Random Energy Model with Compact Distributions

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Abstract
In this paper we study the Random energy model - so called toy model of the spin glass theory - where the underlying distributions are compactly supported. We prove a general theorem on the asymptotics of free energy and obtain formulae in several interesting cases - like uniform distribution, truncated double exponential.

Key words: Spin Glasses; Random Energy Model; Free Energy; Compact Distributions.

1 Introduction
Random Energy Model (REM) is a simple model in the theory of spin glasses, originally proposed by Derrida[3]. For each configuration $\sigma \in \{-1, 1\}^N$ of an $N$ particle system, the Hamiltonian $H_N(\sigma)$ is random and they are i.i.d. over $\sigma$. Apart from the study of distribution of the Gibbs’ distribution, one of the problems in REM is the study of asymptotics of $E \log Z_N(\beta)$, where $Z_N(\beta)$ is the partition function defined as $\sum_\sigma e^{-\beta H_N(\sigma)}$ for $\beta > 0$. This is well studied in the literature[1, 2, 4, 6] when $H_N$ are Gaussian. Guerra’s[5] convexity arguments prove the existence of $\lim_{N \to \infty} \frac{1}{N} E \log Z_N(\beta)$ even when $H_N(\sigma)$ are non Gaussian provide a moment condition holds. Guerra’s treatment was for the SK-Model. See Contucci et al[2] where this is done for REM, though we have trouble following their convexity argument for the Generalized Random Energy Model.

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The methods - powerful as they are - will not help in the evaluation of the limits. In this paper we prove a general theorem when the support of distribution of $H_N(σ)$ is compact. This exercise is undertaken for several reasons. The powerful exponential inequalities are not in general available in the non-Gaussian case. More so, when the distribution is flat, like uniform. The rate at which length of the supports grow should have some influence on the asymptotics. The main theorem specifies certain conditions when there is no phase transition. Our arguments for lower bound are similar in spirit to Talagrand [6](P.11-12).

2 Main Result

For each $N$, let $H_N(σ)$ for $σ ∈ \{−1, 1\}^N$ be $2^N$ i.i.d. symmetric random variables having density $φ_N$ with compact support. Let $[−T_N, T_N]$ be the support of $φ_N$. As expected, large value of $H_N(σ)$ are the most relevant. Accordingly, for $0 < s < 1$, define $a_N(s) = P\{H_N(σ) ≥ sT_N\}$.

**Theorem 2.1** Suppose for any $s$, $0 < s < 1$, there exists $m ≥ 0$ (possibly depending on $s$) such that $0 < \lim_{N→∞} N^m a_N(s) < ∞$ and $\lim_{N→∞} N T_N → α$, $0 ≤ α ≤ ∞$. Then for $β > 0$

$$\lim_{N→∞} \frac{1}{T_N} E \log Z_N(β) = α log 2 + β$$

(when $α = ∞$, the right side is interpreted as $∞$).

**Remark 2.1** Of course, the result can equivalently be stated as

$$\lim_{N→∞} \frac{1}{N} E \log Z_N(β) = log 2 + \frac{β}{α}.$$  

**Proof:** First consider the case $α < ∞$.

Since log is concave, by Jensen’s inequality

$$E \log Z_N(β) ≤ log E Z_N(β). \quad (1)$$

As $H_N$’s are symmetric and i.i.d,

$$E Z_N(β) = 2^N E e^{β H_N} < 2^N e^{β T_N}.$$  

Hence, by assumption and (1),

$$\limsup_{N\to∞} \frac{1}{T_N} E \log Z_N(β) ≤ α log 2 + β. \quad (2)$$
We now show,
\[
\liminf_{N \to \infty} \frac{1}{T_N} \mathbf{E} \log Z_N(\beta) \geq \alpha \log 2 + \beta. \tag{3}
\]

Fix 0 < s < 1 and let \( X_N = \#\{\sigma : -H_N(\sigma) \geq sT_N\} \). Then \( \mathbf{E}X_N = 2^N a_N(s) \) and \( \mathbf{E}X_N^2 = 2^N(2^N - 1)a_N^2(s) + 2^N a_N(s) \), so that
\[
\mathbf{E}(X_N - \mathbf{E}X_N)^2 \leq 2^N a_N(s). \tag{4}
\]

If \( A_N = \{X_N \leq 2^N - 1 a_N(s)\} \) then \( A_N \subset \{(X_N - \mathbf{E}X_N)^2 \geq 2^N a_N(s)\} \), so that by Markov inequality and (4),
\[
\mathbf{P}(A_N) \leq \frac{\mathbf{E}(X_N - \mathbf{E}X_N)^2}{2^N a_N(s)} \leq \frac{4}{2^N a_N(s)},
\]
i.e., \( \mathbf{P}(A_N^c) \geq 1 - \frac{4}{2^N a_N(s)} \). But on \( A_N^c \),
\[
Z_N(\beta) \geq X_N e^{sT_N} > 2^{N - 1} a_N(s) e^{sT_N},
\]
and hence
\[
\mathbf{E} \log Z_N(\beta) 1_{A_N^c} > [(N - 1) \log 2 + \log a_N(s) + sT_N](1 - \frac{4}{2^N a_N(s)}). \tag{5}
\]

Since always, \( Z_N(\beta) \geq 2^N e^{-sT_N} \) and \( \mathbf{P}(X_N = 0) = (1 - a_N(s))2^N \), we have
\[
\mathbf{E} \log Z_N(\beta) 1_{\{X_N = 0\}} \geq (N \log 2 - sT_N)(1 - a_N(s))2^N. \tag{6}
\]

On \( \{1 \leq X_N \leq 2^N - 1 a_N(s)\} \), \( \log Z_N(\beta) > \max_{\sigma \in \sigma} \{-H_N(\sigma)\} \geq \beta sT_N > 0 \) and hence
\[
\mathbf{E} \log Z_N(\beta) 1_{\{1 \leq X_N \leq 2^N - 1 a_N(s)\}} > 0. \tag{7}
\]

Since \( A_N = \{X_N = 0\} \cup \{1 \leq X_N \leq 2^N - 1 a_N(s)\} \) using (5), (6) and (7) we have
\[
\frac{1}{T_N} \mathbf{E} \log Z_N(\beta) \geq \left[ \frac{N - 1}{T_N} \log 2 + \frac{\log a_N(s)}{T_N} + sT_N \right] \left[ 1 - \frac{4}{2^N a_N(s)} \right]
+ \left[ \frac{N}{T_N} \log 2 - \beta sT_N \right] (1 - a_N(s))2^N.
\]

By assumptions,
\[
\frac{\log a_N(s)}{T_N} = \frac{1}{T_N} \log(N^m a_N(s)) - m \frac{\log N}{T_N} \to 0
\]
and \( 2^N a_N(s) \to \infty \) so that \( (1 - a_N(s))2^N \to 0 \) as \( N \to \infty \). Thus, under the assumptions,
\[
\liminf_{N \to \infty} \frac{1}{T_N} \mathbf{E} \log Z_N(\beta) \geq \alpha \log 2 + \beta s.
\]
This being true for any $0 < s < 1$, (3) follows.

If $\alpha = \infty$ the above argument shows that

$$\liminf_{N \to \infty} \frac{1}{T_N} E \log Z_N(\beta) = \infty$$

to complete the proof.

3 Examples

Example 3.1 (Truncated Double Exponential)

Let

$$\phi_N(x) = \frac{\lambda_N}{2(1 - e^{-\lambda_N T_N})} e^{-\lambda_N |x|} 1_{[-T_N, T_N]}.$$ 

Then for each $s$, $0 < s < 1$, we have $\lim_{N \to \infty} a_N(s) > 0$ if $\lambda_N T_N \to \delta(0 < \delta < \infty)$ as $N \to \infty$. Thus, for example, if $T_N = N$ and $\lambda_N = \frac{\delta}{N}(0 \leq \delta < \infty)$ then

$$\lim_{N \to \infty} \frac{1}{N} E \log Z_N(\beta) = \log 2 + \beta,$$

where as, if $T_N = N^2$ and $\lambda_N = \frac{\delta}{N^2}(0 \leq \delta < \infty)$ then

$$\lim_{N \to \infty} \frac{1}{N^2} E \log Z_N(\beta) = \beta.$$

Example 3.2 (Uniform Distribution)

Let

$$\phi_N(x) = \frac{1}{2T_N} 1_{[-T_N, T_N]}.$$ 

Then $a_N(s) = \frac{1-s}{2} > 0$ for all $0 < s < 1$. The theorem now implies,

a) if $T_N = N^\gamma$ with $0 < \gamma < 1$ or $T_N = \log N$ then $\lim_{N \to \infty} \frac{1}{T_N} E \log Z_N(\beta) = \infty$,

b) if $T_N = N$ then $\lim_{N \to \infty} \frac{1}{T_N} E \log Z_N(\beta) = \log 2 + \beta$,

c) if $T_N = N^\gamma$ with $\gamma > 1$ or $T_N = 2^N$ then $\lim_{N \to \infty} \frac{1}{T_N} E \log Z_N(\beta) = \beta$.

Similar remarks apply for the following examples also.

Example 3.3 (Bernoulli)

Let

$$\phi_N(x) = \frac{1}{2}[\delta - T_N + \delta T_N].$$

Here $a_N(s) = \frac{1}{2}$ for all $0 < s < 1$. 

Example 3.4
Let
\[ \phi_N(x) = \frac{\lambda + 1}{2T_N^{\lambda+1}} (T_N - |x|)^\lambda, \quad -T_N \leq x \leq T_N \text{ and } \lambda > 0. \]
Then \( a_N(s) = \frac{1}{2} (1 - s)^{\lambda+1} > 0 \) for all \( 0 < s < 1 \).

Example 3.5
Let
\[ \phi_N(x) = \frac{1}{2T_N} \cos \frac{x}{T_N}, \quad -\frac{\pi T_N}{2} \leq x \leq \frac{\pi T_N}{2}. \]
Then \( a_N(s) = \frac{1}{2} (1 - \sin \frac{\pi s T_N}{2}) > 0 \) for all \( 0 < s < 1 \).

Example 3.6
Let
\[ \phi_N(x) = \frac{N}{2(e^{T_N} - 1)} e^{N|x|} 1_{[-T_N,T_N]}. \]
Then \( a_N(s) \to \frac{1}{2} > 0 \) as \( N \to \infty \) for all \( 0 < s < 1 \).

Remarks 3.1 We are not clear if the convergence in the theorem holds almost everywhere instead of in expectation.

Remarks 3.2 In some cases we have been able to establish limiting law for Gibbs’ distributions, but a general result has not yet emerged.

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References
[1] Bovier, A. (2001). Statistical Mechanics of Disordered Systems, *MaPhySto Lecture Notes*, 192pp.
[2] Contucci, P., Esposti, M. D., Giardinà, C. and Graffi, S. (2003). Thermodynamical Limit for Correlated Gaussian Random Energy Models. *Commun. Math. Phys.*, 236, 55–63.
[3] Derrida, B. (1981). Random Energy Model: An Exactly Solvable Model of Disordered Systems. *Phys. Rev.*, B24, 2613–2626.
[4] Dorlas, T. C. and Wedagedera, J. R. (2001). Large Deviations and The Random Energy Model. *Int. J. of Mod. Phy. B*, 15, No. 1, 1–15.
[5] Guerra, F. and Toninelli, F. L. (2002). The Thermodynamic Limit in Mean Field Spin Glass Models. *Commun. Math. Phys.*, **230**, 71–79.

[6] Talagrand, M. (2003). *Spin Glasses: A Challenge for Mathematicians*, Springer-Verlag, New York.