Knotted Topological Phase Singularities of Electromagnetic Field

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(Dated: February 2, 2008)

In this paper, knotted objects (RS vortices) in the theory of topological phase singularity in electromagnetic field have been investigated in details. By using the \( \phi \)-mapping topological current theory proposed by Prof. Duan, we rewrite the topological current form of RS vortices and use this topological current we reveal that the Hopf invariant of RS vortices is just the sum of the linking and self-linking numbers of the knotted RS vortices. Furthermore, the conservation of the Hopf invariant in the splitting, the mergence and the intersection processes of knotted RS vortices is also discussed.

PACS numbers: 03.65.Vf, 03.50.De, 42.25.-p, 02.40.Xx

I. INTRODUCTION

Topological phase singularities as topological objects of wave fields appear in a variety of physical, chemical, and biological scenarios, such as the quantized vortices in superfluid or superconductor systems\(^1\),\(^2\), the vortices phase singularities in Bose-Einstein condensates\(^3\), the streamlines or singularities in quantum mechanical wave-functions\(^4\),\(^5\), the optical vortices in optical wave systems\(^6\),\(^7\),\(^8\),\(^9\),\(^10\),\(^11\), and the vortex filaments in chemical reaction and molecular diffusion\(^12\),\(^13\).

In particular, the study of phase singularities in electromagnetic field or in optics has evolved into a separate area of research, both theoretical and experimental, called singular optics\(^14\). Various types of phase singularities in optics have been found. In the usual scalar theory of light in optics, when the vector nature of light can be neglected, one may build a single complex scalar field to describe the light. The optical vortices emerge as phase singularities of the complex scalar field, which are located at the zeros of the field. But for a full electromagnetic field which can be described by a complex vector, the phase singularity is very difficult to define because the requirement that all the field components can not simultaneously vanish. However, it is interesting from a fundamental theoretical viewpoint to investigate the phase singularities of the full electromagnetic field. Such singularities are just the Riemann-Silberstein (RS) vortices which associated with the zeros of square of RS vector, and was firstly studied recently by I. Bialynicki-Birula and Z. Bialynicka-Birula\(^15\). Since the first proposal of RS vortices, a great many of works has been focused on the basic character of this phase singularities, especially the geometrical and topological character\(^16\),\(^17\),\(^18\). In the Ref.\(^18\), the authors have derived the topological current structures of the RS vortex line.

On the other hand, the phase singularities usually form a net of lines in three dimensional space, and a very important case is that they are closed and knot-ted curves. The knot-like configurations exist in a variety of physical scenarios, including Bose-Einstein condensations\(^19\), chemical reaction and molecular diffusion systems\(^12\),\(^13\), optical wave systems\(^8\),\(^10\),\(^11\),\(^20\) and field theory\(^22\),\(^23\). For a knotted family, it is well known that there are important characteristic numbers to describe its topology, such as the self-linking and the linking numbers. So in research into knotted configurations in physics, one should pay much attention to these knotted characteristics. In this paper, we will use the topological viewpoint to study the knotted topological phase singularities, i.e. the RS vortices with the Hopf invariant which usually can be used to describe the linkage of the knotted family in mathematics\(^24\), and reveal the inner relationship between the Hopf invariant and the topological knotted characteristic numbers of knotted RS vortices. Furthermore, the conservation of the Hopf invariant in the splitting, the mergence and the intersection processes is also discussed in details.

II. A BRIEF REVIEW OF RS VOR TICES AND THEIR TOPOLOGICAL STRUCTURES

In the theory of the complex form of the Maxwell equations, the electric and magnetic field vectors can be replaced by a complex vector \( \vec{F} \) which called RS vector\(^12\),\(^16\),

\[
\vec{F} = \frac{1}{\sqrt{2}} (\vec{E} + i\vec{B}).
\]

The Maxwell equations in free space written in terms of \( \vec{F} \) read:

\[
i \partial_t \vec{F} = \nabla \times \vec{F},
\]

\[
\nabla \cdot \vec{F} = 0.
\]

The RS vector offers a very convenient representation of the electromagnetic field, especially in the study of phase singularities. Since \( F^2 \) is a complex sum of two electromagnetic invariants, we define the phase of the electromagnetic filed \( \varphi(x) \) as half of the phase of square...
of the RS vector
\[ \vec{F}^2(x) = e^{2i\varphi(x)\|\vec{F}^2(x)\|}. \] (3)
In full analogy with equation of nonrelativistic wave mechanics, we may define a "velocity" four-vector \( v_\mu \) as
\[ v_\mu = \frac{(\vec{F}^2)^* \partial_\mu \vec{F}^2 - \vec{F}^2 \partial_\mu (\vec{F}^2)^*)}{4i\|\vec{F}^2\|^2}. \] (4)
Because \( \vec{F}^2 \) is a complex sum of two electromagnetic invariants, it can also be written as
\[ \vec{F}^2 = S + iP = \frac{1}{2} (\vec{E}^2 - \vec{B}^2) + i \vec{E} \cdot \vec{B}, \] (5)
then the velocity \( v_\mu \) is true relativistic four-vector
\[ v_\mu = \frac{S \partial_\mu P - P \partial_\mu S}{2(S^2 + P^2)}, \] (6)
in three-dimensional space, the spatial component of the four-velocity is
\[ \vec{v} = \frac{S \nabla P - P \nabla S}{2(S^2 + P^2)}. \] (7)

For convenience, we write \( \vec{F}^2 \) as \( \psi \), that is to say
\[ \vec{F}^2 = \psi(\vec{x}, t) = \phi^1(\vec{x}, t) + i \phi^2(\vec{x}, t), \] (8)
where \( \phi^1 = S \) and \( \phi^2 = P \). The velocity Eq. (7) can be rewritten as
\[ \vec{v} = \frac{1}{16i \|\psi\|^2} (\psi^* \nabla \psi - \nabla \psi^* \psi) = \nabla \varphi, \] (9)
it becomes the gradient of the phase factor \( \varphi \). Owing to the works done by Bialynicki-Birula and Bialynicka-Birula, the phase singularities of the electromagnetic field can be well defined by using of \( \varphi \), which is just the phase factor of the electromagnetic field. This singularities are the so-called RS vortex lines which can be found at the intersection of the \( S = 0 \) and \( P = 0 \) surfaces.

Using the definition of the RS vortices, it is interesting to study the topological properties of these topological phase singularities. In Ref. [18], by making use of the topological viewpoint the authors have obtained the topological inner structures of the RS vortices, and they have found that the RS vortices are classified by Hopf index, Brouwer degree in geometry. In the following discussions of this section, for convenience to discuss the knotted topology of RS vortices, we will rewrite the topological form of the RS vortices which have been derived in Ref. [18].

According to the \( \phi \)-mapping theory proposed by Prof. Duan [22, 23], the topological density vector current \( \tilde{\Omega} \) can be defined as
\[ \tilde{\Omega} = \frac{1}{\pi} \nabla \times \vec{v}, \] (10)
Form Eq. (10), we directly obtain a trivial curl-free result: \( \tilde{\Omega} = \frac{1}{\pi} \nabla \times \nabla \varphi = 0 \). But in topology, because of the existence of the phase singularities, i.e. RS vortices, \( \tilde{\Omega} \) does not vanish [1]. So in the following discussions, we will study what the exact expression for \( \tilde{\Omega} \) is in topology.

Introducing the unit vector \( n^a = \phi^a/\|\phi\| (a = 1, 2; n^a n^a = 1) \), one can reexpress the velocity field \( \vec{v} \) as
\[ \vec{v} = \frac{1}{2} \epsilon_{abc} n^a \nabla n^b, \] (11)
and \( \tilde{\Omega} \) is
\[ \tilde{\Omega} = \frac{1}{2\pi} \epsilon^{ijk} \epsilon_{abc} \partial_j n^a \partial_k n^b, \] (12)
Obviously, it is just the topological current of the RS vortices in three dimensional space. It is known that in \( \phi \)-mapping theory, the topological current \( \tilde{\Omega} \) is rewritten as
\[ \Omega^i = \delta^2(\phi) D^i(\frac{\phi}{x}), \] (13)
where the \( D^i(\phi) \) is the vector Jacobians of \( \psi(\vec{r}) \), and it is defined as
\[ D^i(\phi) = \frac{1}{2} \epsilon^{ijk} \epsilon_{abc} \partial_j \phi^a \partial_k \phi^b, \] (14)
we can see from the expression [13] that \( \tilde{\Omega} \) is non-vanishing only if \( \tilde{\phi} = 0 \), i.e., the existence of the RS vortices, so it is necessary to study these zero solutions of \( \phi \). In three dimensions space, these solutions are some isolate zero lines, which are the so-called RS vortices in three dimensional space.

Under the regular condition
\[ D(\phi/x) \neq 0, \]
the general solutions of
\[ \phi^1(t, x^1, x^2, x^3) = 0, \quad \phi^2(t, x^1, x^2, x^3) = 0 \] (15)
can be expressed as
\[ x^1 = x^1_k(s, t), \quad x^2 = x^2_k(s, t), \quad x^3 = x^3_k(s, t), \] (16)
which represent \( N \) isolated singular strings \( L_k \) with string parameter \( s \) \( (k = 1, 2, \ldots, N) \). These singular strings solutions are just the RS vortices solutions in three dimensions space.

In \( \delta \)-function theory [24], one can obtain in three dimensions space
\[ \delta^2(\phi) = \sum_{k=1}^{N} \beta_k \int_{L_k} \frac{\delta^3(\vec{x} - \vec{x}_k(s))}{|D(\phi)_{x_k}|} ds, \] (17)
where
\[ D(\phi)_{x_k} = \frac{1}{2} \epsilon^{ijk} \epsilon_{mn} \partial_m \phi^n \partial_n \phi^i. \]
and $\Sigma_k$ is the $k$th planar element transverse to $L_k$ with local coordinates $(u^1, u^2)$. The $\beta_k$ is the Hopf index of $\phi$ mapping, which means that when $\vec{x}$ covers the neighborhood of the zero point $\vec{x}_k(s)$ once, the vector field $\phi$ covers the corresponding region in $\phi$ space $\beta_k$ times. Meanwhile the direction vector of $L_k$ is given by

$$\frac{dx^i}{ds}|_{x_k} = \frac{D^i(\phi/x)}{D(\phi/u)}|_{x_k}. \quad (18)$$

Then from Eq. (17) and Eq. (18) one can obtain the inner structure of $\Omega$:

$$\Omega^i = \sum_{k=1}^{N} W_k \int_{L_k} \frac{dx^i}{ds} \delta^3(\vec{x} - \vec{x}_k(s)) ds, \quad (19)$$

where $W_k = \beta_k \eta_k$ is the winding number of $\vec{x}$ around $L_k$, with $\eta_k = \text{sgn} D(\phi/u)|_{x_k} = \pm 1$ being the Brouwer degree of $\phi$ mapping. The sign of Brouwer degrees are very important, the $\eta_k = +1$ corresponds to the vortex, and $\eta_k = -1$ corresponds to the antivortex. The integer number $W_k$ measures windings of the phase around the phase singularities, and is also called the topological charge of the RS vortex. Hence the topological charge of the RS vortices $L_k$ is

$$Q_k = \int_{\Sigma_k} \Omega^i d\sigma_i = W_k. \quad (20)$$

The Eq. (20) shows us that the topological current $\vec{Q}$ describes the density of RS vortices in space. Therefore, we call the topological current $\vec{Q}$ the topological charge current density of the RS vortices.

The results in this section show us the topological inner structure of the topological charge current density $\vec{Q}$. The RS vortices are classified by Hopf indices and Brouwer degrees. But we must note that these topological numbers describe the topological properties only for RS vortices, not for the phase singularities of the electromagnetic field. The definition of the topological numbers of phase singularities are different with the RS vortices. Though the sites of RS vortices are just the points of the phase singularities, from the previous definition of the phase of the electromagnetic field, we know that the phase of the electromagnetic filed $\phi(x)$ is half of the phase of square of the RS vector. In other words, the phase of square of the RS vector is $\chi = 2\varphi$. In the above discussions, the winding number of the $\phi$-mapping is relate to the phase $\chi$ i.e.,

$$W_k = \frac{1}{2\pi} \oint \! dx, \quad (21)$$

where the closed path $l$ surrounds the $k$th RS vortex. For the phase singularities, the winding number of the singular point is

$$W'_k = \frac{1}{2\pi} \oint \! d\varphi. \quad (22)$$

Here the winding number $W'_k$ measures the strength of the phase singularities. It is easy to see that there are two different expressions of the winding number relate to a same singular point, but they are not independent with each other because $\chi = 2\varphi$. In our following discussions, we only use the topological numbers which relate to RS vortices, and note that they are also valid for the phase singularities of electromagnetic field.

### III. THE HOPF INVARIANT OF THE KNOTTED RS VORTICES

In this section, let us begin to discuss the topological properties of knotted RS vortices. It is well known that the Hopf invariant is an important topological invariant to describe the topological characteristics of the knot family. In a closed three-manifold $M$, the Hopf invariant is defined as $[13, 24]$

$$H = \frac{1}{2\pi} \int_{M} A_i \Omega^i \delta^3 x, \quad (23)$$

in which $A_i$ is a "induced Abelian gauge potential" constructed with the complex function $F_2$, and

$$A_i = \epsilon_{ab} n^a \partial_n b = 2\nu_i = \partial_i \chi.$$

Substituting Eq. (19) into Eq. (23), one can obtain

$$H = \frac{1}{2\pi} \sum_{k=1}^{N} W_k \int_{L_k} \vec{A} \cdot d\vec{x}, \quad (24)$$

for closed and knotted lines, i.e., a family of knots $\xi_k (k = 1, 2, \ldots, N)$, Eq. (24) becomes

$$H = \frac{1}{2\pi} \sum_{k=1}^{N} W_k \oint_{\xi_k} \vec{A} \cdot d\vec{x}. \quad (25)$$

This is a very important expression. Consider a transformation of $\vec{F}^2$: $\psi' = e^{i\chi} \psi$, this gives the U(1) gauge transformation of $\vec{A}$: $A'_i = A_i + \partial_i \chi$, where $\chi$ is the phase factor denoting the U(1) gauge transformation. It is seen that the $\partial_i \chi$ term in Eq. (25) contributes nothing to the integral $H$ when RS vortices are closed, hence the expression (25) is invariant under the U(1) gauge transformation.

It is well known that many important topological numbers are related to a knot family such as the self-linking number and Gauss linking number. In order to discuss these topological numbers of knotted RS vortices, we define Gauss mapping:

$$\vec{m} : S^1 \times S^1 \to S^2, \quad (26)$$

where $\vec{m}$ is a unit vector

$$\vec{m}(\vec{x}, \vec{y}) = \frac{\vec{y} - \vec{x}}{|\vec{y} - \vec{x}|}. \quad (27)$$
where $\vec{x}$ and $\vec{y}$ are two points, respectively, on the knots $\xi_k$ and $\xi_l$ (in particular, when $\vec{x}$ and $\vec{y}$ are the same point on the same knot $\xi$, $\vec{n}$ is just the unit tangent vector $\vec{T}$ of $\xi$ at $\vec{x}$). Therefore, when $\vec{x}$ and $\vec{y}$, respectively, cover the closed curves $\xi_k$ and $\xi_l$ once, $\vec{n}$ becomes the section of sphere bundle $S^2$. So, on this $S^2$ we can define the two-dimensional unit vector $\vec{e} = \epsilon(\vec{x}, \vec{y})$. $\vec{e}$, $\vec{m}$ are normal to each other, i.e.,
\[ \vec{e}_1 \cdot \vec{e}_2 = \vec{e}_1 \cdot \vec{m} = \vec{e}_2 \cdot \vec{m} = 0, \]
\[ \vec{e}_1 \cdot \vec{e}_1 = \vec{e}_2 \cdot \vec{e}_2 = \vec{m} \cdot \vec{m} = 1. \quad (28) \]

In fact, the gauge potential $\vec{A}$ can be decomposed in terms of this two-dimensional unit vector $\vec{e}$. $A_i = \epsilon_{ab}e^a\partial_i e^b - \partial_i \chi$, where $\chi$ is a phase factor\[2\]. Since one can see from the expression $\Omega = \frac{1}{2} \nabla \times \vec{e} = \frac{1}{2 \pi} \nabla \times \vec{A}$ that the $(\partial_i \chi)$ term does not contribute to the integral $H$, $A_i$ can in fact be expressed as
\[ A_i = \epsilon_{ab}e^a\partial_i e^b. \quad (29) \]

Substituting it into Eq. (14), one can obtain
\[ H = \frac{1}{2 \pi} \sum_{k=1}^{N} \frac{1}{W_k} \int_{\xi_k} \epsilon_{ab}e^a(\vec{x}, \vec{y})\partial_i e^b(\vec{x}, \vec{y})dx^i. \quad (30) \]

Noticing the symmetry between the points $\vec{x}$ and $\vec{y}$ in Eq. (27), Eq. (30) should be reexpressed as
\[ H = \frac{1}{2 \pi} \sum_{k,l=1}^{N} W_k W_l \int_{\xi_k} \int_{\xi_l} \epsilon_{ab} \partial_i e^a \partial_j e^b dx^i \wedge dy^j. \quad (31) \]

In this expression there are three cases: (1) $\xi_k$ and $\xi_l$ are two different RS vortices ($\xi_k \neq \xi_l$), and $\vec{x}$ and $\vec{y}$ are therefore two different points ($\vec{x} \neq \vec{y}$); (2) $\xi_k$ and $\xi_l$ are the same RS vortices ($\xi_k = \xi_l$), but $\vec{x}$ and $\vec{y}$ are two different points ($\vec{x} \neq \vec{y}$); (3) $\xi_k$ and $\xi_l$ are the same RS vortices ($\xi_k = \xi_l$), and $\vec{x}$ and $\vec{y}$ are the same points ($\vec{x} = \vec{y}$). Thus, Eq. (31) can be written as three terms:
\[ H = \sum_{k=1}^{N} \frac{1}{2 \pi} \frac{1}{W_k} \int_{\xi_k} \epsilon_{ab} \partial_i e^a \partial_j e^b dx^i \wedge dy^j \]
\[ + \frac{1}{2 \pi} \sum_{k=1}^{N} W_k \int_{\xi_k} \epsilon_{ab} \partial_i e^a \partial_j e^b dx^i \]
\[ + \sum_{k,l=1}^{N} \frac{1}{2 \pi} W_k W_l \int_{\xi_k} \int_{\xi_l} \epsilon_{ab} \partial_i e^a \partial_j e^b dx^i \wedge dy^j. \quad (32) \]

By making use of the relation $\epsilon_{ab} \partial_i e^a \partial_j e^b = \frac{1}{2} \vec{m} \cdot (\partial_i \vec{m} \times \partial_j \vec{m}) \quad (27)$, the Eq. (32) is just
\[ H = \sum_{k=1}^{N} \frac{1}{2 \pi} W_k \int_{\xi_k} \int_{\xi_k} \vec{m}^* (dS) \]
\[ + \frac{1}{2 \pi} \sum_{k=1}^{N} W_k \int_{\xi_k} \epsilon_{ab} e^a \partial_i e^b dx^i \]
\[ + \sum_{k,l=1}^{N} \frac{1}{4 \pi} W_k W_l \int_{\xi_k} \int_{\xi_l} \vec{m}^* (dS), \quad (33) \]

where $\vec{m}^* (dS) = \vec{m} \cdot (\partial_i \vec{m} \times \partial_j \vec{m}) dx^i \wedge dy^j (\vec{x} \neq \vec{y})$ denotes the pullback of the $S^2$ surface element.

In the following we will investigate the three terms in the Eq. (33) in detail. Firstly, the first term of Eq. (33) is just related to the writhing number \[28\] $WR(\xi_k)$ of $\xi_k$
\[ WR(\xi_k) = \frac{1}{4 \pi} \int_{\xi_k} \int_{\xi_k} \vec{m}^* (dS). \quad (34) \]

For the second term, one can prove that it is related to the twisting number $Tw(\xi_k)$ of $\xi_k$
\[ \frac{1}{2 \pi} \int_{\xi_k} \epsilon_{ab} e^a \partial_i e^b dx^i = \frac{1}{2 \pi} \int_{\xi_k} (\vec{T} \times \vec{V}) \cdot d\vec{V} = Tw(\xi_k), \quad (35) \]

where $\vec{T}$ is the unit tangent vector of knot $\xi_k$ at $\vec{x}$ ($\vec{m} = \vec{T}$ when $\vec{x} = \vec{y}$) and $\vec{V}$ is defined as $e^a = e^{ab}V^b (\vec{V} \perp \vec{T}, \vec{e} = \vec{T} \times \vec{V})$. In terms of the White formula\[28\]
\[ SL(\xi_k) = WR(\xi_k) + Tw(\xi_k), \quad (36) \]

we see that the first and the second terms of Eq. (33) just compose the self-linking numbers of knots.

Secondly, for the third term, one can prove that
\[ \frac{1}{4 \pi} \int_{\xi_k} \int_{\xi_l} \vec{m}^* (dS) \]
\[ = \frac{1}{4 \pi} \epsilon^{ijk} \int_{\xi_k} dx^i \int_{\xi_l} dy^j (x^k - y^k) ||\vec{x} - \vec{y}||^3 \]
\[ = Lk(\xi_k, \xi_l) \quad (k \neq l), \quad (37) \]

where $Lk(\xi_k, \xi_l)$ is the Gauss linking number between $\xi_k$ and $\xi_l$\[28\]. Therefore, from Eqs. (34), (35), (36) and (37), we obtain the important result:
\[ H = \sum_{k=1}^{N} W_k^2 SL(\xi_k) + \sum_{k,l=1}^{N} W_k W_l Lk(\xi_k, \xi_l). \quad (38) \]

This precise expression just reveals the relationship between $H$ and the self-linking and the linking numbers of the RS vortices knots family\[28\]. Since the self-linking and the linking numbers are both the invariant characteristic numbers of the RS vortices knots family in topology, $H$ is an important topological invariant which required to describe the linked RS vortices.
IV. THE CONSERVATION OF HOPF INVARIANT

In this section we will simply discuss the conservation of the Hopf invariant in the branch processes of knotted RS vortices.

In the previous work done in Ref. [15], authors have pointed out that in the evolution of RS vortices, the splitting, the merging, and the intersection processes may occur when the condition $D(\phi/x) \neq 0$ fails. In these branch processes, we note that the sum of the topological charges of final RS vortices must be equal to that of the initial vortices at the bifurcation point. This conclusion is always valid because it is in topological level. So we have,

(a) for the case that one vortex $L$ split into two vortices $L_1$ and $L_2$, we have $W_L = W_{L_1} + W_{L_2}$;

(b) two vortices $L_1$ and $L_2$ merge into one vortex: $W_{L_1} + W_{L_2} = W_L$;

(c) two vortices $L_1$ and $L_2$ meet, then depart as other two vortices $L_3$ and $L_4$: $W_{L_1} + W_{L_2} = W_{L_3} + W_{L_4}$.

In the following we will show that when the branch processes of knotted RS vortices occur as above, the Hopf invariant is preserved:

(A) The splitting case. We consider one knot $\xi$ split into two knots $\xi_1$ and $\xi_2$ which are of the same self-linking number as $\xi$ ($SL(\xi) = SL(\xi_1) = SL(\xi_2)$). And then we will compare the two number $H_\xi$ and $H_{\xi_1+\xi_2}$ (where $H_\xi$ is the contribution of $\xi$ to $H$ before splitting, and $H_{\xi_1+\xi_2}$ is the total contribution of $\xi_1$ and $\xi_2$ to $H$ after splitting. First, from the above text we have $W_\xi = W_{\xi_1} + W_{\xi_2}$ in the splitting process. Second, on the one hand, noticing that in the neighborhood of bifurcation point, $\xi_1$ and $\xi_2$ are infinitesimally displace from each other; on the other hand, for a knot $\xi$ its self-linking number $SL(\xi)$ is defined as $SL(\xi) = Lk(\xi, \xi_V)$, where $\xi_V$ is another knot obtained by infinitely displacing $\xi$ in the normal direction $\bar{V}$. Therefore, $SL(\xi) = SL(\xi_1) = SL(\xi_2) = Lk(\xi_1, \xi_2)$, and $Lk(\xi_1, \xi'_k) = Lk(\xi_2, \xi'_k)$ (where $\xi'_k$ denotes another arbitrary knot in the family $\xi'_k \neq \xi, \xi'_k \neq \xi_{1,2}$). Then, third, we can compare $H_\xi$ and $H_{\xi_1+\xi_2}$ before splitting,

$$H_\xi = W^2_\xi SL(\xi) + \sum_{k=1}^{N} 2W_\xi W_{\xi'_k} Lk(\xi, \xi'_k),$$

where $Lk(\xi, \xi'_k) = Lk(\xi'_k, \xi)$; after splitting,

$$H_{\xi_1+\xi_2} = W^2_{\xi_1} SL(\xi_1) + W^2_{\xi_2} SL(\xi_2) + 2W_{\xi_1} W_{\xi_2} Lk(\xi_1, \xi_2)$$

$$+ \sum_{k=1}^{N} 2W_{\xi_1} W_{\xi'_k} Lk(\xi_1, \xi'_k)$$

$$+ \sum_{k=1}^{N} 2W_{\xi_2} W_{\xi'_k} Lk(\xi_2, \xi'_k).$$

Comparing (35) and (36), we have

$$H_\xi = H_{\xi_1+\xi_2}$$

This means that in the splitting process the Hopf invariant is conserved.

(B) The merging case. We consider two knots $\xi_1$ and $\xi_2$, which are of the same self-linking number, merge into one knot $\xi$ which is of the same self-linking number as $\xi_1$ and $\xi_2$. This is obviously the inverse process of the above splitting case; therefore we have

$$H_{\xi_1+\xi_2} = H_\xi.$$  

(C) The intersection case. This case is related to the collision of two knots. We consider that two knots $\xi_1$ and $\xi_2$, which are of the same self-linking number, meet, and then depart as other two knots $\xi_3$ and $\xi_4$ which are of the same self-linking number as $\xi_1$ and $\xi_2$. This process can be identified to two sub-processes: $\xi_1$ and $\xi_2$ merge into one knot $\xi$, and then $\xi$ split into $\xi_3$ and $\xi_4$. Therefore, from the above two cases (B) and (A) we have

$$H_{\xi_1+\xi_2} = H_{\xi_3+\xi_4}.$$  

Therefore we acquire the result that, in the branch processes during the evolution of RS vortices (splitting, merging, and intersection), the Hopf invariant is preserved.

V. CONCLUSION

In this paper, knotted objects in the theory of topological phase singularity in electromagnetic field have been investigated in details. By making use of the $\phi$-mapping topological current theory, we rewrite the topological inner structure of the RS vortices. Because the phase of the electromagnetic field $\varphi(x)$ is half of the phase of square of the RS vector, we point out that the topological numbers for phase singularities are different with the same numbers of RS vortices. So, for the same singular point where the RS vortex and phase singularity sits, the topological numbers have two definitions, one for RS vortices and another for phase singularities. Because this two definitions are not independent, in this paper we adopt the definition for RS vortices.

Furthermore, we study the knot topology of knotted RS vortices in terms of Hopf invariant. It is revealed that the Hopf invariant $H$ is just the total sum of all the self-linking and linking numbers of knotted family. At last, it is shown that $H$ is preserved in the splitting, the merging and the intersection processes of knotted RS vortices.

Acknowledgments

This work was supported by the National Natural Science Foundation of China and the Cuiying Programme of Lanzhou University.
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