Phonon-mediated Casimir interaction between finite mass impurities

Andrei I. Pavlov, Jeroen van den Brink, and Dmitri V. Efremov
Institute for Theoretical Solid State Physics, Leibniz-Institut für Festkörper- und Werkstoffforschung IFW-Dresden, D-01169 Dresden, Helmholtzstræe 20, Germany

The Casimir effect, a two-body interaction via vacuum fluctuations, is a fundamental property of quantum systems. In solid state physics it emerges as a long-range interaction between two impurity atoms via virtual phonons. In the classical limit for the impurity atoms in D dimensions the interaction is known to follow the universal power-law $U(r) \sim r^{-D}$. However, for finite masses of the impurity atoms on a lattice, it was predicted to be $U(r) \sim r^{-2D-1}$ at large distances. We examine how one power-law can change into another with increase of the impurity mass and in presence of an external potential. We provide the exact solution for the system in one-dimension. At large distances indeed $U(r) \sim r^{-3}$ for finite impurity masses, while for the infinite impurity masses or in an external potential it crosses over to $U(r) \sim r^{-1}$. At short distances the Casimir interaction is not universal and depends on the impurity mass and the external potential.

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Casimir in his pioneering work [1] has shown that a change of the zero-point energy due to a perturbation of the electromagnetic fluctuations by two neutral metallic plates, leads to observable forces between these plates. In fact, this is only one example of the broad class of phenomena, which are based on the concept of perturbation of the long-range fluctuations, e.g. Goldstone modes in the media with broken symmetry. Nowadays this effect, named after Casimir, can be encountered in various fields of physics, chemistry and biology [2-8]. For instance, in high energy physics the existence of Casimir effect sets natural constraints on the Yukawa forces, appearing due to the exchange of light elementary particles and/or extra-dimensional physics [9]. In cosmology the Casimir effect helps to interpret the cosmological constant for a scalar field [10-12]. In chemistry, in particular, it is used to explain the interactions of molecules [13-14]. In biology the Casimir interaction is for instance found to be responsible for organization of the bilayer structure of cell membranes [15].

In condensed matter physics the effect of the Casimir interaction is extensively discussed with respect to interaction of conducting surfaces [16], graphene and conducting plates [17], mesoscopic particles in a critical fluid through critical fluctuations [18] and ultracold atomic gases [19-21]. In the latter case it is possible to study the Casimir interaction in ultraclean bosonic or fermionic gases on tunable lattices with tunable spatial dimensionality and interaction strength. In this context one dimensional setups attract the most attention since the fluctuations are the strongest in 1D.

Precisely this situation was considered in [19-21]. The authors studied the interaction between two static impurities due to perturbation of phonon spectra in a Luttinger liquid. Since the mechanism is similar to the one proposed by Casimir, we hereafter denote it as the Casimir interaction. The examination of the energy of zero-point motion of the Luttinger liquid in the presence of two impurities yielded the Casimir interaction $U(r) \sim -1/r$. This dependence can be easily understood considering the zero-point energy of phonons in a potential well formed by two static impurities. The direct calculation leads to the following expression for the Casimir interaction [22, 23]:

$$U(r) = -\frac{c\pi}{24r},$$  \hspace{1cm} (1)

(here and below we use $\hbar = 1$).

At the same time, for two dynamical impurities which can move inside the medium, Schecter and Kamenev in [24] proposed an essentially different $r$-dependence, $U(r) = -mc^2\frac{G_1 G_2}{2\pi} \frac{1}{r^3}$, where $m$ is the mass of particles in the fluid, $c$ is the sound velocity, $\xi = 1/mc$ and the dimensionless parameters $G_{1,2}$ are impurity-phonon scattering amplitude. How the power law for dynamic impurities transforms to another for the static impurities is an open question.

To address this question we investigate a model of a harmonic crystal lattice with embedded two impurity neutral atoms. It is arguably the simplest model in which one can tune impurities continuously from dynamic to static and keep track of the evolution of the Casimir interaction. In this model the Casimir interaction emerges naturally between two impurity atoms as soon as their mass is different from the masses of the lattice atoms or an external potential is applied. We find that the Casimir interaction has different asymptotics in these two cases. In the former one, for any finite mass of the impurity atoms the Casimir interaction tends to the $1/r^3$-law at large distances in agreement with [24]. At the same time, in the limit of infinity mass the long range asymptotic tends to $1/r$ in agreement with [19-21]. In the case of an external potential the asymptotics is always $1/r$.

Our letter is organized as follows. Firstly we consider two neutral impurity atoms embedded in a harmonic crystal lattice. Using the exact diagonalization method
we show that the power law at short distances strongly deviates from \(1/r^3\) and the characteristic distance of the crossover to \(1/r^3\)-law depends on the masses of the impurity atoms. Then we provide the exact solution for the model of two impurity atoms in an external harmonic potential. We show that the model is nonperturbative and has \(1/r\) asymptotic behavior. Finally, we provide a discussion of the obtained results and conclusion.

The model — We analyze an ideal harmonic cubic lattice described by \(H_0 = \sum_i \frac{\hat{p}_i^2}{2m} + \sum_\langle i,j \rangle (u_i - u_j)^2\), with two embedded impurity atoms, which have mass or external potential different from the mass/potential of the atoms of the lattice. Here \(p_i\) and \(u_i\) are the momentum and coordinate operators, \(m\) is the mass of the atoms of the cubic lattice and \(m\omega_0^2\) is the interaction potential.

The Bogoliubov transformation brings \(H_0\) to the Hamiltonian of noninteracting phonons:

\[
H_0 = \sum_k \omega_k \left( \hat{b}_k^\dagger \hat{b}_k + \frac{1}{2} \right)
\]

with the phonon spectrum: \(\omega_k = \omega_0 \sqrt{Z(1 - \gamma_k)}\). Here \(\gamma_k = \frac{1}{2} \sum_\Delta \epsilon^{ik\Delta}\) with summation over the nearest neighbours and \(Z\) the number of the nearest neighbours. In one-dimensional case it reduces to: \(\omega_k = 2\omega_0 |\sin(ka/2)|\), where \(a\) is the lattice constant. In the low energy limit \(\omega_k = c |k|\) with the phonon velocity \(c = \omega_0 a\). Further for simplicity we put \(a = 1\).

Two impurity atoms having different masses — Firstly, we consider two impurity atoms with masses \(M\) located at the sites \(a\) and \(b\). The resulting Hamiltonian of the system is \(H = H_0 + V\) with the perturbation term of the kinetic energy:

\[
V = -\frac{g}{2m}(p_a^2 + p_b^2)
\]

where the effective coupling constant \(g = (1 - m/M)\).

Exact diagonalization — The implication of two impurity atoms with masses \(M\) breaks the translational invariance and \(H\) can not be reduced to the Hamiltonian of free phonons. However, one can find the Casimir interaction, i.e. the dependence of the total energy of zero point motion \(E = \frac{1}{2} \sum_k \omega_k\) of the all atoms of the lattice on the distance between the impurity atoms. The result of exact diagonalization for 200 atom chain for various masses of impurity atoms is shown in Fig. 1. To check the finite size effect we checked a twice large chain and found no difference.

In general, the energy does not always fall down as \(1/r^3\) as it was proposed in \(\text{[24]}\) for Casimir effect in 1D. Moreover the interaction is not universal and depends on the mass of the impurity atoms (Fig. 1). One can note that the normalized Casimir interaction for masses larger than \(m\) is in the range \(1/r^3 < E < 1/r\) for \(r > 1\), for light impurities \((M < m)\) it is \(E < 1/r^3\). For impurity masses close to \(m\), the Casimir interaction tends to \(1/r^3\) law and in the limit \(M \rightarrow \infty\) (static impurities) one observes the \(1/r\) law.

Perturbation theory — To find the reason of this drastic deviation of the distance dependence of the Casimir interaction from the \(1/r^3\) law we employ the perturbation theory. For the calculation we use the bosonic representation, in which Eq. (3) reads:

\[
V = \sum_{q,q'} (V^{(1)}_{q,q'} \hat{b}_q^\dagger \hat{b}_{q'} + V^{(2)}_{q,q'} \frac{\hat{b}_q^\dagger \hat{b}_{q'}^\dagger}{2} + \text{h.c.}).
\]

Here the vertices are:

\[
V^{(1)}_{q,q'} = -V^{(0)}_{q,q'} \cos \left(\frac{q - q'}{2}\right),
\]

\[
V^{(2)}_{q,q'} = V^{(0)}_{q,q'} \cos \left(\frac{q + q'}{2}\right)
\]

with \(r = r_a - r_b\) and \(V^{(0)}_{q,q'} = g \sqrt{\omega_q \omega_{q'}}\), where \(\omega_q\), \(\omega_{q'}\) are free phonon spectra given above. We choose \(r_a + r_b = 0\) for simplicity.

The first order term of the perturbation theory is \(r\)-independent and therefore do not contribute to the Casimir interaction. The lowest order giving a contribution is the second order of the perturbation theory:

\[
U^{(2)}_{\text{eff}}(r) = -2T \sum_n \frac{|V^{(2)}_{k,k+q}|^2}{(\omega_n^2 + \omega_k^2 + \omega_{k+q}^2)},
\]

Here \(\omega_n = 2\pi T n\) is the Matsubara frequency.

FIG. 1. Normalized Casimir interaction \(U_{\text{eff}}(r)\) calculated for a chain of 200 atoms with two impurity atoms with various masses: Red dots - \(g = 0.1\) \((M/m = 1.1)\), purple - \(g = 0.6\) \((M/m = 2.5)\), green - \(g = 0.875\) \((M/m = 8)\), brown - \(g = 0.95\) \((M/m = 20)\), orange - \(g = 0.99\) \((M/m = 100)\), blue - \(g = 0.998\) \((M/m = 500)\), turquoise - \(g = -0.5\) \((M/m = 0.5)\). The red line shows \(1/r^3\), the blue line - \(1/r\).
At large distances \( r \gg 1 \) the leading contribution comes from the small momenta. At zero temperature the integration Eq. (5) can be performed analytically for the linearized spectrum \( \omega_k = ck \) with use of the substitution \( T \sum_n \to \int d\omega_n/2\pi \). The result is the \( 1/r^3 \)-law:

\[
U^{(2)}_{\text{eff}} (r) = \frac{g^2 \omega_0^2}{32\pi} \frac{1}{r^3}. \tag{6}
\]

This dependence agrees with that previously found in [24], but disagrees with the results of the exact diagonalization.

**Higher order of perturbation theory** — To understand the origin of the deviation from \( 1/r^3 \) law, we explore higher order phonon processes, which correspond to multiple scattering of phonons on the impurities. The result of the perturbation theory up to four-phonon processes for \( g = 0.5 \) is presented in Fig. 2. Here we keep only \( r \)-dependent terms. One immediately notes that the third and fourth orders of the perturbation theory significantly add to the Casimir interaction. Plotting the sum of these contributions up to the fourth order against the exact diagonalization reveals already a good match.

The exact solution is given by the infinite sum of diagrams shown in Fig. 3. We can do this sum (see Supplementary material), and the obtained thermodynamic potential \( \Phi(r) \) contains an \( r \)-independent term, which is related to perturbation of the zero point motion by uncorrelated impurity atoms \( (r \to \infty) \). Defining \( U_{\text{eff}}(r) = \Phi(r) - \Phi(\infty) \) we arrive to the following expression:

\[
U_{\text{eff}}(r) = \frac{1}{2} T \sum_n \ln \left[ 1 - \left( \frac{g G(\omega_n, r)}{1 - g G_0(\omega_n)} \right)^2 \right]. \tag{7}
\]

The direct comparison the results obtained with use of Eq. (7) and Eq. (11) for \( r > 1 \) show excellent matching of the results [25]. From this expression it is clear that \( U_{\text{eff}}(r) \) decays exponentially fast at finite temperature for \( r \gg \omega_0/T \), i.e. thermo-fluctuations prevail on quantum fluctuations. The power law may emerge in some finite range.

To trace the dependence of the Casimir interaction on the coupling constant \( g < 1 \) and distance \( r \) at \( T = 0 \) we
introduce the logarithmic derivative \( \nu = -\frac{d\ln U_{\text{eff}}(r)}{d\ln r} \) as the function of \( r \) and \( g \) of the Casimir interaction between two impurity atoms having different masses.

By power law functions \( 1/r^\nu \) it gives the power \( \nu \). The results are summarized in the Fig. 4. The interval \( 0 < g \leq 1 \) describes the impurity masses \( m < M \leq \infty \). The line \( g = 0 \) is the singular line where \( U_{\text{eff}} = 0 \). And the interval \( -\infty < g < 0 \) corresponds to \( M < m \). One can see from the figure that although for small distances the Casimir interaction cannot be described by the functions \( 1/r^\nu \), at large distances the dependence tends to \( 1/r^3 \). The characteristic distance of the crossover to the \( 1/r^3 \)-law strongly depends on the masses of the impurity atoms. Finally, in the limit \( g \to 1 \) the Casimir interaction depends as \( 1/r \) from the distance between the impurity atoms and coincides with Eq. (11).

External potential — Now we consider two atoms in an external harmonic potential which is defined by the following Hamiltonian:

\[
V = gm\omega_0^2(u_n^2 + u_k^2),
\]

with the interaction constant \( g \geq 0 \).

It leads to the new interaction term \( V_{q,q'}^{(0)} \):

\[
V_{q,q'}^{(0)} = -\frac{g\omega_0^2}{\sqrt{\omega_q^2/\omega_{q'}^2.}}
\]  

The bosonic Green functions are (see [25] for definition):

\[
G(\omega_n, r) = \frac{\omega_n}{c} \frac{\omega_n^2}{\omega_n^2} f(|\omega_n|/2\omega_0, r)
\]

\[
G_0(\omega_n) = \frac{\omega_n}{c} \frac{\omega_n^2}{\omega_n^2} f(|\omega_n|/2\omega_0, 0).
\]

The direct calculation exhibits that all orders of the perturbation theory are divergent at the low energy limit.

But the summation of whole series of the diagrams Fig. 3 leads to cancellation of the singularities and finite expression for the thermodynamic potential Eq. (7). The phononic Green functions are given by Eqs. (14-19).

The correspondent continuous model is different from Eq. (10) and is giving by:

\[
H = \sum_k c|k|b_k^\dagger b_k
\]

\[
+ g \sum_{k,k'}^\nu \omega_0^2 \cos \frac{(k + k')r}{2} \left(b_k^\dagger b_{k'} + b_{k'}^\dagger b_k - \frac{b_k^\dagger b_k}{2} \right)
\]

The Casimir interaction has the form:

\[
U_{\text{eff}}(r) = T \sum_{n > 0} \ln \left[ 1 - \left( \frac{g\omega_0^2}{1 + 2\omega_0^2} \right) \right]
\]

Similar expression was obtained in [20]. To understand the scaling behavior at \( T = 0 \) we plot the logarithmic derivative \( \nu \) of the Casimir interaction \( U_{\text{eff}} \) given by Eq. (11) as a function of \( r \) and \( g \) in Fig. 5. For small values \( gr \) the law is not universal, but \( U_{\text{eff}} \) tends to \( 1/r \) as soon as \( gr \ll 1 \). The integral Eq. (17) in the limit \( gr \gg 1 \) matches the previously found expression for \( M \to \infty \) Eq. (11).

Discussion and conclusions — The obtained long-range interaction can be observed experimentally in ultra cold atomic gases as was shown in [21]. Since the competing Casimir-Polder interaction falls off much faster, namely as \( 1/r^6 \), in the experimental setup of [26] for the impurities at the distance of 1\( \mu \)m the phonon induced Casimir interaction should dominate [27].

Summarizing, we have analyzed the evolution of the Casimir interaction between two impurity atoms embedded into an ideal 1D lattice at \( T = 0 \). We have given...
the exact solution of the model and have studied the evolution of the Casimir interaction with change of the impurity atoms masses and the effect of an external potential. We have shown that multiboson processes change the scaling of the interaction decay with distance and the mass of the considered object plays an important role. As a consequence, the behavior at small distances differs from the power law at large. At large distances between two dynamic impurities the Casimir interaction is universal and obey $1/r^3$ law. For static impurities it tends to the $1/r$ law.

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Supplemental Material

IMPURITY ATOMS WITH MASSES DIFFERENT FROM THE MASS OF THE LATTICE ATOMS

Perturbation theory up to the fourth order

- Second order – The second order perturbation term (Eq.(12) of the main text) integrated over frequency $\omega_n$ reads:

$$U^{(2)}_{\text{eff}}(r) = -\int \frac{dkdq}{(2\pi)^2} \frac{|V_{k,k+q}|^2}{\omega_k + \omega_{k+q}} \cos^2 \left( \frac{qr}{2} \right).$$

https://link.aps.org/doi/10.1103/PhysRevD.67.063515
The phonon Green function in Matsubara formalism reads:

\[
U^{(3)}_{\text{eff}}(r) = -\frac{1}{4} \hbar \omega_0 \int \frac{dq_1 dq_2 dq_3}{(2\pi)^3} \frac{|V^{(0)}_{q_1,q_2} V^{(0)}_{q_2,q_3} V^{(0)}_{q_3,q_1}|}{(\omega_1 + \omega_2)(\omega_2 + \omega_3)} \left( \cos r(q_1 + q_2) + \cos r(q_2 + q_3) + \cos r(q_1 - q_3) \right).
\]

• Forth order reads:

\[
U^{(4,a)}_{\text{eff}}(r) = -\frac{1}{8} \hbar \omega_0 \int \frac{dq_1 dq_2 dq_3 dq_4}{(2\pi)^4} \frac{|V^{(0)}_{q_1,q_2} V^{(0)}_{q_2,q_3} V^{(0)}_{q_3,q_4} V^{(0)}_{q_4,q_1}|}{(\omega_{q_1} + \omega_{q_2})(\omega_{q_2} + \omega_{q_3})(\omega_{q_3} + \omega_{q_4})} \left( \cos r(q_1 + q_2) + \cos r(q_3 + q_4) + \cos r(q_1 + q_3) + \cos r(q_2 + q_4) + \cos r(q_1 + q_4) + \cos r(q_1 - q_3) + \cos r(q_1 - q_4) \right)
\]

\[
U^{(4,b)}_{\text{eff}}(r) = -\frac{1}{8} \hbar \omega_0 \int \frac{dq_1 dq_2 dq_3 dq_4}{(2\pi)^4} \frac{|V^{(0)}_{q_1,q_2} V^{(0)}_{q_2,q_3} V^{(0)}_{q_3,q_4} V^{(0)}_{q_4,q_1}|}{(\omega_{q_1} + \omega_{q_2})(\omega_{q_2} + \omega_{q_3})(\omega_{q_3} + \omega_{q_4})} \left( \cos r(q_1 + q_2) + \cos r(q_3 + q_4) + \cos r(q_2 + q_3) + \cos r(q_1 + q_4) + \cos r(q_1 - q_3) + \cos r(q_1 - q_4) \right)
\]

\[
U^{(4,c)}_{\text{eff}}(r) = -\frac{1}{8} \hbar \omega_0 \int \frac{dq_1 dq_2 dq_3 dq_4}{(2\pi)^4} \frac{|V^{(0)}_{q_1,q_2} V^{(0)}_{q_2,q_3} V^{(0)}_{q_3,q_4} V^{(0)}_{q_4,q_1}|}{(\omega_{q_1} + \omega_{q_2})(\omega_{q_2} + \omega_{q_3})(\omega_{q_3} + \omega_{q_4})} \left( \cos r(q_1 + q_2) + \cos r(q_1 + q_3) + \cos r(q_1 + q_4) + \cos r(q_2 - q_3) + \cos r(q_2 - q_4) + \cos r(q_1 + q_2 + q_3 - q_4) \right)
\]

The forth order contains three nonvanishing at \( T = 0 \) topologically nonequivalent diagrams (Fig. 6).

![Fifth order diagrams](image)

**FIG. 6.** Forth order diagrams

### Analytical solution

We use the definition of the phonon field \( \varphi \) similar to used in \([S1]\):

\[
\varphi(r, t) = \frac{1}{\sqrt{V}} \sum_{q} \sqrt{\omega_q} \left[ b_q e^{iqr - \omega_q t} + b_{q}^{\ast} e^{-iqr + \omega_q t} \right]
\]

The phonon Green function in Matsubara formalism reads:

\[
D(q, \omega_n) = \omega_q \left( \frac{1}{\hbar \omega_n + \omega_q} + \frac{1}{-\hbar \omega_n + \omega_q} \right)
\]

Then the vertices of the phonon scattering on the impurities are \( \Gamma^{(1,2)}_{q,q'} = -g \cos[(q \pm q') \frac{\pi}{2}] \) (Fig. 7). The basic block
of any diagram is depicted in Fig. 8:

\[ g^2 \int_{-\pi}^{\pi} \frac{dk}{2\pi} \cos[(q \pm k)\frac{\pi}{2}] \cos[(k \pm q')\frac{\pi}{2}] \frac{2\omega_k^2}{\omega_n^2 + \omega_k^2} \]
\[ = g^2 \cos \frac{qr}{2} \cos \frac{q'r}{2} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \cos^2 \left( \frac{kr}{2} \right) \frac{2\omega_k^2}{\omega_n^2 + \omega_k^2} + g^2 \sin \frac{qr}{2} \sin \frac{q'r}{2} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \sin^2 \left( \frac{kr}{2} \right) \frac{2\omega_k^2}{\omega_n^2 + \omega_k^2} \]

It's worth to introduce Green functions in the coordinate space:

\[ G_0(\omega_n) - G(r, \omega_n) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \cos^2 \left( \frac{kr}{2} \right) \frac{2\omega_k^2}{\omega_n^2 + \omega_k^2} \]
\[ G_0(\omega_n) + G(r, \omega_n) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \sin^2 \left( \frac{kr}{2} \right) \frac{2\omega_k^2}{\omega_n^2 + \omega_k^2} \]

Then the loop of the \( n \)th order can be expressed in the compact form:

\[ U^{(n)}_{eff} = -\frac{1}{2} g^2 \pi \int_{-\infty}^{\infty} \frac{d\omega_n}{2\pi} \left( (G_0(\omega_n) + G(r, \omega_n))^n + (G_0(\omega_n) - G(r, \omega_n))^n \right), \tag{18} \]

where:

\[ G(\omega_n, r) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \cos(kr) \frac{\omega_k^2}{\omega_n^2 + \omega_k^2} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \cos(kr) \frac{4\omega_0^2 \sin^2(k/2)}{\omega_n^2 + 4\omega_0^2 \sin^2(k/2)} \]
\[ = \delta_{r,0} - \frac{\left| \frac{\omega_n}{2\omega_0} \right|}{\sqrt{1 + \left( \frac{\omega_n}{2\omega_0} \right)^2}} \left( \frac{\omega_n}{2\omega_0} + \sqrt{1 + \left( \frac{\omega_n}{2\omega_0} \right)^2} \right)^{-2r} \frac{\omega_n}{2\omega_0} \sim \frac{\omega_n}{2c} \delta_{r,0} - \frac{\omega_n}{2c} e^{-\frac{\omega_n}{2c}} \]
\[ G_0(\omega_n) = G(0, \omega_n) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \frac{\omega_k^2}{\omega_n^2 + \omega_k^2} = 1 - \frac{\left| \frac{\omega_n}{2\omega_0} \right|}{\sqrt{1 + \left( \frac{\omega_n}{2\omega_0} \right)^2}} \frac{\omega_n}{2\omega_0} \sim 1 - \frac{\omega_n}{2c} \]

The thermodynamic potential at \( T = 0 \) [1]:

\[ \Phi_{total}(r) = -\int_{0}^{\infty} \frac{d\omega_n}{2\pi} \sum_{l=2}^{\infty} \frac{g^l}{l} \left( (G_0(\omega_n) + G(r, \omega_n))^l + (G_0(\omega_n) - G(r, \omega_n))^l \right) \]
\[ = \int_{0}^{\infty} \frac{d\omega_n}{2\pi} \left[ \ln \left( 1 - \frac{g^2 G^2(r, \omega_n)}{(1 - gG_0(\omega_n))^2} \right) + 2 \ln \left( 1 - gG_0(\omega_n) + 2gG_0(\omega_n) \right) \right] \]

The effective Casimir energy goes to 0 when \( r \to \infty \) and should not contain a constant part:

\[ U_{eff}(r) = \Phi_{total}(r) - \Phi_{total}(\infty) = \int_{0}^{\infty} \frac{d\omega_n}{2\pi} \left[ \ln \left( 1 - \frac{g^2 G^2(r, \omega_n)}{(1 - gG_0(\omega_n))^2} \right) \right] \]
\[ = \int_{0}^{\infty} \frac{d\omega_n}{2\pi} \ln \left( 1 - \frac{\left| \frac{\omega_n}{2\omega_0} \right| e^{-\frac{\omega_n}{2c}}}{\left( 1 - g + \frac{\omega_n}{2c} \right)^2} \right), \tag{19} \]
Continuum limit. The Green functions in the continuous limit can be obtained:

\[ G(r, \omega_n) \simeq \int_{-\infty}^{\infty} \frac{dk}{2\pi} \cos kr \frac{k^2}{(\frac{\omega_n}{2})^2 + k^2} = -\frac{\omega_n}{2c} e^{-\frac{\omega_n}{c}r} \]

\[ G_0(\omega_n) = 1 - \int_{-\pi}^{\pi} \frac{dk}{2\pi} \frac{4\omega_n^2 \sin^2 \frac{k}{2}}{\omega_n^2 + 4\omega_0^2 \sin^2 \frac{k}{2}} \simeq 1 - \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{(\frac{\omega_n}{\omega_0})^2}{\omega_n^2 + k^2} = 1 - \frac{\omega_n}{2c} \]

In the second order of the perturbation theory we restore the \(1/r^3\) law:

\[ U^{(2)}_{\text{eff}}(r) = g^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\omega^2 e^{-\frac{\omega}{\omega_0}}}{4\omega_0^2} = -\frac{g^2 \omega_0}{32\pi^3} \]

Casimir Force

The Casimir force reads:

\[ F(r) = -\frac{\partial U_{\text{eff}}}{\partial r} = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\tilde{g}^2 \omega^3 e^{-\frac{\omega}{\omega_0}}}{(1 + \tilde{g}^2 \omega^2)^2} - (\tilde{g}^2 \omega_0 e^{-\frac{\omega}{\omega_0}})^2, \]

where we use a new constant \(\tilde{g} = \frac{g}{\omega_0}\) for convenience. For heavy impurities \((\tilde{g} > 0)\), one can approximate Eq. (20) omitting the exponentially small term from the denominator. It reads then as:

\[ F(r) \simeq -\int_{0}^{\infty} \frac{d\omega}{2\pi} \frac{\tilde{g}^2 \omega^3 e^{-\frac{\omega}{\omega_0}}}{(1 + \tilde{g}^2 \omega^2)^2} = \frac{c}{\tilde{g}^2 \pi} \left( \frac{1}{4\pi^2} - \frac{2}{x} - 4 + (12 + 16x)I(4x) \right), \]

where \(x = r/\tilde{g}\). Here \(I(x) = \int_{0}^{\infty} dt \frac{t^{x-1}}{e^t} \). It can be expressed through the incomplete gamma function: \(I(x) = e^x \Gamma[0, x], \Gamma[\alpha, x] = \int_{x}^{\infty} t^{\alpha-1} e^{-t} dt\).

Integration over \(r\) with condition \(U_{\text{eff}}(r \to \infty) = 0\) gives:

\[ U_{\text{eff}}(x) \simeq \frac{c}{\tilde{g} \pi} \left( -\frac{1}{4x} + 2(1 + 2x)I(4x) - 1 \right) \]

Expression (22) works excellent for small masses, but for infinite masses it gives \(\frac{1}{x^2}\) numerical coefficient instead of \(\frac{1}{x}\) provided by (19) and expected for the Casimir law (Fig. 9). Asymptotically, Eq. (22) for mass ratio \(m/M \to 1 (g \to 0)\) is:

\[ U_{\text{eff}}(x) \simeq \frac{c}{\tilde{g}} \left( -\frac{1}{32\pi^3 x^3} + \frac{1}{16\pi^4 x^4} - \frac{9}{128\pi^5 x^5} + ... \right) \]

The space dependence for this expression is shown at FIG. 10 with three different values of \(g\).

EXTERNAL POTENTIAL

We consider two atoms in an external potential \(V\) given by:

\[ V = gm\omega_0^2 (u_a^2 + u_b^2), \]

\(g \in (0, \infty)\).

For the calculation we use the bosonic representation, in which the perturbation reads:

\[ V = \sum_{q, q'}(V_{q, q'}^{(1)} b_q^\dagger b_{q'} + V_{q, q'}^{(2)} \frac{b_q b_{q'}}{2} + h.c.). \]

Here the vertices are:

\[ V_{q, q'}^{(1)} = V_{q, q'}^{(0)} \cos[(q - q')^2], \]

\[ V_{q, q'}^{(2)} = -V_{q, q'}^{(0)} \cos[(q + q')^2], \]
FIG. 9. Comparison of exact result, result for the linearized vertices and approximate analytical formula. Red color - \( g = 0.9 \), blue - \( g = 0.99 \), brown - \( g = 0.999 \), green - \( g = 1 \). Circles - exact result, lines - linearized vertices, diamonds - approximate formula.

with \( V_{q,q'}^{(0)} \):

\[
V_{q,q'}^{(0)} = \frac{g\omega_0^2}{\sqrt{\omega_q \omega_{q'}}}
\]  

(25)

Now we define a free phonon field as

\[
\tilde{\phi}(r,t) = \frac{1}{\sqrt{V}} \sum_q \omega_0 \sqrt{\frac{1}{\omega_k}} \left[ b_q e^{iqr - i\omega_q t} + b_q^+ e^{-iqr + i\omega_q t} \right]
\]
FIG. 10. Dependence of the logarithmic derivative \( \nu(r) \) on distance for various fixed \( g \).

The Green functions \( G(\omega_n, r), G_0(\omega_n) \) take form:

\[
G(\omega_n, r) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \cos(kr) \frac{\omega_0^2}{\omega_n^2 + \omega_k^2} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \cos(kr) \frac{\omega_0^2}{\omega_n^2 + 4\omega_0^2 \sin^2\left(\frac{k}{2}\right)}
\]

\[
= -\frac{\omega_0^2}{2\omega_n |c|} \left( \frac{\omega_n}{2\omega_0} + \sqrt{1 + \left( \frac{\omega_n}{2\omega_0} \right)^2} \right)^{-2r}
\]

\[
G_0(\omega_n) = G(0, \omega_n) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \frac{\omega_0^2}{\omega_n^2 + \omega_k^2} = \frac{\omega_0^2}{2\omega_n |c| \sqrt{1 + \left( \frac{\omega_n}{2\omega_0} \right)^2}}.
\]

To analyze the Casimir energy, we use the linearized spectrum. Green functions read as

\[
G(\omega_n, r) = \frac{\omega_0^2}{2c|\omega_n|} e^{-|\omega_n|r},
\]

\[
G_0(\omega_n) = \frac{\omega_0^2}{2c|\omega_n|}.
\]
It’s worth to consider the second order term of the perturbation theory for the Casimir interaction:

\[ U^{(2)} = -\frac{g^2}{2} \int_{-\infty}^{\infty} \frac{d\omega_n}{2\pi} \frac{\omega_n^4}{4c^2\omega_n^2} e^{-2\frac{\omega_n}{c}r} \]

This expression diverges at small frequencies. All other terms diverge as well. But the whole sum of the perturbation theory series remains finite and gives us:

\[ U_{\text{eff}}(r) = -\int_{0}^{\infty} \frac{d\omega_n}{2\pi} \ln \left[ 1 - \left( \frac{g\omega_0^2}{2c_0\omega_n^2} e^{-\frac{\omega_n}{c}r} \right)^2 \right] \]

(26)

The distance dependence for various given values of \( g \) is shown at Fig. 11.

At \( g = \infty \), from Eq. (26) follows \( U_{\text{eff}}(r) = \frac{\omega_0^2}{24c^2}r \). To investigate the finite \( g \) case, we use the same approach as before, finding an approximate expression for the Casimir force and integrating it:

\[
F(x) = \int_{0}^{\infty} \frac{d\omega_n}{2\pi} \frac{g^2\omega_n^4}{2c^2\omega_n^2} e^{-2\frac{\omega_n}{c}r} \left( \frac{1 + \frac{g\omega_0^2}{2c_0\omega_n^2} e^{-\frac{\omega_n}{c}r}}{1 + \frac{g\omega_0^2}{2c_0\omega_n^2}} \right)^2 \\
\simeq \frac{g^2\omega_0^4}{4\pi c^3} \left( 1 + (x + 1)I(x) \right),
\]

(27)

where \( x = \frac{gr\omega_0^2}{c^2} \).

\[ E_{\text{Cas}}(r) \simeq \frac{g\omega_0^2}{4\pi c} \left( 1 - xI(x) \right) \]

(28)

At Fig. 5, distribution of energy \( E_{\text{Cas}} \) in relation to \( r \) and \( g \) is shown. The \( r \)-dependence for various given values of \( g \) is also depicted at Fig. 11.

For \( gr \gg 1 \), the expression for the Casimir force reads:

\[ F(r) = \frac{\omega_0^2}{cr^2} \left( \frac{\pi}{24} - \frac{c^2\pi}{6gr\omega_0^2} + \frac{\pi^2}{2\pi} + \frac{3\zeta(3)}{2\pi} \left( \frac{c^2}{gr\omega_0^2} \right)^2 + ... \right) \]
FIG. 11. Dependence of the logarithmic derivative $\nu(r)$ on distance for various fixed $g$. Potential energy case.

[S1] A. A. Abrikosov, L. P. Gorkov, I. Y. Dzyaloshinskii, *Methods of Quantum Field Theory in Statistical Physics* (Dover Publications, New York, 1975).