Monogamy relations and upper bounds for the generalized W-class states using Rényi-\(\alpha\) entropy

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We investigate monogamy relations and upper bounds for generalized W-class states related to the Rényi-\(\alpha\) entropy. First, we present an analytical formula on Rényi-\(\alpha\) entanglement (RaE) and Rényi-\(\alpha\) entanglement of assistance (REoA) of a reduced density matrix for a generalized W-class states. According to the analytical formula, we show monogamy and polygamy relations for generalized W-class states in terms of RoE and REoA. Then we give the upper bounds for generalized W-class states in terms of RoE. Next, we provide tighter monogamy relations for generalized W-class states in terms of concurrence and convex-roof extended negativity and obtain the monogamy relations for RoE by the analytical expression between RoE and concurrence. Finally, we apply our results into quantum games and present a new bound of the nonclassicality of quantum games restricting to generalized W-class states.

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I. INTRODUCTION

In a multipartite quantum system, if a pair of parties share maximal entanglement, according to its restricted sharability, they can share neither entanglement [1][2] and nor classical correlations [3] with the rest. This is known as monogamy of entanglement (MoE) [4]. Since MoE can quantify how much information an eavesdropper could potentially obtain about the secret key to be extracted, MoE is a key ingredient to make quantum cryptography secure [5][6].

The first mathematical characterization of MoE [4] was shown by Coffman, Kundu and Wootters using squared concurrence [7]. It is known as CKW-inequality. Later the CKW-type inequality was shown for arbitrary multiqubit systems [2]. The MoE of squared concurrence can be used to characterize the entanglement structure in multipartite quantum systems and detect the existence of multiqubit entanglement in dynamical procedures [8][11]. Furthermore, there are also many works devoted to the topic of entanglement monogamy [12][15].

However, monogamy relations using concurrence is known to fail in the generalization of CKW inequality for higher dimensional quantum systems [16]. In three-qubit quantum system, it is well-known that there exists two inequivalent classes of genuine tripartite entangled states. The first is the Greenberger-Horne-Zeilinger (GHZ) class [17], the other one is the W-class [18]. The conversion of the states in a same class can be achieved by local operation and classical communication with non-zero probability. CKW inequality is saturated by W-class states and it becomes the most strict inequality with the states in GHZ class [19]. The saturation of the inequality implies a genuine tripartite entanglement could have a complete characterization by the bipartite ones inside it. So in this paper, we are interested in the monogamy relations of the n-qubit generalized W-class states proposed in [19].

Moreover in [19], Kim and Sanders showed that the entanglement of the n-qubit generalized W-class states is fully characterized by their partial entanglements using squared concurrence. In 2014, Kim considered a large class of multi-qubit generalized W-class states, and analytically showed the strong monogamy inequality of multi-qubit entanglement is saturated by this class of states [20]. In 2015, Choi and Kim provided an analytical proof that strong monogamy inequality of squared convex-roof extended negativity is saturated by a large class of multi-qudit states; a superposition of multi-qudit generalized W-class states and vacuums [21]. In 2016, Kim showed some useful properties for a large class of multi-qudit mixed state that are in a partially coherent superposition of a generalized W-class state and the vacuum [22].

Rényi-\(\alpha\) entanglement (RaE) [23] is a well-defined entanglement measure which is the generalization of entanglement of formation (EOF) and has the merits for characterizing quantum phases with differing computational power [24], ground state properties in many-body systems [25], and topologically ordered states [26][27]. Although EoF is known to fail for usual CKW-type characterization of MoE, Rényi-\(\alpha\) entropy can still be shown to have CKW-type monogamy inequality for all case of \(\alpha\) if it exceeds a certain threshold [28]. Kim proved monogamy of entanglement in multi-qubit systems for \(\alpha \geq 2\) using Rényi-\(\alpha\) entanglement to quantify bipartite entanglement [28]. Wei et al presented the squared Rényi-\(\alpha\) entanglement (SRaE) obeys a general monogamy inequality in an arbitrary N-qubit mixed state. In 2016, Wei et al presented lower and upper bounds for Rényi-\(\alpha\) entanglement [30].

Apart from the entertainment value, games among multiplayers often provide an intuitive way to understand...
complex problems. In Ref. [31], Marco et al. investigated the probability that both players in a quantum game simultaneously succeed in guessing the outcome correctly. Their results implies the optimal guessing probability can be achieved without the use of entanglement. In Ref. [32, 38], the authors presented bounds on the difference between multiplayer quantum games and classical games using the monogamy of Tsallis-\(q\) entropy and squashed entanglement, respectively.

This paper is organized as follows. In Sec.II, we give the preliminary knowledge needed in this paper. In Sec.III A, we show monogamy and polygamy relations for \(n\)-qubit GW states using RoE and REoA. We generalize these relations into \(\mu\)-th power of RoE when \(\mu \geq 2\) and REoA when \(0 < \mu \leq 1\). In Sec.III B, we present the upper bounds for \(n\)-qubit GW states in terms of RoE. In Sec.III C, we obtain tighter monogamy relations using concurrence and CREN. We also analysis the general monogamy relations for \(n\)-qubit GW states using RoE according to the analytical expression between the RoE and concurrence. In Sec.IV, we provide a new bound on the difference between multiplayer quantum games and classical games restricting to the \(n\)-qubit GW states using monogamy of RoE. In Sec.V, we end with a conclusion.

II. PRELIMINARY KNOWLEDGE

For a bipartite pure state \(|\psi\rangle_{AB} = \sum_i \sqrt{\alpha_i}|i i\rangle\), the concurrence \(C(|\psi\rangle_{AB})\) is defined as [33]

\[
C(|\psi\rangle_{AB}) = \sqrt{2[1 - \text{Tr}(\rho_A^2)]},
\]

where \(\rho_A = \text{Tr}_B(|\psi\rangle_{AB}\langle\psi|)\) (and analogously for \(\rho_B\)).

For any mixed state \(\rho_{AB}\), the concurrence is given via the so-called convex roof extension

\[
C(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle),
\]

where the minimum is taken over all possible pure decompositions of \(\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB}\langle\psi_i|\).

As the duality of concurrence, the concurrence of assistance (CoA) of any mixed state \(\rho_{AB}\) is defined as [34]

\[
C_a(\rho_{AB}) = \max_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle),
\]

where the maximum is taken over all possible pure state decompositions \(\{p_i, |\psi_i\rangle\}\) of \(\rho_{AB}\).

A well-known quantification of bipartite entanglement is negativity [35], which is based on the positive partial transposition (PPT) criterion. For a bipartite state \(\rho_{AB}\) in a \(d \otimes d\) (\(d \leq d\)) quantum system, its negativity is defined as

\[
N(\rho_{AB}) = \|\rho_{AB}^{T_A}\| - 1,
\]

where \(\rho_{AB}^{T_A}\) is the partial transpose with respect to the subsystem \(A\) and \(\|X\|\) denotes the trace norm of \(X\), \(\|X\| = \text{Tr}\sqrt{XX^\dagger}\).

To overcome the lack of separability criterion, one modification of negativity is convex-roof extended negativity (CREN), which gives a perfect discrimination of PPT bound entangled states and separable states in any bipartite quantum system [35]. For a bipartite mixed state \(\rho_{AB}\), CREN is defined as

\[
\tilde{N}(\rho_{AB}) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i N(|\psi_i\rangle),
\]

where the minimum is taken over all possible pure state decompositions \(\{p_i, |\psi_i\rangle\}\) of \(\rho_{AB}\).

Similar to the duality between concurrence and CoA, we can also define a dual to CREN, namely CRENoA, by taking the maximum value of average negativity over all possible pure state decomposition [16], i.e.

\[
\tilde{N}_a(\rho_{AB}) = \max_{\{p_i, |\psi_i\rangle\}} \sum_i p_i N(|\psi_i\rangle),
\]

where the maximum is taken over all possible pure state decompositions \(\{p_i, |\psi_i\rangle\}\) of \(\rho_{AB}\).
Another well-known quantification of bipartite entanglement is Rényi-α entanglement (RoE) [28]. For a bipartite pure state $|\psi\rangle_{AB} = \sum_i \sqrt{\lambda_i} |ii\rangle$, the RoE is defined as

$$E_\alpha(|\psi\rangle_{AB}) = S_\alpha(\rho_A) = \frac{1}{1-\alpha} \log_2(\text{tr}\rho_A^\alpha),$$

(7)

where the Rényi-α entropy is $S_\alpha(\rho_A) = [\log_2(\sum_i \lambda_i^\alpha)]/(1 - \alpha)$ with $\alpha$ being a nonnegative real number and $\lambda_i$ being the eigenvalue of reduced density matrix $\rho_A$. The Rényi-α entropy $S_\alpha(\rho)$ converges to the von Neumann entropy when the order $\alpha$ tends to 1.

For a bipartite mixed state $\rho_{AB}$, the RoE is defined via the convex-roof extension

$$E_\alpha(\rho_{AB}) = \min \sum_i p_i E_\alpha(|\psi_i\rangle_{AB}),$$

(8)

where the minimum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i|$. As a dual concept to Rényi-α entanglement, we define the Rényi-α entanglement of assistance (REoA) as [28]

$$E_\alpha(\rho_{AB}) = \max \sum_i p_i E_\alpha(|\psi_i\rangle_{AB}),$$

(9)

where the maximum is taken over all possible pure state decompositions of $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB} \langle \psi_i|$. In particular, for any two-qubit pure state with its Schmidt decomposition $|\psi\rangle_{AB} = \sqrt{\lambda_0} |00\rangle_{AB} + \sqrt{\lambda_1} |11\rangle_{AB}$, then we have $C^2(|\psi\rangle_{AB}) = 4\lambda_0\lambda_1$, and

$$E_\alpha(|\psi\rangle_{AB}) = S_\alpha(\rho_A) = \frac{1}{1-\alpha} \log (\lambda_0^\alpha + \lambda_1^\alpha).$$

(10)

from the above equalities [1] and (7). Then we have an analytical expression between the Rényi-α entanglement and concurrence for any two-qubit pure state [28 29 37]

$$E_\alpha(|\psi\rangle_{AB}) = f_\alpha \left[ C^2(|\psi\rangle_{AB}) \right],$$

(11)

where the order $\alpha \geq (\sqrt{7} - 1)/2$ and the function $f_\alpha(x)$ has the form

$$f_\alpha(x) = \frac{1}{1-\alpha} \log_2 \left[ \left( \frac{1 - \sqrt{1 - x}}{2} \right)^\alpha + \left( \frac{1 + \sqrt{1 - x}}{2} \right)^\alpha \right].$$

(12)

Now we present several lemmas on the properties of the function $f_\alpha(x)$ in equality [12].

**Lemma 1 [22]** The function $f_\alpha^\alpha(x)$ with $\alpha \geq (\sqrt{7} - 1)/2$ is monotonically increasing and convex.

**Lemma 2 [22]** The function $f_\alpha(x)$ is monotonically increasing and concave for $\alpha \in [(\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2]$. Set $y = x^2$, $g_\alpha(y) = f_\alpha(x^2)$.

**Lemma 3 [22 37]** The function $g_\alpha(y)$ is a monotonically increasing and convex function for $0 \leq y \leq 1$, and $\alpha \geq (\sqrt{7} - 1)/2$.

Next let us recall a class of multi-qubit generalized W-class (GW) states [19 21]

$$|\psi\rangle_{A_1A_2...A_n} = a_1|10\cdots0\rangle + a_2|01\cdots0\rangle + ... + a_n|00\cdots1\rangle$$

(13)

$$|W^d_n\rangle_{A_1A_2...A_n} = \sum_{i=1}^{d-1} (a_{1i}|i0\cdots0\rangle + a_{2i}|0i\cdots0\rangle + ... + a_{ni}|00\cdots0i\rangle),$$

(14)

with the normalization condition $\sum_{i=1}^{n}|a_{ij}|^2 = 1$ and $\sum_{i=1}^{n} \sum_{j=1}^{d-1} |a_{ij}|^2 = 1$ respectively.

The state in Eq. (14) is a coherent superposition of all $n$-qudit product states with Hamming weight one. Eq. (14) includes $n$-qudit W-class states in Eq. (13) as a special case when $d = 2$.

Next thing we need to do is to present some lemmas for the multi-qubit generalized W-class states which are useful in the proof of our main results.
Lemma 4 [21] Let $|\psi\rangle_{A_1\cdots A_n}$ be a $n$-qudit pure state in a superposition of a $n$-qudit generalized W-class state in Eq. (14) and vacuum, that is,

$$\rho |\psi\rangle_{A_1\cdots A_n} = \sqrt{p} |W_n^d\rangle_{A_1\cdots A_n} + \sqrt{1-p} |0\cdots 0\rangle_{A_1\cdots A_n}$$

(15)

for $0 \leq p \leq 1$. Let $\rho_{A_1A_j\cdots A_{j-1}}$ be a reduced density matrix of $|\psi\rangle_{A_1\cdots A_n}$ onto $m$-qudit subsystems $A_1A_j\cdots A_{j-1}$ with $2 \leq m \leq n-1$. For any pure state decomposition of $\rho_{A_1A_j\cdots A_{j-1}}$ such that

$$\rho_{A_1A_j\cdots A_{j-1}} = \sum_k q_k |\phi_k\rangle_{A_1A_j\cdots A_{j-1}} \langle \phi_k|,$$

(16)

$|\phi_k\rangle_{A_1A_j\cdots A_{j-1}}$ is a superposition of a $m$-qudit generalized W-class state and vacuum.

Lemma 5 [19] For any $n$-qudit W-class states $|\psi\rangle_{A_1\cdots A_n}$ and a partition $P = \{P_1, \ldots, P_m\}$ of the set of subsystems $S = \{A, B_1, \ldots, B_{n-1}\}, m \leq n$

$$C^2_{P_s(P_1\cdots P_s\cdots P_m)} = \sum_{k \neq s} C^2_{P_s P_k} = \sum_{k \neq s} (C_{P_s P_k})^2,$$

(17)

and

$$C_{P_s P_k} = (C_{P_s P_k})^2,$$

(18)

for all $k \neq s$ and $(P_1\cdots P_s\cdots P_m) = (P_1\cdots P_s\cdots P_m) - (P_s)$.

Lemma 6 [22] Let $|\psi\rangle_{A,B_1\cdots B_{n-1}}$ be a $n$-qudit pure state in a superposition of a $n$-qudit generalized W-class state in Eq. (14) and vacuum, then for any partition $P = \{P_1, \ldots, P_m\}$ of the set of subsystems $S = \{A, B_1, \ldots, B_{n-1}\}, m \leq n$, the state $|\psi\rangle_{P_1\cdots P_m}$ is also a superposition of a $n$-qudit generalized W-class state in Eq. (14) and vacuum. Here $P_s \cap P_t = \emptyset$ for $s \neq t$, and $\bigcup_s P_s = S$.

At last, we have one more lemma which is used in the last part of our main results.

Lemma 7 For real numbers $t \in [0,1], x \geq k \geq 1$, we have

$$(1+x)^t \geq 1 + \frac{(1+k)^x - 1}{k^x} x^t.$$  

(19)

[Proof] Consider the function $f_t(x) = \frac{(1+x)^t - 1}{x^t}$. Since

$$\frac{df_t(x)}{dx} = tx^{-(t+1)}[1 - (1+x)^{t-1}] \geq 0,$$

for $t \in [0,1]$ and $x \geq 1$.

In other words, the function $f_t(x)$ is an increasing function with $x \geq 1$. Since $x \geq k \geq 1$, then $f_t(x) \geq f_t(k)$. □

### III. MAIN RESULTS

In this section, we give the main results of this paper. In Sec. III A we show monogamy and polygamy relations for RoE and REoA of GW states, and generalize them into the $\mu$-th power of RoE for $\mu \geq 2$ and $\mu$-th power of REoA for $0 < \mu \leq 1$. In Sec. III B we investigate the upper bounds for GW states using RoE. In Sec. III C we present tighter monogamy relations in terms of concurrence and CREN. We also get the general monogamy relations for RoE using the analytical expression between the RoE and concurrence.

#### A. Monogamy and polygamy relations using Rényi entropy for generalized W-class states

For a pure GW state, we have the following theorem.
Theorem 1 Assume ρ_{A_1,\cdots,A_m} is a reduced density matrix of a pure GW state, then we have

\[ E_\alpha(\rho_{A_1,\cdots,A_m}) = f_\alpha(C^2(\rho_{A_1,\cdots,A_m})), \]

when \( \alpha \geq (\sqrt{7} - 1)/2 \).

The proof is similar to the proof of Theorem 1 in Ref. [28] and Lemma 1 is also needed in the proof. Next we give an analytic formula of REoA for a GW state.

Theorem 2 Assume ρ_{A_1,\cdots,A_m} is a reduced density matrix of a pure GW state, then we have

\[ E_\alpha^a(\rho_{A_1,\cdots,A_m}) = f_\alpha(C^2(\rho_{A_1,\cdots,A_m})), \]

when \( \alpha \in [(\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2] \).

[Proof] For convenience, we denote \( \rho_{A_1,\cdots,A_m} \) as \( \rho_{AB} \). From [19, 20, 21], if \( \rho_{AB} \) is a reduced density matrix of a pure GW state, then we have \( C(\rho_{AB}) = C_\alpha(\rho_{AB}) \). So it is enough for us to show \( E_\alpha^a(\rho_{AB}) = f_\alpha(C^2(\rho_{AB})) \). First we prove \( E_\alpha^a(\rho_{AB}) \leq f_\alpha(C^2(\rho_{AB})) \). Assume \( \{\rho_i, |\psi_i\rangle\} \) is the optimal decomposition for REoA of \( \rho_{AB} \), then we have

\[ E_\alpha^a(\rho_{AB}) = \sum_i p_i E_\alpha(|\psi_i\rangle_{AB}) \]
\[ \leq \sum_i p_i f_\alpha(C^2(|\psi_i\rangle_{AB})) \]
\[ \leq f_\alpha(\sum_i p_i C^2(|\psi_i\rangle_{AB})) \]
\[ \leq f_\alpha(C^2(\rho_{AB})), \]

where the first inequality is due to the concave property of \( f_\alpha(x) \) for \( \alpha \in [(\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2] \) in Lemma 2 and the second inequality is due to the definition of \( C^2(\rho_{AB}) \) and the increasing property of \( f_\alpha(x) \) in Lemma 2.

Next we show \( E_\alpha^a(\rho_{AB}) \geq f_\alpha(C^2(\rho_{AB})) \). Set \( y = x^2 \), \( g_\alpha(y) = f_\alpha(x^2) \). Assume \( \{r_k, |\theta_k\rangle\} \) is the optimal decomposition for \( C_\alpha(\rho_{AB}) \). Then we have

\[ g_\alpha(C_\alpha(\rho_{AB})) = g_\alpha(\sum_k r_k C(|\theta_k\rangle_{AB})) \]
\[ \leq \sum_k r_k g_\alpha(C(|\theta_k\rangle_{AB})) \]
\[ = \sum_k r_k E_\alpha(|\theta_k\rangle_{AB}) \]
\[ \leq E_\alpha^a(\rho_{AB}), \]

where in the first inequality we have used the convex property of \( g_\alpha(y) \) for \( \alpha \geq (\sqrt{7} - 1)/2 \) in Lemma 3. The second inequality is due to the definition of \( E_\alpha^a(\rho_{AB}) \). Since \( y = x^2 \) and let \( x = C_\alpha(\rho_{AB}) \), then we have \( E_\alpha^a(\rho_{AB}) \geq f_\alpha(C^2(\rho_{AB})) \).

Thus combining (22) and (23), we have \( E_\alpha^a(\rho_{AB}) = f_\alpha(C^2(\rho_{AB})) = f_\alpha(C^2(\rho_{AB})) \) which completes the proof.

According to Theorem 1 and Theorem 2, we have the following Theorem 3.

Theorem 3 Assume ρ_{A_1,\cdots,A_m} is a reduced density matrix of a pure GW state, then we have

\[ E_\alpha^a(\rho_{A_1,\cdots,A_m}) = E_\alpha(\rho_{A_1,\cdots,A_m}) = f_\alpha(C^2(\rho_{A_1,\cdots,A_m})), \]

when \( \alpha \in [(\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2] \).

Now we begin to investigate the monogamy relation using Rényi entropy for generalized W-class states.

Theorem 4 Assume ρ_{A_1,\cdots,A_m} is the reduced density matrix of a GW state \( |\psi\rangle_{A_1,\cdots,A_n} \), and here we denote \( \{P_1, P_2, \cdots, P_k\} \) is a partition of the set \( \{A_{i_1}, A_{i_2}, \cdots, A_{i_m}\} \), when \( \alpha \geq (\sqrt{7} - 1)/2 \), we have the following monogamy inequality

\[ E_\alpha^2(\rho_{P_1,\cdots,P_k}) \geq \sum_{i=2}^k E_\alpha^2(\rho_{P_i}). \]
[proof] For \( \alpha \geq (\sqrt{7} - 1)/2 \), we have
\[
E_\alpha^2(\rho_{P_1P_2\cdots P_k}) = f_\alpha(C^2(\rho_{P_1P_2\cdots P_k})) = f_\alpha(\sum_{i=2}^k C^2(\rho_{P_iP_i})) \\
\geq \sum_{i=2}^k f_\alpha(C^2(\rho_{P_iP_i})) = \sum_{i=2}^k E_\alpha^2(\rho_{P_iP_i}),
\]
where in the second equality we use Lemma 5 and the inequality is due to Lemma 1.

Naturally we want to generalize Theorem 4 into the \( \mu \)-th power of R\&E for GW states when \( \mu \geq 2 \). We find that when \( k = 3 \), we can always get \( E_\alpha^2(\rho_{P_1P_2}) \leq E_\alpha^2(\rho_{P_1P_2}) \) through designing the partition \( \{P_1, P_2, P_3\} \), then we get
\[
E_\alpha^2(\rho_{P_1P_2P_3}) \geq (E_\alpha^2(\rho_{P_1P_2}) + E_\alpha^2(\rho_{P_1P_3}))^{\frac{\mu}{2}} = E_\alpha^2(\rho_{P_1P_2})^{\frac{\mu}{2}} \geq E_\alpha^2(\rho_{P_1P_2}) + E_\alpha^2(\rho_{P_1P_3})
\]
where in the first inequality we use Theorem 4. The second inequality is obtained by \( (1 + t)^x \geq 1 + tx \) for any real number \( x \) and \( t, 0 \leq t \leq 1, x \in [1, \infty] \).

Therefore we can have the following corollary by the way of use this operation repeatedly.

**Corollary 1** Assume \( \rho_{A_{j_1}A_{j_2}\cdots A_{j_m}} \) is the reduced density matrix of a GW state \( |\psi\rangle_{A_1\cdots A_n} \), and here we denote \( \{P_1, P_2, \cdots, P_k\} \) is a partition of the set \( \{A_{j_1}, A_{j_2}, \cdots, A_{j_m}\} \), when \( \alpha \geq (\sqrt{7} - 1)/2 \), we have the following monogamy inequality,
\[
E_\alpha^2(\rho_{P_1P_2\cdots P_k}) \geq \sum_{i=2}^k E_\alpha^2(\rho_{P_iP_i}).
\]
for \( \mu \geq 2 \).

As a duality of monogamy relations, polygamy relations using REoA for GW states can also be developed.

**Theorem 5** Assume \( \rho_{A_{j_1}A_{j_2}\cdots A_{j_m}} \) is the reduced density matrix of a GW state \( |\psi\rangle_{A_1\cdots A_n} \), and here we denote \( \{P_1, P_2, \cdots, P_k\} \) is a partition of the set \( \{A_{j_1}, A_{j_2}, \cdots, A_{j_m}\} \), when \( \alpha \in [(\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2] \), we have the following polygamy inequality,
\[
E_\alpha^2(\rho_{P_1P_2\cdots P_k}) \leq \sum_{i=2}^k E_\alpha^2(\rho_{P_iP_i}).
\]
[proof] From Theorem 2 we have
\[
E_\alpha^2(\rho_{P_1P_2\cdots P_k}) = f_\alpha(C^2(\rho_{P_1P_2\cdots P_k})) = f_\alpha(\sum_{i=2}^k C^2(\rho_{P_iP_i})) \\
\leq \sum_{i=2}^k f_\alpha(C^2(\rho_{P_iP_i})) = \sum_{i=2}^k E_\alpha(\rho_{P_iP_i}),
\]
where the inequality is due to Lemma 2.

When \( 0 < \mu \leq 1 \), using \( (1 + t)^x \geq 1 + tx \) with \( 0 \leq t \leq 1, x \in [0, 1] \) and similar method in Corollary 1 we have the following corollary.
Corollary 2 Assume $\rho_{A_1A_2\ldots A_m}$ is the reduced density matrix of a GW state $|\psi\rangle_{A_1\ldots A_n}$, and here we denote $\{P_1, P_2, \ldots, P_k\}$ is a partition of the set $\{A_1, A_2, \ldots, A_m\}$, when $\alpha \in [(\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2]$ we have the following polygamy inequality,

$$ (E_{\alpha}^\mu)^{\mu}(\rho_{P_1|P_2\ldots P_k}) \leq \sum_{i=2}^{k} E_{\alpha}^\mu(\rho_{P_iP_i}). $$

(30)

for $0 < \mu \leq 1$.

The method which has been used so far can be generalized to investigate monogamy and polygamy inequalities using other entanglement measures for GW states, such as Tsallis $q$ entropy [38] and unified entropy [39, 40].

As an example, we consider the 4-qubit generalized W-class state

$$ |\psi\rangle_{A_1A_2A_3A_4} = 0.3|0001\rangle + 0.4|0010\rangle + 0.5|0100\rangle + \sqrt{0.5}|1000\rangle, $$

(31)

Here we choose $\rho_{A_1A_2A_3}$ is the reduced density matrix of $|\psi\rangle_{A_1A_2A_3A_4}$, $P_1 = A_1$, $P_2 = A_2$, $P_3 = A_3$. Then we have

$$ \rho_{A_1A_2A_3} = 0.09|000\rangle\langle 000| + |\phi\rangle\langle \phi| $$

(32)

where $|\phi\rangle = 0.4|001\rangle + 0.5|010\rangle + \sqrt{0.5}|100\rangle$. After calculation, we get $C(\rho_{P_1P_2}) = \sqrt{2}/2$, $C(\rho_{P_1P_3}) = 2\sqrt{2}/5$. Then from Theorem 1, we have

$$ E_{\alpha}(\rho_{P_1P_2}) = f_{\alpha}\left(\left(\frac{\sqrt{2}}{2}\right)^2\right), $$

$$ E_{\alpha}(\rho_{P_1P_3}) = f_{\alpha}\left(\left(\frac{2\sqrt{2}}{5}\right)^2\right), $$

Combining Theorem 3, Theorem 4 and Theorem 5, we have

$$ \sqrt{E_{\alpha}^2(\rho_{P_1P_2}) + E_{\alpha}^2(\rho_{P_1P_3})} \leq E_{\alpha}(\rho_{P_1|P_2P_3}) \leq E_{\alpha}(\rho_{P_1P_2}) + E_{\alpha}(\rho_{P_1P_3}) $$

(33)

In this way, we get the upper and lower bounds for $E_{\alpha}(\rho_{P_1|P_2P_3})$ when $\alpha \in [(\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2], \alpha \neq 1$. See Figure 1.

![Fig. 1: Solid line is the function $\sqrt{E_{\alpha}^2(\rho_{P_1P_2}) + E_{\alpha}^2(\rho_{P_1P_3})}$. Dashed blue line is the function $E_{\alpha}(\rho_{P_1P_2}) + E_{\alpha}(\rho_{P_1P_3})$.](image)

**B. Upper bound for generalized W-class states using Rényi entropy**

The concurrence is related to the linear entropy of a state [42],

$$ T(\rho) = 1 - Tr(\rho^2) $$

(34)
Given a bipartite state $\rho_{AB}$, $T(\rho)$ has the property \[ T(\rho_A) - T(\rho_B) \leq T(\rho_{AB}) \leq T(\rho_A) + T(\rho_B). \] (35)

Assume $|\psi\rangle_{PQR_1R_2\cdots R_{k-2}}$ is a GW state, from the definition of pure state concurrence together with Eq. (35), we have
\[ |C^2(|\psi\rangle_{P|QR_1R_2\cdots R_{k-2}}) - C^2(|\psi\rangle_{Q|PR_1R_2\cdots R_{k-2}})| \leq C^2(|\psi\rangle_{PQ|R_1R_2\cdots R_{k-2}}), \] (36)
\[ C^2(|\psi\rangle_{PQ|R_1R_2\cdots R_{k-2}}) \leq C^2(|\psi\rangle_{P|QR_1R_2\cdots R_{k-2}}) + C^2(|\psi\rangle_{Q|PR_1R_2\cdots R_{k-2}}). \] (37)

**Theorem 6** Assume $|\psi\rangle_{PQR_1R_2\cdots R_{k-2}}$ is a GW state, when $\alpha \in [(\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2]$, we have
\[ E_\alpha(|\psi\rangle_{P|QR_1R_2\cdots R_{k-2}}) \leq 2E_\alpha(\rho_{PP}) + \sum_{i=1}^{k-2} E_\alpha(\rho_{PR_i}) + \sum_{i=1}^{k-2} E_\alpha(\rho_{QR_i}). \] (38)

[Proof] For simplicity, we denote $R = R_1R_2\cdots R_{k-2}$.
In Lemma 2 of Ref. [13], the authors show
\[ f_\alpha(x^2 + y^2) \leq f_\alpha(x^2) + f_\alpha(y^2). \] (39)
with $\alpha \in [(\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2]$ and $0 \leq x, y \leq 1, 0 \leq x^2 + y^2 \leq 1$.
Then we have
\[ E_\alpha(|\psi\rangle_{PQ|R}) = f_\alpha(C^2(|\psi\rangle_{PQ|R})) \leq f_\alpha(C^2(|\psi\rangle_{P|QR}) + C^2(|\psi\rangle_{Q|PR})) \leq f_\alpha(C^2(|\psi\rangle_{P|QR}) + f_\alpha(C^2(|\psi\rangle_{Q|PR})) = f_\alpha(C^2(\rho_{PP}) + f_\alpha(C^2(\rho_{PR}) + f_\alpha(C^2(\rho_{QR}) \leq 2E_\alpha(\rho_{PP}) + f_\alpha(C^2(\rho_{PR}) + f_\alpha(C^2(\rho_{QR})), \) (40)
where in the second inequality we use \[37\] and the monotonically increasing property of $f_\alpha(x)$ in Lemma 2. The third inequality is due to \[39\]. The forth equality is due to Lemma 5. Using \[39\] again, we get the last inequality.

Since $R = R_1R_2\cdots R_{k-2}$, then
\[ f_\alpha(C^2(\rho_{PR})) = f_\alpha\left(\sum_{i=1}^{k-2} C^2(\rho_{PR_i})\right) \leq \sum_{i=1}^{k-2} f_\alpha(C^2(\rho_{PR_i})) = \sum_{i=1}^{k-2} E_\alpha(\rho_{PR_i}). \] (41)
where the first equality is due to Lemma 5 and the second inequality is from the iterative use of \[39\]. Similarly, we get
\[ f_\alpha(C^2(\rho_{QR})) = f_\alpha\left(\sum_{i=1}^{k-2} C^2(\rho_{QR_i})\right) \leq \sum_{i=1}^{k-2} f_\alpha(C^2(\rho_{QR_i})) = \sum_{i=1}^{k-2} E_\alpha(\rho_{QR_i}). \] (42)
Finally, combining \[40\], \[41\] and \[42\], we complete the proof. \[\square\]

**Theorem 7** Assume $\rho_{A_1A_2\cdots A_m}$ is the reduced density matrix of a GW state $|\psi_{A_1\cdots A_m}\rangle$, here we denote $\{P_1, P_2, P_3\}$ is a partition of the set $\{A_1, A_2, \cdots, A_m\}$, when $\alpha \in [(\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2]$, we have the following monogamy inequality, we have
\[ E_\alpha^a(\rho_{P_1|P_2P_3}) \leq E_\alpha^a(\rho_{P_2|P_1P_3}) + E_\alpha^a(\rho_{P_3|P_1P_2}). \] (43)
[proof] Assume $\{p_i, |\psi_i\rangle\}$ is the optimal decomposition for REoA of $\rho_{P_1|P_2P_3}$ such that $E_\alpha^a(\rho_{P_1|P_2P_3}) = \sum_i p_i E_\alpha(|\psi_i\rangle_{P_1|P_2P_3}).$
Let $E(\rho) = 2(1 - Tr(\rho^2))$. For each pure state $|\psi_i\rangle_{P_1P_2P_3}$ in this optimal decomposition with $\rho^1_{P_2P_3} = Tr_{P_1}|\psi_i\rangle_{P_1P_2P_3}\langle\psi_i|$, $\rho^2_{P_2} = Tr_{P_1P_3}|\psi_i\rangle_{P_1P_2P_3}\langle\psi_i|$, $\rho^3_{P_3} = Tr_{P_1P_2}|\psi_i\rangle_{P_1P_2P_3}\langle\psi_i|$, we have

$$E_\alpha(|\psi_i\rangle_{P_1P_2P_3}) = f_\alpha(C^2(|\psi_i\rangle_{P_1P_2P_3}))$$

$$= f_\alpha(E(\rho^1_{P_2P_3}))$$

$$\leq f_\alpha(E(\rho^2_{P_2}) + E(\rho^3_{P_3}))$$

$$= f_\alpha[C^2(|\psi_i\rangle_{P_2P_3}) + C^2(|\psi_i\rangle_{P_3P_3})]$$

$$\leq f_\alpha[C^2(|\psi_i\rangle_{P_2P_3})] + f_\alpha[C^2(|\psi_i\rangle_{P_3P_3})]$$

$$= E_\alpha(|\psi_i\rangle_{P_1P_2P_3}) + E_\alpha(|\psi_i\rangle_{P_1P_2P_3}).$$

(44)

where in the first inequality we use the subadditivity of concurrence \cite{2} and the monotonically increasing property of $f_\alpha(x)$ in Lemma 2. The second inequality is due to \cite{39}.

Then we have

$$E_\alpha^2(\rho_{P_1|P_2P_3}) = \sum_i p_i E_\alpha(|\psi_i\rangle_{P_2P_3})$$

$$\leq \sum_i p_i E_\alpha(|\psi_i\rangle_{P_2P_3}) + \sum_i p_i E_\alpha(|\psi_i\rangle_{P_1P_3})$$

$$\leq E_\alpha^2(\rho_{P_3|P_1P_2}) + E_\alpha^2(\rho_{P_2|P_1P_3}).$$

(45)

where the first inequality is due to inequality \cite{44} and the second inequality is due to the definition of REoA.

According to Theorem 3, we have the following corollary.

**Corollary 3** Assume $\rho_{A_1A_2\cdots A_{j_1}}$ is the reduced density matrix of a GW state $|\psi_{A_1\cdots A_{j_1}}\rangle$, here we denote $\{P_1, P_2, P_3\}$ is a partition of the set $\{A_1, A_2, \cdots, A_{j_1}\}$, when $\alpha \in [((\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2]$, we have the following monogamy inequality, we have

$$E_\alpha(\rho_{P_1|P_2P_3}) \leq E_\alpha(\rho_{P_3|P_1P_2}) + E_\alpha(\rho_{P_2|P_1P_3}).$$

(46)

By Corollary 3, we have the upper bound for RaE.

**Corollary 4** Assume $\rho_{A_{j_1}A_{j_2}\cdots A_{j_m}}$ is the reduced density matrix of a GW state $|\psi_{A_1\cdots A_{j_m}}\rangle$, and here we denote $\{P_1, P_2, Q_1, Q_2, \cdots, Q_k\}$ is a partition of the set $\{A_1, A_2, \cdots, A_{j_m}\}$, when $\alpha \in [((\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2]$, we have the following monogamy inequality,

$$E_\alpha(\rho_{P_1P_2|Q_1\cdots Q_k}) \leq 2E_\alpha(\rho_{P_1P_2}) + \sum_{i=1}^k E_\alpha(\rho_{P_1Q_i}) + \sum_{i=1}^k E_\alpha(\rho_{P_2Q_i}).$$

(47)

As an example, we consider a 4-qubit generalized W-class state

$$|\psi\rangle_{A_1A_2A_3A_4} = a_1|1000\rangle + a_2|0100\rangle + a_3|0010\rangle + a_4|0001\rangle,$$

(48)

with $\sum_{i=1}^4 a_i^2 = 1$.

We choose $P_1 = A_1$, $P_2 = \{A_2, A_3\}$, $P_3 = A_4$, then $|\psi\rangle_{A_1A_2A_3A_4}$ can be rewritten as

$$|\psi\rangle_{P_1P_2P_3} = a_1|1\rangle \otimes |00\rangle \otimes |0\rangle + \sqrt{\frac{a_2^2 + a_3^2}{a_2^2 + a_3^2}}|0\rangle \otimes \left(\frac{a_2}{\sqrt{a_2^2 + a_3^2}}|10\rangle + \frac{a_3}{\sqrt{a_2^2 + a_3^2}}|01\rangle\right) \otimes |0\rangle + a_4|0\rangle \otimes |00\rangle \otimes |1\rangle.$$  

(49)

After calculation, we have $C^2(|\psi\rangle_{P_1P_2P_3}) = 2[1 - Tr(\rho^2_{P_1P_2})]$ with $Tr(\rho^2_{P_1P_2}) = a_1^4 + 2a_2^2(a_2^2 + a_3^2) + (a_2^2 + a_3^2)^2 + a_4^4$, $C^2(\rho_{P_1P_2}) = 4a_1^2(a_2^2 + a_3^2)$, $C^2(\rho_{P_1P_3}) = 4a_2^2a_3^2$, and $C^2(\rho_{P_2P_3}) = 4a_2^2(a_2^2 + a_3^2)$. Set $a_1 = \frac{3}{7}$, $a_2 = \frac{1}{7}$, $a_3 = \frac{\sqrt{7}}{7}$, $a_4 = \frac{1}{7}$, we plot the relation $E_\alpha(|\psi\rangle_{P_1P_2P_3}) \leq 2E_\alpha(\rho_{P_1P_2}) + E_\alpha(\rho_{P_1P_3}) + E_\alpha(\rho_{P_2P_3})$ in Theorem 6 with $\alpha \in [((\sqrt{7} - 1)/2, (\sqrt{13} - 1)/2]$, $\alpha \neq 1$ in Figure 2.
Fig. 2: Solid line is the function $E_\alpha(|\psi\rangle_{P_1P_2P_3})$. Dashed line is the function $2E_\alpha(\rho_{P_1P_2}) + E_\alpha(\rho_{P_1P_3}) + E_\alpha(\rho_{P_2P_3})$, which is the upper bound for $E_\alpha(|\psi\rangle_{P_1P_2P_3})$.

C. Tighter monogamy relations for generalized W-class state

If we set the partition \{P_1, P_2, P_3\} is a subset of the set \{A, B_1, B_2, ..., B_{n-1}\}, then the inequalities \[17\] and \[18\] in Lemma 5 can be written as

\[ C^2_{P_1P_2P_3} = C^2_{P_1P_2} + C^2_{P_1P_3}, \]  
\[ C_{P_1P_2} = C^\alpha_{P_1P_2}. \]  

**Theorem 8** Assume $|\psi\rangle_{AB_1B_2...B_{n-1}}$ is a GW state and set the partition \{P_1, P_2, P_3\} is a subset of the set \{A, B_1, B_2, ..., B_{n-1}\}, if $C^\alpha_{P_1P_3} \geq kC^\alpha_{P_1P_2}$, we have

\[ \left( C^\alpha_{P_1P_2P_3} \right)^\beta \geq h \left( C^\alpha_{P_1P_3} \right)^\beta + \left( C^\alpha_{P_1P_2} \right)^\beta, \]  
with $\beta \in [0, \alpha]$, $\alpha \geq 2$, $h = \frac{(1+k)^2}{k^2} - 1$, $k \geq 1$.

**[Proof]** Since $C^\alpha_{P_1P_3} \geq kC^\alpha_{P_1P_2}$, then we have

\[ \left( C^\alpha_{P_1P_2P_3} \right)^\beta = \left( C^\alpha_{P_1P_2P_3} \right)^\beta \geq \left( C^\alpha_{P_1P_2} + C^\alpha_{P_1P_3} \right)^\beta \]
\[ \geq C^\beta_{P_1P_2} \left( \frac{C^\alpha_{P_1P_3}}{C^\alpha_{P_1P_2}} \right)^\frac{\beta}{\alpha} \]
\[ \geq C^\beta_{P_1P_2} \left[ 1 + \frac{(1+k)^2}{k^2} - 1 \right] \frac{\left( C^\alpha_{P_1P_3} \right)^\beta}{C^\alpha_{P_1P_2}} \]
\[ = C^\beta_{P_1P_2} + \frac{(1+k)^2}{k^2} - 1 C^\beta_{P_1P_3} \]
\[ = \left( C^\alpha_{P_1P_3} \right)^\beta + \frac{(1+k)^2}{k^2} - 1 \frac{C^\alpha_{P_1P_2}}{C^\alpha_{P_1P_3}} \left( C^\alpha_{P_1P_2} \right)^\beta. \]  

Here the first inequality is due to \[50\]. The second inequality is obtained from Lemma 7 and the last equality is due to \[51\].

**Theorem 8** gives us a general monogamy inequality for the GW states using CoA which is tighter than the result in Ref.\[38\]. Next we present an example for Theorem 8.
As an example, we consider a three-qubit generalized state
\[
|\psi\rangle_{A_1 A_2 A_3} = \frac{1}{6} |100\rangle + \frac{1}{6} |010\rangle + \frac{2}{\sqrt{6}} |001\rangle.
\] (54)

Then we have \( C(|\psi\rangle_{A_1 A_2 A_3}) = \frac{\sqrt{7}}{3}, C(\rho_{AB}) = C^a(\rho_{AB}) = \frac{1}{3}, C(\rho_{AC}) = C^a(\rho_{AC}) = \frac{2}{3} \). Choose \( \alpha = 2 \), since \( 1 \leq k \leq 4 \), Set \( k = 2 \), we can see that our result is better than the result in Ref.[38] from Figure.3.

Next we generalize the results to multipartite GW state. This results states all the powers of the GW state in terms of CoA under some restricted conditions.

**Theorem 9** Assume \( \rho_{P_1 \cdots P_m} \) is the reduced density matrix of a GW state \( |\psi\rangle_{AB_1 \cdots B_{n-1}} \), if \( k C_{P_i}^\alpha \leq C_{P_i | P_{i+1} \cdots P_{m-1}}^\alpha \) for \( i = 2, 3, \cdots, n \), and \( C_{P_i}^\alpha \geq k C_{P_i | P_{i+1} \cdots P_m}^\alpha \) for \( j = n + 1, \cdots, m - 1 \), then we have
\[
\left( C_{P_i | P_{i+1} \cdots P_m}^\alpha \right)^\beta \geq \sum_{i=2}^{n} h^{i-2} \left( C_{P_i}^\alpha \right)^\beta + h^n \sum_{i=n+1}^{m-1} \left( C_{P_i}^\alpha \right)^\beta + h^{n-1} \left( C_{P_i}^\alpha \right)^\beta.
\] (55)

with \( \beta \in [0, \alpha], \alpha \geq 2, h = \frac{(1+k)\frac{\beta}{\alpha} - 1}{k\pi}, k \geq 1 \).

**[Proof]** Since \( k C_{P_i}^\alpha \leq C_{P_i | P_{i+1} \cdots P_{m-1}}^\alpha \) for \( i = 2, 3, \cdots, n \), then using Theorem 8 we have
\[
\left( C_{P_i | P_{i+1} \cdots P_m}^\alpha \right)^\beta \geq \left( C_{P_i}^\alpha \right)^\beta + h \left( C_{P_i | P_{i+2} \cdots P_m}^\alpha \right)^\beta \geq \cdots \geq \sum_{i=2}^{n} h^{i-2} \left( C_{P_i}^\alpha \right)^\beta + h^{n-1} \left( C_{P_i | P_{m+1} \cdots P_m}^\alpha \right)^\beta.
\] (56)

Since \( C_{P_i}^\alpha \geq k C_{P_i | P_{i+1} \cdots P_m}^\alpha \) for \( j = n + 1, \cdots, m - 1 \), using Theorem 8 again, we have
\[
\left( C_{P_i | P_{n+1} \cdots P_m}^\alpha \right)^\beta \geq h \left( C_{P_i | P_{n+1}}^\alpha \right)^\beta + \left( C_{P_i | P_{n+2} \cdots P_m}^\alpha \right)^\beta \geq \cdots \geq h \sum_{i=n+1}^{m-1} \left( C_{P_i}^\alpha \right)^\beta + \left( C_{P_i}^\alpha \right)^\beta.
\] (57)
Combining (56) and (57), we have
\[
\left(C_{P_1|P_2...P_m}^\alpha\right)^\beta \geq \sum_{i=2}^{n} h^{i-2} \left(C_{P_1,P_i}^\alpha\right)^\beta + h^{n-1} \left(C_{P_1|P_{n+1}...P_m}^\alpha\right)^\beta
\]
\[
\geq \sum_{i=2}^{n} h^{i-2} \left(C_{P_1,P_i}^\alpha\right)^\beta + h^n \sum_{i=n+1}^{m-1} \left(C_{P_i|P_{i+1}...P_m}^\alpha\right)^\beta + h^{n-1} \left(C_{P_1,P_m}^\alpha\right)^\beta.
\]  
\[
(58)
\]

CREN is equivalent to concurrence for any pure state with Schmidt rank two [10]. So for any two-qubit mixed state \(\rho_{AB}\),
\[
C(\rho_{AB}) = \min_{\{p_i,|\psi_i\}\}} \sum_i p_i C(|\psi_i\rangle) = \min_{\{p_i,|\psi_i\}\}} \sum_i p_i \mathcal{N}(|\psi_i\rangle) = \tilde{\mathcal{N}}(\rho_{AB}),
\]
\[
(59)
\]
\[
C_a(\rho_{AB}) = \max_{\{p_i,|\psi_i\}\}} \sum_i p_i C(|\psi_i\rangle) = \max_{\{p_i,|\psi_i\}\}} \sum_i p_i \mathcal{N}(|\psi_i\rangle) = \tilde{\mathcal{N}}_a(\rho_{AB}),
\]
\[
(60)
\]
For GW states, using the similar methods in Theorem 8 and Theorem 9, we have the results for CREN.

**Theorem 10** Assume \(|\psi\rangle_{AB_1B_2...B_{n-1}}\) is a GW state and set the partition \(\{P_1,P_2,P_3\}\) is a subset of the set \(\{A,B_1,B_2,...,B_{n-1}\}\), if \(\tilde{\mathcal{N}}_{P_1,P_3}^\alpha \geq k\tilde{\mathcal{N}}_{P_1}^\alpha\), we have
\[
\left(\tilde{\mathcal{N}}_{P_1|P_2P_3}^\alpha\right)^\beta \geq \left(\tilde{\mathcal{N}}_{P_1,P_3}^\alpha\right)^\beta + \left(\tilde{\mathcal{N}}_{P_1P_3}^\alpha\right)^\beta.
\]
\[
(61)
\]
with \(\beta \in [0,\alpha], \alpha \geq 2, h = \frac{(1+k)^{\beta - 1}}{k^{\beta}}, k \geq 1\).

**Theorem 11** Assume \(\rho_{P_1...P_m}\) is the reduced density matrix of a GW state \(|\psi\rangle_{AB_1...B_{n-1}}\), if \(k\tilde{\mathcal{N}}_{P_i}^\alpha \leq \tilde{\mathcal{N}}_{P_i|P_{i+1}...P_m}^\alpha\) for \(i = 2, 3, \cdots, n\), and \(\tilde{\mathcal{N}}_{P_1,P_3}^\alpha \geq k\tilde{\mathcal{N}}_{P_1|P_{n+1}...P_m}^\alpha\) for \(j = n+1, \cdots, m-1\), then we have
\[
\left(\tilde{\mathcal{N}}_{P_1|P_2...P_m}^\alpha\right)^\beta \geq \sum_{i=2}^{n} h^{i-2} \left(\tilde{\mathcal{N}}_{P_1,P_i}^\alpha\right)^\beta + h^n \sum_{i=n+1}^{m-1} \left(\tilde{\mathcal{N}}_{P_1,P_i}^\alpha\right)^\beta + h^{n-1} \left(C_{P_1,P_m}^\alpha\right)^\beta.
\]
\[
(62)
\]
with \(\beta \in [0,\alpha], \alpha \geq 2, h = \frac{(1+k)^{\beta - 1}}{k^{\beta}}, k \geq 1\).

Finally, we show the results for \(\mathsf{ReE}\).

**Theorem 12** Assume \(|\psi\rangle_{AB_1B_2...B_{n-1}}\) is a GW state and set the partition \(\{P_1,P_2,P_3\}\) is a subset of the set \(\{A,B_1,B_2,...,B_{n-1}\}\), if \(C_{P_1,P_3}^\mu \geq kC_{P_1,P_2}^\mu\), when \(\alpha \geq (\sqrt{7} - 1)/2\), we have
\[
(E_\alpha(\rho_{P_1|P_2P_3}))^\beta \geq h(E_\alpha(\rho_{P_1,P_3}))^\beta + (E_\alpha(\rho_{P_1,P_2}))^\beta.
\]
\[
(63)
\]
with \(\beta \in [0,\mu], \mu \geq 2, h = \frac{(1+k)^{\beta - 1}}{k^{\beta}}, k \geq 1\).

[Proof] Since
\[
(f_\alpha(x^2 + y^2))^\beta \geq (f_\alpha(x^2) + f_\alpha(y^2))^\beta
\]
\[
\geq (f_\alpha(x^2))^\beta + \frac{(1+k)^{\beta - 1}}{k^{\beta}} (f_\alpha(y^2))^\beta.
\]
\[
(64)
\]
Here the first inequality is due to the convex property of \(f_\alpha(x)\) for \(\alpha \geq (\sqrt{7} - 1)/2\) in Lemma 3. The second inequality is obtained from a similar consideration in the proof of Theorem 8.
Then we have
\[
(E_\alpha(\rho_{P_1|P_2P_3}))^\beta = (f_\alpha(C^2(\rho_{P_1|P_2P_3})))^\beta \\
= (f_\alpha(C^2(\rho_{P_1P_2}) + C^2(\rho_{P_1P_3})))^\beta \\
\geq (f_\alpha(C^2(\rho_{P_1P_2})))^\beta + \frac{(1+k)^\beta - 1}{k^\beta}(f_\alpha(C^2(\rho_{P_1P_3})))^\beta \\
= (E_\alpha(\rho_{P_1P_2}))^\beta + \frac{(1+k)^\beta - 1}{k^\beta}(E_\alpha(\rho_{P_1P_3}))^\beta .
\] (65)

Taking the similar consideration of Theorem 9 to generalize Theorem 12, we have the following results.

**Theorem 13** Assume \(\rho_{P_1\cdots P_m}\) is the reduced density matrix of a GW state \(|\psi\rangle_{AB_1\cdots B_{n-1}}\), if \(kC^\mu_{P_1P_j} \leq C^\mu_{P_1|P_{j+1}\cdots P_{m-1}}\) for \(i = 2, 3, \cdots, n\), and \(C^\mu_{P_1P_j} \geq kC^\mu_{P_1|P_{j+1}\cdots P_{m-1}}\) for \(j = n+1, \cdots, m-1\), when \(\alpha \geq (\sqrt{7} - 1)/2\), then we have
\[
(E_\alpha(\rho_{P_1P_2\cdots P_m}))^\beta \geq \sum_{i=2}^{n} h^{i-2}(E_\alpha(\rho_{P_1P_i}))^\beta + h^{n} \sum_{i=n+1}^{m-1} (E_\alpha(\rho_{P_1P_3}))^\beta + h^{n-1} (E_\alpha(\rho_{P_1P_m}))^\beta .
\] (66)

with \(\beta \in [0, \mu], \mu \geq 2, h = \frac{(1+k)^\beta - 1}{k^\beta}, k \geq 1\).

Kim and Sanders in Ref. [19] propose a class of mixed states
\[
\rho_{A_1\cdots A_n} = p|W^d_{n}|_{A_1\cdots A_n} \langle W^d_{n}| + (1-p)|0\cdots 0\rangle_{A_1\cdots A_n} \langle 0\cdots 0| ,
\] (67)
for \(0 \leq p \leq 1\). They further prove this kind of states satisfy the monogamy relation for concurrence. Since \(\rho_{A_1\cdots A_n}\) is an operator of rank two, we can always have a purification of \(\rho_{A_1\cdots A_n}\) such that
\[
|\psi\rangle_{A_1\cdots A_nA_{n+1}} = \sqrt{p}|W^d_{n}|_{A_1\cdots A_n} \otimes |0\rangle_{A_{n+1}} \\
+ \sqrt{1-p}|0\cdots 0\rangle_{A_1\cdots A_n} \otimes |x\rangle_{A_{n+1}},
\] (68)
with \(|x\rangle_{A_{n+1}} = \sum_{i=1}^{d-1} a_{n+1|i}i\rangle_{A_{n+1}}\) is a 1-qudit quantum state of \(A_{n+1}\). (68) can be rewritten as
\[
|\psi\rangle_{A_1\cdots A_{n+1}} = \sum_{i=1}^{d-1} \left[\sqrt{p}(a_{1|i}i\cdots 00)_{A_1\cdots A_{n+1}} + \cdots + a_{n|0\cdots 0})_{A_1\cdots A_{n+1}} \right] \\
+ \sqrt{1-p}a_{n+1|0\cdots 0}i\rangle_{A_1\cdots A_{n+1}} ,
\] (69)
It is an \((n + 1)\)-qudit W-class state. So we conclude that the above results in this subsection are valid for mixed states in (67).

**IV. APPLICATION IN QUANTUM GAMES**

In this section we reconsider the problem of quantum for GW states considered in Ref. [32, 33] using Rényi entropy.

A two-player game \(G = (A, B, X, Y, \pi, v)\) is played between a referee and two isolated players, Alice and Bob, who communicate only with the referee and not between themselves. \(\pi\) is a probability distribution: \(X \times Y \rightarrow [0, 1]; v\) is a verification function: \(X \times Y \times A \times B \rightarrow [0, 1]\). The referee chooses a question pair \((x, y)\) on the question alphabets \(X \times Y\) according to some probability distribution \(\pi\), then he sends \(x\) to Alice and \(y\) to Bob. Next the two players give their answers \(a\) and \(b\) from the sets \(A\) and \(B\). If \(v(x, y, a, b) = 1\) for the verification function, then they win. The classical value of the game
\[
cv(G) = \sup_{a_x, b_y} \sum_{x, y, a, b} \pi(x, y)v(a, b, x, y) \int_{\Omega} a_x(\omega)b_y(\omega)d\beta(\omega)
\]
is the maximum winning probability when two players can use optimal deterministic strategies $\sum_a a_\omega = \sum \omega b_\omega = 1$ based on some classical correlation $\bar{P}(\omega)$. The quantum value for a bipartite entangled state $\rho_{AB}$ of the game is

$$qv(G) = \sup_{\rho, E^a_x, F^b_y} \sum_{x,y,a,b} \pi(x,y)v(a,b,x,y)tr(\rho E^a_x \otimes F^b_y)$$

where the maximum takes overall the POVMs $E^a_x$ and $F^b_y$, $\sum_a E^a_x = 1$, $\sum_b F^b_y = 1$. It is clear that for all games, $cv(G) \leq qv(G)$.

In Ref. [32], the authors assume Alice has a $d$-dimensional system $A$. She can share quantum or classical correlation with an arbitrary number of players $B_1, B_2, \ldots, B_n$, simultaneously. The referee randomly selects a player $B_i$ and plays the game $G_i = (A, B_i, X_i, Y_i, \pi_i, e_i)$ with Alice and $B_i$. For $\{G_i\}_{1 \leq i \leq n}$, they defined the average entangled value:

$$Av(G) = \frac{1}{n} \sum_{i=1}^n cv(G_i).$$

Here $E^a_x, F^b_y$ are POVMs on $A, B_1, \ldots, B_n$ respectively and $\rho^{AB_1 \cdots B_n}$ is a multipartite state with $|A|$ is at most $d$. Since the classical correlation used for different $G_i$ can be combined, then the average classical value was given by

$$Acv(G) = \frac{1}{n} \sum_{i=1}^n cv(G_i).$$

In Ref. [33], the authors reconsider the bound of the difference between the quantum games and the classical games restricting to GW states using Tsallis $q$-entropy for $q \in (1,2]$. In the following, we get a new bound of the difference between the quantum games and the classical games restricting to GW states using Rényi $\alpha$-entropy for $\alpha \geq 1$. We use the similar method which has been used in Ref. [32] [33]. Let $G = (A, B, X, Y, \pi, v)$ be a quantum game. For fixed auxiliary systems $A, B$ and POVMs $E^a_x, F^b_y$, the value function becomes a positive linear function

$$\text{lin}_G(\rho_{AB}) = \sum_{x,y,a,b} \pi(x,y)v(a,b,x,y)tr(\rho E^a_x \otimes F^b_y).$$

Note that $\text{lin}_G$ is of norm at most 1, then for a separable $\sigma_{AB}$ and an arbitrary $\rho_{AB}$,

$$\text{lin}_G(\rho_{AB}) \leq \text{lin}_G(\rho_{AB} - \sigma_{AB}) + \text{lin}_G(\sigma_{AB}) \leq \|\rho_{AB} - \sigma_{AB}\|_1 + cv(G).$$

For a bipartite pure state $|\psi\rangle_{AB} = \sum_{i=0}^{d-1} \sqrt{\lambda_i}|ii\rangle$, then we show there exists a separable state $|\sigma\rangle_{AB}$ such that

$$\|\psi_{AB} - \sigma_{AB}\|_1 \leq 2\sqrt{2E_\alpha(\rho_{AB})}.$$ (73)

for $\alpha \geq 1$. First we select $\sigma_{AB} = |00\rangle$ then we compute the trace norm $|||\psi\rangle_{AB}(\psi - |00\rangle\langle 00||_1 = 2\sqrt{1 - \lambda_0}$. Then we show $2\sqrt{1 - \lambda_0} \leq 2\sqrt{\frac{-\log \sum_{i=0}^{d-1} \lambda_i}{\alpha}}$.

Since when $\alpha \geq 1$, $\lambda \in [0,1]$, $\sum_{i=0}^{d-1} \lambda_i^{\alpha} \geq \lambda_0^{\alpha} + (1 - \lambda_0)^{\alpha}$, then it is enough for us to show $2\sqrt{1 - \lambda_0} \leq 2\sqrt{-\frac{-\log [\lambda_0^{\alpha} + (1 - \lambda_0)^{\alpha}]}{\alpha}}$.

Let $f_\alpha(\lambda_0) = -2\log [\lambda_0^{\alpha} + (1 - \lambda_0)^{\alpha}] - (1 - \lambda_0)(\alpha - 1)$, we need to show $f_\alpha(\lambda_0) \geq 0$. Since

$$f'_\alpha(\lambda_0) = \frac{-2\alpha \lambda_0^{\alpha-1} + 2\alpha (1 - \lambda_0)^{\alpha-1}}{[\lambda_0^{\alpha} + (1 - \lambda_0)^{\alpha}]\ln 2} + (\alpha - 1),$$

after analysis, we find [74] has only one zero $\epsilon \in [0,1]$. When $\lambda_0 \in [0, \epsilon]$, $f_\alpha(\lambda_0)$ is monotonically increasing while monotonically decreasing when $\lambda_0 \in [\epsilon, 1]$. Note that $\lambda_0 \geq \frac{1}{d}$, then it is enough to show $f_\alpha(0) \geq 0$ and $f_\alpha(\frac{1}{d}) \geq 0$.

$f_\alpha(0) \geq 0$ is clear. After computation, we get $f_\alpha(\frac{1}{d}) = -\log [1 + (d - 1)^\alpha] + \alpha \log d - \frac{(d - 1)(\alpha - 1)}{d}$. It is easy to get $f_\alpha(\frac{1}{d}) \geq 0$ for any pure state with Schmidt rank equal or less than two when $\alpha \geq 1$. 

when $\rho$ is a mixed state, assume $\{p_i, |\psi_i\rangle\}$ is the optimal decomposition of $\rho$ in term of $E_\alpha(\rho_{AB})$, then

$$\|\rho_{AB} - \sigma_{AB}\|_1 \leq \sum_i p_i \|\psi_i\rangle_{AB} \langle \psi_i| - \sum_i p_i |\theta_i\rangle_{AB} \langle \theta_i|\|_1$$

$$\leq 2\sqrt{\frac{2}{\pi}} \sum_i \sqrt{p_i} E_\alpha(|\psi_i\rangle_{AB})$$

$$\leq 2\sqrt{2} \sqrt{E_\alpha(\rho_{AB})}. \quad (75)$$

Here we use the subadditivity of the 1-norm. The second inequality is due to (73) and the last inequality is due to the definition of REoA and Theorem 3.

By monogamy inequality in Theorem 4, we have

$$\sum_{i=1}^n E^2_\alpha(\rho_{AB}) \leq E^2_\alpha(\rho_{A|B_1 \cdots B_n}) \leq \left(\frac{\log d^{1-\alpha}}{1-\alpha}\right)^2 = (\log d)^2. \quad (76)$$

Thus we have

$$Aqv(G) \leq \frac{2\sqrt{2}}{n} \sum_{i=1}^n \sqrt{E_\alpha(\rho_{A|B_1 \cdots B_n})} + Acv(G)$$

$$\leq \frac{2\sqrt{2}}{n^{\frac{1}{2}}} \sqrt{E_\alpha(\rho_{AB})} + Acv(G)$$

$$\leq \frac{2\sqrt{2}}{n^{\frac{1}{4}}} (\log d)^{\frac{1}{2}} + Acv(G), \quad (77)$$

where we use (75) to get the first inequality. The second inequality is obtained from Hölder’s inequality. The last inequality is due to (76).

So we have

$$Aqv(G) - Acv(G) \leq \frac{2\sqrt{2}}{n^{\frac{1}{4}}} (\log d)^{\frac{1}{2}}, \quad (78)$$

We find the bound of the difference between the quantum games and the classical games restricting to GW states using Rényi $\alpha$-entropy for $\alpha \geq 1$ is independent of $\alpha$. When $d = 2$, the bound is the same as the bound obtained by Tsallis $q$-entropy for $q = 2$ in Ref. [38].

Compared the result in Ref. [32]:

$$Aqv(G) - Acv(G) \leq \frac{3}{n^{\frac{1}{4}}} d (\log d)^{\frac{1}{4}}, \quad (79)$$

Our bound is tighter due to $d \geq (\log d)^{\frac{1}{4}}$.

V. CONCLUSION

In this paper, we have investigated the general monogamy inequalities for the GW states using RoE. First, we have shown an analytical formula of RoE and REoA for a reduced density matrix of GW states. According to the analytical formula, we have presented a monogamy inequality in terms of the $\mu$-th power of RoE for density matrices of GW states when $\mu \geq 2$, and a polygamy inequality in terms of the $\mu$-th power of REoA for density matrices of GW states when $0 < \mu < 1$. We also present the upper bounds for the GW states using RoE. Corresponding examples are also given. Then we have provided tighter monogamy relations in terms of concurrence and CREN. By the relation between RoE and concurrence, we also obtain the general monogamy relations for RoE. They are all also valid for a class of mixed states. Finally, we apply the monogamy relations to quantum games when restricting to the GW states. When $d = 2$, our result is the same as the result obtained by Tsallis $q$-entropy for $q = 2$ in Ref. [38]. Our result is also tighter than the result in Ref. [32]. Moreover, our results in this paper will provide a reference for general monogamy and polygamy relations in multipartite higher dimensional quantum systems.
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