AN EXTENDED NOVIKOV-TYPE CRITERION FOR LOCAL MARTINGALES WITH JUMPS

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Abstract. For local martingales with nonnegative jumps, we prove a sufficient criterion for the corresponding exponential martingale to be a true martingale. The criterion is in terms of exponential moments of a convex combination of the optional and predictable quadratic variation. The result extends earlier known criteria.

1. Introduction

In [16], Novikov introduced a sufficient criterion for the exponential martingale of a continuous local martingale to be a uniformly integrable martingale. In this paper, we prove a similar result in the case where the local martingale is not continuous, but is assumed to have nonnegative jumps. The novelty of our criterion rests in that our result is stronger than previously known results, in that it combines optional and predictable components and in that our proof of the criterion demonstrates a straightforward two-step structure. We begin by fixing our notation and recalling some results from stochastic analysis.

Assume given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ satisfying the usual conditions, see [19] for the definition of this and other probabilistic concepts. For any local martingale $M$, we say that $M$ has initial value zero if $M_0 = 0$. For any local martingale $M$ with initial value zero, we denote by $[M]$ the quadratic variation of $M$, that is, the unique increasing adapted process with initial value zero such that $M^2 - [M]$ is a local martingale.

If $A$ is an adapted increasing process with initial value zero, we say that $A$ is integrable if $EA_\infty$ is finite, and we say that $A$ is locally integrable if $A_{T_n}$ is integrable for some localising sequence $(T_n)$, that is, a sequence of stopping times increasing to infinity. If $A$ is an adapted process with initial value zero and paths of finite variation, we say that $A$ is locally integrable if the variation process is locally integrable. Whenever $A$ is adapted, has initial value zero, is of finite

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variation and is locally integrable, there exists a predictable process $\Pi^*_pA$ with those same properties such that $A - \Pi^*_pA$ is a local martingale, see Définition VI.21.3 of [20]. We refer to $\Pi^*_pA$ as the dual predictable projection of $A$, or simply as the compensator of $A$.

If $M$ is locally square integrable, it holds that $[M]$ is locally integrable, and we denote by $\langle M \rangle$ the compensator of $[M]$. We refer to $\langle M \rangle$ as the predictable quadratic variation of $M$. It then holds that $M^2 - \langle M \rangle$ is a local martingale.

For any local martingale with initial value zero, there exists by Theorem 7.25 of [3] a unique decomposition $M = M^c + M^d$, where $M^c$ is a continuous local martingale and $M^d$ is a purely discontinuous local martingale, both with initial value zero. Here, we say that a local martingale with initial value zero is purely discontinuous if it has zero quadratic covariation with any continuous local martingale with initial value zero. We refer to $M^c$ as the continuous martingale part of $M$, and refer to $M^d$ as the purely discontinuous martingale part of $M$.

With $M$ a local martingale with initial value zero and $\Delta M \geq 0$, the exponential martingale of $M$, also known as the Doléans-Dade exponential of $M$, is given by

$$
E(M)_t = \exp \left( M_t - \frac{1}{2} [M]_t \right) \prod_{0 < s \leq t} (1 + \Delta M_s) \exp(-\Delta M_s).$$

The process $E(M)$ is the unique càdlàg solution in $Z$ to the stochastic differential equation $Z_t = 1 + \int_0^t Z_s - dM_s$, see Theorem II.37 of [19]. By Theorem 9.2 of [3], $E(M)$ is always a local martingale with initial value one. We are interested in sufficient criteria to ensure that $E(M)$ is a uniformly integrable martingale. This is a classical question in probability theory, with applications for example in finance, stochastic differential equations and statistical inference for continuously observed stochastic processes, see for example [16], [1], [7], [8] or [12]. For the case when $M$ is continuous, sufficient criteria ensuring that $E(M)$ is a uniformly integrable martingale have been obtained in [16], [2], [9], [10] and [15]. For the case when $M$ has jumps, see [13], [4], [17], [22] and [6].

We now explain the particular result to be obtained in this paper. In [16], the following result was obtained: If $M$ is a continuous local martingale with initial value zero and $\exp(\frac{1}{2}[M]_\infty)$ is integrable, then $E(M)$ is a uniformly integrable martingale. This criterion is known as Novikov’s criterion. In [11], it was shown that for a continuous local martingale $M$ with initial value zero, the condition

$$
\lim_{\varepsilon \to 0} \varepsilon \log E \exp \left( 1 - \varepsilon \frac{1}{2} [M]_\infty \right) < \infty
$$

suffices to ensure that $E(M)$ is a uniformly integrable martingale. This is an extension of the result in [16]. And in [21], optimal constants $\alpha(a)$ and $\beta(a)$ for $a > -1$ were identified such that when $\Delta M_1(\Delta M \neq 0) \geq a$, integrability of $\exp(\alpha(a)[M]_\infty)$ and $\exp(\beta(a)[M]_\infty)$ suffices to ensure that $E(M)$ is a uniformly
integrable martingale, and it was noted that for the case \( a = 0 \), \( \alpha(a) = \beta(a) = \frac{1}{2} \). Thus, the case where \( \Delta M \geq 0 \) presents a higher level of regularity than the general case. In this note, we prove that when \( \Delta M \geq 0 \), the condition

\[
\liminf_{\varepsilon \to 0} \varepsilon \log E \exp \left( (1 - \varepsilon)\frac{1}{2} (\alpha| M |_\infty + (1 - \alpha)\langle M \rangle_\infty) \right) < \infty
\]  

(1.3)

suffices to ensure that \( E(M) \) is a uniformly integrable martingale, thus extending the results of [16] and [11]. Note that while sufficiency of simple Novikov-type criteria such as those given in [21] follow from the results of [13], the condition (1.3) does not. Also, to the best of the knowledge of the author, the condition (1.3) is the first one obtained applying both the quadratic variation and the predictable quadratic variation at the same time.

2. Main results and proofs

In this section, we will prove the following theorem.

**Theorem 2.1.** Let \( M \) be a locally square integrable local martingale with initial value zero and \( \Delta M \geq 0 \). Fix \( 0 \leq \alpha \leq 1 \) and assume that

\[
\liminf_{\varepsilon \to 0} \varepsilon \log E \exp \left( (1 - \varepsilon)\frac{1}{2} (\alpha| M |_\infty + (1 - \alpha)\langle M \rangle_\infty) \right) < \infty
\]  

(2.1)

Then \( E(M) \) is a uniformly integrable martingale. If \( \alpha = 1 \), it is not necessary that \( M \) be locally square integrable. Furthermore, for all \( 0 \leq \alpha \leq 1 \), the constant \( 1/2 \) in (2.1) is optimal.

Optimality of the constant \( 1/2 \) will be shown in Example 2.7. We begin by considering the proof of the case \( \alpha = 1 \), where local square integrability is not required. Our proof in this case rests on the following two elementary martingale lemmas and the following real analysis lemma.

**Lemma 2.2.** Let \( M \) be a local martingale with initial value zero. Let \( C \) denote the set of all bounded stopping times. If there exists \( a > 1 \) such that \( (M_T)_{T \in C} \) is bounded in \( \mathcal{L}^a \), then \( M \) is a uniformly integrable martingale.

**Proof.** This follows from the optional sampling theorem for nonnegative supermartingales. \( \square \)

**Lemma 2.3.** Let \( M \) be a local martingale with initial value zero. Let \( C \) denote the set of all bounded stopping times. If there exists \( a > 1 \) such that \( (M_T)_{T \in C} \) is bounded in \( \mathcal{L}^a \), then \( M \) is a uniformly integrable martingale.

**Proof.** As \( (M_T)_{T \in C} \) is bounded in \( \mathcal{L}^a \), \( (M_T)_{T \in C} \) is uniformly integrable. Let \( (T_n) \) be a localising sequence such that \( M^{T_n} \) is a uniformly integrable martingale for each \( n \geq 1 \). Let \( S \) be a bounded stopping time. Then \( (M_{T_n \wedge S})_{n \geq 1} \) is uniformly
integrable as well. As $M_{T_n,S}$ converges almost surely to $M_S$, we conclude that $M_S$ is integrable and that $M_{T_n,S}$ converges in $\mathcal{L}^1$ to $M_S$. As $M_{T_n}$ is a uniformly integrable martingale, $EM_{T_n}^2 = 0$ by the optional stopping theorem, and thus $EM_S = 0$. By Theorem II.77.6 of [20], $M$ is a martingale. And by our assumptions, $(M_t)_{t \geq 0}$ is uniformly integrable, so $M$ is a uniformly integrable martingale. □

**Lemma 2.4.** Let $x \geq 0$. It then holds that

(2.2) \[ 0 \leq \log \frac{1 + \lambda x}{(1 + x)^\lambda} - \frac{\lambda(1 - \lambda)}{2} x^2 \quad \text{and} \]

(2.3) \[ 0 \leq \log \frac{(1 + x)^a}{1 + ax} - \frac{a(a - 1)}{2} x^2 \]

for $0 \leq \lambda \leq 1$ and $a \geq 1$.

**Proof.** We first prove (2.2). To prove the lower inequality, it suffices to argue the $(1 + \lambda x)/(1 + x)^\lambda \geq 1$, which is equivalent to $1 + \lambda x - (1 + x)^\lambda \geq 0$. Fix $0 \leq \lambda \leq 1$ and define $h_\lambda(x) = 1 + \lambda x - (1 + x)^\lambda$. Then $h_\lambda'(x) = \lambda - \lambda(1 + x)^{\lambda - 1} \geq 0$ and $h_\lambda(0) = 0$. This implies $0 \leq (1 + x)^{\lambda - 1} \leq 1$, as desired, and thus proves the first inequality in (2.2). In order to prove the second inequality, we define $g_\lambda$ by putting $g_\lambda(x) = \frac{1}{2} \lambda(1 - \lambda)x^2 - \log(1 + \lambda x) + \lambda \log(1 + x)$. We then need to prove $g_\lambda(x) \geq 0$. We obtain $g_\lambda(0) = 0$ and

\[ g_\lambda'(x) = \frac{\lambda(1 - \lambda)x - \frac{\lambda}{1 + \lambda x} + \frac{\lambda}{1 + x}}{(1 + \lambda x)(1 + x)} = \frac{\lambda(1 - \lambda)x + \lambda x(1 + \lambda x)(1 + x) - \lambda(1 + x) + \lambda(1 + \lambda x)}{(1 + \lambda x)(1 + x)} \geq 0, \]

(2.4)

so $g_\lambda(x) \geq 0$ for all $0 \leq \lambda \leq 1$ and $x \geq 0$, yielding the second inequality in (2.2).

Next, consider (2.3). For the lower inequality, note that $(1 + x)^a - (1 + ax) \geq 0$, so that $(1 + x)^a/(1 + ax) \geq 1$. For the upper inequality, we may apply (2.2) to obtain

(2.5) \[ \log \frac{(1 + x)^a}{1 + ax} = a \log \frac{1 + x}{(1 + ax)^{1/a}} \leq a \frac{1}{2} \frac{(1 - \frac{1}{a})}{2} (ax)^2 = \frac{a(a - 1)}{2} x^2, \]

for $a \geq 1$.

\[ \square \]

**Proof of Theorem 2.1 for the case $\alpha = 1$.** In this case, we wish to show that when $\liminf_{\epsilon \to 0} \log E \exp((1 - \epsilon)/2[M]_{\infty})$ is finite, $\mathcal{E}(M)$ is a uniformly integrable martingale. We first prove that $\mathcal{E}(M)$ is a uniformly integrable martingale under the stronger condition that $\exp((1 + \epsilon)\frac{1}{2}[M]_{\infty})$ is integrable for some $\epsilon > 0$. Fix
such an $\varepsilon > 0$, and let $a, r > 1$. Applying (2.3) of Lemma 2.4 we then have

$$
\mathcal{E}(M)^a_t = \exp \left( aM_t - \frac{1}{2} a[M^c]_t + \sum_{0 < s \leq t} \log(1 + \Delta M_s)^a - a\Delta M_s \right)
$$

$$
= \mathcal{E}(arM)^{1/r}_t \exp \left( \frac{a(ar - 1)}{2} [M^c]_t + \sum_{0 < s \leq t} \log \left\{ \frac{(1 + a\Delta M_s)^a}{(1 + ar\Delta M_s)^{1/r}} \right\} \right)
$$

(2.6)

$$
\leq \mathcal{E}(arM)^{1/r}_t \exp \left( \frac{a(ar - 1)}{2} [M^c]_t \right).
$$

Now let $T$ be a bounded stopping time. Note that as $arM$ has nonnegative jumps, $\mathcal{E}(arM)$ is a nonnegative supermartingale and so $\mathcal{E}(arM)_T \leq 1$. Let $y = ar$ and let $s$ be the dual exponent to $r$, such that $s = r/(r - 1)$. Applying Hölder’s inequality in (2.6), we obtain

$$
\mathcal{E}(M)^a_T = E \left( \mathcal{E}(arM)^{1/r}_T \right)^{1/s} \leq \left( E \left( \mathcal{E}(arM)^{1/r}_T \right)^{1/s} \right).
$$

(2.7)

Next, note that the mapping $y \mapsto y(y - 1)$ is increasing for $y \geq 1$. Therefore, $\inf_{y > 1} y(y - 1)/(2(r - 1)) = 1/r/2 = 1/2$, and so there exists $y > r > 1$ such that $y(y - 1)/(2(r - 1)) \leq (1 + \varepsilon)/2$. Fixing such $y > r > 1$ and putting $a = y/r$, we obtain $a > 1$ and (2.7) allows us to conclude that with the supremum being over all bounded stopping times, we have

$$
\sup_T E\mathcal{E}(M)^a_T \leq \left( E \left( \mathcal{E}(y(y - 1) [M]_\infty) \right)^{1/s} \right).
$$

(2.8)

where the right-hand side is finite by assumption. By Lemma 2.3, $\mathcal{E}(M)$ is a uniformly integrable martingale.

Next, we merely assume that $\liminf_{\varepsilon \to 0} \varepsilon \log E \exp((1 - \varepsilon)/2[M]_\infty)$ is finite. In particular, for all $\varepsilon > 0$, $\exp((1 - \varepsilon)/2[M]_\infty)$ is integrable. Therefore, $[M]_T$ is integrable, so $M$ is a square-integrable martingale and the limit $M_\infty$ exists. Fix $0 < \lambda < 1$. As $[\lambda M]^c_T = \lambda^2 [M]^c_T$, we have by our earlier results that $\mathcal{E}(\lambda M)$ is a uniformly integrable martingale. Using (2.2) of Lemma 2.4 we have

$$
1 = E \exp \left( \lambda M_\infty - \frac{\lambda^2}{2} [M^c]_\infty + \sum_{0 < t} \log(1 + \lambda\Delta M_t) - \lambda\Delta M_t \right)
$$

$$
= E\mathcal{E}(\lambda M)^\lambda_\infty \exp \left( \frac{\lambda(1 - \lambda)}{2} [M^c]_\infty + \sum_{0 < t} \log \left\{ \frac{1 + \lambda\Delta M_t}{(1 + \lambda\Delta M_t)^{1/\lambda}} \right\} \right)
$$

(2.9)

$$
\leq E\mathcal{E}(\lambda M)^\lambda_\infty \exp \left( \frac{\lambda(1 - \lambda)}{2} [M]_\infty \right).
$$

Now fix $\gamma \geq 0$. Applying Jensen’s inequality in (2.9) with the concave function $x \mapsto x^\lambda$ as well as Hölder’s inequality with the dual exponents $\frac{1}{\lambda}$ and $\frac{1}{1 - \lambda}$, we
obtain, with $F_\gamma = (|M|_\infty > \gamma)$, that
\[
1 \leq EE(M)_{\lambda \gamma} \exp \left( \frac{\lambda(1 - \lambda)}{2} \right) + EE(M)_{\lambda \gamma} \exp \left( \frac{\lambda(1 - \lambda)}{2} |M|_\infty \right) 1_{F_\gamma},
\]
\[
\leq (EE(M)_{\lambda \gamma})^\lambda \exp \left( \frac{\lambda(1 - \lambda)}{2} \right) + (EE(M)_{\lambda \gamma} 1_{F_\gamma})^\lambda \left( E \exp \left( \frac{\lambda}{2} |M|_\infty \right) \right)^{1 - \lambda}.
\]
By our assumptions, we have that $\lim \inf_{\lambda \to 1} (E \exp((\lambda/2)|M|_\infty))^{1 - \lambda}$ is finite. Let $c$ denote the value of the limit inferior. By the above, we then obtain
\[
(2.10) \quad 1 \leq EE(M)_{\lambda \gamma} + cEE(M)_{\lambda \gamma} 1_{(|M|_\infty > \gamma)}.
\]
Letting $\gamma$ tend to infinity, we obtain $1 \leq EE(M)_{\lambda \gamma}$, which by Lemma 2.2 shows that $E(M)$ is a uniformly integrable martingale. 

For the remaining case of $0 \leq \alpha < 1$, we need the following further inequalities.

**Lemma 2.5.** Let $x \geq 0$. It then holds that
\[
(2.11) \quad 0 \leq (1 + \lambda x) - (1 + x)^\lambda \leq \frac{\lambda(1 - \lambda)}{2} x^2 \quad \text{and}
\]
\[
(2.12) \quad 0 \leq (1 + x)^a - (1 + ax) \leq \frac{a(a - 1)}{2} x^2,
\]
for $0 \leq \lambda \leq 1$ and $1 \leq a \leq 2$.

**Proof.** Fix $0 \leq \lambda \leq 1$. The lower inequality in (2.11) is equivalent to the statement that $(1 + \lambda x)/(1 + x)^\lambda \geq 1$, which follows from (2.2) of Lemma 2.4. Next, put $g_\lambda(x) = \frac{\lambda(1 - \lambda)}{2} x^2 + (1 + x)^\lambda - (1 + \lambda x)$. In order to obtain the upper inequality, we need to prove $g_\lambda(x) \geq 0$. To this end, note that
\[
(2.13) \quad g'_\lambda(x) = \lambda(1 - \lambda)x + \lambda(1 + x)^{\lambda - 1} - \lambda \quad \text{and}
\]
\[
(2.14) \quad g''_\lambda(x) = \lambda(1 - \lambda) - \lambda(1 - \lambda)(1 + x)^{\lambda - 2}.
\]
As $g'_\lambda(x) \geq 0$, $g'_\lambda(0) = 0$ and $g''_\lambda(0) = 0$, we conclude that $g_\lambda$ is nonnegative and thus (2.11) holds. Next, consider $a$ with $1 \leq a \leq 2$. Using (2.3) of Lemma 2.4 we find that the lower inequality of (2.12) holds. For the upper inequality, define $h_\alpha(x) = \frac{a(a - 1)}{2} x^2 + 1 + ax - (1 + x)^a$, we need to prove $h_\alpha(x) \geq 0$. To do so, we note that
\[
(2.15) \quad h'_\alpha(x) = a(a - 1)x + a - a(1 + x)^{a - 1} \quad \text{and}
\]
\[
(2.16) \quad h''_\alpha(x) = a(a - 1) - a(a - 1)(1 + x)^{a - 2},
\]
such that $h''_\alpha(x) \geq 0$, $h'_\alpha(0) = 0$ and $h_\alpha(0) = 0$, yielding as in the previous case that $h_\alpha$ is nonnegative and so we obtain (2.12). 

**Lemma 2.6.** Let $x \geq 0$. It then holds that
\[
(2.17) \quad 0 \leq \log \frac{1 + \lambda x + (1 + \sqrt{1 - \alpha x})^\lambda - (1 + \lambda \sqrt{1 - \alpha x})}{{(1 + x)^\lambda}} \leq \frac{\lambda(1 - \lambda)}{2} x^2
\]
for $\alpha, \lambda \in [0, 1]$. 

Proof. Let \( \beta = \sqrt{1 - \alpha} \), such that \( \alpha = 1 - \beta^2 \). We need to prove that for \( x \geq 0 \) and \( \beta, \lambda \in [0, 1] \), it holds that

\[
0 \leq \log \frac{\lambda(1 - \beta)x + (1 + \beta x)^\lambda}{(1 + x)\lambda} \leq (1 - \beta^2)^2 \frac{\lambda(1 - \lambda)}{2} x^2.
\]  

(2.18)

Consider the first inequality in (2.18). To prove this, it suffices to show that for \( x \geq 0 \) and \( \beta, \lambda \in [0, 1] \) it holds that

\[
1 \leq \frac{\lambda(1 - \beta)x + (1 + \beta x)^\lambda}{(1 + x)\lambda},
\]

which is equivalent to \( \lambda(1 - \beta)x + (1 + \beta x)^\lambda - (1 + x)^\lambda \geq 0 \). As this holds for all \( x \geq 0, \lambda \in [0, 1] \) and \( \beta \) equal to one, it suffices to prove that the derivative with respect to \( \beta \) is nonpositive, meaning that we need to prove \( \lambda x \geq x(1 + \beta x)^\lambda - 1 \). However, this follows as \( \lambda x \leq (1 + \beta x)^\lambda - 1 \), it holds for \( x \geq 0 \) and \( \beta, \lambda \in [0, 1] \), it holds that

\[
0 \leq (1 - \beta^2)\frac{\lambda(1 - \lambda)}{2} x^2 - \log \frac{\lambda(1 - \beta)x + (1 + \beta x)^\lambda}{(1 + x)^\lambda}.
\]

By simple substitution, we note that the result holds when \( \beta \) is equal to one, \( x \geq 0 \) and \( 0 \leq \lambda \leq 1 \). It therefore suffices to prove that the derivative with respect to \( \beta \) is nonnegative, meaning that we need to prove that for \( x \geq 0 \) and \( \beta, \lambda \in [0, 1] \),

\[
0 \geq \beta x - x\lambda(1 + \beta x)^{\lambda-1} - \beta x(1 - \lambda)^{\lambda-1}.
\]

(2.21)

Multiplying by the divisor, which is positive, this is equivalent to

\[
0 \leq \beta x - x\lambda(1 + \beta x)^{\lambda-1} - \beta x(1 - \lambda)^{\lambda-1}.
\]

(2.22)

which follows if we can show \( 1 \leq \beta x(1 - \lambda)x(1 - \beta)x + (1 + \beta x)^\lambda - (1 + \beta x)^\lambda - 1 \). As \( \beta - 1 \lambda \leq 0 \), it thus suffices to show that for \( x \geq 0 \) and \( \lambda, \beta \in [0, 1] \), we have \( 1 \leq \beta x(1 - \lambda)x(1 + \beta x)^\lambda - (1 + \beta x)^\lambda - 1 \). However, as this holds for any \( \beta, \lambda \in [0, 1] \) when \( x \) is zero, we find that it suffices to show that the derivative with respect to \( x \) is nonnegative, so that we need to show

\[
0 \leq \beta (1 - \lambda)\frac{(1 + \beta x)^\lambda - x\beta x(1 + \beta x)^{\lambda-1} + \beta (\lambda - 1)(1 + \beta x)^{\lambda-2}}{\lambda(1 - \beta)x + (1 + \beta x)^\lambda}.
\]

(2.23)

for \( x \geq 0 \) and \( \beta, \lambda \in [0, 1] \). To this end, as \( \beta(1 - \lambda) \geq 0 \), it suffices to show that \( 0 \leq (1 + \beta x)^\lambda - x\beta x(1 + \beta x)^{\lambda-1} - (1 + \beta x)^{\lambda-2} \) for \( x \geq 0 \) and \( \beta, \lambda \in [0, 1] \). To this end, simply note that

\[
(1 + \beta x)^\lambda - x\beta x(1 + \beta x)^{\lambda-1} - (1 + \beta x)^{\lambda-2} = (1 + \beta x)^{\lambda-2}\left((1 + \beta x)^{\lambda-2} - x\beta(1 + \beta x) - 1\right)
\]

(2.24)

As this is nonnegative, the result follows. \( \square \)
The upper inequality in Lemma 2.4 is not obvious. However, an indication that the constant $\frac{\alpha(1-\lambda)}{2}$ is the right one may be obtained by a simple argument as follows. By the l'Hôpital rule, we have

$$\lim_{x \to 0} \frac{1}{x^2} \log \frac{1 + \lambda x + (1 + \sqrt{1 - \alpha}x)^\lambda - (1 + \lambda(1 - \alpha)x)}{(1 + x)^\lambda}$$

$$= \lim_{x \to 0} \frac{1}{x^2} \log \frac{(1 - \sqrt{1 - \alpha})x + (1 + \sqrt{1 - \alpha}x)^\lambda}{(1 + x)^\lambda}$$

(2.25) $$= \lim_{x \to 0} \frac{1}{2x} \left( \frac{\lambda(1 - \sqrt{1 - \alpha}) + \sqrt{1 - \alpha}(1 + \sqrt{1 - \alpha}x)^{\lambda - 1}}{\lambda(1 - \sqrt{1 - \alpha})x + (1 + \sqrt{1 - \alpha}x)^\lambda} - \frac{\lambda}{1 + x} \right).$$

Identifying a common divisor and applying the l'Hôpital rule again, we obtain that the above is equal to

$$\frac{1}{2} \lim_{x \to 0} ((1 - \alpha)\lambda(\lambda - 1)(1 + \sqrt{1 - \alpha}x)^{\lambda - 2})(1 + x)$$

$$+ \frac{1}{2} \lim_{x \to 0} (\lambda(1 - \sqrt{1 - \alpha}) + \sqrt{1 - \alpha}(1 + \sqrt{1 - \alpha}x)^{\lambda - 1})$$

(2.26) $$- \frac{1}{2} \lim_{x \to 0} \lambda(1 - \sqrt{1 - \alpha}) + \sqrt{1 - \alpha}(1 + \sqrt{1 - \alpha}x)^{\lambda - 1},$$

which by elementary calculations is equal to $\alpha \frac{(1-\lambda)}{2}$, the factor in front of $x^2$ in Lemma 2.6.

**Proof of Theorem 2.1 for the case** $0 \leq \alpha < 1$. We consider the case $0 < \alpha < 1$, the remaining case of $\alpha = 0$ follows by a similar method.

Fix $\varepsilon > 0$. We first prove that $\mathcal{E}(M)$ is a uniformly integrable martingale under the stronger condition that $\exp((1 + \varepsilon)\frac{1}{4}(\alpha[M]_\infty + (1 - \alpha)(M)_\infty)$ is integrable. Let $a, r > 1$. Defining $U$ by putting $U_t = ar \sum_{0 < s \leq t} \log(1 + \Delta M_s) - \Delta M_s$, we have

$$\mathcal{E}(M)_t^a = \exp \left( arM_t - \frac{1}{2}[arM_t^2 + U_t] \right)^{1/r} \exp \left( \frac{a(ar - 1)}{2}[M^r]_t \right).$$

We wish to decompose the first factor in the right-hand side of (2.27) in two ways, one involving an optional increasing factor and one involving a predictable increasing factor. Put $N^o_t = arM_t$. For the optional decomposition, we note that

$$U_t = \sum_{0 < s \leq t} \log(1 + \Delta N^o_s) - \Delta N^o_s + \sum_{0 < s \leq t} \log \frac{(1 + \Delta M_s)^{ar}}{1 + ar \Delta M_s},$$

which yields

$$\exp \left( arM_t - \frac{1}{2}[arM_t^2 + U_t] \right)^{\alpha/r} = \mathcal{E}(N^o)_t^{\alpha/r} \exp \left( \frac{\alpha}{r} \sum_{0 < s \leq t} \log \frac{(1 + \Delta M_s)^{ar}}{1 + ar \Delta M_s} \right).$$

Next, for $0 \leq \beta < 2$, we define $W^\beta_t = \sum_{0 < s \leq t} (1 + \Delta M_s)^{\beta} - (1 + \beta \Delta M_s)$. Note that the sum is well-defined, increasing and locally integrable by (2.12) of Lemma 2.5 as $[M]$ is locally integrable by our assumptions. Therefore, the compensator
$V^\beta$ of $W^\beta$ is well-defined, and is increasing and locally integrable as well. Also note that $(1 + \Delta M_s)^\beta = 1 + \beta \Delta M_s + \Delta W^\beta$. Further define two local martingales by putting $N_p^r = ar M_t + W^ar_t - V^ar_t$ and $\tilde{N}_t = \int_0^t (1 + \Delta V^ar) dN_p$, where $\tilde{N}$ is well-defined as $\Delta V^ar \geq 0$ and $(1 + \Delta V^ar)^{-1}$ is predictably and locally bounded.

We begin by considering some properties of $\tilde{N}_t$. First, we observe that

$$
\Delta \tilde{N}_t^p = \frac{\Delta N_t^p}{1 + \Delta V^ar_t} = \frac{ar \Delta M_t + \Delta W^ar_t - \Delta V^ar_t}{1 + \Delta V^ar_t} = \frac{(1 + \Delta M_t)^{ar} - (1 + \Delta V^ar_t)}{1 + \Delta V^ar_t} - 1 > -1 
(2.29)
$$

Furthermore, define $A^ar_t = \sum_{0 < s < t} \Delta V^ar_s (1 + \Delta V^ar)^{-1}$. As $\Delta V^ar$ is predictable and nonnegative, the process $A^ar_t$ is well-defined, and is also predictable, increasing and locally bounded, and $[A^ar, N^p]_t = \sum_{0 < s < t} \Delta A^ar_s \Delta N^p_s$. By Proposition 1.4.9 of [5], $[A^ar, N^p]$ is a local martingale. As the two local martingales $\int_0^t A^ar dN^p_p$ and $[A^ar, N^p]$ are purely discontinuous and have the same jumps, they are equal by the uniqueness part of Theorem 7.25 of [3], and we thus obtain

$$
\tilde{N}_t^p = N_t^p - \int_0^t \frac{\Delta V^ar_s}{1 + \Delta V^ar_s} dN^p_s = ar M_t + W^ar_t - V^ar_t - \sum_{0 < s < t} (1 + \Delta V^ar_s)^{-1} \Delta V^ar_s \Delta N^p_s.
(2.30)
$$

Also, as the function $x \mapsto \log(1 + x) - x$ is nonpositive for $x \geq 0$ and $V^ar$ is increasing, we obtain $\log(1 + \Delta V^ar) - \Delta V^ar \leq 0$. Combining our observations, we get

$$
ar \log(1 + \Delta M_s) - ar \Delta M_s - (\log(1 + \Delta \tilde{N}_s^p) - \Delta \tilde{N}_s^p) 
= ar \log(1 + \Delta M_s) - ar \Delta M_s - \left( \log \frac{(1 + \Delta M_s)^{ar}}{1 + \Delta V^ar_s} - \Delta \tilde{N}_s^p \right) 
(2.31)
= \Delta W^ar_s - \frac{\Delta V^ar_s \Delta N^p_s}{1 + \Delta V^ar_s} + \log(1 + \Delta V^ar_s) - \Delta V^ar_s \leq \Delta W^ar_s - \frac{\Delta V^ar_s \Delta N^p_s}{1 + \Delta V^ar_s},
$$

where the logarithm in first expression is well-defined by (2.29). This implies

$$
U_t \leq \left( \sum_{0 < s \leq t} \log(1 + \Delta \tilde{N}_s^p) - \Delta \tilde{N}_s^p \right) + \sum_{0 < s \leq t} \Delta W^ar_s - \frac{\Delta V^ar_s \Delta N^p_s}{1 + \Delta V^ar_s} 
(2.32)
= \tilde{N}_t^p - ar M_t + \left( \sum_{0 < s \leq t} \log(1 + \Delta \tilde{N}_s^p) - \Delta \tilde{N}_s^p \right) + V^ar_t.
$$

Also noting that $[\tilde{N}^p]_t = [N^p]_t = [ar M]_t$, we obtain the relationship

$$
\exp \left( ar M_t - \frac{1}{2} [ar M]_t + U_t \right)^{(1-\alpha)/r} \leq \mathcal{E}([N^p]^{(1-\alpha)/r}) \exp \left( \frac{1-\alpha}{r} V^ar_t \right).
$$
Combining our results with (2.21), we obtain \( \mathcal{E}(M)_t^\alpha \leq \mathcal{E}(N^\alpha)_t^{\alpha/r} \mathcal{E}(N^\alpha)^{(1-\alpha)/r} X_t \), where the process \( X \) is defined by

\[
(2.33) \quad X_t = \exp \left( \frac{a(a-r-1)}{2} [M^r]_t + \frac{\alpha}{r} \sum_{0 < s \leq t} \log \left( \frac{(1 + \Delta M_s)^a}{1 + a\Delta M_s} \right) + \frac{1 - \alpha}{r} V_t \right).
\]

Here, note that by (2.3) of Lemma 2.4 and (2.12) of Lemma 2.5 we have, for \( a, r > 1 \) such that \( 1 \leq ar \leq 2 \), that

\[
(2.34) \quad \sum_{0 < s \leq t} \log \left( \frac{(1 + \Delta M_s)^a}{1 + a\Delta M_s} \right) \leq \frac{ar(a-1)}{2} [M^d]_t \quad \text{and}
\]

\[
(2.35) \quad V_t^ar \leq \frac{ar(a-1)}{2} (M^d)_t,
\]

leading to inequality

\[
(2.36) \quad \mathcal{E}(M)_t^\alpha \leq \mathcal{E}(N^\alpha)^{\alpha/r} \mathcal{E}(N^\alpha)^{(1-\alpha)/r} \exp \left( \frac{a(a-r-1)}{2} (\alpha[M]_t + (1 - \alpha)(M)_t) \right).
\]

Next, as \( \Delta N_t^\alpha \geq 0 > -1 \) and \( \Delta \tilde{N}_t^\alpha > -1 \), \( \mathcal{E}(N^\alpha) \) and \( \mathcal{E}(\tilde{N}^\alpha) \) are nonnegative supermartingales, and so for all bounded stopping times \( T \), \( 0 \leq \mathcal{E}(N^\alpha)_T \leq 1 \) and \( 0 \leq \mathcal{E}(\tilde{N}^\alpha)_T \leq 1 \). Now let \( s \) be the dual exponent of \( r \), such that \( s = r/(r-1) \). Noting that \( \frac{1}{r^\alpha} + \frac{1}{r/(1-\alpha)} + \frac{1}{s} = \frac{1}{r} + \frac{1}{s} = 1 \), we may then apply Hölder’s inequality for triples of functions to the inequality (2.36), yielding for any bounded stopping time \( T \) that

\[
(2.37) \quad E\mathcal{E}(M)_T^\alpha \leq \left( E \exp \left( \frac{y(y-1)}{2(r-1)} (\alpha[M]_\infty + (1 - \alpha)(M)_\infty) \right) \right)^{1/s},
\]

where \( y = ar \). This holds for all \( a, r > 1 \) such that \( ar \leq 2 \), and is a bound similar to (2.7). Proceeding as in the proof of the case \( \alpha = 1 \), we then obtain as a consequence of Lemma 2.4 that \( \mathcal{E}(M) \) is a uniformly integrable martingale.

Now, assume that \( \lim \inf_{\varepsilon \to 0} \varepsilon \log \mathbb{E} \exp((1 - \varepsilon)/2)(M)_\infty \) is finite. In particular, for \( \varepsilon > 0 \), \( \exp((1 - \varepsilon)/2)(M)_\infty \) is integrable. In particular, \( (M)_\infty \) is integrable. Now, as \( M \) is locally square-integrable by assumption, \( [M] \) is locally integrable. Let \( (T_n) \) be a localising sequence such that \( [M]^{T_n} \) is integrable. Then, \( (\langle M \rangle - (M))^{T_n} \) is a uniformly integrable martingale, so \( E[M]^{T_n} = E(M)^{T_n} \). Letting \( n \) tend to infinity, the monotone convergence theorem shows that \( [M]_\infty \) is integrable, so \( M \) is a square-integrable martingale, in particular the limit \( M_\infty \) exists.

Now fix \( 0 < \lambda < 1 \) and define

\[
(2.38) \quad W^\lambda(\alpha)_t = \sum_{0 < s \leq t} (1 + \sqrt{1 - \alpha \Delta M_s})^\lambda - (1 + \lambda \sqrt{1 - \alpha \Delta M_s}).
\]

Note that by Lemma 2.4, the terms in the sum in (2.38) are nonpositive and bounded from below by \( -(1 - \alpha)\frac{1}{2} \lambda (1 - \lambda)(\Delta M_s)^2 \). In particular, we find that \( W^\lambda(\alpha) \) is well-defined, decreasing and integrable. Letting \( V^\lambda(\alpha) \) be the compensator of \( W^\lambda(\alpha) \), \( V^\lambda(\alpha) \) is then increasing integrable as well, and \( W^\lambda(\alpha) - V^\lambda(\alpha) \) is a uniformly integrable martingale. We show that \( V^\lambda(\alpha) \) is continuous. To this
end, let $T$ be some predictable stopping time. By Theorem VI.12.6 of [20] and its proof, we have $E \Delta M_T = 0$ and $E \Delta(W^\lambda(\alpha) - V^\lambda(\alpha))_T = 0$, so that 

$$
EV^\lambda(\alpha)_T = EW^\lambda(\alpha)_T
$$

(2.39) 

$$
= E((1 + \sqrt{1 - \alpha} \Delta M_T)^\lambda - (1 + \lambda \sqrt{1 - \alpha} \Delta M_T))
$$

(2.40) 

$$
= E(1 + \sqrt{1 - \alpha} \Delta M_T)^\lambda - 1 \geq 0,
$$

because of our assumption that $\Delta M \geq 0$. Thus, we know now that as $V^\lambda$ is decreasing, $\Delta V^\lambda_T \leq 0$, and from the above, $EV^\lambda(\alpha)_T \geq 0$. We conclude that $\Delta V^\lambda_T = 0$ for all predictable stopping times. Lemma VI.19.2 of [20] then shows that $V^\lambda(\alpha)$ is continuous.

Let $L^\lambda_t = \lambda M_t + W^\lambda(\alpha) - V^\lambda(\alpha)$. By our previous observations, $L^\lambda$ is a uniformly integrable martingale, in particular the limit $L^\lambda_\infty$ exists. Note that $(L^\lambda)^c = \lambda M^c$, so it holds that $[(L^\lambda)^c]_t = \lambda^2 [M^c]_t$. Also note that by continuity of $V^\lambda(\alpha)$, we have 

$$
\Delta L^\lambda_t = \lambda \Delta M_t + (1 + \sqrt{1 - \alpha} \Delta M_t)^\lambda - (1 + \lambda \sqrt{1 - \alpha} \Delta M_t)
$$

(2.41) 

$$
= (1 - \lambda \sqrt{1 - \alpha}) \Delta M_t + (1 + \sqrt{1 - \alpha} \Delta M_t)^\lambda - 1
$$

$$
\geq (1 - \lambda \sqrt{1 - \alpha}) \Delta M_t \geq 0,
$$

and as $W^\lambda(\alpha)$ has nonpositive jumps, we also have $\Delta L^\lambda_t \leq \Delta M_t$. Combining these observations, we obtain $[L^\lambda]_t \leq \lambda^2 [M^c]_t$, yielding that $L^\lambda$ is square-integrable. We then also obtain $(L^\lambda)^c_\infty \leq \lambda^2 (M^c)_\infty$. This implies 

(2.42) 

$$
\alpha[L^\lambda]_\infty + (1 - \alpha)(L^\lambda)^c_\infty \leq \lambda^2 (\alpha[M]_\infty + (1 - \alpha)(M)_\infty),
$$

so by what we already have shown, $E(L^\lambda)$ is a uniformly integrable martingale. By elementary calculations, we obtain 

$$
E(L^\lambda)_\infty = E(M)_\infty^\lambda \exp \left(\frac{\lambda(1 - \lambda)}{2} [M^c]_\infty + \sum_{0 < t} \log \frac{1 + \Delta N_t}{(1 + M_t)^\lambda} - V^\lambda(\alpha)_\infty\right).
$$

By [211] of Lemma 2.6 and Lemma 2.7, we obtain the two inequalities 

(2.43) 

$$
-V^\lambda(\alpha)_\infty \leq (1 - \alpha) \frac{\lambda(1 - \lambda)}{2} (M^d)_\infty
$$

(2.44) 

$$
\sum_{0 < t} \log \frac{1 + \Delta L^\lambda_t}{(1 + M_t)^\lambda} \leq \alpha \frac{\lambda(1 - \lambda)}{2} [M^d]_\infty,
$$

so that combining our conclusions, we have 

$$
1 = E\mathcal{E}(L^\lambda)_\infty
$$

(2.45) 

$$
\leq E\mathcal{E}(M)_\infty^\lambda \exp \left(\frac{\lambda(1 - \lambda)}{2} ([M^c]_\infty + \alpha [M^d]_\infty + (1 - \alpha)(M^d)_\infty)\right),
$$

$$
= E\mathcal{E}(M)_\infty^\lambda \exp \left(\frac{\lambda(1 - \lambda)}{2} (\alpha [M]_\infty + (1 - \alpha)(M)_\infty)\right),
$$
which is a bound similar to (2.9). Therefore, proceeding as in the proof of the case \( \alpha = 1 \), we obtain as a consequence of Lemma 2.2 that \( \mathcal{E}(M) \) is a uniformly integrable martingale. \( \square \)

We take a moment to reflect on the methods applied in the above proof, and make the following observations. First, while the proof of the case \( 0 \leq \alpha < 1 \) is more complicated than the proof of the case \( \alpha = 1 \), both proofs follow very much the same plan: Use Hölder’s inequality to argue that the result holds in a simple case where \( \frac{1}{2} \) is exchanged with \( (1 + \varepsilon) \frac{1}{2} \) in the exponent, then use Hölder’s inequality again to obtain the general proof. Also, note that the local martingale \( \tilde{N}^p \) used in the first part of the proof of the case \( 0 \leq \alpha < 1 \) is related to general decompositions of exponential martingales, see Lemma II.1 of [14].

The comparatively simple structure of the proof is made possible by three main factors: The factor \( \lambda (1 - \lambda) \) present in the real analysis inequalities allows us to apply Hölder’s inequality in the second parts of the proofs. Some of these inequalities have been noted earlier with a factor \( 1 - \lambda \) instead of \( \lambda (1 - \lambda) \), compare for example (2.11) with (1.2) and (1.3) of [13], where the inequalities follow by a Taylor expansion argument. The more advanced triple-parameter inequality (2.17) allows us to obtain a criterion combining the quadratic variation and the predictable quadratic variation. Finally, the assumption \( \Delta M \geq 0 \), apart from making most of the real analysis inequalities applicable, also ensures that the compensator \( V^\lambda(\alpha) \) in the second part of the proof of the case \( 0 \leq \alpha < 1 \) is continuous.

We conclude by giving an example showing that the constant \( \frac{1}{2} \) obtained in Theorem 2.1 is optimal.

**Example 2.7.** Fix \( 0 \leq \alpha \leq 1 \). Let \( \delta > 0 \). We wish to identify a locally square-integrable local martingale \( M \) with \( \Delta M \geq 0 \) such that

\[
\liminf_{\varepsilon \to 0} \varepsilon \log E \exp \left( (1 - \varepsilon)(1 - \delta) \frac{1}{2} (\alpha [M]_\infty + (1 - \alpha) \langle M \rangle_\infty) \right) < \infty,
\]

while \( \mathcal{E}(M) \) is not a uniformly integrable martingale. To this end, fix \( a, b > 0 \). Let \( N \) be a standard Poisson process and let \( T_b = \inf \{ t \geq 0 \mid N_t - (1 + b)t = -1 \} \).

By the path properties of \( N, T_b \) is almost surely finite, and \( N_{T_b} - (1 + b)T_b = -1 \). Define \( M_t = a(N_t^{T_b} - t \wedge T_b) \). Similarly to [21], by Lemma 2.2 and elementary calculations, we find that \( \mathcal{E}(M) \) is a uniformly integrable martingale if and only if

\[
E \exp(T_b((1 + b) \log(1 + a) - a)) < 1 + a.
\]

Also, for any \( c > 0 \), we have

\[
\exp(c(\alpha [M]_\infty + (1 - \alpha) \langle M \rangle_\infty)) = \exp(ca^2(\alpha N_{T_b} + (1 - \alpha)T_b)) = \exp(ca^2(\alpha((1 + b)T_b - 1) + (1 - \alpha)T_b)) = \exp(T_bca^2(b\alpha + 1)) \exp(-ca^2).
\]
Thus, it suffices to identify $a, b > 0$ such that (2.47) holds and such that
\[ \liminf_{\varepsilon \to 0} \varepsilon \log E \exp \left( T_b \frac{(1 - \varepsilon)(1 - \delta)a^2}{2}(b\alpha + 1) \right) < \infty, \]
and to do so, it suffices to identify $a, b > 0$ such that
\[ E \exp(T_b((1 + b) \log(1 + a) - a)) < 1 + a \] and
\[ E \exp \left( T_b \frac{(1 - \delta)a^2(b\alpha + 1)}{2} \right) < \infty. \] 

Now define $f_b(\lambda) = \exp(-\lambda) + \lambda(1 + b) - 1$. Noting that the process $L^\lambda$ defined by $L^\lambda_t = \exp(-\lambda(N_t - (1 + b)t + t f_b(\lambda)))$ is a nonnegative supermartingale, the optional sampling theorem yields $E \exp(-T_b f_b(\lambda)) \leq \exp(-\lambda)$. By elementary calculations, see [21] for details, we obtain that (2.50) is satisfied whenever $0 < b < a$. Also, note that $f_b$ takes its minimum at $-\log(1 + b)$. Therefore, $-f_b$ takes its maximum at $\log(1 + b)$, and the maximum is $(1 + b) \log(1 + b) - b$. Thus, it suffices to choose $0 < b < a$ such that $(1 - \delta)a^2(b\alpha + 1)/2 \leq (1 + b) \log(1 + b) - b$, in particular it suffices to choose $0 < b < a$ such that
\[ (1 - \delta)a^2 \leq \log(1 + b) - b/(1 + b). \] 

To do so, first note that by elementary inequalities, we may pick $a > 0$ so small that $a^2/2 \leq (1 - \delta)^{-1/2}(\log(1 + a) - a/(1 + a))$. It then suffices to identify $b \in (0, a)$ such that $\sqrt{1 - \delta}(\log(1 + a) - a/(1 + a)) \leq \log(1 + b) - b/(1 + b)$, and this is possible as the mapping $x \mapsto \log(1 + x) - x/(1 + x)$ is continuous. This yields the desired example.

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