Safe Reinforcement Learning via Online Shielding

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Abstract

Reinforcement learning is a promising approach to learning control policies for complex robotics tasks. A key challenge is ensuring safety of the learned control policy—e.g., that a walking robot does not fall over, or a quadcopter does not run into a wall. We focus on the setting where the dynamics are known, and the goal is to prove that a policy learned in simulation satisfies a given safety constraint. Existing approaches for ensuring safety suffer from a number of limitations—e.g., they do not scale to high-dimensional state spaces, or they only ensure safety for a fixed environment. We propose an approach based on shielding, which uses a backup controller to override the learned controller as necessary to ensure that safety holds. Rather than compute when to use the backup controller ahead-of-time, we perform this computation online. By doing so, we ensure that our approach is computationally efficient, and furthermore, can be used to ensure safety even in novel environments. We empirically demonstrate that our approach can ensure safety in experiments on cart-pole and on a bicycle with random obstacles.

1 Introduction

Reinforcement learning is a promising approach to automatically synthesizing control policies for robotics tasks [10, 15]. However, a key challenge to enabling usage on real robots is the inability to provide safety guarantees for learned control policies. For example, we may want to ensure that a walking robot does not fall over, or that a quadcopter does not run into an obstacle.

We consider the setting where a control policy is learned in simulation, and the goal is to use it to control a real robot. We assume that the robot dynamics are known, but the environment may not be known ahead of time. As a concrete example, consider flying a quadcopter. We have very good models of the quadcopter dynamics. However, we may want to use the quadcopter in many different environments, with different configurations of obstacles (e.g., walls, buildings, and trees). We assume that perception is accurate—i.e., we know the positions of the obstacles. Our goal is to ensure that a given learned control policy for the quadcopter is safe when faced with a possibly novel environment.

Existing approaches to safe reinforcement learning rely on ahead-of-time verification of the desired safety property. For example, one approach is to directly verify that the learned control policy is safe [8, 25, 7, 14]. An alternative approach is to use a shield policy, which overrides the learned policy with a safe backup policy as necessary to ensure safety [20, 13, 8, 9, 4]. The goal is to use the learned policy as often as possible. For example, the backup policy may bring the quadcopter to a stop if it goes near an obstacle. This approach implicitly verifies safety of the joint policy (i.e., the combination of the learned policy and the backup policy) ahead-of-time.

These existing approaches have several limitations. First, ahead-of-time verification can be computationally infeasible—it requires checking whether safety holds from every possible state, which can scale exponentially in the dimension of the state space. Many existing approaches only scale to

More generally, the shield can simply constrain the set of allowed actions in a way that ensures safety.
Second, existing algorithms typically focus on verifying a property of the robot dynamics in isolation of its environment (e.g., positions of obstacles) or with respect to a fixed environment. To handle novel environments, they must either verify all possible environments ahead-of-time, or run verification from scratch every time a novel environment is encountered. Finally, many existing approaches focus on stability, which says that the robot remains within some constrained region of the state space, yet we typically want robots that engage in dynamic behavior, such as walking [10], flying [13, 3], or driving [16]. Verifying safety for dynamic behaviors is challenging, and most existing approaches can only do so over a finite time horizon [7, 9, 14] or in some bounded region of the state space [13, 8]. We discuss in more detail in Appendix A.

**Contributions.** Our key insight is that these challenges arise since existing approaches rely on ahead-of-time verification. In this paper, we propose an approach to safe reinforcement learning that ensures safety **online**. Our approach is based on the concept of shielding, where we occasionally override the learned policy with a backup policy to ensure safety. However, in contrast to previous approaches, our algorithm computes whether to use the learned policy or a backup policy on-the-fly. Intuitively, it maintains the invariant that the backup policy can always recover the system—in particular, it only uses the learned policy if it can prove that doing so maintains this invariant.

Our approach assumes given a **safe policy**, which ensures safety, but is only applicable on a restricted portion of the state space. A typical choice for the safe policy is a linear quadratic regular (LQR) synthesized to stabilize the robot near some fixed point. For example, for a walking robot, it may stabilize the robot near an upright, standing position, and for a quadcopter, it may stabilize the quadcopter near a hovering position with zero velocity.

Then, our algorithm learns a **recovery policy** that tries to bring the system to a state where the safe policy is applicable. For example, consider a walking robot, and a learned policy that makes the robot run. The recovery policy may slow down the robot, bringing it close to a standing position where the safe policy can be applied. In summary, the backup policy consists of using the recovery policy until it is close to a standing position, and then using the safe policy.

Finally, our algorithm ensures safety by maintaining the invariant that the recovery policy can safely stop the robot. In particular, to compute whether to use the learned policy or the backup policy, it checks whether using the learned policy maintains this invariant. If so, then it uses the learned policy. Otherwise, it uses the backup policy—by the invariant, doing so is guaranteed to be safe. As long as the robot starts from a state where the invariant holds, then the invariant holds indefinitely.

We empirically evaluate our approach on the cart-pole and a bicycle model. We show how our approach can be used to ensure safety for dynamical robot tasks and when there are random obstacles. We also demonstrate the scalability of our approach compared to ahead-of-time verification. In particular, while our approach incurs runtime overhead during computation of the policy, each computation is efficient. In contrast, ahead-of-time verification suffers from exponential blowup in the computational cost. Even when this computation is offline, it can be infeasible for problems of interest; furthermore, verification may have to be performed from scratch if a novel environment is encountered. Finally, our benchmark is dynamic (i.e., the goal is not stabilization), so unlike our approach, verification can only ensure safety for a finite horizon or bounded region of the state space.

**Related work.** There has been much recent interest in safe reinforcement learning [12, 5]. One approach is to use constrained reinforcement learning to learn policies that satisfy a safety constraint [2, 26]. However, these approaches typically do not guarantee safety.

Another approach is to learn a control policy, and then verify ahead-of-time that the learned policy is safe [25, 7, 14]. However, as we show in our experiments, these approaches typically do not scale well. Also, verification does not provide a way to modify unsafe policies to obtain a safe policy. One solution is to compose policies proven to be safe on different portions of the state space [22, 28]. A special case is shielding, which composes the learned policy with a safe backup policy [20, 13, 4]. However, existing approaches rely on ahead-of-time verification, and thus scale poorly.

We focus on the case where the dynamics are known, and reinforcement learning is used in simulation to learn a policy. Another use of reinforcement learning is when the dynamics are initially unknown, and the goal is to learn a control policy from real-world experience. There has been work on safe exploration that aims to ensure safety in this setting [17, 13, 24, 27, 8, 11]. These approaches also rely on verification; we believe our approach can be integrated with these approaches.
2 Preliminaries

Given an arbitrary control policy \( \hat{\pi} \), our goal is to minimally modify \( \hat{\pi} \) to obtain a safe control policy \( \pi_{\text{shield}} \) for which safety is guaranteed to hold (given some constraints on the initial state).

**Dynamics and control.** We consider a deterministic, discrete time dynamical system with continuous states \( \mathcal{X} \subseteq \mathbb{R}^{n_x} \), continuous actions \( \mathcal{U} \subseteq \mathbb{R}^{n_u} \), dynamics \( f : \mathcal{X} \times \mathcal{U} \to \mathcal{X} \), and initial state distribution \( d_0 \) over \( \mathcal{X} \). Given a control policy \( \pi : \mathcal{X} \to \mathcal{U} \), we let \( f^{(\pi)}(x) = f(x, \pi(x)) \) denote the joint dynamics. Given control policy \( \pi \) and initial state \( x_0 \), the trajectory generated by \( \pi \) from \( x_0 \) is the infinite sequence of states \( x_0, x_1, ... \), where for all \( t \geq 1 \), \( x_t = f^{(\pi)}(x_{t-1}) \). We assume that \( f \) is a polynomial of degree \( d \). We describe the necessary assumptions on initial states in Section 3.

**Remark 2.1.** As in prior work [22], for systems that are not polynomial, we use local degree 5 Taylor approximations. While our theoretical safety guarantees rely on polynomial dynamics, we find that approximations. While our theoretical safety guarantees rely on polynomial dynamics, we find that approximations are very accurate.

**Shielding.** We consider a safety property specified as a safe region \( \mathcal{X}_{\text{safe}} \subseteq \mathcal{X} \)—i.e., the entire trajectory generated by a control policy from an initial state should be contained in \( \mathcal{X}_{\text{safe}} \). We assume that \( \mathcal{X}_{\text{safe}} \) has form \( \mathcal{X}_{\text{safe}} = \{ x \in \mathcal{X} \mid A_{\text{safe}} x \leq \theta_{\text{safe}} \} \), where \( A_{\text{safe}} \in \mathbb{R}^{k \times n_x} \) and \( \theta_{\text{safe}} \in \mathbb{R}^k \) for some \( k \in \mathbb{N} \)—i.e., \( \mathcal{X}_{\text{safe}} \) is the intersection of finitely many half-spaces. Given an initial state \( x_0 \in \mathcal{X} \), a safe region \( \mathcal{X}_{\text{safe}} \), and a control policy \( \pi \), we say \( \pi \) is safe starting from \( x_0 \) if \( x_t \in \mathcal{X}_{\text{safe}} \) for all \( t \geq 0 \).

Consider a control policy \( \hat{\pi} : \mathcal{X} \to \mathcal{U} \). We make no assumptions about \( \hat{\pi} \); e.g., it may be a neural network policy trained using reinforcement learning. Given \( \hat{\pi} \), our goal is to synthesize a control policy \( \pi_{\text{shield}} \) that is (i) safe from as many initial states as possible, and (ii) achieves as high of a reward as possible. Given initial state distribution \( d_0 \) over \( \mathcal{X} \), rewards \( \mathcal{R} : \mathcal{X} \to \mathbb{R} \), and a discount factor \( \gamma \in [0, 1] \), we can formalize (i) as maximizing the safety probability

\[
P_{\text{safe}}^{(\pi_{\text{safe}})} = \mathbb{P}_{x_0 \sim d_0} [\pi_{\text{shield}} \text{ safe starting at } x_0],
\]

and we can formalize (ii) as maximizing the cumulative expected time-discounted reward

\[
f^{(\pi_{\text{shield}})} = \mathbb{E}_{x_0 \sim d_0} \left[ \sum_{t=0}^{\infty} \gamma^t \mathcal{R}(x_t) \right],
\]

where \( x_0, x_1, ... \) is the trajectory generated by \( \pi_{\text{shield}} \) from \( x_0 \).

Our approach is to synthesize a control policy \( \pi_{\text{shield}} \) that equals \( \hat{\pi} \) as often as possible, but overrides \( \hat{\pi} \) when it cannot validate that using \( \hat{\pi} \) is safe. In contrast to prior work on shielding [13, 3, 4], we compute whether to use \( \hat{\pi} \) or the backup policy on-the-fly rather than ahead of time.

**LQR targets.** Our approach uses utilizing control to construct \( \pi_{\text{shield}} \). Given a state-action pair \( (\hat{x}, \hat{u}) \in \mathcal{X} \times \mathcal{U} \), we can use a linear quadratic regulator (LQR), which is a control policy \( \pi_{\text{LQR}} \), to stabilize the system near \( (\hat{x}, \hat{u}) \). Furthermore, we can use LQR verification techniques to compute regions within which the LQR controller is guaranteed to be safe [18, 22].

To leverage these techniques, our algorithm takes as input a LQR target mapping \( \rho : \mathcal{X} \to \mathcal{X} \times \mathcal{U} \). Given a state \( x \in \mathcal{X} \), \( \rho \) returns an LQR target \( \rho(x) = (\hat{x}, \hat{u}) \) around which we can try to stabilize the system. For example, consider the cart-pole, which has states of the form \( x = (z, v, \theta, \omega) \in \mathbb{R}^4 \), where \( z \) is the cart position, \( v \) is cart velocity, \( \theta \) is the pole angle (with \( \theta = 0 \) denoting the upright pole), and \( \omega \) is the pole angular velocity, and actions of the form \( u \in \mathbb{R} \) representing the cart acceleration. Then, we can use \( \rho((z, v, \theta, \omega)) = ((z, 0, 0, 0), 0) \). In other words, \( \rho(x) \) is a state where the pole is upright and both the cart and the pole are not moving, together with an action where the cart acceleration is zero. Other examples of \( \rho(x) \) include a walking robot standing upright, a quadcopter hovering at a position, or a swimming robot treading water.

**Remark 2.2.** Note that not any state \( \hat{x} \in \mathcal{X} \) can be stabilized using LQR control. For example, in the cart-pole system, if \( v \neq 0 \), then it is impossible to stabilize the system, since \( x_{t+1} \) changes regardless of the control policy \( \pi \)—in particular, the dynamics for the cart position are \( z_{t+1} = z_t + v_t \), so \( z_{t+1} \neq z_t \) unless \( v_t = 0 \). While our safety guarantees hold for arbitrary \( \rho(x) \), our shielded control policy performs better if we can stabilize the system near \( \rho(x) \).
While the complexity of this computation is linear in \( T \), it nevertheless adds overhead compared to

### Remark 3.1.

Note that our algorithm has to perform the check \( \text{ISRECOVERABLE} \) on every step. While the complexity of this computation is linear in \( T \), it nevertheless adds overhead compared to

3 Online Shielding Algorithm

**Overview.** At a high level, at each step, the shield \( \pi_{\text{shield}} \) constructed by our algorithm checks whether using the given control policy \( \hat{\pi} \) is safe. If so, then \( \pi_{\text{shield}}(x) = \hat{\pi}(x) \); otherwise, \( \pi_{\text{shield}}(x) = \pi_{\text{backup}}(x) \), where \( \pi_{\text{backup}} \) is a backup controller that is guaranteed to be safe.

More concretely, the backup controller consists of two parts: (i) a recovery policy \( \pi_{\text{rec}} \) that attempts to transition the robot into a state that can be stabilized, and (ii) an LQR \( \pi_{\text{LQR}} \) that can be used once the robot is in a state where it can be stabilized. We refer to a state \( x \in X \) such that an LQR \( \pi_{\text{LQR}} \) can be used to stabilize \( x \) as *stable*. For example, consider a running robot. Stable states are those where the robot is near a motionless, standing position. However, states where the robot is running are not stable, since \( \pi_{\text{LQR}} \) cannot stabilize the robot. Then, \( \pi_{\text{rec}} \) might try to slow the robot down until it reaches a standing position; from there, \( \pi_{\text{LQR}} \) can be used to stabilize the robot.

The challenge is that we have no guarantees about the performance of \( \pi_{\text{rec}} \); instead, \( \pi_{\text{shield}} \) is designed to work with any recovery policy. The key idea is that a state \( x \in X \) is *recoverable* if \( \pi_{\text{rec}} \) can successfully transition the robot from \( x \) to a stable state. Then, to check whether it is safe to use \( \hat{\pi} \), the shield checks if \( x' = \hat{f}(\hat{\pi})(x) \) is recoverable—it does so simply by simulating \( \hat{f}(\pi_{\text{rec}}) \) from \( x' \). If \( x' \) is recoverable, then it is safe to use \( \hat{\pi} \). If not, then since \( x \) is recoverable, it is safe to use \( \pi_{\text{backup}} \)—i.e., either \( \pi_{\text{LQR}} \) if \( x \) is stable, or \( \pi_{\text{rec}} \) otherwise.

Finally, our theoretical guarantee says that \( \pi_{\text{shield}} \) is guaranteed to be safe from any initial state \( x_0 \in X \) that is recoverable. Note that this guarantee depends on the recovery policy \( \pi_{\text{rec}} \). In general, any stable state is also recoverable, but \( \pi_{\text{rec}} \) enlarges the set of recoverable states. Continuing our example, a state is recoverable as long as \( \pi_{\text{rec}} \) can successfully slow the robot down until it is standing upright. Thus, even though the robot is never in a stable state, \( \pi_{\text{shield}} \) can safely continue running using \( \hat{\pi} \) as long as it knows that it can stop itself using \( \pi_{\text{rec}} \).

Our algorithm is shown in Algorithm 1. The subroutine \( \pi_{\text{shield}} \) implements \( \pi_{\text{shield}} \), and the subroutine \( \text{RUNWITHSHIELD} \) controls the system using \( \pi_{\text{shield}} \). The subroutine \( \text{ISRECOVERABLE} \) shown in Algorithm 2 checks if a given state \( x \in X \) is recoverable. We describe the components of our algorithm in more detail in the following sections. Note that \( \pi_{\text{shield}} \) takes an auxiliary input \((\hat{x}, \hat{u})\), which is a current LQR target for the system; we describe the purpose of this auxiliary input below.

#### Algorithm 1 Run \( \pi_{\text{shield}} \) from initial state \( x \).

**procedure** RUNWITHSHIELD(\( \hat{\pi}, \pi_{\text{rec}}, x \))

\[
(x, \hat{u}) \leftarrow \rho(x)
\]

while true do

\[
u, (\hat{x}, \hat{u}) \leftarrow \text{SHIELD}(\hat{\pi}, \pi_{\text{rec}}, x, (\hat{x}, \hat{u}))
x \leftarrow f(x, u)
\]

end while

**end procedure**

#### Algorithm 2 Check if \( x \) is recoverable.

**procedure** ISRECOVERABLE(\( \pi_{\text{rec}}, x \))

for \( t \in \{0, 1, \ldots, T - 1\} \) do

\[
(x, \hat{u}) \leftarrow \rho(x)
\]

\[
\pi_{\text{LQR}}, V \leftarrow \text{LQRCONTROL}(\hat{x}, \hat{u})
\]

\[
G \leftarrow \text{LQRVERIFY}(\hat{x}, \hat{u}, \pi_{\text{LQR}}, V)
\]

if ISRECOVERABLE(\( \pi_{\text{rec}}, f(\hat{\pi})(x) \)) then

\[
\pi_{\text{shield}}(x) = \hat{\pi}(x)
\]

else if \( x \in G \) then

\[
\pi_{\text{shield}}(x) = \pi_{\text{LQR}}(x)
\]

else

\[
\pi_{\text{shield}}(x) = \pi_{\text{rec}}(x)
\]

end if

end for

**end procedure**

Throughout, *polynomial* refers to a multivariate polynomial over \( x \in X \) with real coefficients.
We give a proof in Appendix B. In general, the optimization problem (2) may be intractable. To handle this issue, we can safely add a time out, and have \texttt{ISRECOVERABLE} return false if it runs out of time (i.e., default to using the backup policy).

**LQR control.** Our algorithm uses LQR control to try and stabilize the system when near an LQR target $\rho(x) = (\tilde{x}, \tilde{u})$. In particular, a linear quadratic regulator (LQR) $\pi_{LQR}$, which has form $\pi_{LQR}(x) = Kx$ for some $K \in \mathbb{R}^{n_U \times n_X}$, is the optimal controller for the linear dynamical system $\tilde{f}(x, u) = Ax + Bu$, where $A \in \mathbb{R}^{n_X \times n_X}$ and $B \in \mathbb{R}^{n_X \times n_U}$, with rewards $R(x, u) = x^T Q x + u^T R u$, where $Q \in \mathbb{R}^{n_X \times n_X}$ and $R \in \mathbb{R}^{n_U \times n_U}$. Note that the joint dynamics are $f(\pi_{LQR})(x) = (A + BK)x$. An LQR problem instance is specified by $(A, B, R, Q)$; given such a problem instance, the LQR algorithm either returns $\pi_{LQR}$, or $\emptyset$ if the dynamics cannot be stabilized.

Given an LQR target $\rho(x) = (\tilde{x}, \tilde{u})$, LQRCONTROL constructs an LQR problem instance based on a linear approximation $\tilde{f}$ of $f$ around $(\tilde{x}, \tilde{u})$, and then returns the LQR $\pi_{LQR}$ (or $\emptyset$) for this problem instance. This algorithm is standard [22][23], so we omit details. Intuitively, $\pi_{LQR}$ attempts to stabilize the dynamics $f$ around the given LQR target $(\tilde{x}, \tilde{u})$. Since the linear approximation $\tilde{f}$ of $f$ becomes arbitrarily accurate close to $(\tilde{x}, \tilde{u})$, we intuitively expect $\pi_{LQR}$ to be a good control policy.

As we discuss below, we can compute regions on which $\pi_{LQR}$ is guaranteed to be safe. In addition to returning $\pi_{LQR}$, LQRCONTROL also returns a function $V(x) = x^T P x$, where $P \in \mathbb{R}^{n_X \times n_X}$ is a positive semidefinite matrix. The function $V$ is used to help compute safe regions for $\pi_{LQR}$.

**LQR verification.** We use LQR verification to compute a region in which the LQR $\pi_{LQR}(x) = Kx$ is guaranteed to be safe [18][22][23]. Given a policy $\pi$, a set $G \subseteq \mathcal{X}$ is invariant for $\pi$ if for any initial state $x_0 \in G$, the trajectory generated by $\pi$ from $x_0$ is contained in $G$. Thus, if the system starts in the region $G$, then it remains in $G$ for all time. The following result follows by induction on $t$.

**Theorem 3.2.** Let $\pi$ be a control policy. Suppose that there exists $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ satisfying

$$V(x) \geq V(f(\pi)(x)) \quad (\forall x \in G_\epsilon = \{ x \in \mathcal{X} \mid V(x) \leq \epsilon \}),$$

where $\epsilon \in \mathbb{R}_{\geq 0}$ is some constant. Then, $G_\epsilon$ is an invariant set for $\pi$.

We call $V$ a Lyapunov function. By Theorem 3.2, given a candidate Lyapunov function $V$, we can use optimization to compute $\epsilon$ such that $G_\epsilon$ is invariant. In particular, for a set $\mathcal{F}$ of functions $\mathcal{X} \to \mathbb{R}$, a control policy $\pi$, and a candidate Lyapunov function $V$, consider the following optimization problem:

$$\epsilon = \max_{\lambda \in \mathcal{F}, \mu \in \mathcal{F}^+, \epsilon' \in \mathbb{R}} \quad (\text{subj. to } \quad V(x) \geq V(f(\pi)(x)) + \lambda(x)(V(x) - \epsilon') \geq 0 \quad (\forall x \in \mathbb{R}^{n_X})$$

$$b_{safe} - A_{safe} x + \tilde{\mu}(x)(V(x) - \epsilon') \geq 0 \quad (\forall x \in \mathbb{R}^{n_X})$$

$$\lambda(x), \tilde{\mu}(x), \epsilon' \geq 0 \quad (\forall x \in \mathbb{R}^{n_X}).$$

**Theorem 3.3.** Let $\epsilon$ be the solution to (2): (i) $G_\epsilon$ is invariant for $\pi$, and (ii) $\pi$ is safe from any $x_0 \in G_\epsilon$.

We give a proof in Appendix B. In general, the optimization problem (2) may be intractable. To obtain a tractable variant, we assume that the candidate Lyapunov function $V$ is a polynomial in $x$, and that $\mathcal{F}$ is the set of polynomials in $x$ of degree at most $d'$. Then, for $\pi = \pi_{LQR}$, the left-hand side of each constraint in (2) are also polynomials in $x$, since $\pi_{LQR}(x) = Kx$ is linear.

Finally, we can replace each of these constraints with the stronger constraint that the polynomial is a sum-of-squares (SOS). In particular, a polynomial $p(x)$ is SOS if $p(x) = p_1(x)^2 + \ldots + p_k(x)^2$ for some polynomials $p_1, \ldots, p_k$. If $p(x)$ is SOS, then $p(x) \geq 0$ for all $x \in \mathcal{X}$. With this modification, (2) is an SOS program. For the case where $\mathcal{F}$ is the set of polynomials of bounded degree $d'$, an SOS program can be relaxed to a semidefinite program [18][22][23], which can be solved efficiently.

Together, our approach to solving (2) is sound—i.e., the constraints in (2) are guaranteed to be satisfied by our solution $\lambda, \tilde{\mu}, \epsilon$, so the statement of Theorem 3.3 holds. However, there may exist a solution even if we conclude that (2) is infeasible, or there may exist a better solution than ours (i.e., larger $\epsilon$). Nevertheless, in practice, these approximations appear to be fairly tight [22].

Finally, a standard choice for the candidate Lyapunov function $V$ is the cost-to-go function (i.e., the negative value function) of $\pi_{LQR}$, which is computed as part of the LQR algorithm. In particular, this function has the form $V(x) = x^T P x$, where $P \in \mathbb{R}^{n_X \times n_X}$, so it is a polynomial. Furthermore, we
are guaranteed that $P$ is positive semidefinite, so $V(x) \geq 0$ for all $x \in \mathcal{X}$. It can be shown that $V$ is a Lyapunov function for $\pi_{\text{LQR}}$ for the linear approximation $\tilde{f}$, which makes it a promising choice for a Lyapunov function for the true dynamics $f$.

In summary, LQRVerify solves the SOS variant of (2) for the LQR $\pi_{\text{LQR}}(x) = Kx$ and the candidate Lyapunov function $V(x) = x^TPx$, and returns the safe region $\mathcal{G}$, (or $\emptyset$ if $\pi_{\text{LQR}} = \emptyset$, or if the system cannot be provably stabilized). Finally, we say $x \in \mathcal{X}$ is $(\hat{x}, \hat{u})$-stable if $x \in \mathcal{G}$, where

$$\mathcal{G} = \text{LQRVerify}(\hat{x}, \hat{u}, \pi_{\text{LQR}}, V) \quad \text{where} \quad \pi_{\text{LQR}}, V = \text{LQRCtrl}(\hat{x}, \hat{u}).$$

In other words, $x$ is in the invariant set $\mathcal{G}$ for the LQR $\pi_{\text{LQR}}$ for the LQR target $(\hat{x}, \hat{u})$. We say simply that $x$ is stable if $x$ is $\rho(x)$-stable. We can check stability by computing $\mathcal{G}$ and checking if $x \in \mathcal{G}$.

**Remark 3.4.** In practice, we can often precompute an invariant set for all possible LQR targets $(\hat{x}, \hat{u})$, due to some symmetry in the system. For example, our LQR targets for the cart-pole only vary in the position $z$ of the system. Since the dynamics are equivariant under translation, we can compute the invariant set $\mathcal{G}$ for $z = 0$; then, the invariant set for general $z$ is simply $\mathcal{G}$ translated by $z$.

**Recovery policy.** Our algorithm takes as input a recovery policy $\pi_{\text{rec}}$. Ideally, $\pi_{\text{rec}}$ should try to transition the system to a stable state. Our algorithm works with any given $\pi_{\text{rec}}$. But the cumulative reward achieved depends on the quality of $\pi_{\text{rec}}$. $\pi_{\text{shield}}$ uses $\hat{\pi}$ more frequently if $\pi_{\text{rec}}$ is higher quality.

Given a recovery policy $\pi_{\text{rec}}$, we can define a notion of whether a state $x \in \mathcal{X}$ is recoverable—i.e., whether $\pi_{\text{rec}}$ can safely transition from $x$ to a stable state $x'$. More precisely, given $\pi_{\text{rec}}$ and a time horizon $T \in \mathbb{N}$, a state $x \in \mathcal{X}$ is recoverable if for the trajectory

$$x_0 = x,$$

$$x_{t+1} = f^{(\pi_{\text{ctrl}})}(x_t) \quad (\forall t \in [T]),$$

there exists $t' \in [T] = \{0, 1, ..., T - 1\}$ such that $x_0, ..., x_{t'} \in \mathcal{X}_{\text{safe}}$ and $x_{t'}$ is stable. To check whether a given state $x \in \mathcal{X}$ is recoverable, we simulate the trajectory (3) and check the recoverability condition for each state $x_{t'}$ for $t' \in [T]$. Algorithm 2 returns whether a given state is recoverable.

To train $\pi_{\text{rec}}$, we sample a random initial state $x_0 \sim d_{\text{init}}$ and then sample a random horizon $t \in \mathbb{N}$ (e.g., uniformly on some large horizon $T' \in \mathbb{N}$). Next, we simulate the system using the learned policy $\hat{\pi}$ for $t$ steps to obtain $x_t$. Finally, we reject $x_t$ if $x_t \not\in \mathcal{X}_{\text{safe}}$. In this way, we obtain a distribution $d_{\text{rec}}$ over $\mathcal{X}$. Finally, we use reinforcement learning to train $\pi_{\text{rec}}$ to recover the state using the rewards $R_{\text{rec}}(x) = \mathbb{I}[x \text{ is stable}],$ where $\mathbb{I}$ is the indicator function. We can also use a shaped reward. For example, we can use $R(x) = \|x - \hat{x}\|^2$, where $(\hat{x}, \hat{u}) = \rho(x)$. Assuming points near $\hat{x}$ are stable, this choice rewards states that are “closer” to being stable.

**Shield policy.** Intuitively, our shield $\pi_{\text{shield}}$ maintains safety depending on one of two cases. The first case is when the system is not at a $(\hat{x}, \hat{u})$-stable state. In this case, it maintains the invariant that $x$ is recoverable. In particular, it only uses $\hat{\pi}$ if $f(\hat{\pi})(x)$ is recoverable. Otherwise, it uses $\pi_{\text{rec}}$, which ensures that $f^{(\pi_{\text{ctrl}})}(x)$ is recoverable (assuming $x$ is recoverable).

The second case is when the system reaches a stable state $x$ after using $\pi_{\text{rec}}$ to recover. In principle, since $x$ is stable, we can now use $\pi_{\text{LQR}}$ to stabilize the system, thereby ensuring safety. One subtlety is that we have to continue using $\pi_{\text{LQR}}$ for the same LQR target $(\hat{x}, \hat{u})$—our safety guarantee relies on the fact that the trajectory remains inside the invariant set $\mathcal{G}$, which is only guaranteed to be invariant if $\pi_{\text{LQR}}$ does not change. In particular, the auxiliary input $(\hat{x}, \hat{u})$ to Shield in Algorithm 1 remains unchanged when using $\pi_{\text{LQR}}$. We have the following guarantee, which we prove in Appendix C.

**Theorem 3.5.** If $x_0 \in \mathcal{X}$ is recoverable, then $\pi_{\text{shield}}$ is safe starting from $x_0$.

### 4 Experiments

We evaluate our approach on two benchmarks, the cart-pole and a bicycle, showing how it ensures safety in a scalable way. All experiments are run on a 2.9 GHz Intel Core i9 CPU with 32GB memory.

**Dynamical systems.** First, we consider the cart-pole with continuous actions. Our rewards are for the cart to have positive velocity (i.e., move towards the right), with a target velocity of $v_0 = 0.1$. The safety constraint is that the pole angle does not exceed $\theta_{\text{max}} = 0.15$ (measured in radians) from
where $\phi$ is the current angle of the bicycle (i.e., the direction it travels if $\theta = 0$). We use degree 6 polynomials for $\mathcal{F}$ in the SOS program. For the bicycle, we use

$$\rho(x_f, y_f, x_b, y_b, v) = \left( x_f + v \cdot \cos \phi + v \cdot \sin \phi, x_b + v \cdot \cos \phi, y_b + v \cdot \sin \phi, (0, 0) \right)$$

where $\phi$ is the current angle of the bicycle (i.e., the direction it travels if $\theta = 0$)—i.e., $\rho(x)$ is the point where the bicycle reaches if it moves straight in the direction it is currently moving for one time step, and then comes to a complete stop. Since the bicycle is only moving along one dimension (i.e., $\theta = 0$), we can reparameterize the dynamics so they are linear, so LQR verification is exact.

**Modified dynamical systems.** A common challenge with approaches to safe reinforcement learning based on ahead-of-time verification is that they typically only apply to a specific realization of the dynamical system. If the dynamical system changes—e.g., the positions of obstacles—then verification has to be performed starting from scratch. As we show below, verification can be very intractable to perform ahead-of-time, let alone on-the-fly. Yet, changes in the dynamical systems are a major cause of safety failures, since the learned policy $\hat{\pi}$ is tailored to perform well in the original system. In contrast, our approach ensures safety on-the-fly by construction.

To demonstrate our advantage, we also consider a modification to each of our dynamical systems. First, for the cart-pole, we increase the time horizon—though $\hat{\pi}$ and $\pi_{\text{rec}}$ are trained with a time horizon of $N = 200$, we use them to control the system for a time horizon of $N = 1000$. Second, for the bicycle, we enlarge the obstacles to have radius 0.2, compared to 0.05 originally.

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In other words, we backpropagate through both the neural network and the dynamics.
Results. In Figure 1 (left, middle), we show the reward achieved for both dynamical systems and their respective modifications. The reward shown is the actual performance—\( z \) for cart-pole (i.e., distance traveled by the cart), and \( x \) for the bicycle (i.e., distance traveled towards the target)—rather than the shaped reward used to learn \( \hat{\pi} \). We also show the safety probability—i.e., the probability that a random state visited during a random rollout is safe. We show results for (i) \( \hat{\pi} \) ("No Shield"), (ii) \( \pi_{\text{shield}} \) without \( \pi_{\text{rec}} \)—i.e., using \( T = 0 \) ("No Recovery"), and (iii) \( \pi_{\text{shield}} \) using \( T = 100 \) ("Ours").

All of our shielded policies (i.e., both \( T = 0 \) and \( T = 100 \)) achieve perfect safety. The learned policy \( \hat{\pi} \) achieves good safety probability (1.0 for cart-pole, 0.97 for bicycle) on the original environments. However, as expected, it performs very poorly on the modified environments (0.45 on cart-pole, 0.42 on bicycle), since it was not trained to account for these modifications. While our shielded policy (with \( T = 100 \)) achieves somewhat reduced reward compared to \( \hat{\pi} \), it always achieves perfect safety.

Policy usage. In Figure 1 (right), for \( \pi_{\text{shield}} \) with \( T = 100 \), we show the probability of using \( \hat{\pi} \), \( \pi_{\text{rec}} \), and \( \pi_{\text{LQR}} \) as a function of time \( t \), on the original cart-pole and bicycle environments. For cart-pole, \( \pi_{\text{shield}} \) initially uses \( \pi_{\text{LQR}} \) to upright the pole, and then proceeds to use a combination of \( \hat{\pi} \) and \( \pi_{\text{rec}} \). For the modified environment (plot omitted), \( \pi_{\text{shield}} \) inevitably switches to using \( \pi_{\text{LQR}} \) only—\( \hat{\pi} \) acts pathologically (and unsafely) for states with large \( z \), since it was not trained on these states. For the bicycle, in about half the rollouts, \( \pi_{\text{shield}} \) switches to \( \pi_{\text{LQR}} \) and does not make further progress, likely because the obstacle was blocking the way. For the remaining rollouts, \( \pi_{\text{shield}} \) uses \( \hat{\pi} \) for most of the rollout. For the modified environment (plot omitted), \( \pi_{\text{shield}} \) almost always uses \( \pi_{\text{LQR}} \).

Need for a recovery policy. Our results demonstrate the importance of using \( \pi_{\text{rec}} \)—when \( T = 0 \), \( \pi_{\text{shield}} \) only uses \( \pi_{\text{LQR}} \), and makes no progress. In Figure 1 (left), we additionally show how reward varies as a function of the recovery horizon \( T \) for modified cart-pole. There is a large improvement even for \( T = 25 \); performance then levels off, with \( T = 75 \) and \( T = 100 \) achieving similar reward.

Running time. We study the running time of \( \pi_{\text{shield}} \)—i.e., how long it takes to compute a single action \( u = \pi_{\text{shield}}(x) \). In Figure 2, we show how the average running time varies as a function of the recovery horizon \( T \) on cart-pole. As expected, for \( T = 100 \), the running time is about 100× the running time of \( \hat{\pi} \) (26.0ms vs. 0.2ms), since it simulates the dynamical system for 100 steps. We believe that this overhead is an acceptable cost for guaranteeing safety. The worst case running time is linear in \( T \), but it can be sublinear if \( \pi_{\text{rec}} \) reaches a stable state in fewer than \( T \) steps.

Comparison to ahead-of-time verification. Existing approaches to safety require that the backup policy be verified ahead-of-time [13, 5, 8, 4]. We compare our approach to ahead-of-time verification. One challenge is that these techniques typically only perform verification for a bounded state space or for a bounded time horizon \( \tau \). In Figure 2, we show how the running time of a state-of-the art verification algorithm scales as a function of \( \tau \) for verifying a cart-pole policy [7]. The \( y \)-axis is log-scale—thus, verification is exponential in \( \tau \). Even for \( \tau = 25 \), it takes about 24 minutes to perform verification. Though this computation is offline, it quickly becomes intractable for long horizons. As a rough estimate, the running time grows \( 10^2 \times \) from \( \tau = 10 \) to \( \tau = 20 \); extrapolating this trend, the running time for \( \tau = 200 \) would be over \( 10^{30} \) years. In contrast, our approach not only ensures safety for an unbounded horizon, but is also substantially more computationally feasible.

Conclusion. We leave much room for future work—e.g., improving the reward achieved using our approach, extending our approach to handling uncertainty in the dynamics or partial observability, and reducing the computational cost of our approach.

Figure 2: Reward (left) and time per action (middle) on modified cart-pole for \( \pi_{\text{shield}} \) as a function of \( T \in \{0, 25, 50, 75, 100\} \), and time to verify the cart-pole policy from [7] as a function of \( \tau \) (right). We show means and standard errors over 10 random rollouts (left, middle) or 4 runs (right).
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A  Limitations of Ahead-Of-Time Verification

Existing approaches to ensuring safety rely on ahead-of-time verification to ensure safety. Given a policy \(\pi\), the goal is to prove that

\[
\phi_{\text{safe}} = \bigwedge_{t=0}^{\infty} A_{\text{safe}} x_t \leq b_{\text{safe}},
\]

for every \(x_0 \in X_0 \subseteq X\), where \(x_{t+1} = f^{(\pi)}(x_t)\) for all \(t \geq 0\). Here, \(X_0\) is some set of allowed initial states. Approaches to safe reinforcement learning either learn a policy \(\hat{\pi}\) and try to verify that \(\hat{\pi}\) is safe, or synthesize a safe backup policy \(\pi_{\text{backup}}\), which implicitly verifies that \(\pi_{\text{backup}}\) is safe.

A key challenge to verifying that \(\phi_{\text{safe}}\) holds is the infinite time horizon. One approach is to verify the system over a finite time horizon \([7, 9, 14]\). However, as we show in our experiments, this approach cannot ensure long-term safety. An alternative approach is to compute an invariant set in which the system is guaranteed to stay for all time \([13, 3, 8]\). However, this approach only applies to systems where the goal is to stay in a bounded region of the state space.

In addition, we noted that these approaches have difficulty handling novel environments. The reason is that to verify all possible environments ahead-of-time, we would have to encode the configuration of the environment in the state space. However, doing so quickly causes the dimension of the state space to become very large, and verification algorithms often do not scale well to high-dimensional state spaces, especially those that perform some form of exhaustive search \([13, 3, 8]\). The alternative would be to verify safety every time a novel environment is encountered, but as we show in our experiments, this approach still does not scale well.

Finally, note that our approach ensures safety for \(\phi_{\text{safe}}\) when \(X_0\) is the set of recoverable states.

B  Proof of Theorem 3.3

Consider any \(x\) such that \(V(x) \leq \epsilon\). To see (i), note that in the first constraint in (2), the second term is negative since \(\lambda(x) \geq 0\), so \(V(x) - V(f^{(\pi)}(x)) \geq 0\). Thus, by Theorem 3.2, \(G_x\) is invariant. Similarly, to see (ii), note that in the second constraint in (2), the second term is negative since \(\tilde{\mu}(x) \geq 0\), so \(b_{\text{safe}} - A_{\text{safe}} x \geq 0\). Thus, \(x \in X_{\text{safe}}\) for all \(x \in G_x\). Since \(G_x\) is invariant, it follows that \(\pi\) is safe from any \(x_0 \in G_x\). \(\Box\)

C  Proof of Theorem 3.5

Let \(x_0, x_1, \ldots\) be the trajectory generated using \(\pi_{\text{shield}}\) from \(x_0\). We claim that for all \(t \in \mathbb{N}\), \(x_t\) is either (i) recoverable, or (ii) \((\hat{x}_t, \hat{u}_t)\)-stable, where \((\hat{x}_t, \hat{u}_t)\) is the auxiliary input to \textsc{Shield} in Algorithm 1 on step \(t\). The base case \(t = 0\) holds by assumption. Now, suppose that the inductive hypothesis holds for \(t\); we prove it holds for \(t + 1\). We consider three cases, corresponding to which of the three branches in \textsc{Shield} is taken.

If the first branch is taken, then \(x_{t+1} = f^{(\pi)}(x_t)\) is recoverable, so we are done. If the second branch is taken, then \(x_t = (\hat{x}_t, \hat{u}_t)\)-stable, so \(x_{t+1} = f^{(\pi_{\text{LQR}})}(x_t)\) is also \((\hat{x}_t, \hat{u}_t)\)-stable. Furthermore, in this case, \(\hat{x}_{t+1} = \hat{x}_t\) and \(\hat{u}_{t+1} = \hat{u}_t\), so \(x_{t+1}\) is \((\hat{x}_{t+1}, \hat{u}_{t+1})\)-stable, so we are done.

If the third branch is taken, then \(x_t\) must be recoverable, since it is not \((\hat{x}_t, \hat{u}_t)\)-stable or the second branch would have been taken. Thus, the trajectory \(\zeta = (x'_0, x'_1, \ldots, x'_{t'})\) obtained by using \(\pi_{\text{rec}}\) from \(x'_0 = x\) reaches a stable state after \(t' \in [T]\) steps without leaving \(X_{\text{safe}}\). If \(t' > 0\), then \(x_{t'+1} = f^{(\pi_{\text{LQR}})}(x_t) = x'_{t'}\) is the second state in \(\zeta\). Then, the trajectory \(\zeta' = (x'_{t'}, \ldots, x'_{t'+1})\) reaches a stable state after \(t' - 1\) steps, so \(x_{t'+1}\) is recoverable as well.

We show that we cannot have \(t' = 0\). In particular, by the definition of recoverability, \(t' = 0\) implies that \(x_t\) is stable (i.e., \(\rho(x_t)\)-stable). Since \(x_t\) is not \((\hat{x}_t, \hat{u}_t)\)-stable, we must have \(\rho(x_t) \neq (\hat{x}_t, \hat{u}_t)\). This condition can only happen if \(t > 0\) and \(x_{t-1}\) takes the second branch in \textsc{Shield}. As a consequence, it must be the case that \(x_{t-1} = (\hat{x}_{t-1}, \hat{u}_{t-1})\)-stable, \(x_t = f^{(\pi_{\text{LQR}})}(x_{t-1}) = \hat{x}_{t-1}\), and \(\hat{u}_t = \hat{u}_{t-1}\). Then, \(x_t\) must be \((\hat{x}_t, \hat{u}_t)\)-stable, a contradiction. Thus, we cannot have \(t' = 0\).

Finally, if \(x_t\) is recoverable, then by definition, we have \(x_t \in X_{\text{safe}}\). Alternatively, if \(x_t\) is \((\hat{x}_t, \hat{u}_t)\)-stable, then \(x_t \in G\) for some invariant set \(G\). By Theorem 3.3, \(G \subseteq X_{\text{safe}}\), so \(x_t \in X_{\text{safe}}\). \(\Box\)