Quasinormal resonances of rapidly-spinning Kerr black holes and the universal relaxation bound

Shahar Hod
The Ruppin Academic Center, Emek Hefer 40250, Israel
and
The Hadassah Institute, Jerusalem 91010, Israel
(Dated: February 4, 2022)

The universal relaxation bound suggests that the relaxation times of perturbed thermodynamical systems is bounded from below by the simple time-times-temperature (TTT) quantum relation \( \tau \propto T \). It is known that some perturbation modes of near-extremal Kerr black holes in the regime \( MT_{BH}^{\text{BH}}/\hbar \ll m^{-2} \) are characterized by normalized relaxation times \( \pi \tau \propto T_{BH}^{\text{BH}}/\hbar \) which, in the approach to the limit \( MT_{BH}^{\text{BH}}/\hbar \to 0 \), make infinitely many oscillations with a tiny constant amplitude around 1 and therefore cannot be used directly to verify the validity of the TTT bound in the entire parameter space of the black-hole spacetime (Here \( \{ T_{BH}, M \} \) are respectively the Bekenstein-Hawking temperature and the mass of the black hole, and \( m \) is the azimuthal harmonic index of the linearized perturbation mode). In the present compact paper we explicitly prove that all rapidly-spinning Kerr black holes respect the TTT relaxation bound. In particular, using analytical techniques, it is proved that all black-hole perturbation modes in the complementary regime \( m^{-1} \ll MT_{BH}^{\text{BH}}/\hbar \ll 1 \) are characterized by relaxation times with the simple dimensionless property \( \pi \tau \propto T_{BH}^{\text{BH}}/\hbar \geq 1 \).

I. INTRODUCTION

Perturbed black-hole spacetimes are usually characterized by a relaxation phase which is dominated by exponentially decaying quasinormal oscillations whose discrete complex frequencies \( \{ \omega_n \}_{n=0}^{\infty} \) encode valuable information about the fundamental physical parameters (in particular, the mass and angular momentum) of the perturbed black hole [1–6].

The damped quasinormal oscillations of matter and radiation fields in perturbed black-hole spacetimes, which describe waves with outgoing boundary conditions at spatial infinity and ingoing boundary conditions at the absorbing black-hole horizon [7], reflect the fact that most perturbations fields (fields that conform to existing no-hair theorems [8–14]) are eventually scattered away to infinity or absorbed by the central black hole [15–19].

The timescale \( \tau_{\text{relax}} \) associated with the relaxation dynamics of a newly born black hole may be determined by the imaginary part of the fundamental (least damped) quasinormal resonant frequency:

\[
\tau_{\text{relax}} \equiv 1/3\omega_0 .
\]

Taking cognizance of the fact that black holes have thermodynamic and quantum properties [20, 21], a physically important question naturally arises: How short can the relaxation time (1) of a newly born black hole be?

A mathematically compact answer to this physically intriguing question, which is based on standard ideas from classical thermodynamics and quantum information theory, has been suggested in [22, 23]:

\[
\tau_{\text{relax}} \times T \geq \frac{\hbar}{\pi k_B} ,
\]

where \( T \) is the characteristic temperature of the thermodynamic physical system.

Intriguingly, remembering that the semi-classical Bekenstein-Hawking temperature [20, 21, 24]

\[
T_{\text{BH}} = \frac{\kappa}{2\pi k_B} \cdot \hbar
\]

of black holes depends linearly on the quantum Planck constant (here \( \kappa \) is the characteristic surface gravity of the black-hole horizon), one realizes that the time-times-temperature (TTT) quantum bound (2) provides a classical lower bound [25] on the characteristic relaxation time of a newly born black hole. In particular, taking cognizance of Eqs. (1), (2), and (3), one realizes that all dynamically formed black holes should be characterized by (at least) one relaxation mode with the simple classical property

\[
3\omega_0 \leq \frac{\pi T_{\text{BH}}}{\hbar} = \frac{1}{2} \kappa .
\]
II. RELAXATION DYNAMICS OF RAPIDLY SPINNING KERR BLACK HOLES

Interestingly, the upper bound (4) implies that newly born near-extremal black holes (with the property $T_{BH} \to 0$) should be characterized by extremely long relaxation times [22, 26, 27]. In particular, using the physically important and mathematically elegant Detweiler equation [28], which characterizes the complex resonant spectra of rapidly-spinning (near-extremal) Kerr black holes, one obtains the simple functional relation [29–31]

$$\Im \omega_0 = \pi T_{BH} \cdot \left\{ 1 + C \cdot \sin[2 \delta \ln(MT_{BH})] \right\} \cdot \left[ 1 + O(MT_{BH}) \right]$$

(5)

for the co-rotating perturbation modes of near-extremal black holes, where $C(l, m)$ is a constant [32], $M$ is the black-hole mass, and the physical parameter $\delta(l, m) \in \mathbb{R}$ is closely related to the characteristic eigenvalue of the angular Teukolsky equation [33, 34].

Before proceeding, it is important to stress the fact that the validity of the Detweiler resonance equation [28], and thus also the validity of the analytically derived expression (5) [29, 30], are restricted to the dimensionless regime [33, 35, 36]

$$m^2 \cdot MT_{BH} \ll 1,$$

(6)

where $m$ is the azimuthal harmonic index of the black-hole perturbation mode.

From Eq. (5) one immediately finds that perturbation modes of rapidly-spinning Kerr black holes in the regime (6) are characterized by normalized relaxation times $\pi \times T_{BH}$ that oscillate infinitely many times around 1 as the extremal limit $MT_{BH} \to 0$ is approached. One therefore deduces that, in the near-extremal $MT_{BH} \ll 1$ regime, there are finite intervals of the black-hole temperature for which a perturbation mode in the dimensionless regime (6) cannot be used to prove the general validity of the TTT relaxation bound (2) [or equivalently, the upper bound (4)] in black-hole physics [37].

The main goal of the present compact paper is to prove the physically important fact that all rapidly-spinning Kerr black holes respect the TTT relaxation bound (2). To this end, we shall study below the linearized relaxation dynamics of newly born near-extremal Kerr black holes. In particular, using analytical techniques, we shall explicitly prove that composed black-hole-field perturbation modes in the eikonal large-$m$ regime

$$m^{-1} \ll MT_{BH} \ll 1$$

(7)

are characterized by relaxation times that respect the TTT relaxation bound.

III. RESONANT RELAXATION SPECTRA OF RAPIDLY SPINNING KERR BLACK HOLES IN THE EIKONAL LARGE-FREQUENCY REGIME

Using a physically intuitive and mathematically elegant analysis, Mashhoon [38] has presented a simple analytical technique for calculating the quasinormal resonance spectra of black holes in the large-frequency (geometric-optics) regime. In particular, the physical idea presented in [38] is to relate, in the eikonal large-frequency regime, the real part of the black-hole quasinormal frequencies to the angular velocity which characterizes the motion of massless particles along the equatorial null circular geodesic of the black-hole spacetime and to relate the imaginary part of the black-hole quasinormal frequencies to the instability timescale which characterizes the gradual leakage of the perturbed massless particles (null rays) from the unstable null circular geodesic of the black-hole spacetime.

In particular, using a perturbation scheme for the instability which characterizes the dynamics of equatorial null circular geodesics in the Kerr black-hole spacetime, Mashhoon has provided the remarkably compact formula [38]

$$\omega_n = m \omega_+ - i(n + \frac{1}{2}) \beta \omega_+ \quad ; \quad n = 0, 1, 2, \ldots,$$

(8)

for the discrete quasinormal resonant spectra of spinning Kerr black holes. As emphasized in [38], this resonance formula is valid for perturbation modes in the eikonal large-frequency regime

$$l = m \gg 1,$$

(9)

where $\{l, m\}$ are the (spheroidal and azimuthal) angular harmonic indexes which characterize the linearized field mode.

The various terms in the resonance formula (8) have the following physical interpretations [38]:

(1) The term

$$r_{ph}(a/M) \equiv 2M \cdot \left\{ 1 + \cos \left[ \frac{2}{3} \arccos \left( -\frac{a}{M} \right) \right] \right\}$$

(10)
is the radius of the co-rotating equatorial null circular geodesic, where \( \{M, a\} \) are respectively the mass and angular momentum per unit mass [40] of the central spinning Kerr black hole.

(2) The term
\[
\omega_+ (a/M) \equiv \frac{M^{1/2}}{r_{\text{ph}}^{3/2} + aM^{1/2}}
\]
is the Kepler frequency which characterizes the co-rotating equatorial null circular geodesic of the black-hole spacetime.

(3) The dimensionless functional expression of \( \beta = \beta(M, a) \) is given by [38]
\[
\beta(a/M) = \frac{(12M)^{1/2}(r_{\text{ph}} - r_+)(r_{\text{ph}} - r_-)}{r_{\text{ph}}^3 (r_{\text{ph}} - M)},
\]
where
\[
r_{\pm} = M \pm (M^2 - a^2)^{1/2}
\]
are the (outer and inner) horizon radii of the spinning Kerr black hole.

We shall henceforth focus our attention on rapidly-spinning (near-extremal) Kerr black holes, which are characterized by the simple dimensionless relation
\[
aM = 1 - \epsilon \quad \text{with} \quad 0 \leq \epsilon \ll 1.
\]

From Eqs. (13) and (14) one finds the relation
\[
\frac{r_+}{M} = 1 \pm \sqrt{2} \sqrt{\epsilon + \frac{\epsilon^{3/2}}{2\sqrt{2}}} + O(\epsilon^{5/2}).
\]

Substituting (14) into Eq. (10), one obtains the near-extremal (small-\( \epsilon \)) dimensionless functional expression
\[
M\omega_+ = \frac{1}{2} - \sqrt{3} \cdot \sqrt{\epsilon} + \frac{7}{12} \cdot \epsilon + O(\epsilon^{3/2})
\]
and
\[
\beta = \sqrt{2} \cdot \sqrt{\epsilon} - \frac{4}{3\sqrt{3}} \cdot \epsilon + O(\epsilon^{3/2}).
\]

Substituting Eqs. (17) and (18) into the resonance formula (8), one obtains the dimensionless functional expression
\[
M\omega_n = m \cdot \left( \frac{1}{2} - \sqrt{\frac{3}{2\sqrt{2}}} \cdot \sqrt{\epsilon} + \frac{7}{12} \cdot \epsilon \right) - i\left( n + \frac{1}{2} \right) \cdot \left( \sqrt{\frac{\epsilon}{2\sqrt{2}}} - \frac{13}{6\sqrt{3}} \cdot \epsilon \right) + O(\epsilon^{3/2})
\]
for the complex quasinormal resonant frequencies of rapidly-spinning (near-extremal) Kerr black holes in the eikonal large-frequency regime \( l = m >> 1 \) [see Eq. (9)].

It is physically interesting and mathematically convenient to express the analytically derived Kerr resonance spectrum (19) in terms of the black-hole Bekenstein-Hawking temperature [20, 21]
\[
T_{\text{BH}} = \frac{r_+ - r_-}{4\pi(r_+^2 + a^2)} = \frac{1}{\pi M} \cdot \left[ \sqrt{\frac{\epsilon}{2\sqrt{2}}} - \frac{\epsilon}{2} + O(\epsilon^{3/2}) \right]
\]
and the black-hole angular velocity
\[
\Omega_\text{H} = \frac{a}{r_+^2 + a^2} = \frac{1}{M} \cdot \left[ \frac{1}{2} - \frac{\sqrt{\epsilon}}{\sqrt{2}} + \frac{\epsilon}{2} + O(\epsilon^{3/2}) \right].
\]

Substituting (20) and (21) into Eq. (19), one finds the black-hole resonance formula
\[
\omega_n = \left\{ m\Omega_\text{H} \cdot \left[ 1 + \pi(4 - 2\sqrt{3})MT_{\text{BH}} \right] - i\left( n + \frac{1}{2} \right) \cdot 2\pi T_{\text{BH}} \cdot \left[ 1 - \frac{2\pi}{9} \left( 13\sqrt{3} - 18 \right) MT_{\text{BH}} \right] \right\} \cdot \left[ 1 + O(MT_{\text{BH}}) \right].
\]
Motivated by the suggested time-times-temperature (TTT) universal relaxation bound \( (2) \), we have studied, using analytical techniques, the quasinormal resonance spectra of near-extremal (rapidly-spinning) Kerr black holes.

It was first noted that perturbation modes of near-extremal Kerr black holes in the regime \( m^{-1} \cdot MT_{BH} \ll 1 \) are characterized by the simple near-extremal relation (7) are characterized by the simple near-extremal relation

\[ \Im \omega_{0} = \pi T_{BH} \cdot [1 - \frac{2\pi}{9} (13\sqrt{3} - 18)MT_{BH}] < \pi T_{BH} \quad \text{for} \quad m^{-1} \ll MT_{BH} \ll 1. \tag{23} \]

[It is worth noting that the functional relation (23) characterizes the fundamental \((n = 0)\) resonant mode of the analytically derived black-hole relaxation spectrum (23). We therefore conclude that the eikonal relaxation modes (23), which characterize the composed black-hole-field system in the dimensionless large-frequency regime \( m^{-1} \ll MT_{BH} \ll 1 \), guarantee that newly born near-extremal Kerr black holes respect the suggested TTT universal relaxation bound (2) [41].

ACKNOWLEDGMENTS

This research is supported by the Carmel Science Foundation. I would like to thank Yaël Oren, Arbel M. Ongo, Ayelet B. Lata, and Alona B. Tea for helpful discussions.

[1] H. P. Nollert, Class. Quantum Grav. 16, R159 (1999).
[2] W. H. Press, Astrophys. J. 170, L105 (1971).
[3] V. de la Cruz, J. E. Chase and W. Israel, Phys. Rev. Lett. 24, 423 (1970).
[4] C. V. Misner, Nature 227, 936 (1970).
[5] M. Davis, R. Ruffini, W. H. Press and R. H. Price, Phys. Rev. Lett. 27, 1466 (1971).
[6] E. W. Leaver, Proc. R. Soc. A 402, 285 (1985); B. Mashhoon, Phys. Rev. D 31, 290 (1985); H. P. Nollert, Phys. Rev. D 47, 5253 (1993); S. Hod, Phys. Rev. Lett. 81, 4293 (1998) [arXiv:gr-qc/9812002]; S. Hod, Phys. Rev. D 67, 081501 (2003) [arXiv:gr-qc/0301122]; S. Hod and U. Keshet, Class. Quant. Grav. 22, L71 (2005) [arXiv:gr-qc/0505112]; S. Hod and U. Keshet, Phys. Rev. D 73, 024003 (2006) [arXiv:hep-th/0506214]; U. Keshet and S. Hod, Phys. Rev. D 76, R061501 (2007) [arXiv:0705.1179]; Y. Décanini, A. Folacci, and B. Raffaelli, Phys. Rev. D 84, 084035 (2011); S. Hod, Phys. Rev. D 88, 084018 (2013) [arXiv:1311.3007].
[7] S. L. Detweiler, in Sources of Gravitational Radiation, edited by L. Smarr (Cambridge University Press, Cambridge, England, 1979).
[8] R. Ruffini and J. A. Wheeler, Phys. Today 24, 30 (1971).
[9] B. Carter, in Black Holes, Proceedings of 1972 Session of Ecole d’ete de Physique Theorique, edited by C. De Witt and B. S. De Witt (Gordon and Breach, New York, 1973).
[10] A. E. Mayo and J. D. Bekenstein, Phys. Rev. D 54, 5059 (1996).
[11] J. E. Chase, Commun. Math. Phys. 19, 276 (1970); J. D. Bekenstein, Phys. Rev. Lett. 28, 452 (1972); C. Teitelboim, Lett. Nuovo Cimento 3, 326 (1972); I. Pena and D. Sudarsky, Class. Quant. Grav. 14, 3131 (1997).
[12] J. D. Bekenstein, Phys. Rev. D 5, 1239 (1972); 5, 2403 (1972); M. Heusler, J. Math. Phys. 33, 3497 (1992); D. Sudarsky, Class. Quant. Grav. 12, 579 (1995).
[13] J. Hartle, Phys. Rev. D 3, 2938 (1971); C. Teitelboim, Lett. Nuovo Cimento 3, 397 (1972).
[14] S. Hod, Phys. Rev. D 84, 124030 (2011) [arXiv:1111.3286].
[15] It is interesting to note that some non-decaying matter fields can be supported in asymptotically flat black-hole spacetimes without being absorbed into the central supporting black holes [10, 14].
[16] S. Hod, Phys. Rev. D 86, 104026 (2012) [arXiv:1211.3202]; S. Hod, The Euro. Phys. Journal C 73, 2378 (2013) [arXiv:1311.5208]; S. Hod, Phys. Rev. D 90, 024051 (2014) [arXiv:1406.1179]; S. Hod, Phys. Lett. B 379, 196 (2014) [arXiv:1411.2609]; S. Hod and O. Hod, Phys. Rev. D 81, 061502 Rapid communication (2010) [arXiv:0910.0754]; S. Hod, Phys. Lett. B 708, 320 (2012) [arXiv:1205.1872]; S. Hod, Phys. Lett. B 751, 177 (2015); S. Hod, Class. and Quant. Grav. 33, 114001 (2016) [arXiv:1605.09605]; S. Hod, J. High Energy Phys. 01, 30 (2017) [arXiv:1612.00014].
[17] C. R. Herdeiro and E. Radu, Phys. Rev. Lett. 112, 221101 (2014); C. A. R. Herdeiro and E. Radu, Phys. Rev. D 89, 124018 (2014); C. A. R. Herdeiro and E. Radu, Int. J. Mod. Phys. D 23, 1442014 (2014); Y. Brihaye, C. Herdeiro, and E. Radu, Phys. Lett. B 739, 1 (2014); C. L. Benone, L. C. B. Crispino, C. Herdeiro, and E. Radu, Phys. Rev. D 90, 104024
It is worth mentioning that the late-time relaxation dynamics of perturbed black holes in asymptotically flat spacetimes is also characterized by inverse power-law decaying tails which reflect the late-time backscattering of dynamical perturbation fields by the small curvature potential at asymptotically far regions \([19]\).

It is worth noting again that one learns from the functional relation (5) that some linearized perturbation modes in the late-time regime with the property \(\delta^2 < 0\) [29, 30]. For near-extremal perturbation modes with the property \(\delta^2 < 0\) one finds the relation \(\Delta_{\delta^2} = \pi T_{BH}^4 \left(1 + 2|\delta|\right) \approx \pi T_{BH}^4\) [29, 30].

It is important to note that the quantum relaxation bound (2) provides a quantitative formulation for the fundamental third law of thermodynamics.

We use natural units in which \(G = c = k_B = 1\).

For brevity, we shall henceforth use units in which \(\hbar = 1\).

The amplitude \(C = C(l, m)\) depends on the angular harmonic indexes of the composed black-hole-field perturbation mode \([23, 30]\). It is worth noting that one finds \(|C| \ll 1\) for most perturbation modes \([29, 30]\).

It is worth noting that the simple relation (5) characterizes all near-extremal perturbation modes of rapidly-spinning Kerr black holes with the property \(\delta^2 > 0\) \([23, 30]\). For near-extremal perturbation modes with the property \(\delta^2 < 0\) one finds the relation \(\Delta_{\delta^2} = \pi T_{BH}^4 \left(1 + 2|\delta|\right) \approx \pi T_{BH}^4\) \([23, 30]\).

It is interesting to note that numerical studies indicate that, for a given black-hole temperature, there is at least one black-hole-field perturbation mode in the regime (2) that conforms to the TTT relaxation bound \(\delta^2 \ll 0\) [21, 29, 30]. However, to the best of our knowledge there is no general argument which guarantees that all black holes must be characterized by this property and it is therefore physically important to present a more generic mathematical proof for the validity of the suggested TTT relaxation bound (2) [or equivalently, the upper bound (4)] in black-hole physics.
dimensionless regime (6) may not conform to the TTT relaxation bound (2) for particular black-hole temperatures. The TTT relaxation bound asserts that a dynamically formed black hole should be characterized by (at least) one relaxation mode with the property (3). In the present compact paper we have explicitly proved that the resonant perturbation modes (23) of near-extremal Kerr black holes in the dimensionless regime (7) conform to the suggested TTT relaxation bound (2) (or equivalently, to the upper bound (3)).