A universal framework for entanglement detection

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Abstract

We construct nonlinear multiparty entanglement measures for distinguishable particles, bosons and fermions. In each case properties of an entanglement measures are related to the decomposition of the suitably chosen representation of the relevant symmetry group onto irreducible components. In the case of distinguishable particles considered entanglement measure reduces to the well-known many particle concurrence. We prove that our entanglement criterion is sufficient and necessary for pure states living in both finite and infinite dimensional spaces. We generalize our entanglement measures to mixed states by the convex roof extension and give a non trivial lower bound of thus obtained generalized concurrence.
I. INTRODUCTION

Entanglement is one of the features of quantum systems that makes them different from their classical counterparts. Even since the invention of this concept [1, 2] there has been an ongoing debate how to define precisely and quantify entanglement for various physical systems. Entanglement is usually identified with correlations in the composite quantum system that are stronger then any correlations that can be exhibited by any classical system. In our article we elaborate the method of entanglement detection based on identification of entangled states with particular orbits of underlying group of local transformation [3–5]. For the another approach to entanglement which makes use of the formalism of commuting subalgebras of the algebra of all quantum observables see [6, 7].

The general idea is as follows. We consider a Hilbert space $\mathcal{H}$ of a quantum system and a group $K$ of local, i.e. correlation-preserving transformations, represented on $\mathcal{H}$ irreducibly as a subgroup of the full unitary group $U(\mathcal{H})$. Precise forms of $\mathcal{H}$, $K$ and the representation vary and depend upon a given physical situation. Because the global phase factor of the wave function is irrelevant for physical applications, one considers the corresponding action of $K$ on the complex projective space $\mathbb{P}\mathcal{H}$ rather then on $\mathcal{H}$ itself. Having introduced this language entanglement can be defined as the property of orbits of the group $K$ acting in $\mathbb{P}\mathcal{H}$ and any measure of entanglement should be an invariant of $K$. We present a construction and a general computational scheme for one particular invariant of the action of $K$ on $\mathbb{P}\mathcal{H}$ which can be used for detection of entanglement as it is non negative and vanishes exactly on the the set of states that are coherent or, in other words, “the most classical” [8–10].

In the current paper we focus on three concrete cases: entanglement of finite number of distinguishable, bosonic or fermionic particles. We analyze entanglement in situations when Hilbert spaces considered can be infinite dimensional [11].

What is, obviously, more interesting and at the same time more demanding is to quantify entanglement of mixed states for all three cases, or at least, discriminate between separable and entangled states. In principle, having a good measure of entanglement for pure states we can extend it to mixed ones by the so called convex roof extension utilizing the fact that mixed separable states are convex combinations of pure ones. However, since the convex roof construction of an entanglement measure requires optimization over all convex pure-state decompositions of a given mixed density matrix, the procedure is not effective, or at least
computationally demanding. What might be helpful in discriminating separable and mixed density matrices are various estimates of so constructed convex roof measures. We will show a unified procedure for constructing useful lower bounds for the obtained measures in all three cases.

In Section II we briefly present the general construction of nonlinear entanglement measure in finite dimensional setting with the usage of representation-theoretic tools. In Section III we apply the developed methods of discriminating entangled and non entangled pure states in all three cases of distinguishable particles, bosons and fermions. Number of particles as well as the dimensions of a single particle Hilbert spaces are arbitrary but finite. In the case of distinguishable particles our entanglement measure reduces to the well known multiparty concurrence [12, 13] and we decided to keep this name calling it a generalized concurrence. Section IV is devoted to generalization of results from Section III to systems with arbitrary finite number of particles but with infinite-dimensional single-particle Hilbert spaces. We prove that entanglement criteria that we present in Section III still hold in the infinite-dimensional setting. In Section V we extend our entanglement measures to the general mixed states via the convex roof extension. We provide a systematic way to find lower bounds for generalized concurrences for fermionic and bosonic systems starting from any lower bounds for the concurrence for multiparty distinguishable particles. These lower bounds give a sufficient conditions of entanglement for a mixed fermionic or bosonic state. Relevant mathematical details that we omit during the main discussion are given in the Appendices.

II. NONLINEAR ENTANGLEMENT MEASURES

Let us briefly remind the construction of nonlinear entanglement measure for multiparty pure states space presented in [4]. It is based on an obvious observation that entanglement, or more general, quantum correlations do not change under local transformations. Such local transformations form a (Lie) group $K$ acting in the space of states of the system hence states with the same correlation properties (“equally entangled”) belong to the same orbit of the group $K$, i.e. to the set of states which can be obtained form a particular one by applying all operations of the group $K$. In particular, non-correlated (non-entangled) states form a single particular orbit of $K$. 

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A. Entanglement and orbits of local groups

The notion of a “local transformation” and, consequently, the structure of the group \( K \) depends on the situation at hand. In the case of \( L \) distinguishable particles, the Hilbert space of states is the tensor product of Hilbert spaces of single particles \( \mathcal{H}_1, \ldots, \mathcal{H}_L \) of dimensions \( N_1, \ldots, N_L \) which we conveniently identify with the complex spaces of the same dimensions,

\[
\mathcal{H}_d = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_L = \bigotimes_{i=1}^L \mathbb{C}^{N_i}
\]

In this case the group \( K \) of local transformations leaving correlations invariant is the direct product of special unitary group \( SU(N_k) \) each acting independently in the respective one-particle space \( \mathcal{H}_k = \mathbb{C}^{N_k} \),

\[
K = SU(N_1) \times SU(N_2) \times \cdots \times SU(N_L) = \bigotimes_{i=1}^L SU(N_i),
\]

\[
\Pi^d(U_1 \ldots, U_L)|\psi_1\rangle \otimes \cdots \otimes |\psi_L\rangle = U_1|\psi_1\rangle \otimes \cdots \otimes U_L|\psi_L\rangle
\]

In the notation above we made explicit that the group \( K \) acts on \( \mathcal{H} \) via its particular representation \( \Pi^d \), defined here by its action on simple tensors. For simplicity we assume that the particles are identical (albeit, as stated above, distinguishable), hence their Hilbert space of states are the same, \( N_1 = \cdots = N_S \), hence

\[
\mathcal{H}_d = \bigotimes_1^L (\mathbb{C}^N), \quad K = \bigotimes^L (SU(N)).
\]

For indistinguishable particles the situation differs. The appropriate Hilbert space of the whole system is no longer the full tensor product of one-particle Hilbert spaces but rather its symmetric (for bosons) or antisymmetric (for fermions) part

\[
\mathcal{H}_b = Sym^L (\mathbb{C}^N) = \mathbb{C}^N \vee \cdots \vee \mathbb{C}^N, \quad (3)
\]

\[
\mathcal{H}_f = \bigwedge^L (\mathbb{C}^N) = \mathbb{C}^N \wedge \cdots \wedge \mathbb{C}^N, \quad (4)
\]

where \( \vee \) and \( \wedge \) denote, respectively, the symmetric and antisymmetric part of the full tensor product. To keep symmetry intact, the local group \( K \) must consist of “diagonal” actions of the \( SU(N) \) group, i.e. actions of the same unitary operator in each one-particle space,

\[
K = SU(N), \quad \Pi^b(U)(|\psi_1\rangle \vee \cdots \vee |\psi_S\rangle) = U|\psi_1\rangle \vee \cdots \vee U|\psi_S\rangle, \quad (5)
\]

\[
K = SU(N), \quad \Pi^f(U)(|\psi_1\rangle \wedge \cdots \wedge |\psi_S\rangle) = U|\psi_1\rangle \wedge \cdots \wedge U|\psi_S\rangle, \quad (6)
\]
where we denoted the appropriate representations of $K$ by $\Pi^b$ and $\Pi^f$.

From the mathematical point of view all three cases of a) distinguishable particles (Eq. (2)), b) bosons (Eq. (5)), and c) fermions (Eq. (6)) are instances of the same scheme: a compact (semi)simple group $K$ acts via irreducible representation $\Pi$ on a Hilbert space $\mathcal{H}$ (i.e. there are no proper subspaces of $\mathcal{H}$ preserved by $\Pi$). Each irreducible representation of a compact semisimple group is uniquely determined by the so called highest weight vector in the representation space $\mathcal{H}$ (see Appendix A). In what follows, unless otherwise specified, all representation we considered will be irreducible. From the physical point of view it is more appropriate to consider action of $K$ on the projective space $\mathbb{P}\mathcal{H}$ rather than on $\mathcal{H}$ itself, as in the physical interpretation of vectors from $\mathcal{H}$ their phase does not play a role and we use only vectors normalized to unity. The space $\mathbb{P}\mathcal{H}$ is the space of different complex directions in $\mathcal{H}$, each determined by a normalized vector $|\psi\rangle$. The group $K$ acts naturally on $\mathbb{P}\mathcal{H}$:

$$\tilde{\Pi}(k)([|\psi\rangle]) = [\Pi(k)|\psi\rangle],$$

where $\mathcal{H} \ni |\psi\rangle \mapsto [\psi] \in \mathbb{P}\mathcal{H}$ is the mapping that associates to the unit vector $|\psi\rangle$ the complex direction passing through it. In the above introduced language the set of most classical (coherent or “non-entangled”) states is the orbit of $K$ through the highest weight state $[|\psi_0\rangle]$ [3, 4, 9]. Mathematically, this orbit can be identified as the orbit of Perelomov’s generalized coherent states for the representation $\Pi$ of the group $K$, that are “closest to classical” [9],

$$\mathcal{O}_0 = \{[|\psi\rangle] = [\Pi(k)|\psi_0\rangle] | k \in K\}.$$  

(7)

This definition can be motivated in a threefold way. Firstly, for the case of distinguishable particles (see below), one recovers the standard separable states. Secondly, states that belong to this class minimalise the $K$- invariant uncertainty of the state $|\psi\rangle$:

$$\text{Var} (|\psi\rangle) = \sum_i \left( \langle \psi | X_i^2 |\psi \rangle - \langle \psi | X_i |\psi \rangle^2 \right),$$

where the sum is over generators of the Lie algebra of the group $K$ (see [10] for application of Var ($|\psi\rangle$) in entanglement theory and [14] for application of “minimal uncertainty coherent states” in quantum optics). The third important feature of considered classes of states is
that they are appear naturally when studying classical limits of certain models stemming from quantum optics [14] or condensed matter physics [15].

For the case of distinguishable particles (2), generalized coherent states are precisely separable states,

\[ \mathcal{O}_{sep} = \{ [\psi_1] \otimes [\psi_2] \otimes \ldots \otimes [\psi_L] | [\psi_i] \in \mathcal{H}_i \} . \]  

(8)

One checks that:

\[ \mathcal{O}_b = \{ [\phi] \otimes [\phi] \otimes \ldots \otimes [\phi] | [\phi] \in \mathbb{C}^N \} \]  

and

\[ \mathcal{O}_f = \{ [\phi_1] \land [\phi_2] \land \ldots \land [\phi_L] | [\phi_i] \in \mathbb{C}^N, \langle \phi_i | \phi_j \rangle = \delta_{ij} \} , \]  

(10)

are “coherent states” for system of respectively \( L \) bosons \([5]\) and \( L \) fermions \([6]\). The notion of entanglement for bosons or fermions is not well defined as corresponding Hilbert spaces \( \mathcal{H}_b \) and \( \mathcal{H}_f \) lack the tensor product structure. Nevertheless, we prefer to call states belonging to \( \mathcal{O}_b \) and \( \mathcal{O}_f \) as “least entangled” bosonic and fermionic states. Note that \( \mathcal{O}_b \) and \( \mathcal{O}_f \) consist of simplest tensors available in \( \mathcal{H}_b \) and \( \mathcal{H}_f \) respectively. What is more, these are exactly the sets of pure “separable” states for bosons and fermions analyzed in \([16]\). Classes of states \( \mathcal{O}_b \) and \( \mathcal{O}_f \) are also interesting from the practical point of view. For \( N = 2 \) states from \( \mathcal{O}_b \) are exactly celebrated spin coherent states [14]. On the other hand, \( \mathcal{O}_f \) is, for general \( N \), a very important class of variational states in condensed matter physics.

**B. Characterization of orbits of non-entangled states and generalized concurrence**

It is now clear that identification of a state \([|\psi]\) as a non-entangled one is equivalent to checking whether it belongs to the orbit of the local group \( K \) through the highest weight vector. A constructive way of checking this fact was given in the paper by Lichtenstein [17].

To present it let us go back for a moment to our general setting (for the relevant definitions consult Appendix A). Let

\[ K \ni k \mapsto \Pi^{\lambda_0}(k) \in U(\mathcal{H}^\lambda_0) , \]

be a unitary representation of \( K \) characterized by the highest weight \( \lambda_0 \) (we wrote \( \mathcal{H}^\lambda_0 \) instead of \( \mathcal{H} \) to indicate the parameter encoding the representation). Let us introduce
auxiliary unitary representation of $K$ on the symmetric tensor product $\mathcal{H}^\lambda_0 \vee \mathcal{H}^\lambda_0$

$$K \ni k \rightarrow \Pi^\lambda_0(k) \otimes \Pi^\lambda_0(k) \in U\left(\text{Sym}^2(\mathcal{H}^\lambda_0)\right).$$

(11)

In general $\mathcal{H}^\lambda_0 \vee \mathcal{H}^\lambda_0$ decomposes onto irreducible representations of $K$:

$$\text{Sym}^2(\mathcal{H}^\lambda_0) \approx \mathcal{H}^{2\lambda_0} \oplus \bigoplus_{\beta \neq 2\lambda_0} \mathcal{H}^{\beta},$$

(12)

where $\mathcal{H}^{2\lambda_0}$ is the representation of the highest weight $2\lambda_0$ (one can show that there is only one representation of this kind in the above sum [10]) and the sum on the right side is over other irreducible representations that appear in the decomposition of $\text{Sym}^2(\mathcal{H}^\lambda_0)$. The announced result of Lichtenstein [17] states that $|\psi\rangle$ is a coherent state if in and only if $|\psi\rangle|\psi\rangle \in \mathcal{H}^{2\lambda_0}$. We can write this result in the equivalent form

$$[|\psi\rangle] \in \mathcal{O}_0 \iff \langle\psi| \otimes \langle\psi| \otimes I - \mathbb{P}^{2\lambda_0}|\psi\rangle \otimes |\psi\rangle = 0,$$

(13)

where $\mathbb{P}^{2\lambda_0}$ is the orthogonal projector onto $\mathcal{H}^{2\lambda_0}$ and $I$ stands for the identity operator on $\mathcal{H}^\lambda_0$. Theorem of Lichtenstein written in this form can be used to construct the nonlinear entanglement measure, the generalized concurrence which we define by the following expression

$$C(|\psi\rangle) = \sqrt{\langle\psi| \otimes \langle\psi| \otimes I - \mathbb{P}^{2\lambda_0}|\psi\rangle \otimes |\psi\rangle}.$$ 

(14)

One easily checks that $C(|\psi\rangle)$ is non negative and vanishes exactly for coherent states. Moreover, it is also $K$ invariant. These two conditions allow us to treat $C(|\psi\rangle)$ as an indicator of entanglement.

Although the above construction works only for compact group represented in the finite dimensional Hilbert space in Section [15] we will generalize the concurrence also to systems of distinguishable particles, fermions or bosons described in infinite dimensional Hilbert spaces.

III. EXPLICIT EXPRESSIONS FOR GENERALIZED CONCURRENCES FOR PURE STATES

It possible to compute a detailed form of the projector operator $\mathbb{P}^{2\lambda_0}$ acting $\text{Sym}^2(\mathcal{H}^\lambda_0)$ for the case of distinguishable particles as well as for bosons and fermions. The proofs for formulas for $\mathbb{P}^{2\lambda_0}$ rely on representation theory and are given in Appendix (VII). We obtain
explicit form of the function $C(|\psi\rangle)$. For the case of distinguishable particles we recover the well-known multiparty concurrence [12, 13]. Results we get for bosons and fermions are generalization of the previous paper of one of the authors [4]. The main advantage of our generalization lies in the fact that it reveals a strong connection between concurrences for non-distinguishable particles with the one defined for distinguishable particles. This connection allows for the “physical interpretation” of $C(|\psi\rangle)$ for non-distinguishable particles in terms of the reduced density matrices of the state $|\psi\rangle$. In Section V we use this connection to the problem of detection of mixed entangled states. We obtain non-trivial lower bounds for concurrences for bosons and fermions from any lower bound for the multiparty concurrence. In the Section IV we prove that the formulas for concurrences obtained in this part hold also in the infinite dimensional setting.

A. Distinguishable particles

In the case of $L$ distinguishable particles we have $\mathcal{H}^\lambda_0 = \mathcal{H}_d = \bigotimes_{i=1}^{i=L} \mathcal{H}_i$ and the symmetry group $K = \times_{i=1}^{i=L} SU(N)$ acts on it (2). It is easy to extract give a compact form of $P^{2\lambda_0}$. Let us first introduce some notation:

$$\mathcal{H}_d \otimes \mathcal{H}_d = \left( \bigotimes_{i=1}^{i=L} \mathcal{H}_i \right) \otimes \left( \bigotimes_{i=1'}^{i=L'} \mathcal{H}_i \right),$$

where $L = L'$ and we decided to label spaces from the second copy of the total space with primes in order to avoid ambiguity. Action of $K$ on $\text{Sym}^2(\mathcal{H}_d)$ is given by the restriction to the symmetric (with respect to the interchange of copies of $\mathcal{H}_d$) tensors of the action defined on $\mathcal{H}_d \otimes \mathcal{H}_d$ (11). Let us also introduce the symmetrization operators $P^{+}_{ii'} : \mathcal{H}_d \otimes \mathcal{H}_d \rightarrow \mathcal{H}_d \otimes \mathcal{H}_d$ that project onto the subspace of $\mathcal{H}_d \otimes \mathcal{H}_d$ completely symmetric under interchange spaces $i$ and $i'$ (one can define anti-symmetrization operators $P^{-}_{ii'}$ in the analogous way). Under introduced notation we have the closed expression for the projector operator $P^{2\lambda_0}$

$$P^{2\lambda_0} = P_d = P^{+}_{11} \circ P^{+}_{22} \circ \ldots \circ P^{+}_{LL'}.$$  

We can now write down explicitly our entanglement measure for distinguishable particles. For $|\psi\rangle \in \mathcal{H}_d$ we have
\[ C_d (|\psi\rangle) = \sqrt{\langle \psi | \psi \rangle} = \sqrt{\langle \psi | \psi \rangle} , \]  
\[ C_b (|\psi\rangle) = \sqrt{\langle \psi | \psi \rangle} , \]  
where subscript \( d \) stands from distinguishable particles. Expression above is, up to a multiplicative factor, the well-known multipartite concurrence \[12, 13].

**B. Bosons**

Hilbert space describing \( L \) bosonic particles has the structure \( \mathcal{H}^{\lambda_0} = \mathcal{H}_b = \text{Sym}^L (\mathcal{H}) \), where \( \mathcal{H} \approx \mathbb{C}^N \). The symmetry group represented in this space is \( K = SU(N) \) \[5\]. We embed \( \text{Sym}^L (\mathcal{H}) \) in the Hilbert space of \( L \) identical distinguishable particles, \( \text{Sym}^L (\mathcal{H}) \subset \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_L \), where \( \mathcal{H}_i \approx \mathcal{H} \). We have the analogous embedding of \( \text{Sym}^L (\mathcal{H}) \lor \text{Sym}^L (\mathcal{H}) \),

\[ \text{Sym}^L (\mathcal{H}) \lor \text{Sym}^L (\mathcal{H}) \subset \left( \bigotimes_{i=1}^{i=L} \mathcal{H}_i \right) \otimes \left( \bigotimes_{i=1'}^{i=L'} \mathcal{H}_i \right) = \mathcal{H}_d \otimes \mathcal{H}_d , \]  
where, as before, \( L = L' \). Let \( \mathbb{P}^{\text{sym}}_{\{1, \ldots, L\}} : \mathcal{H}_d \otimes \mathcal{H}_d \rightarrow \mathcal{H}_d \otimes \mathcal{H}_d \) be the projector onto the subspace of \( \mathcal{H}_d \otimes \mathcal{H}_d \) which is completely symmetric with respect to interchange of spaces labeled by indices from the set \( \{1, 2, \ldots, L\} \). We define \( \mathbb{P}^{\text{sym}}_{\{1', \ldots, L'\}} \) in the analogous way.

Under the above notation operator \( \mathbb{P}^{2\lambda_0} \) takes the form:

\[ \mathbb{P}^{2\lambda_0} = \mathbb{P}_b = \left( \bigotimes_{i=1}^{i=L} \mathbb{P}_i^{+} \right) \left( \bigotimes_{i=1'}^{i=L'} \mathbb{P}_i^{+} \right) , \]  
where it is understood that \( \mathbb{P}^{2\lambda_0} \) acts on the space \( \mathcal{H}_d \otimes \mathcal{H}_d \) (see \[19\]). Operators \( \mathbb{P}_i^{+} \) are the same as in the previous section. Let us note that we may write

\[ \mathbb{P}_b |_{\text{Sym}^L (\mathcal{H}) \lor \text{Sym}^L (\mathcal{H})} = \mathbb{P}_1^{+} \circ \mathbb{P}_2^{+} \circ \ldots \circ \mathbb{P}_{LL'}^{+} , \]  
as for any \( |\Psi\rangle \in \text{Sym}^L (\mathcal{H}) \lor \text{Sym}^L (\mathcal{H}) \) we have \( \left( \mathbb{P}^{\text{sym}}_{\{1, \ldots, L\}} \otimes \mathbb{P}^{\text{sym}}_{\{1', \ldots, L'\}} \right) |\Psi\rangle = |\Psi\rangle \). Entanglement measure for bosonic particles takes the same form as for distinguishable particles. For \( |\psi\rangle \in \text{Sym}^L (\mathcal{H}) \) we have

\[ C_b (|\psi\rangle) = \sqrt{\langle \psi | \psi \rangle} \mathbb{P}_1^{+} \circ \mathbb{P}_2^{+} \circ \ldots \circ \mathbb{P}_{LL'}^{+} |\psi\rangle , \]  
where subscript \( b \) stands for bosons.
C. Fermions

Hilbert space describing $L$ fermionic particles is $\mathcal{H}^\lambda = \mathcal{H}_f = \wedge^L (\mathcal{H})$, where $\mathcal{H} \approx \mathbb{C}^N$. The symmetry group is again $K = SU(N)$ (see (6)). Just like in the case of bosons (see (19)) we have

$$\wedge^L (\mathcal{H}) \subset \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_L,$$

(23)

$$\wedge^L (\mathcal{H}) \vee \wedge^L (\mathcal{H}) \subset \left( \bigotimes_{i=1}^{L} \mathcal{H}_i \right) \otimes \left( \bigotimes_{i=1'}^{L} \mathcal{H}_i \right) = \mathcal{H}_d \otimes \mathcal{H}_d.$$

(24)

By $\mathbb{P}^{\text{sym}}_{\{1, \ldots, L\}} : \mathcal{H}_d \otimes \mathcal{H}_d \to \mathcal{H}_d \otimes \mathcal{H}_d$ we denote the projector onto the subspace of $\mathcal{H}_d \otimes \mathcal{H}_d$ which is completely asymmetric with respect to interchange of spaces labeled by indices from the set $\{1, 2, \ldots, L\}$. We define $\mathbb{P}^{\text{sym}}_{\{1', \ldots, L'\}}$ in the analogous way. Under this notation we get

$$\mathbb{P}^{2\lambda} = \mathbb{P}_f = \alpha \left( \mathbb{P}^+_{11'} \circ \mathbb{P}^+_{22'} \circ \ldots \circ \mathbb{P}^+_{LL'} \right) \left( \mathbb{P}^{\text{sym}}_{\{1, \ldots, L\}} \circ \mathbb{P}^{\text{sym}}_{\{1', \ldots, L'\}} \right),$$

(25)

where $\alpha = \frac{2L}{L+1}$ and it is understood that $\mathbb{P}^{2\lambda}$ acts on the space $\mathcal{H}_d \otimes \mathcal{H}_d$. In analogy to the case of bosons we have

$$\mathbb{P}_f|_{\wedge^L (\mathcal{H}) \otimes \wedge^L (\mathcal{H})} = \alpha \mathbb{P}^+_{11'} \circ \mathbb{P}^+_{22'} \circ \ldots \circ \mathbb{P}^+_{LL'},$$

since for any $|\Psi\rangle \in \wedge^L (\mathcal{H}) \otimes \wedge^L (\mathcal{H})$ we have $\left( \mathbb{P}^{\text{sym}}_{\{1, \ldots, L\}} \circ \mathbb{P}^{\text{sym}}_{\{1', \ldots, L'\}} \right) |\Psi\rangle = |\Psi\rangle$. Generalized concurrence for a fermionic state $|\psi\rangle \in \wedge^L (\mathcal{H})$ reads

$$C_f (|\psi\rangle) = \sqrt{\langle \psi | \langle \psi | \mathbb{I} \otimes \mathbb{I} - \alpha \mathbb{P}^+_{11'} \circ \mathbb{P}^+_{22'} \circ \ldots \circ \mathbb{P}^+_{LL'} |\psi\rangle |\psi\rangle},$$

(26)

where subscript $f$ stands for fermions.

D. Physical interpretation of generalized concurrences

Expressions for $C_d$, $C_b$ and $C_f$ depend only upon $\langle \psi | \langle \psi | \mathbb{P}^+_{11'} \circ \mathbb{P}^+_{22'} \circ \ldots \circ \mathbb{P}^+_{LL'} |\psi\rangle |\psi\rangle$. One can show (22) that for arbitrary $L$-particle states the following expression holds,

$$\langle \psi | \langle \psi | \mathbb{P}^+_{11'} \circ \mathbb{P}^+_{22'} \circ \ldots \circ \mathbb{P}^+_{LL'} |\psi\rangle |\psi\rangle = 2^{-L} \left( \sum_k \text{tr} \left( \rho_k^2 \right) + 2 \right),$$

(27)
where the summation is over all different $2^L - 2$ proper subsystems of $L$-particle systems and $\rho_k$ is the reduced density matrix describing the particular subsystem. Notice that the expression (27) is also valid for bosons and fermions because we can formally embed bosonic and fermionic Hilbert spaces in $\bigotimes^L (\mathbb{C}^N)$.

Although in our reasoning we care only whether a given multiparty pure state is “classical” or not, it is tempting to ask what are the “maximally entangled” states corresponding to measures $C_d$, $C_b$ and $C_f$ in each of three considered contexts. Equation (27) enables us to formally answer to this question. Clearly, $C_d (|\psi\rangle)$, $C_b (|\psi\rangle)$ and $C_f (|\psi\rangle)$ will be maximal once for each proper subsystem $k$ the corresponding reduced density matrix will be maximally mixed. For the case of distinguishable particles states $|\psi\rangle$ satisfying this condition are called “absolutely maximally entangled states”. The problem of deciding whether for a given $L$ and $N$ such states at all exist is in general unsolved. Therefore, one cannot hope for an easy characterization of states that maximize $C_d$, $C_b$ or $C_f$ (or equivalently, minimize (27) once $|\psi\rangle \in \mathcal{H}_d$, $\mathcal{H}_b$ or $\mathcal{H}_f$ respectively). Nevertheless, the characterization of “absolutely maximally entangled” bosonic and fermionic states is certainly an interesting open problem.

IV. GENERALIZATION TO INFINITE DIMENSIONAL SYSTEMS

In this section we extend the concept of concurrence for the infinite dimensional setting. We first make a few technical remarks about infinite dimensional setting. In the rest of the section we prove that we can generalize the concept of concurrence introduced in previous two sections also to infinite dimensional Hilbert spaces describing arbitrary finite number of distinguishable particles, fermions or bosons. We prove that criteria for entanglement given by expressions (17), (22) and (26) are also valid in the infinite dimensional setting.

Separable Hilbert space $\mathcal{H}$ is, by definition, a Hilbert space in which it is possible to chose a countable basis. Almost all Hilbert spaces that occur in physics are separable [18]. Examples include all finite dimensional Hilbert spaces or the space space of square integrable (with respect to the Lebesgue measure) functions on $\mathbb{R}^d$, $L^2 (\mathbb{R}^d, dx)$. In this section we consider, unless we indicate otherwise, only general separable Hilbert spaces. The space of pure states of a quantum system is a projective space $\mathbb{P}\mathcal{H}$ which we identify with the collection of rank one orthogonal projectors acting on $\mathcal{H}$. The projective space $\mathbb{P}\mathcal{H}$ is metric space with respect to the Hilbert–Schmidt metric [18]. That is, for $[|\psi\rangle], [|\phi\rangle] \in \mathbb{P}\mathcal{H}$ we have
\[ d (|\psi\rangle, |\phi\rangle) = \sqrt{\text{tr} \left( (|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|)^2 \right)} = \sqrt{2 \left( 1 - |\langle\psi|\phi\rangle|^2 \right)} , \]  

where \( d (\cdot, \cdot) \) denotes the metric. The projective space endowed with the above metric is a complete metric space, i.e. every Cauchy sequence of elements from \( \mathbb{P}\mathcal{H} \) converge.

### A. Distinguishable particles

We first study entanglement of \( L \) distinguishable particles, described by the Hilbert space \( \mathcal{H}_d = \bigotimes_{i=1}^{L} \mathcal{H}_i \), where single particle Hilbert spaces \( \mathcal{H}_i \) are in general infinite dimensional. The notion of the tensor product of infinite dimensional Hilbert spaces involves, by definition, taking into account tensors having infinite rank, i.e. tensors that cannot be written as a finite combination of elements of the form \( |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_L\rangle \). This phenomenon does not occur when dimensions of single particle Hilbert spaces are finite. The set of separable states consists of states having the form of simple tensors from \( \mathcal{H}_d \),

\[ \mathcal{O}_{\text{sep}} = \left\{ [[|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_L\rangle] | |\psi_i\rangle \in \mathcal{H}_i \right\} . \]  

One can identify \( \mathcal{O}_{\text{sep}} \) with the orbit of \( K = U(\mathcal{H}_1) \times U(\mathcal{H}_2) \times \cdots \times U(\mathcal{H}_L) \) through one exemplary separable state \( [|\psi\rangle_{\text{sep}}] \). The main difference with the finite dimensional setting is that the group \( K \) is not a Lie group not to mention it is compact or semisimple. Therefore, methods of representation theory of Lie group cannot be applied to get result of the form (13). Nevertheless we argue that the following holds,

\[ [|\psi\rangle] \in \mathcal{O}_{\text{sep}} \iff \langle \psi | \otimes \langle \psi | \mathbb{I} - \mathbb{P}_{11}^+ \circ \mathbb{P}_{22}^+ \circ \cdots \circ \mathbb{P}_{LL}^+ |\psi\rangle \otimes |\psi\rangle = 0, \]  

where \( \mathbb{P}_{ii}^+ : \mathcal{H}_d \otimes \mathcal{H}_d \rightarrow \mathcal{H}_d \otimes \mathcal{H}_d \) are the symmetrization operators defined as in Part III A. Note that (30) implies that nonzero \( C_d (|\psi\rangle) \) defined as in (17) detects entangled pure states. In order to prove (30) we first observe that \( \langle \psi | \langle \psi | \mathbb{I} - \mathbb{P}_{11}^+ \circ \mathbb{P}_{22}^+ \circ \cdots \circ \mathbb{P}_{LL}^+ |\psi\rangle |\psi\rangle = 0 \) for separable states. Therefore, we only need to prove the inverse implication. Let us denote by \( \mathcal{O}^i_{\text{sep}} \) the set of states that are separable with respect to the bipartition \( \mathcal{H}_d = \mathcal{H}_i \otimes \left( \bigotimes_{j \neq i} \mathcal{H}_j \right) \). That is,

\[ \mathcal{O}^i_{\text{sep}} = \left\{ [[|\psi\rangle \otimes |\phi\rangle] | |\psi\rangle \in \mathcal{H}_i , |\phi\rangle \in \left( \bigotimes_{j \neq i} \mathcal{H}_j \right) \right\} . \]
One checks that $|\psi\rangle \in \mathcal{O}_{sep}$ if and only if $|\psi\rangle \in \mathcal{O}_{sep}^i$ for all $i = 1, \ldots, L$ (in other words $|\psi\rangle$ is separable with respect to any bipartition $\mathcal{H}_d = \mathcal{H}_i \otimes \bigotimes_{j \neq i} \mathcal{H}_j$). Note that in order not to complicate the notation we abuse the notation of the tensor product in (31) (we do not respect the order of terms in the tensor product). For the proof of the above statement see the Appendix B. We can now prove that $\langle \psi | \langle \psi | I \otimes I - P_{11'} \otimes P_{22'} \otimes \ldots \otimes P_{LL'} | \psi \rangle | \psi \rangle = 0$ implies that $|\psi\rangle$ is separable. Assume that $|\psi\rangle$ is entangled. By the discussion above $|\psi\rangle$ is non-separable with respect to some bipartition $\mathcal{H}_{i_0} \otimes \bigotimes_{j \neq i_0} \mathcal{H}_j$. We write the Schmidt decomposition of $|\psi\rangle$ with respect to this bipartition [11],

$$|\psi\rangle = \sum_l \lambda_l |\psi_l\rangle \otimes |\phi_l\rangle, \quad (32)$$

where $|\psi_l\rangle \in H_{i_0}$, $|\phi_l\rangle \in \bigotimes_{j \neq i_0} \mathcal{H}_j$ and $\langle \psi_l | \psi_j \rangle = \langle \phi_l | \phi_j \rangle = \delta_{ij}$. Moreover, we fix the normalization of the state by setting $\sum_i |\lambda_i|^2 = 1$. We have:

$$\langle \psi | \langle \psi | I \otimes I - P_{11'} \circ P_{22'} \circ \ldots \circ P_{LL'} | \psi \rangle | \psi \rangle \geq \langle \psi | \langle \psi | I \otimes I - P_{11'} \otimes I \otimes \ldots \otimes I | \psi \rangle | \psi \rangle.$$

Direct computation based on (32) shows that $\langle \psi | \langle \psi | P_{11'}^{\otimes I} \otimes \ldots \otimes | \psi \rangle | \psi \rangle < 1$ which implies $\langle \psi | \langle \psi | I \otimes I - P_{11'} \circ P_{22'} \circ \ldots \circ P_{LL'} | \psi \rangle | \psi \rangle > 0$. This concludes the proof of (30).

**B. Bosons**

The criterion analogous to (30) holds also for the arbitrary finite number of bosonic particles with infinite dimensional single particle Hilbert space. We have $\mathcal{H}_b = \text{Sym}^L (\mathcal{H})$, where $\mathcal{H}$ is infinite dimensional. In analogy with the finite dimensional case (5) we distinguish bosonic coherent states:

$$O_b = \{ |\phi\rangle \otimes |\phi\rangle \otimes \ldots \otimes |\phi\rangle | |\phi\rangle \in \mathcal{H} \}. \quad (33)$$

We notice that coherent bosonic states are precisely completely symmetric separable states of the system of identical distinguishable particles with single particle Hilbert spaces $\mathcal{H}$. Thus, we can apply criterion (30) restricted to $\text{Sym}^L (\mathcal{H})$ to distinguish coherent bosonic states. More precisely, for $|\psi\rangle \in \text{Sym}^L (\mathcal{H})$ we have

$$|\psi\rangle \in O_b \iff \langle \psi | \langle \psi | I \otimes I - P_{11'} \circ P_{22'} \circ \ldots \circ P_{LL'} | \psi \rangle | \psi \rangle = 0,$$
where operator $P_{11}' \circ P_{22}' \circ \ldots \circ P_{LL}'$, is assumed to act on the Hilbert space $\text{Sym}^L (\mathcal{H}) \otimes \text{Sym}^L (\mathcal{H}) \subset \mathcal{H}_d \otimes \mathcal{H}_d$, with $\mathcal{H}_d$ defined in [IV A] and each single particle Hilbert space $\mathcal{H}_i$ equal to $\mathcal{H}$.

C. Fermions

The case of fermionic particles turns out to be the most demanding, albeit also the most interesting. System of $L$ fermionic particles is described by $\mathcal{H}_f = \bigwedge^L (\mathcal{H})$, where the single particle Hilbert space $\mathcal{H}$ is infinite dimensional. We define “non-entangled” or coherent fermionic states analogously to the finite dimensional case,

$$ O_f = \{ [\phi_1] \wedge [\phi_2] \wedge \ldots \wedge [\phi_L] | [\phi_i] \in \mathcal{H}, \langle \phi_i | \phi_j \rangle = \delta_{ij} \} . \quad (34) $$

In what follows we prove that the criterion based on the generalized concurrence (26) holds also in the infinite dimensional situation. More precisely we show that

$$ [\psi] \in O_f \iff \langle \psi | \langle \psi | I \otimes I - \alpha P_{11}' \circ P_{22}' \circ \ldots \circ P_{LL}' | \psi \rangle | \psi \rangle = 0, \quad (35) $$

where, as before, $\alpha = \frac{2^L}{L+1}$. It is assumed that $P_{11}' \circ P_{22}' \circ \ldots \circ P_{LL}'$ acts on $\bigwedge^L (\mathcal{H}) \otimes \bigwedge^L (\mathcal{H}) \subset \mathcal{H}_d \otimes \mathcal{H}_d$, with $\mathcal{H}_d$ defined in [IV A] and each single particle Hilbert space $\mathcal{H}_i$ equal to $\mathcal{H}$. Let us denote $P_f = \alpha P_{11}' \circ P_{22}' \circ \ldots \circ P_{LL}'$. In order to prove (35) we consider the equivalent problem,

$$ [\psi] \in O_f \iff \langle \psi | \langle \psi | P_f | \psi \rangle | \psi \rangle = 1 , \quad (36) $$

for a normalized $| \psi \rangle \in \bigwedge^L (\mathcal{H})$. Note that if the rank of $| \psi \rangle$ (i.e. the minimal number of elements of the form $| \phi_1 \rangle \wedge | \phi_2 \rangle \wedge \ldots \wedge | \phi_L \rangle$ needed to express $| \psi \rangle$) is finite, we have $| \psi \rangle \in \bigwedge^L (\mathcal{H}_0)$, where $\mathcal{H}_0$ is some finite dimensional subspace of $\mathcal{H}$. Therefore, in this case (36) is proven as we can apply results from Section [II] and Part [III C]. If rank of $| \psi \rangle$ is infinite and $\langle \psi | \langle \psi | P_f | \psi \rangle | \psi \rangle < 1$ there is nothing to prove. The only case left is when $| \psi \rangle$ has infinite rank and $\langle \psi | \langle \psi | P_f | \psi \rangle | \psi \rangle = 1$. In Appendix B we show that such a situation is impossible.
From now on we will assume that dimensions of Hilbert spaces we consider are finite. We will use notation introduced in Sections II and III. Recall that a mixed state of a quantum system described by a Hilbert space \( \mathcal{H} \) is any operator \( \rho \) on \( \mathcal{H} \) satisfying conditions \( \rho \geq 0 \) and \( \text{tr} \rho = 1 \).

We treat separable mixed states for distinguishable particles, bosons and fermions in the unified fashion. Let \( K \) be a semisimple compact Lie group irreducibly represented in the Hilbert space \( \mathcal{H} \). We say that a mixed state \( \rho \in \text{End}(\mathcal{H}) \) (by \( \text{End}(\mathcal{H}) \) we denote the set of all operators on \( \mathcal{H} \)) is a “generalized separable” or “quasi-classical” [3, 5] if and only if \( \rho \) can be written as a convex combination of projectors onto coherent states of the action of \( K \), i.e.

\[
\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|,
\]

where \( p_i > 0, \sum_i p_i = 1 \), and \( |\psi_i\rangle \) are normalized representatives of separable states \( [|\psi_i\rangle] \in \mathcal{O}_0 \) (see (7)). By the appropriate choice of the group and its representation, as discussed in Section II, one recovers usual definition of mixed separable, as well as mixed coherent bosonic and mixed fermionic states [4]. The problem of deciding whether a given mixed state \( \rho \) is coherent is in general very difficult as in the decomposition (37) vectors \( |\psi_i\rangle \) need not to be orthogonal. One way to solve this problem, at least in principle, is to compute the convex roof extension [19] of the generalized concurrence \( C(|\psi\rangle) \) (14). It is defined by

\[
C(\rho) = \inf_{\sum_i p_i |\phi_i\rangle \langle \phi_i| = \rho} \left( \sum_i p_i C(|\phi_i\rangle) \right),
\]

where the infimum is taken over all possible presentations of \( \rho \) as a convex sum of one dimensional projectors. If dimension of \( \mathcal{H} \) is finite the above expression is well defined because \( C(|\phi\rangle) \) is a continuous function [19]. Moreover, \( C(\rho) = 0 \), if and only if \( \rho \) is a coherent state. In what follows we denote by \( C_d(\rho), C_b(\rho) \) and \( C_f(\rho) \) convex roof extensions of generalized concurrences for distinguishable particles, bosons and fermions. The explicit form of the function \( C(\rho) \) is in general not known (see [3] for the general discussion of situations in which \( C(\rho) \) can be explicitly computed). For this reason it is desirable to
have non-trivial lower bounds for $C(\rho)$ that are easy to compute. Such lower bounds give necessary conditions for a given mixed state to be entangled. The problem of finding lower bounds for the concurrence has been intensively studied for distinguishable particles \cite{20,22}. Due to the possible experimental application, lower bounds for $C(\rho)$ that can be expressed as the expectation value of some observable on multiple copies of the state considered, are of greatest interest. An important lower bound for the system of $L$ distinguishable particles \cite{2} is the so-called Mintert-Buchleitner bound \cite{21},

$$C_d(\rho)^2 \geq \text{tr} (\rho \otimes \rho V),$$

where operator $V : \mathcal{H}_d \otimes \mathcal{H}_d \rightarrow \mathcal{H}_d \otimes \mathcal{H}_d$ is given by the formula

$$V = \mathbb{I} \otimes \mathbb{I} - \mathbb{P}_{11}^+ \circ \mathbb{P}_{22}^+ \circ \ldots \circ \mathbb{P}_{LL'}^+ - 2 \left(1 - 2^{-L}\right) \mathbb{P}^-.$$

Operator $\mathbb{P}^-$ in the above expression denotes the projection onto $\bigwedge^2 (\mathcal{H}_d)$, the asymmetric subspace of $\mathcal{H}_d \otimes \mathcal{H}_d$. A necessary condition for a state $\rho$ to be entangled is thus $\text{tr} (\rho \otimes \rho V) > 0$. In what follows we present the systematic way to construct lower bounds for the generalized concurrences for bosons and fermions starting from any lower bound for the concurrence for distinguishable particles. Proofs of our results rely on the similarity of $C_d(|\psi\rangle)$ with $C_b(|\psi\rangle)$ and $C_f(|\psi\rangle)$ (compare \cite{17} with \cite{22} and \cite{20}) and the fact that we can embed $\mathcal{H}_b$ and $\mathcal{H}_f$ in $\mathcal{H}_d$.

A. Bosons

The generalized concurrence for mixed states of $L$ bosons reads,

$$C_b(\rho) = \inf_{\sum_i p_i |\phi_i\rangle \langle \phi_i| = \rho} \left( \sum_i p_i C_b(|\phi_i\rangle) \right) = \inf_{\sum_i p_i |\phi_i\rangle \langle \phi_i| = \rho} \left( \sum_i p_i C_d(|\phi_i\rangle) \right),$$

where the infimum is over all possible presentations of $\rho$ as a convex sum of one dimensional projectors onto normalized $|\phi_i\rangle \in \text{Sym}^L (\mathcal{H})$. In the second equality we used the fact $C_b(|\phi_i\rangle) = C_d(|\phi_i\rangle)$ for $|\phi_i\rangle \in \text{Sym}^L (\mathcal{H})$. Due to \cite{11}, we have: $C_b(\rho) \geq C_d(\rho)$ for $\rho \in \text{Sym}^L (\mathcal{H})$. Indeed,

$$C_b(\rho) = \inf_{\sum_i p_i |\phi_i\rangle \langle \phi_i| = \rho} \left( \sum_i p_i C_d(|\phi_i\rangle) \right) \geq \inf_{\sum_i p_i |\phi_i\rangle \langle \phi_i| = \rho} \left( \sum_i p_i C_d(|\phi_i\rangle) \right) = C_d(\rho),$$
where the second infimum is over all presentations of the state \( \rho \) as a convex sum of one dimensional projectors onto normalized \( |\phi_i\rangle \in \mathcal{H}_d = \otimes^L (\mathcal{H}) \). As a result, any lower bound for the concurrence for distinguishable particles, i.e. a function on mixed states satisfying \( C_d (\rho) \geq f (\rho) \), is a lower bound for concurrence for bosons, \( C_b (\rho) \geq f (\rho) \). In particular we have,

\[
C_b (\rho)^2 \geq \text{tr} \left( \rho \otimes \rho \tilde{V} \right),
\]

where \( \tilde{V} = V|\Lambda^L (\mathcal{H}) \otimes \Lambda^L (\mathcal{H}) \rangle \). In general one can try to exploit the additional symmetries of the bosonic Hilbert space \( \text{Sym}^L (\mathcal{H}) \) to get improved lower bounds for \( C_b (\rho) \) but we do not address this problem here.

### B. Fermions

The generalized concurrence for mixed states of \( L \) fermions is given by

\[
C_f (\rho) = C_\alpha (\rho) = \inf \sum p_i |\phi_i^\alpha \rangle \langle \phi_i^\alpha | = \rho \left( \sum p_i C_f (|\phi_i^\alpha \rangle \langle \phi_i^\alpha |) \right),
\]

where the infimum is over all possible presentations of the state \( \rho \) as a convex sum of one dimensional projectors onto normalized \( |\phi_i^\alpha \rangle \in \Lambda^L (\mathcal{H}) \). We wrote \( C_\alpha (\rho) \) to indicate the dependence on the number \( \alpha \geq 1 \) (itself depending upon the number of particles \( L \)) that appears in (26). We have the following inequality

\[
C_\alpha (\rho) \geq \sqrt{\alpha} C_d (\rho) - \sqrt{\alpha - 1}.
\]

The proof of the above relies on the relation between \( C_d (|\psi\rangle \langle \psi|) \) with \( C_f (|\psi\rangle \langle \psi|) \) (see (17) and (26)). For a normalized vector \( |\psi^\alpha \rangle \in \Lambda^L (\mathcal{H}) \) we have

\[
C_d (|\psi^\alpha \rangle \langle \psi^\alpha |) \geq \frac{1}{\sqrt{\alpha}} C_\alpha (|\psi^\alpha \rangle \langle \psi^\alpha |) + \sqrt{1 - \frac{1}{\alpha}}.
\]

Indeed,
\[ C_d (|\psi^a\rangle) = \sqrt{\langle \psi^a | \psi^a \rangle - \frac{1}{\alpha} \mathbb{P}_f |\psi^a\rangle |\psi^a\rangle} = \sqrt{\frac{1}{\alpha} \langle \psi^a | \psi^a \rangle - \mathbb{P}_f |\psi^a\rangle |\psi^a\rangle} + \left( 1 - \frac{1}{\alpha} \right) \]

\[ \leq \frac{1}{\sqrt{\alpha}} \sqrt{\langle \psi^a | \psi^a \rangle - \mathbb{P}_f |\psi^a\rangle |\psi^a\rangle} + \sqrt{1 - \frac{1}{\alpha}} = \frac{1}{\sqrt{\alpha}} C_a (|\psi^a\rangle) + \sqrt{1 - \frac{1}{\alpha}}, \]

(46)

where we have used (17) and (26) and inequality \( \sqrt{a + b} \leq \sqrt{a} + \sqrt{b} \) for \( a, b \geq 0 \). For a given \( \rho \) we apply to (45) the operation of convex roof extension (43). We get the inequality,

\[ C_a (\rho) + \sqrt{\alpha - 1} \geq \sqrt{\alpha} \inf \sum_i p_i |\phi_i^a\rangle \langle \phi_i^a | \rho = \alpha \left( \sum_i p_i C_d (|\phi_i^a\rangle) \right). \]

(47)

We conclude the proof of (44) by noting that

\[ \inf \sum_i p_i |\phi_i^a\rangle \langle \phi_i^a | \rho = \alpha \left( \sum_i p_i C_d (|\phi_i^a\rangle) \right) = C_d (\rho), \]

where the second infimum is over all presentations of \( \rho \) as a convex sum of one dimensional projectors onto normalized \( |\phi_i^a\rangle \in \mathcal{H}_d = \otimes^L (\mathcal{H}) \).

Consider any lower bound \( f (\rho) \) for the concurrence for \( L \) distinguishable particles. From inequality (44) it follows that we have

\[ C_a (\rho) \geq \sqrt{\alpha} f (\rho) - \sqrt{\alpha - 1}, \]

(48)

where \( f (\rho) \) is restricted to operators on \( \wedge^L (\mathcal{H}) \). Therefore, condition \( \sqrt{\alpha} f (\rho) - \sqrt{\alpha - 1} > 0 \) gives a necessary condition for a mixed fermionic state to be entangled. Assume now that for \( L \) distinguishable particles we have inequality,

\[ C_d (\rho)^2 \geq \text{tr} (\rho \otimes \rho V), \]

(49)

for some operator \( V \) on \( \mathcal{H}_d \). From (43) we have

\[ C_a^2 (\rho) + 2 \alpha C_a (\rho) + \alpha - 1 \geq \alpha C_d^2 (\rho). \]

It follows that

\[ C_a (\rho) (1 + 2 \sqrt{\alpha - 1} C_a (\rho)) \geq \alpha C_d^2 (\rho) - (\alpha - 1). \]
Application of (39) to the above formula results in the inequality,

$$C_\alpha (\rho) \left(1 + 2\sqrt{\alpha - 1}C_\alpha (\rho)\right) \geq \text{tr} \left(\rho \otimes \rho \tilde{V}\right),$$

(50)

where $\tilde{V} = \alpha V - (\alpha - 1) \mathbb{I} \otimes \mathbb{I}$ and acts on $\bigwedge^L (\mathcal{H}) \otimes \bigwedge^L (\mathcal{H})$. From (50) it follows that $\text{tr} \left(\rho \otimes \rho \tilde{V}\right) > 0$ is a necessary condition for a given fermionic state $\rho$ to be “entangled” as $C_\alpha (\rho) \left(1 + 2\sqrt{\alpha - 1}C_\alpha (\rho)\right) > 0$ if and only if $C_\alpha (\rho) > 0$. Application of the above reasoning to the Mintert-Buchleitner bound 50 gives a particularly simple expression for $\tilde{V},$

$$\tilde{V} = \alpha \left(\mathbb{I} \otimes \mathbb{I} - \mathbb{P}^-_{11'V} \otimes \mathbb{P}^+_{11'} \circ \mathbb{P}^+_{22'} \circ \ldots \circ \mathbb{P}^+_{LL'} - 2 \left(1 - 2^{-L}\right) \mathbb{P}^-\right) - (\alpha - 1) \mathbb{I} \otimes \mathbb{I},$$

(51)

$$= (\mathbb{I} \otimes \mathbb{I} - \mathbb{P}_f) - 2 \cdot \alpha \left(1 - 2^{-L}\right) \mathbb{P}^-$$

(52)

where $\mathbb{P}^-$ is the projector onto $\bigwedge^2 \left(\bigwedge^L (\mathcal{H})\right) \subset \bigwedge^L (\mathcal{H}) \otimes \bigwedge^L (\mathcal{H}).$

VI. SUMMARY

We presented a comprehensive discussion of generalized concurrence which is an extension of the usual concurrence to systems consisting of not only distinguishable but also non-distinguishable particles. The generalized concurrence can be used to detect non-coherent or entangled pure states of bosonic or fermionic systems. Using tools of representation theory we gave a closed form expressions for concurrences for systems consisting of arbitrary, albeit finite, number of distinguishable particles (17), bosons (22) or fermions (26). We proved that expressions defining concurrences are valid also when single particle Hilbert spaces are infinite dimensional (Section IV). In the last part of the article we studied mixed states of bosons and fermions with the help of convex roof extensions of appropriate concurrences, $C_b (\rho)$ and $C_f (\rho)$. We used the connection between concurrences for distinguishable and non-distinguishable particles to obtain lower bounds for $C_b (\rho)$ and $C_f (\rho)$ from any lower bound for the concurrence for distinguishable particles, $C_d (\rho)$. This approach allowed us to obtain non-trivial lower bounds for $C_b^2 (\rho)$ and $C_f^2 (\rho)$ in the spirit of Mintert-Buchleitner (see 42 and 50).
VII. ACKNOWLEDGMENTS

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APPENDIX A FINITE DIMENSIONAL CASE

Representation theory of $SU(N)$

Representation theory of semisimple Lie groups and algebras is a rich and beautiful but we will not discuss it here. We refer interested reader to the relevant literature of the subject [24, 25]. For a more elaborate discussion of representation-theoretic methods in the context of entanglement theory see [4, 5]. In this section we briefly describe basic facts from representation theory of $SU(N)$. The group $SU(N)$ is an example of a semisimple Lie group whose representation theory exhibits essentially all the features the general theory.

Let $\text{su}(N)$ be a Lie algebra of $SU(N)$, i.e. a real Lie algebra consisting of all skew-hermitian and traceless $N \times N$ matrices. It is useful to study the complexifications of the group $SU(N)$ and its Lie algebra. The complexified group $SU(N)^C = SL(N)$, consists of all complex $N \times N$ matrices with determinant 1. The complexified algebra $\text{su}(N)^C = \text{sl}(N)$ consists of all complex traceless $N \times N$ matrices with zero trace. Irreducible representations of $SU(N)$, $\text{su}(N)$, $SL(N)$ and $\text{sl}(N)$ are in one to one correspondence - if $\mathcal{H}$ is an irreducible representation of any of four structures specified, then it is necessary an irreducible representation of the remaining three structures. Lie algebra $\text{sl}(N)$ turns is particularly useful in the description of irreducible representations of $SU(N)$. We have the following decomposition of $\text{sl}(N)$:

$$\text{sl}(N) = n_- \oplus \mathfrak{h} \oplus n_+,\n$$

where $\mathfrak{h}$ consists of diagonal traceless matrices and $n_-$ and $n_+$ are respectively strictly lower and upper diagonal matrices. Let $\pi$ be the irreducible representation of $\text{sl}(N)$ in the Hilbert space $\mathcal{H}$. A convenient way of description of the representation $\pi$ uses the notion of weights vectors, i.e., simultaneous eigenvectors of representatives of all elements form the Cartan subalgebra $\mathfrak{h}$. It means that $|\psi_\lambda\rangle \in \mathcal{H}$ is a weight vector if,
\[ \pi(H)|\psi_\lambda\rangle = \lambda(H)|\psi_\lambda\rangle, \]  
for \( H \in \mathfrak{h} \), where a form \( \lambda \in \mathfrak{h}^* \) is called a weight of \( \pi \). We have the decomposition,

\[ \mathcal{H} = \oplus \lambda \mathcal{H}_\lambda, \]  
where summation is over all weights of the considered representation. The subspaces \( \mathcal{H}_\lambda \) are spanned by vectors corresponding to the corresponding weight \( \lambda \). An irreducible representation is uniquely characterized by its highest weight \( \lambda_0 \) determined by the highest weight vector \( |\psi_{\lambda_0}\rangle \), i.e. by the (unique, up to the multiplicative constant) weight vector annihilated by all representatives of \( n_+ \):

\[ \pi(H)|\psi_{\lambda_0}\rangle = \lambda_0(H) \text{ for } H \in \mathfrak{h} \text{ and } \pi(n_+)|\psi_{\lambda_0}\rangle = 0. \]

Given the highest weight vector \( |\psi_{\lambda_0}\rangle \), we can generate the whole \( \mathcal{H} \) by the action of \( \mathfrak{sl}(N) \) (or equivalently by the action of \( \mathfrak{t}, \mathfrak{k} \) or \( G \)): \( \mathcal{H} = \text{span}_\mathbb{C} \{ \pi(X)|\psi_{\lambda_0}\rangle | X \in \mathfrak{sl}(N) \} \). We write \( \mathcal{H}^{\lambda_0} \) instead of \( \mathcal{H} \) when we want to distinguish which irreducible representation of \( \mathfrak{sl}(N) \) is considered.

**Formulas for \( \mathbb{P}^{2\lambda_0} \)**

In this part we prove formulas for \( \mathbb{P}^{2\lambda_0} \) in the case of distinguishable particles, bosons and fermions.

**Distinguishable particles**

Let \( \mathcal{H}^{\lambda_0} = \mathcal{H}_d = \bigotimes_{i=1}^{i=L} \mathcal{H}, \mathcal{H} \approx \mathbb{C}^N \) and \( K = \times_{i=1}^{i=L} SU(N) \) We show here that \( \mathbb{P}_d : \bigotimes^{2L} \mathcal{H} \rightarrow \bigotimes^{2L} \mathcal{H} \) defined by

\[ \mathbb{P}_f = \mathbb{P}^+_{11'} \circ \mathbb{P}^+_{22'} \circ \ldots \circ \mathbb{P}^+_{LL'}, \]  
equals \( \mathbb{P}^{2\lambda_0} \). The proof of this statement is the following. First, notice that for separable \( |\psi\rangle \) we have \( \mathbb{P}_d |\psi\rangle \otimes |\psi\rangle = |\psi\rangle \otimes |\psi\rangle \). Secondly, notice that have the equivalence of representations of \( K \),

\[ \mathbb{P}^+_{11'} \circ \mathbb{P}^+_{22'} \circ \ldots \circ \mathbb{P}^+_{LL'} (\text{Sym}^2 (\mathcal{H}_d)) \approx \text{Sym}^2 (\mathcal{H}_1) \otimes \text{Sym}^2 (\mathcal{H}_2) \otimes \ldots \otimes \text{Sym}^2 (\mathcal{H}_L). \]
Therefore, subspace $P^{+}_{11'} \circ P^{+}_{22'} \circ \ldots \circ P^{+}_{LL'} \left( \text{Sym}^2 (H_d) \right)$ is an irreducible representation of $K$. Talking into account criterion (13) and the fact that separable states are exactly coherent states of $K$ finishes the proof.

**Bosons**

Let $H^{\lambda_0} = H_b = \text{Sym}^L (H), H \approx \mathbb{C}^N$ and $K = SU(N)$. We show that the operator $P_b : \bigotimes^{2L} H \rightarrow \bigotimes^{2L} H$ given by

$$
P_b = \left( P^{+}_{11'} \circ P^{+}_{22'} \circ \ldots \circ P^{+}_{LL'} \right) \left( P_{\{1', \ldots, L'}^\text{sym} \circ P_{\{1', \ldots, L'}^\text{sym} \right),

$$

equals $P^{2\lambda_0}$. The proof is the following. Notice that

$$
P_b \left( \text{Sym}^L (H) \lor \text{Sym}^L (H) \right) \subset \text{Sym}^L (H) \lor \text{Sym}^L (H).

$$

Moreover, $P_b$ is a projector onto $\text{Sym}^{2L} (H)$, a completely symmetric subspace of $H_d \otimes H_d$. Subspace $\text{Sym}^{2L} (H)$ is an irreducible representation of $K$. For a coherent bosonic state $|\psi\rangle \in H_b$, we have $P_b |\psi\rangle \otimes |\psi\rangle = |\psi\rangle \otimes |\psi\rangle$. As a result, by criterion (13), $P_b = P^{2\lambda_0}$.

**Fermions**

Let $H^{\lambda_0} = \bigwedge^L (H), H \approx \mathbb{C}^N, K = SU(N)$ and $\alpha = \frac{2L}{L+1}$. We prove that $P_f : \bigotimes^{2L} H \rightarrow \bigotimes^{2L} H$ defined by

$$
P_f = \alpha \left( P^{+}_{11'} \circ P^{+}_{22'} \circ \ldots \circ P^{+}_{LL'} \right) \left( P_{\{1', \ldots, L'}^\text{asym} \circ P_{\{1', \ldots, L'}^\text{asym} \right)

$$

is precisely $P^{2\lambda_0}$, the projector onto $H^{2\lambda_0} = \bigwedge^L (H) \otimes \bigwedge^L (H) \subset \bigotimes^{2L} H$ (consult Section III C). The full proof relies on the representation theory of $SU(N)$. Main technical tools involved are Young diagrams, Schur-Weyl duality and the theory of plethysms [25–27]. In order to simplify the reasoning we base our argumentation on two simple facts:

1. Operator $P_f$ is the projector onto some irreducible representation of $SU(N)$ in $\bigotimes^{2L} H$.

2. $P_f \left( |\psi_{\lambda_0}\rangle \otimes |\psi_{\lambda_0}\rangle \right) = |\psi_{\lambda_0}\rangle \otimes |\psi_{\lambda_0}\rangle$, where $|\psi_{\lambda_0}\rangle = |\psi_1\rangle \wedge |\psi_2\rangle \wedge \ldots \wedge |\psi_L\rangle$ is the highest weight vector of the representation $H^{\lambda_0}$.
Proof of the Fact 1 can be found in [26]. Before we prove Fact 2 let us assume for the moment that above two facts are true. Because \( P \) preserves \( |\psi_{\lambda_0}\rangle \otimes |\psi_{\lambda_0}\rangle \) and from the vector \( |\psi_{\lambda_0}\rangle \otimes |\psi_{\lambda_0}\rangle \) it is possible to generate (via the action of \( SU(N) \)) the whole \( \mathcal{H}^{2\lambda_0} \subset \bigwedge^L (\mathcal{H}) \otimes \bigwedge^L (\mathcal{H}) \subset \bigotimes^{2L} \mathcal{H} \), one concludes that \( P \big|_{\mathcal{H}^{2\lambda_0}} = P^{2\lambda_0}. \) Let us turn to the proof of the second fact. Let us fix the basis \( \{|\psi_i\rangle\}_{i=1}^N \) of \( \mathcal{H} \) and let \( |\psi_1\rangle \wedge |\psi_2\rangle \wedge \ldots \wedge |\psi_L\rangle = |\psi_{\lambda_0}\rangle \) be the (unnormalized) highest weight vector of the representation \( \bigwedge^L (\mathcal{H}) \). From the definition of the wedge product we have

\[
P_f (|\psi_{\lambda_0}\rangle \otimes |\psi_{\lambda_0}\rangle) = P_f (|\psi_1\rangle \wedge |\psi_2\rangle \wedge \ldots \wedge |\psi_L\rangle \otimes |\psi_1\rangle \wedge |\psi_2\rangle \wedge \ldots \wedge |\psi_L\rangle),
\]

\[
= P_f \left( \sum_{\sigma \in S_L} \sum_{\tau \in S_L} \text{sgn}(\sigma) \text{sgn}(\tau) |\psi_{\sigma(1)}\rangle \otimes |\psi_{\sigma(2)}\rangle \otimes \ldots \otimes |\psi_{\sigma(L)}\rangle \otimes |\psi_{\tau(1)}\rangle \otimes |\psi_{\tau(2)}\rangle \otimes \ldots \otimes |\psi_{\tau(L)}\rangle \right)
\]

\[
= \frac{1}{L+1} \sum_{\sigma \in S_L} \sum_{\tau \in S_L} \text{sgn}(\sigma \tau) \left( |\psi_{\sigma(1)}\rangle \otimes |\psi_{\tau(1)}\rangle + |\psi_{\tau(1)}\rangle \otimes |\psi_{\sigma(1)}\rangle \right) \otimes \ldots \otimes \left( |\psi_{\sigma(L)}\rangle \otimes |\psi_{\tau(L)}\rangle + |\psi_{\tau(L)}\rangle \otimes |\psi_{\sigma(L)}\rangle \right)
\]

(60)

In the above expressions, \( S_L \) denotes permutation group of \( L \) elements and \( \text{sgn}(\cdot) \) denotes the sign of a permutation. In order to simply the notation, we swapped order of terms in the full tensor product \( \bigotimes^{2L} \mathcal{H} \), i.e. we used the isomorphism:

\[
\bigotimes^{2L} \mathcal{H} = \left( \bigotimes_{i=1}^{i=L} \mathcal{H}_i \right) \otimes \left( \bigotimes_{i=1}^{i=L'} \mathcal{H}_i \right) \approx (\mathcal{H}_1 \otimes \mathcal{H}_1') \otimes (\mathcal{H}_2 \otimes \mathcal{H}_2') \otimes \ldots \otimes (\mathcal{H}_L \otimes \mathcal{H}_L'),
\]

for \( \mathcal{H}_i \approx \mathcal{H}. \) Let us introduce the notation

\[
|\Phi_{k,\sigma,\theta}\rangle = (|\psi_{\tau(1)}\rangle \otimes |\psi_{\sigma(1)}\rangle) \otimes \ldots \otimes (|\psi_{\tau(k)}\rangle \otimes |\psi_{\sigma(k)}\rangle) \otimes (|\psi_{\tau(k+1)}\rangle \otimes |\psi_{\sigma(k+1)}\rangle) \otimes \ldots \otimes (|\psi_{\tau(L)}\rangle \otimes |\psi_{\sigma(L)}\rangle) + \ldots
\]

\[
+ (|\psi_{\sigma(1)}\rangle \otimes |\psi_{\tau(1)}\rangle) \otimes (|\psi_{\tau(2)}\rangle \otimes |\psi_{\sigma(2)}\rangle) \otimes \ldots \otimes (|\psi_{\tau(k+1)}\rangle \otimes |\psi_{\sigma(k+1)}\rangle) \otimes (|\psi_{\tau(k+2)}\rangle \otimes |\psi_{\sigma(k+2)}\rangle) \otimes \ldots + \ldots
\]

where \( \ldots \) denotes the summation over remaining \( \binom{L}{k} - 2 \) terms one obtains by the different choice of \( k \) element combinations from \( \{1, \ldots, L\} \). Reordering of terms in (60) gives

\[
\frac{1}{L+1} \sum_{k=0}^{k=L} \left( \sum_{\sigma \in S_L} \sum_{\tau \in S_L} \text{sgn}(\sigma \tau) |\Phi_{k,\sigma,\theta}\rangle \right),
\]

(61)

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Operator $\mathbb{P}_f$ preserves $\wedge^L (\mathcal{H}) \otimes \wedge^L (\mathcal{H})$ and therefore

$$\mathbb{P}_f (|\psi_{\lambda_0} \rangle \otimes |\psi_{\lambda_0} \rangle) = \left( \mathbb{P}_{\text{asym}}^{\{1,\ldots,L\}} \circ \mathbb{P}_{\text{asym}}^{\{1',\ldots,L'\}} \right) \circ \mathbb{P}_f (|\psi_{\lambda_0} \rangle \otimes |\psi_{\lambda_0} \rangle).$$

As a result from (61) we have

$$\frac{1}{L + 1} \sum_{k=0}^{k=L} \left( \sum_{\tau \in S_L} \sum_{\sigma \in S_L} \text{sgn} (\sigma \tau) \left( \mathbb{P}_{\text{asym}}^{\{1,\ldots,L\}} \circ \mathbb{P}_{\text{asym}}^{\{1',\ldots,L'\}} \right) |\Phi_{k,\sigma,\theta} \rangle \right).$$

We claim that for each $k = 0, \ldots, L$ we have

$$\sum_{\tau \in S_L} \sum_{\sigma \in S_L} \text{sgn} (\sigma \tau) \left( \mathbb{P}_{\text{asym}}^{\{1,\ldots,L\}} \circ \mathbb{P}_{\text{asym}}^{\{1',\ldots,L'\}} \right) (|\Phi_{k,\sigma,\theta} \rangle) = |\psi_{\lambda_0} \rangle \otimes |\psi_{\lambda_0} \rangle.$$

Indeed, application of $\mathbb{P}_{\text{asym}}^{\{1,\ldots,L\}} \circ \mathbb{P}_{\text{asym}}^{\{1',\ldots,L'\}}$ gives

$$\frac{1}{(L!)} \sum_{\sigma \in S_L} \sum_{\tau \in S_L} \text{sgn} (\sigma \tau) \left( \left( |\psi_{\tau(1)} \rangle \wedge |\psi_{\tau(2)} \rangle \wedge \ldots \wedge |\psi_{\tau(k)} \rangle \wedge |\psi_{\sigma(k+1)} \rangle \wedge \ldots \right) \otimes \left| \psi_{\sigma(1)} \rangle \wedge |\psi_{\sigma(2)} \rangle \wedge \ldots \wedge |\psi_{\sigma(k)} \rangle \wedge |\psi_{\tau(k+1)} \rangle \wedge \ldots \right) + \ldots \right),$$

where $\ldots$ denotes the summation over remaining $\binom{L}{k} - 1$ terms. Let $S_L (\sigma, k)$ denote the subgroup of $S_L$ consisting of permutations that do not mix sets $\{\sigma(1), \ldots, \sigma(k)\}$ and $\{\sigma(k+1), \ldots, \sigma(L)\}$. We have $S_L (\sigma, k) \approx S_k \times S_{L-k}$. As a result, for the fixed $\sigma \in S_k$ we have

$$\sum_{\tau \in S_L} \text{sgn} (\sigma \tau) \left( |\psi_{\tau(1)} \rangle \wedge |\psi_{\tau(2)} \rangle \wedge \ldots \wedge |\psi_{\tau(k)} \rangle \wedge |\psi_{\sigma(k+1)} \rangle \wedge \ldots \right) \otimes \left( |\psi_{\sigma(1)} \rangle \wedge |\psi_{\sigma(2)} \rangle \wedge \ldots \wedge |\psi_{\sigma(k)} \rangle \wedge |\psi_{\tau(k+1)} \rangle \wedge \ldots \right)$$

$$= \sum_{\tau \in S_L (\sigma, k)} \text{sgn} (\sigma \tau) \text{sgn} (\tau \sigma^{-1}) \left( |\psi_{\sigma(1)} \rangle \wedge |\psi_{\sigma(2)} \rangle \wedge \ldots \wedge |\psi_{\sigma(L)} \rangle \right) \otimes \left( |\psi_{\sigma(1)} \rangle \wedge |\psi_{\sigma(2)} \rangle \wedge \ldots \wedge |\psi_{\sigma(L)} \rangle \right) = (L-k)! \cdot k! |\psi_{\lambda_0} \rangle \otimes |\psi_{\lambda_0} \rangle.$$ 

Treating all other terms in the outer bracket of (64) in the similar fashion gives

$$\frac{1}{(L!)} \left( \sum_{\sigma \in S_L} \binom{L}{k} (L-k)! \cdot k! \right) |\psi_{\lambda_0} \rangle \otimes |\psi_{\lambda_0} \rangle = |\psi_{\lambda_0} \rangle \otimes |\psi_{\lambda_0} \rangle,$$

which proves (63). From (63) and (61) we conclude the proof of the second Fact and therefore prove that $\mathbb{P}_f = \mathbb{P}^{2\lambda_0}$. 

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APPENDIX B INFINITE DIMENSIONAL CASE

Distinguishable particles

We prove here that the state $|\psi\rangle \in \mathcal{P}\mathcal{H}_d$ is separable if and only if $|\psi\rangle \in \mathcal{O}^{i}_{sep}$ for $i = 1,\ldots,L$ (for definition of $\mathcal{O}^{i}_{sep}$ see (31)). First note that a separable state $|\psi\rangle$ clearly belongs to $\mathcal{O}^{i}_{sep}$. On the other hand, if $|\psi\rangle \in \mathcal{O}^{i}_{sep}$, then $|\psi\rangle$ is an eigenvector (with eigenvalue 1) of the operator

$$|\phi_i\rangle\langle\phi_i| \otimes I_i,$$

(65)

where $|\phi_i\rangle \in \mathcal{H}_i$ and $I_i$ is the identity operator on $\bigotimes_{j=1}^{L} \mathcal{H}_j$. Note that in order to do not complicate the notation in (65) we do not respect the order of terms in the tensor product $\bigotimes_{i=1}^{L} \mathcal{H}_i$. Note that we can repeat the above reasoning for all other $i = 1,\ldots,L$. As a result, we get that $|\psi\rangle$ is an eigenvector with the eigenvalue 1 of the operator

$$P|\psi\rangle = |\phi_1\rangle\langle\phi_1| \otimes |\phi_2\rangle\langle\phi_2| \otimes \cdots \otimes |\phi_L\rangle\langle\phi_L|,$$

where $|\psi_i\rangle \in \mathcal{H}_i$. Operator $P|\psi\rangle$ is a projector onto a separable state which concludes the proof that $|\psi\rangle$ is separable.

Fermions

We show here that a normalized fermionic state $|\psi\rangle \in \bigwedge^{L} (\mathcal{H})$ having infinite rank cannot satisfy $\langle\psi|\langle\psi|P_f|\psi\rangle|\psi\rangle = 1$ (see (IV C) for the definition of $P_f$). In the course of argumentation we will need the fact that the set of coherent fermionic states $O_f$ is closed in $\mathcal{P}\mathcal{H}_f$. We prove that $O_f$ is closed directly from the definition. Let $|\psi_k\rangle \in O_f$ be a Cauchy sequence. Let us fix $\epsilon > 0$. Then for $n, m > n_0(\epsilon)$ we have

$$d (|\psi_n\rangle, |\psi_m\rangle) \leq \epsilon .$$

Assuming that vectors $|\psi_k\rangle$ are normalized and making use of (28) we get

$$|\langle\psi_n|\psi_m\rangle|^2 \geq 1 - \frac{\epsilon^2}{2} .$$

(66)

Vectors $|\psi_n\rangle$ and $|\psi_m\rangle$ can be represented (non uniquely) by Slater determinants.
\[ |\psi_n\rangle = |\phi_1^n\rangle \wedge |\phi_2^n\rangle \wedge \ldots \wedge |\phi_L^n\rangle, \quad |\psi_m\rangle = |\phi_1^m\rangle \wedge |\phi_2^m\rangle \wedge \ldots \wedge |\phi_L^m\rangle. \]

Straightforward cancellations show that

\[ \langle \psi_n | \psi_m \rangle = \det (M), \]

where \( M \) is \( L \times L \) density matrix whose entries are given by \( M_{ij} = \langle \phi_i^n | \phi_j^m \rangle \). By the appropriate choice of the basis of \( V_n \), subspace of \( \bigwedge L (\mathcal{H}) \) spanned by vectors \( |\phi_1^n\rangle, \ldots |\phi_L^m\rangle \) matrix \( M \) can be made diagonal. That is we have

\[ \det (M) = \langle \tilde{\phi}_1^n | \phi_i^m \rangle \cdot \ldots \cdot \langle \tilde{\phi}_L^n | \phi_L^m \rangle, \]

where \( |\psi_n\rangle = |\tilde{\phi}_1^n\rangle \wedge |\tilde{\phi}_2^n\rangle \wedge \ldots |\tilde{\phi}_L^n\rangle \). Setting \( |\phi_i^m\rangle = |\tilde{\phi}_i^m\rangle \) and talking into account (66) we get

\[ \max_{i=1,\ldots,L} \left| \langle \tilde{\phi}_i^n | \phi_i^m \rangle \right|^2 \geq 1 - \frac{\epsilon^2}{2}, \]

which means that for each \( i = 1, \ldots, L \) “single particle” states \( \left[ |\tilde{\phi}_i^k\rangle \right] \in \mathbb{P}\mathcal{H} \) form a Cauchy sequence with respect to the metric (28). From the closedness of \( \mathbb{P}\mathcal{H} \) we infer that for each \( i \) we have \( \left[ |\tilde{\phi}_i^k\rangle \right] \xrightarrow{k \to \infty} \left[ |\tilde{\phi}_i^\infty\rangle \right] \in \mathbb{P}\mathcal{H} \). From that we conclude that

\[ \left[ |\tilde{\phi}_1^k\rangle \wedge |\tilde{\phi}_2^k\rangle \wedge \ldots \wedge |\tilde{\phi}_L^k\rangle \right] \xrightarrow{k \to \infty} \left[ |\tilde{\phi}_1^\infty\rangle \wedge |\tilde{\phi}_2^\infty\rangle \wedge \ldots \wedge |\tilde{\phi}_L^\infty\rangle \right], \]

which finishes the proof of closedness of \( \mathcal{O}_f \). We can now return to the original problem.

We introduce the sequence of finite dimensional subspaces

\[ \mathcal{H}_1 \subset \mathcal{H}_2 \subset \ldots \subset \mathcal{H}_k \subset \ldots \subset \mathcal{H}, \quad (67) \]

such that \( \bigcup_{i=1}^{\infty} \mathcal{H}_i = \mathcal{H} \). To the above sequence we associate corresponding sequence of subspaces of \( \bigwedge L (\mathcal{H}) \),

\[ \bigwedge (\mathcal{H}_1) \subset \bigwedge (\mathcal{H}_2) \subset \ldots \subset \bigwedge (\mathcal{H}_k) \subset \ldots \subset \bigwedge (\mathcal{H}). \]

(68)

Obviously we have \( \bigcup_{i=1}^{\infty} \bigwedge L (\mathcal{H}_i) = \bigwedge L (\mathcal{H}) \). We fix the index \( k \) and consider the following orthogonal splittings of \( \bigwedge L (\mathcal{H}) \) and \( \bigwedge L (\mathcal{H}) \otimes \bigwedge L (\mathcal{H}), \)

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\[ \bigwedge^L (\mathcal{H}) = \bigwedge^L (\mathcal{H}_k) \oplus \left[ \bigwedge^L (\mathcal{H}_k) \right]^\perp, \quad (69) \]

\[ \bigwedge^L (\mathcal{H}) \otimes \bigwedge^L (\mathcal{H}) = \left[ \bigwedge^L (\mathcal{H}_k) \otimes \bigwedge^L (\mathcal{H}_k) \right] \oplus \left[ \bigwedge^L (\mathcal{H}_k) \otimes \bigwedge^L (\mathcal{H}_k) \right]^\perp, \quad (70) \]

where orthogonal complements are taken with respect to the usual inner products on \( \bigwedge^L (\mathcal{H}) \) and \( \bigwedge^L (\mathcal{H}) \otimes \bigwedge^L (\mathcal{H}) \) respectively. By \( \mathbb{P}_k : \bigwedge^L (\mathcal{H}) \to \bigwedge^L (\mathcal{H}_k) \) we denote the orthogonal projector on \( \bigwedge^L (\mathcal{H}_k) \). We now prove that the infinite rank of \( |\psi\rangle \) and \( \langle \psi|\langle \psi|\mathbb{P}_f|\psi\rangle|\psi\rangle = 1 \) yield to the contradiction. Let us first note that for normalized \( |\psi\rangle \) condition \( \langle \psi|\langle \psi|\mathbb{P}_f|\psi\rangle|\psi\rangle = 1 \) is equivalent to \( \mathbb{P}_f|\psi\rangle|\psi\rangle = |\psi\rangle|\psi\rangle \). Consider a decomposition

\[ |\psi\rangle|\psi\rangle = |\Psi_k\rangle + |\Psi_k^\perp\rangle, \quad (71) \]

where \( |\Psi_k\rangle \in \bigwedge^L (\mathcal{H}_k) \otimes \bigwedge^L (\mathcal{H}_k) \) and \( |\Psi_k^\perp\rangle \in \left[ \bigwedge^L (\mathcal{H}_k) \otimes \bigwedge^L (\mathcal{H}_k) \right]^\perp \). We have \( \mathbb{P}_f|\psi\rangle|\psi\rangle = |\psi\rangle|\psi\rangle \) and thus

\[ \mathbb{P}_f|\Psi_k\rangle + \mathbb{P}_f|\Psi_k^\perp\rangle = |\Psi_k\rangle + |\Psi_k^\perp\rangle. \quad (72) \]

Because \( \mathbb{P}_f \) preserves \( \bigwedge^L (\mathcal{H}_k) \otimes \bigwedge^L (\mathcal{H}_k) \) we have \( \langle \Psi_k^\perp|\mathbb{P}_f|\Psi_k\rangle = 0 \) and therefore \( \mathbb{P}_f|\Psi_k\rangle = |\Psi_k\rangle \). Notice that \( |\Psi_k\rangle = \mathbb{P}_k \otimes \mathbb{P}_k (|\psi\rangle|\psi\rangle) \) and therefore \( |\Psi_k\rangle \) is a product state. Because \( \mathbb{P}_f|\Psi_k\rangle = |\Psi_k\rangle \) we see that \( \mathbb{P}_k|\psi\rangle \) is actually an (non-normalized) representative of some coherent fermionic state. We can repeat the above construction for the arbitrary number \( k \). We get

\[ 1 = \lim_{k \to \infty} \langle \psi|\mathbb{P}_k|\psi\rangle, \quad (73) \]

where \( \mathbb{P}_k|\psi\rangle \) is the (non-normalized) representative of some coherent state. We therefore get \( \lim_{k \to \infty} [\mathbb{P}_k|\psi\rangle] \) (in a sense of \( \| \psi \| \) ). Since \( [\mathbb{P}_k|\psi\rangle] \in \mathcal{O}_f \), we get that \( [\| \psi \|] \in \mathcal{O}_f \) as the set of coherent fermionic states \( \mathcal{O}_f \) is closed in \( \mathbb{P}\mathcal{H}_f \). This is clearly in contradiction with the assumption that \( |\psi\rangle \) has infinite rank.

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