The Leggett-Garg inequality in electron interferometers

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We consider the violation of the Leggett-Garg inequality in electronic Mach-Zehnder interferometers. This set-up has two distinct advantages over earlier quantum-transport proposals: firstly, the required correlation functions can be obtained without time-resolved measurements. Secondly, the geometry of an interferometer allows one to construct the correlation functions from ideal negative measurements, which addresses the non-invasiveness requirement of the Leggett-Garg inequality.

We discuss two concrete realisations of these ideas: the first in quantum Hall edge-channels, the second in a double quantum dot interferometer.

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Bell inequalities set bounds on the nature of the correlations between spatially-separated entities within local hidden variable theories.¹² In contrast, Leggett-Garg inequalities (LGIs) set bounds on the temporal correlations of a single system,¹³, and are derived under the assumptions of macroscopic realism (MR) and non-invasive measurability (NIM)²⁶.

Bell and Leggett-Garg inequalities are related in that their assumptions both imply the existence of a classical probability distribution that determines experimental outcomes. The probability amplitudes of quantum mechanics allow for violation of these inequalities: with Bell, the violation is due to entanglement between the two systems; with Leggett-Garg, the violation occurs due to the superposition of system states and their collapse under measurement.

The simplest LGI, henceforth referred to as the LGI, reads

\[ K = C_{21} + C_{32} - C_{31} \leq 1, \tag{1} \]

where \(C_{\alpha\beta} = \langle Q(t)_{\alpha} Q(t)_{\beta} \rangle\) is the correlation function of the dichotomous variable \(Q = \pm 1\) at times \(t_{\alpha}\) and \(t_{\beta}\). Since the first experimental violation of this inequality with weak measurements of a superconducting qubit, the Leggett-Garg inequality has been experimentally probed in systems as diverse as photons,²⁹ defects in diamonds,³⁰ nuclear magnetic resonance,¹², and phosphorus impurities in silicon.³¹. Whilst the subjects of these studies may not be macroscopic, the LGI performs a useful role for microscopic systems as an indicator that the device is operating beyond classical probability laws. Moreover, if one accepts that the alternative to classical probabilities is quantum mechanics, the LGI provides a decisive indicator of the “quantumness” on a system.

In this paper, we are interested in the violation of the LGI in quantum transport, and in particular, in electron-interferometers. Although there has been much work on Bell inequalities in electron transport, e.g. Refs ¹⁴–²², the LGI has only relatively recently been considered in this setting.²²,²³. Specifically, the charge flowing through a confined nanostructure, e.g. double quantum dot (DQD), has been shown to violate an inequality similar to Eq. (1) out of equilibrium.²³. Furthermore, the moment-generating function of charge transferred through a device has also been shown to be subject to a set of LG-style inequalities, which are violated for various quantum dot models. The violation of LGIs in excitonic transport has also attracted recent interest.²⁵,²⁶

There are several difficulties which make the investigation of the LGI in electronic transport challenging in practice. Ostensibly, the measurement of Eq. (1) requires time-resolved measurements where the time between successive measurements is smaller than the decoherence time of the system. For the double quantum dot of Ref. ²², for example, this decoherence time is of the order of 1ns,³² which makes the necessary time-resolved measurements very challenging (but, in principle, possible).²³. Furthermore, for the violation of Eq. (1) to be a meaningful indicator of non-classical behaviour, it must be ensured that the measurements are non-invasive. This “clumsiness loophole”²⁶ that allows violations of Eq. (1) to be associated with invasiveness of measurement, along with possible circumventions, have been the subject of much discussion.²²,²⁶,²⁹

The transport set-ups we consider here are based on the electronic Mach-Zehnder Interferometer (MZI), and can overcome both of these problems. The basic idea is that an electron travelling through a MZI can take one of two paths, and this path index defines the variable \(Q = \pm 1\). Unidirectional passage of the electron through the system allows us to map the time indices of Eq. (1) onto positions within the interferometer. As we show below, this removes the need for time-resolved measurements.

We consider two realisations of the MZI in which measurements of \(Q\) are performed in two different ways. In the first, the MZI is formed from quantum Hall edge channels, a set-up which has been realised experimentally and also attracted a large degree of theoretical attention.²⁸,³⁰–³¹. By interrupting the edge channels at various points and diverting electron flow to current meters, we show that \(K\) can be obtained from measurements of mean currents alone. Furthermore, due
to the spatial separation of the \( Q = +1 \) and \( Q = -1 \) channels, our detectors interact with only one of the two \( Q \)-states at any given time. Thus, our scheme provides a natural way to implement \textit{ideal negative measurements}, as advanced by Leggett and Garg as a way to satisfy the NIM criterion.

The second set-up we consider is a MZI with a quantum dot (QD) in each arm. This geometry is similar to several experiments\textsuperscript{52–56} that have investigated transport through Aharonov-Bohm rings with QDs in the arms. The difference here being that the dots are fed by two tunnel-coupled leads\textsuperscript{27}, rather than just one. The QDs are monitored by quantum point contacts, whose transmission is sensitive to the charge state of the QD.\textsuperscript{31,32} In contrast with the first set-up, electrons are not diverted out of the MZI at any point, and the influence of the detectors occurs as a pure dephasing effect. The three correlation functions in Eq. (1) are obtained through a combination of mean currents, both through the MZI and the quantum point contacts, and zero-frequency noise measurements, which cross-correlate current fluctuations in the MZI and quantum point contacts. As in the previous scheme, we construct an ideal negative measurement scheme with this set-up.

Both of these techniques exploit a combination of superpositions of paths through an interferometer combined with a gathering of “which-way” information to violate the LGI. The first set-up is a particularly simple realisation of the LGI, and is by no means restricted to transport, but could be used e.g. with photons, atoms or molecules.

The paper proceeds as follows. In Sec. II we describe the basics of testing the LGI in a MZI. Sec. III describes how this may be translated into experiments with quantum Hall effect edge-channels. Finally, Sec. IV considers the alternative DQD-QPC geometry studied here.

\section{I. MACH-ZEHNDER INTERFEROMETER}

We begin by describing an abstract version of our MZI scheme to outline the basic ideas. The MZI is a two-channel interferometer with two beam-splitters that divide the MZI into three zones which we label: 1, the input ports; 2, the arms of the interferometer; and 3, the output ports (see Fig. 1). We inject one electron at a time into the MZI and the path taken by the electron will be the degree-of-freedom under test with \( Q = +1 \) when the electron is located in the upper channel of the MZI, and \( Q = -1 \) the lower. Since the electron passes sequentially through the three zones, we can map a measurement of \( Q \) at time \( t_0 \) to a “which-way” measurement at any point in the region \( \alpha \) of the interferometer. In particular, \( Q_1 \) and \( Q_4 \) are measured at the input and output ports, and \( Q_2 \) is measured by placing detectors in the arms of the interferometer at \( 2 \pm \), where \( 2 \) refers to the zone, and \( \pm \) the upper or lower channel. In this section we assume that we have ideal single-electron detectors that “click” on detecting an electron, which is then removed from the system (i.e., the detectors act essentially as electronic analogues of photodetectors). More realistic measurements in terms of currents are discussed in section III.

\subsection{A. Ideal negative measurements}

A detector placed in one of the arms interacts strongly with electrons in that path (they are completely removed from the MZI) and has no effect on electrons in the other. With a detector placed at \( Q = +1 \), say, then the absence of a detector response (combined with MR and ideal detectors) allows us to infer the state of the system (\( Q = -1 \)) without any disruption. This is exactly the form of detector required to perform an ideal negative measurement as envisioned in Ref. 21.

To make the measurement scheme as simple as possible, let us inject electrons into the 1+ port, such that the initial state is known. \textsuperscript{64} We do not need to measure in zone 1 and there is no question about the NIM of \( Q_1 \). The correlation function \( C_{21} \) and \( C_{31} \) boil down to measuring \( \langle Q_2 \rangle \) and \( \langle Q_3 \rangle \) respectively.

Let us define \( P_{DQ}(n) \) as the probability that the detector placed at position \( \alpha \pm \) either detects \( n \) or fails to detect \( n = 0 \) the electron. Since no further measurements are made past point 3, it is irrelevant whether we measure non-invasively or not at point 3. Placing detectors at \( 3 \pm \), we measure the probabilities \( P_{DQ}(1) \), and the

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{(Color online) The Mach-Zehnder interferometer with three different detector configurations for the non-invasive measurement of the LGI of Eq. (1). Electrons are injected into the 1+ port. (a) Complete MZI configuration with detectors only at the final outputs \( 3 \pm \). With this set-up we can measure the probabilities \( P_{DQ}(1) \) and construct \( C_{31} \). (b) An additional detector is inserted into the MZI + arm. With this configuration we can measure probabilities \( P_{DQ}(0) \) and \( P_{DQ}(1, -) \). (c) A detector in the ‘-’ arm allows us to obtain \( P_{DQ}(0) \) and \( P_{DQ}(1, -) \). Combining the results of (b) and (c) allows us to construct correlation functions \( C_{21} \) and \( C_{32} \).}
\end{figure}
\( C_{31} \) correlation function can simplify be expressed as
\[
C_{31} = P_{3+}^D(1) - P_{3+}^D(1). 
\]
The set-up for this measurement is shown in Fig. \( \text{I} \).

Since, in measuring \( C_{21} \), no further measurements are made after region 2, it is also not necessary to measure \( C_{21} \) non-invasively. We can measure \( \langle Q_2 \rangle \) (and thus \( C_{21} \)) by running the experiment once with a detector in channel 2+, and once in 2− (Fig. \( \text{I} \) and c) and writing
\[
C_{21} = P_{2+}^D(1) - P_{2+}^D(1). 
\]
It is perhaps instructive to discuss how to obtain this quantity using the ideal negative measurement technique and measure \( C_{21} \) in terms of the probabilities of absence of detector clicks. With the detector at 2+, we can equate the probability that no electron is detected, \( P_{2+}^D(0) \), with the probability that the electron travels the path 2−. Swapping the detector to the other arm, we measure \( P_{2+}^D(0) \) and infer the probability that the electron takes path 2+. Whence, we obtain the non-invasively measured
\[
C_{21} = P_{2+}^D(0) - P_{2+}^D(0). 
\]
Since \( P_{2+}^D(0) = 1 - P_{2+}^D(1) \), Eq. \((4)\) and Eq. \((4)\) give the same result.

We now consider \( C_{32} \), where it is essential that we measure \( Q_2 \) non-invasively, since a subsequent measurement is performed. On the face of it, measuring \( C_{32} \) requires a correlation measurement between two detectors. This, however, is not the case, as we now show.

Let us begin by placing one detector at 2+ and another one at 3+ (Fig. \( \text{I} \)). We can then obtain the four probabilities, \( P_{3+2+}^D(n, n') \), that the detectors at 3+ and 2+ give the results \( n, n' = 0, 1 \) respectively. Of these, the one we are interested in is \( P_{3+2+}^D(1, 0) \), since this allows us to infer (non-invasively) the probability that the electron took path 2− to detector 3+. Moreover, we do not actually need to actively detect at 2+, since, if the electron reaches the 3+ detector, it is clear that it has not entered channel 2+ (because the detector there would have removed the electron from the system). All four probabilities, \( P_{3q,2q'}^D(1, \cdot) \), obtained in this non-invasive way, we can construct
\[
C_{32} = - \sum_{q,q'} q q' P_{3q,2q'}^D(1, \cdot), 
\]
where we have replaced the measurement value at position 2 with a dot to indicate that we do not actually have to measure there (the value is guaranteed to be zero).

In this way we obtain all the required correlation functions, measured non-invasively where necessary. Although we have concentrated on the simplest case here, the above non-invasive techniques are extensible to the case where the input state is unknown and all \( C_{\alpha\beta} \) must be measured in a non-invasive way, or to more complicated LGIs.

### B. Leggett-Garg Inequality

The action of the MZI can be specified by two beam-splitter scattering matrices \( s_X; X = A, B \). With \( a_{\alpha q} \) the annihilation operator for an electron in channel \( \alpha q \), the beam splitter input-output relations read
\[
\begin{pmatrix} a_{2+} \\ a_{2-} \end{pmatrix} = s_A \begin{pmatrix} a_{1+} \\ a_{1-} \end{pmatrix}; \quad \begin{pmatrix} a_{3+} \\ a_{3-} \end{pmatrix} = s_B \begin{pmatrix} a_{2+} \\ a_{2-} \end{pmatrix},
\]
Parameterizing the scattering matrices as
\[
s_X = \begin{pmatrix} \cos\left(\frac{1}{2} \theta_X\right) & \sin\left(\frac{1}{2} \theta_X\right) e^{i \phi_X} \\ -\sin\left(\frac{1}{2} \theta_X\right) e^{-i \phi_X} & \cos\left(\frac{1}{2} \theta_X\right) \end{pmatrix},
\]
we obtain the correlation functions
\[
\begin{align*}
C_{21} &= \cos \theta_A; \\
C_{31} &= \cos \theta_A \cos \theta_B - \sin \theta_A \sin \theta_B \cos \phi; \\
C_{32} &= \cos \theta_B,
\end{align*}
\]
such that the LG correlator reads
\[
K(\theta_A, \theta_B, \phi) = \cos \theta_A + \cos \theta_B - \cos \theta_A \cos \theta_B + \sin \theta_A \sin \theta_B \cos \phi, 
\]
with \( \phi = \frac{1}{2}(\phi_A - \phi_B) \) being the phase difference accumulated between the two paths. This is a familiar expression. If we identify \( \theta_A = \Omega t_1 \) and \( \theta_B = \Omega t_2 \), then Eq. \((11)\) is exactly that obtained for a qubit evolving under the Hamiltonian \( \hat{H} = \frac{1}{2} \Omega \sigma_x \) measured in the \( \sigma_x \) basis at times \( t_1, t_2 = t_1 + \tau_1 \), and \( t_1 = t_2 + \tau_2 \). The properties of Eq. \((11)\) are discussed in Sec. \( \text{II} \).

### II. QUANTUM HALL EDGE-CHANNELS

Quantum Hall edge channels have been shown to allow a direct translation of the MZI into electronic transport experiments\(^{11,\ldots,38}\) and Fig. \( \text{II} \) shows a sketch of the quantum Hall geometry needed to realise our proposal. Each channel in the MZI is realised with a single edge-channel and the electronic beam-splitters are realised by quantum point contacts (QPCs). Backscattering is suppressed between edge-channels such that transport is unidirectional. This set-up is the same as the MZIs of experiment except for the addition of extra contacts to the arms of the interferometer. These contacts are connected to the edge-channels via adjustable quantum point contacts, such that the detectors can be coupled into and out of the MZI as required. This method of coupling probes to the MZI arms has been realised in Ref. \( \text{38} \). Port 1+ is raised to a voltage +\( V \) and electrons are injected into this channel. The output ports (detectors) are all grounded. When the correlation function \( C_{31} \) is being measured, the detectors at 2± are not required and are isolated from the MZI by closing their QPCs. (Fig. \( \text{II} \) shows detector 2− closed off in this way). To measure the remaining correlation functions, the detectors at 2± are, one then the other, connected into the MZI by opening
up their respective QPCs. In Fig. 2, the detector at 2± is connected into the circuit and fully prevents electrons in channel 2± from reaching the outputs 3±.

A. Current measurements

Let \( \langle I_{aq} \rangle \) be the mean stationary current flowing into output \( q \), given that when \( \alpha = 3 \), the detectors at positions 2± are closed off. Further, let \( \langle I_{3q2q'} \rangle \) be the current flowing at output 3q when the output at 2q’ is open. Since, in the linear regime, the current operator for each output is \( I_{aq} = G_{aq} v_{aq} a_{aq} \), with \( G_{aq} = e^2/h \) the conduction quantum, these mean currents are proportional to the probability that an electron travels in the corresponding channel. The correlation functions required for the LGI can then be constructed, as with the CHSH inequality, as

\[
C_{a1} = \frac{\langle I_{a+} \rangle - \langle I_{a-} \rangle}{\langle I_{a+} \rangle + \langle I_{a-} \rangle};
\]

\[
C_{32} = \frac{\sum q' \langle I_{3q2q'} \rangle}{\sum \langle I_{3q2q'} \rangle}.
\]

Division by the sum of detector currents removes proportionality factors and, if all detector are identical, also removes detector inefficiencies. Writing the scattering matrices as

\[
s_X = \begin{pmatrix} r_X & t'_X \\ t_X & r'_X \end{pmatrix},
\]

we obtain the correlation functions

\[
C_{a1} = |r_A|^2 - |t_A|^2;
\]

\[
C_{31} = |r_B r_A + t_B t_A|^2 - |t_B r_A + r_B t_A|^2;
\]

\[
C_{32} = |r_A|^2 \{ |r_B|^2 - |t_B|^2 \} - |t_A|^2 \{ |r_B'|^2 - |r_B|^2 \}.
\]

With scattering matrices as in the previous section, the LG parameter \( K \) obtained from current measurements is the same as Eq. (11). This quantity is plotted in Fig. 3a. A maximum violation of \( K_{\text{max}} = \frac{3}{2} \) is obtained for parameters \( \theta_A = \theta_B = \pi/3 \) and \( \phi = 0 \).

The violation the LGI in this set-up arises because the measurements at 2± remove electrons from the interferometer arms, preventing interference between the two paths. The presence of this interference in \( C_{31} \) combined with its absence in \( C_{32} \) leads to the violation.

B. Dephasing

We can account for the effects of dephasing by allowing the phase \( \phi \) to fluctuate. We replace \( \phi \rightarrow \phi + \delta \phi \) in Eq. (11) and integrate \( \delta \phi \) over a flat distribution in the range \(-\Delta/2 < \delta \phi < \Delta/2\). The resulting LG parameter with dephasing reads

\[
K_{\text{deph}} = \cos \theta_A + \cos \theta_B - \cos \theta_A \cos \theta_B f(\Delta) \sin \theta_A \sin \theta_B \cos \phi,
\]

with \( f(\Delta) = 2\Delta^{-1} \sin(\Delta/2) \) the function containing the dephasing effects. If all angles are freely variable then
the maximum of this function is
\[ K_{\text{max}}^{\text{dep}}(\Delta) = \frac{1 + f(\Delta)(1 + f(\Delta))}{1 + f(\Delta)}, \]
(17)

obtained for \( \cos \theta_A = \cos \theta_B = [1 + f(\Delta)]^{-1} \). Expanding for small \( \Delta \), we find \( K_{\text{max}}^{\text{dep}}(\Delta) = \frac{1}{2} - \frac{1}{4}\Delta^2 \). In the opposite limit, where the dephasing is total, \( \Delta \rightarrow 2\pi \), we have \( f(\Delta) \rightarrow 0 \) and the maximised Leggett-Garg correlator reverts to the classical value, \( \lim_{\Delta \rightarrow 2\pi} K_{\text{max}}^{\text{dep}} = 1 \) as required.

One interesting feature occurs if we assume that the phase \( \phi \) is fixed (e.g., we are not able to vary the magnetic field) and maximise over \( \theta_{A/B} \) (see Fig. 3b). Providing that \( \cos \phi > 0 \), the maximum value is
\[ K_{\text{max}}^{\text{dep}}(\phi, \Delta) = f(\Delta) \cos \phi + \frac{1}{1 + f(\Delta) \cos \phi}, \]
(18)

found by setting \( \cos \theta_A = \cos \theta_B = [1 + f(\Delta) \cos \phi]^{-1} \). If, however, \( \cos \phi \leq 0 \), the maximum value is just the classical value, \( K_{\text{max}}^{\text{dep}}(\theta_{A/B}) = 1 \), found by setting \( \cos \theta_A = \cos \theta_B = 1 \). This reversion to the classical value occurs when the scalar product between the axis of the rotation of beamsplitter \( B \) and that of beamsplitter \( A \) becomes negative.

### C. Multi-channel case

The above scheme is easily modified to include multiple channels. We take the same geometry as before but assume that each lead supports \( M \) channels. The \( M \) channels of the upper lead are all associated with qubit state \( Q = +1 \); the \( M \) channels in the lower lead, with state \( Q = -1 \). The scattering matrices of Eq. (14) are thus generalised to \( 2M \times 2M \) matrices with \( M \times M \) blocks, \( r_X, t_X, r_X', \) and \( t_X' \). Assuming a large source-drain voltage, such that all channels are equally populated, the correlation functions read:

\[ C_{21} = \frac{1}{M} \text{Tr} \left\{ R_A - T_A \right\}; \]
\[ C_{32} = \frac{1}{M} \text{Tr} \left\{ R_A' (R_B - T_B) + T_A' (R_B' - T_B') \right\}; \]
\[ C_{31} = \frac{1}{M^3} \text{Tr} \left\{ R_A' (R_B - T_B) - T_A' (R_B' - T_B') \right\} \]
\[ + \frac{1}{M} \text{Tr} \left\{ t_{A'}^r r_B - t_{B'}^r r_B' \right\} \]
\[ + t_{A'}^t r_B' - t_{B'}^t r_B' \right\}, \]
(19)

with \( R_A = r_{A}^r r_{A}, T_A = t_{A}^t t_{A}, \) etc. The second term in the expression for \( C_{31} \) arises from interference between the paths. In the single channel case, these results reduce to those of Eq. (15).

An important observation can be made about the multi-channel case by considering that the scattering matrices preserve the channel-index, i.e., we essentially have \( M \) independent interferometers. In this case, the LG parameter reads \( K = \frac{1}{M} \sum_{m=1}^{M} K^{(m)} \), where \( K^{(m)} \) is the LG parameter for channel \( m \). If we could tune by hand all the parameters of the scattering matrices, then the maximum violation of \( K_{\text{max}} = \frac{3}{2} \) can be reached. However, in an experiment, there will typically only be a few controllable parameters and this could make violations hard to observe. Let us assume that we can adjust the parameters such that one of the \( K^{(m)} \) is maximised, \( m = 1 \), say. Whether we see a violation or not very much depends on what happens with the parameters of the other channels. If these parameters are all roughly similar to those of channel 1, then violations should still be observed. Generically, however, this will not be the case, and the \( K^{(m)} \) for the other channels will take unrelated values in the range from \(-3 \) to \( \frac{3}{2} \). The negative values are particularly troublesome as they will tend to overwhelm any positive contribution to the violation from other channels. This lack of controllability means that multi-channel geometries are best avoided if violations of the LG are sought.

### III. DOUBLE QUANTUM DOT INTERFEROMETER

The aboveMZI scheme functions by having the detectors remove electrons from the interferometer arms. In this section we study a second MZI realisation which leaves the electrons within the system and the effects of measurement are only felt through dephasing. This second set-up is shown in Fig. 4. As in the foregoing, the basic structure is of two (single-channel) leads that are joined at two beamsplitters. Beamsplitters between non-edge-channel leads can be realised by tunnel junctions, as in the recent experiments by Yamamoto et al.\[^{57}\] In each arm of the interferometer there is a QD and alongside each QD is a QPC charge detector. When connected to a voltage supply, the current flowing through the QPCs serves as read-out of the occupation of their respective QDs. Note that although similar detectors we used in e.g. Refs. 69 and 70, the way in which they are used here is different.

#### A. Model

We first consider the system without detectors. Our MZI model is related to that of, e.g., Refs. 71 and 72 but with different leads. Far from the junctions, we describe the four leads as non-interacting Fermi reservoirs with Hamiltonian \( H_{\text{res}} = \sum \omega_{k\alpha q} c_{k\alpha q}^\dagger c_{k\alpha q} \) with \( \omega \) the wavenumber of the electron, and where \( \alpha = 1, 3 \) and \( q = \pm \) specify the lead (we set \( \hbar = 1 \) and ignore spin). We assume that there is but a single orbital of relevance in each dot and that the DQD system is in the strong Coulomb blockade regime, such that it is restricted to just three states: ‘empty’, \( |0\rangle \); or with one excess electron in either the upper or lower dots, \( |+\rangle = d_+^\dagger |0\rangle \) and \( |\rangle = d_-^\dagger |0\rangle \), respectively. Assuming the dot levels are
detuned by an energy $\epsilon$ from one another, the dot Hamiltonian reads $H_S = \frac{1}{2} \sum_{q} q \rho^\dagger_q \rho_q$. In the following, we set $\epsilon = 0$ for simplicity. We assume that the effect of the beam-splitters is to modify the amplitudes with which the leads couple to the QDs. So, for example, an electron in lead 1+ tunnels into a superposition of lower and lower dot states, with the details of the superposition being determined by the scattering matrix $s_A$. The tunnel Hamiltonian connecting lead and dots therefore reads

$$H_T = \sum_k \left( C_{k1}^\dagger \cdot s_A^\dagger \cdot d + C_{k3}^\dagger \cdot s_B \cdot d + H.C. \right),$$

where $s_{A,B}$ are scattering matrices, assumed to be energy independent, $d = (d_+, d_-)$ is a vector of dot operators, and $C_{k\alpha}^\dagger = (T_\alpha + \epsilon_{k\alpha} T_\alpha) / \epsilon_{k\alpha}$ are vectors of lead operators with tunnel matrix elements $T_\alpha$, also assumed to be energy-independent. The corresponding tunnel rates are $\Gamma_{k\alpha} = 2 \pi |T_{k\alpha}|^2 \mathcal{Q}_{k\alpha}$, where $\mathcal{Q}_{k\alpha}$ is the density-of-states of reservoir $\alpha q$, also assumed constant.

In the infinite-bias limit, the system can be described by a quantum master equation of Lindblad-form. Let us introduce the super-operator notations $\mathcal{J}[d] \rho = dp d^\dagger$ and $A[d] \rho = -\frac{1}{i} \{d^\dagger d, \rho \}$, and introduce the operators

$$\tilde{d}_{1q} = \sqrt{\Gamma_{q}} e_q \cdot s_A \cdot d; \quad \tilde{d}_{3q} = \sqrt{\Gamma_{q}} e_q \cdot s_B \cdot d, \tag{21}$$

with unit vectors $e_+ = (1, 0)$ and $e_- = (0, 1)$. With introduction of counting fields $\chi_{q\alpha}$ to facilitate the calculation of current statistics (see e.g. Refs. [74, 79]), the χ-resolved master equation for the DQD system reads

$$\dot{\rho}(\chi) = -i [H, \rho] + \sum_{q\alpha} \left( e^{i \chi_{q\alpha}} \mathcal{J}[d_{q\alpha}] - A[d_{q\alpha}] \right) \rho. \tag{22}$$

The QDs are monitored by QPCs in a set-up similar to the single dot in an interferometer in the experiment of Ref. [59]. In including the detectors in our theory, we follow Gurvitz [28, 61]. When dot $q$ is unoccupied, the Hamiltonian for QPC $Dq$ reads

$$H_{Dq} = \sum_{k_s} \omega_{k_s}^D a_{k_s}^q a_{k_s}^q + \Omega_q \sum_k a_{kLq}^q a_{kRq}^q + H.c., \tag{23}$$

where $\omega_{k_s}^D$ is the energy of an electron in state $k$ on side $s = L, R$ of the the QPC, and $\Omega_q$ is the coupling amplitude between the two sides, (assumed energy independent). When dot $q$ is occupied, we assume that this Hamiltonian is modified such that the coupling constants shift to different values, $\Omega_q \rightarrow \Omega_q'$. In the limit of large bias across the QPC, the detector at location $q$ gives rise to an extra Liouvillian

$$W_{Dq}(\chi_{Dq}) = e^{i \chi_{Dq}} \mathcal{J}[\tilde{d}_{Dq}] - A[\tilde{d}_{Dq}], \tag{24}$$

which adds to the DQD Liouvillian. Here, $\tilde{d}_{Dq} = \sqrt{\gamma^D_+} |q\rangle \langle q| + \sqrt{\gamma^D_-} (1 - |q\rangle \langle q|)$ with $\gamma^D_+$ the rate of electron transfer through the QPC $q$ when its dot is empty, and $\gamma^D_-$ the rate when the dot is occupied. The counting field $\chi_{Dq}$ here allows us to calculate the statistics of the detector currents. Microscopically, the rates are $\gamma_q = 2 \pi |\Omega^D_q|^2 \rho_{Lq} \rho_{Rq} V_{Dq}$ and $\gamma_q' = 2 \pi |\Omega^D_q'|^2 \rho_{Lq} \rho_{Rq} V_{Dq}$, with $\rho_{sq}$ the density of states of the QPC reservoir $sq$ and $V_{Dq}$ the applied voltage. Detectors may be decoupled or coupled from the MZI-QD system by adjusting the QPC voltages such that the differences between the amplitudes $\Omega_q$ and $\Omega'_q$ is either zero (decoupled) or finite (coupled). Here, we only couple at most one detector to the system at a given time. Furthermore, we assume balanced detectors such that with the $D+$ detector coupled we have $\gamma_+ = \gamma$, $\gamma'_+ = \gamma'$, and $\gamma_- = \gamma'_- = 0$, and when the $D-$ detector is coupled we have $\gamma_- = \gamma$, $\gamma'_- = \gamma'$, and $\gamma_+ = \gamma'_+ = 0$.

### B. Current, correlation functions and probabilities

Our approach to measuring the LGI with this set-up is similar to that with the quantum Hall edge-channels with the main exception being how $C_{32}$ is obtained. We inject electrons into the ‘+’-channel of lead 1 and close the ‘−’-channel: $\Gamma_{1+} \rightarrow \Gamma_L$ and $\Gamma_{1−} \rightarrow 0$. For simplicity, we set the output rates equal: $\Gamma_{3+} = \Gamma_{3−} = \Gamma_R$.

To obtain $C_{31}$, we switch off the QPC detectors and measure the output currents at $3\pm$. Arranging the elements of the density matrix into a vector in the basis $(00, ++, −−, +−, −+, −−)\ldots$, the stationary state of the DQD system reads

$$\rho_{stat} = \frac{1}{2\Gamma} \left( 2 \Gamma_R, \Gamma_L (1 + \cos \theta_A), \Gamma_L (1 - \cos \theta_A), \Gamma_R \sin \theta_A, -e^{i \phi_A / 2} \Gamma_L \sin \theta_A, -e^{-i \phi_A / 2} \Gamma_L \sin \theta_A \right), \tag{25}$$

with total width $\Gamma = \Gamma_L + \Gamma_R$. Here, we have assumed the same scattering matrices as in Eq. (7). The total
The current flowing is \( \langle I \rangle_{3\pm} = \frac{\langle I \rangle_{1+}}{2} (1 \pm \cos \theta_A \cos \theta_B \pm \cos \phi \sin \theta_A \sin \theta_B) \).

Constructing \( C_{31} \) as in Eq. (12) we obtain
\[
C_{31} = \cos \theta_A \cos \theta_B - \cos \phi \sin \theta_A \sin \theta_B,
\]
which agrees with that of Eq. (3).

Next we can obtain \( C_{21} \) by turning on the QPC detectors one at a time. As shown in Ref. [5], the mean current flowing through the QPC can be used to extract the mean current flowing through the corresponding dot. With a detector coupled to dot \( q \), the current through the detector is
\[
\langle I_{Dq} \rangle = \frac{\langle I \rangle_{1+}}{2} \left[ \gamma \left( 1 + 2 \frac{\Gamma_R}{\Gamma_L} \right) + \gamma' + q(\gamma' - \gamma) \cos \theta_A \right].
\]
The current flowing through the QPC when the DQD is empty is \( \langle I_{Dq} \rangle = \gamma \), such that the difference is
\[
\langle \Delta I_{Dq} \rangle = \langle I_{Dq} \rangle - \langle I_{Dq} \rangle = \frac{\gamma' - \gamma}{2\Gamma_R} \langle I \rangle_{1+} (1 + q \cos \theta_A),
\]
which is proportional to the probability that an electron takes the path \( 2q \). Assuming balanced detectors, we obtain
\[
C_{21} = \frac{\langle \Delta I_{D+} \rangle - \langle \Delta I_{D-} \rangle}{\langle \Delta I_{D+} \rangle + \langle \Delta I_{D-} \rangle} = \cos \theta_A,
\]
as in Eq. (12).

Whereas these two correlation functions can be determined with just mean-current measurements, to determine \( C_{32} \) we need to consider current cross-correlations. Let us first imagine that we can measure the current through dot \( q \). Then, in the limit \( \Gamma_L \to 0 \), such that there is only ever at most one electron in the interferometer at a given time, the zero-frequency noise correlator
\[
S_{3q'2q} = \frac{1}{2} \int d\{I_{3q'}, I_{2q}\}_c,
\]
where \( \langle \ldots \rangle_c \) denotes the cumulant average, is proportional to the joint probability, \( P_{3q'2q} \), that the electron travels through dot \( q \) and ends up at output \( 3q' \). This result follows in the same way as in Ref. [13] the difference here being that we correlate the position of a single electron in subsequent regions, as opposed to the correlation of two spatially-separate electrons. Measuring all four such correlators, we obtain the probabilities
\[
P_{3q'2q} = \frac{S_{3q'2q}}{\sum_{r' r} S_{3r'2r}}.
\]
From these directly-obtained probabilities, we construct the ideal-negative-measurement ones as
\[
P_{3q'2q}^{\text{INM}} = P_{3q'} - P_{3q'2q} \bar{q},
\]
where \( \bar{q} = -q \) and the total probability at output \( 3q' \) is obtained from the currents
\[
P_{3q} = \frac{\langle I_{3q'} \rangle}{\sum_{r'} \langle I_{3r'} \rangle}.
\]

These relations follow from charge conservation and the unidirectional nature of the transport.

The QPC detectors couple not the current flowing through the dot, but rather to their occupations. In terms of the zero-frequency correlation function between current fluctuations in the QPC and those in one of the 3± ports,
\[
S_{3q' Dq} = \frac{1}{2} \int d\{I_{3q'}, I_{Dq}\}_c,
\]
the required probabilities read
\[
P_{3q'2q} = \frac{\langle I_{Dq} \rangle S_{3q' Dq}}{\sum_{r' r} \langle I_{Dr} \rangle S_{3r' Dr}}.
\]
This can be understood as follows. Whereas \( S_{3q'2q} \) correlates two delta-function peaks corresponding to the passage of the electron through the regions 2 and 3, \( S_{3q' Dq} \) correlates a delta-function in region 3 with a signal of finite duration in region 2, which corresponds to the finite time for which the dot is occupied. This mean occupation time is proportional to the inverse of mean current through the dot, which can be obtained (up to a proportionality constant) from the mean detector current \( \langle I_{Dq} \rangle \).

Calculating these probabilities, we find that in the limit \( \Gamma_L \to 0 \), the third correlation function reads
\[
C_{32} = \cos \theta_B,
\]
in accordance with Eq. (10). Since, in the \( \Gamma_L \to 0 \) limit, all three correlation function are identical with their ideal counterparts, the LGI for this set-up is identical to that of Eq. (11). In the way that we have described the QPC detectors here, it does not make any difference whether we calculate \( K \) using the ideal negative measurement probabilities or the direct ones since, in our theoretical description, the QPC detectors act as ideal detectors and only influence the system through their dephasing effect. Experimentally, the ideal negative measurement protocol should be used, and actually, the comparison between the case with ideal negative measurement and that without would give an interesting method for studying to what extent the QPC measurements are non-invasive. Let us just add that, whilst the above results were derived in the symmetric case with \( \Gamma_R+\Gamma_- = 1 \) and with balanced detector rates, if these ratios are unequal but known, then the difference can be accounted for by weighting the terms in the correlation functions accordingly.

### C. Dephasing

A simple way to include the effects of dephasing in this model is to “leave the detectors switched on” when
calculating $C_{31}$. With empty and occupied rates $\gamma'$ dephase and $\gamma_{\text{dephase}}$, the measured $K$ function has the form of Eq. (10), with the function $f(\Delta)$ replaced by

$$f = \left[1 + \frac{(\gamma'_{\text{dephase}})^{1/2} - (\gamma_{\text{dephase}})^{1/2}}{2\Gamma_R}\right]^{-1}. \quad (36)$$

$f = 1$ is the ideal no-dephasing case, and $f = 0$ gives the classical limit. To obtain strong violations of the LGI, therefore requires that the difference in rates $\gamma'$ dephase and $\gamma_{\text{dephase}}$ is small compared with the tunnel rate $\Gamma_R$.

### D. Detection errors

Just as the direct relation between the Bell inequalities and noise measurements of, e.g., Refs. [15] and [16] 

$$K' = \frac{1}{(\Gamma_L - \Gamma_R)\Gamma_R} \left\{ \left[-(\Gamma_L + \Gamma_R)^2 + \Gamma_L(\Gamma_L + 3\Gamma_R)\cos^2 \theta_A \right] \cos \theta_B 
+ (\Gamma_L - \Gamma_R)\Gamma_R \left[ 2\cos \theta_A \sin^2 (\theta_B/2) + \cos \phi \sin \theta_A \sin \theta_B \right] \right\}. \quad (37)$$

This expression is again maximised with $\cos \phi = 1$, but, unlike the $\Gamma_L \to 0$ case, the maximizing angles $\theta_A$ and $\theta_B$ are not equal. If we assume that $\Gamma_L/\Gamma_R \ll 1$, we can expand to leading order ($\gamma' - \gamma \to \infty$) to obtain

$$K' = K + 3\frac{\Gamma_L}{\Gamma_R} \cos \theta_B \sin^2 \theta_A + O\left(\frac{\Gamma_L}{\Gamma_R}^2\right),$$

where $K$ is the $\Gamma_L \to 0$ value. We can also calculate the corresponding quantity in the classical limit (this we do by calculating $C_{31}$ in limit $(\gamma'_{\text{dephase}} - \gamma_{\text{dephase}}) \to \infty$). In this case, we obtain $C_{31}^c = \cos \theta_A \cos \theta_B$, and the expansion of $B' = C_{21} + C_{32} - C_{31}$ for small $\Gamma_L$ gives

$$B' = B + 3\frac{\Gamma_L}{\Gamma_R} \cos \theta_B \sin^2 \theta_A + O\left(\frac{\Gamma_L}{\Gamma_R}^2\right), \quad (38)$$

where $B = \cos \theta_A + \cos \theta_B - \cos \theta_A \cos \theta_B$ is the classical value in the ideal case when, which is not, and which we have maximised gives $B_{\text{max}} = 1$, the bound of Eq. (11). Maximising $B'$ over the angles, we obtain a value bigger that unity. To lowest order then, classical and quantum LG correlators are affected in the same way. Fig. 5 shows the maximum values of both quantum and classical correlators.

Thus, assuming that we know the ratio of $\Gamma_L/\Gamma_R$ from current and noise measurements, the effects of a finite tunneling rate $\Gamma_L$ can be included in assessment of whether LGI is violated or not. The conservative approach is say that the quantity $(K'_{\text{max}} - \frac{3}{2})$ represents a systematic error in the measurement, and assuming that this error works against us, we can only conclude that we violate the LGI when the measured value of $K$ exceeds unity by an amount equal to this error. Alternatively, one can say that since one knows how the classical bound behaves at finite $\Gamma/\Gamma_R$, we can simply use $B'_{\text{max}}$ of Eq. (38) as a bound. However, providing that we are in the correct operating limit of $\Gamma_L/\Gamma_R \ll 1$, these modifications will be very small, such that whether they are taken into account or not will only effect the question of violation in marginal cases.

### IV. CONCLUSIONS

We have considered the violation of the LGI in MZ interferometer geometries. The key to the violation is a combination of the interference at the second beamsplitter and the inhibition of this interference by the measurement process. In the two proposals we have considered this inhibition occurs in two different ways. In the first realisation, we physically interrupt transmission through one of the arms of the MZI, obviously preventing interference. On the other hand, in the DQD proposal, the detectors act in a more traditional way and introduce dephasing between the paths.

In this MZ geometry both the state of the electron and measurement time are mapped onto real-space coordinates — the qubit states $Q = \pm$ are physically separate paths, and the regions within the interferometer correspond to different time instances. This mapping has several advantages for seeking a violation of the LGI in transport. The mapping of the time-coordinate means that we do not need to make time-resolved correlation...
measurements. All the measurements required here are either mean stationary currents or zero-frequency noise correlators. Furthermore, the spatial separation of the qubit degrees of freedom facilitates the realisation of ideal negative measurement, since it is relatively easy to couple to just one of the qubit states when they are spatially distinct. In this respect, increasing the separation of the detector arms should decrease the plausibility of claims that detection in one arm is, from a macro-realist point-of-view, influencing the other.

The general principles described here can easily be extended to further systems. Within transport, for example, our second scheme could also be realised with an edge-channel MZI plus QPC detector channel without the quantum dots. An alternative setting for the realisation of our first scheme might be the flying qubit experiment of Ref. which is essentially a MZI away from the quantum Hall regime. Two challenges are obvious with this realisation. Firstly, the leads reported in the experiment have multiple channels, which potentially gives rise to the problems discussed in section. The second problem is that of backscattering at the beamsplitters and detectors, which has (justifiably) been neglected here but probably can not be eliminated in setups such as that of Ref. Applications away from electronic transport are also possible. The application of the first scheme in optics is obvious but the notion of the qubit state is predicated on the source being a single-photon source. So whilst a classical wave might also exceed the right-hand-side of Eq. this would not constitute a violation of the LGI, as it represents an application of the concepts outside their proper realm of definition (i.e., non-dichotomic observables). Going further, the same principles could be used to test the LGI with electrons in free space, neutrons, atoms and molecules, all of which have had interference experiments in the MZI geometry conducted on them. Of these, molecules offer the most exciting prospect, as there the nature of the coherence being tested could potentially be macroscopic, in line with the original goals of Ref.

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