New Value of $m_{\mu}/m_e$ from Muonium Hyperfine Splitting

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The complete contribution to the muonium hyperfine splitting of relative order $\alpha^3 (m_e/m_\mu) \ln \alpha$ is calculated. The result amounts to 0.013 kHz, much smaller than suggested by a previous estimate, and leads to a 2$\sigma$ upward shift of the most precise value for the muon-electron mass ratio, with the error reduced by approximately 30%. Analogous contributions are calculated for the positronium hyperfine splitting: $(217/90 - 17 \ln 2/3) m_e (\alpha^3/\pi) \ln \alpha^{-1} \approx -0.32$ MHz; the remaining theoretical uncertainty is well below experimental error, leaving discrepancies of 2.5$\sigma$ and 3.5$\sigma$ with the two most precise measurements.

Precise measurement of the ground-state muonium ($\mu^+ e^-)$ hyperfine-splitting (HFS), together with the corresponding theoretical analysis, provides a stringent test of bound state theory in Quantum Electrodynamics (QED), and allows a precise determination of the fundamental physical constants $m_{\mu}/m_e$ and $\alpha$. The most precise measurement gives [1]:

$$\Delta \nu_{\text{expt.}} = 4463302.765(53) \text{ kHz} \ [1.2 \times 10^{-8}].$$

(1)

The corresponding theoretical prediction can be expressed as a series expansion in small parameters $\alpha \approx 1/137$ and $m_e/m_\mu \approx 1/207$; terms involving logarithms, $\ln \alpha^{-1} \approx \ln (m_\mu/m_e) \approx 5$, also appear. At leading order in $\alpha$, the splitting is given by the Fermi energy [2]:

$$E_F = \hbar \Delta \nu_F = \frac{16}{3} (\hbar c) R_{\infty} Z^4 \alpha^3 \frac{m_e}{m_\mu} \left[ 1 + \frac{m_e}{m_\mu} \right]^3.$$  

(2)

The complete splitting can be broken into the sum of terms [3] [4] [5]:

$$\Delta \nu_{\text{theory}} = \Delta \nu_D + \Delta \nu_{\text{rad}} + \Delta \nu_{\text{rec}} + \Delta \nu_{\text{t-t}} + \Delta \nu_{\text{weak}} + \Delta \nu_{\text{had}}.$$  

(3)

Here D stands for Dirac, or relativistic corrections, while the other terms are from radiative, recoil, radiative-recoil, weak and hadronic contributions.

Currently, theory is limited by uncalculated or imprecisely-known terms in $\Delta \nu_{\text{rec}}$ and $\Delta \nu_{\text{t-t}}$ of order $E_F \alpha^3 (m_e/m_\mu)$, some of which are enhanced by logarithmic factors; see Table I. This paper presents a calculation of terms of order $E_F \alpha^3 (m_e/m_\mu) \ln \alpha$, with results:

$$\delta(\Delta \nu_{\text{rec}}) = E_F \frac{(Z \alpha)^3 m_e}{\pi m_\mu} \ln(Z \alpha)^{-1} \left( \frac{101}{9} - 20 \ln 2 \right)$$  

(4)

$$\delta(\Delta \nu_{\text{t-t}}) = E_F \frac{\alpha(Z \alpha)^2 m_e}{\pi m_\mu} \ln(Z \alpha)^{-1} \left( -\frac{431}{90} + \frac{32}{3} \ln 2 + Z^2 \right).$$  

(5)

Numerically, these contributions give $-0.034 + 0.047 = 0.013$ kHz. Previous incomplete calculations [2] [4] suggested a contribution of $-0.263(60)$ kHz. The main result of this paper is to show that in fact these contributions are not as large as the previous estimates. Remaining theoretical uncertainty is dominated by terms of order $E_F (Z \alpha)^3 (m_e/m_\mu) \ln(m_\mu/m_e)$ ($\sim 0.06$ kHz); $E_F (Z \alpha)^3 (m_e/m_\mu)$ ($\sim 0.03$ kHz); and $E_F \alpha (Z \alpha)^3 (m_e/m_\mu)$ ($\sim 0.03$ kHz). A discussion of the error due to still uncalculated terms is given at the end of the paper.

Including the complete results of Eqs. (4), (5) does not significantly alter the theoretical prediction for the HFS in physical units, which is in good agreement with the experimental value, Eq. (1). Here we simply quote Ref. [2](Eq.(D14)):

$$\Delta \nu_{\text{theory}} = 4463302.67(27) \text{ kHz} \ [6.1 \times 10^{-8}],$$  

(6)

where the error is due mainly to the measured value of $m_\mu/m_e$ [4]. Likewise, the HFS determination of $\alpha$ is not significantly changed [3]. However, the new results in Eqs. (4), (5) represent a fractional shift of $6.2 \times 10^{-8}$ in the HFS, and hence also in the HFS determination of the mass ratio $m_\mu/m_e$. A recoil term of order $E_F (Z \alpha)^2 (m_e/m_\mu)^2$, which was not included in $\Delta \nu_{\text{rec}}$ of Ref. [4], contributes an additional 0.065(6) kHz or $1.5 \times 10^{-8}$ [4]. The mass ratio is then shifted from the value in Ref. [2](Eq.(161)) to become:

$$\Delta \nu_{\text{theory}} = 4463302.67(27) \text{ kHz} \ [6.1 \times 10^{-8}],$$  

(7)

where the error is due mainly to the measured value of $m_\mu/m_e$ [4]. Likewise, the HFS determination of $\alpha$ is not significantly changed [3]. However, the new results in Eqs. (4), (5) represent a fractional shift of $6.2 \times 10^{-8}$ in the HFS, and hence also in the HFS determination of the mass ratio $m_\mu/m_e$. A recoil term of order $E_F (Z \alpha)^2 (m_e/m_\mu)^2$, which was not included in $\Delta \nu_{\text{rec}}$ of Ref. [4], contributes an additional 0.065(6) kHz or $1.5 \times 10^{-8}$ [4]. The mass ratio is then shifted from the value in Ref. [2](Eq.(161)) to become:
with the errors arising from uncertainty in $\Delta \nu_{\text{theory}}$, due to uncalculated terms, from $\Delta \nu_{\text{expt.}}$, and from the value of $\alpha$, respectively. This represents a shift of 2.5$\sigma$ (in terms of the previous error 3.1 $\times 10^{-8}$), and a 30$\%$ reduction in error.

The positronium ($e^+e^-$) HFS has also been measured precisely, though at present its interest is for testing our knowledge of QED bound states, as opposed to determining fundamental constants. The two most precise values are due to Mills and Bearman ($\Delta \nu (P)_{\text{expt.1, Ref. [1]}}$) and Ritter et. al. ($\Delta \nu (P)_{\text{expt.2, Ref. [10]}}$):

\[
\begin{align*}
\Delta \nu (P)_{\text{expt.1}} & = 203\,387.5(1.6) \text{ MHz } [7.9 \times 10^{-6}] \\
\Delta \nu (P)_{\text{expt.2}} & = 203\,389.10(74) \text{ MHz } [3.6 \times 10^{-6}].
\end{align*}
\]

The theoretical expression is:

\[
\Delta \nu (Ps)_{\text{theory}} = m_r \alpha^4 \left( C_0 + C_1 \frac{\alpha}{\pi} + C_{21} \alpha^2 \ln \alpha^{-1} + C_{20} \alpha^2 \\
+ C_{32} \frac{\alpha^3}{\pi} \ln^2 \alpha^{-1} + C_{31} \frac{\alpha^3}{\pi} \ln \alpha^{-1} + C_{30} \frac{\alpha^3}{\pi} + \mathcal{O}(\alpha^4) \right).
\]

Including the known terms through $C_{20}$ [11] yields $\Delta \nu (Ps)_{\alpha^2} = 203\,392.93 \text{ MHz}$. Coefficient $C_{32} = -7/8$ has been known for some time [12], and in this paper we calculate:

\[
C_{31} = 217/90 - 17 \ln 2/3.
\]

$C_{32}$ and $C_{31}$ contribute $-0.91 \text{ MHZ}$ and $-0.32 \text{ MHz}$ to $\Delta \nu (Ps)_{\text{theory}}$, respectively, bringing the theoretical prediction to:

\[
\Delta \nu (Ps)_{\text{theory}} = 203\,391.70(20) \text{ MHz } [1.0 \times 10^{-6}].
\]

The uncertainty of 0.20 MHz corresponds to a coefficient $C_{30} \approx 4$. For comparison, the numerical values of the other coefficients are: $C_0 = 0.58$, $C_1 = -1.24$, $C_{21} = 0.21$, $C_{20} = -0.39$, $C_{32} = -0.88$, $C_{31} = -1.52$. The discrepancy with experiment is significant: 2.5$\sigma$ and 3.5$\sigma$ for Eqs. (8) and (9) respectively. As with the orthopositronium lifetime [13], [14], a true disagreement between experiment and the predictions of QED would have important consequences.

The calculation is done in the framework of an effective quantum mechanical Hamiltonian theory [14], taking inputs from relativistic QED field theory and from (non-relativistic) NRQED field theory [15]. The results to be derived for muonium can be translated directly to positronium by taking $m_\mu \rightarrow m_e$, and including the additional contributions from virtual $e^+e^-$ annihilation.

The Hamiltonian can be decomposed into the sum:

\[
H = H_0 + V_4 + V_5 + V_6 + V_7,
\]

where $H_0$ is the unperturbed Hamiltonian for the Coulomb problem with reduced mass $m_r = m_e m_\mu/(m_e + m_\mu)$:

\[
H_0 = \frac{p^2}{2m_r} - \frac{(Z\alpha)}{r}.
\]

Potentials $V_4$, $V_5$, $V_6$ and $V_7$ give contributions to the energy of order $ma^4$, $ma^5$, etc. Since non-HFS operators will affect the HFS only in second- or higher-order perturbation theory, it follows that only the HFS parts of potentials $V_6$ and $V_7$ are necessary. Furthermore, any potential not contributing to S-states (in first or second order perturbation theory) may be neglected.

We will write the potentials in terms of a list of standard operators: $(q \equiv l - k)$

\[
\begin{align*}
\langle l | O_1 | k \rangle &= \frac{1}{m_r^2} \\
O_2 &= \frac{1}{\pi(Z\alpha)m_r^2} p^i \left( \frac{p^2}{2m_r} - \frac{Z\alpha}{r} - E \right) \ln \frac{m_r/2}{p^2/m_r} - \frac{Z\alpha}{r} - E p^i \\
\langle l | O_3 | k \rangle &= \frac{1}{m_r} \ln \frac{|q|}{m_r} \\
\langle l | O_4 | k \rangle &= \frac{1}{m_r^2} \frac{|l \times k|^2}{q^2}
\end{align*}
\]
\[ \langle l | \mathcal{O}_5 | k \rangle = \pi (Z\alpha) \frac{|q|}{m_r} \]
\[ \langle l | \mathcal{O}_6 | k \rangle = \frac{q^2}{m_r^2} \ln \frac{|q|}{m_r} \]
\[ \mathcal{O}_7 = \frac{1}{\pi (Z\alpha)} \frac{p^4}{m_r^2} \]
\[ \langle l | \mathcal{O}_8 | k \rangle = \frac{1}{m_r^2} |l \times k|^2 \frac{q^4}{q^4} \]
\[ \langle l | \mathcal{O}_9 | k \rangle = \frac{1}{m_r^2} \left( \sigma_\mu \cdot \sigma_\mu - \frac{3q \cdot \sigma_\mu \cdot \sigma_\mu}{q^2} \right) \]

Note that \( \mathcal{O}_1 = \delta^3(r)/m_r^2 \).

Potential \( V_4 \) is derived from tree-level NRQED diagrams with Fermi, Darwin, Kinetic, and Dipole vertices \([14]\), and contains the leading relativistic corrections:
\[ V_4 = 4\pi (Z\alpha) \left\{ \frac{c_F c_F}{6} \frac{m_r^2}{m_e m_\mu} \left[ \sigma_\mu \cdot \sigma_\mu \mathcal{O}_1 + \frac{1}{2} \mathcal{O}_9 \right] + \frac{1}{8} \left( c_D m_r^2 m_e^2 + c_p m_r^2 m_\mu^2 \right) \mathcal{O}_1 \right\} \]
\[ - \frac{1}{32} \left( \frac{m_r^3}{m_r^3} + \frac{m_\mu^3}{m_\mu^3} \right) \mathcal{O}_7 - \frac{m_r^2}{m_e m_\mu} \mathcal{O}_8 \}
\]

Renormalization constants \( c_F \approx c_D = 1 + \mathcal{O}(\alpha) \) are tabulated below. \( V_5 \) gives the leading radiative corrections:
\[ V_5 = \frac{2\alpha (Z\alpha)}{3} \left( \frac{m_r^2}{m_r^2} + 2Z \frac{m_r^2}{m_e m_\mu} + Z^2 \frac{m_r^2}{m_\mu^2} \right) \mathcal{O}_2 + \frac{14(Z\alpha)^2}{3} \frac{m_r^2}{m_e m_\mu} \mathcal{O}_3 \]
\[ + \left\{ - \frac{4\alpha (Z\alpha)}{3} \left( \frac{m_r^2}{m_r^2} \ln \frac{m_r}{m_e} + Z \frac{m_r^2}{m_\mu} \ln \frac{m_r}{m_\mu} \right) \right. \]
\[ + (Z\alpha)^2 \frac{m_r^2}{m_e m_\mu} \left[ - \frac{2}{m_\mu^2 - m_e^2} \left( \frac{2}{m_r^2 \ln \frac{m_r}{m_e} - m_e^2 \ln \frac{m_\mu}{m_r}} + \frac{20}{9} \right) \right. \]
\[ \left. - \frac{2m_r m_\mu}{m_\mu^2 - m_e^2} \ln \frac{m_r}{m_e} \sigma_\mu \cdot \sigma_\mu \right] - \frac{4\alpha (Z\alpha) m_r^2}{15} \mathcal{O}_1 \}
\]

Using first order perturbation theory, potentials \( V_4 \) and \( V_5 \) correctly reproduce the complete \( S \)-state energy spectrum through \( \mathcal{O}(m^2) \) \([14] [17]\). The contribution from muon vacuum polarization is not relevant to our analysis, and has been excluded from \( V_5 \).

For \( V_6 \), only HFS terms are necessary. These again are taken directly from NRQED diagrams:
\[ V_6 = 4\pi (Z\alpha) \frac{\sigma_\mu \cdot \sigma_\mu}{m_e m_\mu} \left\{ \frac{m_r^2}{m_e m_\mu} \left( \frac{c_F c_F}{48} + \frac{c_F c_D}{6} - \frac{c_F c_F}{12} \right) \right. \]
\[ - \frac{1}{24} \left( c_{\mu p} c_F m_r^2 m_e^2 + c_F c_{h p} m_r^2 m_\mu^2 \right) \mathcal{O}_4 \]
\[ - \frac{1}{48} \left[ c_{\mu F} c_F m_r^2 m_\mu + c_F c_{F h} m_r^2 m_\mu \right] \mathcal{O}_5 \}
\]

Spin-Orbit, retardation, Time-Derivative, \( p'p \), and Seagull interactions have been included. Additional local operator terms, of the form \( -\nabla^2 \delta^3(r) \) and \( \{p^2, \delta^3(r)\} \) are not shown explicitly; these analytic terms do not generate factors of \( \ln \alpha \), and so are not relevant to the present analysis \([13]\).

The necessary renormalization constants have already been calculated \([16] [19]\):
\[ c_F = 1 + a_e, \quad c_D = 1 + \frac{8\alpha}{3\pi} \left( -\frac{3}{8} + \frac{5}{6} \right) + 2a_e, \quad c_S = 1 + 2a_e, \quad c_{\mu p} = a_e \]
\[ \]
Here \( a_e = \alpha/2\pi + \mathcal{O}(\alpha^2) \) is the electron anomalous magnetic moment. For \( c_\mu, m_\mu \) and \( Z^2 \alpha \) are substituted for \( m_e \) and \( \alpha \).
Potential $V_7$ has no non-instantaneous HFS contribution coming from photon momenta $q \approx ma^2$, a consequence of the fact that spin-dependent $M_1$ multipole transitions vanish in the absence of relativistic effects, and are therefore suppressed. The remaining instantaneous part of $V_7$, from momenta $q \approx ma$, is fully determined by requiring that the Hamiltonian correctly reproduce the low-momentum expansion of the 1-loop photon-exchange scattering amplitude. Introducing a photon mass $\lambda$, and ultraviolet cutoff $\Lambda$ on photon momenta, the effective Hamiltonian (without $V_7$) gives \[22\]:

$$\left[ \frac{2\pi(Z\alpha)}{3m_em_\mu} \sigma_e \cdot \sigma_\mu \right] \left( \frac{Z\alpha}{m_em_\mu} \right) \frac{q^2}{m_em_\mu} \left( \frac{\frac{2}{3} \ln \frac{\Lambda}{\lambda} + \cdots}{\frac{2}{3} \ln \frac{\Lambda}{\lambda} + \frac{1}{4} \log \frac{q}{\Lambda} + \cdots} \right),$$  

(20)

where again analytic terms are not shown. The corresponding QED amplitude is (Fig. 1):

$$\left[ \frac{2\pi(Z\alpha)}{3m_em_\mu} \sigma_e \cdot \sigma_\mu \right] \left( \frac{Z\alpha}{m_em_\mu} \right) \frac{q^2}{m_em_\mu} \left( \frac{\frac{2}{3} \ln \frac{\Lambda}{\lambda} + \cdots}{\frac{2}{3} \ln \frac{\Lambda}{\lambda} + \frac{1}{4} \log \frac{q}{\Lambda} + \cdots} \right).$$  

(21)

This result has been checked both in QED Feynman gauge, and in NRQED Coulomb gauge \[2\]. Requiring the effective theory to match QED implies that

$$V_7 = \frac{(Z\alpha)^2}{6} \frac{m_\mu^2}{m_\mu^2} \sigma_e \cdot \sigma_\mu \mathcal{O}_6.$$  

(22)

Contributions to $V_7$ having a dependence on $m_e$, $m_\mu$, $\alpha$ and $Z$ different from Eq.(22) are ruled out by noticing that:

(i) The non-recoil contributions are already present in $V_4$, $V_5$ and $V_6$ (as we will soon verify), so that $V_7$ contains no non-recoil piece; (ii) Masses can enter only as inverse powers $1/m_e$ and $1/m_\mu$, and in particular not as $1/(m_e + m_\mu)$.

This latter result can be seen clearly using time ordered perturbation theory in NRQED: the NRQED vertices are all homogeneous in the masses, leaving only the energy denominators to consider; however, the energy denominators will all have the form $1/|q|^2/m_e + p_q^2/m_\mu$, with photon momentum $q$ and particle momenta $p_1$, $p_2$. (Contributions which are not simply iterations of lower-order potentials must have at least one photon in each intermediate state.)

Such an expression, for $q \approx p_1 \approx p_2 \approx ma$, can be expanded in powers of $p_q^2/m_e |q|$,

$$p_q^2/m_\mu |q|$$—again homogeneous in the masses. Using (i) and (ii), the only possible parameter dependence which is symmetric in $m_e$ and $m_\mu$ is that of Eq.(22).

Having completed the specification of the Hamiltonian, Eqs.(13), (14), (16), (17), (18), (22), we now use the usual expressions from Rayleigh-Schrödinger perturbation theory to solve for the energy shift:

$$\Delta E = (V_6 + V_7) + \left( \langle V_4 + V_5 \rangle \mathcal{G}(V_4 + V_5) \right) + \langle V_4 \rangle \left( \frac{\partial V_5}{\partial E} \right),$$  

(23)

where $\mathcal{G}$ is the Coulomb Green’s function with ground state pole removed, and $\langle V \rangle$ is the expectation value of $V$ in the ground state of the unperturbed $H_0$, Eq.(14). The logarithmic contributions of the necessary matrix elements are:

$$\frac{\langle O_i \rangle}{\delta^3(r)} \rightarrow (Z\alpha)^2 \ln(Z\alpha)^{-1} \begin{cases} 2, & i = 4 \\ 8, & i = 5 \\ 12, & i = 6 \end{cases}$$  

(24)

$$\left( 2 \frac{\langle O_i \mathcal{G} \delta^3(r) \rangle}{\delta^3(r)} + \frac{\partial O_i}{\partial E} \right) \rightarrow \left( \frac{Z\alpha}{\pi} \right) \ln(Z\alpha)^{-1} \times \begin{cases} -2, & i = 1 \\ -4 \ln(Z\alpha)^{-1} + 6 - 8 \ln 2, & i = 2 \\ \ln(Z\alpha)^{-1} + 1 - 2 \ln 2, & i = 3 \\ -16, & i = 7 \\ -1, & i = 8 \end{cases}$$  

(25)

$$\frac{\langle O_9 \mathcal{G} O_9 \rangle}{\delta^3(r)} \rightarrow \frac{10}{m_\mu^2} \frac{(Z\alpha)}{\pi} \ln(Z\alpha)^{-1}.$$  

(26)

where the arrows signify that only logarithmic corrections, and in the case of $\langle O_9 \mathcal{G} O_9 \rangle$, only the HFS part, are shown \[23\]. The pure recoil result for the HFS at order $E_F(Z\alpha)^3(m_e/m_\mu)$ contains the previously known $\ln^2(Z\alpha)$ and $\ln(Z\alpha) \ln(m_\mu/m_e)$ contributions \[13\] \[18\] \[33\]: the new $\ln(Z\alpha)$ term is shown in Eq.(4) \[24\]. For radiative corrections at order $E_F \alpha(Z\alpha)^2$, the non-recoil $\ln^2(Z\alpha)$ and $\ln(Z\alpha)$ terms, and the recoil $(m_e/m_\mu) \ln^2(Z\alpha)$ term \[2\],
agree with previous calculations. A part of the radiative-recoil single-logarithm corresponding to reduced mass factor \( m_e^2/m_\mu m_\mu \approx (1 - 2m_e/m_\mu) \) was included previously [3–4]; the complete contribution is given in Eq.(6). Numerical values are summarized in Table I.

For positronium, there are additional interactions due to virtual annihilation of the electron and positron. The hard annihilation process is described by local operators, which by themselves cannot generate nonanalytic factors. So, for \( \ln \alpha \) contributions, only second order perturbations involving \( V_4 \) and \( V_5 \) need be considered:

\[
\delta V_4 = \frac{\pi \alpha}{2} \left( \frac{3}{4} + \frac{\sigma_e \cdot \sigma_\mu}{4} \right) O_1 \tag{27}
\]

\[
\delta V_5 = \alpha^2 \left( \left( -\frac{22}{9} \right) \left( \frac{3}{4} + \frac{\sigma_e \cdot \sigma_\mu}{4} \right) + (1 - \ln 2) \left( \frac{1}{4} - \frac{\sigma_e \cdot \sigma_\mu}{4} \right) \right) O_1 \tag{28}
\]

\( \delta V_4 \) gives the leading contribution from 1-photon annihilation. The first and second terms of \( \delta V_5 \) come from radiative corrections to \( \delta V_4 \), and from 2-photon virtual annihilation, respectively. \( \mathcal{O}(m\alpha^4 \ln \alpha) \) contributions from these annihilation operators are:

\[
\Delta \nu_{\text{ann.}}(m_e\alpha^2 \ln \alpha) = m_e \frac{\alpha^2}{\pi} \ln \alpha^{-1} \left[ -\frac{3}{8} \ln \alpha^{-1} + \frac{2261}{1080} - 3 \ln 2 \right] . \tag{29}
\]

The non-annihilation contributions for positronium are obtained by taking the limit \( m_\mu \to m_e \) in the muonium analysis (making no expansion in \( m_e/m_\mu \)); the combined result is given in Eq.(3).

The previously most significant sources of error in the muonium HFS were \( \Delta \nu_{\nu-\tau} \) (0.104 kHz) and \( \Delta \nu_{\text{rec}} \) (0.060 kHz) [3]; all other uncertainties are estimated below 0.010 kHz [24]. By calculating the \( \mathcal{O}(E_F\alpha^3(m_e/m_\mu) \ln \alpha) \) contribution to \( \Delta \nu_{\nu-\tau} \), the uncertainty in this quantity should be reduced by a factor \( \sim \ln \alpha^{-1} \approx 5 \); in fact, since there are still uncalculated terms at \( \mathcal{O}(E_F\alpha^2(Z \alpha)(m_e/m_\mu) \ln(m_e/m_e)) \) [27] and \( \mathcal{O}(E_F\alpha^3(Z \alpha)^2 m_e/m_\mu) \) [28], we take this uncertainty as 0.040 kHz. The uncertainty in \( \Delta \nu_{\text{rec}} \) should remain approximately the same, since it is dominated by the still uncalculated terms of order \( \mathcal{O}(E_F(Z \alpha)^3(m_e/m_\mu) \ln(m_e/m_e)) \) [27] and \( \mathcal{O}(E_F(Z \alpha)^3(m_e/m_\mu)) \) [28]. Thus we take 0.070 kHz as an estimate of the total remaining theoretical error.

In the final stages of the calculation, I received word from K. Melnikov and A. Yelkhovsky that they have also performed the calculation of \( \alpha^3 \ln \alpha \) terms, in a dimensional regularization approach [24]. After a detailed comparison, we agree fully on the contributions in both muonium and positronium. The agreement found in different formalisms in two independent calculations lends strong support to the correctness of the results.

This work was motivated in part by, and is an extension of, Ref. [14]. Many ideas used in the calculation originated with G. P. Lepage, who I thank for continued insights and encouragement during the present work. Thanks are also due to P. Labelle, and to K. Melnikov and A. Yelkhovsky for useful conversations. This work was supported by a grant from the National Science Foundation.

| \( \times E_F \cdot \frac{m_e}{m_\mu} \) | Ref. [3] (kHz) | present paper (kHz) |
|-----------------------------------|--------------|---------------------|
| \( (Z \alpha)^2 \ln^2 (Z \alpha) \) | −0.043 | |
| \( (Z \alpha)^3 \ln (Z \alpha) \ln(m_\mu/m_e) \) | −0.210 | |
| \( (Z \alpha)^3 \ln (Z \alpha) \) | −0.257(*) | −0.034 |
| \( (Z \alpha)^3 \ln(m_\mu/m_e) \) | — | −0.035 (*) [27] |
| \( (Z \alpha)^3 \) | 0.107(30) | |
| \( \alpha (Z \alpha)^2 \ln^2 (Z \alpha) \) | 0.344 | |
| \( \alpha (Z \alpha)^2 \ln (Z \alpha) \) | −0.008(*) | 0.034 |
| \( \alpha (Z \alpha)^2 \) | −0.107(30) | |
| \( Z^2 \alpha (Z \alpha)^2 \ln (Z \alpha) \) | — | 0.013 |
| \( \alpha^2 (Z \alpha)^2 \ln (m_\mu/m_e) \) | −0.055 | |
| \( \alpha^2 (Z \alpha)^2 \ln^2 (m_\mu/m_e) \) | 0.010 | |
| \( \alpha^2 (Z \alpha) \ln(m_\mu/m_e) \) | 0.009(*) | |
| \( \alpha^2 (Z \alpha) \) | — | |

**TABLE I.** Contributions of order \( E_F \alpha^3(m_e/m_\mu) \) to the muonium HFS. The second column lists the contributions used in Ref.[3]; the third column gives new or modified values from the present paper. Asterisks denote partial results.
The constant terms at $O(K)$. Melnikov and A. Yelkhovsky, to be published, hep-ph/0010131.

The $(Z\alpha)$ factor is determined by the coefficient of $(1/m_e/m_e)\ln(m_e/m_e)$ in the HFS part of the 3-loop threshold scattering amplitude: $\sigma_e \cdot \sigma_r [Z\alpha^2/(m_e/m_e)] \ln(m_e/m_e) - 8(2\ln 2 + 1)(m_e^2/\lambda^2) - 2\ln(m_e/\lambda) + C + O(\lambda/m_e)$. This result was derived directly from results for positronium HFS in P. Labelle, Cornell University Ph.D. Thesis (1994), and has also been checked using the techniques of Ref. [14]. The gauge invariant contribution, $3\ln 2 - 9/2$ would contribute $-0.035$ kHz to the HFS. We do not include this in the final numerical values, but simply note that the size of this term does not contradict our suggested error estimates.

The constant terms at $O(E_F\alpha Z\alpha^2/m_e/m_e)$ and $O(E_F\alpha Z\alpha^3/m_e/m_e)$ were estimated in the NRQED framework [5] [5].

K. Melnikov and A. Yelkhovsky, to be published, hep-ph/0001013.
Figure 1: Feynman diagrams for the QED amplitude necessary to determine $V_T$. 