Some remarks on the \(\varepsilon\)-expansion of dimensionally regulated Feynman diagrams

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Some problems related to construction of the \(\varepsilon\)-expansion of dimensionally regulated Feynman integrals are discussed. For certain classes of diagrams, an arbitrary term of the \(\varepsilon\)-expansion can be expressed in terms of log-sine integrals related to the polylogarithms. It is shown how the analytic continuation of these functions can be constructed in terms of the generalized Nielsen polylogarithms.

1. Dimensional regularization \(\varepsilon\) is one of the most powerful tools used in loop calculations. In some cases, one can derive results valid for an arbitrary space-time dimension \(n\), usually in terms of various hypergeometric functions. However, for practical purposes the coefficients of the expansion in \(\varepsilon\) are important, where the regulator \(\varepsilon\) corresponds to the difference between \(n\) and the \((\text{integer})\) number of dimensions of interest. Below we shall usually imply that \(n = 4 - 2\varepsilon\). In multi-loop calculations higher terms of the \(\varepsilon\)-expansion of one- and two-loop functions are needed, since they may get multiplied by poles in \(\varepsilon\), not only due to factorizable loops, but also as a result of applying the well-known reduction techniques \(\varepsilon\).

In refs. \[6,7\], it was shown that the log-sine integral functions (see, e.g., in \[8\], chapter 7.9),

\[
\text{Ls}_j(\theta) \equiv - \int_0^\theta d\theta' \ln^{j-1} \left| 2 \sin \frac{\theta'}{2} \right|
\]

happen to be very useful to represent results for higher terms of the \(\varepsilon\)-expansion.

For instance, for the one-loop two-point function \(J^{(2)}(n; \nu_1, \nu_2)\) with external momentum \(k\), masses \(m_1\) and \(m_2\) and unit powers of propagators, the following result for an arbitrary term of the \(\varepsilon\)-expansion has been obtained in \[6,8\]:

\[
J^{(2)}(4-2\varepsilon; 1, 1) = \text{ln}^2 \frac{\Gamma(1 + \varepsilon)}{2(1 - 2\varepsilon)} \\
\times \left\{ \frac{m_1^{-2\varepsilon} + m_2^{-2\varepsilon}}{\varepsilon} + \frac{m_1^{2} - m_2^{2}}{\varepsilon k^2} (m_1^{-2\varepsilon} - m_2^{-2\varepsilon}) \right\} \\
+ \left[ \Delta(m_1^2, m_2^2, k^2) \right]^{1/2-\varepsilon} \sum_{j=0}^{\infty} \frac{(2\varepsilon)^j}{j!} \\
\times \sum_{i=1}^{2} \left[ \text{Ls}_{j+1}(\pi) - \text{Ls}_{j+1}(2\tau_{02}^i) \right]
\]

where

\[
\cos \tau_{01}^i = \left( m_1^2 - m_2^2 + k^2 \right) / \left( 2m_1 \sqrt{k^2} \right), \\
\cos \tau_{02}^i = \left( m_2^2 - m_1^2 + k^2 \right) / \left( 2m_2 \sqrt{k^2} \right), \\
\cos \tau_{12} = \left( m_1^2 + m_2^2 - k^2 \right) / \left( 2m_1 m_2 \right).
\]

whereas the “triangle” function \(\Delta\) is defined as

\[
\Delta(x, y, z) = 2xy + 2yz + 2zx - x^2 - y^2 - z^2.
\]

One can see that \(\tau_{12} + \tau_{01}^1 + \tau_{02}^2 = \pi\). In fact, these angles can be associated with a triangle whose sides are \(m_1, m_2\) and \(\sqrt{k^2}\). Moreover, the area of this triangle is \(\frac{1}{4} \sqrt{\Delta(m_1^2, m_2^2, k^2)}\). For details of geometrical description, see in \[8\].

Note that the values of \(\text{Ls}_j(\pi)\) can be expressed in terms of Riemann’s \(\zeta\) function, see
Eqs. (7.112)–(7.113) of [8]. The infinite sum with \( L_s(\pi) \) in [8] can be converted into \( \Gamma \) functions,
\[
\sum_{j=0}^{\infty} \frac{(2\pi)^j}{j!} L_{s,j+1}(\pi) = -\pi \frac{\Gamma(1+2\pi)}{\Gamma^2(1+\pi)}.
\] (5)

2. The \( \varepsilon \)-expansion of [8] is directly applicable in the region where \( \Delta(m_1^2, m_2^2, k^2) \geq 0 \), i.e. when \( (m_1 - m_2)^2 \leq k^2 \leq (m_1 + m_2)^2 \). In other regions, the proper analytic continuation of the occurring \( L_s(\theta) \) should be constructed. To do this, it is convenient to introduce the variable (cf. in ref. [8])
\[
z = e^{i\sigma \theta}, \quad \ln(-z) = \ln(z) - iP\sigma,
\] (6)
where the choice of the sign \( \sigma = \pm 1 \) is related to the causal “+i0” prescription for the propagators.

Since \( L_{s,1}(\theta) = -\theta \), we get
\[
is [L_{s,1}(\pi) - L_{s,1}(\theta)] = \ln(-z).
\] (7)

For the next order, we can use the fact that \( L_{s,2}\theta) = Cl_2(\theta), \) where (see in [8])
\[
Cl_j(\theta) = \left\{ \begin{array}{ll}
\frac{1}{2\pi} \left[ Li_j (e^{i\theta}) - Li_j (e^{-i\theta}) \right], & j \text{ even} \\
\frac{1}{2} \left[ Li_j (e^{i\theta}) + Li_j (e^{-i\theta}) \right], & j \text{ odd}
\end{array} \right.
\] (8)
is the Clausen function, whereas \( Li_j \) is the polylogarithm. In other words, \( Cl_j(\theta) \) corresponds either to the imaginary part or to the real part of \( Li_j (e^{i\theta}) \), depending on whether \( j \) is even or odd.

Therefore, the analytic continuation reads
\[
is [L_{s,2}(\pi) - L_{s,2}(\theta)] = -\frac{1}{2} \left[ Li_2 (z) - Li_2 (1/z) \right], \quad (9)
\] where \( L_{s,2}(\pi) = 0 \). The result for the \( \varepsilon \)-term of the two-point function was obtained in [8].

To proceed further, we need similar relations between higher \( L_{s,j}(\theta) \) and the imaginary (or real) parts of the polylogarithms. For \( j = 3 \), \( L_{s,3}(\theta) \) can be expressed in terms of the imaginary part of \( Li_3 (1 - e^{i\theta}) \). Then, the imaginary part of \( Li_4 (1 - e^{i\theta}) \) is already a mixture of \( L_{s,4}(\theta) \) and \( Cl_4(\theta) \), whereas its real part involves the generalized log-sine integral \( L_{s,4}(1)(\theta) \). All these relations can be found in [8] (some misprints are mentioned in [8]). However, attempts to generalize these results to higher functions show that the relations get more and more cumbersome.

Instead of going that way, we suggest to consider how the higher \( L_{s,j} \) functions are generated by the imaginary (and real) parts of the generalized Nielsen polylogarithms (see, e.g., in [10]),
\[
S_{s,b}(\xi) = \frac{(-1)^{a+b-1}}{(a+1)!} \int_0^1 d\xi \frac{\ln^{a-1} \xi \ln^b(1 - \xi)}{\xi}, \quad (10)
\] where \( S_{s,1}(z) = Li_{s,1}(z) \). We obtain
\[
Re \, S_{1,2}(e^{i\theta}) = \frac{1}{2} Cl_3 (\theta) + \frac{1}{2} \zeta_3 - \frac{1}{2} (\pi - \theta) Ls_2 (\theta),
\]
\[
Im \, S_{1,2}(e^{i\theta}) = -\frac{1}{2} Ls_3 (\theta) - \frac{1}{24} \theta (\theta^2 - 3 \pi \theta + 3 \pi^2),
\]
\[
Re \, S_{1,3}(e^{i\theta}) = -\frac{1}{2} Ls_4^{(1)} (\theta) + \frac{1}{4} \pi Ls_3 (\theta)
\]
\[
+ \frac{11 \pi^4}{48} - 3 \pi^3 \theta - \frac{1}{32} \pi^2 \theta^2 + \frac{1}{48} \pi \theta^3 - \frac{1}{192} \theta^4,
\]
\[
Im \, S_{1,3}(e^{i\theta}) = \frac{1}{6} Ls_4 (\theta) + \frac{1}{4} Cl_4 (\theta) - \frac{1}{4} \pi \zeta_3
\]
\[
+ \frac{1}{2} (\pi - \theta) Cl_3 (\theta) - \frac{1}{8} (\pi - \theta)^2 Ls_2 (\theta),
\]
where the generalized log-sine integral (see in [8]) is defined as
\[
L_{s,2}^{(k)}(\theta) = -\int_0^\theta d\theta' \theta'^{k-1} \left| \frac{2 \sin \theta'}{2} \right|. \quad (11)
\]

In particular, \( L_{s,4}^{(0)}(\theta) = L_{s,1}(\theta) \). Using these relations, we can express the \( L_{s,3} \) and \( L_{s,4} \) functions,
\[
is [L_{s,3}(\pi) - L_{s,3}(\theta)] =

S_{1,2}(z) - S_{1,2}(1/z) - \frac{1}{4} \ln^3(-z), \quad (12)
is [L_{s,4}(\pi) - L_{s,4}(\theta)] = -3 \left[ S_{1,3}(z) - S_{1,3}(1/z) \right]
\]
\[
+ \frac{1}{2} \left[ Li_4 (z) - Li_4 (1/z) \right]
\]
\[
- \frac{3}{4} \left[ Li_3 (z) + Li_3 (1/z) \right] \ln(-z)
\]
\[
+ \frac{1}{8} \left[ Li_2 (z) - Li_2 (1/z) \right] \ln^2(-z),
\] (13)
where \( L_{s,3}(\pi) = -\frac{1}{12} \pi^3, L_{s,4}(\pi) = \frac{3}{2} \pi \zeta_3 \).

We have also constructed further expressions,
\[
is [L_{s,5}(\pi) - L_{s,5}(\theta)] = 12 \left[ S_{1,4}(z) - S_{1,4}(1/z) \right]
\]
\[
- 3 \left[ S_{2,3}(z) - S_{2,3}(1/z) \right]
\]
\[
+ 3 \left[ S_{1,3}(z) + S_{1,3}(1/z) \right] \ln(-z)
\]
\[
+ \frac{1}{24} \ln^5(-z), \quad (14)
is [L_{s,6}(\pi) - L_{s,6}(\theta)] = -60 \left[ S_{1,5}(z) - S_{1,5}(1/z) \right]
\]
\[
+ 15 \left[ S_{2,4}(z) - S_{2,4}(1/z) \right]
\]
\[
- \frac{15}{4} \left[ Li_6 (z) - Li_6 (1/z) \right]
\]
\[
- 15 \left[ S_{1,4}(z) + S_{1,4}(1/z) \right] \ln(-z)
\]
\[
+ \frac{15}{4} \left[ Li_5 (z) + Li_5 (1/z) \right] \ln(-z)
\]
\[-\frac{15}{8} [\text{Li}_4(z) - \text{Li}_4(1/z)] \ln^2(-z)\\ + \frac{5}{4} [\text{Li}_3(z) + \text{Li}_3(1/z)] \ln^3(-z)\\ - \frac{37}{2} [\text{Li}_2(z) - \text{Li}_2(1/z)] \ln^4(-z), \quad (15)\]

with \(\text{Li}_k(z)\) being the polylogarithmic functions.

Substituting (12) into Eq. (4) (taking \(\theta = \tau_{01} \equiv \tau'_{02}\), denoting the \(z\)'s from Eq. (3) as \(z_1\) and \(z_2\), and setting \(\sigma = 1\), we arrive at the analytical continuation of the terms of the \(\varepsilon\)-expansion, up to order \(\varepsilon^3\). In fact, we have also obtained results for higher \(L_j\) functions, up to \(j = 10\), which allowed us to reach the order \(\varepsilon^9\).

It is instructive to consider the limit \(m_2 \to 0\). Introducing the variables \(x = m_2^2/k^2\) and \(y = m_2^2/k^2\) (and remembering that \(\sigma = 1\)), we get

\[z_1 \to \frac{y}{(1-x)^2} + O(y^2), \quad z_2 \to \frac{2xy}{1-x} + O(y^2), \]

\[\ln [\Delta(m_1^2, m_2^2, k^2)/(k^2)^2] \to -i\pi + 2\ln(1-x)\\ - \frac{2y(1+x)}{(1-x)^2} + O(y^2). \]

Then, \(S_{a,b}(1/z_1)\) can be converted into \(S_{a,b}(z_1)\) by means of known relations given in Ref. [10]. After this, the limit \(y \to 0\) (\(m_2 \to 0\)) can be taken, since all \(\ln y\) terms cancel. The obtained expressions can be simplified by transforming \(S_{a,b}(z_2)\) into \(S_{a,b}(1/z_2)\). In such a way, we also avoid appearance of terms like \(S_{a,b}(-1)\). Note that for this limit the terms up to order \(\varepsilon^3\) can be extracted from Eq. (A.3) of ref. [10]. Our expressions are in agreement with their results.

In fact, using hypergeometric representation

\[J^{(2)}(4-2\varepsilon; 1, 1)\bigg|_{m_1 = 0, m_2 = m} = \frac{i\pi^{2-\varepsilon} m^{-2\varepsilon}}{\varepsilon(1-\varepsilon)} \Gamma(1+\varepsilon) \Gamma(1)\]

(see, e.g., Eq. (10) of [12]), an arbitrary term of the \(\varepsilon\)-expansion can be obtained. Employing Kummer's relations for contiguous functions, one can transform the \(2F_1\) function from Eq. (10) into

\[1 - \varepsilon \left\{ \frac{1 + u - (1-u)^2}{2u} - \frac{1 - \varepsilon}{2u} \right\} 2F_1 \left( 1, 1+\varepsilon \left| 1 - \varepsilon \right) u \right\}, \]

with \(u \equiv k^2/m^2\). This (transformed) \(2F_1\) function can be expressed in terms of a simple one-fold parametric integral,

\[(1-u)^{-1-2\varepsilon} \left\{ 1 - \varepsilon \int_0^1 \frac{dt}{t} t^{-\varepsilon} \left[ (1-ut)^{2\varepsilon} - 1 \right] \right\}. \]

Expanding the integrand in \(\varepsilon\), we arrive at

\[J^{(2)}(4-2\varepsilon; 1, 1)\bigg|_{m_1 = 0, m_2 = m} = \frac{i\pi^{2-\varepsilon} m^{-2\varepsilon}}{\varepsilon(1-\varepsilon)} \Gamma(1+\varepsilon) \Gamma(1)\]

\[\times \frac{1}{2} \sum_{j=1}^{\infty} \sum_{k=1}^{j} (-1)^{j-k} S_{k,j-k+1}(u) \]

(17)

which agrees with the results discussed earlier.

3. In ref. [10] it was shown that similar explicit results can be constructed for the off-shell massless one-loop three-point function with external momenta \(p_1, p_2\) and \(p_3\) \((p_1 + p_2 + p_3 = 0)\),

\[J(n; \nu_1, \nu_2, \nu_3| p_{12}^2, p_{23}^2, p_{31}^2) = \int \frac{d^n r}{(p_2 - r)^{1+\nu_2} \Gamma(1+\nu_2) \Gamma(1+\nu_3)} \Leftarrow \]

as well as for the two-loop vacuum diagram with arbitrary masses \(m_1, m_2\) and \(m_3\),

\[I(n; \nu_1, \nu_2, \nu_3| m_{12}^2, m_{23}^2, m_{31}^2) = \int \frac{d^n q}{(q^2 - m_{12}^2)^{1+\nu_2} \Gamma(1+\nu_2) \Gamma(1+\nu_3)} \]

(19)

Results for general \(n\) and \(\nu_i\) (in terms of hypergeometric functions of two variables) are available in Refs. [13, 14]. According to the magic connection [10], these integrals are closely related to each other. For example, in the case \(\nu_1 = \nu_2 = \nu_3 = 1\) this connection (see Eq. (16) of [8]) yields

\[J(4-2\varepsilon; 1, 1, 1) = 4 \pi^{-3\varepsilon} \Gamma(1+\varepsilon) \Gamma(1) \]

\[\times \Gamma(1+\varepsilon) \Gamma(1) \int (2 + 2\varepsilon; 1, 1, 1), \quad (20)\]

where we assume that \(p_{12}^2 \leftrightarrow m_{12}^2\). Below we shall omit the arguments \(p_{12}^2\) and \(m_{12}^2\) in the integrals \(J\) and \(I\), respectively.

Then, using exact results in terms of \(2F_1\) functions \([14, 15]\), in combination with the formula

\[\sum_{j=0}^{\infty} \frac{(-2\varepsilon)^j}{j!} L_{j+1}(2\phi) = -2\pi \frac{\Gamma(1-2\varepsilon)}{\Gamma^2(1-\varepsilon)} \theta(-\cos \phi) \]

(21)
whose sides are analogy with the two-point case (3), the angles following results have been obtained in [5]:
\[
J(4 - 2\varepsilon; 1, 1, 1) = 2\pi^2 - \varepsilon i^{1+2\varepsilon} \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1 - 2\varepsilon)} \times \left[ \frac{\Delta(p_1^2, p_2^2, p_3^2)}{(p^2_1 p^2_2 p^2_3)^{1/2}} \right]^{1/2+\varepsilon} \sum_{j=0}^{\infty} (-2\varepsilon)^j \left( \frac{\varepsilon^2}{(j+1)!} \right) \\
\times \left[ L_{j+2}(\pi) - \sum_{i=1}^{3} [L_{j+i}(\pi) - L_{j+2}(2\phi_i)] \right],
\] (22)

where the angles \( \phi_i \) (i = 1, 2, 3) are defined via
\[
\cos \phi_1 = \frac{(p_1^2 + p_2^2 - p_3^2)}{2\sqrt{p_1^2 p_2^2}}, \quad \cos \phi_2 = \frac{(p_2^2 + p_3^2 - p_1^2)}{2\sqrt{p_2^2 p_3^2}}, \quad \cos \phi_3 = \frac{(p_3^2 + p_1^2 - p_2^2)}{2\sqrt{p_3^2 p_1^2}}.
\] (24)

(remember that \( p_1^2 \leftrightarrow m_1^2 \) for the integrals \( I \)), so that \( \phi_1 + \phi_2 + \phi_3 = \pi \). Note that the angles \( \theta_i \) from \([14, 13]\) are related to \( \phi_i \) as \( \theta_i = 2\phi_i \). By analogy with the two-point case \([3]\), the angles \( \phi_i \) can be understood as the angles of a triangle whose sides are \( \sqrt{p_1^2} \), \( \sqrt{p_2^2} \) and \( \sqrt{p_3^2} \), whereas its area is \( \frac{1}{2}\sqrt{\Delta(p_1^2, p_2^2, p_3^2)} \).

For the lowest orders (\( \varepsilon^0 \) and \( \varepsilon^1 \)), we reproduce eqs. (9)–(10) from \([8]\). Useful representations for the \( \varepsilon^0 \) terms of both types of diagrams can also be found in \([14]\). Moreover, in Eq. (26) of \([14]\), a one-fold integral representation for \( J(4 - 2\varepsilon; 1, 1, 1) \) is presented (for its generalization, see Eq. (7) of \([8]\)). Expanding the integrand in \( \varepsilon \), we were able to confirm the \( \varepsilon \)-expansion \([22]\) numerically.

To construct the analytic continuation of the terms of the \( \varepsilon \)-expansion \([22]\) and \([23]\), we need just to apply substitutions \([7]\), \([6]\), \([32, 33]\), with \( \theta = 2\phi_i \) (i = 1, 2, 3). The remaining terms \( L_{j+2}(\pi) \) can actually be treated in the same way, if we substitute \( \theta = 0 \). In any case, their values are known (in terms of \( \zeta \) function) and can be summed into a combination of \( \Gamma \) functions, Eq. \([3]\). We get three variables \( z_i = \varepsilon^{2i}\phi_i \), see Eq. \([3]\), such that \( z_1 z_2 z_3 = 1 \). The causal prescription requires to take \( \sigma = +1 \) for the \( J \)-integrals and \( \sigma = -1 \) for the \( I \)-integrals. Using the substitutions presented above, we obtain the analytic continuation of the results \([22]\) and \([23]\) up to \( \varepsilon^4 \). We note that the result for the \( \varepsilon \)-term was known \([14, 13]\) (in terms of \( L_{i+3} \)).

When the masses are equal, \( m_1 = m_2 = m_3 = m \) (this also applies to the symmetric case \( p_1^2 = p_2^2 = p_3^2 \equiv p^2 \)), the three angles \( \phi_i \) are all equal to \( \pi/3 \), whereas \( \Delta(m^2, m^2, m^2) = 3m^4 \). Therefore, in this case the r.h.s. of Eq. \([23]\) becomes
\[
\pi^4 - 2\varepsilon^4 \frac{\Gamma^2(1 + \varepsilon) m^{2 - 4\varepsilon}}{(1 - \varepsilon)(1 - 2\varepsilon)} \left\{ - \frac{3}{2\varepsilon^2} + \frac{\sqrt{3}}{3\varepsilon} \sum_{j=0}^{\infty} \frac{(2\varepsilon)^j}{(j+1)!} \times \left[ 3L_{j+2}(\pi) - 2L_{j+2}(2\phi_i) \right] \right\}.
\] (25)

For instance, in the contribution of order \( \varepsilon \) the transcendental constant \( L_{i+3}(2\pi/3) \) occurs. This constant was discussed in detail in \([8]\). The fact that \( L_{i+3}(2\pi/3) \) occurs in certain two-loop on-shell integrals and three-loop vacuum integrals has been noticed in \([15, 20]\). Moreover, in \([10]\) it was observed that the higher-\( j \) terms from \([23]\) form a basis for certain on-shell integrals with a single mass parameter. Connection of \( L_{i+3}(2\pi/3) \) with multiple binomial sums is discussed in \([21]\). We also note that in \([22]\) the constant \( L_{i+3}(\pi/2) \) appeared.

4. One of the interesting problems is to construct terms of the \( \varepsilon \)-expansion for the one-loop three-point function with general masses. In this sense, the geometrical description seems to be rather instructive. The geometrical approach to the three-point function is discussed in section V of \([8]\) (see also in \([3]\)). This function can be represented as an integral over a spherical (or hyper-
bolic) triangle, as shown in Fig. 6 of [7], with a weight factor $1/\cos^{1-2\varepsilon}\theta$ (see eqs. (3.38)–(3.39) of [7]). This triangle 123 is split into three triangles 012, 023 and 031. Then, each of them is split into two rectangular triangles, according to Fig. 9 of [7]. We consider the contribution of one of the six resulting triangles, namely the left rectangular triangle in Fig. 9. Its angle at the vertex 0 is denoted as $1/2\varepsilon_1^+$, whereas the height dropped from the vertex 0 is denoted $\eta_2$.

The remaining angular integration is (see eq. (5.16) of [7])

$$\frac{1}{2\varepsilon} \int_0^{\varepsilon_1^{1/2}} d\varphi \left[ 1 - \left( 1 + \frac{\tan^2 \eta_2}{\cos^2 \varphi} \right)^{-\varepsilon} \right] = \frac{1}{\varepsilon} \sum_{j=0}^{\infty} \frac{(-\varepsilon)^j}{j!} \int_0^{\varepsilon_1^{1/2}} d\varphi \ln^{j+1} \left( 1 + \frac{\tan^2 \eta_2}{\cos^2 \varphi} \right). \quad (26)$$

First of all, we note that the l.h.s. of Eq. (26) yields a representation valid for an arbitrary $\varepsilon$ (i.e., in any dimension). To get the result for the general three-point function, we need to consider a sum of six such integrals. The resulting representation is closely related to the representation in terms of hypergeometric functions of two arguments [23] (see also in [24] for some special cases).

In the limit $\varepsilon \to 0$ we get a combination of Cl$_2$ functions, eq. (5.17) of [7]. Collecting the results for all six triangles, we get the result for the three-point function with arbitrary masses and external momenta, corresponding (at $\varepsilon = 0$) to the analytic continuation of the well-known formula presented in [23]. The higher terms of the $\varepsilon$-expansion correspond to the angular integrals on the r.h.s. of Eq. (26). The problem of constructing closed representations for these terms, as well as their analytic continuation, is very important. We note that the $\varepsilon$-term of the three-point function with general masses has been calculated in [9] in terms of Li$_3$.

5. We have shown that the compact structure of the coefficients of the $\varepsilon$-expansion of the two-point function (1), the massless off-shell three-point function (22) and two-loop massive vacuum diagrams (23), in terms of log-sine integrals, allows to perform analytic continuation in terms of generalized Nielsen polylogarithms (10), in some cases (17) even for an arbitrary order of the $\varepsilon$-expansion. It is likely that a further generalization of these results is possible, e.g. for the three-point function with different masses, two-point integrals with two (and more) loops and three-loop vacuum integrals. In particular, numerical analysis of the coefficients of the expansion of certain two-point on-shell integrals and three-loop vacuum integrals [19] shows that in some cases the values of generalized log-sine integrals $L_{s_j}^{(k)}$, Eq. (11), may be involved. For instance, in ref. [19] it was shown that $L_{s4}^{(1)}(2\pi/3)$ is connected with $V_{5,1}$ from [20].

The fact that the generalization of $L_{s2} = \text{Cl}_2$ goes in the $L_{s_j}$ direction, rather than in $\text{Cl}_j$ direction (see Eq. (8)), is very interesting. There is another example [27,28], the off-shell massless ladder three- and four-point diagrams with an arbitrary number of loops, when such a generalization went in the $\text{Cl}_j$ direction (for details, see [7]). It could be also noted that the the two-loop non-planar (crossed) three-point diagram gives in this case the square of the one-loop function, $(\text{Cl}_2(\theta))^2$ (cf. Eq. (23) of [17]), leading to the structure $(\text{Cl}_2(\pi/3))^2$ in the symmetric ($p_1^2 = p_2^2$) case. Recently, these constants have been also found in massive three-loop calculations [27,28,29].

The construction of analytic continuation of the generalized log-sine functions should be investigated in more detail. In fact, it may require including some other generalizations of polylogarithms (see, e.g., in Ref. [31]).

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