A NEW ALGORITHM FOR THE SUBTRACTION GAMES

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Abstract. Subtraction games is a class of combinatorial games. It was solved since the Sprague-Grundy Theory was put forward. This paper described a new algorithm for subtraction games. The new algorithm can find win or lost positions in subtraction games. In addition, it is much simpler than Sprague-Grundy Theory in one pile of the games.

1. Introduction
Subtraction games is a class of impartial combinatorial games. The subtraction games is satisfies the following conditions.

- There are two-player game involved a pile of chips.
- a finite set $S$ of positive integers called the subtraction set. Subtraction sets $S = s_1, s_2, ..., s_k$ will be ordered $s_1 < s_2 < ... < s_k$.
- The two players move alternately, subtracting some $s$ chips such that $s \in S$.
- The game ends when a position is reached from which no moves are possible for the player whose turn it is to move. The last player to move wins.

Since the Sprague-Grundy Theory was proposed by Roland P.Sprague (in 1935) and independently Patrick M.Grundy (in 1939)([1]), people realized how to judge the game’s position which is a $P$-position or an $N$-position (in [2] and [3]), and how to move it to make sure win by mathematical methods.

However, the theory is so special that make us difficult to understand how to solve the problem. This paper described a new algorithm to solve subtraction games, and the algorithm make us simple to understand what happened. It used $P$-position to find $N$-position, so we named it PTFN algorithm.

2. Some Definitions
Before presenting our results, we need to recall some notations and terminologies.
In section 3, this paper need through the definitions of $P$-position and $N$-position to prove the correctness of the PTFN algorithm. $P$-positions’ and $N$-positions’ definition from the [2]. It first noted by Ernst Zermelo [3] in 1912. $P$-positions and $N$-positions are defined recursively by the following three statements.

**Definition 2.1.** $P$-position and $N$-position

- All terminal positions are $P$-positions.
- From every $N$-position, there is at least one move to a $P$-position.
- From every $P$-position, every move is to an $N$-position.

The terminal position means the game has no chip to move.

Sprague-Grundy theory use a very special way to find all positions of the subtraction games. Sprague-Grundy theory was represented by Sprague-Grundy function $G(x)$, as $G(x) = 0$ it means the point $x$ is $P$-position ; as $G(x) \neq 0$, the point $x$ is
that. And it was named PTFN, which is used to find N-position, namely $X[0]$, $X[0+1]$, $X[0+3]$, $X[0+7]$, $X[0+8]$ must be N-position. they can be moved to $X[0]$. Then $X[2]$ is a P-position, it legal move to $X[2+1]$, $X[2+3]$, $X[2+7]$, $X[2+8]$ , so these points must be N-positions. Just like that. It wouldn’t finish this work until all positions of elements were found. Now, it used pseudocode to describe the PTFN algorithm (Algorithm 1).

A simple proof. It is simple to vindicate the PTFN algorithm. This paper can make sure its correctness, only need to satisfy the conditions of the P-position’s and N-position’s definition (define 1).

(1) In the subtraction games, $X[0]$ must be a terminal position (P-position), our deduction begin at this position.

(2) In PTFN algorithm, it gets N-position by one step which moved from P-position.

(3) PTFN means use P-position to find N-position, so from every P-position, every step move to N-position.
Algorithm 1 PTFN for one pile

1: \( \text{SIZE}_\text{OF}_\text{PILE} \leftarrow n \) \{\( n \) is size of pile\}
2: \( \text{SIZE}_\text{OF}_\text{SET} \leftarrow s \) \{\( s \) is size of the set \( s \)\}
3: \( X[\text{SIZE}_\text{OF}_\text{PILE} + 1] \) \{Define a Array of \( X \)\}
4: \( \text{CheckSet}[\text{SIZE}_\text{OF}_\text{SET} + 1] \) \{The array save the check set such as \( \{1, 3, 7, 8\} \}\}

5: for \( i = 0 \) to \( \text{SIZE}_\text{OF}_\text{PILE} + 1 \) do
6: \( X[i] \leftarrow 0 \)
7: end for

8: for \( i = 0 \) to \( \text{SIZE}_\text{OF}_\text{PILE} + 1 \) do
9: \( \text{if} \ X[i] == 0 \) then
10: \( \text{for} \ j = 0 \) to \( j < \text{SIZE}_\text{OF}_\text{SET} + 1 \) do
11: \( X[i + \text{CheckSet}[j]] \leftarrow 1 \)
12: end for
13: end if
14: end for

Through the above analysis, this paper vindicated the PTFN algorithms correctness.

Example Next, using the method of PTFN analyzes Example 1 and tests the algorithm (in Figure 2).

Figure 2. The New Algorithm of the way with \( S = \{1, 3, 7, 8\} \)

From the results obtained so far. The results of PTFN algorithm and Sprague-Grundy function are same.

4. Some Application

For one pile of the subtraction games, the PTFN algorithm is available. On the more than one pile subtraction games, the algorithm can be used, such as more than one pile subtraction games, Whyof’s Game and misère play rule subtraction games.

Sum of subtraction games, first this paper give a formal description of a sum of games and then show how the PTFN algorithm for the component games.

Given more than one pile subtraction games, one can form a new game played according to the following rules.
• A given initial position is set up in each of the games ($S_1, S_2, S_3, ...$).
• Players alternate moves.
• A move for a player consists in selecting any one of the games and making a legal move in that game, leaving all other games untouched. Play continues until all of the games have reached a terminal position, when no more moves are possible.
• The player who made the last move is the winner.

As an example, this paper used PTFN algorithm for the 2-pile game of subtraction games. It only changes 1-dimension array $X[SIZE\_OF\_PILE]$ to 2-dimension array $X[SIZE\_OF\_PILE\_A][SIZE\_OF\_PILE\_B]$, and row array presents one game, raw array presents the other game. The Algorithm 2 shows main pseudocode.

**Algorithm 2 PTFN for two piles**

1: for $i = 0$ to $SIZE\_OF\_PILE\_A + 1$ do
2:   for $j = 0$ to $SIZE\_OF\_PILE\_B + 1$ do
3:     if $X[i][j] == 0$ then
4:       for $k = 0$ to $k < SIZE\_OF\_SET\_A + 1$ do
5:         $X[i + CheckSetA[k]][j] ← 1$
6:       end for
7:     end for
8:   end for
9: end for

Consider the sum of two subtraction games. One is $S = \{1, 3, 7, 8\}$ (Example 1) and the pile has 15 chips. The other is $S = \{1, 2, 3, 4\}$ and the pile has 15 chips. Thus, it obtains Figure 3 from different methods which PTFN algorithm and Sprague-Grundy theory.

![Figure 3. The New Algorithm of the way](image)

The Figure 3 results show that it is correct to use PTFN algorithm in the 2-pile game. Wythoffs Game introduced and solved by Willem A.Wythoff [5] in 1907. And we can solve the game in the same way which solved 2-pile of subtraction games. And the only change is that they may take an equal number from both.

**Misère play rule for subtraction games** For the games, if subtractions set is $S = \{s_1, s_2, ..., s_k\}$, we only set is $x[0]$ to $X[s_1]$ as $N$-position when the initialization of the PTFN algorithm.
5. Epilogue

This paper used a new algorithm to analyse the subtraction games. By comparison, this paper knows that for sums of subtraction games, Sprague-Grundy theory is better than PTFN algorithm, but for one pile subtraction games, PTFN has certain advantages over Sprague-Grundy theory.

In fact, $P$-position to find $N$-position is a good idea in many of games. We will use the PTFN algorithm to study these games in the future.

References

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