FROM LOW-RANK APPROXIMATION TO AN EFFICIENT RATIONAL KRYLOV SUBSPACE METHOD FOR THE LYAPUNOV EQUATION

D. A. KOLESNIKOV ‡ AND I. V. OSELEDETS¶

Abstract. We propose a new method for the approximate solution of the Lyapunov equation with rank-1 right-hand side, which is based on extended rational Krylov subspace approximation with adaptively computed shifts. The shift selection is obtained from the connection between the Lyapunov equation, solution of systems of linear ODEs and alternating least squares method for low-rank approximation. The numerical experiments confirm the effectiveness of our approach.

Key words. Lyapunov equation, rational Krylov subspace, low-rank approximation, model order reduction

AMS subject classifications. 65F10, 65F30, 15A06, 93B40.

1. Introduction. Let $A$ be an $n \times n$ stable matrix (i.e. $(A + A^*) < 0$) and $y_0$ is a vector of length $N$. We consider the continuous-time Lyapunov equation with rank-1 right-hand side:

$$AX + XA^\top = -y_0y_0^\top,$$

For large $n$ it is impossible to store $X$, thus a low-rank approximation of the solution is sought:

$$X \approx UZU^\top, \quad U \in \mathbb{R}^{n \times r}, \quad Z \in \mathbb{R}^{r \times r}.$$  

Lyapunov equation has fundamental role in many application areas such as signal processing and system and control theory [25, 1, 35, 8, 20]. There are many approaches for the solution of the Lyapunov equation. Alternating directions implicit (ADI) methods are powerful techniques that arise from the solution methods for elliptic and parabolic partial differential equations [19, 34, 22, 17, 5, 21, 2].

Krylov subspace methods have been successful in solving linear systems and eigenvalues problems. They utilizes Arnoldi-type or Lanczos-type algorithms to construct low-rank approximation using Krylov subspaces [16, 18, 16, 27, 15, 26, 23, 31, 30]. Krylov subspace methods have advantage in simplicity but the convergence can be slow for ill-conditioned $A$.

Rational Krylov subspace methods (extended Krylov subspace method [6, 28], adaptive rational Krylov [30, 7, 8, 9], Smith method [13, 24]) are often the method of choice. Manifold-based approaches have been proposed in [33, 82] where the solution is been sought directly in the low-rank format (1.2). The main computational cost in such algorithms is the solution of linear systems with matrices of the form $A + \lambda I$.

A comprehensive review on the solution of linear matrix equations in general and Lyapunov equation in particular can be found in [29].

In this work we start from the Lyapunov equation and a simple method that doubles the size of $U$ at each step using the solution of an auxiliary Sylvester equation. The convergence of this approach is not bad but still too many linear system

‡Skolkovo Institute of Science and Technology, Novaya St. 100, Skolkovo, Odintsovsky district, 143025 Moscow Region, Russia (denis.kolesnikov@skoltech.ru, i.oseledets@skoltech.ru)

¶Institute of Numerical Mathematics, Gubkina St. 8, 119333 Moscow, Russia
solvers are required. Using the rank-1 approximation to the correction equation, we obtain a simple formula for the new vector. In our experiments we also found that it is a good idea to add a Krylov vector to the subspace. This increases the accuracy significantly at almost no additional cost. We compare the effectiveness of the new method with publicly available implementations of the extended rational Krylov method and adaptive rational Krylov approach on several model examples with symmetric and non-symmetric matrices \( A \) coming from discretizations of two-dimensional elliptic PDEs on different grids.

2. Minimization problem. How do we define what is the best low-rank approximation to the Lyapunov equation? A natural way is to formulate the initial problem as a minimization problem

\[ R(X) \rightarrow \min, \]

and then reduce this problem to the minimization over the manifold of low-rank matrices. A popular choice is the residual:

\[ R(X) = \|AX + XA^\top + y_0y_0^\top\|_2, \tag{2.1} \]

which is easy to compute for a low-rank matrix \( X \). Disadvantage of the functional (2.1) is well known: it may lead to the large condition numbers. For the symmetric positive definite case another functional is often used:

\[ R(X) = \text{tr}(XAX) + \text{tr}(Xy_0y_0^\top). \]

For a non-symmetric case we can use a different functional, which is based on the connection of the low-rank solution to the Lyapunov equation and low-dimensional subspace approximation to the solution of a system of linear ODEs. Consider an ODE with the matrix \( A \):

\[ \frac{dy}{dt} = Ay, \quad y(0) = y_0. \]

It is natural to look for the solution in the low-dimensional subspace form

\[ y(t) \approx \tilde{y}(t) = Uc(t), \]

where \( U \) is an \( n \times r \) orthogonal matrix and \( c(t) \) is an \( r \times 1 \) vector. Then the columns of the matrix \( U \) that minimizes

\[ \min_{U,c(t)} \int_0^\infty \|y(t) - \tilde{y}(t)\|^2 \, dt = \min_U \int_0^\infty \|y(t) - UU^\top y(t)\|^2 \, dt. \]

are the eigenvectors, corresponding to the largest eigenvalues of the matrix \( X \) that solves the Lyapunov equation (1.1). However, the computation of \( U^\top y(t) \) requires the knowledge of the true solution \( y(t) \) which is not known. Instead, we can consider the Galerkin projection:

\[ \tilde{y} = Uc(t), \]

\[ \frac{dc}{dt} = U^\top AUc, \quad c(0) = U^\top y_0, \tag{2.2} \]
The final approximation is then
\[ \tilde{y} = U e^{Bt} U^T y_0, \]  
(2.3)

where \( B = U^T A U \). The functional to be minimized is
\[ F(U) = \int_0^\infty \| y - \tilde{y} \|^2 dt. \]  
(2.4)

Note that the functional depends only on \( U \). Given \( U \), the approximation to the solution of the Lyapunov equation can be recovered from the solution of the “small” Lyapunov equation
\[ X \approx UZU^T, \quad BZ + ZB^T = -c_0c_0^T. \]  
(2.5)

The functional (2.4) cannot be efficiently computed. However, a simple expansion of the norm gives
\[ F(U) = \int_0^\infty \| y \|^2 dt - 2 \int_0^\infty \langle y, \tilde{y} \rangle dt + \int_0^\infty \| \tilde{y} \|^2 dt. \]  
(2.6)

The first term in (2.6) does not depend on \( U \), so it can be omitted in the minimization. Then the resulting functional is
\[ F(U) = F_1(U) - 2F_2(U), \]

where
\[ F_1(U) = \int_0^\infty (\| y \|^2 - \| \tilde{y} \|^2) dt, \quad F_2(U) = \int_0^\infty (\langle y, \tilde{y} \rangle - \| \tilde{y} \|^2) dt. \]

**Lemma 2.1.** The functionals \( F_1(U), F_2(U) \) can be calculated as follows:
\[ F_1(U) = \text{tr } X - \text{tr } Z, \]
\[ F_2(U) = \text{tr } U^T (P - UZ), \]  
(2.7)

where \( P \) is the solution of the Sylvester equation and \( Z \) is solution of the Lyapunov equation:
\[ AP + PB^T = -y_0c_0^T, \]
\[ BZ + ZB^T = -c_0c_0^T. \]  
(2.8)

*Proof:*

It is easy to see, that
\[ - \int_0^\infty \langle y, \tilde{y} \rangle dt = - \text{tr } \int_0^\infty e^{At} y_0 c_0^T e^{B^T t} U^T dt = - \text{tr } \left( \begin{array}{c} e^{At} P e^{B^T t} U^T \end{array} \right) \bigg|_0^\infty = \text{tr } U^T P, \]
\[ \int_0^\infty \| \tilde{y} \|^2 dt = \text{tr } \int_0^\infty U e^{Bt} c_0 c_0^T e^{B^T t} U^T dt = - \text{tr } \left( e^{Bt} Z e^{B^T t} \right) \bigg|_0^\infty = \text{tr } Z. \]
In the same way \( \int_0^\infty \|\tilde{y}\|^2 dt = \text{tr} X \) and we can write

\[
F_1(U) = \int_0^\infty (\|y\|^2 - \|\tilde{y}\|^2) dt = \text{tr} X - \text{tr} Z,
\]

\[
F_2(U) = \int_0^\infty (y, \tilde{y}) - \|\tilde{y}\|^2 dt = \text{tr}(U^T P - Z) = \text{tr} U^T (P - UZ).\]

The Sylvester equation can be solved using a standard method since the matrix \( B \) is \( r \times r, r \ll n \). We compute the Schur decomposition of \( B^T \) and the equation is reduced to \( r \) linear systems with the matrices \( A + \lambda_i I, \ i = 1, \ldots, r. \)

**Lemma 2.2.** The gradient of \( F(U) \) can be computed as:

\[
\text{grad} F(U) = -2P + 2y_0(c_0^T Z_1 - y_0^T P_U) + 2AU(ZZ_1 - P^T P_U) + 2A^T U(ZZ_1 - P^T P_U),
\]

where \( P, Z \) are defined by \((2.8)\) and

\[
A^T P_U + P_U B = -U,
\]

\[
B^T Z_1 + Z_1 B = -I_r.
\]

**Proof:**

Variation of \( Z \) can be expressed as a solution of the Lyapunov equation with another right hand side:

\[
B \delta Z + \delta Z B^T = -\delta c_0 c_0^T - c_0 \delta c_0^T - \delta BZ - Z \delta B^T.
\]

Using the well-known integral form of the solution of the Lyapunov equation we get that:

\[
- \text{tr} \delta Z = - \text{tr} \int_0^\infty e^{Bt}(\delta c_0 c_0^T + c_0 \delta c_0^T + \delta BZ + Z \delta B^T)e^{B^T t} dt =
\]

\[
= - \text{tr} \int_0^\infty e^{B^T t} B e^{Bt} dt)(\delta c_0 c_0^T + c_0 \delta c_0^T + \delta BZ + Z \delta B^T)
\]

\[
= - \text{tr} Z_1(\delta c_0 c_0^T + c_0 \delta c_0^T + \delta BZ + Z \delta B^T)
\]

\[
= -2 \text{tr} U^T(y_0 c_0 c_0^T Z_1 + A U ZZ_1 + A^T U Z_1 Z).
\]

Similarly for \( P \):

\[
A \delta P + \delta P B^T = -y_0 \delta c_0^T - P \delta B^T,
\]

therefore,

\[
\delta \text{tr}(U^T P) = \text{tr}(\delta U^T P + U^T \delta P) =
\]

\[
= \text{tr}(\delta U^T P + U^T \int_0^\infty e^{at}(y_0 \delta c_0^T + P \delta B^T)e^{B^T t} dt)
\]

\[
= \text{tr} \delta U^T P + \text{tr}(\int_0^\infty e^{B^T t}(U^T e^{At} dt)(y_0 \delta c_0^T + P \delta B^T))
\]

\[
= \text{tr} \delta U^T P + \text{tr} P_U^T (y_0 \delta c_0^T + P \delta B^T)
\]

\[
= \text{tr} \delta U^T (P + y_0 \delta c_0^T P_U + A^T U P_U^T X + A^T U P_U^T P_U).\]
Finally,
\[
\delta F(U) = \text{tr} \delta U^\top (\text{grad} F) = \text{tr} \delta (-2U^\top P + Z),
\]
\[
\text{grad} F(U) = -2P + 2y_0(c_0^\top Z I - y_0^\top P_U)) + 2AU(ZZ^\top - P^\top P_U) + 2A^\top U(ZI - P_U^\top P).
\]

Now denote by \( R_1(U) \) and \( R_2(U) \) residuals of the Lyapunov and Sylvester equations:
\[
\begin{align*}
R_1(U) &= \| A(UZU^\top) + (UZU^\top)A^\top + y_0y_0^\top \|, \\
R_2(U) &= \| A(UZ) + (UZ)B^\top + y_0c_0^\top \|. 
\end{align*}
\]

**Lemma 2.3.** Assume that \( y_0 \) lies in the column space of \( U \). Then the next equality holds:
\[
R_1(U) = \sqrt{2}R_2(U) = \sqrt{2}\| (AU - UB)Z \|. 
\]

**Proof:**
Since \( Z \) is the solution of the Lyapunov equation, we get that
\[
R_1(U)^2 = \| A(UZU^\top) + (UZU^\top)A^\top + y_0y_0^\top \|^2 = \\
= \| y_0y_0^\top - UU^\top y_0y_0^\top UU^\top + (AU - UB)ZU^\top + UZ(AU - UB)^\top \|^2 = \\
= \| (AU - UB)ZU^\top \|^2 + \| UZ(AU - UB)^\top \|^2 = 2\| (AU - UB)Z \|^2.
\]

We can use the same trick for the residual of the Sylvester equation:
\[
R_2(U)^2 = \| A(UZ) + (UZ)B^\top + y_0c_0^\top \|^2 = \\
= \| (I - UU^\top)y_0c_0^\top + (AU - UB)Z \|^2 = \| (AU - UB)Z \|^2.
\]

**Lemma 2.3** is valid if \( y_0 \in \text{span} U \) and we will always make sure that \( y_0 = UU^\top y_0 \).

The next Lemma shows that if the residual of the Lyapunov equation goes to zero, so does the values of the functional \( F(U) \).

**Lemma 2.4.** Assume that \( y_0 \) lies in the column space of \( U \). Then,
\[
F(U) \leq CR_1(U),
\]
with a constant \( C \) that depends only on \( A \) and \( r \).

**Proof:**
The matrix \( X - UZU^\top \) satisfies the Lyapunov equation
\[
A(X - UZU^\top) + (X - UZU^\top)A^\top = -(AU - UB)ZU^\top - UZ(AU - UB)^\top,
\]
therefore
\[
(I \otimes A + A \otimes I) \text{ vec } \left( X - UZU^\top \right) = -\text{ vec } \left( (AU - UB)ZU^\top + UZ(AU - UB)^\top \right),
\]
and
\[
\| X - UZU^\top \| \leq \| \left( I \otimes A + A \otimes I \right)^{-1} \| R_1(U)
\]
Thus,
\[ |\text{tr} X - \text{tr} Z| \leq C_1 R_1(U). \]
In the same way,
\[ \|P - UZ\| \leq \left\| \left(I \otimes A + B \otimes I\right)^{-1}\right\| R_2(U) = C_2 R_2(U), \]
therefore
\[ |F_2(U)| = |\text{tr} U^\top (P - UZ)| \leq \|U\| \|P - UZ\| \leq \sqrt{r} C_2 R_2(U). \]
Finally,
\[ F(U) \leq |F_1(U)| + 2|F_2(U)| \leq (C_1 + \sqrt{2r} C_2) R_1(U). \]

Lemma 2.4 shows that \( R_1(U) \), the residual of the Lyapunov equation, is a viable error bound for our functional \( F(U) \).

3. Methods for basis enrichment. Notice that the column vectors of the gradient are a linear combination of the column vectors of \( P, AU, A^\top U \) and \( y_0 \). We need a method to enlarge the basis \( U \) so the first idea is to use matrix \( P \) to extend the basis. Note, that the matrix \( UZ \) can be also considered as an approximation to the solution of Sylvester equation. So to enrich the basis we will use \( P_1 = P - UZ \) instead, and the matrix \( P_1 \) satisfies the equation:
\[
\begin{align*}
A(P - UZ) + (P - UZ)B^\top &= -y_0 c_0^\top - A U Z - UZB^\top = \\
= -(I - UU^\top) y_0 c_0^\top - (AU - UB)Z,
\end{align*}
\]
(3.1)
where we have used that \( y_0 = UU^\top y_0 \).

The method is summarised in Algorithm 1. The convergence of Algorithm 1 is not bad, however the computational cost grows at each step. If \( U \) has \( r \) columns, the next step will require \( r \) solutions of \( n \times n \) linear systems with matrices of the form \( A + \lambda I \).

**Algorithm 1: The doubling method**

**Data:** Input matrix \( A \in \mathbb{R}^{n \times n} \), vector \( y_0 \in \mathbb{R}^{n \times 1} \), maximal rank \( r_{\text{max}} \), accuracy parameter \( \varepsilon \).

**Result:** Orthonormal matrix \( U \in \mathbb{R}^{n \times r} \).

**begin**

1. set \( U = \frac{y_0}{\|y_0\|} \) \( \triangleright \) **Initialization**

2. for rank \( U \leq r_{\text{max}} \) do

3. Compute \( c_0 = U^\top y_0 \), \( B = U^\top AU \)

4. Compute \( Z \) as Lyapunov equation solution: \( BZ + ZB^\top = -c_0 c_0^\top \)

5. Compute error estimate \( \delta = \| (AU - UB)Z \| \)

6. if \( \delta \leq \varepsilon \) then

7. Stop

8. Compute \( P_1 \) as Sylvester equation solution:
\[
AP_1 + P_1 B^\top = -(AU - UB)Z
\]

9. Update \( U = \text{orth}(U, P_1) \)

**end**
Algorithm 1 is similar to the IRKA method \cite{11,12,10} for the computation of shifts in the rational Krylov subspaces.

To reduce the number of solvers required by the algorithm, we propose two improvements. The first is to add the last Krylov vector to the subspace. In this case, as we will show, the residual will always have rank-1. The second improvement is to add only one vector each time. In order to do so, we will use a simple rank-1 approximation to $P_1$.

3.1. Adding a Krylov vector and a rational Krylov vector to the subspace. In the following section we will show, that under a special basis enrichment strategy, the rank-1 approximation to $P_1$ can be replaced by adding one vector of the form $(A + sI)^{-1}w$ to the subspace. In this case, as it was already described in in \cite{28}, \cite{30} adding an additional Krylov vector preserves the rank-1 structure of the residual of the equation \eqref{3.1} is a rank-1 matrix.

Lemma 3.1. Let $A$ be an $n \times n$ matrix and $y_0$ is a vector of size $n$. Assume that an $n \times r$ orthogonal matrix $U$ and vectors $w, v$ of size $n$ satisfy the following equations:

$$(I - UU^\top)AU = wq^\top, \quad v = (A + sI)^{-1}w.$$ 

Let us denote by $U_1$ the basis of the span of the columns of the matrix $[U, w, v]$. Then the following equality holds

$$(I - U_1U_1^\top)AU_1 = (I - U_1U_1^\top)Awq^\top.$$ 

Proof:

Due to the fact that $(I - UU^\top)AU = wq^\top$ we get that $(I - U_1U_1^\top)AU = 0$. On the other hand we have

$$(I - U_1U_1^\top)Av = (I - U_1U_1^\top)((A + sI)v - sv) = (I - U_1U_1^\top)(w - sv) = 0.$$ 

Therefore $(I - U_1U_1^\top)AU_1 = (I - U_1U_1^\top)Awq^\top$.

Lemma 3.1 shows that if the approximation algorithm starts from $U_0 = \frac{y_0}{\|y_0\|}$ and adds a vector from the Krylov subspace and a corresponding vector from the rational Krylov subspace at each step then the residual matrix $(AU - UB)Z = (I - UU^\top)AUZ$ is rank-1 at any step. That is the main benefit from using the Krylov subspaces in our approach. Moreover, the residual has the form $\hat{w}q^\top$, where $\hat{w} = (I - U_1U_1^\top)Aw$ is the next Krylov vector. This fact is important and will be used later.

3.2. Rank-1 approximation to the correction equation. Suppose we have the matrix $U$ constructed by sequential adding Krylov and rational Krylov vectors to the subspace and the equation for $P_1$ has the form Note, that due to the equality

$$(AU - UB) = (I - UU^\top)AU,$$ 

the matrix $(AU - UB)$ has only one non-zero column. If the new vector are added as the rightmost vectors,

$$(AU - UB)Z = wz^\top,$$ 

where $z^\top$ is the last row of the matrix $Z$. Therefore, the equation for $P_1$ takes the form

$$AP_1 + P_1B^\top = -(AU - UB)Z = wz^\top.$$

(3.2)
The last step from the doubling method to the final algorithm is find to a rank-1 approximation to the solution of (3.2). If $U$ is known, we apply one step of alternating iterations, looking for the solution in the form $P_1 \approx vq^\top$, where $q = \frac{z}{\|z\|}$ is the normalized last row of the matrix $Z$. The Galerkin condition for $v$ leads to the equation

$$(A + (q^\top B^\top q) I) v = w.$$ 

Due to the simple rank-1 structure of the residual its norm can be efficiently computed as

$$R_1(U) = \sqrt{2\|w\|\|z\|}.$$ 

The final algorithm which we call alternating low rank (ALR) method is presented in Algorithm 2. The main computational cost is solving one linear system at each step.

**Algorithm 2**: The Adaptive low-rank method

**Data**: Input matrix $A \in \mathbb{R}^{n \times n}$, vector $y_0 \in \mathbb{R}^{n \times 1}$, maximal rank $r_{\text{max}}$, accuracy parameter $\varepsilon$.

**Result**: Orthonormal matrix $U \in \mathbb{R}^{n \times r}$.

**begin**

1. set $U = \frac{y_0}{\|y_0\|}, w_0 = \frac{y_0}{\|y_0\|}$ \hspace{1cm} $\triangleright$ Initialization

2. for rank $U \leq r_{\text{max}}$ do

3. Compute $w_k = (I_n - UU^\top)Aw_{k-1}$

4. Compute $c_0 = U^\top y_0, \quad B = U^\top AU$

5. Compute $Z$ as Lyapunov equation solution: $BZ + ZB^\top = -c_0c_0^\top$

6. Compute $z$ as the last row of the matrix $Z$.

7. Compute error estimate $\delta = \|w_k\| \|z\|$

8. if $\delta \leq \varepsilon$ then

9. Stop

10. Compute shift $s = q^\top Bq, q = \frac{z}{\|z\|}$

11. Compute $v_k = (A + sI)^{-1}w_k$

12. Update $U := \text{orth}[U, w_k, v_k]$.

**end**

4. Numerical experiments. We have implemented the ALR method in Python using scipy and numpy packages available in the Anaconda Python distribution. The implementation is available online at [https://github.com/dkolesnikov/alr](https://github.com/dkolesnikov/alr). The matrices, Matlab code and IPython notebooks which reproduce all the figures in this work are available at [https://github.com/dkolesnikov/alr-paper](https://github.com/dkolesnikov/alr-paper) where the .mat files with test matrices and vectors can be found as well. We have compared the ALR method with two methods with publicly available implementations.

The first method uses the Extended Krylov Subspaces appproach which was proposed in [6]. Its main idea is to use as the basis the extended Krylov subspace of the form

$$\text{span} \left( A^{-k}y_0, \ldots, y_0, \ldots, A^l y_0 \right).$$

Note, that the residual in this approach also has rank-1 and can be cheaply computed. This approach was implemented as the Krylov plus Inverted Krylov algorithm (hereafter KPIK) in [28] and convergence estimate also was obtained.
The second approach is the Rational Krylov Subspace Method (RKS M) which was proposed in [7]. Its main idea is to compute vectors step by step from the rational Krylov subspaces
\[ \text{span} \left( (A + s_i I)^{-1} y_0, \quad i = 1, \ldots \right). \]
The shifts \( s_i \) are selected by a special procedure. There are different algorithms to compute the shift (and the method proposed in this paper falls into this class, also there is a recent algorithm [9] based on tangential interpolation). We use the RKSM method described in [8] which has the publicly available implementation. The MATLAB code of both methods can be downloaded from \url{http://www.dm.unibo.it/~simoncin/software.html}.

Note that it is not fully fair to compare the efficiency of the ALR and KPIK with RKSM. The first two methods use vectors from Krylov subspace and have \( 2r + 1 \) size of approximation subspaces at \( r \) iteration step, in but “pure” RKSM method has a basis of size \( (r + 1) \) after \( r \) iterations. We tried to “extend” the RKSM method by adding Krylov vectors. It is not a trivial task due to a special functional, optimized in the RKSM method to compute the shifts. However, we can mimic the Krylov vectors by adding very large shift \( s_i \) at each odd iteration. This introduces an additional empirical parameter \( s_i \) which have to be selected. For our numerical examples we have found that the effect of such approach is twofold. For small \( r \), the convergence is improved. For larger \( r \) the method reaches a plateau. The exact reason for this behaviour is to be investigated. This extended approach we will call ERKSM.

4.1. Model problem 1: Laplace equation. The first problem is the discretization of the two-dimensional Laplace operator with Dirichlet boundary conditions on the unit square using 5-point stencil operator. The vector \( y_0 \) is obtained by the discretization on the grid of the function
\[ f(x, y) = e^{-((x-0.5)^2 + 1.5(y-0.7)^2)}. \]

On Figure 4.1 the convergence of different methods is presented, and the ALR and the ERKSM methods have better convergence (in terms of the size of the rational Krylov subspace) than other methods.

4.2. Model problem 2: Convection-diffusion equation. It is interesting to study the convergence of the ALR method for the case of non-symmetric \( A \), so we have performed several experiments on the convection diffusion problem of the form
\[ F(\phi) = D\nabla^2 \phi - \text{vec} v \cdot \nabla \phi, \quad (4.1) \]
with Dirichlet boundary conditions on the unit square. The discretization was done on an $n \times n$ mesh using the finite volume scheme provided by the FiPy package [14]. The vector $y_0$ is obtained from the discretization of non-homogeneous boundary conditions: it is equal to 1 on the left and the top sides of the unit square and is equal to zero in all other points. For the first problem the parameters are $D = 1.0$, $\text{vec } v = (1.0, -1.0)$. The results are presented on Figure 4.2 and the ALR and the ERKSM methods have better convergence than two other methods.

For the second problem parameters are $D = 0.01$, $\text{vec } v = (1.0, -1.0)$. The results are presented on Figure 4.3.

For the third problem we take the convection-dominated case: $D = 0.0001$, $\text{vec } v = (1.0, -1.0)$. The results are presented on Figure 4.4. The ERKSM, RKSM and KPIK methods stagnate, and ALR method has much better convergence. This behaviour has to be studied in more details, but it is probably due to the large Peclet number.
5. Conclusions and future work. In this paper we propose a new projection method for low-rank approximation to the solution of large-scale Lyapunov equations. Numerical experiments confirm the efficiency of our method: it adapts the rank of the approximation, has cheap error estimation and simple stopping criterion.

The main computational cost of the ALR method is the solution of linear systems with shifted matrices of the form $A + \lambda I$. In the current version, these systems are solved with high accuracy using the direct sparse solver, however for most of the shifts the condition number is small, and iterative methods can be used. We plan to investigate this issue (including the effect of inexact solves) in our future work.

The algorithm described in this paper formally applies to rank-1 right-hand sides, however it is not difficult to generalize it to the rank-$r$ case. It would be also very interesting to obtain theoretical estimates on the convergence of the method.

The ALR method for rank-1 right-hand side can be used to approximate the action of the matrix exponential and we plan to compare its efficiency with other well-established ODE solvers.

Acknowledgements. This work was supported by Russian Science Foundation grant 14-11-00659. We thank Dr. Vladimir Druskin for helpful comments on the RKS approach and the idea how to incorporate Krylov vectors into the existing RKSM code. We also thank Dr. Bart Vandreycken for his comments on the initial draft of the paper.

REFERENCES

[1] C. A. Beattie and S. Gugercin, Krylov-based minimization for optimal $H^2$ model reduction, in 46th IEEE conference on decision and control, 2007, pp. 4385–4390.
[2] P. Benner, P. Kürschner, and J. Saak, Self-generating and efficient shift parameters in ADI methods for large Lyapunov and Sylvester equations, GAMM-Mitteilungen, 17 (2013), pp. 123–143.
[3] P. Benner, J.-R. Li, and T. Penzl, Numerical solution of large-scale Lyapunov equations, Riccati equations, and linear-quadratic optimal control problems, Numer. Linear Algebr., 15 (2008), pp. 755–777.
[4] P. Bliman, Lyapunov equation for the stability of linear delay systems of retarded and neutral type, Automatic Control, IEEE Transactions on, 47 (2002), pp. 327–335.
[5] T. Damm, Direct methods and adi-preconditioned Krylov subspace methods for generalized Lyapunov equations, Numer. Linear Algebr., 15 (2008), pp. 853–871.
[6] V. Druskin and L. Knizhnerman, Extended Krylov subspaces: approximation of the matrix square root and related functions, SIAM J Matrix Anal A, 19 (1998), pp. 755–771.
[7] V. Druskin, C. Lieberman, and M. Zaslavsky, *On adaptive choice of shifts in rational Krylov subspace reduction of evolutionary problems*, SIAM J. Sci. Comput., 32 (2010), pp. 2485–2496.

[8] V. Druskin and V. Simoncini, *Adaptive rational Krylov subspaces for large-scale dynamical systems*, Systems & Control Letters, 60 (2011), pp. 546–560.

[9] V. Druskin, V. Simoncini, and M. Zaslavsky, *Adaptive tangential interpolation in rational Krylov subspaces for MIMO dynamical systems*, SIAM J Matrix Anal A, 35 (2014), pp. 476–498.

[10] G. Flagg, C. Beattie, and S. Gugercin, *Convergence of the iterative rational Krylov algorithm*, Syst. Control. Lett., 61 (2012), pp. 688–691.

[11] S. Gugercin, *An iterative rational Krylov algorithm (IRKA) for optimal $H^2$ model reduction*, in Householder Symposium XVI, Seven Springs Mountain Resort, PA, USA, 2005.

[12] S. Gugercin, A. C. Antoulas, and C. Beattie, *$H^2$ model reduction for large-scale linear dynamical systems*, SIAM J. Matrix Anal. A., 30 (2008), pp. 609–638.

[13] S. Gugercin, D. C. Sorensen, and A. C. Antoulas, *A modified low-rank Smith method for large-scale Lyapunov equations*, Numer Algorithms, 32 (2003), pp. 27–55.

[14] J. E. Guyer, D. Wheeler, and J. A. Warren, *FiPy: partial differential equations with Python*, Computing in Science & Engineering, 11 (2009), pp. 6–15.

[15] M. Hochbruck and G. Starke, *Preconditioned Krylov subspace methods for Lyapunov matrix equations*, SIAM J. Matrix Anal. Appl., 16 (1995), pp. 156–171.

[16] I. M. Jaimoukha and E. M. Kasenally, *Krylov subspace methods for solving large Lyapunov equations*, SIAM J. Numer. Anal., 31 (1994), pp. 227–251.

[17] K. Jbilou, *ADI preconditioned Krylov methods for large Lyapunov matrix equations*, Linear Algebra Appl., 432 (2010), pp. 2473–2485.

[18] K. Jbilou and A. Riquet, *Projection methods for large Lyapunov matrix equations*, Linear Algebra Appl., 415 (2006), pp. 344–358.

[19] V. I. Lebedev, *On a Zolotarev problem in the method of alternating directions*, USSR Computational Mathematics and Mathematical Physics, 17 (1977), pp. 58–76.

[20] J.-R. Li, F. Wang, and J. K. White, *An efficient Lyapunov equation-based approach for generating reduced-order models of interconnect*, in Proceedings of the 36th annual ACM/IEEE Design Automation Conference, ACM, 1999, pp. 1–6.

[21] J.-R. Li and J. White, *Low rank solution of Lyapunov equations*, SIAM J. Matrix Anal. and Appl., 24 (2002), pp. 260–280.

[22] A. Lu and E. L. Wachspress, *Solution of Lyapunov equations by alternating direction implicit iteration*, Comput. Math. Appl., 21 (1991), pp. 43–58.

[23] C. C. K. Mikkelsen, *Any positive residual curve is possible for the Arnoldi method for Lyapunov matrix equations*, tech. rep., Tech. Rep. UMINF 10.03, Department of Computing Science and HPC2N, Umeå University, 2010.

[24] T. Penzl, *A cyclic low-rank Smith method for large sparse Lyapunov equations*, SIAM J. on Sci. Comput., 21 (1999), pp. 1401–1418.

[25] J.-B. Pomet and L. Praly, *Adaptive nonlinear regulation: estimation from the Lyapunov equation*, Automatic Control, IEEE Transactions on, 37 (1992), pp. 729–740.

[26] Y. Saad, *Numerical solution of large Lyapunov equations*, Research Institute for Advanced Computer Science, NASA Ames Research Center, 1989.

[27] ———, *Overview of Krylov subspace methods with applications to control problems*, Research Institute for Advanced Computer Science, NASA Ames Research Center, 1989.

[28] V. Simoncini, *A new iterative method for solving large-scale Lyapunov matrix equations*, SIAM J. Sci. Comput., 29 (2007), pp. 1268–1288.

[29] V. Simoncini, *Computational methods for linear matrix equations*, tech. rep., 2013.

[30] V. Simoncini and V. Druskin, *Convergence analysis of projection methods for the numerical solution of large Lyapunov equations*, SIAM J. Numer. Anal., 47 (2009), pp. 826–843.

[31] T. Stykel and V. Simoncini, *Krylov subspace methods for projected Lyapunov equations*, Appl. Numer. Math., 62 (2012), pp. 35–50.

[32] B. Vanderleycken, *Riemannian and multilevel optimization for rank-constrained matrix problems*, PhD thesis, Katholieke Universiteit Leuven, 2010.

[33] B. Vanderleycken and S. Vandewalle, *A Riemannian optimization approach for computing low-rank solutions of Lyapunov equations*, SIAM J. Matrix Anal. Appl., 31 (2010), pp. 2553–2579.

[34] E. L. Wachspress, *Iterative solution of the Lyapunov matrix equation*, Appl.Math. Lett., 1 (1988), pp. 87–90.

[35] B. Zhou, G. Duan, and Z. Lin, *A parametric Lyapunov equation approach to the design of low gain feedback*, Automatic Control, IEEE Transactions on, 53 (2008), pp. 1548–1554.