Commenting on “Revisiting the 2PN Pericenter Precession in View of Possible Future Measurements”, by L. Iorio, and on “The orbital pericenter precession in the 2PN approximation”, by S.M. Kopeikin

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Abstract

Recently, the secular pericentre precession was analytically computed to the second post-Newtonian (2PN) order by the present author with the Gauss equations in terms of the osculating Keplerian orbital elements in order to obtain closer contact with the observations in astronomical scenarios of potential interest. A discrepancy with previous results by other authors was found. Moreover, some of such findings by the same authors were deemed as mutually inconsistent. In this paper, it is demonstrated that, in fact, two calculational errors plagued the most recent calculation. They are explicitly disclosed and corrected. As a result, all the examined approaches mutually agree yielding the same analytical expression for the total 2PN pericentre precession once the appropriate conversions from the adopted parameterizations are made.

keywords general relativity and gravitation; celestial mechanics

1. Introduction

The analytical calculation of the secular 2PN pericentre precession $\dot{\omega}^{2\text{PN}}$ of a gravitationally bound two-body system made of two mass monopoles $M_A, M_B$ with the perturbative Gauss equations for the variation of the osculating Keplerian orbital elements (e.g. Kopeikin, Efroimsky & Kaplan 2011; Soffel & Han 2019) was the subject of Iorio (2020). For the sake of simplicity, the test particle limit will be considered in most of the paper. In the following, $c$ is the speed of light in vacuum, $\mu = GM$ is the gravitational parameter of the primary whose mass is $M$, $G$ is the Newtonian constant of gravitation, $v, v_r, v$ are the test particle’s speed, radial velocity and velocity, respectively, $r$ is the test particle’s distance from the primary, $\hat{r}$ is the position unit vector of the test particle with respect to the primary, $f_0, a, e$ are the osculating numerical values of the true anomaly, semimajor axis and eccentricity, respectively, at the same arbitrary moment of time $t_0$, and $n_b = \sqrt{\mu/a^3}$ is the osculating mean motion.

The expression for the total 2PN pericentre precession derived by Iorio (2020) consists of the sum of three contributions. The first one, dubbed as “direct”, is (Iorio 2020, Equation (8))

$$\dot{\omega}^{2\text{PN}}_{\text{dir}} = \frac{n_b \mu^2 (28 - e^2)}{4 a^2 e^2 (1 - e^2)^2}, \quad (1)$$

arising straightforwardly from the 2PN acceleration

$$A^{2\text{PN}} = \frac{\mu^2}{c^6 r^5} \left[ (2 v_r^2 - 9 \mu / r) \hat{r} - 2 v \hat{r} v \right]. \quad (2)$$

There are also two further contributions, labeled as “mixed” or “indirect”. They account for the fact that, when the Gauss equation for the rate of change of the pericentre induced by the 1PN
acceleration
\[ A^{1\text{PN}} = \frac{\mu}{c^2 r^2} \left[ \left( \frac{4\mu}{r} - v^2 \right) \hat{r} + 4vr \right] \]  
(3)
is averaged over one orbital period \( P_b \), the latter one has to be considered as the time interval between two consecutive crossings of the moving pericentre. Moreover, also the instantaneous shifts of the other orbital elements due to Equation (3) itself are to be taken into account when the orbital average is performed. Both such effects contribute the total pericentre precession to the 2PN level. The first indirect effect yields (Iorio 2020, Equation (14))
\[ \dot{\omega}^{2\text{PN}(\text{I})}_{\text{indir}} = \frac{n_b \mu^2 \left( 9 + 37e^2 + e^4 \right)}{2c^4 a^2 e^3 (1 - e^2)^2}, \]
(4)
while the second indirect contribution reads (Iorio 2020, Equation (22))
\[ \dot{\omega}^{2\text{PN}(\text{II})}_{\text{indir}} = -\frac{n_b \mu^2 \left( 9 - 87e^2 - 136e^4 + 19e^6 - 6e^3 \left[ (34 + 26e^2) \cos f_0 + 15e \cos 2f_0 \right] \right)}{2c^4 e^2 a^2 (1 - e^2)^3}. \]
(5)
Thus, the sum of Equation (1), Equation (4), and Equation (5) gives the total 2PN pericentre precession
\[ \dot{\omega}^{2\text{PN}}_{\text{tot}} = \frac{3n_b \mu^2 \left[ 86 + 57e^2 - 13e^4 + 8e \left( 17 + 13e^2 \right) \cos f_0 + 60e^2 \cos 2f_0 \right]}{4c^4 a^2 (1 - e^2)^3}. \]
(6)
Iorio (2020) compared his results with those by Kopeikin & Potapov (1994), who used the perturbative approach relying upon the Gauss equations as well, and those by Damour & Schafer (1988), obtained with the Hamilton-Jacobi method. A discrepancy with such authors was found since Iorio (2020) demonstrated that the total 2PN pericentre precession inferred by Kopeikin & Potapov (1994, Equation (5.2)) can be cast into the form
\[ \dot{\omega}^{2\text{PN}}_{\text{tot}} = \frac{3n_b \mu^2 \left( 2 + e^2 - 32e^2 \cos f_0 \right)}{4c^4 a^2 (1 - e^2)^2}, \]
(7)
which, however, was not shown by Kopeikin & Potapov (1994). Furthermore, Iorio (2020) claimed that Damour & Schafer (1988, Equation (3.12)), which was shown to be coincident with Equation (7), and Damour & Schafer (1988, Equation (5.18)) would be mutually inconsistent.

Here, it will be proven that, actually, a mere calculational error occurred in the derivation of \( \dot{\omega}^{2\text{PN}(\text{II})}_{\text{indir}} \) by Iorio (2020) which prevented to obtain Equation (7) instead of the incorrect Equation (6). Once such an error is corrected, both the approaches by Iorio (2020) and Kopeikin & Potapov (1994), which differ in how obtaining just \( \dot{\omega}^{2\text{PN}(\text{II})}_{\text{indir}} \), agree yielding the same total 2PN pericentre precession of Equation (7). Instead, Kopeikin (2020) incorrectly claimed that the whole approach by Iorio (2020) would be flawed by serious and fundamental errors. Moreover,
it will be shown that also the alleged inconsistency of Equation (3.12) and Equation (5.18) by Damour & Schafer (1988) is, in fact, due to another error by Iorio (2020), as correctly pointed out by Kopeikin (2020).

To the benefit of the reader, it is noted that Damour & Schafer (1988); Kopeikin & Potapov (1994) usually dealt with the fractional pericentre advance per orbit, i.e., $\Delta \omega/2\pi$; in order to obtain the corresponding precession, it is sufficient to multiply it by $n_b$.

The paper is organized as follows. In Section 2, the calculational error in working out $\dot{\omega}_{\text{indir}}^{2\text{PN}(II)}$ is explicitly disclosed and corrected. Section 3 is devoted to showing that Equation (3.12) and Equation (5.18) of Damour & Schafer (1988) are, actually, mutually consistent yielding both the same total 2PN pericentre precession as Equation (7). Some aspects of the critique by Kopeikin (2020) are discussed in Section 4. Section 5 summarizes the present findings and offers concluding remarks.

## 2. Disclosing and correcting the error for $\dot{\omega}_{\text{indir}}^{2\text{PN}(II)}$

In Iorio (2020), it turned out that Equation (1) and Equation (4) agree with the corresponding calculation by Kopeikin & Potapov (1994), despite such authors did neither recur to the schematization by Iorio (2020) nor explicitly display their intermediate results.

Instead, Iorio (2020) realized that the discrepancy among his results and those by Kopeikin & Potapov (1994) resides in $\dot{\omega}_{\text{indir}}^{2\text{PN}(II)}$, i.e., in that part of the indirect precession arising from the fact that the semimajor axis and the eccentricity do change instantaneously during an orbital revolution due to Equation (3). The resulting 1PN instantaneous shifts are

$$\Delta a(f_0, f)^{1\text{PN}} = -\frac{2\epsilon \mu (\cos f - \cos f_0) \left[7 + 3e^2 + 5e(\cos f + \cos f_0)\right]}{c^2 (1-e^2)^2}, \quad (8)$$

$$\Delta e(f_0, f)^{1\text{PN}} = \frac{\mu (\cos f_0 - \cos f) \left[3 + 7e^2 + 5e(\cos f + \cos f_0)\right]}{c^2 a (1-e^2)}. \quad (9)$$

The calculation of $\dot{\omega}_{\text{indir}}^{2\text{PN}(II)}$ by Kopeikin & Potapov (1994) can be reproduced as follows (Iorio 2020, pp. 13). Evaluate the Gauss equation of the pericentre for a perturbing in-plane acceleration

$$\frac{d\omega}{df} = \frac{r^2}{\mu e} \left[-A_\rho \cos f + \left(1 + \frac{p}{p}\right) \sin f A_\tau\right], \quad (10)$$

where $p \equiv a \left(1-e^2\right)$, with the radial and transverse components of the 1PN acceleration of
Equation (3)

\[ A_{1PN}^1 = \frac{\mu^2 (1 + e \cos f)^2 \left(3 + e^2 + 2e \cos f - 2e^2 \cos 2f\right)}{c^2 a^3 (1 - e^2)^3}, \]  

(11)

\[ A_{1PN}^2 = \frac{4e \mu^2 (1 + e \cos f)^2 \sin f}{c^2 a^3 (1 - e^2)^3}. \]  

(12)

Then, make the replacement

\[ a \rightarrow a + \Delta a (f_0, f)^{1PN}, \]  

(13)

\[ e \rightarrow e + \Delta e (f_0, f)^{1PN}, \]  

(14)

by means of Equations (8)-(9), expand \( d\omega / df \) to the order of \( O\left(c^{-4}\right) \), and integrate the resulting expression

\[ \frac{d\omega}{df}_{\text{indir}}^{2PN} = -\frac{\mu^2 (\cos f - \cos f_0)}{2c^4 a^2 e^2 (1 - e^2)^2} \left\{ e \left[ 15 - 43e^2 + 5 \left(3 + 17e^2\right) \cos 2f\right] + 2 \cos f \left[9 + 48e^2 - e^4 + 5e \left(3 + e^2\right) \cos f_0\right]\right\} \]  

(15)

from \( f_0 \) to \( f_0 + 2\pi \). As a result, the following formula is obtained

\[ \dot{\omega}_{\text{indir}}^{2PN} = \frac{n_b \mu^2 \left(-9 - 48e^2 + e^4 - 48e^3 \cos f_0\right)}{2c^4 a^2 e^2 (1 - e^2)^2}. \]  

(16)

so that the sum of Equation (14) and Equation (16) gives the total 2PN indirect precession (Iorio 2020, Equation (55))

\[ \dot{\omega}_{\text{indir}}^{2PN} = \frac{n_b \mu^2 \left(-11 + 2e^2 - 48e \cos f_0\right)}{2c^4 a^2 e^2 (1 - e^2)^2}. \]  

(17)

which is, actually, correct. It turns out that adding Equation (17) to Equation (1) yields just the total 2PN precession of Equation (7).

Iorio (2020, pag. 5) followed another approach in calculating \( \dot{\omega}_{\text{indir}}^{2PN} \). In the specific case of the pericentre and of Equation (3), starting from Equation (10), calculated with Equations (11)-(12), the net 2PN pericentre shift per orbit due to the 1PN instantaneous variations of Equations (8)-(9) is worked out as

\[ \Delta \omega_{\text{indir}}^{2PN} = \int_{f_0}^{f_0 + 2\pi} \frac{\partial (d\omega / df)}{\partial a} \Delta a (f_0, f)^{1PN} + \frac{\partial (d\omega / df)}{\partial e} \Delta e (f_0, f)^{1PN}. \]  

(18)
A calculational error\(^1\) in the first addend of Equation (18) yielded the wrong result of Equation (5). After correcting it, it is possible to show that the function to be integrated in Equation (18) agrees with Equation (15). Thus, Equation (16) can be correctly obtained also with the method for calculating \(\dot{\omega}_{\text{indir}}^{2\text{PN}(\Pi)}\) used by Iorio (2020).

By repeating the calculation by Iorio (2020), corrected for the aforementioned error, one obtains, for the full two-body system,

\[
\dot{\omega}_{\text{indir}}^{2\text{PN}} = \frac{n_b \mu^2}{8 e^4 a^2 (1 - e^2)^2} \left[ -44 + 8 \nu (-8 + 7 \nu) + e^2 \left( 8 + 39 \nu + 48 \nu^2 \right) + 96 e (-2 + \nu) \cos f_0 \right]
\] (19)

which, added to Iorio (2020, Equation (32)), returns

\[
\dot{\omega}_{\text{tot}}^{2\text{PN}} = \frac{3 n_b \mu^2}{4 e^4 a^2 (1 - e^2)^2} \left[ 2 - 4 \nu + e^2 (1 + 10 \nu) + 16 e (-2 + \nu) \cos f_0 \right].
\] (20)

In Equations (19)-(20), it is

\[
\nu = \frac{M_A M_B}{(M_A + M_B)^2},
\] (21)

\[
\mu \equiv G (M_A + M_B).
\] (22)

As far as Mercury is concerned, for which it is Iorio (2020, Fig. 1)

\[
\dot{\omega}_{\text{dir}}^{2\text{PN}} = 2.6 \mu\text{as cty}^{-1},
\] (23)

where \(\mu\text{as cty}^{-1}\) stands for microarcseconds per century, Equation (17) yields

\[
-4 \mu\text{as cty}^{-1} \leq \dot{\omega}_{\text{indir}}^{2\text{PN}} \leq -0.2 \mu\text{as cty}^{-1},
\] (24)

for \(0^\circ \leq f_0 \leq 360^\circ\); Equation (24) corrects Iorio (2020, Equation (24)).

For the double pulsar PSR J0737-3039A/B, for which it is Iorio (2020, Equation (33))

\[
\dot{\omega}_{\text{dir}}^{2\text{PN}} = 0.00019^\circ \text{yr}^{-1},
\] (25)

from Equation (19), it turns out

\[
-0.00022^\circ \text{yr}^{-1} \leq \dot{\omega}_{\text{indir}}^{2\text{PN}} \leq -0.00013^\circ \text{yr}^{-1}
\] (26)

\(^1\)To be more specific, \(\mu^2\) entering Equations (11)-(12) was expressed in \(d\omega/df\) as \(n_b^4 a^6\), thus altering the partial derivative of \(d\omega/df\) with respect to \(a\).
for $0^\circ \leq f_0 \leq 360^\circ$. For the Hulse-Taylor binary pulsar PSR B1913+16, for which it is \textsuperscript{Iorio 2020} Equation (35))

$$\dot{\omega}_{2\text{PN}} = 0.000038 \text{ yr}^{-1},$$  \tag{27}

Equation (19) yields

$$-0.00009 \text{ yr}^{-1} \leq \dot{\omega}_{\text{indir}} \leq 0.000034 \text{ yr}^{-1}$$  \tag{28}

for $0^\circ \leq f_0 \leq 360^\circ$. Equation (26) and Equation (28) correct \textsuperscript{Iorio 2020} Equations (48)-(49).

For the supermassive binary black hole in OJ 287, for which it is \textsuperscript{Iorio 2020, pag. 10}

$$\dot{\omega}_{2\text{PN}} = 11.0^\circ \text{ cty}^{-1},$$  \tag{29}

Equation (19) returns an indirect 2PN perinigricon precession ranging within

$$-33.4^\circ \text{ cty}^{-1} \leq \dot{\omega}_{\text{indir}} \leq 17^\circ \text{ cty}^{-1}$$  \tag{30}

for $0^\circ \leq f_0 \leq 360^\circ$. Equation (30) corrects the figures yielded in \textsuperscript{Iorio 2020, pag. 12}. In retrospect, they should have been a wake-up call concerning the validity of \textsuperscript{Iorio 2020, Equation (47)} since the reported maximum value of $516^\circ \text{ cty}^{-1}$ is even larger than than the 1PN precession itself amounting to \textsuperscript{Iorio 2020, Equation (37)} $\dot{\omega}_{1\text{PN}} = 206.8^\circ \text{ cty}^{-1}$.

The discussion in \textsuperscript{Iorio 2020} concerning the measurability of the 2PN pericentre precessions of Mercury and of the binary pulsars will not be repeated here.

### 3. Correcting the error for $e_T$

As correctly pointed out by \textsuperscript{Kopeikin 2020, Iorio 2020} erroneously claimed that Equation (3.12) of \textsuperscript{Damour & Schafer 1988}

$$\frac{\Delta \omega_{2\text{PN}}^{\text{tot}}}{2\pi} = \frac{3}{c^2 h^2} \left[ 1 + \left( \frac{5}{2} - y \right) E c^2 + \left( \frac{35}{4} - 5y \right) \frac{1}{c^2 h^2} \right],$$  \tag{31}

and Equation (5.18) of \textsuperscript{Damour & Schafer 1988}

$$\frac{\Delta \omega_{2\text{PN}}^{\text{tot}}}{2\pi} = \frac{3 (\mu n)^{2/3}}{c^2 (1 - e_T^2)} \left[ 1 + \frac{(\mu n)^{2/3}}{c^2 (1 - e_T^2)} \left( \frac{39}{4} x_A^2 + \frac{27}{4} x_B^2 + 15 x_A x_B \right) - \frac{(\mu n)^{2/3}}{c^2} \left( \frac{13}{4} x_A^2 + \frac{1}{4} x_B^2 + \frac{13}{3} x_A x_B \right) \right]$$  \tag{32}
would be mutually inconsistent after being expressed in terms of $a$, $e$, $f_0$. In Equations (31)-(32), $h$ and $E$ are the coordinate-invariant, reduced orbital angular momentum and energy, respectively,

\[ x_A \doteq \frac{M_A}{M_A + M_B}, \quad (33) \]

\[ x_B \doteq \frac{M_B}{M_A + M_B} = 1 - x_A, \quad (34) \]

$n$ is the PN mean motion (Damour & Deruelle 1985), and $e_T$ is one of the several Damour-Deruelle (DD) parameters (Damour & Deruelle 1986). More precisely, in Iorio (2020, pp. 14-15), it was correctly demonstrated that Equation (31) yields Equation (7). On the other hand, in Iorio (2020, pag. 15), it was erroneously claimed that Equation (32) could not reduce to Equation (7). The error consists of the fact that Iorio (2020) confused $e_T$ (Damour & Deruelle 1986) entering Equation (32) with $e_t$, another member of the DD parameterization (Damour & Deruelle 1985). Instead, it is (Damour & Deruelle 1986, pag. 272)

\[ e_T = e_t (1 + \delta) + e_\theta - e_r. \quad (35) \]

The parameters entering Equation (35) are defined as (Damour & Deruelle 1985, Equation (3.8 b))

\[ e_t = \frac{e_R}{1 + \frac{\mu}{c^2 a_R} \left(4 - \frac{3}{2} \nu\right)}. \quad (36) \]

(Damour & Deruelle 1986, Equation (20))

\[ \delta = \frac{\mu}{c^2 a_R} \left(x_A x_B + 2x_B^2\right). \quad (37) \]

(Damour & Deruelle 1985, Equation (4.13))

\[ e_\theta = e_R \left(1 + \frac{\mu}{2 c^2 a_R} \nu\right). \quad (38) \]

(Damour & Deruelle 1985, Equation (6.3 b))

\[ e_r = e_R \left[1 - \frac{\mu}{2 c^2 a_R} \left(x_A^2 - \nu\right)\right]. \quad (39) \]

In Equations (36)-(39), the DD “semimajor axis” of the relative motion $a_R$ can be expressed as

\[ a_R = \frac{M_B a_R}{(M_A + M_B)}. \]

\[ ^2\text{In Iorio (2020), it is designed as } a_r: \text{ in fact, such a choice may be confusing since, in Damour & Deruelle (1985, Equation (6.3 a)), such a quantity is meant as } a_r = \frac{M_B a_R}{(M_A + M_B)}. \]
\(4\left(1 - e^2\right)^2 a_R = 4 \left\{ a\left(1 - e^2\right)^2 - \frac{\mu}{c^2} \left[-3 + \nu + e^4 \left(1 + 2\nu + e^2 \left(-13 + 7\nu\right)\right)\right] + \right.\)
\[\left. + e \frac{\mu}{c^2} \left[56 + e^2 \left(24 - 31\nu\right) - 24\nu\right] \cos f_0 + \right.\]
\[\left. + e \left[4 \left(5 - 4\nu\right) \cos 2f_0 - e\nu \cos 3f_0\right]\right\} \tag{40}\]

and the DD “eccentricity” \(e_R\) is given by
\[8a \left(-1 + e^2\right) e_R = 4e \left\{ 2a \left(-1 + e^2\right) + \frac{\mu}{c^2} \left[-17 + 6\nu + e^2 \left(2 + 4\nu\right)\right] + \right.\]
\[\left. + \frac{\mu}{c^2} \left[8 \left(-3 + \nu\right) + e^2 \left(-56 + 47\nu\right)\right] \cos f_0 + \right.\]
\[\left. + e \left[4 \left(-5 + 4\nu\right) \cos 2f_0 + e\nu \cos 3f_0\right]\right\} \tag{41}\]

It turns out that, using Equations (35)-(41) and \cite{Damour:1985}, Equation (3.7))
\[n = \sqrt{\frac{\mu}{a_R^3} \left[1 + \frac{\mu}{2c^2a_R} \left(-9 + \nu\right)\right]} \tag{42}\]
in Equation (32) and expanding all to the order of \(O(c^{-4})\) removes the previously mentioned alleged discrepancy. Indeed, now, the corresponding 2PN precession can be cast just into the form of Equation (7), inasmuch the same way as Equation (31) did.

4. Commenting on Kopeikin’s critique

Apart from correctly pointing out the error concerning \(e_T\) treated in Section 3, Kopeikin \cite{Kopeikin:2020} failed his main goal, i.e., demonstrating that Equation (6) is wrong by disclosing and amending the error dealt in Section 2. Indeed, the very same fact that the correction of a mere calculational error in Section 2 allowed to obtain the same results by \cite{Damour:1988}, \cite{Kopeikin:1994} demonstrates that the method by Iorio \cite{Iorio:2020} is correct and is not flawed at a fundamental level by any misconception, as erroneously repeated several times by Kopeikin \cite{Kopeikin:2020}. Moreover, apart from Kopeikin \cite{Kopeikin:2020, point 1, pag.8}, the rest of Kopeikin

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\footnote{In Iorio \cite{Iorio:2020}, it is denoted as \(e_r\), but, in view of Equation (39), such a choice is misleading.}
is fraught with errors, sometimes rather trivial, and contradictions making it utterly useless and untrustworthy.

Below, just some examples of the confusion and unreliability of the Kopeikin’s arguments are provided.

The total pericentre shift, in unit of $2\pi$, obtained by Kopeikin & Potapov (1994, Equation 5.2) and reported in Kopeikin (2020, Equation 27) is, in the test particle limit,

$$k = \frac{3\mu}{c^2 a (1 - e^2)} \left[ 1 + \frac{3\mu}{4c^2 a (1 - e^2)} - \frac{\mu}{4c^2 a} \right]. \quad (43)$$

Kopeikin (2020, pag. 6) wrote that “$a$, $e$ are constants of integrations which are the mean values of the perturbed orbital elements, $a = a(f)$ and $e = e(f)$, over one orbital period with respect to the (perturbed) true anomaly $f$,

$$a \equiv \frac{1}{2\pi} \int_0^{2\pi} \frac{da}{df} \, df, \quad e \equiv \frac{1}{2\pi} \int_0^{2\pi} \frac{de}{df} \, df. \quad (44)$$

” However you understand it, such a statement is meaningless, if not erroneous. Indeed, it is well known that the 1PN field of a mass monopole does not induce any net shift of the semimajor axis and eccentricity, as a straightforward application of the Gauss equations for such orbital elements to Equation (44) easily shows. It suffices to calculate Equations (8)-(9) with $f = f_0 + 2\pi$ getting zero. On the other hand, should the aforementioned statement be understood in the sense that they are averages over the orbital period meant as the time interval between two consecutive passages at the moving perigee, then Equation (44) would return, in principle, 2PN formulas. Indeed, the modified expression for $dt/df$, which is (Brumberg 1991; Poisson & Will 2014)

$$\frac{dt}{df} = \frac{r^4}{\sqrt{\mu^3 p}} \left[ -\cos f A_\rho + \left( 1 + \frac{r}{p} \right) \sin f A_\tau \right], \quad (45)$$

contains just the radial and transverse components $A_\rho, A_\tau$ of the perturbing acceleration itself which, in this case, is just Equation (3). By the way, an analytical calculation performed with Equation (45) shows that, also in this case, there are no net shifts of the semimajor axis and the eccentricity. Be that as it may, later, Kopeikin (2020) wrote that “[...] a and e, [...] do not depend on the initial phase $f_0 at all.” Such a statement is, in fact, contradicted just by Kopeikin (2020, Equations (29)-(31)) which are

$$a_0 = a + da_0, \quad e_0 = e + de_0, \quad (46)$$

where

$$da_0 = \frac{\mu e}{c^2 (1 - e^2)} \left( -14 - 6e^2 \right) \cos f_0 - 5e \cos 2f_0, \quad (47)$$

$$de_0 = \frac{\mu}{c^2 a (1 - e^2)} \left( -3 - 7e^2 \right) \cos f_0 - \frac{5}{2}e \cos 2f_0. \quad (48)$$
In Equations (46)-(48), $a_0$, $e_0$ are “the values $a_0 \equiv a(f_0)$ and $e_0 \equiv e(f_0)$ of the osculating elements taken at the initial instant of time $t_0$ corresponding to the initial value of the true anomaly $f_0$”. By solving Equations (46)-(48) for $a$, $e$, it is apparent that the latter ones do depend on $f_0$. Moreover, it is hardly necessary to point out that Equations (47)-(48) are manifestly erroneous. Indeed, Equation (47), which refers to a length, is made of two terms: the first one is dimensionally a length, while the second one is dimensionless and independent of $\mu$ and $c$. Also in Equation (48), the second term is independent of the source of the gravitational field, i.e. $\mu$, and of $c$, which is meaningless since, to the Newtonian level, the eccentricity of a Keplerian ellipse does not depend on the initial phase.

As if that were not enough, Kopeikin (2020, point 2, pag. 9), wrote that the fractional pericentre shift corresponding to Equation (7) can be directly reproduced by Equation (43) by means of Equations (46)-(48). In fact, this is incorrect. Indeed, by replacing $a$, $e$ with $a_0$, $e_0$ given by Equations (46)-(48), which are in themselves wrong, in the fractional pericentre advance corresponding to Equation (7) and expanding it to the 2PN order, the result differs from the 2PN part of Equation (43) by the amount

$$ \frac{3 \mu^2}{4 c^4} \left\{ \frac{2 + e^2}{a^2 (1 - e^2)^2} + \frac{2 - 32 \cos f_0 + e^2 \left(1 - \frac{5}{2} \cos 2 f_0\right)^2}{\left(a - 5 e \cos 2 f_0\right)^2 \left(-1 + e^2 \left(1 - \frac{5}{2} \cos 2 f_0\right)^2\right)^2} \right\}, $$

as it can be straightforward inferred. Incidentally, Equation (49) is not even dimensionally correct. Even if, for some reasons, one rewrote Equations (47)-(48) in the dimensionally correct form

$$ da_0 = \frac{\mu \theta}{c^2 (1 - e^2)^2} \left[(-14 - 6 e^2) \cos f_0 - 5 e \cos 2 f_0\right], $$

$$ de_0 = \frac{\mu}{c^2 a (1 - e^2)} \left[(-3 - 7 e^2) \cos f_0 - \frac{5}{2} e \cos 2 f_0\right], $$

the discrepancy would still be in place being equal to

$$ \frac{-24 \mu^2 \cos f_0}{c^4 a^2 (1 - e^2)^2}. $$

Instead, the correct way to obtain the fractional pericentre advance corresponding to Equation (7) from Equation (43) was described in Iorio (2020, pp. 12-13).

To further increase the confusion, it is noted that, in Kopeikin & Potapov (1994), $a$, $e$ are denoted as $a_0$, $e_0$ and defined as constants of integrations. From Kopeikin & Potapov (1994, Equations (4.5)-(4.6)), it seems apparent that, in fact, $a$, $e$ are just Iorio (2020).
Equations (51)-(52)), reproduced here as

\[
a = a + \frac{e \mu \left[(14 + 6 e^2) \cos f_0 + e (4 + 5 \cos 2f_0)\right]}{c^2 (1 - e^2)^2}, \tag{53}
\]

\[
e = e + \frac{\mu \left[(6 + 14 e^2) \cos f_0 + e (2 + 5 \cos 2f_0)\right]}{2 c^2 a (1 - e^2)}, \tag{54}
\]

which, indeed, led from Equation (43) to Equation (7), as already pointed out.

Other contradictory and incorrect statements by Kopeikin (2020) consist in that he, first, affirmed the correctness of Equation (7), and, then, claimed without any explicit proof that both the 1PN and the 2PN precessions would conspire in changing with \( f_0 \) in such a way that the total 1PN+2PN pericentre precession would not depend on \( f_0 \) at all. Actually, it is wrong since, to the 1PN order, Equation (43), written in terms of the proper constants of integration of Equations (53)-(54), yields just the Einstein pericentre precession written in terms of the unperturbed\(^4\) Keplerian values \( a, e \); it is notoriously independent of \( f_0 \). Moreover, Equation (7) comes just from an expansion of Equation (43) to the 2PN order by means of Equations (53)-(54), as described in Iorio (2020, pp. 12-13).

As far as it is possible to understand, another incorrect statement by Kopeikin (2020) is that the 2PN part of his Equation (47) would allegedly reproduce the sum of Equation (8) and Equation (23) of Iorio (2020), i.e. Equation (6). In fact, by identifying \( a, e \) with \( a_0, e_0 \), Equation (6) turns out to be manifestly different from the 2PN part of Kopeikin (2020, Equation (47)).

It may be worth noticing that the part of Kopeikin, Efroimsky & Kaplan (2011) devoted to the conversion from the osculating orbital elements to the DD parameters contain some errors. Indeed, in Kopeikin, Efroimsky & Kaplan (2011, Equation (6.143), pag. 509), there are even two mistakes in the same place: the denominator should be \( (1 - e^2)^2 \) (Klioner & Kopeikin 1994, Equation (28)), while it erroneously reads \( a \left(1 - e^2\right) \). Moreover, \( e_R \) incorrectly enters the formula for \( e_T \) displayed in Kopeikin, Efroimsky & Kaplan (2011, pag. 510) instead of \( e_r \) (Damour & Deruelle 1986, pag. 272).

Rather surprisingly, after having repeated several times that the method by Iorio (2020) would be allegedly plagued by any sort of misconceptions and fundamental flaws, Kopeikin (2020, pag. 10) wrote that “In case of BepiColombo mission, the correct formula for measuring the secular 2PN perihelion advance is given by” the fractional pericentre shift corresponding to Equation (7), thus implicitly admitting that, apart from the previously examined calculational errors, Iorio (2020) was right. Incidentally, it would have been interesting to know by Kopeikin...
what would be the correct formula for the secular 1PN advance in the Solar System arena in light of his previous claims concerning the mutual cancellation of $f_0$ in the 1PN+2PN precession.

5. Summary and conclusions

After having disclosed and corrected the calculational errors affecting Iorio (2020), it was demonstrated that the approaches by Damour & Schafer (1988); Kopeikin & Potapov (1994); Iorio (2020) are, in fact, equivalent in analytically calculating the 2PN pericentre precession. Indeed, they yield the same result once the appropriate conversions from the adopted parameterizations are made. This demonstrates that, contrary to what erroneously claimed by Kopeikin (2020), the approach by Iorio (2020) is correct being not plagued by any fundamental misconception.
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