Apollo’s Voyage: A New Take on Dynamics in Rotating Frames

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Abstract

We first demonstrate how our general intuition of pseudoforces has to navigate around several pitfalls in rotating frames. And then, we proceed to develop an intuitive understanding of the different components of the pseudoforces in most general accelerating (rotating and translating) frames: we show that it is not just a sum of the contributions coming from translation and rotation separately, but there is yet another component that is a more complicated combination of the two. Finally, we demonstrate using a simple example, how these dynamical equations can be used in such frames.

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I. INTRODUCTION

The diurnal motion of the Sun across the sky from east to west has perhaps been the most significant and most universal natural occurrence in the lives of human beings since antiquity. There was no lack of theories among our ancients, on the Sun’s apparent motion across the firmament during the day, and absence at night. The ancient Egyptians pictured the Sun god Ra sailing his mighty ship across the ethereal waters over the earth at day, and in the underworld at night,\(^\text{2}\) while the Greeks believed that Apollo had been tasked with pulling the enchained Sun across the skies everyday, on his golden chariot drawn by four mighty stallions.\(^\text{3}\) The east did not lack theories either: in ancient Hindu mythology, it was Surya riding across the heavens in his chariot drawn by the seven fiery steeds, with Aruna as his charioteer,\(^\text{4}\) while for the Shinto of Japan, it was the goddess Amaterasu coming out of her divine palace to preside over the mortals.\(^\text{5}\) It was not until the early Renaissance, that the heliocentric model of the solar system was founded:

“Nicolaus Copernicus Thorunensis, terrae motor, solis caelique stator”

(“Nicolaus Copernicus, son of Toruń, mover of the earth, stopper of the sun and heavens”).\(^\text{6}\) Here, we show that a reflection on the apparent diurnal motion of the Sun might help modify some of our intuition regarding dynamics in accelerating frames.

II. AN APPARENT PARADOX

A. A Problem with our Intuition

We begin with a problem, and an observation that goes against the common intuition regarding “pseudoforces”. We observe from the Earth, that the Sun appears to move around the Earth once in about 24 hours, but know that actually the Earth rotates on its axis once in the same time. The Earth also revolves around the Sun in about 365\(\frac{1}{4}\) days, but this motion due to the gravitational attraction between the two would be ignored at all points in our discussion for simplicity. One can see directly that even its inclusion would not affect our conclusion qualitatively, and a back-of-envelope calculation reveals that the quantitative changes are negligible, so this approximation really isn’t remotely similar to the spherical chicken approximation, but rather akin to considering the spherical earth (rather than the
oblate spheroid earth) approximation. Of course, the entire observation can be explained quite simply from a space-fixed (inertial) frame. But, suppose an observer on the rotating Earth (a non-inertial frame) decides to address the situation using “pseudoforces”. This should be possible without much ado using the tools one learns in a basic undergraduate course on Newtonian mechanics. So, the following question was posed to quite a large number of students who had taken such a course in the same level as the textbook by Kleppner and Kolenkow\textsuperscript{7} and also to some students who had also taken a more advanced course on the same, based on the celebrated textbook by Goldstein\textsuperscript{8}. For such an observer (someone on the Earth), which force on the Sun causes it to revolve around the Earth? A common answer received was, the “centripetal” force. This answer is technically correct, but not at all enlightening. It is a bit like the saying: “If you know when a tree was planted, you can determine its age quite accurately”. Because, “centripetal” is a name assigned to just any force directed centrally inwards: in other words, a force that is responsible for the centripetal acceleration is the centripetal force. It is the dynamical origin of such a force that we are interested in here: how it arises through physical interactions, or as a pseudoforce. However, in spite of being formally trained in the basic physics of non-inertial frames, quite a large number of students failed to correctly identify the origin of this “centripetal” force. The gravitational attraction between the Earth and the Sun is not really relevant in this particular problem, since we do not take consider effects due to the rotation of the Earth round the Sun. So, the only forces relevant in this scenario are the pseudoforces. While accounting for the forces of relevance in this scenario, most students successfully identified only the “centrifugal” (pseudo)force. This is perhaps because it is common intuition built up from day-to-day experience (e.g. when a moving vehicle brakes suddenly), that for an observer in an accelerated frame, the pseudoforce on an object is equal in magnitude to the mass of the object multiplied by the acceleration of the frame, and directed opposite to the acceleration of the frame. And this recipe yields the centrifugal force in this case. However, observe that though the centrifugal force is equal in magnitude to the required “centripetal” force, the direction is opposite. The centrifugal force on the Sun is directed radially outward, so it cannot support the observation of the Sun moving around the Earth once in a day, just by itself. It appears that we have here a paradox: one has an object rotating around the observer, but, cannot identify the dynamical origin of the centripetal force (the force that provides the centripetal acceleration)!
Another example, of a different physical setting, but of essentially the same nature, where we encounter the same “paradox”: an observer sitting at the centre of a rotating merry-go-round (a carousel). Perhaps this might appear more appropriate to some readers, since we do not have to make approximations like neglecting the revolution of the Earth around the Sun. For simplicity, let us assume that the origin of the merry-go-round-fixed-frame is a point on its axis of rotation, and that the angular velocity is constant. Our observer shall find that the surroundings execute circular motion around him or her. The centrifugal force would be directed outwards from the axis. Along the same lines as in the previous paragraph, one might think that this is the only force of interest here (neglecting gravity and all other such forces, which are orders of magnitude less). So, the acceleration of the surrounding bodies can be expected to be horizontal and radially outward from the centre of the merry-go-round. But this is evidently false: in the rotating frame, the surroundings revolving around the axis must have an acceleration directed towards the axis itself, and not outwards. So, who provides the centripetal force?

The answer to both problems, is that the pseudoforce in a rotating frame does not constitute of the centrifugal force alone. There is yet another force which we usually ignore in most day-to-day scenarios. It is the “Coriolis force”. It does not occur to us immediately, perhaps because in most observations to which we apply our intuition of pseudoforces, Coriolis forces do not affect the dynamics significantly over short time scales. For example, in most cases, students imagine Coriolis forces as those which deflect winds and ocean currents, or decide which way a whirlpool is going to swirl on different sides of the Equator. It usually does not cross one’s mind that the same Coriolis forces might be crucially relevant for observations as simple as the diurnal motion of the Sun. In the discussion below, we see how the Coriolis force resolves the situation completely, and try to build an intuition about the true nature of this force (there is good reason why this intuition is often lacking).

B. The Resolution

As a first step towards a resolution, we look at the following result from Goldstein. When we take the derivative of vector $G$ with respect to time in a rotating frame, we have,

$$
\left( \frac{dG}{dt} \right)_s = \left( \frac{dG}{dt} \right)_r + \omega \times G
$$

(1)
where $s$ stands for the space-fixed frame, and $r$ for the frame rotating with angular velocity $\omega$. We see that the time derivative in a rotating frame, $\left(\frac{dG}{dt}\right)_r$, is different from its counterpart in a non-rotating frame, $\left(\frac{dG}{dt}\right)_s$. This can be rationalized in the following fashion: the unit vectors in a rotating frame, unlike their translating counterparts, are not constant. Hence, once has to account for that in the derivative. One also sees that for any translating but non-rotating frame, $\omega = 0$, and hence, $\left(\frac{dG}{dt}\right)_\text{translating} = \left(\frac{dG}{dt}\right)_s$. Thus, instead of an inertial frame, $s$ can be any non-rotating frame having arbitrary translational motion for the above Eq. (1) to be satisfied.

We address both the merry-go-round problem and the Earth’s rotation problem simultaneously, as the difference between the two is essentially superficial. Consider two frames, with origins coinciding on the axis of the merry-go-round (or, the axis of the Earth’s rotation, which means that our observer is positioned on the North or the South Pole, or that we have chosen to ignore the radius of the Earth, an omission the reader will undoubtedly pardon). One is $s$, non-rotating, the other is $r$, fixed to the merry-go-round (or the Earth), and rotating with it. Then, let $r$ be the coordinate of some the particle whose motion is under study (or the Sun, which we treat here as a particle). In both frames $r$ and $s$, the coordinate $r$ reads the same, since the origin is the same. Furthermore, let us assume that $\omega$ is constant. Then, taking the derivative of $r$ twice, each time making adjustments by Eq. (1), we get the following equation governing the motion in the rotating frame $r$:

$$F - 2m (\omega \times v_r) - m\omega \times (\omega \times r) = ma_r$$  \hspace{1cm} (2)

where $v_r$ and $a_r$ represent the particle’s velocity and acceleration respectively in the rotating frame, and we have made the substitution $F = m \left(\frac{d^2r}{dt^2}\right)_s$, by Newton’s laws. A detailed calculation is shown in the appendix (A).

Now, if $F_{\text{pseudo}}$ be the pseudoforces in the rotating frame $r$, we should demand $F + F_{\text{pseudo}} = ma_r$. Hence, we have: $F_{\text{pseudo}} = F_{\text{co}} + F_{\text{ce}}$. The second term is the all-too-familiar centrifugal term, $F_{\text{ce}} = -m\omega \times (\omega \times r)$. The first term is the one of greater interest here (even if only because it often escapes our intuition): $F_{\text{co}} = -2m (\omega \times v_r)$, the Coriolis term. The Coriolis term is the one responsible for counter-intuitive behaviour. For the ‘surroundings’, stationary in the $s$ frame, $v_r = -\omega \times r$, and hence, the Coriolis term equals twice of the centrifugal term with a reversed sign. This provides a resolution of the problem.
Thus, we see that the folklore of pseudoforces being equal to the mass of the particle times the negative of the frame’s acceleration does not really hold true in rotating frames. Why does the Coriolis term not appeal to our intuition as much as the ‘negative of frame’s acceleration term’? After all, all of us are in a rotating frame (on the surface of the Earth)! The answer lies in the fact that the Coriolis force is proportional to the cross product of the angular velocity of the frame \( \omega \), and the \textit{velocity of the observed particle} in the frame, \( v_r \). For an observer on the surface of the Earth, \( \omega \) is approximately \( 7.3 \times 10^{-5} \text{s}^{-1} \). So, except for bodies moving at very, very high \( v_r \), its effects are practically negligible over short intervals of time. Of course, over long intervals of time, even relatively small \( v_r \)'s do cause significant deviations, depending on the angle between \( \omega \) and \( v_r \), e.g. for ocean currents and winds. But for objects moving at sufficiently high \( v_r \), the Coriolis term is definitely not negligible, compared to the centrifugal term. In fact, as demonstrated, for objects fixed respect to that frame, with respect to which our concerned frame is rotating at angular velocity \( \omega \), the Coriolis term is twice the centrifugal term with an opposite sign, and their resultant provides the centripetal. This is exactly why an observer on the surface of the Earth observes the Sun to orbit the Earth diurnally, in spite of the centrifugal term directed radially outward.

III. DYNAMICAL EQUATIONS, IN THE MOST GENERAL ACCELERATING FRAME

Having discussed some of the kinematic consequences of an observer being in a rotating frame, we now turn to the dynamics in a most general accelerating frame: a frame that is rotating (no longer at a constant \( \omega \)) as well as translating. One may think, shouldn’t the dynamics in such a frame simply be the sum of the effects due to translation without rotation, and due to rotation without translation? Though naively this appears to be a plausible solution, we shall see that this is not entirely correct. The force equation concerned can still be visualized as a combination of the effects due to translation and rotation, but it is slightly more involved than just their sum. We demonstrate below, that, in the pseudoforce, considering the contributions due to the rotation in absence of translation (\( F_{\text{rot}} = F_{\text{ang}} + F_{\text{co}} + F_{\text{ce}} \)) and the contribution due to translation in absence of translation (\( F_{\text{trans}} = -mA_r \)), we still need to add yet another term \( F_{\text{comb}} \), which is zero if the frame is either stationary (coinciding with \( s \)) or non-rotating.
In what follows, $s$ will denote a stationary / inertial frame, while $r$ will be an accelerating frame.

**A. The Force Equations, in the Most General Case**

We now can no longer assume that the origins of the $s$ frame and the $r$ frame coincide. Let $r_s$ and $r_r$ be the coordinates of the particle under study in the $s$ (inertial) and $r$ (non-inertial) frames respectively. Let $R$ the (time-dependent) vector between the origins, that is, $r_s = r_r + R$. Then, $R$ and its derivatives $V_r = \left(\frac{dR}{dt}\right)_r$ and $A_r = \left(\frac{d^2R}{dt^2}\right)_r$ capture the translational dependence.

Let $\omega_r$ be the angular velocity of the frame, \((\frac{d\omega_r}{dt})_r = (\frac{d\omega_r}{dt})_s = \dot{\omega}_r\).

We have,

$$F_{pseudo} = F_{trans} + F_{rot} + F_{comb}$$

where

$$F_{trans} = -mA_r$$

$$F_{rot} = F_{ang} + F_{co} + F_{ce}$$

with

$$F_{ang} = -m\dot{\omega}_r \times r_r$$

$$F_{co} = -2m(\omega_r \times v_r)$$

$$F_{ce} = -m\omega_r \times (\omega_r \times r_r)$$

and

$$F_{comb} = -2m\omega_r \times V_r - m\omega_r \times (\omega_r \times R) - m\dot{\omega}_r \times R$$

Then,

$$F + F_{pseudo} = ma_r$$

The derivations are in the appendix (B).

**B. Considering a Few Special Cases**

The last stated Eq. (10) may be considered to be the working form of Newton’s law in a most general accelerating frame. All this might appear a bit cumbersome. It is easier to
analyze a few simple cases:

1. There is a complete agreement with our common intuition when the frame is non-rotating ($\omega = 0 = \dot{\omega}$): $F_{pseudo} = -mA_r$.

2. When it is rotating uniformly ($\dot{\omega}_r = 0$) but not translating, it is reasonable to set $R = 0$, and then $F_{pseudo} = F_{ce} + F_{co}$. We see that this too is in complete agreement, with Eq. (2).

3. When it is rotating (perhaps non-uniformly) but not translating, again, it is reasonable to set $R = 0$, and then we have $F_{pseudo} = F_{ang} + F_{co} + F_{ce}$. That is, we have the minimal modification to Eq. (2), simply by addition of the term $F_{ang}$, just as we might expect.

4. When there is both translation and rotation, the situation is more involved. However, it might be interesting to pose the following questions: given $R$, $V_r$, does there exist $\omega_r$, $\dot{\omega}_r$ such that $F_{comb} = 0$, and vice versa? Perhaps this question is similar in spirit somewhat to Chasles’ theorem.

We provide an interesting and instructive example below to demonstrate how these equations might be used.

C. An Example: the Merry-Go-Round Archer

Suppose an archer decides to try out shooting an arrow from a merry-go-round. What additional factors should be taken into account?

The first thing to observe, is that once the arrow leaves the bow (literally), there is no relevant force acting on it (we are only interested in the horizontal motion). Gravity does not affect the horizontal motion of the arrow, except for limiting the range, so we leave that consideration to the archer’s usual skill. The vertical motion of the arrow (due to gravity / the angle decided upon by the archer) is not affected by the rotation, as $\omega$ has a non-zero component only in the vertical direction. Of course, we also ignore the drag force due to the air, and assume the arrow doesn’t go “whichever way the wind doth blow”. So a stationary observer should observe the arrow to fly off in a straight line (we ignore the vertical motion of the arrow). But what does the archer see?
Suppose, we consider the archer to be sitting on the rotating merry-go-round at a distance of \( r_0 \) from its axis of rotation. And, the merry-go-round is rotating on its axis at a constant angular speed \( \omega \), anticlockwise. We consider the inertial frame \( s \) to have origin on the axis of the merry-go-round, and for the non-inertial frame \( r \) (the archer) to be using cylindrical coordinates \( (\hat{\rho}, \hat{\theta}, \hat{z}) \) where \( \hat{\rho} \) is directed towards the axis of the merry-go-round, and \( \hat{z} \) is directed upwards. So, \( R = -r_0 \hat{\rho}, \omega_r = \omega \hat{z}, V_r = 0, \) and \( A_r = 0 \). Here, \( F_{\text{comb}} = -m\omega r \times (\omega \times R) = -m\omega^2 r_0 \hat{\rho}, F_{\text{ang}} = F_{\text{trans}} = 0, F_{\text{co}} = 2m\omega v_r \times \hat{z}, \) and \( F_{ce} = m\omega^2 r_r \). Hence, we have

\[
a_r = \omega^2 (r_r - r_0 \hat{\rho}) + 2\omega v_r \times \hat{z} \tag{11}
\]

Of course, one may solve this second order differential equation with suitable initial conditions to get the full trajectory. However, a qualitative analysis of what happens just at the instant the arrow is shot, is quite easy. At that instant, \( r_r = 0 \), and so \( a_r \big|_{t=0} = -\omega^2 r_0 \hat{\rho} + 2\omega v_r \big|_{t=0} \times \hat{z} \). \( -\omega^2 r_0 \hat{\rho} \) looks the acceleration due to a centrifugal (pseudo)force, but it actually follows from \( F_{\text{comb}} \). When \( r_0 = 0 \) (that is, the archer is at the center of the merry-go-round), the arrow is observed to be simply deflected to the right by the Coriolis term \( 2\omega v_r \times \hat{z} \), as expected.

### IV. CONCLUSION

We have demonstrated some of the limitations of our common intuition when dealing with accelerating frames, in particular, failure in rotating frames. That being done, we have provided a natural way to extend our intuition to such scenarios. In this article, we have restricted ourselves to dynamics of point particles, that is, equations involving force and acceleration only. It is interesting to note that similar studies can be carried out for systems of particles, in particular, in the setting of rigid body dynamics. One may study the relations between torque, angular momenta, and angular velocities from general accelerating frames as well, in the same spirit as of this article. We are presently compiling such an article\textsuperscript{10}. This leads us to obtain an alternate proof (and more importantly, a new physical realization) of Euler’s equations for rigid body motion, in the same lines as in Mott’s 1966 deduction\textsuperscript{11}.
Appendix A: Deduction of the Dynamical Equation in a Frame Rotating at a Constant Angular Velocity:

From Eq. (1), with $r$ as $G$:

$$\left(\frac{dr}{dt}\right)_s = \left(\frac{dr}{dt}\right)_r + \omega \times r$$  \hspace{1cm} (A1)

And again, with $\left(\frac{dr}{dt}\right)_s$ as $G$:

$$\left(\frac{d^2r}{dt^2}\right)_s = \left(\frac{d^2r}{dt^2}\right)_r + 2\omega \times \left(\frac{dr}{dt}\right)_r + \omega \times (\omega \times r)$$  \hspace{1cm} (A2)

Appendix B: Deduction of the Dynamical Equation in a Most General Accelerating Frame:

For a translating-cum-rotating frame, a straightforward calculation by considering the time derivative of $r_s$ in the two frames by Eq. (1) yields:

$$v_s = (v_r + V_r) + \omega_r \times (r_r + R)$$  \hspace{1cm} (B1)

$$a_s = (a_r + A_r) + 2\omega_r \times (v_r + V_r) + \omega_r \times (\omega_r \times (r_r + R)) + \dot{\omega}_r \times (r_r + R)$$  \hspace{1cm} (B2)

Using $ma_s = F$, this can be written as:

$$ma_r = F - mA_r - 2m(\omega_r \times v_r) - m\omega_r \times (\omega_r \times r_r) - m\dot{\omega}_r \times r_r - 2m(\omega_r \times V_r) - m\omega_r \times (\omega_r \times R) - m\dot{\omega}_r \times R$$  \hspace{1cm} (B3)

We can immediately identify the second term on the right hand side as arising out of the translational motion of the frame, the third through fifth terms as arising out of the rotational motion of the frame, and the last three terms as a combination of both translation and rotation.
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