GENERALIZATIONS OF THE DIRAC EQUATION AND THE MODIFIED BARGMANN-WIGNER FORMALISM∗

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Abstract. We present various generalizations of the Dirac formalism. The different-parity solutions of the Weinberg’s 2(2J + 1)-component equations are found. On this basis, generalizations of the Bargmann-Wigner (BW) formalism are proposed. Relations with modern physics constructs are discussed.

I. INTRODUCTION

In this work I am going to discuss the following matters:

• Generalizations of the Dirac equation. Why? Who? How far?

• The theory in the (J,0) ⊕ (0,J) representation of the Lorentz group (the Weinberg 2(2J + 1) theory).

• A lot of antisymmetric tensor (AST) fields. The Proca theory is generalized.

• (The standard Bargmann-Wigner (2J +1)-formalism) ⇔ (The Proca-Duffin-Kemmer formalism.)

• (Modifications of the Bargmann-Wigner formalism.) ⇔ (Modifications of the Proca theory.)

• Conclusions. What further?

What is the purpose of my research? I am sure that the modern physics constructs have much deeper relations with the space-time and discrete symmetries than it was believed before.

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II. GENERALIZATIONS OF THE DIRAC FORMALISM

First of all, I am going to present some preliminary material (perhaps, not so known as it deserves). The equations below can be derived from the first principles and they are used for modifications of the Bargmann-Wigner formalism.

A. The Tokuoka-Sen Gupta-Fushchich formalism

It is based on the equation [1–4]:

\[ [i \gamma_\mu \partial_\mu + m_1 + m_2 \gamma^5] \Psi = 0 \]  (1)

- If \( m_1^2 \neq m_2^2 \) it was claimed [1] that this is simply the change of the representation of \( \gamma \)'s.
- Physical consequences are:
  1. The equation can describe bradyonic, massless and tachyonic particles depending on the choice of the parameters \( m_1 \) and \( m_2 \), ref. [3].
  2. The equation we get under the charge conjugation operation is not the same equation; the sign before \( \gamma^5 \) term is changed to the opposite one [1], provided that \( m_1, m_2 \in \mathbb{R} \).
  3. It is impossible to construct a Lagrangian unless we introduce the second field, e.g., \( \Psi^c \).
  4. In the massless case the solutions are no longer eigenstates of the \( \gamma_5 \) operator: neither they are eigenstates of the helicity operator of the \((1/2, 0) \oplus (0, 1/2)\) representation [1].
  5. The commutation relations are quite unusual [2], e.g., for massless particles \( m_1 = \pm m_2 \) one has \( \{ \Psi_\sigma, \overline{\Psi}_{\sigma'} \} \neq 0, \{ \Psi_\sigma, \Psi_{\sigma'} \} \neq 0 \), but \( \{ \Psi_\sigma, \overline{\Psi}_{\sigma'} \} = 0 \). However, even in the massive case we can have nonlocality, i.e. the presence of the even Pauli-Jordan function [5].

- Fushchich [6] generalized the formalism even further in 1970-72, and, in fact, he connected it with the Gelfand-Tsetlin-Sokolik idea [7] of the 2-dimensional representation of the inversion group.\(^1\)

- I derived the above parity-violating equation [4] (and its charge-conjugate) by the Sakurai-Gersten method from the first principles, see the Appendix A.

\(^1\)This type of theories is frequently called the Wigner-type. However, the Wigner lectures turn out to have been presented later (1962-64).
B. The Barut Formalism

It is based on the equation [8,9]:

\[
[i \gamma_\mu \partial_\mu + \alpha_2 \frac{\partial_\mu \partial_\mu}{m} + \alpha \xi] \Psi = 0 .
\] (2)

It was re-derived from the first principles in [10,11].

- It represents a theory with the conserved current that is linear in the generators of the 4-dimensional representation of the \(O(4,2)\) group.

- Instead of 4 solutions it has 8 solutions with the correct relativistic dispersion \(E = \pm \sqrt{p^2 + m^2}\); and, in fact, describes two mass states \(m_\mu = m_e (1 + \frac{3}{2} \alpha)\), \(\alpha\) is the fine structure constant, provided that the certain physical condition is imposed on the \(\alpha_2\) parameter [8].

- One can also generalize the formalism to include the third state, \(\tau\)-lepton, see refs. [8d,11].

- Barut also indicated the possibility of including \(\gamma^5\) term. For instance, the equation can look something like this:

\[
[i \gamma_\mu \partial_\mu + a + b \Box + \gamma^5 (c + d \Box)] \psi = 0 ,
\] (5)

which cannot yet be factorized as a product of two Dirac equations with different masses.

C. The Weinberg-Tucker-Hammer (WTH) Formalism

The basic principles of constructing the theory in the \((J,0) \oplus (0, J)\) representation [12,13] can be seen in Appendix B.

For spin 1 we start from

2The Ryder relation between zero-momentum left- and right- 2-spinors has been generalized [10]:

\[
\phi_L^h(p^\mu) = a(-1)^{1/2-h} e^{i(\vartheta_1 + \vartheta_2)} \Theta_{1/2}[\phi_L^{h-}(p^\mu)]^* + be^{2i\vartheta_h} \Xi_{1/2}[\phi_L^{h+}(p^\mu)]^* .
\] (3)

see for the notation therein. As a result, in the Majorana representation we have different equations for real and imaginary parts of the field function:

\[
\left[ a \frac{i \gamma_\mu \partial_\mu}{m} + b + 1 \right] \Psi_1(x^\mu) = 0 ,
\] (4a)

\[
\left[ a \frac{i \gamma_\mu \partial_\mu}{m} - b + 1 \right] \Psi_2(x^\mu) = 0 .
\] (4b)
\[ [\gamma_{\alpha\beta} p_{\alpha} p_{\beta} + A p_{\alpha} p_{\alpha} + B m^2] \Psi = 0 , \]

where \( p_{\mu} = -i \partial_{\mu} \) and \( \gamma_{\alpha\beta} \) are the Barut-Muzinich-Williams covariantly defined 6x6 matrices given in Appendix B, \( \sum_{\mu} \gamma_{\mu\mu} = 0 \). The determinant of \( [\gamma_{\alpha\beta} p_{\alpha} p_{\beta} + A p_{\alpha} p_{\alpha} + B m^2] \) is of the 12th order in \( p_{\mu} \). Solutions with \( E^2 - p^2 = m^2 \), \( c = \hbar = 1 \) can be obtained if and only if

\[
\frac{B}{A+1} = 1, \quad \frac{B}{A-1} = 1. \tag{7}
\]

The particular cases are:

- \( A = 0, B = 1 \) if and only if we have Weinberg’s equation for \( J = 1 \) with 3 solutions \( E = +\sqrt{p^2 + m^2} \), 3 solutions \( E = -\sqrt{p^2 + m^2} \), 3 solutions \( E = +\sqrt{p^2 - m^2} \) and 3 solutions \( E = -\sqrt{p^2 - m^2} \).

- \( A = 1, B = 2 \) if and only if we have the Tucker-Hammer equation for \( J = 1 \). The solutions are only with \( E = \pm \sqrt{p^2 + m^2} \).

So, the addition of the Klein-Gordon equation may change the physical content even on the free level.

What are the corresponding equations for antisymmetric tensor field? Proca? Maxwell? Recently we have shown \([14,16]\) that one can obtain four different equations for antisymmetric tensor fields from the Weinberg 2(2 + 1) component formalism. First of all, we note that \( \Psi \) is, in fact, bivector, \( E_i = -i F_{4i}, B_i = \frac{1}{2} \epsilon_{ijk} F_{jk}, \) or \( E_i = -\frac{1}{2} \epsilon_{ijk} \tilde{F}_{jk}, B_i = -i \tilde{F}_{4i} \), or their combination. One can single out the four cases:

- \( \Psi^{(I)} = \left( \begin{array}{c} E + i B \\ E - i B \end{array} \right), P = -1 \), where \( E_i \) and \( B_i \) are the components of the tensor.

- \( \Psi^{(II)} = \left( \begin{array}{c} B - i E \\ B + i E \end{array} \right), P = +1 \), where \( E_i \) and \( B_i \) are the components of the tensor.

- \( \Psi^{(III)} = \Psi^{(I)} \), but (!) \( E_i \) and \( B_i \) are the corresponding vector and axial-vector components of the dual tensor \( \tilde{F}_{\mu\nu} \).

- \( \Psi^{(IV)} = \Psi^{(II)} \), where \( E_i \) and \( B_i \) are the components of the dual tensor \( \tilde{F}_{\mu\nu} \).

The mappings of the WTH equations are:

\[
\partial_{\alpha} \partial_{\mu} F_{\mu\beta}^{(I)} - \partial_{\beta} \partial_{\mu} F_{\mu\alpha}^{(I)} + \frac{A - 1}{2} \partial_{\mu} F_{\alpha\beta}^{(I)} = \frac{B}{2} m^2 F_{\alpha\beta}^{(I)} = 0 , \tag{8a}
\]

\[
\partial_{\alpha} \partial_{\mu} F_{\mu\beta}^{(II)} - \partial_{\beta} \partial_{\mu} F_{\mu\alpha}^{(II)} - \frac{A + 1}{2} \partial_{\mu} F_{\alpha\beta}^{(II)} = \frac{B}{2} m^2 F_{\alpha\beta}^{(II)} = 0 , \tag{8b}
\]

\[
\partial_{\alpha} \partial_{\mu} F_{\mu\beta}^{(III)} - \partial_{\beta} \partial_{\mu} F_{\mu\alpha}^{(III)} - \frac{A + 1}{2} \partial_{\mu} F_{\alpha\beta}^{(III)} = \frac{B}{2} m^2 F_{\alpha\beta}^{(III)} = 0 , \tag{8c}
\]

\[
\partial_{\alpha} \partial_{\mu} F_{\mu\beta}^{(IV)} - \partial_{\beta} \partial_{\mu} F_{\mu\alpha}^{(IV)} + \frac{A - 1}{2} \partial_{\mu} F_{\alpha\beta}^{(IV)} = \frac{B}{2} m^2 F_{\alpha\beta}^{(IV)} = 0 . \tag{8d}
\]

In the Tucker-Hammer case \((A = 1, B = 2)\) we can recover the Proca theory from (8a):

\[
\partial_{\alpha} \partial_{\mu} F_{\mu\beta} - \partial_{\beta} \partial_{\mu} F_{\mu\alpha} = m^2 F_{\alpha\beta} , \tag{9}
\]
We also noted [15] that the massless limit of this theory does not coincide in full with the Maxwell theory, while it contains the latter as a particular case. In [17,18] we showed that it is possible to define various massless limits for the Proca-Duffin-Kemmer theory. Another is the Ogievetskii-Polubarinov notoph (which in the US literature is called the Kalb-Ramond field), ref. [19]. Transverse components of the AST field can be removed from the corresponding Lagrangian by means of the “new gauge transformation” \( A_{\mu \nu} \rightarrow A_{\mu \nu} + \partial_{\mu} \Lambda_{\nu} - \partial_{\nu} \Lambda_{\mu} \), with the vector gauge function \( \Lambda_{\mu} \).

The second case is

\[
\partial_\alpha \partial_\mu F_{\mu \beta} - \partial_\beta \partial_\mu F_{\mu \alpha} = [\partial_\mu \partial_\mu - m^2] F_{\alpha \beta}.
\]

So, on the mass shell \([\partial_\mu \partial_\mu - m^2] F_{\alpha \beta} = 0\), and, hence,

\[
\partial_\alpha \partial_\mu F_{\mu \beta} - \partial_\beta \partial_\mu F_{\mu \alpha} = 0,
\]

which rather corresponds to the Maxwell-like case. However, if we calculate dispersion relations for the second case (11) it appears that the equation has solutions even if \( m \neq 0 \).

The interesting case is \( B = 8 \) and \( A = B - 1 = 7 \). In this case we can describe various mass states which are connected by the relation \( m^2 = \frac{4}{3} m^2 \). One can get

\[
\partial_\alpha \partial_\mu F_{\mu \beta} - \partial_\beta \partial_\mu F_{\mu \alpha} = -3(\partial_\mu \partial_\mu) F_{\alpha \beta} + 4m^2 F_{\alpha \beta} \quad \text{(originated from } P = -1),
\]

\[
\partial_\alpha \partial_\mu F_{\mu \beta} - \partial_\beta \partial_\mu F_{\mu \alpha} = +4(\partial_\mu \partial_\mu) F_{\alpha \beta} - 4m^2 F_{\alpha \beta} \quad \text{(originated from } P = +1).
\]

If we consider \( \partial^2_{\mu} = m^2 \), the first equation will give us only causal solutions with the mass \( m \) which are compatible with the Proca theory; the second case reduces to (11).

Let us consider also \( \partial^2_{\mu} = \alpha m^2 \). If one does not want to have neither tachyonic solutions nor old Proca-like solutions with \( m \), one can find another possibility: set from the first equation \((4 - 3\alpha)m^2 = 0\) and, hence, \( \alpha = \frac{4}{3} \). On using this value of \( \alpha \) in the second equation we observe that the right-hand side comes to be equal

\[
(4\alpha - 4)m^2 = \frac{4}{3} m^2,
\]

i.e., this is the compatible solution with the mass \( m^2 = \frac{4}{3} m^2 \):

\[
\partial_\alpha \partial_\mu F_{\mu \beta} - \partial_\beta \partial_\mu F_{\mu \alpha} = \frac{4}{3} m^2.
\]

Now we are interested in parity-violating equations for antisymmetric tensor fields. We also investigate the most general mapping of the Weinberg-Tucker-Hammer formulation to the antisymmetric tensor field formulation. Instead of \( \Psi(I-IV) \) we shall try to use now

\[
\Psi^{(A)} = \left( \begin{array}{c} E + iB \\ B + iE \end{array} \right) = \frac{1 + \gamma^5}{2} \Psi(I) + \frac{1 - \gamma^5}{2} \Psi(II).
\]

As a result, the equation for the AST fields is

\[
\partial_\alpha \partial_\mu F_{\mu \beta} - \partial_\beta \partial_\mu F_{\mu \alpha} = \frac{1}{2} (\partial_\mu \partial_\mu) F_{\alpha \beta} + \left[ -\frac{A}{2} (\partial_\mu \partial_\mu) + \frac{B}{2} m^2 \right] \bar{F}_{\alpha \beta}.
\]
Of course, \( \Psi^{(A)\prime} = \left( \begin{array}{c} B - iE \\ E - iB \end{array} \right) = -i\Psi^{(A)} \), and the equation is unchanged. The different choice is

\[
\Psi^{(B)} = \left( \begin{array}{c} E + iB \\ -B - iE \end{array} \right) = \frac{1 + \gamma^5}{2} \Psi^{(I)} - \frac{1 - \gamma^5}{2} \Psi^{(II)}.
\]  

(17)

Thus, one has

\[
\partial_\alpha \partial_\mu F_{\mu\beta} - \partial_\beta \partial_\mu F_{\mu\alpha} = \frac{1}{2} (\partial_\mu \partial_\mu) F_{\alpha\beta} + \left[ \frac{A}{2} (\partial_\mu \partial_\mu) - \frac{B}{2} \right] F_{\alpha\beta}.
\]

(18)

Of course, one can also use the dual tensor \( \mathbf{E}^i = -\frac{1}{2} \epsilon_{ijk} \tilde{F}_{jk} \) and \( \mathbf{B}^i = -i \tilde{F}_{4i} \) and obtain analogous equations:

\[
\partial_\alpha \partial_\mu \tilde{F}_{\mu\beta} - \partial_\beta \partial_\mu \tilde{F}_{\mu\alpha} = \frac{1}{2} (\partial_\mu \partial_\mu) \tilde{F}_{\alpha\beta} + \left[ \frac{A}{2} (\partial_\mu \partial_\mu) + \frac{B}{2} \right] \tilde{F}_{\alpha\beta}.
\]

(19a)

\[
\partial_\alpha \partial_\mu \tilde{F}_{\mu\beta} - \partial_\beta \partial_\mu \tilde{F}_{\mu\alpha} = \frac{1}{2} (\partial_\mu \partial_\mu) \tilde{F}_{\alpha\beta} + \left[ \frac{A}{2} (\partial_\mu \partial_\mu) - \frac{B}{2} \right] \tilde{F}_{\alpha\beta}.
\]

(19b)

They are connected with (16,18) by the dual transformations.

The states corresponding to the new functions \( \Psi^{(A)} \), \( \Psi^{(B)} \) etc are not the parity eigenstates. So, it is not surprising that we have \( F_{\alpha\beta} \) and its dual \( \tilde{F}_{\alpha\beta} \) in the same equations. In total we have already eight equations.

One can also consider the most general case

\[
\Psi^{(W)} = \left( \begin{array}{c} aF_{4i} + b\tilde{F}_{4i} + c\epsilon_{ijk} \tilde{F}_{jk} + d\epsilon_{ijk} \tilde{F}_{jk} \\ eF_{4i} + f\tilde{F}_{4i} + g\epsilon_{ijk} \tilde{F}_{jk} + h\epsilon_{ijk} \tilde{F}_{jk} \end{array} \right).
\]

(20)

So, we shall have dynamical equations for \( F_{\alpha\beta} \) and \( \tilde{F}_{\alpha\beta} \) with additional parameters \( a, b, c, d, \ldots \in \mathbb{C} \). We have a lot of antisymmetric tensor fields here. However,

- the covariant form preserves if there are some restrictions on the parameters. Alternatively, we have some additional terms of \( \partial^2 \) or \( \nabla^2 \);
- both \( F_{\mu\nu} \) and \( \tilde{F}_{\mu\nu} \) are present in the equations;
- under the definite choice of \( a, b, c, d \ldots \) the equations can be reduced to the above equations for the tensor \( H_{\mu\nu} \) and its dual:

\[
H_{\mu\nu} = c_1 F_{\mu\nu} + c_2 \tilde{F}_{\mu\nu} + \frac{c_3}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta} + \frac{c_4}{2} \epsilon_{\mu\nu\alpha\beta} \tilde{F}_{\alpha\beta};
\]

(21)

- the parity properties of \( \Psi_W \) are very complicated.
D. The Bargmann-Wigner Formalism

The way for constructing equations of high-spin particles has been given in [20,21]. However, they claimed explicitly that they constructed \((2J + 1)\) states (the Weinberg-Tucker-Hammer theory has essentially \(2(2J + 1)\) components). In this subsection we present the standard Bargmann-Wigner formalism for \(J = 1\):

\[
[\gamma_{\mu}\partial_{\mu} + m]_{\alpha\beta} \Psi_{\beta\gamma} = 0, \tag{22a}
\]

\[
[\gamma_{\mu}\partial_{\mu} + m]_{\gamma\beta} \Psi_{\alpha\beta} = 0, \tag{22b}
\]

If one has

\[
\Psi_{\{\alpha\beta\}} = (\gamma_{\mu} R)_{\alpha\beta} A_{\mu} + (\sigma_{\mu\nu} R)_{\alpha\beta} F_{\mu\nu}, \tag{23}
\]

with

\[
R = e^{i\varphi} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{24}
\]

in the spinorial representation of \(\gamma\)-matrices we obtain the Duffin-Proca-Kemmer equations:

\[
\partial_{\alpha} F_{\alpha\mu} = \frac{m}{2} A_{\mu}, \tag{25a}
\]

\[
2m F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}. \tag{25b}
\]

(In order to obtain this set one should add the equations (22a,22b) and compare functional coefficients before the corresponding commutators, see [21]). After the corresponding renormalization \(A_{\mu} \rightarrow 2mA_{\mu}\), we obtain the standard textbook set:

\[
\partial_{\alpha} F_{\alpha\mu} = m^2 A_{\mu}, \tag{26a}
\]

\[
F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}. \tag{26b}
\]

It gives the equation (9) for the antisymmetric tensor field. How can one obtain other equations following the Weinberg-Tucker-Hammer approach?

The recipe for the third equation is simple: use, instead of \((\sigma_{\mu\nu} R) F_{\mu\nu}\), another symmetric matrix \((\gamma^5 \sigma_{\mu\nu} R) F_{\mu\nu}\), see [22] and Appendix C. And what about the second and the fourth equations? One can modify the Dirac equation and form the direct product \(\Psi_{\alpha\beta} = \Psi_{\alpha} \otimes \Psi_{\beta}\), see [16]. So, I suggest:

- by means of specific similarity transformation, see above and [1]:

\[
[\gamma_{\mu}\partial_{\mu} + m]_{\alpha\beta} \Psi = 0 \Rightarrow [\gamma_{\mu}\partial_{\mu} + m + m_1 + m_2 \gamma_5] \Psi = 0; \tag{27}
\]

- to use the Barut extension too:

\[
[\gamma_{\mu}\partial_{\mu} + m]_{\alpha\beta} \Psi = 0 \Rightarrow [\gamma_{\mu}\partial_{\mu} + a \frac{\partial_{\mu}\partial_{\mu}}{m} + \in\Psi = 0. \tag{28}
\]

In such a way we can enlarge the set of possible states.
III. MODIFIED BARGMANN-WIGNER FORMALISM

We begin with
\[ [i\gamma_\mu \partial_\mu + a - b \Box + \gamma_5(c - d \Box)]_{\alpha\beta} \Psi_{\beta\gamma} = 0, \]  
(29a)
\[ [i\gamma_\mu \partial_\mu + a - b \Box - \gamma_5(c - d \Box)]_{\alpha\beta} \Psi_{\gamma\beta} = 0, \]  
(29b)
\[ \Box \] 
is the d’Alembertian. Thus, we obtain the Proca-like equations:
\[ \partial_\nu A_\lambda - \partial_\lambda A_\nu - 2(a + b \partial_\mu \partial_\mu) F_{\nu\lambda} = 0, \]  
(30a)
\[ \partial_\mu F_{\mu\lambda} = \frac{1}{2}(a + b \partial_\mu \partial_\mu) A_\lambda + \frac{1}{2}(c + d \partial_\mu \partial_\mu) \tilde{A}_\lambda, \]  
(30b)
\[ \tilde{A}_\lambda \] 
is the axial-vector potential (analogous to that used in the Duffin-Kemmer set for \( J = 0 \)).

Additional constraints are:
\[ i\partial_\lambda A_\lambda + (c + d \partial_\mu \partial_\mu) \tilde{\phi} = 0, \]  
(31a)
\[ \epsilon_{\mu\lambda\sigma\tau} \partial_\mu F_{\lambda\sigma\tau} = 0, \]  
(31b)
\[ (c + d \partial_\mu \partial_\mu) \phi = 0. \]  
(31c)

The spin-0 Duffin-Kemmer equations are:
\[ (a + b \partial_\mu \partial_\mu) \phi = 0, \]  
(32a)
\[ i\partial_\mu \tilde{A}_\mu - (a + b \partial_\mu \partial_\mu) \tilde{\phi} = 0, \]  
(32b)
\[ (a + b \partial_\mu \partial_\mu) \tilde{A}_\nu + (c + d \partial_\mu \partial_\mu) A_\nu + i(\partial_\nu \tilde{\phi}) = 0. \]  
(32c)

The additional constraints are:
\[ \partial_\mu \phi = 0 \]  
(33a)
\[ \partial_\nu \tilde{A}_\lambda - \partial_\lambda \tilde{A}_\nu + 2(c + d \partial_\mu \partial_\mu) F_{\nu\lambda} = 0. \]  
(33b)

In such a way the spin states are mixed through the 4-vector potentials. For higher-spin equations similar calculations and conclusions have been reached by M. Moshinsky et al. [23] and S. Kruglov [24]. After elimination of the 4-vector potentials we obtain the equation for the AST field of the second rank:
\[ [\partial_\mu \partial_\nu F_{\nu\lambda} - \partial_\lambda \partial_\nu F_{\nu\mu}] + \left[ (c^2 - a^2) - 2(ab - cd) \partial_\mu \partial_\mu + (d^2 - b^2)(\partial_\mu \partial_\mu)^2 \right] F_{\mu\lambda} = 0, \]  
(34)
which should be compared with our previous equations which follow from the Weinberg-like formulation. Just put:
\[ c^2 - a^2 \Rightarrow \frac{-Bm^2}{2}, \quad c^2 - a^2 \Rightarrow \frac{Bm^2}{2}, \]  
(35a)
\[ -2(ab - cd) \Rightarrow \frac{A - 1}{2}, \quad +2(ab - cd) \Rightarrow \frac{A + 1}{2}, \]  
(35b)
\[ b = \pm d. \]  
(35c)
algebraic equations have solutions in terms $A$ and $B$. We found them and restored the equations:

$$\left[\partial_{\mu}\partial_{\nu}F_{\nu\lambda} - \partial_{\lambda}\partial_{\nu}F_{\nu\mu}\right] - \frac{A + 1}{2} \partial_{\nu}\partial_{\mu}F_{\mu\lambda} + \frac{B}{2} m^2 F_{\mu\lambda} = 0$$  \hspace{1cm} (36)

and

$$\left[\partial_{\mu}\partial_{\nu}F_{\nu\lambda} - \partial_{\lambda}\partial_{\nu}F_{\nu\mu}\right] + \frac{A - 1}{2} \partial_{\nu}\partial_{\mu}F_{\mu\lambda} - \frac{B}{2} m^2 F_{\mu\lambda} = 0$$  \hspace{1cm} (37)

Thus, the procedure which we carried out is the following: The Modified Bargmann-Wigner formalism $\rightarrow$ The AST field equations $\rightarrow$ The Weinberg-Tucker-Hammer approach.

The parity violation and the spin mixing are intrinsic possibilities of the Proca-like theories. One can go in a different way: instead of modifying the equations, consider the (parity) basis. In the helicity basis we have (see also [25,26], where it was claimed explicitly that helicity states cannot be parity eigenstates):

$$\epsilon_\mu(p, \lambda = +1) = \frac{1}{\sqrt{2}} e^{i\phi} \left( 0, \frac{p_x p_x - ip_y p_y}{\sqrt{p_x^2 + p_y^2}}, \frac{p_y p_x + ip_x p_y}{\sqrt{p_x^2 + p_y^2}}, -\sqrt{p_x^2 + p_y^2} \right),$$  \hspace{1cm} (38a)

$$\epsilon_\mu(p, \lambda = -1) = \frac{1}{\sqrt{2}} e^{-i\phi} \left( 0, \frac{p_x p_x + ip_y p_y}{\sqrt{p_x^2 + p_y^2}}, \frac{-p_y p_x + ip_x p_y}{\sqrt{p_x^2 + p_y^2}}, +\sqrt{p_x^2 + p_y^2} \right),$$  \hspace{1cm} (38b)

$$\epsilon_\mu(p, \lambda = 0) = \frac{1}{m} \left( p_x, -\frac{E}{p} p_x, -\frac{E}{p} p_y, -\frac{E}{p} p_z \right),$$  \hspace{1cm} (38c)

$$\epsilon_\mu(p, \lambda = 0_t) = \frac{1}{m} \left( E, -p_x, -p_y, -p_z \right).$$  \hspace{1cm} (38d)

and

$$E(p, \lambda = +1) = -\frac{iE p_x}{\sqrt{2} p p_t} \tilde{p} - \frac{E}{\sqrt{2} p_t} \tilde{p}, \hspace{1cm} B(p, \lambda = +1) = -\frac{p_x}{\sqrt{2} p_t} \tilde{p} + \frac{i p}{\sqrt{2} p_t} \tilde{p},$$  \hspace{1cm} (39a)

$$E(p, \lambda = -1) = +\frac{iE p_x}{\sqrt{2} p p_t} \tilde{p} - \frac{E}{\sqrt{2} p_t} \tilde{p}^*, \hspace{1cm} B(p, \lambda = -1) = -\frac{p_x}{\sqrt{2} p_t} \tilde{p} - \frac{i p}{\sqrt{2} p_t} \tilde{p}^*,$$  \hspace{1cm} (39b)

$$E(p, \lambda = 0) = \frac{i m}{p} \tilde{p}, \hspace{1cm} B(p, \lambda = 0) = 0,$$  \hspace{1cm} (39c)

with $\tilde{p} = \left( \begin{array}{c} p_y \\ -p_x \end{array} \right)$. See Appendix D for the 4-potentials and fields in the more common (parity) basis.

In fact, there are several modifications of the BW formalism. Thanks to Professor Z. Oziewicz I came to the following set:

$$[i \gamma_\mu \partial_\mu + \epsilon_1 m_1 + \epsilon_2 m_2 \gamma_5]_{\alpha\beta} \Psi_{\beta\gamma} = 0,$$  \hspace{1cm} (40a)

$$[i \gamma_\mu \partial_\mu + \epsilon_3 m_3 + \epsilon_4 m_4 \gamma_5]_{\alpha\beta} \Psi_{\beta\gamma} = 0,$$  \hspace{1cm} (40b)

where $\epsilon_i$ are the sign operators. So, at first sight, we have 16 possible combinations for the AST fields. First, we come to
\[
[i\gamma_\mu \partial_\mu + m_1 A_1 + m_2 A_2 \gamma_5]_{\alpha\beta} \{ (\gamma_\lambda R)_{\beta\gamma} A_\lambda + (\sigma_{\lambda\epsilon} R)_{\beta\gamma} F_{\lambda\epsilon} \} + \\
+ [m_1 B_1 + m_2 B_2 \gamma_5] \{ R_{\beta\gamma} \varphi + (\gamma_5 R)_{\beta\gamma} \tilde{\phi} + (\gamma_5 \gamma_\lambda R)_{\beta\gamma} \tilde{A}_\lambda \} = 0, \\
[i\gamma_\mu \partial_\mu + m_1 A_1 + m_2 A_2 \gamma_5]_{\gamma_\beta} \{ (\gamma_\lambda R)_{\alpha\beta} A_\lambda + (\sigma_{\lambda\epsilon} R)_{\alpha\beta} F_{\lambda\epsilon} \} - \\
- [m_1 B_1 + m_2 B_2 \gamma_5] \{ R_{\alpha\beta} \varphi + (\gamma_5 R)_{\alpha\beta} \tilde{\phi} + (\gamma_5 \gamma_\lambda R)_{\alpha\beta} \tilde{A}_\lambda \} = 0, 
\]

where \( A_1 = \frac{\epsilon_1 + \epsilon_2}{2} \), \( A_2 = \frac{\epsilon_2 - \epsilon_1}{2} \), \( B_1 = \frac{\epsilon_1 - \epsilon_2}{2} \), and \( B_2 = \frac{\epsilon_2 - \epsilon_1}{2} \). Thus for spin 1 we have

\[
\partial_\mu A_\lambda - \partial_\lambda A_\mu + 2m_1 A_1 F_{\mu\lambda} + im_2 A_2 \epsilon_{\alpha\beta\mu\lambda} F_{\alpha\beta} = 0, \\
\partial_\lambda F_{\alpha\epsilon\lambda} - \frac{m_1}{2} A_1 A_\epsilon - \frac{m_2}{2} B_2 \tilde{A}_\epsilon = 0,
\]

with constraints

\[
-i \partial_\mu A_\mu + 2m_1 B_1 \tilde{\phi} + 2m_2 B_2 \tilde{\phi} = 0, \\
i \epsilon_{\mu\nu\alpha\lambda} \partial_\mu F_{\nu\epsilon\lambda} - m_2 A_2 A_\lambda - m_1 B_1 \tilde{A}_\lambda = 0, \\
m_1 B_1 \tilde{\phi} + m_2 B_2 \phi = 0.
\]

If we remove \( A_\lambda \) and \( \tilde{A}_\lambda \) from this set, we come to the final results for the AST field.

Actually, we have twelve equations (another four equations coincide with some of those indicated below):

(I) : \[
\begin{align*}
\partial_\mu A_\lambda - \partial_\lambda A_\mu + 2im_2 \tilde{F}_{\mu\lambda} &= 0, \\
\partial_\lambda F_{\alpha\epsilon} &= 0;
\end{align*}
\]

(II) : \[
\begin{align*}
\partial_\mu A_\lambda - \partial_\lambda A_\mu + 2m_1 F_{\mu\lambda} &= 0, \\
\partial_\lambda F_{\alpha\epsilon} + \frac{m_1}{2} A_\epsilon - \frac{m_2}{2} \tilde{A}_\epsilon &= 0;
\end{align*}
\]

(III) : \[
\begin{align*}
\partial_\mu A_\lambda - \partial_\lambda A_\mu - 2m_1 F_{\mu\lambda} - 2im_2 \tilde{F}_{\mu\lambda} &= 0, \\
\partial_\lambda F_{\alpha\epsilon} - \frac{m_1}{2} A_\epsilon &= 0;
\end{align*}
\]

(IV) : \[
\begin{align*}
\partial_\mu A_\lambda - \partial_\lambda A_\mu - 2im_2 \tilde{F}_{\mu\lambda} &= 0, \\
\partial_\lambda F_{\alpha\epsilon} &= 0;
\end{align*}
\]

(V) : \[
\begin{align*}
\partial_\mu A_\lambda - \partial_\lambda A_\mu - 2m_1 F_{\mu\lambda} &= 0, \\
\partial_\lambda F_{\alpha\epsilon} - \frac{m_1}{2} A_\epsilon + \frac{m_2}{2} \tilde{A}_\epsilon &= 0;
\end{align*}
\]

10
In the general case we obtain the equation where both $F_{\mu\nu}$ and its dual $\tilde{F}_{\mu\nu}$ present, as above we obtain from the WTH formalism

$$\frac{m_1 B_1}{m_1^2 A_1 B_1 - m_2^2 A_2 B_2} [\partial_\mu \partial_\nu F_{\alpha \lambda} - \partial_\lambda \partial_{\alpha \nu} F_{\nu \mu}] + \frac{im_2 B_2}{m_1^2 A_1 B_1 - m_2^2 A_2 B_2} [\partial_\mu \partial_\nu \tilde{F}_{\alpha \lambda} - \partial_\lambda \partial_{\alpha \nu} \tilde{F}_{\nu \mu}] = m_1 A_1 F_{\mu \lambda} + im_2 A_2 \tilde{F}_{\mu \lambda}$$

One can go even further. One can use the Barut equations for the BW input. So, we can get $16 \times 16$ combinations (depending on the eigenvalues of the corresponding sign operators), and we have different eigenvalues of masses due to $\partial_\mu^2 = \alpha m^2$. 

Some of them can coincide each other.
What is their physical sense? Why do I think that the shown arbitrariness of equations for the AST fields is related to 1) spin basis rotations; 2) the choice of normalization? In the common-used basis the three 4-potentials have parity eigenvalues $-1$ and one time-like (or spin-0 state), $+1$; the fields $E$ and $B$ have also definite parity properties in this basis. If we transfer to other basis, e.g., to the helicity basis we can see that the 4-vector potentials and the corresponding fields are superpositions of the vector and the axial-vector. Of course, they can be expanded in the fields in the “old” basis. We are going to investigate this surprising fact in other publications.

IV. CONCLUSIONS

- The addition of the Klein-Gordon equation to the $(J,0) \oplus (0,J)$ equations may change physical content even on the free level.
- In the $(1/2,0) \oplus (0,1/2)$ representation it is possible to introduce the parity-violating frameworks.
- We found the mappings between the Weinberg-Tucker-Hammer formalism for $J = 1$ and the AST fields of the 2nd rank of at least eight types. Four of them include both $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$, which tells us that the parity violation may occur during the study of the corresponding dynamics.
- If we want to take into account the $J = 1$ solutions with different parity properties, the Bargmann-Wigner (BW), the Proca and the Duffin-Kemmer-Petiau (DKP) formalisms are to be generalized.
- We considered the most general case, introducing eight scalar parameters. In order to have covariant equations for the AST fields, one should impose constraints on the corresponding parameters.
- It is possible to get solutions with mass splitting.
- We found the 4-potentials and fields in the helicity basis. They have different parity properties comparing with the standard (“parity”) basis (cf. [25,26]).
- The discussion induced us to generalize the BW, the Proca and the Duffin-Kemmer-Petiau formalisms. Higher-spin equations may actually describe various spin, mass, helicity and parity states. The states of different parity, helicity, and mass may be present in the same equation.
- On the basis of generalizations of the BW formalism, finally, we obtained twelve equations for the AST fields.
- A hypothesis was presented that the obtained results are related to the spin basis rotations and to the choice of normalization.
APPENDIX A. DERIVATION OF THE GENERALIZED DIRAC EQUATION

I use the following equation for the two-component spinor wave function [27]:

\[(E^2 - c^2 p^2)\Phi^{(2)} = \left[ EI^{(2)} - cp \cdot \sigma \right] \left[ EI^{(2)} + cp \cdot \sigma \right] \Psi = \mu_2^2 c^4 \Psi. \quad (58)\]

One can define two-component ‘right’ and ‘left’ wave functions

\[
\phi_R = \frac{1}{\mu_1 c} (i\hbar \frac{\partial}{\partial x_0} - i\hbar \sigma \cdot \nabla) \Psi, \quad \phi_L = \Psi
\]

with an additional mass parameter \(\mu_1\). In such a way we come to the set of equations

\[
(i\hbar \frac{\partial}{\partial x_0} + i\hbar \sigma \cdot \nabla) \phi_R = \frac{\mu_2^2 c}{\mu_1} \phi_L, \quad (60a)
\]

\[
(i\hbar \frac{\partial}{\partial x_0} - i\hbar \sigma \cdot \nabla) \phi_L = \mu_1 c \phi_R, \quad (60b)
\]

which can be written in the 4-component form:

\[
\begin{pmatrix}
  i\hbar (\partial/\partial x_0) & i\hbar \sigma \cdot \nabla \\
  -i\hbar \sigma \cdot \nabla & -i\hbar (\partial/\partial x_0)
\end{pmatrix}
\begin{pmatrix}
  \psi_A \\
  \psi_B
\end{pmatrix} = \frac{c}{2} \begin{pmatrix}
  (\mu_2^2/\mu_1 + \mu_1) & (-\mu_2^2/\mu_1 + \mu_1) \\
  (-\mu_2^2/\mu_1 + \mu_1) & (\mu_2^2/\mu_1 + \mu_1)
\end{pmatrix}
\begin{pmatrix}
  \psi_A \\
  \psi_B
\end{pmatrix}
\]

(61)

for the function \(\psi = \text{column}(\phi_R + \phi_L, \phi_R - \phi_L)\). The equation (61) can be written in the covariant form (as one can see, the standard representation of \(\gamma^\mu\) matrices was used here):

\[
[i\gamma_\mu \partial_\mu + \frac{\mu_2^2 c}{\mu_1 \hbar} \frac{1 - \gamma^5}{2} + \frac{\mu_1 c}{\hbar} \frac{1 + \gamma^5}{2}] \psi = 0.
\]

(62)

If \(\mu_1 = \mu_2\) we can recover the standard Dirac equation. As noted in ref. [1] this procedure can be viewed as simply changing the representation of \(\gamma^\mu\) matrices (unless \(\mu_1 \neq 0, \mu_2 \neq 0\)).

It is interesting that we also can repeat this procedure for the definition (or even more general):

\[
\phi_L = \frac{1}{\mu_3 c} (i\hbar \frac{\partial}{\partial x_0} + i\hbar \sigma \cdot \nabla) \Psi, \quad \phi_R = \Psi
\]

(63)

since in the two-component equation the parity properties of the two-component spinor are undefined. The resulting equation is

\[
[i\gamma_\mu \partial_\mu + \frac{\mu_2^2 c}{\mu_3 \hbar} \frac{1 + \gamma^5}{2} + \frac{\mu_3 c}{\hbar} \frac{1 - \gamma^5}{2}] \bar{\psi} = 0.
\]

(64)

The above procedure can be generalized to any Lorentz group representation, i. e., to any spin.
APPENDIX B. WEINBERG 2(2J + 1)-COMPONENT FORMALISM

I reviewed the Weinberg 2(2J + 1)-component formalism in ref. [13]. I found the following postulates [12]:

- The fields transform according to the formula:
  \[ U[\Lambda, a] \Psi_n(x) U^{-1}[\Lambda, a] = \sum_m D_{nm}[\Lambda^{-1}] \Psi_m(\Lambda x + a) , \]  
  where \( D_{nm}[\Lambda] \) is some representation of \( \Lambda \); \( x^\mu \rightarrow \Lambda^\nu_{\mu} x^\nu + a^\mu \), and \( U[\Lambda, a] \) is a unitary operator.

- For \((x - y)\) spacelike one has
  \[ [\Psi_n(x), \Psi_m(y)]_\pm = 0 \]  
  for fermion and boson fields, respectively.

- The interaction Hamiltonian density is said by S. Weinberg to be scalar, and it is constructed out of the creation and annihilation operators for the free particles described by the free Hamiltonian \( H_0 \).

- The \( S \)-matrix is constructed as an integral of the \( T \)-ordering product of the interaction Hamiltonians by Dyson’s formula.

Weinberg wrote: “In order to discuss theories with parity conservation it is convenient to use 2(2J + 1)-component fields, like the Dirac field. These do obey field equations, which can be derived as... consequences of (65,66).” In such a way he proceeds to form the 2(2J + 1)-component object

\[ \Psi = \begin{pmatrix} \Phi_{\sigma} \\ \Xi_{\sigma} \end{pmatrix} \]

transforming according to the Wigner rules. They are the following ones:

\[ \Phi_{\sigma}(p) = \exp(+\Theta \hat{p} \cdot \mathbf{J}) \Phi_{\sigma}(0) , \]
\[ \Xi_{\sigma}(p) = \exp(-\Theta \hat{p} \cdot \mathbf{J}) \Xi_{\sigma}(0) \]  

from the zero-momentum frame. \( \Theta \) is the boost parameter, \( \tanh \Theta = |p|/E, \hat{p} = p/|p| \), \( p \) is the 3-momentum of the particle, \( \mathbf{J} \) is the angular momentum operator. For a given representation the matrices \( \mathbf{J} \) can be constructed. In the Dirac case (the \((1/2, 0) \oplus (0, 1/2)\) representation) \( \mathbf{J} = \sigma/2 \); in the \( J = 1 \) case (the \((1, 0) \oplus (0, 1)\) representation) we can choose \((J_i)_{jk} = -i\epsilon_{ijk}\), etc. Hence, we can explicitly calculate (67a,67b), see \( J = 1/2, 1, 3/2, 2 \) cases in the Table (cf. ref. [33]):

\[ \text{In the (2J+1) formalism, fields obey only the Klein-Gordon equation, according to the Weinberg wisdom.} \]
Spin 0 1
Spin 1/2 \[ E + m \pm \sigma \cdot p \left/ [2m(E + m)] \right|^{1/2} \]
Spin 1 \[ m(E + m) \pm (E + m)(\mathbf{J} \cdot \mathbf{p}) + (\mathbf{J} \cdot \mathbf{p})^2 \left/ [m(E + m)] \right| \]
Spin 3/2 \[ -m^2(E - 5m)/(3(E - 13m)(\mathbf{J} \cdot \mathbf{p}) + 4(\mathbf{J} \cdot \mathbf{p})^2 + (13/3)(E + m)^{-1}(\mathbf{J} \cdot \mathbf{p})^2 \]
Spin 2 \[ m^2(E + m)^2 \Xi(1/3)(E - 4m)(E + m)^2(\mathbf{J} \cdot \mathbf{p}) - (1/6)(E - 7m)(E + m)(\mathbf{J} \cdot \mathbf{p})^2 + (1/3)(E + m)(\mathbf{J} \cdot \mathbf{p})^3 + (1/6)(\mathbf{J} \cdot \mathbf{p})^4 \]

The task is now to obtain relativistic equations for higher spins. Weinberg uses the following procedure (see also [28,29]). Firstly, he defined the scalar matrix

\[ \Pi^{(j)}_{\sigma}\sigma(q) = (-)^{2j} t_{\sigma\sigma}^{\mu_1 \mu_2 \cdots \mu_{2j}} q_{\mu_1} q_{\mu_2} \cdots q_{\mu_{2j}} \]  

(68)

for the \((J,0)\) representation of the Lorentz group \((q_{\mu}q_{\mu} = -m^2)\), with the tensor \(t\) being defined by [12a,Eqs.(A4-A5)]. Hence,

\[ D^{(j)}[\Lambda] \Pi^{(j)}(q) D^{(j)\dagger}[\Lambda] = \Pi^{(j)}(\Lambda q) \]  

(69)

Since at rest we have \([J^{(j)},\Pi^{(j)}(m)] = 0\), then according to the Schur’s lemma \(\Pi^{(j)}_{\sigma\sigma'}(m) = m^{2j} \delta_{\sigma\sigma'}\). After the substitution of \(D^{(j)}[\Lambda]\) in Eq. (69) one has

\[ \Pi^{(j)}(q) = m^{2j} \exp(2\Theta \mathbf{q} \cdot \mathbf{J}^{(j)}) \]  

(70)

One can construct the analogous matrix for the \((0,J)\) representation by the same procedure:

\[ \mathbf{\Pi}^{(j)}(q) = m^{2j} \exp(-2\Theta \mathbf{q} \cdot \mathbf{J}^{(j)}) \]  

(71)

Finally, by direct verification one has in the coordinate representation

\[ \Pi_{\sigma\sigma'}(-i\partial) \Phi_{\sigma'} = m^{2j} \Xi_{\sigma} \]  

(72a)

\[ \Pi_{\sigma\sigma'}(-i\partial) \Xi_{\sigma'} = m^{2j} \Phi_{\sigma} \]  

(72b)

if \(\Phi_{\sigma}(0)\) and \(\Xi_{\sigma}(0)\) are indistinguishable.\(^5\)

As a result one has

\[ [\gamma^{\mu_1 \mu_2 \cdots \mu_{2j}} \partial_{\mu_1} \partial_{\mu_2} \cdots \partial_{\mu_{2j}} + m^{2j}] \Psi(x) = 0 \]  

(73)

with the Barut-Muzinich-Williams covariantly-defined matrices [28,32]. For the spin-1 they are:

\[ \gamma_{ij} = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \]  

(74a)

\[ \gamma_{44} = \left( \begin{array}{cc} 0 & iJ_i \\ -iJ_i & 0 \end{array} \right) \]  

(74b)

\[ \gamma_{ij} = \left( \begin{array}{cc} 0 & \delta_{ij} - J_i J_j - J_j J_i \\ \delta_{ij} - J_i J_j - J_j J_i & 0 \end{array} \right) \]

\(^5\)Later, this fact has been incorporated in the Ryder book [30]. Truely speaking, this is an additional postulate. It is also possible that the zero-momentum-frame \(2(2J+1)\)-component objects (the 4-spinor in the \((1/2,0)\oplus(0,1/2)\) representation, the bivector in the \((1,0)\oplus(0,1)\) representation, etc.) are connected by an arbitrary phase factor [31].
Later Sankaranarayanan and Good considered another version of this theory [32]. For the $J = 1$ case they introduced the Weaver-Hammer-Good sign operator, ref. [33], $m^2 \rightarrow m^2 (i\partial/\partial t)/E$, which led to the different parity properties of an antiparticle with respect to a boson particle. Next, Hammer et al [34] introduced another higher-spin equation. In the spin-1 case it is:

$$[\gamma_{\mu\nu}\partial_\mu\partial_\nu + \partial_\mu\partial_\mu - 2m^2]\Psi^{(j=1)} = 0. \tag{75}$$

In fact, they added the Klein-Gordon equation to the Weinberg equation.

Weinberg considered massless cases too. He claimed that there is no problem [12b] to put $m \rightarrow 0$ in propagators and field functions of the $(J,0)$ and $(0,J)$ fields. But, there are indeed problems for the fields of the $(J/2, J/2)$-types, e.g., for the 4-vector potential. The Weinberg theorem says [12b,p.B885]: “A massless particle operator $a(p, \lambda)$ of helicity $\lambda$ can only be used to construct fields which transform according to representations $(A, B)$ such that $B - A = \lambda$. For instance, a left-circularly polarized photon with $\lambda = -1$ can be associated with $(1,0)$, $(3/2, 1/2)$, $(2,1)$ fields but not with the 4-vector potential $(1/2, 1/2)$, at least until we broaden our notion of what we mean by Lorentz transformations”. He indicated that this is a consequence of the non-semi-simple structure of the little group. In his book [35, §5.9], he gave additional details of what he meant in the above statement. Indeed, divergent terms of the 4-vector potential ($\lambda = \pm 1$; under certain choice of the normalization) in the $m \rightarrow 0$ limit may be removed by gauge transformations, see Appendix D.

**APPENDIX C. ADDITIONAL FIELDS IN THE BARGMANN-WIGNER FORMALISM**

I noted [22] that it is possible to use another matrices in the expansion of the Bargmann-Wigner symmetric spinor [20,21]. If

$$\Psi_{(\alpha\beta)} = (\gamma^\mu R)_{\alpha\beta}(c a m A_\mu + c f F_\mu) + (\sigma^{\mu\nu} R)_{\alpha\beta}(c a m \gamma^5 A_{\mu\nu} + c f F_{\mu\nu}), \tag{76}$$

we have “new” Proca equations

$$c a m(\partial_\mu A_\nu - \partial_\nu A_\mu) + c f(\partial_\mu F_\nu - \partial_\nu F_\mu) = ic a m^2 \epsilon_{\alpha\beta\mu\nu}A_{\alpha\beta} + 2mc F F_{\mu\nu} \tag{77a}$$

$$c a m^2 A_\mu + c f m F_\mu = -ic a m\epsilon_{\mu\nu\alpha\beta}\partial_\nu A_{\alpha\beta} - 2c f \partial_\nu F_{\mu\nu}. \tag{77b}$$

Other generalizations of the Bargmann-Wigner formalism have been presented therein and in ref. [16].

**APPENDIX D. FIELD FUNCTIONS IN THE MOMENTUM SPACE**

The 4-potentials in the momentum space are [29,35,17,18] (the standard basis is used):

$$u^\mu(p, +1) = -\frac{N}{\sqrt{2m}} \begin{pmatrix} p_r \\ m + \frac{p_r p_r}{E_p + m} \\ im + \frac{p_3 p_r}{E_p + m} \\ \frac{p_3 p_3}{E_p + m} \end{pmatrix}, \quad u^\mu(p, -1) = \frac{N}{\sqrt{2m}} \begin{pmatrix} p_t \\ m + \frac{p_t p_t}{E_p + m} \\ -im + \frac{p_3 p_3}{E_p + m} \\ \frac{p_3 p_3}{E_p + m} \end{pmatrix}, \tag{78a}$$

16
\[
u^\mu(p, 0) = \frac{N}{m} \pmatrix{ \frac{p_3}{E_{p+m}} \\ \frac{p_1 p_3}{E_{p+m}} \\ \frac{p_2 p_3}{E_{p+m}} \\ m + \frac{p_3^2}{E_{p+m}}} , \quad u^\mu(p, 0_t) = \frac{N}{m} \pmatrix{ E_p \\ p_1 \\ p_2 \\ p_3 },
\]

(\(p_{r,l} = p_1 \pm ip_2\)). The corresponding fields are:

\[
B^{(+)}(p, +1) = -\frac{iN}{2\sqrt{2m}} \pmatrix{ -ip_3 \\ p_3 \\ ip_r } = +e^{-i\alpha_{-1}} B^{(-)}(p, -1) ,
\]

\[
B^{(+)}(p, 0) = \frac{iN}{2m} \pmatrix{ p_2 \\ -p_1 \\ 0 } = -e^{-i\alpha_0} B^{(-)}(p, 0) ,
\]

\[
B^{(+)}(p, -1) = \frac{iN}{2\sqrt{2m}} \pmatrix{ ip_3 \\ p_3 \\ -ip_l } = +e^{-i\alpha_{+1}} B^{(-)}(p, +1) ,
\]

and

\[
E^{(+)}(p, +1) = -\frac{iN}{2\sqrt{2m}} \pmatrix{ E_p - \frac{p_1 p_3}{E_{p+m}} \\ iE_p - \frac{p_2 p_3}{E_{p+m}} \\ -\frac{p_3^2}{E_{p+m}} } = +e^{-i\alpha'_{-1}} E^{(-)}(p, -1) ,
\]

\[
E^{(+)}(p, 0) = \frac{iN}{2m} \pmatrix{ -\frac{p_3 p_1}{E_{p+m}} \\ -\frac{p_3 p_2}{E_{p+m}} \\ E_p - \frac{p_3^2}{E_{p+m}} } = -e^{-i\alpha'_{0}} E^{(-)}(p, 0) ,
\]

\[
E^{(+)}(p, -1) = \frac{iN}{2\sqrt{2m}} \pmatrix{ E_p - \frac{p_3 p_1}{E_{p+m}} \\ -iE_p - \frac{p_3 p_2}{E_{p+m}} \\ -\frac{p_3^2}{E_{p+m}} } = +e^{-i\alpha'_{+1}} E^{(-)}(p, +1) ,
\]

where we denoted a normalization factor appearing in the definitions of the potentials (and/or in the definitions of the physical fields through potentials) as \(N\) (it is usually to be chosen equal to 1 for 4-potentials). The signs (\(\pm\)) refer to the positive- and negative-frequency solutions, respectively.

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