Asymptotic completeness of wave operators for Schrödinger operators with time-periodic magnetic fields

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Abstract Under the effect of suitable time-periodic magnetic fields, the velocity of a charged particle grows exponentially in $t$; this phenomenon provides the asymptotic completeness for wave operators with slowly decaying potentials. These facts were shown under some restrictions for time-periodic magnetic fields and the range of wave operators. In this study, we relax these restrictions and finally obtain the asymptotic completeness of wave operators. Additionally, we show them under generalized conditions, which are truly optimal for time-periodic magnetic fields. Moreover, we provide a uniform resolvent estimate for the perturbed Floquet Hamiltonian.

Keywords: quantum scattering theory; time-periodic magnetic fields; Floquet Hamiltonian; Time-periodic systems

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1 Introduction

We study a scattering problem for a charged particle under the effect of a time-periodic magnetic field. In this study, we assume that the charged particle moves on a plane $\mathbb{R}^2$ in the presence of a time-periodic magnetic field $B(t) = (0, B(t))$ with $B(t + T) = B(t)$, which is always perpendicular to the plane. Subsequently, the free Hamiltonian for this system is given by

$$H_0(t) = \frac{(p - qA(t, x))^2}{2m}, \quad A(t, x) = (-B(t)x_2, B(t)x_1)/2,$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $p = (p_1, p_2) = -i(\partial_1, \partial_2)$, $m > 0$ and $q \in \mathbb{R}\{0\}$ are position, momentum, mass, and charge of a particle, respectively. $B(t) \in L^\infty(\mathbb{R})$ is the intensity of the magnetic field at time $t$. The wave function $\psi(t, x)$ described by this system satisfies the following time-dependent Schrödinger equation;

$$\begin{cases} 
  i\partial_t \psi(t, x) = H_0(t)\psi(t, x), \\
  \psi(0, x) = \psi_0
\end{cases},$$

By defining a propagator for $H_0(t)$ as $U_0(t, s)$, the wave function $\psi(t, x)$ is denoted by $\psi(t, x) = U_0(t, 0)\psi_0$, which we refer to as family of unitary operators $\{U_0(t, s)\}_{(t, s) \in \mathbb{R}^2}$ a propagator for $H_0(t)$ if each component satisfies

$$\begin{align*}
  i\partial_t U_0(t, s) &= H_0(t)U_0(t, s), \quad i\partial_s U_0(t, s) = -U_0(t, s)H_0(s) \\
  U_0(t, \theta)U_0(\theta, s) &= U_0(t, s), \quad U_0(s, s) = \text{Id}_{L^2(\mathbb{R}^2)}.
\end{align*}$$

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Under these settings, the classical trajectory $x(t)$ and $p(t)$ of this system can be expressed in the form

$$x(t) := U_0(0, t)xU_0(t, 0), \quad p(t) := U_0(0, t)pU_0(t, 0).$$

Let us define $L := x_1 p_2 - x_2 p_1$, $\omega(t) = qB(t)/m$, and $\Omega(t) = \int_0^t \omega(s) ds$. It can be noted that $\tilde{U}_0(t, 0) := e^{-i\Omega(t)}L U_0(t, 0)$ is a propagator for $\tilde{H}_0(t) := p^2/(2m) + q^2 B^2(t) x^2/(8m)$ because $L$ commutes with $p^2$ and $x^2$. By defining $\tilde{x}(t)$ and $\tilde{p}(t)$ as $U_0(0, t)x\tilde{U}_0(t, 0)$ and $\tilde{U}_0(0, t)p\tilde{U}_0(t, 0)$, respectively, the straightforward calculation shows that

$$\tilde{x}'(t) = \tilde{U}_0(0, t) i[\tilde{H}_0(t), x]\tilde{U}_0(t, 0) = \tilde{p}(t)/m,$$
$$\tilde{p}'(t)/m = \tilde{U}_0(0, t) i[\tilde{H}_0(t), p/m]\tilde{U}_0(t, 0) = -(qB(t)/(2m))^2 \tilde{x}(t).$$

hold on $\mathscr{S}(\mathbb{R}^2)$, where $[\cdot, \cdot]$ denotes the commutator of operators, and these equations yield Hill’s equation

$$\tilde{x}''(t) + \left(\frac{qB(t)}{2m}\right)^2 \tilde{x}(t) = 0, \quad \begin{cases} \tilde{x}(0) = 0, \\ \tilde{x}'(0) = \tilde{p}(0)/m. \end{cases}$$

and a differential equation

$$\tilde{p}(t) = m\tilde{x}'(t),$$

refer to Kawamoto \[7\] §3. Hence, by introducing the fundamental solutions of the Hill’s equation, $\zeta_1(t)$ and $\zeta_2(t)$ as

$$\zeta_1''(t) + \left(\frac{qB(t)}{2m}\right)^2 \zeta_1(t) = 0, \quad \begin{cases} \zeta_1(0) = 1, \\ \zeta_1'(0) = 0, \end{cases}$$
$$\zeta_2''(t) + \left(\frac{qB(t)}{2m}\right)^2 \zeta_2(t) = 0, \quad \begin{cases} \zeta_2(0) = 0, \\ \zeta_2'(0) = 1, \end{cases}$$

we obtain

$$\begin{pmatrix} \tilde{x}(t) \\ \tilde{p}(t) \end{pmatrix} = \begin{pmatrix} \zeta_1(t) & \zeta_2(t)/m \\ m\zeta_1'(t) & \zeta_2'(t) \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}.$$

Noting

$$e^{-i\Omega(t)L} \begin{pmatrix} x \\ p \end{pmatrix} e^{i\Omega(t)L} = \begin{pmatrix} \hat{R}(\Omega(t)/2)x \\ \hat{R}(\Omega(t)/2)p \end{pmatrix}, \quad \hat{R}(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix},$$

(refer to Adachi-Kawamoto \[1\] or \[7\]), it can be deduced that

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} \zeta_1(t) & \zeta_2(t)/m \\ m\zeta_1'(t) & \zeta_2'(t) \end{pmatrix} \begin{pmatrix} \hat{R}(\Omega(t)/2)x \\ \hat{R}(\Omega(t)/2)p \end{pmatrix}$$
holds. Here we let

$$\mathcal{L} := \begin{pmatrix} \zeta_1(T) & \zeta_2(T)/m \\ m\zeta_1'(T) & \zeta_2'(T) \end{pmatrix}.  \quad 2$$
Then, for \( t = NT, N \in \mathbb{Z} \), we have
\[
\begin{pmatrix}
  x(NT) \\
  p(NT)
\end{pmatrix} = \mathcal{L}^N \begin{pmatrix}
  \hat{R}(\Omega(NT)/2)x \\
  \hat{R}(\Omega(NT)/2)p
\end{pmatrix},
\]
(2)

Using \( \zeta_1(t)\zeta_2'(t) - \zeta_1'(t)\zeta_2(t) = 1 \) for all \( t \in \mathbb{R} \), we have
\[
\det(\mathcal{L} - \lambda) = \lambda^2 - D\lambda + 1,
\]
where \( D = \zeta_1(T) + \zeta_2'(T) \) is referred to as the discriminant of the Hill’s equation, and together with \( (2) \), we find the following lemma:

**Lemma 1.1.** Let \( N \in \mathbb{Z}, |N| \gg 1 \). Then, for all \( \phi \in C_0^\infty(\mathbb{R}^2) \), there exist constants \( \lambda_c, C_e, C_p, C_m > 0 \) such that
\[
(C_e)^{-1}e^{\lambda_cN} \leq \|xU_0(NT,0)\phi\|_{(L^2(\mathbb{R}^2))^2} \leq Ce^{\lambda_cN}, \quad \text{if } D^2 > 4,
\]
\[
(C_p)^{-1}N \leq \|xU_0(NT,0)\phi\|_{(L^2(\mathbb{R}^2))^2} \leq C_pN, \quad \text{if } D^2 = 4,
\]
\[
\|xU_0(NT,0)\phi\|_{(L^2(\mathbb{R}^2))^2} \leq C_m, \quad \text{if } D^2 < 4,
\]
hold.

This lemma implies that for \( D^2 = 4 \), the particle assumes a uniform linear motion; however, for \( D^2 > 4 \), the asymptotic velocity of the particle grows exponentially in \( t \). Such physical phenomena were reported by Korotyaev [10] and the scattering theory for the case where \( D^2 = 4 \) with dimensions \( n = 3 \) and \( D^2 > 4 \) with dimension \( n = 2, 3 \) have been considered. Conversely, [7] found a relationship between the repulsive Hamiltonian \( p^2 - x^2 \) and \( D^2 > 4 \); Schrödinger operator \( p^2 \) and \( D^2 = 4 \), as defined by the Floquet operator.

In this study, we focus on the case \( D^2 > 4 \) and prove the asymptotic completeness of the wave operators. The scattering theory for a time-periodic magnetic field was first considered by [10], who proved the asymptotic completeness under some technical conditions of the magnetic field; thereafter, Adachi-Kawamoto [1] proved the asymptotic completeness of wave operators for the case where the magnetic field is pulsed. In the study of [1], they discovered an explicit formula for the integral kernel of free propagators, which indicated that \( D^2 > 4 \) and \( \zeta_2(T) \neq 0 \) is the best possible condition for showing asymptotic completeness in the pulsed case; if \( \zeta_2(T) = 0 \), the absolute value of the integral kernel of \( U_0(nT,0)\phi, n \in \mathbb{Z} \), diverges infinity for any \( \phi \in L^2(\mathbb{R}^2) \). Hence, it still remains important to determine the asymptotic completeness for general time-periodic magnetic fields with only two conditions \( D^2 > 4 \) and \( \zeta_2(T) \neq 0 \); this factor is considered herein.

**Assumption 1.2.** Suppose that \( \zeta_1(t), \zeta_2(t), \zeta_2'(t) \), and \( \zeta_2''(t) \) are continuous functions on \( t \in [0, T) \), and that for all \( t \in [0, T) \) and \( N \in \mathbb{Z} \) there exist \( A_{1,N}, A_{2,N}, A_{3,N}, A_{4,N} \) such that the solutions to \( (1) \) satisfy
\[
\begin{pmatrix}
  \zeta_1(t + NT) \\
  \zeta_2(t + NT)
\end{pmatrix} = \begin{pmatrix}
  A_{1,N} & A_{2,N} \\
  A_{3,N} & A_{4,N}
\end{pmatrix} \begin{pmatrix}
  \zeta_1(t) \\
  \zeta_2(t)
\end{pmatrix}.
\]
Moreover, for some \( \lambda > 0, \tilde{\lambda} \leq \lambda \), and for \( |N| \gg 1 \), there exist \( 0 < c_3 < C_3 \) and \( 0 < c_4 < C_4 \) such that
\[
c_3e^{\lambda N} \leq |A_{3,N}| \leq C_3e^{\lambda N}, \quad c_4e^{\tilde{\lambda} N} \leq |A_{4,N}| \leq C_4e^{\tilde{\lambda} N},
\]
holds.
Remark 1.3. Owing to Lemma 8 in [7], this assumption will be true for $D^2 = (\zeta_1(T) + \zeta_2(T))^2 > 4$ and $\zeta_2(T) \neq 0$, and according to Kargel-Korotyaev [5], the model of $B(t)$ such that $\zeta_2(T) \neq 0$ is known if $B(t)$ is even. To simplify the proofs, there is need to handle this assumption. In this case, where $D^2 > 4$ and $\zeta_2(T) \neq 0$, it is possible that $A_{1,N} \sim e^{-\lambda N}$ as $|N| \gg 1$ (refer also to [10]). The assumption on $A_{1,N}$ is stated to admit such cases.

We assume the following on the potential $V$;

Assumption 1.4. $V$ is a multiplication operator of $V(x)$, and $V(x) = \rho_1(x)\rho_2(x)$, where $\rho_1(x) = |V(x)|^{1/2}$ and $\rho_2(x) = \text{sign}(V)|V|^{1/2}$ satisfies the following: $V$ is in $C(\mathbb{R}^2)$ and is bounded. Moreover, there exists $p > 4$ such that $\rho_1, \rho_2 \in L^p(\mathbb{R}^2)$.

Remark 1.5. To simplify the proof, we used this assumption in this study. In the case where $|V(x)| \leq C \langle x \rangle^{-\rho}$, $\rho > 0$, Assumption 1.4 demands $\rho > 2/p$ and noting that we can consider $p > 4$ to be sufficiently large, this assumption allows any small $\rho > 0$. In this sense, we can consider the scattering theory even if $V$ decays slowly in $x$.

Here, let us define $U(t,0)$ as a propagator for $H(t) = H_0(t) + V$, and the unique existence of propagator $U(t,0)$ under this assumption is guaranteed by Yajima [13].

Under these settings, we can define the Floquet Hamiltonian associated with $H_0$ and $H$. Let $\mathcal{K} := L^2([0,T]; L^2(\mathbb{R}^2))$ and define the self-adjoint operators acting on $\mathcal{K}$ as

$$\hat{H} = \hat{H}_0 + V, \quad \hat{H}_0 = -i\partial_t + H_0$$

and term $\hat{H}$ and $\hat{H}_0$ as the Floquet Hamiltonian associated with $H_0(t)$ and $H(t) = H_0(t) + V$, respectively. Under these assumptions, we can prove the compactness of $V(H - i)^{-1}$ on $\mathcal{K}$ (refer to, e.g., [7] (refer also to [10] and [1])). However, our scheme does not demand this property.

As a theorem of this study, we first obtain the following Theorem:

Theorem 1.6. Under assumptions 1.2 and 1.4 for all $\phi \in \mathcal{K}$, there exists $C > 0$ such that

$$\sup_{\lambda \in \mathbb{R} \setminus \sigma_{pp}(\hat{H}), \mu > 0} \left\| |V|^{1/2}(\hat{H} - \lambda \mp i\mu)^{-1}|V|^{1/2}\phi \right\|_{\mathcal{K}} \leq C \|\phi\|_{\mathcal{K}}$$

holds, where $\sigma_{pp}(\hat{H})$ denotes the set of pure point spectra of $\hat{H}$.

Thanks to this theorem, the following Theorem immediately follows:

Theorem 1.7. Under assumptions 1.2 and 1.4 $\sigma_{\text{sing}}(\hat{H}) = \emptyset$, where $\sigma_{\text{sing}}(A)$ denotes the set of the singular continuous spectrum of $A$.

Owing to the correspondence of the spectrum sets between $\hat{H}$ and the monodromy operator $U(T,0)$, refer to, for example, Proposition 3.3 of Møller [11], we also have the following corollary:

Corollary 1.8. Under Assumptions 1.2 and 1.4 $\sigma_{\text{sing}}(U(T,0)) = \emptyset$. 

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Remark 1.9. Recently, [7] showed the absence of singular continuous spectra of $U(T, 0)$ using the Mourre theory. However, in this study, $V \in C^2(R^2)$ is required, and hence, our result is successful in relaxing this condition.

Under these assumptions, we can obtain the existence and completeness of wave operators:

Theorem 1.10. Under the assumption 1.2 and the assumption 1.4, the wave operators

$$W^\pm = s- \lim_{t \to \pm \infty} U(t, 0)^* U_0(t, 0)$$

exist and complete, that is,

$$\text{Ran} \left( W^\pm \right) = L^2_{ac}(U(T, 0))$$

holds, where $L^2_{ac}(U(T, 0)) \subset L^2(R)$ indicates the subspace of the absolutely continuous spectrum of $U(T, 0)$.

Remark 1.11. For the case where $V$ has singularities, by denoting $V = V^r + V^{\text{sing}}$, we can show all theorems if both $V^r$ and $V^{\text{sing}}$ are included in $L^\tilde{p}(R^2)$ with some $\tilde{p} > 2$. However, in this case, the potential decays slowly, and the singularity is also weak. In this sense, we do not discuss this issue.

The first approach of the proof is to show the uniform resolvent estimate (URE) for $\hat{H}_0$, in which we imitate the approach in [10]. Owing to the URE and Kato’s smooth perturbation method, Kato [6], we have

$$\int_{\mathbb{R}} \left\| |V|^{1/2} e^{-i\sigma \hat{H}_0} \phi \right\|^2_{\mathcal{K}} d\sigma \leq C \| \phi \|^2_{\mathcal{K}}. \quad (4)$$

Here, Korotaev’s strong propagation estimate (Proposition 2.1) enables us to extend URE for $\hat{H}_0$ to that of $\hat{H}$. By employing Kato’s smooth perturbation method in URE for $\hat{H}$, we obtain a resolvent estimate for $\hat{H}$, and using this, we can prove the nonexistence of singular continuous spectra of $\hat{H}$. Moreover, two estimates (1) and (4) with replacement $\hat{H}_0 \to \hat{H}$ immediately prove the asymptotic completeness of the wave operators. For URE of $\hat{H}$, we employ the approach of Herbst [3]. Here, we employ the strong propagation estimate (Proposition 2.1).

The key approach for characterizing the range of wave operators for time periodic systems is the Howland-Yajima method; if the wave operators $W^\pm$ exist and that wave operators in the sense of the Floquet Hamiltonian

$$\hat{W}^\pm := s- \lim_{\sigma \to \pm \infty} e^{i\sigma \hat{H}} e^{-i\sigma \hat{H}_0}$$

exist and satisfy $\text{Ran} \left( \hat{W}^\pm \right) = \mathcal{K}_{ac}(\hat{H})$, then the asymptotic completeness

$$\text{Ran} \left( W^\pm \right) = L_{ac}(U(T, 0))$$

holds, where $\mathcal{K}_{ac}(\hat{H}) \subset \mathcal{K}$ is the space of the absolutely continuous spectrum of $\hat{H}$. Hence, to prove Theorem 1.10, we should show the existence of $W^\pm$, $\hat{W}^\pm$, and $\text{Ran} \left( \hat{W}^\pm \right) = \mathcal{K}_{ac}(\hat{H})$. Therefore, for the time periodic case, we need to extend URE for $\hat{H}_0$ to that of $\hat{H}$.
The equivalence

\[ L^2(\mathbb{R}^2) = L_{sc}(U(T, 0)) \oplus L_{ac}(U(T, 0)) \oplus L_{pp}(U(T, 0)). \]

where \( L_{sc}(U(T, 0)) \), \( L_{ac}(U(T, 0)) \), and \( L_{pp}(U(T, 0)) \) denote the space of the singular continuous spectral, absolutely continuous spectral, and pure point spectra, respectively.

## 2 Uniform resolvent estimates for \( \hat{H}_0 \)

In this section, we consider the URE for \( \hat{H}_0 \) under assumption 1.2. In the following, \( \| \cdot \|_p \), \( 1 \leq p \leq \infty \) denotes \( \| \cdot \|_{L^p(\mathbb{R}^2)} \). The estimation key for showing the URE is the following dispersive estimates for the free propagator \( U_0(t, s) \):

**Proposition 2.1.** For all \( \phi \in L^1(\mathbb{R}^2) \), the dispersive estimates

\[ \| U_0(\tau, s)\phi \|_\infty \leq \frac{C}{|\Gamma(\tau, s)|} \| \phi \|_1 \]  

holds, where

\[ \Gamma(\tau, s) = |\zeta_1(s)\zeta_2(\tau) - \zeta_1(\tau)\zeta_2(s)| \]  

Moreover, for \( \eta_1, \eta_2 \in L^{\hat{p}}(\mathbb{R}) \) with \( 2 \leq \hat{p} < \infty \) and \( \psi \in L^2(\mathbb{R}^2), \)

\[ \| \eta_1 U_0(\tau, s)\eta_2 \psi \|_2 \leq C|\Gamma(\tau, s)|^{-2/\hat{p}} \| \eta_1 \|_{\hat{p}} \| \eta_2 \|_{\hat{p}} \| \psi \|_2. \]  

**Proof.** This proposition was proven by [10] for a special case of magnetic fields. Inequality (5) can be shown using (35) in [7] (or refer also to (7.7) of [1] and Lemma 2.3 in Kawamoto [8]). Hence, we only show (7). Owing to the Riesz-Thorin interpolation theorem, we have for \( 2 \leq Q \leq \infty \) and \( \phi \in L^{Q/(Q-1)}(\mathbb{R}^2) \)

\[ \| U_0(\tau, s)\phi \|_Q \leq C|\Gamma(\tau, s)|^{-2(1/2-1/Q)} \| \phi \|_{(Q/(Q-1))} \]

holds. Hence, for \( \psi \in L^2(\mathbb{R}^2) \) and \( Q = 2\hat{p}/(\hat{p} - 2) \),

\[ \| \eta_1 U_0(\tau, s)\eta_2 \psi \|_2 \leq C \| \eta_1 \|_{\hat{p}} \| U_0(\tau, s)\eta_2 \psi \|_Q \]

\[ \leq C|\Gamma(\tau, s)|^{-2/\hat{p}} \| \eta_1 \|_{\hat{p}} \| \eta_2 \psi \|_{2\hat{p}/(\hat{p}+2)} \]

\[ \leq C|\Gamma(\tau, s)|^{-2/\hat{p}} \| \eta_1 \|_{\hat{p}} \| \eta_2 \|_{\hat{p}} \| \psi \|_2. \]  

\[ \square \]

Before we show the URE for \( \hat{H}_0 \), we remark some properties for \( \zeta_j(t), j = 1, 2 \). By the definition of \( \zeta_j(t) \), one discovery

\[ \zeta_1(t)\zeta_2'(t) - \zeta_1'(t)\zeta_2(t) = 1. \]
Owing to this equation, one sees that zero points for $\zeta(t)$ and $t \in [0, T)$ are unique. Indeed, if $t_0 \in [0, T)$ exists such that $\zeta_1(t_0) = 0$, then (9) yields $\zeta_1'(t_0)\zeta_2(t_0) = -1 \neq 0$, that is, $\zeta_1'(t_0) \neq 0$. The same is true for $\zeta_2(t)$. Now, we divide $[0, T)$ into

$$[0, T) = \Omega_1^l \cup \Omega_2^l \cup \Omega_{j_l}^l \cup \{0, t_1^{(l)}, t_2^{(l)}, \ldots, t_{j_l-1}^{(l)}\}, \quad J_l \in \mathbb{N}, \quad l = 1, 2 \quad (10)$$

where $\zeta(t_0^{(l)}) = 0$, $k \in \{1, 2, \ldots, J_l - 1\}$, and $\zeta(t) \neq 0$ for all $t \in \Omega_{j_l}^l$, $j \in \{1, \ldots, J_l\}$.

Noting (11), we also notice that $\zeta_1(t)/\zeta_2(t)$ (resp. $\zeta_2(t)/\zeta_1(t)$) is a monotonically increasing function (resp. the monotone decreasing function) on $\Omega_2^l$ (resp. $\Omega_1^l$) and satisfies

$$\frac{d}{dt} \zeta_1(t) = \frac{1}{(\zeta_2(t))^2} > 0, \quad \text{(resp. } \frac{d}{dt} \zeta_2(t) = -\frac{1}{(\zeta_1(t))^2} < 0 \text{).}$$

**Proposition 2.2.** For all $\phi \in \mathcal{X}$, there exists a constant $C > 0$ such that

$$\sup_{\lambda \in \mathbb{R}, \mu > 0} \left\| (\rho_1(\hat{H}_0 - \lambda \mp i\mu)^{-1}\rho_2\phi) \right\|_{\mathcal{X}} \leq C\|\phi\|_{\mathcal{X}} \quad (12)$$

**Proof.** The fundamental proof is based on [10]. Owing to the Laplace transform, we have:

$$(\rho_1(\hat{H}_0 - \lambda - i\mu)^{-1}\rho_2\phi)(t, x) = -i\rho_1(x) \int_0^\infty e^{-i\sigma(\hat{H}_0 - \lambda - i\mu)}(t, x) d\sigma. \quad (11)$$

Then, using the formula of $e^{-i\sigma\hat{H}_0}$, consider, for example, Yajima [12], we have

$$(\rho_1(\hat{H}_0 - \lambda - i\mu)^{-1}\rho_2\phi)(t, x) = i\rho_1(x) \sum_{N=1}^\infty \int_0^T e^{i(t+NT-s)(\lambda+i\mu)}U_0(t + NT, s)(\rho_2\phi)(s, x) ds$$

$$+ i\rho_1(x) \int_0^T e^{i(t-s)(\lambda+i\mu)}U_0(t, s)(\rho_2\phi)(s, x) ds.$$

Then, for $p > 4$, $\lambda \in \mathbb{R}$ and $\mu > 0$, using Proposition 2.1 we have

$$\| (\rho_1(\hat{H}_0 - \lambda \mp i\mu)^{-1}\rho_2\phi) \|_{\mathcal{X}}$$

$$\leq C \sum_{N=0}^\infty \left\| \int_0^T \| (\rho_1U_0(t + NT, s)\rho_2\phi(s)) \|_2 L^2([0, T]) \right\| ds$$

$$\leq C \sum_{N=0}^\infty \| \rho_1 \|_p \| \rho_2 \|_p \left( \int_0^T \Gamma(t + NT, s)^{-2/2} \| \phi(s) \|_2 L^2([0, T]) \right)$$

$$\leq C \sum_{N=0}^\infty \| \rho_1 \|_p \| \rho_2 \|_p \| \phi \|_\mathcal{X} \left( \int_0^T \| \Gamma(t + NT, s) \|^{-4/p} ds \right)^{1/2}. \quad (12)$$

By (13),

$$\int_0^T \int_0^T |\Gamma(t + NT, s)|^{-4/p} ds dt$$

$$= \int_0^T |\zeta_2(t + NT)|^{-4/p} \int_0^T |\zeta_2(s)|^{-4/p}|(\zeta_1(s)/\zeta_2(s) - \alpha)|^{-4/p} ds dt$$
holds, where \( \alpha = \alpha(t) = \zeta_1(t + NT)/\zeta_2(t + NT) \). We decompose

\[
\int_0^T |\zeta_2(s)|^{-4/p}(\zeta_1(s)/\zeta_2(s) - \alpha)]^{-4/p} \, ds = \sum_{j=1}^{J_2} \int_{\Omega_j^2} |\zeta_2(s)|^{-4/p}(\zeta_1(s)/\zeta_2(s) - \alpha)]^{-4/p} \, ds,
\]

where \( J_2 \) and \( \Omega_j^2 \) are defined by the same rules as in (10). Here, we remark that \( J_2 \) can be considered a finite integer. Then, by denoting \( \tau = \zeta_1(s)/\zeta_2(s) \) and employing \( d\tau/ds = (\zeta_2(s))^{-2} \), we have

\[
\int_{\Omega_j^2} |\zeta_2(s)|^{-4/p}(\zeta_1(s)/\zeta_2(s) - \alpha)]^{-4/p} \, ds \leq I_1 + I_2 + I_3
\]

with

\[
I_1 := \int_{|\tau - \alpha| \leq 1} |\zeta_2(s)|^{2-4/p}|\tau - \alpha]^{-4/p} \, d\tau \leq C \int_{|\tau - \alpha| \leq 1} |\tau - \alpha]^{-4/p} \, d\tau \leq C,
\]

\[
I_2 := \int_{|\tau - \alpha| > 1, |\tau| \leq 1} |\zeta_2(s)|^{2-4/p}|\tau - \alpha]^{-4/p} \, d\tau \leq C \int_{|\tau| \leq 1} |\tau]^{-4/p} \, d\tau \leq C
\]

and

\[
I_3 := \int_{|\tau - \alpha| > 1, |\tau| \geq 1} |\zeta_2(s)|^{2-4/p}|\tau - \alpha]^{-4/p} \, d\tau
\]

where we use \( p > 4 \). Furthermore, \( |\zeta_2(s)|^{-1} \leq |\tau]^{-1}|\zeta_1(s)| \leq C|\tau]^{-1} \), we also have

\[
I_3 \leq C \int_{|\tau - \alpha| > 1, |\tau| \geq 1} |\tau]^{-2+4/p}|\tau - \alpha]^{-4/p} \, d\tau \leq C
\]

by \( p > 4 \). Combining all, one can obtain a constant \( C > 0 \), independent of \( t \), such that

\[
\sum_{j=1}^{J_2} \int_{\Omega_j^2} |\zeta_2(s)|^{-4/p}(\zeta_1(s)/\zeta_2(s) - \alpha)]^{-4/p} \, ds \leq C.
\]

Hence, one has

\[
\| (\rho_1(\hat{H}_0 - \lambda \mp i\mu)^{-1}\rho_2\phi) \|_x \leq C \| |V|^{1/2} \parallel \phi \|_x \| \sum_{N=0}^{\infty} \left( \int_0^T |\zeta_2(t + NT)|^{-4/p} dt \right) \right)^{1/2}.
\]

We now calculate the integral in \( t \). By \( \zeta_2(t + NT) = A_{3,N}\zeta_1(t) + A_{4,N}\zeta_2(t) \), we have

\[
\int_0^T |\zeta_2(t + NT)|^{-4/p} \, dt \leq |A_{3,N}|^{-4/p} \int_0^T |\zeta_1(t)|^{-4/p} |1 + r_0(\zeta_2(t)/\zeta_1(t))|^{-4/p} \, dt \quad (14)
\]

where \( r_0 = A_{4,N}/A_{3,N} \). We again decompose

\[
\int_0^T |\zeta_1(t)|^{-4/p} |1 + r_0(\zeta_2(t)/\zeta_1(t))|^{-4/p} \, dt = \sum_{j=1}^{J_1} \int_{\Omega_j^1} |\zeta_1(t)|^{-4/p} |1 + r_0(\zeta_2(t)/\zeta_1(t))|^{-4/p} \, dt,
\]
where $J_1$ and $\Omega^1_J$ are defined by the same rules as in (10). By denoting $\sigma = \zeta_2(t)/\zeta_1(t)$, we have $d\sigma/(dt) = -(\zeta_1(t))^{-2}$ and

$$\int_{\Omega^1_J} |\zeta_1(t)|^{-4/p} |1 + r_0(\zeta_2(t)/\zeta_1(t))|^{-4/p} dt \leq I_4 + I_5 + I_6$$

with

$$I_4 := \int |1 + r_0\sigma| \geq 1/2, |\sigma| \leq 1 |\zeta_1(t)|^{2-4/p} |1 + r_0\sigma|^{-4/p} d\sigma \leq C,$$

$$I_5 := \int |1 + r_0\sigma| \geq 1/2, |\sigma| \geq 1 |\sigma|^{-2+4/p} |1 + r_0\sigma|^{-4/p} d\sigma \leq C.$$

and

$$I_6 := \int |1 + r_0\sigma| \leq 1/2 |\zeta_1(t)|^{2-4/p} |1 + r_0\sigma|^{-4/p} d\sigma.$$

Now, we estimate $I_6$. Based on assumption 1.2 there are $\lambda > 0$ and $\tilde{\lambda} \leq \lambda$ such that

$$(c_4/C_3)e^{-(\lambda-\tilde{\lambda})N} \leq |r_0| \leq (C_4/c_3)e^{-(\lambda-\tilde{\lambda})N}.$$ holds. Then, it can be calculated that $|\sigma| \geq Ce^{(\lambda-\tilde{\lambda})N}$, that is, $|\sigma|^{-1} \leq Ce^{-(\lambda-\tilde{\lambda})N}$ on the support of $|1 + r_0\sigma| \leq 1/2$, which yields $|\zeta_1(t)| \leq |\sigma|^{-1}|\zeta_2(t)| \leq Ce^{-(\lambda-\tilde{\lambda})N}$. Thus, we also have

$$I_6 \leq Ce^{-N(\lambda-\tilde{\lambda})(2-4/p)} \int_{|1 + r_0\sigma| \leq 1/2} |1 + r_0\sigma|^{-4/p} d\sigma$$

$$\leq Ce^{-N(\lambda-\tilde{\lambda})(2-4/p)} |r_0|^{-1}$$

$$\leq Ce^{-N(\lambda-\tilde{\lambda})(1-4/p)}$$

$$\leq C.$$

Thus, we finally obtain that for some $C > 0$,

$$\int_0^T |\zeta_2(t + NT)|^{-4/p} dt \leq Ce^{-4\lambda N/p}$$

holds for $p > 4$. Finally, from (12), (13), and (15), we obtain

$$\left\| \rho_1(\hat{H}_0 - \lambda + i\mu)^{-1}\rho_2\phi \right\|_X \leq C \left\| V^{1/2} \phi \right\|_X \sum_{N=0}^\infty e^{-2\lambda N/p} \leq C \left\| \phi \right\|_X$$

It proves Proposition 2.2.
3 Uniform resolvent estimate for $\hat{H}$

In this section, we present the URE for $\hat{H}$. To demonstrate this, we employ the approach according to Herbst [3]. However, in [3], among the specific conditions, only the Stark Hamiltonian has been fully used, and imitating this approach may be difficult. To overcome this difficulty, we should find the alternative condition of $\hat{H}$, as well as the following Lipschitz continuity for resolvent of $\hat{H}_0$, which plays a crucial role in mimicking the approach of [3]:

**Theorem 3.1.** Let $z_\pm, w_\pm \in C_\pm$. Then, for all $\phi \in X$,

$$\left\| \left( \rho_1(\hat{H}_0 - z_\pm)^{-1}\rho_2 - \rho_1(\hat{H}_0 - w_\pm)^{-1}\rho_2 \right) \phi \right\|_X \leq C|z_\pm - w_\pm|\|\phi\|_X.$$ 

holds.

**Remark 3.2.** In the case where $k_0 = -\Delta$, the Hölder continuity

$$\left\| \langle x \rangle^{-s}( (k_0 - z_\pm)^{-1} - (k_0 - w_\pm)^{-1} ) \langle x \rangle^{-s} \phi \right\|_2 \leq C|z_\pm - w_\pm|^{(2s-1)/(2s+1)}\|\phi\|_2 \tag{16}$$

holds. Here, formally, we consider $s \to \infty$; then, the power on $|z_\pm - w_\pm|$ in (16) tends to 1 (that is, the Lipschitz continuity). In our model, compared with the exponential growth of $|x(t)|$ in $t$, the decay for the potential $\rho_1 \in L^p(\mathbb{R}^2)$ is fast. In the words of $k_0$, the potential decays exponentially in $x$. Thus, it is not surprising that Lipschitz continuation (16) holds.

This theorem can be obtained as the sub-consequence of the following lemma;

**Lemma 3.3.** Suppose Assumption [12]. For a sufficiently small $\varepsilon > 0$, let $\tau \in [-\varepsilon, \varepsilon]$. Define

$$\Sigma_R = \int_0^R \left\| \sigma \rho_1 e^{-i(H_0 + \lambda \tau i)\sigma}(\rho_2 \phi) \right\|_X d\sigma$$

for $R > 0$. Then, there exists $\varepsilon > 0$ such that for all $\tau \in [-\varepsilon, \varepsilon]$ and $\lambda \in \mathbb{R}$, the limit $\lim_{R \to \infty} \Sigma_R$ exists, satisfying

$$\lim_{R \to \infty} \Sigma_R < C\|\phi\|_X.$$ 

Furthermore,

$$s - \lim_{\sigma \to \infty} \sigma \rho_1 e^{-i(H_0 + \lambda \tau i)\sigma}\rho_2 = 0$$

holds.

**Proof.** Let $R = N_0T + s$ with $N_0 \in \mathbb{N}$ and $s \in [0, T]$. Then, the same calculations in the proof of Proposition [22] yield

$$\Sigma_R \leq C \sum_{N=0}^{N_0+1} \int_0^T NT e^{\varepsilon NT} \|\rho_1 U_0(t + NT, s)(\rho_2 \phi)\|_X ds$$

$$\leq C \sum_{N=0}^{N_0+1} |NT| e^{\varepsilon NT} e^{-2\lambda N/p} \|\phi\|_X$$

with $p > 4$. Then, if $|\varepsilon|$ is sufficiently small, there exists $\varepsilon_1 > 0$ such that $(|NT| e^{\varepsilon NT} e^{-2\lambda N/p}) \leq Ce^{-\varepsilon_1 N}$. This implies that $\lim \Sigma_R$ exists and $\lim \Sigma_R < \infty.$
Because

\[
p_1(\hat{H}_0 - z_\pm)^{-1}p_2\phi - p_1(\hat{H}_0 - w_\pm)^{-1}p_2\phi = \mp i \int_0^\infty \left( e^{\mp i \sigma (\hat{H}_0 - z_\pm)} - e^{\mp i \sigma (\hat{H}_0 - w_\pm)} \right) \rho_2 \phi d\sigma
\]

\[
= \mp i \int_0^\infty \sigma \rho_1 e^{\mp i \sigma \hat{H}_0} \rho_2 \left( \frac{e^{\mp i \sigma z_\pm} - e^{\mp i \sigma w_\pm}}{\sigma} \right) \phi d\sigma,
\]

and Lemma \text{3.3} with \( \lambda = \tau = 0 \), Theorem \text{3.4} can be proven.

### 3.1 Proof of Theorem \text{1.6}

Now, we prove Theorem \text{1.6}. The fundamental approach is based on the approach described by Herbst \cite{Herbst1974}. By the resolvent formula, we have that

\[
p_1(\hat{H} - \lambda - i\epsilon)^{-1}p_2 = (1 + p_1(\hat{H} - \lambda - i\epsilon)^{-1}p_2)^{-1} \cdot p_1(\hat{H}_0 - \lambda - i\epsilon)^{-1}p_2
\]

holds. Hence, we prove that for any \( \epsilon \geq 0 \), if \( \psi = \psi_\epsilon \in \mathcal{K} \) such that \( (1 + p_1(\hat{H} - \lambda - i\epsilon)^{-1}p_2)\psi = 0 \), then \( \psi \equiv 0 \). Therefore, we have that \( (1 + p_1(\hat{H} - \lambda - i\epsilon)^{-1}p_2) \) is invertible for any \( \epsilon \geq 0 \), which proves that Theorem \text{1.6} holds.

Let \( \varphi = (\hat{H}_0 - \lambda - i\epsilon)^{-1}p_2\psi \), then

\[
(\hat{H} - \lambda - i\epsilon)\varphi = 0
\]

holds. Here, remarking \( \hat{H} \) is a selfadjoint operator and \( \lambda \not\in \sigma_{pp}(\hat{H}) \), \text{(17)} implies \( \varphi \equiv 0 \), (i.e., \( \psi \equiv 0 \)) if \( \varphi \in \mathcal{D}(\hat{H}) \). Thus, we prove \( \varphi \in \mathcal{D}(\hat{H}) \), as follows: Using the resolvent formula

\[
\varphi = (\hat{H}_0 - i)^{-1}p_2\psi + (\lambda + \epsilon - i)(\hat{H}_0 - i)^{-1}\varphi,
\]

we notice that it suffices to show that \( \varphi \in \mathcal{K} \) shows \( \varphi \in \mathcal{D}(\hat{H}) \). Hence, we show \( \varphi \in \mathcal{K} \).

In the case where \( \epsilon > 0 \), it is clear that \( \varphi \in \mathcal{K} \); hence, we only consider the case where \( \epsilon \to 0 \), that is, \( \psi = -p_1(\hat{H}_0 - \lambda - i\delta)^{-1}p_2\psi \) and \( \varphi = (\hat{H}_0 - \lambda - i\delta)^{-1}p_2\psi \):

\[
\|\varphi\|^2_{\mathcal{K}} = \lim_{\epsilon \to 0} \left\| (\hat{H}_0 - \lambda - i\epsilon)^{-1}p_2\psi \right\|^2_{\mathcal{K}}
\]

\[
= \lim_{\epsilon \to 0} \left\{ \frac{-i}{2\epsilon} \left( p_2(\hat{H}_0 - \lambda + i\epsilon)^{-1} - (\hat{H}_0 - \lambda - i\epsilon)^{-1}p_2\psi, \psi \right) \right\}
\]

\[
= \lim_{\epsilon \to 0} \frac{-i}{2\epsilon} \left( \rho_2 \left( \hat{H}_0 - \lambda - i\epsilon)^{-1} - (\hat{H}_0 - \lambda + i\epsilon)^{-1} \right) \rho_2\psi, \psi \right)
\]

\[
= \lim_{\epsilon \to 0} \frac{-i}{2\epsilon} \left( \rho_2 \left( \hat{H}_0 - \lambda - i\epsilon)^{-1} - (\hat{H}_0 - \lambda + i\epsilon)^{-1} \right) \rho_2\psi, \psi \right) - \frac{i}{2\epsilon} (\psi, (\psi, (\psi, \psi)))
\]

\[
= \lim_{\epsilon \to 0} \lim_{\delta_1 \to 0} \left\{ \frac{-i}{2\epsilon} \left( \rho_2(\hat{H}_0 - \lambda - i\epsilon)^{-1}p_2 - p_1(\hat{H}_0 - \lambda - i\delta_1)^{-1}p_2\psi, \psi \right) \right\}
\]

\[
- \lim_{\epsilon \to 0} \lim_{\delta_2 \to 0} \left\{ \frac{-i}{2\epsilon} (\psi, (\rho_2(\hat{H}_0 - \lambda - i\epsilon)^{-1}p_2 - p_1(\hat{H}_0 - \lambda - i\delta_2)^{-1}p_2)\psi) \right\}
\]

\[
=: \lim_{\epsilon \to 0} \frac{-i}{2\epsilon} (I_1(\epsilon) - I_2(\epsilon)).
\]
We have

\[ I_1(\varepsilon) = \lim_{\delta_1 \to 0} \left( \langle \rho_2 \left( \hat{H}_0 - \lambda - i\varepsilon \right)^{-1} - \left( \hat{H}_0 - \lambda - i\delta_1 \right)^{-1} \rangle \rho_2 \right) \psi, \psi \rangle + \left( \frac{\rho_1 - \rho_2}{\rho_1} \right) \psi, \psi \rangle. \]

and

\[ I_2(\varepsilon) = \lim_{\delta_2 \to 0} \left( \psi, \left( \rho_2 \left( \hat{H}_0 - \lambda - i\varepsilon \right)^{-1} - \left( \hat{H}_0 - \lambda - i\delta_2 \right)^{-1} \right) \rho_2 \psi \right) \langle + \left( \psi, \frac{\rho_1 - \rho_2}{\rho_1} \psi \right). \]

These and (16) yield

\[ \| \varphi \|_2^2 \leq C \left( \lim_{\varepsilon \to 0} \lim_{\delta_1 \to 0} \frac{|\varepsilon - \delta_1|}{\varepsilon} + \lim_{\varepsilon \to 0} \lim_{\delta_2 \to 0} \frac{|\varepsilon - \delta_2|}{\varepsilon} \right) \| \psi \|_2^2 \leq C \| \psi \|_2^2, \]

where we use

\[ \left( \frac{\rho_1 - \rho_2}{\rho_1} \psi, \psi \right) - \left( \psi, \frac{\rho_1 - \rho_2}{\rho_1} \psi \right) = 0. \]

Therefore, we obtain \( \varphi \equiv 0 \) and which implies \( \psi \equiv 0 \).

### 3.2 Proof of Theorem 1.7

Next, we show the absence of a singular continuous spectrum of \( \hat{H} \). In the case where \( V \in C^2(\mathbb{R}^2) \), [7] showed this issue using the Mourre inequality. Hence, we relaxed this condition. From Theorem 1.6 we have that

\[ \sup_{\lambda \in \mathbb{R} \setminus \sigma_{pp}(\hat{H}), \varepsilon > 0} \| |V|^{1/2} \left( (\hat{H} - \lambda - i\varepsilon)^{-1} - (\hat{H} - \lambda + i\varepsilon)^{-1} \right) |V|^{1/2} \phi \|_\mathcal{K} \leq C \| \phi \|_\mathcal{K}. \]

Here, we let \((\cdot, \cdot)\) be an inner product of \( L^2(\mathbb{R}^2) \). Then, for all \( \lambda_1, \lambda_2 \in \mathbb{R} \),

\[ \left| \left( (E_{\hat{H}}(\lambda_1) - E_{\hat{H}}(\lambda_2)) |V|^{1/2} \phi, |V|^{1/2} \phi \right) \right| = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\lambda_2}^{\lambda_1} \left| \left| \left( (\tau - \lambda - i\varepsilon)^{-1} - (\tau - \lambda + i\varepsilon)^{-1} \right) |V|^{1/2} \phi, \phi \right| \right| d\tau \leq C|\lambda_2 - \lambda_1| \| \phi \|_\mathcal{K}^2 \]

holds, where \( E_{\hat{H}}(\cdot) \) is the spectral decomposition of \( \hat{H} \), implying that for the connected Borel set \( I := (\lambda_1, \lambda_2) \subset \mathbb{R} \setminus \sigma_{pp}(\hat{H}) \), \( E(I)|V|^{1/2} \phi \in \mathcal{K}_{ac}(\hat{H}) \). Here, \( |V|^{1/2} \) is the closed operator, which yields for all \( u \in \mathcal{K} \), \( E(I)u \in \mathcal{K}_{ac}(\hat{H}) \). Because we can arbitrarily take \( \lambda_1, \lambda_2 \), we find \( \mathcal{K} = \mathcal{K}_{ac}(\hat{H}) \), that is, \( \mathcal{K}_{sing}(\hat{H}) = \emptyset \).

### 4 Asymptotic completeness

Finally, the main theorem is presented. By Proposition 2.2 Theorem 1.6 and Kato’s smooth perturbation method, we get for all \( \varphi \in C^\infty_0(\mathbb{R} \setminus \sigma_{pp}(\hat{H})) \) and \( \phi, \psi \in \mathcal{K} \),

\[ \int_{-\infty}^{\infty} \| |V|^{1/2} \varphi(\hat{H}) e^{-i\sigma \hat{H}} \phi \|_\mathcal{K}^2 d\sigma \leq C \| \varphi(\hat{H}) \phi \|_\mathcal{K}, \quad \int_{-\infty}^{\infty} \| |V|^{1/2} e^{-i\sigma \hat{H_0}} \psi \|_\mathcal{K}^2 d\sigma \leq C \| \psi \|_\mathcal{K}^2 \]

(18)
holds. Hence for all $\sigma_1, \sigma_2 \in \mathbb{R}_\pm := \{a \in \mathbb{R} \mid \pm a \geq 0\}$,

$$\left\| e^{i\sigma_1 \hat{H}_0} e^{-i \sigma_1 \hat{H}} \varphi(\hat{H}) \phi - e^{i\sigma_2 \hat{H}_0} e^{-i \sigma_2 \hat{H}} \varphi(\hat{H}) \phi \right\|_{\mathcal{X}} \leq \sup_{\|\psi\|_{\mathcal{X}} = 1} \left( \int_{\sigma_2}^{\sigma_1} \left( |\rho_1 e^{-i\sigma \hat{H}} \varphi(\hat{H}) \phi, \rho_2 e^{-i\sigma \hat{H}_0} \psi \rangle \right) \frac{d\sigma}{\|\psi\|_{\mathcal{X}}} \right)^{1/2} \left( \int_{\sigma_2}^{\sigma_1} \left( |V|^{1/2} e^{-i\sigma \hat{H}_0} \psi \right) \frac{d\sigma}{\mathcal{X}} \right)^{1/2} \to 0, \quad \text{as} \quad \sigma_1, \sigma_2 \to \infty$$

holds, where we use (18). Hence, there is a strong limit: $s- \lim_{\sigma \to \pm \infty} e^{i\sigma \hat{H}_0} e^{-i \sigma \hat{H}} \varphi(\hat{H})$ exists for all $\varphi \in C_0^\infty(\mathbb{R} \setminus \sigma_{pp}(\hat{H}))$. Similarly, we have a strong limit: $s- \lim_{\sigma \to \pm \infty} e^{i\sigma \hat{H}_0} e^{-i \sigma \hat{H}_0}$. Consequently, we obtain

$$\text{Ran} \left( s- \lim_{\sigma \to \pm \infty} e^{i\sigma \hat{H}_0} e^{-i \sigma \hat{H}_0} \right) = \mathcal{K}_{ac}(\hat{H}).$$

Using the Howland-Yajima method, the proof for Theorem 1.6 is completed by showing the existence of $W^\pm$; more precisely, refer to §4 of [12] or the end §6 of [1]. We now show the existence of $W^\pm$. Owing to the Cook-Kuroda method, it suffices to show that for all $\phi \in \mathcal{S}(\mathbb{R}^2)$,

$$\int_1^\infty \left\| |V|^{1/2} U(t, 0) \phi \right\|_{L^2(\mathbb{R}^2)} dt \leq C. \quad (19)$$

Hence, we show that

$$\sum_{N=1}^{\infty} \int_0^T \left\| |V|^{1/2} U(t + NT, 0) \phi \right\|_2^2 dt \leq C. \quad (20)$$

holds. Then, (19) holds. By noting (3), for $p > 4$, the left hand side of (20) is smaller than

$$C \left\| |V|^{1/2} \right\|_p \| \phi \|_{2^p/(p+2)} \sum_{N=0}^{\infty} \int_0^T |\zeta_2(t + NT)|^{-2/p} dt,$$

where we also use (5) and (7) with $\tau = t + NT$ and $s = 0$. From (15), we obtain (20).

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