Supersymmetric Boundary Conditions for the $\mathcal{N} = 2$ Sigma Model

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Abstract

We clarify the discussion of $\mathcal{N} = 2$ supersymmetric boundary conditions for the classical $d = 2$, $\mathcal{N} = (2, 2)$ Non-Linear Sigma Model on an infinite strip. Our conclusions about the supersymmetric cycles match the results found in the literature. However, we find a constraint on the boundary action that is not satisfied by many boundary actions used in the literature.
1 Introduction

Weakly coupled Type II string theory compactified on a Calabi-Yau manifold provides a tractable setting for understanding (or at least observing) aspects of stringy geometry [1]. Our ability to glean insights into such non-trivial issues is largely due to the comparative tractability of the worldsheet approach in this compactification. Although the $d = 2, \mathcal{N} = (2,2)$ Non-Linear Sigma Model with Calabi-Yau target space is a complicated theory with non-trivial IR dynamics, the existence of a well-defined classical limit, the presence of topological sectors, and mirror symmetry allow us to draw some rigorous conclusions about the theory. Using these techniques, much has been learned about how closed strings probe the geometry. Naturally, we would like to learn how objects other than fundamental strings probe the geometry. This is, in principle, a very difficult task, as it requires a study of complicated solitonic objects in the theory. However, as is well known, it is our inexplicable bit of fortune to have access to a large class of non-perturbative objects whose fluctuations can be described in terms of perturbative degrees of freedom: namely, D-branes, which can be thought of as “places where open strings can end.” This statement is a bit too naive in general curved backgrounds. However, it is the right point of view at the large radius limit, where classical analysis is valid. This connection means that in the $g_s \to 0$ limit we can study D-branes by examining the open string NLSM. In this paper we will study the classical worldsheet theory for an open string whose endpoints are attached to D-branes in a Calabi-Yau manifold. The problem of interest here is to classify all stable BPS configurations of D-branes. As a first step, one could classify the BPS configurations. In the worldsheet description a BPS configuration is a set of boundary conditions and a boundary action preserving an $\mathcal{N} = 2$ superconformal worldsheet symmetry with integral $U(1)_R$ charges. What we will do here is to classify all boundary conditions and boundary actions which classically preserve an $\mathcal{N} = 2$ superconformal symmetry. A set of boundary conditions for the NLSM includes a choice of a submanifold on which the open string ends, and the BPS conditions single out particular (minimal) representatives of equivalence classes under homology (cycles). A cycle that has a representative preserving $\mathcal{N} = 2$ worldsheet supersymmetry will be called a supersymmetric cycle.

It is simple to extend the NLSM description to include open strings—one needs to consider worldsheets with boundaries and to introduce additional background fields that couple to the string endpoints. We will work in the $H = dB = 0$ background. On a worldsheet with boundaries, the familiar $\mathcal{N} = (2,2)$ NLSM can be modified by adding a local boundary action constrained by boundary reparametrization invariance and (classical) scale invariance. In addition, one needs to specify boundary conditions for the fields. These will be chosen to eliminate the surface term in the variation of the action. Physically, this ensures that the bulk equations of motion continue smoothly to the boundary. This constraint has sometimes been called “locality” [2]. To preserve an $\mathcal{N} = 2$ SUSY we will need to ensure that the boundary conditions and the action are invariant under the SUSY variations.
Over the years, this analysis has been carried out by several groups \cite{3,2,4,5} with results that more or less agree with an earlier spacetime analysis in \cite{6}. An exception is the finding in \cite{4} that special Lagrangian cycles must, in general, be extended to co-isotropic submanifolds for a complete classification. More careful analyses in \cite{7,8,9} extended this work to nontrivial spaces. Our results agree with the consensus, but we clarify several points. Namely, we discuss the role of the “locality” constraint, the necessity for the action and the boundary conditions to be invariant under the preserved supersymmetry, and constraints on the boundary couplings. Surprisingly, we find that the standard supersymmetric boundary coupling that dates back to Callan et al. \cite{10} does not satisfy the locality constraint! Although our analysis does not uniquely determine this the boundary coupling, we do suggest a very natural candidate.

The rest of the paper is organized as follows. We begin with a general discussion of classical bulk symmetries in a field theory with boundaries. Next, we apply this general discussion to the NLSM. We introduce the open string NLSM by modifying the familiar bulk action by boundary terms, and we find the general boundary conditions required for “locality” of this improved action. Next, we find the conditions for SUSY invariance of the boundary conditions and the action. After giving the geometric interpretation of the SUSY conditions, we wrap up with a discussion of our results.

2 Symmetries and Boundaries in Classical Field Theory

2.1 Review of the classical Noether theorem.

In order to set notation, we briefly review the connection between symmetries and conserved charges in classical field theory. Consider a field theory defined on $\Sigma = \mathbb{R}^2$ by the action

$$S_\Sigma = \int_\Sigma d^2x \mathcal{L} (\varphi^i, \partial_\mu \varphi^i).$$

(1)

Let $\delta_\eta \varphi^i = \eta f^i(\varphi, \partial_\mu \varphi, \ldots)$ be an internal infinitesimal symmetry of $S_\Sigma$. In other words, the variation of the Lagrangian is a boundary term: $\delta_\eta \mathcal{L} = -\eta \partial_\mu F^\mu$ for some $F^\mu(\varphi, \ldots)$. But,

$$\delta_\eta \mathcal{L} = \eta f^i \frac{\partial \mathcal{L}}{\partial \varphi^i} + \eta \partial_\mu f^i \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi^i)}.$$  

(2)

Hence, the current

$$j^\mu = F^\mu + f^i \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi^i)}$$

(3)

is conserved up to the equations of motion. The corresponding conserved charge is $Q = \int dx^1 j^0$. That is one direction of the Noether theorem: each continuous symmetry of the
Lagrangian corresponds to the existence of a conserved current. Now let us see how this statement is modified in the presence of a boundary.

### 2.2 Field theory on a strip

We consider a theory with the same bulk Lagrangian but on a strip, with \(-\infty < x^0 < \infty\), and \(0 \leq x^1 \leq \pi\). The addition of a boundary allows the introduction of a boundary action:

\[
S_{\partial \Sigma} = \int_{\partial \Sigma} dx^0 M \left( \varphi^i, \partial_0 \varphi^i, \partial_1 \varphi^i \right).
\]

In order to have a well-posed initial value problem, we must specify a set of boundary conditions for the fields: \(B_n(\varphi^i|_{\partial \Sigma}, \partial_\mu \varphi^i|_{\partial \Sigma}) = 0\). The choice of \(B_n\) will, in general, restrict both the values and the variations of \(\varphi^i\) on the boundary: the equations of motion follow by performing only variations satisfying \(\delta B_n = 0\).

We will refer to this restricted set of variations as **allowed** variations. The boundary conditions are required to satisfy a locality constraint. For any allowed variation, we have

\[
\delta S = \int_{\Sigma} d^2 x \left[ \frac{\partial L}{\partial \varphi^i} - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \varphi^i)} \right) \right] \delta \varphi^i \\
+ \int_{\partial \Sigma} dx^0 \left\{ \left[ \frac{\partial M}{\partial \varphi^i} - \partial_0 \left( \frac{\partial M}{\partial (\partial_0 \varphi^i)} \right) + \frac{\partial L}{\partial (\partial_1 \varphi^i)} \right] \delta \varphi^i + \frac{\partial M}{\partial (\partial_1 \varphi^i)} \delta_1 \delta \varphi^i \right\}.
\]

If the bulk equations of motion (the first line) are to extend smoothly to the boundary, the \(B_n\) must be chosen so that the boundary term in \(\delta S\) (the second line) is zero. If this does not hold, the classical equations of motion and their solutions will be discontinuous. The last term, involving \(\partial_1 \delta \varphi\), cannot lead to local equations of motion, so we require that it vanish; rather than imposing boundary conditions on \(\partial_1 \varphi\) we will restrict attention to boundary actions of the form \(M(\varphi, \partial_0 \varphi)\). The second line will now vanish provided \(B_n\) are chosen so that

\[
\delta \varphi^i \left[ \frac{\partial M}{\partial \varphi^i} - \partial_0 \left( \frac{\partial M}{\partial (\partial_0 \varphi^i)} \right) + \frac{\partial L}{\partial (\partial_1 \varphi^i)} \right] = 0
\]

for any allowed variation.

One may be tempted to think of \(B_n\) as “boundary equations of motion.” This is quite misleading, as the boundary conditions are much stronger than equations of motion. In quantizing the problem via path integral techniques, we expect that the boundary conditions are to be imposed on all field configurations. Thus, the boundary conditions hold as operator equations, and they can be used in the boundary action.

The symmetry of the bulk theory will persist in the presence of boundaries if (in classical mechanics) the symmetry transform of a classical trajectory is another classical trajectory.

\(^1\)In other words, the action is varied over field configurations that obey the boundary conditions.
To define the symmetry transform, the variation \( \delta \eta \phi^i = \eta f^i \) must be an allowed variation about any classical trajectory. Boundary conditions for which a given symmetry variation is allowed in this sense will be called *classically symmetric* boundary conditions. Classical trajectories will transform into others if the equations of motion are invariant under the symmetry, which means the symmetry variation of the action vanishes.

The symmetry variation of the action is given by

\[
\delta \eta (S_\Sigma + S_{\partial \Sigma}) = \int_{\partial \Sigma} dx^0 \left( -\eta F^1 + \eta f^i \left\{ \frac{\partial M}{\partial \phi^i} - \partial_0 \left( \frac{\partial M}{\partial (\partial_0 \phi^i)} \right) \right\} \right).
\]

(7)

Suppose that the boundary conditions are symmetric so that the symmetry variation is an allowed variation. Then we can use Eqs. 3, 6, to conclude that

\[
\delta \eta S = -\eta \int_{\partial \Sigma} dx^0 j^1.
\]

(8)

If the action is invariant, then \( j^1|_{\partial \Sigma} = \partial_0 K \), for some \( K(\phi, \partial_0 \phi) \) defined on the boundary. It is then clear that \( \tilde{Q} = Q + K \) is conserved:

\[
\partial_0 \tilde{Q} = \partial_0 \int dx^1 j^0 + \partial_0 K|_{\partial \Sigma} = -j^1|_{\partial \Sigma} + \partial_0 K|_{\partial \Sigma} = 0.
\]

(9)

Let us contrast \( K \) with a similar bulk quantity, \( F^\mu \). The crucial difference between these two is that the latter is determined by the bulk action, while the former is determined by the choice of boundary conditions. The presence of a boundary will, in general, break the bulk symmetry. By choosing boundary conditions appropriately, one can ensure that the symmetry is preserved, with the conserved charge given as above: \( \tilde{Q} = Q + K \). In principle, one could imagine that different boundary conditions could lead to the same preserved bulk symmetry but with a different conserved charge: \( \tilde{Q}' = Q + K' \).

With this background in mind, we will now proceed to the \( \mathcal{N} = (2, 2) \) NLSM. We will add boundaries, consider locality, and ensure that the boundary conditions are supersymmetric under an \( \mathcal{N} = 2 \) subalgebra.

### 3 The NLSM with Boundaries

#### 3.1 The Action

The NLSM is a field theory of maps \( \Phi : \Sigma \rightarrow X \) from the worldsheet \( \Sigma \) to the target space \( X \). Locally (both on the worldsheet and target space), we can specify such a map by a set of functions \( \phi^I(x) : \Sigma \rightarrow \mathbb{R} \), \( I = 1 \ldots d \), where \( d \) is the (real) target space dimension, and we can think of \( \phi^i \) as coordinates on a patch of the target space. In order for the action to have \( \mathcal{N} = (1, 1) \) supersymmetry, we must add worldsheet Majorana-Weyl fermions, \( \psi^I_\pm(x) \),
sections of $K^{\pm \frac{1}{2}} \otimes \Phi^* (TX)$, where $K$ is the canonical bundle of $\Sigma$, and $\Phi^*$ is the pull-back by the map $\Phi : \Sigma \rightarrow X$. We take our action to be $S = S_\Sigma + S_{\partial \Sigma}$. The bulk piece is the familiar closed string $\mathcal{N} = (2, 2)$ NLSM:

$$S_\Sigma = \int_\Sigma d^2 x \left\{ \frac{1}{2} G_{IJ} \left[ -\partial_\mu \phi^I \partial^\mu \phi^J + i \left( \psi_+^I D_+ \psi_+^J + \psi_-^I D_- \psi_-^J \right) \right] + \frac{1}{4} R_{IJKL} \psi_+^I \psi_+^J \psi_-^K \psi_-^L - \frac{1}{2} B_{IJ} \epsilon^{\mu \nu} \partial_\mu \phi^I \partial_\nu \phi^J \right\}. \quad (10)$$

We work on a worldsheet with a flat Minkowski metric of signature $(-, +)$ and $\epsilon^{01} = +1$. We define $\partial_\pm = \partial_0 \pm \partial_1$. The covariant derivatives are defined by

$$D_\pm \psi^I = \partial_\pm \psi^I + \partial_\pm \phi^J \Gamma^I_{JK} \psi^K.$$

The $\Gamma^I_{JK}$ are Christoffel symbols for the Levi-Civita connection associated with the target space metric. If the target space metric is Kähler and $H = dB = 0$, this action has $\mathcal{N} = (2, 2)$ supersymmetry. In addition, this NLSM is classically conformally invariant. One should keep in mind that in the full string theory, the model would include free fields representing the noncompact directions in spacetime. Since the worldsheet field theory factorizes, we will restrict attention to the internal degrees of freedom.

Now we consider the boundary action $S_{\partial \Sigma}$. The most general boundary action invariant (classically) under boundary reparametrizations and scale transformations is

$$S_{\partial \Sigma} = \int_{\partial \Sigma} d\mu \mu (\phi, \psi_-, \psi_+) + \int_{\partial \Sigma} \sqrt{|d\mu d\nu \eta_{\mu \nu}|} W (\phi, \psi_+, \psi_-), \quad (11)$$

where

$$V_- = A_I (\phi) \partial_- \phi^I + i C_{IJ} (\phi) \psi_-^I \psi_-^J,$$

$$V_+ = A_I (\phi) \partial_+ \phi^I + i \tilde{C}_{IJ} (\phi) \psi_+^I \psi_+^J,$$

$$W = i D_{IJ} (\phi) \psi_+^I \psi_-^J. \quad (12)$$

The $A_I (\phi), C_{IJ} (\phi), \tilde{C}_{IJ} (\phi), D_{IJ}$ are tensors on the target space.\(^2\) If we take the worldsheet to be a strip as above, then

$$S_{\partial \Sigma} = \int_{-\infty}^{\infty} d\phi \left\{ W + \frac{1}{2} (V_+ + V_-) \right\} \bigg|_{x^1 = \pm \pi}.$$

We will restrict attention to this case in what follows.

\(^2\)The same field $A_I (\phi)$ couples to $\partial_+ \phi$ and $\partial_- \phi$ to ensure that only the tangential derivative $\partial_0 \phi$ appears in the boundary action.
3.2 The Boundary Conditions

Let us vary the action with respect to $\phi$ and $\psi_{\pm}$. In computing the variation it is important to note that when we vary the map $\Phi$ the Fermi fields, which as noted above are sections of bundles determined by $\Phi$, cannot be held “constant.” We can think of $\psi_{\pm}$ as sections of a bundle over the space of maps $\Sigma \to X$ with connection given by pulling back $\Gamma$. Parallel transport then determines the variation

$$\delta \psi^I_{\pm} = \delta \psi^I_{\pm} + \Gamma^I_{JK} \psi^J_{\pm} \delta \phi^K,$$

(14)

where $\delta \psi^I_{\pm}$ is the variation of $\psi_{\pm}$ independent of $\delta \phi$.

$$\delta S = \int_\Sigma d^2x \left\{ \delta \phi^K E_K + \delta \psi^I_{+} E^+_{I} + \delta \psi^I_{-} E^-_{I} \right\}
+ \int_{\partial \Sigma} dx^0 \left\{ F_{KI} \delta \phi^I - G_{KI} \partial_\Sigma \phi^I + \frac{1}{2} V'_K \right\} \delta \phi^K
+ \frac{i}{2} \int_{\partial \Sigma} dx^0 \left\{ \delta \psi^I_{-} \left[ (2C_{IJ} - G_{IJ}) \psi^J_{-} - 2D_{JI} \psi^J_{+} \right]
+ \delta \psi^I_{+} \left[ (2\tilde{C}_{IJ} + G_{IJ}) \psi^J_{+} + 2D_{IJ} \psi^J_{-} \right] \right\}$$

(15)

where $F_{KI} = A_{I,K} - A_{K,I} + B_{KI}$, $V'_K = i \left( C_{IJ,K} \psi^I_{+} \psi^J_{-} + 2D_{IJ,K} \psi^I_{+} \psi^J_{+} + \tilde{C}_{IJ,K} \psi^I_{+} \psi^J_{+} \right)$, and $C_{IJ,K} = \nabla_K C_{IJ}$. The bulk term corresponds to the bulk equations of motion for the fields:

$$E^I = D^2 \phi^I - \frac{i}{2} R^I_{JKL} \left( \partial_\Sigma \phi^J \psi^K_{+} \psi^L_{+} + \partial_\Sigma \phi^J \psi^K_{-} \psi^L_{-} \right) - \frac{1}{4} G^{IA} R_{JKLM} A_{I,J,K}^I \psi^J_{+} \psi_+^M \psi^L_{-} \psi_-^M,$$

$$E^-_I = D_+ \psi_{-I} - \frac{i}{2} R_{IJKL} \psi^-_+ \psi^K_+ \psi^L_+,$$

$$E^+_I = D_- \psi_{+I} - \frac{i}{2} R_{IJKL} \psi^+_+ \psi^-_+ \psi^L_+.$$

(16)

As discussed in the previous section, we must choose boundary conditions such that the boundary term in $\delta S$ vanishes. We use the standard Ansatz for the fermion boundary conditions:

$$\psi^I_{+} = \tilde{R}^I_{J} (\phi) \psi^J_{-}.$$

(17)

As previously noted in [9], this form is unique provided that we demand that it respects classical conformal invariance and is non-singular in field-space. These boundary conditions constrain the variations of $\psi_{\pm}$. The allowed variations must satisfy

$$\delta \psi^I_{+} = \tilde{R}^I_{J;K} \delta \phi^K \psi^-_+ + \tilde{R}^I_{J} \delta \psi^J_{-}.$$

(18)
Plugging these expressions into the variation, we find that the boundary term takes the form

\[ \delta S = \int_{\partial \Sigma} d\tau^0 \left\{ F_{KI} \partial_0 \phi^I - G_{KI} \partial_1 \phi^I + \frac{1}{2} V_K \delta \phi^K + \frac{i}{2} \delta \psi^I \right\} \left[ 2C_{IJ} - G_{IJ} \right. \\
+ \tilde{R}^K_i \left( 2\tilde{C}_{KL} + G_{KL} \right) \tilde{R}^L_J + 2\tilde{R}^K_i D_{KJ} - 2\tilde{R}^K_J D_{KI} \left. \psi^J \right\}, \] (19)

where

\[ V_K = i \left\{ C_{IJ,K} + R^L_{J,K} \left( 2D_{LJ} + \left( 2\tilde{C}_{LM} + G_{LM} \right) R^M_J \right) \right\} \psi^J \] (20)

Since we do not wish to constrain \( \psi^- \) on the boundary, we find that in order for locality to hold we must have

\[ \tilde{R}^T G \tilde{R} = G, \quad \tilde{C} = \tilde{R} D^T - D \tilde{R}^T - \tilde{R}^T C \tilde{R}. \] (21)

We use an obvious notation: \( \tilde{R}^A_I G_{AB} \tilde{R}^B_J = G_{IJ} \) is written as \( \tilde{R}^T G \tilde{R} = G \), etc. We now need to choose boundary conditions for the bosons. We assume that the end-point of the string moves along \( M \), a submanifold of \( X \). This represents a D-brane wrapping \( M \subseteq X \). Thus, the allowed \( \delta \phi \) are tangent to \( M \) on the boundary, and the boson boundary conditions we need to impose for locality are

\[ \partial_1 \phi^K = F^K_I \partial_0 \phi^I + \frac{1}{2} V^K - n^K, \] (22)

everywhere on \( \partial \Sigma \), where \( n^K \) is an arbitrary vector normal to \( M \). In an open neighborhood \( U \subset M \) of \( \Phi(z) \) for any \( z \in \partial \Sigma \) we can choose a basis \( e^I_\alpha \) for \( TX|_U \) (a vielbein) adapted to the splitting of \( TX|_U = TM \oplus NM \) into tangent and normal directions. We split the index \( \alpha \) into tangential indices (labelled by \( a, b, c, \ldots \)) and normal indices (labelled by \( z, y, x, \ldots \)). In this basis the boson boundary conditions are

\[ \partial_+ \phi^\alpha = R^\alpha_\beta \partial_- \phi^\beta + T^\alpha_\beta V^\beta, \] (23)

where \( R \) is an orthogonal matrix

\[ R = \begin{pmatrix} (1+F)^{-1} & 0 \\ 0 & -\delta_{zy} \end{pmatrix}, \] (24)

and \( T^\alpha_\beta = \frac{1}{2} \left( \delta^\alpha_\beta + R^\alpha_\beta \right) \). It will be important later that in the vielbein, the only nonvanishing components of \( T \) are \( T^a_b = (1 - F)^{-1} \). Furthermore, the daunting \( V^\alpha \) is greatly simplified by the use of the fermion boundary conditions and Eq. (21):

\[ V^\alpha = i \tilde{R}^{\mu\beta}_\alpha \tilde{R}_{\mu\delta} \psi^\beta \psi^\delta. \] (25)
Note that $V$ is independent of the boundary fermion couplings $C, \tilde{C},$ and $D.$ This reflects the general result that once the fermion boundary conditions are imposed, then the fermion bilinear term in the boundary action is identically zero, hence trivially SUSY invariant, for any $C, \tilde{C}, D$ satisfying Eq. (21). The locality constraint Eq. (21), which can be solved for $\tilde{C}$ in terms of $C, D,$ and $\tilde{R},$ is thus the only constraint on these couplings. As we will see shortly, $\tilde{R}$ is fixed by supersymmetry.

4 Conditions for $\mathcal{N} = 2$ Supersymmetry

In this section we find the restrictions on $M$ and the boundary couplings which lead to an unbroken $\mathcal{N} = 2$ SUSY. We will express these as equations to be satisfied everywhere along $M$ by the various background fields. In the next section we will study these equations and interpret them geometrically.

4.1 SUSY Variation of the NLSM Fields

The bulk action is supersymmetric up to a boundary term under $\mathcal{N} = (2, 2)$ supersymmetry. Since the target space is a complex manifold, and we are working locally, we can choose a set of coordinates where the complex structure $J^A_B$ is constant, in addition to being covariantly constant. In these coordinates the SUSY variations take a particularly simple form:

$$
\delta \phi^I = i \left( \epsilon_+^2 \psi_-^I + \epsilon_+^1 J^I_J \psi_-^J - \epsilon_-^2 \psi_+^I + \epsilon_-^1 J^I_J \psi_+^J \right),
$$

$$
\delta \psi_-^I = \delta \psi_-^I + \Gamma_{JK} \psi_-^J \delta \phi^K,
$$

$$
\delta \psi_+^I = \delta \psi_+^I + \Gamma_{JK} \psi_+^J \delta \phi^K,
$$

(26)

where

$$
\delta \psi_-^I = -\epsilon_-^2 \partial_- \phi^I + \epsilon_+^1 J^I_K \partial_- \phi^K,
$$

$$
\delta \psi_+^I = \epsilon_-^2 \partial_+ \phi^I - \epsilon_+^1 J^I_K \partial_+ \phi^K.
$$

(27)

Up to an irrelevant phase, there are two choices for the $\mathcal{N} = 2$ subalgebra of $\mathcal{N} = (2, 2)$ SUSY preserved by the boundary. These are commonly labelled as $A$ and $B,$ and they are parametrized by $\epsilon$ and $\epsilon',$ with the $\mathcal{N} = (2, 2)$ parameters given by

$$
\epsilon_-^2 = -\epsilon_-^2 = \epsilon,
$$

$$
\epsilon_+^1 = \eta \epsilon_-^1 = \epsilon',
$$

(28)

where $\eta = +1$ for $A$ SUSY and $\eta = -1$ for $B$ SUSY.

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3Essentially, these are the two sets of SUSY generators that anticommute to $\partial_0,$ the unbroken translation generator.
The A/B SUSY variations are
\[
\delta \phi^I = i \epsilon (\psi^I_+ + \psi^I_-) + i \epsilon' J^I_J (\psi^J_- - \eta_\psi^J_+) ,
\]
\[
\delta \psi^I_- = -\epsilon \partial_- \phi^I + \epsilon' J^I_K \partial_- \phi^K ,
\]
\[
\delta \psi^I_+ = -\epsilon \partial_+ \phi^I - \eta \epsilon' J^I_K \partial_+ \phi^K.
\]

4.2 SUSY of the Fermion Boundary Condition
Now we will study the supersymmetry variations of the fermion boundary conditions. Equating coefficients of expressions in the independent boundary fields, in variance of the boundary conditions will lead to geometric constraints on background fields.

4.2.1 The \( \epsilon \) variation
Plugging the \( \epsilon \) SUSY variation into Eq. (18) we get the condition
\[
-(\partial_+ \phi^I - \tilde{R}^I_J \partial_- \phi^J) = 2i \tilde{R}^I_J T^K \psi^L_- \psi^L_+ .
\]
(30)
Using the boson boundary conditions, Eq. (23), we find
\[
-(R^I_J - \tilde{R}^I_J) \partial_- \phi^J = T^I_J V^J - 2i \tilde{R}^I_J T^K \psi^L_+ \psi^L_- .
\]
(31)
Since we do not wish to impose any further constraints on \( \partial_- \phi^I \) and \( \psi^I_+ \), the coefficients of \( \partial_- \phi^I \) and \( \psi^I_- \psi^L_- \) must vanish separately. Thus,
\[
\tilde{R} = R .
\]
(32)
The rest of the computation is simplified if we replace \( \tilde{R} \) by \( R \), so we will do so.

The vanishing of the \( \psi_- \psi_- \) term implies
\[
-T_{ab} V_a = 4i T_{a \beta;a} T_{ab} \psi^b_- \psi^b_- ,
\]
(33)
where we have used \( R^I_{J,K} = 2T^I_{J,K} \), and written the expression in the vielbein basis.

To avoid confusion, let us be explicit about the form of the covariant derivatives in the vielbein basis:
\[
A_{\alpha;\beta} = A_I e^I_{\alpha} e^J_{\beta}.
\]
More explicitly,
\[
A_{\alpha;\beta} = A_{\alpha,\beta} + \omega_{\alpha \gamma} A_{\gamma},
\]
where \( A_{\alpha,\beta} = e^I_{\beta} \nabla_I A_{\alpha} \), and \( \omega_{\alpha \gamma} = e^I_{\beta} e^J_{\alpha} \nabla_I e^J_{\gamma} \) is the spin connection.
Using the form of $V_\alpha$ given in Eq. (25), we find the following restriction on $R_{\alpha\beta}$:

$$T_{\alpha a} \left( T_{\mu\delta,a} R_{\mu\beta} - T_{\mu\beta,a} R_{\mu\delta} \right) = 2 \left( T_{\alpha\delta,a} T_{a\beta} - T_{\alpha\beta,a} T_{a\delta} \right).$$  \hspace{1cm} (34)

To extract the consequences of this equation, we split the free indices into the normal and tangent directions, and carry out a case-by-case analysis. This is aided by a few helpful and easily verifiable facts:

$$T_{xya} = 0, \quad T_{xba} = w_{axc} T_{eb}, \quad T_{bxa} = w_{axc} T_{bc}, \quad T_{abc} = T_{ad} H_{dec} T_{eb},$$  \hspace{1cm} (35)

where $H_{dec} = -H_{edc} = F_{de} + w_{eza} F_{zd} - w_{aze} F_{zd}$. Also, the number of cases to consider is lessened if we note the antisymmetry of Eq. (34) in the $\beta, \delta$ indices. We find that Eq. (34) is equivalent to

$$\omega_{eza} = \omega_{aze},$$  \hspace{1cm} (36)

$$T_{ae} \left( T_{fde} R_{fb} - T_{fbd} R_{fe} \right) = 2 \left( T_{ad} e T_{eb} - T_{abc} T_{ed} \right).$$  \hspace{1cm} (37)

So, the fermion boundary conditions are invariant under the $\epsilon$ SUSY if and only if these two requirements are met.

### 4.2.2 The $\epsilon'$ variation

Plugging the $\epsilon'$ variation into Eq. (18) and using the boson boundary conditions, we find

$$- \left( \eta_{IJ} J^I J^J_K + R^I J^J_K \right) \partial_a \phi^K = \eta_{IJ} J^I J^J_K V^J + 4i T_{J^I} T_{J^K} T^K L_P J^L_P \psi^P \psi^J.$$

(38)

Again, the two sides must vanish separately. The left-hand side gives the condition

$$RJ + \eta JR = 0.$$  \hspace{1cm} (39)

The left-hand side gives

$$\eta J_{ae} T_{ef} T_{\mu\delta,f} R_{\mu\beta} - 2 T_{a\delta,f} T_{ef} J_{f\beta} = (\delta \leftrightarrow \beta).$$  \hspace{1cm} (40)

It turns out that it is not necessary to do the case-by-case analysis as above. Using $J^2 = -1$, $JR + \eta JR = 0$, and the covariant constancy of $J$, it is easy to show that this is equivalent to Eq. (37). Thus, $\mathcal{N} = 2$ invariance follows from $\mathcal{N} = 1$ and Eq. (39).
4.3 SUSY of the Boson Boundary Condition

Now we explore the supersymmetry of the boson boundary conditions. As a warm-up, let us consider what happens in flat space with $F = 0$. This is an easy case, since the matrix $R$ is now constant, and the two fermion term in the boson boundary conditions can be taken to be zero. Consider the $\epsilon$ SUSY, for which $\delta \phi^I = i\epsilon (\psi^I_+ + \psi^I_-)$. Plugging this variation into the bosonic boundary conditions, we have the requirement

$$\partial_+ (\psi^I_- + \psi^I_+) = R^I_J \partial_- (\psi^J_- + \psi^J_+) . \quad (41)$$

This is not an algebraic condition on the $\psi_{\pm}$, and we cannot satisfy it by using just the algebraic conditions we already have. One is tempted to differentiate the fermion boundary conditions. Since $\psi^I_+(x^0, 0) = R^I_J \psi^J_+(x^0, 0)$ must hold for all $x^0$, we are allowed to differentiate this relation with respect to $x^0$. Unfortunately, it does not make sense to differentiate it with respect to $x^1$. We recall, however, the fermion equations of motion: $\partial_{\mp} \psi^I_{\pm} = 0$, which allow us to relate $\partial_1 \psi_{\pm}$ to $\partial_0 \psi_{\pm}$. By using the equations of motion, we see that the variation of the bosonic boundary conditions becomes

$$\partial_0 \psi^I_+ = R^I_J \partial_0 \psi^J_-, \quad (42)$$

which is satisfied if the fermion boundary conditions are supersymmetric. Classically, the use of the equations of motion is justified. Since we are eventually interested in the quantum problem, we will need to find a suitable modification. We discuss this below. Here we will show that the boson boundary conditions are supersymmetric up to the fermion equations of motion.

The computation is greatly simplified if we use the identity $T^I_J V^J = 4i T^I_{J;K} T^K_L \psi^J_L - \psi^L_J$. This reduces to Eq. (33) when we use the properties of $T$. Since this holds for any values of the Fermi fields satisfying the (SUSY) boundary conditions, its SUSY variation is also an identity. We will use this to write the boson boundary conditions as

$$\partial_+ \phi^I - R^I_J \partial_- \phi^J = i S^I_{JL} \psi^J_L, \quad (43)$$

where $S^I_{JL} = 2 (T^I_{J;K} T^K_L - T^I_{L;K} T^K_J)$. The variation will take the form

$$\partial_+ \delta \phi^I - R^I_J \delta \phi^K \partial_- \phi^J - R^I_J \partial_- \delta \phi^J = i \delta \left( S^I_{JL} \psi^J_L \psi^L_+ \right) . \quad (44)$$

4.3.1 The $\epsilon$ variation

Let us first work out the left-hand side of Eq. (44).

$$\text{LHS} = i \epsilon \left\{ \partial_+ (\psi^I_- + \psi^I_+) - R^I_J \partial_- (\psi^J_- + \psi^J_+) \right\} - 2i \epsilon R^I_{J,K} \partial_- \phi^J T^K_L \psi^L_. \quad (45)$$
We will use the fermion equations of motion (Eq. (16)) to simplify the \( \ldots \) above. We find

\[
\text{LHS} = i \epsilon \left\{ 2 S^I_{JL} \psi^J \partial_\phi^L \\
+ i \left[ (R^I_{C:L} - 2 \Gamma^I_{JL} T^I_C) S^L_{AB} \\
+ T^I_J R^J_{DEF} \left( \delta^D_A R^E_B R^F_C - R^D_A \delta^E_B \delta^F_C \right) \right] \psi^A \psi^B \psi^C \right\}.
\]

The variation of the right-hand side is straightforward.

\[
\text{RHS} = 2 i \epsilon \left\{ S^I_{JL} \psi^J \partial_\phi^L + i \left( S^I_{AB;K} - \Gamma^I_{JL} S^J_{AB} \right) T^K_C \psi^A \psi^B \psi^C \right\}.
\]

Setting LHS = RHS, we see that the one fermion terms cancel, and we are left with the three fermion terms. The boson boundary conditions are invariant under the \( \epsilon \) SUSY variation if

\[
Q^I_{[ABC]} = \frac{1}{2} T^I_J R^J_{DEF} \left( \delta^D_A R^E_B R^F_C - R^D_A \delta^E_B \delta^F_C \right),
\]

where \( Q^I_{ABC} = S^I_{AB;K} T^K_C - S^K_{AB} T^I_{C;K} \). We will spare the reader the details, but roughly, the equality can be shown as follows. There are terms of the form \( [\nabla_L, \nabla_M] T \) in \( Q^I_{[ABC]} \). By the definition of the Riemann tensor, these can be written as sums of contractions of the Riemann tensor with \( T \). Then Eq. (48) can be shown to hold by repeated use of \( R^I_{JKL} = 0 \).

We see that, up to the equations of motion, there are no new constraints from the SUSY of the boson boundary conditions.

### 4.3.2 The \( \epsilon' \) variation

We begin the same way as for the \( \epsilon \) variation. Eliminating the \( \partial_1 \psi_\pm \) by the equations of motion, and using the boundary conditions, we find that the variation of the left-hand side is

\[
\text{LHS} = 2 i \epsilon' \left\{ S^I_{AB} J^A_{\partial_\phi^D \psi^B} + i \left[ (T^I_{J:L} - \Gamma^I_{JL} T^I_K) J^I_A S^L_{BC} \right] + \frac{1}{2} T^I_J J^J_K R^K_{DEF} \left( \delta^D_A R^E_B R^F_C + \eta R^D_A \delta^E_B \delta^F_C \right) \right\}.
\]

The variation of the right-hand side gives

\[
\text{RHS} = 2 i \epsilon' \left\{ S^I_{AB} J^A_{\partial_\phi^D \psi^B} + i \left( S^I_{AB;K} - \Gamma^I_{JL} S^J_{AB} \right) T^K_M J^M_C \psi^A \psi^B \psi^C \right\}
\]

To show that LHS = RHS, we will first show that the one fermion terms match. This is so if

\[
S_{\gamma\alpha\beta} = J_{\rho\alpha} S_{\gamma\rho\nu} J_{\nu\beta}.
\]

But, SUSY of the fermion boundary conditions implies (Eq. (34))

\[
S_{\gamma\rho\nu} = T_{\gamma\epsilon} (T_{\rho\nu;e} R_{\mu\rho} - T_{\mu\nu;e} R_{\mu\rho}).
\]
Using $T_{\mu\lambda;e} = \eta J_{\mu\chi} T_{\chi\kappa;e} J_{\kappa\lambda}$, and $JR + \eta RJ = 0$, we find

\[ J_{\rho\alpha} S_{\gamma\rho\nu} J_{\nu\beta} = T_{\gamma e} (T_{\rho\alpha;e} R_{\mu\beta} - T_{\mu\beta;e} R_{\rho\alpha}) = S_{\gamma\alpha\beta} \quad (53) \]

So, we see that the one fermion terms do indeed agree.

What about the three fermion terms? The remaining condition to be satisfied is

\[ (S^I_{AB;K} T^K_J - T^I_{J,L;S^L_{AB}}) \psi_+^A \psi_B^J \psi_-^C = \frac{1}{2} T^I_J J^K_R R^K_{DEF} (\delta^D_A R^E_B R^F_C + \eta R^K_A \delta^E_B R^K_C) \psi_+^A \psi_B^J \psi_-^C. \quad (54) \]

Using $S = JSJ$, the left-hand side takes the form

\[ \text{LHS} = (S^I_{AB;K} T^K_J - T^I_{J,L;S^L_{AB}}) \widehat{\psi}_+^A \widehat{\psi}_B^J \widehat{\psi}_-^C, \quad (55) \]

where $\widehat{\psi}_+^A = J^A_J \psi_+^J$.

With a little work we can rewrite the right-hand side as

\[ \text{RHS} = \frac{1}{2} R^K_{DEFG} (\delta^K_A R^K_B R^K_C + \eta R^K_A \delta^K_B R^K_C) \widehat{\psi}_+^A \widehat{\psi}_B^B \widehat{\psi}_-^C. \quad (56) \]

Setting LHS = RHS, and extracting the piece totally antisymmetric in $(ABC)$, we find Eq. (58).

So, we have shown that, up to the fermion equations of motion, the boson boundary condition is SUSY provided that the fermion boundary condition is SUSY.

### 4.4 SUSY of the Action

We will now examine the conditions for the action to be supersymmetric. This computation is simplified by the realization that the fermion bilinear boundary couplings drop out of the action once we use the boundary conditions.

Thus,

\[ \delta S_{\partial \Sigma} = -\frac{i}{2} \int_{\partial \Sigma} dx^0 \left( \partial_+ \phi^I + \partial_- \phi^I \right) F_{IK} \left[ \epsilon T^K_L + \epsilon' T^K_M J^M_L \right] \psi_-^L \quad (57) \]

The variation of the bulk action is given by:

\[ \delta S_{\Sigma} = \frac{i}{2} \int_{\partial \Sigma} dx^0 \left\{ \epsilon \left( \partial_- \phi^I R^I_{IL} - \partial_+ \phi^I G^I_{IL} \right) - \epsilon' \left( \eta \partial_- \phi^I J^I_{JL} + \partial_+ \phi^I J^I_{JL} \right) \right\} \psi_-^L. \quad (58) \]
4.4.1 The $\epsilon$ variation

Extracting the $\epsilon$ term from above, and using the boson boundary conditions, we find

$$\delta S = \frac{i\epsilon}{2} \int_{\partial \Sigma} dx^0 \left\{ \partial_- \phi^I \left( -4T^I_J F_{JK} T^K_L + R_{IL} - R_{LI} \right) \partial_- \phi^I \psi_-^L - V^I T^I_J \left( G_{JL} + 2F_{JK} T^K_L \right) \right\}. \quad (59)$$

It is easy to show that $R - R^T = 4T^T F T$, so that the one fermion term is zero. Since, $T^T(1 + 2FT) = T$, it follows that

$$\delta S = -\frac{i\epsilon}{2} \int_{\partial \Sigma} dx^0 V_I T^I_L \psi_-^L. \quad (60)$$

Using the expression for $V$ (Eq. (25)) the condition for the action to be invariant under the $\epsilon$ SUSY is

$$T_{e\alpha} T_{\mu\beta;e} R_{\mu\delta} \psi_-^\alpha \psi_-^\beta \psi_-^\delta = 0. \quad (61)$$

This leads to one non-trivial requirement:

$$T_{\epsilon f;e} [T_{e\alpha} \delta_{f\beta} R_{\epsilon\delta} + (bda) + (dab)] = 0. \quad (62)$$

4.4.2 The $\epsilon'$ variation

The $\epsilon'$ variation of the action is

$$\delta S = \frac{i\epsilon'}{2} \int_{\partial \Sigma} dx^0 \left\{ \partial_- \phi^I \left( -4T^I_J F_{JK} T^K_L + R_{IL} - R_{LI} \right) J^M_L \psi_-^L - V^I T^I_J \left( G_{JM} + 2J_{JK} T^K_M \right) J^M_L \psi_-^L \right\}. \quad (63)$$

As in the $\epsilon'$ variation, the one fermion term is zero, and we are left with the condition

$$T_{\alpha;e} T_{\beta;\alpha} R_{\mu\delta} J^\gamma_{\gamma\alpha} \psi_-^\alpha \psi_-^\beta \psi_-^\delta = 0. \quad (64)$$

To show that this holds, we use $JR + \eta RJ = 0$, and $(JTJ)_{e\alpha} = \eta T_{\alpha}$. The condition becomes

$$T_{\mu\alpha;\alpha} R_{\mu\beta} T_{\alpha\gamma} \psi_-^\alpha \psi_-^\beta \psi_-^\gamma = 0, \quad (65)$$

which is satisfied provided that Eq. (61) holds.
5 Satisfying the Constraints

5.1 Algebraic Conditions on $R, J, F$

Here we will explore the geometric meaning of the condition $RJ + \eta JR = 0$. Using the explicit form of $R$ in the vielbein (Eq. (24)), we find that for A-type supersymmetry, $J$ and $F$ must satisfy the following:

\begin{align*}
J_{zy} &= 0, \\
J_{ab} &= F_{ac}J_{cd}F_{db}, \\
F_{ab}J_{bc} &= 0. \\
\end{align*}

(66)

The first equation implies that the A-type supersymmetric cycle is locally a co-isotropic submanifold. The last two equations have solutions if and only if $\dim M = \frac{1}{2}\dim X + 2k$, where $k$ is a non-negative integer. A cycle is Lagrangian if and only if $F_{ab} = 0$. Note that if $B \neq 0$, $F$ is not a curvature associated to a connection on a line bundle!

For B-type supersymmetry, we find a different set of conditions:

\begin{align*}
J_{az} &= 0, \\
J_{ab}F_{bc} &= F_{ab}J_{bc}. \\
\end{align*}

(67)

The first of these means that $M$ is a holomorphic submanifold of $X$, while the second requires $F$ to be a $(1,1)$ form with respect to the complex structure $M$ inherits from $X$. These conditions are familiar from both earlier worldsheet analyses as well as the world-volume analysis of Becker et al [6].

5.2 Differential Geometry Constraints

5.2.1 Constraint on the Spin Connection

The first constraint we will address is Eq. (36). This is a condition on the curvature, $\omega_{a zb} = \omega_{bza}$. We recall that $\omega_{a zb} = e^I_oe^J_z\nabla_I e_b = (e_z, \nabla e_a e_b)$, where $(.,.)$ denotes the Riemannian metric. We recall a basic result about the Levi-Civita connection. Since it is, by definition, symmetric, it satisfies $\nabla_X Y - \nabla_Y X = [X, Y]$ for any differentiable vector fields $X, Y$. So, we see that the condition is equivalent to

\begin{align*}
(e_z, [e_a, e_b]) = 0. \\
\end{align*}

(68)

This is a necessary and sufficient condition that the collection of vector spaces $T_pM \subset T_pX$ for $p \in U \subset M$ spanned by the vector fields $e_a$ contains integral submanifolds! If the brane wraps a submanifold of the target space, then this condition is met.
5.2.2 Constraints on $R$

Now we will study the other two constraints, Eqs. (37,62). Let us start with the latter. Remembering that $T_{cf;e} = T_{cg} H_{ghe} T_{hf}$, and $T^T R = T$, we write it as

$$[H_{ghe} + H_{egh} + H_{heg}] T_{gd} T_{hb} T_{ea} = 0. \quad (69)$$

But, it is trivial to show that $H_{[abc]} = F_{[ab,c]}$. Hence, SUSY is implied by the Bianchi identity for the $B$-gauge invariant field-strength $F_{ab}$ on the brane.

Now let us show that Eq.(37) is satisfied as well. Writing $T_{fd;e}$ as above, and using $T^T R = T$, we get

$$T_{ae} T_{ib} H_{ije} T_{jd} - 2 T_{ai} H_{ije} T_{jd} T_{eb} = (b \leftrightarrow d). \quad (70)$$

Since $T_{ae}$ is invertible, we get

$$(H_{ijk} - 2 H_{kji}) T_{ib} T_{jd} = (b \leftrightarrow d). \quad (71)$$

Using the antisymmetry of $H$ on the first two indices, we finally have

$$(H_{ijk} + 2 H_{jki} - H_{jik} - 2 H_{ikj}) T_{ib} T_{jd} = 0, \quad (72)$$

which is exactly $H_{[ijk]} = 0$. Again, the condition reduces to the Bianchi identity for $F$. Since $F$ clearly satisfies the Bianchi identity, we conclude that classically there are no constraints on the local geometry of supersymmetric cycles except for the usual co-isotropic/holomorphic conditions—Eqs. (66,67).

6 Discussion

We have given a careful treatment of SUSY boundary conditions for the $\mathcal{N} = (2, 2)$ NLSM. Starting from simple classical mechanics notions we reproduced the well-known conditions on the supersymmetric cycles. Before we wrap up, we would like to reward the reader’s patience with a discussion of some interesting issues raised in the text.

6.1 Locality constraint on the boundary action

We have stressed that the fermion boundary couplings have no effect on the supersymmetric cycles. In fact, provided that the boundary conditions are chosen to satisfy locality (Eqs. (21,23)), the fermion boundary coupling drops out of the action entirely. However, in order for locality to be satisfied, $C, D, \bar{C}$ must satisfy

$$\bar{C} = RD^T - DR^T - R^T CR, \quad (73)$$
where we have used the SUSY condition $\tilde{R} = R$. A common fermion boundary coupling used in the literature is

$$F_{IJ} \left( \psi_I^+ + \psi_I^- \right) \left( \psi_J^+ + \psi_J^- \right).$$

(74)

It was motivated by Callan et al as the exponentiation of the open string photon vertex operator, and more recently it has been used in, for example, [10]. This is a very natural term to write. Not only does it look like the exponentiation of the photon vertex operator, but in addition, by using the boundary conditions it may be written as $4(T^T F T)_{IJ} \psi_I^+ \psi_J^-$, a form that ensures that only the tangential components of $F$ enter into the action. Unfortunately, this natural term does not satisfy the locality requirement. Consider a space-filling brane. Setting $C = C = D = F$, the locality constraint is written as

$$F(2 + R + R^T) = 0.$$  

(75)

It is easy to check that this is satisfied if and only if $F = 0$. So, although there are few constraints on the boundary fermion action, the boundary actions in the literature do not seem to satisfy them!

Upon careful consideration, it is clear that there are subtleties associated with the “exponentiation” of the photon vertex. Quite simply, the naive exponentiation procedure does not take into account the change in the boundary conditions that accompanies turning on a non-trivial $A(\phi)$ background. This should be compared with the exponentiation of closed string massless states into coherent state backgrounds, where such subtleties do not arise.

6.2 Superfield Considerations

In studying the SUSY of the boson boundary condition, we came across a vexing problem: the condition was SUSY, but only up to the equations of motion! This is fine for classical mechanics, but it is certainly not satisfactory for quantum considerations. How are we to remedy this? There is one case where this is familiar to superspace aficionados: the supersymmetry algebra does not close off-shell once the auxiliary fields are integrated out of the theory. Indeed, the algebra closes up to the equations of motion! Here a similar situation holds, and the most optimistic way to interpret this is that there is a superspace formulation of this discussion where the boson boundary conditions are supersymmetric off-shell. Thus, it would be very satisfactory to express this entire discussion in superspace. By doing this, one might hope to obtain a description where the supersymmetry algebra closes off-shell and the boson boundary conditions are supersymmetric off-shell. Even more fundamentally, one might hope that superspace would naturally provide a unique boundary action.

6.2.1 Adding auxiliary fields: a toy example

Let us try to include the auxiliary fields. To simplify matters, let us consider the case of flat target space and constant $F$. Also, let us just worry about preserving an $\mathcal{N} = 1$ subalgebra.
of the manifest \( \mathcal{N} = (1, 1) \) SUSY. The \( \mathcal{N} = (1, 1) \) superfield has the component expansion

\[ \Phi = \phi + i\theta^- \psi_\perp + i\theta^+ \psi_\perp + i\theta^- \theta^+ Y. \]  

(76)

If we demand that the fermions obey the boundary condition \( \psi_\perp = R \psi_\perp \) (we suppress target space indices for this discussion), and that the boundary condition is supersymmetric under the \( \epsilon \) SUSY, we find that the bosons must satisfy

\[ \partial_+ \phi = R \partial_- \phi - (1 + R) Y. \]  

(77)

This is supersymmetric since the preserved supercharge squares to \( i\partial_0 \). Furthermore, there is an elegant superspace expression for the boundary conditions:

\[ [\mathcal{D}_+ \Phi - R \mathcal{D}_- \Phi]_{\theta^+=\theta^-} = 0, \]  

(78)

where \( \mathcal{D}_\pm \) are the superspace derivatives. This looks nice, but the rub is in trying to write down a sensible boundary action which will produce the above as a locality constraint without spoiling the supersymmetry of the action itself. In fact, it is easy to convince oneself that this is impossible. A way out of this conundrum was suggested by Lindström \textit{et al} in [12].

Since the local boson boundary conditions are \( \partial_+ \phi = R \partial_- \phi \), it is natural to reconcile locality with Eq. (77) by introducing an additional boundary condition for the auxiliary field:

\[ (1 + R) Y = 0. \]  

(79)

This is not quite enough. We must also demand that \( \delta_\epsilon [(1 + R) Y] = 0. \) This leads to a final boundary condition:

\[ (1 + R) (\partial_- \psi_\perp - \partial_+ \psi_\perp) = 0. \]  

(80)

These subsidiary boundary conditions can be written in superspace as

\[ [(1 + R) \mathcal{D}_+ \mathcal{D}_- \Phi]_{\theta^+=\theta^-} = 0. \]  

(81)

At first sight these look quite strange since they do not follow from locality, and since the second one involves derivatives of the fermions on the boundary. One might worry that such a term spoils the initial value problem. However, as pointed out in [12], these conditions are trivially satisfied on-shell since Eq. (79) is proportional to the \( Y \) equations of motion, and Eq. (80) is proportional to the fermion equations of motion. It is easy to check explicitly that these boundary conditions are supersymmetric.

Thus, at least in this toy example, we can find a supersymmetric version of the boundary conditions. The price to pay is the (expected) introduction of auxiliary fields \( Y \) for the closure of the supersymmetry algebra and additional boundary conditions—Eqs. (79,80) which do not follow from locality.
6.2.2 A superspace action

The toy example above suggests that it is possible to introduce auxiliary fields and supplementary boundary conditions so that the action and the boundary conditions are supersymmetric off-shell. To study the more general curved background, it is convenient to work in superspace. Consider once again the bulk $\mathcal{N} = (1, 1)$ NLSM (see, for example, [13] for superspace conventions). In the presence of a boundary the action transforms under the diagonal $\mathcal{N} = 1$ supersymmetry generated by the unbroken supercharge $Q = Q_+ + Q_-$. Of course, one can choose boundary conditions that ensure the invariance of the action.

Alternatively, it is fairly straightforward to add a boundary Lagrangian to make the action $B$-gauge invariant and of the form

$$ S = \int_\Sigma [Q, O] + \int_{\partial \Sigma} [Q, O], $$

where $[Q, O]$ denotes the SUSY action on $O$. The “improved” total action is given by

$$ S = -\frac{1}{4} \int_\Sigma d^2x \left[ \left( G_{IJ}(\Phi) + B_{IJ}(\Phi) \right) D_+ \Phi^I D_- \Phi^J \right]_{\theta^\pm=0} $$

$$ - \frac{i}{2} \int_{\partial \Sigma} dx^0 D \left[ A_I(\Phi) D\Phi^I \right]_{\theta^\pm=0}. $$

Here $D = D_+ + D_-$ and $\bar{D} = D_+ - D_-$ correspond to the preserved and to the broken linear combinations of supercharges respectively. Since $Q^2 = 2i\partial_0$, this action is obviously SUSY invariant. We can consider the variation of action under the $\epsilon'$ SUSY as well. One can show that the total action is invariant without the use of boundary conditions under the $\epsilon'$ variation for $\mathbf{B}$-type SUSY, but not for $\mathbf{A}$-type SUSY, where boundary conditions are needed for invariance. This is to be expected: $\mathbf{B}$-type SUSY is compatible with the holomorphic structure of $\mathcal{N} = (2, 2)$ superspace, while $\mathbf{A}$-type SUSY requires reality conditions that are, in a sense, “unnatural” from the holomorphic $\mathcal{N} = (2, 2)$ point of view.

To investigate how locality is affected by the addition of this improvement term, we write the variation of the action as

$$ \delta S = \int_\Sigma d^2x \left[ D_+ \Phi^I \delta \Phi^J \right]_{\theta^\pm=0} $$

$$ + \frac{i}{2} \int_{\partial \Sigma} dx^0 D \left[ F_I(\Phi) \delta \Phi^I \right]_{\theta^\pm=0} $$

$$ - i \int_{\partial \Sigma} dx^0 \left[ (G_{IJ} D_+ \Phi^I + G_{IJ} \Gamma^I LM D_- \Phi^L D_+ \Phi^M) \delta \Phi^J \right]_{\theta^\pm=0}. $$

Note that in superspace the algebraic statement $[Q, O]$ is geometrized into $D(O)$.

One can add a different boundary term that ensures invariance of the action under $\mathbf{A}$-type SUSY without the use of boundary conditions. In components, it is of the form $G_{IJ} \phi^I \phi^J + \frac{i}{2} \partial_1 (\phi^I G_{IJ} \phi^J)$. Unfortunately, this term is incompatible with locality.
The first line encodes the bulk equations of motion for the fields. The second and third lines encode the boundary terms in the variation that need to be eliminated by an appropriate choice of boundary conditions. The term in the second line are explicitly supersymmetric, since they are expressed as $[Q,s]$. The third line precisely contains the non-supersymmetric contribution to the boundary conditions. One can show that these contributions will vanish if we impose the following subsidiary boundary conditions in addition to locality:

$$T^I_J \left( Y^J - i \Gamma^J_{KL} \psi^K_+ \psi^L_+ \right) = 0, \quad (85)$$

and

$$T^{IJ} \left( E^+_J - E^-_J \right) = 0, \quad (86)$$

where $E^\pm$ are the fermion equations of motion (Eq. [16]). As in the toy example, the two additional boundary conditions are trivially satisfied on-shell: Eq. (85) follows from the $Y$ equations of motion, while Eq. (86) follows from the fermion equations of motion.

The subsidiary boundary conditions can be imposed dynamically by introducing boundary superfield Lagrange multipliers, $\Lambda^I(x^0) = \lambda^I(x^0) + \theta^I(x^0)$ into the action:

$$\int_{\partial \Sigma} dx^0 \mathcal{D} \left[ \Lambda^I \left( G_{IJ} \mathcal{D}_- \mathcal{D}_+ \Phi^J + G_{IJ} \Gamma^J_{LM} \mathcal{D}_- \Phi^L \mathcal{D}_+ \Phi^M \right) \right] \big|_{\theta^\pm=0}. \quad (87)$$

$\Lambda^I$ is a fermionic superfield with non-zero components in the directions tangent to the cycle.

As promised above, the superspace approach also suggests a natural form for the fermion bilinear coupling on the boundary. It is

$$-\frac{i}{4} \left[ F_{IJ} \left( \psi^I_+ + \psi^I_- \right) \left( \psi^J_+ + \psi^J_- \right) + 2G_{IJ} \psi^I_+ \psi^J_- \right], \quad (88)$$

which automatically satisfies the locality condition (Eq. [21]). Though this form is not uniquely determined by requirements of locality, it arises so naturally from superspace that it would not be too surprising if this is precisely what one would obtain by a careful exponentiation of the photon vertex operator. This action has also been obtained by a different method in [5].

7 Acknowledgments

We would like to thank P.S. Aspinwall, R. Bryant, C. Curto, D. Fox, A. Kapustin, R. Karp, D.R. Morrison, K. Narayan, M. Roček and M. Stern for useful discussions. One of us (IVM) would like to thank the organizers of TASI 2003, where some of this work was completed. MRP and SR thank KITP and the organisers of the Geometry, Topology, and Strings workshop (MP03) where some of this work was completed. Their participation was supported in part by the National Science Foundation under Grant No. PHY99-07949. This work is also supported by NSF grant DMS-0074072.
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