + \beta_0^2 g_{ab}(\mu) \log^2 \frac{p^2}{\mu^2}\right] \nonumber
\]

\[
= -\left(1 - \frac{1}{N^2}\right) \beta_0 g_{ab}(\mu) \frac{p^4}{4\pi^2} \left(1 + \frac{\beta_1}{\beta_0} g_{ab}(\mu)\right) \nonumber
\]

\[
\times \left[1 + \frac{\beta_1}{\beta_0} g_{ab}(\mu) \right] \left(1 + \frac{\beta_1}{\beta_0} g_{ab}(\mu)\right) \log \frac{p^2}{\mu^2} \nonumber
\]

\[
\times \left[1 - \beta_0 g_{ab}(\mu) \log \frac{p^2}{\mu^2} - 2\beta_1 g_{ab}(\mu) \log \frac{p^2}{\mu^2} + \beta_0^2 g_{ab}(\mu) \log^2 \frac{p^2}{\mu^2}\right] \quad (3.28) \nonumber
\]

where we have multiplied and divided by $\left(1 + \frac{\beta_1}{\beta_0} g_{ab}(\mu)\right)$ in order to exploit the two-loop relation:

\[
\beta_0 \left(1 + \frac{\beta_1}{\beta_0} g_{ab}(\mu)\right) \log \frac{p^2}{\mu^2} = \frac{1}{g_{ab}(p)} - \frac{1}{g_{ab}^2(\mu)} \quad (3.29) \nonumber
\]

We now evaluate separately:

\[
\frac{\left(1 + \frac{\beta_1}{\beta_0} g_{ab}(\mu)\right)}{\left(1 + \frac{\beta_1}{\beta_0} g_{ab}(\mu)\right)} \nonumber
\]

\[
= \left(1 + \frac{\beta_1}{\beta_0} g_{ab}(\mu)\right) \left(1 - \frac{\beta_1}{\beta_0} g_{ab}(\mu)\right) + O(g^6 \log \frac{p^2}{\mu^2}) \nonumber
\]

\[
= 1 + \beta_1 g_{ab}^4(\mu) \log \frac{p^2}{\mu^2} + O(g^6 \log \frac{p^2}{\mu^2}) \quad (3.30) \nonumber
\]

Putting all together we get:

\[
\langle \beta(g_{ab}) \rangle \frac{tr F^2(p)}{tr F^2(-p)} \bigg| \text{conn} \nonumber
\]

\[
= -\left(1 - \frac{1}{N^2}\right) \beta_0 g_{ab}(\mu) \frac{p^4}{4\pi^2} \nonumber
\]

\[
\times \left(1 + \beta_1 g_{ab}(\mu) \log \frac{p^2}{\mu^2}\right) \left(1 + \frac{\beta_1}{\beta_0} g_{ab}(\mu)\right) \left(\frac{1}{g_{ab}(p)} - \frac{1}{g_{ab}^2(\mu)}\right) \nonumber
\]

\[
\times \left[1 - \beta_0 g_{ab}(\mu) \log \frac{p^2}{\mu^2} - 2\beta_1 g_{ab}(\mu) \log \frac{p^2}{\mu^2} + \beta_0^2 g_{ab}(\mu) \log^2 \frac{p^2}{\mu^2}\right] \nonumber
\]

\[
= -\left(1 - \frac{1}{N^2}\right) \beta_0 g_{ab}(\mu) \frac{p^4}{4\pi^2} \left(1 + \frac{\beta_1}{\beta_0} g_{ab}(\mu)\right) \left(\frac{1}{g_{ab}(p)} - \frac{1}{g_{ab}^2(\mu)}\right) \nonumber
\]

\[
\times \left[1 - \beta_0 g_{ab}(\mu) \log \frac{p^2}{\mu^2} - \beta_1 g_{ab}(\mu) \log \frac{p^2}{\mu^2} + \beta_0^2 g_{ab}(\mu) \log^2 \frac{p^2}{\mu^2}\right] \quad (3.31) \nonumber
\]
The factor in square brackets in the last line is now precisely the renormalization factor for two powers of $g_{ab}$. Hence the correlator reads:

$$\langle \frac{\beta(g_{ab})}{N g_{ab}} tr F^2(p) \frac{\beta(g_{ab})}{N g_{ab}} tr F^2(-p) \rangle_{conn}$$

$$= -(1 - \frac{1}{N^2}) \frac{p^4}{4\pi^2} \frac{g_{ab}^2(\mu)}{\beta_0} \left( 1 + \frac{\beta_1}{\beta_0} g_{ab}(\mu) \right) \left( \frac{1}{g_{ab}(p)} - \frac{1}{g_{ab}(\mu)} \right)$$

$$= -(1 - \frac{1}{N^2}) \frac{p^4}{4\pi^2} \left( 1 + \frac{\beta_1}{\beta_0} g_{ab}(\mu) \right) \left( g_{ab}^2(\mu) - g_{ab}(p) \right)$$

$$= (1 - \frac{1}{N^2}) \frac{p^4}{4\pi^2} \left[ g_{ab}^2(p) \left( 1 + \frac{\beta_1}{\beta_0} g_{ab}(\mu) \right) - g_{ab}^2(\mu) \left( 1 + \frac{\beta_1}{\beta_0} g_{ab}(\mu) \right) \right] \quad (3.32)$$

The second term in the last line is a contact term, but the first term depends on $g_{ab}(\mu)$, therefore it is not $RGI$. Hence Eq.(3.32) is not exactly $RGI$ even after subtracting the contact terms. The scale dependence in the physical term is due to the fact that the correlator is not exact but it is computed to a finite order of perturbation theory. We notice that the scale dependence occurs at order of $g^4$ only and in any case it does not affect the structure of the universal $UV$ behavior but only the overall coefficient in the $RG$ estimate. Yet it is interesting to determine the precise overall coefficient of the asymptotic behavior. This is done for the correlator of $\frac{\beta(g)}{N} tr F^2$ in $SU(3)$ $QCD$ in sect.(3.7) by assuming its $RG$-invariance, instead of checking it to a finite order of perturbation theory as we just did.

### 3.3 Correlator of $\frac{g^2}{N} tr F^2$ in $SU(N)$ $YM$ (three loops)

We now present the result for the correlator of $\frac{g^2}{N} tr F^2$. We recall that in this case we do not expect to get a $RGI$ function to all orders in perturbation theory. We start from Eq.(3.27) and we write it as:

$$\langle \frac{g^2}{N} tr F^2(p) \frac{g^2}{N} tr F^2(-p) \rangle_{conn}$$

$$= -\left( 1 - \frac{1}{N^2} \right) \frac{p^4}{4\pi^2} \frac{g_{ab}^4(\mu)}{\beta_0} \left( 1 + \frac{\beta_1}{\beta_0} g_{ab}(\mu) \right) \left( \frac{1}{g_{ab}(p)} - \frac{1}{g_{ab}(\mu)} \right)$$

$$\times \left[ 1 - \beta_0 g_{ab}^2(\mu) \log \frac{p^2}{\mu^2} - 2\beta_1 g_{ab}^4(\mu) \log \frac{p^2}{\mu^2} + \beta_0^2 g_{ab}(\mu) \log^2 \frac{p^2}{\mu^2} \right]$$

$$= -\left( 1 - \frac{1}{N^2} \right) \frac{p^4}{4\pi^2} \frac{g_{ab}^4(\mu)}{\beta_0} \left( 1 + \frac{\beta_1}{\beta_0} g_{ab}(\mu) \right) \left( \frac{1}{g_{ab}(p)} - \frac{1}{g_{ab}(\mu)} \right)$$

$$\times \left[ 1 - \beta_0 g_{ab}^2(\mu) \log \frac{p^2}{\mu^2} - \beta_1 g_{ab}^4(\mu) \log \frac{p^2}{\mu^2} + \beta_0^2 g_{ab}(\mu) \log^2 \frac{p^2}{\mu^2} \right]$$

$$= -\left( 1 - \frac{1}{N^2} \right) \frac{p^4}{4\pi^2} \frac{g_{ab}^4(\mu)}{\beta_0} \left( 1 - \frac{\beta_1}{\beta_0} g_{ab}(\mu) + \frac{\beta_1^2}{\beta_0^2} g_{ab}(\mu) \right) \left( g_{ab}(\mu) - g_{ab}(p) \right)$$

$$= (1 - \frac{1}{N^2}) \frac{p^4}{4\pi^2} \frac{1}{\beta_0} \left[ g_{ab}(p) - g_{ab}(\mu) + \frac{\beta_1}{\beta_0} g_{ab}(\mu) - \frac{\beta_1^2}{\beta_0^2} g_{ab}(p) g_{ab}(\mu) \right] \quad (3.33)$$

Surprisingly we notice that the term that depends on the product $g_{ab}(\mu) g_{ab}(p)$, that is not $RGI$, is of the same order of $g^4$ as the non-$RGI$ terms in the correlator in Eq.(3.32), that must be $RGI$. 
3.4 Correlator of $\frac{g^2}{N} trF\tilde{F}$ in $SU(N)$ YM (three loops)

We repeat the same steps to find the $RG$-improved expression for the correlator of $\frac{g^2}{N} trF\tilde{F}$, that is $RGI$. The three-loop correlator reads:

$$
\langle \frac{g^2}{N} trF\tilde{F}(p) \frac{g^2}{N} trF\tilde{F}(-p) \rangle_{\text{conn}} = - \left(1 - \frac{1}{N^2}\right) \frac{p^4}{4\pi^2} g^4_{\text{MS}}(\mu) \log \frac{p^2}{\mu^2}
$$

$$
\times \left[ 1 + g^2_{\text{MS}}(\mu) \left( \tilde{f}_0 - \beta_0 \log \frac{p^2}{\mu^2} \right) + g^4_{\text{MS}}(\mu) \left( \tilde{f}_1 + \tilde{f}_2 \log \frac{p^2}{\mu^2} + \beta_0^2 \log^2 \frac{p^2}{\mu^2} \right) \right]
$$

(3.34)

Now we perform a generic change of scheme as in Eq. (3.23):

$$
g^2_{ab} = g^2_{\text{MS}}(\mu) \left( 1 + \tilde{a} g^2_{\text{MS}}(\mu) + \tilde{b} g^4_{\text{MS}}(\mu) \right)
$$

(3.35)

The correlator becomes:

$$
\langle \frac{g^2}{N} trF\tilde{F}(p) \frac{g^2}{N} trF\tilde{F}(-p) \rangle_{\text{conn}} = - \left(1 - \frac{1}{N^2}\right) \frac{p^4}{4\pi^2} g^4_{\text{MS}}(\mu) \log \frac{p^2}{\mu^2}
$$

$$
\times \left[ 1 + (\tilde{f}_0 - 2\tilde{a}) g^2_{ab}(\mu) - \beta_0 g^2_{ab}(\mu) \log \frac{p^2}{\mu^2} + (\tilde{f}_1 + 5\tilde{a}^2 + 2\tilde{b} - \tilde{a} \tilde{f}_0) g^4_{ab}(\mu)
$$

$$
+ (\tilde{f}_2 + 3\beta_0 \tilde{a}) g^4_{ab}(\mu) \log \frac{p^2}{\mu^2} + \beta_0^2 g^4_{ab}(\mu) \log^2 \frac{p^2}{\mu^2} \right]
$$

(3.36)

Again we impose the conditions to eliminate the finite terms:

$$
\tilde{a} = \frac{\tilde{f}_0}{2}
$$

$$
\tilde{f}_1 + 5(\frac{\tilde{f}_0}{2})^2 + 2\tilde{b} - \frac{\tilde{f}_0}{2} = 0
$$

$$
\Rightarrow \tilde{b} = \frac{3}{8} \tilde{f}_0 - \frac{\tilde{f}_1}{2}
$$

(3.37)

With this choice of $\tilde{a}$ the coefficient of the $g^4 \log^2 \frac{p^2}{\mu^2}$ term becomes:

$$
\tilde{f}_2 + 3\beta_0 \tilde{a} = \tilde{f}_2 + \frac{3}{2} \tilde{f}_0 \beta_0 = - \frac{34}{3(4\pi)^4} = -\beta_1
$$

(3.38)

Substituting in the correlator we get:

$$
\langle \frac{g^2}{N} trF\tilde{F}(p) \frac{g^2}{N} trF\tilde{F}(-p) \rangle_{\text{conn}} = - \left(1 - \frac{1}{N^2}\right) \frac{p^4}{4\pi^2} g^4_{\text{MS}}(\mu) \log \frac{p^2}{\mu^2}
$$

$$
\times \left[ 1 - \beta_0 g^2_{ab}(\mu) \log \frac{p^2}{\mu^2} - \beta_1 g^4_{ab}(\mu) \log \frac{p^2}{\mu^2} + \beta_0^2 g^4_{ab}(\mu) \log^2 \frac{p^2}{\mu^2} \right]
$$

(3.39)
We notice that the expression in square brackets is the two-loop $Z$ factor implied by the anomalous dimension of $tr F \tilde{F}$ computed in Eq. (2.39). It renormalizes two powers of $g(\mu)$. Therefore, the correlator reads:

\[
\langle \frac{g^2}{N} tr F \tilde{F}(p)\rangle_{\text{conn}} = \langle \frac{g^2}{N} tr F \tilde{F}(-p)\rangle_{\text{conn}} = \frac{-1}{N^2} \left( \frac{p^4}{4\pi^2} \frac{1}{\beta_0} (g_{ab}(\mu) - g_{ab}(p)) \log \frac{p^2}{\mu^2} \right)
\]

\[
= \left( 1 - \frac{1}{N^2} \right) \left( \frac{p^4}{4\pi^2} \frac{1}{\beta_0} (g_{ab}(\mu) - g_{ab}(p)) \frac{1}{\beta_0} \left( g_{ab}(\mu) - g_{ab}(p) \right) \right)
\]

\[
= \left( 1 - \frac{1}{N^2} \right) \left( \frac{p^4}{4\pi^2} \frac{1}{\beta_0} (g_{ab}(\mu) - g_{ab}(p)) \frac{1}{1 + \frac{2\beta_1}{\beta_0} g_{ab}(p)} \right)
\]

\[
= \left( 1 - \frac{1}{N^2} \right) \left( \frac{p^4}{4\pi^2} \frac{1}{\beta_0} \left( g_{ab}(\mu) - g_{ab}(p) \right) \left( 1 - \frac{\beta_1}{\beta_0} g_{ab}(p) + \frac{\beta_2}{\beta_0} g_{ab}^4(p) \right) \right)
\]

\[
= \left( 1 - \frac{1}{N^2} \right) \left( \frac{p^4}{4\pi^2} \frac{1}{\beta_0} \left( g_{ab}(\mu) - g_{ab}(p) \right) \left( 1 + \frac{\beta_1}{\beta_0} g_{ab}^2(p) - \frac{\beta_1}{\beta_0} g_{ab}^4(p) - g_{ab}(\mu) \right) \right)
\]

(3.40)

The second term in the last line is scale dependent to the order of $g^4$ as the term that occurs in the correlator of $\frac{g^2}{N} tr F^2$ in Eq. (3.33).

### 3.5 Correlator of $\frac{g^2}{N} tr F^{-2}$ in SU(N) YM (three loops)

We now sum the two results for the correlators of $tr F^2$ and of $tr F \tilde{F}$ to obtain the correlator of $tr F^{-2}$. Indeed, we recall that:

\[
\frac{1}{2} \langle tr F^{-2}(p) tr F^{-2}(-p) \rangle_{\text{conn}} = 2 \langle tr F^2(p) tr F^2(-p) \rangle_{\text{conn}} + 2 \langle tr F \tilde{F}(p) tr F \tilde{F}(-p) \rangle_{\text{conn}}
\]

(3.41)

Summing the two results in Eq. (3.40) and in Eq. (3.33) we obtain:

\[
\frac{1}{2} \langle \frac{g^2}{N} tr F^{-2}(p) \rangle_{\text{conn}} = \left( 1 - \frac{1}{N^2} \right) \left( \frac{p^4}{2\pi^2} \frac{1}{\beta_0} \left( g_{ab}(\mu) + g_{ab}(p) - g_{ab}(\mu) - g_{ab}^2(\mu) + \frac{\beta_1}{\beta_0} g_{ab}^4(p) \right) \right)
\]

\[
- \frac{\beta_1}{\beta_0} g_{ab}^4(p) + \frac{\beta_1}{\beta_0} g_{ab}^2(p) g_{ab}(\mu) - g_{ab}(\mu) g_{ab}^2(p) + \frac{\beta_1}{\beta_0} g_{ab}^4(p) - g_{ab}(\mu) g_{ab}^2(p) \right)
\]

\[
= \left( 1 - \frac{1}{N^2} \right) \left( \frac{p^4}{2\pi^2} \frac{1}{\beta_0} \left( g_{ab}(\mu) + g_{ab}(p) - g_{ab}(\mu) - g_{ab}(\mu) + \frac{\beta_1}{\beta_0} g_{ab}^4(p) \right) \right)
\]

(3.42)

where surprisingly the mixed terms $g^2(p) g^2(\mu)$ cancel to the order of $g^6$. There is no perturbative explanation for such cancellation, but conjecturally the cancellation occurs because of the RG invariance of the non-perturbative formula Eq.(1.8) in the TFT for the $L = 2$ ground state [4] [5] of the large-$N$ one-loop integrable sector of Ferretti-Heise-Zarembo (see sect.(4)). We can express the last result in terms of the coupling constant in
the $\overline{\text{MS}}$ scheme:

$$\frac{1}{2} \frac{g^2}{N} trF^{-2}(p) \frac{g^2}{N} trF^{-2}(-p)_{\text{conn}}$$

$$= (1 - \frac{1}{N^2}) \frac{p^4}{2\pi^2} \left( \frac{g^2}{\overline{\text{MS}}} - 2g^2_{\overline{\text{MS}}} + (a + \tilde{a} - \frac{\beta_1}{\beta_0}) g^4_{\overline{\text{MS}}} \right)$$

$$+ \left( \frac{\beta_1}{\beta_0} - a - \tilde{a} \right) g^4_{\overline{\text{MS}}} + O(\alpha^6) \quad (3.43)$$

that coincides with Eq.(1.4).

### 3.6 Scalar correlators in $SU(3)$ QCD with $n_l$ massless Dirac fermions

In this section we derive the RG-improved expression for the correlators of $trF^2$ and of $\frac{\beta(g_{YM})}{g_{YM}} trF^2$ in QCD.

The three-loop perturbative result for the imaginary part of the correlator of $trF^2$ in QCD with $n_l$ massless Dirac fermions is [14]:

$$\text{Im} \langle trF^2(p) trF^2(-p) \rangle_{\text{conn}}$$

$$= \frac{2}{\pi} \frac{p^4}{\pi} \left\{ 1 + \frac{\alpha_s(\mu)}{\pi} \left[ \frac{73}{4} - \frac{11}{2} \log \frac{\mu^2}{\mu^2} \right] - n_l \left( \frac{7}{6} - \frac{1}{3} \log \frac{\mu^2}{\mu^2} \right) \right\}$$

$$+ \frac{\alpha_s(\mu)}{\pi} \left[ \left( \frac{37631}{96} - \frac{363}{8} \zeta(2) - \frac{5}{8} \zeta(3) - \frac{2871}{16} \log \frac{\mu^2}{\mu^2} + \frac{2817}{16} \log \frac{\mu^2}{\mu^2} \right) \right.$$

$$+ n_l \left( -\frac{7189}{144} + \frac{11}{2} \zeta(2) + \frac{5}{4} \zeta(3) + \frac{263}{12} \log \frac{\mu^2}{\mu^2} - \frac{11}{4} \log \frac{\mu^2}{\mu^2} \right)$$

$$+ n_l \left( \frac{127}{108} - \frac{1}{6} \zeta(2) - \frac{7}{12} \log \frac{\mu^2}{\mu^2} + \frac{1}{12} \log \frac{\mu^2}{\mu^2} \right) \left\} \right\}$$

(3.44)

We write the correlator in terms of the coupling $g_{YM}$ in the $\overline{\text{MS}}$ scheme instead of $\alpha_s$:

$$\text{Im} \langle trF^2(p) trF^2(-p) \rangle_{\text{conn}}$$

$$= \frac{2}{\pi} \frac{p^4}{\pi} \left\{ 1 + g_{YM}^2(\mu) \left[ \frac{73}{4} - \frac{22}{3} \log \frac{\mu^2}{\mu^2} \right] - n_l \left( \frac{14}{3} - \frac{4}{3} \log \frac{\mu^2}{\mu^2} \right) \right\} \frac{1}{(4\pi)^2}$$

$$+ g_{YM}^4(\mu) \left[ \left( \frac{37631}{6} - \frac{7266}{3} \zeta(2) - 990 \zeta(3) - 2871 \log \frac{\mu^2}{\mu^2} + 363 \log \frac{\mu^2}{\mu^2} \right) \right.$$

$$+ n_l \left( -\frac{7189}{9} + 88 \zeta(2) + 20 \zeta(3) + \frac{1052}{3} \log \frac{\mu^2}{\mu^2} - 44 \log \frac{\mu^2}{\mu^2} \right)$$

$$+ n_l^2 \left( \frac{508}{27} - \frac{8}{3} \zeta(2) - \frac{28}{3} \log \frac{\mu^2}{\mu^2} + \frac{4}{3} \log \frac{\mu^2}{\mu^2} \right) \right\} \frac{1}{(4\pi)^2}$$

(3.45)

If we suppose the correlator to be of the form:

$$\langle trF^2(p) trF^2(-p) \rangle_{\text{conn}}$$

$$= -\frac{2}{\pi} \frac{p^4}{\pi} \log \frac{\mu^2}{\mu^2} \left[ 1 + g_{YM}^2(\mu) \left( h_0 + h_1 \log \frac{\mu^2}{\mu^2} \right) \right.$$

$$+ g_{YM}^4(\mu) \left( h_2 + h_3 \log \frac{\mu^2}{\mu^2} + h_4 \log \frac{\mu^2}{\mu^2} \right) \right\]$$

(3.46)
its imaginary part is:
\[
\text{Im} \{ tr F^2(p) tr F^2(-p) \}_{\text{conn}}
\]
\[
= \frac{2}{\pi} p^4 \left[ 1 + h_0 g_{YM}^2(\mu) + 2 h_1 g_{YM}^2(\mu) \log \frac{p^2}{\mu^2}
\right.
\]
\[
+ (h_2 - \pi^2 h_4) g_{YM}^4(\mu) + 2 h_3 g_{YM}^4(\mu) \log \frac{p^2}{\mu^2} + 3 h_4 g_{YM}^4(\mu) \log^2 \frac{p^2}{\mu^2}\]
\]
(3.47)

Comparing Eq. (3.47) and Eq. (3.45) we get:
\[
h_0 = \left( 73 - \frac{14}{3} n_l \right) \frac{1}{(4\pi)^2}
\]
\[
2h_1 = \left( -22 + \frac{4}{3} n_l \right) \frac{1}{(4\pi)^2}
\]
\[
\Rightarrow h_1 = \left( -11 + \frac{2}{3} n_l \right) \frac{1}{(4\pi)^2} = -\bar{\beta}_0
\]
\[
h_2 = \pi^2 h_4 = \left[ \frac{37361}{6} - 726 \zeta(2) - 990 \zeta(3) + n_l \left( \frac{7189}{9} + 88 \zeta(2) + 20 \zeta(3) \right) \right.
\]
\[
+ n_l^2 \left( \frac{508}{27} - \frac{8}{3} \zeta(2) \right) \left( 1 + \frac{1}{(4\pi)^2} \right)
\]
\[
2h_3 = \left[ -2817 + \frac{1052}{3} n_l - \frac{28}{3} n_l^2 \right] \frac{1}{(4\pi)^4}
\]
\[
\Rightarrow h_3 = \left[ -\frac{2817}{2} + \frac{526}{3} n_l - \frac{14}{3} n_l^2 \right] \frac{1}{(4\pi)^4}
\]
\[
3h_4 = \left[ 363 - 44 n_l + \frac{4}{9} n_l^2 \right] \frac{1}{(4\pi)^4}
\]
\[
\Rightarrow h_4 = \left[ 121 - \frac{44}{3} n_l + \frac{4}{9} n_l^2 \right] \frac{1}{(4\pi)^4} = \bar{\beta}_0^2
\]
(3.48)

Now we repeat the same steps as in the \( n_l = 0 \) case. We change renormalization scheme in order to cancel the finite parts:
\[
g_{\text{un}}^2(\mu) = g_{YM}^2(\mu) \left( 1 + u g_{YM}^2(\mu) + v g_{YM}^4(\mu) \right)
\]
(3.49)

We use the perturbative expression for the renormalized coupling constant with two-loop accuracy:
\[
g_{YM}^2(p) = g_{YM}^2(\mu) \left( 1 - \tilde{\beta}_0 g_{YM}^2(\mu) \log \frac{p^2}{\mu^2} - \tilde{\beta}_1 g_{YM}^4(\mu) \log \frac{p^2}{\mu^2} \right.
\]
\[
+ \tilde{\beta}_0^2 g_{YM}^4(\mu) \log^2 \frac{p^2}{\mu^2}\)
\]
where the tilde refers to the QCD coefficients of the \( \beta \) function:
\[
\tilde{\beta}_0 = \left( 11 - \frac{2}{3} n_l \right) \frac{1}{(4\pi)^2}
\]
\[
\tilde{\beta}_1 = \left( 102 - \frac{38}{3} n_l \right) \frac{1}{(4\pi)^2}
\]
(3.50)
We consider now the correlator of $g_{YM}^2 trF^2$:

$$
\langle g_{YM}^2 trF^2(p)g_{YM}^2 trF^2(-p) \rangle_{conn} = - \frac{2g_{YM}^4(\mu)}{\pi^2} p^4 \log \frac{p^2}{\mu^2} \left[ 1 + g_{YM}^2(\mu) \left( h_0 - 3\log \frac{p^2}{\mu^2} \right) + g_{YM}^4(\mu) \left( h_2 + h_3 \log \frac{p^2}{\mu^2} + g_{YM}^2(\mu) \log \frac{p^2}{\mu^2} \right) \right] = - \frac{2g_{YM}^4(\mu)}{\pi^2} p^4 \log \frac{p^2}{\mu^2} \times \left[ 1 + (h_0 - 2u)g_{av}^2(\mu) - \frac{\tilde{\beta}_0 g_{av}^2(\mu)}{g_{YM}^2(\mu)} \log \frac{p^2}{\mu^2} + (h_2 + 5u^2 + 2v - uh_0)g_{av}^4(\mu) + (h_3 + 3\tilde{\beta}_0 u)g_{av}^4(\mu) \log \frac{p^2}{\mu^2} + \frac{\tilde{\beta}_0 g_{av}^4(\mu) \log \frac{p^2}{\mu^2}}{g_{YM}^2(\mu)} \right]
$$

Choosing $u = \frac{h_0}{2}$ to cancel the finite term of order of $g^2$ in the square brackets we get for the coefficient of the term of order of $g^4 \log \frac{p^2}{\mu^2}$:

$$
h_3 + 3\tilde{\beta}_0 u = h_3 + \frac{3}{2} \tilde{\beta}_0 h_0 = \left( \frac{-2817}{2} + \frac{526}{3} n_l - \frac{14}{3} n_l^2 \right) \frac{1}{(4\pi)^4} + \frac{3}{2} \left( \frac{73}{3} - \frac{14}{3} n_l \right) \left( \frac{11}{3} - \frac{2}{3} n_l \right) \frac{1}{(4\pi)^4} = -204 + \frac{76}{3} n_l = -2\tilde{\beta}_1
$$

as predicted by Eq. (2.35) and by the computational experience gained in the pure $YM$ case. To cancel the finite term of order of $g^4$ we put:

$$
h_2 + \frac{5}{2} h_0^2 + 2v - \frac{h_0^2}{2} = 0
$$

Therefore, the correlator reads:

$$
\langle g_{YM}^2 trF^2(p)g_{YM}^2 trF^2(-p) \rangle_{conn} = - \frac{2g_{YM}^4(\mu)}{\pi^2} p^4 \log \frac{p^2}{\mu^2} \left[ 1 - \frac{\tilde{\beta}_0 g_{av}^2(\mu) \log \frac{p^2}{\mu^2} + 2\tilde{\beta}_1 g_{av}^4(\mu) \log \frac{p^2}{\mu^2}}{g_{YM}^2(\mu) \log \frac{p^2}{\mu^2}} \right]
$$

Now we follow the same steps as in the $n_l = 0$ case. The only differences are the coefficients of the $\beta$ function and the parameters $u, v$ that define the new renormalization scheme. The
Hence:

\[
\left( \frac{\beta(g_{uv})}{g_{uv}} \right)^2 \Pi \left( \frac{p}{\mu} \right) = \frac{2\beta_0}{\pi^2} \left( \frac{\beta}{\beta_0} g_{uv}^2(p) \right) \left( g_{uv}^2(p) + \frac{\beta_1}{\beta_0} g_{uv}^2(\mu) g_{uv}(p) - \frac{\beta_1}{\beta_0} g_{uv}^4(\mu) \right)
\]  

(3.56)

We recall that \( u = \frac{\hbar}{2} \), therefore:

\[
g_{uv}^2(\mu) = g_{YM}^2(\mu) \left( 1 + \frac{\hbar_0}{2} g_{YM}^2(\mu) + v g_{YM}^4(\mu) \right)
\]  

(3.57)

Hence we get:

\[
\left( \frac{\beta(g_{uv})}{g_{uv}} \right)^2 \Pi \left( \frac{p}{\mu} \right) = \frac{2\beta_0}{\pi^2} \left( g_{YM}^2(p) + \frac{\hbar_0}{2} g_{YM}^4(\mu) + \frac{\beta_1}{\beta_0} g_{YM}^2(\mu) g_{YM}(p) + \frac{\beta_1}{\beta_0} g_{YM}^4(\mu) \right)
\]  

(3.58)

Now we multiply the RHS of Eq.(3.58) by \( \frac{(\beta(g_{YM}))^2}{\beta(g_{uv})} \). Indeed, this is necessary to take into account the change of scheme performed to compute Eq.(3.58). The additional factor is:

\[
\frac{(\beta(g_{YM}))^2}{\beta(g_{uv})} = \frac{\left( 1 + \frac{\beta_1}{\beta_0} g_{YM}^2(\mu) \right)^2}{\left( 1 + \frac{\beta_1}{\beta_0} g_{uv}^2(\mu) \right)^2} = \left( 1 + \frac{\beta_1}{\beta_0} g_{YM}^2(\mu) \right) \left( 1 - 2 \frac{\beta_1}{\beta_0} g_{uv}^2(\mu) + 3 \frac{\beta_1^2}{\beta_0^2} g_{YM}^4(\mu) \right)
\]  

(3.59)

Therefore, the correlator in Eq.(3.58) becomes:

\[
\left( \frac{\beta(g_{YM})}{g_{YM}} \right)^2 \Pi \left( \frac{p}{\mu} \right) = \frac{2\beta_0}{\pi^2} \left( g_{YM}^2(p) + \frac{\hbar_0}{2} g_{YM}^4(\mu) + \frac{\beta_1}{\beta_0} g_{YM}^2(\mu) g_{YM}(p) + \frac{\beta_1}{\beta_0} g_{YM}^4(\mu) \right)
\]  

(3.60)
that has some dependence on the scale $\mu$ even after subtracting the contact terms. In the next section we get rid of this dependence by using a different method, that assumes the RG invariance of the correlator instead of checking it.

In any case the universal UV asymptotic behavior is in agreement with the RG estimate, i.e.:

$$\langle \frac{\beta(g_{YM})}{g_{YM}} tr F^2(p) \frac{\beta(g_{YM})}{g_{YM}} tr F^2(-p) \rangle_{\text{conn}} \sim \frac{p^4}{\tilde{\beta}_0 \log \frac{p^2}{\Lambda^2_{\text{MS}}}} \left( 1 - \frac{\tilde{\beta}_1}{\tilde{\beta}_0} \frac{\log \log \frac{p^2}{\Lambda^2_{\text{MS}}}}{\log \frac{p^2}{\Lambda^2_{\text{MS}}}} \right)$$

(3.61)

3.7 RG-invariant scalar correlator in SU(3) QCD with $n_l$ massless Dirac fermions

Firstly, we check the correctness of the finite parts of the scalar correlator in QCD, reconstructed in sect.(3.6) from its imaginary part, thanks to another result reported in [31]:

$$p^2 \frac{d}{dp^2} \Pi(p) \bigg|_{\log \frac{p^2}{\mu^2} = 0} = \frac{1}{\pi^2} \left[ -2 + \frac{\alpha_s}{\pi} \left( -\frac{73}{2} + \frac{7}{3} n_l \right) + \frac{\alpha_s^2}{\pi^2} \left( -\frac{37631}{48} + \frac{495}{4} \zeta(3) + n_l \left( \frac{7189}{72} - \frac{5}{2} \zeta(3) \right) - \frac{127}{54} n_l^2 \right) \right]$$

(3.62)

with:

$$p^4 \Pi(p) = \langle tr F^2(p) tr F^2(-p) \rangle$$

(3.63)

We recall that:

$$h_0 = \left( \frac{73}{3} - \frac{14}{3} n_l \right) \frac{1}{(4\pi)^2}$$

$$h_2 = \frac{37361}{6} - 990 \zeta(3) + n_l \left( -\frac{7189}{9} + 20 \zeta(3) \right) + \frac{508}{27} n_l^2$$

(3.65)
It is easy to verify that Eq. (3.62) and Eq. (3.65) are in agreement. Indeed:

\[
\frac{1}{\pi^2} \left[ -2 + \frac{\alpha_s}{\pi} \left( -\frac{73}{2} + \frac{7}{3} n_l \right) + \frac{\alpha_s^2}{\pi^2} \left( -\frac{37631}{48} + \frac{495}{4} \zeta(3) + n_l \left( \frac{7189}{72} - \frac{5}{2} \zeta(3) \right) - \frac{127}{34} n_l^2 \right) \right] \\
= -\frac{2}{\pi^2} \left[ 1 + \frac{g_Y^2 M}{4\pi^2} \left( \frac{73}{4} - \frac{7}{6} n_l \right) + \frac{g_Y^4 M^2}{(4\pi^2)^2} \left( \frac{37631}{96} - \frac{495}{8} \zeta(3) + n_l \left( \frac{7189}{144} + \frac{5}{4} \zeta(3) \right) + \frac{127}{108} n_l^2 \right) \right]
\]

From Eq. (3.64) it follows the derivative of the correlator of \( \frac{\beta(g Y M)}{g Y M} \) with two-loop accuracy:

\[
p^2 \frac{d}{dp^2} \frac{(\beta(g Y M))}{g Y M} \left( \Pi \left( \frac{p^2}{\mu^2} \right) \right)_{\log \frac{p^2}{\mu^2} = 0} = -\frac{2}{\pi^2} \beta_0^2 g_Y^4 M(\mu) \left( 1 + \frac{\beta_1}{\beta_0} g_Y^2 M(\mu) \right)^2 \left[ 1 + g_Y^2 M(\mu) h_0 + g_Y^4 M(\mu) h_2 \right] \tag{3.67}
\]

Secondly, we write \( g_Y M(p) \) instead of \( g_Y M(\mu) \) in Eq. (3.67) since \( \log \frac{p^2}{\mu^2} = 0 \Rightarrow p^2 = \mu^2 \) in order to get the large-momentum correlator in a manifestly RGI form. Exploiting the definition of the \( \beta \) function we can express \( d \log p^2 \) in terms of \( dq(p) \):

\[
\frac{dq}{d \log p} = \beta(g) \Rightarrow d \log(p^2) = 2 \frac{dq}{\beta(g)} = \frac{d(g^2)}{g \beta(g)} \tag{3.68}
\]

We integrate Eq. (3.67) to obtain:

\[
\frac{d}{d \log p^2} \left( \frac{\beta(g)}{g} \right)^2 \Pi \left( \frac{p^2}{\mu^2} \right)_{\log \frac{p^2}{\mu^2} = 0} = -\frac{2}{\pi^2} \beta_0^2 g_Y^4 M(p) \left( 1 + \frac{\beta_1}{\beta_0} g_Y^2 M(p) \right)^2 \left[ 1 + g_Y^2 M(p) h_0 + g_Y^4 M(p) h_2 \right]
\]

\[
\Rightarrow \left( \frac{\beta(g_Y M)}{g_Y M} \left( p \right) \right)^2 \Pi \left( \frac{p}{\mu} \right) - \left( \frac{\beta(g_Y M)}{g_Y M} \left( \mu \right) \right)^2 \Pi(1)
\]

\[
= \frac{2}{\pi^2} \beta_0^2 \int g_Y^2 M(\mu) \frac{g_Y^4 M(\mu)^2}{g_Y^2 M(\mu)} \left[ 1 + g_Y^2 M(\mu) h_0 + g_Y^4 M(\mu) h_2 \right] \frac{d(g_Y^2 M)}{\beta_0 g_Y^4 M(1 + \frac{\beta_1}{\beta_0} g_Y^2 M)}
\]

\[
= \frac{2}{\pi^2} \beta_0 \left[ g_Y^2 M(p) - g_Y^2 M(\mu) + \left( \frac{\beta_1}{2 \beta_0} \right) \left( g_Y^4 M(p) - g_Y^4 M(\mu) \right) + O(g^6) \right] \tag{3.69}
\]

Eq. (3.69) gives the manifestly RGI form of the correlator after subtracting the \( \mu \)-dependent contact terms.
3.8 Correlators in the coordinate representation

In this section we find the RG-improved expression for the perturbative correlators in the coordinate representation. This procedure has the main advantage that in the coordinate representation the contact terms do not occur, since they are eliminated by the Fourier transform. Indeed, the Fourier transform of \( p^4 \) is:

\[
\int p^4 e^{ip \cdot x} \frac{d^4 p}{(2\pi)^4} = \Delta^2 \delta(x)
\]  

(3.70)

that is supported only at \( x = 0 \). This implies that at points different from zero the contact terms do not occur. The RG improvement and the Fourier transform must commute up to perhaps finite scheme-dependent terms. Therefore, in this way we get another check of the asymptotic behavior. In appendix A we compute the Fourier transforms necessary to pass from the momentum to the coordinate representation. In particular we use the following results:

\[
\int (p^2)^2 \log \frac{p^2}{\mu^2} e^{ip \cdot x} \frac{d^4 p}{(2\pi)^4} = -\frac{2^6 \cdot 3}{\pi^2 x^8}
\]

\[
\int (p^2)^2 \left( \log \frac{p^2}{\mu^2} \right)^2 e^{ip \cdot x} \frac{d^4 p}{(2\pi)^4} = \frac{2^7 \cdot 3}{\pi^2 x^8} \left( -\frac{10}{3} + 2\gamma_E - \log \frac{4}{x^2 \mu^2} \right)
\]

\[
\int (p^2)^2 \left( \log \frac{p^2}{\mu^2} \right)^3 e^{ip \cdot x} \frac{d^4 p}{(2\pi)^4} = \frac{2^6 \cdot 3}{\pi^2 x^8} \left( -\frac{51}{2} + 40\gamma_E - 12\gamma_E^2 - (20 - 12\gamma_E) \log \frac{4}{x^2 \mu^2} - 3 \log^2 \frac{4}{x^2 \mu^2} \right)
\]

(3.71)

Using these formulae to compute the Fourier transform of the two-loop perturbative result in Eq.(3.13) we get, disregarding the finite parts in Eq.(3.13):

\[
- \int g_{\text{MS}}^4(\mu) p^4 \log \frac{p^2}{\mu^2} \left[ 1 - \beta_0 g_{\text{MS}}^2(\mu) \log \frac{p^2}{\mu^2} \right] e^{ip \cdot x} \frac{d^4 p}{(2\pi)^4}
\]

\[
= \frac{3 \cdot 2^6}{\pi^2 x^8} g_{\text{MS}}^4(\mu) + \beta_0 \frac{2^6}{\pi^2 x^8} \left[ -2 \log \frac{4}{x^2 \mu^2} + 4\gamma_E - \frac{20}{3} \right]
\]

\[
= \frac{3 \cdot 2^6}{\pi^2 x^8} g_{\text{MS}}^4(\mu) \left[ 1 + \left( -\beta_0 \frac{20}{3} + 4\beta_0 \gamma_E \right) g_{\text{MS}}^2(\mu) - 2\beta_0 \frac{2}{\pi^2 x^8} \left( \log \frac{4}{x^2 \mu^2} \right) \right]
\]

(3.72)

Firstly, the Fourier transform has produced a new finite part. Secondly, the coefficient of the logarithm in the square brackets is multiplied by two after the Fourier transform. This implies that the factor in the square brackets renormalizes four powers of \( g_{\text{MS}}^2(\mu) \), as opposed to the momentum representation, where only two powers of the coupling constant were renormalized. This is as expected, since in the coordinate representation the correlator is multiplicatively renormalizable as implied by Eq.(2.41).

To eliminate the finite term arising from the Fourier transform we change scheme defining:

\[
g_s^2(\mu) = g^2(\mu) \left[ 1 + \frac{1}{2} \left( -\beta_0 \frac{20}{3} + 4\beta_0 \gamma_E \right) g^2(\mu) \right]
\]

(3.73)
The integral in Eq. (3.72) reads:

\[
- \int \frac{g^4_{\text{MS}}(\mu) p^4}{\mu^2} \left[ 1 - \beta_0 \frac{g^2_{\text{MS}}(\mu) \log \frac{p^2}{\mu^2}}{p^2} \right] e^{ip \cdot x} \frac{d^4 p}{(2\pi)^4} = \frac{3 \cdot 2^6}{\pi^2 x^8} g^4(x)
\]

(3.74)

where \(g(x)\) is the one-loop running coupling in the coordinate scheme [30]:

\[
g^2(x) = g^2(\mu) \left[ 1 - \beta_0 g^2(\mu) \log \frac{4}{x^2 \mu^2} \right]
\]

(3.75)

Therefore, the renormalization group improved one-loop asymptotic expression for the correlator is:

\[
\langle \frac{g^2}{N} \text{tr} F^2(x) \frac{g^2}{N} \text{tr} F^2(0) \rangle_{\text{conn}} \sim \left( 1 - \frac{1}{N^2} \right) \frac{3 \cdot 2^6}{\pi^2 x^8} \log \frac{1}{x^2 \mu^2}
\]

(3.76)

The Fourier transform provides automatically the change in sign necessary to obtain a positive expression. This is due to the fact that in the coordinate representation contact terms do not occur. We now go one step further performing the Fourier transform of the three-loop propagators in Eq. (3.8) and in Eq. (3.10). We start with the scalar correlator up to the overall normalization:

\[
- \int \frac{g^4_{ab}(\mu) p^4}{\mu^2} \left[ 1 - \beta_0 g^2_{ab}(\mu) \log \frac{p^2}{\mu^2} - 2 \beta_1 g^4_{ab}(\mu) \log \frac{p^2}{\mu^2} + \beta_2 g^4_{ab}(\mu) \log \frac{p^2}{\mu^2} \right] e^{ip \cdot x} \frac{d^4 p}{(2\pi)^4}
\]

\[
= \frac{2^6 \cdot 3}{\pi^2 x^8} g^4_{ab}(\mu) \left[ 1 + \left( -\beta_0 \frac{20}{3} + 4 \beta_0 \gamma_E \right) g^2_{ab}(\mu) - 2 \beta_0 g^2_{ab}(\mu) \log \frac{4}{x^2 \mu^2} \right.
\]

\[
+ \left( 8 \beta_1 \gamma_E - \frac{40}{3} \beta_1 + \frac{51}{2} \beta_0^2 - 40 \beta_0^2 \gamma_E + 12 \beta_0^2 \gamma_E \right) g^4_{ab}(\mu) - 4 \beta_1 g^4_{ab}(\mu) \log \frac{4}{x^2 \mu^2}
\]

\[
- \beta_2 g^4_{ab}(\mu) \log \frac{4}{x^2 \mu^2} + 3 \beta_0 g^4_{ab}(\mu) \log \frac{4}{x^2 \mu^2} \]

(3.77)

The following scheme redefinition:

\[
g^2_{\text{st}}(\mu) = g^2_{ab}(\mu) \left( 1 + (2 \beta_0 \gamma_E - \frac{10}{3} \beta_0) g^2_{ab} + g^4_{ab}(\mu) \right)
\]

(3.78)

cancels the finite term of order of \(g^2\) in the square brackets and some terms of order of \(g^4 \log \frac{4}{x^2 \mu^2}\), leaving only the term proportional to \(-4 \beta_1\). Moreover, the finite term of order of \(g^4\) in the square brackets is cancelled by a suitable choice of \(t\), as in the previous section.
Eq. (3.77) now reads:  

\[ - \int g_{ab}^4(\mu) p^4 \log \frac{p^2}{\mu^2} \]

\[ \times \left[ 1 - \beta_0 g_{ab}^2(\mu) \log \frac{p^2}{\mu^2} - 2\beta_1 g_{ab}^4(\mu) \log \frac{p^2}{\mu^2} + \beta_0^2 g_{ab}^2(\mu) \log \frac{p^2}{\mu^2} \right] e^{ip \cdot \frac{d^4 p}{(2\pi)^4}} \]

\[ = \frac{2^6 \cdot 3}{\pi^2 x^8} g_{st}^4(\mu) \left[ 1 + \frac{2\beta_1}{\beta_0} g_{st}(x) \right] \]

\[ \times \left[ 1 - 2\beta_0 g_{st}^2(\mu) \log \frac{4}{x^2 \mu^2} - 4\beta_1 g_{st}^4(\mu) \log \frac{4}{x^2 \mu^2} + 3\beta_0^2 g_{st}^2(\mu) \log \frac{4}{x^2 \mu^2} \right] \]

\[ \times \left[ 1 + \frac{2\beta_1}{\beta_0} g_{st}(x) - \frac{2\beta_1}{\beta_0} g_{st}(\mu) \right] \]

\[ (3.79) \]

The scale dependent term in Eq. (3.79) occurs now at the order of \( g^6 \), while in the momentum representation occurred at the order of \( g^4 \). Now we multiply both sides of Eq. (3.79) by \((1 + \frac{\beta_1}{\beta_0} g_{st}^2(\mu))^2\), i.e. by the factor necessary to make the correlator \( RGI \). Reinserting the overall normalization, we obtain:

\[ \int \langle \frac{\beta(g)}{N} tr F^2(p) \frac{\beta(g)}{N} tr F^2(-p) \rangle_{conn} e^{ip \cdot \frac{d^4 p}{(2\pi)^4}} \]

\[ = \frac{1}{4\pi^2} \left[ 1 - \frac{1}{N^2} \right] \frac{2^6 \cdot 3}{\pi^2 x^8} g_{st}^4(x) \left( 1 + \frac{\beta_1}{\beta_0} g_{st}(\mu) \right)^2 \left( 1 - 2\beta_0 g_{st}^2(x) - 2\beta_1 g_{st}^4(x) \right) \]

\[ (3.80) \]

As a result the possible scale dependence is of order of \( g^8 \).

Performing the same steps for the pseudoscalar correlator in Eq. (3.10) we get:

\[ \int \langle \frac{g_{st}^2}{N} tr F \tilde{F}(p) \frac{g_{st}^2}{N} tr F \tilde{F}(-p) \rangle_{conn} e^{ip \cdot \frac{d^4 p}{(2\pi)^4}} \]

\[ = \frac{1}{4\pi^2} \left[ 1 - \frac{1}{N^2} \right] \frac{2^6 \cdot 3}{\pi^2 x^8} g_{st}^4(\mu) \left[ 1 - 2\beta_0 g_{st}^2(\mu) \log \frac{4}{x^2 \mu^2} - 2\beta_1 g_{st}^4(\mu) \log \frac{4}{x^2 \mu^2} + 3\beta_0^2 g_{st}^4(\mu) \log \frac{4}{x^2 \mu^2} \right] \]

\[ = \frac{1}{4\pi^2} \left[ 1 - \frac{1}{N^2} \right] \frac{2^6 \cdot 3}{\pi^2 x^8} g_{st}^4(x) \]

\[ (3.81) \]

The correlator in Eq. (3.81) is \( RGI \) in the coordinate representation with three-loop accuracy, while in the momentum representation scale-dependent terms of the order of \( g^4 \) occurred in Eq. (3.40). As in the momentum representation we find the correlator of \( tr F^{-2} \) summing the double of the scalar Eq. (3.79) and pseudoscalar Eq. (3.81) correlators. We
obtain:
\[
\frac{1}{2} \int \left( \frac{g^2}{N} \text{tr} F^{-2}(p) \frac{g^2}{N} \text{tr} F^{-2}(-p) \right)_{\text{conn}} e^{ip \cdot x} \frac{d^4 p}{(2\pi)^4}
\]
\[
= \frac{1}{4\pi^2} \left( 1 - \frac{1}{N^2} \right) \frac{2^7 \cdot 3}{\pi^2 x^8} \left( g_{st}^4(x) + g_{st}^4(x) + 2 \frac{\lambda_1}{\lambda_0} g_{st}^6(x) - 2 \frac{\lambda_1}{\lambda_0} g_{st}^2(\mu) g_{st}^4(x) \right)
\]

(3.82)

The scale dependence enters the term of order of \( g^6 \) as in the momentum representation in Eq.(3.42).

We check the correctness of the separation of the contact terms performed in the momentum representation. We verify to the order of the leading logarithm that the Fourier transform of the RG-improved expression in the momentum representation in Eq.(3.43) is equal to Eq.(3.82) in the coordinate representation. Within the leading logarithmic accuracy it is sufficient to put \( g^2(p) \):
\[
g^2(p) = \frac{g^2(\mu)}{1 + \beta_0 g^2(\mu) \log \frac{\mu^2}{\mu^2}}
\]

(3.83)

Therefore, the Fourier transform of the correlator in Eq.(3.43) can be computed reducing it to a series of positive powers of logarithms:
\[
\left( 1 - \frac{1}{N^2} \right) \frac{1}{\pi^2 \beta_0} \int \frac{g^2(\mu)}{1 + \beta_0 g^2(\mu) \log \frac{\mu^2}{\mu^2}} e^{ip \cdot x} \frac{d^4 p}{(2\pi)^4}
\]
\[
= \left( 1 - \frac{1}{N^2} \right) \frac{1}{\pi^2 \beta_0} \sum_{l=0}^{\infty} (-1)^l \int p^4 g^2(\mu)(\beta_0 g^2(\mu) \log \frac{p^2}{\mu^2})^l e^{ip \cdot x} \frac{d^4 p}{(2\pi)^4}
\]

(3.84)

We extract the leading logarithms of this Fourier transform. By leading we mean terms that have the highest power of logarithm with the power of \( g \) fixed. We use Eq.(A.20) that furnishes the leading logarithm of the Fourier transform:
\[
\int p^4 \left( \log \frac{p^2}{\mu^2} \right)^l e^{ip \cdot x} \frac{d^4 p}{(2\pi)^4} = - \frac{l! \Gamma(4)^2 5}{\pi^2} \frac{1}{x^8} \left( \log \frac{4}{x^2 \mu^2} \right)^{l-1} + \cdots
\]

(3.85)

Inserting it in Eq.(3.84) we obtain for the leading logarithms:
\[
\left( 1 - \frac{1}{N^2} \right) \frac{1}{\pi^2 \beta_0} \int \frac{g^2(\mu)}{1 + \beta_0 g^2(\mu) \log \frac{\mu^2}{\mu^2}} e^{ip \cdot x} \frac{d^4 p}{(2\pi)^4}
\]
\[
= \left( 1 - \frac{1}{N^2} \right) \frac{1}{\pi^2 \beta_0} \sum_{l=0}^{\infty} (-1)^l \frac{g^2(\mu)(\beta_0 g^2(\mu) \log \frac{p^2}{\mu^2})^l \Gamma(4)^2 5}{\pi^2} \frac{1}{x^8} \left( \log \frac{4}{x^2 \mu^2} \right)^{l-1}
\]

(3.86)

We compare it with the \( ASD \) correlator in the coordinate representation Eq.(3.82):
\[
\left( 1 - \frac{1}{N^2} \right) \frac{1}{\pi^2 \beta_0} \frac{2^5 \Gamma(4) g^4(x)}{\pi^2 x^8} \sim \left( 1 - \frac{1}{N^2} \right) \frac{1}{\pi^2 \beta_0} \frac{2^5 \Gamma(4)}{\pi^2 x^8} \frac{g^2(\mu)}{1 + \beta_0 g^2(\mu) \log \frac{4}{x^2 \mu^2}}^2
\]
\[
= \left( 1 - \frac{1}{N^2} \right) \frac{1}{\pi^2 \beta_0} \frac{2^5 \Gamma(4) g^4(\mu)}{\pi^2 x^8} \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} (-1)^{n+l}(\beta_0 g^2(\mu) \log \frac{x^2 \mu^2}{4})^{n+l}
\]

(3.87)
We want to prove that the two series in Eq.(3.86) and in Eq.(3.87) are equal. The proof is by induction. We prove it for the first non trivial term, i.e. for \( l = 1 \) in Eq.(3.86):

\[
(1 - \frac{1}{N^2}) \frac{1}{\pi^2 \beta_0} \frac{2^5}{\pi^2 x^8} \Gamma(4) g^4(\mu) = 0
\]  

(3.88)

that is equal to the term obtained from Eq.(3.87) putting \( n = l = 0 \). Assuming that the equality is valid up to the order of \( \left( \log \frac{x^2 \mu^2}{4} \right)^{m-1} \), we show that it holds at the order of \( \left( \log \frac{x^2 \mu^2}{4} \right)^{m-1} \). Indeed, the \( m \)-power of the logarithm occurs in Eq.(3.86) for \( l = m + 1 \):

\[
(1 - \frac{1}{N^2}) \frac{1}{\pi^2 \beta_0} \frac{2^5}{\pi^2 x^8} \Gamma(4) g^4(x) \quad \sim \quad (1 - \frac{1}{N^2}) \frac{1}{\pi^2 \beta_0} \left[ \frac{2^5}{\pi^2 x^8} \Gamma(4) g^4(\mu) \sum_{n,l=0}^{n+l \leq m} (-1)^{n+l} \beta_0 g^2(\mu) \log \frac{4}{x^2 \mu^2} \right]^{n+l} + \quad + (-1)^{m+1} \frac{2^5}{\pi^2 x^8} g_0^m \Gamma(4) g^{2m+1}(\mu) \left( \log \frac{4}{x^2 \mu^2} \right)^m \]  

(3.89)

The \( m \)-th power of the logarithm in Eq.(3.87) occurs for the \( m + 1 \) couples \( (n,l) \) such that \( l + n = m \):

\[
(1 - \frac{1}{N^2}) \frac{1}{\pi^2 \beta_0} \frac{2^5}{\pi^2 x^8} \Gamma(4) g^4(x) \quad \sim \quad (1 - \frac{1}{N^2}) \frac{1}{\pi^2 \beta_0} \left[ \frac{2^5}{\pi^2 x^8} \Gamma(4) g^4(\mu) \sum_{n,l=0}^{n+l \leq m} (-1)^{n+l} \beta_0 g^2(\mu) \log \frac{4}{x^2 \mu^2} \right]^{n+l} + \quad + \quad + \frac{2^5}{\pi^2 x^8} g_0^m \Gamma(4) g^{2m+1}(\mu) \left( \log \frac{4}{x^2 \mu^2} \right)^m \]  

(3.90)

For the inductive hypothesis the first term in the last expression is equal to the first one in Eq.(3.89), i.e.:

\[
(1 - \frac{1}{N^2}) \frac{1}{\pi^2 \beta_0} \frac{2^5}{\pi^2 x^8} \Gamma(4) g^4(\mu) \sum_{n,l=0}^{n+l \leq m} (-1)^{n+l} \beta_0 g^2(\mu) \log \frac{4}{x^2 \mu^2} \]  

(3.91)

The remaining terms, i.e. the terms of order of \( \log \frac{m}{x^2 \mu^2} \), are equal and therefore the proof by induction is complete.
4 ASD correlator in the Topological Field Theory

We briefly summarize the results for the glueball propagators in the TFT underlying large-$N$ YM [2] [3] [4] [5] [6]. For the ASD glueball propagator [5][4] 6:

$$\frac{1}{2} \langle g^2 \frac{1}{N} tr (F^{-2}(p)) \rangle \frac{g^2}{N} tr (F^{-2}(-p)) \rangle_{\text{conn}} = \frac{1}{\pi^2} \sum_{k=1}^{\infty} \frac{k^2 g^4_k \Lambda^6_W}{p^2 + k \Lambda^2_W} + \ldots$$  (4.1)

Besides, in the TFT the two-point correlators of certain scalar operators $O_{2L}$ of naive dimension $D = 2L$ that are homogeneous polynomials of degree $L$ in the ASD curvature $F^{-}[5][4]$ can be computed asymptotically for large $L$:

$$\langle O_{2L}(p) O_{2L}(-p) \rangle_{\text{conn}} = \text{const} \sum_{k=1}^{\infty} \frac{k^{2L-2} Z_k^{-L} \Lambda^2_W \Lambda^{4L-4}_W}{p^2 + k \Lambda^2_W}$$  (4.2)

The operators $O_{2L}$ occur as the ground state in the integrable sector of large-$N$ YM of Ferreti-Heise-Zarembo [29] asymptotically for large $L$. Ferreti-Heise-Zarembo have computed their one-loop anomalous dimension for large $L$ [29]:

$$\gamma_{0(O_{2L})} = \frac{1}{(4\pi)^2} \frac{5}{3} L + O(\frac{1}{L})$$  (4.3)

The ground state for $L = 2$ is the ASD operator that occurs in Eq.(4.1) for which $\gamma_{0(O_4)} = 2\beta_0$ exactly.

In Eq.(4.1) and in Eq.(4.2) $\Lambda_W$ is the RG invariant scale in the scheme in which it coincides with the mass gap. The functions $g^2(\frac{p^2}{\Lambda^2_W})$ and $Z(\frac{p^2}{\Lambda^2_W})$ are the solutions of the differential equations:

$$\frac{\partial g}{\partial \log p} = -\beta_0 g^3 + \frac{1}{(4\pi)^2} g^3 \frac{\partial \log Z}{\partial \log p}$$

$$\frac{\partial \log Z}{\partial \log p} = 2\gamma_0 g^2 + \cdots$$

$$\gamma_0 = \frac{1}{(4\pi)^2} \frac{5}{3}$$  (4.4)

where $p$ is equal to the square root of $p^2$. The definitions of $g_k$ and $Z_k$ are:

$$g_k = g(k)$$  (4.5)

$$Z_k = Z(k)$$  (4.6)

In [2] it is shown that Eq.(4.4) reproduces the correct universal one-loop and two-loop coefficients of the perturbative $\beta$ function of pure YM. Indeed, substituting in Eq.(4.4) we

---

6We use here a manifestly covariant notation as opposed to the one in the TFT [5][4].
\[
\frac{\partial g}{\partial \log p} = -\beta_0 g^3 + \frac{2\gamma_0}{(4\pi)^2} g^5 + \cdots
\]
\[
= (-\beta_0 g^3 + \frac{2\gamma_0}{(4\pi)^2} g^5)\left(1 + \frac{4}{(4\pi)^2} g^2\right) + \cdots
\]
\[
= -\beta_0 g^3 + \frac{2\gamma_0}{(4\pi)^2} g^5 - \frac{4\beta_0}{(4\pi)^2} g^5 + \cdots
\]
\[
= -\beta_0 g^3 + \frac{1}{(4\pi)^4} \frac{10}{3} g^5 - \frac{44}{3} \frac{1}{(4\pi)^4} \frac{1}{3} g^5 + \cdots
\]
\[
= -\beta_0 g^3 - \beta_1 g^5 + \cdots \tag{4.7}
\]

where:
\[
\beta_0 = \frac{1}{(4\pi)^2} \frac{11}{3} \tag{4.8}
\]
\[
\beta_1 = \frac{1}{(4\pi)^4} \frac{34}{3} \tag{4.9}
\]

These are the correct one- and two-loop coefficients that arise in perturbation theory of pure YM for the 't Hooft coupling. Therefore, the renormalization-group improved universal asymptotic behavior of \(g_k\) is:
\[
g_k^2 \sim \frac{1}{\beta_0 \log \frac{k}{c}} \left(1 - \frac{\beta_1}{\beta_0^2} \log \log \frac{k}{c}\right) + O\left(\frac{1}{\log^2 \frac{k}{c}}\right) \tag{4.10}
\]

and the renormalization group improved universal asymptotic behavior of \(Z_k^{-1}\) is:
\[
Z_k^{-1} \sim (g_k^2) \frac{\gamma_0}{\beta_0} \sim \left(\frac{1}{\beta_0 \log \frac{k}{c}} \left(1 - \frac{\beta_1}{\beta_0^2} \log \log \frac{k}{c}\right) + O\left(\frac{1}{\log^2 \frac{k}{c}}\right)\right)^{\frac{\gamma_0}{\beta_0}} \tag{4.11}
\]

In this section we find the asymptotics of the ASD propagator in Eqs.(4.1) and of the large-\(L\) propagator in Eq.(4.2) at the order of the leading and of the next-to-leading logarithms following the technique employed in [5] at the order of the leading logarithm.

To find the asymptotics of the glueball propagator for large \(L\) in Eq.(4.2) we follow the strategy explained in sect.(1.4) for the ASD correlator. Firstly, we highlight the physical terms contained in Eq.(4.2) neglecting the non-physical contact terms. Secondly, we extract the asymptotic behavior writing the sum in Eq.(4.2) as an integral [5]. Finally, we use the leading and next-to-leading expression for \(Z_k^{-1}\) in Eq.(4.11) to compare Eq.(4.2) with RG-improved perturbation theory.
We write Eq.(4.2) as [5][4]:

\[
\sum_{k=1}^{\infty} \frac{k^{2(L-1)}Z_k^{-L}A_k^2 \Lambda^4}{p^2 + k\Lambda^2 W} \\
= \sum_{k=1}^{\infty} \frac{((k\Lambda^2 W + p^2)(k\Lambda^2 W - p^2) + p^4)^{L-1}Z_k^{-L}A_k^2 \Lambda^4}{p^2 + k\Lambda^2 W} \\
= p^{4L-4} \sum_{k=1}^{\infty} \frac{Z_k^{-L}A_k^2 \Lambda^4}{p^2 + k\Lambda^2 W} \\
+ \sum_{k=1}^{\infty} \sum_{m=1}^{L-1} \frac{(L-1)}{m} p^{4(L-1-m)}(k\Lambda^2 W + p^2)^{m-1}(k\Lambda^2 W - p^2)^m Z_k^{-L}A_k^2 \Lambda^4 \\
\sim p^{4L-4} \sum_{k=1}^{\infty} \frac{Z_k^{-L}A_k^2 \Lambda^4}{p^2 + k\Lambda^2 W} + \ldots 
\tag{4.12}
\]

where the dots stand for contact terms.

As in sect.(1.4) we use the Euler-McLaurin formula to approximate the sum to an integral [5][4]:

\[
\sum_{k=k_1}^{\infty} G_k(p) = \int_{k_1}^{\infty} G_k(p) dk - \sum_{j=1}^{\infty} B_j \left[ \frac{\partial^{j-1} G_k(p)}{\partial k^{j-1}} \right]_{k=k_1} 
\tag{4.13}
\]

In our case the terms proportional to the Bernoulli numbers involve negative powers of \(p\) and they are therefore subleading with respect to the first term, hence we ignore them.

We obtain:

\[
\sum_{k=1}^{\infty} \frac{Z_k^{-L}A_k^2 \Lambda^4}{p^2 + k\Lambda^2 W} \sim \int_{1}^{\infty} \frac{Z_k^{-L}A_k^2 \Lambda^4}{k + \frac{p^2}{\Lambda^2 W}} dk
\tag{4.14}
\]

In order to compare Eq.(4.14) to the RG-improved perturbation theory, we substitute for \(Z_k^{-1}\) its leading and next-to-leading logarithmic behavior given by Eq.(4.11). We define:

\[
\gamma' = \frac{\gamma_0 L}{\beta_0} 
\tag{4.15}
\]

and:

\[
\nu = \frac{p^2}{\Lambda^2 W} 
\tag{4.16}
\]

The integral that determines the leading asymptotic behavior is:

\[
I_1^1(\nu) = \int_{1}^{\infty} \frac{1}{\beta_0 \log \left( \frac{k}{c} \right)} \gamma' \frac{dk}{k + \nu} 
\tag{4.17}
\]

The next-to-leading logarithmic behavior is determined by:

\[
I_2^2(\nu) = \int_{1}^{\infty} \left( \frac{1}{\beta_0 \log \left( \frac{k}{c} \right)} \left( 1 - \frac{\beta_1 \log \log \left( \frac{k}{c} \right)}{\beta_0 \log \left( \frac{k}{c} \right)} \right) \right) \gamma' \frac{dk}{k + \nu} 
\tag{4.18}
\]
\[ \gamma' = 2 \text{ for the ASD correlator and } \gamma' = \frac{2\eta}{\beta_0} L \text{ for the large-} L \text{ correlator. We show in the following that the leading and next-to-leading behavior of } I^2_c(\nu) \text{ is:} \]

\[ I^2_c(\nu) \sim \frac{1}{\gamma_0 L - \beta_0} \left[ \frac{1}{\beta_0 \log \frac{\nu^2}{\Lambda^2}} \left( 1 - \frac{\beta_1}{\beta_0} \log \log \frac{\nu^2}{\Lambda^2} \right) \right]^{\frac{2\eta}{\beta_0} L - 1} \]

(4.19)

Therefore, the asymptotic behavior of the correlator of the TFT for large- \( L \) is:

\[ \langle O_{2L}(p)O_{2L}(-p) \rangle_{\text{conn}} \sim p^{4L - 4} \frac{1}{\gamma_0 L - \beta_0} (g^2(p))^{\frac{2\eta}{\beta_0} L - 1} \]

(4.20)

It agrees with the naive RG estimate Eq. (2.43).

4.1 Asymptotic series to the order of the leading logarithm

We now perform an explicit expansion in series of \( I^1_c(\nu) \). Firstly, we change variables from \( k \) to \( k + \nu \):

\[ I^1_c(\nu) = \int_{1+\nu}^{\infty} \left( \frac{1}{\beta_0 \log \left( \frac{k - \nu}{\epsilon} \right)} \right)^{\gamma'} \frac{dk}{k} \]

(4.21)

We have that:

\[ \left[ \log \left( \frac{k' - \nu}{\epsilon} \right) \right]^{-\gamma'} = \left[ \log \left( \frac{k'}{\epsilon} \right) \right]^{-\gamma'} \left[ 1 + \frac{\log(1 - \frac{\nu}{k'})}{\log \left( \frac{k'}{\epsilon} \right)} \right]^{-\gamma'} \]

(4.22)

It is easy to see that if \( c < 1 \):

\[ \frac{\log(1 - \frac{\nu}{k'})}{\log \left( \frac{k'}{\epsilon} \right)} < 1 \]

(4.23)

We define:

\[ \epsilon = \frac{\log(1 - \frac{\nu}{k'})}{\log \left( \frac{k'}{\epsilon} \right)} \]

(4.24)

and we exploit the binomial formula [32]:

\[ (1 + \epsilon)^{-\gamma'} = \sum_{r=0}^{\infty} \binom{\gamma' + r - 1}{r} (-1)^r \epsilon^r \]

(4.25)

to obtain a series expansion. We proceed order by order in \( \epsilon \). At the order of \( \epsilon^1 \) the only contribution is:

\[ - \gamma' \epsilon \]

(4.26)

\( \epsilon \) can be further expanded in powers of \( \eta = \frac{\nu}{k'} \), since in the integration domain \( \eta < 1 \):

\[ \log(1 - \eta) = \sum_{m=1}^{\infty} \frac{(-1)^{2m+1}}{m} \eta^m \]

(4.27)

Up to the order of \( \eta^1 \) this expansion reads:

\[ - \gamma' \epsilon \sim - \gamma' \frac{\nu}{k' \log \left( \frac{k'}{\epsilon} \right)} \]

(4.28)
Substituting in $I^1_c(\nu)$ we get:

$$\int_{1+\nu}^{\infty} \frac{1}{k'} \left[ \beta_0 \log\left( \frac{k'}{c} \right) \right]^{-\gamma'} \left[ 1 + \frac{\log(1 - \frac{k'}{c})}{\log\left( \frac{k'}{c} \right)} \right]^{-\gamma'} dk'$$

$$\sim \int_{1+\nu}^{\infty} \frac{1}{k'} \left[ \beta_0 \log\left( \frac{k'}{c} \right) \right]^{-\gamma'} \left[ 1 + \gamma' \frac{\nu}{k' \log\left( \frac{k'}{c} \right)} \right] dk'$$

$$= \int_{1+\nu}^{\infty} \frac{1}{k'} \left[ \beta_0 \log\left( \frac{k'}{c} \right) \right]^{-\gamma'} dk' + \gamma' \nu \int_{1+\nu}^{\infty} \frac{1}{k'^2} \beta_0^{-\gamma'} \left[ \log\left( \frac{k'}{c} \right) \right]^{-\gamma'} dk' \quad (4.29)$$

From the first integral it follows the leading asymptotic behavior [5]:

$$\int_{1+\nu}^{\infty} \frac{1}{k'} \left[ \beta_0 \log\left( \frac{k'}{c} \right) \right]^{-\gamma'} dk' = \frac{1}{\gamma' - 1} \beta_0^{-\gamma'} \left[ \log\left( \frac{1 + \nu}{c} \right) \right]^{-\gamma' + 1} \quad (4.30)$$

Since for large $\nu$:

$$\left( \log\left( \frac{1 + \nu}{c} \right) \right)^{-1} \sim (\log \nu)^{-1} \quad (4.31)$$

it follows the leading asymptotic behavior of Eq.(4.2) [5]:

$$\langle O_{2L}(p)O_{2L}(-p) \rangle_{\text{conn}} \sim \frac{p^{4L-4}}{\gamma_0 L - \beta_0} \left[ \frac{1}{\beta_0 \log \left( \frac{p^2}{W^2} \right)} \right] \quad (4.32)$$

Performing the same steps for the ASD correlator, i.e. for $L = 2$ and $\gamma' = 2$, we get:

$$\frac{1}{\pi^2} \int_1^{\infty} \frac{(\beta_0 \log \left( \frac{k}{c} \right))^2}{k + \nu} \sim \frac{1}{\pi^2 \beta_0} \left( \beta_0 \log\left( \frac{1 + \frac{c^2}{W^2}}{c} \right) \right)^{-1} \sim \frac{1}{\pi^2 \beta_0} \frac{1}{\beta_0 \log \left( \frac{p^2}{W^2} \right)} \quad (4.33)$$

that agrees with the leading logarithm of the asymptotic behavior in Eq.(1.3).

We now compute the second term in the last line of Eq.(4.29), that is the first subleading term. We write it as:

$$\frac{\gamma' \nu \beta_0^{-\gamma'}}{c} \int_{1+\nu}^{\infty} \frac{1}{k^2} \left[ \log(k) \right]^{-\gamma'-1} dk \quad (4.34)$$

and we integrate by parts:

$$\frac{\gamma' \nu \beta_0^{-\gamma'}}{c} \int_{1+\nu}^{\infty} \frac{1}{k^2} \left[ \log(k) \right]^{-\gamma'-1} dk$$

$$= \frac{\gamma' \nu \beta_0^{-\gamma'}}{c} \left[ \frac{\log(k)^{-\gamma'-1}}{k} \right]_{1+\nu}^{\infty} - (\gamma' + 1) \int_{1+\nu}^{\infty} \frac{dk}{k^2} \left[ \log(k) \right]^{-\gamma'-2}$$

$$= \frac{\gamma' \nu \beta_0^{-\gamma'}}{c} \left[ \frac{c}{1 + \nu \log\left( \frac{1 + \nu}{c} \right)} \right]^{-\gamma'-1} - (\gamma' + 1) \int_{1+\nu}^{\infty} \frac{dk}{k^2} \left[ \log(k) \right]^{-\gamma'-2} \quad (4.35)$$

We notice that the second term in the last line has the same structure as the original integral but with a more negative power of the logarithm. This implies that it is a less relevant term.
Furthermore, since performing integration by parts repeatedly we always obtain integrals with the same structure, we can derive a possibly asymptotic series expansion for Eq. (4.34):

\[
\frac{\gamma' \nu \beta_0^{-\gamma'}}{c} \int_{\frac{1}{1+\nu}}^{\infty} \frac{dk}{k^\nu \log(k)^{\gamma'-1}}
\]

\[
= \beta_0^{-\gamma'} \frac{\nu}{1+\nu} \sum_{s=0}^{\infty} (-1)^s \left( \prod_{t=0}^{s} (\gamma'+t) \right) \log \left( \frac{1+\nu}{c} \right)^{-\gamma'-1-s}
\]

\[
= \beta_0^{-\gamma'} \frac{\nu^2}{p^2 + A^2} \sum_{s=0}^{\infty} (-1)^s \left( \prod_{t=0}^{s} (\gamma'+t) \right) \left[ \log \left( \frac{1+\nu^2}{c} \right) \right]^{-\gamma'-1-s}
\]  

(4.36)

Now that we have understood the technique, we derive a complete expression taking into account all the terms coming from the expansion of the logarithm in Eq. (4.27), simply substituting it in \( I_1^1(\nu) \):

\[
\int_{1+\nu}^{\infty} \frac{dk'}{k'} \left[ \beta_0 \log \left( \frac{k'}{c} \right) \right]^{-\gamma'} \left[ 1 - \gamma' \sum_{m=1}^{\infty} \frac{(-1)^{2m+1} \nu^m}{m \log \left( \frac{k'}{c} \right)^{\gamma'-1}} \right]
\]

\[
= \int_{1+\nu}^{\infty} \frac{dk'}{k'} \left[ \beta_0 \log \left( \frac{k'}{c} \right) \right]^{-\gamma'}
\]

\[
- \gamma' \beta_0^{-\gamma'} \sum_{m=1}^{\infty} \frac{(-1)^{2m+1} \nu^m}{m} \int_{1+\nu}^{\infty} \frac{dk'}{k'^{m+1}} \left[ \log \left( \frac{k'}{c} \right) \right]^{-\gamma'-1}
\]  

(4.37)

Focusing on the second term:

\[
\gamma' \beta_0^{-\gamma'} \sum_{m=1}^{\infty} \frac{(-1)^{2m+1} \nu^m}{m} \int_{1+\nu}^{\infty} \frac{dk'}{k'^{m+1}} \left[ \log \left( \frac{k'}{c} \right) \right]^{-\gamma'-1}
\]

\[
= \beta_0^{-\gamma'} \gamma' \sum_{m=1}^{\infty} \frac{(-1)^{2m+1} \nu^m}{m \log \left( \frac{k'}{c} \right)^{\gamma'-1}}
\]

\[
- \gamma' \beta_0^{-\gamma'} \sum_{m=1}^{\infty} \frac{(-1)^{2m+1} \nu^m}{m \log \left( \frac{k'}{c} \right)^{\gamma'-1} (\gamma'+1) \int_{1+\nu}^{\infty} \frac{dk}{k^m \log \left( \frac{k}{c} \right)^{\gamma'-2}}
\]

\[
= \beta_0^{-\gamma'} \gamma' \sum_{m=1}^{\infty} \frac{(-1)^{2m+1} \nu^m}{m \log \left( \frac{k'}{c} \right)^{\gamma'-1}}
\]

\[
- \sum_{s=0}^{\infty} (-1)^s \mu^m \int_{1+\nu}^{\infty} \frac{dk'}{k'^{m+1}} \left[ \log \left( \frac{1+\nu}{c} \right) \right]^{-\gamma'-1-s}
\]

\[
= \beta_0^{-\gamma'} \sum_{m=1}^{\infty} \sum_{s=0}^{\infty} (-1)^{s+1} \left( \frac{\nu}{1+\nu} \right)^m \int_{1+\nu}^{\infty} \frac{dk'}{k'^{m+1}} \left[ \log \left( \frac{1+\nu}{c} \right) \right]^{-\gamma'-s-1}
\]  

(4.38)

Therefore, at the first order in \( \epsilon \) we get:

\[
\int_{1+\nu}^{\infty} \frac{dk'}{k'} \left[ \beta_0 \log \left( \frac{k'}{c} \right) \right]^{-\gamma'} \left[ 1 + \frac{\log(1+\nu)}{\log \left( \frac{k'}{c} \right)} \right]^{-\gamma'}
\]

\[
\sim \int_{1+\nu}^{\infty} \frac{dk'}{k'} \left[ \beta_0 \log \left( \frac{k'}{c} \right) \right]^{-\gamma'}
\]

\[
+ \beta_0^{-\gamma'} \sum_{m=1}^{\infty} \sum_{s=0}^{\infty} (-1)^s \left( \frac{\nu}{1+\nu} \right)^m \int_{1+\nu}^{\infty} \frac{dk'}{k'^{m+1}} \left[ \log \left( \frac{1+\nu}{c} \right) \right]^{-\gamma'-s-1}
\]  

(4.39)
We find the subleading behavior keeping only the terms with \( s = 0 \) in Eq. (4.39). We obtain in the large \( \nu \) limit:

\[
\int_{1+\nu}^{\infty} dk' \frac{1}{k'} [\beta_0 \log(k' c)]^{-\gamma'} \left( 1 + \frac{\log(1 - \frac{k'}{c} \nu)}{\log(k' c)} \right)^{-\gamma'} \\
\sim \int_{1+\nu}^{\infty} dk' \frac{1}{k'} [\beta_0 \log(k' c)]^{-\gamma'} + \gamma' \beta_0^{-\gamma'} [\log \nu]^{-\gamma'-1} \sum_{m=0}^{\infty} \frac{1}{m^2} \\
\sim \frac{1}{\gamma' - 1} \beta_0^{-\gamma'} [\log \nu]^{-\gamma'+1} + \gamma' \beta_0^{-\gamma'} \zeta(2) [\log \nu]^{-\gamma'-1} \tag{4.40}
\]

It is interesting to notice that the transcendental function \( \zeta(2) = \frac{\pi^2}{6} \) occurs, as it often does in Feynman-graph computations.

4.2 Asymptotic series to the order of the next-to-leading logarithm

We now perform a series expansion of \( I_\nu^2(\nu) \):

\[
I_\nu^2(\nu) = \int_{1}^{\infty} \beta_0^{-\gamma'} \left( \frac{1}{\log(k' c)} \left( 1 - \frac{\beta_1 \log \log(k' c)}{\beta_0} \right) \right)^{\gamma'} \frac{dk'}{k + \nu} \\
= \beta_0^{-\gamma'} \int_{1+\nu}^{\infty} \left( \frac{1}{\log(k' c)} \left( 1 - \frac{\beta_1 \log \log(k' c)}{\beta_0} \right) \right)^{\gamma'} \frac{dk'}{k} \\
\sim \beta_0^{-\gamma'} \int_{1+\nu}^{\infty} \left[ \log(k' c) \right]^{\gamma'} \frac{dk'}{k} \\
\sim \beta_0^{-\gamma'} \int_{1+\nu}^{\infty} \left[ \log(k' c) \right]^{\gamma'} \frac{dk'}{k} \\
- \gamma' \frac{\beta_1}{\beta_0^2} \beta_0^{-\gamma'} \int_{1+\nu}^{\infty} \left[ \log(k' c) \right]^{\gamma'-1} \log \log \frac{k' c}{k} \frac{dk'}{k} \tag{4.41}
\]

The first integral has been evaluated in the previous section and the second term is the new contribution. We evaluate it at the leading order by changing variables and integrating by parts:

\[
\gamma' \frac{\beta_1}{\beta_0^2} \beta_0^{-\gamma'} \int_{1+\nu}^{\infty} \left[ \log(k' c) \right]^{\gamma'-1} \log \log \frac{k' c}{k} \frac{dk'}{k} \\
\sim \gamma' \frac{\beta_1}{\beta_0^2} \beta_0^{-\gamma'} \int_{1+\nu}^{\infty} \left[ \log(k' c) \right]^{\gamma'-1} \log \frac{k' c}{k} \frac{dk'}{k} \\
= \gamma' \frac{\beta_1}{\beta_0^2} \beta_0^{-\gamma'} \int_{\log \frac{k' c}{k}}^{1+\nu} t^{\gamma'-1} \log(t) dt \\
= \gamma' \frac{\beta_1}{\beta_0^2} \beta_0^{-\gamma'} \frac{1}{\gamma} \left( \log \left( \frac{1 + \nu}{c} \right) \right)^{-\gamma'} \log \log \frac{1 + \nu}{c} + \frac{1}{\gamma' \gamma} \left( \log \left( \frac{1 + \nu}{c} \right) \right)^{-\gamma'} \tag{4.42}
\]
The second term in brackets is subleading with respect to the first one. Putting together Eq. (4.42) and Eq. (4.30) we get for $I_{2}^{c}(\nu)$:

$$
\beta_{0}^{-\gamma'} \int_{1}^{\infty} \left( \frac{1}{\log\left(\frac{k}{c}\right)} \left( 1 - \frac{\beta_{1} \log \log \left(\frac{k}{c}\right)}{\beta_{0}^{2}} \right) \right) \frac{d\nu}{k + \nu} \\
\sim \frac{1}{\gamma' - 1} \beta_{0}^{-\gamma'} \left( \log \frac{1 + \nu}{c} \right)^{-\gamma' + 1} - \frac{\beta_{1}}{\beta_{0}^{2}} \beta_{0}^{-\gamma'} \left( \log \left(\frac{1 + \nu}{c}\right) \right)^{-\gamma'} \log \log \left(\frac{1 + \nu}{c}\right) \\
= \frac{\beta_{0}^{-\gamma'}}{\gamma' - 1} \left( \frac{\beta_{0} \log \left(\frac{1 + \nu}{c}\right)}{c} \right)^{-\gamma' + 1} - \frac{\beta_{1} (\gamma' - 1)}{\beta_{0}^{2}} \left( \log \left(\frac{1 + \nu}{c}\right) \right)^{-1} \log \log \left(\frac{1 + \nu}{c}\right) \\
\sim \frac{1}{\beta_{0}(\gamma' - 1)} \left( \frac{\beta_{0} \log \left(\frac{1 + \nu}{c}\right)}{c} \right)^{-\gamma' + 1} - \frac{\beta_{1} (\gamma' - 1)}{\beta_{0}^{2}} \left( \log \left(\frac{1 + \nu}{c}\right) \right)^{-1} \log \log \left(\frac{1 + \nu}{c}\right) \gamma' - 1 \\
\sim \frac{1}{\beta_{0}(\gamma' - 1)} (g^{2}(p))^{\gamma' - 1} + O \left( \frac{1}{\log^{2} \frac{k}{\Lambda_{W}}} \right) \quad (4.43)
$$

This result agrees with the RGI perturbative estimate in Eq. (2.43). Repeating the same steps for the ASD correlator we get:

$$
\frac{1}{\pi^{2}} \int_{1 + \nu}^{\infty} \frac{1}{k^{2}} \left[ \beta_{0} \log \left(\frac{k'}{c}\right) \right]^{-2} d\nu' - \frac{\beta_{1}}{\pi^{2} \beta_{0}^{4}} \int_{1 + \nu}^{\infty} \left[ \log \left(\frac{k - \nu}{c}\right) \right]^{-3} \log \log \left(\frac{k - \nu}{c}\right) d\nu' \\
\sim \frac{1}{\pi^{2} \beta_{0}} g^{2}(p) + O \left( \frac{1}{\log^{2} \frac{k}{\Lambda_{W}}} \right) \quad (4.44)
$$

Again this result agrees with the universal behavior of the RG-improved perturbation theory in Eq. (3.21).
4.3 Link with the Lerch transcendent and the polylogarithmic function

We may obtain the asymptotic behavior by a different method as an independent check, relating the relevant integrals to special functions and employing the known asymptotic behavior of the special functions.

We briefly recall the definition of the Lerch Zeta function [33, 34]:

\[ L(\lambda, s, a) = \sum_{n=0}^{\infty} \frac{e^{2\pi i \lambda n}}{(n + a)^s} \tag{4.45} \]

Setting \( z = e^{2\pi i \lambda} \), we obtain the Lerch transcendent [33, 34]:

\[ \Phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n + a)^s} \tag{4.46} \]

The Lerch transcendent admits the integral representation:

\[ \Phi(z, s, a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}e^{-at}}{1 - ze^{-t}} \, dt \tag{4.47} \]

which is valid for \( \text{Re}(a) > 0 \land \text{Re}(s) > 0 \land |z| < 1 \) or \( \text{Re}(a) > 0 \land \text{Re}(s) > 1 \land |z| = 1 \). The Lerch transcendent can be analytically continued to the region [35]:

\[ \mathcal{M} = \{(z, s, a) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{C} \times (\mathbb{C} \setminus \mathbb{Z})\} \tag{4.48} \]

Moreover, we exploit the following recursive formula:

\[ \Phi(z, s, a) = z^l \Phi(z, s, a + l) + \sum_{k=0}^{l-1} \frac{z^k}{(a + k)^s} \tag{4.49} \]

Finally, we use the relationship between the Lerch transcendent and the polylogarithmic function [34, 36]:

\[ \text{Li}_s(z) = z \Phi(z, s, 1) \tag{4.50} \]

where the polylogarithmic function is defined by:

\[ \text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} \tag{4.51} \]

4.4 Asymptotic behavior and polylogarithmic function

We start performing the change of variables \( t = \log \frac{k}{c} \) in the integral in Eq.(4.17):

\[ I_1(\nu) = \int_{1}^{\infty} \frac{[\beta_0 \log(k)]^{-\gamma'}}{k + \nu} \, dk = c\beta_0^{-\gamma'} \int_{\log \frac{1}{c}}^{\infty} \frac{t^{-\gamma'}}{c + \nu e^{-t}} \, dt \tag{4.52} \]

Setting \( c = e^{-\epsilon} \) in the limit \( \epsilon \rightarrow 0 \) we get the upper bound:

\[ I_1(\nu) = \beta_0^{-\gamma'} \int_{\log \frac{1}{c}}^{\infty} \frac{t^{-\gamma'}}{1 + \nu e^{-t}} \, dt \leq \beta_0^{-\gamma'} \int_{\epsilon}^{\infty} \frac{t^{-\gamma'}}{1 + \nu e^{-t}} \, dt = I_1(\nu) \tag{4.53} \]
but the upper bound is in fact asymptotic since varying $c$ is equivalent to a change of scheme. Therefore, we take the limit $\epsilon \to 0$ in order to express $I_1^1$ in terms of the integral representation of the Lerch transcendent in Eq.(4.47). We get:

$$I_1^1(\nu) = \beta_0^{-\gamma'} \Gamma(-\gamma' + 1) \Phi(-\nu, -\gamma' + 1, 0)$$

(4.54)

We now exploit the relation in Eq.(4.49) with $n = 1$, $a = 0$, $z = -\nu$ and $s = -\gamma' + 1$:

$$\Phi(-\nu, -\gamma' + 1, 0) = z \Phi(-\nu, -\gamma' + 1, 1)$$

(4.55)

Finally, we find the relation with the polylogarithmic function:

$$I_1^1(\nu) = \beta_0^{-\gamma'} \Gamma(-\gamma' + 1) \text{Li}_{-\gamma' + 1}(-\nu)$$

(4.56)

Now we use the following asymptotic expansion of $\text{Li}_s$ [36]:

$$\text{Li}_s(z) = \sum_{j=0}^{\infty} (-1)^j (1 - 2^{1-2j})(2\pi)^{2j} B_{2j} \frac{[\log(-z)]^{s-2j}}{(2j)! \Gamma(s + 1 - 2k)}$$

(4.57)

to find an asymptotic expansion for $I_1^1(\nu)$:

$$I_1^1(\nu) = \beta_0^{-\gamma'} \Gamma(-\gamma' + 1) \sum_{j=0}^{\infty} (-1)^j (1 - 2^{1-2j})(2\pi)^{2j} B_{2j} \frac{[\log \nu]^{-\gamma' + 1 - 2j}}{(2j)! \Gamma(-\gamma' + 2 - 2j)}$$

(4.58)

We get the leading behavior of $I_1^1(\nu)$ from the $j = 0$ term in Eq.(4.58):

$$I_1^1(\nu) \sim \frac{[\beta_0 \log \nu]^{-\gamma' + 1}}{\beta_0(\gamma' - 1)}$$

(4.59)

Keeping also the $j = 1$ term we obtain:

$$I_1^1(\nu) \sim \frac{[\beta_0 \log \nu]^{-\gamma' + 1}}{\beta_0(\gamma' - 1)} + \gamma' \beta_0^{-\gamma'} \frac{\pi^2}{6} \log \nu \sim -\gamma' \beta_0^{-\gamma'} \frac{\pi^2}{6} \log \nu$$

(4.60)

in perfect agreement with Eq.(4.40) since $\zeta(2) = \frac{\pi^2}{6}$. Reinserting the momentum $p$ in the definition of $\nu$ the asymptotic result is:

$$I_1^1 \left( \frac{p^2}{\Lambda^2} \right) \sim \frac{[\beta_0 \log(\frac{p^2}{\Lambda^2})]^{-\frac{\pi^2}{6} \frac{L+1}{L-\beta_0}}}{\gamma_0 L - \beta_0} + \gamma_0 L \frac{\pi^2}{6} [\beta_0 \log(\frac{p^2}{\Lambda^2})]^{-\frac{\pi^2}{6} \frac{L-1}{L-\beta_0}}$$

(4.61)

Using the same technique we find the next-to-leading logarithmic behavior of $I_2^2$. Indeed, also in this case we obtain an upper bound putting $c = e^{-\epsilon}$ and taking the limit $\epsilon \to 0$:

$$I_2^2(\nu) = c \beta_0^{-\gamma'} \int_{-\log \frac{1}{c}}^{\infty} \left( \frac{1}{t} \left( 1 - \frac{\beta_1 \log t}{\beta_0^2 t} \right) \right)^{\gamma'} \frac{dt}{c + \nu e^{-t}}$$

(4.62)

$$\leq \beta_0^{-\gamma'} \int_{-\infty}^{\infty} \left( \frac{1}{t} \left( 1 - \frac{\beta_1 \log t}{\beta_0^2 t} \right) \right)^{\gamma'} \frac{dt}{1 + e^t e^{-t}} = I_1^{2-\epsilon}(\nu)$$

(4.63)
but the upper bound is in fact asymptotic since varying \( c \) is equivalent to a change of scheme. We now expand \( I^2_{1-\epsilon}(\nu) \):

\[
I^2_{1-\epsilon}(\nu) \sim \beta_0^{-\gamma'} \int_{\epsilon}^{\infty} \frac{1}{t^{\gamma'}} \left( 1 - \frac{\beta_1 \gamma' \log t}{\beta_0^2} \right) \frac{dt}{1 + \frac{\nu + e^{-t}}{e^{-t}}}
\]

The first term is equal to \( I^1_{1-\epsilon}(\nu) \), while the second one is the new contribution. This new term can be linked again to the polylogarithmic function using the relation:

\[
\int_{\epsilon}^{\infty} t^{-\gamma' - 1} \log t = -\frac{\partial}{\partial \alpha} t^{-\alpha} \bigg|_{\alpha = \gamma' + 1}
\]

We find:

\[
I^2_{1-\epsilon}(\nu, -\gamma') \sim I^1_{1-\epsilon}(\nu, -\gamma') + \frac{\beta_1 \gamma'}{\beta_0^2} \frac{\partial}{\partial \alpha} I^1_{1-\epsilon}(\nu, -\alpha) \bigg|_{\alpha = \gamma' + 1}
\]

We take the limit \( \epsilon \to 0 \) and we perform the derivative in the asymptotic expression of \( I^1_{1}(\nu, -\alpha) \) in Eq. (4.58). Keeping only the leading contribution we obtain:

\[
\frac{\partial}{\partial \alpha} I^1_{1}(\nu, -\alpha) \bigg|_{\alpha = \gamma' + 1} = \beta_0^{-\gamma'} \frac{\Gamma(-\gamma')}{\Gamma(-\gamma' + 1)} (\log \nu)^{-\gamma'} \log \log \nu = -\beta_0^{-\gamma'} (\log \nu)^{-\gamma'} \log \log \nu
\]

Thus the asymptotic behavior to the next-to-leading logarithmic order is:

\[
I^2_{\epsilon}(\nu) \sim \frac{[\beta_0 \log \left( \frac{p^2}{\Lambda^2} \right)]^{-\frac{\beta_1}{\beta_0} L + 1}}{\gamma_0 L - \beta_0} - \frac{\beta_1}{\beta_0^2} (\beta_0 \log \frac{p^2}{\Lambda^2} \log p^2 \Lambda^2) \Gamma^{-\frac{\beta_1}{\beta_0} L + 1} \log \log \frac{p^2}{\Lambda^2}
\]

\[
\sim \frac{1}{\gamma_0 L - \beta_0} \left[ \frac{1}{\beta_0 \log \frac{p^2}{\Lambda^2}} \left( 1 - \frac{\beta_1 \log \log \frac{p^2}{\Lambda^2}}{\beta_0 \log \frac{p^2}{\Lambda^2}} \right) \right]^{\frac{\beta_1}{\beta_0} L - 1}
\]

that agrees perfectly with the RG estimate Eq. (4.43).

### A Fourier Transforms

In this appendix we compute Fourier transforms of the kind:

\[
\int (p^2)^L \left( \log \frac{\mu^2}{p^2} \right)^n e^{ip \cdot x} \frac{d^4p}{(2\pi)^4}
\]

for \( L \) and \( n \) positive integers. We start writing:

\[
(p^2)^L \left( \log \frac{\mu^2}{p^2} \right)^n = \left( \mu^2 \right)^L \frac{\partial^n}{\partial \alpha^n} \left( \frac{p^2}{\mu^2} \right)^{\alpha} \bigg|_{\alpha = L}
\]

and:

\[
\left( \frac{p^2}{\mu^2} \right)^{\alpha} = \frac{1}{\Gamma(-\alpha)} \int_0^\infty e^{-\frac{p^2}{\mu^2} t^{-\alpha - 1}} dt
\]
Substituting in Eq. (A.1) and exchanging the order of integration we get:

\[
\left. \left( \frac{\mu^2}{2\pi^4} \frac{\partial^n}{\partial \alpha^n} \left( \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha - 1} \int e^{-\frac{p^2}{\mu^2} t + ip \cdot x} d^4 p d t \right) \right|_{\alpha = \lambda} \right) \tag{A.4}
\]

The integral on \( p \) is now Gaussian and we obtain:

\[
\left. \left( \frac{\mu^2}{2\pi^4} \frac{\partial^n}{\partial \alpha^n} \left( \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha - 1} t^{-\frac{1}{2} \frac{\mu^2}{4} \frac{p^2}{4} - \frac{1}{2} \frac{\mu^2}{4} \frac{t^2}{4} dt \right) \right|_{\alpha = \lambda} \right) \tag{A.6}
\]

We compute the last integral reducing it to a \( \Gamma \) function by the substitution \( t' = \frac{1}{4} \frac{x^2 \mu^2}{t} \):

\[
\int_0^\infty t^{-\alpha - 3} e^{-\frac{1}{2} x^2 \mu^2} dt \tag{A.7}
\]

\[
= \int_0^\infty e^{-t'} \left[ \frac{x^2 \mu^2}{4} (t')^{-1} \right]^{-\alpha - 3} \frac{x^2 \mu^2}{4} (t')^{-2} dt' \tag{A.8}
\]

\[
= \left( \frac{x^2 \mu^2}{4} \right)^{-\alpha - 2} \Gamma (\alpha + 2) \tag{A.9}
\]

The Fourier transform now reads:

\[
\left. \left( \frac{\mu^2}{2\pi} \right)^L \left( \log \frac{p^2}{\mu^2} \right)^n e^{i p \cdot x} d^4 p \right|_{\alpha = \lambda} \tag{A.10}
\]

We evaluate the Fourier transform in some cases by means of Mathematica. In particular we are interested in the cases \( L = 2 \) with \( n = 1, 2, 3 \). We obtain:

\[
\int (p^2)^2 \log \frac{p^2}{\mu^2} e^{i p \cdot x} d^4 p = -\frac{2^6 \cdot 3}{\pi^2 x^8}
\]

\[
\int (p^2)^2 \left( \log \frac{p^2}{\mu^2} \right)^2 e^{i p \cdot x} d^4 p = \frac{2^7 \cdot 3}{\pi^2 x^8} \left( -\frac{10}{3} + 2 \gamma_E - \log \frac{4}{x^2 \mu^2} \right)
\]

\[
\int (p^2)^2 \left( \log \frac{p^2}{\mu^2} \right)^3 e^{i p \cdot x} d^4 p = \frac{2^9 \cdot 3}{\pi^2 x^8} \left( -\frac{51}{2} + 40 \gamma_E - 12 \gamma_E + \right.
\]

\[\left. - (20 - 12 \gamma_E) \log \frac{4}{x^2 \mu^2} - 3 \log^2 \frac{4}{x^2 \mu^2} \right) \tag{A.11}
\]

where \( \gamma_E \) is Euler-Mascheroni constant.
Moreover, again by means of Mathematica, we evaluate Eq. (A.10) for generic \( L \) and \( n = 1 \) or \( n = 2 \):

\[
\int (p^2)^L \left( \log \frac{p^2}{\mu^2} \right) e^{i p \cdot x} \frac{d^4 p}{(2\pi)^4} = -\frac{(-4)^L L! \Gamma(2 + L)}{\pi^2} x^{-2(2 + L)}
\]

(A.12)

\[
\int (p^2)^L \left( \log \frac{p^2}{\mu^2} \right)^2 e^{i p \cdot x} \frac{d^4 p}{(2\pi)^4} = 2\frac{(-4)^L L! \Gamma(2 + L)}{\pi^2} \left( \gamma_E - H(L) - \psi(2 + L) + \log \left( \frac{x^2 \mu^2}{4} \right) \right) x^{-2(2 + L)}
\]

(A.13)

where \( H(L) \) is the harmonic number defined by:

\[
H(L) = \sum_{i=1}^{L} \frac{1}{i}
\]

(A.14)

and \( \psi \) is the digamma function defined by:

\[
\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}
\]

(A.15)

Inverting Eqs. (A.12-A.13) we obtain:

\[
\int x^{-2(2 + L)} e^{-i p \cdot x} d^4 x = -\frac{\pi^2}{(-4)^L L! \Gamma(2 + L)} (p^2)^L \left( \log \frac{p^2}{\mu^2} \right)
\]

(A.16)

\[
\int x^{-2(2 + L)} \log \left( \frac{x^2 \mu^2}{4} \right) e^{-i p \cdot x} d^4 x = \frac{\pi^2}{(-4)^L L! \Gamma(2 + L)} (p^2)^L \log \frac{p^2}{\mu^2} + \frac{\pi^2}{2 (-4)^L L! \Gamma(2 + L)} (\gamma_E - H(L) - \psi(2 + L)) (p^2)^L \log \frac{p^2}{\mu^2}
\]

(A.17)

We are also interested in extracting the leading logarithms in Eq. (A.10) in the generic case. We obtain the leading logarithm from the terms that contain \( n - 1 \) derivatives with respect to \( \alpha \) of \( \left( \frac{x^2 \mu^2}{4} \right)^{-\alpha - 2} \) and one derivative of \( \frac{1}{\Gamma(-\alpha)} \), since otherwise we get zero because \( \frac{1}{\Gamma(-\alpha)} = 0 \) for \( L \) a positive integer:

\[
\int (p^2)^L \left( \log \frac{p^2}{\mu^2} \right) e^{i p \cdot x} \frac{d^4 p}{(2\pi)^4} = n \frac{\Gamma(L + 2 \mu^2)}{\pi^2} \left( \frac{\Gamma'(-\alpha)}{\Gamma^2(-\alpha)} \right) \left. \frac{1}{(x^2 \mu^2)^{L+2}} \right|_{\alpha \to L} \left( \log \frac{4}{x^2 \mu^2} \right)^{n-1} + \cdots
\]

(A.18)

The factor of \( n \) occurs because there are \( n \) such terms performing the \( n \)-th derivative.

The limit \( \left( \frac{\Gamma'(-\alpha)}{\Gamma^2(-\alpha)} \right) \bigg|_{\alpha \to -L} \frac{(-1)^L L!}{L!} \). The result is:

\[
\left. \frac{\Gamma'(-\alpha)}{\Gamma^2(-\alpha)} \right|_{\alpha \to L} = (-1)^{L+1} L!
\]

(A.19)
Therefore, the leading logarithm of the Fourier transform is:

\[
\int (p^2)^L \left( \log \frac{p^2}{\mu^2} \right)^n \frac{d^4 p}{(2\pi)^4} e^{ip \cdot x} = n \Gamma(L + 2) 2^{2L} \pi^2 (-1)^{L+1} \frac{1}{(x^2)^{L+2}} \left( \log \frac{4}{x^2 \mu^2} \right)^{n-1} + \cdots \tag{A.20}
\]

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