Dual algebraic structures for pairing models

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Abstract. Duality relations for pairing models are discussed in an unified framework for bosons and fermions. Quantum phase transitions for pairing systems are studied and analogies and differences between bosonic and fermionic systems are highlighted.

1. Introduction
Pairing models acquired an important role in physics when the theory of superconductivity was introduced (1957) [1]. In 1958, they were introduced in the description of finite systems: atomic nuclei [2]. In recent years they have become again of interest for applications to mesoscopic systems: atomic condensates, atomic clusters, metallic grains, etc. For finite systems, the Hamiltonian operator for pairing models can be written as

\[ H = \sum_{km} \varepsilon_{km} a_{km}^\dagger a_{km} + \frac{1}{4} \sum_{kk'mm'} G_{k,k'} (-)^{j_{k'} - m' - m} a_{k',m'}^\dagger a_{k,m}^\dagger (-)^{j_{k' - m} - m} a_{k,m} a_{k,m}, \]  

(1)

where \( k = 1, 2, ..., j_k \) = integer (bosons) and \( j_k \) = half-integer (fermions). It is of importance to provide a classification scheme for pairing models, and to find the eigenvalues and eigenvectors of \( H \). Duality relations provide a great simplification in solving this problem.

2. Dual algebraic structures
The fundamental algebra, \( G \), of many-boson and many-fermion systems is composed of bilinear products of creation and annihilation operators

\[ a_{m',m}^\dagger a_{m',m}^\dagger a_{m} a_{m} ; \ (m', m) = 1, ..., n ; \ G \equiv \left\{ \begin{array}{c} \text{Sp}(2n, \mathbb{R}) \\ \text{SO}(n) \end{array} \right\} \]  

(2)

If bosons transform as irreps of SO(3) and fermions as irreps of SU(2) labelled by \( j \) (rotationally invariant systems), it is convenient to use \( m = j, j - 1, ..., -j \), and time-reversed creation and annihilation operators \( a_{m}^\dagger \rightarrow a_{j,m}^\dagger \) and \( a_{m}^\dagger \rightarrow (-)^{j-m} a_{j,-m}^\dagger \). For a single value of \( j \), the number of allowed states is \( n = 2j + 1 \).

The algebra \( G \) has two commuting subalgebras:

1. Number-conserving, \( G_1 \). This is the unitary algebra \( U(n) \) composed of elements \( G_{m'm} = a_{m'}^\dagger a_{m} \) and classification scheme

\[ \left\{ \begin{array}{c} \text{Sp}(2n, \mathbb{R}) \\ \text{SO}(2n) \end{array} \right\} \supset U(n) \supset \left\{ \begin{array}{c} \text{SO}(n) \\ \text{Sp}(n) \end{array} \right\} \]

(3)
Here SO($n$) and Sp($n$) leave invariant $\sum m a_m^+ a_m^\dagger |0\rangle$. The representations of U($n$), characterized by the total number of particles $N$, are totally symmetric for bosons and totally antisymmetric for fermions, with Young tableaux

$$\begin{array}{cccc}
\Box & \Box & \ldots & \Box \\
\Box & \Box & \ldots & \Box \\
\Box & \Box & \ldots & \Box \\
\end{array}$$

$$N_B \text{(bosons)} \quad N_F \text{(fermions)}$$

The representations of SO($n$) and Sp($n$) contained in U($n$) are also totally symmetric (bosons) and antisymmetric (fermions). They are characterized by the quantum number $v$, called seniority, introduced by Racah in 1949 [3]. The branching of U($n$) into SO($n$) and Sp($n$) was worked out years ago and is given by

$$\begin{align*}
U(n) &\supset SO(n) \quad v = (N \mod 2), \ldots, N - 2, N \\
U(n) &\supset Sp(n) \quad v = (N' \mod 2), \ldots, N' - 2, N' \\
N' &\equiv \min(N, n - N)
\end{align*}$$

(2) Non-numberconserving, G2. This is the so-called quasi-spin algebra composed of three elements

$$\begin{align*}
S^+ &= \frac{1}{2} \sum \bar{m} a_m^+ a_{\bar{m}}, \\
S^- &= \frac{1}{2} \sum m a_{\bar{m}} a_m, \\
S_0 &= \frac{1}{4} \sum m \left( a_m^+ a_{\bar{m}} + \vartheta a_{\bar{m}} a_m^\dagger \right),
\end{align*}$$

where $\vartheta = +1$ for bosons, SU(1,1), and $\vartheta = -1$ for fermions, SU(2). The representations of SU(1,1)$\supset$U(1) and SU(2)$\supset$U(1) are characterized by $S$ (quasi-spin) and $M$ (quasi-spin projection). This gives the alternate decomposition

$$\begin{align*}
\text{Sp}(2n,\mathbb{R}) &\supset \text{SU}(1,1) \supset \text{U}(1) & \text{bosons} \\
\text{SO}(2n) &\supset \text{SU}(2) \supset \text{U}(1) & \text{fermions}
\end{align*}$$

(7)

States of pairing models can thus be classified in terms of two commuting algebras G1 and G2 with labels in one-to-one correspondence (duality relations)

$$\begin{align*}
S &= \frac{1}{2} (\Omega + \vartheta v), \\
M &= \frac{1}{2} (N + \vartheta \Omega),
\end{align*}$$

where $\Omega = (j + \frac{1}{2}) = \frac{n}{2}$. For a complete classification scheme of many-body bosonic and fermionic systems, the algebras SO($n$) and Sp($n$) need to be further reduced

$$\begin{align*}
\text{Sp}(2n,\mathbb{R}) &\supset \left\{ \begin{array}{c} U(n) \supset \text{SO}(n) \supset \ldots \supset \text{SO}(3) \supset \text{SO}(2) \\
\text{SU}(1,1) \supset \text{U}(1) \otimes \text{SO}(n) \supset \ldots \supset \text{SO}(3) \supset \text{SO}(2) \end{array} \right\} \\
\text{SO}(2n) &\supset \left\{ \begin{array}{c} U(n) \supset \text{Sp}(n) \supset \ldots \supset \text{SO}(3) \supset \text{SO}(2) \\
\text{SU}(2) \supset \text{U}(1) \otimes \text{Sp}(n) \supset \ldots \supset \text{Spin}(3) \supset \text{Spin}(2) \end{array} \right\}
\end{align*}$$

(9)

3. Two-level systems

Two-level systems have a variety of applications in physics, especially those in which $j_1 = 0$ (scalar boson) and $j_2 = \ell \ (2\ell + 1$-fold degenerate boson), called s-b boson systems [4]. The group theory of generic two-level systems has now been completely worked out [5]. Duality relations provide a great simplification in the study of two-level systems. The fundamental algebra for these systems is still $G = \left\{ \begin{array}{c} \text{Sp}(2n,\mathbb{R}) \\
\text{SO}(2n) \end{array} \right\}$ but with $n = n_1 + n_2$. 


3.1. Dual algebras for two-level models

The dual algebras for two-level models are:

(1) The number-conserving algebra \( U(n_1 + n_2) \) with elements \( G^{(g)}_{a_1a_2} = (a_1^+ \times \tilde{a}_1)^{(g)} \), \( G^{(g)}_{a_2a_1} = (a_1^+ \times \tilde{a}_1)^{(g)} \), \( G^{(g)}_{a_2a_2} = (\tilde{a}_2^+ \times a_2)^{(g)} \). The algebra \( U(n_1 + n_2) \) has in turn several (non-dual) algebras, two of which are of importance for the pairing problem. [The two subalgebras correspond to the two physically relevant cases of \( \varepsilon = \varepsilon_2 - \varepsilon_1 = 0 \) (strong coupling limit) and \( \varepsilon = \text{large (weak coupling limit)} \)]. This can be summarized in the schemes

\[
\begin{align*}
U(n_1 + n_2) & \supset \binom{\SO(n_1 + n_2)}{v} \bigg/ \binom{\U_1(n_1) \otimes \U_2(n_2)}{v_1} \otimes \SO_2(n_2) \supset \ldots \quad (10) \\
U(n_1 + n_2) & \left\{ \begin{array}{c}
\supset \binom{\SP(n_1 + n_2)}{v} \\
\U_1(n_1) \otimes \U_2(n_2)
\end{array} \bigg/ \binom{\SP_1(n_1)}{v_1} \otimes \SP_2(n_2) \supset \ldots \quad (11)
\end{align*}
\]

with branching \( U(n_1 + n_2) \supset \SO(n_1 + n_2) \) given by \( v = (N \text{ mod } 2), \ldots, N - 2, N \) and branching \( \SO(n_1 + n_2) \supset \SO(n_1) \otimes \SO(n_2) \) given by \( v = v_1 + v_2 + 2n_v \), \( (n_v = 0, 1, \ldots, \lfloor v/2 \rfloor) \), for bosons, and

\[
\begin{align*}
U(n_1 + n_2) & \supset \binom{\SP(n_1 + n_2)}{v} \bigg/ \binom{\U_1(n_1) \otimes \U_2(n_2)}{v_1} \otimes \SP_2(n_2) \supset \ldots \quad (11)
\end{align*}
\]

with branching \( U(n_1 + n_2) \supset \SP(n_1 + n_2) \) given by \( v = (N' \text{ mod } 2), \ldots, N' - 2, N' \), where \( N' = \min(N, n - N) \), and branching \( \SP(n_1 + n_2) \supset \SP(n_1) \otimes \SP(n_2) \) given by \( v = v_1 + v_2 + 2n_v \), \( (n_v = 0, 1, \ldots, \lfloor v/2 \rfloor) \), \( |v_1 - v_2| = \frac{1}{2}(n_1 - n_2) \leq \frac{1}{2}(n_1 + n_2) - v \), for fermions.

Eigenvalues of the quadratic Casimir operators of algebras appearing in the chains above in the symmetric representations of \( \SO(n) \) and antisymmetric representations of \( \SP(n) \) can be obtained from the basic relations

\[
\bigg\langle \binom{C_2[\SO(n)]}{v} \bigg\rangle = 2v (\vartheta v + n - 2\vartheta). \quad (12)
\]

For \( U(n) \) the linear Casimir operator has trivial eigenvalues \( \langle C_1[U(n)] \rangle = N \).

(2) Number-nonconserving subalgebras \( \SU(1,1) \otimes \SU_2(1,1) \) for bosons and \( \SU(2) \otimes \SU_2(2) \) for fermions. These are constructed from the quasi-spin algebra

\[
[S_0, S_+] = +S_+, \quad [S_0, S_-] = -S_-, \quad [S_+, S_-] = -2\vartheta S_0, \quad (13)
\]

with quadratic Casimir operator \( S^2 \) and eigenvalues \( \langle S^2 \rangle = S(S - \vartheta) \), specifically

\[
\langle S^2 \rangle = \begin{cases} 
S(S - 1) & \text{with } \begin{cases} S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \\
S = 0, \frac{1}{2}, 1, \ldots 
\end{cases} \\
S_0 = 0, S, S + 1, S + 2, \ldots \\
S_0 = -S, \ldots, +S 
\end{cases} 
\quad (14)
\]

for bosons \( (\vartheta = 1) \) and fermions \( (\vartheta = -1) \) respectively. [The boson representations are projective representations of \( \SU(1,1) \) [6], [7]]. Introducing a quasi-spin algebra for each of the two systems, \( k = 1, 2 \), satisfying duality relations

\[
S_k = \frac{1}{2} (\Omega_k + \vartheta v_k) \\
M_k = \frac{1}{2} (N_k + \vartheta \Omega_k) \quad (15)
\]
where $M_k = \langle S_{k0} \rangle$, $\Omega_k = \frac{1}{2}(2j_k + 1)$, and $\nu_k = 0, 1, 2, ...$ [$\nu_k \leq N_k$ for bosons and $\nu_k \leq \min(N_k, 2\Omega_k - N_k)$ for fermions], one can obtain the elements of the combined algebra by simple addition

$$S_+ = S_{1+} + \sigma S_{2+}, \quad S_- = S_{1-} + \sigma S_{2-}, \quad S_0 = S_{10} + S_{20},$$

(16)

where $\sigma = \pm 1$ signifies an inner automorphism. The label $S$ characterizing the representations of the combined algebras is given by the rules

$$\begin{cases} S \geq S_1 + S_2 \\ |S_1 - S_2| \leq S \leq |S_1 + S_2| \end{cases}$$

for $\text{SU}(1,1)$ bosons and

$$\begin{cases} S \geq S_1 + S_2 \\ |S_1 - S_2| \leq S \leq |S_1 + S_2| \end{cases}$$

for $\text{SU}(2)$ fermions.

Thus, in the dual space we have

$$\text{SU}_1(1,1) \otimes \text{SU}_2(1,1) \supset \left\{ \begin{array}{c} \text{SU}_1(1,1) \\ S \\ \text{U}_1(1) \otimes \text{U}_2(1) \\ M_1 \otimes M_2 \end{array} \right\} \supset \text{U}_{12}(1) \quad (18)$$

for bosons and

$$\text{SU}_1(2) \otimes \text{SU}_2(2) \supset \left\{ \begin{array}{c} \text{SU}_1(2) \\ S \\ \text{U}_1(1) \otimes \text{U}_2(1) \\ M_1 \otimes M_2 \end{array} \right\} \supset \text{U}_{12}(1) \quad (19)$$

for fermions.

3.2. Hamiltonian duality relations

The generic pairing Hamiltonian for multi-level models is given in (1). This Hamiltonian can be easily rewritten in terms of the dual non-number conserving quasi-spin algebra

$$H = \sum_k \varepsilon_k (2S_{k0} - \vartheta \Omega_k) + \sum_{k'k} G_{kk'} S_{k'} S_{k'\mp} S_{k,\mp}.$$  

(20)

$H$ conserves quasispin $S_k$ (or seniority $\nu_k$) and $S_0 = \frac{1}{2}(N + \vartheta \Omega)$. It can be easily diagonalized in the basis $|S_1 M_1, S_2 M_2, ...\rangle$.

3.3. Dynamic symmetries of the two-level pairing Hamiltonian

(1) Number-conserving subalgebra $\text{U}(n_1 + n_2)$. The generic two-level pairing Hamiltonian $H$ can be re-written in terms of all the invariants in (10)

$$H = aC_1 [U(n_1 + n_2)] + b_1 C_1 [U_1(n_1)] + b_2 C_1 [U_2(n_2)] + b \left\{ \begin{array}{c} C_2 [\text{SO}(n_1 + n_2)] \\ C_2 [\text{Sp}(n_1 + n_2)] \end{array} \right\}$$  

(21)

for bosons and fermions respectively. Dynamical symmetries are situations in which $H$ contains only invariants of a chain [8, Ch.11]. We have two possible dynamical symmetries:

(I) $b = 0$ (weak coupling limit) with chain $U(n_1 + n_2) \supset U(n_1) \otimes U(n_2)$ and eigenvalues

$$E^{(I)} = aN + b_1 N_1 + b_2 N_2.$$  

(22)

(II) $b_1 = b_2 = 0$ (strong coupling limit) with chains $U(n_1 + n_2) \supset \text{SO}(n_1 + n_2)$ for bosons and $U(n_1 + n_2) \supset \text{Sp}(n_1 + n_2)$ for fermions, and eigenvalues

$$E^{(II)} = aN + 2b \vartheta (\nu n_1 + n_2 - 2\vartheta).$$  

(23)
To the Hamiltonian in (21), also invariants of the common subalgebras can be added, still retaining the dynamical symmetry

\[ H' = H + c_1 \left\{ \frac{C_2[\text{SO}(n_1)]}{C_2[\text{Sp}(n_1)]} \right\} + c_2 \left\{ \frac{C_2[\text{SO}(n_2)]}{C_2[\text{Sp}(n_2)]} \right\}. \]

(24)

The energy formulas are modified to:

(I) \( b = 0 \)

\[ E^{(I)} = aN + b_1 N_1 + b_2 N_2 + 2c_1 v_1 (\vartheta v_1 + n_1 - 2\vartheta) + 2c_2 v_2 (\vartheta v_2 + n_2 - 2\vartheta), \quad (25) \]

(II) \( b_1 = b_2 = 0 \)

\[ E^{(II)} = aN + 2bv (\vartheta v + n_1 + n_2 - 2\vartheta) + 2c_1 v_1 (\vartheta v_1 + n_1 - 2\vartheta) + 2c_2 v_2 (\vartheta v_2 + n_2 - 2\vartheta). \quad (26) \]

Spectra of pairing systems in the strong coupling limit are shown in Fig.1, illustrating the difference between bosons and fermions.

**Figure 1.** Energy diagram for the bosonic \( \text{SO}(n_1 + n_2) \) and fermionic \( \text{Sp}(n_1 + n_2) \) dynamical symmetries of the two-level pairing model. The degeneracy within irreps of the two-level algebra (brackets labeled by \( v \)) is split according to the single-level algebra irrep \( (v_1 v_2) \). (a) Energy levels for the bosonic \( \text{SO}(6) \oplus \text{SO}(3) \oplus \text{SO}(3) \) dynamical symmetry Hamiltonian \( (j_1 = j_2 = 1) \). (b) Energy levels for the fermionic \( \text{Sp}(8) \oplus \text{Sp}(4) \oplus \text{Sp}(4) \) dynamical symmetry Hamiltonian \( (j_1 = j_2 = \frac{3}{2}) \). The Hamiltonain in each case is chosen as in (24) with \( b = -\frac{1}{2} \) and \( c_1 = c_2 = +\frac{1}{2} \). The total occupation in both cases is \( N = 4 \), which for the fermionic example gives half-filling.

(2) Number-nonconserving subalgebras \( \text{SU}_1(1,1) \oplus \text{SU}_2(1,1) \) for bosons and \( \text{SU}_1(2) \oplus \text{SU}_2(2) \) for fermions. The generic two-level pairing Hamiltonian in the dual space is

\[ H = \varepsilon_1 (2S_{10} - \vartheta \Omega_1) + \varepsilon_2 (2S_{20} - \vartheta \Omega_2) + G_{11} S_{1+} S_{1+} + G_{22} S_{2+} S_{2+} + G_{12} (S_{1+} S_{2+} + S_{2+} S_{1+}). \]

(27)

Dynamical symmetries can be easily constructed in this space.

(I) Weak coupling limit, \( G_{11} = G_{22} = G_{12} = 0 \), with algebra chains \( \text{SU}_1(1,1) \oplus \text{SU}_2(1,1) \oplus \text{U}_1(1) \oplus \text{U}_2(1) \) (bosons) and \( \text{SU}_1(2) \oplus \text{SU}_2(2) \oplus \text{U}_1(1) \oplus \text{U}_2(1) \) (fermions), and eigenvalues

\[ E^{(I)} = \varepsilon_1 (2M_1 - \vartheta \Omega_1) + \varepsilon_2 (2M_2 - \vartheta \Omega_2). \]
By duality this can be rewritten as in (22) (with $\alpha = 0$)

$$E^{(I)} = \varepsilon_1 N_1 + \varepsilon_2 N_2.$$ (29)

(II) Strong coupling limit, $\varepsilon_1 = \varepsilon_2 = 0$, $G_{11} = G_{22} = G$, with chains $SU_1(1,1) \otimes SU_2(1,1) \supset SU_{12}(1,1)$ (bosons) and $SU_1(2) \otimes SU_2(2) \supset SU_{12}(2)$ (fermions), Hamiltonian $H = GS_+ S_-$, and eigenvalues

$$E^{(II)} = G [S(S - \vartheta) - S_o(S_o - \vartheta)].$$ (30)

By duality the Hamiltonian can be rewritten as

$$H = G \frac{1}{4} \left[ -\vartheta C_1[U(n_1 + n_2)] + C_2[U(n_1 + n_2)] + \frac{1}{2} \left\{ \begin{array}{c} C_2[SO(n_1 + n_2)] \\ C_2[Sp(n_1 + n_2)] \end{array} \right\} \right]$$ (31)

with eigenvalues

$$E^{(II)} = G \frac{1}{4} \vartheta [N(N + 2\vartheta \Omega - 2) - \nu(\nu + 2\vartheta \Omega - 2)],$$ (32)

a variation of (25). [Cases when $G_{11} \neq G_{22} \neq G_{12}$ can also be solved]. The eigenvalues of $H$ in the strong coupling limit (symmetry II) are shown in Fig.2, again emphasizing the difference between bosons and fermions.

Figure 2. Eigenvalues of the pairing interaction term $4S_+ S_-$, which determines the energy spectrum of the two-level pairing model in the strong-coupling limit, shown for (a) bosonic and (b) fermionic systems, as a function of filling $N$. The eigenvalues are given by (30). The axes are labeled generically, to indicate the asymptotic (large-$\Omega$) dependence, but the specific points shown for illustration are calculated for $\Omega = 50$.

4. Pairing phase transitions
Quantum phase transitions (QPT) are qualitative changes in the ground state of a system induced by a change in one (or more) parameters appearing in the quantum Hamiltonian describing the system. For two phases, $\alpha$ and $\beta$, it is convenient to write

$$H = (1 - \xi) H_\alpha + \xi H_\beta,$$ (33)

where $\xi = [0, 1]$ is the control parameter. For pairing models the two phases are the weak coupling phase, $\alpha$, and the strong coupling phase, $\beta$. QPTs in pairing models are induced by a change in the level splitting, $\varepsilon/G$. They can be studied both classically and quantum mechanically [9]. Duality relations are useful also in the study of QPTs in pairing models.
4.1. Quantal analysis

A convenient way to analyze simultaneously pairing phase transitions in both bosonic and fermionic systems is to introduce the Hamiltonian

\[ H = \frac{(1 - \xi)}{N} N^2 \pm \frac{\xi}{N^2} 4S_+ S_-, \]  

and diagonalize it as a function of \( \xi \), thus constructing the so-called correlation diagram. The \( \pm \) signs in (34) correspond to repulsive or attractive interactions. For purposes of presentation it is convenient to consider, for repulsive pairing, the Hamiltonian

\[ H_+ = \frac{(1 - \xi)}{N} N^2 + \frac{\xi}{N^2} \left[ 4S_+ S_- - \left\{ \frac{N(N + 2\Omega - 2)}{N(2\Omega - N + 2)} \right\} \right], \]  

and, for attractive pairing,

\[ H_- = \frac{(1 - \xi)}{N} N^2 - \frac{\xi}{N^2} 4S_+ S_- \]  

The correlation diagram of two-level pairing models is shown in Fig. 3.

![Figure 3](image-url)

**Figure 3.** Eigenvalues of the bosonic two-level pairing model, with level degeneracies \( n_1 = n_2 = 5 \) (at left), and fermionic two-level pairing model, with level degeneracies \( n_1 = n_2 = 50 \) (at right), for repulsive (at top) and attractive (at bottom) pairing interactions, shown for the \( (v_1 v_2) \) subspace, as functions of the control parameter \( \xi \) between the weak- and strong-coupling limits. All calculations are for \( N = 50 \), thus for \( \Omega \ll N \) in the boson case and \( \Omega = N \) (half-filling) in the fermionic case. The alternative regime in which the bosonic system also has \( \Omega \approx N \) is shown (specifically, for \( n_1 = n_2 = 51 \) and \( N = 50 \)) in the inset to panel (a). The Hamiltonians \( H_\pm \) of (35) and (36) are used in the calculations.
4.2. Classical analysis
In order to do the classical analysis, one needs to introduce the geometry of pairing models. In s-b models this has been done by introducing the coset \( U(n_2+1)/U(n_2) \otimes U(1) \) with dimension \( \text{dim} = 2n_2 \). For general two-level pairing models, one needs to introduce cosets \( U(n_1+n_2)/U(n_1) \otimes U(n_2) \) with dimension \( \text{dim} = 2n_2n_1 \). The classical analysis in this multidimensional space is rather difficult. Instead, in the dual space, the classical analysis can be done easily by introducing cosets \( SU(1,1)/SO(1,1) \) for bosons or \( SU(2)/SO(2) \) for fermions. This analysis will not be discussed here.

4.3. Differences and similarities between bosonic and fermionic systems
The onset of criticality in pairing systems depends on the number of particles, \( N \), and the available states, \( \Omega \). The ratio \( N/\Omega \) is called filling. For bosonic systems, one can take the limit \( N \to \infty \). The onset of critical phenomena occurs when \( N \gg \Omega \). For fermionic systems the occupancy is limited to \( n_1 + n_2 \). The limit \( N \to \infty \) must be taken simultaneously as \( \Omega \to \infty \). The case of \( N = 2\Omega \) (complete filling) is trivial, since all states are occupied. Phase transitions in fermionic systems are best studied at half-filling, \( N = \Omega \). Also, as one can see from Fig. 3, a fermionic system with attractive interactions has similar critical properties of a bosonic system with repulsive interactions. Finally, even for bosonic and fermionic systems with the same number of available states, fixed \( n_1 + n_2 \), there are differences, as shown in Fig. 4. While for bosonic systems the critical behavior does not depend on the individual degeneracies, \( n_1 \) and \( n_2 \), for fermionic systems there is a dramatic dependence.

![Figure 4. Eigenvalues for the bosonic two-level pairing model, with level degeneracies (a) \( n_1 = 1 \) and \( n_2 = 5 \) or (b) \( n_1 = 3 \) and \( n_2 = 3 \), and for the fermionic two-level pairing model, with level degeneracies (c) \( n_1 = 2 \) and \( n_2 = 18 \) or (d) \( n_1 = 10 \) and \( n_2 = 10 \), as a function of the control parameter \( \xi \), between the weak- and strong-coupling limits. All calculations are for \( N = 10 \). Eigenvalues are shown only for the lowest seniority subspaces (\( v_1 v_2 \)), specifically those with \( v_1 + v_2 \leq 2 \). The Hamiltonians \( H_\pm \) of (35) and (36) are used in the calculations.](image-url)
5. Conclusions
The analysis of two-level pairing models has been completed. This analysis can be easily
generalized in the dual space to multi-level systems by combining quasi-spin. The only Wigner-
Racah algebra needed in this space is that of SU(2) for fermions or SU(1,1) for bosons. Multilevel
systems are of interest in a variety of fields. Two examples are: (a) Nuclear physics, where usually
one has to deal with ~5-6 levels with \( j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots \) and (b) metallic grains where one
has to deal with ~100 levels but with \( j = \frac{1}{2} \).

For many-body finite systems with interactions other that pairing, one needs further explicit
breaking of SO(\( n \)) and Sp(\( n \)). Duality relations are also of help in these cases, since they allow
a simple construction of a basis for the diagonalization of \( H \), as for example, in the Interacting
Boson Model of nuclei [10].

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References
[1] Bardeen J, Cooper L N and Schrieffer J R 1957 *Phys. Rev.* **106** 162; **108** 1175
[2] Bohr A, Mottelson B R and Pines D 1958 *Phys. Rev.* **110** 936
[3] Racah G 1949 *Phys. Rev.* **76** 1352
[4] Cejnar P and Iachello F 2007 *J. Phys.* A: Math. Theor. **40** 581
[5] Caprio M A, Skrabacz J H and Iachello F 2011 *J. Phys.* A: Math. Theor. **44** 075303
[6] Ui H 1968 *Ann. Phys.* **49** 69
[7] Arima A and Iachello F 1976 *Ann. Phys.* (N.Y.) **99** 253
[8] Iachello F 2006 *Lie Algebras and Applications*, Lecture Notes in Physics, Vol. 708 (Berlin: Springer)
[9] Iachello F 2011 *Rivista del Nuovo Cimento* **34** 617
[10] Iachello F and Arima A 1987 *The Interacting Boson Model* (Cambridge: Cambridge University Press)