Asymptotic behavior of solutions toward the rarefaction waves to the Cauchy problem for the generalized Benjamin-Bona-Mahony-Burgers equation with dissipative term

Natsumi Yoshida

Abstract
In this paper, we investigate the asymptotic behavior of solutions to the Cauchy problem with the far field condition for the generalized Benjamin-Bona-Mahony-Burgers equation with a fourth-order dissipative term. When the corresponding Riemann problem for the hyperbolic part admits a Riemann solution which consists of single rarefaction wave, it is proved that the solution of the Cauchy problem tends toward the rarefaction wave as time goes to infinity. We can further obtain the same global asymptotic stability of the rarefaction wave to the generalized Korteweg-de Vries-Benjamin-Bona-Mahony-Burgers equation with a fourth-order dissipative term as the former one.

Keywords: Benjamin-Bona-Mahony-Burgers equation, Korteweg-de Vries-Benjamin-Bona-Mahony-Burgers equation, convex flux, asymptotic behavior, rarefaction wave

AMS subject classifications: 35K55, 35B40, 35L65

1. Introduction and main theorems
In this paper, we consider the asymptotic behavior of solutions to the Cauchy problem for the generalized Benjamin-Bona-Mahony-Burgers equation with a fourth-order dissipative term

\[
\begin{aligned}
\partial_t u + \partial_x \left( f(u) \right) - \alpha \partial_t \partial_x^2 u - \beta \partial_x^2 u + \gamma \partial_x^4 u &= 0 \quad (t > 0, \ x \in \mathbb{R}), \\
u(0, x) &= u_0(x) \to u_{\pm} \quad (x \to \pm\infty).
\end{aligned}
\]

(1.1)
Here, \( u = u(t, x) \) is the unknown function of \( t > 0 \) and \( x \in \mathbb{R} \), and \( \alpha, \beta, \gamma \) are positive constants \( u_0 \) is the initial data, and \( u_{\pm} \in \mathbb{R} \) are the prescribed far field states. We suppose that \( f \) is a smooth function.

There are many results concerning with the mathematical structure, such as the global existence and time-decay properties of solutions, of the generalized Benjamin-Bona-Mahony-Burgers equation with the dissipative terms (see Kondo-Webler [21], [22], [23], [24], Wang [49], Xu-Li [51], Zhao-Xuan [64] and so on). The model (1.1) is closely related to the following Benjamin-Bona-Mahony-Burgers equation

\[
\partial_t u + \delta \partial_x u + u \partial_x u - \alpha \partial_t \partial_x^2 u - \beta \partial_x^2 u = 0,
\]  

(1.2)

where \( \delta \in \mathbb{R} \). The mathematical structure of (1.2) have also been investigated by Amick-Bona-Schonbek [1], Mei [37], [38], Mei-Schmeiser [39], Naumkin [40] and so on. When \( \beta = 0 \), then (1.2) becomes the following Benjamin-Bona-Mahony equation

\[
\partial_t u + \delta \partial_x u + u \partial_x u - \alpha \partial_t \partial_x^2 u = 0,
\]  

(1.3)

which was advocated by [3] as a refinement of the following Korteweg de-Vries equation

\[
\partial_t u + \delta \partial_x u + u \partial_x u - \alpha \partial_t^3 u = 0.
\]  

(1.4)

For the case \( \alpha = 1 \) and \( \beta = 0 \) of (1.2), (1.2) is the so-called regularized long wave equation, which was proposed by Peregrine [44] and [3], as follows.

\[
\partial_t u + \delta \partial_x u + u \partial_x u - \partial_t \partial_x^2 u = 0.
\]  

(1.5)

We note that (1.3) and (1.5) are known as the approximated models for the long waves of small amplitude.

We are going to obtain the rarefaction stability of the solution to (1.1). Therefore we deal with the case where the flux function \( f \) is fully convex, that is,

\[
f''(u) > 0 \quad (u \in \mathbb{R}),
\]  

(1.6)

and \( u_- < u_+ \). Then, since the corresponding Riemann problem (cf. [27])

\[
\begin{cases}
\partial_t u + \partial_x (f(u)) = 0 & (t > 0, \ x \in \mathbb{R}),
\\
u(0, x) = u_0^R(x) := \begin{cases}
    u_- & (x < 0),
    u_+ & (x > 0)
\end{cases}
\end{cases}
\]  

(1.7)

turns out to admit a single rarefaction wave solution, we expect that the solution of the Cauchy problem (1.1) tends toward the rarefaction wave as time goes to infinity. Here, the rarefaction wave connecting \( u_- \) to \( u_+ \) is given by

\[
u'(\frac{x}{t}; u_-, u_+ ) = \begin{cases}
    u_- & (x \leq f'(u_-)t),
    (f')^{-1}(\frac{x}{t}) & (f'(u_-)t \leq x \leq f'(u_+)t),
    u_+ & (x \geq f'(u_+)t).
\end{cases}
\]  

(1.8)
In particular, we also expect that if \( u_- = u_+ =: \tilde{u} \), then the solution of the Cauchy problem (1.1) tends toward the constant state \( \tilde{u} \) as time goes to infinity.

There are many results concerning with the rarefaction stabilities. For viscous conservation law,

\[
\begin{align*}
\partial_t u + \partial_x \left( f(u) - \mu \partial_x u \right) &= 0 \quad (t > 0, \ x \in \mathbb{R}), \\
u(0, x) &= u_0(x) \to u_\pm \quad (x \to \pm \infty),
\end{align*}
\]

with the condition (1.6), Il’in-OleĂnik [18] showed that the solution tends toward the single rarefaction wave under the condition \( u_- < u_+ \) (and the one does the single viscous shock wave under the condition \( u_- > u_+ \), for further studies, see [34], [35], [53], [61] and so on). Hattori-Nishihara [16] also obtained the pointwise and time-decay estimates of the difference \( |u - u^r| \). Harabetian [14] further considered the following rarefaction problem for a quasilinear parabolic equation

\[
\begin{align*}
\partial_t u + \partial_x \left( f(u) - A'(u) \partial_x u \right) &= 0 \quad (t > 0, \ x \in \mathbb{R}), \\
u(0, x) &= u_0(x) \to u_\pm \quad (x \to \pm \infty),
\end{align*}
\]

where \( A'(u) \geq 0 \ (u \in \mathbb{R}) \), and obtained the precise time-decay estimates of global stability of the rarefaction wave with the aid of the arguments on monotone semigroups by Osher-Ralston [42]. For the following Cauchy problem of the Matsumura-Nishihara model

\[
\begin{align*}
\partial_t u + \partial_x \left( f(u) - \mu |\partial_x u|^{p-1} \partial_x u \right) &= 0 \quad (t > 0, \ x \in \mathbb{R}), \\
u(0, x) &= u_0(x) \to u_\pm \quad (x \to \pm \infty),
\end{align*}
\]

where \( p > 1 \) and the viscosity \( \mu |\partial_x u|^{p-1} \partial_x u \) is the so-called Ostwald-de Waele-type viscosity advocated by de Waele [9] and Ostwald [43] (which is a typical example for the non-Newtonian viscosity, see also [3], [4], [5], [10], [25], [30], [31], [48] and so on), Matsumura-Nishihara [33] first investigated and proved the global stability of the rarefaction wave by using the technical energy method. Yoshida [54] further obtained its precise time-decay estimates by using the time-weighted energy method (for the stabilities of the multiwave pattern, see [55], [56], [57]). Furthermore, Matsumura-Yoshida [36] considered the following Cauchy problem of the non-Newtonian viscous conservation law

\[
\begin{align*}
\partial_t u + \partial_x \left( f(u) - \sigma(\partial_x u) \right) &= 0 \quad (t > 0, \ x \in \mathbb{R}), \\
u(0, x) &= u_0(x) \to u_\pm \quad (x \to \pm \infty),
\end{align*}
\]

where viscosity function \( \sigma \) satisfies the conditions

\[
\begin{align*}
\sigma(0) &= 0, \quad \sigma'(v) > 0 \quad (v \in \mathbb{R}), \\
|\sigma(v)| &\sim |v|^p, \quad |\sigma'(v)| \sim |v|^{p-1} \quad (|v| \to \infty),
\end{align*}
\]

and obtained the global stability of the rarefaction wave for the case \( 0 < p < 3/7 \). Recently, Yoshida [60] obtained this rarefaction stability for more general case.
0 < p < 1/3 and its precise time-decay estimates. For the rarefaction problem of the Korteweg-de Vries equation

\[
\begin{aligned}
\partial_t u + \partial_x (-3u^2 + \partial_x^2 u) &= 0 \quad (t > 0, \ x \in \mathbb{R}), \\
u(0, x) = u_0(x) \rightarrow 0 \quad (x \to \infty), \\
&\rightarrow u_- > 0 \quad (x \to -\infty),
\end{aligned}
\]

Egorova-Grunert-Teschl \cite{12} and Egorova-Teschl \cite{13} obtained the existence and uniqueness in some class of the classical solution, and Andreiev-Egorova-Lange-Teschl \cite{2} also obtained the valid asymptotic formula of the solution. For diffusive dispersive conservation laws, Wang-Zhu \cite{50} obtained the local stability of the rarefaction wave for the following Cauchy problem of the generalized Korteweg-de Vries-Burgers equation

\[
\begin{aligned}
\partial_t u + \partial_x (f(u) - \mu \partial_x u + \delta \partial_x^2 u) &= 0 \quad (t > 0, \ x \in \mathbb{R}), \\
u(0, x) = u_0(x) \rightarrow u_\pm \quad (x \to \pm \infty).
\end{aligned}
\]

Duan-Zhao \cite{11} and Yoshida \cite{59} further obtained the global stabilities of the rarefaction wave under some growth conditions for \( f \) (for stability and time-decay properties of a travelling wave, see \cite{4}, \cite{9}, \cite{41}). For the Cauchy problem of the generalized Korteweg-de Vries-Burgers-Kuramoto equation

\[
\begin{aligned}
\partial_t u + \partial_x (f(u) - \mu \partial_x u + \delta \partial_x^2 u + \nu \partial_x^3 u) &= 0 \quad (t > 0, \ x \in \mathbb{R}), \\
u(0, x) = u_0(x) \rightarrow u_\pm \quad (x \to \pm \infty),
\end{aligned}
\]

Ruan-Gao-Chen \cite{47} first obtained the local stability of the rarefaction wave. Duan-Fan-Kim-Xie \cite{10} and Yoshida \cite{59} also obtained the global stability of the rarefaction wave under some growth conditions for \( f \). Recently, Yoshida \cite{63} (see also \cite{62}) investigated the following Cauchy problem for a diffusive dispersive conservation wave, that is, the generalized Korteweg-de Vries-Burgers-Kuramoto equation without the viscosity term as

\[
\begin{aligned}
\partial_t u + \partial_x (f(u) + \delta \partial_x^2 u + \nu \partial_x^3 u) &= 0 \quad (t > 0, \ x \in \mathbb{R}), \\
u(0, x) = u_0(x) \rightarrow u_\pm \quad (x \to \pm \infty),
\end{aligned}
\]

and obtained the global stability of the rarefaction wave under some growth conditions for \( f \).

Our main results of the present paper are as follows.

**Theorem 1.1** (Main Theorem 1). Assume the far field states \( u_\pm \) satisfy \( u_- = u_+ = \tilde{u} \), and the convective flux \( f \in C^2(\mathbb{R}) \). Further assume the initial data satisfy \( u_0 - \tilde{u} \in L^2 \) and \( \partial_x u_0 \in H^2 \). Then the Cauchy problem (1.1) has a unique global in time solution \( u \) satisfying

\[
\begin{aligned}
u u \in C^0 \left( \{0, \infty\}; H^3 \right), \\
\partial_x u \in L^2 \left( \{0, \infty\}; H^3 \right), \\
\partial_t u \in L^2 \left( \{0, \infty\}; H^2 \right),
\end{aligned}
\]
and the asymptotic behavior

\[
\lim_{t \to \infty} \left( \sup_{x \in \mathbb{R}} |u(t, x) - \bar{u}| + \sup_{x \in \mathbb{R}} |\partial_x u(t, x)| + \sup_{x \in \mathbb{R}} |\partial_x^2 u(t, x)| \right) = 0.
\]

**Theorem 1.2** (Main Theorem II). Assume the far field states \(u_\pm\) satisfy \(u_- < u_+\), and the convective flux \(f \in C^0(\mathbb{R})\) satisfy (1.6). Further assume the initial data satisfy \(u_0 - u_0^R \in L^2\) and \(\partial_x u_0 \in H^2\). Then the Cauchy problem (1.1) has a unique global in time solution \(u\) satisfying

\[
\begin{align*}
&\begin{cases}
 u - u_0^R \in C^0([0, \infty); H^3), \\
 \partial_x u \in L^2_{\text{loc}}(0, \infty; H^3), \\
 \partial_t u \in L^2_{\text{loc}}(0, \infty; H^2),
\end{cases} \\
\end{align*}
\]

and the asymptotic behavior

\[
\begin{align*}
&\begin{cases}
 \lim_{t \to \infty} \sup_{x \in \mathbb{R}} |u(t, x) - u^r \left( \frac{x}{t} ; u_-, u_+ \right)| = 0, \\
 \lim_{t \to \infty} \sup_{x \in \mathbb{R}} |\partial_x u(t, x) - \partial_x u^r(t, x ; u_-, u_+)| = 0, \\
 \lim_{t \to \infty} \sup_{x \in \mathbb{R}} |\partial_x^2 u(t, x) - \partial_x^2 u^r(t, x ; u_-, u_+)| = 0,
\end{cases}
\end{align*}
\]

where \(\partial_x u^r\) and \(\partial_x^2 u^r\) are given by

\[
\begin{align*}
\partial_x u^r(t, x ; u_-, u_+) &= \begin{cases}
 0 & (x \geq f'(u_+) t), \\
 \frac{1}{f''(f')^{-1} \left( \frac{x}{t} \right)} \frac{1}{t} & (f'(u_-) t \leq x \leq f'(u_+) t), \\
 0 & (x \leq f'(u_-) t),
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\partial_x^2 u^r(t, x ; u_-, u_+) &= \begin{cases}
 0 & (x \leq f'(u_-) t), \\
 \frac{1}{f''(f')^{-1} \left( \frac{x}{t} \right)} \frac{x}{t^3} & (f'(u_-) t \leq x \leq f'(u_+) t), \\
 0 & (x \geq f'(u_+) t),
\end{cases}
\end{align*}
\]

Furthermore, for the Cauchy problem for the generalized Benjamin-Bona-Mahony-Burgers equation with third-order dispersive and fourth-order dissipative terms, the so-called generalized Korteweg-de Vries-Benjamin-Bona-Mahony-Burgers equation with a fourth-order dissipative term (see [40])

\[
\begin{align*}
&\begin{cases}
 \partial_t u + \partial_x \left( f(u) \right) - \alpha \partial_t^2 u - \beta \partial_x^2 u + \delta \partial_x^3 u + \gamma \partial_x^4 u = 0 & (t > 0, x \in \mathbb{R}), \\
 u(0, x) = u_0(x) \to u_\pm & (x \to \pm \infty),
\end{cases}
\end{align*}
\]
where $\delta \in \mathbb{R}$, we can obtain the same stabilities as Theorems 1.1 and 1.2 in the next theorems.

**Theorem 1.3** (Main Theorem III). Assume the far field states $u_\pm$ satisfy $u_- = u_+ = \tilde{u}$, and the convective flux $f \in C^2(\mathbb{R})$. Further assume the initial data satisfy $u_0 - \tilde{u} \in L^2$ and $\partial_x u_0 \in H^2$. Then for the Cauchy problem (1.9), the same result as in Theorem 1.1 holds true.

**Theorem 1.4** (Main Theorem IV). Assume the far field states $u_\pm$ satisfy $u_- < u_+ = \tilde{u}$, and the convective flux $f \in C^5(\mathbb{R})$ satisfy (1.6). Further assume the initial data satisfy $u_0 - \tilde{u} \in L^2$ and $\partial_x u_0 \in H^2$. Then for the Cauchy problem (1.9), the same result as in Theorem 1.2 holds true.

We finally remark that when $\gamma = \delta = 0$, the problem (1.9) becomes the following Cauchy problem for usual generalized Benjamin-Bona-Mahony-Burgers equation.

\[
\begin{aligned}
\partial_t u + \partial_x \left( f(u) \right) - \alpha \partial_t \partial_x^2 u - \beta \partial_x^2 u &= 0 \quad (t > 0, x \in \mathbb{R}), \\
\partial_x^2 u(x) &= 0. 
\end{aligned}
\]  

We can also obtain the similar stabilities as Theorems 1.1 and 1.2 in the next theorems.

**Theorem 1.5**. Assume the far field states $u_\pm$ satisfy $u_- = u_+ = \tilde{u}$, and the convective flux $f \in C^2(\mathbb{R})$. Further assume the initial data satisfy $u_0 - \tilde{u} \in L^2$ and $\partial_x u_0 \in H^2$. Then the Cauchy problem (1.10) has a unique global in time solution $u$ satisfying

\[
\begin{aligned}
\left\{ 
\begin{array}{c}
\partial_t u(x) + \partial_x \left( f(u) \right) = 0 \\
u(0, x) = u_0(x) \to u_\pm 
\end{array} \right. \\
(t > 0, x \in \mathbb{R}),
\end{aligned}
\]

and the asymptotic behavior

\[
\lim_{t \to \infty} \left( \sup_{x \in \mathbb{R}} |u(t, x) - \tilde{u}| + \sup_{x \in \mathbb{R}} |\partial_x u(t, x)| + \sup_{x \in \mathbb{R}} |\partial_x^2 u(t, x)| \right) = 0.
\]

**Theorem 1.6**. Assume the far field states $u_\pm$ satisfy $u_- < u_+ = \tilde{u}$, and the convective flux $f \in C^5(\mathbb{R})$ satisfy (1.6). Further assume the initial data satisfy $u_0 - \tilde{u} \in L^2$ and $\partial_x u_0 \in H^2$. Then the Cauchy problem (1.10) has a unique global in time solution $u$ satisfying

\[
\left\{ 
\begin{array}{c}
\partial_t u(x) + \partial_x \left( f(u) \right) = 0 \\
u - u_0(x) \in L^2(0, \infty; H^2),
\end{array} \right. \\
\lim_{t \to \infty} \left( \sup_{x \in \mathbb{R}} |u(t, x)| \right) = 0.
\]
and the asymptotic behavior
\[
\lim_{t \to \infty} \sup_{x \in \mathbb{R}} \left| u(t, x) - u^r \left( \frac{x}{t} ; u_-, u_+ \right) \right| = 0,
\]
\[
\lim_{t \to \infty} \sup_{x \in \mathbb{R}} |\partial_x u(t, x) - \partial_x u^r (t, x ; u_-, u_+) | = 0,
\]
\[
\lim_{t \to \infty} \sup_{x \in \mathbb{R}} |\partial_x^2 u(t, x) - \partial_x^2 u^r (t, x ; u_-, u_+) | = 0,
\]
where \( \partial_x u^r \) and \( \partial_x^2 u^r \) are given in Theorem 1.2.

Because the proofs of Theorems 1.1 and 1.3-1.6 are similarly given as or easier than that for Theorem 1.2, we only show Theorem 1.2 in the following sections.

This paper is organized as follows. In Section 2, we construct the approximation of the rarefaction wave and prepare the basic properties of the rarefaction wave and the approximated one. We reformulate the problem in terms of the deviation from the asymptotic state in Section 3. In order to show the asymptotics, we establish the a priori estimates by using the technical energy method in Section 4. Finally in Section 5, we give several uniform estimates by using the a priori estimates in Sections 3 and 4.

**Some Notation.** We denote by \( C \) generic positive constants unless they need to be distinguished. In particular, use \( C_{\alpha, \beta, \cdots} \) when we emphasize the dependency on \( \alpha, \beta, \cdots \).

For function spaces, \( L^p = L^p(\mathbb{R}) \) and \( H^k = H^k(\mathbb{R}) \) denote the usual Lebesgue space and \( k \)-th order Sobolev space on the whole space \( \mathbb{R} \) with norms \( || \cdot ||_{L^p} \) and \( || \cdot ||_{H^k} \), respectively.

2. Preliminaries

In this section, we prepare the several lemmas concerning with the basic properties of the rarefaction wave for the proof of the main Theorem 1.2. Since the rarefaction wave \( u^r \) is not smooth enough, we construct a smooth approximated one. To do that, we first consider the rarefaction wave solution \( w^r \) to the Riemann problem for the non-viscous Burgers equation
\[
\begin{cases}
\partial_t w + \partial_x \left( \frac{1}{2} w^2 \right) = 0 & (t > 0, x \in \mathbb{R}), \\
w(0, x) = w^R (x ; w_-, w_+) := \begin{cases} w_+ & (x > 0), \\
w_- & (x < 0),
\end{cases}
\end{cases}
\]
(2.1)
where \( w_\pm \in \mathbb{R} (w_- < w_+) \) are the prescribed far field states. The unique global weak solution \( w = w^r (x/t ; w_-, w_+) \) of (2.1) is explicitly given by
\[
w^r \left( \frac{x}{t} ; w_-, w_+ \right) = \begin{cases} w_- & (x \leq w_- t), \\
\frac{x}{t} & (w_- t \leq x \leq w_+ t), \\
w_+ & (x \geq w_+ t).
\end{cases}
\]
(2.2)
Lemma 2.1. Assume that the far field states satisfy \( w \) classical solution \( (2) \) for any \( w = w^r \left( \frac{x}{t}; u_-, u_+ \right) \) of the Riemann problem \( (1.2) \) for hyperbolic conservation law is exactly given by

\[
  w^r \left( \frac{x}{t}; u_-, u_+ \right) = (\lambda)^{-1} \left( w^r \left( \frac{x}{t}; \lambda_-, \lambda_+ \right) \right)
\]  

(2.3)

which is nothing but \( (1.6) \), where \( \lambda_\pm := \lambda(u_\pm) = f'(u_\pm) \). We define a smooth approximation of \( w^r(\frac{x}{t}; w_-, w_+) \) by the unique classical solution

\[
w = w(t, x; w_-, w_+)
\]

to the Cauchy problem for the following non-viscous Burgers equation as

\[
  \begin{align*}
  \partial_t w + \partial_x \left( \frac{1}{2} w^2 \right) &= 0 \\
  w(0, x) &= w_0(x) := \frac{w_- + w_+}{2} + \frac{w_+ - w_-}{2} K_q \int_0^{\epsilon x} \frac{dy}{(1 + y^2)^q} 
  \end{align*}
\]  

(2.4)

where \( K_q \) is a positive constant such that

\[
  K_q \int_0^{\infty} \frac{dy}{(1 + y^2)^{q'}} = 1 \quad \left( q > \frac{1}{2} \right).
\]

By applying the method of characteristics, we get the following formula

\[
  \begin{align*}
  w(t, x) &= w_0(x_0(t, x)) = \frac{\lambda_- + \lambda_+}{2} + \frac{\lambda_+ - \lambda_-}{2} K_q \int_0^{x_0(t, x)} \frac{dy}{(1 + y^2)^q} \\
  x &= x_0(t, x) + w_0(x_0(t, x)) t.
  \end{align*}
\]  

(2.5)

By making use of \( (2.5) \) similarly as in \([22]\), we can obtain the properties of the smooth approximation \( w(t, x; w_-, w_+) \) in the next lemma.

**Lemma 2.1.** Assume that the far field states satisfy \( w_- < w_+ \). Then the classical solution \( w(t, x) = w(t, x; w_-, w_+) \) given by \( (2.4) \) satisfies the following properties:

1. \( w_- < w(t, x) < w_+ \) and \( \partial_x w(t, x) > 0 \) \( (t > 0, x \in \mathbb{R}) \).
2. For any \( r \in [1, \infty] \), there exists a positive constant \( C_{q, r} \) such that

\[
  \begin{align*}
  \| \partial_x w(t) \|_{L^r} &\leq C_{q, r} \min \left\{ \epsilon r^{-1} \tilde{w}^r, \tilde{w} (1 + t)^{-r+1} \right\} \quad (t \geq 0), \\
  \| \partial_t w(t) \|_{L^r} &\leq C_{q, r} \tilde{w}^r \min \left\{ \epsilon r^{-1} \tilde{w}^r, \tilde{w} (1 + t)^{-r+1} \right\} \quad (t \geq 0), \\
  \| \partial^2 w(t) \|_{L^r} &\leq C_{q, r} \min \left\{ \epsilon^{2r-1} \tilde{w}^r, \epsilon^{(r-1)(1-\frac{1}{q'})} \tilde{w} - \frac{2 - q}{q} (1 + t)^{-r-\frac{2}{q}} \right\} \quad (t \geq 0), \\
  \| \partial^3 w(t) \|_{L^r} &\leq C_{q, r} \min \left\{ \epsilon^{3r-1} \tilde{w}^r, a(1 + t, \epsilon, \tilde{w}) \right\} \quad (t \geq 0), \\
  \| \partial^4 w(t) \|_{L^r} &\leq C_{q, r} \min \left\{ \epsilon^{4r-1} \tilde{w}^r, b(1 + t, \epsilon, \tilde{w}) \right\} \quad (t \geq 0),
  \end{align*}
\]
where
\[ \hat{w} := \frac{w_+ - w_-}{2} > 0, \quad \hat{w} := \max\{|w_-|, |w_+|\}, \]

\[ a(t, \epsilon, \hat{w}) := e^{3r} \hat{w}^r \left( 1 + \epsilon \hat{w} t \right)^{1-4r} + e^{2(r-1)(1-\frac{1}{4r})} \hat{w}^{-\frac{1}{4r}} t^{-1-(r-1)(1+\frac{4}{r})} + e^{(2r-1)(1-\frac{1}{4r})} \hat{w}^{-\frac{2r-1}{2q}} t^{-r-\frac{2r-1}{2q}}, \]

and
\[ b(t, \epsilon, \hat{w}) := e^{3r} \hat{w}^r \left( 1 + \epsilon \hat{w} t \right)^{1-5r} + e^{(3r-2)(1-\frac{1}{4r})} \hat{w}^{-\frac{3r-2}{2q}} t^{-r(1+\frac{3}{2q})+\frac{1}{4r}} + e^{(3r-1)(1-\frac{1}{4r})} \hat{w}^{-\frac{3r-1}{2q}} t^{-r-\frac{3r-1}{2q}}. \]

(3) It follows that
\[ \lim_{t \to \infty} \sup_{x \in \mathbb{R}} \left| w(t, x) - w^r \left( \frac{x}{t} \right) \right| = 0, \]
\[ \lim_{t \to \infty} \sup_{x \in \mathbb{R}} |\partial_x w(t, x) - \partial_x w^r(t, x)| = 0, \]

where \( \partial_x w^r \) is given by
\[ \partial_x w^r(t, x; w_-, w_+) = \begin{cases} 0 & (x \leq w_- t), \\ \frac{1}{t} & (w_- t \leq x \leq w_+ t), \\ 0 & (x \geq w_+ t). \end{cases} \]

We now define the approximation for the rarefaction wave \( u^r(x/t; u_-, u_+) \) by
\[ U^r(t, x; u_-, u_+) = (\lambda)^{-1} \left( w(t, x; \lambda, \lambda) \right). \quad (2.6) \]

Using Lemma 2.1, we also have the next lemma.

**Lemma 2.2.** Let \( q = 1 \). Assume that the far field states satisfy \( u_- < u_+ \), and the flux function \( f \in C^6(\mathbb{R}), f''(u) > 0 \ (u \in [u_-, u_+]) \). Then we have the following properties.

1. \( U^r(t, x) \) defined by (2.6) is the unique \( C^4 \)-global solution in space-time of the Cauchy problem

\[
\begin{cases}
\frac{\partial_t U^r + \partial_x (f(U^r))}{(1 + y^2)^{q}} = 0 & \ (t > 0, \ x \in \mathbb{R}), \\
U^r(0, x) = (\lambda)^{-1} \left( \frac{\lambda_+ + \lambda_-}{2} + \frac{\lambda_+ - \lambda_-}{2} K_q \int_0^x dy (1 + y^2)^q \right) & \ (x \in \mathbb{R}), \\
\lim_{x \to \pm \infty} U^r(t, x) = u_\pm & \ (t \geq 0).
\end{cases}
\]

2. \( u_- < U^r(t, x) < u_+ \) and \( \partial_x U^r(t, x) > 0 \ (t > 0, \ x \in \mathbb{R}) \).
For any generalized Benjamin-Bona-Mahony-Burgers equation

It follows that

3. Reformulation of the problem

Because the proofs of Lemmas 2.1 and 2.2 are given in [15], [16], [29], [32], [33], [60], [63] and so on, we omit the proofs here.

3. Reformulation of the problem

In this section, we reformulate our problem (1.1) in terms of the deviation from the asymptotic state. Putting \( \phi \) as

\[
    u(t, x) = U^r(t, x) + \phi(t, x),
\]

we reformulate the problem (1.1) in terms of the deviation \( \phi \) from \( U^r \) as

\[
\begin{align*}
    \partial_t \phi + \partial_x \left( f(\phi + U^r) - f(U^r) \right) &= 0 \\
    -\alpha \partial_t \partial_x^2 \phi - \beta \partial_x \partial_t \phi + \gamma \partial_x^4 \phi &= F(U^r) \quad (t > 0, x \in \mathbb{R}), \\
    \phi(0, x) = \phi_0(x) &= u_0(x) - U^r(0, x) \to 0 \quad (x \to \pm \infty),
\end{align*}
\]

where

\[
    F(U^r) := \alpha \partial_t \partial_x^2 U^r + \beta \partial_x \partial_t U^r - \delta \partial_x^4 U^r - \gamma \partial_x^6 U^r.
\]

Then we look for the unique global in time solution \( \phi \) which has the asymptotic behavior

\[
    \sum_{k=1}^{3} \sup_{x \in \mathbb{R}} |\partial_x^k \phi(t, x)| \to 0 \quad (t \to \infty).
\]

Here we note that \( \phi_0 \in H^3 \) by the assumptions on \( u_0 \) and Lemma 2.2. Then the corresponding theorems for \( \phi \) to Theorem 1.2 we should prove is as follows.
Theorem 3.1. (Global Existence). Assume the far field states \( u_- < u_+ \), and the convective flux \( f \in C^5(\mathbb{R}) \) satisfy (1.6). Further assume the initial data satisfy \( \phi_0 \in H^3 \). Then the Cauchy problem (3.2) has a unique global in time solution \( \phi \) satisfying

\[
\begin{align*}
\phi & \in C^0([0, \infty); H^3), \\
\partial_x \phi & \in L^2(0, \infty; H^3), \\
\partial_t \phi & \in L^2(0, \infty; H^2),
\end{align*}
\]

and the asymptotic behavior

\[
\lim_{t \to \infty} \sum_{k=0}^{2} \sup_{x \in \mathbb{R}} \left| \partial_x^k \phi(t, x) \right| = 0.
\]

In order to obtain Theorem 3.1, we prepare the local existence precisely, we formulate the problem (3.2) at general initial time \( \tau \geq 0 \):

\[
\begin{align*}
\partial_t \phi + \partial_x \left( f \left( \phi + U^r \right) - f(U^r) \right) \\
- \alpha \partial_t \partial_x^2 \phi - \beta \partial_x^3 \phi + \gamma \partial_t^4 \phi = F(U^r) \quad (t > \tau, x \in \mathbb{R}), \\
\phi(\tau, x) = \phi_r(x) := u_r(x) - U^r(\tau, x) \to 0 \quad (x \to \pm \infty).
\end{align*}
\]

Theorem 3.2 (Local Existence). For any \( M > 0 \), there exists a positive constant \( t_0 = t_0(M) \) not depending on \( \tau \) such that if \( \phi_r \in H^3 \) and

\[
\| \phi_r \|_{H^3} \leq M,
\]

then the Cauchy problem (3.4) has a unique solution \( \phi \) on the time interval \([\tau, \tau + t_0(M)]\) satisfying

\[
\begin{align*}
\phi & \in C^0([\tau, \tau + t_0]; H^3), \\
\partial_x \phi & \in L^2(\tau, \tau + t_0; H^3), \\
\partial_t \phi & \in L^2(\tau, \tau + t_0; H^2), \\
\sup_{t \in [\tau, \tau + t_0]} \| \phi(t) \|_{H^3} & \leq 2M.
\end{align*}
\]

Because the proof of Theorem 3.2 is standard, we omit the details here (cf. [52], [64]). The a priori estimates we establish in Section 4 are the following.

Theorem 3.3 (A Priori Estimates). Under the same assumptions as in Theorem 3.1, for any initial data \( \phi_0 \in H^3 \), there exists a positive constant \( C_{\phi_0} \) such that if the Cauchy problem (3.1) has a solution \( \phi \) on the time interval \([0, T]\) satisfying

\[
\begin{align*}
\phi & \in C^0([0, T]; H^3), \\
\partial_x \phi & \in L^2(0, T; H^3), \\
\partial_t \phi & \in L^2(0, T; H^2),
\end{align*}
\]
for some positive constant $T$, then it holds that

$$
\| \phi(t) \|_{H^3}^2 + \int_0^t \| (\sqrt{\partial_x U^r} \phi(t)) \|_{L^2}^2 \, d\tau \\
+ \int_0^t \| \partial_x \phi(\tau) \|_{H^3}^2 \, d\tau + \int_0^t \| \partial_t \phi(\tau) \|_{H^2}^2 \, d\tau \leq C_{\phi_0} \quad (t \in [0, T]).
$$

Combining the local existence Theorem 3.2 together with the a priori estimates, Theorem 3.3, we can obtain global existence Theorem 3.1. In fact, we can obtain the unique global in time solutions $\phi$ to (3.2) in Theorem 3.1 satisfying

$$
\begin{cases} 
\phi \in C^0([0, \infty) ; H^3), \\
\partial_x \phi \in L^2(0, \infty ; H^3), \\
\partial_t \phi \in L^2(0, \infty ; H^2),
\end{cases}
$$

and

$$
\sup_{t \geq 0} \| \phi(t) \|_{H^3}^2 + \int_0^\infty \| (\sqrt{\partial_x U^r} \phi(t)) \|_{L^2}^2 \, dt \\
+ \int_0^\infty \| \partial_x \phi(t) \|_{H^3}^2 \, dt + \int_0^\infty \| \partial_t \phi(t) \|_{H^2}^2 \, dt < \infty
$$

which yields

$$
\int_0^\infty \frac{d}{dt} \| \partial_x \phi(t) \|_{L^2}^2 \, dt < \infty, \quad \int_0^\infty \frac{d}{dt} \| \partial_t^2 \phi(t) \|_{L^2}^2 \, dt < \infty.
$$

In fact, by using (3.6), direct computation shows that

$$
\int_0^\infty \frac{d}{dt} \| \partial_x \phi(t) \|_{L^2}^2 \, dt = 2 \int_0^\infty \left| \int_{-\infty}^\infty \partial_x \phi(t) \partial_t \partial_x \phi \, dx \right| \, dt \\
\leq \int_0^\infty \left( \| \partial_x \phi(t) \|_{L^2}^2 + \| \partial_t \partial_x \phi(t) \|_{L^2}^2 \right) \, dt < \infty,
$$

$$
\int_0^\infty \frac{d}{dt} \| \partial_t^2 \phi(t) \|_{L^2}^2 \, dt = 2 \int_0^\infty \left| \int_{-\infty}^\infty \partial_t^2 \phi(t) \partial_t \partial_x \phi \, dx \right| \, dt \\
\leq \int_0^\infty \left( \| \partial_t^2 \phi(t) \|_{L^2}^2 + \| \partial_t \partial_t \phi(t) \|_{L^2}^2 \right) \, dt < \infty.
$$

From (3.8) and (3.9), we get (3.7). We immediately have from (3.7) that

$$
\lim_{t \to \infty} \| \partial_x \phi(t) \|_{L^2} = 0, \quad \lim_{t \to \infty} \| \partial_t^2 \phi(t) \|_{L^2} = 0.
$$
Further from (3.10), by using the Sobolev inequality, we obtain the desired asymptotic behavior (3.3) as follows.

\[
\sup_{x \in \mathbb{R}} |\phi(t, x)| \leq \sqrt{2} \left\| \phi(t) \right\|_{L^2}^{\frac{1}{2}} \left\| \partial_x \phi(t) \right\|_{L^2}^{\frac{1}{2}} \rightarrow 0 \quad (t \to \infty),
\]

\[
\sup_{x \in \mathbb{R}} |\partial_x \phi(t, x)| \leq \sqrt{2} \left\| \partial_x \phi(t) \right\|_{L^2}^{\frac{1}{2}} \left\| \partial_x^2 \phi(t) \right\|_{L^2}^{\frac{1}{2}} \rightarrow 0 \quad (t \to \infty),
\]

\[
\sup_{x \in \mathbb{R}} |\partial_x^2 \phi(t, x)| \leq \sqrt{2} \left\| \partial_x^2 \phi(t) \right\|_{L^2}^{\frac{1}{2}} \left\| \partial_x^3 \phi(t) \right\|_{L^2}^{\frac{1}{2}} \rightarrow 0 \quad (t \to \infty).
\]

Thus Theorems 3.1 is proved.

4. A priori estimates

In this section, we show the following a priori estimate for \( \phi \) in Theorem 3.3. To do that, we prepare the following basic estimate.

Proposition 4.1. There exists a positive constant \( C_{\phi_0} \) such that

\[
\left\| \phi(t) \right\|_{H^1}^2 + \int_0^t \int_{-\infty}^{\infty} \left( f'(\eta + U^r) - f'(U^r) \right) d\eta \partial_x U^r \, dx \, d\tau \\
+ \int_0^t \left\| \partial_x \phi(\tau) \right\|_{H^1}^2 \, d\tau \leq C_{\phi_0} \quad (t \in [0, T]).
\]

Proof of Proposition 4.1. Multiplying the equation in (3.2) by \( \phi \) and integrating it with respect to \( x \), we have, after integration by parts, that

\[
\frac{1}{2} \frac{d}{dt} \left\| \phi(t) \right\|_{L^2}^2 + \frac{\alpha}{2} \frac{d}{dt} \left\| \partial_x \phi(t) \right\|_{L^2}^2 \\
+ \int_{-\infty}^{\infty} \int_0^t \left( f'(\eta + U^r) - f'(U^r) \right) d\eta \partial_x U^r \, dx \\
+ \beta \left\| \partial_x \phi(t) \right\|_{L^2}^2 + \gamma \left\| \partial_x^2 \phi(t) \right\|_{L^2}^2 = \int_{-\infty}^{\infty} \phi F(U^r) \, dx.
\] (4.1)

By making use of the Sobolev inequality and the Young inequality, we estimate the right-hand-side of (4.1) as follows.

\[
\left| \int_{-\infty}^{\infty} \phi F(U^r) \, dx \right| \leq \sqrt{2} \left\| \phi \right\|_{L^2}^{\frac{3}{2}} \left\| \partial_x \phi \right\|_{L^2}^{\frac{3}{2}} \left\| F(U^r) \right\|_{L^1} \\
\leq \frac{\beta}{2} \left\| \partial_x \phi \right\|_{L^2}^2 + C \beta \left\| \phi \right\|_{L^2} \left\| F(U^r) \right\|_{L^1}^{\frac{2}{3}} \\
\leq \frac{\beta}{2} \left\| \partial_x \phi \right\|_{L^2}^2 + C \beta \left( 1 + \left\| \phi \right\|_{L^2}^3 \right) \left\| F(U^r) \right\|_{L^1}^{\frac{2}{3}}.
\] (4.2)
Substituting (4.2) into (4.1), integrating the resultant inequality with respect to $t$, noting
\[
\partial_t \partial_x U^r = -f''(U^r) |\partial_x U^r|^2 - f'(U^r) \partial_x^2 U^r,
\]
\[
\|F(U^r)\|_{L^1}^4 \leq C_{\alpha,u_2} \|\partial_x U^r\|_{L^2}^8 + C_{\alpha,\beta,u_2} \|\partial_x^2 U^r\|_{L^1}^4 + \gamma \|\partial_x^4 U^r\|_{L^1}^4 \in L^1_t(0, \infty)
\]
from Lemma 2.2, and using the Gronwall inequality, we obtain the desired estimate.

Thus, we complete the proof of Proposition 4.1.

Only from Proposition 4.1, using the Sobolev inequality, we can easily get the uniform boundedness of $\phi$ in the next lemma (cf. [17], [20], [26]).

**Lemma 4.2.** There exists a positive constant $C_{\phi_0}$ such that
\[
\sup_{t \in [0,T], x \in \mathbb{R}} |\phi(t,x)| \leq C_{\phi_0}.
\]

By the uniform boundedness of $\phi$, Lemma 4.2, we note that the second term on the left-hand side of the a priori estimate in Proposition 4.1 can be replaced by the left-hand side of the following inequality as
\[
\begin{align*}
\int_0^t \int_{-\infty}^\infty \int_0^\infty \left( f'(\eta + U^r) - f'(U^r) \right) d\eta \partial_x U^r \, dx \, d\tau \\
\geq C_{\phi_0}^{-1} \int_0^t \left\| \sqrt{\partial_x U^r} \phi \right\|_{L^2}^2 d\tau.
\end{align*}
\]

We now prepare the following properties for the a priori estimates for the derivatives of $\phi$ in the next lemma.

**Lemma 4.3.** There exists a positive constant $C_{\phi_0}$ such that
\[
\int_0^t \left\| (f(\phi + U^r) - f(U^r))(\tau) \right\|_{L_2}^2 \, d\tau \leq C_{\phi_0} \quad (t \in [0,T]),
\]
\[
\int_0^t \left\| F(U^r)(\tau) \right\|_{L_2}^2 \, d\tau \leq C_{\phi_0} \quad (t \in [0,T]).
\]

**Proof of Lemma 4.3.** The second estimate, that is, the time-integrability of $\| F(U^r) \|_{L_2}^2$, is obtained by Lemma 2.2. Noting Proposition 4.1 and using Lagrange’s mean-value theorem, we can also obtain the first estimate, that is, the time-integrability of $\| f(\phi + U^r) - f(U^r) \|_{L_2}^2$.

Thus, the proof of Lemma 4.3 is completed.

Next, we show the a priori estimate for $\partial_x \phi$ and $\partial_x^2 \phi$ as follows.

**Proposition 4.4.** There exists a positive constant $C_{\phi_0}$ such that
\[
\| \partial_x \phi(t) \|_{H^1}^2 + \int_0^t \left( \| \partial_x \phi(\tau) \|_{H^1}^2 + \| \partial_x^2 \phi(\tau) \|_{H^1}^2 \right) \, d\tau \leq C_{\phi_0} \quad (t \in [0,T]).
\]
Proof of Proposition 4.4. Multiplying the equation in (3.2) by $\partial_t \phi - \partial_x^2 \phi$ and integrating it with respect to $x$, we have, after integration by parts, that

$$
\frac{\beta + 1}{2} \frac{d}{dt} \| \partial_x \phi(t) \|_{L^2}^2 + \frac{\gamma + 1}{2} \frac{d}{dt} \| \partial_x^2 \phi(t) \|_{L^2}^2 \\
+ \| \partial_t \phi(t) \|_{L^2}^2 + \alpha \| \partial_t \partial_x \phi(t) \|_{L^2}^2 + \beta \| \partial_x^2 \phi(t) \|_{L^2}^2 + \gamma \| \partial_x^2 \phi(t) \|_{L^2}^2 \\
= - \int_{-\infty}^{\infty} (\partial_t \phi - \partial_x^2 \phi) \left( f(\phi + U^r) - f(\phi) \right) dx \\
+ \int_{-\infty}^{\infty} (\partial_t \phi - \partial_x^2 \phi) F(U^r) dx.
$$

(4.4)

By using the Young inequality, we estimate the each terms on the right-hand side of (4.4) as follows.

$$
\left| \int_{-\infty}^{\infty} (\partial_t \phi - \partial_x^2 \phi) \left( f(\phi + U^r) - f(\phi) \right) dx \right| \leq \epsilon \left( \| \partial_t \phi \|_{L^2}^2 + \| \partial_x^2 \phi \|_{L^2}^2 \right) + C_{c, \beta} \| f(\phi + U^r) - f(\phi) \|_{L^2}^2,
$$

(4.5)

$$
\left| \int_{-\infty}^{\infty} (\partial_t \phi - \partial_x^2 \phi) F(U^r) dx \right| \leq \epsilon \left( \| \partial_t \phi \|_{L^2}^2 + \| \partial_x^2 \phi \|_{L^2}^2 \right) + C_{c, \beta} \| F(U^r) \|_{L^2}^2,
$$

(4.6)

for $\epsilon > 0$. Choosing $\epsilon$ suitably small, substituting (4.5) and (4.6) into (4.4), and using Lemma 4.3, we obtain the desired estimate.

Thus, we complete the proof of Proposition 4.4.

Remark 4.5. Similarly to Lemma 4.2, we have the uniform boundedness of $\partial_x \phi$ by using Proposition 4.4 that

$$
\sup_{t \in [0, T], x \in \mathbb{R}} | \partial_x \phi(t, x) | \leq C_{\phi_0}.
$$

We further show the a priori estimate for $\partial_x^2 \phi$ and $\partial_x^3 \phi$ as follows.

Proposition 4.6. There exists a positive constant $C_{\phi_0}$ such that

$$
\| \partial_x^2 \phi(t) \|^2_{H^1} + \int_0^t \left( \| \partial_t \partial_x \phi(\tau) \|^2_{H^1} + \| \partial_x^3 \phi(\tau) \|^2_{H^1} \right) d\tau \leq C_{\phi_0} \quad (t \in [0, T]).
$$

Proof of Proposition 4.6. Multiplying the equation in (3.2) by $\partial_x^4 \phi - \partial_t \partial_x^2 \phi$
and integrating it with respect to $x$, we have, after integration by parts, that
\[
\frac{\beta + 1}{2} \frac{d}{dt} \| \partial_x^2 \phi(t) \|_{L^2}^2 + \frac{\alpha + \gamma}{2} \frac{d}{dt} \| \partial_t^2 \phi(t) \|_{L^2}^2 + \| \partial_t \partial_x \phi(t) \|_{L^2}^2 + \alpha \| \partial_t \partial_x^2 \phi(t) \|_{L^2}^2 + \beta \| \partial_x^2 \phi(t) \|_{L^2}^2 + \gamma \| \partial_t \phi(t) \|_{L^2}^2 \\
= - \int_{-\infty}^{\infty} \left( \partial_x^2 \phi - \partial_t \partial_x^2 \phi \right) \left( f(\phi + U^r) - f(U^r) \right) dx \\
+ \int_{-\infty}^{\infty} \left( \partial_t^4 \phi - \partial_t \partial_x^2 \phi \right) F(U^r) dx.
\]
(4.7)

The each terms on the right-hand side of (4.7) can be estimated quite similarly to (4.5) and (4.6). Therefore, we obtain the desired estimate by using Lemma 4.3.

Thus, we complete the proof of Proposition 4.6.

**Remark 4.7.** Similarly to Lemma 4.2 and remark 4.5, we have the uniform boundedness of $\partial_x^2 \phi$ by using Proposition 4.6 that
\[
\sup_{t \in [0,T], x \in \mathbb{R}} | \partial_x^2 \phi(t, x) | \leq C_{\phi_0}.
\]

5. Remarks on the uniform estimates

It is worthwhile to mention the uniform estimates of the solution $\phi$ to (3.2). By using the Sobolev inequality to $\phi, \partial_x \phi, \partial_t \phi, \partial_x^2 \phi, \partial_x^3 \phi, \partial_t \phi$ and $\partial_t \partial_x \phi$, and integrating by parts, we obtain
\[
\sup_{x \in \mathbb{R}} | \phi(t, x) | \leq \sqrt{2} \| \phi(t) \|_{L^2}^{\frac{3}{2}} \| \partial_x \phi(t) \|_{L^2}^{\frac{1}{2}}, \quad \phi(t) \in H^1
\]
(5.1)

\[
\sup_{x \in \mathbb{R}} | \partial_x \phi(t, x) | \leq \sqrt{2} \| \partial_x \phi(t) \|_{L^2}^{\frac{3}{2}} \| \partial_x^2 \phi(t) \|_{L^2}^{\frac{1}{2}}
\]
(5.2)

\[
\sup_{x \in \mathbb{R}} | \partial_t \phi(t, x) | \leq \sqrt{2} \| \partial_t \phi(t) \|_{L^2}^{\frac{3}{2}} \| \partial_x \phi(t) \|_{L^2}^{\frac{1}{2}} \| \partial_x \phi(t) \|_{H^1}
\]
(5.3)

\[
\sup_{x \in \mathbb{R}} | \partial_x^2 \phi(t, x) | \leq \sqrt{2} \| \partial_x^2 \phi(t) \|_{L^2}^{\frac{3}{2}} \| \partial_t \phi(t) \|_{L^2}^{\frac{1}{2}}
\]

\[
\sup_{x \in \mathbb{R}} | \partial_x^3 \phi(t, x) | \leq \sqrt{2} \| \partial_x^3 \phi(t) \|_{L^2}^{\frac{3}{2}} \| \partial_t \phi(t) \|_{L^2}^{\frac{1}{2}}
\]

\[
\sup_{x \in \mathbb{R}} | \partial_t^2 \phi(t, x) | \leq \sqrt{2} \| \partial_t^2 \phi(t) \|_{L^2}^{\frac{3}{2}} \| \partial_x \phi(t) \|_{L^2}^{\frac{1}{2}} \| \partial_x \phi(t) \|_{H^1}
\]
Proposition 5.1. There exists a positive constant $C_{\phi_0}$ such that

$$\int_0^t \left( \sup_{x \in \mathbb{R}} |\phi(\tau, x)| \right)^k d\tau \leq C_{\phi_0}, \quad (2 \leq k \leq 32),$$

$$\int_0^t \left( \sup_{x \in \mathbb{R}} |\partial_x \phi(\tau, x)| \right)^k d\tau \leq C_{\phi_0}, \quad (2 \leq k \leq 16),$$

$$\int_0^t \left( \sup_{x \in \mathbb{R}} |\partial_x^2 \phi(\tau, x)| \right)^k d\tau \leq C_{\phi_0}, \quad (2 \leq k \leq 8),$$

$$\int_0^t \left( \sup_{x \in \mathbb{R}} |\partial_x^3 \phi(\tau, x)| \right)^k d\tau \leq C_{\phi_0}, \quad (2 \leq k \leq 4),$$

$$\int_0^t \left( \sup_{x \in \mathbb{R}} |\partial_t \phi(\tau, x)| \right)^2 d\tau \leq C_{\phi_0},$$

$$\int_0^t \left( \sup_{x \in \mathbb{R}} |\partial_t \partial_x \phi(\tau, x)| \right)^2 d\tau \leq C_{\phi_0},$$

for $t \in [0, T]$. 

Noting the first term on the \textit{a priori} estimates in Theorem 3.3, that is,

\begin{align*}
\sup_{x \in \mathbb{R}} |\partial_x^3 \phi(t, x)| &\leq \sqrt{2} \left\| \partial_x^3 \phi(t) \right\|_{L^2}^{\frac{1}{2}} \left\| \partial_x^4 \phi(t) \right\|_{L^2}^{\frac{1}{2}} \leq \left\| \partial_x^4 \phi(t) \right\|_{H^1}, \quad (5.4) \\
\sup_{x \in \mathbb{R}} |\partial_x \phi(t, x)| &\leq \sqrt{2} \left\| \partial_x \phi(t) \right\|_{L^2}^{\frac{1}{2}} \left\| \partial_t \partial_x \phi(t) \right\|_{L^2}^{\frac{1}{2}} \leq \left\| \partial_t \phi(t) \right\|_{H^1}, \quad (5.5) \\
\sup_{x \in \mathbb{R}} |\partial_t \partial_x \phi(t, x)| &\leq \sqrt{2} \left\| \partial_t \partial_x \phi(t) \right\|_{L^2}^{\frac{1}{2}} \left\| \partial_t \partial_x^2 \phi(t) \right\|_{L^2}^{\frac{1}{2}} \leq \left\| \partial_t \partial_x \phi(t) \right\|_{H^1}, \quad (5.6)
\end{align*}

and using (5.1)-(5.6), we arrive at

\begin{align*}
\sup_{x \in \mathbb{R}} |\phi(t, x)| &\leq C_{\phi_0} \min \left\{ \min_{k=1,2,3,4} \left\| \partial_x^k \phi(t) \right\|_{L^2}^{\frac{1}{2}}, \left\| \phi(t) \right\|_{H^1} \right\}, \quad (5.8) \\
\sup_{x \in \mathbb{R}} |\partial_x \phi(t, x)| &\leq C_{\phi_0} \min \left\{ \min_{k=1,2,3} \left\| \partial_x^{k+1} \phi(t) \right\|_{L^2}^{\frac{1}{2}}, \left\| \partial_x \phi(t) \right\|_{H^1}, \left\| \partial_x \phi(t) \right\|_{H^1} \right\}, \quad (5.9) \\
\sup_{x \in \mathbb{R}} |\partial_x^2 \phi(t, x)| &\leq C_{\phi_0} \min \left\{ \left\| \partial_x^2 \phi(t) \right\|_{L^2}^{\frac{1}{2}}, \left\| \partial_x^3 \phi(t) \right\|_{L^2}^{\frac{1}{2}}, \left\| \partial_x^2 \phi(t) \right\|_{L^2}^{\frac{1}{2}}, \left\| \partial_x^3 \phi(t) \right\|_{H^1} \right\}, \quad (5.10) \\
\sup_{x \in \mathbb{R}} |\partial_x^3 \phi(t, x)| &\leq C_{\phi_0} \min \left\{ \left\| \partial_x^3 \phi(t) \right\|_{L^2}^{\frac{1}{2}}, \left\| \partial_x^4 \phi(t) \right\|_{L^2}^{\frac{1}{2}} \right\}, \quad (5.11)
\end{align*}

for some $C_{\phi_0} > 0$. Then, by (5.4), (5.5) and (5.8)-(5.11), noting the interpolation, we obtain the uniform estimates as follows.
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