An analysis of variational wave function for the pairing problem in strongly correlated system

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Abstract.
We report a theoretical analysis of variational wave functions for the BCS pairing problem. Starting with a Jastrow–Feenberg (or, in a more recent language “fixed–node”) wave function for the superfluid state, we develop the full optimized Fermi-Hypernetted Chain (FHNC-EL) equations which sum a local approximation of the parquet–diagrams. Close examination of the procedure reveals that it is essential to go beyond the usual Jastrow–Feenberg approximation to guarantee the correct stability range.

1. Motivation
Jastrow–Feenberg theory has been established as a very effective method for calculating ground state properties of strongly interacting systems. We assume a non-relativistic many-body Hamiltonian

$$H = -\sum_i \frac{\hbar^2}{2m} \nabla_i^2 + \sum_{i<j} V(i, j),$$

or, in second quantized form,

$$\hat{H} = \sum_{\alpha, \beta} \langle \alpha | T | \beta \rangle a_\alpha^\dagger a_\beta + \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta} \langle \alpha \beta | V | \gamma \delta \rangle a_\alpha^\dagger a_\beta^\dagger a_\delta a_\gamma.$$

The method starts with an ansatz for the wave function, \[1\]

$$\Psi_0(r_1, \ldots, r_N) = \frac{1}{\sqrt{I_{0,0}}} F_N(r_1, \ldots, r_N) \Phi_0(r_1, \ldots, r_N),$$

$$F_N(r_1, \ldots, r_N) = \exp \frac{1}{2} \left[ \sum_{i<j} u_2(r_i, r_j) + \cdots + \sum_{i_1 < \cdots < i_n} u_n(r_{i_1}, \ldots, r_{i_n}) + \cdots \right],$$

where \(I_{0,0} = \langle \Phi_0 | F_N^2 | \Phi_0 \rangle\) is the normalization constant. Here \(\Phi_0(r_1, \ldots, r_N)\) denotes a model state, which for normal Fermi systems is a Slater-determinant, and \(F_N\) is an \(N\)-body correlation operator written in general form, but to be truncated at the two-body \(u_2\) term in a standard Jastrow calculation. The computationally most effective way to deal with expectation values of correlated wave functions of the type \[4\] are diagrammatic methods, specifically the optimized Euler-Lagrange Fermi-hypernetted chain.
(FHNC-EL) method, which is well suited for calculation of physically interesting quantities. These diagrammatic methods have been successfully applied to such highly correlated Fermi systems as $^3$He at $T = 0 \ [2]$. We have shown in recent work \[3\] that even the simplest version of the FHNC-EL theory is accurate within better than one percent at densities less than 25% of the saturation density of liquid $^3$He and similarly for nuclear systems \[4\]. It is one of the most attractive features of the FHNC-EL method that the Euler equation (See Eq. (9) below) has no solution if the assumed ground state is incorrect.

Following up on previous work \[5, 6\] and partly motivated by interest in the BCS-BEC crossover in cold gases (see Refs. 7 and 8 for review articles), we have recently examined pairing phenomena in both model Fermi systems \[9\] and neutron matter interacting via the $^1S_0$ components of the Reid soft-core $V_6 \ [10]$ and the Argonne $V'\,_4 \ [11]$ two-nucleon interactions. These calculations were based on a generalization of the BCS wave function

$$|\text{BCS}\rangle = \prod_k \left[ u_k + v_k \alpha_{k \uparrow} \alpha_{k \downarrow}^\dagger \right] |0\rangle$$

of the form

$$|\text{CBCS}\rangle = \sum_{m,N} |\Psi_m^{(N)}\rangle \langle m^{(N)}| \text{BCS}$$

where the correlated states $N$-body $|\Psi_m^{(N)}\rangle$ are

$$|\Psi_m^{(N)}\rangle = \left[ I_m^{(N)} \right]^{-1/2} P_N |m^{(N)}\rangle, \quad I_m^{(N)} = \langle m^{(N)}| P_N^\dagger P_N |m^{(N)}\rangle.$$  

(7)

Here the $\{ |m^{(N)}\rangle \}$ form complete sets of $N$-particle Slater determinants of single-particle orbitals. We have commented on alternative choices of the correlated BCS wave function in Ref. \[9\], the arguments do not need to be repeated here.

Physically interesting quantities like the free energy of the superfluid system

$$\langle \hat{H}' \rangle_s = \frac{\langle \text{CBCS}| \hat{H}'|\text{CBCS}\rangle}{\langle \text{CBCS}| \text{CBCS}\rangle}, \quad \hat{H}' \equiv \hat{H} - \mu \hat{N}.$$  

(8)

are then calculated by cluster expansion and resummation techniques; the correlation functions are determined by the variational principle

$$\frac{\delta \langle \hat{H}' \rangle_s}{\delta u_n}(r_1, \ldots, r_n) = 0.$$  

(9)

The combination of the (Fermi-)hypernetted-chain summation technique for the energy with the optimization of the ground state correlations is equivalent to a local approximation of the so–called “parquet-diagrams” \[12, 13, 14\].

In Refs. \[9\] and \[4\] we have simplified the problem by expanding the free energy \[(8)\] in the deviation of the Bogoliubov amplitudes $u_k, v_k$ from their normal state values $u_k^{(0)} = \theta(k - k_F), v_k^{(0)} = \theta(k_F - k)$. In that case, the pair correlation functions $u_2(r)$ can be optimized for the normal system. With that, we have arrived at the energy expression of the superfluid state

$$\langle \hat{H}' \rangle_s = H_{00}^{(N)} - \mu N + 2 \sum_{k, |k| > k_F} v_k^2 (e_k - \mu) - 2 \sum_{k, |k| < k_F} u_k^2 (e_k - \mu) + \sum_{k,k'} u_k v_k u_{k'} v_{k'} \mathcal{P}_{kk'}.$$  

(10)
Above, $H^{(N)}_{00}$ is the energy of the normal $N$-particle system, $\mu$ is the chemical potential. The $e_k$ are the single particle energies derived in correlated basis function (CBF) theory \[15\], and the paring interaction has the form

\[ P_{kk'} = W_{kk'} + (|e_k - \mu| + |e_{k'} - \mu|)N_{kk'}, \] (11)

\[ W_{kk'} = \langle k \uparrow, -k \downarrow | W(1,2) | k' \uparrow, -k' \downarrow \rangle_a, \] (12)

\[ N_{kk'} = \langle k \uparrow, -k \downarrow | N(1,2) | k' \uparrow, -k' \downarrow \rangle_a. \] (13)

where the effective interaction $W(1,2)$ and the correlation corrections $N(1,2)$ are then given by the compound-diagrammatic ingredients of the FHNC-EL method for off-diagonal quantities in CBF theory \[15\].

The Bogoliubov amplitudes $u_k, v_k$ are obtained in the standard way by variation of the energy expectation (10). This leads to the familiar gap equation

\[ \Delta_k = -\frac{1}{2} \sum_{k'} P_{kk'} \frac{\Delta_k'}{\sqrt{(|e_k - \mu|^2 + \Delta_k'^2)}}. \] (14)

In what follows, we shall denote the value of the gap function $\Delta_k$ at the Fermi surface with $\Delta \equiv \Delta_{|k|=k_F}$.

The conventional (i.e. “uncorrelated” or “mean-field”) BCS gap equation \[16\] is retrieved by replacing the effective interaction $P_{kk'}$ by the pairing matrix of the bare interaction.

In the calculations of Refs. 4 and 9 we have, however, encountered two problems indicating that the “weak coupling approximation” can be problematic:

- We have examined in Ref. 9 a large array of model systems described by a Lennard-Jones potential

\[ V_{LJ} = 4\varepsilon \left[ \left( \frac{\sigma}{r} \right)^{12} - \left( \frac{\sigma}{r} \right)^{6} \right] \] (15)

and an attractive square well (SW) potential

\[ V_{SW}(r) = \begin{cases} -\varepsilon & \text{if } r < \sigma, \\ 0 & \text{if } r > \sigma. \end{cases} \] (16)

Both potentials are parametrized by a characteristic length $\sigma$ and the depth $\varepsilon$ of the attractive well. For a large array of model systems with net attractive interactions (measured by a negative vacuum scattering length $a_0$) we found that the in-medium scattering length $a$ diverges, as a function of interaction strength, well before the vacuum scattering length $a_0$ diverges, as shown in Fig. 1. The divergence of the in-medium scattering length appears to be an indication for the formation of bound dimers (in other words the BCS-BEC “crossover”) which is enhanced by phonon exchange. Such a dimerized state is not described by the Jastrow–Feenberg wave function of a normal system. On the other hand, the simple BCS wave function \[5\] can describe the crossover \[17,18\] in situations where many–body correlations are negligible. It is therefore expected that the correlated BCS state \[6\] can describe the system through the BCS-BEC crossover, taking many–body correlations like phonon–exchange properly into account.

- For systems like neutron matter that have no many-body bound state, many–body correlations are less important. Nevertheless we found in Ref. 4 – in agreement with very many previous calculations and also more recent fixed–node Monte-Carlo calculations \[19,20\] at low densities, a pairing gap of the order of half of the Fermi energy $e_F$, see Fig. 2. The derivations of the correlated BCS theory \[9,5,6,21\] assume, on the other hand, that the paring gap is much smaller than the Fermi energy; the validity of the approximations leading to the formulation (10-13) must therefore be questioned.
Figure 1. The left plot shows the ratio between the in-medium scattering length $a$ and the vacuum scattering length $a_0$ as function of $-k_Fa_0$, for the Lennard-Jones (red lines) and the square-well potential (blue lines). The curves correspond to values of $a_0/\sigma = -0.5, -1.0, \ldots, -4.0$ (SW) and $a_0/\sigma = -1.5, -2.0, \ldots, -4.0$ (LJ), with the higher curves corresponding to larger $|a_0|$. The right plot shows the dependence of the Landau parameter $F_0^s$ for the Lennard-Jones model of the interaction. The blue (dashed) lines correspond to coupling strengths $\varepsilon = 7.0, 9.0, 9.2, 9.3, \ldots 9.9$. The curves for the interaction strengths $\varepsilon = 7.0$ ($a_0/\sigma = -1.12$) (blue, dashed), $\varepsilon = 7.51$ ($a_0/\sigma = -1.5$) and $\varepsilon = 8.01$ ($a_0/\sigma = -2$) (both red, solid) are discontinuous at high density. The curve that ends at the lowest density corresponds to the strongest interaction. From Ref. 9.

Figure 2. The figure shows the ratio $\Delta/e_F$ in neutron matter for the Reid $V_6$ and the Argonne $V'_4$ potentials. From Ref. 4.

2. BCS theory with local correlations
The divergence of the Euler equation observed in Ref. 9 – recall that we have assumed there a normal ground state and treated BCS correlations perturbatively – is therefore expected to be a signature of a crossover to a strongly dimerized state: The disappearance of solutions of the Euler equation for the normal system tells us that our assumption that the system is normal is incorrect, and the pairing is so strong that a perturbative expansion in terms of the deviation from the normal state as used previously is not legitimate.

To clarify the situation, we have developed the cluster expansion and resummation procedures for the fully correlated state (6).

The central quantity in the development of the method is the free energy (8) of the superfluid system.
which we can write as

$$
\langle H' \rangle_s = \frac{\sum_{N,m,n} \langle \text{BCS} | m^{(N)} \rangle \langle \Psi_m^{(N)} | H | \Psi_n^{(N)} \rangle \langle n^{(N)} | \text{BCS} \rangle}{\langle \text{BCS} | \text{BCS} \rangle}. \tag{17}
$$

It is helpful in the development of the formal theory to utilize methods developed for the cluster expansions of the normal system \cite{15}. Besides the normalization integrals $I_{m,m}$ defined in Eq. (7) we need the overlap integrals

$$
M_{m,n}^{(N)} = \langle \Psi_m^{(N)} | \Psi_n^{(N)} \rangle \equiv \delta_{m,n} + N_{m,n}^{(N)}. \tag{18}
$$

and matrix elements of the Hamiltonian

$$
H_{m,n}^{(N)} \equiv \langle \Psi_m^{(N)} | H | \Psi_n^{(N)} \rangle, \tag{19}
$$

$$
W_{m,n}^{(N)} \equiv H_{m,n}^{(N)} - \frac{1}{2} \left( H_{m,m}^{(N)} + H_{n,n}^{(N)} \right) M_{m,n}^{(N)}. \tag{20}
$$

Using these definitions, we can write the superfluid free energy as

$$
\langle H' \rangle_s = E_{\text{diag}} + \frac{\sum_{N,m,n} \langle \text{BCS} | m^{(N)} \rangle W_{m,n}^{(N)} \langle n^{(N)} | \text{BCS} \rangle}{\langle \text{BCS} | \text{BCS} \rangle} + \frac{1}{2} \frac{\sum_{N,m \neq n} \langle \text{BCS} | m^{(N)} \rangle \left( H_{m,m}^{(N)} + H_{n,n}^{(N)} - 2E_{\text{diag}} \right) N_{m,n} \langle n^{(N)} | \text{BCS} \rangle}{\langle \text{BCS} | \text{BCS} \rangle} \equiv E_{\text{diag}} + E_{\text{offd}} + E_{\text{enum}} \tag{21}
$$

where

$$
E_{\text{diag}} = \sum_{N,m} \langle \text{BCS} | m^{(N)} \rangle H_{m,m}^{(N)} \langle m^{(N)} | \text{BCS} \rangle. \tag{22}
$$

The actual expansions are tedious matter \cite{22} and will be reported elsewhere, we display here only the most important results and discuss, in a simplified example, the salient conclusions of our analysis. The physical interpretation of the three terms in Eq. (21) is revealed by examining cluster expansions for the individual quantities. It turns out that the term $E_{\text{diag}}$ is obtained simply the expectation value of a state where the momentum distribution of the ground state given by $v_k$ instead of the normal Fermi distribution $v_k^{(0)} = \theta(k_F - k)$. The second term in Eq. (21) contains only off–diagonal matrix elements, we can order this term according to the number of orbitals in which the states $|m^{(N)}\rangle$ and $|n^{(N)}\rangle$ differ, which will turn out, in the diagrammatic analysis, the number of Cooper pairs. If pairing is weak as examined in our previous work, then $|m^{(N)}\rangle$ and $|n^{(N)}\rangle$ differ by only two states. In other words we consider the interaction of only one Cooper pair at a time.

The last term, $E_{\text{enum}}$ gives rise to the “energy numerator corrections” shown in (11). A discussion of the significance of these terms is found in Refs. 9 and 4.

The purpose of the decomposition (21) is that cluster expansion and resummation techniques of ($E_{\text{diag}} + E_{\text{offd}}$) can be derived directly from the corresponding expression for the normal system. We cite here only the final result:

In the normal system, the cluster expansion of the generating function $G \equiv \ln I_{o.o}$ is diagrammatically expressed by the sum of all irreducible diagrams constructed by the following rules \cite{23}:

(i) Small circles depict particle coordinates. Filled circles imply integration over the particle coordinate and a density factor.

(ii) Every point is attached to the diagram by at least one correlation line $h(r) \equiv \exp(u_2(r)) - 1$. 

(iii) Every pair of points is connected by at most one correlon line.

(iv) Exchange lines $\ell(r_{kF})$ always appear in non-overlapping closed loops, or polygons. An $n$-sided exchange polygon contributes a weight factor $(-1/v)^{n-1}$ to the corresponding analytic expression, where $v$ is the degree of degeneracy of single–particle states.

(v) No point can be attached to more than two exchange lines.

The corresponding expansion for the superfluid system is generated from that by the following rule:

(i) Interpret the density factor as

$$\rho = \frac{v}{\Omega} \sum_{k} v_{k}^{2}, \quad (23)$$

where $\Omega$ is the normalization volume.

(ii) Re-interpret all exchange lines $\ell(r_{kF})$ as

$$\ell_{v}(r) = \frac{v}{\rho \Omega} \sum_{k} v_{k}^{2} e^{i\mathbf{k} \cdot \mathbf{r}} = \frac{v}{\rho} \int \frac{d^{3}k}{(2\pi)^{3}} v_{k}^{2} e^{i\mathbf{k} \cdot \mathbf{r}}. \quad (24)$$

(iii) In each exchange loop, replace, in turn, each pair of exchange lines $\ell_{v}(r_{ij})\ell_{v}(r_{kl})$ by a pair $-\ell_{u}(r_{ij})\ell_{u}(r_{kl})$, where

$$\ell_{u}(r) = \frac{v}{\rho \Omega} \sum_{k} u_{k} v_{k} e^{i\mathbf{k} \cdot \mathbf{r}} = \frac{v}{\rho} \int \frac{d^{3}k}{(2\pi)^{3}} u_{k} v_{k} e^{i\mathbf{k} \cdot \mathbf{r}}. \quad (25)$$

The FHNC theory for the terms $E_{\text{diag}} + E_{\text{offd}}$ in the energy expression (21) can therefore be read off from the diagrammatic expansions for the normal system. As an example, we show in Fig. the diagrammatic representation of some leading potential energy contributions to $E_{\text{diag}} + E_{\text{offd}}$. 
Figure 3. (color online) The figure shows the leading diagrams contributing to the potential energy. The black and red exchange lines depict the exchange functions $\ell_v(r_{ij})$ and $\ell_u(r_{ij})$, cf. Eqs. (24) and (25), respectively, and the red wavy line an interaction $V(r_{ij})$. Otherwise we follow the usual diagrammatic conventions [23]. The uniform limit approximation amounts to keeping only diagrams 1-5, 10-12, and 25. The term $E_{\text{diag}}$ is represented by the subset of diagrams containing only black exchange lines.

The energy numerator terms must be treated separately, we will spell out in the next section the simplest possible form.

2.1. Uniform limit approximation

We shall not implement the full method here but rather spell out what is known as the “uniform limit” approximation because it shows already the most important features of the theory and implications of using locally correlated wave functions in the superfluid system. Technically, the uniform limit approximation amounts to assuming that a product of any two correlation functions is negligible when it appears in coordinate space, but not negligible in momentum space, in other words we assume $\hbar^2(r) \ll \hbar(r)$ but not that $\tilde{\hbar}^2(k)$ is small compared to $\tilde{\hbar}(k)$. Note that we define here the dimensionless Fourier transform

$$\tilde{f}(k) = \rho \int d^3 r e^{i \mathbf{k} \mathbf{r}} f(r).$$

In the uniform limit approximation, only those exchange diagrams are retained that contain either a
loop \ell^2(r_{ij}) or \ell^2_u(r_{ij}), see Fig. 3. It is then convenient to define
\[
\sigma_v(r) = \frac{1}{\rho} \delta(r) - \frac{1}{\sqrt{\nu}} \ell^2_v(r)
\]
(27)
\[
\sigma_u(r) = \frac{1}{\sqrt{\nu}} \ell^2_u(r).
\]
(28)

Then, the static structure function of the “non–interacting” system is
\[
S_F(k) = \frac{1}{\langle N \rangle_0} \langle \text{BCS} | \rho_k \rho_{-k} | \text{BCS} \rangle = \bar{\sigma}_u(k) + \bar{\sigma}_v(k)
\]
(29)
\[
= \int \frac{d^3k}{(2\pi)^3} \rho^3 [S(k) - 1] \tilde{\nu}(k)
\]
(30)
where \[\ldots\] \tilde{\nu}(k) is used as an alternative notation for the Fourier–transform \[[26] and \langle N \rangle_0 = \nu \sum_k \nu_k^2 \text{ is the expectation value of the number operator } N \text{ with respect to the BCS state} [5].

The evaluation of the energy numerator expression \(E_{\text{enum}}\) must be done in an independent cluster expansion. These terms lead, in the weakly coupled limit considered in previous work, to the second term in the pairing matrix element \([15\). Recall, however, the derivation of the pairing matrix elements in Ref. [15] and the analysis of Ref. 9 that the matrix elements \(\mathcal{M}_{k,k'}\) should be identified with a static approximation of the 2-body scattering matrix which contains ring– and ladder–diagrams. In the uniform limit approximation, the pairing interaction, on the other hand, does not contain ladder diagrams, and the energy numerator term cancels the overcounted ladder diagrams in the scattering matrix. The analysis is analogous to the one found in Ref. [24] where the gap equation is expressed in terms of the \(T\)-matrix.

Now the “uniform limit” approximation contains only chain diagrams, hence there is no double-counting of ladder diagrams and the energy numerator term should, in that approximation, not be retained for the derivation of the Euler equation. We have therefore now two optimization conditions, one for the Bogoliubov amplitudes and one for the pair correlations. The gap equation \[[14\ remains the same, only the pairing matrix element depends implicitly on the Bogoliubov amplitudes. The analysis of the Euler equation for the spatial amplitudes is more complicated and will be carried out in the next section.

### 3. Euler equation for local correlations

The analysis of the Euler equation for the spatial correlations turns out to be much more subtle: Carrying out the variation of the energy \[[31\ yields the familiar answer
\[
S(k) = \frac{S_F(k)}{1 - \hbar(k) S_F(k)}.
\]
(33)
Retaining the energy numerator terms simply supplements the bare interaction with a small additive term. At the first glance, the result looks innocuous. In the normal system, we have $S_F(k) \propto k$ as $k \to 0^+$ and the condition that the expression under the square root be positive

$$1 + \frac{3}{4} \frac{3m}{\hbar^2 k_F^2} \tilde{V}(0^+) > 0. \quad (34)$$

which is — apart from the numerical factor $3/4$ which will be discussed momentarily — recognized as the condition $F_0 > -1$. In the superfluid system, we have instead

$$S_F(0+) = 2 \frac{\sum_k \omega_k^2 v_k^2}{\Sigma_k v_k^2} > 0. \quad (35)$$

This says that a locally correlated wave function for a super fluid system has a spurious instability whenever $\tilde{V}(0+) < 0$, no matter how small the gap is. This is evidently only a consequence of the approximations implicit to the wave function (6). Our observation applies, of course, equally to “fixed–node” Monte Carlo calculations which may see this instability only in large stochastic fluctuations. In view of the fact that the BCS-BEC crossover is driven by a net attractive interaction (see also the right panel of Fig. 1), it is worth examining this problem in detail.

The source of the problem can be identified as follows: In the normal system, the expression (33) can be derived from an ordinary random phase approximation $S(k) = -\int_0^\infty \frac{d\omega}{\pi} \frac{\chi_0(k, \omega)}{1 - \tilde{V}(q)\chi_0(k, \omega)} \chi_0(k, \omega)$ where $\chi_0(k, \omega)$ is the Lindhard function. Consistent with the convention (26) according to which $\tilde{V}(q)$ has the dimension of an energy, we have defined the density-density response function slightly different than usual [16] namely such that it has the dimension of an inverse energy. Eq. (33) can be obtained by approximating the Lindhard function $\chi_0(q, \omega)$ by a “collective” Lindhard function (occasionally also referred to as “one–pole approximation” or “mean spherical approximation”) $\chi_{\text{coll}}^0(q, \omega)$ which is constructed by approximating the particle–hole band by an effective single pole such that the $m_0$ and $m_1$ sum rules are satisfied [2]

$$-\Im \int_0^\infty \frac{d\omega}{\pi} \frac{\chi_{\text{coll}}(k, \omega)}{1 - \tilde{V}(q)\chi_0(k, \omega)} = -\Im \int_0^\infty \frac{d\omega}{\pi} \frac{\chi_0(k, \omega)}{1 - \tilde{V}(q)\chi_0(k, \omega)} = S_F(k) \quad (37)$$

$$-\Im \int_0^\infty \frac{d\omega}{\pi} \omega \chi_{\text{coll}}(k, \omega) = -\Im \int_0^\infty \frac{d\omega}{\pi} \omega \chi_0(k, \omega) = t(k). \quad (38)$$

This leads to

$$\chi_{\text{coll}}^0(k, \omega) = \frac{2t(k)}{(\omega + i\eta)^2 - (\frac{t(k)}{S_F(k)})^2}. \quad (39)$$

The frequency integration (36) then can be carried out analytically and leads to the equation (33). Experience with electronic systems and the very strongly interaction $^3$He has shown that the expression (33) is accurate within 1-2 percent; this is the reason why the Jastrow–Feenberg theory for fermions has been so successful. Of course, this is relevant only for integrated or Fermi–sea averaged quantities; we have pointed out already a long time ago [25] that the approximation should be particularly poor for
observables that are determined by the dynamics close to the Fermi surface. Among others, the stability condition (34) is replaced by

$$1 + \frac{3m}{\hbar^2 k_F^2} \bar{V}(0+) > 0.$$  
(40)

which is the correct stability condition.

Returning to the superfluid case, one can expect that the correct Lindhard function removes the spurious instability. There have been several suggestions [26, 27, 28, 29]. The most frequently used form for $T = 0$ is, in terms of the usual relationships of BCS theory,

$$u_k^2 = \frac{1}{2} \left( 1 + \frac{\xi_k}{E_k} \right)$$
$$v_k^2 = \frac{1}{2} \left( 1 - \frac{\xi_k}{E_k} \right).$$  
(41)

with $\xi_k = t(k) - \mu$ and $E_k = \sqrt{\xi_k^2 + \Delta_k^2}$ we have [30, 31, 32, 26]

$$\chi_0(q, \omega) = \frac{\nu}{\langle N \rangle_0} \sum_k \sum_q b_{-k,q} \left[ \frac{1}{\omega - E_{k+q} - E_k + i\eta} - \frac{1}{\omega + E_{k+q} + E_k + i\eta} \right]$$  
(42)

where

$$b_{-k,q} = \frac{1}{4} \left[ 1 - \frac{\xi_{k+q}}{E_{k+q}} \frac{\xi_k}{E_k} + \frac{\Delta_k \Delta_{k+q}}{E_k E_{k+q}} \right].$$  
(43)

In the limit of a normal system, the coefficients $b_{-k,q}$ become

$$b_{-k,q} \rightarrow n_k (1 - n_{k+q}),$$  
(44)

where $n_k$ is the Fermi distribution, as it should come out.

This superfluid Lindhard function is consistent with the $S_F(k)$ as defined in Eq. (29).

$$S_F(k) = - \int_0^{\omega} \frac{d\omega}{\pi} 3m \chi_0(k, \omega) = \frac{\nu}{\langle N \rangle_0} \int_0^{\omega} d\omega \sum_q b_{-k,q} \delta(\omega - E_q - E_{q+k})$$

$$= \frac{\nu}{\langle N \rangle_0} \sum_q b_{-k,q} = \tilde{\sigma}_d(k) + \tilde{\sigma}_c(k)$$  
(45)

We can now return to the frequency integration (36). All we need to show is that this expression exists, for small gaps, for $-1 < \frac{3m}{\hbar^2 k_F^2} \bar{V}(0+)$. For that, it is sufficient to look at the limit $q \rightarrow 0$ of the static response function and the static Lindhard function.

For $\omega = 0$ we get for the static Lindhard function

$$\lim_{q \rightarrow 0} \chi_0(q, 0) = - \frac{\nu}{2\pi} \int \frac{d^3k}{(2\pi)^3} \frac{\Delta_k^2}{E_k^2}.$$  
(46)

To estimate this limit for small gap energies, we add and subtract a function that can be integrated
exactly such that the remainder vanishes as $\Delta \to 0$.

\[
\lim_{q \to 0} \chi_0(q,0) = -\frac{4\pi \nu}{2\rho} \int \frac{dk}{(2\pi)^3} \frac{\Delta_k^2}{((t(k) - \mu)^2 + \Delta_k^2)^{3/2}}
\]

\[
= -\frac{3}{2k_F^2} \int dk \left[ \frac{k^2 \Delta_k^2}{((t(k) - \mu)^2 + \Delta_k^2)^{3/2}} + \frac{k\Delta_k^2}{((t(k) - \mu)^2 + \Delta_k^2)^{3/2}} \right]
\]

\[
= \mathcal{O}(\Delta) - \frac{3}{4} \left[ \frac{1}{\sqrt{\mu^2 + \Delta^2}} + \frac{1}{\mu} \right] \to -\frac{3}{2\mu} \quad \text{as} \quad \Delta \to 0. \tag{47}
\]

This is identical to the same limit of the Lindhard function of the normal system, see Fetter-Walecka Eq. (12.46b), considering that the Lindhard function defined here contains a factor $1/\rho$.

With that we get for the stability condition

\[
1 + \frac{3\bar{V}(0+)}{2\mu} > 0 \tag{48}
\]

4. Discussion

We have formulated in this contribution the beginnings of a theoretical method to deal with strongly interacting superfluids that promises the same accuracy as what was achieved for the normal helium liquids. We have studied the Euler equation for the local correlations in a simplified example that does not require the formulation of the full FHNC theory. We have derived the interesting, yet disturbing, result that the Euler equation has no physically meaningful solution for net attractive interactions $\bar{V}(0+) < 0$. We hasten to point out that this result is not specific to the “uniform limit approximation”, in the fully correlated theory the bare interaction $\bar{V}(q)$ is simply replaced by the particle–hole interaction. This is a rather profound observation since it applies to any local correlation operator $F$; including the “fixed-node” approximation used in Quantum Monte Carlo calculations where it would, of course, be hard to discover.

We have also outlined a pathway to the solution of this problem: One must go beyond local correlations. The rigorous way to do that has been carried out in Ref. [33]. One must add, to the variational energy expectation value, an infinite sum of terms in CBF perturbation theory. That has the effect that the “collective approximation” for $S(k)$, Eq. [33] is replaced by the RPA expression [36]. The calculations of Ref. [33] have been rather tedious; to be rigorous one would have to re-do all of this work for the superfluid system. The formulation of Coupled Cluster theory with correlated wave functions [6] would be a way to carry this through, but, on the other hand, our result is sufficiently plausible to be applicable without a rigorous proof.

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References

[1] Feenberg E 1969 *Theory of Quantum Fluids* (New York: Academic)
[2] Krotscheck E 2000 *J. Low Temp. Phys.* **119** 103–145
[3] Egger J, Krotscheck E and Zillich R E 2011 *J. Low Temp. Phys.* **165** 275–291
[4] Fan H H, Krotscheck E and Clark J W 2017 *J. Low Temp. Phys.* **189** 470–494
[5] Krotscheck E and Clark J W 1980 *Nucl. Phys. A* **333** 77–115
[6] Krotscheck E, Smith R A and Jackson A D 1981 *Phys. Rev. B* **24** 6404–6420
[7] Duine R and Stoof H T C 2004 *Phys. Rep.* **396** 115–195
[8] Chen Q, Stajic J, Tan S and Levin K 2005 *Phys. Rep.* **412** 1–88
[9] Reid, Jr R V 1968 *Ann. Phys. (NY)* **50** 411–448
[10] Wiringa R B, Stoks V G J and Schiavilla R 1995 *Phys. Rev. C* **51** 38–51
[11] Jackson A D, Lande A and Smith R A 1982 *Phys. Rep.* **86** 55–111
[12] Jackson A D, Lande A and Smith R A 1985 *Phys. Rev. Lett.* **54** 1469–1471
[13] Krotscheck E, Smith R A and Jackson A D 1986 *Phys. Rev. A* **33** 3535–3536
[14] Krotscheck E and Clark J W 1979 *Nucl. Phys. A* **328** 73–103
[15] Krotscheck E and Mateo D 2015 *Phys. Rev. A* **92** 023640
[16] Cooper L N, Mills R L and Sessler A M 1959 *Phys. Rev.* **114** 1377–1382
[17] Nozières P and Schmitt-Rink S 1985 *J. Low Temp. Phys.* **59** 195–211
[18] Gezerlis A and Carlson J 2008 *Phys. Rev. C* **77** 032801
[19] Gezerlis A and Carlson J 2010 *Phys. Rev. C* **81** 025803
[20] Krotscheck E 2002 *Introduction to Modern Methods of Quantum Many–Body Theory and their Applications* (Advances in Quantum Many–Body Theory vol 7) ed Fabrocini A, Fantoni S and Krotscheck E (Singapore: World Scientific) pp 267–330
[21] Fan H H 2018 *Pairing Phenomena from Low-Density Fermi Gases to Neutron Star Matter* Ph.D. thesis University at Buffalo SUNY
[22] Clark J W 1979 *Progress in Particle and Nuclear Physics* vol 2 ed Wilkinson D H (Oxford: Pergamon Press Ltd.) pp 89–199
[23] Pethick C J and Smith H 2008 *Bose–Einstein Condensation in Dilute Gases* second edition ed (Cambridge, UK: Cambridge University Press)
[24] Jackson A D, Krotscheck E, Meltzer D and Smith R A 1982 *Nucl. Phys. A* **386** 125–165
[25] Voo K K, Wu W C, Li J X and Lee T K 2000 *Phys. Rev. B* **13** 9095–9100
[26] Combescot R, Kagan M Y and Stringari S 2006 *Phys. Rev. A* **74** 042717
[27] Steiner A W and Reddy S 2009 *Phys. Rev. C* **79** 015802
[28] Vitali E, Shi H, Qin M and Zhang S 2017 *J. Low Temp. Phys.* **189** 312–327
[29] Schrieffer J R 1999 *Theory Of Superconductivity* (Advanced Books Classics) revised ed (Perseus Books)
[30] Kee H Y and Varma C M 1998 *Phys. Rev. B* **58** 15035–15044
[31] Kee H Y and Kim Y B 1999 *Phys. Rev. B* **59** 4470–4474
[32] Krotscheck E 1982 *Phys. Rev. A* **26** 3536–3556