A Single-Letter Capacity Formula for MIMO Gauss-Markov Rayleigh Fading Channels

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Abstract

Over the past decades, the problem of communication over finite-state Markov channels (FSMCs) has been investigated in many researches and the capacity of FSMCs has been studied in closed form under the assumption of the availability of partial/complete channel state information at the sender and/or the receiver. In our work, we focus on infinite-state Markov channels by investigating the problem of message transmission over time-varying single-user multiple-input multiple-output (MIMO) Gauss-Markov Rayleigh fading channels with average power constraint and with complete channel state information available at the receiver side (CSIR). We completely solve the problem by giving a single-letter characterization of the channel capacity in closed form and by providing a proof of it.

Index Terms

Gauss-Markov Rayleigh fading channels, channel capacity, multiple-antenna channels

I. INTRODUCTION

In many new applications in modern wireless communications such as several machine-to-machine and human-to-machine systems, the tactile internet [1] and industry 4.0 [2], robust and ultra-reliable low latency information exchange is required. These applications impose challenges on the robustness requirement because of the time-varying nature of the channel conditions caused by the mobility and the changing wireless medium.

Several accurate tractable channel models are employed to model the channel variations appearing in wireless communications including the Markov model, often employed in flat fading and inter-symbol interference [3]. The Markov model is widely used for modeling wireless flat-fading channels due to its low memory and its consolidated theory.

The availability and quality of channel state information (CSI) has a high influence on the capacity of the Markov channels. Over the past decades, many researchers have addressed the problem of communication over finite-state Markov channels (FSMCs) [4] and extensive studies have been performed to analyze the capacity of FSMCs in closed form under the assumption of the availability of partial/complete channel state information at the sender and/or the receiver side [5]–[11].

In our work, the focus is on discrete-time continuous-valued time-varying Markov channels, which are of high relevance for practical systems. In particular, we are concerned with the time-varying single-user multiple-input multiple-output (MIMO) Rayleigh fading channels, where we assume that the statistics of the gain sequence are known to both the sender and the receiver and that the actual realization of the channel state sequence is completely known to the receiver only (CSIR). Therefore, the state sequence is viewed as a second output sequence of the channel. We further assume that the channel fades are modeled as a first-order Gauss-Markov process, which is widely used to describe the time-varying aspect of the channel [12]–[15]. The focus is on the multiple-antenna setting which has drawn considerable attention in the area of wireless communications because MIMO systems offer higher rates and more reliability and resistance to interference, compared to single-input single-output (SISO) systems [16].

To the best of our knowledge, no capacity formula for MIMO Gauss-Markov fading channels with CSIR is provided in the literature. A single-letter expression for the capacity is provided in [17] in the case when the channel fades are independent and identically distributed (i.i.d.). Other than that, only the proof of a general formula based on the inf-information rate for the capacity which can be generalized for arbitrary channels with abstract alphabets is provided in [18].

The main contribution of our work is to give a single-letter expression of the capacity of MIMO Gauss-Markov fading channels with average power constraint and to provide a proof of it.

Paper Outline: The rest of the paper is organized as follows. In Section II, we present the channel model, provide the key definitions and the main and auxiliary results. In Section III, we provide a rigorous proof of the capacity of time-varying multi-antenna Rayleigh fading channels with CSIR. Section IV is devoted to deriving an upper-bound on the variance of the normalized information density between the inputs and the outputs of the time-varying MIMO Rayleigh fading channel. This auxiliary result is used in the proof of the capacity formula. Section V contains concluding remarks and proposes potential future research in this field. Several auxiliary lemmas are collected in the Appendix.
Clearly, the statement of the Lemma holds for

Show that for any

For

\( N \)

where

\( t \)

the notation

\( S \)

cardinality of

\( A \). Channel Model

law has a density

\( p_{X,Y}(x,y) \)

and

\( p_{Y|X}(y|x) \)

For any random variables \( X, Y \) whose joint probability law has a density \( p_{X,Y}(x,y) \) and \( p_{Y|X}(y|x) \), respectively, and their conditional probability density functions by \( p_X(x) \) and \( p_Y(y) \), respectively, and their conditional probability density functions by \( p_{X,Y}(x|y) \)

For any random variables \( X, Y \) and \( Z \), we use the notation \( X \oplus Y \oplus Z \) to indicate a Markov chain.

For any random variable \( X \), \( var(X) \) refers to the variance of \( X \) and for any random variables \( X \) and \( Y \), \( cov(X,Y) \) refers to the covariance between \( X \) and \( Y \). For any set \( S \), \( |S| \) stands for the cardinality of \( S \). For any matrix \( A \), \( tr(A) \) refers to the trace of \( A \). \( ||A|| \) stands for the operator norm of \( A \) with respect to the Euclidean norm, \( A^H \) stands for the standard Hermitian transpose of \( A \), \( vec(A) \) refers to the vectorization of \( A \), \( \lambda_{max}(A) \) refers to the maximum eigenvalue of \( A \) and \( \lambda_{min}(A) \) refers to its minimum eigenvalue. For any matrix \( A \) and \( B \), we use the notation \( A \preceq B \) to indicate that \( B - A \) is positive semi-definite. For any vector \( X \), \( X^T \) refers to its transpose. For any random matrix \( A \in \mathbb{C}^{m \times n} \) with entries \( A_{i,j} \) \( i = 1, \ldots, m, j = 1, \ldots, n \), we define

\[
E[|A|] = \begin{bmatrix}
E[A_{11}] & E[A_{12}] & \cdots \\
\vdots & \ddots & \\
E[A_{m1}] & E[A_{m2}] & \cdots
\end{bmatrix}.
\]

For any integer \( m \), \( Q_{(P,m)} \) is defined to be the set of positive semi-definite Hermitian matrices which are elements of \( \mathbb{C}^{m \times m} \) and whose trace is smaller than or equal to \( P \).

II. CHANNEL MODEL, DEFINITIONS AND RESULTS

A. Channel Model

For any block-length \( n \), we consider the following channel model for the time-variant fading channel \( W_{G_i} \)

\[
z_i = G_i t_i + \xi_i \quad i = 1 \ldots n,
\]

where \( t^n = (t_1, \ldots, t_n) \in \mathbb{C}^{N_T \times n} \) and \( z^n = (z_1, \ldots, z_n) \in \mathbb{C}^{N_R \times n} \) are channel input and output blocks, respectively, and where \( N_T \) and \( N_R \) refer to the number of transmit and receive antennas, respectively.

Here, \( G^n = G_1 \ldots G_n \), where \( G_i \) models the gain for the \( i^{th} \) channel use. We consider the following model for the gain. For \( 0 \leq \alpha < 1 \):

\[
G_i = \sqrt{\alpha} G_{i-1} + \sqrt{1 - \alpha} W_i, \quad i = 2 \ldots n.
\]

For any \( i = 2 \ldots n \), are i.i.d. such that \( vec(G_1) \), \( vec(W_i) \), \( i = 2 \ldots n \) are drawn from \( \mathcal{N} \left( 0_{N_R N_T}, K \right) \), where \( K \) is any \( N_R N_T \times N_R N_T \) covariance matrix. Therefore, the sequence of \( G_i, i = 1 \ldots n \), forms a Markov chain. \( \xi_i = (\xi_i, \ldots, \xi_n) \in \mathbb{C}^{N_R \times n} \) models the noise sequence. We further assume that the \( \xi_i \) s are i.i.d., where \( \xi_i \sim \mathcal{N} \left( 0_{N_R}, \sigma^2 I_{N_R} \right) \), \( i = 1 \ldots n \), that \( G^n \) and \( \xi^n \) are mutually independent. Furthermore, we assume that the random input sequence \( T^n = (T_1, \ldots, T_n) \) is independent of \( (G^n, \xi^n) \). It is also assumed that both the sender and the receiver know the statistics of the random gain sequence \( G^n \) and that only the receiver knows its actual realization (CSIR). Therefore, \( G^n \) is viewed as a second output sequence of the fading channel.

B. Properties of the random gain sequence

In the following lemmas, we present some properties of the random gain in (2).

Lemma 1. For \( 0 < \alpha < 1 \) and \( i \in \{1 \ldots n \} \),

\[
G_i = \sqrt{\alpha}^{-1} G_{i-1} + \sqrt{1 - \alpha} \sum_{j=2}^{i} \sqrt{\alpha}^{i-j} W_j.
\]

Proof. We will proceed by induction. Base Case: Clearly, the statement of the Lemma holds for \( i = 1 \)

Inductive step: Show that for any \( k \geq 2 \), if the statement of the lemma holds for \( i = k \) then it holds for \( i = k + 1 \).

Assume that the statement of the lemma holds for \( i = k \), then we have

\[
G_k = \sqrt{\alpha}^{k-1} G_1 + \sqrt{1 - \alpha} \sum_{j=2}^{k} \sqrt{\alpha}^{k-j} W_j.
\]
It follows that

\[
G_{k+1}^{(a)} = \sqrt{\alpha} G_k^{(a)} + \sqrt{1 - \alpha} W_{k+1}^{(a)}
\]

\[
= \sqrt{\alpha} \left[ \sqrt{\alpha^{k-1}} G_1^{(a)} + \sqrt{1 - \alpha} \sum_{j=2}^{k} \sqrt{\alpha^{k-j}} W_j^{(a)} \right] + \sqrt{1 - \alpha} W_{k+1}^{(a)}
\]

\[
= \sqrt{\alpha} G_1^{(a)} + \sqrt{1 - \alpha} \sum_{j=2}^{k} \sqrt{\alpha^{k-j}} W_j^{(a)} + \sqrt{1 - \alpha} W_{k+1}^{(a)}
\]

\[
= \sqrt{\alpha} G_1^{(a)} + \sqrt{1 - \alpha} \sum_{j=2}^{k+1} \sqrt{\alpha^{k+1-j}} W_j^{(a)}
\]

where \((a)\) follows from (2) and \((b)\) follows from the induction assumption. Thus, the statement of the lemma holds for \(i = k+1\).

**Conclusion:** Since both the base case and the inductive step have been proved as true, by mathematical induction the statement of the lemma holds for every \(i = 1 \ldots n\).

\(\square\)

**Lemma 2.** \(\forall i \in \{1, \ldots, n\}, \) it holds that

\[
\text{vec} (G_i) \sim \mathcal{N}_C (0_{N_R N_T}, K),
\]

where \(G_i, i = 1 \ldots n\) is defined in (2) with \(0 \leq \alpha < 1\).

**Proof.** Clearly, the statement of the lemma holds for \(\alpha = 0\). Now, let \(0 < \alpha < 1\). The statement of the lemma is valid for \(i = 1\). Let \(i \in \{2, \ldots, n\}\) be fixed arbitrarily. Let \(G_1' = \sqrt{\alpha^{-1}} G_1\) and \(W_j' = \sqrt{1 - \alpha} \sqrt{\alpha^{-1-j}} W_j\) for every \(j \in \{2, \ldots, i\}\). Since \(G_1\) and \(W_j, j = 2, \ldots, n\) are independent, it follows that \(G_1'\) and \(W_j', j = 2, \ldots, n\) are also independent. Since \(\text{vec} (G_1) \sim \mathcal{N}_C (0_{N_R N_T}, K)\) and \(\text{vec} (W_j) \sim \mathcal{N}_C (0_{N_R N_T}, K)\) for every \(j \in \{2, \ldots, i\}\), it follows that

\[
\text{vec} (G_1') \sim \mathcal{N}_C (0_{N_R N_T}, \alpha^{i-1} K)
\]

and that for every \(j \in \{2, \ldots, i\}\)

\[
\text{vec} (W_j') \sim \mathcal{N}_C (0_{N_R N_T}, (1 - \alpha) \alpha^{i-j} K).
\]

Now, from Lemma 1, it follows that

\[
G_i = G_1' + \sum_{j=2}^{i} W_j'.
\]

As a result,

\[
\text{vec} (G_i) \sim \mathcal{N}_C \left( 0_{N_R N_T}, \left[ \alpha^{i-1} + (1 - \alpha) \sum_{j=2}^{i} \alpha^{i-j} \right] K \right).
\]

For \(0 < \alpha < 1\), we have

\[
\sum_{j=2}^{i} \alpha^{i-j} = \alpha \sum_{j=2}^{i} \left( \frac{1}{\alpha} \right)^j
\]

\[
= \alpha \left( \frac{1}{\alpha} \right)^2 \frac{1 - (\frac{1}{\alpha})^{i-1}}{1 - \frac{1}{\alpha}}
\]

\[
= \frac{\alpha^i - \alpha}{\alpha^2 - \alpha}
\]

\[
= \frac{1 - \alpha^{i-1}}{1 - \alpha}.
\]
It follows that
\[ \alpha^{i-1} + (1 - \alpha) \sum_{j=2}^{i} \alpha^{j-i} = 1. \]

This yields
\[ \text{vec}(G_i) \sim \mathcal{N}_C \left( 0_{N_R \times N_T}, K \right) \quad \forall i \in \{1, \ldots, n\}. \]

**Lemma 3.** Let \( i_1, i_2 \in \{1, \ldots, n\} \). Assume without loss of generality that \( i_1 < i_2 \). Consider the gain model presented in (2). Then, for \( 0 < \alpha < 1 \), it holds that
\[ G_{i_2} = \sqrt{\alpha^{i_2-i_1}} G_{i_1} + \sqrt{1 - \alpha} \sum_{j=i_1+1}^{i_2} \sqrt{\alpha^{j-i_1}} W_j. \]

**Proof.** By Lemma 1, it holds that
\[ G_{i_2} = \sqrt{\alpha^{i_2-1}} G_{i_1} + \sqrt{1 - \alpha} \sum_{j=2}^{i_2} \sqrt{\alpha^{j-1}} W_j \]
and that
\[ G_{i_1} = \sqrt{\alpha^{i_1-1}} G_{i_1} + \sqrt{1 - \alpha} \sum_{j=2}^{i_1} \sqrt{\alpha^{j-1}} W_j. \]
Thus
\[
G_{i_2} - \sqrt{\alpha^{i_2-i_1}} G_{i_1} \\
= \sqrt{\alpha^{i_2-1}} G_{i_1} + \sqrt{1 - \alpha} \sum_{j=2}^{i_2} \sqrt{\alpha^{j-1}} W_j - \sqrt{\alpha^{i_1-1}} G_{i_1} + \sqrt{1 - \alpha} \sum_{j=2}^{i_1} \sqrt{\alpha^{j-1}} W_j \\
= \sqrt{\alpha^{i_2-1}} G_{i_1} + \sqrt{1 - \alpha} \sum_{j=2}^{i_2} \sqrt{\alpha^{j-1}} W_j - \sqrt{\alpha^{i_1-1}} G_{i_1} + \sqrt{1 - \alpha} \sum_{j=2}^{i_1} \sqrt{\alpha^{j-1}} W_j \\
= \sqrt{1 - \alpha} \sum_{j=i_1+1}^{i_2} \sqrt{\alpha^{j-i_1}} W_j.
\]

**C. Achievable Rate and Capacity**

Next, we define an achievable rate for the channel \( W_{G^n} \) and the corresponding capacity. For this purpose, we begin by providing the definition of a transmission-code for \( W_{G^n} \).

**Definition 1.** A transmission-code \( \Gamma \) of length \( n \) and size\(^1\) \( ||\Gamma|| \) with average power constraint \( P \) for the channel \( W_{G^n} \) is a family of pairs \( \left\{ (t_{\ell}, D_{\ell}(g^n)) : g^n \in (\mathbb{C}^{N_R \times N_T})^n, \ \ell = 1, \ldots, ||\Gamma|| \right\} \) such that for all \( \ell, j \in \{1, \ldots, ||\Gamma||\} \) and all \( g^n \) for which \( g^n \in (\mathbb{C}^{N_R \times N_T})^n \), we have:
\[
t_{\ell} \in \mathbb{C}^{N_T \times n}, \quad D_{\ell}(g^n) \subset \mathbb{C}^{N_R \times n},
\]
\[
\frac{1}{n} \sum_{i=1}^{n} t_{\ell,i}^H t_{\ell,i} \leq P \quad t_{\ell} = (t_{\ell,1}, \ldots, t_{\ell,n}),
\]
\[
D_{\ell}(g^n) \cap D_{j}(g^n) = \emptyset, \quad \ell \neq j.
\]

Here, \( t_{\ell}, \ \ell = 1, \ldots, ||\Gamma|| \) and \( D_{\ell}(g^n), \ \ell = 1, \ldots, ||\Gamma|| \), are the codewords and the decoding regions, respectively.

\(^1\)This is the same notation used in [19].
Remark 1. Since we do not assume any channel state information at the transmitter side, the codewords $t_\ell, \ell = 1, \ldots, ||\Gamma||$, do not depend on the gain sequence.

Definition 2. A real number $R$ is called an achievable rate of the channel $W_{G^n}$ if for every $\theta, \delta > 0$ there exists a code sequence $(\Gamma_n)_{n=1}^\infty$, where each code $\Gamma_n$ of length $n$ is defined according to Definition 1, such that
\[
\frac{\log ||\Gamma_n||}{n} \geq R - \delta
\]
and
\[
\epsilon_{\max}(\Gamma_n) = \max_{\ell \in \{1, \ldots, ||\Gamma_n||\}} \mathbb{E} \left[ W_{G^n}(D_\ell^{(G^n)}, |t_\ell|) \right] \leq \theta
\]
for sufficiently large $n$.

Definition 3. The supremum of all achievable rates defined according to Definition 2 is called the capacity of the fading channel $W_{G^n}$ and is denoted by $C(P, N_R \times N_T)$.

D. Main Result

In this section, we present the main result of our work, which is a single-letter characterization of the time-varying MIMO Gauss-Markov Rayleigh fading channel. This is illustrated in the following theorem.

Theorem 1. Let $G$ be any random matrix such that $\text{vec}(G) \sim \mathcal{N}(0_{N_R N_T}, K)$. A single-letter characterization of the capacity of the channel in (1) with gain model in (2) with $0 \leq \alpha < 1$ is
\[
C(P, N_R \times N_T) = \max_{Q \in \mathcal{Q}(P, N_T)} \mathbb{E} \left[ \log \det \left( I_{N_T} + \frac{1}{\sigma^2} G Q G^H \right) \right].
\]

The proof of Theorem 1 is provided in Section III.

Remark 2. The capacity in (4) is the same as the capacity of i.i.d. MIMO Rayleigh fading channels. Under the CSIR assumption, the memory factor $\alpha$ has no influence on the capacity. The CSIR assumption is critical in deriving the closed-form expression for the capacity. In the absence of CSIR, no single-letter formula for the capacity exists even for SISO finite-state Markov fading channels [20] [21].

E. Auxiliary Result

For the proof of Theorem 1, we require the following auxiliary result on the normalized information density of $W_{G^n}$.

Lemma 4. Let $T^n = (T_1, \ldots, T_n)$ be an $n$-length input sequence of the channel $W_{G^n}$ in (1) with gain model in (2) such that $0 \geq \alpha < 1$ and such that the $T_i$s are i.i.d., where $T_i \sim \mathcal{N}(0_{N_T}, Q), i = 1, \ldots, n$, and $Q \in \mathcal{Q}(P, N_T)$. Let $Z^n = (Z_1, \ldots, Z_n)$ be the corresponding output sequence. Then, it holds that
\[
\frac{\text{var} \left( i(T^n; Z^n, G^n) \right)}{n} \leq \kappa(n, \alpha),
\]
where
\[
\kappa(n, \alpha) = \left\{ \begin{array}{ll}
\frac{\mu'}{n} & \text{for } \alpha = 0 \\
\frac{2c'\mu}{n(1-\sqrt{\alpha})} + \frac{\kappa'}{n} & \text{for } 0 < \alpha < 1
\end{array} \right.
\]
for some $c', \kappa' > 0$ and where $\lim_{n \to \infty} \kappa(n, \alpha) = 0$ for any $0 \leq \alpha < 1$.

The proof of Lemma 4 is provided in Section IV.

III. PROOF OF THEOREM 1

A. Direct Proof

Let
\[
R_{\max} = \max_{Q \in \mathcal{Q}(P, N_T)} \mathbb{E} \left[ \log \det \left( I_{N_T} + \frac{1}{\sigma^2} G Q G^H \right) \right],
\]
where $G \in \mathbb{C}^{N_R \times N_T}$ is any random matrix satisfying $\text{vec}(G) \sim \mathcal{N}_\mathbb{C}(0_{N_R N_T}, K)$. We are going to show that
\[
C(P, N_R \times N_T) \geq R_{\max} - \epsilon,
\]
with $\epsilon$ being an arbitrarily small positive constant. Let $\theta, \delta > 0$ and
\[ E_n = \{ t^n = (t_1, \ldots, t_n) \in \mathbb{C}^{N_T \times n} : \frac{1}{n} \sum_{i=1}^{n} \| t_i \|^2 \leq P \}. \] (5)

We define for any $Q \in Q_{(P, N_T)}$,
\[ \phi(Q) = \mathbb{E} \left[ \log \det \left( I_{N_R} + \frac{1}{\sigma^2} G Q G^H \right) \right]. \]

Now notice that any $Q \in Q_{(P, N_T)}$, we have
\[
\log \det \left( I_{N_R} + \frac{1}{\sigma^2} G Q G^H \right) \\
\leq \log \det \left( I_{N_R} + \frac{1}{\sigma^2} \| G Q G^H \| I_{N_R} \right) \\
= \log \det \left( 1 + \frac{1}{\sigma^2} \| G Q G^H \| \right) I_{N_R} \\
= N_R \log \left( 1 + \frac{1}{\sigma^2} \| G Q G^H \| \right) \\
\leq \frac{N_R}{\ln(2) \sigma^2} \| G Q G^H \| \\
\leq \frac{N_R}{\ln(2) \sigma^2} \| Q \| \| G \|^2. \]

where $(a)$ follows because $A \preceq \| A \| I_n$ for any Hermitian $A \in \mathbb{C}^{n \times n}$ (by Lemma 10 in the Appendix) and $(b)$ follows because $\| Q \| = \lambda_{\text{max}}(Q) \leq \text{tr}(Q) \leq P$. Now, it holds that $\mathbb{E} \left[ \frac{P N_R}{\ln(2) \sigma^2} \| G \|^2 \right] < \infty$ since $\mathbb{E} \left[ \| G \|^2 \right] < \infty$ (by Lemma 13 in the Appendix). Therefore, it follows from the dominated convergence theorem that $\phi$ is continuous on the compact set $Q_{(P, N_T)}$. Therefore, one can find a $Q \in Q_{(P, N_T)}$ such that $\text{tr}(Q) = P - \beta$ for some $\beta > 0$ and such that
\[ \phi(Q) \geq R_{\text{max}} - \epsilon. \] (6)

We define
\[ \hat{P} = P - \beta \]
and
\[ \hat{\beta} = \frac{\beta}{\ln(2) P} - \log(1 + \frac{\beta}{P}) > 0. \] (7)

Let us now introduce the following well-known lemma:

**Lemma 5.** (Feinstein’s Lemma with input constraints) [22] Let $n > 0$ be fixed arbitrarily. Consider any channel with random input sequence $T^n$, with corresponding random channel output sequence $Z^n$ and with information density $i(T^n; Z^n)$. Then, for any integer $\tau > 0$, real number $\gamma > 0$, and measurable set $E_n$, there exists a code with cardinality $\tau$, maximum error probability $\epsilon_n$ and block-length $n$, whose codewords are contained in the set $E_n$, where $\epsilon_n$ satisfies
\[ \epsilon_n \leq \mathbb{P} \left[ \frac{1}{n} i(T^n; Z^n) \leq \frac{\log \tau}{n} + \gamma \right] + \mathbb{P} \left[ T^n \notin E_n \right] + 2^{-n \gamma}. \]

Let $T^n = (T_1, \ldots, T_n) \in \mathbb{C}^{N_T \times n}$ to be the random input sequence of the channel $W_G$, where the $T_i$s are i.i.d. such that $T_i \sim \mathcal{N}(0_{N_T}, \hat{Q})$, $i = 1, \ldots, n$. We denote its corresponding random output sequence by $Z^n = (Z_1, \ldots, Z_n)$. Now, we apply Lemma 5 for the set $E_n$ defined in (5) and for $\gamma = \frac{\delta}{4}$. It follows that there exists a code sequence $(\Gamma_n)_{n=1}^{\infty}$, where each code $\Gamma_n$ is defined according to Definition 1 such that
\[ \epsilon_{\text{max}}(\Gamma_n) \leq \mathbb{P} \left[ \frac{1}{n} i(T^n; Z^n, G^n) \leq \frac{1}{n} \log \| \Gamma_n \| + \frac{\delta}{4} \right] + \mathbb{P} \left[ T^n \notin E_n \right] + 2^{-n \frac{\delta}{4}}, \]
Lemma 6. \( \epsilon_{\text{max}}(\Gamma_n) = \max_{\ell \in \{1, \ldots, |\Gamma_n|\}} \mathbb{E}[W_{G^n}(D^G_n^{(\ell)} | t^\ell)] \)
\[ = \max_{\ell \in \{1, \ldots, |\mathcal{M}|\}} \mathbb{E}\left[ \mathbb{P}\left[ \hat{M} \neq \ell | M = \ell, G^n \right] \right] \]
\[ = \max_{\ell \in \{1, \ldots, |\mathcal{M}|\}} \mathbb{P}\left[ \hat{M} \neq \ell | M = \ell \right]. \]

with \( M, \hat{M} \) being the random message and the random decoded message and with \( \mathcal{M} \) being the set of messages.

Choose \( \|\Gamma_n\| \) such that for sufficiently large \( n \)
\[
R_{\text{max}} - \epsilon - \delta \leq \frac{\log \|\Gamma_n\|}{n} \leq R_{\text{max}} - \epsilon - \frac{\delta}{2}.
\]

It follows that
\[
\epsilon_{\text{max}}(\Gamma_n) \leq \mathbb{P}\left[ \frac{1}{n} i(T^n; Z^n, G^n) \leq R_{\text{max}} - \epsilon - \frac{\delta}{2} \right] + \mathbb{P}[T^n \notin E_n] + 2^{-n\frac{\delta}{2}}
\]
\[
\leq \mathbb{P}\left[ \frac{1}{n} i(T^n; Z^n, G^n) \leq \phi(\hat{Q}) - \frac{\delta}{2} \right] + \mathbb{P}[T^n \notin E_n] + 2^{-n\frac{\delta}{2}}, \tag{8}
\]

where we used (6) in the last step. It remains to find upper-bounds for \( \mathbb{P}\left[ \frac{1}{n} i(T^n; Z^n, G^n) \leq \phi(\hat{Q}) - \frac{\delta}{2} \right] \) and for \( \mathbb{P}[T^n \notin E_n] \) that vanish as \( n \) goes to infinity.

1) Upper-bound for \( \mathbb{P}[T^n \notin E_n] \): We will prove that
\[
\mathbb{P}[T^n \notin E_n] \leq 2^{-n\beta},
\]
where \( \beta \) is defined in (7). For this purpose, we will introduce and prove the following lemma:

**Lemma 6.** Let \( X_i, i = 1, \ldots, n \) be i.i.d. \( N \)-dimensional complex Gaussian random vectors with mean \( 0_N \) and covariance matrix \( O \) whose trace is smaller than or equal to \( \rho \). Then, for any \( \delta > 0 \)
\[
\mathbb{P}\left[ \sum_{i=1}^{n} \|X_i\|^2 \geq n(\rho + \delta) \right] \leq \left( 1 + \frac{\delta}{\rho} \right)^2 2^{-n\delta/2},
\]

where
\[
\|X_i\|^2 = \sum_{j=1}^{N} |X_{ij}|^2
\]
and
\[
X_i = (X_i^1, \ldots, X_i^N)^T.
\]

**Proof.** Let \( X \) be a random vector with the same distribution as each of the \( X_i \). Then
\[
\mathbb{P}\left[ \sum_{i=1}^{n} \|X_i\|^2 \geq n(\rho + \delta) \right]
\]
\[= \mathbb{P}\left[ \sum_{i=1}^{n} \|X_i\|^2 - n(\rho + \delta) \geq 0 \right]
\]
\[\leq \mathbb{E}\left[ \exp\left( \beta \left( \sum_{i=1}^{n} \|X_i\|^2 - n(\rho + \delta) \right) \right) \right]
\]
\[= \left[ \exp(-|\rho + \delta| \beta) \mathbb{E}\left[ \exp(\beta \|X\|^2) \right] \right]^{n},
\]
where we used the \( X_i \)'s are i.i.d.. By a standard calculation which follows below, one can show that
\[
\mathbb{E}\left[ \exp(\beta \|X\|^2) \right] = \mathbb{E}\left[ \exp(\beta X^H X) \right]
\]
\[= \prod_{j=1}^{N} (1 - \beta \mu_j)^{-1} \quad \beta < \beta_0,
\]
where \( \mu_1, \ldots, \mu_N \) are the eigenvalues of \( O \), and for \( \beta_0 = \frac{1}{\rho} \leq \beta_{1, \ldots, \rho N} \leq \min_{j \in \{1, \ldots, N\}} \frac{1}{\mu_j} \) so that all the factors are positive, whether \( O \) is non-singular or singular.

To prove this, we let \( r \) be the rank of \( O \). It holds that \( r \leq N \). We make use of the spectral decomposition theorem to express \( O \) as \( S^* \Lambda^* S \Lambda^* H \), where \( \Lambda^* \) is a diagonal matrix whose first \( r \) diagonal elements are positive and where the remaining diagonal elements are equal to zero. Next, we let \( V^* = S \Lambda^* S \), and remove the \( N - r \) last columns of \( V^* \), which are null vectors to obtain the matrix \( V \). Then, it can be verified that \( O = V V^H \). We can write \( X = VU^* \) where \( U^* \sim N_C(0, I_r) \).

As a result:

\[
X^H X = (U^*)^H V V^H U^*.
\]

Let \( S \) be a unitary matrix which diagonalizes \( V V^H \) such that \( S^* V V^H VS = \text{Diag}(\mu_1, \ldots, \mu_r) \) with \( \mu_1, \ldots, \mu_r \) being the positive eigenvalues of \( O = V V^H \) in decreasing order. One defines \( U = S^* U^* \). We have

\[
\text{cov}(U) = S^* \text{cov}(U^*) S = S^* S = I_r.
\]

Therefore, it holds that \( U \sim N_C(0, I_r) \). Since \( S \) is unitary, we have

\[
X^H X = ((S^*)^{-1} U)^H V V^H (S^*)^{-1} U
= U^H S^* V V^H S U
= U^H \text{Diag}(\mu_1, \ldots, \mu_r) U
= \sum_{j=1}^r \mu_j |U_j|^2.
\]

Then, we have

\[
\mathbb{E} \left[ \exp \left( \beta \|X\|^2 \right) \right] = \mathbb{E} \left[ \prod_{j=1}^r \exp \left( \frac{1}{2} \beta \mu_j^2 |U_j|^2 \right) \right]
= \prod_{j=1}^r \mathbb{E} \left[ \exp \left( \frac{1}{2} \beta \mu_j^2 |U_j|^2 \right) \right]
= \prod_{j=1}^N (1 - \beta \mu_j)^{-1},
\]

where we used that all the \( U_j \)'s are independent, that \( \forall j \in \{1, \ldots, r\}, 2|U_j|^2 \) is chi-square distributed with \( k = 2 \) degrees of freedom and with moment generating function equal to \( \mathbb{E} \left[ \exp(2t|U_j|^2) \right] = (1 - 2t)^{-k/2} \) for \( t < \frac{1}{2} \) and that \( \forall j \in \{1, \ldots, r\} \) and for \( \beta < \beta_0, \frac{1}{2} \beta \mu_j \leq \frac{1}{2} \). This proves (9).

Now, it holds that

\[
\prod_{i=1}^N (1 - \beta \mu_i) \geq 1 - \beta (\mu_1 + \ldots + \mu_N) \geq 1 - \beta \rho.
\]

This yields

\[
\mathbb{P} \left[ \sum_{i=1}^n \|X_i\|^2 \geq n(\rho + \delta) \right] \leq \exp(-\rho + \delta) \mathbb{E} \left[ \exp \left( \beta \|X\|^2 \right) \right]
\leq \frac{\exp(-\rho + \delta) \beta}{1 - \beta \rho},
\]

where \( 0 < \beta < \frac{1}{\rho} = \beta_0 \). Putting \( \beta = \frac{\delta}{\rho(\delta + \rho)} < \frac{1}{\rho} \) yields

\[
\mathbb{P} \left[ \sum_{i=1}^n \|X_i\|^2 \geq n(\rho + \delta) \right] \leq \left( 1 + \frac{\delta}{\rho} \right) \exp \left( -\frac{\delta}{\rho} \right)
= \left( 1 + \frac{\delta}{\rho} \right) 2^{(-\frac{\delta}{\rho + \delta})}.\]
This completes the proof of Lemma 6.

By Lemma 6, it holds that
\[
\mathbb{P}\left[ \sum_{i=1}^{n} \|T_i\|^2 \geq n(\hat{P} + \beta) \right] \leq \left( 1 + \frac{\beta}{\hat{P}} \right) 2^{\left( \frac{-n\beta}{\log(1 + \frac{\beta}{\hat{P}})} \right)} = 2^{-n\beta}.
\]

As a result, we have
\[
\mathbb{P}\left[ T_n \notin E_n \right] = \mathbb{P}\left[ \sum_{i=1}^{n} \|T_i\|^2 > nP \right] \\
\leq \mathbb{P}\left[ \sum_{i=1}^{n} \|T_i\|^2 \geq n(\hat{P} + \beta) \right] \\
\leq 2^{-n\beta}.
\]

2) Upper-bound for \( \mathbb{P}\left[ \frac{1}{n} i(T^n; Z^n, G^n) \leq \phi(\hat{Q}) - \frac{\delta}{2} \right] \): Let us introduce the following lemma:

Lemma 7.
\[
i(T^n; Z^n, G^n) = \sum_{i=1}^{n} i(T_i; Z_i, G_i).
\]

Proof. We have
\[
i(T^n, Z^n, G^n) = \log \left( \frac{p_{T^n, Z^n, G^n}(T^n, Z^n, G^n)}{p_{Z^n, G^n}(Z^n, G^n) p_{T^n}(T^n)} \right) \\
= \log \left( \frac{p_{Z^n, G^n|T^n}(Z^n, G^n|T^n)}{p_{Z^n, G^n}(Z^n, G^n)} \right).
\]

Since \( G^n \) and \( T^n \) are independent, we have
\[
\log \left( \frac{p_{Z^n, G^n|T^n}(Z^n, G^n|T^n)}{p_{Z^n, G^n}(Z^n, G^n)} \right) = \log \left( \frac{p_{Z^n|G^n,T^n}(Z^n|G^n,T^n)}{p_{Z^n|G^n}(Z^n|G^n)} \right).
\]

Furthermore, since conditioned on \((G^n, T^n)\), the outputs are independent, we have
\[
\log \left( \frac{p_{Z^n|G^n,T^n}(Z^n|G^n,T^n)}{p_{Z^n|G^n}(Z^n|G^n)} \right) = \log \left( \prod_{i=1}^{n} \frac{p_{Z_i|G_i,T_i}(Z_i|G_i,T_i)}{p_{Z_i|G_i}(Z_i|G_i)} \right).
\]

This yields
\[
i(T^n, Z^n, G^n) \\
= \log \left( \prod_{i=1}^{n} \frac{p_{Z_i|G_i,T_i}(Z_i|G_i,T_i)}{p_{Z_i|G_i}(Z_i|G_i)} \right) \\
\overset{(a)}{=} \log \left( \prod_{i=1}^{n} \frac{p_{Z_i|G_i,T_i}(Z_i|G_i,T_i)}{p_{Z_i|G_i}(Z_i|G_i)} \right),
\]

where \((a)\) follows because
\( G_1 T_1 \ldots G_{i-1} T_{i-1} G_{i+1} T_{i+1} \ldots G_n T_n Z^{i-1} \circ G_i T_i \circ Z_i \)
forms a Markov chain.
Now since conditioned on $G^n$ and for independent inputs, the outputs are independent, we have

$$
\log \left( \prod_{i=1}^{n} \frac{p_{Z_i|G_i, T_i} G_i (Z_i|G_i, T_i)}{p_{Z_i|G^n} G_i (Z^n|G^n)} \right)
= \log \left( \prod_{i=1}^{n} \frac{p_{Z_i|G_i, T_i} G_i (Z_i|G_i, T_i)}{\prod_{i=1}^{n} p_{Z_i|G^n} G_i (Z_i|G^n)} \right).
$$

It follows that

$$
i(T^n; Z^n, G^n)
= \log \left( \prod_{i=1}^{n} \frac{p_{Z_i|G_i, T_i} G_i (Z_i|G_i, T_i)}{\prod_{i=1}^{n} p_{Z_i|G^n} G_i (Z_i|G_i)} \right)
= \log \left( \prod_{i=1}^{n} \frac{p_{Z_i|G_i, T_i} G_i (Z_i|G_i, T_i)}{p_{Z_i|G_i} G_i (Z_i|G_i)} \right)
= \sum_{i=1}^{n} \log \left( \frac{p_{Z_i|G_i, T_i} G_i (Z_i|G_i, T_i)}{p_{Z_i|G_i} G_i (Z_i|G_i)} \right)
= \sum_{i=1}^{n} \log \left( \frac{p_{Z_i|G_i, T_i} G_i (Z_i|G_i, T_i) p_{G_i, T_i} G_i (G_i, T_i)}{p_{Z_i|G_i} G_i (Z_i|G_i) p_{G_i, T_i} G_i (G_i, T_i)} \right)
= \sum_{i=1}^{n} \log \left( \frac{p_{Z_i|G_i, T_i} G_i (Z_i, G_i, T_i)}{p_{Z_i, G_i} (Z_i, G_i) p_{T_i} (T_i)} \right)
= \sum_{i=1}^{n} i(T_i; Z_i, G_i),
$$

where (a) follows because conditioned on $G_i$, $Z_i$ is independent of $G_1, \ldots, G_{i-1}, G_{i+1}, \ldots, G_n$ since $(T_i, \xi_i)$ is independent of $G_1, \ldots, G_{i-1}, G_{i+1}, \ldots, G_n$ and (b) follows because $T_i$ and $G_i$ are independent for $i = 1 \ldots n$. \hfill \Box

Now, recall that we chose the inputs $T^n$ of $W_{G^n}$ to be i.i.d such that $T_i \sim \mathcal{N}_C \left( 0_{g^2}, \hat{Q} \right)$, $i = 1 \ldots n$. We have using Lemma 7

$$
E \left[ \frac{1}{n} i(T^n; Z^n, G^n) \right]
= \frac{1}{n} E \left[ \sum_{i=1}^{n} i(T_i; Z_i, G_i) \right]
= \frac{1}{n} \sum_{i=1}^{n} E [i(T_i; Z_i, G_i)]
= \frac{1}{n} \sum_{i=1}^{n} I(T_i; Z_i, G_i)
= \frac{1}{n} \sum_{i=1}^{n} (I(T_i; Z_i|G_i) + I(T_i, G_i))
= \frac{1}{n} \sum_{i=1}^{n} I(T_i; Z_i|G_i)
\overset{(a)}{=} \frac{1}{n} \sum_{i=1}^{n} E \left[ \log \det \left( I_{N_R} + \frac{1}{\sigma^2} G_i \tilde{Q} G_i^H \right) \right]
\overset{(b)}{=} E \left[ \log \det \left( I_{N_R} + \frac{1}{\sigma^2} G \tilde{Q} G^H \right) \right]
= \phi(Q),
$$

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where \( a \) follows because \( \xi_i \sim \mathcal{N}(0_{N_R}, \sigma^2 I_{N_R}) \), \( i = 1, \ldots, n \) and because all the \( T_i \)'s are i.i.d. such that \( T_i \sim \mathcal{N}(0_{N_T}, \hat{Q}) \), \( i = 1 \ldots n \). and \( b \) follows because from Lemma 2, we know that \( \text{vec}(G_i) \sim \mathcal{N}(0_{N_{R,T}}, I_{N_{R,T}}) \), \( i = 1 \ldots n \) and because \( \text{vec}(G) \sim \mathcal{N}(0_{N_{R,T}}, I_{N_{R,T}}) \). It follows that

\[
\Pr \left[ \frac{1}{n} i(T^n; Z^n, G^n) \leq \phi(\hat{Q}) - \frac{\delta}{2} \right] \\
= \Pr \left[ \frac{1}{n} i(T^n; Z^n, G^n) \leq \mathbb{E} \left[ \frac{1}{n} i(T^n; Z^n, G^n) \right] - \frac{\delta}{2} \right] \\
\leq \Pr \left[ \frac{1}{n} i(T^n; Z^n, G^n) - \mathbb{E} \left[ \frac{1}{n} i(T^n; Z^n, G^n) \right] \geq \frac{\delta}{2} \right]
\]

\[(a) \quad 4\text{var} \left( \frac{i(T^n; Z^n, G^n)}{n} \right) \leq \frac{4\kappa(n, \alpha)}{\delta^2}, \]

\[(b) \quad 4\kappa(n, \alpha) \leq \frac{\delta^2}{\delta}, \]

(11)

where \( a \) follows from the Chebyshev’s inequality and \( b \) follows because \( \text{var} \left( \frac{i(T^n; Z^n, G^n)}{n} \right) \leq \kappa(n, \alpha) \) for some \( \kappa(n, \alpha) > 0 \) with \( \lim_{n \to \infty} \kappa(n, \alpha) = 0 \) for any \( 0 \leq \alpha < 1 \) (from the auxiliary result of Lemma 4).

From (8), (10) and (11), we obtain

\[ e_{\text{max}}(\Gamma_n) \leq \frac{\kappa(n)}{\delta^2} + 2^{-n^4} + 2^{-n^2} \]

where \( \lim_{n \to \infty} 4\kappa(n) + 2^{-n^4} + 2^{-n^2} = 0 \). Therefore, for sufficiently large \( n \), it holds that \( e_{\text{max}}(\Gamma_n) \leq \theta \). This completes the direct proof of Theorem 1.

**B. Converse Proof**

Let \( R \) be any achievable rate for the channel \( W_{\hat{G}^n} \) in (1). So, for every \( \theta, \delta > 0 \), there exists a code sequence \( (\Gamma_n)_{n=1}^\infty \) such that

\[ \frac{\log \| \Gamma_n \|}{n} \geq R - \delta \]

and

\[ e_{\text{max}}(\Gamma_n) = \max_{\ell \in \{1 \ldots \| \Gamma_n \| \}} \mathbb{E} \left[ W_{\hat{G}^n}(D_{\ell}^{\hat{G}^n}; \ell_t) \right] \leq \theta \]

(12)

for sufficiently large \( n \).

Notice that from (12), it follows that the average error probability is also bounded from above by \( \theta \). The uniformly-distributed message \( \hat{M} \) is mapped to the random input sequence \( T^n = (T_1, \ldots, T_n) \) of the channel in (1), where the covariance matrix of each input \( T_i \) is denoted by \( \hat{Q}_i \). Let \( (Z^n, \hat{G}^n) \) the corresponding outputs, where \( Z^n = (Z_1, \ldots, Z_n) \). We define \( Q^* \) such that \( Q^* = \frac{1}{n} \sum_{i=1}^n \hat{Q}_i \). We model the random decoded message by \( \hat{M} \). The set of messages is denoted by \( M \).

**Lemma 8.**

\[ \text{tr}(Q^*) \leq P \]

**Proof.** From (3), it holds that

\[ \frac{1}{n} \sum_{i=1}^n T_i^H T_i \leq P, \quad \text{almost surely.} \]

This implies that

\[ \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n T_i^H T_i \right] = \frac{1}{n} \sum_{i=1}^n \mathbb{E} [T_i^H T_i] \leq P. \]
This yields
\[
\text{tr} [Q^*] = \text{tr} \left[ \frac{1}{n} \sum_{i=1}^{n} Q_i \right] \\
= \frac{1}{n} \sum_{i=1}^{n} \text{tr} [Q_i] \\
\leq \frac{1}{n} \sum_{i=1}^{n} \text{tr} (\mathbb{E} [T_i T_i^H]) \\
= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \text{tr} (T_i T_i^H) \right] \\
= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \text{tr} (T_i^H T_i) \right] \\
= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} [T_i^H T_i] \\
\leq P,
\]
where we used \( r = \text{tr}(r) \) for scalar \( r \), \( \text{tr} (AB) = \text{tr} (BA) \) and the linearity of the expectation and of the trace operators.

By using \( \Gamma_n \) as a transmission-code for the channel \( W_{G^n} \), it follows using the fact that \( M \) and \( G^n \) are independent that
\[
P \left[ \hat{M} \neq M \right] = \mathbb{E} \left[ P \left[ M \neq \hat{M} | G^n \right] \right] \\
= \mathbb{E} \left[ \sum_{\ell=1}^{|M|} P[M = \ell] \mathbb{P} \left[ \hat{M} \neq \ell | M = \ell, G^n \right] \right] \\
= \sum_{\ell=1}^{|M|} P[M = \ell] \mathbb{E} \left[ P \left[ \hat{M} \neq \ell | M = \ell, G^n \right] \right] \\
= \sum_{\ell=1}^{|M|} P[M = \ell] \mathbb{E} \left[ W_{G^n} (D(G)^c_{\ell} | t_{\ell}) \right] \\
\leq e_{\text{max}}(\Gamma_n) \\
\leq \theta.
\]

By applying Fano’s inequality, we obtain
\[
H(M|\hat{M}) \leq 1 + P \left[ M \neq \hat{M} \right] \log |M| \\
\leq 1 + \theta \log |M| \\
= 1 + \theta H(M).
\]

Now, on the one hand, it holds that
\[
I(M; \hat{M}) = H(M) - H(M|\hat{M}) \\
\geq (1 - \theta) H(M) - 1.
\]

Since
\[
H(M) = \log |M| \\
= \log ||\Gamma_n|| \\
\geq n(R - \delta),
\]

it follows that
\[
n(R - \delta) \leq \frac{1 + I(M; \hat{M})}{1 - \theta}.
\]
On the other hand, we have

\[
\frac{1}{n} I(M; \hat{M}) \\
\leq \frac{1}{n} I(T^n; Z^n, G^n) \\
= \frac{1}{n} I(T^n; Z^n|G^n) + \frac{1}{n} I(T^n; G^n) \\
= \frac{1}{n} I(T^n; Z^n|G^n) \\
= \frac{1}{n} \sum_{i=1}^{n} I(Z_i; T^n|G^n, Z_i^{i-1}) \\
= \frac{1}{n} \sum_{i=1}^{n} h(Z_i|G^n, Z_i^{i-1}) - h(Z_i|G^n, T^n, Z_i^{i-1}) \\
\leq \frac{1}{n} \sum_{i=1}^{n} h(Z_i|G_i) - h(Z_i|G_i, T_i) \\
= \frac{1}{n} \sum_{i=1}^{n} I(T_i; Z_i|G_i) \\
\leq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \log \det(\mathbf{I}_{N_R} + \frac{1}{\sigma^2} \mathbf{G}_i \mathbf{Q}_i \mathbf{G}_i^H) \right] \\
\leq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ \log \det \left( \mathbf{I}_{N_R} + \frac{1}{\sigma^2} \mathbf{G}_i \mathbf{Q}_i \mathbf{G}_i^H \right) \right] \\
\leq \mathbb{E} \left[ \log \det \left( \mathbf{I}_{N_R} + \frac{1}{\sigma^2} \mathbf{G} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbf{Q}_i \right) \mathbf{G}^H \right) \right] \\
\leq \mathbb{E} \left[ \log \det \left( \mathbf{I}_{N_R} + \frac{1}{\sigma^2} \mathbf{G} \mathbf{Q}^* \mathbf{G}^H \right) \right] \\
\leq \mathbb{E} \left[ \log \det \left( \mathbf{I}_{N_R} + \frac{1}{\sigma^2} \mathbf{G} \mathbf{Q}^* \mathbf{G}^H \right) \right],
\]

where (a) follows from the Data Processing Inequality because \( M \circ T^n \circ G^n, Z^n \circ \hat{M} \) forms a Markov chain, (b) follows because \( G^n \) and \( T^n \) are independent, (c) follows from the chain rule for mutual information, (d) follows because \( G_1, T_1, \ldots, G_{i-1}, T_{i-1}, G_{i+1}, T_{i+1}, \ldots, G_n, T_n, Z_i^{i-1} \circ G_i, T_i \circ Z_i \)
forms a Markov chain, (e) follows because conditioning does not increase entropy, (f) follows because \( \xi_i \sim \mathcal{N}_C(\mathbf{0}_{N_R}, \sigma^2 \mathbf{I}_{N_R}), i = 1 \ldots n \), (g) follows because the \( G_i, s \) are identically distributed from Lemma 2 where \( G \) is a random matrix that has the same distribution as each of the \( G_i \) and (h) follows from Jensen’s Inequality since the function \( \log \circ \det \) is concave on the set of Hermitian positive semidefinite matrices and since \( \mathbf{I}_{N_R} + \frac{1}{\sigma^2} \mathbf{G} \mathbf{Q}^* \mathbf{G}^H \) is Hermitian positive semidefinite for \( i = 1 \ldots n \), (i) follows because \( \mathbf{Q}^* = \frac{1}{n} \sum_{i=1}^{n} \mathbf{Q}_i, \mathbf{Q}_i \in \mathcal{Q}_{(P, N_R)} \) from Lemma 8.

As a result, we have

\[
n(R - \delta) \leq \frac{n \max_{\mathbf{Q} \in \mathcal{Q}_{(P, N_R)}} \mathbb{E} \left[ \log \det \left( \mathbf{I}_{N_R} + \frac{1}{\sigma^2} \mathbf{G} \mathbf{Q} \mathbf{G}^H \right) \right] + 1}{1 - \theta}.
\]
This implies that

\[
R \leq \max_{Q \in \mathcal{Q}(P, N_T)} \frac{\mathbb{E} \left[ \log \det \left( \mathbf{I}_{N_R} + \frac{1}{\sigma^2} \mathbf{Q} \mathbf{G} \mathbf{G}^H \right) \right] + \frac{1}{n}}{1 - \theta} + \delta.
\] (13)

In particular, we can choose \(\delta, \theta > 0\) to be arbitrarily small such that the right-hand side of (13) is equal to

\[
\max_{Q \in \mathcal{Q}(P, N_T)} \mathbb{E} \left[ \log \det \left( \mathbf{I}_{N_R} + \frac{1}{\sigma^2} \mathbf{Q} \mathbf{G} \mathbf{G}^H \right) \right] + \delta' \quad \text{for} \quad n \to \infty,
\]

with \(\delta'\) being an arbitrarily small positive constant. This completes the converse proof of Theorem 1.

IV. PROOF OF LEMMA 4

Let \(T^n = (T_1, \ldots, T_n)\) be an \(n\)-length input sequence of the channel \(W_{G^n}\) such that the \(T_i\)'s are i.i.d., where \(T_i \sim \mathcal{N}(0_{N_T}, \tilde{Q})\), \(i = 1 \ldots n\) and where \(\tilde{Q} \in \mathcal{Q}(P, N_T)\). Let \(Z^n\) be the corresponding output sequence, where \(Z^n = (Z_1, \ldots, Z_n)\).

By Lemma 7, it holds that

\[
i(T^n; Z^n, G^n) = \sum_{i=1}^{n} i(T_i; Z_i, G_i).\] (14)

We have

\[
\operatorname{var} \left( \frac{i(T^n; Z^n, G^n)}{n} \right) = \frac{1}{n^2} \mathbb{E} \left[ i(T^n; Z^n, G^n)^2 \right] - \frac{1}{n^2} \mathbb{E} \left[ i(T^n; Z^n, G^n) \right]^2.
\] (15)

Now

\[
\frac{1}{n^2} \mathbb{E} \left[ i(T^n; Z^n, G^n)^2 \right] = \frac{1}{n^2} \mathbb{E} \left[ \left( \sum_{i=1}^{n} i(T_i; Z_i, G_i) \right)^2 \right]
\]

\[
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1, k \neq i}^{n} \mathbb{E} \left[ i(T_i; Z_i, G_i) i(T_k; Z_k, G_k) \right] + \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E} \left[ i(T_i; Z_i, G_i)^2 \right]
\]

\[
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1, k \neq i}^{n} \mathbb{E} \left[ \mathbb{E} \left[ i(T_i; Z_i, G_i) i(T_k; Z_k, G_k) \right] | G_i, G_k \right] + \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E} \left[ i(T_i; Z_i, G_i)^2 \right]
\]

\[
(a) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1, k \neq i}^{n} \mathbb{E} \left[ \mathbb{E} \left[ i(T_i; Z_i, G_i) G_i, G_k \right] \mathbb{E} \left[ i(T_k; Z_k, G_k) G_i, G_k \right] \right] + \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E} \left[ i(T_i; Z_i, G_i)^2 \right]
\]

\[
(b) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1, k \neq i}^{n} \mathbb{E} \left[ \mathbb{E} \left[ i(T_i; Z_i, G_i) G_i \right] \mathbb{E} \left[ i(T_k; Z_k, G_k) G_k \right] \right] + \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E} \left[ i(T_i; Z_i, G_i)^2 \right]
\]

\[
(c) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1, k \neq i}^{n} \mathbb{E} \left[ \log \det(\mathbf{I}_{N_R} + \frac{1}{\sigma^2} \mathbf{G}_i \mathbf{Q} \mathbf{G}_i^H) \log \det(\mathbf{I}_{N_R} + \frac{1}{\sigma^2} \mathbf{G}_k \mathbf{Q} \mathbf{G}_k^H) \right] + \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E} \left[ i(T_i; Z_i, G_i)^2 \right],
\] (16)

where (a) follows because for independent inputs and conditioned on \((G_i, G_j), i(T_i; Z_i, G_i)\) and \(i(T_j; Z_j, G_j)\) are independent, (b) follows because for independent inputs and conditioned on \(G_i, i(T_i; Z_i, G_i)\) and \(G_k\) are independent, and because for independent inputs and conditioned on \(G_k, i(T_k; Z_k, G_k)\) and \(G_i\) are independent, and (c) follows because \(\xi_i \sim \mathcal{N}_{C}(0_{N_R}, \sigma^2 \mathbf{I}_{N_R}), i = 1, \ldots, n\) and because all the \(T_i\)s are i.i.d. such that \(T_i \sim \mathcal{N}(0_{N_T}, \tilde{Q}), i = 1 \ldots n\).
It holds using (14) that
\[
\frac{1}{n^2} \mathbb{E} \left[ i(\mathbf{T}^n; \mathbf{Z}^n, \mathbf{G}^n) \right]^2 = \frac{1}{n^2} \mathbb{E} \left[ \sum_{i=1}^{n} i(\mathbf{T}_i; \mathbf{Z}_i, \mathbf{G}_i) \right]^2 \\
= \frac{1}{n^2} \left( \sum_{i=1}^{n} \mathbb{E} \left[ i(\mathbf{T}_i; \mathbf{Z}_i, \mathbf{G}_i) \right] \right)^2 \\
\geq \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1, k \neq i}^{n} \mathbb{E} \left[ i(\mathbf{T}_i; \mathbf{Z}_i, \mathbf{G}_i) \right] \mathbb{E} \left[ i(\mathbf{T}_k; \mathbf{Z}_k, \mathbf{G}_k) \right] \\
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1, k \neq i}^{n} \mathbb{E} \left[ \mathbb{E} \left[ i(\mathbf{T}_i; \mathbf{Z}_i, \mathbf{G}_i) | \mathbf{G}_i \right] \mathbb{E} \left[ \mathbb{E} \left[ i(\mathbf{T}_k; \mathbf{Z}_k, \mathbf{G}_k) | \mathbf{G}_k \right] \right] \right] \\
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1, k \neq i}^{n} \mathbb{E} \left[ \log \det(\mathbf{I}_{N_R} + \frac{1}{\sigma^2} \mathbf{G}_i \tilde{\mathbf{Q}} \mathbf{G}_i^H) \right] \mathbb{E} \left[ \log \det(\mathbf{I}_{N_R} + \frac{1}{\sigma^2} \mathbf{G}_k \tilde{\mathbf{Q}} \mathbf{G}_k^H) \right] \\
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1, k \neq i}^{n} \mathbb{E} \left[ \log \det \left( \mathbf{I}_{N_R} + \frac{1}{\sigma^2} \tilde{\mathbf{Q}} \tilde{\mathbf{Q}}^H \right) \right]. \tag{17}
\]

It follows from (15), (16) and (17) that
\[
\text{var} \left( \frac{i(\mathbf{T}^n; \mathbf{Z}^n, \mathbf{G}^n)}{n} \right) \\
\leq \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1, k \neq i}^{n} \mathbb{E} \left[ \log \det(\mathbf{I}_{N_R} + \frac{1}{\sigma^2} \mathbf{G}_i \tilde{\mathbf{Q}} \mathbf{G}_i^H) \log \det(\mathbf{I}_{N_R} + \frac{1}{\sigma^2} \mathbf{G}_k \tilde{\mathbf{Q}} \mathbf{G}_k^H) \right] \\
+ \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E} \left[ i(\mathbf{T}_i; \mathbf{Z}_i, \mathbf{G}_i)^2 \right] - \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1, k \neq i}^{n} \mathbb{E} \left[ \log \det \left( \mathbf{I}_{N_R} + \frac{1}{\sigma^2} \tilde{\mathbf{Q}} \tilde{\mathbf{Q}}^H \right) \right]^2 \\
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1, k \neq i}^{n} \left( \mathbb{E} \left[ \log \det(\mathbf{I}_{N_R} + \frac{1}{\sigma^2} \mathbf{G}_i \tilde{\mathbf{Q}} \mathbf{G}_i^H) \log \det(\mathbf{I}_{N_R} + \frac{1}{\sigma^2} \mathbf{G}_k \tilde{\mathbf{Q}} \mathbf{G}_k^H) \right] - \mathbb{E} \left[ \log \det \left( \mathbf{I}_{N_R} + \frac{1}{\sigma^2} \tilde{\mathbf{Q}} \tilde{\mathbf{Q}}^H \right) \right] \right) \\
+ \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E} \left[ i(\mathbf{T}_i; \mathbf{Z}_i, \mathbf{G}_i)^2 \right]. \tag{18}
\]

By defining for any \( i, k \in \{1, \ldots, n\} \) with \( i \neq k \),
\[
m(i, k) = \mathbb{E} \left[ \log \det(\mathbf{I}_{N_R} + \frac{1}{\sigma^2} \mathbf{G}_i \tilde{\mathbf{Q}} \mathbf{G}_i^H) \log \det(\mathbf{I}_{N_R} + \frac{1}{\sigma^2} \mathbf{G}_k \tilde{\mathbf{Q}} \mathbf{G}_k^H) \right] - \mathbb{E} \left[ \log \det \left( \mathbf{I}_{N_R} + \frac{1}{\sigma^2} \tilde{\mathbf{Q}} \tilde{\mathbf{Q}}^H \right) \right]^2,
\]
we obtain using (18)
\[
\text{var} \left( \frac{i(\mathbf{T}^n; \mathbf{Z}^n, \mathbf{G}^n)}{n} \right) \\
\leq \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1, k \neq i}^{n} m(i, k) + \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E} \left[ i(\mathbf{T}_i; \mathbf{Z}_i, \mathbf{G}_i)^2 \right].
\]
Proof. A. Upper-bound for $i$th term

Now, the goal is to find a suitable upper-bound for each term in (19). Let $\tilde{G}$ be any random matrix independent of $G_i$, such that $\text{vec}(\tilde{G}) \sim N_C(0_{N_RN_T},K)$. By Lemma 2, it follows that $G$ has the same distribution as $G_i$, $i=1,...,n$. Furthermore, since $G$ is independent of $G_1$ and $W_i$, $i=2,...,n$, it is also independent of all the $G_i$s. By Lemma 3, we know that

$$G_{i_2} = \sqrt{\alpha^{2-i_1}}G_{i_1} + \sqrt{1-\alpha} \sum_{j=i_1+1}^{i_2} \sqrt{\alpha^{2-j}}W_j.$$  

By defining

$$S = \sqrt{1-\alpha} \sum_{j=i_1+1}^{i_2} \sqrt{\alpha^{2-j}}W_j,$$

it follows that

$$G_{i_2} = \sqrt{\alpha^{2-i_1}}G_{i_1} + S.$$  

Define

$$W = S + \sqrt{\alpha^{2-i_1}}G.$$

Notice that $W$ is independent of $G_{i_1}$, since $G_{i_1}$ is independent of $(S, \tilde{G})$. Analogously to the proof of Lemma 2, one can show that $\text{vec}(W) \sim N_C(0_{N_RN_T},K)$.

First, we introduce and prove the following claims:

**Lemma 9.** Let $i_1, i_2 \in \{1,...,n\}$. Assume without loss of generality that $i_1 < i_2$, then

$$\mathbb{E} \left[ \log \det(\mathbf{I}_{N_R} + \frac{1}{\sigma^2}G_{i_2}G_{i_1}^H) \right] \leq \mathbb{E} \left[ \log \det(\mathbf{I}_{N_R} + \frac{1}{\sigma^2}G_{i_1}G_{i_1}^H) \right] + c' \sqrt{\alpha^{2-i_1}},$$

for some $c' > 0$, where $G$ is a random matrix such that $\text{vec}(G) \sim N_C(0_{N_RN_T},K)$.

**Proof.** Let $\tilde{G}_i$ be any random matrix independent of $G_1$ and $W_i$, $i=2,...,n$ such that $\text{vec}(\tilde{G}_i) \sim N_C(0_{N_RN_T},K)$. By Lemma 2, it follows that $G$ has the same distribution as $G_i$, $i=1,...,n$. Furthermore, since $G$ is independent of $G_1$ and $W_i$, $i=2,...,n$, it is also independent of all the $G_i$s. By Lemma 3, we know that

$$G_{i_2} = \sqrt{\alpha^{2-i_1}}G_{i_1} + \sqrt{1-\alpha} \sum_{j=i_1+1}^{i_2} \sqrt{\alpha^{2-j}}W_j.$$  

By defining

$$S = \sqrt{1-\alpha} \sum_{j=i_1+1}^{i_2} \sqrt{\alpha^{2-j}}W_j,$$

it follows that

$$G_{i_2} = \sqrt{\alpha^{2-i_1}}G_{i_1} + S.$$  

Define

$$W = S + \sqrt{\alpha^{2-i_1}}G.$$

Notice that $W$ is independent of $G_{i_1}$, since $G_{i_1}$ is independent of $(S, \tilde{G})$. Analogously to the proof of Lemma 2, one can show that $\text{vec}(W) \sim N_C(0_{N_RN_T},K)$.

First, we introduce and prove the following claims:
Claim 1. It holds that
\[
\log \det(I_{N_R} + \frac{1}{\sigma^2} G_{i_2} Q G_{i_2}^H) \leq \log \det \left( I_{N_R} + \frac{1}{\sigma^2} \tilde{W} Q \tilde{W}^H \right) + \frac{N_R}{\ln(2) \sigma^2} \sqrt{\alpha^{2-i_1}} \left( P\|G_{i_1}\|^2 + P\|\tilde{G}\|^2 + 2\|\tilde{W} Q G_{i_2}^H\| + 2P\|G_{i_1}\|\|S\| \right).
\]

Claim 2. It holds that
\[
\log \det(I_{N_R} + \frac{1}{\sigma^2} G_{i_1} \tilde{Q} G_{i_1}^H) \leq \frac{P N_R}{\ln(2) \sigma^2} \|G_{i_1}\|^2.
\]

Proof of Claim 1. From (20), we have
\[
\frac{1}{\sigma^2} G_{i_2} \tilde{Q} G_{i_2}^H = \frac{1}{\sigma^2} \left[ \sqrt{\alpha^{2-i_1}} G_{i_1} + S \right] \tilde{Q} \left[ \sqrt{\alpha^{2-i_1}} G_{i_1}^H + S^H \right]
\]
\[
= \frac{1}{\sigma^2} \alpha^{2-i_1} G_{i_1} \tilde{Q} G_{i_1}^H + \frac{1}{\sigma^2} \sqrt{\alpha^{2-i_1}} G_{i_1} \tilde{Q} S^H + \frac{1}{\sigma^2} \sqrt{\alpha^{2-i_1}} S \tilde{Q} G_{i_1}^H + \frac{1}{\sigma^2} S \tilde{Q} S^H.
\]

We will prove first that
\[
\frac{1}{\sigma^2} \sqrt{\alpha^{2-i_1}} G_{i_1} \tilde{Q} S^H + \frac{1}{\sigma^2} \sqrt{\alpha^{2-i_1}} S \tilde{Q} G_{i_1}^H \preceq \frac{2P}{\sigma^2} \sqrt{\alpha^{2-i_1}} \|G_{i_1}\|\|S\| I_{N_R}.
\]

Notice now that the matrix
\[
\frac{1}{\sigma^2} \sqrt{\alpha^{2-i_1}} G_{i_1} \tilde{Q} S^H + \frac{1}{\sigma^2} \sqrt{\alpha^{2-i_1}} S \tilde{Q} G_{i_1}^H
\]
is a Hermitian matrix since it is equal to its Hermitian transpose. Since \( A \preceq \|A\| I_n \) for any Hermitian matrix \( A \in \mathbb{C}^{n \times n} \) (by Lemma 10 in the Appendix), it follows that
\[
\frac{1}{\sigma^2} \sqrt{\alpha^{2-i_1}} G_{i_1} \tilde{Q} S^H + \frac{1}{\sigma^2} \sqrt{\alpha^{2-i_1}} S \tilde{Q} G_{i_1}^H \preceq \frac{2N}{\sigma^2} \sqrt{\alpha^{2-i_1}} \|G_{i_1}\|\|\tilde{Q}\||\|S\| I_{N_R}.
\]

This proves (22).

Next, we will prove that
\[
\frac{1}{\sigma^2} \left[ \alpha^{i_2-i_1} G_{i_1} \tilde{Q} G_{i_1}^H + S \tilde{Q} S^H \right] \preceq \frac{1}{\sigma^2} \tilde{W} \tilde{Q} \tilde{W}^H + \frac{P}{\sigma^2} \sqrt{\alpha^{2-i_1}} \left( \|G_{i_1}\|^2 + \|\tilde{G}\|^2 \right) I_{N_R} + \frac{2}{\sigma^2} \sqrt{\alpha^{2-i_1}} \|\tilde{W} \tilde{Q} G_{i_2}^H\| I_{N_R}.
\]

(23)
Since $\tilde{W} = S + \sqrt{\alpha^{2-i_1}} \tilde{G}$, it holds that

$$S\tilde{Q}S^H$$

$$= \left( S + \sqrt{\alpha^{2-i_1}} \tilde{G} - \sqrt{\alpha^{2-i_1}} \tilde{G} \right) \tilde{Q} \left( S^H + \sqrt{\alpha^{2-i_1}} \tilde{G}^H - \sqrt{\alpha^{2-i_1}} \tilde{G}^H \right)$$

$$= \left( \tilde{W} - \sqrt{\alpha^{2-i_1}} \tilde{G} \right) \tilde{Q} \left( \tilde{W}^H - \sqrt{\alpha^{2-i_1}} \tilde{G}^H \right)$$

$$= \tilde{W}\tilde{Q}\tilde{W}^H - \sqrt{\alpha^{2-i_1}} \tilde{W}\tilde{Q}\tilde{G}^H - \sqrt{\alpha^{2-i_1}} \tilde{G}\tilde{Q}\tilde{W}^H + \alpha^{i_2-i_1} \tilde{G}\tilde{Q}\tilde{G}^H.$$

This yields

$$\frac{1}{\sigma^2} \left[ \alpha^{i_2-i_1} G_i, QG_i^H + S\tilde{Q}S^H \right]$$

$$= \frac{1}{\sigma^2} \left[ \alpha^{i_2-i_1} G_i, \tilde{Q}G_i^H + \tilde{W}\tilde{Q}\tilde{W}^H - \sqrt{\alpha^{2-i_1}} \tilde{W}\tilde{Q}\tilde{G}^H - \sqrt{\alpha^{2-i_1}} \tilde{G}\tilde{Q}\tilde{W}^H + \alpha^{i_2-i_1} \tilde{G}\tilde{Q}\tilde{G}^H \right]$$

$$= \frac{1}{\sigma^2} \tilde{W}\tilde{Q}\tilde{W}^H + \frac{1}{\sigma^2} \alpha^{i_2-i_1} \left[ G_i, \tilde{Q}G_i^H + \tilde{G}\tilde{Q}\tilde{G}^H \right] - \frac{1}{\sigma^2} \sqrt{\alpha^{2-i_1}} \tilde{W}\tilde{Q}\tilde{G}^H - \frac{1}{\sigma^2} \sqrt{\alpha^{2-i_1}} \tilde{G}\tilde{Q}\tilde{W}^H.$$

Now notice that

$$\frac{1}{\sigma^2} \alpha^{i_2-i_1} \left[ G_i, \tilde{Q}G_i^H + \tilde{G}\tilde{Q}\tilde{G}^H \right]$$

is a Hermitian matrix. This implies that

$$\frac{1}{\sigma^2} \alpha^{i_2-i_1} \left[ G_i, \tilde{Q}G_i^H + \tilde{G}\tilde{Q}\tilde{G}^H \right] \leq \frac{1}{\sigma^2} \alpha^{i_2-i_1} \| G_i, \tilde{Q}G_i^H + \tilde{G}\tilde{Q}\tilde{G}^H \| \| I_{N_R} \|

\leq \frac{1}{\sigma^2} \alpha^{i_2-i_1} \left\| \tilde{Q} \left( \| G_i \|^2 + \|\tilde{G}\|^2 \right) I_{N_R} \right\| \| I_{N_R} \|

\leq \frac{P}{\sigma^2} \alpha^{i_2-i_1} \left( \| G_i \|^2 + \|\tilde{G}\|^2 \right) I_{N_R}. \quad (24)$$

Notice also that $-\frac{1}{\sigma^2} \sqrt{\alpha^{2-i_1}} \tilde{W}\tilde{Q}\tilde{G}^H - \frac{1}{\sigma^2} \sqrt{\alpha^{2-i_1}} \tilde{G}\tilde{Q}\tilde{W}^H$ is a Hermitian matrix. It follows that

$$-\frac{1}{\sigma^2} \sqrt{\alpha^{2-i_1}} \tilde{W}\tilde{Q}\tilde{G}^H - \frac{1}{\sigma^2} \sqrt{\alpha^{2-i_1}} \tilde{G}\tilde{Q}\tilde{W}^H$$

$$\leq \left\| -\frac{1}{\sigma^2} \sqrt{\alpha^{2-i_1}} \tilde{W}\tilde{Q}\tilde{G}^H - \frac{1}{\sigma^2} \sqrt{\alpha^{2-i_1}} \tilde{G}\tilde{Q}\tilde{W}^H \right\| I_{N_R} \| I_{N_R} \|

\leq \frac{2}{\sigma^2} \sqrt{\alpha^{2-i_1}} \| \tilde{W}\tilde{Q}\tilde{G}^H \| I_{N_R}. \quad (25)$$

As a result, we have using (24) and (25)

$$\frac{1}{\sigma^2} \tilde{W}\tilde{Q}\tilde{W}^H + \frac{1}{\sigma^2} \alpha^{i_2-i_1} \left[ G_i, \tilde{Q}G_i^H + \tilde{G}\tilde{Q}\tilde{G}^H \right] - \frac{1}{\sigma^2} \sqrt{\alpha^{2-i_1}} \tilde{W}\tilde{Q}\tilde{G}^H - \frac{1}{\sigma^2} \sqrt{\alpha^{2-i_1}} \tilde{G}\tilde{Q}\tilde{W}^H$$

$$\leq \frac{1}{\sigma^2} \tilde{W}\tilde{Q}\tilde{W}^H + \frac{P}{\sigma^2} \alpha^{i_2-i_1} \left( \| G_i \|^2 + \|\tilde{G}\|^2 \right) I_{N_R} + \frac{2}{\sigma^2} \sqrt{\alpha^{2-i_1}} \| \tilde{W}\tilde{Q}\tilde{G}^H \| I_{N_R}.$
This proves (23). We deduce using (22) and (23) that
\[
\frac{1}{\sigma^2} \left[ G_{i_1} Q G_{i_1}^H + S Q S \right] + \frac{1}{\sigma^2} \sqrt{\alpha^{i_2 - i_1}} G_{i_2} Q S + \frac{1}{\sigma^2} \sqrt{\alpha^{i_2 - i_1}} S Q G_{i_1}^H \\
\leq \frac{1}{\sigma^2} \tilde{W} Q \tilde{W}^H + \frac{P}{\sigma^2} \sqrt{\alpha^{i_2 - i_1}} \left( \| G_{i_1} \|^2 + \| \tilde{G} \|^2 \right) I_{N_R} + \frac{2}{\sigma^2} \sqrt{\alpha^{i_2 - i_1}} \| \tilde{W} Q \tilde{G} \| I_{N_R} + \frac{2P}{\sigma^2} \sqrt{\alpha^{i_2 - i_1}} \| G_{i_1} \| \| S \| I_{N_R}
\]
\[
= \frac{1}{\sigma^2} \tilde{W} Q \tilde{W}^H + \frac{P}{\sigma^2} \sqrt{\alpha^{i_2 - i_1}} \left( \| G_{i_1} \|^2 + \| \tilde{G} \|^2 \right) I_{N_R} + \frac{2}{\sigma^2} \sqrt{\alpha^{i_2 - i_1}} \left( \| \tilde{W} Q \tilde{G} \| + P \| G_{i_1} \| \| S \| \right) I_{N_R}
\]
\[
\leq \frac{1}{\sigma^2} \tilde{W} Q \tilde{W}^H + \frac{1}{\sigma^2} \sqrt{\alpha^{i_2 - i_1}} \left( P \| G_{i_1} \|^2 + P \| \tilde{G} \|^2 + 2 \| \tilde{W} Q \tilde{G} \|^2 \right) I_{N_R} - \frac{1}{\sigma^2} \tilde{W} Q \tilde{W}^H + \frac{1}{\sigma^2} \sqrt{\alpha^{i_2 - i_1}} \left( P \| G_{i_1} \|^2 + P \| \tilde{G} \|^2 + 2 \| \tilde{W} Q \tilde{G} \|^2 \right) I_{N_R},
\]
where (a) follows because \( \alpha < \sqrt{\alpha} \) for \( 0 < \alpha < 1 \).

Therefore, it follows from (21) and (26) that
\[
\frac{1}{\sigma^2} G_{i_2} \tilde{Q} G_{i_2}^H
\]
\[
\leq \frac{1}{\sigma^2} \tilde{W} Q \tilde{W}^H + \frac{1}{\sigma^2} \sqrt{\alpha^{i_2 - i_1}} \left( P \| G_{i_1} \|^2 + P \| \tilde{G} \|^2 + 2 \| \tilde{W} Q \tilde{G} \|^2 \right) I_{N_R}.
\]

This yields
\[
\log \det (I_{N_R} + \frac{1}{\sigma^2} G_{i_2} \tilde{Q} G_{i_2}^H)
\]
\[
\leq \log \det \left( I_{N_R} + \frac{1}{\sigma^2} \tilde{W} Q \tilde{W}^H + \frac{1}{\sigma^2} \sqrt{\alpha^{i_2 - i_1}} \left( P \| G_{i_1} \|^2 + P \| \tilde{G} \|^2 + 2 \| \tilde{W} Q \tilde{G} \|^2 \right) I_{N_R} \right).
\]

Now by Lemma 11 in the Appendix, we know that for any positive-definite Hermitian matrix \( A \in \mathbb{C}^{n \times n} \) with smallest eigenvalue \( \lambda_{\min}(A) \) and for any positive semi-definite Hermitian matrix \( B \in \mathbb{C}^{n \times n} \), the following is satisfied:
\[
\log \det (A + B) \leq \log \det (A) + \log \det (I_n + \frac{1}{\lambda_{\min}(A)} B).
\]

By applying Lemma 11 in the Appendix for
\[
A = I_{N_R} + \frac{1}{\sigma^2} \tilde{W} Q \tilde{W}^H
\]
and for
\[
B = \frac{1}{\sigma^2} \sqrt{\alpha^{i_2 - i_1}} \left( P \| G_{i_1} \|^2 + P \| \tilde{G} \|^2 + 2 \| \tilde{W} Q \tilde{G} \|^2 \right) I_{N_R},
\]
it follows from (27) that
\[
\log \det (I_{N_R} + \frac{1}{\sigma^2} G_{i_2} \tilde{Q} G_{i_2}^H)
\]
\[
\leq \log \det \left( I_{N_R} + \frac{1}{\sigma^2} \tilde{W} Q \tilde{W}^H \right)
\]
\[
+ \log \det \left( I_{N_R} + \frac{1}{\sigma^2} \sqrt{\alpha^{i_2 - i_1}} \left( P \| G_{i_1} \|^2 + P \| \tilde{G} \|^2 + 2 \| \tilde{W} Q \tilde{G} \|^2 \right) I_{N_R} \right) \frac{\log \det (I_{N_R} + \frac{1}{\sigma^2} \tilde{W} Q \tilde{W}^H)}{\lambda_{\min}(I_{N_R} + \frac{1}{\sigma^2} \tilde{W} Q \tilde{W}^H) I_{N_R}}.
\]

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It follows using Claim 1 and Claim 2 that

\[ \log \det \left( I_{N_R} + \frac{1}{\sigma^2} \hat{W} \hat{Q} \hat{W}^H \right) \leq \log \det \left( I_{N_R} + \frac{1}{\sigma^2} \hat{W} \hat{Q} \hat{W}^H \right) + \frac{1}{\ln(2)} \text{tr} \left[ \frac{1}{\sigma^2} \sqrt{\alpha^{2-i} \left( P \|G_i\|^2 + P\|G\|^2 + 2\|\hat{W} \hat{Q} \hat{G}^H\| + 2P\|G_i\|\|S\|\right)} \lambda_{\min} \left( I_{N_R} + \frac{1}{\sigma^2} \hat{W} \hat{Q} \hat{W}^H \right) \right] \]

\( (b) \) follows because \( \ln \det(I_n + A) \leq \text{tr}(A) \) for positive semi-definite \( A \), \( (b) \) follows because \( \lambda_{\min} \left( I_{N_R} + \frac{1}{\sigma^2} \hat{W} \hat{Q} \hat{W}^H \right) \geq 1 \) and \( (c) \) follows because \( \text{tr}(c I_{N_R}) = c N_R \) for any constant \( c \). This completes the proof of Claim 1.

**Proof of Claim 2.** Since \( A \leq \|A\|I_n \) for any Hermitian \( A \in \mathbb{C}^{n \times n} \) (by Lemma 10 in the Appendix) and since the matrix \( G_i \hat{Q} G_i^H \) is Hermitian, we have

\[ \log \det \left( I_{N_R} + \frac{1}{\sigma^2} G_i \hat{Q} G_i^H \right) \leq \log \det \left( I_{N_R} + \frac{1}{\sigma^2} \|G_i\|\|\hat{Q}\| I_{N_R} \right) \]

\[ \leq \frac{1}{\ln(2)} \text{tr} \left[ \frac{1}{\sigma^2} \|G_i\|\|\hat{Q}\| I_{N_R} \right] \]

\[ \leq \frac{1}{\ln(2)} \text{tr} \left[ \frac{1}{\sigma^2} \|G_i\|^2 \|\hat{Q}\| I_{N_R} \right] \]

\[ = \frac{N_R}{\ln(2)\sigma^2} \|G_i\|^2 \|\hat{Q}\| \]

\[ \leq \frac{PN_R}{\ln(2)\sigma^2} \|G_i\|^2 \]

This completes the proof of Claim 2.

Now that we proved the two claims, we let

\[ \Lambda \left( G_i, S, \hat{G}, \hat{W} \right) = \|G_i\|^2 \left( P\|G_i\|^2 + P\|\hat{G}\|^2 + 2\|\hat{W} \hat{Q} \hat{G}^H\| + 2P\|G_i\|\|S\|\right) . \]

It follows using Claim 1 and Claim 2 that

\[ \mathbb{E} \left[ \log \det \left( I_{N_R} + \frac{1}{\sigma^2} G_i \hat{Q} G_i^H \right) \log \det \left( I_{N_R} + \frac{1}{\sigma^2} G_i \hat{Q} G_i^H \right) \right] \]

\[ \leq \mathbb{E} \left[ \log \det \left( I_{N_R} + \frac{1}{\sigma^2} \hat{W} \hat{Q} \hat{W}^H \right) \log \det \left( I_{N_R} + \frac{1}{\sigma^2} G_i \hat{Q} G_i^H \right) \right] \]

20
Lemma 9 implies that

\[ i < k \]

If

\[ k < i \]

Now, from Lemma 12 in the Appendix, we know that \( \mathbb{E} \left[ \Lambda \left( G_{k}, S, \tilde{G}, \tilde{W} \right) \right] \) is bounded from above by some \( c > 0 \). Therefore it follows that for \( i_1 < i_2 \)

\[
\mathbb{E} \left[ \log \det (I_{N_R} + \frac{1}{\sigma^2} G_{i_2} \tilde{Q} G_{i_2}^H) \log \det (I_{N_R} + \frac{1}{\sigma^2} G_{i_1} \tilde{Q} G_{i_1}^H) \right] \leq \mathbb{E} \left[ \log \det \left( I_{N_R} + \frac{1}{\sigma^2} \tilde{G} \tilde{Q} \tilde{G}^H \right) \right]^2 + \frac{P N_R^2}{\ln(2)^2 \sigma^4} \sqrt{\alpha^2 - i_1}
\]

\[
= \mathbb{E} \left[ \log \det \left( I_{N_R} + \frac{1}{\sigma^2} \tilde{G} \tilde{Q} \tilde{G}^H \right) \right]^2 + \frac{P N_R^2}{\ln(2)^2 \sigma^4} \sqrt{\alpha^2 - i_1} + c^2 \sqrt{\alpha^2 - i_1},
\]

for some \( c > 0 \), where \( c' = \frac{P N_R^2 c}{\ln(2)^2 \sigma^4} > 0 \). This completes the proof of Lemma 9.

Now that we have proved Lemma 9, we will use that lemma to prove that

\[
\frac{1}{i^2} \sum_{i=1}^{n} \sum_{k=1}^{i} m(i, k) + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=i+1}^{n} m(i, k) \leq \frac{2c^2}{n(1 - \sqrt{\alpha})}.
\]

We recall that for any \( i, k \in \{1, \ldots n\} \) with \( i \neq k \),

\[
m(i, k) = \mathbb{E} \left[ \log \det (I_{N_R} + \frac{1}{\sigma^2} G_i \tilde{Q} G_i^H) \log \det (I_{N_R} + \frac{1}{\sigma^2} G_k \tilde{Q} G_k^H) \right] - \mathbb{E} \left[ \log \det \left( I_{N_R} + \frac{1}{\sigma^2} G_i \tilde{Q} G_i^H \right) \right]^2.
\]

If \( k < i \): Lemma 9 implies that

\[
m(i, k) \leq c' \sqrt{\alpha^{i-k}}.
\]

If \( i < k \): Lemma 9 implies that

\[
m(i, k) \leq c' \sqrt{\alpha^{k-i}}.
\]

Therefore, we have

\[
\frac{1}{i^2} \sum_{i=1}^{n} \sum_{k=1}^{i-1} m(i, k) \leq \frac{c'^2}{n^2} \sum_{i=1}^{n} \sum_{k=1}^{i-1} \sqrt{\alpha^{i-k}} \leq \frac{c'^2}{n(1 - \sqrt{\alpha})}.
\]

(28)
because by Lemma 14 in the Appendix, we have for any \(0 < \alpha < 1\)
\[
\sum_{i=1}^{n} \sum_{k=1}^{i-1} \alpha^{i-k} \leq \frac{n}{1-\alpha}.
\]
Furthermore, it holds that
\[
\frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=i+1}^{n} m(i, k) \leq \frac{c'}{n^2} \sum_{i=1}^{n} \sum_{k=i+1}^{n} \sqrt{\alpha^{k-i}}
\]
\[
\leq \frac{c'}{n(1-\sqrt{\alpha})}
\]
(29)
because by Lemma 15 in the Appendix, we have for any \(0 < \alpha < 1\)
\[
\sum_{i=1}^{n} \sum_{k=i+1}^{n} \alpha^{k-i} \leq \frac{n}{1-\alpha}.
\]
From (28) and (29), we deduce that
\[
\frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=i+1}^{n} m(i, k) + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=i+1}^{n} m(i, k) \leq \frac{2c'}{n(1-\sqrt{\alpha})}.
\]

**B. Upper-bound for** \(\frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E} \left[i(T; Z_i, G_i)^2\right]\) **for** \(0 \leq \alpha < 1\)

We are going to prove that
\[
\frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E} \left[i(T; Z_i, G_i)^2\right] \leq \frac{c''}{n}
\]
for some \(c'' > 0\). It suffices to show that \(\mathbb{E} \left[i(T; Z_i, G_i)^2\right]\) is bounded from above for \(i = 1, \ldots, n\). Recall that
\[
Z_i = G_i T_i + \xi_i, \quad i = 1, \ldots, n
\]
and that for \(i = 1, \ldots, n\)
\[
\xi_i \sim \mathcal{N}(0_{N_R}, \sigma^2 I_{N_R}).
\]
By Lemma 16 in the Appendix, we know that for \(i = 1, \ldots, n\)
\[
i(T; Z_i, G_i)
\]
\[
= \log \det(I_{N_R} + \frac{1}{\sigma^2} G_i \tilde{Q} \tilde{G}_i^H) - \frac{1}{\ln(2)\sigma^2} (Z_i - G_i T_i)^H (Z_i - G_i T_i) + \frac{1}{\ln(2)\sigma^2} Z_i^H \left(I_{N_R} + \frac{1}{\sigma^2} G \tilde{Q} \tilde{G}^H\right)^{-1} Z_i.
\]
We have
\[
|i(T; Z_i, G_i)|
\]
\[
= \left| \log \det(I_{N_R} + \frac{1}{\sigma^2} G_i \tilde{Q} \tilde{G}_i^H) - \frac{1}{\ln(2)\sigma^2} (Z_i - G_i T_i)^H (Z_i - G_i T_i) + \frac{1}{\ln(2)\sigma^2} Z_i^H \left(I_{N_R} + \frac{1}{\sigma^2} G \tilde{Q} \tilde{G}^H\right)^{-1} Z_i \right|
\]
\[
\leq \left| \log \det(I_{N_R} + \frac{1}{\sigma^2} G_i \tilde{Q} \tilde{G}_i^H) + \frac{1}{\ln(2)\sigma^2} Z_i^H \left(I_{N_R} + \frac{1}{\sigma^2} G_i \tilde{Q} \tilde{G}_i^H\right)^{-1} Z_i \right| + \frac{1}{\ln(2)\sigma^2} \| Z_i - G_i T_i \| (Z_i - G_i T_i)
\]
\[
= \left| \log \det(I_{N_R} + \frac{1}{\sigma^2} G_i \tilde{Q} \tilde{G}_i^H) + \frac{1}{\ln(2)\sigma^2} Z_i^H \left(I_{N_R} + \frac{1}{\sigma^2} G_i \tilde{Q} \tilde{G}_i^H\right)^{-1} Z_i \right| + \frac{1}{\ln(2)\sigma^2} \| \tilde{\xi}_i \| \| \tilde{\xi}_i \|^2.
\]
Since \( i(T_i; Z_i, G_i) \in \mathbb{R} \), we have

\[
\begin{align*}
    \log i(T_i; Z_i, G_i)^2 &= |i(T_i; Z_i, G_i)|^2 \\
    &\leq \left( \log \det(I_{N_R} + \frac{1}{\sigma^2} G_i \tilde{Q} G_i^H) + \frac{1}{\ln(2)\sigma^2} Z_i^H \left( I_{N_R} + \frac{1}{\sigma^2} G_i \tilde{Q} G_i^H \right)^{-1} Z_i \right) + \frac{1}{\ln(2)\sigma^2} \| \xi_i \|^2 \\
    &\leq 2 \left( \log \det(I_{N_R} + \frac{1}{\sigma^2} G_i \tilde{Q} G_i^H) + \frac{1}{\ln(2)\sigma^2} Z_i^H \left( I_{N_R} + \frac{1}{\sigma^2} G_i \tilde{Q} G_i^H \right)^{-1} Z_i \right)^2 + \frac{2}{\ln(2)\sigma^2} \| \xi_i \|^4 \\
    &\leq 4 \left( \log \det(I_{N_R} + \frac{1}{\sigma^2} G_i \tilde{Q} G_i^H) \right)^2 + \frac{4}{\ln(2)\sigma^2} \left( Z_i^H \left( I_{N_R} + \frac{1}{\sigma^2} G_i \tilde{Q} G_i^H \right)^{-1} Z_i \right)^2 + \frac{2}{\ln(2)\sigma^2} \| \xi_i \|^4 \\
    &\leq 4 \left( \log \det(I_{N_R} + \frac{1}{\sigma^2} G_i \tilde{Q} G_i^H) \right)^2 + \frac{4}{\ln(2)\sigma^2} \| G_i \|_F \| T_i \| + \frac{2}{\ln(2)\sigma^2} \| \xi_i \|^4 \\
    &\leq 4 \left( \log \det(I_{N_R} + \frac{1}{\sigma^2} G_i \tilde{Q} G_i^H) \right)^2 + \frac{4}{\ln(2)\sigma^2} \| G_i \|_F \| T_i \| + \frac{2}{\ln(2)\sigma^2} \| \xi_i \|^4 \\
    &\leq 4 \left( \log \det(I_{N_R} + \frac{1}{\sigma^2} G_i \tilde{Q} G_i^H) \right)^2 + \frac{4}{\ln(2)\sigma^2} \| G_i \|_F \| T_i \| + \frac{2}{\ln(2)\sigma^2} \| \xi_i \|^4 \\
    &\leq 4 \left( \log \det(I_{N_R} + \frac{1}{\sigma^2} G_i \tilde{Q} G_i^H) \right)^2 + \frac{4}{\ln(2)\sigma^2} \| G_i \|_F \| T_i \| + \frac{2}{\ln(2)\sigma^2} \| \xi_i \|^4 \\
    &\leq 4 \left( \log \det(I_{N_R} + \frac{1}{\sigma^2} G_i \tilde{Q} G_i^H) \right)^2 + \frac{4}{\ln(2)\sigma^2} \left( \| G_i \|_F \| T_i \| + \frac{2}{\ln(2)\sigma^2} \| \xi_i \|^4 \right) + \frac{2}{\ln(2)\sigma^2} \| \xi_i \|^4 \\
    &\leq \frac{4}{\ln(2)\sigma^2} \left( \| G_i \|_F \| T_i \| + \frac{2}{\ln(2)\sigma^2} \| \xi_i \|^4 \right) + \frac{2}{\ln(2)\sigma^2} \| \xi_i \|^4 \\
    &\leq \frac{4}{\ln(2)\sigma^2} \| G_i \|_F \| T_i \| + \frac{2}{\ln(2)\sigma^2} \| \xi_i \|^4 \\
    &\leq \frac{4}{\ln(2)\sigma^2} \| G_i \|_F \| T_i \| + \frac{2}{\ln(2)\sigma^2} \| \xi_i \|^4 \\
    &\leq \frac{4}{\ln(2)\sigma^2} \| G_i \|_F \| T_i \| + \frac{2}{\ln(2)\sigma^2} \| \xi_i \|^4 \\
    &\leq \frac{4}{\ln(2)\sigma^2} \| G_i \|_F \| T_i \| + \frac{2}{\ln(2)\sigma^2} \| \xi_i \|^4 \\
    &\leq \frac{4}{\ln(2)\sigma^2} \| G_i \|_F \| T_i \| + \frac{2}{\ln(2)\sigma^2} \| \xi_i \|^4 \\
\end{align*}
\]

where \( a(b) \) follow because for \( K_1, K_2 \geq 0 \), \( (K_1 + K_2)^2 \leq 2K_1^2 + 2K_2^2 \), \( c \) follows because \( \| I_{N_R} + \frac{1}{\sigma^2} G_i \tilde{Q} G_i^H \|^{-1} \| = \frac{1}{\lambda_{\min}(I_{N_R} + \frac{1}{\sigma^2} G_i \tilde{Q} G_i^H)} \leq 1 \), \( d \) follows because \( A \preceq \| A \|_F I_n \) for any Hermitian \( A \in \mathbb{C}^{n \times n} \) (by Lemma 10 in the Appendix), \( e \) follows because \( \ln \det(I_n + A) \leq \text{tr}(A) \) for \( A \) positive semi-definite and because for \( K_1, K_2 \geq 0 \), \( (K_1 + K_2)^2 \leq 2K_1^2 + 2K_2^2 \) and \( f \) follows because \( \| Q \| = \lambda_{\max}(Q) \leq \text{tr}(Q) \leq P \). This implies using the fact that \( G_i \) and \( T_i \) are independent that

\[
\begin{align*}
    \mathbb{E} [ i(T_i; Z_i, G_i)^2 ] &\leq \frac{4P^2}{\ln(2)\sigma^2} \| G_i \|^4 \mathbb{E} \| \xi_i \|^4 + \frac{32}{\ln(2)\sigma^2} \mathbb{E} \| G_i \|^4 \mathbb{E} \| T_i \|^4 + \mathbb{E} \| \xi_i \|^4 + \frac{2}{\ln(2)\sigma^2} \| \xi_i \|^4 \\
    &\leq \frac{4P^2}{\ln(2)\sigma^2} \| G_i \|^4 \mathbb{E} \| \xi_i \|^4 + \frac{16}{\ln(2)\sigma^2} (c_1 c_2 + c_3) + \frac{2}{\ln(2)\sigma^2} c_3
\end{align*}
\]
for some \( c_1, c_2, c_3 > 0 \), where we used that \( \mathbb{E} \left[ \| G_i \|^4 \right] < \infty \), \( \mathbb{E} \left[ \| T_i \|^4 \right] < \infty \) and \( \mathbb{E} \left[ \| \xi_i \|^4 \right] < \infty \) (by Lemma 13 in the Appendix) and where \( c'' > 0 \).

As a result, we have

\[
\frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E} \left[ i(T_i; Z_i, G_i)^2 \right] \leq \frac{c''}{n}.
\]

To summarize, we have proved that for \( 0 < \alpha < 1 \)

\[
\frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1}^{n-1} m(i, k) + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=i+1}^{n} m(i, k) \leq \frac{2c''}{n(1 - \sqrt{\alpha})}
\]

and that for \( 0 \leq \alpha < 1 \)

\[
\frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E} \left[ i(T_i; Z_i, G_i)^2 \right] \leq \frac{c''}{n}.
\]

Now, from (19), we know that

\[
\text{var} \left( \frac{i(T^n_i; Z^n_i, G^n_i)}{n} \right)
\]

\[
\leq \begin{cases}
\frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E} \left[ i(T_i; Z_i, G_i)^2 \right], & \text{for } \alpha = 0 \\
\frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1}^{n-1} m(i, k) + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=i+1}^{n} m(i, k) + \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E} \left[ i(T_i; Z_i, G_i)^2 \right] & \text{for } 0 < \alpha < 1.
\end{cases}
\]

To conclude, it follows that

\[
\text{var} \left( \frac{i(T^n_i; Z^n_i, G^n_i)}{n} \right) \leq \begin{cases}
\frac{c''}{n}, & \text{for } \alpha = 0 \\
\frac{2c''}{n(1 - \sqrt{\alpha})} + \frac{c''}{n}, & \text{for } 0 < \alpha < 1
\end{cases}
\]

where \( \lim_{n \to \infty} \kappa(n, \alpha) = 0 \). This completes the proof of Lemma 4.

V. CONCLUSION

In this paper, we studied the problem of message transmission over time-varying MIMO first-order Gauss-Markov Rayleigh fading channels with average power constraint and with CSIR, as an example of infinite-state Markov fading channels. The Gauss-Markov model is widely used to describe the time-varying aspect of the channel. The novelty of our work lies in establishing a single-letter characterization of the channel capacity. The capacity formula that we proved coincides with the one of i.i.d. MIMO Rayleigh fading channels. Therefore, the memory factor has no influence on the capacity. The CSIR assumption is critical in deriving the single-letter expression for the capacity. In the absence of that assumption, no single-letter formula for the capacity exists even for SISO finite-state Markov fading channels. As a future work, it would be interesting to study the capacity of time-varying MIMO Rayleigh fading channels when a higher-order Gauss-Markov model is used to describe the channel variations over the time.

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A. Auxiliary Lemmas

Lemma 10. For any Hermitian matrix $A \in \mathbb{C}^{n \times n}$, the matrix

$$||A|| I_n - A$$

is positive semi-definite.

Proof. Since $A$ is Hermitian, we know that for any $x \in \mathbb{C}^n$, $x^H A x$ is real. Therefore, for any $x \in \mathbb{C}^n \setminus \{0\}$,

$$x^H A x \leq |x^H A x|$$

$$\leq ||A|| ||x||^2.$$

It follows that

$$x^H (||A|| I_n - A) x = x^H ||A|| I_n x - x^H A x$$

$$= ||A|| ||x||^2 - x^H A x$$

$$\geq 0.$$

Lemma 11. Let $A \in \mathbb{C}^{n \times n}$ be any positive-definite Hermitian matrix with $\lambda_{\min}(A)$ being its smallest eigenvalue and let $B \in \mathbb{C}^{n \times n}$ be any positive semi-definite matrix, then

$$\log \det(A + B) \leq \log \det(A) + \log \det(I_n + \frac{1}{\lambda_{\min}(A)} B).$$

Proof.

$$\det(A + B) = \det(A) \det(I_n + A^{-1} B)$$

$$= \det(A) \det(I_n + B A^{-1})$$

$$= \det(A) \det(I_n + B^\frac{1}{2} B^\frac{1}{2} A^{-1})$$

$$= \det(A) \det(I_n + B^\frac{1}{2} A^{-1} B^\frac{1}{2})$$

$$\leq \det(A) \det(I_n + B^\frac{1}{2} \frac{1}{\lambda_{\min}(A)} I_n B^\frac{1}{2})$$

$$= \det(A) \det(I_n + \frac{1}{\lambda_{\min}(A)} I_n B)$$

$$= \det(A) \det(I_n + \frac{1}{\lambda_{\min}(A)} B),$$

where (a) follows from the following properties [23]:

1) For any positive semi-definite Hermitian matrices $M_1$ and $M_2$, if $M_1 - M_2$ is Hermitian positive semi-definite then

$$\det(M_1) \geq \det(M_2).$$

2) For any Hermitian matrix $M \in \mathbb{C}^{n \times n}$, with minimum eigenvalue $\lambda_{\min}(M)$, it holds that

$$M - \lambda_{\min}(M) I$$

is positive semi-definite,

3) For any positive definite Hermitian matrices $M$ and $\tilde{M}$, $M - \tilde{M}$ is positive semi-definite if and only if $\tilde{M}^{-1} - M^{-1}$ is positive semi-definite.

Therefore, it follows that

$$\log \det(A + B) \leq \log \det(A) + \log \det(I_n + \frac{1}{\lambda_{\min}(A)} B).$$
Lemma 12. \( \mathbb{E} \left[ \Lambda \left( G_{i_1}, S, \tilde{G}, \tilde{W} \right) \right] \leq c \) for some \( c > 0 \).

Proof. We recall that
\[
\Lambda \left( G_{i_1}, S, \tilde{G}, \tilde{W} \right) = \| G_{i_1} \|^2 \left( P \| G_{i_1} \|^2 + P \| \tilde{G} \|^2 + 2 \| \tilde{W} \tilde{Q} \tilde{G}^H \| + 2 P \| G_{i_1} \| \| S \| \right),
\]
where
\[
\tilde{W} = S + \sqrt{\alpha}^{i_2-i_1} \tilde{G}
\]
with \( i_1 < i_2 \), where \( S = \sqrt{1-\alpha} \sum_{j=i_1+1}^{i_2} \sqrt{\alpha}^{j-i_1} W_j \), and where \( \tilde{G} \) is a random matrix independent of \( G_1 \) and \( W_i, i = 2, \ldots, n \) such that \( \text{vec}(\tilde{G}) \sim \mathcal{N} \left( \mathbf{0}_{NN^T}, K \right) \).

We have
\[
\mathbb{E} \left[ \Lambda \left( G_{i_1}, S, \tilde{G}, \tilde{W} \right) \right]
= \mathbb{E} \left[ \| G_{i_1} \|^2 \left( P \| G_{i_1} \|^2 + P \| \tilde{G} \|^2 + 2 \| \tilde{W} \tilde{Q} \tilde{G}^H \| + 2 P \| G_{i_1} \| \| S \| \right) \right]
= P \mathbb{E} \left[ \| G_{i_1} \|^4 \right] + P \mathbb{E} \left[ \| G_{i_1} \|^2 \| \tilde{G} \|^2 \right] + 2 \mathbb{E} \left[ \| G_{i_1} \|^2 \| \tilde{W} \tilde{Q} \tilde{G}^H \| \right] + 2 P \mathbb{E} \left[ \| G_{i_1} \|^3 \| S \| \right]
= P \mathbb{E} \left[ \| G_{i_1} \|^4 \right] + P \mathbb{E} \left[ \| \tilde{G} \|^2 \right] + 2 \mathbb{E} \left[ \| G_{i_1} \|^2 \right] \mathbb{E} \left[ \| \tilde{W} \tilde{Q} \tilde{G}^H \| \right] + 2 P \mathbb{E} \left[ \| G_{i_1} \|^3 \| S \| \right],
\]
where we used that \( G_{i_1} \) is independent of \( (\tilde{W}, \tilde{G}) \) and that \( \tilde{G} \) has the same distribution as \( G_{i_1} \).

By Lemma 13, we know that \( \mathbb{E} \left[ \| \tilde{G} \|^\ell \right] < \infty \) for \( \ell \in \{2, 3, 4\} \). Therefore, to complete the proof, we have to show that \( \mathbb{E} \left[ \| \tilde{W} \tilde{Q} \tilde{G}^H \| \right] \) and \( \mathbb{E} \left[ \| S \| \right] \) are both bounded from above.

It holds that
\[
\mathbb{E} \left[ \| \tilde{W} \tilde{Q} \tilde{G}^H \| \right]
= \mathbb{E} \left[ \left\| \left( S + \sqrt{\alpha}^{i_2-i_1} \tilde{G} \right) \tilde{Q} \tilde{G}^H \right\| \right]
= \mathbb{E} \left[ \left\| S \tilde{Q} \tilde{G}^H + \sqrt{\alpha}^{i_2-i_1} \tilde{G} \tilde{Q} \tilde{G}^H \right\| \right]
\leq \mathbb{E} \left[ \| S \| \mathbb{E} \left[ \| \tilde{Q} \| \| S \| \right] + \| \tilde{G} \|^2 \right]
\leq P \mathbb{E} \left[ \| S \| \mathbb{E} \left[ \| \tilde{G} \| \right] + \mathbb{E} \left[ \| \tilde{G} \|^2 \right] \right],
\]
where we used that \( \tilde{G} \) and \( S \) are independent in the last step, since \( \tilde{G} \) and \( W_{i_1+1}, \ldots, W_{i_2} \) are independent.
Now, from Lemma 13, we know that $\mathbb{E}[\|\hat{G}\|] < \infty$. Therefore, to complete the proof, it suffices to show that $\mathbb{E}[\|S\|]$ is bounded from above.

We have

$$
\mathbb{E}[\|S\|] = \mathbb{E}\left[\left\|\sqrt{1-\alpha} \sum_{j=i_1+1}^{i_2} \sqrt{\alpha^{i_2-j}} W_j\right\|\right] \\
\leq \mathbb{E}\left[\sqrt{1-\alpha} \sum_{j=i_1+1}^{i_2} \sqrt{\alpha^{i_2-j}} \mathbb{E}[\|W_j\|]\right] \\
= \sqrt{1-\alpha} \mathbb{E}[\|\hat{G}\|] \sum_{j=i_1+1}^{i_2} \sqrt{\alpha^{i_2-j}} \\
= \sqrt{1-\alpha} \mathbb{E}[\|\hat{G}\|] \left(1 - \frac{\sqrt{\alpha^{i_2-i_1}}}{\sqrt{\alpha}}\right) \mathbb{E}[\|\hat{G}\|] \\
\leq \sqrt{1-\alpha} \mathbb{E}[\|\hat{G}\|],
$$

where we used that

$$
\sum_{j=i_1+1}^{i_2} \sqrt{\alpha^{i_2-j}} = \sqrt{\alpha^{i_2} \sum_{j=i_1+1}^{i_2} \left(\frac{1}{\sqrt{\alpha}}\right)^j} \\
= \sqrt{\alpha^{i_2}} \left(\frac{1}{\sqrt{\alpha}}\right)^{i_1+1} \frac{1 - \left(\frac{1}{\sqrt{\alpha}}\right)^{i_2-i_1}}{1 - \frac{1}{\sqrt{\alpha}}} \\
= \sqrt{\frac{\alpha^{i_2-i_1} - 1}{\alpha - 1}} \\
= 1 - \sqrt{\frac{\alpha^{i_2-i_1}}{\alpha - 1}} \\
= \frac{1}{\sqrt{\alpha}} - 1
$$

and that $\hat{G}$ has the same distribution as each of the $W_i$. Therefore, $\mathbb{E}[\|S\|]$ is bounded from above. This proves that $\mathbb{E}[\Lambda(G_{i_1}, S, G, \hat{G})] \leq c$ for some $c > 0$. \qed

**Lemma 13.** For any $K \in \mathbb{C}^{N_R N_T \times N_R N_T}$, let $G \in \mathbb{C}^{N_R \times N_T}$ a random matrix such that

$$
\text{vec}(G) \sim \mathcal{N}_\mathbb{C}(0_{N_R N_T}, K).
$$

Then for $\ell \in \{1, 2, 3, 4\}$, it holds that $\mathbb{E}[\|G\|^{\ell}] < \infty$.

**Proof.** Let $G^{(1)}, \ldots, G^{(N_R N_T)}$ be the entries of vec($G$).

For $\ell = 2$, we have

$$
\|G\|^2 = \lambda_{\text{max}}(GG^H) \\
\leq \text{tr}(GG^H) \\
= \sum_{i=1}^{N_R N_T} |G^{(i)}|^2.
$$

Therefore

$$
\mathbb{E}[\|G\|^2] \leq \sum_{i=1}^{N_R N_T} \mathbb{E}[|G^{(i)}|^2] \\
< \infty,
$$

(31)
where we used that the second moment of each $G^{(i)}$, $i = 1, \ldots, NRNT$, is finite.

For $\ell = 4$, it follows using (30) that

$$
\|G\|_4^4 \leq \left( \sum_{i=1}^{NRNT} NRT \sum_{j=1, j \neq i} \|G^{(i)}\|^2 \|G^{(j)}\|^2 + \sum_{i=1}^{NRNT} \|G^{(i)}\|^4 \right)^2
$$

This yields

$$
E[\|G\|^4] \leq \sum_{i=1}^{NRNT} NRT \sum_{j=1, j \neq i} E[\|G^{(i)}\|^2 \|G^{(j)}\|^2] + \sum_{i=1}^{NRNT} E[\|G^{(i)}\|^4]
$$

(a)

$$
\leq \sum_{i=1}^{NRNT} NRT \sum_{j=1, j \neq i} \sqrt{E[\|G^{(i)}\|^4] E[\|G^{(j)}\|^4]} + \sum_{i=1}^{NRNT} E[\|G^{(i)}\|^4]
$$

(b)

$$
< \infty,
$$

where (a) follows from Cauchy Schwarz’s inequality and (b) follows because the fourth moment of each $G^{(i)}$, $i = 1, \ldots, NRNT$, is finite.

For $\ell = 1$, we know that

$$
E[\|G\|] \leq \sqrt{E[\|G\|^2]}
$$

< \infty,

where we used (31) in the last step.

For $\ell = 3$, it holds that

$$
E[\|G\|^3] = E[\|G\|\|G\|^2]
$$

(a)

$$
\leq \sqrt{E[\|G\|^2] E[\|G\|^4]}
$$

(b)

$$
< \infty,
$$

where (a) follows from Cauchy Schwarz’s inequality and (b) follows from (31) and (32).

\[\square\]

**Lemma 14.** For any $0 < \alpha < 1$, it holds that

$$
\sum_{i=1}^{n} \sum_{k=1}^{i-1} \alpha^{i-k} \leq \frac{n}{1-\alpha}.
$$

**Proof.** We have

$$
\sum_{i=1}^{n} \sum_{k=1}^{i-1} \alpha^{i-k}
$$

$$
= \sum_{i=1}^{n} \alpha^i \sum_{k=1}^{i-1} \left( \frac{1}{\alpha} \right)^k
$$

$$
= \sum_{i=1}^{n} \alpha^i \frac{1 - \left( \frac{1}{\alpha} \right)^{i-1}}{\alpha - 1}
$$

$$
= \sum_{i=1}^{n} \alpha^i \frac{1 - \left( \frac{1}{\alpha} \right)^{i-1}}{\alpha - 1}
$$

$$
= \sum_{i=1}^{n} \frac{\alpha^i - \alpha}{\alpha - 1}
$$
\[ \sum_{i=1}^{n} \frac{\alpha - \alpha^i}{1 - \alpha} = \frac{n\alpha}{1 - \alpha} - \sum_{i=1}^{n} \frac{\alpha^i}{1 - \alpha} \leq \frac{n\alpha}{1 - \alpha} \leq \frac{n}{1 - \alpha}. \]

**Lemma 15.** For any \(0 < \alpha < 1\) it holds that
\[ \sum_{i=1}^{n} \sum_{k=i+1}^{n} \alpha^{k-i} \leq \frac{n}{1 - \alpha}. \]

**Proof.** We have
\[
\begin{align*}
\sum_{i=1}^{n} \sum_{k=i+1}^{n} \alpha^{k-i} &= \sum_{i=1}^{n} \left( \frac{1}{\alpha} \right)^i \sum_{k=i+1}^{n} \alpha^k \\
&= \sum_{i=1}^{n} \left( \frac{1}{\alpha} \right)^i \alpha^{i+1} \frac{(1 - \alpha^{n-i})}{1 - \alpha} \\
&= \sum_{i=1}^{n} \frac{\alpha (1 - \alpha^{n-i})}{1 - \alpha} \\
&= \frac{n\alpha}{1 - \alpha} - \frac{\alpha^{n+1}}{1 - \alpha} \sum_{i=1}^{n} \left( \frac{1}{\alpha} \right)^i \\
&\leq \frac{n\alpha}{1 - \alpha} \leq \frac{n}{1 - \alpha}.
\end{align*}
\]

**Lemma 16.** \(\forall i \in \{1, \ldots, n\}\)
\[ i(T_i; Z_i, G_i) \]
\[ = \log \det(I_{NR} + \frac{1}{\sigma^2} G_i \tilde{Q} G_i^H) - \frac{1}{\ln(2)\sigma^2} (Z_i - G_i T_i)^H (Z_i - G_i T_i) + \frac{1}{\ln(2)\sigma^2} Z_i^H \left( I_{NR} + \frac{1}{\sigma^2} G_i \tilde{Q} G_i^H \right)^{-1} Z_i, \]
where \(T_i \sim \mathcal{N}_C \left( 0_{N_T}, \tilde{Q} \right), i = 1 \ldots n. \)

**Proof.** Notice that
\[ i(T_i; Z_i, G_i) = \log \left( \frac{p_{z_i|G_i,T_i}(Z_i,G_i,T_i)}{p_{z_i|G_i}(Z_i|G_i)p_{T_i}(T_i)} \right) = \log \left( \frac{p_{Z_i|G_i,T_i}(Z_i|G_i,T_i)}{p_{Z_i|G_i}(Z_i|G_i)} \right), \]
where we used that \(T_i\) and \(G_i\) are independent.
It holds that
\[ Z_i|G_i, T_i \sim \mathcal{N}_C \left( G_i T_i, \sigma^2 I_{NR} \right) \]

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and that

\[ Z_i | G_i \sim \mathcal{N}_C \left( 0_{N_R}, G_i \hat{Q}_i G_i^H + \sigma^2 I_{N_R} \right). \]

It follows that

\[
\log \frac{p_{Z_i | G_i, T_i}(Z_i | G_i, T_i)}{p_{Z_i | G_i}(Z_i | G_i)} = \log \frac{1}{\pi^N \det(\sigma^2 \mathbf{I}_{N_R})^{1/2}} \exp \left( -\frac{1}{\sigma^2} (Z_i - G_i T_i)^H (Z_i - G_i T_i) \right) \\
= \log \frac{\det(G_i \hat{Q}_i G_i^H + \sigma^2 \mathbf{I}_{N_R})}{\det(\sigma^2 \mathbf{I}_{N_R})} \exp \left( -\frac{1}{\sigma^2} (Z_i - G_i T_i)^H (Z_i - G_i T_i) + \frac{1}{\ln(2) \sigma^2} Z_i^H (\mathbf{I}_{N_R} + \frac{1}{\sigma^2} G_i \hat{Q}_i G_i^H)^{-1} Z_i \right) \\
= \log \det(\mathbf{I}_{N_R} + \frac{1}{\sigma^2} G_i \hat{Q}_i G_i^H) - \frac{1}{\ln(2) \sigma^2} (Z_i - G_i T_i)^H (Z_i - G_i T_i) + \frac{1}{\ln(2) \sigma^2} Z_i^H (\mathbf{I}_{N_R} + \frac{1}{\sigma^2} G_i \hat{Q}_i G_i^H)^{-1} Z_i.
\]

\[
\square
\]

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