Non degeneracy of critical points of the Robin function with respect to deformations of the domain

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Abstract

We show a result of genericity for non degenerate critical points of the Robin function with respect to deformations of the domain.

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1 Introduction

Let Ω be a smooth bounded domain in \( \mathbb{R}^N \), \( N \geq 2 \). The Green function of the Laplace operator vanishing at the boundary \( \partial \Omega \) is of the form

\[
G_y(x) = \frac{1}{\omega_N} \left[ \Gamma_y(x) - H_y(x) \right], \quad x, y \in \Omega,
\]

with where \( \omega_N \) denotes the surface area of the unit sphere in \( \mathbb{R}^N \). The singular part \( \Gamma_y \) is given by \( \Gamma_y(x) = \Gamma(|x-y|) \)

\[
\Gamma(|x-y|) = -\ln |x-y| \quad \text{if} \quad N = 2 \quad \text{and} \quad \Gamma(|x-y|) = \frac{1}{N-2} |x-y|^{2-N} \quad \text{if} \quad N \geq 3.
\]

The regular part \( H_y \) is a harmonic function with the same boundary value as the singular part, i.e. for any \( y \in \Omega \)

\[
\begin{cases}
\Delta_x H_y(x) = 0 & \text{if } x \in \Omega \\
H_y(x) = \Gamma_y(x) & \text{if } x \in \partial \Omega.
\end{cases}
\]

The Robin function of \( \Omega \) is defined by \( t(x) := t^\Omega(x) := H_x(x), \quad x \in \Omega \).

This function plays an important role in various fields of the mathematics, e.g., geometric function theory, capacity theory, concentration problems (see [2] and the references therein).

In particular, existence and uniqueness of solutions of some critical problems is strictly dependent on the non degeneracy of critical points of the Robin function (see, for example, [1, 7, 12, 14]. Non degenerate critical points of the Robin function plays also a crucial role in studying existence and uniqueness of solutions of the Gelfand’s problem (see for example [3, 5, 6, 9, 11]).

As far as we know, the only results about non degeneracy of critical points of the Robin function are in [4] and [10]. In [4] the authors show that the Robin function of a smooth bounded and convex domain of \( \mathbb{R}^2 \) has a unique critical point which is non degenerate. In [10] the author proves that the origin is a non degenerate critical point of the Robin function of a smooth bounded domain of \( \mathbb{R}^N \) which is symmetric with respect to the origin and convex in any directions \( x_1, \ldots, x_N \).

Here we prove that for most domains the critical points of the Robin function are non degenerate.
Let $\Omega \subset \mathbb{R}^N$ be a domain of class $C^k$ with $k \geq 4$ and $N \geq 2$. We consider the domain $\Omega_\theta := (I + \theta)\Omega$ given by the deformation $I + \theta$. Here $I$ is the identity map on $\mathbb{R}^N$.

We are interested in studying the non degeneracy of the critical points of the Robin function of the domain $\Omega_\theta$ with respect to the parameter $\theta$.

Let $\mathcal{E}^k$ be the vector space of all the $C^k$ applications $\theta : \mathbb{R}^N \to \mathbb{R}^N$ such that
\[
\|\theta\|_k := \sup_{x \in \mathbb{R}^N} \max_{0 \leq |\alpha| \leq k} \left| \frac{\partial^\alpha \theta(x)}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}} \right| < +\infty.
\]
\(\mathcal{E}^k\) is a Banach space equipped with the norm $\| \cdot \|_k$. Let $\mathcal{B}_\rho := \{ \theta \in \mathcal{E}^k : \|\theta\|_k \leq \rho \}$ be the ball in $\mathcal{E}^k$ centered at 0 with radius $\rho$. We will prove the following result.

**Theorem 1.1.** The set $\mathcal{A} := \{ \theta \in \mathcal{B}_\rho : \text{all the critical points of the Robin function of the domain } \Omega_\theta \text{ are non degenerate} \}$ is a residual (hence dense) subset of $\mathcal{B}_\rho$, provided $\rho$ is small enough.

To get Theorem [11] we use an abstract transversality theorem previously used by Quinn [13], Saut and Temam [15] and Uhlenbeck [16]. The strategy in our work is similar to the one used by Saut and Temam in [15] to get some generic property with respect to the domain of the solutions to certain semilinear elliptic equations. In our case we need some new delicate estimates which involve the derivative of Robin function with respect to the variation of the domain.

The paper is organized as follows. In Section 2 we set the problem and we prove the main result. All the technical results are proved in Section 3 and in Section 4.

## 2 Setting of the problem and proof of the main result

First of all let us recall some useful properties of the Robin function (see [2]).

**Remark 2.1.** If $\Omega$ is of class $C^{2,\alpha}$ then the Robin function $t \in C^{2,\alpha}(\overline{\Omega})$ and it holds
\[
\nabla t(x) = 2\nabla_x H_\rho(x)_{|y=x} \quad \text{and} \quad \frac{\partial^2 t}{\partial x_i \partial x_j}(x) = 4 \frac{\partial^2 H_\rho}{\partial x_i \partial x_j}(x)_{|y=x}. (5)
\]

Given $\Omega \subset \mathbb{R}^N$, $N \geq 2$ of class $C^k$ with $k \geq 3$, we consider the domain $\Omega_\theta := (I + \theta)\Omega$ with $\theta \in \mathcal{B}_\rho$. It is well known that we can choose $\rho$ positive and small enough such that if $\theta \in \mathcal{B}_\rho$ then the map $I + \theta : \overline{\Omega} \to (I + \theta)\overline{\Omega}$ is a diffeomorphism of class $C^k$. We set $I + \gamma = (I + \theta)^{-1}$.

**Remark 2.2.** Since, by definition $(I + \theta) \circ (I + \gamma) = I$ we have that $\gamma(z) = -\theta(z + \gamma(z))$. Moreover, it holds
\[
[I + \theta'](z + \gamma(z))(h + \gamma'(z)(h)) = h \quad \forall \, h \in \mathbb{R}^N.
\]
Then we have
\[
[I + \theta'(z + \gamma(z))] \circ \gamma'(z) = -\theta'(z + \gamma(z))
\]
which implies
\[
\gamma'(z) = -[I + \theta'(x)]^{-1} \circ \theta'(x) = \sum_{i \geq 0} (-1)^{i+1} [\theta'(x)]^{i+1} \quad \text{where } x = z + \gamma(z). (7)
\]

Moreover by (6)
\[
[I + \theta'(z + \gamma(z))] \circ \gamma''(z)(h)(k) = -\theta''(z + \gamma(z)) (h + \gamma'(z)(h))(k + \gamma'(z)(k)) \quad \forall \, h, k \in \mathbb{R}^N
\]
Then we have if $x = z + \gamma(z)$
\[
\gamma''(z)(h)(k) = \sum_{i \geq 0} (-1)^i [\theta'(x)]^i [-\theta''(x) (h + \gamma'(z)(h))(k + \gamma'(z)(k))] = -\theta''(x) (h + \gamma'(z)(h))(k + \gamma'(z)(k)) - \sum_{i \geq 1} (-1)^i [\theta'(x)]^i \theta''(x) (h + \gamma'(z)(h))(k + \gamma'(z)(k)).
\]

2
Given $y \in \Omega_\theta$, we consider the unique function $v_\theta^y$ solution of the problem
\[
\begin{cases}
\Delta v_\theta^y(z) = 0 & \text{if } z \in \Omega_\theta \\
v_\theta^y(z) = \Gamma (|z - y|) & \text{if } z \in \partial \Omega_\theta.
\end{cases}
\tag{8}
\]

More precisely, $v_\theta^y$ is the regular part of the Green’s function of the domain $\Omega_\theta$. We have that $v_\theta^y \in C^{2,\alpha}(\overline{\Omega})$ because $k \geq 3$. If $\xi \in \Omega$ is such that $\xi + \theta(\xi) = y$, we define the function $\tilde{v}_\xi^0 \in C^{2,\alpha}(\Omega)$ by
\[\tilde{v}_\xi^0(x) := v_\theta^y(x + \theta(x)) = v_\theta^y(z).
\tag{9}
\]

The function $\tilde{v}_\xi^0$ is the unique solution of the following problem
\[
\begin{cases}
\sum_{i,j,s=1}^N \frac{\partial^2 \tilde{v}_\xi^0}{\partial x_i \partial x_j} \left[ \delta_{s+i} + \frac{\partial \gamma_i}{\partial z_s} x + \theta(x) \right] = 0 & \text{if } x \in \Omega \\
\tilde{v}_\xi^0(x) = \Gamma (|x - \xi + \theta(x) - \theta(\xi)|) & \text{if } x \in \partial \Omega.
\end{cases}
\tag{10}
\]

When $\theta = 0$ we obviously have that $\tilde{v}_\xi^0$ is the unique solution of the problem
\[
\begin{cases}
\Delta x \tilde{v}_\xi^0 = 0 & \text{if } x \in \Omega \\
\tilde{v}_\xi^0(x) = \Gamma (|x - \xi|) & \text{if } x \in \partial \Omega.
\end{cases}
\tag{11}
\]

Remark 2.3. It is easy to see that there exists a $C^3$-extension of the function $\Gamma(|z - y|)$ for $z \in \partial \Omega_\theta$ on the domain $\Omega_\theta$, $z \to \Gamma(|z - y|)\chi_d(|z - y|)$. Here the smooth cut off function $\chi_d$ is such that
\[\chi_d(s) = 0 \text{ if } 0 < s < d, \quad \chi_d(s) = 1 \text{ if } s > 2d, \quad |\chi'(s)| < \frac{c}{d}, \quad |\chi''(s)| < \frac{c}{d^2}
\]
where $d = \text{dist}(y, \partial \Omega_\theta)/3$, for some constant $c > 1$.

Since $\tilde{v}_\xi^0$ solves (10), by maximum principle and standard elliptic regularity theory (see Theorem 6.6, [8]) we get
\[
\|\tilde{v}_\xi^0\|_{C^{2,\alpha} (\overline{\Omega})} \leq c \left[ \|\tilde{v}_\xi^0\|_{C^0 (\overline{\Omega})} + \|\varphi\|_{C^{2,\alpha} (\overline{\Omega})} \right] \leq c \sup_{x \in \partial \Omega} \Gamma (|x - \xi + \theta(x) - \theta(\xi)|) + \|\varphi\|_{C^{2,\alpha} (\overline{\Omega})},
\]
where
\[\varphi(x) := \Gamma (|x - \xi + \theta(x) - \theta(\xi)|) \chi_d(|x - \xi + \theta(x) - \theta(\xi)|).
\]

It is important to point out that by standard regularity theory (see Theorem 6.6, [8]) we also get that $\tilde{v}_\xi^0 \in C^{3,\alpha}(\Omega)$ if $k \geq 4$.

Let us establish some properties of the function $\tilde{v}_\xi^0$.

It is useful to point out that when $\theta = 0$, for any $p = 1, \ldots, N$ the function $w_\xi^0 := \frac{\partial}{\partial x_p} \tilde{v}_\xi^0$ is the unique solution of the following problem
\[
\begin{cases}
\Delta x w_\xi^0 = 0 & \text{if } x \in \Omega \\
w_\xi^0(x) = \frac{x_p - \xi_p}{|x - \xi|^N} & \text{if } x \in \partial \Omega.
\end{cases}
\tag{12}
\]

We are interested in studying the non degeneracy of the critical points of the Robin function of the domain $\Omega_\theta$, namely by (5) and (8) the points $z \in \Omega_\theta$ such that
\[0 = \nabla_z \xi \xi^\theta (z) = 2 \nabla_z v_\theta^y(z)|_{y=z} \tag{13}\]

This is equivalent to study the non degeneracy of $x \in \Omega$ such that $0 = \nabla_x \tilde{v}_\xi^0(x)|_{\xi=x}$. Thus, we are led to consider the map $F : \Omega \times \mathcal{B}_p \to \mathbb{R}^N$ defined by
\[F(x, \theta) := \nabla_x \tilde{v}_\xi^0(x)|_{\xi=x} \tag{14}\]

By Remark 2.3 and Lemma 4.2 $F$ is a $C^1$—map.

We shall apply the following abstract transversality theorem to the map $F$ (see [13, 15, 16]).
Theorem 2.4. Let $X,Y,Z$ be three Banach spaces and $U \subset X$, $V \subset Y$ open subsets. Let $F : U \times V \to Z$ be a $C^\alpha$-map with $\alpha \geq 1$. Assume that

1) for any $y \in V$, $F(\cdot,y) : U \to Z$ is a Fredholm map of index $l$ with $l \leq \alpha$;

2) $0$ is a regular value of $F$, i.e. the operator $F'(x_0,y_0) : X \times Y \to Z$ is onto at any point $(x_0,y_0)$ such that $F(x_0,y_0) = 0$;

3) the map $\pi \circ \iota : F^{-1}(0) \to Y$ is $\sigma$-proper, i.e. $F^{-1}(0) = \cup_{n=1}^\infty C_n$ where $C_n$ is a closed set and the restriction $\pi \circ i_{|C_n}$ is proper for any $n$; here $i : F^{-1}(0) \to Y$ is the canonical embedding and $\pi : X \times Y \to Y$ is the projection.

Then the set $\Theta := \{ \eta \in V : 0$ is a regular value of $F(\cdot,y) \}$ is a residual subset of $V$, i.e. $V \setminus \Theta$ is a countable union of closed subsets without interior points.

Proof of the main result. We are going to apply the transversality theorem [2,4] to the map $F$ defined by (14). In this case we have $X = Z = \mathbb{R}^N$, $Y = \mathcal{E}^k$, $U = \Omega \subset \mathbb{R}^N$ and $V = \mathcal{B}_\rho \subset \mathcal{C}^k$, where $\rho$ is small enough. Since $X = Z$ is a finite dimensional space, it is easy to check that for any $\theta \in \mathcal{B}_\rho$, the map $x \to F(x,\theta)$ is a Fredholm map of index $0$ and then assumption 1) holds. As far as it concerns assumption 3), we have that

$$F^{-1}(0) = \cup_{n=1}^\infty C_n,$$

where $C_n := \{ \Omega_n \times \mathcal{B}_{\rho-\frac{1}{n}} \} \cap F^{-1}(0)$ and $\Omega_n := \{ x \in \Omega : \text{dist}(x,\partial \Omega) \leq 1/n \}$.

Using the compactness of $\Omega_n$, we can show that the restriction $\pi \circ i_{|C_n}$ is proper, namely if the sequence $(\theta_n) \subset \mathcal{B}_{\rho-\frac{1}{n}}$ converges to $\psi_0$ and the sequence $(x_n) \subset \Omega_n$ is such that $F(x_n,\theta_n) = 0$ then there exists a subsequence of $(x_n)$ which converges to $x_0 \in \Omega_n$ and $F(x_0,\psi_0) = 0$.

To prove that assumption 2) holds we will show in Lemma 3.1 that if $(\bar{x},\bar{\theta}) \in \Omega \times \mathcal{B}_\rho$ is such that $F(\bar{x},\bar{\theta}) = \nabla_x \bar{v}_{\bar{\theta}}(\bar{x})|_{x=\bar{x}} = 0$ the map $F'_\theta(\bar{x},\bar{\theta}) : \mathcal{E}^k \to \mathbb{R}^N$ defined by $\theta \to D_\theta \nabla_x \bar{v}_{\bar{\theta}}(x)|_{\theta=\bar{\theta},x=\bar{x}}[\theta]$ is surjective.

Finally, we can apply the transversality theorem [2,4] and we get that the set

$$\mathcal{A} := \{ \theta \in \mathcal{B}_\rho : F'_\theta(x,\theta) : \mathbb{R}^N \to \mathbb{R}^N \text{ is invertible at any point } (x,\theta) \text{ such that } F(x,\theta) = 0 \} = \{ \theta \in \mathcal{B}_\rho : \text{the critical points of the Robin function of the domain } \Omega_\theta \text{ are nondegenerate} \}$$

is a residual, and hence dense, subset of $\mathcal{B}_\rho$.

3 0 is a regular value of $F$

In this section we show that 0 is a regular value of the map $F$ defined by (14).

Lemma 3.1. The map $\theta \to F'_\theta(\bar{x},\bar{\theta})[\theta]$ is onto on $\mathbb{R}^N$ for any $(\bar{x},\bar{\theta}) \in \Omega \times \mathcal{B}_\rho$ such that $F(\bar{x},\bar{\theta}) = 0$.

Proof. Let us fix $(\bar{x},\bar{\theta}) \in \Omega \times \mathcal{B}_\rho$ such that $F(\bar{x},\bar{\theta}) = 0$. We want to show that given $e(1),\ldots,e(N)$ the canonical base in $\mathbb{R}^N$, for any $i = 1,\ldots,N$ there exists $\theta \in \mathcal{C}^e$ such that $F'(\bar{x},\bar{\theta})[\theta] = e_i$. We point out that the oneteness of the map $\theta \to F'_\theta(\bar{x},\bar{\theta})[\theta]$ is invariant with respect to the change of variables $\eta = (I + \bar{\theta})(x)$. We have that

$$F'_\theta(\bar{x},\bar{\theta})[\theta] = \left( \frac{\partial}{\partial x_1} D_\theta \bar{v}_{\bar{\theta}}[\theta](\bar{x}),\ldots,\frac{\partial}{\partial x_N} D_\theta \bar{v}_{\bar{\theta}}[\theta](\bar{x}) \right)$$

because $\frac{\partial}{\partial x_\eta} D_\theta \bar{v}_{\bar{\theta}}[\theta](\bar{x}) = D_\theta \frac{\partial}{\partial x_\eta} \bar{v}_{\bar{\theta}}[\theta](\bar{x})$ as it is easy to verify.

Let $\bar{\eta} = \bar{x} + \bar{\theta}(\bar{x}) \in \Omega_{\bar{\theta}}$. By (8), (9), (10) and Lemma 4.2 we deduce that $\bar{v}_{\bar{\theta}}(\eta) = \bar{v}_{\bar{\theta}}(x + \bar{\theta}(x)) = \bar{v}_{\bar{\theta}}(x)$ is the unique solution of

$$\begin{cases}
\Delta_{\bar{\eta}} \bar{v}_{\bar{\theta}} = 0 & \text{if } \eta \in \Omega_{\bar{\theta}} \\
v_{\bar{\theta}}(\eta) = \Gamma(|\eta - \bar{\eta}|) & \text{if } \eta \in \partial \Omega_{\bar{\theta}}.
\end{cases}$$
and \( \nabla_\eta \psi^\theta_{\bar{\eta}}(\eta)|_{\eta=\bar{\eta}} = 0 \).

We consider the deformation \( I + \bar{\theta} + \theta = (I + \alpha)(I + \bar{\theta}) \), where \( \theta = \alpha(I + \bar{\theta}) \) and the domain \((I + \bar{\theta} + \theta)\Omega = (I + \alpha)(I + \bar{\theta})\Omega \). We set
\[
\bar{\eta} := (I + \bar{\theta})\bar{x} \quad \text{and} \quad \bar{z} := (I + \alpha)\bar{\eta}.
\]

Let \( \psi^\theta_{\bar{\eta}} \) be the unique solution of
\[
\begin{cases}
\Delta_z w(z) = 0 & \text{if } z \in \Omega_{\bar{\eta} + \theta} \\
w(z) = \Gamma(\eta - z) & \text{if } z \in \partial\Omega_{\bar{\eta} + \theta}.
\end{cases}
\]

Then we set
\[
\psi^\theta_{\bar{\eta}}(\eta) = \psi^\theta_{\bar{\eta}}(\eta + \alpha(\eta)) = \psi^\theta_{\bar{\eta}}(\eta) = \psi^\theta_{\bar{\eta}}(x + \bar{\theta}(x)) = \psi^\theta_{\bar{\eta}}(x).
\]

We immediately obtain that
\[
D_{\theta}\psi^\theta_{\bar{\eta}}|_{\theta=0}[\beta](\eta) = D_{\theta}\psi^\theta_{\bar{\eta}}|_{\theta=0}[\beta](x) \quad \text{with } \eta = x + \bar{\theta}(x).
\]

By (18) we have that given \( \theta^{(1)}, \ldots, \theta^{(N)} \) the \( N \) vectors
\[
\nabla_x D_{\theta}\psi^\theta_{\bar{\eta}}|_{\theta=0}[\theta^{(1)}](\bar{x}), \ldots, \nabla_x D_{\theta}\psi^\theta_{\bar{\eta}}|_{\theta=0}[\theta^{(N)}](\bar{x})
\]
are linearly independent if and only if the \( N \) vectors
\[
\nabla_x D_{\theta}\psi^\theta_{\bar{\eta}}|_{\theta=0}[\theta^{(1)}](\bar{\eta}), \ldots, \nabla_x D_{\theta}\psi^\theta_{\bar{\eta}}|_{\theta=0}[\theta^{(N)}](\bar{\eta})
\]
are linearly independent.

At this stage our aim is to find \( \theta^{(1)}, \ldots, \theta^{(N)} \) so that the \( N \) vectors
\[
\nabla_x D_{\theta}\psi^\theta_{\bar{\eta}}|_{\theta=0}[\theta^{(1)}](\bar{\eta}), \ldots, \nabla_x D_{\theta}\psi^\theta_{\bar{\eta}}|_{\theta=0}[\theta^{(N)}](\bar{\eta})
\]
are linearly independent. First of all we point out that by Lemma 41 the function \( w^\theta_{\bar{\eta}}(\alpha)(\cdot) := D_{\theta}\psi^\theta_{\bar{\eta}}|_{\theta=0}[\theta](\cdot) \) is the unique solution of the problem
\[
\begin{cases}
\Delta_\eta \psi^\theta_{\bar{\eta}} - \sum_{i,j=1}^N \frac{\partial^2 \psi^\theta_{\bar{\eta}}}{\partial \eta_i \partial \eta_j}(\eta) \left[ \frac{\partial \alpha_j}{\partial \eta_i}(\eta) + \frac{\partial \alpha_i}{\partial \eta_j}(\eta) \right] - \sum_{j=1}^N \frac{\partial \psi^\theta_{\bar{\eta}}}{\partial \eta_j}(\eta) \Delta_\eta \alpha_j(\eta) = 0 & \text{if } \eta \in \Omega_{\bar{\eta}} \\
w(\eta) = \sum_{i=1}^N \frac{\eta_i - \bar{\eta}_i}{|\eta - \bar{\eta}|^N} (\alpha_i(\eta) - \alpha_i(\bar{\eta})) & \text{if } \eta \in \partial\Omega_{\bar{\eta}}.
\end{cases}
\]

Here \( \alpha = \theta(I + \bar{\theta})^{-1} \) and \( \psi^\theta_{\bar{\eta}} \) is the unique solution of
\[
\begin{cases}
\Delta_\eta \psi^\theta_{\bar{\eta}} = 0 & \text{if } \eta \in \Omega_{\bar{\eta}} \\
\psi^\theta_{\bar{\eta}}(\eta) = \Gamma(\eta - \bar{\eta}) & \text{if } \eta \in \partial\Omega_{\bar{\eta}}.
\end{cases}
\]

We remark that by standard regularity theory (see also Remark 3) it follows
\[
\|\psi^\theta_{\bar{\eta}}\|_{C^3(\Omega_{\bar{\eta}})} \leq c(\bar{\theta}, \bar{\eta}),
\]
for some positive constant depending only on \( \bar{\theta} \) and \( \bar{\eta} \).

Moreover, we also get that the function \( \eta \rightarrow \frac{\partial}{\partial \eta_p} D_{\theta}\psi^\theta_{\bar{\eta}}|_{\theta=0}[\theta](\eta) = \frac{\partial}{\partial \eta_p} w^\theta_{\bar{\eta}}(\alpha)(\eta) \) for \( p = 1, \ldots, N \) is the unique solution of the problem
\[
\begin{cases}
\Delta_\eta \frac{\partial}{\partial \eta_p} w^\theta_{\bar{\eta}}(\alpha)(\eta) = \frac{\partial}{\partial \eta_p} \left\{ \sum_{i,j=1}^N \frac{\partial^2 \psi^\theta_{\bar{\eta}}}{\partial \eta_i \partial \eta_j}(\eta) \left[ \frac{\partial \alpha_j}{\partial \eta_i}(\eta) + \frac{\partial \alpha_i}{\partial \eta_j}(\eta) \right] - \frac{\partial \psi^\theta_{\bar{\eta}}}{\partial \eta_j}(\eta) \Delta_\eta \alpha_j(\eta) \right\} & \text{if } \eta \in \Omega_{\bar{\eta}} \\
\frac{\partial}{\partial \eta_p} w^\theta_{\bar{\eta}}(\alpha)(\eta) = \frac{\partial}{\partial \eta_p} \left\{ \sum_{i=1}^N \frac{\eta_i - \bar{\eta}_i}{|\eta - \bar{\eta}|^N} (\alpha_i(\eta) - \alpha_i(\bar{\eta})) \right\} & \text{if } \eta \in \partial\Omega_{\bar{\eta}}.
\end{cases}
\]
Therefore we look for $\alpha^{(1)}, \ldots, \alpha^{(N)}$ such that the $N$ vectors
\[
\nabla_\eta w_\eta^b(\alpha^{(1)})[\bar{\eta}], \ldots, \nabla_\eta w_\eta^b(\alpha^{(N)})[\bar{\eta}]
\]
are linearly independent. Using the Green’s representation formula by (21) we get
\[
\frac{\partial}{\partial \eta_p} w_\eta^b(\alpha)[\bar{\eta}] = \int_{\partial \Omega_\theta} \frac{\partial}{\partial \eta_p} \left\{ \sum_{i=1}^{N} \frac{\bar{\eta}_i - \eta_i}{|\eta - \bar{\eta}|^N} \left( \alpha_i(\eta) - \alpha_i(\bar{\eta}) \right) \right\} \frac{\partial G}{\partial \eta_p}(\eta, \bar{\eta})d\sigma \\
+ \int_{\Omega_\theta} \frac{\partial}{\partial \eta_p} \left\{ \sum_{i=1}^{N} \frac{\partial^2 \bar{v}_\eta^b}{\partial \eta_i \partial \eta_j}(\eta) \left[ \frac{\partial \alpha_j}{\partial \eta_i}(\eta) + \frac{\partial \alpha_i}{\partial \eta_j}(\eta) \right] - \sum_{j=1}^{N} \frac{\partial \bar{v}_\eta^b}{\partial \eta_j}(\eta) \Delta_\eta \alpha_j(\eta) \right\} G(\eta, \bar{\eta})d\eta. \tag{22}
\]
We now choose $\alpha^{(1)}$ so that
\[
\alpha_1^{(1)}(\eta) = |\eta - \bar{\eta}|^N \chi \left((\text{dist}(\eta, \partial \Omega_\theta))^a\right) \text{ and } \alpha_2^{(1)}(\eta) = \cdots = \alpha_N^{(1)}(\eta) = 0.
\]
Since $\partial \Omega_\theta$ is smooth, the function $\eta \to \text{dist}(\eta, \partial \Omega_\theta)$ is of class $C^3$ when $\eta$ is close enough to the boundary. Here the cut off function $\chi$ is of class $C^3$ and satisfies
\[
\chi(s) = 1 \text{ if } s \in (0, \bar{\rho}), \chi(s) = 0 \text{ if } s \in (2\bar{\rho}, \infty), \text{ and } |\chi'(s)| \leq \frac{1}{\bar{\rho}}, |\chi''(s)| \leq \frac{1}{\bar{\rho}^2}, |\chi'''(s)| \leq \frac{1}{\bar{\rho}^3}
\]
where $\bar{\rho} > 0$ is such that $4\bar{\rho} \leq \text{dist}(\bar{\eta}, \partial \Omega_\theta)$ and $\bar{\rho}$ will be chosen small enough. The positive number $a$ will be chosen $a \geq 4$ (so that estimate (30) holds).

By the definition of $\alpha^{(1)}$ and (22) we have
\[
\int_{\partial \Omega_\theta} \frac{\partial}{\partial \eta_p} \left\{ \sum_{i=1}^{N} \frac{\bar{\eta}_i - \eta_i}{|\eta - \bar{\eta}|^N} \left( \alpha_i(\eta) - \alpha_i(\bar{\eta}) \right) \right\} \frac{\partial G}{\partial \eta_p}(\eta, \bar{\eta})d\sigma = \int_{\partial \Omega_\theta} \frac{\partial}{\partial \eta_p}(\eta_1 - \bar{\eta}_1) \frac{\partial G}{\partial \eta_p}(\eta, \bar{\eta})d\sigma = \delta_1 \int_{\partial \Omega_\theta} \frac{\partial G}{\partial \eta_p}(\eta, \bar{\eta})d\sigma \tag{24}
\]
Moreover we have
\[
\int_{\Omega_\theta} \frac{\partial}{\partial \eta_p} \left\{ \sum_{i=1}^{N} \frac{\partial^2 \bar{v}_\eta^b}{\partial \eta_i \partial \eta_j}(\eta) \left[ \frac{\partial \alpha_j}{\partial \eta_i}(\eta) + \frac{\partial \alpha_i}{\partial \eta_j}(\eta) \right] - \sum_{j=1}^{N} \frac{\partial \bar{v}_\eta^b}{\partial \eta_j}(\eta) \Delta_\eta \alpha_j(\eta) \right\} G(\eta, \bar{\eta})d\eta = \int_{\Omega_\theta} \frac{\partial}{\partial \eta_p} \left\{ \sum_{i=1}^{N} \frac{\partial \bar{v}_\eta^b}{\partial \eta_i}(\eta) \frac{\partial \alpha_i^{(1)}}{\partial \eta_1}(\eta) - \frac{\partial \bar{v}_\eta^b}{\partial \eta_1}(\eta) \Delta_\eta \alpha_1^{(1)}(\eta) \right\} G(\eta, \bar{\eta})d\eta =: \sigma_1^{(1)}(\bar{\rho}), \tag{25}
\]
where $\Omega_\theta^\rho := \{ \eta \in \Omega_\theta : \text{dist}(\eta, \partial \Omega_\theta) < 2\bar{\rho} \}$. We now establish an accurate estimate of $\sigma^{(1)}(\bar{\rho})$. By Lemma 3.2 proved at the end of this section, for $\bar{\rho}$ small enough we have that there exists $c_1 > 0$ such that
\[
|G(\eta, \bar{\eta})| \leq c_1 \bar{\rho} \text{ for any } \eta \in \Omega_\theta^\rho. \quad (26)
\]
Moreover, it is easy to check that there exists $c_2 > 0$ such that for any $t = (t_1, \ldots, t_N)$ with $|t| \leq 3$
\[
\frac{|\partial^t |\eta - \bar{\eta}|^N|}{|\partial \eta_1 \cdots \partial \eta_N^N|} \leq \begin{cases} 
 c_2 & \text{if } N \geq 3, \ c_2 \bar{\rho}^{-1} & \text{if } N = 2
\end{cases} \text{ for any } \eta \in \Omega_\theta^\rho. \tag{27}
\]
By (25), (26), (27) and (26) it follows that
\[
\sigma_1^{(1)}(\bar{\rho}) \leq c \int_{\Omega_\theta^\rho} \left\{ \left| \frac{\partial \alpha_i^{(1)}}{\partial \eta_1} \right| + \left| \frac{\partial^2 \alpha_i^{(1)}}{\partial \eta_1 \partial \eta_p} \right| + \left| \Delta_\eta \alpha_i^{(1)} \right| + \left| \frac{\partial \Delta_\eta \alpha_i^{(1)}}{\partial \eta_p} \right| \right\} d\eta \\
\leq c \int_{\Omega_\theta^\rho} \left\{ \left| \frac{\partial}{\partial \eta_1} \chi (d^a(\eta)) \right| + \left| \frac{\partial^2}{\partial \eta_1 \partial \eta_p} \chi (d^a(\eta)) \right| + \left| \Delta_\eta \chi (d^a(\eta)) \right| + \left| \frac{\partial \Delta_\eta \chi}{\partial \eta_p} \chi (d^a(\eta)) \right| \right\} d\eta \tag{28}
\]
where \( d^a(\eta) := \text{dist}(\eta, \partial \Omega_\eta)^a \).

Let us estimate \( A_p(\eta) \) when \( \eta \in \Omega_\eta^\delta \). We recall that \( 0 \leq d(\eta) \leq \bar{\rho} \) since \( \eta \in \Omega_\eta^\delta \) and (23) holds. By a simple calculation of the derivatives of the function \( \eta \to \chi(d^a(\eta)) \) we easily get that there exists \( c_1 > 0 \) such that
\[
0 \leq A_p(\eta) \leq c_1 \left( \bar{\rho}^{2a-2} + \bar{\rho}^{2a-3} + \bar{\rho}^{2a-4} + \bar{\rho}^{2a-5} + \bar{\rho}^{3a-6} \right) \quad \text{for any } \eta \in \Omega_\eta^\delta. \tag{29}
\]

Then choosing \( a \geq 4 \) we have that there exists \( c_4 > 0 \) such that
\[
0 \leq A_p(\eta) \leq c_2 \quad \text{for any } \eta \in \Omega_\eta^\delta. \tag{30}
\]

By (25), (26) and (29) we deduce that \( \lim_{\bar{\rho} \to 0} \sigma_p(\bar{\rho}) = 0 \). Therefore, by (22), (21) and (24) we get
\[
\nabla_{\bar{\eta}} D_{\bar{\eta}} \vec{v} + \bar{\rho} \cdot \nabla_{\bar{\eta}} |_{\theta = 0} [\alpha(1)](\bar{\eta}) = \nabla_{\bar{\eta}} w_{\bar{\eta}}^\delta [\alpha(1)](\bar{\eta}) = \left( \sigma_0 + \sigma_1(\bar{\rho}), \sigma_2(\bar{\rho}), \ldots, \sigma_N(\bar{\rho}) \right),
\]
where \( \sigma_0 := \int_{\partial \Omega_\eta} \frac{\partial G}{\partial \eta}(\eta, \bar{\eta}) d\sigma \neq 0 \).

In a similar way, for any \( q = 1, \ldots, N \) we can choose \( \alpha^{(q)} \) such that
\[
\alpha^{(q)}_\eta(\eta) = |\eta - \bar{\eta}|^N \chi \left( \text{dist}(\eta, \partial \Omega_\eta) \right)^a \quad \text{and} \quad \alpha^{(q)}_\eta(\eta) = 0 \quad \text{if } i \neq q.
\]

Arguing as above, for any \( q = 1, \ldots, N \) we get
\[
\nabla_{\bar{\eta}} D_{\bar{\eta}} \vec{v} + \bar{\rho} \cdot \nabla_{\bar{\eta}} |_{\theta = 0} [\alpha^{(q)}](\bar{\eta}) = \nabla_{\bar{\eta}} w_{\bar{\eta}}^\delta [\alpha^{(q)}](\bar{\eta}) = \left( \sigma_1^{(q)}(\bar{\rho}), \ldots, \sigma_0^{(q)}(\bar{\rho}), \ldots, \sigma_N^{(q)}(\bar{\rho}) \right),
\]
where \( \lim_{\bar{\rho} \to 0} \sigma_p^{(q)}(\bar{\rho}) = 0 \) for any \( p = 1, \ldots, N \).

Finally, we choose \( \bar{\rho} \) small enough so that the \( N \) vectors \( \nabla_{\bar{\eta}} w_{\bar{\eta}}^\delta [\alpha^{(1)}](\bar{\eta}), \ldots, \nabla_{\bar{\eta}} w_{\bar{\eta}}^\delta [\alpha^{(N)}](\bar{\eta}) \) are linearly independent and the claim follows.

Next, we prove Lemma 3.2 used in the proof of Lemma 3.1.

**Lemma 3.2.** Given \( y \in \Omega \), there exist \( \tau_0 > 0 \) and \( c_1 > 0 \) such that for any \( \tau \in (0, \tau_0) \)
\[
|G(x, y)| \leq c_1 \tau \quad \forall x \in \Omega_\tau := \{ x \in \Omega : \text{dist}(x, \partial \Omega) \leq \tau \}.
\]

**Proof.** Let us fix \( y \in \Omega \). First of all, if \( \tau \) is small enough, for any \( x \in \Omega_\tau \) there exists a unique \( p_x \in \partial \Omega \) such that
\[
\text{dist}(x, \partial \Omega) = |x - p_x| \leq \tau. \tag{31}
\]

By mean value theorem we get for some \( t \in (0, 1) \)
\[
G(x, y) = G(x, y) - G(p_x, y) = \langle \nabla_x G(tx + (1 - t)p_x, y), x - p_x \rangle.
\]

Therefore, taking into account that (31) holds and also \( tx + (1 - t)p_x \in \Omega_\tau \) for any \( x \in \Omega_\tau \), we get
\[
|G(x, y)| \leq \tau \max_{x \in \Omega_\tau} |\nabla_x G(x, y)|.
\]

The claim will follow if we prove that
\[
\max_{x \in \Omega_\tau} |\nabla_x G(x, y)| \leq c(y), \tag{32}
\]
for some positive constant \( c \) depending on \( y \).

Let us recall that (see (1)) \( G(x, y) = \gamma [\Gamma(|x - y|) - H(x, y)] \). If we choose \( \tau < \frac{\text{dist}(y, \partial \Omega)}{2} \) then
\[
|x - y| \geq \text{dist}(y, \partial \Omega) - \text{dist}(x, \partial \Omega) \geq \frac{\text{dist}(y, \partial \Omega)}{2}
\]
and so by the expression of $\Gamma$ in (2) we get
\[
\max_{x \in \Omega, t} |\nabla_x \Gamma(x, y)| \leq c(y),
\]
for some positive constant $c$ depending on $y$. Moreover, by (3) and by standard regularity theory (see Remark 2.3), we also have that
\[
\max_{x \in \Omega, t} |\nabla_x H(x, y)| \leq c(y),
\]
for some positive constant $c$ depending on $y$. Finally, by (33) and (34) and (1), we get (32) and so the claim is proved.

4 The dependence on $\theta$ of $\partial^\theta x \xi$ and $\nabla_x \partial^\theta x \xi$

In the following we calculate the Frechet derivative with respect to $\theta$ of $\partial^\theta x \xi$ and $\partial^\theta x \xi$ for $p = 1, \ldots, N$. Moreover, we prove that the map $\theta \mapsto \partial^\theta x \xi$ is of class $C^1$ for any $p = 1, \ldots, N$.

Lemma 4.1. For any $\xi \in \Omega$ the map $T : \aleph_\rho \rightarrow C^2,\alpha (\Omega)$ defined by $T(\theta) = \partial^\theta x \xi$ is of class $C^1$. Moreover
\[
T^*_{0}(0) [\theta] = D_{\theta} \partial^\theta x |_{\theta=0} [\theta] = u[\theta]
\]
is the unique solution of the problem
\[
\begin{aligned}
\Delta_x u[\theta] + \sum_{i,j=1}^N \partial^2 \partial^\theta x \xi \partial_{x_i x_j} [\partial \theta_j + \partial \theta_i] - \sum_{j=1}^N \partial^\theta x \xi \Delta_x \theta_j = 0 & \quad \text{if } x \in \Omega \\
u[\theta] |_{x \in \partial \Omega} = - \sum_{i=1}^N \frac{x_i - \xi_i}{|x - \xi|^N} (\theta_i(x) - \theta_i(\xi)) & \quad \text{if } x \in \partial \Omega.
\end{aligned}
\]

Proof. First, we prove that the Gateaux derivative of the map $T$ at 0 is the unique solution of the problem (35).

It holds
\[
\begin{aligned}
\Delta_x \left( \frac{\partial^\theta x \xi - \partial^\theta x \xi}{t} \right) + \sum_{i,j=1}^N \partial^2 \partial^\theta x \xi \partial_{x_i x_j} \left[ \partial \gamma_j + \partial \gamma_i \right] \\
+ \sum_{j=1}^N \frac{\partial \gamma_j}{\partial x_j} \left( \frac{\partial^\theta x \xi}{t} \right) \frac{1}{t} \sum_{s=1}^N \frac{\partial^2 \gamma_s}{\partial z_s^2} + f^t = 0 & \quad \text{if } x \in \Omega \\
\left( \frac{\partial^\theta x \xi - \partial^\theta x \xi}{t} \right) (x) = \frac{\Gamma (|x - \xi + t \theta (x) - t \theta (\xi)|) - \Gamma (|x - \xi|)}{t} & \quad \text{if } x \in \partial \Omega,
\end{aligned}
\]
where $\gamma^t$ is such that $I + \gamma^t = (I + t \theta)^{-1}$ so $\gamma^t (z) = - t (z + \gamma^t (z))$ and
\[
f^t := \frac{1}{t} \left\{ \sum_{i,j=1}^N \frac{\partial^2 \partial^\theta x \xi}{\partial x_i \partial x_j} \left[ \frac{\partial \gamma_j}{\partial z_i} + \frac{\partial \gamma_i}{\partial z_j} \right] + \sum_{s=1}^N \frac{\partial^2 \gamma_s}{\partial z_s^2} \right\}.
\]

By the fact that $(I + t \theta) \circ (I + \gamma^t) = I$ and by Remark 2.2 we deduce that
\[
(\gamma^t)' (z) [h] = - t (I + t \theta (x))^{-1} (\theta' (x) [h])
\]
and
\[
(\gamma^t)'' (z) [h][k] = - t (I + t \theta (x))^{-1} \left( \theta'' (x) \left[ h + (\gamma^t)' (z) [h] \right] \right) \left[ k + (\gamma^t)' (z) [k] \right],
\]

8
where \( x = z + \gamma^t(z) \). Then we get that as \( t \to 0 \)

\[
f^t \to - \sum_{i,j=1}^{N} \frac{\partial^2 \overline{v}_\xi}{\partial x_i \partial x_j} \left[ \frac{\partial \theta_j}{\partial x_i} + \frac{\partial \theta_i}{\partial x_j} \right] - \sum_{j=1}^{N} \frac{\partial \overline{v}_\xi}{\partial x_j} \sum_{s=1}^{N} \frac{\partial^2 \theta_j}{\partial x_s^2} \text{ in } C^{0,\alpha}(\Omega),
\]

(37)

because \( \|v^\theta - \overline{v}_\xi\|_{C^{2,\alpha}(\Omega)} \to 0 \). Recalling that for \( x \neq \xi \) we have

\[
\lim_{t \to 0} \frac{\Gamma(|x - \xi + t\theta(x) - t\theta(\xi)|) - \Gamma(|x - \xi|)}{t} = - \sum_{i=1}^{N} \frac{x_i - \xi_i}{|x - \xi|^N} (\theta_i(x) - \theta_i(\xi)),
\]

by (36) and (37), using the standard regularity theory, we get that \( \|\overline{v}^\theta - \overline{v}_\xi\|_{C^{2,\alpha}(\Omega)} \) is bounded. Then for any sequence \( (t_n) \) such that \( t_n \to 0 \), the sequence of functions \( \overline{v}^\theta - \overline{v}_\xi \), up to a subsequence, is convergent in \( C^2(\Omega) \) and by (36) and (37) it converges to the unique solution \( u[\theta]^t \) of problem (35). In fact, by Remark 2.1 we have that

\[
\left\| \sum_{i,j=1}^{N} \frac{\partial^2 \overline{v}^\theta - \overline{v}_\xi}{\partial x_i \partial x_j} \left[ \frac{\partial \gamma_i^t}{\partial z_i} + \frac{\partial \gamma^t_j}{\partial z_j} + \sum_{s=1}^{N} \frac{\partial \gamma^t_i \partial \gamma^t_j}{\partial z_s} \right] + \sum_{j=1}^{N} \frac{\partial \overline{v}^\theta - \overline{v}_\xi}{\partial x_j} \sum_{s=1}^{N} \frac{\partial^2 \gamma^t_j}{\partial z_s^2} \right\|_{C^{0,\alpha}(\Omega)} \to 0
\]

as \( t \to 0 \). Next, it is easy to check that the Gateaux derivative exists and is continuous. Then the claim follows. \( \square \)

**Lemma 4.2.** Let \( p = 1, \ldots, N \). For any \( \xi \in \Omega \) the map \( G : \mathcal{B}_p \to C^{2,\alpha}(\Omega) \) defined by \( G(\theta) = \frac{\partial \overline{v}^\theta}{\partial x_p} \) is of class \( C^1 \).

Moreover

\[
G(\theta)(\theta) = D_\theta \frac{\partial \overline{v}^\theta}{\partial x_p}|_{\theta=0}[\theta] = u_p[\theta]
\]

is the unique solution of the problem

\[
\begin{align*}
\Delta_p u_p[\theta] &= \frac{\partial}{\partial x_p} \sum_{i,j=1}^{N} \frac{\partial^2 \overline{v}^\theta}{\partial x_i \partial x_j} \left[ \frac{\partial \theta_j}{\partial x_i} + \frac{\partial \theta_i}{\partial x_j} \right] - \frac{\partial}{\partial x_p} \sum_{j=1}^{N} \frac{\partial \overline{v}^\theta}{\partial x_j} \Delta_x \theta_j = 0 & \text{if } x \in \Omega \\
u_p[\theta](x) &= - \frac{\partial}{\partial x_p} \sum_{i=1}^{N} \frac{x_i - \xi_i}{|x - \xi|^N} (\theta_i(x) - \theta_i(\xi)) & \text{if } x \in \partial \Omega.
\end{align*}
\]

(38)

**Proof.** First, we prove that the Gateaux derivative of the map \( G \) at 0 is the unique solution of the problem (35). The function \( \frac{w^\theta_p - w_0^p}{t} \) is a solution of the problem

\[
\begin{align*}
\Delta_x \left( \frac{w^\theta_p - w_0^p}{t} \right) + \sum_{i,j=1}^{N} \frac{\partial^2 (w^\theta_p - w_0^p)}{\partial x_i \partial x_j} \left[ \frac{\partial \gamma_i^t}{\partial z_i} + \frac{\partial \gamma^t_j}{\partial z_j} + \sum_{s=1}^{N} \frac{\partial \gamma^t_i \partial \gamma^t_j}{\partial z_s} \right] \\
+ \sum_{j=1}^{N} \frac{\partial (w^\theta_p - w_0^p)}{\partial x_j} \sum_{s=1}^{N} \frac{\partial^2 \gamma^t_j}{\partial z_s^2} + \left[ t^\theta(x) + \frac{1}{t} \overline{v}^\theta \right](x) &= 0 & \text{if } x \in \Omega
\end{align*}
\]

(39)

\[
\left( \frac{w^\theta_p - w_0^p}{t} \right)(x) = \frac{\overline{v}^\theta(x)}{t} & \text{if } x \in \partial \Omega,
\]
Then we have \( g^0(x) := \sum_{i,j=1}^{N} \frac{\partial^2 v_x^0}{\partial x_i \partial x_j} \frac{\partial}{\partial x_p} \left[ \frac{\partial \gamma_j}{\partial z_i} + \frac{\partial \gamma_i}{\partial z_j} + \sum_{s=1}^{N} \frac{\partial \gamma_i}{\partial z_s} \frac{\partial \gamma_j}{\partial z_s} \right] + \sum_{j=1}^{N} \frac{\partial v_x^0}{\partial x_j} \frac{\partial}{\partial x_p} \sum_{s=1}^{N} \frac{\partial^2 \gamma_j}{\partial z_s^2}, \) (40)

\( g^0(x) := \sum_{i,j=1}^{N} \frac{\partial w_p^0}{\partial x_i \partial x_j} \frac{\partial}{\partial x_p} \left[ \frac{\partial \gamma_j}{\partial z_i} + \frac{\partial \gamma_i}{\partial z_j} + \sum_{s=1}^{N} \frac{\partial \gamma_i}{\partial z_s} \frac{\partial \gamma_j}{\partial z_s} \right] + \sum_{j=1}^{N} \frac{\partial w_p^0}{\partial x_j} \sum_{s=1}^{N} \frac{\partial^2 \gamma_j}{\partial z_s^2}, \) (41)

\( \varphi^0_p(x) := \frac{x_p - \xi_p + \theta_p(x) - \theta_p(\xi)}{|x - \xi + \theta(\xi) - \theta(\xi)|^N} - \frac{x_p - \xi_p}{|x_p - \xi|^N} + \sum_{i=1}^{N} \frac{x_i - \xi_i + \theta_i(x) - \theta_i(\xi)}{|x - \xi + \theta(\xi) - \theta(\xi)|^N} \frac{\partial \theta_i}{\partial x_p} \) (42)

Moreover \( \gamma^t \) is such that \( I + \gamma^t = (I + t\theta)^{-1} \) so \( \gamma^t(z) = -t\theta(z + \gamma^t(z)) \). We point out that \( t^0 \theta \) in (40) and \( g^t \theta \) in (41) also contain \( \gamma^t \). By Remark 2.2 we deduce that if \( z := x + \theta(x) \)

\[ \gamma^t(z) = -t\theta(x) + \sum_{i \geq 2} (-1)^{i} (t\theta(x))^i \] and \( (\gamma^t)^\prime(z) = -t\theta^\prime(x) + \sum_{i \geq 2} (-1)^{i} (t\theta^\prime(x))^i. \] (43)

Then we have

\[ \lim_{t \to 0} \left\| \frac{1}{t} f^\theta + \frac{1}{t} g^\theta + \sum_{i,j=1}^{N} \frac{\partial^2 \varphi^0_p}{\partial x_i \partial x_j} \frac{\partial}{\partial x_p} \left[ \frac{\partial \theta_j}{\partial x_i} + \frac{\partial \theta_i}{\partial x_j} \right] + \sum_{j=1}^{N} \frac{\partial \varphi^0_p}{\partial x_j} \frac{\partial}{\partial x_p} \Delta \theta_j \right\|_{C^0[\Omega]} = 0. \] (44)

Since \( w^\theta - w^0 / t \) solves problem (39), by estimate (44) we deduce that \( \left\| \frac{w^\theta - w^0}{t} \right\|_{C^2(\Omega)} \) is bounded as \( t \to 0 \). Then for any sequence \( (t_n) \) such that \( t_n \to 0 \), the sequence of functions \( w^\theta - w^0 / t_n \), up to a subsequence, is convergent in \( C^2(\Omega) \). Moreover, arguing as in Remark 2.3 we can prove that

\[ \left\| \frac{\partial \varphi^0_p}{\partial x_p} - \frac{\partial \varphi^0_0}{\partial x_p} \right\|_{C^2(\Omega)} \to 0 \text{ as } \left\| \theta \right\|_{k} \to 0. \] (45)

Finally, by (45), (43) and (44), passing to the limit in (39) as \( t \to 0 \) we get the claim. Next, it is easy to check that the Gateaux derivative exists and is continuous. Then the claim follows. \( \square \)

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