OPTIMAL LAYER REINSURANCE ON THE MAXIMIZATION OF THE ADJUSTMENT COEFFICIENT

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Abstract. In this paper, we study the optimal retentions for an insurance company, which intends to transfer risk by means of a layer reinsurance treaty. Under the criterion of maximizing the adjustment coefficient, the closed form expressions of the optimal results are obtained for the Brownian motion risk model as well as the compound Poisson risk model. Moreover, we conclude that under the expected value principle there exists a special layer reinsurance strategy, i.e., excess of loss reinsurance strategy which is better than any other layer reinsurance strategies. Whereas, under the variance premium principle, the pure excess of loss reinsurance is not the optimal layer reinsurance strategy any longer. Some numerical examples are presented to show the impacts of the parameters as well as the premium principles on the optimal results.

1. Introduction. With reinsurance, insurers are able to transfer some of their risks to another party at the expense of making less potential profit, and hence finding optimal reinsurance strategy to balance their risk and profit is of great interest to them. In the past few years, optimal reinsurance problems have gained much interest in the actuarial literature, and the technique of stochastic control theory and the Hamilton-Jacobi-Bellman equation are frequently used to cope with these problems. See, for example, [22], [13], [21], [19], and [16].

In the study of optimal reinsurance contracts, a few objective functions are commonly seen in the literature. [4], [23], and [20] consider the objective function that minimizes ruin probability. [14, 15] study the optimal reinsurance problem under various mean-variance premium principles of the reinsurer. Expected utility, as another important objective function in the financial and actuarial literature, has attracted a great deal of interest. Therefore, some papers including [13], [2], [19], and [16] focus on constructing optimal contracts to maximize the expected utility of terminal wealth. Moreover, [5], [6], and [3] adopt the criteria of minimizing tail risk measures such as value at risk and conditional tail expectation. In this paper, our objective is to maximize the adjustment coefficient which is another popular criterion for modern risk theory since the explicit expression for the ruin probability

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is difficult to derive when the underlying risk follows a compound Poisson process. For example, see [12], [7], [17], [11] and references therein.

For the optimization problem with combined quota-share and excess of loss reinsurance, [8] obtains some analytical results for the single period compound Poisson risk model. [24] gets the explicit expressions of the optimal dynamic reinsurance for the Brownian motion risk model. These results are extended in [18] to the compound Poisson risk model, and the closed form expressions of the optimal retentions and the maximal adjustment coefficient are also derived.

The contributions of this paper are given as follows. Under the criterion of maximizing the adjustment coefficient, we study the optimal layer reinsurance for both Brownian motion case and the compound Poisson case. Under the expected value premium principle, we derive the closed-form expressions of the optimal results, and prove that there exists a pure excess of loss reinsurance strategy which is better than any other layer reinsurance strategies, not only in the Brownian motion risk model but also in the compound Poisson risk model. However, under variance premium principle, we can show that the pure excess of loss reinsurance is not the optimal layer reinsurance strategies any longer.

The paper is organized as follows. In section 2, the model and assumptions are given. The closed-form expressions of the optimal values are obtained in sections 3 and 4. In section 5, some numerical examples are presented, which demonstrate the results of this paper, and show that the pure excess of loss reinsurance is not the optimal layer reinsurance strategies for the variance premium principle case.

2. The model. Under the classical risk model, the surplus process \( \{X_t\} \) is given by

\[
X_t = u + ct - S(t),
\]

where \( u \geq 0 \) is the initial capital, \( c \) is the premium income rate, \( S(t) = \sum_{i=1}^{N_t} Y_i \) is the aggregate claims process, where \( N_t \) is a Poisson process with rate \( \lambda \). The claim sizes \( \{Y_i\} \) are i.i.d. strictly positive random variables, and independent of the claim number process \( N_t \).

We assume that \( F(y) \), the distribution function of \( Y_i \), is such a function that \( F(0) = 0, 0 < F(y) < 1 \) for \( 0 < y < N \) and \( F(y) = 1 \) for \( y \geq N \), here \( N = \inf\{ y : F(y) = 1 \} \), and \( 0 < N \leq +\infty \); that \( dF(y)/dy := f(y) \) exists and is continuous; that the moment generating function of \( F(y) \), \( M_Y(r) \), exists for \( r \in (-\infty, r_\infty) \) for some \( 0 < r_\infty \leq \infty \) and that \( \lim_{r \to r_\infty} M_Y(r) = +\infty \). The last assumption is exactly the condition needed for the Cramèr-Lundberg approximation in the classical risk model (see, for instance, the book by [1] or [9]). Let \( \mu \) be the expected value of \( Y_i \).

In this paper, we suppose that the insurer has a choice of reinsuring his (or her) risk by a layer reinsurance treaty. Let \( d_1 \) and \( d_2 \) be the decision variables representing the layer reinsurance retention. The ceded loss function is the layer reinsurance in the form of

\[
f(x) = \min\{(x - d_1)_+, \quad d_2 - d_1\} = (x - d_1)_+ - (x - d_2)_+,
\]

where \( \{x\}_+ = \max\{x, 0\} \), and \( 0 < d_1 \leq d_2 \leq N \). So by layer reinsurance treaty, the insurer will retain, from the \( i \)th claim,

\[
Y_i(d_1, d_2) = (Y_i \wedge d_1) + (Y_i - d_2)_+ \quad (i = 1, \ldots, N_t).
\]
Assume that the premium is calculated according to the expected value principle, then the premium income rate becomes
\[ c(d_1, d_2) = \lambda \mu (1 + \theta) - (1 + \eta) \lambda E(Y_t - Y_i(d_1, d_2)) \]
\[ = (\theta - \eta) \lambda \mu + (1 + \eta) \lambda E(Y_i(d_1, d_2)) \]
where \( \theta = \frac{\lambda \mu}{\lambda \sigma} - 1 \) denotes the safety loading of the insurer, and \( \eta \) is the safety loading of the reinsurer. Without loss of generality, we assume that \( \eta > \theta \).

Then \( \{Y_i(d_1, d_2)\} \) are i.i.d. strictly positive random variables, and independent of the claim number \( N_t \). It is not difficult to get
\[
\left\{
\begin{align*}
EY_i(d_1, d_2) &= \int_0^{d_1} (1 - F(y))dy + \int_{d_2}^{N} (1 - F(y))dy := \mu(d_1, d_2); \\
E(Y_i(d_1, d_2))^2 &= 2 \int_0^{d_1} y(1 - F(y))dy + 2 \int_{d_2}^{N} (y - d_2)(1 - F(y))dy \\
&+ 2d_1 \int_{d_2}^{N} (1 - F(y))dy \\
&:= \sigma^2(d_1, d_2); \\
MY_i(d_1, d_2)(r) &= r \int_0^{d_1} (1 - F(y))e^{ry}dy + r \int_{d_2}^{N} (1 - F(y))e^{(y+d_1-d_2)r}dy + 1.
\end{align*}
\right.
\]
Thus, surplus follows the process
\[ X^{d_1, d_2}_t = u + c(d_1, d_2)t - S_1(t), \] (1)
here \( S_1(t) = \sum_{i=1}^{N(t)} Y_i(d_1, d_2) \) is another aggregate claims process.

Now define
\[ \tau^{d_1, d_2} = \inf\{t \geq 0 : X^{d_1, d_2}_t < 0\} \]
be the ruin time, and
\[ \psi^{d_1, d_2}(u) = P\{\tau^{d_1, d_2} < \infty | X^{d_1, d_2}_0 = u\} \]
be the probability of ultimate ruin.

3. **Optimal results for the diffusion approximation risk model.** In this section, we assume that the aggregate claims process of model (1) follows a Brownian motion with drift, i.e.,
\[ \hat{S}_1(t) = \lambda \mu (d_1, d_2)t - \sqrt{\lambda \sigma} (d_1, d_2)W_t, \]
where \( \{W_t, t \geq 0\} \) is a standard Brownian motion. \( \hat{S}_1(t) \) can be seen as the diffusion approximation of the compound Poisson process \( S_1(t) \) ( see [10]).

Replacing \( S_1(t) \) of (1) with \( \hat{S}_1(t) \) and simplifying yield a new surplus process which is given by
\[ \hat{X}^{d_1, d_2}_t = u + [(\theta - \eta) \lambda \mu + \lambda \eta \mu (d_1, d_2)]t + \sqrt{\lambda \sigma} (d_1, d_2)W_t, \] (2)
where \( 0 < d_1 \leq d_2 \leq N. \)

Let \[ \tau^{d_1, d_2}_D = \inf\{t \geq 0 : \hat{X}^{d_1, d_2}_t \leq 0\} \]
be the ruin time, and
\[ \psi^{d_1,d_2}_D(u) = P(r^{d_1,d_2}_D < \infty | X^{d_1,d_2}_0 = u) \]
be the probability of ultimate ruin.

By the “martingale approach”, we get the following

**Theorem 3.1.** The process \( \{e^{-R_D(d_1,d_2)}X^{d_1,d_2}_t, t \geq 0\} \) is a martingale. Furthermore,

\[ \psi^{d_1,d_2}_D(u) = e^{-R_D(d_1,d_2)u}, \tag{3} \]

where
\[ R_D(d_1,d_2) = \frac{2[(\theta - \eta)\lambda \mu + \lambda \eta \mu(d_1,d_2)]}{\lambda \sigma^2(d_1,d_2)}. \]

**Proof.** From (2), it is not difficult to see that
\[ E(e^{-r(X^{d_1,d_2}_t - u)}) = e^{-r[(\theta - \eta)\lambda \mu + \lambda \eta \mu(d_1,d_2)] + \frac{1}{2}r^2\lambda \sigma^2(d_1,d_2)t}. \]

Let
\[ \tilde{g}(r) = r[(\theta - \eta)\lambda \mu + \lambda \eta \mu(d_1,d_2)] - \frac{1}{2}r^2\lambda \sigma^2(d_1,d_2), \]
then equation \( \tilde{g}(r) = 0 \) has a positive root
\[ R_D(d_1,d_2) = \frac{2[(\theta - \eta)\lambda \mu + \lambda \eta \mu(d_1,d_2)]}{\lambda \sigma^2(d_1,d_2)}. \]

Thus \( E(e^{-R_D(d_1,d_2)(X^{d_1,d_2}_t - u)}) = 1 \) and the process \( \{e^{-R_D(d_1,d_2)}X^{d_1,d_2}_t, t \geq 0\} \) is a martingale. Here \( R_D(d_1,d_2) \) is an adjustment coefficient, or the so-called Lundberg exponent of the risk model (2).

By the standard martingale approach as in [10], we can directly derive that
\[ \psi^{d_1,d_2}_D(u) = e^{-R_D(d_1,d_2)u}, \]
and the proof is completed. \( \square \)

Now we are to find the optimal strategy \( (d_1^*, d_2^*) \) to minimize the ruin probability, i.e.,
\[ \psi_D(u) := \psi^{d_1^*,d_2^*}_D(u) = \inf_{d_1,d_2} \psi^{d_1,d_2}_D(u). \]

From (3), we can see that, finding \( (d_1^*, d_2^*) \) to minimize the ruin probability is equivalent to finding \( (d_1^*, d_2^*) \) to maximize the adjustment coefficient. So we try to find \( R_D \) such that
\[ R_D = R_D(d_1^*, d_2^*) = \sup_{d_1,d_2} R_D(d_1,d_2). \]

Note that the function \( \tilde{g}(r) \) is non-positive at \( r = R_D \), i.e., \( R_D \) is the solution to
\[ \sup_{d_1,d_2} \{r[(\theta - \eta)\lambda \mu + \lambda \eta \mu(d_1,d_2)] - \frac{1}{2}r^2\lambda \sigma^2(d_1,d_2)\} = 0. \tag{4} \]

Now the question is transformed into seeking the maximizer \( (\bar{d}_1, \bar{d}_2) \) of the function \( \tilde{g}(r) \) w.r.t. \( d_1 \) and \( d_2 \), which will be shown in the following Lemma 3.2.

**Lemma 3.2.** The maximizer of \( \tilde{g}(r) \) w.r.t. \( d_1 \) and \( d_2 \) is
\[ (\bar{d}_1, \bar{d}_2) = \left( \frac{\eta}{r}, N \right). \]
Proof. Differentiating $\tilde{g}(r)$ w.r.t. $d_1$ yields
\[
\frac{\partial \tilde{g}(r)}{\partial d_1} = r\lambda \left[ \eta(1 - F(d_1)) - rd_1(1 - F(d_1)) - r \int_{d_2}^{N} (1 - F(y))dy \right],
\]
and thus the maximizer $\bar{d}_1$ satisfies the following equation
\[
(\eta - r\bar{d}_1)(1 - F(\bar{d}_1)) = r \int_{d_2}^{N} (1 - F(y))dy \tag{5}
\]
for any fixed $d_2$.

And then differentiating $\tilde{g}(r)$ w.r.t. $d_2$ when $d_1 = \bar{d}_1$, by (5), we have
\[
\frac{\partial \tilde{g}(r)}{\partial d_2} = r\lambda \left[ \eta F(d_2) - 1 + r \int_{d_2}^{N} (1 - F(y))dy + rd_1(1 - F(d_2)) \right]
\]
\[
= r^2 \lambda \left[ \frac{(F(d_2) - 1) \int_{d_2}^{N} (1 - F(y))dy + \int_{d_2}^{N} (1 - F(y))dy}{1 - F(d_1)} \right]
\]
\[
= r^2 \lambda \left( \frac{F(d_2) - 1}{1 - F(d_1)} + 1 \right) \int_{d_2}^{N} (1 - F(y))dy.
\]

Since $d_1 \leq d_2$, we have
\[
\frac{1 - F(d_2)}{1 - F(d_1)} \leq 1,
\]
and thus
\[
r^2 \lambda \left( \frac{F(d_2) - 1}{1 - F(d_1)} + 1 \right) \int_{d_2}^{N} (1 - F(y))dy \geq 0.
\]

Then, we derive
\[
\bar{d}_2 = N.
\]
Substituting $\bar{d}_2 = N$ into (5), we can get
\[
\bar{d}_1 = \frac{\eta}{r}.
\]
Therefore, the maximizer of $\tilde{g}(r)$ w.r.t. $d_1$ and $d_2$ is
\[
(\bar{d}_1, \bar{d}_2) = \left( \frac{\eta}{r}, N \right).
\]
\[
\square
\]

Replacing $(\bar{d}_1, \bar{d}_2) = \left( \frac{\eta}{r}, N \right)$ back into (4) yields
\[
\left[ (\theta - \eta)\lambda \mu + \lambda \eta \mu \left( \frac{\eta}{r}, N \right) \right] r - \frac{1}{2} \lambda \sigma^2 \left( \frac{\eta}{r}, N \right) r^2 = 0.
\]
Since $r = \frac{\eta}{\sigma^2}$, we have a new equation for $\bar{d}_1$, i.e.,
\[
\left[ (\theta - \eta)\mu + \eta \mu (\bar{d}_1, N) \right] \bar{d}_1 - \frac{1}{2} \sigma^2 (\bar{d}_1, N) \eta = 0. \tag{6}
\]

Then we have

**Lemma 3.3.** Equation (6) has a unique positive root $\bar{d}_{1D}$.

Proof. Since the layer reinsurance with $(d_1, d_2) = (d_1, N)$ is exactly the pure excess of loss reinsurance, following from Lemma 3.2 of [18], we may prove the results. $\square$
Therefore, we have
\[(d^*_1, d^*_2) = (\bar{d}_{1D} \wedge N, N)\].
By Theorem 3.1, we can directly get the following result.

**Theorem 3.4.** Let \(\bar{d}_{1D} < N\) be the unique positive root of (6). Then the optimal retention level of model (2) to minimize the ruin probability is \((\bar{d}_{1D}, N)\), and the minimum ruin probability is
\[\psi_D(u) = e^{-R_D u},\]
where \(R_D = \frac{\eta}{\bar{d}_{1D}}\).

**Remark 1.** From Lemma 3.3, we can easily get that in the case of \(\bar{d}_{1D} = N\), the ceded function is zero, which means that no reinsurance strategy is considered.

**Remark 2.** The results in Theorem 3.4 coincide with those in [18] and [24].

4. **Optimal results for the compound Poisson risk model.** In this section, we discuss the optimal layer reinsurance strategy of an insurer with a compound Poisson risk process (1). Since it is not easy to get the explicit expression of the ruin probability in the compound Poisson case, we consider here the optimal layer reinsurance to maximize the adjustment coefficient.

Let \(R_C(d_1, d_2)\) be the adjustment coefficient of risk model (1). Then \(R_C(d_1, d_2)\) satisfies the equation
\[c(d_1, d_2)r = \lambda \left[ M_Y(d_1, d_2)(r) - 1 \right],\]
or, alternatively,
\[\left[(\theta - \eta)\lambda\mu + (1 + \eta)\lambda\mu(d_1, d_2)\right]r - \lambda \left[ M_Y(d_1, d_2)(r) - 1 \right] = 0. \tag{7}\]
Our goal is to maximize \(R_C(d_1, d_2)\), i.e., to find the optimal strategy \((d^*_1, d^*_2)\), such that
\[R_C := R_C(d^*_1, d^*_2) = \sup_{d_1, d_2} R_C(d_1, d_2).\]
Note that the left hand side of (7) is non-positive at \(r = R_C\), i.e., \(R_C\) is the solution to
\[\sup_{d_1, d_2} \left\{ \left[(\theta - \eta)\lambda\mu + (1 + \eta)\lambda\mu(d_1, d_2)\right]r - \lambda \left[ M_Y(d_1, d_2)(r) - 1 \right] \right\} = 0, \tag{8}\]
or, equivalently,
\[- r\lambda \left[ \int_0^{d_1} (1 - F(y))e^{ry} dy + \int_{d_2}^N (1 - F(y))e^{(y+d_1-d_2)r} dy \right] = 0.\]
Let
\[g(d_1, d_2) = \left[(\theta - \eta)\lambda\mu + (1 + \eta)\lambda\mu(d_1, d_2)\right]r - r\lambda \left[ \int_0^{d_1} (1 - F(y))e^{ry} dy + \int_{d_2}^N (1 - F(y))e^{(y+d_1-d_2)r} dy \right].\]
Then we have
Lemma 4.1. The maximizer of $g(d_1, d_2)$ w.r.t. $d_1$ and $d_2$ is

$$(\bar{d}_1, \bar{d}_2) = \left( \frac{\ln(1 + \eta)}{r}, N \right).$$

Proof. Differentiating $g(d_1, d_2)$ with respect to $d_1$ yields

$$\frac{\partial g(d_1, d_2)}{\partial d_1} = r\lambda[(1 + \eta)(1 - F(d_1)) - (1 - F(d_1))e^{rd_1} - r \int_{d_2}^{N} (1 - F(y))e^{(y + \bar{d}_2 - d_2)r} dy],$$

which means that the maximizer $\bar{d}_1$ satisfies the following equation

$$(1 + \eta) - e^{rd_1} (1 - F(d_1)) = r \int_{d_2}^{N} (1 - F(y))e^{(y + \bar{d}_2 - d_2)r} dy \quad (9)$$

for any fixed $d_2$.

Then differentiating $g(\bar{d}_1, d_2)$ w.r.t. $d_2$, by (9), we have

$$\frac{\partial g(\bar{d}_1, d_2)}{\partial d_2} = -r\lambda(1 + \eta)(1 - F(d_2)) - r\lambda \left( (F(d_2) - 1)e^{rd_1} - r \int_{d_2}^{N} (1 - F(y))e^{(y + \bar{d}_2 - d_2)r} dy \right),$$

$$= r^2\lambda \left( \frac{(F(d_2) - 1) \int_{d_2}^{N} (1 - F(y))e^{(y + \bar{d}_2 - d_2)r} dy + \int_{d_2}^{N} (1 - F(y))e^{(y + \bar{d}_2 - d_2)r} dy} {1 - F(d_1)} \right),$$

$$= r^2\lambda \left( \frac{F(d_2) - 1}{1 - F(d_1)} + 1 \right) \int_{d_2}^{N} (1 - F(y))e^{(y + \bar{d}_2 - d_2)r} dy.$$

By the same manner as in Section 3, we can see that the inequality

$$r^2\lambda \left( \frac{F(d_2) - 1}{1 - F(d_1)} + 1 \right) \int_{d_2}^{N} (1 - F(y))e^{(y + \bar{d}_2 - d_2)r} dy \geq 0$$

holds for any $d_2 \geq d_1$, and thus

$$\bar{d}_2 = N.$$

Substituting $\bar{d}_2 = N$ into (9), we can derive

$$\bar{d}_1 = \frac{\ln(1 + \eta)}{r}.$$

Therefore, the maximizer of $g(d_1, d_2)$ w.r.t. $d_1$ and $d_2$ is

$$(\bar{d}_1, \bar{d}_2) = \left( \frac{\ln(1 + \eta)}{r}, N \right).$$

Replacing $(\bar{d}_1, \bar{d}_2) = \left( \frac{\ln(1 + \eta)}{r}, N \right)$ back into (8) yields

$$(\theta - \eta)\mu + (1 + \eta) \int_{0}^{d_1} (1 - F(y))dy - \int_{0}^{\bar{d}_1} (1 - F(y))(1 + \eta)\frac{1}{\bar{d}_1} dy = 0, \quad (10)$$

which is exactly the same equation as (3.10) of [18]. Then following from Lemma 3.4 of [18], we have

Lemma 4.2. Equation (10) has a unique positive root $\bar{d}_1 C$. 
Therefore, we obtain the optimal retention level
\[ (d_1^*, d_2^*) = (\bar{d}_{1C} \wedge N, N), \]
and thus we have

**Theorem 4.3.** Let \( \bar{d}_{1C} < N \) be the unique positive root of (10). Then the optimal retention level of model (1) to maximize the adjustment coefficient is \((\bar{d}_{1C}, N)\), and the maximal adjustment coefficient is
\[ R_C = \frac{\ln(1 + \eta)}{\bar{d}_{1C}}. \] (11)

By the standard martingale approach as in [10], we can directly derive

**Theorem 4.4.** Let \( \bar{d}_{1C} < N \) be the unique positive root of (10), and \( R_C \) be the maximal adjustment coefficient of model (1). Then the process \( \{e^{-R_C X_1^{d_{1C} \wedge N}}\} \) is a martingale and
\[ \psi^{\bar{d}_{1C}, N}(u) \leq e^{-R_C u} \leq e^{-R_C (d_1, d_2)u}, \] (12)
where \((d_1, d_2)\) is an arbitrary layer reinsurance strategy.

**Remark 3.** From Theorem 4.2, we obtain an exact analogue of the estimate for the ruin probability of model (1), i.e., an exponential inequality:
\[ \psi^{\bar{d}_{1C}, N}(u) \leq e^{-R_C u}. \] Since the exponent \( R_C \) is the largest, we get the smallest upper bound on the ruin probability of compound Poisson risk model.

**Remark 4.** Comparing the optimal results in this paper with those in [18], we can see that, under the criterion of minimizing the ruin probability or maximizing the adjustment coefficient and with the expected value premium principle, the optimal layer reinsurance strategy or the optimal combining quota-share and excess of loss reinsurance is exactly the pure excess of loss reinsurance. However, when we extend the problem to the case with variance premium principle, we will find that the optimal layer reinsurance is not necessary the pure excess of loss reinsurance (see Section 5.2).

5. **Numerical example.** In this section, by some numerical examples, we will show that the optimal layer reinsurance strategy with expected value principle is indeed the pure excess of loss reinsurance. However, for the case with variance premium principle, the optimal layer reinsurance is not necessary the pure excess of loss reinsurance.

5.1. **Expected value principle.** In this subsection, we assume that the claim sizes \( Y_i \) has a uniform distribution on the interval \([0, 4]\), which is denoted by \( Y_i \sim U[0, 4] \). Then \( \mu = 2 \). We first give the following results

**Lemma 5.1.** Assume that the claim sizes \( Y_i \sim U[0, 4] \). Let \( \bar{d}_{1D} \) and \( \bar{d}_{1C} \) be the unique positive solution of equations (6) and (10), respectively, then the optimal retention levels of models (2) and (1) are
\[ (d_1^*, d_2^*) = (\bar{d}_{1D} \wedge 4, 4) \]
and
\[ (d_1^*, d_2^*) = (\bar{d}_{1C} \wedge 4, 4), \]
respectively, where
\[ \bar{d}_{1D} = \frac{6\eta - 2\sqrt{12\eta \theta - 3\eta^2}}{\eta}, \]
\[ \bar{d}_{1C} = \frac{6\eta - 2\sqrt{12\eta \theta - 3\eta^2}}{\eta}, \]
and

\[ \bar{d}_{1C} = \frac{8(1 + \eta) \ln(1 + \eta) - 8\eta - 8 \ln(1 + \eta) \cdot M}{2(1 + \eta)(\ln(1 + \eta))^2 - 2(1 + \eta) \ln(1 + \eta) + 2\eta) }, \]

where,

\[ M = \sqrt{(1 + \eta)(1 + \theta)(\ln(1 + \eta))^2 - 2\theta(1 + \eta) \ln(1 + \eta) - \eta^2 + 2\eta\theta}. \]

Lemma 5.2. Assume that the claim sizes \( Y_i \sim U[0, 4] \). Then

i) When \( \bar{d}_{1D} \leq 4 \), the maximal adjustment coefficient of model (2) is

\[ R_D = \frac{\eta^2}{6\eta - 2\sqrt{12\eta\theta - 3\eta^2}}. \]

ii) When \( \bar{d}_{1C} \leq 4 \), the maximal adjustment coefficient of model (1) is

\[ R_C = \frac{2 \ln(1 + \eta)((1 + \eta)(\ln(1 + \eta))^2 - 2(1 + \eta) \ln(1 + \eta) + 2\eta)}{8(1 + \eta) \ln(1 + \eta) - 8\eta - 8 \ln(1 + \eta) \cdot M}. \]

Example 1. Let \( \theta = 0.4, \eta = 0.5 \), we consider the ruin probability in the diffusion approximation case and the upper bound of the ruin probability in the compound Poisson case. The results are shown in Figures 1, 2, 3 and 4.

![Figure 1](image_url). The effect of \( u \) on the \( \psi_{d_1,d_2}^D(u) \) with \( d_1 = \bar{d}_{1D} \).
From Figure 1, we can see that when \( d_1 = \bar{d}_{1D} = 0.8620 \), the ruin probability attains its minimum at \( d_2^* = 4 \). Moreover, a higher retention level \( d_2 \) yields a lower ruin probability. From Figure 2, we can see that when \( d_2 = d_2^* = 4 \), the ruin probability attains its minimum at \( \bar{d}_{1D} = 0.8620 \). Therefore, we can conclude that the ruin probability of model (2) indeed attains its minimum at \( (\bar{d}_{1D}, 4) \), which is the natural consequence of Theorem 3.4.
From Figures 3 and 4, we can see the same results as those in Figures 1 and 2. That is, when \( d_1 = \bar{d}_1C = 0.8053 \), the upper bound attains its minimum at \( d_1^* = 4 \). Moreover, a higher retention level \( d_2 \) yields a lower upper bound. When \( d_2 = \bar{d}_2C = 0.8053 \), the upper bound attains its minimum at \( d_1 = d_1^* = \bar{d}_1C = 0.8053 \).

Therefore, we can conclude that the upper bound of the ruin probability of model (1) indeed attains its minimum at \( (\bar{d}_1C, 4) \), which is the natural consequence of Theorem 4.4.

5.2. Variance premium principle. In this subsection, we may extend our results to the case with variance premium principle. Under the variance premium principle with layer reinsurance treaty, the premium income rate becomes:

\[
\tilde{c}(d_1, d_2) = \lambda \mu + \theta_1 \lambda \sigma^2 - \lambda(\mu - E(Y_i(d_1, d_2))) - \theta_2 \lambda E(Y_i - Y_i(d_1, d_2))^2 \\
= \lambda \theta_1 \sigma^2 + \lambda \mu(d_1, d_2) - \lambda \theta_2 [\sigma^2 + \sigma^2(d_1, d_2) - \beta(d_1, d_2)] \\
= \lambda(\theta_1 - \theta_2) \sigma^2 + \lambda \mu(d_1, d_2) - \lambda \theta_2 \sigma^2(d_1, d_2) + \lambda \theta_2 \beta(d_1, d_2),
\]

where \( \theta_1 \) and \( \theta_2 \) denote the safety loading of the insurer and the reinsurer, respectively, and

\[
\beta(d_1, d_2) = 2E[Y_i(d_1, d_2)Y_i]
\]

\[
= 2\left[ \int_0^{d_1} y^2 dF(y) + \int_{d_1}^{d_2} ydF(y) + \int_N^{d_2} y(y - d_2 + d_1)dF(y) \right]
\]

\[
= 4 \int_0^{d_1} y(1 - F(y))dy + 2d_1 \int_{d_1}^{d_2} (1 - F(y))dy \\
+ 2 \int_N^{d_2} (2y - d_2 + d_1)(1 - F(y))dy.
\]

Without loss of generality, we assume that \( \theta_2 > \theta_1 \).

Thus, the surplus process is given by

\[
X_i^{d_1, d_2} = u + \tilde{c}(d_1, d_2)t - S_1(t).
\]
By the same manner as in Section 3, we can get the diffusion approximation risk model of (13) as follows:
\[ \hat{X}_t^{d_1,d_2} = u + [\lambda (\theta_1 - \theta_2) \sigma^2 - \lambda \theta_2 \sigma^2 (d_1, d_2) + \lambda \theta_2 \beta (d_1, d_2)] t + \sqrt{\lambda} \sigma (d_1, d_2) W_t. \quad (14) \]

Let
\[ g_1(r, d_1, d_2) = r [\lambda (\theta_1 - \theta_2) \sigma^2 - \lambda \theta_2 \sigma^2 (d_1, d_2) + \lambda \theta_2 \beta (d_1, d_2)] - \frac{1}{2} r^2 \lambda \sigma^2 (d_1, d_2). \]

For the diffusion approximation risk model (14), finding \((d_1^*, d_2^*)\) to minimize the ruin probability is equivalent to finding \((d_1^*, d_2^*)\) to maximize the adjustment coefficient. Here the maximal adjustment coefficient \(R_D\) is the solution to
\[ \sup_{d_1, d_2} g_1(r, d_1, d_2) = 0. \]

Now, we assume that the claim sizes \(Y_i \sim U[0, 1]\), then \(\mu = \frac{1}{2}\), \(\sigma = \frac{1}{\sqrt{2}}, f(y) = 1\) and \(F(y) = y\) for \(0 \leq y \leq 1\). Let \(\theta_1 = 0.3\), \(\theta_2 = 0.4\), \(\lambda = 1\).

By solving the equations
\[
\begin{align*}
\frac{\partial g_1(d_1, d_2)}{\partial d_1} &= 2 \theta_2 \int_{d_1}^{d_2} (1 - F(x)) dx - r d_1 (1 - F(d_1)) - r \int_{d_1}^{d_2} (1 - F(x)) dx = 0, \\
\frac{\partial g_1(d_1, d_2)}{\partial d_2} &= 2 (d_1 - d_2) \theta_2 (1 - F(d_2)) + r \int_{d_2}^{d_1} (1 - F(x)) dx + r d_1 (1 - F(d_1)) = 0,
\end{align*}
\]

we may obtain one of the solutions \((d_1, d_2) = (0.0423, 1)\) and the corresponding adjustment coefficient \(R_D(d_1, d_2) = 9.0648\). Note that the layer reinsurance with \((d_1, d_2) = (0.0423, 1)\) is a pure excess of loss reinsurance.

On the other hand, we have
\[
\begin{align*}
\frac{\partial^2 g_1(d_1, d_2)}{\partial d_1^2} &= r \lambda [-2 \theta_2 (1 - F(d_1)) - r (1 - F(d_1)) + r d_1 f(d_1)], \\
\frac{\partial^2 g_1(d_1, d_2)}{\partial d_1 \partial d_2} &= r \lambda [2 \theta_2 (1 - F(d_2)) + r (1 - F(d_2))], \\
\frac{\partial^2 g_1(d_1, d_2)}{\partial d_2^2} &= r \lambda [-2 \theta_2 d_1 f(d_2) + 2 \theta_2 d_2 f(d_2) - d_1 f(d_2)],
\end{align*}
\]

and thus
\[
\begin{align*}
\frac{\partial^2 g(d_1, d_2)}{\partial d_1^2} \bigg|_{(d_1, d_2) = (0.0423, 1)} &= -82.1641, \\
\frac{\partial^2 g(d_1, d_2)}{\partial d_1 \partial d_2} \bigg|_{(d_1, d_2) = (0.0423, 1)} &= 0, \\
\frac{\partial^2 g(d_1, d_2)}{\partial d_2^2} \bigg|_{(d_1, d_2) = (0.0423, 1)} &= 3.4693.
\end{align*}
\]

Since
\[
\begin{vmatrix}
\frac{\partial^2 g(d_1, d_2)}{\partial d_1 \partial d_2} & \frac{\partial^2 g(d_1, d_2)}{\partial d_1^2} \\
\frac{\partial^2 g(d_1, d_2)}{\partial d_1 \partial d_2} & \frac{\partial^2 g(d_1, d_2)}{\partial d_2^2}
\end{vmatrix}
= \begin{vmatrix}
-82.1641 & 0 \\
0 & 3.4693
\end{vmatrix} < 0,
\]

we prove that \((d_1, d_2) = (0.0423, 1)\) is not the maximizer of the function \(g_1(r, d_1, d_2)\) w.r.t. \(d_1\) and \(d_2\). Therefore, the optimal layer reinsurance with variance premium principle is not the pure excess of loss reinsurance any longer in this case. This result is shown in the following Figure 5.
Figure 5. the effect of $u$ on the ruin probability $\psi_{d_1,d_2}(u)$

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