Scaling treatment of the random-field Ising model

R. B. Stinchcombe\textsuperscript{a}, E. D. Moore\textsuperscript{a}\textsuperscript{*} and S. L. A. de Queiroz\textsuperscript{\dagger}

\textsuperscript{a} Department of Physics, Theoretical Physics, University of Oxford,
1 Keble Road, Oxford OX1 3NP, United Kingdom

\textsuperscript{b} Instituto de Física, Universidade Federal Fluminense,
Outeiro de São João Batista s/n, 24210-130 Niterói RJ, Brazil

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Analytic phenomenological scaling is carried out for the random field Ising model in general dimensions \(d\) using a bar geometry. Domain wall configurations and their decorated profiles and associated wandering and other exponents (\(\zeta, \gamma, \delta, \mu\)) are obtained by free energy minimization. Scaling between different bar widths provides the renormalization group (RG) transformation. Its consequences are (i) criticality at \(h = T = 0\) in \(d \leq 2\) with correlation length \(\xi(h, T)\) diverging like \(\xi(h, 0) \propto h^{-2/(2-d)}\) for \(d < 2\) and \(\xi(h, 0) \propto \exp[1/(c_1 h^\gamma)]\) for \(d = 2\), where \(c_1\) is a decoration constant; (ii) criticality in \(d = 2+\epsilon\) dimensions at \(T = 0, h^* = (\epsilon/c_1)^{1/\gamma}\), where \(\xi \propto [(s-s^*)/s]^{-2\epsilon/\gamma}, s \equiv h^\gamma\). Finite temperature generalizations are outlined. Numerical transfer matrix calculations and results from a ground state algorithm adapted for strips in \(d = 2\) confirm the ingredients which provide the RG description.

\textsuperscript{*}Electronic address: dmoore@thphys.ox.ac.uk
\textsuperscript{\dagger}Electronic address: sldq@portela.if.uff.br
The random field Ising model (RFIM), which is closely related to dilute antiferromagnetic Ising systems in a uniform field [1], has provided a sequence of challenges to theoreticians as well as experimentalists [2, 3]. The eventual resolution [3] of the puzzle concerning its lower critical dimension confirmed the validity of domain wall pictures [1, 2]. Attempts to elucidate the critical behaviour at the lower critical dimension, \( d = 2 \), by normally very powerful numerical techniques [4, 5] have encountered difficulties related to the apparently anomalously severe divergence of the correlation length there. The phase diagram boundaries are not known for general dimensions, and the critical behaviour in three dimensions [1, 2] is presently not understood.

Yet methods for describing individual domain walls have been provided [4, 6], and their exploitation in standard geometries [15, 16] has been very effective, within the limits of a scheme without renormalization.

The aim of the present work is to use domain wall pictures in a bar geometry to build up, through finite size scaling [17, 18], a renormalization group (RG) description from which the critical properties can be obtained. For strip geometries in \( d = 2 \), we also present preliminary free energy results from a numerical transfer matrix procedure [19] from which quantities such as the free energy and correlation length can be estimated. Though presently short of providing a full self-contained numerical phenomenological RG description, these together with and a direct numerical solution for the ground state [20] allow a check of the analytical domain wall arguments from which the renormalization scheme is built up.

The method exploits RG transformations obtained by finite size scaling in the bar geometry shown in Fig. 1.

We begin with the case of zero temperature \( T \), which most easily illustrates the procedure and already includes several important results and considerations. Here the basic quantity is the energy \( U \) per site. Throughout the letter the no-wall term is subtracted, and hypercubic or continuum cases are considered. For flat domain walls with average spacing \( \xi_L \), \( U \) becomes \( U^F = WL^{1-d}/\xi_L \) where

\[
W = J(2L^{d-1} - 4c_0 h(\xi_L L^{d-1})^{1/2}) .
\]  

This is made up of a surface term and one proportional to \((\text{volume})^{1/2}\). The variable \( h \) is the rms random field in units of the exchange interaction \( J \), \( d \) is the dimensionality and \( c_0 \) is a constant of order 1. Minimizing \( U^F \) with respect to \( \xi_L \) gives the equilibrium values \( \xi_L^F, U^F \):

\[
\xi_L^F = L^{d-1}(c_0 h)^{-2}, \quad U^F = -2(c_0 h)^2/L^{d-1}.
\]  

As discussed later, these primitive results are asymptotically correct for small \( h \), where \( \xi_L \gg L \), and are confirmed by comparison with the numerical transfer matrix (TM) results in the appropriate limit.

The RG transformation of parameters (here \( h \to h' = R_h(h) \)) under rescaling of \( L \) by \( b \) is obtained from the phenomenological scaling relation [17, 18]

\[
\xi_L(h')/L = \xi_{bL}(h)/(bL) .
\]  

Using \( \xi_L^F \), as in Eq. 3 gives

\[
h' = R_h^b(h) \equiv b^{(2-d)/2} h .
\]  

This result has been previously obtained in the field-theoretic scheme of Grinstein and Ma [21], see also Ref. [22]. Eq. 3 has an unstable fixed point at \( h = 0 \) for \( d < 2 \). Here the eigenvalue \( b^{(2-d)/2} \) implies that the bulk correlation length diverges for small \( h \) at \( T = 0 \) in \( d < 2 \) like

\[
\xi \propto h^{-2/(2-d)} .
\]  

The transformation [4] is marginal for \( d = 2 \). So, subdominant terms are required, even at small \( h \), \( T = 0 \), for \( d = 2 \), to deal with the marginality. These come from “decoration” perturbations of the flat walls [4–16]. These can be built up from smooth shape modifications of base length \( a \) and height \( b(a) \), superimposed on all scales \( a \) between \( L \) and a cutoff value determined by the lattice constant (=1), see Fig. 2. The minimum energy \( \varepsilon(a) \) and minimizing height \( b \) for a smooth decoration of base scale \( a \) are of the form

\[
\varepsilon(a) \propto -h^\gamma a^\delta, \quad b(a) \propto h^{\mu} a^\zeta \equiv a^\zeta \varepsilon(a),
\]  

where \( \zeta \) is a wandering exponent [14, 21, 23, 24], and the exponent \( \gamma \) will be particularly significant below. The minimization gives (general \( d \)):

\[
(\gamma, \delta, \mu, \zeta) = \left( \frac{4}{3} \frac{d+1}{3}, \frac{2}{3} - \frac{5-d}{3} \right) \text{ or } (2, 1, 2, 3-d) ,
\]  

where the first result applies for the continuum case if \( b \ll a \), and the second one applies to the continuum if \( b \gg a \), and to the lattice case. So, in the continuum case, if \( a \) is large the first result applies for \( d > 2 \) (largest \( \delta \), i.e., lowest energy) and the second for \( d < 2 \).

Since in Eq. 3 \( \varepsilon(a) \) is negative, decorations on decorations occur down to a smallest base length \( a_m \) such that \( \min(a_m, b(a_m)) \approx 1 \). The resulting contribution of all decorations to the energy of a single wall of base length \( L \) is, for \( d = 2 \),

\[
\Sigma = -Jc_1 h^\gamma L \ln(Lh^\mu c_2) ,
\]  

where \( c_1 \) and \( c_2 \) are constants of order 1. The effect of decorations is therefore to replace the wall energy \( W \) by \( W + \Sigma \). Minimizing the energy per site with respect to wall separation then gives equilibrium values \( U^D, \xi_L^D \) for the decorated wall generalization, and hence (via \( \xi_L^D / L \))
the generalized phenomenological scaling equation. For $d$ near 2 (the marginal case), this is:

$$h' = h^{(2-d)/2}[1 + c_1 h^\gamma \ln b] .$$

(9)

From Eq. 3 $\gamma = 4/3$, 2 for continuum and lattice cases respectively. The transformation given by Eq. 3 has the proper semigroup character, in its range of validity ($h$ small, $d$ near 2, $T = 0$). As will be seen, this range includes the critical effects for $d$ at or just above 2.

For $d = 2$, Eq. 6 shows that the $h = 0$ fixed point is marginally unstable, as expected at the lower critical dimension. Analyzing the equation by standard RG procedures gives the two-dimensional bulk correlation length diverging for small $h$ and $T = 0$ as

$$\xi = A \exp \left( \frac{1}{c_1 h^\gamma} \right) .$$

(10)

In $2+\epsilon$ dimensions, $\epsilon$ small and positive, the RG transformation Eq. 3 yields an unstable zero temperature fixed point at small non-zero field $h^\star$, as well as the associated divergence of $\xi$:

$$h^* = (\epsilon/2c_1)^{1/\gamma}, \quad \xi \propto [s - s^*]/s^{-2/\gamma}, \quad s \equiv h^\gamma .$$

(11)

This divergence crosses over to the behaviour of Eq. 10 as $\epsilon \to 0$. In Eqs. 10 and 11, the exponent $\gamma$ differs between lattice and continuum systems. This feature has previously been noted in the interface context [14].

For larger fields, the flat wall description requires more drastic generalizations. Increasing $h$ at fixed $L$ would, according to Eq. 3, eventually give $\xi_L < L$. So, by then the effects of domain boundaries in all directions should have been allowed for. This generalization yields a complicated form which confirms Eq. 11 and its consequences in the low field regime where $\xi_L \gg L$, but which for $\xi_L \ll L$ gives

$$U = J[\xi L^{1-d} - c_2 h \xi_L^{d(\phi-1)}] ,$$

(12)

where $c_1$, $c_2$ are geometric constants and $\phi$ crosses over between 1/2 and 1 as $\xi_L$ approaches order 1. For $d = 2$, $T = 0$ the result of allowing for this, and the decoration effects, is as follows. As $h$ increases the low field result for $U$ arising from Eq. 2 (namely $-2(c_0 h^2)/L$) picks up a factor $A(h, L) \equiv [1 + c_3 h^\gamma \ln(Lh^\phi c_0)]$, then goes through a complicated intermediate regime, and finally for $h$ larger than about 1 becomes linear in $h$ and independent of $L$.

A check of the basic (flat wall) ingredients in the domain scaling description is provided by comparing these results for $U$ with those provided by a numerical transfer matrix calculation for the $d = 2$ RFIM [13]. Here the configurationally averaged free energy is enumerated exactly for very long ($\sim 10^5$ lattice parameters) strips of finite width $L$. Fig. 3 gives the field dependence of the negative of the free energy per spin $f$, for relatively small widths $L = 2, \ldots, 7$, at temperature $T/J = 0.1$ which is sufficiently low that $f$ is essentially $U$. The main characteristics of the domain wall results for $-U (h^2/L$ crossing over to $h/L^0$), may be seen to be present. Indeed the analytical results, taking $c_0 = 0.4614$ in Eq. 2, fit the numerical data for $L = 4 - 9$ within the very small error estimated from the data fluctuations in the low field regime $h \leq 0.5$ for $T/J = 0.1$, and also for higher temperatures using extensions to the domain wall theory outlined immediately below. However, the fit is not sufficiently sensitive to the decoration terms to confirm the theoretical value $\gamma = 2$. Other comparisons discussed later show more direct effects of the decoration terms.

The finite temperature generalization involves entropic terms. At the most primitive level (flat walls only) the entropy per site is:

$$S^F = -k_B L^{(1-d)}[x \ln x + (1 - x) \ln(1 - x)], \quad x \equiv 1/\xi_L ,$$

(13)

coming from the number of ways of selecting $Lx$ out of $L$ sites for the placement of the walls. Adding $-TS^F$ to the flat wall internal energy $U^F$ and minimizing gives the equilibrium flat wall separation $\xi_L^F$ and the free energy $F^F$ per site. $\xi_L^F$ satisfies

$$c_0 h(\xi_L^{d(1-d)}/L^{1-d})^{1/2} + (L^{1-d}/2K) \ln(\xi_L^F - 1) = 1 ,$$

(14)

where $K \equiv J/k_B T$.

This result is valid if $\xi_L \gg L$, which is the case if both $hU^{(2-d)/2}$ and $(1/K)L^{1-d} \ln L$ are small, sufficient to encompass Eqs. 2 and 3 and their low-temperature generalizations. Otherwise domain walls in all directions are needed, and they increase the total entropy, asymptotically to $c_3 \xi_L^{-d} \ln(c_4 \xi_L^d)$ where $c_3,4$ are constants. And in $d = 2$ decoration terms are required to resolve the marginality at low $T$. They affect both the energy as discussed above (taking $K$ in Eq. 14 to $K\lambda(h, L)$), and the entropy. Their entropy contribution for each vertical wall of base length $L$ in $d = 2$ is of the form $c_5 L h^{\phi} \ln(c_6 L)$.

The resulting temperature-dependent generalizations of Eqs. 2, 3, 6, 11 and 12 will be given elsewhere [25], together with phase boundaries and comparisons between analytic and numeric transfer matrix and Monte Carlo [26] results for finite temperature free energies.

However, we briefly present here evidence for the marginality–breaking domain wall roughening given in equation (8) and used to derive key results such as $\xi$, (10) and (12). Specifically, we demonstrate that, for a gaussian distribution of random fields, the exponent $\mu$ approaches 2 as $L$ grows large. The max–flow algorithm of Ogilski [24] was implemented on a strip geometry of dimensions $L \times 1000$ and the ground state was generated for 100 independent random field configurations of fixed standard deviation $h$. For each spin configuration, the rms width of the domain walls was measured, and this statistic was averaged over all the domain
walls of the 100 field configurations. We expect $b_{rms}(L)$ to scale like $b(a)$ in equation (6). Figure 4 justifies this prediction for large $L$.

Detailed results, discussions of the operational procedures will be given elsewhere [19, 25, 26]. The results are here used merely to support the domain scaling approach.

In conclusion, an analytic phenomenological scaling approach has been constructed using domain considerations. The resulting RG transformations have been used to find critical properties, including the correlation length behaviour at and near the marginal (lower critical) dimension $d = 2$. Numerical transfer matrix free energy comparisons have confirmed, in $d = 2$, the basic domain picture used in the scaling. And numerical ground state wall roughening investigations have quantitatively tested a particular theoretical prediction (the exponent $\mu$) for the decoration ingredient, which is so crucial for the criticality in 2 and $2 + \epsilon$ dimensions (see Eqs. 10, 11).

Generalizations of the domain scaling approach to related vector model systems and to dynamics are being pursued starting from existing domain wall pictures (respectively Refs. [7] and [28]).

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[1] S. Fishman and A. Aharony, J. Phys. C 12, L729 (1979).
[2] H. Yoshizawa et al., Phys. Rev. Lett. 48, 438 (1982).
[3] R. Birgeneau, R. Cowley, G. Shirane, and H. Yoshizawa, Phys. Rev. Lett. 54, 2147 (1985).
[4] D. Belanger, A. King, V. Jaccarino, and J. Cardy, Phys. Rev. B 28, 2522 (1983).
[5] A. Aharony, Y. Imry, and S. Ma, Phys. Rev. Lett. 37, 1364 (1976).
[6] J. Imbrie, Phys. Rev. Lett. 53, 1747 (1984).
[7] Y. Imry and S. Ma, Phys. Rev. Lett. 35, 1399 (1975).
[8] E. Pytte and J. Fernandez, J. Appl. Phys. 57, 3274 (1985).
[9] J. Fernandez, Phys. Rev. B 31, 2886 (1985).
[10] U. Glaus, Phys. Rev. B 34, 3203 (1986).
[11] H. Rieger and A. Young, J. Phys. A 26, 5279 (1993).
[12] J. Gofman et al., Phys. Rev. Lett. 71, 1569 (1993).
[13] B. Derrida and J. Vannimenus, Phys. Rev. B 27, 4401 (1983).
[14] T. Natterman, Phys. Stat. Sol. (b) 131, 563 (1985).
[15] J. Villain, J. Phys. (Paris) 43, L551 (1982).
[16] K. Binder, Z. Phys. B. 50, 343 (1983).
[17] M. Barber, in Phase Transitions, edited by Domb and Lebowitz (Academic Press, London, 1983), Vol. 8.
[18] M. Nightingale, in Finite Size Scaling and Numerical Simulations of Statistical Systems, edited by V. Privman (World Scientific, Singapore, 1990).
[19] S. de Queiroz, E. Moore, and R. Stinchcombe, preprint (unpublished).
[20] A. Ogielski, Phys. Rev. Lett. 57, 1251 (1986).
[21] G. Grinstein and S. Ma, Phys. Rev. Lett. 49, 685 (1982).
[22] A. Bray and M. Moore, J. Phys. C 18, L927 (1985).
[23] T. Halpin-Healy, Phys. Rev. A 42, 711 (1990).
[24] M. Mézard and G. Parisi, J. Phys. A 23, L1229 (1990).
[25] E. Moore, R. Stinchcombe, and S. de Queiroz (unpublished).
[26] E. Moore and R. Stinchcombe, preprint (unpublished).
[27] S. L. A. de Queiroz and R. B. Stinchcombe, Phys. Rev. B 46, 6635 (1992).
[28] D. Huse and D. Fisher, Phys. Rev. B 35, 6841 (1987).
FIG. 1. Domain walls of separation $\xi_L$, in a $d$-dimensional bar of transverse scale $L$.

FIG. 2. Domain wall decorations ($d = 2$ for simplicity) of base length $a$, height $b(a)$: (i) for continuum; (ii) for lattice; (iii) superimposed at successively smaller scale.

FIG. 3. Free energy for the $d = 2$ RFIM, from numerical transfer matrix calculations for long strips of widths $L = 2, 3, \ldots, 7$, as a function of $h/J$ at $T/J = 0.1$. The behaviour crosses over from $h^2/L$ to $h/L^\beta$, as predicted by domain wall arguments and used in scaling.

FIG. 4. The domain wall roughening exponent $\mu$. This has been determined by least-squares fits to semi-log plots of $b_{r.m.s}(L)$ vs $h$. 
