Transferring entangled states through spin chains by boundary-state multiplets

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Quantum spin chains may be used to transfer quantum states between elements of a quantum information processing device. A scheme discovered recently [Phys. Rev. A 85, 022312 (2012)] was shown to have favorable transfer properties for single-qubit states even in the presence of built-in static disorder caused by manufacturing errors. We extend that scheme in a way suggested already in Bruderer et al. [Phys. Rev. A 85, 022312 (2012)] and study the transfer of the four Bell states which form a maximally entangled basis in the two-qubit Hilbert space. We show that perfect transfer of all four Bell states separately and of arbitrary linear combinations may be achieved for chains with hundreds of spins. For simplicity we restrict ourselves to systems without disorder.

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Quantum information processing [1] relies on quantum bits and quantum gates between which information must be transferred, for example, by quantum spin chains, that is, linear arrays of coupled qubits. Research on quantum information transfer by spin chains started roughly a decade ago [2] and has developed into a highly active field [3]. However, most of the research up to now has focused on the transfer of single-qubit states, although entangled multiqubit states are crucial in most quantum algorithms. In this Brief Report we show how a natural extension of a single-qubit state transfer protocol [4] can be used to achieve high-fidelity transmission of arbitrary pure two-qubit states along spin chains with up to hundreds of sites.

Two main classes of spin chains for quantum information transfer may be distinguished by the degree of “engineering,” or fine-tuning, of the nearest-neighbor couplings along the chain. Spin chains showing perfect state transfer (PST) have a spatial symmetry and a spectrum with commensurate energy differences [5–8]. To achieve that, the chains must be “fully engineered,” that is, all nearest-neighbor couplings must be tuned to specific values. In contrast, good (although not perfect) state transfer is achieved by modifying only the nearest-neighbor couplings along the chain. Spin chains showing perfect state transfer (PST) have a spectrum (2), the boundary-dominated approach to solving inverse eigenvalue problems; here we follow Ref. [4] for details. A prominent example [5] is given by $J_i = \sqrt{i(N-i)}$, leading to an equidistant ladder of $\varepsilon_\nu$ values. However, it is even possible to prescribe a set of eigenvalues $\varepsilon_\nu$ and find the corresponding couplings $J_i$ by solving a structured inverse eigenvalue problem [14,15]. There are several algorithms for solving inverse eigenvalue problems; here we follow Ref. [4] in using an algorithm by de Boor and Golub [16]. A PST chain with particularly benign properties [4] is defined by the “inverted quadratic spectrum,”

$$\varepsilon_\nu = \nu(N-1-|\nu|), \quad \nu = -\frac{N-1}{2}, \ldots, \frac{N-1}{2}.$$

Note that the differences between successive $\varepsilon_\nu$’s are largest close to the center of the spectrum, i.e., to $\nu = 0$, for odd $N$, the case on which we concentrate from now on. The spectrum (2) superficially resembles the cosine-shaped spectrum of the homogeneous ($J_i \equiv J$) XX chain; the couplings $J_i$ corresponding to (2), however, are roughly constant only in the central region of the chain, oscillating significantly towards the boundaries [4].

In contrast to the fully engineered approach exemplified by the spectrum (2), the boundary-dominated approach to quantum state transfer employs a simple pattern of couplings,

$$J_1 = J_{N-1} = \alpha J, \quad J_i = J$$

for $i \neq 1, N-1$. (3)
where $\alpha < 1$ is an adjustable parameter. For small $\alpha$ a perturbation calculation shows three closely spaced energy eigenvalues close to the center of the spectrum ($N$ is odd); the corresponding eigenvectors are dominantly localized on the boundary sites and thus can be used to transfer information back and forth between the ends of the chain. As $\alpha$ gets smaller, the influence of the interior spins decreases, and the fidelity of the state transfer increases, but so does the transfer time which is inversely proportional to the energy splitting between the states of the dominant triplet.

The combination [4] of PST and boundary-dominated state transfer rests on the following key observation: A PST chain with spectrum (2) may be equipped with a triplet of closely spaced energies by simply "contracting" it towards the center:

$$\epsilon'_\nu = \epsilon_\nu - (N-3) \text{sgn} \epsilon_\nu \quad (4)$$

($\text{sgn} 0$ is to be interpreted as zero). The new spectrum $\epsilon'_\nu$ is still commensurate, ensuring PST, but the triplet of states with energies close to zero are strongly boundary dominated as with the couplings (3) for small $\alpha$. As a bonus, the temporal behavior of the probability to collect the transmitted state at the receiving end of the chain changes from an extremely spiky shape with a needlelike maximum for the original spectrum (2) to a single broad and smooth maximum, thus making it much easier to measure the transmitted state at the right instant. This feature can be intuitively understood from the fact that the states dominating the transfer involve only small energy differences, i.e., long time scales.

The state transfer scheme just discussed can be extended quite naturally as already suggested in Ref. [4]: Collecting additional closely spaced eigenvalues in the center of the spectrum generates boundary states on more sites. A contraction, such as (4) with $(N-5)$ in place of $(N-3)$ and acting on all levels except the previously created triplet, generates a quintuplet of levels with unit spacing. A look at the coupling-constant distribution shows that the region of nearly constant $J_i$ around $i = \frac{N-1}{2}$ shrinks with every contraction step, giving way to oscillatory behavior near the ends of the chain.

The spatial structure of the energy eigenstates is shown in Fig. 1. The states of the quintuplet show large weights on the two first and last sites of the chain, respectively, whereas all other states have negligible weight there.

We study the transfer properties of the system for the set of Bell states, which form a maximally entangled basis in the two-spin Hilbert space,

$$|\psi_1\rangle_\pm = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle),$$
$$|\psi_2\rangle_\pm = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle). \quad (5)$$

In terms of Jordan-Wigner fermions, $|\psi_1\rangle_\pm$ belongs to the one-particle subspace of the full chain Hamiltonian (1), whereas $|\psi_2\rangle_\pm$ contains two-particle and zero-particle components, respectively, the latter having trivial dynamics. We denote by $|\phi_i\rangle$ the state with the first two spins in one of the states (5) with all other spins down and by $|\phi_f\rangle$ the spatial mirror image of $|\phi_i\rangle$. Then, a convenient measure for the fidelity of transmission is

$$|f(t)| = |\langle \phi_f | e^{-iHt} | \phi_i \rangle|. \quad (6)$$

Figure 2 shows $|f(t)|$ for the Bell states $|\psi_1\rangle_\pm$ with the $N = 31$ chain discussed above. ($|f(t)|$) is identical for the two states. Since all energies (2) and (4) are integers, perfect transfer occurs at time $t = \pi$; however, small high-frequency oscillations are visible. They are due to higher-energy states. They become stronger for longer chains, making PST difficult beyond $N \approx 60$ because the fidelity maximum at $t = \pi$ becomes extremely narrow. This can be remedied by another contraction of the spectrum. Subtracting $\Delta \text{ sgn} \epsilon_\nu$ with $\Delta = 60$ from all energies outside the quintuplet leads to the spectrum $\epsilon_\nu = 0, \pm 1, \pm 2, \pm 7, \pm 70, \ldots$ for the $N = 71$ chain, which displays a smooth fidelity vs time curve, see Fig. 3. The corresponding eigenvectors are shown in Fig. 4.

Comparison to Fig. 1 shows that the quintuplet of boundary-dominated states is on its way to develop into a septuplet. Furthermore, the quintuplet contains almost the complete weight of states localized on the first two lattice sites. This is definitely different for $\Delta = 0$ (not shown here) where the...
position eigenstate on site 2 contains significant weight from energy eigenstates close to the boundaries of the spectrum. This is what causes small high-frequency oscillations in the transfer fidelity of the single-qubit state initially located at site 2, and consequently, of the Bell states $|\psi_1\rangle_\pm$ as shown in Fig. 2 for $N = 31$.

The recipe just demonstrated for $N = 71$ is also applicable for longer chains. For $N \approx 200$ the optimal contraction parameter turns out to be $\Delta = N - 7$, meaning that the quintuplet of levels with unit spacing has grown into a septuplet. Continuation of the process generates a nonuplet of states which, for example, leads to decent behavior for $N = 321$ as shown in Fig. 5.

The Bell states $|\psi_2\rangle_\pm$ involve the dynamics of two-particle states. Since the vacuum component of those states is trivial, the transfer fidelities of $|\psi_2\rangle_\pm$ are equal. It turns out that the fidelity of $|\psi_2\rangle_\pm$ is slightly more delicate than that of $|\psi_1\rangle_\pm$ with narrower maxima and stronger oscillations. Nevertheless, the configuration with $N = 71$ and additional contraction by $\Delta = 60$ yields similar excellent transfer properties for states $|\psi_1\rangle_\pm$ (Fig. 3) and $|\psi_2\rangle_\pm$ (Fig. 6).

Since all Bell states can be transferred perfectly, one is tempted to conclude that PST is possible for arbitrary two-spin states, the Bell states forming a basis in the two-spin Hilbert space. However, this is not so due to the presence of different phase factors [17]. We discuss those in terms of Jordan-Wigner fermions. Phase differences occur between states with different numbers of up spins (Jordan-Wigner fermions). Two mechanisms are responsible for these phase differences. First, the perfect transfer of every up spin is accompanied by a fixed phase shift, and second, sign changes occur due to Fermi statistics.

The phase $\varphi$ involved in the PST of single-spin states [5–8] is defined by

$$e^{-iH_\tau}|i\rangle = e^{i\varphi}|N + 1 - i\rangle. \quad (7)$$

Here, $|i\rangle$ denotes the state with a single up spin at site $i$ and down spins everywhere else; $\tau$ is the perfect transfer time. The phase $\varphi$ does not depend on the site index $i$; for the PST chain with $J_i = \sqrt{i(N - i)}$ it is [18]

$$\varphi = -\frac{\pi}{2}(N - 1). \quad (8)$$

Equation (8) also holds for general PST chains of odd length $N$ by the general properties of the eigenvectors and eigenvalues of the tridiagonal single-particle Hamiltonian matrix. For even $N$ two cases have to be distinguished. For all PST chains, differences between neighboring single-particle energies must be odd numbers (in appropriate units). Thus, for even $N$ the two smallest (in absolute value) eigenvalues must be $\pm|l + \frac{1}{2}|$ with some integer $l$, and (8) changes to

$$\varphi = -\frac{\pi}{2}(N - 1) - \pi l. \quad (9)$$

![FIG. 3. Same as Fig. 2, for a chain with a quintuplet of closely spaced energies and $N = 71$ and an additional contraction by $\Delta = 60$.](image)

![FIG. 4. (Color online) Same as Fig. 1 for a chain with a quintuplet of closely spaced energies and $N = 71$ and an additional contraction by $\Delta = 60$. Seven states in the center of the energy spectrum are concentrated near the boundaries, whereas all other states extend through the whole system.](image)

![FIG. 5. Same as Fig. 2 for a chain with a nonuplet of closely spaced energies and $N = 321$.](image)

![FIG. 6. Same as Fig. 3 for the Bell states $|\psi_2\rangle_\pm$.](image)
The single-particle phase $\varphi$ may be adjusted by adding a constant magnetic field in the $z$ direction, corresponding to a nonzero constant diagonal in the Hamiltonian matrix.

A two-particle state with up spins at sites $i$ and $j > i$ is equivalent to a two-fermion Fock state,

$$|i, j\rangle = c_{i}^\dagger c_{j}^\dagger |0\rangle,$$

where $|0\rangle$ is the vacuum (all spins down) state. The PST property (7) in combination with Fermi statistics determines the time evolution of that state,

$$e^{-iH\tau}|i, j\rangle = e^{-iH\tau}c_{i}^\dagger c_{j}^\dagger |0\rangle = e^{i2\varphi}c_{N+1-i}^\dagger c_{N+1-j}^\dagger |0\rangle = e^{i(2\varphi+\pi)}|N+1-j, N+1-i\rangle.$$

The generalization to a state with $n$ fermions is straightforward; the phase picked up during PST then is

$$\phi_n = n\varphi + (n-1)\pi/2.$$

For the Bell states this means that $|\varphi_1\rangle_\pm$ and $|\varphi_2\rangle_\pm$ are transferred with different phase factors, even if $\varphi = 0$(mod 2$\pi$) can be achieved by choosing $N = 4k+1$ for example. Furthermore, if the spins at sites 3 through $N$ are not initialized to the down state, particle-number-dependent phase factors will mix up the transferred state.

However, if initialization is possible, there is a protocol that achieves PST with equal phases for all Bell states. The single-particle phase should be adjusted to $\varphi = \frac{\pi}{2}$ (either by making $N$ even or by an external field). Then $\phi_n$ is effectively zero for even particle number $n$ and $\frac{\pi}{2}$ for odd $n$. The first two sites then are initialized to an arbitrary superposition of the four Bell states, whereas all other sites stay initialized to the down state. Subsequently two controlled-X gates are applied with qubits 1 and 2 as the control and (say) qubit 3 as target qubits: $CX(1,3)CX(2,3)$. This creates a superposition of states with only even total particle numbers, which can be transferred without additional phase factors.

To summarize, we have extended the quantum information transfer scheme of Ref. [4] from single-qubit states to arbitrary two-qubit states for chains of up to hundreds of spins in the absence of built-in disorder. Phase factors causing unwanted interference between states containing even and odd numbers of up spins can be compensated for. The transfer of two-qubit states is achieved by creating a quintuplet of closely spaced equidistant energy eigenvalues in the center of the spectrum. Additional manipulations of the energy spectrum may be used to transfer Bell states along progressively longer chains. These additional modifications essentially create higher multiplets of energy eigenstates which are concentrated on more boundary sites and thus may in addition serve to transfer states of three or more qubits. The protocol compensating for unwanted phase factors may be naturally extended to states involving three or more spins.

The role of disorder (random perturbations in the couplings $J_i$) has been studied [4] for single-spin state transfer between sites 1 and $N$, showing the robustness of the modified inverse quadratic energy spectrum. The analysis from Ref. [4] will also apply to the transfer of single-spin states from site 2 to site $N-1$, provided the energy spectrum is modified so as to generate eigenstates localized predominantly not only on one, but on two boundary sites at each end of the chain. Since spin states are mapped to noninteracting fermion states, the only many-body effect to be expected is the fermionic sign change in Eq. (11). That sign change, however, does not depend on the presence or absence of randomness in the couplings. Consequently, coupling randomness will affect the state transfer only on the single-particle level already studied earlier [4].

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