The Positive and Negative Deficiency Indices of Formally Self-Adjoint Difference Equations

Guojing Ren

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Abstract
This paper is concerned with formally self-adjoint difference equations and their positive and negative deficiency indices. It is shown that the order of any formally self-adjoint difference equation is even, and some characterizations of formally self-adjoint difference equations are established. Further, we show that the positive and negative deficiency indices are always equal, which implies the existence of the self-adjoint extensions of the minimal linear relations generated by the difference equations. This is an important and essential difference between formally self-adjoint difference equations and their corresponding differential equations in the spectral theory.

Keywords Positive and negative deficiency indices · Hermitian operator · Formally self-adjoint · Difference equation · $K$-real

Mathematics Subject Classification 39A70 · 47B39 · 34B20

1 Introduction

Difference equations are usually regarded as the discretization of the corresponding differential equations. According to the existing results, most of the properties in spectral theory of difference equations coincide with those of the corresponding differential equations; only a few, but important, are different. It has been found that the maximal operator corresponding to a formally self-adjoint difference equation may be multi-valued, and the corresponding minimal operator may be non-densely defined.
Therefore, the classical spectral theory for symmetric operators, i.e., densely defined and Hermitian single-valued operators, are not available in the studying of the spectral properties of difference equations in general. Due to this reason, some researchers focus on extending the spectral theory of linear operators to linear non-densely defined or multi-valued operators (which are called linear relations or linear subspaces), and many good results have been obtained (See [6, 7, 23, 24] and their references).

According to the generalized von Neumann theory and the GKN theory, a Hermitian linear relation, has a self-adjoint extension if and only if its positive and negative deficiency indices are equal, and in this case the domain of the self-adjoint extension has a close relationship with the deficiency indices [6, 23]. So, the positive and negative deficiency indices of Hermitian linear relations play a key role in the study of spectral theory of linear relations.

Now, we briefly recall some important results about the deficiency indices of formally self-adjoint differential and difference equations, respectively.

The theory of positive and negative deficiency indices of Hermitian differential equations has been well developed. The canonical form of any formally self-adjoint differential expression $\tau$ of order $m$ on some interval $J$ is given by

$$\tau y := \sum_{j=0}^{\lfloor m/2 \rfloor} (-1)^j [a_j y^{(j)}]^{(j)} + i \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} \left[ (b_k y^{(k+1)})^{(k)} + (b_k y^{(k)})^{(k+1)} \right], \quad (1.1)$$

where $a_j$ and $b_k$ are all real-valued functions belonging to $C^\infty(J)$, and $i = \sqrt{-1}$.

The following equation

$$\tau y(x) = \lambda w(x)y(x), \quad x \in J \quad (1.2)$$

is called the formally self-adjoint differential equation corresponding to $\tau$ [8, XIII,2], where $w \geq 0$ is called the weighted function, $\lambda \in \mathbb{C}$ is the spectral parameter. In the case that both endpoints of $J$ are singular, we can divided $J$ into two subintervals such that each subinterval has at least one regular endpoint. So we take $J := [0, +\infty)$ without loss generality.

Since (1.1) is formally self-adjoint, the minimal linear relation generated by (1.2) in the corresponding Hilbert space is Hermitian, and consequently its deficiency index $d_\lambda(\tau)$ is constant when $\lambda$ is in the upper half-plane and lower half-plane. Denote $d_\pm(\tau) := d_{\pm i}(\tau)$, which are called the positive and negative deficiency indices of (1.2).

In addition, by $n_\lambda(\tau)$ denote the number of linearly independent solutions of (1.2) which satisfies

$$\int_J w(x)|y(x)|^2 dx < +\infty.$$
(A1) there exists a bounded interval $\mathcal{J}_0 \subset \mathcal{J}$ such that for any $\lambda \in \mathbb{C}$ and for any non-trivial solution $y$ of (1.2), the following always holds

$$\int_{\mathcal{J}_0} w(x)|y(x)|^2dx > 0.$$  

It is evident that (A1) holds when $w(t) > 0$ on $\mathcal{J}$. It is to be noted that the assumption (A1) is independent of $\lambda$, i.e., it is either satisfied for all $\lambda \in \mathbb{C}$ or for none of them (See [12, Section 2]). Under the assumption (A1), it has been shown that $d_+(\tau) = n_+(\tau)$ for all $\lambda \in \mathbb{C}$. This equivalence does not hold without the assumption (A1) (See [12, Proposition 2.19]).

In the case that all the coefficients of $\tau$ are real-valued, i.e., all $b_k(t) \equiv 0$ on $\mathcal{J}$, it has been easily shown that $n_+(\tau) = n_-(\tau)$. This, together with (A1), implies $d_+(\tau) = d_-(\tau)$. However, the values of $d_\pm(\tau)$ may differ when $\tau$ has complex-valued coefficients. Mcleod [14] first gave an example of a fourth-order formally self-adjoint differential equation with $d_+(\tau) = 3$ and $d_-(\tau) = 2$. Later, Kogan and Rofe-Beketov [10, 11] showed that

$$|d_+(\tau) - d_-(\tau)| = 1$$

happens for any $m \geq 3$.

Many authors are interested in the positive and negative deficiency indices of formally self-adjoint difference equations, and have got many excellent results. A formally self-adjoint difference expression is necessary to be even, and it has the following form (see Theorem 3.4):

$$\mathcal{L}y := \sum_{j=0}^n F^j(A_jy) + \sum_{j=1}^n A_j F^{-j}y, \quad (1.3)$$

where $A_j$, $j = 1, \ldots, n$, are complex-valued functions, and $A_0$ is real-valued on $\mathcal{I}$; $A_n \neq 0$ on $\mathcal{I}$; $F$ is the forward shift operator, i.e., $Fy(t) = y(t + 1)$. The following equation

$$(\mathcal{L}y)(t) = \lambda w(t)y(t), \quad t \in \mathcal{I} \quad (1.4)$$

is called the formally self-adjoint equation generated by $\mathcal{L}$, where $w(t) \geq 0$ and $\lambda \in \mathbb{C}$. We take $\mathcal{I} := \{t\}_{t=0}^{+\infty}$ in the following. It is worth noting that the order of any formally self-adjoint difference equation is even. This is a difference between formally self-adjoint difference equations and differential equations.

In addition, $\mathcal{L}$ defined by (1.3) can be rewritten as:

$$\mathcal{L}y = \sum_{j=0}^n (-1)^j \Delta^j(p_j \nabla^j y) + i \sum_{k=1}^n [(-1)^{k+1} \Delta^k(q_k y) + q_k \nabla^k y], \quad (1.5)$$
where $\Delta$ and $\nabla$ are the forward and backward difference operators, respectively, i.e., $\Delta y(t) = y(t + 1) - y(t)$ and $\nabla y(t) = y(t) - y(t - 1)$; the coefficients $p_j$ and $q_k$ are all real-valued on $I$. The coefficients between (1.3) and (1.5) have the following relationship:

\[
A_0(t) = P_0(t), \quad A_j(t) = P_j(t) + i Q_j(t), \quad 1 \leq j \leq n, \\
P_j(t) = (-1)^j \sum_{s=j}^{n} \sum_{k=0}^{s-j} C_s^k C_{s-j-k}^k p_s(t + k), \quad 0 \leq j \leq n, \\
Q_j(t) = (-1)^{j+1} \sum_{k=j}^{n} C_j^k q_k(t), \quad 1 \leq j \leq n,
\]

where $C_s^k = \frac{s!}{k!(s-k)!}$.

The definiteness condition for (1.4) is given by:

(A2) there exists a bounded integer interval $I_0 \subset I$ such that for any $\lambda \in \mathbb{C}$ and for any non-trivial solution $y$ of (1.4), the following always holds

\[
\sum_{t \in I_0} w(t) |y(t)|^2 > 0.
\]

The same remark holds as that for the assumption (A1), that is, the assumption (A2) does not depend on $\lambda \in \mathbb{C}$ (See [17, Remark 4.2] or [27, Lemma 2.6]).

Similarly as the continuous case, the minimal linear relation generated by (1.4) is Hermitian and consequently its deficiency index $d_\lambda(L)$ is constant when $\lambda$ is in the upper and lower half-planes, respectively [20]. Denote $d_\pm(L) := d_{\pm i}(L)$. Under the assumption (A2), $d_\lambda(L) = n_\lambda(L)$ holds, where $n_\lambda(L)$ is the number of linearly independent solutions of (1.4), which satisfies

\[
\sum_{t \in I} w(t) |y(t)|^2 < +\infty.
\]

When $n = 1$, it is well known that the positive and negative deficiency indices of (1.4) are always equal, and either $d_+(L) = d_-(L) = 1$ or $d_+(L) = d_-(L) = 2$. When (1.4) is with real-valued coefficients, it follows that

\[
n \leq d_+(L) = d_-(L) \leq 2n
\]  

(1.6)

holds and all the values in this range can be realized. When (1.4) is with complex-valued coefficients, it has been shown that all the values in (1.6) can be realized. The readers are referred to [2, 4, 5, 9, 16, 17, 25] for more details.

Up to now, all the existing results on the the positive and negative deficiency indices of formally self-adjoint difference equations coincide with those of their corresponding differential equations. So, one may think there would exist some special case of (1.4), corresponding to Mcleod’s example mentioned above, satisfying $d_+(L) \neq d_-(L)$. 
In this manuscript, we show that the positive and negative deficiency indices of any formally self-adjoint difference equations are equal (Theorem 4.2). This is a new and important difference in the spectral theory between the formally self-adjoint difference equations and their corresponding differential equations.

The rest of the paper is organized as follows. In Sect. 2, we introduce some basic concepts of the spectral theory in Hilbert space and give some sufficient and necessary conditions for an Hermitian operator to have equal positive and negative deficiency indices. In Sect. 3, we establish two characterizations of formally self-adjoint difference expressions. In Sect. 4, we prove the positive and negative deficiency indices of any formally self-adjoint difference equations are equal. To illustrate the main results in this paper, we give an example in Sect. 5.

2 Fundamental Results about Linear Relations and Linear Operators

In this section, we introduce some basic concepts of linear relations and linear operators, and show some results on the deficiency indices of Hermitian linear relations and linear operators, and on their self-adjoint extensions.

By $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{Z}$ denote the sets of all complex numbers, real numbers and integers, separately. By $\bar{a}$ denote the conjugation of $a \in \mathbb{C}$, and $i := \sqrt{-1}$.

Let $X$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $T_1$ and $T_2$ be two linear relations (briefly, relation) in $X^2 := X \times X$. Denote

$\mathcal{D}(T) := \{ x \in X : (x, f) \in T \text{ for some } f \in X \}$,

$\mathcal{R}(T) := \{ f \in X : (x, f) \in T \text{ for some } x \in X \}$,

$\mathcal{N}(T) := \{ x \in X : (x, 0) \in T \}$,

$T^* := \{ (x, f) \in X^2 : (x, g) = \langle f, y \rangle \text{ for all } (y, g) \in T \}$,

$T(x) := \{ f \in X : (x, f) \in T \}$,

$T_1 + T_2 := \{ (x, f_1 + f_2) \in X^2 : (x, f_i) \in T_i, i = 1, 2 \}$.

It is clear that $T(0) = \{0\}$ if and only if there exists a unique linear operator, denoted by $S_T : \mathcal{D}(T) \to X$, such that its graph $G(S_T) = T$. In this case, $S_T x = f$ for any $(x, f) \in T$.

In addition, $T$ is said to be Hermitian if $T \subset T^*$; $T$ is said to be symmetric if $T \subset T^*$ and $\mathcal{D}(T)$ is dense in $X$; $T$ is said to be self-adjoint if $T = T^*$.

If $T$ is a closed linear relation in $X^2$, then $T$ can be decomposed as [1]:

$T = T_s \oplus T_\infty$,

where $T_\infty = \{ (0, f) \in X^2 : f \in T(0) \}$ is called the pure multi-valued parts of $T$, and

$T_s = T \ominus T_\infty$

is called the operator part of $T$. 

Lemma 2.1 [24, Proposition 2.1, 3.1] Let $T$ be a closed Hermitian relation in $X^2$. Then
\[ T_{\infty} = T \cap (T(0))^2, \quad T_s = T \cap (T(0)^\perp)^2, \]
and $T_s$ is closed Hermitian in $(T(0)^\perp)^2$. Further, if $T$ is self-adjoint in $X^2$, then $T_s$ is self-adjoint in $(T(0)^\perp)^2$.

Let $T$ be a linear relation in $X^2$ or a linear operator on $X$. The subspace $R(T - \lambda I)^\perp$ and the number $d_{\lambda}(T) := \dim R(T - \lambda I)^\perp$ are called the deficiency space and deficiency index of $T$ and $\lambda$, respectively. By $\overline{T}$ denote the closure of $T$. It can be easily verified that $d_{\lambda}(T) = d_{\lambda}(\overline{T})$ for all $\lambda \in \mathbb{C}$.

Let $T$ be a linear relation in $X^2$. The set
\[ \Gamma(T) := \{ \lambda \in \mathbb{C} : \exists c(\lambda) > 0 \text{ s.t. } \|f - \lambda x\| \geq c(\lambda)\|x\|, \forall (x, f) \in T \} \]
is called the regularity domain of $T$.

It has been shown by [23, Theorem 2.3] that the deficiency index $d_{\lambda}(T)$ is constant in each connected subset of $\Gamma(T)$. If $T$ is Hermitian, then $d_{\lambda}(T)$ is constant in the upper and lower half-planes. So, we can denote $d_{\pm}(T) := d_{\pm i}(T)$ for an Hermitian linear relation $T$, and call $d_{\pm}(T)$ the positive and negative deficiency indices of $T$, respectively.

Lemma 2.2 [6, Theorem 15] Let $T$ be a closed Hermitian relation in $X^2$. Then $T$ has self-adjoint extensions if and only if $d_{\pm}(T) = d_{\pm}(T)$.

Lemma 2.3 [24, Corollary 2.1] Let $T$ be a closed Hermitian relation in $X^2$. Then $d_{\pm}(T) = d_{\pm}(T_s)$, and consequently, $T$ has a self-adjoint extension in $X^2$ if and only if $T_s$ has a self-adjoint extension in $(T(0)^\perp)^2$.

A relation $T$ is said to be bounded from below (above) if there exists a number $c \in \mathbb{R}$ such that
\[ \langle x, f \rangle \geq c\|x\|^2, \quad \langle x, f \rangle \leq c\|x\|^2, \quad \forall (x, f) \in T, \]
while such a constant $c$ is called a lower (upper) bound of $T$.

Lemma 2.4 [3, Proposition 1.4.6] Let $T$ be an Hermitian relation and be bounded from below with lower bound $c$. Then $\mathbb{C} \setminus [c, +\infty) \subset \Gamma(T)$, and the deficiency index $d_{\lambda}(T)$ is constant for all $\lambda \in \mathbb{C} \setminus [c, +\infty)$.

Combining Lemmas 2.2 and 2.4, one can get the following result.

Lemma 2.5 Let $T$ be an Hermitian relation and be bounded from below. Then $d_{\pm}(T) = d_{\pm}(T)$ and $T$ has self-adjoint extensions.

Next, we pay attention to the self-adjointness of an operator by using the concept of $K$-real. Let $S$ be an operator on $X$. 

Definition 2.6 [26, Section 8.1] Let $X$ be a complex Hilbert space. A mapping $K$ of $X$ onto itself is called a conjugation if

1. $K(ax + by) = \bar{a}K(x) + \bar{b}K(y)$ for all $x, y \in X, a, b \in \mathbb{C}$,
2. $K^2 = I$,
3. $\langle Kx, Ky \rangle = \langle y, x \rangle$ for all $x, y \in X$.

An operator $S$ on $X$ is said to be a $K$-real if

1. $KD(S) \subset D(S)$,
2. $SKx = KSx$ for $x \in D(S)$.

A sufficient condition for a symmetric operator to have equal positive and negative deficiency indices is given by [26, Theorem 8.9]. Since $S$ is not required to be densely defined in the proof, the assertion is still true for Hermitian operators.

Lemma 2.7 [26, Theorem 8.9] Let $X$ be a complex Hilbert space, and let $K$ be a conjugation on $X$. If $S$ is a $K$-real Hermitian operator on $X$, then $d_+(S) = d_-(S)$.

The following is a necessary condition for an operator $S$ to be self-adjoint.

Theorem 2.8 Let $X$ be a complex Hilbert space, and let $S$ be a self-adjoint operator on $X$. Then there exists a conjugation $K$ for which $S$ is $K$-real.

To prove Theorem 2.8, we introduce several notations and a lemma first. The readers are referred to [26].

Let $\{\rho_\alpha : \alpha \in A\}$ be a family of right continuous non-decreasing function defined on $\mathbb{R}$. By $L^2(\mathbb{R}, \rho_\alpha)$ denote the set of all the square integrable functions with respect to $\rho_\alpha$ on $\mathbb{R}$. It has been shown that $L^2(\mathbb{R}, \rho_\alpha)$ is a Hilbert space with inner

$$\langle f, g \rangle_\alpha = \int_\mathbb{R} \overline{g}(t)f(t)d\rho_\alpha(t).$$

By $\bigoplus_{\alpha \in A} L^2(\mathbb{R}, \rho_\alpha)$ denote the orthogonal sum of the spaces $L^2(\mathbb{R}, \rho_\alpha)$. Then $\bigoplus_{\alpha \in A} L^2(\mathbb{R}, \rho_\alpha)$ is a Hilbert space with inner

$$\langle (x_\alpha), (y_\alpha) \rangle = \sum_{\alpha \in A} \langle x_\alpha, y_\alpha \rangle_\alpha, \quad (x_\alpha), (y_\alpha) \in \bigoplus_{\alpha \in A} L^2(\mathbb{R}, \rho_\alpha).$$

By $K_0$ denote the natural conjugation on $\bigoplus_{\alpha \in A} L^2(\mathbb{R}, \rho_\alpha)$, i.e.,

$$K_0(x_\alpha) := (\bar{x}_\alpha), \quad (x_\alpha) \in \bigoplus_{\alpha \in A} L^2(\mathbb{R}, \rho_\alpha).$$

The following result is a combination of Theorems 7.16-7.18 of [26].

Lemma 2.9 Let $S$ be a self-adjoint operator on $X$. Then there exists exactly one spectral family $E$ for which $S = \int t \, dE(t)$. Moreover, for this spectral family $E$, there exists a family $\{\rho_\alpha : \alpha \in A\}$ of right continuous non-decreasing functions (the cardinality of
A is at most the dimension of $X$) and a unitary operator $U : X \to \bigoplus_{\alpha \in A} L^2(\mathbb{R}, \rho_\alpha)$ for which

$$S = U^{-1}S_{id}U,$$

(2.2)

where $S_{id}$ denotes the maximal operator of multiplication by the function $id$ on $\bigoplus_{\alpha \in A} L^2(\mathbb{R}, \rho_\alpha)$.

Now we give the proof of Theorem 2.8.

**Proof** Let $S$ be a self-adjoint operator on $X$. Then it follows from Lemma 2.9 that there exists a family $\{\rho_\alpha : \alpha \in A\}$ of right continuous non-decreasing functions (the cardinality of $A$ is at most the dimension of $X$) and a unitary operator $U : X \to \bigoplus_{\alpha \in A} L^2(\mathbb{R}, \rho_\alpha)$ such that (2.2) holds. Based on the proof of [26, Theorems 7.16], the unitary operator $U$ can be constructed to be linear, i.e.,

$$U(ax + by) = aU(x) + bU(y), \quad x, y \in X, \ a, b \in \mathbb{C}.$$

Let

$$K = U^{-1}K_0U,$$

(2.3)

where $K_0$ is the natural conjugation on $\bigoplus_{\alpha \in A} L^2(\mathbb{R}, \rho_\alpha)$ defined by (2.1). Then it follows that

$$K(ax + by) = \bar{a}K(x) + \bar{b}K(y)$$

for all $x, y \in X, \ a, b \in \mathbb{C}$. Further, by the facts that $K_0$ is a conjugation and $U$ is a unitary operator, it follows that

$$K^2 = U^{-1}K_0UU^{-1}K_0U = I,$$

$$(Kx, Ky) = (K_0Ux, K_0Uy) = \langle Uy, Ux \rangle = \langle y, x \rangle, \quad x, y \in X.$$

These yield that $K$ defined by (2.3) is a conjugation on $X$.

Further, it can be verified that $S_{id}$ is $K_0$-real, and $UD(S) \subset D(S_{id})$. Therefore it follows that $KD(S) \subset D(S)$, and

$$SKx = U^{-1}S_{id}K_0Ux = U^{-1}K_0S_{id}Ux = Kx, \quad x \in D(S).$$

This implies that $S$ is $K$-real. The proof is complete.

\[\Box\]

### 3 Characterization of Formally Self-Adjoint Difference Expressions

In this section we pay attention to the characterization of formally self-adjoint difference expressions.
First, we introduce some notations and concepts. Let \( X \) be a vector space over \( \mathbb{C} \). A mapping \( s : X \times X \to \mathbb{C} \) is called a sesquilinear form on \( X \) if it follows
\[
\begin{align*}
s[x, ay + bz] &= \bar{a}s[x, y] + \bar{b}s[x, z], \\
s[ay + bz, x] &= as[y, x] + bs[z, x],
\end{align*}
\]
for all \( x, y, z \in X \) and \( a, b \in \mathbb{C} \).

Let \( I = \{ t \}_{t=0}^{\infty} \). For some \( n \in \mathbb{Z} \), denote
\[
l(I, n) = \{ x = \{ x(t) \}_{t=-n}^{\infty} : x(t) \in \mathbb{C} \}.
\]

Let \( L \) be an arbitrary difference expression of order \( m \) defined on \( l(I, m) \):
\[
(Ly)(t) := \sum_{j=0}^{k} A_j(t + j)y(t + j) + \sum_{j=1}^{s} A_{-j}(t)y(t - j),
\]
where all \( A_j(t), -s \leq j \leq k \), are complex valued functions; \( k \) and \( s \) are non-negative integer satisfying \( k + s = m \), and \( A_k(t + k) \neq 0, A_{-s}(t) \neq 0 \) for \( t \in I \). By \( F \) denote the forward shift operator, i.e.,
\[
(Fy)(t) = y(t + 1),
\]
and define \( (F^{-1}y)(t) = y(t - 1), F^j = FF^{j-1} \). Thus \( L \) can be rewritten briefly as
\[
Ly = \sum_{j=0}^{k} F^j(A_jy) + \sum_{j=1}^{s} A_{-j}F^{-j}y. 
\tag{3.1}
\]

Now, we give the definition of the formal adjoint of \( L \) and try to establish the characterization of its formal adjoint.

**Definition 3.1** Let \( L \) be an \( m \)th-order difference expression. An \( m \)th-order difference expression, denoted by \( L^+ \), is said to be a formal adjoint of \( L \), if there exists a sesquilinear form \( s[\cdot, \cdot] \) on \( l(I, m) \) such that
\[
\bar{y}(t)(Lx)(t) - (L^+y)(t)x(t) = \Delta s[x, y](t), \quad t \in I.
\]
Moreover, \( L \) is said to be formally self-adjoint if \( L = L^+ \).

**Remark 3.2** Definition 3.1 is enlightened by [13, II, Definition 2.1], which gives the definition of the formal adjoint of a formal differential expression. It guarantees that a difference expression with formally self-adjointness generates a Hermitian operator in the corresponding space.

For \( n \geq 0 \), let \( L_n \) be a difference expression defined as the following:
\[
(L_ny)(t) := \sum_{j=0}^{n} F^j(B_jy)(t), \tag{3.2}
\]
where all \(B_j(t), 0 \leq j \leq n\), are complex-valued functions, \(B_n(t) \neq 0\) on \(\mathcal{I}\). We get the following result on the formal adjoint of \(L_n\).

**Lemma 3.3** Let \(L_n (n \geq 1)\) be an nth-order difference expression defined by (3.2). Then

(1) *(Green Formula)* For any \(x, y \in l(\mathcal{I}, n), 0 \leq k \leq r,

\[
\sum_{t=k}^{r} [\bar{y}(t)(L_n x(t)) - (L_n^+ y(t))x(t)] = s[x, y](r + 1) - s[x, y](k)\tag{3.3}
\]

where \(L_n^+\) is given by

\[
(L_n^+ y)(t) = \sum_{j=0}^{n} \bar{B}_j(t) F^{-j} y(t)\tag{3.4}
\]

and \(s[x, y]\) is a sesquilinear on \(l(\mathcal{I}, n)\) with the form

\[
s[x, y](t) = \sum_{k=1}^{n} \sum_{j=0}^{k-1} \bar{y}(t + j - k) B_k(t + j) x(t + j)\tag{3.5}
\]

(2) \(L_n + L_n^+\) is a 2nth-order formally self-adjoint difference expression.

**Proof** (1) Let \(L_n^+\) and \(s[x, y]\) be defined as (3.4) and (3.5), respectively. It suffices to show

\[
\bar{y}(t)(L_n x(t)) - (L_n^+ y(t))x(t) = \Delta s[x, y](t), \quad t \in \mathcal{I}\tag{3.6}
\]

We will proof (3.6) by induction. For \(n = 1\), it follows that

\[
\bar{y}(t)(L_1 x(t)) - (L_1^+ y(t))x(t) = \bar{y}(t) B_1(t + 1) x(t + 1) - \bar{y}(t - 1) B_1(t) x(t)
\]

\[
= \Delta [\bar{y}(t - 1) B_1(t) x(t)].
\]

This yields (3.6) with \(n = 1\). Assume that (3.6) hold for \(n - 1\). It can be easily verified

\[
\bar{y}(t) B_n(t + n) x(t + n) - \bar{y}(t - n) B_n(t) x(t)
\]

\[
= \Delta \sum_{j=0}^{n-1} [\bar{y}(t + j - n) B_n(t + j) x(t + j)].
\]

Then it follows that

\[
\bar{y}(t)(L_n x(t)) - (L_n^+ y(t))x(t)
\]

\[
= [\bar{y}(t)(L_{n-1} x(t)) - (L_{n-1}^+ y(t))x(t)] + \bar{y}(t) B_n(t + n) x(t + n) - \bar{y}(t - n) B_n(t) x(t)
\]

\[
= \Delta s[x, y](t).
\]

(3.7)
\[
\Delta \sum_{k=1}^{n-1} \sum_{j=0}^{k-1} \bar{y}(t + j - k)B_k(t + j)x(t + j) + \Delta \sum_{j=0}^{n-1} [\bar{y}(t + j - n)B_n(t + j)x(t + j)] = \Delta \sum_{k=1}^{n} \sum_{j=0}^{k-1} \bar{y}(t + j - k)B_k(t + j)x(t + j).
\]

Thus (3.6), and consequently, assertion (1) is proved.

(2) Assertion (1) yields that \((L_n^+)^+ = L_n\). So, \((L_n + L_n^+) = L_n + L_n^+\), which implies that \(L_n + L_n^+\) is formally self-adjoint. The proof is complete. \(\square\)

Based on the above discussion, we now establish a characterization of formally self-adjoint difference expressions.

**Theorem 3.4** Let \(\mathcal{L}\) be a formally self-adjoint difference expression of order \(m\). Then \(m\) is even, saying \(m = 2n\), and \(\mathcal{L}\) has the the following form

\[
\mathcal{L}y = \sum_{j=0}^{n} F^j(A_j y) + \sum_{j=1}^{n} \bar{A}_j F^{-j} y,
\]

where \(A_j, j = 1, \ldots, n\), are complex-valued functions, and \(A_0\) is real-valued.

**Proof** First, we shown \(\mathcal{L}\) defined by (3.7) is self-adjoint. Denote

\[
L_n y = \sum_{j=1}^{n} F^j(A_j y) + \frac{1}{2} A_0 y.
\]

Then one has that \(\mathcal{L} = L_n + L_n^+\) by (1) of Lemma 3.2, and consequently, \(\mathcal{L}\) is formally self-adjoint by (2) of Lemma 3.2.

On the other hand, we proof \(\mathcal{L}\) has form (3.7) if \(\mathcal{L}\) is self-adjoint. Let \(L\) be any \(m\)th-order difference expression with the form of (3.1). Then it can be written as \(L = L_k + L_s\), where

\[
(L_k y)(t) = \sum_{j=0}^{k} A_j(t + j)y(t + j), \quad (L_s y)(t) = \sum_{j=1}^{s} \bar{A}_{-j}(t)y(t - j).
\]

It follows from (1) of Lemma 3.2 that

\[
(L^+ y)(t) = (L_k^+ y)(t) + (L_s^+ y)(t) = \sum_{j=1}^{s} \bar{A}_{-j}(t + j)y(t + j) + \sum_{j=0}^{k} \bar{A}_j(t) y(t - j).
\]

This yields that \(L = L^+\) if and only if \(s = k\), which implies that \(m\) is even, saying \(m = 2n\); and \(A_j(t) \equiv \bar{A}_{-j}(t)\) for \(j = 0, 1, \ldots, n\), which implies that \(A_0(t)\) is real-valued. The proof is complete. \(\square\)
Theorem 3.5 For any nth-order forward difference expression \( L_n \), \( L_n L_n^+ \) is a formally self-adjoint difference expression of order \( 2n \).

Proof Let \( L_n \) be defined as (3.2). Then it can be verified that

\[
(L_n L_n^+)y = L_n(L_n^+y) = \sum_{j=0}^{n} F^j (D_j y) + \sum_{j=1}^{n} D_j F^{-j} y,
\]

where

\[
D_j(t) = \sum_{k=j}^{n} B_k(t + k - j) \overline{B}_{k-j}(t + k - j), \quad j = 0, 1, \ldots, n, \quad t \in \mathcal{I}. \quad (3.8)
\]

It is clear \( D_0(t) = \sum_{k=0}^{n} |B_k(t + k)|^2 \) is real-valued on \( \mathcal{I} \). So, \( L_n L_n^+ \) is formally self-adjoint by Theorem 3.4. The proof is complete. \( \square \)

Now we can give another characterization of formally self-adjoint difference expression \( \mathcal{L} \) by using \( L_n \).

Theorem 3.6 For any formally self-adjoint difference expression \( \mathcal{L} \) with order \( 2n \), there exist two difference expressions \( L_n \) and \( L_0 \) with the form of (3.2) such that

\[
\mathcal{L} = L_n L_n^+ + L_0.
\]

Proof Let \( \mathcal{L} \) be formally self-adjoint difference expression defined as (3.7). By Theorem 3.5 and (3.8), it suffices to show there exists a set of complex-valued functions \( B_j(t), 0 \leq j \leq n \), and a real-valued function \( C(t) \) such that

\[
(L_n y)(t) = \sum_{j=0}^{n} F^j (B_j y)(t), \quad (L_0 y)(t) = C(t) y(t), \quad (3.9)
\]

and

\[
\sum_{k=j}^{n} B_k(t + k - j) \overline{B}_{k-j}(t + k - j) \equiv A_j(t), \quad j = 1, 2, \ldots, n, \quad (3.10)
\]

\[
\sum_{k=0}^{n} B_k(t + k) \overline{B}_k(t) + C(t) \equiv A_0(t). \quad (3.11)
\]

By taking \( B_0(t) \equiv 1 \) on \( \mathcal{I} \), it can be easily verified that \( B_j(t) = A_n(t) \) on \( \mathcal{I} \) yields (3.10) with \( j = n \). In addition, by giving the following initial values

\[
B_1(0), \quad B_2(0), \quad B_2(1), \quad \ldots \quad B_{n-1}(0), \quad B_{n-1}(1), \quad \ldots \quad B_{n-1}(n - 2),
\]
the values of $B_j(t), t \geq j, 1 \leq j \leq n - 1,$ can be determined uniquely by Eq. (3.10). At last, take

$$C(t) = A_0(t) - \sum_{k=0}^{n} |B_k(t+k)|^2, \quad t \in \mathcal{I}.$$ 

Then (3.11) is satisfied. The proof is complete. \qed

4 Main Result

Let $l(I, n)$ be defined as that in Sect. 3, and $\mathcal{L}$ be a formally self-adjoint difference expression of order $2n$ defined on $l(I, n)$. Denote

$$l^2_w := \left\{ x \in l(I, n) : \sum_{t=0}^{+\infty} w(t)|x(t)|^2 < +\infty \right\}.$$ 

and

$$\langle x, y \rangle_w := \sum_{t=0}^{+\infty} \bar{y}(t)w(t)x(t), \quad \|x\|_w := \langle x, x \rangle_w^{1/2}.$$ 

For any $x, y \in l^2_w$, we say $x = y$ if $\|x\| = \|y\|$. Then $l^2_w$ is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_w$ (cf. [21, Lemma 2.5]).

Denote

$$l^2_{w,0}(I) := \left\{ y \in l^2_w(I) : y(t) = 0, -n \leq t \leq n - 1 \text{ and } t \geq k \text{ for some } k \in \mathcal{I} \right\}.$$ 

and

$$T := \left\{ (x, f) \in (l^2_w(I))^2 : \mathcal{L}x(t) = w(t)f(t) \text{ on } \mathcal{I} \right\},$$

$$T_0 := \left\{ (x, f) \in T : x \in l^2_{w,0}(I) \right\}.$$ 

$T$, $T_0$, and the closure of $T_0$, denoted by $\overline{T_0}$, are called the maximal, the pre-minimal, and the minimal linear relations corresponding to $\mathcal{L}$, separately. Thus, $d_\lambda(\mathcal{L})$, given in Sect. 1, just refers to $d_\lambda(\overline{T_0})$, which is equal to $d_\lambda(T_0)$.

It follows from Theorem 3.6 there exist two difference expressions $L_n$ and $L_0$ defined as (3.9) such that $\mathcal{L} = L_n L_n^+ + L_0$. Similarly, denote

$$H := \left\{ (x, f) \in (l^2_w(I))^2 : (L_n L_n^+)x(t) = w(t)f(t) \text{ on } \mathcal{I} \right\},$$

$$H_0 := \left\{ (x, f) \in H : x \in l^2_{w,0}(I) \right\}.$$ 

Lemma 4.1 Both $T_0$ and $H_0$ are Hermitian linear relations.
Proof Since both \( \mathcal{L} \) and \( L_n L_n^+ \) are formally self-adjoint, it follows that both \( T_0 \) and \( H_0 \) are Hermitian linear relations by using (3.3).

\[ \mathbf{\Box} \]

Theorem 4.2 For any formally self-adjoint difference expression \( \mathcal{L} \),

\[ d_+ (\mathcal{L}) = d_- (\mathcal{L}). \] \hfill (4.1)

Proof Based on the notations given above, (4.1) is equivalent to

\[ d_+ (T_0) = d_- (T_0). \]

For any \((x, f) \in H_0\), it follows from (3.3) that

\[ \langle x, f \rangle_w = \sum_{t \in I} x(t) L_n L_n^+ x(t) = \sum_{t \in I} L_n x(t) L_n^+ x(t) \geq 0. \]

This yields that \( H_0 \) is Hermitian and bounded from below with lower bound 0. Hence, it follows from Lemma 2.5 that \( H_0 \) has self-adjoint extensions. Let \( H_1 \) be a self-adjoint extension of \( H_0 \). By \( H_{1,s} \) denote the operator part of \( H_1 \). It follows from Lemma 2.1 that \( H_{1,s} \) is self-adjoint in \( (H_1^+) (0))^2 \), and there exists a unique self-adjoint operator, denoted by \( S_H \) in \( H_1^+ (0) \) such that \( G(S_H) = H_{1,s} \). Then by Theorem 2.8, there exists a conjugate \( K \) on \( H_1^+ (0) \) such that \( S_H \) is \( K \)-real.

Corresponding to difference expression \( L_0 \), we define

\[ \mathcal{D}(S_0) = \{ x \in l_w^2 (I) \cap H_1^+ (0) : \exists f \in l_w^2 (I) \text{ s.t. } C(t) x(t) = w(t) f(t) \text{ on } I \}, \]

\[ S_0 x = f. \]

It can be easily verified that \( S_0 \) is a well-defined operator on \( l_w^2 (I) \). Moreover, \( S_0 \) is \( K \)-real. In fact, for any \( x \in \mathcal{D}(S_0) \cap H_1^+ (0) \), there exists uniquely a \( f \in l_w^2 (I) \) such that \( C(t) x(t) = w(t) f(t) \) on \( I \). Since both \( C(t) \) and \( w(t) \) are real-valued, it follows that

\[ C(t) (Kx)(t) = w(t) (Kf)(t), \quad t \in I. \]

This implies that \( K \mathcal{D}(S_0) \subset \mathcal{D}(S_0) \) and \( S_0 K x = K f = K S_0 x \). Thus \( S_0 \) is \( K \)-real, and consequently, \( S_H + S_0 \) is \( K \)-real. So, by Lemma 2.7 one has that

\[ d_+ (S_H + S_0) = d_- (S_H + S_0), \]

and consequently, \( S_H + S_0 \) has self-adjoint extensions by Lemma 2.2.

On the other hand, it is clear \( \overline{T}_0 \) is closed and Hermitian. Let \( \overline{T}_{0,s} \) be the operator part of \( \overline{T}_0 \) and operator \( S_T \) be the operator which satisfies \( G(S_T) = \overline{T}_{0,s} \). Then one has \( S_T \subset S_H + S_0 \). Thus, all the self-adjoint extensions of \( S_H + S_0 \) are still the self-adjoint extensions of \( S_T \). Again by Lemma 2.2, one has \( d_+ (S_T) = d_- (S_T) \). This, together with \( \mathcal{R}(\overline{T}_0) = \mathcal{R}(S_T) \), yields (4.1). The proof is complete.

\[ \mathbf{\Box} \]
Remark 4.3 Based on the above discussion, we make some explanations of the fact that the situation in the discrete case is different than that in the continuous case.

(1) In comparison with the continuous case, a formally self-adjoint difference expression is necessary to be even-order. This is an important and essential difference between the discrete and the continuous cases. And this leads to that any formally self-adjoint difference expression is dealing simultaneously with \( \Delta \) and \( \nabla \) as displayed in (1.5).

(2) Any formally self-adjoint difference expression is the sum of some difference expressions with the form

\[
A_j(t + j)y(t + j) + \overline{A}_j(t)y(t - j).
\]

(4.2)

Note that any difference expression with form (4.2) is formally self-adjoint and its positive and negative deficiency indices are always equal [18, Theorem 3.1].

(3) As it has been shown by Theorem 3.6, any formally self-adjoint difference expression can be decomposed into the sum of two formally self-adjoint difference expressions, one of which is bounded from below and another is real-valued. This is another important and essential difference between the discrete and the continuous cases. And this property contributes mainly to the proof of Theorem 4.2.

(4) Another fact which is worth noting is that odd-order formally self-adjoint differential expressions with \( d_+(\tau) \neq d_-(\tau) \) exist to a considerable extent. For example, let \( \tau y = iy' \). Then \( \tau \) is first-order formally self-adjoint differential expression and \( (d_+(\tau), d_-(\tau)) = (0, 1) \). An example of third-order \( \tau \) with \( (d_+(\tau), d_-(\tau)) = (2, 1) \) is given by [15].

5 An Example

In this section, we will give an example to illustrate the main results of this paper. To do that, we need some basic results on the limit circle case and limit point case for formally self-adjoint difference Eq. (1.4). The readers are referred to [18, 21, 25] for more details.

By letting \( u(t) = (u_1(t), u_2(t), \ldots, u_{2n}(t))^T \) with

\[
u_j(t) = \Delta^{j-1}y(t - j), \quad j = 1, 2, \ldots, n,
\]

\[
u_{n+j}(t) = \sum_{k=j}^{n}(-1)^{k+j}\Delta^{k-j}(p_k(t)\Delta^ky(t - k)) + i\sum_{k=j}^{n}(-1)^{k+1}\Delta^{k-j}(q_k(t)y(t)),
\]

(5.1)

for \( j = 1, 2, \ldots, n \), (1.4) can be converted into the following discrete linear Hamiltonian system

\[
(\mathcal{L}u)(t) := J\Delta u(t) - P(t)R(u)(t) = \lambda W(t)R(u)(t), \quad t \in I,
\]

(5.2)
where \( R(u)(t) = (u_1(t+1), \ldots, u_n(t+1), u_{n+1}(t), \ldots, u_{2n}(t))^T \) is the partial right shift operator; \( J \) is the \( 2n \times 2n \) canonical symplectic matrix, i.e.,

\[
J = \begin{pmatrix}
0 & -I_n \\
I_n & 0
\end{pmatrix},
\]

and \( I_n \) is the \( n \times n \) unit matrix;

\[
P(t) = \begin{pmatrix}
-C(t) A^*(t) \\
A(t) & B(t)
\end{pmatrix}, \quad W(t) = \text{diag}\{w(t), 0, \ldots, 0\},
\]

\[
A(t) = \begin{pmatrix}
0 & I_{n-1} \\
\frac{i q_n(t)}{p_n(t)} & 0
\end{pmatrix}, \quad C(t) = \begin{pmatrix}
p_0(t) + \frac{q_n(t)}{p_n(t)} \alpha(t) \\
\alpha^*(t) & D(t)
\end{pmatrix},
\]

and

\[
B(t) = \text{diag}\{0, \ldots, 0, p_n^{-1}(t)\}, \quad \alpha(t) = (i q_{n-1}(t), i q_{n-2}(t), \ldots, i q_1(t))^T, \quad D(t) = \text{diag}\{p_1(t), p_2(t), \ldots, p_{n-1}(t)\}.
\]

Similarly to the scalar case, we denote

\[
l^2_W(I) := \left\{ u = \{u(t)\}_{t=0}^{+\infty} \subset \mathbb{C}^{2n} : \sum_{t=0}^{+\infty} R^*(u)(t) W(t) R(u)(t) < \infty \right\}.
\]

Then \( l^2_W(I) \) is a Hilbert space with the inner product

\[
\langle u, v \rangle_W = \sum_{t=0}^{+\infty} R^*(v)(t) W(t) R(u)(t).
\]

By \( d_\pm(\mathcal{L}) \) denote the positive and negative deficiency indices of \( H_0 \), where \( H_0 \) is the minimal operator corresponding to system (5.2). The discrete Hamiltonian system (5.2) is said to be in the limit point case at \( t = +\infty \) if \( d_+(\mathcal{L}) = d_-(\mathcal{L}) = n \), and it is said to be in the limit circle case at \( t = +\infty \) if \( d_+(\mathcal{L}) = d_-(\mathcal{L}) = 2n \). It has been shown that \( \mathcal{L} \) is in limit circle case at \( t = +\infty \) if for some \( \lambda_0 \in \mathbb{C} \), every solution of \( (\mathcal{L}u)(t) = \lambda_0 W(t) R(u)(t) \) belongs to \( l^2_W(I) \). It is well-known that \( d_\pm(\mathcal{L}) = d_\pm(\mathcal{L}) \), and the limit case of \( \mathcal{L} \) coincides with that of \( \mathcal{L} \).

The following is criteria of the limit point case for (5.2).

**Lemma 5.1** [21, Theorem 6.14] The Hamiltonian system (5.2) is in limit point case at \( t = +\infty \) if and only if the following holds

\[
\lim_{t \to +\infty} u^*(t) J v(t) = 0,
\]
for any pair of \( \lambda_1, \lambda_2 \in \mathbb{C} \) with \( \text{Im}\lambda_j \neq 0 \) \((j = 1, 2)\), and for each pair of solutions \( u \in l^2_w(\mathcal{I}) \) and \( v \in l^2_w(\mathcal{I}) \) of (5.2) with \( \lambda_1, \lambda_2 \in \mathbb{C} \), respectively.

To be more explicit, we consider the following fourth-order formally self-adjoint difference equation:

\[
(\mathcal{L}y)(t) := A_2(t + 2)y(t + 2) + A_1(t + 1)y(t + 1) + A_0(t)y(t) \\
+ \bar{A}_1(t)y(t - 1) + \bar{A}_2(t)y(t - 2) = \lambda w(t)y(t).
\]  

(5.3)

Let \( y_j, 1 \leq j \leq 4 \), be four solutions of (5.3) with some \( \lambda \in \mathbb{C} \). Denote

\[
W[y_1, y_2, y_3, y_4](t) := \det\begin{pmatrix}
y_1(t) & y_2(t) & y_3(t) & y_4(t) \\
y_1(t + 1) & y_2(t + 1) & y_3(t + 1) & y_4(t + 1) \\
y_1(t + 2) & y_2(t + 2) & y_3(t + 2) & y_4(t + 2) \\
y_1(t + 3) & y_2(t + 3) & y_3(t + 3) & y_4(t + 3)
\end{pmatrix}.
\]

Note that \( A_2(t) \neq 0 \) on \( \mathcal{I} \), it can be easily verified that

\[
W[y_1, y_2, y_3, y_4](t) = \prod_{j=0}^{t+1} \frac{\bar{A}_2(j)}{A_2(j + 2)} W[y_1, y_2, y_3, y_4](-2),
\]

and \( y_j(t), 1 \leq j \leq 4 \), are linearly independent if and only if \( W[y_1, y_2, y_3, y_4](-2) \neq 0 \). Further, we require \( y_j, 1 \leq j \leq 4 \), satisfy the following initial values

\[
y_j(t) = \begin{cases} 1, & t = j - 3, \\ 0, & t \in [-2, 1] \setminus \{j - 3\}, \end{cases} \quad j = 1, 2, 3, 4. \tag{5.4}
\]

Then \( y_j, 1 \leq j \leq 4 \), are linearly independent solutions of (5.3) and satisfy

\[
|W[y_1, y_2, y_3, y_4](t)| = \frac{1}{|A_2(t + 3)A_2(t + 2)|}, \quad t \geq -2. \tag{5.5}
\]

We get the following property of limit circle case for (5.3).

**Lemma 5.2** If \( \mathcal{L} \) defined by (5.3) is in limit circle case at \( t = +\infty \), then

\[
\sum_{t=0}^{+\infty} \frac{(w(t)w(t + 1)w(t + 2)w(t + 3))^{1/2}}{|A(t + 3)A(t + 2)|} < +\infty. \tag{5.6}
\]

**Proof** If \( \mathcal{L} \) defined by (5.3) is in limit circle case at \( t = +\infty \), then all solutions of (5.3) with any \( \lambda \in \mathbb{C} \) belong to \( l^2_w(\mathcal{I}) \). Specially \( y_j \in l^2_w(\mathcal{I}), j = 1, 2, 3, 4 \), where \( y_j \) are defined by (5.4). Thus, it follows from (5.5) that

\[
\frac{(w(t)w(t + 1)w(t + 2)w(t + 3))^{1/2}}{|A_2(t + 3)A_2(t + 2)|}.
\]
\[ (w(t)w(t+1)w(t+2)w(t+3))^{1/2}|W[y_1, y_2, y_3, y_4](t)|. \quad (5.7) \]

It is clear that

\[ |W[y_1, y_2, y_3, y_4](t)| \leq \sum_{j_1, j_2, j_3, j_4} |y_{j_1}(t)y_{j_2}(t+1)y_{j_3}(t+2)y_{j_4}(t+3)|, \]

where \( \sum_{j_1, j_2, j_3, j_4} \) means the sum of all the permutations of \( \{1, 2, 3, 4\} \). By Hôlder inequality and each \( y_j \in l^2_w(\mathcal{I}), j = 1, 2, 3, 4 \), we have

\[
\sum_{t=0}^{+\infty} (w(t)w(t+1)w(t+2)w(t+3))^{1/2}|y_{j_1}(t)y_{j_2}(t+1)y_{j_3}(t+2)y_{j_4}(t+3)| \\
\leq \left( \sum_{t=0}^{+\infty} (w^{1/2}(t)y_{j_1}(t))^4 \right)^{1/4} \left( \sum_{t=0}^{+\infty} (w^{1/2}(t+1)y_{j_2}(t+1))^4 \right)^{1/4} \\
\cdot \left( \sum_{t=0}^{+\infty} (w^{1/2}(t+2)y_{j_3}(t+2))^4 \right)^{1/4} \left( \sum_{t=0}^{+\infty} (w^{1/2}(t+3)y_{j_4}(t+3))^4 \right)^{1/4} \\
< +\infty.
\]

This, together with (5.7), yields (5.6). The proof is complete. \( \square \)

**Remark 5.3** Lemma 5.2 can be regarded as an analogue of [25, Theorem 3.1] which is for second-order difference equations.

Now, we give an example to illustrate our main results.

**Example 5.4** Consider the following fourth-order formally self-adjoint difference equation with complex coefficients

\[
(\mathcal{L}y)(t) = y(t+2) + \frac{1-a^{2t+2}}{a^{2t+2}(1-a^2)} + i \left( -\frac{1}{a^2} \right)^{t+1} y(t+1) + \left( a^2 + \frac{1}{a^2} \right) y(t) \\
+ \left[ \frac{1-a^t}{a^{2t}(1-a^2)} - i \left( -\frac{1}{a^2} \right)^{t} \right] y(t-1) + y(t-2) = \lambda y(t), \quad t \in \mathcal{I}, \quad (5.8)
\]

where \( a \) is real constant with \( 0 < |a| < 1 \). Based on the existing results, the possible pairs of value of \( (d_+ (\mathcal{L}), d_- (\mathcal{L})) \) are \( (2, 2) \), \( (2, 3) \), \( (3, 2) \), \( (3, 3) \) and \( (4, 4) \).

Firstly, since \( A_2(t) = w(t) \equiv 1 \) on \( \mathcal{I} \), one can get that

\[
\sum_{t=0}^{+\infty} \frac{(w(t)w(t+1)w(t+2)w(t+3))^{1/2}}{|A(t+3)A(t+2)|} = +\infty.
\]

Hence, it follows from Lemma 5.2 that \( \mathcal{L} \) defined by (5.8) is not in limit circle case at \( t = +\infty \). That is \( (d_+ (\mathcal{L}), d_- (\mathcal{L})) \neq (4, 4) \).
Secondly, it can be easily checked that \( L \) defined by (5.8) can be written as

\[
(Ly)(t) = \Delta^2 \nabla^2 y(t) - \Delta(p_1(t) \nabla y(t)) + p_0(t)y(t) + i[\Delta(q_1(t)y(t)) + q_1(t)\nabla y(t)]
\]

\[
= \lambda y(t) \quad t \in I,
\]

with

\[
p_1(t) = \frac{a^{2t} - 1}{a^{2t}(1 - a^2)} - 4,
\]

\[
p_0(t) = a^2 + \frac{1}{a^2} + \frac{1 - a^{2t}}{a^{2t}(1 - a^2)} + \frac{1 - a^{2t+2}}{a^{2t+2}(1 - a^2)} + 2,
\]

\[
q_1(t) = (-1)^t \frac{1}{a^{2t}}.
\]

In addition, it is easy to check that \( y(t) = (ia)^t \) is a solution of

\[
(Ly)(t) = \frac{i}{a} y(t), \quad t \in I.
\]

and \( y \in l^2(I) \). By transformation (5.1) one has that

\[
u(t) = \begin{pmatrix}
y(t - 1) \\
\nabla y(t - 1) \\
- \Delta \nabla^2 y(t) + p_1(t) \nabla y(t) - iq_1(t)y(t)
\end{pmatrix},
\]

and further by using \( y(t) = (ia)^t \to 0 \) as \( t \to +\infty \) one can get that

\[
\lim_{t \to +\infty} u^*(t)Ju(t)
\]

\[
= \lim_{t \to +\infty} \left[p_1(t)\nabla y(t) - iq_1(t)y(t)\right]y(t - 1) - \left[p_1(t)\nabla y(t) - iq_1(t)y(t)\right]y(t - 1)
\]

\[
= \frac{2i}{a(1 - a^2)} \neq 0.
\]

Hence \( L \) is not in limit point case at \( t = +\infty \) by Lemma 5.1. That is \((d_+(L), d_-(L)) \neq (2, 2)\).

Thirdly, by taking

\[
(L_2y)(t) = y(t + 2)
\]

\[
+ \left[1 + (-1)^t a^{2t+2} \over a^{2t}(1 - a^2) + \mu(t) \over 1 - a^2 + i \left(-1\right)^{t+1} - a^{2t+2} \over a^{2t}(1 + a^2)\right]y(t + 1) + y(t),
\]

where \( \mu(t) \) satisfies \( \mu(t) + \mu(t + 1) = -1 \) and \( \mu(0) = 0; \)

\[
(L_0y)(t)
\]
\[ a^2 + \frac{1}{a^2} - 2 - \left( \frac{1 - (-1)^t a^{2t+2}}{a^{2t}(1 - a^4)} + \frac{\mu(t)}{1 - a^2} \right)^2 - \left( \frac{(-1)^t+1 - a^{2t+2}}{a^{2t}(1 + a^2)} \right)^2 \]
y(t),

one can get that by directly calculating

\[ \mathcal{L} = L_2 L_2^+ + L_0. \]

Then it follows from Theorem 4.2 that \(d_+(\mathcal{L}) = d_-(\mathcal{L})\). This implies that \((d_+(\mathcal{L}), d_-(\mathcal{L})) \neq (2, 3)\) and \((d_+(\mathcal{L}), d_-(\mathcal{L})) \neq (3, 2)\). Consequently, one can get that

\[ (d_+(\mathcal{L}), d_-(\mathcal{L})) = (3, 3). \]

**Data Availability** Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

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