GENERALIZED LONG-MOODY REPRESENTATIONS OF BRAID GROUPS

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Abstract. Long and Moody give a method of constructing representations of the braid groups $B_n$. We discuss some ways to generalize their construction. One of these gives representations of subgroups of $B_n$, including the Gassner representation of the pure braid group as a special case. Another gives representations of the Hecke algebra.

1. Introduction

The braid group $B_n$ is the group with presentation

$$\langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| > 1, \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \text{ if } |i-j| = 1 \rangle.$$ 

It has a rich and interesting representation theory [Bir94] [FLSV04].

D. D. Long, in joint work with J. Moody, gave a construction that inputs a representation of a certain semidirect product $F_n \rtimes B_n$ and outputs a new representation of $B_n$. At first this construction may not sound useful, since its input seems to be harder to come by than its output. However there are at least two factors in its favor. First, the output seems to be in some sense more sophisticated than the input. For example, one can start with a one dimensional representation and obtain the Burau representation. Second, the semidirect product $F_n \rtimes B_n$ is a subgroup of $B_{n+1}$. Thus one can recursively apply the algorithm to construct a sequence of increasingly complicated representations of $B_n$ for all $n$.

Let $F_n$ be the free group $\langle g_1, \ldots, g_n \rangle$. There is a well known action of $B_n$ on $F_n$, which gives rise to a semidirect product $F_n \rtimes B_n$. This requires an arbitrary choice of convention. In this paper, the relations are as follows.

- $g_{i+1} \sigma_i = \sigma_i g_i$,
- $g_i \sigma_i = \sigma_i g_i g_{i+1} g_i^{-1}$,
- $g_j \sigma_i = \sigma_i g_j$ if $j \not\in \{i, i+1\}$.

The following is Theorem 2.1 in [Lon94], which Long credits to be a joint work with Moody.

**Theorem 1.1.** Given a representation $\rho: F_n \rtimes B_n \to \text{GL}(V)$ we may construct a representation $\rho^+: B_n \to \text{GL}(V^{\otimes n})$.

The method of construction can be summarized as follows. The representation $\rho$ determines a system of local coefficients on the $n$-times punctured disk $D_n$. The first cohomology of $D_n$ with these local coefficients then turns out to be isomorphic to $V^{\otimes n}$. The braid group $B_n$ is the mapping class group of $D_n$, so it has a natural induced action on this vector space. For details, see [Lon94].
Long also describes how to explicitly compute the matrices for $\rho^+$ in terms of the matrices for $\rho$. In this paper we will only use this more concrete (but less well motivated) definition.

In Section 2 we give a proof of Theorem 1.1. In Section 3 we generalize the Long-Moody construction to subgroups of $B_n$. We then discuss important special cases of this construction. In Section 4 we apply the construction to the pure braid group $P_n$, obtaining the Gassner representation as a special case. In Section 5 we describe how these methods can be used to obtain Lawrence's representations. These are of interest because they contain all representations of the Hecke algebra.

Finally, in Section 6 we give the “reduced” version of the Long-Moody construction.

For convenience, we will work with finite dimensional representations over the field $\mathbb{C}$. The construction works equally well for infinite dimensional representations, and for representations over any field or commutative ring.

Note that, while the main results of this paper are stated as Theorems and Corollaries and so on, they are really constructions. Taken literally, a statement of the form “Given one representation, we may construct another representation” is trivial. Of course, the interesting thing is the method of construction.

2. The universal Long-Moody representation

The aim of this section is to give a proof of Theorem 1.1. Our proof is not exactly new, but it does make explicit an idea that is implicit in [Lon94]. This is the idea of a representation we call the “universal Long-Moody representation”.

Proof of Theorem 1.1. Recall that we are given a representation $\rho: F_n \rtimes B_n \to \text{GL}(V)$, and must construct a representation $\rho^+: B_n \to \text{GL}(V^{\otimes n})$. First we define the “universal Long-Moody representation” of $B_n$. This is a homomorphism from $B_n$ to the group of invertible $n$ by $n$ matrices with entries in the group ring of $F_n \rtimes B_n$.

Let $\phi$ be the following homomorphism from $B_n$ to the group of invertible $n$ by $n$ matrices with entries in $\mathbb{Z}[F_n \rtimes B_n]$.

$$\phi(\sigma_i) = \sigma_i \begin{bmatrix} I_{i-1} & R_i & I_{n-i-1} \end{bmatrix}.$$ 

Here, $I_{i-1}$ and $I_{n-i-1}$ are the appropriately sized identity matrices, and $\sigma_i$ acts on the matrix by scalar multiplication on the left.

It remains to check that $\phi$ satisfies the braid relations. The most difficult is the braid relation

$$\phi(\sigma_i)\phi(\sigma_{i+1})\phi(\sigma_i) = \phi(\sigma_{i+1})\phi(\sigma_i)\phi(\sigma_{i+1}).$$

One can verify this by direct calculation, using the relations in $F_n \rtimes B_n$. 


Having defined $\phi$, it is now easy to construct $\rho^+$. Choose a finite basis for $V$. (This is not strictly necessary - we leave it to the reader to fill in the details when $V$ is infinite dimensional.) The image of $\rho$ now consists of square matrices. For any $\beta \in B_n$, let $\rho^+(\beta)$ be the block matrix obtained by applying $\rho$ to each entry of $\phi(\beta)$. This is an element of $\text{GL}(V^\oplus n)$, as required. \hfill \Box

The simplest nontrivial example of this construction is stated in the following proposition.

**Proposition 2.1.** If $\rho : F_n \rtimes B_n \to \mathbb{C}^*$ is a one dimensional representation then $\rho^+ : B_n \to \text{GL}(n, \mathbb{C})$ is the unreduced Burau representation, up to specializing and rescaling.

**Proof.** Suppose $\rho : F_n \rtimes B_n \to \mathbb{C}^*$ is a one dimensional representation. The generators $\sigma_i$ of $B_n$ are all conjugate to each other, so must all be mapped to the same value $s \in \mathbb{C}^*$. Similarly, the generators $g_i$ are all conjugate to each other in $F_n \rtimes B_n$, so must all be mapped to the same value $t \in \mathbb{C}^*$. Then

$$
\rho^+(\sigma_i) = s \begin{bmatrix}
I_{i-1} & 0 & t \\
0 & 1 & 1-t \\
1 & 1/t & I_{n-i-1}
\end{bmatrix}.
$$

This is the familiar matrix of the unreduced Burau representation, rescaled by a factor of $s$. \hfill \Box

3. THE LONG-MOODY CONSTRUCTION FOR SUBGROUPS OF $B_n$.

Suppose $G$ is a subgroup of $B_n$. Then $F_n \rtimes G$ is a subgroup of $F_n \rtimes B_n$. The aim of this section is to prove the following theorem.

**Theorem 3.1.** Given a subgroup $G$ of $B_n$ and a representation $\rho : F_n \rtimes G \to \text{GL}(V)$, we may construct a representation $\rho^+ : G \to \text{GL}(V^\oplus n)$.

The construction is the same as that of Long and Moody. Given an element $\beta \in G$, we let $\rho^+(\beta)$ be the result of applying $\rho$ to each of the entries in the matrix $\phi(\beta)$. We need only check that these entries lie in the domain of $\rho$. Thus it remains only to prove the following lemma.

**Lemma 3.2.** If $\beta \in B_n$ then every entry of the matrix $\phi(\beta)$ is of the form $a\beta$ for some $a \in \mathbb{Z}[F_n]$. 

**Proof.** Write $\beta$ as a word in the generators $\sigma_i$. Recall that $\phi(\sigma_i)$ is the scalar $\sigma_i$ times a matrix with entries in $\mathbb{Z}[F_n]$. Thus $\phi(\beta)$ is a product of terms, each of which is either a scalar $\sigma_i^{\pm 1}$ or a matrix with entries in $\mathbb{Z}[F_n]$. The product of the terms $\sigma_i^{\pm 1}$ in this product, taken in order, is equal to $\beta$.

Observe that if $A$ is any matrix with entries in $\mathbb{Z}[F_n]$ and $\sigma_i$ is a generator of $B_n$ then $\sigma_i A$ is equal to $A' \sigma_i$ for some matrix $A'$ with entries in $\mathbb{Z}[F_n]$. To see this, apply the relations of $F_n \rtimes B_n$ to each term in each entry of $A$. By using this observation repeatedly, we can move the terms $\sigma_i^{\pm 1}$ to the right of the product expression for $\phi(\beta)$. We obtain an expression of the form $A \beta$, where $A$ is a matrix with entries in $\mathbb{Z}[F_n]$. \hfill \Box
4. The Gassner representation

Let \( P_n \) denote the pure braid group. The aim of this section is to prove the following theorem.

**Theorem 4.1.** Given a representation \( \rho: P_{n+1} \to \text{GL}(V) \) we may construct an \( n \) parameter family of representations \( \rho^+_{t_1, \ldots, t_n} : P_n \to \text{GL}(V^{\otimes n}) \).

Long achieved a similar result by different methods in [Lon89a] and [Lon89b]. The construction there uses an action of the braid group by diffeomorphisms on a certain representation variety. To obtain a linear representation, he takes the derivative of this action at any point that is fixed by the action of \( B_n \), or \( P_n \).

Long was most interested in the Gassner representation, but the approach is quite general, and could give a result similar to Theorem 4.1.

Our construction is more along the lines of [Lon94], especially the following theorems, which are [Lon94, Theorem 2.4] and [Lon94, Corollary 2.6] respectively.

**Theorem 4.2.** Given a representation \( \rho \) of \( B_{n+1} \), we may construct a new representation \( \rho^+ \) of \( B_n \).

**Theorem 4.3.** Given a representation \( \rho \) of \( F_n \ltimes B_{n} \), we may construct a one parameter family of representations \( \rho^+_{t_1, \ldots, t_n} \) of \( B_{n} \).

The first follows immediately from the fact that \( F_n \ltimes B_{n} \) is a subgroup of \( B_{n+1} \).

To prove Theorem 4.1, we apply the same ideas in the context of the pure braid group.

**Proof of Theorem 4.1.** Let \( F_n \) be the subgroup of \( P_{n+1} \) consisting of all braids in which the first \( n \) strands are straight. This is the free group with generators

\[ g_i = (\sigma_n \ldots \sigma_{i+1})\sigma_i^2(\sigma_n \ldots \sigma_{i+1})^{-1}. \]

Let \( P_n \) be a subgroup of \( P_{n+1} \) in the usual way. It is well known that \( P_n \) and \( F_n \) together generate \( P_{n+1} \). Indeed, \( P_{n+1} \) is isomorphic to the subgroup \( F_n \ltimes P_n \) of our semidirect product \( F_n \ltimes B_n \).

Fix \( t_1, \ldots, t_n \in \mathbb{C}^* \). Let \( \rho_{t_1, \ldots, t_n} : P_{n+1} \to \text{GL}(V) \) be the representation such that

\[ \rho_{t_1, \ldots, t_n}(g_i) = t_i \rho(g_i) \]

for any of the generators \( g_i \) of \( F_n \), and

\[ \rho_{t_1, \ldots, t_n}(\beta) = \rho(\beta) \]

for any \( \beta \in P_n \).

To see that \( \rho_{t_1, \ldots, t_n} \) is well defined, note that if \( \beta \in P_n \) and \( i \in \{1, \ldots, n\} \) then

\[ \beta g_i \beta^{-1} = wg_i w^{-1} \]

for some \( w \in F_n \). These relations determine the semidirect product structure of \( P_{n+1} \), and they are preserved by \( \rho_{t_1, \ldots, t_n} \).

Now simply apply Theorem 3.1 to \( \rho_{t_1, \ldots, t_n} \) to obtain the desired representation of \( P_n \). \( \square \)
As an example, we compare the results of Theorems 4.3 and 4.1 when \( \rho \) is the trivial representation. If \( \rho \) is the trivial representation of \( F_n \rtimes B_n \), then Theorem 4.3 gives the unreduced Burau representation \( \rho_1^+ \) of \( B_n \) (by Proposition 2.1). If \( \rho \) is the trivial representation of \( P_{n+1} \) then \( \rho_1^+, \ldots, \rho_n^+ \) is the unreduced Gassner representation of \( P_n \).

For more sophisticated examples, we could think of Theorem 4.1 as giving a “Gassner” version of representations of \( B_n \) other than the Burau representation.

5. Lawrence’s construction.

The Hecke algebra is a certain quotient of the group algebra \( \mathbb{C}B_n \). For our purposes, a representation of the Hecke algebra is any representation \( \rho \) of \( B_n \) such that the matrices \( \rho(\sigma_i) \) all satisfy a quadratic relation. Such representations are the subject of a large body of ongoing research [Mat99] [SB06].

The following proposition suggests a connection between the representation \( \phi \) from Section 2 and representations of the Hecke algebra.

**Proposition 5.1.** \((\phi(\sigma_i) + \sigma_i g_i I)(\phi(\sigma_i) - \sigma_i I) = 0\).

This can be checked by direct calculation. However it does not qualify as a representation of the Hecke algebra, since the “scalars” of this quadratic relation lie in a non-commutative ring.

By rescaling, it suffices to consider representations of \( B_n \) that satisfy \((\rho(\sigma_i) + q I)(\rho(\sigma_i) - I) = 0\) for some \( q \in \mathbb{C}^* \). For generic values of \( q \), such representations of \( B_n \) can be enumerated by the partitions of \( n \). In [Law93], Lawrence gives a topological construction of the representations of the Hecke algebra that correspond to partitions of \( n \) of the form \((n - m, m)\). This was generalized to all partitions in [Law96].

In [Lon94, Corollary 2.10], Long describes how his construction can be used to parallel that of [Law93]. The methods in this section are intended to parallel those of [Law96] (which had not appeared at the time of [Lon94]). We will not prove any concrete connection. To do that would require a thorough treatment of the homological details behind the Long-Moody construction. Instead, this section could be viewed as a self-contained description of a way to construct representations of the Hecke algebra.

Let \( B_{n,m} \) denote the set of braids in \( B_{n+m} \) such that the first \( n \) nodes at the bottom are connected to the first \( n \) nodes at the top, and (hence) the last \( m \) nodes at the bottom are connected to the last \( m \) nodes at the top. This is generated by \( \sigma_1, \ldots, \sigma_{n-1}, \sigma_n^2, \sigma_{n+1}, \ldots, \sigma_{n+m-1} \).

**Lemma 5.2.** Given a representation \( \rho: B_{n+1} \to \text{GL}(V) \), we may construct a representation \( \rho^+: B_{n,m} \to \text{GL}(V^{\oplus(n+m)}) \).

**Proof.** Let \( B_{n,m} \) be a subgroup of \( B_{n+m} \) in the obvious way. Let \( F_{n+m} \) be the set of braids in \( B_{n+m} \) such that the first \( n + m \) strands are straight. This is the free group with generators \( y_i = (\sigma_{n+m} \ldots \sigma_{i+1})\sigma_i^2(\sigma_{n+m} \ldots \sigma_{i+1})^{-1} \).

Then \( B_{n,m} \) and \( F_{n+m} \) together generate a subgroup \( F_{n+m} \rtimes B_{n,m} \) of \( B_{n+m} \). The result now follows from Theorem 3.1. \( \square \)
Theorem 5.3. Given a representation $\rho: B_m \to \text{GL}(V)$, we may construct a one parameter family of representations $\rho^+_q: B_n \to \text{GL}(V \oplus k)$, where $q \in \mathbb{C}^*$ and $k = (n + m - 1)(n + m - 2) \ldots (n + 1)(n)$.

Proof. Suppose we are given $k = (\ldots n + m - 1)$; now apply Lemma 5.2 to $\rho$ to construct a representation $B$ of $\text{GL}(V \oplus k)$, where $\rho$ is the Hecke algebra as a subrepresentation of $B$. Lawrence still has to do some more work to obtain a representation of $\text{GL}(V \oplus k)$. Theorem 5.4. Lawrence representation will be a homomorphism $\phi$ denoted the infinite cyclic group generated by a formal variable $e$. Our universal Lawrence representation will be a homomorphism $\phi$ from $B_n$ to the group of invertible $n \times n$ matrices with entries in $\mathbb{Z}[[q]]$. Enumerate the rows and columns of this matrix by basis vectors $e_{a_1}, \ldots, e_{a_m}$ for $1 \leq a_1 < \cdots < a_m \leq n$.

To make the definition more readable, we first fix some notation. Let $a$ denote a sequence of the form $a_1, \ldots, a_j$, where $1 \leq a_1 < \cdots < a_j < i$. Let $b$ denote a sequence of the form $b_1, \ldots, b_k$, where $i + 1 < b_1 < \cdots < b_k \leq n$. In each of the following, assume that $j$ and $k$ are such that the length of the whole sequence is $n$.

- $\phi(\sigma_i)(e_{a,b}) = e_{a,b}$
- $\phi(\sigma_i)(e_{a,i,b}) = e_{a,i+1,b}$
- $\phi(\sigma_i)(e_{a,i+1,b}) = qe_{a,i,b} + (1 - q)e_{a,i+1,b}$
- $\phi(\sigma_i)(e_{a,i+1,b}) = \sigma_j + 1 e_{a,i+1,b}$. We must check that $\phi$ satisfies the braid relations, so that it gives a well defined representation of $B_n$. This is done by direct computation and checking cases. We will only describe some special cases that give a flavor of the general proof.

If $m = 0$ then $\phi$ is the trivial representation.

If $m = 1$ then $\phi$ is the unreduced Burau representation.

If $m = n$ then $\phi$ maps each braid $\beta$ to the one by one matrix $[\beta]$. 
Finally, consider the case \( m = 2 \) and \( n = 3 \). The matrices for \( \phi \) with respect to the basis \((e_{2,3}, e_{1,3}, e_{1,2})\) are as follows.

\[
\begin{align*}
\phi(\sigma_1) & = \begin{bmatrix} 1 - q & 1 & 0 \\ q & 0 & 0 \\ 0 & 0 & \sigma_1 \end{bmatrix}, \\
\phi(\sigma_2) & = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & 1 - q & 1 \\ 0 & q & 0 \end{bmatrix}.
\end{align*}
\]

One can check that these matrices satisfy the braid relation \( ABA = BAB \).

To prove that \( \phi \) satisfies the braid relations in general, we can break down the image into blocks, each of which resembles one of the above cases.

Having defined \( \phi \), we can define the specific representation \( \rho^+ \) in the same way as we did for the Long-Moody construction. First choose a basis for \( V \), and an ordering for our basis vectors \( e_{a_1, \ldots, a_k} \). The images of \( \rho \) and \( \phi \) can then be written as square matrices. Given \( \beta \in B_n \), let \( \rho^+(\beta) \) be the block matrix obtained by applying \( \rho \) to each entry of \( \phi(\beta) \) and substituting a value \( q \in \mathbb{C}^* \).

By some further computation and checking of cases, every nonzero entry of the matrix \( (\phi(\sigma_i) + qI)(\phi(\sigma_i) - I) \) is of the form \( (q + \sigma_j)(1 - \sigma_j) \) for some \( j \). Now suppose \( \rho \) satisfies the relation \( (\rho(\sigma_i) + qI)(\rho(\sigma_i) - I) = 0 \).

If we apply \( \rho \) to each entry of \( (\phi(\sigma_i) + qI)(\phi(\sigma_i) - I) \) then we obtain the zero matrix. Thus \( \rho^+ \) satisfies the same quadratic relation as \( \rho \). \(\square\)

6. THE REDUCED LONG-MOODY CONSTRUCTION.

The unreduced Burau representation obtained in Proposition \( 2.1 \) is reducible. It is the direct sum of a one dimensional representation with the reduced Burau representation. In [Lon94, Theorem 2.11], Long shows that this happens in general. We record here how to compute the “reduced” version of the Long-Moody construction.

For \( i = 1, \ldots, n - 1 \), let \( S_i \) be the following three by three matrix with entries in \( \mathbb{Z}[F_n \rtimes B_n] \).

\[
S_i = \begin{bmatrix} 1 & g_i & 0 \\ 0 & -g_i & 0 \\ 0 & 1 & 0 \end{bmatrix}.
\]

This is invertible.

\[
S_i^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -g_i^{-1} & 0 \\ 0 & g_i^{-1} & 1 \end{bmatrix}.
\]

Let \( \phi_r \) be the following homomorphism from \( B_n \) to the group of invertible \( n - 1 \) by \( n - 1 \) matrices with entries in \( \mathbb{Z}[F_n \rtimes B_n] \).

\[
\phi_r(\sigma_i) = \sigma_i \begin{bmatrix} I_{n-2} & S \\ & I_{n-i-2} \end{bmatrix}.
\]
If \( i \) is 1 or \( n - 1 \) then part of \( S \) is “cut off” by the edge of the matrix. In other words,

\[
\phi_r(\sigma_1) = \sigma_1 \begin{bmatrix} -g_1 & 0 \\ 1 & 1 \\ I_{n-3} \end{bmatrix},
\]

\[
\phi_r(\sigma_{n-1}) = \sigma_{n-1} \begin{bmatrix} I_{n-3} & 1 & g_{n-1} \\ 0 & -g_{n-1} \end{bmatrix}.
\]

REFERENCES

[Sb06] S. Bigelow, Braid groups and the Iwahori-Hecke algebras, in Problems on mapping class groups and related topics, proc. Sympos. Pure Math., 74 (2006), 285–299. MR MR2264547 (2007f:20008)

[Bir94] J. Birman, D. D. Long, J. Moody, Finite-dimensional representations of Artin’s braid group, in The Mathematical Legacy of Wilhelm Magnus, Contemporary Math., 169 (1994), 123–132. MR MR1292900 (95h:20046)

[FLSV04] Edward Formanek, Woo Lee, Inna Sysoeva, Monica Vasirani, The irreducible complex representations of the braid group on \( n \) strings of degree \( \leq n \), J. Algebra Appl. 2 (2003), no. 3, 317–333. MR MR1997750 (2004f:20068)

[Kra01] Daan Krammer, The braid group \( B_4 \) is linear, Invent. Math. 142 (2000), no. 3, 451–486. MR MR1804157 (2001k:20078)

[Law93] R. J. Lawrence, A functorial approach to the one-variable Jones polynomial, J. Differential Geom. 37 (1993), no. 3, 689–710. MR MR1217166 (95f:57013)

[Law96] Ruth J. Lawrence, Braid group representations associated with \( sl_3 \), J. Knot Theory Ramifications 5 (1996), no. 5, 637–660. MR MR1414092 (98j:57011)

[Lon89a] D. D. Long, On the linear representation of braid groups, Trans. Amer. Math. Soc. 311 (1989), no. 2, 535–560. MR MR0943606 (89h:20054)

[Lon89b] D. D. Long, On the linear representation of braid groups II, Duke J. of Math. vol. 59 (1989), 443 - 460. MR MR0943606 (89h:20054)

[Lon94] D. D. Long, Constructing representations of braid groups, Comm. Anal. Geom. 2 (1994), no. 2, 217–238. MR MR1312687 (95m:20044)

[Mat99] Andrew Mathas, Iwahori-Hecke algebras and Schur algebras of the symmetric group, American Mathematical Society, Providence, RI, 1999, University Lecture Series, 15. MR MR1711316 (2001g:20006)