Application of quasi-rotation surface segments in architectural prototyping

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Abstract. The present paper details an approach to modeling of quasi-rotation surface segments with predefined conditions. The desired surfaces and their segments are formed parametrically through Maple computer algebra system. Various analytical approaches to trimming of the excessive parts of quasi-rotation surfaces are indicated. Geometric modeling is performed with use of a previously developed algorithm. An example illustrating the initial stage of design of an overpass between two separate buildings is provided. The design objective contains the basic geometric parameters of the desired architectural element. Three control models were created with the same initial parameters through basic CAD software methods. All of the constructed models are thin-walled. Mechanical properties of the scale 3D models of the target element were compared in equal conditions by means of CAD software. The prototype created through the use of a quasi-rotation surface segment was the most resistant to load. The paper also features a rendered concept of a complex of three towers connected with three overpasses. The acquired results allow us to conclude that quasi-rotation is an effective method of formation of models of architectural elements.

1. Introduction
Papers [1] consider theoretical basics of the geometrical correspondence known as “quasi-rotation”. Its capability to form cyclic surfaces is covered in papers [1-3]. The analytical entry of the quasi-rotation apparatus presented in paper [4] was used to create an algorithm in the environment the computer algebra system (Maple). The images of the quasi-rotation surfaces presented in papers [1-3] demonstrate the possibility to create surfaces of various shapes through application of the quasi-rotation apparatus to various generating curves. The present paper demonstrates the possibility of application of quasi-rotation surface segments in architectural form creation. Parametric architecture has been seeing stable development in industrial design for twenty years. The combined development in computer and construction technologies allows one to realize even the most daring architectural concepts. Papers [4-8] consider problems and perspectives of this area of engineering and emphasize the value of the modern methods in parametric design of architectural forms. The authors of various studies in this area consider the development of parametric architecture appropriate to the current level of technological progress. The problems of realization of hi-tech projects, including projects in the area of architecture, are highlighted in paper [9]. Stable development in this area is one of the priorities of many countries across the globe.

2. Problem definition
The objective of the present paper is to develop methods of geometric modeling of segments of quasi-rotation surfaces shaped according to certain parameters. It is also required to demonstrate on a specific example the applicability of such segments in parametric architecture. Figure 1 depicts a model of an overpass between two cylindrical buildings and specifies its basic parameters. Dimensions are in meters. It is required to create a quasi-rotation surface segment conforming to the specified
parameters. It is also important to compare mechanical properties of the acquired element of construction and its counterparts acquired through alternative methods.

3. Theory

The first stage of modeling of the desired geometric object is construction of a 3D model of its surface; the second stage is trimming of the excess if necessary. If one works with surfaces of finite area, i.e. closed surfaces, the second stage might not be needed, since the desired object can be a body formed by the sheet of a quasi-rotation surface. In other cases, it is required to extract a surface segment given the parameters of its boundary.

Let us consider the existing capabilities of realizing the first stage on the example of a one sheet of quasi-rotation surface (Figure 2). The particular feature of this surface is location of the generating circle center at the center of an ellipse. Determinant $\gamma$ of such surface is of the following form:

$$\gamma(l(d1), i_e(a, b))|li_e = QRT_{i_e}(l)|,$$

(1)

where $d1$ represents diameter of generating circle $l$; $a$ and $b$ represent major and minor half-axes of elliptical axis $i_e$ respectively; $QRT_{i_e}$ represents the Quasi-Rotation Transformation with respect to axis $i_e$. In order to acquire a 3D model of this surface, it is required to introduce the basic parameters $(d1, a, b)$ of the surface into the initial data for the algorithm developed in paper [3].

Figure 1. The basic parameters of an architectural element

Figure 2. A sheet of a quasi-rotation surface and some of its parameters
Even though the parameters of the elliptic axis take part in calculations, they are not geometric parameters of the surface. In other words, determining quasi-rotation axis parameters given an existing 3D model of a surface is a separate problem. There is, however, an opportunity to operate on a number of parameters that are not represented in the determinant, but can have an impact on the desired shapes. Let us consider such parameters on example (Figure 2).

Only one parameter \( dl \) is represented in the determinant. The other dimensions of the surface specified on Figure 1 are not represented in the determinant at all. However, they can be changed with constant generating circle diameter \( dl \) through variation of parameters of the quasi-rotation axis. For example, one of the overall dimensions – diameter \( D \) – depends on the value of the major half-axis \( a \) of the ellipse \( i_e \) and is calculated through formula

\[
D = 2a + dl.
\]  
(2)

At that, parameter \( d \) takes a certain value according to formula

\[
d = D - 2dl.
\]  
(3)

As we can see, the three parameters \( D, d, dl \) are linearly dependent. Any two of the three parameters can be assigned arbitrarily, and determine the major half-axis value of the ellipse \( i_e \).

The remaining overall dimension \( G \) depends on the value of focal distance \( c \) of ellipse \( i_e \). This dimension can be expressed as two times the maximum value of ordinate \( y \):

\[
y_{max} = \frac{G}{2}.
\]  
(4)

The parametric equations of a quasi-rotation surface that served as the basis for algorithm of construction of the respective 3D plots in paper [3] are of the following form:

\[
\begin{align*}
x &= r(\cos \beta + 1) \cos \theta + x_1, \\
y &= r(\cos \beta + 1) \sin \theta + y_1, \\
z &= r \sin \beta 
\end{align*}
\]  
(5)

According to the equations (5), the ordinate \( y \) values of the points of the modeled surface conform to the following parametric equation:

\[
y = r(\cos \beta + 1) \sin \theta + y_1.
\]  
(6)

Obviously, \( y \) takes the maximum value at \( \beta = \pi \). Angle \( \theta \) and parameters \( y_i \) and \( r \) are functions of a single parameter that defines a certain point of the generating circle, that is:

\[
\begin{align*}
\theta &= \theta(\tau, c), & y_1 &= y_1(\tau), & r &= r(\tau, c).
\end{align*}
\]  
(7)

By substitution of (7) into (6), and considering that \( \beta = \pi \), we acquire:

\[
y = y(\tau, c).
\]  
(8)

At \( y = y_{max} \), the derivative of function (8) with respect to \( \tau \) equals zero.

\[
\frac{\partial y}{\partial \tau} = 0 \rightarrow \tau = \tau(c).
\]  
(9)

One can express \( c \) from equations (4), (6)-(9) and acquire the formula for quasi-rotation axis focal distance:

\[
\frac{G}{2} = 2r(c) \sin \theta(c) + y_1(c) \rightarrow c = c(G).
\]

An algorithm of quasi-rotation surface construction based on the above reasoning can feature the initial data in the form of parameters of the desired surface, namely \( D \) and \( G \) (Figure 2) instead of elliptic axis parameters. It is important to note that the studied quasi-rotation surface is described by three parameters included in the geometric part of its determinant (1). Any of the three parameters can be replaced with one of the parameters of the desired surface specified on Figure 1. Therefore, the studied quasi-rotation surface can be described by a triplet of parameters, such as \((dl, a, c), (dl, D, c), (d, D, G), (dl, D, d2)\), etc. Regardless, the calculation of the surface model is reduced to calculating a triplet of values included in the geometric part of the determinant (1).

Once the model of the desired surface is acquired, it may require trimming in order to extract the desired surface segment. Such operation can be performed in with specific CAD software tools. However, it is possible to export a segment from a computer algebra system that would otherwise be a challenge to construct using a CAD system.
Boundaries of the target segments can be specified through ranges of parameters $\tau$ and $\beta$ for equations describing the target surface. Figure 3(a) depicts a segment of a surface presented on Figure 2 and generated by quasi-rotation of a circular arc upon variation of parameter $\tau$ in interval $\tau \in [\pi/6, 5\pi/4]$.

![Figure 3](image)

**Figure 3.** A segment of a sheet of a surface generated by quasi-rotation of a circular arc: (a) on angle $\beta = 360^\circ$, (b) on angle $\beta = 90^\circ$

The second parameter for the equations describing the quasi-rotation surface is the angle $\beta$ determining rotation of a point of the generating curve. In order to acquire the desirable surface segment, it is sufficient to constrain this parameter within interval. Figure 3(b) depicts a segment of a surface presented on Figure 3(a) within interval of parameter $\beta \in [0., \pi/2]$. Such segment is generated by means of quasi-rotation of each point of a generating circle on a specified angle. For example, a point $A$ has moved into position $A_\beta$ generating an arc of a circle, in the same way as every point of the arc of the generating circle $l$.

The above reasoning is an example and can be applied to modeling of the desired surfaces and surface segments with pre-defined parameters. It is worth noting that the above problems can be solved by the means of constructive modeling realized in CAD software through methods described in study [10].

4. Results of experiments

The element A presented on Figure 1 can be modeled through various methods with different results. A segment with specified parameters can be acquired through quasi-rotation of arcs of various algebraic curves around any conic. Since a parabolic dome is considered the most stable, as the shape of a parabola is close to a catenary line, let us acquire the desired model through quasi-rotation of an arc of parabola around a co-focused parabolic axis.

A 3D plot of the target model can be acquired through the algorithm described in paper [3]. In order to do that, it is required to know the focal parameters of the parabolic axis and the parabola arc generatrix.

Figure 4 represents both curves in plane $XY$. Focuses of parabolas are at the center of coordinates. The equations of these curves are of the following form:

$$x = \frac{y^2}{2p_l} - \frac{p_l}{2}, \quad x = \frac{y^2}{2p_i} - \frac{p_i}{2},$$

(10)

where $p_l$ and $p_i$ are focal parameters of the parabolic generatrix $l$ and the parabolic axis $i_p$ respectively. The values of these parameters determine the shape of the quasi-rotation surface. It is worth mentioning that the surfaces of quasi-rotation typically include four sheets, however, in order to solve our task, only a single sheet $\gamma^l$ generated through quasi-rotation of a generating curve around the first focus $F_l$ along the closer trajectory is required. Note that here we apply the terminology accepted in paper [3].

Paper [3] presents a constructive scheme of formation of quasi-rotation surfaces using of parabolic axis. According to this scheme, the angle $\theta$ of incline of plane of quasi-rotation trajectory to line $OX$.
equals 45° only when the generating point belongs to axis $OY$ ($\phi = 90^\circ$). As a result, the end point $L$ of the generatrix $l$ has coordinates $L(0, p_l)$. According to the initial conditions, the generating arc chord length is 24 m. According to the scheme displayed on Figure 4, the focal parameter $p_i$ equals half the length of this chord ($p_i = 12$ m).

![Figure 4. Calculation of parameters of the generating arc of parabola l and parabolic axis i_p](image)

It is known from the initial conditions that the diameter of the neck circle $k_{1'}$ of the target cyclic surface has to be 14 m. Therefore, the focal parameter $p_i$ of the axis $i_p$ equals the sum of the neck circle $k_{1'}$ diameter and focal parameter $p_l$ of the generating parabola $l$ ($p_i = 26$ m).

The acquired values of the sought parameters of the generatrix and the axis were used to construct a 3D plot of one sheet of the usable surface (the closer sheet, with respect to the first focus). The arc of the generating parabola $l$ is constrained within range of ordinate values ($-12 \leq y \leq 12$).

The model containing the segment depicted on Figure 5 was exported from Maple software and applied in further design of the desired architectural element.

5. Consideration of the results

The solution to the problem of modeling of the element presented on Figure 1 can be realized in a number of ways. However, labor input into the optimization process depends on mechanical properties of the initial model. It is worth noting that the presented shell is not considered bearing load in this project; it should only have sufficient strength to support its own weight. Calculation of a metal frame supporting the shell as well as the bearing construction of the overpass is a separate task. In any case, the resultant model has to pass a number of stages of optimization through CAD systems, including deep optimization through methods documented in paper [14]. With that in mind, a comparative analysis of the mechanical characteristics of the quasi-rotation surface segment and its counterparts has been performed. The model depicted on Figure 5 was compared to three control models designed in CAD systems with application of the standard methods of 3D modeling. All four surfaces are cyclic and conform to the initial conditions.

The control surfaces were created through the use of two methods. One of the surfaces was constructed through three sections and a sixth-degree Bezier curve generatrix. The other two surfaces were constructed through rotation of a cubic Bezier curve with subsequent Free Form Deformation (FFD). The surfaces created through the second method were different in curve and FFD parameters. The studied surfaces received reference designations: C1 for the quasi-rotation surface sheet; C2, C3 and C4 for the control surfaces.

Strength analysis was performed on thin-walled shells (0.01 m) modeled from the initial surfaces in scale 1:10. The assigned material of the surfaces is stainless steel AISI 440C.
The strength analysis was performed in a CAD system. The results of the study are listed in table 1. Figure 6 illustrates the maximum principal tensile stress under static load (gravity) and deformation of the initial shape under static load (gravity) for models C1 and C2.

![Visualization of stress in models: (a) C1, (b) C2](image)

**Figure 6. Visualization of stress in models: (a) C1, (b) C2**

| Object | Principal tensile stress under static load (gravity). | Deformation of the initial form under static load (gravity). |
|--------|-----------------------------------------------------|----------------------------------------------------------|
|        | Max, MPa                                            | Max, mm                                                  |
| C1     | 0.744,6                                             | 0.006,318                                                |
| C2     | 0.843,5                                             | 0.008,442                                                |
| C3     | 0.842,9                                             | 0.008,182                                                |
| C4     | 0.814,7                                             | 0.008,931                                                |

According to the acquired data, mechanical properties of the model created through use of the quasi-rotation apparatus exceed those of the counterparts by 9% and higher.

### 6. Conclusion

The present paper describes an approach to modeling of quasi-rotation surface segments with pre-defined parameters. An example of application of the described methods is presented. The performed studies are primary and do not allow one to draw a conclusion on definitive advantages of the quasi-rotation apparatus over other approaches to surface modeling. The research of the thin-walled shells conducted in the present paper only marginally indicates the properties of framed constructs on their basis. However, the acquired results allow one to conclude that quasi-rotation as a method of formation is effective at creating models of architectural elements. Figure 7 depicts a rendered 3D model of a complex of three buildings connected by three overpasses. This 3D model utilized the quasi-rotation surface segment depicted on Figure 5.
The variety of shapes of quasi-rotation surfaces can serve as a source of inspiration for the architect as well as yield a specific solution to a problem. In this case, as with other cyclic surfaces in architecture described in papers [12-14], the design process is simplified by the fact that the basic element of the load bearing construction is a circular frame.

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