Quantum Residual Correlation: Interpreting through State Merging

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In this brief report we revisit the concept of “quantum dissension”, which was introduced as a natural extension of quantum discord for three qubit system. Here we bring in new expression for quantum dissensions and more interestingly we name one such expression as residual correlation. The basic objective behind the introduction of such a quantity is to capture the extra amount of correlation generated by doing measurement in a correlated system from a situation where we do not bring in a correlated system in the measurement process. Apart from this we also present an operational interpretation of this correlation in context of state merging. Remarkably, we find that the residual correlation (merging the quantum information of two parties into one) is captured by the residual correlation. In addition to that we found that this quantity can be negative for mixed states. This indeed opens up a new dimension in the tripartite scenario where we can observe situations where there is a decrease in the cost of state merging on discarding relevant prior information. We claim that this result establishes a reconceptualization of information processing tasks in tripartite situations where we can use suitable measurement and states to bring down the cost of the protocol.

PACS numbers:

INTRODUCTION

For a long time studying quantum entanglement was of only foundational interest and the researchers were mainly addressing the questions that were related with the quantum mechanical understanding of various fundamental notions like reality and locality [1,2]. However, for the last two decades we had seen that quantum entanglement is not only a foundational question but also a reality as far as the laboratory preparation of entangled qubits are concerned [3]. The researches that were conducted during these decades were not all concerned about its existence but mostly about its usefulness as a resource to carry out information processing protocols like quantum teleportation [4], cryptography [5], superdense coding [6], remote state preparation [7], quantum state merging [8] etc.

Quantifying the amount of quantum correlation present in a pure bipartite system is straightforward. This is quantified by the amount of entanglement present in the system. There are certain standard measures, more specifically entanglement monotones like negativity [9] and concurrence [10,11], quantifying the amount of entanglement for both pure as well as for mixed states. However, there are certain open issues in understanding the nature of correlations present in a mixed state, or a multiqubit state. In addition to that it is also not totally evident whether all the information-processing tasks that can be done more efficiently with quantum systems require entanglement as resource. There are several instances where even in the absence or near absence of entanglement one can carry out some information-processing tasks more efficiently [12]. Then, the natural question arises if not entanglement then what is responsible for such a behavior. Researchers argued that the amount of correlation present in the composite system is not entirely captured by the entanglement. They came up with different quantities for capturing the quantum correlation and giving a meaningful explanation of such behavior [13,14]. One such measure based on information theoretic approach to quantify correlation is quantum discord [13,14]. This aims to capture the non classical correlations present in a system, and more specifically that part of the correlation that cannot be witnessed with the standard measures of entanglement. These new measures are also extended for many qubit systems by adopting different approaches [15,16].

In addition to these standard approaches of quantifying the correlation in quantum mechanical systems we have number of geometric approaches [17] to quantify it.

An important information processing tasks is quantum state merging [8]. In normal classical scenario when we talk about state merging we refer to a situation, where we have two parties Alice and Bob like most of two party communication protocols. Bob has some prior information Y and the other party Alice has some missing information X (where X and Y are random variables). Now here one can ask a very simple question that if Bob wants to learn about X, how much additional information Alice needs to send him. It had been seen that only $H(X|Y)$ bits of information will suffice [18]. This quantity is known as the conditional entropy of X.
given that $Y$ has occurred. This partial information that Alice needs to send to Bob is always a positive quantity as also sending negative information classically doesn’t make sense. In the context of quantum information, Alice and Bob each possess a system in some unknown quantum state with the total density operator being $\rho_{AB}$ and each party having states with density operators $\rho_A$ and $\rho_B$ respectively. The interesting case is where Bob is correlated with Alice, so that he has some prior information about her state. We now ask how much additional quantum information Alice needs to send him, so that he has the full state (with density operator $\rho_{AB}$). The amount of partial quantum information that Alice needs to send Bob is given by the quantum conditional entropy, which is exactly the same quantity as in the classical case but with the Shannon entropy changed to the von Neumann entropy: $S(A|B) = S(AB) - S(B)$. Another elegant way of visualizing this state as a purification of a state in a higher dimensional Hilbert space. The notion of state-merging was extended to n-parties by [19]. In another work authors have given an operational interpretation of quantum discord based on the quantum state merging protocol [20]. Quantum discord is shown as the markup in the cost of quantum communication in the process of quantum state merging, if one discards relevant prior information.

In this work, first of all, we revisit the notion of “quantum dissension” which happens to be a multipartite extension of quantum discord. We introduce new expressions of quantum dissension and have named one of them as residual correlation. The basic motivation of introducing this quantity is to quantify the amount of extra correlation generated by doing measurement in a correlated system from a situation where we do not bring in extra correlated system in the measurement process. In other words, this quantifies the residual leftover correlation generated of a two particle measurement which is not captured by single particle measurement. Secondly, in this work we also give an operational interpretation of residual correlation. Remarkably, we find that the change in the cost of state merging by discarding relevant prior information is given by this residual correlation. Not only that for mixed states we found this residual correlation as well as the change in the state merging cost can take negative values. This is indeed a remarkable feature because of its counter intuitive nature as compared to a bipartite situation. Being negative signifies the decrease in the cost of state merging as a result of measurement. We believe that this phenomenon of discarding relevant information can be used in bringing down the cost of state merging process in a quantum network (QNET).

The organization of our work is as follows. In the second section, we introduce new forms of quantum dissension for three qubit system in line with the definitions given in reference [15]. In particular, we introduce a new measure called residual correlation. In third section we analytically show that the change in the cost of state merging is given by the residual correlation.

**QUANTUM CORRELATION: BEYOND ENTANGLEMENT**

In this section we introduce new expression for quantum dissensions other than what have been proposed in reference [15]. More, specifically we introduce one such expression, which we refer as the residual quantum correlation. The key is to quantify the difference in the amount of correlation generated by bringing in a correlated system in the measurement process.

The notion of quantum discord [13] has been generalised to three parties from different perspectives [15, 16]. One such possible way of extending depending on the nature of projective measurement is given by the quantity called “Quantum Dissension” [15]. Quantum dissensions are obtained by taking the difference of different expressions for mutual information. These are not equivalent in quantum domain. The difference of these three definitions can capture various aspects of the quantum correlations. Let us consider a three-qubit state $\rho_{ABC}$ where $A, B$ and $C$ refer to first, second and third qubit respectively. The initial quantum mutual information $I_0(A:B:C)$ when no measurement is carried out is given by

$$I_0(A:B:C) = S(\rho_A) + S(\rho_B) + S(\rho_C)$$

$$- [S(\rho_{AB}) + S(\rho_{BC}) + S(\rho_{AC})] + S(\rho_{ABC}) \quad (1)$$

The final quantum mutual information $I_1(A:B:C)$, obtained by carrying out 1-particle projective measurement in the qubit $B$ is given by,

$$I_1(A:B:C) = S(\rho_A) + S(\rho_C) - S(\rho_{A|B}^\rho)$$

$$- S(\rho_{AC}) - S(\rho_{BC}) + S(\rho_{ABC}) \quad (2)$$

Here $S(\rho_{A|B}^\rho) = \sum_j p_j S(\rho_{A|\pi_j}^B)$, $\rho_{A|\pi_j}^B = \frac{\pi_j^B \rho_{AB} \pi_j^B}{\text{Tr}(\pi_j^B \rho_{AB})}$ and $p_j$ is the probability of obtaining the $j$th outcome. Here, $S(\rho_{A|B}^\rho)$ is the average Von Neumann entropy of the qubit $A$, when the projective measurement is done on the subsystem $B$ in the general basis $\{|u_1\rangle = \cos(t)|0\rangle + e^{i\phi} \sin(t)|1\rangle, |u_2\rangle = \sin(t)|0\rangle - e^{i\phi} \cos(t)|1\rangle\}$ (where $t, \phi \in [0,2\pi]$). Similarly, the final quantum mutual information $I_2(A:B:C)$, obtained after bringing in the correlated system $A$ with $B$ and then carrying out the
measurement, is given by,

\[ I_2(A : B : C) = [S(\rho_A) + S(\rho_B) + S(\rho_C)] - [S(\rho_{CB}) + S(\rho_{CA}) + S(\rho_{C|AB})] \tag{3} \]

The two-particle projective measurement ought to be performed in the general basis: \( |v_1\rangle = \cos \theta |00\rangle + e^{i\phi} \sin \theta |11\rangle \), \( |v_2\rangle = \sin \theta |00\rangle - e^{i\phi} \cos \theta |11\rangle \), \( |v_3\rangle = \cos \theta |01\rangle + e^{i\phi} \sin \theta |10\rangle \), \( |v_4\rangle = \sin \theta |01\rangle - e^{i\phi} \cos \theta |10\rangle \), where \( \theta, \phi \in [0, 2\pi] \). In this case, the quantum conditional entropy is given as: \( S(\rho_{C|AB}) = \sum_j p_j S(\rho_{C|AB}^{i\phi}) \) where \( p_j \) is the probability of \( j \) th outcome and \( S(\rho_{C|AB}^{i\phi}) \) is the average Von Neumann entropy of the qubit \( C \), when the projective measurement is done on qubits \( AB \). It is interesting to note that since in principal there can be various expressions of mutual informations classically, consequently we will have various definitions of quantum dissension. All these definitions individually captures various aspect of quantum correlation in multi-party scenario. They together establish the fact that quantum correlation is indeed a vector like quantity and each measure represents some component of it.

We give the first definition of quantum dissension as the difference of \( I_1(A : B : C) \) and \( I_0(A : B : C) \), i.e.

\[ D_{1C}(A : B : C) = I_1(A : B : C) - I_0(A : B : C) = S(\rho_{AB}) - S(\rho_B) - S(\rho_{AB|C}) \tag{4} \]

The above expression must be minimized above all one-particle measurement projects in order to reveal the maximum possible quantum correlations. This is given by, \( D_{1C} = \min(D_{1C}(A : B : C)) \). This expression \( D_{1C} \) gives the amount of quantum correlation generated as a result of carrying out a projective measurement on \( B \). Similarly, the second definition of quantum dissension is given by the difference, i.e.

\[ D_{2C}(A : B : C) = I_2(A : B : C) - I_0(A : B : C) = S(\rho_{C|AB}) + S(\rho_{AB}) - S(\rho_{ABC}) \]

One can minimize \( D_{2C} \) over all two-particle measurement projectors and obtain \( D_{2C} = \min(D_{2C}(A : B : C)) \). This quantity in principle gives the amount of quantum correlation generated by bringing in a correlated system \( A \) with \( B \) and then carrying out projective measurement. The third definition of quantum dissension is given by the difference of \( I_2(A : B : C) \) and \( I_1(A : B : C) \), i.e.

\[ D_{RC}(A : B : C) = I_2(A : B : C) - I_1(A : B : C) = S(\rho_{C|AB}) + S(\rho_{AB}) - S(\rho_{ABC}) - S(\rho_{AB}) + S(\rho_B) + S(\rho_{AB|C}) \tag{6} \]

The above expression must be minimized above all one-particle measurement projects in order to reveal the maximum possible quantum correlations. Mathematically, this is \( D_{RC} = \min(D_{RC}(A : B : C)) \).

**Lemma 1:** For a three qubit state \( \rho_{ABC} \), we have \( D_{2C} - D_{1C} = D_{RC} \)

**Proof:** From the definition itself, \( D_{RC} = \min(D_{RC}(A : B : C)) = \min(I_2(A : B : C) - I_1(A : B : C)) = \min(I_2(A : B : C) - I_0(A : B : C)) = \min(I_2(A : B : C) - I_1(A : B : C) - I_0(A : B : C)) = \min(D_{2C}(A : B : C) - D_{1C}(A : B : C)) = D_{2C} - D_{1C} \)

We call this quantity \( D_{RC} \) as a separate correlation measure and name it as Residual Correlation. The main purpose for this naming is that this correlation intends to capture the residual amount of correlation that is present between the qubits that has undergone the measurement process. This is given by the difference in the amount of correlation generated in a system \( \rho_{ABC} \) as a result of single particle measurement on a qubit \( B \) from the correlation generated in the system by attaching another party say \( A \) with fixed qubit \( B \).

**Lemma 2:** For arbitrary pure three qubit state \( \rho_{ABC} \), the residual correlation is given by \( D_{RC} = S(B) + \min(S(\rho_{A|B})). \)

**Proof:** For arbitrary pure three qubit state \( \rho_{ABC} \), \( S(\rho_{C|AB}) = 0 \). This is because after measurement the system is in a product state of the state \( A \) and the projected state of the \( BC \) subsystem, which is a pure state. There fore,

\[ D_{RC} = D_{2C} - D_{1C} = S(C|AB) - S(C|AB) - S(A|B) + S(A|B) = S(A|B) - S(A|B) + \min(S(\rho_{A|B})) \tag{7} \]

**Lemma 3:** For pure three qubit state of the form \( \rho_{ABC} = |\psi\rangle_B \otimes |\phi\rangle_A \otimes \rho_{C} \) (full separable pure state), the total residual correlation is zero.

**Proof:** Since we know that for arbitrary pure three qubit state \( \rho_{ABC} \) we have the total residual correlation as \( D_{RC} = S(B) + \min(S(\rho_{A|B})) \) (from Lemma 2). Since \( \rho_{A|B} = \frac{\pi_{j\rho_{AB}}}{\text{Tr}(\pi_{j\rho_{AB}})} \), \( S(\rho_{A|B}) = S(\pi_{j\rho_{AB}}(\psi|\pi_{j\rho_{AB}}) + S(\phi|A(\phi)) = 0 \) This is because the state \( |\psi\rangle_B \) is in a product form \( |\phi\rangle_A \), and they are pure states.

**A: Some Examples of Pure States**
Next we present some examples of pure three qubit states and plotted their residual correlation function $D_{RC}$ against the state parameters and measurement basis parameter.

**Example 1: GHZ class states**

As a first example we consider the generalized GHZ (GGHZ) states $|\psi_g\rangle$ given by,

$$|\psi_g\rangle = \cos(\theta)|000\rangle + \sin(\theta)|111\rangle. \quad (8)$$

From this we can always get back the well known GHZ state for $\theta = \frac{\pi}{2}$. In Figure 1 we plot the residual correlation function against the input state parameter $\theta$ and the basis parameter $\phi$.

**Example 2: W states**

Next we consider the example of W state $|\phi\rangle$ against the basis parameter $\theta$ in Figure 2.

**FIG. 1: The Residual Correlation function $D_{RC}$ of GGHZ state is plotted against the input state parameter $\theta$ and the basis parameter $\phi$.**

**FIG. 2: The Residual Correlation function $D_{RC}$ of W state is plotted against the basis parameter $\phi$.**

**INTERPRETATION OF RESIDUAL CORRELATION: STATE MERGING PERSPECTIVE**

In this section we give an operational interpretation of quantum residual correlation in terms of the total state merging cost required to merge the state of two parties on third party. Let us consider a hypothetical situation where we have three parties $A$, $B$ and $C$. They share an ensemble of three qubit states $\rho_{ABC}$ between themselves. Now if both $A$ and $C$ want to merge their information to $B$, the entire process can be carried out in following steps.

**Step I:** $A$ sends the additional information to $B$. This is given by the quantity $S(A|B)$.

**Step II:** Now $C$ sends the additional information to $B$. Now since $B$ is already having $A$’s information. Now this additional information is given by the quantity $S(C|AB)$.

**Step III:** The total cost of state merging in this scenario is given by $S(A|B) + S(C|AB)$

Here we relate the change in the cost of state merging as a result of discarding a quantum system through a measurement process with quantum residual correlation. First of all we simulate an arbitrary quantum operation $\epsilon$ (including measurement) on $B$. For that, we initially bring in a pure state $D(|0\rangle)$ in proximity to the qubit $B$ (see figure 1). We assume that $U$ to be the unitary interaction between $B$ and $D$. Here primes are used denote the state of the systems after $U$ has acted upon. We have $S(AC,B) = S(AC,BD)$ as $D$ starts with product state with $ACB$. We also have $I(AC : BD) = I(AC : BD)$ as there is no change in the total correlation of the system as result of unitary interaction. Since discarding quantum systems cannot increase the mutual information, $I(AC : B) \leq I(AC : BD) = I(AC : BD) = I(AC : B)$. In other words in terms of conditional entropy we can say at most, $S(AC|B) \geq S(AC|B)$. However, from this we can not conclude, whether $S(A|B) \leq S(A|B)$ or $S(A|B) \geq S(A|B)$. At most we can say that the change in the cost of state merging is captured by the quantity $\Delta_1 = S(A|B) - S(A|B)$. The state of $\rho_{ABC}$ after measurement on the sub system $B$ changes to $\rho_{ABC} = \sum_j p_j \rho_{Aij} \otimes \pi_j \otimes \rho_{Cij}$ (where $\pi_j$ is the projection operator). The unconditional post measurement states of $A$, $B$ and $C$ are, $\rho_A = \sum_j p_j \rho_{Aij}$, $\rho_B = \sum_j p_j \pi_j$ and
\( \rho_C = \sum_j p_j \rho_{C|j} \).

Next we consider the case when instead of one particle measurement, we carry out two particle measurement. For that we simulate an arbitrary quantum operation \( \varepsilon \) (including measurement) on \( B \) and \( A \). Similarly over here we bring in ancilla \( D \) which is initially in a pure state \( |0\rangle \) (see figure 2). Here \( U \) once again is the unitary interaction between \( B, C \) and \( D \). Since \( D \) is in a complete product state with the rest of the system, so it does not contribute to the entropy of the system. We have \( S(C,AB) = S(C,ABD) \). Primes are used to indicate that the particles have undergone unitary reactions. Since unitary interaction does not change the total correlation of the system we have, \( I(C : ABD) = I(C : AB) \). As we know, by discarding quantum system can not increase the mutual information, we have \( I(C : AB) \leq I(C : ABD) = I(C : AB) \). Equivalently, we can write \( S(C|AB) \geq S(C|AB) \). Hence, the change in the cost of state merging \( \Delta_2 = S(C|AB) - S(C|AB) \) is a positive quantity. In other words, discarding prior information increases the cost of state merging. Similarly, the state of \( \rho_{ABC} \) after measurement on the sub system \( AB \) changes to \( \rho_{ABC} = \sum_j p_j \pi_j \otimes \rho_{C|j} \) (where \( \pi_j \) are two particle projection operators). The unconditional post measurement states are, \( \rho_{AB} = \sum_j p_j \pi_j \) and \( \rho_C = \sum_j p_j \rho_{C|j} \). Therefore, the total change in the cost of state merging after carrying out measurement is given by, \( \Delta = \Delta_1 + \Delta_2 = S(A|B) - S(A|B) + S(C|AB) - S(C|AB) \). From previous section we have seen that the total mutual information after measurement is done on Bob’s qubit is given by.

\[
I_2(A : B : C) = [S(\rho_A) + S(\rho_B) + S(\rho_C)] - [S(\rho_{CB}) + S(\rho_{CA})] + S(\rho_{C|\Pi^A_B})
\]

\[
= \text{min}(\mathcal{D}_{RC} (A : B : C)) = \text{min}(I_2(A : B : C) - I_1(A : B : C)) = \text{min}(S(\rho_{C|\Pi^A_B}) + S(\rho_{AB})) - S(\rho_{ABC}) - S(\rho_{C|\Pi^A_B}) + S(\rho_{ABC})
\]

\[
= S(C|AB) - S(C|AB) - S(A|B) + S(A|B) + \Delta_1 + \Delta_2
\]

This is indeed a remarkable feature as the change in the cost of state merging is captured by the residual correlation present in the system. In figure 5 we have plotted the total change of the cost of state merging (the residual correlation) with respect to the mixedness of the system (which is given by \( Tr(\rho^2) \)) and found that this quantity in principle can take positive, negative and zero values. More the leftover correlation present in the system more is the increase in the cost of state merging by discarding prior information. For states with zero residual correlation, there is no increase in the cost of state merging by discarding information. Negative residual correlation indicates the decrease in the cost of state merging. This is quite counter intuitive as far as the cost of state merging is concerned. However this is possible in tripartite system where actually measurement can reduce the cost of state merging. This result gives a new dimension where in three qubit we can...
use suitable measurements to reduce the cost of state merging in a larger setting of quantum network (QNET).

FIG. 5: The total change of the cost of state merging $\Delta_1 + \Delta_2$ (the residual correlation $D_{RC}$) vs $Tr(\rho^2)$ vs Basis parameter $(j)$. The total change of the cost of state merging $\Delta_1 + \Delta_2$ (the residual correlation $D_{RC}$) of randomly generated $2 \otimes 10^5$ two qubit mixed states $\rho$ is plotted against $Tr(\rho^2)$ and basis parameter $(j)$.

Lemma 4: For three qubit state $\rho_{ABC}$, the total change in the cost of state merging $\Delta_1 + \Delta_2$ is equal to the difference between dissensions $D_{2C}$ and $D_{1C}$.

Proof: From Lemma 1, we already have the residual correlation $D_{RC}$ as the difference of two dissensions $D_{2C}$ and $D_{1C}$ i.e $D_{RC} = D_{2C} - D_{1C}$. Again we have seen that the residual correlation present in the system accounts for the change in the cost of state merging i.e $D_{RC} = \Delta_1 + \Delta_2$. Combining these two results we can say that $\Delta_1 + \Delta_2 = D_{2C} - D_{1C}$.

Lemma 5: For arbitrary pure three qubit state $\rho_{ABC}$, the total change in the cost of state merging $\Delta_1 + \Delta_2$ is positive.

Proof: For arbitrary pure three qubit state $\rho_{ABC}$, the residual correlation is given by $D_R = S(B) + \min S(\rho^A|\Pi^B)$, Since, the quantity $S(\rho^A|\Pi^B) > 0$ and $S(B)$ being the Von Neumann entropy is always positive, we have the residual correlation $D_R$ which is equal to the total change in the cost of state merging $\Delta_1 + \Delta_2$ both as positive quantities. This implies that there is always an increase in the cost of state merging on discarding the relevant prior information for a pure three qubit state.

CONCLUSION

In a nutshell, this work is all about introducing a new facet of quantum correlation in form of residual correlation and then interpreting the measure in terms of state merging capacity of a resource state. This quantity captures the correlation created between the qubits on which two particle measurement is carried out. In other words, it can be viewed as the difference of the correlation generated in a one particle measurement from the correlation generated in a two particle measurement. Interestingly, we find that this correlation exactly becomes equal to the total change in the cost of state merging which is a consequence of discarding quantum information. A positive value of residual correlation indicates a hike up in the cost, whereas a zero residual correlation is a signature of the fact that there is no change in the cost by discarding prior information. We had also seen that this quantity can also take up negative values. This is a clear indication of the fact that indeed a measurement can bring down the cost of state merging in a tripartite situation.

Acknowledgement: Authors acknowledge Prof. Nilanjana Dutta for useful and illuminating discussions. Sourav Chatterjee acknowledges Mr. Avijit Misra for his help in carrying out few calculations.

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