INVERSION OF ADJUNCTION FOR QUOTIENT SINGULARITIES II: NON-LINEAR ACTIONS

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Abstract. We prove the precise inversion of adjunction formula for quotient singularities. As an application, we prove the semi-continuity of minimal log discrepancies for hyperquotient singularities. This paper is a continuation of [NS22], and we generalize the previous results to non-linear group actions.

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1. INTRODUCTION

The minimal log discrepancy is an invariant of singularities defined in birational geometry. The importance of this invariant is that two conjectures on the invariant, the LSC (lower semi-continuity) conjecture and the ACC (ascending chain condition) conjecture, imply the conjecture of termination of flips [Sho01]. This paper is a continuation of [NS22], and we focus on the LSC conjecture. We always work over an algebraically closed field \( k \) of characteristic zero.

The LSC conjecture is proposed by Ambro [Amb99] and the conjecture predicts that the minimal log discrepancies satisfy the lower semi-continuity.

Conjecture 1.1 (LSC conjecture, [Amb99]). Let \( (X, \mathfrak{a}) \) be a log pair with an \( \mathbb{R} \)-ideal \( \mathfrak{a} \), and let \( |X| \) be the set of all closed points of \( X \) with the Zariski topology. Then the function

\[ |X| \to \mathbb{R}_{\geq 0} \cup \{-\infty\}; \quad x \mapsto \text{mld}_x(X, \mathfrak{a}) \]
is lower semi-continuous.

The LSC conjecture is proved when dim $X \leq 3$ by Ambro [Amb99]. Ein, Mustaţă, and Yasuda in [EMY03] prove the conjecture when $X$ is smooth. Ein and Mustaţă in [EM04] generalize the argument to the case where $X$ is a locally complete intersection variety. In [Nak16], the first author proves the conjecture when $X$ has quotient singularities, more generally when $X$ has a crepant resolution in the category of the Deligne-Mumford stacks. In [NS22], the authors study the conjecture when $X$ has hyperquotient singularities and prove the conjecture for the following $X$:

- Suppose that a finite subgroup $G \subset \text{GL}_N(k)$ acts linearly on $\mathbb{A}^N_k$ freely in codimension one. Let $Y := \mathbb{A}^N_k/G$ be the quotient variety. Let $X \subset Y$ be a klt subvariety of codimension $c$ that is locally defined by $c$ equations in $Y$.

The main purpose of this paper is to generalize the result in [NS22] to non-linear group actions. See Definition 8.1 for the definition of quotient singularity in this paper.

**Theorem 1.2 (Theorem 9.2).** Let $Y$ be a variety with only quotient singularities. Let $X$ be a klt subvariety of $Y$ of codimension $c$ that is locally defined by $c$ equations in $Y$. Let $a$ be an $\mathbb{R}$-ideal sheaf on $X$. Then the function

$$|X| \to \mathbb{R}_{\geq 0} \cup \{-\infty\}; \quad x \mapsto \operatorname{mld}_x(X, a)$$

is lower semi-continuous.

In this paper, we also treat the PIA (precise inversion of adjunction) conjecture.

**Conjecture 1.3** (PIA conjecture, [92 17.3.1]). Let $(X, a)$ be a log pair and let $D$ be a normal Cartier prime divisor. Let $x \in D$ be a closed point. Suppose that $D$ is not contained in the cosupport of the $\mathbb{R}$-ideal sheaf $a$. Then

$$\operatorname{mld}_x(X, a\mathcal{O}_X(-D)) = \operatorname{mld}_x(D, a\mathcal{O}_D)$$

holds.

Ein, Mustaţă, and Yasuda in [EMY03] prove the PIA conjecture when $X$ is smooth. Ein and Mustaţă in [EM04] generalize the argument to the case where $X$ is a locally complete intersection variety. The authors in [NS22] prove the conjecture for the following $X$ and $D$:

- Suppose that a finite subgroup $G \subset \text{GL}_N(k)$ acts linearly on $\mathbb{A}^N_k$ freely in codimension one. Let $Y := \mathbb{A}^N_k/G$ be the quotient variety, and let $x \in Y$ be the image of the origin. Let $X$ be a subvariety of $Y$ of codimension $c$ that has only klt singularities and is locally defined by $c$ equations in $Y$ at $x$. Let $D$ be a Cartier prime divisor on $X$ through $x$ with a klt singularity at $x \in D$.

In this paper, this result in [NS22] is generalized to non-linear group actions.

**Theorem 1.4 (Corollary 9.1).** Suppose that a variety $Y$ has a quotient singularity at a closed point $x \in Y$. Let $X$ be a subvariety of $Y$ of codimension $c$ that is locally defined by $c$ equations at $x$. Suppose that $X$ is klt at $x$. Let $a$ be an $\mathbb{R}$-ideal sheaf on $X$. Let $D$ be a prime divisor on $X$ through $x$ that is klt and Cartier at $x$. Suppose that $D$ is not contained in the cosupport of the $\mathbb{R}$-ideal sheaf $a$. Then it follows that

$$\operatorname{mld}_x(X, a\mathcal{O}_X(-D)) = \operatorname{mld}_x(D, a\mathcal{O}_D).$$

Due to Theorem 1.4, Theorem 1.2 can be reduced to the known case where $X$ has quotient singularities. Hence, this paper is mainly devoted to proving Theorem 1.4. If $X$ has a quotient singularity at a closed point $x \in X$, then the completion $\mathcal{O}_{X,x}$ is isomorphic to $k[[x_1, \ldots, x_N]]^G$ for some linear group action $G \subset \text{GL}_N(k)$. Therefore, Theorem 1.4 can be proved by extending the proofs in [NS22] to the case of the formal power series ring. In what follows, we shall explain the main differences.
from the proofs in [NS22] and the difficulties that arise when dealing with formal power series rings.

The key ingredient of the proofs in [NS22] is the theory of arc spaces of quotient singularities established by Denef and Loeser in [DL02]. Suppose that a finite group $G \subset \text{GL}_N(k)$ of order $d$ acts on $A = \text{Spec} k[x_1, \ldots, x_N]$. Let $\overline{X} \subset A$ be a $G$-invariant closed subvariety and let $X := \overline{X}/G$ be its quotient. Let $I \subset k[x_1, \ldots, x_N]^G$ be the defining ideal of $X \subset \overline{A}/G$. For each $\gamma \in G$, $\gamma$ can be diagonalized with some new basis $x_1^{(\gamma)}, \ldots, x_N^{(\gamma)}$. Let $\text{Diag}(\xi_{e_1}^{(\gamma)}, \ldots, \xi_{e_N}^{(\gamma)})$ be the diagonal matrix with $0 \leq e_i \leq d - 1$, where $\xi$ is a primitive $d$-th root of unity in $k$. Then we define the ring homomorphism $\overline{X}_\gamma$ by

$$\overline{X}_\gamma : k[x_1, \ldots, x_N]^G \rightarrow k[t][x_1, \ldots, x_N]; \quad x_i^{(\gamma)} \mapsto t^{e_i}x_i^{(\gamma)},$$

and define a $k[t]$-scheme $\overline{X}^{(\gamma)}$ by

$$\overline{X}^{(\gamma)} = \text{Spec}(k[t][x_1, \ldots, x_N]/\overline{T}^{(\gamma)}),$$

where $\overline{T}^{(\gamma)}$ is the ideal generated by the elements of $\overline{X}_\gamma(I)$. Then the theory of Denef and Loeser in [DL02] allows us to compare the spaces $X_\infty \cup \bigsqcup_{\gamma \in G} X_\infty^{(\gamma)}$. In [NS22], using this theory, $X_\infty$ is studied through each $X_\infty^{(\gamma)}$.

In this paper, we deal with the case of formal power series rings, i.e. when $\overline{A} = \text{Spec} k[[x_1, \ldots, x_N]].$ Let $G$, $\gamma$, $\overline{X}$, and $X$ be as above. In this case, we take $e_i$’s above to satisfy $1 \leq e_i \leq d$. Then we can define the following two natural ring homomorphisms

$$\overline{X}_\gamma : k[[x_1, \ldots, x_N]]^G \rightarrow k[[x_1, \ldots, x_N]][[t]]; \quad x_i^{(\gamma)} \mapsto t^{e_i}x_i^{(\gamma)},$$

$$\overline{X}_\gamma : k[[x_1, \ldots, x_N]]^G \rightarrow k[[x_1, \ldots, x_N]]; \quad x_i^{(\gamma)} \mapsto t^{e_i}x_i^{(\gamma)},$$

and we define $k[t]$-schemes

$$\overline{X}^{(\gamma)} = \text{Spec}(k[[x_1, \ldots, x_N]][[t]]/\overline{T}^{(\gamma)}), \quad \overline{X}^{(\gamma)} = \text{Spec}(k[[x_1, \ldots, x_N]]/\overline{T}^{(\gamma)}),$$

where $\overline{T}^{(\gamma)}$ is the ideal generated by $\overline{X}_\gamma(I)$. In this paper, we will use both arc spaces $\overline{X}_\infty^{(\gamma)}$ and $\overline{X}_\infty^{(\gamma)}$ to study $X_\infty$. We shall also explain below how to use $\overline{X}_\infty^{(\gamma)}$ and $\overline{X}_\infty^{(\gamma)}$ differently.

First, the theory of Denef and Loeser in [DL02] can be generalized to the formal power series rings, and $X_\infty$ can be compared with $\overline{X}_\infty^{(\gamma)}$. Indeed, we shall see in Proposition 6.3 that $\overline{X}_\gamma$ gives a map $\bigsqcup_{\gamma \in G} \overline{X}_\infty^{(\gamma)} \rightarrow X_\infty$ that is surjective outside a thin set. We note that it is not enough to consider $\overline{X}_\infty^{(\gamma)}$ in this respect (see Remark 6.4 for the detail).

On the other hand, $\overline{X}_\infty^{(\gamma)}$ will be used with the following motivation. In [NS22], a $k$-arc $\beta \in \overline{X}_\infty^{(\gamma)}$ is called the “trivial arc” when it corresponds to the $k[t]$-ring homomorphism $\beta^* : k[t][x_1, \ldots, x_N] \rightarrow k[t]$ satisfying $\beta^*(x_i) = 0$ for each $i$. Another key point of the argument in [NS22] is to show the fact that the trivial arc always has a lift on a resolution $W$ of $\overline{X}^{(\gamma)}$. The existence of such a lift is proved by combining the result by Graber, Harris, and Starr [GHS03] and the rational chain connectedness of the fibers of the resolution proved by Hacon and McKernan [HM07]. In our formal power series ring setting, this argument does not work directly on $\overline{X}^{(\gamma)}$ because each closed fiber $\overline{X}^{(\gamma)} \rightarrow \text{Spec} k[t]$ over $t = a \in k^\times$ is empty. Whereas, the same argument works on $\overline{X}_\infty^{(\gamma)}$ and proves that the trivial arc has a lift on a desingularization $W'$ of $\overline{X}^{(\gamma)}$ (see Claim 8.3). It should be noted that it is not clear whether the results [GHS03] and [HM07] can be applied to the formal power series ring setting.
However, this difficulty can be avoided by using the functorial desingularization due to Temkin [Tem12].

A large part of this paper (Sections 2, 4, and 5) is devoted to proving and listing the basic facts for dealing with the arc spaces $X^{(\gamma)}$ and $X'^{(\gamma)}$. Firstly, $X^{(\gamma)}$ can be seen as the arc space (Greenberg scheme) of a formal $k[[t]]$-scheme, and the theory developed by Sebag in [Seb04] can be applied (cf. [CLNS18, Yas24]). On the other hand, as far as we know, the arc spaces of the form $X'^{(\gamma)}$ have not been fully discussed so far. In Subsection 5.1, we discuss the basic facts on the arc spaces of $k[[x_1, \ldots, x_N]]$-schemes of finite type. Furthermore, we discuss in Section 2 the theory of sheaves of special differentials introduced by de Fernex, Ein, and Mustaţă in [dFEM11], and the theory of derivations which are needed in Section 5. The theory of the arc spaces of $k[[x_1, \ldots, x_N]]$-schemes of finite type has the following technical difficulties (see Remark 5.16 for the detail). For a $k$-variety $X$, it is almost trivial that $Z_{\infty}$ is a thin set of $X_{\infty}$ for the closed subscheme $Z \subset X$ defined by the Jacobian ideal $\text{Jac}_X$. This fact is also valid for $k[t]$-schemes $X$ of finite type dealt with in [NS22] and for formal $k[[t]]$-schemes of finite type dealt with in [Seb04]. However, it is not clear to us whether the same statement is valid for $k[[x_1, \ldots, x_N]]$-schemes of finite type. For avoiding this difficulty, many propositions in Subsection 5.1.3 are proved under stronger assumptions.

The paper is organized as follows. In Section 2, we discuss the theory of sheaves of special differentials introduced by de Fernex, Ein, and Mustaţă in [dFEM11] and the theory of derivations. In Section 3, we review some definitions on log pairs. In Section 4, we discuss the theory of arc spaces of $k[[x_1, \ldots, x_N]]$-schemes and see that the formula in [EMY03] and [EM09] representing the minimal log discrepancies of $k$-varieties in terms of arc spaces can be generalized to the formal power series ring setting (Theorem 4.11). In Section 5, we discuss the theory of arc spaces for $k[[t]][[x_1, \ldots, x_N]]$-schemes and affine formal $k[[t]]$-schemes in the formal power series ring setting. In Section 6, we review the minimal log discrepancies of quotient singularities and describe them by the codimension of cylinders in arc spaces of the $k[[t]]$-schemes using the theories in Sections 5 and 3. In Section 7, we prove the PIA conjecture for quotient singularities, where the group action may be non-linear (Theorem 7.2). In Section 8, we prove the main theorems Corollary 8.1 and Theorem 8.2.

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Notation

- We basically follow the notations and the terminologies in [Har77] and [Kol13].
- Throughout this paper, $k$ is an algebraically closed field of characteristic zero. We say that $X$ is a variety over $k$ or a $k$-variety if $X$ is an integral scheme that is separated and of finite type over $k$.

2. Sheaves of special differentials

Let $R_0$ be a ring. In this section, following [dFEM11] Appendix A], we define the sheaf $\Omega^\prime_{X/R_0}$ of special differentials for a scheme $X$ of finite type over $\text{Spec} R_0[[x_1, \ldots, x_N]]$. In [dFEM11] Appendix A], the sheaf $\Omega^\prime_{X/R_0}$ of special differentials is defined for $R_0 = k$. This definition can be generalized to an arbitrary ring $R_0$. We are interested in the case where $R_0 = k$ or $R_0 = k[[t]]$ for our later use.
Let $R = R_0[[x_1, \ldots, x_N]]$.

**Definition 2.1.** (cf. [dFEM11 Appendix A]).

1. Let $M$ be an $R$-module. Then an $R_0$-derivation $D : R \to M$ is called a special $R_0$-derivation if $D$ satisfies

$$D(f) = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i} D(x_i)$$

for any $f \in R$.

2. For an $R$-algebra $A$ and an $A$-module $M$, an $R_0$-derivation $D : A \to M$ is called a special $R_0$-derivation if its restriction to $R$ is a special $R_0$-derivation. We denote by $\text{Der}'_{R_0}(A, M)$ the set of all special $R_0$-derivations. Then $\text{Der}'_{R_0}(A, M)$ is an $A$-submodule of $\text{Der}_{R_0}(A, M)$.

**Lemma 2.2.** Let $M$ be an $A$-module that is separated in the $(x_1, \ldots, x_N)$-adic topology, i.e. $M$ satisfies $\bigcap_{\ell \geq 1} (x_1, \ldots, x_N)^{\ell} M = 0$. Then

$$\text{Der}'_{R_0}(A, M) = \text{Der}_{R_0}(A, M)$$

holds. In particular, it holds in the following two cases.

1. When $M$ is a finite $R$-module.
2. When $M = A$ and $A$ is an integral domain such that $(x_1, \ldots, x_N)A \neq A$.

**Proof.** By the definition of special derivations, it is sufficient to show that $\text{Der}'_{R_0}(R, M) = \text{Der}_{R_0}(R, M)$. Let $D \in \text{Der}_{R_0}(R, M)$. If we set $D' : R \to M$ by $D'(f) := \sum_{i=1}^{N} \frac{\partial f}{\partial x_i} D(x_i)$, then $D' \in \text{Der}'_{R_0}(R, M)$ holds. Hence it is sufficient to show that $D = 0$ if $D(x_i) = 0$ holds for all $i$. Let $f \in R$. For any $\ell \geq 0$, we may write $f = f_1 + f_2$ with $f_1 \in R_0[x_1, \ldots, x_n]$ and $f_2 \in (x_1, \ldots, x_n)^{\ell+1} R$. Then we have

$$D(f) = D(f_1) + D(f_2) = D(f_2) \in (x_1, \ldots, x_n)^{\ell} M.$$

It shows that $D(f) \in \bigcap_{\ell \geq 1} (x_1, \ldots, x_N)^{\ell} M = 0$.

In both cases (1) and (2), $M$ is separated by [Mat89 Theorem 8.9].

**Proposition 2.3.** (cf. [dFEM11 Appendix A]). For any $R$-algebra $A$, there exists an $A$-module $\Omega'_{A/R_0}$ with a special $R_0$-derivation $d'_{A/R_0} : A \to \Omega'_{A/R_0}$ such that the induced map

$$\text{Hom}_A(\Omega'_{A/R_0}, M) \to \text{Der}'_{R_0}(A, M); \quad f \mapsto f \circ d'_{A/R_0}$$

is an isomorphism for any $A$-module $M$.

**Proof.** The same proof as in [dFEM11 A.1-A.4] works.

An $A$-module $\Omega'_{A/R_0}$ satisfying Proposition 2.3 is unique up to an isomorphism commuting with $d'_{A/R_0}$ and $\Omega'_{A/R_0}$ is called the module of special differentials. We sometimes abbreviate $d'_{A/R_0}$ to $d'$ when no confusion can arise. We note that $\Omega'_{A/R_0}$ depends on the choice of $R$. We list some properties on $\Omega'_{A/R_0}$ from [dFEM11 Appendix A].

**Proposition 2.4.** (cf. [dFEM11 Appendix A]).

1. If $A = R[y_1, \ldots, y_m]$, then $\Omega'_{A/R_0}$ is a free $A$-module of rank $N + m$ with basis

$$d'_{A/R_0}(x_1), \ldots, d'_{A/R_0}(x_N), d'_{A/R_0}(y_1), \ldots, d'_{A/R_0}(y_m).$$

2. Let $f : A \to B$ be a homomorphism of $R$-algebras. Then we have an exact sequence

$$\Omega'_{A/R_0} \otimes_A B \xrightarrow{\alpha} \Omega'_{B/R_0} \xrightarrow{\beta} \Omega_{B/A} \to 0$$

of $B$-modules, where the maps $\alpha$ and $\beta$ are defined by $\alpha(d'_{A/R_0}(g) \otimes 1) = d'_{B/R_0}(f(g))$ for $g \in A$ and $\beta(d'_{B/R_0}(g)) = d_{B/A}(g)$ for $g \in B$.
Remark

sequences in (2) and (3) are derived from the following corresponding exact sequences

\[ I/I^2 \xrightarrow{\delta} \Omega'_{A/R_0} \otimes_A B \cong \Omega'_{B/R_0} \to 0 \]

of \( B \)-modules, where the map \( \delta \) is defined by \( \delta(\overline{g}) = d'_{A/R_0}(g) \otimes 1 \) for \( g \in I \).

Proof. The same proofs as in \cite{Kun86} Lemmas A.1-3, A.6-7 work. The exact sequences in (2) and (3) are derived from the following corresponding exact sequences

\[
0 \to \text{Der}_A(B, M) \to \text{Der}'_{R_0}(B, M) \to \text{Der}'_{R_0}(A, M), \\
0 \to \text{Der}'_{R_0}(B, M) \to \text{Der}'_{R_0}(A, M) \to \text{Hom}_B(I/I^2, M),
\]

for any \( B \)-module \( M \). The isomorphism in (4) is derived from the following isomorphisms

\[
\text{Hom}_{S^{-1}A}(\Omega'_{A/R_0} \otimes_A S^{-1}A, M) \cong \text{Hom}_A(\Omega'_{A/R_0}, M) \\
\cong \text{Der}'_{R_0}(A, M) \cong \text{Der}'_{R_0}(S^{-1}A, M)
\]

for any \( S^{-1}A \)-module \( M \).

Remark 2.5. (1) The usual module \( \Omega_{A/R_0} \) of differentials is not a finite \( A \)-module in general when \( A \) is an \( R \)-algebra of finite type. However, the module \( \Omega'_{A/R_0} \) of special differentials becomes a finite \( A \)-module and has similar properties as the module of differentials defined for \( R_0 \)-algebras of finite type.

(2) The universal finite module of differentials is also a module defined with the same motivation (see \cite{Kun86} Section 11). For an \( R_0 \)-algebra \( A \), a finite \( A \)-module \( \tilde{\Omega}_{A/R_0} \) with an \( R_0 \)-derivation \( \tilde{d}_{A/R_0} : A \to \tilde{\Omega}_{A/R_0} \) is called the universal finite module of differentials if it satisfies the following universal property.

- For any \( R_0 \)-derivation \( D : A \to M \) to a finite \( A \)-module \( M \), there exists a unique homomorphism \( \alpha : \tilde{\Omega}_{A/R_0} \to M \) of \( A \)-modules satisfying

\[
D = \alpha \circ \tilde{d}_{A/R_0}.
\]

In other words, \( \tilde{\Omega}_{A/R_0} \) and \( \tilde{d}_{A/R_0} \) satisfy

\[
\text{Hom}_A(\tilde{\Omega}_{A/R_0}, M) \xrightarrow{\cong} \text{Der}_{R_0}(A, M); \ \alpha \mapsto \alpha \circ \tilde{d}_{A/R_0}
\]

for any finite \( A \)-module \( M \).

In contrast to the module \( \Omega'_{A/R_0} \) of special differentials, the universal finite module \( \tilde{\Omega}_{A/R_0} \) of differentials does not necessarily exist.

(3) \( A \) is called an analytic \( R_0 \)-algebra if there exists \( R = R_0[[x_1, \ldots, x_N]] \) for some \( N \geq 0 \) such that \( A \) is a finite \( R \)-algebra. If \( A \) is an analytic \( R_0 \)-algebra, then the universal finite module \( \tilde{\Omega}_{A/R_0} \) of differentials exists. Furthermore, if \( A \) is a finite \( R_0[[x_1, \ldots, x_N]] \)-algebra and if \( \Omega'_{A/R_0} \) is the module of special differentials with respect to \( R = R_0[[x_1, \ldots, x_N]] \), then we have \( \tilde{\Omega}_{A/R_0} \cong \Omega'_{A/R_0} \). This is because we have

\[
\text{Der}'_{R_0}(A, M) = \text{Der}_{R_0}(A, M)
\]

for any finite \( A \)-module \( M \) by Lemma 2.2 in this case. Therefore, we can also see that \( \Omega'_{A/R_0} \) does not depend on the choice of \( R \) as long as \( A \) is finite as an \( R \)-module.
(4) Even if $A$ is an algebra of finite type over $R = R_0[[x_1, \ldots, x_N]]$, $\Omega_{A/R_0}$ does not necessarily exist despite the fact that $\Omega_{\bar{A}/R_0}'$ is a finite $A$-module. We shall see that $\Omega_{A/k}$ and $\Omega_{B/k}$ do not exist for

$$A = k[[x]][y], \quad B = k[[x]][y]/(1 - xy) \simeq k((x)).$$

In fact, since

$$\dim_k(k((x))) = \dim_k(k((x)) \operatorname{Hom}_k(k((x))/k, k((x))) = \operatorname{trdeg}_k k((x)) = \infty,$$

$\operatorname{Der}_k(B)$ is not a finite $B$-module and hence $\tilde{\Omega}_{B/k}$ does not exist. Furthermore, since there is a natural injective map $\operatorname{Der}_k(B, B) \to \operatorname{Der}_k(A, B)$, $\operatorname{Der}_k(A, B)$ is not a finite $A$-module and hence $\tilde{\Omega}_{A/k}$ does not exist (cf. [Kun80 Corollary 11.10]).

Remark 2.6. (1) Let $A$ be a ring. For a non-negative integer $\ell$, and for subsets $F \subset A$ and $\Delta \subset \operatorname{Der}(A)$, we denote by $J_\ell(F; \Delta)$ the ideal of $A$ generated by the determinants $\det(D_i(f_j))_{1 \leq i, j \leq \ell}$ of all the matrices $(D_i(f_j))_{1 \leq i, j \leq \ell}$ of size $\ell$ with $D_i \in \Delta$ and $f_j \in F$.

If $I$ is an ideal of $A$ generated by $f_1, \ldots, f_\ell$ and the $A$-submodule $A\Delta$ of $\operatorname{Der}(A)$ is generated by $D_1, \ldots, D_\ell$ as an $A$-module, then we have

$$J_\ell(I; \Delta) + I = J_\ell(\{f_1, \ldots, f_\ell\}; \{D_1, \ldots, D_\ell\}) + I.$$

(2) Let $A$ be a regular ring and let $P$ be a prime ideal. For an ideal $I$ of $A$ such that $I \subset P$, the following hold (cf. [Mat89 Theorem 30.4]).

(a) $J_\ell(I; \operatorname{Der}(A)) \subset P$ holds for any $\ell > \operatorname{ht}(IA_P)$.

(b) $A_P/IA_P$ is regular if $J_\ell(I; \operatorname{Der}(A)) \not\subset P$ holds for $\ell = \operatorname{ht}(IA_P)$.

Some regular rings satisfy the inverse implication of (b), and such rings are said to satisfy the weak Jacobian condition (WJ) (cf. [Mat89 Section 30]). Rings of finite type over $k$ are classically known to satisfy (WJ), and this is known as the Jacobian criterion for regularity. Matsumura proved in [Mat77] that $R$-algebras of finite type satisfy (WJ)$_k$ when $R = k[[x_1, \ldots, x_N]]$ (see [Mat77 Theorem 9] for a more general result).

(c) Let $A = k[[x_1, \ldots, x_N]][y_1, \ldots, y_m]$ and let $P$ and $Q$ be prime ideals of $A$ such that $Q \subset P$. Then $A/Q$ is regular at $P$ if and only if $J_\ell(Q; \operatorname{Der}_k(A)) \not\subset P$ holds for $\ell = \operatorname{ht}(Q)$. Note that $\operatorname{Der}_k(A) = \operatorname{Der}_k(A)$ holds for $A = k[[x_1, \ldots, x_N]][y_1, \ldots, y_m]$ (cf. Lemma 2.2), and this is a free $A$-module generated by $\frac{\partial}{\partial x_i}$'s and $\frac{\partial}{\partial y_j}$'s.

In [dFEM11 Proposition A.8], the local freeness of $\Omega_{\bar{A}/k}'$ is proved for regular rings $A$ when $R_0 = k$.

Proposition 2.7 ([dFEM11 Proposition A.8]). Suppose that $R_0 = k$. Let $A$ be an $R$-algebra of finite type, and let $\mathfrak{q}$ be a prime ideal of $A$. If $A_\mathfrak{q}$ is regular, then $\Omega_{\bar{A}_\mathfrak{q}/k}'$ is a free $A_\mathfrak{q}$-module of rank $\dim A_\mathfrak{q} + \dim_k(k((q))/k)$.

In the proof of [dFEM11 Proposition A.8], the following statement is proved.

Proposition 2.8 ([dFEM11 Proposition A.8]). Suppose that $R_0 = k$. Let $S := R[y_1, \ldots, y_m]$, and let $P$ and $Q$ be prime ideals of $S$ with $P \subset Q$. Let $A := S/P$ and let $\mathfrak{q}$ be the prime ideal of $A$ corresponding to $Q$. If $A_\mathfrak{q}$ is regular, then the sequence

$$0 \to PS_Q/P^2S_Q \to \Omega_{\bar{A}_\mathfrak{q}/k}' \otimes_S A_\mathfrak{q} \to \Omega_{\bar{A}_\mathfrak{q}/k}' \to 0$$

obtained by Proposition 2.4.3 is exact and splits.
Remark 2.9. (1) By Propositions 2.7 and 2.8, we have
\[ \dim S - \text{ht} P = \dim A_q + \dim_{k(q)}(\Omega'_{k(q)/k}). \]
It shows that \( \text{coht} P = \dim S - \text{ht} P \) is independent of the choice of \( S \).
Lemma 2.11(1) below also proves this independence. Lemma 2.11(1) also gives a ring-theoretic interpretation of this value without using \( \Omega' \).

(2) In Proposition 2.7, we note that the rank of \( \Omega'_{A/k} \) is not equal to \( \dim A \) in general.

If we set \( S = k[[x]][y] \) and \( P = Q = (xy - 1) \) in Proposition 2.8, then we have \( A_q = A = k((x)) \). (1) shows that the rank of \( \Omega'_{A/k} \) is equal to one even though \( \dim A = 0 \).

Definition 2.10. Suppose that \( R_0 \) is a Noetherian domain. Let \( X \) be an irreducible scheme of finite type over \( R \) and let \( X_{\text{red}} \) be its underlying reduced subscheme. Let \( X_{\text{red}} \to \text{Spec} R \) be the structure morphism, and let \( p \in \text{Spec} R \) be the image of the generic point of \( X_{\text{red}} \). Then we define
\[ \dim' X := \text{trdeg}_{k(p)} K(X_{\text{red}}) + \dim R - \text{ht} p, \]
where \( k(p) := R_p/pR_p \) and \( K(X_{\text{red}}) \) is the function field of \( X_{\text{red}} \). When \( X = \text{Spec} A \) is an affine scheme, we also define \( \dim' A := \dim' X \).

Lemma 2.11. Suppose that \( R_0 \) is a Noetherian domain. Let \( X = \text{Spec} A \) be an irreducible affine scheme of finite type over \( R \). Then the following hold.

(1) If \( A \) is a domain, then \( \dim S - \text{ht} P = \dim' A \) holds for any representation \( A \simeq S/P \) with \( S = R[y_1, \ldots, y_m] \) and a prime ideal \( P \) of \( S \).
(2) \( \dim A \leq \dim' A \) holds.
(3) Suppose that \( R_0 = k \) and \( A \) is a domain. Then \( \dim' A = \dim_K(\Omega'_{K/k}) \) holds for \( K = \text{Frac} A \).
(4) Suppose that \( R_0 = k \) or \( R_0 = k[t] \). Then \( \dim' A = \dim A \) holds if \( A/m = k \) holds for some maximal ideal \( m \) of \( A \).
(5) Suppose that \( R \) is a universally catenary ring. If \( I = (f_1, \ldots, f_c) \) is an ideal of \( A \) generated by \( c \) elements, then \( \dim'(A/I) \geq \dim' A - c \) holds for any minimal prime \( Q \) of \( I \).
(6) Suppose that \( R \) is a universally catenary ring. Let \( p \) and \( q \) be prime ideals of \( A \) such that \( p \subset q \). Let \( B := A/p \) and \( \overline{q} := q/p \in \text{Spec} B \). Then we have \( \text{ht} p = \dim A_q - \dim B_{\overline{q}} = \dim' A - \dim' B \).

Proof. First, we prove (1). Let \( p \) be the image of \( P \) according to the map \( \text{Spec} S \to \text{Spec} R \). Then by [Mat89], Theorem 15.5] (and the definition below in [Mat89]), it follows that
\[ \text{trdeg}_{\text{Frac} R}(\text{Frac} S) - \text{ht} P = \text{trdeg}_{k(p)}(\text{Frac}(S/P)) - \text{ht} p. \]
Therefore, the assertion follows from \( \text{trdeg}_{\text{Frac} R}(\text{Frac} S) = m = \dim S - \dim R \).
(2) follows from (1) and the inequality \( \dim(S/P) + \text{ht} P \leq \dim S \). (3) follows from (1) and Remark 2.9(1).

Next, we prove (4) for \( R_0 = k[t] \) (the case when \( R_0 = k \) is similar). We may assume that \( A \) is a domain. Take \( S \) and \( P \) as in (1). Then it is sufficient to prove
\[ \dim S - \text{ht} P = \dim(S/P). \]
Let \( M \) be the maximal ideal of \( S \) corresponding to \( m \). Since \( S/M = k \), \( M \) is of the form
\[ M = (t - a, x_1, \ldots, x_N, y_1 - b_1, \ldots, y_m - b_m), \]
with \( a, b_i \in k \). Therefore we have \( \dim S = N + m + 1 = \text{ht} M \). Since \( S \) is a catenary ring, we also have \( \dim(S/P) = \text{ht} M - \text{ht} P \), which proves \( \dim S - \text{ht} P = \dim(S/P) \).
(5) follows from (1) and Krull’s height theorem. (6) also follows from (1). □
Proposition 2.12. Suppose that $R_0 = k$. Let $A$ be a domain of finite type over $R$, and let $q_1$ and $q_2$ be prime ideals of $A$ such that $q_1 \subset q_2$. Let $B := A/q_1$ and $\overline{q}_2 := q_2/q_1 \in \text{Spec } B$. If both $A_{q_2}$ and $B_{\overline{q}_2}$ are regular, then the sequence
\[ 0 \to q_1 A_{q_2}/q_1^2 A_{q_2} \to \Omega'_{A_{q_2}/k} \otimes A_{q_2} B_{\overline{q}_2} \to \Omega'_{B_{\overline{q}_2}/k} \to 0 \]
induced by Proposition 2.8(3) is exact and splits.

Proof. By Proposition 2.3(3), the sequence
\[ q_1 A_{q_2}/q_1^2 A_{q_2} \to \Omega'_{A_{q_2}/k} \otimes A_{q_2} B_{\overline{q}_2} \to \Omega'_{B_{\overline{q}_2}/k} \to 0 \]
is exact. Since $A_{q_2}$ and $B_{\overline{q}_2}$ are regular, it follows from Proposition 2.7 that
\[ \Omega'_{A_{q_2}/k} \otimes A_{q_2} B_{\overline{q}_2} \cong B_{\overline{q}_2}^{\text{dim} A} \quad \text{and} \quad \Omega'_{B_{\overline{q}_2}/k} \cong B_{\overline{q}_2}^{\text{dim} B}. \]
Therefore $\text{Ker}(\delta)$ is a free $B_{\overline{q}_2}$-module of rank equal to $\text{dim} A - \text{dim} B$. On the other hand, since $A_{q_2}$ and $B_{\overline{q}_2}$ are regular, $q_1 A_{q_2}/q_1^2 A_{q_2}$ is also a free $B_{\overline{q}_2}$-module of rank equal to $\text{ht}(q_1)$, which is equal to $\text{dim} A - \text{dim} B$ by Lemma 2.11(6). Hence, the induced surjective map $q_1 A_{q_2}/q_1^2 A_{q_2} \to \text{Ker}(\delta)$ should be an isomorphism. We complete the proof. \(\square\)

Definition 2.13. (1) Let $X$ be a scheme over $\text{Spec } R$. Then due to Proposition 2.4(3), there exists a quasi-coherent sheaf $\Omega'_{X/R_0}$ satisfying $\Omega'_{X/R_0}(U) \cong \Omega'_{\mathcal{O}_X(U)/R_0}$ for any affine open subset $U \subset X$. Note that $\Omega'_{X/R_0}$ is coherent by Proposition 2.4(3) when $X$ is of finite type over $\text{Spec } R$. The sheaf $\Omega'_{X/R_0}$ is called the sheaf of special differentials. We denote $\Omega^n_{X/R_0} := \Lambda^n \Omega'_{X/R_0}$ for a non-negative integer $n$.

(2) Suppose that $R_0 = k$ and $X$ is a scheme of finite type over $\text{Spec } R$. Let $n$ be a non-negative integer. Suppose that any irreducible component $X_i$ of $X$ satisfies $\text{dim } X_i = n$, where $\text{dim } X_i$ is defined in Definition 2.10. Then we denote $\text{Jac}^i_{X/k} := \text{Fitt}^n(\Omega'_{X/k})$ and it is called the special Jacobian ideal of $X$ (see [Eis95, Section 20.2] for the definition of the Fitting ideal). We note that $\text{dim } X_i = \text{dim } X_i$ holds if $X_i$ contains a $k$-point by Lemma 2.11(4).

(3) Suppose that $R_0 = k$ and $X$ is an integral normal scheme of finite type over $\text{Spec } R$. Let $n = \text{dim } X$ and let $i : X_{\text{reg}} \to X$ be the inclusion map from the regular locus $X_{\text{reg}}$ of $X$. Then the special canonical sheaf $\omega^i_{X/k}$ is defined by $\omega^i_{X/k} = i_*(\Omega^n_{X_{\text{reg}}/k})$.

(4) Under the same setting in (3), a Weil divisor $K_X$ on $X$ satisfying $\mathcal{O}_X(K_X)|_{X_{\text{reg}}} \cong \Omega^n_{X_{\text{reg}}/k}$ is called the canonical divisor on $X$. The canonical divisor $K_X$ is defined up to linear equivalence. Note that $\omega^i_{X/k} \cong \mathcal{O}_X(K_X)$ holds as usual. In fact, since we have $\text{codim}_X(X \setminus X_{\text{reg}}) \geq 2$ by the normality of $X$, it follows that
\[ \Gamma(V, \mathcal{O}_X(K_X)) = \Gamma(V \cap X_{\text{reg}}, \mathcal{O}_X(K_X)) = \Gamma(V, i_*i^*\mathcal{O}_X(K_X)) \]
for any open subset $V \subset X$.

(5) Under the same setting in (3), we say that $X$ is $Q$-Gorenstein if $\omega^{[r]}_{X/k} := (\omega^i_{X/k})^{\otimes r} \cong \mathcal{O}_X(rK_X)$ is an invertible sheaf for some $r \in \mathbb{Z}_{>0}$. In this case, we have a canonical map $\eta_r : (\Omega^n_{X/k})^{\otimes r} \to \omega^{[r]}_{X/k}$.

Since $\omega^{[r]}_{X/k}$ is an invertible sheaf, an ideal sheaf $\mathfrak{n}_{r,X} \subset \mathcal{O}_X$ is uniquely determined by $\text{Im}(\eta_r) = \mathfrak{n}_{r,X} \otimes \mathcal{O}_X \omega^{[r]}_{X/k}$. The ideal sheaf $\mathfrak{n}_{r,X}$ is called the $r$-th Nash ideal of $X$. 
Remark 2.14. As with the usual Jacobian ideals for varieties over $k$, the special Jacobian ideal can be locally described by the Jacobian matrix. Let $S = R[y_1, \ldots, y_m]$ with $R = R_0[[x_1, \ldots, x_N]]$ and let $A = S/I$ for some ideal $I = (f_1, \ldots, f_r) \subset S$. Then by Proposition 2.4(3), we have an exact sequence

$$I/I^2 \to \Omega^1_{S/R_0} \otimes_S A \to \Omega^1_{A/R_0} \to 0.$$ 

Here, we have $\Omega^1_{S/R_0} \simeq S^{\oplus \dim R} + m$ with basis $d'x_i$'s and $d'y_j$'s. Furthermore, for $f \in I$, we have

$$d'f = \sum_{i=1}^N \frac{\partial f}{\partial x_i} d'x_i + \sum_{j=1}^m \frac{\partial f}{\partial y_j} d'y_j.$$ 

Therefore, we have

$$Fitt^1(\Omega^1_{A/R_0}) = (\mathcal{J}_{N+m-n}(I; \text{Der}_{R_0}(S)) + I)/I.$$ 

Note here that $\text{Der}_{R_0}(S) = \text{Der}_{R_0}'(S)$ is a free $S$-module generated by $\partial/\partial x_i$'s and $\partial/\partial y_j$'s.

This observation shows that if $R_0 = k$ and $X$ is an integral scheme of finite type over $R$, then $\text{Jac}_{X/k}'$ defines the singular locus of $X$ by Remark 2.14(2)(c). Note here that $\text{ht}I = N + m - \dim X$ holds by Lemma 2.11(1).

3. Log pairs

A log pair $(X, a)$ is a normal $\mathbb{Q}$-Gorenstein $k$-variety $X$ and an $\mathbb{R}$-ideal sheaf $a$ on $X$. Here, an $\mathbb{R}$-ideal sheaf $a$ on $X$ is a formal product $a = \prod_{i=1}^s a_i^{r_i}$, where $a_1, \ldots, a_s$ are non-zero coherent ideal sheaves on $X$ and $r_1, \ldots, r_s$ are positive real numbers. For a morphism $Y \to X$ and an $\mathbb{R}$-ideal sheaf $a = \prod_{i=1}^s a_i^{r_i}$ on $X$, we denote by $aO_Y$ the $\mathbb{R}$-ideal sheaf $\prod_{i=1}^s (a_iO_Y)^{r_i}$ on $Y$.

Let $(X, a = \prod_{i=1}^s a_i^{r_i})$ be a log pair. Let $f : X' \to X$ be a proper birational morphism from a normal variety $X'$ and let $E$ be a prime divisor on $X'$. We denote by $K_{X'/X} := K_{X'} - f^*K_X$ the relative canonical divisor. Then the log discrepancy of $(X, a)$ at $E$ is defined as

$$a_E(X, a) := 1 + \text{ord}_E(K_{X'/X}) - \text{ord}_E a,$$

where we define $\text{ord}_E a := \sum_{i=1}^s r_i \text{ord}_E a_i$. The image $f(E)$ is called the center of $E$ on $X$ and we denote it by $c_X(E)$. For a closed point $x \in X$, we define the minimal log discrepancy at $x$ as

$$\text{mld}_x(X, a) := \inf_{c_X(E) = \{x\}} a_E(X, a)$$

if $\dim X \geq 2$, where the infimum is taken over all prime divisors $E$ over $X$ with center $c_X(E) = \{x\}$. It is known that $\text{mld}_x(X, a) \in \mathbb{R}_{\geq 0} \cup \{-\infty\}$ in this case (cf. [KM98, Corollary 2.31]). When $\dim X = 1$, we define $\text{mld}_x(X, a) := \inf_{c_X(E) = \{x\}} a_E(X, a)$ if the infimum is non-negative, and we define $\text{mld}_x(X, a) := -\infty$ otherwise.

Let $R = k[[x_1, \ldots, x_N]]$. By [dFEM11, Appendix A], we can extend the definition above to normal $R$-schemes of finite type. Let $X$ be an integral normal scheme of finite type over $R$. Then the canonical divisor $K_X$ is defined in Definition 2.13(4). Suppose that $X$ is $\mathbb{Q}$-Gorenstein, that is, $rK_X$ is Cartier for some positive integer $r$. Let $f : Y \to X$ be a proper birational morphism over $R$ from a regular scheme $Y$. Then the relative canonical divisor $K_{Y/X}$ of $f$ is defined as the $Q$-divisor supported on the exceptional locus of $f$ such that $rK_Y - f^*(rK_X)$ and $rK_{Y/X}$ are linearly equivalent. We note that $K_{Y/X}$ is uniquely defined as a $\mathbb{Q}$-divisor (cf. [dFEM11, Lemma A.11(ii)]). Therefore, the log discrepancies and the minimal log discrepancies for $k$-varieties defined above can be extended to $\mathbb{Q}$-Gorenstein normal schemes of finite type over $R$, and we use the same notation.
Remark 3.1. Let $X$ be a normal $k$-variety and let $x \in X$ be a closed point. Let $\hat{O}_{X,x}$ be the completion of the local ring $O_{X,x}$ at its maximal ideal. Let $\hat{X} := \text{Spec}(\hat{O}_{X,x})$ and let $\hat{x} \in \hat{X}$ be the closed point. Then for the induced flat morphism $f : \hat{X} \to X$, it follows that
\[
f^*(K_X) = K_{\hat{X}}, \quad f^*(\omega_X) = \omega_{\hat{X}/k},
\]
by [dFEM11] Proposition A.14. Furthermore, for an integer $r$, if $rK_X$ is Cartier, then so is $rK_{\hat{X}}$.

Suppose further that $X$ is $\mathbb{Q}$-Gorenstein. Let $a$ be an $\mathbb{R}$-ideal sheaf on $X$. Then it follows from [Kaw21] Remark 2.6 (cf. [dFEM11] Proposition 2.11) that
\[
mld_{\hat{x}}(\hat{X}, \hat{a}) = mld_x(X, a),
\]
where $\hat{a} := aO_{\hat{X}}$.

4. Arc spaces of $k[[x_1, \ldots, x_N]]$-schemes

In this section, we suppose $R_0 = k$ and $R = k[[x_1, \ldots, x_N]]$, and we discuss the jet schemes and the arc spaces of $R$-schemes of finite type. We refer the reader to [EM09] and [CLNS18] for the theory of jet schemes and arc spaces of $k$-varieties. In this section, we see that the codimensions of cylinders of arc spaces can be defined in the same way as with $k$-varieties.

Let $X$ be a scheme over $k$. Let $(\text{Sch}/k)$ be the category of $k$-schemes and $(\text{Sets})$ the category of sets. For a non-negative integer $m$, we define a contravariant functor $F^X_m : (\text{Sch}/k) \to (\text{Sets})$ by
\[
F^X_m(Y) = \text{Hom}_k(Y \times_{\text{Spec} k} \text{Spec} k[t]/(t^{m+1}), X).
\]
It is known that the functor $F^X_m$ is always represented by a scheme $X_m$ over $k$ (cf. [CLNS18] Ch.3, Proposition 2.1.3).

For $m \geq n \geq 0$, the canonical surjective ring homomorphism $k[t]/(t^{m+1}) \to k[t]/(t^{n+1})$ induces a morphism $\pi^X_{mn} : X_m \to X_n$, which is called the truncation morphism. There exist the projective limit and the projections
\[
X_\infty := \lim_{\longrightarrow} X_m, \quad \psi^X_m : X_\infty \to X_m,
\]
and $X_\infty$ is called the arc space of $X$. Then there is a bijective map
\[
\text{Hom}_k(\text{Spec} K, X_\infty) \simeq \text{Hom}_k(\text{Spec} K[[t]], X)
\]
for any field $K$ with $k \subset K$. For $m \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, we denote by $\pi^X_m : X_m \to X$ the canonical truncation morphism. For $m \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and a morphism $f : Y \to X$ of schemes over $k$, we denote by $f_m : Y_m \to X_m$ the morphism induced by $f$. We often abbreviate $\pi^X_{mn}$, $\pi^X_m$ and $\psi^X_m$ to $\pi_{mn}$, $\pi_m$ and $\psi_m$, respectively when no confusion can arise.

If $X$ is a scheme of finite type over $k$, then so is $X_m$ (cf. [EM09] Proposition 2.2). In this paper, we deal with a scheme of finite type over $R = k[[x_1, \ldots, x_N]]$.

Proposition 4.1 (cf. [Ish09] Corollary 4.2)]. Let $X$ be a scheme of finite type over $R = k[[x_1, \ldots, x_N]]$. Then the following hold.

1. $X_m$ is a scheme of finite type over $R$.
2. For any $m \geq n \geq 0$, the truncation map $\pi_{mn} : X_m \to X_n$ is a morphism of finite type.

Proof. We omit the proof because we will give a complete proof for Proposition 5.4 which deals with a more complicated case. See also Remark 4.2 below.

Remark 4.2. The same arguments in Lemmas 5.1 and 5.2 give a local description of $X_m$ as follows.
Lemma 4.4. Let $S := k[[x_1, \ldots, x_N]][y_1, \ldots, y_M]$ and $A := \text{Spec} S$. Then we have $A_m \simeq \text{Spec} S_m$, where 

$$S_m := k[[x_1^{(0)}, \ldots, x_N^{(0)}], y_{j}^{(0)}, \ldots, y_{j'}^{(0)} | 1 \leq j \leq N, 1 \leq j' \leq M].$$

Furthermore, for $m \geq n \geq 0$, the truncation map $\pi_{mn} : A_m \to A_n$ is induced by the ring inclusion $S_n \hookrightarrow S_m$.

(2) Let $X = \text{Spec}(S/I)$ be the closed subscheme of $A$ defined by an ideal $I = (f_1, \ldots, f_r) \subset S$. For $1 \leq i \leq r$ and $0 \leq \ell \leq m$, we define $F^{(\ell)}_i \in S_m$ as follows:

$$f_i \left( \sum_{\ell=0}^{m} x_1^{(\ell)} t^\ell, \ldots, \sum_{\ell=0}^{m} x_N^{(\ell)} t^\ell, \sum_{\ell=0}^{m} y_1^{(\ell)} t^\ell, \ldots, \sum_{\ell=0}^{m} y_M^{(\ell)} t^\ell \right) \equiv \sum_{\ell=0}^{m} F^{(\ell)}_i t^\ell \pmod{t^{m+1}}.$$

Let

$$I_m := \left( F^{(s)}_i | 1 \leq i \leq r, 0 \leq s \leq m \right) \subset S_m$$

be the ideal of $S_m$ generated by $F^{(s)}_i$’s. Then we have $X_m \simeq \text{Spec}(S_m/I_m)$. Furthermore, for $m \geq n \geq 0$, the truncation map $\pi_{mn} : X_m \to X_n$ is induced by the ring homomorphism $S_n/I_n \to S_m/I_m$.

A subset $C \subset X_\infty$ is called a cylinder if $C = \psi^{-1}_m(S)$ holds for some $m \geq 0$ and a constructible subset $S \subset X_m$. Typical examples of cylinders appearing in this paper are the contact loci $\text{Cont}^m(a)$ and $\text{Cont}^{2m}(a)$ defined as follows.

**Definition 4.3.**

1. For an arc $\gamma \in X_\infty$ and an ideal sheaf $a \subset \mathcal{O}_X$, the order of $a$ measured by $\gamma$ is defined as follows:

$$\text{ord}_\gamma(a) := \sup \{ r \in \mathbb{Z}_{\geq 0} | \gamma^*(a) \subset (t^r) \},$$

where $\gamma^* : \mathcal{O}_X \to K[[t]]$ is the induced ring homomorphism by $\gamma$. Here $K$ is the field extension of $k$.

2. For $m \in \mathbb{Z}_{\geq 0}$, we define $\text{Cont}^m(a), \text{Cont}^{2m}(a) \subset X_\infty$ as follows:

$$\text{Cont}^m(a) := \{ \gamma \in X_\infty | \text{ord}_\gamma(a) = m \},$$

$$\text{Cont}^{2m}(a) := \{ \gamma \in X_\infty | \text{ord}_\gamma(a) \geq m \}.$$

By definition, we have

$$\text{Cont}^{2m}(a) = \psi^{-1}_{m-1}(Z(a)_{m-1}),$$

where $Z(a)$ is the closed subscheme of $X$ defined by the ideal sheaf $a$. Therefore, $\text{Cont}^m(a)$ and $\text{Cont}^{2m}(a)$ are cylinders.

For $m \leq n + 1$, we also define the subsets $\text{Cont}^m(a)_n$ and $\text{Cont}^{2m}(a)_n$ of $X_n$ in the same way.

We denote by $a_X \subset \mathcal{O}_X$ the ideal sheaf

$$a_X := (x_1, \ldots, x_N) \mathcal{O}_X \subset \mathcal{O}_X$$

generated by $x_1, \ldots, x_N \in R$. In this paper, we are interested in arcs contained in the contact locus $\text{Cont}^{2m}(a_X)$. Due to the following lemma, the contact locus $\text{Cont}^{2m}(a_X)_m$ is a scheme of finite type over $k$.

**Lemma 4.4.** Let $X$ be a scheme of finite type over $R = k[[x_1, \ldots, x_N]]$. Then for each $m \geq 0$, the contact locus $\text{Cont}^{2m}(a_X)_m \subset X_m$ is a scheme of finite type over $k$.

**Proof.** The assertion follows from Proposition 1.12. \qed

For the proof of Lemma 4.4, we state Hensel’s lemma in several variables.
Lemma 4.5. Let $K$ be a field. Let $N, s$ and $r$ be non-negative integers with $N + s \geq r$. Let $f_1, \ldots, f_r \in K[[x_1, \ldots, x_N]][x_{N+1}, \ldots, x_{N+s}]$ and let $a_1, \ldots, a_{N+s} \in K[[t]]$. Let $S \subset \{1, \ldots, N + s\}$ be a subset with cardinality $\# S = r$. Let $m$ and $e$ be non-negative integers with $m \geq e$. Suppose that

- $a_1, \ldots, a_N \in (t)$,
- $f_i(a_1, \ldots, a_{N+s}) \in (t^{m+e+1})$ for each $1 \leq i \leq r$, and
- $\det \left( \frac{\partial f_i}{\partial y_j}(a_1, \ldots, a_{N+s}) \right)_{1 \leq i \leq r, \ j \in S} \notin (t^{e+1})$.

Then the following hold.

1. There exist $b_1, \ldots, b_{N+s} \in K[[t]]$ such that
   - $f_i(b_1, \ldots, b_{N+s}) = 0$ for each $1 \leq i \leq r$, and
   - $a_j - b_j \in (t^{m+1})$ for each $1 \leq j \leq N + s$.

Furthermore, for $b_1, \ldots, b_{N+s} \in K[[t]]$ and $b'_1, \ldots, b'_{N+s} \in K[[t]]$ with the above two conditions, if

- $b_j - b'_j \in (t^{m+2})$ holds for each $j \in \{1, \ldots, N + s\} \setminus S$,

then

- $b_j - b'_j \in (t^{m+2})$ holds also for each $j \in S$.

2. Moreover, for any sequence $(a'_j \in K[[t]] \mid j \in \{1, \ldots, N + s\} \setminus S)$ satisfying $a'_j - a_j \in (t^{m+1})$, there exist $b_1, \ldots, b_{N+s} \in K[[t]]$ satisfying the following conditions:
   - $f_i(b_1, \ldots, b_{N+s}) = 0$ for each $1 \leq i \leq r$,
   - $a'_j - b_j \in (t^{m+2})$ for each $j \in \{1, \ldots, N + s\} \setminus S$, and
   - $a_j - b_j \in (t^{m+2})$ for each $j \in S$.

Proof. When $N = 0$ and $f_1, \ldots, f_r \in K[x_1, \ldots, x_s]$, the assertions are proved in the proof of [DL99, Lemma 4.1] (cf. [EM09, Proposition 4.1]). The same proof works in our setting. 

Remark 4.6. The same statement as in Lemma 4.5 holds even when we replace $K[[t]][x_1, \ldots, x_N][x_{N+1}, \ldots, x_{N+s}]$ with $K[x_{N+1}, \ldots, x_{N+s}][[t]]$. This version will be used in the proof of Proposition 4.31.

Lemma 4.7. Let $N, s$ and $r$ be non-negative integers with $N + s \geq r$. Let $R = k[[x_1, \ldots, x_N]]$ and let $I = R[y_1, \ldots, y_s]$. Let $J = (F_1, \ldots, F_r)$ be the ideal generated by elements $F_1, \ldots, F_r \in S$, and let $M = \text{Spec}(S/I)$. Let $\mathfrak{o}_M \subset \mathcal{O}_M$ be the ideal sheaf generated by $x_1, \ldots, x_N \in R$. Let $\mathfrak{T}_i = \text{Fitt}_i(N^{N+s-r}Y_{M/k})$. Then, for non-negative integers $m$ and $e$ with $m \geq e$, the following hold.

1. It follows that
   \[
   \psi_m \left( \text{Cont}_c(\mathfrak{T}) \cap \text{Cont}_{\geq 1}(\mathfrak{o}_M) \right) = \pi_{m+e, m} \left( \text{Cont}^c(\mathfrak{T})_{m+e} \cap \text{Cont}_{\geq 1}(\mathfrak{o}_M)_{m+e} \right).
   \]

2. $\pi_{m+1, m} : M_{m+1} \to M_m$ induces a piecewise trivial fibration
   \[
   \psi_{m+1} \left( \text{Cont}_c(\mathfrak{T}) \cap \text{Cont}_{\geq 1}(\mathfrak{o}_M) \right) \to \psi_m \left( \text{Cont}^c(\mathfrak{T}) \cap \text{Cont}_{\geq 1}(\mathfrak{o}_M) \right)
   \]
   with fiber $\mathfrak{A}_{N+s-r}$.

Proof. Let $J := J(I; \text{Der}_k(S)) \subset S$. Then we have $\mathfrak{T} = (J + I)/I$ by Remark 2.14. Note here that $\text{Der}_k(S) = \text{Der}_k'(S)$ is generated by $\partial/\partial x_i$’s and $\partial/\partial y_j$’s. Therefore, (1) follows from the first assertion of Lemma 4.5(1). Furthermore, (2) follows from Lemma 4.5(2) and the second assertion of Lemma 4.5(1).

The following proposition is a formal power series ring version of [EM09, Proposition 4.1].

Proposition 4.8. Let $X$ be an integral scheme of finite type over $R = k[[x_1, \ldots, x_N]]$ of $\text{dim} X = n$. Then there exists a positive integer $c$ such that the following hold for non-negative integers $m$ and $e$ with $m \geq ce$. 

We have
\[
\psi_m \left( \text{Cont}^e(\text{Jac}_{X/k}) \cap \text{Cont}^{\geq 1}(\mathfrak{a}_X) \right) = \pi_{m+e, m} \left( \text{Cont}^e(\text{Jac}_{X/k})_{m+e} \cap \text{Cont}^{\geq 1}(\mathfrak{a}_X)_{m+e} \right).
\]

(2) \(\pi_{m+1, m} : X_{m+1} \to X_m\) induces a piecewise trivial fibration
\[
\psi_{m+1} \left( \text{Cont}^e(\text{Jac}_{X/k}) \cap \text{Cont}^{\geq 1}(\mathfrak{a}_X) \right) \to \psi_m \left( \text{Cont}^e(\text{Jac}_{X/k}) \cap \text{Cont}^{\geq 1}(\mathfrak{a}_X) \right)
\]
with fiber \(k^n\).

**Proof.** We omit the proof. See the proof of Proposition 5.9 to see how it can be reduced to the complete intersection case proved in Lemma 4.7. Note that we may assume that \(X\) has a \(k\)-point and hence we have \(\dim' X = \dim X = n\) by Lemma 2.11 (4). Otherwise, we have \(\text{Cont}^{\geq 1}(\mathfrak{a}_X) = \emptyset\) (cf. Lemma 4.3), and the assertions are clear. \(\Box\)

**Remark 4.9.** Proposition 4.8 is a formal power series ring version of [EM09, Proposition 4.1]. Note that in [EM09, Proposition 4.1], they prove that \(c = 1\) satisfies the statement. However, the weaker statement as in Proposition 4.8 using \(c\) is enough for our later use.

By Proposition 4.8, the codimension of cylinder contained in \(\text{Cont}^{\geq 1}(\mathfrak{a}_X)\) is well-defined as follows.

**Definition 4.10.** Let \(X\) be an integral scheme of finite type over \(R = k[[x_1, \ldots, x_N]]\) and let \(C \subset X_\infty\) be a cylinder contained in \(\text{Cont}^{\geq 1}(\mathfrak{a}_X)\).

1. Assume that \(C \subset \text{Cont}^e(\text{Jac}_{X/k})\) for some \(e \in \mathbb{Z}_{\geq 0}\). Then we define the codimension of \(C\) in \(X_\infty\) as
   \[
   \text{codim}(C) := (m + 1) \dim X - \dim(\psi_m(C))
   \]
   for any sufficiently large \(m\). This definition is well-defined by Proposition 4.8.

2. In general, we define the codimension of \(C\) in \(X_\infty\) as follows:
   \[
   \text{codim}(C) := \min_{e \in \mathbb{Z}_{\geq 0}} \text{codim} \left( C \cap \text{Cont}^e(\text{Jac}_{X/k}) \right).
   \]

By convention, \(\text{codim}(C) = \infty\) if \(C \cap \text{Cont}^e(\text{Jac}_{X/k}) = \emptyset\) for any \(e \geq 0\).

The following theorem is a formal power series ring version of [EM09, Theorem 7.4].

**Theorem 4.11.** Let \(X\) be a \(\mathbb{Q}\)-Gorenstein integral normal scheme of finite type over \(R = k[[x_1, \ldots, x_N]]\). Let \(x\) be a \(k\)-point of \(X\) and let \(m_x \subset \mathcal{O}_X\) be the corresponding maximal ideal sheaf. Let \(r\) be a positive integer such that \(rK_X\) is Cartier. Let \(a\) be a non-zero ideal sheaf on \(X\) and \(\delta\) a positive real number. Then we have

\[
\mld_x(X, a^\delta) = \inf_{w, m \in \mathbb{Z}_{\geq 0}} \left\{ \text{codim}(C_{w, m}) - \frac{m}{r} - \delta w \right\}
\]
\[
= \inf_{w, m \in \mathbb{Z}_{\geq 0}} \left\{ \text{codim}(C'_{w, m}) - \frac{m}{r} - \delta w \right\},
\]
where
\[
C_{w, m} := \text{Cont}^w(a) \cap \text{Cont}^m(n_{r, X}) \cap \text{Cont}^{\geq 1}(m_x),
\]
\[
C'_{w, m} := \text{Cont}^{\geq w}(a) \cap \text{Cont}^m(n_{r, X}) \cap \text{Cont}^{\geq 1}(m_x).
\]

**Proof.** The assertions are formal power series ring versions of Theorem 7.4 and Remark 7.5 in [EM09], and their proofs also work in this setting by making the following modifications:
• Replacing $\Omega_{-/k}$ with $\Omega'_{-/k}$, and Jac$_{-/k}$ with Jac'$_{-/k}$.
• Considered cylinders $C$ are contained in $\text{Cont}^{\geq 1}(\mathfrak{a}_X)$.

Theorem 7.4 in [EM09] is a consequence of Lemma 7.3 in [EM09]. The key ingredients of the proof of Lemma 7.3 in [EM09] are
• Theorem 6.2 and Corollary 6.4 in [EM09], and
• Proposition 5.11 in [EM09].

Theorem 6.2 and Corollary 6.4 in [EM09] are the codimension formula as in Proposition 5.43, and they are formal consequences of Proposition 4.4(i) in [EM09]. Proposition 4.4(i) in [EM09] is still valid in our setting by replacing $\Omega_X$ with $\Omega'_{X/k}$ due to Lemma 2.2 (see Lemma 5.13 for the detailed argument).

Proposition 5.11 in [EM09] is a proposition on codimensions as in Proposition 5.36, and it is a consequence of Lemma 5.1 and Corollary 5.2 in [EM09] (Corollary 5.2 is a corollary of Lemma 5.1). The proof of Lemma 5.1 in [EM09] still works in our setting by replacing Jac$_{-/k}$ with Jac'$_{-/k}$. The only important point is that the ideal Jac'$_{X/k}$ defines the singular locus of $X$ even when $X$ is an integral scheme of finite type over $R$ (cf. Remark 2.14).

Besides, Lemma 6.1 in [EM09] is used in the proof of Lemma 7.3 in [EM09], and Proposition 3.2 in [EM09] is used in the proof of Corollary 5.2 in [EM09]. Proposition 3.2 and Lemma 6.1 in [EM09] are formal consequences of the valuative criterion of properness, and their proofs work in our setting.

5. Arc spaces of $k[t]$-schemes

In this section, we deal with the arc spaces of $k[t]$-schemes. Let $X$ be a scheme over $k[t]$. For a non-negative integer $m$, we define a contravariant functor $F^X_m : (\text{Sch}/k) \to (\text{Sets})$ by

$$F^X_m(Y) = \text{Hom}_{k[t]}(Y \times_{\text{Spec } k \text{ Spec } k[t]/(t^{m+1})} X).$$

By the same argument as in [CLNS18] Ch.4. Theorem 3.2.3], we can see that the functor $F^X_m$ is always represented by a scheme $X_m$ over $k$. We shall use the same symbols $X_\infty$, $\pi_{mn}$, $\psi_m$ and $\pi_m$ as in Section 4 also for this setting.

In this section, we deal with the following two categories of $k[t]$-schemes:

1. $X$ is a scheme of finite type over $k[t][[x_1, \ldots, x_N]]$.
2. $X$ is an affine scheme of the form $X = \text{Spec}(k[x_1, \ldots, x_N][[t]]/I)$.
3. $X$ is a scheme of finite type over $k[t]$.

In [NS22], we dealt with the arc spaces of $X$ in (3). However, in this paper, we need to work on the arc spaces of $X$ in (1) and (2). We also note that Sebag in [Seb04] deals with formal $k[[t]]$-schemes of finite type, and this theory can be applied to (2) and (3) (see also [CLNS18] for this theory).

In Subsection 5.1, we discuss case (1). In Subsection 5.2, we discuss case (2), where we will not deal with formal $k[[t]]$-schemes in general but deal with only affine schemes in a minimum way.

5.1. Arc spaces of $k[t][[x_1, \ldots, x_N]]$-schemes. In this subsection, we suppose $R_0 = k[t]$ and $R = k[t][[x_1, \ldots, x_N]]$, and we discuss the arc spaces of $R$-schemes of finite type.
5.1.1. Arc spaces. First, we prove that if $X$ is a scheme of finite type over $R$, then so is $X_m$.

**Lemma 5.1.** Let $S := k[t][[x_1, \ldots, x_N]][y_1, \ldots, y_M]$ and let $A := \text{Spec } S$. Then we have $A_m \cong \text{Spec } S_m$, where 

$$S_m := k[[x_1^{(0)}, \ldots, x_N^{(0)}], \ldots, x_j^{(m)}, \ldots, y_j^{(0)}, \ldots, y_j^{(m)} | 1 \leq j \leq N, 1 \leq j' \leq M].$$

Furthermore, for $m \geq n \geq 0$, the truncation map $\pi_{nm} : A_m \to A_n$ is induced by the ring inclusion $S_m \hookrightarrow S_n$.

**Proof.** Let $Y := \text{Spec } C$ be an affine scheme over $k$. We shall give a natural bijective map 

$$\Phi : \text{Hom}_{k[t]}(S, C[t]/(t^{m+1})) \to \text{Hom}_k(S_m, C).$$

For each $0 \leq i \leq m$, we denote by $p_i$ the projection 

$$p_i : C[t]/(t^{m+1}) \to C; \quad c_0 + c_1 t + \cdots + c_m t^m \mapsto c_i.$$ 

For $\alpha \in \text{Hom}_{k[t]}(S, C[t]/(t^{m+1}))$, we define $\Phi(\alpha) \in \text{Hom}_k(S_m, C)$ as follows. First, we define the ring homomorphism $\alpha'_0 : S_0 \to C$ as the composition 

$$S_0 \hookrightarrow S \xrightarrow{\alpha} C[t]/(t^{m+1}) \xrightarrow{p_0} C.$$ 

Then we define $\alpha' : S_m \to C$ as the ring homomorphism uniquely determined by the following conditions:

- $\alpha'(f) = \alpha'_0(f)$ holds for any $f \in S_0$.
- $\alpha'(x_j^{(s)}) = p_s(\alpha(x_j))$ holds for each $1 \leq j \leq N$ and $1 \leq s \leq m$.
- $\alpha'(y_j^{(s)}) = p_s(\alpha(y_j))$ holds for each $1 \leq j \leq M$ and $1 \leq s \leq m$.

Then we define $\Phi(\alpha) = \alpha'$.

Next, we define the inverse map 

$$\Psi : \text{Hom}_k(S_m, C) \to \text{Hom}_{k[t]}(S, C[t]/(t^{m+1})).$$

We set 

$$S'_m := k[x_1^{(1)}, \ldots, x_j^{(m)} | 1 \leq j \leq N][[x_1^{(0)}, \ldots, x_N^{(0)}]],$$

$$S''_m := k[[x_1^{(0)}, \ldots, x_N^{(0)}], x_j^{(1)}, \ldots, x_j^{(m)} | 1 \leq j \leq N].$$

We define a ring homomorphism $\Lambda_1 : k[t][x_1, \ldots, x_N] \to S'_m[t]/(t^{m+1})$ by 

$$\Lambda_1(t) = t, \quad \Lambda_1(x_j) = x_j^{(0)} + x_j^{(1)} t + \cdots + x_j^{(m)} t^m.$$ 

Since $\Lambda_1((x_1, \ldots, x_N)) \subset (x_1^{(0)}, \ldots, x_N^{(0)}, t)$ holds, $\Lambda_1$ induces a ring homomorphism $\Lambda_2 : k[t][[x_1, \ldots, x_N]] \to S'_m[t]/(t^{m+1})$. Note here that its image is contained in $S''_m[t]/(t^{m+1})$. Furthermore, $S''_m[t]/(t^{m+1})$ is a subring of $S_m[t]/(t^{m+1})$. Therefore, we have a ring homomorphism $\Lambda_3 : k[t][[x_1, \ldots, x_N]] \to S_m[t]/(t^{m+1})$. Then we define $\Lambda : S \to S_m[t]/(t^{m+1})$ as the ring homomorphism uniquely determined by the following conditions:

- $\Lambda(f) = \Lambda_3(f)$ holds for any $f \in k[t][[x_1, \ldots, x_N]]$.
- $\Lambda(y_j) = y_j^{(0)} + y_j^{(1)} t + \cdots + y_j^{(m)} t^m$ holds for each $1 \leq j \leq M$.

Then $\Lambda$ is a $k[t]$-ring homomorphism.

For $\beta \in \text{Hom}_k(S_m, C)$, we define $\Psi(\beta) \in \text{Hom}_{k[t]}(S, C[t]/(t^{m+1}))$ as the composition 

$$S \xrightarrow{\Lambda} S_m[t]/(t^{m+1}) \xrightarrow{\beta} C[t]/(t^{m+1}),$$

where $S_m[t]/(t^{m+1}) \to C[t]/(t^{m+1})$ is the $k[t]$-ring homomorphism induced by $\beta : S_m \to C$.

By the construction of $\Phi$ and $\Psi$, if $\alpha' = \Phi(\alpha)$ and $\beta' = \Psi(\beta)$, then they satisfy the following:
Therefore, we have \( \Psi = \text{id} \) and \( \Phi \circ \Psi = \text{id} \). Hence, \( F_m^X \) is represented by \( \text{Spec} S_m \).
The second assertion follows from the construction of \( A_m \).

**Lemma 5.2.** We take over the notation in Lemma 5.1. Let \( X = \text{Spec}(S/I) \) be the closed subscheme of \( A \) defined by an ideal \( I = (f_1, \ldots, f_r) \subset S \). For \( 1 \leq i \leq r \) and \( 0 \leq m \leq n \), we define \( F_i^{(m)} \in S_m \) as follows:

\[
\sum_{\ell=0}^{m} x_i^{(m)} t^{\ell}, \ldots, \sum_{\ell=0}^{m} x_N^{(m)} t^{\ell}, \sum_{\ell=0}^{m} y_1^{(m)} t^{\ell}, \ldots, \sum_{\ell=0}^{m} y_M^{(m)} t^{\ell} \equiv \sum_{\ell=0}^{m} F_i^{(m)} t^{\ell} \pmod{t^{m+1}}.
\]

Let \( I_m := (F_i^{(s)} \mid 1 \leq i \leq r, \ 0 \leq s \leq m) \subset S_m \) be the ideal of \( S_m \) generated by \( F_i^{(s)} \)'s. Then we have \( X_m = \text{Spec}(S_m/I_m) \). Furthermore, for \( n \geq m \geq 0 \), the truncation map \( \pi_{m,n} : X_m \to X_n \) is induced by the ring homomorphism \( S_n/I_n \to S_m/I_m \).

**Proof.** Let \( Y := \text{Spec} C \) be an affine scheme over \( k \). We can see that the bijective map \( \Phi \) in the proof of Lemma 5.1 induces the bijective map

\[
\text{Hom}_{k[t]}(S/I, C[t]/(t^{m+1})) \to \text{Hom}_k(S_m/I_m, C).
\]

Therefore, the functor \( F_m^X \) is represented by \( \text{Spec}(S_m/I_m) \). The second assertion follows from the construction of \( X_m \).

**Remark 5.3.** More precisely,

\[
\sum_{\ell=0}^{m} x_i^{(m)} t^{\ell}, \ldots, \sum_{\ell=0}^{m} x_N^{(m)} t^{\ell}, \sum_{\ell=0}^{m} y_1^{(m)} t^{\ell}, \ldots, \sum_{\ell=0}^{m} y_M^{(m)} t^{\ell}
\]

in Lemma 5.2 is defined as \( \Lambda(f_i) \in S_m[t]/(t^{m+1}) \), where \( \Lambda \) is defined within the proof of Lemma 5.1.

**Proposition 5.4.** If \( X \) is a scheme of finite type over \( R = k[t][[x_1, \ldots, x_N]] \). Then the following hold.

1. \( X_m \) is a scheme of finite type over \( R \).
2. For any \( m \geq n \geq 0 \), the truncation map \( \pi_{m,n} : X_m \to X_n \) is a morphism of finite type.

**Proof.** Take an affine cover \( X = U_1 \cup \cdots \cup U_s \). Then, \( F_m^X \) is represented by the scheme obtained by gluing the schemes \( (U_i)_m \) constructed in Lemma 5.2 (cf. [EM09, Proposition 2.2]). Therefore, the assertions follow from Lemma 5.2.

Cylinders and the contact loci

\( \text{Cont}^m(a), \text{Cont}^{\geq m}(a) \subset X_{\infty}, \quad \text{Cont}^m(a)_n, \text{Cont}^{\geq m}(a)_n \subset X_n \)

are also defined in this setting in the same way.

We denote by \( a_X \subset O_X \) the ideal sheaf

\[ (x_1, \ldots, x_N) O_X \subset O_X \]

generated by \( x_1, \ldots, x_N \in R \). From the next subsection, we basically work on arcs contained in the contact locus \( \text{Cont}^{\geq 1}(a_X) \). Due to the following lemma, the contact locus \( \text{Cont}^{\geq 1}(a_X)_m \) is a scheme of finite type over \( k \).
Lemma 5.5. Let $X$ be a scheme of finite type over $R = k[t][[x_1, \ldots, x_N]]$. Then the follow hold.

1. For each $m \geq 0$, the contact locus $\mathrm{Cont}^\geq 1(\mathfrak{o}_X)_m \subset X_m$ is a scheme of finite type over $k$.

2. Any $k$-arc of $X$ is contained in $\mathrm{Cont}^\geq 1(\mathfrak{o}_X)$.

Proof. (1) follows from Proposition 2.14.

We shall prove (2). Let $\gamma \in X_\infty$ be a $k$-arc. We may assume that $X$ is affine, and we may write $X = \text{Spec } A$ with $A = S/I$, where

$$S := k[t][[x_1, \ldots, x_N]][y_1, \ldots, y_m]$$

and $I$ is an ideal of $S$. Let $\gamma^* : A \rightarrow k[[t]]$ be the corresponding $k[[t]]$-ring homomorphism. Let $M$ be the kernel of the composite map $S \rightarrow A \xrightarrow{\gamma^*} k[[t]] \rightarrow k$. Since $S/M = k$, $M$ is of the form

$$(t, x_1, \ldots, x_N, y_1 - a_1, \ldots, y_m - a_m)$$

for some $a_i \in k$. It shows that $\gamma^*(\mathfrak{o}_X) \subset (t)$ and hence $\gamma \in \mathrm{Cont}^\geq 1(\mathfrak{o}_X)$. \hfill \Box

Lemma 5.6. Let $n$ be a non-negative integer and let $X$ be a scheme of finite type over $R = k[t][[x_1, \ldots, x_N]]$. Suppose that each irreducible component $X_i$ of $X$ has $\dim' X_i \geq n + 1$ (see Definition 2.10). Let $\gamma \in X_\infty$ be a $k$-arc with $\text{ord}_x(\text{Fitt}^n(\Omega_{X/k[t]}')) < \infty$. Then we have

$$\gamma^* \Omega_{X/k[t]}/T \simeq k[[t]]^\oplus n,$$

where $T$ is the torsion part of $\gamma^* \Omega_{X/k[t]}'$. \hfill \Box

Proof. We may assume that $X$ is affine, and we may write $X = \text{Spec } A$ with $A = S/I$, where

$$S := k[t][[x_1, \ldots, x_N]][y_1, \ldots, y_m]$$

and $I$ is an ideal of $S$. If $P$ is a minimal prime of $I$, then we have

$$\text{ht } P = \dim S - \dim'(S/P) \leq (N + m + 1) - (n + 1) = N + m - n$$

by Lemma 2.11(1). Therefore we have $\text{ht } I \leq N + m - n$.

Let $\gamma^* : A \rightarrow k[[t]]$ be the corresponding $k[[t]]$-ring homomorphism, and let $\bar{\gamma}^* : A \rightarrow k((t))$ be its composition with $k[[t]] \rightarrow k((t))$. Let $q \subset A$ be the kernel of $\bar{\gamma}^*$ and $Q \subset S$ the corresponding prime ideal. Since $\bar{\gamma}^*$ factors through $A_q$, it is sufficient to show that $\Omega_{A_q/k[t]}' \otimes A_q k((t))$ has dimension $n$ as a $k((t))$-vector space.

Let $w_1, \ldots, w_{l'} \in I$ be generators of $I$. Let $M \in M_{N + m, l'}(A_q)$ be the Jacobian matrix with respect to $w_1, \ldots, w_{l'} \in I$ and derivations $\frac{\partial}{\partial x_i}$'s and $\frac{\partial}{\partial y_i}$'s. Then $M$ defines a map $M : A_q^l \rightarrow A_q^{N + m}$ and its cokernel is isomorphic to $\Omega_{A_q/k[t]}'$ by Proposition 2.13(1). Since $\text{ord}_x(\text{Fitt}^n(\Omega_{X/k[t]}')) < \infty$, $M$ has an $(N + m - n)$-minor which is not contained in $qA_q$ (cf. Remark 2.14). Furthermore, since we have

$$\text{ht}(IS_Q) \leq \text{ht } I \leq N + m - n,$$

any $(N + m - n + 1)$-minor of $M$ is contained in $qA_q$ by Remark 2.10(2)(a) (cf. [Mat89] Theorem 30.4). Therefore, the image of $M$ in $M_{N + m, l'}(k((t)))$ has rank $N + m - n$, and it follows that $\Omega_{A_q/k[t]}' \otimes A_q k((t))$ has dimension $n$ as a $k((t))$-vector space. \hfill \Box

Lemma 5.7. Let $n$ and $e$ be non-negative integers and let $X$ be a scheme of finite type over $R = k[t][[x_1, \ldots, x_N]]$. Suppose that each irreducible component $X_i$ of $X$ has $\dim' X_i \geq n + 1$. Let $\gamma \in \text{Cont}^e(\text{Fitt}^n(\Omega_{X/k[t]}'))$ be a $k$-arc. Then we have

$$\gamma^* \Omega_{X/k[t]}' \simeq k[[t]]^\oplus n \oplus \bigoplus_{i} k[t]/(t^e_i)$$
as \( k[[t]] \)-modules with \( \sum_i e_i = e \).

**Proof.** The same proof as in [NS22, Lemma 2.13(1)] works due to Lemma 5.6. \( \square \)

5.1.2. Cylinders and Codimension. In this subsection, we define and discuss the codimensions of cylinders of the arc space of an \( R \)-scheme \( X \) of finite type. We define the codimension only for cylinders contained in the contact locus \( \text{Cont}^{\geq 1}(\sO_X) \), where \( \sO_X \subset \sO_X \) is the ideal sheaf generated by \( x_1, \ldots, x_N \in R \). Due to Lemma 5.5, the contact locus \( \text{Cont}^{\geq 1}(\sO_X)_m \subset X_m \) is a scheme of finite type over \( k \), and hence cylinders contained in \( \text{Cont}^{\geq 1}(\sO_X) \) are easier to handle than the general cylinders.

First, we prove Proposition 5.9 which is necessary for defining the codimension of cylinders.

**Lemma 5.8.** Let \( N, s \) and \( r \) be non-negative integers with \( N + s \geq r \). Let \( R = k[[t]][[x_1, \ldots, x_N]] \) and let \( S = R[y_1, \ldots, y_r] \). Let \( I = (F_1, \ldots, F_r) \) be the ideal generated by elements \( F_1, \ldots, F_r \in S \), and let \( M = \text{Spec}(S/I) \). Let \( \sO_M \subset \sO_X \) be the ideal sheaf generated by \( x_1, \ldots, x_N \in R \). Let \( J = \text{Fitt}^{N+s-r}(\Omega^I_M/S/k[t]) \). For non-negative integers \( m \) and \( e \) with \( m \geq e \), the following hold.

1. It follows that
   \[
   \psi_m \left( \text{Cont}^e(J) \cap \text{Cont}^{\geq 1}(\sO_M) \right) = \pi_{m+e,m} \left( \text{Cont}^e(J)_{m+e} \cap \text{Cont}^{\geq 1}(\sO_M)_{m+e} \right).
   \]
2. \( \pi_{m+1,m} : M_{m+1} \to M_m \) induces a piecewise trivial fibration
   \[
   \psi_{m+1} \left( \text{Cont}^e(J) \cap \text{Cont}^{\geq 1}(\sO_M) \right) \to \psi_m \left( \text{Cont}^e(J) \cap \text{Cont}^{\geq 1}(\sO_M) \right)
   \]
   with fiber \( A^{N+s-r} \).

**Proof.** Let \( J := J_I(\text{Der}_{k[t]}(S)) \subset S \). Then we have \( J = (J + I)/I \) by Remark 2.14. Note here that \( \text{Der}_{k[t]}(S) = \text{Der}_{k[t]}(S) \) is generated by \( \partial/\partial x_i \)’s and \( \partial/\partial y_j \)’s. Therefore, (1) follows from the first assertion of Lemma 4.5(1). Furthermore, (2) follows from Lemma 4.5(2) and the second assertion of Lemma 4.5(1). \( \square \)

**Proposition 5.9.** Let \( n \) be a non-negative integer and let \( X \) be a scheme of finite type over \( R = k[[t]][[x_1, \ldots, x_N]] \). Suppose that each irreducible component \( X_i \) of \( X \) has \( \dim X_i \geq n+1 \). Then there exists a positive integer \( c \) such that the following hold for non-negative integers \( m \) and \( e \) with \( m \geq ce \).

1. It follows that
   \[
   \psi_m \left( \text{Cont}^e(\text{Fitt}^n(\Omega^I_X/S/k[t])) \cap \text{Cont}^{\geq 1}(\sO_X) \right)
   \]
   \[
   = \pi_{m+e,m} \left( \text{Cont}^e(\text{Fitt}^n(\Omega^I_X/S/k[t]))_{m+e} \cap \text{Cont}^{\geq 1}(\sO_X)_{m+e} \right).
   \]
2. \( \pi_{m+1,m} : X_{m+1} \to X_m \) induces a piecewise trivial fibration
   \[
   \psi_{m+1} \left( \text{Cont}^e(\text{Fitt}^n(\Omega^I_X/S/k[t])) \cap \text{Cont}^{\geq 1}(\sO_X) \right)
   \]
   \[
   \to \psi_m \left( \text{Cont}^e(\text{Fitt}^n(\Omega^I_X/S/k[t])) \cap \text{Cont}^{\geq 1}(\sO_X) \right)
   \]
   with fiber \( A^n \).

**Proof.** The same proof as in [NS22, Proposition 2.17] works. We shall give a sketch of the proof.

We may assume that \( X \) is affine, and we may write \( X = \text{Spec}(S/I_X) \), where
\[
S := k[[t]][[x_1, \ldots, x_N]][y_1, \ldots, y_m]
\]
and \( I_X \) is an ideal of \( S \). By the assumption and Lemma 2.11(1), we have
\[
\text{ht} P = \dim S - \dim'(S/P) \leq (N + m + 1) - (n + 1) = N + m - n
\]
for any minimal prime \( P \) of \( I_X \). We set \( r := N + m - n \).
Let $f_1, \ldots, f_d$ be generators of $I_X$. For $1 \leq i \leq r$, we set $F_i := \sum_{j=1}^d a_{ij} f_j$ for general $a_{ij} \in k$. Let $M \subset \text{Spec} \ S$ be the closed subscheme defined by the ideal $I_M := (F_1, \ldots, F_r)$. We denote
\[ I' := (I_M : I_X) \subset S, \quad J := \mathcal{J}_r(I_M; \text{Der}[t](S)) \subset S. \]
Here, we claim that
\[ (\star) \ J \subset \sqrt{I' + I_X^\times} \text{ holds.} \]
We note that if $(\star)$ is true, then the assertions for $X$ can be reduced to those for $M$ by the same argument as in [NS22, Proposition 2.17]. Therefore, the assertions follow from Lemma 5.8.

Let $p$ be a prime ideal satisfying $I_X + I_X' \subset p$. To prove $(\star)$, it is sufficient to show that $S/I_M$ is not regular at $p$. Indeed, if $S/I_M$ is not regular at $p$, then we have
\[ J = \mathcal{J}_r(I_M; \text{Der}[t](S)) \subset \mathcal{J}_r(I_M; \text{Der}[S](S)) \subset p \]
by $ht(I_M S_p) \leq r$ and the Jacobian criterion of regularity (Remark 2.6(a)(b)).

Suppose that the contrary is true. Since any minimal prime $P$ of $I_X$ satisfies $ht P = r$ and $a_{ij} \in k$ are general, for any irreducible component $X_0$ of $X$, there exists an irreducible component $M_0$ of $M$ such that $X_0 \subset M_0$ and $(X_0)_{\text{red}} = (M_0)_{\text{red}}$. Therefore, since $I_M \subset I_X \subset p$ and $M$ is regular at $p$, we have $(I_M)_p = (I_X)_p$. It shows that
\[ (I')_p = (I_M : I_X)_p = ((I_M)_p : (I_X)_p) = S_p, \]
which contradicts $I_X' \subset p$. We complete the proof of $(\star)$.

For an $R$-scheme $X$, a subset $C \subset X_\infty$ is called a cylinder if $C = \psi_m^{-1}(S)$ holds for some $m \geq 0$ and a constructible subset $S \subset X_m$.

**Proposition 5.10.** Let $n$ be a non-negative integer and let $X$ be a scheme of finite type over $R = k[t][[x_1, \ldots, x_N]]$. Suppose that each irreducible component $X_i$ of $X$ has dim $X_i \geq n + 1$. Let $C$ be a cylinder in $X_\infty$ which is contained in $\text{Cont}^{\geq 1}(a_X) \cap \text{Cont}^e(\text{Fitt}^n(\Omega_{X/k[t]}^1))$ for some $e \geq 0$. Then its image $\psi_m(C) \subset X_m$ is a constructible subset for any $m \geq 0$.

**Proof.** Let $S \subset X_\ell$ be a constructible subset such that $\psi_\ell^{-1}(S) = C$. For $m \geq \ell$, we have
\[ \pi_m^{-1}(S) \cap \psi_m(C) = \psi_m(C) = \pi_m^{-1}(S) \cap \psi_m(X_\infty). \]
By the assumption $C \subset \text{Cont}^{\geq 1}(a_X) \cap \text{Cont}^e(\text{Fitt}^n(\Omega_{X/k[t]}^1))$, we also have
\[ \psi_m(C) = \pi_m^{-1}(S) \cap \psi_m(\text{Cont}^{\geq 1}(a_X) \cap \text{Cont}^e(\text{Fitt}^n(\Omega_{X/k[t]}^1))). \]
Let $c$ be the positive integer appearing in Proposition 5.9. Then the constructibility of $\psi_m(C)$ follows from Proposition 5.9(1) if $m \geq \max\{ce, \ell\}$. When $m < \max\{ce, \ell\}$, the constructibility follows from that for $m = \max\{ce, \ell\}$ and Chevalley’s theorem.

We define the codimensions of cylinders $C$ when they satisfy $C \subset \text{Cont}^{\geq 1}(a_X)$.

**Definition 5.11.** Let $n$ be a non-negative integer and let $X$ be a scheme of finite type over $R = k[t][[x_1, \ldots, x_N]]$. Suppose that each irreducible component $X_i$ of $X$ has dim $X_i \geq n + 1$. Let $C \subset X_\infty$ be a cylinder contained in $\text{Cont}^{\geq 1}(a_X)$.

1. Assume that $C \subset \text{Cont}^e(\text{Fitt}^n(\Omega_{X/k[t]}^1))$ for some $e \in \mathbb{Z}_{\geq 0}$. Then we define the codimension of $C$ in $X_\infty$ as
\[ \text{codim}(C) := (m + 1)n - \text{dim}(\psi_m(C)) \]
for any sufficiently large $m$. This definition is well-defined by Proposition 5.9(2).
(2) In general, we define the codimension of $C$ in $X_\infty$ as follows:

$$\text{codim}(C) := \min_{e \in \mathbb{Z}_{\geq 0}} \text{codim}(C \cap \text{Cont}^e(\text{Fitt}^n(\Omega_{X/k[t]}'))).$$

By convention, $\text{codim}(C) = \infty$ if $C \cap \text{Cont}^e(\text{Fitt}^n(\Omega_{X/k[t]}')) = \emptyset$ for any $e \geq 0$.

**Remark 5.12.** The definition of the codimension above depends on the choice of $n$. In Subsection 5.1.3 we fix a non-negative integer $n$, and we use the codimension defined for this $n$.

**Lemma 5.13.** Let $X$ be a scheme of finite type over $R = k[[x_1, \ldots, x_N]]$. Let $p$ and $m$ be non-negative integers with $2p + 1 \geq m \geq p$. Let $\gamma \in X_p(k)$ be a jet. If $\pi_{m,p}^{-1}(\gamma) \neq \emptyset$, it follows that

$$\pi_{m,p}^{-1}(\gamma) \cong \text{Hom}_{k[t]/(t^{p+1})}(\gamma^* \Omega_{X/k[t]}', (t^{p+1})/(t^{m+1})).$$

**Proof.** We may assume that $X$ is affine, and we may write $X = \text{Spec} A$ with an $R$-algebra $A$ of finite type. Let $\gamma^* : A \to k[t]/(t^{p+1})$ be the corresponding $k[t]$-ring homomorphism to $\gamma$. Take any $\alpha \in \pi_{m,p}^{-1}(\gamma)$. Let $\alpha^* : A \to k[t]/(t^{m+1})$ be the corresponding $k[t]$-ring homomorphism. Then for the same reason as in the case of $k$-schemes (cf. [EM09, Proposition 4.4]), we have an isomorphism

$$\pi_{m,p}^{-1}(\gamma) \cong \text{Der}_{k[t]}(A, (t^{p+1})/(t^{m+1}) ; \beta \mapsto \beta^* - \alpha^*).$$

Here, $(t^{p+1})/(t^{m+1})$ in the right-hand side has an $A$-module structure via $\gamma^*$. Then the assertion follows from the isomorphisms

$$\text{Der}_{k[t]}(A, (t^{p+1})/(t^{m+1})) \cong \text{Hom}_A(\Omega_{A/k[t]}', (t^{p+1})/(t^{m+1})).$$

Here, the first equality follows from Lemma 2.2.

5.1.3. Thin and very thin cylinders. We fix a non-negative integer $n$ throughout this subsection.

**Definition 5.14.** Let $X$ be a scheme of finite type over $R = k[[x_1, \ldots, x_N]]$. Suppose that each irreducible component $X_i$ of $X$ has dim $X_i \geq n + 1$. A subset $A \subset X_\infty$ is called thin if $A \subset Z_\infty$ holds for some closed subscheme $Z$ of $X$ with dim $Z \leq n$. A is called very thin if $A \subset Z_\infty$ holds for some closed subscheme $Z$ of $X$ with dim $Z \leq n - 1$.

The term “very thin” is used only in this paper. In Question 5.15(1) and Remark 5.16(1), we shall explain the motivation to introduce this terminology.

**Question 5.15.** Let $R$ and $X$ be as in Definition 5.14

1. Suppose that $C$ is a thin cylinder of $X_\infty$. Then, does $C \cap \text{Cont}^e(\text{Fitt}^n(\Omega_{X/k[t]}')) = \emptyset$ hold for any $e \geq 0$?

2. Suppose that $X$ is an integral scheme and $Y \subset X$ is the closed subscheme defined by the ideal $\text{Fitt}^n(\Omega_{X/k[t]}')$. Then, is $Y_\infty$ a thin set of $X_\infty$?

3. Let $S = k[[x_1, \ldots, x_N]][y_1, \ldots, y_m]$, and let $P$ be a prime ideal of $S$ of height $r$. Suppose that $P \cap k[t] = (0)$. Then, does $J_r(P; \text{Der}_{k[t]}(S)) \not\subset P$ hold?

**Remark 5.16.** (1) Note that Question 5.15(1) is true for the arc spaces of $k$-varieties $X$:

- If $C$ is a thin cylinder of $X_\infty$, then $C \cap \text{Cont}^e(\text{Jac}_{X/k}) = \emptyset$ holds for any $e \geq 0$ (cf. [EM09, Lemma 5.1]).
The same statement is true for schemes $X$ of finite type over $k[[x_1, \ldots, x_N]]$ by replacing $\text{Jac}_X/k$ with $\text{Jac}'_X/k$. Furthermore, Question 5.15(1) is true also for schemes $X$ of finite type over $k[t]$ (cf. [NS22, Lemma 2.23]). However, the same proofs do not work for schemes $X$ of finite type over $R = k[t][[x_1, \ldots, x_N]]$, and hence it is not clear to us whether Question 5.15(1) is true for this setting (see also the discussion in (3) below). This is why we introduce the term “very thin” and we will prove weaker statements instead in Lemma 5.18 for very thin cylinders and Proposition 5.23 for $X$ with an additional assumption.

(2) Due to the proof of [EM09, Lemma 5.1], Question 5.15(1) can be reduced to Question 5.15(2) by the Noetherian induction on dimension. Furthermore, Question 5.15(3) implies Question 5.15(2).

(3) Question 5.15(3) is related to the weak Jacobian condition (WJ) explained in Remark 2.6(2). Indeed, if $N = 0$, then Question 5.15(3) can be proved using Remark 2.6(2)(c) as follows. We denote

$$S' := (k[t] \setminus \{0\})^{-1} S = k(t)[y_1, \ldots, y_m]$$

the localization. Then by the assumption $P \cap k[t] = (0)$, we have $PS' \neq S'$ and hence $PS'$ is a prime ideal of height $r$. Since $S'$ satisfies (WJ)$_{(k(t))}$, we have

$$J_r(P; \text{Der}_{k[t]}(S))S' + PS' = J_r(PS'; \text{Der}_{k(t)}(S')) + PS' \not\subset PS',$$

which proves $J_r(P; \text{Der}_{k[t]}(S)) \not\subset P$. Note that the same proof does not work when $N > 0$ because we have

$$S' := (k[t] \setminus \{0\})^{-1} S \neq k(t)[[x_1, \ldots, x_N]][y_1, \ldots, y_m],$$

and it is not clear whether $J_r(PS'; \Delta) \not\subset PS'$ holds for $\Delta = \{ \partial/\partial x_i, \partial/\partial y_j \mid i, j \}$. Question 5.15(1) is also true for the arc spaces (Greenberg schemes) of formal $k[[t]]$-schemes of finite type, which will be dealt with in Subsection 5.2 (see [CLNS18, Ch.6, Proposition 2.4.3]). Actually, Question 5.15(3) is true for this setting:

- Let $S = k[x_1, \ldots, x_N][[t]]$, and let $P$ be a prime ideal of $S$ of height $r$. Suppose that $P \cap k[[t]] = (0)$. Then we have $J_r(P; \text{Der}_{k[[t]]}(S)) \not\subset P$. We denote

$$S' := S_t = k[x_1, \ldots, x_N][[(t)]]$$

the localization. We note that the assumption $P \cap k[[t]] = (0)$ is equivalent to $t \not\subset P$, and hence $PS'$ is a prime ideal of height $r$. Since $S'$ satisfies (WJ)$_{(k(t))}$ by [Nag62, Theorem 46.3], the same proof as in (3) above works and we have $J_r(P; \text{Der}_{k[[t]]}(S)) \not\subset P$. Note here that both $\text{Der}_{k[[t]]}(S)$ and $\text{Der}_{k((t))}(S')$ are generated by $\partial/\partial x_i$’s.

In Lemmas 5.17 and 5.18 below, for a scheme $X$ over $R = k[t][[x_1, \ldots, x_N]]$, we also consider the jet schemes and the arc space in the sense of Section 4. To avoid confusion, we denote them by $L_m(X)$ and $L_\infty(X)$, that is, $L_m(X)$ is the scheme representing the functor

$$F_m : \text{(Sch}/k) \to (\text{Set}); \quad Y \mapsto \text{Hom}_k(Y \times_{\text{Spec} k} \text{Spec} k[t]/(t^{m+1}), X)$$

and $L_\infty(X) = \lim_{\longrightarrow m} L_m(X)$ is the projective limit.

**Lemma 5.17.** Let $X$ be a scheme of finite type over $R = k[t][[x_1, \ldots, x_N]]$. Then the following hold.

1. There exist natural closed immersions $X_m \to L_m(X)$ for $m \geq 0$ which commute with the truncation morphisms.
(2) Let $x \in X$ be a $k$-point and let $\hat{O}_{X,x}$ be the completion of the local ring $O_{X,x}$ at its maximal ideal. Let $X' := \text{Spec}(\hat{O}_{X,x})$ and let $x' \in X'$ be the corresponding $k$-point. Then for the truncation morphisms $\pi_m^X : \mathcal{L}_m(X) \to X$ and $\pi_m^{X'} : \mathcal{L}_m(X') \to X'$, we have $(\pi_m^X)^{-1}(x) \cong (\pi_m^{X'})^{-1}(x)$.

Proof. We may assume that $X$ is affine, and we may write $X = \text{Spec}(S/I)$ with

$$S := k[[x_1, \ldots, x_N]][y_1, \ldots, y_M]$$

and $I := (f_1, \ldots, f_r)$ an ideal of $S$. We set $A := \text{Spec } S$.

Then, by the same argument as in Lemma 5.1 we have $\mathcal{L}_m(A) \cong \text{Spec } T_m$, where

$$T_m := k[[u^{(i)}]] \left[ \left[ x_1^{(0)}, \ldots, x_N^{(0)} \right] \left[ u^{(s)}, x_j^{(s)}, y_{j'}^{(s')} \right] \text{ for each } 1 \leq j \leq N, 1 \leq j' \leq M, 1 \leq s \leq m, 0 \leq s' \leq m \right].$$

In the same way as in Lemma 5.1 we can define a ring homomorphism $\Lambda' : S \to T_m[t]/(t^{m+1})$ that satisfies

- $\Lambda'(t) = u^{(0)} + u^{(1)} t + \cdots + u^{(m)} t^m$,
- $\Lambda'(x_j) = x_j^{(0)} + x_j^{(1)} t + \cdots + x_j^{(m)} t^m$ for each $0 \leq j \leq N$, and
- $\Lambda'(y_{j'}) = y_{j'}^{(0)} + y_{j'}^{(1)} t + \cdots + y_{j'}^{(m)} t^m$ for each $0 \leq j' \leq M$.

For $1 \leq i \leq r$ and $0 \leq \ell \leq m$, we define $G_i^{(\ell)} \in T_m$ as

$$\Lambda'(f_i) = \sum_{\ell=0}^m G_i^{(\ell)} t^\ell \pmod{t^{m+1}}.$$ 

Let

$$J_m := \left( G_{i_r}^{(s)} \mid 1 \leq i \leq r, 0 \leq s \leq m \right) \subset T_m$$

be the ideal generated by $G_{i_r}^{(s)}$s. Then by the same argument as in Lemma 5.2 we have $\mathcal{L}_m(X) \cong \text{Spec } (T_m/J_m)$.

Let $S_m$ be the ring defined in Lemma 5.1. Let $\Xi : T_m \to S_m$ be a surjective ring homomorphism defined by

- $\Xi(u^{(i)}) = 1$, and $\Xi(u^{(s)}) = 0$ for each $s = 0$ and $2 \leq s \leq m$.
- $\Xi(x_j^{(s)}) = x_j^{(s)}$ for each $1 \leq j \leq N$ and $0 \leq s \leq m$.
- $\Xi(y_{j'}^{(s)}) = y_{j'}^{(s)}$ for each $1 \leq j' \leq M$ and $0 \leq s \leq m$.

We note that $\Lambda : S \to S_m[t]/(t^{m+1})$ defined in the proof of Lemma 5.1 coincides with the composition

$$S \overset{\Lambda'}{\to} T_m[t]/(t^{m+1}) \overset{\Xi}{\to} S_m[t]/(t^{m+1}),$$

where $T_m[t]/(t^{m+1}) \to S_m[t]/(t^{m+1})$ is the $k[t]$-ring homomorphism induced by $\Xi$. Therefore, $F^{(s)}$ in Lemma 5.2 coincides with $\Xi(G_{i_r}^{(s)})$ for each $i$ and $s$.

Let $I_m \subset S_m$ be the ideal defined in Lemma 5.2. Then $\Xi$ induces a surjective ring homomorphism $T_m/J_m \to S_m/I_m$. It gives a closed immersion

$$X_m \cong \text{Spec } (S_m/I_m) \hookrightarrow \text{Spec } (T_m/J_m) \cong \mathcal{L}_m(X),$$

which completes the proof of (1).

Since $x \in X$ is a $k$-point, the corresponding maximal ideal of $S$ is of the form

$$\left( t - a, x_1 - b_1, \ldots, x_N - b_M \right)$$

with $a, b_1, \ldots, b_M \in k$. For simplicity, we assume that $a = b_1 = \cdots = b_M = 0$. Then $(\pi_m^X)^{-1}(x) \subset \mathcal{L}_m(X)$ is isomorphic to the closed subscheme of Spec $T_m$ defined by

$$J_m + (u^{(0)}, x_1^{(0)}, \ldots, x_N^{(0)}, y_1^{(0)}, \ldots, y_M^{(0)}).$$
Then by [Ish09, Proposition 4.1], we have \( C \subset L \). Furthermore, by [Ish09] Corollary 4.2, we have \( \mathcal{L}_m(X') \approx \text{Spec}(T''_m/J_mT''_m) \). Therefore, \( (\pi^X_m)^{-1}(x') \subset \mathcal{L}_m(X') \) is isomorphic to the closed subscheme of \( \text{Spec}T''_m \) defined by

\[
J_mT''_m + (u(0), x_1(0), \ldots, x_N(0), y_1(0), \ldots, y_M(0)).
\]

Therefore we have \( (\pi^X_m)^{-1}(x) \approx (\pi^X_m)^{-1}(x') \), which completes the proof of (2). \( \square \)

**Lemma 5.18.** Let \( X \) be a scheme of finite type over \( R = k[t][[x_1, \ldots, x_N]] \). Suppose that each irreducible component \( X_1 \) of \( X \) has \( \dim X_1 \geq n + 1 \). Let \( C \subset X_\infty \) be a cylinder contained in \( \text{Cont}^\pm_\infty(\mathfrak{a}_X) \). If \( C \) is very thin, then \( C \cap \text{Cont}^\varepsilon(\text{Fitt}(\Omega^0_{X/k[t]})) = \emptyset \) holds for any \( \varepsilon \geq 0 \).

**Proof.** Suppose the contrary that \( C \cap \text{Cont}^\varepsilon(\text{Fitt}(\Omega^0_{X/k[t]})) \neq \emptyset \) for some \( \varepsilon \geq 0 \). By replacing \( C \) with \( C \cap \text{Cont}^\varepsilon(\text{Fitt}(\Omega^0_{X/k[t]})) \), we may assume that \( \emptyset \neq C \subset \text{Cont}^\varepsilon(\text{Fitt}(\Omega^0_{X/k[t]})) \). Pick a \( k \)-arc \( \gamma \in C \). Let \( \pi^X_m(\gamma) \in X \) be the \( k \)-point of \( X \). Then, by replacing \( C \) with \( \cap (\pi^X_m)^{-1}(x) \), we may assume that \( C \subset (\pi^X_m)^{-1}(x) \).

Since \( C \) is a very thin set, there exists a closed subscheme \( Z \subset X \) such that \( C \subset Z \) and \( \dim Z \leq n - 1 \). Since \( \gamma \in Z \), it follows that \( x \in Z \). Let \( \hat{O}_{Z,x} \) be the completion of the local ring \( O_{Z,x} \) at its maximal ideal. Let \( Z' := \text{Spec}(\hat{O}_{Z,x}) \), and let \( x' \in Z' \) be the corresponding \( k \)-point. Then, since \( C \subset (\pi^X_m)^{-1}(x) \), we may identify \( C \) with a subset of \( \mathcal{L}_\infty(Z') \) by Lemma 3.17(1)(2). Note that \( C \subset \mathcal{L}_\infty(Z') \) is not necessarily a cylinder of \( \mathcal{L}_\infty(Z') \) under this identification.

Let \( S \) be the set of the closed subschemes \( Y' \) of \( Z' \) with the following condition:

- There exists a cylinder \( C' \) of \( X_\infty \) such that \( \emptyset \neq C' \subset C \) and \( C' \subset \mathcal{L}_\infty(Y') \).

Here, the inclusion \( C' \subset \mathcal{L}_\infty(Y') \) is considered by the identifications \( \mathcal{L}_\infty(Y') \subset \mathcal{L}_\infty(Z') \) and \( C' \subset \mathcal{L}_\infty(Z') \). Let \( Y' \) be a minimal element of \( S \), and let \( C' \) be a corresponding cylinder of \( X_\infty \). Then \( Y' \) is reduced by the minimality.

We shall prove that \( Y' \) is irreducible. Suppose the contrary that \( Y' = Y'_1 \cup \cdots \cup Y'_\ell \) is the irreducible decomposition with \( \ell \geq 2 \). By the minimality of \( Y' \), it follows that \( C' \not\subset \mathcal{L}_\infty(Y'_1) \) and hence we have

\[
C'' := C' \cap ((\psi^Z_q)^{-1}(\mathcal{L}_q(Y'_1)) \setminus (\psi^Z_{q+1})^{-1}(\mathcal{L}_{q+1}(Y'_1))) \neq \emptyset
\]

for some \( q \geq -1 \), where we set \( (\psi^Z_q)^{-1}(\mathcal{L}_q(Y'_1)) = \mathcal{L}_\infty(Z') \) for \( q = -1 \) by abuse of notation. Here, we have taken the intersection in the space \( \mathcal{L}_\infty(Z') \). Since \( C'' \cap \mathcal{L}_\infty(Y'_1) = \emptyset \), we have

\[
C'' \subset C \setminus \mathcal{L}_\infty(Y'_1) \subset \mathcal{L}_\infty(Y') \setminus \mathcal{L}_\infty(Y'_1) \subset \mathcal{L}_\infty(Y'_1 \cup \cdots \cup Y'_\ell).
\]

To get a contradiction by the minimality of \( Y' \), it is sufficient to show that \( C'' \) is a cylinder of \( X_\infty \). For this purpose, we shall see that

\[
C'' := C' \cap ((\psi^Z_q)^{-1}(\mathcal{L}_q(Y'_1)))
\]
is a cylinder of \( X_\infty \). Under the following identifications

\[
\begin{align*}
X_\infty \cap (\pi_\infty^X)^{-1}(x) & \xrightarrow{\psi_q^X} X_q \cap (\pi_q^X)^{-1}(x) \\
\cup & \\
Z_\infty \cap (\pi_\infty^Z)^{-1}(x) & \xrightarrow{\cup} Z_q \cap (\pi_q^Z)^{-1}(x) \\
\cap & \\
L_\infty(Z) \cap (\pi_\infty^Z)^{-1}(x) & \xrightarrow{\cong} L_q(Z) \cap (\pi_q^Z)^{-1}(x) \\
C' & \subset L_\infty(Y') \cap (\pi_\infty^{Y'})^{-1}(x') \xrightarrow{\psi_q^{Y'}} L_q(Y') \cap (\pi_q^{Y'})^{-1}(x')
\end{align*}
\]

we can consider the intersection \( F = L_q(Y'_1) \cap Z_q \cap (\pi_q^Z)^{-1}(x) \) and it can be identified with a closed subset of \( X_\infty \cap (\pi_\infty^Z)^{-1}(x) \). Then we have \( C'_q = C' \cap (\pi_q^X)^{-1}(F) \) since \( C'_q \subset Z_\infty \). Therefore, \( C'_q \) turns out to be a cylinder of \( X_\infty \), and hence so is \( C'' = C'_q \setminus C'_{q+1} \). We have proved that \( Y' \) is integral.

Let \( Y'' \subset Y' \) be the subscheme defined by \( \Jac_{Y'/k} \). Since \( Y' \) is reduced, we have \( Y'' \subset Y' \) by the Jacobian criterion of regularity (cf. Remark 2.6(2)(c)). By the minimality of \( Y' \), we have \( C' \not\subset Y''_{\infty} \) and hence \( C' \cap \text{Cont}^{e'}(\Jac_{Y'/k}) \neq \emptyset \) holds for some \( e' \geq 0 \). Take a \( k \)-arc \( \beta \in C' \cap \text{Cont}^{e'}(\Jac_{Y'/k}) \). For \( m \geq 0 \), we set

\[
D_{m,\beta} := (\psi_m^{Y'})^{-1}(\psi_m^X(\beta)) \subset L_\infty(Y'), \quad E_{m,\beta} := (\psi_m^X)^{-1}(\psi_m^X(\beta)) \subset X_\infty.
\]

Then by applying Proposition 4.3(2) to the map

\[
\psi_{m+1}^{Y'}(D_{m,\beta}) \to \psi_m^{Y'}(D_{m,\beta}) = \{\psi_m^Y(\beta)\},
\]

we have

\[
\dim(\psi_{m+1}^{Y'}(D_{m,\beta})) = \dim Y'
\]

for sufficiently large \( m \). On the other hand, by applying Proposition 5.2(2) to the map

\[
\psi_{m+1}^X(E_{m,\beta}) \to \psi_m^X(E_{m,\beta}) = \{\psi_m^X(\beta)\},
\]

we have

\[
\dim(\psi_{m+1}^X(E_{m,\beta})) = n
\]

for sufficiently large \( m \).

Since \( \dim Y' \leq \dim Z \leq n - 1 \), to get a contradiction, it is enough to show \( E_{m,\beta} \subset D_{m,\beta} \) for sufficiently large \( m \). Since \( C' \) is a cylinder of \( X_\infty \), there exists a constructible subset \( V \subset X_p \) for some \( p \geq 0 \) such that \( C' = (\psi_p^X)^{-1}(V) \). We shall prove the inclusion \( E_{m,\beta} \subset D_{m,\beta} \) for any \( m \geq p \).

Let \( X' := \text{Spec}(\mathcal{O}_{X,x}) \). Then by Lemma 5.17(1)(2), we have the following diagram:

\[
\begin{align*}
X_\infty \cap (\pi_\infty^X)^{-1}(x) & \subset L_\infty(X') \cap (\pi_\infty^{X'})^{-1}(x') \supset L_\infty(Y') \cap (\pi_\infty^{Y'})^{-1}(x') \\
\psi_m^X & \quad \quad \psi_m^Y \\
X_m \cap (\pi_m^X)^{-1}(x) & \subset L_m(X') \cap (\pi_m^{X'})^{-1}(x') \supset L_m(Y') \cap (\pi_m^{Y'})^{-1}(x')
\end{align*}
\]
Let $\beta_m := \psi_m^X(\beta)$. For $m \geq p$, we have

$$E_{m,\beta} = (\psi_m^X)^{-1}(\beta_m) = (\psi_m^X)^{-1}(\pi_{\infty}^X)^{-1}(x) = (\psi_m^X)^{-1}(\beta_m) \cap C'. $$

On the other hand, we have

$$D_{m,\beta} = (\psi_m^Y)^{-1}(\beta_m) = (\psi_m^Y)^{-1}(\beta_m) \cap \mathcal{L}_\infty(Y') \cap (\pi_m^Y)^{-1}(x').$$

Since $C' \subset \mathcal{L}_\infty(Y') \cap (\pi_m^Y)^{-1}(x')$, we have $E_{m,\beta} \subset D_{m,\beta}$ for $m \geq p$. We complete the proof. \(\Box\)

We prove a much weaker version of [NS22, Lemma 2.26(1)].

**Lemma 5.19.** Let $X$ be a scheme of finite type over $R = k[t][[x_1, \ldots, x_N]]$. For each $a \in k$, we denote by $X_a$ the closed subscheme of $X$ defined by $(t-a)\mathcal{O}_X$. Suppose for any $a \in k^\times$ that $X_a$ is an integral regular scheme and has $\dim X_a = n$. Then there exists a positive integer $\ell$ such that $\text{ord}_\gamma(o_X + \text{Fitt}^n(\Omega'_{X/k[t]})) \leq \ell$ holds for all $k$-regular $\gamma \in X_\infty$. In particular, if $\gamma$ satisfies $\text{ord}_\gamma(o_X) = \infty$, then $\text{ord}_\gamma(\text{Fitt}^n(\Omega'_{X/k[t]})) \leq \ell$.

**Proof.** We may assume that $X$ is affine, and we may write $X = \text{Spec}(S/I)$, where

$$S := k[t][[x_1, \ldots, x_N]][y_1, \ldots, y_m]$$

and $I$ is an ideal of $S$. We set $I_a := (I + (t-a))/(t-a)$, which is an ideal of the ring $S/(t-a) \simeq k[[x_1, \ldots, x_N]][y_1, \ldots, y_m]$. We have $\text{ht}(I_a) = N + m - n$ for any $a \in k^\times$ since $\dim X_a = n$.

We set

$$J := \mathcal{J}_{N+m-n}(I; \text{Der}_k(t)[S]) \subset S.$$ 

Note that $\text{Der}_k(t)[S] = \text{Der}'_k(t)[S]$ is generated by $\partial/\partial x_i$'s and $\partial/\partial y_i$'s. Then we have $\text{Fitt}^n(\Omega'_{X/k[t]}) = (J + I)/I$ by Remark [2.14]. Let $a \in k^\times$. Since $\text{ht}(I_a) = N + m - n$, we have

$$\text{Jac}_{X_a/k} = (J + I + (t-a))/(I + (t-a)).$$

Since $X_a$ is regular, we have

$$J + I + (t-a) = S$$

by the Jacobian criterion of regularity (cf. Remark [2.6](2)(c)). Therefore, for any $a \in k^\times$, we have

$$(J + I + (t-a))S' = S',$$

where $S' := S/(x_1, \ldots, x_N) \simeq k[t][y_1, \ldots, y_m]$. Then by Hilbert’s nullstellensatz, we have $t^\ell \in J + I + (x_1, \ldots, x_N)$ for some $\ell \geq 0$, which proves the assertions. \(\Box\)

**Lemma 5.20.** Let $P$ be a prime ideal of $S = k[[x_1, \ldots, x_N]]$ of height $r$, and let $I$ be an ideal of $S$ satisfying $P \subseteq I$. If $S/P$ is regular, then $\mathcal{J}_{r+1}(I; \text{Der}_k(S)) \not\subseteq I$.

**Proof.** Note that $\text{Der}_k(S) = \text{Der}'_k(S)$ is generated by $\partial/\partial x_i$'s. First, we prove the assertion when $r = 0$. Let $f \in I \setminus \{0\}$ be an element with the minimum order $a$. Suppose that $x_i$ appears in the lowest order term of $f$. Then it follows from the minimality of $a$ that $\partial f/\partial x_i \not\in I$, which proves the assertion when $r = 0$.

Suppose $r > 0$. Since $S/P$ is regular, by the Jacobian criterion of regularity (cf. Remark [2.6](2)(c)), there exist $D_1, \ldots, D_r \in \text{Der}_k(S)$ and $f_1, \ldots, f_r \in P$ such that

$$u := \det(D_i(f_j))_{1 \leq i,j \leq r} \not\in (x_1, \ldots, x_N).$$

Since $S/P$ is a complete regular local ring with the coefficient field $k$, $S/P$ is isomorphic to $k[[y_1, \ldots, y_{N-r}]]$. Therefore, by what we have already proved, there exist $D' \in \text{Der}_k(S/P)$ and $f' \in I/P$ such that $D'(f') \not\in I/P$. Let $f_{r+1} \in I$ be a lift of $f'$. By [Mat89, Theorem 30.8], there exists a lift $D_{r+1} \in \text{Der}_k(S)$ of $D'$ too. Since $D_{r+1}(P) = 0$, we have

$$\det(D_i(f_j))_{1 \leq i,j \leq r+1} = uD_{r+1}(f_{r+1}) \not\in I,$$
which shows that \( J_{r+1}(I;\text{Der}_k(S)) \not\subset I \).

\[ \square \]

Remark 5.21. We are interested in the case where \( \text{ht} I = r + 1 \). If \( I \) is a prime ideal, then it is true more generally that \( J_{\ell}(I;\text{Der}_k(S)) \not\subset I \) for \( \ell = \text{ht} I \). It is true because \( S \) satisfies the weak Jacobian condition \((WJ)_{k}\) (cf. Remark 2.6(2)). If \( I \) is not a prime ideal, then \( J_{\ell}(I;\text{Der}_k(S)) \not\subset I \) does not hold in general.

Lemma 5.22. Let \( P \) be a prime ideal of \( S = k[[x_1,\ldots,x_N]][y_1,\ldots,y_m] \) of height \( r \), and let \( I \) be an ideal of \( S \) satisfying \( P \not\subset I \). Suppose that \( S/P \) is regular and \( I + (x_1,\ldots,x_N) \not\subset S \). Then it follows that \( J_{r+1}(I;\text{Der}_k(S)) \not\subset I \).

Proof. Note that \( \text{Der}_k(S) = \text{Der}_k^r(S) \) is generated by \( \partial/\partial x_i 's \) and \( \partial/\partial y_i 's \). Since \( I + (x_1,\ldots,x_N) \not\subset S \), there exists a maximal ideal \( m \) containing \( I \) of the form

\[ m = (x_1,\ldots,x_N, y_1 - a_1,\ldots,y_m - a_m), \]

where \( a_i \in k \). Let \( \widehat{S} \) be the completion of \( S \) at \( m \). Let \( Y_i \in \widehat{S} \) be the image of \( y_i - a_i \).

Then we have \( \widehat{S} \simeq k[[x_1,\ldots,x_N,Y_1,\ldots,Y_m]] \), and \( \text{Der}_k(\widehat{S}) \) is generated by \( \partial/\partial x_i 's \) and \( \partial/\partial Y_i 's \). Therefore we have

\[ J_{r+1}(I\widehat{S};\text{Der}_k(\widehat{S})) + I\widehat{S} = J_{r+1}(I;\text{Der}_k(S))\widehat{S} + I\widehat{S}. \]

We also note that \( \widehat{S}/P\widehat{S} \) is regular and \( P\widehat{S} \not\subset I\widehat{S} \). Therefore by Lemma 5.20 we have \( J_{r+1}(I\widehat{S};\text{Der}_k(\widehat{S})) \not\subset I\widehat{S} \), which shows the assertion \( J_{r+1}(I;\text{Der}_k(S)) \not\subset I \).

\[ \square \]

Proposition 5.23. Let \( X \) be a scheme of finite type over \( R = k[t][[x_1,\ldots,x_N]] \). Suppose that each irreducible component \( X_k \) of \( X \) has \( \dim X_k \geq n + 1 \). For each \( a \in k \), we denote by \( X_a \) the closed subscheme of \( X \) defined by \( (t-a)O_X \). Suppose for any \( a \in k^\times \) that \( X_a \) is an integral regular scheme and has \( \dim X_a = n \). Then, there is no thin cylinder \( C \) of \( X_a \) containing a \( k \)-arc \( \gamma \) with \( \text{ord}_\gamma(O_X) = \infty \).

Proof. We may assume that \( X \) is affine, and we may write \( X = \text{Spec} A \) with \( A = S/I \), where

\[ S := k[t][[x_1,\ldots,x_N]][y_1,\ldots,y_m] \]

and \( I \) is an ideal of \( S \).

Suppose the contrary that there exists a thin cylinder \( C \) containing a \( k \)-arc \( \gamma \) with \( \text{ord}_\gamma(O_X) = \infty \). Replacing \( C \) with \( C \cap \text{Cont}^{\geq 1}(O_X) \), we may assume that \( C \subset \text{Cont}^{\geq 1}(O_X) \). By Lemma 5.19 it follows that \( e := \text{ord}_c(\text{Fitt}^{n}(\Omega_{X/k[t]}^{\gamma})) < \infty \).

By replacing \( C \) with \( C \cap \text{Cont}^{\varepsilon}(\text{Fitt}^{n}(\Omega_{X/k[t]}^{\gamma})) \), we may assume that \( \emptyset \not\subset C \subset \text{Cont}^{\varepsilon}(\text{Fitt}^{n}(\Omega_{X/k[t]}^{\gamma})) \).

Let \( \mathcal{S} \) be the set of the closed subschemes \( W \) of \( X \) with the following condition:

- There exists a cylinder \( C' \) of \( X_a \) such that \( \gamma \in C' \subset C \) and \( C' \subset W \).

Let \( W \) be a minimal element of \( \mathcal{S} \), and let \( C' \) be a corresponding cylinder of \( X \). Then \( W \) is reduced by the minimality. Let \( W = W_1 \cup \cdots \cup W_\ell \) be its irreducible decomposition. Since \( C \) is thin, we may assume \( \dim W \leq n \). Here, we claim as follows:

Claim 5.24. (1) \( \gamma \in (W_i)_\infty \) holds for each \( 1 \leq i \leq \ell \).

(2) \( \dim W_i = n \) holds for each \( 1 \leq i \leq \ell \).

(3) We denote by \( Z_i \subset W_i \) the closed subscheme defined by \( \text{Fitt}^{n-1}(\Omega_{W_{i/k[t]}^{\gamma}}^{\gamma}) \).

Then \( Z_i \not\subset W_i \) holds for each \( 1 \leq i \leq \ell \).

First, we assume this claim and finish the proof. By Claim 5.24(3) and the minimality of \( W \), we have

\[ C' \not\subset (Z_1 \cup \cdots \cup Z_\ell)_\infty = (Z_1)_\infty \cup \cdots \cup (Z_\ell)_\infty. \]
Take a $k$-arc $\beta \in C' \setminus ((Z_1)_\infty \cup \cdots \cup (Z_\ell)_\infty)$. For each $i$, we denote by $I_{Z_i} \subset A$ the ideal corresponding to $Z_i$, and we set $q_i := \text{ord}_\beta(I_{Z_i}) < \infty$. Then
\[ C'' := C' \cap \bigcap_i \text{Cont}^{q_i}(I_{Z_i}) \]
is a non-empty cylinder of $X_\infty$. By applying Proposition 5.9(2) to $W_i$ and its cylinder
\[ C'' \cap (W_i)_\infty \subset \text{Cont}^{q_i}(\text{Fitt}^{n-1}(\Omega_{W_i/k[\ell]})) \],
it follows that the truncation map
\[ \psi_{m+1}(C'' \cap (W_i)_\infty) \rightarrow \psi_m(C'' \cap (W_i)_\infty) \]
has $(n-1)$-dimensional fibers for sufficiently large $m$. Therefore,
\[ \psi_{m+1}(C'') = \bigcup_i \psi_{m+1}(C'' \cap (W_i)_\infty) \rightarrow \psi_m(C'') = \bigcup_i \psi_m(C'' \cap (W_i)_\infty) \]
also has $(n-1)$-dimensional fibers for sufficiently large $m$. However, by Proposition 5.9(2), it should have $n$-dimensional fibers because $\emptyset \neq C'' \subset \text{Cont}^{q_i}(\text{Fitt}^{n}(\Omega_{X/k[l]}))$. We get a contradiction. Therefore, it is sufficient to prove Claim 5.24(3).

**Proof of Claim 5.24.** We shall prove (1). Suppose the contrary that $\gamma \not\in (W_1)_\infty$. Let $I_{W_1} \subset A$ be the ideal corresponding to $W_1$ and let $q := \text{ord}_\gamma(I_{W_1}) < \infty$. Then the cylinder
\[ C'' := C' \cap \text{Cont}^{q}(I_{W_1}) \]
contains $\gamma$ and satisfies
\[ C'' \subset W_\infty \setminus (W_1)_\infty \subset (W_2 \cup \cdots \cup W_\ell)_\infty, \]
which contradicts the minimality of $W$.

We shall prove (2). Suppose the contrary that $\dim W_1 \leq n - 1$. Let $W' := W_2 \cup \cdots \cup W_\ell$ and let $I_{W'} \subset A$ be the ideal corresponding to $W'$. By the minimality of $W$, it follows that $C' \not\subset W_\infty$. Therefore, we have
\[ C' := C' \cap \text{Cont}^{q}(I_{W'}) \neq \emptyset \]
for some $q \geq 0$. Since $C' \cap W_\infty = \emptyset$, we have $C'' \subset (W_1)_\infty$ and $C''$ turns out to be a very thin cylinder of $X_\infty$. It contradicts $\emptyset \neq C'' \subset \text{Cont}^{q}(\text{Fitt}^{n}(\Omega_{X/k[l]}))$ by Lemma 5.18.

We shall prove (3). Let $H$ be one of $W_i$'s. Let $Q$ be the prime ideal of $S$ corresponding to $H$. Since $H_\infty$ contains a $k$-arc, $H$ contains a $k$-point. Therefore, by Lemma 2.11(4), we have
\[ \text{ht} Q = \dim S - \dim' H = \dim S - \dim H = N + m - n + 1. \]

First, we prove that
\[ Q + (x_1, \ldots, x_N) + (t - a) \neq S \]
for some $a \in k\n$. Suppose the contrary that $Q + (x_1, \ldots, x_N) + (t - a) = S$ holds for any $a \in k\n$. Then by Hilbert's nullstellensatz, it follows that $t^\ell \in Q + (x_1, \ldots, x_N)$ for some $\ell \geq 0$. This contradicts $\gamma \in H_\infty$ and $\text{ord}_\gamma(x_N) = \infty$, and we get (♠) for some $a \in k\n$.

(♠) implies $Q + (t - a) \neq S$. Furthermore, we have $t - a \not\in Q$ because $H_\infty$ contains a $k$-arc. Therefore, we have $\text{ht}(Q + (t - a)) = \text{ht} Q + 1 = N + m - n + 2$.

We set
\[ S_a := S/(t - a) \cong k[[x_1, \ldots, x_N]][y_1, \ldots, y_m], \]
\[ I_a := (I + (t - a))/(t - a), \quad Q_a := (Q + (t - a))/(t - a). \]
Then we have $\text{ht}(Q_a) = N + m - n + 1$. Furthermore, we have $\text{ht}(I_a) = N + m - n$ by the assumption $\dim' X_a = n$. Therefore it follows that $Q_a \supset I_a$. Let $J := \ldots$
prove that \( J \) is contained in \( P \). Therefore by Lemma 5.22, we have

\[
\frac{J + (t - a)}{(t - a)} = \frac{J_{N+m-n+1}(Q_a; \text{Der}_k(S_a))}{Q_a} \not\subseteq Q_a = \frac{(Q + (t - a))/(t - a)}{(t - a)}.
\]

In particular, we have \( J \not\subseteq Q \).

Since \( \text{Fitt}^{n-1}(\Omega'_H/k[t]) = (J + Q)/Q \), we complete the proof of (3). \( \square \)

**Lemma 5.25.** Let \( R = k[t][[x_1, \ldots, x_N]] \). Let \( e_1, \ldots, e_N \) and \( d \) be integers satisfying \( 0 < e_i \leq d \) for each \( i \). For each \( c \in k^x \), let \( T_c : R \rightarrow R \) be the ring isomorphism defined by \( T_c(t) = c^{-d}t \) and \( T_c(x_i) = c^{e_i}x_i \). Let \( I \) be an ideal of \( R \) that is \( T_c \)-invariant (i.e., \( T_c(I) = I \)) for any \( c \in k^x \). Let \( P \) be a minimal prime of \( I \). Then \( P \) satisfies one of the following conditions.

1. \( P \cap k[t] \neq (0) \) and \( t \in P \).
2. \( P \cap k[t] = (0) \), and \( P + (t - a) \neq R \) holds for any \( a \in k^x \).
3. \( P \cap k[t] = (0) \), and there exists \( f \in P \) such that

\[
f - t^\ell \in (t^{\ell+1}x_1, \ldots, t^{\ell+1}x_N) + (x_1, \ldots, x_N)^{\ell+1}
\]

holds for some \( \ell \geq 0 \).

**Proof.** First, we prove that \( P \) is also \( T_c \)-invariant for any \( c \in k^x \). Let \( P_1, \ldots, P_m \) be the minimal primes of \( I \). Since \( T_c \) is an isomorphism, \( T_c \) induces a permutation on \( P_1, \ldots, P_m \). Let \( p : k^x \rightarrow \mathfrak{S}_m \) be the induced group homomorphism, where \( \mathfrak{S}_m \) is the symmetric group of degree \( m \). For any \( c \in k^x \), we can take \( b \in k^x \) such that \( c = b^{m^!} \). Therefore, we have \( p(c) = p(b^{m^!}) = (p(b))^{m^!} = 1 \). It shows that \( T_c(P_i) = P_i \) for any \( c \in k^x \) and \( 1 \leq i \leq m \).

Suppose \( P \cap k[t] \neq (0) \). Then \( t - a \in P \) for some \( a \in k \). Since \( P \) is \( T_c \)-invariant for any \( c \in k^x \), it follows that \( a = 0 \). Therefore, \( P \) satisfies (1).

Suppose that \( P \cap k[t] = (0) \) and \( P + (t - a) = R \) holds for some \( a \in k^x \). We shall prove that \( P \) satisfies (3). Since \( P \) is \( T_c \)-invariant for any \( c \in k^x \), it follows that \( P + (t - a) = R \) holds for any \( a \in k^x \). Then by Hilbert’s nullstellensatz, it follows that

- \( t^\ell \in P + (x_1, \ldots, x_N) \) for some \( \ell \geq 0 \).

Therefore there exists \( g \in P \) such that \( g - t^\ell \in (x_1, \ldots, x_N) \).

We denote \( M := (x_1, \ldots, x_N) \subset k[t][[x_1, \ldots, x_N]] \). Since \( M \) is \( T_c \)-invariant, \( T_c \) induces an automorphism on \( k[t][[x_1, \ldots, x_N]]/M^{\ell+1} \). Hence, \( k[t][[x_1, \ldots, x_N]]/M^{\ell+1} \) has a graded ring structure satisfying \( \deg t = -d \) and \( \deg x_i = e_i \). Then \( P + M^{\ell+1}/M^{\ell+1} \) is a homogeneous ideal. Therefore, the term \( g_{-d\ell} \) of \( g \) with degree \(-d\ell \) is contained in \( P + M^{\ell+1} \). We may write \( g_{-d\ell} = f - h \) with \( f \in P \) and \( h \in M^{\ell+1} \). On the other hand, since \( g_{-d\ell} - t^\ell \in M \), we have \( g_{-d\ell} - t^\ell \in (t^{\ell+1}x_1, \ldots, t^{\ell+1}x_N) \) by looking at the degrees of its terms. Therefore, the condition (3) holds for this \( f \). \( \square \)

**Remark 5.26.** Let \( I \) and \( P \) be as in Lemma 5.25. Then the following hold for \( Y := \text{Spec}(R/P) \):

- \( Y_{\infty} = \emptyset \) if \( P \) is of the form (1).
- \( Y_{\infty} \cap \text{Cont}^{\geq 1}(\omega_Y) = \emptyset \) holds if \( P \) is of the form (3).

### 5.2. Arc spaces of affine formal \( k[[t]] \)-schemes.

In this subsection, we discuss the arc space of \( X \) of the form \( X = \text{Spec}(k[x_1, \ldots, x_N][[t]]/I) \). As we will mention in Remark 5.27, the arc space of \( X \) can be seen as the Greenberg scheme of the corresponding affine formal scheme. In this subsection, we do not deal with general formal \( k[[t]] \)-schemes.
Remark 5.27. Sebag in [Seb04] investigates the theory of arc spaces of formal \( k[[t]] \)-schemes with \( k \) a perfect field, and the theory can be applied to \( X = \text{Spec}(k[x_1, \ldots, x_N][[t]]/I) \) dealt with in this subsection. The reader is also referred to [CLNS18] to this theory.

For a scheme \( X = \text{Spec}(k[x_1, \ldots, x_N][[t]]/I) \), we can associate the formal affine scheme \( X = \text{Spf}(k[x_1, \ldots, x_N][[t]]/I) \). Then the Greenberg schemes \( \text{Gr}_m(X) \) and \( \text{Gr}(X) \) defined in [Seb04] are isomorphic to \( X_m \) and \( X_\infty \), respectively. Therefore, the theory of Greenberg schemes developed in [Seb04] and [CLNS18] can be applied to the arc space \( X_\infty \) of \( X \).

**Definition 5.28** (cf. [CLNS18, Appendix 3.3]). Let \( I \) be an ideal of \( S = k[x_1, \ldots, x_N][[t]] \) and let \( A := S/I \). Then we denote by \( \hat{\Omega}_{A/k[[t]]} \) the completion of the \( A \)-module \( \Omega_{A/k[[t]]} \) with respect to the \((t)\)-adic topology, i.e.

\[
\hat{\Omega}_{A/k[[t]]} := \lim_{\to} (\Omega_{A/k[[t]]}/(t^n)\Omega_{A/k[[t]]}).
\]

The canonical derivation \( d_{A/k[[t]]} : A \to \Omega_{A/k[[t]]} \) induces a derivation

\[
\hat{d}_{A/k[[t]]} : A \to \hat{\Omega}_{A/k[[t]]}.
\]

We sometimes abbreviate \( \hat{d}_{A/k[[t]]} \) to \( \hat{d} \).

When \( X = \text{Spec} A \), we denote by \( \hat{\Omega}_{X/k[[t]]} \) the sheaf on \( X \) associated to the \( A \)-module \( \hat{\Omega}_{A/k[[t]]} \).

**Remark 5.29** (cf. [CLNS18, Example 3.3.5 in Appendix]).

1. \( \hat{\Omega}_{S/k[[t]]} \) is a free \( S \)-module of rank \( N \) with basis

\[
\hat{d}_{S/k[[t]]}(x_1), \ldots, \hat{d}_{S/k[[t]]}(x_N).
\]

Furthermore, we have an exact sequence

\[
I/I^2 \xrightarrow{\delta} \hat{\Omega}_{S/k[[t]]} \otimes_S A \xrightarrow{\alpha} \hat{\Omega}_{A/k[[t]]} \to 0
\]

of \( A \)-modules, where \( \alpha \) is the map satisfying \( \alpha(\hat{d}_{S/k[[t]]}(g) \otimes 1) = \hat{d}_{A/k[[t]]}(\overline{g}) \) for \( g \in S \), and \( \delta \) is the map satisfying \( \delta(\overline{g}) = \hat{d}_{S/k[[t]]}(g) \otimes 1 \) for \( g \in I \). In particular, \( \hat{\Omega}_{A/k[[t]]} \) is a finite \( A \)-module.

2. The canonical derivation \( \hat{d} : A \to \hat{\Omega}_{A/k[[t]]} \) has the following universal property:

   (1) The induced map

\[
\text{Hom}_A(\hat{\Omega}_{A/k[[t]]}, M) \to \text{Der}_{k[[t]]}(A, M); \quad f \mapsto f \circ \hat{d}
\]

is an isomorphism for any \( A \)-module \( M \) that is complete with respect to the \((t)\)-adic topology. In particular, this map is an isomorphism for any finite \( A \)-module (cf. [Mat89, Theorem 8.7]). This follows from the following general fact from [Gro64, 20.4.8.2]:

   (2) Let \( B \) be a topological ring and \( C \) a topological \( B \)-algebra. Let \( N \) be a topological \( C \)-module. Then we have an isomorphism

\[
\text{Hom}_C^c(\Omega_{C/B}, N) \xrightarrow{\sim} \text{Der}_B^c(C, N); \quad f \mapsto f \circ d_{C/B}.
\]

Here, \( \text{Hom}_C^c(\Omega_{C/B}, N) \) denotes the set of the continuous homomorphisms \( \Omega_{C/B} \to N \) of \( C \)-modules, and \( \text{Der}_B^c(C, N) \) denotes the set of the continuous \( B \)-derivations \( C \to N \). The topology on \( \Omega_{C/B} = I/I^2 \) is defined as the quotient topology, where \( I \) is the kernel of the augmentation map \( C \otimes_B C \to C \); \( a \otimes b \mapsto ab \).
Lemma 5.30. Let $X$ be an affine scheme of the form $X = \text{Spec}(k[x_1, \ldots, x_N][[t]]/I)$. Suppose that each irreducible component $X_i$ of $X$ has dim $X_i \geq n + 1$. Let $\gamma \in \text{Cont}^e(\text{Fitt}^n(\hat{\Omega}_{X/k[[t]]}))$ be a $k$-arc. Then we have
\[
\gamma \ast \hat{\Omega}_{X/k[[t]]} \simeq k[[t]]^{\oplus n} \oplus \bigoplus_i k[t]/(t^{e_i})
\]
as $k[[t]]$-modules with $\sum_i e_i = e$.

Proof. The same proofs as in Lemmas 5.6 and 5.7 work. Note that any minimal prime $P$ of $I$ satisfies $ht P \leq N - n$. This is because
\[
ht P = \dim S - \dim(S/P) \leq (N + 1) - (n + 1) = N - n,
\]
where we set $S := k[x_1, \ldots, x_N][[t]]$. The first equality follows from the facts that any maximal ideal $M$ of $S$ has $ht M = N + 1$ and $S$ is a catenary ring.

Proposition 5.31. Let $n$ be a non-negative integer, and let $X$ be an affine scheme of the form $X = \text{Spec}(k[x_1, \ldots, x_N][[t]]/I)$. Suppose that each irreducible component $X_i$ of $X$ has dim $X_i \geq n + 1$. Then, there exists a positive integer $c$ such that the following hold for non-negative integers $m$ and $e$ with $m \geq c$.

1. We have
\[
\psi_m\left(\text{Cont}^e(\text{Fitt}^n(\hat{\Omega}_{X/k[[t]]}))\right) = \pi_{m+e,m}\left(\text{Cont}^e(\text{Fitt}^n(\hat{\Omega}_{X/k[[t]]}))\right)_{m+e}.
\]

2. $\pi_{m+1,m} : X_{m+1} \to X_m$ induces a piecewise trivial fibration
\[
\psi_{m+1}\left(\text{Cont}^e(\text{Fitt}^n(\hat{\Omega}_{X/k[[t]]}))\right) \to \psi_m\left(\text{Cont}^e(\text{Fitt}^n(\hat{\Omega}_{X/k[[t]]}))\right)
\]
with fiber $k^n$.

Proof. The same proof as in Proposition 5.9 works (cf. Remark 1.6).

Remark 5.32. When $X$ is flat over $k[[t]]$, Proposition 5.31 (1) is proved in CLNS18, Ch.5, Proposition 2.3.4, and Proposition 5.31 (2) is proved in Sech04, Lemme 4.5.4 (cf. CLNS15, Ch.5, Theorem 2.3.11). We also note that Proposition 5.31 (2) can be reduced to the flat case by the argument in NS22, Remark 2.14(3)].

We define cylinders and their codimensions.
Definition 5.33. Let $n$ be a non-negative integer, and let $X$ be an affine scheme of the form $X = \text{Spec}(k[x_1, \ldots, x_N][[t]]/I)$. Suppose that each irreducible component $X_i$ of $X$ has $\dim X_i \geq n + 1$. A subset $C \subset X_\infty$ is called a cylinder if $C = \psi_m^{-1}(S)$ holds for some $m \geq 0$ and a constructible subset $S \subset X_m$. We define the codimension of $C$ as follows:

1. Assume that $C \subset \text{Cont}^e(\text{Fitt}^n(\hat{\Omega}_{X/k[[t]]}))$ for some $e \in \mathbb{Z}_{\geq 0}$. Then we define the codimension of $C$ in $X_\infty$ as

$$\text{codim}(C) := (m + 1)n - \dim(\psi_m(C))$$

for any sufficiently large $m$. This definition is well-defined by Proposition 5.34.

2. In general, we define the codimension of $C$ in $X_\infty$ as follows:

$$\text{codim}(C) := \min_{e \in \mathbb{Z}_{\geq 0}} \text{codim}(C \cap \text{Cont}^e(\text{Fitt}^n(\hat{\Omega}_{X/k[[t]]}))) .$$

By convention, $\text{codim}(C) = \infty$ if $C \cap \text{Cont}^e(\text{Fitt}^n(\hat{\Omega}_{X/k[[t]]})) = \emptyset$ for any $e \geq 0$.

Remark 5.34. As in Remark 5.12, the definition of the codimension above depends on the choice of $n$.

Definition 5.35. Let $n$ be a non-negative integer, and let $X$ be an affine scheme of the form $X = \text{Spec}(k[x_1, \ldots, x_N][[t]]/I)$. Suppose that each irreducible component $X_i$ of $X$ has $\dim X_i \geq n + 1$. A subset $A \subset X_\infty$ is called thin if $A \subset Z_\infty$ holds for some closed subscheme $Z$ of $X$ with $\dim Z \leq n$.

Proposition 5.36 (cf. [Seb04 Théorème 6.3.5]). Let $n$ be a non-negative integer, and let $X$ be an affine scheme of the form $X = \text{Spec}(k[x_1, \ldots, x_N][[t]]/I)$. Suppose that each irreducible component $X_i$ of $X$ has $\dim X_i \geq n + 1$. Let $C$ be a cylinder in $X_\infty$. Let $\{C_\lambda\}_{\lambda \in \Lambda}$ be a set of countably many disjoint subcylinders $C_\lambda \subset C$. If $C \setminus \bigcup_{\lambda \in \Lambda} C_\lambda \subset X_\infty$ is a thin set, then it follows that

$$\text{codim}(C) = \min_{\lambda \in \Lambda} \text{codim}(C_\lambda).$$

Proof. This follows from [CLNS18, Ch.6. Lemma 3.4.1] and [CLNS18, Ch.6. Example 3.5.2].
5.3. Codimension formulae. In this subsection, we discuss a $k[t]$-morphism $f : X \to Y$ of affine $k[t]$-schemes in the following two cases.

(a) $X$ and $Y$ are affine schemes of the forms $X = \text{Spec}(k[x_1, \ldots, x_N][[t]]/I)$ and $Y = \text{Spec}(k[x_1, \ldots, x_M][[t]]/J)$.

(b) $X$ and $Y$ are affine schemes of the forms $X = \text{Spec}(k[t][x_1, \ldots, x_L]/[t])/I$ and $Y = \text{Spec}(k[t][x_1, \ldots, x_L]/[J])/J$. Furthermore, $f$ satisfies $(x_1, \ldots, x_L)O_X \subset (t)$.

Lemma 5.38. In case (a) above, the canonical map $f^*\hat{\Omega}_{Y/k[[t]]} \to \hat{\Omega}_{X/k[[t]]}$ is induced. In case (b), the canonical map $f^*\Omega_{Y/k[t]}^* \to \hat{\Omega}_{X/k[[t]]}$ is induced.

Proof. Let $A = O_X$ and $B = O_Y$ be the corresponding rings, and $g : B \to A$ the corresponding $k[t]$-ring homomorphism.

First, we deal with case (a). Since $g : B \to A$ is a $k[t]$-ring homomorphism, $\hat{\Omega}_{A/k[[t]]}$ is a complete $B$-module with respect to the $(t)$-adic topology. Therefore, by the universal property of $\hat{\Omega}_{B/k[[t]]}$ (cf. Remark 5.29(2)), the derivation $\hat{d}_{A/k[[t]]} \circ g : B \to \hat{\Omega}_{A/k[[t]]}$ factors through $\hat{\Omega}_{B/k[[t]]}$. We complete the proof in case (a).

Next, we deal with case (b). By the universal property of $\Omega_{B/k[t]}^*$, it is sufficient to show that the composition $B \xrightarrow{g} A \xrightarrow{\hat{d}_{A/k[[t]]}} \hat{\Omega}_{A/k[[t]]}$ is a special $B$-derivation. Note that $\hat{\Omega}_{A/k[[t]]}$ is a complete $A$-module with respect to the $(t)$-adic topology. Since $g((x_1, \ldots, x_L)) \subset (t)$ holds by assumption, $\hat{\Omega}_{A/k[[t]]}$ is a separated $B$-module with respect to the $(x_1, \ldots, x_L)$-adic topology. Therefore $\hat{d}_{A/k[[t]]} \circ g$ is a special $B$-derivation by Lemma 2.23.

We define the order of the Jacobian for a morphism.

Definition 5.39. Let $f : X \to Y$ be a morphism of affine $k[t]$-schemes of the form (a) above. Then $f$ induces a homomorphism $f^*\hat{\Omega}_{Y/k[[t]]} \to \hat{\Omega}_{X/k[[t]]}$ by Lemma 5.38. Let $\gamma \in X_\infty$ be a $k$-arc and let $\gamma' := f_\infty(\gamma)$. Let $S$ be the torsion part of $\gamma^*\hat{\Omega}_{X/k[[t]]}$. Then we define the order $\text{ord}_\gamma(\text{jac}_f)$ of the Jacobian of $f$ at $\gamma$ as the length of the $k[[t]]$-module

$$\text{Coker}(\gamma^*\hat{\Omega}_{Y/k[[t]]} \to \gamma^*\hat{\Omega}_{X/k[[t]]}/S).$$

In particular, if $\text{ord}_\gamma(\text{jac}_f) < \infty$, then we have

$$\text{Coker}(\gamma^*\hat{\Omega}_{Y/k[[t]]} \to \gamma^*\hat{\Omega}_{X/k[[t]]}/S) \simeq \bigoplus_i k[t]/(t^{e_i})$$

as $k[[t]]$-modules with some positive integers $e_i$ satisfying $\sum_i e_i = \text{ord}_\gamma(\text{jac}_f)$.

(2) Let $f : X \to Y$ be a morphism of affine $k[t]$-schemes of the form (b) above. Then $f$ induces a homomorphism $f^*\Omega_{Y/k[t]}^* \to \hat{\Omega}_{X/k[[t]]}$ by Lemma 5.38. Let $\gamma \in X_\infty$ be a $k$-arc and let $\gamma' := f_\infty(\gamma)$. Let $S$ be the torsion part of $\gamma^*\hat{\Omega}_{X/k[[t]]}$. Then we also define the order $\text{ord}_\gamma(\text{jac}_f)$ of the Jacobian of $f$ at $\gamma$ as the length of the $k[[t]]$-module

$$\text{Coker}(\gamma^*\Omega_{Y/k[t]}^* \to \gamma^*\hat{\Omega}_{X/k[[t]]}/S).$$

(3) By abuse of notation, we define

$$\text{Cont}^e(\text{jac}_f) := \{ \gamma \in X_\infty \mid \text{ord}_\gamma(\text{jac}_f) = e \}$$

for $e \geq 0$. 
Lemma 5.40. Let \( n \) be a non-negative integer. Let \( f : X \to Y \) be a morphism of affine \( k[t] \)-schemes of the form (a) above. Let \( g : Y \to Z \) be a morphism of affine \( k[t] \)-schemes of the form (b). Suppose that each irreducible component \( W_i \) of \( X, Y \) and \( Z \) has \( \dim W_i \geq n + 1 \). Let \( \gamma \in X_\infty \) be a \( k \)-arc and let \( \gamma' := f_\infty(\gamma) \). Suppose that

\[
\text{ord}_\gamma(\text{Fitt}^n(\hat{\Omega}_{X/k[[t]]})) < \infty, \quad \text{ord}_{\gamma'}(\text{Fitt}^n(\hat{\Omega}_{Y/k[[t]]})) < \infty.
\]

Then we have

\[
\text{ord}_\gamma(\text{Jac}_{g\circ f}) = \text{ord}_\gamma(\text{Jac}_f) + \text{ord}_{\gamma'}(\text{Jac}_g).
\]

Proof. The same proof as in [NS22, Lemma 2.10] works due to Lemma 5.30. \( \square \)

Remark 5.41. The same statement holds if \( g \) is of the form (a).

Remark 5.42. We note in this remark that all propositions (Proposition 2.29, Lemmas 2.31, 2.32 and Proposition 2.33) in Subsection 2.6 in [NS22] are also true for a \( k[t] \)-morphism \( f \) of the form (a) and (b) by making the following modifications:

- Replacing the conditions \((*)_n \) and \((**)_n \) in [NS22] on \( X \) with the following condition:
  - "Each irreducible component \( X_i \) of \( X \) has \( \dim X_i \geq n + 1 \)."

- Replacing \( \Omega_{-/k[t]} \) with \( \Omega_{-/k[[t]]} \) and \( \Omega'_{-/k[t]} \) and replacing \( \text{Jac}_{-/k[t]} \) with \( \text{Fitt}^n(\hat{\Omega}_{-/k[[t]]}) \) and \( \text{Fitt}^n(\hat{\Omega}'_{-/k[t]}) \).

We note that

- Proposition 2.29(2), Lemmas 2.31, 2.32, and Proposition 2.33 in [NS22] are formal consequences of Proposition 2.29(1), Lemma 2.13(1) and Proposition 2.17 in [NS22]. The formal power series ring versions of Proposition 2.29(1), Lemma 2.13(1) and Proposition 2.17 in [NS22] are proved in Lemmas 6.13 and 6.37 Lemmas 5.7 and 5.30 and Propositions 5.9 and 5.31 in this paper.

Furthermore, Lemma 2.34 in [NS22] is also true in the formal power series ring setting by replacing \( A = \text{Spec} k[t][x_1, \ldots, x_N] \) with \( \text{Spec} k[x_1, \ldots, x_N][[t]] \). Indeed, the same proof of Lemma 2.34 in [NS22] works in this setting.

Proposition 5.43. Let \( n \) be a non-negative integer. Let \( f : X \to Y \) be a morphism of affine \( k[t] \)-schemes of the form (b) above. Suppose that each irreducible component \( W_i \) of \( X \) and \( Y \) has \( \dim W_i \geq n + 1 \). Let \( e, e', e'' \in \mathbb{Z}_{\geq 0} \). Let \( A \subset X_\infty \) be a cylinder and let \( B = f_\infty(A) \). Assume that

\[
A \subset \text{Cont}^{e''}(\hat{\Omega}_{X/k[[t]]}) \cap \text{Cont}^e(\text{Jac}_f), \quad B \subset \text{Cont}^{e'}(\hat{\Omega}'_{Y/k[[t]]}).
\]

Then, \( B \) is a cylinder of \( Y_\infty \) contained in \( \text{Cont}^{\geq 1} (\mathfrak{a}_Y) \), where \( \mathfrak{a}_Y \subset \mathcal{O}_Y \) is the ideal sheaf generated by \( x_1, \ldots, x_L \in \mathcal{O}_Y \). Moreover, if \( f_\infty \mid A \) is injective, then it follows that

\[
\text{codim}(A) + e = \text{codim}(B).
\]

Proof. By Remark 5.42, the same proofs as in Subsection 2.6 in [NS22] work by making the following modifications:

- Replacing \( \Omega_{X/k[t]} \) with \( \hat{\Omega}_{X/k[[t]]} \) and \( \Omega_{Y/k[t]} \) with \( \Omega'_{Y/k[t]} \).

- Replacing \( \text{Jac}_{X/k[t]} \) with \( \hat{\Omega}_{X/k[[t]]} \) and \( \text{Jac}_{Y/k[t]} \) with \( \hat{\Omega}'_{Y/k[t]} \).

\( \square \)

Proposition 5.44. Suppose that a finite group \( G \) acts on the ring \( k[x_1, \ldots, x_N] \) over \( k \). Let \( I \subset k[x_1, \ldots, x_N]^G[[t]] \) be an ideal, and let \( I' \subset k[x_1, \ldots, x_N][[t]] \) be the ideal generated by \( I \). We denote

\[
X := \text{Spec}(k[x_1, \ldots, x_N][[t]]/I'), \quad Y := \text{Spec}(k[x_1, \ldots, x_N]^G[[t]]/I),
\]
and denote \( f : X \to X/G = Y \) the quotient morphism. Suppose that each irreducible component \( W_i \) of \( X \) and \( Y \) has \( \dim W_i \geq n + 1 \). Let \( A \subset X_{\infty} \) be a \( G \)-invariant cylinder and let \( B = f_\infty(A) \). Let \( e, e', e'' \in \mathbb{Z}_{\geq 0} \). Assume that

\[
A \subset \text{Cont}^e\left( \text{Fitt}^n\left( \Omega_{X/k[t]} \right) \right) \cap \text{Cont}^{e'}(\text{jac}_f), \quad B \subset \text{Cont}^e\left( \text{Fitt}^n\left( \Omega_{Y/k[t]} \right) \right).
\]

Then \( B \) is a cylinder of \( Y_{\infty} \) with

\[
\text{codim}(A) + e = \text{codim}(B).
\]

**Proof.** Note that \( f \) is a morphism of the form (a). By Remark 5.42 the same proof of Proposition 2.35 in [NS22] works. \( \square \)

**Remark 5.45.**

1. Proposition 5.43 is true also for \( k[t] \)-morphisms of the form (a). Indeed, this is known for morphisms \( f : X \to Y \) of formal \( k[[t]] \)-schemes (not necessarily affine). When \( X \) is smooth over \( k[[t]] \), this is proved in [Seb04] Lemme 7.1.3 (cf. [CLNS18] Ch.5, Theorem 3.2.2). The general case is proved in [Yas24] Lemmas 10.19, 10.20. We also note that Yasuda proves it in the more general setting, for formal Deligne-Mumford stacks of arbitrary characteristics.

2. Proposition 5.43 is true also for a \( k[t] \)-morphism \( f : X \to Y \) of formal \( k[[x_1, \ldots, x_N]] \)-schemes of finite type (not necessarily affine) by making the following modification:

- Replacing the condition “each irreducible component \( W_i \) of \( X \) and \( Y \) has \( \dim W_i \geq n + 1 \)” in Proposition 5.43 with “each irreducible component \( W_i \) of \( X \) and \( Y \) has \( \dim W_i \geq n + 1 \)”.

6. **Denef and Loeser’s theory for quotient singularities**

In this section, we review the theory of arc spaces of quotient varieties established by Denef and Loeser [DL12] (cf. [Yas16], [NS22] Section 3). We explain their theory in the formal power series ring setting.

Let \( d \) be a positive integer and \( \xi \in k \) a primitive \( d \)-th root of unity. Let \( G \subset \text{GL}_N(k) \) be a finite subgroup with order \( d \) that linearly acts on \( \overline{A} := \hat{k}_k^N := \text{Spec} k[[x_1, \ldots, x_N]] \). Let \( \overline{X} \subset \overline{A} \) be a \( G \)-invariant subscheme. We denote by

\[
A := \overline{X}/G, \quad X := \overline{X}/G
\]

the quotient schemes. Let \( Z \subset A \) be the minimum closed subset such that \( \overline{A} \to A \) is étale outside \( Z \).

Fix \( \gamma \in G \). Since \( G \) is a finite group, \( \gamma \) can be diagonalized with some new basis \( x_1^{(\gamma)}, \ldots, x_N^{(\gamma)} \). Let \( \text{diag}(\xi^{e_1}, \ldots, \xi^{e_N}) \) be the diagonal matrix with \( 0 < e_i \leq d \). Then we define a \( k[t] \)-ring homomorphism

\[
\lambda^\gamma_t : k[t][[x_1, \ldots, x_N]]^G \to k[x_1, \ldots, x_N]^{C_\gamma}[[t]]; \quad x_i^{(\gamma)} \mapsto t^{e_i/d}x_i^{(\gamma)},
\]

where \( C_\gamma \) is the centralizer of \( \gamma \) in \( G \). Let \( i : k[x_1, \ldots, x_N]^{C_\gamma}[[t]] \to k[x_1, \ldots, x_N][[t]] \) be the inclusion map, and let \( \overline{\lambda} = i \circ \lambda^\gamma_t \) be the composite map.

Let \( I_X \subset k[[x_1, \ldots, x_N]]^G \) be the defining ideal of \( X \) in \( A \). Let \( I_X' \subset k[t][[x_1, \ldots, x_N]]^G \) be the ideal generated by \( I_X \). Let

\[
\overline{I}_X^{(\gamma)} \subset k[x_1, \ldots, x_N]^{C_\gamma}[[t]], \quad \overline{I}_X^{(\gamma)} \subset k[x_1, \ldots, x_N][[t]]
\]
be the ideals generated by $\lambda^*_\gamma(I^*_X)$ and $\overline{\lambda}^*_\gamma(I^*_X)$, respectively. Then we have the following diagram:

```
\[ \begin{array}{c}
\xymatrix{
\mathcal{X}_\gamma \ar[r]^-{\lambda^*_\gamma} & \mathcal{X}'_\gamma \ar[d]^-{\mu^*_\gamma} \ar[r]^-{\overline{\lambda}^*_\gamma} & \mathcal{X}^*_\gamma \ar[d]^-{\overline{\mu}^*_\gamma} \\
\mathcal{X}_\infty \ar[r]^-{\lambda_\gamma} & \mathcal{X}'_\infty \ar[r]^-{\overline{\lambda}_\gamma} & \mathcal{X}^*_\infty
}\end{array} \]
```

We define $k[t]$-schemes $\overline{\mathcal{X}}^\gamma$, $\tilde{\mathcal{X}}^\gamma$, and $\mathcal{X}^\gamma$ as follows:

- $\overline{\mathcal{X}}^\gamma := \text{Spec } k[x_1, \ldots, x_N][[t]]$,
- $\tilde{\mathcal{X}}^\gamma := \text{Spec } \left( k[x_1, \ldots, x_N]^{C^\gamma}[[t]]/\mathfrak{T}^\gamma_X \right)$,
- $\mathcal{X}^\gamma := \text{Spec } \left( k[x_1, \ldots, x_N][[t]]/\mathfrak{T}^\gamma_X \right)$.

Let $\overline{\mathcal{X}}^{\gamma}_\infty$, $\tilde{\mathcal{X}}^{\gamma}_\infty$, and $\mathcal{X}^{\gamma}_\infty$ be their arc spaces as $k[t]$-schemes defined in Section 5.

Then we have the following diagram of arc spaces:

```
\[ \begin{array}{c}
\xymatrix{
\overline{\mathcal{X}}^\gamma \ar[r]^-{\lambda^*_\gamma} & \overline{\mathcal{X}}^*_\gamma \ar[d]^-{\overline{\mu}^*_\gamma} \\
\mathcal{X}^\gamma \ar[r]^-{\lambda_\gamma} & \mathcal{X}^*_\gamma \ar[d]^-{\overline{\lambda}_\gamma} \\
\mathcal{X}_\infty \ar[r]^-{\lambda_\gamma} & \mathcal{X}^*_\infty
}\end{array} \]
```

We denote by $\mu_\gamma$ and $\overline{\mu}_\gamma$ the restrictions of $\lambda_\gamma$ and $\overline{\lambda}_\gamma$ to $\mathcal{X}^\gamma_\infty$ and $\mathcal{X}^*_\infty$, respectively.

**Remark 6.1.** (1) Here, we have used the fact that the arc spaces of

\[ \text{Spec } k[t][x_1, \ldots, x_N]^{C^\gamma} \text{ and } \text{Spec } (k[t][x_1, \ldots, x_N][t]/I^*_X) \]

as $k[t]$-schemes (defined in Section 5) are isomorphic to the arc spaces of $A$ and $X$ as $k$-schemes (defined in Section 4).

(2) Furthermore, the vertical arrows are closed immersions. Under these identifications, we have

\[ \lambda^{-1}_\gamma(X_\infty) = \tilde{X}^\gamma_\infty, \quad \overline{\lambda}^{-1}_\gamma(X_\infty) = \mathcal{X}^*_\infty. \]

**Proposition 6.2** ([DL02 Section 2], cf. [NS22 Subsections 3.1, 3.2]). The ring homomorphism $\lambda^*_\gamma$ induces the maps $\lambda_\gamma : (\overline{\mathcal{X}}^\gamma/C^\gamma)_\infty \to A_\infty$ and $\overline{\lambda}_\gamma : \overline{\mathcal{X}}^\gamma_\infty \to A_\infty$, and the following hold.

1. There is a natural inclusion $\overline{\mathcal{X}}^{\gamma}_\infty/C^\gamma \hookrightarrow (\overline{\mathcal{X}}^\gamma/C^\gamma)_\infty$.
2. The composite map $\overline{A}^{\gamma}_\infty/C^\gamma \xrightarrow{\lambda_\gamma} A_\infty$ is injective outside $Z_\infty$.
3. $\left[ \gamma \right]_{\text{Conj}(G)}(\overline{\lambda}_\gamma(\overline{\mathcal{X}}^{\gamma}_\infty) \setminus Z_\infty) = A_\infty \setminus Z_\infty$ holds, where $\text{Conj}(G)$ denotes the set of the conjugacy classes of $G$.

**Proof.** In [DL02] and [NS22], the assertions are proved for the polynomial ring $k[t][x_1, \ldots, x_N]$, and their proofs work in the formal power series ring setting. □
By Remark 6.1(2), we can deduce the same statement for $X$.

**Proposition 6.3** (cf. [NS22, Subsection 3.3]). The ring homomorphism $\lambda^\gamma_*$ induces the maps $\mu^\gamma : \tilde{X}^{(\gamma)}_\infty \to X_\infty$ and $\overline{\mu}_\gamma : \overline{X}^{(\gamma)}_\infty \to X_\infty$, and the following hold.

1. There is a natural inclusion $\overline{X}^{(\gamma)}_\infty / C_\gamma \to \tilde{X}^{(\gamma)}_\infty$.
2. The composite map $\overline{X}^{(\gamma)}_\infty / C_\gamma \to \tilde{X}^{(\gamma)}_\infty$ is injective outside $Z_\infty$.
3. $\bigsqcup_{(\gamma) \in \text{Conj}(G)} (\overline{\mu}_\gamma (\overline{X}^{(\gamma)}_\infty) \setminus Z_\infty) = X_\infty \setminus Z_\infty$ holds.

**Remark 6.4**. (1) In [NS22], $e_i$ is taken to satisfy $0 \leq e_i \leq d - 1$. Note that the ring homomorphism $\lambda^\gamma_*$ cannot be defined in this way of taking in our formal power series ring setting.

(2) It is also natural to define a $k[t]$-ring homomorphism

$$\overline{\lambda}^\gamma_* : k[t][[x_1^1, \ldots, x_N^1]]^G \to k[t][[x_1^1, \ldots, x_N^1]]; \quad x^{(\gamma)}_i \mapsto t^{e_i/d}_i x^{(\gamma)}_i,$$

and schemes

$$\overline{A}^{(\gamma)} := \text{Spec} k[t][[x_1, \ldots, x_N]], \quad \overline{X}^{(\gamma)} := \text{Spec} \left(k[t][[x_1, \ldots, x_N]] / \overline{I}^{(\gamma)}_X\right),$$

where $\overline{I}^{(\gamma)}_X$ is the ideal of $k[t][[x_1, \ldots, x_N]]$ generated by $\overline{X}^{(\gamma)}_\gamma (I X)_\gamma$. Then by the same argument as in this section, $\overline{X}^{(\gamma)}_\gamma$ induces maps

$$\overline{X}^{(\gamma)}_\gamma : \overline{A}^{(\gamma)}_\infty \to A_\infty, \quad \overline{\mu}^{(\gamma)} : \overline{X}^{(\gamma)}_\infty \to X_\infty.$$

However, as we can see in the discussion below, if $\overline{A}^{(\gamma)}$ and $\overline{X}^{(\gamma)}$ are replaced with $\overline{A}^{(\gamma)}$ and $\overline{X}^{(\gamma)}$, then Propositions 6.2 and 6.3 are no longer valid.

First, we note that

$$\overline{A}^{(\gamma)}_m \simeq \text{Spec} (k[[x_1^0, \ldots, x_N^0]] [x_1^1, \ldots, x_N^1 \mid 1 \leq s \leq m]),$$

$$\overline{X}^{(\gamma)}_m \simeq \text{Spec} (k[x_1^1, \ldots, x_N^1 \mid 0 \leq s \leq m]),$$

and we have a natural morphism $\overline{A}^{(\gamma)}_m \to \overline{A}^{(\gamma)}_m$ induced by the ring inclusion

$$k[x_1^1, \ldots, x_N^1 \mid 0 \leq s \leq m] \hookrightarrow k[[x_1^0, \ldots, x_N^0]] [x_1^1, \ldots, x_N^1 \mid 1 \leq s \leq m].$$

Since the morphisms $\overline{A}^{(\gamma)}_m \to \overline{A}^{(\gamma)}_m$ are compatible with the truncation maps, they induce a map $\overline{A}^{(\gamma)}_\infty \to \overline{A}^{(\gamma)}_\infty$. The map $\overline{X}^{(\gamma)}_\infty \to \overline{X}^{(\gamma)}_\infty$ is also induced, and we have the following commutative diagrams:

$$\begin{array}{ccc}
A_\infty & \xrightarrow{A} & \overline{A}^{(\gamma)}_\infty \\
\downarrow{A}^{(\gamma)}_\gamma & & \downarrow{A}^{(\gamma)}_\gamma \\
X_\infty & \xrightarrow{X} & \overline{X}^{(\gamma)}_\infty
\end{array}$$

Furthermore, by contraction, $\overline{A}^{(\gamma)}_\infty \to \overline{A}^{(\gamma)}_\infty$ induces isomorphisms

$$\overline{A}^{(\gamma)}_\infty \cap \text{Cont}^{\geq 1} ((x_1, \ldots, x_N)) \simeq \overline{A}^{(\gamma)}_\infty \cap \text{Cont}^{\geq 1} ((x_1, \ldots, x_N)),$$

$$\overline{X}^{(\gamma)}_\infty \cap \text{Cont}^{\geq 1} ((x_1, \ldots, x_N)) \simeq \overline{X}^{(\gamma)}_\infty \cap \text{Cont}^{\geq 1} ((x_1, \ldots, x_N)).$$

We also note that $\overline{A}^{(\gamma)}_\infty (k) \hookrightarrow \overline{A}^{(\gamma)}_\infty (k)$, $\overline{X}^{(\gamma)}_\infty (k) \hookrightarrow \overline{X}^{(\gamma)}_\infty (k)$ on $k$-points. However, these two maps are not surjective in general (see (3) below).
(3) Suppose that \( N = 2 \) and \( G = \langle \gamma \rangle \), where \( \gamma : k[[x_1, x_2]] \to k[[x_1, x_2]] \) is the involution defined by \( \gamma(x_i) = -x_i \) for \( i \in \{1, 2\} \). Then we have \( d = 2 \), \( e_1 = e_2 = 1 \), and \( k[[x_1, x_2]]^G = k[x_1^2, x_1 x_2, x_2^2] \). We denote by \( \alpha \in A_\infty \) the \( k \)-arc corresponding to the \( k[t] \)-ring homomorphism

\[
\alpha^* : k[t][[x_1, x_2]] \to k[[t]]; \quad x_1 \mapsto t, \quad x_2 \mapsto t.
\]

Then, \( \alpha \) is contained in the image of \( \overline{\lambda}_\gamma : \overline{A}_\infty^{(\gamma)} \to A_\infty \). Indeed, if \( \beta \in A_\infty \) is the \( k \)-arc defined by

\[
\beta^* : k[x_1, x_2][[t]] \to k[[t]]; \quad x_1 \mapsto 1, \quad x_2 \mapsto 1,
\]

then we have \( \alpha^* = \beta^* \circ \overline{\lambda}_\gamma \) and hence \( \alpha = \overline{\lambda}_\gamma(\beta) \).

On the other hand, \( \alpha \) is not contained in the image of \( \overline{\lambda}_\gamma : \overline{A}_\infty^{(\gamma)} \to A_\infty \) because there is no \( k[t] \)-ring homomorphism \( \beta^* : k[t][[x_1, x_2]] \to k[[t]] \) satisfying \( \beta^*(x_1) = \beta^*(x_2) = 1 \).

7. Arc Spaces of Hyperquotient Singularities

In this section, we prove in Theorem 7.3 that [16] Theorem 4.8] is still valid in the formal power series ring setting.

7.1. Minimal log discrepancies of hyperquotient singularities. Let \( d \) be a positive integer and let \( \xi \in k \) be a primitive \( d \)-th root of unity. Let \( G \subset GL_N(k) \) be a finite group with order \( d \) that linearly acts on \( \mathcal{A} := \hat{\mathcal{A}}_k^N = \text{Spec } k[[x_1, \ldots, x_N]] \). We denote by

\[
A := \mathcal{A}/G
\]

the quotient scheme. Let \( Z \subset A \) be the minimum closed subset such that \( \mathcal{A} \to A \) is étale outside \( Z \). We assume that \( \text{codim } Z \geq 2 \), and hence the quotient morphism \( \overline{A} \to A \) is étale in codimension one. We note that \( A \) is Q-Gorenstein (cf. Remark 3.1). We fix a positive integer \( r \) such that \( \omega_{A/k}^{[r]} \) is invertible.

We fix \( \gamma \in G \). Let \( C_\gamma \) be the centralizer of \( \gamma \) in \( G \). Since \( G \) is a finite group, \( \gamma \) can be diagonalized with a suitable basis \( x_1, \ldots, x_N \). Let \( \text{diag}(\xi^{e_1}, \ldots, \xi^{e_N}) \) be the diagonal matrix with \( 0 < e_i \leq d \). We define a \( k[t] \)-ring homomorphism

\[
\lambda^*_\gamma : k[[x_1, \ldots, x_N]]^G \to k[x_1, \ldots, x_N][C_\gamma][[t]]; \quad x_i \mapsto \xi^{e_i} x_i,
\]

and define \( \overline{\lambda}^*_\gamma : k[[x_1, \ldots, x_N]]^G \to k[x_1, \ldots, x_N][[t]] \) as the composition of \( \lambda^*_\gamma \) and the inclusion \( i : k[x_1, \ldots, x_N][C_\gamma][[t]] \to k[x_1, \ldots, x_N][[t]] \).

Let \( f_1, \ldots, f_c \in k[[x_1, \ldots, x_N]]^G \) be a regular sequence. We set

\[
B := \text{Spec } (k[[x_1, \ldots, x_N]]^G/(f_1, \ldots, f_c)), \quad \overline{B} := \text{Spec } (k[[x_1, \ldots, x_N]]/(f_1, \ldots, f_c)).
\]

We denote by \( n := N - c \) their dimensions.

Suppose that \( B \) is normal. Then it follows that \( \overline{B} \to B \) is also étale in codimension one, and \( \overline{B} \) is also normal. Indeed, since \( \text{codim}_A Z \geq 2 \), we have \( A_{\text{sing}} = Z \) by the purity of the branch locus (cf. [Nag59]). Therefore we have \( B \cap Z \subset B_{\text{sing}} \) since \( f_1, \ldots, f_c \) is a regular sequence (cf. [Stack] tag 00NU)). Then it follows that \( \text{codim}_B (Z \cap B) \geq \text{codim}_B (B_{\text{sing}}) \geq 2 \) by the normality of \( B \). Therefore, \( B \to B \) is also étale in codimension one, and hence \( \overline{B} \) is also normal by Serre’s criterion of normality.

Note that \( \omega_{B/k}^{[r]} \) is invertible. Indeed, we have the adjunction formula

\[
\omega_{B/k}^' \simeq \det^{-1}(I/I^2) \otimes_O \omega_{A/k}^,'\]
where $I := (f_1, \ldots, f_c) \subset k[[x_1, \ldots, x_N]]^G$, since the sequence

$$0 \to I/I^2 \to \Omega_{A/k}^1 \otimes \mathcal{O}_A \to \Omega_{B/k}^2 \to 0$$

is exact at a regular point of $B$ by Proposition \ref{prop:exact_sequence}.

We define ideals $I'$, $\overline{I}(\gamma)$ and $T(\gamma)$ by

$$I' := (f_1, \ldots, f_c) \subset k[[x_1, \ldots, x_N]]^G,$$
$$\overline{I}(\gamma) := (\lambda^*_i(f_1), \ldots, \lambda^*_i(f_c)) \subset k[[x_1, \ldots, x_N]]^C_i[[t]],$$
$$T(\gamma) := (\overline{x}_i(f_1), \ldots, \overline{x}_i(f_c)) \subset k[[x_1, \ldots, x_N]][[t]].$$

Then we have the following diagram.

![Diagram](https://via.placeholder.com/150)

We define $k[t]$-schemes $A'$, $\overline{A}(\gamma)$, $\overline{A}(\gamma)$, $B'$, $\overline{B}(\gamma)$ and $\overline{B}(\gamma)$ as follows:

$$A' := \text{Spec } k[t][[x_1, \ldots, x_M]]^G,$$
$$B' := \text{Spec } k[t][[x_1, \ldots, x_M]]^G/I',$$
$$\overline{A}(\gamma) := \text{Spec } k[[x_1, \ldots, x_N]]^C_i[[t]],$$
$$\overline{B}(\gamma) := \text{Spec } k[[x_1, \ldots, x_N]]^C_i[[t]]/\overline{I}(\gamma),$$
$$\overline{A}(\gamma) := \text{Spec } k[[x_1, \ldots, x_N]][[t]],$$
$$\overline{B}(\gamma) := \text{Spec } k[[x_1, \ldots, x_N]][[t]]/\overline{T}(\gamma).$$

Then we have the following morphisms between the corresponding $k[t]$-schemes.

![Diagram](https://via.placeholder.com/150)

**Remark 7.1.**

1. Note that
   - $A'$ and $B'$ are affine schemes of the form $\text{Spec } k[t][[x_1, \ldots, x_M]]/J$, and
   - $\overline{A}(\gamma)$, $\overline{A}(\gamma)$, $\overline{B}(\gamma)$ and $\overline{B}(\gamma)$ are of the form $\text{Spec } k[[x_1, \ldots, x_N]]/[J]$.  

   We will use the notion of the sheaf $\Omega_{X/k[t]}^1$ of special differentials for $X = A'$, $B'$ defined in Section \ref{sec:special_differentials} and use the notion of the sheaf $\Omega_{X/k[t]}^1$ for $X = \overline{A}(\gamma), \overline{A}(\gamma), \overline{B}(\gamma), \overline{B}(\gamma)$ defined in Subsection \ref{subsec:special_differentials}.

   Since $I'$, $\overline{I}(\gamma)$ and $\overline{T}(\gamma)$ are generated by $c$ elements, each irreducible component $W_i$ of $B'$, $\overline{B}(\gamma)$ and $\overline{B}(\gamma)$ has $\dim W_i \geq n + 1$. Therefore, we can apply lemmas and propositions in Subsections \ref{subsec:irreducible_components} and \ref{subsec:irreducible_components} to their arc spaces.

2. Lemma \ref{lem:irreducible_components}(4) below show that $\overline{B}(\gamma)$ has only one irreducible component that is flat over $k[[t]]$. Furthermore, the component $V$ has $\dim V = n + 1$.

**Lemma 7.2.** Let $f : C_1 \to C_2$ be a flat ring homomorphism of Noetherian rings. Then the following hold.
Let $P$ be a prime ideal of $C_1$ such that $PC_2 \neq C_2$. Then $ht P = ht(PC_2)$ holds.

(2) Suppose that $f$ is faithfully flat. If $I \subseteq C_1$ is a proper ideal of $C_1$, then $ht I = ht(I(C_2))$ holds.

Proof. We shall prove (1). Let $Q$ be a minimal prime of $PC_2$. Then by the going-down theorem, we have $P = Q \cap C_1$. Therefore, we have $ht Q = ht P$ by [Mat89, Theorem 15.1], which proves (1).

We shall prove (2). Note that $IC_2 \neq C_2$ holds by the faithfully flatness. First, the inequality $ht I \leq ht(I(C_2))$ follows from the going-down theorem. We shall prove the opposite inequality. Take a minimal prime $P$ of $I$ such that $ht I = ht P$. Then by (1), it follows that

$$ht I = ht P = ht(PC_2) \geq ht(I(C_2)),$$

which completes the proof. □

Lemma 7.3. We denote $F_i := \overline{X}_i(f_i)$ for each $1 \leq i \leq c$. Consider the following diagram of rings.

$$
\begin{array}{cccc}
S_1 := k[x_1, \ldots, x_N][t] & \rightarrow & C_1 := k[x_1, \ldots, x_N][t]/(F_1, \ldots, F_c) \\
\downarrow h_1 & & \downarrow g_1 \\
S_2 := k[x_1, \ldots, x_N][(t)] & \rightarrow & C_2 := k[x_1, \ldots, x_N][(t)]/(F_1, \ldots, F_c) \\
\downarrow h_2 & & \downarrow g_2 \\
S_3 := k[x_1, \ldots, x_N][(t^{1/d})] & \rightarrow & C_3 := k[x_1, \ldots, x_N][(t^{1/d})]/(F_1, \ldots, F_c) \\
\downarrow h_3 & & \downarrow g_3 \\
S_4 := k[x_1, \ldots, x_N] & \rightarrow & C_4 := k[x_1, \ldots, x_N]/(f_1, \ldots, f_c)
\end{array}
$$

We denote $I_i := (F_1, \ldots, F_c) \subset S_i$ for $i \in \{1, 2, 3\}$, and $I_4 := (f_1, \ldots, f_c) \subset S_4$. Then the following hold.

(1) $h_1$, $h_2$ and $h_3$ are regular, and hence so are $g_1$, $g_2$ and $g_3$.
(2) $h_2$ and $h_3$ are faithfully flat, and hence so are $g_2$ and $g_3$.
(3) $C_2$ and $C_3$ are normal domains. In particular, $I_2$ and $I_3$ are prime ideals.
(4) $ht(I_2) = ht(I_3) = c$.

Proof. We shall prove (1) and (2). Since $h_1$ is the localization by $t \in S_1$, it is regular. Note that the inclusion map

$$h_2' : S_2' := k[x_1, \ldots, x_N][t, t^{-1}] \rightarrow S_3' := k[x_1, \ldots, x_N][t^{1/d}, t^{-1/d}]$$

is étale and faithfully flat. Since $h_2$ is the base change $- \otimes_{S_2} S_2$ of $h_2'$, it follows that $h_2$ is étale (in particular, regular) and faithfully flat.

Let $P$ be a prime ideal of $S_1$ and let $Q$ be a prime ideal of $S_4$ such that $Q \cap S_4 = P$. To see that $h_3$ is regular, it is sufficient to show the following two conditions:

- $ht(P(S_3)_Q) = ht P$.
- $S_3/PS_3$ is regular at $Q$.

Note here that the first condition is equivalent to the flatness by [Mat89, Theorem 23.1] since $S_3$ is regular and $S_3$ is Cohen-Macaulay. We also note that the inequality $ht(P(S_3)_Q) \leq ht P$ always holds by [Mat89, Theorem 15.1].

Let $r := ht P$. Since $S_3/P$ is regular at $P$, by the Jacobian criterion of regularity, there exist $D_1, \ldots, D_r \in Der_k(S_4)$ and $a_1, \ldots, a_r \in P$ such that $c := det(D_i(a_j))_{ij} \notin P$. Since $Der_k(S_4)$ is generated by $\partial/\partial x_i$'s, we may assume that $D_i = \partial/\partial x_i$ holds.
Proof. by Lemma 7.3(2) and Lemma 7.2. We complete the proof. □

Since $I$ is an integral extension, we have $h_t^c I$.

Lemma 7.4. Suppose the contrary that $Q$ holds for some $t$. Then we have $t^s = (t^{s+1})^{1/(t^s)}$ for $s > 0$. To get a contradiction, we shall prove (3). Note that $C_i$ is normal by our assumption. Since the normality is preserved under faithfully flat regular ring homomorphisms (cf. [Mat89] Theorem 32.2]), $C_2$ and $C_3$ are normal. Therefore, it is sufficient to show that $C_2$ and $C_3$ are domains. In what follows, we shall only prove that $C_2$ is a domain since the same proof works for $C_3$. Suppose the contrary that $I_2$ has minimal primes $P_1$ and $P_2$ with $P_1 \neq P_2$. Then by the normality of $C_2$, we have $P_1 + P_2 = S_2$ (cf. [Eis95] Proposition 2.20]. Set $Q_1 := P_1 \cap S_1$ and $Q_2 := P_2 \cap S_1$. Then $P_1 + P_2 = S_1$, it follows that $t^s \in Q_1 + Q_2$ for some $s \geq 0$. To get a contradiction, we shall prove

- $Q_1, Q_2 \subset (x_1, \ldots, x_N, t^{s+1})$.

Suppose the contrary that

- there exist $g \in k[[t]]$ and $h \in (x_1, \ldots, x_N)$ such that $g + h \in Q_1$ and $g \notin (t^{s+1})$.

Let $0 \leq a \leq s$ be the minimum $a$ such that $g \notin (t^{a+1})$. Then $g - bt^a \in (t^{a+1})$ holds for some $b \in k^x$. We may assume $b = 1$ by replacing $g$ and $h$ with $b^{-1}g$ and $b^{-1}h$. For $c \in k^x$, we denote by $T_c : S_1 \to S_1$ the ring isomorphism

$$T_c : k[x_1, \ldots, x_N][[t]] \to k[x_1, \ldots, x_N][[t]] : t \mapsto c^{-d}t, \ x_i \mapsto c^i x_i.$$ 

Since $I_1$ is $T_c$-invariant, so is its minimal prime $Q_1$. Since the ideal $(t^{s+1})$ of $S_1$ is also $T_c$-invariant, $T_c$ induces the ring isomorphism $T'_c : S_1/(t^{s+1}) \to S_1/(t^{s+1})$. Hence,

$$(Q_1 + (t^{s+1}))/((t^{s+1}))$$

is a homogeneous ideal. Therefore, the term $f_{-d}$ of $f := g + h$ with degree $-d$ is contained in $Q_1 + (t^{s+1})$. Since $f_{-d} - t^a = (x_1, \ldots, x_N)$, we have $f_{-d} - t^a \in (t^{a+1})$ by looking at the degrees of its terms. Therefore $f_{-d} = t'(1 + f')$ holds for some $f' \in (t)$. Since $1 + f' \in S_1^x$, we have $t^a \in Q_1 + (t^{s+1})$, and hence $t^a \in Q_1$. Therefore, we have $P_1 = S_2$ and we get a contradiction.

Note that $ht(I_4) = c$ by our assumption. Then, (4) follows from (2) and Lemma 7.2. □

Lemma 7.4. Let $S_i$ and $h_i$ be as in Lemma 7.3. Let $\mathfrak{c} \subset S_1^c$ be an ideal of $S_1^c$. Let $c_i$ be the ideal of $S_1$ generated by the image of $\mathfrak{c}$ by the ring homomorphism $S_1^c \to S_1^c : x_i \mapsto t^{c_i/d} x_i$. We denote $c_2 := c_1 S_2$. Then we have $ht(c_2) = ht(\mathfrak{c})$.

Proof. We define $c_3 := c_2^3 S_3$ and $c_4 := c S_4$. Then we have $c_3 = c_4 S_3$. Since $S_1^c \to S_4$ is an integral extension, we have $ht(\mathfrak{c}) = ht(c S_1^c) = ht(c_4)$. Furthermore, we have

$$ht(c_3) = ht(c_1 S_2^3) = ht(c_4), \quad ht(c_3) = ht(c_2 S_3) = ht(c_2)$$

by Lemma 7.3(2) and Lemma 7.2. We complete the proof. □

Definition 7.5. (1) We shall define sheaves $\omega'_A/k[t]$ on $A'$ and $\omega'_B/k[t]$ on $B'$ using the special canonical sheaves $\omega_A/k$ and $\omega_B/k$ defined in Section 2. Let
The canonical map

\[ \Omega_{B'}^n/k[t] \to \omega_{B'/k[t]} \]

is induced by \( \Omega_{B'}^n \to \omega_{B'/k} \) and the isomorphism \( \Omega_{B'/k[t]}^n \simeq \Omega_{B/k}^n \otimes_{O_B} \mathcal{O}_{B'} \).

The canonical map \( \Omega_{A'/k[t]}^n \to \omega_{A'/k[t]} \) is also defined. We define an ideal sheaf \( \mathfrak{n}_{r,B'} \subset \mathcal{O}_{B'} \) by

\[ \text{Im} \left( (\Omega_{B'}^n/k[t])^\otimes r \to \omega_{B'/k[t]}^r \right) = \mathfrak{n}_{r,B'} \otimes_{\mathcal{O}_{B'}} \omega_{B'/k[t]}^r. \]

Then it satisfies \( \mathfrak{n}_{r,B'} = \mathfrak{n}_{r,B} \mathcal{O}_{B'} \).

We define ideal sheaves \( \text{Jac}_{B'/k[t]} \), \( \text{Jac}_{\mathcal{O}_B^{(\gamma)}}/k[[t]] \) and \( \text{Jac}_{\mathcal{O}_B^{(\gamma)}}/k[[t]] \) by

\begin{align*}
\text{Jac}_{B'/k[t]} &:= \text{Fitt}^n(\Omega_{B'/k[t]}^n) \subset \mathcal{O}_{B'}, \\
\text{Jac}_{\mathcal{O}_B^{(\gamma)}}/k[[t]] &:= \text{Fitt}^n(\hat{\Omega}_{B^{(\gamma)}/k[[t]]}^n) \subset \mathcal{O}_{B^{(\gamma)}}, \\
\text{Jac}_{\mathcal{O}_B^{(\gamma)}}/k[[t]] &:= \text{Fitt}^n(\hat{\Omega}_{B^{(\gamma)}/k[[t]]}^n) \subset \mathcal{O}_{B^{(\gamma)}}.
\end{align*}

Then \( \mathcal{O}_A^{(\gamma)} \) and \( \mathcal{O}_B^{(\gamma)} \) are not necessarily equidimensional.

We define an invertible sheaf \( L_{\mathcal{O}_B^{(\gamma)}} \) on \( \mathcal{O}_B^{(\gamma)} \) by

\[ L_{\mathcal{O}_B^{(\gamma)}} := \mathcal{O}_\gamma^r \left( \det^{-1}(I'/I'^2) \right) \otimes_{\mathcal{O}_B^{(\gamma)}} \tau^* \hat{\Omega}_{A^{(\gamma)}}^n/k[[t]]. \]

Then there exist canonical homomorphisms

\[ \hat{\Omega}_{B^{(\gamma)}}^n/k[[t]] \to L_{\mathcal{O}_B^{(\gamma)}}, \quad \tau^* \omega_{B'/k[t]}^r \to L_{\mathcal{O}_B^{(\gamma)}}^r, \]

such that the following diagram commutes (cf. [NS22] Lemma 4.5(2)).

\[ \begin{CD}
\mathcal{O}_\gamma(\Omega_{B'/k[t]}^n)^\otimes r @>>> (\hat{\Omega}_{B^{(\gamma)}}^n/k[[t]])^\otimes r \\
@VVV @VVV \\
\tau^* \omega_{B'/k[t]}^r @>>> L_{\mathcal{O}_B^{(\gamma)}}^r
\end{CD} \]

Furthermore, by the same argument as in [NS22] Lemma 4.5(1), we have

\[ \text{Im} \left( \hat{\Omega}_{B^{(\gamma)}}^n/k[[t]] \to L_{\mathcal{O}_B^{(\gamma)}} \right) = \text{Jac}_{\mathcal{O}_B^{(\gamma)}}/k[[t]] \otimes_{\mathcal{O}_B^{(\gamma)}} L_{\mathcal{O}_B^{(\gamma)}}. \]

We define ideal sheaves \( \mathfrak{n}'_{1,p} \) and \( \mathfrak{n}'_{\tau,B} \) on \( \mathcal{O}_B^{(\gamma)} \) by

\[ \begin{align*}
\text{Im} \left( \hat{\Omega}_{B^{(\gamma)}}^n/k[[t]] \to L_{\mathcal{O}_B^{(\gamma)}} \right) &\subseteq \mathfrak{n}'_{1,p} \otimes_{\mathcal{O}_B^{(\gamma)}} L_{\mathcal{O}_B^{(\gamma)}}, \\
\text{Im} \left( \tau^* \Omega_{B'/k[t]}^n \to L_{\mathcal{O}_B^{(\gamma)}} \right) &\subseteq \mathfrak{n}'_{\tau,B} \otimes_{\mathcal{O}_B^{(\gamma)}} L_{\mathcal{O}_B^{(\gamma)}}.
\end{align*} \]

We define \( \text{age}'(\gamma) := \sum_{i=1}^N e_i \). Note that we took \( e_i \) to satisfy \( 0 < e_i \leq d \). The age of \( \gamma \) is usually defined by \( \text{age}(\gamma) = \text{age}'(\gamma) - \# \{ 1 \leq i \leq N \mid e_i = d \} \).
Lemma 7.6. Let $\alpha \in \overline{B}^-_\infty$ be a $k$-arc with $\text{ord}_\alpha(\text{Jac}(\overline{B}^-_\infty)/k[[t]]) < \infty$. Then the following hold.

1. $\text{ord}_\alpha(\text{Jac}(\overline{B}^-_\infty)/k[[t]]) = \text{ord}_\alpha(\text{Jac}(\overline{B}^-_\infty)/k[[t]])$.
2. $\text{ord}_\alpha(\text{Jac}(\overline{B}^-_\infty)/k[[t]]) = \text{ord}_\alpha(\text{Jac}(\overline{B}^-_\infty)/k[[t]])$.

Proof. The same proof as in [NS22, Lemma 4.6] works due to Lemma 5.30.

Lemma 7.7. Let $\alpha \in \overline{B}^-_\infty$ be a $k$-arc. Set $\alpha' := \overline{B}_\infty(\alpha)$. Suppose that $\alpha' \notin Z_\infty$. Then it follows that

$$\text{ord}_\alpha(\text{Jac}(\overline{B}^-_\infty)/k[[t]]) = \frac{1}{r} \text{ord}_\alpha(\text{Jac}(\overline{B}^-_\infty)/k[[t]]) + \text{ord}_\alpha(\text{Jac}(\overline{B}^-_\infty)/k[[t]])$$

Proof. The same proof as in [NS22, Lemma 4.7] works.

Lemma 7.8. Let $I_Z \subset \mathcal{O}_A$ be the ideal sheaf defining $Z \subset A$. Let $J$ be one of the following ideal sheaves on $\overline{B}^-_\infty$:

$$I_Z \mathcal{O}_{\overline{B}^-_\infty}, \quad \text{Jac}(\overline{B}^-_\infty)/k[[t]], \quad \text{Jac}(\overline{B}^-_\infty)/k[[t]], \quad \mathcal{O}_{\overline{B}^-_\infty},$$

$$\text{Jac}(\overline{B}^-_\infty)/k[[t]], \quad \text{Jac}(\overline{B}^-_\infty)/k[[t]], \quad \mathcal{O}_{\overline{B}^-_\infty},$$

Let $W \subset \overline{B}^-_\infty$ be the closed subscheme defined by $J$. Then $W_\infty$ is a thin subset of $\overline{B}^-_\infty$.

Proof. We set

$$J_1 := I_Z \mathcal{O}_{\overline{B}^-_\infty}, \quad J_2 := \text{Jac}(\overline{B}^-_\infty)/k[[t]], \quad J_3 := \text{Jac}(\overline{B}^-_\infty)/k[[t]], \quad J_4 := \mathcal{O}_{\overline{B}^-_\infty},$$

and we denote by $W_i \subset \overline{B}^-_\infty$ the closed subscheme defined by $J_i$.

Since $B \cap Z \subset B_{\text{sing}}$, we have $(W_1)_\text{red} \subset (W_4)_\text{red}$. Since the map $\eta_r : (\Omega^0_{B/k})^\otimes r \to \omega^r_{B/k}$ in Definition 2.13(5) is an isomorphism on the regular locus $B_{\text{reg}}$, we have $(W_7)_\text{red} \subset (W_4)_\text{red}$. By Lemma 7.7 we have

$$(W_6)_\infty \cup (W_1)_\infty = (W_7)_\infty \cup (W_1)_\infty.$$

By Lemmas 7.6 and 7.40 we have

$$(W_5)_\infty \subset (W_2)_\infty \cup (W_3)_\infty \cup (W_6)_\infty.$$

Therefore, it is sufficient to show the assertion for

$$J_2 = \text{Jac}(\overline{B}^-_\infty)/k[[t]], \quad J_3 = \text{Jac}(\overline{B}^-_\infty)/k[[t]], \quad J_4 = \mathcal{O}_{\overline{B}^-_\infty}.$$

We set

$$S_1 := k[x_1, \ldots, x_N][t], \quad S_2 := k[x_1, \ldots, x_N][t],$$

$$T_1 := k[x_1, \ldots, x_N][t], \quad T_2 := k[x_1, \ldots, x_N][t].$$

The ideal $J_2$ is $\text{Jac}(\overline{B}^-_\infty)/k[[t]]$ corresponds to the ideal

$$\mathfrak{r} := \mathcal{J}_c(\mathfrak{t}^{\otimes r}; \text{Der}_{k[[t]]}(S_1)) + \mathfrak{t}^{\otimes r} \subset S_1$$

of $S_1$. To show that $(W_2)_\infty$ is a thin set, it is sufficient to show $h(t) \geq c + 1$. Since $\mathfrak{t}^{\otimes r} S_2$ is a prime ideal of height $c$ by Lemma 7.33(3)(4), it is sufficient to show that $tS_2 \notin \mathfrak{t}^{\otimes r} S_2$. Since $S_2$ satisfies the weak Jacobian condition (WJ) over $k(t(t))$ by [Nag62, Theorem 46.3], we have

$$\mathcal{J}_c(\mathfrak{t}^{\otimes r}; \text{Der}_{k[[t]]}(S_1)) + \mathfrak{t}^{\otimes r} S_2 = \mathcal{J}_c(\mathfrak{t}^{\otimes r} S_2; \text{Der}_{k(t(t))}(S_2)) + \mathfrak{t}^{\otimes r} S_2 \notin \mathfrak{t}^{\otimes r} S_2.$$
Here, the first equality follows from the fact that both \( \text{Der}_{k[[t]]}(S_1) \) and \( \text{Der}_{k((t))}(S_2) \) are generated by the derivations \( \partial/\partial x_i \)'s. We complete the proof of the assertion for \( J_2 \).

Let \( \mathfrak{r} \subset T_1 \) be the ideal of \( T_1 \) corresponding to \( \text{Jac'}_{B^{(\gamma)}/k[[t]]} \subset T_1/\tilde{T}^{(\gamma)} \). Then it is sufficient to show \( \text{ht}(\mathfrak{r}S_2) \geq c + 1 \). Note that \( \tilde{T}^{(\gamma)} \) is \( T_2 \) holds. Therefore, \( \tilde{T}^{(\gamma)}T_2 \) is a prime ideal of height \( c \). By the same argument as above, we have \( tT_2 \not\subset \tilde{T}^{(\gamma)}T_2 \), and hence \( \text{ht}(tT_2) \geq c + 1 \). Since \( T_2 \subset S_2 \) is an integral extension, we have \( \text{ht}(tS_2) = \text{ht}(tT_2) \geq c + 1 \), which completes the proof of the assertion for \( J_3 \).

Let \( \mathfrak{r} \subset k[[x_1, \ldots, x_N]]^G \) be the ideal corresponding to \( \text{Jac'}_{B/k} \subset k[[x_1, \ldots, x_N]]^G/I \). Since \( J_4 = tO_{B^{(\gamma)}} \), it is sufficient to show \( \text{ht}(\mathfrak{r}S_2) \geq c + 1 \) and \( \mathfrak{r} \) by the normality of \( B \). Therefore, we have \( \text{ht}(tS_2) = \text{ht}(t \mathfrak{r}) \geq c + 2 \) by Lemma 7.3. We complete the proof of the assertion for \( J_4 \).

**Theorem 7.9.** Let \( x = 0 \in B \) be the origin and let \( \mathfrak{m}_x \subset O_B \) be the corresponding maximal ideal. Let \( a \subset O_B \) be a non-zero ideal sheaf and \( \delta \) a positive real number. Then

\[
\text{mld}_x(B, a^\delta) = \inf_{w,b_i \in \mathbb{Z}_{\geq 0}, \gamma, b \in G} \left\{ \text{codim}(C_{w,\gamma,b_1}) + \text{age}^\delta(\gamma) - b - \delta w \right\}
\]

\[
= \inf_{w,b_i \in \mathbb{Z}_{\geq 0}, \gamma, b \in G} \left\{ \text{codim}(C'_{w,\gamma,b_1}) + \text{age}^\delta(\gamma) - b - \delta w \right\}
\]

holds for

\[
C_{w,\gamma,b_1} := \text{Cont}^w_{-1}(aO_{B^{(\gamma)}}) \cap \text{Cont}^\geq_{-1}(I_{aO_{B^{(\gamma)}}}) \cap \text{Cont}^b_{1}(\text{Jac'}_{B^{(\gamma)}/k[[t]]}),
\]

\[
C'_{w,\gamma,b_1} := \text{Cont}^w_{-1}(aO_{B^{(\gamma)}}) \cap \text{Cont}^\geq_{-1}(I_{aO_{B^{(\gamma)}}}) \cap \text{Cont}^b_{1}(\text{Jac'}_{B'/k[[t]]}).
\]

**Proof.** The formula for \( C_{w,\gamma,b_1} \) is the formal power series ring version of [NS22, Theorem 4.8]. The same proof as in [NS22, Theorem 4.8] works. First, [EM09, Theorem 7.4] plays an important role in the proof of [NS22, Theorem 4.8] and it can be substituted by Theorem 7.11. Furthermore, Propositions 2.25, 2.33, 2.35, 3.4 and 3.8, and Lemmas 2.10, 4.6 and 4.7 in [NS22, which are also the key ingredients of the proof of [NS22, Theorem 4.8]], are substituted by Propositions 5.36, 5.43, 5.44 and 6.3 and Lemmas 5.40, 7.4 and 7.7 in this paper.

We also note that Proposition 5.36 is applied to \( Z_{\infty} \) and \( W_{\infty} \), where \( W \) is the closed subscheme of \( B^{(\gamma)} \) corresponding to one of the following ideals:

\[
\text{Jac'}_{B^{(\gamma)}/k[[t]]}, \quad \text{Jac'}_{B'/k[[t]]} \quad \text{O}_{B^{(\gamma)}}, \quad \text{Jac'}_{B'/k[[t]]} \quad \text{O}_{B^{(\gamma)}}, \quad \text{n}_1, \quad \text{n}_1', \quad \text{n}_{r,B'} \quad \text{O}_{B^{(\gamma)}}.
\]

By Lemma 7.8, they are actually thin sets.

The formula for \( C'_{w,\gamma,b_1} \) is the formal power series ring version of [NS22, Corollary 4.9], and the same proof works.

**Remark 7.10.** Theorem 7.9 can be easily extended to \( \mathbb{R} \)-ideals \( a = \prod_{i=1}^r a_i^{\delta_i} \), where \( a_1, \ldots, a_r \) are non-zero ideal sheaves on \( B \) and \( \delta_1, \ldots, \delta_r \) are positive real numbers. In this setting, we have

\[
\text{mld}_x(B, \prod_{i=1}^r a_i^{\delta_i}) = \inf_{w,b_i \in \mathbb{Z}_{\geq 0}, \gamma, b \in G} \left\{ \text{codim}(C_{w_1,\ldots,w_r,\gamma,b_1}) + \text{age}^\delta(\gamma) - b - i=1 \right. \sum_{i=1}^r \delta_i w_i \right\}
\]

\[
= \inf_{w,b_i \in \mathbb{Z}_{\geq 0}, \gamma, b \in G} \left\{ \text{codim}(C'_{w_1,\ldots,w_r,\gamma,b_1}) + \text{age}^\delta(\gamma) - b - i=1 \right. \sum_{i=1}^r \delta_i w_i \right\}
\]
for
\[ C_{w_1,\ldots,w_r,\gamma,\beta} := \left( \bigcap_{i=1}^{r} \text{Cont}^{w_i}(a_i \mathcal{O}_{\mathcal{O}_B^{\gamma}}) \right) \cap \text{Cont}^{\geq 1}(m_x \mathcal{O}_{\mathcal{O}_B^{\gamma}}) \cap \text{Cont}^b(\text{Jac}^{\prime}\mathcal{O}_{\mathcal{O}_B^{\gamma}}/k[[t]]) , \]

\[ C_{w_1,\ldots,w_r,\gamma,\beta} := \left( \bigcap_{i=1}^{r} \text{Cont}^{w_i}(a_i \mathcal{O}_{\mathcal{O}_B^{\gamma}}) \right) \cap \text{Cont}^{\geq 1}(m_x \mathcal{O}_{\mathcal{O}_B^{\gamma}}) \cap \text{Cont}^b(\text{Jac}^{\prime}\mathcal{O}_{\mathcal{O}_B^{\gamma}}/k[[t]]) . \]

7.2. Properties on $\mathcal{B}^{(\gamma)}$. In the remainder of this section, we define a scheme $\mathcal{B}^{(\gamma)}$ and investigate its properties, which will be used in Section 8.

We denote by $\overline{\mathcal{X}}_{\gamma}$ the $k[t]$-ring homomorphism
\[ \overline{\mathcal{X}}_{\gamma} : k[t][x_1,\ldots,x_N]^G \to k[t][x_1,\ldots,x_N] : x_i \mapsto t^{e_i}x_i. \]

We set
\[ \overline{\mathcal{T}}^{(\gamma)} := (\overline{\mathcal{X}}_{\gamma}(f_1),\ldots,\overline{\mathcal{X}}_{\gamma}(f_c)) \subset k[t][x_1,\ldots,x_N], \]
\[ \mathcal{B}^{(\gamma)} := \text{Spec}(k[t][x_1,\ldots,x_N]/\overline{\mathcal{T}}^{(\gamma)}). \]

Then $\mathcal{B}^{(\gamma)}$ is a scheme of finite type over $R = k[t][x_1,\ldots,x_N]$. Let $\Omega_{\mathcal{B}^{(\gamma)}}^1/k[t]$ be the sheaf of special differentials defined in Definition (2.13(1)) with respect to $R$ and $R_0 = k[t]$. We set
\[ \text{Jac}^{\prime}\mathcal{O}_{\mathcal{B}^{(\gamma)}}/k[t] := \text{Fitt}^n(\Omega_{\mathcal{B}^{(\gamma)}}^1/k[t]). \]

First, we study the dimensions of the irreducible components of $\mathcal{B}^{(\gamma)}$.

Lemma 7.11. We denote $F_i := \overline{\mathcal{X}}_{\gamma}(f_i)$ for each $1 \leq i \leq c$. Consider the following diagram of rings.
\[ \begin{array}{ccc}
S_1 := k[t][x_1,\ldots,x_N] & \to & C_1 := k[t][x_1,\ldots,x_N]/(F_1,\ldots,F_c) \\
\downarrow h_1 & & \downarrow g_1 \\
S_2 := k[t, t^{-1}][x_1,\ldots,x_N] & \to & C_2 := k[t, t^{-1}][x_1,\ldots,x_N]/(F_1,\ldots,F_c) \\
\downarrow h_2 & & \downarrow g_2 \\
S_3 := k[t^{1/d}, t^{-1/d}][x_1,\ldots,x_N] & \to & C_3 := k[t^{1/d}, t^{-1/d}][x_1,\ldots,x_N]/(F_1,\ldots,F_c) \\
\simeq & & \simeq \\
S_4 := k[t^{1/d}, t^{-1/d}][x_1,\ldots,x_N] & \to & C_4 := k[t^{1/d}, t^{-1/d}][x_1,\ldots,x_N]/(f_1,\ldots,f_c) \\
\downarrow h_3 & & \downarrow g_3 \\
S_5 := k[x_1,\ldots,x_N] & \to & C_5 := k[x_1,\ldots,x_N]/(f_1,\ldots,f_c)
\end{array} \]

We denote $I_i := (F_1,\ldots,F_c) \subset S_i$ for $i \in \{1,2,3\}$, and $I_i := (f_1,\ldots,f_c) \subset S_i$ for $i \in \{4,5\}$. Then the following hold.

(1) $h_1$, $h_2$ and $h_3$ are regular, and hence so are $g_1$, $g_2$ and $g_3$.
(2) $h_2$ and $h_3$ are faithfully flat, and hence so are $g_2$ and $g_3$.
(3) $C_2$, $C_3$ and $C_4$ are normal domains. In particular, $I_2$, $I_3$ and $I_4$ are prime ideals.
(4) $\text{ht}(I_2) = \text{ht}(I_3) = \text{ht}(I_4) = c$.
(5) There exists only one minimal prime $P$ of $I_1$ of the form (2) in Lemma 5.28. Furthermore, it satisfies $\text{ht}P = c$ and $P = I_2 \cap S_1$. 


Proof. We shall prove (1). Since the ring inclusion $k[t, t^{-1}] \to k[t^{1/d}, t^{-1/d}]$ is étale, so is its base change

$$k[t, t^{-1}][x_1, \ldots, x_N] \to k[t, t^{-1}][x_1, \ldots, x_N][t^{1/d}, t^{-1/d}].$$

Furthermore, the ring inclusion

$$k[t, t^{-1}][x_1, \ldots, x_N][t^{1/d}, t^{-1/d}] \to k[t^{1/d}, t^{-1/d}][x_1, \ldots, x_N]$$

is regular since it can be seen as the completion at the prime ideal $(x_1, \ldots, x_N)$ and the ring on the left side is an excellent ring, in particular a G-ring (cf. [Mat80, Theorem 79]). Therefore, their composition $h_2$ turns out to be regular.

We shall prove (2) for $h_2$. Any maximal ideal $M$ of $S_2$ is of the form $M = (t - a, x_1, \ldots, x_N)$, where $a \in k^\times$. Therefore, we have $MS_3 \neq S_3$ and hence $h_2$ is faithfully flat. The same proof works for $h_3$.

We shall prove (3). Note that $C_3$ is normal by our assumption. Therefore, the normality of $C_2, C_3$ and $C_4$ follows from (1), (2) and the fact that the normality is preserved under faithfully flat regular ring homomorphisms (cf. [Mat89, Theorem 32.2]). In what follows, we prove that $C_2, C_3$ and $C_4$ are domains. Since $h_2$ is faithfully flat, $g_2$ is injective (cf. [Mat89, Theorem 7.5]). Therefore, it is sufficient to show that $C_4$ is a domain. Let $P_1, \ldots, P_\ell$ be the minimal primes of $I_4$. Suppose the contrary that $\ell \geq 2$. Since $C_4$ is normal, we have $P_1 + P_2 = S_4$ (cf. [Eis95, Proposition 2.20]). Take maximal ideals $M_1$ and $M_2$ of $S_4$ such that $P_i \subset M_i$ for each $i \in \{1,2\}$. We may write $M_i = (t^{1/d} - a_i, x_1, \ldots, x_N)$ with $a_i \in k^\times$. For each $c \in k^\times$, we denote by $T_c$ the ring isomorphism

$$T_c : S_4 \to S_4 : i^{1/d} \mapsto c i^{1/d}, \quad x_i \mapsto x_i.$$

Then, $I_4$ is $T_c$-invariant for any $c \in k^\times$. Therefore, its minimal primes $P_1$ and $P_2$ are also $T_c$-invariant for any $c \in k^\times$. Therefore $P_1 \subset M_2$ holds, and hence $P_1 + P_2 \subset M_2$, a contradiction.

Note that $h(I_5) = c$ by our assumption. Therefore, (4) follows from (2) and [Mat89, Theorem 7.2].

We shall prove (5). Let $P_1, \ldots, P_\ell$ be the minimal primes of $I_1$. Then $P_1^{a_1} \cap \cdots \cap P_\ell^{a_\ell} \subset I_1$ holds for some $a_1, \ldots, a_\ell \geq 1$. Since $h_1$ is flat, we have

$$I_2 = I_1 S_2 \supset (P_1^{a_1} \cap \cdots \cap P_\ell^{a_\ell}) S_2 = P_1^{a_1} S_2 \cap \cdots \cap P_\ell^{a_\ell} S_2$$

by [Mat89, Theorem 7.4]. If $P_i$ is of the form (1) or (3) in Lemma 5.25, then we have $P_i S_2 = S_2$. Since $I_2 \neq S_2$, some $P_i$ has to be of the form (2) in Lemma 5.25. Suppose that $P$ is a minimal prime of $I_1$ of the form (2) in Lemma 5.25. Then $P + (t - a) \neq S_1$ holds for any $a \in k^\times$, and hence we have $PS_2 \neq S_2$. Since we have

$$c = h(I_2) \leq h(PS_2) = h(P) \leq c$$

by Lemma 7.2 and Krull’s height theorem, it follows that $h(P) = c$ and $I_2 = PS_2$. Since we have

$$c = h(P) \leq h(I_2 \cap S_1) \leq h(I_2) = c$$

by the going-down theorem, we have $P = I_2 \cap S_1$, which also shows the uniqueness of $P$.

**Remark 7.12.** Let $\overline{B}^{(\gamma)} = V_1 \cup \cdots \cup V_t$ be the irreducible decomposition. By Lemma 7.11(3), $\overline{B}^{(\gamma)}$ has the unique irreducible component $V$ of the form (2) in Lemma 5.25 and it satisfies $\dim V = \dim' V = n + 1$. Furthermore, any irreducible component $V'$ other than $V$ satisfies

$$V'_\infty \cap \text{Cont}^1(\sigma_{\overline{B}^{(\gamma)}}) = \emptyset$$

by Remark 5.26. Here, $\sigma_{\overline{B}^{(\gamma)}} \subset \overline{B}^{(\gamma)}(\gamma)$ denotes the ideal sheaf generated by $x_1, \ldots, x_N$. 
Next, we see the relationship between \( \overline{B}^{(\gamma)}_\infty \) and \( \overline{B}^{(\gamma)}_\infty \).

**Lemma 7.13.** Let \( a^{(\gamma)}_{\overline{B}} \subset \mathcal{O}_{\overline{B}^{(\gamma)}} \) and \( a^{(\gamma)}_{\overline{B}} \subset \mathcal{O}_{\overline{B}^{(\gamma)}} \) be the ideal sheaves generated by \( x_1, \ldots, x_N \). Then the following hold.

1. For \( m \geq 0 \), there exist canonical morphisms \( \overline{B}^{(\gamma)}_m \to \overline{B}^{(\gamma)}_m \) which commute with the truncation morphisms. In particular, they induce \( \overline{B}^{(\gamma)}_\infty \to \overline{B}^{(\gamma)}_\infty \).

2. The map \( \overline{B}^{(\gamma)}_\infty \to \overline{B}^{(\gamma)}_\infty \) induces an isomorphism
   \[
   \overline{B}^{(\gamma)}_\infty \cap \text{Cont}^{\geq 1}(a^{(\gamma)}_{\overline{B}}) \cong \overline{B}^{(\gamma)}_\infty \cap \text{Cont}^{\geq 1}(a^{(\gamma)}_{\overline{B}}).
   \]

3. For a \( k \)-arc \( \gamma \in \overline{B}^{(\gamma)}_\infty \cap \text{Cont}^{\geq 1}(a^{(\gamma)}_{\overline{B}}) \) and the corresponding arc \( \gamma' \in \overline{B}^{(\gamma)}_\infty \cap \text{Cont}^{\geq 1}(a^{(\gamma)}_{\overline{B}}) \), it follows that
   \[
   \text{ord}_\gamma \left( \text{Jac}_{\overline{B}^{(\gamma)}/k[t]} \right) = \text{ord}_{\gamma'} \left( \text{Jac}_{\overline{B}^{(\gamma)}/k[t]} \right).
   \]

**Proof.** (1) and (2) follow from the discussion in Remark 5.29(2). (3) follows from Remarks 2.14 and 5.29(3). \( \square \)

**Lemma 7.14.** Let \( W \subset \overline{B}^{(\gamma)}_\infty \) be the closed subscheme defined by \( \text{Jac}_{\overline{B}^{(\gamma)}/k[t]} \). Then \( W_\infty \cap \text{Cont}^{\geq 1}(a^{(\gamma)}_{\overline{B}}) \) is a thin subset of \( \overline{B}^{(\gamma)}_\infty \).

**Proof.** Let \( T_c : k[t][x_1, \ldots, x_N] \to k[t][x_1, \ldots, x_N] \) be the ring isomorphism defined in Lemma 5.25.

Set \( J := \text{Jac}_{\overline{B}^{(\gamma)}/k[t]} \). Let \( J \subset k[t][x_1, \ldots, x_N] \) be the corresponding ideal. Since \( \overline{B}^{(\gamma)}_\infty \) is \( T_c \)-invariant, \( J \) is also \( T_c \)-invariant for each \( c \in k \times \). Therefore by Lemma 5.25 each minimal prime \( P \) of \( J \) satisfies one of the conditions in Lemma 5.25. By Remark 5.26 it is sufficient to show that \( h \text{t}(P) \geq c + 1 = N - n + 1 \) for \( P \) satisfying (2) in Lemma 5.25. Since \( P + (t - 1) \neq k[t][x_1, \ldots, x_N] \) in this case, it is sufﬁcient to show that
   \[
   h \text{t}(P + (t - 1)) \geq N - n + 2.
   \]

Under the identiﬁcation \( k[t][x_1, \ldots, x_N]/(t - 1) \simeq k[x_1, \ldots, x_N] \), the ideal \( (\overline{J} + (t - 1))/(t - 1) \) corresponds to \( \text{Jac}_{\overline{B}/k} \). Since \( \overline{B} \) is normal and hence regular in codimension one, it follows that
   \[
   h \text{t}(\overline{J} + (t - 1)) \geq N - n + 3
   \]
by the Jacobian criterion of regularity. Therefore, we get the desired inequality for \( P \). \( \square \)

8. PIA Formula for Quotient Singularities of Non-linear Action

In this section, we generalize Theorem 5.1 in [NS22] to non-linear group actions (Theorem 8.2). First, we clarify the deﬁnition of quotient singularities in this paper.

**Deﬁnition 8.1.** Let \( X \) be a variety over \( k \) and \( x \in X \) a closed point. We say that \( X \) has a quotient singularity at \( x \) if there exist a quasi-projective variety \( \overline{M} \) over \( k \), a ﬁnite subgroup \( G \subset \text{Aut}(\overline{M}) \), and a smooth closed point \( \overline{y} \in \overline{M} \) such that \( \mathcal{O}_{X,x} \simeq \mathcal{O}_{\overline{M},y} \) holds, where \( M := \overline{M}/G \) is the quotient variety and \( y \in M \) is the image of \( \overline{y} \).

**Theorem 8.2.** Suppose that a variety \( X \) has a quotient singularity at a closed point \( x \in X \). Let \( Y \) be a subvariety of \( X \) of codimension \( c \) that is locally deﬁned by \( c \) equations \( h_1, \ldots, h_c \in \mathfrak{m}_{X,x} \) at \( x \). Suppose that \( Y \) is klt at \( x \). Let \( a \subset \mathcal{O}_X \) be an
ideal sheaf and let $\delta$ be a positive real number. Suppose that $b := aO_Y \neq 0$. Then it follows that

$$\mld_x(X, (h_1 \cdots h_c) a^\delta) = \mld_y(Y, b^\delta).$$

Proof. Since $X$ has a quotient singularity at $x$, there exist a variety $\overline{M}$ with a smooth closed point $y \in \overline{M}$ and a finite subgroup $G' \subset \text{Aut}(\overline{M})$ such that $\mathcal{O}_{X,x} \simeq \mathcal{O}_{M',y'}$ holds, where $M' := \overline{M}/G'$ and $y' \in M'$ is the image of $y$.

We denote $G := \{g \in G \mid g(y) = y\}$ the stabilizer group of $y$, $M := \overline{M}/G$ the quotient variety, and $y \in M$ the image of $y$. Then we note that $M = \overline{M}/G \rightarrow M' = \overline{M}/G'$ is étale at $y$ (cf. [Kol13, 3.17]). Furthermore, $G$ acts on $m_{\overline{M},y}$ for each $i \geq 0$, and hence the projection $s : m_{\overline{M},y} \rightarrow m_{M,y}/m_{M,y}^2$ becomes a $G$-equivariant $k$-linear map. Let $u$ be any $k$-linear section of $s$. Then the map $u' := \frac{1}{[G]} \sum_{g \in G} g^{-1} \circ u \circ g$ gives a $G$-equivariant $k$-linear section $u' : m_{\overline{M},y}/m_{\overline{M},y}^2 \rightarrow m_{\overline{M},y}$ of $s$. Let $N := \dim X$.

Then $u'$ induces a ring homomorphism

$$k[x_1, \ldots, x_N] \rightarrow \mathcal{O}_{\overline{M},y}$$

which is étale. Furthermore, $G$ acts linearly on $k[x_1, \ldots, x_N]$ and the ring homomorphism above becomes $G$-equivariant. Since the ring homomorphism above is étale, we get an isomorphism

$$k[[x_1, \ldots, x_N]] \cong \mathcal{O}_{\overline{M},y}$$

Note that $k[[x_1, \ldots, x_N]]^{G_{pr}} \simeq k[[x_1, \ldots, x_N]]$ holds for $G_{pr} \subset G$, where $G_{pr}$ is the subgroup generated by the pseudo-reflections (cf. [Kol13 3.18]). Hence $\overline{M}/G_{pr}$ is smooth at the image of $y$. Therefore, by replacing $G$ with $G/G_{pr}$ and $\overline{M}$ with $\overline{M}/G_{pr}$, we may assume that $G$ does not contain a pseudo-reflection. Then we have the following diagram of rings.

$$\begin{array}{ccc}
\mathcal{O}_{X,x} & \xrightarrow{\cong} & \mathcal{O}_{\overline{M},y} \\
\mathcal{O}_{M',y'} & \text{étale} & \\
\mathcal{O}_{M,y} & \cong & \mathcal{O}_{\overline{M},y} \\
\mathcal{O}_{\overline{M},y} & \cong & k[[x_1, \ldots, x_N]]^G \\
\mathcal{O}_{\overline{M},y} & \cong & k[[x_1, \ldots, x_N]]
\end{array}$$

We denote by $(\overline{N}, \overline{y}) \subset (\overline{M}, y)$ the germ defined by the images of $h_1, \ldots, h_c \in \mathfrak{m}_{X,x}$ in $\mathcal{O}_{\overline{M},y}$. Let $f_1, \ldots, f_c \in k[[x_1, \ldots, x_N]]^G$ be the images of $h_1, \ldots, h_c \in \mathfrak{m}_{X,x}$. Then we have an isomorphism

$$\mathcal{O}_{\overline{N},\overline{y}} \cong k[[x_1, \ldots, x_N]]/(f_1, \ldots, f_c).$$

We set

$$A := \text{Spec } k[[x_1, \ldots, x_N]]^G, \quad B := \text{Spec}(k[[x_1, \ldots, x_N]]^G/(f_1, \ldots, f_c)),$$

and $x' \in A$ the origin. Let $a' \subset \mathcal{O}_A$ and $b' \subset \mathcal{O}_B$ be the ideal sheaves corresponding to $a \subset \mathcal{O}_X$ and $b \subset \mathcal{O}_Y$. Since

$$\mld_x(X, (h_1 \cdots h_c) a^\delta) = \mld_{x'}(A, (f_1 \cdots f_c) a'^\delta), \quad \mld_x(Y, b^\delta) = \mld_{x'}(B, b'^\delta)$$

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hold (cf. Remark 3.1), it is sufficient to show that

\[(\diamond) \quad \text{mld}_x(A, (f_1 \cdots f_s)^{d_0}) = \text{mld}_y(B, b^{d_0}).\]

Note that \(G, A\) and \(B\) satisfy all assumptions in Section 7. Therefore we take all notation. Since \(Y\) is klt at \(x\), so is \(B\). Since \(\overline{B} \rightarrow B\) is étale in codimension one, it follows that the germ \((\overline{X}, \overline{y})\) is also klt. This fact will be used in the proof of Claim 8.3.

\[(\diamond)\] is proved for \(k\)-varieties in [NS22] Theorem 5.1. The key ingredients of the proof of [NS22] Theorem 5.1 are Corollary 4.9, Lemma 2.34 and Claim 5.2 in [NS22]. First, Corollary 4.9 in [NS22] can be substituted by Theorem (6.2) in the formal power series ring setting. Second, Lemma 2.34 in [NS22] is still true in our setting by replacing \(k[t][x_1, \ldots, x_N]\) with \(k[x_1, \ldots, x_N][[t]]\) (cf. Remark 5.42). On the other hand, the proof of Claim 5.2 in [NS22] does not work directly because they use the result on the rational connectedness proved by Hacon and McKernan [HM07], which is not clear for the formal power series ring setting. We also note the lack of [NS22] Lemma 2.27 in our setting. It will be substituted by Proposition 5.23.

In what follows, we shall only prove Claim 5.2 in [NS22] in the formal power series ring setting.

**Claim 8.3 (cf. [NS22] Claim 5.2)].** Let \(C \subset \overline{A}_{\infty}^{(\gamma)}\) be a cylinder that is the intersection of finitely many cylinders of the form \(\text{Cont}^2(\epsilon O_A^{(\gamma)})\), where \(\epsilon \subset O_A\) is an ideal sheaf on \(A\) and \(\ell\) is a non-negative integer. Let \(C'\) be an irreducible component of \(C\). Then \(C' \cap \overline{B}_{\infty}^{(\gamma)}\) contains a \(k\)-arc \(\delta\) such that \(\text{ord}_\delta(\text{Jac}_{\overline{B}_{\infty}^{(\gamma)}}/k[[t]]) < \infty\).

**Proof of Claim 8.3** First, we introduce a \(k\)-action on the arc space \(\overline{A}_{\infty}^{(\gamma)}\) as follows. For a \(k\)-arc \(\alpha \in \overline{A}_{\infty}^{(\gamma)}\), we denote \(g^\alpha_t := \alpha^*(x_i) \in k[[t]]\), where \(\alpha^*: k[x_1, \ldots, x_N][[t]] \rightarrow k[[t]]\) is the corresponding \(k[t]\)-ring homomorphism. Let \(\alpha \in \overline{A}_{\infty}^{(\gamma)}\) and \(a \in k\). Then we define \(a \cdot \alpha \in \overline{A}_{\infty}^{(\gamma)}\) by

\[g^\alpha_t(t) := a^\epsilon g^\alpha_t(a^d t).\]

Then for \(f \in k[[x_1, \ldots, x_N]]^G\), we have \(v(t) = u(a^d t)\) for \(u(t) := \alpha^*(\overline{X}_\gamma(f))\), \(v(t) := (a \cdot \alpha)^*(\overline{X}_\gamma(f)) \in k[[t]]\).

Therefore, we have

\[\text{ord}_a(\overline{X}_\gamma(f)) = \text{ord}_{a \cdot \alpha}(\overline{X}_\gamma(f))\]

if \(a \in k^\times\). Hence, any cylinder of the form \(\text{Cont}^2(\epsilon O_A^{(\gamma)})\) with an ideal \(\epsilon \subset O_A = k[[x_1, \ldots, x_N]]^G\) is invariant under the \(k\)-action. Therefore, \(C\) in the statement and its irreducible component \(C'\) are also invariant under the \(k\)-action.

We denote by \(\beta \in \overline{A}_{\infty}^{(\gamma)}\) the trivial arc determined by \(g^\beta_i = 0\) for each \(i\). Note that \(\beta = 0 \cdot \alpha\) holds for any \(k\)-arc \(\alpha \in \overline{A}_{\infty}^{(\gamma)}\). Therefore, we have \(\beta \in C'\) and hence

\[\beta \in C' \cap \text{Cont}^{\geq 1}(\alpha_{\overline{B}_{\infty}^{(\gamma)}}) \neq \emptyset.\]

By Lemma 7.13(1)(2), there exists a cylinder \(D \subset \overline{B}_{\infty}^{(\gamma)}\) that is isomorphic to \(C' \cap \text{Cont}^{\geq 1}(\alpha_{\overline{B}_{\infty}^{(\gamma)}})\) under the map in Lemma 7.13(2). Then by Lemma 7.13(3), it is sufficient to show that \(D\) contains a \(k\)-arc \(\delta\) such that \(\text{ord}_\delta(\text{Jac}_{\overline{B}_{\infty}^{(\gamma)}}/k[[t]]) < \infty\).

Therefore, Claim 8.3 follows from Claim 8.4 below and Lemma 7.14. □

**Claim 8.4.** Let \(D \subset \overline{B}_{\infty}^{(\gamma)}\) be a cylinder contained in \(\text{Cont}^{\geq 1}(\alpha_{\overline{B}_{\infty}^{(\gamma)}})\). If \(D\) contains the trivial arc \(\beta\), then \(D\) is not a thin subset of \(\overline{B}_{\infty}^{(\gamma)}\).
Proof of Claim 8.2. Let $T \subset \overline{B}^{(\gamma)}$ be the closed subscheme defined by $\mathcal{O}_{\overline{m}^{\gamma}}$. First, we prove the following claim.

(♣) $\overline{B}^{(\gamma)}$ has a desingularization $r : W \to \overline{B}^{(\gamma)}$ with the following conditions.

1. For each $a \in k^\times$, the closed subscheme $W_a \subset W$ defined by the ideal $(t-a)\mathcal{O}_W \subset \mathcal{O}_W$ is an integral regular scheme with $\dim W_a = n$.

2. $r|_{T'} : T' \to T$ has a section, where $T' := r^{-1}(T)$.

We set $F_i := \mathbb{A}_y^r(f_i) = f_i(t^{e_i}x_1, \ldots, t^{e_i}x_N)$.

Then, we have the following natural morphisms.

$$V_1 := \overline{B}^{(\gamma)} = \text{Spec}(k[t][x_1, \ldots, x_N]/(F_1, \ldots, F_c))$$

$$V_2 := \text{Spec}(k[t, t^{-1}][x_1, \ldots, x_N]/(F_1, \ldots, F_c))$$

$$V_3 := \text{Spec}(k[t^{1/d}, t^{-1/d}][x_1, \ldots, x_N]/(f_1, \ldots, f_c))$$

$$V_4 := \text{Spec}(\mathcal{O}_{\overline{m}^{\gamma}}[t^{1/d}, t^{-1/d}])$$

$$V_5 := \text{Spec}(\mathcal{O}_{\overline{m}^{\gamma}})$$

Note that these four morphisms are regular morphisms (cf. Lemma 7.11(1)). Hence by the functorial desingularization by Temkin [Tem12, Theorem 1.2.1], there exist desingularizations $r_i : W_i \to V_i$ with the following Cartesian diagram.

$$W_1 \xrightarrow{r_1} V_1 := \text{Spec}(k[t][x_1, \ldots, x_N]/(F_1, \ldots, F_c))$$

$$W_2 \xrightarrow{r_2} V_2 := \text{Spec}(k[t, t^{-1}][x_1, \ldots, x_N]/(F_1, \ldots, F_c))$$

$$W_3 \xrightarrow{r_3} V_3 := \text{Spec}(k[t^{1/d}, t^{-1/d}][x_1, \ldots, x_N]/(f_1, \ldots, f_c))$$

$$W_4 \simeq W_5 \times (\mathbb{A}_k^{1} \setminus \{0\}) \xrightarrow{r_4} V_4 := \text{Spec}(\mathcal{O}_{\overline{m}^{\gamma}}[t^{1/d}, t^{-1/d}]) = V_5 \times (\mathbb{A}_k^{1} \setminus \{0\})$$

$$W_5 \xrightarrow{r_5} V_5 := \text{Spec}(\mathcal{O}_{\overline{m}^{\gamma}})$$

We shall prove that $r_1$ satisfies the conditions (1) and (2) in (♣).

We shall prove (1). For each $i \in \{1, 2\}$ and $a \in k^\times$, we denote by $(W_i)_a \subset W_i$ and $(V_i)_a \subset V_i$ the closed subschemes defined by $(t-a)\mathcal{O}_{W_i} \subset \mathcal{O}_{W_i}$ and $(t-a)\mathcal{O}_{V_i} \subset \mathcal{O}_{V_i}$, respectively. Similarly, for each $i \in \{3, 4\}$ and $a \in k^\times$, we denote by $(W_i)_a \subset W_i$ and $(V_i)_a \subset V_i$ the closed subschemes defined by $(t^{1/d}-a)\mathcal{O}_{W_i} \subset \mathcal{O}_{W_i}$ and $(t^{1/d}-a)\mathcal{O}_{V_i} \subset \mathcal{O}_{V_i}$, respectively. Then, the Cartesian diagram above induces the
following Cartesian diagram for each \( a \in k^\times \).

\[
\begin{array}{ccc}
(W_1)_a & \rightarrow & (V_1)_a \\
\downarrow & & \downarrow \\
(W_2)_a & \rightarrow & (V_2)_a \\
\downarrow & & \downarrow \\
\bigcup_{b' = a}(W_3)_b & \rightarrow & \bigcup_{b' = a}(V_3)_b \\
\downarrow & & \downarrow \\
\bigcup_{b' = a}(W_4)_b & \rightarrow & \bigcup_{b' = a}(V_4)_b
\end{array}
\]

Here, by construction, we have \((V_1)_a \simeq (V_2)_a \simeq (V_3)_b\) for any \( a, b \in k^\times \) with \( b'^d = a \). Therefore, we also have \((W_1)_a \simeq (W_2)_a \simeq (W_3)_b\). Since \( W_4 \simeq W_5 \times (A^1_k \setminus \{0\})\), \((W_4)_b\) turns out to be regular for any \( b \in k^\times \). Since \((W_3)_b \rightarrow (W_4)_b\) is a regular morphism, \((W_3)_b\) is also regular (cf. [Mat99] Theorem 32.2). Note that the morphism \((V_3)_b \rightarrow (V_4)_b \simeq V_5\) is isomorphic to the completion map

\[
\mathcal{B} = \text{Spec}(\mathcal{O}_{N, \mathcal{B}}) \rightarrow \text{Spec}(\mathcal{O}_{N, \mathcal{F}}) = V_5.
\]

Since \( W_5 \rightarrow V_5\) is a birational map, so is \((W_3)_b \rightarrow (V_3)_b\). Since \((V_3)_b \simeq \mathcal{B}\) is integral, so is \((W_3)_b\). Furthermore, we have \(\dim'(W_3)_b = \dim'(V_3)_b = \dim(V_3)_b = n\). Therefore, for any \( a \in k^\times\), \((V_1)_a\) is an integral regular scheme with \(\dim'(V_1)_a = n\).

We shall prove (2). Let \( T_i \subset V_i \) be the closed subschemes defined by the ideals \((x_1, \ldots, x_N)\mathcal{O}_{V_i}\) for \( i \in \{1, 2, 3\} \) and by the ideal \( m_{\mathcal{N}, \mathcal{B}} \mathcal{O}_{V_4}\) for \( i = 4\). Let \( T'_i := r_i^{-1}(T_i)\). Then, we have the following Cartesian diagram.

\[
\begin{array}{ccc}
T'_1 & \rightarrow & T_1 \simeq A^1_k \\
\downarrow & & \downarrow \\
T'_2 & \rightarrow & T_2 \simeq A^1_k \setminus \{0\} \\
\downarrow & & \downarrow \\
T'_3 & \rightarrow & T_3 \simeq A^1_k \setminus \{0\} \\
\uparrow & & \uparrow \\
T'_4 & \rightarrow & T_4 \simeq A^1_k \setminus \{0\}
\end{array}
\]

Since the above diagram forms a Cartesian diagram, any closed fiber of \( T'_2 \rightarrow T_2 \) is isomorphic to some fiber of \( T'_4 \rightarrow T_4 \). Since the germ \((N, \mathcal{F})\) is klt, so is \( V_4\). Therefore, \( T'_4 \rightarrow T_4\) has rationally connected fibers by [HM07] Corollary 1.7(1)], and so does \( T_2 \rightarrow T_2\). Therefore \( T'_2 \rightarrow T_2\) has a section by [GHS03]. Hence by the properness of \( r_1\), the morphism \( T'_1 \rightarrow T_1\) also has a section. We have proved the claim (\(\spadesuit\)).

Note that \( T_\infty = \{\beta\}\) consists of only one arc \( \beta\). By claim (\(\spadesuit\)), there exists \( \beta' \in T'_\infty \subset W_\infty\) such that \( r_\infty(\beta') = \beta\). Suppose the contrary that \( D\) is a thin subset of \( B^\infty(\gamma)\). Since \( \beta' \in r_\infty^{-1}(D)\) satisfies \(\text{ord}_{\beta'}(a_W) = \infty\), to get a contradiction by Proposition [5.23], it is sufficient to show that \( r_\infty^{-1}(D)\) is also a thin subset of \( W_\infty\).

By Lemma [7.11(5)], there exists the unique irreducible component \( U\) of \( B^\infty(\gamma)\) of the form (2) in Lemma [5.25]. We also note that \( U_\infty \cap D = \emptyset\) holds for any other irreducible component \( U'\) of \( B^\infty(\gamma)\), since \( D \subset \text{Cont}^{\geq 1}(a_{\mathcal{F}(\gamma)})\) (cf. Remark [5.26]). Therefore, since \( D\) is a thin set, there exists a closed subscheme \( F \subset U\) such that \( D \subset F_\infty\) and \( \dim F \leq n\). Let \( W'\) be the connected component of \( W\) that dominates \( U\). We note
that \( \dim W' = \dim U = n + 1 \) by Lemma 7.11.15. We set \( F' := r^{-1}(F) \cap W' \). Then we have \( r^{-\infty}_x(D) \subset F'_x \) and \( \dim F' \leq n \), and hence \( r^{-\infty}_x(D) \) is also a thin set. We complete the proof of Claim 8.4.

We complete the proof of Theorem 8.2.

Remark 8.5. Theorem 8.2 can be generalized to \( \mathbb{R} \)-ideals due to Remark 7.10. We have

\[
\operatorname{mld}_x(X, (h_1 \cdots h_c)a) = \operatorname{mld}_x(Y, b).
\]

for an \( \mathbb{R} \)-ideal \( a \) on \( X \) and \( b := aO_Y \).

9. PROOF OF THE MAIN THEOREMS

As a corollary of Theorem 8.2, we prove the PIA conjecture for quotients of locally complete intersection singularities.

Corollary 9.1. Suppose that a variety \( X \) has a quotient singularity at a closed point \( x \in X \). Let \( Y \) be a subvariety of \( X \) of codimension \( c \) that is locally defined by \( c \) equations at \( x \). Suppose that \( Y \) is klt at \( x \). Let \( a \) be an \( \mathbb{R} \)-ideal sheaf on \( Y \). Let \( D \) be a prime divisor on \( Y \) through \( x \) that is klt and Cartier at \( x \). Suppose that \( D \) is not contained in the cosupport of the \( \mathbb{R} \)-ideal sheaf \( a \). Then it follows that

\[
\operatorname{mld}_x(Y, aO_Y(−D)) = \operatorname{mld}_x(D, aO_D).
\]

Proof. Take an \( \mathbb{R} \)-ideal sheaf \( b \) on \( X \) such that \( a = bO_Y \), and take local equations \( h_1, \ldots, h_c \in O_{X,x} \) of \( Y \) in \( X \) at \( x \). Furthermore, take \( g \in O_{X,x} \) such that its image \( \overline{g} \) in \( O_{Y,x} \) defines \( D \). Then we have

\[
\operatorname{mld}_x(Y, aO_Y(−D)) = \operatorname{mld}_x(X, (h_1 \cdots h_c \cdot g)b) = \operatorname{mld}_x(D, aO_D)
\]

by applying Theorem 8.2 twice.

Theorem 9.2. Let \( X \) be a variety with only quotient singularities. Let \( Y \) be a klt subvariety of \( X \) of codimension \( c \) that is locally defined by \( c \) equations in \( X \). Let \( a \) be an \( \mathbb{R} \)-ideal sheaf on \( Y \). Then the function

\[
|Y| \rightarrow \mathbb{R}_{\geq 0} \cup \{-\infty\}; \quad x \mapsto \operatorname{mld}_x(Y, a)
\]

is lower semi-continuous, where we denote by \( |Y| \) the set of all closed points of \( Y \) with the Zariski topology.

Proof. We take over the notation in the proof of Corollary 9.1. Then we have

\[
\operatorname{mld}_x(Y, a) = \operatorname{mld}_x(X, (h_1 \cdots h_c)b)
\]

by Theorem 8.2. Then the assertion follows from the fact proved in [Nak16] Corollary 1.3 that the lower semi-continuity holds for the variety \( X \) with only quotient singularities.

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