Modified gravity with vacuum polarization

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Abstract

A brief review of cosmology in some generalized modified gravity theories with vacuum polarization is presented. Stability question of de Sitter solution is investigated.

DGP brane dynamics with vacuum polarization

The acceleration universe force us to explain this phenomena and this stimulate an interest to a number theories of modified gravity. One of such theories is the well-known Dvali-Gabadadze-Porrati (DGP) braneworld model [1,2], for instance, can lead to an accelerating universe without the presence of either a cosmological constant or some other form of dark energy. Generalizations of the DGP model can result in a phantom-like acceleration of the universe at late times, which is not excluded by observational data. We consider the simplest generic braneworld model with action of the form

\[ S_{DGP} = M^3 \left[ \int_{\text{bulk}} (R_5 - 2\Lambda)\sqrt{-GD^5}x - 2\int_{\text{brane}} K\sqrt{-g}\,dx \right] 
+ \int_{\text{brane}} (m^2 R_4 - 2\lambda)\sqrt{-g}\,dx + \int_{\text{brane}} L(g_{ab}, \rho, \phi)\sqrt{-g}\,dx. \]  

(1)

Here, \( R_5 \) is the scalar curvature of the metric \( G_{ab} \) in the five-dimensional bulk, and \( R_4 \) is the scalar curvature of the induced metric \( g_{ab} = G_{ab} - n_a n_b \) on the brane, where \( n^a \) is the vector field of the inner unit normal to the brane, which is assumed to be a boundary of the bulk space, and the notation and conventions of [3] are used. The quantity \( K = g^{ab}K_{ab} \) is the trace of the symmetric tensor of extrinsic curvature \( K_{ab} = g^c_a \nabla_c n_b \) of the brane. The symbol \( L(g_{ab}, \phi) \) denotes the Lagrangian density of the four-dimensional matter fields \( \phi \) whose dynamics is restricted to the brane so that they interact only with the induced metric \( g_{ab} \). All integrations over the bulk and brane are taken with the corresponding natural volume elements. The symbols \( M \) and \( m \) denote the five-dimensional and four-dimensional Planck masses, respectively, \( \Lambda \) is the bulk cosmological constant, and \( \lambda \) is the brane tension (it may be interpreted as cosmological constant on the brane). Note also that original DGP model apply \( \lambda = 0 \).

In \( Z_2 \) symmetry case cosmological constant is equal from two side of the brane \( \Lambda_1 = \Lambda_2 = \Lambda \), and corresponding dynamical equation in FRW background looks like [4,5]:

\[ H^2 + \frac{k}{a^2} = \frac{\rho + \lambda}{3m^2} + \frac{2}{l^2} \left[ 1 \pm \sqrt{1 + \frac{1}{l^2} \left( \frac{\rho + \lambda}{3m^2} - \frac{\Lambda}{6} - \frac{C}{a^4} \right)} \right]. \]  

(2)

Here \( l = 2m^2/M^3 \), \( \Lambda \equiv \dot{a}/a \) is the Hubble parameter, and \( \rho \) is the matter energy density on the brane. Here and below, the overdot derivative is taken with respect to the cosmological time \( t \) on the brane. The expression under square root tend to zero during the evolution. It mean that in some time all time derivative from Hubble parameter \( H, \dot{H}, ... \) tends to infinity while \( H \) is finite. It’s a new specific type of singularity and it was studied very well in the literature [6]. The term containing the constant \( C \) describes the so-called “dark radiation.” We don’t take into account this term, also we consider a spatially flat universe \( (k = 0) \). The “±” signs in the solution correspond to two branches defined by the two possible ways of bounding the Schwarzschild–(anti)-de Sitter bulk space by the brane [7,8].

Now let us briefly consider dynamic of the DGP brane. Classical dynamics depends significantly on the bulk cosmological constant \( \Lambda \) and the full picture is sufficiently complicate so we refer you to the original paper focusing on only essential point for further statement (\( \Lambda > 0 \)). The case \( \Lambda > 0 \) is shown in Fig[1]. The graph of \( (H^2, \rho_{\text{tot}}) \) in the \((H^2, \rho_{\text{tot}})\) plane in Fig[1] illustrates that in an expanding universe the matter density \( \rho \) decreases (except for a “phantom matter” which we do not consider in the present paper), and the point in the plane \((H^2, \rho_{\text{tot}})\) moves from right to left in Fig[1].
A striking feature of Fig. 1 is that the value of the Hubble parameter in the braneworld can never drop to zero. In other words, the Friedmann asymptote $H \to 0$ is absent in our case. The upper and lower branches in Fig. 1 describe the two complementary braneworld models: branches AB and DB are associated with Brane 2 and Brane 1 of [9], respectively, while branches AC and DC correspond to the lower and upper signs in (2), respectively, and describe the two branches with different embedding in the bulk. It should be noted that, in many important cases, the behaviour of the braneworld does not have any parallel in conventional Friedmannian dynamics (by this we mean standard GR in a FRW universe).

Now let us consider some quantum effects. In general, quantum effects in curved space-time can arise on account of the vacuum polarization as well as particle production. It is well known that the latter is absent for conformally invariant fields (which we shall consider here) and that, in this case, quantum corrections to the equations of motion are fully described by the renormalized vacuum energy–momentum tensor which has the form [10]:

$$\langle T_{ik} \rangle = \left( \frac{m_2}{2880\pi^2} \right) \left( R_{ik} R_{kl} - \frac{3}{2} R R_{ik} - \frac{1}{2} g_{ik} R_{lm} R^{lm} + \frac{1}{4} g_{ik} R^2 \right)$$

$$+ \left( \frac{m_3}{2880\pi^2} \right) \frac{1}{6} \left( 2 R_{;ik} - 2 g_{ik} R_{;jl} - 2 R R_{ik} + \frac{1}{2} g_{ik} R^2 \right),$$

where $m_1, m_2$ depend upon the spin weights of the different fields contributing to the vacuum polarization. This effect is known for a long time in cosmology. For example, it was demonstrated the possibility of singularity problem solution by some specific changes of $m_2$ and $m_3$ [11].

Since we work in flat FRW space-time, it is comfortable to use next relation:

$$\rho_q = k_2 H^4 + k_3 (2 \dot{H} H + 6 \dot{H} H^2 - \dot{H}^2).$$

Where parameters $k_2$ and $k_3$ take the next form [12, 13, 14]:

$$k_2 = \frac{m_2}{60(4\pi)^2} = \frac{N + 11 N_{1/2} + 62 N_1 + 1411 N_2 - 28 N_{HD}}{60(4\pi)^2},$$

$$k_3 = \frac{m_3}{60(4\pi)^2} = \frac{N + 6 N_{1/2} + 12 N_1 + 611 N_2 - 8 N_{HD}}{60(4\pi)^2}.$$  

Here $N_i$ is number of the fields with spin $i$ contributing to the vacuum polarization: $N$ – a number of scalar fields, $N_{1/2}$ – a number of fermion fields, $N_1$ – a number of vector fields, $N_2$ ($= 0$ or $1$) – a number of gravitons, $N_{HD}$ – a number of conformal scalar fields. Note also that there is strong restriction on $k_2$ and $k_3$ in usual 4D-space-time. At least need $k_3 < 0$ with any $k_2$ for de Sitter (and in particular Minkovsky!) solution stability. Since we see that Minkovsky solution is stable (there isn’t particle production in conformal flat vacuum) we may use condition $k_3 < 0$ as observational data.
In order to assess the effects of the vacuum polarization on the dynamics of the braneworld, one must add $\rho_q$ to the matter density in (2) so that $\rho \rightarrow \rho + \rho_q$ in those equations. An important consequence of this operation is that the form of the equation of motion changes dramatically — the original algebraic equation changes to a differential equation! The dynamical equation (2) now takes the form

$$\ddot{H} = \frac{1}{2} \dot{H}^2 - 3\dot{H}H^2 + (2k_3)^{-1}\left(-k_3H^4 + 3m^2H^2 - \rho_{\text{tot}} \pm 3M^3 \sqrt{H^2 - \Lambda/6}\right).$$  \hspace{1cm} (7)

The goal of the present paper is to study the stability of the classical solutions when vacuum polarization terms are taken into account. The $k_2$-term in (4), which does not contain time derivatives of $H$, can only change the position of the future stable points. On the contrary, due to the $k_3$-term in (4), some classical solutions can lose stability. Therefore, for simplicity (and without loss of generality), we set $k_2 = 0$ in our calculations. If the brane has nonzero tension $\lambda$, the stationary points of (7) in the case $k_2 = 0$ can be found by substituting $\lambda$ into (2) and setting $\rho = 0$. After that, we linearize Eq. (7) at these stationary points and find the eigenvalues of the corresponding linearized system. The condition of stability of the stationary point is that its eigenvalues’s real parts are negative. The eigenvalues at the stationary points where $\dot{H} = 0$ are given by

$$\mu_{1,2} = \frac{1}{2} \left(f_1 \pm \sqrt{f_1^2 + 4f_2}\right),$$  \hspace{1cm} (8)

where we have made the notation

$$f_1 = -3H,$$  \hspace{1cm} (9)

$$f_2 = \frac{1}{2k_3} \left(1 + \frac{\lambda}{3m^2H^2} \pm \frac{2l^{-1}\Lambda/6}{H^2 \sqrt{H^2 - \Lambda/6}}\right).$$  \hspace{1cm} (10)

Two different signs in Eq. (10) correspond to two different equations of motion, while, in Eq. (8), we have two different eigenvalues of a single equation.

Since $f_1$ is negative, the eigenvalue $\mu_2$ corresponding to the “−” sign in Eq. (8) is also always negative. Moreover, $\mu_1$ is positive if and only if $f_2$ is positive. As a result, the stability of a fixed point is equivalent to the condition $f_2 < 0$. More careful investigation of this condition (for details see original paper [15]) show that part BC on the Fig. 1 corresponding to effective phantom behavior is unstable with respect vacuum polarization.

**General bran dynamics with vacuum polarization**

Now let us generalize obtaining result. We investigate any theory, which lead to dynamical equation in the next form:

$$\rho = F(H),$$  \hspace{1cm} (11)

where $F(H)$ may be any algebraical function, which is don’t contain time derivative of $H$. First of all we investigate the case $k_2 = 0$. Substituting in (11) expression (4) for $\rho_q$ we may rewrite equation as dynamical system:

$$\dot{H} = C, \hspace{1cm} \dot{C} = \frac{C^2}{2m} - 3CH + \frac{1}{2k_3H}F(H) \equiv f(H, C).$$  \hspace{1cm} (12)

Linearizing this system at the fixed point $(H_0, 0)$

$$\dot{H} = C, \hspace{1cm} \dot{C} = (\frac{\partial f}{\partial C})_0 C + (\frac{\partial f}{\partial H})_0 H,$$  \hspace{1cm} (13)

we find its eigenvalues

$$\mu_{1,2} = \frac{1}{2} \left[(\frac{\partial f}{\partial C})_0 \pm \sqrt{(\frac{\partial f}{\partial C})_0^2 + 4(\frac{\partial f}{\partial H})_0}\right].$$  \hspace{1cm} (14)
Now let us consider effects do not influence on de Sitter stability condition. Note also that vacuum polarization effects to account of vacuum polarization effect can’t change denominator’s sign. It means that in this case quantum stability, but it may change dynamical equation and fantom regimes disappear at all [16].

To (16) is given by expression (21) for the model [22] for this class of theories may be written in the next form [17] (for a more general review of $f(R)$-theories which more popular in recent time):

$$S = S_m + \frac{1}{2\chi} \int d^4x \sqrt{-g} f(R),$$

(16)

here $\chi = 8\pi G$ and for the sake of simplicity we set $2\chi = 1$. The general equation of motion corresponding to (16) is given by

$$g_{ik} f^{il} - f_{il;k} + f'R_{ik} - \frac{1}{2} fg_{ik} = T_{ik}.$$  

(17)

The trace of equation of motion (17) reads

$$3 f^{il}_l + f'R - 2f = T.$$  

(18)

So existence condition of de Sitter solution in vacuum is [20] [21]

$$2f(R_0) - R_0 f'(R_0) = 0,$$

(19)

and condition of its stability following from [18] is given by

$$\frac{f'(R_0)}{R_0 f''(R_0)} - 1 > 0.$$  

(20)

Now let us take into account the vacuum polarization effects. For the sake of simplicity we investigate only $k_2 = 0$ case. So we find from [3] ($T^i_l = -\frac{1}{2\chi} R^i_l$). Substituting this relation into (18) we find that this equation is restored to a normal state (without ($T^i_l$) contribute) by the $f \rightarrow f - \frac{k_3}{2\chi} R^2$ transformation. And substituting this one into (20) we find new de Sitter stability condition:

$$\frac{f'(R_0) - R_0 f''(R_0)}{R_0 (f''(R_0) - \frac{k_3}{2\chi})} > 0,$$

(21)

which is turn into the expression (20) by the limit $k_3 = 0$. For instance, let us consider meaning of new expression [24] for the model [22] $f(R) = R + R^{-m} + R^n$. We see that only denominator of the (21) has change when vacuum polarization take into account, while numerator is unchangeable. From another hand for this class of theories $f'' = m(m + 1)R^{-m-2} + n(n + 1)R^{n-2} > 0$ and since $k_3 < 0$ taking into account of vacuum polarization effect can’t change denominator’s sign. It means that in this case quantum effects do not influence on de Sitter stability condition. Note also that vacuum polarization effects to above $f(R)$ gravity model may suppress instabilities as it was noted in [23].
Conclusion

Some modified gravity theories was studied with respect to vacuum polarization. It was demonstrate instability of effective fantom regimes in the brane cosmology caused by vacuum polarization. Modified condition of the vacuum de Sitter solution stability in $f(R)$ theories has been derived.

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