The ranks of alternating string C-groups

Mark Mixer

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Abstract

In this paper, string C-groups of all ranks $3 \leq r < \frac{n}{2}$ are provided for each alternating group $A_n$, $n \geq 12$. As the string C-group representations of $A_n$ have also been classified for $n \leq 11$, and it is known that larger ranks are impossible, this paper provides the exact values of $n$ for which $A_n$ can be represented as a string C-group of a fixed rank.

1 Introduction

In Problem 32 of [18], Hartley asks “Find regular, chiral, or other polytopes whose automorphism groups are alternating groups $A_n$. In particular, given a rank $r$, for which $n$ does $A_n$ occur as the automorphism group of a regular or chiral polytope of rank $r$?” In [3], the maximum achievable rank for each group $A_n$ was found in the regular case. In this paper we finish the solution to Hartley’s question in the regular case, by finding string C-groups (regular polytopes) of all achievable ranks $r$ for each $n$.

The paper is organized as follows. In section 2 we briefly outline any necessary definitions and background. Sections 3 contains many families of string C-groups that are needed to prove the main theorem. Sections 4 and 5 consider string C-groups of rank at least seven, for odd and even $n$ respectively. In Section 6, we provide the string C-groups with ranks less than or equal to six for all possible $n$. Finally, in section 7, we summarize the main theorem.

2 Background and Basic Notions

The automorphism group of an abstract regular polytope, along with a distinguished set of generators $\{\rho_0, \ldots, \rho_{r-1}\}$, is called a rank $r$ string C-group. In general, we say that a group $\Gamma$ is a rank $r$ string group generated by involutions (or an sgg for short) if $\Gamma$ is generated by $\{\rho_0, \ldots, \rho_{r-1}\}$ which satisfy the following conditions.

$$(\rho_i \rho_j)^{p_{ij}} = \epsilon \quad (0 \leq i, j \leq r - 1),$$

(1)
where \( p_{ii} = 1 \) for all \( i \), \( 2 \leq p_{ji} = p_{ij} \) if \( j = i - 1 \), and
\[
p_{ij} = 2 \quad \text{for} \quad |i - j| \geq 2. \tag{2}
\]

Moreover, if \( \Gamma \) has the following intersection property, then it is considered to be a string C-group.

\[
\langle \rho_i \mid i \in I \rangle \cap \langle \rho_i \mid i \in J \rangle = \langle \rho_i \mid i \in I \cap J \rangle \quad \text{for} \quad I, J \subseteq \{0, \ldots, r - 1\} \tag{3}
\]
The automorphism group \( \Gamma(P) \) of an abstract regular polytope \( P \) is a string C-group, and conversely, it is known (see [12, Sec. 2E]) that an abstract regular \( n \)-polytope can be constructed uniquely from any string C-group.

We will often use the fact that not all of the intersections from Equation 3 need to be verified.

**Proposition 2.1.** Let \( \Gamma \) be a rank \( r \) string group generated by involutions, and suppose that \( \Gamma_0 \) and \( \Gamma_{r-1} \) are both string C-groups. If \( \Gamma_0 \cap \Gamma_{r-1} = \Gamma_{0,r-1} \), then \( \Gamma \) is a string C-group. Moreover, if \( \Gamma_{0,r-1} \) is a maximal subgroup of either \( \Gamma_0 \) or \( \Gamma_{r-1} \) then this condition is satisfied.

**Proof.** This combines Proposition 2E16 of [12] and Lemma 2.2 of [7]. \( \square \)

Let \( \Gamma = \langle \rho_0, \ldots, \rho_{r-1} \rangle \) be a string group generated by involutions acting as a permutation group on a set \( \{1, \ldots, n\} \). We can construct the permutation representation graph \( X \) of \( \Gamma \) as the \( r \)-edge-labeled graph with \( n \) vertices, and with a single \( i \)-edge \( \{a, b\} \) whenever \( a \rho_i = b \) with \( a < b \). When \( \Gamma \) is a string C-group that acts faithfully on \( \{1, \ldots, n\} \), the graph \( X \) is called a CPR graph, as defined in [15].

If \( P \) and \( Q \) are string C-groups, then we say that \( P \) covers \( Q \) if there is a well-defined surjective homomorphism from \( P \) to \( Q \) that respects the canonical generators. In other words, if \( P = \langle \rho_0, \ldots, \rho_{r-1} \rangle \) and \( Q = \langle \rho'_0, \ldots, \rho'_{r-1} \rangle \), then \( P \) covers \( Q \) if there is a homomorphism that sends each \( \rho_i \) to \( \rho'_i \).

Given string C-groups \( P \) and \( Q \), the mix of \( P \) and \( Q \), denoted \( P \odot Q \), is the subgroup of the direct product \( P \times Q \) that is generated by the elements \( (\rho_i, \rho'_i) \). This group is the minimal string group generated by involutions that covers both \( P \) and \( Q \) - where again, we only consider homomorphisms that respect the generators; see [13, Section 5] for more details.

It is possible to mix a rank \( r \) string C-group \( P \) with a rank \( s \) string C-group \( Q \). In particular we often mix a string C-group with the automorphism group of an edge \( e \) (which is a rank 1 regular polytope). To do so, we take \( e = \langle \rho_0, \ldots, \rho_{r-1} \rangle \) with defining relations \( \rho_0^2 = e \) and \( \rho_i = e \) for \( 1 \leq i \leq r - 1 \), and then use the same definition as before. In general, to mix two string C-groups of different ranks, we add trivial generators to the group of smaller rank.

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The comix of $P$ and $Q$, denoted $P \square Q$, is the largest string group generated by involutions that is covered by both $P$ and $Q$ [4]. A presentation for $P \square Q$ can be obtained from that of $P$ by adding all of the relations of $Q$, rewriting the relations to use the generators of $P$ instead. The size of the comix of $P$ and $Q$ is the index of the mix in the full direct product. Throughout the paper we will rely on the following results about mixing of string C-groups.

**Proposition 2.2.** (Theorem 5.12 of [13]) Suppose $P$ and $Q$ are rank $r$ string C-groups, and that $P_{r-1}$ covers $Q_{r-1}$ then $P \diamond Q$ is a string C-group.

**Proposition 2.3.** (Theorem 7A7 of [12]) If $P$ is a rank $r$ string C-group then $P \diamond P_{r-1}$ is a string C-group.

**Definition 2.4.** Let $\Gamma = \langle \rho_0, \ldots, \rho_{r-1} \rangle$ be an sggi, and let $\tau$ be an involution in a super-group of $\Gamma$ such that $\tau \not\in \Gamma$ and $\tau$ commutes with all of $\Gamma$. For fixed $k$, we define the group $\Gamma^* = \langle \rho_i \tau^{\eta_i} \mid i \in \{0, \ldots, r-1\} \rangle$ where $\eta_i = 1$ if $i = k$ and 0 otherwise, the sesqui-extension of $\Gamma$ with respect to $\rho_k$ and $\tau$.

A sesqui-extension of a group $\Gamma$, with respect to its first generator can be seen as a mix of $\Gamma$ with the automorphism group of an edge, and thus we have the following.

**Proposition 2.5.** (Proposition 5.3 of [5]) If $\Psi$ is a sesqui-extension of a string C-group $\Gamma$ with respect to $\rho_0$, then $\Psi$ is a string C-group.

The following lemma shows that we can often use this result in a more general setting.

**Lemma 2.6.** (Lemma 5.4 of [5]) If $\Gamma = \langle \rho_i \mid i = 0, \ldots, r-1 \rangle$ and $\Psi = \langle \rho_i \tau^{\eta_i} \mid i \in \{0, \ldots, r-1\} \rangle$ is a sesqui-extension of $\Gamma$ with respect to $\rho_k$, then:

1. $\Psi \cong \Gamma$ or $\Psi \cong \Gamma \times \langle \tau \rangle \cong \Gamma \times 2$.

2. If the identity element of $\Gamma$ can be written as a product of generators involving an odd number of $\rho_k$’s, then $\Psi \cong \Gamma \times \langle \tau \rangle$.

3. if $\Gamma$ is a finite permutation group, $\tau$ and $\rho_k$ are odd permutations, and all other $\rho_i$ are even permutations, then $\Psi \cong \Gamma$.

4. whenever $\tau \not\in \Psi$, $\Gamma$ is a string C-group if and only if $\Psi$ is a string C-group.

**Proposition 2.7.** Let $\Gamma = \langle \rho_i \mid i = 0, \ldots, r-1 \rangle$ and $\Psi = \langle \rho_i \tau^{\eta_i} \mid i \in \{0, \ldots, r-1\} \rangle$ be a sesqui-extension of $\Gamma$ with respect to $\rho_k$. If either $\Psi_0 \cong \Gamma_0$ or $\Psi_{r-1} \cong \Gamma_{r-1}$ as string C-groups, then $\Psi$ is a string C-group.

**Proof.** This is a consequence of part (b) of Proposition 2E16 in [12]. Assume $\Psi_{r-1} \cong \Gamma_{r-1}$ as string C-groups. The intersection condition of part (b) holds as $\tau$ is not in any of the groups $\Psi_{r-1} \cap \langle \rho_k, \ldots, \rho_{r-1} \rangle$. □
For any permutation group $\Gamma$ of degree $n$, we will use the notation $S_n$ to represent the full symmetric group, $A_n$ to represent the full alternating group, and $G^+$ to denote $G \cap A_n$. If $\Gamma := \langle \rho_0, \ldots, \rho_{n-1} \rangle$, then for each $i$ we denote $\Gamma_i = \langle \rho_j \mid j \neq i \rangle$, where each $\Gamma_i$ is itself a string C-group. Similarly, we will denote $\Gamma_{i,j} = \langle \rho_k \mid k \notin \{i,j\} \rangle$. The dual $\Gamma^*$ of a string C-group $\Gamma$ is the group generated by the same involutions, but with the indexing reversed.

Finally, we will occasionally use the following rank reduction technique of [2]. We will frequently need to apply this construction to the dual of a string C-group, and then take the dual again. When we do this, we simply call it the dual rank reduction.

**Proposition 2.8.** Let $\Gamma = \langle \rho_0, \ldots, \rho_{r-1} \rangle$ be a rank $r$ string C-group, where $|\rho_i\rho_{i+1}| > 2$ for all $0 \leq i \leq r - 2$. If $\rho_0 \in \langle \rho_0 \rho_2, \rho_3 \rangle$, then $\Gamma \cong \langle \rho_1, \rho_0 \rho_2, \rho_3, \ldots, \rho_{r-1} \rangle$ is a string C-group of rank $r - 1$. Furthermore, if $\rho_2 \rho_3$ has odd order, then this condition is satisfied.

**Proof.** This combines the results of Theorem 1.1 and Corollary 1.2 of [2].

### 3 Building Blocks

In this section, we provide some examples of families of string C-groups which appear as subgroups of the groups in our main theorem.

**Lemma 3.1.** For each $r \geq 3$ the permutation group $\text{FL}(r,k)$ given by the following graph is a string C-group isomorphic to $S_{r+1+k}$, for all odd $k \geq 0$, and

for each $r \geq 3$ the permutation group $\text{FL}(r,k)$ given by the following graph is a string C-group isomorphic to $S_{r+1+k}$, for all even $k \geq 0$

We point out that there are $k + 2$ edges of labels 0 or 1 in each graph.

**Proof.** This combines the results of Theorem 1, Theorem 2, and Lemma 21 from [6].

**Lemma 3.2.** For each $r \geq 5$ the permutation group $\text{R}(r,k)$ given by the following graph is a string C-group isomorphic to $S_{r+3+k}$, for all odd $k \geq 0$, and

We point out that there are $k + 2$ edges of labels 0 or 1 in each graph.
for each \( r \geq 5 \) the permutation group \( R(r,k) \) given by the following graph is a string C-group isomorphic to \( S_{r+3+k} \), for all even \( k \geq 0 \).

![Graph](image)

**Proof.** We will prove this by induction on \( k \). To clarify the notation, we point out that there are \( k + 2 \) edges of labels 0 or 1 in each graph. When \( k = 0 \), \( R(r,k) \) is a string C-group isomorphic to \( S_{r+3} \) as it is the dual of \( FL(r,2) \) from Lemma 3.1. We will assume by induction that \( R(r+1,k-1) \) is a string C-group isomorphic to \( S_{r+1+3+k-1} \), and then applying the rank reduction from Proposition 2.8 to \( R(r+1,k-1) \) we get \( R(r,k) \), which is thus a string C-group isomorphic to \( S_{r+3+k} \).

\[ \square \]

**Lemma 3.3.** For each \( r \geq 4 \) the permutation group \( Sh(r,k) \) given by the following graph is a string C-group isomorphic to \( (S_2 \wr S_{r+\frac{k}{2}})^+ \), for all \( k \equiv 2 \pmod{4} \), with \( k \geq 0 \).

![Graph](image)

Note that there are \( 1 + \frac{k}{2} \) edges of label 0 in each such graph. We also note that when \( r = 3 \) this graph still provides a string C-group.

**Proof.** We prove that the group is a string C-group by induction on \( r \), where the proof of the base case will be nearly the same as the proof of the inductive step. Let \( \Gamma = \langle \rho_0, \ldots, \rho_{r-1} \rangle = Sh(r,k) \). When \( r = 4 \), the group \( \Gamma_{r-1} \) is a string C-group by Theorem 4.4 of [15]. Furthermore, when \( r \) is greater than 4, we can assume that \( \Gamma_{r-1} \) is a string C-group as \( \Gamma_{r-1} = Sh(r-1,k) \).

For all \( r \geq 4 \), the group \( \Gamma_0 \) is isomorphic to the string C-group \( FL(r-1,\frac{k}{2}) \circ FL(r-1,\frac{k}{2}) \) which is isomorphic to \( S_{r+\frac{k}{2}} \). It remains to show that \( \Gamma_0 \cap \Gamma_{r-1} = \Gamma_{0,r-1} \).

The group \( \Gamma \) is an imprimitive group with a natural block structure, having \( r + \frac{k}{2} \) blocks of size two. The generator \( \rho_0 \) is the only generator acting within a block, and thus \( \Gamma_0 \) gives the action on the blocks.

Let \( \alpha \in \Gamma_0 \cap \Gamma_{r-1} \). Since \( \alpha \in \Gamma_0 \) it only acts on the blocks, and since \( \alpha \in \Gamma_{r-1} \) it fixes the “last block” (the block in the support of \( \rho_{r-1} \)). Since \( \Gamma_{0,r-1} \) gives the action on these \( r + \frac{k}{2} - 1 \) blocks, we know that \( \alpha \in \Gamma_{0,r-1} \). Thus by Proposition 2.1, \( \Gamma \) is a string...
C-group. Finally, to prove that the group is the collection of all even permutations in the wreath product, observe that the group $\Gamma_0$ gives the full symmetric group acting on the blocks, and the element $(\rho_0\rho_1)^2$ is a product of two disjoint transpositions, each swapping two elements within a block.

Lemma 3.4. For each $r \geq 6$ the permutation group $Bl(r, k)$ given by the following graph is a string C-group isomorphic to $(S_2 \wr S_{r+2+\frac{k}{2}})^+$, for all $k \equiv 2 \pmod{4}$, with $k \geq 0$.

We note that there are $1 + \frac{k}{2}$ edges of label 0 in each graph.

Proof. The group $Bl(r, k)$ is obtained by applying the dual rank reduction of Lemma 2.8 to $Sh(r + 2, k)$ and then again to the result. This shows that $Bl(r, k)$ is a string C-group, and also that $Bl(r, k)$ is isomorphic to $(S_2 \wr S_{r+2+\frac{k}{2}})^+$.

Lemma 3.5. For each $r \geq 4$ the permutation group $P(r, k)$ given by the following graph is a string C-group isomorphic to $S_{r+2+k}$, for all odd $k \geq 0$, and

for each $r \geq 4$ the permutation group $P(r, k)$ given by the following graph is a string C-group isomorphic to $S_{r+2+k}$, for all even $k \geq 0$, and

There are $k + 2$ edges of labels 0 or 1 in each graph. Also, note that when $r = 3$ this graph still provides a string C-group, see for example Theorem 4.5 of [15] when $k$ is odd or Theorem 4.4 of [15] when $k$ is even.

Proof. Let $\Gamma = \langle \rho_0, \ldots, \rho_{r-1} \rangle = P(r, k)$. The group $\Gamma_0$ is a string C-group isomorphic to $S_2 \times S_{r+1}$ as it is a sesqui-extension of string C-group $FL(r - 1, 1)$ from Lemma 3.1. The group $\Gamma_{r-1}$ is a string C-group isomorphic to $S_2 \times S_{r+k}$ as it also is a sesqui-extension of string C-group $FL(r - 1, k)$. Thus $\Gamma$ is isomorphic to $S_{r+2+k}$. Finally, the group $\Gamma_{0,r-1}$ is isomorphic to $S_2 \times S_{r-1} \times S_2$, which is maximal in $\Gamma_0$, and thus, by Proposition 2.1, $\Gamma$ is also a string C-group.
Lemma 3.6. For each $r \geq 4$ the permutation group $Sp(r, k)$ given by the following graph is a string $C$-group isomorphic to $S_{r+2+\frac{k}{2}} \times S_{r+1+\frac{k}{2}}$ for $k \equiv 2 \pmod{4}$ with $k \geq 0$.

We note that there are $k + 4$ edges of labels 0 or 1 in each such graph.

Proof. Let $\Gamma = \langle \rho_0, \ldots, \rho_{r-1} \rangle = Sp(r, k)$. Then, $\Gamma = P \circ Q$, where $P = P(r, \frac{k}{2})$ from Lemma 3.5, and $Q$ is $FL(r, \frac{k}{2})$ from Lemma 3.1. Furthermore, the facets of $P$ cover the facets of $Q$; as the facets of $P$ are isomorphic to $S_{r+\frac{k}{2}} \times S_2$ where as the facets of $Q$ are isomorphic to $S_{r+\frac{k}{2}}$. Thus by Proposition 2.2, $\Gamma \cong P \circ Q$ is a string $C$-group.

To determine that $\Gamma$ is the full direct product, we consider the comix of $P$ and $Q$, $C = P \square Q$. We will write $C$ as generated by $\rho_i$, along with all the relations from both $P$ and $Q$. Thus, in $C$, $(\rho_{r-1}\rho_{r-2})^3 = 1$ from $Q$ and $1 = (\rho_{r-1}\rho_{r-2})^4$ from $P$, and thus in $C$, $\rho_{r-1} = \rho_{r-2}$.

Then, it follows that in $C$, $(\rho_{r-2}\rho_{r-3})^3 = 1 = (\rho_{r-1}\rho_{r-3})^3 = (\rho_{r-1}\rho_{r-3})^2$, and thus $\rho_{r-1} = \rho_{r-3}$. Furthermore in $C$, we know that $(\rho_{r-3}\rho_{r-2}\rho_{r-1})^5 = 1$ from $P$, and so for instance $\rho_{r-1}^5 = \rho_{r-1}^2 = 1$, and thus $\rho_{r-1} = \rho_{r-2} = \rho_{r-3} = 1$.

Then, it will follow that all $\rho_i = 1$ in $C$. For example in $C$, $(\rho_{r-3}\rho_{r-4})^3 = 1$ and thus in $C \rho_{r-4} = 1$. This argument works for showing that in $C$, $1 = \rho_2 = \rho_3 = \cdots = \rho_{r-1}$.

Finally, $(\rho_0\rho_1\rho_2)^{\frac{k}{2}+4} = 1$ and $(\rho_0\rho_1)^{\frac{k}{2}+3} = 1$, and so in $C$, $\rho_0 = \rho_1$.

We have showed that $C = P \square Q$ has size at most two, and therefore $P \circ Q$ is the full direct product or an index two subgroup of the full direct product. There are three index two subgroups of $S_{r+2+\frac{k}{2}} \times S_{r+1+\frac{k}{2}}$; namely $A_{r+2+\frac{k}{2}} \times S_{r+1+\frac{k}{2}}$, $S_{r+2+\frac{k}{2}} \times A_{r+1+\frac{k}{2}}$, and $(S_{r+2+\frac{k}{2}} \times S_{r+1+\frac{k}{2}})^+$. As there are odd permutations in $P$, odd permutations in $Q$, and odd permutations in $P \circ Q$, we can rule out all three cases and conclude that $\Gamma \cong S_{r+2+\frac{k}{2}} \times S_{r+1+\frac{k}{2}}$.

□

Lemma 3.7. For each rank $r \geq 4$ the permutation group $Sm(r)$ given by the following graph is a string $C$-group isomorphic to $(S_r \times S_{r+3})^+$.

\[\begin{array}{c}
0 & 1 & 2 & r-2 & r-1 & r-3
\end{array}\]

\[\begin{array}{c}
0 & 1 & 2 & r-2 & r-3 & r-1
\end{array}\]
Proof. Let $\Gamma = \langle \rho_0, \ldots, \rho_{r-1} \rangle = Sm(r)$. We proceed by induction on the rank $r$, where the base case of $r = 4$ can be verified using Magma. The group $\Gamma_0$ is isomorphic to $Sm(r - 1)$, and by induction we may assume that $\Gamma_0$ is a string C-group isomorphic to $(S_{r-1} \times S_{r+2})^+$. Therefore, it can be seen that $\Gamma \cong (S_r \times S_{r+3})^+$, and it remains to show that $\Gamma$ is a string C-group. The group $\Gamma_{r-1}$ is isomorphic to the sesqui-extension of a group $Sp(r - 1, 0)$ through $\rho_{r-3}$. By parts (3) and (4) of Lemma 2.6, we can see that $\Gamma_{r-1}$ is a string C-group. Finally, $\Gamma$ is a string C-group by Lemma 2.1 as $\Gamma_{0,r-1} \cong (S_{r-2} \times S_{r+1} \times S_2)^+$ which is maximal in $\Gamma_0$.

Lemma 3.8. For each rank $r \geq 6$ the permutation group $Sy(r,k)$ given by the following graph is a string C-group isomorphic to $(S_{r+\frac{k}{2}} \times S_{r+3+\frac{k}{2}})^+$ for all $k \equiv 2 \pmod{4}$, with $k \geq 0$, and

![Graph 1](image1)

for each rank $r \geq 6$ the permutation group $Sy(r,k)$ given by the following graph is a string C-group isomorphic to $(S_{r+\frac{k}{2}} \times S_{r+3+\frac{k}{2}})^+$ for all $k \equiv 0 \pmod{4}$, with $k \geq 0$, and

![Graph 2](image2)

Proof. This can be shown by induction on $k$. When $k = 0$ this representation is given in Lemma 3.7. Assuming by induction that $Sy(r,k)$ is a string C-group, it follows that $Sy(r,k+2)$ is a string C-group by applying the dual rank reduction of Proposition 2.8 to $Sy(r + 1, k)$; in order to apply the dual rank reduction, we need that $r + 1 \geq 7$.

Lemma 3.9. For each rank $r \geq 6$ the permutation group $L(r,k)$ given by the following graph is a string C-group isomorphic to $S_{r+3+k}$, for all odd $k \geq 0$, and

![Graph 3](image3)
for each rank \( r \geq 6 \) the permutation group \( L(r,k) \) given by the following graph a string C-group isomorphic to \( S_{r+3+k} \), for all even \( k \geq 0 \).

Furthermore \( L_{r-1} \) is isomorphic to \( S_{r+2+k} \).

To clarify, we note that there are \( k + 2 \) edges of labels either 0 or 1 in each such graph.

**Proof.** This can be shown by induction on \( k \). When \( k = 0 \) this representation was shown to be a string C-group for all \( r \geq 6 \) in [8]. Assuming by induction that \( L(r,k) \) is a string C-group, it follows that \( L(r,k + 1) \) is a string C-group by applying the rank reduction of Proposition 2.8 to \( L(r+1,k) \). The structure of \( L_{r-1}(r,k) \) follows, again using Proposition 2.8, from its relationship to \( FL(r−1,0) \) from Lemma 3.1.

**Lemma 3.10.** For each rank \( r \geq 6 \) the permutation group \( M(r,k) \) given by the following graph is a string C-group, for all \( k \equiv 2 \) (mod 4), with \( k \geq 0 \), and

for each rank \( r \geq 6 \) the permutation group \( M(r,k) \) given by the following graph is a string C-group, for all \( k \equiv 0 \) (mod 4), with \( k \geq 0 \).

We note that permutation degree of \( M(r,k) \) is \( 2r + 5 + k \), and there are \( k + 4 \) edges of labels either 0 or 1 in such a graph.

**Proof.** The group \( M(r,k) \) is the mix of \( L(r,k) \) with \( L_{r-1}(r,k) \). Thus \( M(r,k) \) is a string C-group by Proposition 2.3.

**Lemma 3.11.** For each rank \( r \geq 6 \) the permutation group \( Sl(r,k) \) given by the following graph is a string C-group isomorphic to \( S_{2r+1+k} \), for \( k \equiv 2 \) (mod 4) with \( k \geq 0 \).
We note that there are a total of $\frac{k}{2} + 1$ edges of label 0 in such a graph.

**Proof.** Let $\Gamma = \langle \rho_0, \ldots, \rho_{r-1} \rangle = SL(r,k)$. The group $\Gamma_0$ is a string C-group, as it is isomorphic to $Sp(r-1,\frac{k}{2})$ from Lemma 3.6, and thus $\Gamma_0$ is isomorphic to $S_{r+1+\frac{k}{2}} \times S_{r+\frac{k}{2}}$. Since $\Gamma_0$ is maximal in $S_{2r+1+k}$, we conclude that $\Gamma \cong S_{2r+1+k}$.

Let $Sh(r-1,k) = \langle a_0, \ldots, a_{r-2} \rangle$, from Lemma 3.3. By part (2) of Lemma 2.6, since $(a_{r-3}a_{r-2})^3 = 1$, we know that $\Gamma_{r-1} \cong Sh(r-1,k) \times S_2$, and thus $\Gamma_{r-1}$ is a string C-group isomorphic to $(S_2 \wr S_{r-1+\frac{k}{2}})^+ \times S_2$. Notice that although $\Gamma_{r-1}$ is not transitive, it still has an imprimitive block structure, with blocks of size two, now with one more block, and also a block of size one. It remains to show that $\Gamma_{r-1} \cap \Gamma_0 \cong \Gamma_{0,r-1}$ which we will do by analyzing the orbits of $\Gamma_{r-1} \cap \Gamma_0$.

If $\alpha \in \Gamma_{r-1} \cap \Gamma_0$ then: $\alpha$ preserves the two orbits of $\Gamma_0$; $\alpha$ preserves the three orbits of $\Gamma_{r-1}$, namely, it fixes the vertex of the graph which is incident to only an edge of label $r-1$, and it either fixes the vertex of the graph which is incident only to an edge of label $r-2$, or it it interchanges it with the other vertex on that edge of label $r-2$. Finally, $\alpha$ preserves the block structure of $\Gamma_{r-1}$. Thus, we can see $\alpha$ acting only on the blocks, since each block consists of two elements in different $\Gamma_0$ orbits. Therefore, $\Gamma_{r-1} \cap \Gamma_0 \leq (S_{r-1+\frac{k}{2}} \times S_2)$. It is easy to check that $\Gamma_{0,r-1} \equiv (S_{r-1+\frac{k}{2}} \times S_2)$, and thus $\Gamma$ is a string C-group.

4 Odd degree and high rank

In this section we deal with alternating groups of odd permutation degree, represented as string C-groups of rank at least seven.

**Theorem 4.1.** For each $r \geq 7$ the permutation group $S(r,k)$ given by the following graph is a string C-group isomorphic to $A_{2r+1+k}$ for all $k \equiv 2 \pmod{4}$, with $k \geq 0$.

![Graph](image)

**Proof.** Let $\Gamma = \langle \rho_0, \ldots, \rho_{r-1} \rangle = S(r,k)$. The group $\Gamma_0 \cong Sy(r-1,k)$ is a string C-group isomorphic to $(S_{r-1+\frac{k}{2}} \times S_{r+2+\frac{k}{2}})^+$ by Lemma 3.8. The group $\Gamma_{r-1} \cong P \circ Q$ where $P = SL(r-1,k)$ is a string C-group isomorphic to $S_{2r-1+k}$, and $Q$ is a single involution extending $\rho_{r-3}$. Since $\Gamma_{r-1}$ only contains even permutations, by parts (3) and (4) of Lemma 2.6, $\Gamma_{r-1}$ is a string C-group isomorphic to $S_{2r-1+k}$. Finally, by Lemma 3.6, the
group $\Gamma_{0,r-1} \cong Sp(r-2, k)$ is isomorphic to $S_{\frac{r+k}{2}} \times S_{\frac{r-1+k}{2}}$, which is maximal in $\Gamma_0$. Thus $\Gamma$ is a string C-group by Lemma 2.1.

It is clear that $\Gamma$ is a subgroup of $A_{2r+1+k}$. The main theorem of [10] shows that $(S_{\frac{r-1+k}{2}} \times S_{\frac{r+2+k}{2}})$ is maximal in $A_{2r+1+k}$ and thus $\Gamma \cong A_{2r+1+k}$.

**Theorem 4.2.** For each $r \geq 7$ the permutation group $B(r, k)$ given by the following graph is a string C-group isomorphic to $A_{2r+3+k}$ for all $k \equiv 2 \pmod{4}$, with $k \geq 0$.

Proof. Let $\Gamma = \langle \rho_0, \ldots, \rho_{r-1} \rangle = B(r, k)$.

First let us show that $\Gamma \cong A_{2r+3+k}$. This can be checked using MAGMA for $B(7, 2)$, $B(7, 6)$, $B(8, 2)$, and $B(9, 2)$ where $n = 2r + 3 + k \leq 23$. For all the other cases, $n > 24$, and we can use Corollary 1.2 of [11]. In these cases we know that, $|\Gamma| < 2^n$ or $\Gamma \cong A_n$. The group $\Gamma_0$ contains a symmetric group acting on one of its orbits, and thus $|\Gamma| > (\frac{n+1}{2})!$; however for all $n \geq 25$, $(\frac{n+1}{2})! > 2^n$, and so $\Gamma \cong A_n$.

The group $\Gamma_0$ is a string C-group $M(r-1, k)$ from Lemma 3.10. The group $\Gamma_{r-1}$ is a string C-group $Bl(r-1, k)$ by Lemma 3.4. Note that although $\Gamma_{r-1}$ is not transitive, it still has an imprimitive block structure, with blocks of size two, and also a block of size one.

It remains to show that $\Gamma_{r-1} \cap \Gamma_0 \cong \Gamma_{0,r-1}$ which we will do by analyzing the orbits of $\Gamma_{r-1} \cap \Gamma_0$. If $\alpha \in \Gamma_{r-1} \cap \Gamma_0$ then: $\alpha$ preserves the two orbits of $\Gamma_0$; $\alpha$ preserves the block structure of $\Gamma_{r-1}$, and $\alpha$ preserves the fixed point of $\Gamma_{r-1}$. Thus $\Gamma_{r-1} \cap \Gamma_0$ can be seen acting on the $r + 1 + \frac{k}{2}$ blocks of size two, and is isomorphic to a subgroup of $S_{\frac{r+1+k}{2}}$.

Finally, the group $\Gamma_{0,r-1} \cong P \circ P$ where $P$ is a string C-group $R(r-2, \frac{k}{2})$ isomorphic to $S_{\frac{r+1+k}{2}}$, by Lemma 3.2, and thus $\Gamma$ is a string C-group.

**Corollary 4.3.** For each $r \geq 7$, and each $n \geq 2r + 1$, there is a string C-group representation of $A_n$ of rank $r$.

Proof. When $n = 2r + 1$ this follows from Theorem 7.2 of [5]. When $n = 2r + 3 + 4j$ for some integer $j$, it follows from Theorem 4.1. Finally, when $n = 2r + 5 + 4j$ for some integer $j$, it follows from Theorem 4.2.
5 Even degree and high rank

In this section we deal with alternating groups of even permutation degree, represented as string C-groups of rank at least seven.

**Theorem 5.1.** For each rank \( r \geq 6 \) the permutation group \( D(r, k) \) given by the following graph is a string C-group isomorphic to \( A_{2r+2+k} \), for all \( k \equiv 2 \pmod{4} \), with \( k \geq 0 \), and for each rank \( r \geq 6 \) the permutation group \( M(r, k) \) given by the following graph is a string C-group isomorphic to \( A_{2r+2+k} \), for all \( k \equiv 0 \pmod{4} \), with \( k \geq 0 \).

To clarify this notation we both include an example \( D(7, 4) \) and remark that there are \( k + 4 \) edges with labels 0 or 1 in each such graph.

![Graph](image)

**Figure 1:** The group \( D(7, 4) \)

**Proof.** This is proved by induction on \( k \). When \( k = 0 \), \( D(r, k) \) was shown to be a string C-group isomorphic to \( A_{2r+2} \) in Theorem 7.1 of [5]. To see that \( D(r, k+2) \) is a string C-group isomorphic to \( A_{2r+2+k+2} \), we notice that you obtain \( D(r, k+2) \) from \( D(r+1, k) \) using the rank reduction of Proposition 2.8, which is assumed to be a string C-group isomorphic to \( A_{2r+2+k+2} \) by induction. We can apply the rank reduction to \( D(r+1, k) \) as long as \( r + 1 \geq 7 \). □

**Corollary 5.2.** For all \( r \geq 7 \), the alternating group \( A_n \) can be represented as a rank \( r \)-string if \( n \) is even and \( n \geq 2r+2 \).
The group $\Gamma$ is an string group generated by involutions. Then $\Gamma^t$ is a group with one more generator and larger permutation degree.

## 6 Low Ranks and Extension Construction

In this section we deal with alternating groups represented as rank 4, 5, or 6 string C-groups.

In [14] it was showed that $A_n$ is generated by three involutions two of which commute if and only if $n \notin \{3, 4, 6, 7, 8\}$. Additionally, in [15], permutation representations for the string C-groups for each of these $A_n$ were provided. Thus it is known exactly which $A_n$ are rank 3 string C-groups.

**Proposition 6.1.** The alternating group $A_n$ can be represented as a rank 3 string C-group if and only if $n = 5$ or $n \geq 9$.

In order to show which $A_n$ can be represented as a rank $r$ string C-group (for $r = 4, 5, 6$) we mainly rely on the following construction, which is similar to one of Pellicer [16] or of Schulte [17].

**Definition 6.2.** Given an sggi $\Gamma$ whose permutation representation graph $X$ has vertices $[1, \ldots, m]$ and edges labeled $[1, \ldots, r - 1]$, we define $\Gamma^t$ (for each $t > 0$) as the group generated by involutions (with one more generator) whose permutation representation graph has vertices $[1, \ldots, m + t]$ and is obtained by adjoining an alternating path of length $t$ of edges labeled 0,1 to the vertex $m$ of $X$.

As a matter of notation, recall that for an sggi $G = \langle \rho_0, \ldots, \rho_{r-1} \rangle$ and an integer $k$ we define the groups $G_{\leq k} = \langle \rho_0, \ldots, \rho_k \rangle$.

**Proposition 6.3.** Let $\Gamma$ be an sggi acting on the points $[1, \ldots, m]$, and for each positive integer $t$ construct $\Gamma^t$ as described above. If there is an integer $b \geq 2$ such that $\Gamma^b$ is a string C-group, and for each $2 \leq k \leq r - 2$, the group $\Gamma^b_{\leq k}$ acts as a symmetric group on $\text{Orbit}(\Gamma^b_{\leq k}, m)$, then for each $t > b$, the group $\Gamma^t$ is also a string C-group.
Proof. Assume that for some $b \geq 2$, $\Gamma^b = \langle \rho_0, \ldots, \rho_{r-1} \rangle$ is a string C-group. Then, by the commuting relationship between the generators, only the elements $\rho_0$ and $\rho_1$ can have the point $m$ in their support, and thus $m$ is of degree 2 in the natural CPR graph of $\Gamma^b$. Let $\Gamma^t = \langle \rho_0, \ldots, \rho_{r-1} \rangle$. To show that $\Gamma^t$ is a string C-group we rely on part (b) of Proposition 2E16 of [12]. There is an isomorphism $\phi$ that maps generators of $\Gamma^t_0$ to generators of $\Gamma^t_0$, and thus $\Gamma^t_0$ is also a string C-group. It remains to show that for each $k = 0, \ldots, r - 2$ that $\Gamma^t_0 \cap \Gamma^t_{\leq k} = \langle \rho_1, \ldots, \rho_k \rangle$.

We first deal with small values of $k$. If $k = 0$, it is clear by construction that $\rho_0 \not\in \Gamma^t_0$ and thus $\Gamma^t_0 \cap \langle \rho_0 \rangle = \langle 1 \rangle$. Let $k = 1$ and $\alpha \in \Gamma^t_0 \cap \langle \rho_0, \rho_1 \rangle$. We assume, without loss of generality, that $\alpha$ has an even number of factors of $\rho_1$ and thus (as it is in $\Gamma^t_0$) fixes all points greater than or equal to $m$. Since $\alpha \in \langle \rho_0, \rho_1 \rangle$ with an even number of factors of $\rho_1$ it also fixes all points less than $m$. Therefore $\alpha$ is identity and $\Gamma^t_0 \cap \langle \rho_0, \rho_1 \rangle = \langle \rho_1 \rangle$.

Now assume that $k > 2$ and let $\alpha \in \Gamma^t_0 \cap \Gamma^t_{\leq k}$ again fixing all points greater than or equal to $m$. Consider $\alpha$ as a word in $\Gamma^t_0$. Using the isomorphism $\phi$, we get a word $\alpha' \in \Gamma^b_0$ by changing all the factors of $\rho_j$ in $\alpha$ to $\rho_j$. Under the conditions of the proposition, we assume that the group $\Gamma^b_{\leq k}$ acts as a symmetric group on $\text{Orbit}(\Gamma^b_{\leq k}, m)$. Additionally, the action of $\alpha$ on all other orbits $\text{Orbit}(\Gamma^t_{\leq k}, p)$ is identical to the action of $\alpha'$ on $\text{Orbit}(\Gamma^b_{\leq k}, p)$, and thus any permutation in $\Gamma^t_{\leq k}$ that fixes the points greater than or equal to $m + b + 1$ can also be written as a word in $\Gamma^b_{\leq k}$. Thus the permutation associated with the word $\alpha'$ can also be written as a word in $\Gamma^b_{\leq k}$.

Finally, as $\Gamma^b$ is a string C-group, $\alpha' \in \Gamma^b_{\leq k}$ and $\alpha' \in \Gamma^b_0$ implies that $\alpha' \in \langle \rho_1, \ldots, \rho_k \rangle$. Again using the isomorphism $\phi$ we get $\alpha \in \langle \rho_1, \ldots, \rho_k \rangle$ as required.

\[ \square \]

If one can find appropriate groups $\Gamma^b$ which fit the conditions of the proposition above, then these can be used to build new string C-groups of the same rank for groups of larger permutation degree.

In particular, for each rank $r \in \{4, 5, 6\}$, the proof that $A_n$ can be represented as a rank $r$ string C-group (for all large $n$) is done by giving examples of string C-groups $\Gamma^2$ (see Figures 4, 5, and 6) that satisfy the conditions of Proposition 6.3, and such that for some $j$, the group $\Gamma^j$ is isomorphic to $A_n$. These examples then yield families of string C-groups $\Gamma^{j+4k} \cong A_{n+4k}$. Thus, for each rank, we need to find an example of $\Gamma^t \cong A_n$ for each value of $n \mod 4$.

**Theorem 6.4.** The alternating group $A_n$ has a rank 4 string C-group representation if and only if $n = 9$, $n = 10$, or $n \geq 12$. The alternating group $A_n$ has a rank 5 string C-group representation if and only if $n = 10$, or $n \geq 12$. Finally, The alternating group $A_n$ has a rank 6 string C-group representation if and only if $n = 11$ or $n \geq 13$.

**Proof.** All of the string C-group representations of $A_n$ for $n \leq 10$ were classified in [7], with $n = 11, 12, 13,$ and 14 subsequently classified in [9]. It remains to show that, for all $n \geq 15$, there are rank 4, rank 5, and rank 6 string C-group representations of $A_n$. 

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For each rank, by listing the generators $\rho_i$, we now give four examples of string C-groups $\Gamma^2$ that fit the conditions of Proposition 6.3, and show which value of $t$ gives $\Gamma^t \cong A_n$. To clarify the notation, we include an image of the first family below.

Figure 3: $\Gamma$, $\Gamma_2$, $\Gamma_3$, and $\Gamma_7$ from Family 1 of rank 4 string C-group representations. The group $\Gamma_3$ is isomorphic to $A_9$, and the group $\Gamma_7$ is isomorphic to $A_{13}$.

Family 1: $\Gamma^{3+4k} \cong A_{9+4k}$.
\[
\begin{align*}
\rho_0 &= (6, 7). \\
\rho_1 &= (5, 6)(7, 8). \\
\rho_2 &= (2, 3)(4, 5). \\
\rho_3 &= (1, 2)(3, 4).
\end{align*}
\]

Family 2: $\Gamma^{3+4k} \cong A_{18+4k}$.
\[
\begin{align*}
\rho_0 &= (15, 16). \\
\rho_1 &= (4, 5)(6, 7)(14, 15)(16, 17). \\
\rho_2 &= (1, 2)(3, 4)(6, 8)(9, 10)(11, 12)(13, 14). \\
\rho_3 &= (2, 3)(4, 6)(5, 7)(8, 9)(10, 11)(12, 13).
\end{align*}
\]

Family 3: $\Gamma^{4+4k} \cong A_{15+4k}$.
\[
\begin{align*}
\rho_0 &= (11, 12). \\
\rho_1 &= (3, 4)(5, 7)(6, 11)(10, 9)(12, 13). \\
\rho_2 &= (2, 3)(4, 6)(5, 8)(7, 10). \\
\rho_3 &= (1, 2)(3, 5)(4, 7)(10, 9).
\end{align*}
\]

Family 4: $\Gamma^{4+4k} \cong A_{16+4k}$.
\[
\begin{align*}
\rho_0 &= (12, 13). \\
\rho_1 &= (3, 4)(5, 7)(6, 9)(10, 12)(13, 14). \\
\rho_2 &= (2, 3)(4, 6)(5, 8)(7, 10). \\
\rho_3 &= (1, 2)(3, 5)(4, 7)(8, 11).
\end{align*}
\]

Figure 4: Rank groups $\Gamma^2$ satisfying the conditions of Lemma 6.3

First let us prove that $\Gamma^t \cong A_n$ for all given families. To do this, we rely on the fact that if $\Gamma^t$ is primitive subgroup of $S_n$ and $\Gamma^t$ contains a 3-cycle then $\Gamma^t \geq A_n$. All twelve given families of groups $\Gamma^t$ are constructed so that for all $t \geq 2$

\[
(\rho_2\rho_1\rho_0\rho_1)^2
\]

is a 3-cycle. Thus we will only need to show why the groups are primitive. All twelve given families of groups $\Gamma^t$ are also constructed so that $\langle \rho_0, \rho_1, \rho_2 \rangle$ acts as a symmetric group on the orbit of the point $m$. Therefore, once $k \geq 5$, in all cases $\Gamma^t$ will contain a symmetric group acting on more than half its permutation degree, and thus cannot be imprimitive. The isomorphisms between $\Gamma^t$ and $A_n$ have been found in MAGMA for $k \leq 4$. Therefore,
Family 1: $\Gamma^{3+4k} \cong A_{13+4k}$.
\begin{align*}
\rho_0 &= (10, 11). \\
\rho_1 &= (9, 10)(11, 12). \\
\rho_2 &= (4, 5)(8, 9). \\
\rho_3 &= (1, 2)(3, 4)(5, 6)(7, 8). \\
\rho_4 &= (2, 3)(6, 7).
\end{align*}

Family 2: $\Gamma^{3+4k} \cong A_{14+4k}$.
\begin{align*}
\rho_0 &= (11, 12). \\
\rho_1 &= (10, 11)(12, 13). \\
\rho_2 &= (5, 6)(9, 10). \\
\rho_3 &= (2, 3)(4, 5)(6, 7)(8, 9). \\
\rho_4 &= (1, 2)(3, 4)(5, 6)(7, 8).
\end{align*}

Family 3: $\Gamma^{3+4k} \cong A_{15+4k}$.
\begin{align*}
\rho_0 &= (12, 13). \\
\rho_1 &= (9, 12)(13, 14). \\
\rho_2 &= (3, 4)(5, 7)(6, 9)(10, 11). \\
\rho_3 &= (2, 3)(4, 6)(5, 8)(7, 10). \\
\rho_4 &= (1, 2)(3, 5)(4, 7)(10, 11).
\end{align*}

Family 4: $\Gamma^{3+4k} \cong A_{12+4k}$.
\begin{align*}
\rho_0 &= (9, 10). \\
\rho_1 &= (8, 9)(10, 11). \\
\rho_2 &= (1, 2)(3, 4)(5, 6)(7, 8). \\
\rho_3 &= (2, 3)(6, 7). \\
\rho_4 &= (3, 5)(4, 6).
\end{align*}

Figure 5: Rank 5 groups $\Gamma^2$ satisfying the conditions of Lemma 6.3

all the given families of groups yield alternating groups, and using Proposition 6.3, they have been shown to be string C-groups, with the base case of $\Gamma^2$ checked in Magma.

We note that for the remaining values of $t$ each of these groups yields a string C-group representation of a symmetric group, as either $\rho_0$ or $\rho_1$ will be odd, and $\Gamma$ will still contain an alternating group.

### 7 Main Theorem

In this section, we put together all of our results to summarize the relationship between alternating groups and string C-groups.

**Theorem 7.1.** For all ranks $r \geq 3$, and all $n \geq 2r + 1$, if $n \geq 12$, the alternating group $A_n$ has a representation as a rank $r$ string C-group.

**Proof.** The small values of $n = 12$, $n = 13$, and $n = 14$ follow from [9]. Ranks 3, 4, 5, and 6, were considered in Proposition 6.1 and Theorem 6.4. For $r \geq 6$, when $n$ is even this follows from Corollary 5.2, and for $r \geq 7$ when $n$ is odd it follows from Corollary 4.3.

**Corollary 7.2.** For each rank $r$ it is known exactly which alternating groups can be represented as a string C-group of that rank. Similarly, for each alternating group $A_n$, the exact set of ranks of string C-groups for which it can be represented is known.

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Figure 6: Rank 6 groups $\Gamma^2$ satisfying the conditions of Lemma 6.3

Proof. This follows from the previous theorem in addition to Theorem 1.1 of [3], which states that the highest rank of a string C-group representation of $A_n$ is 3 if $n = 5$, 4 if $n = 9$, 5 if $n = 10$, 6 if $n = 11$, and $(n-1)/2$ if $n \geq 12$. Moreover, if $n = 3, 4, 6, 7, or 8$, the group $A_n$ is not a string C-group.

Corollary 7.3. The group $A_{11}$ is the only alternating group that has string C-group representations of two ranks $r_1$ and $r_2$, but not all ranks $r_i$ in between $r_1$ and $r_2$.

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