ANALYSIS OF NONCONFORMING VIRTUAL ELEMENT METHOD FOR THE CONVECTION DIFFUSION REACTION EQUATION WITH POLYNOMIAL COEFFICIENTS

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Abstract. In this paper we discuss the application of nonconforming virtual element methods (VEM) for the second order diffusion dominated convection diffusion reaction equation. Stability of the virtual element methods has been proved for the symmetric bilinear form. But the same analysis cannot be carried out for the non-symmetric case. In this work we present the external virtual element methods using $L^2$ projection operator and prove the well-posedness of VEM for non-symmetric bilinear form. We also proved polynomial consistency of discrete bilinear form assuming $H^2$ regularity of approximate solution on each triangle. We have shown optimal convergence estimate in the broken sobolev norm.

Key words. convection-diffusion, mimetic finite difference, virtual element methods

1. INTRODUCTION

In recent times the virtual element method has been successfully applied to a variety of problems [11, 6]. The basic principle of virtual element method has been discussed in [13, 3]. A mimetic discretization method with arbitrary polynomial order is presented recently in [5], but the classical finite element framework with arbitrary polynomial is still making the presentation cumbersome [8]. The idea of the virtual element methods is very similar to mimetic finite difference methods [15, 12]. Virtual element space is a unisolvent space of smooth functions containing a polynomial subspace. In other way we can say non conforming virtual element method is a generalization of classical nonconforming finite element methods. Very recently it has been clearly understood that the degrees of freedom associated to trial/test functions is enough to construct finite element framework, which lead to the study of virtual element method. Unlike classical nonconforming FEM [8, 17] [4, 16] VEM has the advantage that virtual element space together with polynomial consistency property allows us to approximate the bilinear form without explicit knowledge of basis function.

Stability analysis of virtual element method irrespective of conforming or non-conforming is quite different from classical finite element method. If the bilinear form is symmetric then we divide it into two parts [13, 3], one is responsible for polynomial consistency property and other one for stability analysis. The framework for stability analysis for conforming and nonconforming virtual element is almost same. A pioneering work using elliptic projection operator has been introduced to approximate symmetric bilinear form by Brezzi et al in their papers [11, 3]. If the bilinear form is not symmetric like convection diffusion reaction form, we can not extend this idea directly which may be considered as the drawback of using elliptic projection operator [1, 5]. In this case we may use $L^2$ projection operator for the modified approximation of bilinear form. The name virtual comes from the fact that the local approximation space in each mesh either polygon or polyhedra...
contains the space of polynomials together with some non-polynomial smooth function satisfying the weak formulation of model problem. The novelty of this method is to take the spaces and the degrees of freedom in such a way that the elementary stiffness matrix can be computed without actually computing non-polynomial functions, but just using the degrees of freedom.

In this paper we have approximated non-symmetric bilinear form using $L^2$ projection operator. We did not approximate the bilinear form same as [13] to avoid difficulty. Conforming virtual element method using $L^2$ projection operator has been already discussed for convection diffusion reaction equation with variable coefficient. We have considered the nonconforming discretization same as defined in [20]. In two dimension, the design of schemes of order of accuracy $k \geq 1$ was guided by the patch test [18, 16] which enforces continuity at $k$ Gauss-Legendre points on edge. Over the last few years, further generalization of non-conforming elements have been still considered by several authors, one such generalization to stokes problem is considered in [21, 2, 9].

This article is organized as follows. In Section 1.1 we discuss continuous setting of the model problem (1). In Section 1.5 we have discussed construction of non-conforming virtual element briefly since it has been discussed in the following paper [13]. Also, the global and local setting of discrete bilinear formulation has been presented explicitly. In Section 2 we have discussed the construction of bilinear form, stability analysis and polynomial consistency property of the discrete bilinear form. Section 3 discusses the construction of source term, boundary term and Section 5 provides well-posedness and convergence analysis. Stability analysis discussed in this paper is applicable only for the diffusion dominated problem. Finally in the last section we have shown the optimal convergence rate in broken norm $\| \cdot \|_{1,h}$.

### 1.1. Continuous Problem.

In this section we present the basic setting and describe the continuous problem. Throughout the paper, we use the standard notation of Sobolev spaces [8]. Moreover, for any integer $l \geq 0$ and a domain $D \in \mathbb{R}^m$ with $m \leq d, d = 2, 3$, $P^l(D)$ is the space of polynomials of degree at most $l$ defined on $D$. We also assume the convention that $P^{-1} = \{0\}$. Let the domain $\Omega$ in $\mathbb{R}^d$ with $d = 2, 3$ be a bounded open polygonal domain with straight boundary edges for $d = 2$ or a polyhedral domain with flat boundary faces for $d = 3$. Let us consider the model problem:

$$-
abla \cdot (K(x) \nabla (u)) + \beta(x) \cdot \nabla u + c(x) u = f(x) \quad \text{in} \quad \Omega,$$

$$u = g \quad \text{on} \quad \partial \Omega,$$

(1)

where $K \in (C^1(\Omega))^{d \times d}$ is the diffusive tensor, $\beta(x) \in (C(\Omega))^d$ is the convection field, $c \in C(\Omega)$ is the reaction field and $f \in L^2(\Omega)$. We assume that $(c(x) - \frac{1}{2} \nabla \cdot \beta(x)) \geq c_0$, where $c_0$ is a positive constant. This assumption guarantees that (1) admits a unique solution. The diffusive tensor is a full symmetric $d \times d$ sized matrix and strongly elliptic, i.e. there exists two strictly positive real constants $\xi$ and $\eta$ such that

$$\eta |v|^2 \leq v K v \leq \xi |v|^2$$

for almost every $x \in \Omega$ and for any sufficiently smooth vector field $v$ defined on $\Omega$, where $| \cdot |$ denotes the standard euclidean norm on $\mathbb{R}^d$. $K, \beta, c$ are chosen to be polynomials for the present problem.

The weak formulation of the model problem (1) reads:
Find \( u \in V_g \) such that
\[
A(u, v) = <f, v> \quad \forall \ v \in V
\]
where the bilinear form \( A(\cdot, \cdot) : V_g \times V \rightarrow \mathbb{R} \) is given by
\[
A(u, v) = \int_{\Omega} K \nabla u \cdot \nabla v + \int_{\Omega} (\beta \cdot \nabla u) v + \int_{\Omega} c uv
\]
\[
<f, v> = \int_{\Omega} fv
\]
\(<\cdot, \cdot>\) denotes the duality product between the functional space \( V' \) and \( V \), where the space \( V_g \) and \( V \) are defined by
\[
V_g = \{ v \in H^1(\Omega) : v|_{\partial \Omega} = g \}
\]
\[
V = H^1_0(\Omega)
\]
We define the elemental contributions of the bilinear form \( A(\cdot, \cdot) \) by
\[
a(u, v) = \int_{\Omega} K \nabla u \cdot \nabla v \ d\Omega
\]
\[
b(u, v) = \int_{\Omega} (\beta \cdot \nabla u) v \ d\Omega
\]
\[
c(u, v) = \int_{\Omega} c uv \ d\Omega
\]
Now we can estimate
\[
A(v, v) = \int_{\Omega} K \nabla v \cdot \nabla v + \int_{\Omega} (\beta \cdot \nabla v) v + \int_{\Omega} c v^2
\]
\[
\geq \eta \|\nabla v\|^2 + \int_{\Omega} (c - \frac{1}{2} \nabla \cdot \beta) v^2
\]
\[
\geq C \|v\|^2_1
\]
Combining inequality (7) with the continuity of \( A(\cdot, \cdot) \) it follows that there exists a unique solution to the variational form of equation (1).

1.2. Basic Setting. We describe now the basic assumptions of the mesh partitioning and related function spaces. Let \( \{\tau_h\} \) be a family of decompositions of \( \Omega \) into elements, \( T \) and \( \varepsilon_h \) denote a single element and set of edges of a particular partition respectively. By \( \varepsilon_h^0 \) and \( \varepsilon_h^\partial \) we refer to the set of interior and boundary edges/faces respectively. We will follow the same assumptions on the family of partitions as [13].

1.3. Assumptions on the family of partitions \( \{\tau_h\} \). There exists a positive \( \rho > 0 \) such that
(A1) for every element \( T \) and for every edge/face \( e \subset \partial T \), we have \( h_e \geq \rho h_T \).
(A2) every element \( T \) is star-shaped with respect to all the points of a sphere of radius \( \geq \rho h_T \).
(A3) for \( d = 3 \), every face \( e \in \varepsilon_h \) is star-shaped with respect to all the points of a disk having radius \( \geq \rho h_e \).
The maximum of the diameters of the elements \( T \in \tau_h \) will be denoted by \( h \). For every \( h > 0 \), the partition \( \tau_0 \) is made of a finite number of polygons or polyhedra.
We introduce the broken sobolev space for any \( s > 0 \)
\[
H^s(\tau_h) = \bigoplus_{T \in \tau_h} H^s(\Omega) = \{ v \in L^2(\Omega) : v|_T \in H^s(T) \}
\]
and define the broken $H^s$-norm

\[ \|v\|_{s,T_h}^2 = \sum_{T \in T_h} \|v\|_{s,T}^2 \quad \forall v \in H^s(T_h) \]

In particular for $s = 1$

\[ \|v\|_{1,T_h}^2 = \sum_{T \in T_h} \|v\|_{1,T}^2 \quad \forall v \in H^1(T_h) \]

Let $e \in \varepsilon_h^0$ be an interior edge and let $T^+, T^-$ be two triangles which share $e$ as a common edge. We denote the unit normal on $e$ in the outward direction with respect to $T^\pm$ by $n^\pm_e$. We then define the jump operator as:

\[
[v] := v^+ n^+_e + v^- n^-_e \quad \text{on} \quad e \in \varepsilon_h^0
\]

and \([v]\) := $vn_e$ on $e \in \varepsilon_h^\partial$

### 1.4. Discrete space.

In this section we will introduce discrete space same as [13]. For an integer $k \geq 1$ we define

\[ H^{1,nc}(\tau_h; k) = \left\{ v \in H^1(\tau_h) : \int_e [v] \cdot n_e q ds = 0 \quad \forall q \in P^{k-1}(e), \quad \forall e \in \varepsilon_h \right\} \]

We mention that if $v \in H^{1,nc}(\tau_h; k)$ then $v$ in said to satisfy patch test [20] of order $k$. To approximate second order problems we must satisfy the patch test. The space $H^{1,nc}(\tau_h; 1)$ is the space with minimal required order of patch test to ensure convergence analysis.

### 1.5. Nonconforming virtual element methods.

In this section we explain discrete nonconforming VEM framework for the equation (1). Before passing from the weak formulation to discrete problem, we first apply integration by parts to the convective term $(\beta \cdot \nabla u, v)$ to obtain

\[
\int_\Omega (\beta \cdot \nabla u)v = \frac{1}{2} \left[ \int_\Omega (\beta \cdot \nabla u)v - \int_\Omega (\beta \cdot \nabla v)u - \int_\Omega \nabla \cdot \beta uv \right]
\]

for $u \in H^1(\Omega), v \in H^1_0(\Omega)$

Bilinear form (2) can be written as

\[ A(u, v) = \sum_{T \in \tau_h} A_T(u, v) \]

where

\[ A_T(u, v) = \int_T K \nabla u \cdot \nabla v + b^\text{conv}_T(u, v) + \int_T c uv \]

and \[ b^\text{conv}_T(u, v) = \frac{1}{2} \left[ \int_T (\beta \cdot \nabla u)v - \int_T (\beta \cdot \nabla v)u - \int_T \nabla \cdot \beta uv \right] \]

We want to construct a finite dimensional space $V^k_h \subset H^{1,nc}(\tau_h; k)$, a bilinear form $A_h(\cdot, \cdot) : V^k_{h,g} \times V^k_h \to \mathbb{R}$, and an element $f_h \in (V^k_h)'$ such that the discrete problem:

Find $u_h \in V^k_{h,g}$ such that

\[ A_h(u_h, v_h) = \langle f_h, v_h \rangle \quad \forall v_h \in V^k_h \]

has a unique solution $u_h$. 

1.6. Local nonconforming virtual element space: We define for \( k \geq 1 \) the finite dimensional space \( V^k_h(T) \) on \( T \) as

\[
V^k_h(T) = \left\{ v \in H^1(T) : \frac{\partial v}{\partial n} \in \mathbb{P}^{k-1}(e), v|_e \in \mathbb{P}^k(e) \forall e \subset \partial T, \Delta v \in \mathbb{P}^{k-2}(T) \right\}
\]

with the usual convention that \( \mathbb{P}^{-1}(T) = \{0\} \).

From the definition, it immediately follows that \( \mathbb{P}^k(T) \subset V^k_h(T) \). The non-conforming VEM is formulated through the \( L^2 \) projection operators,

\[
\Pi_k : V^k_h(T) \rightarrow \mathbb{P}^k(T)
\]

\[
\Pi_{k-1} : \nabla (V^k_h(T)) \rightarrow (\mathbb{P}^{k-1}(T))^d.
\]

For \( k = 1 \), degrees of freedom are defined same as Crouzeix-Raviart element [10, 11]. In general we can say normal derivative \( \frac{\partial v}{\partial n} \) of an arbitrary element of \( V^1_h \) is constant on each edge \( e \subset \partial T \) (and different on each edges) and inside \( T \) are harmonic (i.e., \( \Delta v = 0 \)). It can be easily concluded that the total no of degrees of freedom of a particular element \( T \) with \( n \) edges/faces is \( n \) which is explained in detail in [13].

For \( k = 2 \), the space \( V^2_h(T) \) consists of functions whose normal derivative \( \frac{\partial v}{\partial n} \) is a polynomial of degree 1 on each edge/face, i.e. \( \frac{\partial v}{\partial n} \subset \mathbb{P}^1(e) \) and is a polynomial of degree 0 on interior region, i.e. a constant function. The dimension of \( V^2_h(T) \) is \( d(n+1) \) where \( n, d \) denote number of edges/face associated with an element \( T \) and spatial dimension of \( T \) respectively.

For each element \( T \), the dimension of \( V^k_h(T) \) is given by

\[
N_T = \begin{cases} 
  nk + \frac{(k-1)k}{2} & \text{for } d = 2, \\
  nk(k+1)^2 + \frac{(k-1)k(k+1)}{6} & \text{for } d = 3
\end{cases}
\]

which is explained in detail in [13, 6]. We need to introduce some further notation to define degrees of freedom same as [13]. Let us define space of scaled monomials \( M^l(e) \) and \( M^l(T) \) on \( e \) and \( T \) as

\[
M^l(e) = \left\{ \left( \frac{(x - x_e)}{h_e} \right)^s, |s| \leq l \right\}
\]

\[
M^l(T) = \left\{ \left( \frac{(x - x_T)}{h_T} \right)^s, |s| \leq l \right\}
\]

where \( s = (s_1, s_2, \ldots, s_d) \) be a \( d \)-dimensional multi index notation with \( |s| = \sum_{i=1}^{d} s_i \) and \( x^s = \Pi_{i=1}^{d} x_i^{s_i} \) where \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) and \( l \geq 0 \) be an integer. In \( V^k_h(T) \) we will choose same degrees of freedom as defined in [13]. On each edge \( e \subset \partial T \)

\[
\mu^{k-1}_e(v_h) = \left\{ \frac{1}{e} \int_{e} v_h q \, ds, \quad \forall q \in M^{k-1}(e) \right\}
\]

on each element \( T \)

\[
\mu^{k-2}_T(v_h) = \left\{ \frac{1}{|T|} \int_{T} v_h q, \quad \forall q \in M^{k-2}(T) \right\}
\]

The set of functional defined in (15) and (16) are unisolvent for the space \( V^k_h(T) \).
Lemma 1.1. Let \( T \) be a simple polygon/polyhedra with \( n \) edges/faces, and let \( V_h^k(T) \) be the space defined in (14) for any integer \( k \geq 1 \). The degrees of freedom (15) and (16) are unisolvent for \( V_h^k(T) \).

Proof. See the details in [13]. \( \square \)

The degrees of freedom equation (15) and (16) are defined using the monomials in \( \mu^{k-1} \) and \( \mu^{k-2} \) as basis functions for the polynomial spaces \( \mathbb{P}^{k-1}(e) \) and \( \mathbb{P}^{k-2}(T) \). This special choice of the basis functions gives advantages to implement the nonconforming VEM on arbitrary polygonal domain. Implementation part is described explicitly in articles [1, 11].

1.7. Global nonconforming virtual element space. We now introduce the nonconforming (global) virtual element space \( V_h^k \) of order \( k \). We have already defined local nonconforming virtual element space \( V_h^k(T) \) on each element \( T \) of partition \( \tau_h \). The global nonconforming virtual element space \( V_h^k \) of order \( k \) is defined by

\[
V_h^k = \{ v_h \in H^{1,nc}(\tau_h; k) : v_h|_T \in V_h^k(T) \quad \forall T \in \tau_h \}
\]

1.8. Interpolation error. We can define an interpolation operator in \( V_h^k \) having optimal approximation properties using same idea as described in [6, 7, 8, 13]. We can define an operator \( \chi_i \) which associates each function \( \phi \) to the \( i^{th} \) degree of freedom and virtual basis functions \( \psi_i \) of global virtual element space satisfies the condition \( \chi_i(\psi_i) = \delta_{ij} \) for \( i,j = 1,2,\ldots,N \) where \( N \) denotes the number of degrees of freedom of global space. Then for any \( v \in H^{1,nc}(\tau_h, k) \), there exists unique \( v_I \in V_h^k \) such that

\[
\chi_i(v - v_I) = 0 \quad \forall i = 1,2,\ldots,N
\]

Using all these properties we can claim that there exists a constant \( C > 0 \), independent of \( h \) such that for every \( h > 0 \), every \( K \in \tau_h \), every \( s \) with \( 2 \leq s \leq k+1 \) and every \( v \in H^s(K) \) the interpolant \( v_I \in V_h^k \) satisfies:

\[
\|v - v_I\|_{0,T} + h_T\|v - v_I\|_{1,K} \leq Ch^s_T\|v\|_{s,T}
\]

Technical detail of the above approximation is described in [13].

2. Construction of \( A_h \)

The goal of this section is to define the nonconforming virtual element discretization [13]. If the discretized bilinear form is symmetric then we can easily prove the good stability and nice approximation property and ensure the computability of the defined bilinear form \( A_h(\cdot, \cdot) \) over functions in \( V_h^k \). But since the bilinear form of the convection diffusion reaction equation is not symmetric we will accept certain assumptions on the model problem [1]. Diffusive and reactive part are symmetric and therefore we will split these two terms as a sum of polynomial part or consistency part and stability part. Convective part is not symmetric therefore we will take only polynomial approximation of this part. The present framework is only applicable to the diffusion-dominated case when the Peclet number is sufficiently small.

\[
A_h(u_h, v_h) = \sum_{T \in \tau_h} A^T_h(u_h, v_h) \quad \forall u_h, v_h \in V_h^k,
\]
where \( A^T_h : V_h^k \times V_h^k \to \mathbb{R} \) denoting the restriction to the local space \( V_h^k(T) \). The bilinear form \( A^T_h \) can be decomposed into sum of element terms. Thus defining the approximate bilinear form

\[
A^T_h(u_h, v_h) := a^T_h(u_h, v_h) + b^T_h(u_h, v_h) + c^T_h(u_h, v_h)
\]

for each element \( T \), we define the element contributions to \( A^T_h \) by

\[
a^T_h(u_h, v_h) := \int_T \mathbf{K} \Pi_{k-1}(\nabla u_h) \cdot \Pi_{k-1}(\nabla v_h) \, dT \\
+ S^T_a ((I - \Pi_k)u_h, (I - \Pi_k)v_h)
\]

\[
b^T_h(u_h, v_h) := \frac{1}{2} \left( \int_T \beta \Pi_{k-1}(\nabla u_h) \Pi_k(v_h) \, dT \\
- \int_T \beta \cdot \Pi_{k-1}(\nabla v_h) \Pi_k(u_h) \, dT \\
- \int_T (\nabla \cdot \beta) \Pi_k(u_h) \Pi_k(v_h) \, dT \right)
\]

\[
c^T_h(u_h, v_h) := \int_T c \Pi_k(u_h) \Pi_k(v_h) \, dT \\
+ S^T_c ((I - \Pi_k)u_h, (I - \Pi_k)v_h)
\]

where \( S^T_a \) and \( S^T_c \) are the stabilising terms. These terms are symmetric and positive definite on the quotient space \( V_h^k(T)/\mathbb{P}_k(T) \) and satisfy the stability property:

\[
\alpha_a a^T(v_h, v_h) \leq S^T_a(v_h, v_h) \leq \alpha^* a^T(v_h, v_h), \\
\gamma_c c^T(v_h, v_h) \leq S^T_c(v_h, v_h) \leq \gamma^* c^T(v_h, v_h),
\]

for all \( v_h \in V_h^k \) with \( \Pi_k(v_h) = 0 \). The first term ensures polynomial consistency property and second term ensures stability property of the corresponding bilinear form \( a^T_h(u_h, v_h) \) and \( c^T_h(u_h, v_h) \).

2.1. Consistency.

**Lemma 2.1.** Let \( u_h|_T \in \mathbb{P}_k(T) \) and \( v_h|_T \in H^2(T) \), then the bilinear forms \( a^T_h, b^T_h, c^T_h \) defined in equation (21) satisfy the following consistency property for all \( h > 0 \) and for all \( T \in \mathcal{T}_h \).

**Proof.** Whenever either \( u_h \) or \( v_h \) or both are elements of the polynomial space \( \mathbb{P}_k(T) \), the following consistency property satisfy

\[
a^T_h(u_h, v_h) = a^T(u_h, v_h) \\
b^T_h(u_h, v_h) = b^T(u_h, v_h) \\
c^T_h(u_h, v_h) = c^T(u_h, v_h)
\]

The consistency property in (22) follows immediately since \( \Pi_k(\mathbb{P}_k(T)) = \mathbb{P}_k(T) \) which implies \( S^T_a (p - \Pi_k(p), v_h - \Pi_k(v_h)) = 0 \) and \( S^T_c (p - \Pi_k(p), v_h - \Pi_k(v_h)) = 0 \). Now we will prove \( a^T_h(p, v_h) = a^T(p, v_h), b^T_h(p, v_h) = b^T(p, v_h), c^T_h(p, v_h) = c^T(p, v_h) \) for all \( p \in \mathbb{P}_k(T) \) and for all \( v_h \in V_h^k(T) \).
\[ a_T^T(p, v_h) = \int_T K \nabla p \cdot \Pi_{k-1}(\nabla v_h) dT \]
\[ = \int_T (\Pi_{k-1}(\nabla v_h) - \nabla v_h) \cdot K \nabla p dT + \int_T \nabla v_h \cdot K \nabla p dT \]
\[ = \int_T K \nabla p \nabla v_h dT \]
\[ = a_T^T(p, v_h) \]  
(23)

\[ b_h^T(p, v_h) = \frac{1}{2} \left( \int_T \beta \cdot \nabla p \Pi_k(v_h) dT \right) \]
\[ - \int_T \beta \cdot \Pi_{k-1}(\nabla v_h) p dT - \int_T (\nabla \cdot \beta) p \Pi_k(v_h) dT \]
\[ = \int_T \beta \cdot \nabla p \Pi_k(v_h) dT \]
\[ = \int_T \beta \cdot \nabla p v_h \]  
(24)

\[ \int_T \beta \cdot \Pi_{k-1}(\nabla v_h) p = \int_T \Pi_{k-1}(\nabla v_h) \cdot \beta p \]
\[ + \int_T \beta \cdot \nabla v_h p \]
\[ \leq \| \beta \|_{\infty,T} \| p \|_{0,T} \| \nabla v_h - \Pi_{k-1}(\nabla v_h) \| + \int_T \beta \cdot \nabla v_h p \]
\[ \leq C \| \beta \|_{\infty,T} h_T|\nabla v_h|_{1,T} \| p \|_{0,T} \]
\[ + \int_T \beta \cdot \nabla v_h p \]  
(25)

\[ \approx \int_T \beta \cdot \nabla v_h p \quad \text{(for small values of } h_T) \]

\[ \int_T (\nabla \cdot \beta) p \Pi_k(v_h) dT = \int_T (\nabla \cdot \beta) (\Pi_k(v_h) - v_h) p + \int_T (\nabla \cdot \beta) p v_h \]
\[ = \int_T (\nabla \cdot \beta) p v_h \]  
(27)

Putting estimations (25), (26) and (27) in (24) we get \( b_h^T(p, v_h) = b^T(p, v_h) \)

Similarly,

\[ c_h^T(p, v_h) = c^T(p, v_h) \]  
(28)

Hence we proved required polynomial consistency of local discrete bilinear form \( A_T^T(u_h, v_h) \), i.e.

\[ A_h^T(p, v_h) = A^T(p, v_h) \]  
(29)
for \( p \in \mathbb{P}^k \) and \( v_h \in V_h^k \)

### 2.2. Discrete stability

Before discussing stability property of the discrete bilinear form \( A_h^T(u_h, v_h) \) we reveal that the following framework is applicable for diffusion dominated case. The stabilizing part \( \mathbf{S}_d^T \) and \( \mathbf{S}_e^T \) of the discrete bilinear form ensure the stability of the bilinear form, precisely we can conclude that there exist two pairs of positive constants \( \alpha_* \), \( \alpha^* \) and \( \gamma_* \), \( \gamma^* \) that are independent of \( h \) such that

\[
\alpha_* a^T(v_h, v_h) \leq a^T_h(v_h, v_h) \leq \alpha^* a^T(v_h, v_h)
\]

\[
\gamma_* c^T(v_h, v_h) \leq c^T_h(v_h, v_h) \leq \gamma^* c^T(v_h, v_h)
\]

for all \( v_h \in V_h^k(T) \) and mesh elements \( T \).

### 3. Construction of right hand side term

In order to build the the right hand side \( < f_h, v_h > \) for \( v_h \in V_h^k \) we need polynomial approximation of degree \((k-2) \geq 0 \), that is \( f_h = \mathbf{P}_{k-2}^T f \) on each \( T \in \tau_h \), where \( \mathbf{P}_{k-2}^T \) is \( L^2(T) \) projection operator on \( \mathbb{P}^{k-2}(T) \) for each element \( T \in \tau_h \). We define \( f_h \) locally by:

\[
(f_h)_T := \begin{cases} 
\mathbf{P}_{k}^T(f) & \text{for } k = 1 \\
\mathbf{P}_{k-2}^T(f) & \text{for } k \geq 2 
\end{cases}
\]

The projection operator is orthogonal to the polynomial space \( \mathbb{P}^k(T) \). Therefore we can write as

\[
<f_h, v_h> := \sum_T \int_T \mathbf{P}_{k-2}^T(f) v_h \, dT = \sum_T \int_T f P_{k-2}^T(v_h) \, dT
\]

Now we can prove the error estimates using orthogonality property of projection operator, Cauchy-Schwarz inequality and standard approximates \([8, 7]\) for \( k \geq 2 \), \( s \geq 1 \).

\[
|<f, v_h> - <f_h, v_h>| = \left| \sum_T \int_T (f - \mathbf{P}_{k-2}^T(f)) \, v_h \, dT \right|
\]

\[
\leq \left| \sum_T \int_T (f - \mathbf{P}_{k-2}^T(f))(v_h - \mathbf{P}_{k}^T(v_h)) \, dT \right|
\]

\[
\leq \| (f - \mathbf{P}_{k-2}^T(f)) \|_{0, \tau_h} \| (v_h - \mathbf{P}_{k}^T(v_h)) \|_{0, \tau_h}
\]

(33)

For \( k = 1 \) the above analysis is not applicable, so we do the following

\[
\bar{v}_h|_K := \frac{1}{n} \sum_{e \in \partial K} \frac{1}{|e|} \int_e v_h \, ds \approx \mathbf{P}_0^T(v_h),
\]

(34)

\[
<f_h, \bar{v}_h> := \sum_T \int_T \mathbf{P}_0^T(f) \, \bar{v}_h \approx \sum_T \int_T \mathbf{P}_0^T(f) \mathbf{P}_0^T(v_h)
\]

(35)
Then, there exists a constant $C > 0$ be the solution of the model problem (1). Let $\mathbf{H}$ finite element space.

**Lemma 3.1.**

\[(40)\]

The above estimation guides us to compute the boundary terms easily using degrees of freedom.

**3.2. Estimation of the jump term.**

**Lemma 3.1.** Let $g := P_{k-1}^e(g)$ where $g$ is non-homogeneous Dirichlet boundary value.

\[(37) \int g v_h \ ds := \sum_{e \in e_h^k} \int_{e} P_{k-1}^e(g) v_h \ ds = \sum_{e \in e_h^k} \int_{e} g P_{k-1}^e(v_h) \ ds \forall \ v_h \in V_h^k\]

The above estimation guides us to compute the boundary terms easily using degrees of freedom.

**3.2. Estimation of the jump term.**

\[(38) J_h(u, v_h) = \sum_{T \in \tau_h} \int_{\partial T} \frac{\partial u}{\partial \mathbf{n}_T} v_h \ ds = \sum_{e \in e_h} \int_{e} \nabla u \cdot [v_h] \]

\[|< f, v_h > - < f_h, \tilde{v}_h >| = \left| \int_T (f v_h - P_0^T f \tilde{v}_h) \right| \]

\[\leq \sum_{T} \int_T (f - P_0^T f) v_h + \int_{K} P_0^T f (v_h - \tilde{v}_h) \]

\[\leq \sum_{T} \int_T (f - P_0^T f) v_h \]

\[\leq C h \|f\|_{0,h} \|v_h\|_{1,h} \]

\[|\mathbf{J}_h(u, v_h)| \leq Ch^{|\min(s, k)} \|u\|_{s+1, \Omega} \|v_h\|_{1,h} \]

**Proof.** Let $v_h \in H^{1,nc}(\tau_h; k)$ be an arbitrary element. From the definition of the finite element space $H^{1,nc}(\tau_h; k)$, we can say $v_h$ satisfies patch test of order $k$. Hence the following equality holds

\[\int_{e} [v_h] q \ ds = 0, \ \forall q \in P_{k-1} \]

Let $P_{k}^e : L^2(e) \rightarrow P_{k}^e(e)$ is the $L^2$-orthogonal projection operator onto the space $P_{k}^e(e)$ for $k \geq 1$. Using patch test of order $k$, Cauchy-Schwarz inequality and $L^2(e)$ orthogonal projection operator $P_{k}^e$ we can find

\[J_h(u, v_h) = \sum_{T \in \tau_h} \int_{\partial T} \frac{\partial u}{\partial \mathbf{n}_T} v_h \ ds \]

\[= \sum_{e \in e_h} \int_{e} \nabla u \cdot [v_h] \]

\[= \sum_{e \in e_h} \int_{e} (\nabla u - P_{k-1}^e(\nabla u)) \cdot [v_h] \ ds \]

\[= \sum_{e \in e_h} \int_{e} (\nabla u - P_{k-1}^e(\nabla u)) \cdot ([v_h] - P_0^e([v_h])) \]

\[\leq \sum_{e \in e_h} \| \nabla u - P_{k-1}^e(\nabla u) \|_{0,e} \| [v_h] - P_0^e([v_h]) \|_{0,e} \]
using standard polynomial approximation on edge $e$

\begin{equation}
\| \nabla u - P_{k-1}^e (\nabla u) \|_{0,e} \leq Ch^{\text{min}(m,k)-\frac{1}{2}} \| u \|_{m+1,T}
\end{equation}

\begin{equation}
\| ||v_h|| - P_0^m(||v_h||) \|_{0,e} \leq Ch^{\frac{1}{2}} ||v_h||_{1,T}
\end{equation}
we can easily bound the above two terms. Hence, putting the estimation equation (42), (43) in (41) we obtain required result. \hfill \Box

4. Well-posedness of nonconforming-virtual element methods

In this section we will discuss the well-posedness of nonconforming-virtual element method. Let the assumption (A1,A2,A3), polynomial consistency, stability defined in equation (30). (31) holds then the bilinear form $A_h$ is coercive with respect to broken-norm $\| \cdot \|_{1,h}$, i.e.

\begin{equation}
A_h(v_h, v_h) \geq \alpha \| v_h \|^2_{1,h} \quad \forall v_h \in V_h
\end{equation}

where $\alpha$ is a positive constant.

Using the stability properties equation (30), (31) for diffusion and reaction parts of discrete bilinear form we can bound

\begin{equation}
A_h(v_h, v_h) \geq \alpha_s \alpha^T (v_h, v_h) + b_h^T (v_h, v_h) + \gamma_s c^T (v_h, v_h)
\end{equation}

\begin{equation}
\geq \alpha_s \eta \| v_h \|^2_{1,T} + b_h^T (v_h, v_h) + \gamma_s c^T (v_h, v_h) + [b_h^T (v_h, v_h) - b^T (v_h, v_h)]
\end{equation}

\begin{equation}
\geq \alpha_s \eta \| v_h \|^2_{1,T} + \min(1, \gamma_s) (|b_h|_{0,t}) + c^T (v_h, v_h) \geq \alpha_s \eta \| v_h \|^2_{1,T} + \min(1, \gamma_s) c_0 \| v_h \|^2_{0,T}
\end{equation}

\begin{equation}
\geq \alpha_s \eta \| v_h \|^2_{1,T} + \min(1, \gamma_s) c_0 \| v_h \|^2_{0,T}
\end{equation}

\begin{equation}
\geq \alpha_s \eta \| v_h \|^2_{1,T} - \min(1, \gamma_s) c_0 \| v_h \|^2_{0,T}
\end{equation}

\begin{equation}
\geq \alpha_s \eta \| v_h \|^2_{1,T} - \min(1, \gamma_s) c_0 \| v_h \|^2_{0,T}
\end{equation}

\begin{equation}
\geq \alpha_s \eta \| v_h \|^2_{1,T} - |b_h^T (v_h, v_h) - b^T (v_h, v_h)| \quad \forall v_h \in V_h^k(T)
\end{equation}

\begin{equation}
|b_h^T (v_h, v_h) - b^T (v_h, v_h)| = \frac{1}{2} \left| \int_T (\nabla \cdot \beta)(\Pi_k(v_h))^2 dT - \int_T (\nabla \cdot \beta)v_h^2 dT \right|
\end{equation}

\begin{equation}
\leq C \| \nabla \cdot \beta \|_{\infty,T} \| v_h \|_{0,T} \| v_h - \Pi_k(v_h) \|_{0,T}
\end{equation}

Since $\| v_h \|_{0,T} \leq \| v_h \|_{1,T}$ and $|v_h|_{1,T} \leq \| v_h \|_{1,T}$ inequalities hold [7] we can estimate

\begin{equation}
|b_h^T (v_h, v_h) - b^T (v_h, v_h)| \geq - C \| \nabla \cdot \beta \|_{\infty,T} h_T \| v_h \|_{1,T}
\end{equation}
Therefore
\[ A_h(v_h, v_h) = \sum_T A_h^T(v_h, v_h) \]
\[ \geq \sum_T \alpha_s \eta |v_h|_{1,T}^2 + \sum_T \min(1, \gamma_s) c_0 \|v_h\|_{0,T}^2 \]
\[ - \sum_T C \|\nabla \cdot \beta\|_{\infty,T} h_T \|v_h\|_{1,T} \]
\[ \geq \sum_T \alpha_T \|v_h\|_1^2 \]
\[ \geq \alpha \sum_T \|v_h\|_1^2 \]
\[ = \alpha \|v_h\|_{1,h} \]
where \( \alpha = \min(\alpha_T) \), and
\[ \alpha_T = \min\{\alpha_s \eta - C \|\nabla \beta\|_{\infty} h_T, \min(1, \gamma_s)c_0 - C \|\nabla \beta\|_{\infty} h_T\} \]

4.1. Continuity of discrete bilinear form.

**Lemma 4.1.** Under the assumption of the polynomial consistency and stability along with the coefficients \( K, \beta, c \) the bilinear form \( A_h \) defined in equation (20) is continuous.

**Proof.** Diffusive part \( a_h^T(u_h, v_h) \) and reactive part \( c_h^T(u_h, v_h) \) of the bilinear form \( A^T(u_h, v_h) \) are symmetric and hence they can be viewed as inner product in VE space \( V_h^T \) over each element \( T \). Convective part \( b_h^T(u_h, v_h) \) is not symmetric and hence we cannot bound it like diffusive part and reactive part, but using properties of the projection operator we can simply bound it. Hence we conclude

\[ a_h^T(u_h, v_h) \leq (a_h^T(u_h, u_h))\frac{1}{2} (a_h^T(v_h, v_h))\frac{1}{2} \]
\[ \leq \alpha^* (a^T(u_h, u_h))\frac{1}{2} (a^T(v_h, v_h))\frac{1}{2} \]
\[ \leq \alpha^* \|K\|_{\infty} \|\nabla u_h\|_{0,T} \|\nabla v_h\|_{0,T} \]

(48)

similarly

\[ c_h^T(u_h, v_h) \leq \gamma^* |c|_{\infty} \|u_h\|_{0,T} \|v_h\|_{0,T} \]

(49)

\[ b_h^T(u_h, v_h) = \frac{1}{2} \left( \int_T \beta \Pi_{k-1}(\nabla u_h) \Pi_k(v_h) \right.) \]
\[ \int_T \beta \cdot \Pi_{k-1}(\nabla v_h) \Pi_k(u_h) \]
\[ - \int_T (\nabla \cdot \beta) \Pi_k(u_h) \Pi_k(v_h) \]

Again

(50)

\[ \int_T \beta \Pi_{k-1}(\nabla u_h) \Pi_k(v_h) \leq C \|\beta\|_{\infty} \|\nabla u_h\|_{0,T} \|v_h\|_{0,T} \]

(51)

\[ \int_T \beta \cdot \Pi_{k-1}(\nabla v_h) \Pi_k(u_h) \leq C \|\beta\|_{\infty} \|\nabla v_h\|_{0,T} \|u_h\|_{0,T} \]

(52)

\[ \int_T (\nabla \cdot \beta) \Pi_k(u_h) \Pi_k(v_h) \leq C \|\beta\|_{1,\infty} \|u_h\|_{0,T} \|v_h\|_{0,T} \]
Thus,
\begin{align*}
A_T(u_h, v_h) &= a_T(u_h, v_h) + b_T(u_h, v_h) + c_T(u_h, v_h) \\
&\leq \alpha \|K\|_\infty \|\nabla u_h\|_{0,T} \|\nabla v_h\|_{0,T} + C\|\beta\|_{1,\infty} \|u_h\|_{1,T} \|v_h\|_{0,T} + \gamma \|c\|_\infty \|u_h\|_{0,T} \|v_h\|_{0,T} \\
&\leq C_T \|v_h\|_{1,T} \|u_h\|_{1,T} \\
(53)
\end{align*}

Therefore the bilinear form is continuous. □

The bilinear form $A_T$ is *discrete coercive* and *bounded or continuous* in discrete norm $\| \cdot \|_{1,h}$ on nonconforming virtual element space $V_h^k$, defined in equation (17). Hence the bilinear form has unique solution in $V_h^k$ by Banach Necas Babuska (BNB) theorem [14].

5. Convergence analysis and apriori error analysis in $\| \cdot \|_{1,h}$ norm

In this section we reveal nonconforming convergence analysis of discrete solution $u_h \in V_h^k$ which satisfy the discrete bilinear form equation (13). The basic idea is same as non conforming error analysis of convection diffusion problem proposed by Tobiska et al [19, 20]. It is well known that non conforming virtual element space $V_h^k \subset H^{1,nc}(\tau_h;k) \not\subset H^1(\Omega)$ and introduce consistency error. The finite element solution $v_h \in V_h^k$ is not continuous along interior edge $e$ except certain points which implies an additional jump term $J(u, v_h)$ defined in equation (38). Nonconforming virtual element space $V_h^k$ satisfy ‘patch-test’ [18] of order $k$. Using this property we can easily bound the jump term.

**Theorem 5.1.** Let $u$ be the exact solution problem (2) with polynomial coefficients $K, \beta, c$. Let $u_h \in V_h^k$ be the solution of the non conforming virtual element approximation (13). Let $f \in L^2(\Omega)$ and $u \in H^{k+1}(\Omega)$ $(k \geq 1)$ then

\begin{align*}
\|u - u_h\|_{1,h} \leq C h^k \|u\|_{k+1,h} + C h |f|_{0,h} \\
(55)
\end{align*}

where $\| \cdot \|_{1,h}$ denote broken norm in the space $H^{1,NC}(\tau_h,k)$.

**Proof.** We consider $u_I$ be the approximation of $u$ in $V_h^k$ and $u_{II}$ be polynomial approximation of $u$ in $P^k(\tau_h)$. Define $\delta := u_h - u_I$. using coercivity of the virtual element form, we can write
\[
\alpha \|\delta\|^2 \leq A_h(\delta, \delta) = A_h(u_h, \delta) - A_h(u_I, \delta) = <f_h, \delta> - A_h(u_I, \delta)
\]

(56)

Now we shall analyse the local bilinear form \(A_T^h(u_I, \delta)\) term by term

\[
A_T^h(u_I, \delta) = A_T^h(u_I - u_{II}, \delta) + A_T^h(u_{II}, \delta)
\]

(57)

Discrete bilinear form \(A_T^h(u_h, v_h)\) is polynomial consistent, hence \(A_T^h(u_{II}, \delta) = A^T(u_{II}, \delta)\)

Therefore

\[
A_T^h(u_I, \delta) = A_T^h(u_I - u_{II}, \delta) + A^T(u_{II} - u, \delta) + A^T(u, \delta)
\]

(58)

Now using Green’s theorem on the triangle \(T\), we get

\[
\int_T (\beta \cdot \nabla u) \delta dT = \frac{1}{2} \left( \int_T (\beta \cdot \nabla u) \delta dT - \int_T (\beta \cdot \nabla \delta) u dT \right) - \int_T (\nabla \cdot \beta) u \delta dT + \int_{\partial T} (\beta \cdot n) u \delta ds
\]

(59)

Rearranging terms we can write

\[
\frac{1}{2} \left( \int_T (\beta \cdot \nabla u) \delta dT - \int_T (\beta \cdot \nabla \delta) u dT - \int_T (\nabla \cdot \beta) u \delta dT \right) = \int_T (\beta \cdot \nabla u) \delta dT - \frac{1}{2} \int_{\partial T} (\beta \cdot n_{\partial T}) u \delta ds
\]

(60)

Taking sum over all element \(T \in \tau_h\) we get

\[
\sum_T A^T(u, \delta) = \sum_T (-\nabla(K \nabla u) + (\beta \cdot \nabla u) + cu) \delta
\]

(61)

In the above equation we get jump term corresponding to \(\delta\) only, since \(\delta\) is discontinuous along interior edge \(e \in \varepsilon^0_h\).

Globally the equation (58) can be written as
\[ A_h(u_I, \delta) = \sum_T A_h^T(u_I, \delta) \]
\[ = \sum_T A_h^T(u_I - u_{II}, \delta) + \sum_T A^T(u_{II} - u, \delta) \]
\[ + \langle f, \delta \rangle + \sum_{e \in \mathcal{E}_h} \int_e (K \nabla u \cdot n_e) [||\delta||] ds - \frac{1}{2} \sum_{e \in \mathcal{E}_h} \int_e (\beta \cdot n_e) u [||\delta||] \]

Let us denote
\[ M_1 = A_h^T(u_I - u_{II}, \delta) \]
\[ M_2 = A^T(u_{II} - u, \delta) \]

\[ |A_h^T(u_I - u_{II}, \delta)| \leq C \|u_I - u_{II}\|_{1,T} \|\delta\|_{1,T} \]
\[ |A^T(u_{II} - u, \delta)| \leq C \|u_{II} - u\|_{1,T} \|\delta\|_{1,T} \]

using (63), (64) and interpolation error estimation (13) we can bound

\[ |M_1| + |M_2| \leq C \|\delta\|_{1,T} (||u_I - u_{II}||_{1,T} + ||u_{II} - u||_{1,T}) \]
\[ \leq C \|\delta\|_{1,T} h^k \|u\|_{k+1,T} \]

we use equation (63) for \( k \geq 2 \) and (36) for \( k = 1 \) to bound the right hand side

\[ |<f_h, \delta> - <f, \delta>| = |\sum_T \int_T (f_h - f) \delta| \]
\[ \leq Ch^k |f|_{k-1,T_h} \|\delta\|_{1,h} \]

In particular for \( k = 1 \)

\[ |<f_h, \delta> - <f, \delta>| \leq C h |f|_{0,T_h} \|\delta\|_{1,h} \]

using estimation (39) we bound consistency error

\[ \left| \int_e (K \nabla u \cdot n_e)[||\delta||] \right| \leq ||K||_{\infty} \int_e (\nabla u \cdot n_e)[||\delta||] \]
\[ \leq Ch^k \|u\|_{k+1,T_T^+} \|\delta\|_{1,T_T^+} \|\delta\|_{1,T_T^-} \]

Edge \( e \) is an interior edge shared by triangles \( T^+ \) and \( T^- \). Therefore

\[ \left| \sum_{e \in \mathcal{E}_h} \int_e (K \nabla u \cdot n_e)[||\delta||] \right| \leq C h^k \|u\|_{k+1,h} \|\delta\|_{1,h} \]
again
\[
\left| \sum_{e \in e_h} \int_e (\mathbf{\beta} \cdot \mathbf{n}_e) u \| \delta \| \right| \leq \sum_{e \in e_h} \| \mathbf{\beta} \cdot \mathbf{n}_e \|_\infty \int_e u \| \delta \|
\]
\[
\leq C \sum_{e \in e_h} \int_e (u - P^{k-1}(u)) \| \delta \|
\]
\[
= C \sum_{e \in e_h} \int_e (u - P^{k-1}(u)) (|\delta| - P^0(|\delta|))
\]
\[
\leq C \sum_{e \in e_h} \| u - P^{k-1}(u) \|_{0,e} \| |\delta| - P^0(|\delta|) \|_{0,e}
\]
using standard approximation [2]
\[
\| u - P^{k-1}(u) \|_{0,e} \leq Ch^{\min(k,s)-1/2} \| u \|_{s,T+\cup T^{-}}
\]
\[
\| |\delta| - P^0(|\delta|) \|_{0,e} \leq Ch^{1/2} \| |\delta| \|_{1,T+\cup T^{-}}
\]
we can bound
\[
\left| \sum_{e \in e_h} \int_e (\mathbf{\beta} \cdot \mathbf{n}_e) u \| |\delta| \| \right| \leq Ch^{\min(k,s)} \sum_{T} \| u \|_{s,T} \| |\delta| \|_{1,T}
\]
\[
\leq Ch^{\min(k,s)} \| u \|_{s+1,h} \| |\delta| \|_{1,h}
\]
In particular for \( s = k \)
\[
(68) \quad \left| \sum_{e \in e_h} \int_e (\mathbf{\beta} \cdot \mathbf{n}_e) u \| |\delta| \| \right| \leq Ch^k \| u \|_{k+1,h} \| |\delta| \|_{1,h}
\]
Using (65), (66), (67), (68) we bound
\[
|\delta|_{1,h} \leq (Ch^k \| u \|_{k+1,h} + C \| f \|_{0,h}) |\delta|_{1,h}
\]
(69)
\[
|\delta|_{1,h} \leq Ch^k \| u \|_{k+1,h} + C \| f \|_{0,h}
\]
We can write
\[
(70) \quad \| u - u_h \|_{1,h} \leq \| (u - u^I) \|_{1,h} + \| (u^I - u_h) \|_{1,h}
\]
The first term can be estimated using the standard approximation (15) and second term can be estimated using (69). Hence we obtain
\[
(71) \quad \| u - u_h \|_{1,h} \leq Ch^k \| u \|_{k+1,h} + C \| f \|_{0,h}
\]
\[
\square
\]
6. Conclusions
In this work we presented the analysis of nonconforming virtual element method for convection diffusion reaction equation with polynomial coefficients using \( L^2 \) projection. \( L^2 \) projection can be partially computed using degrees of freedom of the finite element. The external virtual element method using \( L^2 \) projection operator is not fully computable which may be considered as a drawback of this method. We have proved stability of the method assuming that the model problem is diffusion dominated. If the model problem is convection dominated then the present analysis is not applicable and hence we require a new framework for convection dominated problem, which may be carried out as a future work.
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