Nonconvex-Nonconcave Min-Max Optimization with a Small Maximization Domain

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Abstract

We study the problem of finding approximate first-order stationary points in optimization problems of the form \( \min_{x \in X} \max_{y \in Y} f(x, y) \), where the sets \( X, Y \) are convex and \( Y \) is compact. The objective function \( f \) is smooth, but assumed neither convex in \( x \) nor concave in \( y \). Our approach relies upon replacing the function \( f(x, \cdot) \) with its \( k \)-th order Taylor approximation (in \( y \)) and finding a near-stationary point in the resulting surrogate problem. To guarantee its success, we establish the following result: let the Euclidean diameter of \( Y \) be small in terms of the target accuracy \( \varepsilon \), namely \( O(\varepsilon^{2k+1}) \) for \( k \in \mathbb{N} \) and \( O(\varepsilon) \) for \( k = 0 \), with the constant factors controlled by certain regularity parameters of \( f \); then any \( \varepsilon \)-stationary point in the surrogate problem remains \( O(\varepsilon) \)-stationary for the initial problem. Moreover, we show that these upper bounds are nearly optimal: the aforementioned reduction provably fails when the diameter of \( Y \) is larger. For \( 0 \leq k \leq 2 \) the surrogate function can be efficiently maximized in \( y \); our general approximation result then leads to efficient algorithms for finding a near-stationary point in nonconvex-nonconcave min-max problems, for which we also provide convergence guarantees.

1 Introduction

In the past few years, min-max optimization has become popular among practitioners due to its relevance for machine learning applications—in particular, when training generative adversarial networks (GANs) [1, 2, 3, 4], robust machine learning [5, 6, 7, 8, 9, 10], in fair statistical inference [11, 12, 13, 14, 15, 16], in reinforcement learning [17], distributed optimization and learning over networks [18, 19, 20, 21, 22], and for optimal resource allocation in multi-agent systems [23, 24]. The common task arising in these applications, as well as in many others, is solving optimization problems of the general form

\[
\min_{x \in X} \max_{y \in Y} f(x, y),
\]

where \( X, Y \) are some convex sets in the corresponding high-dimensional Euclidean spaces and \( f \) is smooth in both variables, possibly in a heterogeneous manner. Min-max optimization has been an active area of research since the early works of Nash, von Neumann, and Morgenstern [25, 26, 27]. A very important subclass of such problems where \( f \) is convex-concave—i.e., \( f(\cdot, y) \) is convex and \( f(x, \cdot) \) is concave for all \((x, y) \in X \times Y\)—has been studied extensively from the algorithmic viewpoint starting from the work of Nemirovski [28], who proposed a first-order algorithm with \( O(1/T) \)

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dimension-independent convergence rate, generalizing the extragradient algorithm from the early work of Korpelevich [29]. This has been followed by many extensions (e.g., [30, 31, 32, 33, 34]) and applications in machine learning, statistics, and signal processing; see [35, 36, 37] and references therein. Convex-concave min-max optimization remains an active area up to this day: see, e.g., the recent works [38, 39] establishing the matching upper and lower complexity bounds under strong convexity-concavity; the recent proliferation of works on last-iterate convergence [40, 41, 42, 43]; the work [44] extending the functionality of CVX [45] (a.k.a. “disciplined convex programming”) to convex-concave min-max problems and monotone variational inequalities [46].

Meanwhile, many modern applications where (P) appears fall beyond the convex-concave scenario. For example, in the standard formulation of GANs $f$ is neither convex in $x$ nor concave in $y$ [1]; this is also the case in the task of adversarially-robust deep neural network training [47]; other applications of nonconvex-nonconcave min-max optimization can be found in the recent review article [48]. Problems in which $f$ is concave $y$ but nonconvex in $x$ arise in fair inference “à la Rényi” [12] and when minimizing the maximum of smooth functions [49] (in the latter case $f(x, \cdot)$ is even affine). Moreover, machine learning under distributional uncertainty [10, 50, 9], power control for wireless communication [51], and special formulations of the task of learning from multiple domains [52] all lead to nonconvex-nonconcave min-max optimization. In the past couple of years, these new applications reignited a substantial interest across the optimization theory community for analyzing (P) in the general nonconvex-nonconcave setup, the nonconvex-concave setup as an approachable stepping stone towards it, and in other “intermediate” scenarios, where $f(\cdot, y)$ is nonconvex, and $f(x, \cdot)$ is not concave but has some other special structure allowing to efficiently solve the nested maximization problem—that is, evaluate at any point $x \in X$ the primal function\footnote{Note also that with $f(\cdot, y)$ nonconvex, the minimax theorem does not apply, and the order of min and max in (P) becomes important. As such, there is a crucial difference between nonconvex-concave and convex-nonconcave instances of (P): the former ones are easier, as in them the primal function $\varphi(\cdot)$, cf. (1) is merely hard to minimize, while in the convex-nonconcave case we typically cannot even evaluate it at a point.}

$$\varphi(x) := \max_{y \in Y} f(x, y). \quad (1)$$

The nonconvex-concave case, first addressed in [53], is by now quite well understood. Generally, we lose all hope of actually solving (P)—or minimizing $\varphi(\cdot)$—when $f(\cdot, y)$ is nonconvex, as we then deal with a nonconvex minimization problem even for a singleton $Y$. Hence, one has to be satisfied here with approximating a local minimizer of $\varphi(\cdot)$, or a local saddle point (also called a Nash equilibrium) in (P)—i.e., a point $(x^\text{NE}, y^\text{NE}) \in X \times Y$ such that

$$f(x^\text{NE}, y) \leq f(x^\text{NE}, y^\text{NE}) \leq f(x, y^\text{NE})$$

for all $(x, y) \in X \times Y$ in a neighborhood of $(x^\text{NE}, y^\text{NE})$.

In fact, even these two tasks can be too ambitious, and most of recent studies have been focused on the tasks of approximating a first-order stationary point in (P) up to accuracy $\varepsilon > 0$, as measured by the norm of either the proximal gradient of $f$ or the gradient of the standard Moreau envelope of $\varphi(\cdot)$. (The notion of the Moreau envelope, essential in the context of our work, shall be recalled in Section 2.) Painting with a broad brush, these two accuracy measures turn out to be (essentially) equivalent in the nonconvex-concave case, and settling on one of them is largely a matter of personal preference; see [54, Section 5] for a technical discussion of this equivalence. In this line of research, the works [54, 55, 56, 57] demonstrated that an $\varepsilon$-first-order stationary point with respect to the Moreau envelope criterion, referred to as $\varepsilon$-FOSP from now on, can be found in $O(\varepsilon^{-3})$ queries of the
gradient of $f$, and in $\widetilde{O}(\varepsilon^{-2})$ queries when $f(x, \cdot)$ is strongly concave.$^2$ The latter of these estimates was recently shown to be worst-case optimal [58, 59, 60], and it is widely believed—although, to the best of our knowledge, not yet proved—that the former estimate is also tight. Furthermore, in the absence of concavity, but assuming access to an abstract maximization oracle evaluating $\varphi(x)$, the authors of [61] observed that an algorithm earlier proposed in [62] allows to compute an $\varepsilon$-FOSP in $O(\varepsilon^{-4})$ queries of the maximization oracle and of the $x$-gradient.

Nonconvex-nonconcave problems. By contrast, our understanding of the general nonconvex-nonconcave setup is rather fragmentary. One delicate issue here is that the two viewpoints on local optimality in $(P)$—the one focusing on finding a (local) minimizer $x^*$ of $\varphi(x) = \max_{y \in Y} f(x, y)$, historically due to Stackelberg [63], and the game-theoretic paradigm of Nash [25], where one aims at finding a local Nash equilibrium—might result in quite different notions of (local) optima and stationary point when $f(x, \cdot)$ is not concave [61]. Indeed: consider, as an illustration, the problem

$$\min_{x \in \mathbb{R}} \max_{y \in [-2, 2]} \{ f(x, y) := xy - \frac{1}{3} y^3 \}. \quad (2)$$

By computing $\varphi(x)$ one can verify that $x^* = 1$ is a unique local (hence also global) minimizer of $\varphi(x)$, and $\max_{y \in [-2, 2]} f(1, y)$ is attained at $y^* = -2$; in other words, $(1, -2)$ is a unique solution to (2). On the other hand, a unique first-order Nash equilibrium, that is $(x^{\text{FNE}}, y^{\text{FNE}})$ such that

$$\langle \nabla_x f(x^{\text{FNE}}, y^{\text{FNE}}), x - x^{\text{FNE}} \rangle \geq 0 \text{ and } \langle \nabla_y f(x^{\text{FNE}}, y^{\text{FNE}}), y - y^{\text{FNE}} \rangle \geq 0 \text{ for all } (x, y) \in X \times Y, \quad (3)$$

is the point $(0, 0)$. Finally, (2) does not have any (local) Nash equilibrium, and these conclusions remain valid if $x$ restricted to a compact set (e.g., $X = [0, 4]$). Meanwhile, $\inf_{x \in X} \varphi(x)$ is always finite in $(P)$, and is attained whenever $\varphi(x)$ is bounded from below on $X$ (in particular, whenever $X$ is compact), and must satisfy the necessary condition $\partial \varphi(x^*) \ni 0$ in terms of the weak subdifferential. (To make the paper self-contained, we provide background on weak convexity and weak subdifferentials in Appendix D; for the present non-technical discussion this is irrelevant.) Together with the above example, this observation makes a case for the Stackelberg approach as the one better suited for the general nonconvex-nonconcave scenario, and we adhere to this approach in our work. Another argument in favor of the Stackelberg approach is that in most of the modern applications of nonconvex-nonconcave min-max optimization we mentioned, the actual practical goal is to minimize the primal function, not to find a local Nash equilibrium.

The second challenge posed by nonconvex-nonconcave min-max optimization is an exceptional algorithmic difficulty of finding (approximately) even a local first-order stationary point or Nash equilibrium when no additional structure is imposed on $(P)$. In particular, in [64] it is shown that if $X, Y = [0, 1]^d$ are hypercubes and $f$ is $L$-smooth, $G$-Lipschitz, and takes values in $[-B, B]$, then:

(i) The problem of exhibiting $(x, y) \in [0, 1]^{2d}$ such that $\| \nabla f(x, y) \| \leq \frac{1}{21}$ or detecting that no such point exists,$^3$ is FNP-complete (in $d$) in the regime $L = d$, $G = \sqrt{d}$, $B = 1$ [64, Theorem 4.1].$^4$

(ii) At least one $(\epsilon, r)$-approximate local Nash equilibrium, defined as $(x^*, y^*) \in [0, 1]^{2d}$ such that

$$f(x^*, y) - \epsilon < f(x^*, y^*) < f(x, y^*) + \epsilon \quad \forall (x, y) \in [0, 1]^{2d} : \max \{ \| x - x^* \|, \| y - y^* \| \} \leq r,$$

---

$^2$We use the standard “big-O” notation: $g = O(h)$ or $g \preceq h$ both hide an absolute constant, i.e., means that $g \leq c h$ for some $c > 0$ uniformly over all allowed pairs of values of $g, h$; similarly, $g = \widetilde{O}(h)$ is a shorthand for $g = O(h \log(h + c))$.

$^3$In what follows, $\| \cdot \|$ stands for the standard Euclidean norm on a given Euclidean space (defined by the context).

$^4$See [65, 66] for a technical background on the complexity classes FNP, PPAD—in particular, in the context of $(P)$.,
is guaranteed to exist in the local regime \( r < \sqrt{2L/\varepsilon} \); however exhibiting it is PPAD-hard already when \( r \geq \sqrt{L/\varepsilon} \) and \( \max\{L, G, B, 1/\varepsilon, 1/r\} = O(\text{poly}(d)) \) [64, Theorems 4.3 and 4.4].

These two negative results demonstrate to us that approximating even a stationary point, or a local min-max point for large enough neighborhood, is virtually impossible without imposing additional structure on \((P)\). As such, existing positive results rely on adding this structure in one way or another. In particular, a common methodology is to “mimic” the case of a VI with a monotone operator [28] by imposing pseudomonotonicity, the Minty condition, or contraction of the best response mappings [67, 68, 69, 70, 71, 72, 73, 74, 75, 76]. However, these assumptions are restrictive and rarely satisfied in modern applications; see [48] for a detailed discussion. In addition, they are tied to the Nashian paradigm, whereas applications tend to call for the Stackelbergian one. Other interesting approaches rely upon restricting the nature and level of coupling between the variables [77] or the power of the max-player [78]. Yet another recent line of research advocated an alternative to the local Nash equilibria—“greedy adversarial equilibria” that are computationally feasible, but at the expense of a certain loss of transparency and interpretability [79, 80]. Finally, a growing body of literature is devoted to the asymptotic behavior of algorithms [81, 82, 83, 84, 85], and structural results for GANs and adversarial training [86, 87, 88, 89].

Going back to the hardness results (i)–(ii) established in [64], the latter of them hints at a possibility to control the complexity of \((P)\) through the size of a feasible set. Our work explores this possibility; let us now present the main ideas behind our approach.

**Outline of our approach.** Our work is motivated by a trivial observation: finding an approximate first-order stationary point of \((P)\) becomes a computationally feasible task when \(Y\) is a singleton, as in this case we deal with a smooth minimization problem, so an approximate stationary point can be found by running projected gradient descent. Furthermore, one can hope that the computational tractability is preserved when \(Y\) is not singleton but is small. Then, letting \(\hat{f}_k(x,\cdot)\) be the \(k\)th-order Taylor approximation of \(f(x,\cdot)\) for some \(k \geq 0\), one can advocate the following two-step approach.

1°. Prove that any \(\varepsilon\)-FOSP in the surrogate min-max problem is also an \(O(\varepsilon)\)-FOSP in \((P)\), due to \(\hat{f}_k(x,\cdot)\) approximating \(f(x,\cdot)\) over the (small) set \(Y\) well enough for our purposes.

2°. Find \(\varepsilon\)-FOSP in the surrogate problem—and thus \(O(\varepsilon)\)-FOSP in \((P)\)—by exploiting the structure of \(\hat{f}_k(x,\cdot)\) such as, e.g., linearity for \(k = 1\) or quadratic structure for \(k = 2\).

A natural question immediately arising in connection with this strategy, namely with 1°, is:

*How small the diameter of \(Y\) has to be in order for the reduction in step 1° to be valid?*

One would expect this question to have a nontrivial answer: indeed, it seems unlikely that a Taylor approximation would work over a set of a constant diameter.\(^5\) As such, one may expect that

\[
\text{diam}(Y) \leq \varepsilon^\rho(k)
\]

\(^5\)Note, however, that this would not immediately contradict the results of [64], as the \(\ell_2\)-diameter of \([0,1]^d\) is \(\sqrt{d}\).
allows for the reduction in 1° to work, with \( p(k) > 0 \) defined by the order \( k \) of Taylor approximation, and the hidden constant depending on the regularity properties of \( f \). Later on we shall verify this hypothesis, proving explicit bounds of the form (4) for arbitrary \( k \), under the natural regularity assumptions on \( f \), and showing that they are not only sufficient, but also necessary—that is, the reduction in 1° may fail for \( \text{diam}(Y) \) beyond the allowed threshold. Finally, we shall also implement step 2° of the strategy by designing efficient algorithms for solving the surrogate min-max problem.

**Applications of nonconvex-nonconcave min-max optimization with a small maximization domain.** In some applications in machine learning and signal processing, it is quite natural to assume that \( \text{diam}(Y) \) is relatively small. Such is the case, for example, in the task of training neural networks robust against adversarial attacks, where \( \text{diam}(Y) \) corresponds to the magnitude of the attack performed, and typically has to be small in order for the attack to remain undetected; meanwhile, the target accuracy often does not have to be optimized very accurately [5, 7]. More recently, the work [90] demonstrated that adding a small adversarial perturbation to the trained statistical model allows to avoid sharp local minima of the training loss; this approach, termed \textit{sharpness-aware training}, results in optimization problems of the form \( \min_{w} \max_{\|u\| \leq r} L(w + u) \), where \( L(\cdot) \) is the training loss functional, \( u \) is the perturbation of a model \( w \), and \( r \) should be small as the analysis relies upon linearizing \( L(\cdot) \) near \( w \). More generally, robust design of any system against small perturbations of certain parameters may lead to min-max problems of form (P) with a small maximization domain \( Y \).

**Paper organization and summary of contributions.** In a nutshell, our work is concerned with implementing both steps 1°–2° of the proposed approach; combined together, they result in efficient algorithms for finding an \( \varepsilon \)-FOSP in (P). The rest of the paper is organized as follows.

In Section 2 we set off by stating our assumptions about \( f \). In a nutshell, we grant Lipschitzness of the \( x \)-gradient \( \nabla_x f \) and of the order-\( k \) differential tensor \( \nabla^k_{y\to y} f \), for given \( k \in \mathbb{N} \setminus \{0\} \), in accordance with our intention of using the \( k \)-order Taylor expansion of \( f(x, \cdot) \). We then recall some mathematical background on (P), including the definitions of the Moreau envelope and approximate first-order stationary points, and finally define the reduction in 1° in a rigorous way.

In Section 3 we formulate, discuss, and prove our first result: the general bound of the form (4), holding for arbitrary \( k \in \mathbb{N} \cup \{0\} \) under the matching regularity assumption (i.e., with the same \( k \)). Informally, we show (cf. Theorem 3.1) that, for any \( k \in \mathbb{N} \cup \{0\} \), an arbitrary \( \varepsilon \)-FOSP in the counterpart of (P) with \( f_k \) instead of \( f \) as objective function, is also an \( \varepsilon \)-FOSP in (P) whenever

\[
\text{diam}(Y) \lesssim \max \left\{ \frac{\varepsilon^2 (k + 1)!}{\lambda \rho_k}, \frac{\varepsilon}{\mu} \right\}, \tag{5}
\]

where \( \lambda, \mu, \) and \( \rho_k \) are the uniform over \((x, y) \in X \times Y\) Lipschitz constants of \( \nabla_x f(\cdot, y), \nabla_x f(x, \cdot), \) and \( \nabla^k_{y\to y} f(x, \cdot) \) respectively. This gives (4) with \( p(k) = \frac{2^{k+1}}{k+1} \) for \( \varepsilon \) below a certain (constant) level.

In Section 4 we present our second series of results, showing that the bound (5) is nearly tight (namely, up to the \((k+1)\)-fold inflation of the term \( \varepsilon/\mu \)). We do this by constructing specific instances of (P), carefully choosing the center \( \hat{y} \in Y \) of the Taylor approximation, and then exhibiting a point \( x \in X \) which is \textit{stationary} in the approximated problem, while \textit{not} being an \( \varepsilon \)-FOSP in (P).

In Section 5 we carry out step 2° of the strategy. To this end, we suggest three algorithms based on the Taylor approximation with \( k \in \{0, 1, 2\} \). For \( k = 0 \) and \( k = 1 \) they are based on gradient
descent and descent-ascent; the resulting oracle complexity is \( O(\varepsilon^{-2}) \), and \( \text{diam}(Y) \leq \varepsilon \) is allowed, cf. Theorems 5.1 and 5.2. For \( k = 2 \), our algorithm allows for a larger diameter \( O(\varepsilon^{2/3}) \) as per (5). Yet, this improvement has a price: \( Y \) has to be a Euclidean ball, we require access to higher-order derivatives of \( f \), and the oracle complexity grows to \( O(\varepsilon^{-3}) \); see Theorem 5.3 for the precise claim.

**Notation.** We use the standard \( O(\cdot) \) and \( \Omega(\cdot) \) notation: given two functions \( g, h > 0 \) we use \( g = O(h) \) and \( g \leq h \) as shorthands for saying that \( g \leq ch \) for some \( c > 0 \) uniformly over all simultaneously allowed pairs of values of \( g, h \); similarly, \( g = \Omega(h) \) is a shorthand for \( g = O(h \log(h + e)) \). The symbol \( c \) denotes a numerical constant whose value is unimportant and might change from line to line. We use abridged notation \( \nabla_x f(x,y), \nabla_y f(x,y) \) for the partial gradients of \( f \) in the first and second argument evaluated at some \( (x,y) \in X \times Y \), and similarly for higher-order tensors of partial derivatives (see Section 2). \( \| \cdot \| \) stands for the standard Euclidean norm on a given Euclidean space (defined by the context) when the argument is a vector, and for the induced operator norm when the argument is a \( k \)-tensor (such tensors shall arise from the Taylor expansion of \( f(x, \cdot) \)). Other notation shall be introduced when necessary.

Most of our results are rather technical, and we defer the proofs to the appendix. An exception is made for Theorem 3.1, as its proof is not so technical and illuminates the mechanism behind (5).

## 2 Standing assumptions, definitions, and technical background

We shall focus on (P) assuming that \( X,Y \) are two convex sets with non-empty interior in the corresponding Euclidean spaces \( E_X, E_Y \); \( Y \) is compact and its Euclidean diameter is bounded by \( D \):

\[
D \geq \text{diam}(Y) := \max_{y,y' \in Y} \| y' - y \|.
\]

Let \( \| \cdot \| \) be the (Euclidean) operator norms on the spaces of multilinear forms on \( E_X^i \times E_Y^j \) for \( 0 \leq i \leq 2 \) and \( 0 \leq j \leq k \), where \( k \) is a non-negative integer. We grant two assumptions on the regularity of \( f \).

**Assumption 1** (First-order smoothness in \( x \)). The partial gradient \( \nabla_x f(x,y) \) exists and is Lipschitz-continuous on \( X \times Y \). More precisely, for some \( \lambda > 0 \), \( \mu \geq 0 \), any \( x, x' \in X \) and \( y, y' \in Y \), it holds that

\[
\| \nabla_x f(x', y') - \nabla_x f(x, y) \| \leq \lambda \| x' - x \| + \mu \| y' - y \|.
\]  

(A1)

**Assumption 2** (\( k \)-order smoothness in \( y \)). For given integer \( k \geq 0 \), the tensor \( \nabla_y^k f(x,y) \) of \( k \)-order partial derivatives of \( f \) in \( y \) exists and is Lipschitz-continuous on \( X \times Y \): for some \( \rho_k, \sigma_k, \tau_k \geq 0 \), any \( x, x' \in X \) and \( y, y' \in Y \), it holds that

\[
\| \nabla_y^k f(x', y') - \nabla_y^k f(x, y) \| \leq \rho_k \| y' - y \| + \sigma_k \| x' - x \|.
\]  

(A2)

Moreover, the tensor \( \nabla_{xy}^{k+1} f \) incorporating the partial derivatives of order \( k \) in \( y \) and \( 1 \) in \( x \) (in this order) exists and is Lipschitz continuous in \( x \): for some \( \tau_k \geq 0 \), any \( x', x \in X \) and \( y \in Y \), one has

\[
\| \nabla_{xy}^{k+1} f(x', y) - \nabla_{xy}^{k+1} f(x, y) \| \leq \tau_k \| x' - x \|.
\]  

(A3)

Assumption 2 merits some discussion concerning the cases \( k \in \{0,1,2\} \) that are most interesting from the algorithmic point of view (cf. Section 5).
• When $k = 0$, (A3) follows from (A1) with $\tau_0 = \lambda$, and so does not give any extra restrictions. As for (A2), it then requires that $f$ is $\rho_0$-Lipschitz in $y$ and $\sigma_0$-Lipschitz in $x$. In fact, such Lipschitzness conditions are not necessary for our approximation results to hold, although they can possibly improve the approximation bounds in Theorem 3.1 in the case $k = 0$, which is of marginal practical interest. Meanwhile, the Lipschitzness in $x$ is used in one of the algorithms proposed in Section 5. Even so, we have the variational bounds $\sigma_0 \leq \min \{\lambda \text{diam}(X), \mu D\}$ provided that $X$ contains in its interior a first-order stationary point of $f(\cdot, y)$ for any $y \in Y$.

• When $k = 1$, condition (A2) reduces to the Lipschitzness of the partial gradient $\nabla_y f$, namely

$$\| \nabla_y f(x', y') - \nabla_y f(x, y) \| \leq \rho_1 \|y' - y\| + \sigma_1 \|x' - x\|. \quad (6)$$

Moreover, due to (A3) we can assume, without loss of generality, that $\sigma_1 = \mu$. Meanwhile, (A3) is a second order condition: it implies that $\nabla^2_y f$ is differentiable in $y$ almost everywhere on $X$, and allows to preserve weak convexity in $x$ after the Taylor expansion in $y$. Note that (A3) holds with $\tau_1 = 0$ (in fact, with $\tau_k = 0$ for $k \geq 1$) for bilinearly-coupled (BC) objectives:

$$f(x, y) = p(x) + (Ax, y) + q(y). \quad (BC)$$

• When $k = 2$, condition (A2) reduces to the Lipschitzness of the partial Hessian $\nabla^2_y f$, namely

$$\| \nabla^2_y f(x', y') - \nabla^2_y f(x, y) \| \leq \rho_2 \|y' - y\|^2 + \sigma_2 \|x' - x\|. \quad (7)$$

Clearly, under (BC) for $k \geq 2$ one has $\sigma_k = 0$, and also $\rho_k = 0$ if, in addition, $q(y)$ is quadratic.

Finally, we grant a mild regularity assumption allowing to differentiate $\nabla^{k+1} f$ in $x$ under the integral (we use Lebesgue measures on $E_x$ and $E_y$); it is trivially satisfied if $\nabla^{k+2} f$ exists everywhere.

**Assumption 3.** Define the set-valued map $x \mapsto Y_x' \subseteq Y$ where $Y_x'$ is the set of $y$ for which $\nabla^{k+2} f(x, y)$ does not exist. Its graph, i.e., the subset of $X \times Y$ defined as $\{(x, y) : x \in X, y \in Y_x'\}$, is measurable.

Moreau envelope and the notion of FOSP. In this work we use a stationarity criterion based on the Moreau envelope of the primal function. To define it formally, we first have to remind some standard definitions; readers familiar with the notion of Moreau envelope for weakly convex functions can safely skip this part, while those looking for more details may refer to [62, 91] or [54, Sec. 5]. Recall that $\varphi(x) := \max_{y \in Y} f(x, y)$ is called the primal function for (P), cf. (1). Typically $\varphi$ is non-smooth (unless $f(x, \cdot)$ is strictly concave), and so its gradient is not defined on $X$. However, under Assumption 1 $\varphi$ is $\lambda$-weakly convex—that is, for any $x \in X$ the function given by $\varphi(\cdot) + \frac{\lambda}{2} \|\cdot\|^2$ is convex, and $(\lambda - \lambda)$-strongly convex if regularization is with $\frac{\lambda}{2} \|\cdot\|^2$ for $\lambda > \lambda$. (In order to streamline the presentation while keeping the paper self-contained, we provide necessary background on weakly convex functions and weak subdifferentials in Appendix D.) This allows to define the Moreau envelope of $\varphi$, as a function of $x \in X$, by minimizing $\varphi(\cdot) + \frac{\lambda}{2} \|\cdot\|^2$ for given $x$.

**Definition 1** (Moreau envelope). Let $\phi : X \to \mathbb{R}$ be $\lambda$-weakly convex. Given an $\tilde{\lambda} > \lambda$, the function

$$\phi_{\lambda}(x) := \min_{u \in X} \left\{ \phi(u) + \frac{\lambda}{2} \|u - x\|^2 \right\} \quad (8)$$

is called the $\tilde{\lambda}$-Moreau envelope of $\phi$, and the unique solution $x_{\phi/\lambda}(x)$ to (8) is called the proximal mapping of $x$ (corresponding to $\phi$ with stepsize $1/\tilde{\lambda}$).
It is well known (see, e.g., [91, Lemma 2.2]) that \( \varphi \) is \( C^1 \)-smooth when \( \tilde{\lambda} > \lambda \), and moreover it has \( O(\lambda) \)-Lipschitz gradient whenever \( \tilde{\lambda} = (1 + c)\lambda \) for \( c > 0 \). The standard practice (see, e.g., [62]) is to simply choose \( \tilde{\lambda} = 2\lambda \) and use the gradient norm \( \| \nabla \varphi_{2\lambda}(\cdot) \| \) as the accuracy measure. Indeed, from the first-order optimality conditions for (8) one can easily conclude (cf., e.g., [91, p. 4]) that

\[
\nabla \varphi_{2\lambda}(x) = 2\lambda(x - x^*) + \partial(\varphi + I_X)(x^*),
\]

where \( x^* = x^*/\bar{\lambda}(2\lambda)(x) \) for arbitrary \( x \in X \), \( I_X \) is the indicator function of \( X \), and \( \partial(\cdot) \) is the Fréchet (or weak) subdifferential (see Appendix D.1 for details). As a result, in the \( x \)-unconstrained case \( (X = E_{\lambda}) \) the inequality \( \| \nabla \varphi_{2\lambda}(x) \| \leq \varepsilon \) for some \( x \in X \) implies the existence of a point \( x^* = x^*/\lambda(x) \) within \( O(\varepsilon) \) distance from \( x \), such that \( \varphi \) has a subgradient at \( x^* \) with the norm at most \( \varepsilon \). A similar result holds in the constrained case if the subgradient norm of \( \varphi \) is replaced with an appropriate inaccuracy measure based on projection onto \( X \); see [54, Prop. 5.1] also given as Proposition D.1 in our paper. These results motivate us to introduce the following definition.

**Definition 2** (Approximate first-order stationary points). The point \( x \in X \) is called \((\varepsilon, \lambda')\)-first-order stationary \((\varepsilon, \lambda')\)-FOSP for (P) if \( \| \nabla \varphi_{\lambda}(x) \| \leq \varepsilon \).

**Taylor expansion and the surrogate problem.** Let us formally define the Taylor expansion of \( f(x, \cdot) \) to be used from now on. Recall that our approach relies on replacing \( f(x, \cdot) \) with its \( k \)-order Taylor approximation \( \hat{f}_k(x, \cdot) \) for a non-negative integer \( k \) and a fixed “center” point \( \hat{y} \in Y \):

\[
\hat{f}_k(x, y) = \sum_{j=0}^{k} \frac{1}{j!} \nabla^{j} f(x, \hat{y}) [(y - \hat{y})^{j}] . \tag{TE}
\]

Here \( T[y^j] \) is the diagonal evaluation of tensor \( T : E_{\bar{\lambda}}^{\otimes j} \rightarrow \mathbb{R} \) on \( y \in E_{\lambda} \)—that is, \( T[y^j] := T[y, \ldots, y] \).

In particular, in the most practically important cases \( k \in \{0, 1, 2\} \), (TE) reduces to \( \hat{f}_0(x, y) = f(x, \hat{y}), \)

\[
\hat{f}_1(x, y) = f(x, \hat{y}) + \nabla_y f(x, \hat{y}) \cdot (y - \hat{y}), \quad \text{and}
\]

\[
\hat{f}_2(x, y) = \hat{f}_1(x, y) + \frac{1}{2} (y - \hat{y}, \nabla^2_y \hat{f}(x, \hat{y})(y - \hat{y})). \tag{10}
\]

Note that for \( k \leq 2 \) we can efficiently maximize \( \hat{f}_k(x, \cdot) \); indeed, \( \hat{f}_0(x, \cdot) = f(x, \hat{y}) \) is constant, \( \hat{f}_1(x, \cdot) \) is affine, and \( \hat{f}_2(x, \cdot) \) for \( k = 2 \) is a (nonconcave) quadratic that can be globally maximized using the gradient oracle \( \nabla_y \hat{f}_2(x, \cdot) \) when \( Y \) is a ball [92]. From now on, we focus on the surrogate problem

\[
\min_{x \in X} \max_{y \in Y} \hat{f}_k(x, y) \tag{P_k}
\]

and let \( \hat{\varphi}(x) := \max_{y \in Y} \hat{f}_k(x, y) \) be the corresponding primal function, with \( k \) omitted for brevity.

### 3 Upper bounds on the admissible diameter

We are now in the position to rigorously formulate the question we first posed in the introduction:

**For** \( k \geq 0 \), **what** \( D \) **allows to guarantee the existence of** \( \tilde{\lambda} \leq \lambda \) **and** \( c > 0 \) **such that any** \((c \varepsilon, 2\tilde{\lambda})\)-FOSP **for** (P) **for** (P) **regardless of the choice of** \( \hat{y} \), **is** \((\varepsilon, 2\tilde{\lambda})\)-FOSP **for** (P)?
In this section, our goal is to answer this question, and such an answer will be given in Theorem 3.1. But before, let us clarify a subtle point about it: namely, note that we should expect $c < 1$ and $\bar{\lambda} > \lambda$. Indeed, replacing $f$ with its Taylor approximation is likely to cause some deterioration of accuracy as measured by the gradient norm of the Moreau envelope, so we cannot expect, say, $(2\varepsilon, 2\bar{\lambda})$-FOSP for $(P_k)$ to also be an $(\varepsilon, 2\lambda)$-FOSP for $(P)$. Similarly, considering $\bar{\lambda} = \lambda$ would imply that the surrogate primal function $\varphi(\cdot)$ is $\lambda$-weakly convex (cf. Definition 1), but in fact such a guarantee is not available. Fortunately, it is not hard to prove (cf. Lemma 3.3 below) that under the regularity assumptions granted in Section 2 and a weaker bound on $D$ than the one imposed in Theorem 3.1, the function $\varphi(\cdot)$ is $\bar{\lambda}$-weakly convex with $\bar{\lambda} = (1 + o(\varepsilon))\lambda$.

To prepare the ground for proving Theorem 3.1, we first obtain the bounds for approximating $f$ with $\hat{f}_k$ uniformly over $(x, y) \in X \times Y$, in terms of the function value, the $x$-gradient, and $x$-Hessian.

**Lemma 3.1.** Grant (A2) with some $k \geq 0$ and possibly $\sigma_{k} = \infty$. Then for any $x \in X$, $y \in Y$ one has

$$|f(x, y) - \hat{f}_k(x, y)| \leq \frac{\sigma_k D^{k+1}}{(k+1)!}.$$

**Lemma 3.2.** Let $k \geq 0$ be given. Grant (A1) if $k = 0$ (possibly with $\lambda = \infty$) and grant (A2) (possibly with $\rho_k = \infty$, but with $\nabla_y^k f(\cdot, y)$ absolutely continuous $\forall y \in Y$). Then for any $x \in X$, $y \in Y$ one has

$$\|\nabla_x f(x, y) - \nabla_x \hat{f}_k(x, y)\| \leq \begin{cases} \min\{\mu D, 2\sigma_0\} & \text{for } k = 0, \\ \frac{2\sigma_k D^k}{k!} & \text{for } k \geq 1. \end{cases}$$

**Lemma 3.3.** Given $k \geq 0$, grant (A1) (possibly with $\mu = \infty$). If $k \geq 1$ grant (A3), Assumption 3, and continuity of $\nabla_y^k f(x, \cdot)$ for all $x \in X$. Then $\nabla_x \hat{f}_k(\cdot, y)$ is $\bar{\lambda}_k$-Lipschitz in $x$ for any $y \in Y$ with

$$\bar{\lambda}_k := \lambda + \frac{2\tau_k D^k}{k!} \cdot 1\{k \geq 1\}.$$ (11)

Note that an immediate corollary of (11) is that $\varphi$ is $\bar{\lambda}_k$-weakly convex. Lemmas 3.1 to 3.3 are proved by integrating the remainder term of the Taylor expansion; however, in the case of Lemma 3.3 some technicalities arise; while they could be easily resolved by imposing an extra order of regularity (namely by requiring that $\nabla_x^{k+2} f$ exists everywhere), we manage to avoid this condition via a careful application of Assumption 3; see Appendix A.

Next we present the first of our main results: a general upper bound on the admissible diameter of $Y$—i.e., the one allowing to replace $(P)$ with $(P_k)$ when searching for FOSPs.

**Theorem 3.1.** Given $k \geq 0$, grant Assumptions 1 to 3, and let $x^*$ be $(\frac{1}{6} \varepsilon, 2\bar{\lambda}_k)$-FOSP for $(P_k)$ with $\bar{\lambda}_k$ defined in (11). Then $x^*$ is $(\varepsilon, 2\lambda_k)$-FOSP for $(P)$ as long as

$$\min\left\{ \mu D, 2\sigma_0, \sqrt{\frac{\lambda \rho_0 D}{50}} \right\} \leq \frac{\varepsilon}{24} \text{ when } k = 0, \quad (12)$$

$$\min\left\{ \mu D + \frac{2\sigma_k D^k}{k!}, \sqrt{\frac{\lambda_k \rho_k D^{k+1}}{50 \cdot (k + 1)!}} \right\} \leq \frac{\varepsilon}{24} \text{ when } k \geq 1. \quad (13)$$

We shall present the proof of Theorem 3.1 in Section 3.2 and investigate its tightness in Section 4. But before doing all this, let us discuss the implications of this result.
3.1 Discussion of Theorem 3.1

Zeroth-order approximation. Condition (12) can be satisfied even without Assumption 2, i.e., when \( f \) is not Lipschitz (recall that Assumption 1 already implies (A3) with \( \tau_0 = \lambda \)). Indeed, in this case (12) still gives a non-trivial condition \( 4\mu \Gamma \leq \varepsilon \). In the “\( x \)-FOSP effectively unconstrained” scenario where \( f(\cdot, y) \) has a stationary point in the interior of \( X \) for any \( y \in Y \), we have the bounds

\[
\sigma_0 \leq \min \{ \mu \Gamma, \lambda \operatorname{diam}(X) \}. \tag{14}
\]

Typically, there is no reason to believe the set \( X \) to be small or even bounded (in contrast to \( Y \)), so the second bound above is usually non-restrictive. However, the first bound is not vacuous; in particular, in the scenario in question it shows that (12) may indeed benefit from \( \sigma_0 \) replacing \( \mu \Gamma \).

Another interesting scenario is the “\( y \)-FOSP effectively unconstrained” one, where \( f(x, \cdot) \) has a stationary point in the interior of \( Y \) for any \( x \in X \). Assuming that \( \nabla_y f \) is \( \rho_1 \)-Lipschitz (cf. (A2)) we then have a variational bound \( \rho_0 \leq 2\rho_1 \Gamma \), so (12) gets weaker than (13) with \( k = 1 \), which reads

\[
\min \{ \mu, \sqrt{1 + \lambda \Gamma \rho_1} \} \Gamma \leq \varepsilon. \tag{15}
\]

Note, that this scenario is not too “pathological”: the interior stationary point does not have to be a maximizer of \( f(x, \cdot) \). In particular, nonconcave polynomials, while maximized on the boundary of a compact domain, tend to have interior saddle points (e.g., \( f = y_1^2 - y_2^2 \) on \( Y = \{ y \in \mathbb{R}^2 : \| y \| < 1 \} \)).

Leading-order terms and bilinearly-coupled objectives. In the case \( k \geq 1 \), condition (13) improves over the baseline \( \mu \Gamma \leq \varepsilon \) via the square-root term arising due to Assumption 2. In fact, omitting higher-order in \( \Gamma \) additive terms we reduce (13) to the independent of \( \sigma_0 \) and \( \tau_0 \) condition

\[
\min \{ \mu \Gamma, \sqrt{\lambda \rho_0 \Gamma^{k+1}} \} \leq k \varepsilon, \tag{16}
\]

where we hide a \( k \)-dependent constant factor in \( \lesssim_k \) for brevity. When solved for \( \Gamma \) this amounts to

\[
\Gamma \lesssim_k \max \left\{ \frac{\varepsilon}{\mu}, \left( \frac{\varepsilon^2}{\lambda \rho_k} \right)^{1/k} \right\}. \tag{17}
\]

This approximation of (13) becomes exact for BC objectives (cf. (BC)), since in this case for \( k \geq 1 \) one has \( \tau_k = 0 \) and \( \sigma_k = \mu \mathbb{E}\{ k = 1 \} \), so higher-order in \( \Gamma \) additive terms in (13) simply vanish. In Section 4 we demonstrate that the approximation in (16)–(17) is also tight for BC problems, by exhibiting the worst-case examples on which the bound (16) is attained. Moreover, approximation (16)–(17) applies beyond the class (BC), in the (typical) scenario where the smoothness parameters in Assumptions 1 and 2 are fixed, as long as \( \varepsilon \) is below a certain threshold. Indeed, in this case \( \Gamma \) must shrink with \( \varepsilon \) in order to guarantee (13), and for \( \varepsilon \) small enough the higher-order (in \( \Gamma \)) additive terms in (13) are dominated by the lower-order ones. More precisely, in Appendix A we show that it suffices to have

\[
\varepsilon \lesssim_k \begin{cases} 
\left( \frac{\lambda \rho_1}{\tau_k^2} \right)^{1/2} & \text{when } k = 1, \\
\min \left\{ \frac{\mu^k}{\sigma_k}, \left( \frac{\lambda \rho_k^k}{\tau_k^{k+1}} \right)^{1/k} \right\} & \text{when } k > 1.
\end{cases}
\tag{18}
\]
**Coupling-independent behavior for small** $\varepsilon$. Observe that condition (17) reads $D = O(\varepsilon)$ in the case $k = 1$, and simplifies to $D = O(\varepsilon^{\frac{2}{1}})$ when $k > 1$ and for small enough $\varepsilon$: namely, whenever

$$\varepsilon \leq k \left( \frac{\mu^k + 1}{\lambda \rho_k} \right)^{\frac{1}{k-1}}.$$  

(19)

Thus, increasing the approximation order above $k = 1$ allows to gain in terms of the range of $D$ for which Theorem 3.1 applies and (P) can be replaced with (P_k) when searching for FOSPs. Moreover, in this “high-accuracy” regime the critical $D$ becomes *coupling-independent* for $k > 1$, defined solely by the “homogeneous” parameters $\lambda, \rho_k$ and the target $\varepsilon$; meanwhile, $\mu$ defines the moment of transition to this regime from the initial $D = O(\varepsilon)$ one, as $\varepsilon$ is driven below the threshold in (19).

For $k = 1$ there is no such “elbow effect.” Here the critical diameter is $D = O(\varepsilon)$ over the *whole range of* $\varepsilon$, with $\mu$ appearing in the hidden constant factor $1/\min\{\mu, \sqrt{\lambda \rho_1}\}$. Thus, here we benefit from low interaction levels ($\mu \leq \sqrt{\lambda \rho_1}$) while not suffering from higher ones ($\mu \geq \sqrt{\lambda \rho_1}$), regardless of $\varepsilon > 0$. Of course, beyond the BC class interaction does manifest in higher order, via $\tau_1$ in (18).

For $k = 0$ the “elbow” is “in reverse”: we start with $D = O(\varepsilon^{1/2})$ for large $\varepsilon$, and switch to $D = O(\varepsilon)$ critical diameter as $\varepsilon$ passes the threshold $\lambda \rho_0/\mu$ which corresponds to (19) with $k = 0$. In the scenario where Assumption 2 holds simultaneously for $k \in \{0, 1\}$, and we can choose between approximations with these orders, the gain for $k = 1$ is marginal, only in the constant factor: namely, $1/\min\{\mu, \sqrt{\lambda \rho_1}\}$ instead of $1/\mu$—and this effect only manifests on high interaction levels; $\mu^2 \geq \lambda \rho_1$. In fact, even this marginal comparative disadvantage of zeroth-order approximation disappears in the “$y$ unconstrained as per FOSP” scenario, where $f(x, \cdot)$ has a stationary point inside $Y$ for any $x \in X$: indeed, in this case (12) automatically follows from (15) due to the variational bound $\rho_0 \leq 2\rho_1 D$.

### 3.2 Proof of Theorem 3.1

The result follows by combining Propositions 3.1 and 3.2 which we formulate and prove next. These propositions correspond to the two choices for the minimum in (13), and we prove each of them under minimal assumptions; the full Assumptions 1 and 2 are required to have both results simultaneously.

**Proposition 3.1.** For $k \geq 0$ and $\tilde{\lambda}_k$ given by (11), under the premise of Lemmas 3.1–3.3 one has

$$\| \nabla \check{\varphi}_{2\tilde{\lambda}_k}(x) - \nabla \varphi_{2\tilde{\lambda}_k}(x) \| \leq \sqrt{\frac{8\tilde{\lambda}_k \rho_k D^{k+1}}{(k + 1)!}}, \quad \forall x \in X.$$  

(19)

**Proof.** Clearly, $\varphi(\cdot)$ is $\lambda$-weakly convex (cf. Assumption 1), hence also $\tilde{\lambda}_k$-weakly convex. Moreover, by Lemma 3.3 the function $\check{\varphi}(\cdot) = \max_{y \in Y} \hat{f}_k(\cdot, y)$ is also $\tilde{\lambda}_k$-weakly convex. Whence by (9) we have

$$\nabla \check{\varphi}_{2\tilde{\lambda}_k}(x) = 2\tilde{\lambda}_k (x - x^+), \quad \nabla \varphi_{2\tilde{\lambda}_k}(x) = 2\tilde{\lambda}_k (x - \hat{x}^+),$$

with $x^+, \hat{x}^+$ being the associated proximal-point operators:

$$x^+ = \arg\min_{x' \in X} \{ \varphi(x') + \tilde{\lambda}_k \| x' - x \|^2 \}, \quad \hat{x}^+ = \arg\min_{x' \in X} \{ \check{\varphi}(x') + \tilde{\lambda}_k \| x' - x \|^2 \}.$$
As a result, $\|\nabla \varphi_{2\lambda_k}(x) - \nabla \hat{\varphi}_{2\lambda_k}(x)\| = 2\lambda_k \|\hat{x}^+ - x^+\|$, and we can focus on bounding $\|\hat{x}^+ - x^+\|$. To this end, since the function $\varphi(\cdot) + \lambda_k \cdot -x\|^2$ is $\lambda_k$-strongly convex and minimized at $x^+$, we have that

$$\frac{1}{2} \lambda_k \|\hat{x}^+ - x^+\|^2 \leq \varphi(\hat{x}^+) + \lambda_k \|\hat{x}^+ - x\|^2 - \varphi(x^+) - \lambda_k \|\hat{x}^+ - x\|^2. \quad (20)$$

On the other hand, since the function $\hat{\varphi}(\cdot) + \lambda\cdot -x\|$ is $\lambda$-strongly convex and minimized at $\hat{x}^+$,

$$\frac{1}{2} \lambda_k \|\hat{x}^+ - x^+\|^2 \leq \hat{\varphi}(x^+) + \lambda_k \|\hat{x}^+ - x\|^2 - \hat{\varphi}(\hat{x}^+) - \lambda_k \|\hat{x}^+ - x\|^2. \quad (21)$$

Summing the two inequalities we get

$$\lambda_k \|\hat{x}^+ - x^+\|^2 \leq \varphi(x^+) - \varphi(x^+) + \varphi(\hat{x}^+) - \hat{\varphi}(\hat{x}^+) \leq 2 \sup_{x \in X} |\varphi(x) - \varphi(x)|.$$

Finally, let $x \in X$ be arbitrary and $\hat{y}^* = \hat{y}^*(x)$ be such that $\varphi(x) = \hat{f}(x, \hat{y}^*)$. Then Lemma 3.1 gives

$$\varphi(x) - \varphi(x) \leq \hat{f}_k(x, \hat{y}^*) - f(x, \hat{y}^*) \leq \frac{\rho_k D^{k+1}}{(k+1)!}, \quad \forall x \in X.$$

The same estimate holds for $\varphi(x) - \varphi(x)$ which is bounded from above by $f(x, y^*) - \hat{f}_k(x, y^*)$ with $y^* = y^*(x)$ such that $\varphi(x) = f(x, y^*)$, and thus for $\sup_{x \in X} |\varphi(x) - \varphi(x)|$. The result follows.  

Proposition 3.2. Let $k \geq 0, \lambda_k$ be as in (11). Grant Assumption 1 and the assumptions of Lemmas 3.2 and 3.3. Then for any $x^* \in X$ such that $\|\nabla \hat{\varphi}_{2\lambda_k}(x^*)\| \leq \varepsilon$, one has

$$\|\nabla \varphi_{2\lambda_k}(x^*) - \nabla \hat{\varphi}_{2\lambda_k}(x^*)\| \leq \begin{cases} 
4 \left( \mu D + \frac{2\sigma_k D^k}{k!} + \varepsilon \right) & \text{when } k \geq 1, \\
4 \left( \min \{\mu D, 2\sigma_0\} + \varepsilon \right) & \text{when } k = 0.
\end{cases}$$

Proof. First observe that $\varphi(\cdot)$ and $\hat{\varphi}(\cdot)$ are $\lambda_k$-weakly convex by Lemma 3.3. Hence, $\varphi(\cdot) + \lambda_k \cdot -x^\cdot$ and $\hat{\varphi}(\cdot) + \lambda_k \cdot -x^\cdot$ are $\lambda_k$-strongly convex, and their corresponding minimizers $x^+, \hat{x}^+$ satisfy (cf. (20))

$$\frac{1}{2} \lambda_k \|\hat{x}^+ - x^+\|^2 \leq \lambda_k \|\hat{x}^+ - x^\cdot\|^2 + \varphi(\hat{x}^+) - \varphi(x^+) - \lambda_k \|\hat{x}^+ - x^\cdot\|^2 \leq 4 \lambda_k \|\hat{x}^+ - x^\cdot\|^2 + \varphi(\hat{x}^+) - \varphi(x^+) - \frac{3}{4} \lambda_k \|\hat{x}^+ - x^\cdot\|^2. \quad (21)$$

Here in the final step we used the inequality

$$\|\hat{x}^+ - x^\cdot\|^2 \leq \frac{4}{3} \|\hat{x}^+ - x^\cdot\|^2 + 4 \|\hat{x}^+ - x^\cdot\|^2$$

which can be easily deduced from the triangle inequality. Furthermore, by Proposition D.1 we have

$$\nabla \varphi_{2\lambda_k}(x^*) = 2\lambda_k (x^* - x^\cdot), \quad \nabla \hat{\varphi}_{2\lambda_k}(x^*) = 2\lambda_k (x^* - \hat{x}^\cdot).$$

Hence $\|\nabla \hat{\varphi}_{2\lambda_k}(x^*) - \nabla \varphi_{2\lambda_k}(x^*)\| = 2\lambda_k \|\hat{x}^+ - x^\cdot\|$ so we can focus on bounding $\|\hat{x}^+ - x^\cdot\|$ using (21). Now observe that $x^+$, being an $(\varepsilon, 2\lambda_k)$-FOSP for $(P_k)$ with a $\lambda_k$-weakly convex primal function $\hat{\varphi}(\cdot)$, satisfies the premise of Proposition D.1, so that

$$2\lambda_k \|\hat{x}^+ - x^\cdot\| \leq \varepsilon, \quad \min_{\xi \in \partial \hat{\varphi}(\hat{x}^\cdot)} S_X(\hat{x}^\cdot, \xi, 2\lambda_k) \leq \varepsilon, \quad (22)$$

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cf. (90), with functional $S_X(x, \xi, \lambda')$ defined by

$$S_X^2(x, \xi, \lambda') := 2\lambda' \max_{u \in X} \left\{ -\langle \xi, u - x \rangle - \frac{\lambda'}{2} \|u - x\|^2 \right\}$$

for arbitrary $x \in X$, $\xi \in E_X$, and $\lambda' > 0$. By (21) and the first bound in (22) we get

$$\frac{1}{2}(\lambda_k \|\hat{x}^+ - x^+\|^2) \leq \varepsilon^2 + \lambda_k \left( \varphi(\hat{x}^+) - \varphi(x^+) - \frac{\lambda_k}{2} \|\hat{x}^+ - x^+\|^2 \right).$$

Meanwhile, convexity of $\varphi(\cdot) + \frac{\lambda_k}{2} \cdot \cdot \cdot \|^2$ implies that $\varphi(\hat{x}^+) - \varphi(x^+) = \lambda_k \|\hat{x}^+ - x^+\|^2$ for any $\xi^+ \in \partial \varphi(\hat{x}^+)$. Using this fact and letting $\hat{\xi}_X \in \text{Argmin}_{\xi \in \partial \varphi(\hat{x}^+)} S_X(\hat{x}^+, \xi, 2\lambda_k)$, cf. (22), we get

$$\lambda_k \|\hat{x}^+ - x^+\|^2 \leq 2\varepsilon^2 + 2\lambda_k \left( -\langle \xi^+, x^+ - \hat{x}^+ \rangle - \frac{\lambda_k}{4} \|\hat{x}^+ - x^+\|^2 \right)$$

$$\leq 2\varepsilon^2 + 2\lambda_k \left( -\langle \hat{\xi}_X, x^+ - \hat{x}^+ \rangle - \frac{\lambda_k}{4} \|\hat{x}^+ - x^+\|^2 + \|\hat{x}^+ - x^+\| \cdot \|\hat{\xi}_X - \xi^+\| \right)$$

$$\leq 2\varepsilon^2 + 2S_X(\hat{x}^+, \hat{\xi}_X, \frac{\lambda_k}{2}) + 2\lambda_k \|\hat{x}^+ - x^+\| \cdot \|\hat{\xi}_X - \xi^+\|. \tag{23}$$

We furthermore have $S_X(\hat{x}^+, \hat{\xi}_X, \lambda_k) \leq \varepsilon$ due to (22) and the well-known fact that $S_X(x, \xi, \lambda)$ is non-decreasing in $\hat{\lambda} > 0$. (This monotonicity property follows from the proximal Polyak-Lojasiewicz inequality—see, e.g., [93, Lem. 1].) Meanwhile, by a version of Danskin’s theorem (cf. Lemma D.1 in the appendix), $\hat{\xi}_X$ belongs to the closed convex hull of the set of active $x$-gradients of $\hat{f}_k$ at $\hat{x}^+$:

$$\hat{\xi}_X \in \text{conv} \left( \left\{ \frac{\nabla_x \hat{f}_k(\hat{x}^+, y), \ y \in \text{Argmax}_{y \in X} \hat{f}_k(\hat{x}^+, y) \right\} \right).$$

Similarly, we can choose $\xi^* = \nabla_x f(\hat{x}^+, y^*)$ for $y^* \in \text{Argmax}_{y \in X} f(\hat{x}^+, y)$, whence by convexity of $\| \cdot \|$,

$$\|\hat{\xi}_X - \xi^*\| \leq \max_{y \in X} \|\nabla_x \hat{f}_k(\hat{x}^+, y) - \nabla_x f(\hat{x}^+, y^*)\|.$$  

Thus, in the case $k = 0$ we have $\|\hat{\xi}_X - \xi^*\| \leq \min \{ \mu D, 2\sigma_0 \}$ by Lemma 3.2. On the other hand, in the case $k \geq 1$ we take arbitrary $\bar{y} \in \text{Argmax}_{y \in X} \nabla \hat{f}_k(\hat{x}^+, y) - \nabla_x f(\hat{x}^+, y^*)$ and use Lemma 3.2 to get

$$\|\hat{\xi}_X - \xi^*\| \leq \frac{\mu D}{k} + \frac{2\sigma_k D^k}{k!}.$$  

Finally, plugging the obtained estimates for $S_X(\hat{x}^+, \hat{\xi}_X, \lambda_k)$ and $\|\hat{\xi}_X - \xi^*\|$ into (23) we have that

$$(\lambda_k \|\hat{x}^+ - x^+\|^2) \leq 4\varepsilon^2 + 2\lambda_k \|\hat{x}^+ - x^+\| \left( \min \{ \mu D, 2\sigma_0 \} \ 1 \{ k = 0 \} + \left( \mu D + \frac{2\sigma_k D^k}{k!} \right) \ 1 \{ k \geq 1 \} \right).$$

Solving this inequality for $\lambda_k \|\hat{x}^+ - x^+\| = \frac{1}{2} \langle \nabla \varphi_{2\lambda_k}(x^*), \nabla \varphi_{2\lambda_k}(x^*) \rangle$ we conclude the proof. \hfill \Box

4 Nearly matching lower bounds on the admissible diameter

Our next goal is to prove that conditions (12)–(13) in Theorem 3.1 are tight in leading-order terms. Namely, for any $k \in \mathbb{N} \cup \{0\}$ we exhibit instances of (P) such that the corresponding approximate problem (P) has an exact FOSP which is not $(\varepsilon, 2\lambda)$-FOSP in (P) for any accuracy $\varepsilon$ in the range

$$\varepsilon \leq \frac{1}{k+1} \min \left\{ \mu D, \sqrt{\frac{\lambda_p D^{k+1}}{k!}} \right\}. \tag{24}$$
(For simplicity, in this section we let $D = \text{diam}(Y)$; this is anyway the case in all instances to be exhibited.) The objectives in these instances satisfy Assumption 1, Assumption 2 with appropriate $k$, and Assumption 3. Moreover, for $k \geq 1$ we use bilinearly-coupled objectives (cf. (BC)), so Assumption 2 is satisfied with $\tau_k = 0$ (hence, $\tilde{\lambda}_k$ in (13) simplifies to $\lambda$, cf. (11)) and $\sigma_k = \mu 1_{\{k = 0\}}$ (hence, the additive to $\mu D$ term in (13) disappears). Thus, for $k \geq 1$ our lower bound (24) is tight over the BC subclass up to a $O(1/k)$ factor; for $k = 0$ it misses the $\sigma_0$ term in the minimum, typically anyway large (cf. (14) and the accompanying discussion). In this sense, the condition Theorem 3.1 turns out to be nearly tight.

Let us now specify our problem instances. Consider a family of functions on $\mathbb{R} \times \mathbb{R}$ in the form

$$F_{k,s,\lambda,\mu,\rho}(x,y) := -\frac{\lambda x^2}{2} + \mu xy + s\rho|y|^{k+1}/(k+1)! \quad \text{for} \quad k \in \mathbb{N} \cup \{0\}, \ s \in \{\pm 1, 0\}, \ \lambda > 0, \ \mu > 0, \ \rho > 0. \quad (25)$$

Note that $F_{k,s,\lambda,\mu,\rho}$ is BC, concave in $x$, and convex or concave in $y$ depending on $s$. We also consider

$$S_{\lambda,\rho,D}(x,y) := -\frac{\lambda x^2}{4} + \frac{\lambda y}{2} \left( \tanh \left( \sqrt{\frac{\lambda}{\rho D}} x \right) - 1 \right) \quad \text{for} \quad \lambda > 0, \ \rho > 0, \ D \geq 0. \quad (26)$$

Each of our hard instances of the form (P) is specified by choosing one of these functions as the objective, $X \subseteq \mathbb{R}$, and $Y = [a, a + D]$ with some shift $a \in \mathbb{R}$, while allowing for varying $\lambda, \mu, \rho$, and $D$. In the next lemma (proved in Appendix A) we establish the smoothness properties of these functions.

**Lemma 4.1.** For any $k \in \mathbb{N} \cup \{0\}, \ s = \pm 1, \ \lambda > 0, \ \mu > 0, \ \rho > 0, \ \text{and} \ D \geq 0$, the following claims hold:

1. Function $F_{k,s,\lambda,\mu,\rho}$ satisfies Assumption 1 on $\mathbb{R} \times \mathbb{R}$ (and thus also on $\mathbb{R} \times [a, a + D]$ for any $a \in \mathbb{R}$). Moreover, function $S_{\lambda,\rho,D}$ satisfies Assumption 1 on $\mathbb{R} \times [0, D]$ provided that $\mu \geq \sqrt{2\lambda \rho / D}$.

2. Let $r = \frac{\mu D}{2 \lambda}$. Assumption 2 with $k = 0$, $\rho_0 = \rho$, and $\sigma_0 = \mu D$ is satisfied by function $F_{0,0,\lambda,\mu,0}$ on $[-r, r] \times [-\frac{1}{2} D, \frac{1}{2} D]$ if $\mu \leq \sqrt{2\lambda \rho / D}$, and by function $S_{\lambda,\rho,D}$ on $[-r, r] \times [0, D]$ if $\mu \geq \sqrt{2\lambda \rho / D}$.

3. When $k \geq 1$ and $a \in \mathbb{R}$, function $F_{k,s,\lambda,\mu,\rho}$ on $\mathbb{R} \times [a, a + D]$ satisfies Assumption 2 with $\rho_k = \rho, \ \sigma_k = \mu 1_{\{k = 1\}}$, and $\tau_k = 0$ — in other words, Assumption 2 restricted to the objective class (BC).

Due to significant differences in the statements and analyses, we separately consider the cases $k = 0$, $k = 1$, and $k \geq 2$. We begin with the case $k = 0$, where we use $F_{0,0,\lambda,\mu,\rho}$ or $S_{\lambda,\rho,D}$ depending on the level of coupling (cf. claims 1 and 2 of Lemma 4.1) and obtain the following result.

**Proposition 4.1.** For $\lambda, \mu, \rho, D > 0$, let $X = [-\frac{\mu D}{2\lambda}, \frac{\mu D}{2\lambda}], \hat{f} \equiv \hat{f}_0$, cf. (TE), with $f, Y, \hat{y}$ to be defined.

1. For $f = F_{0,0,\lambda,\mu,0}$, $Y = [-\frac{1}{2} D, \frac{1}{2} D]$ one can find $\hat{y} \in Y$ and $x^* \in X$ such that $\varphi'_{2\lambda}(x^*) = 0$ while $|\varphi_{2\lambda}'(x^*)| \geq \frac{\mu D}{2}$.

2. If $\mu \geq \sqrt{2\lambda \rho / D}$, then for $f = S_{\lambda,\rho,D}$, $Y = [0, D]$ there exist $\hat{y} \in Y$, $x^* \in X$ such that $\varphi_{2\lambda}'(x^*) = 0$, $|\varphi_{2\lambda}'(x^*)| \geq \frac{\sqrt{\lambda \rho D}}{3}$. 

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When combined with the first two claims of Lemma 4.1, Proposition 4.1 establishes (24) for \( k = 0 \). As a result, we verify tightness of Theorem 3.1 in the case of zeroth-order approximation. Note, however, that Lemma 4.1 is restricted to the regime \( \sigma_0 \approx \mu D \); hence, the term \( \sigma_0 \) in the left-hand side of (12)—which is beneficial when \( \sigma_0 \ll \mu D \)—is not captured in the result we have just obtained.

Next we address the case of first-order approximation. Here we consider instances of (P) with objective given by \( F_{1,-1,\lambda,\mu,\rho} \) or \( F_{1,1,\lambda,\bar{\mu},\bar{\rho}} \) for some \( \bar{\mu} \leq \mu \) and \( \bar{\rho} \leq \rho \), depending on the region of parameters (as controlled by the relative level of coupling \( \mu \) compared to the geometric mean \( \sqrt{\lambda \rho} \) of the “homogeneous” Lipschitz constants, cf. (A2) with \( k = 1 \)). Here we obtain the following result.

**Proposition 4.2.** For \( \lambda, \mu, \rho, D > 0 \), set \( Y = [-\frac{1}{2}D, \frac{1}{2}D] \), \( \hat{\mu} \equiv \hat{f}_1 \), cf. (TE), with \( f, \hat{y} \) to be defined.

1. If \( \mu \leq \sqrt{\lambda \rho}/2 \), then for \( f = F_{1,-1,\lambda,\mu,\rho} \), one can find \( \hat{y} \in Y, x^* \in \mathbb{R} \) such that \( \varphi'_{2\lambda}(x^*) = 0 \) while

\[
|\varphi'_{2\lambda}(x^*)| \geq \frac{\mu D}{3}.
\]

2. If \( \mu \geq \sqrt{\lambda \rho}/2 \), then for \( f = F_{1,1,\lambda,\bar{\mu},\bar{\rho}} \) with some \( \bar{\mu} \leq \mu \) and \( \bar{\rho} \leq \rho \), one can find \( \hat{y} \in Y, x^* \in \mathbb{R} \) such that \( \varphi'_{2\lambda}(x^*) = 0 \) while

\[
|\varphi'_{2\lambda}(x^*)| \geq \sqrt{\lambda \rho D^2} \frac{1}{8}.
\]

By combining Proposition 4.2 with claims 1 and 3 of Lemma 4.1, we establish (24) for \( k = 1 \), and thus verify tightness of Theorem 3.1 in the case of first-order approximation (cf. (13)), without imposing any restrictions on the problem parameters. More precisely, our lower bound \( \varepsilon \geq \min\{\mu D, \sqrt{\lambda \rho D^2}\} \) on the approximation accuracy matches the upper bound (13) for bilinearly-coupled objectives (BC), and replaces \( \lambda_1 = \lambda + 2\tau_2 D \) with \( \lambda \) in the general case; this is only a minor modification since \( D = O(\varepsilon) \) is anyway required in order for the guarantees in Theorem 3.1 and Proposition 4.2 to be applicable.

It remains to cover the case of approximation with \( k \geq 2 \). To this end, we focus on the instances of (P) with \( f = F_{1,-1,\lambda,\bar{\mu},\rho} \) or \( f = F_{1,1,\lambda,\bar{\mu},\rho} \) for some \( \bar{\mu} \leq \mu \) depending on the level of interaction \( \mu \).

**Proposition 4.3.** For \( \lambda, \mu, \rho, D > 0 \) and \( k > 2 \), set \( \hat{f} = \hat{f}_k \), cf. (TE), with \( f, Y, \hat{y} \) to be defined, and let

\[
\mu_{cr} := \sqrt[2k]{\frac{\lambda \rho D^{k-1}}{k!}}. \tag{27}
\]

1. If \( \mu \leq \mu_{cr} \), then for \( Y = [0, D] \) and \( f = F_{k,-1,\lambda,\bar{\mu},\rho} \) with some \( \bar{\mu} \leq \mu \) one can find \( \hat{y} \in Y, x^* \in \mathbb{R} \) such that \( \varphi'_{2\lambda}(x^*) = 0 \) while

\[
|\varphi'_{2\lambda}(x^*)| \geq \frac{\mu D}{2k}
\]

2. If \( \mu \geq \mu_{cr} \), then for \( Y = [-\frac{1}{2}D, \frac{1}{2}D] \) and \( f = F_{k,1,\lambda,\bar{\mu},\rho} \) with some \( \bar{\mu} \leq \mu \) one can find \( \hat{y} \in Y, x^* \in \mathbb{R} \) such that \( \varphi'_{2\lambda}(x^*) = 0 \) while

\[
|\varphi'_{2\lambda}(x^*)| \geq \frac{\mu_{cr} D}{2k}
\]

By combining Proposition 4.3 with claims 1 and 3 of Lemma 4.1, we establish (24) for \( k \geq 2 \). Thus, we verify that for BC objectives, the bound in Theorem 3.1 is tight, up to the two terms in (13) being divided by \( O(k) \) and \( O(\sqrt{k}) \) correspondingly; removing this gap is left for future work.
On the proofs of Propositions 4.1 to 4.3. While our choice of objective functions used in Propositions 4.1 and 4.3 is quite natural (except, perhaps, for (25) used in the case \( k = 0 \)), the actual proofs of these propositions, as given in Appendix B, require quite a lot of technical work, and are way less straightforward than the proof of Theorem 3.1. In all three cases, the analysis relies upon carefully choosing the center \( \hat{y} \) of Taylor expansion and exhibiting \( x^* \) which is stationary for \( \hat{\varphi}_{2\lambda} \), yet such that \( |\varphi'_{2\lambda}(x^*)| \) is large. The choices of \( \hat{y}, x^* \) depend on the problem parameters and approximation order; in particular, they are different in the two regimes of strong/weak coupling. The analysis in Proposition 4.3 is especially delicate, notably due to our ambitious goal of matching the bound of Theorem 3.1 up to a polynomial in \( k \) gap. (For example, the reader may verify that merely replacing \( Y = [0,D] \) with \([\frac{-1}{2}D,\frac{1}{2}D]\) in the case \( \mu < \mu_{cr} \) would result in the extra \( 2^k \) factor.)

We hypothesize that the extra \( O(1/k) \) factor in Proposition 4.3, as compared to (13), can be removed. However, this will require analyzing a different family of objectives, as our analysis of (25) seems to be tight. In any case, such an improvement is of little practical interest: approximation with \( k \gg 1 \) does not lead to efficient algorithms for searching FOSPs: as we shall see next, such schemes rely on solving the nested maximization problem, which becomes a daunting task for \( k > 2 \).

5. Efficient algorithms for the search of first-order stationary points

Our goal in this section is to implement step \( 2^o \) of the strategy outlined in the introduction. To this end, we exploit the guarantee in Theorem 3.1 by replacing the task of finding FOSPs in \( (P) \) with that of finding FOSPs in \( (P_k) \) with \( k \in \{0,1,2\} \), and propose efficient algorithms for solving the latter task. Overall, as \( k \) increases, the proposed algorithms require access to higher derivatives of \( f \) in \( y \) and their oracle complexity estimates also deteriorate (especially when transitioning from \( k = 1 \) to \( k = 2 \)). On the other hand, increasing \( k \) allows us to handle larger diameter of \( Y \) for the same \( \varepsilon \).

Note that, despite the fact that our “approximation theory” in Sections 3 and 4 carefully handles the general case of arbitrary \( k \), we do not propose algorithms based on Taylor approximations of order \( k \geq 3 \). This is because already for cubic approximation (i.e., when \( k = 3 \)), solving the nested maximization problem becomes a daunting task: as it is shown in [94, Theorem 4], maximization of a general trilinear form is NP-hard even when it is available explicitly (in its tensor representation).

5.1 Algorithms based on the constant and linear approximations

In the case \( k = 0 \), we fix arbitrary \( \hat{y} \in Y \) and consider \( (P_k) \) with objective \( \hat{f}_0(x,y) = f(x,\hat{y}) \), that is

\[
\min_{x \in X} f(x, \hat{y}).
\]

This is a nonconvex minimization problem with a smooth objective \( \hat{\varphi}(x) = f(x, \hat{y}) \), so we can find a near-stationary point via projected gradient descent. This approach is summarized in Algorithm 1.

It produces a point \( x_T \) satisfying \( \|\nabla \hat{\varphi}(x_T)\| \leq \varepsilon \) in \( O(1/\varepsilon^2) \) iterations, with one projected gradient step (in \( x \)) per iteration. Using Lemma C.1 stated below, this implies \( |\nabla \varphi_{2\lambda}(x_T)| \leq \varepsilon \) in terms of the approximate Moreau envelope, which then results in the desired guarantee \( |\nabla \varphi_{2\lambda}(x_T)| \leq \varepsilon \) by applying Theorem 3.1 with \( k = 0 \). We shall rigorously state these results later on in Theorem 5.1.
Algorithm 1 FOSP search based on the constant approximation of $f(x, \cdot)$

**Input:** $x_0 \in X$, $\hat{y} \in Y$, $\gamma_x > 0$, $T \in \mathbb{N}$

1. $x^* = x_0$; $\varepsilon^* = +\infty$
2. for $t \in \{0, 1, ..., T - 1\}$ do
   3. $\tilde{x}_{t+1} = x_t - \gamma_x \nabla_x f(x_t, \hat{y})$
   4. $x_{t+1} = \Pi_X[\tilde{x}_{t+1}]$
   5. ▷ Maintain the best iterate so far
   6. $\varepsilon_t^2 = \frac{\|\nabla_x f(x_t, \hat{y})\|^2}{\gamma_x} - \frac{1}{\gamma_x} \|\tilde{x}_{t+1} - x_{t+1}\|^2$
   7. if $\varepsilon_t < \varepsilon^*$ then
      8. $x^* = x_t$; $\varepsilon^* = \varepsilon_t$
   9. end if
10. end for
**Output:** $x^*$

Another approach we advocate here is based upon focusing on the nonconvex-affine problem

$$
\min_{x \in X} \max_{y \in Y} \hat{f}_1(x, y),
$$

which corresponds to $(P_k)$ with $k = 1$ (the choice of $\hat{y} \in Y$ is again arbitrary). General nonconvex-concave problems can be solved by a simple gradient descent-ascent (GDA) scheme combined with quadratic regularization in $y$, with iteration complexity $O(\varepsilon^{-5})$ \cite{53}. More elaborate algorithmic schemes based on the proximal-point method have $O(\varepsilon^{-3})$ iteration complexity \cite{54, 95, 55, 57}. Here we propose a GDA-type scheme in the form of Algorithm 2. With it we manage to guarantee $O(\varepsilon^{-2})$ iteration complexity, by exploiting the special properties of $(P_1)$:

- $\hat{f}_1(x, \cdot)$ is affine, so $\hat{f}_1(x, \cdot) + \frac{\rho}{2} \|y\|^2$ is maximized via a single projected gradient ascent step.

- $(P_1)$ has to be solved in the regime $\varepsilon \gtrsim \mathcal{D} \min\{\mu, (\bar{\lambda}_1 \rho_1)^{1/2}\}$ where $\varepsilon$-FOSPs for $(P_1)$ translate to $\varepsilon$-FOSPs for $(P)$ via Theorem 3.1 (cf. (13) with $k = 1$). This allows to regularize with $\rho = \rho_1$ and results in $\varepsilon$-independent smoothness $O((\bar{\lambda}_1 + \mu^2)/\rho_1)$ of the corresponding primal function.

\footnote{This estimate follows from the complexity $O(\varepsilon_x^{-2}\varepsilon_y^{-3/2})$ of finding an $(\varepsilon_x, \varepsilon_y)$-approximate first-order Nash equilibrium by such method, see \cite[p. 3]{54}, combined with \cite[Proposition 5.5]{54} which verifies that the $x$-component of such a point is an $O(\varepsilon_x^2)$-FOSP as long as $\varepsilon_y = O(\varepsilon_x^k)$. Note that the well-known result \cite[Proposition 4.12]{96} commonly used for such a reduction in recent works, is erroneous—see the discussion immediately following \cite[Proposition 5.2]{54}.}
Algorithm 2 FOSP search based on the linear approximation of $f(x, \cdot)$

Input: $x_0 \in X$, $\tilde{y} \in Y$, $\gamma_x > 0$, $T \in \mathbb{N}$, $\text{Coupled } \in \{0, 1\}$, $\gamma_y > 0$

1: $x^* = x_0$; $y^* = y_t$; $\varepsilon^* = +\infty$
2: for $t \in \{0, 1, ..., T - 1\}$ do
3: \hspace{1em} if Coupled then \hspace{2em} $\triangleright \gamma_y \text{ needed in this case only}$
4: \hspace{2em} $y_t = \Pi_Y[\tilde{y} + \gamma_y \nabla_y f(x_t, \tilde{y})]$
5: \hspace{2em} $\overline{x}_{t+1} = x_t - \gamma_x [\nabla_x f(x_t, \tilde{y}) + \nabla^2_y f(x_t, \tilde{y})(y_t - \tilde{y})]$ \hspace{1em}
6: \hspace{2em} else
7: \hspace{2em} \hspace{1em} Choose $y_t \in \text{Argmax}_{y \in Y} \langle \nabla_y f(x_t, \tilde{y}), y \rangle$
8: \hspace{2em} $\overline{x}_{t+1} = x_t - \gamma_x \nabla_x f(x_t, y_t)$
9: \hspace{2em} end if
10: $x_{t+1} = \Pi_X[\overline{x}_{t+1}]$
11: \hspace{1em} $\triangleright$ Maintain the best iterate so far
12: $\varepsilon_t^2 = \frac{1}{\gamma_x} \|\overline{x}_{t+1} - x_t\|^2 - \frac{1}{\gamma_x} \|\overline{x}_{t+1} - x_{t+1}\|^2$
13: \hspace{1em} if $\varepsilon_t < \varepsilon^*$ then
14: \hspace{1em} $x^* = x_t$; $y^* = y_t$; $\varepsilon^* = \varepsilon_t$
15: \hspace{1em} end if
16: end for

Output: $x^*$

Algorithm 2 admits $O(\varepsilon^{-2})$ iteration complexity estimate as in the case of Algorithm 1. That said, compared to Algorithm 2, the new algorithm has two advantages: a slightly increased range of available accuracies in the strongly-coupled regime $\mu^2 \geq \lambda_2 \rho_1$; a smaller leading factor in the complexity estimate – depending on the primal gap, rather than the full duality gap. However, these improvements come at a price: computing $\nabla f_1(x, y)$ requires access to a partial Hessian-vector product oracle for $f$, namely

\[
(x, y) \mapsto \nabla^2_y f(x, \tilde{y})(y - \tilde{y}),
\]

(28)

since $\nabla f_1(x_t, y_t) = \nabla_x f(x_t, \tilde{y}) + \nabla^2_y f(x_t, \tilde{y})(y_t - \tilde{y})$. On the other hand, for weakly-coupled problems – i.e., when $\mu^2 \leq \lambda_2 \rho_1$ – Algorithm 2 uses a simplified approach: (i) the descent step is performed in the negative direction of $\nabla_x f(x_t, y_t)$ instead of $\nabla f_1(x_t, y_t)$; (ii) the gradient ascent step is replaced by the full maximization of the linear model $f_1(x_t, \cdot)$. These two properties allow us to avoid Hessian-vector product (28), and also to access $Y$ through the (weaker) linear maximization oracle.

Next we state convergence guarantees for Algorithms 1 and 2 (see Appendix C for the proofs).

**Theorem 5.1.** Grant Assumption 1. Running Algorithm 1 with $\gamma_x = \frac{1}{\lambda}$ and number of iterations

\[
T \geq \frac{300\lambda[\varphi(x_0) - \psi(\tilde{y})]}{\varepsilon^2},
\]

(29)

where $\psi(y) := \min_{x \in X} f(x, y)$ is the dual function of (P), guarantees $\|\nabla \hat{\varphi}_{2\lambda}(x^*)\| \leq \varepsilon/6$. Moreover, we have $\|\nabla \hat{\varphi}_{2\lambda}(x^*)\| \leq \varepsilon$ (i.e., in terms of initial problem (P)) as long as

\[
24\mu D \leq \varepsilon.
\]

(30)
Note that the factor $\varphi(x_0) - \psi(\hat{y})$ in (29) is the duality gap for the point $(x_0, \hat{y})$; by weak duality, it is lower-bounded by the sum of the dual gap $\max_{y \in Y} \psi(y) - \psi(\hat{y})$ and the primal gap
\[ \Delta := \varphi(x_0) - \min_{x \in X} \varphi(x). \] (31)

As we shall see next, Algorithm 2 admits a slightly different (and typically better) complexity estimate, in which the full duality gap is replaced with the primal gap, and $\lambda$ with $O(\bar{\lambda}_1 + \mu^2/\rho_1)$. In addition, we relax condition (30) by replacing $\mu$ with $O(\min\{\mu, (\bar{\lambda}_1 \rho_1)^{1/2}\})$.

**Theorem 5.2.** Grant Assumptions 1 to 3 for $k = 1$, let $\bar{\lambda}_1 = \lambda + 2\tau_1 D$ (cf. (11)), and assume that
\[ 200 \min\{\mu, (\bar{\lambda}_1 \rho_1)^{1/2}\} D \leq \varepsilon. \] (32)

Running Algorithm 2 with $\gamma_x = \frac{3\lambda_1 + \mu^2}{3\lambda_1 + \mu^2 \rho_1}$, $\text{Coupled} = 1\{ \mu \geq (\bar{\lambda}_1 \rho_1)^{1/2} \}$ and $\gamma_y = \frac{1}{\rho_1}$ if Coupled = 1, for
\[ T \geq \left(3 + \frac{\mu^2}{\lambda_1 \rho_1}\right) \left(\frac{700\bar{\lambda}_1 \Delta}{\varepsilon^2} + 1\right) \] (33)
iterations, with $\Delta$ being the initial primal gap (cf. (31)), results in $x^* \in X$ for which $\|\nabla \varphi_{2\lambda_1}(x^*)\| \leq \varepsilon$.

**Remark 1.** As a criterion for selecting the “best” iterate, in Algorithms 1 and 2 we use the quantity
\[ \varepsilon_t^2 = \frac{1}{\gamma_x} \left(\|\bar{x}_{t+1} - x_t\|^2 - \|\bar{x}_{t+1} - x_{t+1}\|^2\right), \]
where $\bar{x}_{t+1}$ is the result of the gradient descent step from $x_t$ prior to projection (i.e., $x_{t+1} = \Pi_X[\bar{x}_{t+1}]$). In fact, $\varepsilon_t = S_X(x_t, \frac{1}{\gamma_x}(x_t - \bar{x}_{t+1}), \frac{1}{\gamma_x})$, where $S_X$ is the functional used in the proof of Proposition 3.2. Using this criterion, instead of the gradient norm (which is a weaker criterion, cf. [97, Theorem 4.3]) allows to work with the Moreau envelope (cf., in particular, Lemma C.1 in Appendix C), and seems to be necessary already in the nonconvex-concave setup (see [54, Proposition 5.5]).

In what follows next, we propose another algorithm based on the quadratic approximation of $f$.

### 5.2 An algorithm based on the quadratic approximation

Finally, we propose a more sophisticated method in which we focus on the quadratic approximation
\[ \min_{x \in X} \max_{y \in Y} \hat{f}_2(x, y). \] (P2)

(Recall that $\hat{f}_2(x, \cdot)$ is the quadratic approximation of $f(x, \cdot)$ in $y$ with arbitrary choice of $\hat{y} \in Y$, cf. (10).) Compared to the previous ones, the approach we are about to present has an additional limitation: $Y$ must be a Euclidean ball in $\mathbb{R}^d$. For simplicity, we shall assume that $Y$ has diameter precisely $D$, and is origin-centered—i.e., $Y = B_d(D)$ with
\[ B_d(D) := \{ y \in \mathbb{R}^d : \| y \| \leq \frac{1}{2} D \}. \]

Note that centering $Y$ in the origin is not a limitation: Assumptions 1 and 2 are preserved under shifts of $y$. The construction of our method rests upon the following two observations.
We first recall, following [62], that an \((\varepsilon/6, 2\tilde{\lambda}_2)\)-FOSP in \((P_2)\) can be found by running \(O(\varepsilon^{-4})\) iterations of a (projected) subgradient scheme on the associated to \((P_2)\) primal function \(\tilde{\varphi}(x)\), which is \(\tilde{\lambda}_2\)-weakly convex (cf. Lemma 3.3). By Theorem 3.1, this also gives an \((\varepsilon, 2\tilde{\lambda}_2)\)-FOSP in \((P_2)\). Each iteration of the subgradient scheme amounts to alternating between a maximization step in \(y\), i.e., finding \(y^* = y^*(x) \in \text{Argmax}_{y \in Y} \hat{f}_2(x, y)\), and a projected gradient descent step on \(\hat{f}_2(\cdot, y^*)\). Moreover, the analysis in [62] (cf. also [61, Theorem 31]) shows that this complexity is preserved under objective value errors of up to \(O(\varepsilon^2/\tilde{\lambda}_2)\) in the maximization step.

Our second observation is that, despite the corresponding objective \(\hat{f}_2(x, \cdot)\) being nonconcave, the maximization steps can be efficiently performed by running a first-order algorithm on \(\hat{f}_2(x, \cdot)\). To this end, we make use of the recent result of [92], who showed that the Krylov subspace of dimension \(\tilde{O}(D\delta^{-1/2})\) contains a \(\delta\)-accurate maximizer of a nonconcave quadratic form on a Euclidean ball. Krylov-type schemes can usually be efficiently implemented via a Lanczos-type method (see [98]), with \(\tilde{O}(D\delta^{-1/2})\) matrix-vector products to find a \(\delta\)-accurate maximizer.

Below we present Algorithm 3 which adapts the general subgradient scheme [62] to the present situation, assuming access to an abstract maximization oracle \(\text{ApproxMax}(\hat{f}_2(x, \cdot), Y, \delta)\) returning a \(\delta\)-accurate maximizer of \(\hat{f}_2(x, \cdot)\) over \(Y\). Efficient implementation of this oracle, in the form of Algorithm 4 is discussed in Appendix E. Observe that Algorithm 3 can be run in a simplified (or “naive”) regime, in which the descent step is performed using \(\nabla_x f(\cdot, y_t)\) rather than \(\nabla_x \hat{f}_2(\cdot, y_t)\), so there is no need of higher-order oracles used otherwise (cf. line 7 of Algorithm 3).

**Algorithm 3** FOSP search based on the quadratic approximation of \(f(x, \cdot)\)

**Input:** \(x_0 \in X\), \(y \in Y\), \(\gamma_x > 0\), \(T \in \mathbb{N}\), \(\delta > 0\), Naive \(\in \{0, 1\}\)

1. for \(t \in \{0, 1, \ldots, T-1\}\) do
2. \(y_t = \text{ApproxMax}(\hat{f}_2(x_t, \cdot), Y, \delta)\) \(\triangleright \text{implemented in Algorithm 4 (cf. Appendix E)}\)
3. if Naive then
4. \(x_{t+1} = \Pi_X [x_t - \gamma_x \nabla_x f(x_t, y_t)]\)
5. else
6. \(x_{t+1} = \Pi_X [x_t - \gamma_x \nabla_x \hat{f}(x_t, y_t)]\),
7. where \(\nabla_x \hat{f}(x_t, y_t) = \nabla_x f(x_t, \hat{y}) + \nabla^2_{xx} f(x_t, \hat{y})(y_t - \hat{y}) + \nabla^3_{xxy} f(x_t, \hat{y})[\cdot, y_t - \hat{y}, y_t - \hat{y}]\)
8. end if
9. end for

**Output:** \(x_s\), where \(s \in \{0, \ldots, T-1\}\) is sampled uniformly at random.

We now present a convergence guarantee for Algorithm 3.

**Proposition 5.1.** Grant Assumptions 1 to 3 for \(k = 2\), let \(\tilde{\lambda}_2 = \lambda + \tau_2 D^2\) (cf. (11)), and assume that

\[
24 \min \left\{ \mu D + \sigma_2 D^2, \sqrt{\frac{\lambda_2 \rho_2 D^3}{300}} \right\} \leq \varepsilon. \tag{34}
\]

Furthermore, assume that \(f(\cdot, y)\) is \(\sigma_0\)-Lipschitz for any \(y \in Y\). Finally, for \(\delta > 0\) and any \(x \in X\), let \(\text{ApproxMax}(\nabla_y \hat{f}_2(x, \cdot), Y, \delta)\) output \(y_\delta = y_\delta(x) \in Y\) such that \(\hat{f}_2(x, y_\delta) \geq \max_{y \in Y} \hat{f}_2(x, y) - \delta\). Then:

1. Algorithm 3 run with Naive = 0 and the choice of parameters (for fixed \(p \in (0, 1)\))
   \[
   \gamma_x = \frac{1}{\sigma_0 + \sigma_2 D^2} \sqrt{\frac{\Delta + \rho_2 D^3}{\lambda_2 T}}, \quad \delta = \frac{4p}{10^4} \cdot \frac{\varepsilon^2}{\lambda_2}, \quad T \geq \frac{6 \cdot 10^6}{p^2} \cdot \frac{\tilde{\lambda}_2 (\Delta + \rho_2 D^3)(\sigma_0 + \sigma_2 D^2)^2}{\varepsilon^4}, \tag{35}
   \]

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for $\Delta$ defined in (31), with probability at least $1 - p$ outputs $x_s \in X$ such that $\| \nabla \varphi_{2\lambda_2}(x_s) \| \leq \varepsilon$.

2. Moreover, the output of Algorithm 3 run with Naive = 1 has the same property if $24\sigma_2 D^2 \leq \varepsilon \sqrt{p}$.

We prove Proposition 5.1 in Appendix C. The first claim is proved by following the footsteps of [61, Theorem 31] up to minor modifications: first, parameters $\sigma_0$ and $\Delta$ have to be adjusted for the use of $\hat{\varphi}$ instead of $\varphi$; second, the bound in expectation is replaced with a fixed-probability one. The argument proceeds by establishing $O(\varepsilon^{-4})$ complexity in terms of the surrogate $\| \hat{\nabla} \varphi_{2\lambda_2}(x_s) \|$, and then applying Theorem 3.1 under the high-probability event, which allows to control the resulting perturbation of $\| \hat{\varphi}_{2\lambda_2}(x_s) \|$. This perturbation turns out to be $O(p^{-1/2}\sigma_2 D^2)$, so requiring that $\sigma_2 D^2 \leq \varepsilon \sqrt{p}$ suffices for the “naive” approach to work. Note that under (34), this requirement is very weak: it is either satisfied right away if the minimum in (34) is attained on the first argument, or follows from (34) when $D$ is smaller than an $\varepsilon$-independent threshold—namely, when $\sigma_2 D \leq \lambda_2 \rho_2$. Moreover, in the latter case—which is of main interest, as otherwise there is no advantage in using Algorithm 3 over Algorithms 1 and 2 anyway—the number of iterations as per (35) becomes

$$
\frac{1}{p} O \left( \frac{\lambda_2 \Delta \sigma_0^2}{\varepsilon^4} + \frac{\lambda_2 \Delta + \sigma_0^2}{\varepsilon^2} + 1 \right).
$$

Implementation of the max-oracle. Next we show how to implement $\text{ApproxMax}(\hat{f}_2(x, \cdot), Y, \delta)$ in the case where $Y$ is a Euclidean ball. To this end, for $g \in \mathbb{R}^d$ and a symmetric $H \in \mathbb{R}^{d \times d}$, we let

$$
\Psi_{H,g}(y) = \frac{1}{2} y^T H y + g^T y
$$

be the corresponding quadratic form, and we aim at efficiently solving problems of the form

$$
\max_{y \in B_d(D)} \Psi_{H,g}(y)
$$

up to accuracy $\delta > 0$ in objective value given access to $g$ and the matrix-vector multiplication oracle $y \mapsto Hy$. In order to accomplish this goal, we shall exploit the following result from [92].

Proposition 5.2 ([92, Corollary 5.2]). Consider the joint Krylov subspace

$$
K_{2m}(H, \{g, \xi\}) := \text{span} \left( \{ H^j g, H^j \xi \}_{j = \{0, \ldots, m-1\}} \right)
$$

of dimension $\min\{2m, d\}$ with $\xi \sim \text{Uniform}(S^{d-1})$. For any $R \geq 0$ and $q \in (0, 1)$, w.p. $\geq 1 - q$ one has

$$
\max_{\|y\| \leq R} \Psi_{H,g}(y) - \max_{y \in K_{2m}(H, \{g, \xi\}) : \|y\| \leq R} \Psi_{H,g}(y) \leq \frac{4\|H\| R^2}{m^2} \left( 2 + \log^2 \left( \frac{2 \sqrt{d}}{q} \right) \right).
$$

Proposition 5.2 immediately implies that whenever

$$
m \geq D \sqrt{ \frac{\|H\|}{\delta} \left( 2 + \log^2 \left( \frac{2 \sqrt{d}}{q} \right) \right)}.
$$
we provide a formal algorithm (following the footsteps of \[5.1\]) with

\[\hat{g}(x) = \nabla_y f(x, \hat{y}) \quad \text{and} \quad \hat{H}(x) = \nabla^2_y f(x, \hat{y}),\]

we conclude that, granted Assumption 2 with \(k = 1\) (more precisely, finiteness of \(\rho_1\), cf. (A2)), any optimal solution to the problem

\[
\max_{y \in \mathbb{K}(\hat{H}(x), \hat{g}(x), \xi))} \Psi_{\hat{H}(x), \hat{g}(x)}(y) \quad \text{with} \quad \overline{\mathcal{m}} = \left( \min \left\{ D \sqrt{\frac{\rho_1}{\delta}} \left( 2 + \log^2 \left( \frac{2\sqrt{\delta}}{q} \right) \right), \frac{d}{2} \right\} \right),
\]

implies the query \(\text{ApproxMax}(f_2(\cdot, \cdot), B_d(D), \delta)\) with probability at least \(1 - q\). On the other hand, as discussed in \[92\], the computational burden of solving (40) to machine precision is dominated by \(O(\overline{\mathcal{m}})\) calls of the oracle \((x, y) \mapsto [\hat{g}(x), \hat{H}(x)y]\), inner products, and elementwise vector operations on \(E_y\) (typically \(y \mapsto \hat{H}(x)y\) is the most expensive of these operations).\(^7\)

Now, by recalling Proposition 5.1 and plugging in the value of \(\delta\) from (35), we arrive at the following result.

**Theorem 5.3.** Grant the premise of Proposition 5.1—that is, Assumptions 1 to 3 with \(k = 2\); \(\sigma_0\)-Lipschitzness of \(f(\cdot, y)\) for any \(y \in Y\), and condition (34) on \(D\). Moreover, assume that \(Y = B_d(D)\), and \(\nabla_y f(x, \cdot)\) is \(\rho_1\)-Lipschitz for all \(x\), i.e., \(\|\nabla_y f(x, y') - \nabla_y f(x, y)\| \leq \rho_1 \|y' - y\| \; \forall x \in X\) and \(y, y' \in Y\).

Choosing \(p \in (0, 1)\) and \(q = (0, 1 - p)\), run Algorithm 3 with \(\text{Naive} = 0\), parameters \(\gamma, \delta, T\) chosen according to (35), and the oracle \(x \mapsto \text{ApproxMax}(f_2(\cdot, \cdot), Y, \delta)\) implemented by running Algorithm 4 with \(g = \hat{g}(x), H = \hat{H}(x)\) (cf. (39)), \(R = \frac{1}{2} D\), and \(m = \left\lfloor \min\{M, d/2\} \right\rfloor\) with

\[
M = \frac{50 D}{\varepsilon} \left( 2 + \log^2 \left( \frac{2T\sqrt{\delta}}{q} \right) \right) \frac{\rho_1 \lambda_2}{p} \left( \frac{80}{\min\{(\lambda_2 \rho_2), 1/3\}} \right),
\]

Then the resulting point \(x_s \in X\) satisfies \(\|\nabla \varphi_2(x_s)\| \leq \varepsilon\) with probability at least \(1 - (p + q)\), and is constructed by performing \(O(T)\) calls of the oracle

\[(x, y) \mapsto \nabla_x f_2(x, y) \quad \left[ = \nabla_x f(x, \hat{y}) + \nabla^2_{xy} f(x, \hat{y})(y - \hat{y}) + \nabla^2_{\hat{y}y} f(x, \hat{y})[\cdot, y - \hat{y}, y - \hat{y}] \right],\]

and projections onto \(X\), and \(O(M T)\) calls of the oracle \((x, y) \mapsto (\nabla_y f(x, \hat{y}), \nabla^2_{\hat{y}y} f(x, \hat{y})(y - \hat{y}))\), inner products on \(E_y\), and elementwise vector operations on \(E_y\).

Moreover, if in addition assume that \(24 \rho_2 D^2 \leq \sqrt{\varepsilon}\) (cf. the second claim of Proposition 5.1), then running Algorithm 3 with \(\text{Naive} = 1\) produces \(x_s \in X\) with the same property while using the oracle \((x, y) \mapsto \nabla_x f(x, y)\) instead of (41).

Comparing this result with Theorems 5.1 and 5.2 we see that for Algorithm 3 the allowed range of \(D\) improves from \(O(\varepsilon)\) to \(O(\varepsilon^{2/3})\), but this happens at the price of a significantly deteriorated complexity—from \(O(\varepsilon^{-2})\) to \(O(\varepsilon^{-13/10})\). We leave open the questions of whether the latter complexity estimate can be improved, and whether one can smoothly interpolate between the two complexities.

\(^7\)Such an implementation is discussed in [92, Appendix A], but somewhat informally, and no pseudocode of an algorithm is given. For this reason, in Appendix E we provide a formal algorithm (following the footsteps of [92]) and analyze its complexity. Note that this can also be useful in the broader context of nonconvex quadratic optimization.
A Deferred proofs for Section 3

A.1 Proof of Lemma 3.1

Take arbitrary \( \hat{y}, y \in Y \) and \( x \in X \). Let \( y_t = (1-t)\hat{y} + ty \) for \( t \in [0,1] \), and define \( \psi_x: [0,1] \to \mathbb{R} \) by \( \psi_x(t) := f(x,y_t) \). Clearly, \( f(x,y) = \psi_x(1) \). Moreover, \( \psi_x(\cdot) \) has the first \( k \) derivatives in the form

\[
\psi_x^{(j)}(t) = \nabla^j_y f(x,y_t) [(y - \hat{y})]^j, \quad 0 \leq j \leq k;
\]

thus by (TE) we have

\[
\hat{f}_k(x,y) = \sum_{j=0}^{k} \frac{1}{j!} \psi_x^{(j)}(0).
\]

Now, it follows from (A2) with \( x' = x \) that \( \psi_x^{(k)} \) is absolutely continuous on \([0,1]\), hence its derivative exists almost everywhere on \([0,1]\) and is given by

\[
\psi_x^{(k+1)}(t) = \nabla^k_y f(x,y_t) [(y - \hat{y})]^{k+1}.
\]

Thus, expressing the Taylor expansion remainder \( \psi_x(1) - \sum_{j=0}^{k} \frac{1}{j!} \psi_x^{(j)}(0) \) in the integral form we get

\[
f(x,y) - \hat{f}_k(x,y) = \int_0^1 \frac{(1-t)^k}{k!} \psi_x^{(k+1)}(t) \, dt = \int_0^1 \frac{(1-t)^k}{k!} \nabla^k_y f(x,y_t) [(y - \hat{y})]^{k+1} \, dt. \quad (44)
\]

Whence we arrive at

\[
|f(x,y) - \hat{f}_k(x,y)| \leq \|y - \hat{y}\|^{k+1} \int_0^1 \frac{(1-t)^k}{k!} \|\nabla^k_y f(x,y_t)\| \, dt \leq \rho_k \mathcal{D}^{k+1} \int_0^1 \frac{(1-t)^k}{k!} \, dt = \frac{\rho_k \mathcal{D}^{k+1}}{(k+1)!}.
\]

The lemma is proved. \( \square \)

A.2 Proof of Lemma 3.2

For \( k = 0 \) the result is obvious: we have \( \nabla_x \hat{f}_0(x,y) = \nabla_x f(x,\hat{y}) \), so \( \| \nabla_x f(x,y) - \nabla_x \hat{f}(x,\hat{y}) \| \) can be bounded via Assumption 1 or via triangle inequality and Lipschitzness of \( f(\cdot, y) \). For \( k \geq 1 \), fix \( \hat{y} \) and arbitrary \( y \in Y \) and \( x \in X \), and let \( y_t = (1-t)\hat{y} + ty \) for \( t \in [0,1] \). As in the proof of Lemma 3.1 we define \( \psi_x: [0,1] \to \mathbb{R} \) as \( \psi_x(t) := f(x,y_t) \), and observe that (42)–(44) are still valid. (Indeed, imposing (A2) with \( \sigma_k = \infty \) suffices for \( \psi_x^{(k)} \) to be absolutely continuous on \([0,1]\), and hence for its derivative to exists almost everywhere on \([0,1]\) and to be given by (43).) As a result, we have that

\[
f(x,y) - \hat{f}_k(x,y) = \int_0^1 \frac{(1-t)^k}{k!} \psi_x^{(k+1)}(t) \, dt = -\frac{\psi_x^{(k)}(0)}{k!} + \int_0^1 \frac{(1-t)^k}{(k-1)!} \psi_x^{(k)}(t) \, dt
\]

\[
= \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} \left( \nabla^k_y f(x,y_t) - \nabla^k_y f(x,\hat{y}) \right) [(y - \hat{y})]^{k} \, dt, \quad (45)
\]

where we first used integration by parts and then (42). Taking the partial gradient in \( x \) we have

\[
\langle \nabla_x f(x,y) - \nabla_x \hat{f}(x,y), u \rangle = \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} \left( \nabla_{xy}^k f(x,\hat{y}) [(y - \hat{y})]^k u - \nabla_{xy}^{k+1} f(x,y_t) [(y - \hat{y})]^k u \right) \, dt,
\]

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where \( u \in E_X \) is arbitrary, we used the abridged notation \([(y - \hat{y})^k; u] := [(y - \hat{y}), \ldots, (y - \hat{y}); u]\) for tensor evaluation, and the right-hand side is well-defined by the premise of the lemma. Taking supremum over the unit ball in \( E_X \) and combining Jensen’s inequality with (A3), we arrive at

\[
\|\nabla_x f(x, y) - \nabla_x \hat{f}_k(x, y)\| \leq \|y - \hat{y}\|^k \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} \left( \|\nabla^{k+1}_{xy} f(x, \hat{y})\| + \|\nabla^{k+1}_{xy} f(x, y_t)\| \right) dt \leq \frac{2\tau_k D^k}{k!}.
\]

The lemma is proved. \( \square \)

### A.3 Proof of Lemma 3.3

For \( k = 0 \) the result is clear from \( \nabla_x \hat{f}_0(x, y) = \nabla_x f(x, \hat{y}) \). When \( k \geq 1 \), it suffices to prove that

\[
\|\nabla^2_{xx} \hat{f}_k(x, y) - \nabla^2_{xx} f(x, y)\| \leq \frac{2\tau_k D^k}{k!}
\]

for all \( y \in Y \) almost everywhere on \( X \). Indeed, (A1) with \( \mu = \infty \) is equivalent to \( \|\nabla^2_{xx} f(x, y)\| \leq \lambda \) holding for all \( y \in Y \) almost everywhere on \( X \), whence (46) would imply (by the triangle inequality) that \( \|\nabla^2_{xx} \hat{f}_k(x, y)\| \leq \lambda_k \) for all \( y \in Y \) almost everywhere on \( X \), which is equivalent to (11). Hence it only remains to verify (46). This can be done via (45) (which is valid by continuity of \( \nabla^k_y f(x, \cdot) \)):

\[
f(x, y) - \hat{f}_k(x, y) = \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} \left( \nabla^k_y f(x, y_t) - \nabla^k_y f(x, \hat{y}) \right) dy_t dt.
\]

Now observe that by (A3), for any \( y \in Y \) tensor \( \nabla^k_{xx} f(x, y) \) exists and satisfies \( \|\nabla^k_{xx} f(x, y)\| \leq \tau_k \) almost everywhere on \( X \). Fix \( y \in Y, \hat{y} \in Y, \) and \( x \in X \), and assume w.l.o.g. that \( x \in X \) is such that \( \nabla^k_{xx} f(x, y_t) \) exists (and hence \( \|\nabla^k_{xx} f(x, y_t)\| \leq \tau_k \) for all \( y_t \in [\hat{y}, y] \). Then, for any \( u, v \in E_X \),

\[
\{u, (\nabla^2_{xx} f(x, y) - \nabla^2_{xx} \hat{f}_k(x, y)) u\} = \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} \left( \nabla^2_{xx} f(x, y_t) - \nabla^2_{xx} \hat{f}_k(x, \hat{y}) \right) [(y - \hat{y})^k; u, v] dt.
\]

Whence, taking supremum over \( u, v \) on the unit sphere, by Jensen’s inequality and (A3) we arrive at

\[
\|\nabla^2_{xx} f(x, y) - \nabla^2_{xx} \hat{f}_k(x, y)\| \leq \|y - \hat{y}\|^k \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} \left( \|\nabla^{k+2}_{xx} f(x, y_t)\| + \|\nabla^{k+2}_{xx} \hat{f}_k(x, \hat{y})\| \right) dt \\
\leq 2\tau_k D^k \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} dt = \frac{2\tau_k D^k}{k!}.
\]

Finally, observe that these estimates remain valid, for any \( y \in Y \) and almost all \( x \in X \), even if \( \nabla^{k+2}_{xx} f \) is not guaranteed to exist everywhere on \( X \times Y \). Indeed, for any \( x \in X \) define \( Y_x' \) as the set of all \( y' \in Y \) where \( \nabla^{k+2}_{xx} f(x, y') \) does not exist. Consider the graph of the set-valued map \( x \mapsto Y_x' \), that is,

\[
\Gamma := \{(x, y') : x \in X, y' \in Y_x'\} \subset X \times Y.
\]

By Assumption 3, \( \Gamma \) is \((m_X \times m_Y)\)-measurable (here \( m_X, m_Y \) are the Lebesgue measures on \( X, Y \)). Hence its restriction \( \Gamma^* \) on \( X \times [\hat{y}, y] \) is measurable with respect to the induced measure \( m^* = m_X \times [\hat{y}, y] \), and we can apply Fubini’s theorem:

\[
m^*(\Gamma^*) = \int_{x \in X} m_{[\hat{y}, y]}(Y_x') dx = \int_{y' \in [\hat{y}, y]} m_X(X_{y'}) dy', \text{ where } X_{y'} := \{x \in X : y' \in Y_x'\}.
\]

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By (A3) we have \( m_X(X_{y'}) = 0 \) for any \( y' \in Y \), whence \( m^*(\Gamma^+) = 0 \) by the second representation of \( m^*(\Gamma^+) \), and therefore \( m_{[\hat{y},\hat{y}]}(Y_{x}) = 0 \) for almost all \( x \in X \) (by the first representation of \( m^*(\Gamma^+) \)). This shows that, for any choice of \( y, \hat{y} \in Y \), formula (47) is valid for almost all \( x \) (the integrand exists almost everywhere on \([0,1]\)), and so the final estimate is preserved.

A.4 Justification of (18)

We first consider the case \( k > 1 \). Recall that we have to show that condition \( \min \{ \mu D, \sqrt{\lambda \rho_k D^{k+1}} \} \leq \varepsilon \), for suitable constant factors that might depend on \( k \), implies (13) under (18)—i.e., provided that

\[
\varepsilon \leq \varepsilon_k \min \left\{ \left( \frac{\mu}{\sigma_k} \right)^{1/1}, \left( \frac{\lambda^{2k+1} \rho_k}{\tau_k^{k+1}} \right)^{1/1} \right\}.
\]

It suffices to show that (18) implies \( \sigma_k D^k \leq \varepsilon_k \mu D \) when \( \mu D \leq \varepsilon \), and \( \tau_k D^k \leq \varepsilon_k \lambda \) when \( \sqrt{\lambda \rho_k D^{k+1}} \leq \varepsilon_k \).

The first of these implications follows from the first part of (18): indeed, under its premise we have

\[
\mu D \leq \varepsilon \leq \varepsilon_k \left( \frac{\mu}{\sigma_k} \right)^{1/1},
\]

whence \( \sigma_k D^k \leq \varepsilon_k \mu D \) follows by taking power \( k-1 > 0 \). For the second implication, the premise gives

\[
\lambda \rho_k D^{k+1} \leq \varepsilon^2 \leq \varepsilon_k \left( \frac{\lambda^{2k+1} \rho_k}{\tau_k^{k+1}} \right)^{1/1},
\]

whence \( D^{k+1} \leq \left( \lambda/\tau_k \right)^{1/1} \), that is \( \tau_k D^k \leq \varepsilon_k \lambda \) by taking power \( k/\tau_k^{k+1} > 0 \). Both implications are proved.

Finally, in the case \( k = 1 \) the first implication holds trivially, as \( \sigma_1 = \mu \) w.l.o.g. On the other hand, our previous argument for the second implication applies here as well (since \( k/\tau_k^{k+1} > 0 \)).

B Proofs for Section 4

B.1 Proof of Lemma 4.1

The first claim is straightforward for \( F_{k,s,\lambda,\mu,\rho} \). For \( S_{\lambda,\rho,D} \), as \( 0 \leq \tanh'(x) \leq 1 \) and \( -1 \leq \tanh''(x) \leq 1 \),

\[
\frac{\partial^2}{\partial x^2} S_{\lambda,\rho,D}(x,y) = \frac{\lambda}{2} \left( -1 + \frac{y}{D} \tanh'' \left( \sqrt{\frac{\lambda}{\rho D}} x \right) \right) \in [-\lambda, 0],
\]

\[
\frac{\partial^2}{\partial x \partial y} S_{\lambda,\rho,D}(x,y) = \frac{1}{2} \sqrt{\frac{\lambda \rho}{D}} \tanh' \left( \sqrt{\frac{\lambda}{\rho D}} x \right) \in \left[ 0, \frac{1}{2} \sqrt{\frac{\lambda \rho}{D}} \right] \subset \left[ 0, \frac{\mu}{2 \sqrt{2}} \right].
\]

For the second claim, we first note that \( \frac{\partial}{\partial y} F_{0,s,\lambda,\mu,0} = \mu x \) and \( \frac{\partial}{\partial x} F_{0,s,\lambda,\mu,0} = \mu y - \lambda x \). So if \( \mu \leq \sqrt{2\lambda \rho/D} \), then we have on \([-\tau, \tau] \times [-D/2, D/2] \) that

\[
\left| \frac{\partial}{\partial y} F_{0,0,\lambda,\mu,0} \right| \leq \frac{\mu^2 D}{2 \lambda} \leq \rho, \quad \left| \frac{\partial}{\partial x} F_{0,0,\lambda,\mu,0} \right| \leq \mu D.
\]

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On the other hand, if \( \mu \geq \sqrt{2\lambda \rho/D} \), we have on \([-r, r] \times [0, D] \) that

\[
\frac{\partial}{\partial y} S_{\lambda, \rho, D} = \frac{\rho}{2} \left( \tanh \left( \sqrt{\frac{\lambda}{\rho D}} x \right) - 1 \right) \in [-\rho, 0],
\]

\[
\frac{\partial}{\partial x} S_{\lambda, \rho, D} = \frac{y}{2} \sqrt{\frac{\rho D}{\lambda}} \tanh' \left( \sqrt{\frac{\lambda}{\rho D}} x \right) - \frac{\lambda x}{2} \leq \frac{1}{2} \sqrt{\lambda \rho D} + \frac{\mu D}{4} < \mu D,
\]

where we used that \(-1 \leq \tanh(x) \leq 1\) and \(0 \leq \tanh'(x) \leq 1\). The third claim is straightforward. \(\square\)

**B.2 Proof of Proposition 4.1**

1°. As per the first claim, let \( f(x, y) = -\frac{1}{2} \lambda x^2 + \mu xy, X = [-r, r], Y = [R, R] \) for \( R = D/2 \) and \( r := \mu R/\lambda \). (48)

Clearly, \( \varphi(x) = -\frac{1}{2} \lambda x^2 + \mu R|x| = \lambda (-\frac{1}{2} x^2 + r|x|) \), whence \( \varphi_{2\lambda}(x) = \lambda \min_{u \in \mathbb{X}} \{(u - x)^2 - \frac{1}{2} u^2 + r|u|\} \). Let \( x^*(x) \) be the (constrained) minimizer. The unconstrained minimizer is given by \([2x]_r\), where

\[
[z]_r := (\max\{|z|, r\} - r) \text{sign}(z)
\]

is the soft thresholding operator. Now, observe that whenever \(|x| \leq r\), one has \([2x]_r| \leq r\), and therefore \( x^*(x) = [2x]_r \). Whence by (9) (cf. also Proposition D.1 in appendix) for any \( x \in X \) we have

\[
\varphi'_{2\lambda}(x) = 2\lambda(x - [2x]_r).
\]

(50)

On the other hand, we have that \( \varphi(x) = f(x, y) \) and thus \( \varphi_{2\lambda}(x) = \lambda \min_{u \in \mathbb{X}}\{(u - x)^2 - \frac{1}{2} u^2 + u\mu y/\lambda\} \). Here the unconstrained minimizer is given by \( 2x - \mu y/\lambda \), hence \( \hat{x}^*(x) = 2x - \mu y/\lambda \) for the actual (constrained) minimizer as long as \( 2x - \mu y/\lambda \leq r \). Now, let us choose

\[
x^* = r/2 \quad \hat{y} = y/2.
\]

Clearly, \( 2x^* - \mu \hat{y}/\lambda = r/2 \), therefore \( \varphi'_{2\lambda}(x^*) = 2\lambda(x^* - \hat{x}^*(x^*)) = 0 \). Meanwhile, due to (50) we have

\[
\varphi'_{2\lambda}(x^*) = 2\lambda(x^* - [2x^*]_r) = 2\lambda\left( \frac{1}{2} r - [r]_r \right) = \lambda r = \mu R.
\]

2°. Now consider \( f(x, y) = S_{\lambda, \rho, D}(x, y) \), cf. (26), on \([-r, r] \times [0, D] \). Since \( |\tanh(\cdot)| \leq 1 \) on \( \mathbb{R} \), we have \( \varphi(x) = f(x, 0) = -\frac{1}{4} \lambda x^2 \) and \( \varphi_{2\lambda}(x) = \lambda \min_{-r \leq u \leq r}\{(u - x)^2 - u^2\} \), with unconstrained minimizer given by \( \frac{1}{3} x \). Hence, as long as \(|x| \leq 3r/4\), we have \( \varphi_{2\lambda}(x) = -\lambda x^2/3 \) and \( \varphi'_{2\lambda}(x) = -2\lambda x/3 \).

On the other hand, for the choice \( \hat{y} = 2D/3 \) we have

\[
\varphi(x) = f(x, \hat{y}) = -\frac{\lambda x^2}{4} + \frac{\rho D}{3} \left( \tanh \left( \sqrt{\frac{\lambda}{\rho D}} x \right) - 1 \right).
\]

Since \( \varphi(\cdot) \) is differentiable on \( X = [-r, r] \), the set of its points on \( X \) with vanishing derivative coincides with such set for \( \varphi_{2\lambda}(\cdot) \). Let us now find \( x^* \in [-r, r] \) for which \( \varphi'(x^*) = 0 \), i.e., solutions to

\[
\sqrt{\frac{\lambda}{\rho D}} x^* \cosh^2 \left( \sqrt{\frac{\lambda}{\rho D}} x^* \right) = \frac{2}{3}.
\]

26
The unique solution is \( x^* = c\sqrt{\rho D/\lambda} \) with \( c \in (0.51, 0.52) \). Using that \( \mu \geq \sqrt{2\lambda\rho/D} \), we conclude that
\[
x^* < \frac{0.52\mu D}{\sqrt{2\lambda}} < \frac{3\mu D}{8\lambda} = \frac{3r}{4},
\]
whence \( \varphi'_{2\lambda}(x^*) = -2\lambda x^*/3 < -\sqrt{\lambda\rho D}/3 \).

**B.3 Proof of Proposition 4.2**

1°. Assume that \( \mu \leq \sqrt{\lambda\rho/2} \). Recall that \( f(x, y) = -\frac{1}{2}\lambda x^2 + \mu xy - \frac{1}{2}\rho y^2 \). Hence for \( \hat{y} = 0 \) we have \( \hat{f}(x, y) = -\frac{1}{2}\lambda x^2 + \mu xy \) and \( \hat{\varphi}(x) = \lambda(-\frac{1}{2}x^2 + r|x|) \) where \( r = \mu R/\lambda \) with \( R = \frac{1}{2}D \) (cf. (48)). This results in \( \hat{\varphi}'_{2\lambda}(x) = 2\lambda([2x], -x) \) and \( x^* = r \) being a stationary point for \( \hat{\varphi}_{2\lambda} \) (cf. (50)).

Meanwhile, the maximum in \( \varphi(x) = -\frac{1}{2}\lambda x^2 + \max_{|y|\leq R} \{\mu xy - \frac{1}{2}\rho y^2\} \) is effectively unconstrained—attained at \( \mu x/\rho \)—whenever \( |x| \leq \rho R/\mu \); thus, for such \( x \) we have \( \varphi(x) = \frac{1}{2} (\mu^2/\rho - \lambda) x^2 \) and hence
\[
\varphi'(x) = (\mu^2/\rho - \lambda) x.
\]

Moreover, by the first-order optimality condition \( 2\lambda(x - x^*)(x) \in \partial \varphi'(x^*(x)) \), cf. (8), we express the proximal mapping \( x^*(x) = x^+_{\varphi/2\lambda}(x) \) as the solution to \( 2\lambda(x^+ - x) + (\mu^2/\rho - \lambda) x^+(x) = 0 \), that is
\[
x^+(x) = \frac{2\lambda\rho x}{\lambda\rho + \mu^2}, \quad \text{whenever} \quad \frac{2\lambda\rho}{\lambda\rho + \mu^2} \leq \frac{\rho R}{\mu}.
\]

In particular, the above expression is valid for \( x^* = r \); indeed, recalling that \( \mu \leq \sqrt{\lambda\rho/2} \), we obtain
\[
\frac{2\lambda\rho}{\lambda\rho + \mu^2} < 2r = \frac{2\mu R}{\lambda} \leq \frac{\rho R}{\mu}.
\]

To verify the first claim of the proposition, it remains to observe that
\[
-\varphi'_{2\lambda}(r) = 2\lambda(x^+(r) - r) = 2\lambda r \left( \frac{2\lambda\rho}{\lambda\rho + \mu^2} - 1 \right) = 2\lambda r \left( \frac{\lambda \rho - \mu^2}{\lambda\rho + \mu^2} \right) \geq \frac{2\lambda r}{3} = \frac{2\mu R}{3},
\]
where the inequality is due to \( \mu^2 \leq \lambda\rho/2 \).

2°. We shall now prove the second claim. Define \( \overline{\rho} = \rho/4, \overline{\mu} := \sqrt{\lambda\rho/2} = \sqrt{2\lambda\overline{\rho}}, \) and \( \overline{r} := \overline{\mu} R/\lambda \) (cf. (48)), so that \( f(x, y) = F_{1,1,\lambda, \overline{\rho}, \overline{\mu}, \overline{r}} = -\frac{1}{2}\lambda x^2 + \overline{\mu} x y + \frac{1}{2} \overline{\rho} y^2 \). As such, \( \varphi(x) = \lambda(-\frac{1}{2}x^2 + \overline{r}|x|) + \frac{1}{2} \overline{\rho} R^2 \) and \( \varphi_{2\lambda}(x) = \lambda \min_{u\in\mathbb{R}} \{ (u-x)^2 - \frac{1}{2} u^2 + r|u| \} + \frac{1}{2} \overline{\rho} R^2 \). This implies the same result as in (50), namely
\[
\varphi'_{2\lambda}(x) = 2\lambda(x - [2x]_R).
\]

On the other hand, \( \hat{f} \equiv \hat{f}_1 \) at any \( \hat{y} \in Y \) is given by \( \hat{f}(x, y) = -\frac{1}{2}\lambda x^2 + (\overline{\mu} x + \overline{\rho} y) y - \frac{1}{2} \overline{\rho} y^2 \); in particular, for \( \hat{y} = R \) we have \( \hat{f}(x, y) = -\frac{1}{2}\lambda x^2 + (\overline{\mu} x + \overline{\rho} R) y - \frac{1}{2} \overline{\rho} R^2 \) and \( \hat{\varphi}(x) = -\frac{1}{2}\lambda x^2 + R|\overline{\mu} x + \overline{\rho} R| - \frac{1}{2} \overline{\rho} R^2 \), thus
\[
\hat{\varphi}'_{2\lambda}(x) = \lambda \min_{u\in\mathbb{R}} \left\{ (u-x)^2 - \frac{1}{2} u^2 + \overline{r} \left| u + \frac{\overline{\rho} R}{\overline{\mu}} \right| \right\} - \frac{\overline{\rho} R^2}{2}.
\]

By the optimality condition, the minimizer is given by \( \hat{x}^+(x) = [2x + \overline{\rho} R/\overline{\mu}]_R - \overline{\rho} R/\overline{\mu} \), so we arrive at
\[
\hat{\varphi}'_{2\lambda}(x) = 2\lambda \left( x + \frac{\overline{\rho} R}{\overline{\mu}} - \left[ 2x + \frac{\overline{\rho} R}{\overline{\mu}} \right]_R \right).
\]
We conclude that \( x^* = -\overline{r}R/\overline{\mu} \) is stationary for \( \dot{\varphi}_{2\lambda}(\cdot) \): indeed, plugging in \( \overline{\mu} = \sqrt{2\lambda \overline{\rho}} \) we have
\[
-\dot{\varphi}'_{2\lambda}(x^*) = 2\lambda [-\overline{r}R/\overline{\mu}] = 2\lambda [-\bar{r}/2] = 0.
\]
Meanwhile, due to (51) we conclude that
\[
-\dot{\varphi}'_{2\lambda}(x^*) = 2\lambda \left( \frac{\overline{r}R}{\overline{\mu}} + \frac{2\overline{r}R}{\overline{\mu}} \right) = 2\lambda \left( \frac{\overline{r}R}{\overline{\mu}} + [-\bar{r}] \right) = \frac{2\lambda \overline{\rho} R^2}{\overline{\mu}} = \frac{\lambda \rho D^2}{8}.
\]
This concludes the proof. \( \square \)

### B.4 Proof of Proposition 4.3

#### B.4.1 Case \( \mu \leq \mu_{cr} \)

Recall that here we use \( Y = [0, D] \) and
\[
f(x, y) = F_{k, -1, \lambda, \mu, \rho}(x, y) = -\frac{\lambda x^2}{2} + \bar{\mu}xy - \frac{\rho|y|^{k+1}}{(k+1)!},
\]
with some \( \bar{\mu} \leq \mu \) yet to be chosen. Let \( \hat{y} = 0 \); then \( \dot{f}(x, y) = -\frac{1}{2} \lambda x^2 + \bar{\mu}xy \) is maximized on \( \{0, D\} \), so
\[
\dot{\varphi}(x) = -\frac{\lambda x^2}{2} + \bar{\mu}D \max\{x, 0\}.
\]
Clearly, the point
\[
x^* = \frac{\bar{\mu}D}{\lambda}
\]
is stationary for \( \dot{\varphi}(\cdot) \), and thus for \( \dot{\varphi}_{2\lambda}(\cdot) \) as well. It remains to lower-bound \( |\dot{\varphi}'_{2\lambda}(x^*)| \). To this end, note that \( \frac{\partial}{\partial y} f(x, y) = \bar{\mu}x - \frac{1}{k+1} \rho |y|^k \sign(y) \), so \( f(x, \cdot) \) has a unique unconstrained maximizer \( \bar{y} = \bar{y}(x) \) which is given as the solution to \( \bar{\mu}x - \frac{1}{k+1} \rho |y|^k \sign(y) = 0 \); in other words, \( \bar{y}(x) = (|x|\bar{\mu}k!/\rho)^{1/k} \sign(x) \).

Clearly, we have that \( \varphi(x) = f(x, \bar{y}(x)) \) for any \( x \in \mathbb{R} \) such that \( \bar{y}(x) \in [0, D] \)—in other words, when
\[
0 \leq \bar{\mu} x \leq \frac{\rho D^k}{k!}.
\]
As a result, for such \( x \) function \( \varphi(x) \) is differentiable, and we have
\[
\varphi(x) = -\frac{\lambda x^2}{2} + \frac{k}{k+1} \left( \frac{x\bar{\mu}k!}{\rho} \right)^{1/k} \mu x, \quad \varphi'(x) = -\lambda x + \bar{\mu} \left( \frac{x\bar{\mu}k!}{\rho} \right)^{1/k}.
\]

Now, recall (cf. (9)) that we have \( 2\lambda(x - x^+ (x)) \in \partial \varphi(x^+(x)) \) for the proximal mapping \( x^+(x) \) corresponding to \( \varphi \) with stepsize \( \frac{1}{2\lambda} \). Therefore, we can compute \( x^+(x) \) for given \( x \geq 0 \) by solving
\[
x^* + \frac{\bar{\mu}}{\lambda} \left( \frac{x^+ \bar{\mu}k!}{\rho} \right)^{1/k} = 2x
\]
for \( x^* \geq 0 \) (such a solution is clearly unique) and verifying that the solution satisfies \( \bar{\mu} x^+ \leq \frac{1}{k+1} \rho D^k \) (cf. (54)). It is clear that, for any \( x \geq 0 \), the corresponding solution \( x^+(x) \) to (55) satisfies the bounds
\[
2x - \frac{\bar{\mu}}{\lambda} \left( \frac{2x\bar{\mu}k!}{\rho} \right)^{1/k} \leq x^+(x) \leq 2x.
\]
To this end, let \( x^+ = x^+(x^*) \) for \( x^* = \frac{\mu D}{\lambda} \), and \( \bar{\mu} = \frac{1}{2} \mu \left[ \leq \frac{1}{2} \mu_{cr} \right] \). By (27) and the upper bound in (56),

\[
\bar{\mu} x^+ \leq \frac{2 \bar{\mu}^2 D}{\lambda} \leq \frac{\rho D^k}{2 \cdot k!},
\]

which verifies our characterization of \( x^+ \) as the solution to (55) for chosen \( x^* \). On the other hand,

\[
\frac{\bar{\mu}}{\lambda} \left( \frac{2 x^+ \bar{\mu} k!}{\rho} \right)^{1/k} = \frac{\bar{\mu} D}{\lambda} \left( \frac{2 \bar{\mu}^2 k!}{\lambda \rho D^{k-1}} \right)^{1/k} \leq \frac{x^+}{2^{1/k}},
\]

whence by the lower bound in (56) we get \( x^+ > (2 - 2^{-1/k}) x^* \) and, using (9), arrive at

\[
-\varphi_2^\prime(x^*) = 2\lambda (x^+ - x^*) \geq (1 - 2^{-1/k}) \mu D > \frac{\mu D}{2k}.
\]

Here the final step uses \( k \geq 2 \) and the fact that the function \( t \mapsto (1 - \frac{1}{t})^t \) increases on \([1, +\infty]\).

### B.4.2 Case \( \mu \geq \mu_{cr} \)

Denote \( R = D/2 \) and let

\[
\bar{r} := \frac{\bar{\mu} R}{\lambda}
\]

for some \( \bar{\mu} \leq \mu_{cr} \left[ \leq \mu \right] \) yet to be chosen. Recall that here we use \( Y = [-R, R] \) and

\[
f(x, y) = F_{k, 1, \lambda, \bar{\mu}, \rho}(x, y) = -\frac{\lambda x^2}{2} + \bar{\mu} x y + \frac{\rho |y|^{k+1}}{(k + 1)!}.
\]

Clearly \( \varphi(x) = \lambda \left( -\frac{1}{2} x^2 + \bar{r}|x| \right) + \frac{\rho R^{k+1}}{(k+1)!} \), therefore \( \varphi_2 \lambda(x) = \lambda \min_{u \in R} \{ (u - x)^2 - \frac{1}{2} u^2 + \bar{r}|u| \} + \frac{\rho R^{k+1}}{(k+1)!} \), and

\[
\varphi_2^\prime \lambda(x) = 2\lambda (x - [2x] x),
\]

cf. (51). Now, note that \( g(y) = \frac{1}{(k+1)!} |y|^{k+1} \) is \( k \) times continuously differentiable with \( j \)-th derivative

\[
g^{(j)}(y) = \frac{|y|^{k+1-j} \text{sign}(y)^j}{(k + 1 - j)!}, \quad j \leq k.
\]

Whence by the binomial formula we conclude that, for any \( \tilde{y} \in [0, R] \),

\[
\hat{f}(x, y) = -\frac{\lambda x^2}{2} + \bar{\mu} x y + \frac{\rho \left( y^{k+1} + (y - \tilde{y})^{k+1} \right)}{(k + 1)!}.
\]

From now on, we consider two cases depending on the parity of \( k \) (the case of odd \( k \) being harder).

**Case of even \( k \).** In this case we choose \( \bar{\mu} = \mu_{cr} \), \( \tilde{y} = R \), and observe that the resulting function \( \hat{f}(x, \cdot) \) is convex on \([-R, R]\). Indeed, in terms of the rescaled variable \( z = y/R \) we have that \( \hat{f}(x, y) = h(x, z) \) with \( h(x, z) = -\frac{1}{2} x^2 + \bar{\mu} R x z + \rho R^{k+1} p(z) \) and function \( p : [-1, 1] \to \mathbb{R} \) given by

\[
p(z) = \frac{z^{k+1} - (z - 1)^{k+1}}{(k + 1)!}.
\]
Let us verify that $p$ is convex on $[-1, 1]$. Indeed: on one hand, for $z \in [-1, 1]$ one has $\frac{z}{z-1} \leq 1/2$, thus $(\frac{z}{z-1})^{k-1} \leq 1$ using that $k-1$ is odd; on the other hand, $(z-1)^{k-1} \leq 0$ for $z \in [-1, 1]$. As a result,

$$p''(z) = \frac{z^{k-1} - (z - 1)^{k-1}}{(k-1)!} \geq 0, \quad \forall z \in [-1, 1].$$

As such, $p(\cdot)$ is convex on $[-1, 1]$; $h(x, \cdot)$ is convex on $[-1, 1]$ and maximized at an endpoint, so that

$$\hat{\varphi}(x) = \max_{z=\pm 1} h(x, z) = \lambda \left( -\frac{x^2}{2} + \max \left\{ -\sqrt{\frac{\rho R+1}{\mu(k+1)!}}, -\sqrt{\frac{2k-1 - \rho R+1}{\mu(k+1)!}} \right\} \right)$$

$$= \lambda \left( -\frac{x^2}{2} + \sqrt{\frac{2k-1 - \rho R+1}{\mu(k+1)!}} \right) + \frac{2k\rho R+1}{\mu(k+1)!}$$

and (cf. (52)–(53))

$$\hat{\varphi}_{2\lambda}'(x) = 2\lambda \left( x - \frac{(2k-1 - \rho R)}{\mu(k+1)!} \left[ 2x - \frac{(2k-1 - \rho R)}{\mu(k+1)!} \right]_F \right).$$

Now, observe that, due to (27), the point

$$x^* = \left( \frac{2k-1 - \rho R}{\mu(k+1)!} \right)_F$$

satisfies

$$x^* \leq \frac{2\bar{\bar{r}}}{k+1}.$$

As a result, we have $[x^*]_F = 0$ and $0 \leq [2x^*]_F \leq [2x^*]_F = \frac{1}{2}x^* = \frac{1}{2}x^*; \therefore \hat{\varphi}_{2\lambda}'(x^*) = 0$ and

$$\hat{\varphi}_{2\lambda}'(x^*) = 2\lambda(x^* - [2x^*]_F) \geq 2\lambda x^* = \lambda x^* = \frac{(2k-1 - \rho R)}{\mu(k+1)!} = \frac{(2k-1 - \rho R)}{\mu(k+1)!}.$$ 

**Case of odd $k$.** Note that here we have $k \geq 3$ by the premise of the theorem. We choose $\hat{y} = (1 - \frac{1}{k})R$, so that $\hat{f}(x, y) = h(x, z)$ with $h(x, z) = -\frac{1}{2}k x^2 + \bar{\bar{r}} R x + \rho R k+1 q(z)$, for $z = y/R$ and $q(z)$ given by

$$q(z) = \frac{1}{(k+1)!} \left( z^{k+1} - \left( z - \left( 1 - \frac{1}{k} \right) \right)^{k+1} \right).$$  

(60)

We choose $\bar{\bar{\mu}}$ as follows:

$$\bar{\bar{\mu}} = \frac{\sqrt{2\lambda D^{k-1}}}{k!} \left( 1 - \frac{1}{2k} \right) \left( 1 - \frac{1}{k} \right)^{k-1}$$

(61)

Since $k > 2$, we have $\frac{1}{\sqrt{2\mu_{cr}}} \leq \bar{\bar{\mu}} \leq \mu_{cr}$ (cf. (27)). Now, let us show that

$$x^* = -\left( 1 - \frac{1}{k} \right) \bar{\bar{r}}$$

is a stationary point for $\hat{\varphi}$ (and thus also for $\hat{\varphi}_{2\lambda}$).
We first observe that
\[
\frac{\partial}{\partial z} h(x^*, z) = \bar{\mu} Rx^* + \rho R^{k+1} q'(z) = -\left(1 - \frac{1}{k}\right) \frac{\bar{\mu}^2 R^2}{\lambda} + \frac{\rho R^{k+1}}{k!} \left(z^k - \left(1 - \frac{1}{k}\right) z^{k-1}\right). \tag{62}
\]

Let us find all stationary points \(z^*\) of \(h(x^*, \cdot)\) on \(\mathbb{R}\). Plugging the expression for \(\bar{\mu}\) (cf. (61)) into the right-hand side of (62) and dividing over \(1/k! R^{k+1} \left(1 - \frac{1}{k}\right)^k > 0\), we arrive at the equation
\[
w^k - (w - 1)^k = 2^k - 1 \tag{63}
\]
in terms of \(w = \frac{k}{k-1} z\), and guess two solutions: \(w_1^* = -1\) and \(w_2^* = 2\). Moreover, \(w \mapsto w^k - (w - 1)^k\) is a strictly convex function (to see this, note that \(k - 2 > 0\) is odd, and the function \(u \mapsto u^{k-2}\) increases on \(\mathbb{R}\) modulo the sole fixed point \(u = 0\)), therefore (63) has no other (real) solutions.

- Since \(k \geq 3\), we have \(1 < \frac{k}{k-1} \leq \frac{3}{2}\), so only one of the two solutions, namely \(w_1^* = -1\), belongs to the range \([-\frac{k}{k-1}, \frac{k}{k-1}]\) of \(w = w(z)\) corresponding to \(z \in [-1, 1]\). As such,
\[
z^* = -\left(1 - \frac{1}{k}\right)
\]
is a unique stationary point of \(h(x^*, \cdot)\) on \([-1, 1]\). Moreover, since \(k - 1\) is even, we have that
\[
\frac{\partial^2}{\partial z^2} h(x^*, z^*) = \frac{\rho R^{k+1}}{(k-1)!} \left((z^*)^{k-1} - \left(z^* - \left(1 - \frac{1}{k}\right)\right)^{k-1}\right) = (1 - 2^{k-1}) \left(1 - \frac{1}{k}\right)^{k-1} \frac{\rho R^{k+1}}{(k-1)!} < 0,
\]
so \(z^*\) is a local maximizer—and hence also a unique global maximizer—of \(h(x^*, \cdot)\) on \([-1, 1]\).

- Finally, by a version of Danskin’s theorem (see, e.g., Lemma D.1 in appendix) we have that
\[
\varphi'(x^*) = \frac{\partial}{\partial x} h(x^*, z^*) = \bar{\mu} R x^* - \lambda x^* = \lambda (\bar{\mu} - x^*) = 0.
\]

We have just verified that \(x^*\) is a stationary point for \(\varphi(\cdot)\). On the other hand, we observe that
\[
-\varphi_{2\lambda}(x^*) = 2\lambda [2x^* - x^*] = 2\lambda \left(\left(1 - \frac{1}{k}\right) \bar{\mu} + \left[-2 \left(1 - \frac{1}{k}\right) \bar{\mu}\right]\right) = \frac{2\bar{\mu} \lambda}{k} > \frac{\mu \alpha D}{2k}.
\]
This concludes the proof. \(\Box\)

### C Proofs for Section 5

The proofs of Theorems 5.1 and 5.2 rely on the following technical result extracted from [54].

**Lemma C.1** (cf. [54, Proposition 5.5]). For \(x \in X, \xi \in E_x, \bar{\lambda} > 0\) define functionals \(S_X, S_Y\) by
\[
S_X(x, \xi, \bar{\lambda}) := 2\bar{\lambda} \max_{u \in X} \left\{ -\langle \xi, u - x \rangle - \frac{\bar{\lambda}}{2} \| u - x \|^2 \right\} \quad \text{for} \quad x \in X, \xi \in E_x, \bar{\lambda} > 0,
\]
\[
S_Y(y, \eta, \bar{\rho}) := 2\bar{\rho} \max_{v \in Y} \left\{ -\langle \eta, v - y \rangle - \frac{\bar{\rho}}{2} \| v - y \|^2 \right\} \quad \text{for} \quad y \in Y, \eta \in E_y, \bar{\rho} > 0. \tag{64}
\]
1. Under Assumption 1, for \( \hat{f} \equiv \hat{f}_0 \), at any \( x \in X \) (and regardless of the choice of \( \hat{y} \in Y \)) we have
\[
\| \nabla \varphi_{2\lambda}(x) \| \leq 2S_X(x, \nabla_x f(x, \hat{y}), \lambda).
\] (65)

2. Under Assumptions 1 to 3 with \( k = 1 \), for \( \hat{f} \equiv \hat{f}_1 \), at any \( x \in X \) and \( y \in Y \) we have
\[
\| \nabla \varphi_{2\lambda}(x) \| \leq 2\left( S_X(x, \nabla_x \hat{f}(x, y), \hat{\lambda}_1) + \sqrt{\frac{\lambda}{\rho_1}} S_Y(y, -\nabla_y \hat{f}(x, y), \rho_1) + \sqrt{\lambda \rho_1} D \right).
\] (66)

Moreover, for \( y^* \equiv y^*(x) \in \text{Argmax}_{y \in Y} \hat{f}(x, y) \) we have \( \| \nabla \varphi_{2\lambda}(x) \| \leq 2S_X(x, \nabla_x \hat{f}(x, y^*), \hat{\lambda}_1) \).

**Proof.** Inequality (66) follows from [54, Proposition 5.5] after recovering the constant factors from the proof (see [54, Appendix A]) via Cauchy-Schwarz, and extracting the square root. For \( y^* = y^*(x) \) the corresponding term vanishes, and we get \( \| \nabla \varphi_{2\lambda}(x) \| \leq 2S_X(x, \nabla_x \hat{f}(x, y^*), \hat{\lambda}_1) \) directly from [54, Eq. (5.6)]. We obtain (65) by the same argument, as in this case \( \hat{f} \equiv \hat{f}_0 \) is formally maximized at \( \hat{y} \) for any \( x \in X \). Note that the argument cannot be extended to \( k \geq 2 \): in this case \( \hat{f}_k(x, \cdot) \) is not concave, so [54, Proposition 5.5] cannot be applied anymore. □

### C.1 Proof of Theorem 5.1

We first observe that, with \( \gamma_x = 1/\lambda \), the quantity \( \varepsilon_t \) computed in line 6 of Algorithm 1 is nothing else but \( S_X(x_t, \nabla_x \hat{f}(x_t, \hat{y}), \lambda) \). Indeed:
\[
S_X^2(x_t, \nabla_x f(x_t, \hat{y}), \lambda) = 2\lambda \max_{x \in X} \left\{ -\langle \nabla_x f(x_t, \hat{y}), x - x_t \rangle - \frac{1}{\gamma_x} \| x - x_t \| ^2 \right\}
\]
\[
= \frac{1}{\gamma_x} \max_{x \in X} \left\{ -2\langle \gamma_x \nabla_x f(x_t, \hat{y}), x - x_t \rangle - \| x - x_t \| ^2 \right\}
\]
\[
= \| \nabla_x f(x_t, \hat{y}) \|^2 - \frac{1}{\gamma_x} \min_{x \in X} \| x - \gamma_x \nabla_x f(x_t, \hat{y}) - x \|^2 = \varepsilon_t^2.
\] (67)

Thus, Algorithm 1 maintains \( x^* \) such that \( S_X(x^*, \nabla_x f(x^*, \hat{y}), \lambda) = \min_{0 \leq t \leq T - 1} S_X(x_t, \nabla_x f(x_t, \hat{y}), \lambda) \). On the other hand, by the descent lemma (cf. Assumption 1), for each \( t \in \{0, \ldots, T - 1\} \) we have that
\[
f(x_{t+1}, \hat{y}) \leq f(x_t, \hat{y}) + \langle \nabla_x f(x_t, \hat{y}), x_{t+1} - x_t \rangle + \frac{1}{2} \| x_{t+1} - x_t \|^2
\]
\[
= f(x_t, \hat{y}) + \min_{x \in X} \left\{ \langle \nabla_x f(x_t, \hat{y}), x - x_t \rangle + \frac{1}{2} \| x - x_t \|^2 \right\}
\]
\[
= f(x_t, \hat{y}) - \frac{1}{\lambda} S_X^2(x_t, \nabla_x f(x_t, \hat{y}), \lambda),
\]
where the first equality follows from the definition of \( x_{t+1} \). Whence, via telescoping and (29) we get
\[
S_X^2(x^*, \nabla_x f(x^*, \hat{y}), \lambda) = \min_{0 \leq t \leq T - 1} S_X^2(x_t, \nabla_x f(x_t, \hat{y}), \lambda) \leq \frac{1}{T} \sum_{t=0}^{T-1} S_X^2(x_t, \nabla_x f(x_t, \hat{y}, \lambda)
\]
\[
\leq \frac{2\lambda}{T} [f(x_0, \hat{y}) - f(x_T, \hat{y})] \leq \frac{2\lambda [\varphi(x_0) - \psi(\hat{y})]}{T} < \varepsilon^2/144.
\]

By the first claim of Lemma C.1 (cf. (65)) this gives \( \| \nabla \varphi_{2\lambda}(x^*) \| \leq 2\varepsilon^* \leq \varepsilon/6 \). Finally, (30) implies the premise of Theorem 3.1 with \( k = 0 \); applying it we arrive at \( \| \nabla \varphi_{2\lambda}(x^*) \| \leq \varepsilon. \) □

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C.2 Proof of Theorem 5.2

1\textsuperscript{o}. We first consider the case \( \mu \leq (\bar{\lambda}_1 \rho_1)^{1/2} \), so that Coupled = 0 and condition (32) reduces to

\[
200\mu D \leq \varepsilon. \tag{68}
\]

As \( \hat{f} := \hat{f}_1 \), line 7 of Algorithm 2 reads \( y_t \in \text{Argmax}_{y \in Y} \hat{f}(x_t, y) \); then the second claim of Lemma C.1 (with \( y_t = y_0 \)) guarantees that

\[
\| \nabla \hat{\varphi}_{2\lambda_1}(x_t) \| \leq 2S_X(x_t, \nabla \hat{f}(x_t, y_t), \bar{\lambda}_1). \tag{69}
\]

Let us now upper-bound \( S_X(x_t, \nabla \hat{f}(x_t, y_t), \bar{\lambda}_1) \). To this end, for \( \delta > 0 \) to be chosen later, consider

\[
f_{\text{reg}}(x, y) := f(x, y) - \frac{\delta}{2} \| y - y \|^2, \quad \varphi_{\text{reg}}(x) := \max_{y \in Y} f_{\text{reg}}(x, y). \tag{70}
\]

Since \( f_{\text{reg}}(x, \cdot) \) is \( \delta \)-strongly concave, by Danskin’s theorem (cf. [53, Lemma 24]) \( \varphi_{\text{reg}} \) is differentiable; \( \nabla \varphi_{\text{reg}}(x) = \nabla f_{\text{reg}}(x, y_{\text{reg}}(x)) \) with \( y_{\text{reg}}(x) := \text{argmax}_{y \in Y} f_{\text{reg}}(x, y) \), and is \( \lambda_{\text{reg}} \)-Lipschitz with \( \lambda_{\text{reg}} := \lambda + \frac{\mu^2}{\delta} \). Now, let us choose \( \delta = \frac{\mu^2}{\bar{\lambda}_1} \), so that \( \lambda_{\text{reg}} = \lambda + \bar{\lambda}_1 \in [\bar{\lambda}_1, 2\bar{\lambda}_1] \). By the descent lemma

\[
\varphi_{\text{reg}}(x_{t+1}) - \varphi_{\text{reg}}(x_t) \leq \langle \nabla \varphi_{\text{reg}}(x_t), x_{t+1} - x_t \rangle + \frac{\lambda_{\text{reg}}}{4} \| x_{t+1} - x_t \|^2 = \langle \nabla_x f(x_t, y_{\text{reg}}(x_t)), x_{t+1} - x_t \rangle + \frac{\lambda_{\text{reg}}}{4} \| x_{t+1} - x_t \|^2
\]

\[
\leq \langle \nabla_x f(x_t, y_t), x_{t+1} - x_t \rangle + \frac{3\lambda_{\text{reg}}}{4} \| x_{t+1} - x_t \|^2 + \frac{1}{\lambda_{\text{reg}}} \| \nabla_x f(x_t, y_{\text{reg}}(x_t)) - \nabla_x f(x_t, y_t) \|^2
\]

\[
\leq \langle \nabla_x f(x_t, y_t), x_{t+1} - x_t \rangle + \frac{3\lambda_{\text{reg}}}{4} \| x_{t+1} - x_t \|^2 + \frac{\mu^2 D^2}{\lambda_{\text{reg}}}
\]

\[
= \min_{x \in X} \left\{ \langle \nabla_x f(x_t, y_t), x - x_t \rangle + \frac{1}{2} \left( 3\bar{\lambda}_1 + \frac{\mu^2}{\rho_1} \right) \| x - x_t \|^2 + \frac{\mu^2 D^2}{\bar{\lambda}_1} \right\}
\]

\[
= -\frac{1}{6\bar{\lambda}_1 + 2\mu^2/\rho_1} S_X^2 \left( x_t, \nabla_x f(x_t, y_t), 3\bar{\lambda}_1 + \mu^2/\rho_1 \right) + \frac{\mu^2 D^2}{\bar{\lambda}_1}
\]

\[
\leq \frac{1}{8\bar{\lambda}_1} S_X^2 \left( x_t, \nabla_x f(x_t, y_t), 3\bar{\lambda}_1 + \mu^2/\rho_1 \right) + \frac{\mu^2 D^2}{\bar{\lambda}_1}. \tag{71}
\]

Here the second inequality is by Cauchy-Schwarz, the next one is via Assumption 1, and the subsequent identities are by the definitions of \( x_{t+1} \) and \( \gamma_x \) (cf. line 10 of Algorithm 2). (Note that the factor \( \mu^2/\rho_1 \) can be upper-bounded with \( \bar{\lambda}_1 \) by the standing assumption, but we avoid this in order for our estimates to have a similar form as in the strongly coupled case (cf. 2\textsuperscript{o}) which is yet to be considered.) We now proceed as in (67): by telescoping (71) we get

\[
S_X^2 \left( x^*, \nabla_x f(x^*, y^*), 3\bar{\lambda}_1 + \frac{\mu^2}{\rho_1} \right) = \min_{0 \leq t \leq T-1} S_X^2 \left( x_t, \nabla_x f(x_t, y_t), 3\bar{\lambda}_1 + \frac{\mu^2}{\rho_1} \right)
\]

\[
\leq \frac{1}{T} \sum_{t=0}^{T-1} S_X^2 \left( x_t, \nabla_x f(x_t, y_t), 3\bar{\lambda}_1 + \frac{\mu^2}{\rho_1} \right) \leq \frac{8\bar{\lambda}_1[\varphi_{\text{reg}}(x_0) - \varphi_{\text{reg}}(x_T)]}{T} + 8\mu^2 D^2
\]

\[
\leq \frac{8\bar{\lambda}_1[\varphi(x_0) - \varphi(x_T)]}{T} + \frac{4\bar{\lambda}_1 \delta D^2}{T} + 8\mu^2 D^2 \leq \frac{8\bar{\lambda}_1[\varphi(x_0) - \varphi(x_T)]}{T} + 12\mu^2 D^2. \tag{72}
\]
As a result,

\[ S_X^2(x^*, \nabla f(x^*, y^*), \bar{\lambda}_1) \leq S_X^2(x^*, \nabla f(x^*, y^*), 4\bar{\lambda}_1) \]

\[ = 8\bar{\lambda}_1 \max_{x \in X} \left\{ \nabla f(x^*, y^*) \right\} - 2\bar{\lambda}_1 \| x - x^* \|^2 \]

\[ \leq 8\bar{\lambda}_1 \max_{x \in X} \left\{ \nabla f(x^*, y^*) \right\} - 3\bar{\lambda}_1 \| x - x^* \|^2 \]

\[ + 4\| \nabla f(x^*, y^*) - \nabla f(x^*, y^*) \|^2 \]

\[ = \frac{4}{3} S_X^2(x^*, \nabla f(x^*, y^*), 3\bar{\lambda}_1) + 4\| \nabla f(x^*, y^*) - \nabla f(x^*, y^*) \|^2 \]

\[ = \frac{4}{3} S_X^2(x^*, \nabla f(x^*, y^*), 3\bar{\lambda}_1) + 16\mu_D^2D^2 \]

\[ \leq \frac{32\bar{\lambda}_1 [\varphi(x_0) - \varphi(x_T)]}{3T} + 32\mu_D^2D^2 \leq 11\varepsilon^2 \left( \frac{1}{2100} + \frac{3}{40000} \right) < \frac{\varepsilon^2}{144} \]

Here the first estimate relies on the proximal Polyak-Lojasiewicz (PL) inequality that ensures that \( S_X(x, \xi, \bar{\lambda}) \) is non-decreasing in \( \bar{\lambda} \) (cf. [93, Lemma 1]); the second one is by Cauchy-Schwarz, the third one is by Lemma 3.2, the fourth one is by (72), and the last one is by (33)–(68). Finally, returning to (69) we get \( |\nabla \varphi(x^*)| \leq \varepsilon/6 \), and by Theorem 3.1 this results in \( |\nabla \varphi(x^*)| \leq \varepsilon \).

**2°.** We now consider the case \( \mu \geq (\bar{\lambda}_1\rho_1)^{1/2} \), where Algorithm 2 is run with Coupled = 1. By (32),

\[ 200\sqrt{\bar{\lambda}_1\rho_1} \leq \varepsilon. \]  

(73)

Casting the update in line 4 of Algorithm 2 as \( y_t = \arg\max_{y \in Y} \{ \nabla f(x_t, y) - \rho_1\| y - \hat{y} \|^2 \} \) gives

\[ \{ \nabla f(x_t, \hat{y}) - \rho_1(y_t - \hat{y}), y_t - y \} \leq 0, \quad \forall y \in Y, \]

from to the first-order optimality condition. As a result, for any \( 0 \leq t \leq T - 1 \) we have

\[ S_X^2(y_t, -\nabla f(x_t, y_t), \rho_1) = S_X^2(y_t, -\nabla f(x_t, \hat{y}), \rho_1) = 2\rho_1 \max_{y \in Y} \left\{ \nabla f(x_t, \hat{y}, y - y_t) - \rho_1 \| y - y_t \|^2 \right\} \]

\[ \leq 2\rho_1 \max_{y \in Y} \left\{ \| y - y_t \|^2 + \langle y_t - \hat{y}, y - y_t \rangle \right\} \leq \rho_1^2D^2. \]

where in the final step we maximized over the whole \( E_y \). Whence by Lemma C.1 (cf. (66)) we get

\[ \| \nabla \varphi(h)(x_t) \| \leq 2S_X(x_t, \nabla f(x_t, y_t), \bar{\lambda}_1) + 4\sqrt{\bar{\lambda}_1D}. \]

(74)

Our next goal is to estimate \( S_X(x_t, \nabla f(x_t, y_t), \bar{\lambda}_1) \) via a telescoping argument. To this end, let

\[ \tilde{f}_\text{reg}(x, y) := \hat{f}_1(x, y) - \frac{\rho_1}{2} \| y - \hat{y} \|^2, \quad \bar{\varphi}_\text{reg}(x) := \max_{y \in Y} \tilde{f}_\text{reg}(x, y). \]

(75)

Recall that \( \nabla f(x,y) \) is \( \bar{\lambda}_1 \)-Lipschitz for any \( y \in Y \) (cf. Lemma 3.3); moreover, \( \tilde{f}_\text{reg}(x, \cdot) \) is \( \rho_1 \)-strongly concave. Therefore by Danskin’s theorem (cf. [53, Lemma 24]), \( \nabla \bar{\varphi}_\text{reg}(x) = \nabla f(x, \bar{y}_\text{reg}(x)) \) with \( \bar{y}_\text{reg}(x) := \arg\max_{y \in Y} \tilde{f}_\text{reg}(x, y) \), and is \( \bar{\lambda}_1 \)-Lipschitz with \( \bar{\lambda}_1 := \lambda_1 + \mu^2/\rho_1 \). Moreover, we have

\[ y_t = \bar{y}_\text{reg}(x_t), \]

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as seen by looking at line 4 of Algorithm 2 again. Whence, by the descent lemma (cf. (71)) we get

\[
\bar{\varphi}_{\text{reg}}(x_{t+1}) - \bar{\varphi}_{\text{reg}}(x_t) \leq \langle \nabla \bar{\varphi}_{\text{reg}}(x_t), x_{t+1} - x_t \rangle + \frac{\bar{\lambda}_{\text{reg}}}{2} \|x_{t+1} - x_t\|^2 \\
= \langle \nabla_x \hat{f}(x_t, y_t), x_{t+1} - x_t \rangle + \frac{\bar{\lambda}_{\text{reg}}}{2} \|x_{t+1} - x_t\|^2 \\
\leq \langle \nabla_x \hat{f}(x_t, y_t), x_{t+1} - x_t \rangle + \frac{1}{2} \left( 3\bar{\lambda}_1 + \frac{\mu^2}{\rho_1} \right) \|x_{t+1} - x_t\|^2 \\
= \min_{x \in \mathcal{X}} \left\{ \langle \nabla_x \hat{f}(x_t, y_t), x - x_t \rangle + \frac{1}{2} \left( 3\bar{\lambda}_1 + \frac{\mu^2}{\rho_1} \right) \|x - x_t\|^2 \right\} \\
\leq -\frac{1}{6\bar{\lambda}_1 + 2\mu^2/\rho_1} S_X^2 \left( x_t, \nabla_x \hat{f}(x_t, y_t), 3\bar{\lambda}_1 + \frac{\mu^2}{\rho_1} \right).
\]

Here the penultimate line follows by recasting the update in line 5 of Algorithm 2 as

\[
\bar{x}_{t+1} = x_t - \gamma_x \nabla_x \hat{f}(x_t, \hat{y}) - \gamma_x \nabla_x^2 \hat{f}(x_t, \hat{y})(y_t - \hat{y}) = x_t - \gamma_x \nabla_x \hat{f}(x_t, y_t).
\]

Moreover, for the same reason we have (cf. (67))

\[
\varepsilon_t^2 = \|\nabla_x \hat{f}(x_t, y_t)\|^2 - \frac{1}{\gamma_x} \min_{x \in \mathcal{X}} \|x_t - \gamma_x \nabla_x \hat{f}(x_t, y_t) - x\|^2 = S_X^2 \left( x_t, \nabla_x \hat{f}(x_t, y_t), 3\bar{\lambda}_1 + \frac{\mu^2}{\rho_1} \right).
\]

Whence, proceeding as in (72) we get

\[
S_X^2 \left( x^*, \nabla_x \hat{f}(x^*, y^*), 3\bar{\lambda}_1 + \frac{\mu^2}{\rho_1} \right) = \min_{0 \leq t < T} S_X^2 \left( x_t, \nabla_x \hat{f}(x_t, y_t), 3\bar{\lambda}_1 + \frac{\mu^2}{\rho_1} \right) \\
\leq \frac{1}{T} \sum_{t=0}^{T-1} S_X^2 \left( x_t, \nabla_x \hat{f}(x_t, y_t), 3\bar{\lambda}_1 + \frac{\mu^2}{\rho_1} \right) \\
\leq \left( \frac{6\bar{\lambda}_1 + 2\mu^2/\rho_1}{\rho_1} \right) \frac{\bar{\varphi}_{\text{reg}}(x_0) - \bar{\varphi}_{\text{reg}}(x_T)}{T} \\
\leq \frac{2}{T} \left( 3\bar{\lambda}_1 + \frac{\mu^2}{\rho_1} \right) \left( \bar{\varphi}(x_0) - \bar{\varphi}(x_T) + \frac{\rho_1 D^2}{2} \right) \\
\leq \frac{2}{T} \left( 3\bar{\lambda}_1 + \frac{\mu^2}{\rho_1} \right) \left( \varphi(x_0) - \varphi(x_T) + \frac{3\rho_1 D^2}{2} \right).
\]

Here for (i) we used the two inequalities \( \bar{\varphi}_{\text{reg}}(x_0) - \bar{\varphi}_{\text{reg}}(x_T) \leq \bar{\varphi}(x_0) - \max_{y \in \mathcal{Y}} \{ \hat{f}_1(x_T, y) - \frac{\rho_1}{2} \|y - \hat{y}\|^2 \} \) and \( \bar{\varphi}(x_T) \leq \max_{y \in \mathcal{Y}} \{ \hat{f}_1(x_T, y) - \frac{\rho_1}{2} \|y - \hat{y}\|^2 \} + \frac{\rho_1 D^2}{2} \) (cf. (75)); for (ii) we applied Lemma 3.1 to the
right-hand side of $|\hat{v}(x) - \varphi(x)| \leq \max_{y \in Y} |f(x,y) - \tilde{f}_1(x,y)|$. Returning to (74) we now obtain that

$$\|\nabla \hat{\varphi}_{2\lambda_1}(x^*)\| \leq 2S\left(x_t, \nabla_x \hat{f}(x_t, y_t), 3\lambda_1 + \frac{\mu^2}{\rho_1}\right) + 4\sqrt{\lambda_1 D}$$

$$(73) \leq \sqrt{\frac{8}{T}} \left(3\lambda_1 + \frac{\mu^2}{\rho_1}\right) \left(\varphi(x_0) - \varphi(x_T) + \frac{3\rho_1 D^2}{2} + 4\sqrt{\lambda_1 D}\right)$$

$$\leq \sqrt{\frac{8}{T}} \left(3\lambda_1 + \frac{\mu^2}{\rho_1}\right) \left(\varphi(x_0) - \varphi(x_T) + \frac{3\varepsilon^2}{80000\lambda_1}\right) + \frac{\varepsilon}{50}$$

$$(33) \leq \varepsilon \left(\sqrt{\frac{8}{700}} + \frac{2\sqrt{3}}{100} + \frac{1}{50}\right) < \frac{\varepsilon}{6}.$$!

Finally, by applying Theorem 3.1 we conclude that $\|\nabla \hat{\varphi}_{2\lambda_1}(x^*)\| \leq \varepsilon$ as required.

\[\square\]

C.3 Proof of Proposition 5.1

We first observe that $\tilde{f}(\cdot, y) \equiv \hat{f}_2(\cdot, y)$ is $(\sigma_0 + \sigma_2 D^2)$-Lipschitz for any $y \in Y$ as can be seen from

$$\|\nabla_x \tilde{f}(x,y)\| \leq \|\nabla_x f(x,y)\| + \|\nabla_x \hat{f}(x,y) - \nabla_x f(x,y)\| \leq \sigma_0 + \sigma_2 D^2,$$

where the last step is by Lemma 3.2. Moreover, by Lemma 3.3 $\nabla_x \tilde{f}(\cdot, y)$ is $\lambda_2$-Lipschitz, and thus $\hat{v}$ is $\lambda_2$-weakly convex. These two observations allow us to adapt the analysis of the projected subgradient method from [61, Theorem 31] (initially carried out in [62]) to Algorithm 3 in order to establish that $x^*$ is an approximate FOSP for $(\text{P}_2)$ – and after that allude to Theorem 3.1.

For brevity, let $\hat{f} \equiv f_2$ and $\hat{v} = \max_{y \in Y} \tilde{f}(x, y)$. Observe that, for any $x \in X$ and iterate $(x_t, y_t)$,

$$\hat{v}(x) \geq \hat{f}(x, y_t) \geq \hat{f}(x_t, y_t) + \langle \nabla_x \hat{f}(x_t, y_t), x - x_t \rangle - \frac{\lambda_2}{2}\|x - x_t\|^2$$

$$\geq \hat{v}(x_t) - \delta + \langle \nabla_x \tilde{f}(x_t, y_t), x - x_t \rangle - \frac{\lambda_2}{2}\|x - x_t\|^2,$$  

where the last step is by definition of ApproxMax. Moreover, let $x_t^+ = \arg\min_{x \in X} \{\hat{v}(x + \lambda_2\|x - x_t\|^2)\}$ be the proximal mapping of $x_t$, so that (cf. (9))

$$\nabla \hat{v}_{2\lambda_2}(x_t) = 2\lambda_2(x_t - x_t^+).$$

1°. Let Algorithm 3 be run with Naive = 0, i.e., $x_{t+1} = \Pi_X [x_t - \gamma_x \nabla_x \hat{f}(x_t, y_t)]$ (cf. line 6). Then

$$\hat{v}_{2\lambda_2}(x_{t+1}) \leq \hat{v}(x_{t+1}^+) + \lambda_2\|x_{t+1} - x_{t+1}^+\|^2$$

$$\leq \hat{v}(x_t^+) + \lambda_2\|x_t - \gamma_x \nabla_x \hat{f}(x_t, y_t) - x_t^+\|^2$$

$$\leq \hat{v}(x_t^+) + \lambda_2\|x_t - x_t^+\|^2 + 2\gamma_x \lambda_2(\nabla_x \tilde{f}(x_t, y_t), x_t^+ - x_t) + \gamma_x^2 \lambda_2^2(\sigma_0 + \sigma_2 D^2)^2$$

$$= \hat{v}_{2\lambda_2}(x_t) + 2\gamma_x \lambda_2(\nabla_x \tilde{f}(x_t, y_t), x_t^+ - x_t) + \gamma_x^2 \lambda_2^2(\sigma_0 + \sigma_2 D^2)^2$$

$$\leq \hat{v}_{2\lambda_2}(x_t) + 2\gamma_x \lambda_2 \left(\hat{v}(x_t^+) - \hat{v}(x_t) + \delta + \frac{\lambda_2}{2}\|x_t^+ - x_t\|^2\right) + \gamma_x^2 \lambda_2^2(\sigma_0 + \sigma_2 D^2)^2,$$

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where the second line relies on the projection lemma. Repeating this for \( t \in \{0, \ldots, T-1 \} \) we get
\[
\hat{\varphi}_{2\lambda_2}(x_T) \leq \hat{\varphi}_{2\lambda_2}(x_0) + 2\gamma_\lambda \lambda_2 \sum_{t=0}^{T-1} \left( \varphi(x_t^+) - \varphi(x_t) + \delta + \frac{\lambda_2}{2} \|x_t^+ - x_t\|^2 \right) + \gamma^2 T \lambda_2 (\sigma_0 + \sigma_2 D^2)^2,
\]
whence, by rearranging and dividing over \( 2\gamma_\lambda \lambda_2 T \),
\[
\frac{1}{T} \sum_{t=0}^{T-1} \varphi(x_t) - \hat{\varphi}(x_t^+) - \frac{\lambda_2}{2} \|x_t^+ - x_t\|^2 \leq \frac{\hat{\varphi}_{2\lambda_2}(x_0) - \hat{\varphi}_{2\lambda_2}(x_T)}{2\gamma_\lambda \lambda_2 T} + \frac{\gamma_\lambda (\sigma_0 + \sigma_2 D^2)^2}{2} + \delta. \tag{81}
\]
On the other hand, by \( \lambda_2 \)-strong convexity of \( \hat{\varphi}(\cdot) + \lambda_2 \| \cdot \|_2^2 \) and the definition of \( x_t^+ \) we get
\[
\hat{\varphi}(x_t) - \hat{\varphi}(x_t^+) - \frac{\lambda_2}{2} \|x_t^+ - x_t\|^2 = \hat{\varphi}(x_t) + \lambda_2 \|x_t - x_t\|^2 - (\hat{\varphi}(x_t^+) + \lambda_2 \|x_t^+ - x_t\|^2) + \frac{\lambda_2}{2} \|x_t^+ - x_t\|^2
\]
\[
= \hat{\varphi}(x_t) + \lambda_2 \|x_t - x_t\|^2 - \min_{x \in \mathcal{X}} \{ \hat{\varphi}(x_t^+) + \lambda_2 \|x_t^+ - x_t\|^2 \} + \frac{\lambda_2}{2} \|x_t^+ - x_t\|^2
\]
\[
\geq \lambda_2 \|x_t^+ - x_t\|^2 \geq \lambda_2 \|x_t^+ - x_t\|^2 \tag{79} \geq \frac{\|\nabla \hat{\varphi}_{2\lambda_2}(x_t)\|^2}{4\lambda_2}. \tag{82}
\]
Plugging this into (81) we arrive at
\[
E[\|\nabla \hat{\varphi}_{2\lambda_2}(x_s)\|^2] = \frac{1}{T} \sum_{t=0}^{T-1} \|\nabla \hat{\varphi}_{2\lambda_2}(x_t)\|^2 \leq \frac{2\lambda_2 \left[ \hat{\varphi}_{2\lambda_2}(x_0) - \hat{\varphi}_{2\lambda_2}(x_T) \right]}{\gamma_\lambda T} + 2\lambda_2 \gamma_\lambda (\sigma_0 + \sigma_2 D^2)^2 + 4\lambda_2 \delta \tag{a}
\]
\[
\leq \frac{2\lambda_2 \left[ \hat{\varphi}(x_0) - \hat{\varphi}(x_T) \right]}{\gamma_\lambda T} + 2\lambda_2 \gamma_\lambda (\sigma_0 + \sigma_2 D^2)^2 + 4\lambda_2 \delta \tag{b}
\]
\[
\leq \frac{2\lambda_2 \left( \Delta + \rho_2 D^3 \right)}{\gamma_\lambda T} + 2\lambda_2 \gamma_\lambda (\sigma_0 + \sigma_2 D^2)^2 + 4\lambda_2 \delta; \tag{83}
\]
here for (a) we used that \( \hat{\varphi}_{2\lambda_2}(x_0) - \min_{x \in \mathcal{X}} \{ \hat{\varphi}(x) + \lambda_2 \|x - x_0\|^2 \} \leq \hat{\varphi}(x_0) \); for (b) we used Lemma 3.1. Whence by Markov’s inequality, for any \( p \in (0, 1) \) with probability at least \( 1 - p \) one has
\[
\|\nabla \hat{\varphi}_{2\lambda_2}(x_s)\|^2 \leq \frac{4}{p} \left( \frac{\lambda_2 \left( \Delta + \rho_2 D^3 \right)}{\gamma_\lambda T} + \gamma_\lambda \lambda_2 (\sigma_0 + \sigma_2 D^2)^2 + 2\lambda_2 \delta \right) \tag{84}
\]
\[
\leq \frac{8}{p} \left( (\sigma_0 + \sigma_2 D^2) \sqrt{\frac{\lambda_2 \left( \Delta + \rho_2 D^3 \right)}{T} + \lambda_2 \delta} \right) < \frac{\varepsilon^2}{150},
\]
where for the second line we substituted \( \gamma_\lambda \) from (35) (note that this choice of \( \gamma_\lambda \) balances the two terms) and then performed an explicit calculation by plugging in \( T \) and \( \delta \) from (35). Finally, under this event (34) allows to apply Theorem 3.1 (replacing \( \varepsilon \) with \( \varepsilon/6 \)) and conclude that \( \|\nabla \varphi_{2\lambda_2}(x_s)\| \leq \varepsilon \).

2°. Conforming with the second claim we intend to prove, from now on we shall assume that
\[
24\sigma_2 D^2 \leq \varepsilon \sqrt{p},
\]
and consider the iterates of Algorithm 3 with Naive = 1—i.e., $x_{t+1} = \Pi_X [x_t - \gamma_x \nabla_x f(x_t, y_t)]$ (cf. line 4).

First of all, let us correct (80) for the discrepancy between $\nabla_x f(x_t, y_t)$ and $\nabla_x \tilde{f}(x_t, y_t)$: to this end,

$$\bar{\phi}_{2\lambda_2}(x_{t+1}) \leq \bar{\phi}(x_t) + \bar{\lambda}_2 2 \|x_{t+1} - x_t\|^2$$

(a) $\leq \bar{\phi}(x_t) + \bar{\lambda}_2 \|x_t - \gamma_x \nabla_x f(x_t, y_t) - x_t\|^2$

(b) $\leq \bar{\phi}(x_t) + \bar{\lambda}_2 (\|x_t^+ - x_t\|^2 + 2\gamma_x (\nabla_x f(x_t, y_t), x_t^+ - x_t) + \gamma_x^2 \sigma_0^2)$

(c) $= \bar{\phi}_{2\lambda_2}(x_t) + 2\bar{\lambda}_2 \gamma_x (\nabla_x f(x_t, y_t), x_t^+ - x_t) + \bar{\lambda}_2 \gamma_x^2 \sigma_0^2$

(d) $\leq \bar{\phi}_{2\lambda_2}(x_t) + 2\bar{\lambda}_2 \gamma_x (\nabla_x \tilde{f}(x_t, y_t), x_t^+ - x_t) + \frac{\bar{\lambda}_2}{2} \|x_t^+ - x_t\|^2 + \frac{\sigma_2 D^4}{2\lambda_2} + \bar{\lambda}_2 \gamma_x^2 \sigma_0^2$

(78) $\leq \bar{\phi}_{2\lambda_2}(x_t) + 2\bar{\lambda}_2 \gamma_x (\bar{\phi}(x_t^+ - \bar{\phi}(x_t)) + \delta + \bar{\lambda}_2 \|x_t^+ - x_t\|^2 + \frac{\sigma_2 D^4}{2\lambda_2} + \bar{\lambda}_2 \gamma_x^2 \sigma_0^2$.

Here (a) is by the projection lemma; (b) by the Lipschitz assumption; (c) by Cauchy-Schwarz and Lemma 3.2; (d) by $uv \leq (u^2 + v^2)/2$ for $u, v \in \mathbb{R}$. Whence we arrive at a counterpart of (81):

$$\frac{1}{T} \sum_{t=0}^{T-1} \bar{\phi}(x_t) - \bar{\phi}(x_t^+) - \bar{\lambda}_2 \|x_t^+ - x_t\|^2 \leq \frac{\bar{\phi}_{2\lambda_2}(x_0) - \bar{\phi}_{2\lambda_2}(x_T)}{2\gamma_x \lambda_2 T} + \frac{\gamma_x \sigma_0^2}{2} + \frac{\sigma_2 D^4}{2\lambda_2}. \quad (85)$$

On the other hand, in the same spirit as for (82) we get

$$\bar{\phi}(x_t) - \bar{\phi}(x_t^+) - \bar{\lambda}_2 \|x_t^+ - x_t\|^2 = \bar{\phi}(x_t) + \bar{\lambda}_2 \|x_t - x_t\|^2 - (\bar{\phi}(x_t^+) + \bar{\lambda}_2 \|x_t^+ - x_t\|^2)$$

$$= \bar{\phi}(x_t) + \bar{\lambda}_2 \|x_t - x_t\|^2 - \min_{x \in X} \{\bar{\phi}(x_t^+) + \bar{\lambda}_2 \|x_t^+ - x_t\|^2\} \quad (86)$$

$$\geq \frac{\bar{\lambda}_2}{2} \|x_t^+ - x_t\|^2$$

Plugging (86) into (85) and proceeding in the same way as in (83), we conclude that w.p. $\geq 1 - p$

$$\|\nabla \bar{\phi}_{2\lambda_2}(x_s)\|^2 \leq \frac{8}{p} \left( \frac{\bar{\lambda}_2 (\Delta + \rho_2 D^3)}{\gamma_x T} + \gamma_x \bar{\lambda}_2 \sigma_0^2 + 2\bar{\lambda}_2 \delta + \sigma_2 D^4 \right) \leq \frac{16}{p} \left( \sigma_0 \sqrt{\frac{\bar{\lambda}_2 (\Delta + \rho_2 D^3)}{T}} + \bar{\lambda}_2 \delta \right) + \frac{8\sigma_2 D^4}{p} \leq \frac{\varepsilon^2}{75} + \frac{\varepsilon^2}{72} < \frac{\varepsilon^2}{36}.$$  

In order to verify the claim, it remains to apply Theorem 3.1.  

\[\square\]

D Background on weak convexity and the Moreau envelope notion

D.1 Characterization of weakly convex functions

Let $E_x$ be a finite-dimensional Euclidean space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|x\| = \sqrt{\langle x, x \rangle}$. Moreover, we identify with $E_x$ with its dual space $E_x^*$ (in particular $\| \cdot \|^* = \| \cdot \|$). Also we let $\mathbb{R} = (-\infty, +\infty]$ and tacitly assume that all arising convex functions are lsc and proper (not identically $+\infty$).  

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Recall that any such function \( \phi : E_x \to \bar{\mathbb{R}} \) admits a variational representation as the pointwise supremum over a non-empty family of affine functions, and this is in fact a criterion (see, e.g., [99, Thm. 8.13]). The set of all affine minorants of \( \phi \) at given \( x \in E_x \) generates the corresponding set of linear functionals, called the subdifferential of \( \phi \) at \( x \) and denoted \( \partial \phi(x) \)—i.e.,

\[
\partial \phi(x) := \{ \xi \in E_x : \phi(x') \geq \phi(x) + \langle \xi, x' - x \rangle \ \forall x' \in E_x \}.
\] (87)

This set is non-empty at any interior point of effective domain \( \text{dom}(\phi) := \{ x \in E_x : \phi(x) < +\infty \} \), and its elements are called subgradients of \( \phi \) at \( x \). (For simplicity, we assume int(\( \text{dom}(\phi) \)) is non-empty.)

**Definition 3.** \( \phi : E_x \to \bar{\mathbb{R}} \) is called \( \lambda \)-weakly convex (for \( \lambda > 0 \)) if the function \( \phi(\cdot) + \frac{\lambda}{2} \| \cdot \|^2 \) is convex.

From the above characterization of (proper, lsc) convex functions it immediately follows that \( \phi \) is \( \lambda \)-weakly convex if and only if the function \( \phi_{\lambda,x}(\cdot) := \phi(\cdot) + \frac{\lambda}{2} \| \cdot - x \|^2 \) is convex for any \( x \in E_x \). Indeed, by Definition 3 \( \phi_{\lambda,0} \) is convex; meanwhile, for any \( x \in E_x \) we have

\[
\phi_{\lambda,x}(u) = \phi_{\lambda,0}(u) + \frac{\lambda}{2} \left( \| u - x \|^2 - \| u \|^2 \right).
\]

By applying (87) to \( \phi_{\lambda,0} \) we see that \( \phi_{\lambda,x} \) has subdifferential \( \partial \phi_{\lambda,x}(u) = \partial \phi_{\lambda,0}(u) - \lambda x \) (the difference being in the Minkowski sense), and thus is convex with \( \text{dom}(\phi_{\lambda,x}) = \text{dom}(\phi) \). It is then natural to define the weak subdifferential of \( \phi \) at \( x \) by

\[
\partial \phi(x) = \partial \phi_{\lambda,x}(x).
\]

Equivalently, \( \partial \phi(u) = \partial \phi_{\lambda,x}(u) + \lambda(x - u) \) for any \( u, x \in E_x \)—or, in other words,

\[
\partial \phi(x) = \{ \xi \in E_x : \phi(x') \geq \phi(x) + \langle \xi, x' - x \rangle - \frac{\lambda}{2} \| x' - x \|^2, \ \forall x, x' \in E_x \}.
\] (88)

Thus, \( \partial \phi(x) \) belongs to the Fréchet (local) subdifferential at \( x \), defined as the set of \( \xi \in E_x \) satisfying

\[
\phi(x') \geq \phi(x) + \langle \xi, x' - x \rangle + o(\| x' - x \|) \quad \text{as } x' \to x.
\] (89)

cf. [99, Def. 8.3]. Generally, the property in (89) is much weaker than that in (88); however, for weakly convex functions the weak and Fréchet subdifferentials coincide (see [99, Theorem 12.17]).

**Hessian-based criterion.** Recall Alexandrov’s theorem ([100], see also [101, Thm. 2.1]): a convex \( \phi : E_x \to \bar{\mathbb{R}} \) is twice differentiable almost everywhere (and hence \( \nabla^2 \phi \geq 0 \)) in the interior of its domain. Under \( C^1 \) regularity it is a criterion: if \( \phi : E_x \to \bar{\mathbb{R}} \) is continuously differentiable on int(dom(\( \phi \))), and \( \nabla^2 \phi \geq 0 \) almost everywhere on int(dom(\( \phi \))), then \( \phi \) is convex. From Definition 3 we see that this criterion extends to \( \lambda \)-weakly convex functions if we replace \( \nabla^2 \phi \geq 0 \) with \( \nabla^2 \phi \geq -\lambda I \).

**D.2 Upper envelope as a weakly convex function**

Let \( X \subseteq E_x \) be convex with non-empty interior. If \( \phi \to X \) is \( \lambda \)-smooth, i.e., such that \( \nabla \phi \) exists and

\[
\| \nabla \phi(x') - \nabla \phi(x) \| \leq \lambda \| x' - x \| \quad \forall x, x' \in X,
\]

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then $\phi$ is $\lambda$-weakly convex. (This follows from the Hessian-based criterion above, as $\nabla \phi(x)$ exists almost everywhere on $X$ and its eigenvalues are in $[-\lambda, \lambda]$.) More generally, the upper envelope of a family of such functions is $\lambda$-weakly convex. That is, given $f : X \times Y$ (in the setup of (P)) satisfying Assumption 1 with $\lambda < \infty$, the primal function $\varphi(x) := \max_{y \in Y} f(x, y)$ is $\lambda$-weakly convex. Indeed, note that $f_{\lambda, x'}(\cdot, y) = f(\cdot, y) + \frac{\lambda}{2} \| x - x' \|^2$ is convex for any $y \in Y$ (cf. Definition 3), therefore

$$\varphi(x) + \frac{\lambda}{2} \| x - x' \|^2 = \max_{y \in Y} f_{\lambda, x'}(x, y)$$

is convex, i.e., $\varphi$ is $\lambda$-weakly convex. Moreover, $\partial \varphi(x)$ can be expressed explicitly through $\nabla_x f(x, y)$.

**Lemma D.1.** Let $f : X \times Y \to \mathbb{R}$ be continuous and satisfy Assumption 1 with $\lambda < \infty$, $X$ be convex, and $Y$ be convex and compact. Then $\partial \varphi(x)$ is the closed convex hull of the set of active gradients:

$$\partial \varphi(x) = \overline{\text{conv}}\{\nabla_x f(x, y^*), \forall y^* \in \text{Argmax}_{y \in Y} f(x, y)\}.$$  

**Proof.** We apply Danskin’s theorem ([102, Sec. B.5]) to the function $\varphi_{\lambda, x'}(x) = \max_{y \in Y} f_{\lambda, x'}(x, y)$,

$$\partial \varphi_{\lambda, x'}(x) = \overline{\text{conv}}\{\nabla_x f_{\lambda, x'}(x, y^*), y^* \in \text{Argmax}_{y \in Y} f_{\lambda, x'}(x, y)\},$$

$$= \overline{\text{conv}}\{\nabla_x f(x, y^*) + \lambda(x - x'), y^* \in \text{Argmax}_{y \in Y} f(x, y)\}.$$  

This holds because each $f_{\lambda, x'}(\cdot, y)$ is convex, and the set of maximizers for $f_{\lambda, x'}(x, y)$ (over $y \in Y$) is the same as for $f(x, y)$. It remains to note that $\partial \varphi(x) = \partial \varphi_{\lambda, x'}(x) - \lambda(x - x')$ by Definition 3. \qed

### D.3 Characterization of $(\varepsilon, 2\lambda)$-FOSP in terms of the primal function

The following property of $(\varepsilon, 2\lambda)$-FOSP for (P) has been leveraged in the proof Proposition 3.2. It extends [91, Lem. 2.2] to the case $X \neq E_x$ with guarantee (90) in terms of $\partial \varphi$ rather than $\partial(\varphi + I_X)$.

**Proposition D.1 ([54, Proposition 5.1]).** Let $\phi : X \to \mathbb{R}$ be $\lambda$-weakly convex, then it holds that

$$\nabla \phi_{2\lambda}(x) = 2\lambda(x - x^+(x)), \text{ where } x^+(x) = \frac{1}{2\lambda} \nabla \phi(x) \text{ for any } x \in X.$$  

where $\phi_{2\lambda}$ is the $2\lambda$-Moreau envelope of $\phi$ (cf. Definition 1). Moreover, if $\|\nabla \phi_{2\lambda}(x)\| \leq \varepsilon$, then

$$\min_{\xi \in \partial \phi(x^+)} S_X(x^+, \xi, 2\lambda) \leq \varepsilon,$$  

(90)

where we define the functional $S_X(x, \xi, \lambda) := 2\lambda \max_{u \in X} \left\{ -\langle \xi, u - x \rangle - \frac{\lambda}{2} \| u - x \|^2 \right\}$ for $x, \xi \in E_x, \lambda > 0$.

### E Solving (40): quadratic optimization on a joint Krylov subspace

Following [92, Appendix A], in this section we describe a procedure for maximizing a general quadratic function $\Psi_{H, g}(y)$, cf. (36), over the intersection of $B_d(2\mathbb{R})$ and the joint Krylov subspace

$$K_{2m} = K_{2m}(H, \{g, \xi\}) := \text{span}\{H^j g, H^j \xi\}_{j=0, \ldots, m-1}\]$$

in $O(m)$ computations of the matrix-vector product $y \mapsto Hy$ and elementwise vector operations on $E_y$ (including the inner product). This procedure is given in pseudocode in Algorithm 4. It consists of two steps which we are now about to outline and discuss.
Finding the right basis for $\mathcal{K}_{2m}$. In the first stage of the algorithm (up to line 16) we construct an orthonormal basis of $\mathcal{K}_{2m}$ in which the associated linear operator (corresponding to $H$ in the initial basis), when restricted to $\mathcal{K}_{2m}$, is represented by a block tridiagonal matrix $\tilde{H}$ with blocks of size 2 (which pertains to the two vectors $\{g, \xi\}$ generating $\mathcal{K}_{2m}$). In other words, we construct $\tilde{H} \in \mathbb{R}^{2m \times 2m}$ and $Q \in \mathbb{R}^{d \times 2m}$ such that $\tilde{H} = Q^T HQ$, where $Q$ has orthonormal columns (i.e., $Q^T Q = I_{2m}$) and

$$\tilde{H} = \begin{bmatrix}
\alpha_1 & \beta_1^T \\
\beta_2 & \ddots & \ddots \\
& \ddots & \ddots & \ddots \\
& & \beta_m & \alpha_m
\end{bmatrix},$$

where each $\alpha_i \in \mathbb{R}^{2 \times 2}$ is symmetric and each $\beta_i \in \mathbb{R}^{2 \times 2}$ is upper-triangular (so $\tilde{H}$ is pentadiagonal).

In the large-scale context, where $H$ is accessed through the $y \mapsto Hy$ oracle, the one can construct such a decomposition in a computationally feasible manner via the block Lanczos process [103, 104]:

1. Form $q_1 \in \mathbb{R}^{d \times 2}$ as the Gram-Schmidt orthonormalization of $\{g, \xi\}$.
2. Compute $\alpha_1 = q_1^T H q_1$ and put $\beta_1 = 0_{2 \times 2}$. (We let $0_{a \times b}$ be the $a \times b$ zero matrix.)
3. For $t \in \{1, m - 1\}$ iterate:

$$q_{t+1} = Hq_t - q_t \alpha_t - q_{t-1} \beta_t^T,$$

$$(q_{t+1}, \beta_{t+1}) = QR(q_{t+1}^T),$$

$$\alpha_{t+1} = q_{t+1}^T H q_{t+1}.$$  

Here $QR(A)$ is the QR decomposition of a squared matrix $A$—i.e., the ordered pair $(U, R)$ of square matrices such that $A = UR$, $U$ is orthogonal, and $R$ is upper-triangular. To see that the above process is sound, observe that it incrementally builds up $q_t, \alpha_t, \beta_t$ that satisfy the matrix relations

$$H[q_1 \cdots q_t] = [q_1 \cdots q_t | q_{t+1}] egin{bmatrix} \alpha_1 & \beta_1^T \\
\beta_2 & \ddots & \ddots \\
& \ddots & \ddots & \ddots \\
& & \beta_m & \alpha_m
\end{bmatrix}$$

for $t \in \{1, \ldots, m - 1\}$, and $HQ_m = [q_1 \cdots q_m] \begin{bmatrix} 0 \\
\vdots \\
0 & \beta_m^T \\
\alpha_m \end{bmatrix}$.

while ensuring that $\beta_t$'s are upper-triangular and $q_t$'s form an orthonormal system. These relations amount to the identity $HQ = Q\tilde{H}$ with $Q = [q_1 \cdots q_m]$—and thus to $Q^T HQ = \tilde{H}$ as desired. The first part of Algorithm 4 (until line 16) implements the block Lanczos process with an explicitly rendered QR factorization step.

Passing to the equivalent problem and solving it. Once $\tilde{H}$ and $Q$ have been constructed, we compute $\tilde{g} = Q^T g$ and the top eigenvalue $\omega_0$ of $\tilde{H}$ (cf. line 18). The first of these two steps amounts to $O(m)$ inner products, and the second one can be done in $O(m)$ arithmetic operations since $\tilde{H}$ is pentadiagonal. Using these data, we recast the initial maximization problem (cf. (40)) by changing the variable $y$ to $z = Q^T y \in \mathbb{R}^{2m}$—i.e., we pass to

$$\max_{z \in \mathbb{B}_{2m}(2\tilde{g})} \frac{1}{2} z^T \tilde{H} z + \tilde{g}^T z.$$  

(91)
The new problem is equivalent to (40) in the following sense: if \( z \) is \( \delta \)-suboptimal in (91), then \( y = Qz \) is \( \delta \)-suboptimal in (40). Thus we can focus on (91) and then recover a candidate solution to (40) by multiplying with \( Q \) (which has essentially the same computational cost as forming \( \bar{g} = Q^\top g \) from \( g \)).

The final part of Algorithm 4 (after line 18) consists of solving (91). Here we proceed as follows.

1. We first test if \( \bar{H} \) is negative-semidefinite (by inspecting \( \text{sign}(\omega_0) \)) and, if yes, whether \( B_{2m}(2\mathbb{R}) \) contains an unconstrained maximizer \( z_0 \)—an optimal solution to the linear system \( \bar{H}z + \bar{g} = 0 \).

If the answers to both questions are positive, we in fact found a maximizer on \( B_{2m}(2\mathbb{R}) \), so we output \( Qz_0 \) and terminate. Note that here if \( \bar{H} \) is negative-definite, then \( z_0 \) exists, is unique, and is given by \( -\bar{H}^{-1}\bar{g} \); otherwise (i.e., when \( \omega_0 = 0 \), \( \bar{H} \) is rank-deficient, so \( \bar{H}z + \bar{g} = 0 \) is solvable whenever \( \bar{g} \) is in the range of \( \bar{H} \), and then the least-norm solution is given by \( z_0 = -\bar{H}^\top \bar{g} \), where \( \bar{H}^\top \) is the generalized inverse of \( \bar{H} \) (i.e., \( \bar{H}^\top \) has the same eigenspaces as \( \bar{H} \) and reciprocal nonzero eigenvalues). Note that \( \bar{H}^\top \) can be computed in \( O(m) \) arithmetic operations (a.o.).

2. If the above test failed (and thus we have not terminated), then \( \Psi_{\bar{H},\bar{g}} \) has a unique maximizer at the boundary of \( B_{2m}(2\mathbb{R}) \), and this maximizer also solves the strictly concave problem

\[
\arg\max_{z \in B_{2m}(2\mathbb{R})} 0.5 z^\top \bar{H} - \omega I z + \bar{g}^\top z
\]

for the unique value of \( \omega > \max\{\omega_0, 0\} \) such that \( \|z_\omega\| = R \) (see, e.g., [105, Corollary 7.2.2]).

Thus, we can proceed by a root-finding method of choice: evaluation of the mapping \( \omega \mapsto \|z_\omega\| \) amounts to solving a well-defined linear system with a pentadiagonal matrix and thus can be done in \( O(m) \) a.o.; thus, we find a high-accuracy maximizer—exact one in floating point arithmetic—in \( O(m) \) a.o. In particular, a good option is practice is to seek for the root of \( \psi(\omega) - 1/R \) with \( \psi(\omega) = 1/\|z_\omega\| \) by Newton’s method as discussed in [105, Chapter 7].

**Time and memory expenses.** By carefully inspecting Algorithm 4 one may verify that the Lanczos process takes \( O(1) \) matrix-vector products and elementwise vector operations in \( E_y \) per iteration, so \( O(m) \) such operations in total. Computation of \( \bar{g} \) takes \( O(m) \) inner products, and those of \( \omega_0, \bar{H}^\top, \) and \( (\bar{H} - \omega I)^{-1} \) each take \( O(m) \) a.o. Finally, as we previously discussed, a root finder performs \( O(1) \) evaluations of \( \omega \) in the floating-point model of computation. Hence, the total **computational cost** of Algorithm 4 in a.o. is

\[
O(m(d_y + \text{time}(HVP))),
\]

where \( \text{time}(HVP) \) is the cost of computing \( y \mapsto Hy \), and \( d_y = \text{dim}(E_y) \). As for the **memory cost**, it is dominated by

\[
O(md_y + \text{mem}(HVP)),
\]

where the first term allows to explicitly store \( Q \) (which we need in order to report the final solution \( Qz_0 \) or \( Qz_\omega \)), and the second term is the memory needed for computing \( y \mapsto Hy \) (i.e., \( \text{mem}(HVP) \leq d_y^2 \) or less if \( H \) has special structure). Note also that we can replace the term \( md_y \) in (93) with \( m+d_y = O(d_y) \) if Algorithm 4 in the situation where there is no need to report the solution in \( E_y \) (e.g., if only the optimal value of is of interest). Indeed, apart from reporting the final solution, we only need \( Q \) when computing \( \bar{g} = Q^\top g = [q_1^\top g; \ldots; q_m^\top g] \)—but this can be done incrementally during the Lanczos process, without ever having to memorize more than \( O(1) \) columns of \( Q \) at once.
Algorithm 4 Approximate maximization of a quadratic $\Psi_{H,g}(y) = \frac{1}{2}y^THy + g^Ty$ on a ball $Y = B_d(2R)$

Input: $g \in \mathbb{R}^d$, oracle $y \mapsto Hy$, $R > 0$, $m \in \mathbb{N}$

1: $\triangleright$ Restrict search to the joint Krylov subspace $\mathcal{K}_{2m}(H,\{g,\xi\})$, cf. (38), by using Proposition 5.2.

2: $\triangleright$ Blockwise Lanczos iterations for $\mathcal{K}_{2m} = \mathcal{K}_{2m}(H,\{g,\xi\})$ with blocks of size 2 ([92, Appendix A]); results in a block tridiagonal matrix $\tilde{H} = Q^THQ \in \mathbb{R}^{2m \times 2m}$ with $Q \in \mathbb{R}^{d \times 2m}; Q^TQ = I$.

3: $\triangleright$ $q_t \in \mathbb{R}^{d \times 2}$, $\alpha_t, \beta_t \in \mathbb{R}^{2 \times 2}$

4: $q_0 = 0_{d \times 2}$

5: Draw $\xi \sim \text{Uniform}(S^{d-1})$; $u = \frac{1}{\|g\|}g$; $w = \xi - (\xi,u)u$; $v = \frac{1}{\|w\|}w$

6: $q_1 = [u,v]$; $\alpha_1 = q_1^THq_1$ $\triangleright q_1$ is an orthonormalization of $[g,\xi]$

7: for $t \in \{1, \ldots, m-1\}$ do

8: $q_{t+1} = Hq_t - q_t\alpha_t - q_t\beta_t^T$ $\triangleright \beta_1$ never used since $q_0 = 0$

9: $\triangleright$ Compute $(q_{t+1}^T, \beta_{t+1}) = QR(q_{t+1}^T)$ via Gram-Schmidt:

10: $[u',v'] = q_{t+1}^T$ $\triangleright$ extract columns from $q_{t+1}^T$

11: $u = \frac{1}{\|u'\|}u'$; $w = v' - (v',u)u$; $v = \frac{1}{\|w\|}w$

12: $q_{t+1} = [u,v]$; $\beta_{t+1} = \begin{bmatrix} (u',u) & (v',u) \\ 0 & (v',v) \end{bmatrix}$ $\triangleright q_{t+1}$ is an orthonormalization of $q_{t+1} = q_{t+1}^T\beta_{t+1}$

13: $\alpha_{t+1} = q_{t+1}^THq_{t+1}$

14: end for

15: $Q = [q_1 \ldots q_m]$ $\triangleright$ columns of $Q$ form an orthonormal basis for $\mathcal{K}_{2m}$

16: $\tilde{H} = \begin{bmatrix} \alpha_1 & \beta_2^T & \cdots & \beta_m^T \\ \beta_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \alpha_m \end{bmatrix}$ $\triangleright \tilde{H} = Q^THQ \in \mathbb{R}^{2m \times 2m}$ is block tridiagonal with $2 \times 2$ blocks

17: $\tilde{g} = Q^Tg$

18: $\omega_0 = \lambda_{\max}(\tilde{H})$

19: $\triangleright$ Eliminate the interior case: $\Psi_{H,g}$ restricted to $\mathcal{K}_{2m}$ is concave and maximized inside the ball:

20: if $\omega_0 < 0$ then $\triangleright \tilde{H}$ is full-rank, so the unconstrained maximizer $z_0$ exists and is unique

21: $z_0 = -\tilde{H}^{-1}\tilde{g}$

22: end if

23: if $\omega_0 = 0$ and $\tilde{H}(\tilde{H}^T)^+\tilde{H}\tilde{g} = \tilde{g}$ then $\triangleright \tilde{H}$ is rank-deficient, but $\tilde{g}$ is in its range

24: $z_0 = -\tilde{H}^+\tilde{g}$

25: end if

26: if $z_0$ is defined and $\|z_0\| \leq R$ then

27: return $Qz_0$

28: end if

29: $\triangleright$ If we have not terminated yet, the maximizer is on the boundary ([105, Corollary 7.2.2])

30: Find $\omega > \max\{\omega_0, 0\}$ such that the norm of $z_\omega = -(\tilde{H} - \omega I)^{-1}\tilde{g}$ is $R$

31: return $Qz_\omega$
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