An improved lower bound for the Seidel energy of tree graphs

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Abstract

Let \(G\) be a graph with the vertex set \(\{v_1, \ldots, v_n\}\). The Seidel matrix of \(G\) is an \(n \times n\) matrix whose diagonal entries are zero, \(ij\)-th entry is \(-1\) if \(v_i\) and \(v_j\) are adjacent and otherwise is \(1\). The Seidel energy of \(G\), denoted by \(E(S(G))\), is defined to be the sum of absolute values of all eigenvalues of the Seidel matrix of \(G\). In [1], the authors proved that the Seidel energy of any graph of order \(n\) is at least \(2n - 2\). In this study, we improve the aforementioned lower bound for tree graphs.

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1 Introduction and Terminology

Throughout this paper all graphs we consider are simple and finite. For a graph \(G\), we denote the set of vertices and edges of \(G\) by \(V(G)\) and \(E(G)\), respectively. The order of a graph is its number of vertices and the size of a graph is its number of edges. The complement of \(G\) is denoted by \(\overline{G}\). Also, the \(n\)-vertex complete graph, path graph and star graph are denoted by \(K_n\), \(P_n\) and \(S_n\), respectively. The distance between two vertices in a graph is the number of edges in a shortest path connecting them. Moreover, \(\delta(G)\) and \(\Delta(G)\) represent the minimum degree and the maximum degree of \(G\), respectively.

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For every Hermitian matrix $A$ the energy of $A$, $\mathcal{E}(A)$, is defined to be sum of the absolute values of the eigenvalues of $A$. The well-known concept of energy of a graph $G$, denoted by $\mathcal{E}(G)$, is the energy of its adjacency matrix.

Let $G$ be a graph and $V(G) = \{v_1, \ldots, v_n\}$. The Seidel matrix of $G$, denoted by $S(G)$, is an $n \times n$ matrix whose diagonal entries are zero, $ij$-th entry is $-1$ if $v_i$ and $v_j$ are adjacent and otherwise is 1 (It is noteworthy that at first, van Lint and Seidel introduced the concept of Seidel matrix for the study of equiangular lines in [6]). The Seidel energy of $G$ is defined to be $\mathcal{E}(S(G))$. Moreover, the Seidel switching of $G$ is defined as follows: Partition $V(G)$ into two subsets $V_1$ and $V_2$, delete the edges between $V_1$ and $V_2$ and join all vertices $v_1 \in V_1$ and $v_2 \in V_2$ which are not adjacent. Therefore, if we call the new graph by $G'$, then we have $S(G') = D S(G) D$, where $D$ is a diagonal matrix with entries 1 (resp. -1) corresponding to the vertices of $V_1$ (resp. $V_2$) ([3]). Hence, $S(G)$ and $S(G')$ are similar and they have the same Seidel energy. Note that if one of the $V_1$ or $V_2$ is empty, then $G$ remains unchanged and also, for every $v \in V(G)$, using a Seidel switching on $G$, one can convert $v$ to an isolated vertex. Two graphs $G_1$ and $G_2$ are called SC-equivalent if $G_2$ is obtained from $G_1$ or $\overline{G_1}$ by a Seidel switching and is denoted by $G_1 \cong G_2$. Note that in either cases, $S(G_2)$ is similar to $S(G_1)$ or $-S(G_1)$, hence $\mathcal{E}(S(G_1)) = \mathcal{E}(S(G_2))$.

If $X$ and $Y$ are two disjoint subsets of $V(G)$, the set of edges of $G$ with one endpoint in $X$ and another in $Y$ is denoted by $E(X, Y)$. An ordered pair $(X, Y)$ of disjoint subsets of $V(G)$ with $|X| = |Y| = 2$, is called an odd pair if $|E(X, Y)|$ is an odd number (which is either 1 or 3). One can easily see that applying a Seidel switching on an arbitrary graph $G$ does not change its odd pair(s). A subset $\{u, v\} \subseteq V(G)$ is called an odd set if $\{u, v\}$ is the first component of an odd pair of $G$. We denote the number of odd pairs in $G$ by $s(G)$.

From a graph $G$, we construct a graph denoted by $\Lambda(G)$, as follows: $V(\Lambda(G)) = V(G)$ and $E(\Lambda(G))$ consists of all the edges $e = uv$ such that $\{u, v\}$ is an odd set of $G$. By $\lambda(v)$ we denote the degree of vertex $v \in V(G)$ in the graph $\Lambda(G)$.

\textbf{Example 1.} The following identities hold which can be easily checked:

1. $\Lambda(C_4) = \overline{K_4}$.
2. $\Lambda(P_4) = C_4$.
3. $\Lambda(C_5) = K_5$.
4. $\Lambda(K_n) = \overline{K_n}$.

Haemers in [3] introduced the concept of Seidel energy of a graph and he conjectured that for every graph $G$ of order $n$,

\[ \mathcal{E}(S(G)) \geq \mathcal{E}(S(K_n)) = 2n - 2. \]
Several attempts were made to prove the conjecture (see for example [2] and [4]) and the conjecture was proved in [1]. Here, we strengthen inequality [1] for tree graphs.

2 Main Theorem

In this section, we propose some lemmas to prove the main result of the paper. First, we express the following lemma which deals with the non-adjacent vertices in $\Lambda(T)$, where $T$ is a tree of order at least 4.

Lemma 2. Let $T$ be a tree of order $n \geq 4$. Assume that $u$ and $v$ are two vertices of $T$ which are non-adjacent in $\Lambda(T)$. Then, exactly one of the following cases occurs:

Case 1. $u$ and $v$ are two leaves, both connected to another vertex of $T$.

Case 2. $T$ is the tree graph depicted in Figure 1.

Case 3. $T$ is the tree graph depicted in Figure 2.

Proof. Let $d$ be the distance between $u$ and $v$ in $T$. Then there exists a path $P$ as an induced subgraph of $T$ which connects $u$ and $v$.

If $d \geq 4$, then suppose that $w$ is the vertex adjacent to $u$ in $P$ and $x$ is another vertex in $P$ which is neither connected to $u$ nor $v$ (since $d \geq 4$, $x$ exists). Therefore, $(\{u, v\}, \{w, x\})$ is an odd pair, a contradiction.

If $d = 3$, then every vertex of $V(T) \setminus V(P)$ is adjacent to exactly one of the two vertices $u$ and $v$; because if a vertex $w$ is connected to both $u$ and $v$, we have cycle in $T$, a contradiction.
On the other hand, if there exist some vertices in $T$ which are neither connected to $u$ nor $v$, without loss of generality, there is a vertex $x$ which has distance 2 from $u$ ($x$ and $v$ are non-adjacent). Let $w$ be the vertex joining $u$ and $x$ together. Then $\{\{u, v\}, \{w, x\}\}$ is an odd pair of $T$, a contradiction. Hence, $T$ is the tree depicted in Figure 2.

Next, assume that $d = 2$. Hence, there is a vertex in $T$ which connects $u$ to $v$. Name it $w$. Then every vertex of $T$ is either connected to both $u$ and $v$ or is neither connected to $u$ nor $v$; otherwise, without loss of generality, there exists a vertex $x$ which is connected to $u$ and, $x$ and $v$ are non-adjacent, which implies that $\{\{u, v\}, \{w, x\}\}$ is an odd pair, a contradiction. Moreover, since $T$ is a tree, $u$ and $v$ have exactly one common neighbor (the vertex $w$). Therefore the first case occurs.

Finally, suppose that $d = 1$. So, $u$ and $v$ are adjacent. Now, similar to the discussion in case $d = 3$, it can be verified that the assumptions of lemma imply that $T$ is the tree depicted in Figure 1.

\[\square\]

**Definition 3.** Let $T$ be a tree graph. Then, by $D(T)$ or simply $D$, we mean the maximum number of leaves which are connected to a vertex of $T$. Obviously,

\[1 \leq D(T) \leq \Delta(T),\]

which are achieved (for example) by the path graphs and the star graphs, respectively.

Now, we define two types of trees which will be used in the sequel:

**Type 1:** The family of trees of order $a + b + 2$ depicted in Figure 3 where $a \geq b \geq 1$ are integers.

![Figure 3: Type 1](image)

**Type 2:** The family of trees of order $a + b + 4$ depicted in Figure 4 where $a \geq b$ are non-negative integers and $(a, b) \neq (0, 0)$. (Note that here, the case $(a, b) = (0, 0)$ yields a Type 1 tree with parameters $a = b = 1$.)
Lemma 4. Let $T$ be a tree of order $n \geq 4$ which is not $S_n$, $P_4$, $P_5$ and $P_6$. Then we have

$$|E(\Lambda(T))| \geq \frac{1}{2} n(n - D(T)).$$

**Proof.** For simplicity, we denote $D(T)$ by $D$. Let $u$ and $v$ be arbitrary vertices of $T$. By Lemma 2 if $T$ is not any of the star graph or graph of Type 1 or graph of Type 2, then $uv \notin E(\Lambda(T))$ implies that $u$ and $v$ are two leaves, both connected to another vertex of $T$. Hence, if $w$ is not a leaf, then $\lambda(w) = n - 1$. On the other hand, if $w$ is a leaf, assume that $w$ is connected to a vertex $v$. Then, $\lambda(w) = n - 1 - (k - 1) = n - k$, where $k$ is the number of leaves adjacent to $v$. Therefore, in either case, $\lambda(w) \geq n - D$, which implies that

$$|E(\Lambda(T))| \geq \frac{1}{2} n(n - D).$$

Next, suppose that $T$ is of Type 1. Therefore $D(T) = a$. Since $T \neq P_4$, we have $a > 1$. In this case, $D = a$ and

$$xy \notin E(\Lambda(T)) \iff \{x, y\} = \{u, v\} \text{ or } x, y \in A \text{ or } x, y \in B.$$ 

Hence,

$$2|E(\Lambda(T))| = n(n - 1) - 2 - a(a - 1) - b(b - 1)$$

$$\geq n(n - 1) - 2(a - 1) - a(a - 1) - b(a - 1)$$

$$\geq n(n - 1) - n(a - 1) = n(n - a),$$

which yields

$$|E(\Lambda(T))| \geq \frac{1}{2} n(n - a),$$

as desired.
Finally, suppose that \( T \) is of Type 2 and \( T \neq P_5, P_6 \). So, \( a \geq 2 \) and \( D = a \). Therefore,

\[ xy \notin E(\Lambda(T)) \iff (\{x, y\} = \{u, v\} \; \text{or} \; x, y \in A \; \text{or} \; x, y \in B). \]

Hence,

\[
2|E(\Lambda(T))| \geq n(n - 1) - 2 - a(a - 1) - b(b - 1) \\
\geq n(n - 1) - (a + b + 2)(a - 1) \\
> n(n - 1) - n(a - 1) = n(n - a),
\]

which implies

\[ |E(\Lambda(T))| \geq \frac{1}{2}n(n - a), \]

as desired.

The next lemma provides a lower bound for the number of odd pairs in a graph \( G \) which have the same first component.

**Lemma 5.** Let \( G \) be a graph of order \( n \) and \( e = uv \) be an edge in \( E(\Lambda(G)) \). Then, there exist at least \( n - 3 \) odd pairs in \( G \) such that their first component is \( X = \{u, v\} \).

**Proof.** Assume that \( (X, Y) \) is an odd pair of \( G \), where \( Y = \{u', v'\} \). Then, \( |E(X, Y)| \) is an odd number and the parity of \( E(X, \{u'\}) \) and \( E(X, \{v'\}) \) are different. Hence, for every \( w \in V(G) \setminus (X \cup Y) \), exactly one of the two pairs

\[
(X, \{u', w\}), \quad (X, \{v', w\})
\]

is an odd pair. Therefore, \( G \) has at least \( n - 4 \) odd pairs different from \( (X, Y) \) and the proof is complete.

**Remark 6.** We notify that in the procedure of the proof of Theorem 2 of [1], for a graph \( G \) of order \( n \geq 4 \), the inequality

\[
\mathcal{E}(\mathcal{S}(G)) \geq n - 4 + \sqrt{n^2 - 2n + 4 + 4\sqrt{\frac{3}{4}n^2 + s(G)}}.
\]

was obtained. With this inequality in hand and the above lemmas, we are ready to prove our main theorem:

**Theorem 7.** Let \( T \) be a tree of order \( n \). If \( D = D(T) \), then

\[
\mathcal{E}(\mathcal{S}(T)) \geq 2n - 6 + \sqrt{2(n - D)}.
\]
Proof. First, note that if $T$ is a star graph or one of the paths $P_4$, $P_5$ or $P_6$, then by [1],
$\mathcal{E}(S(T)) \geq 2n - 2$ which is greater than the lower bound given in theorem.

So, assume that $n \geq 4$ and $T$ is not a star graph or a path of order less than 7. By Lemmas
[4] and [5] we have

$$s(T) \geq |E(\Lambda(T))|(n - 3) \geq \frac{1}{2}n(n - 3)(n - D)$$
$$\geq \frac{1}{2}(n - 2)^2(n - D). \quad (n \geq 4) \quad (2)$$

Now, inequality [2] and Remark [6] imply that

$$\mathcal{E}(S(T)) \geq n - 4 + \sqrt{n^2 - 2n + 4 + 4\sqrt{s(T)}}$$
$$\geq n - 4 + \sqrt{(n - 2)^2 + 2n + 4\sqrt{\frac{1}{2}(n - 2)^2(n - D)}}$$
$$\geq n - 4 + \sqrt{(n - 2)^2 + 2(n - D) + 2(n - 2)\sqrt{2(n - D)}}$$
$$= n - 4 + (n - 2) + \sqrt{2(n - D)}$$
$$= 2n - 6 + \sqrt{2(n - D)},$$

which completes the proof. \hfill \square

We close the paper by a numerical discussion about the average value of $D(T)$. Let $T$ be a random tree of order $n$, where by random we mean a uniformly chosen spanning tree of $K_n$. Thanks to Gordon Royle answer [1] and using the software SageMath [5], we have the following table:

| n  | Average value of $D(T)$       | n  | Average value of $D(T)$       |
|-----|-------------------------------|----|-------------------------------|
| 6   | 1.5692592592592598           | 15 | 1.8093535353535353           |
| 7   | 1.6018325697625999           | 16 | 1.83426034689951             |
| 8   | 1.6152038574218757           | 17 | 1.8576535568845751           |
| 9   | 1.6481679057505914           | 18 | 1.8801276901061494           |
| 10  | 1.6749189                    | 19 | 1.9017080817999203           |
| 11  | 1.7038043317654221           | 20 | 1.9224240041946314           |
| 12  | 1.7314162738607985           | 21 | 1.9423085461667031           |
| 13  | 1.7585077425737015           | 22 | 1.9613962948137142           |
| 14  | 1.7846671875609421           | 23 | 1.979725829436216           |

Table 1: The average value of $D(T)$.

1https://mathoverflow.net/a/402550/125843
Table 1 shows clearly that the average value of $D(T)$ is much less than $n$. At the time of writing this paper, we do not know the value of $\lim_{n \to \infty} \frac{D(T)}{n}$. We therefore encourage motivated readers to calculate the limit as a future study.

References

[1] S. Akbari, M. Einollahzadeh, M.M. Karkhaneei, M. A. Nematollahi, Proof of a conjecture on the Seidel energy of graphs, European J. Combin. 86 (2020): 103078.

[2] E. Ghorbani, On eigenvalues of Seidel matrices and Haemers’ conjecture, Designs, Codes and Cryptography, 84. 1–2 (2017) 189–195.

[3] W.H. Haemers, Seidel switching and graph energy, MATCH Commun. Math. Comput. Chem. 68 (2012) 653–659.

[4] M.R. Oboudi, Energy and Seidel energy of graphs, MATCH Commun. Math. Comput. Chem, 75 (2016) 291–303.

[5] SageMath, the Sage Mathematics Software System (Version 9.4), The Sage Developers, 2021, http://www.sagemath.org.

[6] J.H. van Lint, J.J. Seidel, Equilateral point sets in elliptic geometry, Indag. Math. 28 (1966) 335–348.