Enumeration of simple random walks and tridiagonal matrices.

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Abstract
We present some old and new results in the enumeration of random walks in one dimension, mostly developed in works of enumerative combinatorics. The relation between the trace of the $n$-th power of a tridiagonal matrix and the enumeration of weighted paths of $n$ steps allows an easier combinatorial enumeration of paths. It also seems promising for the theory of tridiagonal random matrices.

1 Introduction.

Already at the foundation of the theory of random matrices by E.Wigner, the relevance of the combinatorics of random walks was recognized. The following decades witnessed an explosion of the theory and the applications of random matrices. A number of specific techniques were devised and the relation with the combinatorics of random walks was almost forgotten, with few remarkable exceptions. An interesting and recently investigated open problem, is the spectral density of non-hermitian tridiagonal random matrices. Here the enumeration of one dimensional random walks, where to each step an independent random variable is associated, plays a dominant role. Since the subject of random walks is very basic and useful, it has been thoroughly studied both by mathematicians and physicists for a long time, yet with techniques and aims so different that the relevant literature about inhomogeneous walks of one group ignores that of the other group. The aim of this paper is to review some old and recent results in enumerative combinatorics of random walks pertinent to the analysis of tridiagonal matrices in a language accessible to physicists, with examples, comments and relations that are not easily avail-
able in the literature.

A simple walk of \( p \) steps on the lattice of integers, may be coded by a sequence \( \{ \mu_1, \ldots, \mu_p \} \), where each \( \mu_k \) may take the values 1 (right move) or \(-1\) (left move). The walk is simple by the fact that we exclude the move \( \mu_k = 0 \). The index \( k \) can be regarded as discrete time. Since our walks start from the origin, at time \( k \) the walker is on site \( s(k) = \mu_1 + \ldots + \mu_k \). It is very convenient to view the walk as a continuous broken line in the lattice \((k, s)\) of time and site occupancy. Most of the terminology is based on this picture. An illustration of a simple walk we consider is given in Fig.1; the discrete time that numbers the steps is measured in the horizontal axis, and the positions of the walker on the line are recorded in the vertical axis.

![Fig.1](image)

A little terminology is useful. Simple walks where also the end site is the origin, that is \( s(p) = 0 \), are "closed". A simple closed walk is "weakly positive" if \( s(k) \geq 0 \) for all \( k \). If we further require that it returns to the origin only at the end, the simple walk is "strictly positive". Weakly positive closed walks are also known as Dyck paths. The width of a walk is the number of different visited sites.

Simple walks can be classified and then counted according to different properties. A large number of results in the enumeration of simple walks have been obtained by a method which we call "iteration by length": the counting of walks with a given property is related to the counting of similar but shorter ones, attached sequentially. In terms of generating functions, the method leads to a system of equations which, in several important cases, can be solved explicitly. One of the best known examples is the enumeration of returns to the origin in closed positive walks. In sect.2 we illustrate the method by evaluating the number of peaks and valleys in several ensembles of walks. It is a very simple exercise given the known methods in enumerative combinatorics [2], [3], [4], and the method is familiar to physicists as it recalls a Dyson equation in quantum field theory.
Recently we developed a technique, \[1\], to count the number of visits of each site, for an ensemble of simple walks. The method is again recursive but here properties of longer and wider walks are computed in terms of shorter and narrower ones. We call this method "iteration by top insertions". It leads to explicit results in the form of sums of products of binomial coefficients, the sum being actually in several indices taking values in the compositions of an integer, occasionally with further restrictions. Since the results are involved, techniques to simplify them would be very useful. We think that this method is promising in the analysis of the spectrum of tridiagonal random matrices and the related random hopping problem in one dimension, where few exact results are available, especially in the non-hermitian case. In this introduction we shall describe in detail the connection of this counting problem for simple walks with tridiagonal matrices.

We then found that our method 'iteration by top insertions' had already been discovered in the form of enumeration of vertices of planar rooted trees, \[13\], \[21\], \[22\]. We present this relation in sect.3.

Occasionally a problem may be analyzed both by the simpler method of sect.2 as well as by the more complex one of sect.3. We provide an example by evaluating the area under Dyck paths, recently discussed by Jonsson and Wheater \[23\] with the simple method. While our discussion does not add to their solution, the interpretation in terms of a tridiagonal matrix facilitates its determination.

The method in sect.2 is most useful for homogeneous (or almost homogeneous) random walks, where all steps (or almost all of them) have the same probability. Some early papers are \[15\], \[16\]. We remark that even the completely inhomogeneous walks where each step is associated with an independent random variable were enumerated \[17\] by methods similar to the one we describe in sect.2.

As it is often the case, computing the same quantities in two different ways originates identities which could be hard to prove in a direct way. For example, the sums over compositions of an integer of products of binomials introduced in sect. 3, will be shown to yield simple trigonometric sums. An early example of this type of sums is provided by eq.(22) in a paper by D.A.Klarner \[18\].

Most of the methods described in this paper have been developed for walks where also the move $\mu_k = 0$, that is an horizontal step, is allowed. Dyck paths properly generalized to include such moves are called Motzkin paths. Enumeration formulas become more complicated and, for the introductory purpose of this paper, we avoid these walks. Some results for generating functions for Dyck and Motzkin paths and their relation to continued fractions were recently summarized by C.Krattenthaler \[19\].

Ensembles of walks are among the most basic topics in probability theory and their statistical properties are relevant in a large class of models in statis-
tical mechanics.
In one dimension, to each step $e_k$ of a walk $\gamma$ one may assign a weight $w(e_k)$. The whole path is then attributed a weight $w(\gamma) = w(e_1)w(e_2)\ldots w(e_p)$. Finally one is interested in summing over walks in some set $\Gamma$, weighted with their own factor $w(\gamma)$. One encounters this procedure in the discrete approximation of functional integrals $\int Dx \, f[x(t)]$, where the integration over continuous positive functions $x(t)$ with boundary conditions $x(t_0) = 0 = x(t_f)$ is replaced by an ensemble of Dyck paths. The enumeration methods described here may be useful for the evaluation of functional integrals where the paths may have various restrictions. In the example 3 of sect.3, few cases of weighted sums of Dyck paths are provided.

**Random walks and tridiagonal matrices**

There is a very close relation between random walks and tridiagonal matrices, and the latter are known to be a powerful tool for counting walks with various specifications. Tridiagonal matrices are frequently used to describe the motion of a particle in a one-dimensional chain.

Since the walks we are considering are made of steps $\pm 1$, we actually consider bidiagonal matrices of size $N \times N$

$$M(a, b) = \begin{pmatrix} 0 & b_1 & & \\ a_1 & 0 & b_2 & \\ & a_2 & 0 & \ldots \\ & & \ddots & \ddots \\ & & & a_{N-1} & b_{N-1} \\ & & & & a_N \end{pmatrix}$$

(1.1)

The starting point of our discussion is the fact that the explicit expansion of $M(a, b)^p_{i,j}$ immediately suggests to order the various terms in the sum according to some property of the random walk that connects site $i$ to site $j$ in $p$ steps.

In more detail, if we identify the matrix element $M(a, b)_{ij}$ with the symbol $(i, j)$, and agree that repeated neighboring indices are summed, we have: $(M^p)_{ij} = (i, i_1)(i_1, i_2)\ldots(i_{p-1}, j)$. Since the matrix is bidiagonal, either $i_{k+1} = i_k - 1$ or $i_{k+1} = i_k + 1$. Because of this, the pair $(i_k, i_{k+1})$ can also be identified with a move from site $i_k$ to a neighboring site $i_{k+1}$ in a simple walk. Therefore, each sequence of $p$ factors $(i, i_1)\ldots(i_{p-1}, j)$ not only represents a product of matrix elements, but also a random walk of $p$ steps that connects site $i$ to site $j$. The corresponding product of matrix elements may be interpreted as the "weight" of the walk.

In this picture, the evaluation of $M(a, b)^p_{i,j}$ consists in summing the weights of all walks of $p$ steps from $i$ to $j$. If we sum over lengths we get the resolvent, which is the generating function for all weighted simple paths from $i$ to $j$:

$$F(z, a, b; i, j) = \sum_p z^p M(a, b)^p_{i,j} = [I - zM(a, b)]^{-1}_{i,j} =$$
Here \([A; i, j]\) is the matrix \(A\) with row \(i\) and column \(j\) removed. This important result is very well known (see for instance theor.4.7.2 in \([14]\)).

It is very useful to note that the matrix \(M(a, b)\) is similar to the matrix \(M(1, x)\) where \(x_i = a_i b_i\),

\[
M(1, x) = \begin{pmatrix}
0 & x_1 \\
1 & 0 & x_2 \\
& 1 & 0 & \ldots \\
\end{pmatrix}
\]  

(1.3)

In the relation \(M(a, b) = S M(1, x) S^{-1}\) the matrix \(S\) is diagonal, with entries \(s_1 = 1, s_2 = a_1, \ldots, s_N = a_1 a_2 \ldots a_{N-1}\). Therefore

\[
[M(a, b)^p]_{ij} = \frac{s_i}{s_j} [M(1, x)^p]_{ij}
\]  

(1.4)

From this point, we exploit the similarity (1.4), where we note that the factor \(s_i/s_j\) does not depend on \(p\). To consider the matrix \(M(1, x)\), as we shall do hereafter, is a great simplification. In each walk, only upward steps \((k, k+1)\) correspond to nontrivial factors \(x_k\). Several walks may have the same weight since they contain, but with different order, the same intermediate steps \((k, k+1)\). By collecting these equal contributions, we have a useful expression in terms of certain counting numbers of random walks. The simplest example is

\[
[M(1, x)^{2p}]_{1,1} = \sum_{t=1}^{N-1} \sum_{\kappa(p, t)} N(n_1, \ldots, n_t) x_1^{n_1} x_2^{n_2} \ldots x_t^{n_t}
\]  

(1.5)

Here \(N(n_1, \ldots, n_t)\) is the number of Dyck paths that make \(n_j\) upward steps \((j-1, j)\) from site \(j-1\) to site \(j, j = 1 \ldots t\). Their length is \(2p = 2(n_1 + \ldots + n_t)\) and \(t\) is the height, which is bounded by the size of the matrix.

The multiple sums in \(n_1, \ldots, n_t \geq 1\) such that \(n_1 + \ldots + n_t = p\) are summarized as the sum on the compositions of \(p\) into \(t\) integers. We denote the set of compositions of \(p\) into \(t\) integers as \(\kappa(p, t)\).

This formula gives the explicit representation of \(M(1, x)^{2p}\) as a polynomial in the entries of the matrix.

In the end of sect. 3 we describe the general case \([M(1, x)^p]_{i,j}\) which corresponds to the counting of random paths of \(p\) steps from site \(i\) to site \(j\), without positivity restriction. In the case \(x = 1\) each walk has unit weight and one obtains the
counting number of simple walks from site \(i\) to site \(j\) of length \(p\), restricted in the lattice of points from 1 to \(N\), in terms of a simple matrix quantity. Since eigenvalues and eigenvectors are known, the multiple sums of the counting numbers in eq. (3.49) are equal to the trigonometric sum:

\[
[M(1,1)^p]_{i,j} = \frac{2^{p+1}}{N+1} \sum_{k=1}^{N} \cos^p \left( \frac{k\pi}{N+1} \right) \sin \left( i \frac{k\pi}{N+1} \right) \sin \left( j \frac{k\pi}{N+1} \right) 
\]  

One can use random bidiagonal matrices to count random walks with restrictions. The trick is to assign the variables \(x_i\) a probability density such that the contribution of unwanted walks is made to vanish in the average. For example, in [11] we studied the "q-root of unity" matrix ensemble, generalizing a problem first suggested by Zee [5]. A random matrix \(M(1,x)\) in the ensemble is characterized by independent random complex entries \(x_k\) chosen in the set of the \(q\) - roots of unity, with uniform probability. All moments \(\langle x_k^r \rangle\) have values 1 or 0, according to \(r\) being a multiple of \(q\) or not. Therefore, only walks that visit each site a number multiple of \(q\) do contribute, and with weight one, to \(\text{tr} M(1,x)^p\). This non local constraint strongly modifies the statistical properties of the random walks, as it was shown Noh et al. [9] in a model of surface growth. The case \(q = 2\) of "even-visiting walks" has been investigated in great detail by Bauer et al. [10], while the limit case \(q = \infty\) was considered by Derrida et al. [8].

On the other hand, given an ensemble of bidiagonal random matrices \(M(a,b)\), the enumeration methods may be useful for the study of the spectral density, usually in the limit \(N \to \infty\). If all random numbers \(x_j = a_j b_j\) are positive, the matrices are similar to real symmetric ones, \(M(c,c)\) with \(c_j = \sqrt{a_j b_j}\). The spectral density has support on the real axis and can be achieved by the usual resolvent approach. The average resolvent can be obtained by evaluating the ensemble average \(\text{tr} M(1,x)^p\) for every \(p\). To this end, the random walk description could prove to be a valuable tool, even for approximate results, if one were able to acknowledge and count the dominant walks. In the case where the random variables \(x_j\) are unrestricted, as in ref.[11], the spectral density has complex support, and the evaluation of the resolvent may only allow the determination of the boundary of the support.

2 Counting by iterations along the length of the path.

One is often interested in the enumeration of paths according to a parameter having the additive property: the parameter of a sequence of Dyck paths is the sum, or a linear combination, of the parameters of the single Dyck paths in the sequence, see for instance ref. [12]. Perhaps the best known example of additive
parameter is the number of returns to the origin, see ref. [26]. The method of enumeration by iteration along the length is suitable for the case of additive parameters, and it is here described following the pattern of paper [26].

**Statistics of peaks and valleys.**

Given a walk we say that an ‘inversion’ occurs at time $k$ if $\mu_k \neq \mu_{k+1}$. One may further partition inversions into peaks ($\mu_k = 1$, $\mu_{k+1} = -1$) and valleys ($\mu_k = -1$, $\mu_{k+1} = 1$). We now show that the evaluation of the number of inversions in simple random walks is a straightforward combinatorial exercise rather similar to the evaluation of the number of visits to the origin, which is known in the literature. The counting of inversions in long walks will be expressed in terms of the counting in shorter and simpler ones.

It is useful to code a simple random walk of $n$ steps as a sequence of $m$ positive integers $n_1 \ldots n_m$, that count consecutive equal steps. A walk with initial step $\mu_1 = 1$ and described by the sequence $(n_1, \ldots, n_m)$ begins with $n_1$ steps $\mu = +1$, followed by $n_2$ steps in opposite direction $\mu = -1$, and so on. The sum $n_1 + \ldots + n_m$ is the length $n$ of the walk, and the number of inversions is $m - 1$. For instance, the walk of 12 steps and 3 inversions in Fig.1 is coded by the sequence $(3, 4, 2, 3)$. The same sequence also describes a second walk, with opposite signs of steps. Each sequence $(n_1, n_2, \ldots, n_m)$ of positive integers is a composition of the integer $n$ in $m$ parts and the number of such compositions is $C(n, m)$

\[
C(n, m) = \binom{n-1}{m-1}
\]  

(2.1)

It follows that the number of random paths of $n$ steps and $m$ inversions is $2C(n, m + 1)$. By summing over inversions one reproduces the total number of random paths with $n$ steps, $2\sum_m C(n, m) = 2^n$.

Our first evaluation is the number of walks of given length and given number of inversions. It is convenient to consider first walks returning to the origin (therefore the number of steps $n$ is even) and introduce the counting numbers:

- $c_1(n, m)$ is the number of strictly positive closed walks of $n$ steps and $m$ inversions, $m = 1, 3, \ldots, n - 3$
- $c_2(n, m)$ is the number of weakly positive closed walks of $n$ steps and $m$ inversions, $m = 1, 3, \ldots, n - 1$
- $c_3(n, m)$ is the number of closed walks of $n$ steps and $m$ inversions, with no positivity requirement, $m = 1, 2, \ldots, n - 1$.

Since for strictly positive walks it is $\mu_1 = \mu_2 = 1$ and $\mu_{n-1} = \mu_n = -1$, and for weakly positive walks it is $\mu_1 = 1$ and $\mu_n = -1$, it follows that
\[ c_1(n, m) = c_2(n - 2, m) \quad \text{for} \quad n \geq 2 \]
\[ c_1(2, m) = c_2(2, m) = \delta_{m,1} \]  

(2.2)

A walk contributing to \( c_2(n, m) \) either is in the class contributing to \( c_1(n, m) \) or has at least one visit to the origin before the last site, (Fig.2). If a walk contributing to \( c_2 \) but not to \( c_1 \) first returns to the origin after \( n_1 \) steps, we have

\[ c_2(n, m) = c_1(n, m) + \sum_{n_1=2,4,..,n-2} \sum_{m_1=3,5,..,n_1-3} c_1(n_1, m_1) c_2(n-n_1, m-m_1-1) \]  

(2.3)

In terms of generating functions

\[ C_i(x, y) = \sum_{n\geq2,4,..,m\geq1,2,..,n-1} c_i(n, m) x^n y^m \]  

(2.4)

the eqs.(2.2), (2.3) translate into

\[ C_1(x, y) = x^2 y + x^2 C_2(x, y) \]
\[ C_2(x, y) = C_1(x, y) + y C_1(x, y) C_2(x, y) \]  

(2.5)

The resulting algebraic quadratic equations lead to

\[ C_1(x, y) = \frac{1 - x^2 + x^2 y^2 - \sqrt{(1 - x^2 + x^2 y^2)^2 - 4x^2 y^2}}{2y} \]  

(2.6)

\[ C_2(x, y) = \frac{1 - x^2 - x^2 y^2 - \sqrt{(1 - x^2 + x^2 y^2)^2 - 4x^2 y^2}}{2x^2 y} \]  

(2.7)

The generating functions are closely related to the function \( u(x, y) \) evaluated by Narayana [27]
\[ u(x, y) = 1 - \sqrt{(1 - x + xy)^2 - 4xy} = \]
\[ = x + xy + \sum_{n=2}^{\infty} \sum_{r=1}^{n-1} x^n y^r \frac{2}{n-1} \binom{n-1}{r} \binom{n-1}{n-r} \] (2.8)

For example, it is simple to obtain

\[ C_2(x, y) = \sum_{n=2}^{\infty} \sum_{m=1}^{n-1} x^{2n-2} y^{2m-1} \frac{1}{n-1} \binom{n-1}{m} \binom{n-1}{n-m} \] (2.9)

A walk contributing to \( c_3(n, m) \) (an example is given in Fig.3) either it is strictly positive or negative, or it has a first return to the origin after \( n_1 \) steps, where it may or may not have an inversion.

![Fig.3](image_url)

This leads to the equation

\[ C_3(x, y) = 2C_1(x, y) + (1 + y)C_1(x, y)C_3(x, y) \] (2.10)

The solution is found using the previous results, and is written in the convenient form

\[ C_3(x, y) = -1 + \frac{1 - x^2(1 - y)^2}{\sqrt{(1 - x^2 + x^2y^2)^2 - 4x^4y^2}} \] (2.11)

To extract the coefficients of the power expansion we use the generating function of Legendre polynomials, also found in paper [27]

\[ \frac{1}{\sqrt{(1 - x^2 - x^2y^2)^2 - 4x^4y^2}} = \sum_{n \geq 0} x^{2n} (y^2 - 1)^n P_n \left( \frac{y^2 + 1}{y^2 - 1} \right) = \]
\[ = \sum_{n,m} \binom{2n}{m}^2 x^{2n} y^{2m} \] (2.12)
Then:

\[ C_3(x, y) = 2x^2y + 2 \sum_{n \geq 2, m \geq 1} x^{2n} \left[ y^{2m-1} \binom{n-1}{m-1}^2 + y^{2m} \binom{n-1}{m} \binom{n-1}{m-1} \right] \]  

(2.13)

Eq. (2.13) has a simple interpretation when the paths are coded by the sequences mentioned at the beginning of this section. A random path with \( n \) steps and \( m \) inversions contributing to \( c_3(n, m) \) is coded by a sequence \( (n_1, n_2, \ldots, n_{m+1}) \) such that \( \sum_{j \text{ odd}} n_j = \sum_{j \text{ even}} n_j = n/2 \). The two subsequences \( s_{\text{odd}} = (n_1, n_3, \ldots) \) and \( s_{\text{even}} = (n_2, n_4, \ldots) \) both have \( (m+1)/2 \) terms if \( m \) is odd, whereas if \( m \) is even \( s_{\text{odd}} \) has \( \frac{m+1}{2} \) terms and \( s_{\text{even}} \) has \( m/2 \) terms. Then if \( m \) is odd, the number of sequences \( (n_1, n_2, \ldots, n_{m+1}) \) is given by \( C(\frac{n}{2}, \frac{m}{2} + 1)^2 \), whereas if \( m \) is even it is given by the product \( C(\frac{n}{2}, \frac{m}{2} + 1) \cdot C(\frac{n}{2}, \frac{m}{2}) \). Since each sequence describes two walks, one obtains the coefficients \( c_3(n, m) \) in eq. (2.13). Analogous results are quoted in the monograph [28].

Let us now consider random walks of \( n \) steps and \( m \) inversions without the restriction that the end point of the path is the origin, and no restriction about positivity. Such walks correspond to all sequences \( (n_1, n_2, \ldots, n_{m+1}) \) with \( n_1 + \ldots + n_{m+1} = n \). From eq. (2.1) we obtain the generating function of the counting numbers:

\[ C_4(x, y) = 2 \sum_{n,m} x^n y^m \binom{n-1}{m} = \frac{2}{(1+y)(1-x(1+y))} \]  

(2.14)

Finally we evaluate the number \( c_5(n, m) \) of weakly positive walks of \( n \) steps and \( m \) inversions, which may not be closed; an example is given in Fig. 4. The corresponding generating function is \( C_5(x, y) \).

![Graph](image)

We may determine \( c_5(n, m) \) in terms of shorter walks, by partitioning the set of weakly positive walks in three disjoint sets: a) the set of walks that after an
initial upward step never return to level one, b) the set of walks with an initial upward step followed by a walk in the ensemble described by the generating function $C_2(x, y)$; c) the set of walks of type b followed by a generic walk of type $C_5(x, y)$. This translates into the equation

$$C_5(x, y) = x + x C_5(x, y) + x C_2(x, y) + xy C_2(x, y) C_5(x, y)$$

(2.15)

Then

$$C_5(x, y) = -\frac{1 + x - xy}{2} + \frac{1}{2} \sqrt{\frac{(1 - x^2 + x^2 y^2)^2 - 4x^2 y^2}{1 - x - xy}} = xy + \frac{1 + x + xy}{2} C_3(x, y)$$

(2.16)

The last equality allows an easy derivation of the counting numbers $c_5(n, m)$. As a simple check, we notice that the function $C_5(x, y = 1)$, enumerates walks of fixed length and any number of inversions:

$$C_5(x, y = 1) = \frac{1}{2} \left( -1 + \sqrt{\frac{1 + 2x}{1 - 2x}} \right)$$

(2.17)

The numbers are well known, see for instance [26].

Statistics of the area.

The approach illustrated in the previous paragraph can also be used to obtain equations for the generating functions that enumerate the positive random walks of $T$ steps returning to the origin and enclosing with the time axis a fixed area $A$. This problem was recently discussed by Jonsson and Wheater, and we refer to their paper [23] for the physics motivations of the study and the analysis of the functional equation for the generating function.

Let $c_1(A, T)$ and $c_2(A, T)$ be the numbers of, respectively, strictly and weakly positive walks that return to the origin after $T$ steps, such that the area between the walk and the horizontal axis is $A$. In both cases $T$ is an even integer. Just as in Fig.2 and in eq.(2.3) we have

$$c_2(A, T) = c_1(A, T) + \sum_{A_1 = 1, 2, \ldots} \sum_{T_1 = T - T_1} c_1(A_1, T_1) c_2(A - A_1, T - T_1)$$

(2.18)

Any weakly positive path may be elevated by adding a step at the beginning and another at the end, thus transforming it into a strictly positive path, then
\[ c_2(A, T) = c_1(A + T + 1, T + 2) \quad \text{for} \quad A \geq 2 ; \]
\[ c_2(A = 1, T) = c_1(A = 1, T) = \delta_{T,2} \quad \text{(2.19)} \]

We define the generating functions

\[ F_i(x, y) = \sum_{A,T} c_i(A, T) x^A y^T \quad \text{for} \quad i = 1, 2. \quad \text{(2.20)} \]

Then eqs. (2.18), (2.19) translate into

\[
\begin{align*}
F_2(x, y) &= F_1(x, y) + F_1(x, y) F_2(x, y) ; \\
F_1(x, y) &= x y^2 + x y^2 F_2(x, xy) 
\end{align*}
\quad \text{(2.21)}
\]

One obtains the nonlinear functional equation of Jonsson and Wheater

\[ F_1(x, y) = x y^2 + F_1(x, y) F_1(x, xy) \quad \text{(2.22)} \]

which has a formal solution as an infinite continued fraction

\[ F_1(x, y) = \frac{xy^2}{1 - \frac{x^3y^2}{1 - \frac{x^5y^2}{1 - \ldots}}} \quad \text{(2.23)} \]

In a similar way we obtain the functional equation for \( F_2(x, y) \)

\[ 1 - x y^2[1 + F_2(x, y)] = \frac{1}{1 + F_2(x, xy)} \quad \text{(2.24)} \]

\[ F_2(x, y) = -1 + \frac{1}{1 - \frac{x y^2}{1 - \frac{x^3y^2}{1 - \ldots}}} \quad \text{(2.25)} \]

From eqs. (2.22), (2.24), one has \( F_1(1, y) = \frac{1 - \sqrt{1 - 4y^2}}{2} \) and \( F_2(1, y) = \frac{1 - 2y^2 - \sqrt{1 - 4y^2}}{2y^2} \).

As shown in [23], eq. (2.22) allows the evaluation of \( \frac{\partial}{\partial x} F_1(x, y)_{|x=1} \), which is the generating function for the area associated to the set of strictly positive paths of length \( T \)

\[ \frac{\partial}{\partial x} F_1(x, y)_{|x=1} = \sum_f y^T \sum_A A c_1(A, T) = \frac{y^2}{1 - 4y^2} \quad \text{(2.26)} \]
In the same way we obtain the generating function for the area associated to the set of weakly positive paths of length $T$

$$\frac{\partial}{\partial x} F_2(x,y)|_{x=1} = \sum_T y^T \sum_A A c_2(A,T) = \frac{1 - 2y^2 - \sqrt{1 - 4y^2}}{2y^2(1 - 4y^2)}$$ (2.27)

The results (2.26), (2.27) were obtained in [24] and extended to higher moments of the area in [25] by a different method. We do not proceed to the analysis of eq.(2.22), whose main properties were found in paper [23]. In the next section we describe a different approach to enumerate random walks, which will provide a different viewpoint of this problem.

3 Counting by insertions which increase the width of the path.

The counting method described in sect. 2 was based on the construction of the ensemble of random walks by adding new random walks at the end of previously considered ones. We now present a different approach where the ensemble is generated by inserting new random walks on the vertices of previously considered ones.

To evaluate the number of Dyck paths of $2n$ steps, height $t$, that visit prescribed numbers $N_j$ of times the sites $j = 0, 1 \ldots t$, it is useful to consider the one-to-one map between Dyck paths and planar rooted trees [20]. The correspondence is shown in Fig.5 where the rooted planar tree on the right side is mapped to the compressed ”mountain range” of the left side, where each edge of the tree arises from one matched up-down pair.

| $j$ | $N_j$ | $n_j$ |
|-----|-------|-------|
| 5   | 2     | 2     |
| 4   | 4     | 2     |
| 3   | 5     | 3     |
| 2   | 5     | 2     |
| 1   | 3     | 1     |
| 0   | 2     | 0     |

Fig.5
Given a Dyck path, let $n_j$ denote the number of up steps between level $j - 1$ and level $j$. Since $N_j$ is the sum of the number of up steps from level $j - 1$ and level $j$ plus the number of down steps between level $j$ and level $j + 1$, we have

$$N_j = n_j + n_{j+1}, \quad j = 1, ..., t; \quad n_0 = 0, \quad N_0 = n_1 + 1 \quad (3.1)$$

The following table recalls corresponding elements:

| Dyck path | planar rooted tree |
|-----------|--------------------|
| one pair of matched up-down steps | one edge of the tree |
| length of path=2n | sum of degrees of vertices at level j |
| $N_j, \quad j = 1, .., t$ | number of edges between level $j - 1$ and level $j$ |
| $n_j, \quad j = 1, .., t$ | $N_j - n_{j+1} = v_j = \text{number of vertices at level } j$ |
| height $t$ | height $t$ |

The number $N(v_1, v_2, ..., v_t)$ of planar rooted trees with given set of non-root vertices $\{v_1, ..., v_t\}$ is evaluated in a recursive way because planar rooted trees with a set of non-root vertices $\{v_1, ..., v_t, v_{t+1}\}$ are obtained by adding $v_{t+1}$ new vertices at level $t + 1$ and using $t + 1$ new edges to connect all of them to some of the $v_t$ vertices at level $t$. This may be done in $\binom{v_{t+1} + v_t - 1}{v_{t+1}}$ ways. Then

$$N(v_1, v_2, ..., v_t) = \left( \frac{v_2 + v_1 - 1}{v_2} \right) \left( \frac{v_3 + v_2 - 1}{v_3} \right) \cdots \left( \frac{v_t + v_{t-1} - 1}{v_t} \right) \quad (3.2)$$

Since $v_j = N_j - n_{j+1} = n_j$ the evaluation (3.2) provides the number $N(n_1, n_2, ..., n_t)$ of Dyck paths of height $t$, with specified numbers $n_j$ of up steps between level $j - 1$ and level $j$

$$N(n_1, n_2, ..., n_t) = \left( \frac{n_1 + n_2 - 1}{n_2} \right) \left( \frac{n_2 + n_3 - 1}{n_3} \right) \cdots \left( \frac{n_{t-1} + n_t - 1}{n_t} \right) \quad (3.3)$$

Such paths have total length $2n$, where $n = n_1 + \ldots + n_t$.

We remark that it might seem more natural to enumerate Dyck paths of fixed length in terms of the integers $N_j$ enumerating the visits at each level, rather than in terms of the positive integers $n_j$ enumerating the up steps from each level. The second coding is superior because the positive numbers $n_j$ are independent variables, save the restriction $\sum n_j = n$.

We obtained the result (3.3) by inserting new mountain ranges on top of the highest vertices of Dyck paths [11]. We were unaware of the previous works on the enumeration of vertices in planar rooted trees [13], [21].

We now proceed to discuss some properties of the enumeration numbers (3.3) by introducing generating functions. They have a very direct relation
with general tridiagonal matrices. This fruitful and interesting connection was
described long ago by P.Flajolet, in his basic paper apparently almost ignored
in later investigations [32].

Let \( F_t(x_1, x_2, \ldots x_t) \) be the generating function for the numbers \( N(n_1, \ldots, n_t) \):

\[
F_t(x_1, x_2, \ldots, x_t) = \sum_{n_j=1, \ldots, \infty, j=1,\ldots} N(n_1, n_2, \ldots, n_t) x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t} \quad (3.4)
\]

We here derive various useful properties of the function, that originate from a
recursive relation.

**Proposition 1**

\[
F_t(x_1, x_2, \ldots, x_t) = F_{t-1}(x_1, x_2, \ldots, \frac{x_{t-1}}{1-x_t}) - F_{t-1}(x_1, x_2, \ldots, x_{t-1}) \quad (3.5)
\]

**Proof:**

\[
\sum_{n_j=1, \ldots, \infty, j=1,\ldots} N(n_1, n_2, \ldots, n_t) x_1^{n_1} \cdots x_n^{n_t} =
\]

\[
= \sum_{n_j=1, \ldots, \infty, j=1,\ldots} N(n_1, n_2, \ldots, n_{t-1}) x_1^{n_1} x_2^{n_2} \cdots x_{t-1}^{n_{t-1}} \sum_{n_t=1}^{\infty} \left( \frac{n_t + n_{t-1} - 1}{n_t} \right) x_t^{n_t} =
\]

\[
= \sum_{n_j=1, \ldots, \infty, j=1,\ldots} N(n_1, n_2, \ldots, n_{t-1}) x_1^{n_1} x_2^{n_2} \cdots x_{t-1}^{n_{t-1}} \left[ (1 - x_t)^{-n_{t-1} - 1} \right] =
\]

\[
= F_{t-1}(x_1, x_2, \ldots, x_{t-1}) - F_{t-1}(x_1, x_2, \ldots, x_{t-1}) \cdot
\]

Beginning with \( N(n) = 1 \), one iteratively constructs the first few generating
functions:

\[
F_1(x) = \frac{x}{1-x}; \quad F_2(x, y) = \frac{xy}{(1-x)(1-x-y)};
\]

\[
F_3(x, y, z) = \frac{xyz}{(1-x-y)(1-x-y-z+xz)} \ldots
\]

They suggest the following formal solution to equation (3.5):

**Proposition 2**

\[
F_t(x_1, x_2, \ldots, x_{t-1}, x_t) = \frac{x_1 x_2 \cdots x_{t-1} x_t}{P_t(x_1, \ldots, x_t) P_{t-1}(x_1, \ldots, x_{t-1})} =
\]

\[
= \frac{P_{t-1}(x_2, \ldots, x_t)}{P_t(x_1, \ldots, x_t)} \frac{P_{t-1}(x_1, \ldots, x_{t-1})}{P_{t-1}(x_1, \ldots, x_{t-1})} \quad (3.6)
\]
where $P_t(x_1, \ldots, x_t)$ is the polynomial generated through the recursion

$$
P_t(x_1, \ldots, x_t) = P_{t-1}(x_1, \ldots, x_{t-1}) - x_t P_{t-2}(x_1, \ldots, x_{t-2})
$$

$P_0 = 1$ , $P_1(x) = 1 - x$ \hspace{1cm} (3.7)

Proof: the first equality is proven by direct substitution of the solution into the recursive formula (3.5) and by using the following identity, which is proven by repeated use of eq. (3.7):

$$
P_t(x_1, \ldots, x_t) = (1 - x_t)P_{t-1}(x_1, \ldots, x_{t-2}, \frac{x_{t-1}}{1 - x_t}) \hspace{1cm} (3.8)
$$

To prove the second equality in eq.(3.6), we formally solve the recurrence relation for polynomials, with the given initial conditions, by means of a transfer matrix:

$$
\begin{pmatrix}
P_t(x_1, \ldots, x_t) \\
P_{t-1}(x_1, \ldots, x_{t-1})
\end{pmatrix} =
\begin{pmatrix}
P_t(x_1, \ldots, x_t) & P_{t-1}(x_2, \ldots, x_t) \\
P_{t-1}(x_1, \ldots, x_{t-1}) & P_{t-2}(x_2, \ldots, x_{t-1})
\end{pmatrix} =
\begin{pmatrix}
1 & -x_t \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & -x_{t-1} \\
1 & 0
\end{pmatrix} \ldots
\begin{pmatrix}
1 & -x_1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix} \hspace{1cm} (3.9)
$$

By taking the determinant of both sides, we obtain a useful identity:

$$
P_t(x_1, \ldots, x_t)P_{t-2}(x_2, \ldots, x_{t-1}) - P_{t-1}(x_1, \ldots, x_{t-1})P_{t-1}(x_2, \ldots, x_t) = -x_1 x_2 \ldots x_t \hspace{1cm} (3.10)
$$

The equality follows by dividing both terms by $P_t(x_1, \ldots, x_t)P_{t-1}(x_1, \ldots, x_{t-1})$. 

Note that the recursive property implies that the polynomial $P_t(x_1, \ldots, x_t)$ coincides with the determinant of a matrix of size $t + 1$:

$$
P_t(x_1, \ldots, x_t) = \det[I + M(1, x)] \hspace{1cm} (3.11)
$$

The generating function for walks with height not greater than $t$ is

$$
\Phi_t(x_1, \ldots, x_t) = 1 + F_1(x_1) + F_2(x_1, x_2) + \ldots + F_t(x_1, \ldots, x_t) = \frac{P_{t-1}(x_2, \ldots, x_t)}{P_t(x_1, \ldots, x_t)} \hspace{1cm} (3.12)
$$

This function has a simple representation in terms of the bidiagonal matrix $M(1, x)$, of size $t + 1$:

$$
\Phi_t(x_1, x_2, \ldots, x_t) = [I + M(1, x)]_{1,1}^{-1} \hspace{1cm} (3.13)
$$

By replacing the last term in the sum (3.12) with the right side of eq. (3.5) and resumming, one obtains the recursive property

$$
\Phi_t(x_1, x_2, \ldots, x_{t-1}, x_t) = \Phi_{t-1}(x_1, x_2, \ldots, \frac{x_{t-1}}{1 - x_t}) \hspace{1cm} (3.14)
$$
and, since \( \Phi_1(x) = (1 - x)^{-1} \), the following finite continued fraction representation:

\[
\Phi_t(x_1, x_2, \ldots, x_t) = \frac{1}{1 - \frac{x_1}{1 - \frac{x_2}{1 - \ldots \frac{x_{t-1}}{1 - x_t}}}} \tag{3.15}
\]

The numbers \( N(n_1, \ldots, n_t) \) bring a very detailed information about random walks. One is often interested in reduced information, which require restricted summations over the variables \( n_1 \ldots n_t \). This is, in general, a rather formidable task. We here provide some interesting examples. In the first we count closed weakly positive walks of fixed height and length and obtain identities involving sums of products of binomials. In the second example, we again discuss the problem of the area, by relating it to certain tridiagonal matrices. The third example addresses the issue of sums of weighted walks.

**Example 1: enumeration of walks with fixed length and height.**

Let \( C(n, t) \) be the number of weakly positive closed walks of length \( 2n \) and height \( t \). Evidently, the parameter \( t \) cannot be larger than \( n \). The sum over possible \( t \) values gives the well known number of weakly positive walks of length \( 2n \):

\[
\sum_{t=1}^{n} C(n, t) = \frac{1}{n+1} \binom{2n}{n} \tag{3.16}
\]

The generating function for the numbers \( C(n, t) \) can be obtained by setting all arguments \( x_i \) equal to \( x \) in the function (3.4):

\[
F_t(x) = \sum_{n=t, \ldots, \infty} x^n C(n, t) = \frac{x^t}{P_t(x) P_{t-1}(x)} = \frac{P_{t-1}(x)}{P_t(x)} - \frac{P_{t-2}(x)}{P_{t-1}(x)}
\]

\[
C(n, t) = \sum_{\kappa(n; t)} N(n_1 \ldots n_t), \quad (n \geq t) \tag{3.17}
\]

where, for shortness, we adopt the notation for the sum over the \( t \) integers \( n_i > 1 \) such that \( n_1 + \ldots + n_t = n \) as a sum over the set \( \kappa(n; t) \) of compositions of \( n \) into \( t \) integers \( n_i \geq 1 \). The polynomials \( P_t(x) \) are easily evaluated from the recursion property

\[
P_t(x) = P_{t-1}(x) - x P_{t-2}(x), \quad P_0 = 1, \quad P_1(x) = 1 - x \tag{3.18}
\]

and have a simple expression in terms of the roots of the equation \( z^2 - z + x = 0 \):

\[
P_t(x) = \frac{\rho_{t+2}^{t+2} - \rho_2^{t+2}}{\rho_1 - \rho_2} = \frac{1}{\sqrt{1 - 4x}} \left[ \left( \frac{1 + \sqrt{1 - 4x}}{2} \right)^{t+2} - \left( \frac{1 - \sqrt{1 - 4x}}{2} \right)^{t+2} \right]
\]
$$\prod_{k=1}^{[(t+1)/2]} \left( 1 - 4x \cos^2 \frac{k\pi}{t+2} \right) \quad (3.19)$$

For the last equality we used the known eigenvalues of the matrix $M(1, 1)$. $P_t(x)$ is related to a Chebyshev polynomial of the second kind.

The multiple sums over the compositions in eq. (3.17) are obtained by extracting the coefficient of $x^n$ from the generating function $F_t(x)$. This is simple for small $t$; for example:

$$C(n, 2) = \sum_{n=2} \binom{n_2 + n_1 - 1}{n_2} = 2^{n-1} - 1 ;$$

$$C(n, 3) = \sum_{n=3} \binom{n_2 + n_1 - 1}{n_2} \binom{n_3 + n_2 - 1}{n_3} = -2^{n-1} +$$

$$+ \left( \frac{3 + \sqrt{5}}{2} \right)^{n-2} \left( 1 + \frac{2}{\sqrt{5}} \right) + \left( \frac{3 - \sqrt{5}}{2} \right)^{n-2} \left( 1 - \frac{2}{\sqrt{5}} \right) \quad (3.20)$$

We now consider the general case.

**Proposition 3**

$$C(n, t) = -\frac{1}{t+2} \sum_j \frac{1 - 4a_j}{a_j^{n+1}} + \frac{1}{t+1} \sum_j \frac{1 - 4b_j}{b_j^{n+1}} \quad (3.21)$$

where $n \geq t$, $\{a_j\}$ and $\{b_j\}$ are, respectively, the zeros of $P_t(x)$ and $P_{t-1}(x)$.

Proof: we use the following identity for the inverse of a polynomial $P(x)$ with simple roots $x_j$:

$$\frac{1}{P(x)} = \sum_{j=1}^n \frac{1}{P'(x_j) x - x_j}$$

to obtain an expression for the inverse of the polynomial $P_t(x)P_{t-1}(x)$ in terms of the known roots of the factors:

$$\frac{1}{P_t(x)P_{t-1}(x)} = \sum_j \frac{1}{P_t'(a_j)P_{t-1}(a_j) x - a_j} + \sum_j \frac{1}{P_t'(b_j)P_{t-1}'(b_j) x - b_j}$$

We then use the following results, which will be proven

$$P_t'(a_j)P_{t-1}(a_j) = (t+2) \frac{a_j^t}{1 - 4a_j}, \quad P_t'(b_j)P_{t-1}'(b_j) = -(t+1) \frac{b_j^t}{1 - 4b_j} \quad (3.22)$$
and obtain the power expansion of \(F_t(x)\) we searched for:

\[
\frac{x^t}{P_t(x)P_{t-1}(x)} = \frac{1}{t+2} \sum_{j} \frac{1 - 4a_j}{a_j^t} x^t - \frac{1}{t+1} \sum_{j} \frac{1 - 4b_j}{b_j^t} x^t = \sum_{n=t}^{\infty} x^n \left[ \frac{1}{t+2} \sum_{j} \frac{1 - 4a_j}{a_j^{n+1}} + \frac{1}{t+1} \sum_{j} \frac{1 - 4b_j}{b_j^{n+1}} \right]
\]

To derive the first identity in eq. (3.22) one evaluates \(P'(x)\) in terms of the roots \(z_1\) and \(z_2\) of the equation \(z^2 - z + x = 0\). For \(x = a_j\) it is

\[
P'(a_j) = -(t + 2) \frac{z_1^{t+1} + z_2^{t+1}}{(z_1 - z_2)^2}
\]

The condition \(P_t(a_j) = 0\) implies that \(z_1^{t+2} = z_2^{t+2}\); then:

\[
P'(a_j)P_{t-1}(a_j) = -(t + 2) \frac{z_1^{2t+2} - z_2^{2t+2}}{z_1 - z_2} = -(t + 2) \frac{[z_1 z_2]^t [z_2 - z_1]}{[z_1 - z_2]^3}
\]

This, together with \(z_1 + z_2 = 1\) and \(z_1z_2 = a_j\), allows the simplification that yields the result. The second identity is proven along the same line.

The structure of the coefficients \(C(n,t)\) as a two term difference suggests to introduce the numbers \(C(n, \leq t)\) of walks of length \(2n\) and height not larger than \(t\):

\[
C(n, \leq t) = C(n, 1) + C(n, 2) + \ldots + C(n, t) = -\frac{1}{t+2} \sum_{j} \frac{1 - 4a_j}{a_j^{n+1}} = \frac{2}{t+2} \sum_{j=1}^{t+1} \left( 4 \cos^2 \frac{j\pi}{t+2} \right)^n \sin^2 \frac{j\pi}{t+2}
\]

\[
= 2 \sum_{\ell} \sum_{n=\ell(t+2)}^{n+\ell(t+2)} \frac{2n+2}{n+\ell(t+2)} - \frac{1}{2} \sum_{\ell} \sum_{n=\ell(t+2)}^{n+\ell(t+2)} \frac{2n+2}{n+1+\ell(t+2)}
\]

(3.24)

where we inserted the explicit form for \(a_j\) from eq. (3.19). The form of eq. (3.23) may be recognized from the property \(C(n, \leq t) = [M(1,1)^{2n}]_{11}\), where \(M(1,1)\) has size \(t+1\). Of course \(C(n, t) = C(n, \leq t) - C(n, \leq t-1)\). Note that \(C(n, \leq n)\) is precisely the number in eq. (3.16).

The generating function \(\Phi_t(x)\) for the numbers is obtained by setting all arguments equal to \(x\) in the function (3.13).

\[
\Phi_t(x) = \sum_{n=0}^{\infty} C(n, \leq t) x^n = \frac{2}{t+2} \sum_{j=1}^{t+1} \frac{\sin^2 \frac{j\pi}{t+2}}{1 - 4x \cos^2 \frac{j\pi}{t+2}}
\]

(3.25)
Example 2: enumeration of walks of fixed length and area.
The area of closed weakly positive walks with numbers of upward step $s_{n_1}, \ldots, s_{n_t}$ is $A = n_1 + 3n_2 + 5n_3 + \ldots + (2t-1)n_t$, while the total length is $T = 2n$, $n = n_1 + \ldots + n_t$. With the position $x_1 = xy$, $x_2 = xy^3$, ..., $x_t = xy^{2t-1}$ in equation (3.4) we obtain the generating function $F_t(x, y)$ of counting numbers $C(n, t, A)$ of positive closed walks with fixed length $2n$, height $t$ and area $A$:

$$F_t(x, y) = \sum_{n,A} x^n y^A C(n, t, A)$$

$$= \frac{P_t(xy^3, \ldots, xy^{2t-1})}{P_t(xy, \ldots, xy^{2t-1})} - \frac{P_{t-1}(xy^3, \ldots, xy^{2t-3})}{P_{t-1}(xy, \ldots, xy^{2t-3})},$$

where the $\delta$ function imposes the restriction that the integers $n_1 \ldots n_t$ of the compositions of $n$ should correspond to walks with fixed area $A$.

If we relax the restriction on height, being interested on counting numbers $C(n, A)$ for the length and the area, the generating function is

$$\Phi_\infty(x, y) = 1 + \sum_{n,A} C(n, A)x^n y^A = 1 + \sum_{t \geq 1} F_t(x, y) = \frac{P_\infty(xy^3, xy^5, \ldots)}{P_\infty(xy, xy^4, \ldots)}$$

(3.27)

We shall solve the counting problem by giving first the explicit expression of the polynomial $P_t(x, y) = P_t(xy, xy^3, \ldots, xy^{2t-1})$, which coincides with the determinant of the tridiagonal matrix of size $t + 1$:

$$I + M(1, xy) = \begin{pmatrix} 1 & xy & 1 & xy^3 & 1 & xy^5 & \cdots & \cdots & \cdots & 1 & 1 & xy^{2t-1} \\ 1 & 1 & \ddots & & & & & & & 1 & 1 & 1 \end{pmatrix}$$

(3.28)

Proposition 4

$$P_t(x, y) = 1 + \sum_{n=1}^{[t+1)/2]} (-1)^{n+1} x^n y^{n(2n-1)} \prod_{k=1}^{n} \frac{1 - y^{2(t-n+2-k)}}{1 - y^{2k}}$$

(3.29)

Proof: the polynomials solve the recurrence relation $P_t(x, y) = P_{t-1}(x, y) - xy^{2t-1}P_{t-2}(x, y)$ with $P_0 = 1$ and $P_1(x, y) = 1 - xy$. The explicit computation of the first few ones suggests the following general structure: $P_t(x, y) =$
\[
\sum_{n=0}^{[t+1/2]}(-x)^n c_{n,t}(y), \text{ where } c_{n,t}(y) \text{ are polynomials for which recurrence relations are easily written. By solving the first few ones, one easily arrives to conjecture the form stated in the proposition. A formal proof then follows by induction, but we omit it.}
\]

For the purpose of the statistics of the area, we need the limit \( t \to \infty \):
\[
P_\infty(x, y) = 1 + \sum_{n>0} (-x)^n \frac{y^{n(2n-1)}}{(1-y^2)(1-y^4)\ldots(1-y^{2n})} \tag{3.30}
\]

From this expression, we derive the following functional equation:
\[
P_\infty(x, y) - P_\infty(xy^2, y) + xy P_\infty(xy^4, y) = 0 \tag{3.31}
\]

Proof:
\[
P_\infty(xy^2, y) = 1 + \sum_{n>0} (-x)^n \frac{y^{n(2n-1)}}{(1-y^2)(1-y^4)\ldots(1-y^{2n-2})} \frac{y^{2n}}{1-y^{2n}} =
\]
\[
= P_\infty(x, y) - \sum_{n>0} (-x)^n \frac{y^{n(2n-1)}}{(1-y^2)(1-y^4)\ldots(1-y^{2(n-1)})} =
\]
\[
= P_\infty(x, y) + xy P_\infty(xy^4, y) \quad \bullet
\]

Since \( \Phi_\infty(x, y) = P_\infty(xy^2, y)/P_\infty(x, y) \), we obtain a functional equation which corresponds to eq. (2.24) for \( F_2(x, y) \):
\[
\Phi_\infty(x, y) = 1 + xy \Phi_\infty(x, y) \Phi_\infty(xy^2, y) \tag{3.32}
\]

The formula (3.27) allows to compare \( \Phi_\infty(x, y) \) with the generating function \( F_2(x, y) \) discussed in sect. 2, for weakly positive walks. There, the variable \( y \) was conjugated to the total length \( T = 2n \) and \( x \) was conjugated to the area. Here, for uniformity with the rest of the paper, \( x \) is conjugated to \( n \) and \( y \) to \( A \). It turns out that \( \Phi_\infty(x^2, y) = 1 + F_2(y, x) \).

Remark. From the continued fraction representation of \( \Phi_\infty(x_1, x_2, x_3, \ldots) \), see eq. (3.15)
\[
\Phi_\infty(x_1, x_2, x_3, \ldots) = \frac{1}{1 - x_1 \frac{x_2}{1 - x_3 / 1 - \ldots}} \tag{3.33}
\]

it is easy to see that it solves the formal equation
\[
\Phi_\infty(x_1, x_2, x_3, \ldots) = 1 + x_1 \Phi_\infty(x_1, x_2, x_3, \ldots) \Phi_\infty(x_2, x_3, x_4, \ldots) \tag{3.34}
\]

Let us suppose that each \( x_k \) is a function of two variables \( x_k = x_k(x, y) \). The above formal equation will turn into a functional equation for a function

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$\Phi(\infty, x, y)$ if $x_k(x, y) = x_{k-1}(x, y)$, where the replacements $x \to \mathfrak{x}$ and $y \to \mathfrak{y}$ do not depend on $k$. This is possible for the case

$$x_k = x y^{p_k + r}, \quad p \text{ and } r \text{ real} \quad (3.35)$$

Then eq.(3.34) becomes

$$\Phi(\infty, x, y) = 1 + x y^{p + r} \Phi(\infty, x, y) \Phi(\infty, x y^{p}, y) \quad (3.36)$$

Example 1 in this section corresponds to $p = 0$ and $r = 0$, Example 2 corresponds to $p = 2, r = -1$. However the more general equation (3.36) does not add substantially to the previous examples since it provides the generating function for counting Dyck paths with given number of steps and given

$$f(p, r) = \sum_{j} (pj + r) n_j$$

Another interesting case is

$$x_k = x y^{a_k} \quad (3.37)$$

and will be discussed in the end of the next example.

As this work was being written we became aware of recent papers which have some overlap with this section [29] [30] [31].

**Example 3 : Toy functional integrals.**

In the introduction we mentioned the subject of weighted Dyck paths and speculated about the possibility of an operative definition of path integral in the frame of Dyck paths. One may consider a variety of functionals $G[\gamma]$ where $\gamma$ is a Dyck path of length $2n$, and write

$$\int \mathcal{D}\gamma G[\gamma] = \frac{1}{C(n)} \sum_{\gamma \in \Gamma(n)} G[\gamma]$$

where $\{s_k\}_{k=0}^{2n}$, with $s_0 = s_{2n} = 0$, is the sequence of sites visited by the path $\gamma$ and $C(n)$ is the cardinality of the set $\Gamma(n)$ of Dyck paths with $2n$ steps. It is given by the Catalan number $C(n) = \frac{1}{n+1} \binom{2n}{n}$, see eq.(3.16). Since we shall not consider the continuum limit, our finite sums are here called toy functional integrals.

We are also interested in the set $\Gamma(n, t)$ of Dyck paths of length $2n$, height $t$ and cardinality $C(n, t)$, see Proposition 3, whence one obtains the self-explanatory set $\Gamma(n, \leq t)$. We discuss with some detail the class of functionals $G[\gamma] = \int d\tau g(\gamma(\tau))$ and consider the toy integral

$$\int_{\Gamma(n, t)} \mathcal{D}\gamma \int d\tau g(\gamma(\tau)) = \frac{1}{C(n, t)} \sum_{\gamma \in \Gamma(n, t)} \sum_{k=1}^{2n} g(s_k) \quad (3.38)$$

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Two interesting examples are

\[ g(s) = s^r, \quad g(s) = c a^s, \quad a > 0 \]  \hspace{1cm} (3.39)

In particular, the toy integral with \( g(s) = s \) evaluates the average area enclosed by Dyck paths of height \( t \) and length \( 2n \). The second example leads to explicit equations, and will be discussed after the general case.

Eq. (3.38) is first written in terms of the numbers \( N_j \) of visits of site \( j \) of the Dyck path \( \gamma \), next in terms of the familiar numbers of upward steps \( n_j \)

\[
\sum_{k=1}^{2n} g(s_k) = \sum_{j=0}^{t} g(j) N_j = \sum_{j=0}^{t} g(j) (n_j + n_{j+1}) = \sum_{j=0}^{t} [g(j) + g(j-1)] n_j
\]  \hspace{1cm} (3.40)

The sum over paths is performed by introducing the counting numbers \( N(n_1, n_2, ..., n_t) \)

\[
\sum_{\gamma \in \Gamma(n,t)} \sum_{j=1}^{t} [g(j) + g(j-1)] n_j = \sum_{\kappa \in \kappa(n,t)} N(n_1, n_2, ..., n_t) \sum_{j=1}^{t} [g(j) + g(j-1)] n_j = \sum_{G \in G} G \cdot C(n, t, G)
\]  \hspace{1cm} (3.41)

where \( C(n, t, G) \) is the number of Dyck paths of length \( 2n \), height \( t \) and fixed value \( G \)

\[
C(n, t, G) = \sum_{\kappa \in \kappa(n,t)} N(n_1, n_2, ..., n_t) \delta \left( G - \sum_{i} [g(i) + g(i-1)] n_i \right)
\]  \hspace{1cm} (3.42)

and the sum \( G \in G \) runs over the set \( G \) of the possible values of \( G \). The generating function

\[
F_t(x, y) = \sum_{n,G} C(n, t, G) x^n y^G
\]

is obtained from the general theory by setting \( x_k = x y^{g(k)+g(k-1)} \) in eq. (3.6). It may be evaluated from ratios of the determinants \( P_t(x, y) \).

The function

\[
\Phi_\infty(x, y) = 1 + F_1(x, y) + \ldots + F_t(x, y) + \ldots = \sum_{n,G} C(n, G) x^n y^G
\]  \hspace{1cm} (3.43)

is the generating function for the numbers \( C(n, G) = \sum_{i=1}^{n} C(n, t, G) \) that count Dyck paths of length \( 2n \) and fixed value \( G \). The generating function \( f(x) \) for the numbers \( G_n = \sum_{G \in G} GC(n, G) \) can be written as the first derivative of \( \Phi_\infty(x, y) \),

\[
f(x) = \sum_{n=1}^{\infty} G_n x^n = \frac{\partial}{\partial y} \Phi_\infty(x, y) \bigg|_{y=1}
\]  \hspace{1cm} (3.44)

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While we have been unable to find a ”quadratic” functional equation for the function $\Phi_\infty(x, y)$ related to the toy integral (3.38) with $g(s) = s^r$ for $r \geq 2$, we were successful with the exponential function $g(s) = c a^s$. In this case the functional evaluated on a Dyck path is, see eq. (3.40),

$$G = c \sum_{k=0}^{2n} a^{sk} = c \frac{a + 1}{a} \sum_{s=1}^{t} a^n n_s$$

The generating function for counting numbers is obtained with the position $x_k = x y^{(a^s)}$, so that

$$x_1^{n_1} \ldots x_t^{n_t} = x^{n_1 + \ldots + n_t} y^{n_1 a + n_2 a^2 + \ldots + n_t a^t} = x^n y^G$$

and, for the general theory, it is given by the continued fraction

$$\Phi_\infty(x, y) = 1 \frac{1}{1 - \frac{xy}{1 - \frac{xy}{1 - \frac{xy}{1 - \ldots}}}}$$

which corresponds to the functional equation

$$\Phi_\infty(x, y) = 1 + xy^a \Phi_\infty(x, y) \Phi_\infty(x, y^a) \quad (3.45)$$

From this equation, since $\Phi_\infty(x, 1) = (1 - \sqrt{1 - 4x})/(2x)$, one easily obtains the generating function for the numbers $G_n$, see eq. (3.44),

$$\frac{\partial}{\partial y} \Phi_\infty(x, y) \bigg|_{y=1} = \frac{a}{x} \frac{(1 - 2x - \sqrt{1 - 4x})}{(1 - a + (1 + a)\sqrt{1 - 4x})}$$

It is interesting that the counting numbers $C(n, A)$, see eq. (3.27), allow to define a partition function for Dyck paths of any length $L = 2n$, weighted with their length $L$ and area $A$:

$$\int_{\Gamma} D \gamma e^{-aL[\gamma] - bA[\gamma]} = \sum_{L, A} e^{-aL - bA} C(L/2, A) = \Phi_\infty(e^{-2a}, e^{-b}) \quad (3.46)$$

where $\Gamma$ is the set of all Dick paths and a different normalization has been used.

**Enumeration of closed paths.**

The counting numbers $N(n_1, \ldots, n_t)$ apply to weakly positive closed paths, and are useful for expressing the matrix element $(1, 1)$ of $M(1, x)^{2p}$ as a polynomial in the entries $x_i$ of the matrix, see eq. (1.1).
The generalization of the counting numbers to arbitrary closed paths is straightforward, and is given in

**Proposition 5.**

The number of closed paths of height \(t\), depth \(s\) and length \(2p\), containing \(n_{-i}\) upward steps \((-i - 1, -i), i = s - 1, \ldots, 0\), and \(n_{j}\) upward steps \((j - 1, j), j = 1, \ldots, t\), is

\[
N_{s,t}(2p; n_{-s+1} \ldots n_{0} | n_{1} \ldots n_{t}) = N(n_{0}, n_{-1} \ldots n_{-s+1}) (n_{0} + n_{1}) N(n_{1}, n_{2}, \ldots, n_{t}) = \prod_{i=1}^{s-1} \left( \frac{n_{-i} + n_{-i+1} - 1}{n_{-i}} \right) \left( \frac{n_{0} + n_{1}}{n_{0}} \right) \prod_{i=1}^{t} \left( \frac{n_{i} + n_{i+1} - 1}{n_{i+1}} \right)
\]

(3.47)

Proof: Closed paths contributing to the above counting number are sequences of \(n_{1}\) closed positive paths and \(n_{0}\) closed negative paths. The factors \(N(n_{0}, \ldots, n_{-s+1})\) and \(N(n_{1}, \ldots, n_{t})\) respectively count the negative and positive closed paths that are obtained by joining the negative or positive subsequences. The intermediate binomial factor counts the ways into which the subsequences can be ordered. •

The length \(2p\) is specified in the symbol in view of a further generalization to open paths; for closed paths \(p = n_{-s+1} + \ldots + n_{t}\).

The counting numbers allow to write the expression for the matrix element \((i, i), i = 1 \ldots N\) of the powers of the bidiagonal matrix \(M(\mathbf{1}, \mathbf{x})\) as a polynomial in the entries:

\[
M(\mathbf{1}, \mathbf{x})^{2p}_{ii} = \sum_{t=0}^{N-i} \sum_{s+t=1}^{N-1} \sum_{\alpha(2p,s+t)} N_{s,t}(2p; n_{-s+1} \ldots n_{0} | n_{1} \ldots n_{t}) x_{t-s}^{n_{-s+1}} \ldots x_{i-1}^{n_{0}} x_{i}^{n_{1}} \ldots x_{i+t-1}^{n_{t}}
\]

(3.48)

### Enumeration of general paths

The enumeration is extended to paths that may not return to the origin. For open paths the counting numbers \(N_{s,t}(p; n_{-s+1} \ldots n_{0} | n_{1} \ldots n_{t})\) correspond to an expression which is more general than the above one.

The paths have length \(p\) and make the specified set of upward steps between sites \(-s\) and \(t\). Since the total number of upward steps is \(u = n_{-s+1} + \ldots + n_{t}\) and the number of downward steps is \(d = p - u\), the endpoint of the walks is site (level) \(y = u - d = 2u - p\). If \(2u - p \neq 0\) the paths are open.

For all paths it is \(n_{1}, \ldots, n_{t} \geq 1\), because the walk has to reach the highest site \(t\). However, for open paths some numbers \(n_{i}\) with \(i < 0\) can be zero. More precisely, if \(y \geq 0\) all \(n_{i} \geq 1\) because the walk has to ascend from \(-s\) to the level \(y\); if \(y < 0\) it may be that \(n_{k} = 0\), for \(k \leq y\).
With this notation and postponing the problem of the evaluation of the counting numbers, we write a general expression for the power of a matrix $M(1, x)$ as a polynomial in the entries $x_1, \ldots, x_{N-1}$:

**Proposition 6**

\[ [M(1, x)^p]_{i,j} = \sum_{s,t} \sum_{n_{s+1} \ldots n_t} N_{s,t}(p; n_{s+1} \ldots n_0 | n_1 \ldots n_t) x_i^{n_{s+1}} \ldots x_i^{n_0} x_j^{n_1} \ldots x_j^{n_t} \]

In the first sum $s$ and $t$ have the following restrictions:
1) the finite size of the matrix, which constrains walks inside the strip of integers $1 \ldots N - 1$ that label the entries. For this reason: $0 \leq s \leq i - 1$ and $0 \leq t \leq N - 1 - i$.
2) walks have to visit all sites between $i$ and $j$ included. Therefore, if $j \geq i$ it is $t \geq j - i$; if $j \leq i$ it is $s \geq i - j$. If $i = j$ both $s$ and $t$ cannot be zero.
3) the sum on $n_{s+1} \ldots n_t$ has the restriction that, being $u = m_s + \ldots n_t$ the number of upward moves, the constraint $2u = j - i + p$ must be satisfied. (Note that for a bidiagonal matrix $[M^p]_{i,j} = 0$ if $p$ and $|j-i|$ have opposite parities).

The general expression for the counting numbers has been given by Kratenthaler [19], including the case of the null step. We here discuss the case of paths that, starting from the origin, only visit sites $i > 0$.

**Proposition 7**

The number of positive walks of length $p$, that never return to the origin and with prescribed numbers of upward steps is:

\[ N_{0,t}(p; 1, n_2, \ldots n_t) = \prod_{j=2}^{y} \binom{n_j + n_{j-1} - 2}{n_j - 1} N(n_y, \ldots, n_t) \]

where $y = 2u - p$ is the height of the walk, $u = 1 + n_2 + \ldots n_t$ is the total number of upward steps (see comment in proposition 5).

To obtain this counting number it is convenient to generalize the planar rooted tree by using edges of two types, with solid or dotted lines, as in Fig.6. There is a one-to-one map between the "open mountain range" and the corresponding generalized planar rooted tree: to any solid line edge of the tree correspond, as before, a pair of matched up-and-down sides of the mountain range, while a dotted line of the tree corresponds to a up side of the mountain range without the matching down side.
The correspondence between the number of visits at each level and the number of up steps is

\[ N_j = n_j + n_{j+1} - 1, \quad 1 \leq j \leq y \]

\[ N_j = n_j + n_{j+1}, \quad y < j \leq t \]  

(3.51)

Each vertex of the planar rooted tree is connected to a vertex at the lower level by a full line or a dotted line. \( n_j \) counts the full and dotted lines in the tree between level \( j-1 \) and level \( j \); then

\[ v_j = n_j, \quad j = 1, \ldots, t \]  

(3.52)

The number of planar rooted trees is evaluated first by drawing the root and a dotted path from the root to level \( y \), with one vertex at each level \( 1, \ldots, y \), next by adding a number \( \tilde{v}_j \) of new vertices at each level \( j \), and connecting each vertex with a full line to a vertex at the lower level. If we call \( \tilde{v}_j = v_j \) for \( j = y + 1, \ldots, t \) and \( \tilde{v}_j = v_j - 1 \) for \( j = 1, \ldots, y \), we obtain that the number of planar rooted trees with given set of non-root vertices:

\[ N(v_1, v_2, \ldots, v_t) = \frac{\tilde{v}_t + v_{t-1} - 1}{\tilde{v}_t} \frac{\tilde{v}_{t-1} + v_{t-2} - 1}{\tilde{v}_{t-1}} \cdots \frac{\tilde{v}_2 + v_1 - 1}{\tilde{v}_2} \]  

(3.53)

From this, eq. (3.52) and the relation between \( \tilde{v}_j \) and \( v_j \) we obtain the number in proposition 7. ✷

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