Approximate inertial manifolds for a shallow water model with varying bottom topography

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Abstract. We construct approximate inertial manifolds for the dynamical system associated with a two-dimensional shallow water model, which describes the motion of an incompressible fluid confined to shallow basin with varying bottom topography. Approximate inertial manifolds are Lipschitz manifolds with a thin neighbourhood that surrounds it and where each orbit of the dynamical system must enter into a finite time. The order of the manifold is the width of the thin neighbourhood and it is exponentially small compared to the dimension of the manifold. We estimate the order of the approximate inertial manifolds, giving the explicit dependence of all the constants with respect to the physical parameters.

1. Introduction

In [13] was derived a two-dimensional shallow water model, which describes the motion of an incompressible fluid confined to shallow basin with varying bottom topography. The shallow water model has been derived from a three-dimensional anisotropic eddy viscosity model of an incompressible fluid confined to shallow basin with varying bottom topography. To derive the shallow water model, the authors assume that the basin is shallow, i.e. the depth is much smaller than the typical horizontal length, and moreover they assume that the typical velocity of the fluid is much smaller than the velocity of the gravity waves. This last assumption is equivalent to consider the fluid motion on time scales much longer than the period of the gravity waves so that averaging on time suppresses gravity waves. The viscous shallow water model refines the lake system [1] and the great lake system [2]. In [13] was proved also the well posedness of the model. In this paper
we construct approximate inertial manifolds whose order decreases exponentially fast with respect to the dimension of the manifold. We give the dependence of all the constants with respect to physical parameters and in particular we give explicitly the order of the approximate inertial manifolds. The concept of inertial manifold was introduced [9], as part of the theory of dissipative differential equation. An inertial manifold for an evolutionary semigroup associated to a dissipative dynamical system, is a finite dimensional Lipschitz manifold which is positively invariant, and attracts all the orbits exponentially [16, 20, 21]. There is a substantial difference between the concept of inertial manifold and the concept of attractor for a dynamical system. The first one attracts the orbits exponentially, unlike the attractor attracts the orbits of the dynamical system arbitrarily. Therefore, if a dissipative dynamical system admits an inertial manifold, this will contain the attractor. To determinate the existence of the inertial manifold it is necessary that the so called spectral gap condition [21] is verified. Unfortunately, this spectral gap condition is not verified for Navier-Stokes equations. For that reason, it was subsequently introduced the notion of approximate inertial manifolds (AIM) [3, 5, 7, 8, 16, 17, 18, 21]. The existence of these manifolds does not require the spectral gap condition and therefore there exist for a broader class of dissipative dynamic systems. The AIM can be defined as a Lipschitz manifold with a thin neighbourhood that surrounds it and where each orbit of the system must enter into a finite time. The order of the manifold is the width of the thin neighbourhood and it is exponentially small compared to the size of AIM, hence the AIM gives an approximation of the attractor of exponential order. The AIM theory plays an important role in the development of new numerical algorithms suitable to the approximation of dissipative systems for long times [4, 7, 11, 12]. In the next section we introduce the equations of the shallow water model and the mathematical settings. In section 3 we prove the existence of the AIM and in section 4 we give the thickness of the thin neighbourhood in terms of the data.
2. The mathematical setting

The bi-dimensional equations that we consider are:

\[
\frac{du}{dt} + u \cdot \nabla u + \nabla p + \eta u = b^{-1} \nabla \cdot \left[ b \nu (\nabla u + (\nabla u)^T) - \mathbf{I} \cdot \nabla \cdot u \right] + f,
\]

\[
\nabla \cdot (bu) = 0,
\]

\[
u \cdot u = 0 \quad x \in \partial \Omega,
\]

\[
\tau \cdot (\nabla u + (\nabla u)^T) \cdot \nu = -\beta u \cdot \tau \quad x \in \partial \Omega.
\]  

(2.1)

\[u(x, t = 0) = u_0,\]

\[\Omega \subset \mathbb{R}^2\] is a bounded domain, with boundary \(\partial \Omega\), and we suppose that they are sufficiently regular. \(u(x, t)\) is the velocity of the fluid, \(x \in \Omega\), and \(t\) the temporal variable, \(0 < c \leq b(x) \leq C\) is a smooth positive bounded function which describes the bottom topography, \(\nu(x)\) is the viscosity, \(\eta(x)\) is a positive smooth bounded function defined in \(\Omega\) which represents the combined actions of the friction at the bottom and the wind pressure, \(I\) is the bi-dimensional identity \(\mathbf{I}\) and \(\nu\) are respectively the unity tangent and normal vector to the boundary \(\partial \Omega\), \(\beta(z)\) is a regular function defined in \(\partial \Omega\) which represents the friction coefficient at the boundary, and \(f(x)\) is the force term which describes the wind stress at the upper boundary. In [13] it was proved the well-posedness of (2.1). For the mathematical setting, we consider the following Hilbert spaces:

\[H = \{u : u \in L^2_b, \nabla \cdot (bu) = 0, \nu \cdot u = 0 \quad x \in \partial \Omega\}\]

(2.2)

\[V = \{u : u \in H^1_b, \nabla \cdot (bu) = 0, \nu \cdot u = 0 \quad x \in \partial \Omega\}\]

(2.3)

where \(L^2_b\) and \(H^1_b\) are Sobolev spaces with scalar products and weight norms

\[\langle u, v \rangle_b = \int_\Omega b u \cdot v dx, \quad |u|_b = \left( \int_\Omega b|u|^2 dx \right)^{1/2},\]

\[((u, v))_b = \int_\Omega b \nabla u : \nabla v dx, \quad \|u\|_b = \left( \int_\Omega b|\nabla u|^2 dx \right)^{1/2} .\]

The following Poincaré inequality holds:

\[|u|_b \leq \Pi \|u\|_b,\]

(2.4)

where \(\Pi = \Pi(\Omega)\). We write (2.1) in weakly form:

\[\frac{d}{dt} u + A_{bu} u + B(u, u) + \eta u = f\]

(2.5)
where $A_{b\nu}$ is the elliptic operator obtained by the Leray projection of the diffusive term. As usually, we denote by

$$D(A_{b\nu}) = \{ u \in H^2(\Omega), \nabla \cdot b u = 0 \text{ in } \Omega, \ u \cdot \nu = 0, \ \tau \cdot (\nabla u + (\nabla u)^T) \cdot \nu = -\beta u \cdot \tau \text{ in } \partial\Omega \},$$

and $D(A_b) \subset V \subset H \subset V'$, where the inclusions are continuous and dense. Moreover $V$ is compactly embedded in $H$. If $\beta(x) \geq \kappa(x)$, where $\kappa$ is the curvature of $\partial\Omega$, then the elliptic operator $A_{b\nu}$ is coercive (see [13]):

$$(A_{b\nu} u, u)_{b} \geq \frac{c_{\nu}}{C} \|u\|_{b}^2,$$

where $\bar{\nu}$ is the minimum of $\nu(x)$ in $\Omega$. The spectral problem associated to the compact self-adjoint operator $A_{b\nu}$ admits solution in $H$ [6], and from the coercivity (2.6) derives the existence of a non-decreasing sequence of positive eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$ with (see [14])

$$\lambda_n \sim d\lambda_1 n,$$

for $n \to \infty$, and a sequence of eigenfunctions which form an orthonormal basis in $H$. We denote by $P_n$ the projection onto the finite dimensional space generated by the first $n$ eigenfunctions and $Q_n = I - P_n$:

$$(2.8) \quad P_n u = y, \quad Q_n u = z \quad \text{and} \quad u = y + z.$$ 

It is easy to verify that [10]:

$$(2.9) \quad |e^{-A_{b\nu} t} Q_n|_{\mathcal{L}(H,V)} \leq \left(\frac{C}{c}\right)^{1/2} \left(\tilde{\nu} t + \lambda_{n+1}^2\right) e^{-\lambda_{n+1} t}, \quad t > 0,$$

$$(2.10) \quad |(I + \tau A_{b\nu}) P_n|_{\mathcal{L}(V)} \leq (1 + \tau \lambda_n) \leq e^{\tau \lambda_n},$$

$$(2.11) \quad |I|_{\mathcal{L}(P_n H, P_n V)} \leq \left(\frac{\lambda_n C}{c\nu}\right)^{1/2}.$$ 

If we consider an initial datum $u_{in}$ in a ball of $H$ with center in the origin and radius $R$, then (see [15, 19]) there exists a time $t_0(R)$, depending to $R$ and to $\nu, f, \Lambda, b$, such that $t \geq t_0$ results

$$(2.12) \quad |u(t)|_b \leq \rho_0, \quad \|u(t)\|_b \leq \rho_1,$$

where $\rho_0 = \frac{C_{11}}{c\nu} |f|_b$ is the radius of the absorbing balls in $H$, instead $\rho_1$ is the radius of the absorbing balls in $V$, whose explicit expression is given in [15, 19]. From (2.12) and (2.12) derives the existence of a compact global attractor $\mathcal{A}$, connected and maximal in $H$ and its Hausdorff dimension $\tilde{m}$ is bounded by

$$\tilde{m} - 1 \leq \frac{C^3}{c^{1/2}\Omega} \frac{|f|_b \Pi^{1/2}}{\nu^2} < \tilde{m}.$$
Moreover, it is possible to prove that:

\[ |\frac{d^k u}{dt^k}|_b \leq \frac{2k!}{\alpha^k} \rho_0, \quad \|\frac{d^k u}{dt^k}\|_b \leq \frac{2^{2k} k!}{\alpha^k} \rho_1, \]

for \( t \geq 2\alpha \), and \( \alpha = \alpha(\Omega, |f|_b, \|u_n\|_b, \nu) \) define the domain of analyticity \( \Delta = \{ \xi \in \mathbb{C} : |\Im \xi| \leq \Re \xi \text{ and } \Re \xi \leq \alpha \text{ or } \Re \xi \geq \alpha \text{ and } |\Im \xi| \leq \alpha \} \). The non-linear term \( B(u, v) = u \cdot \nabla v \) defines a continuous bilinear operator from \( V \times V \) in \( V' \) and from \( D(A_b) \times D(A_b) \) in \( H \). As it is usually in the theory of inertial manifold, we consider the prepared equation which consists to truncate the non linear term of (2.1) outside the absorbing ball, where essentially there are not dynamical effects. This trick is used to overcome the fact that the non linear term \( B(u, u) \) is not globally Lipschitz. We consider a \( C^1 \) function \( \theta \), defined from \( \mathbb{R}_+ \) in \( [0, 1] \); which is 1 in \( [0, 1] \) and 0 in \( [2, +\infty] \). We denote by

\[ B_\theta u = B_\theta(u, u) = \theta(\frac{\|u\|^2_2}{\rho_1}) B(u, u). \]

We consider in the sequel the system:

\[ \frac{du}{dt} + A_{bu} u + \eta u = B_\theta u + f \]

with clear meaning of the terms. Clearly, also (2.15) is well posed, and has the same dynamics and the same attractor of (2.1). Moreover, there exist two constants \( M_0 \) and \( M_1 \) such that, for every \( u, v \in V \),

\[ |B_\theta u|_b \leq M_0, \quad |B_\theta u - B_\theta v|_b \leq M_1\|u - v\|_b. \]

3. Existence of Approximate Inertial Manifolds

An Inertial Manifold \( \mathcal{M} = \{y, \Phi(y)\} \) is a positively invariant manifold defined as the graph of a Lipschitz function \( \Phi \), defined from \( P_n H \) to \( Q_n H \), which attracts all trajectories of (2.15) exponentially. To prove the existence of the approximate inertial manifolds (AIM) for (2.15) we follow the ideas of [3, 5, 17, 21], where the AIM are constructed approximating the iteration map of the Lyapunov-Perron method. The equation (2.15) for a solution in \( \mathcal{M} \), using the projections \( P_n \) and \( Q_n \), can be decomposed in

\[ \frac{dy}{dt} + A_{bu} y + \eta y = P_n B_\theta(y + \Phi(y)) + P_n f \]

(3.1)

\[ \frac{d\Phi(y)}{dt} + A_{bu} \Phi(y) + \eta \Phi(y) = Q_n B_\theta(y + \Phi(y)) + Q_n f. \]

(3.2)

The finite dimensional system of ordinary differential equations (3.1) is called the inertial system associated to \( \mathcal{M} \). As \( \Phi \) is a Lipschitz
function, then (3.1) determinate uniquely \( y(t) = y(t; y_0, \Phi) \) for every \( t \in \mathbb{R} \), where \( u_0 = y_0 + \Phi(y_0) \) is the initial condition. Assuming that \( \Phi \) is bounded, to determinate the function \( \Phi \), we integrate in time the system (3.2) and we construct the function \( \Phi \) as the fixed point of the map \( \phi \rightarrow F \phi \) defined by

\[
F \phi (y_0) = \int_{-\infty}^{0} e^{A_{bw}s} \left[ Q_n (B_\theta (y(s) + \phi(y(s))) + f) - \eta \phi(y(s)) \right] ds,
\]

where \( \phi \) is a bounded Lipschitz function from \( P_nH \) to \( Q_nH \). The proof of the existence of an inertial manifold consists to show that the map \( F \) is a contractive map in the complete metric space

\[
F_{l,L} = \{ \phi: P_nV \rightarrow Q_nV : \text{Lip}(\phi) \leq l, \quad |\phi|_\infty = \sup_{y \in P_nV} \|\phi(y)\|_b \leq L \},
\]

and \( y \) is a solution of the system (3.1) with \( y(t = 0) = y_0 \). We recall that the proof of the existence of the Inertial Manifold is based to the spectral gap condition. In the case where the spectral gap condition is not verified, it is possible to approximate the map \( F \) using an Eulerian explicit scheme. This permits to construct a sequence of AIM. Now, we fix \( n \) and we choose a positive integer \( N \) and a time step \( \tau > 0 \) and we denote by \( y_0 = y_{in} \in P_nV \), then we define recursively a family of \( y_k \), with \( k \geq 0 \), using

\[
y_{k+1} = A_{bw} y_k + P_n B_\theta (y_k + \phi(y_k)) - \eta y_k + P_n f,
\]

where \( y_k \) approximates \( y(-k\tau) \). To approximate \( y \), we construct the function \( y_\tau \) as

\[
y_{\tau}(s) = y_k \quad \text{for} \quad -(k+1)\tau < s \leq -k\tau, \quad k = 0, \ldots, N - 1,
\]

\[
(y_{\tau})(s) = y_N \quad \text{for} \quad s \leq -N\tau.
\]

The approximation \( F^N_{\tau} \) of \( F \) is defined substituting \( y \) with \( y_\tau \) into (3.3). Explicitly

\[
F^N_{\tau} \phi(y_0) = -(A_{bw})^{-1} (I - e^{-A_{bw} \tau}) \sum_{k=0}^{N-1} e^{-kA_{bw}\tau} \left[ Q_n (B_\theta (y_k + \phi(y_k)) + f) - \eta \phi(y_k) \right]
\]

\[
= -(A_{bw})^{-1} e^{-NA_{bw}\tau} \left[ Q_n (B_\theta (y_N + \phi(y_N)) + f) - \eta \phi(y_N) \right].
\]

To construct the family of AIM, we consider a sequence \((N_k)_{k \in \mathbb{N}}\) of integers and \((\tau_k)_{k \in \mathbb{N}}\) and we define the manifolds \( M_k \) as the graph of
the functions $\Phi_k$ constructed recursively, for $k \geq 0$, by

$$\Phi_0 = 0, \quad \Phi_{k+1} = F_{\tau_k}^N(\Phi_k).$$

The main result of this section is to prove the existence of $\Phi_k$ in $\mathcal{F}_{i,L}$, for every $k \geq 0$. Before to proceed with the formulation of the main theorem of this section and its proof, we recall some preliminary properties, which guaranties the consistence of the approximation scheme described before. We can write (3.5) as

$$y(k + 1)\tau = (I + \tau A_{bc})y(-k\tau) - \tau P_n(B_\theta(y(-k\tau) + z(-k\tau)) + f) + \tau \eta y(-k\tau),$$

and

$$\epsilon_k = y(-(k + 1)\tau) - y(-k\tau) - \frac{d}{dt}(-k\tau)$$

the approximation error. The following Lemmas hold:

**Lemma 1.** Suppose that $u(t)$ is a complete trajectory inside the global attractor $\mathcal{A}$, and that (2.13) holds, then:

$$\|\epsilon_k\|_b \leq \tau^2 \beta_1, \quad k = 0, \ldots, N - 1,$$

$$\left\|\frac{du}{dt}\right\|_b \leq \beta_2, \quad t < 0,$$

with $\beta_1 \leq \frac{8\alpha_1}{\sigma_1}$ and $\beta_2 \leq \frac{2\rho_1}{\alpha}$. 

**Proof.** If the trajectory $u(t)$ is a complete trajectory inside the global attractor $\mathcal{A}$, one can obtains (3.11) where $\beta_1 = \sup_{u(t) \in \mathcal{A}} \sup_{t \in \mathbb{R}} \left\|\frac{d^2 u}{dt^2}\right\|_b$. Moreover $\beta_2 = \sup_{u(t) \in \mathcal{A}} \sup_{t \in \mathbb{R}} \left\|\frac{du}{dt}\right\|_b$, and thanks to (2.13) we have the explicit expressions of $\beta_1$ and $\beta_2$. \hfill \blacksquare

**Lemma 2.** Let be $i = 1, 2$, and let be $y_0^i \in P_n V$. Let define $y_k^i$, $k = 0, \ldots, N$ by (3.5) and (3.9) with $y_0 = y_0^i$ and let construct $y^i_\tau(s)$ using (3.6). Then, for every $s \leq 0$,

$$\left\|y^1_\tau(s) - y^2_\tau(s)\right\|_b \leq e^{-s[A_n + \left(\frac{C_{\lambda_n}}{\sigma_n}\right)^2(M_1 + \Pi)(1 + l)]} \left\|y_0^1 - y_0^2\right\|_b,$$

where $\bar{\eta} = \sup_{\Omega} \eta$.

**Proof.** Denoting by $y_k = y_k^1 - y_k^2$ and subtracting (3.5) or (3.9) for $i = 1, 2$, and using (2.16), (2.10) and the Lipschitz constant $l$ of $\phi$,
we obtain

\[
\|y_{k+1}\|_b \leq (1 + \tau \lambda_n)\|y_k\|_b + \tau \left(\frac{C\lambda_n}{c\bar{\nu}}\right)^{\frac{1}{2}}.
\]

\[
\cdot \left[ |B_{y_k^1}(y_k^1 + \phi(y_k^1))| - B_{y_k^2}(y_k^2 + \phi(y_k^2))|_b + \bar{\eta} |y_k + \phi(y_k)|_b \right]
\]

\[
\leq (1 + \tau \lambda_n)\|y_k\|_b + \tau \left(\frac{C\lambda_n}{c\bar{\nu}}\right)^{\frac{1}{2}} (M_1 + \Pi \bar{\eta})(1 + l)\|y_k\|_b
\]

\[
\leq \exp\{k\tau[\lambda_n + \left(\frac{C\lambda_n}{c\bar{\nu}}\right)^{\frac{1}{2}} (M_1 + \Pi \bar{\eta})(1 + l)]\}\|y_0\|_b.
\]

for \( k = 0, \ldots, N \). From the definition of \( y^i(s) \) by (3.6) and (3.6), we obtain (3.13).

We are now ready to give the following main theorem. In the sequel we denote with \( \gamma = \int_{-\infty}^{0} |s|^{-1/2} e^s ds \).

**Theorem 1.** If \( L \geq L_0 \), with

\[
L_0 = \left(\frac{C}{c}\right)^{\frac{1}{2}} (|f|_b + M_0 + \bar{\eta} \rho_0) (\gamma \bar{\nu}^{\frac{1}{2}} + 1) \lambda_{n+1}^{-\frac{1}{2}},
\]

and if \( l \) and \( \delta \) satisfy

\[
l \geq \left(\frac{C}{c}\right)^{\frac{1}{2}} (l + 1) e^{\delta (l+1)} \left[ 2 (M_1 + \Pi \bar{\eta})^{\frac{1}{2}} \left(\frac{c}{C\bar{\nu} \lambda_n}\right)^{\frac{1}{2}} + 
\right.
\]

\[
+ (M_1 + \Pi \bar{\eta})^2 (\nu \lambda_{n+1})^{-\frac{1}{2}} \] \[
\left. + e^{\delta (l+1)} \left(\frac{\nu \lambda_{n+1}}{\lambda_n}\right)^{\frac{1}{2}} \right),
\]

then, for every \( \tau \) and \( N \) such that

\[
(N + 1)\tau \leq \frac{\delta}{(M_1 + \Pi \bar{\eta})} \left(\frac{c}{C\lambda_n}\right)^{\frac{1}{2}},
\]

results that \( F^N_{\tau} : F_{l,L} \rightarrow F_{l,L} \).

**Proof.** Let \( \phi \in F_{l,L} \), we consider \( y_0 \in P_n V \) and \( (y_k)_{k=0,\ldots,N} \) and \( y_{\tau}(s) \) given by (3.5), (3.6). We obtain the constant \( L \) of the space \( F_{l,L} \).
Using (2.16) and (2.9), we have:

\[
(3.17) \left\| F^N \phi(y_0) \right\|_b \leq \\
\leq \int_{-\infty}^{0} \left| e^{A_{bc} s} Q_n \right| L(\mathcal{H},\mathcal{V}) |f + B_\theta(y(s) + \phi(y(s))) - \eta(y(s) + \phi(y(s)))|_b ds \\
\leq \left( \frac{C}{c} \right)^{1/2} (|f|_b + M_0 + \bar{\eta}\rho_0) \int_{-\infty}^{0} \left( |\bar{\nu}s|^{-1/2} + \lambda_{n+1}^{1/2} \right) e^{\lambda_{n+1}s} ds \\
\leq \left( \frac{C}{c} \right)^{1/2} (|f|_b + M_0 + \bar{\eta}\rho_0)(\gamma\bar{\nu}^{-1/2} + 1)\lambda_{n+1}^{-1/2}.
\]

From the previous inequality we deduce that \( \left\| F^N \phi(y_0) \right\|_b \leq L \), for every

\[
(3.18) \quad L \geq L_0 = \left( \frac{C}{c} \right)^{1/2} (|f|_b + M_0 + \bar{\eta}\rho_0)(\gamma\bar{\nu}^{-1/2} + 1)\lambda_{n+1}^{-1/2}.
\]

Now, we try to determine the Lipschitz constant \( l \). For this scope, let \( y_0^i \in P_n \) and \( (y_k^i)_{k=0,...,N} \) and \( y_\tau^i(s) \) constructed by (3.5) and (3.6), for \( i = 1, 2 \). Therefore, we write

\[
F^N_\tau \phi(y_0^i) - F^N_\tau \phi(y_0^2) = \int_{-(N+1)\tau}^{0} e^{A_{bc} s} [Q_n(B_\theta(y_\tau^1(s) + \phi(y_\tau^1(s)))) \\
- B_\theta(y_\tau^2(s) + \phi(y_\tau^2(s)))) - \eta(\phi(y_\tau^1(s)) - \phi(y_\tau^2(s)))]|ds \\
+ (A_{bc})^{-1} e^{-(N+1)A_{bc}\tau} [Q_n(B_\theta(y_N^1 + \phi(y_N^1)) \\
- B_\theta(y_N^2 + \phi(y_N^2))) - \eta(\phi(y_N^1) - \phi(y_N^2))].
\]

Using again (2.16), (2.16) and (2.9), we have:

\[
\left\| F^N_\tau \phi(y_0^1) - F^N_\tau \phi(y_0^2) \right\|_b \leq \left( \frac{C}{c} \right)^{1/2} (M_1 + \Pi\bar{\eta})(l + 1) \cdot \\
\cdot \int_{-(N+1)\tau}^{0} \left( |\bar{\nu}s|^{-1/2} + \lambda_{n+1}^{1/2} \right) e^{\lambda_{n+1}s} \left\| y_\tau^1(s) - y_\tau^2(s) \right\|_b ds \\
+ \left( \frac{C}{c} \right)^{1/2} (M_1 + \Pi\bar{\eta})(l + 1)\bar{\nu}^{-1/2}\lambda_{n+1}^{-1/2} e^{-\lambda_{n+1}(N+1)\tau} \left\| y_N^1 - y_N^2 \right\|_b.
\]
Using (3.13), we obtain \( \| F_N^t \phi(y_0) - F_N^t \phi(y_0') \|_b \leq l \| y_0 - y_0' \|_b \), with

\[
l = \left( \frac{C}{c} \right)^{1/2} (M_1 + \Pi \bar{\eta})(l + 1) \cdot \left( \int_{(N+1)\tau}^{0} \left( \tilde{\nu}^2 \lambda_n^{1/2} e^{[(\lambda_{n+1} - \lambda_n) - \left( \frac{C_\lambda n}{c} \right)^{1/2}}(M_1 + \Pi \bar{\eta})(1+l) \right) ds \right)
\]

Because \( \lambda_{n+1} - \lambda_n \geq 0 \) and using (3.16), we deduce (3.15).

\[\square\]

4. Approximation of the attractor

In this section we will prove that the approximate inertial manifolds \( \mathcal{M}_N \) built in the previous section as a graph of the \( \Phi_N \), approximate the global attractor \( \mathcal{A} \). We will prove that the attractor is on a thin neighbourhood of \( \mathcal{M}_N \) and its thickness decreases as \( N \) grows, converging to a small number (see \[3, 5, 17, 21\]). We will try to estimate the semi-distance in \( V \) of \( \mathcal{A} \) to \( \mathcal{M}_N \)

\[
(4.1) \quad \varrho_N = d_N(\mathcal{A}, \mathcal{M}_N) = \sup_{\nu \in \mathcal{A}} \inf_{w \in \mathcal{M}_N} \| v - w \|_b.
\]

As a first step we need a lemma that will allow us to estimate the thickness of the thin neighbourhood. We continue to use the notations of the previous sections.

**Lemma 3.** Suppose that (3.11), (3.12), (3.14), (3.15) and (3.16), of the previous section, are satisfied. Then for every \( \phi \in \mathcal{F}_{l,l} \) results that

\[
\| F_N^t \phi(y_0) - z_0 \|_b \leq \left( \frac{C}{c} \right)^{1/2} (M_1 + \Pi \bar{\eta}) \left[ l(\lambda_n \tilde{\nu}^{-1/2} + \left( \frac{\lambda_n \tilde{\nu}}{\lambda_{n+1}} + 1 \right)^{1/2} \right] \sup_{\nu \in \mathcal{A}} \| \phi(y) - z \|_b
\]

\[
+ \left[ \beta_1 \lambda_n^{-1} + \beta_2 \left( \frac{C}{c} \right)^{1/2} (M_1 + \Pi \bar{\eta})(1 + l) \left( \frac{\lambda_n \tilde{\nu}}{\lambda_{n+1}} + 1 \right)^{1/2} \right] \tau
\]

\[
(4.2) \quad + 2 \left( \frac{C}{c} \right)^{1/2} (M_0 + \bar{\eta} \rho_0) \left[ \frac{\tilde{\nu}(N + 1)\tau^{-1/2} + \lambda_{n+1}^{1/2}}{\lambda_{n+1}} \right] e^{-\lambda_{n+1}(N + 1)\tau}.
\]

**Proof.** Take \( \phi \in \mathcal{F}_{l,l} \) and \( u_0 = y_0 + z_0 \in \mathcal{A} \) a point in the global attractor. Denote with \( (u(t))_{t \in \mathbb{R}} \) the trajectory in \( \mathcal{A} \) which pass to \( u_0 \) at \( t = 0 \). Consider \( y(t) = P_n u(t) \), \( z(t) = Q_{\mathcal{A}} u(t) \). Define \( \tilde{y}_k = y(-k\tau) \) e \( (y_k)_{k=0,\ldots,N} \) with (3.5); and consider \( y_{\tau} \) constructed with (3.6). Using
(2.16) and (2.9), the Lipschitz property of $\phi$, the Poincaré inequality (2.4) and (2.12), we have:

$$\|F_N^\tau \phi(y_0) - z_0\|_b \leq \left(\frac{C}{c}\right)^{\frac{1}{2}} (M_1 + \Pi \bar{\eta})(1 + l) \cdot$$

$$\cdot \int_{-(N+1}\tau}^{0} (|\bar{v}s|^{-\frac{1}{2}} + \lambda_{n+1}^{\frac{1}{2}}) \|y_\tau(s) - y(s)\|_b ds$$

$$+ (M_1 + \Pi \bar{\eta}) \left(\frac{C}{c}\right)^{\frac{1}{2}} (\gamma \bar{v}^{-\frac{1}{2}} + \lambda_n^{-1}) \lambda_{n+1}^{-1} \|\phi(y) - z\|_b$$

$$+ 2 \left(\frac{C}{c}\right)^{\frac{1}{2}} (M_0 + \bar{\eta}) \left[\bar{v}(N + 1)\tau\right]^{-\frac{1}{2}} + \lambda_{n+1}^{\frac{1}{2}} e^{-\lambda_{n+1}(N+1)\tau}.$$

To estimate the integral on (4.3), from (3.12), for every $s$ in $(-k\tau, -\tau)$, we have

$$\|y_\tau(s) - y(s)\|_b \leq \|e_k\|_b + k\tau + s \sup_{\xi \leq 0} \left|\frac{d}{dt}(\xi)\right|_b \leq \|e_k\|_b + \tau \beta_2,$$

with $e_k = y_k - \bar{y}_k$. Using (3.9) and (3.10), we have

$$\|e_{k+1}\|_b \leq (1 + \tau \lambda_n) \|e_k\|_b + \tau \left(\frac{C\lambda_n}{c\bar{v}}\right)^{1/2} (M_1 + \Pi \bar{\eta})(1 + l) \|e_k\|_b$$

$$+ \tau \left(\frac{C\lambda_n}{c\bar{v}}\right)^{1/2} (M_1 + \Pi \bar{\eta}) \|\phi(\bar{y}_k) - z\|_b + \|\epsilon_k\|_b,$$

and from (3.11), we have

$$\|e_k\|_b \leq \left[(M_1 + \Pi \bar{\eta}) \left(\frac{C}{c\bar{v}\lambda_n}\right)^{\frac{1}{2}} \sup_{y + \Sigma \in A} \|\phi(y) - z\|_b + \tau \beta_1 \lambda_n^{-1}\right]$$

$$\cdot \exp\{k\tau[\lambda_n + \left(\frac{C\lambda_n}{c\bar{v}}\right)^{\frac{1}{2}} (M_1 + \Pi \bar{\eta})(1 + l)\}\}.$$

Now we are ready to estimate the first integral on (4.3):

$$\sqrt{\frac{C}{c}} (M_1 + \Pi \bar{\eta})(1 + l) \int_{-(N+1)\tau}^{0} (|\bar{v}s|^{-\frac{1}{2}} + \lambda_{n+1}^{\frac{1}{2}}) \|y_\tau(s) - y(s)\|_b ds$$

$$\leq l[(M_1 + \Pi \bar{\eta}) \left(\frac{C}{c\lambda_n\bar{v}}\right)^{\frac{1}{2}} \sup_{y + \Sigma \in A} \|\phi(y) - z\|_b + \tau \beta_1 \lambda_n^{-1}\]$$

$$+ \tau \beta_2 \left(\frac{C}{c}\right)^{\frac{1}{2}} (M_1 + \Pi \bar{\eta})(1 + l) (\gamma \bar{v}^{-\frac{1}{2}} + 1) \lambda_{n+1}^{\frac{1}{2}}.$$
Recalling (3.15) and (3.16) and collecting the previous estimate with (4.3), we have the (4.2).

In the next theorem we give an estimate on the number $n$ of modes in such a way to have an exponential approximation of $\mathcal{M}_N$ of the attractor, for $N$ large.

**Theorem 2.** Suppose that the hypothesis of theorem 1 hold and that (3.11), (3.12), (3.14) and (3.15) hold. Assuming that $\lambda_n$ and $n$ satisfy the following estimates:

\begin{align*}
(4.4) \quad & \left(\frac{C^*}{c}\right)^{\frac{1}{2}} \bar{\nu}^{-\frac{1}{2}} [2(M_1 + \Pi \bar{\eta})]^3 \left(\frac{C\lambda_n}{c\bar{\nu}}\right)^{-\frac{1}{4}} + (M_1 + \Pi \bar{\eta})^2 \lambda_n^{-\frac{1}{2}} \leq \frac{1}{2}, \\
(4.5) \quad & \left(\frac{C^*}{c}\right)^{\frac{1}{2}} (M_1 + \Pi \bar{\eta}) \left[ \left(3 + 6 \sup_n \left(\frac{\rho n+1}{\lambda_n}\right)^{\frac{1}{2}}\right) \frac{c\bar{\nu}}{(\bar{\nu} \lambda_n)^{\frac{1}{2}}} + (\gamma \bar{\nu}^{-\frac{1}{2}} + 1) \lambda^{-\frac{1}{2}} n+1 \right] \leq \frac{1}{2}.
\end{align*}

Suppose, moreover, that the sequence $\tau_N$ and $N \in \mathbb{N}$ satisfies

\begin{align*}
(4.6) \quad & \chi \leq \tau_N (N+1) \leq \frac{\delta}{(M_1 + \Pi \bar{\eta})} \left(\frac{c\bar{\nu}}{C\lambda_n}\right)^{1/2},
\end{align*}

where $\chi$ is a fixed constant. Then the approximate inertial manifolds $\mathcal{M}_N$, constructed in theorem 1, satisfy, for $N$ sufficiently large,

\begin{align*}
(4.7) \quad & d_V(\mathcal{A}, \mathcal{M}_N) \leq 4 \left(\frac{C^*}{c}\right)^{1/2} (M_0 + \bar{\eta} \rho_0) \frac{1}{\lambda_{n+1}^{1/2}} e^{-\lambda_{n+1} \chi}.
\end{align*}

**Proof.** Using the expression (4.2) in the previous Lemma, we have $\varepsilon_{n+1} \leq \xi \varepsilon_N + \sigma_N$, where

\begin{align*}
(4.8) \quad & \xi = \left(\frac{C^*}{c}\right)^{\frac{1}{2}} (M_1 + \Pi \bar{\eta}) \left[ l(\lambda_n \bar{\nu})^{-\frac{1}{2}} + (\gamma \bar{\nu}^{-\frac{1}{2}} + 1) \lambda_n^{-\frac{1}{2}} \right],
\end{align*}

with $l$ given by (3.15), and

\begin{align*}
(4.9) \quad & \sigma_N = \tau_N \left[ \beta_1 l \lambda_n^{-1} + \beta_2 \left(\frac{C^*}{c}\right)^{\frac{1}{2}} (M_1 + \Pi \bar{\eta})(1 + l)(\gamma \bar{\nu}^{-\frac{1}{2}} + 1) \lambda_n^{-\frac{1}{2}} \right] \\
& + 2 \left(\frac{C^*}{c}\right)^{\frac{1}{2}} (M_0 + \bar{\eta} \rho_0) \left[ \bar{\nu}(N+1) \tau_N \right]^{-\frac{1}{2}} + \lambda_n^{\frac{3}{2}} e^{-\lambda_{n+1} \tau_N}.
\end{align*}
Iterating, we obtain $\varrho_n \leq \xi^N \varrho_0 + \sum_{j=1}^{N-1} \sigma_{N-j-1} \xi^j$, with $\varrho_0 = \sup_{u_0 \in A} \|z_0\|_b$.

By (2.12), (2.16), (2.9):

\begin{equation}
\|z\|_b \leq \left( \frac{C}{c} \right)^{1/2} (M_0 + |f|_b + \bar{\eta}\rho_0) (\gamma \bar{\nu}^{-1/2} + 1) \lambda_{n+1}^{1/2}.
\end{equation}

Using (4.6) for $N \geq (\tau_N \bar{\nu})^{-1}$ and supposing that $\xi \leq \frac{1}{2}$, we have

\[ \sum_{j=0}^{N-1} \beta_{N-j-1} \xi^j = 4 \left( \frac{C}{c} \right)^{1/2} (M_0 + \bar{\eta}\rho_0) \frac{1}{\lambda_{n+1}^{1/2}} e^{-\lambda_{n+1} \chi} \]

\[ + 2 \left[ \beta_1 t \lambda_n^{-1} + \beta_2 \left( \frac{C}{c} \right)^{1/2} (M_1 + \Pi \bar{\eta})(1 + l) \frac{(\gamma \bar{\nu}^{-\frac{1}{2}} + 1)}{\lambda_{n+1}^{1/2}} \right] \sup_{0 \leq j \leq N-1} \tau_{N-j-1}. \]

Combining (4.10) and (4.11) we obtain that

\begin{equation}
d_V(A, M_N) \leq 2^{-N} \left( \frac{C}{c} \right)^{1/2} (M_0 + \bar{\eta}\rho_0 + |f|_b) (\gamma \bar{\nu}^{-1/2} + 1) \lambda_{n+1}^{1/2} \\
+ 4 \left( \frac{C}{c} \right)^{1/2} (M_0 + \bar{\eta}\rho_0) \frac{1}{\lambda_{n+1}^{1/2}} e^{-\lambda_{n+1} \chi} \\
+ 2 \left[ \beta_1 t \lambda_n^{-1} + \beta_2 \left( \frac{C}{c} \right)^{1/2} (M_1 + \Pi \bar{\eta})(1 + l) \frac{(\gamma \bar{\nu}^{-\frac{1}{2}} + 1)}{\lambda_{n+1}^{1/2}} \right] \sup_{0 \leq j \leq N-1} \tau_{N-j-1}. \end{equation}

Moreover, from (4.6) results that $\tau_N \to 0$ as $N \to \infty$, then from (4.12) we obtain (4.7) for $N \to \infty$. To complete the proof we may determinate $\lambda_n$ in such a way that the previous estimates are satisfied. Choosing $\lambda_n$ such that (4.4) is satisfied, we write (3.15) as

\[ \left( \frac{1 + l}{2} + \left( \frac{\lambda_{n+1}}{\lambda_n} \right)^{1/2} \right) e^{\delta(l+1)} \leq l. \]

The last inequality is verified if we choose $l = 3 + 6 \sup_{n} (\frac{\lambda_{n+1}}{\lambda_n})^{1/2}$. In this way, condition $\xi \leq \frac{1}{2}$ can be written as (4.5) and the proof is complete.

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