On the use of circulant matrices for the stability analysis of recent weakly compressible SPH methods

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Abstract

In this study, a linear stability analysis is performed for different Weakly Compressible Smooth Particle Hydrodynamics (WCSPH) methods on a 1D periodic domain describing an incompressible base flow. The perturbation equation can be vectorized and written as an ordinary differential equation where the coefficients are circulant matrices. The diagonalization of the system is equivalent to apply a spatial discrete Fourier transform. This leads to stability conditions expressed by the discrete Fourier transform of the first and second derivatives of the kernel. Although spurious modes are highlighted, no tensile nor pairing instabilities are found in the present study, suggesting that the perturbations of the stresses are always damped if the base flow is incompressible. The perturbations equation is solved in the Laplace domain, allowing to derive an analytical solution of the transient state. Also, it is demonstrated analytically that a positive background pressure combined with the uncorrected gradient operator leads to a reordering of the particle lattice. Finally, the dispersion curves for inviscid and viscous flows are plotted for different WCSPH methods and compared to the continuum solution. It is observed that a background pressure equal to $\rho c^2$ for inviscid flows and $\rho c^2/2$ for viscous flows, gives the best fidelity to predict the propagation of a sound wave.

Keywords: weakly compressible SPH, background pressure, $\delta$-SPH, Transport Velocity SPH, circulant matrix, Laplace transforms, dispersion relation

1. Introduction

The Smoothed Particles Hydrodynamics (SPH) method was initiated by Gingold and Monaghan (1977) and Lucy (1977) to study star formation. It is a second-order (Monaghan, 1992) Lagrangian method resolving and storing physical quantities (mass, momentum, energy) on spatial discretization points, so-called particles, which move with the local fluid velocity. The interaction between several particles is taken into account with a weighting function which promotes the influence of closer particles. This function is called the kernel. SPH was later extended to fluid mechanics (Monaghan, 1994), based on a weakly compressible approach that is commonly used nowadays. In the context of multiphase flow simulation, the main advantage of SPH over traditional grid-based methods is the intrinsic capturing of the phase interface by the natural rearrangement of the particles. In contrast to Eulerian methods, no interface capturing algorithms nor local mesh refinement is required. Hence, the SPH method is broadly used for simulating free surface flow configurations where the liquid motion is driven by means of gravity, inertia or pressure (Liu and Liu, 2010). However, despite its advantages, SPH suffers from peculiar weaknesses. First, mostly in shear driven flows, an instability can lead to voids in the particle lattice, which in turns, lead to numerical divergence. Second, due to the motion of the particles with the fluid velocity, a disorder of the particle arrangement may be encountered, that continuously changes during the simulation and compromises the accuracy of the spatial operators. Therefore, a correction algorithm has to be applied at every time step, which significantly

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increases the computational costs. On the other hand, it was observed by Colagrossi et al. (2012) that the
inaccuracy induced by the particle disorder counteracts the onset of the instability mentioned above. This
phenomenon will be demonstrated analytically in the present study.

Several authors investigated the stability of the SPH method. Fulk (1994) extensively studied the stability
in a 1D configuration on an infinite domain for different sets of SPH equations. He emphasized the onset of
the instability at the highest wavenumber $\pi/\Delta x$ ($\Delta x$ being the interparticle distance), when the summation
of the second derivative of the kernel ($W''$) over the neighbors and the local pressure ($P$) have different
signs. The author proposed many strategies to overcome this instability, such as using (i) concave up/down
kernels, (ii) a background pressure or (iii) a particle motion correction. It is worth to note that the two latter
tweaks are still among the most popular. At the same time, Swegle et al. (1995) conducted a von-Neumann
linear stability analysis (LSA) of viscous 1D SPH with an equation of state (EoS) similar to the Tait’s
EoS. They derived their analysis by supposing only two neighbors, and they found the same condition as
Fulk (1994) for the onset of the instability at highest wavenumber. They finally extended the results to an
arbitrary number of neighbors and found the same condition on $\sum W''$ to determine the stability. Balsara
(1995) conducted a von-Neumann LSA of 1D compressible SPH where different time integration schemes
and different interparticle spacings were investigated. The author derived the dispersion relation for medium
to large wavelength, contrary to Fulk (1994) and Swegle et al. (1995) who focused on the largest wavenumber.
The author advised to keep the ratio of smoothing length to interparticle distance between 1.0 and
1.4 to minimize unrealistic acoustic wave dispersion. However, no Taylor series expansion of the kernel was
taken into account, and no tensile instability was found. Morris (1996) applied LSA to 1D smooth particles
magneto-hydrodynamics (SPMHD) on an infinite domain. The author considered a weakly compressible
approach and both a viscous and inviscid flow. For the first time, a non-zero background pressure was taken
into account, and different approximations of the pressure gradient were investigated. The author found that
the same instability as Fulk (1994) for the highest wavenumber, but he attributed its origin to a negative
background pressure and not depending on the product ($\sum W''$) $\times$ $P$. Hence, he implied that the sum of $W''$
has always the same sign. He found this instability independent of the viscosity. In a similar study, Rasio
(2000) derived the numerical sound speed depending on several parameters, among which are the number
of neighbors and the first and second derivative of the kernel. Belytschko et al. (2000) proposed a unified
framework to study the stability of meshless particle methods including SPH. They found new conditions
of instability based on the relation between the wavenumber and the number of neighbors, and attributed
it to the tensile instability found by Swegle et al. (1995). Like Morris (1996), they highlighted a spurious
mode at $\pi/\Delta x$ that they assigned to a rank deficiency of the stiffness matrix in the linearized equations.
Berve et al. (2004) applied a LSA to 2D SPMHD without background pressure in the EoS. They found too
that the ratio of smoothing length to interparticle distance should be between 1.0 and 1.4. More recently,
Dehnen and Aly (2012) conducted a LSA in 3D of WCSPH and showed the superiority of the Wendland
kernel. They highlighted the difference between tensile and pairing instability and explored the link between
negative values of the kernel Fourier transform and the pairing instability. Due to their multidimensional
analysis, they were able to separate the longitudinal and transverse modes. Furthermore, they stressed the
importance of the density estimator in SPH method, especially with regards to the pairing instability.

As pointed out by Dehnen and Aly (2012), the estimation of the density is an important issue that in-
fluences the stability of SPH method. Classically, there are two different methods to estimate the density
of the fluid in WCSPH. First, the density is algebraically determined by the ratio of the particle mass and
its effective volume, based on the position of its neighbors (Monaghan, 2005). This method is referred to
as sum-SPH subsequently. Second, the density is estimated from the SPH formulation of the continuity
equation (Monaghan, 2005). This method will be referred to as div-SPH in the subsequently. Recently, a
method called $\delta$-SPH was presented by Antuono et al. (2010). It was successfully applied in the field of free
surface flows (Marrone et al., 2013). It is based on div-SPH with additional dissipation terms in each of the
Navier-Stokes equation, which causes the modification of the estimation of density. Finally, a method
based on an artificial transport velocity formulation was developed by Adami et al. (2013). It is referred to
as TV-SPH in the following. Since this method is based on sum-SPH with artificially convected particles,
its density estimator is different from sum-SPH.
The objective of the present study is to investigate the stability of the four methods for estimating the density. An new treatment is applied to the perturbation equations from the 1D-SPH equations. The equations of perturbations are casted into a matrix form of an ordinary differential equation in time. Its coefficients are circulant matrices, whose eigenvalues and eigenvectors are easily obtained using the circulant matrix theory. The matrix form of the perturbations equations is then diagonalized, which corresponds to a transformation from a geometrical to a modal space. These equations are solved in the Laplace domain, which features the advantage over the Fourier transform to capture the transient state of the motions, in addition to the stability and the dispersion relation. Then, after simplification of the transfer function, the solution of the perturbation equation is transformed back into the temporal modal form. Finally, the solution is transformed into the temporal geometrical domain.

In Section 2, the method described-above is applied to the classical sum-SPH approach, and every steps are explained thoroughly. Then the same method is applied to the div-SPH, δ-SPH and TV-SPH density estimator in Sections 3, 4 and 5, respectively. The dispersion curve of all methods, both for inviscid and viscous flows are discussed in Section 6. An illustration of the spurious mode is presented in Section 7. The expression of the perturbations equation with circulant matrices is illustrated in Appendix A, and the validation of the derivation is provided in Appendix B where the analytical temporal solutions of the perturbations are compared to the resolution by a numerical scheme.

2. Application of LSA on sum-SPH density estimator

2.1. Preliminary definitions

The 1D domain consists of \( N \) particles equidistantly placed along a line, with an interparticle distance of \( \Delta x \). This is representative of the equilibrium state (Fig. 1). In order to avoid any boundary effects, the domain is periodic. The number of neighbors on one side of the particle is labelled \( M \). Hence, a complete sphere of influence consists of \( 2M + 1 \) particles. The smoothing length \( h \) is chosen so that its multiple exactly covers the neighbors included in the sphere of influence as in (Quinlan et al., 2006). Therefore, \( h, \Delta x \) and \( M \) are related by:

\[
kh = (2M + 1) \Delta x
\]  

(1)

where \( k \) is an integer and \( kh \) is the diameter of the sphere of influence. Due to the periodicity, the sphere of influence of particles located near the boundaries may be split into two parts located at either boundary of the domain, as illustrated on Fig. 1 (bottom).

![Periodicity](image)

Figure 1: Top: 1D domain. Bottom: illustration of split sphere of influence

In this study, two different kernels were investigated. The so-called quintic kernel (Morris et al., 1997) is defined in 1D by:

\[
W(r,h) = \frac{1}{120h} \begin{cases}
      (3 - r)^5 - 6 (2 - r)^5 + 15 (1 - r)^5 & \text{for } 0 \leq r \leq 1 \\
      (3 - r)^5 - 6 (2 - r)^5 & \text{for } 1 < r \leq 2 \\
      (3 - r)^5 & \text{for } 2 < r \leq 3
\end{cases}
\]

(2)

where \( r = |x_a - x_b|/h \) is the normalized distance between particle \( a \) and \( b \). The second investigated kernel belongs to the family of Wendland kernels (Wendland, 1995). It is expressed in 1D as:

\[
W(r, h) = \frac{1}{2h} (1 - r)^5 (1 + 5r + 8r^2)
\]

(3)
In this case, the particle distance is normalized by the radius of the sphere of influence \( r = |x_a - x_b|/3h \).

In the rest of this paper, the dependency of the kernel on \( h \) will be dropped in the notation: \( W(r) \equiv W(r, h) \) for the sake of simplicity. The radius of the sphere of influence is set constant to \( 3h \), which leads to discrete values of \( \Delta x/h \) (Eq. 1):

\[
\frac{\Delta x}{h} = \frac{6}{2M+1}
\]

Furthermore, the letter \( a \) refers to the particle under consideration whereas \( b \) refers to its neighbors. Hence \( \sum_b \) is a summation over the neighbors of particle \( a \). The difference between a quantity at particle \( a \) and at particle \( b \) is indexed \( ab \). For instance, \( x_a - x_b \) is abbreviated by \( x_{ab} \). For the kernel, \( W(x_a - x_b) \) is abbreviated by \( W_{ab} \).

2.2. System of equations

In the absence of gravity, the 1D WCSPH approach with the algebraic density formulation (sum-SPH) is defined by the following system of equation:

\[
\begin{align*}
\rho_a &= m_a \sum_b W_{ab} \\
\frac{d^2 x_a}{dt^2} &= -\frac{1}{\rho_a} \sum_b V_b (p_b + p_a) W'_{ab} + 2\nu \sum_b \frac{u_a - u_b}{r_{ab}} W''_{ab} \\
p_a &= \frac{\rho_0 c^2}{\gamma} \left[ \left( \frac{\rho_a}{\rho_0} \right)^\gamma - 1 \right] + p_{\text{back}}
\end{align*}
\]

In the momentum conservation (Eq. 5b), the term \( W'_{ab} \) is the derivative \( \partial W_{ab}/\partial x_a \) which corresponds to the kernel gradient \( \nabla W_{ab} \) in 1D. The pressure gradient is expressed by the so-called \( G_+ \) operator where the pressure \( p_a \) and \( p_0 \) are added, ensuring conservation of linear momentum. The second term on the RHS of Eq. 5b represents the viscous effects. It exhibits the kinematic viscosity \( \nu \) multiplied by the Laplacian of the velocity in the SPH formalism (Cleary and Monaghan, 1999). Another frequent expression of the Laplacian was proposed by Morris et al. (1997), but in 1D, its expression reduces to the same as the one of Cleary and Monaghan (1999). Equation 5c is the Tait equation of state, valid for weakly compressible flows (Batchelor, 2000). The term \( \gamma \) is the polytropic ratio, usually set to 7.

2.3. Linearization

The linearization of the physical variables (density, position, velocity and pressure) is written:

\[
\rho = \bar{\rho} + \delta \rho, \quad x = \bar{x} + \delta x, \quad u = \bar{u} + \delta u, \quad p = \bar{p} + \delta p
\]

where overlined quantities are constant and \( \delta X \) is a small perturbation of the quantity \( X \). In the rest of the paper, the mean velocity \( \bar{u} \) is set to a constant, so that the base flow is incompressible. The particle volume \( V \) is defined as \( V = m/\rho \) and linearized to the first order using Eq. 6:

\[
V = m/\rho \approx \bar{V} \left( 1 - \frac{\delta \rho}{\bar{\rho}} \right)
\]

The value of the kernel is linearized by a Taylor expansion of first order as depicted in Fig. 2:

\[
\begin{align*}
W(x_a - x_b) &\approx W(\bar{x}_a - \bar{x}_b) + (\delta x_a - \delta x_b) W'(\bar{x}_a - \bar{x}_b) \\
W'(x_a - x_b) &\approx W'(\bar{x}_a - \bar{x}_b) + (\delta x_a - \delta x_b) W''(\bar{x}_a - \bar{x}_b)
\end{align*}
\]
For the sake of clarity in the following, Eqs. 8 are written as:

\[
W_{ab} \approx W_{ab} + \delta x_{ab} W'_{ab}
\]

and

\[
W'_{ab} \approx W'_{ab} + \delta x_{ab} W''_{ab}
\]

Inserting Eqs. 6, 7 and 9 into the Eqs. 5, the linearized equations of perturbations around an equilibrium state is expressed as:

\[
\begin{align}
\delta \rho_a &= -m \sum_b \delta x_b W'_{ab} \\
\frac{d^2 \delta x_a}{dt^2} &= V \sum_b \left[ \frac{\nabla \rho_b W'_{ab} - \frac{p}{\rho} \delta x_{ab} W''_{ab} - \frac{\rho_{b}}{p} W'_{ab} + 2 \nu \frac{d}{dt} (\delta x_{ab}) W''_{ab}}{x_{ab}} \right] \\
\delta p_a &= c_s^2 \delta \rho_a
\end{align}
\]

Note that in the derivation of Eqs. 10, the perturbations at particle \(a\) vanish when they are multiplied by the kernel gradient (e.g. in Eq. 10a, \( \sum \delta x_a W'_{ab} = \delta x_a \sum W'_{ab} = 0 \)). This is because the kernel gradient is an odd function and the pairwise summation over the neighbors leads to zero: \( W'(\Delta x) + W'(-\Delta x) = 0 \). The motion of particle \(a\) (Eq. 10b) depends on the perturbations of its neighbors via four different terms on the right-hand side. The first term depends on the neighbor density due to the mass conservation (Eq. 7) and represents the cross effects of compressibility and constant pressure. The second term depends on the particle distance from their equilibrium position and is proportional to the equilibrium pressure. It has no physical meanings and stems from the inaccuracy of the SPH gradient operator on disordered particle lattices. The third term results from the pressure perturbations and the last term comprises the viscous effects.

2.4. Converting the perturbation equation into a circulant matrix system

In this section the transformation of the perturbation equations (Eqs. 10) to a matrix system is explained. All perturbed variables \( \delta x, \delta \rho, \delta p \) of Eq. 10 are written down for all particles i.e. \( \delta x_0, \delta x_1, \ldots, \delta \rho_0, \delta \rho_1, \ldots, \delta p_0, \delta p_1, \ldots \) and then gathered into perturbation vectors:

\[
\begin{align}
X &= (\delta x_0, \delta x_1, \ldots, \delta x_{N-1})^T \\
R &= (\delta \rho_0, \delta \rho_1, \ldots, \delta \rho_{N-1})^T \\
P &= (\delta p_0, \delta p_1, \ldots, \delta p_{N-1})^T
\end{align}
\]

As illustrated in Appendix A, Eqs. 10 can be expressed as a matrix system:

\[
\begin{align}
R &= m A' X \\
\frac{d^2 (X)}{dt^2} &= -2 \frac{p V}{\rho} A' R + 2 \frac{p V}{\rho} A'' X + \frac{V}{\rho} A' P + 2 \nu A''' \frac{d}{dt} X \\
P &= c_s^2 R
\end{align}
\]

where the matrices \( A', A'' \) and \( A''' \) enjoy interesting properties. First, \( A' \) is antisymmetric while \( A'' \) and \( A''' \) are symmetric. Second, and most important for the rest of the study, all three matrices are circulant matrices.
Introducing the non-dimensional background pressure as $p_{bg}$ to include any background pressure term. The system 12 is further simplified with $p = p_{bg}$ and $\rho = \rho_{bg}$, where the constant part of the pressure is the background pressure. Note that originally, the Tait EoS does not include any non-dimensional background pressure term. Therefore, this approach can be regarded as an expression of the linearized SPH operators in a matrix form. The linearized gradient is expressed by $\mathbf{A}'$, and the linearized Laplacian is expressed by $\mathbf{A}''$. The matrix $\mathbf{A}'''$ comes from the linearization of the gradient of a constant term. It should be zero if the SPH gradient operator was first order consistent.

Expressing the pressure fluctuation with the help of the EoS (Eq. 5c) leads to $p = p_{bg}$. Therefore, the constant part of the pressure is the background pressure. Note that originally, the Tait EoS does not include any background pressure term. The system 12 is further simplified with $V = \Delta x$ and $m = \rho \Delta x$. Introducing the non-dimensional background pressure as $p_{bg} = p_{bg}/\rho c^2$ and substituting the perturbed pressure and density into Eq. 12b finally leads to:

$$
\frac{d^2}{dt^2}(X) = \Delta x \cdot c^2 \left[ \Delta x \mathbf{A}'^2 (1 - 2p_{bg}) + 2p_{bg} \mathbf{A}'' \right] X + 2 \nu \Delta x \mathbf{A}''' \frac{d}{dt}(X)
$$

Equation 16 can be written in a non-dimensional form using:

$$
\mathbf{X}' = X/h, \quad t' = ct/h, \quad \Delta x' = \Delta x/h, \quad \mathbf{A}' = h^2 \mathbf{A}', \quad \mathbf{A}'' = h^3 \mathbf{A}'', \quad \mathbf{A}''' = h^3 \mathbf{A}''', \quad \nu' = \nu/(hc)
$$

It leads to:

$$
\frac{1}{\Delta x} \frac{d^2}{dt^2}(X) = \left[ \Delta x \mathbf{A}'^2 (1 - 2p_{bg}) + 2p_{bg} \mathbf{A}'' \right] X + 2 \nu \mathbf{A}''' \frac{d}{dt}(X)
$$

where the variables are written in their non-dimensional form without the asterisk (*) for the sake of simplicity. In the following, all variables are written in the non-dimensional form unless it is explicitly mentioned. Equation 18 is a matrix ordinary differential equation where the unknown is the vector $\mathbf{X}$.

2.5. Properties of matrices $\mathbf{A}'$, $\mathbf{A}''$ and $\mathbf{A}'''$

Since matrices $\mathbf{A}'$, $\mathbf{A}''$ and $\mathbf{A}'''$ are circulant matrices, they are diagonalizable in the same basis (Davis, 1979). Their eigenvalues are labeled $\lambda'_j$, $\lambda''_j$ and $\lambda'''_j$, respectively:

$$
\lambda'_j = 2i \sum_{k=1}^{M} c_k \sin \left( \frac{2\pi}{N} k j \right)
$$

$$
\lambda''_j = c_0'' + 2 \sum_{k=1}^{M} c_k'' \cos \left( \frac{2\pi}{N} k j \right)
$$
\[
\lambda''_j = c''_0 + 2 \sum_{k=1}^{M} c''_k \cos \left( \frac{2\pi}{N} kj \right)
\]

(21)

where \( \tilde{i} \) is the imaginary number. These eigenvalues depend only on (i) the kernel, (ii) the total number of particles \( N \), and (iii) the number of neighbours \( M \). The matrix \( A'' \) is also a circulant matrix. Since the eigenvalues of \( A' \) are pure imaginary, the eigenvalues of \( A'' \) are negative. In addition, eigenvalues are symmetric in the sense:

\[
\lambda'_j = \lambda''_{N-j}, \quad \lambda''_j = \lambda''_{N-j}, \quad \lambda'''_j = \lambda''_{N-j} \quad \text{for} \quad j \in [1, N_{mid}]
\]

(22)

where

\[
N_{mid} = \begin{cases} 
(N-1)/2 & \text{for } N \text{ odd} \\
N/2 & \text{for } N \text{ even}
\end{cases}
\]

(23)

Furthermore, the eigenvalues exhibit particular values for the mode \( j=0 \) with \( \lambda'_0=\lambda''_0=\lambda'''_0=0 \). At the largest mode \( j=N/2 \), \( \lambda''_{N/2} \) is equal to zero, meaning that matrix \( A' \) has a rank deficiency that may lead to spurious modes (Belytschko et al., 2000). As highlighted by Fulk (1994), the eigenvalue of \( A'' \) for the largest mode is different from zero, i.e. \( \lambda''_{N/2} \neq 0 \). Furthermore, it is found in the present study that the eigenvalue of \( A''' \) at the largest mode is also non-zero, \( \lambda'''_{N/2} \neq 0 \). Finally, the eigenvalues of a circulant matrix are equal to the coefficients of the discrete Fourier transform of the its first row (Davis, 1979). Hence, \( \lambda'_j, \lambda''_j \) and \( \lambda'''_j \) are the coefficients of the discrete Fourier transform of \( W'(x), W''(x) \) and \( W'''(x)/x \), respectively.

Figure 3 shows an illustration of the eigenvalues, for \( N=63, M=3 \) (i.e. \( \Delta x=0.86 \)) and different kernels. It is observed that all eigenvalues are always negative. At largest modes, \( \lambda''_j \) goes to zero whereas the other values remain significantly negative.

2.6. Projection in the modal space

Since \( A'^2, A'' \) and \( A''' \) are circulant matrices, they can be diagonalized on the same basis. By expressing \( A'^2, A'' \) and \( A''' \) as diagonal matrices:

\[
A'^2 = P^{-1} A'^2 P \quad \text{and} \quad A'' = P^{-1} A'' P \quad \text{and} \quad A''' = P^{-1} A''' P
\]

(24)

and defining \( Y = P \hat{X} \), Eq. 18 is expressed:

\[
\frac{1}{\Delta x} \frac{d^2}{dt^2} (Y) = \left[ \Delta x A'^2 + 2 p_{bg} (A'' - \Delta x A'^2) \right] Y + 2 \nu A''' \frac{d}{dt} (Y)
\]

(25)
As $\Delta^2$, $\Delta''$ and $\Delta'''$ are diagonal matrices composed of the eigenvalues, Eq. 25 can be reduced to $N_{mid} + 1$ non-coupled ordinary differential equations:

$$\frac{1}{\Delta x} \frac{d^2}{dt^2}(y_i) - 2 \nu \lambda'' \frac{d}{dt}(y_i) - \left[ \Delta x \lambda'^2 + 2p_{bg}(\lambda'' - \Delta x \lambda'^2) \right] y_i = 0$$

(26)

with $i$ being the mode number and $i \in [0, N_{mid}]$. Hence, Eq. 26 represents the perturbations expressed in the modal space. Furthermore, the change-of-basis rules from the matrix $\Delta'$ to its eigenvalue is the classical definition of a discrete Fourier transform (Davis, 1979). This means that expressing the equation of motion in the modal space is equivalent to apply a discrete spatial Fourier transform. Thus, studying the stability of SPH with $\lambda'_i$, $\lambda''_i$ and $\lambda'''_i$ is equivalent to study the values of the discrete Fourier transform of the different expressions involving the kernel derivatives. This result is similar to those of (i) Fulk (1994) and Swegle et al. (1995) because it involves derivatives of the kernel, and similar to those of Dehnen and Aly (2012) because it involves the Fourier transform of the kernel, here in the discrete form. Equation 26 is formally rewritten as:

$$\frac{d^2}{dt^2}(y_i) + \sigma_i \frac{d}{dt}(y_i) + \phi_i^2 y_i = 0$$

(27)

with

$$\sigma_i = -2 \nu \Delta x \lambda'' \quad \text{and} \quad \phi_i^2 = -\Delta x^2 \lambda'^2 - 2 \Delta x p_{bg}(\lambda'' - \Delta x \lambda'^2)$$

(28)

In the following, the two terms appearing in the expression of $\phi_i^2$ are labeled $\psi_{1,i}$ and $\psi_{2,i}$, and they are given by:

$$\psi_{1,i} = (\Delta x \lambda'_i)^2(1 - 2p_{bg}) \quad \text{and} \quad \psi_{2,i} = 2p_{bg}\Delta x \lambda''_i$$

(29)

Note that these terms resemble to those labeled $D_1$ and $D_2$ by Fulk (1994). However, $D_1$ and $D_2$ were defined with an infinite summation so they cannot be numerically determined. In the present study, $\psi_{1,i}$ and $\psi_{2,i}$ are exactly represented by a summation from 1 to $M$.

Equation 27 is an ordinary differential equation of second order with a damping term $\sigma_i$ and an undamped angular frequency $\phi_i$. Some values of the damping factor $\sigma_i$ are plotted in Fig. 4 for $N=1000$ and $\nu=1$, with different $\Delta x$. The values are always positive, ensuring a damping for any selection of the parameters. It was observed (but not presented here) that $\sigma_i$ is very little dependent on the type of kernel. It can be shown that the asymptote of the damping term in Fig. 4 for high modes is given by:

$$\lim_{N \to \infty} \sigma_{N/2} = \int_{\Omega} \frac{W''(x)}{x} \, dx$$

(30)

In contrast, $\phi_i^2$ shows a more complex behavior with regards to $p_{bg}$, $\Delta x$ and the type of kernel. A map of $\phi_i^2$ is shown in Fig. 5, for a larger number of particles ($N=10000$), $p_{bg} = 1$, and different number of neighbors.
First, $\phi_i^2$ is always $\geq 0$, implying that $\phi_i$ is real. Hence, any onset of linear instability will be cancelled. However, for the quintic kernel, when $M \geq 9$ (i.e. below $\Delta x \leq 0.32$), some modes lower than $N_{mid}$ lead to $\phi_i^2=0$, generating a spurious mode which may trigger marginal instabilities at certain conditions. Such particular values are not observed with the Wendland kernel, suggesting at better stability.

2.6.1. Effect of the background pressure on the particle disorder

This subsection details the effect of background pressure on one single particle and demonstrates the reordering effect of background pressure. Considering only the contribution of the background pressure to the acceleration of the perturbation (Eq. 26), one obtains:

$$\frac{d^2y_i}{dt^2} = 2p_{bg}(\lambda''_i - \Delta x \lambda'^2_i)\Delta x y_i$$

Equation 31 states that the contribution of the background pressure to the acceleration of any perturbation is of the sign of the term between bracket $\phi_i = \lambda''_i - \Delta x \lambda'^2_i$. This term depends on $\Delta x$ and on the type of kernel. Figure 6 shows its dependence on $\Delta x$ for the quintic and Wendland kernel. The black crosses in Fig. 6 correspond to negative values that could not be shown in a log-log plot. Hence, the $y$ position of black crosses corresponds to $\phi_i$ instead of $-\phi_i$. They show the presence of positive values for lower modes, both for the quintic and the Wendland kernels. Apart from the points indicated by the black crosses, $\phi_i$ is always negative, meaning that the acceleration due to the background pressure is acting most of time
against the perturbation, which demonstrates the reordering effect of the background pressure. This result was observed by Colagrossi et al. (2012), Marrone et al. (2013), and Litvinov et al. (2015), but to the authors knowledge, there was no formal proof of the feature. According to Eq. 31, the time scales associated with the reordering effect of background pressure are: \[ \tau_{bg} = \frac{1}{\sqrt{2} p_{bg} \Delta x (\lambda_{i}' - \Delta x \lambda_{i}'')} \].

2.7. Solutions of the perturbation equation

This subsection aims to solve the equation of perturbation for the general case. It is organized as follow: the equations are transformed into the Laplace domain, where the transient and steady state as well as the stability can be investigated. Another advantage of the Laplace domain is to express ordinary differential equations by polynomials, which simplifies the analysis. Indeed, other WCSPH methods are ruled by a system of differential equations that cannot be directly integrated. After a simplification of the rational function, the solution are converted back into a temporal form of the modal space. Finally, the perturbations are written in the original geometric space.

2.7.1. Projection into the Laplace domain

Equation 27 is transformed into a Laplace formulation. The rules of Laplace transformations are:

\[
\begin{align*}
\mathcal{L}(y_i) &= Y_i \\
\mathcal{L}\left(\frac{dy_i}{dt}\right) &= p Y_i - y_{i,0} \\
\mathcal{L}\left(\frac{d^2y_i}{dt^2}\right) &= p^2 Y_i - p y_{i,0} - y_{i,0}'
\end{align*}
\]

where \( y_{i,0} \) and \( y_{i,0}' \) are the initial perturbation of position and velocity, respectively. The term \( p \) is the Laplace variable and must not be mistaken for the pressure term. In the Laplace domain, the perturbation of positions (Eq. 27) is given by:

\[
Y_i = \frac{p + \sigma_i}{p^2 + \sigma_i p + \phi_i^2} y_{i,0} + \frac{1}{p^2 + \sigma_i p + \phi_i^2} y_{i,0}'
\]

Hence, \( Y_i \) depends on two polynomials in \( p \) proportional to the initial perturbations. Their poles determine the stability of the system. The discriminant of the denominator is:

\[
\Delta_i = \sigma_i^2 - 4 \phi_i^2
\]

and the poles yield:

\[
\pi_i^{\pm} = -\frac{\sigma_i}{2} \left(1 \pm \sqrt{1 - 4(\phi_i/\sigma_i)^2}\right)
\]

The real part of \( \pi_i^{\pm} \) is always of the sign of \( -\sigma_i \), (i.e. the sign of \( \lambda_i'' \)) because the square root in Eq. 35 is either imaginary or lower than one. This is always true when the background pressure is positive. This implies that all modes will be damped in time, so that no instability is present, even when \( \Delta_i \) becomes positive. However, even though the real part of \( \pi_i^{\pm} \) remains negative, it can be very close to zero, leading to an insufficient damping of the perturbation and a spurious mode may arise.

2.7.2. Temporal solution in the modal space

In order to transform the solution back into the temporal modal space, the denominator of the polynomials in Eq. 33 are written as:

\[
p^2 + \sigma_i p + \phi_i^2 = \left(p + \frac{\sigma_i}{2}\right)^2 + \omega_i^2 \quad \text{with} \quad \omega_i^2 = -\frac{\Delta_i}{4} = \phi_i^2 - \frac{\sigma_i^2}{4}
\]
so that Eq. 33 can be expressed as:

$$Y_i = \left[ \frac{p + \sigma_i/2}{(p + \sigma_i/2)^2 + \omega_i^2} + \frac{\sigma_i}{2\omega_i(p + \sigma_i/2)^2 + \omega_i^2} \right] y_{i,0} + \left[ \frac{1}{\omega_i (p + \sigma_i/2)^2 + \omega_i^2} \right] y_{i,0}'$$  \hspace{1cm} (37)

If $\Delta_i$ is negative, then $\omega_i^2$ is positive. Applying the inverse Laplace transform rules leads to:

$$\begin{align}
y_i(t) &= f_i(t) y_{i,0} + g_i(t) y_{i,0}' \\
f_i(t) &= \exp\left(-\frac{\sigma_i}{2} t\right) \left[ \cos(\omega_i t) + \frac{\sigma_i}{2\omega_i} \sin(\omega_i t) \right] \\
g_i(t) &= \exp\left(-\frac{\sigma_i}{2} t\right) \left[ \frac{1}{\omega_i} \sin(\omega_i t) \right]
\end{align}$$  \hspace{1cm} (38a)

When $\Delta_i > 0$, the solution is obtained by replacing the sine and cosine functions by their hyperbolic expressions. When $\Delta_i = 0$, which is equivalent to $\omega_i = 0$, then $f_i(t)$ and $g_i(t)$ reduce to:

$$f_i(t) = \exp\left(-\frac{\sigma_i}{2} t\right) \left[ 1 + \frac{\sigma_i}{2} t \right] \quad \text{and} \quad g_i(t) = t \exp\left(-\frac{\sigma_i}{2} t\right)$$  \hspace{1cm} (39)

which correspond to a spurious mode. When $\nu > 0$, this mode is damped. However, in Eq. 39, $g_i(t)$ has a maximum at $t = 2/\sigma_i$, which is equal to $2/(\sigma_i \nu)$. This might cause instabilities in case of small damping ratio. When $\nu = 0$, the spurious mode is characterized by $f_i(t) - 1$ and $g_i(t) - t$. Hence, any initial perturbation of velocity grows linearly with time.

Finally, the zero mode ($i = 0$) is similar to a non-damped spurious mode:

$$f_0(t) = 1 \quad \text{and} \quad g_0(t) = t$$  \hspace{1cm} (40)

Therefore, an initial perturbation of the velocity always leads to a constant velocity of all the particles. This is related to the conservation of linear momentum as illustrated in the next subsection. It is interesting to note that the stability of the system depends also on which physical value is perturbed.

### 2.7.3. Temporal solution in the geometrical space

The matrix $Y(t)$ in Eq. 25 is a diagonal matrix whose diagonal elements are the right hand side of Eq. 38a. Multiplying $Y(t)$ on the left by $P^{-1}$ leads to the expression of $X$:

$$X(t) = F(t) X_0 + G(t) U_0$$  \hspace{1cm} (41)

where $X_0$ and $U_0$ are the vector of initial perturbations for the position and velocity, respectively. The terms $F(t)$ and $G(t)$ are time-dependent matrices expressed by their general term:

$$F_{ij}(t) = \frac{1}{N} \left[ f_0(t) + 2 \sum_{k=1}^{N_{mod}} f_k(t) \cos \left( \frac{2\pi}{N} k(i - j) \right) \right]$$  \hspace{1cm} (42a)

$$G_{ij}(t) = \frac{1}{N} \left[ g_0(t) + 2 \sum_{k=1}^{N_{mod}} g_k(t) \cos \left( \frac{2\pi}{N} k(i - j) \right) \right]$$  \hspace{1cm} (42b)

Equations 42 highlights the coupling between the temporal solutions and the nodal oscillations. On the other hand, if only the position of the particle $i$ is perturbed, then its own response is given by $\delta x_i(t) = F_{ii}(t) \delta x_0$ which is a pure combination of time-oscillations independent of the oscillations of the neighbors.

Let us consider an initial velocity perturbation of $\delta u_0$ applied to the particle $0$ with $\nu > 0$. After a long time, due to the damping (Eq. 38c), all $g_k(t)$ are zero except $g_0(t) = t$. Therefore, the position of each particle follow the same law $\delta x_i = \delta u_0 t/N$ which means a constant momentum of $m \delta u_0 / N$. After summing the velocity of all particles, one obtains $m \delta u_0$ which corresponds to the initial momentum of the system. Consequently the linear momentum of the system is conserved. The validation of Eqs. 38 and 42 is given in Appendix B.
2.8. Towards the dispersion relation

As of now it was not possible to derive any dispersion equation because no wavelength was present in the equation of motion. However, it is possible to find a link between the time oscillations and the spatial modes by considering an initial spatial sinusoidal perturbation of mode $p$:

$$\delta x_{i,0} = \delta_0 \cos \left( \frac{2\pi i}{N} \right)$$

where $\delta_0$ is the amplitude of the perturbation. Because of the aliasing effect, the largest mode that can be resolved on the particle lattice is $N_{\text{mid}}$. Thus, $p \in [0, N_{\text{mid}}]$. The wavelength associated to mode $p$ is

$$\lambda_p = \frac{L}{p}$$

where $L = N \Delta x$ is the total length of the domain. Therefore, the initial sinusoidal perturbation can be written in terms of $\lambda_p$:

$$\delta x_{i,0} = \delta_0 \cos \left( 2\pi i \frac{\Delta x}{\lambda_p} \right)$$

and the smallest resolved wavelength is $\lambda_{\text{min}} = 2\Delta x$ if $N$ is even. Figure 7 shows an illustration of different initial sinusoidal perturbations.

![Figure 7: Illustration of different initial sinusoidal perturbations for $N=11$.](image)

Given these initial perturbations, the motion of particle $i$ is given by:

$$\delta x_i(t) = \sum_{j=0}^{N-1} F_{ij}(t) \delta x_{j,0}$$

which simplifies, after some algebra, to:

$$\delta x_i(t) = f_p(t) \delta x_{i,0}$$

where $f_p$ is given by Eq. 38b. This means that when a particle lattice is excited with by a spatial mode $p$, the particles will oscillate with the corresponding temporal mode $p$. Therefore, Eq. 46 establishes a link between spatial and temporal modes, i.e. a dispersion relation which will be investigated in Section 6.

3. Application of LSA to div-SPH

In this section the motion of perturbations with div-SPH will be investigated, i.e. when the density is estimated with the divergence of the velocity. The equations of motion are:
The rational functions using the velocity divergence to estimate the density is the non-damped perturbation of position and density. Thus, the only particularity of \( Q \) component signal. In other words, an initial perturbation of position or density will remain unaffected. The poles of \( \psi_{1,i} \) and \( \psi_{2,i} \) are defined by Eqs. 28 and 29. The term \( z_i \) is the general term of the density perturbations vector in the modal space \( Z = P \tilde{R} \). The system is transformed into the Laplace domain:

\[
\begin{align*}
\frac{d\psi_i}{dt} &= \Delta x \lambda_i \frac{d\psi_i}{dt} \\
\frac{d^2\psi_i}{dt^2} &= \psi_{2,i} \sigma_i \frac{d\psi_i}{dt} + \frac{\psi_{1,i}}{\Delta x \lambda_i^2} \psi_i & (47a)
\end{align*}
\]

which leads, after linearization, non-dimensionalization and projection into the mode basis, to:

\[
\begin{align*}
\frac{d\psi_{i1}}{dt} &= \Delta x \lambda_{1i} \frac{d\psi_{i1}}{dt} \\
\frac{d^2\psi_{i1}}{dt^2} &= \psi_{2,i1} \sigma_{i1} \frac{d\psi_{i1}}{dt} + \frac{\psi_{1,i1}}{\Delta x \lambda_{1i}^2} \psi_{i1} & (48a)
\end{align*}
\]

where \( \sigma_{1,i} \) and \( \psi_{2,i} \) are defined by Eqs. 28 and 29. The term \( z_i \) is the general term of the density perturbations vector in the modal space \( Z = P \tilde{R} \). The system is transformed into the Laplace domain:

\[
\begin{align*}
\frac{d\psi_{i1}}{dt} &= \psi_{2,i} \psi_{i1} - \phi_{1,0} \frac{d\psi_{i1}}{dt} - \frac{\psi_{1,i}}{\Delta x \lambda_{1i}^2} \phi_{i1} & (49a)
\end{align*}
\]

where \( \psi_{1}(p) \) and \( \psi_{2}(p) \) are the Laplace transforms of \( \psi_{i1}(t) \) and \( \psi_{i2}(t) \). The initial conditions of position, velocity and density are denoted by \( y_{i0}, \dot{y}_{i0} \) and \( z_{i0} \), respectively. Substituting Eq. 49a into Eq. 49b leads to:

\[
\begin{align*}
Y_i &= P_1(p) y_{i0} + P_2(p) \dot{y}_{i0} + P_3(p) z_{i0} & (50a) \\
P_1(p) &= |p^2 + \sigma_i p - \psi_{1,i}|/Q(p) & (50b) \\
P_2(p) &= p/Q(p) & (50c) \\
P_3(p) &= (\psi_{1,i}/\Delta x \lambda_i^2)/Q(p) & (50d) \\
Q(p) &= p[p^2 + \sigma_i p + \sigma_i^2] & (50e)
\end{align*}
\]

The poles of \( P_1, P_2, P_3 \) are the roots of \( Q(p) \). They are given by:

\[
r_1 = 0 \quad \text{and} \quad r_{2,3} = -\frac{\sigma_i}{2} \left( 1 \pm \sqrt{1 - 4(\phi_{1,i}/\sigma_i)^2} \right) & (51)
\]

The roots \( r_{2,3} \) are equal to \( \pi_x^2 \) as given by Eq. 35. Hence, their real part is either negative or zero, leading to a decreasing exponential or continuous component in the temporal domain. In addition, the value \( p = 0 \) is also a pole of \( P_1 \) and \( P_3 \), which means that the decomposition of \( P_1 \) and \( P_3 \) into rational functions will exhibit the term \( 1/p \). The inverse Laplace transform of \( 1/p \) is the Heaviside function, i.e. a continuous component signal. In other words, an initial perturbation of position or density will remain unaffected. Since the roots \( r_2, r_3 \) of \( Q(p) \) are the same as \( \pi_x^2 \), it means that, apart from the continuous mode, the dispersion and the damping curves of div-SPH are the same as for sum-SPH. Thus, the only particularity of using the velocity divergence to estimate the density is the non-damped perturbation of position and density.

The rational functions \( P_1, P_2 \) and \( P_3 \) are simplified to apply the inverse Laplace transforms:

\[
\begin{align*}
P_1 &= -\frac{1}{\sigma_i^2} \left( \psi_{1,i} \frac{1}{p} + \psi_{2,i} \frac{p + \sigma_i/2}{(p + \sigma_i/2^2 + \omega_i^2)} + \frac{\psi_{1,i}}{2 \omega_i} \frac{\omega_i}{(p + \sigma_i/2)^2 + \omega_i^2} \right) & (52a) \\
P_2 &= \frac{1}{\omega_i} \frac{1}{(p + \sigma_i/2)^2 + \omega_i^2} & (52b) \\
P_3 &= \frac{-\psi_{1,i}}{\Delta x \lambda_i^2 \phi_{1,i}^2} \left( \frac{p + \sigma_i/2}{(p + \sigma_i/2)^2 + \omega_i^2} + \frac{\sigma_i}{2 \omega_i} \frac{\omega_i}{(p + \sigma_i/2)^2 + \omega_i^2} - \frac{1}{p} \right) & (52c)
\end{align*}
\]

\[
13
\]
respectively. In their original paper, Antuono et al. (2010) omitted the background pressure in the EoS but the mass diffusivity, the energy diffusivity and the internal energy response to compressibility, respectively. When \( \Delta_i \) (Eq. 34) is negative, the functions can be expressed as:

\[
\begin{align*}
    f_i(t) &= -\frac{1}{\omega_i^2} \left[ \psi_{1,i} + \psi_{2,i} \exp \left( -\frac{\sigma_i}{2} t \right) \left( \cos(\omega_i t) + \frac{\sigma_i}{2 \omega_i} \sin(\omega_i t) \right) \right] \\
    g_i(t) &= \frac{1}{\omega_i} \exp \left( -\frac{\sigma_i}{2} t \right) \sin(\omega_i t) \\
    h_i(t) &= -\frac{1}{\Delta x} \frac{\lambda_i}{\phi_i^2} \left[ \exp \left( -\frac{\sigma_i}{2} t \right) \left( \cos(\omega_i t) + \frac{\sigma_i}{2 \omega_i} \sin(\omega_i t) \right) - 1 \right]
\end{align*}
\] (53a-c)

When \( \Delta_i > 0 \), the sinusoidal functions are to be replaced by their hyperbolic variants. When \( \Delta_i = 0 \), the cosine is replaced by 1 and \( \sin(\omega_i t)/\omega_i \) is replaced by \( t \). The zero mode are \( f_0(t) = 1, g_0(t) = t \) and \( h_0(t) = 0 \). Note that \( h_i(t) \) is a pure imaginary function due to the term \( \lambda_i' \) in its prefactor. The solution of the perturbation equation in the modal space yields:

\[
y_i(t) = f_i(t) y_{i,0} + g_i(t) y_{i,0}' + h_i(t) z_{i,0}
\] (54)

Transforming \( y_i(t) \) to \( X(t) \) is done using Eqs. 42 for the position and velocity. For the density, due to the property \( \lambda_k' = -\lambda_{n-k}' \), the inverse transformation of \( h_i(t) \) is:

\[
H_{ij}(t) = \frac{1}{N} \left[ h_0(t) + 2 \sum_{k=1}^{N_{\text{cell}}} h_k(t) \sin \left( \frac{2\pi}{N} k(i-j) \right) \right]
\] (55)

which is a real function given that \( h_k(t) \) are purely imaginary. Equations. 53 and 55 are validated in Appendix B.

4. Application of LSA to \( \delta \)-SPH

The same analysis is now applied to \( \delta \)-SPH. This scheme, originally proposed by Antuono et al. (2010), is based on div-\( \delta \)-SPH, in which additional diffusion terms are added in the density and energy equation. The equations of motion are recalled in 1D:

\[
\begin{align*}
    \frac{d\rho_a}{dt} &= \rho_a \sum_b V_b (u_a - u_b) W_{ab}' + \xi h \sum_b V_b \psi_{ab} W_{ab}' \\
    \rho_a \frac{dx_a}{dt^2} &= -\sum_b V_b (p_b + p_a) W_{ab}' + \alpha h^2 \sum_b V_b \pi_{ab} W_{ab}' \\
    \rho_a \frac{dc_a}{dt} &= \rho_a \sum_b V_b (u_a - u_b) W_{ab}' - \frac{\alpha h^2}{2} \sum_b V_b \pi_{ab} (u_a - u_b) W_{ab}' + \chi h \sum_b V_b \phi_{ab} W_{ab}' \\
    p_a &= c^2 (\rho_a - \bar{p}) \left[ 1 + \Gamma \left( \frac{e_a}{\bar{e}} - 1 \right) \right] + p_{\text{back}}
\end{align*}
\] (56a-d)

where:

\[
\psi_{ab} = 2 \frac{\rho_a - \rho_b}{x_{ab}} \quad \phi_{ab} = 2 \frac{e_a - e_b}{x_{ab}} \quad \pi_{ab} = \frac{u_a - u_b}{x_{ab}}
\] (57)

The system of equations 56 shows, in order of appearance, the conservation of mass, momentum and energy, and the EoS. The terms \( \alpha, \xi, \chi \) and \( \Gamma \) are the non-dimensional coefficients of (i) the viscosity (equal to \( 2\nu \)), (ii) the mass diffusivity, (iii) the energy diffusivity and (iv) the internal energy response to compressibility, respectively. In their original paper, Antuono et al. (2010) omitted the background pressure in the EoS but
where, in this case, the term dimensionalization and projection into the mode basis lead to the following system:

is second order, so it does not appear in the density and motion equation after linearization. Non-dimensionalization and projection into the mode basis lead to the following system:

As mentioned by Antuono et al. (2010), the energy contribution inside the equation of state (Eq. 56d) is second order, so it does not appear in the density and motion equation after linearization. Non-dimensionalization and projection into the mode basis lead to the following system:

where, in this case, the term \( \sigma_i \) defined in Eq. 28 is now equal to \(-\alpha \Delta x \lambda_i''\). It is worth to note that this system of equations is equal to the system of div-SPH (Eqs. 48) plus an additive diffusion term in the density equation. Translating this system into the Laplace domain and expressing the particle motion \( Y_i \) leads to:

\[
Y_i = P_1(p) y_{i,0} + P_2(p) y_{i,0} + P_3(p) z_{i,0} \quad \text{with} \quad P_1(p) = [p^2 - (\alpha + 2\xi) \Delta x \lambda_i'' p + 2\alpha \xi (\Delta x \lambda_i'')^2 - \psi_{1,i}]/Q(p) \quad \text{(60a)}
\]

\[
P_2(p) = (p - 2\xi \Delta x \lambda_i'')/Q(p) \quad \text{(60b)}
\]

\[
P_3(p) = (\psi_{1,i}/\Delta x \lambda_i')/Q(p) \quad \text{(60c)}
\]

and

\[
Q(p) = p^3 - (\alpha + 2\xi) \Delta x \lambda_i'' p^2 + [2\alpha \xi (\Delta x \lambda_i'')^2 + \psi_{1,i}^2] p + 4\Delta x^2 \xi \rho_{bg} \lambda_i'' \lambda_i''' \quad \text{(61)}
\]

In this case, the roots of \( Q(p) \) cannot be analytically expressed in a compact form. However, it is worth to note that because of the constant term \(4\Delta x^2 \xi \rho_{bg} \lambda_i'' \lambda_i'''\), \( p = 0 \) is not a root of \( Q \), which means that no undamped continuous mode can be found, provided that both \( \xi \) and \( \rho_{bg} \) are non zero. Consequently, for \( \xi > 0 \) and \( \rho_{bg} > 0 \), any initial perturbation of any physical type is damped. This is a clear advantage of \( \delta \)-SPH over div-SPH when background pressure is added. In order to investigate the poles of \( P_1, P_2 \) and \( P_3 \), the equation \( Q(p) = 0 \) is solved numerically and the roots are discussed below.

Figure 8 shows the roots of \( Q \) for different parameters. Their real part represents the stability while the imaginary part is the angular frequency of the oscillation. When the viscosity increases (Fig. 8 right), a second regime is observed for higher modes. It is an aperiodic regime where one of the damping ratio decreases to zero (\( r_1 \) in Fig. 8 right), as also pointed out by Antuono et al. (2012). In this case, the continuous solution \( (r_3) \) is damped faster than the sinusoidal modes. This means that physical oscillations have a larger amplitude that the unphysical continuous perturbations in \( \delta \)-SPH. It was also observed (but not illustrated here) that when \( \xi > \alpha/2 \), the real part of \( r_1 \) and \( r_2 \) becomes positive for the first modes, leading to the slow divergence of a perturbation of large wavelength. The value of \( \xi = \alpha/2 \) was found by Antuono et al. (2010) as an optimal value, but not a limiting one, probably due to the assumption of zero background pressure in their analysis.

Figure 9 shows the roots of \( Q \) for the same parameters as in Fig. 8 except that background pressure is set to zero. It is observed that the angular frequency decreases to zero at higher modes, and that the aperiodic mode starts at lower modes. As mentioned earlier, the continuous mode is not damped \( (r_3=0) \), for any values of \( \alpha \) Therefore to fully benefit of the advantage of \( \delta \)-SPH, it is demonstrated here that a strictly positive background pressure is mandatory. However, for a large viscosity (Fig. 9 right), the damping of the aperiodic mode is larger than when the background pressure is positive. This means that high frequency perturbations are faster damped without background pressure, which could be an advantage.
Figure 8: Real (top) and imaginary (bottom) parts of the roots of $Q$ for $\alpha = 0.2$ (left) and 2 (right). Other parameters are $p_{bg}=1$ and $\Delta x=0.86$, $N=63$, $\xi = \alpha/4$ and the quintic kernel was used.

In the present case where $p_{bg}=0$, the value of $\xi$ is not limited by $\alpha$, as mentioned by Antuono et al. (2010), as it appears as a regular viscosity coefficient in the expression of $Q$ (Eq. 61). In this context, Antuono et al. (2012) studied the stability of the $\delta$-SPH method when the background pressure is equal to zero. They applied a 1D linear stability analysis with a modified diffusive term in the density equation ($\psi_{ab}$ in Eq. 56a) which is also valid on an incomplete sphere of influence. For a given $\alpha$, they provided a range of $\xi$ to avoid the aperiodic regime. However, they did not mention the undamped continuous mode. Applying the same type of analysis with the diffusion operator presented in Eq. 56a leads to the condition on the diffusion constants $\xi$ and $\alpha$:

$$\frac{\alpha}{2} + \frac{\phi_i}{\Delta x \lambda_i'''} \leq \xi \leq \frac{\alpha}{2} - \frac{\phi_i}{\Delta x \lambda_i'''}$$

(62)

which is similar to the one derived by Antuono et al. (2012), in the sense that the limiting value depends on $\phi_i/\Delta x \lambda_i'''$ which is related to the maximum resolved wavenumber. Finally, even though the temporal solution of the linearized $\delta$-SPH cannot be expressed as an analytical function, Eqs. 60 and 61 are validated in Appendix B.

5. Application of LSA to the Transport Velocity Formulation (TV-SPH)

The Transport-Velocity Formulation of SPH (referred to as TV-SPH in the following) was developed by Adami et al. (2013). The basic idea is to (i) advect particles at a modified transport velocity $\tilde{u}$ that reduces the particle disorder, and (ii) to compute the physical source terms from the original fluid velocity $u$. The lower particle disorder increases the accuracy of the operators compared to traditional methods. This approach is based on the assumption of incompressible flow even though it is applied to WCSPH. The
The set of equations is, in 1D:

\[
\begin{align*}
\rho_a &= m_a \sum_b W_{ab} \\
\frac{\tilde{d}u_a}{dt} &= \frac{1}{m_a} \sum_b (V_a^2 + V_b^2) \left[ -\tilde{p}_{ab}W_{ab}' + \frac{1}{2}(T_a + T_b)W_{ab} + \tilde{p}_{ab} \frac{u_{ab}}{x_{ab}} W_{ab}' \right] \\
p_a &= c^2 (\rho_a - \overline{\rho}) \\
\frac{d\tilde{u}_a}{dt} &= \frac{\tilde{d}u_a}{dt} - \frac{\tilde{p}_{back}}{m_a} \sum_b (V_a^2 + V_b^2)W_{ab}' \\
\frac{dx_a}{dt} &= \tilde{u}_a 
\end{align*}
\]

where \( \tilde{d}/dt \) is the material derivative of a particle moving with the modified transport velocity and \( T = \rho u(\tilde{u} - u) \) is the convection of the particle momentum with the relative velocity \( (\tilde{u} - u) \). The dynamic viscosity and pressure are expressed with an inter-particle approach as \( \tilde{\mu}_{ab} = 2\mu_a \mu_b / (\mu_a + \mu_b) \) and \( \tilde{p}_{ab} = (\rho_b \rho_a + \rho_a \rho_b) / (\rho_a + \rho_b) \), respectively. It is important to notice that even though \( d/dt \) and \( d/dt \) are two material derivatives with different advection velocities for the Eulerian point of view, they are equivalent to \( \partial/\partial t \) in the Lagrangian framework (Belytschko et al., 2000). In Eqs. 63, the two independent variables are \( u_a \) and \( x_a \).

It is assumed that the viscosity is homogenous, i.e. \( \tilde{\mu}_{ab} = \mu \). The intermediate quantities are linearized to the first-order as:

\[
\begin{align*}
V_a^2 &\approx \overline{V}^2 \left( 1 - 2\frac{\delta \rho_a}{\rho} \right) \\
\tilde{p}_{ab} &\approx \overline{\rho}_{b} \delta \rho_a + \overline{\rho}_{a} \delta \rho_b \\
T_a &\approx \overline{\rho} \overline{\mu} \overline{\mu} \delta u_a + \overline{\rho} \overline{\mu} \overline{\mu} \delta \rho_a
\end{align*}
\]
where the relative velocity is decomposed as \( w = \tilde{w} - u = \overline{w} + \delta w \). Due to the quadratic dependency of \( \frac{\partial}{\partial \nu} \) on \( u \), the mean velocity is considered positive only (\( \overline{w} \geq 0 \)) to apply the linearization. The non-dimensionalization and the projection into the mode basis leads to:

\[
\begin{align}
\frac{1}{\Delta x} \frac{dv_i}{dt} &= [\overline{w} \lambda_i' - (1 + \overline{w}) \Delta x \lambda_i'^2] y_i + [(\overline{w} - \nu) \lambda_i' + 2 \nu \lambda_i''] v_i + \overline{w} \lambda_i' \frac{dy_i}{dt} \\
\frac{d^2 y_i}{dt^2} &= \frac{dv_i}{dt} + 2p_{bg} \Delta x(\Delta x \lambda_i'^2 - \lambda_i'')v_i
\end{align}
\]

(65a)

(65b)

where \( v_i \) is the general term of the projection of \( \mathbf{U} \) in the mode basis. Expressing the coordinate \( y_i \) in the Laplace domain yields:

\[
Y_i = P_1(p) y_i,0 + P_2(p) y_i',0 + P_3(p) v_i,0 \quad \text{with} \quad Y_i = \mathcal{L}^{-1} \left[ \frac{dv_i}{dt} \right] = \frac{1}{\Delta x} \frac{dv_i}{dt}
\]

(66a)

\[
P_1(p) = \frac{1}{p - (\overline{w} \lambda_i' + 2 \nu \lambda_i'') \Delta x} \left( \frac{Q}{p} \right)
\]

(66b)

\[
P_2(p) = \frac{1}{p - (\overline{w} \lambda_i' + 2 \nu \lambda_i'') \Delta x} \left( \frac{Q}{p} \right)
\]

(66c)

\[
P_3(p) = \left( \frac{\overline{w} - \nu}{} \right) \Delta x \left( \frac{Q}{p} \right)
\]

(66d)

and

\[
Q(p) = p^3 + \Delta x(\overline{w} \lambda_i' - 2 \nu \lambda_i'') p^2
\]

\[
- \Delta x[2 \pi \nu \varphi_i + \Delta x \lambda_i'^2] p
\]

\[
- 2 \Delta x \pi \nu \varphi_i (\overline{w} - \nu \lambda_i') p
\]

(67)

where \( \pi_{bg} = \frac{\overline{w}}{2} \varphi_i = \lambda_i' - \Delta x \lambda_i'' \). As in \( \delta \)-SPH, the constant part of \( Q(p) \) ensures that \( p=0 \) is not a root. Therefore, \( p_{\text{shake}} > 0 \) is a sufficient condition to ensure damping of any perturbations in the case of viscous flow (\( \nu > 0 \)). The Laplace variable of the physical velocity \( v \) is labeled \( Y \) and yields:

\[
Y_i = P_1(p) y_i,0 + P_2(p) y_i',0 + P_3(p) v_i,0 \quad \text{with} \quad Y_i = \mathcal{L}^{-1} \left[ \frac{dv_i}{dt} \right] = \frac{1}{\Delta x} \frac{dv_i}{dt}
\]

(68a)

\[
P_1(p) = \frac{1}{p - (\overline{w} \lambda_i' + 2 \nu \lambda_i'') \Delta x} \left( \frac{Q}{p} \right)
\]

(68b)

\[
P_2(p) = \frac{1}{p - (\overline{w} \lambda_i' + 2 \nu \lambda_i'') \Delta x} \left( \frac{Q}{p} \right)
\]

(68c)

\[
P_3(p) = \frac{1}{p - (\overline{w} \lambda_i' + 2 \nu \lambda_i'') \Delta x} \left( \frac{Q}{p} \right)
\]

(68d)

with \( Q(p) \) as defined in Eq. 67. Equations 66, 67 and 68 are validated in Appendix B.

Figure 10 displays the real and imaginary part of the roots of \( Q(p) \), for \( \nu=0.1 \) (left) and \( \nu=1 \) (right). Other parameters are \( p_{bg}=1 \), \( \Delta x=0.86 \), \( N=63 \), \( (\overline{w}, \overline{w})= (0,0) \), and the quintic kernel was used. A continuous solution (\( r_1 \)) is observed for both values of the viscosity. The damping ratio is significantly larger than the oscillating solutions, for a mode number \( r_1 > 10 \). For a high mode numbers, the oscillating solutions show a damping ratio very close to zero, suggesting that high frequency disturbances might be weakly damped. Further investigations, especially concerning the influence of the velocities \( \overline{w} \) and \( \overline{w} \) are provided when discussing the dispersion curves in the next section.

6. Dispersion curves

In this section, the dispersion curves of the WCSPH methods are investigated as outlined in the previous sections. First, the viscosity is set to zero in order to study the non-damped sound propagation and second, the influence of the viscosity is studied.

In order to quantify the deviation of discrete WCSPH to a continuum, the dispersion curves will be compared to the ones obtained from the linearized Navier-Stokes equations. In this approach, the linearization of mass and momentum conservation leads to:

\[
\frac{\partial^2 \delta u}{\partial t^2} - c^2 \frac{\partial^2 \delta u}{\partial x^2} = \nu \frac{\partial^3 \delta u}{\partial x \partial t}
\]

(69)
Non-dimensionalizing Eq. 69 with the variables $h$ and $c$, and assuming a solution of the form $\delta u(x,t) = \delta_0 \exp[i(\omega t - kx)]$ leads to dispersion relation:

$$\omega^2 - iv\omega k^2 - k^2 = 0 \tag{70}$$

These solutions will compared to the SPH solutions in the following. The stability analysis is made in a temporal sense. The location is arbitrarily fixed at $x=0$ and the temporal evolution is investigated. Note that in this case, the stability is obtained for $\text{Im}(\omega)>0$.

6.1. Propagation in an inviscid medium

6.1.1. Classical WCSPH: sum-SPH

The dispersion curves are obtained by plotting the angular frequency $\omega_p$ defined in Eq. 36 versus the non-dimensional wavenumber $\kappa_p = 2\pi h/\lambda_p$. When the viscosity is zero, the angular frequency is given by $\omega_p = \phi_p$. Figure 11 displays the dispersion curve for the quintic kernel (left) and the Wendland kernel (right), and different $p_{bg}$. The domain parameters are $N=1000$ and $M=3$. The solid vertical line is related to the diameter of the sphere of influence $D = (2M+1)\Delta x$ and is labeled $\kappa_D$ while the dashed vertical line corresponds to the smoothing length $h$ and is located at $\kappa_h = 2\pi$. The highest resolved wavenumber is $\pi/\Delta x$.

The line of equation $y = x$ is the dispersion curve obtained from Eq. 70 and characterizes a non-dispersive medium. First, it is observed that both kernels have a similar behavior. Hence, the superiority of one kernel compared to the other is not obvious here. Second, due to a low number of neighbors, $M = 3$, the wavenumber $\kappa_h$ related to the smoothing length cannot be resolved by the particle lattice. This remark highlights the lack of physical meaning of the smoothing length $h$ as pointed out by Dehnen and Aly (2012). The ideal non-dispersive behavior is partly retrieved by the SPH scheme for low wavenumbers up to $\approx \kappa_D$. The upper bound of the ideal behavior depends on the background pressure, and the best case is observed for $p_{bg}=1$, independently of the kernel. For $\kappa > \kappa_D$, the angular frequency is not proportional to the wavenumber anymore. Finally, the spurious mode is observed at large wavenumber ($\kappa = \pi/\Delta x$) for $p_{bg}=0$.

From the dispersion relation, it is possible to extract the phase velocity $c_\phi = \omega/\kappa$ and the group velocity $c_g = \partial\omega/\partial\kappa$. They are both shown for the quintic kernel in Fig. 12. On the left, the phase velocity is
equal to 1 up to $\approx \kappa_D$, where it depends on $p_{bg}$, the best case being $p_{bg}=1$ as previously mentioned. Contrary to the phase velocity $c_\phi$, the group velocity $c_g$ (Fig. 12, right) becomes negative at high wavenumbers. It is observed that a larger background pressure increases the wavenumber at which $c_g$ becomes negative. Note that for a wavenumber leading to $c_g<0$, the previous results (e.g. Fig. 11) show no instability. Therefore, a negative group velocity does not imply an instability. On the other hand, setting $p_{bg}>1$ increases the group velocity for $\kappa \lesssim \kappa_D$, meaning that the pressure information propagates at a speed faster than $c$. As for the phase velocity, the best compromise is found for $p_{bg}=1$.

The dispersion curve for $\Delta x=0.29$ ($M=10$) is shown in Fig. 13 for the quintic kernel (left) and the Wendland kernel (right). The same characteristics with regards to the background pressure are observed. However, in this case due to the larger number of neighbors, the particle lattice can resolve wavelengths smaller than $h$. The spurious modes when $p_{bg}=0$ occurs several time for the quintic kernel, while it occurs only at $\kappa_{max}$ for the Wendland kernel. This suggests a better stability of the Wendland kernel with a large number of neighbors when $p_{bg}=0$. The corresponding phase and group velocities are depicted in Fig. 14 for the quintic kernel. The same findings as for $\Delta x=0.86$ are valid. In addition, for $\kappa>\kappa_h$, the group velocity is almost zero.
Figure 13: Dispersion curve for \( N=1000 \) and \( \Delta x=0.29 \). Left: quintic kernel. Right: Wendland kernel.

Figure 14: Phase (left) and group (right) velocity for \( N=1000, \Delta x=0.29 \) and the quintic kernel.

6.1.2. Other methods without viscosity

When the viscosity is set to zero in div-SPH and \( \delta \)-SPH, the roots of \( Q(p) \), labeled as \( \pi^{\pm} \), are zero and \( \pm \phi_i \), and the dispersion curve of these methods is the same as the one obtained from sum-SPH, with an additional continuous component. Concerning TV-SPH, it also depends on the mean velocity \( \overline{v} \) and the mean velocity difference \( \overline{w} \). When both of them are zero, the roots of \( Q(p) \) are also 0 and \( \pm \phi_i \), leading to the regular dispersion curve plus a continuous component. However, when \( \overline{w} \neq 0 \) and \( \overline{u} > 0 \), the polynomial \( Q \) is:

\[
Q(p) = p^3 + \phi_i^2 p + 2 \Delta x^2 \rho y \phi_i \overline{v} \xi_i + \Delta x^2 \rho y (\overline{w} - \overline{v}) \phi_i \overline{v} \xi_i
\]

(71)

The roots of Eq. 71 were numerically computed for \( (\overline{v}, \overline{w})=(0.05,0), \rho y=1, N=1000 \) and \( M=3 \). All roots have a real part equal to zero so no damping is found. The imaginary part of the roots of Eq. 71 are depicted in Fig. 15. It is obvious that, in addition to the regular oscillations following the line \( y=x \), there are additional oscillations of lower frequency. This means that when both the physical and transport velocity are close (i.e. \( \overline{w} \approx 0 \)), a spatial excitation results in a superposition of several frequencies, which can be regarded as a non-linear behavior.

When \( \overline{w} \neq 0 \) and \( \overline{u} > 0 \), the polynomial \( Q \) is:

\[
Q(p) = p^3 + \Delta x \overline{v} \xi_i^2 p^2 - \Delta x^2 [2 \pi y \phi_i + \Delta x \phi_i^2] p - 2 \Delta x^2 \rho y \phi_i (\overline{w} - \overline{v}) \xi_i
\]

(72)
The roots of Eq. 72 were numerically computed for \((\pi, \overline{\pi})=(0.05, -0.05)\), the other parameters being identical to the previous case. Like in the previous case, all roots have a real part equal to zero, and an additional mode of lower frequency is also found. Therefore this is not illustrated. The particular case \(\pi = \overline{\pi} \neq 0\) leads to a polynomial \(Q\):

\[
Q(p) = p (p^2 + \Delta x \pi \lambda_i p + \phi_i^2)
\]  

(73)

Although this polynomial is different from the one obtained for \(\pi = \overline{\pi} = 0\), the roots are almost the same and the real part is always zero.

6.2. Effects of viscosity

6.2.1. sum-SPH

Due to the damping ratio in the differential equation of motion, not only the imaginary part of \(\pi_i\) but also its real part is presented and compared to the continuum solution. The solutions of Eq. 36 are presented in Fig. 16 for the quintic kernel, \(N=500\), \(p_{bg}=0.5\), \(\Delta x=0.86\) (left) and \(\Delta x=0.29\) (right), and compared to the continuum solution, labeled CON in figures. For \(\Delta x=0.86\), \(p_{bg}=0.5\) leads to a good prediction in terms of angular frequency and damping ratio. For \(\Delta x=0.29\), a large viscosity \((\nu=1)\) shows a good agreement with the continuum solutions, whereas with \(\nu=0.1\), the sum-SPH method predicts damping without oscillations starting at \(\kappa \approx \kappa_h\) although the continuum solution shows damped oscillations. The risk of instability is also observed for \(\Delta x=0.29\) at \(\kappa \approx \kappa_h\) where the spurious mode \((\omega_i=0)\) shows a very low damping ratio. Finally, it is to be mentioned that the prefactor of the Laplacian operator in Eq. 5b was set to 2 and not to 6 as proposed by Macià et al. (2011). In this case, the damping ratio of sum-SPH would be overestimated by a factor of 3.

The same investigation with the Wendland kernel is shown in Fig. 17. When \(\Delta x=0.86\), the Wendland kernel is very similar to the quintic kernel. When \(\Delta x=0.29\), at high wavenumber \((\kappa \approx \kappa_h)\), the Wendland kernel shows a monotonic decrease and a larger damping ratio than for the quintic kernel. For both kernels, the best value of \(p_{bg}\) seems to be 0.5.

6.2.2. \(\delta\)-SPH

The dispersion curve of \(\delta\)-SPH method is presented in this subsection. As the poles of the transfer function cannot be analytically expressed in a compact form, they are computed numerically. Figure 18 shows the imaginary and real part of the solution for the quintic kernel, \(p_{bg}=0.5\), \(\Delta x=0.86\) and \(\xi = \alpha/4\). Different values of the viscosity \(\alpha=0.2\) (left) and \(\alpha=2\) (right) were investigated. The imaginary part of the damped continuous mode identified in Section 4 cannot be shown in a log-log plot because it is equal to zero. Its real part, \(i.e.\) its damping ratio, is approximately two times larger than the damping ratio of the oscillating modes. It is observed that \(p_{bg}=0.5\) leads to a good agreement with the continuous solution for
Figure 16: Imaginary and real part of angular frequency $\pi^\pm$ versus the wavenumber $\kappa$ for the quintic kernel and $p_{bg}=0.5$, superimposed with the continuum solution. For SPH, solid and dashed lines represent $\pi^+$ and $\pi^-$, respectively, whereas they are represented by dashed and dot-dashed lines for the continuum solution. Left: $\Delta x=0.86$. Right: $\Delta x=0.29$.

Figure 17: Imaginary and real part of angular frequency $\pi^\pm$ versus the wavenumber $\kappa$ for the Wendland kernel and $p_{bg}=0.5$, superimposed with the continuum solution. $\Delta x=0.86$ (left) and $\Delta x=0.29$ (right). For SPH, solid and dashed lines represent $\pi^+$ and $\pi^-$, respectively, whereas they are represented by dashed and dot-dashed lines for the continuum solution.

the two values of $\alpha$, both in terms of angular frequency and damping ratio. Investigations with the Wendland kernel showed very little difference to the quintic kernel. With $\Delta x=0.29$, the findings of sum-SPH apply also to $\delta$-SPH.
6.2.3. TV-SPH

The poles of the transfer function (Eq. 67) are computed numerically. They are depicted in Fig. 19 for the quintic kernel, \( p_{bg} = 0.5 \) and \( \Delta x = 0.86 \), and \( (\pi, \tau) = (0,0) \). The artificial continuous mode is still observed (not depicted in the log-log plot of the imaginary part) but it is damped. The damping ratio is significantly lower than that of the continuum solution up to \( \kappa_D \), independently of the viscosity. For \( \nu = 1 \) (Fig. 19 right), large wavenumbers show weakly damped oscillations whereas for the continuum solution a damped aperiodic regime is predicted. This phenomenon is always observed for other values of \( (\pi, \tau) \). It was also observed but not depicted here, that a lower background pressure decreases the damping ratio of the non-physical mode and increases the damping ratio of the physical oscillations at high wavenumber, and vice versa.

This suggests that there exist an optimal background pressure for the prediction of acoustic waves with the TV-SPH method.

Solutions for \( (\pi, \tau) = (0,0.1) \) and \( \nu = 0.1 \) are depicted in Fig. 20 (left). In this case, the non-physical continuous mode is turned into a low-frequency mode and shows the same damping characteristics as in Fig. 19. The result of a positive fluid velocity \( (\pi, \tau) = (0.05,0.1) \) is shown in Fig. 20 (right) and highlights the presence of instabilities at large wavenumber. They are depicted by cross symbols in Fig. 20 and represent \(+\text{Re}(\pi_i)\) instead of \(-\text{Re}(\pi_i)\). This instability is illustrated in Appendix B.

7. Illustration of the spurious mode with a large number of neighbors

This section illustrates the difference between the quintic and the Wendland kernel if a large number of neighbor particles are present (M=9 \( \leftrightarrow \Delta x = 0.32 \)) with \( p_{bg} = 1 \) and \( \nu = 0 \). Special emphasis is put on the spurious mode as observed in Fig. 5. The number of particle \( N \) is 100. The dispersion curves \( c_p \) and \( c_g \) are plotted for the quintic and the Wendland kernel in Fig. 21 with several modes that represent different states. Mode 6 corresponds to a phase velocity close to the continuum solution and a group velocity \( \approx 0 \), mode 9 shows a negative group velocity of magnitude \( \approx 1 \). Mode 20 and 30 have a phase velocity \( \ll 1 \). In particular, mode 20 is the spurious mode for the quintic kernel. Mode 30 shows a group velocity \( \approx 0 \).
In order to compare the LSA results with a real simulation, 1D sum-SPH simulations were run. The same numbers \((N,M)\) were used and \(p_{bg}=1\) and \(\nu=0\) were set. An initial excitation velocity of 1% of the speed of sound was applied, and the simulations were run for a long non-dimensional time. The focus is put on the long time instability with a large deviation from the equilibrium state. Hence, the temporal solution of the linearized perturbation may not apply.
Wave number $\kappa_p$ [-]

\begin{align*}
10^{-3} & \\
10^{-2} & \\
10^{-1} & \\
10^0 & \\
10^1 & \\
\end{align*}

c\phi [-]

\begin{align*}
\text{Mode 6 9 20 30} & \\
quintic kernel & \\
Wendland kernel & \\
\end{align*}

Figure 21: Dispersion curve for $(N,M)=(50,9)$. Left: Phase velocity. Right: Group velocity. The vertical lines correspond to mode 6, 9, 20 and 30.

In Fig. 22 to 24 the normalized position of each particle is plotted versus non-dimensional time, for the quintic kernel (left figures) and the Wendland kernel (right figures). With the mode 6, both kernels show a stable behavior even though the group velocity is almost zero. Therefore, the trajectories are not shown. At mode 9 (Fig. 22), the quintic kernel shows long time instability whereas the Wendland kernel is stable. At mode 20 (Fig. 23), the spurious mode of the quintic kernel is visible with all particles moving at their initial velocity until \( t \approx 1000 \). Even though the temporal solutions of the linearized equations (Eq. 39) may not apply, the behavior highlighted in Fig. 23 corresponds to the linearized solution. As observed in Fig. 5, the Wendland kernel is not subject to the spurious mode, and provides a stable solution. For mode 30 (Fig. 24), both kernels shows regular oscillations until \( t \approx 500 \). For \( t \gtrsim 500 \), the quintic kernel shows an unstable behavior with particle trajectories crossing each other whereas the Wendland kernel shows stable oscillations. These plots underline the more stable behavior of the Wendland kernel in case of large initial deviation from equilibrium. It also shows one more time that a negative group velocity does necessary lead to instability.

Figure 22: Mode 9: Graph of the particles position for $N=50$. Left: Quintic kernel. Right: Wendland Kernel.

8. Conclusion

In this study, a unified framework for studying the linear stability of the SPH method was presented. This method was applied to different WCSPH methods on a 1D periodic domain. It appeared that the
equation of perturbations can be written as a matrix ordinary differential equation (in time) where the coefficients are circulant matrices. With this approach, the different SPH operators such as the gradient and the Laplacian can be represented by circulant matrices. Due to the properties of circulant matrices, the diagonalization of the system is equivalent to applying a spatial discrete Fourier transform. Thus, the equation of perturbations can be transformed from a geometrical-temporal to a modal-temporal domain. It is shown that the stability of the system depends on the Fourier transform of the first and second derivative of the kernel, and the damping is determined by the Fourier transform of $W'(x)/x$. In a second step, the equation of perturbations is transformed from the modal-temporal to the modal-Laplace domain, where the stability of the system is assessed. The system is also solved for transient and steady-state. Finally, it is transformed back into the geometrical-temporal domain for further validation.

Four different WCSPH methods were studied, namely sum-SPH, div-SPH, $\delta$-SPH and TV-SPH. When the background pressure is positive, no tensile nor pairing instability was found. However, under particular conditions, the TV-SPH method showed a slowly growing instability. The dispersion curve of the four investigated methods was plotted for inviscid and viscous flows. It was observed that the sum-SPH method provides the best agreement to the continuum solution. For all methods,
it was also found that the background pressure must be set to $\rho c^2$ for inviscid flows and to $\rho c^2/2$ for viscous flows to obtain the largest fidelity when simulating sound wave propagation. With a large number of neighbors, the Wendland kernel showed a more stable behavior featuring less spurious modes than the quintic kernel.

It was also demonstrated in this paper that the background pressure in combination with the uncorrected gradient operator counteracts any perturbations of equilibrium. In other words, particles will rearrange in a manner to counteract the particle disorder. Also, the present results disagree with those of Fulk (1994) and Swegle et al. (1995) who found that the stability depends on the second derivative of the kernel only, whereas our study showed that the stability depends on the first and second derivatives of the kernel.

The fact that no tensile nor pairing instabilities were found with sum-SPH and div-SPH can be explained by the fact that the base mean flow is incompressible. The next step of this study is to apply the circulant matrix decomposition on a 2D configuration, in order to study the effect of a transverse instability.

9. Acknowledgement

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10. Nomenclature

| Roman letter | Unit         | Description                              |
|--------------|--------------|------------------------------------------|
| $a$          | $[-]$        | Particle of interest                     |
| $b$          | $[-]$        | Neighbor particle                        |
| $c$          | $[\text{m/s}]$ | Speed of sound                           |
| $f$          | $[-]$        | Temporal function of the initial position |
| $g$          | $[-]$        | Temporal function of the initial velocity |
| $h$          | $[\text{m}]$ | Temporal function of the initial density  |
| $i,j$        | $[-]$        | Indices                                  |
| $i$          | $[-]$        | Imaginary number                         |
| $k$          | $[1/\text{m}]$ | Wave vector                              |
| $m$          | $[\text{kg}]$ | Mass                                     |
| $p$          | $[\text{Pa}]$ | Pressure                                 |
| $\Delta p$   | $[-]$        | Laplace variable                         |
| $r$          | $[-]$        | Roots of a polynomial                    |
| $u$          | $[\text{m/s}]$ | Velocity                                 |
| $w$          | $[\text{m/s}]$ | Relative velocity in TV-SPH              |
| $x$          | $[\text{m}]$ | Position                                 |
| $y$          | $[-]$        | Position perturbation in the modal space  |
| $y_i$        | $[-]$        | General term of the position perturbation vector |
| $A'$         | $[1/\text{m}]$ | Circulant matrix related compressibility |
| $A_m'$       | $[1/\text{m}^2]$ | Circulant matrix related to background pressure |
| $A_v'$       | $[1/\text{m}]$ | Circulant matrix related to viscosity     |
| $M$          | $[-]$        | Number of neighbors on one side          |
| $N$          | $[-]$        | Total number of particles                |
| $P$          | $[-]$        | Change-of-basis matrix                   |
| $P(p)$       | $[-]$        | Numerator of a rational function         |
| $Q(p)$       | $[-]$        | Denominator of a rational function       |
| $R$          | $[\text{kg/m}^3]$ | Vector of density perturbation           |
\[ V \quad [m^3] \quad \text{Volume} \]
\[ W \quad [1/m] \quad \text{Kernel} \]
\[ \Delta x \quad [m] \quad \text{Vector of position perturbation} \]
\[ \overline{Y} \quad [\cdot] \quad \text{Vector of position perturbation in the modal space} \]
\[ Y_i \quad [\cdot] \quad \text{Position perturbation in the Laplace domain} \]
\[ Z_i \quad [\cdot] \quad \text{Density perturbation in the Laplace domain} \]

**Greek letter** | **Unit** | **Description**
--- | --- | ---
\[ \gamma \] | [-] | Polytropic ratio \[ \delta \] | [-] | Perturbation \[ \kappa \] | [-] | Wave number \[ \lambda_p \] | [-] | Wave length \[ \lambda' \quad [1/m] \quad \text{Eigenvalue of } A' \]
| \[ \lambda'' \quad [1/m^2] \quad \text{Eigenvalue of } A'' \]
| \[ \lambda''' \quad [1/m] \quad \text{Eigenvalue of } A''' \]
| \[ \nu \quad [m^2/s] \quad \text{Kinematic viscosity} \]
| \[ \phi_i \] | [-] | Undamped angular frequency \[ \psi_{1,i} \] | [-] | First term of \( \phi_i^2 \)
| \[ \psi_{2,i} \] | [-] | Second term of \( \phi_i^2 \)
| \[ \omega \] | [-] | Angular frequency \[ \Delta_1 \] | [-] | Discriminant \[ \Delta t \quad [m] \quad \text{Time step} \]
| \[ \Delta x \quad [m] \quad \text{Particle spacing} \]
| \[ \overline{\Delta x} \quad [\cdot] \quad \text{Diagonal matrix equivalent to } A' \]
| \[ \overline{\Delta x}'' \quad [\cdot] \quad \text{Diagonal matrix equivalent to } A'' \]
| \[ \overline{\Delta x}''' \quad [\cdot] \quad \text{Diagonal matrix equivalent to } A''' \]
| \[ \Upsilon_i \quad [\cdot] \quad \text{Velocity perturbation in the Laplace domain} \]

11. References

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Litvinov, S., Hu, X., and Adams, N. (2015). Towards consistence and convergence of conservative SPH approximations. *Journal of Computational Physics*, 301:394–401.
Appendix A. Expressing perturbation equation as a matrix system

This section illustrates how the perturbation equations can be cast into a matrix system for a simplified case. Let us consider Eq. 10b where only the term proportional to $W''$ is present:

$$\frac{d^2 \delta x_a}{dt^2} = -2 \frac{p V}{p} \sum_b (\delta x_{ab}) W''_{ab}$$  \hspace{1cm} (A.1)

Focusing on particle 0 at the left of the domain (see Fig. 1) and considering a sphere of influence containing 5 particles (i.e. M=2), the equation of motion is explicitly expressed as:

$$\frac{p}{2p V} \frac{d^2 \delta x_0}{dt^2} = + \delta x_{N-2} W''(2 \Delta x)$$

$$+ \delta x_{N-1} W''(\Delta x)$$

$$- \delta x_0 \left[ W''(2 \Delta x) + W''(\Delta x) + W''(-\Delta x) + W''(-2 \Delta x) \right]$$

$$+ \delta x_1 W''(-\Delta x)$$

$$+ \delta x_2 W''(-2 \Delta x)$$  \hspace{1cm} (A.2)

Gathering all particle perturbations into the vector $X = (\delta x_0, \delta x_1, \delta x_2, ..., \delta x_{n-1})^T$, Eq. A.1 can be expressed as a matrix system:

$$\frac{d^2}{dt^2} (X) = 2 \frac{p V}{p} \tilde{A}'' X$$  \hspace{1cm} (A.3)

where the matrix $\tilde{A}''$ is given by:

$$\tilde{A}'' = \begin{bmatrix}
    c_0 & W''(-\Delta x) & W''(-2 \Delta x) & 0 & ... & 0 & W''(2 \Delta x) & W''(\Delta x) \\
    W''(\Delta x) & c_0 & W''(-\Delta x) & W''(-2 \Delta x) & 0 & ... & 0 & W''(2 \Delta x) \\
    W''(2 \Delta x) & W''(\Delta x) & c_0 & W''(-\Delta x) & W''(-2 \Delta x) & 0 & ... & 0 \\
    ... & ... & ... & ... & ... & ... & ... & ... \\
    W''(-2 \Delta x) & 0 & ... & 0 & W''(2 \Delta x) & W''(\Delta x) & c_0 & W''(-\Delta x) \\
    W''(-\Delta x) & W''(-2 \Delta x) & 0 & ... & 0 & W''(2 \Delta x) & W''(\Delta x) & c_0
\end{bmatrix}$$

with:

$$c_0 = -[W''(-2 \Delta x) + W''(-\Delta x) + W''(\Delta x) + W''(2 \Delta x)]$$  \hspace{1cm} (A.4)
\[ A'' \] is a circulant matrix which is entirely defined by one line or one column. For instance, the vector of the first row:

\[ C = (c_0, W''(-\Delta x), W''(-2\Delta x), 0, ..., 0, W''(2\Delta x), W''(\Delta x)) \]  

contains all the matrix coefficients. Circulant matrices enjoy the property to be diagonalizable in the same basis, composed of the n-th root of unity (Davis, 1979). Therefore, \[ A'' \] can written as:

\[ A'' = P^{-1} \Delta'' P \]  

with \[ P \] is the change-of-basis matrix of general term:

\[ p_{ij} = \frac{1}{\sqrt{N}} \exp \left( i \frac{2\pi}{N} ij \right) \]  

where \( i \) is the imaginary number. The diagonal matrix \( \Delta'' = \text{diag}(\lambda_0'', \lambda_1'', \lambda_2'', ..., \lambda_{N-1}'') \) in Eq. A.6 is expressed in the general case by:

\[ \lambda_j'' = \sum_{k=0}^{N-1} c_k'' \exp \left( i \frac{2\pi}{N} kj \right) \]  

### Appendix B. Validation of temporal solutions

This section provides a validation of the derivation of the perturbations equation for each of the four methods. Therefore the temporal solutions provided for each methods are compared to the results of a simple numerical scheme. Two types of perturbations are investigated. First, an inviscid flow (\( \nu = \alpha = 0 \)), where only the particle at the center of the domain is perturbed. Second, a viscous flow (\( \nu = 0.1 \), equivalently \( \alpha = 0.2 \) for \( \delta \)-SPH), where all particles are following an initial sinusoidal perturbation. In this case, two different modes are investigated.

In both cases, the initial perturbed value as well as the intensity of the perturbation was varied depending on the method (Table B.1). This was done in order to illustrate different initial configurations, but every SPH method was extensively tested with all types of initial perturbations. The domain is a periodic line composed of 63 particles (\( N=63 \)) and each particle has 6 neighbors (\( M=3 \)). The time integration of the numerical scheme is a simple Euler explicit scheme, and the time step \( \Delta t \) is selected according to:

\[ \Delta t = C_{\Delta t} \min(\Delta t_{\nu}, \Delta t_{CFL}) \quad \text{where} \quad \Delta t_{\nu} = \frac{1}{8} \frac{\Delta x^2}{\nu} \quad \text{and} \quad \Delta t_{CFL} = \frac{1}{4} \frac{\Delta x}{c + u_{\text{max}}} \]  

The constant \( C_{\Delta t} \) was introduced because the different SPH methods show different accuracies depending on the time integration. Its values are recalled in Table B.1. Please note that these differences of accuracy depicts the sensitivity of each method to the numerical scheme used to produce the reference temporal solution. The temporal solution provided by our derivation do not require any time integration. Other parameters are \( p_{bg}=1, \ c=1 \) and \( \gamma=7 \).

Specific parameters for \( \delta \)-SPH are \( \alpha = 2\nu, \ \xi = \alpha/4, \ \chi = \xi \) and \( \Gamma = 0.151 \). The latter was estimated as proposed by Antuono et al. (2010) by taking into account the background pressure in the equation of state. For TV-SPH, both physical and transport velocities are set to zero unless it is mentioned.

| Parameter | sum-SPH | div-SPH | \( \delta \)-SPH | TV-SPH |
|-----------|---------|---------|----------------|--------|
| Perturbed value | \( u \) | \( x \) | \( \rho \) | \( x \) |
| \( \delta_0 \) | 0.1 \( c \) | 0.1 \( \Delta x \) | 0.1 \( p \) | 0.1 \( \Delta x \) |
| \( C_{\Delta t} \) | 1 | 0.5 | 0.1 | 1 |
For sum-SPH and div-SPH, the analytical solution is directly compared to the results of the simulation. Therefore, Eqs. 38, 41 and 42 are compared to the numerical solution of the system 5 for sum-SPH, and Eqs. 53, 54 and 55 are compared to the numerical solution of the system 47 for div-SPH. Concerning δ-SPH and TV-SPH, no analytical solutions were derived. Therefore, the transfer functions in the Laplace domain were decomposed into simple complex rational functions using partial fraction decomposition for each temporal mode $i$:

$$P_i(p) = \frac{z_1}{p-z_2} + \frac{z_3}{p-z_4} + \frac{z_5}{p-z_6}$$

(B.2)

where $z_j$ are complex numbers. Then, each simple rational function is transformed into a temporal function using the identity:

$$\mathcal{L}^{-1} \left[ \frac{\alpha}{p-\beta} \right] = \alpha \exp(\beta t) \quad \text{for} \quad p > \text{Re}(\beta) \quad \text{and} \quad (\alpha, \beta) \in \mathbb{C}^2$$

(B.3)

The temporal functions are then transformed into the geometrical space using the general change-of-basis formula for circulant matrix:

$$F_{ij}(t) = \frac{1}{N} \left[ \sum_{k=0}^{N} f_k(t) \exp \left( i \frac{2\pi}{N} k(i-j) \right) \right]$$

(B.4)

This last operation is equivalent to expressing the matrices $A'$, $A''$ and $A'''$ into their original basis. One can observe that Eqs. 42 and 55 are particular cases of Eq. B.4. The functions $F_{ij}(t)$ are finally compared to the numerical solutions of the original system, Eqs. 56 for δ-SPH and Eqs. 63 for TV-SPH.

Appendix B.1. Case 1: inviscid flow with a single perturbed particle

In this subsection, the position of the initially perturbed particle ($\delta x_{31}$) and the particle which is located furthest away ($\delta x_0$) are shown. Figure B.25 (left) shows the comparison with sum-SPH, where the initial velocity was perturbed by 0.1c. The comparison with div-SPH is shown in Fig. B.25 (right). It is observed that the perturbed particle ($\delta x_{31}$) does not oscillate around the equilibrium position, but around a position shifted by $\approx -0.5$, as predicted by the constant term in Eq. 53a. The temporal solution shows a superposition of several modes, which are retrieved by our derivations. For both methods, the agreement is excellent, even for a large perturbation of 0.1c and $0.1\Delta x$. The time delay of the information propagation, visible on particle 0, is well taken into account.

Figure B.25: Comparison of LSA solution with a 1D SPH simulation. Left: sum-SPH. Right: div-SPH.

The comparison with δ-SPH and TV-SPH are shown in Fig. B.26. For δ-SPH, the perturbed position $\delta x_{31}$ of particle 31 was constant over the time so the position $\delta x_{30}$ of the next particle is presented. The slight
discrepancy for $\delta x_{30}$ is attributed to the sensitivity of the $\delta$-SPH method to the time integration. For instance, an initial perturbation of 0.01 $\rho g$ gives an excellent agreement. In the comparison with TV-SPH (Fig. B.26 right), $\delta x_0$ was artificially shifted by +0.5 for the sake of clarity. It also shows a good agreement between LSA and SPH simulation.

Figure B.26: Comparison of LSA solution with a 1D SPH simulation. Left: $\delta$-SPH. Right: TV-SPH, $(\pi, \bar{w})=(0,0)$, $\delta x_0$ was shifted of +0.5 for the sake of clarity.

Appendix B.2. Case 2: viscous flow with a spatial sinusoidal perturbation

For this test case, all particles are spatially perturbed according to a cosine function. Two different modes are investigated. With a wavelength of almost two time the diameter of the sphere of influence, mode 5 represents resolved oscillations. Mode 31 is investigated as it represents high frequency spatial oscillations which are prone to instability. Since in mode 31, every consecutive particle is shifted in the opposite 'direction', the perturbation amplitude is set to $\delta_0/2$ in this case compared to $\delta_0$ for mode 5.

The comparison between LSA and SPH simulation is presented in Figs. B.27 and B.28 where the agreement is excellent for all methods. For TV-SPH (Fig. B.28 right), the mode 31 is not damped, as it was highlighted by the dispersion curve discussion in Section 6.

Finally, in Fig. B.29 mode 5 and 31 are shown for TV-SPH with $(\pi, \bar{w})=(0.05,0.1)$ during a long time simulation. The instabilities exhibited in Fig. 20 are visible but with a larger growth rate in the 1D simulation compared to LSA. This deviation of growth rate is probably due to the first-order time integration scheme of the numerical methods.

Figure B.27: Left: sum-SPH. Right: div-SPH
As a conclusion of this validation part, the temporal solutions provided by our derivations perfectly match the solutions provided by the numerical resolution of the original systems of equations. In the case of a single perturbed particle, which is equivalent to an excitation by a white noise, the temporal solutions shows a superposition of all modes. This means that all modes predicted by our derivation are well retrieved in the numerical simulation. In addition, the caveats highlighted by the dispersion curves are also well illustrated by the numerical simulations. Indeed, the div-SPH method fails to damp a perturbation of position (Figs. B.25 right and B.27 right), while the TV-SPH method do not damp the mode of higher frequency (Figs. B.28 right).