Delta Potentials Supported by Hybrid Geometries and Their Deformations

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Abstract

We study the hybrid type of delta potentials in $\mathbb{R}^2$ and $\mathbb{R}^3$, where the delta potentials supported by a circle/sphere are considered together with the delta potentials supported by a point outside of the circle/sphere. The construction of the self-adjoint Hamiltonian operator associated with the formal expressions for the circle and point delta potentials is explicitly given. The bound state energies and scattering properties for each problem are also studied. Finally, we consider the delta potentials supported by deformed circle/sphere and show that the first order change in the bound state energies under small deformations of circle/sphere has a simple geometric interpretation.

Keywords: Delta potentials supported by curves and surfaces, resolvent, Krein’s resolvent formula, bound states, scattering.

1 Introduction

Dirac delta potentials, also known as point interactions in general, are among the exactly solvable classes of potentials studied from both physical and mathematical point of views. A detailed review of the subject together with their mathematically rigorous constructions as well as spectral properties are given in the monographs \cite{1, 2}.

Extension of such singular class of delta potentials to the ones supported by a sphere is formally studied in several quantum mechanics textbooks \cite{3, 4, 5}. However, a precise mathematical treatment of the so-called delta shell potentials is first given in \cite{6}. In this work, an additional delta potential supported by a point at the center of the sphere is also discussed in the zero angular momentum sector $l = 0$. Actually, more general and systematic studies are later developed, where the support of the delta potentials are considered to be a curve or surface \cite{7, 8, 9, 10}. The generalizations to delta functions supported on curves and surfaces embedded in manifolds are presented in the works \cite{11, 12}. Such circular/spherical singular interactions, considered to be models for circular/spherical quantum billiards are studied analytically and numerically recently \cite{13, 14}. Higher dimensional delta shell type of interactions have been also worked out in the literature from the point of view of differential equations \cite{15} using the partial wave analysis. Small geometric deformations of the support of the delta potentials recently attract some attention \cite{16}, where the area preserving small deformations can give rise to the isolated eigenvalues. Furthermore, the scattering theory for delta-potentials supported by locally deformed planes is constructed in \cite{17}.

In this paper, we consider the Schrödinger operators with the following type of interactions:

(i) Delta potential supported by a circle and delta potential supported by a point outside of the circle,
(ii) Delta potential supported by a sphere and delta potential supported by a point outside of the sphere,
(iii) Delta potential supported by a small deformation in the normal direction of a circle,
(iv) Delta potential supported by a small deformation in the normal direction of a sphere,
and study their bound state and scattering spectrum.

It is well known that the resolvent of such singular interactions (delta potential supported by a point, a curve, or a surface) can be expressed by some explicit formulae involving the resolvent of the free Hamiltonian. These expressions are commonly known as Krein’s formula in the literature \cite{11,12,10}. To find the resolvent of the above hybrid type of potentials, we first regularize the ill-defined interaction terms by finite rank projections acting on Hilbert space and then find the regularized resolvent associated with these regularized Hamiltonians. Then, considering the strong limit of these regularized resolvents, as we remove the regularization parameter, allows us to define a self-adjoint operator corresponding to their limits thanks to the Trotter-Kato theorem. If the support of the interaction is codimension two or three, then it is well-known that we need to renormalize the problem (see e.g., \cite{13,19} for the point interactions in two and three dimensions). In this case, we need to choose the coupling constants or strengths as functions of regularization parameter such that the limit converges. For an alternative treatment of scattering applied to coexisting point and line defects see the recent work \cite{20}.

For the sake of brevity, we only present the construction of the self-adjoint operator associated with the first system (i) and skip the technical details of the construction of the self-adjoint Hamiltonian associated with the other systems (ii)-(iv) since the idea of construction is essentially the same. The main results of the paper are to give explicitly the bound state energies and differential cross sections for systems (i)-(ii). Moreover, we show that the change in the bound state energies under small deformations in the normal directions of the circle/sphere can be similarly studied to first order in the deformation. An interesting observation here is that the first order perturbative calculation of the bound state energy gives the same result as the delta potential supported by a circle/sphere with radius increased by the average of the deformation. The method developed in this paper is in fact rather general and can be applied also to delta potentials supported by curves and surfaces in principle.

The paper is organized as follows. In Section 2, we explicitly show that there exists a self-adjoint operator associated with the initial formal Hamiltonian where the interaction contains a delta potential supported by a circle centered at the origin and a point outside of this circle (point being inside does not present any difficulties, it can equally be considered). Then, we briefly discuss the bound state analysis as well as scattering solutions. Section 3 deals with the bound state spectrum and scattering states for the delta potential supported by a sphere centered at the origin and a point outside of the sphere. Moreover, we study how the small deformations of the circle and sphere in the normal directions change the bound state spectrum and scattering properties in Sections 4 and 5. Finally, Appendix A is devoted to the Trotter-Kato theorem, which is needed to prove the self-adjointness of the Hamiltonian.

**Notation.** In our formulae we often use the Dirac notation for the inner products, however this notation is particularly designed for self-adjoint operators (often the distinction between symmetric and self-adjoint is ignored in physics literature), here we need to keep in mind that the operators appearing in our expressions always act on the right, unless specified otherwise. We shall also use the notation $*$ and $\dagger$ for the complex conjugation of a complex number and adjoint of an operator, respectively.

Dirac delta function supported by a point $a$ is defined on the test functions $\psi$ by $\langle \delta_a | \psi \rangle = \langle a | \psi \rangle := \psi(a)$. Similarly, the Dirac delta function $\delta_\Gamma$ supported by the curve $\Gamma$ and the Dirac delta function $\delta_\Sigma$ supported by the surface $\Sigma$ are defined by their action on $\psi$ \cite{27}:

\begin{align}
\langle \delta_\Gamma | \psi \rangle &= \langle \Gamma | \psi \rangle := \frac{1}{L(\Gamma)} \int_\Gamma \psi \, ds , \\
\langle \delta_\Sigma | \psi \rangle &= \langle \Sigma | \psi \rangle := \frac{1}{A(\Sigma)} \int_\Sigma \psi \, dA ,
\end{align}

where $ds$ is the integration element over the curve $\Gamma$ and $dA$ is the integration element over the surface. For the circle $\Gamma = S^1$, $ds = R \, d\theta$ and $L(\Gamma) = 2\pi R$. For the sphere $\Sigma = S^2$, $dA = R^2 \sin \theta \, d\theta \, d\phi$ and $A(\Sigma) = 4\pi R^2$.

## 2 Delta Potential Supported by a Circle and a Point

We first consider the delta potential supported by a circle and a point, given formally in Dirac notation by

\begin{equation}
H = H_0 - \lambda_1 |a| \langle a | - \lambda_2 |\Gamma| \langle \Gamma | ,
\end{equation}

where $\Gamma$ is the circle centered at the origin with radius $R$. We shall use units such that $\hbar = 2m = 1$ for simplicity. In order to make sense of the above expression, we first regularize the Hamiltonian $H$ by heat kernel $K_{\epsilon/2}$ in the following way:

\begin{equation}
H_\epsilon = H_0 - \lambda_1 (\epsilon) |a^*| \langle a^* | - \lambda_2 |\Gamma^*| \langle \Gamma^* | ,
\end{equation}
where
\[
\langle \mathbf{a}' | \psi \rangle = \int_{\mathbb{R}^2} K_{\epsilon/2}(\mathbf{r}, \mathbf{a}) \psi(\mathbf{r}) \, d^2 r,
\]
\[
\langle \Gamma' | \psi \rangle = \frac{1}{L(S^1)} \int_{S^1} \left( \int_{\mathbb{R}^2} K_{\epsilon/2}(\mathbf{r}, \gamma(s)) \psi(\mathbf{r}) \, d^2 r \right) \, ds.
\] (2.3) (2.4)

Here \( \epsilon > 0 \) is the regularization parameter or cut-off and \( \gamma(s) = (R \cos(s/R), R \sin(s/R)) \) is the parametrization of the circle \( S^1 \). The explicit form of the heat kernel in \( \mathbb{R}^n \) is given by \[22\]
\[
K_{\epsilon/2}(\mathbf{r}, \mathbf{r}') = \frac{1}{(2\pi \epsilon)^{n/2}} e^{-|\mathbf{r}-\mathbf{r}'|^2/2\epsilon}.
\] (2.5)

The strength or coupling constant of the point Dirac delta interaction, denoted by \( \lambda_1 \), is considered to be a function of \( \epsilon > 0 \), whose explicit form will be determined later. In this way, \( H_\epsilon \) becomes a finite rank perturbation of the free Hamiltonian so that it is self-adjoint on the domain of \( H_0 \) thanks to the Kato-Rellich theorem \[23\]. This choice of the regularization is based on the fact that the heat kernel converges to the Dirac delta function as \( \epsilon \to 0^+ \) in the distributional sense. This is an especially natural choice if we consider delta potentials in manifolds \[24, 25\] and here it is not only useful for the regularization but also allows us to approximate a singular interaction supported by a circle with a more regular one.

Next, we find the resolvent of the regularized Hamiltonian \[22\]. For this, we need to solve the following inhomogenous Schrödinger equation
\[
(H_\epsilon - E) |\psi\rangle = |\rho\rangle,
\] (2.6)

for a given function \( |\psi\rangle = \rho(\mathbf{r}) \in L^2(\mathbb{R}^2) \). The existence of the solution is guaranteed by the basic self-adjointness criteria \( \text{Re}(H_\epsilon - E) = L^2(\mathbb{R}^2) \) for at least one \( E \) in the upper half-plane and one in the lower half-plane \[23\]. By defining \( |\mathbf{a}'\rangle = |f_1(\epsilon)\rangle \) and \( |\Gamma'\rangle = |f_2(\epsilon)\rangle \), we can express the interaction as the sum of the rescaled projection operators:
\[
H_\epsilon = H_0 - \sum_{j=1}^{2} |f_j(\epsilon)\rangle \langle f_j(\epsilon)|,
\] (2.7)

where \( |f_j(\epsilon)\rangle = \sqrt{\lambda_j(\epsilon)} |f_j(\epsilon)\rangle \) in \( L^2(\mathbb{R}^2) \). Then, applying the free resolvent \( R_0(E) = (H_0 - E)^{-1} \) defined on the resolvent set \( \rho(H_0) = \mathbb{C} \setminus [0, \infty) \) to the equation \[2.6\], we find
\[
|\psi\rangle = R_0(E)|\rho\rangle + R_0(E) \sum_{j=1}^{2} |f_j(\epsilon)\rangle \langle f_j(\epsilon)| |\psi\rangle.
\] (2.8)

The right hand side of this expression involves unknown complex numbers \( \langle f_j(\epsilon)| \psi \rangle \). In order to find them, we project this equation onto the \( \langle f_i(\epsilon)| \) and isolating the \( j = i \) term in the summation to get the following matrix equation
\[
\sum_{j=1}^{2} \hat{\Phi}_{ij}(\epsilon, E) |f_j(\epsilon)\rangle |\psi\rangle = \langle f_i(\epsilon)| R_0(E)|\rho\rangle,
\] (2.9)

where
\[
\hat{\Phi}_{ij}(\epsilon, E) = \begin{cases} 
1 & i = j \\
-\langle f_i(\epsilon)| R_0(E)|f_j(\epsilon)\rangle & i \neq j
\end{cases}.
\] (2.10)

As emphasized in the introduction, here the Dirac’s notation \( \langle f_i(\epsilon)| R_0(E)|f_j(\epsilon)\rangle \) should be interpreted as \( \langle f_i(\epsilon)| R_0(E)|f_j(\epsilon)\rangle \). Assume that the matrix \( \hat{\Phi} \) is invertible for some subset of the free resolvent set to be determined below. Then, the solution of \[2.9\] exists and unique. Substituting this solution into \[2.8\], we get
\[
|\psi\rangle = R_0(E)|\rho\rangle + R_0(E) \sum_{i,j=1}^{2} |f_i(\epsilon)\rangle \left( \hat{\Phi}^{-1}(\epsilon, E) \right)_{ij} \langle f_j(\epsilon)| R_0(E)|\rho\rangle.
\] (2.11)

The resolvent of the regularized Hamiltonian can be directly read from the above result
\[
R(\epsilon, E) = R_0(E) + R_0(E) \sum_{i,j=1}^{2} |f_i(\epsilon)\rangle \left( \hat{\Phi}^{-1}(\epsilon, E) \right)_{ij} \langle f_j(\epsilon)| R_0(E)\rangle.
\] (2.12)
It is convenient to express the above sum in the following way
\[
\sum_{i,j=1}^{2} |\hat{f}_i(e)| \left( \Phi^{-1}(e, E) \right)_{ij} |\hat{f}_j(e)| = \text{Tr} \left( \hat{F}(e)\Phi^{-1}(e, E) \right),
\]  
(2.13)
where we have defined the matrix \( \hat{F}_{ij} := |\hat{f}_i(e)|^2 \). If we define the diagonal matrix \( D_{ij}(e) := \sqrt{\lambda_i(e)} \delta_{ij} \) we can decompose \( \hat{F} = DFD \), where \( F_{ij} = |\hat{f}_i(e)|^2 \). This helps us to write the summation term as \( \text{Tr} \left( \hat{F}\Phi^{-1} \right) = \text{Tr}(DFD\Phi^{-1}) = \text{Tr}(FD\Phi^{-1}D) = \text{Tr}(F\Phi^{-1}) \), where \( \Phi \) is related to \( \hat{\Phi} \) by a similarity transformation \( \Phi = D^{-1}\hat{\Phi}D^{-1} \) and given by
\[
\Phi_{ij}(e, E) = \begin{cases} 
\sqrt{\lambda_i(e)} & i = j \\
-(|\hat{f}_i(e)|R_0(E)|\hat{f}_j(e)|) & i \neq j . 
\end{cases} 
\]  
(2.14)
Hence, we explicitly find the resolvent formula for the regularized Hamiltonian
\[
R(e, E) = R_0(E) + R_0(E) \sum_{i,j=1}^{2} |\hat{f}_i(e)| \left( \Phi^{-1}(e, E) \right)_{ij} (|\hat{f}_j(e)|R_0(E)) .
\]  
(2.15)
We now claim that \( E \in \rho(H_0) \) lies in the resolvent set for \( H_\epsilon - E \) if and only if the matrix \( \Phi(e, E) \) is invertible. To prove this, we first assume that \( \Phi(e, E) \) is invertible for some values of \( E \in \rho(H_0) \). From the triangle inequality, we have
\[
||R(e, E)|\psi|| \leq ||R_0(E)|\psi|| + \frac{4}{\max} |\left( \Phi^{-1}(e, E) \right)_{ij} |(|\hat{f}_j(e)|R_0(E)|\phi||) ||R_0(E)|\phi|| .
\]  
(2.16)
We need to show that the right hand side of this inequality is a bounded function of \( E \) where \( E \) must lie in \( \rho(H_0) \) and satisfy \( \Phi(e, E) \neq 0 \). Moreover, this bound must also be a regular function of \( \epsilon \) since we will consider the limiting case as \( \epsilon \to 0^+ \) by appropriately choosing \( \lambda_1(e) \), as we will show later on.

A direct application of Cauchy-Schwarz inequality to the inner product in the right hand side of the above inequality does not yield a regular estimate in \( \epsilon \) since the norm of the function \( f_\epsilon(e) \) is not regular. For this reason, we may think that the adjoint of the bounded free resolvent operator acts on the first entry in the inner product and then apply the Cauchy-Schwarz inequality
\[
|\langle f_\epsilon(e)|R_0(E)|\phi|| \leq ||R_0(E^*)||\phi|| ||\phi|| < \infty ,
\]  
(2.17)
where we have used the fact that \( R_0^*(E^*) = R_0(E^*) \). Since \( E \) is inside the resolvent set of \( H_0 \), the expression \( ||R_0(E)|\phi|| \) and the inner product in the right hand side of the inequality lies in \( L^2(\mathbb{R}^2) \). However, we must also show that their bounds must be regular in \( \epsilon \) as \( \epsilon \to 0^+ \). It is easy to see that
\[
||R_0(E^*)|\phi||^2 = \int_{\mathbb{R}^2} \frac{|\langle p|a^\nu \rangle|^2}{(p^2 - E)(p^2 - E^*)} \frac{d^2p}{(2\pi)^2}.
\]  
(2.18)
Using (2.4) and the explicit form of the heat kernel (2.5), we find
\[
\langle p|a^\nu \rangle = \frac{e^{-|p| \rho}}{2\pi\epsilon} \int_{\mathbb{R}^2} e^{-|p| \rho} e^{-|p|\alpha^2} d^2p .
\]  
(2.19)
By writing the integral in polar coordinates and using the integral representation of the Bessel function of the first kind \( J_0(x) \)
\[
J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ix \cos \theta} d\theta
\]  
(2.20)
and the formula (20)
\[
\int_0^\infty x^{\nu+1} e^{-ax^2} J_\nu(\beta x) \ dx = \frac{\beta^\nu}{(2\alpha)^{\nu+1}} e^{-\frac{\beta^2}{4\alpha}},
\]  
(2.21)
we get
\[
\langle p|a^\nu \rangle = \frac{e^{-|p| \rho}}{2\pi\epsilon} e^{-|p|^2/2} .
\]  
(2.22)
Substituting this result into (2.15) yields the following bound
\[
||R_0(E^*)|\alpha^*||^2 \leq \frac{1}{2\pi} \int_0^\infty \frac{e^{-\epsilon v^2}}{|v^4 - 2\epsilon v^2 \text{Re}(E) + (\text{Re}(E))^2 + (\text{Im}(E))^2|} \, dv .
\] (2.23)

Except for the positive real \(E\) axis, the above integral converges and one can estimate its upper bound if \(\text{Re}(E) < 0\) by
\[
||R_0(E^*)|\alpha^*||^2 \leq \frac{1}{2\pi} \int_0^\infty \frac{e^{-\epsilon v^2}}{v^4 + \text{Re}(E)} \, dv ,
\] (2.24)

where \(A = \text{Re}(E)^2 + \text{Im}(E)^2\). Thanks to the result (3.354) given in [26], we can evaluate the above integral so that
\[
||R_0(E^*)|\alpha^*||^2 \leq \frac{1}{4\pi \sqrt{A}} \left( \text{ci}(\epsilon \sqrt{A}) \sin(\epsilon \sqrt{A}) - \text{si}(\epsilon \sqrt{A}) \cos(\epsilon \sqrt{A}) \right) ,
\] (2.25)

where \(\text{si}(x) = -\int_1^\infty \sin t / t \, dt\) is the sine integral function, and \(\text{ci}(x) = -\int_1^\infty \cos t / t \, dt\) is the cosine integral function. It is easy to see that this bound is a regular function of \(\epsilon\) for all \(A \neq 0\).

If \(\text{Re}(E) > 0\),
\[
||R_0(E^*)|\alpha^*||^2 = \frac{1}{4\pi} \int_0^\infty \frac{e^{-\epsilon u}}{(u - \text{Re}(E))^2 + \text{Im}(E)^2} \, du
\]
\[
= \frac{1}{4\pi} \int_{-\text{Re}(E)}^{0} \frac{e^{-\epsilon (u + \text{Re}(E))}}{u^2 + \text{Im}(E)^2} \, dv + \frac{1}{4\pi} \int_0^\infty \frac{e^{-\epsilon (u + \text{Re}(E))}}{v^2 + \text{Im}(E)^2} \, dv
\]
\[
\leq \frac{1}{4\pi} \int_{-\text{Re}(E)}^{0} \frac{1}{v^2 + \text{Im}(E)^2} \, dv + \frac{1}{4\pi} \int_0^\infty \frac{e^{-\epsilon v}}{v^2 + \text{Im}(E)^2} \, dv ,
\] (2.26)

which are finite and regular in \(\epsilon\) by the same reason given above. We can similarly show that the following norm
\[
||R_0(E^*)|\Gamma^*||^2 = \int_{\mathbb{R}^2} \frac{|\langle p |\Gamma^* \rangle|^2}{(p^2 - E)(p^2 - E^*)} \, d^2p / (2\pi)^2 ,
\] (2.27)

is a bounded function of \(E\) on \(\rho(H_0)\) and regular in \(\epsilon\). For this, we need to find
\[
\langle p |\Gamma^* \rangle = \frac{1}{L} \int_{\mathbb{R}^2} e^{i\rho \cdot r} \left( \int_{S^1} K_{\epsilon/2}(r, \gamma(s)) \, ds \right) \, d^2r.
\] (2.28)

Using the explicit expression of the heat kernel (2.24) and the integral representation of the modified Bessel function of the first kind [27]
\[
I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{\rho \cos \theta} \, d\theta ,
\] (2.29)

we get
\[
\int_{S^1} K_{\epsilon/2}(r, \gamma(s)) \, ds = \frac{R}{\rho} e^{-\frac{(x^2 + y^2)}{2\epsilon}} I_0 \left( \frac{R}{\epsilon} \right) .
\] (2.30)

Then, from the result (6.633) in [26]
\[
\int_0^\infty x e^{-\alpha x^2} I_\nu(\beta x) J_\nu(\gamma x) \, dx = \frac{1}{2\alpha} e^{-\frac{\beta^2 - \gamma^2}{4\alpha}} J_\nu \left( \frac{\beta \gamma}{2\alpha} \right) ,
\] (2.31)

we obtain
\[
\langle p |\Gamma^* \rangle = e^{-\frac{x^2}{4\alpha}} J_0(p \rho) .
\] (2.32)

Combining all these results yield
\[
||R_0(E^*)|\Gamma^*||^2 = \frac{1}{2\pi} \int_0^\infty \frac{p e^{-\epsilon p^2}}{(p^2 - E)(p^2 - E^*)} \, dp .
\] (2.33)
Since $J_0^2(pR) \leq 1$, we obtain the same form of the estimate (2.25) and (2.26) as above. All these establish that $R(\epsilon, E)$ is bounded if $E \in \rho(H_0)$ and satisfies $\det(\Phi(\epsilon, E)) \neq 0$, then $E \in \rho(H_\epsilon)$.

Conversely, if $E \in \rho(H_\epsilon)$ then $\det(\Phi(\epsilon, E)) \neq 0$. For this, suppose that $E \in \mathbb{C} \setminus [0, \infty)$ satisfies $\det(\Phi(\epsilon, E)) = 0$. We need to show that $E \notin \rho(H_\epsilon)$ or $E$ lies in the spectrum of $H_\epsilon$, or in particular $E$ is an eigenvalue of $H_\epsilon$: 

$$H_\epsilon |\psi\rangle = E |\psi\rangle ,$$  

(2.34)

for some non-zero $|\psi\rangle \in L^2(\mathbb{R}^2)$. The above eigenvalue problem for the regularized Hamiltonian is equivalent to the problem of finding non-trivial solution $\langle \tilde{f}_j(\epsilon) |\psi\rangle$ of the equation (2.35) with $|\rho\rangle = |0\rangle$:

$$\sum_{j=1}^{2} \Phi_{ij}(\epsilon, E)\langle \tilde{f}_j(\epsilon) |\psi\rangle = 0 .$$  

(2.35)

Since the equation (2.35) is derived from the eigenvalue equation (2.34) of the regularized Hamiltonian, the set of $E$ in (2.34) must also satisfy the equation (2.35). To prove the converse, we first need to show that $\langle f_i(\epsilon) |R_0(E) |f_j(\epsilon)\rangle \neq 0$ for all $i, j$. Otherwise, $\Phi$ would be an identity matrix, which is clearly invertible. Then, the equation (2.35) implies that $\langle \tilde{f}_j(\epsilon) |\psi\rangle = 0$ for all $j$. Expanding explicitly the form of the matrix $\Phi$ in (2.35) and using the above fact, it follows that $E$ must satisfy the eigenvalue equation for the regularized Hamiltonian. Hence, we have a non-trivial solution of the above linear equation (2.35) for $\langle \tilde{f}_j(\epsilon) |\psi\rangle$ with some $|\psi\rangle \in L^2(\mathbb{R}^2)$ if and only if $\det(\Phi(\epsilon, E)) = \det(\Phi(\epsilon, E)) = 0$. Let us summarize this short result as the following lemma:

**Lemma 1.** Let $\lambda_1(\epsilon)$ be a continuous function of $\epsilon$, which converges to zero as $\epsilon \to 0^+$ and $\lambda_2 > 0$ be an arbitrary positive real number. The resolvent of the regularized Hamiltonian

$$H_\epsilon = H_0 - \sum_{i=1}^{2} \lambda_i(\epsilon) |f_i(\epsilon)\rangle \langle f_i(\epsilon)|$$

is given by

$$R(\epsilon, E) = R_0(E) + R_0(E) \sum_{i,j=1}^{2} |f_i(\epsilon)\rangle \langle \Phi^{-1}(\epsilon, E)_{ij} |f_j(\epsilon)|R_0(E) ,$$

where

$$\Phi_{ij}(\epsilon, E) = \frac{\delta_{ij}}{\lambda_1(\epsilon)} - \langle f_i(\epsilon) |R_0(E) |f_j(\epsilon)\rangle ,$$  

(2.36)

and its resolvent set is given by $\rho(H_\epsilon) = \{E \in \rho(H_0) : \det(\Phi(\epsilon, E)) \neq 0 \text{ for all } \epsilon > 0\}$.

Now, we consider the limiting case as $\epsilon \to 0$ to properly define the initial formal Hamiltonian. For this reason, we choose $\lambda_1(\epsilon)$ in such a way that the regularized Hamiltonian has a reasonable and non-trivial limit as we remove the cut-off parameter, that is, as $\epsilon \to 0^+$. The off-diagonal elements of the matrix $\Phi(\epsilon, E)$ for $E = -\nu^2$ in the limit $\epsilon \to 0^+$ can be directly calculated using the Lebesgue dominated convergence theorem and the integral [26]

$$\int_0^{\infty} J_\xi(ax) J_\xi(bx) \frac{x}{x^2 + c^2} dx = \begin{cases} K_\xi(ac) I_\xi(bc) & 0 < b < a \\ K_\xi(bc) I_\xi(ac) & 0 < a < b \end{cases} ,$$  

(2.37)

for $\text{Re}(\xi) > -1$, so that

$$\lim_{\epsilon \to 0^+} \Phi_{12}(\epsilon, -\nu^2) = \lim_{\epsilon \to 0^+} \Phi_{21}(\epsilon, -\nu^2) = -\lim_{\epsilon \to 0^+} \langle a^c |R_0(-\nu^2)|G^\epsilon\rangle = -\frac{1}{2\pi} K_0(\nu R) I_0(R^\epsilon) .$$  

(2.38)

The limit of the second diagonal term of the matrix $\Phi(\epsilon, -\nu^2)$ as $\epsilon \to 0^+$ can be evaluated easily thanks to the Lebesgue dominated convergence theorem so we have

$$\lim_{\epsilon \to 0^+} \Phi_{22}(\epsilon, -\nu^2) = \frac{1}{\lambda_2} - \langle \Gamma^\epsilon |R_0(\epsilon, -\nu^2)|G^\epsilon\rangle = \frac{1}{\lambda_2} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^2} \frac{|p|^2}{p^2 + \nu^2} \frac{d^2p}{(2\pi)^2} = \frac{1}{\lambda_2} - \frac{1}{2\pi} I_0(\nu R)K_0(\nu R) ,$$  

(2.39)
where we have used the result (2.32) and the continuity of the integral (2.37) in the limiting case \( a \to b \). The \( \epsilon \to 0^+ \) limit of the first diagonal element of the matrix \( \Phi(\epsilon, E) \) in (2.14) includes the following term

\[
\lim_{\epsilon \to 0^+} (a^\nu R_0(-\nu^2)) |a^\nu| = \int_{0}^{\infty} K_{t}(a, a) e^{-t\nu^2} dt .
\] (2.40)

However, this is divergent due to the singular behaviour of the heat kernel around \( t = 0 \). To obtain a well-defined expression, we apply a renormalization process, an idea that is well-known in field theory models. In line with this idea, we introduce a new parameter \( \mu > 0 \) and make the following choice of the coupling constant \( \lambda_1 \) as a function of the regularization parameter \( \epsilon \):

\[
\frac{1}{\lambda_1(\epsilon)} := \int_{0}^{\infty} K_{t+\epsilon}(a, a) e^{-t\mu^2} dt .
\] (2.41)

Applying the method of renormalization to find well-defined results for point Dirac delta potentials in two and three dimensions is actually not new, see e.g. [18, 19]. After substituting (2.41) for the first diagonal element of the matrix (2.14) for \( E = -\nu^2 \) and then taking the limit as \( \epsilon \to 0^+ \), we get

\[
\lim_{\epsilon \to 0^+} \Phi_{11}(\epsilon, -\nu^2) = \frac{1}{4\pi} \log(\nu^2/\mu^2) .
\] (2.42)

Hence, we denote the limit of the matrix \( \Phi(\epsilon, -\nu^2) \) as \( \Phi(-\nu^2) \), it is given by

\[
\Phi(-\nu^2) := \begin{pmatrix}
\frac{1}{4\pi} \log \left( \frac{\nu}{\mu} \right) & -\frac{1}{2\pi} K_0(\nu a) I_0(\nu R) \\
-\frac{1}{2\pi} K_0(\nu a) I_0(\nu R) & \frac{1}{2\pi} - \frac{1}{2\pi} K_0(\nu R) I_0(\nu R)
\end{pmatrix} .
\] (2.43)

The formal limit of the resolvent (2.15) of the regularized Hamiltonian as we take \( \epsilon \to 0^+ \), is given by

\[
R(E) := R_0(E) + R_0(E) \sum_{i,j=1}^{2} \langle f_i | (\Phi^{-1}(E))_{ij} | f_j \rangle R_0(E) ,
\] (2.44)

for \( E = -\nu^2 \), which satisfies \( \det \Phi(E) \neq 0 \) and the matrix \( \Phi \) is given by (2.43). Here \( |f_1| = |a⟩ \) and \( |f_2⟩ = |Γ⟩ \). Actually, one can show that the above regularized resolvent (2.15) converges strongly to the expression (2.44) as \( \epsilon \to 0^+ \) for real negative values of \( E \) that satisfy \( \det \Phi(E) \neq 0 \), that is,

\[
\lim_{\epsilon \to 0^+} \| (R(\epsilon, E) - R(E)) |f_i⟩ \| = 0 ,
\] (2.45)

for any \( |f⟩ \in L^2(\mathbb{R}^2) \) and \( E \in \rho(H_ε) \). Since \( E \) is assumed to satisfy \( \det \Phi(E) \neq 0 \), we conclude that \( \det \Phi(\epsilon, E) \neq 0 \) for sufficiently small \( \epsilon > 0 \). Then, if we show

\[
\lim_{\epsilon \to 0^+} \| R_0(E)|f_i(\epsilon)⟩ - R_0(E)|f_i⟩ \| = 0 ,
\] (2.46)

then the strong convergence of the resolvent easily follows. The above condition is easy to check once we write it explicitly

\[
\lim_{\epsilon \to 0^+} \| R_0(E)|f_i(\epsilon)⟩ - R_0(E)|f_i⟩ \|^2 = \int_{\mathbb{R}^2} \left( 1 + e^{-\epsilon^2} - 2e^{-\epsilon^2/2} \right) \frac{d^2 p}{(2\pi)^2} \left( \frac{d^2 p}{(2\pi)^2} \right)^2 \text{ for } i = 1
\]

\[
\int_{\mathbb{R}^2} \left( 1 + e^{-\epsilon^2} - 2e^{-\epsilon^2/2} \right) \frac{d^2 p}{(2\pi)^2} \left( \frac{d^2 p}{(2\pi)^2} \right)^2 \text{ for } i = 2 ,
\] (2.47)

where we have used the equations (2.22) and (2.32). Then, the Lebesgue dominated convergence theorem implies that this limit is zero. Hence, we prove that

**Lemma 2.** Let \( E \) be a real negative number that satisfies \( \det \Phi(E) \neq 0 \). Then, the resolvent \( R(\epsilon, E) \) of the regularized Hamiltonian \( H_ε \) converges strongly to the expression \( R(E) \), given by (2.44) as \( \epsilon \to 0^+ \).

However, it is now natural to ask whether the above limiting expression is the resolvent of some self-adjoint operator. This can be answered affirmatively by following the ideas given in [11] or in [23]. Here we essentially follow a similar argument presented in [20] developed for the point interactions in the plane.

To establish this claim, we first show that the limit operator \( R(E) \) for the real negative values of \( E \) that satisfies \( \det \Phi(E) \neq 0 \) is invertible (equivalently, \( \text{Ker}(R(E)) = \{ |0⟩ \} \)). Suppose that \( R(E)|f⟩ = |0⟩ \) for some
$|f\rangle \in L^2(\mathbb{R}^2)$. From the explicit expression of the operator $R(E)$ given by (2.44) and writing the equation in momentum representation, we find

$$
\hat{f}(p) = -\sum_{i,j=1}^{2} \frac{\langle p|f_i \rangle}{p^2 - E} \left[ \Phi^{-1}(E) \right]_{ij} \int_{\mathbb{R}^2} \langle f_j(R_0(E))|q \rangle \langle q|f \rangle \frac{d^2q}{(2\pi)^2}.
$$

(2.48)

By Cauchy-Schwarz inequality, we have

$$
\int_{\mathbb{R}^2} \langle a|R_0(E)|q \rangle \langle q|\psi \rangle \frac{d^2q}{(2\pi)^2} = \int_{\mathbb{R}^2} \frac{e^{iq\cdot a}}{q^2 - E} f(q) \frac{d^2q}{(2\pi)^2} \leq \left( \int_{0}^{\infty} \frac{q}{(q^2 - E)^2} \frac{dq}{2\pi} \right)^{1/2} ||f|| < \infty,
$$

(2.49)

and

$$
\int_{\mathbb{R}^2} \langle T|R_0(E)|q \rangle \langle q|\psi \rangle \frac{d^2q}{(2\pi)^2} = \int_{\mathbb{R}^2} J_0(qR) f(q) \frac{d^2q}{(2\pi)^2} \leq \left( \int_{0}^{\infty} \frac{q}{(q^2 - E)^2} \frac{dq}{2\pi} \right)^{1/2} ||f|| < \infty.
$$

(2.50)

With the above bounds (2.49) and (2.50), we show that

$$
\hat{f}(p) = -\left[ e^{-ip\cdot a} \left[ \Phi^{-1}(E) \right]_{11} C_1 + e^{-ip\cdot a} \left[ \Phi^{-1}(E) \right]_{12} C_2 + J_0(pR) \left[ \Phi^{-1}(E) \right]_{21} C_2 + J_0(pR) \left[ \Phi^{-1}(E) \right]_{22} C_1 \right],
$$

(2.51)

where $C_1, C_2$ are finite real numbers and $E$ is a negative real number that satisfies $\det \Phi(E) \neq 0$. However, this solution $\hat{f}(p)$ can not be in $L^2(\mathbb{R}^2)$ unless $|f| = |0\rangle$. This allows us to define an operator $H$ depending on the parameter $\mu$ via

$$
R(E) := (H(\mu) - E)^{-1}
$$

(2.52)

for the above values of $E$. From now on we suppress the dependence of the Hamiltonian on the parameter $\mu$ for simplicity. Hence, we have

**Lemma 3.** Let $E$ be a real negative number that satisfies $\det \Phi(E) \neq 0$. Then, $R(E)$ is invertible.

After all these preliminary steps together with a version of Trotter-Kato theorem, quoted in Appendix A (see also [29]), it follows that the limit $R(\epsilon, E)$ converges strongly to $R(E)$ as $\epsilon \to 0^+$ for all complex numbers $E$ except for the interval $[0, \infty]$ and for the values satisfying $\det \Phi(E) \neq 0$. Moreover, there exists a self-adjoint operator $H$ such that $R(E) = (H - z)^{-1}$ and the matrix $\Phi$ for complex values are defined through its analytic continuation, given by

$$
\Phi(k^2) = \begin{pmatrix}
\frac{1}{2\pi} \log \left( \frac{-k^2}{\mu} \right) & -\frac{1}{2\pi} K_0(-ika) I_0(-ikR) \\
-\frac{1}{2\pi} K_0(-ika) I_0(-ikR) & \frac{1}{\sqrt{2}} - \frac{1}{2\pi} K_0(-ikR) I_0(-ikR)
\end{pmatrix},
$$

(2.53)

where we parametrize $E = k^2$ with unambiguous square root $k$ with $\text{Im}(k) > 0$ for convenience. We shall call this matrix as principal matrix from now on. Let us summarize the last result as the following:

**Theorem 1.** For complex $E$ not in $\det \Phi(E) = 0$ and $[0, \infty)$, the resolvent $R(\epsilon, E)$ of regularized Hamiltonian $H_\epsilon$ converges strongly to $R(E)$. Furthermore, there exists a self-adjoint operator $H$ such that $R(E) = (H - E)^{-1}$.

Suppose $E = k^2$ with unambiguous square root $k$ where $\text{Im}(k) > 0$ and $\phi_k(r) \in D(H_0) = H^2(\mathbb{R}^2)$. Thanks to the self-adjointness of $H$, we have

$$
D(H) = (H - k^2)^{-1} L^2(\mathbb{R}^2) = (H - k^2)^{-1}(H_0 - k^2)D(H_0).
$$

(2.54)
Then, using the explicit form of the resolvent formula (2.44), we have the following characterization of the domain of $H$:

$$D(H) = \left( 1 + \sum_{i,j=1}^{2} R_0(k^2) |f_i, j \rangle \langle f_j | \right) D(H_0) .$$  \hfill (2.55)

This means that the domain of $H$ consists of all functions of the following form

$$\psi(r) = \phi_k(r) + \sum_{i,j=1}^{2} \langle r | R_0(k^2) | f_i \rangle \langle f_j | \phi_k \rangle ,$$  \hfill (2.56)

where $\langle a | \phi_k \rangle = \phi_k(a)$, $\langle \Gamma | \phi_k \rangle = \frac{1}{L} \int_{S^1} \phi_k(\gamma(s)) \, ds$, and

$$\langle r | R_0(k^2) | f_1 \rangle = \langle r | R_0(k^2) | a \rangle = \int_{\mathbb{R}^2} \frac{e^{ip(r-a)}}{p^2-k^2} \, \frac{d^2p}{(2\pi)^2} = \frac{i}{4} H_0^{(1)}(k | r-a \rangle , \hfill (2.57)

$$\langle r | R_0(k^2) | f_2 \rangle = \langle r | R_0(k^2) | \Gamma \rangle = \int_{\mathbb{R}^2} \frac{e^{ipr}}{p^2-k^2} \, J_0(pR) \, \frac{d^2p}{(2\pi)^2} = \int_0^\infty \frac{p J_0(pr) J_0(pR) \, dp}{p^2-k^2}$$

$$= \frac{i}{4} \left( H_0^{(1)}(kr) J_0(kR) \theta(R-r) + H_0^{(1)}(kR) J_0(kr) \theta(r-R) \right) .$$  \hfill (2.58)

We have evaluated the last integral by the analytic continuation of the result (2.54) and used the fact that $K_0(z) = \frac{i}{2} H_0^{(1)}(e^{i\pi/2} z)$ and $J_0(z) = e^{-i\pi/2} J_0(e^{i\pi/2} z)$ for $-\pi < \text{arg}(z) < \pi/2$ [27], where $H_0^{(1)}$ is the zeroth order Hankel function of the first kind. Hence, we obtain

$$\psi(r) = \phi_k(r) + \frac{i}{4} H_0^{(1)}(k | r-a \rangle \left[ \Phi^{-1}(k^2) \right]_{11} \phi_k(a) + \left[ \Phi^{-1}(k^2) \right]_{12} \left( \frac{1}{L} \int_{S^1} \phi_k(\gamma(s)) \, ds \right)$$

$$+ \frac{i}{4} \left( H_0^{(1)}(kr) J_0(kR) \theta(R-r) + H_0^{(1)}(kR) J_0(kr) \theta(r-R) \right) \times \left( \left[ \Phi^{-1}(k^2) \right]_{21} \phi_k(a) + \left[ \Phi^{-1}(k^2) \right]_{22} \left( \frac{1}{L} \int_{S^1} \phi_k(\gamma(s)) \, ds \right) \right) .$$  \hfill (2.59)

Indeed, this decomposition (2.59) is unique. For this, let us set $\psi(r) = 0$ identically. Then, it follows from the above decomposition that

$$\phi_k(r) = -\frac{i}{4} H_0^{(1)}(k | r-a \rangle \left[ \Phi^{-1}(k^2) \right]_{11} \phi_k(a) + \left[ \Phi^{-1}(k^2) \right]_{12} \left( \frac{1}{L} \int_{S^1} \phi_k(\gamma(s)) \, ds \right)$$

$$- \frac{i}{4} \left( H_0^{(1)}(kr) J_0(kR) \theta(R-r) + H_0^{(1)}(kR) J_0(kr) \theta(r-R) \right) \times \left( \left[ \Phi^{-1}(k^2) \right]_{21} \phi_k(a) + \left[ \Phi^{-1}(k^2) \right]_{22} \left( \frac{1}{L} \int_{S^1} \phi_k(\gamma(s)) \, ds \right) \right) .$$  \hfill (2.60)

Since the functions $H_0^{(1)}(k | r-a \rangle$ and $H_0^{(1)}(kr) J_0(kR) \theta(R-r) + H_0^{(1)}(kR) J_0(kr) \theta(r-R)$ in each term are discontinuous at $r = a$ and $r = R$, the function $\phi_k(r)$ can only be continuous if

$$\left[ \Phi^{-1}(k^2) \right]_{11} \phi_k(a) + \left[ \Phi^{-1}(k^2) \right]_{12} \left( \frac{1}{L} \int_{S^1} \phi_k(\gamma(s)) \, ds \right) = 0 ,$$  \hfill (2.61)

$$\left[ \Phi^{-1}(k^2) \right]_{21} \phi_k(a) + \left[ \Phi^{-1}(k^2) \right]_{22} \left( \frac{1}{L} \int_{S^1} \phi_k(\gamma(s)) \, ds \right) = 0 .$$  \hfill (2.62)

Therefore, these conditions imply that the decomposition (2.59) is unique. It is also straightforward to show that $(H - k^2)^{-1}(H_0 - k^2)|\phi_k \rangle = |\psi \rangle$, which is equivalent to $(H - k^2)|\psi \rangle = (H_0 - k^2)|\phi_k \rangle$.

After showing the existence of a self-adjoint operator $H$ associated with the resolvent $R(E)$, we may not guarantee that $H$ must be of the form $H_0 + V$ with some operator $V$. Nevertheless, we can show that $H$ is a local operator in the sense that $\psi(r) = 0$ in an open set $U \subseteq \mathbb{R}^2$ implies that $H \psi(r) = \langle r | H | \psi \rangle = 0$. For this, let $\psi(r) = 0$ for all $r \in U$. Then, the function $\phi_k(r)$ for $r \in U$ is given by equation (2.60).
The solution of this in momentum representation is given by

\[ \psi(r) = \phi_k(r) + \sum_{i,j=1}^2 F_i(r) \left[ \Phi^{-1}(k^2) \right]_{ij} \langle f_j | \phi_k \rangle, \tag{2.63} \]

where \( F_i(r) = \langle r | R_0(k^2) | f_i \rangle \), given explicitly by the equations (2.77) and (2.68). Here \( \phi_k \in D(H_0) = H^2(\mathbb{R}^2) \) and \( k^2 \in \rho(H) \) with \( \text{Im}(k) > 0 \). The above decomposition is unique and \((H - k^2)|\psi\rangle = (H_0 - k^2)|\phi_k\rangle\). Moreover, suppose that \( D(H) \ni \psi(r) = 0 \) in an open set \( U \subseteq \mathbb{R}^2 \). Then, \( H\psi(r) = 0 \) for all \( r \in U \).

### 2.1 Bound State Analysis

As it is well-known that the point spectrum \( \sigma_p \) of an operator \( H \) consists of the set of complex numbers \( E \) such that \( \ker(H - E) \neq \{0\} \). From the explicit expression of the resolvent \( R(k^2) \) given by (2.51) for \( E = k^2 \), the poles of the resolvent for \( k^2 < 0 \) can only appear if the matrix \( \Phi(k^2) \) is singular, that is, if

\[ \det(\Phi(k^2)) = 0. \tag{2.64} \]

Let \( |\psi_{ev} \rangle \) be an eigenvector of \( H \) with corresponding eigenvalue \( E_{ev} = k^2_{ev} \), i.e.,

\[ H|\psi_{ev} \rangle = E_{ev}|\psi_{ev} \rangle, \tag{2.65} \]

where \( |\psi_{ev} \rangle \in D(H) \). Since any function in the domain of \( H \) can be decomposed according to Theorem 2, we have

\[ |\psi_{ev} \rangle = |\phi_k \rangle + \sum_{i,j=1}^2 R_0(k^2) |f_i \rangle \left[ \Phi^{-1}(k^2) \right]_{ij} \langle f_j | \phi_k \rangle, \tag{2.66} \]

for some \( k^2 \in \rho(H) \) with \( \text{Im}(k) > 0 \) and \( |\phi_k \rangle \in D(H_0) \). Actually, Theorem 2 provides us another relation between \( |\psi_{ev} \rangle \) and \( |\phi_k \rangle \):

\[ |\phi_k \rangle = (k^2_{ev} - k^2) R_0(k^2) |\psi_{ev} \rangle. \tag{2.67} \]

Substituting equation (2.66) into (2.67), we find

\[ |\phi_k \rangle = (k^2_{ev} - k^2) \left( R_0(k^2) |\phi_k \rangle + \sum_{i,j=1}^2 R_0(k^2) R_0(k^2) |f_i \rangle \left[ \Phi^{-1}(k^2) \right]_{ij} \langle f_j | \phi_k \rangle \right). \tag{2.68} \]

By acting \( H_0 - k^2 \) on this vector, it yields

\[ (H_0 - k^2_{ev}) |\phi_k \rangle = (k^2_{ev} - k^2) \sum_{i,j=1}^2 R_0(k^2) |f_i \rangle \left[ \Phi^{-1}(k^2) \right]_{ij} \langle f_j | \phi_k \rangle. \tag{2.69} \]

The solution of this in momentum representation is given by

\[ \hat{\phi}_k(p) = \frac{(k^2_{ev} - k^2)}{p^2 - k^2_{ev}} \sum_{i,j=1}^2 \frac{\langle p | f_i \rangle}{p^2 - k^2} \left[ \Phi^{-1}(k^2) \right]_{ij} \langle f_j | \phi_k \rangle. \tag{2.70} \]

If \( E_{ev} = k^2_{ev} \geq 0 \), then this equation has no nontrivial solution since \( \hat{\phi}_k(p) \) cannot lie in \( L^2(\mathbb{R}^2) \) unless it is identically zero. This implies that \( |\phi_k \rangle \notin L^2(\mathbb{R}^2) \) thanks to the Plancherel theorem. Hence, \( \psi_{ev}(r) = 0 \) for all \( r \in \mathbb{R}^2 \), which proves that there is no nonnegative eigenvalue of \( H \).
However, if $E_{ev} = k_{ev}^2 = -\nu_{ev}^2 < 0$ with $\nu > 0$, it is legitimate to apply $R_{0}(\nu_{ev}^2)$ on each side of the equation (2.69) and get

$$|\phi_k\rangle = (R_{0}(\nu_{ev}^2) - R_{0}(k^2)) \sum_{i,j=1}^{2} |f_{i}\rangle [\Phi^{-1}(k^2)]_{ij} \langle f_{j}|\phi_{k}\rangle .$$

(2.71)

Inserting this solution into (2.69), we formally find the eigenfunctions of $H$

$$\psi_{ev}(r) = \sum_{i,j=1}^{2} \langle r|R_{0}(\nu_{ev}^2)|f_{i}\rangle [\Phi^{-1}(k^2)]_{ij} \langle f_{j}|\phi_{k}\rangle .$$

(2.72)

This formal solution includes unknown factors $\langle f_{j}|\phi_{k}\rangle$. In order to find them, we first note that the principal matrix $\Phi$ can also be expressed purely in terms of the free resolvent kernels, that is,

$$\Phi(k^2) = \begin{pmatrix}
    \langle f_{1}|(R_{0}(\mu^2) - R_{0}(k^2))|f_{1}\rangle & -\langle f_{1}|R_{0}(k^2)|f_{2}\rangle \\
    -\langle f_{2}|R_{0}(k^2)|f_{1}\rangle & \frac{1}{k^2} - \langle f_{2}|R_{0}(k^2)|f_{2}\rangle
\end{pmatrix} .$$

(2.73)

Then, it is easy to check that

$$\langle f_{1}|(R_{0}(\nu_{ev}^2) - R_{0}(k^2))|f_{j}\rangle = \Phi_{ij}(k^2) - \Phi_{ij}(-\nu_{ev}^2) .$$

(2.74)

Using this result in (2.71) after taking the projection onto $\langle f_{j}|$, we obtain

$$\sum_{j=1}^{2} \Phi_{ij}(-\nu_{ev}^2)A_{j} = 0 ,$$

(2.75)

where $A_{j} = \sum_{i=1}^{2} [\Phi^{-1}(k^2)]_{ji} \langle f_{i}|\phi_{k}\rangle$. This equation tells us that $A_{i}$ is an eigenvector of the matrix $\Phi(-\nu_{ev}^2)$ with a zero eigenvalue.

Conversely, let us suppose that

$$|\psi_{ev}\rangle = \sum_{i=1}^{2} \frac{R_{0}(\nu_{ev}^2)}{2} f_{i} A_{i} ,$$

(2.76)

where $A_{i} = \sum_{j=1}^{2} [\Phi^{-1}(k^2)]_{ji} \langle f_{j}|\phi_{k}\rangle$ is an eigenvector of $\Phi(-\nu_{ev}^2)$ with eigenvalue zero. We will show that $|\psi_{ev}\rangle \in D(H)$ and $H|\psi_{ev}\rangle = -\nu_{ev}^2|\psi_{ev}\rangle$. First, we need to show that $|\psi_{ev}\rangle \in D(H)$. For this, we define

$$|\phi_{k}\rangle = (k_{ev}^2 - k^2)R_{0}(k^2)|\psi_{ev}\rangle ,$$

(2.77)

for some $k^2 \in \rho(H)$ with $\text{Im}(k) > 0$. Then, it follows easily that $|\phi_{k}\rangle \in D(H_{0})$ and inserting (2.76) into (2.77) and using the first resolvent identity for the free resolvent we obtain

$$|\phi_{k}\rangle = (R_{0}(\nu_{ev}^2) - R_{0}(k^2)) \sum_{i=1}^{2} A_{i} .$$

(2.78)

or

$$|\phi_{k}\rangle + R_{0}(k^2) \sum_{i=1}^{2} |f_{i}\rangle A_{i} = |\psi_{ev}\rangle .$$

(2.79)

Moreover, by taking the projection of (2.74) onto $\langle f_{j}|$ and using the above result (2.77), $|\psi_{ev}\rangle \in D(H)$ by the Theorem 2. Finally, using the result $(H_{0} - k^2)|\phi_{k}\rangle = (H - k^2)|\psi_{ev}\rangle$ in Theorem 2 for the eigenstate $|\psi_{ev}\rangle$, and the equation (2.77) we deduce that

$$H|\psi_{ev}\rangle = (H_{0} - k^2)|\phi_{k}\rangle + k^2|\psi_{ev}\rangle = -\nu_{ev}^2|\psi_{ev}\rangle .$$

(2.80)

It is useful to express the condition (2.64) in terms of a real positive parameter $\nu$, defined by $\nu = -ik > 0$. Then, the solutions of the equation $\det \Phi(-\nu^2) = 0$ determine the point spectrum of $H$ or bound state spectrum of $H$. However, finding the roots of the equation (2.64) is analytically not possible. Nevertheless,
we may obtain some information about the bound states as follows. First, suppose that the principal matrix \( \Phi \) has an eigenvector \( A \) associated with the eigenvalue \( \omega \),

\[
\Phi A = \omega A .
\]  

The eigenvalues can be explicitly calculated

\[
\omega_1(\nu) = \frac{1}{4\pi\lambda} \left\{ 2\pi + \lambda \log \left( \frac{\nu}{\mu} \right) - \lambda I_0(\nu R)K_0(\nu R) - \left[ \lambda^2 I_0^2(\nu R) (4K_0^2(\nu a) + K_0^2(\nu R)) \right]^{1/2} \right\},
\]  

and

\[
\omega_2(\nu) = \frac{1}{4\pi\lambda} \left\{ 2\pi + \lambda \log \left( \frac{\nu}{\mu} \right) - \lambda I_0(\nu R)K_0(\nu R) + \left[ \lambda^2 I_0^2(\nu R) (4K_0^2(\nu a) + K_0^2(\nu R)) \right]^{1/2} \right\}.
\]

Finding zeroes of the determinant of the matrix \( \Phi \) is equivalent to finding the zeroes of its eigenvalues. We will show that these are increasing functions of \( \nu \) by expressing the principal matrix \( \Phi \) in its closed form. Suppose for simplicity that the eigenvectors \( A \) are normalized. Then we can determine how the eigenvalues change with respect to \( \nu \) according to the Feynman-Hellman theorem [30]

\[
\frac{\partial \omega}{\partial \nu} = \lambda^* T \frac{\partial \Phi}{\partial \nu} A , \tag{2.86}
\]

where \( * \) and \( T \) denote the complex conjugation and transpose, respectively. Here, it is convenient to express the derivative of the principal matrix in the following form,

\[
\frac{\partial \Phi_{11}}{\partial \nu} = \frac{1}{2\pi\nu}, \tag{2.87}
\]

\[
\frac{\partial \Phi_{12}}{\partial \nu} = \frac{\partial \Phi_{21}^*}{\partial \nu} = (2\nu) \int_{\mathbb{R}^2} e^{ip^a} J_0(pR) \frac{d^2 p}{(2\pi)^2}, \tag{2.88}
\]

\[
\frac{\partial \Phi_{22}}{\partial \nu} = (2\nu) \int_{\mathbb{R}^2} J_0^2(pR) \frac{d^2 p}{(2\pi)^2}, \tag{2.89}
\]

by taking the derivative of \( \Phi \) under the integral sign thanks to the Lebesgue dominated convergence theorem.

Then, one can show that

\[
\frac{\partial \omega}{\partial \nu} = (2\nu) \int_{\mathbb{R}^2} \left| A_1 e^{ip^a} + A_2 J_0(pR) \right|^2 \frac{1}{(p^2 + \nu^2)^2} \frac{d^2 p}{(2\pi)^2} > 0 , \tag{2.90}
\]

for all \( \nu > 0 \), that is, all the eigenvalues \( \omega \) of the principal matrix \( \Phi \) are strictly increasing functions of \( \nu \). Fig. 1 below shows how the eigenvalues change with respect to \( \nu \) for the particular values of parameters. The solutions of (2.64) can also be considered as the zeroes of the eigenvalues of the principal matrix \( \Phi \) so all the bound state energies can be found from the zeroes of the eigenvalues, say \( \nu_* \), for which

\[
E = -\nu_*^2 . \tag{2.91}
\]

The positivity condition (2.90) implies that there are at most two bound state energies since each eigenvalue can cross the \( \nu \) axis only once. In Fig. 1 the zeroes of the eigenvalue \( \omega_1 \) corresponds to the ground state energy. This bound state always exists for all values of the parameter since \( \lim_{\nu \to 0^+} \omega_1 = -\infty \) and it is an increasing function of \( \nu \) and positive for sufficiently large values of \( \nu \). However, the second eigenvalue \( \omega_2 \) may not have any zeroes if it is not negative around \( \nu = 0 \). One can also numerically calculate the bound state energies and plot them as a function of \( a \) and \( R \), respectively for the fixed given values of the parameters, as shown in Fig. 2 and Fig. 3.

It follows from Weyl’s theorem [31] that the essential spectra of \( H \) and \( H_0 \) coincide, that is, \( \sigma_{ess}(H) = \sigma_{ess}(H_0) = [0, \infty) \) if we show that \( R(E) - R_0(E) \) is compact for some \( E \in \rho(H) \cap \rho(H_0) \). Note that we have
\[ \omega_2 \]
\[ \omega_1 \]

Figure 1: Eigenvalues of the principal matrix \( \Phi \) for \( \lambda = 10, \mu = 1, R = 1, a = 2 \) units.

\[ E_B \]

(a) Ground state energy versus \( a \)

(b) Excited state energy versus \( a \)

Figure 2: Bound state energies versus \( a \) for \( \lambda = 10, R = 1, \mu = 1 \) units.

\[ E_B \]

(a) Ground state energy versus \( R \)

(b) Excited state energy versus \( R \)

Figure 3: Bound state energies \( E_B \) versus \( R \) for \( \lambda = 10, a = 5.1, \mu = 1 \) units.

\[ R(E) - R_0(E) \] given by an explicit formula (2.44), here \( \Phi_{ij} \) is invertible for a sufficiently negative \( E_* \) on the real axis and all its eigenvalues become positive. Therefore

\[ R(E_*) - R_0(E_*) = R_0(E_*) \sum_{i,j=1}^2 |f_j \rangle \Phi_{ij}^{-1} (f_j | R_0(E_*) , \]

indeed becomes a finite rank operator. For this note that the principal matrix has a spectral decomposition

\[ \Phi^{-1}(E_*) = \sum_k \omega_k^{-1}(E_*) A^{(k)}(E_*) A^{(k)*}(E_*) \] with \( A^{(k)} \) representing the \( k \)th eigenvector of \( \Phi(E) \) and \( \omega_k \) is the corresponding eigenvalue. All the eigenvalues become positive for \( E_* \). We therefore need to observe that all the vectors

\[ \sum_{i=1}^2 \omega_k^{-1/2}(E_*) A_i^{(k)}(E_*) R_0(E_*) | f_i \rangle , \]

indeed becomes a finite rank operator. For this note that the principal matrix has a spectral decomposition

\[ \Phi^{-1}(E_*) = \sum_k \omega_k^{-1}(E_*) A^{(k)}(E_*) A^{(k)*}(E_*) \] with \( A^{(k)} \) representing the \( k \)th eigenvector of \( \Phi(E) \) and \( \omega_k \) is the corresponding eigenvalue. All the eigenvalues become positive for \( E_* \). We therefore need to observe that all the vectors

\[ \sum_{i=1}^2 \omega_k^{-1/2}(E_*) A_i^{(k)}(E_*) R_0(E_*) | f_i \rangle , \]
for \( k = 1, 2 \) have a finite norm, as can be seen as follows;

\[
\| \sum_{i=1}^{2} \omega_k^{-1/2}(E_i)A_i^{(k)}R_0(E_i)|f_i\| \leq \sum_{i=1}^{2} |\omega_k^{-1/2}(E_i)A_i^{(k)}|| |R_0(E_i)|f_i|| .
\]  

(2.94)

Hence, we have shown that \( R(E) - R_0(E) \) is a trace class operator, which is sufficient for compactness.

To summarize the above results, we have

**Theorem 3.** Let \( a \in \mathbb{R}^2 \) and \( \Gamma \) be the circle centered at the origin with radius \( R < a \). Then, the essential spectrum of \( H \) associated with the point delta and delta potential supported by \( \Gamma \) coincides with the essential spectrum of the free Hamiltonian, i.e., \( \sigma_{ess}(H) = \sigma_{ess}(\mathcal{H}_0) = [0, \infty) \). Furthermore, the point spectrum \( \sigma_p(H) \) of \( H \) lies in the negative real axis and \( H \) has at least one and at most two negative eigenvalues (counting multiplicity). Let \( \text{Re}(k) = 0 \) and \( \text{Im}(k) > 0 \), then \( k^2 \in \sigma_p(H) \) if and only if \( \det \Phi(k^2) = 0 \) and multiplicity (degeneracy) of the eigenvalue \( k^2 \) is the same as the multiplicity of the zero eigenvalue of the matrix \( \Phi(k^2) \).

Moreover, let \( E = -\nu^2 < 0 \) be an eigenvalue of \( \mathcal{H} \), then the eigenfunction \( |\psi_{ev}\rangle \) associated with this eigenvalue is given by

\[
|\psi_{ev}\rangle(r) = \sum_{i=1}^{2} (r|R_0(-\nu^2)|f_i)A_i ,
\]

where \( (A_1, A_2) \) are eigenvectors with zero eigenvalue of \( \Phi(-\nu^2) \).

### 2.2 Stationary Scattering Problem

Stationary scattering problem for such singular potentials is well-defined, that is, the wave operators \( \Omega_{\pm} \) exist and are complete thanks to Birman-Kuroda theorem [32]. This theorem states that if the difference between the resolvent \((H - E)^{-1}\) of the self-adjoint operator \( H \) and the resolvent of the free self-adjoint Hamiltonian \((\mathcal{H}_0 - E)^{-1}\), this difference being defined on their common resolvent set, is trace class, then wave operators exist and are complete. We have already shown above that \( R(E) - R_0(E) \) is trace class. Therefore, the wave operators for defining scattering phenomena exist. Once we have well-defined wave operators, we can study physically measurable quantities (e.g., cross section) about a scattering experiment by finding the scattering amplitudes. For this reason, we first need to determine the boundary values of the operator \( T(E) \) as \( E \) approaches to the positive real axis from above. This is accomplished from the explicit formula of the resolvent written on the complex plane. For convenience, let \( E = E_k + i\epsilon \) where \( E_k = k^2 \) with \( k > 0 \). The relation between the resolvent and operator \( T(E) \) is given by

\[
R(E) = R_0(E) - R_0(E)T(E)R_0(E) .
\]

(2.95)

Since we have the explicit expression for the resolvent \((2.44)\) extended onto the complex plane, we can read off the boundary values of operator \( T(E) \) on the positive real axis:

\[
T(E_k + i\epsilon) = -\sum_{i,j=1}^{2} |f_i\rangle \left( \Phi^{-1}(E_k + i\epsilon) \right)_{ij} \langle f_j| ,
\]

(2.96)

where

\[
\Phi(E_k + i\epsilon) = \begin{pmatrix}
\frac{1}{i\pi} \left( -\frac{i\pi}{2} + \log \left( \frac{\epsilon}{\nu} \right) \right) & -\frac{1}{i} \mathcal{H}_0^{(1)}(ka) J_0(kR) \\
-\frac{1}{i} \mathcal{H}_0^{(1)}(ka) J_0(kR) & \frac{1}{\epsilon^2} \mathcal{H}_0^{(1)}(ka) J_0(kR)
\end{pmatrix} .
\]

(2.97)

Here we have used \( \mathcal{K}_0(z) = \frac{i\pi}{2} \mathcal{H}_0^{(1)}(e^{i\pi/2}z) \) and \( J_0(z) = e^{-iz/2} J_0(e^{iz/2}z) \) for \( -\pi < \arg(z) < \pi/2 \) [27]. The scattering amplitude denoted by \( f \) and the boundary values of the operator \( T \) in two dimensions is related by

\[
f(k \rightarrow k') = -\frac{e^{i\pi/4}}{\pi k} \sqrt{\frac{\pi}{2}} (k'|T(E_k + i\epsilon)|k) ,
\]

(2.98)

where \(|k|\) is the generalized Dirac ket vector and \(|k'| = |k|\). (We note that, there is another choice for the scattering amplitude by ignoring the factor \( \sqrt{k} \) to get some desirable properties [34], here we use the conventional version). Substituting the result \((2.96)\) into

\[
|k'|T(E_k + i\epsilon)|k) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{ikx - ik'x'} \langle x'|T(E_k + i\epsilon)|x\rangle d^2x d^2x'
\]

(2.99)
and using the integral representation of the Bessel function $J_0(x)$ given in (2.20) we find

$$
\langle k' | T(E_k + i0) | k \rangle = e^{i(k-k') \cdot a} \left( \Phi^{-1}(E_k + i0) \right)_{11} J_0(kR) \left( \Phi^{-1}(E_k + i0) \right)_{22} ,
$$

(2.100)

where $(\Phi^{-1}(E_k + i0))_{ij}$ is the $ij$th element of the inverse of the matrix $\Phi(E_k + i0)$ given in equation (2.97).

**Theorem 4.** The differential cross section for the delta potential supported by a circle of radius $R$ centered at the origin and by the point at $a$ outside of the circle is given by

$$
\frac{d\sigma}{d\theta} = \left| f(k \rightarrow k') \right|^2 = \frac{1}{8\pi k} e^{i(k-k') \cdot a} \left( \Phi^{-1}(E_k + i0) \right)_{11} J_0(kR) \left( \Phi^{-1}(E_k + i0) \right)_{22} .
$$

(2.101)

The differential cross section is plotted as a function of $\theta$ in Fig. 4. Here we assume that the support of the point defect is at $x = a$ without loss of generality and $\theta$ is the angle between $k'$ and $k$ chosen to be along the positive $x$ axis. One can also plot the differential cross section as a function of $k$ for different choice of parameters, as shown in Fig. 5a and Fig. 5b. The behaviour near $k = 0$ of the differential cross section is consistent with the fact that the differential cross section for two dimensional low energy scatterings blows up with decreasing energy, as emphasized in [35].

### 3 Delta Potential Supported by a Sphere and a Point

In this section, we will consider the spherical shell delta potential perturbed by point like delta potential in three dimensions. Since all the techniques and results are similar to the case discussed in the previous section, we will summarize some results without giving detailed proofs.

The regularized Hamiltonian for this model is given by

$$
H_\epsilon = H_0 - \lambda_1(\epsilon) |a^\epsilon \rangle \langle a^\epsilon| - \lambda_2(\Sigma^\epsilon) |\Sigma^\epsilon| ,
$$

(3.1)

where $\Sigma$ is the sphere centered at the origin with radius $R$ and

$$
\langle a^\epsilon | \psi \rangle = \int_{\mathbb{R}^3} K_{\epsilon/2}(r, a) \psi(r) \, d^3r ,
$$

(3.2)

$$
\langle \Sigma^\epsilon | \psi \rangle = \frac{1}{A(S^2)} \int_{S^2} \int_{\mathbb{R}^3} K_{\epsilon/2}(r, \sigma(\theta, \phi)) \psi(r) \, d^3r \, dA .
$$

(3.3)
Here $\sigma : (0, 2\pi) \times (0, \pi) \to S^2$ is the local parametrization given by

$$\sigma(\theta, \phi) := (R \sin \theta \cos \phi, R \sin \theta \sin \phi, R \cos \theta).$$

(3.4)

Proceeding analogously to the previous construction of the resolvent for the circular defect accompanied by a point defect problem, we obtain the resolvent essentially in the same form (2.44). In this case, the first diagonal element of the matrix $\Phi$ for $E = -\nu^2$ can be calculated similarly:

$$\Phi_{11}(-\nu^2) = \lim_{\epsilon \to 0^+} \int_0^\infty K_{t+}(a, a) \left( e^{-t\nu^2} - e^{-t\nu^2_0} \right) dt = \frac{(\nu - \mu)}{4\pi},$$

(3.5)

by choosing the bare coupling constant $\lambda_1(\epsilon)$ of the point interaction to be of the same type as (2.41) except the heat kernel here is written in three dimensions. Choosing the support of the point defect along the $z$ axis, we find the off-diagonal matrix elements of $\Phi$ by going to spherical coordinates and evaluating the radial part of the integral by the residue theorem:

$$\Phi_{12}(-\nu^2) = \Phi_{21}(-\nu^2) = -\langle a | R_0(-\nu^2) | \Sigma \rangle = -\int_{\mathbb{R}^3} \frac{e^{ip \cdot a}}{(p^2 + \nu^2)} \frac{\sin(pR)}{pR} \frac{d^3p}{(2\pi)^3},$$

(3.6)

where we have used

$$\langle p | \Sigma \rangle = \frac{\sin(pR)}{pR}.$$  

(3.7)

Similarly,

$$\Phi_{22}(-\nu^2) = \frac{1}{\lambda_2} \langle \Sigma | R_0(-\nu^2) | \Sigma \rangle = \frac{1}{\lambda_2} - \int_{\mathbb{R}^3} \frac{1}{p^2 + \nu^2} \frac{\sin^2(pR)}{(pR)^2} \frac{d^3p}{(2\pi)^3},$$

(3.8)

The resolvent of the model is formally given by the same equation (2.44), where $|f_2| = |\Sigma|$ and the matrix $\Phi$ can be defined on the complex plane by an analytic continuation of the above expressions. It is easy to see that the matrix elements of above matrix looks similar to our two dimensional version if we express its entries in terms of the Bessel functions using $I_{1/2}(z) = \sqrt{\frac{z}{\pi}} \sinh z$ and $K_{1/2}(z) = \sqrt{\frac{z}{\pi}} e^{-z}$:

$$\Phi(-\nu^2) = \begin{pmatrix}
\frac{1}{4\pi}(\nu - \mu) & -\frac{1}{4\pi\sqrt{\alpha}} K_{1/2}(\nu a) I_{1/2}(\nu R) \\
-\frac{1}{4\pi\sqrt{\alpha}} K_{1/2}(\nu a) I_{1/2}(\nu R) & \frac{1}{\lambda_2} - \frac{1}{4\pi\sqrt{\alpha}} K_{1/2}(\nu R) I_{1/2}(\nu R) 
\end{pmatrix}.$$  

(3.9)

### 3.1 Bound State Problem

Bound state analysis of this problem is performed exactly in the same manner as in the case of delta potential supported by a circle and a point. For this reason, we are not going to derive the analogous expressions for
the flow of the eigenvalues with respect to $\nu$. Positivity of the flow of eigenvalues equally holds in this case so we conclude that there are at most two bound states (and at least one bound state).

One can plot the eigenvalues as a function of $\nu$ for particular values of the parameters. As shown in the previous section, zeroes $\nu_*$ of the eigenvalues correspond to the bound state energies $E = -\nu_*^2$. It is interesting to notice that there is only one bound state if we choose the same values of the parameters for the circular defect perturbed by a point defect problem, as shown in Fig. 6(a). The reason for this may be based on the fact that particle has more freedom to escape from the spherical defect compared to the circular defect. If we increase the strength of the spherical defect potential, the second bound state appears as shown in Fig. 6(b).

Figure 6: Eigenvalues of $\Phi$ versus $\nu$

One can find how the bound state energies change with respect to the parameters $R$ and $a$ by numerically solving the zeroes of the eigenvalues $\omega_1$ and $\omega_2$. They are plotted in Figs. 7 and 8. By following the same line of arguments, we have the following theorem:

**Theorem 5.** Let $a \in \mathbb{R}^3$ and $\Sigma$ be the sphere centered at the origin with radius $R < a$. Then, the essential spectrum of $H$ associated with the point delta and delta potential supported by $\Sigma$ coincides with the essential
spectrum of the free Hamiltonian, \( \sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0) = [0, \infty) \). Furthermore, the point spectrum \( \sigma_p(H) \) of \( H \) lies in the negative real axis and \( H \) has at most two negative eigenvalues (counting multiplicity) and always has one. Let Re(\( k \)) = 0 and Im(\( k \)) > 0, then \( k^2 \in \sigma_p(H) \) if and only if \( \det \Phi(k^2) = 0 \) and multiplicity (degeneracy) of the eigenvalue \( k^2 \) is the same as the multiplicity of the zero eigenvalue of the matrix \( \Phi(k^2) \). Moreover, let \( E = -\nu_2^2 < 0 \) be an eigenvalue of \( H \), then the eigenfunction \( |\psi_{\text{ev}}\rangle \) associated with this eigenvalue is given by

\[
|\psi_{\text{ev}}\rangle = \sum_{i=1}^{2} \langle r | R_0(-\nu_2^2)|f_i\rangle A_i ,
\]

where \( (A_1, A_2) \) are eigenvectors with zero eigenvalue of \( \Phi(-\nu_2^2) \) and \( |f_1\rangle = |a\rangle, |f_2\rangle = |\Sigma\rangle \).

### 3.2 Stationary Scattering Problem

For the scattering problem, we similarly find the boundary values of the principal operator by analytical continuation

\[
\Phi(E_k + i0) = \begin{pmatrix}
\frac{1}{4\pi} (-ik - \mu) & -\frac{1}{4\pi aR} e^{ika} \sin(kR) \\
-\frac{1}{4\pi aR} e^{ika} \sin(kR) & \frac{1}{\lambda_2} - \frac{aR^k}{4\pi R^k} \sin(kR)
\end{pmatrix},
\]

(3.10)

and

\[
\langle k'|T(E_k + i0)|k\rangle = -\sum_{i,j=1}^{2} \langle k'| f_i \rangle \left( \Phi^{-1}(E_k + i0) \right)_{ij} \langle f_j|k\rangle
\]

\[
= -\left( e^{i(k-k') a} \left( \Phi^{-1}(E_k + i0) \right)_{11} + \frac{e^{-i k' a} + e^{i k' a}}{kR} \left( \Phi^{-1}(E_k + i0) \right)_{12} \right) + \frac{\sin^2(kR)}{k^2 R^2} \left( \Phi^{-1}(E_k + i0) \right)_{22}.
\]

(3.11)

Hence, we find the scattering amplitude from the formula \( f(k \to k') = -\frac{1}{2\pi} \langle k'|T(E_k + i0)|k\rangle \), and the graph of the differential cross section \( \frac{d\sigma}{d\Omega} = |f(k \to k')|^2 \) as a function of \( \theta \) is given in Fig. 9. Let us summarize the result:

**Theorem 6.** The differential cross section for the delta potential supported by a sphere of radius \( R \) centered
at the origin and by the point at a outside of the sphere is given by

\[
\frac{d\sigma}{d\Omega} = |f(k \rightarrow k')|^2 = \left| \frac{1}{16\pi^2} e^{i(k-k') \cdot a} \left( \Phi^{-1}(E_k + i0) \right)_{11} + \left( e^{-ik' \cdot a} + e^{ik \cdot a} \right) \sin(kR) \left( \Phi^{-1}(E_k + i0) \right)_{12} \right|^2 + \frac{\sin^2(kR)}{k^2 R^2} \left( \Phi^{-1}(E_k + i0) \right)_{22}^2 .
\]

(3.12)

For the forward scattering, the differential cross section is plotted as a function of \( k \) for the below values of the parameters, as shown in Figs. 10a and 10b.

(a) Differential cross section versus \( k \) for \( \theta = 0, \lambda_2 = 5, \mu = 1 \) units.

(b) Differential cross section versus \( k \) for \( \theta = 0, \lambda_2 = 20, \mu = 1 \) units.

Figure 10: Differential cross section versus \( k \).

4 Small Deformations of a Circle

It would be interesting to ask how the bound state spectrum and scattering cross section for above (or similar) potentials change under small deformation of the support of the potentials. Let us first briefly define the normal deformations of a general curve in two dimensions. We consider a regular planar curve \( \Gamma \) parametrized with its arc length \( s \) with finite length. The Serret-Frenet equations for this curve are given by \( t = \frac{ds}{d\gamma}, \frac{dt}{ds} = \kappa n, \) and \( \frac{dn}{ds} = -\kappa t \), where \( t, n \) are the tangent and normal vectors and \( \kappa \) is the curvature of the curve \( \Gamma \). The small deformation along a normal direction of the curve \( \Gamma \) is defined by

\[
\tilde{\gamma}(s) = \gamma(s) + \epsilon h(s)n(s) ,
\]

(4.1)

where \( h \) is assumed to be a smooth function of \( s \). It is worth pointing out that \( \epsilon \) here is a small deformation parameter, not the same parameter used for regularization.

The length of the deformed curve \( \tilde{\Gamma} \) up to order \( \epsilon \) is given by

\[
L(\tilde{\Gamma}) = \int_0^L \frac{d\tilde{s}}{ds} ds = \int_0^L \left( \frac{d\gamma}{ds} \cdot \frac{d\tilde{\gamma}}{ds} \right)^{1/2} ds = \int_0^L \left( 1 - \epsilon \kappa(s)h(s) \right)^2 + \epsilon^2 \left( \frac{dh(s)}{ds} \right)^2 \right)^{1/2} ds
\]

\[= \int_0^L \left( 1 - \epsilon \kappa(s)h(s) + O(\epsilon^2) \right) ds = L(\Gamma) - \epsilon \int_0^L \kappa(s)h(s) ds + O(\epsilon^2) .
\]

(4.2)

If \( \Gamma \) is a circle of radius \( R, \kappa = 1/R \) so that

\[
L(\tilde{\Gamma}) = 2\pi R - \frac{\epsilon}{R^2} \int_0^L h(s) ds + O(\epsilon^2) .
\]

(4.3)

4.1 Perturbative First Order Calculation of the Bound State Energy

Since the support of the defect has codimension one, the renormalization is not required for this model and the resolvent of the Hamiltonian \( H \) associated with deformed circular defect potential can be found by using similar arguments summarized previously, as a result we find

\[
R(E) = R_0(E) \left( k |\tilde{\Gamma} \rangle \langle R_0(E) ,
\]

(4.4)
where we denote the deformation of the circle $\tilde{S}^1$ by $\tilde{\Gamma}$ for notational simplicity. For bound state, we need to calculate

$$\Phi(-\nu^2) = \frac{1}{\lambda} - \frac{1}{\lambda} \langle \tilde{\Gamma} | R_0(-\nu^2) | \tilde{\Gamma} \rangle = \frac{1}{\lambda} - \int_{\mathbb{R}^2} \frac{|(\tilde{\Gamma} | p)^2}{p^2 + \nu^2} \frac{d^2 p}{(2\pi)^2}. \quad (4.5)$$

Using

$$\langle \tilde{\Gamma} | p \rangle = \frac{1}{L(\tilde{\Gamma})} \int_0^L e^{ip\tilde{\gamma}(s)} |\tilde{\gamma}'(s)| ds,$$ \hspace{1cm} (4.6)

and expanding the exponential $e^{i \nu h(\theta)p_n(\theta)}$ in $\epsilon$ and the fact $|\tilde{\gamma}'(s)| = 1 - \nu h(s) + O(\epsilon^2)$, it is easy to show that

$$\Phi(-\nu^2) = \frac{1}{\lambda} - \frac{1}{\lambda} \left(1 + \frac{\epsilon}{\pi R} \int_0^{2\pi} h(\theta) d\theta \right) \left[ \int_{\mathbb{R}^2} \left( \int_0^{2\pi} \int_0^{2\pi} \frac{e^{i p \cdot (\gamma(\theta_1) - \gamma(\theta_2))}}{p^2 + \nu^2} \right) d\theta_1 d\theta_2 \right] \frac{d^2 p}{(2\pi)^2}$$

$$+ O(\epsilon^2). \quad (4.7)$$

Let us consider the first integral in the square bracket above:

$$\int_{\mathbb{R}^2} \left( \int_0^{2\pi} \int_0^{2\pi} \frac{e^{i p \cdot (\gamma(\theta_1) - \gamma(\theta_2))}}{p^2 + \nu^2} d\theta_1 d\theta_2 \right) \frac{d^2 p}{(2\pi)^2}. \quad (4.8)$$

The uniformly convergent plane wave expansion in two dimensions $\sum_{m=0}^\infty \epsilon^m e^{imn} J_m(p r) \cos(m\theta)$, \hspace{1cm} (4.9)

with $\epsilon_0 = 1$, $\epsilon_m = 2$ if $m > 0$, and $\theta$ being the angle between $p$ and $r$ helps us to compute the above integral with respect to the angle variables easily and left with the integration over the variable $p$ only:

$$(2\pi) \int_0^\infty \frac{J_n^2(pR)}{p^2 + \nu^2} p dp,$$ \hspace{1cm} (4.10)

where we have used $\int_0^{2\pi} \cos(m(\theta - \theta_k)) d\theta = 2\pi \delta_{mk}$. Thanks to the integral representation \hspace{1cm} $\int_0^\infty \frac{x}{x^2 + a^2} J_0(x) dx = I_0(a) K_0(a)$, \hspace{1cm} (4.11)

we find

$$\int_{\mathbb{R}^2} \left( \int_0^{2\pi} \int_0^{2\pi} \frac{e^{i p \cdot (\gamma(\theta_1) - \gamma(\theta_2))}}{p^2 + \nu^2} d\theta_1 d\theta_2 \right) \frac{d^2 p}{(2\pi)^2} = (2\pi) I_0(\nu R) K_0(\nu R). \quad (4.12)$$

For the second integral in equation \hspace{1cm} (4.7), \hspace{1cm} it is sufficient to consider

$$\int_{\mathbb{R}^2} \left( \int_0^{2\pi} \int_0^{2\pi} \frac{e^{i p \cdot (\gamma(\theta_1) - \gamma(\theta_2))}}{p^2 + \nu^2} h(\theta_1) d\theta_1 d\theta_2 \right) \frac{d^2 p}{(2\pi)^2}. \quad (4.13)$$

With the help of the plane wave expansion \hspace{1cm} (4.9) and the formula \hspace{1cm} (4.11), \hspace{1cm} the above integral becomes

$$I_0(\nu R) K_0(\nu R) \left( \int_S h(\theta) d\theta \right). \quad (4.14)$$

The last integral in \hspace{1cm} (4.7) \hspace{1cm} can be computed similarly by first rewriting the expression $i(p \cdot n(\theta))e^{iP \gamma(\theta)} = \frac{\partial}{\partial R} (e^{iP \gamma(\theta)})$ and $\frac{dJ_0(\nu)}{dx} = -J_1(x)$ we find

$$\int_{\mathbb{R}^2} \left( \int_0^{2\pi} \int_0^{2\pi} \frac{e^{i p \cdot (\gamma(\theta_1) - \gamma(\theta_2))}}{p^2 + \nu^2} i (p \cdot n(\theta_1)) h(\theta_1) d\theta_1 d\theta_2 \right) \frac{d^2 p}{(2\pi)^2}$$

$$= - \left( \int_S h(\theta) \right) \int_0^\infty J_0(pR) J_1(pR) \frac{p^2}{p^2 + \nu^2} dp. \quad (4.15)$$
Rewriting $\frac{p^2}{p^2 + \nu^2}$ as $1 - \frac{\nu^2}{p^2 + \nu^2}$, and using the formula (6.512) in [20]

$$\int_0^\infty J_\nu(\alpha x)J_{\nu-1}(\alpha x)dx = \frac{1}{2\alpha},$$

(4.16)

and the formula (6.577) in [20],

$$\int_0^\infty \frac{J_0(pR)J_1(pR)}{p^2 + \nu^2} dp = \frac{1}{\nu} I_1(\nu R) K_0(\nu R),$$

(4.17)

it follows that

$$\int_{\mathbb{R}^2} \left( \int_0^{2\pi} \int_0^{2\pi} \frac{\mathbf{p} \cdot \mathbf{n}(\theta_1) \mathbf{h}(\theta_1) d\theta_1 d\theta_2}{p^2 + \nu^2} \right) d^2 p = - \left( \frac{1}{2R} - \nu I_1(\nu R) K_0(\nu R) \right) \left( \int_0^{2\pi} h(\theta) d\theta \right).$$

(4.18)

After combining all the above results (4.12), (4.14), and (4.18), we finally obtain

$$\tilde{\Phi}(-\nu^2) = \frac{1}{\lambda} - \frac{1}{2\pi} I_0(\nu R) K_0(\nu R) + \epsilon \frac{1}{2\pi^2} \left( \frac{1}{2R} + \nu I_0(\nu R) K_1(\nu R) \right) \left( \int_0^{2\pi} h(\theta) d\theta \right) + O(\epsilon^2),$$

(4.19)

where we have used $I_1(x) K_0(x) + I_0(x) K_1(x) = 1/x$.

When there is no deformation ($\epsilon = 0$), we have only one bound state. This can be seen by simply expressing the second term $I_0(\nu R) K_0(\nu R)$ using its integral representation (4.11):

$$\frac{1}{\lambda} = \frac{1}{2\pi} I_0(\nu R) K_0(\nu R) = \frac{1}{2\pi} \int_0^\infty \frac{x}{x^2 + \nu^2 R^2} J_0^2(x) dx.$$

(4.20)

Then, by taking the derivative of the right hand side with respect to $\nu$ under the integral sign, it is easy to see that the right hand side of the above equation is a decreasing function of $\nu$ for given parameters $\lambda$ and $R$. Therefore, there is a unique solution, say $\nu_0$, to the above equation.

It is important to notice that deformations satisfying $\int_0^{2\pi} h(\theta) d\theta = 0$ do not change the bound state energies up to first order in $\epsilon$. Since we evaluate the deformation to order $\epsilon$ we can actually solve the bound state energy for the deformed curve to the same order. In [25, 37] we derived a general formula for perturbations of eigenvalues for small distortions of the principal matrix $\Phi$, here we have a one-dimensional version of this formula so we can use directly the expansion above. Let $\nu = \nu_0 + \epsilon \nu_1 + O(\epsilon^2)$, where $\nu_0$ denotes the solution to the original unperturbed circle case. Then, the bound state energy $E_B = -(\nu_0 + \epsilon \nu_1)^2$ for the deformed circular defect can be found by the zeroes of $\tilde{\Phi}$. This is achieved up to order $\epsilon$ by simply expanding its first term around $\nu_0$

$$\frac{1}{\lambda} - \frac{1}{2\pi} I_0((\nu_0 + \epsilon \nu_1) R) K_0((\nu_0 + \epsilon \nu_1) R) + \epsilon \frac{1}{2\pi^2} \left( \frac{1}{2R} + \nu_0 I_0(\nu_0 R) K_1(\nu_0 R) \right) \left( \int_0^{2\pi} h(\theta) d\theta \right) = 0,$$

(4.21)

and using the fact that the zeroth order term cancels out $\frac{1}{\lambda}$ above to get the solution $\nu_1$. Hence, we obtain an explicit formula for the bound state energy up to order $\epsilon$

$$E_B = -\nu_0^2 - \epsilon \frac{2\nu_0}{\pi R} \left( \frac{1}{I_1(\nu_0 R) K_0(\nu_0 R) - I_0(\nu_0 R) K_1(\nu_0 R)} \right) \left( \int_0^{2\pi} h(\theta) d\theta \right) + O(\epsilon^2),$$

(4.22)

which can be further simplified into

$$E_B = -\nu_0^2 - \epsilon \frac{\nu_0^2}{\pi R} \left( \int_0^{2\pi} h(\theta) d\theta \right) + O(\epsilon^2).$$

(4.23)

The simplicity of the first order result is remarkable, and hints at a geometric interpretation. Suppose that instead of the original circle with radius $R$ we replace the circle with a circle of radius $R - \epsilon R_1$ where $\epsilon R_1 = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{ch}(\theta) Rd\theta$ (note that the normal in the curvature description is inward). Because we are now using a delta function supported by a circle, we do have the same eigenvalue equation,

$$\frac{1}{\lambda} - \frac{1}{2\pi} I_0((\nu_0 + \epsilon \nu_1)(R - \epsilon R_1)) K_0((\nu_0 + \epsilon \nu_1)(R - \epsilon R_1)) = 0.$$

(4.24)

If we expand all the terms to order $\epsilon$, we find the relation $R \nu_1 = \nu_0 R_1$. By using $E_B = -(\nu_0 + \epsilon \nu_1)^2 = -\nu_0^2 - 2\nu_0 \nu_1$ we find exactly the above result. So we state this observation as,
Lemma 4. A small deformation in the normal direction of a given circle, which supports an attractive delta function, leads to a perturbation of the original bound state energy, to first order the resulting change can be obtained as follows: increase the initial radius by an amount equal to the average of the deformation over the given circle, then compute the first order perturbation of the bound state energy corresponding to this new circle with the same coupling constant.

Remark 1. It is tempting to push this to the second order and search for, if there is any, a geometric interpretation of the result. But the calculations are rather involved so we postpone it for future work. Note that the circle problem per se can be solved by elementary methods, that is by choosing polar coordinates at the center and writing the delta function along the radial direction. However, a general curve cannot be solved by this approach as there is no natural coordinate system to choose. In the case of a small deformation, one can think of delta potential supported on this curve as a delta function supported on the original circle plus a series of perturbations. This idea leads to, even to first order, a term of the form of

\[ \int_{R}^{2R} h(\theta) d\theta \]

and some additional ones coming from the change of arc-length as well as the change of total length. Here the derivative of delta function term is important since the wave function is of the form (disregarding the normalization)

\[ I_0(vR_0)K_0(vR_0)v(R - r) + I_0(Rv_0)K_0(rv_0)v(r - R), \]

and the usual first order perturbation of energy, which is found by evaluating the expectation value in the state of interest, leads to a divergence (here we need to use the symmetric choice for the theta function as often used in distribution theory).

The single bound state energy \( E_B \) for the original circular defect can numerically be plotted as a function of \( R \) with fixed values of \( \lambda \). For a particular deformation \( h(\theta) = \sin^2 \theta \), we can plot how the bound state energy \( E_B \) changes with respect to \( R \) numerically with the help of Mathematica, as shown in Fig. 11. For

![Figure 11: Bound state energy for the circular defect and for the deformed circular defect versus R with \( \epsilon = 0.1, \lambda = 10 \) units.](image)

4.2 Perturbative First Order Stationary Scattering Problem

The function \( \hat{\Phi} \) can be analytically continued onto the complex plane using (4.19) and \( \hat{\Phi}(E_k + i0) \) can be evaluated in terms of the variable \( k > 0 \)

\[ \hat{\Phi}(E_k + i0) = \frac{1}{\lambda} - \frac{i}{4} J_0(kR)H_1^{(1)}(kR) \]

\[ + \frac{\epsilon}{2\pi^2} \left( -\frac{1}{2R} + \frac{i\pi k}{2} J_0(kR)H_1^{(1)}(kR) \right) \left( \int_0^{2\pi} h(\theta) d\theta \right) + O(\epsilon^2). \]  (4.25)

Let \( \theta' \) be the angle between \( k' \) and \( k \), which is the momentum of the incoming particle chosen to be parallel to the \( x \) axis for simplicity. Then, we get

\[ \langle k' | \hat{\Gamma} | k \rangle = \left( 1 + \frac{\epsilon}{2\pi R} \int_0^{2\pi} h(\theta) d\theta \right) \left( J_0(kR) - \frac{\epsilon}{2\pi R} \int_0^{2\pi} e^{-ikR \cos(\theta - \theta')} h(\theta) d\theta \right) \]

\[ - \frac{ik\epsilon}{2\pi} \int_0^{2\pi} e^{-ikR \cos(\theta - \theta')} \cos(\theta - \theta') h(\theta) d\theta \]  (4.26)
1. Of arguments discussed above. The explicit form of the resolvent operator is given by
\[ \theta d \theta d \phi \sin H \]
where \( \theta \) and its surface area up to order \( \epsilon \) is a small deformation parameter,
\( S \) is the mean curvature of the sphere. To simplify the notation, we will write
\( \lambda \rightarrow k \).

2. Suppose that \( \tilde{\Sigma} \) is the small deformation of the sphere along its normal direction, defined by
\[ \tilde{\sigma}(\theta, \phi) := \sigma(\theta, \phi) + \epsilon h(\theta, \phi) N(\theta, \phi), \]
where \( \epsilon \) is a small deformation parameter, \( N \) is the normal vector field on the sphere, and \( h \) is a smooth function on the sphere. If \( |\epsilon| \) is sufficiently small, it is well-known that the deformed sphere \( \Sigma \) is a regular surface and its surface area up to order \( \epsilon \) is given by
\[ A(\tilde{\Sigma}) = A(\Sigma) - 2 \epsilon \int_0^{2\pi} \int_0^\pi H(\theta, \phi) h(\theta, \phi) R^2 \sin \theta d\theta d\phi + O(\epsilon^2), \]
where \( H = 1/R \) is the mean curvature of the sphere. To simplify the notation, we will write \( d\Omega \) instead of \( \sin \theta d\theta d\phi \), and \( \Omega \) as the argument of the functions on the sphere.

The differential cross sections as a function of \( k \) for the circular defect and deformed circular defect for a particular deformation \( h(\theta) = \sin^2 \theta \) is plotted in Fig. 12.

5. Small Deformations of a Sphere

We consider a particular regular surface, a sphere \( S^2 \) centered at the origin with radius \( R \). Let \( \sigma : (0, \pi) \times (0, 2\pi) \rightarrow S^2 \) be a local chart, given by (3.3). Suppose that \( \tilde{\Sigma} \) is the small deformation of the sphere along its normal direction, defined by
\[ \tilde{\sigma}(\theta, \phi) := \sigma(\theta, \phi) + \epsilon h(\theta, \phi) N(\theta, \phi), \]
where \( \epsilon \) is a small deformation parameter, \( N \) is the normal vector field on the sphere, and \( h \) is a smooth function on the sphere. If \( |\epsilon| \) is sufficiently small, it is well-known that the deformed sphere \( \Sigma \) is a regular surface and its surface area up to order \( \epsilon \) is given by
\[ A(\tilde{\Sigma}) = A(\Sigma) - 2 \epsilon \int_0^{2\pi} \int_0^\pi H(\theta, \phi) h(\theta, \phi) R^2 \sin \theta d\theta d\phi + O(\epsilon^2), \]
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where \( H = 1/R \) is the mean curvature of the sphere. To simplify the notation, we will write \( d\Omega \) instead of \( \sin \theta d\theta d\phi \), and \( \Omega \) as the argument of the functions on the sphere.

The differential cross sections as a function of \( k \) for the circular defect and deformed circular defect for a particular deformation \( h(\theta) = \sin^2 \theta \) is plotted in Fig. 12.
where
\[ \tilde{\Phi}(E) = \frac{1}{\lambda} - \langle \tilde{\Sigma} | R_0(E) | \tilde{\Sigma} \rangle. \] (5.4)

### 5.1 Perturbative First Order Calculation of the Bound State Energy

For this part, we assume that the sphere problem has a bound state solution. We will choose \( E = -\nu^2 \), as we will be interested in a bound state to begin with. If we use the realization in the Fourier domain, the resolvent kernel is given by

\[ R_0(r, r') - \nu^2 = \int_{\mathbb{R}^3} \frac{e^{i\mathbf{p} \cdot (r-r')}}{p^2 + \nu^2} \left( \frac{d^3p}{(2\pi)^3} \right) \] (5.5)

Our aim is to calculate the function \( \tilde{\Phi}(-\nu^2) \) up to order \( \epsilon \). Using (5.5) and expanding the terms up to order \( \epsilon \), we have

\[ \tilde{\Phi}(-\nu^2) = \frac{1}{\lambda} - \frac{1}{(4\pi)^2} \left( 1 + \frac{\epsilon R}{\pi} \int S^2 h(\Omega) \, d\Omega \right) \times \int_{S^2 \times S^2} R_0(\tilde{\sigma}(\Omega), \tilde{\sigma}(\Omega')) - \nu^2 \) \left( 1 - \frac{2\epsilon R}{\pi} (h(\Omega) + h(\Omega')) \right) \, d\Omega \, d\Omega' + O(\epsilon^2). \] (5.6)

The resolvent kernel up to order \( \epsilon \) can be calculated using (5.5) and (5.6).

Substituting this into (5.6), and keeping the first order terms in \( \epsilon \) for the surface integrals of the resolvent kernel, we obtain

\[ \tilde{\Phi}(-\nu^2) = \frac{1}{\lambda} - \frac{1}{(4\pi)^2} \left( 1 + \frac{\epsilon R}{\pi} \int S^2 h(\Omega) \, d\Omega \right) \left[ \int_{\mathbb{R}^3} \left( \int_{S^2 \times S^2} e^{i\mathbf{p} \cdot (\sigma(\Omega) - \sigma(\Omega'))} \, d\Omega \, d\Omega' \right) \right] \times \frac{d^3p}{p^2 + \nu^2} \left( \frac{d^3p}{(2\pi)^3} \right) \]

\[ \left. + \frac{1}{R} \int_{\mathbb{R}^3} \left( \int_{S^2 \times S^2} e^{i\mathbf{p} \cdot (\sigma(\Omega) - \sigma(\Omega'))} h(\Omega) \, d\Omega \, d\Omega' \right) \frac{1}{p^2 + \nu^2} \frac{d^3p}{(2\pi)^3} \right] + O(\epsilon^2). \] (5.7)

We have already computed the above first integral in evaluating the second diagonal element of the matrix \( \Phi \) in equation (3.9), and the result can be expressed as

\[ \langle \Sigma | R_0(-\nu^2) | \Sigma \rangle = \frac{1}{(4\pi)^2} \int_{\mathbb{R}^3} \left( \int_{S^2 \times S^2} e^{i\mathbf{p} \cdot (\sigma(\Omega) - \sigma(\Omega'))} \, d\Omega \, d\Omega' \right) \frac{1}{p^2 + \nu^2} \frac{d^3p}{(2\pi)^3} \]

\[ = \frac{1}{4\pi R} K_{1/2}(\nu R) I_{1/2}(\nu R). \] (5.9)

For the second integral, we will use the identity \( (i \mathbf{p} \cdot \mathbf{N}(\Omega)) e^{i\mathbf{p} \cdot \sigma(\Omega)} = \frac{\partial}{\partial R} e^{i\mathbf{p} \cdot \sigma(\Omega)} \). The exponential factors can be expressed in terms of the spherical Bessel functions of first kind and spherical harmonics using the well-known expansion of the plane waves into spherical harmonics (5.9):

\[ e^{i\mathbf{p} \cdot \sigma(\Omega)} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^l j_l(pR) Y_{lm}(\Omega_p) Y_{lm}(\Omega). \] (5.10)

Here \( \Omega_p \) and \( \Omega \) are the polar angles of the vector \( \mathbf{p} \) and \( \sigma \), respectively. Hence, we obtain

\[ \int_{\mathbb{R}^3} \left( \int_{S^2 \times S^2} e^{i\mathbf{p} \cdot (\sigma(\Omega) - \sigma(\Omega'))} (i \mathbf{p} \cdot \mathbf{N}(\Omega)) h(\Omega) \, d\Omega \, d\Omega' \right) \frac{1}{p^2 + \nu^2} \frac{d^3p}{(2\pi)^3} \]

\[ = (4\pi)^2 \int_0^\infty \int_{S^2} \left( \int_{S^2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^l j_l(pR) Y_{lm}(\Omega_p) Y_{lm}(\Omega) h(\Omega) \right) \frac{1}{p^2 + \nu^2} \frac{d^3p}{(2\pi)^3} \]

\[ \times \sum_{l'=0}^{\infty} \sum_{m'=0}^{l'} (-i)^l \frac{\partial^{l'} (pR)}{\partial R^l} Y_{l'm'}(\Omega_p) Y_{l'm'}(\Omega) \, d\Omega \, d\Omega' \, d\Omega \, d\Omega' \right) \frac{d\Omega_p \, dp \, d\Omega' \, d\Omega}{(2\pi)^3}. \] (5.11)
By the orthonormality of the spherical harmonics \( \int_{S^2} Y_{lm}^{\dagger}(\Omega)Y_{l'm'}(\Omega) d\Omega = \delta_{ll'}\delta_{mm'} \), integrations over \( \Omega \) lead to
\[
\int_{\mathbb{R}^3} \left( \int_{S^2 \times S^2} e^{i \mathbf{p} \cdot (\sigma(\Omega) - \sigma(\Omega'))} (\mathbf{i} \cdot \mathbf{N}(\Omega)) h(\Omega) d\Omega d\Omega' \right) \frac{1}{p^2 + \nu^2} \frac{d^3 p}{(2\pi)^3} = \frac{(4\pi)^2}{(2\pi)^3} \int_0^\infty j_0(pR)(-j_1(pR)) \frac{p^3}{p^2 + \nu^2} dp \left( \int_{S^2} h(\Omega) d\Omega \right),
\]
where we have used \( Y_{00}(\Omega) = 1/\sqrt{4\pi} \) and the relation \( \frac{d\ln(x)}{dx} = -j_1(x) \). We now use \( j_1(x) = \sqrt{\frac{2}{\pi x}} J_{1/2}(x) \) and decompose \( \frac{p^3}{p^2 + \nu^2} \) as \( 1 - \frac{\nu^2}{p^2} \) together with the formulas (6.512) and (6.577) in [26] for the integrals of the Bessel functions.
\[
\int_0^\infty J_{1/2}(pR)J_{3/2}(pR) dp = \frac{1}{2R},
\]
\[
\int_0^\infty J_{3/2}(pR)J_{1/2}(pR) \frac{dp}{p^2 + \nu^2} = \frac{1}{\nu} I_{3/2}(\nu R)K_{1/2}(\nu R),
\]
to get
\[
\int_{\mathbb{R}^3} \left( \int_{S^2 \times S^2} e^{i \mathbf{p} \cdot (\sigma(\Omega) - \sigma(\Omega'))} (\mathbf{i} \cdot \mathbf{N}(\Omega)) h(\Omega) d\Omega d\Omega' \right) \frac{1}{p^2 + \nu^2} \frac{d^3 p}{(2\pi)^3} = -\frac{1}{R} \left( \int_{S^2} h(\Omega) d\Omega \right) \left( \frac{1}{2R} - \nu K_{1/2}(\nu R)I_{3/2}(\nu R) \right).
\]
By applying similar arguments, we can find easily for the last integral
\[
\int_{\mathbb{R}^3} \left( \int_{S^2 \times S^2} e^{i k \cdot (\sigma(\Omega) - \sigma(\Omega'))} h(\Omega) d\Omega d\Omega' \right) \frac{1}{k^2 + \nu^2} \frac{d^3 k}{(2\pi)^3} = \frac{2}{R} K_{1/2}(\nu R)I_{1/2}(\nu R) \left( \int_{S^2} h(\Omega) d\Omega \right)
\]
Combining all these results (5.10), (5.15) and (5.16), we obtain
\[
\Phi(-\nu^2) = \frac{1}{\lambda} - \frac{1}{4\pi R} I_{1/2}(\nu R)K_{1/2}(\nu R) + \frac{\epsilon}{8\pi^2 R} \left( \frac{1}{2R} + \nu I_{1/2}(\nu R)K_{3/2}(\nu R) \right) \left( \int_{S^2} h(\Omega) d\Omega \right),
\]
where we have used \( I_{1/2}(\nu R)K_{1/2}(\nu R) + I_{3/2}(\nu R)K_{3/2}(\nu R) = 1/\nu \).

It is important to notice that the formula for the function \( \Phi \) is very similar to the one obtained for the deformed circular defect case, however there is a difference. The eigenvalue flow can be obtained again by writing \( I_{1/2}(\nu R)K_{1/2}(\nu R) \) as (from the formula (6.577) in [26]):
\[
\frac{1}{\lambda} = \frac{1}{4\pi R} I_{1/2}(\nu R)K_{1/2}(\nu R) = \frac{1}{4\pi R} \int_0^\infty \frac{x}{x^2 + \nu^2 R^2} J_{1/2}^2(x) dx
\]
As one can see, the right hand side of the above equation is a decreasing function of \( \nu \) for given parameters \( \lambda \) and \( R \). Yet the product \( I_{1/2}(\nu R)K_{1/2}(\nu R) \) is finite as \( \nu \to 0^+ \), so there may not always be a solution if \( \lambda \) is small enough. If there is a solution then it is unique, say \( \nu_0 \), to the above equation. We assume that this is the case.

Let \( \nu = \nu_0 + \epsilon \nu_1 + O(\epsilon^2) \), then the bound state energy up to order \( \epsilon \) can be found by solving the zeroes of \( \Phi \) by expanding terms around \( \nu = \nu_0 \). Hence, we find
\[
E_B = -\nu_0^2 - \epsilon \nu_0^2 \left( \frac{1}{\nu_0 R} - I_{1/2}(\nu_0 R)K_{3/2}(\nu_0 R) - I_{3/2}(\nu_0 R)K_{1/2}(\nu_0 R) \right) \times \left( \frac{1}{\pi R} \int_{S^2} h(\Omega) d\Omega \right) + O(\epsilon^2).
\]
Not surprisingly, this result has the same geometric interpretation as in the case of circle, we replace the original sphere with another sphere of slightly different radius \( R - \epsilon R_1 \), with \( R_1 = \frac{1}{\nu_0 R} \int_{S^2} h(\Omega) R^2 d\Omega \) and then look for the small change in the energy because of this alteration, as a result of this computation, we recover the above expression. Hence,
Lemma 5. A small deformation in the normal direction of a given sphere, which supports an attractive delta function, leads to a perturbation of the original bound state energy, to first order the resulting change can be obtained as follows: increase the initial radius by an amount equal to the average of the deformation over the given sphere, then compute the first order perturbation of the bound state energy corresponding to this new sphere with the same coupling constant.

For a particular deformation \( h(\theta) = \sin \theta \), one can numerically plot how the bound state energies change with respect to \( R \) for a given \( \lambda \), as shown in Fig. 13.

![Figure 13: Bound state energy for the spherical defect and for the deformed spherical defect (red curve) versus \( R \) with \( \epsilon = 0.1 \), \( \lambda = 10 \) units.](image)

### 5.2 Perturbative First Order Stationary Scattering Problem

For the deformed spherical defect, the function \( \Phi \) can be analytically continued onto the complex plane using (5.17) and \( \Phi(E_k + i0) \) can then be evaluated in terms of the variable \( k > 0 \)

\[
\Phi(E_k + i0) = \frac{1}{\lambda} - \frac{i}{8R} J_{1/2}(kR) H^{(1)}_{1/2}(kR) + \frac{\epsilon}{8\pi^2 R} \left(-\frac{1}{2R} + \frac{i\pi k}{2} J_{1/2}(kR) H^{(1)}_{3/2}(kR) \right) \left( \int_{S^2} h(\Omega) d\Omega \right) + O(\epsilon^2). \tag{5.20}
\]

For the scattering amplitude, we need to find the expression \( \langle \tilde{\Sigma} | k \rangle \) in terms of the deformation function \( h(\Omega) \):

\[
\langle \tilde{\Sigma} | k \rangle = \frac{1}{A(\tilde{\Sigma})} \int_{S^2} e^{ik\cdot\tilde{\sigma}(\Omega)} R^2 \left(1 - \frac{2\epsilon}{R} h(\Omega) \right) d\Omega \tag{5.21}
\]

By expanding the exponential \( e^{i\epsilon h(\Omega) k \cdot N(\Omega)} \) in \( \epsilon \) and expanding \( A(\tilde{\Sigma}) \) in \( \epsilon \) from the formula (5.2), it is easy to show that

\[
\langle \tilde{\Sigma} | k \rangle = \left(1 + \frac{\epsilon}{2\pi R} \int_{S^2} h(\Omega) d\Omega \right) \left( \frac{\sin(kR)}{kR} + \frac{\epsilon}{2\pi R} \int_{S^2} e^{ik\sigma(\Omega)} h(\Omega) d\Omega \right)
+ \frac{i\epsilon}{4\pi} \int_{S^2} e^{ik\sigma(\Omega)} (k \cdot N(\Omega)) h(\Omega) d\Omega \right) + O(\epsilon^2). \tag{5.22}
\]

For simplicity, we consider a particular class of deformations, where \( h(\Omega) = h(\theta) \). In this case, let \( \theta' \) be the angle between \( k' \) and \( k \), which is the momentum of the incoming particle chosen to be parallel to the \( z \) axis.
Then, we get the explicit expression for the scattering amplitude for a given deformation $h$, given by

$$\tilde{f}(k \to k') = -\frac{1}{4\pi} (k'|\tilde{\Sigma}|(\tilde{\Phi}(E_k + i0))^{-1}|\tilde{\Sigma}|k)$$

$$= -\frac{1}{4\pi} \left[ 1 + \frac{\epsilon}{R} \int_0^\pi h(\theta) \sin \theta d\theta \right] \left( \frac{\sin kR}{kR} - \frac{\epsilon}{R} \int_0^\pi e^{-ikR \cos(\theta - \theta')} h(\theta) \sin \theta d\theta \right. $$

$$\left. - \frac{i k \epsilon}{2} \int_0^\pi e^{-ikR \cos(\theta - \theta')} \cos(\theta - \theta') h(\theta) \sin \theta d\theta \right) + O(\epsilon^2)$$

$$\times \left[ \frac{1}{\lambda} \frac{i}{8R} J_{1/2}(kR) H_{1/2}^{(1)}(kR) + \frac{\epsilon}{4\pi R} \left( - \frac{1}{2R} + \frac{i \pi k}{2} J_{1/2}(kR) H_{1/2}^{(1)}(kR) \right) \left( \int_0^\pi h(\theta) \sin \theta d\theta \right) + O(\epsilon^2) \right]^{-1} \times \left[ 1 + \frac{\epsilon}{R} \int_0^\pi h(\theta) \sin \theta d\theta \right] \left( \frac{\sin kR}{kR} - \frac{\epsilon}{R} \int_0^\pi e^{ikR \cos(\theta)} h(\theta) \sin \theta d\theta \right. $$

$$\left. + \frac{i k \epsilon}{2} \int_0^\pi e^{ikR \cos(\theta)} \cos(\theta) \sin \theta h(\theta) d\theta \right) + O(\epsilon^2) \right). \quad (5.23)$$

The differential cross sections as a function of $k$ for the spherical defect and deformed spherical defect for a particular deformation $\psi(\theta) = \sin \theta$ is plotted in Fig. 14.

**Appendix A: Trotter-Kato Theorem**

This is a slightly different version of Trotter-Kato theorem than typically found in the literature, as stated also in [29]:

**Theorem 7.** Suppose that $H_n$ be a sequence of self-adjoint operators with resolvents $R_n(E) = (H_n - E)^{-1}$ defined for all complex numbers $E$ except a closed proper subset $U$ of $\mathbb{R}$. Furthermore, assume that $R_n(E)$ converges strongly for some $E \not\in U$ and this limit is invertible. Then, there exists a self-adjoint operator $H$ with resolvents $R(E) = (H - E)^{-1}$ such that $R_n(E)$ converges strongly to $R(E)$ for all complex numbers $E \not\in U$.

The idea of the proof is essentially the same as the original Trotter-Kato theorem. In our problem, we choose a sequence $\epsilon_n = 1/n$. If $n$ is sufficiently large $\det \Phi(\epsilon, E) \neq 0$ if $E$ satisfies $\det \Phi(E) \neq 0$. Then, $R(\epsilon_n, E)$ is defined for all complex $E$ except a closed proper subset $U$ of $\mathbb{R}$, namely $U = \{E \in [0, \infty) : \det \Phi(E) = 0\}$. Since we have shown that $R(\epsilon_n, E)$ converges strongly for some $E \not\in U$ (e.g., choose $E$ to be sufficiently large negative real number) and the limit is invertible, we conclude that there exists a self-adjoint operator $H$ with resolvents $R(E) = (H - E)^{-1}$ such that $R(\epsilon_n, E)$ converges strongly to $R(E)$ for all complex numbers $E \not\in U$ thanks to the above theorem.
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