DISCRETE ANALOGUES IN HARMONIC ANALYSIS: A THEOREM OF
STEIN-WAINGER

BEN KRAUSE

Abstract. For $d \geq 2$, $D \geq 1$, let $\mathcal{P}_{d,D}$ denote the set of all degree $d$ polynomials in $D$ dimensions with real coefficients without linear terms. We prove that for any Calderón-Zygmund kernel, $K$, the maximally modulated and maximally truncated discrete singular integral operator,

$$\sup_{P \in \mathcal{P}_{d,D}, \, N} \left| \sum_{0 < |m| \leq N} f(x - m)K(m)e^{2\pi i P(m)} \right|,$$

is bounded on $\ell^p(\mathbb{Z}^D)$, for each $1 < p < \infty$. Our proof introduces a stopping time based off of equidistribution theory of polynomial orbits to relate the analysis to its continuous analogue, introduced and studied by Stein-Wainger:

$$\sup_{P \in \mathcal{P}_{d,D}} \left| \int_{\mathbb{R}^D} f(x - t)K(t)e^{2\pi i P(t)} \, dt \right|.$$

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1. Introduction

This paper will be concerned with a so-called discrete Carleson-type operator, namely the maximally modulated and maximally truncated discrete singular integral operator,

$$\sup_{P \in \mathcal{P}_{d,D}, \, N} \left| \sum_{0 < |m| \leq N} f(x - m)K(m)e^{2\pi i P(m)} \right|,$$

where $K : \mathbb{R}^D \to \mathbb{C}$ is a normalized Calderón-Zygmund kernel: $K \in C^1(\mathbb{R}^D \setminus 0)$ with $\|K\|_{CZ(\mathbb{R}^D)} \leq 1$, i.e.

$$\sup_{0 < r < R} \left| \int_{r \leq |x| \leq R} K(x) \, dx \right| + \sup_{x \neq 0} |x|^D \cdot |K(x)| + \sup_{x \neq 0} |x|^{D+1} \cdot |\nabla K(x)| \leq 1$$

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years, significant attention has been devoted to exploring their arguments, and a number of papers
modulation-invariant tools, but rather by relying on oscillatory integral techniques. In recent
They were able to establish full $L^p$ coefficients without linear terms, and
where, now,
they were interested in understanding the following operator:

$$\mathcal{D}_{d, D} := \left\{ \sum_{2 \leq |\alpha| \leq d} \lambda_{\alpha} x^\alpha \in \mathbb{R}[x_1, \ldots, x_D] \right\}$$

denote the set of all degree $d$ polynomials in $D$ dimensions with real coefficients without linear terms; see (1.16) below for multi-index notation.

1.1. History. The study of maximally polynomially modulated singular integrals has a long and
trich history, most notably encompassing Carleson’s celebrated theorem on convergence of Fourier
series [5]; recent work of Lie [13] (also see Zorin-Kranich [28]) have essentially concluded this line
of research: for every $1 < p < \infty$, the operator

$$\mathcal{C}_{d, D} f(x) := \sup_{P, R} \left| \int_{|t| > R} f(x - t) K(t) e^{2\pi i P(t)} \, dt \right|$$
satisfies

$$\|\mathcal{C}_{d, D} f\|_{L^p(\mathbb{R}^D)} \lesssim_{d, D, p} \|f\|_{L^p(\mathbb{R}^D)},$$

where the supremum is taken over all polynomials in $D$ variables, of degree at most $d$, and all
$0 < R < \infty$.

The major challenge presented in developing this theory is that the operator (1.5) is invariant
under modulation by any polynomial of degree $\leq d$, a point which propagates throughout the
analysis, and limits the efficacy of the standard Calderón-Zygmund/Littlewood-Paley approach, in
which the zero frequency has a distinguished role in the analysis. Indeed, if $Q(t)$ is a polynomial
of degree $\leq d$, then for any $P$ of degree $\leq d$

$$\int_{|t| > R} \left( f(x - t) e^{2\pi i Q(x - t)} \right) K(t) e^{2\pi i P(t)} \, dt$$

$$= e^{2\pi i Q(x)} \cdot \int_{|t| > R} \left( f(x - t) e^{2\pi i (Q(x - t) - Q(x))} \right) K(t) e^{2\pi i P(t)} \, dt$$

$$= e^{2\pi i Q(x)} \cdot \int_{|t| > R} f(x - t) K(t) e^{2\pi i P_x(t)} \, dt$$

where $t \mapsto P_x(t)$ has degree $\leq d$, uniformly in $x$. Consequently,

$$\left| \int_{|t| > R} \left( f(x - t) e^{2\pi i Q(x - t)} \right) K(t) e^{2\pi i P(t)} \, dt \right| \leq \mathcal{C}_{d, D} f(x);$$

taking supra, we see that

$$\mathcal{C}_{d, D} \left( f \cdot e^{2\pi i Q(\cdot)} \right)(x) = \mathcal{C}_{d, D} f(x).$$

Earlier, Stein and Wainger [24] investigated the analogue of (1.5) when this modulation invariance
is eliminated, and the role of the zero frequency remains appropriately distinguished. In particular,
they were interested in understanding the following operator:

$$C_{d, D} f(x) := \sup_{P \in \mathcal{P}_{d, D}} \left| \int f(x - t) K(t) e^{2\pi i P(t)} \, dt \right|$$

where, now, $\mathcal{P}_{d, D}$ denotes (1.3), the set of all degree $d$ polynomials in $D$ dimensions with real
coefficients without linear terms, and $K$ is a normalized Calderón-Zygmund kernel satisfying (1.2).
They were able to establish full $L^p$ estimates, for $1 < p < \infty$, for $C_{d, D}$ without resorting to
modulation-invariant tools, but rather by relying on oscillatory integral techniques. In recent
years, significant attention has been devoted to exploring their arguments, and a number of papers
studying oscillatory integrals without modulation invariance in a wide variety of contexts have been
written, see for instance [7] [8] [10] [11] [12] [19] [20] [21]. Indeed, this work was explicitly discussed in [6], a review of Stein’s major mathematical contributions.

1.2. Discrete Harmonic Analysis. Recently, the study of maximally modulated singular integrals without modulation invariance has been conducted in the integer setting. This study was initiated in [10], where the analogue of Stein’s purely quadratic Carleson operator [22] was introduced and studied:

\[ (1.7) \quad \sup_{\lambda} \left| \sum_{m \neq 0} f(x - m) e^{2\pi i \lambda m^2} \right|. \]

At this point, the theory of polynomial Radon transforms, initiated by Jean Bourgain [1] [2] [3], had become well-developed, see [15] [16] [17]; the operator (1.7) was the first example of a discrete analogue in harmonic analysis that was not of this form. Accordingly, it proved surprisingly resistant to early attempts to bound it, even on \( \ell^2(\mathbb{Z}) \), see [10]. These difficulties were later resolved in [11], where a full \( \ell^2(\mathbb{Z}) \) theory was developed, and more recently in [12], where (1.7) was shown to be bounded on \( \ell^p(\mathbb{Z}) \) for each \( 1 < p < \infty \) as a special case of broader work concerning suprema over one parameter families of modulation parameters.

1.3. Main Results. In this paper, we establish a discrete analogue of Stein-Wainger’s result:

**Theorem 1.** For every \( 1 < p < \infty \), the operator (1.1) is bounded on \( \ell^p \):

\[ \| \sup_{P \in \mathcal{B}_{d,D}} \sum_{0 < |m| \leq N} f(x - m) K(m) e^{2\pi i P(m)} \|_{\ell^p(\mathbb{Z}^D)} \lesssim_{d,D,p} \| f \|_{\ell^p(\mathbb{Z}^D)}. \]

An immediate application of our theorem is to variable coefficient singular integrals. For \( V = (v_1, \ldots, v_k), \quad v_1, \ldots, v_k : \mathbb{Z}^D \to \mathbb{Z}^D \), consider the variable coefficient singular integral operator

\[ T_V f(x_0, x_1, \ldots, x_k) : \mathbb{Z}^D(k+1) \to \mathbb{C} \]

\[ := \sum_{m \neq 0, m \in \mathbb{Z}^D} f(x_0 - m, x_1 - v_1(x_0)P_1(m), \ldots, x_k - v_k(x_0)P_k(m)) \cdot K(m), \]

where \( K \) is a normalized Calderón-Zygmund kernel, see (1.2).

By taking a partial Fourier transform in the final \( k \) variables, and applying Plancherel’s theorem and the \( p = 2 \) case of Theorem 1, we arrive at the following Corollary.

**Corollary 2.** Suppose that \( f \in \ell^2(\mathbb{Z}^D(k+1)) \) and that \( \{ P_i(m) \} \subset \mathbb{Z}[m_1, \ldots, m_D] \) are a collection of polynomials without linear terms and maximal degree \( d \). Then for each \( d, D, k \),

\[ \sup_{V = (v_1, \ldots, v_k)} \| T_V f \|_{\ell^2(\mathbb{Z}^D(k+1))} \lesssim_{d,D,k} \| f \|_{\ell^2(\mathbb{Z}^D(k+1))}. \]

Theorem 1, along with the recent preprint [4], is the first multi-parameter result in discrete analogues in harmonic analysis: one can express

\[ \sup_{P \in \mathcal{B}_{d,D}} \sum_{0 < |m| \leq N} f(x - m) K(m) e^{2\pi i P(m)} \]

\[ = \sup_{\lambda, \alpha \in \Gamma} \sum_{0 < |m| \leq N} f(x - m) K(m) e^{2\pi i \sum \lambda_\alpha m^\alpha} \]

see (1.13); the multi-parameter nature of the supremum necessitated a different set of techniques than those used in [10] [11] [12]. The analysis here is much closer in spirit to that of Stein-Wainger,
with the Kolmogorov-Seliverstov method of $TT^*$ playing a crucial role. The technical ingredient needed to address (1.6) was the oscillatory integral bound

\begin{equation}
\left| \int_{[0,1]^D} e^{2\pi i P(t)} \, dt \right| \lesssim (1 + \|P\|)^{-\theta}
\end{equation}

and the corresponding sub-level estimate

\begin{equation}
|\{ t \in [0,1]^D : |P(t)| \leq \epsilon \}| \leq \left( \frac{\epsilon}{\|P\|} \right)^\theta
\end{equation}

for real-valued polynomials

$$P(t) = \sum_{\alpha} \lambda_\alpha t^\alpha$$

equipped with the coefficient norm

\begin{equation}
\|P\| := \sum_{|\alpha| \neq 0} |\lambda_\alpha|.
\end{equation}

(1.9) is often known as a non-concentration estimate, as it says that it is very hard for the image of a polynomial to cluster disproportionately near a single value. The optimal bound $\theta = 1/d$ is established in [24], but the existence of any $\theta > 0$ would still be sufficient to establish their main result.

In this work, we develop an analogous mechanism for estimating exponential sums and use it to establish appropriate non-concentration estimates.

Classical equidistribution theory dictates that any polynomial $P : \mathbb{Z}^D \to \mathbb{R}$ of degree $\leq d$, and any (large) integer $A$, there exists some $M \lesssim A, d$ so that on the interval

$$\{1, \ldots, N - 1, N\}$$

one may decompose

\begin{equation}
P = P_{\text{Smooth}} + P_{\text{Equi}} + P_{\text{Rat}},
\end{equation}

see [26 Proposition 1.1.17], so that – essentially –

\begin{equation}
\left| \sum_{n=1}^N e^{2\pi i P(n)} \right| \lesssim M^2 \max_{\frac{N}{M^T} \leq T \leq N} \left| \sum_{n=1}^T e^{2\pi i P_{\text{Equi}}(n)} \right| + M^{-A}
\end{equation}

(say), where $P_{\text{Equi}}$ is $M^A$-equidistributed, in that

$$\sup_{\frac{N}{M^T} \leq T \leq N} \left| \frac{1}{T} \sum_{n=1}^T e^{2\pi i P_{\text{Equi}}(n)} \right| = o_M(1)$$

while $P_{\text{Smooth}}$ is smooth,

$$\|P_{\text{Smooth}}\|_{\text{Lip}} \leq \frac{M}{N}$$

and $P_{\text{Rat}}$ has rational coefficients of bounded denominator, in that there exists $Q \leq M$ so that

$$Q \cdot P_{\text{Rat}} \equiv 0 \mod 1.$$

The equation (1.12) speaks to a negotiation: the faster that $P$ equidistributes $\mod 1$, the smaller that the pertaining exponential sums become, but the longer it takes to achieve (1.11), the less control over (1.12) we enjoy. With this in mind, we introduce the scale-dependent stopping time

\begin{equation}
N_N(P) := \min \{ M : P_{\text{Equi}} \equiv 0 \text{ vanishes entirely} \}
\end{equation}
and note that it is well-defined (a trivial upper bound is $N^{\deg(P)}$). The significance of this quantity is that we may use inverse theorems to bound

\[
\left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i P(n)} \right| \lesssim N_N(P)^{-\theta}
\]

for some $\theta = \theta_{A,d} > 0$, which acts as a substitute to (1.8); we use (1.13) to deduce appropriate arithmetic analogues of the estimate (1.9), which contains the key analytical input behind our approach.

The key quantitative estimate (1.14) essentially appears as [26, Lemma 1.16]. We will need a higher-dimensional version, which first appeared in [25]; we provide an alternative proof for completeness in our Appendix [A].

1.4. Structure. The structure of the paper is as follows:

We begin by reviewing [24] in §2 with a focus on the central role of the estimates (1.8) and (1.9);

In §3 we introduce and discuss (1.13), our analogue of the Euclidean coefficient norm. We use this technology to deduce some non-concentration estimates for polynomials;

In §4 we reduce the study of (1.7) to the special case where the coefficients of $P$ live very close to cyclic subgroups with small denominators;

In §5 we use the circle method to approximate our operators, subject to the constraint on the coefficients of the polynomials. The arguments here are similar to those in [11, §5];

In §6 we complete the proof.

Our Appendix [A] contains the proofs of our exponential sums estimates.

1.5. Notation. Throughout, we let $e(t) := e^{2\pi it}$ denote the complex exponential. For real $K > 0$, let

\[
\mu_K(s) := \frac{1}{K} \cdot (1 - \frac{|s|}{K})_+
\]

be the one-dimensional Fejér kernel at scale $K$, so that

\[
\hat{\mu}_K(\xi) = \left( \frac{\sin(\pi K \xi)}{\pi K \xi} \right)^2.
\]

We use the induced norm on the Torus

\[
\|x\|_T := \min \{|x - n| : n \in \mathbb{Z}\}.
\]

For multi-indices

\[
\alpha = (\alpha_1, \ldots, \alpha_D), \quad \alpha_i \in \mathbb{Z}_{\geq 0}
\]

we define

\[
\alpha^\alpha := \prod_{i=1}^{D} x_i^{\alpha_i}.
\]

We let

\[
e_j := (0, \ldots, 0, 1, 0, \ldots, 0)
\]

denote the coordinate vector with 1 in the $j$th component.

We define the ordering on multi-indices $\beta \leq \alpha$ if $\beta_i \leq \alpha_i$, $1 \leq i \leq D$. If at least one strict inequality holds, we use $\beta < \alpha$.

We let $|\alpha| := \sum_i \alpha_i$, and let

\[
\Gamma = \Gamma_{d,D} := \{ \alpha : 2 \leq |\alpha| \leq d \}
\]
so that

\[ P \in \mathcal{P}_{d,D} \]

precisely when \( P \) is a linear combination of monomials with exponents in \( \Gamma \). Note the upper bound

\[ |\Gamma| \leq \binom{D + d}{D}. \]

For \( \lambda = \{\lambda_\alpha : \alpha \in \Gamma\} \in \mathbb{R}^{|\Gamma|} \), we use the notation

\[ P_\lambda(x) := \sum_\alpha \lambda_\alpha x^\alpha. \tag{1.19} \]

We will let \([R] := [-R, R]\) and \((R) := (0, R]\), with context determining whether we restrict to integers or not.

We let \( \vec{R} = (R_1, \ldots, R_D) \)

and define

\[ \vec{R}^\alpha := \prod_{i=1}^D R_i^{\alpha_i}, \]

and

\[ |\vec{R}| := \prod_{i=1}^D R_i. \]

We also set

\[ [\vec{R}] = [R_1] \times \cdots \times [R_D]. \tag{1.20} \]

We will make use of the modified Vinogradov notation. We use \( X \lesssim Y \), or \( Y \gtrsim X \) to denote the estimate \( X \leq CY \) for an absolute constant \( C \) and \( X, Y \geq 0 \). If we need \( C \) to depend on a parameter, we shall indicate this by subscripts, thus for instance \( X \lesssim_p Y \) denotes the estimate \( X \leq C_p Y \) for some \( C_p \) depending on \( p \). We use \( X \asymp Y \) as shorthand for \( Y \lesssim X \lesssim Y \). We reserve the notation

\[ X \sim Y := \left( X \in [Y/2, Y] \right) \tag{1.21} \]

to denote the inequality \( Y/2 \leq X < Y \).

We also make use of big-O notation: we let \( O(Y) \) denote a quantity that is \( \lesssim Y \), and similarly \( O_p(Y) \) a quantity that is \( \lesssim_p Y \).

2. A Review of Stein-Wainger

Recall the Stein-Wainger maximal operator,

\[ C_{d,D} f(x) := \sup_{\mathcal{P}_{d,D}} \left| \int f(x - t) K(t) e(P(t)) \, dt \right|. \tag{2.1} \]

In this section we review the argument of [24] to prove the following Theorem with the key analytic estimates (1.8) and (1.9) as the departure point.

**Theorem 2.1 (Stein-Wainger).** For each \( 1 < p < \infty \),

\[ \|C_{d,D} f\|_{L^p(\mathbb{R}^D)} \lesssim_{d,D,p} \|f\|_{L^p(\mathbb{R}^D)}. \]
2.1. **Using The Estimates.** Since $K$ is a normalized Calderón-Zygmund kernel, by [23 §13] there exist a collection of mean zero $C^1$ functions $\{\psi_j\}$ so that

\[
|\psi_j(x)| \cdot 2^{Dj} + |\nabla \psi_j(x)| \cdot 2^{D(j+1)} \leq C
\]

for each $j$, so that each $\psi_j$ is supported in $\{2^{i-2} \leq |x| \leq 2^i\}$, and so that

\[
K(x) = \sum_j \psi_j(x), \quad x \neq 0.
\]

For $P_\lambda(t) = \sum_\alpha \lambda_\alpha t^\alpha$, decompose

\[
\int f(x - t)K(t)e(P_\lambda(t)) \, dt
\]

\[
= \sum_{j > j(\lambda)} \int f(x - t)\psi_j(t)e(P_\lambda(t)) \, dt + \sum_{j \leq j(\lambda)} \int f(x - t)\psi_j(t)e(P_\lambda(t)) \, dt
\]

\[
= \sum_{l \geq 1} \sum_{j > j(\lambda)} \int f(x - t)\psi_j(t)e(P_\lambda(t)) \, dt \cdot 1_{\|P_\lambda(2^j \cdot )\| \sim 2^l}
\]

\[
+ \sum_{j \leq j(\lambda)} \int f(x - t)\psi_j(t) \, dt + \sum_{j \leq j(\lambda)} O\left( \int \|f(x - t)\|\psi_j(t)\|P_\lambda(t)\| \, dt \right),
\]

see [12,21]. Above, $j(\lambda)$ is the maximal integer, $j$, so that

\[
\|P_\lambda(2^j \cdot )\| = \sum_\alpha 2^{j|\alpha|} \cdot |\lambda_\alpha| \leq 1,
\]

In particular, we may bound

\[
\left| \int f(x - t)K(t)e(P(t)) \, dt \right|
\]

\[
\leq \sum_{l \geq 1} \sup_P \left| \int f(x - t)\psi_{j(\lambda,l)}(t)e(P(t)) \, dt \right| + T^*f(x) + M_{HL}f(x)
\]

\[
\leq \sum_{l \geq 1} A_l f(x) + T^*f(x) + M_{HL}f(x),
\]

where $T^*f$ is a maximally truncated singular integral, and $j(\lambda,l)$ is defined to be the unique $j$ (if it exists) so that

\[
\|P(2^j \cdot )\| \sim 2^l,
\]

see [23]; if no such $j$ exists, set $\psi_{j(\lambda,l)} = 0$.

We will prove

**Proposition 2.2.** For each $1 < p < \infty$, there exists an absolute $c_p > 0$ so that

\[
\|A_l f\|_{L^p(\mathbb{R}^D)} \lesssim 2^{-c_p l} \|f\|_{L^p(\mathbb{R}^D)}.
\]

Theorem 2.1 then follows from a summation over $l \geq 1$.

By interpolation, it suffices to establish Proposition 2.2 at the $L^2$ level. If we linearize the supremum, we may express $A_l$ as an integral operator,

\[
A_l f(x) = \int K(x,t)f(t) \, dt
\]

with kernel

\[
K(x,t) = e(P_\lambda(x - t))\psi_{j(\lambda,l)}(x - t)
\]

\[
\leq 2^{-c_p l} \|f\|_{L^p(\mathbb{R}^D)}.
\]
where importantly $\lambda = \lambda(x)$ varies measurably with $x$, see (1.19).

By the Kolmogorov-Seliverstov method of $TT^*$, it suffices to show that

$$\| \int \mathcal{K}(x, y) f(y) \, dy \|_{L^2(\mathbb{R}^D)} \lesssim 2^{-cl} \| f \|_{L^2(\mathbb{R}^D)},$$

where

$$\mathcal{K}(x, y) = \int K(x, t) \overline{K(y, t)} \, dt$$

$$= \int e(P_{\mu}(x - t) - P_{\mu}(y - t)) \phi_k(x - t) \psi_r(y - t) \, dt$$

and $\lambda = \lambda(x), \mu = \mu(y)$, and $k = k(\lambda, l), r = r(\mu, l)$. Indeed

$$\| A_t f \|_{L^2(\mathbb{R}^D)} = \| \int K(\cdot, t)f(t) \, dt \|_{L^2(\mathbb{R}^D)}$$

(2.5)

$$\leq \| K \|_{\text{Ker}(\mathbb{R}^D)} \cdot \| f \|_{L^2(\mathbb{R}^D)}$$

$$= \| K \|_{\text{Ker}(\mathbb{R}^D)} \cdot \| f \|_{L^2(\mathbb{R}^D)}$$

$$\lesssim 2^{-c/2l} \cdot \| f \|_{L^2(\mathbb{R}^D)}.$$

Above, we have used the operator norm

$$\| K \|_{\text{Ker}(\mathbb{R}^D)} := \sup_{\| f \|_{L^2(\mathbb{R}^D)} = 1} \| \int K(x, t)f(t) \, dt \|_{L^2(\mathbb{R}^D)}.$$

The key properties of $\mathcal{K}(x, y)$ are contained in the following Lemma.

**Lemma 2.3.** There exists an absolute constant $c > 0$ so that

$$|\mathcal{K}(x, y)| \lesssim 2^{-kD-cl} 1_{|x-y| \leq 2^k} + 2^{-rD-cl} 1_{|x-y| \leq 2^r}$$

$$+ 2^{-kD} 1_{E(x)}(y) + 2^{-rD} 1_{E(y)}(x),$$

where $E(x) \subset \{ |y| \leq 2^k \}, E(y) \subset \{ |x| \leq 2^r \}$ depend only on the identified variables and have

$$2^{kD} |E(x)| + 2^{rD} |E(y)| \lesssim 2^{-cl}.$$

In particular, if we set the $h$-small set maximal function

$$M_h f(x) := \sup_k \sup_E \frac{1}{2^{kD}} \int_E |f(x - y)| \, dy,$$

where the supremum is over

$$E \subset \{ |x| \leq 2^k \} : |E| \lesssim 2^{kD-h},$$

then by interpolating the pointwise inequality $M_h f \lesssim M_{HL} f$ against the $L^\infty$ bound,

$$\| M_h f \|_{L^\infty(\mathbb{R}^D)} \lesssim 2^{-h} \| f \|_{L^\infty(\mathbb{R}^D)}$$

one deduces that $M_h$ is a contraction on each $L^p$ space, and we may bound, for an appropriate $\| g \|_{L^2(\mathbb{R}^D)} = 1$,

$$\| \int \mathcal{K}(x, y) f(y) \, dy \|_{L^2(\mathbb{R}^D)} \leq \int |g(x)| |\mathcal{K}(x, y)||f(y)| \, dx \, dy$$

$$\lesssim 2^{-cl} \int M_{HL} f(x)|g(x)| \, dx + \int M_{df} f(x)|g(x)| \, dx + \int |f(y)| M_d g(y) \, dy$$

$$\lesssim 2^{-c/2l} \| f \|_{L^2(\mathbb{R}^D)}.$$
It remains only to prove Lemma 2.3, which follows from the following estimate, after changing variables appropriately, $v = x - y$, and using symmetry to reduce to the case where $r \leq k$,

\begin{equation}
\left| \int e(P_\lambda(v - 2^{-r_0}t) - P_\mu(-t))\psi(v - 2^{-r_0}t)\psi(-t) \, dt \right| \lesssim 2^{-cl} + 1_{E_\lambda(v)}, \quad r_0 \geq 0
\end{equation}

where $E_\lambda \subset \{|v| \lesssim 1\}$ depends only on $\lambda$ and has $|E_\lambda| \lesssim 2^{-cl}$, and we think of $r_0 = k - r$.

There are two cases. If $2^{-r_0} \leq \eta \ll 1$, then

\[ t \mapsto P_\lambda(v - 2^{-r_0}t) - P_\mu(-t) = P_\lambda(v) + ((P_\lambda(v - 2^{-r_0}t) - P_\lambda(v)) - P_\mu(-t)) \]

has a coefficient norm that is $\gtrsim 2^l$, as the coefficient norm of

\[ t \mapsto P_\lambda(v - 2^{-r_0}t) - P_\lambda(v) \]

is $O(2^{l-r_0})$. By (1.8), the bound (2.6) holds. In the other case, we can bound the coefficient norm of

\[ t \mapsto P_\lambda(v - 2^{-r_0}t) - P_\lambda(-t) \]

from below by the coefficient norm of the linear terms in the above difference; the presence of these linear terms is due to the lack of linear terms in $P_\mu$. In particular

\[ \|P_\lambda(v - 2^{-r_0}t) - P_\mu(-t)\| \geq \left\| \sum_{j=1}^{D} \lambda_\alpha \alpha_j v^{\alpha-\epsilon_j} \right\| \]

\[ \gtrsim \sum_{j=1}^{D} \left| \sum_{\alpha > \epsilon_j} \lambda_\alpha \alpha_j v^{\alpha-\epsilon_j} \right| =: \sum_{j=1}^{D} |P_j(v)|; \]

by another application of (1.8), it suffices to show that

\[ \{|v| \lesssim 1 : \sum_{j=1}^{D} |P_j(v)| \lesssim 2^{c_0l} \} \lesssim 2^{-c_0l} \]

for some $c_0 > 0$. But this just follows from (1.9):

\[ \{|v| \lesssim 1 : |P_j(v)| \lesssim 2^{c_0l} \} \lesssim \left( \frac{2^{c_0l}}{\|P_j\|} \right)^\theta \lesssim 2^{\theta(c_0-1)l} \]

for some $\theta > 0$ (which can be taken to be $1/d$).

2.1.1. *Stein-Wainger: Continuous Summary and Discrete Preliminaries.* Aside from (1.9), which follows directly from (1.8), the techniques needed to establish Theorem 2.1 are fairly modest:

- The Hardy-Littlewood maximal function, $M_{HL}$;
- Maximally truncated singular integrals;
- $TT^*$ arguments;
- Interpolation.

The clarity of this scheme suggests that a similar approach should extend to the discrete situation, namely to Theorem 1. As is characteristic of the field, our analysis requires more delicacy as the positive Hardy-Littlewood maximal function is insensitive to destructive interference arising from arithmetic considerations. This difficulty can be partially resolved by earlier work [11, 12], and by applying further $TT^*$ arguments, matters to formulating an arithmetic version of the Stein-Wainger argument.
Accordingly, we apply a discrete analogue of (1.8) to derive the analogue of the crucial sublevel bound (1.9).

3. Exponential Sums and Sublevel Estimates

In this section we introduce the relevant analogues of (1.8) and (1.9) in the discrete context. As this result has been appeared [25], we defer its proof to Appendix A.

We begin by defining scale-dependent coefficient norms in full generality:

**Definition 3.1.** Suppose $P(x) = \sum_{\alpha} \lambda_\alpha x^\alpha \in \mathbb{R}[x_1, \ldots, x_D]$, that $R_i \geq 1$, and set

$$\vec{R} = (R_1, \ldots, R_D).$$

Let $s_0$ denote the minimal $s$ so that either there exists some $Q \leq 2^s$, $s \geq 1$ so that

$$\sum_{\alpha} \|Q\lambda_\alpha\|_T \cdot \vec{R}^\alpha \leq 2^s,$$

or

$$\sum_{\alpha} \|\lambda_\alpha\|_T \cdot \vec{R}^\alpha \leq 2^s$$

if $s \leq 0$.

Then, define the coefficient norm of $P$ at scale $\vec{R}$,

$$N_{\vec{R}}(P) := 2^{s_0}.$$

If $\vec{R} = (R, \ldots, R)$ is cubic, we will abbreviate

$$N_{R}(P) = N_{\vec{R}}(P).$$

**Example 1.** Suppose that $\lambda_\alpha = \frac{A_\alpha}{Q} \in \mathbb{Q}$ where $(\{A_\alpha\}, Q) = 1$ are reduced. If $R_i \geq Q^{1+\delta}$ where $1 > \delta > 0$ is small, then

$$N_{R}(P) \geq Q^\delta.$$  

**Reason.** Suppose that $N_{R}(P) \leq 2^t$, where $2^t < \frac{Q}{10}$ (say). This means that there exists $Q_1 \leq 2^t$ so that

$$\sum_{\alpha} \|\frac{A_\alpha}{Q}Q_1\|_T \cdot \vec{R}^\alpha \leq 2^t;$$

since $(\{A_\alpha\}, Q)$ are reduced, there must be some $\alpha_0$ so that $\frac{A_{\alpha_0}}{Q}Q_1 \not\equiv 0 \mod 1$, leading to the lower bound

$$Q^{-1} \leq \|\frac{A_{\alpha_0}}{Q}Q_1\|_T,$$

and thus

$$\frac{R_i}{Q} \leq Q^{-1} \vec{R}^{\alpha_0} \leq \|\frac{A_{\alpha_0}}{Q}Q_1\|_T \cdot \vec{R}^{\alpha_0} \leq 2^t,$$

so $2^t \geq Q^\delta$. 

The significance of the quantity $N_{R}(P)$ is that it controls various exponential sums, analogous to the way the coefficient norms control oscillatory integrals in Euclidean space; note the trivial upper bound

$$(3.2) \quad N_{R}(P) \leq \sum_{\alpha} \vec{R}^\alpha,$$
and the multiplicative property
\[ N_R(P) \leq 2^{\lceil \log_2 k \rceil} \cdot N_R(k \cdot P) \quad k \geq 1 \]
which is occasionally sharp, as can be seen for polynomials with coefficients in \( \mathbb{Z}/k\mathbb{Z} \). To see this, just observe that whenever \( N_R(kP) = 2^s \), there exists some multiple of \( k \), \( Qk \), with \( Q \leq 2^s \), so that
\[ \sum_{\alpha} ||Qk\lambda_\alpha||_T \cdot \bar{R}^\alpha \leq 2^s \leq 2^{\lceil \log_2 k \rceil} 2^s. \]

Finally, we record the following convexity lemma concerning the coefficient norms.

**Lemma 3.1.** For any \( P \), for each \( s \geq 1 \) the set 
\[ \{ k : N_{2^k}(P) = 2^s \} \]
is an interval. When \( s \leq 0 \), there exists at most one \( k \) so that \( N_{2^k}(P) = 2^s \).

**Proof.** The case where \( s \leq 0 \) is clear, so we consider the more interesting case where \( s \geq 1 \). We need to show that if 
\[ N_{2^k}(P) = N_{2^h}(P) = 2^s, \quad k \leq h, \]
then \( N_{2^{k'}}(P) = 2^s \) as well, for all \( k \leq k' \leq h \). Since \( N_{2^k}(P) = 2^s \), for any \( Q \leq 2^s - 1 \),
\[ 2^{s-1} < \sum_{\alpha} ||Q\lambda_\alpha||_T \cdot 2^{k'^{[\alpha]}}, \]
which says that \( N_{2^{k'}}(P) \geq 2^s \); since \( N_{2^h}(P) = 2^s \), there exist some \( Q' \leq 2^s \) so that
\[ 2^s \geq \sum_{\alpha} ||Q'\lambda_\alpha||_T \cdot 2^{k'^{[\alpha]}}, \]
for the reverse inequality. \( \square \)

The key property of the coefficient norm is captured by the following Theorem, see [25, Proposition 8]; a complete proof can be found in Appendix A below.

**Theorem 3.2 (Coefficient Norms Control Exponential Sums).** There exists an absolute \( \theta = \theta_{d,D} > 0 \) so that the following holds whenever \( N_R(P) = 2^s \geq 2 \):

For every multi-dimensional arithmetic progression,
\[ \vec{P} = \mathcal{P}_1 \times \cdots \times \mathcal{P}_D, \quad \mathcal{P}_i \subset [R_i] \]
\[
\left| \frac{1}{|R|} \sum_{n \in \mathcal{P}} e(k \cdot P(n)) \phi(n) \right| \lesssim \left( \frac{|k|}{2^s} \right)^\theta + \sum_i R_i^{-\theta},
\]
whenever
\[ \| \phi \|_{\ell^\infty} + \sum_{j=1}^D R_j \cdot \| \phi - \phi(-e_j) \|_{\ell^\infty} \leq 1. \]

If \( N_R(k \cdot P) = 2^s \leq 1 \), then
\[
\frac{1}{|R|} \sum_{n \in \mathcal{P}} e(k \cdot P(n)) \phi(n) = \frac{1}{|R|} \sum_{n \in \mathcal{P}} \phi(n) + O(N_R(k \cdot P)).
\]

The second point just follows from the mean-value theorem; the content of Theorem 3.2 concerns the case where the coefficient norm is large; the condition on the amplitudes, \( \phi \), is standard, as one can quickly reduce to the case of constant amplitudes, see for instance [18, §A.1].
3.1. Sublevel Estimates. In the Euclidean setting, the sublevel estimate (1.9) follows directly from the coefficient-norm bound (1.8); in our present context, we can use Theorem 3.2 to derive appropriate sublevel estimates as well.

The following lemmas will be used to bound the percentage of time a polynomial can cluster extremely close to cyclic subgroups with small denominators; these fill the role of (1.9).

Our first lemma will be used in the case where the polynomial in question has a fairly large coefficient norm.

Lemma 3.3 (Non-Concentration for Polynomials with Large Coefficient Norms). Suppose $A \leq N_R(P)^{\theta}$, where $N_R(P) \geq 2$ and $\theta = \theta_{d,D} > 0$ is sufficiently small. There exists a $C = C_{d,D}$ so that whenever $B \geq 100$ (say)

$$\sum_{|v| \leq R} 1_{\|P(v)q\|_T \leq B^{-1}}(P(v)) \lesssim R^D \cdot A \cdot \left( N_R(P)^{-\theta} + B^{-\theta} + R^{-\theta} \right)$$

Proof. By the union bound, it suffices to consider a single $q \leq A$, but without the factor of $A$ on (3.3). Suppose that $N_R(P) = 2^s$. Assume, as we may, that $B$ is an integer. We dominate the indicator function by a Fejér kernel and bound

$$\sum_{|v| \leq R} 1_{\|P(v)q\|_T \leq B^{-1}}(P(v)) \leq \sum_{|v| \leq R} \left( \sum_k \mu_B(k)e(k \cdot P(v)) \right),$$

see (A.3). In particular, if we set $\bar{R} = [R] \times [B]$, and define

$$P_0(v,k) := k \cdot P(v)q \in \mathbb{R}[v_1, \ldots, v_D, k]$$

to be a polynomial of $D + 1$ many variables, it suffices to show that

$$\left| \frac{1}{R^D} \sum_{|v| \leq R, k} \mu_B(k)e(P_0(v,k)) \right| \lesssim \left( N_R(P)^{-\theta} + B^{-\theta} + R^{-\theta} \right).$$

If $N_R(P_0) \geq 2^{s-1}/A \gtrsim 2^{s(1-\theta)}$, the bound is clear, so assume otherwise. This says that there exist $Q_1 < 2^{s-1}/A$ so that

$$\sum_{\alpha} \|Q_1(q\lambda_\alpha)\|_T \cdot R^{\alpha} \leq B^{-1} \cdot 2^{s-1};$$

let $Q_0 := Q_1q < 2^{s-1}$ be minimal subject to this constraint.

On the other hand, since $N_R(P) = 2^s$, we know that for every $Q \leq 2^{s-1}$

$$\sum_{\alpha} \|Q\lambda_\alpha\|_T \cdot R^{\alpha} > 2^{s-1};$$

specializing to $Q = Q_0$ yields the contradiction. \qed

The following sub-level set estimate will serve as a substitute to Lemma 3.3 when $P$ has a very large coefficient norm: $B = N_R(P)^{O(1)}$.

Lemma 3.4 (Non-Concentration for Polynomials with Large Coefficient Norms). Suppose that $N_R(P) \geq R^0$ where $0 < \eta \ll 1$ is bounded away from zero, and that $0 < \kappa \ll \eta$ is sufficiently small.

Then there exists some absolute $\kappa_0 = \kappa_0(\eta) > 0$ so that

$$\sum_{|v| \leq R} 1_{\|P(v)q\|_T \leq R^{\kappa-1}} \lesssim R^{D-\kappa_0}. $$
Proof. We use a union bound; thus, it suffices exhibit \( \kappa_0 > 0 \) so that for each \( q \leq R^\kappa \),

\[
|\{ |v| \leq R : \| P(v)q \|_T \leq R^{\kappa-1} \}| \lesssim R^{D-\kappa_0-\kappa}.
\]

By another Fejér kernel argument, it suffices to exhibit a \( \kappa_0 > 0 \) so that

\[
\left| \frac{1}{R^D} \sum_{|v| \leq R, n} \mu_{R^1-\kappa}(n) \cdot e(P_0(v, n)) \right| \lesssim R^{-\kappa_0-\kappa},
\]

(3.4)

where \( P_0(v, n) = \sum_{\alpha} (q \cdot \lambda_\alpha) \cdot (nv^\alpha) \in \mathbb{R}[v_1, \ldots, v_D, n] \)

is a polynomial of \( D + 1 \) variables. Set \( \vec{R} = (R, \ldots, R, R^{1-\kappa}) \); we claim that \( N_\vec{R}(P_0) \geq R^\eta/2 \); this allows us to bound

\[
\text{(3.4)} \leq R^\kappa \cdot (R^{-\theta \eta} + R^{-\theta}),
\]

which would yield the result.

So, suppose otherwise, and extract a minimal \( Q_1 \leq R^{\eta/2} \) so that

\[
\sum_{\alpha} \| (Q_1 q)_{\lambda_\alpha} \|_{T R^{\alpha+1-\kappa}} \leq R^{\eta/2}.
\]

On the other hand, since \( N_{\vec{R}}(P) \geq R^n \), for every \( Q \leq \frac{R^n}{2} \)

\[
\sum_{\alpha} \| Q_{\lambda_\alpha} \|_{T R^{\alpha}} > R^n;
\]

the contradiction arises by specializing \( Q = Q_1 q \leq R^{\eta/2+\kappa} \ll R^n \).

In what follows, we will use the machinery developed above to prove Theorem 1.

4. The Discrete Stein-Wainger Operator

Regarding \( \| K \|_{C^1([R^D])} \) as given, let \( \{ \psi_j \} \) be as in (2.2).

We introduce a large real parameter,

\[
A_0 = A_0(d, D, p)
\]

(4.1)

which we are free to adjust upwards finitely many times as needed.

Define

\[
C_{d,D}f(x) := \sup_{P \in \mathscr{P}_{d,D}, k_0} \left| \sum_{m \in \mathbb{Z}^D} f(x-m) \psi_k(m) e(P(m)) \right|.
\]

Theorem 4.1 follows directly from the following result after an argument with the Hardy-Littlewood maximal function.

**Theorem 4.1.** For each \( 1 < p < \infty \) and \( d, D \),

\[
\| C_{d,D}f \|_{L^p(\mathbb{Z}^D)} \lesssim \| f \|_{L^p(\mathbb{Z}^D)}
\]

By Proposition 4.2 below, our attention will be focused on the “oscillatory” truncated singular integrals

\[
\mathscr{A}_k f(x) := \sup_{k_0 \geq 2^k/4A_0} \left| \sum_{k=2^k/4A_0} f(x-m) \psi_k(m) e(P_{\lambda(x)}(m)) f(x-m) \cdot 1_{n : N_{\psi_k}(P_{\lambda(x)}) = 2^k(x)} \right|.
\]

(4.2)
for \( A_0 \) as in (4.1) a sufficiently large absolute constant, and \( \lambda : \mathbb{Z}^D \to [0,1]^{|\Gamma|} \) an appropriate linearizing function. Note that for each \( s \geq 1 \), the set

\[ \{ k : N_{2^k}(P) = 2^s \} \]

is an interval, see Lemma 3.1.

We present this reduction in the form of a proposition.

**Proposition 4.2.** The following pointwise bound holds:

\[
C_{d,D} f \lesssim \sum_{s \geq 1} \mathcal{A}_s f + \mathcal{E} f + H^* f + M_{HL} f, 
\]

where \( H^* f \) is a maximally truncated singular integral, \( \mathcal{A}_s \) are as in (4.2), and

\[ \mathcal{E} f = \sum_{k \geq 1} \mathcal{E}_k f \]

is a sum of single scale operators with

\[
\| \mathcal{E}_k f \|_{\ell^p(\mathbb{Z}^D)} \lesssim k^{-2} \cdot \| f \|_{\ell^p(\mathbb{Z}^D)}. 
\]

The proof of Proposition 4.2 will take up the early part of this section, with (4.4) being the crucial point. We accordingly defer the estimate (4.4) to Proposition 4.3 below.

**Proof of Proposition 4.2 Assuming (4.4).** With \( \lambda(x) \) an appropriate linearizing function, foliate

\[
\sum_{k=1}^{k_0} \sum_{s \in \mathbb{Z}} \sum_{m \in \mathbb{Z}^D} f(x - m) \psi_k(m) e(P_{\lambda(x)}(m))
\]

\[
= \sum_{k=1}^{k_0} \sum_{s \in \mathbb{Z}} \sum_{m \in \mathbb{Z}^D} f(x - m) \psi_k(m) e(P_{\lambda(x)}(m)) \cdot 1_{n:N_{2^k}(P_{\lambda(n)}) \sim \gamma^2}(x)
\]

according to the size of the pertaining coefficient norms.

We bound the foregoing above by the sum of two terms, which we will address individually: the stationary component

\[
\sum_{k=1}^{k_0} \sum_{m \in \mathbb{Z}^D} f(x - m) \psi_k(m) e(P_{\lambda(x)}(m)) \cdot 1_{n:N_{2^k}(P_{\lambda(n)}) \leq 1}(x);
\]

and the oscillatory component

\[
\sum_{k=1}^{k_0} \sum_{s \geq 1} \sum_{m \in \mathbb{Z}^D} f(x - m) \psi_k(m) e(P_{\lambda(x)}(m)) \cdot 1_{n:N_{2^k}(P_{\lambda(n)}) \sim \gamma^2}(x).
\]
We begin by bounding (4.5). Since the coefficient norm of \( P_{\lambda} (x) \) is small in this case, we just Taylor expand the phase:

\[
\tag{4.5} \leq \left| \sum_{k=1}^{k_0} \sum_{m \in \mathbb{Z}^D} f(x - m) \psi_k(m) \cdot 1_{n;N_{2k}(P_{\lambda(n)}) \leq 1}(x) \right|
\]

where the scale \( k \) error terms are defined, explicitly:

\[
\tag{4.6} \leq \sum_{s \geq 0} \sum_{m \in \mathbb{Z}^D} \left| f(x - m) \psi_k(m) e(P_{\lambda(x)}(m)) \cdot 1_{n;N_{2k}(P_{\lambda(n)}) \sim 2^s}(x) \right|
\]

where \( H^* \) is a truncated singular integral, as

\[
\{k : N_{2k}(P) \leq 1\}
\]

is an interval, see Lemma 3.1 and we crucially used that for each \( s \leq 0 \), there is at most one \( k \) so that

\[
N_{2k}(P) = 2^s,
\]

see Lemma 3.1. This concludes the estimate of (4.5).

We now address (4.6). To do so, bound

\[
\tag{4.6} \leq \sum_{k=1}^{k_0} \sum_{s;2^s \leq k_0} \sum_{m \in \mathbb{Z}^D} f(x - m) \psi_k(m) e(P_{\lambda(x)}(m)) \cdot 1_{n;N_{2k}(P_{\lambda(n)}) \sim 2^s}(x)
\]

where the scale \( k \) error terms are defined, explicitly:

\[
\mathcal{E}_k f(x) := \sup_{P;N_{2k}(P) \geq k_0} \left| \sum_{m \neq 0} e(P(m)) \psi_k(m) f(x - m) \right|.
\]

In particular, we have reduced the proof of Proposition 4.2 to establishing (4.4). This will be the focus of the following Proposition.
Proposition 4.3. There exists an absolute $c = c_{d,D} > 0$ so that for each $1 < p < \infty$

$$\|E_kf\|_{\ell^p(Z^D)} \lesssim k^{-cA_0} \frac{2^k}{p^*} \cdot \|f\|_{\ell^{p^*}(Z^D)}, \quad p^* = \max\{p, p'\}.$$  

We prove Proposition 4.3 by showing that

\begin{align*}
\sup_{P : N_{2k}(P) = 2^s} \left| \sum_m e(P(m)) \psi_k(m) f(x-m) \right|_{\ell^2(Z^D)} & \lesssim \left( 2^{-\theta s} + 2^{-\theta k} \right) \cdot \|f\|_{\ell^2(Z^D)} \\
& \lesssim 2^{-cs} \|f\|_{\ell^2(Z^D)}
\end{align*}

(4.7)

and interpolating against a trivial single-scale estimate, see (3.2) for the final inequality.

We prove (4.7) by the Kolmogorov-Seliverstov method of $TT^\ast$. Regarding $s$ and $k$ as fixed, subject to the constraint $k^A_0 \leq 2^s \leq 2^A_{d,D} k$, see (3.2), it suffices to prove that, uniformly in measurable maps

$$Z^D \to \{ P \in \mathcal{P}_{d,D} : N_{2k}(P) = 2^s \}$$

the kernel

$$K(x,m) := e(P_\lambda(x-m)) \psi_k(x-m),$$

satisfies

$$\|K\|_{\text{Ker}(Z^D)} := \sup_{\|f\|_{\ell^2(Z^D)} = 1} \left\| \sum_m K(\cdot,m) f(m) \right\|_{\ell^2(Z^D)} \lesssim 2^{-cs}.$$  

(4.8)

By arguing as in (2.5) above, it suffices to instead bound

$$\|K\|_{\text{Ker}(Z^D)} \lesssim 2^{-2cs}$$

(4.9)

where

$$K(x,n) = \sum_m e(P_\lambda(x-m) - P_\mu(n-m)) \psi_k(x-m) \psi_k(n-m)$$

(4.10)

where

$$N_{2k}(P_\lambda) = N_{2k}(P_\mu) = 2^s.$$  

The key point is the following Lemma.

Lemma 4.4. There exists an absolute $c_0 > 0$ so that the following inequality holds pointwise:

$$|K(x,n)| \lesssim 2^{-c_0 s-Dk} \cdot 1_{|x-n| \leq 2^k} + 2^{-kD} \cdot 1_{O(x)}(n)$$

where

$$O(x) \subset \{ |v| \leq 2^k \}, \quad |O(x)| \lesssim 2^{-c_0 s+kD}$$

depends only on the $x$ variable.

In particular,

$$\sup_x \sum_n |K(x,n)| \lesssim 2^{-c_0 s}, \quad \sup_n \sum_x |K(x,n)| \lesssim 1,$$

so (4.10) follows from Schur’s test, with $c = \frac{c_0}{2}$.  

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4.1. The Proof of Lemma 4.4. With \( v = x - n, \lambda = \lambda(x) \) and \( \mu = \mu(n) \), we need to show that

\[
|\mathcal{K}(x, n)| = |\sum_m \psi_k(v - m)\psi_k(-m)e(P_\lambda(v - m) - P_\mu(-m))|
\]

\[
\lesssim 2^{-kD - \kappa s} \cdot 1_{|v| \leq 2^k} + 2^{-kD} \cdot 1_{\mathcal{U}(\lambda)}(v)
\]

where \( \mathcal{U}(\lambda) \subset [2^k]^D \) has \( |\mathcal{U}(\lambda)| \leq 2^{kD - \kappa s} \), and is independent of \( \mu \).

The coefficient norm of the phase

\[
m \mapsto P_\lambda(v - m) - P_\mu(-m)
\]

is bounded below by that of the linear terms in \( m \):

\[
m \mapsto \sum_{j=1}^D \left( \sum_{\alpha} \lambda_{\alpha} \alpha_j v^{\alpha - \epsilon_j} \right) m^{\epsilon_j} =: \sum_{j=1}^D P_j(v) \cdot m^{\epsilon_j}
\]

We distinguish between two cases according to the relationship between \( s \) and \( k \). Below, we let \( \eta = \eta_{d, D} \ll 1 \) denote a sufficiently small constant.

**Case One:** \( s \leq \eta k \).

Let \( \kappa_0 > 0 \) be a very small constant, depending on \( d, D \).

Collect

\[
\mathcal{G} := \{|v| \lesssim 2^k : N_{2^k} \left( \sum_{j=1}^D P_j(v) m^{\epsilon_j} \right) \geq 2^{\kappa_0 s} \}
\]

and observe that for \( v \in \mathcal{G} \),

\[
|\sum_m \psi_k(v - m)\psi_k(-m)e(P_\lambda(v - m) - P_\mu(-m))| \lesssim 2^{-\theta \kappa_0 s} + 2^{-\theta k} \lesssim 2^{-\theta \kappa_0 s}.
\]

We set

\[
\mathcal{U}(\lambda) := [2^k]^D \setminus \mathcal{G} \subset \bigcap_{j=1}^D B_j
\]

where

\[
B_j := \{|v| \lesssim 2^k : \min_{q \leq 2^{\kappa_0 s}} \|q P_j(v)\|_\mathbb{T} \leq 2^{\kappa_0 s - k} \}
\]

In the language of Lemma 3.3, \( A = 2^{\kappa_0 s}, \ B = 2^{k - \kappa_0 s}, \) and \( R = 2^k \), so by that lemma

\[
|B_j| \lesssim 2^{kD} \cdot 2^{-cs}.
\]

**Case Two:** \( \eta k < s \lesssim d, D \ k \).

Collect

\[
\mathcal{G} := \{|v| \lesssim 2^k : N_{2^k} \left( \sum_{j=1}^D P_j(v) m^{\epsilon_j} \right) \geq 2^{\kappa k} \}
\]

and observe that for \( v \in \mathcal{G} \),

\[
|\sum_m \psi_k(v - m)\psi_k(-m)e(P_\lambda(v - m) - P_\mu(-m))| \lesssim 2^{-Dk - \theta s} \lesssim 2^{-Dk - \theta' s}.
\]
Collect

\[ U(\lambda) := [2^k]^D \setminus \mathcal{G} \subset \bigcap_{j=1}^D \mathcal{B}_j \]

where

\[ \mathcal{B}_j := \{|v| \lesssim 2^k : \min_{q \leq 2^\kappa k} \|qP_j(v)\|_T \leq 2^{(\kappa-1)k}\} \]

By our second sub-level set estimate, Lemma 3.4,

\[ |\mathcal{B}_j| \lesssim 2^{k(D-\kappa_0)} \lesssim 2^{kD-\kappa_1 s}, \]

which completes the proof.

With Proposition 4.2 in mind, in the following section we will apply the circle method to approximate our oscillatory operators \( \{\mathcal{A}_s\} \), see (4.2), by more tractable family of analytically-defined operators.

5. Approximations

We now construct analytic approximates to the multipliers

\[ m_{j,\lambda}(\beta) := \sum_m \hat{\psi}_j(m)e(-P_{\lambda}(m) - \beta \cdot m), \]

where \( \lambda \in [0,1]|\Gamma| \),

\[ P_{\lambda}(m) = \sum_{\alpha} \lambda_{\alpha} m^\alpha, \]

see (1.19), and \( \beta \in [0,1]^D \).

\[ \beta = (\beta_1, \ldots, \beta_D). \]

For \( (\frac{A}{Q}, \frac{B}{Q}) \in \mathbb{Q}^{|\Gamma|} \times \mathbb{Q}^D \), define the complete Gauss sum

\[ S(A/Q, B/Q) = \frac{1}{Q^D} \sum_{r \in (Q)^D} e(-P_{A/Q}(r) - \frac{B}{Q} \cdot r) \]

Lemma 5.1. Suppose that \( (A, B, Q) = 1 \), but \( (A, Q) = v > 1 \). Then

\[ S(A/Q, B/Q) = 0. \]

Proof. Express \( \frac{A}{Q} = \frac{a_0}{R} \) and \( \frac{B}{Q} = \frac{B}{Rv} \). Expressing

\[ r = pR + l \]

we have

\[ P_{A/Q}(r) + \frac{B}{Q} \cdot r \equiv P_{A/Q}(l) + \frac{B}{Rv} \cdot l + \left( \frac{B}{v} \cdot p \right) \]

In particular, since \( v \) does not divide at least one of the \( B_i \),

\[ S(A/Q, B/Q) = \frac{1}{R^D} \sum_{l \in (R)^D} e(-P_{A/Q}(l) - \frac{B}{Q} \cdot l) \times \left( \frac{1}{v^D} \sum_{p \in (v)^D} e(-\frac{B}{v} \cdot p) \right) \]

\[ = 0. \]

\[ \square \]
Next, define
\[ \Phi_{j,\nu}(\beta) := \int e(-P_\nu(t) - \beta \cdot t) \psi_j(t) \, dt, \]
and
\[ \Phi^*_{j,\nu}(\beta) = \Phi_{j,\nu}(\beta) \cdot 1_{|\nu_\alpha| \leq j^{-A_0} 2^{-j|\alpha|}}. \]

With
\[ L^s_{j,\lambda}(\beta) = \sum_{Q \sim 2^s} \sum_{B \in (Q)^D} S(A/Q, B/Q) \Phi^*_{j,\lambda - A/Q}(\beta - B/Q) \chi_s(\beta - B/Q) \]
for \( \chi_s \) a Schwartz function which satisfies
\[ 1_{|\beta_i| \leq 2^{-2^{10\rho s}}} \leq \chi_s(\beta) \leq 1_{|\beta_i| \leq 10^{-2^{10\rho s}}} \]
for \( \rho > 0 \) an extremely small constant determined below, consolidate
\[ L_{j,\lambda}(\beta) := \sum_{s: 2^s \leq j^{-A_0}} L^s_{j,\lambda}(\beta). \]

By arguing similarly to [11, 12], we show that
\[ \sup_{\lambda} |L^\vee_{j,\lambda} \ast f| \]
well approximates
\[ \sup_{\lambda} |m^\vee_{j,\lambda} \ast f| \]
provided that
\[ \lambda \in X_j := \prod_{\alpha} \left( \bigcup_{q \leq j^{-A_0}} \mathbb{Z}/q\mathbb{Z} + O(j^{-A_0} 2^{-j|\alpha|}) \right) \]
\[ (5.2) \]

Lemma 5.2. Let \( 2^{-j} \leq \delta \leq 1 \) be a small constant, and suppose that \( |\lambda_\alpha - \frac{A_\alpha}{Q}| \leq \delta \cdot 2^{-j(|\alpha|-1)} \) for each \( \alpha \in \Gamma \), and that \( |\beta_i - \frac{B_i}{Q}| \leq \delta \) for each \( 1 \leq i \leq D \).

Then
\[ m_{j,\lambda}(\beta) = S(A/Q, B/Q) \Phi_{j,\lambda - A/Q}(\beta - B/Q) + O(Q \delta). \]

Proof. With \( m = pQ + r \), express
\[ P_\lambda(pQ + r) + \beta \cdot (pQ + r) \equiv P_{A/Q}(r) + B/Q \cdot r \]
\[ + (P_{A/Q}(pQ) + (\beta - B/Q) \cdot pQ) + O(Q \delta) \mod 1 \]
Summing yields
\[ m_{j,\lambda}(\beta) = \sum_{p,r} \psi_j(pQ) e(-P_\lambda(pQ + r) - \beta \cdot (pQ + r)) + O(Q \cdot 2^{-j}) \]
\[ = S(A/Q, B/Q) \cdot \sum_p Q^D \psi_j(pQ) e(-P_{A/Q}(pQ) - (\beta - B/Q) \cdot pQ) + O(Q \delta) \]
\[ = S(A/Q, B/Q) \cdot \Phi_{j,\lambda - A/Q}(\beta - B/Q) + O(Q \delta) \]
by a Riemann sum approximation. \( \square \)

Lemma 5.3. Suppose that \( L_{j,\lambda}(\beta) \neq 0 \). Then there exists precisely one \( (A/Q, B/Q) \) with \( Q \leq j^{-A_0} \) so that
\[ L_{j,\lambda}(\beta) = L^s_{j,\lambda}(\beta) = S(A/Q, B/Q) \Phi_{j,\lambda - A/Q}(\beta - B/Q) \chi_s(\beta - B/Q) \]
if \( Q \sim 2^s \).

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Proof. Suppose that there exists some \((A'/Q', B'/Q')\) so that
\[
|\lambda_\alpha - A'_\alpha/Q'| \leq j^{A_0} 2^{-j|\alpha|}
\]
If \(A'/Q' \neq A/Q\), we would have the following chain of inequalities.
\[
j^{-2A_0} \leq \frac{1}{QQ'} \leq \left| \frac{A_\alpha}{Q} - \frac{A'_\alpha}{Q'} \right| \leq j^{A_0} 2^{-2j}
\]
Now, if \(Q' \sim 2^{s_0}\) with \(s_0 > s\), then \(S(A'/Q', B'/Q') = 0\), as \(A'/Q'\) would not be in reduced form. The only case to check is when there exist \(B'/Q' \neq B/Q\) so that
\[
|\beta_i - B_i/Q|, \ |\beta_i - B'_i/Q'| \leq 2^{-210\rho s}.
\]
If \(B/Q \neq B'/Q'\), then
\[
\left| \frac{B_i}{Q} - \frac{B'_i}{Q'} \right| \geq \frac{1}{QQ'} \approx 2^{-2s},
\]
for the desired contradiction. \(\square\)

5.1. Major Arcs. Let \(\epsilon_0 = 2^{-10}\), and for \(Q \leq 2^{\epsilon_0j}\), define
\[
\mathfrak{M}_j(A/Q, B/Q) := \{ (\lambda, \beta) \in \mathbb{T}^{T_1} \times \mathbb{T}^D : |\lambda_\alpha - A_\alpha/Q| \leq 2^{(\epsilon_0-|\alpha|)j}, \ |\beta_i - B_i/Q| \leq 2^{(\epsilon_0-1)j} \},
\]
and collect
\[
\mathfrak{M}_j = \bigcup_{Q \leq 2^{\epsilon_0j}} \mathfrak{M}_j(A/Q, B/Q)
\]

Proposition 5.4. Suppose \(\lambda \in X_j\). Then there exists some \(c_0 = c_0(d, D) > 0\) so that
\[
|m_{j, \lambda}(\beta) - L_{j, \lambda}(\beta)| \leq 2^{-c_0\epsilon_0j}.
\]

Proof. First suppose that \((\lambda, \beta) \notin \mathfrak{M}_j\). Then for all \(Q \leq 2^{\epsilon_0j}\),
\[
\sum_{\alpha \in \Gamma} \|Q\lambda_\alpha\| \cdot 2^{j|\alpha|} + \sum_{i=1}^D \|Q\beta_i\| \cdot 2^j > 2^{\epsilon_0j}.
\]
So, \(|m_{j, \lambda}(\beta)| \lesssim 2^{-\epsilon_0j} + 2^{-\epsilon_0j}\) for some appropriate \(\epsilon = \epsilon(d, D) > 0\).
We next observe that for each \(Q \leq j^{A_0}\) the Euclidean coefficient norm of the phase
\[
(5.3) \quad t \mapsto P_{\lambda-\lambda/Q}(2^j t) + (\beta - B/Q) \cdot 2^j t
\]
is \(\geq 2^{\epsilon_0j}\), see (1.10). Consequently,
\[
|L_{j, \lambda}(\beta)| \lesssim 2^{-\epsilon_0j}\theta, \quad \theta = 1/d
\]
by stationary phase estimates, see (1.8). Above, we used the fact that \(L_{j, \lambda}(\beta) = L_{j, \lambda}(\beta)_s\) for some unique \(s = s(\lambda, \beta)\), and once again the fact that for each \(Q \leq 2^{\epsilon_0j}\)
\[
\|5.3\| = \sum_{\alpha} |\lambda_\alpha - A_\alpha/Q| \cdot 2^{j|\alpha|} + \sum_i |\beta_i - B_i/Q| \cdot 2^j \geq 2^{\epsilon_0j},
\]
see (1.10).
Next, suppose that \((\lambda, \beta) \in \mathfrak{M}_j(A/Q, B/Q)\) with \(Q \sim 2^{s_0} \leq j^{A_0}\).
Since $\lambda \in X_j$, there exists some $\frac{\lambda}{Q}$ so that

$$|\lambda - \frac{A}{Q}| \leq j^{A_0}2^{-j|\alpha|},$$

which forces $A'/Q' = A/Q$, as otherwise one would arrive at the following chain of inequalities for some $\alpha$:

$$j^{-2A_0} \leq \frac{|A - A'|}{Q'} \leq |\lambda - \frac{A}{Q}| + |\lambda - \frac{A'}{Q'}| \leq 2^{(e_0 - |\alpha|)}j + j^{A_0}2^{-j|\alpha|},$$

Since $\beta \in \mathfrak{M}_j(A/Q, B/Q)$

$$|\beta - B_i/Q| \leq 2^{(e_0 - 1)}j \ll 2^{-2^e_0^*} \ll 2^{-2^{10\rho}}$$

so

$$L_{j,\lambda}^{\mathfrak{M}}(\beta) = S(A/Q, B/Q)\Phi_{j,\lambda - A/Q}(\beta - B/Q)$$

while

$$m_{j,\lambda}(\beta) = S(A/Q, B/Q)\Phi_{j,\lambda - A/Q}(\beta - B/Q) + O(2^{2^e_0 - 1}j)$$

where we have recalled the bounds: $Q \leq 2^{e_0j}$ and $\delta \leq 2^{(e_0 - 1)}j$.

Finally if $(\lambda, \beta) \in \mathfrak{M}_j(A/Q, B/Q)$ with $j^{A_0} < Q \leq 2^{e_0j}$, we would necessarily have $A/Q = A'/Q'$ for some $A'/Q' \in X_j$ (so $Q' \leq j^{A_0}$). This would force $(A/Q) > 1$, and thus

$$L_{j,\lambda}^{\mathfrak{M}}(\beta), m_{j,\lambda}(\beta) = O(2^{2^e_0 - 1}j)$$

\[\square\]

**Lemma 5.5.** Set

$$E_{j,\lambda}(\beta) := m_{j,\lambda}(\beta) - L_{j,\lambda}(\beta)$$

Then there exists $c_0 > 0$ so that

$$\sup_{\lambda} \|\partial^{\alpha_1, \ldots, \alpha_m}_{j,\lambda} E_{j,\lambda}^\vee \ast f\|_{L^p(\mathbb{Z}^d)} \lesssim_{d, D} 2^{-c_0e_0^*j} \cdot 2^{j(|\alpha_1| + \ldots + |\alpha_m|)}\|f\|_{L^p(\mathbb{Z}^d)}$$

for each $\alpha_1, \ldots, \alpha_m, m \leq |\Gamma|$, $1 < p < \infty$.

**Proof.** The $\ell^2$ estimate without any derivatives was just proven. To handle derivatives on $\ell^2$, just observe that

$$\partial^{\alpha_1, \ldots, \alpha_m}_{j,\lambda} m_{j,\lambda} = 2^{j(|\alpha_1| + \ldots + |\alpha_m|)} m^{\alpha_1, \ldots, \alpha_m}_{j,\lambda}$$

where $m^{\alpha_1, \ldots, \alpha_m}_{j,\lambda}$ is like $m_{j,\lambda}$, except the amplitude $\psi_j(t)$ is replaced by

$$\prod_{i=1}^{m} \frac{\mu_{\alpha_i}}{2^{j|\alpha_i|}} \psi_j(t),$$

which satisfies all of the same differential inequalities as does $\psi_j$ up to an absolute constant depending on $d, D$; similarly for $L_{j,\lambda}$.

To handle the $\ell^p$ estimates, bound

\begin{equation}
(5.4) \quad |\partial^{\alpha_1, \ldots, \alpha_m}_{j,\lambda} L_{j,\lambda}^\vee(x)| \lesssim j^{O(d, D)(A_0) \cdot 2^{j(|\alpha_1| + \ldots + |\alpha_m|)} \cdot 2^{-jD} \cdot (1 + 2^{-j|x|})^{-100}},
\end{equation}

and

$$|\partial^{\alpha_1, \ldots, \alpha_m}_{j,\lambda} m_{j,\lambda}^\vee(x)| \lesssim 2^{j(|\alpha_1| + \ldots + |\alpha_m|)} \cdot |\psi_j(x)|.$$
To see (5.4), we just bound
\[
|L_{j,\lambda}^\gamma(x)| \lesssim 2^j \cdot \max_{2^j \leq j \leq 2^j A_0} |(L_{j,\lambda}^\gamma(x)|
\]
where
\[
\lesssim 2^j \cdot \max_{B/Q \leq j \leq 2^j A_0} \max_{2^j \leq j \leq 2^j A_0} \big| \big( \Phi_{j,\lambda}^{\ast} \cdot A/Q (\beta - B/Q) \chi_s (\beta - B/Q) \big) \big( (\Phi_{j,\lambda}^{\ast} - A/Q (\beta - B/Q) \chi_s (\beta - B/Q) \big) \big)^\gamma \big( (\Phi_{j,\lambda}^{\ast} - A/Q (\beta - B/Q) \chi_s (\beta - B/Q) \big) \big)^\gamma(x)
\]
where the final estimate follows since the spatial scale of $2^j \gg 2^{10 \alpha s}$ is so large compared to that of $\chi_s$. 

**Proposition 5.6.** There exists some $c = c(d, D) > 0$ so small that
\[
\| \sup_{\lambda \in X_j} |\mathcal{E}_{j,\lambda}^\gamma * f\|_p \lesssim 2^{C_{\omega_D} - \gamma} \cdot a(p)
\]
where $p^* := \max\{p, p'\}$.

**Proof.** By decomposing $X_j$ into $j^{2 \cdot A_0 |\Gamma|}$ many boxes of dimensions
\[
\{ [2^{-j} \alpha] : \alpha \in \Gamma \},
\]
it suffices to prove the estimate for a single box.

The proof is by induction on $|\Gamma|$. Thus, let
\[
\mathfrak{P}(R)
\]
denote the statement that for all $F : Q \times \mathbb{Z}^D \to \mathbb{C}$ with $Q$ a box of side-lengths $\{[L_i] : 1 \leq i \leq R\}$ satisfying
\[
\sup_{\lambda} \| \partial_{\lambda}^\gamma F(\lambda; x) \|_{L^p(\mathbb{R}^x)} \leq 100^{|\gamma|} \cdot E^{-\gamma} \cdot a(p)
\]
where $\max_i \gamma_i \leq 1$, the following estimate holds for some absolute constant $C_R < \infty$:
\[
\| \sup_{\lambda} |F(\lambda; x)\|_{L^p(\mathbb{R}^x)} \leq C_R \cdot a(p).
\]

Note that once we have established $\mathfrak{P}(R)$ for $R = |\Gamma|$, we may specialize
\[
F(\lambda; x) := \mathcal{E}_{j,\lambda}^\gamma * f(x),
\]
where $a(p) = 2^{C_{\omega_D} - \gamma} \cdot \| f \|_{L^p(\mathbb{Z}^D)}$.

Note that since the statement is translation invariant, we can and will assume that each box is centered at the origin; by dilation invariance we may assume that $L_i = 1$ for each $i$.

When $R = 1$, so $Q = [1]$, the result follows from the pointwise bound
\[
|F(\lambda; x)|^p \lesssim |F(\mu; x)|^p + \left| \int_{[\mu, \lambda]} \partial_t F(t; x) \, dt \right|^p
\]
\[
\leq |F(\mu; x)|^p + \int_{[\mu, \lambda]} |\partial_t F(t; x)|^p \, dt
\]
\[
\leq |F(\mu; x)|^p + \int_{[1]} |\partial_t F(t; x)|^p \, dt
\]
where $\mu \in [1]$ is arbitrary and
\[
\sup_{t \in [1]} \| \partial_t F(t; x) \|_{L^p(\mathbb{Z}^D)} \leq a(p).
\]
For the inductive statement, express \( \lambda = (\lambda', \lambda_R) \), and pointwise bound
\[
|F(\lambda', \lambda_R; x)|^p \lesssim \sup_{\lambda'} |F(\lambda', \mu_R; x)|^p + \left| \int_{[\mu_R, \lambda_R]} \partial_t F(\lambda', t; x) \ dt \right|^p
\]
\[
\leq \sup_{\lambda'} |F(\lambda', \mu_R; x)|^p + \int_{[1]} |\sup_{\lambda'} |\partial_t F(\lambda', t; x)|^p \ dt
\]
\[
\leq \sup_{\lambda'} |F(\lambda', \mu_R; x)|^p + \int_{[1]} |\sup_{\lambda'} |\partial_t F(\lambda', t; x)|^p \ dt,
\]
where \( \mu_R \in [1] \) is arbitrary.

But, \( \mathcal{P}(R - 1) \) applies to both
\[
\lambda' \mapsto F(\lambda', \mu_R; x), \quad \lambda' \mapsto \partial_t F(\lambda', t; x)
\]
uniformly in \( \mu_R, t \), with the same constant \( a(p) \), which closes the induction.

\[\square\]

6. Analytic Estimates

In the previous section, we reduced matters to estimating
\[
\sup_{j_0, \lambda} \left| \sum_{j=1}^{j_0} L^r_{j, \lambda} * f \right|.
\]
We will decompose this maximal function by pigeon-holing in the sizes of the denominators of our rational approximates to \( \{\lambda_\alpha\} \).

Define the operator
\[
L^s_{\lambda, j_0} := \sum_{2^s A_0 \leq j \leq j_0} L^s_{j, \lambda},
\]
and, for any given \( \lambda, j_0 \) majorize
\[
\left| \sum_{j=1}^{j_0} L^r_{j, \lambda} * f \right| \leq \sum_{s=1}^{j_0} \sum_{j: j_0 A_0 \geq 2^s} \left( L^s_{j, \lambda} \right) * f \left| L^s_{\mu, j_0} \right| * f,
\]
\[
\leq \sum_{s=1}^{j_0} \sup_{\mu, j_0} \left| \left( L^s_{\mu, j_0} \right) * f \left| L^r_{\lambda, j_0} \right| \right|.
\]

In particular, the proof will be complete once we have proven the following proposition.

**Proposition 6.1.** There exists an absolute \( c = c(d, D, p) \) so that for each \( s \geq 1 \),
\[
\| \sup_{\lambda, j_0} \left| \left( L^s_{\lambda, j_0} \right) * f \right| \|_{L^p(D^p)} \leq 2^{-cs} \| f \|_{L^p(D^p)}.
\]

We introduce some further notation to streamline the proof.
For a bounded multiplier, \( m \), define
\[
\mathcal{L}_{s, A/Q}[m](\beta) := \sum_{B \in (Q)^p} S(A/Q, B/Q)m(\beta - B/Q)\chi_s(\beta - B/Q)
\]
where \( \chi_s \) is as above. We next recall the Ionescu-Wainger exhaustion of the rationals: there exists a function
\[
h : \mathbb{Q}^D \rightarrow 2^N
so that

\[ h(B/Q) \leq Q \]

if \((B, Q) = 1\), and if

\[ \mathcal{U}_s := \{ B/Q : h(B/Q) = 2^s \} \]

then the following holds:

- \( \mathcal{U}_s \subset \{ B'/Q' : Q' \leq 2^{2r} \} \), where \( \rho > 0 \) is chosen sufficiently small relative to all other parameters introduced; and
- If \( \chi'_s \) is like \( \chi_s \), but is one on its support, then multipliers

\[ \Pi_{m,s}(\beta) = \sum_{\theta \in \mathcal{U}_s} m(\beta - \theta) \cdot \chi'_s(\beta - \theta) \]

satisfy

\[ \|\Pi_{m,s}\|_{M_p(Z^D)} \lesssim \|m\|_{M_{2r}(\mathbb{R}^D)} \]

for any \((2r)' \leq p \leq 2r\), where \( M_p(X) \) denotes the multiplier norm

\[ \|M\|_{M_p(X)} := \sup_{\|f\|_{L^p(X)} = 1} \|M \ast f\|_{L^p(X)}, \quad X = Z^D, \mathbb{R}^D. \]

This construction is ultimately due to Tao, \[27\], building on breakthrough work of Ionescu and Wainger \[9\] and subsequent refinements \[15, 18\].

Note that we can factor

\[ L_{s,A/Q}[m] = L_{s,A/Q}[1] \cdot \Pi_{m,s}, \]

which is a key point in establishing Lemma \[6.4\].

We now observe the following identity, which we capture in the following lemma.

**Lemma 6.2.** The following identity holds:

\[ L_{s,A/Q}[m]^\vee(n) = \sum_{B \in (Q)^D} S(A/Q, B/Q)e(B/Qn)(m\chi_s)^\vee(n) \]

\[ = e(-P_{A/Q}(n)) \cdot (m\chi_s)^\vee(n) \]

and in particular,

\[ \sup_{A/Q} |L_{s,A/Q}[m]^\vee \ast f(n)| \leq |(m\chi_s)^\vee| \ast |f| (n) \]

pointwise.

**Proof.** The proof is just computation:

\[ \sum_{B \in (Q)^D} S(A/Q, B/Q) \int m(\beta - B/Q) \chi_s(\beta - B/Q)e(\beta n) \, d\beta \]

\[ = \sum_{B \in (Q)^D} S(A/Q, B/Q) \cdot e(B/Qn) \cdot (m\chi_s)^\vee(n) \]

\[ = \sum_{r \in (Q)^D} e(-P_{A/Q}(r)) \cdot \frac{1}{Q^D} \sum_{B \in (Q)^D} e(-B/Q \cdot (r - n)) \cdot (m\chi_s)^\vee(n) \]

\[ = e(-P_{A/Q}(n)) \cdot (m\chi_s)^\vee(n), \]

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where we used the relationship
\[
\frac{1}{Q^D} \sum_{B \in (Q)^D} e(B/Q \cdot x) = \begin{cases} 
1 & \text{if } x \equiv 0 \mod Q \\
0 & \text{otherwise.}
\end{cases}
\] 

With this notation in mind, we may express
\[
L^{s;j_0}_\lambda = L_{s,A/Q}^{\Phi^{s;j_0}_\lambda}
\]
where
\[
\Phi^{s;j_0}_\lambda = \sum_{2^s/A \leq j \leq j_0} \Phi^{s,\lambda}_{j_0}
\]
and \(A/Q\) is the unique element with \(Q \sim 2^s\) so that
\[
|\lambda_\alpha - A_\alpha/Q| \leq 2^{-10s-10},
\]
or an arbitrary element of the complement otherwise (note that in this case \(L^{s,\lambda}_\lambda(\beta) = 0\)).

Lemma 6.3. There exists an absolute \(c = c(d, D, p) > 0\) so that for every \(s \geq 1\)
\[
\| \sup_{A/Q, Q \sim 2^s} |L_{s,A/Q}[1]| \|_{\ell^2} \lesssim 2^{-cs} \|f\|_{\ell^2(Z^D)}.
\]

Proof. By interpolation with Lemma 6.2, it suffices to prove the estimate at \(\ell^2\). We apply a \(TT^*\) argument, and are left to consider the kernel,
\[
K_0(x, n) = \sum_{B \in (Q)^D, B' \in (Q')^D} S(A/Q, B/Q)S(A'/Q', B'/Q')
\times \sum_m e(B/Q(x - m))e(-B'/Q'(n - m))(\chi^\vee_s)(x - m)(\chi^\vee_{s'})\overline{(n - m)}
\]
where \(A/Q = A(x)/Q(x)\) and \(A'/Q' = A'(n)/Q'(n)\); our job is to show that
\begin{equation}
\label{eq:6.2}
\|K_0\|_{\text{ker}(Z^D)} \lesssim 2^{-cs}.
\end{equation}

We will do so by bounding
\[
|K_0(x, n)| \lesssim 2^{-cs} \rho_s(v) + (1_{E_{\text{Per}}(A/Q)} \cdot \rho_s)(v)
\]
where
\[
\rho_s(x) := \sum_n (\chi^\vee_s)(x - n) \cdot (\chi^\vee_{s'})\overline{(n)}
\]
has spatial scale \(2^{10s} \rho_s\), and \(E_{\text{Per}}(A/Q)\) is the \(Q\)-periodic extension of some subset, \(E(A/Q) \subset (Q)^D\) depending only on \(A/Q\) which has density \(2^{-cs}\).

In particular, we bound
\[
\sup_x \sum_n |K_0(x, n)| \lesssim 2^{-cs}, \quad \sup_n \sum_x |K_0(x, n)| \leq 1,
\]
so (6.2) follows from Schur’s test.

Turning to \(K_0(x, n)\), since \(\chi^\vee_s\) has such a small Fourier support, the sum vanishes unless \(B/Q = B'/Q'\), as can be seen by applying Poisson summation. The only way that can happen is if \(Q|Q'\)
or vice versa; in either event we would find $Q = Q'$, since both have size $\sim 2^s$. This leads to the diagonalization

$$K_0(x, n) = \sum_{B \in (Q)^D} S(A/Q, B/Q)S(A'/Q, B/Q) \cdot e(B/Q(x-n)) \cdot \rho_s(x-n)$$

$$= \frac{1}{Q^D} \sum_{r \in (Q)^D} e(-P_{A/Q}(x-n+r) + P_{A'/Q}(r)) \cdot \rho_s(x-n).$$

We claim that there exists an absolute $c_0 > 0$ so that

$$(6.3) \quad \frac{1}{Q^D} \sum_{r \in (Q)^D} e(-P_{A/Q}(v+r) + P_{A'/Q}(r)) \leq 2^{-c_0 s} + 1_{E(A/Q)}(v)$$

where $E(A/Q) \subset (Q)^D$ has density $2^{-c_0 s}$; consequently

$$|K_0(x, n)| \leq 2^{-c_0 s} \rho_s(x-n) + (1_{E_{\text{Per}}(A/Q)} \cdot \rho_s)(x-n).$$

where $E_{\text{Per}}(A/Q)$ is the $Q$-periodic extension of $E(A/Q)$. But, since $P$ has no linear terms

$$N_Q(P_{A/Q}) \geq 2^{s-1}$$

since for any $|\alpha| \geq 2$

$$\min_{q \leq 2^{s-2}} Q^{|\alpha|} \cdot \|q \frac{A_{\alpha}}{Q}\|_T \geq Q^{|\alpha|-1} > 2^{s-2};$$

similarly, $N_Q(P_{A'/Q}) \geq 2^{s-1}$. So, (6.3) follows from Lemma 3.4. The details are as follows: with

$$P_j(v) = \sum_{\alpha} \frac{A_{\alpha}}{Q} \alpha_j v^{\alpha - \epsilon_j},$$

a polynomial with coefficient norm $\gtrsim a 2^s$, it suffices to show that

$$(6.4) \quad |\{ v \in (Q)^D : N_Q(\sum_{j=1}^{D} P_j(v) \cdot r^{\epsilon_j}) \leq Q^s \}| \lesssim Q^{D-c_0 s},$$

but the left-hand side of (6.4) is contained in

$$\bigcap_{j=1}^{D} \{ v \in (Q)^D : \min_{q \leq Q^s} \| q \cdot P_j(v) \|_T \leq Q^{s-1} \},$$

which has measure bounded by $Q^{D-c_0 s}$ by Lemma 3.4.

We next recall the following Lemma from [12].

**Lemma 6.4.** For $j \geq 1$ let $K_j$ denote a mean-zero $C^1$ function supported on $\{|x| \approx 2^j\}$, with

$$2^j |D| |K_j(x)| + 2^j (D+1) |\nabla K_j(x)| \leq C$$

uniformly in $j \geq 1$. Let $K_{a,b} := \sum_{a \leq j < b} K_j$. Then there exists $c = c(d, D, p) > 0$ so that

$$\| \sup_{A/Q, J \geq 1} |\mathcal{L}_{A/Q}[K_{0,J}]^\vee * f|_{L^p(Z^D)} \lesssim 2^{-cs} \| f \|_{L^p(Z^D)}.$$  

**Proof.** See [12], Lemma 4.4, noting that the argument is invariant under rearrangement of the polynomial phase in the appropriate Gauss sums,

$$S(A/Q, B/Q) = \frac{1}{Q^D} \sum_{r \in (Q)^D} e(-\sum_{\alpha} \frac{A_{\alpha}}{Q} r^{\alpha} - B/Q \cdot r),$$
subject to the estimate from Lemma 6.3 and the factorization (6.1). □

We now prove Proposition 6.1.

6.1. The Proof of Proposition 6.1 For \( s \geq 1 \) fixed, let

\[ \mathcal{J}_{l,\mu} := \{ j : j^{A_0} \geq 2^s, \| P_\mu(2^j \cdot) \| \sim 2^l \}, \]

see (1.10). Note that \( |\mathcal{J}_{l,\lambda}| = O_d(1) \), see [13] Lemma 2.1 for details; the key point is that there are only \( O(d^2 \log A) = O_d(1) \) many scales \( j \) so that

\[ A^{-1} \leq \frac{\sum_{|\alpha|=k} 2^{j|\alpha|} |\lambda_\alpha|}{\sum_{|\alpha|=k'} 2^{j'|\alpha|} |\lambda_\alpha|} \leq A \]

for \( 2 \leq k \neq k' \leq d \). By sparsifying our scales into \( O_d(1) \) many sub-families, we will assume that

\[ \sup_l |\mathcal{J}_{l,\lambda}| \leq 1, \]

which we will index

\[ \mathcal{J}_{l,\lambda} = \{ j_l \}. \]

We collect

\[ \mathbb{I}(j_0) := \{ l \leq -Cs : j_l \leq j_0 \}, \]

where \( C = C_{d,D,p} \) is a sufficiently large constant,

It suffices to bound

\[ \| \sup_{A/Q,\mu,j_0} \left| \sum_{l \in \mathbb{I}(j_0)} \mathcal{L}_{s,A/Q}[\Phi_{j_l,\mu}]^\vee * f \right| \|_{L^p(Z^D)} \lesssim 2^{-cs} \| f \|_{L^p(Z^D)}, \]

and

\[ \| \sup_{A/Q,\mu} \left| \mathcal{L}_{s,A/Q}[\Phi_{j_0,\mu}]^\vee * f \right| \|_{L^p(Z^D)} \lesssim 2^{-cs} \min\{ 1, 2^{-cl} \} \cdot \| f \|_{L^p(Z^D)}. \]

We begin with the low frequency case; by direct computation, for any \( A/Q \), we can express

\[ \sum_n f(x-n) \sum_{l \in \mathbb{I}(j_0)} \sum_{B \in [Q]^D} S(A/Q, B/Q) \cdot e(B/Q n) \cdot \int \chi^\vee_s(n-t)e(-P_\mu(t))\psi_{j_l}(t) \, dt \]

\[ = \sum_n f(x-n) \sum_{B \in [Q]^D} S(A/Q, B/Q) \cdot e(B/Q n) \cdot \int \chi^\vee_s(n-t) \left( \sum_{l \in \mathbb{I}(j_0)} \psi_{j_l}(t) \right) \, dt \]

\[ + O\left( \sum_{l \leq -Cs} Q^D \cdot \sum_n |f(x-n)| \int |\chi^\vee_s(n-t)||\psi_{j_l}(t)||P_\mu(t)| \, dt \right) \]

\[ = \mathcal{L}_{s,A/Q}[\widetilde{K_{\mathcal{J}^-}}*f(x)] \]

\[ + O\left( Q^D \cdot \sum_{l \leq -Cs} 2^l \cdot \sum_n |f(x-n)| \cdot \int |\chi^\vee_s(n-t)||\psi_{j_l}(t)| \, dt \right) \]

\[ = \mathcal{L}_{s,A/Q}[\widetilde{K_{\mathcal{J}^-}}*f(x)] + O\left( 2^{SD} \cdot 2^{-Cs} \cdot M_{HL} f(x) \right), \]

where \( \mathcal{J}_- := \min\{ j : j^{A_0} \geq 2^s \} \) and

\[ J_+(\mu) := \min \{ j_0, \max\{ j : \| P_\mu(2^j \cdot) \| \leq 2^{-Cs} \} \}, \]

see (1.10).
By Lemma 6.4, we bound
\[ \| \sup_{A/Q,\mu,j_0} \mathcal{L}_{s,A/Q}^\ast \Phi_{j_1,\mu} \|_{\ell^p(Z^D)} \lesssim 2^{-cs} \| f \|_{\ell^p(Z^D)}, \]
provided that \( C \) is chosen sufficiently large.

We now prove that for each \( l \),
\[ \| \sup_{A/Q,\mu} \mathcal{L}_{s,A/Q}^\ast \Phi_{j_1,\mu} \|_{\ell^p(Z^D)} \lesssim 2^{-cs} \cdot \| f \|_{\ell^p(Z^D)}. \]

First, we observe that
\[ \sup_{A/Q,\mu} \mathcal{L}_{s,A/Q}^\ast \Phi_{j_1,\mu} \| \lesssim M_{HL} f(x) \]
by Lemma 6.2. So, it suffices to exhibit the decay at the \( \ell^2 \) level.

We use the method of \( TT^\ast \): for an appropriate choice of linearizing functions,
\[ A(x)/Q(x) : Z^D \rightarrow \{ A/Q : Q \sim 2^s \} \]
and
\[ P_{\mu(x)} : Z^D \rightarrow \mathcal{P}_{d,D} \]
we may bound
\[ \sup_{A/Q,\mu} \| \sum_{l \in L} \mathcal{L}_{s,A/Q}^\ast \Phi_{j_1,\mu} \| \lesssim \| K(x,n) f(n) \|, \]
where
\[ K(x,n) = \sum_{B \in (Q)^D} S(A(x)/Q(x), B/Q)e(B/Q(x-n)) \]
\[ \times \int \chi_s((x-n)-t)e(-P_{\mu(x)}(t))\psi_{j_1}(t) \, dt \]
We exhibit an absolute \( c > 0 \) so that
(6.6) \[ \| K \|_{\text{Ker}(Z^D)} \lesssim 2^{-cs} \]
by \( TT^\ast \). In particular, we will show that the integral operator with kernel
\[ \mathcal{K}(x,z) = \sum_n K(x,n)K(z,n) \]
\[ = \sum_{B \in (Q)^D, B' \in (Q')^D} S(A(x)/Q(x), B/Q)S(A'(z)/Q'(z), B'/Q')e(B/Qx - B'/Q'z) \]
\[ \times \int \left( \sum_n e(-(B/Q - B'/Q')n)\chi_s((x-n)-t)\chi_s(z-n-u) \right) \]
\[ \times e(-P_{\mu(x)}(t) + P_{\mu'(z)}(u))\psi_{j_1}(t)\psi_{j_1}(u) \, dtdu \]
satisfies
(6.7) \[ \sup_x \sum_z |\mathcal{K}(x,z)| \lesssim 2^{-c_0s}, \quad \sup_z \sum_x |\mathcal{K}(x,z)| \lesssim 1, \]
at which point we can bound \( \| \mathcal{K} \|_{\text{Ker}(Z^D)} \lesssim 2^{-c_0/2s} \) by Schur’s test, from which (6.6) follows with \( c = c_0^4 \).
But, since $\chi_s$ has such small support, $K(x, z)$ diagonalizes:

$$K(x, z) = \sum_{B \in (Q)^D} S(A/Q, B/Q) \overline{S(A'/Q, B'/Q)} e(B/Q(x - z))$$

$$\times \int \rho_s(x - z)e(-P_{\mu(x)}(t) + P_{\mu'(z)}(u))\psi_{ji}(t)\overline{\psi_{ji}(u)} \, dtdu$$

$$+ O(2^{-2^{10} \rho_s} \cdot 2^{-j_D 1_{|x-z| \leq 2^j}})$$

where the error term arises from approximating

$$\sum_n \chi^\vee_s(a - n)\chi^\vee_s(b - n)$$

by

$$\rho_s(a - b) = \sum_n \chi^\vee_s(a - b - n)\chi^\vee_s(n),$$

using the smoothness of $\chi^\vee_s$ at spatial scales of the order $2^{-10 \rho_s}$. By (6.3), we may bound

$$|\sum_{B \in (Q)^D} S(A/Q, B/Q) \overline{S(A'/Q, B'/Q)} e(B/Q(x - z))| \lesssim 2^{-c_0s} + 1_{E_{Per}(A/Q)}(x - z)$$

where $E_{Per}(A/Q)$ is the $Q$-periodic extension of a subset $E(A/Q) \subset (Q)^D$ which depends only on $A/Q$, and has relative density $2^{-cs}$. Consequently

$$|K(x, z)| \leq 2^{-c_0s} \cdot \int |\rho_s((x - z) - (t - u))|\psi_{ji}(t)\psi_{ji}(u)| \, dtdu$$

$$+ \int |\rho_s((x - t) - (z - u))|\psi_{ji}(t)\psi_{ji}(u)| \, dtdu \cdot 1_{E_{Per}(A/Q)}(x - z).$$

which establishes (6.7), given the small Lipschitz norm of $\lesssim 2^{-10 \rho_s}$ of

$$v \mapsto \int |\rho_s(v - t + u)|\psi_{ji}(t)\psi_{ji}(u)| \, dtdu.$$

Finally, we just observe that for each $\theta \in \mathbb{T}^D$

$$\| \sup_{\mu} \int \Phi_{I, \mu}(\beta - \theta)\chi_s(\beta - \theta)\hat{f}(\beta)e(\beta x) \|_{L^p(\mathbb{Z}^D)}$$

$$= \| \sup_{\mu} \int \Phi_{I, \mu}(\beta)\chi_s(\beta)\hat{f}(\beta + \theta)e(\beta x) \|_{L^p(\mathbb{Z}^D)}$$

$$\lesssim 2^{-c_d} \| f \|_{L^p(\mathbb{Z}^D)}, \quad c = c_{d, D, p} > 0$$

by applying Magyar-Stein-Wainger transference [17] and the continuous result of Stein-Wainger [21]. Summing appropriately

$$\| \sup_{A/Q, \mu} |\mathcal{L}_{s, A/Q}[\Phi_{I, \mu}]|^{\vee} \ast f \|_{L^p(\mathbb{Z}^D)}$$

$$\leq \sum_{A/Q, 2^s} \| \sup_{\mu} |\mathcal{L}_{s, A/Q}[\Phi_{I, \mu}]|^{\vee} \ast f \|_{L^p(\mathbb{Z}^D)}$$

$$\leq 2^{s(|I| + D)} \cdot \sup_{\theta} \| \sup_{\mu} \int \Phi_{I, \mu}(\beta - \theta)\chi_s(\beta - \theta)\hat{f}(\beta)e(\beta x) \|_{L^p(\mathbb{Z}^D)}$$

$$\lesssim 2^{s(|I| + D)} \cdot 2^{-c_d} \| f \|_{L^p(\mathbb{Z}^D)}.$$
In particular, for any \( l \geq 1 \), we may bound
\[
\| \sup_{A/Q,\mu} |z_{s,A/Q}[\Phi_{j,l},\mu]|^* f \|_{\ell_p(\mathbb{Z}^D)} \lesssim \min\{2^{-c_s}, 2^{s(\Gamma+D)} \cdot 2^{-cl}\} \cdot \|f\|_{\ell_p(\mathbb{Z}^D)}
\]
\[
\lesssim 2^{-c_0(s+l)} \cdot \|f\|_{\ell_p(\mathbb{Z}^D)},
\]
after interpolating appropriately.

**APPENDIX A. THE PROOF OF THEOREM 3.2**

In this appendix we provide a full proof of Theorem 3.2 by establishing the following inverse theorem, Theorem A.1.

Multi-dimensional arithmetic progressions inside of \([\vec{N}]\) will be indexed as
\[
P = P_1 \times \cdots \times P_D \subset [\vec{N}]
\]
provided that \( P_i \subset [N_i] \) are arithmetic progressions. We will use
\[
\sigma_i := \min_{p \neq p' \in P_i} |p - p'|
\]
to denote the gap sizes of \( \{P_i\} \).

Then our result is as follows.

**Theorem A.1.** Suppose that
\[
\left| \frac{1}{|N_i|} \sum_{n_i \in P_i} e(P(n_i)) \right| \geq \delta
\]
for some arithmetic progressions \( P_i \subset [N_i] \) with gap sizes \( \sigma_i \leq \delta^{-1} \), see (A.2). Then either

- For some \( i \), \( N_i = \delta^{-O(1)} \); or
- There exists some \( Q \lesssim \delta^{-O(1)} \) so that for each \( \lambda_\alpha \)
\[
\|Q\lambda_\alpha\|_T \leq \frac{\delta^{-O(1)}}{N^\alpha}.
\]

The proof we provide proceeds by a double induction on the degree of \( P \), \( d \), and the on the dimension of the ambient space, \( D \), as well. The \( D = 1 \) case of Theorem A.1 appears in [26 §1], and in any event follows the inductive arguments used below (the base case \( d = D = 1 \) is again trivial); since the \( d = 1 \) case holds for any \( D \) by direct computation, we may assume that Theorem A.1 holds for all polynomials of degree \( < d \) in every dimension, and that Theorem A.1 holds for all polynomials of degree \( d \) in \( < D \) dimensions.

The following general Hilbert-space lemma provides the main mechanism to induct downwards. We recall the Fejér kernel at scale \( K \):
\[
\mu_K(n) := \frac{1}{K} (1 - \left| \frac{n}{K} \right|)_+.
\]

**Lemma A.2** (van der Corput’s inequality, Special Case). The following estimate holds for any phase \( P \), and any \( 0 \leq H \leq |I| \).
\[
\left| \frac{1}{|I|} \sum_{n \in I} e(P(n)) \right|^2 \lesssim \sum_k \mu_k \cdot \left| \frac{1}{|I|} \sum_{k \in I - k} e(P(n + k) - P(n)) \right| + \left( \frac{H}{|I|} \right)^2.
\]
Proof. Set \( F(n) := e(P(n)) \cdot 1_f(n) \), and observe that
\[
I := \frac{1}{|I|} \sum_{n \in I} F(n) = \frac{1}{|I|} \sum_{n \in I} F(n + h) \, dt + O\left(\frac{H}{|I|}\right)
\]
for any \( h \leq H \). In particular,
\[
I = \frac{1}{|I|} \sum_{n \in I} \left( \frac{1}{H} \sum_{h \leq H} F(n + h) \right) + O\left(\frac{H}{|I|}\right),
\]
so by Cauchy-Schwartz
\[
\left| \frac{1}{|I|} \sum_{n \in I} F(n) \right|^2 \leq \frac{1}{|I|} \sum_{n \in I} \left| \frac{1}{H} \sum_{h \leq H} F(n + h) \right|^2 \, dt + O\left(\frac{H^2}{|I|^2}\right);
\]
note how we used that the support constraint on \( F \) implies that
\[
n \mapsto \frac{1}{H} \sum_{h \leq H} F(n + h)
\]
is supported in \( 3I \). We expand the integral and change variables to conclude. \( \square \)

In the discrete setting, passing to appropriate subsets of arithmetic progressions plays the role of rescaling. Since the mechanism of passing to a small sub-interval of an arithmetic progression with small gap size will be used often, we introduce the following definition.

**Definition A.5.** Suppose that \([\vec{N}]\) is given, and let \( \delta > 0 \) be a small number. Given two multi-dimensional arithmetic progressions, \( P', P \subset [\vec{N}] \), we say that \( P' \) is a \( \delta \)-rescaling of \( P \) if \( P' \subset P \), and
\[
|P'_i| \leq \delta \cdot |P_i|, \quad \sigma'_i \geq \delta^{-1} \sigma_i
\]
for each \( 1 \leq i \leq D \), see \([A.2]\).

The following lemma will be used often after rescaling in various Taylor expansion arguments along arithmetic progressions.

**Lemma A.3.** Suppose that \( Q \) is such that
\[
||\lambda_\alpha Q||_T \leq \Delta \cdot \vec{N}^{-\alpha}.
\]
Suppose that \( l_i \in [M_i] \) with \( M_i \leq N_i \). Then
\[
P(t_0 + lQ) = P(t_0) + O(\Delta \cdot \sum_{i=1}^D M_i/N_i \cdot Q^{d-1}).
\]

Proof. Set \( \mu := \sum_{i=1}^D \frac{M_i}{N_i} \). For each \( \alpha \),
\[
\lambda_\alpha(t_0 + lQ)^\alpha = \lambda_\alpha l_0^\alpha + \lambda_\alpha Q \left( \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} l_0^{\alpha-\beta} Q^{\alpha-\beta-1} \right)
\]
\[
= \lambda_\alpha l_0^\alpha + O(\Delta \vec{N}^{-\alpha} \cdot Q^{d-1} \vec{N}^\alpha \cdot \sum_{0<|\beta|\leq\alpha} O(\frac{\vec{M}^\beta}{\vec{N}^\beta}))
\]
\[
= \lambda_\alpha l_0^\alpha + O(\Delta \cdot Q^{d-1} \cdot \mu),
\]
so the result follows by summing. \( \square \)
We will also require a “condensation of singularities” lemma, which appears as [26, Lemma 1.1.14]. The content is that if one begins with a frequency which lives relatively close to many cyclic subgroups of not-too-large height, then it must live extremely close to some cyclic subgroup with extremely small height.

Lemma A.4. Suppose that \(0 < \epsilon \ll \delta \ll 1\), and that \(N \gg \delta^{-1}\). Suppose that there exists a subset \(H \subset [N]\) with \(|H| \geq \delta N\) so that for all \(n \in H\),

\[
\|n\alpha_0\|^T \leq \epsilon.
\]

Then there exists some \(q \leq \delta^{-1}\) so that

\[
\|q\alpha_0\|^T \lesssim \epsilon \cdot \frac{q}{\delta N}.
\]

With these reductions in mind, we are prepared to prove Theorem A.1.

A.0.1. The Proof of Theorem A.1. We begin by reducing our attention to top order degrees.

Lemma A.5. It suffices to establish Theorem A.1 only for coefficients \(\{\lambda_\alpha : |\alpha| = d\}\)

Proof. Let \(P\) be an arbitrary degree \(d\) polynomial, which we decompose as above as

\[
P(n) = \sum_{j=1}^{d} P_j(n) = \sum_{j=1}^{d} \left( \sum_{|\alpha| = j} \lambda_\alpha n^\alpha \right).
\]

We induct downwards on \(|\alpha|\). Thus, let \(d > j_0\) be arbitrary, and assume that Theorem A.1 holds for \(|\alpha| > j_0\). Thus, we will assume that there exist \(Q \leq \delta^{-C}\) so that

\[
\|Q\lambda_\alpha\|^T \lesssim \frac{\delta^{-C}}{N^\alpha}
\]

for all \(|\alpha| > j_0\). Set

\[
K := Q \cdot \prod_{i=1}^{D} \sigma_i = \delta^{-O(1)},
\]

and subdivide \(P\) into

\[
P = \bigcup_{k \leq \delta^{-O(DA)}} Q_k \cup I, \quad |I| \leq \delta^{AD}|\tilde{N}|
\]

where each \(Q_k\) is a \(\delta^A\)-rescaling of \([\tilde{R}]\) all with common gap size \(\sigma_i = K\). By the pigeon-hole principle, there exists some \(Q\) with lengths \(N'_i = \delta^A N_i\) so that

\[
(A.6) \quad \delta^2 \leq \left| \frac{1}{|N'|} \sum_{n_i \in [N'_i]} e\left( \sum_{k=j_0+1}^{d} P_k(r_Q + Kn) + \sum_{k=1}^{j_0} P_k(r_Q + Kn) \right) \right|
\]

for some \(r_Q \in [\tilde{N}]\). By Lemma A.3 provided that \(A = O_{d,D}(1)\) is sufficiently large

\[
\sum_{k=j_0+1}^{d} P_k(r_Q + Kn) \equiv \sum_{k=j_0+1}^{d} P_k(r_Q) + O(\delta^{A/2}),
\]

so (A.6) becomes

\[
\delta^2 \lesssim \left| \frac{1}{|N'|} \sum_{n_i \in [N'_i]} e\left( \sum_{k=1}^{j_0} P_k(r_Q + Kn) \right) \right|.
\]
The polynomial
\[ n \mapsto \sum_{k=1}^{j_0} P_k(r_Q + K n) = P_{j_0}(r_Q + K n) + \text{Lower order terms in } n \]
\[ = \sum_{|\alpha|=j_0} \lambda_\alpha K^{j_0} n^\alpha + \text{Lower order terms in } n; \]
by hypothesis, there exist \( Q' \leq \delta^{-O(1)} \) so that
\[ \|Q' \lambda_\alpha K^j\| \lesssim \frac{\delta^{-O(1)} N^\alpha}{N^\alpha}; \]
setting \( Q_0 = Q' K^j \lesssim \delta^{-O(1)} \) for \( |\alpha| = j \) completes the proof. \( \square \)

To close the induction, we decompose
\[ (A.7) \quad P(n) = P_{\neq D}(n) + \sum_{j=1}^{d} P_{j,D}(n), \]
where
\[ (A.8) \quad P_{j,D}(n) := \sum_{|\alpha|=j: \alpha_D \neq 0} \lambda_\alpha n^\alpha \]
and \( P_{\neq D} \) is defined by subtraction and is independent of the \( D \)th variable.

Below, with \( \tilde{N} \) fixed, we call a coefficient approximable, or \( \delta \)-approximable, if there exists an absolute \( C \) so that
\[ \min_{q \leq \delta^{-C}} \|\lambda_\alpha q\|_T \lesssim \frac{\delta^{-C}}{N^\alpha}. \]

We will complete the proof of Theorem [A.1] by completing the following program:

- **Base Case:** The coefficients of \( P_{d,D} \) are approximable;
- **Downwards Inductive Step:** The coefficients of each \( P_{j,D}, 1 \leq j < d \) are approximable;
- **Second Inductive Step:** The degree \( d \) coefficients of \( P_{\neq D} \) are approximable as well.

The second inductive step is the least involved, so we dispose of it quickly.

*The Second Inductive Step:* Assume that we have established the existence of \( \{q_\alpha : \alpha_D \neq 0\} \) bounded above by \( \delta^{-C} \), so that
\[ \|q_\alpha \lambda_\alpha\|_T \lesssim \frac{\delta^{-O(1)}}{N^\alpha} \]
for each \( \lambda_\alpha : \alpha_D \neq 0 \). Set
\[ (A.9) \quad Q_0 := \prod_{\alpha: \alpha_D \neq 0} q_\alpha \lesssim \delta^{-O(1)} \]
and, for \( A \) sufficiently large, use the pigeon-hole principle to extract a \( \delta^A \)-rescaling of \( [\tilde{N}] \) with gap size \( Q_0 \), call it \( \mathcal{P} \), so that
\[ (A.10) \quad \delta^C \lesssim \left| \frac{1}{|\mathcal{P}|} \sum_{p \in Q_0 p + r \in \mathcal{P}} e(P(Q_0 p + r)) \right|. \]
By Lemma A.3

\[ P(Q_0p + r) = P_{\neq D}(Q_0p_1 + r_1, \ldots, Q_0p_D + r_D) + \sum_{j=1}^{d} P_{j,D}(Q_0p_1 + r_1, \ldots, Q_0p_D + r_D) \]

\[ = P_{\neq D}(Q_0p_1 + r_1, \ldots, Q_0p_D + r_D) + \sum_{j=1}^{d} P_{j,D}(r_1, \ldots, r_D) + O(\delta^{A/2}) , \]

and thus the lower bound (A.10) implies

\[ \delta^C \lesssim \left| \frac{1}{|P|} \sum_{p; Q_0p+r \in P} e\left( P_{\neq D}(Q_0p + r) \right) \right| + O(\delta^{A/2}) \]

at which point the inductive hypothesis kicks in, as \( p \mapsto P_{\neq D}(Q_0p + r) \) is a degree \( \leq d \) polynomial in at most \( D - 1 \) many variables with leading order coefficients the same as

\[ n \mapsto Q_0^d \cdot P_{\neq D}(n), \]

and \( Q_0 = \delta^{-O(1)} \).

\[ \square \]

We now turn to the main argument.

The Base Case. Our goal is to prove that the coefficients of \( P_{d,D} \) are approximable, see (A.1).

With \( K = c_0\delta^{N_D/\sigma_D} \) for a sufficiently small constant \( c_0 \), we may express

\[ \frac{1}{|N|} \sum_{n_i \in P_i} e(P(n)) = \frac{1}{|N|} \sum_{n_i \in P_i} e(P(n + h\sigma_D \cdot e_D)) + O(c_0\delta) \]

uniformly in \( h \in (K) \), see (1.17). Averaging in \( h \in (K) \) and applying Cauchy-Schwartz, we deduce a lower bound,

\[ \delta^2 \lesssim \sum_{h} \mu_K(h) \cdot \left| \frac{1}{|N|} \sum_{n_i \in P_i, n_D \in (P_D \cap P_D - \sigma_D h)} e(P(n + h\sigma_D \cdot e_D) - P(n)) \right| \]

see (A.10).

By the pigeon-hole principle, there exists some subset \( H \subset [K] \subset \left[ N_D/\sigma_D \right] \) of size

\[ |H| \gtrsim K \approx \delta N_D/\sigma_D \gtrsim \delta^2 N_D \]

so that for all \( h \in H \)

\[ \delta^2 \lesssim \left| \frac{1}{|N|} \sum_{n_i \in P_i, n_D \in (P_D \cap P_D - \sigma_D h)} e(P(n + h\sigma_D \cdot e_D) - P(n)) \right| . \]

The polynomials

\[ n \mapsto P(n + h\sigma_D \cdot e_D) - P(n) \]

are polynomials of degree \( d - 1 \); by our inductive hypothesis, we know that for each \( h \in H \) there exists some \( q_\alpha(h) \lesssim \delta^{-C} \) so that

\[ \| q_\alpha(h) \cdot (\lambda_\alpha_{\sigma_D} \cdot \sigma_D) \cdot h \| _T \lesssim \frac{\delta^{-C}}{N^{\alpha - e_D}} \]

for each \( |\alpha| = d \), since the monomials

\[ n \mapsto \lambda_\alpha_{\sigma_D} h \cdot n^{\alpha - e_D} \]

with \( |\alpha| = d \) will appear as top order terms in (??).
By pigeon-holing appropriately, there exists some subset $H' \subset H$ of size $\gtrsim \delta^C N_D$ so that for each $h \in H'$, there exists a single $q_\alpha \lesssim \delta^{-C}$ so that

$$\|q_\alpha \cdot (\lambda_\alpha \alpha D \sigma D) \cdot h\|_T \lesssim \frac{\delta^{-C}}{\tilde{N}_{\alpha - e D}}$$

for each $|\alpha| = d_0$.

Set $Q = \prod_{\alpha : \alpha D \neq 0} q_\alpha \cdot \alpha D$, so that $Q \lesssim \delta^{-O_d D(1)}$, and for each $|\alpha| = d$ so that $\alpha D \neq 0$,

$$\|Q \cdot (\lambda_\alpha \sigma D) \cdot h\|_T \lesssim \frac{\delta^{-C}}{\tilde{N}_{\alpha - e D}}$$

We now apply Lemma A.4 specifically, with

$$\epsilon := \frac{\delta^{-C}}{N_{\alpha - e D}},$$

and

$$\alpha_0 = Q \cdot (\lambda_\alpha \sigma D),$$

we deduce the existence of an integer $q_0 \lesssim \delta^{-O_d D(1)}$ so that

$$\|q_0 \alpha_0\|_T = \|(q_0 Q \sigma D) \cdot \lambda_\alpha\|_T \leq \frac{\epsilon q_0}{\epsilon N_{\alpha e D}} = \frac{\delta^{-O_d D(1)}}{\tilde{N}_\alpha}.$$

Since $q_0 Q \sigma D = \delta^{-O_d D(1)}$, we have shown that for every $|\alpha| = d_0$ with $\alpha D \neq 0$, there exists some $q = q_0 Q \sigma D \lesssim \delta^{-O(1)}$ so that

$$\|q_\alpha\|_T \leq \frac{\delta^{-O(1)}}{\tilde{N}_{\alpha}}.$$

□

We now complete the proof by establishing our main inductive step.

The Downwards Inductive Step. We here assume the existence of some $Q = \delta^{-O(1)}$ so that

$$\|Q \alpha\|_T \lesssim \frac{\delta^{-O(1)}}{N_{\alpha}}$$

for all $|\alpha| > j_0$ with $\alpha D \neq 0$, and our job is to extract some $\{q_\alpha\} \lesssim \delta^{-O(1)}$ so that

$$\|q_\alpha\|_T \leq \frac{\delta^{-O(1)}}{N_{\alpha}}$$

for all $|\alpha| = j_0$ with $\alpha D \neq 0$.

By the pigeon-hole principle, we can find a $\delta^A$-rescaling, $P' \subset P$, with $\sigma_i' = Q \sigma_i$, so that

$$\delta^C \lesssim \frac{1}{|P'|} \sum_{n_i \in P'_i} \epsilon(P'_{\neq D}(n) + \sum_{j=1}^d P_j(n)),$$

Expressing

$$P'_i \ni n_i = (\sigma_i Q) \cdot p + k_i, \quad p \leq \delta^A N_i, \quad k_i \in [N_i]$$

we compute that for each $j > j_0$

$$P_{j, D}(n) = P_{j, D}(n_1, \ldots, n_D) = P_{j, D}((\sigma_1 Q) \cdot p_1 + k_1, \ldots, (\sigma_D Q) \cdot p_D + k_D)$$

$$= P_{j, D}(k_1, \ldots, k_D) + O(\delta^{A/2})$$
provided that $A$ is sufficiently large, by another application of Lemma A.3. We therefore deduce

\begin{equation}
\delta^C \lesssim \frac{1}{|Q'|} \sum_{n_i \in Q_i'} e(P_{\neq D}(n) + \sum_{j=1}^{j_0} P_j(n));
\end{equation}

we now argue as above, differencing, pigeon-holing, and then applying Lemma A.4 to exhibit a $Q_0 \lesssim \delta^{-O(1)}$ so that

$$
\norm{Q_0 \lambda_\alpha}_T \lesssim \frac{\delta^{-O(1)}}{N^\alpha}
$$

for all $|\alpha| = j_0$ where $\alpha_D \neq 0$, closing the induction and completing the proof. □

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BK: Department of Mathematics, King's College London, WC2R 2LS, UK

Email address: ben.krause@kcl.ac.uk