Graviton scattering from classical matter

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Abstract

The low-energy scattering of gravitons from a composite extended system, which is made of classical massive bodies, is considered; by using the Feynman rules of effective quantum gravity, the corresponding cross-section is computed to lowest order in powers of the gravitational coupling constant. For the gravitons scattering from a rotating planet or a star, it is shown that the classical limit of the matter–graviton coupling in the effective quantum gravity Lagrangian leads to a low-energy scattering amplitude which coincides with the expression obtained in classical general relativity.

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1. Introduction

In the low-energy limit, the scattering of electromagnetic waves from free charged particles can be approximated by the Thomson scattering, in which the outcoming radiation can be interpreted as the radiation emitted due to the particle’s acceleration which is induced by the incoming wave. The resulting total cross-section is given by the Thomson formula \( \sigma = \frac{8\pi}{3} r_c^2 \), where \( r_c = \frac{q^2}{mc^2} \) represents the classical radius of the charged particles. Instead, the gravitons scattering from classical free massive bodies is expected to be dominated—in the same limit—by the Newtonian scattering, which is due to the gravitational attraction between gravitons and massive bodies. When the interaction potential vanishes at large distances as the inverse power of the distance, the total cross-section is divergent.

The low-energy dominance of the Newtonian scattering is in agreement with all the results—which have been obtained by means of classical arguments—concerning, for instance, the graviton scattering from black holes [1–5] and the low-energy gravitons scattering from a planet or a star [6–14]. The classical arguments are essentially based on the study of the linearized gravitational equations in a nontrivial background, possibly with the introduction...
One of the purposes of this paper is to produce the expression of the cross-section for the gravitons scattering from a composite extended system which is made of classical free massive bodies. The effective quantum field theory description of gravity will be used to derive the corresponding scattering amplitude. It will be shown that, in the appropriate semiclassical limit, the cross-section for the graviton scattering from a rotating star is also recovered; the result agrees with classical Peters formula [6] and with the expression obtained in classical general relativity.

In effective quantum gravity, the Newtonian contribution to the Compton graviton–scalar amplitude is described, in the Born approximation, by a single Feynman diagram containing one 3-graviton vertex and one graviton propagator in the $t$-channel. This single diagram (with the appropriate modifications in the external legs) is expected to give the dominant part of the low-energy scattering amplitude also in the case of gravitons scattering from a classical massive body such as a planet or a star. More precisely, the classical limit of the matter–gravitons coupling in the effective quantum gravity Lagrangian leads to a scattering amplitude which corresponds to a single diagram containing one 3-graviton vertex. However, the computation of this Feynman diagram which has been presented in [23] is not in agreement with Peters formula [6]; it is also not in agreement with the results obtained in classical general relativity and doubts [23] on the gauge invariance of the result have been raised. The second purpose of this paper is to clarify this issue and to show that, really, the contribution of the 3-graviton Feynman diagram (with modified external legs) to the transition amplitude is in complete agreement with Peters equation [6], it is in complete agreement with the results which have been obtained by means of classical arguments and represents the low-energy approximation of a gauge-invariant expression. This subject presents some interest because, since the 1-graviton exchange process contains one 3-graviton vertex, the gravitons scattering from classical matter at low energy provides a test of the non-linear structure of the equations which describe the dynamics of the gravitational field in general relativity.

This paper is organized as follows. In section 2, the scattering of gravitons from a classical matter system made of a set of dust particles is considered and the corresponding cross-section is computed, at the first order in powers of the gravitational coupling constant, by means of the effective quantum field theory formalism. The gauge-invariant transition amplitude of the process is written as a linear combination of the amplitudes which refer to the gravitational scattering from the elementary constituents of the dust, which are approximated by spinless massive particles. The low-energy behaviour of the amplitude is considered and its expression is produced as a function of the velocities of the massive bodies. The corrections to the geometric optics approximation are computed at the first order in powers of the graviton momentum transfer and at the first order in powers of the velocities of the massive bodies; the corresponding transition amplitude only depends on the total energy and total angular momentum of the matter system and coincides with the amplitude for the graviton scattering from a rotating planet or star. In section 3 it is shown that the same result can also be obtained by considering, in the quantum field theory approach, the classical limit for the Lagrangian matter–graviton coupling; in this case, the transition amplitude is given by a single Feynman diagram with precisely one 3-graviton vertex and the result coincides with the expression obtained in classical general relativity.
2. Cross-section for the graviton scattering

Let us consider the gravitons scattering from a classical matter system made of dust particles; as depicted in figure 1, this system can be represented by a collection of free moving classical massive objects. It is assumed that the motion of each elementary constituent of the dust is influenced only by the gravitational field. It is also assumed that the typical size $L$ of each particle is small compared to the characteristic wavelength $\lambda$ of the gravitational wave, $L/\lambda \ll 1$, so that, as far as the graviton scattering is concerned, each constituent of the dust can be approximated by a pointlike particle. Since we are interested in low-frequency gravitational waves, depending on the value of $\lambda$ the massive particles system could consist of a Boltzmann molecular gas, or it could be made by a large number of massive bodies with the size varying from the micron scale up to a few kilometres or, possibly, up to the planet length scale.

The gravitational scattering amplitude can be written as a sum of amplitudes for the scattering of one graviton from a single classical particle of dust. The expression of the amplitude for the elementary graviton–particle scattering can be obtained by taking the low-energy limit of the so-called Compton graviton–scalar amplitude computed in the Born approximation.

2.1. A single particle scattering

The coupling of gravitons with massive scalar particles is described by the action $S$ for the gravitational and matter fields; $S$ is the sum of the action of a minimally coupled massive scalar field and the Einstein–Hilbert action of general relativity

$$ S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} [g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - m^2 \varphi^2] - \frac{16 \pi G}{\lambda^2} R(x) \right\}. $$

(1)

One can put $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$, where $\eta_{\mu\nu}$ denotes the Minkowski flat metric and $h_{\mu\nu}(x)$ represents the small fluctuation of the metric. The expansion of the functional (1) in powers of $h_{\mu\nu}$ provides the interaction vertices which can be used to compute the amplitude for the graviton scattering [24–26]. The term of this expansion which is quadratic in $h_{\mu\nu}$ together with the gauge-fixing Lagrangian determines the form of the graviton propagator. To the lowest order in powers of the gravitational coupling constant $G$, the Compton amplitude $A_C$ is given by the sum of the contributions which are associated with the Feynman diagrams shown in figure 2

$$ A_C = A_{(a)} + A_{(b)} + A_{(c)} + A_{(d)}. $$

(2)
For generic values of the particle momenta, in order to recover a gauge-invariant amplitude one has to sum the contributions of all the diagrams shown in figure 2.

In what follows, we shall use the standard units in which \( \hbar = c = 1 \). Let us denote by \( p_1 = (E_1, \vec{p}_1) \) and \( p_2 = (E_2, \vec{p}_2) \) the initial and final momenta of the massive scalar particle; the incoming graviton has momentum \( k_1 = (\omega_1, \vec{k}_1) \) and polarization tensor \( \epsilon_{\mu \nu} \), with \( \epsilon^\nu_\mu = 0 = k_1^\mu \epsilon_{\mu \nu} \), whereas the outgoing graviton has momentum \( k_2 = (\omega_2, \vec{k}_2) \) and polarization \( \pi_{\nu \mu} \), with \( \pi^\nu_\mu = 0 = k_2^\nu \pi_{\nu \mu} \). The amplitude \( A(a) + A(b) \), which is associated with the diagrams (a) and (b) of figure 2, is given by

\[
A(a) + A(b) = 4G \left\{ \frac{(p + k)^\mu (p + k)^\nu (p - k)^\xi (p - k)^\tau}{2(p \cdot q + k^2)} \pi^{*}_\mu \epsilon_{\nu \tau} \right. \\
- \left. \frac{(p - k)^\mu (p - k)^\nu (p + k)^\xi (p + k)^\tau}{2(p \cdot q - k^2)} \pi^{*}_\nu \epsilon_{\mu \tau} \right\}; 
\tag{3}
\]

the amplitude \( A(c) \) corresponding to the diagram (c) is

\[
A(c) = 8G \left\{ \frac{p^\mu p^\nu \pi^{*}_\mu \epsilon_{\nu \sigma} - \frac{3}{4} \eta^{\mu \nu} k^2 \pi^{*}_\nu \epsilon_{\mu \sigma}}{k^2} \right\}; 
\tag{4}
\]

finally the amplitude \( A(d) \), which is represented by the diagram (d), reads

\[
A(d) = 4G \left\{ \frac{(p \cdot q)^2}{4k^2} \pi^{*}_\nu \epsilon_{\mu \sigma} \right. \\
- \left. \frac{k^\mu k^\nu - \frac{3}{4} \eta^{\mu \nu} k^2}{k^2} \pi^{*}_\mu \epsilon_{\nu \sigma} - \frac{p^\mu p^\nu k^\sigma k^\tau}{k^2} \left[ \frac{p^\nu k^\sigma k^\tau}{k^2} \pi^{*}_\sigma \epsilon_{\tau \mu} + \pi^{*}_\tau \epsilon_{\sigma \mu} \right] \right\}; 
\tag{5}
\]

where the three independent momenta \( p, q \) and \( k \) are defined by

\[
p = \frac{1}{2}(p_1 + p_2), \quad q = \frac{1}{2}(k_1 + k_2), \quad k = \frac{1}{2}(p_1 - p_2) = \frac{1}{2}(k_2 - k_1), 
\tag{6}
\]

and the scattering angle \( \theta \) is determined by \( k^2 = -\omega_1 \omega_2 \sin^2 \theta / 2 \). The individual terms \( A(a), A(b), A(c) \) and \( A(d) \) are not gauge invariant but their sum is. In the present context, gauge invariance means that \( A_C \) vanishes when \( \epsilon^{\mu \nu} \) is replaced by \( k_1^\mu \xi^\nu + \xi^\mu k_1^\nu \) with arbitrary \( \xi^\mu \) (and similarly for \( \pi_{\mu \nu} \)). To sum up, the Lorentz-invariant and gauge-invariant scattering amplitude \( A_C \) turns out to be (see for instance \([18]\))

\[
A_C = 4G \left\{ \frac{(p \cdot q)^2}{4k^2} \pi^{*}_\mu \epsilon_{\nu \sigma} \right. \\
+ \left. \frac{(p + k)^\mu (p + k)^\nu (p - k)^\xi (p - k)^\tau}{2(p \cdot q + k^2)} \pi^{*}_\mu \epsilon_{\nu \tau} + \frac{2 p^\mu p^\nu k^\sigma k^\tau}{k^2} \pi^{*}_\nu \epsilon_{\sigma \mu} \right\}; 
\tag{7}
\]
2.2. Low-energy limit

We are interested in the case in which the graviton momenta $k_1$ and $k_2$ are small compared to the momentum of the massive particle, so that in the semiclassical—or large quantum numbers—limit for each particle of dust, one can neglect the variation of the momentum of the massive particle and one can put $p_1 \simeq p_2 \simeq p = E(1, \vec{v})$, where $E$ represents the energy of the particle and $\vec{v}$ denotes its velocity. We shall also concentrate on the coherent component of the gravitational scattering, in which the frequency of the outgoing gravitational wave is equal to the frequency of the incoming wave. Then one has $k_1 \simeq \omega(1, \vec{n}_1)$ and $k_2 \simeq \omega(1, \vec{n}_2)$, where $\vec{n}_1$ and $\vec{n}_2$ are the unit vectors representing the directions along which the incoming and outgoing gravitational waves propagate.

For fixed energy $E$ and fixed scattering angle $\theta$, let us consider the low energy ($\omega/E \to 0$) limit. In the Taylor expansion of the scattering amplitude (7) in powers of $(\omega/E)$, the leading term $B$ does not depend on $\omega$; $B$ represents the effective amplitude which describes the low-energy graviton scattering from a classical particle of dust. When the nontrivial components of the polarization tensors $\pi_{\mu\nu}$ and $\epsilon_{\mu\nu}$ are of spatial type, the expression of $B$ up to the second order in powers of the velocity components $\vec{v}$ of the massive particle, is given by

$$B(E, \vec{v}) = \frac{G E^2}{\sin^2 \theta/2} \left\{ \pi_{ij}^* e^{ij} - \vec{v} \cdot (\vec{n}_1 + \vec{n}_2) \pi_{ij}^* e^{ij} + 2v' n_i^* n_j^* \epsilon_{jk} + 2v' n_i^* \pi_{jk}^* \epsilon_{ik} \right. \right.$$

$$+ \frac{1}{2} (\vec{v} \cdot \vec{n}_1)^2 + (\vec{v} \cdot \vec{n}_2)^2 \pi_{ij}^* e^{ij} - 2(\vec{v} \cdot \vec{n}_1)v' n_i^* \pi_{jk}^* \epsilon_{jk}$$

$$- 2(\vec{v} \cdot \vec{n}_1)v' n_i^* \pi_{jk}^* \epsilon_{ik} - 4v' v' \pi_{jk}^* \epsilon_{jk} \sin^2 \theta/2 + 2v' v' n_i^* n_j^* \pi_{jk}^* \epsilon_{jp}$$

$$+ \left. v' v' \left[ n_i^* n_j^* \pi_{jk}^* \epsilon_{kp} + n_i^* n_j^* \pi_{kp}^* \epsilon_{ij} \right] \right\}.$$  

(8)

Latin indices $i, j, \ldots$ are used to denote the spatial components of the Lorentz vectors or tensors and take the values $i, j, \ldots = 1, 2, 3$. The sum over repeated Latin indices is Euclidean, i.e. $a_i b^i = a^i b_j = a_i b_j = a^j b^i = a_1 b_1 + a_2 b_2 + a_3 b_3 = \vec{a} \cdot \vec{b}$.

In order to display the low-energy contribution of each Feynman diagram to the scattering amplitude, let us denote by $B_{(a)}$ the leading term of $A_{(a)}$ in the $(\omega/E) \to 0$ limit. One finds

$$B_{(a)} + B_{(b)} = 16 G E^2 \sin^2 \theta/2 \left[ \frac{v' v' v' v'}{2 - \vec{v} \cdot (\vec{n}_1 + \vec{n}_2)} \right] \pi_{ij}^* e^{ij} = O(v^3),$$

(9)

$$B_{(c)} = -8 G E^2 v' v' n_i^* \pi_{jk}^* \epsilon_{jk} = O(v^3),$$

(10)

$$B_{(d)} = \frac{G E^2}{\sin^2 \theta/2} \left\{ \pi_{ij}^* e^{ij} - \vec{v} \cdot (\vec{n}_1 + \vec{n}_2) \pi_{ij}^* e^{ij} + 2v' n_i^* \pi_{jk}^* \epsilon_{jk} + 2v' n_i^* \pi_{jk}^* \epsilon_{ik} \right. \right.$$

$$+ \frac{1}{2} (\vec{v} \cdot \vec{n}_1)^2 + (\vec{v} \cdot \vec{n}_2)^2 \pi_{ij}^* e^{ij} - 2(\vec{v} \cdot \vec{n}_1)v' n_i^* \pi_{jk}^* \epsilon_{jk}$$

$$- 2(\vec{v} \cdot \vec{n}_1)v' n_i^* \pi_{jk}^* \epsilon_{ik} + 4v' v' \pi_{jk}^* \epsilon_{jk} \sin^2 \theta/2 + 2v' v' n_i^* n_j^* \pi_{jk}^* \epsilon_{jp}$$

$$+ \left. v' v' \left[ n_i^* n_j^* \pi_{jk}^* \epsilon_{kp} + n_i^* n_j^* \pi_{kp}^* \epsilon_{ij} \right] \right\}$$

$$= \frac{G E^2}{\sin^2 \theta/2} \left\{ \pi_{ij}^* e^{ij} - \vec{v} \cdot (\vec{n}_1 + \vec{n}_2) \pi_{ij}^* e^{ij} + 2v' n_i^* \pi_{jk}^* \epsilon_{jk} + 2v' n_i^* \pi_{jk}^* \epsilon_{ik} \right. \right.$$}

$$+ 2v' n_i^* \pi_{jk}^* \epsilon_{jk} + 2v' n_i^* \pi_{jk}^* \epsilon_{ik} \right\} + O(v^3).$$  

(11)
The sum $B(a) + B(b) + B(c) + B(d)$, up to the second order in powers of the velocity components $\vec{v}$, coincides with expression $B(E, \vec{v})$ shown in equation (8). It is important to note that the amplitude contribution $B(a) + B(b)$, which is associated with the exchange (in the $s$- and $u$-channels) of the massive particle, is at least of the fourth order in powers of the velocity. The contact term $B(c)$ is quadratic in the velocity. The zero-order term and first-order term—in powers of the velocity—originate from the graviton exchange in the $t$-channel exclusively, amplitude $B(d)$. Finally, the $[\sin^2 \theta/2]^{-1}$ factor in front of expression (11), which is due to the 1-graviton exchange, is strictly connected with the presence of the Newtonian potential in gravitational interactions.

2.3. Many-particle scattering

In order to produce the amplitude $A$ for the graviton scattering from a many-particle system, it is convenient to consider first, in the case of a single-particle scattering, the wave packets representing the quantum-mechanical states of the massive particle. The wavefunction of the initial state of the particle can be represented, for instance, by a Gaussian wave packet that (at a fixed time) corresponds to an average momentum $\vec{p}_1$ and an average position $\vec{x}$,

$$\psi_{\text{in}}(\vec{p}) \sim e^{-a|\vec{p}-\vec{p}_1|^2} e^{-i\vec{p} \cdot \vec{x}}.$$  

(12)

Similarly, the wavefunction of the final state is of the type

$$\psi_{\text{out}}(\vec{p}) \sim e^{-b|\vec{p}-\vec{p}_2|^2} e^{-i\vec{p} \cdot \vec{x}}.$$  

(13)

In the limit in which the Gaussian wavefunctions become delta functions concentrated on the momenta $\vec{p}_1$ and $\vec{p}_2$, the two phase factors which are related to the position of the particle give origin to the following contribution:

$$e^{i(\vec{k}_2 - \vec{k}_1) \cdot \vec{x}} = e^{i(\vec{k}_1 - \vec{k}_2) \cdot \vec{x}},$$  

(14)

which acts as a multiplicative factor on amplitude (8). For a single-particle scattering, the presence of factor (14) can be ignored, but in the case of a many-particle scattering the position-dependent phase (14) has to be taken into account.

It should be noted that, for our purposes, the relativistic covariant generalization of the multiplicative phase factor (14) is not needed. In fact, for the low-energy coherent scattering of gravitons, the nonvanishing components of the momentum transfer are of spatial type, $k_2 - k_1 \simeq \omega(\hat{n}_2 - \hat{n}_1)$. The same conclusion can also be obtained by means of the following argument. The leading order of the low-energy approximation in which $\omega_1 \simeq \omega_2$ is based on the assumption that, during the scattering process, the relevant global variables which are associated with the many-particle system are essentially (or, can essentially be considered) constant in time; consequently, the scattering can be understood as a stationary process in which the graviton energy is (in the first approximation) conserved. In the stationary process, then, only the relative spatial positions of the different dust particles, as described by the phase factor (14), can appear in the expression of the amplitude.

Let each particle of dust be labelled by the index $a$; $E_a$ and $\vec{v}_a$ represent the energy and the velocity of the $a$th particle. By taking into account the position-dependent phase factor (14) and the normalization factor $E_a^{-1}$ which must multiply expression (8), the resulting amplitude for the many-particle scattering is

$$A = \sum_a \frac{1}{E_a} B(E_a, \vec{v}_a) e^{i(\vec{k}_1 - \vec{k}_2) \cdot \vec{x}_a}. $$  

(15)

The coordinate system can always be chosen so that

$$\sum_a E_a \vec{v}_a = 0 = \sum_a E_a \vec{x}_a. $$  

(16)
and the graviton cross-section takes the form

\[ \frac{d\sigma}{d\Omega} = |A|^2 = \left| \sum_{a} \frac{1}{E_a} B(E_a, \tilde{v}_a) e^{i(k_1 - k_2) \cdot \tilde{x}_a} \right|^2. \]  

(17)

Equation (17) gives the required expression of the cross-section for the gravitons scattering from a composite classical matter system.

When the spatial extension \( D \) of the system is bigger than the wavelength of the gravitational wave, \( D > \lambda \), the exponential factor in (17) has large fluctuations and the total cross-section can be approximated by a sum of cross-sections for the different parts of the system. On the other hand, when \( D < \lambda \), one can use the approximation \( e^{i(k_1 - k_2) \cdot \tilde{x}_a \sim 1} \). In this case, one obtains

\[ A = \frac{GE_0}{\sin^2 \theta/2} \{ \pi^*_i \epsilon^{ij} \left[ 1 + \frac{1}{2} W_{ij} (n^i_1 n^j_1 + n^i_2 n^j_2) + \Lambda (1 - 4 \sin^2 \theta/2) \right] \]

\[ - 2 W^i_0 n^i_1 n^j_1 \pi^{*j}_1 \epsilon^{ik} - 2 W^j_0 n^j_1 n^i_1 \pi^{*i}_1 \epsilon^{jk} + 2 W^i_0 n^i_2 n^j_2 \pi^{*j}_2 \epsilon^{ik} + 2 \Lambda n^i_1 n^j_2 \pi^{*j}_2 \epsilon^{ik} \]

\[ - 4 W^i_0 \pi^{*j}_1 \epsilon^{jk} \sin^2 \theta/2 + W^j_0 \left[ n^i_2 n^j_2 \pi^{*j}_2 \epsilon^{ik} + n^i_1 n^j_2 \pi^{*j}_1 \epsilon^{ij} \right], \]

(18)

where

\[ \sum_a E_a = E_0, \quad \sum_a E_a v^j_a = E_0 (W^{ij} + \Lambda \delta^{ij}), \]

(19)

with \( W^{ij} = W^{ij} \) and \( W^j = 0 \).

2.4. Corrections of the first order in momentum transfer

Let us now take into account the corrections to the scattering amplitude (18) which are of the first order in powers of the graviton momentum transfer; that is, let us put \( e^{i(k_1 - k_2) \cdot \tilde{x}_a \sim 1} + i \) in equation (15). The part of the amplitude which is proportional to the momentum transfer gives the first correction to the geometric optics approximation. For slowly moving particles, one can neglect the terms with two or more powers of the velocity components \( \tilde{v}_a \), and from equation (15) one finds

\[ A = \frac{GE_0}{\sin^2 \theta/2} \{ \pi^*_i \epsilon^{ij} \left[ 1 + i \omega \varepsilon_{ikl} S^k \right] \]

\[ - i \omega \varepsilon_{ikl} S^n \left( n^i_2 - n^i_1 \right) \left[ n^j_2 \pi^{*j}_2 \epsilon^{ik} + n^j_1 \pi^{*j}_1 \epsilon^{ik} \right] \}, \]

(20)

where \( \varepsilon_{ijk} \) denotes the completely antisymmetric 3-tensor and \( J^i = E_0 S^i \) represents the angular momentum of the many-particle system,

\[ \sum_a E_a = E_0, \quad \sum_a E_a x^i_a v^j_a = \frac{1}{2} \varepsilon^{ijk} J^k = \frac{1}{2} E_0 \varepsilon^{ijk} S^k. \]

(21)

It should be noted that, in addition to the antisymmetric component (21), the sum \( \sum_a E_a x^i_a v^j_a \) could also contain a symmetric part \( Q^{ij} = Q^{ji} \). But since \( Q^{ij} \) corresponds to a total time derivative, \( Q^{ij} = (d/dt) \frac{1}{2} \sum_a E_a x^i_a x^j_a \), it gives a vanishing contribution to the scattering amplitude in the low-energy stationary approximation.

When \( \tilde{v}_1 \) is directed as \( \hat{J} \), the effect of the gravitational elicity interaction [12] is maximal. In fact, from equation (20) it follows that the difference of the cross-sections for the two different helicities \( \pm \) of the graviton becomes (in the \( \theta \to 0 \) limit)

\[ \frac{(d\sigma_+/d\Omega) - (d\sigma_-/d\Omega)}{(d\sigma_+/d\Omega) + (d\sigma_-/d\Omega)} \simeq - \frac{2 \omega J \theta^2}{E_0}. \]

(22)
The important point now is that amplitude (20) only depends on the total energy \( E_0 \) of the dust system—which is globally at rest—and on its total angular momentum \( J \). Therefore, expression (20) should also represent the amplitude—computed in the Born approximation—for the scattering of gravitons from a rotating star or planet of mass \( M = E_0 \) and angular momentum \( J \). In fact, equation (20) is in agreement with the results which have been obtained, by means of classical methods (see for instance \([2, 5, 6, 12–14]\)), for the transition amplitude associated with the gravitons scattering from a rotating star. As a check, let us consider the scattering of unpolarized gravitons from a massive body with \( J = 0 \); equations (17) and (20) give

\[
\frac{d\sigma}{d\Omega} = \frac{G^2 M^2}{\sin^4 \theta/2} (\cos^8 \theta/2 + \sin^8 \theta/2), \tag{23}
\]

which coincides precisely with Peters formula [6].

### 3. Classical matter coupling

By construction, amplitude (20) represents the semiclassical approximation of a complete gauge-invariant transition amplitude for the low-energy scattering of gravitons. Expression (20) is the sum of the terms of the zero-order and first-order in powers of the velocity components of the dust particles. It has been shown in section 2.2 that these two contributions originate from the 1-graviton exchange diagram of figure 2(d) exclusively. So one expects that, by taking the classical limit of the matter–gravity coupling in the Lagrangian of effective quantum gravity, amplitude (20) for the gravitational scattering of gravitons from a rotating massive body could also be obtained by means of a single Feynman diagram containing a 1-graviton exchange.

This possibility has already been considered in the literature by De Logi and Kovács [23], but the result produced in [23] is not in agreement with equation (20) and with Peters formula (23). As admitted in equation (5.7) of [23], the cross-section \((d\sigma/d\Omega)_{DLK}\) of De Logi and Kovács is related to the cross-section \((d\sigma/d\Omega)_P\) of Peters according to the equation

\[
(d\sigma/d\Omega)_{DLK} = \cos^2 \theta (d\sigma/d\Omega)_P.
\]

We shall now discuss this subject and do the calculation again; firstly, the classical matter coupling in the effective quantum gravity Lagrangian will be considered, then the corresponding scattering amplitude will be computed. As a matter of fact, it turns out that the final result coincides with expression (20) and is in complete agreement with Peters formula (23), as should be.

In the large distance limit and to the first order in \( v/c \), the coupling of the fluctuation field \( h_{\mu\nu}(x) \) with a classical heavy rotating body, which is subject to stationary conditions and is placed in position \( \vec{r} = \vec{0} \), is given—at the first order in \( h_{\mu\nu} \)—by the action term

\[
S_m = \frac{1}{2} \int d^4x \Theta^{\mu\nu}(\vec{x}) h_{\mu\nu}(x) = \frac{1}{2} \int d^4x [Mh_{00}(x) + \epsilon^{ijk} J_i \partial_j h_{0k}(x)] \delta^3(\vec{x}), \tag{24}
\]

where \( M \) denotes the mass of the body and \( \{J_i\} \) are the components of its total angular momentum. Coupling (24) is in agreement with the expression of the metric which is induced by the presence of a rotating massive body [27, 28]. Since the energy–momentum tensor \( \Theta^{\mu\nu} \) of the massive body which appears in equation (24) is conserved, the interaction term \( S_m \) is invariant under infinitesimal gauge transformations acting on \( h_{\mu\nu} \).

When the tensor \( \Theta^{\mu\nu} \) represents a fixed classical background field, \( \Theta^{\mu\nu} \) is no longer a dynamical variable and then it does not transform under diffeomorphisms; consequently, the action \( S_m \) together with the correction terms with higher orders in powers of \( h_{\mu\nu} \) are
not invariant under general coordinate transformations. This is just what happens in any non-Abelian gauge theory in the presence of a generic nontrivial background. In order to clarify the connection between gauge invariance and semiclassical limit, let us recall the two possibilities:

(1) one firstly computes the complete gauge-invariant transition amplitude, and then takes the classical limit in the appropriate variables;

(2) one takes the appropriate classical limit directly in the interaction Lagrangian, and subsequently computes the transition amplitude.

Method (a) always gives the correct answer for the quantities which are really observed in laboratories; this is precisely the way in which expression (20) has been derived in section 2. Method (b) generally breaks gauge invariance and leads to wrong conclusions, but there are exceptions. In fact, in this section it will be shown that, as far as the computation of the transition amplitude (at the first order in $\frac{v}{c}$) for the gravitons scattering from a rotating star is concerned, method (b) produces the correct answer (20). As discussed in section 2.2 and at the beginning of this section, this is essentially a consequence of the low-energy behaviour of the amplitude components (9)–(11).

The expansion of Lagrangian (1) in powers of $h_{\mu\nu}$ determines the 3-graviton vertex

$$S_{EH}^{(3)} = \frac{1}{32\pi G} \int d^4x \left[ \partial_\nu h_\mu^\nu \partial^\mu h_{\sigma\tau} h^{\sigma\tau} + \frac{1}{4} \partial_\mu h_{\sigma\tau} \partial^\mu h^{\sigma\tau} h_\nu^\nu \right. $$

$$- \frac{1}{2} \partial_\mu h_{\nu\sigma} \partial^\mu h_\nu^\nu - \frac{1}{4} \partial_\mu h_\nu^\nu \partial^\mu h_\sigma^\sigma - \partial_\mu h_\nu^\nu \partial^\nu (h^{\mu\sigma} h_\sigma^\sigma) + \frac{1}{2} \partial_\mu h_{\nu\sigma} \partial^\nu h_\nu^\nu + \frac{1}{2} \partial_\mu h_{\nu\sigma} \partial^\nu h_\sigma^\sigma - \frac{1}{2} \partial_\mu h_{\nu\sigma} \partial^\nu h_\sigma^\sigma \left. \right].$$

With a covariant gauge fixing, the graviton propagator is given by

$$h_{\mu\nu}(x)h_{\sigma\tau}(y) = i(16\pi G) \int \frac{d^4p}{(2\pi)^4} \frac{e^{-i(q-x-y)}}{p^2 + i\epsilon} \left\{ \delta^\mu_\nu \delta^\sigma_\tau + \delta^\mu_\sigma \delta^\nu_\tau - \eta^\mu_\nu \eta^\sigma_\tau \right. $$

$$+ \frac{\alpha - 1}{p^2 + i\epsilon} \left[ \delta^\mu_\nu p^\sigma p_\sigma + \delta^\mu_\sigma p^\nu p_\tau + \delta^\nu_\tau p^\mu p_\sigma + \delta^\sigma_\tau p^\nu p_\mu \right] \right\},$$

where $\alpha$ represents the gauge parameter; the choice $\alpha = 0$ is the analogue of the Landau gauge in electrodynamics, whereas $\alpha = 1$ is the analogue of the Feynman gauge. In our computations, $\alpha$ is left free. With the classical matter coupling (24), the amplitude $A$ for the gravitons scattering is determined by the 1-graviton exchange diagram shown in figure 3,

$$A = i\phi^2(k_2, \pi)(S_{EH}[h, h, H]|k_1, \epsilon),$$

and takes the form

$$A = \frac{i2GM\omega_1 \delta(\omega_1 - \omega_2)}{\pi} \left\{ \pi_{\epsilon^j}^i e^{ij} \left[ 1 + i(J^k / \omega_1 M)\epsilon_{k\ell m} k_2^\ell k_1^m \right] \right. $$

$$- i(J^n / \omega_1 M)(\vec{k}_2 - \vec{k}_1)n\pi_{\epsilon^j}^i \epsilon_{\ell}^j \left( (\vec{k}_2 - \vec{k}_1) \epsilon_{\ell m n} - (\vec{k}_2 - \vec{k}_1) \epsilon_{\ell n m} \right) \right\}{(28)}.$$

By means of the definition

$$A = \frac{i\phi(\omega_1 - \omega_2)}{2\pi \omega_1} A, \quad (29)$$

and the relationships

$$\epsilon_{\ell m n} = \epsilon_{\ell n m},$$

$$\epsilon_{\ell m n} = -\epsilon_{\ell n m},$$

$$\epsilon_{\ell m n} = \epsilon_{\ell n m},$$
the graviton scattering cross-section is given by

\[
\frac{d\sigma}{d\Omega} = |A|^2, \tag{30}
\]

and, from expression (28), one finds that the reduced amplitude \(A\) is given by

\[
A = \frac{GM}{\sin^2(\theta/2)} \left[ \pi^{\alpha\beta} e^{ij} \left\{ 1 + i(S^m/\omega) e_{mjk} k^j_1 k^k_1 \right\} \right.
- i(S_n/\omega)(\vec{k}_2 - \vec{k}_1)_m \pi^{\alpha\beta} e^{ij} (\vec{k}_2 - \vec{k}_1)_j e_{imn} - (\vec{k}_2 - \vec{k}_1)_j e_{imn} \left\} \right], \tag{31}
\]

where \(S_k = J_k/M\) and \(|\vec{k}_2 - \vec{k}_1|^2 = 4\omega^2 \sin^2(\theta/2)\). For the scattering of gravitons from a classical rotating body, there are no additional contributions to the transition amplitude. In fact, the classical matter coupling at the second order in powers of \(h_{\mu\nu}\) takes the form

\[
\frac{1}{4} \int d^4x \left\{ \left( M/2 \right) h^2_{00}(x) + \epsilon^{ijk} J_i \partial_j (h_{00}(x) h_{0k}) \right\} \delta^3(\vec{x})
\]

and gives a vanishing contribution for on-shell gravitons. Similarly, there are no nontrivial corrections to the amplitude coming from the gauge-fixing Lagrangian.

By taking into account the relations \(\omega_1 = \omega_2 = \omega\), \(\vec{k}_1 = \omega \vec{\hat{n}}_1\) and \(\vec{k}_2 = \omega \vec{\hat{n}}_2\), expression (31) can be written as

\[
A = \frac{GM}{\sin^2(\theta/2)} \left[ \pi^{\alpha\beta} e^{ij} \left\{ 1 + i\omega e_{klp} S^k n^p_1 \right\} \right.
- i\omega e_{klp} S^p \left( n^k_2 - n^k_1 \right) \left\{ n^j_1 \pi^{\alpha\beta} e^{ij} + n^j_1 \pi^{\alpha\beta} e^{ik} \right\} \right], \tag{32}
\]

which coincides with equation (20) in which \(E_0 = M\).

To sum up, because of the low-energy dominance of the Newtonian scattering, the transition amplitude (20) for the gravitons scattering from a macroscopic rotating massive body can also be obtained by means of the single Feynman diagram shown in figure 3. Therefore, the amplitude which corresponds to the Feynman diagram of figure 3 is really in complete agreement with Peters formula (23) and represents the semiclassical approximation of a gauge-invariant transition amplitude for the low-energy scattering of gravitons.

4. Conclusions

In this paper, the cross-section for the scattering of gravitons from an extended macroscopic system made of classical massive bodies has been derived. By means of the effective quantum gravity formalism, the gauge-invariant scattering amplitude has been computed in the Born approximation and the low-energy limit has been discussed. With the inclusion of the corrections of the first order in powers of the graviton momentum transfer, the transition amplitude of the first order in velocities only depends on the total energy and total angular momentum of the matter system and coincides also with the amplitude for the graviton...
scattering from a rotating planet or star. It has been shown that the same result can also be obtained by considering, in the effective quantum gravity approach, the classical limit for the Lagrangian matter–graviton coupling. In this case, the transition amplitude corresponds to a single Feynman diagram with one 3-graviton vertex and the result coincides with the expression obtained in classical general relativity.

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