QUASIDIAGONALITY OF $C^*$-ALGEBRAS OF SOLVABLE LIE GROUPS

INGRID BELITIȚĂ AND DANIEL BELITIȚĂ

Abstract. We characterize the solvable Lie groups of the form $\mathbb{R}^m \rtimes \mathbb{R}$, whose $C^*$-algebras are quasidiagonal. Using this result, we determine the connected simply connected solvable Lie groups of type I whose $C^*$-algebras are strongly quasidiagonal. As a by-product, we give also examples of amenable Lie groups with non-quasidiagonal $C^*$-algebras.

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1. Introduction

Quasidiagonality properties have proved to be very important in the study of $C^*$-algebras and their applications. In the case of group $C^*$-algebras, the proof of Rosenberg’s conjecture was recently completed: The amenability of a countable discrete group is equivalent to quasidiagonality of its reduced $C^*$-algebra (see [Hd87 and [TWW16], and also [Da96, Prop. VII.7.8]). However, this statement does not carry over to general locally compact groups, and quasidiagonality properties of their $C^*$-algebras still remain to be understood. Interesting results in this direction were obtained in the paper [SW01].

In the present paper we address the above problems for connected solvable Lie groups, continuing our quest for understanding topological aspects of their unitary dual (see [BB16a, BB16b, BBG16, BBL17]). We show that a connected simply connected solvable Lie group of type I has quasidiagonal $C^*$-algebra if and only if it is CCR. We will also see below that there are many connected simply connected solvable Lie groups of type I that have quasidiagonal $C^*$-algebras. We also construct pretty large and precisely described classes of Lie groups which are amenable and yet their $C^*$-algebras are not quasidiagonal. More specifically, we characterize the generalized ax + b-groups, that is, solvable Lie groups of the form $\mathbb{R}^m \rtimes \mathbb{R}$, whose $C^*$-algebras are quasidiagonal. The $C^*$-algebras of these groups have been recently studied in [LiLu13].

Main results and structure of this paper. Let us state our main result, which will be proved in Section 3.

Theorem 1.1. For any connected simply connected solvable Lie group $G$ of type I with its Lie algebra $\mathfrak{g}$, the following properties are equivalent:

(i) The $C^*$-algebra $C^*(G)$ is strongly quasidiagonal.

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(ii) For every $A \in \mathfrak{g}$ the eigenvalues of the linear map $\mathfrak{g} \ni X \mapsto [A, X] \in \mathfrak{g}$ are purely imaginary or zero.

(iii) For every $[\pi] \in \hat{G}$ one has $\pi(C^*(G)) = \mathcal{K}(\mathcal{H}_\pi)$.

If $G$ is an exponential solvable Lie group, then the above properties are further equivalent to

(iv) The Lie group $G$ is nilpotent.

(v) The coadjoint orbits of $G$ are closed.

The proof of Theorem 1.1 is based on quasidiagonality properties of some special generalized $ax + b$-groups.

We give a thorough treatment of $C^*$-algebras of generalized $ax + b$-groups in Section 2. Namely, we first establish necessary and sufficient conditions for groups in this class to have isomorphic $C^*$-algebras (see Theorems 2.2 and 2.7). We thus generalize the results of [L11a,B] using a completely different method, based on topological equivalence of linear dynamical systems. Then we characterize the generalized $ax + b$-groups whose $C^*$-algebras are quasidiagonal (Theorem 2.15). As a by-product we show that among the isomorphism classes of the $C^*$-algebra of these amenable Lie groups there is exactly one class which contains non-quasidiagonal $C^*$-algebras (Corollary 2.17).

As already mentioned, Section 3 is devoted to the proof of the above Theorem 1.1. In Section 3 we prove that if $G$ is a connected locally compact solvable group of type I such that $C^*(G)$ has a faithful tracial state, then $G$ is commutative. This shows that the study of quasidiagonality of connected solvable Lie groups of type I is in some sense complementary to the case of discrete groups as seen in [TWW16], and therefore we had to use here completely different methods for establishing when their $C^*$-algebras are quasidiagonal.

General notation and terminology. For any $C^*$-algebra $\mathcal{A}$ we define $\mathcal{A}^1 := \mathcal{A}$ if $\mathcal{A}$ is unital, while if $\mathcal{A}$ has no unit element, then $\mathcal{A}^1 := \mathcal{A} \oplus \mathbb{C}1 \subseteq M(\mathcal{A})$ is the unitization of $\mathcal{A}$, and $M(\mathcal{A})$ is the multiplier algebra of $\mathcal{A}$.

We denote by $\hat{\mathcal{A}}$ the set of equivalence classes of irreducible $*$-representations of $\mathcal{A}$, endowed with its usual topology. If $\pi: \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi)$ is an irreducible $*$-representation, we denote its equivalence class by $[\pi] \in \hat{\mathcal{A}}$.

We say that a $*$-representation $\pi: \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi)$ of a separable $C^*$-algebra $\mathcal{A}$ is quasidiagonal, if one has

$$\left(\forall a \in \mathcal{A}\right) \lim_{n \to \infty} \|\pi(a)P_n - P_n\pi(a)\| = 0$$

for a suitable sequence $P_n = P_n^* = P_n^2 \in \mathcal{B}(\mathcal{H}_\pi)$ with rank $P_n < \infty$ for every $n \geq 1$ and $P_n \to 1$ in the strong operator topology as $n \to \infty$. The separable $C^*$-algebra $\mathcal{A}$ is called quasidiagonal if it has a quasidiagonal faithful $*$-representation, and $\mathcal{A}$ is called strongly quasidiagonal if every irreducible $*$-representation of $\mathcal{A}$ is quasidiagonal (see e.g., [Y93, Da96, BrOz08]).

For $G$ a locally compact group, let $\hat{G}$ be its unitary dual, that is, the set of all equivalence classes $[\pi]$ of unitary irreducible representations $\pi: G \to \mathcal{B}(\mathcal{H}_\pi)$. Then, as usually, we identify $\hat{G}$ with $\hat{\mathcal{A}}$, where $\mathcal{A} = C^*(G)$.

By $C^*$-dynamical system we mean a triple $(\mathcal{A}, G, \alpha)$, where $\mathcal{A}$ is a $C^*$-algebra, $G$ is a locally compact group, and $\alpha: G \times \mathcal{A} \to \mathcal{A}$, $(g, a) \to \alpha_g(a)$ is a continuous action of $G$ by $*$-automorphisms of $\mathcal{A}$. Sometimes we denote a $C^*$-dynamical system...
simply as $G \times A \to A$, $(g, a) \mapsto g \cdot a$, especially when the notation for the action $\alpha$ is not essential.

In this paper by a Lie group we mean a finite dimensional real Lie group. We recall that a connected Lie group $G$ is solvable if its Lie algebra $\mathfrak{g}$ is solvable, that is,

$$\mathfrak{g}^{(n)} = \{0\} \text{ for } n \text{ large enough,}$$

where

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}] \text{ for } k \geq 1.$$

This is further equivalent with the fact that $G$ is solvable as a discrete group.

An exponential Lie group is a Lie group $G$ whose exponential map $\exp_G: \mathfrak{g} \to G$ is a bijection, where $\mathfrak{g}$ is the Lie algebra of $G$. All exponential Lie groups are solvable. See for instance [FuLu15] for more details.

2. ON THE $C^*$-ALGEBRAS OF GENERALIZED $ax + b$-GROUPS

In this section, we fix a finite-dimensional real vector space $\mathcal{V}$.

For any $D \in \text{End}(\mathcal{V})$ we define the corresponding generalized $ax + b$-group as the connected simply connected Lie group $G_D := \mathcal{V} \rtimes_{\alpha_D} \mathbb{R}$, where the semidirect product of the abelian Lie groups $(\mathbb{R}, +)$ and $(\mathcal{V}, +)$ involves the action

$$\alpha_D: \mathcal{V} \times \mathbb{R} \to \mathcal{V}, \quad \alpha_D(v, t) := e^{tD}v,$$

so the group operation is given by $(v_1, t_1) \cdot (v_2, t_2) = (v_1 + e^{t_1D}v_2, t_1 + t_2)$ for all $v_1, v_2 \in \mathcal{V}$ and $t_1, t_2 \in \mathbb{R}$.

We also denote by $\mathcal{V}/\alpha_D$ the orbit space of the group action $\alpha_D$, endowed with its quotient topology.

We consider the $\mathbb{C}$-linear extension $D: \mathcal{V}_\mathbb{C} \to \mathcal{V}_\mathbb{C}$, and we denote its spectrum by $\sigma(D)$, where $\mathcal{V}_\mathbb{C} := \mathcal{V} \otimes_{\mathbb{R}} \mathbb{C}$. For $\mu \in \sigma(D) \cap (\mathbb{C} \setminus \mathbb{R})$, the real generalized eigenspace for $\mu$, $\overline{\mathfrak{V}}$ is the linear subspace of $\mathcal{V}$ given by

$$E^D(\mu) := \{v_1, v_2 \in \mathcal{V} | v_1 + iv_2 \in \text{Ker}(D - \mu 1)^m\},$$

where $m$ is the dimension of the largest Jordan block for $\mu$. (See [CK14, Ex. 1.6.9].) If $\mu \in \sigma(D) \cap \mathbb{R}$, then the real generalized eigenspace for $\mu$ is $E^D(\mu) := \text{Ker}(D - \mu 1)^m$, where $m$ is again the dimension of the largest Jordan block for $\mu$. For every $\mu \in \mathbb{C} \setminus \sigma(D)$ we also define $E^D(\mu) := \{0\}$.

We also define $\mathcal{V}_+^D := \bigoplus_{\mu \in \mathbb{R}, \mu > 0} E^D(\mu)$ and $\mathcal{V}_0^D := \bigoplus_{\mu \in \mathbb{R}, \mu = 0} E^D(\mu)$, and then one has $\mathcal{V}^D = \mathcal{V}_+^D \oplus \mathcal{V}_0^D \oplus \mathcal{V}_-^D$. (See for instance [CK14, §1.4].) With this notation we put

$$n^D_+ := \text{dim } \mathcal{V}_+^D.$$  \hfill (2.1)

2.1. Isomorphism classes of $C^*$-algebras of generalized $ax + b$-groups.

**Lemma 2.1.** If $D_1, D_2 \in \text{End}(\mathcal{V})$, then the following assertions are equivalent:

(i) There exists a homeomorphism $\Phi: \mathcal{V} \to \mathcal{V}$ satisfying $\Phi \circ e^{tD_1} = e^{tD_2} \circ \Phi$ for every $t \in \mathbb{R}$.

(ii) One has $n^D_+ = n^{D_2}_+$ and $TD_1|_{n^D_+} = D_2T$ for a suitable linear isomorphism $T: \mathcal{V}_0^{D_1} \to \mathcal{V}_0^{D_2}$.

**Proof.** See [La73, Th. B'].
A proof of Lemma 2.1 in the special case of hyperbolic maps (that is, under the additional assumption $\dim V_0^{D_1} = \dim V_0^{D_2} = \{0\}$) can also be found for instance in [CK13, Th. 2.2.5].

**Theorem 2.2.** Let $D \in \text{End} (V)$ and define

$$\bar{D} = \begin{pmatrix} -1_{n_-} & 0 & 0 \\ 0 & D_0 & 0 \\ 0 & 0 & 1_{n_+} \end{pmatrix}$$

with respect to the direct sum decomposition $V = V_0^D \oplus V_0^D \oplus V_0^D$, where $n_{\pm} := n_{\pm}^D$, we have denoted by $1_{n_{\pm}}$ the identity map of $V_0^{D_{\pm}}$, and $D_0 := D|_{V_0^D}$.

Then the $C^*$-algebras $C^*(G_D)$ and $C^*(G_{\bar{D}})$ are $*$-isomorphic.

**Proof.** We denote $W := V^*$ and also let $T$ be the identity map on $V_0^D$.

Then one has $\dim W_0^D = \dim W_0^{\bar{D}}$ and $T^* \bar{D}^* = D^*T^*$, where $T^*: W_0^D \rightarrow W_0^{D^*}$ is a linear isomorphism. Then, by Lemma 2.1, there exists a homeomorphism $\Phi: V^* \rightarrow V^*$ satisfying $\Phi \circ e^{tD^*} = e^{t\bar{D}^*} \circ \Phi$ for every $t \in \mathbb{R}$. The map

$$\varphi: C_0(V^*) \rightarrow C_0(V^*), \quad \phi(f) := f \circ \Phi$$

is an equivariant isomorphism between the $C^*$-dynamical systems $(C_0(V^*), \mathbb{R}, \alpha^D)$ and $(C_0(V^*), \mathbb{R}, \alpha^{\bar{D}})$, where $\alpha^D: \mathbb{R} \times C_0(V^*) \rightarrow C_0(V^*)$, $\alpha^{\bar{D}}(t, f) := f \circ e^{tD^*}$, and $\alpha^{\bar{D}}$ is similarly defined with $D$ replaced by $\bar{D}$.

We now obtain a $*$-isomorphism

$$\text{id} \times \varphi: \mathbb{R} \ltimes_{\alpha^D} C_0(V^*) \rightarrow \mathbb{R} \ltimes_{\alpha^{\bar{D}}} C_0(V^*)$$

by [W07, Lemma 2.65]. We now recall from [W07, Ex. 3.16] that there are $*$-isomorphisms $C^*(G_D) \simeq \mathbb{R} \ltimes_{\alpha^D} C_0(V^*)$ and $C^*(G_{\bar{D}}) \simeq \mathbb{R} \ltimes_{\alpha^{\bar{D}}} C_0(V^*)$, and we are done. \hfill $\square$

**Remark 2.3.** Theorem 2.2 is equivalent to the following fact: If $D_1, D_2 \in \text{End} (V)$ with $n_{\pm}^{D_1} = n_{\pm}^{D_2}$ and $TD_1 = D_2T$ for a suitable linear isomorphism $T: V_0^{D_1} \rightarrow V_0^{D_2}$, then the $C^*$-algebras $C^*(G_{D_1})$ and $C^*(G_{D_2})$ are $*$-isomorphic.

We point out that a related idea was suggested in [Ro70].

We now turn to establishing a converse to Theorem 2.2. The following lemma replaces the method of coadjoint orbits for the particular class of solvable Lie groups we are interested in here.

**Lemma 2.4.** Let $D \in \text{End} (V)$. If $V_0^D = \text{Ker} D$, then $G_D$ is an exponential solvable Lie group and its unitary dual is homeomorphic to the quotient space $(V^* \times \mathbb{R})/ \sim$, where we define an equivalence relation $\sim$ on $V^* \times \mathbb{R}$ whose equivalence classes are given by

$$[(\xi, s)] = \begin{cases} \{(\xi, s)\} & \text{if } \xi \in (V_0^- \oplus V_0^+) / \sim, \\ (e^{it\alpha^D} \xi | t \in \mathbb{R}) \times \mathbb{R} & \text{if } \xi \in V^* \setminus (V_0^- \oplus V_0^+) / \sim \end{cases}$$

then the unitary dual space of $G_D$ is homeomorphic to $(V^* \times \mathbb{R})/ \sim$.

**Proof.** The hypothesis $V_0^D = \text{Ker} D$ implies that $\sigma(D) \cap i\mathbb{R} \subseteq \{0\}$, hence $G_D$ is an exponential solvable Lie group by an application of [FuLu15, Def. 5.2.11 and Th. 5.2.16]. In particular, $C^*(G_D)$ is type I. Using [W07, Th. 6.2 and Th. 8.43] it then follows that the orbits of $\alpha^D: \mathbb{R} \times V^* \rightarrow V^*$ are locally closed subsets of $V^*$. 
Moreover, for any $\xi \in \mathcal{V}^*$ its stability group $H_\xi$ with respect to the group action $\alpha^D$ is a closed connected subgroup of $H := (\mathbb{R}, +)$ by [FulLu15] Th. 5.3.2 and Prop. 5.2.13, hence that stability group is equal to $\mathbb{R}$ if $\xi \in (\mathcal{V}_D^D \oplus \mathcal{V}_D^D)^\perp$ and is equal to $\{0\}$ if $\xi \in \mathcal{V}^* \setminus (\mathcal{V}_D^D \oplus \mathcal{V}_D^D)^\perp$. Here we have used again the hypothesis $\mathcal{V}_0^D = \text{Ker} \, D$. We then obtain

$$H_\xi^\perp = \begin{cases} \{0\} & \text{if } \xi \in (\mathcal{V}_D^D \oplus \mathcal{V}_D^D)^\perp \\ \mathbb{R} & \text{if } \xi \in \mathcal{V}^* \setminus (\mathcal{V}_D^D \oplus \mathcal{V}_D^D)^\perp. \end{cases}$$

It then follows by [Wi07] Th. 8.3.9 that $(\mathcal{V}^* \times \mathbb{R})/\sim$ is homeomorphic to the primitive ideal space of $\mathbb{R} \ltimes_{\alpha^D} C_0(\mathcal{V}^*)$, which is further homeomorphic to the unitary dual space of $G_D$, and this completes the proof. □

**Lemma 2.5.** Let $D \in \text{End} (\mathcal{V})$ and denote $\mathcal{V}_\epsilon := \mathcal{V}_\epsilon^D$ for $\epsilon \in \{-, 0, +\}$. There exist a norm $\| \cdot \|$ on $\mathcal{V}$ and a constant $a > 0$ with the properties:

(i) If $v \in \mathcal{V}_-$ then $\|e^{\epsilon tD}v\| \leq e^{-at}\|v\|$ and $\|e^{-\epsilon tD}v\| \geq e^{at}\|v\|$ for all $t \geq 0$.

(ii) If $v \in \mathcal{V}_+$ then $\|e^{\epsilon tD}v\| \geq e^{at}\|v\|$ and $\|e^{-\epsilon tD}v\| \leq e^{-at}\|v\|$ for all $t \geq 0$.

(iii) For every $v \in \mathcal{V}_- \setminus \{0\}$ (respectively, $v \in \mathcal{V}_+ \setminus \{0\}$) the function $\mathbb{R} \to (0, \infty)$, $t \mapsto \|e^{\epsilon tD}v\|$ is strictly decreasing (respectively, increasing) and bijective.

**Proof.** Using [CK14] Prop. 2.2.7 and proof of Prop. 2.2.8 we first define $\| \cdot \|$ on $\mathcal{V}_-$ and on $\mathcal{V}_+$ satisfying the conditions in the statement. Then we define $\| \cdot \|$ on $\mathcal{V}_0$ as an arbitrary norm, and finally we define $\|v_- + v_0 + v_+\| := \max(\|v_-\|, \|v_0\|, \|v_+\|)$ for any $v_\pm \in \mathcal{V}_\pm$ and $v_0 \in \mathcal{V}_0$, using the direct sum decomposition $\mathcal{V} = \mathcal{V}_- \oplus \mathcal{V}_0 \oplus \mathcal{V}_+$. This completes the proof of Assertions (i–iii).

For (iv) we note that if $v \in \mathcal{V}_- \setminus \{0\}$ then by (i) the function $h_v : \mathbb{R} \to [0, \infty)$, $t \mapsto \|e^{\epsilon tD}v\|$, is strictly decreasing and $\lim_{t \to 0} h_t(v) = 0$ while $\lim_{t \to \infty} h_t(v) = \infty$. On the other hand, if $v \in \mathcal{V}_+ \setminus \{0\}$ then by (ii) the function $h_v$ is strictly increasing and $\lim_{t \to \infty} h_t(v) = \infty$ while $\lim_{t \to 0} h_t(v) = 0$. □

**Lemma 2.6.** Let $D \in \text{End} (\mathcal{V})$ and denote $\mathcal{V}_\epsilon := \mathcal{V}_\epsilon^D$ for $\epsilon \in \{-, 0, +\}$.

Assume $\mathcal{V}_0 = \text{Ker} \, D$ and let $q : \mathcal{V} \to \mathcal{V}/\alpha_D$ denote the quotient map associated to the group action $\alpha_D$. We also define

$$A := \mathcal{V} \setminus ((\mathcal{V}_- + \mathcal{V}_0) \cup (\mathcal{V}_+ + \mathcal{V}_0)) \quad \text{and} \quad A_\pm := (\mathcal{V}_\pm + \mathcal{V}_0) \setminus \mathcal{V}_0.$$

Then the following assertions hold:

(i) One has the partition $\mathcal{V} = A \cup A_- \cup A_+ \cup \mathcal{V}_0$ into invariant subsets with respect to the group action $\alpha_D$.

(ii) If $A \neq \emptyset$, then $q(A)$ is open, dense and Hausdorff in $\mathcal{V}/\alpha_D$.

(iii) The sets $q(A_+)$ and $q(A_-)$ are closed subsets of $(\mathcal{V}/\alpha_D) \setminus q(\mathcal{V}_0)$.

(iv) The set $q(A_\pm)$ is homeomorphic to the Cartesian product of $\mathcal{V}_0$ and unit sphere of $\mathcal{V}_\pm$.

(v) If $\mathcal{V}_\pm = \text{Ker} \, (D \mp 1)$, then the set $q(A)$ is the set of all separated points of $\mathcal{V}/\alpha_D$.

**Proof.** Assertion (i) is clear, using the direct sum decomposition $\mathcal{V} = \mathcal{V}_- \oplus \mathcal{V}_0 \oplus \mathcal{V}_+$, in which all summands are invariant to the group action $\alpha$.

For (ii–iii) we recall that the quotient map $q$ is continuous and open, while $A$ is open and dense in $\mathcal{V}$ if $A \neq \emptyset$. Hence $q(A)$ is open and dense in $\mathcal{V}/\alpha_D$. Similarly,
we note that $A_{\pm}$ are disjoint open subsets of $\mathcal{V} \setminus \{A \cup \mathcal{V}_0\}$, hence $q(A_{\pm})$ and $q(A_{\pm})$ are closed subsets of $(\mathcal{V}/\alpha_D) \setminus q(\mathcal{V}_0)$.

It remains to prove that $q(A)$ is Hausdorff and that (iv) and (v) hold true. To this end we note that

$$A = \{v_0 + v_+ \in \mathcal{V}_+ \cup \mathcal{V}_0 \cup \mathcal{V}_+ \mid v_0 \neq 0 \neq v_+\}$$

and

$$A_{\pm} = \{v_{\pm} + v_0 \in \mathcal{V}_{\pm} \cup \mathcal{V}_0 \mid v_{\pm} \neq 0\}.$$  

We endow $\mathcal{V}$ with the norm given by Lemma 2.3 and we define

$$\Delta := \{v_0 + v_+ + v_- \in \mathcal{V}_+ \cup \mathcal{V}_0 \cup \mathcal{V}_+ \mid \|v_0\| = \|v_+\| = 0\}$$

and

$$\Delta_{\pm} = \{v_{\pm} + v_0 \in \mathcal{V}_{\pm} \cup \mathcal{V}_0 \mid \|v_{\pm}\| = 1\}.$$  

Then the maps

$$\Psi := \alpha|_{\mathcal{R}_D}: \mathcal{R} \times \Delta \to A$$

and

$$\Psi_{\pm} := \alpha|_{\mathcal{R}_D}: \mathcal{R} \times \Delta_{\pm} \to A_{\pm}$$

are homeomorphisms as a direct consequence of Lemma 2.3 (iii). One thus obtains the homeomorphisms $q|_{\Delta} : \Delta \to q(A)$ and $q|_{\Delta_{\pm}} : \Delta_{\pm} \to q(A_{\pm})$, which entail (iv) and the fact that the open set $q(A) \subseteq \mathcal{V}/\alpha_D$ is Hausdorff in its relative topology. Therefore every point of $q(A)$ is separated in $\mathcal{V}/\alpha_D$.

It remains to check that no point in $(\mathcal{V}/\alpha_D) \setminus q(A)$ is separated in $\mathcal{V}/\alpha_D$. One has $\mathcal{V}/\alpha_D = q(A) \cup q(A_{-}) \cup q(A_{+}) \cup q(\mathcal{V}_0)$. To this end we note that, for arbitrary $v_{\pm} \in \Delta_{\pm} \cap \mathcal{V}_{\pm}$ and $v_0 \in \mathcal{V}_0$, the sequence

$$q(e^{-j}v_- + v_0 + v_+) = \{e^{-j}e^{t}v_- + v_0 + e^{t}v_+ \mid t \in \mathbb{R}\}$$

contains in its limit set both $v_- + v_0$ (for $t_j = -j$) and $v_0 + v_+$ (for $t_j = 0$), hence these points cannot be separated from each other by disjoint open neighborhoods. This completes the proof.

\begin{theorem}
Let $D_1, D_2 \in \text{End}(\mathcal{V})$ for which the $C^*$-algebras $C^*(G_{D_1})$ and $C^*(G_{D_2})$ are $*$-isomorphic and $\mathcal{V}^{D_j}_0 = \text{Ker} D_j$ for $j = 1, 2$. Then $\dim \mathcal{V}^{D_j}_\pm = \dim \mathcal{V}^{D_1}_\pm$ and $\{n_{\pm}\mid \epsilon = \pm\} = \{n_{\pm}^{D_j}\mid \epsilon = \pm\}$.
\end{theorem}

\begin{proof}
The solvable Lie group $G_{D_j}$ is exponential by Lemma 2.4. It then follows by [BH16a, Th. 3.5] that the real rank of $C^*(G_{D_j})$ can be computed in the following way:

$$\text{RR}(C^*(G_{D_j})) = \dim \mathcal{V}^{D_j}_0 - \dim \text{Ran} D_j$$

$$= 1 + \dim \mathcal{V} - \dim \text{Ran} D_j = 1 + \dim \text{Ker} D_j.$$  

Since $C^*(G_{D_1}) \simeq C^*(G_{D_2})$, we obtain $\dim \text{Ker} D_j = \dim \text{Ker} D_2$, hence by hypothesis $\dim \mathcal{V}^{D_1}_\pm = \dim \mathcal{V}^{D_2}_\pm$.

It remains to prove that $n_{\pm}^{1, j} = n_{\pm}^{2, j}$, where $n_{\pm}^{j} := n_{\pm}^{D_j}$. By Theorem 2.2 we may assume $\mathcal{V}^{D_j}_\pm = \text{Ker}(D_j \mp 1)$, that is, one has

$$D_j = \begin{pmatrix} -1_{n_{\pm}^{j}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1_{n_{\pm}^{j}} \end{pmatrix}$$

with respect to the direct sum decomposition $\mathcal{V} = \mathcal{V}^{D_j}_- \cup \mathcal{V}^{D_j}_0 \cup \mathcal{V}^{D_j}_+$, where $1_{n_{\pm}^{j}}$ is the identity map of $\mathcal{V}^{D_j}_\pm$ for $j = 1, 2$. \hfill \square
Then we use the fact that the unitary dual spaces of $G_{D_j}$ and $G_{D_2}$ are homeomorphic, and by Lemmas 2.4 and 2.6 one obtains \( \{ n^1_\epsilon \mid \epsilon = \pm \} = \{ n^2_\epsilon \mid \epsilon = \pm \} \) by a direct application of an argument in the proof of [LiLu13, Prop. 2.1].

\( \square \)

**Remark 2.8.** Theorems 2.2 and 2.7 are generalizations of [LiLu13, Prop. 2.4], and of [LiLu13, Prop. 2.1], respectively, where one additionally assumed that $D_j \in \text{End}(V)$ is semisimple (which is stronger than our assumption $V^{D_j} = \text{Ker} D_j$) for $j = 1, 2$.

### 2.2. Quasidiagonality of Generalized $ax+b$-Groups

The main results of this section are Theorem 2.15 and its corollaries, and their proofs require some basic notions on topological dynamical systems, which we now recall.

Let $X \times \mathbb{R} \to \mathbb{R}$, $(x, t) \mapsto x \cdot t$, be a continuous right action of the group $(\mathbb{R}, +)$ on a compact space $X$. For every subset $Y \subseteq X$ we define the closed subsets of $X$

\[
\omega(Y) := \bigcap_{t \in \mathbb{R}} \text{Cl}(Y \cdot [t, \infty)) \\
\omega^*(Y) := \bigcap_{t \in \mathbb{R}} \text{Cl}(Y \cdot (-\infty, t])
\]

where Cl\((\cdot)\) stands for the closure of a subset of $X$. Equivalently

\[
\omega(Y) = \{ x \in X \mid (\exists \{y_n\}_{n \in \mathbb{N}} \subseteq Y, \{t_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}) \lim_{n \to \infty} t_n = \infty, \lim_{n \to \infty} y_n \cdot t_n = x \}
\]

and there is a similar description of $\omega^*(Y)$ in which \( \lim_{n \to \infty} t_n = -\infty \).

For every singleton set $\{x\} \subseteq X$ we denote $\omega(\{x\}) =: \omega^*(\{x\}) =: \omega^*(x)$.

A subset $A \subseteq T$ is said to be an attractor, respectively repeller, if it has a neighborhood $N$ with $\omega(N) = A$, respectively $\omega^*(N) = A$. It is clear that attractors and repellers are closed invariant subsets of $X$.

If $A \subseteq X$ is an attractor, then we define its corresponding complementary repeller

\[A^* := \{ x \in X \mid \omega(x) \cap A = \emptyset \} = X \setminus \tilde{o(A)}\]

(see [CK14, Lemma 8.2.5]) where the set $\tilde{o}(A) := \{ x \in X \mid \omega(x) \cap A \neq \emptyset \}$ is called the *domain of influence of $A$*. The pair of sets $(A, A^*)$ is said to be a nontrivial attractor-repeller pair if $A \cup A^* \subseteq X$. (We point out that an attractor-repeller pair $(A, A^*)$ was called trivial in [CK14] page 161) if either $A = X$ and $A^* = \emptyset$, or $A = \emptyset$ and $A = X$, which leads to a different notion of nontrivial attractor-repeller pair. If however $X$ is connected then, using the fact that both $A$ and $A^*$ are closed, it follows that our notion of nontriviality agrees with the terminology of [CK14]. Incidentally, in our applications of these notions in Lemma 2.13 and Proposition 2.14 below, the space $X$ is the one-point compactification of a finite-dimensional real vector space hence is connected.)

**Notation 2.9.** For any locally compact space $S$ we denote by $\overline{S} := S \cup \{\infty\}$ its one-point compactification if $S$ is non-compact, and for every homeomorphism $\theta : S \to S$ we denote again by $\theta : \overline{S} \to \overline{S}$ its extension to a homeomorphism with $\theta(\infty) = \infty$. If $S$ is compact, then we denote $\overline{S} := S$ and $C_0(S) := C(S)$.

**Lemma 2.10.** Let $M_0$ be a separable $C^*$-algebra without unit, with its unitization $\mathcal{M} := M_0 \oplus \mathbb{C}$. Assume that $G$ is an amenable separable locally compact group for which $C^*(G)$ is quasidiagonal. If one has a continuous action $G \times M_0 \to M_0$, canonically extended to an action $G \times \mathcal{M} \to \mathcal{M}$, then $G \times M_0$ is quasidiagonal if and only if $G \ltimes \mathcal{M}$ is quasidiagonal.
Proof. One has the split exact sequence $0 \to \mathcal{M}_0 \to \mathcal{M} \to \mathbb{C}1 \to 0$ which, by [Pl87, Lemma 2.82], leads to the split exact sequence

$$0 \to G \rtimes \mathcal{M}_0 \to G \rtimes \mathcal{M} \to C^*(G) \to 0.$$  

If $G \rtimes \mathcal{M}$ is quasidiagonal then also its ideal $G \rtimes \mathcal{M}_0$ is quasidiagonal.

Conversely, let us assume that $G \rtimes \mathcal{M}_0$ is quasidiagonal. The $C^*$-algebra $G \rtimes \mathcal{M}_0$ is separable, hence is $\sigma$-unital. Moreover, in the above split exact sequence, $C^*(G)$ is a separable nuclear $C^*$-algebra since $G$ is an amenable separable locally compact group (see [Pe79, p. 393]). Since $C^*(G)$ is quasidiagonal, it then follows by [BrDa04 Prop. 2.5] that $G \rtimes \mathcal{M}$ is quasidiagonal, and this completes the proof. \hfill \Box

**Proposition 2.11.** Let $\alpha : S \times \mathbb{R} \to \mathbb{R}$, $(x, t) \mapsto x \cdot t$, be a continuous right action of the group $(\mathbb{R}, +)$ on a separable, locally compact, metrizable space $S$. We extend this action to $\alpha : \mathbb{S} \times \mathbb{R} \to \mathbb{S}$ as above. Then the following assertions are equivalent:

(i) The flow $\alpha$ on $S$ has no nontrivial attractor-repeller pair.
(ii) The $C^*$-algebra $\mathbb{R} \rtimes_\alpha C_0(S)$ is quasidiagonal.
(iii) There exists an embedding of $\mathbb{R} \rtimes_\alpha C_0(S)$ into an AF-algebra.

Proof. Since the locally compact space $S$ is metrizable and separable, the compact space $\mathbb{S}$ is also metrizable. On the other hand, let $\mathcal{R} := \bigcap_A (A \cup A^*)$, where the intersection is taken over all attractors $A$ of the flow $\alpha$ on $S$. It follows by [CK14 Def. 3.1.7 and Th. 8.3.3] that the flow $\alpha$ on $\mathbb{S}$ is chain recurrent if and only if $\mathcal{R} = \mathbb{S}$. By the above definition of $\mathcal{R}$, the equality $\mathcal{R} = \mathbb{S}$ holds if and only if for every attractor $A$ of $\mathbb{S}$ one has $A \cup A^* = \mathbb{S}$, which by our definition prior to Notation 2.9 means that the attractor-repeller pair $(A, A^*)$ is trivial. Thus, the flow $\mathbb{S}$ has no nontrivial attractor-repeller pair if and only if it is chain-recurrent. Therefore, by [Pi99 Th. 4.7], the following conditions are equivalent:

1. The flow $\alpha$ on $\mathbb{S}$ has no nontrivial attractor-repeller pair.
2. The $C^*$-algebra $\mathbb{R} \rtimes_\alpha C(\mathbb{S})$ is quasidiagonal.
3. There exists an embedding of $\mathbb{R} \rtimes_\alpha C(\mathbb{S})$ into an AF-algebra.

By Lemma 2.10 one has $\mathbb{II} \iff \mathbb{II}$. Since $\mathbb{R} \rtimes_\alpha C(S)$ is an ideal of $\mathbb{R} \rtimes_\alpha C(\mathbb{S})$, we obtain $\mathbb{II} \implies \mathbb{III}$, and moreover it is well known that every AF-embeddable $C^*$-algebra is quasidiagonal, hence $\mathbb{III} \implies \mathbb{II}$. This completes the proof that actually $\mathbb{II} \iff \mathbb{III} \iff \mathbb{II} \iff \mathbb{II} \implies \mathbb{III}$. \hfill \Box

**Remark 2.12.** Here we collect a few basic facts to be used in the proof of Lemma 2.13 and Proposition 2.14 below. For more details, see [CK14].

Let $V$ be a finite-dimensional real vector space and $D \in \text{End}(V)$ and let $E(\mu_k) := E^D(\mu_k)$, $k = 1, \ldots, r$ be the real generalized eigenspaces for $D$, where $\mu_k, k = 1, \ldots, r$, are the eigenvalues of $\sigma(D)$.

We denote by $\lambda_1 > \cdots > \lambda_r$ the distinct real parts of the eigenvalues $\mu_k, k = 1, \ldots, r$, of $\sigma(D)$, and define the Lyapunov space $V(\lambda_j)$ to be the direct sum of all real generalized eigenspaces $E(\mu_k)$ associated to eigenvalues $\mu_k$ with $\text{Re} \mu_k = \lambda_j$. Then one has the direct sum decomposition

$$V = V(\lambda_1) \oplus \cdots \oplus V(\lambda_r)$$  

(see [CK14 page 13]). One can prove that

$$V(\lambda_j) = \{0\} \cup \left\{ v \in V \mid \lim_{t\to\pm\infty} \frac{1}{t} \log \| \exp(tD)v \| = \lambda_j \right\}$$  

(2.2)
(see [CK14 Th. 1.4.3]). With \( V_j := V(\lambda_j) \oplus \cdots \oplus V(\lambda_k) \) and \( W_j := V(\lambda_1) \oplus \cdots \oplus V(\lambda_j) \), it follows by [CK14 Th. 1.4.4] that if \( v \in V \setminus \{0\} \), then
\[
\lim_{t \to \infty} \frac{1}{t} \log \| \exp(tD)v \| = \lambda_j \iff v \in V_j \setminus V_{j+1}
\]
and
\[
\lim_{t \to -\infty} \frac{1}{t} \log \| \exp(tD)v \| = -\lambda_j \iff v \in W_j \setminus W_{j-1},
\]
where \( V_{j+1} = W_0 := \{0\} \).

By (2.3), one obtains
\[
v \in V(\lambda_j), \pm \lambda_j > 0 \implies \lim_{t \to \pm \infty} \exp(tD)v = \infty \text{ and } \lim_{t \to \mp \infty} \exp(tD)v = 0. \quad (2.6)
\]
It follows by (2.4)–(2.5) that
\[
v \in V_j \setminus V_{j+1}, \lambda_j > 0 \implies \lim_{t \to \infty} \exp(tD)v = \infty \quad (2.7)
\]
and
\[
v \in W_j \setminus W_{j-1}, \lambda_j < 0 \implies \lim_{t \to -\infty} \exp(tD)v = \infty. \quad (2.8)
\]

**Lemma 2.13.** If \( V \) is a finite-dimensional real vector space and \( D \in \text{End}(V) \) whose corresponding flow \( \alpha_D : \mathbb{R} \times V \to V \), \( \alpha_D(v, t) := \exp(tD)v \), has a nontrivial attractor-repeller pair, then there exists no \( \tau \in \mathbb{R} \setminus \{0\} \) with \( \tau \in \sigma(D) \).

**Proof.** Let \( \nabla = A \sqcup Z \sqcup A^* \) with \( Z \neq \emptyset \), where \((A, A^*)\) is a nontrivial attractor-repeller pair, and assume that there exists \( \tau \in \mathbb{R} \setminus \{0\} \) with \( \tau \in \sigma(D) \). We will argue by contradiction, in a few steps, repeatedly using the fact that by [CK14 Prop. 8.2.7],
\[
\omega(v) \subseteq A \text{ and } \omega^*(v) \subseteq A^*, \text{ hence } \omega(v) \cap \omega^*(v) = \emptyset, \text{ for all } v \in Z. \quad (2.9)
\]
This implies that, since 0 and \( \infty \) are fixed points of the flow \( \alpha_D \), we cannot have \( 0, \infty \in Z \) hence \( Z \subseteq V \setminus \{0\} \).

**Step 1:** We first prove that \( V = \omega(0) \), that is, \( \sigma(D) \subseteq i\mathbb{R} \).

To this end we note that since \( \tau \in \sigma(D) \), there exist \( v_1, v_2 \in V \) with \( 0 \neq v_1 + iv_2 \in \text{Ker}(D - i\tau) \subseteq \mathbb{C} \otimes \mathbb{R} \). Since \( D(V) \subseteq V \), it is easy to check that \( v_1, v_2 \in V \setminus \{0\} \) are linearly independent vectors. It also follows that \( \exp(tD)v_1 = (\cos(t\tau))v_1 - (\sin(t\tau))v_2 \) for every \( t \in \mathbb{R} \), and, since \( \tau \neq 0 \), this directly implies \( \omega(v_1) = \omega^*(v_1) = \exp(\mathbb{R}D)v_1 \simeq \mathbb{T} \). Then, by (2.9), one has \( v_1 \notin Z \).

Therefore \( v_1 \in A \sqcup A^* \), and one similarly obtains \( sv \in A \sqcup A^* \) for every \( s \in \mathbb{R} \). (If \( s = 0 \) then \( \omega(0) = \omega^*(0) = \{0\} \), so \( 0 \notin Z \) by the above argument.) Thus \( \mathbb{R}v_1 \subseteq A \sqcup A^* \), and since \( \mathbb{R}v_1 \) is connected, we must have either \( \mathbb{R}v_1 \subseteq A \) or \( \mathbb{R}v_1 \subseteq A^* \). We may assume \( \mathbb{R}v_1 \subseteq A \). Then, since \( A \) is a closed subset of \( \nabla \), we obtain \( \mathbb{R}v_1 \cup \{\infty\} \subseteq A \), hence \( 0, \infty \in A \). On the other hand, if there exists \( \lambda \in \sigma(D) \) with \( \text{Re} \lambda \neq 0 \), then (2.6) implies that we have either \( 0 \in A \) and \( \infty \in A^* \), or \( 0 \in A^* \) and \( \infty \in A \), which is a contradiction with \( 0, \infty \in A \). Consequently, \( \text{Re} \lambda = 0 \) for every \( \lambda \in \sigma(D) \), that is, \( V = \omega(0) \).

**Step 2:** We now prove that the map \( D \) is semisimple.

To this end let us assume that \( D \) is not semisimple. We fix a real Jordan basis in \( V \) (see for instance [CK14 Th. 1.2.3]), and then we define the Euclidean scalar product on \( V \) for which that Jordan basis is an orthonormal basis. Let \( D = S + N \) be the Jordan decomposition, where \( S \) is semisimple, \( N \) is nilpotent, and \( SN = NS \).

The assumption that \( D \) has only purely imaginary eigenvalues implies that for...
every $t \in \mathbb{R}$ the map $\exp(tS)$ is an isometry, while the assumption that $D$ is not semisimple implies that there exists an integer $k \geq 2$ with $N^k = 0 \neq N^{k-1}$. Since $Z$ is an open nonempty subset of $\mathcal{V}$ while $N^{k-1} \neq 0$, it follows that $Z \nsubseteq \ker N^{k-1}$, hence there exists $v \in Z$ with $N^{k-1}v \neq 0$. Then it is easily checked that

$$\exp(Nv) = \sum_{j=0}^{k-1} \frac{N^j v}{j!} \to \infty \text{ as } t \to \pm \infty$$

hence

$$\lim_{t \to \pm \infty} \|\exp(t)v\| = \lim_{t \to \pm \infty} \|\exp(tS)\exp(Nv)\| = \lim_{t \to \pm \infty} \|\exp(tNv)\| = \infty$$

and this shows that $\infty \in \omega(v) \cap \omega^*(v)$, which is a contradiction with (2.9). Consequently, $D$ must be semisimple.

**Step 3:** We now show that if $D \in \text{End}(\mathcal{V})$ is semisimple and $\sigma(D) \subseteq i\mathbb{R}$, then

$$(\forall v \in \mathcal{V}) \quad \omega(v) = \omega^*(v). \tag{2.10}$$

For $v \in Z$, this is a contradiction with (2.9), which completes the proof.

To prove (2.10), we will use a real Jordan basis and its corresponding Euclidean structure on $\mathcal{V}$ as in Step 2 above. By the current assumption on $D$ we obtain as above that $\exp(tD) \in \text{SO}(\mathcal{V})$ for every $t \in \mathbb{R}$, where $\text{SO}(\mathcal{V})$ is the group of all isometries of $\mathcal{V}$ that can be joined by a continuous path to the identity operator $1$. It is well known that $\text{SO}(\mathcal{V})$ is a compact group.

Now let $v \in \mathcal{V}$ and $x \in \omega(v)$ arbitrary. Then there exists a sequence of positive real numbers $\{t_n\}_{n \geq 1}$ with $\lim_{n \to \infty} t_n = \infty$ and $\lim_{n \to \infty} \exp(t_nD)v = x$. Since $\exp(t_nD) \in \text{SO}(\mathcal{V})$ for all $n \geq 1$, one has $\|x\| = \|v\| < \infty$, hence $x \neq \infty$.

We choose an arbitrary sequence $\{s_n\}_{n \geq 1}$ in $(0, \infty)$ with $s_{n+1} - s_n \geq 2t_n$ for every $n \geq 1$. Since $\text{SO}(\mathcal{V})$ is compact, there exists a sequence of positive integers $n_1 < n_2 < \cdots$ for which the sequence $\{\exp(s_{n_q}D)\}_{q \geq 1}$ is convergent. In particular, for $r_q := s_{n_{q+1}} - s_{n_q} \geq 2t_{n_q}$ we obtain

$$\lim_{q \to \infty} r_q = \infty \quad \text{and} \quad \lim_{q \to \infty} \exp(r_qD) = 1.$$  

We also note that for all $q \geq 1$,

$$r_q = s_{n_{q+1}} - s_{n_q} = \sum_{j=n_q}^{n_{q+1}-1} s_{j+1} - s_j \geq \sum_{j=n_q}^{n_{q+1}-1} 2t_j \geq 2t_{n_q}.$$  

For $a_q := r_q - t_{n_q} \geq t_{n_q}$ we have $\lim_{q \to \infty} a_q = \infty$ and, by $\exp(-a_qD) \in \text{SO}(\mathcal{V})$,

$$\|\exp(-a_qD)v - x\| \leq \|\exp(-a_qD)v - \exp(t_{n_q}D)v\| + \|\exp(t_{n_q}D)v - x\| = \|v - \exp((a_q + t_{n_q})D)v\| + \|\exp(t_{n_q}D)v - x\| = \|v - \exp(r_qD)v\| + \|\exp(t_{n_q}D)v - x\| \to 0 \text{ as } q \to \infty$$

hence $x \in \omega^*(v)$. This shows that $\omega(v) \subseteq \omega^*(v)$. The converse inclusion can be proved similarly, hence (2.10) is completely proved, and we are done.

**Proposition 2.14.** Let $\mathcal{V}$ be a finite-dimensional real vector space. If $D \in \text{End}(\mathcal{V})$ then its corresponding flow

$$\alpha_D : \overline{\mathcal{V}} \times \mathbb{R} \to \overline{\mathcal{V}}, \quad \alpha_D(v, t) := \exp(tD)v$$
has a nontrivial attractor-repeller pair if and only if either \( \Re z > 0 \) for every \( z \in \sigma(D) \) or \( \Re z < 0 \) for every \( z \in \sigma(D) \). If this is the case, then the only nontrivial attractor-repeller pair is \((\{\infty\}, \{0\})\) or \((\{0\}, \{\infty\})\), respectively.

**Proof.** If \( \Re z < 0 \) for every \( z \in \sigma(D) \), then \( 0 > \lambda_1 > \cdots > \lambda_t \), hence by (2.4) we easily obtain that for arbitrary \( v \in V \) there exists \( t_0 > 0 \) such that if \( t \in (t_0, \infty) \), then \( \frac{1}{t} \log \|exp(tD)v\| < \lambda_1/2 \), that is, \( \|exp(tD)v\| < \exp(t\lambda_1/2) \), and then \( \lim_{t \to \infty} \exp(tD)v = 0 \). This shows that \( \omega(v) = \{0\} \) for every \( v \in V \), which directly implies that there exists a unique nontrivial attractor-repeller pair, namely \((\{0\}, \{\infty\})\).

In the case when \( \Re z > 0 \) for every \( z \in \sigma(D) \), we obtain just as above, using however (2.5) instead of (2.4), that for every \( v \in V \{0\} \) one has \( \lim_{t \to \infty} \exp(tD)v = \infty \), hence \( \omega(v) = \{\infty\} \subset V^* \), and this implies that there exists a unique nontrivial attractor-repeller pair, namely \((\{\infty\}, \{0\})\).

By Lemma 2.13 it remains to show that if \( \lambda_1 > 0 > \lambda_t \), then there exists no nontrivial attractor-repeller pair. Reasoning by contradiction, let us assume that there exists an attractor \( A \) with \( Z := \overline{V} \setminus (A \cup A^*) \neq \emptyset \).

The inequalities \( \lambda_1 > 0 > \lambda_t \) imply in particular \( \ell \geq 2 \), and then \( V_2 \subseteq V \) and \( W_{\ell-1} \subseteq V \).

We note that \( Z = \overline{V} \setminus (A \cup A^*) \) is an open subset of \( V^* \) and \( Z \neq \emptyset \) by assumption. As noted at the beginning of the proof of Lemma 2.13 one has \( Z \subseteq V \{0\} \). Then \( Z \) is an open nonempty subset of \( V \), hence one has neither \( Z \subseteq V_2 \) nor \( Z \subseteq W_{\ell-1} \). That is, there exist \( z_1 \in Z \setminus V_2 \) and \( z_2 \in Z \setminus W_{\ell-1} \). Then, using (2.7), we obtain \( \{\infty\} \subseteq \omega(z_1) \subseteq \omega(Z) \subseteq A \), and on the other hand, by (2.8) we obtain \( \{\infty\} \subseteq \omega^*(z_2) \subseteq \omega^*(Z) \subseteq A^* \), hence \( \infty \in A \cap A^* \), which is a contradiction with the fact that always \( A \cap A^* = \emptyset \). This completes the proof. \( \square \)

**Theorem 2.15.** Let \( V \) be a finite-dimensional real vector space, fix \( D \in \text{End}(V) \). The following assertions are equivalent:

(i) One has either \( \Re z > 0 \) for every \( z \in \sigma(D) \) or \( \Re z < 0 \) for every \( z \in \sigma(D) \).

(ii) The \( C^* \)-algebra \( C^*(G_D) \) is not quasidiagonal.

(iii) There exists no embedding of \( C^*(G_D) \) into any AF-algebra.

**Proof.** We denote \( G := G_D \) for simplicity.

The implication (iii) \( \implies \) (iii) holds true with \( C^*(G) \) replaced by any \( C^* \)-algebra.

For (iii) \( \iff \) (ii) we note that Assertion (ii) is equivalent to the similar property of the linear map \( D^* \in \text{End}(V^*) \). By Proposition 2.14 the Assertion (ii) is equivalent to existence of a nontrivial attractor-repeller pair for the flow \( \alpha_D^* : V^* \times \mathbb{R} \to V^* \), and by Lemma 2.11 this is further equivalent to any of the following assertions:

(a) The \( C^* \)-algebra \( \mathbb{R} \ltimes_{\alpha_D^*} M \) fails to be quasidiagonal.

(b) There exists no embedding of \( \mathbb{R} \ltimes_{\alpha_D^*} M \) into any AF-algebra.

Here we denoted \( M := \mathbb{C}1 + \mathcal{C}_0(V^*) \simeq \mathcal{C}(V^*) \). Recall from [Wi07] Ex. 3.16 that there is a \(*\)-isomorphism

\[
C^*(G) \simeq \mathbb{R} \ltimes_{\alpha_D^*} \mathcal{C}_0(V^*).
\] (2.11)

Then, by Lemma 2.10 and (2.11), the above Assertion (ii) is equivalent to the fact that \( C^*(G) \) fails to be quasidiagonal, and this completes the proof of (iii) \( \iff \) (iii).

For (iii) \( \iff \) (i) we note that if (i) does not hold true, then the above assertion (i) also fails to be true (by the above discussion), hence there exists an
embedding of \( \mathbb{R} \rtimes \alpha_D, \mathcal{M} \) into some AF-algebra. In particular, the ideal \( C^*(G) \simeq \mathbb{R} \rtimes \alpha_D, C_0(V^*) \) of \( \mathbb{R} \rtimes \alpha_D, \mathcal{M} \) embeds into some AF-algebra, hence \( \textup{iii} \) fails to be true, and this completes the proof. \( \square \)

We now draw a corollary of the above theorem that also needs Theorem 1.1 proved in the next section. We point out however that the proof of Theorem 1.1 relies only on Theorem 2.15 and not on this corollary. We decided to give this result here in order to draw a more complete picture of the quasidiagonality properties of the generalized ax + b-groups.

**Corollary 2.16.** Let \( V \) be a finite-dimensional real vector space and select any \( D \in \text{End}(V) \) whose spectrum does not contain any nonzero purely imaginary eigenvalues. Then \( G_D \) is an exponential solvable Lie group, and moreover \( C^*(G_D) \) is quasidiagonal but not strongly quasidiagonal if and only if \( D \) satisfies the following spectral condition:

- One has \( \sigma(D) \neq \{0\} \) and there exist \( z_1, z_2 \in \sigma(D) \) with \( \text{Re } z_1 \leq 0 \leq \text{Re } z_2 \).

**Proof.** Existence of \( z_1, z_2 \in \sigma(D) \) with \( \text{Re } z_1 \leq 0 \leq \text{Re } z_2 \) is equivalent to the fact that Assertion \( \textup{ii} \) in Theorem 2.15 does not hold true, and this is equivalent to the fact that \( C^*(G_D) \) is quasidiagonal.

Moreover, the spectrum of \( D \in \text{End}(V) \) does not contain nonzero purely imaginary eigenvalues if and only if the solvable Lie group \( G_D \) is exponential (see e.g., \( \textup{FuLu15} \)). Then, by Theorem 2.15, \( C^*(G_D) \) is not strongly quasidiagonal if and only if the Lie group \( G_D \) is not nilpotent, which is further equivalent to the fact that the map \( D \) is not nilpotent, that is, \( \sigma(D) \neq \{0\} \). \( \square \)

**Corollary 2.17.** Let \( V \) be a real vector space with \( m := \dim V < \infty \), and denote by \( \text{End}_0(V) \) the set of all endomorphisms \( D \in \text{End}(V) \) with \( V^D = \text{Ker } D \).

There exist exactly

\[
N(m) := \sum_{n_0=0}^m \left( 1 + \left\lceil \frac{m-n_0}{2} \right\rceil \right) \geq m + 1
\]

\(*\)-isomorphism classes in the set of \( C^*\)-algebras \( \{ C^*(G_D) \mid D \in \text{End}_0(V) \} \), and exactly one of these \( *\)-isomorphism classes contains non-quasidiagonal \( C^*\)-algebras.

**Proof.** For every \( D \in \text{End}_0(V) \) we denote \( n_0^D := \text{dim}(\text{Ker } D) \). It follows by Theorem 2.15 and Theorem 2.17 that for \( D_1, D_2 \in \text{End}_0(V) \) one has a \( *\)-isomorphism \( C^*(G_{D_1}) \simeq C^*(G_{D_2}) \) if and only if one has both \( n_0^{D_1} = n_0^{D_2} \) and \( \{ n_0^{D_1} | \epsilon = \pm \} = \{ n_0^{D_2} | \epsilon = \pm \} \). Therefore, it easily follows that, with \( N(m) \) defined in the statement, there exist exactly \( N(m) \) \( *\)-isomorphism classes in the set of \( C^*\)-algebras \( \{ C^*(G_D) \mid D \in \text{End}_0(V) \} \). By Theorem 2.15, exactly one of these \( *\)-isomorphism classes corresponds to non-quasidiagonal \( C^*\)-algebras, namely for \( D \in \text{End}_0(V) \) with \( n_0^D = n_0^{D_2} n_0^{D_2} = 0 \).

As an example of the above Theorem 2.15, we obtain that the \( C^*\)-algebra of the Mautner group \( \mathbb{R}^4 \rtimes \alpha_D, \mathbb{R} \) is quasidiagonal, where

\[
D = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{C}) \simeq \text{End}(\mathbb{R}^4)
\]

with \( \theta \in \mathbb{R} \setminus \mathbb{Q} \). It is well known that it is a solvable Lie group that is not of type I (see for instance \( \textup{AuMo66} \) or \( \textup{AbAiSc04} \) Prop. 2.1).
Existence of solvable Lie groups whose $C^*$-algebras are not quasidiagonal, which follows by Corollaries 2.16, 2.17 shows that the analogue of Rosenberg’s conjecture (see [TWW16, Cor. C]) does not carry over directly to non-discrete groups.

3. Proof of Theorem 1.1

We first establish the most direct implications between the properties from the statement.

(iii) $\equiv$ (i): This is well known. See for instance [Hd87, Prop. 8(2)].

(ii) $\iff$ (iii): This follows by [AnMo66, Ch. V, Thms. 1–2].

For any exponential solvable Lie group $G$, one has (iii) $\iff$ (iv) by [BBG16, Rem. 2.11]. Finally, (iv) $\iff$ (v) by [Plu15, Th. 5.3.31].

We now begin the preparations for the remaining step in the proof of Theorem 1.1, namely (i) $\implies$ (iii).

**Proposition 3.1.** Let $G$ be a locally compact group with a $C^*$-dynamical system $G \times A \to A$, $(g,a) \mapsto g \cdot a$ and a normal subgroup $K \subseteq G$ for which we define the fixed-point subalgebra $A^K := \{ a \in A \mid (\forall k \in K) \, k \cdot a = a \}$. Then the following assertions hold:

(i) $A^K$ is a $G$-invariant closed *-subalgebra of $A$.

(ii) If $K$ is additionally assumed to be a compact subgroup of $G$, $S$ is another amenable closed subgroup of $G$ with $G = SK$ and $S \cap K = \{1\}$, and, the probability Haar measure of $K$ is invariant under the map $k \mapsto sks^{-1}$ for every $s \in S$, then there exists an injective *-morphism $S \ltimes A^K \hookrightarrow G \times A$.

**Proof.** For every $a \in A$, $g \in G$ and $k \in K$ one has

$$k \cdot (g \cdot a) = g \cdot ((g^{-1}k) \cdot a)$$

where $g^{-1}k \in K$ since $K$ is a normal subgroup of $G$. Therefore, if $a \in A^K$ then $g \cdot a \in A^K$ for arbitrary $g \in G$.

We will now prove Assertion (iii). Since $K$ is a compact group, one has the injective *-morphism

$$\eta: A^K \to L^1(K,A) \hookrightarrow K \ltimes A, \quad (\eta(a))(k) = a \text{ for all } a \in A^K, k \in K.$$ 

(See [Ro79].)

We now make explicit certain natural actions of $S$ on $A^K$ and on $K \ltimes A$, and check that $\eta$ is $S$-equivariant with respect to these actions. It follows by Assertion (i) that the action of $G$ on $A$ induces an action of $G$ on $A^K$, which further gives by restriction an action of $G$ on $A^K$. To describe the action of $S$ on $K \ltimes A$, we first note the semidirect product decomposition $G = SK$, which follows by the hypothesis. We then obtain by [Wi07, Prop. 3.11] a $C^*$-dynamical system $S \times (K \ltimes A) \to K \ltimes A$ given by

$$(s \cdot f)(k) = s \cdot (f(s^{-1}ks)) \text{ for all } s \in S \text{ and } f \in L^1(K,A) \hookrightarrow K \ltimes A,$$

where we have used the fact that $K$ is a normal subgroup of $G$ and the action of $S$ on $A$ given by the $C^*$-dynamical system $G \times A \to A$. With respect to these actions of $S$ on $A^K$ and on $K \ltimes A$, one clearly has

$$((\forall s \in S)(\forall a \in A^K) \quad \eta(s \cdot a) = s \cdot (\eta(a)).$$
Since $S$ is an amenable group, crossed products by $L$ are canonically isomorphic to their corresponding reduced crossed products, hence by the above $S$-equivariance property of the injective $*$-morphism $\eta$ obtain an injective $*$-morphism

$$\tilde{\eta}: S \rtimes A^K \to S \rtimes (K \rtimes A)$$

for instance by [Pe79, Prop. 7.7.9]. Using the $*$-isomorphism $S \rtimes (K \rtimes A) \simeq (S \rtimes K) \rtimes A$ given by [Wi07, Prop. 3.11], the above map $\tilde{\eta}$ gives an injective $*$-morphism $S \rtimes A^K \to G \rtimes A$, we are done. \hfill \square

**Corollary 3.2.** Let $G$ be a locally compact group two closed subgroups $K$ and $S$ satisfying $G = SK$, $S \cap K = \{1\}$, and $sk = ks$ for all $s \in S$ and $k \in K$. We also assume that $K$ is compact and we fix a $C^*$-dynamical system $G \times A \to A$, $(g, a) \mapsto g \cdot a$.

Then $A^K := \{a \in A \mid (\forall k \in K) k \cdot a = a\}$ is an $S$-invariant closed $*$-subalgebra of $A$ whose corresponding crossed product $S \rtimes A^K$ is a quasidiagonal $C^*$-algebra (respectively, is AF-embeddable) if $G \rtimes A$ is.

**Proof.** It follows directly from Proposition 3.1 leave that $S \rtimes A^K$ is $*$-isomorphic to a closed $*$-subalgebra of $G \rtimes A$. \hfill \square

**Example 3.3.** For each integer $n \geq 1$, we denote by $SO(n)$ its corresponding special orthogonal group, that is, the group of all orthogonal matrices $T \in M_n(\mathbb{R})$ with $\det T = 1$. It is well known that $SO(n)$ is a connected compact Lie group.

Since the tautological action of $K := SO(n)$ on $\mathbb{R}^n$ is an action by linear maps, it commutes with the action of the multiplicative group $(0, \infty)$ on $\mathbb{R}^n$ by $(r, x) \mapsto rx$. One has the group isomorphism $S := (\mathbb{R}, +) \simeq ((0, \infty), \cdot)$, $s \mapsto e^s$, hence one thus obtains an action of the group $G := K \times S$ on $\mathbb{R}^n$,

$$\mathbb{R}^n \times G \to \mathbb{R}^n, \quad (x, (T, s)) \mapsto T(e^sx) = e^s(Tx) \quad (3.1)$$

for $(T, s) \in K \times S = G$ and $x \in \mathbb{R}^n$. This defines a $C^*$-dynamical system $G \times A \to A$ for the commutative $C^*$-algebra $A := C_0(\mathbb{R}^n)$.

In the notation of Corollary 3.2, one clearly has the $*$-isomorphism

$$C_0([0, \infty)) \to A^K, \quad f \mapsto f(\cdot | \cdot),$$

where $\cdot | \cdot$ is the Euclidean norm on $\mathbb{R}^n$. One can transport the action of $S$ from $A^K$ to $C_0([0, \infty))$ via this $*$-isomorphism, and one thus obtains the $C^*$-dynamical system

$$S \times C_0([0, \infty)) \to C_0([0, \infty)), \quad (s, h) \mapsto s \cdot h,$$

where $(s \cdot \varphi)(r) := \varphi(e^sr)$ for all $s \in \mathbb{R}$, $r \in [0, \infty)$, and $\varphi \in C_0([0, \infty))$.

On the other hand, the unitization of the commutative $C^*$-algebra $C_0([0, \infty))$ is $*$-isomorphic to $C([0, \infty])$, and one has a canonical extension of the above action of $S$ on $C_0([0, \infty))$ to an action of $S$ on $C([0, \infty))$. The corresponding crossed product $S \times C([0, \infty])$ is the $C^*$-algebra of the flow

$$S \times [0, \infty] \to [0, \infty], \quad (s, r) \mapsto e^sr.$$

It is clear that the pair $([\infty], \{0\})$ is a nontrivial attractor-repeller pair for this flow, hence it follows by Proposition 2.11 that the $C^*$-algebra $S \times C_0([0, \infty))$ is not quasidiagonal. Therefore $S \times A^K$ is not quasidiagonal, and then by Corollary 3.2 we obtain that the $C^*$-algebra $G \rtimes A$, that is, $(SO(n) \times \mathbb{R}) \rtimes C_0(\mathbb{R}^n)$, is not quasidiagonal.
Example 3.4. In connection with Example [22] we recall that the Euclidean motion group $E(n) := \mathbb{R}^n \rtimes SO(n)$ is the semidirect product defined via the tautological action of $K := SO(n)$ on $\mathbb{R}^n$, hence in particular

$$C^*(E(n)) \simeq SO(n) \ltimes C_0(\mathbb{R}^n).$$

This is a liminal algebra for instance by [Wi81, Th. 3.1], therefore it is strongly quasidiagonal. The $C^*$-algebras of such groups were recently studied in [AraH15].

On the other hand, there is also the action of $S = (\mathbb{R}, +)$ on $\mathbb{R}^n$ as in Example [3.3] which defines the following action of $\mathbb{R}$ by automorphisms of $E(n)$,

$$\mathbb{R} \times E(n) \to E(n), \quad (s, (x, T)) \mapsto (e^s x, T)$$

and which further defines a natural action of $\mathbb{R}$ on $C^*(E(n))$. We will show that the corresponding crossed product $\mathbb{R} \rtimes C^*(E(n))$ is not quasidiagonal. (We recall however from [Br04, Cor. 11.2] that the crossed product of any quasidiagonal $C^*$-algebra by a separable compact group remains quasidiagonal.)

To this end we first note the Lie group isomorphism

$$E(n) \rtimes \mathbb{R} \to \mathbb{R}^n \rtimes (SO(n) \rtimes \mathbb{R}), \quad ((x, T), s) \mapsto (x, (T, s))$$

which defines a $*$-isomorphism $C^*(E(n) \rtimes \mathbb{R}) \simeq C^*(\mathbb{R}^n \rtimes (SO(n) \rtimes \mathbb{R}))$. Since $C^*(E(n) \rtimes \mathbb{R}) \simeq \mathbb{R} \rtimes C^*(E(n))$ and $C^*(\mathbb{R}^n \rtimes (SO(n) \rtimes \mathbb{R})) \simeq (SO(n) \rtimes \mathbb{R}) \rtimes C_0(\mathbb{R}^n)$, it then follows by Example [22] that $\mathbb{R} \rtimes C^*(E(n))$ is not quasidiagonal.

We now prove that if (iii) in Theorem [1.1] fails to be true, then (i) is not true. To this end we show that there exists a closed 2-sided ideal $\mathcal{J} \subseteq C^*(G)$ for which the quotient $C^*(G)/\mathcal{J}$ is not strongly quasidiagonal, and then $C^*(G)$ cannot be strongly quasidiagonal. The proof of this fact is divided in three steps. The first two show how to reduce the problem to certain low-dimension groups, while the third step treats the case of those low-dimension groups.

Step 1. It follows by [AuMo66, Th. 1–2 and Prop. 2.2, Ch. V] that if $G$ is a connected simply connected solvable Lie group of type I, then $G$ fails to be a liminal group if and only if there exists a connected simply connected normal subgroup $N \subseteq G$ for which the quotient Lie group $G/N$ is isomorphic to one of the following Lie groups:

1. $S_2 := \mathbb{R} \rtimes \mathbb{R}$, the connected real $ax + b$-group, defined via

   $$\alpha : (\mathbb{R}, +) \to \text{Aut}(\mathbb{R}, +), \quad \alpha(t)s = e^t s$$

2. $S_3 := \mathbb{R}^2 \rtimes_{\sigma^\epsilon} \mathbb{R}$, defined for $\sigma \in \mathbb{R} \setminus \{0\}$ via

   $$\alpha^\sigma : (\mathbb{R}, +) \to \text{Aut}(\mathbb{R}^2, +), \quad \alpha^\sigma(t) = e^{at} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

3. $S_4 := \mathbb{R}^2 \rtimes_\beta \mathbb{R}^2$, defined via

   $$\beta : (\mathbb{R}^2, +) \to \text{Aut}(\mathbb{R}^2, +), \quad \beta(t, s) = e^t \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}$$
Step 2. Every short exact sequence of amenable locally compact groups
\[ 1 \rightarrow N \rightarrow G \rightarrow G/N \rightarrow 1 \]
leads to a short exact sequence of C*-algebras
\[ 0 \rightarrow J \rightarrow C^*(G) \rightarrow C^*(G/N) \rightarrow 0 \]
for a suitable closed 2-sided ideal J of C*(G). This shows that in order to implement the method of proof described above, it suffices to check that the C*-algebra of the above groups S2, S3 with \( \sigma \in \mathbb{R} \setminus \{0\} \), and S4 are not strongly quasidiagonal.

Step 3. Using Theorem 2.15 it follows directly that the C*-algebra of any of the groups S2 and S3 with \( \sigma \in \mathbb{R} \setminus \{0\} \) is not strongly quasidiagonal.

We now discuss the case of the group S4. It is easily seen that the surjective map
\[ S_4 = \mathbb{R}^2 \rtimes_{\sigma} \mathbb{R} \rightarrow \mathbb{R}^2 \rtimes_{\alpha} (\mathbb{R} \times \text{SO}(2)), \quad (x, (t, s)) \mapsto (x, (t, \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix})) \]
is a group homomorphism (whose kernel is isomorphic to \( \mathbb{Z} \)), where \( \alpha \) is the group action from (3.1) for \( n = 2 \). As in Step 2 above, one then obtains a short exact sequence of C*-algebras
\[ 0 \rightarrow I \rightarrow C^*(S_4) \rightarrow C^*(\mathbb{R}^2 \rtimes_{\alpha} (\mathbb{R} \times \text{SO}(2))) \rightarrow 0 \]
for a suitable closed 2-sided ideal I of C*(S4). The special case \( n = 2 \) of Example 3.3 shows that the C*-algebra \( C^*(\mathbb{R}^2 \rtimes_{\alpha} (\mathbb{R} \times \text{SO}(2))) \) is not quasidiagonal, and then the above short exact sequence shows that \( C^*(S_4) \) cannot be strongly quasidiagonal.

This completes the proof of Theorem 1.1.

4. ON FAITHFUL TRACIAL STATES OF C*-ALGEBRAS OF SOLVABLE GROUPS

In this final section we show that the C*-algebras of the groups we have been studied in the previous sections do not admit faithful tracial states. Thus we are in a situation that is complementary to the situation in [TWW16]: The reduced C*-algebra of any discrete group admits a canonical faithful tracial state which is quasidiagonal when the group is countable and amenable, which in turn implies the quasidiagonality of the reduced C*-algebra of the group.

**Lemma 4.1.** If \( G \) is a connected topological solvable group and \( \pi: G \rightarrow \mathcal{B}(\mathcal{H}) \) is a continuous irreducible representation with \( \dim \mathcal{H} < \infty \), then \( \dim \mathcal{H} \leq 1 \).

**Proof.** See [BaRa86, Ch. 8, §1, Th. 1].

The following Lemma is well known, but we insert its proof for the sake of completeness.

**Lemma 4.2.** Let \( \mathcal{A} \) be a C*-algebra with a faithful state \( \varphi: \mathcal{A} \rightarrow \mathbb{C} \). Assume that \( \pi_\varphi: \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_\varphi) \) is a *-representation with a cyclic vector \( x_\varphi \in \mathcal{H}_\varphi \) with \( \|x_\varphi\| = 1 \) and \( \varphi(a) = (\pi_\varphi(a)x_\varphi | x_\varphi) \) for every \( a \in \mathcal{A} \). Then the following assertions hold:
(i) One has \( \text{Ker} \pi_\varphi = \{0\} \).
(ii) If the faithful state \( \varphi \) is moreover a tracial state, then the functional
\[ \tau_\varphi: \pi_\varphi(\mathcal{A})'' \rightarrow \mathbb{C}, \quad \tau_\varphi(T) := (Tx_\varphi | x_\varphi) \]
is a normal faithful tracial state on the von Neumann algebra \( \pi_\varphi(\mathcal{A})'' \).
Proof. For the first assertion, if \( \pi_\varphi(a) = 0 \) then \( (\pi_\varphi(a^*a)x_\varphi \mid x_\varphi) = \varphi(a^*a) = 0 \).
Since the state \( \varphi \) is faithful, we then obtain \( a^*a = 0 \), hence \( a = 0 \).

For the second assertion, let us assume that the faithful state \( \varphi \) is tracial. The functional \( \tau_\varphi \) in the statement is clearly a state of \( \pi_\varphi(A)'\)' and is continuous with respect to the weak operator topology, hence it is in particular a normal state of \( \pi_\varphi(A)'' \).

In order to check that \( \tau_\varphi \) is a tracial state, we first note that \( \tau_\varphi(\pi_\varphi(a)) = \varphi(a) \) for every \( a \in A \). Therefore, if \( T_j = \pi_\varphi(a_j) \) for \( j = 1, 2 \), then, using the fact that \( \varphi \) is a tracial state, we obtain

\[
\tau_\varphi(T_1T_2) = \tau_\varphi(\pi_\varphi(a_1a_2)) = \varphi(a_1a_2) = \varphi(a_2a_1)\tau_\varphi(\pi_\varphi(a_2a_1)) = \tau_\varphi(T_2T_1).
\]

Since we have already seen that \( \tau_\varphi \) is a normal state and on the other hand the bicommutant theorem ensures that \( \pi_\varphi(A) \) is dense in \( \pi_\varphi(A)'' \) in the strong operator topology, the equality \( \tau_\varphi(T_1T_2) = \tau_\varphi(T_2T_1) \) extends to all \( T_1, T_2 \in \pi_\varphi(A)'' \).

It remains to prove that \( \tau_\varphi \) is faithful. To this end, let \( T \in \pi_\varphi(A)'' \) with \( \tau_\varphi(T^*T) = 0 \), which is equivalent to \( Tx_\varphi = 0 \). Then for every \( a \in A \) one has

\[
\|T\pi_\varphi(a)x_\varphi\|^2 = (\pi_\varphi(a)^*T^*T\pi_\varphi(a)x_\varphi \mid x_\varphi) = \tau_\varphi(\pi_\varphi(a)^*T^*T\pi_\varphi(a)).
\]

We have already seen that \( \tau_\varphi \) is a tracial state of \( \pi_\varphi(A)'' \), hence we further obtain

\[
\|T\pi_\varphi(a)x_\varphi\|^2 = \tau_\varphi(\pi_\varphi(a)^*\pi_\varphi(a)^*T^*T) = (\pi_\varphi(a)^*\pi_\varphi(a)^*T^*T)x_\varphi \mid x_\varphi) = 0
\]

where we used \( Tx_\varphi = 0 \). Thus \( T\pi_\varphi(a)x_\varphi = 0 \) for every \( a \in A \). Since \( x_\varphi \) is a cyclic vector for the representation \( \pi_\varphi \), it follows that \( T = 0 \), hence the state \( \tau_\varphi \) is faithful, and we are done. \( \Box \)

Lemma 4.3. If \( A \) is a \( C^* \)-algebra of type I which has a faithful tracial state, then the following assertions hold:

(i) If \( \{\pi_j\}_{j \in J} \) is a complete system of distinct representatives of the unitary equivalence classes of finite-dimensional irreducible \( * \)-representations of \( A \), then the \( * \)-representation \( \bigoplus_{j \in J} \pi_j \) is faithful.

(ii) If for every irreducible \( * \)-representation \( \pi : A \to B(H) \) one has \( \dim H \leq 1 \), then \( A \) is commutative.

Proof. The first assertion is a more detailed statement of \([\text{BrOz08}, \text{Prop. 7.1.8}]\), whose proof also requires the above Lemma 4.2. The second assertion follows directly from the first assertion. \( \Box \)

Proposition 4.4. Let \( G \) be a connected locally compact solvable group of type I. If \( C^*(G) \) has a faithful tracial state, then \( G \) is commutative.

Proof. It is well known that the group \( G \) is commutative if and only if the Banach algebra \( L^1(G) \) is commutative. This is further equivalent to the property that \( C^*(G) \) be commutative, since \( L^1(G) \) is dense in \( C^*(G) \).

On the other hand, \( C^*(G) \) is a \( C^* \)-algebra of type I with a faithful tracial state, and for every irreducible \( * \)-representation of \( \pi : C^*(G) \to B(H) \) one has \( \dim H \leq 1 \) as a direct consequence of Lemma 4.3. Therefore \( C^*(G) \) is commutative by Lemma 4.3 and this concludes the proof. \( \Box \)
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