MINI-MAX THEORY, SPECTRAL INVARIANTS AND GEOMETRY OF THE HAMILTONIAN Diffeomorphism Group

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Abstract. In this paper, we first develop a mini-max theory of the action functional over the semi-infinite cycles via the chain level Floer homology theory and construct spectral invariants of Hamiltonian diffeomorphisms on arbitrary compact symplectic manifold \((M, \omega)\). To each given time dependent Hamiltonian function \(H\) and quantum cohomology class \(0 \neq a \in \mathbb{Q}H^*(M)\), we associate an invariant \(\rho(H; a)\) which varies continuously over \(H\) in the \(C^0\)-topology. This is obtained as the mini-max value over the semi-infinite cycles whose homology class is ‘dual’ to the given quantum cohomology class \(a\) on the covering space \(\tilde{\Omega}_0(M)\) of the contractible loop space \(\Omega_0(M)\). We call them the Novikov cycles. We then use the spectral invariants to construct a new invariant norm on the Hamiltonian diffeomorphism group and a partial order on the set of time-dependent Hamiltonian functions of arbitrary compact symplectic manifolds. As some applications, we obtain a new lower bound of the Hofer norm of non-degenerate Hamiltonian diffeomorphisms in terms of the symplectic area of certain pseudo-holomorphic curves and prove the semi-global \(C^1\)-flatness of the Hofer norm.

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§1. Introduction and the main results

The group $\text{Ham}(M,\omega)$ of (compactly supported) Hamiltonian diffeomorphisms of the symplectic manifold $(M,\omega)$ carries a remarkable invariant norm defined by

$$
\|\phi\| = \inf_{H \mapsto \phi} \|H\|
$$

$$
\|H\| = \int_0^1 (\max H_t - \min H_t) \, dt
$$

which was introduced by Hofer [Ho]. Here $H \mapsto \phi$ means that $\phi$ is the time-one map $\phi^1_H$ of the Hamilton’s equation $\dot{x} = X_H(x)$ of the Hamiltonian $H : [0,1] \times M \to \mathbb{R}$, where the Hamiltonian vector field is defined by

$$
\omega(X_H, \cdot) = dH.
$$

This norm can be easily defined on arbitrary symplectic manifolds although proving non-degeneracy is a non-trivial matter (See [Ho], [Po1] and [LM] for its proof of increasing generality. See also [Ch] for a Floer theoretic proof and [Oh5] for a simple proof of the non-degeneracy in tame symplectic manifolds).

On the other hand Viterbo [V] defined another invariant norm on $\mathbb{R}^{2n}$. This was defined by considering the graph of the Hamiltonian diffeomorphism $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ and compactifying the graph in the diagonal direction in $\mathbb{R}^{4n} = \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ into $T^*S^{2n}$. He then applied the critical point theory of generating functions of any Lagrangian submanifold $L$ Hamiltonian isotopic to the zero section of $T^* N$ and proved that they depend only on the Lagrangian submanifold but not on the generating functions, at least up to normalization.

The present author [Oh4,5] and Milinković [MO1,2, M] developed a Floer theoretic approach to construction of Viterbo’s invariants using the canonically defined action functional on the space of paths, utilizing the observation made by Weinstein [W] that the action functional is a generating function of the given Lagrangian submanifold defined on the path space. This approach is canonical including normalization and provides a direct link between Hofer’s geometry and Viterbo’s invariants.
in a transparent way. One of the key points in our construction in [Oh4] is the emphasis on the usage of the existing group structure on the space of Hamiltonians defined by
\[(H, K) \rightarrow H \# K := H + K \circ (\phi'_H)^{-1}\]
(1.3) in relation to the pants product and the triangle inequality. However we failed to fully exploit this structure and fell short of proving the triangle inequality at the time of writing [Oh4,5].

This construction can be carried out for the Hamiltonian diffeomorphisms as long as the action functional is single valued, e.g., on weakly-exact symplectic manifolds. Schwartz [Sc] carried out this construction in the case of symplectically aspherical \((M, \omega)\), i.e., for \((M, \omega)\) with \(c_1|_{\pi_2(M)} = \omega|_{\pi_2(M)} = 0\). Among other things he proved the triangle inequality for the invariants constructed using the notion of Hamiltonian fibration and (flat) symplectic connection on it. It turns out that the proof of this triangle inequality [Sc] is closely related to the notion of the \(K\)-area of the Hamiltonian fibration [Po2] with connections [GLS], [Po2], especially to the one with fixed monodromy studied by Entov [En1]. In this context, the choice of the triple \((H, K; H \# K)\) we made in [Oh4] can be interpreted as the one which makes infinity the \(K\)-area of the corresponding Hamiltonian fibration over the Riemann surface of genus zero with three punctures equipped with the given monodromy around the punctures. Entov [En1] develops a general framework of Hamiltonian connections with fixed boundary monodromy and relates the \(K\)-area with various quantities of the given monodromy which are of the Hofer length type. This framework turns out to be particularly useful for our construction of spectral invariants in the present paper.

On non-exact symplectic manifolds, the action functional is not single valued and the Floer homology theory has been developed as a circle-valued Morse theory or a Morse theory on a covering space \(\tilde{\Omega}_0(M)\) of the space \(\Omega_0(M)\) of contractible (free) loops on \(M\) in the literature related to Arnold’s conjecture which was initiated by Floer himself [Fl2]. The Floer theory now involves quantum effects and uses the Novikov ring in an essential way [HoS]. The presence of quantum effects had been the most serious obstacle that plagued the study of family of Hamiltonian diffeomorphisms, until the author [Oh6] recently developed a general framework of the mini-max theory over natural semi-infinite cycles on the covering space \(\tilde{\Omega}_0(M)\) which we call the Novikov cycles. In the present paper, we will exploit the ‘finiteness’ condition in the definitions of the Novikov ring and the Novikov cycles in a crucial way for the proofs of various existence results of pseudo-holomorphic curves that are needed in the proofs of the axioms of spectral invariants and nondegeneracy of the norm that we construct. Although the Novikov ring is essential in the definition of the Floer homology and the quantum cohomology in the literature, as far as we know, it is the first time for the finiteness condition to be explicitly used beyond the purpose of giving the definition of the quantum cohomology and the Floer homology.

A brief description of the setting of the Floer theory [HoS] is in order, partly to fix our convention: Let \((\gamma, w)\) be a pair of \(\gamma \in \Omega_0(M)\) and \(w\) be a disc bounding \(\gamma\). We say that \((\gamma, w)\) is \(\Gamma\)-equivalent to \((\gamma, w')\) iff
\[\omega([w' \# \overline{w}]) = 0 \quad \text{and} \quad c_1([w' \# \overline{w}]) = 0\]
(1.4) where \(\overline{w}\) is the map with opposite orientation on the domain and \(w' \# \overline{w}\) is the
obvious glued sphere. Here $\Gamma$ stands for the group

$$\Gamma = \frac{\pi_2(M)}{\ker (\omega|_{\pi_2(M)}) \cap \ker (c_1|_{\pi_2(M)})}.$$ 

We denote by $[\gamma, w]$ the $\Gamma$-equivalence class of $(\gamma, w)$ and by $\pi : \widetilde{\Omega}_0(M) \to \Omega_0(M)$ the canonical projection. We also call $\widetilde{\Omega}_0(M)$ the $\Gamma$-covering space of $\Omega_0(M)$. The action functional $A_0 : \widetilde{\Omega}_0(M) \to \mathbb{R}$ is defined by

$$A_0([\gamma, w]) = -\int w^*\omega. \quad (1.5)$$

Two $\Gamma$-equivalent pairs $(\gamma, w)$ and $(\gamma', w')$ have the same action and so the action is well-defined on $\widetilde{\Omega}_0(M)$. When a $t$-periodic Hamiltonian $H : (\mathbb{R}/\mathbb{Z}) \times M \to \mathbb{R}$ is given, we consider the functional $A_H : \Omega(M) \to \mathbb{R}$ by

$$A_H([\gamma, w]) = -\int w^*\omega - \int H(t, \gamma(t))dt. \quad (1.6)$$

Our convention is chosen to be consistent with the classical mechanics Lagrangian on the cotangent bundle with the symplectic form

$$\omega_0 = -d\theta, \quad \theta = \sum_i p_i dq_i$$

when (1.2) is adopted as the definition of Hamiltonian vector field. See the remark in the end of this introduction on other conventions in the symplectic geometry. The conventions in the present paper coincide with our previous papers [Oh4,5,7] and Entov’s [En1,2] but different from many other literature on the Floer homology one way or the other. (There was a sign error in [Oh4,5] when we compare the Floer complex and the Morse complex for a small Morse function, which was rectified in [Oh6]. In our convention, the positive gradient flow of $\epsilon f$ corresponds to the negative gradient flow of $A_{\epsilon f}$.)

The mini-max theory of this action functional on the $\Gamma$-covering space has been implicitly used in the proof of Arnold’s conjecture. Recently the present author has further developed this mini-max theory via the Floer homology and applied it to the study of Hofer’s geometry of Hamiltonian diffeomorphism groups [Oh6]. We also outlined construction of spectral invariants of Hamiltonian diffeomorphisms of the type [V], [Oh4], [Sc] on arbitrary non-exact symplectic manifolds for the classical cohomological classes. The main purpose of the present paper is to further develop the chain level Floer theory introduced in [Oh6] and to carry out construction of spectral invariants for arbitrary quantum cohomology classes, and also to apply the invariants to the study of geometry of the Hamiltonian diffeomorphism group. The organization of the paper is now in order.

In §2, we briefly review various facts related to the action functional and its action spectrum. Some of these may be known to the experts, but precise details for the action functional on the covering space $\widetilde{\Omega}_0(M)$ of general $(M, \omega)$ first appeared in our paper [Oh7] especially concerning the normalization and the loop effect on the action spectrum: We define the action spectrum of $H$ by

$$\text{Spec}(H) := \{A_H([z, w]) \in \mathbb{R} \mid [z, w] \in \widetilde{\Omega}_0(M), dA_H([z, w]) = 0\}$$
i.e., the set of critical values of $A_H : \tilde{\Omega}_0(M) \to \mathbb{R}$. In [Oh7], we have shown that once we normalize the Hamiltonian $H$ on compact $M$ by

$$\int_M H_t \, d\mu = 0$$

with $d\mu$ the Liouville measure, $\text{Spec}(H)$ depends only on the equivalence class $\phi = [\phi, H]$ (see §2 for the definition) and so $\text{Spec}(\phi) \subset \mathbb{R}$ is a well-defined subset of $\mathbb{R}$ for each $\phi \in \tilde{\mathcal{H}}\mathcal{am}(M, \omega)$. Here

$$\pi : \tilde{\mathcal{H}}\mathcal{am}(M, \omega) \to \mathcal{H}am(M, \omega)$$

is the universal covering space of $\mathcal{H}am(M, \omega)$. Note that $\text{Spec}(\phi)$ is a principal homogeneous space modelled by the group $\Gamma_\omega := \omega(\Gamma) \subset \mathbb{R}$. We also proved that for the natural action of $\Omega(\mathcal{H}am(M, \omega), id)$, the space of based loops $h$, on $\tilde{\mathcal{H}}\mathcal{am}(M, \omega)$ defined by

$$(h, \phi) \mapsto h \cdot \phi$$

we have

$$\text{Spec}(h \cdot \phi) = \text{Spec}(\phi) + I_\omega([h, \tilde{h}])$$

(1.15)

where $\tilde{h}$ is any lift $\tilde{h} : \tilde{\Omega}_0(M) \to \Omega_0(M)$

of the action $h : \Omega_0(M) \to \Omega_0(M)$. Furthermore $L_\omega : \pi_0(\tilde{G}) \to \mathbb{R}$ defines a group homomorphism (see [Oh7] or §2 for more details). This kind of normalization of the action spectrum is a crucial point for systematic study of the spectral invariants of the Viterbo type in general. Schwarz [Sc] previously proved that in the aspherical case where the action functional is single valued already on $\Omega_0(M)$, this normalization can be made on $\mathcal{H}am(M, \omega)$, not just on $\tilde{\mathcal{H}}\mathcal{am}(M, \omega)$, and also proved that $I_\omega \equiv 0$.

In §3, we review the quantum cohomology and its Morse theory realization of the corresponding complex. We emphasize the role of the Novikov ring in relating the quantum cohomology and the Floer homology and the reversal of upward and downward Novikov rings in this relation. In §4, we review the standard operators in the Floer homology theory and explain the filtration naturally present in the Floer complex and how it changes under the Floer chain map. In §5, we give the definition of our spectral invariants and prove finiteness of the mini-max values $\rho(H; a)$. We also give the description of $\rho(H; a)$ in terms of the Hamiltonian fibration over the disc with connection. In §6, we prove all the basic properties of the spectral invariants. We summarize these into the following theorem. We denote by $C^0_m([0, 1] \times M)$ the set of normalized continuous functions on $[0, 1] \times M$.

**Definition & Theorem I.** For any given quantum cohomology class $0 \neq a \in QH^*(M)$, we have a continuous function denoted by

$$\rho_a = \rho(\cdot; a) : C^0_m([0, 1] \times M) \to \mathbb{R}$$

such that for two $C^1$ functions $H \sim K$ we have

$$\rho(H; a) = \rho(K; a)$$
for all $a \in QH^*(M)$. We call the subset $\text{spec}(\tilde{\phi}) \subset \text{Spec}(\tilde{\phi})$ defined by

$$\text{spec}(\tilde{\phi}) = \{ \rho(\tilde{\phi}; a) \mid a \in QH^*(M) \}$$

the (homologically) essential spectrum of $\tilde{\phi}$. Similarly the essential spectrum $\text{spec}(H)$ of the Hamiltonian $H$ is defined. For each given degree $k$ of the quantum cohomology we also define

$$\text{spec}_k(\tilde{\phi}) = \{ \rho(\tilde{\phi}; a) \mid a \in QH^k(M) \}.$$

Let $\tilde{\phi}, \tilde{\psi} \in \tilde{\text{Ham}}(M, \omega)$ and $a \neq 0 \in QH^*(M)$. We define the map

$$\rho : \tilde{\text{Ham}}(M, \omega) \times QH^*(M) \to \mathbb{R}$$

by $\rho(\tilde{\phi}; a) := \rho(H; a)$. Then $\rho$ satisfies the following axioms:

1. **(Spectrality)** For each $a \in QH^*(M)$, $\rho(\tilde{\phi}; a) \in \text{Spec}(\tilde{\phi})$.

2. **(Projective invariance)** $\rho(\tilde{\phi}; \lambda a) = \rho(\tilde{\phi}; a)$ for any $0 \neq \lambda \in \mathbb{Q}$.

3. **(Normalization)** For $a = \sum_{A \in \mathcal{R}} a_A q^{-A}$, we have $\rho(\tilde{\phi}; a) = \nu(a)$ where $\nu$ is the identity in $\tilde{\text{Ham}}(M, \omega)$ and

$$\nu(a) := \min \{ \omega(-A) \mid a_A \neq 0 \} = -\max \{ \omega(A) \mid a_A \neq 0 \}. \quad (1.16)$$

is the (upward) valuation of $a$.

4. **(Symplectic invariance)** $\rho(\eta \tilde{\phi} \eta^{-1}; a) = \rho(\tilde{\phi}; a)$ for any symplectic diffeomorphism $\eta$.

5. **(Triangle inequality)** $\rho(\tilde{\phi} \cdot \tilde{\psi}; a \cdot b) \leq \rho(\tilde{\phi}; a) + \rho(\tilde{\psi}; b)$.

6. **(C0-continuity)** $|\rho(\tilde{\phi}; a) - \rho(\tilde{\psi}; a)| \leq \|\tilde{\phi} \circ \tilde{\psi}^{-1}\|$ where $\|\cdot\|$ is the Hofer’s pseudo-norm on $\tilde{\text{Ham}}(M, \omega)$. In particular, the function $\rho_a : \tilde{\phi} \mapsto \rho(\tilde{\phi}; a)$ is $C^0$-continuous.

7. **(Monodromy shift)** Let $[h, \tilde{h}] \in \pi_0(\tilde{G})$ act on $\tilde{\text{Ham}}(M, \omega) \times QH^*(M)$ by the map

$$\tilde{\phi} \mapsto (h \cdot \tilde{\phi}, \tilde{h} \cdot a)$$

where $\tilde{h}^a$ is the image of the (adjoint) Seidel’s action $[\tilde{\phi}]$ by $[h, \tilde{h}]$ on the quantum cohomology $QH^*(M)$. Then we have

$$\rho([h, \tilde{h}] \cdot (\tilde{\phi}, a)) = \rho(\tilde{\phi}; a) + L_\omega([h, \tilde{h}]). \quad (1.17)$$

It would be an interesting question to ask whether these axioms together with (1.37) below characterize the spectral invariants $\rho$. It is related to the question whether the graph of the sections

$$\rho_a : \tilde{\phi} \mapsto \rho(\tilde{\phi}; a) ; \quad \tilde{\text{Ham}}(M, \omega) \to \text{Spec}(M, \omega)$$

can be split into other ‘branch’ in a way that the other branch can also satisfy all the above axioms or not. Here the action spectrum bundle $\hat{\text{Spec}}(M, \omega)$ is defined by

$$\hat{\text{Spec}}(M, \omega) := \bigcup_{\tilde{\phi} \in \tilde{\text{Ham}}(M, \omega)} \text{Spec}(\tilde{\phi}) \subset \tilde{\text{Ham}}(M, \omega) \times \mathbb{R}.$$
We will investigate this question elsewhere.

In the classical mini-max theory for the indefinite functionals [Ra], [BnR], there was implicitly used the notion of ‘semi-infinite cycles’ to carry out the mini-max procedure. There are two essential ingredients needed to prove existence of actual critical values out of the mini-max values: one is the finiteness of the mini-max value, or the linking property of the (semi-infinite) cycles associated to the class $a$ and the other is to prove that the corresponding mini-max value is indeed a critical value of the action functional. In our case, the latter is precisely the spectrality axiom. When the global gradient flow of the action functional exists as in the classical critical point theory [BnR], this point is closely related to the well-known Palais-Smale condition and the deformation lemma which are essential ingredients needed to prove the criticality of the mini-max value. Partly because we do not have the global flow, we need to geometrize all these classical mini-max procedures. It turns out that the Floer homology theory in the chain level is the right framework for this purpose.

We would like to mention that for the exact case as in [Oh4,5], [Sc] or more generally for the rational case where the period group $\Gamma_\omega \subset \mathbb{R}$ and so the action spectrum is discrete, it is rather immediate from our definition of $\rho$ that the mini-max value $\rho(H; a)$ is indeed a critical value once the finiteness of the mini-max value $\rho(\tilde{\phi}; a)$ is proven, at least for nondegenerate Hamiltonians. However in the non-rational case when the action spectrum is a dense subset of $\mathbb{R}$, proof of this fact (the spectrality axiom) is highly non-trivial and heavily relies on the finiteness condition in the definition of Novikov ring (see §5) and also on the idea of non-pushing down lemma which the author introduced in [Oh6]. The other parts of the theorem are direct analogs to the ones in [Oh4,5] and [Sc]. Proof of the continuity is a refinement of the arguments used in [Oh4,5]. Proof of the triangle inequality uses the concept of Hamiltonian fibration with fixed monodromy and the $K$-area [Po2], [En1], which is an enhancement of the arguments used in [Oh4], [Sc].

In §6, we focus on the invariant $\rho(\tilde{\phi}; 1)$ for $1 \in \mathbb{Q}H^*(M)$. We first recall the following quantities

$$E^-(\tilde{\phi}) = \inf_{[\phi, H] = \tilde{\phi}} \int_0^1 -\min H_t \, dt$$
$$E^+(\tilde{\phi}) = \inf_{[\phi, H] = \tilde{\phi}} \int_0^1 \max H_t \, dt$$

(1.18)

(See [EP1], [Po3], [Mc2] for example). Note that we have

$$E^- (\tilde{\phi}^{-1}) = E^+ (\tilde{\phi})$$

and

$$0 \leq E^+(\tilde{\phi}) + E^-(\tilde{\phi}) \leq \inf_{[\phi, H] = \tilde{\phi}} \int (\max H_t - \min H_t) \, dt$$

(1.19)

and in particular Hofer’s norm $\|\phi\|$ satisfies

$$\|\phi\|_{\text{mid}} := \inf_{\pi(\tilde{\phi}) = \phi} (E^+(\tilde{\phi}) + E^-(\tilde{\phi})) \leq \|\phi\|.$$  

(1.20)

McDuff [Mc2] proved that the medium Hofer pseudo-norm $\|\cdot\|_{\text{mid}}$ is nondegenerate. There is another smaller pseudo-norm which we call the small Hofer pseudo-norm
defined by
\[ \|\phi\|_{sm} = \inf_{H \rightarrow \phi} \left( \int_0^1 \max H_t \, dt \right) + \inf_{F \rightarrow \phi} \left( \int_0^1 - \min F_t \, dt \right). \]

McDuff [Mc2] proved that this is nondegenerate either for the weakly exact case or for \( \mathbb{C}P^n \). Whether this is non-degenerate is still open in general.

We also denote
\[ \|\tilde{\phi}\| = \inf_{[\phi, H] = \tilde{\phi}} \int (\max H_t - \min H_t) \, dt \]
and call it the Hofer pseudo-norm on \( \widetilde{\text{Ham}}(M, \omega) \).

In §7, we launch our construction of invariant norm. We define \( \tilde{\gamma} : \widetilde{\text{Ham}}(M, \omega) \rightarrow \mathbb{R} \) by
\[ \tilde{\gamma}(\tilde{\phi}) = \rho(\tilde{\phi}; 1) + \rho(\tilde{\phi}^{-1}; 1), \]
on \( \widetilde{\text{Ham}}(M, \omega) \). Obviously we have
\[ \tilde{\gamma}(\tilde{\phi}) = \tilde{\gamma}(\tilde{\phi}^{-1}) \]
for any \( \tilde{\phi} \) and the triangle inequality implies \( \tilde{\gamma}(\tilde{\phi}) \geq 0 \). We also have the following theorem

**Theorem II.** We have
\[ \rho(\tilde{\phi}; a) \leq E^- (\tilde{\phi}) + v(a). \]  

In particular, we have
\[ \rho(\tilde{\phi}; 1) \leq E^- (\tilde{\phi}), \quad \rho(\tilde{\phi}^{-1}; 1) \leq E^+ (\tilde{\phi}). \]

and
\[ \tilde{\gamma}(\tilde{\phi}) \leq E^+ (\tilde{\phi}) + E^- (\tilde{\phi}) \]

We note that \( v(a) = 0 \) for any classical cohomology class \( a \in H^*(M) \) by the definition (1.16) of the valuation \( v(a) \) and hence (1.23) reduces to
\[ \rho(\tilde{\phi}; a) \leq E^- (\tilde{\phi}) \]
for this case. (1.25) was previously proven in [Oh4], [Sc] for the exact case. (1.25) shows that the same inequality still holds for the classical cohomology classes in arbitrary symplectic manifolds, but (1.23) shows that there is a quantum correction for the general quantum cohomology class \( a \in QH^*(M) \).

We would like to emphasize that \( \rho(\tilde{\phi}; 1) \) could be negative in general (see [Os] for examples on the aspherical case), although the sum (1.22) cannot and so at least one of \( \rho(\tilde{\phi}; 1) \) and \( \rho(\tilde{\phi}^{-1}; 1) \) must be non-negative. This leads us to introduce the following definition
Definition 1.1. We call \( \tilde{\phi} \in \tilde{\text{Ham}}(M, \omega) \) positive if \( \rho(\tilde{\phi}; 1) \leq 0 \). We also call a normalized Hamiltonian \( H \) or the corresponding Hamiltonian path \( \{\phi_H^t\}_{0 \leq t \leq 1} \) homologically positive if \([\phi, H] = \tilde{\phi} \) is positive. We define
\[
\tilde{\text{Ham}}_+(M, \omega) = \{ \tilde{\phi} | \tilde{\phi} \text{ positive} \}
\]
\[
C^+_m([0,1] \times M, \mathbb{R}) = \{ H | H \text{ homologically positive} \}
\]
and denote
\[
\mathcal{P}^+(\text{Ham}(M, \omega), id) = \{ f : [0, 1] \to \text{Ham}(M, \omega) | f(0) = id, \ f(t) = \phi_H^t, H \in C^+_m([0,1] \times M, \mathbb{R}) \}
\]
for the set of positive Hamiltonian paths issued at the identity.

It follows from the triangle inequality, symplectic invariance and the normalization axiom that the subset \( C := \tilde{\text{Ham}}_+(M, \omega) \) forms a normal cone in \( D := \tilde{\text{Ham}}(M, \omega) \) in the sense of Eliashberg-Polterovich [ElP2], i.e., satisfies

1. If \( f, g \in C \), \( fg \in C \)
2. If \( f \in C \) and \( h \in D \), \( hfh^{-1} \in C \)
3. \( id \in C \)

Furthermore the corresponding partial order on \( \tilde{\text{Ham}}(M, \omega) \) is defined by
\[
f \geq g \text{ on } D \text{ iff } fg^{-1} \in C.
\]
The question whether this is non-trivial, i.e., satisfies the axiom
\[
f \leq g \land g \leq f \text{ iff } f = g
\]
is a non-trivial problem to answer in general and is related to the study of Hamiltonian loops \( h \) and the corresponding spectral invariants \( \rho(h; 1) \) (see [§7, Po3] for some related question in terms of Hofer’s length). We refer readers to [ElP2] for a general discussion on the partially ordered groups and the definition of the normal cone. Viterbo [V] had earlier introduced the notion of positive Hamiltonians and a similar partial order for the set of compactly supported Hamiltonians on \( \mathbb{R}^{2n} \) and proved non-degeneracy of the partial order. In our normalization of having zero mean value as in Viterbo’s case, normalized Hamiltonian can never have non-negative values everywhere unless it is identically zero. This is the reason why we call the Hamiltonians in Definition 1.1 homologically positive. We have shown in [Oh7] that either normalization of Hamiltonians removes ambiguity of constant of defining the action spectrum of Hamiltonian diffeomorphisms. It appears that there is some sort of ‘duality’ between the two normalization.

Now we define a non-negative function \( \gamma : \tilde{\text{Ham}}(M, \omega) \to \mathbb{R}_+ \) by
\[
\gamma(\phi) = \inf_{\pi(\tilde{\phi}) = \phi} \gamma(\tilde{\phi}). \tag{1.26}
\]

In §8, we give the proof of the following theorem except the non-degeneracy which we prove in §9.
Theorem III. \( \gamma : \text{Ham}(M, \omega) \to \mathbb{R}_+ \) is well-defined and satisfies the following properties

1. \( \phi = \text{id} \) if and only if \( \gamma(\phi) = 0 \)
2. \( \gamma(\eta \phi \eta^{-1}) = \gamma(\phi) \) for any symplectic diffeomorphism \( \eta \)
3. \( \gamma(\psi \phi) \leq \gamma(\psi) + \gamma(\phi) \)
4. \( \gamma(\phi^{-1}) = \gamma(\phi) \)
5. \( \gamma(\phi) \leq \|\phi\|_{\mid \mid} \leq \|\phi\| \).

In particular, it defines a symmetric (i.e., \( \gamma(\phi) = \gamma(\phi^{-1}) \)) invariant norm on \( \tilde{\text{Ham}}(M, \omega) \).

This norm reduces to the norm Schwarz constructed in [Sc] following [V] and [Oh4] for the symplectically aspherical case where the norm \( \gamma \) is defined by

\[ \gamma(H) = \rho(H; 1) - \rho(H; \mu) \]  

(1.27)

where \( \mu \) is the volume class in \( H^*(M) \). The reason why the quantity (1.27) coincides with the norm (1.26) is that we have the additional identity

\[ \rho(\overline{H} : 1) = -\rho(H; \mu) \]

in the aspherical case. But Polterovich observed [Po4] that this latter identity fails in the non-exact case due to the quantum contribution. In fact, it seems that even positiveness of (1.27) fails in the non-exact case.

As was shown by Ostrover [Os] in the aspherical case, \( \gamma \) is a different norm from the Hofer norm. By the same reason, it can be shown to be also different from the medium Hofer norm \( \| \cdot \|_{\mid \mid} \).

The proof of Theorem III reveals new lower estimates of the Hofer norm. To describe these results, we need some preparation. Let \( \phi \) be a Hamiltonian diffeomorphism that has only finite number of fixed points (e.g., non-degenerate ones). We denote by \( J_0 \) a compatible almost complex structure on \((M, \omega)\) and by \( J_\omega \) the set of compatible almost complex structures on \( M \). For given \((\phi, J_0)\), we consider paths \( J : [0, 1] \to J_\omega \) with

\[ J(0) = J_0, \quad J(1) = \phi^* J_0 \]  

(1.28)

and denote the set of such paths by

\[ j(\phi, J_0) \]

For each given \( J' \in j(\phi, J_0) \), we define the constant

\[ A_S(\phi, J_0; J') = \inf \{ \omega([u]) \mid u : S^2 \to M \text{ non-constant and } \partial_t u = 0 \text{ for some } t \in [0, 1] \} \]  

(1.29)

and then

\[ A_S(\phi, J_0) = \sup_{J \in j(\phi, J_0) \setminus j(\phi, J_0)} A_S(\phi, J_0; J') \].  

(1.30)

As usual, we set \( A_S(\phi, J_0) = \infty \) if there is \( J' \in j(\phi, J_0) \) for which there is no \( J'_t \)-holomorphic sphere for any \( t \in [0, 1] \) as in the weakly exact case. The positivity
$A_S(\phi, J_0; J') > 0$ and so $A_S(\phi, J_0) > 0$ is an immediate consequence of the one parameter version of the uniform \( \epsilon \)-regularity theorem (see [SU], [Oh1]).

Next for each given \( J' \in j(\phi, J_0) \), we consider the equation of \( v : \mathbb{R} \times [0, 1] \to M \)
\[
\begin{cases}
\frac{\partial v}{\partial \tau} + J_t \frac{\partial v}{\partial t} = 0 \\
\phi(v(\tau, 1)) = v(\tau, 0), \quad \int |\frac{\partial v}{\partial \tau}|_J^2 < \infty.
\end{cases}
\] (1.31)

This equation itself is analytically well-posed and (1.28) enables us to interpret solutions of (1.31) as pseudo-holomorphic sections of the mapping cylinder of \( \phi \) with respect to suitably chosen almost complex structure on the mapping cylinder. See §6 and 7 for the explanation.

Note that any such solution of (1.31) has the limit \( \lim_{\tau \to \pm} v(\tau) \). Now it is a crucial matter to produce a non-constant solution of (1.31). For this purpose, using the fact that \( \phi \neq \text{id} \), we choose a symplectic ball \( B_p(r) \) such that \( \phi(B_p(r)) \cap B_p(r) = \emptyset \) (1.32)

where \( B_p(r) \) is the image of a symplectic embedding into \( M \) of the standard Euclidean ball of radius \( r \). We then study (1.31) together with \( v(0, 0) \in B_p(r) \). (1.33)

Because of (1.32), it follows
\[ v(\pm \infty) \in \text{Fix } \phi \subset M \setminus B_p(r). \] (1.34)

Therefore such solution cannot be constant because of (1.33) and (1.34).

We now define the constant
\[ A_D(\phi, J_0; J') := \inf_v \left\{ \int v^* \omega, \mid v \text{ non-constant solution of (1.31)} \right\} \] (1.35)

for each \( J \in j(\phi, J_0) \). Again we have \( A_D(\phi, J_0; J') > 0 \). We also define
\[ A(\phi, J_0; J') = \min \{ A_S(\phi, J_0; J'), A_D(\phi, J_0; J') \}. \]

We will prove that \( 0 < A(\phi, J_0; J') < \infty \) in §7 as a consequence of some existence theorem, Proposition 7.10 which is proven in [Oh8]. Finally we define
\[ A(\phi, J_0) := \sup_{J' \in j(\phi, J_0)} A(\phi, J_0; J') \] (1.36)

and
\[ A(\phi) = \sup_{J_0} A(\phi, J_0). \] (1.37)

Note when \( (M, \omega) \) is weakly exact and so \( A_S(\phi, J_0; J') = \infty \), \( A(\phi, J_0) \) is reduced to
\[ A(\phi, J_0) = \sup_{J' \in j(\phi, J_0)} \{ A_D(\phi, J_0; J') \}. \]

Because of the assumption that \( \phi \) has only finite number of fixed points, it is clear that \( A(\phi; \omega, J_0) > 0 \) and so we have \( A(\phi) > 0 \). We will also use the more standard invariant of \( (M, \omega) \)
\[ A(\omega, J_0) = \inf \{ \omega([u]) \mid u \text{ non-constant } J_0\text{-holomorphic} \} \]
\[ A(\omega) = \sup_{J_0} A(\omega, J_0). \]

In §7, we prove the following lower estimate
Theorem IV. Suppose that $\phi$ has non-degenerate fixed points and let $A(\phi)$ be the constant (1.37). Then we have

$$\gamma(\phi) \geq A(\phi)$$

and in particular the Hofer norm $\|\phi\|$ satisfies

$$\|\phi\| \geq A(\phi).$$

This together with Theorem III (5) will also immediately prove non-degeneracy of $\gamma$ noting that the null-set

$$\text{null}(\gamma) = \{ \phi \in \text{Ham}(M,\omega) \mid \gamma(\phi) = 0 \}$$

is a normal subgroup of $\text{Ham}(M,\omega)$, while $\text{Ham}(M,\omega)$ is simple by Banyaga’s theorem [Ba].

In §8, we will prove a stronger lower bound than (1.32) exploiting the fact that the definition of $\tilde{\gamma}$ involves only the identity class $1 \in QH^*(M)$. We will exploit this fact and refine the definition of the lower bound by replacing $A(\phi)$ by another stronger bound, which we denote by $A(\phi;1)$. We refer to §8 for the detailed description of the lower bound $A(\phi;1)$. Using this bound, we study the Hamiltonian diffeomorphisms $\phi$ whose graph is ‘engulfed’ by a Darboux neighborhood of the diagonal $\Delta \subset (M,-\omega) \times (M,\omega)$. More precisely, let

$$\Phi : U \subset M \times M \to V \subset T^* \Delta$$

be a Darboux chart satisfying

1. $\Phi^*\omega_0 = -\omega \oplus \omega$
2. $\Phi|_{\Delta} = \text{id}_\Delta$ and $d\Phi|_{T\Delta} : T\Delta \to T\Phi|_{\Delta}$ is the obvious symplectic bundle map from $T(M \times M)|_{\Delta} \cong N\Delta \oplus T\Delta$ to $T(T^*\Delta)|_{\Delta} \cong T^*\Delta \oplus T\Delta$ which is the identity on $T\Delta$ and the natural bundle map from $N\Delta$ to $T^*\Delta$ induced by the symplectic form $\omega$.

Given such a chart, we consider any Hamiltonian diffeomorphism $\phi : M \to M$ such that

$$\Delta_\phi := \text{graph } \phi \subset U$$
$$\Phi(\Delta_\phi) = \text{graph } dS_\phi$$

for the unique function $S_\phi : \Delta \to \mathbb{R}$ with $\int_{\Delta} S_\phi = 0$. Motivated by the paper [La], we call $\phi$ an *engulfable* Hamiltonian diffeomorphism, if we can find such Darboux chart $\Phi$. To state the next result, let

$$U' \subset \overline{U} \subset U$$

be another neighborhood of $\Delta$ and assume that

$$\Delta_\phi \subset U' \subset U.$$

Then we can define a constant $A(J_\theta : U' \subset U)$ depending only on $U'$, $U$ and $J_\theta$, which is roughly the minimal possible area of $J_\theta$-holomorphic rectangle with lower boundary lying on $\Delta$ and upper boundary lying on $\Delta_\phi$, and the side boundaries mapped into $\Delta \cap \Delta_\phi$. We refer to §9 for the precise definition of the constant.
Theorem V. Let $\Phi : U \to V$ be a Darboux chart as above. Suppose that $\phi$ is engulfable and let $S_\phi : \Delta \cong M \to \mathbb{R}$ be the unique function given by (1.40) and suppose that (1.41) holds for some $U' \subset U$ and
\[ \text{osc}(S_\phi) \leq A(J_0; U' \subset U) \tag{1.42} \]
for some almost complex structure $J_0$. Then we have
\[ \gamma(\phi) = A(\phi; 1) = \text{osc}(S_\phi) = ||\phi||. \tag{1.43} \]

It is easy to see from the example of the two sphere $S^2$ with the standard symplectic form that (1.42) is an optimal upper bound for $\text{osc}(S_\phi)$ for (1.43) to hold.

Corollary 1.2. Let $\phi$ be engulfable and satisfy (1.41) and (1.42), and let $\phi^t$ be the Hamiltonian path determined by the equation
\[ \Phi(\text{graph } \phi^t) = \text{graph } t dS_\phi. \]
Then the path $t \in [0, 1] \mapsto \phi^t$ is a Hofer geodesic which is length minimizing among all paths from the identity to $\phi$.

This corollary extends McDuff’s recent result [Proposition 1.8, Mc2] which proves the same minimizing property for the $\phi = \phi_H^1$ for sufficiently $C^2$-small $H$’s. The corollary above gives a quantitative estimate how large the $C^1$ distance of $\phi$ from the identity can be so that the property holds. As in [Proposition 1.8, Mc2], this also proves the following flatness property of the neighborhood of the identity of $\mathcal{Ham}(M, \omega)$ of a definite size provided by the invariant $A(J_0; U' \subset U)$.

Corollary 1.3. Let $N \subset \mathcal{Ham}(M, \omega)$ be a $C^1$-neighborhood of the identity such that

1. any $\phi \in N$ is engulfable
2. $\text{osc}(S_\phi) \leq A(J_0; U' \subset U)$ for $\phi \in N$ for some $J_0$.

Then the assignment
\[ \phi \mapsto S_\phi; \quad N \to C^2_m(M) \tag{1.44} \]
is an isometry with respect to the Hofer norm on $N$ and the norm on $C^2_m(M)$ provided by $\text{osc}(S_\phi) = \max S - \min S$.

Note that the norm function
\[ \phi \mapsto \text{osc}(S_\phi); \quad N \to \mathbb{R}_+ \]
can be extended to the completion of $N$ with respect to the topology induced by the Hofer norm. It would be interesting to see whether we can enlarge the space $C^2_m(M)$ so that (1.44) extends as an isometry.

The proof of Theorem V involves the thick and thin decomposition of the Floer moduli space when $H$ is $C^2$-small and then Conley’s continuation argument [Fl2] using the isolatedness of the ‘thin’ part of the relevant moduli space [Oh2]. The details are given in §9 and the Appendix 1.

Finally one pedagogical remark is in order for those who are not familiar with the virtual moduli cycle machinery. To get the main stream of ideas in this paper
without getting bogged down with technicalities related with transversality question of various moduli spaces, we suggest them to presume that \((M, \omega)\) is strongly semi-positive in the sense of [Se], [En1]. Under this condition, the transversality problem concerning various moduli spaces of pseudo-holomorphic curves is standard. We will not mention this generic transversality question at all in the main body of the paper unless it is absolutely necessary. In §10, we will briefly explain how this general framework can be incorporated in our proofs in the context of Kuranishi structure [FOn] all at once. In the Appendix 2, we introduce the notion of bounded quantum cohomology and explain how to extend our definition of spectral invariants to the bounded quantum cohomology classes. We also leave the proof of Proposition 7.11 to [Oh8] because its proof is purely analytic in nature and is independent of the main arguments in the present paper and can be refined to prove a stronger existence theorem for the perturbed Cauchy-Riemann equation with discontinuous Hamiltonian perturbation term.

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Convention.

1. The Hamiltonian vector field \(X_f\) associated to a function \(f\) on \((M, \omega)\) is defined by \(df = \omega(X_f, \cdot)\).
2. The addition \(F \# G\) and the inverse \(\overline{G}\) on the set of time periodic Hamiltonians \(C^\infty(M \times S^1)\) are defined by
   \[
   F \# G(x, t) = F(x, t) + G((\phi^t_F)^{-1}(x), t) \\
   \overline{G}(x, t) = -G(\phi^t_F(x), t).
   \]

There is another set of conventions which are used in the literature (e.g., in [Po3]):

1. \(X_f\) is defined by \(\omega(X_f, \cdot) = -df\)
2. The action functional has the form
   \[
   \mathcal{A}_H([z, w]) = - \int w^*\omega + \int H(t, z(t)) \, dt. \tag{1.45}
   \]

Because our \(X_f\) is the negative of \(X_f\) in this convention, the action functional is the one for the Hamiltonian \(-H\) in our convention. While our convention makes the positive Morse gradient flow correspond to the negative Cauchy-Riemann flow, the
other convention keeps the same direction. The reason why we keep our convention is that we would like to keep the definition of the action functional the same as the classical Hamilton’s functional

$$\int pdq - H dt$$

(1.46)

on the phase space and to make the negative gradient flow of the action functional for the zero Hamiltonian become the pseudo-holomorphic equation.

It appears that the origin of the two different conventions is the choice of the convention on how one defines the canonical symplectic form on the cotangent bundle $T^*N$ or in the classical phase space: If we set the canonical Liouville form

$$\theta = \sum_i p_i dq^i$$

for the canonical coordinates $q^1, \cdots, q^n, p_1, \cdots, p_n$ of $T^*N$, we take the standard symplectic form to be

$$\omega_0 = -d\theta = \sum dq^i \wedge dp_i$$

while the people using the other convention (see e.g., [Po3]) takes

$$\omega_0 = d\theta = \sum dp_i \wedge dq^i.$$ 

As a consequence, the action functional (1.45) in the other convention is the negative of the classical Hamilton’s functional (1.46). It seems that there is not a single convention that makes everybody happy and hence one has to live with some nuisance in this matter one way or the other.

§2. The action functional and the action spectrum

Let $(M, \omega)$ be any compact symplectic manifold and $\Omega_0(M)$ be the set of contractible loops and $\tilde{\Omega}_0(M)$ be its the covering space mentioned before. We will always consider normalized functions $f : M \to \mathbb{R}$ by

$$\int_M f d\mu = 0$$

(2.1)

where $d\mu$ is the Liouville measure of $(M, \omega)$.

When a periodic normalized Hamiltonian $H : M \times (\mathbb{R}/\mathbb{Z}) \to \mathbb{R}$ is given, we consider the action functional $A_H : \tilde{\Omega}(M) \to \mathbb{R}$ by

$$A_H(\gamma, w) = -\int w^* \omega - \int H(\gamma(t), t) dt$$

We denote by $\text{Per}(H)$ the set of periodic orbits of $X_H$. 

**Definition 2.1.** We define the *action spectrum* of $H$, denoted as $\text{Spec}(H) \subset \mathbb{R}$, by

$$\text{Spec}(H) := \{ A_H(z, w) \in \mathbb{R} \mid (z, w) \in \tilde{\Omega}_0(M), z \in \text{Per}(H) \},$$

i.e., the set of critical values of $A_H : \tilde{\Omega}(M) \to \mathbb{R}$. For each given $z \in \text{Per}(H)$, we denote

$$\text{Spec}(H; z) := \{ A_H(z, w) \in \mathbb{R} \mid (z, w) \in \pi^{-1}(z) \}.$$ 

Note that $\text{Spec}(H; z)$ is a principal homogeneous space modelled by the period group of $(M, \omega)$

$$\Gamma_\omega := \{ \omega(A) \mid A \in \pi_2(M) \} = \omega(\Gamma) \subset \mathbb{R}$$

and

$$\text{Spec}(H) = \bigcup_{z \in \text{Per}(H)} \text{Spec}(H; z).$$

Recall that $\Gamma_\omega$ is either a discrete or a countable dense subset of $\mathbb{R}$. The following was proven in [Oh6].

**Lemma 2.2.** $\text{Spec}(H)$ is a measure zero subset of $\mathbb{R}$ for any $H$.

For given $\phi \in \text{Ham}(M, \omega)$, we denote $F \mapsto \phi$ if $\phi^1_F = \phi$, and denote

$$\mathcal{H}(\phi) = \{ F \mid F \mapsto \phi \}.$$ 

We say that two Hamiltonians $F$ and $G$ are equivalent and denote $F \sim G$ if they are connected by one parameter family of Hamiltonians $\{ F^s \}_{0 \leq s \leq 1}$ such that $F^s \mapsto \phi$ for all $s \in [0, 1]$. We write $[F]$ for the equivalence class of $F$. Then the universal covering space $\tilde{\text{Ham}}(M, \omega)$ of $\text{Ham}(M, \omega)$ is realized by the set of such equivalence classes. Note that the group $G := \Omega(\text{Ham}(M, \omega), \text{id})$ of based loops naturally acts on the loop space $\Omega(M)$ by

$$(h \cdot \gamma)(t) = h(t)(\gamma(t))$$

where $h \in \Omega(\text{Ham}(M, \omega))$ and $\gamma \in \Omega(M)$. An interesting consequence of Arnold’s conjecture is that this action maps $\Omega_0(M)$ to itself (see e.g., [Lemma 2.2, Se]). Seidel [Lemma 2.4, Se] proves that this action can be lifted to $\tilde{\Omega}_0(M)$. The set of lifts $(\tilde{h}, \tilde{h})$ forms a covering group $\tilde{G} \to G$

$$\tilde{G} \subset G \times \text{Homeo}(\tilde{\Omega}_0(M))$$

whose fiber is isomorphic to $\Gamma$. Seidel relates the lifting $(\tilde{h}, \tilde{h})$ of $h : S^1 \to \text{Ham}(M, \omega)$ to a section of the Hamiltonian bundle associated to the loop $h$ (see §2 [Se]).

When a Hamiltonian $H$ generating the loop $h$ is given, the assignment

$$z \mapsto h \cdot z$$

provides a natural one-one correspondence

$$h : \text{Per}(F) \mapsto \text{Per}(H \# F) \quad (2.2)$$
where $H \# F = H + F \circ (\phi_t^H)^{-1}$. Let $F, G$ be normalized Hamiltonians with $F, G \mapsto \phi$ and $H$ be the Hamiltonian such that $G = H \# F$, and $f_t, g_t$ and $h_t$ be the corresponding Hamiltonian paths as above. In particular the path $h = \{h_t\}_{0 \leq t \leq 1}$ defines a loop. We also denote the corresponding action of $h$ on $\Omega_0(M)$ by $h$. Let $\tilde{h}$ be any lift of $h$ to Homeo($\tilde{\Omega}_0(M)$). Then a straightforward calculation shows (see [Oh7])

$$\tilde{h}^*(dA_F) = dA_G$$

(2.3)

as a one-form on $\tilde{\Omega}_0(M)$. In particular since $\tilde{\Omega}_0(M)$ is connected, we have

$$\tilde{h}^*(A_F) - A_G = C(F, G, \tilde{h})$$

(2.4)

where $C = C(F, G, \tilde{h})$ is a constant a priori depending on $F, G, \tilde{h}$.

**Theorem 2.3** [Theorem II, Oh7]. Let $h$ be the loop as above and $\tilde{h}$ be a lift. Then the constant $C(F, G, \tilde{h})$ in (2.4) depends only on the homotopy class $[h, \tilde{h}] \in \pi_0(G)$.

If we define a map $I_\omega: \pi_0(G) \to \mathbb{R}$ by $I_\omega([h, \tilde{h}]) := C(F, G, \tilde{h})$, then

$$A_G \circ \tilde{h} = A_F + I_\omega([h, \tilde{h}])$$

(2.5)

and $I_\omega$ defines a group homomorphism. In particular if $F \sim G$, we have $A_F \circ \tilde{h} = A_G$ and hence

$$\text{Spec } F = \text{Spec } G$$

as a subset of $\mathbb{R}$. For any $\tilde{\phi} \in \tilde{\text{Ham}}(M, \omega)$, we define

$$A_\tilde{\phi} := A_F, \quad \text{Spec } (\tilde{\phi}) := \text{Spec } F$$

for a (and so any) normalized Hamiltonian $F$ with $[\phi, F] = \tilde{\phi}$.

**Definition 2.4** [Action Spectrum Bundle]. We define the action spectrum bundle of $(M, \omega)$ by

$$\mathcal{Spec}(M, \omega) = \{(\tilde{\phi}, A(\tilde{\phi}, [z, w])) \mid dA_\tilde{\phi}([z, w]) = 0 \} \subset \tilde{\text{Ham}}(M, \omega) \times \mathbb{R}$$

and denote by $\pi: \mathcal{Spec}(M, \omega) \to \tilde{\text{Ham}}(M, \omega)$ the natural projection.

The spectral invariants we define in §5 will provide canonical sections of this bundle which are continuous with respect to the $C^0$-topology of $\tilde{\text{Ham}}(M, \omega)$.

**§3. Quantum cohomology in the chain level**

We first recall the definition of the quantum cohomology ring $QH^*(M)$. As a module, it is defined as

$$QH^*(M) = H^*(M, \mathbb{Q}) \otimes \Lambda^\uparrow$$

where $\Lambda^\uparrow$ is the (upward) Novikov ring

$$\Lambda^\uparrow = \left\{ \sum_{A \in \Gamma} a_A q^{-A} \mid a_A \in \mathbb{Q}, \#\{A \mid a_i \neq 0, \int_A \omega < \lambda \} < \infty, \forall \lambda \in \mathbb{R} \right\}.$$
Due to the finiteness assumption on the Novikov ring, we have the natural (upward) valuation \( v : QH^*(M) \to \mathbb{R} \) defined by

\[
v(\sum_{A \in \Gamma} a_A q^{-A}) = \min\{\omega(-A) : a_A \neq 0\} : \quad (3.1)
\]

It satisfies that for any \( a, b \in QH^*(M) \)

(i) \( v(a \cdot b) = v(a) + v(b) \)
(ii) \( v(a + b) \geq \min\{v(a), v(b)\} \).

**Definition 3.1.** For each homogeneous element

\[
a = \sum_{A \in \Gamma} a_A q^{-A} \in QH^k(M), \quad a_A \in H^*(M, \mathbb{Q}) \quad (3.2)
\]

of degree \( k \), we also call \( v(a) \) the *level* of \( a \) and the corresponding term in the sum the *leading order term* of \( a \) and denote by \( \sigma(a) \). Note that the leading order term \( \sigma(a) \) of a homogeneous element \( a \) is unique among the summands in the sum by the definition (1.4) of \( \Gamma \).

The product on \( QH^*(M) \) is defined by the usual quantum cup product, which we denote by \( \cdot \) and which preserves the grading, i.e., satisfies

\[
QH^k(M) \times QH^\ell(M) \to QH^{k+\ell}(M).
\]

Often the homological version of the quantum cohomology is also useful, sometimes called the quantum homology, which is defined by

\[
QH_*(M) = H_*(M) \otimes \Lambda^\downarrow_{\omega}
\]

where \( \Lambda^\downarrow_{\omega} \) is the (downward) Novikov ring

\[
\Lambda^\downarrow_{\omega} = \{ \sum_{B_j \in \Gamma} b_j q^{B_j} | b_j \in \mathbb{Q}, \#\{B_j | b_j \neq 0, \int_{B_j} \omega > \lambda\} < \infty, \forall \lambda \in \mathbb{R} \}.
\]

We define the corresponding (downward) valuation by

\[
v(\sum_{B \in \Gamma} a_B q^{B}) = \max\{\omega(B) : a_B \neq 0\} : \quad (3.3)
\]

It satisfies that for \( f, g \in QH_*(M) \)

(i) \( v(f \cdot g) = v(f) + v(g) \)
(ii) \( v(f + g) \leq \max\{v(f), v(g)\} \).

We like to point out that the summand in \( \Lambda^\downarrow_{\omega} \) is written as \( b_B q^B \) while the one in \( \Lambda^\uparrow_{\omega} \) as \( a_A q^{-A} \) with the minus sign. This is because we want to clearly show which one we use. Obviously \(-v\) in (3.1) and \( v\) in (3.3) satisfy the axiom of non-Archimedean norm which induce a topology on \( QH^*(M) \) and \( QH_*(M) \) respectively. In each case the finiteness assumption in the definition of the Novikov ring allows us to numerate the non-zero summands in each given Novikov chain (3.2) so that

\[
\lambda_1 > \lambda_2 > \cdots > \lambda_j > \cdots
\]
with $\lambda_j = \omega(B_j)$ or $\omega(A_j)$.

Since the downward Novikov ring appears mostly in this paper, we will just use $\Lambda_\omega$ or $\Lambda$ for $\Lambda_\omega^\downarrow$, unless absolutely necessary to emphasize the direction of the Novikov ring. We define the level and the leading order term of $b \in QH_*(M)$ similarly as in Definition 3.1 by changing the role of upward and downward Novikov rings. We have a canonical isomorphism

$$\flat : QH^*(M) \rightarrow QH_*(M); \quad \sum a_i q^{-A_i} \rightarrow \sum PD(a_i)q^{A_i}$$

and its inverse

$$\sharp : QH_*(M) \rightarrow QH^*(M); \quad \sum b_j q^{B_j} \rightarrow \sum PD(b_j)q^{-B_j}.$$

We denote by $a^\flat$ and $b^\sharp$ the images under these maps.

There exists the canonical non-degenerate pairing

$$\langle \cdot, \cdot \rangle : QH^*(M) \otimes QH_*(M) \rightarrow \mathbb{Q}$$

defined by

$$\langle \sum a_i q^{-A_i}, \sum b_j q^{B_j} \rangle = \sum (a_i, b_j)\delta_{A_iB_j} \quad (3.4)$$

where $\delta_{A,B}$ is the delta-function and $(a_i, b_j)$ is the canonical pairing between $H^*(M, \mathbb{Q})$ and $H_*(M, \mathbb{Q})$. Note that this sum is always finite by the finiteness condition in the definitions of $QH^*(M)$ and $QH_*(M)$ and so is well-defined. This is equivalent to the Frobenius pairing in the quantum cohomology ring. However we would like to emphasize that the dual vector space $(QH_*(M))^*$ of $QH_*(M)$ is not isomorphic to $QH^*(M)$ even as a $\mathbb{Q}$-vector space. Rather the above pairing induces an injection

$$QH^*(M) \hookrightarrow (QH_*(M))^*$$

whose images lie in the set of bounded linear functionals on $QH_*(M)$ with respect to the non-Archimedean norm (3.3) on $QH_*(M)$. We refer to [Br] for a good introduction to non-Archimedean analytic geometry. In fact, the description of the standard quantum cohomology in the literature is not really a ‘cohomology’ but a ‘homology’ in that it never uses linear functionals in its definition. To keep our exposition consistent with the standard literature in the Gromov-Witten invariants and the quantum cohomology, we prefer to call them the quantum cohomology rather than the quantum homology as some authors did (e.g., [Se]) in the symplectic geometry community. In Appendix 2, we will introduce a genuinely cohomological version of quantum cohomology which we call bounded quantum cohomology using the bounded linear functionals on the quantum chain complex below with respect to the topology induced by the non-Archimedean norm.

Let $(C_*, \partial)$ be any chain complex on $M$ whose homology is the singular homology $H_*(M)$. One may take for $C_*$ the usual singular chain complex or the Morse chain complex of a Morse function $f : M \rightarrow \mathbb{R}$, $(C_*(-\epsilon f), \partial_{-\epsilon})$ for some sufficiently small $\epsilon > 0$. However since we need to take a non-degenerate pairing in the chain level, we should use a model which is finitely generated. We will always prefer to use the Morse homology complex because it is finitely generated and avoids some technical issue related to singular degeneration problem (see [FOh1,2]). The negative sign in
(\(C_*(-\epsilon f), \partial_{-\epsilon f}\)) is put to make the correspondence between the Morse homology and the Floer homology consistent with our conventions of the Hamiltonian vector field (1.2) and the action functional (1.6). In our conventions, solutions of negative gradient of \(-\epsilon f\) correspond to ones for the negative gradient flow of the action functional \(\mathcal{A}_\epsilon f\). We denote by

\[(C^*(-\epsilon f), \delta_{-\epsilon f})\]

the corresponding cochain complex, i.e,

\[C^k := \text{Hom}(C_k, \mathbb{Q}), \quad \delta_{-\epsilon f} = \partial_{-\epsilon f}.\]

Now we extend the complex \((C_*(-\epsilon f), \partial_{-\epsilon f})\) to the quantum chain complex, denoted by

\[(CQ_*(-\epsilon f), \partial_Q)\]

\[CQ_*(-\epsilon f) := C_*(-\epsilon f) \otimes \Lambda_\omega, \quad \partial_Q := \partial_{-\epsilon f} \otimes \Lambda_\omega. \quad (3.5)\]

This coincides with the Floer complex \((CF_*(\epsilon f), \partial)\) as a chain complex if \(\epsilon\) is sufficiently small. Similarly we define the quantum cochain complex \((CQ^*(-\epsilon f), \delta_Q)\) by changing the downward Novikov ring to the upward one. In other words, we define

\[CQ^*(-\epsilon f) := CM_{2n-\epsilon}(\epsilon f) \otimes \Lambda^\uparrow, \quad \delta_Q := \partial_{\epsilon f} \otimes \Lambda^\uparrow_\omega.\]

Again we would like to emphasize that \(CQ^*(-\epsilon f)\) is not isomorphic to the dual space of \(CQ_*(-\epsilon f)\) as a \(\mathbb{Q}\)-vector space. We refer to Appendix 2 for some further discussion on this issue.

It is well-known that the corresponding homology of this complex is independent of the choice \(f\) and isomorphic to the above quantum cohomology (resp. the quantum homology) as a ring (see [PSS], [LT2], [Lu] for its proof). However, to emphasize the role of the Morse function in the level of complex, we denote the corresponding homology by \(HQ^*(-\epsilon f) \cong QH^*(M)\).

With these definitions, we have the obvious non-degenerate pairing

\[CQ^*(-\epsilon f) \otimes CQ_*(-\epsilon f) \to \mathbb{Q} \quad (3.6)\]

in the chain level which induces the pairing (3.4) above in homology.

We now choose a generic Morse function \(f\). Then for any given homotopy \(\mathcal{H} = \{H^s\}_{s \in [0,1]}\) with \(H^0 = \epsilon f\) and \(H^1 = H\), we denote by

\[h_\mathcal{H} : CQ_*(-\epsilon f) = CF_*(\epsilon f) \to CF_*(H) \quad (3.7)\]

the standard Floer chain map from \(\epsilon f\) to \(H\) via the homotopy \(\mathcal{H}\). This induces a homomorphism

\[h_\mathcal{H} : HQ_*(-\epsilon f) \cong HF_*(\epsilon f) \to HF_*(H). \quad (3.8)\]

Although (3.7) depends on the choice \(\mathcal{H}\), (3.8) is canonical, i.e, does not depend on the homotopy \(\mathcal{H}\). One confusing point in this isomorphism is the issue of grading. See the next section for a review of the construction of this chain map and the issue of grading of \(HF_*(H)\).
§4. Filtered Floer homology

For each given generic non-degenerate $H : S^1 \times M \to \mathbb{R}$, we consider the free $\mathbb{Q}$ vector space over

$$\text{Crit}_{A_H} = \{ [z, w] \in \tilde{\Omega}_0(M) \mid z \in \text{Per}(H) \}. \quad (4.1)$$

To be able to define the Floer boundary operator correctly, we need to complete this vector space downward with respect to the real filtration provided by the action $A_H([z, w])$ of the element $[z, w]$ of (4.1). More precisely,

**Definition 4.1.** We call the formal sum

$$\beta = \sum_{[z, w] \in \text{Crit}_{A_H}} a_{[z, w]} [z, w], \quad a_{[z, w]} \in \mathbb{Q} \quad (4.2)$$

a Novikov chain if there are only finitely many non-zero terms in the expression (4.2) above any given level of the action. We denote by $CF_*(H)$ the set of Novikov chains. We call those $[z, w]$ with $a_{[z, w]} \neq 0$ generators of the chain $\beta$ and just denote as

$$[z, w] \in \beta$$

in that case. Note that $CF_*(H)$ is a graded $\mathbb{Q}$-vector space which is infinite dimensional in general, unless $\pi_2(M) = 0$.

We briefly review construction of basic operators in the Floer homology theory [Fl2]. Let $J = \{J_t\}_{0 \leq t \leq 1}$ be a periodic family of compatible almost complex structure on $(M, \omega)$.

For each given pair $(J, H)$, we define the boundary operator

$$\partial : CF_*(H) \to CF_*(H)$$

considering the perturbed Cauchy-Riemann equation

$$\begin{cases} \frac{\partial u}{\partial \tau} + J \left( \frac{\partial u}{\partial t} - X_H(u) \right) = 0 \\
\lim_{\tau \to -\infty} u(\tau) = z^-, \lim_{\tau \to \infty} u(\tau) = z^+ \end{cases} \quad (4.3)$$

This equation, when lifted to $\tilde{\Omega}_0(M)$, defines nothing but the negative gradient flow of $A_H$ with respect to the $L^2$-metric on $\tilde{\Omega}_0(M)$ induced by the family of metrics on $M$

$$g_{J_t} = (\cdot, \cdot)_{J_t} := \omega(\cdot, J_t \cdot) :$$

This $L^2$-metric is defined by

$$\langle \xi, \eta \rangle_{J_t} := \int_0^1 (\xi, \eta)_{J_t} \, dt.$$  

We will also denote

$$\|v\|_{J_0}^2 = \langle v, v \rangle_{J_0} = \omega(v, J_0 v) \quad (4.4)$$

for $v \in TM$. 
For each given \([z^-, w^-]\) and \([z^+, w^+]\), we define the moduli space
\[
\mathcal{M}(H, J; [z^-, w^-], [z^+, w^+])
\]
of solutions \(u\) of (4.3) satisfying
\[
w^- \# u \sim w^+ \tag{4.5}
\]
\(\partial\) has degree \(-1\) and satisfies \(\partial \circ \partial = 0\).

When we are given a family \((j, H)\) with \(H = \{H^s\}_{0 \leq s \leq 1}\) and \(j = \{J^s\}_{0 \leq s \leq 1}\), the chain homomorphism
\[
h_{(j, H)} : CF_*(J^0, H^0) \to CF_*(J^1, H^1)
\]
is defined by the non-autonomous equation
\[
\begin{cases}
\frac{\partial u}{\partial \tau} + J^1\rho_1(\tau)\left(\frac{\partial u}{\partial \tau} - X_{H^2(\tau)}(u)\right) = 0 \\
\lim_{\tau \to -\infty} u(\tau) = z^-, \lim_{\tau \to \infty} u(\tau) = z^+
\end{cases} \tag{4.6}
\]
also with the condition (4.5). Here \(\rho_i, i = 1, 2\) is the cut-off functions of the type \(\rho : \mathbb{R} \to [0, 1]\),
\[
\rho(\tau) = \begin{cases}
0 & \text{for } \tau \leq -R \\
1 & \text{for } \tau \geq R
\end{cases}
\]
\(\rho'(\tau) \geq 0\)
for some \(R > 0\). \(h_{(j, H)}\) has degree 0 and satisfies
\[
\partial_{(J^1, H^1)} \circ h_{(j, H)} = h_{(j, H)} \circ \partial_{(J^0, H^0)}.
\]

Finally when we are given a homotopy \((\bar{j}, \bar{H})\) of homotopies with \(\bar{j} = \{j_\kappa\}, \bar{H} = \{H_\kappa\}\), consideration of the parameterized version of (4.6) for \(0 \leq \kappa \leq 1\) defines the chain homotopy map
\[
H_{\bar{H}} : CF_*(J^0, H^0) \to CF_*(J^1, H^1) \tag{4.7}
\]
which has degree +1 and satisfies
\[
h_{(j_\kappa, H_\kappa)} - h_{(j_0, H_0)} = \partial_{(J^1, H^1)} \circ H_{\bar{H}} + H_{\bar{H}} \circ \partial_{(J^0, H^0)}.
\]

By now, construction of these maps using these moduli spaces has been completed with rational coefficients (See [FOn], [LT1] and [Ru]) using the techniques of virtual moduli cycles. We will suppress this advanced machinery from our presentation throughout the paper. The main stream of the proof is independent of this machinery except that it is implicitly needed to prove that various moduli spaces we use are non-empty. Therefore we do not explicitly mention these technicalities in the main body of the paper until §11, unless it is absolutely necessary. In §11, we will provide justification of this in the general case all at once.

Now we consider a Novikov chain
\[
\beta = \sum a_{[p, w]} [p, w], \quad a_{[p, w]} \in \mathbb{Q} \tag{4.9}
\]
As in [Oh6], we introduce the following which is a crucial concept for the mini-max argument we carry out later.
**Definition 4.2.** Let \( \beta \) be a Novikov chain in \( \mathrm{CF}_*(H) \). We define the level of the cycle \( \beta \) and denote by

\[
\lambda_H(\beta) = \max_{[p,w]} \{A_H([p,w]) \mid a_{[p,w]} \neq 0 \text{ in } (4.9)\}
\]

if \( \beta \neq 0 \), and just put \( \lambda_H(0) = +\infty \) as usual.

The following upper estimate of the action change can be proven by the same argument as that of the proof of [Theorem 5.2, Oh3]. We would like to emphasize that in general there does not exist a lower estimate of this type. The upper estimate is just one manifestation of the ‘positivity’ phenomenon in symplectic topology through the existence of pseudo-holomorphic curves that was first discovered by Gromov [Gr]. We would like to point out that the equations (4.3), (4.6) can be studied for any \( H \) or (\( \mathcal{H},j \)) which are not necessarily non-degenerate or generic, although the Floer complex or the operators may not be defined for such choices.

**Proposition 4.3.** Let \( H, K \) be any Hamiltonian not necessarily non-degenerate and \( j = \{J^x\}_{x \in [0,1]} \) be any given homotopy and \( \mathcal{H}^{\text{lin}} = \{H^x\}_{0 \leq s \leq 1} \) be the linear homotopy \( H^x = (1-s)H + sK \). Suppose that (4.6) has a solution satisfying (4.5). Then we have the identity

\[
\mathcal{A}_F([z^+,w^+]) - \mathcal{A}_H([z^-,w^-]) = -\int_0^1 \left( \rho_2(t) - \rho_1(t) \right) dt. \tag{4.10}
\]

**Proof.** Let \([z^+, w^+] \in \text{Crit}_A_K \) and \([z^-, w^-] \in \text{Crit}_A_H \) be given. As argued in [Oh3], for any given solution \( u \) of (4.5) and (4.6), we compute

\[
\mathcal{A}_K([z^+, w^+]) - \mathcal{A}_H([z^-, w^-]) = \int_{-\infty}^{\infty} \frac{d}{d\tau}(\mathcal{A}_{H^\tau}(u(\tau))) d\tau.
\]

Here we have

\[
\frac{d}{d\tau}(\mathcal{A}_{H^\tau}(u(\tau))) = \frac{d}{d\tau}(\mathcal{A}_{H^\tau}(\mathcal{A}_{H^\tau}(u(\tau)))) - \int_0^1 \left( \frac{\partial H^\tau(\tau)}{\partial \tau} \right)(u(t)) dt. \tag{4.13}
\]

However since \( u \) satisfies (4.6), we have

\[
d(\mathcal{A}_{H^\tau}(u(\tau))) \frac{\partial u}{\partial \tau} = \int_0^1 \omega \left( \frac{\partial u}{\partial t} - X_{H^\tau}(u), \frac{\partial u}{\partial \tau} \right) dt
\]

\[
= -\int_0^1 \left( \frac{\partial H^\tau(\tau)}{\partial \tau} \right)(u(t)) dt \tag{4.14}
\]

and

\[
\int_0^1 \left( \frac{\partial H^\tau(\tau)}{\partial \tau} \right)(u(t)) dt = -\int_0^1 \rho_2(\tau)(K-H)(u,t) dt \tag{4.15}
\]

Substituting (4.14), (4.15) into (4.13) and integrating (4.13) over \(-\infty < \tau < \infty\), we have obtained (4.10). For (4.11), we just note \( \rho_2(\tau) \geq 0 \). (4.12) is obvious from (4.11). This finishes the proof. \( \square \)

By considering the case \( K = H \), we immediately have
Corollary 4.4. For a fixed $H$ and for a given one parameter family $j = \{J^s\}_{s \in [0,1]}$, let $u$ be as in Proposition 4.3. Then we have

\[ A_H([z^+, w^+]) - A_H([z^-, w^-]) = -\int \left| \frac{\partial u}{\partial \tau} \right|^2_{\rho_1(\tau)} \leq 0. \]

Remark 4.5. We would like to remark that similar calculation proves that there is also an uniform upper bound $C(j, H)$ for the chain map over general homotopy $(j, H)$ or for the chain homotopy maps (4.7). Existence of such an upper estimate is crucial for the construction of these maps. This upper estimate depends on the choice of homotopy $(j, H)$ and is related to the curvature estimates of the relevant Hamiltonian fibration (see [Po2], [En1]). By taking the infimum over all such paths with fixed ends, whenever $[z^+, w^+]$, $[z^-, w^-]$ allow a solution of (4.6) for some choice of $(j, H)$, we can derive

\[ A_K([z^+, w^+]) - A_K([z^-, w^-]) \leq C(K, H) \]

where $C(K, H)$ is a constant depending only on $K, H$ but not on the solution or the homotopies.

Now we recall that $CF_*(H)$ is also a $\Lambda$-module: each $A \in \Gamma$ acts on $Crit A_H$ and so on $CF_*(H)$ by "gluing a sphere"

\[ A : [z, w] \to [z, w \# A]. \]

Then $\partial$ is $\Lambda$-linear and induces the standard Floer homology $HF_*(H; \Lambda)$ with $\Lambda$ as its coefficients. However the action does not preserve the filtration we defined above. Whenever we talk about filtration, we will always presume that the relevant coefficient ring is $\mathbb{Q}$.

For each given pair of real numbers $[\lambda, \mu]$, we define

\[ CF^{[\lambda, \mu]} := CF^\mu / CF^\lambda. \]

Then for each triple $\lambda < \mu < \nu$ where $\lambda = -\infty$ or $\nu = \infty$ are allowed, we have the short-exact sequence of the complex of graded $\mathbb{Q}$ vector spaces

\[ 0 \to CF_k^{[\lambda, \mu]}(H) \to CF_k^{[\mu, \nu]}(H) \to CF_k^{[\lambda, \nu]}(H) \to 0 \]

for each $k \in \mathbb{Z}$. This then induces the long exact sequence of graded modules

\[ \cdots \to HF_k^{[\lambda, \mu]}(H) \to HF_k^{[\mu, \nu]}(H) \to HF_k^{[\lambda, \nu]}(H) \to HF_k^{[\lambda, \mu]}(H) \to \cdots \]

whenever the relevant Floer homology groups are defined.

We close this section by fixing our grading convention for $HF_*(H)$. This convention is the analog to the one we use in [Oh3.5] in the context of Lagrangian submanifolds. We first recall that solutions of the negative gradient flow equation of $-f$, (i.e., of the positive gradient flow of $f$)

\[ \dot{\chi} - \text{grad } f(\chi) = 0 \]
corresponds to the *negative* gradient flow of the action functional $A_\epsilon f$. This gives rise to the relation between the Morse indices $\mu_{Morse}(p)$ and the Conley-Zehnder indices $\mu_{CZ}([p, \tilde{p}] ; \epsilon f)$ (see [Lemma 7.2, SZ] but with some care about the different convention of the Hamiltonian vector field. Their definition of $X_H$ is $-X_H$ in our convention):

$$\mu_{CZ}([p, \tilde{p}] ; \epsilon f) = n - \mu_{Morse}(p)$$

(4.17)

in our convention. On the other hand, obviously we have

$$n - \mu_{Morse}(p) = n - (2n - \mu_{Morse}(p)) = \mu_{Morse}(p) - n$$

Because of this reason, we will grade $HF_* (H)$ by the integer

$$k = \mu_{CZ} + n.$$  

(4.18)

This grading convention makes the degree $k$ of $[q, \hat{q}]$ in $CF_*(\epsilon f)$ coincides with the Morse index of $q$ of $\epsilon f$ for each $q \in \text{Crit}\epsilon f$. Recalling that we chose the Morse complex

$$CM_*(-\epsilon f) \otimes \Lambda$$

for the quantum chain complex $CQ_*(-\epsilon f)$, it also coincides with the standard grading of the quantum cohomology via the map

$$\flat : QH^k(M) \to QH_{2n-k}(M).$$

Form now on, we will just denote by $\mu_H([z, w])$ the Conley-Zehnder index of $[z, w]$ for the Hamiltonian $H$. Under this grading, we have the following grading preserving isomorphism

$$QH^k(M) \to QH_{2n-k}(M) \cong HQ_{2n-k}(-\epsilon f) \to HF_k(\epsilon f) \to HF_k(H).$$

(4.19)

We will also show in §6 that this grading convention makes the pants product, denoted by $\ast$, preserves the grading:

$$\ast : HF_k(H) \otimes HF_\ell(K) \to HF_{k+\ell}(H \# K)$$

just as the quantum product does

$$\cdot : QH^k(M) \otimes QH^\ell(M) \to QH^{k+\ell}(M).$$

Remark 4.6. If we give the grading on $FH_* (H)$ by the Conley-Zehnder index $\mu_{CZ}$, then our grading convention means that we consider the ‘shifted graded complex’

$$HF(H)[n]$$

in the notation of homological algebra. In the standard homological algebra, the shifted complex $C[n]$ of the graded complex $(C, \partial C)$ is defined by the identity

$$(C[n])_k = C_{n+k}, \quad \partial_{C[n]}^C = (-1)^n \partial_{n+k}^C$$

and the corresponding homology $H(C[n], \partial C[n])$ is written as $H(C)[n]$. This point of view seems to be more natural than our convention. But we will stick to this convention at least in this paper to simplify the labelling of degrees in relation to the pants product.
§5. Definition of the spectral invariants

In this section, we associate some homologically essential critical values of the action functional $\mathcal{A}_g$ to each $\tilde{\phi} \in \widetilde{\text{Ham}}(M, \omega)$ and quantum cohomology class $a$, and call them the spectral invariants. We denote this assignment by

$$\rho: \widetilde{\text{Ham}}(M, \omega) \times QH^*(M) \to \mathbb{R}$$

as described in the introduction of this paper. Before launching our construction, some overview of our construction of spectral invariants is necessary.

5.1. Overview of the construction

We recall the canonical isomorphism $h_{\alpha\beta}: FH_*(H_{\alpha}) \to FH_*(H_{\beta})$ which satisfies the composition law

$$h_{\alpha\gamma} = h_{\alpha\beta} \circ h_{\beta\gamma}.$$ 

We denote by $FH_*(M)$ the corresponding model $\mathbb{Q}$-vector space. We also note that $FH_*(H)$ is induced by the filtered chain complex $(CF_*^\lambda(H), \partial)$ where

$$CF_*^\lambda(H) = \text{span}_\mathbb{Q}\{\alpha \in CF_*(H) \mid \lambda_H(\alpha) \leq \lambda\}$$

i.e., the sub-complex generated by the critical points $[z, w] \in \text{Crit}_{A_H}$ with $A_H([z, w]) \leq \lambda$.

Then there exists a canonical inclusion

$$i_\lambda: CF_*^\lambda(H) \to CF_*^\infty(H) := CF_*(H)$$

which induces a natural homomorphism $i_\lambda: HF_*^\lambda(H) \to HF_*(H)$. For each given element $\ell \in FH_*(M)$ and Hamiltonian $H$, we represent the class $\ell$ by a Novikov cycle $\alpha$ of $H$ and measure its level $\lambda_H(\alpha)$ and define

$$\rho(H; \ell) := \inf\{\lambda \in \mathbb{R} \mid \ell \in \text{Im} i_\lambda\}$$

or equivalently

$$\rho(H; \ell) := \inf_{\alpha, i_\lambda(\alpha) = \ell} \lambda_H(\alpha).$$

The crucial task then is to prove that for each (homogeneous) element $\ell \neq 0$, the value $\rho(H; \ell)$ is finite, i.e., “the cycle $\alpha$ is linked and cannot be pushed away to infinity by the negative gradient flow of the action functional”. In the classical critical point theory (see [BnR] for example), this property of semi-infinite cycles is called the linking property. The problem with the above definition, though simple and a posteriori correct (see Theorem 5.5), is that there is no way to see the linking property or the criticality of the mini-max value $\rho(H; \ell)$ out of the definition.
We will prove this finiteness first for the Hamiltonian $\epsilon f$ where $f$ is a Morse-Smale function and $\epsilon$ is sufficiently small. This finiteness relies on the facts that the Floer boundary operator $\partial_{\epsilon f}$ in this case has the form

$$\partial_{\epsilon f} = \partial_{Morse}^{\epsilon f} \otimes \Lambda$$

i.e., “there is no quantum contribution on the Floer boundary operator”, and that the classical Morse theory proves that $\partial_{Morse}^{\epsilon f}$ cannot push down the level of a non-trivial cycle more than $-\epsilon \max f$ (see [Oh6]).

Once we prove the finiteness for $\epsilon f$, we compare the cycles in $CF_*(\epsilon f)$ and the transferred cycles in $CF_*(H)$ by the chain map $h_H: CF_*(\epsilon f) \to CF_*(H)$ where $H$ is a homotopy connecting $\epsilon f$ and $H$. The change of the level then can be measured by judicious use of (4.7) and Remark 4.5 which will prove the finiteness for any $H$. However at this stage, $\rho(H; \ell)$ appears to depend on the initial Hamiltonian $\epsilon f$. So we need to prove that $\rho(H; \ell)$ is indeed independent of the choice. This will be done by considering the level change between arbitrary pair $(H, K)$ using (4.7) and proving that the limit

$$\lim_{\epsilon \to 0} \rho(\epsilon f; \ell)$$

is independent of the choice of Morse function $f$. In this procedure, we can avoid considering the ‘singular limit’ of the ‘chains’ (See the Appendix 1 for some illustration of the difficulty in studying such limits). We only need to consider the limit of the values $\rho(H; \ell)$ as $H \to 0$ which is a much simpler task than considering the limit of chains which involves highly non-trivial analytical work (we refer to the forthcoming work [FOh2] for the consideration of this limit in the chain level). Our way of defining the value $\rho(H; \ell)$ is also crucial to prove that these mini-max values are indeed critical values of $A_H$.

5.2. Finiteness; the linking property of semi-infinite cycles

With this overview, we now start with our construction. We first recall the natural pairing

$$\langle \cdot, \cdot \rangle: CQ^*(-\epsilon f) \otimes CQ_*(-\epsilon f) \to \mathbb{Q}:$$

where we have

$$CQ_k(-\epsilon f) := (CM_k(-\epsilon f), \partial_{-\epsilon f}) \otimes \Lambda^\dagger$$

$$CQ^k(-\epsilon f) := (CM_{2n-k}(\epsilon f), \partial_{\epsilon f}) \otimes \Lambda^\dagger.$$

**Remark 5.1.** We would like to emphasize that in our definition $CQ^k(-\epsilon f)$ is not isomorphic to $\text{Hom}_\mathbb{Q}(CQ_k(-\epsilon f), \mathbb{Q})$ in general. However there is a natural homomorphism

$$CQ^k(-\epsilon f) \to \text{Hom}_\mathbb{Q}(CQ_k(-\epsilon f), \mathbb{Q}); \quad a \mapsto \langle a, \cdot \rangle$$

whose image lies in the subset of bounded linear functionals

$$\text{Hom}_{\text{bdd}}(CQ_k(-\epsilon f), \mathbb{Q}) := CQ^k_{\text{bdd}}(-\epsilon f) \subset \text{Hom}_\mathbb{Q}(CQ_k(-\epsilon f), \mathbb{Q}).$$

See Appendix 2 for more discussions on this aspect. We would like to emphasize that (5.2) is well-defined because of the choice of directions of the Novikov rings $\Lambda^\dagger$.
and $\Lambda^\perp$. In general, the map (5.2) is injective but not an isomorphism. Polterovich [Po4, [EnP] observed that this point is closely related to certain failure of “Poincaré duality” of the Floer homology with Novikov rings as its coefficients.

Let $\tilde{\phi} \in \widetilde{\text{Hom}}(M, \omega)$ and $\tilde{\phi} = [\phi, H]$ for some Hamiltonian $H$. We consider paths $\mathcal{H} = \{H^s\}$ with

$$H^0 = \epsilon f, \quad H^1 = H.$$ Morally one might want to consider this path as a path from $0$ to $\tilde{\phi}$ on $\widetilde{\text{Hom}}(M, \omega)$.

Each such path defines the Floer chain map

$$h_{\mathcal{H}} : CF_*(\epsilon f) \to CF_*(H)$$

Now we are ready to give the definition of our spectral invariants. Previously in [Oh6], the author outlined this construction for the classical cohomology class in $H^*(M) \subset QH^*(M)$.

**Definition 5.2.** Let $H$ be a generic non-degenerate Hamiltonian and $\mathcal{H}$ be a generic path from $\epsilon f$ to $H$. We denote by $h_{\mathcal{H}} : CF_*(\epsilon f) \to CF_*(H)$ be the Floer chain map. Then for each given $a \in QH^k(M) \cong HQ_k(-\epsilon f)$, we define

$$\rho(H, a : \epsilon, f) = \inf_H \inf_\alpha \{ \lambda_H(h_{\mathcal{H}}(\alpha)) \mid [\alpha] = a^\perp, \alpha \in CF_k(\epsilon f) \}$$

**Proposition 5.3.** Let $H$ be as above. For each given Morse function $f$ and for sufficiently small $\epsilon_0 = \epsilon_0(f) > 0$, the number $\rho(H, a; \epsilon, f)$ is finite and the assignment

$$(H, \epsilon) \to \rho(H, a; \epsilon, f)$$

is continuous over $\epsilon \in (0, \epsilon_0)$ and with respect to $C^0$-topology of $H$. Furthermore the limit

$$\rho(H, a; f) := \lim_{\epsilon \to 0} \rho(H, a; \epsilon, f)$$

is independent of $f$.

**Proof.** We fix $\epsilon_0 > 0$ so small that there is no quantum contribution for the Floer boundary operator $\partial_{\epsilon f}$ i.e, we have

$$\partial_{\epsilon f} \simeq \partial_{-\epsilon f} \otimes \Lambda^\perp.$$ (5.3)

Then by considering the Morse homology of $-\epsilon f$, we have the identity

$$QH^*(M) \cong \ker \partial_{-\epsilon f} \otimes \Lambda^\perp / \text{Im} \partial_{-\epsilon f} \partial_{-\epsilon f} \otimes \Lambda^\perp = HM_*(\epsilon f) \otimes \Lambda^\perp$$

$$QH_*(M) \cong \ker \partial_{-\epsilon f} \partial_{-\epsilon f} \otimes \Lambda^\perp / \text{Im} \partial_{-\epsilon f} \partial_{-\epsilon f} \otimes \Lambda^\perp = HM_*(-\epsilon f) \otimes \Lambda^\perp.$$ (5.4)

For the first isomorphism, we use the fact

$$(C^*(-\epsilon f), \delta_{-\epsilon f}) \cong (C_*(\epsilon f), \partial_{\epsilon f})$$

by taking the gradient flow upside down. We represent $a^\perp \in QH_{2n-k}(M)$ by a Novikov cycle of $\epsilon f$ where

$$a = \sum_A a_p \otimes q^A$$ (5.4)
with \( a_p \in \mathbb{Q} \) and \( p \in \text{Crit}_f(-\epsilon f) \) and
\[
2n - k = \mu_{-\epsilon f}^{\text{Morse}}(p) + c_1(-A).
\] (5.5)

Recalling that
\[
\mu_{-\epsilon f}^{\text{Morse}}(p) = n - \mu_f([p, \hat{p}])
\]
in our convention, (5.5) is equivalent to
\[
k = \mu_f(p \otimes q^A) + n
\] (5.6)
where \( \mu_H \) is the Conley-Zehnder index of the element \( p \otimes q^A = [p, \hat{p}#A] \). Therefore we have
\[
\text{CF}_k(\epsilon f) \cong C\mathbb{Q}_{2n-k}(-\epsilon f)
\]
as a chain in \( \text{CF}_k(\epsilon f) \). \( \alpha \) has the level
\[
\lambda_f(\alpha) = \max \{-\epsilon f(p) - \omega(A) \mid a_p \otimes q^A \neq 0\}
\] (5.7)
because \( A_{-\epsilon f}([p, \hat{p}#A]) = -\epsilon f(p) - \omega(A) \). Now the most crucial point in our construction is to prove the finiteness
\[
\inf_{[\alpha] = a} \lambda_f(\alpha) > -\infty.
\] (5.8)

The following lemma proves this linking property.

**Lemma 5.4.** Let \( a \neq QH^k(M) \) and \( a^b \in QH_{2n-k}(M) \) be its dual. Suppose that
\[
a^b = \sum_j a_j q^{A_j}
\]
with \( 0 \neq a_j \in H_{2n-k+2c_1(A_j)}(M) \) and
\[
\lambda_1 > \lambda_2 > \lambda_3 > \cdots
\] (5.9)
where \( \lambda_j = -\omega(A_j) \). Denote by \( \alpha \) a Novikov cycle of \( \epsilon f \) with \( [\alpha] = a^b \in HF_k(\epsilon f) \cong QH_{2n-k}(M) \) and define the ‘gap’
\[
c(a) := \min \{\lambda_1 - \lambda_2, |\lambda_1|\}.
\]

Then we have
\[
v(a) - \frac{1}{2} c(a) \leq \inf_\alpha \{\lambda_f(\alpha) \mid [\alpha] = a^b\} \leq v(a) + \frac{1}{2} c(a)
\] (5.10)
for any sufficiently small \( \epsilon > 0 \) and in particular, (5.8) holds. We also have
\[
\lim_{\epsilon \to 0} \inf_\alpha \{\lambda_f(\alpha) \mid [\alpha] = a^b\} = v(a)
\] (5.11)
and so the limit is independent of the choice of Morse functions \( f \).

**Proof.** We first like to point out that the quantum cohomology class
\[
a = \sum_A a_A q^{-A}
\]
uniquely determines the set
\[ \Gamma(a) := \{ A \in \Gamma \mid a_A \neq 0 \}. \]

By the finiteness condition in the formal series, we can numerate \( \Gamma(a) \) as in (5.9) without loss of generalities.

We represent \( a^{\flat} \) by a Novikov cycle
\[ \alpha = \sum_A \alpha_A q^A, \quad \alpha_A \in CM_*(-\epsilon f) \]
of \( \epsilon f \). Because of (5.3), all the coefficient Morse chains in this sum must be cycles and if \( A \notin \Gamma(a) \), the corresponding coefficient cycle must be exact. Therefore we can write \( \alpha \) as
\[ \alpha = \sum_j \alpha_j q^{A_j} + \partial \epsilon f(\gamma). \]

By removing the exact term \( \partial \epsilon f(\gamma) \) when we take the infimum over the cycles \( \alpha \) with \([\alpha] = a^{\flat}\) for the definition of \( \rho(H, a; \epsilon, f) \), we may always assume that \( \alpha \) has the form
\[ \alpha = \sum_j \alpha_j q^{A_j} \quad (5.12) \]
with \( A_j \in \Gamma(a) \). Then again by (5.3), we have
\[ [\alpha_j] = a_j \in H_*(M) \]
and hence we have \( v(a) = -\omega(A_1) = \lambda_1 \). Furthermore we have
\[ \lambda_{\epsilon f}(a) = \max\{\lambda_{\epsilon f}(a_1 q^{A_1}), \lambda_{\epsilon f}(a_2 q^{A_2})\} \]
provided \( \epsilon \) is sufficiently small. Therefore we derive from (5.7)
\[ \max\{-\omega(A_1) - \epsilon \max f, -\omega(A_2) + \epsilon \min f\} \leq \lambda_{\epsilon f}(a) \]
\[ \leq \max\{-\omega(A_1) + \epsilon \max f, -\omega(A_2) + \epsilon \min f\}. \quad (5.13) \]

(5.10) follows from (5.13) if we choose \( \epsilon \) so that \( \epsilon \| f \| < \frac{c(a)}{2} \). (5.11) also immediately follows from (5.13). \( \square \)

Now we go back to the proof of Proposition 5.3. Let \( \alpha \in CF_*(\epsilon f) \) be as above with \([\alpha] = a^{\flat}\), and \( H \) be a homotopy connecting \( \epsilon f \) and \( H \). We also denote by \( H^{lin} \)
the linear homotopy
\[ H^{lin} : s \mapsto (1 - s)\epsilon f + \epsilon H. \]

Obviously we have
\[ \inf_{H} \lambda_H(h_H(\alpha)) \leq \lambda_H(h_{H^{lin}}(\alpha)). \quad (5.14) \]

On the other hand for the linear homotopy, we have the inequality
\[ \lambda_H(h_{H^{lin}}(\alpha)) \leq \lambda_{\epsilon f}(a) + \int_0^1 -\min(H - \epsilon f) \, dt \quad (5.15) \]
by (4.8) (see [§8, Oh6] for detailed verification). To obtain a lower bound, we consider the composition $h_{(H^\text{lin})^{-1}} \circ h_H$. By the same calculation as for (5.15), we have

$$
\lambda_{ef}(h_{(H^\text{lin})^{-1}} \circ h_H(\alpha)) \leq \lambda_H(h_H(\alpha)) + \int_0^1 \min(\epsilon f - H) \, dt \\
\leq \lambda_H(h_H(\alpha)) + \int_0^1 \max(H - \epsilon f) \, dt
$$

(5.16)

However since $h_{(H^\text{lin})^{-1}} \circ h_H$ is chain homotopic to the identity map we have

$$
h_{(H^\text{lin})^{-1}} \circ h_H(\alpha) = \alpha + \partial_{ef}(\beta)
$$

(5.17)

for some $\beta$. If we write

$$
\alpha = q^A_1(a_1 + \sum_{k \geq 2} a_k q^A_k - A_1)
$$

with $a_1 \neq 0 \in C_*(-\epsilon f)$, then we have

$$
\alpha + \partial_{ef}(\beta) = q^A_1\left(a_1 + \sum_{k \geq 2} a_k q^A_k - A_1 + \partial_{ef}(q^{-A_1}\beta)\right).
$$

We recall $\partial_{ef} = \partial_{\text{Morse}}^{\text{Morse}} \otimes \Lambda$ and decompose

$$
q^{-A_1}\beta = \beta_0 + \beta'
$$

where $\beta_0$ is the sum of critical points of $[p, \tilde{p}]$ while $\beta'$ is the sum of those $[p, w]$ with $[w] \neq 0$. Since $a_1$ is a sum of critical points of trivial homotopy class, it follows that $\partial_{ef}(\beta')$ cannot cancel terms in $a_1$. On the other hand, the finite dimensional Morse theory implies

$$
\lambda_{ef}(a_1 + \partial_{\text{Morse}}^{\text{Morse}}(\beta_0)) \geq \lambda_{ef}(a_1) - \epsilon \max f
$$

and hence

$$
\lambda_{ef}(\alpha + \partial_{ef}(\beta)) \geq \lambda_{ef}(a_1) - \epsilon \max f - \omega(A_1)
$$

(5.18)

It follows from (5.17) and (5.18) that

$$
\lambda_{ef}(h_{(H^\text{lin})^{-1}} \circ h_H(\alpha)) \geq \lambda_{ef}(\alpha) - \epsilon \max f - \omega(A_1).
$$

(5.19)

Combining (5.16)-(5.19), we derive

$$
\lambda_H(h_H(\alpha)) \geq \lambda_{ef}(\alpha) - \epsilon \max f - \omega(A_1) - \int_0^1 \max(H - \epsilon f) \\
\geq \lambda_{ef}(\alpha) + \int_0^1 - \max H \, dt - \epsilon \|f\| - \omega(A_1)
$$

for all $H$ and so have proven that $\rho(H, a; \epsilon, f)$ is finite.
To prove the continuity statement, we consider general pairs $H, K$. Let $\delta > 0$ be any given number. We choose homotopies $\mathcal{H}_H$ from $\epsilon f$ to $H$ and $\mathcal{H}_K$ from $\epsilon f$ to $K$ such that
\[
\lambda_H(\mathcal{H}_H(\alpha)) \leq \rho(H, a; \epsilon, f) + \delta
\]
\[
\lambda_K(\mathcal{H}_K(\alpha)) \leq \rho(K, a; \epsilon, f) + \delta
\]
(5.20)
By considering the linear homotopy $h_{HK}^{lin}$ from $H$ to $K$, we derive
\[
\lambda_K(h_{HK}^{lin} \circ h_{H}(\alpha)) \leq \lambda_H(h_{H}(\alpha)) + \int_{-\min(K-H)}^1 dt.
\]
(5.21)
On the other hand (5.20) implies
\[
\lambda_H(h_{H}(\alpha)) + \int_{-\min(K-H)}^1 dt \leq \rho(H, a; \epsilon, f) + \delta + \int_{-\min(K-H)}^1 dt.
\]
Since $h_{HK}^{lin} \circ h_{H}$ is homologous to $h_{H}$ and by the gluing of chain maps we have
\[
h_{HK}^{lin} \circ h_{H} = h_{H}^{lin} \#_{R} h_{H}
\]
in the chain level for sufficiently large $R > 0$, we have
\[
\lambda_K(h_{HK}^{lin} \circ h_{H}(\alpha)) = \lambda_K(h_{H}^{lin} \#_{R} h_{H}(\alpha)) \geq \rho(K, a; \epsilon, f)
\]
(5.23)
by the definition of $\rho(K, a; \epsilon, f)$. Combining (5.21)-(5.23), we have derived
\[
\rho(K, a; \epsilon, f) - \rho(H, a; \epsilon, f) \leq \delta + \int_{0}^{1} -\min(K-H) dt.
\]
By changing the role of $H$ and $K$, we also derive
\[
\rho(H, a; \epsilon, f) - \rho(K, a; \epsilon, f) \leq \delta + \int_{0}^{1} -\min(H-K) dt.
\]
Hence, we have the inequality
\[
-\delta + \int_{0}^{1} -\max(K-H) dt \leq \rho(K, a; \epsilon, f) - \rho(H, a; \epsilon, f)
\]
\[
\leq \delta + \int_{0}^{1} -\min(K-H) dt.
\]
Noting that $\delta$ can be chosen arbitrarily once $H, K$ are fixed, we have proven
\[
\int_{0}^{1} -\max(K-H) dt \leq \rho(K, a; \epsilon, f) - \rho(H, a; \epsilon, f) \leq \int_{0}^{1} -\min(K-H) dt.
\]
(5.24)
This immediately implies that $\rho(H, a; \epsilon, f)$ is a continuous function of $H$ in $C^0$-topology (or more precisely with respect to Hofer’s norm).

Continuity of $\rho(H, a; \epsilon, f)$ with respect to $\epsilon$ or $f$, existence of the limit as $\epsilon \to 0$ and independence of the limit of $f$ can be proven similarly which we omit. This finishes the proof of Proposition 5.3. $\square$

Proposition 5.3 enables us to extend the definition of $\rho$ by continuity to arbitrary $C^0$-Hamiltonians.
**Definition & Theorem 5.5.** Let $C^0_m(M \times [0,1], \mathbb{R})$ be the set of normalized $C^0$-Hamiltonians on $M$. We define

$$
\rho : C^0_m(M \times [0,1], \mathbb{R}) \times QH^*(M) \to \mathbb{R}
$$

by the value

$$
\rho(H; a) := \rho(H, a; f)
$$

for a (and so any) Morse function $f$. Then $\rho_a = \rho(\cdot; a)$ is continuous over $H$ whose value lie in Spec($H$). Therefore $\rho(H; a)$ depends only on the homotopy class $\tilde{\phi} = [\phi, H]$ and hence we can define

$$
\rho : \tilde{\text{Ham}}(M, \omega) \times QH^*(M) \to \mathbb{R} ; \quad \rho(\tilde{\phi}; a) := \rho(H; a)
$$

(5.25)

for any $H$ not just for generic $H$ with $\tilde{\phi} = [\phi, H]$. In addition, we have the formula

$$
\rho(H; a) = \inf_H \inf_\alpha \{ \lambda_H(h_{\tilde{\phi}}^a)(\alpha) \mid [\alpha] = a^b, \alpha \in CF_k(\epsilon f) \}
$$

(5.26)

$$
= \inf_\alpha \{ \lambda_H(h_{\tilde{\phi}}^a)(\alpha) \mid [\alpha] = a^b, \alpha \in CF_k(\epsilon f) \}
$$

(5.27)

$$
= \inf_\alpha \{ \lambda \mid h_{\tilde{\phi}}^a(\alpha^b) \in \text{Im } i_\lambda \}
$$

for any Morse-Smale function $f$ as long as $\epsilon > 0$ is sufficiently small. In other words, the right hand sides of (5.26) are independent of $f$ and $\epsilon > 0$. In (5.27), $H_0$ is any fixed homotopy from $\epsilon f$ to $H$, e.g., can be fixed to the linear homotopy $H_{lin} : s \mapsto (1 - s)\epsilon f + sH$.

**Proof.** $C^0$-continuity of $\rho$ over $H$ is already proven. We will prove in §6 that $\rho(H; a)$ is indeed a critical value of $A_H$, i.e., lies in Spec($H$). This assumed for now, the well-definedness of the definition (5.25), i.e, independence of $H$ with $\tilde{\phi} = [\phi, H]$ is an immediate consequence of the facts that $H \mapsto \rho(H; a)$ is continuous, Spec($H$) is of measure zero subset of $\mathbb{R}$ [Lemma 2.2, Oh6] and Spec($H$) depends only on its homotopy class $\tilde{\phi}$ [Theorem 3.1, Oh7].

For the proofs of (5.26), (5.27), Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two homotopies from $\epsilon f$ to $H$, denote by $\mathcal{H}_2^{-1}$ the obvious the inverse homotopy from $H$ to $\epsilon f$. Since $h_{\mathcal{H}} : HF_*(\epsilon f) \to HF_*(H)$ is independent of the homotopy $\mathcal{H}$ and an isomorphism, for any Novikov cycle $\alpha$ of $\epsilon f$ with $[\alpha] = a^b$, we can write

$$
h_{\mathcal{H}_2}(\alpha) = h_{\mathcal{H}_1}(\gamma)
$$

as chains for another Novikov cycle $\gamma$ of $\epsilon f$ with $[\gamma] = [\alpha] = a^b$. This proves that (5.26) and (5.27) have the same values. Similar arguments also prove that (5.26) or (5.27) are independent of $f$ as long as $\epsilon > 0$ is sufficiently small. This finishes the proof. \(\square\)

By the spectrality of $\rho(\tilde{\phi}; a)$ for each $a \in QH^*(M)$, we have constructed continuous ‘sections’ of the action spectrum bundle

$$
\text{Spec}(M, \omega) \to \tilde{\text{Ham}}(M, \omega)
$$
We define the essential spectrum of $\tilde{\phi}$ by

$$\text{spec}(\tilde{\phi}) := \{\rho(\tilde{\phi}; a) \mid 0 \neq a \in \text{QH}^*(M)\}$$

$$\text{spec}_k(\tilde{\phi}) := \{\rho(\tilde{\phi}; a) \mid 0 \neq a \in \text{QH}^k(M)\}$$

and the bundle of essential spectra by

$$\text{spec}(M, \omega) = \bigcup_{\tilde{\phi} \in \text{Ham}(M, \omega)} \text{spec}(\tilde{\phi})$$

and similarly for $\text{spec}_k(M, \omega)$. For each normalized Hamiltonian $H$, we also define the essential spectrum $\text{spec}(H)$ in the obvious way.

5.3. Description in terms of the Hamiltonian fibration

As far as the definition of $\rho$ is concerned, it would seem more natural to use the Piunikhin-Salamon-Schwarz map [PSS] which is supposed to directly relate $\text{QH}_*(M)$ and $\text{HF}_*(H)$. In fact Theorem 5.5, especially (5.27), enables us to give another way of writing $\rho(H; a)$ in terms of the Hamiltonian fibration and the connection.

Consider $\tilde{\phi}$ and let $\tilde{\phi} = [\phi, H]$ for $t$-periodic Hamiltonian $H$. We assume that

$$H \equiv 0 \quad \text{near} \quad t = 0 \equiv 1$$

which can be always achieved by reparameterizing the Hamiltonian path $\phi^t_H$ without changing any relevant information (see Lemma 5.2, Oh6 for justification of this adjustment). One good thing about considering such $t$-periodic Hamiltonians is that it breaks the $S^1$-symmetry and provides a canonical (asymptotic) marking on the circle $S^1$. We denote this canonical marking with $0 \in S^1 = \mathbb{R}/\mathbb{Z}$.

Consider the Riemann surface of genus zero $\Sigma$ with one puncture, denoted by $x_0$. We fix a holomorphic identification of a neighborhood $D$ of $x_0$

$$\varphi : D \setminus \{x_0\} \to [0, \infty) \times S^1$$

with the standard complex structure on the cylinder $[0, \infty) \times S^1$. Using the canonical marking on $S^1$ mentioned above, there is the unique such identification if we impose the condition

$$\varphi(\{\theta = 0\}) = [0, \infty) \times \{0\} \quad (5.28)$$

We consider cut-off functions $\rho : [0, \infty) \to [0, 1]$ of the form

$$\rho = \begin{cases} 1 & \tau \geq 2 \\ 0 & \tau \leq 1 \end{cases}$$

and fix a trivialization

$$\Phi : P|_{D \setminus \{x_0\}} \to (D \setminus \{x_0\}) \times M \cong [0, \infty) \times S^1 \times M$$

We canonically identify the fiber $P|_{\varphi^{-1}((\tau, 0))}$ with $M$ for any $\tau$ using the above mentioned canonical marking.
With this preparation, we fix the fibration $P|_{D \setminus \{x_0\}}$ so that it becomes the mapping cylinder

$$E_0 = [1, \infty) \times \mathbb{R} \times M/\{\tau, 0, x\} \sim (\tau, 0, x) \to [1, \infty) \times \mathbb{R} / \mathbb{Z}$$

over $[1, \infty) \times S^1$. Considering the linear homotopy

$$H^{lin} : s \in [0, 1] \mapsto sH,$$

we interpolate the zero Hamiltonian and $H$ using the cut-off function $\rho : [0, \infty) \times S^1 \to [0, 1]$

$$(\tau, t) \to \rho(\tau)H_t.$$

(5.30)

The Hamiltonian $H$ will provide a canonical connection $\nabla_H$ on $[1, \infty) \times \mathbb{R} / \mathbb{Z}$ whose monodromy becomes the diffeomorphism $\phi : M \to M$ on $[1, \infty) \times \mathbb{R} / \mathbb{Z}$ (see [En1] for a nice exposition of this correspondence in general), when we measure it with the above identification of the fiber $P_{\phi^{-1}(\tau, 0)}$ with $M$. Obviously the connection is flat there, and trivial in a neighborhood of the marked line $[1, \infty) \times \{0\}$. We consider the two form $\omega + d(\rho H_t dt)$ on $[0, \infty) \times S^1 \times M$ and pull it back to $P|_{D}$ which we denote

$$\omega_D = \Phi^*(\omega + d(\rho H_t dt)).$$

This induces a natural symplectic connection $\nabla_D$ on $P|_{D}$ which coincides with $\nabla_H$ on the portion of the mapping cylinder over $\phi^{-1}(1, \infty) \times S^1$.

Now we consider all possible Hamiltonian connection $\nabla$ on $\Sigma$ in a fixed homotopy class in $L(\phi_H)$ (see §3.6 & 3.7, En1 for the description of this homotopy class) which extends $\nabla_D$ and denote by $\omega_{\nabla}$ the coupling form of $\nabla$ [GLS]. With this preparation, we are ready to translate Theorem 5.5, in particular (5.27) in terms of Hamiltonian fibration. The following lemma is an immediate consequence of [Lemma 4.5, Sc] or [Sublemma 5.0.2, En1] whose proof we omit.

**Lemma 5.6.** Let $[z, w] \in \text{Crit}_{A_H}$. and $v : \Sigma \to P$ be any section such that

1. it has the asymptotic condition

$$\lim_{\tau \to +\infty} u(\tau, t) = z(t)$$

uniformly in $C^1$-topology (e.g., exponentially fast)

2. $[u^\#v] = 0$ in $\Gamma$

in the coordinates $v(\tau, t) = (\tau, t, u(\tau, t))$ over the trivialization $\Phi$. Then we have

$$A_H([z, w]) = -\int v^*\omega_{\nabla}$$

(5.31)

Entov considered the moduli space $M(H, J; \hat{z})$ in general whose description we refer to [En1], or §6 below for the case with 3 punctures is studied. In the current case, the number of punctures is 1. The dimension of $M(H, J; \hat{z})$ for this case with $z$ incoming has

$$2n - (\mu_H([z, w]) + n)$$

(5.32)
Define
\[ M(H, J) := \bigcup_{\hat{z} \in \text{Crit} \mathcal{A}_H} M(H, J; \hat{z}) \]
We define the evaluation map by
\[ ev : M(H, J) \to M; \quad ev(v) = v(0) \]
where 0 is the center of \( \Sigma \setminus D \) with \( \Sigma \setminus D \) conformally identified with the unit disc, which has the canonical meaning because we already have the holomorphic chart near the boundary \( \partial D \).

We now consider a classical cohomology class \( a \in H^k(M) \) and denote by \( PD(a) \in H_{2n-k}(M) \) its Poincaré dual. Represent \( PD(a) \) by a singular cycle \( C \) in \( M \). Consider \( \hat{z} = [z, w] \in \text{Crit} \mathcal{A}_H \) with the Conley-Zehnder index satisfying
\[ 2n - (\mu_H([z, w]) + n) = k \quad \text{i.e.,} \quad \mu_H([z, w]) = n - k \quad (5.33) \]
and the fiber product
\[ M(H, J, C; \hat{z}) := M(H, J; \hat{z}) \times_{ev} C \]
which has virtual dimension 0. We define the matrix coefficient
\[ \langle C, [z, w] \rangle = \# \left( M(H, J; \hat{z}) \times_{ev} C \right) \quad (5.34) \]
and form the chain
\[ h_{P, \nabla, \tilde{J}}(C) := \sum_{[z, w]} \langle C, [z, w] \rangle [z, w] \quad (5.35) \]
We call those \([z, w]\) for which \( \langle C, [z, w] \rangle \neq 0 \) generators of \( h_{P, \nabla, \tilde{J}}(C) \) as before. This map obviously extends to the quantum complex
\[ h_{P, \nabla, \tilde{J}} : C_{2n-k}(M) \otimes \Lambda \to CF_k(H). \quad (5.36) \]
We recall our convention of grading (4.17) and (4.18). This map (5.36) can be considered as the limit map of the chain map
\[ h_{P, \nabla, \tilde{J}} : CF_k(\epsilon f) \cong CM_{2n-k}(-\epsilon f) \otimes \Lambda \to CF_k(H) \]
used in (5.27). Although proving this latter fact is technically non-trivial, the following is an easy translation of (5.27) by realizing the singular cycle by a Morse cycle (See [HaL] for an elegant description of this realization). Because we do not use this theorem in this paper, we will give the details of the proof elsewhere or leave them to the interested readers.

**Theorem 5.7.** Let \( 0 \neq a \in H^*(M) \) and \( C \) denote cycle representing \( PD(a) \). Then we define
\[ \rho(H, C : P, \nabla, \tilde{J}) = \inf_{v \in M(H, J, C; \hat{z}) : \langle C, \hat{z} \rangle \neq 0} \left( - \int v^* \omega_{\nabla} \right) \quad (5.37) \]
\[ \rho(H, a : P, \nabla, \tilde{J}) = \inf_{C : [C] = PD(a)} \rho(H, C : P, \nabla, \tilde{J}). \quad (5.38) \]
(1) Then (5.38) lies in Spec(H) = Spec(declaring) and is independent of ∇ and J as long as the homotopy class [∇] is fixed, i.e., ∇ ∈ L([h_H]) in the sense of Entov [3, En1]. Furthermore the common value is exactly our ρ(H; a).

(2) We define

\[ \rho(H, C) := \rho(H, C : P, \nabla, J). \]  

Then we also have

\[ \rho(H; a) = \inf_{C : |C| = PD(a)} \rho(H, C). \]  

This description of ρ in terms of the Hamiltonian fibration does not seem to contain any other extra information than our original dynamical description, and is not as flexible to use as the latter in practice. However it gives a link between our spectral invariants and the invariants of the Hofer type defined over Hamiltonian fibrations, e.g., the K-area [Po2], [En1] or the area [Mc1,2] which have been studied in the literature on the symplectic topology. We refer to [Po2,3], [En1] and [Mc1,2] and references therein for the study of Hofer’s geometry in this respect. Our spectral invariants are of intrinsically different nature from these. Even the closest cousin ρ(·;1) of the Hofer type invariant was shown to behave differently from them as illustrated by Ostrover [Os]. We hope to further investigate the relation between the two elsewhere in the future.

For the rest of this paper, we will always use the dynamical description of the spectral invariants in our investigation.

§6. Basic properties of the spectral invariants

In this section, we will prove all the remaining properties stated in Theorem III in the introduction which we re-state below.

**Theorem 6.1.** Let \( \tilde{\phi}, \tilde{\psi} \in \widehat{\text{Ham}}(M, \omega) \) and \( a \neq 0 \in QH^*(M) \) and let

\[ \rho : \widehat{\text{Ham}}(M, \omega) \times QH^*(M) \to \mathbb{R} \]

be as defined in §5. Then ρ satisfies the following properties:

1. **(Spectrality)** \( \rho(\tilde{\phi}; a) \in \text{Spec}(\tilde{\phi}) \) for any \( a \in QH^*(M) \).
2. **(Projective invariance)** \( \rho(\tilde{\phi}; \lambda a) = \rho(\tilde{\phi}; a) \) for any \( 0 \neq \lambda \in \mathbb{Q} \).
3. **(Normalization)** For \( a = \sum_{A \in \mathbb{Z}} a_A \otimes q^A \), \( \rho(\tilde{\phi}; a) = v(a) \) the valuation of \( a \).
4. **(Symplectic invariance)** \( \rho(\eta \tilde{\phi} \eta^{-1}; a) = \rho(\tilde{\phi}; a) \) for any symplectic diffeomorphism \( \eta \).
5. **(Triangle inequality)** \( \rho(\tilde{\phi} \tilde{\psi}; a \cdot b) \leq \rho(\tilde{\phi}; a) + \rho(\tilde{\psi}; b) \)
6. **(C0-Continuity)** \( |\rho(\tilde{\phi}; a) - \rho(\tilde{\psi}; a)| \leq \|\phi \circ \psi^{-1}\| \) and in particular \( \rho(\cdot, a) \) is continuous with respect to the C0-topology of \( \widehat{\text{Ham}}(M, \omega) \).

One more important property concerns the effect of ρ under the action of \( \pi_0(\widetilde{G}) \). We first explain how \( \pi_0(\widetilde{G}) \) acts on \( \widetilde{\text{Ham}}(M, \omega) \times QH^*(M) \) following (and adapting into cohomological version) Seidel’s description of the action on \( QH_*(M) \). According to [Se], each element \( \tilde{h}, \tilde{h} \in \pi_0(\widetilde{G}) \) acts on \( QH_*(M) \) by the quantum product of an even element \( \Psi([h, \tilde{h}]) \) on \( QH_*(M) \). We take the adjoint action of it on \( a \in QH^*(M) \) and denote it by \( \tilde{h}^* a \). More precisely, \( \tilde{h}^* a \) is defined by the identity

\[ \langle \tilde{h}^* a, \beta \rangle = \langle a, \Psi([h, \tilde{h}]) \cdot \beta \rangle \]

with respect the non-degenerate pairing \( \langle \cdot, \cdot \rangle \) between \( QH^*(M) \) and \( QH_*(M) \).
Theorem 6.2. Let $\pi_0(G)$ act on $\hat{\text{Ham}}(M,\omega) \times QH^*(M)$ as above, i.e,

$$[h,\tilde{h}] : (\tilde{\phi}, a) = (h \cdot \tilde{\phi}, \tilde{h}^*a) \quad (6.2)$$

Then we have

$$\rho([h,\tilde{h}] \cdot (\tilde{\phi}, a)) = \rho(\tilde{\phi}, a) + I_\omega([h,\tilde{h}])$$

Proof. This is immediate from the construction of $\Psi([h,\tilde{h}])$ in [Se]. Indeed, the map

$$[h,\tilde{h}]_* : CF_*(F) \mapsto CF_*(H\#F) \quad (6.3)$$

is induced by the map $[z,w] \mapsto \tilde{h}([z,w])$ and we have

$$A_H(\tilde{h}([z,w])) = A_F([z,w]) + I_\omega([h,\tilde{h}])$$

by (2.5). Furthermore the map (6.3) is a chain isomorphism whose inverse is given by $([h,\tilde{h}]^{-1})_*$. This immediately implies the theorem from the construction of $\rho$. □

Remark 6.3. Strictly speaking, $\tilde{h}^*a$ may not lie in the standard quantum cohomology $QH^*(M)$ because it is defined as the linear functional on the complex $CQ^*(M)$ that is dual to the Seidel element $\Psi([h,\tilde{h}]) \in CQ^*(M)$ under the canonical pairing between $CQ_*(M)$ and $CQ^*(M)$. A priori, the bounded linear functional

$$\tilde{h}^*a = \langle \Psi([h,\tilde{h}]), \cdot \rangle$$

may not lie in the image of (5.2) in general. In that case, one should consider $\tilde{h}^*a$ as a bounded quantum cohomology class in the sense of Appendix 2. We refer readers to Appendix 2 for the explanation on how to extend the definition of our spectral invariants to the bounded quantum cohomology classes.

We have already proven the properties of normalization and $C^0$ continuity in the course of proving the linking property of the Novikov cycles in §5. The symplectic invariance is easy to check by construction. The only non-trivial parts remaining are the spectrality and the triangle inequality.

6.1. Proof of the spectrality.

We start with $\tilde{\phi} = [\phi, H]$ where $\phi$ is non-degenerate in the sense of Lefschetz fixed point theory. We will deal with the general degenerate $C^2$-Hamiltonians (or more generally Lipschitz Hamiltonians) afterwards. By the non-degeneracy of $H$ there are only finitely many one-periodic orbits of $H$. We need to show that the mini-max value $\rho(H; a)$ is a critical value, i.e., that there exists $[z,w] \in \Omega_0(M)$ such that

$$A_H([z,w]) = \rho(H; a)$$

$$dA_H([z,w]) = 0, \quad \text{i.e.,} \quad \dot{z} = X_H(z). \quad (6.4)$$

This is trivial to see from the definition of $\rho(H; a)$ when $(M,\omega)$ is rational, once the finiteness of the value is proven. For the non-rational case, the obvious argument for the rational case fails because the set of critical values of $A_H$ is not closed but dense in $\mathbb{R}$. In the classical mini-max theory [BuR] where the global gradient
flow of the functional exists, such a statement heavily relies on the Palais-Smale type condition and the deformation lemma. In our case the global flow does not exist. Instead we will prove criticality of the mini-max value \( \rho(H; a) \) by a geometric argument using the chain level Floer theory. Here again the finiteness condition in the definition of the Novikov ring or the Floer complex plays a crucial role.

We recall from (5.27) that for a given quantum cohomology class

\[
a = \sum a_A q^{-A}, \quad a_A \in H^*(M)
\]

we have

\[
\rho(H; a) = \inf_\alpha \{ \lambda_H(\mathcal{H}_0(\alpha)) \mid \alpha \in CF_*(\epsilon f), \text{ with } [\alpha] = a^b \} \quad (6.5)
\]

where \( \mathcal{H}_0 \) is a fixed homotopy, say, the linear homotopy \( \mathcal{H}^{lin} \) from \( \epsilon f \) to \( H \).

As in (5.12), we may assume that the representative \( \alpha \) of \( a^b \) has the form

\[
\alpha = \sum_{A \in \Gamma(a)} \alpha_A q^A \quad (6.6)
\]

With this preparation, we proceed with the proof of spectrality. It follows from (6.5) and (6.6) that there exists a sequence \( \alpha_j \) of Novikov cycles of \( \epsilon f \) in the form (6.6) with

\[
\lim_{j \to \infty} \lambda_H(\mathcal{H}_0(\alpha_j)) = \rho(H; a).
\]

In particular, choosing \([z_j, w_j] \in \text{Crit } \mathcal{A}_H \) with

\[
\lambda_H([z_j, w_j]) = \lambda_H(\mathcal{H}_0(\alpha_j)), \quad [z_j, w_j] \in \mathcal{H}_0(\alpha_j) \quad (6.7)
\]

which exists by the definition of \( \lambda_H \), we have

\[
\lim_{j \to \infty} \lambda_H([z_j, w_j]) = \rho(H; a). \quad (6.8)
\]

The main task is to prove that the sequence \([z_j, w_j] \in \tilde{\Omega}_0(M) \) is pre-compact. Once the pre-compactness is proven, the limit of a subsequence of \([z_j, w_j] \) will be a critical point of \( \mathcal{A}_H \) that realizes the mini-max value \( \rho(H; a) \).

Recall from the definition of \( \rho(H; a) \) that

\[
\lambda_H([z_j, w_j]) = \lambda_H(\mathcal{H}_0(\alpha_j)) \geq \rho(H; a).
\]

By definition of the transferred cycle \( \mathcal{H}_0(\alpha_j) \), there exists

\[
[p_j, \tilde{p}_j \# A] = p_j \otimes q^A_j \in \alpha_j \in CF_*(\epsilon f)
\]

such that the moduli space of (4.6) is non-empty

\[
\mathcal{M}((\mathcal{H}_0, j); [p_j, \tilde{p}_j \# A_j], [z_j, w_j]) \neq \emptyset. \quad (6.9)
\]

Therefore the upper estimate in Proposition 4.3, after multiplying by -1, implies

\[
\mathcal{A}_f([p_j, \tilde{p}_j \# A_j]) - \mathcal{A}_H([z_j, w_j]) \geq \int_0^1 \min(H - \epsilon f) \, dt
\]
and so
\[
\mathcal{A}_f([p_j, \hat{p}_j # A_j]) \geq \mathcal{A}_H([z_j, w_j]) + \int_0^1 \min(H - \epsilon f) \, dt
\]
\[
\geq \rho(H; a) + \int_0^1 \min(H - \epsilon f) \, dt =: \lambda_0
\]
for all \( j \). We recall that \( \alpha_j \) is in the form of (6.6) and that there are only finitely many \( A \)'s in \( \Gamma(a) \) above the level
\[
-\omega(A) \geq -\epsilon \max f + \lambda_0 \geq \lambda_0 - \max f =: \lambda_1.
\]

Now we examine the image of \( h_{\mathcal{H}_0} \) of the span over the set
\[
\{ [p, \hat{p} # A] : p \in \text{Crit}(-\epsilon f), A \in \Gamma(a), \mathcal{A}_f([p, \hat{p} # A]) \geq \lambda_1 \}.
\]  

**Lemma 6.4.** Let \( \delta > 0 \) be given. There are only finitely many \([z, w] \in \text{Crit} \mathcal{A}_H \) such that
\[
\begin{align*}
(1) & \quad \rho(H; a) - \delta \leq \mathcal{A}_H([z, w]) \\
(2) & \quad \mathcal{M}((\mathcal{H}_0, j); [p, \hat{p} # A], [z, w]) \neq \emptyset \text{ for } A \in \Gamma(a)
\end{align*}
\]

**Proof.** Let \( u \in \mathcal{M}((\mathcal{H}_0, j); [p, \hat{p} # A], [z, w]). \) Then from (1), we have the uniform energy bound
\[
\int \left| \frac{\partial u}{\partial \tau} \right|^2 \, dt \leq \mathcal{A}_f([p, \hat{p} # A]) - \mathcal{A}_H([z, w]) + C(\mathcal{H}_0) \]
\[
\leq \mathcal{A}_f([p, \hat{p} # A]) - \rho(H; a) + \delta + C(\mathcal{H}_0)
\]  

which is independent of \([z, w]\). Here \( C(\mathcal{H}_0) \) is the constant mentioned in Remark 4.5. Therefore for each given \([p, \hat{p} # A]\) in the set (6.10), it follows from a standard compactness argument that there are only finitely many \( \omega \)-limits \([z, w]\) of \([p, \hat{p} # A]\). Since there are only finitely many \( A \)'s in the set (6.10), the lemma is proved. \( \Box \)

Now we go back to examine the sequence \([z_j, w_j]\) in (6.8). Since there are only finitely many periodic orbits, we may assume \( z_j = z_\infty \) for all \( j \geq j_0, j_0 \) sufficiently large, by choosing a subsequence if necessary. And since it follows from (6.7)-(6.9) that \([z_j, w_j]\) satisfies (1) and (2) in Lemma 6.4 for sufficiently large \( j \), Lemma 6.4 implies we also have \( A_j = A_\infty \) for a subsequence. Therefore the sequence \([p_j, \hat{p}_j # A_j]\) and in turn \([z_j, w_j]\) stabilizes as \( j \to \infty \) after taking a subsequence if necessary, and in particular we have
\[
\lim_{j \to \infty} [z_j, w_j] = [z_\infty, w_\infty]
\]

in the \( C^\infty \)-topology for \( z_\infty \in \text{Per } H \). Since \( z_j \) satisfies \( \dot{z}_j = X_{H_j}(z_j) \) and \( H_j \to H \) in \( C^2 \)-topology, it follows that \( \dot{z} = X_H(z) \), i.e., \([z, w]\) is a critical point of \( \mathcal{A}_H \) and
\[
\mathcal{A}_H([z_\infty, w_\infty]) = \lim_{j \to \infty} \mathcal{A}_H([z_j, w_j]) = \rho(H; a).
\]

For the first identity, we use the fact that \([z_j, w_j]\) stabilizes. This finishes the proof of spectrality for the non-degenerate \( \phi \).
Now consider arbitrary \( \tilde{\phi} \) and let \( \tilde{\phi} = [\phi, H] \). We approximate \( H \) by a sequence of non-degenerate Hamiltonians \( H_j \) in the \( C^2 \)-topology. By the \( C^0 \)-continuity property of \( \rho(\cdot; a) \), we have

\[
\lim_{j \to \infty} \rho(H_j; a) = \rho(H; a).
\] (6.12)

By the definition of \( \rho \), for each \( j \) we can choose \([z_j, w_j] \in \text{Crit} A_{H_j}\)

such that

\[
A_{H_j}([z_j, w_j]) = \rho(H_j, a), \quad \dot{z}_j = X_{H_j}(z_j).
\] (6.13)

Since \( M \) is compact and \( \dot{z}_j = X_{H_j}(z_j) \) and \( H_j \to H \) in the \( C^2 \)-topology, it follows from the Ascoli-Arzela theorem that there exists a subsequence, again denoted by \( z_j \), with

\[
\lim_{j \to \infty} z_j = z_\infty
\]

in the \( C^1 \)-topology where \( \dot{z}_\infty = X_H(z_\infty) \). It remains to prove that we can choose a subsequence for which \( w_j \) also converges in \( C^1 \)-topology. Noting that since all \( z_j \)'s are in a small \( C^1 \)-neighborhood of \( z_\infty \), there is a canonical one-one correspondence

\[
\pi^{-1}(z_j) \to \pi^{-1}(z_\infty)
\]

such that the correspondence converges to the identity as \( j \to \infty \) in \( C^1 \)-topology. This is provided by the (homotopically) unique ‘thin’ cylinder between \( z_j \) and \( z_\infty \).

Here \( \pi : \tilde{\Omega}_0(M) \to \Omega_0(M) \) is the projection. Once we have this, it is easy to modify the proof of Lemma 6.4 to allow \( C^2 \)-variation of Hamiltonians ("\( H_j \)") near a given arbitrary Hamiltonian ("\( H \)") to conclude that \([z_j, w_j] \to [z_\infty, w_\infty] \) uniformly in the \( C^1 \)-topology as \( j \to \infty \), choosing a subsequence if necessary. Then (6.12) and (6.13) imply

\[
A_H([z_\infty, w_\infty]) = \lim_{j \to \infty} A_{H_j}([z_j, w_j]) = \rho(H; a), \quad \dot{z}_\infty = X_H(z_\infty).
\]

This finishes the proof that \( \rho(H; a) \) is indeed a critical value of \( A_H \) and finishes the proof of the spectrality axiom.

### 6.2. Proof of the triangle inequality

To start with the proof of the triangle inequality, we need to recall the definition of the “pants product”

\[
HF_*(\tilde{\phi}) \otimes HF_*(\tilde{\psi}) \to HF_*(\tilde{\phi} \cdot \tilde{\psi}).
\] (6.14)

We also need to straighten out the grading problem of the pants product.

For the purpose of studying the effect on the filtration under the product, we need to define this product in the chain level in an optimal way as in [Oh4], [Sc] and [En1]. For this purpose, we will mostly follow the description provided by Entov [En1] with few notational changes and different convention on the grading. As pointed out before, our grading convention preserves the grading under the pants product. Except the grading convention, the conventions in [En1,2] on the
definition of Hamiltonian vector field and the action functional coincide with our conventions in [Oh3,5,7] and here.

Let $\Sigma$ be the compact Riemann surface of genus 0 with three punctures. We fix a holomorphic identification of a neighborhood of each puncture with either $[0, \infty) \times S^1$ or $(-\infty, 0] \times S^1$ with the standard complex structure on the cylinder. We call punctures of the first type negative and the second type positive. In terms of the “pair-of-pants” $\Sigma \setminus \bigcup_i D_i$, the positive puncture corresponds to the outgoing ends and the negative corresponds to the incoming ends. We denote the neighborhoods of the three punctures by $D_i$, $i = 1, 2, 3$ and the identification by
\[ \varphi^+_i : D_i \to (-\infty, 0] \times S^1 \]
for positive punctures and
\[ \varphi^-_3 : D_3 \to [0, \infty) \times S^1 \]
for negative punctures. We denote by $(\tau, t)$ the standard cylindrical coordinates on the cylinders.

We fix a cut-off function $\rho^+ : (-\infty, 0] \to [0, 1]$ defined by
\[ \rho = \begin{cases} 1 & \tau \leq -2 \\ 0 & \tau \geq -1 \end{cases} \]
and $\rho^- : [0, \infty) \to [0, 1]$ by $\rho^-(\tau) = \rho^+(-\tau)$. We will just denote by $\rho$ these cut-off functions for both cases when there is no danger of confusion.

We now consider the (topologically) trivial bundle $P \to \Sigma$ with fiber isomorphic to $(M, \omega)$ and fix a trivialization $\Phi_i : P_i := P|_{D_i} \to D_i \times M$ on each $D_i$. On each $P_i$, we consider the closed two form of the type
\[ \omega^\Sigma_{P_i} := \Phi_i^*(\omega + d(\rho H_t dt)) \] (6.15)
for a time periodic Hamiltonian $H : [0, 1] \times M \to \mathbb{R}$. The following is an important lemma whose proof we omit (see [En1]).

**Lemma 6.5.** Consider three normalized Hamiltonians $H_i$, $i = 1, 2, 3$. Then there exists a closed 2-form $\omega_P$ such that
\begin{enumerate}
  \item $\omega_P|_{P_i} = \omega^\Sigma_{P_i}$
  \item $\omega_P$ restricts to $\omega$ in each fiber
  \item $\omega^p_{P}^{n+1} = 0$
\end{enumerate}

Such $\omega_P$ induces a canonical symplectic connection $\nabla = \nabla_{\omega_P}$ [GLS], [En1]. In addition it also fixes a natural deformation class of symplectic forms on $P$ obtained by those
\[ \Omega_{P, \lambda} := \omega_P + \lambda \omega^\Sigma \]
where $\omega^\Sigma$ is an area form and $\lambda > 0$ is a sufficiently large constant. We will always normalize $\omega^\Sigma$ so that $\int_{\Sigma} \omega^\Sigma = 1$.

Next let $\tilde{J}$ be an almost complex structure on $P$ such that
\begin{enumerate}
  \item $\tilde{J}$ is $\omega_P$-compatible on each fiber and so preserves the vertical tangent space
  \item the projection $\pi : P \to \Sigma$ is pseudo-holomorphic, i.e, $d\pi \circ \tilde{J} = j \circ d\pi$.
\end{enumerate}
When we are given three $t$-periodic Hamiltonian $H = (H_1, H_2, H_3)$, we say that $\tilde{J}$ is $(H, J)$-compatible, if $\tilde{J}$ additionally satisfies

(3) For each $i$, $(\Phi_i)_* \tilde{J} = j \oplus J_H$, where

$$J_H(\tau, t, x) = (\phi_{H_i}^t)^* J$$

for some $t$-periodic family of almost complex structure $J = \{J_t\}_{0 \leq t \leq 1}$ on $M$ over a disc $D'_i \subset D_i$ in terms of the cylindrical coordinates. Here $D'_i = \varphi_i^{-1}((-\infty, -K_i] \times S^1)$, $i = 1, 2$ and $\varphi_i^{-1}([K_i, \infty) \times S^1)$ for some $K_i > 0$. Later we will particularly consider the case where $J$ is in the special form adapted to the Hamiltonian $H$. See (7.23).

The condition (3) implies that the $\tilde{J}$-holomorphic sections $v$ over $D'_i$ are precisely the solutions of the equation

$$\frac{\partial u}{\partial \tau} + J_t \left( \frac{\partial u}{\partial t} - X_{H_t}(u) \right) = 0$$

(6.16)

if we write $v(\tau, t) = (\tau, t, u(\tau, t))$ in the trivialization with respect to the cylindrical coordinates $(\tau, t)$ on $D'_i$ induced by $\phi_i^t$ above.

Now we are ready to define the moduli space which will be relevant to the definition of the pants product that we need to use. To simplify the notations, we denote

$$\tilde{\Sigma} = [z, w]$$

in general and $\tilde{\Sigma} = ([\tilde{z}_1, \tilde{z}_2, \tilde{z}_3])$ where $\tilde{z}_i = [z_i, w_i] \in \text{Crit}A_H$, for $i = 1, 2, 3$.

**Definition 6.6.** Consider the Hamiltonians $H = \{H_i\}_{1 \leq i \leq 3}$ with $H_3 = H_1 \# H_2$, and let $\tilde{J}$ be a $H$-compatible almost complex structure. We denote by $M(H, \tilde{J}; \tilde{\Sigma})$ the space of all $\tilde{J}$-holomorphic sections $u : \Sigma \to P$ that satisfy

1. The maps $u_i := u \circ (\phi_i^{-1}) : (-\infty, K_i] \times S^1 \to M$ which are solutions of (6.16), satisfy

$$\lim_{\tau \to -\infty} u_i(\tau, \cdot) = z_i, \quad i = 1, 2$$

and similarly for $i = 3$ changing $-\infty$ to $+\infty$.

2. The closed surface obtained by capping off $pr_M \circ u(\Sigma)$ with the discs $w_i$ taken with the same orientation for $i = 1, 2$ and the opposite one for $i = 3$ represents zero in $\Gamma$.

Note that $M(H, \tilde{J}; \tilde{\Sigma})$ depends only on the equivalence class of $\tilde{\Sigma}$'s: we say that $\tilde{\Sigma}' \sim \tilde{\Sigma}$ if they satisfy

$$z_i' = z_i, \quad w_i' = w_i \# A_i$$

for $A_i \in \pi_2(M)$ and $\sum_{i=1}^3 A_i$ represents zero in $\Gamma$. The (virtual) dimension of $M(H, \tilde{J}; \tilde{\Sigma})$ is given by

$$\dim M(H, \tilde{J}; \tilde{\Sigma}) = 2n - (\mu_{H_1}(z_1) + n) - (\mu_{H_2}(z_2) + n) - (\mu_{H_3}(z_3) + n)$$

$$= n - (\mu_{H_3}(z_3) - \mu_{H_1}(z_1) - \mu_{H_2}(z_2)).$$

(6.17)
Note that when \( \dim \mathcal{M}(H, \tilde{J}; \tilde{\xi}) = 0 \), we have

\[
n = \mu_{H_3}(\tilde{z}_3) - \mu_{H_1}(\tilde{z}_1) - \mu_{H_2}(\tilde{z}_2)
\]

which is equivalent to

\[
k_3 = k_1 + k_2
\]

if we write

\[
k_i = n + \mu_{CZ}(\tilde{z}_i)
\]

which is exactly the grading of the Floer complex we adopt in the present paper. Now the pair-of-pants product \( * \) for the chains is defined by

\[
\tilde{z}_1 * \tilde{z}_2 = \sum \#(\mathcal{M}(H, \tilde{J}; \tilde{\xi})) \tilde{z}_3
\]

for the generators \( \tilde{z}_i \) and then by linearly extending over the chains in \( CF_*(H_1) \otimes CF_*(H_2) \). Our grading convention makes this product is of degree zero. Now with this preparation, we are ready to prove the triangle inequality.

**Proof of the triangle inequality.** Let \( f_1, f_2 \) be generic Morse functions and let \( \tilde{\phi} = [\phi, H] \) and \( \psi = [\psi, F] \). Let \( \mathcal{H}_i \) be paths from \( \epsilon f_i \) to \( H \) and \( F \) respectively. We denote by \( \mathcal{H}_1 \# \mathcal{H}_2 := \{ H_1 \# H_2 \} \) be the path which connects \( \epsilon(f_1 + f_2) \) and \( H \# F \). Then by construction we have the identity

\[
h_{\mathcal{H}_1 \# \mathcal{H}_2}((a \cdot b)^\rho) = h_{\mathcal{H}_1}(a^\rho) * h_{\mathcal{H}_2}(b^\rho)
\]

in homology. In the level of chains, for any chains \( \alpha, \beta \) and \( \gamma \) with

\[
[a] = a^\rho, \ [\beta] = b^\rho, \ [\gamma] = (a \cdot b)^\rho,
\]

we have

\[
h_{\mathcal{H}_1}(\alpha) * h_{\mathcal{H}_2}(\beta) = h_{\mathcal{H}_1 \# \mathcal{H}_2}(\gamma) + \partial \eta
\]

for some \( \eta \in CF_*(H \# F) \). Since \( h_{\mathcal{H}_1 \# \mathcal{H}_2} : HF_*(\epsilon(f_1 + f_2)) \to HF_*(H \# F) \) is an isomorphism, we may assume, by re-choosing \( \gamma \),

\[
h_{\mathcal{H}_1}(\alpha) * h_{\mathcal{H}_2}(\beta) = h_{\mathcal{H}_1 \# \mathcal{H}_2}(\gamma).
\]

By definition of \( \rho \), we have

\[
\rho(\tilde{\phi} \cdot \tilde{\psi}; a \cdot b) \leq \lambda_{H \# F}(h_{\mathcal{H}_1 \# \mathcal{H}_2}(\gamma)). \tag{6.22}
\]

Let \( \delta > 0 \) be any given number. We choose \( \alpha \in CF(\epsilon f_1) \) and \( \beta \in CF(\epsilon f_2) \) so that

\[
\lambda_H(h_{\mathcal{H}_1}(\alpha)) \leq \rho(H; a) + \delta \tag{6.23}
\]

\[
\lambda_H(h_{\mathcal{H}_2}(\beta)) \leq \rho(F; b) + \delta. \tag{6.24}
\]

i.e., we have the expressions

\[
h_{\mathcal{H}_1}(\alpha) = \sum_i a_i[z_i, w_i] \text{ with } \mathcal{A}_H([z_i, w_i]) \leq \rho(H; a) + \delta
\]

and

\[
h_{\mathcal{H}_2}(\beta) = \sum_j a_j[z_j, w_j] \text{ with } \mathcal{A}_H([z_j, w_j]) \leq \rho(H; b) + \delta.
\]

Now using the pants product (6.19), we would like to estimate the level of the chain \( h_{\mathcal{H}_1}(\alpha) \# h_{\mathcal{H}_2}(\beta) \in CF_*(H \# F) \). The following is a crucial lemma whose proof we omit but refer to [Sect. 4.1, Sc] or [Sect. 5, En1].
Lemma 6.7. Suppose that $\mathcal{M}(H, \tilde{J}; \tilde{z})$ is non-empty. Then we have the identity

$$\int v^*\omega_P = -A_{H_1}(z_3, w_3) + A_{H_1}(z_1, w_1) + A_{H_2}(z_2, w_2)$$

(6.25)

for any $v \in \mathcal{M}(H, \tilde{J}; \tilde{z})$.

Now since $\tilde{J}$-holomorphic and $\tilde{J}$ is $\Omega_{P, \lambda}$-compatible, we have

$$0 \leq \int v^*\Omega_{P, \lambda} = \int v^*\omega_P + \lambda \int v^*\omega_S = \int v^*\omega_P + \lambda.$$

Lemma 6.8 [Theorem 3.6.1 & 3.7.4, En1]. Let $H_i$’s be as in Lemma 6.5. Then for any given $\delta > 0$, we can choose a closed 2-form $\omega_P$ so that $\Omega_{P, \lambda} = \omega_P + \lambda\omega_S$ becomes a symplectic form for all $\lambda \geq \delta$. In other words, the size $\text{size}(H)$ (see [Definition 3.1, En1]) is $\infty$.

Applying the above two lemmata to $H$ and $F$ for $\lambda$ arbitrarily close to 0, and also applying (6.22) and (6.23), we immediately derive

$$\mathcal{A}_{H#F}(h_{H_1}(\alpha)) \leq \mathcal{A}_H(h_{H_1}(\alpha)) + \mathcal{A}_F(h_{H_2}(\beta)) + \delta$$

$$\leq \rho(H; a) + \rho(F; b) + 2\delta$$

(6.26)

provided that $\mathcal{M}(\tilde{J}, H; \tilde{z})$ is non-empty. However non-emptiness of $\mathcal{M}(\tilde{J}, H; \tilde{z})$ immediately follows from the definition of $\ast$ and the identity

$$[h_{H_1}(\alpha)] \ast [h_{H_2}(\beta)] = [h_{H_1#H_2}(\gamma)]$$

in $HF_*(H#F)$. Combining (6.21) and (6.26), we derive

$$\rho(H#F; a \cdot b) \leq \rho(H; a) + \rho(F; b) + 2\delta$$

Since this holds for any $\delta$ and $\rho(H; a)$ is independent of $H$ with $\tilde{\phi} = [\phi, H]$ [Corollary, Oh7], we have proven the triangle inequality. □

§7. Definition of the invariant norm

In this section, we prove Theorem IV in the introduction. The fact that

$$\tilde{\gamma} : \tilde{\text{Ham}}(M, \omega) \to \mathbb{R}$$

has non-negative values is clear by the triangle inequality,

$$\rho(\tilde{\phi} \cdot \tilde{\psi}; 1) \leq \rho(\tilde{\phi}; 1) + \rho(\tilde{\psi}; 1).$$

(7.1)

Indeed, applying (7.1) to $\tilde{\psi} = (\tilde{\phi})^{-1}$, we get

$$\tilde{\gamma}(\tilde{\phi}) = \rho(\tilde{\phi}; 1) + \rho(\tilde{\phi}^{-1}; 1) \geq \rho(0; 1) = v(1) = 0.$$

Here the identity second to the last follows from the normalization axiom in Theorem 6.1.
Theorem 7.1. For any \( \tilde{\phi} \) and \( 0 \neq a \in QH^*(M) \), we have
\[
\rho(\tilde{\phi}; a) \leq E^-(\tilde{\phi}) + v(a)
\]
and in particular
\[
\rho(\tilde{\phi}; 1) \leq E^-(\tilde{\phi}). \tag{7.2}
\]

Proof. This is an immediate consequence of (5.24) with substitution of \( H = 0 \) and the normalization axiom \( \rho(0; a) = v(a) \). \( \Box \)

On the other hand, we have
\[
\rho(\tilde{\phi} - 1; 1) \leq E^-(\tilde{\phi} - 1) = E^+(\tilde{\phi}).
\]
This finishes the proof of all the properties of \( \rho(\tilde{\phi}; 1) \) and \( \tilde{\gamma} \) stated in Theorem III in the introduction and also enables us to define

Definition 7.2. We define \( \gamma : \text{Ham}(M , \omega) \to \mathbb{R}_+ \) by
\[
\gamma(\phi) := \inf_{\pi(\phi) = \phi} \tilde{\gamma}(\phi). \tag{7.3}
\]

Theorem 7.3. Let \( \gamma \) be as above. Then \( \gamma : \text{Ham}(M , \omega) \to \mathbb{R}_+ \) defines an invariant norm i.e., it has the following properties

1. \( \phi = \text{id} \) if and only if \( \gamma(\phi) = 0 \)
2. \( \gamma(\eta \phi \eta^{-1}) = \gamma(\phi) \) for any symplectic diffeomorphism \( \eta \).
3. \( \gamma(\psi \phi) \leq \gamma(\psi) + \gamma(\phi) \)
4. \( \gamma(\phi^{-1}) = \gamma(\phi) \)
5. \( \gamma(\phi) \leq \| \phi \|_{\text{mid}} \leq \| \phi \| \)

Except the non-degeneracy, all the properties stated in this theorem are obvious by now from the construction. The rest of this section and the next will be occupied by the proof of non-degeneracy of the semi-norm
\[
\gamma : \text{Ham}(M , \omega) \to \mathbb{R}_+.
\]

For the proof of this, we first need some preparation. Let \( \phi \) be a Hamiltonian diffeomorphism that has only finite number of fixed points (e.g., non-degenerate ones). Denote by \( J_0 \) a compatible almost complex structure on \( (M , \omega) \). For given \( \phi \), we consider paths \( J' : [0 , 1] \to J_\omega \) with
\[
J'(0) = J_0 , \quad J'(1) = \phi^* J_0 \tag{7.4}
\]
and define by \( j(\phi, J_0) \) the set of such paths.

For each given \( J' \in j(\phi, J_0) \), we define the constant
\[
A_S(\phi, J_0; J') = \inf \{ \omega([u]) | u : S^2 \to M \text{ non-constant and satisfying} \overline{\partial}_{J_t} u = 0 \text{ for some} t \in [0 , 1] \}. \tag{7.5}
\]
As usual, we set \( A_S(\phi, J_0'; J') = \infty \) if there is no \( J_t' \)-holomorphic sphere for any \( t \in [0, 1] \) as in the weakly exact case. When \( A_S(\phi, J_0'; J') \neq \infty \), the positivity
\[
A_S(\phi, J_0; J') > 0 \tag{7.6}
\]
is an immediate consequence of the one parameter version of the uniform \( \epsilon \)-regularity theorem (see [SU], [Oh1]).

Next for each given \( J' \in j(\phi, J_0) \), we consider the equation of \( v : \mathbb{R} \times [0, 1] \to M \)
\[
\begin{cases}
\frac{\partial v}{\partial \tau} + J_t' \frac{\partial v}{\partial t} = 0 \\
\phi(v(\tau, 1)) = v(\tau, 0), \\
\int |\frac{\partial v}{\partial \tau}|^2 J_t' < \infty.
\end{cases}
\]
\tag{7.7}
This equation itself is analytically well-posed and (7.4) enables us to interpret solutions of (7.7) as pseudo-holomorphic sections of the mapping cylinder of \( \phi \) with respect to suitably chosen almost complex structure on the mapping cylinder.

Note that any such solution of (7.7) also satisfies that the limit \( \lim_{\tau \to \pm} v \) exists and
\[
v(\pm \infty) \in \text{Fix} \phi \tag{7.8}
\]
Now it is a crucial matter to produce a non-constant solution of (7.7). For this purpose, using the fact that \( \phi \neq \text{id} \), we choose a symplectic ball \( B_p(r) \) such that
\[
\phi(B_p(r)) \cap B_p(r) = \emptyset \tag{7.9}
\]
where \( B_p(r) \) is the image of a symplectic embedding into \( M \) of the standard Euclidean ball of radius \( r \). We then study (7.7) together with
\[
v(0, 0) \in B_p(r) \tag{7.10}
\]
Because of (7.8), it follows
\[
v(\pm \infty) \in \text{Fix} \phi \subset M \setminus B_p(r). \tag{7.11}
\]
Therefore such solution cannot be constant because of (7.9) and (7.11).

**Definition 7.4.** We define the constant
\[
A_D(\phi, J_0; J') := \inf_v \left\{ \int v^* \omega, \ | \ v \text{ non-constant solution of (7.7)} \right\} \tag{7.12}
\]
for each \( J \in j(\phi, J_0) \). Again we have \( A_D(\phi, J_0; J') > 0 \). We then define
\[
A(\phi, J_0; J') = \min\{A_S(\phi, J_0; J'), A_D(\phi, J_0; J')\}.
\]
Proposition 7.11 will prove
\[
0 < A(\phi, J_0; J') < \infty.
\]
Finally we define
\[
A(\phi, J_0) := \sup_{J' \in j(\phi, J_0)} A(\phi, J_0; J') \tag{7.13}
\]
and
\[
A(\phi) = \sup_{J_0} A(\phi, J_0). \tag{7.14}
\]

Note when \( (M, \omega) \) is weakly exact and so \( A_S(\phi, J_0; J) = \infty, A(\phi, J_0) \) is reduced to
\[
A(\phi, J_0) = \sup_{J' \in j(\phi, J_0)} \{A(\phi, J_0; J')\}.
\]
Because of the assumption that \( \phi \) has only finite number of fixed points, it is clear that \( A(\phi; \omega, J_0) > 0 \) and so we have \( A(\phi) > 0 \).

With these definitions, the following is the main theorem we prove in this and next sections.
**Theorem 7.5.** Suppose that $\phi$ has non-degenerate fixed points. Then for any $J_0$ and $J' \in j(\phi, J_0)$, we have

$$\gamma(\phi) \geq A(\phi, J_0; J')$$

(7.15)

and hence

$$\gamma(\phi) \geq A(\phi).$$

We have the following two immediate corollaries

**Corollary 7.6.** The pseudo-norm is non-degenerate, i.e., $\gamma(\phi) = 0$ if and only if $\phi = id$.

**Proof.** Because the null set

$$\text{null}(\gamma) := \{ \phi \in \text{Ham}(M, \omega) \mid \gamma(\phi) = 0 \}$$

is a normal subgroup of $\text{Ham}(M, \omega)$ and the latter group is simple by Banyaga’s theorem [Ba], it is enough to exhibit one $\phi$ such that $\gamma(\phi) \neq 0$. Theorem 7.5 says that any nondegenerate $\phi$ satisfies $\gamma(\phi) \neq 0$. This finishes the proof. □

**Corollary 7.7.** Let $\phi$ be as in Theorem 7.5. Then we have

$$\|\phi\| \geq \|\phi\|_{\text{mid}} \geq A(\phi)$$

(7.16)

where $\|\phi\|_{\text{mid}}$ is the medium Hofer norm of $\phi$.

The rest of this section and the next will be occupied by the proof of Theorem 7.5.

Let $\phi$ be a non-degenerate Hamiltonian diffeomorphism with $\phi \neq id$. In particular, we can choose a small symplectic ball $B_p(r)$ such that

$$B_p(r) \cap \phi(B_p(r)) = \phi$$

(7.17)

for some point $p \in M$ and $r > 0$. Now the proof will be by contradiction. Suppose the contrary that $\gamma(\phi) < A(\phi)$ and fix $\delta > 0$ such that

$$\gamma(\phi) \leq A(\phi) - 3\delta.$$  

(7.18)

By definition of $\gamma$, we can find $\tilde{\phi}$ such that

$$\rho(\tilde{\phi}; 1) + \rho(\tilde{\phi}^{-1}; 1) \leq A(\phi) - 2\delta.$$  

(7.19)

Let $H$ be a Hamiltonian representing $\tilde{\phi}$ in general. We already know that $H$ represents $(\tilde{\phi})^{-1}$. However a more useful Hamiltonian representing $(\tilde{\phi})^{-1}$ in the study of the duality and the pants product is the following Hamiltonian

$$\tilde{H}(t, x) := -H(1 - t, x).$$
Lemma 7.8. Let $H$ be a Hamiltonian representing $\tilde{\phi}$ i.e, $\tilde{\phi} = [\phi, H]$. Then $H \sim \tilde{H}$ and so we also have

$$(\tilde{\phi})^{-1} = [\phi^{-1}, \tilde{H}]$$

Proof. A direct calculation shows that the Hamiltonian path generated by $\tilde{H}$ is given by the path

$$t \mapsto \phi_H^{(1-t)} \circ \phi_H^{-1}. \quad (7.20)$$

The following composition of homotopies shows that this path is homotopic to the path $t \mapsto (\phi_H^t)^{-1}$. Consider the homotopy

$$\phi_s^t : = \begin{cases} 
\phi_H^{s-t} \circ (\phi_H^t)^{-1} & \text{for } 0 \leq t \leq s \\
(\phi_H^s)^{-1} & \text{for } s \leq t \leq 1.
\end{cases}$$

It is easy to check that the Hamiltonian path for $s = 1$ is (7.20) and the one for $s = 0$ is $t \mapsto (\phi_H^t)^{-1}$ and satisfies $\phi_s^t \equiv \phi_H^{-1}$ for all $s \in [0, 1]$. This finishes the proof. □

One advantage of using the representative $\tilde{H}$ over $H$ is that the time reversal $(\tau, t) \mapsto (-\tau, 1-t)$ (7.21) induces a natural one-one correspondence between $\text{Crit}(H)$ and $\text{Crit}(\tilde{H})$, and between the moduli spaces $M(H, J)$ and $M(\tilde{H}, \tilde{J})$ of the perturbed Cauchy-Riemann equations corresponding to $(H, J)$ and $(\tilde{H}, \tilde{J})$ respectively, where $\tilde{J}_t = J_{1-t}$. This correspondence reverses the flow and satisfies

$$A_{\tilde{H}}([z', w']) = -A_{H}([\tilde{z}', \tilde{w}']). \quad (7.22)$$

The following estimate of the action difference is a crucial ingredient in our proof of non-degeneracy. The proof here is similar to the analogous one for a similar non-triviality proof for the Lagrangian submanifolds studied in $[\S 6-7, \text{Oh}4]$. We would like to point out that for any $J' \in j(\phi, J_0)$ and $H \mapsto \phi$, the family

$$J_t = (\phi_H^t)_* J'_t \quad (7.23)$$

is $t$-periodic.

Proposition 7.9. Let $J_0$ be any compatible almost complex structure, $J' \in j(\phi, J_0)$ and $J$ be the $t$-periodic family in (7.23). Let $H$ be any Hamiltonian with $H \mapsto \phi$. Consider the family (7.23) and the equation

$$\begin{cases} 
\frac{\partial u}{\partial \tau} + J_t \left( \frac{\partial u}{\partial \tau} - X_H(u) \right) = 0 \\
u(\infty) = [z_-, w_-], \quad u(\infty) = [z_+, w_+] \\
w_- \# u \sim w_+, \quad u(0, 0) = q \in B_p(r)
\end{cases} \quad (7.24)$$

If (7.24) has a cusp-solution

$$u_1 \# u_2 \# \cdots \# u_N$$
which is a connected union of Floer trajectories for $H$ that satisfies the asymptotic condition

\[ u_N(\infty) = [\tilde{z}', \tilde{w'}], \quad u_1(-\infty) = [z, w] \]

\[ u_j(0, 0) = q \text{ for some } j. \]

for some $[z, w] \in \text{Crit } \cdot A_H$ and $[z', w'] \in \text{Crit } \cdot \tilde{A}_H$, then we have

\[ A_H(u(-\infty)) - A_H(u(\infty)) \geq A_D(\phi, J_0 : J'). \quad (7.25) \]

**Proof.** Suppose $u$ is such a solution. Opening up $u$ along $t = 0$, we define a map $v: \mathbb{R} \times [0, 1] \to M$ by

\[ v(\tau, t) = (\phi_t^H)^{-1}(u(\tau, t)). \]

It is straightforward to check that $v$ satisfies (7.11). Moreover we have

\[ \int \left| \frac{\partial v}{\partial \tau} \right|^2 J' \leq \int v^* \omega \geq A_D(\phi, J_0; J'). \quad (7.27) \]

Combining (7.26) and (7.27), we have proven

\[ A_H(u(-\infty)) - A_H(u(\infty)) = \int \left| \frac{\partial u}{\partial \tau} \right|^2 J \geq A_D(\phi, J_0; J'). \]

This finishes the proof. \qed

Next we have the following non-pushing down lemma.

**Lemma 7.10 [Lemma 7.8, Oh6].** Consider the cohomology class $1 \in QH^*(M) = HQ^*(-\epsilon f)$. Then there is the unique Novikov cycle $\gamma$ of the form

\[ \gamma = \sum_j c_j [x_j, \hat{x}_j] \quad (7.28) \]

with $x_j \in \text{Crit } (-\epsilon f)$ that represents the class $1^\phi = [M]$. Furthermore for any Novikov cycle $\beta$ with $[\beta] = [M]$, i.e.,

\[ \beta = \gamma + \partial_{(-\epsilon f)}(\delta) \]

with $\delta$ a Novikov chain, we have

\[ \lambda_{\epsilon f}(\beta) \geq \lambda_{\epsilon f}(\gamma). \quad (7.29) \]

**Proof.** Note that Morse index $\mu_{-\epsilon f}(x_j)$ in (7.28) must be $2n$ to represent the fundamental cycle $[M]$. Standard Morse theory shows

\[ \text{Im } \partial_{-\epsilon f} \cap C_{2n}(-\epsilon f) = 0. \]
This proves that there exists the unique Morse cycle that represents the fundamental class \([M]\).

For the second statement, because \(\partial = \partial_{-\epsilon f} \otimes \Lambda_{\omega}\), there cannot exist any Floer trajectory of \(\epsilon f\) as well as the gradient trajectory of \(-\epsilon f\) that lands at any \(x_j\) in (7.28). In other words, the term \(\partial \delta\) cannot kill the leading term, of \(\gamma\). Hence follows (7.29). ☐

Let \(\delta > 0\) be any given number and choose \(H \mapsto \phi\) so that

\[
0 \leq \rho(H; 1) + \rho(\overline{H}; 1) \leq A(\phi) - 2\delta. \tag{7.30}
\]

Such \(H\) exists by (7.19).

By the definition of \(\rho\), there exist homotopies \(\mathcal{H}_1\) and \(\mathcal{H}_2\) from \(\epsilon f\) to \(H\) and to \(\overline{H}\) respectively, and \(\alpha, \beta \in CF_0(\epsilon f)\) representing \(1^\flat = [M]\) such that

\[
\rho(H; 1) \leq \lambda_H(h_{\mathcal{H}_1}(\alpha)) \leq \rho(H; 1) + \frac{\delta}{2} \tag{7.31}
\]

\[
\rho(\overline{H}; 1) \leq \lambda_H(h_{\mathcal{H}_2}(\beta)) \leq \rho(\overline{H}; 1) + \frac{\delta}{2}. \tag{7.32}
\]

By adding (7.31) and (7.32) we have

\[
0 \leq \rho(H; 1) + \rho(\overline{H}; 1) \leq \lambda_H(h_{\mathcal{H}_1}(\alpha)) + \lambda_{\overline{H}}(h_{\mathcal{H}_2}(\beta)) \leq \rho(H; 1) + \rho(\overline{H}; 1) + \delta. \tag{7.33}
\]

Combining (7.30) and (7.33), we have

\[
0 \leq \lambda_H(h_{\mathcal{H}_1}(\alpha)) + \lambda_{\overline{H}}(h_{\mathcal{H}_2}(\beta)) \leq A(\phi) - \delta. \tag{7.34}
\]

We write

\[
h_{\mathcal{H}_1}(\alpha) = \sum_j a_j'[z_j, w_j], \quad \lambda_1 > \lambda_2 > \cdots
\]

where \(\lambda_j = \mathcal{A}_H([z_j, w_j])\) and

\[
h_{\mathcal{H}_2}(\beta) = \sum_k b_k'[z_k', w_k'], \quad \mu_1 > \mu_2 > \cdots
\]

where \(\mu_j = \mathcal{A}_{\overline{H}}([z_j', w_j']).\) Of course, we assume that \(a_1' \neq 0\) and \(b_1' \neq 0\) and so

\[
\lambda_1 = \lambda_H(h_{\mathcal{H}_1}(\alpha)), \quad \mu_1 = \lambda_{\overline{H}}(h_{\mathcal{H}_2}(\beta)).
\]

Now we recall the identity (6.24)

\[
\int_{\Sigma} v^* \omega_P = \mathcal{A}_{\mathcal{H}_1}([z_1, w_1]) + \mathcal{A}_{\mathcal{H}_2}([z_2, w_2]) - \mathcal{A}_{\mathcal{H}_1, \# \mathcal{H}_2}([z_3, w_3]) \tag{7.35}
\]

for any \(\tilde{J}\)-holomorphic sections \(v \in \mathcal{M}(H, \tilde{J}; \tilde{z})\). Note that this provides the uniform energy bound for the pseudo-holomorphic sections \(v \in \mathcal{M}(H, \tilde{J}; \tilde{z})\).
For the discussion below, morally we would like to apply (7.35) to the case

\[ H_1 = H, \ H_2 = \tilde{H}, \ H_3 = 0 \]  

(7.36)

for a pseudo-holomorphic section of an appropriate Hamiltonian bundle \( P \to \Sigma \) that has the boundary condition

\[
\begin{align*}
  u|_{\partial_1 \Sigma} &= [z, w] \in h_{H_1}(\alpha), \ u|_{\partial_2 \Sigma} = [z', w'] \in h_{H_2}(\beta) \\
  u|_{\partial_3 \Sigma} &= [q, \hat{q}] \in \gamma
\end{align*}
\]  

(7.37)

with

\[ A_H([z, w]) + A_{\tilde{H}}([z', w']) \geq \rho(H; 1) + \rho(\tilde{H}; 1) - \delta. \]

(7.38)

Here one should, a priori, consider all possible cycles \( \gamma \) from \( CF_0(\epsilon f) \) that represents \([M]\), but Lemma 7.7 guarantees that the choice of the unique Morse cycle provided in (7.28) is enough to consider and so can be fixed throughout the later discussions.

Note that because \( H_3 = 0 \) in the monodromy condition (7.36) and the outgoing end with monodromy \( H_2 = \tilde{H} \) is equivalent to the incoming end with monodromy \( H_2 = H \), we can fill-up the hole \( z_3 \in \Sigma \) and consider the cylinder with one outgoing and one incoming end with the same monodromy \( H \). In other words our Hamiltonian bundle \( P \to \Sigma \) becomes just the mapping cylinder

\[ E_\phi := \mathbb{R} \times \mathbb{R} \times M/(\tau, t + 1, \phi(x)) \sim (\tau, t, x) \to \mathbb{R} \times S^1. \]

\( \phi \) with the canonical Hamiltonian connection \( \nabla_H \) induced by \( H \) (see [En1] for a detailed explanation of the relation between Hamiltonians and connections). Then with the \( H \)-compatible almost complex structure \( \tilde{J} \), \( J \)-pseudo-holomorphic sections of \( E_\phi \) becomes nothing but solutions of (7.24). This heuristic reasoning motivates the following proposition.

**Proposition 7.11 [Oh8].** Let \( H \) and \( J_0 \) be as in Proposition 7.9 and let \( \delta_1 > 0 \) be given. Then for any \( J' \in j(\phi, J_0) \), there exist some \([z, w] \in h_{H_1}(\alpha) \) and \([z', w'] \in h_{H_2}(\beta) \) that satisfy (7.38) such that the following alternative holds:

1. (7.23) has a cusp-solution

\[ u_1 \# u_2 \# \cdots \# u_N \]

which is a connected union of Floer trajectories for \( H \) that satisfies the asymptotic condition

\[ u_N(\infty) = [\tilde{z}', \tilde{w}'], \ u_1(-\infty) = [z, w], \ u_j(0, 0) = q \in B_p(r). \]

(7.39)

for some \( 1 \leq j \leq N \),

2. or

\[ A_H([z, w]) - A_{\tilde{H}}([\tilde{z}', \tilde{w}']) \geq A_S(\phi, J_0; J') - \delta_1. \]

(7.40)

This in particular implies

\[ 0 < A(\phi, J_0) \leq A(\phi) < \infty \]
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for any \( \phi \) and \( J_0 \).

Referring the readers to [Oh8] for the proof of this proposition, we proceed with the non-degeneracy proof.  

Finish-up of the proof of non-degeneracy. Because of (7.37) and (7.30)-(7.32), we have

\[
A_H([z, w]) \leq A_H(h_{H_1}(\alpha)) \leq \rho(H; 1) + \frac{\delta}{2}, \tag{7.41}
\]

\[
A_{\tilde{H}}([z', w']) \leq A_{\tilde{H}}(h_{H_1}(\beta)) \leq \rho(\tilde{H}; 1) + \frac{\delta}{2}.
\]

Combining (7.27), (7.35) and (7.38), we obtain

\[
-\delta \leq A_H([z, w]) + A_{\tilde{H}}([z', w']) \leq A(\phi) - \delta. \tag{7.42}
\]

On the other hand, (7.22) and (7.25) imply

\[
A_{\tilde{H}}([z', w']) + A_H([z, w]) = -A_H([\tilde{z}', \tilde{w}']) + A_H([z, w]) \geq A_D(\phi, J_0).
\]

for the case (1). For the case (2), we have (7.40). In either case, we have

\[
A_H([z, w]) - A_H([\tilde{z}', \tilde{w}']) \geq A(\phi, J_0) - \delta. \tag{7.43}
\]

Therefore choosing \( \delta_1 < \frac{\delta}{3} \) and \( J_0 \) with

\[
A(\phi, J_0) \geq A(\phi) - \frac{\delta}{3},
\]

we obtain

\[
A(\phi) - \delta \geq A(\phi) - \frac{2\delta}{3} \tag{7.44}
\]

from (7.42) and (7.43). This is absurd and finishes the proof. \( \square \)

Remark 7.12. We would like to note that (7.38) and (7.10) together also imply the following lower bounds for \( A_H([z, w]) \) and \( A_{\tilde{H}}([z', w']) \)

\[
-\rho(\tilde{H} : 1) - 2\delta \leq A_H([z, w]) \leq \rho(H; 1) + \frac{\delta}{2} \tag{7.45}
\]

\[
-\rho(\tilde{H} : 1) - 2\delta \leq A_{\tilde{H}}([z', w']) \leq \rho(\tilde{H}; 1) + \frac{\delta}{2} \tag{7.46}
\]

§8. Lower estimates of the Hofer norm

In this section, we will apply the results in the previous sections to the study of the Hofer norm of Hamiltonian diffeomorphisms. We start with improving the lower bound (7.42) (and hence (7.43)). For this purpose, we re-examine our set-up of the definition

\[
\gamma(\phi) = \inf_{\pi(\phi) = \phi} \tilde{\gamma}(\tilde{\phi}) = \inf_{\pi(\phi) = \phi} (\rho(\tilde{\phi}; 1) + \rho(\tilde{\phi}^{-1}; 1)).
\]
This involves only the class $1 \in QH^*(M)$ and only the orbits $[z, w]$ of $H$ and $[z', w']$ of $\tilde{H}$ of the Conley-Zehnder indices

$$\mu_H([z, w]) = -n, \quad \mu_{\tilde{H}}([z', w']) = -n$$  \hfill (8.1)

If we reverse the flow of $\tilde{H}$ by considering the time reversal map (7.19)

$$[z', w'] \mapsto [\tilde{z}', \tilde{w}']$$

then the cusp-solution of $u$ constructed in Proposition 7.24 satisfy the asymptotic condition

$$u(\infty) = [z', w'] \in CF_{2n}(H) \quad \text{i.e., } \mu_H([z', w']) = n$$

$$u(-\infty) = [z, w] \in CF_0(H) \quad \text{i.e., } \mu_H([z', w']) = n$$

with respect to our grading convention (4.18). Here is the main definition

**Definition 8.1.** Let $J_0$ be a given almost complex structure and $J' \in j(\phi, J_0)$. Let $h_{H_1}(\alpha)$ and $h_{H_2}(\beta)$ be as before in §7, 8 with $[\alpha] = [\beta] = 1^p = [M]$. Consider all cusp-solutions

$$u = u_1 \# u_2 \# \cdots \# u_N$$

of

$$\left\{ \begin{array}{l}
\frac{\partial u}{\partial \tau} + J_t \left( \frac{\partial u}{\partial t} - X_H(u) \right) = 0 \\
u_1(-\infty) \in h_{H_1}(\alpha), \quad u_N(\infty) \in h_{H_2}(\beta), \\
u_j(0,0) = q \in B_p(r) \quad \text{for some } j = 1, \cdots, N
\end{array} \right. \quad \text{(8.2)}$$

with $J_t = (\phi^t_H)_* J'_t$. We define

$$A(\phi, J_0; J': h_{H_1}(\alpha), h_{H_2}(\beta)) = \inf \left\{ \int \left| \frac{\partial u}{\partial \tau} \right|^2 | u \text{ satisfies (8.2)} \right\}$$

$$= \inf \left\{ \int v^* \omega | v \text{ satisfies (1.31) with} \\
z^- \in h_{H_1}(\alpha) : z^-(t) = \phi^t_H(v(-\infty)) \\
z^+ \in h_{H_2}(\beta) : z^+(t) = \phi^t_H(v(+\infty)) \right\} \quad \text{(8.3)}$$

$$A(\phi, J_0; J': 1) = \inf_{\alpha, \beta, h_{H_1}, h_{H_2}} A(\phi, J_0; J': h_{H_1}(\alpha), h_{H_2}(\beta))$$

and then

$$A(\phi, J_0; 1) = \sup_{J' \in j(\phi, J_0)} A(\phi, J_0; J': 1) \quad \text{(8.4)}$$

and finally

$$A(\phi; 1) = \sup_{J_0} A(\phi, J_0; 1) \quad \text{(8.5)}$$

With this definition, the proof of §8 indeed proves the following stronger estimates
Theorem 8.2. For any non-degenerate \( \phi \neq id \), we have
\[
0 < A(\phi; 1) < \infty \quad \text{and} \quad \gamma(\phi) \geq A(\phi; 1).
\]
(8.7)

Now we will try to give an estimate \( A(\phi; 1) \) when the Hamiltonian diffeomorphism \( \phi \) has the property that its graph
\[
\Delta_\phi := \text{graph } \phi \subset (M, -\omega) \times (M, \omega)
\]
is close to the diagonal \( \Delta \subset M \times M \) in the following sense: Let \( o_\Delta \) be the zero section of \( T^*\Delta \) and
\[
\Phi : (U, \Delta) \subset M \times M \rightarrow (V, o_\Delta) \subset T^*\Delta
\]
be a Darboux chart such that
\[\begin{array}{l}
(1) \; \Phi^*\omega_0 = -\omega \oplus \omega \\
(2) \; \Phi|_\Delta = id_\Delta \quad \text{and} \quad d\Phi|_{T|U_\Delta} : T(U|_\Delta) \rightarrow T(V|_{o_\Delta}) \quad \text{is the obvious symplectic bundle map from } T(M \times M)_\Delta \cong N_\Delta \oplus T\Delta \text{ to } T(T^*\Delta)_{o_\Delta} \cong T^*\Delta \oplus T\Delta \text{ which is the identity on } T\Delta \text{ and the canonical map}
\end{array}\]
\[
\tilde{\omega} : N\Delta \rightarrow T^*\Delta; \quad X \mapsto X|\omega
\]
on the normal bundle \( N\Delta = T(M \times M)_\Delta/T\Delta \).

Definition 8.3. A Hamiltonian diffeomorphism \( \phi : M \rightarrow M \) is called *engulfable* if there exists a Darboux chart \( \Phi : U \rightarrow V \) as above such that
\[
\Phi(\Delta_\phi) = \text{graph } dS_\phi
\]
for the unique smooth function \( S_\phi : \Delta \rightarrow \mathbb{R} \) with \( \int_\Delta S_\phi \, dt = 0 \).

The term ‘engulfable’ has origin from topology and comes more directly from Laudenbach’s paper [La] where a version of engulfing was introduced in the similar context for Lagrangian submanifolds. Any \( \phi \) that is \( C^1 \)-close to \( id \) is obviously engulfable and the corresponding \( S_\phi \) is \( C^2 \)-small. Consider the path
\[
t \mapsto \text{graph } tdS_\phi
\]
which is a Hamiltonian isotopy of the zero section and define \( \phi^t : M \rightarrow M \) to be the Hamiltonian diffeomorphism satisfying
\[
\Phi(\Delta_{\phi^t}) = \text{graph } tdS_\phi
\]
for \( 0 \leq t \leq 1 \). By the requirement (1) and (2) for the Darboux chart \( \Phi \) (8.9), it follows that the path \( t \mapsto \phi^t \) is a Hofer geodesic, i.e., a quasi-autonomous Hamiltonian path which has no non-constant one periodic orbits. We denote by \( H = H_\phi \) the corresponding Hamiltonian with \( \phi^t = \phi^t_{H_\phi} \).

Theorem II immediately implies
\[
\rho(\phi; 1) + \rho(\phi^{-1}; 1) \leq \int_0^1 (\max H_t - \min H_t) \, dt
\]
(8.12)
On the other hand, if $H$ is the one obtained from (8.11), we also have
\[ \int_0^t - \min H_u \, du = t \max S_\phi, \quad \int_0^t - \max H_u \, du = t \min S_\phi. \tag{8.13} \]
and so
\[ \gamma(\phi) \leq \rho(H; 1) + \rho(\overline{T}; 1) \leq \max S_\phi - \min S_\phi := osc(S_\phi). \tag{8.14} \]

We will prove some estimates of $A(\phi; 1)$ in the next section. To describe this estimate, we need some general discussion on the topology of the pair $(L_0, L_1)$ of Lagrangian submanifolds of a symplectic manifold $(P, \omega)$. Consider a map
\[ u : [0, 1] \times [0, 1] \to P \]
satisfying
\[ u(s, 0) \in L_0, \quad u(s, 1) \in L_1 \]
\[ u(0, t) = p_0, \quad u(1, t) = p_1 \quad \text{for } p_0, p_1 \in L_0 \cap L_1 \tag{8.15} \]
We denote by $\pi_2(L_0, L_1; p_0, p_1)$ the set of homotopy classes of $u$ satisfying (8.15).

Note that the symplectic action of $u$ defines a well-defined map
\[ I_\omega : \pi_2(L_0, L_1; p_0, p_1) \to \mathbb{R}. \]

Now let us restrict to the case of our main interest:
\[ \Delta, \Delta_\phi \subset \mathcal{U} \subset (M, -\omega) \times (M, \omega) \]
In this case, there are distinguished classes in $\pi_2(\Delta, \Delta_\phi; p_0, p_1)$ for each given $p_0, p_1 \in \Delta \cap \Delta_\phi$ defined by
\[ u : (s, t) \mapsto t \, dS_\phi(\chi(s)) \]
where $\chi : [0, 1] \to \Delta$ is a curve connecting $p_0, p_1$ on $\Delta$. Furthermore for each such map, the action of $u$ becomes
\[ \int u^*(-\omega \oplus \omega) = S_\phi(p_0) - S_\phi(p_1) \]
which are independent of the choice of $\chi$.

Now we consider the family of compatible almost complex structures on $(M, -\omega) \times (M, \omega)$
\[ \overline{T}(t) := -J^t_1 \oplus J^t_\Delta = -J^t_{1- \Delta} \oplus J^t_\Delta \]
We consider the family $J^t = \{ J^t_1 \}$ on $M \times M$ such that
\[ J^t_1 = \begin{cases} \overline{T}(t) & \text{on } \mathcal{U}' \\ J_\Delta & \text{on } \mathcal{U} \setminus \overline{T} \end{cases} \tag{8.16} \]
where $J_\Delta$ is an almost complex structure with respect to which the neighborhood of $\partial \mathcal{U}$ is pseudo-convex on $\mathcal{U}'' \subset \mathcal{U}$ with
\[ \overline{T} \subset \mathcal{U}' \subset \overline{T}' \subset \mathcal{U} \tag{8.17} \]
We can always produce such an almost complex structure by pulling back one from $V \subset T^* \Delta$. We suitably interpolate $J'$ on $U'$ and $J_\Delta$ on $U \setminus U''$ and extend to $M \times M$ arbitrarily outside of $U$ to obtain $J^\mu$ on $M \times M$. We will make the region $U \setminus U''$ grow in the end until it exhausts $U \setminus U'$.

We then decompose the moduli space $M(\Delta, \Delta_\phi; J^\mu)$ into

$$M(\Delta, \Delta_\phi; J^\mu) = M^\mu(\Delta, \Delta_\phi; J^\mu) \text{∐} M'(\Delta, \Delta_\phi; J^\mu)$$

where the first is the set of those for which the image of $V$ as defined in (9.27) below is contained in $U$ and the second is that of those not. We then define

$$A'(J_0; J'; U) = \inf \{ \omega(u) \mid u \in M'(\Delta, \Delta_\phi; J^\mu) \}$$

$$A'(J_0; U) = \sup_{J'} A'(J_0; J'; U)$$

$$A'(U) = \sup_{J_0} A'(J_0; U)$$

With this definition, we will prove the following theorem in the next section.

**Theorem 8.4.** Let $\phi$ and $S_\phi$ as above and assume

$$\text{osc}(S_\phi) \leq A'(U).$$

Then we have

$$A(\phi; 1) \geq \text{osc}(S_\phi)$$

and hence

$$\gamma(\phi) = \text{osc}(S_\phi) = A(\phi; 1).$$

Assuming this theorem for the moment, we state several consequences of it.

**Theorem 8.5.** Let $\Phi : U \to V$ be a Darboux chart along the diagonal $\Delta \subset M \times M$, and $H = H^\phi$ is the Hamiltonian generating $\phi^t : M \times M$ as in (8.9). Then the path $t \in [0, 1] \to \phi^t$ is length minimizing among all paths from the identity to $\phi$. In this case, we also have $\gamma(\phi) = \|\phi\|$.

**Proof.** Let $K \to \phi$ and $\phi^t_K$ be the corresponding Hamiltonian path. By the canonical adjustment [Lemma 5.2, Oh6], we may assume that $K$ is one-periodic. It follows from (7.3) that

$$\tilde{\gamma}([\phi, K]) \leq \|K\|.$$  

In particular, we have

$$\tilde{\gamma}(\phi) \leq \|K\|. \quad (8.19)$$

Combining Theorem 8.4, (8.13)-(8.15), we have obtained

$$\gamma(\phi) \leq \|H^\phi\| = \text{osc}(S_\phi) = A(\phi; 1) = \gamma(\phi)$$

and so all the inequalities become the equality. In particular, we have

$$\gamma(\phi) = \|H^\phi\| \quad (8.20)$$
Therefore we derive $\|H\phi\| \leq \|K\|$ from (8.19) and (8.20) which is exactly what we wanted to prove. The identity $\gamma(\phi) = \|\phi\|$ follows from (8.20) and the inequality $\|H\phi\| \geq \|\phi\|$ by definition. This finishes the proof. □

We recall from [Oh4] that for any Morse function $f : N \to \mathbb{R}$ the path

$$t \mapsto \text{graph } t df$$

is globally length minimizing geodesic as the Hamiltonian path of Lagrangian submanifolds on any cotangent bundle $T^*N$. Therefore the above theorem provides an estimate of the diameter of $\text{Ham}(M, \omega)$ in terms of the size of the Darboux neighborhood of the diagonal in $(M, -\omega) \times (M, \omega)$ which is an invariant of the symplectic manifold $(M, \omega)$. It will be interesting to further investigate this aspect in the future.

An immediate corollary of Theorem 8.5 is the following McDuff's result for the $\phi = \phi_1^H$ for $C^2$-small Hamiltonian $H$.

**Corollary 8.6 [Proposition 1.8, Mc2].** There is a path connected neighborhood $N \subset \text{Ham}(M, \omega)$ of the identity in the $C^1$-topology such that any element in $N$ can be joined to the identity by a path that minimizes the Hofer length. Moreover $(N, \|\cdot\|)$ is isometric to a neighborhood of $\{0\}$ in a normed vector space which is nothing but the space of $\{S_\phi\}$.

**Proof.** This is an immediate consequence of Theorem 8.4 since any Hamiltonian diffeomorphism sufficiently $C^1$-close to the identity has its graph is contained in a Darboux chart and we can assume $\text{osc}(S_\phi) \leq A'(U)$. In fact, in this case of $C^1$-close to the identity, we can replace $A'(U)$ by the constant $A(\omega)$ (see Appendix 1). □

§9. Area estimates of pseudo-holomorphic curves

In this section, we will prove Theorem 8.4. For this purpose, we need to analyze structure of the Floer boundary operator and the pants product when the relevant Hamiltonians are of the type $H = H_\phi$ as defined in (8.9) and (8.10). In particular for the pants product, we will be particularly interested in the case

$$H_1 = H_\phi, H_2 = \bar{H}_\phi, H_3 = 0$$

which we regard as the limit of the case (8.1). Since we will use a concept of ‘isolated continuation’ as in [Fl2], [Oh2,7], we first study the case of $\phi$’s which are $C^1$-close to the identity.

9.1. The case $C^1$-close to id

In this case, the relevant moduli space is decomposed into the thin part (the “classical” contributions) and the thick part (the “quantum” contributions). We will prove this decomposition result as $\|H\|_{C^2} \to 0$ for the moduli space with the number of ends 2 and 3. The first case is relevant to the boundary operator and the second to the pants product. Similar decomposition exists for arbitrary number of ends, of course. Since the case of boundary operator is easier and has been previously studied in [Oh2,7], we will focus on the pants product in this section.
To motivate our discussion, we recall the quantum product of \( a, b \in H^*(M) \) can be written as
\[
a \circ b = a \cap b + \sum_{A \neq 0} (a, b)_A q^{-A}
\] (9.1)
where \((a, b)_A \in H^*(M)\) is the cohomology class defined by
\[
(a, b)_A = PD([\mathcal{M}_3(J; A) \times_{(ev_1, ev_2)} (Q_a \times Q_b), ev_0]).
\] (9.2)
Here \(\mathcal{M}_3(J; A)\) is the set of stable maps with three marked points in class \(A \in H_2(M)\). We denote by \(ev_i : \mathcal{M}_3(J; A) \to M\) for \(i = 0, 1, 2\) the evaluation maps. Then
\[
\mathcal{M}_3(J; A) \times_{(ev_1, ev_2)} (Q_a \times Q_b)
\]
is the fiber product of \(\mathcal{M}_3(J; A)\) and \(Q_a, Q_b\) via the evaluation maps \((ev_1, ev_2)\).
\([\mathcal{M}_3^A \times_{(ev_1, ev_2)} (Q_a \times Q_b), ev_0]\) is the homology class of the fiber product as a chain via the map
\[
ev_0 : \mathcal{M}_3(J; A) \times_{(ev_1, ev_2)} (Q_a \times Q_b) \to M.
\]
Geometrically it is provided by the image (by \(ev_0\)) of the holomorphic spheres intersecting the cycles \(Q_a\) and \(Q_b\) at the first and second marked points. The term corresponding to \(A = 0\) provides the classical cup product \(a \cup b\).

Pretending this case as the one for \(H = 0\), we now study the pants product in the Floer homology for \(C^2\)-small Hamiltonians \(H\). We recall the pants product \(\gamma * \delta\) of the Novikov cycles \(\gamma\) and \(\delta\) of \(H_1\) and \(H_2\) are defined by replacing the above \(\mathcal{M}_3(J; A)\) by the moduli space
\[
\mathcal{M}(H, \tilde{J}; \tilde{z})
\]
for \(\tilde{z} = (\tilde{z}_1, \tilde{z}_2, \tilde{z}_3)\) with \(\tilde{z}_i = [z_i, w_i]\). We note that when \(H = (H_1, H_2, H_3)\) are all \(C^2\)-small quasi-autonomous and in particular when all the time-one periodic orbits are constant, any periodic orbit \(z = p\) has the canonical lifting \([p, \tilde{p}]\) in \(\hat{\Omega}_0(M)\).

Therefore we can write the generator \([z, w]\) as
\[
p \otimes q^A := [p, \tilde{p} \# A].
\] (9.3)
We fix a trivialization \(P = \Sigma \times M\) and write \(u = pr_2 \circ v\).

For each given \(p = (p_1, p_2, p_3)\) with \(p_i\) are (constant) periodic orbits of \(H_i\)’s respectively, we denote by \(\mathcal{M}_3(H, \tilde{J}; p)\) the set of all \(\tilde{J}\)-holomorphic sections \(v\) over \(\Sigma\) with the obvious asymptotic conditions
\[
u(x_1) = p_1, \ u(x_2) = p_2, \ u(x_3) = p_3
\] (9.4)
where \(u\) is the fiber component of \(v\) under the trivialization \(\Phi : P \to \Sigma \times M\) and \(x_i\)’s are the given punctures in \(\Sigma\) as before. If we denote by \(\mathcal{M}_3(H, \tilde{J})\) as the set of all \textit{finite energy} \(\tilde{J}\)-holomorphic sections, then it has the natural decomposition
\[
\mathcal{M}_3(H, \tilde{J}) = \cup_p \mathcal{M}_3(H, \tilde{J}; p).
\]
Furthermore, because of the asymptotic condition (9.4) for any element \(v \in \mathcal{M}_3(H, \tilde{J})\), \(u = pr_M \circ v\) naturally compactifies and so defines a homotopy class \([u] \in \pi_2(M)\). In this way, \(\mathcal{M}_3(H, \tilde{J}; p)\) is further decomposed into
\[
\mathcal{M}_3(H, \tilde{J}; p) = \cup_{A \in H_2(M)} \mathcal{M}_3^A(H, \tilde{J}; p)
\]
where $\mathcal{M}_3^A(H, \tilde{J}; p)$ is the subset of $\mathcal{M}_3(H, \tilde{J}; p)$ in class $A$. We denote

$$\mathcal{M}_3^A(H, \tilde{J}; p) = \cup_{A \neq 0} \mathcal{M}_3^A(H, \tilde{J}; p)$$

and then have

$$\mathcal{M}_3(H, \tilde{J}; p) = \mathcal{M}_3^0(H, \tilde{J}; p) \coprod \mathcal{M}_3^A(H, \tilde{J}; p)$$

Assuming $\phi = \phi^1_H$ is sufficiently $C^1$-small, we will prove the following decomposition of the moduli space in the Appendix 1.

**Proposition 9.1.** Let $H$ be $C^2$-small and fix any constants $\alpha_i$, $i = 1, 2$ with

$$0 < \alpha_1 < A(\omega, J_0), \quad \alpha_1 + \alpha_2 < A(\omega, J_0).$$

Consider the Hamiltonian fibration $P \to \Sigma$ of $H$ equipped with $H$-compatible almost complex structure $\tilde{J}$ with respect to the symplectic form

$$\Omega_{P, \lambda} = \omega_P + \lambda \omega_\Sigma.$$

We denote by $v$ an element in $\mathcal{M}_3(H, \tilde{J})$. Then there exists a constant $\delta = \delta(J_0, \alpha)$ such that if $\|H\|_{C^2} < \delta$, we can find a closed 2 form $\omega_P$ satisfying (6.5) and a sufficiently small constant $\lambda > 0$ for which the following alternative holds:

1. **and all those $v$ with** $\int v^* \omega_P = 0$ **are ‘very thin’**

$$\int |Dv|^2_{\tilde{J}} \alpha_1$$

**and the fiber class** $[u] = 0$.

2. **all the elements** $u$ **with** $\int v^* \omega_P \neq 0$ **are ‘thick’ i.e.,**

$$\int |Dv|^2_{\tilde{J}} > A(\omega, J_0) - \alpha_2.$$

For the trivial generators $\hat{p}_i = [p_i, \hat{p}_i], i = 1, 2$, the pants product is given by

$$\hat{p}_1 * \hat{p}_2 = \hat{p}_1 *_0 \hat{p}_2 + \sum_{A \neq 0} \hat{p}_1 *_A \hat{p}_2$$

where $\hat{p}_1 *_A \hat{p}_2$ is defined as

$$\hat{p}_1 *_A \hat{p}_2 = \sum_{p_3 \in \text{Per}(H_3)} (p_1, p_2; p_3)_A \hat{p}_3 \otimes q^A.$$

Here

$$(p_1, p_2; p_3)_A = \#(\mathcal{M}_3^A(H, \tilde{J}; p)).$$

(9.9) induces the formula for arbitrary generators $[z, w] = [p, \hat{p} # A]$ in an obvious way. Note that (9.9) is the analog to (9.1) for $H \neq 0$ but $C^2$-small.

The main point of Proposition 9.1 is that the thin part $\mathcal{M}_3^0(H, \tilde{J})$ of the moduli space $\mathcal{M}_3(H, \tilde{J})$ is ‘far’ from the thick part $\mathcal{M}_3^A(H, \tilde{J})$, and isolated under the
continuation of the whole moduli space \( \mathcal{M}_3(H, \tilde{J}) \) via \( C^2 \)-small deformations of Hamiltonians. This in particular implies that the classical term of the pants product induced by \( \tilde{z}_1 * \tilde{z}_2 \) is invariant in homology under such continuation of \( H \) and induces the classical cup product

\[
\cup : H^k(M) \times H^\ell(M) \to H^{k+\ell}(M).
\]

In particular the product by the identity 1 induces an isomorphism 1 : \( H^0(M) \to H^0(M) \). In the dual description of this multiplication by 1 is given by the cap action

\[
[M]\cap : H^{2n}(M) \to H^0(M)
\]

and its homological description provides an isomorphism

\[
([M]\cap)^* : H_0(M) \to H_{2n}(M)
\]

which is one of the key ingredients to prove Theorem 8.6. This operation can be defined for the local Floer complex of \( H \)'s as long as \( H \) is \( C^2 \)-small (see [Oh2] for this kind of argument). We now recall the following homologically essential property of \( [x^-, \tilde{x}^-] \) in the local Floer complex of \( H \) with \( x^- \) is the minimum point of the quasi-autonomous Hamiltonian.

**Lemma 9.2 [Proposition 4.4, Oh6].** Suppose that \( \|G\|_{C^2} < \delta \) with \( \delta \) so small that graph \( \phi_G \subset \mathcal{U} \) lies in the given Darboux neighborhood of \( \Delta \subset M \times M \). Suppose that \( G \) is quasi-autonomous with the unique maximum point \( x^+ \) and minimum point \( x^- \). Then both critical points \( x^\pm \) are homologically essential in the local Floer complex \( CF(J, G : \mathcal{U}) \).

The interpretation of the isomorphism (9.12) in the local Floer complex immediately implies the following existence theorem of ‘thin’ solutions. The proof is the same as that of [Lemma 7.4, Oh4] which deals with the case of Morse theory but works for this case of the local Floer complex verbatim.

**Proposition 9.3.** Let \( J_0 \) and \( H \) be as before. Assume that \( \phi \) is \( C^1 \)-small so that the graph \( \phi \) is contained in the Darboux neighborhood \( \mathcal{U} \). Let \( \mathcal{H}_1 = 1, 2 \) be any homotopies connecting \( \epsilon f \), and \( H, \tilde{H} \) respectively. Then there must be a ‘thin’ cusp solution of the equation (8.2)

\[
u = u_1 \# u_2 \# \cdots \# u_N
\]

of

\[
\begin{align*}
\frac{\partial u}{\partial \tau} + J_t \left( \frac{\partial u}{\partial t} - X_H(u) \right) &= 0 \\
u_1(-\infty) &= h_{\mathcal{H}_1}(\alpha), \quad u_N(\infty) = h_{\mathcal{H}_2}(\beta), \\
u_j(0,0) &= p \in B_p(r) \quad \text{for some } j = 1, \cdots, N
\end{align*}
\]

(9.13)

with \( J_t = (\phi_t^{\mathcal{H}})_* J_0 \), that satisfy

\[
A_H(u(\infty)) = \int_0^1 \max H \, dt = \int_0^1 H_1(x^+) \, dt
\]

\[
A_H(u(-\infty)) = \int_0^1 - \min H \, dt = \int_0^1 - H_1(x^-) \, dt
\]

(9.14)
Proof. By the Morse theory realization of the cap action (see [BzC], [Oh5]), we may assume, by re-choosing the ball $B_p(r)$ if necessary, that there is no cusp gradient trajectory of $-S_\phi$ that passes through the ball $B_p(r)$, unless the trajectory connects a (global) maximum and a (global) minimum points of $-S_\phi$. This is a consequence of the homologically essentialness of maximum and minimum values and isomorphism property of the cap action. We will assume this in the rest of the proof.

Using the description of critical points of $A_H$, we write

$$h_{H_1}(\alpha) = \sum_A a_A p_A \otimes q^A,$$

$$h_{H_2}(\beta) = \sum_A b_A \tilde{p}_A \otimes q^A$$

and compute the pants product

$$h_{H_1}(\alpha) \ast h_{H_2}(\beta) = \sum_{A,B} (\hat{p}_A + \hat{p}_B)q^{A+B}$$

$$= \sum_{A,B,C} (\hat{p}_A \ast_C \hat{p}_B)q^{A+B}$$

$$= \sum_{A,B,C} (p_A, \tilde{p}_B; q_C)_{A+B-C} \tilde{q}_C \otimes q^{(A+B-C)}. \quad (9.15)$$

On the other hand, we recall

$$h_{H_1}(\alpha) \ast h_{H_2}(\beta) = \gamma + \partial \delta \quad (9.16)$$

where $\gamma$ is the unique Morse cycle of $-\epsilon f$ that realizes the class $[M]$ (see Lemma 7.7). Comparing (9.15) and (9.16), we now prove

**Lemma 9.4.** Suppose that $\phi$ is sufficiently $C^1$-close to the identity. Then for any $H_i$ above, we can find classes $A$, $B$ with $A + B = 0$ and $p_A \in \text{Per}(H)$, $\tilde{p}_B \in \text{Per}(\tilde{H})$ such that

$$A_H([p_A; \tilde{p}_A]) = -\min H \#(\epsilon f), \quad A_H([p_B; \tilde{p}_B]) = \max H \quad (9.17)$$

and $M^s_3(p_A, \tilde{p}_B; q) \neq \emptyset$ for some $q \in \text{Crit}(-\epsilon f)$ with $\mu_{-\epsilon f}^\text{Morse}(q) = 2n$.

Proof. From (9.15) and (9.16), we have

$$\sum_{A,B,C} (p_A, \tilde{p}_B; q_C)_{A+B-C} \tilde{q}_C \otimes q^{(A+B-C)} = \gamma + \partial \delta \quad (9.18)$$

contains a non-zero contribution from a global maximum point $q$ of $-\epsilon f$. Therefore there must by $A$, $B$ and $C$ and corresponding $p_A$ and $p_B$ such that

$$A + B - C = 0$$

and $(p_A, \tilde{p}_B; q_C)_{A+B-C} \neq 0$, in particular $M^s_3(p_A, \tilde{p}_B; q_C) \neq \emptyset$. It remains to show that we can indeed find these so that (9.17) is also satisfied. However this follows from the condition imposed on $\phi$ in the beginning of the proof of Proposition 9.3, and when $\phi$ is $C^1$-close to $S_\phi$ and then if we choose $\epsilon$ sufficiently small. Hence the proof of Proposition 9.3. $\square$
Once we have the lemma, we can repeat the adiabatic limit argument employed in [Oh8] and produces a cusp-solution required in Proposition 9.3. This finishes the proof. □

9.2. The case of engulfing

Now we consider $\phi$ not necessarily $C^1$-small but still satisfying
\[ \Phi(\Delta \phi) = \text{graph } dS_\phi \]
for the unique generating function $S_\phi : M \to \mathbb{R}$ with $\int_\Delta S_\phi \, d\mu = 0$. Let $H = H_\phi$ be as before and consider the homotopy
\[ G : s \mapsto H^s, \quad s \in [s_0, 1] \]
where $H^s$ is the Hamiltonian defined by $H^s_t := H_{st}$. Starting from a sufficiently small $s_0 > 0$, we will apply continuation argument of the local Floer complex along the family $G$. We write
\[ \phi_s := \phi^1_{H^s} = \phi^s_{H^s} \]
and recall
\[ \Phi(\Delta \phi_s) = \text{graph } sdS_\phi. \]

We modify the definition of the local Floer complex from [Fl2], [Oh2,7] for the engulfable Hamiltonian diffeomorphisms. For given almost complex structure $J_0$ and an engulfable $\phi \in \text{Ham}(M, \omega)$, we consider the Cauchy-Riemann equation
\[ \begin{cases}
\frac{\partial u}{\partial \tau} + J_t \frac{\partial u}{\partial t} = 0 \\
\phi(v(\tau, 1)) = v(\tau, 0)
\end{cases} \quad (9.19) \]
for each $J' \in j(\phi, J_0)$.

When a $t$-periodic Hamiltonian $H$ is given, (9.19) is equivalent to
\[ \begin{cases}
\frac{\partial u}{\partial \tau} + J_t \left( \frac{\partial u}{\partial t} - X_H(u) \right) = 0 \\
u : \mathbb{R} \times S^1 \to M
\end{cases} \quad (9.20) \]
with $J_t = (\phi^1_H)_{*} J'_t$. The moduli spaces of finite energy solutions of (9.19) and (9.20) are in one-one correspondence via the map
\[ \mathcal{M}(\phi, J') \to \mathcal{M}(H, J); \quad v \mapsto u; \quad u(\tau, t) = (\phi^1_H)(v(\tau, t)). \quad (9.21) \]

When there exists no non-constant contractible periodic orbits of $H$, (9.21) also canonically induces a chain isomorphism from the corresponding Floer complexes defined over the covering spaces of the path spaces relevant to (9.20) and (9.21), which we do not explicitly describe because we do not need it here.

From now on we will focus on the case $H = H^\phi$ for an engulfable $\phi$. Recall that $H^\phi$ does not carry any non-constant one-periodic orbits. Now we define the moduli space that we use to define the local Floer complex of $\phi$ or $H$. For this purpose, we need to compare solutions of (9.19) with those for the Lagrangian Floer complex for the pair $(\Delta, \Delta_\phi)$ in $(M, -\omega) \times (M, \omega)$. We recall the family $J^U$ of almost complex structures on $M \times M$ as defined in (8.17) for each given $U' \subset U$. We will make
the region $U \setminus U''$ grow in the end until it exhausts $U \setminus U'$. For each fixed $J^U$, we consider the local Lagrangian Floer complex

$\mathcal{M}^U(\Delta, \Delta_\phi; J^U)$

as in [Oh2] and §8. This local complex is isolated in $U$ under the continuation

$s \mapsto \mathcal{M}(\Delta, \Delta_\phi; J^U)$. \hfill (9.22)

Under the Darboux chart $\Phi : U \to V$, we can identify the local complex with the one

$\mathcal{M}^V(o_\Delta, \text{graph } dS_\phi; \Phi^*J^U)$ \hfill (9.23)

which is isolated in $T^*\Delta$ under the global continuation

$s \mapsto \mathcal{M}(o_\Delta, \text{graph } s dS_\phi)$.

Therefore the (local) cap action (9.12) can be defined and induces an isomorphism

$HF_0(\Delta; U) \cong H_0(\Delta) \to H_2^n(\Delta) \cong HF_0(\Delta; U)$ \hfill (9.24)

(see [Oh4] for the precise description of this cap action), and in particular gives rise to

**Lemma 9.5.** For any $J^U$ above, there exists a $J^U$-holomorphic cusp-map

$V = V_1 # \cdots # V_N$

with

$V_j : \mathbb{R} \times [0, 1] \to \mathbb{R} \subset U \subset M \times M$

for all $j = 1, \cdots, N$ and

$V_j(\tau, 0) \in \Delta, V_j(\tau, 1) \in \text{graph } \phi$

$V_1(-\infty) \in S_\phi^{-1}(-s_{\text{min}}), V_N(\infty) \in S_\phi^{-1}(-s_{\text{max}})$,

$V_j(0, 0) = (q, q)$ for some $j = 1, \cdots, N$ \hfill (9.25)

where $s_{\text{max}}, s_{\text{min}}$ are the maximum and the minimum values of $S_\phi$ respectively.

Going back to the moduli space $\mathcal{M}(\phi, J')$, we define

$\mathcal{M}(\phi, J' : U') = \{ v \in \mathcal{M}(\phi, J') \mid \text{Im } V \subset U' \}$ \hfill (9.26)

where $V$ is defined as below. It is easy to see that for any given $v \in \mathcal{M}(\phi, J'; U')$ the map $V : \mathbb{R} \times [0, 1] \to (M, -\omega) \times (M, \omega)$ defined by

$V(\tau, t) := (v\left(\frac{\tau}{2}, 1 - \frac{t}{2}\right), v\left(\frac{\tau}{2}, \frac{t}{2}\right))$ \hfill (9.27)

lies in $\mathcal{M}^U(\Delta, \Delta_\phi; J^U)$ and so defines a natural map

$\mathcal{M}(\phi, J'; U') \hookrightarrow \mathcal{M}^U(\Delta, \Delta_\phi; J^U)$.

(9.28)
Lemma 9.6. Any solution $V \in \mathcal{M}^{u}(\Delta, \Delta\phi; J^{t})$ is in the image of the embedding (9.28), provided

$$\text{Image} V \subset U'.' \tag{9.29}$$

Proof. We write $V(\tau, t) = (v_{1}(\tau, t), v_{2}(\tau, t))$. Due to the boundary condition of $V$

$$V(\tau, 0) \in \Delta, V(\tau, 1) \in \Delta\phi$$

we have

$$v_{1}(\tau, 0) = v_{2}(\tau, 0), \tag{9.30}$$

$$\phi(v_{2}(\tau, 1)) = v_{1}(\tau, 1). \tag{9.31}$$

Furthermore $v_{1}$ satisfies

$$\frac{\partial v}{\partial \tau} - J'_{1} \frac{\partial v}{\partial t} = 0$$

and $v_{2}$ satisfies

$$\frac{\partial v}{\partial \tau} + J'_{t} \frac{\partial v}{\partial t} = 0.$$  

We define $v : \mathbb{R} \times [0, 1] \to M$ by

$$v(\tau, t) = \begin{cases} v_{1}(\tau, 1 - t) & 0 \leq t \leq \frac{1}{2} \\ v_{2}(\tau, t) & \frac{1}{2} \leq t \leq 1. \end{cases} \tag{9.32}$$

It follows from (9.30) that $v$ is indeed smooth along $t = \frac{1}{2}$ and from (9.31) that we have

$$v(\tau, 0) = v_{1}(\tau, 1) = \phi(v_{2}(\tau, 1)) = \phi(v(\tau, 1))$$

and so $v \in \mathcal{M}(\phi, J'; U')$. This finishes the proof. □

We remark that there is the obvious one to one correspondence

$$x \in \text{Fix} \phi \to (x, x) \in \Delta \cap \Delta\phi \tag{9.33}$$

which induces a natural homomorphism

$$CF(\phi; U') \to CF(\Delta, \Delta\phi; U) \tag{9.34}$$

with suitable Novikov rings as coefficients, since $H^{\phi}$ does not have non-constant periodic orbits. The boundary maps in $CF(\phi; U')$ and $CF(\Delta, \Delta\phi; U)$ are defined by the moduli spaces $\mathcal{M}(\phi, J_{0}; U')$ and $\mathcal{M}_{ju}(\Delta, \Delta\phi; U)$ respectively. We would like to emphasize that (9.34) is not a chain map in general. Because of this failure of chain property of (9.34), Lemma 9.5 does not immediately give rise to the existence result Theorem 8.4. Instead we need to use Lemma 9.6 and apply limiting argument to produce the existence result which we now explain.

Since Lemma 9.5 is true for any $J^{t}$ satisfying (9.22) for any $U'$ and $U''$, we let the interpolating region $U \setminus \overline{U}$ smaller and smaller. Because the uniform (symplectic) area bound, we can take the limit and produce a cusp-solution

$$V = V_{1} \# \cdots \# V_{N},$$
satisfying (9.25) whose image is contained in $\mathcal{U}'$. In particular, $V$ satisfies
\[ \frac{\partial V}{\partial \tau} + \mathcal{T}_t \frac{\partial V}{\partial t} = 0. \]

Then Lemma 9.5 produces a cusp-solution of (8.2). Now we need to compare the area of those in $\mathcal{M}^{\mu}(\Delta, \Delta_\phi; J^{\mu})$ and $\mathcal{M}'^{\mu}(\Delta, \Delta_\phi; J^{\mu})$. Since those in $\mathcal{M}^{\mu}(\Delta, \Delta_\phi; J^{\mu})$ have area always less than $\text{osc}(S_\phi)$ by (8.18), if we assume
\[ \text{osc}(S_\phi) \leq A'(\mathcal{U}) \] (9.35)
then for any $J' \in \mathcal{j}(\phi, J_0)$ $A(\phi, J_0; J'; 1)$ is realized by a curve in $\mathcal{M}^{\mu}(\Delta, \Delta_\phi; J^{\mu})$ which connects a maximum and minimum points of $S_\phi$. Therefore we have proven
\[ A(\phi, J_0; 1; J') = \text{osc}(S_\phi) \leq A'(\mathcal{U}). \]

for any $J_0$ and $J' \in \mathcal{j}(\phi, J_0)$. By taking the supremum over $J' \in \mathcal{j}(\phi, J_0)$ and then over $J_0$, we have finally finished the proof of Theorem 8.4.

§10. Remarks on the transversality

Our construction of various maps in the Floer homology works as they are in the previous section for the strongly semi-positive case [Se], [En1] by the standard transversality argument. On the other hand in the general case where constructions of operations in the Floer homology theory requires the machinery of virtual fundamental chains through multi-valued abstract perturbation [FOn], [LT1], [Ru], we need to explain how this general machinery can be incorporated in our construction. We will use the terminology ‘Kuranishi structure’ adopted by Fukaya and Ono [FOn] for the rest of the discussion.

One essential point in our proofs is that various numerical estimates concerning the critical values of the action functional and the levels of relevant Novikov cycles do not require transversality of the solutions of the relevant pseudo-holomorphic sections, but depends only on the non-emptiness of the moduli space
\[ \mathcal{M}(H, \bar{J}; \bar{z}) \]
which can be studied for any, not necessarily generic, Hamiltonian $H$. Since we always have suitable a priori energy bound which requires some necessary homotopy assumption on the pseudo-holomorphic sections, we can compactify the corresponding moduli space into a compact Hausdorff space, using a variation of the notion of stable maps in the case of non-degenerate Hamiltonians $H$. We denote this compactification again by
\[ \mathcal{M}(H, \bar{J}; \bar{z}). \]

This space could be pathological in general. But because we assume that the Hamiltonians $H$ are non-degenerate, i.e, all the periodic orbits are non-degenerate, the moduli space is not completely pathological but at least carries a Kuranishi structure in the sense of Fukaya-Ono [FOn] for any $H$-compatible $J$. This enables us to apply the abstract multi-valued perturbation theory and to perturb the compactified moduli space by a Kuranishi map $\Xi$ so that the perturbed moduli space
\[ \mathcal{M}(H, \bar{J}; \bar{z}, \Xi) \]
is transversal in that the linearized equation of the perturbed equation
\[ \overline{\mathcal{D}}_J(v) + \Xi(v) = 0 \]
is surjective and so its solution set carries a smooth (orbifold) structure. Furthermore the perturbation \( \Xi \) can be chosen so that as \( ||\Xi|| \to 0 \), the perturbed moduli space \( \mathcal{M}(H, \tilde{J}; \tilde{z}, \Xi) \) converges to \( \mathcal{M}(H, \bar{J}; \bar{z}) \) in a suitable sense (see [FO] for the precise description of this convergence). We refer to [Mc1] for some discussion on the virtual moduli cycle in the setting of pseudo-holomorphic sections of symplectic fibration.

Now the crucial point is that non-emptiness of the perturbed moduli space will be guaranteed as long as certain topological conditions are met. For example, the followings are the prototypes that we have used in this paper:

1. \( h_{H_1}(\epsilon f) \to CF_0(H) \) is an isomorphism in homology and so \( [h_{H_1}(1^p)] \neq 0 \). This is immediately translated as an existence result of solutions of the perturbed Cauchy-Riemann equation.

2. The identity
\[ h_{H_1}(a^\flat) * h_{H_2}(b^\flat) = h_{H_1}#_{H_2}((a \cdot b)^\flat) \]
holds in homology, which guarantees non-emptiness of the relevant perturbed moduli space \( \mathcal{M}(H, \tilde{J}; \tilde{z}) \).

Once we prove non-emptiness of \( \mathcal{M}(H, \tilde{J}; \tilde{z}, \Xi) \) and an a priori energy bound for the non-empty perturbed moduli space and if the asymptotic conditions \( \tilde{z} \) are fixed, we can study the convergence of a sequence \( v_j \in \mathcal{M}(H, \tilde{J}; \tilde{z}, \Xi_j) \) as \( \Xi_j \to 0 \) by the Gromov-Floer compactness theorem. However a priori there are infinite possibility of asymptotic conditions for the pseudo-holomorphic sections that we are studying, because we typically impose that the asymptotic limit lie in certain Novikov cycles like
\[ \tilde{z}_1 \in h_{H_1}(\alpha) \text{, } \tilde{z}_2 \in h_{H_2}(\beta) \text{, } \tilde{z}_3 \in h_{H_3}(\gamma) \]
Because the Novikov cycles are generated by an infinite number of critical points \([z, w]\) in general, one needs to control the asymptotic behavior to carry out compactness argument. For this purpose, the kind of the bound (7.45)-(7.46), especially the lower bound for the actions enables us to consider only finite possibilities for the asymptotic conditions because of the finiteness condition in the definition of Novikov chains. With such a bound for the actions, we may then assume, by taking a subsequence if necessary, that the asymptotic conditions are fixed when we take the limit and so we can safely apply the Gromov-Floer compactness theorem to produce a (cusp)-limit lying in the compactified moduli space \( \mathcal{M}(H, \bar{J}; \bar{z}) \). This then justifies all the statements and proofs in the previous sections for the complete generality.

Appendix 1; Thick and thin decomposition of the moduli space

In this appendix, we will prove Proposition 9.1. We first need a decomposition formula for the moduli space with \( k = 2 \). We recall a similar result from [Proposition 4.1, Oh2] in the context of Lagrangian intersection Floer theory for \( k = 2 \).
The following is a translation of [Proposition 4.1, Oh2] in the present context of Hamiltonian diffeomorphisms. Since we also need the result for $k = 2$, for readers’ convenience and also to motivate the case with $k = 3$, we give a complete proof in the present context of Hamiltonian diffeomorphisms rather than leaving the translation from the context of Lagrangian submanifolds studied in [Oh2] to readers.

**Proposition A.1.1.** Let $J_0$ be an almost complex structure on $M$ and a constant $\alpha_i, i = 1, 2$ with

$$0 < \alpha_i < A(\omega, J_0), \quad \alpha_1 + \alpha_2 < A(\omega, J_0)$$

Let $J_t = (\phi_t^*)^* J_0$ as before. Let $u$ be any cusp-solution of (8.2). Then there exists a constant $\delta = \delta(J_0, \alpha) > 0$ such that if $\|H\|_{C^2} < \delta$, the following alternative holds:

1. either $u$ is ‘very thin’
   $$\int \left| \frac{\partial u}{\partial \tau} \right|^2_{J_t} = \int \left| \frac{\partial v}{\partial \tau} \right|^2_{J_t'} = \int v^* \omega < \alpha_1$$

2. or it is ‘thick’
   $$\int \left| \frac{\partial u}{\partial \tau} \right|^2_{J_t} = \int \left| \frac{\partial v}{\partial \tau} \right|^2_{J_t'} > A(\omega, J_0) - \alpha_2.$$

**Proof.** The proof is by contradiction as in [Oh2, 7]. Suppose the contrary that for some $\alpha$ with $0 < \alpha < A(\omega, J_0)$ such that there exists a sequence $\delta_j \to 0$, $H_j$ with $\|H_j\|_{C^2} \leq \delta_j$ and $u_j$ that satisfies (8.2) for $H_j$ and $J'_t = (\phi_{t_j}^*)^* J'_t$, but with the bound

$$\alpha_1 \leq \int \left| \frac{\partial u_j}{\partial \tau} \right|^2_{J'_t} = \int \left| \frac{\partial v_j}{\partial \tau} \right|^2_{J'_t} \leq A(\omega, J_0) - \alpha_2. \quad (A.4)$$

Note that since $H_j \to 0$ in $C^2$-topology and so $J_0 \sim \phi^* J_0$, we can choose the path $J' \in j(\phi, J_0)$ so that it is close to the constant map $J_0$ and hence $J'$ close to $J_0$. Because of the energy upper bound in (A.4), we can apply Gromov’s type of compactness argument. Note that if $H_j$ is $C^2$-small, any one-periodic trajectory must be constant. Therefore the homotopy class $[u_j] \in \pi_2(M)$ is canonically defined for any finite energy solution $u_j$ and in particular for $u_j$. A straightforward computation shows

$$\int \left| \frac{\partial u}{\partial \tau} \right|^2_{J'_t} = \int \omega \left( \frac{\partial u}{\partial \tau}, \frac{\partial u}{\partial t} \right) dt dt - \int_0^1 (H(u(\infty)) - H(u(-\infty))) dt \quad (A.5)$$

and so for the finite energy solution $u_j$, we derive

$$\int \left| \frac{\partial u}{\partial \tau} \right|^2_{J'_t} = \omega([u_j]) - \int_0^1 (H(u(\infty)) - H(u(-\infty))) dt \leq \omega([u_j]) + \|H\|. \quad (A.6)$$

Because of the energy bound (A.4), choosing a subsequence, we may assume that the homotopy class $[u_j] = A$ fixed. If $A = 0$, then (A.6) implies

$$\int \left| \frac{\partial u}{\partial \tau} \right|^2_{J'_t} \leq \|H\| \leq \|H\|_{C^2} \leq \delta_j$$
which is a contradiction to the lower bound in (A.4) if \( j \) is sufficiently large so that \( \delta_j < \alpha \). Now assume that \( A \neq 0 \). In this case, there must be some \( C > 0 \) and \( \tau_j \in \mathbb{R} \) with

\[
\text{diam}(t \mapsto u(t, \tau_j)) \geq C > 0
\]  

where \( C \) is uniform over \( j \). By translating \( u_j \) along the direction of \( \tau \), we may assume that \( \tau_j = 0 \). Now if bubbling occurs, we just take the bubble to produce a non-constant \( J_0 \)-holomorphic sphere. If bubbling does not occur, we take a local limit around \( \tau = 0 \) using the energy bound (A.4), we can produce a map \( u : \mathbb{R} \times S^1 \to M \) satisfying

\[
\frac{\partial u}{\partial \tau} + J_0 \frac{\partial u}{\partial t} = 0.
\]  

(A.8)

Furthermore if there occurs no bubbling and so if the sequence converges uniformly to the local limit around \( \tau = 0 \), then the local limit cannot be constant because of (A.7) and so it must be a non-constant \( J_0 \)-holomorphic cylinder with the energy bound

\[
\frac{1}{2} \int |Dv|^2 \leq A(\omega, J_0) - \alpha_2 < \infty
\]

By the removal of singularity, \( u \) can be compactified to a \( J_0 \)-holomorphic sphere that is non-constant.

Therefore whether the sequence bubbles off or not, we have

\[
\limsup_j \int \left| \frac{\partial u}{\partial \tau} \right|^2 \geq A(\omega, J_0).
\]

This then contradicts to the upper bound of (A.4). This finishes the proof of the alternative (A.2)-(A.3). \( \square \)

Now consider the case with \( k = 3 \) and give the proof of Proposition 9.1. The main idea of the proof for this case is essentially the same as the case \( k = 2 \) except that we need to accommodate the set-up of Hamiltonian fibration in our proof.

For given \( H = (H_1, H_2, H_3) \) with \( H_1 \neq H_2 = H_3 \), we choose a Hamiltonian fibration \( P \to \Sigma \) with connection \( \nabla \) whose monodromy at the ends are given by \( H_i \)'s respectively for \( i = 1, 2, 3 \). We recall that as a topological fibration, \( P = \Sigma \times M \). We note that as \( \|H\|_{C^2} \to 0 \), we may assume that the coupling two form \( \omega_P \) of the connection \( \nabla \omega \) can be made arbitrarily small. Let \( \bar{J} \) be a \( H \)-compatible almost complex structure which is also compatible to the symplectic form \( \Omega_{P,\lambda} \). As usual, if we denote \( |\cdot| = |\cdot|_{\bar{J}} \), we have

\[
\frac{1}{2} \int |Dv|^2 = \int v^* (\Omega_{P,\lambda}) + \int |\bar{J}v|^2
\]

for arbitrary map \( v : \Sigma \to P \). For a \( \bar{J} \)-holomorphic section, this reduces to

\[
\frac{1}{2} \int |Dv|^2 = \int v^* \omega_P + \lambda.
\]  

(A.9)
We decompose $Dv = (Dv)^v + (Dv)^h$ into vertical and horizontal parts and write

$$|Dv|^2 = |(Dv)^v|^2 + |(Dv)^h|^2 + 2\langle (Dv)^v, (Dv)^h \rangle.$$  

Now it is straightforward to prove

$$|(Dv)^h|^2 = 2(K(v) + \lambda) \tag{A.10}$$

by evaluating

$$\sum_{i=1}^{2} |(Dv)^h(e_i)|^2 = \sum_{i=1}^{2} \Omega_{P,\lambda}((Dv)^h(e_i), \tilde{J}(Dv)^h(e_i))$$

$$= \sum_{i=1}^{2} (\omega_P + \lambda \omega_{\Sigma})((Dv)^h(e_i), \tilde{J}(Dv)^h(e_i))$$

for an orthonormal frame $\{e_1, e_2\}$ of $T\Sigma$. Here $K : P \to \mathbb{R}$ is the function such that $K(v)\,d\tau \wedge dt$ is the curvature of the connection $\nabla$. See the curvature identity from [(1.12), GLS].

We note that (A.10) can be made arbitrarily small as $\|H\|_{C^2} \to 0$. This can be directly proven or follows from [Theorem 3.6.1, En1]. Therefore using the inequality

$$2ab \leq \frac{\epsilon^2 a^2}{2} + \frac{1}{\epsilon^2} b^2$$

we have

$$|Dv|^2 \geq (1 - \epsilon^2)|Dv|^v|^2 + (1 - \frac{1}{\epsilon \delta})|(Dv)^h|^2$$

for any $\epsilon > 0$.

If $\int v^* \omega_P = 0$, then we derive from (A.9), (A.10) by setting $\epsilon = \frac{1}{2}$

$$\frac{3}{4} \int |(Dv)^v|^2 \leq 2\lambda + 3 \int |(Dv)^h|^2.$$

But it follows from (A.10) that $2\lambda + 3 \int |(Dv)^h|^2$ can be made arbitrarily small as $\|H\|_{C^2} \to 0$ and so

$$\int |(Dv)^v|^2 \to 0 \tag{A.11}$$

as $\delta \to 0$. Combining (A.10) and (A.11), we have established

$$\int |Dv|^2 \leq \alpha_1$$

in this case if $\delta > 0$ is sufficiently small.

On the other hand, if $\int v^* \omega_P \neq 0$, then $pr_{\Sigma} \circ v : \Sigma \to \Sigma$ has degree one and the fiber homotopy class $[u]$ of $v$ satisfies

$$[u] =: A \neq 0 \in \pi_2(M). \tag{A.12}$$
Furthermore noting that as $\|H\|_{C^2} \to 0$, the connection can be made closer and closer to the trivial connection in the trivial fibration $P = \Sigma \times M$ and the $H$-compatible $\bar{J} = \bar{J}(J_0)$ also converges to the product almost complex structure $J \times J_0$ and hence the image of $J$-holomorphic sections cannot be completely contained in the neighborhood of one of the obvious horizontal section

$$\Sigma \times \{q\}$$

for any one fixed $q \in M$. Now consider a sequence $H_j$ with $\|H_j\|_{C^2} \to 0$, and $H_j$-compatible almost complex structure $\bar{J}_j$, and let $v_j$ be a sequence of $\bar{J}_j$-holomorphic sections in the fixed fiber class (A.12). In other words, if we write

$$v_j(z) = (z, u_j(z))$$

in the trivialization $P = \Sigma \times M$, then we have $[u_j] = A \neq 0$. Since we assume $A \neq 0$, there is a constant $C > 0$ such that

$$\operatorname{diam}(u_j) \geq C > 0$$

for all sufficiently large $j$ after choosing a subsequence. By applying a suitable conformal transformation on the domain, either by taking a bubble if bubble occurs or by choosing a limit when bubbling does not occur, we can produce at least one non-constant $J_0$-holomorphic map

$$u_\infty : S^2 \to M$$

out of $u_j$’s as in the case $k = 2$ before. Furthermore we also have the energy bound

$$\limsup_{j \to \infty} \int |(Dv_j)^v|_{\bar{J}_j}^2 \geq \int |Du_\infty|_{\bar{J}_0}^2.$$ 

Therefore we have

$$\int |Dv_j|^2 \geq \int |(Dv_j)^v|^2 \geq \int |Du_\infty|^2_{\bar{J}_0} \geq A(\omega, J_0) - \alpha_2$$

if $\delta > 0$ is sufficiently small. This finishes the proof of Proposition 9.1.

**Remark A.1.2.** In the above proof, the readers might be wondering why we are short of stating

"By the Gromov compactness theorem, the sequence $v_j$ converges to

$$u_h + \sum_{k=1}^{n} w_k, \quad n \neq 0$$

(A.13)

as $j \to \infty$ or $\|H_j\| \to 0$, where $u_h$ is an obvious horizontal section and each $w_j$ is a $J_0$-holomorphic sphere into a fiber $(M, \omega)$ of $P = \Sigma \times M$.”

The reason is because such a convergence result fails in general by two reasons: First unless we specify how the limiting sequence $H_j$ converges to 0, the sequence $v_j$ cannot have any limit in any reasonable topology. This is because the case $H = 0$
is a singular situation in the study of the Floer moduli space $\mathcal{M}(H, \tilde{J}; \tilde{z})$. Secondly even if we specify a good sequence, e.g., consider the ‘adiabatic’ sequence

$$H_{1,j} = \epsilon_j f_1, \ H_{2,j} = \epsilon_j f_2, \ H_{3,j} = \epsilon_j f_3$$  \hspace{1cm} (A.14)

for Morse functions $f_1, f_2, f_3$ with the same sequence $\epsilon_j \to 0$, we still have to deal with the degenerate limit, i.e., the limit that contains components of Hausdorff dimension one as we studied in §8.

What we have proved in the above proof is that we can always produce at least one non-constant $J_0$-holomorphic sphere as $j \to \infty$ without using such a strong convergence result, when the homotopy class of the $v_j$ is not trivial (in the fiber direction).

### Appendix 2: Bounded quantum cohomology

In this appendix, we define the genuinely cohomological version of the quantum cohomology and explain how we can extend the definition of the spectral invariants to the classes in this cohomological version.

We call this \textit{bounded quantum cohomology} and denote by

$$QH^*_\text{bdd}(M).$$

In this respect, we call the usual quantum cohomology ring $QH^*(M) = H^*(M) \otimes \Lambda^\uparrow$ the \textit{finite quantum cohomology}. We call elements in $QH^*_\text{bdd}(M)$ and $QH^*(M)$ bounded (resp. finite) quantum cohomology classes.

We first define the chain complex associated to $QH^*_\text{bdd}(M)$. Let $f$ be a Morse function and consider the complex of Novikov chains

$$CQ_{2n-k}(-\epsilon f) = CM_{2n-k}(-\epsilon f) \otimes \Lambda^\uparrow (= CF_k(\epsilon f)).$$  \hspace{1cm} (A.15)

On non-exact symplectic manifolds, this is typically \textit{infinite dimensional} as a $Q$-vector space. Therefore it is natural to put some topology on it rather than to consider it just as an \textit{algebraic} vector space. For this purpose, we recall the definition of the level $\lambda(\alpha) = \lambda_f(\alpha)$ of an element

$$\alpha = \sum A \alpha_A q^A :$$

$$\lambda(\alpha) = \max \{ A_f(\alpha_A q^A) \mid \alpha_A \neq 0 \}$$

$$= \max \{ \lambda_{Morse}^*(-\epsilon f)(\alpha_A) - \omega(A) \}.$$

As we saw before, the level provides a natural filtration on $CQ_{2n-k}(-\epsilon f)$ and so defines a topology in an obvious way. One can easily see that the Morse boundary operator

$$\partial_{-\epsilon f}^{Morse} : CQ_{2n-k}(-\epsilon f) \to CQ_{2n-k-1}(-\epsilon f)$$

is continuous with respect to this topology. Now we define
Definition A.2.1. A linear functional

\[ a : CQ_{2n-k}(\epsilon f) \to Q \]

is called continuous (or bounded) if it is so with respect to the topology induced by the above filtration. We denote by \( CQ^*_\text{bd}(\epsilon f) \) the set of bounded linear functionals on \( CQ_{2n-k}(\epsilon f) \).

It is easy to see from the definition of Novikov chains that a linear functional \( \mu \) is bounded if and only if there exists \( \lambda_\mu \in \mathbb{R} \) such that

\[ \mu(\alpha_Aq^A) = 0 \quad (A.16) \]

for all \( A \) with \( -\omega(A) \leq \lambda_\mu \). It follows that

\[ \partial^*_Q = \partial^*_{\epsilon f} : (CQ^*(\epsilon f))^* \to (CQ_{\epsilon+1}(\epsilon f))^* \]

maps bounded linear functionals to bounded ones and so defines the canonical complex

\[ (CQ^*_\text{bd}(\epsilon f), \partial^*_Q) \]

and hence defines the homology

\[ QH^*_\text{bd}(M) := H^*(CQ^*_\text{bd}(\epsilon f), \partial^*_Q). \]

We recall the canonical embedding

\[ \sigma : CQ^l(\epsilon f) = CM_{2n-\epsilon f} \otimes \Lambda^l \hookrightarrow CQ^l_{\text{bd}}(\epsilon f); a \mapsto \langle a, \cdot \rangle \quad (A.17) \]

mentioned in Remark 5.1. We have the following proposition which is straightforward to prove. We refer to the proof of [Proposition 2.2, Oh4] for the details.

Proposition A.2.2. The map \( \sigma \) in (A.17) is a chain map from \( (CQ^l(\epsilon f), \partial^Q) \) to \( (CQ^l_{\text{bd}}(\epsilon f), \partial^*_Q) \). In particular we have a natural degree preserving homomorphism

\[ \sigma : QH^*(M) \cong HQ^*(\epsilon f) \to HQ^*_\text{bd}(\epsilon f) \cong QH^*_\text{bd}(M). \quad (A.18) \]

Now we can define the notion of bounded Floer cohomology \( HF^*_\text{bd}(H) \) for any given Hamiltonian in a similar way. Then the co-chain map

\[ (h_H)^* : CF^k(H) \to CF^k(\epsilon f) \]

restricts to the co-chain map

\[ (h_H)^* : CF^k_{\text{bd}}(H) \to CF^k_{\text{bd}}(\epsilon f). \]

Once we have defined the bounded quantum cohomology and the bounded Floer cohomology, it is straightforward to define the spectral invariants for the bounded cohomology class in the following way.

Definition A.2.3. Let \( \mu \in QH^*_\text{bd}(M) \). Then we define

\[ \rho(H; \mu) := \lim_{\epsilon \to 0} \inf \{ \lambda \in \mathbb{R} \mid \mu \in \text{Im} (i_\lambda \circ h_H)^* \} \quad (A.19) \]

Now it is straightforward to generalize all the axioms in Theorem I to the bounded quantum cohomology class. The only non-obvious axiom is the triangle inequality. But the proof will be a verbatim modification of [Theorem II (5), Oh4] incorporating the argument in the present paper that uses the Hamiltonian fibration and pseudo-holomorphic sections. We leave the details to the interested readers. We will investigate further properties of the bounded quantum cohomology and its applications elsewhere.
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