Elementary proofs of infinite families of congruences for Merca’s cubic partitions

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Abstract
Recently, using modular forms and Smoot’s Mathematica implementation of Radu’s algorithm for proving partition congruences, Merca proved the following two congruences: for all \( n \geq 0 \),

\[
A(9n + 5) \equiv 0 \pmod{3},
\]
\[
A(27n + 26) \equiv 0 \pmod{3}.
\]

Here, \( A(n) \) is closely related to the function which counts the number of cubic partitions, partitions wherein the even parts are allowed to appear in two different colors. Indeed, \( A(n) \) is defined as the difference between the number of cubic partitions of \( n \) into an even numbers of parts and the number of cubic partitions of \( n \) into an odd numbers of parts. In this brief note, we provide elementary proofs of these two congruences via classical generating function manipulations. We then prove two infinite families of non-nested Ramanujan-like congruences modulo 3 satisfied by \( A(n) \) wherein Merca’s original two congruences serve as the initial members of each family.

Keywords Partitions · Cubic partitions · Congruences · Generating functions

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1 Introduction

In a recent work, Merca [7] extensively studied the function which he called $A(n)$ which is defined to be the difference between the number of cubic partitions of $n$ into an even numbers of parts and the number of cubic partitions of $n$ into an odd numbers of parts. (Cubic partitions were introduced by Chan [2, 3] to be integer partitions in which even parts are allowed to appear in two different colors. Chan introduced these in connection with Ramanujan’s cubic continued fraction.) Merca notes that the generating function for $A(n)$ is given by

$$
\sum_{n=0}^{\infty} A(n)q^n = (q; q^2)_{\infty}(q^2; q^4)_{\infty} = \frac{f_1}{f_4}
$$

where the $q$-Pochhammer symbol $(a; q)_{\infty}$ is defined by

$$(a; q)_{\infty} = (1 - a)(1 - aq)(1 - aq^2)(1 - aq^3)\ldots$$

and $f_{a}^{b}$ is defined by

$$f_{a}^{b} = (q^a; q^a)_{\infty}$$

for positive integers $a$ and any integer $b$.

In [7], Merca proved the following two Ramanujan-like congruences satisfied by $A(n)$:

**Theorem 1.1** [7, Theorem 1.10] For all $n \geq 0$,

$$A(9n + 5) \equiv 0 \pmod{3}, \quad (1)$$

$$A(27n + 26) \equiv 0 \pmod{3}. \quad (2)$$

Merca’s proof of these two congruences relies solely on Smoot’s Mathematica implementation of Radu’s algorithm for proving partition congruences (which relies heavily on the machinery of modular forms). Indeed, Merca’s proof of (1) involves finding a generating function for $A(9n + 5)$ which is produced by Smoot’s Mathematica package. In this case, the generating function in question turns out to be a non-trivial linear combination of four ratios of eta products (in essence, ratios of products whose terms involve $q$-Pochhammer symbols as defined above), while the generating function for $A(27n + 26)$ contains a dozen such terms.

Our goal in the work below is to provide elementary proofs of Merca’s two congruences via classical generating function manipulations and dissections. We then significantly extend Merca’s work on such divisibility properties for $A(n)$ by proving two infinite families of non-nested Ramanujan-like congruences modulo 3 satisfied by $A(n)$ wherein Merca’s two congruences above serve as the initial members of each family. Indeed, we will prove the following:
For all \( j \geq 0 \) and all \( n \geq 0 \),
\[
A \left( 9^{j+1}n + \frac{39 \cdot 9^j + 1}{8} \right) \equiv 0 \pmod{3}
\]
and
\[
A \left( 3 \cdot 9^{j+1}n + \frac{23 \cdot 9^j + 1}{8} \right) \equiv 0 \pmod{3}.
\]

In order to accomplish the above goals, we require a few classical tools. First, we recall Ramanujan’s theta functions

\[
f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)} b^{n(n-1)}, \quad \text{for} \ |ab| < 1,
\]
\[
\phi(q) := f(q, q) = \sum_{n=\infty}^{\infty} q^{n^2} = \frac{(q^2; q^2)_{\infty}^5}{(q; q)_{\infty}^2(q^4; q^4)_{\infty}^2},
\]
\[
\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}}.
\]

These functions satisfy many interesting properties (see Entries 18, 19, and 22 in [1]), including

\[
\phi(-q) = \frac{(q; q)_{\infty}^2}{(q^2; q^2)_{\infty}},
\]
\[
\psi(-q) = \frac{(q; q)_{\infty}(q^4; q^4)_{\infty}}{(q^2; q^2)_{\infty}}.
\]

As noted above, the generating function for \( A(n) \) is given by

\[
\sum_{n=0}^{\infty} A(n)q^n = (q; q^2)_{\infty}(q^2; q^4)_{\infty}
\]
\[
= \frac{f_1}{f_4}
\]
\[
= \frac{f_1^2}{f_2} \cdot \frac{f_2}{f_1 f_4}
\]
\[
= \frac{\phi(-q)}{\psi(-q)}. \quad (3)
\]

In order to prove the congruences in question, we require a few well-known \( q \)-series dissections.
Lemma 1.2 We have
\[ \phi(-q) = \frac{f_9^2}{f_{18}} - 2q \frac{f_9 f_{18}}{f_6 f_9}. \]

Proof A proof of this identity can be seen in [4, Eq. 14.3.4]. \(\square\)

Lemma 1.3 We have
\[ \frac{1}{\psi(-q)} = \frac{f_{18}^9}{f_3^2 f_9 f_{12} f_{36}} + q \frac{f_6^2 f_9^3 f_{18}}{f_3 f_{12}^3} + q^2 \frac{f_6^4 f_9 f_{36}^3}{f_3 f_{12}^4 f_{18}}. \]

Proof A proof of this identity appears in [8, Lemma 2.1]. \(\square\)

Lemma 1.4 We have
\[ \frac{1}{\phi(-q)} = \frac{f_6^4 f_9^6}{f_3^2 f_9 f_{18}^3} + 2q \frac{f_6^3 f_9^3 f_{18}}{f_3^4} + 4q^2 \frac{f_6^2 f_9 f_{18}^3}{f_3^4}. \]

Proof A proof of this result can be seen in [5, Theorem 1]. \(\square\)

Lemma 1.5 We have
\[ \frac{1}{\psi(q)} = \frac{f_3^2 f_9^3}{f_6^3} - q \frac{f_3 f_{18}^3}{f_6^3} + q^2 \frac{f_3^2 f_9 f_{18}^3}{f_6^3}. \]

Proof A proof of this result can be seen in [6, Lemma 2.2]. \(\square\)

These are all of the tools that we need in order to prove Merca’s congruences in an elementary fashion. We now transition to providing these proofs.

2 Elementary proof of Theorem 1.1

Initially, we use Lemmas 1.2 and 1.3 to extract the terms involving \(q^{3n+2}\) in (3):
\[ \sum_{n=0}^{\infty} A(3n + 2) q^{3n+2} = q^2 \frac{f_6^4 f_9^3 f_{36}^3}{f_3 f_{12}^4 f_{18}^4} - 2q^2 \frac{f_6 f_{18}^5}{f_3 f_{12}^4 f_{18}^4} . \]

Dividing by \(q^2\) and replacing \(q^3\) by \(q\) yields
\[ \sum_{n=0}^{\infty} A(3n + 2) q^n = \frac{f_6^4 f_3 f_{12}^3}{f_1^4 f_4^4 f_6^4} - 2 \frac{f_2 f_6^5}{f_1^2 f_3 f_4^4}. \]
\[ \equiv \frac{f_2}{f_1 f_4} \frac{f_3 f_{12}^2}{f_6^3} - 2 \frac{f_2}{f_1^2} \frac{f_6^5}{f_3 f_{12}} \quad (\text{mod } 3). \]
Using Lemmas 1.3 and 1.4, we extract the terms of the form \( q^{3n+1} \) from both sides of the last congruence to obtain

\[
\sum_{n=0}^{\infty} A(9n+5)q^{3n+1} \equiv q \frac{f_3 f_{18}^3}{f_6 f_{12}} - 4q \frac{f_8 f_9}{f_3 f_{12}} \quad (\text{mod } 3)
\]

\[
\equiv q \frac{f_3 f_{18}^3}{f_6 f_{12}} - 4q \frac{f_3}{f_6 f_{12}} \frac{f_6^2 f_9}{f_3^9} \quad (\text{mod } 3)
\]

\[
\equiv -3q \frac{f_3 f_{18}^3}{f_6 f_{12}} \quad (\text{mod } 3)
\]

\[
\equiv 0 \quad (\text{mod } 3),
\]

which proves (1).

Using Lemmas 1.3 and 1.4, we extract the terms of the form \( q^{3n+2} \) from both sides of (4):

\[
\sum_{n=0}^{\infty} A(9n+8)q^{3n+2} \equiv q^2 \frac{f_6 f_9 f_{18}^3}{f_{12}^2 f_{18}^3} - 8q^2 \frac{f_7 f_9}{f_3 f_{12}} \quad (\text{mod } 3).
\]

Dividing by \( q^2 \) and replacing \( q^3 \) by \( q \), we obtain

\[
\sum_{n=0}^{\infty} A(9n+8)q^n \equiv \frac{f_2 f_3 f_{12}^3}{f_4^2 f_6^3} - 8 \frac{f_2^7 f_9}{f_1^3 f_4} \quad (\text{mod } 3)
\]

\[
\equiv \frac{f_2}{f_4^2} \frac{f_3 f_{12}^3}{f_6^3} - 8 \frac{f_2}{f_1 f_4} \frac{f_6^5}{f_3^2} \quad (\text{mod } 3). \quad (5)
\]

Now we employ Lemmas 1.3 and 1.5 to extract the terms of the form \( q^{3n+2} \) from the congruence above. The resulting congruence after division by \( q^2 \) and replacing \( q^3 \) by \( q \) is given by

\[
\sum_{n=0}^{\infty} A(27n+26)q^n \equiv - \frac{f_1^3 f_{12}^3}{f_4^4} - 8 \frac{f_2^9 f_3 f_{12}^3}{f_1^6 f_4 f_6^3} \quad (\text{mod } 3)
\]

\[
\equiv - \frac{f_1^3 f_{12}^3}{f_4^4} - 8 \frac{f_1^3 f_{12}^3}{f_4^4} \quad (\text{mod } 3)
\]

\[
\equiv 0 \quad (\text{mod } 3),
\]

which proves (2).
3 Infinite families of congruences modulo 3

While the elementary proofs of Merca’s original congruences that we have provided above are very satisfying, it turns out that much more is true about $A(n)$ modulo 3, and our elementary approach to proving these congruences yields the insights needed in order to see this. In this light, we now proceed to proving two infinite families of non-nested Ramanujan-like congruences modulo 3 satisfied by $A(n)$.

We begin by proving the following internal congruence satisfied by $A(n)$ which will serve as an important component in our remaining proofs.

**Theorem 3.1** For all $n \geq 0$,

$$A(27n + 8) \equiv A(3n + 1) \pmod{3}. \quad (6)$$

**Proof** We prove this result by showing that both $A(27n + 8)$ and $A(3n + 1)$ have the same generating function modulo 3. Using Lemmas 1.3 and 1.5, we extract the terms of the form $q^{3n}$ from (5):

$$\sum_{n=0}^{\infty} A(27n + 8)q^{3n} = \frac{f_5^3 f_{18}^3}{f_6 f_{12}^3} - 8 \frac{f_6^5 f_{18}^9}{f_3 f_5^3 f_2^2 f_{36}^3} \pmod{3},$$

which after replacement of $q^3$ by $q$ yields

$$\sum_{n=0}^{\infty} A(27n + 8)q^n = \frac{f_5^3 f_6^3}{f_2 f_4^3} - 2 \frac{f_6^5 f_6^9}{f_4 f_5^3 f_4^2 f_{12}^3} \pmod{3}. \quad (7)$$

The generating function for $A(3n + 1)$ is obtained in the same way using (3) and Lemmas 1.2 and 1.3:

$$\sum_{n=0}^{\infty} A(3n + 1)q^{3n+1} = q \frac{f_6^2 f_9^2 f_{18}^2}{f_3 f_{12}^3} - 2q \frac{f_{18}^{11}}{f_3 f_6 f_9^4 f_{12}^2 f_{36}^3}.$$

Dividing this expression by $q$ and replacing $q^3$ by $q$, we are left with

$$\sum_{n=0}^{\infty} A(3n + 1)q^n = \frac{f_2^2 f_3^2 f_6^2}{f_1^3 f_4^3} - 2 \frac{f_{18}^{11}}{f_1 f_2 f_3^3 f_4^2 f_{12}^3}$$

$$\equiv \frac{f_5^3 f_6^3}{f_2 f_4^3} - 2 \frac{f_2^5 f_6^9}{f_1^4 f_3^2 f_4^2 f_{12}^3} \pmod{3} \quad (mod\ 3)$$

which coincides with (7) and the proof is complete. \qed
We can now prove the following two theorems.

**Theorem 3.2** For all \( j \geq 0 \) and all \( n \geq 0 \),

\[
A \left( 9^{j+1} n + \frac{39 \cdot 9^j + 1}{8} \right) \equiv 0 \pmod{3}.
\]

**Proof** This theorem follows from a straightforward proof by induction on \( j \). First, we note that the basis case, \( j = 0 \), is simply the statement that, for all \( n \geq 0 \),

\[
A(9n + 5) \equiv 0 \pmod{3}.
\]

This is Theorem 1.1 Eq. (1) above, the first of Merca’s original congruences.

Next, we assume that the statement is true for some \( j \geq 0 \). We then wish to prove that

\[
A \left( 9^{j+2} n + \frac{39 \cdot 9^{j+1} + 1}{8} \right) \equiv 0 \pmod{3}
\]

for all \( n \geq 0 \) as well. Note that

\[
9^{j+2} n + \frac{39 \cdot 9^{j+1} + 1}{8} = 27 \left( 3 \cdot 9^j n + \frac{13 \cdot 9^j}{8} - \frac{7}{24} \right) + 8.
\]

Thanks to (6), we know that, for all \( n \geq 0 \),

\[
A(27n + 8) \equiv A(3n + 1) \pmod{3}.
\]

Thus,

\[
A \left( 9^{j+2} n + \frac{39 \cdot 9^{j+1} + 1}{8} \right) \\
= A \left( 27 \left( 3 \cdot 9^j n + \frac{13 \cdot 9^j}{8} - \frac{7}{24} \right) + 8 \right) \\
= A \left( 3 \left( 3 \cdot 9^j n + \frac{13 \cdot 9^j}{8} - \frac{7}{24} \right) + 1 \right) \pmod{3} \\
= A \left( 9^{j+1} n + \frac{39 \cdot 9^j + 1}{8} \right) \\
\equiv 0 \pmod{3}
\]

by the induction hypothesis. The result follows. \( \square \)
Theorem 3.3 For all $j \geq 0$ and all $n \geq 0$,

$$A \left( 3 \cdot 9^{j+1} n + \frac{23 \cdot 9^{j+1} + 1}{8} \right) \equiv 0 \pmod{3}.$$  

Proof As above, this theorem follows from a straightforward proof by induction on $j$. Note that the basis case, $j = 0$, is simply the statement that, for all $n \geq 0$,

$$A(27n + 26) \equiv 0 \pmod{3}.$$  

This is Theorem 1.1 Eq. (2) above, the second of Merca’s original congruences.

Next, we assume that the statement is true for some $j \geq 0$. We then wish to prove that

$$A \left( 3 \cdot 9^{j+2} n + \frac{23 \cdot 9^{j+2} + 1}{8} \right) \equiv 0 \pmod{3}$$  

for all $n \geq 0$ as well. Note that

$$3 \cdot 9^{j+2} n + \frac{23 \cdot 9^{j+2} + 1}{8} = 27 \left( 9^{j+1} n + \frac{3(23) \cdot 9^j}{8} - \frac{7}{24} \right) + 8.$$  

Thanks to (6), we know that, for all $n \geq 0$,

$$A(27n + 8) \equiv A(3n + 1) \pmod{3}.$$  

Thus,

$$A \left( 3 \cdot 9^{j+2} n + \frac{23 \cdot 9^{j+2} + 1}{8} \right) = A \left( 27 \left( 9^{j+1} n + \frac{3(23) \cdot 9^j}{8} - \frac{7}{24} \right) + 8 \right)$$

$$\equiv A \left( 3 \left( 9^{j+1} n + \frac{3(23) \cdot 9^j}{8} - \frac{7}{24} \right) + 1 \right) \pmod{3}$$

$$= A \left( 3 \cdot 9^{j+1} n + \frac{23 \cdot 9^{j+1} + 1}{8} \right)$$

$$\equiv 0 \pmod{3}$$

by the induction hypothesis. The result follows.  

\[\square\]

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