Iterating the recursively Mahlo operations

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Abstract

In this paper we address a problem: How far can we iterate lower recursively Mahlo operations in higher reflecting universes? Or formally: How much can lower recursively Mahlo operations be iterated in set theories for higher reflecting universes?

It turns out that in $\Pi_1^N$-reflecting universes the lowest recursively Mahlo operation can be iterated along towers of $\Sigma_1$-exponential orderings of height $N - 3$, and that all we can do is such iterations. Namely the set theory for $\Pi_1^N$-reflecting universes is proof-theoretically reducible to iterations of the operation along such a tower.

For set-theoretic formulas $\varphi$,

$$P \models \varphi : \iff (P, \in) \models \varphi.$$  

In what follows, let $L$ denote a transitive set, which is a universe in discourse. $P, Q, \ldots$ denotes transitive sets in $L \cup \{L\}$ such that $\omega \in P$.

Let $\mathcal{X}$ be a first-order class of transitive sets. This means that there exists a first-order sentence $\varphi$ such that $P \in \mathcal{X} \iff P \models \varphi$. Then a set theory $T$ is said to prove $L \in X$ iff $T \vdash \varphi$.

A $\Pi_i$-recursively Mahlo operation for $2 \leq i < \omega$, is then defined through a universal $\Pi_i$-formula $\Pi_i(a)$:

$$P \in M_i(\mathcal{X}) \iff \forall b \in P [P \models \Pi_i(b) \rightarrow \exists Q \in \mathcal{X} \cap P(b \in Q \models \Pi_i(b))].$$

(read: $P$ is $\Pi_i$-reflecting on $\mathcal{X}$.)

Its iteration is defined by transfinite recursion on ordinals $\beta$:

$$M_\beta^i := \bigcap \{M_i(M_\nu^i) : \nu < \beta\}.$$  

Observe that $M_i(\mathcal{X})$ is $\Pi_{i+1}$, i.e., there exists a $\Pi_{i+1}$-sentence $m_i(\mathcal{X})$ such that $P \in M_i(\mathcal{X})$ iff $P \models m_i(\mathcal{X})$ for any transitive (and admissible) set $P$.

A transitive set $P$ is said to be $\Pi_i$-reflecting if $P \in M_i = M_1^i$.

Let us denote

$$\mathcal{X} \prec_i \mathcal{Y} : \iff \mathcal{Y} \subseteq M_i(\mathcal{X}),$$

i.e., $\forall P \in \mathcal{Y}(P \in M_i(\mathcal{X}))$.  

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$P \in M_{i+1}$ is much stronger than $P \in M_i$: Assume $P \in M_{i+1}$ and $P \models \Pi_i(b)$ for $b \in P$. Then $P \in M_i$ and $P \models m_i \land \Pi_i(b)$ for the $\Pi_{i+1}$-sentence $m_i$ such that $P \in M_i$ iff $P \models m_i$. Hence there exists a $Q \in P$ such that $Q \models m_i \land \Pi_i(b)$, i.e., $Q \in M_i \land Q \models \Pi_i(b)$. This means $P \in M_i^2 = M_i(M_i)$, i.e., $M_i \prec_i M_{i+1}$. Moreover $P \in M_i^\Delta$, i.e., $P \in \bigcap\{M_i^\beta : \beta \in \text{ord}(P)\}$, $M_i^\Delta \prec_i M_{i+1}$, and so on.

In particular a set theory $\text{KPI}_{i+1}$ for universes in $M_{i+1}$ proves the consistency of a set theory for universes in $M_i^\Delta$.

In this paper we address a problem: How far can we iterate lower recursively Mahlo operations in higher reflecting universes? Or formally: How much can lower recursively Mahlo operations be iterated in set theories for higher reflecting universes? Specifically: What kind of iterations of the lowest operations $M_2$ do we need to obtain equiconsistent theories for set theories for higher reflecting universes?

1 **Iterations of the operation $M_i$ in $\Pi_{i+1}$-reflectings**

In this section we see that iterations of the operation $M_i$ along $\Sigma_1$-relations on $\omega$ are too short to resolve $\Pi_{i+1}$-reflecting universes provided that the $\Sigma_1$-relations are provably wellfounded in $\text{KPI}_{i+1}$.

**Definition 1.1**

1. $\text{KP}_\ell$ denotes a set theory for limits of admissibles. $\text{KPI}_N$ denotes a set theory for universes in $M_N$.

2. For a definable relation $\prec$ and set-theoretic universe $P$ (admissibility suffices) let

$$P \in M_i(a; \prec) :\iff P \in \bigcap\{M_i(M_i(b; \prec)) : b \prec^P a\},$$

where $b \prec^P a :\iff P \models b \prec a$.

Note that $M_i(a; \prec)$ is a $\Pi_{i+1}$-class for (set-theoretic) $\Sigma_{i+1} \prec$.

3. We say that a theory $T$ is **proof-theoretically reducible** to another theory $S$ if $T$ is a $\Pi^1_1$ (on $\omega$)-conservative extension of $S$, and the fact is provable in a weak arithmetic, e.g., the elementary recursive arithmetic EA.

4. For a relation $\prec$ on $\omega$, $TI(a, \prec)$ denotes the transfinite induction schema up to $a \in \omega$:

$$\{\forall x \in \omega[\forall y < x \varphi(y) \rightarrow \varphi(x)] \rightarrow \forall x < a \varphi(x) : \varphi \text{ is a set-theoretic formula}\}$$

and $TI(a, \prec, \Pi_n)$ its restriction to $\Pi_n$-formulas $\varphi$.

Using a universal $\Pi_n$-formula, $TI(a, \prec, \Pi_n)$ is equivalent to a single $\Pi_{n+2}$-formula.

5. A relation $\prec$ on $\omega$ is said to be **almost wellfounded** in $\text{KP}_\ell$ if $\text{KP}_\ell$ proves the transfinite induction schema $TI(a, \prec)$ up to each $a \in \omega$. 

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It is easy to see the following lemma using the fact that $M_i(a;\prec)$ is $\Pi_{i+1}$.

**Lemma 1.2** Let $\prec$ be a $\Sigma_1$ relation on $\omega$. Then $\text{KP} \Pi_{i+1}$ ($i \geq 2$) proves

\[ \forall a \in \omega [TI(a, \prec, \Pi_{i+1}) \rightarrow L \in M_i(a; \prec)] . \]

A fortiori $\text{KP} \Pi_{i+1}$ proves $\forall a \in \omega [TI(a, \prec, \Pi_{i+1}) \rightarrow L \in M_2(M_i(a; \prec))]$.

In other words, $\text{KP} \ell$ proves $P \in M_{i+1} \rightarrow \forall a \in \omega [TI(a, \prec P, \Pi_{i+1}^P) \rightarrow P \in M_i(a; \prec)]$.

Therefore $\forall a \in \omega [L \in M_i(a; \prec)]$ is too weak to reduce $\text{KP} \Pi_{i+1}$ proof-theoretically for any $\Sigma_1$ relation $\prec$ on $\omega$, for example $\text{KP} \Pi_{i+1} \vdash \text{CON}(\forall a \in \omega [L \in M_i(a; \prec)])$ if $\forall a \in \omega [TI(a, \prec)]$ is provable in $\text{KP} \Pi_{i+1}$.

Nonetheless $\Pi_{i+1}$-reflecting universes can be approximated by iterations of the operation $M_i$ along well founded $\Sigma_1$ relations on $\omega$.

**Theorem 1.3** For each $i$ ($2 \leq i < \omega$) there exists a $\Sigma_1$ almost wellfounded relation $<_i$ in $\text{KP} \ell$ such that $\text{KP} \Pi_{i+1}$ is proof-theoretically reducible to the theory

\[ \text{KP} \ell + \{L \in M_i(a; <_i) : a \in \omega\} . \]

Theorem 13 follows from Lemma 12 and Theorem 3.5 below.

The case $i = 2$ means that the set theory $\text{KP} \Pi_3$ for $\Pi_3$-reflecting universes can be resolved by iterations of the recursively Mahlo operations $M_2$.

**Remark.** Although $\text{KP} \ell$ is weaker than $\text{KP} \Pi_{i+1}$, $\text{KP} \Pi_{i+1}$ does not prove the soundness of $\text{KP} \ell$: Let $\text{Fund}$ denote the axiom schema for Foundation. Then for a $\varphi \in \Sigma_{i+2}$ and a standard provability predicate $\text{Pr}_\text{Fund}$ of $\text{Fund}$

\[ \text{KP} \Pi_{i+1} \not| \forall n \in \omega [\text{Pr}_\text{Fund}([\varphi(n)]) \rightarrow \varphi(n)] \]

since $\text{KP} \Pi_{i+1} \setminus \text{Fund} \subseteq \Pi_{i+2}$ ($i \geq 2$).

Hence even if $\text{KP} \Pi_{i+1} \vdash \forall a \in \omega [\text{Pr}_\text{KP} \ell([TI(a, <_i, \Pi_{i+1}))])$, this does not imply $\text{KP} \Pi_{i+1} \vdash \forall a \in \omega TI(a, <_i, \Pi_{i+1})$.

## 2 $\Pi_3$-reflecting on $\Pi_3$-reflectings

Our goal is to approximate $\Pi_{i+1}$-reflecting universes by iterations of the lowest recursively Mahlo operations $M_2$. Let us consider first the simplest case: $\Pi_3$-reflecting universes on $\Pi_3$-reflectings, $M^2_3 = M_3(M_3)$. Universes in $M^2_3$ are seen to be resolved in terms of iterations of the operation $M_2$ along a lexicographic ordering on pairs.

**Definition 2.1** 1. For a $\Sigma_1$ relation $\prec$ on $\omega$, $W(\prec)$ denotes the well-founded part of $\prec$:

\[ a \in W(\prec) :\iff \forall f \in \omega \exists n \in \omega [f(0) = a \rightarrow f(n + 1) \neq f(n)]. \]
$W(\prec)$ is $\Pi_1$.

Note that $W(<^Q)$ is a set in limits of admissibles $P$ for any transitive set $Q \in P$.

2. For two transitive relations $<_1, <_0$ on $\omega$, $<_L := L(<_1, <_0)$ denotes the lexicographic ordering:

$\langle n_1, n_0 \rangle <_L \langle m_1, m_0 \rangle \iff n_1 <_1 m_1 \text{ or } (n_1 = m_1 \& n_0 <_0 m_0)$.

$L(<_1, <_0)$ is $\Sigma_1$ if $<_1$ and $<_0$ are $\Sigma_1$.

$<_{LW}$ denotes the restriction of $<_L$ to the wellfounded part in the second component:

$\langle n_1, n_0 \rangle <_{LW} \langle m_1, m_0 \rangle \iff \langle n_1, n_0 \rangle <_L \langle m_1, m_0 \rangle \& n_0, m_0 \in W(\prec)$.

$<_{LW}$ is $\Delta_2$ if $<_1$ and $<_0$ are $\Sigma_1$.

**Proposition 2.2** Let $P$ be a limit of admissibles and $<$ be a $\Sigma_1$ relation on $\omega$. Suppose $P \models a \in W(<)$. Then $a \in W^P(<^Q) = W(<^Q)$ and $Q \models TI(a, <)$ for any $Q \in P$, where

$$a \in W^P(<^Q) :\iff \forall f \in \omega \& P \exists n \in \omega [f(0) = a \to f(n + 1) \not<^Q f(n)].$$

**Proof.** Since $<$ is $\Sigma_1$ and $Q \subseteq P$, we have $<^Q \subseteq <^P$. Hence $a \in W^P(<^P) \subseteq W^R(<^Q)$ for any $R \subseteq P$. Therefore $a \in W^P(<^Q) = W^{Q^+}(<^Q) = W(<^Q)$ for the set $<^Q$ in $P$, and the next admissible $Q^+ \in P$ above $Q$. This yields the transfinite induction schema $TI(a, <^Q)$ up to $a$. \qed

**Lemma 2.3** Let $<_1, <_0$ be two $\Sigma_1$ transitive relations on $\omega$, and $<_{LW}$ the restriction of the lexicographic ordering defined from these to the wellfounded part in the second component.

Then KPi3(\Pi_3) proves

$$\forall a, \alpha \in \omega [TI(a, <_1, \Pi_3) \to L \in M_2(\langle a, \alpha \rangle; <_{LW})].$$

**Proof.** Let $L \in M_3(M_3)$. By transfinite induction on $a$ along $<_1$ we show

$$\forall \alpha \in \omega [L \in M_2(\langle a, \alpha \rangle; <_{LW})]$$

where

$$P \in M_2(\langle a, \alpha \rangle; <_{LW}) \iff P \in \bigcap \{M_2(M_2(\langle b, \beta \rangle; <_{LW}) : \langle b, \beta \rangle <_{LW}^P \langle a, \alpha \rangle \}$$

and

$$\langle b, \beta \rangle <_{LW}^P \langle a, \alpha \rangle \iff \langle b, \beta \rangle <_L^P \langle a, \alpha \rangle \& P \models \alpha, \beta \in W(<_0).$$

Suppose that $\forall b <_1 a \forall \beta \in \omega [L \in M_2(\langle b, \beta \rangle; <_{LW})]$, and $\langle b, \beta \rangle <_{LW}^P \langle a, \alpha \rangle$. We show $L \in M_2(M_2(\langle b, \beta \rangle; <_{LW})].$
III yields the case \( b <_1 a \). Assume \( b = a \) and \( \beta <_0 \alpha \in W(<_0) \). Suppose a \( \varphi \in \Pi_2 \) holds in \( L \in M_3(M_3) \). Pick a \( Q \in L \cap M_3 \) so that \( Q \models \varphi \) and \( Q \in \{ M_2(M_2((b, \gamma); <_{LW})) : Q \models b <_1 a \land \gamma \in W(<_0) \} \) by III.

We claim that \( Q \in M_2((a, \beta); <_{LW}) \). By Proposition 2.2 we have \( Q \models TI(\beta, <_0) \). Hence we have \( Q \in M_2((a, \beta); <_{LW}) \) by transfinite induction on \( \beta \).

**Theorem 2.4** There exist \( \Sigma_1 \) transitive relations \( <_1, <_0 \) on \( \omega \) such that \( <_1 \) is almost wellfounded in \( KP \ell \), and \( KP\Pi_3(\Pi_3) \) is proof-theoretically reducible to the theory

\[
KP \ell + \{ L \in \bigcap \{ M_2(M_2((a, \alpha); <_{LW})) : \alpha \in W(<_0) \} : a \in \omega \}
\]

for the restriction \( <_{LW} \) of the lexicographic ordering \( <_L = L(<_1, <_0) \) defined from these to the wellfounded part in the second components.

For a proof of Theorem 2.4 see [A]\(\infty\)b.

### 3 \( \Pi_N \)-reflection

As you expected, an exponential structure involves in resolving \( \Pi_N \)-reflecting universes \( L \).

**Definition 3.1** Let \( <_1, <_0 \) be two transitive relations on \( \omega \).

1. The relation \( <_E = E(<_1, <_0) \) is on sequences \( \langle n_i^1, n_i^0 \rangle : i < \ell \) of pairs with \( <_1 \)-decreasing first components \( n_{i+1}^1 < n_i^1 \), and is defined by

\[
\langle (n_i^1, n_i^0) : i < \ell_0 \rangle <_E \langle (m_i^1, m_i^0) : i < \ell_1 \rangle \iff
\]

either

\[
\exists k \forall i < k \forall j < 2[n_i^j = m_i^j \& (n_k^1, n_k^0) <_L (m_k^1, m_k^0)]
\]

or

\[
\ell_0 < \ell_1 \& \forall i < \ell_0 \forall j < 2[n_i^j = m_i^j]
\]

where \( <_L = L(<_1, <_0) \) in Definition 2.12.

Write \( \sum_{i<\ell} \pi^{n_i^1} n_i^0 \) for \( \langle (n_i^1, n_i^0) : i < \ell \rangle \).

2. Let \( \text{dom}(<_E) \) denote the domain of the relation \( _E \):

\[
\text{dom}(<_E) := \{ \sum_{i<\ell} \pi^{n_i^1} n_i^0 : \forall i < \ell - 1(n_{i+1}^1 < n_i^1) \& n_i^1, n_i^0, \ell \in \omega \}.
\]

3. \( <_{EW} \) denotes the restriction of \( <_E \) to the wellfounded part in the second components:

\[
\alpha = \sum_{i<\ell_0} \pi^{n_i^1} n_i^0 <_{EW} \sum_{i<\ell_1} \pi^{m_i^1} n_i^0 = \beta \iff
\]

\[
\alpha <_E \beta \& \{ n_i^0 : i < \ell_0 \} \cup \{ m_i^0 : i < \ell_1 \} \subseteq W(<_0).
\]
Lemma 3.2 Let $<_1,<_0$ be two transitive relations on $\omega$, $<_1$ is $\Delta_2$, $<_0$ is $\Sigma_1$, and $<_{EW}$ the restriction of the exponential ordering defined from these to the wellfounded part of the second component. Then KPL proves for each $i \geq 2$

$$\forall P \in L \cup \{L\} \forall a \in \omega \forall \alpha <^P a [P \in M_{i+1}(M_{i+1}(a;<_1)) \rightarrow P \in M_i(\alpha;<_{EW})]$$

where for $\alpha = \sum_{i<\ell} \pi_i^1 n_i^0 \in \text{dom}(<^P_E), \alpha <^P a :\iff n_0^1 <^P a$.

Proof. We show for any $a \in \omega$ and any $\beta \in \text{dom}(<^P_{EW} \uparrow a)$

$$P \in M_{i+1}(M_{i+1}(a;<_1)) \& P \in M_i(\beta;<_{EW}) \rightarrow \forall \alpha <^P a \{P \in M_i(\beta+\alpha;<_{EW})\}$$

by main induction on $P \in L \cup \{L\}$ with respect to the relation $\in$, where for $\beta = \sum_{i<\ell} \pi_i^1 n_i^0$ and $\alpha = \sum_{i<\ell_0} \pi_i^1 n_i^0$,

$$\beta \in \text{dom}(<^P_{EW} \uparrow a) :\iff \beta \in \text{dom}(<^P_{EW}) \& (\ell_1 > 0 \rightarrow a <^P 1 \cdot m_{\ell_1-1})$$

and $\beta + \alpha = \sum_{i<\ell_1} \pi_i^1 n_i^0 + \sum_{i<\ell_0} \pi_i^1 n_i^0$.

Suppose $\beta \in \text{dom}(<^P_{EW} \uparrow a)$, $P \in M_{i+1}(M_{i+1}(a;<_1))$ and $P \in M_i(\beta;<_{EW})$. Pick an $\alpha = \pi^b x + \alpha_0 \in \text{dom}(<^P_{EW})$ so that $\alpha_0 <^P b <^P a$ and $x \in W^P(<^0_E)$.

We show $P \in M_i(\beta+\alpha;<_{EW})$. It suffices to show $P \in M_i(\beta+\gamma;<_{EW})$ for any $\gamma <^P_{EW} \alpha$ by $P \in M_i(\beta;<_{EW})$.

If $\gamma$ is the empty sequence, then $P \in M_i(M_i(\beta;<_{EW}))$ follows from $P \in M_i(\beta;<_{EW})$, which is $\Pi_{i+1}$, and $P \in M_{i+1}(M_{i+1}(a;<_1)) \subseteq M_{i+1}$.

Let $\gamma = \pi^c y + \gamma_0$ with $\gamma_0 <^P c \leq^P b$, and $P \models \theta$ for a $\theta \in \Pi_i$. It suffices to find a $Q \in P$ so that $Q \models M_i(\gamma+\beta;<_{EW})$ and $Q \models \theta$.

First consider the case when $c <^P b$. By $P \in M_{i+1}(M_{i+1}(a;<_1))$, pick a $Q \in P$ so that $Q \models M_{i+1}(a;<_1)$, $Q \models \theta$, $\beta \in \text{dom}(<^Q_{EW} \uparrow a)$, $Q \models M_i(\beta;<_{EW})$ and $\text{dom}(<^Q_{EW}) \ni \gamma <^Q y <^Q a$.

Then $Q \models M_{i+1}(a;<_1) \subseteq M_{i+1}(M_{i+1}(b;<_1))$, and hence MIH yields $Q \in M_i(\beta+\gamma;<_{EW})$.

Thus we have shown $P \in \bigcap \{M_i(\beta+\delta;<_{EW}) : \delta <^P b\}$, which is $\Pi_{i+1}$, and hence

$$P \in M_i(M_{i+1}(a;<_1) \cap \bigcap \{M_i(\beta+\delta;<_{EW}) : \delta < b\}) \quad (1)$$

Second consider the case when $c = b$.

We can find a $Q \in P$ so that $Q \models M_{i+1}(a;<_1)$, $Q \models \theta$, $\beta \in \text{dom}(<^Q_{EW} \uparrow a)$, $Q \models \bigcap \{M_i(\beta+\delta;<_{EW}) : \delta <^Q b\}$ by (1) and $\text{dom}(<^Q_{EW}) \ni \gamma \& b <^Q a$. We have $x \in W^P(<^0_E) \subseteq W(<^0_Q)$ by Proposition 2.2.

Therefore it suffices to show

$$\forall x \in W(<^0_Q) \forall b \in \omega \forall \beta \models \text{dom}(<^Q_{EW} \uparrow b)[Q \models P \& Q \models M_{i+1}(M_{i+1}(b;<_1)) \& Q \models \bigcap \{M_i(\beta+\delta;<_{EW}) : \delta <^Q b\}]$$

$$\implies \forall \gamma \models <^Q b[Q \models M_i(\beta+\pi^b x + \gamma_0;<_{EW})]$$

by subsidiary induction on $x \in W(<^0_Q)$. 

First assume \( \beta + \pi^b y + \delta_0 <_Q^W \beta + \pi^b x + \gamma_0 \) with \( y <_Q^x \). SIH yields \( Q \in M_i(\beta + \pi^b y + \delta_0; <_E^W) \), and this implies \( Q \in M_i(M_i(\beta + \pi^b y + \delta_0; <_E^W)) \) by \( Q \in M_i+1 \).

Therefore we have shown \( Q \in M_i(\beta + \pi^b x; <_E^W) \) with \( \gamma_0 = 0 \). Now let \( \gamma_0 = \pi^c y + \gamma_1 \) with \( c <_1^Q b \). We have \( \beta + \pi^b x \in \text{dom}(<_Q^E \cup c) \), \( Q \in M_i+1(M_i+1(b; <_1)) \) & \( Q \in M_i(\beta + \pi^b x; <_E^W) \) and \( Q \in P \). Hence MIH yields \( Q \in M_i(\beta + \pi^b x + \gamma_0; <_E^W) \) for \( \gamma_0 <_Q^0 b \). \( \square \)

**Definition 3.3** Let \( <_i \) (\( 2 \leq i \leq N - 1 \)) be \( \Sigma_1 \) relations on \( \omega \). Define a *tower* relation \( <_T \) from these as follows.

Define inductively relations \( <_{E_i} \) (\( 2 \leq i \leq N - 1 \)).

1. \( <_{E_{N-1}} : \equiv <_{N-1} \).

2. \( <_{E_i} : \equiv E(<_{E_{i+1}}, <_i) \) for \( 2 \leq i \leq N - 2 \), cf. Definition 3.1.

Then let

\[
>_T : \equiv <_{E_2} .
\]

\( >_T \) denotes the restriction of \( <_T \) to the wellfounded parts in the second components hereditarily. Namely \( >_T \equiv >_{E_2} \) and

\[
\sum_{n < \ell} \alpha^a_n x_n \in \text{dom}(<_{E_i} W) \iff \\
\forall n < \ell-1(\alpha_{n+1} <_{E_{i+1}} W \alpha_n) \& \forall n < \ell(x_i \in W(<_i))
\]

with \( <_{E_{N-1}} W \equiv <_{N-1} \).

For \( a \in \omega \) and \( \alpha = \sum_{n < \ell} \alpha^a_n x_n \in \text{dom}(<_T) \), define inductively

\[
\alpha < a : \equiv \forall n < \ell(x_n < a)
\]

with \( \alpha_n < a : \equiv \alpha < a \) for \( \alpha_n \in \omega \).

Lemmas 3.2 and 1.2 yield the following for the set theory KP\( \Pi \Pi \) for universes in \( M_N \).

**Theorem 3.4** Let \( <_i \) (\( 2 \leq i \leq N - 1 < \omega \)) be \( \Sigma_1 \) transitive relations on \( \omega \). Let \( >_T \) denote the restriction of the tower \( >_T \) of the exponential orderings \( >_{E_i} \) defined from these to the wellfounded parts in the second components hereditarily.

Then KP\( \Pi \Pi \) proves that

\[
\forall a \in \omega \forall \alpha < a[TI(a, <_{N-1}, \Pi_N) \rightarrow L \in M_2(\alpha; >_T W)]
\]

and hence

\[
\forall a \in \omega \forall \alpha < a[TI(a, <_{N-1}, \Pi_N) \rightarrow L \in M_2(\alpha; >_T W)]
\]

We see an optimality of this resolving of \( \Pi_N \)-reflecting universes in terms of the lowest recursively Mahlo operation \( M_2 \).
Theorem 3.5 For each $N (2 < N < \omega)$ there exist $\Sigma_1$ transitive relations $<_i (2 \leq i \leq N - 1)$ on $\omega$ such that $<_N$ is almost wellfounded in KP$\ell$, and KP$\Pi_N$ is proof-theoretically reducible to the theory

$$\text{KP}\ell + \{L \in \bigcap (M_2(M_2(\alpha; <_{TW})) : \text{dom}(<_{TW}) \ni \alpha < a) : a \in \omega\}$$

for the restriction $<_{TW}$ of the tower $<_T$ of the exponential orderings $<_{E_i}$ defined from these to the wellfounded parts in the second components hereditarily.

Theorem 3.5 is extracted from proof-theoretic analyses of KP$\Pi_N$ in [A$\infty$] and [A$\infty$]. Let me spend some words on ordinal analyses, an ordinal informative proof-theoretic investigations in generalities.

4 Background materials from proof theory

Let $T$ be a recursive theory containing ACA$_0$, the predicative (and hence conservative) extension of the first order arithmetic PA, and $\Pi_1$-sound, i.e., any $T$-provable $\Pi^1_1$-sentence is true in the standard model.

Then its proof-theoretic ordinal $|T|$ is defined to be the supremum of the order types of the provably recursive well orderings:

$$|T| := \sup\{\alpha < \omega^\CK : T \vdash WO[<] \& \alpha = \text{order type } < \text{ of } < \text{ for a recursive ordering } <\}$$

Remark. The ordinal $|T|$ is stable if we consider $\Sigma^1_1$-orderings and/or add true $\Sigma^1_1$-sentences to $T \supseteq \text{ACA}_0$, an analogue to the C. Spector’s boundedness theorem. For a proof see [A98].

It is seen that $|T|$ is recursive, i.e., $|T| < \omega^\CK$, and easy to cook up a recursive well ordering $<_T$ whose order type is equal to $|T|$.

For each $p \in \omega$ let $<_p$ denote a recursive well ordering defined as follows:

1. The case when $p$ is a G"odel number of a proof in $T$ whose endformula is $WO[<]$ for a recursive binary relation $\prec$. Then put $<_p := \prec$.

2. Otherwise, let $<_p$ denote an empty ordering, i.e., $\text{dom}(<_p) = \emptyset$.

Glue these orderings together to get a recursive ordering $<_T$:

$$\langle n, p \rangle <^T \langle m, q \rangle : \iff p = q \& n <_p m \lor p < q$$

for a bijective pairing function $\langle n, p \rangle$.

Then $<_T$ is a recursive well ordering by the assumptions, and $|<_T| \leq |T| = \sup\{<_p : p \in \omega\} \leq |<_T| < \omega^\CK$ as desired.

Gentzen’s celebrated pioneering work yields $|\text{ACA}_0| = \varepsilon_0$. The first achievement for proof theory of impredicative theory was done by G. Takeuti. He
designed a recursive notation system of ordinals, which describes the proof theoretic ordinal of, e.g., $\Pi_1$-Comprehension Axiom. Nowadays Takeuti’s proof is understood as for set theories of $\Pi_2$-reflecting universes, i.e., for the Kripke-Platek set theory with the Axiom of Infinity, $\text{KP}_\omega$.

Ordinal analyses for stronger theories are now obtained. Let $\langle O(T), <_T \rangle$ denote a notation system of proof-theoretic ordinal of $T = \text{ACA}_0$, $\text{KP}_\omega$, $\text{KPM}$, $\text{KP}\Pi_N$, etc.

Ordinal analyses of theories $T$ show not only the fact $|O(T)| = |T|$ but also more, i.e., some conservative extension results.

**Theorem 4.1** Let $\text{EA}$ denote the elementary recursive arithmetic, a fragment $I\Delta_0 + \forall x\exists y(2^x = y)$ of $\text{PA}$.

1. If $<$ is an irreflexive, transitive and provably well founded relation in $T$ (not necessarily a total ordering), then there exists an ordinal term $\alpha \in O(T)$ and an elementary recursive function $f$ so that $\text{EA} + \forall n, m, k [n \neq n & (n < m < k \rightarrow n < k)]$ proves that
   $$\forall n, k ([n < k \rightarrow f(n) <_T f(k)) \& f(n) <_T \alpha]$$

2. Over $\text{EA}$, $\text{WO}[<_T]$ is equivalent to the uniform reflection principle $\text{RFN}_{\Pi_1}(T)$ of $T$ for $\Pi_1$-formulas.

3. $T$ is $\Pi_1$-conservative over the theory $\text{ACA}_0 \cup \{\text{WO}[<_T | n] : n \in \omega\}$, which is an extension of $\text{ACA}_0$ by augmenting the wellfoundedness of each initial segment $<_T | n$ of the ordering $<_T$.

4. Over $\text{EA}$, the $I$-consistency $\text{RFN}_{\Pi_2}(T)$ of $T$ is equivalent to the fact $\text{ERWO}[<_T]$ that there is no elementary recursive descending chain of ordinals in $O(T)$.

5. $T$ is $\Pi_0$-conservative over the theory $\text{EA} \cup \{\text{ERWO}[<_T | n] : n \in \omega\}$.

   Therefore provably recursive functions in $T$ are exactly the functions defined by ordinal recursions along initial segments $<_T | n (n \in \omega)$.

6. Over $\text{EA}$, finitely iterated consistency statements $\text{CON}^{(n)}(T)$ of $T$
   $$\text{CON}^{(0)}(T) :\Leftrightarrow \forall x (0 = 0); \text{CON}^{(n+1)}(T) :\Leftrightarrow \text{CON}(T + \text{CON}^{(n)}(T))$$

   is equivalent to the inference rule
   $$\begin{align*}
   \frac{[q(\alpha) <_T \alpha \rightarrow A(q(\alpha))] \rightarrow A(\alpha)}{A(\alpha)}
   \end{align*}$$

   where $\alpha$ denotes a variable ranging over $O(T)$, and $A[q]$ is an elementary recursive relation [function], resp.
For a proof of Theorem 4.1.1 see [A98]. Theorem 4.1.6 is seen from Theorem 4.1.4 through an Herbrand analysis and a result due to W. Tait [Tait65]. The rest of Theorem 4.1 is seen from Lemma 4.2 below, cf. [A96a], [A96b], [A97a], [A97b], [A99], [A00a], [A00b], [A03a], [A03b], [A04a], [A04b], [A∞a] and [A∞b]. Also cf. [A02], [A03a], [A05a], [A05b] and [A06] for proof theory based on epsilon substitution method.

**Lemma 4.2**  1. $T$ proves that each initial segment $<_T|n$ is wellfounded. The proof is uniform in the sense that

$$\text{EA} \vdash \text{Proof}_T(p(x),WO[<_T|x])$$

for an elementary recursive function $p(x)$ and a canonical proof predicate $\text{Proof}_T(x,y)$ (read: $x$ is a (code of a) $T$-proof of a (code of a) formula $y$).

2. We can define a rewrite rule (cut-elimination step) $r(p,n)$ on (finite) $T$-proofs $p$ of $\Pi^1_1$-formulas, and an ordinal assignment $\alpha : p \mapsto \alpha(p) \in O(T)$ so that $\text{EA}$ proves

$$\forall n[\alpha(r(p,n))<_T\alpha(p) \rightarrow \text{Tr}_{\Pi^1_1}(\text{end}(r(p,n)))] \rightarrow \text{Tr}_{\Pi^1_1}(\text{end}(p))$$

where $\text{Tr}_{\Pi^1_1}$ denotes a partial truth definition for $\Pi^1_1$-sentences, and $\text{end}(p)$ the end-formula of a proof $p$.

For proofs $p$ of $\Sigma^0_1$-sentences, the rewrite rule degenerates to be unary, $r(p,n) = r(p,m)$.

**NB.**

The size of proof-theoretic ordinals is by no means related to consistency strengths of theories. Only when we restrict to initial segments of notation systems $O(T)$, the sizes are relevant. Cf. [Beklemishev00] and [Beckmann02] for some pathological examples on provably well orderings.

Let $\text{CON}(T,n) :\equiv \forall x \leq n\neg \text{Proof}_T(x,[0 = 1])$ denote a partial consistency of $T$ up to $n$.

1. ([Kreisel77])

Let $n < m$ denote a recursive relation defined as follows:

$$n < m :\equiv [\text{CON}(T,\min\{n,m\}) \land n < m] \lor [\neg \text{CON}(T,\min\{n,m\}) \land n > m].$$

Even though $|<| = \omega$ since $T$ is assumed to be consistent, $WO[<]$ implies $\text{CON}(T)$ finitistically.

2. Modifying the above Kreisel’s pathological example, one sees that for any recursive and $\text{Bool}(\Pi^1_1)$-sound theory $T$ ($\text{Bool}(\Pi^1_1)$ denotes the Boolean
combinations of $\Pi_1$-sentences), there exists a recursive and $\text{Bool}(\Pi_1)$-sound theory $T'$ such that $|T| < |T'|$ but $T' \not\vdash \text{CON}(T)$: let $<_T$ be any recursive well ordering of type $|T|$, and let

$$n <' m :\iff \text{CON}(T, \max\{n, m\}) \& n <_T m.$$  

Although $|<'| = |<_T|$, $<'$ is a finite ordering if $T$ is inconsistent. A fortiori $\text{EA} \vdash \neg \text{CON}(T) \rightarrow \text{WO}[<']$. Hence $T' := T \cup \{\text{WO}[<']\}$ is a desired one.

Note that if each initial segment of $<_T$ is provably wellfounded in $T$, then so is for $<'$.

5 Collapsing functions iterated

The essential step in cut-elimination for a set theory $T$ is to analyse the axiom expressing an ordinal $\sigma$ reflects any $\Pi_2$-formula $\varphi$:

$$\varphi^{L_\sigma}(a) \land a \in L_\sigma \rightarrow \exists \beta < \sigma \varphi^{L_\beta} \land a \in L_\beta.$$  

This means that given a proof figure $P$ of the premise, we have to find an ordinal term $\beta < \sigma$:

$$\vdash P \,
\,
\vdash \varphi^{L_\sigma}(a) \land a \in L_\sigma \implies \varphi^{L_\beta}(a) \land a \in L_\beta.$$  

This is done by putting $\beta = d_\sigma \alpha < \sigma (o(P) = \alpha \in Od(T))$ for a (Mostowski) collapsing function $d$.

Let $C(\alpha)$ ($\alpha = o(P)$) denote the set of ordinals which may occur in the reducts of $P$. Ordinals in $C(\alpha)$ are on the solid lines with gaps here and there in the following figure:

$$\begin{array}{cccc}
0 & d_\sigma \alpha & \sigma & \sigma + d_\sigma \alpha \\
\hline
\end{array}$$

By stuffing the gap below $\sigma$ in the set $C(\alpha)$ up, $\sigma$ is collapsed down to the least indescribable ordinal $d_\sigma \alpha$. Then ordinals in $C(\alpha)$ cannot discriminate between $\sigma$ and $d_\sigma \alpha$.

Thus the ordinal $\beta = d_\sigma \alpha$ can be a substitute for $\sigma$.

To analyse larger ordinals, e.g., $\Pi_3$-reflecting ordinals, the collapsing process has to be iterated.

A $\Pi_3$-reflecting ordinal $K$ is understood to be $< \varepsilon_{K+1}$-recursively Mahlo, $L_K \cap \bigcap_{\mu < \varepsilon_{K+1}} M_\mu^\mu$. First $K$ is collapsed to a $\mu_0$-recursively Mahlo ordinal for a $\mu_0 < \varepsilon_{K+1}$: $\kappa_1 = d_{\mu_0}^\mu \alpha_0 < K$. Then $L_{\kappa_1} \in M_{\mu_0}^{\mu_0}$ is collapsed to a $\mu_1$-recursively Mahlo ordinal: $\kappa_2 = d_{\kappa_1}^{\mu_1} \alpha_1 < \kappa_1 (\mu_1 < \mu_0)$, etc. In this way a
possibly infinite collapsing process is generated: \( K = \kappa_0 > d^K_{\mu_0} \alpha_0 = \kappa_1 > d^K_{\mu_1} \alpha_1 = \kappa_2 > \cdots (\varepsilon_{K+1} > \mu_0 > \mu_1 > \cdots). \)

We have designed a recursive notation system \( \langle O\!d(\Pi_N), < \rangle \) of ordinals for proof theoretical analysis of KP\( \Pi_N \), and showed in [A∞b] that KP\( \Pi_N \) is proof-theoretically reducible to the theory ACA\(_0\) + \( \{W\!O[< | \alpha] : \Omega > \alpha \in O\!d(\Pi_N)\} \), where \( \Omega \in O\!d(\Pi_N) \) denotes the least \( \Pi_2 \)-reflecting ordinal \( \omega_1^{CK} \) and \( < | \alpha \) the restriction of the ordering \( < \) in \( O\!d(\Pi_N) \) to \( \alpha \). Thus \( O(KP\Pi_N) = O(\Pi_N)) \Omega \).

On the other side in [A∞b] we have shown that KP\( \Pi_N \) proves \( W\!O[< | \alpha] \) for each \( \alpha < \Omega \). Indeed, this wellfoundedness proof is essentially formalizable in a theory \( KP\ell + \{L \in \bigcap\{M_2(M_2(\alpha; <TW)) : \text{dom}(<TW) \ni \alpha < a \} : a \in \omega\} \) for some \( \Sigma_1 \) relations \( <_i \) \((2 \leq i \leq N - 1)\) on \( \omega \) such that \( <_{N-1} \) is almost wellfounded in KP\( \ell \). This shows Theorem 3.5.

In the next section we give a sketch of the wellfoundedness proof.

## 6 Wellfoundedness proof

Our wellfoundedness proof of \( O\!d(\Pi_N) \) is based on the maximal distinguished class \( W \) [Buchholz75], a \( \Sigma_1 \)-definable set of integers, and a proper class in KP\( \Pi_N \).

To formalize the proof in KP\( \Pi_N \), we have to show for each \( \eta \in O\!d(\Pi_N) \) there exists an \( \eta \)-Mahlo set on which the maximal distinguished class enjoys the same closure properties as \( W \) up to the given \( \eta \). The \( \eta \)-Mahlo sets are defined through a ramification process to resolve the reflecting universes in terms of iterations of lower Mahlo operations [A∞b].

### 6.1 The notation system \( O\!d(\Pi_N) \)

The notation system \( O\!d(\Pi_N) \) (an element of \( O\!d(\Pi_N) \)) is called an ordinal diagram, which is abbreviated o.d.) contains the constants \( \Omega \) for \( \omega_1^{CK} \) and \( \pi \) for the least \( \Pi_N \)-reflecting ordinal.

The main constructor is to form an o.d. \( d^a_\sigma \alpha < \sigma \) from a symbol \( d \) and o.d.'s \( \sigma, q, \alpha \), where \( \sigma \) denotes a recursively regular ordinal and \( q \) a finite sequence of o.d.'s.

\( \gamma \prec_2 \sigma \) denotes the transitive closure of \( \{ (\beta, \sigma) : \exists \alpha, q(\beta = d^a_\sigma \alpha) \} \). The set \( \{ \tau : \sigma \prec_2 \tau \} \) is finite and linearly ordered by \( \prec_2 \) for each \( \sigma \), namely \( \{ \sigma : \sigma \preceq_2 \pi \} \) is a tree with its root \( \pi \).

In the diagram \( d^a_\sigma \alpha \), \( q \) includes some data telling us how the diagram \( d^a_\sigma \alpha \) is constructed from \( \{ \tau : d^a_\sigma \alpha \prec_2 \tau \} = \{ \tau : \sigma \preceq_2 \tau \} \).

The main task in wellfoundedness proofs is to show the tree \( \{ \sigma : \sigma \preceq_2 \pi \} \) to be wellfounded.

Specifically \( q \) in \( \eta = d^a_\sigma \alpha \) includes some data \( st_i(\eta), pd_i(\eta), rg_i(\eta) \) for \( 2 \leq i < N. \) \( st_{N-1}(\eta) \) is an o.d. less than \( \varepsilon_{\pi+1} \), and \( pd_2(\eta) = \sigma \).

A relation \( \prec_i \) is defined from \( pd_i(\eta) \) as the transitive closure of \( \{ (\eta, \kappa) : \kappa = pd_i(\eta) \} \). This enjoys \( \prec_{i+1} \subseteq \prec_i \). Therefore the diagram \( pd_i(\eta) \) is a proper
subdiagram of $\eta$. $st_i(\eta)$ is an o.d. less than the next admissible $\kappa^+$ to a \( \kappa = rg_i(\eta) \leq pd_{i+1}(\eta) \). $rg_{N-1}(\eta) = \pi$ for any such $\eta = d_3 \alpha$.

$q$ determines a sequence \( \{\eta^m_i : m < lh_i(\eta)\} \) of subdiagrams of $\eta$ with its length $lh_i(\eta) = n + 1 > 0$. The sequence enjoys the following property:

\[
\eta \preceq_i \eta_i^0 \prec_i \eta_i^1 \prec_i \cdots \prec_i \eta_i^n < \pi
\]

with $st_i(\eta_i^m) < (rg_i(\eta_i^m))^+$.

### 6.2 Towers derived from ordinal diagrams

Define relations $\ll_i$ for $2 \leq i \leq N - 1$ by

\[
\eta \ll_i \rho :\Leftrightarrow \eta \prec_i \rho \& rg_i(\eta) = rg_i(\rho) \& st_i(\eta) < st_i(\rho).
\]

Extend $\ll_i$ by augmenting the least element $1$:

\[
1 \ll_i \eta.
\]

$\pi^\alpha$ denotes $\pi^\alpha \cdot 1$.

Let $\ll_i :\equiv \ll_{E_i}$ be exponential ordering defined from $\ll_i$ ($2 \leq i \leq N - 1$). Namely $\ll_{N-1} :\equiv \ll_{N-1}$ and $\ll_i :\equiv E(\ll_{i+1}, \ll_i)$, cf. Definition 3.1.

Extend $\ll_i$ to $\ll^+_i$ by adding the successor function $+1$. Namely the domain is expanded to $dom(\ll^+_i) := dom(\ll_i) \cup \{a + 1 : a \in dom(\ll_i)\}$, and define for $a, b \in dom(\ll_i)$

\[
a + 1 \ll^+_i b + 1 :\Leftrightarrow a \ll_i b
\]

\[
a + 1 \ll^+_i b :\Leftrightarrow a \ll_i b
\]

\[
a \ll^+_i b + 1 :\Leftrightarrow a \ll_i b \text{ or } a = b
\]

From the sequence \( \{\eta_i^m : 2 \leq i < N - 1, m < lh_i(\eta)\} \) we define a tower $T(\eta) = E_2(\eta)$. The elements of the form $E_i(\eta)(+1)$ are understood to be ordered by $\ll^+_i$. Let $\ll_T :\equiv \ll^+_2$.

\[
E_{N-1}(\eta) := \eta
\]

\[
E_i(\eta) := \sum_{1 \leq m < lh_i(\eta)} \pi^E_{i+1}(\eta_i^m) \eta_i^m - 1 + \pi^E_{i+1}(\eta_i^{m+1}) + \pi^E_{i+1}(\eta)
\]

The sequence \( \{\eta_i^m : m < lh_i(\eta)\} \) is defined so that, cf. \[A \infty b\] for a proof,

\[
\gamma \ll_i \eta \Rightarrow E_i(\gamma) \ll^+_i E_i(\eta).
\]

In particular

\[
\gamma \ll_2 \eta \Rightarrow T(\gamma) \ll_T T(\eta)
\]
6.3 Distinguished classes

An elementary fact on the maximal distinguished class \( W \) says that \( W \) is well ordered by \(<\) on \( \text{Od}(\Pi_N) \), and \( W|\Omega \) is included in the wellfounded part of \( \text{Od}(\Pi_N) \). Therefore it suffices to show \( \eta \in W \) for each \( \eta \in \text{Od}(\Pi_N) \).

\( W \) is defined to be the union of the distinguished sets,

\[
W = \bigcup \{ X \subseteq \text{Od}(T) : D[X] \}
\]

where \( D[X] \) (read: \( X \) is a distinguished set) is a \( \Delta_1 \) formula on limits of admissibles. Hence \( W \) is a \( \Sigma_1 \)-definable set of integers, and a proper class in \( KP\Pi_N \).

Since \( D[X] \) is \( \Delta_1 \) on limits of admissibles, it is absolute: \( D[X] \Leftrightarrow P \models D[X] \) for any \( X \in P \cap P(\omega) \). Let \( W^P = \bigcup \{ X \in P : P \models D[X] \} \) denote the maximal distinguished class on \( P \).

The following is a key on distinguished sets.

**Lemma 6.1** There exists a \( \Pi_2 \)-formula \( g(\eta) (\eta \in \text{Od}(\Pi_N)) \) for which the following holds for any limits \( Q \) of admissibles: Assume \( g(\eta)^Q \) and

\[
\forall \gamma <_2 \eta \{ g(\gamma)^Q \Rightarrow \gamma \in W^Q \}
\]

Then there exists a distinguished class \( X \) such that \( \eta \in X \) and \( X \) is definable in \( Q \).

For some \( \Sigma_1 \) classes \( U_i \) on \( \omega \), the \( \Sigma_1 \) transitive relations on \( \omega \), \( <_i \) mentioned in Theorem 3.5 are now defined to be

\[
\eta <_i \rho \Leftrightarrow \eta <_i \rho \& \eta, \rho \in U_i.
\]

By definition 1 \( \in U_i \) for any \( i \). \( <_{N-1} \) is seen to be almost wellfounded in \( KP\ell \).

Let \( <_{TW} \) denote the restriction of the tower \( <_T \) of the exponential orderings \( <_{E_i} \), defined from these \( \Sigma_1 \) relations \( <_i \) (\( 2 \leq i \leq N-1 \)) to the wellfounded parts in the second components hereditarily.

In other words,

\[
T(\eta) <_T T(\rho) \Leftrightarrow T(\eta) <_T T(\rho) \& \forall i[K_i(\eta) \cup K_i(\rho) \subseteq U_i]
\]

and

\[
T(\eta) <_{TW} T(\rho) \Leftrightarrow T(\eta) <_T T(\rho) \& \forall i > 0[K_i(\eta) \cup K_i(\rho) \subseteq W(<_i)]
\]

where

1. \( K_2(\eta) := \{ \eta^m_2 : m < lh_2(\eta) \} \).
2. For \( 2 < i < N-1 \), \( K_i(\eta) := \{ \rho^m_i : m < lh_i(\rho), \rho \in K_{i-1}(\eta) \} \).

**Lemma 6.2** If \( P \in M_2(M_2(T(\eta); <_{TW})) \), then \( g(\eta)^P \rightarrow \eta \in W^P \).
Proof by induction on $\epsilon$. Suppose $P \in M_2(M_2(T(\eta); <_{TW}))$ and $g(\eta)^P$. Pick a $Q \in P \cap M_2(T(\eta); <_{TW})$ so that $g(\eta)^Q$.

We show (3). Assume $\gamma \prec \eta$ and $g(\gamma)^Q$. (2) yields $T(\gamma) <_T T(\eta)$. On the other side the $\Pi_2$ formula $g(\gamma)$ is defined so that

$$g(\gamma)^Q \rightarrow \forall i[K_i(\gamma) \subseteq U_i^Q] \& \forall i > 0[K_i(\gamma) \subseteq W_i^Q(\eta^Q)].$$

Since $\bigcup_i K_i(\eta)$ is finite, we can assume $\forall i[K_i(\eta) \subseteq U_i^Q]$, and hence $T(\gamma) <_{TW}^Q T(\eta)$. Therefore $Q \in M_2(M_2(T(\gamma); <_{TW}))$. IH yields $\gamma \in W^Q$. This shows (3).

By Lemma 6.1 let $X$ be a distinguished class definable over $Q$ such that $\eta \in X$. Thus $X \in P \& D[X]$, and $\eta \in W^P$.

Assuming $L \in M_2(M_2(T(\eta); <_{TW}))$ for each $\eta$, we have $g(\eta)^L \rightarrow \eta \in W^L = W$ by Lemma 6.2. On the other side, it is not hard to show $g(\eta)^L$ for each $\eta$ in KP$\ell$.

Therefore the wellfoundedness of $Od(\Pi_N)$ up to each $\eta < \Omega$ follows from $\{L \in M_2(M_2(T(\eta); <_{TW})) : \eta \in Od(\Pi_N)\}$ over KP$\ell$.

References

[A96a] T. Arai, Systems of ordinal diagrams, draft, 1996.

[A96b] T. Arai, Proof theory for theories of ordinals I: Reflecting ordinals, draft, 1996.

[A97a] T. Arai, Proof theory for theories of ordinals II: $\Sigma_1$-stability, draft, 1997.

[A97b] T. Arai, Proof theory for theories of ordinals III: $\Pi_1$-collection, draft, 1997.

[A98] T. Arai, Some results on cut-elimination, provable well-orderings, induction and reflection, Ann. Pure Appl. Logic 95 (1998) 93-184.

[A99] T. Arai, Introduction to finitary analyses of proof figures, In: Sets and Proofs. Invited papers from Logic Colloquium '97-European Meeting of the Association for Symbolic Logic, Leeds, July 1997. Ed. by S. B. Cooper and J. K. Truss, London Mathematical Society Lecture Notes, vol. 258, Cambridge University Press (1999), pp.1-25.

[A00a] T. Arai, Ordinal diagrams for recursively Mahlo universes, Arch. Math. Logic 39 (2000) 353-391.

[A00b] T. Arai, Ordinal diagrams for $\Pi_3$-reflection, Jour. Symb. Logic 65 (2000) 1375-1394.

[A02] T. Arai, Epsilon substitution method for theories of jump hierarchies, Arch. Math. Logic 41 (2002) 123-153.
[A03a] T. Arai, Epsilon substitution method for $ID_1(\Pi_0^1 \vee \Sigma_1^0)$, Ann. Pure Appl. Logic 121 (2003) 163-208.

[A03b] T. Arai, Proof theory for theories of ordinals I: recursively Mahlo ordinals, Ann. Pure Appl. Logic 122 (2003) 1-85.

[A04a] T. Arai, Proof theory for theories of ordinals II: $\Pi_3$-Reflection, Ann. Pure Appl. Logic 129 (2004) 39-92.

[A04b] T. Arai, Wellfoundedness proofs by means of non-monotonic inductive definitions I: $\Pi_2^0$-operators, Jour. Symb. Logic 69 (2004) 830-850.

[A05a] T. Arai, Ideas in the epsilon substitution method for $\Pi_0^1$-FIX, Ann. Pure Appl. Logic 136 (2005) 3-21.

[A05b] T. Arai, Epsilon substitution method for $[\Pi_0^0, \Pi_0^1]$-FIX, Arch. Math. Logic 44 (2005) 1009-1043.

[A06] T. Arai, Epsilon substitution method for $\Pi_2^0$-FIX, Jour. Symb. Logic 71 (2006) 1155-1188.

[A∞a] T. Arai, Proof theory for theories of ordinals III: $\Pi_\infty$-reflection, submitted.

[A∞b] T. Arai, Wellfoundedness proofs by means of non-monotonic inductive definitions II: first order operators, submitted.

[Beckmann02] A. Beckmann, A non-well-founded primitive recursive tree provably well-founded for co-r.e. sets, Arch. Math. Logic 41(2002) 251-257.

[Beklemishev00] L. Beklemishev, Another pathological well-ordering, in Logic Colloquium 98(Prague), 105-108, Lect. Notes Logic 13, Assoc. Symb. Logic, 2000.

[Buchholz75] W. Buchholz, Normalfunktionen und konstruktive Systeme von Ordinalzahlen. In: Diller, J., Müller, G.H.(eds.) Proof Theory Symposium, Kiel 1974 (Lecture Notes in Mathematics, vol.500, pp.4-25). Berlin: Springer 1975

[Kreisel77] G. Kreisel, Wie die Beweistheoire zu ihren Ordinalzahlen kam und kommt, Jber. Deutsch. Math.-Verein 78(1977), 177-223.

[Richter-Aczel74] W.H. Richter and P. Aczel, Inductive definitions and reflecting properties of admissible ordinals, Generalized Recursion Theory, Studies in Logic, vol.79, North-Holland, 1974, pp.301-381.

[Tait65] W. W. Tait, Functionals defined by transfinite recursion, Jour. Symb. Logic 30 (1965) 155-174.