First-Principles Elastic Constants for the hcp Transition Metals Fe, Co, and Re at High Pressure

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(February 1, 2008)

The elastic constant tensors for the hcp phases of three transition metals (Co, Re, and Fe) are computed as functions of pressure using the Linearized Augmented Plane Wave method with both the local density and generalized gradient approximations. Spin-polarized states are found to be stable for Co (ferromagnetic) and Fe (antiferromagnetic at low pressure). The elastic constants of Co and Re are compared to experimental measurements near ambient conditions and excellent agreement is found. Recent measurements of the lattice strain in high pressure experiments when interpreted in terms of elastic constants for Re and Fe are inconsistent with the calculated moduli.

62.20.Dc,62.50.+p,71.20.Be,71.15.Ap

I. INTRODUCTION

The effect of pressure on the propagation of elastic waves in materials is essential for understanding interatomic interactions, mechanical stability of solids, phase transition mechanisms, material strength, and the internal structure of the Earth and other planets. However, little is known of the elasticity of solids at high pressure. The experimental study of the elasticity of materials under high pressure is challenging, as traditional methods have been applied only to moderate pressures. Ultrasonic measurements are generally limited to a few GPa, while Brillouin spectroscopy has been applied up to 25 GPa.

We investigate the elasticity of three hexagonal transition metals at high pressure: iron, rhenium, and cobalt. High pressure properties of iron are of considerable geophysical interest as the Earth’s solid inner core is composed primarily of this element. The elasticity of hcp iron is important for understanding the elastic anisotropy of the inner core and its super-rotation. Rhenium is the strongest metal known at high pressure and is widely used as a gasket material in diamond anvil cell experiments. We have chosen cobalt for this study because of its proximity to iron in the periodic table and as an example of a ferromagnetic hcp metal.

All three of these metals have been studied experimentally under high pressure and their equations-of-state are well known. Iron transforms from the bcc phase at ambient conditions to hcp near 13 GPa; the equation-of-state of the hcp phase has been measured up to 300 GPa. Recent advances in diamond anvil cell techniques have made it possible to evaluate the lattice strain in a polycrystal subjected to a non-hydrostatic stress field which can be associated with elastic constants. The elasticity of iron has been inferred by this method at high pressure (up to 210 GPa), and its elastic constants were obtained at zero pressure using traditional ultrasonic methods. In the case of rhenium the equation-of-state is known to 215 GPa, its elastic constants and their pressure derivatives have been ultrasonically measured at low pressure. The same experimental method for evaluating lattice strains as in hcp iron has been applied to rhenium in the pressure range 18–37 GPa.

Iron has been studied widely with first-principles theoretical approaches because of its geophysical importance and the well known failure of the local density approximation (LDA) to the exchange-correlation potential to predict the ferromagnetic bcc ground state. This failure was a major impetus in the development of the generalized gradient approximation (GGA). The equation-of-state of hcp iron under LDA and GGA is well known to high pressures, and its elastic constants have been calculated by the full-potential linearized muffin-tin orbital method (FP-LMTO) and a total energy tight-binding (TB) method. For hcp cobalt calculations have been performed with the LMTO method in the atomic sphere approximation for LDA and the linearized combination of atomic orbital method (LCAO) for GGA. There is no previous theoretical work on the elastic constants of hcp cobalt. For rhenium only one study has focused on the hcp phase at high pressure, using FP-LMTO with LDA the equation-of-state and the elastic constants at zero pressure have been calculated.

We organize the paper as follows. Section elaborates the computational details of our first principles calculations and our approach to calculating the elastic constants, the elastic wave velocities, and the acoustic anisotropy. It is followed by a section presenting our results on the magnetic state of the materials studied, their c/a ratios,
equation-of-state, and the elastic constants as functions of pressure. We compare our results in terms of the elastic wave velocities to high pressure experiments and the Earth’s inner core. In section IV we analyze the elastic anisotropy resulting from our calculations, recent experimental and theoretical results, and the predictions from a central nearest neighbor force model. Finally, we present our conclusions in section V.

II. METHOD

LAPW Total Energy Calculations

We investigate the energetics of hcp iron, cobalt, and rhenium using the full-potential linearized-augmented plane-wave method (LAPW) with both LDA and GGA approximations to the exchange-correlation potential. For LDA the form of Hedin and Lundqvist and von Barth and Hedin are used for non-magnetic and spin-polarized calculations, respectively. For GGA we adopt the efficient formulation of Perdew, Burke, and Ernzhoefer.

Core states are treated self-consistently using the full Dirac equation for the spherical potential neglecting spin-orbit coupling. We investigate ferromagnetic alignment in spin-polarized calculations for all metals and antiferromagnetism for iron. For consistency of the results all parameters in the calculations except for spin-polarization are kept fixed.

For the 3d metals 3s, 3p, 3d, 4s, and 4p states are treated as valence electrons for all volumes. For rhenium we treat all electrons up to 4f as core, 5d and 6s as valence states. For rhenium we also have tested other configurations, such as including the 4f as valence states, which did not change our results significantly. The muffin-tin radii $R_{MT}$ are 2.0 Bohr for the 3d metals, and 2.3 Bohr for rhenium. As spin-orbit coupling of the valence electrons is important for the band structure and other properties of heavy element, we consider the influence of the spin-orbit term on the equation-of-state for Re by including it in a variational step.

We carry out total energy calculations over a wide range of volumes for all three metals (0.7 $V_0$ - 1.2 $V_0$, with $V_0$ the zero pressure volume). At each volume we determine the equilibrium ratio of the lattice constants $c/a$ by performing calculations for several different values of this ratio. The equation-of-state is obtained by describing the energy-volume curve with a third order expansion in the Eulerian finite strain.

We have performed convergence tests with respect to Brillouin zone sampling and the size of the basis set, $R_{MT}K_{max}$, where $K_{max}$ is the largest reciprocal space wave-vector in the basis set. Converged results are achieved with a 12x12x12 special k-point mesh, yielding 114 k-points in the irreducible wedge of the Brillouin zone for the hcp lattice, and up to 468 k-points for the monoclinic lattice used in elastic constants calculations. The number of k-points in the full Brillouin zone is well above the convergence criterion for elastic constant calculations established by Fast et al. The size of the basis set is given by $R_{MT}K_{max} = 9.0$, yielding 158 to 311 basis functions, depending on volume. Careful convergence tests show that with these parameters relative energies are converged to better than 0.1 mRy/atom, magnetic moments to better than 0.05 $\mu_B$/atom, and $c/a$ to within 0.025.

Elastic Constants

We calculate the elastic constants as the second derivatives of the internal energy with respect to the strain tensor ($\varepsilon$). We choose the applied strains to be isochoric (volume-conserving) which has several important consequences: First, we assure the identity of our calculated elastic constants with the stress-strain coefficients, which are appropriate for the calculation of elastic wave velocities; this identity is non-trivial for finite applied pressure. Second, the total energy depends on the volume much more strongly than on strain; by choosing volume conserving strains we obviate the separation of these two contributions to the total energy. Third, the change in the basis set associated with the applied strain is minimized, thereby minimizing computational uncertainties.

We obtain the elastic constants at the equilibrium relaxed structure at any volume $V$ by straining the lattice, relaxing the symmetry allowed internal degrees of freedom, and evaluating the total energy changes due to the strain as a function of its magnitude $\delta$.

The bulk modulus $K$ is calculated by differentiating the equation-of-state. For hexagonal crystals $K$ is the combination of elastic constants

$$K = \left[ C_{33} (C_{11} + C_{12}) - 2C_{13}^2 \right] / C_S,$$  \hspace{1cm} (1)

with

$$C_S = C_{11} + C_{12} + 2C_{33} - 4C_{13}. $$  \hspace{1cm} (2)
The volume dependence of the optimized $c/a$ is related to the difference in the linear compressibilities along the $a$- and $c$-axes ($k_a$ and $k_c$). The dimensionless quantity $R$ describes this as

$$ R = K(k_a - k_c) = -\frac{d\ln(c/a)}{d\ln V}. \tag{3} $$

In terms of the elastic constants,

$$ R = (C_{33} - C_{11} - C_{12} + C_{13})/C_S. \tag{4} $$

We calculate $C_S$ by varying the $c/a$ ratio at a given volume, according to the isochoric strain

$$ \varepsilon(\delta) = \begin{pmatrix} \delta & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & (1 + \delta)^{-2} - 1 \end{pmatrix}. \tag{5} $$

The corresponding energy change is

$$ E(\delta) = E(0) + C_S V \delta^2 + O(\delta^3). \tag{6} $$

In the expressions for $C_S$, $K$, and $R$, $C_{11}$ and $C_{12}$ occur only as a sum. To separate these constants we determine their difference, $C_{11} - C_{12} = 2C_{66}$ by applying an orthorhombic strain, space group $Cmcm$. For the strained lattice we use the two atom primitive unit cell, with the atoms in the Wyckoff position 4c, coordinates $(y, -y, 1/4)$. The strain is

$$ \varepsilon(\delta) = \begin{pmatrix} \delta & 0 & 0 \\ 0 & -\delta & 0 \\ 0 & 0 & \delta^2/(1 - \delta^2) \end{pmatrix}, \tag{7} $$

leading to a change in total energy:

$$ E(\delta) = E(0) + 2C_{66} V \delta^2 + O(\delta^4). \tag{8} $$

In the unstrained lattice the atomic coordinate is $y = 1/3$, but varies under strain. We relax our calculations with respect to this internal degree of freedom.

To determine $C_{44}$ we use a monoclinic strain, space group $C2/m$. The atomic positions in the two atom primitive unit cell are $(1/6, 5/6, 1/4)$. The strain applied

$$ \varepsilon(\delta) = \begin{pmatrix} 0 & 0 & \delta \\ 0 & \delta^2/(1 - \delta^2) & 0 \\ \delta & 0 & 0 \end{pmatrix}, \tag{9} $$

results in an energy change

$$ E(\delta) = E(0) + 2C_{44} V \delta^2 + O(\delta^4). \tag{10} $$

The equilibrium positions of the atoms are unaffected by this strain and do not need to be redetermined.

While for $C_{66}$ and $C_{44}$ the leading error term is of the order $\delta^4$, for $C_S$ it is of third order in $\delta$. It is therefore crucial to include positive and negative strains in the calculation for $C_S$. The strain amplitudes applied are typically nine values of $\delta$ covering $\pm 4\%$ for $C_S$; for $C_{66}$ and $C_{44}$, seven values of $\delta$ ranging to $6\%$ are applied. The elastic constants are then given by the quadratic coefficient of polynomial fits to the total energy results; the order of the polynomial fit is determined by a method outlined by Mehl. From the full elastic constant tensor we can determine the shear modulus $\mu$ according to the Voigt-Reuss-Hill scheme and hence the isotropically averaged aggregate velocities for compressional ($v_p$) and shear waves ($v_s$)

$$ v_p = \sqrt{\left(K + \frac{4}{3}\mu\right)/\rho}, \quad v_s = \sqrt{\mu/\rho}, \tag{11} $$

with $\rho$ the density.

More generally, the acoustic velocities are related to the elastic constants by the Christoffel equation

$$ (C_{ijkl}n_in_k - M\delta_{il}) u_i = 0, \tag{12} $$
where $C_{ijkl}$ is the fourth rank tensor description of elastic constants, $n_\text{ij}$ is the propagation direction, $u$ the polarization vector, $M = \rho v^2$ is the modulus of propagation and $v$ the velocity.

The acoustic anisotropy can be described as

$$\Delta_i = \frac{M_i(n \times 1)}{M_i(100)},$$  \hspace{1cm} (13)

where $n_x$ is the extremal propagation direction other than [100] and $i$ is the index for the three types of elastic waves (one longitudinal and the two polarizations of the shear wave). Solving the Cristoffel equation for the hexagonal lattice one can calculate the anisotropy of the compressional ($P$) wave as

$$\Delta_P = \frac{C_{33}}{C_{11}},$$  \hspace{1cm} (14)

For the shear waves the wave polarized perpendicular to the basal plane ($S_1$) and the one polarized in the basal plane ($S_2$) have the anisotropies

$$\Delta_{S1} = \frac{C_{11} + C_{33} - 2C_{13}}{4C_{44}}, \quad \Delta_{S2} = \frac{C_{14}}{C_{66}}.$$  \hspace{1cm} (15)

While for $S_2$- and $P$-waves the extremum occurs along the $c$-axis, for $S_1$ it is at an angle of $45^\circ$ from the $c$-axis in the $a$-$c$-plane. We note that an additional extremum may occur for the compressional wave propagation at intermediate directions depending on the values of the elastic constants.

**III. RESULTS**

**Magnetism**

We find a stable ferromagnetic state only in cobalt. It is stabilized over a wide volume range with the magnitude of the moment decreasing with pressure in agreement with previous theoretical results on the pressure dependence of magnetic moments in other transition metals. Only at the smallest volume considered (50 Bohr$^3$, 180 GPa) is the moment vanishingly small (Fig. 1). LDA and GGA yield consistent results and predict a zero pressure magnetic moment of 1.55 $\mu_B$, in excellent agreement with experiment (1.58 $\mu_B$).

In the case of hcp iron, we also investigate two antiferromagnetic states. The first consists of atomic layers of opposing spin perpendicular to the $c$-axis (afmI). The other arranges the planes of opposite spins normal to the [100] direction in the hcp lattice; this can be described by the orthorhombic representation of the hcp unit cell (space group $Pmna$) with spin up in the $(1/4, 0, 1/3)$ and spin down in the $(1/4, 1/2, 5/6)$ position (afmII). We find that both structures are more stable than the non spin-polarized state and that afmII is energetically favored over afmI. For both antiferromagnetic states the moment is strongly pressure dependent. For afmI it vanishes at volumes larger than $V_0$ (Fig. 1), in excellent agreement with results of Asada and Terakura. The other structure, afmII, possesses a magnetic moment well into the stable pressure regime of hcp iron, up to $\sim 40$ GPa. (Fig. 1). Because of frustration on the triangular lattice, it is possible that more complex spin arrangements such as incommensurate spin waves as for fcc iron or a spin glass are still more energetically favorable than afmII.

Diamond anvil cell in situ Mössbauer measurements of hcp iron have shown no evidence of magnetism in the hcp phase. The low antiferromagnetic moment we calculate in the stable hcp regime and the significant hysteresis of the bcc-hcp transition might explain that no magnetism in hcp iron has been detected in the high pressure Mössbauer experiment. In this context it may be relevant that indirect evidence for magnetism exists at low pressure. Epitaxially grown iron-ruthenium superlattices have shown magnetism occurring in hcp iron multilayers. Its character, however, is still controversial.

**c/a Ratios**

For all materials studied the $c/a$ ratio agrees with experimental data to within 2% and is essentially independent of the exchange correlation potential (GGA or LDA). Equilibrium $c/a$ ratios for iron range from 1.58 at zero pressure to 1.595 at 320 GPa. This is consistent with experimental measurements in the range of 15 to 300 GPa, which have shown considerable scatter. For cobalt the zero pressure $c/a$ ratio is calculated as 1.615, increasing to 1.62 at a
pressure of almost 200 GPa. The zero pressure \( c/a \) is slightly lower than the experimental value of 1.623. Diamond anvil cell experiments have found a higher value of \( c/a \), as much as the ideal value (1.633)\( ^{12} \), this discrepancy might be due to the coexistence of hcp and metastable fcc cobalt in the polycrystalline sample.\( ^{13} \) The \( c/a \) ratio for rhenium (1.615) does not change over the whole pressure range studied - and is in good agreement with experimental results (1.613)\( ^{4} \).

**Equation-of-State**

For the equation-of-state of rhenium, LDA shows better agreement with experimental data than does GGA (Fig. 2, Table I). GGA overestimates the zero pressure volume and softens the bulk modulus, supporting a general pattern seen in prior density functional calculations using GGA for other 5d metals.\( ^{14} \) \( ^{15} \) Including spin-orbit coupling in the calculation has little effect on the equation-of-state parameters, resulting in less than 1% change in the zero pressure volume and 2% in the bulk modulus. For cobalt, as for other 3d metals GGA is superior to LDA and reproduces the experimental equation-of-state to within 2% in volume and 10% in bulk modulus (Fig. 2, Table I).

The discrepancy in the equation-of-state parameters of hcp iron between non spin-polarized calculations and experiment is significantly larger than for the other two metals studied here (Table I) or other transition metals.\( ^{11} \) The zero pressure volume is underestimated by \( \sim \) 9%, and the zero pressure bulk modulus is too stiff by 75% (Table I). Especially at low pressure the non-magnetic equation-of-state deviates considerably from experimental values, while at high pressure the agreement is very good (Fig. 2). The stabilization of antiferromagnetic states at low pressure can account for some of the discrepancy. For afmII magnetism persists to volumes smaller than \( V_0 \), resulting in a larger zero pressure volume, reducing the difference with experiment to 5%, and lowering the bulk modulus considerably (Table I). This is still larger than the difference in \( V_0 \) for cobalt and for cubic iron phases \(< 3\% \)\( ^{15} \). We attribute the remaining discrepancy between low pressure experimental data and the afmII equation-of-state (Fig. 2) to the approximations in GGA and the possible stabilization of more complex spin arrangements than those considered here.

**Elasticity**

The agreement of the calculated elastic constants for cobalt and rhenium with zero pressure experimental results\( ^{13} \)\( ^{15} \) is excellent with a root mean square error of better than 20 GPa for both metals and both exchange-correlation potentials (Fig. 3, Tables I and II). The initial pressure derivative of the elastic constants for rhenium is also well reproduced by the calculations (Fig. 3). LDA and GGA exchange-correlation potentials give almost equally good agreement, the minor differences arising primarily from differences in the bulk modulus (Tables I, II, and III).

Our elastic constant calculations for rhenium and iron do not agree with the results of lattice strain experiments (Fig. 3, Tables II and IV). For rhenium the overall agreement between these experiments and our elastic constants is better than for iron. \( C_{11} \) and \( C_{12} \) agree well over the pressure range of the experiments, while the other longitudinal \( (C_{13}) \) and off-diagonal constant \( (C_{13}) \) differ significantly (Fig. 3, Table III). The shear elastic modulus \( (C_{44}) \) shows the largest discrepancy of all elastic constants (factor of 1.5). For iron the results of the lattice strain experiments and our calculations are in reasonable agreement for the off-diagonal constants only. The longitudinal moduli we obtain at 60 Bohr\(^3\) and 50 Bohr\(^3\) (\( \sim \) 50 GPa and \( \sim \) 200 GPa, respectively) are larger by approximately 50%. This is partly related to the overestimated bulk modulus in the calculations. The largest discrepancy, as in the case of rhenium, occurs in the shear elastic constants \( (C_{44} \) and \( C_{66}) \).

Aggregate properties such as the bulk and shear modulus, and the compressional and shear wave velocity are in somewhat better agreement between the theoretical results and the lattice strain experiment for both rhenium and iron (Figs. 3 and 4). For rhenium, theory and experiment differ by less than 15% in bulk and shear modulus (Fig. 4). For iron the discrepancy is considerable at intermediate pressure but becomes smaller with increasing pressure, as already seen for the equation-of-state (Figs. 2 and 3). At \( \sim \) 200 GPa the difference in bulk modulus between GGA and experiment is less than 5% and the elastic wave velocities differ by \( \sim \) 10%. The shear modulus differs by 25% even at high pressure.

For iron the comparison with previous theoretical results gives a more coherent picture. While the longitudinal elastic constants from our calculations are larger by 10-20% compared to TB\( ^{24} \) and FP-LMTO results\( ^{24} \) (Table IV), the elastic anisotropy is similar: the pairs of longitudinal, shear, and off-diagonal elastic moduli display similar values. For the TB study this is true over the whole pressure range considered, for the FP-LMTO calculations only at low pressure; the ratio of the off-diagonal constants \( (C_{12}/C_{13}) \) is strongly pressure dependent in that study, varying from 0.9 at zero pressure to 0.6 at 400 GPa.
IV. DISCUSSION

We find that the elastic anisotropy (eqs. 14 and 15) is similar for all three metals studied here. The magnitude of the anisotropy is $10 \pm 2\%$ for the longitudinal anisotropy and $\Delta S_1$, and $30 \pm 3\%$ for $\Delta S_2$ and is nearly independent of pressure (Fig. 6). This is consistent with the experimentally observed behavior of other hcp transition metals, all of which - except for the filled d-shell metals zinc and cadmium - show anisotropy of similar magnitude (Fig. 6).

These results can be understood by comparison to a hcp crystal interacting with central nearest neighbor forces (CNNF). For this model the elastic anisotropy is independent of the interatomic potential to lowest order in $P/C_{11}$, hence the anisotropy is dependent on the symmetry of the crystal only. Born and Huang have shown that from this CNNF model the elastic constants scale as $32:29:11:8:8$ for $C_{33}:C_{11}:C_{12}:C_{13}:C_{44}$, yielding $\Delta P = 32/29$, $\Delta S_1 = 8/9$, and $\Delta S_2 = 45/32$ (Fig. 6).

The experimentally determined elastic anisotropies of rhenium and hcp iron at high pressure from lattice-strain measurements differ substantially from our theoretical predictions, previous theoretical calculations, the behavior of all other hcp transition metals, and the simple CNNF model (Fig. 6). The shear anisotropy in particular is very different in the high pressure experiments as compared with all other relevant results. We suggest that this discrepancy may arise from assumptions made in the data analysis. In particular, the assumption that the state of stress on all crystallographic planes is identical. This condition may not be satisfied in a material undergoing anisotropic deformation (e. g. dominated by basal slip), behavior that is observed for many hcp transition metals.

Theory shows much better agreement with lattice-strain experiments in terms of the isotropically averaged moduli. Even so, the agreement in the case of rhenium is much better than for iron. In this context it is important to point out that part of the discrepancy in the case of iron is due to the mutual inconsistency of the experimentally reported elastic constants and isotropic moduli. We have found that the elastic constants reported in Ref. 11 do not yield the values of $K$ and $\mu$ reported in the same paper. The reason for this discrepancy is unknown.

V. CONCLUSIONS

The equations-of-state and the elastic constant tensor at zero pressure and under compression for two ambient condition hcp transition metals, cobalt and rhenium, and for the high pressure phase of iron, hcp, are calculated by means of the first principles LAPW method. We find a ferromagnetic ground state for cobalt and an antiferromagnetic one for iron, with the antiferromagnetic moment vanishing at 60 Bohr$^3$. The equations-of-state for the metals are in good agreement with experiment, as are the elastic constants and pressure derivatives of the elastic constants for cobalt and rhenium at ambient pressure.

Elastic constants for iron under high pressure as inferred from lattice-strain experiments differ significantly from our theoretical results. Similarly large discrepancies are also found between theory and high pressure static experiments on rhenium. The lattice-strain experiments also lead to large values of the shear anisotropy that differ from that of all other open shell hcp transition metals. Given the excellent agreement of the theoretical elastic constants for cobalt and rhenium with experiment at zero pressure, we suggest that a re-examination of the lattice-strain experiments for rhenium and iron is warranted.

ACKNOWLEDGEMENT

We greatly appreciate helpful discussions with Tom Duffy, Rus Hemley, Dave Mao, and Per Söderlind. The work was supported by the National Science Foundation under grant EAR-9614790, and by the Academic Strategic Alliances Program of the Accelerated Strategic Computing Initiative (ASCI/ASAP) under subcontract no B341492 of DOE contract W-7405-ENG-48 (REC). Computations were performed on the SGI Origin 2000 at the Department of Geological Sciences at the University of Michigan and the Cray J90/16-4069 at the Geophysical Laboratory, support by NSF grant EAR-9304624 and the Keck Foundation.

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FIG. 1. Magnetic moment per atom within the muffin-tin sphere for the two antiferromagnetic states of iron considered here and the ferromagnetic moment for cobalt as a function of volume.

FIG. 2. Equations-of-state for the hcp metals considered. The upper panel compares the GGA non-magnetic (solid line) with the afmII structure (dotted line) for iron; Static experimental data is from Ref. [8] (open circles) and Ref. [9] (filled circles). The lower two figures show the equations-of-state for ferromagnetic cobalt and non-magnetic rhenium, GGA results are shown in solid, LDA in dashed curves. The static experimental data for cobalt are from Ref. [12], for rhenium static (open circles) and reduced shock wave data (filled circles) are from Ref. [7] and Ref. [48], respectively.
FIG. 3. The elastic constants of hcp iron from our calculations are shown in the upper figure. The lines are Eulerian finite strain fits to the theoretical results at 45, 50, and 60 Bohr$^3$: solid (GGA), dashed (LDA). Lattice strain experiments from Ref. 10 and 11 are shown by the open symbols: $C_{11}$ ($\Theta$), $C_{33}$ ($\Delta$), $C_{12}$ ($\nabla$), $C_{13}$ ($\diamond$), and $C_{44}$ ($\Box$). In the middle panel elastic constants for hcp cobalt are shown as a function of volume. The curve is again a fit to the calculations at 65, 70, and 75 Bohr$^3$. GGA is shown in solid, LDA in dashed lines. At the zero pressure volume they are compared to ultrasonic experiments from Ref. 13 (filled symbols as above). The lower figure shows the equivalent for rhenium with calculations at 85, 93, and 100 Bohr$^3$. The thick dotted lines indicate the initial pressure derivatives as determined from ultrasonic measurements. For lattice strain experiments from Ref. 16 open symbols are used again.

FIG. 4. Isotropic properties for hcp rhenium in comparison to experiments. The lower panel shows the bulk ($K$) and shear modulus ($\mu$) of our calculations (GGA) in solid lines. The ultrasonic experiments at ambient condition from Ref. 15 are shown in filled circles with the initial pressure dependence in thick dotted lines. Lattice strain experiments from Ref. 16 are shown in open symbols. The upper panel uses the same symbols as the lower one for the compressional ($v_p$) and shear wave velocity ($v_s$).

FIG. 5. Bulk properties for hcp iron in comparison to experiments and the Earth’s inner core. The lower panel shows the bulk ($K$) and shear modulus ($\mu$) of our calculations (GGA) in solid lines. Diamond anvil cell experimental results are from Refs. 10 ($\bullet$) and 10 ($\Theta$ and $\Box$, denoting two different approaches). Ultrasonic measurements in a multianvil experiment ($\triangle$) are from Ref. 14 as well. The crosses display seismic observations of the inner core. The lower figure uses the same symbols as the upper one for the compressional ($v_p$) and shear wave velocity ($v_s$).

FIG. 6. As a measure of anisotropy the elastic constant ratios $C_{11}/C_{33}$, $(C_{11}+C_{33}-2C_{13})/4C_{44}$, and $C_{44}/C_{66}$, which govern the compressional ($\Delta P$) and shear wave anisotropy ($\Delta S_1$ and $\Delta S_2$, respectively) of the single crystal, are shown as a function of the number of d-electrons. The upper figure shows the shear elastic anisotropy $\Delta S_1$, the middle $\Delta S_2$, and the lower the ratio of the longitudinal elastic constants $\Delta P$. For all transition metals crystallizing in the hcp phase filled circles are used. The dashed lines show the CNNF model predictions. High pressure lattice strain results for iron from Ref. 14 and for rhenium from Ref. 16 are displayed with gray squares. Our results are the open diamonds with the pressure dependence shown in solid lines connected to the symbols.

TABLE I. Equation-of-state parameters from a third order finite Eulerian strain expansion of the energy-volume relation for the hcp transition metals. $V_0$, $K_0$, are the zero pressure volume and bulk modulus, respectively; $K'_0$ the pressure derivative of the bulk modulus. For experimental values the bulk modulus is calculated from the elastic constants at ambient pressure.

| Metal | $E_0$ [Ry/atom] | $V_0$ [Bohr$^3$] | $K_0$ [GPa] | $K'_0$ |
|-------|----------------|-----------------|-------------|--------|
| Fe    | exp            | 75.4            | 165         | 5.3    |
|       | LDA nm         | -2541.1046      | 64.7        | 344    | 4.4    |
|       | GGA nm         | -2545.6188      | 69.0        | 292    | 4.4    |
|       | GGA afmI       | -2546.6195      | 70.5        | 210    | 5.5    |
|       | GGA afmII      | -2546.6212      | 71.2        | 209    | 5.2    |
|       | LMTO GGA       | 65.5            | 340         |        |
| Co    | exp            | 74.9            | 190         | 3.6(2) |
|       | LDA fm         | -2782.1081      | 68.0        | 255    | 4.0    |
|       | GGA fm         | -2786.7364      | 73.6        | 212    | 4.2    |
|       | LCAO GGA       | 76.2            | 214         |        |
|       | LMTO LDA       | -2782.173       | 71.1        | 276    |        |
| Re    | exp            | 99.3            | 365         |        |
|       | LDA nm         | -33416.1921     | 98.2        | 382    | 3.9    |
|       | GGA nm         | -33436.2502     | 103.0       | 344    | 3.9    |
|       | LMTO LDA       | 98.8            | 447         |        |
TABLE II. Elastic constants of hcp cobalt from theory (GGA, LDA) and experiment. $C_{66} = \frac{1}{2}(C_{11} - C_{12})$ is added for comparison with $C_{44}$.

| Volume (Bohr$^3$) | $C_{11}$ (GPa) | $C_{33}$ (GPa) | $C_{12}$ (GPa) | $C_{13}$ (GPa) | $C_{44}$ (GPa) | $C_{66}$ (GPa) |
|-------------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 74.9              | 306            | 357            | 165            | 102            | 75             | 71             |
| 75.0              | 325            | 365            | 165            | 105            | 90             | 80             |
| 70.0              | 440            | 485            | 210            | 140            | 125            | 115            |
| 65.0              | 580            | 640            | 290            | 195            | 160            | 145            |
| 75.0              | 295            | 340            | 135            | 85             | 95             | 80             |
| 70.0              | 390            | 440            | 170            | 115            | 125            | 110            |
| 65.0              | 515            | 575            | 245            | 175            | 160            | 135            |

TABLE III. Elastic constants of hcp rhenium from theory (present work: GGA, LDA) and experiment. $C_{66} = \frac{1}{2}(C_{11} - C_{12})$ is added for comparison with $C_{44}$.

| Volume (Bohr$^3$) | $C_{11}$ (GPa) | $C_{33}$ (GPa) | $C_{12}$ (GPa) | $C_{13}$ (GPa) | $C_{44}$ (GPa) | $C_{66}$ (GPa) |
|-------------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 99.3              | 616            | 683            | 273            | 206            | 161            | 172            |
| 93.5              | 760(65)        | 735(165)       | 370(40)        | 355(50)        | 320(60)        | 195(60)        |
| 100.0             | 640            | 695            | 280            | 220            | 170            | 180            |
| 93.0              | 815            | 900            | 385            | 300            | 205            | 215            |
| 85.0              | 1075           | 1200           | 555            | 435            | 265            | 260            |
| 100.0             | 605            | 650            | 235            | 195            | 175            | 185            |
| 93.0              | 780            | 855            | 350            | 280            | 200            | 215            |
| 85.0              | 1040           | 1150           | 510            | 400            | 250            | 265            |
| 98.7              | 837            | 895            | 293            | 217            | 223            | 272            |

TABLE IV. Elastic constants for non-magnetic hcp Fe under compression (present work: GGA, LDA); the pressure range covered corresponds to approximately 50 GPa to 350 GPa, almost the pressure in the Earth’s center. For comparison results of other studies at $\sim 60$ Bohr$^3$ are included. $C_{66} = \frac{1}{2}(C_{11} - C_{12})$ is added for comparison with $C_{44}$.

| Volume (Bohr$^3$) | $C_{11}$ (GPa) | $C_{33}$ (GPa) | $C_{12}$ (GPa) | $C_{13}$ (GPa) | $C_{44}$ (GPa) | $C_{66}$ (GPa) |
|-------------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 60                | 640(55)        | 650(85)        | 300(55)        | 255(40)        | 420(25)        | 170(55)        |
| 60                | 930            | 1010           | 320            | 295            | 260            | 305            |
| 50                | 1675           | 1835           | 735            | 645            | 415            | 475            |
| 45                | 2320           | 2545           | 1140           | 975            | 400            | 590            |
| 60                | 860            | 950            | 280            | 260            | 235            | 290            |
| 50                | 1560           | 1740           | 720            | 595            | 415            | 420            |
| 45                | 2210           | 2435           | 1090           | 915            | 535            | 560            |
| 60                | 845            | 900            | 350            | 235            | 245            |                |
| 60                | 870            | 810            | 255            | 320            | 235            | 310            |
Volume (Bohr$^3$) vs. Magnetic Moment ($\mu_B$)

- Co
- Fe afm I
- Fe afm II
The graphs show the elastic moduli (GPa) as a function of volume (Bohr$^3$) for different materials:

- **Fe** (Iron)
- **Co** (Cobalt)
- **Re** (Rhenium)

The moduli are denoted by $C_{ij}$ where $i$ and $j$ represent the crystallographic directions.

- $C_{33}$
- $C_{11}$
- $C_{12}$
- $C_{13}$
- $C_{44}$

The graphs display a downward trend in elastic modulus with increasing volume, indicating a decrease in stiffness as the material expands.
Number of d-electrons

Sc, Y
Ti, Zr, Hf
Re
Fe, Ru
Co
Zn, Cd

Longitudinal Anisotropy $C_{33}/C_{11}$

Shear Anisotropy $C_{44}/C_{66}$

Shear Anisotropy $(C_{11} + C_{33} - 2C_{13})/C_{44}$