On the degenerated Arnold-Givental conjecture

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Abstract
We present another view dealing with the Arnold-Givental conjecture on a real symplectic manifold \((M, \omega, \tau)\) with nonempty and compact real part \(L = \text{Fix}(\tau)\). For given \(\Lambda \in (0, +\infty)\) and \(m \in \mathbb{N} \cup \{0\}\) we show the equivalence of the following two claims: (i) \(#(L \cap \phi^t_H(L)) \geq m\) for any Hamiltonian function \(H \in C_0^\infty([0, 1] \times M)\) with Hofer’s norm \(\|H\| < \Lambda\); (ii) \(#\mathcal{P}(H, \tau) \geq m\) for every \(H \in C_0^\infty(\mathbb{R}/\mathbb{Z} \times M)\) satisfying \(H(t, x) = H(-t, \tau(x)) \forall (t, x) \in \mathbb{R} \times M\) and with Hofer’s norm \(\|H\| < 2\Lambda\), where \(\mathcal{P}(H, \tau)\) is the set of all 1-periodic solutions of \(\dot{x}(t) = X_H(t, x(t))\) satisfying \(x(-t) = \tau(x(t))\) \(\forall t \in \mathbb{R}\) (which are also called brake orbits sometimes). Suppose that \((M, \omega)\) is geometrical bounded for some \(J \in \mathcal{J}(M, \omega)\) with \(\tau^*J = -J\) and has a rationality index \(r_\omega > 0\) or \(r_\omega = +\infty\). Using Hofer’s method we prove that if the Hamiltonian \(H\) in (ii) above has Hofer’s norm \(\|H\| < r_\omega\) then \(#(L \cap \phi^t_H(L)) \geq \#\mathcal{P}_0(H, \tau) \geq \text{Cuplength}_F(L)\) for \(F = \mathbb{Z}_2\), and further for \(F = \mathbb{Z}\) if \(L\) is orientable, where \(\mathcal{P}_0(H, \tau)\) consists of all contractible solutions in \(\mathcal{P}(H, \tau)\).

1 Introduction

A real symplectic manifold is a triple \((M, \omega, \tau)\) consisting of a symplectic manifold \((M, \omega)\) and an anti-symplectic involution \(\tau\) on \((M, \omega)\), i.e. \(\tau^*\omega = -\omega\) and \(\tau^2 = \text{id}_M\). The Marsden-Weinstein quotients of real Hamiltonian systems provide a great deal of examples of such manifolds. Let \(\mathcal{J}(M, \omega)\) denote the space of all \(\omega\)-compatible smooth almost complex structures on \(M\), and

\[\mathbb{R}\mathcal{J}(M, \omega) = \{J \in \mathcal{J}(M, \omega) \mid J \circ d\tau = -d\tau \circ J\},\]

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that is, \( J \in \mathbb{R}J(M, \omega) \) if and only if \( \tau \) is anti-holomorphic with respect to \( J \). With the standard trick of Sévennec (see [McSa1, p.64]) one can prove that \( \mathbb{R}J(M, \omega) \) is a separable Frechét submanifold of \( J(M, \omega) \) which is nonempty and contractible (cf. [Wel Prop. 1.1]). The fixed point set \( L := \text{Fix}(\tau) \) is called the real part of \( M \). Since \( \tau \) is an isometry of the natural Riemann metric \( g_J = \omega \circ (id_M \times J) \) for any \( J \in \mathbb{R}J(M, \omega) \), \( L \) is either empty or a Lagrange submanifold (cf. [Wel, Prop. 1.1]). The fixed point set \( \text{Fix}(\tau) \) is called the real part of \( M \).

Arnold-Givental conjecture ([Gi]): Let \((M, \omega, \tau)\) be a real symplectic manifold of dimension \( 2n \), and \( L = \text{Fix}(\tau) \) be a nonempty compact submanifold without boundary. Then for every Hamiltonian diffeomorphism \( \phi \) on \((M, \omega)\), it holds that

\[
\sharp(L \cap \phi(L)) \geq \sum_{k=0}^{n} b_k(L, \mathbb{Z}_2) \quad \text{or} \quad \sum_{k=0}^{n} b_k(L, \mathbb{Z})
\]

(1.1)

provided that \( L \) and \( \phi(L) \) intersect transversally, and that

\[
\sharp(L \cap \phi(L)) \geq \text{Cuplength}_{\mathbb{Z}_2}(L) \quad \text{or} \quad \text{Cuplength}_{\mathbb{Z}}(L)
\]

(1.2)

generally. Hereafter the \( F \)-cuplength of a paracompact topological space \( X \) over an integral domain \( F \), \( \text{Cuplength}_{F}(X) \), is defined the supremum of natural numbers \( k \) such that there exist cohomology classes \( \alpha_1, \ldots, \alpha_{k-1} \) in \( H^*(X, F) \) of positive degree satisfying \( \alpha_1 \cup \cdots \cup \alpha_{k-1} \neq 0 \).

This conjecture is a special case of Arnold’s conjecture on Lagrangian intersections ([Ar1, Ar2]). If \( M \) is closed, the estimate (1.1) in \( \mathbb{Z}_2 \)-coefficients follows from Floer [Fl1] if \( \pi_2(M, L) = 0 \), [Oh] if \( L \) is the strongly negative monotone, and [FuOOO] Theorem H] if \( L \) is the semipositive, and [Fr] if \( L \) is in Marsden-Weinstein quotients. The estimate (1.2) in \( \mathbb{Z}_2 \)-coefficients follows from Floer and Hofer [Fl2, Ho2], and Liu [Liu] if \((M, \omega)\) has positive rationality index \( r_\omega \) and \( \phi \) may be generated by \( H \in C^\infty([0, 1] \times M) \) with Hofer’s norm \( \|H\| < r_\omega/2 \). The estimates in (1.1) and (1.2) were obtained for \((M, L) = (CP^n, \mathbb{R}P^n) \) [ChJi, Gi]. (The author [Lu2] also generalized the arguments in [ChJi] to the case of weighted complex projective spaces, which are symplectic orbifolds).

Arnold-Givental conjecture contains Arnold conjecture for the symplectic fixed points ([Ar1, Ar2]), which stated that for every Hamiltonian diffeomorphism \( \phi \) on a closed symplectic manifold \((M, \omega)\) the following estimates hold true,

\[
\sharp\text{Fix}(\phi) \geq \text{Cuplength}_{\mathbb{Z}}(M),
\]

(1.3)

\[
\sharp\text{Fix}(\phi) \geq \sum_{k=0}^{\dim M} b_k(M; \mathbb{F})
\]

(1.4)

if each \( x \in \text{Fix}(\phi) \) is nondegenerate in the sense that the tangent map \( d\phi(x) : T_x M \rightarrow T_x M \) has no eigenvalue 1. After Floer [Fl3], first invented Floer homologies to prove (1.4) for monotone \((M, \omega)\) and \( \mathbb{F} = \mathbb{Z} \), Fukaya-Ono [FnO] and Liu-Tian [LiuT] developed Floer homologies to affirm it for any closed symplectic manifold \((M, \omega)\) and \( \mathbb{F} = \mathbb{Q} \).
Recall Hofer’s norm of a Hamiltonian function $H: \mathbb{R} \times M \to \mathbb{R}$, $(t, x) \mapsto H(t, x) = H_t(x)$ satisfying

$$H_t(x) = H_{t+1}(x) \quad \text{and} \quad H(t, x) = H(-t, \tau(x)) \forall (t, x) \in \mathbb{R} \times M. \quad (1.5)$$

Such a Hamiltonian function $H$ is said to be 1-periodic in time and symmetric.

Let $X_{H_t}$ be defined by $\omega(X_{H_t}, \cdot) = -dH_t(\cdot)$. Then $X_{H_t} = X_{H_{t+1}}$ and

$$X_{H_{t-1}}(x) = -d\tau(x)X_{H_t}(\tau(x)) \forall (t, x) \in \mathbb{R} \times M. \quad (1.6)$$

For $x_0 \in M$ let $x: \mathbb{R} \to M$ be the solution of

$$\dot{x}(t) = X_{H_t}(x(t)) \quad (1.7)$$

through $x_0$ at $t = 0$. Then both $y(t) := x(-t)$ and $z(t) := \tau(x(t))$ are solutions of

$$\dot{x}(t) = d\tau(x(t))X_{H_t}(\tau(x(t))).$$

So $y = z$ if and only if $x_0 = y(0) = z(0) = \tau(x(0)) = \tau(x_0)$. We are interested in those 1-periodic solutions $x$ of the equation (1.7) which satisfy

$$x(-t) = \tau(x(t)) \forall t \in \mathbb{R}. \quad (1.8)$$

Clearly, such a solution $x$ satisfies: $x(0), x(1/2) \in L$ and $x(1-t) = \tau(x(t)) \forall t$. A loop $x: S^1 = \mathbb{R}/\mathbb{Z} \to M$ satisfying (1.8) is called a $\tau$-reversible. ($\tau$-reversible 1-periodic solutions are also called brake orbits in literature.) Denote by

$$\mathcal{P}(H, \tau) \text{ (resp. } \mathcal{P}_0(H, \tau) \text{)}$$

the set of all $\tau$-reversible 1-periodic solutions (resp. contractible $\tau$-reversible 1-periodic solutions) of (1.7). Clearly, $\mathcal{P}(H, \tau)$ must be empty if $L = \emptyset$. Let $\phi_t^H : M \to M$ be the Hamiltonian diffeomorphisms defined by

$$\frac{d}{dt}\phi_t^H = X_{H_t} \circ \phi_t^H, \quad \phi_0^H = id_M.$$

From (1.6) it easily follows that $\phi_t^H \circ \tau = \tau \circ \phi_t^H \forall t \in \mathbb{R}$. Moreover, it always holds that $\phi_{t+1}^H = \phi_t^H \circ \phi_1^H \forall t \in \mathbb{R}$. So we get that

$$\phi_1^H \circ \tau = \tau \circ (\phi_1^H)^{-1}. \quad (1.9)$$

One also easily checks that the elements of $\mathcal{P}(H, \tau)$ are one-to-one correspondence with points in $L \cap \text{Fix}(\phi_1^H)$. So we have

$$\sharp(L \cap \text{Fix}(\phi_1^H)) = \sharp\mathcal{P}(H, \tau) \geq \sharp\mathcal{P}_0(H, \tau). \quad (1.10)$$

Recall Hofer’s norm of a Hamiltonian function $H \in C_0^\infty([0, 1] \times M)$ is defined by

$$||H|| = \int_0^1 [\sup_x H_t(x) - \inf_x H_t(x)] dt.$$
Theorem 1.1 Let $(M, \omega, \tau)$ be a real symplectic manifold of dimension $2n$, and the fixed point set $L = \text{Fix}(\tau)$ be nonempty. Let $\Lambda \in (0, +\infty]$ and $m \in \mathbb{N} \cup \{0\}$. Then the following two claims are equivalent.

(i) Every Hamiltonian diffeomorphism $\phi$ on $M$ generated by a Hamiltonian function $H \in C^\infty_0([0, 1] \times M)$ with $\|H\| < \Lambda$, satisfies

$$\sharp(L \cap \phi(L)) \geq m.$$ 

(ii) Every 1-periodic in time and symmetric $H \in C^\infty_0(\mathbb{R}/\mathbb{Z} \times M)$ with $\|H\| < 2\Lambda$, satisfies

$$\sharp \mathcal{P}(H, \tau) \geq m.$$ 

Remark 1.2 The proof of “(i)$\implies$(ii)” in the proof of Theorem 1.1 actually shows

$$\mathcal{P}(H, \tau) = \{ x(t) = \phi^H_t(x_0) \mid x_0 \in L \cap (\phi^H_{1/2})^{-1}(L) \}$$

and so $$\sharp \mathcal{P}(H, \tau) = \sharp(L \cap \phi^H_{1/2}(L)).$$

So using the results obtained for the Arnold conjecture on Lagrangian intersections one may get the estimates of the lower bound of $\sharp \mathcal{P}(H, \tau)$ under certain assumptions. For example, it follows from Theorem 1.1 and [FuOOO, Theorem H] that if $M$ is closed, $L$ is semipositive, and $L \pitchfork \phi^H_{1/2}(L)$ then $\sharp \mathcal{P}(H, \tau) \geq \sum \text{rank} H_\ast(L; \mathbb{Z}_2)$.

Recall that a symplectic manifold $(M, \omega)$ without boundary is said to be geometrically bounded if there exist a geometrically bounded Riemannian metric $\mu$ on $M$ (i.e., its sectional curvature is bounded above by some constant $K > 0$ and injectivity radius $i(M, \mu) > 0$) and a $\omega$-compatible almost complex structure $J$ such that such that

$$\omega(X, JX) \geq \alpha_0 \|X\|_\mu^2 \quad \text{and} \quad |\omega(X, Y)| \leq \beta_0 \|X\|_\mu \|Y\|_\mu \quad \forall X, Y \in TM$$

for some positive constants $\alpha_0$ and $\beta_0$ (cf. [Gr], [AuLaPo], [CGK], [Lu1]). For a real symplectic manifold $(M, \omega, \tau)$ without boundary, if the almost complex structure $J$ above can be chosen in $\mathbb{R}J(M, \omega)$ we say $(M, \omega, \tau)$ to be real geometrically bounded (with respect to $(J, \mu)$).

The rationality index of a symplectic manifold $(M, \omega)$ is defined by

$$r_\omega = r(M, \omega) := \inf \{ \langle \omega, A \rangle \mid A \in \pi_2(M), \langle \omega, A \rangle > 0 \} \in [0, +\infty],$$

where we use the convention that the infimum over the empty set is equal to $+\infty$. Since $\{\omega(A) \mid A \in \pi_2(M)\}$ is a subgroup of $(\mathbb{R}, +)$, it is easily checked that $r_\omega$ is a finite positive number if and only if $\omega(\pi_2(M)) = r_\omega \mathbb{Z}$. For $J \in \mathcal{J}(M, \omega)$ let $m(M, \omega, J) \in [0, +\infty]$ denote the infimum of the area of all nonconstant $J$-holomorphic spheres in $M$, where as usual we understand $m(M, \omega, J) = \infty$ if no nonconstant $J$-holomorphic sphere exists. Clearly, $r_\omega \leq m(M, \omega, J) \forall J \in \mathcal{J}(M, \omega)$. As showed by (ii)-(iii) of Example 1.2 there exist closed symplectic manifolds $(M, \omega)$ such that

$$0 < r_\omega < \sup_{J \in \mathcal{J}(M, \omega)} m(M, \omega, J) = +\infty.$$
If $M$ is compact, it directly follows from the Gromov compactness theorem that

$$m(M, \omega, J) > 0.$$ 

If $(M, \omega, J)$ is only geometrically bounded as above, this may be derived from the monotonicity principle (Sik, Prop.4.3.1(ii)): For $r_0 = \min\{i(M, \mu), \pi/\sqrt{K}\}$, a compact Riemann surface with boundary $S$ and a $J$-holomorphic map $f : S \to M$, assume that there exists a $\mu$-metric ball $B(x, r)$ with $r \leq r_0$ and with $x \in f(S)$ such that $f(\partial S) \subset \partial B(x, r)$, then

$$\pi \frac{\alpha_0}{4\beta_0} r^2 \leq \text{Area}_\mu(f(S)) \leq \frac{1}{\alpha_0} \int_f^* \omega.$$ 

In fact, put $\delta = \min\{\pi \frac{\alpha_0}{4\beta_0} r_0^2, \pi \frac{\alpha_0}{4\beta_0} (i(M, \mu))^2/9\}$. It follows that $\int_{\Sigma} u^* \omega \geq \delta$ for every nonconstant $J$-holomorphic map $u$ from a closed Riemann surface $\Sigma$ to $M$. (See the proof of FuO, Lemma 8.1 below Lemma 8.10 therein).

Based on Hofer’s method in [Ho2] we can get our second result.

**Theorem 1.3** Let $(M, \omega, \tau)$ be a real geometrical bounded symplectic manifold with respect to $J \in \mathbb{R} J(M, \omega)$ and a Riemannian metric $\mu$, and $L = \text{Fix}(\tau)$ be a nonempty compact submanifold without boundary. Let $H \in C^\infty_0(\mathbb{R}/\mathbb{Z} \times M)$ be a symmetric Hamiltonian function. If $r_\omega > 0$ and $\|H\| < r_\omega$, then

$$\sharp(L \cap \text{Fix}(\phi^H)) \geq \sharp P_0(H, \tau) \geq \text{Cuplength}_{\mathbb{Z}^2}(L), \quad (1.11)$$ 

and $\sharp P_0(H, \tau) \geq \text{Cuplength}_{\mathbb{Z}^2}(L)$ if $L$ is orientable.

Note that $r_\omega \in (0, +\infty)$ (resp. $= +\infty$) implies $\omega(\pi^2(M, L)) = \frac{r_\omega^2}{2 \pi} \mathbb{Z}$ (resp. $= 0$). As a direct consequence of (1.10) and Theorems 1.1, 1.3 we get

**Theorem 1.4** Let $(M, \omega, \tau, J, L)$ be as in Theorem 1.3. If $r_\omega > 0$, then every Hamiltonian diffeomorphism $\phi$ on $M$ generated by a Hamiltonian function $H \in C^\infty_0([0, 1] \times M)$ with $\|H\| < r_\omega/2$, satisfies the estimates

$$\sharp(L \cap \phi(L)) \geq \text{Cuplength}_{\mathbb{Z}^2}(L), \quad (1.12)$$

and $\sharp(L \cap \phi(L)) \geq \text{Cuplength}_{\mathbb{Z}^2}(L)$ if $L$ is orientable.

**Remark 1.5** If $M$ is closed, (1.12) is a special case of the main result in [Lin] proved with Floer homology; the latter and Theorems 1.1 can only lead to $\sharp P(H, \tau) \geq \text{Cuplength}_{\mathbb{Z}^2}(L)$, which is weaker than the second inequality in (1.11). Actually, the main result in [Lin] can also be proved by refining Hofer’s arguments in [Ho2] as done in this paper. Hofer’s method does not involve Floer and Morse homologies (and thus complicated transversality arguments). Recently, Albers and Hein [AH] gave an abstract result based on Morse cohomology. As in the proof of [AH, Theorem 5.1], it may lead to (1.11), but no better result.
The twisted product \((\hat{M}, \hat{\omega}) = (M \times M, \omega \times (-\omega))\) of a symplectic manifold \((M, \omega)\) and itself with anti-symplectic involution given by

\[
\tau : M \times M \to M \times M, \ (x, y) \mapsto (y, x),
\]

is a real symplectic manifold with \(\text{Fix}(\tau) = \Delta_M\). For any \(J \in \mathcal{J}(M, \omega)\) it is easily checked that \(J \times (-J) \in \mathbb{R} \mathcal{J}(M \times M, \omega \times (-\omega))\) and

\[
m(M \times M, \omega \times (-\omega), J \times (-J)) = 2m(M, \omega, J).
\]  \hspace{1cm} (1.13)

If \(H \in C^\infty(\mathbb{R}/\mathbb{Z} \times M)\), then

\[
\hat{H} : \mathbb{R} \times M \times M \to \mathbb{R}, \ (t, x, y) \mapsto H_t(x) + H_{-t}(y),
\]

is 1-periodic in time and symmetric. Note that \(X_{\hat{H}_t}(x, y) = (X_{H_t}(x), -X_{H_{-t}}(y))\) by the definition of \(X_{\hat{H}}\) above \([1.6]\). One easily proves that \(z = (x, y) : \mathbb{R}/\mathbb{Z} \to \mathbb{R}\) belongs to \(\mathcal{P}(\hat{H}, \tau)\) (resp. \(\mathcal{P}_0(\hat{H}, \tau)\)) if and only if \(x \in \mathcal{P}(H)\) (resp. \(x \in \mathcal{P}_0(H)\)) and \(y(t) = x(-t) \forall t \in \mathbb{R}\). Here \(\mathcal{P}(H)\) (resp. \(\mathcal{P}_0(H)\)) always denote the set of 1-periodic solutions (resp. contractible 1-periodic solutions) of the equation \(\dot{x} = X_H(t, x)\). Moreover,

\[
\|\hat{H}\| = \int_0^1 \left[ \sup_{(x, y)} H_t(x, y) - \inf_{(x, y)} H_t(x, y) \right] dt = 2\|H\|
\]

and \(r_{\hat{\omega}} = r_\omega\) are clear. Using this and \([1.13]\) we derive from Theorem \([1.3]\)

**Theorem 1.6** \(\text{Let } (M, \omega) \text{ be a closed symplectic manifold, and } H \in C^\infty(\mathbb{R}/\mathbb{Z} \times M) \text{ satisfy } \|H\| < r_\omega. \text{ Then}\)

\[
\sharp\text{Fix}(\phi_t^H) \geq \text{Cuplength}_{\mathbb{Z}_2}(M) \quad \text{and} \quad \sharp\text{Fix}(\phi_t^H) \geq \text{Cuplength}_{\mathbb{Z}}(M).
\]

The first inequality was proved in \([\text{Sch}]\) Theorem 1.1 by Floer homology method. It is a generalization of the result in \([\text{Fl2}, \text{Ho2}]\). Without the assumption “\(\|H\| < r_\omega\)”, Le and Ono \([\text{LeO}]\) got the estimates \([1.3]\) for \(F = \mathbb{Z}_2\) if \((M, \omega)\) is negative monotone and has minimal Chern number \(N \geq \dim M/2\) (cf. Example \([1.2]\)(iii) for these two notions).

The cotangent bundle of a manifold \(N\), \((T^*N, \omega_{\text{can}} = -d\lambda_{\text{can}})\), is a real symplectic manifold with the anti-symplectic involution given by

\[
\tau : T^*N \to T^*N, \ (q, p) \mapsto (q, -p),
\]

where \(q \in N\) and \(p \in T^*_qN\). Recall that the Liouville 1-form \(\lambda_{\text{can}}\) on \(T^*N\) is defined by \(\lambda_{\text{can}}(\xi) = p(T\pi^*\xi) \forall \xi \in T_pT^*N\), where \(\pi^* : T^*N \to N\) is the natural projection. The fixed point set \(\text{Fix}(\tau)\) is the zero section \(0_N\) which can be identified with \(N\). Assume now that \(N\) is closed. As in \([\text{CGK}, \text{Lu}]\) we can prove that \((T^*N, \omega_{\text{can}}, \tau)\) is geometrically bounded for some \(J \in \mathbb{R} \mathcal{J}(T^*N, \omega_{\text{can}})\) and some metric \(G\) on \(T^*N\). Applying Theorem \([1.3]\) to \((T^*N, \omega_{\text{can}}, \tau)\) we immediately obtain:
Corollary 1.7 Let $N$ be a closed manifold, and $H \in C_0^\infty(\mathbb{R}/\mathbb{Z} \times T^*N)$ satisfy $H(-t,q,p) = H(t,q,-p)$ for all $t \in \mathbb{R}$ and $(q,p) \in T^*N$. Then

$$\sharp(0_N \cap \text{Fix}(\phi^H_1)) \geq \sharp P_0(H,\tau) \geq \text{Cuplength}_{\mathbb{Z}_2}(N),$$

and $\sharp P_0(H,\tau) \geq \text{Cuplength}_{\mathbb{Z}}(N)$ if $N$ is orientable.

This and Theorem 1.1 immediately lead to

Corollary 1.8 ([Ho1, LaSi]) Let $N$ be a closed manifold. Then any Hamiltonian diffeomorphism $\phi$ on $(T^*N, \omega_{can})$ generated by a Hamiltonian function $H \in C_0^\infty([0,1] \times T^*N)$ satisfies estimates: $\sharp(N \cap \phi(N)) \geq \text{Cuplength}_{\mathbb{Z}_2}(N)$, and $\sharp(N \cap \phi(N)) \geq \text{Cuplength}_{\mathbb{Z}}(N)$ if $N$ is orientable.

The arrangements of the paper as follows. In Section 2.1 we first prove Theorem 1.1. Then in Section 2.2 we complete the proof of Theorem 1.3 by improving the arguments in [HoZe, §6.4] (also see [Ho2]). Unlike they consider the space of all bounded trajectories we here only use a subset of it. Another different point is to introduce a definition of topological degree for maps from a Banach Fredholm bundle to a manifold, not using the $\mathbb{Z}_2$-degree for Fredholm section having Fredholm index zero as in [HoZe, §6.4]. The final Section 3 gives two examples and a further programme.

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2 Proofs of Theorems 1.1, 1.3

2.1 Proof of Theorem 1.1

(i) $\implies$ (ii): Let $\phi_t$ be the Hamiltonian flow generated by $H$. Define $Q : [0,1] \times M \to \mathbb{R}$ by $Q(t,x) = H(t/2,x)$, and denote by $\varphi_t$ the flow of $X_Q$. It is easily proved that $\phi_{1/2} = \varphi_1$ and $\|Q\| = \frac{1}{2}\|H\| < \Lambda$. (2.1)

It follows from (i) that

$$\sharp(L \cap \phi^\frac{1}{2}(L)) \geq m.$$

For any $x_0 \in L \cap \phi^{-1}_{1/2}(L)$, $x(t) := \phi_t(x_0)$ satisfies $\dot{x}(t) = X_{H_{1/2}}(x(t)) \ \forall t$ and $x(\frac{1}{2}) = \phi_{1/2}(x_0) \in L$. Since $H_{1/2} = H_{1-t} \circ \tau$, for $\frac{1}{2} \leq t \leq 1$ we have

$$\dot{x}(t) = X_{H_{1/2}}(x(t)) = -d\tau(\tau(x(t)))X_{H_{1-t}}(\tau(x(t))) \text{ or equivalently}$$

$$\frac{d}{dt}\tau(x(t)) = -X_{H_{1-t}}(\tau(x(t))).$$

It follows that $y(t) := \tau(x(1-t))$ on $[0,\frac{1}{2}]$ satisfies $\dot{y}(t) = X_{H_{1/2}}(y(t))$. Note that $x(\frac{1}{2}) \in L$ implies $y(\frac{1}{2}) = \tau(x(\frac{1}{2})) = x(\frac{1}{2})$, i.e., $\phi_{1/2}(y(0)) = \phi_{1/2}(x_0)$. Hence $y(0) = x_0$. Hence $y(0) = x_0$.
and thus \( \tau(x(1 - t)) = y(t) = x(t) \forall 0 \leq t \leq \frac{1}{2} \). This implies \( x(1 - t) = \tau(x(t)) \forall t \in [0, 1] \). In particular, we get \( x(1) = \tau(x(0)) = x_0 = x(0) \). Moreover, since \( H_0 = H_1 \), one has \( \dot{x}(1) = \dot{x}(0) \). Hence \( x \) is a 1-periodic solution of \( \dot{x}(t) = X_{H_t}(x(t)) \) satisfying \( x(1 - t) = \tau(x(t)) \forall t \), that is, \( x \in \mathcal{P}(H, \tau) \). It is also clear that two different \( x_0, x_0^* \in L \cap \phi^{-1}_{\frac{1}{2}}(L) \) give two different elements in \( \mathcal{P}(H, \tau) \), \( x(t) = \phi_t(x_0) \) and \( x^*(t) = \phi^*_t(x_0) \).

Conversely, each \( x \in \mathcal{P}(H, \tau) \) determines a point \( x(0) \in L \cap \phi^{-1}_{\frac{1}{2}}(L) \) uniquely. So we get

\[
\mathcal{P}(H, \tau) = \{ x(t) = \phi_t(x_0) \mid x_0 \in L \cap \phi^{-1}_{\frac{1}{2}}(L) \} \tag{2.2}
\]

which implies \( \sharp \mathcal{P}(H, \tau) = \sharp(L \cap \phi^{-1}_{\frac{1}{2}}(L)) \geq m \).

(ii) \( \implies \) (i): By the assumption there exists a Hamiltonian \( H \in C_0^\infty([0, 1] \times M) \) with \( \|H\| < \Lambda \), such that its Hamiltonian flow \( \phi_t \) satisfies \( \phi_1 = \phi \). The proof will be finished along the line of proof of [BiPoSa, Proposition 2.1.3]. Take a small \( \delta > 0 \) so that \( 2\|H\| + 2\delta < 2\Lambda \). Then choose a smooth function \( \lambda : [0, 1] \rightarrow [0, 1] \) such that for a given small \( 0 < \epsilon \ll 1/2 \),

\[
\begin{align*}
\lambda(t) &= 0 \text{ for } t \in [0, \epsilon], \\
\lambda(t) &= 0 \text{ for } t \in [1 - \epsilon, 1], \\
\lambda'(t) &> 0 \text{ for } t \in (\epsilon, 1 - \epsilon).
\end{align*}
\tag{2.3}
\]

Clearly, \( \int_0^1 \lambda'(t) dt = 1 \). Take a time independent compactly supported function \( F : M \rightarrow \mathbb{R} \) which is \( \tau \)-invariant, such that \( \|F\|_{C^0} < \delta/4 \). Let \( f_t \) be the Hamiltonian flow generated by \( F \). Then the Hamiltonian isotopy \( \varphi_t := f_{t-\lambda(t)} \circ \phi_{\lambda(t)} \) is generated by the Hamiltonian function

\[
\overline{H}_t := F + \lambda'(t)(H_{\lambda(t)} - F) \circ f_{\lambda(t)-t}.
\]

The function \( \overline{H}_t \) equals \( F \) near \( t = 0 \) and \( t = 1 \) and hence defines a smooth Hamiltonian on \( S^1 \times M \). Moreover, \( \varphi_1 = \phi_1 \). Denote by

\[
A_H(t) = \sup_{x \in M} H_t(x) - \inf_{x \in M} H_t(x) \quad \forall t \in [0, 1].
\]

Then \( \|H\| = \int_0^1 A_H(t) dt \), and it is easily computed that

\[
A_{\overline{H}}(t) := \sup_{x \in M} \overline{H}_t(x) - \inf_{x \in M} \overline{H}_t(x) \\
\leq \lambda'(t)(\sup_{x \in M} H_{\lambda(t)}(x) - \inf_{x \in M} H_{\lambda(t)}(x)) + 2\|F\|_{C^0} + 2\lambda'(t)\|F\|_{C^0}.
\]

From this and (2.3) we arrive at

\[
\|\overline{H}\| = \int_0^1 A_{\overline{H}}(t) dt \\
\leq \int_0^1 \lambda'(t) A_H(\lambda(t)) dt + 4\|F\|_{C^0} \\
= \int_\epsilon^{1-\epsilon} A_H(\lambda(t)) d\lambda(t) + 4\|F\|_{C^0} \\
= \int_0^1 A_H(t) dt + 4\|F\|_{C^0} \\
= \|H\| + 4\|F\|_{C^0}.
\]
Let us define a smooth Hamiltonian $G : [0,1] \times M \to \mathbb{R}$ by

$$G_t(x) = \begin{cases} 2\overline{H}_{2t}(x) & \text{if } 0 \leq t \leq 1/2, \\ 2\overline{H}_{2(1-t)}(\tau x) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

It is easy to see that $G_t = F$ near $t = 0, 1/2, 1$, and $G_{t-1}(x) = G_t(\tau x)$ for any $(t,x) \in [0,1] \times M$. Extend $G$ to $\mathbb{R} \times M$ 1-periodically in $t$, still denoted by $G$, we easily see that $G$ satisfies $\|G\| = 2\|\overline{H}\| < 2\|H\| + 2\delta < 2\Lambda$ and (1.5), i.e.,

$$G_{t+1} = G_t \quad \text{and} \quad G_{t+1}(x) = G_t(\tau x) \quad \forall (t,x) \in \mathbb{R} \times M.$$ 

It follows that

$$X_{G_t}(x) = \begin{cases} 2X_{\overline{P}_{2t}}(x) & \text{if } 0 \leq t \leq 1/2, \\ -2d\tau(\tau x)X_{\overline{P}_{2(1-t)}}(\tau x) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

and thus the flow $\psi_t$ of $X_G$ and the flow $\varphi_t$ of $X_{\overline{P}}$ satisfy

$$\psi_{1/2}(x) = \varphi_t(x) \quad \text{for } (t,x) \in [0,1] \times M.$$ 

Specially, we have $\psi_{1/2} = \varphi_1 = \phi$. Now for any $y \in \mathcal{P}(G,\tau)$, the map $x : [0,1] \to M$ defined by $x(t) = y(t/2)$ satisfies $\dot{x}(t) = X_{\overline{P}_t}(x(t))$. Note that both $x(0) = y(0) = y(1)$ and $x(1) = y(1/2)$ belong to $L = \text{Fix}(\tau)$. Hence $x(1) = \varphi_1(x(0)) \in L \cap \varphi_1(L) = L \cap \phi(L)$ since $\varphi_1 = \phi = \phi_1$.

Moreover, for two different $y_1, y_2 \in \mathcal{P}(G,\tau)$ we have $y_1(t_0) \neq y_2(t_0)$ for some $t_0 \in [0,1/2]$. For $t \in [0,1]$ let $x_i(t) = y_i(t/2)$, $i = 1, 2$. Both satisfy $\dot{x}(t) = X_{\overline{P}_t}(x(t))$. Since $x_1(2t_0) \neq x_2(2t_0)$, that is, $\varphi_{2t_0}(x_1(0)) \neq \varphi_{2t_0}(x_2(0))$, we obtain $x_1(0) \neq x_2(0)$ and thus $x_1(1) \neq x_2(1)$.

In summary, we have proved $\sharp(L \cap \phi(L)) \geq \sharp\mathcal{P}(G,\tau)$. Applying Theorem 1.3(ii) to $G$ we have also that $\sharp\mathcal{P}(G,\tau) \geq \text{Cuplength}_{\sharp_2}(L)$. The desired claim is proved. \qed

2.2 Proof of Theorem 1.3

Let $(M,\omega,\tau)$ be real geometrical bounded for $J \in \mathcal{R}^J(M,\omega)$ and a Riemannian metric $\mu$ on $M$. By the assumptions of Theorem 1.3 there exists a compact subset $K \subset M$ such that

$$\text{supp}(H_t) \subset K \forall t \in \mathbb{R}, \quad L \subset K \quad \text{and} \quad \bigcup_{x \in \mathcal{P}(H,t)} x(\mathbb{R}) \subset K. \quad (2.4)$$

From now on, we assume $(M,g,J) \subset (\mathbb{R}^N,\langle \cdot,\cdot \rangle)$ by the Nash embedding theorem. Consider the standard Riemannian sphere $(S^2 = \mathbb{C} \cup \{\infty\},j)$ and the submanifold of the Banach manifolds $W^{1,p}(S^2, M)$ for a fixed $p > 2$,

$$\mathcal{B} = \{w \in W^{1,p}(S^2, M) \mid w \text{ is contractible} \}.$$ 

Let $E_j \to S^2 \times M$ be the vector bundle, whose fiber over $(z,m) \in S^2 \times M$ consists of all linear maps $\phi : T_zS^2 \to T_mM$ such that $J(m)\phi = -\phi \circ j$. Due to the
inclusion $W^{1, p}(S^2, M) \hookrightarrow C^0(S^2, M)$, for given $w \in W^{1, p}(S^2, M)$, we can denote by $ar{w}: S^2 \to S^2 \times M$ the “graph map” $\bar{w}(z) = (z, w(z))$ and write $\bar{w}^* E_j \to S^2$ for the pull back bundle. There exists a natural Banach space bundle $\mathcal{E} \to \mathcal{B}$ whose fiber $\mathcal{E}_w = L^p(\bar{w}^* E_j)$ at $w \in \mathcal{B}$ consists of all $L^p$ sections of the vector bundle $\bar{w}^* E_j \to S^2$. The nonlinear Cauchy-Riemannian operator $\bar{\partial}_j$,

$$\bar{\partial}_j(w) = dw + J \circ dw \circ j,$$

can be considered as a smooth section of the bundle $\mathcal{E} \to \mathcal{B}$.

Denote by $Z_T = [-T, T] \times S^1$ for $T > 1$. Take a smooth function $\gamma : \mathbb{R} \to [0, 1]$ such that $\gamma(s) = 1$ for $s \leq -1$, $\gamma(s) = 0$ for $s \geq 0$, and $\gamma'(s) \leq 0$ and for $s \in \mathbb{R}$. Define

$$\gamma_T(s) = \begin{cases} 
1, & s \in [-T + 1, T - 1], \\
\gamma(s - T), & s \geq T - 1, \\
\gamma(-s - T), & s \leq -T + 1.
\end{cases}$$

(2.5)

Then $\gamma'_T(s) \leq 0$ for $s \geq T - 1$, and $\gamma'_T(s) \geq 0$ for $s \leq -T + 1$. Denote by $\nabla$ the Levi-Civita connection with respect to the metric $\langle \cdot, \cdot \rangle = g_j(\cdot, \cdot)$. By the definition of $X_{H_t}$ above (1.6), $\nabla H_t = -JX_{H_t}$. For $(z, m) \in (S^2 \setminus \{0, \infty\}) \times M$ let us define

$$h^T_j(z, m)\left(\xi \frac{\partial}{\partial x}, \eta \frac{\partial}{\partial y}\right) = \xi \left(\frac{-T(s) e^{-2\pi s \cos(2\pi t)}}{2\pi} \nabla H_t(m)
- \gamma_T(s) e^{-2\pi s \sin(2\pi t)} J(m) \nabla H_t(m)
- \eta \frac{-T(s) e^{-2\pi s \sin(2\pi t)}}{2\pi} \nabla H_t(m)
- \gamma_T(s) e^{-2\pi s \cos(2\pi t)} J(m) \nabla H_t(m)\right)$$

for $\xi, \eta \in \mathbb{R}$ and $z = e^{2\pi(s + it)} \in \mathbb{C}$. It is easily checked that $h^T_j(z, m) \circ j = -J \circ h^T_j(z, m)$, i.e., $h^T_j(z, m) \in (E_j)(z, m)$. Note that

$$0 < |z| = e^{2\pi s} \leq e^{-2\pi(T + 1)} \iff s \in (-\infty, -T] \implies \gamma_T(s) = 0,$$

$$\infty > |z| = e^{2\pi s} \geq e^{2\pi(T + 1)} \iff s \in [T + 1, +\infty) \implies \gamma_T(s) = 0.$$ 

Hence we can define $h^T_j(0, m) = 0, h^T_j(\infty, m) = 0$ and get a smooth family of sections $h^T_j : S^2 \times M \to E_j, T > 1$. These give rise to a smooth family of sections of the Banach bundle $\mathcal{B} \to \mathcal{E}$, $g^T_j : \mathcal{B} \to \mathcal{E}, T > 1$, where

$$g^T_j(w)(z) = h^T_j(z, w(z)) \quad \forall z \in S^2.$$ 

For $\lambda \in [0, 1]$ we define

$$\mathcal{F}_{T, \lambda} : \mathcal{B} \to \mathcal{E}, \ w \mapsto \bar{\partial}_j w + \lambda g^T_j(w).$$

(2.6)

Note that $\tau$ and the standard complex conjugate $c_S$ on $(S^2, j)$ induce an involution

$$\tau_B : \mathcal{B} \to \mathcal{B}, \ w \mapsto \tau \circ w \circ c_S^{-1},$$

(2.7)
and its lifting involution
\[ \tau_E : \mathcal{E} \to \mathcal{E}, \]
where for \( \xi \in \mathcal{E}_u, \tau_E(\xi) \in \mathcal{E}_{\tau_B(w)} \) is given by
\[ \tau_E(\xi)(z, \tau_B(w)(z)) = d\tau(w(\bar{z})) \circ \xi(\bar{z}, w(\bar{z})) \circ dc_S(z) \quad \forall z \in S^2. \]

Let \( B^r \) be the set of fixed points of \( \tau_B \). It is a Banach submanifold in \( B \), and \( w \in B \) sits in \( B^r \) if and only if \( w(\bar{z}) = \tau(w(z)) \) for any \( z \in S^2 = \mathbb{C} \cup \{\infty\} \). Moreover, the involution \( \tau_E \) induces bundles homomorphisms on \( \mathcal{E}|_{B^r} \). Denote by \( \mathcal{E}_{+1} \) (resp. \( \mathcal{E}_{-1} \)) the eigenspace associated to the eigenvalue +1 (resp. -1) of this homomorphism. Then both \( \mathcal{E}_{+1} \) and \( \mathcal{E}_{-1} \) are Banach subbundles of \( \mathcal{E}|_{B^r} \), and \( \mathcal{E}|_{B^r} = \mathcal{E}_{+1} \oplus \mathcal{E}_{-1} \). Note also that
\[ \bar{\partial}_J(\tau_B(w)) = \tau_E(\bar{\partial}_J(w)) \quad \forall w \in B. \] (2.9)

So the restriction \( \bar{\partial}_J|_{B^r} \) gives rise to a section of the bundle \( \mathcal{E}^+ \to B^r \).

Since \( c_S(0) = 0 \) and \( c_S(\infty) = \infty \), we compute
\[ g^T_J(\tau_B(w))(z) = h^T_J(z, \tau_B(w)(z)) = h^T_J(z, \tau(w(\bar{z}))) \quad \text{for} \quad z \in S^2. \] (2.10)

Note that (1.6) implies that for \( x \in M \),
\[ \nabla H_{-1}(x) = d\tau(\tau(x))\nabla H_{+}(\tau(x)) \quad \text{and} \quad d\tau(x) \circ J(x) = -J(\tau(x)) \circ d\tau(x). \]

From the expression of \( h^T_J(z, m)\left(\xi \frac{\partial}{\partial z} + \eta \frac{\partial}{\partial \bar{z}}\right) \) above one easily checks
\[ h^T_J(z, \tau(w(\bar{z})))\left(\xi \frac{\partial}{\partial z} + \eta \frac{\partial}{\partial \bar{z}}\right) = d\tau(w(\bar{z}))h^T_J(z, w(\bar{z}))\left(\xi \frac{\partial}{\partial z} - \eta \frac{\partial}{\partial \bar{z}}\right), \]
that is, \( h^T_J(z, \tau(w(\bar{z}))) = d\tau(w(\bar{z})) \circ h^T_J(z, w(\bar{z})) \circ dc_S(z). \) So (2.8) and (2.10) lead to
\[ g^T_J(\tau_B(w)) = \tau_E(g^T_J(w)) \quad \forall w \in B. \] (2.11)

It follows from (2.9) and (2.11) that \( \mathcal{F}_\lambda \) in (2.6) satisfies
\[ \mathcal{F}_{T, \lambda}(\tau_B(w)) = \tau_E(\mathcal{F}_{T, \lambda}(w)) \quad \forall w \in B, \]
that is, each \( \mathcal{F}_{T, \lambda} \) is equivariant with respect to the involutions in (2.7) and (2.8). Hence the restrictions \( \mathcal{F}_{T, \lambda}|_{B^r} \) are the sections of the bundle \( \mathcal{E}^+ \to B^r \). It is easy to prove that all \( \mathcal{F}_{T, \lambda}|_{B^r} \) are Fredholm sections of index \( n = \dim L \) (by Lemma 2.4 the proof of [Ho2 Prop.6]). Define
\[ \mathcal{Z}^r_{T, \lambda} := \{ w \in B^r \mid \mathcal{F}_{T, \lambda}(w) = 0 \} \quad \text{and} \quad \mathcal{Z}^r_T := \{ (\lambda, w) \in [0, 1] \times B^r \mid \mathcal{F}_{T, \lambda}(w) = 0 \}. \]

The elliptic regularity arguments show that \( \mathcal{Z}^r_{T, \lambda} \) is contained in \( C_c^\infty(S^2, M) := \{ w \in C^\infty(S^2, M) \mid w \text{ is contractible} \} \).
Lemma 2.1 For $w \in \mathcal{Z}^T_{T,\lambda}$, define $u : Z_\infty \to M$ by $u = w \circ \phi$, where
\[
\phi : Z_\infty = \mathbb{R} \times S^1 \to S^2 \setminus \{0, \infty\}, \quad (s, t) \mapsto e^{2\pi(s+it)}
\]
is the biholomorphism. Then $u$ satisfies
\[
\partial_s u(s, t) + J(u(s, t)) (\partial_t u(s, t) - \lambda \gamma_T(s) X_{H_t}(u(s, t))) = 0,
\]
(2.12)
\[
E(u) : = \int_{Z_\infty} |\partial_s u|^2 dt ds \leq \|H\| \leq 2\|H\|_{C^0}.
\]
(2.13)

Proof. The equation $dw(z) + J(w) \circ dw(z) \circ j + h^T_{j,T}(z, w(z)) = 0$ yields
\[
dw(z) (\partial_s + J(w) \partial_g) + J(w) \circ dw(z) \circ j (\partial_s + J(w) \partial_g) = 0,
\]
that is
\[
\partial_s w + J(w) \partial_g w + \frac{\lambda \gamma_T(s) e^{-2\pi s} \cos(2\pi t)}{2\pi} \nabla^J H_t(w)
\]
\[- \frac{\lambda \gamma_T(s) e^{-2\pi s} \sin(2\pi t)}{2\pi} J(w) \nabla^J H_t(w) = 0.
\]
Since
\[
\partial_s w + J(w) \partial_g w = \frac{e^{-2\pi s} \cos(2\pi t)}{2\pi} (\partial_s u + J(u) \partial_t u) - \frac{e^{-2\pi s} \sin(2\pi t)}{2\pi} J(u) (\partial_s u + J(u) \partial_t u),
\]
it follows that $u(s, t) = w(e^{2\pi(s+it)})$ satisfies
\[
\frac{e^{-2\pi s} \cos(2\pi t)}{2\pi} (\partial_s u + J(u) \partial_t u) - \frac{e^{-2\pi s} \sin(2\pi t)}{2\pi} J(u) (\partial_s u + J(u) \partial_t u)
\]
\[+ \frac{\lambda \gamma_T(s) e^{-2\pi s} \cos(2\pi t)}{2\pi} \nabla^J H_t(u) - \frac{\lambda \gamma_T(s) e^{-2\pi s} \sin(2\pi t)}{2\pi} J(u) \nabla^J H_t(u)
\]
\[= \frac{e^{-2\pi s} \cos(2\pi t)}{2\pi} (\partial_s u + J(u) \partial_t u + \lambda \gamma_T(s) \nabla^J H_t(u))
\]
\[+ \frac{e^{-2\pi s} \sin(2\pi t)}{2\pi} J(u) (\partial_s u + J(u) \partial_t u + \lambda \gamma_T(s) \nabla^J H_t(u)) = 0.
\]
This is equivalent to (2.12) since $\nabla H_t = -j X_{H_t}$ and $g_J(X, JX) = 0$ for any $X \in TM$.

As to (2.13), note that the contractility of $w : S^2 \to M$ implies
\[
0 = \int_{S^2} w^* \omega = \int_{Z_\infty} u^* \omega = \int_{Z_\infty} |\partial_s u|_{g_J}^2 + \lambda \gamma_T(s) dH_t(\partial_s u) ds dt
\]
\[= \int_{Z_\infty} |\partial_s u|_{g_J}^2 ds dt + \lambda \int_0^1 dt \int_{-T-1}^{T+1} \gamma_T(s) \frac{d}{ds} H_t(u) ds
\]
\[= \int_{Z_\infty} |\partial_s u|_{g_J}^2 ds dt - \lambda \int_0^1 dt \int_{-T-1}^{T+1} \gamma_T(s) H_t(u) ds.
\]

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Hence

\[ E(u) = \int_{Z_\infty} |\partial_x u|^2_g \, ds \, dt \]
\[ = \lambda \int_0^1 dt \int_{T-1}^T \gamma_T'(s) H_t(u(s)) \, ds + \lambda \int_0^1 dt \int_{T-1}^{T+1} \gamma_T'(s) H_t(u(s)) \, ds \]
\[ \leq \lambda \int_0^1 \sup_p H_t(p) \, dt \int_{T-1}^{T+1} \gamma_T'(s) \, ds + \lambda \int_0^1 \inf_p H_t(p) \, dt \int_{T-1}^T \gamma_T'(s) \, ds \]
\[ = \lambda \int_0^1 \sup_p H_t(p) \, dt - \lambda \int_0^1 \inf_p H_t(p) \, dt \leq \lambda \|H\| \leq 2\|H\|_{C^0}, \]

where the first inequality is because $\gamma_T'(s) \geq 0$ for $-T \leq s \leq -T + 1$, and $\gamma_T'(s) \leq 0$ for $T - 1 \leq s \leq T$. \hfill \Box

**Lemma 2.2** Suppose that $\|H\| < +\infty$. Then there exists a compact subset $W \subset M$ such that $w(S^2) \subset W$ for any $(\lambda, w) \in Z_T$, and this $W$ can be assumed to be a compact submanifold of codimension zero and to contain $K$ in its interior.

**Proof.** Define $\Delta(w) := w^{-1}(M \setminus K) \subset S^2$. As in Lemma 2.1 let $u : Z_\infty \to M$ be defined by $u = w \circ \phi$. By (2.13) we may derive

\[ \int_{\Delta(w)} w^* \omega \leq E(u) \leq \|H\|. \]

Then one can complete the proof as in the proof of [Lu1, Theorem 2.9] or as in the proof of Lemma 2.3(i) below. There exists also another method to prove this. Each $(\lambda, w) \in Z_T$ satisfies $\bar{\partial}_J w(z) + \lambda h_T'(z, w(z)) = 0$ for $z \in S^2$. Thus the “graph map” $\bar{w} : S^2 \to S^2 \times M$ given by $\bar{w}(z) = (z, w(z))$ is holomorphic with respect to the almost complex structure $J_H$ on $S^2 \times M$ by

\[ J_{H,\lambda}(z, m)(X_1, X_2) = (iX_1, -\lambda J(m) \circ h_T'(z, m)X_1 + J(m)X_2). \]

Then fixing a metric $\tau$ on $S^2$ and applying [Sik, Prop.4.4.1] to $\bar{w} : S^2 \to (S^2 \times M, J_{H,\lambda}, \tau_0 \oplus \mu)$, the desired conclusion can be obtained. \hfill \Box

Let $C^\infty_c(S^1, M)$ denote the set of all contractible smooth loops $x : S^1 \to M$, and

\[ \mathcal{L}(M, \tau) := \{ x \in C^\infty_c(S^1, M) \mid x(-t) = \tau(x(t)) \ \forall t \in \mathbb{R} \}. \]

In the following we always assume that $C^\infty(\mathbb{R} \times S^1, M)$ is equipped with the compact open $C^\infty$-topology. Then it is not necessarily path connected even if $M$ is so. For $u \in C^\infty(\mathbb{R} \times S^1, M)$ and $s \in \mathbb{R}$ we write $u(s) : S^1 \to M$ by $u(s)(t) := u(s, t)$. It is clear that $u(s) \in C^\infty_c(S^1, M)$ for all $s \in \mathbb{R}$ if and only if $u(s) \in C^\infty_c(S^1, M)$ for some $s \in \mathbb{R}$. When $u \in C^\infty(\mathbb{R} \times S^1, M)$ satisfies the equation

\[ \partial_s u(s, t) + J(u(s, t))(\partial_t u(s, t) - X_{H_{\lambda}}(u(s, t))) = 0, \tag{2.14} \]

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we define its energy by $E(u) = \int_{Z_{\infty}} |\partial_s u|^2_{g_j} dsdt < +\infty$. Denote by
\begin{equation}
C^\tau := \{ u \in C^\infty(\mathbb{R} \times S^1, M) \mid u(s) \in \mathcal{L}(M, \tau) \forall s \in \mathbb{R} \}, \quad (2.15)
\end{equation}
\begin{equation}
X^\tau_{\infty} := \{ u \in C^\tau \mid u \text{ satisfies } (2.14), \ E(u) \leq \|H\| \}. \quad (2.16)
\end{equation}
Both are equipped with the topology induced from $C^\infty(\mathbb{R} \times S^1, M)$.

**Lemma 2.3** (i) The compact submanifold $W$ in Lemma 2.2 can be enlarged so that $u(\mathbb{R} \times S^1) \subset W$ for all $u \in X^\tau_{\infty}$.
(ii) $X^\tau_{\infty}$ is a compact metrisable space provided that $\|H\| < n(M, \omega, J)$.
(iii) If $\mathcal{P}_0(H, \tau)$ is finite, then for every $u \in C^\tau$ satisfying (2.14) and $E(u) < +\infty$ there exist $x^+, x^- \in \mathcal{P}_0(H, \tau)$ such that
\begin{equation*}
\lim_{s \to \pm \infty} u(s, t) = x^+(t) \quad \text{and} \quad \lim_{s \to \pm \infty} \partial_s u(s, t) = 0,
\end{equation*}
where both limits are uniform in the $t$-variable.

**Proof.** (i) We may assume that $M$ is noncompact. Let $u \in C^\tau$ satisfy (2.14) and $E(u) < +\infty$. Then
\begin{equation*}
\int_{-\infty}^{+\infty} \left( \int_{S^1} |\partial_t u(s, t) - X_{Ht}(u(s, t))|^2_{g_j} dt \right) ds = E(u) < +\infty.
\end{equation*}
Hence there exist sequences $s^+_k \uparrow +\infty$ and $s^-_k \downarrow -\infty$ such that
\begin{equation}
\lim_{k \to +\infty} \left\| \frac{\partial u}{\partial t}(s^+_k, \cdot) - X_{Ht}(u(s^+_k, \cdot)) \right\|^2_{L^2} = 0. \quad (2.17)
\end{equation}
Clearly, we may assume $0 < s^+_1 < s^+_2 < \cdots$ and $0 > s^-_1 > s^-_2 > \cdots$. Since $X_{Ht}$ vanishes outside the compact subset $K$, it follows from (2.17) that there exists a constant $C > 0$ such that
\begin{equation}
\left\| \frac{\partial u}{\partial t}(s^\pm_k, \cdot) \right\|^2_{L^2} \leq C, \quad \forall k = 1, 2, \ldots. \quad (2.18)
\end{equation}
These imply that for all $t \in [0, 1]$,
\begin{equation*}
d_{g_j}(u(s^+_k, t), u(s^-_k, t)) \leq \int_0^t \left| \frac{\partial u}{\partial t}(s^\pm_k, \tau) \right|_{g_j} d\tau \\
\leq \sqrt{t} \left( \int_0^t \left| \frac{\partial u}{\partial t}(s^\pm_k, \tau) \right|^2_{g_j} d\tau \right)^{1/2} \\
\leq \sqrt{C}, \quad \forall k = 1, 2, \ldots.
\end{equation*}
Since $u(s^\pm_k, t) \in L$, it follows that all $u_k(\{s^\pm_k\} \times S^1)$ are contained in a compact subset $K$ of $M$. Clearly, we can assume that $\hat{K}$ is a compact submanifold of codimension zero and with boundary and that $K$ is contained the interior of $\hat{K}$. 

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Now let us assume that this $u$ belongs to $X_\infty^s$. Define
\[ w = u|_{[s_1^-, s_1^+]} \times S^1, \quad w_j^+ = u|_{[s_j^+, s_{j+1}^+] \times S^1}, \quad w_j^- = u|_{[s_j^-, s_k^-] \times S^1}, \quad k = 1, 2, \ldots. \]

Then each connected component $\Sigma$ of
\[ w^{-1}(M \setminus \bar{K}) \text{ or } (w_j^+)^{-1}(M \setminus \bar{K}) \text{ or } (w_j^-)^{-1}(M \setminus \bar{K}) \] 

has compact closure and is the increasing union of connected compact Riemannian surfaces with smooth boundary $\Sigma_j$, $j = 1, 2, \ldots$. For sufficiently large $j$ we have always $\partial \Sigma_j \subset \bar{K}_1 = \{ x \in M \mid d(x, \bar{K}) \leq 1 \}$. Note that the restriction of $w$ or $w_j^+$ or $w_j^-$ to each $\Sigma_j$ is $J$-holomorphic and has the energy $\leq \| H \|$. By [Sik] Prop.4.4.1 we may deduce that this restriction has the image contained in the $\tau$-neighborhood of $(\bar{K}_1)_{\tau}$ for some $\tau > 0$ only depending on $(M, \omega, \mu, J)$.

(ii) By (i) we may assume that $M$ is compact below. As in [HoZe, page 236], it suffices to prove that there exists a constant $C > 0$ such that
\[ |\nabla u(s,t)|_{g_j} \leq C \quad \forall u \in X_\infty^s \text{ and } (s,t) \in Z_\infty. \tag{2.19} \]

Arguing indirectly, as on pages 236-238 in [HoZe], we find sequences $\varepsilon_k \downarrow 0$, $\{t_k\}_k \subset [0,1]$ and $\{u_k\}_k \subset X_\infty^s$ such that
\[
\begin{align*}
& t_k \to t_0 \in [0,1], \quad \varepsilon_k R_k \to +\infty \quad \text{for } R_k = |\nabla u_k(0,t_k)|_{g_j} \to +\infty, \\
& |\nabla u_k(s,t)|_{g_j} \leq 2|\nabla u_k(0,t_k)|_{g_j} \quad \text{if } |s|^2 + |t - t_k|^2 \leq \varepsilon_k^2, \quad 0 \leq t_k \leq 1
\end{align*}
\]

where we consider the $u_k$ as maps defined on $\mathbb{R} \times \mathbb{R}$ by a 1-periodic continuation in the $t$-variable. It follows that the new sequence $v_k \in C^\infty(\mathbb{R}^2, M)$ defined by
\[ v_k(s,t) = u_k\left(\frac{s}{R_k}, t_k + \frac{t}{R_k}\right) \text{ for } s^2 + t^2 \leq (\varepsilon_k R_k)^2 \]

converges, in $C^\infty(\mathbb{R}^2, M)$, to $v \in C^\infty(\mathbb{R}^2, M)$ which satisfies
\[ |\nabla v(0)|_{g_j} = 1, \quad \sup_{x \in \mathbb{R}^2} |\nabla v(x)|_{g_j} \leq 2, \quad v_s + J(v)v_t = 0. \tag{2.20} \]

Denote by $B(p,r) \subset \mathbb{R}^2$ the disk centred at $p$ and of radius $r$. Then
\[
\int_{B(0,\varepsilon_k R_k)} |\partial_s v_k|^2_{g_j} dsdt = \int_{B(0,\varepsilon_k R_k)} \frac{1}{R_k^2} |\partial_s u_k\left(\frac{s}{R_k}, t_k + \frac{t}{R_k}\right)|^2_{g_j} dsdt
\]
\[
= \int_{B(0,t_k,\varepsilon_k)} |\partial_s u_k(s,t)|^2_{g_j} dsdt
\]
\[
\leq E(u_k) \leq \| H \|
\]

for sufficiently large $k$ (so that $\varepsilon_k < 1/2$). It easily follows that
\[ \int_C |\partial_s v|^2_{g_j} dsdt \leq \| H \| < m(M, \omega, J). \]
However, \((2.20)\) and Gromov’s removable singularity allow us to extend \(v\) to a non-constant \(J\)-holomorphic sphere \(v_\infty : S^2 \to M\) with
\[
\int_{S^2} v_\infty^* = \int_C |\partial_s v|^2_{g_J} dsdt \leq \|H\| < m(M, \omega, J)
\]
which contradicts to the definition of \(m(M, \omega, J)\). \((2.19)\) is proved.

(iii). Since the condition \(\# \mathcal{P}(H) < +\infty\) is actually sufficient for “(i)\(\Rightarrow\)(ii)” in the proof of \(\text{[Sa], Prop.1.21}\), we may complete the proof with the same reason. \(\square\)

**Lemma 2.4** Suppose that \(\|H\| < m(M, \omega, J)\). Then \(Z_{T,\lambda}^\tau\) and \(Z_T^\tau\) are compact in \(C^\infty(S^2, M)\) and \([0, 1] \times C^\infty(S^2, M)\), respectively.

**Proof.** By Lemma 2.2 we may assume \(M\) to be compact. Using \((2.13)\) we can, as in the proof of Lemma 2.3, prove that there exists a constant \(C_T > 0\) such that for every \((\lambda, w) \in Z_T^\tau\) and \(u = w \circ \phi : Z_\infty \to M\) as in Lemma 2.1
\[
\sup_{(s,t) \in Z_\infty} |\nabla u(s,t)|_{g_J} \leq C_T. \quad (2.21)
\]
It implies that for each multi-index \(\alpha \in \mathbb{N}^2\) one can find a constant \(C_{T,\alpha} > 0\) such that for all \(u\) as above,
\[
\sup_{(s,t) \in Z_\infty} |(D^\alpha u)(s,t)|_{g_J} \leq C_{T,\alpha}. \quad (2.22)
\]

Now suppose that \(Z_T^\tau\) is noncompact. Then there exists sequences \(\{(\lambda_k, w_k)\}_k \subset Z_T^\tau\) and \(\{z_k\}_k \subset S^2 = \mathbb{C}P^1\) such that
\[
\lambda_k \to \lambda_0 \quad \text{and} \quad |dw_k(z_k)| = \|dw_k\| := \max_{z \in S^2} |dw_k(z)| \to +\infty, \quad (2.23)
\]
where \(|dw_k(z)|\) is the norm of the tangent map \(dw_k(z) : T_zS^2 \to T_{w_k(z)}M\) induced by \(g_J\) and the standard Riemannian metric on \(S^2\). We may assume that \(z_k \to z_0 \in S^2 = \mathbb{C}P^1\). By \((2.21)\) this \(z_0\) must be 0 or \(\infty\) in \(\mathbb{C}P^1\). (Otherwise, passing to a subsequence we may assume \(inf_k d(z_k,0) > 0\) and \(inf_k d(z_k,\infty) > 0\). Thus there exists a large \(T_0 > 0\) such that \((s_k, t_k) = \phi^{-1}(z_k) \subset S^1 \times [-T_0, T_0]\) for all \(k\). It follows from \((2.23)\) that \(u_k = w_k \circ \phi\) satisfies \(|du_k(s_k, t_k)| \to \infty\), which contradicts to \((2.22)\).) By the Gromov compactness theorem the sequence \(\{w_k\}_k\) has a subsequence, still denoted by \(\{w_k\}_k\), converges weakly to a connected union of \(N \geq 1\) nonconstant \(J\)-holomorphic spheres \(v_1, \cdots, v_N : S^2 \to M\) and a smooth map \(w_\infty : S^2 = \mathbb{C}P^1 \to M\) satisfying
\[
\partial_J w + \lambda_0 g_J^{-1}(w) = 0. \quad (2.24)
\]
In particular, \([v_1^* \cdots v_N^* w_\infty] = 0 \in \pi_2(M)\). Let \(u_\infty = w_\infty \circ \phi : Z_\infty \to M\). Then as
in the proof of Lemma 2.1 we have
\[
0 = \sum_{k=1}^{N} \int_{S^2} v_k^* \omega + \int_{S^2} w^*_\infty \omega = \sum_{k=1}^{N} \int_{S^2} v_k^* \omega + \int_{\tau} u^*_\infty \omega \\
= \sum_{k=1}^{N} \int_{S^2} v_k^* \omega + \int_{\tau} \left(\|\partial_s u_\infty\|_{g^2} + \lambda_0 \gamma_T(s) dH_t(\partial_s u_\infty)\right) dsdt \\
= \sum_{k=1}^{N} \int_{S^2} v_k^* \omega + \int_{\tau} \|\partial_s u_\infty\|_{g^2} dsdt + \lambda_0 \int_{0}^{T} \int_{-T}^{T+1} \gamma_T(s) \frac{d}{ds} H_t(u_\infty) ds.
\]
It follows that
\[
m(M, \omega, J) \leq Nm(M, \omega, J) + E(u_\infty) \\
\leq \sum_{k=1}^{N} \int_{S^2} v_k^* \omega + E(u_\infty) \\
= -\lambda_0 \int_{0}^{1} dt \int_{-T}^{T+1} \gamma_T(s) \frac{d}{ds} H_t(u_\infty) ds \\
\leq \lambda_0 \|H\| \leq \|H\| < m(M, \omega, J)
\]
as in the proof of Lemma 2.1. This contradiction gives the desired conclusion. \(\square\)

For \(T > 1\) we set
\[
X_T^\tau := \{u \in C^\infty(Z_T, M) \mid u(0) \in \mathcal{L}(M, \tau) \text{ and } \int_{Z_T} |\partial_s u|^2_{g^2} \leq \|H\|\}, \\
X_T^{\tau, J} := \{u \in X_T^\tau \mid \partial_s u + J(u) \partial_t u + \nabla H_t(u) = 0 \text{ on } Z_T\}.
\]
As in the proofs of (i)-(ii) of Lemma 2.3 we may get

**Lemma 2.5** The compact submanifold \(W\) in Lemma 2.3 can be furthermore enlarged so that \(u(Z_T) \subset W\) for all \(u \in X_T^\tau\). Moreover, there exists a constant \(\tilde{C} > 0\) such that for every \(T > 2\),
\[
\sup \left\{ |\nabla u(s, t)|_{g^2} \mid (s, t) \in Z_T \right\} \leq \tilde{C} \quad \forall u \in X_T^\tau. \tag{2.25}
\]
Let \(\gamma_T(s)\) be as in (2.5). Define
\[
\sigma_T : X_T^\tau \to C^\tau, \quad u \mapsto \sigma_T(u) \tag{2.26}
\]
by \(\sigma_T(u)(s, t) = u(\gamma_T(s), s, t)\). Then \(\sigma_T(u)(s, t) = u(s, t) \forall (s, t) \in Z_T+1\).

**Theorem 2.6** Suppose that \(\|H\| < m(M, \omega, J)\). Then for a given open neighborhood \(U\) of \(X_\infty^\tau\) in \(C^\tau\) there exists \(T_0 > 1\) such that
\[
\sigma_T(X_T^{\tau, J}) \subset U \text{ for any } T \geq T_0.
\]
Furthermore, this \(T_0\) can be enlarged so that
\[
\sigma_T(u|Z_T^\tau) \subset U \quad \forall T > T_0
\]
for any \(u = w \circ \phi\) with \(w \in Z_T^{\tau, 1}\), where \(Z_T^{\tau, 1}\) is as above Lemma 2.1.
Proof. Since (2.25) implies that for each multi-index $\alpha \in \mathbb{N}^2$ one can find a constant $\tilde{C}_\alpha > 0$ such that for every $T > 6$,

$$
\sup \left\{ \left| (D^\alpha u)(s,t) \right|_{g_J} \left| (s,t) \in Z_{T-3} \right\} \leq \tilde{C}_\alpha \quad \forall u \in X^T_T.
$$

As in the arguments on pages 244-245 of [HoZe], suppose that there exist an open neighborhood $U$ of $X^T_\infty$ in $C^T$ and sequences $T_k \to +\infty$ and $u_k \in X^T_{T_k}$ such that $u_k \notin U$ for all $k$. From (2.27) we may choose a subsequence $\{u_{k_j}\}_j$ of $\{u_k\}_k$ such that $u_{k_j}$ converges to $u$ in $C^\infty_0(\mathbb{R} \times S^1, M)$. Clearly, $u$ satisfies

$$
\partial_s u + J(u)\partial_t u + \nabla H_t(u) = 0 \quad \text{on } Z_\infty,
$$

$$
u(0,\cdot) \in C^\infty_c(S^1, M) \quad \text{and} \quad u(s,-t) = \tau(u(s,t)) \quad \forall (s,t) \in Z_\infty,
$$

$$
E(u) = \int_{Z_\infty} \left| \partial_s u(s,t) \right|_{g_J}^2 dsdt \leq \|H\|.
$$

That is, $u \in X^T_T$. Moreover, all $u_{k_j}$ belong to the closed subset $C^T \setminus U$ of $C^T$. Hence $u \notin U$, which contradicts $u \in X^T_T \subset U$. \hfill \Box

For $C^T$ in (2.15) we define an evaluation map

$$
\pi : C^T \to L, \ u \mapsto u(0,0),
$$

and denote $\tilde{H}^*$ by the Alexander-Spanier cohomology. Then Theorem 1.3 can be derived from the following result.

**Theorem 2.7** Under the assumptions, for every open neighborhood $U$ of $X^T_\infty$ in $C^T$ the restriction $\pi|_U$ induces an injection

$$
(\pi|_U)^* : \tilde{H}^*(L, \mathbb{Z}_2) \to \tilde{H}^*(U, \mathbb{Z}_2).
$$

So the continuity property of the Alexander-Spanier cohomology implies

$$
\pi|_{X^T_\infty} : \tilde{H}^*(L, \mathbb{Z}_2) \to \tilde{H}^*(X^T_\infty, \mathbb{Z}_2)
$$

is injective. If $L$ is orientable, $\pi|_{X^T_\infty} : \tilde{H}^*(L, \mathbb{Z}) \to \tilde{H}^*(X^T_\infty, \mathbb{Z})$ is also injective.

Consequently, Cuplength$_{\mathbb{Z}_2}(X^T_\infty) \geq$ Cuplength$_{\mathbb{Z}_2}(L)$, and Cuplength$_{\mathbb{Z}}(X^T_\infty) \geq$ Cuplength$_{\mathbb{Z}}(L)$ if $L$ is orientable.

**Proof of Theorem 1.3** Clearly, we may assume $P_0(H, \tau)$ to be a finite set under the assumptions of Theorem 1.3. Consider the closed 1-form $\alpha$ on $\mathcal{L}(M)$ given by

$$
\alpha(x) = \int_0^1 \omega(\dot{x}(t) - X_{H_t}(x(t),\xi(t))) dt \quad \forall (x, \xi) \in TL(M, \tau).
$$

It restricts to a closed 1-form $\alpha^\tau$ on $\mathcal{L}(M, \tau)$. Let $\phi_\omega : \pi_2(M) \to \mathbb{R}$ be the homomorphism defined by integration of $\omega$. Denote by $\hat{\mathcal{L}}(M)$ the set of all pairs $\hat{x} = (x, [w])$, where $x \in \mathcal{L}(M)$ and $[w]$ is an equivalence class of smooth discs $w : D^2 \to M$ with $w(e^{2\pi it}) = x(t)$ $\forall t$ for the equivalence relation $\sim$: $w \sim w'$ if and only if the sphere
$w_\tau w'$ being vanished by $\omega$. Then $\Pi : \tilde{L}(M) \to L(M)$, $(x, [w]) \mapsto x$ is a covering whose desk group is the quotient $\Gamma(\omega) = \pi_2(M)/\ker(\phi_\omega)$. The symplectic action functional

$$A_H : \tilde{L}(M) \to \mathbb{R}, \quad (x, [w]) \mapsto -\int_{D^2} w^*\omega + \int_0^1 H(t, x(t)) \, dt$$

(2.30)
is a primitive of $\Pi^*\alpha$, i.e., $dA_H(x, [w])[\xi] = (\Pi^*\alpha)(x, [w])(\xi) = \alpha_x(\xi)$ for any $\xi \in C^\infty(x^*TM)$. Let $A_{H,\tau}$ be the restriction of $A_H$ to $\tilde{L}(M, \tau) := \Pi^{-1}(L(M, \tau))$. Then $dA_{H,\tau} = \alpha^\tau$.

By the assumption the rationality index $r_\omega$ of $(M, \omega)$ is positive. If $r_\omega = +\infty$, i.e., $\omega|\pi_2(M) = 0$, then $\tilde{L}(M) = L(M)$. If $r_\omega \in (0, +\infty)$, $A_{H,\tau}$ descends to a map $A^*_H : L(M, \tau) \to \mathbb{R}/r_\omega\mathbb{Z}$, which is a primitive of $\alpha^\tau$. Let $p : \mathbb{R} \to \mathbb{R}/r_\omega\mathbb{Z}$ be the canonical projection in the latter case. Define $a_H : L(M, \tau) \to \mathbb{R}$ by

$$a_H(x) = \begin{cases} A_{H,\tau}(x) & \text{if } r_\omega = +\infty, \\ (p|_{[0, r_\omega)})^{-1} \circ A^*_H(x) & \text{if } r_\omega \in (0, +\infty). \end{cases}$$

(2.31)

This is continuous and satisfies

$$\frac{d}{ds} a_H(u(s)) = \int_0^1 |\partial_s u(s, t)|_{g_J}^2 \, dt \quad \forall u \in X^r_\infty,$$

(2.32)

where $u(s)(t) = u(s, t)$. By Lemma 2 on [HoZe, page 225] (or its proof)

$$\frac{d}{ds} a_H(u(s))|_{s=s_0} = 0 \quad \text{for some } s_0 \in \mathbb{R}$$

implies that $u(s) = u(s_0) \forall s \in \mathbb{R}$ and $x := u(s_0) = u(s_0, \cdot)$ belongs to $P_0(H, \tau)$. This shows that the natural flow on the compact metric space $X^r_\infty$ defined by

$$\Phi : \mathbb{R} \times X^r_\infty \to X^r_\infty : (\sigma, u) \mapsto \sigma \cdot u,$$

(2.33)

where $(\sigma \cdot u)(s, t) = u(\sigma + s, t)$, is gradient-like and has $a_H$ as a Ljapunov function. Thus Corollary on [CoZe, page 42] yields

$$\#P_0(H, \tau) \geq \text{Cuplength}_{\mathbb{Z}_2}(L) \quad \text{(or } \geq \text{Cuplength}_{\mathbb{Z}}(L) \text{ if } L \text{ is orientable}).$$

(2.34)

This and Theorem [2.7] give the desired conclusion immediately. $\square$

## 3 Proof of Theorem [2.7]

In order to prove this result let us recall that a Banach Fredholm bundle of index $r$ and with compact zero sets is a triple $(X, E, S)$ consisting of a Banach manifold $X$, a Banach vector bundle $E \to X$ and a Fredholm section $S$ of index $r$ and with compact zero sets. If the determinant bundle $\det(S) \to Z(S)$ is oriented, i.e., it is trivializable and is given a continuous section nowhere zero, we said $(X, E, S)$ to be oriented. One has the following standard result (cf. [Lu1, Theorem 1.5]).
Theorem 3.1 Let \((X, E, S)\) be a Banach Fredholm bundle of index \(r\). Then there exist finitely many smooth sections \(\sigma_1, \sigma_2, \ldots, \sigma_m\) of the bundle \(E \to X\) such that for the smooth sections

\[
\Phi : X \times \mathbb{R}^m \to \Pi_1^*E, \quad (y, t) \mapsto S(y) + \sum_{i=1}^m t_i \sigma_i(y),
\]

\[
\Phi_t : X \to E, \quad y \mapsto S(y) + \sum_{i=1}^m t_i \sigma_i(y),
\]

where \(t = (t_1, \ldots, t_m) \in \mathbb{R}^m\) and \(\Pi_1\) is the projection to the first factor of \(X \times \mathbb{R}^m\), the following holds: There exist an open neighborhood \(\mathcal{W} \subset \mathcal{O}(Z(S))\) of \(Z(S)\) and a small \(\varepsilon > 0\) such that:

(A) The zero locus of \(\Phi\) in \(Cl(\mathcal{W} \times B_\varepsilon(\mathbb{R}^m))\) is compact. Consequently, for any given small open neighborhood \(\mathcal{U}\) of \(Z(S)\) there exists a \(\varepsilon \in (0, \varepsilon']\) such that \(Cl(\mathcal{W}) \cap \Phi_t^{-1}(0) \subset \mathcal{U}\) for any \(t \in B_\varepsilon(\mathbb{R}^m)\). In particular, each set \(\mathcal{W} \cap \Phi_t^{-1}(0)\) is compact for \(t \in B_\varepsilon(\mathbb{R}^m)\) sufficiently small.

(B) The restriction of \(\Phi\) to \(\mathcal{W} \times B_\varepsilon(\mathbb{R}^m)\) is (strong) Fredholm and also transversal to the zero section. So

\[
U_\varepsilon := \{(y, t) \in \mathcal{W} \times B_\varepsilon(\mathbb{R}^m) \mid \Phi(y, t) = 0\}
\]

is a smooth manifold of dimension \(m + \text{Ind}(S)\), and for \(t \in B_\varepsilon(\mathbb{R}^m)\) the section \(\Phi_t|_\mathcal{W} : X \to E\) is transversal to the zero section if and only if \(t\) is a regular value of the (proper) projection

\[
P_\varepsilon : U_\varepsilon \to B_\varepsilon(\mathbb{R}^m), \quad (y, t) \mapsto t,
\]

and \(\Phi_t^{-1}(0) \cap \mathcal{W} = P_\varepsilon^{-1}(t)\). (Specially, \(t = 0\) is a regular value of \(P_\varepsilon\) if \(S\) is transversal to the zero section). Then the Sard theorem yields a residual subset \(B_\varepsilon(\mathbb{R}^m)_{\text{res}} \subset B_\varepsilon(\mathbb{R}^m)\) such that:

(B.1) For each \(t \in B_\varepsilon(\mathbb{R}^m)_{\text{res}}\) the set \((\Phi_t|_\mathcal{W})^{-1}(0) \approx (\Phi_t|_\mathcal{W})^{-1}(0) \times \{t\} = P_\varepsilon^{-1}(t)\) is a compact smooth manifold of dimension \(\text{Ind}(S)\) and all \(k\)-boundaries

\[
\partial^k(\Phi_t|_\mathcal{W})^{-1}(0) = (\partial^k X) \cap (\Phi_t|_\mathcal{W})^{-1}(0)
\]

for \(k = 1, 2, \ldots\). Specially, if \(Z(S) \subset \text{Int}(X)\) one can shrink \(\varepsilon > 0\) so that \((\Phi_t|_\mathcal{W})^{-1}(0)\) is a closed manifold for each \(t \in B_\varepsilon(\mathbb{R}^m)_{\text{res}}\).

(B.2) If the Banach Fredholm bundle \((X, E, S)\) is oriented, i.e., the determinant bundle \(\det(DS) \to Z(S)\) is given a nowhere vanishing continuous section over \(Z(S)\), then it determines an orientation on \(U_\varepsilon\). In particular, it induces a natural orientation on every \((\Phi_t|_\mathcal{W})^{-1}(0)\) for \(t \in B_\varepsilon(\mathbb{R}^m)_{\text{res}}\).

(B.3) For any \(l \in \mathbb{N}\) and two different \(t^{(1)}, t^{(2)} \in B_\varepsilon(\mathbb{R}^m)_{\text{res}}\) the smooth manifolds \((\Phi_{t^{(1)}}|_\mathcal{W})^{-1}(0)\) and \((\Phi_{t^{(2)}}|_\mathcal{W})^{-1}(0)\) are cobordant in the sense that for a generic \(C^l\)-path \(\gamma : [0, 1] \to B_\varepsilon(\mathbb{R}^m)\) with \(\gamma(0) = t^{(1)}\) and \(\gamma(1) = t^{(2)}\) the set

\[
\Phi^{-1}(\gamma) := \bigcup_{t \in [0, 1]} \{t\} \times (\Phi_{\gamma(t)}|_\mathcal{W})^{-1}(0)
\]
is a compact smooth manifold with boundary
\[ \{0\} \times (\Phi_t(\cdot)|_{W})^{-1}(0) \cup \{-1\} \times (\Phi_t(\cdot)|_{W})^{-1}(0). \]

In particular, if \( Z(S) \subset \text{Int}(X) \) and \( \varepsilon > 0 \) is suitably shrunk so that \( (\Phi_t|_{W})^{-1}(0) \subset \text{Int}(X) \) for any \( t \in B_\varepsilon(\mathbb{R}^m) \) then \( \Phi^{-1}(\gamma) \) has no corners.

(B.4) The cobordant class of the manifold \( (\Phi_t|_{W})^{-1}(0) \) above is independent of all related choices.

Now we furthermore assume that \( N \) is a connected manifold of dimension \( r \) and \( f : X \to N \) is a smooth map. When \( X \) has no boundary, by Theorem 3.1(B.1), for each \( t \in B_\varepsilon(\mathbb{R}^m)_{\text{res}} \) the section \( \Phi_t : X \to E \) is transversal to the zero section and the set \( (\Phi_t|_{W})^{-1}(0) \subset X \) is a compact smooth manifold of dimension \( r \) and without boundary. So we may consider the \( \mathbb{Z}_2 \)-Brouwer degree
\[ \deg_{\mathbb{Z}_2}(f|_{\Phi_t|_{W}}^{-1}(0)) \]
of the restriction \( f|_{\Phi_t|_{W}}^{-1}(0) : (\Phi_t|_{W})^{-1}(0) \to N \). The elementary properties and Theorem 3.1(B.3) show that \( \deg_{\mathbb{Z}_2}(f|_{\Phi_t|_{W}}^{-1}(0)) \in \mathbb{Z}_2 \) is independent of the choice of \( t \in B_\varepsilon(\mathbb{R}^m)_{\text{res}} \). Moreover, it is claimed in Theorem 3.1(B.4) that the cobordant class of the manifold \( (\Phi_t|_{W})^{-1}(0) \) above is independent of all related choices. Namely, suppose that \( \sigma_1', \sigma_2', \ldots, \sigma_{m'} \) are another group of smooth sections of the bundle \( E \to X \) such that the section
\[ \Psi : \mathcal{W}' \times B_\varepsilon(\mathbb{R}^{m'}) \to \Pi_1^*E, \quad (y, t') \mapsto S(y) + \sum_{i=1}^{m'} t'_i \sigma_i'(y), \]
is Fredholm and transversal to the zero and that the set \( \Psi_{t'}^{-1}(0) \) is compact for each \( t' \in B_\varepsilon(\mathbb{R}^{m'})_{\text{res}} \), where the section \( \Psi_{t'} : \mathcal{W}' \to E \) is given by \( \Psi_{t'}(y) = \Psi(y, t') \). Let \( B_\varepsilon(\mathbb{R}^{m'})_{\text{res}} \subset B_\varepsilon(\mathbb{R}^{m'}) \) be the corresponding residual subset such that for each \( t' \in B_\varepsilon(\mathbb{R}^{m'})_{\text{res}} \) the section \( \Psi_{t'} \) is transversal to the zero section and that any two \( t', s' \in B_\varepsilon(\mathbb{R}^{m'})_{\text{res}} \) yield cobordant manifolds \( (\Psi_{t'})^{-1}(0) \) and \( (\Psi_{s'})^{-1}(0) \). Then it was shown in the proof of [Lau1] Theorem 1.5(B.4)] that there exist a compact submanifold \( \Theta_{t, t'}^{-1}(0) \subset X \times [0, 1] \) of dimension \( r + 1 \) for any \( t \in B_\varepsilon^{reg}(\mathbb{R}^m) \) and \( t' \in B_\varepsilon^{reg}(\mathbb{R}^{m'}) \) such that \( \partial \Theta_{t, t'}^{-1}(0) = (\Phi_t|_{W})^{-1}(0) \times \{0\} \cup \Psi_{t'}^{-1}(0) \times \{1\} \). This implies that
\[ \deg_{\mathbb{Z}_2}(f|_{\Phi_t|_{W}}^{-1}(0)) = \deg_{\mathbb{Z}_2}(f|_{\Psi_{t'}|_{W}}^{-1}(0)). \]

Hence we have a well-defined \( \mathbb{Z}_2 \)-value degree
\[ \deg_{\mathbb{Z}_2}(f, N, X, E, S) := \deg_{\mathbb{Z}_2}(f|_{\Phi_t|_{W}}^{-1}(0)) \in \mathbb{Z}_2 \] (3.1)
for any \( t \in B_\varepsilon^{reg}(\mathbb{R}^m) \), and call it the \( \mathbb{Z}_2 \)-degree of \( f : X \to N \) relative to \( (X, E, S) \).

Of course, when both \((X, E, S)\) and \( N \) are oriented, we may define the \( \mathbb{Z} \)-degree of \( f : X \to N \) relative to \((X, E, S)\).

Let \( \{S_\lambda\}_{\lambda \in [0, 1]} \) be a smooth family of smooth Fredholm sections of the bundle \( E \to X \) of index \( r \) and with compact zero sets. Then we can still choose finitely
many smooth sections $\sigma_1, \sigma_2, \ldots, \sigma_m$ of the bundle $E \to X$, an open neighborhood $W_\lambda$ of each $Z(S_\lambda) \subset X$, and a residual subset $B_\varepsilon(\mathbb{R}^m)_{\text{res}}$ for some small $\varepsilon > 0$, such that for each $t \in B_\varepsilon(\mathbb{R}^m)_{\text{res}}$ the restrictions of the smooth sections

$$
\Phi_t^0 : X \to E, \ y \mapsto S_0(y) + \sum_{i=1}^m t_i \sigma_i(y),
$$

$$
\Phi_t^1 : X \to E, \ y \mapsto S_1(y) + \sum_{i=1}^m t_i \sigma_i(y),
$$

$$
\Phi_t : X \times [0, 1] \to \Pi_1^* E, \ (y, \lambda) \mapsto S_\lambda(y) + \sum_{i=1}^m t_i \sigma_i(y)
$$
to $W_0$, $W_1$ and $W = \cup_{\lambda \in [0, 1]} W_\lambda$ are transversal to the zero sections respectively. In particular, we get

$$
\partial(\Phi_t|_W)^{-1}(0) = (\Phi_t^0|_{W_0})^{-1}(0) \times \{0\} \bigcup (\Phi_t^1|_{W_1})^{-1}(0) \times \{1\}.
$$

It follows that

$$
\text{deg}_{Z_2}(f, N, X, E, S_0) = \text{deg}_{Z_2}(f, N, X, E, S_1)
$$

and thus $\text{deg}_{Z_2}(f, N, X, E, S_\lambda)$ is independent of $\lambda \in [0, 1]$.

Similarly, if $(X, E, S_\lambda)$ and $N$ are oriented, $\text{deg}_{Z_2}(f, N, X, E, S_\lambda)$ is independent of $\lambda \in [0, 1]$ as well.

**Proof of Theorem 2.7** Define the evaluation map

$$
\Theta : B^r \to L, \ u \mapsto u(1),
$$

where $1 \in \mathbb{C} \subset \mathbb{C} \cup \{\infty\} = S^2$. Applying the arguments above to the Banach Fredholm bundle $(B^r, \mathcal{E}^+, \mathcal{F}_{T,\lambda}|_{B^r})$, $\lambda \in [0, 1]$, we arrive at

$$
\text{deg}_{Z_2}(\Theta, L, B^r, \mathcal{E}^+, \mathcal{F}_{T,1}|_{B^r}) = \text{deg}_{Z_2}(\Theta, L, B^r, \mathcal{E}^+, \mathcal{F}_{T,0}|_{B^r})
$$

by (3.2). Since each $w \in B$ is contractible, $Z_{T,0}^r = (\mathcal{F}_{T,0}|_{B^r})^{-1}(0_{\mathcal{E}^+})$ precisely consists of the constant maps $S^2 \to L$. It is easily proved that $\mathcal{F}_{T,0}|_{B^r} : B^r \to \mathcal{E}^+$ is transversal to the zero section, and that (3.1) yields

$$
\text{deg}_{Z_2}(\Theta, L, B^r, \mathcal{E}^+, \mathcal{F}_{T,0}|_{B^r}) = 1.
$$

Let $F$ be a smooth perturbation section of $\mathcal{F}_{T,1}|_{B^r}$ as $\Phi_t^1$ above. Choose $l_0 \in L$ to be a regular value for the evaluations

$$
\Theta|_{F^{-1}(0_{\mathcal{E}^+})} : F^{-1}(0_{\mathcal{E}^+}) \to L.
$$

Then (3.4) and (3.5) show that

$$
\text{deg}_{Z_2}(\Theta|_{F^{-1}(0_{\mathcal{E}^+})}, l_0) = 1.
$$

Hence $\Theta|_{F^{-1}(0_{\mathcal{E}^+})} : F^{-1}(0_{\mathcal{E}^+}) \to L$ induces an injection map

$$
(\Theta|_{F^{-1}(0_{\mathcal{E}^+})})^* : \tilde{H}^*(L, \mathbb{Z}) \to \tilde{H}^*(F^{-1}(0_{\mathcal{E}^+}), \mathbb{Z}^2).
$$
Note that $F^{-1}(0_{E^+})$ can be chosen so close to $Z_{T,1}^T$ that it is contained in a given small neighborhood of $Z_{T,1}^T$ for which Theorem 2.6 implies for $T \geq T_0 > 6$

$$\sigma_T(u|Z_T) \in U \quad \forall w \in F^{-1}(0_{E^+}) \text{ and } u = w \circ \phi.$$ (3.7)

Here we use $F^{-1}(0_{E^+}) \subset C^\infty(S^2, M)$ due to the arguments above Lemma 2.1. Define

$$\Xi : F^{-1}(0_{E^+}) \to X_{T,d}, \ w \mapsto u|Z_T \text{ for } u = w \circ \phi,$$

by (2.26), (2.28), (3.3) and (3.7) it is easy to see that we have for $T \geq T_0$ the commutative diagram

$$\begin{array}{ccc}
X_{T} & \xrightarrow{\sigma_T} & U \\
\Xi \downarrow & & \downarrow \pi|_U \\
F^{-1}(0_{E^+}) & \xrightarrow{\Theta|F^{-1}(0_{E^+})} & L
\end{array}$$

By (3.6) we get the injectiveness of the map

$$(\pi|_U)^* : \tilde{H}^*(L, \mathbb{Z}_2) \to \tilde{H}^*(U, \mathbb{Z}_2).$$

If $L$ is orientable, the Banach Fredholm bundles $(\mathcal{B}^r, \mathcal{E}^+, \mathcal{F}_{T,0}|_{B^r})$, and therefore $(\mathcal{B}^r, \mathcal{E}^+, \mathcal{F}_{T,\lambda}|_{B^r})$, $\lambda \in [0,1]$, are orientable. In this case we can define $\mathbb{Z}$-degree $\text{deg}_\mathbb{Z}(\Theta, L, \mathcal{B}^r, \mathcal{E}^+, \mathcal{F}_{T,\lambda}|_{B^r})$ and get $\text{deg}_\mathbb{Z}(\Theta, L, \mathcal{B}^r, \mathcal{E}^+, \mathcal{F}_{T,\lambda}|_{B^r}) \in \{1,-1\}$. The desired conclusion follows immediately. \hfill \Box

4 Examples and further programme

Example 4.1 Consider the torus $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ with the standard symplectic form $\omega = dx \wedge dy$. There exist three natural anti-symplectic involutions on it given by

- $\tau_1 : T^{2n} \to T^{2n}, \ [x,y] \mapsto [-x,y],$
- $\tau_2 : T^{2n} \to T^{2n}, \ [x,y] \mapsto [x,-y],$
- $\tau_3 : T^{2n} \to T^{2n}, \ [x,y] \mapsto [y,x].$

Clearly, $\text{Fix}(\tau_1) = [0] \times T^n \subset T^{2n}$, $\text{Fix}(\tau_2) = T^n \times [0] \subset T^{2n}$ and $\text{Fix}(\tau_3) = \{[x,x] \in T^{2n} | x \in \mathbb{R}^n \}$. Let $H \in C^\infty(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$ be 1-periodic in all its variables so that it may be viewed as a Hamiltonian function on the standard torus which is 1-periodic in time. Corresponding to the three cases above, suppose that Hamiltonian $H$ also, respectively, satisfies

- $H(-t,x,y) = H(t,-x,y)$ for any $t \in \mathbb{R}$ and $z = (x,y) \in \mathbb{R}^{2n},$
- $H(-t,x,y) = H(t,x,-y))$ for any $t \in \mathbb{R}$ and $z = (x,y) \in \mathbb{R}^{2n},$
- $H(-t,x,y) = H(t,y,x)$ for any $t \in \mathbb{R}$ and $z = (x,y) \in \mathbb{R}^{2n}.$

Then in each case Theorem 1.3 gives at least $n+1$ contractible 1-periodic solutions $\gamma : \mathbb{R} \to T^{2n}$ of $\dot{\gamma}(t) = X_{H}(t, \gamma(t))$ satisfying $\gamma(-t) = \tau_i(\gamma(t))$ for any $t \in \mathbb{R}$, $i = 1, 2, 3,$
respectively. It is the contractibility of $\gamma$ that there exists a lift loop $z = (x, y)$: $\mathbb{R}/\mathbb{Z} \to \mathbb{R}^{2n}$ of it satisfying the associated Hamiltonian system on $\mathbb{R}^{2n}$

$$\dot{z} = J \nabla H(t, z) \quad \text{with} \quad J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

and the following conditions

- $x(-t) + x(t) \in \mathbb{Z}^n$ and $y(-t) - y(t) \in \mathbb{Z}^n$ for any $t \in \mathbb{R}$,
- $x(-t) - x(t) \in \mathbb{Z}^n$ and $y(-t) + y(t) \in \mathbb{Z}^n$ for any $t \in \mathbb{R}$,
- $x(-t) - y(t) \in \mathbb{Z}^n$ and $y(-t) - x(t) \in \mathbb{Z}^n$ for any $t \in \mathbb{R}$, respectively.

Clearly, the conclusions in Example 4.1 cannot be derived from [CoZe, Th.1] though the latter yields at $2n + 1$ periodic solutions of $\dot{z} = J \nabla H(t, z)$ of period 1.

**Example 4.2 (i)** Consider the standard complex projective space $\mathbb{C}P^n$ with the Fubini-Study form $\omega_{FS}$ satisfying $\int_{\mathbb{C}P^n} \omega_{FS} = \pi$. Then the rationality index of $(\mathbb{C}P^n, \omega_{FS})$ is equal to $\pi$. Let $H \in C^\infty(\mathbb{R} \times \mathbb{C}P^n, \mathbb{R})$ be 1-periodic in the first variable, and also satisfy $H(-t, [z]) = H(t, \sigma([z]))$ for any $t \in \mathbb{R}$ and $[z] \in \mathbb{C}P^n$, where $\sigma$ is the standard complex conjugation on $\mathbb{C}P^n$ with $\text{Fix}(\sigma) = \mathbb{R}P^n$. Then the associated Hamiltonian system $\dot{z} = X_H(t, z)$ on $(\mathbb{C}P^n, \omega_{FS})$ has at least $n + 1$ contractible periodic solutions $z : \mathbb{R} \to \mathbb{C}P^n$ of period 1 satisfying $z(-t) = \sigma(z(t))$ for any $t \in \mathbb{R}$ provided the Hofer norm $\|H\| < \pi$ by Theorem 1.3. (Actually, the final restriction “$\|H\| < \pi$” may be removed out with Fortune’s method in [Fo].)

(ii) Let $(P, \omega)$ be a simply connected closed symplectic manifold of dimension 4 and with $c_1(TP)|_{\pi_2(P)} = 0$. By the Hurewicz isomorphism theorem and the Poincaré dual theorem there exists a class $A \in \pi_2(P)$ such that $\beta(A) > 0$. So $r_\omega \in [0, +\infty)$. It easily follows from [McSa2, Theorem 3.1.5] that for generic $J \in \mathcal{J}(P, \omega)$ there is no nonconstant $J$-holomorphic spheres in $P$, and thus $m(P, \omega, J) = +\infty$. Moreover, if $(P, \omega)$ is also real symplectic, by [FuOOO, Proposition 11.10] for generic $J \in \mathbb{R} \mathcal{J}(P, \omega)$ there is no nonconstant $J$-holomorphic sphere in $P$ and so $m(P, \omega, J) = +\infty$. A well-known example of such real symplectic manifolds is the $K3$-surface

$$X = \{ [z_0 : \cdots : z_3] \in \mathbb{C}P^3 \mid \sum_{j=0}^3 z_j^4 = 0 \}$$

with the canonical symplectic structure $\omega_{\text{can}}$ induced by the form $\omega_{FS}$ on $\mathbb{C}P^n$ as in (i) above and with the anti-symplectic involution induced by the standard complex conjugation on $\mathbb{C}P^n$ (cf. [McSa1, Example 4.27]). Hence $\pi \leq r_{\omega_{\text{can}}} < +\infty$. Note that the real part of $X$ is empty!

(iii) A symplectic manifold $(M, \omega)$ of dimension $2n$ is said to be **negative monotone** if $c_1(TM)|_{\pi_2(M)} = \lambda \cdot \omega|_{\pi_2(M)}$ for some negative constant $\lambda$, and **semipositive** if either $\omega(M)|_{\pi_2(M)} = \mu \cdot c_1|_{\pi_2(M)}$ for some constant $\mu \geq 0$, or $c_1|_{\pi_2(M)} = 0$ or the minimal Chern number $N \geq n - 2$, see [McSa2, Exercise 6.4.3]. Here the minimal Chern number $N$ of $(M, \omega)$ is the positive generator of $c_1(M)(\pi_2(M))$ if $c_1|_{\pi_2(M)} \neq 0$, and
if \( c_1|_{\pi_2(M)} = 0 \). Note that a simply connected and closed symplectic manifold has always finite rationality index by the Hurewicz isomorphism theorem and the universal coefficient theorem. In a negative monotone symplectic manifold \((M, \omega)\) with minimal Chern number \( N \geq \text{dim } M/2 \), for generic \( J \in \mathcal{J}(M, \omega) \) there is no nonconstant \( J \)-holomorphic sphere and hence \( m(M, \omega, J) = +\infty \) by [McSa2], and \( m(M, \omega, J) = +\infty \) for generic \( J \in \mathbb{R}\mathcal{J}(M, \omega) \) by [FuOOO] Proposition 11.10 if \((M, \omega)\) is also real. Here are some concrete examples, which were in details discussed in [Laz, Appendix A]. For an integer \( n \geq 4 \) and an odd integer \( d \) let 

\[ M_{n,d} = \{ [z_0 : \cdots : z_n] \in \mathbb{C}P^n \mid \sum_{j=0}^{3} z_j^d = 0 \} \]

equipped with a canonical symplectic structure \( \omega_{n,d} \) induced by the form \( \omega_{FS} \) on \( \mathbb{C}P^n \) as in (i) above. Then \( \pi \leq r_{\omega_{n,d}} < +\infty \). The standard complex conjugation on \( \mathbb{C}P^n \) induces an anti-symplectic \( \tau \) on \( M_{n,d} \) with \( \text{Fix}(\tau) = M_{n,d} \cap \mathbb{R}P^n \) which is homeomorphic to \( \mathbb{R}P^{n-1} \). So if \( H \in C^\infty(\mathbb{R} \times M_{n,d}, \mathbb{R}) \) is 1-periodic in the first variable, and also satisfy \( H(-t, [z]) = H(t, \tau([z])) \) for any \( t \in \mathbb{R} \) and \([z] \in M_{n,d} \), we have \( \sharp P_0(H, \tau) \geq n \) provided \( \|H\| < \pi \).

It was shown in [Laz, Appendix A] that \( M_{n,d} \) is simply connected, has a minimal Chern number \( N_{n,d} = |n + 1 - d| \), and satisfies

\[ c_1(M_{n,d})|_{\pi_2(M_{n,d})} = \frac{n + 1 - d}{r} \cdot \omega_{n,d}|_{\pi_2(M_{n,d})} \]

for some \( r > 0 \). Since \( \text{dim } M_{n,d} = 2n - 2 \), \( M_{n,d} \) is negative monotone if and only if \( n + 1 < d \), and \( N_{n,d} \geq \frac{1}{2} \text{dim } M_{n,d} \) if and only if \( d \geq 2n \) or \( d = 1 \). Hence the arguments above show that each \( M_{n,d} \) with \( n \geq 4 \) and odd integer \( d \geq 2n \) or \( d = 1 \) satisfies

\[ m(M_{n,d}, \omega_{n,d}, J) = +\infty \]

for generic \( J \in \mathcal{J}(M_{n,d}, \omega_{n,d}) \) and for generic \( J \in \mathbb{R}\mathcal{J}(M_{n,d}, \omega_{n,d}) \). Nonsingular algebraic subvarieties of \( \mathbb{C}P^n \) defined by real equations may provide more examples.

Our programme [Lu3] is to construct a real Floer homology \( FH_*(M, \omega, \tau, H) \) for a real symplectic manifold \((M, \omega, \tau)\) with nonempty compact \( L = \text{Fix}(\tau) \) only using \( P_0(H, \tau) \), which may be viewed as an intermediate between the Floer homology for Hamiltonian maps and the Floer homology for Lagrangian intersections, to prove that it is isomorphic to \( H_*(L) \otimes R_\omega \) for some Novikov ring \( R_\omega \), and then to relate it to some possible open GW-invariants and something as in [FuOOO], [BiCo] and Auroux’s talk at Montreal, May 19-24, 2008.

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