Quasi-bound states of massive scalar fields in the Kerr black-hole spacetime: Beyond the hydrogenic approximation

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Rotating black holes can support quasi-stationary (unstable) bound-state resonances of massive scalar fields in their exterior regions. These spatially regular scalar configurations are characterized by instability timescales which are much longer than the timescale $M$ set by the geometric size (mass) of the central black hole. It is well-known that, in the small-mass limit $\alpha \equiv M\mu \ll 1$ (here $\mu$ is the mass of the scalar field), these quasi-stationary scalar resonances are characterized by the familiar hydrogenic oscillation spectrum: $\omega_R/\mu = 1 - \alpha^2/2\bar{n}_0$, where the integer $\bar{n}_0(l, n; \alpha \to 0) = l + n + 1$ is the principal quantum number of the bound-state resonance (here the integers $l = 1, 2, 3, ...$ and $n = 0, 1, 2, ...$ are the spheroidal harmonic index and the resonance parameter of the field mode, respectively). As it depends only on the principal resonance parameter $\bar{n}_0$, this small-mass ($\alpha \ll 1$) hydrogenic spectrum is obviously degenerate. In this paper we go beyond the small-mass approximation and analyze the quasi-stationary bound-state resonances of massive scalar fields in rapidly-spinning Kerr black-hole spacetimes in the regime $\alpha = O(1)$. In particular, we derive the non-hydrogenic (and, in general, non-degenerate) resonance oscillation spectrum $\omega_R/\mu = \sqrt{1 - (\alpha/\bar{n})^2}$, where $\bar{n}(l, n; \alpha) = (l + 1/2)^2 - 2\alpha n + 2\alpha^2 + 1/2 + n$ is the generalized principal quantum number of the quasi-stationary resonances. This analytically derived formula for the characteristic oscillation frequencies of the composed black-hole-massive-scalar-field system is shown to agree with direct numerical computations of the quasi-stationary bound-state resonances.

I. INTRODUCTION

Recent analytical [1] and numerical [2] studies of the coupled Einstein-scalar equations have revealed that rotating black holes can support spatially regular configurations of massive scalar fields in their exterior regions. These bound-state resonances of the composed black-hole-scalar-field system owe their existence to the well-known phenomenon of superradiant scattering [3,4] of integer-spin (bosonic) fields in rotating black-hole spacetimes.

The stationary black-hole-scalar-field configurations [1,2] mark the boundary between stable and unstable bound-state resonances of the composed system. In particular, these stationary scalar field configurations are characterized by azimuthal frequencies $\omega_{\text{field}}$ which are in resonance with the black-hole angular velocity $\Omega_H$ [5]:

$$\omega_{\text{field}} = m\Omega_H,$$

(1)

where $m = 1, 2, 3, ...$ is the azimuthal quantum number of the field mode. Bound-state field configurations in the superradiant regime $\omega_{\text{field}} < m\Omega_H$ are known to be unstable (that is, grow in time), whereas bound-state field configurations in the regime $\omega_{\text{field}} > m\Omega_H$ are known to be stable (that is, decay in time) [2,3].

The bound-state scalar resonances of the rotating Kerr black-hole spacetime are characterized by at least two different time scales: (1) the typical oscillation period $\tau_{\text{oscillation}} \equiv 2\pi/\omega_R \sim 1/\mu$ of the bound-state massive scalar configuration (here $\mu$ is the mass of the scalar field [6]), and (2) the instability growth time scale $\tau_{\text{instability}} \equiv 1/\omega_I$ associated with the superradiance phenomenon. Former studies [7,11] of the Einstein-massive-scalar-field system have revealed that these two time scales are well separated. In particular, it was shown [7,11] that the composed system is characterized by the relation

$$\tau_{\text{instability}} \gg \tau_{\text{oscillation}},$$

(2)

or equivalently

$$\omega_I \ll \omega_R.$$

(3)

The strong inequality [2] implies that the bound-state massive scalar configurations may be regarded as the quasi-stationary resonances of the composed system.

As shown in [12,13], the physical significance of the characteristic black-hole-scalar-field oscillation frequencies $\{\omega_R(n)\}_{n=0}^{n=\infty}$ [14] lies in the fact that the corresponding quasi-stationary scalar resonances dominate the dynamics of massive scalar fields in curved black-hole spacetimes. In particular, recent numerical simulations [12,13] of the...
dynamics of massive scalar fields in the Kerr black-hole spacetime have demonstrated explicitly that these quasi-stationary bound-state resonances dominate the characteristic Fourier power spectra \( P(\omega) \) of the composed black-hole-massive-scalar-field system \[15\].

II. THE SMALL-MASS HYDROGENIC SPECTRUM

As shown by Detweiler \[7\], the massive scalar resonances can be calculated analytically in the small-mass regime \( M\mu \ll 1 \). In particular, one finds \[7\] that the quasi-stationary bound-state scalar resonances are characterized by the familiar hydrogenic spectrum

\[
\frac{\omega_R(\tilde{n}_0)}{\mu} = 1 - \frac{\alpha^2}{2\tilde{n}_0} \quad \text{for} \quad \alpha \equiv M\mu \ll 1 ,
\]

where the integer

\[
\tilde{n}_0(l, n; \alpha \to 0) = l + 1 + n
\]

is the principal quantum number of the quasi bound-state resonances. Here the integer \( l \geq |m| \) is the spheroidal harmonic index of the field mode and \( n = 0, 1, 2, \ldots \) is the resonance parameter.

It is worth emphasizing that the hydrogenic spectrum \[11\] depends only on the principal resonance parameter (quantum number) \( \tilde{n}_0 = l + 1 + n \). This small-mass oscillation spectrum is therefore degenerate. That is, two different modes, \((l, n)\) and \((l', n')\) with \( l + n = l' + n' \), are characterized by the same resonant frequency: \( \omega_R(l, n) = \omega_R(l', n') \) for \( l + n = l' + n' \).

To the best of our knowledge, the oscillation frequency spectrum which characterizes the quasi-stationary bound-state resonances of massive scalar fields in the rotating Kerr black-hole spacetime has not been studied analytically beyond the hydrogenic regime \[11\] of small \( \alpha \ll 1 \) field masses. The main goal of the present paper is to analyze the oscillation spectrum of the composed black-hole-massive-scalar-field system in the \( \alpha = O(1) \) regime. To that end, we shall use the resonance equation \[see Eq. (11) below\] derived in \[10\] for the bound-state resonances of massive scalar fields in rapidly-rotating (near-extremal) Kerr black-hole spacetimes. As we shall show below, this resonance equation can be solved analytically to yield the characteristic oscillation spectrum \( \{\omega_R(n)\}_{\tilde{n} = 0}^{\infty} \) of the quasi-stationary bound-state scalar resonances in the regime \( \alpha \lesssim 1 \).

III. DESCRIPTION OF THE SYSTEM

We consider a scalar field \( \Psi \) of mass \( \mu \) linearly coupled to a rapidly-rotating (near-extremal) Kerr black hole of mass \( M \) and dimensionless angular momentum \( a/M \to 1^- \). The dynamics of the scalar field in the black-hole spacetime is governed by the Klein-Gordon (Teukolsky) wave equation

\[
(\nabla^a \nabla_a - \mu^2) \Psi = 0 .
\]

Substituting the field decomposition \[10\] \[18\]

\[
\Psi(t, r, \theta, \phi) = \int \sum_{l,m} e^{im\phi} S_{lm}(\theta) R_{lm}(r) e^{-i\omega t} d\omega
\]

into the wave equation \[10\], one finds \[19\] \[20\] that the radial function \( R \) and the angular function \( S \) obey two ordinary differential equations of the confluent Heun type \[21\] \[22\].

The angular eigenfunctions, known as the spheroidal harmonics, are determined by the angular Teukolsky equation \[19\] \[23\]

\[
\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial S_{lm}}{\partial \theta} \right) + \left[ \frac{2\omega^2 - \mu^2}{\sin^2 \theta} \cos^2 \theta - \frac{m^2}{\sin^2 \theta} + A_{lm} \right] S_{lm} = 0 .
\]

The regularity requirements of these functions at the two boundaries \( \theta = 0 \) and \( \theta = \pi \) single out a discrete set of angular eigenvalues \( \{A_{lm}\} \) \[see Eq. (14) below\] labeled by the integers \( l \) and \( m \).

The radial Teukolsky equation is given by \[19\] \[20\] \[24\]

\[
\Delta \frac{d}{dr} \left( \Delta \frac{dR_{lm}}{dr} \right) + \left\{ (\nu^2 + a^2) \omega - am \right\}^2 - \Delta (a^2 \omega^2 - 2ma\omega + \mu^2 r^2 + A_{lm}) \right\} R_{lm} = 0 ,
\]
where $\Omega_H$ is the angular velocity of the black-hole horizon [see Eq. (12) below]. The boundary conditions (10) imposed on the radial eigenfunctions single out a discrete set of eigenfrequencies $\{\omega_n(a/M, l, m, \alpha)\}_{n=0}^{\infty}$ which characterize the quasi-stationary bound-state resonances of the massive scalar fields in the Kerr black-hole spacetime [7–9].

IV. THE CHARACTERISTIC RESONANCE EQUATION AND ITS REGIME OF VALIDITY

Solving analytically the radial Klein-Gordon (Teukolsky) equation (9) in two different asymptotic regions and using a standard matching procedure for these two radial solutions in their common overlap region [see Eq. (15) below], we have derived in [10] the characteristic resonance equation

$$\frac{1}{\Gamma\left(\frac{1}{2} + \beta - \kappa\right)} = \left[\frac{\Gamma(-2\beta)}{\Gamma(2\beta)}\right]^2 \frac{\Gamma\left(\frac{1}{2} + \beta - ik\right)}{\Gamma\left(\frac{1}{2} - \beta - ik\right)} \left[8iMr_+\sqrt{\mu^2 - \omega^2(m\Omega_H - \omega)}\right]^{2\beta}$$

(11)

for the bound-state resonances of the composed Kerr-black-hole-massive-scalar-field system. Here

$$\Omega_H = \frac{a^2}{r_+^2 + a^2}$$

(12)

is the angular velocity of the black-hole horizon, and

$$k \equiv 2\omega r_+ \quad , \quad \kappa \equiv \frac{\omega k - \mu^2 r_+}{\sqrt{\mu^2 - \omega^2}} \quad , \quad \beta^2 \equiv a^2 \omega^2 - 2ma \omega + \mu^2 r_+^2 + A_{lm} - k^2 + \frac{1}{4} \quad ,$$

(13)

where $\{A_{lm}\}$ are the angular eigenvalues which couple the radial Teukolsky equation (9) to the angular (spheroidal) equation (8). These angular eigenvalues can be expanded in the form [23]

$$A_{lm} = l(l + 1) + \sum_{k=1}^{\infty} c_k a^{2k}(\mu^2 - \omega^2)^k \quad ,$$

(14)

where the expansion coefficients $\{c_k\}$ are given in [23].

Before proceeding, it should be emphasized that the validity of the resonance equation (11) is restricted to the regime

$$\tau \ll M(m\Omega - \omega) \ll x_0 \ll \frac{1}{M\sqrt{\mu^2 - \omega^2}} \quad ,$$

(15)

where $\tau \equiv (r_+ - r_-)/r_+ \ll 1$ is the dimensionless temperature of the rapidly-rotating (near-extremal) Kerr black hole, and the dimensionless coordinate $x_0 \equiv (r_0 - r_+)/r_+$ belongs to the overlap region in which the two different solutions of the radial Teukolsky equation (hypergeometric and confluent hypergeometric radial wave functions) can be matched together, see [14, 27] for details. The inequalities in (15) imply that the resonance condition (11) should be valid in the regime [28]

$$M^2(m\Omega_H - \omega)\sqrt{\mu^2 - \omega^2} \ll 1 \quad .$$

(16)

V. THE QUASI-STATIONARY BOUND-STATE RESONANCES OF THE COMPOSED BLACK-HOLE-MASSIVE-SCALAR-FIELD SYSTEM

As we shall now show, the resonance condition (11) can be solved analytically in the physical regime (16). In particular, in the present section we shall derive a (remarkably simple) analytical formula for the discrete spectrum
of oscillation frequencies, \( \{\omega_{R}(l, m, \alpha; n)\}_{n=0}^{\infty} \), which characterize the quasi-stationary bound-state resonances of the composed Kerr-black-hole-massive-scalar-field system.

Our analytical approach is based on the fact that the right-hand-side of the resonance equation (11) is small in the regime (16) with \( \beta \in \mathbb{R} \). The resonance condition can therefore be approximated by the simple zeroth-order equation

\[
\frac{1}{\Gamma\left(\frac{1}{2} + \beta - \kappa\right)} = 0.
\]

(17)

As we shall now show, this zeroth-order resonance condition can be solved analytically to yield the real oscillation frequencies which characterize the bound-state scalar resonances. We first use the well-known pole structure of the Gamma functions [23] in order to write the resonance equation (17) in the form [10]

\[
\frac{1}{2} + \beta - \kappa = -n,
\]

(18)

where the integer \( n = 0, 1, 2, \ldots \) is the resonance parameter of the field mode.

Defining the dimensionless variable [30]

\[
\epsilon \equiv M \sqrt{\mu^2 - \omega^2},
\]

(19)

one finds from Eq. (13)

\[
\beta^2 = \beta_0^2 + O(\epsilon^2, \tau) \quad \text{and} \quad \kappa = \frac{\alpha^2}{\epsilon} - 2\epsilon + O(\tau),
\]

(20)

where

\[
\beta_0^2 \equiv (l + 1/2)^2 - 2m\alpha - 2\alpha^2.
\]

(21)

Substituting (20) into the resonance condition \( \beta^2 = (\kappa - (n + 1/2))^2 \) [see Eq. (18)], one obtains the characteristic equation

\[
\epsilon^2 \cdot [(2l + 1)^2 - 8m\alpha + 8\alpha^2 - (2n + 1)^2] + \epsilon \cdot 4(2n + 1)\alpha^2 - 4\alpha^4 + O(\tau, \epsilon^3) = 0
\]

(22)

for the dimensionless parameter \( \epsilon \). This resonance equation can easily be solved to yield

\[
\epsilon(l, m; n) = \frac{2\alpha^2}{\sqrt{(2l + 1)^2 - 8m\alpha + 8\alpha^2 + 1 + 2n}}.
\]

(23)

Finally, taking cognizance of the relation (19), one finds the discrete spectrum of oscillation frequencies

\[
\omega_{R}(n)/\mu = \sqrt{1 - \left(\frac{\alpha}{\ell + 1 + n}\right)^2}
\]

(24)

which characterize the quasi-stationary bound-state resonances of the composed black-hole-massive-scalar-field system. Here

\[
\ell \equiv \frac{1}{2} \left[ \sqrt{(2l + 1)^2 - 8m\alpha + 8\alpha^2} - 1 \right]
\]

(25)

is the generalized (finite-mass) spheroidal harmonic index. Note that \( \ell \to l \) in the small mass \( \alpha \ll 1 \) limit, in which case one recovers from (24) the well-known hydrogenic spectrum [4] of [7].

It is worth noting that, in general, the parameter \( \ell(\alpha) \) is not an integer. This implies that, for generic values of the dimensionless mass parameter \( \alpha \), the non-hydrogenic oscillation spectrum (24) is not degenerate [31].

VI. NUMERICAL CONFIRMATION

It is of physical interest to test the accuracy of the analytically derived formula (24) for the characteristic oscillation frequencies \( \omega_{R}(n)/\mu \) of the quasi-stationary massive scalar configurations. The quasi bound-state resonances can be computed using standard numerical techniques, see [9, 12] for details. In Table I, we present a comparison between the
analytically derived oscillation frequencies \([24]\) and the numerically computed resonances \([12]\). The data presented is for the fundamental \(l = m = 1\) mode with \(a/M = 0.99\), \(\alpha \equiv M\mu = 0.42\), and \(n = 0, 1, 2, 3, 4\). We display the dimensionless ratio between the analytically derived oscillation frequencies \(\omega_R^{\text{ana}}(n)\) [see Eq. (24)] and the numerically computed resonances \(\omega_R^{\text{num}}(n)\) of \([12]\). One finds a good agreement between the analytical formula \([24]\) and the numerical data of \([12]\).

| Resonance parameter \(n\) | 0     | 1     | 2     | 3     | 4     |
|---------------------------|-------|-------|-------|-------|-------|
| \(\omega_R^{\text{ana}}(n)/\omega_R^{\text{num}}(n)\) | 0.9999 | 1.0006 | 1.0004 | 1.0002 | 0.9994 |

TABLE I: Quasi-stationary resonances of massive scalar fields in the rotating Kerr black-hole spacetime. The data shown is for the fundamental \(l = m = 1\) mode with \(a/M = 0.99\), \(\alpha \equiv M\mu = 0.42\), and \(n = 0, 1, 2, 3, 4\). We display the dimensionless ratio between the analytically derived oscillation frequencies \(\omega_R^{\text{ana}}(n)\) [see Eq. (24)] and the numerically computed resonances \(\omega_R^{\text{num}}(n)\) of \([12]\). One finds a good agreement between the analytical formula \([24]\) and the numerical data of \([12]\).

In order to compare the accuracy of the newly derived analytical formula \([24]\) with the accuracy of the familiar hydrogenic (small-mass) spectrum \([4]\), we display in Table II the physical quantity \(\epsilon(n)\) [see Eq. (19)] which provides a quantitative measure for the deviation of the resonant oscillation frequency \(\omega_R(n)\) from the field mass parameter \(\mu\). In particular, we present the dimensionless ratios \(\epsilon^{\text{ana}}/\epsilon^{\text{num}}\) and \(\epsilon^{\text{ana-hydro}}/\epsilon^{\text{num}}\), where \(\epsilon^{\text{ana}}(n)\) is given by the analytical formula \([24]\), \(\epsilon^{\text{ana-hydro}}(n)\) is defined from the hydrogenic spectrum \([4]\), and \(\epsilon^{\text{num}}(n)\) is obtained from the numerically computed resonances of \([12]\). One finds that, in general, the newly derived formula \([24]\) performs better than the hydrogenic formula \([4]\) \([33]\).

| Resonance parameter \(n\) | 0     | 1     | 2     | 3     | 4     |
|---------------------------|-------|-------|-------|-------|-------|
| \(\epsilon^{\text{ana}}(n)/\epsilon^{\text{num}}(n)\) | 1.001 | 0.974 | 0.971 | 0.974 | 1.140 |
| \(\epsilon^{\text{ana-hydro}}(n)/\epsilon^{\text{num}}(n)\) | 0.910 | 0.916 | 0.928 | 0.939 | 1.107 |

TABLE II: Quasi-stationary resonances of massive scalar fields in the rotating Kerr black-hole spacetime. The data shown is for the fundamental \(l = m = 1\) mode with \(a/M = 0.99\), \(\alpha \equiv M\mu = 0.42\), and \(n = 0, 1, 2, 3, 4\). We display the dimensionless ratios \(\epsilon^{\text{ana}}/\epsilon^{\text{num}}\) and \(\epsilon^{\text{ana-hydro}}/\epsilon^{\text{num}}\), where \(\epsilon^{\text{ana}}(n)\) is given by the analytical formula \([24]\), \(\epsilon^{\text{ana-hydro}}(n)\) is defined from the hydrogenic spectrum \([4]\), and \(\epsilon^{\text{num}}(n)\) is obtained from the numerically computed resonances of \([12]\). One finds that, in general, the newly derived analytical formula \([24]\) performs better than the hydrogenic formula \([4]\) \([33]\).

VII. SUMMARY

In summary, we have studied the resonance spectrum of quasi-stationary massive scalar configurations linearly coupled to a near-extremal (rapidly-rotating) Kerr black-hole spacetime. In particular, we have derived a compact analytical expression [see Eq. (24)] for the characteristic oscillation frequencies \(\omega_R^{\text{ana}}(n)/\mu\) of the bound-state massive scalar fields. It was shown that the analytically derived formula \([24]\) agrees with direct numerical computations \([12]\) of the black-hole-scalar-field resonances.

It is well known that the characteristic hydrogenic spectrum \([4]\) in the small-mass \(\alpha \ll 1\) limit is highly degenerate – it depends only on the principal resonance parameter \(n_0 \equiv l + 1 + n\) \([30]\). Thus, according to \([4]\), two different modes which are characterized by the integer parameters \((l, n)\) and \((l', n')\) with \(l + n = l' + n'\) share the same resonant frequency \(\omega_R(n)\) in the \(\alpha \rightarrow 0\) limit]. On the other hand, the newly derived resonance spectrum \([24]\), which is valid in the \(\alpha = O(1)\) regime, is no longer degenerate. That is, for generic values of the dimensionless mass parameter \(\alpha\), two quasi-stationary modes with different sets of the integer parameters \((l, n)\) are characterized, according to \([24]\), by different oscillation frequencies \([57]\).

Finally, it is worth emphasizing again that the physical significance of the characteristic oscillation frequencies \([24]\) lies in the fact that these quasi-stationary (long-lived) resonances dominate the dynamics [and, in particular, dominate the characteristic Fourier power spectra \(P(\omega)\) \([12]\)] of the massive scalar fields in the black-hole spacetime.

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This is the largest black-hole spin for which we have exact (numerical) data \[\text{[12]}\] for the characteristic oscillation frequencies.

Compare this property of the non-hydrogenic spectrum \[\text{[24]}\] with the familiar hydrogenic spectrum \[\text{[4]}\] which is known to be degenerate.

As discussed above, our resonance equation \[\text{[11]}\] is restricted to the regime \[M^2(\Omega H - \omega)\sqrt{\mu^2 - \omega^2} \lesssim 10^{-2}\] \[\text{[28]}\]. Substituting our solution \[\text{[24]}\] into the left-hand-side of \[\text{[10]}\], one finds that, for the fundamental \(l = m = 1\) mode, the requirement \[M^2(\Omega H - \omega)\sqrt{\mu^2 - \omega^2} \lesssim 10^{-2}\] is respected in the regime \(\alpha \lesssim 0.58\). Note that this mass regime is consistent with our previous requirement \(\beta \in \mathbb{R}\) \[\text{[29]}\], which implies the weaker restriction \(\alpha \lesssim 0.67\) for the fundamental \(l = m = 1\) mode.
Note that Eqs. (21) and (23) yield $\beta_0^2 \simeq 1.057$ and $\epsilon(n = 0) \simeq 0.097$ for the fundamental $l = m = 1$ mode with $\alpha = 0.42$. One therefore finds the characteristic small ratio $\epsilon^2/\beta_0^2 \simeq 0.009 \ll 1$. This strong inequality justifies our previous assumption $\epsilon^2/\beta_0^2 \ll 1$ [see Eq. (20)]. Moreover, since $\epsilon(n)$ is a decreasing function of the resonance parameter $n$ [see Eq. (23)], one finds that the approximation $\epsilon^2/\beta_0^2 \ll 1$ becomes even better for the excited $(n \geq 1)$ resonant modes.

The only exception is the excited $n = 4$ mode, for which the agreement between the numerically computed deviation-parameter $\epsilon_{num}(n = 4)$ and the analytical formulas (4) and (24) is quite poor.

Note that this parameter is an integer.

Note that, for generic values of the mass parameter $\alpha$, the generalized (finite-mass) spheroidal harmonic index $\ell(\alpha)$ that appears in the resonance spectrum (24) is not an integer.