NECESSARY AND SUFFICIENT CONDITIONS FOR A FUNCTION INVOLVING DIVIDED DIFFERENCES OF THE DI- AND TRI-GAMMA FUNCTIONS TO BE COMPLETELY MONOTONIC

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Abstract. In the present paper, necessary and sufficient conditions are established for a function involving divided differences of the digamma and trigamma functions to be completely monotonic. Consequently, necessary and sufficient conditions are derived for a function involving the ratio of two gamma functions to be logarithmically completely monotonic, and some double inequalities are deduced for bounding divided differences of polygamma functions.

1. Introduction

Recall [22, Chapter XIII] and [43, Chapter IV] that a function $f$ is said to be completely monotonic (CM) on an interval $I$ if $f$ has derivatives of all orders on $I$ and

$$(-1)^n f^{(n)}(x) \geq 0 \quad (1)$$

for $x \in I$ and $n \geq 0$. The well-known Bernstein-Widder’s Theorem [43, p. 160, Theorem 12a] states that a function $f(x)$ on $[0, \infty)$ is CM if and only if there exists a bounded and non-decreasing function $\alpha(t)$ such that

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t) \quad (2)$$

converges for $x \in [0, \infty)$. This expresses that a CM function $f$ on $[0, \infty)$ is a Laplace transform of the measure $\alpha$.

Recall also [6, 31] that a function $f$ is said to be logarithmically completely monotonic (LCM) on an interval $I \subseteq \mathbb{R}$ if it has derivatives of all orders on $I$ and its logarithm $\ln f$ satisfies

$$(-1)^k [\ln f(x)]^{(k)} \geq 0 \quad (3)$$

for $k \in \mathbb{N}$ on $I$. The terminology “logarithmically completely monotonic function” was first put forward in [6] without an explicit definition, but it seems to have been ignored until recently by the mathematical community. In early 2004, this notion was recovered in [31, 38]. Since the class of LCM functions is a subclass of the CM functions, this definition is significant and meaningful. For more information on
basic properties of LCM functions, please refer to [12, 18, 37] and related references therein.

It is well-known that the classical Euler’s gamma function
\[ \Gamma(x) = \int_{0}^{\infty} t^{x-1}e^{-t} \, dt \]  
for \( x > 0 \), the psi function \( \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \) and the polygamma functions \( \psi^{(i)}(x) \) for \( i \in \mathbb{N} \) are a series of important special functions and have much extensive applications in many branches such as statistics, probability, number theory, theory of 0-1 matrices, graph theory, combinatorics, physics, engineering, and other mathematical sciences. In particular, the functions \( \psi(x) \) and \( \psi'(x) \) for \( x > 0 \) are also called the digamma and trigamma functions respectively, see [1] and [13, p. 71].

By using the double inequalities
\[ \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} < \psi'(x) < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3}, \]  
see [17, p. 860, Theorem 4], and
\[ -\frac{1}{x^2} - \frac{1}{x^3} - \frac{1}{2x^4} < \psi''(x) < -\frac{1}{x^2} - \frac{1}{x^3}, \]
a special cases of [2, Theorem 9], for \( x > 0 \), in order to show that the double inequality
\[ (n - 1)! \exp \left[ \frac{\alpha}{x} - n\psi(x) \right] < |\psi^{(n)}(x)| < (n - 1)! \exp \left[ \frac{\beta}{x} - n\psi(x) \right] \]
holds for \( x > 0 \) if and only if \( \alpha \leq -n \) and \( \beta \geq 0 \), it was established in the proof of [3, Theorem 4.8] that
\[ |\psi'(x)|^2 + \psi''(x) > \frac{p(x)}{900x^4(x + 1)^{10}} \]  
for \( x > 0 \), where
\[ p(x) = 75x^{10} + 900x^9 + 4840x^8 + 15370x^7 + 31865x^6 + 45050x^5 + 44101x^4 + 29700x^3 + 13290x^2 + 3600x + 450. \]

From (8), the inequality
\[ |\psi'(x)|^2 + \psi''(x) > 0 \]
for \( x > 0 \) was deduced and used to present a double inequality
\[ \exp \left\{ \alpha \left[ e^{\psi(x)}\psi(x) - e^{\psi(x)} + 1 \right] \right\} \leq \frac{\Gamma(x)}{\Gamma(c)} \leq \exp \left\{ \beta \left[ e^{\psi(x)}\psi(x) - e^{\psi(x)} + 1 \right] \right\} \]
\[ \text{for } x > c, \text{ where } c = 1.4616 \cdots \text{ is the only positive zero of the psi function } \psi(x) \text{ on } (0, \infty). \]

In [10, Theorem 2.1] and [11, Theorem 2.1], in order to prove that the inequality
\[ \psi(x) > \ln \frac{x^2}{6} - \gamma - \ln \left( e^{1/x} - 1 \right) \]
holds for \( x \geq 2 \), the inequality (10) was recovered in [10, Lemma 1.1] and [11, Lemma 1.1] elegantly.
In [34], the inequality (10) was used to give a simple proof for the increasing property of the function
\[ \phi(x) = \psi(x) + \ln(e^{1/x} - 1) \] (13)
on (0, \infty).

In [9, Remark 1.3], it was pointed out that the inequality (10) is a special case of the inequality
\[ (-1)^n \psi^{(n+1)}(x) < \frac{n}{\sqrt{(n-1)!}} [(-1)^{n-1} \psi^{(n)}(x)]^{1+1/n} \] (14)
for \( x > 0 \) and \( n \in \mathbb{N} \).

In [5, Theorem 4.3], the inequality (10) was applied to provide a sharp and generalized version of (11): For \( 0 < a < b \leq \infty \) and \( x \in (a, b) \), the inequality (11) is valid with the best possible constant factors
\[ \alpha = \begin{cases} Q(b), & \text{if } b < \infty \\ 1, & \text{if } b = \infty \end{cases} \text{ and } \beta = Q(a), \] (15)
where
\[ Q(x) = \begin{cases} \frac{\ln \Gamma(x) - \ln \Gamma(c)}{[\psi(x) - 1]e^{\psi(x)} + 1}, & x \neq c; \\ \frac{1}{\psi'(c)}, & x = c. \end{cases} \] (16)

In [5, Lemma 4.6] and [5, Theorem 4.8], the inequalities (10) and (11) were respectively generalized to \( q \)-analogues.

In [33, Theorem 2], the inequality (10) was used to show that the function \( e^{\psi(x+1)} - x \) is strictly decreasing and strictly convex on \((-1, \infty)\).

In [32], among other things, it was proved that the function
\[ \Delta_\lambda(x) = [\psi'(x)]^2 + \lambda \psi''(x) \] (17)
is CM on \((0, \infty)\) if and only if \( \lambda \leq 1 \).

In [16, Theorem 1], it was proved that the function
\[ z_{s,t}(x) = \begin{cases} \frac{\Gamma(x+t)}{\Gamma(x+s)} \left[ \frac{[\psi(x+t) - \psi(x+s)]}{t-s} - x \right], & s \neq t \\ \frac{e^{\psi(x+s)} - x}{\psi'(s)}, & s = t \end{cases} \] (18)
on \((-\alpha, \infty)\) for real numbers \( s \) and \( t \) and \( \alpha = \min\{s, t\} \) is either convex and decreasing for \( |t-s| < 1 \) or concave and increasing for \( |t-s| > 1 \). In order to provide an alternative proof for [16, Theorem 1], the function
\[ \Delta_{s,t}(x) = \begin{cases} \frac{\psi'(x+t) - \psi'(x+s)}{t-s} - x, & s \neq t \\ \frac{\psi'(x+t) - \psi'(x+s)}{t-s}, & s = t \end{cases} \] (19)
for \( |t-s| < 1 \) and \(-\Delta_{s,t}(x)\) for \( |t-s| > 1 \) are proved in [23, 26] to be CM on \((-\alpha, \infty)\).

Using the complete monotonicity of the function (19), the inequality (11) and [5, Theorem 4.3] mentioned on page 3 were generalized in [29, Theorem 5] to a
monotonic property as follows: For real numbers \( s \) and \( t \), \( \alpha = \min\{s, t\} \) and \( c \in (-\alpha, \infty) \), let

\[
g_{s,t}(x) = \begin{cases} 
\int_c^x \frac{1}{t-s} \ln \left[ \frac{\Gamma(u+t) \Gamma(c+s)}{\Gamma(u+s) \Gamma(c+t)} \right] \, du, & s \neq t \\
\int_c^x [\psi(u+s) - \psi(c+s)] \, du, & s = t
\end{cases}
\]  

(20)
on \( (-\alpha, \infty) \). Then the function

\[
f_{s,t}(x) = \begin{cases} 
g_{s,t}(x), & x \neq c \\
\frac{1}{g'_{s,t}(c)}, & x = c
\end{cases}
\]  

(21)
on \( (-\alpha, \infty) \) is decreasing for \(|s - t| < 1\) and increasing for \(|s - t| > 1\).

In [29, 36], some other applications of the complete monotonicity of the function (19) were also demonstrated.

For real numbers \( s, t, \alpha = \min\{s, t\} \) and \( \lambda \), define

\[
\Delta_{s,t;\lambda}(x) = \begin{cases} 
\left[ \frac{\psi(x+t) - \psi(x+s)}{t-s} \right]^2 + \lambda \frac{\psi'(x+t) - \psi'(x+s)}{t-s}, & s \neq t \\
\left[ \psi(x+s)^2 + \lambda \psi''(x+s) \right], & s = t
\end{cases}
\]  

(22)
on \( (-\alpha, \infty) \). It is clear that \( \Delta_{s,t;\lambda}(x) = \Delta_{s,\lambda}(x) \) and \( \Delta_{s,t;\lambda}(x) = \Delta_{t,s;\lambda}(x) \).

The aim of this paper is to present necessary and sufficient conditions for the function \( \Delta_{s,t;\lambda}(x) \) to be CM on \( (-\alpha, \infty) \).

Our main results can be stated as the following Theorem 1.

**Theorem 1.** The function \( \Delta_{s,t;\lambda}(x) \) has the following CM properties:

1. For \( 0 < |t-s| < 1 \),
   - (a) the function \( \Delta_{s,t;\lambda}(x) \) is CM on \( (-\alpha, \infty) \) if and only if \( \lambda \leq 1 \),
   - (b) so is the function \(-\Delta_{s,t;\lambda}(x)\) if and only if \( \lambda \geq \frac{1}{1-1/t-s} \);
2. For \( |t-s| > 1 \),
   - (a) the function \( \Delta_{s,t;\lambda}(x) \) is CM on \( (-\alpha, \infty) \) if and only if \( \lambda \leq \frac{1}{1-1/t-s} \),
   - (b) so is the function \(-\Delta_{s,t;\lambda}(x)\) if and only if \( \lambda \geq 1 \);
3. For \( s = t \), the function \( \Delta_{s,s;\lambda}(x) \) is CM on \( (-s, \infty) \) if and only if \( \lambda \leq 1 \);
4. For \( |t-s| = 1 \),
   - (a) the function \( \Delta_{s,s;\lambda}(x) \) is CM if and only if \( \lambda < 1 \),
   - (b) so is the function \(-\Delta_{s,s;\lambda}(x)\) if and only if \( \lambda > 1 \),
   - (c) and \( \Delta_{s,s;1}(x) \equiv 0 \).

As a consequence of Theorem 1, the following logarithmically complete monotonicity of a function involving the ratio of two gamma functions is deduced.

**Theorem 2.** For real numbers \( s, t, \alpha = \min\{s, t\} \) and \( \lambda \), let

\[
\mathcal{H}_{s,t;\lambda}(x) = \begin{cases} 
\frac{(x+t)^{1/(t-s)-\lambda/2}}{(x+s)^{1/(t-s)+\lambda/2}} \left[ \frac{\Gamma(x+t)}{\Gamma(x+s)} \right]^{1/(t-s)}, & s \neq t \\
\frac{1}{(x+s)^{\lambda}} \exp \left[ \psi(x+s) + \frac{1}{2(x+s)} \right], & s = t
\end{cases}
\]  

(23)
on \( (-\alpha, \infty) \).
(1) For $0 < |t - s| < 1$,
   (a) the function $\mathcal{H}_{s,t;\lambda}(x)$ is LCM on $(-\alpha, \infty)$ if and only if $\lambda \geq \frac{1}{|t-s|}$,
   (b) so is the function $[\mathcal{H}_{s,t;\lambda}(x)]^{-1}$ if and only if $\lambda \leq 1$;

(2) For $|t - s| > 1$,
   (a) the function $\mathcal{H}_{s,t;\lambda}(x)$ is LCM on $(-\alpha, \infty)$ if and only if $\lambda \geq 1$,
   (b) so is the function $[\mathcal{H}_{s,t;\lambda}(x)]^{-1}$ if and only if $\lambda \leq \frac{1}{|t-s|}$;

(3) For $s = t$, the function $[\mathcal{H}_{s,s;\lambda}(x)]^{-1}$ is LCM on $(-\alpha, \infty)$ if and only if $\lambda \leq 1$;

(4) For $|t - s| = 1$,
   (a) the function $\mathcal{H}_{s,t;\lambda}(x)$ is LCM on $(-\alpha, \infty)$ if and only if $\lambda > 1$,
   (b) so is the function $[\mathcal{H}_{s,s;\lambda}(x)]^{-1}$ if and only if $\lambda < 1$,
   (c) and $\mathcal{H}_{s,t;1}(x) \equiv 1$.

As consequences of Theorem 2, some inequalities for the ratio of two gamma functions and divided differences of polygamma functions are derived as follows.

**Theorem 3.** For positive numbers $a$ and $b$, the inequality

$$\left[\frac{\Gamma(b)}{\Gamma(a)}\right]^{1/(b-a)} < \sqrt{ab} \left(\frac{b}{a}\right)^{1/(2(b-a))}$$

holds for $0 < |b - a| < 1$ and reverses for $|b - a| > 1$; For $0 < |b - a| < 1$, the double inequality

$$\frac{(k-1)!}{2} \left[\left(\frac{1}{b-a} + \beta\right)\frac{1}{a^k} + \left(\beta - \frac{1}{b-a}\right)\frac{1}{b^k}\right]$$

$$\leq \frac{(-1)^{k-1}}{b-a} \left[\psi^{(k-1)}(b) - \psi^{(k-1)}(a)\right]$$

$$\leq \frac{(k-1)!}{2} \left[\left(\frac{1}{b-a} + \gamma\right)\frac{1}{a^k} + \left(\gamma - \frac{1}{b-a}\right)\frac{1}{b^k}\right]$$

(25)

on $(-\alpha, \infty)$ holds if and only if $\beta \leq 1$ and $\gamma \geq \frac{1}{|b-a|}$; For $|b - a| > 1$, the inequalities in (25) are valid if and only if $\beta \leq \frac{1}{|b-a|}$ and $\gamma \geq 1$.

After proving the above theorems in next section, we would also like to give some remarks on the above theorems and related results to the inequality (12) and the increasing property of $\phi(x)$ defined by (13) in the final section of this paper.

2. **Proofs of theorems**

Now we are in a position to prove the above theorems.

**Proof of Theorem 1.** For $s = t$, it has been proved in [32] that the function $\Delta_{s,s;\lambda}(x)$ is CM on $(-s, \infty)$ if and only if $\lambda \leq 1$.

It is easy to calculate by integrating in part in (4) that

$$\Gamma(x + 1) = x\Gamma(x), \quad x > 0.$$ (26)

Taking the logarithm of equation (26) and differentiating $k \in \mathbb{N}$ times consecutively on both sides give

$$\psi^{(k-1)}(x + 1) = \psi^{(k-1)}(x) + (-1)^{k-1}(k-1)! \frac{1}{x^k}, \quad x > 0, \quad k \in \mathbb{N}.$$ (27)
For $s - t = \pm 1$, using (27) gives

$$
\Delta_{s,s \mp 1; \lambda}(x) = \begin{cases} 
1 - \lambda & \frac{1}{(x + s - 1)^2}; \\
1 - \lambda & \frac{1}{(x + s)^2}.
\end{cases}
$$

(28)

As a result, the function $\Delta_{s,s \mp 1; \lambda}(x)$ is CM if and only if $\lambda < 1$, so is the function $-\Delta_{s,s \mp 1; \lambda}(x)$ if and only if $\lambda > 1$, and $\Delta_{s,s \pm 1; \lambda}(x) \equiv 0$.

For $0 < |s - t| \neq 1$, direct calculation and utilization of (27) yield

$$
\Delta_{s,t; \lambda}(x) - \Delta_{s,t; \lambda}(x + 1) = \frac{1}{(t - s)^2} \left\{ 2[\psi(x + t) - \psi(x + s)] + \left( \frac{1}{x + s} - \frac{1}{x + t} \right) \right\}
+ \frac{\lambda}{t - s} \left[ \frac{1}{(x + t)^2} - \frac{1}{(x + s)^2} \right]
- \frac{2}{(x + s)(x + t)} \left[ \frac{\psi(x + t) - \psi(x + s)}{t - s} - \frac{1}{2(x + s)(x + t)} - \frac{\lambda(2x + s + t)}{2(x + s)(x + t)} \right]
\Delta \frac{2\theta_{s,t; \lambda}(x)}{(x + s)(x + t)}.
$$

(29)

From (4), it is easy to deduce that

$$
\frac{1}{x^\omega} = \frac{1}{\Gamma(\omega)} \int_0^\infty t^{\omega - 1} e^{-xt} dt
$$

(30)

for real numbers $x > 0$ and $\omega > 0$. For $x > 0$, it was listed in [1, p. 259, 6.3.22] that

$$
\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt.
$$

By virtue of formulas (29) and (30), it follows that

$$
\theta_{s,t; \lambda}(x) = \frac{\psi(x + t) - \psi(x + s)}{t - s} - \frac{1}{2} \left[ \frac{1}{(t - s)(1 - e^{-u})} e^{-xu} du - \frac{1}{2(t - s)} \int_0^\infty (e^{-su} - e^{-tu}) e^{-xu} du \right]
+ \frac{\lambda}{t - s} \int_0^\infty \frac{e^{-su} - e^{-tu}}{1 - e^{-u}} du
\Delta \frac{1}{2} \int_0^\infty \frac{\tanh((t - s)u/2)}{t - s} \tanh(u/2) - \frac{\lambda}{2} (e^{-su} + e^{-tu}) e^{-xu} du.
$$
It is not difficult to obtain that
\[
\lim_{t \to -\infty} \frac{\tanh((t - s)u/2)}{(t - s) \tanh(u/2)} = \frac{1}{|t - s|} \quad \text{and} \quad \lim_{t \to 0^+} \frac{\tanh((t - s)u/2)}{(t - s) \tanh(u/2)} = 1. \tag{31}
\]

Straightforward differentiation gives
\[
\frac{d}{du} \left[ \left( \tanh((t - s)u/2) \right) \left( \tanh(u/2) \right) \right] = \frac{u}{4 \sinh^2(u/2) \cosh^2((t - s)u/2)} \left[ \frac{\sinh u}{u} - \frac{\sinh((t - s)u)}{(t - s)u} \right].
\]

Since the function \( \frac{\sinh u}{u} \) is even and increasing on \( (-\infty, \infty) \setminus \{0\} \), then
\[
\frac{d}{du} \left[ \tanh((t - s)u/2) \right] = \begin{cases} < 0, & \text{if } |t - s| > 1; \\ > 0, & \text{if } 0 < |t - s| < 1. \end{cases}
\]

Hence, the function \( \frac{\tanh((t - s)u/2)}{(t - s) \tanh(u/2)} \) on \( (0, \infty) \) is increasing for \( 0 < |t - s| < 1 \) and decreasing for \( |t - s| > 1 \). Therefore,

1. the function \( \theta_{s,t;\lambda}(x) \) is CM on \( (-\alpha, \infty) \) if
   (i) either \( 0 < |t - s| < 1 \) and \( \lambda \leq 1 \)
   (ii) or \( |t - s| > 1 \) and \( \lambda \leq \frac{1}{|t-s|} \)
2. the function \( -\theta_{s,t;\lambda}(x) \) is CM on \( (-\alpha, \infty) \) if
   (iii) either \( 0 < |t - s| < 1 \) and \( \lambda \geq \frac{1}{|t-s|} \)
   (iv) or \( |t - s| > 1 \) and \( \lambda \geq 1 \).

Further, since the product of any finite CM functions is still CM and the function
\[
\frac{2}{(x+t)(x+t)} \quad \text{is CM, then the function} \quad \Delta_{s,t;\lambda}(x) - \Delta_{s,t;\lambda}(x+1) \quad \text{is CM on} \quad (-\alpha, \infty) \quad \text{under the above condition (i) or (ii) and the function} \quad \Delta_{s,t;\lambda}(x+1) - \Delta_{s,t;\lambda}(x) \quad \text{is CM on} \quad (-\alpha, \infty) \quad \text{under the condition (iii) or (iv) above.}
\]

For \( x > 0 \) and \( k \in \mathbb{N} \), it was listed in [1, p. 260, 6.4.1] that
\[
\psi^{(k)}(x) = (-1)^{k+1} \int_{0}^{\infty} \frac{t^k}{1 - e^{-t}} e^{-xt} \, dt. \tag{32}
\]

This implies \( \lim_{x \to \infty} \psi^{(k)}(x) = 0 \), and so
\[
\Delta_{s,t;\lambda}^{(k-1)}(x) = \left\{ \left[ \frac{1}{t-s} \int_{s}^{t} \psi^{(k)}(x+u) \, du \right]^{2^{(k-1)}} + \frac{\lambda \psi^{(k)}(x+t) - \psi^{(k)}(x+s)}{t-s} \right\} \right. \\
= \frac{1}{t-s} \sum_{i=0}^{k-1} \binom{k-1}{i} \int_{s}^{t} \psi^{(i+1)}(x+u) \, du \int_{s}^{t} \psi^{(k-i)}(x+u) \, du \\
+ \lambda \frac{\psi^{(k)}(x+t) - \psi^{(k)}(x+s)}{t-s} \right. \\
\to 0
\]

for \( k \in \mathbb{N} \) as \( x \to \infty \).

If \( \Delta_{s,t;\lambda}(x) - \Delta_{s,t;\lambda}(x+1) \) is CM, then
\[
(-1)^{k-1} [\Delta_{s,t;\lambda}(x) - \Delta_{s,t;\lambda}(x+1)]^{(k-1)} \\
= (-1)^{k-1} \Delta_{s,t;\lambda}^{(k-1)}(x) - (-1)^{k-1} \Delta_{s,t;\lambda}^{(k-1)}(x+1) \\
\geq 0
\]
for \( k \in \mathbb{N} \) and \( x \in (-\alpha, \infty) \). Thus, in virtue of the mathematical induction and the verified fact that \( \lim_{x \to -\infty} \Delta_{s,t;\lambda}^{(k-1)}(x) = 0 \) for \( k \in \mathbb{N} \), it follows that

\[
(-1)^{k-1} \Delta_{s,t;\lambda}^{(k-1)}(x) \geq (-1)^{k-1} \Delta_{s,t;\lambda}^{(k-1)}(x + 1) \geq (-1)^{k-1} \Delta_{s,t;\lambda}^{(k-1)}(x + 2)
\]

\[
(-1)^{k-1} \Delta_{s,t;\lambda}^{(k-1)}(x + 3) \geq \cdots \geq (-1)^{k-1} \Delta_{s,t;\lambda}^{(k-1)}(x + m) \to 0
\]
as \( m \to \infty \). This means that the function \( \Delta_{s,t;\lambda}(x) \) is CM on \((-\alpha, \infty)\).

Similarly, if \( \Delta_{s,t;\lambda}(x + 1) - \Delta_{s,t;\lambda}(x) \) is CM on \((-\alpha, \infty)\), then the function \(-\Delta_{s,t;\lambda}(x)\) is also CM on \((-\alpha, \infty)\).

As a consequence of either \([15, \text{Theorem 2}], [19, \text{Theorem 2.1}], [27, \text{Theorem 1.3}]\) or \([28, \text{Theorem 3}]\), the double inequality

\[
\frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} < (-1)^{k+1} \psi^{(k)}(x) < \frac{(k-1)!}{x^k} + \frac{k!}{x^{k+1}} \tag{33}
\]

for \( k \in \mathbb{N} \) on \((0, \infty)\) can be derived easily. This implies that

\[
\lim_{x \to -\infty} \frac{(-1)^{k+1} x^k \psi^{(k)}(x)}{x^k} = (k-1)! \tag{34}
\]

If the function \( \Delta_{s,t;\lambda}(x) \) is CM on \((-\alpha, \infty)\), then \( \Delta_{s,t;\lambda}(x) \geq 0 \) on \((-\alpha, \infty)\), which is equivalent to

\[
\lambda \leq -\frac{[\psi(x + t) - \psi(x + s)]^2}{(t - s)[\psi'(x + t) - \psi'(x + s)]} = \frac{[\psi'(x + \xi)]^2}{\psi''(x + \xi)} \to 1 \tag{35}
\]
as \( x \to \infty \) by making use of the mean value theorem for derivative and \((34)\), where \( \xi \) is between \( s \) and \( t \). On the other hand, since \( \lim_{x \to 0^+} \frac{(-1)^k \psi^{(k)}(x)}{x^k} = \infty \) for \( k \in \mathbb{N} \), L'Hôpital's rule gives

\[
\lim_{x \to (-\alpha)^+} \frac{[\psi(x + t) - \psi(x + s)]^2}{(t - s)[\psi'(x + t) - \psi'(x + s)]} = \lim_{x \to (-\alpha)^+} \frac{[\psi(x + t) - \psi(x + s)]^2}{(t - s)[\psi'(x + t) - \psi'(x + s)]}
\]

\[
= \frac{2}{t - s} \lim_{x \to (-\alpha)^+} \left[ \frac{\psi'(x + t)\psi'(x + s)}{\psi''(x + s)} - \frac{\psi(x + s)\psi'(x + s)}{\psi''(x + s)} \right]
\]

\[
= \frac{2}{t - s} \lim_{x \to 0^+} \left[ \frac{\psi(u + t - s)\psi'(u)}{\psi''(u)} - \frac{\psi(u)\psi'(u)}{\psi''(u)} \right],
\]

where \( t > s \) and \( t - s \neq 1 \) are assumed without loss of generality. From

\[
\lim_{u \to 0^+} \frac{\psi'(u)}{\psi''(u)} = \lim_{u \to 0^+} \frac{\psi'(u)}{\psi''(u)} = 0
\]

by \((27)\) for \( k = 1 \) and the fact obtained in \([3, \text{p. 182, Lemma 2.1}]\) that the function \( \frac{\psi^{(k+1)}(x)}{\psi''(x)} \) for \( k \in \mathbb{N} \) is strictly increasing from \([0, \infty)\) onto \([-k, -k)\), it is deduced readily that

\[
\lim_{u \to 0^+} \frac{\psi'(u)}{\psi''(u)} = \lim_{u \to 0^+} \frac{\psi'(u)}{\psi''(u)} = 0
\]

and

\[
\lim_{u \to 0^+} \frac{\psi(u)\psi'(u)}{\psi''(u)} = \lim_{u \to 0^+} \frac{\psi(u)\psi'(u)}{\psi''(u)} = 1/2.
\]

So

\[
\lim_{x \to (-\alpha)^+} \frac{[\psi(x + t) - \psi(x + s)]^2}{(t - s)[\psi'(x + t) - \psi'(x + s)]} = -\frac{1}{t - s}. \tag{39}
\]

As a result, the necessities for the function \( \Delta_{s,t;\lambda}(x) \) to be CM on \((-\alpha, \infty)\) is proved. The left proofs are similar and so omitted. The proof of Theorem 1 is complete. \( \square \)
Proof of Theorem 2. For the case $t - s = 1$, it is easy to see that
\[ H_{s,s+1;\lambda}(x) = [(x + s)(x + s + 1)]^{(1-\lambda)/2}. \] (40)
Therefore, the logarithmically complete monotonicity of the function $H_{s,s+1;\lambda}(x) = H_{s+1,s;\lambda}(x)$ is proved.

For $s = t$, it is equivalent to showing the logarithmically complete monotonicity of the function
\[ \frac{1}{x^\lambda} \exp \left[ \psi(x) + \frac{1}{2x} \right] \] (41)
on $(0, \infty)$, which is a direct consequence of the complete monotonicity of the function
\[ \psi'(x) - \frac{1}{2x^2} - \frac{\lambda}{x} \] (42)
on $(0, \infty)$, whose sufficiency has been verified in the proof of In order to show the necessity, it is sufficient to deduce $\lambda \leq 1$ from the positivity of the function $\lambda \leq 1$ from the positivity of the function (42), which is equivalent to
\[ \lambda \leq x \left[ \psi'(x) - \frac{1}{2x^2} \right] = x\psi'(x) - \frac{1}{2x} \to 1 \] (43)
as $x \to \infty$ by making use of (34).

For $0 < |t - s| \neq 1$, taking the logarithm of $H_{s,t;\lambda}(x)$ and differentiating yields $[\ln H_{s,t;\lambda}(x)]' = \theta_{s,t;\lambda}(x)$, where $\theta_{s,t;\lambda}(x)$ is the function defined in the proof of Theorem 1 on page 6. Hence, the function $H_{s,t;\lambda}(x)$ is LCM on $(-\alpha, \infty)$ when the condition (iii) or (vi) on page 7 is satisfied and so is the function $[H_{s,t;\lambda}(x)]^{-1}$ when the condition (i) or (ii) on page 7 in the proof of Theorem 1 is satisfied.

If the function $H_{s,t;\lambda}(x)$ is LCM on $(-\alpha, \infty)$, then $\theta_{s,t;\lambda}(x) \leq 0$ on $(-\alpha, \infty)$, which can be rewritten as
\[ \lambda \geq \frac{2(x + s)(x + t)}{2x + s + t} \left[ \psi(x + t) - \psi(x + s) \right] - \frac{1}{2(x + s)(x + t)} \triangleq \Lambda_{s,t}(x). \]

It is easy to see that
\[
\lim_{x \to \infty} \Lambda_{s,t}(x) = \lim_{x \to \infty} \frac{2(x + s)(x + t)[\psi(x + t) - \psi(x + s)]}{(t - s)(2x + s + t)} - \lim_{x \to \infty} \frac{1}{2(x + s)(x + t)} \\
= \lim_{x \to \infty} \frac{2(x + s)(x + t)}{(t - s)(2x + s + t)} \lim_{x \to \infty} \{x[\psi(x + t) - \psi(x + s)]\} \\
= \frac{1}{t - s} \lim_{x \to \infty} \left[ x \int_s^t \psi'(x + u) \, du \right] \\
= \frac{1}{t - s} \int_s^t \lim_{x \to \infty} [x\psi'(x + u)] \, du \\
= 1
\]
by (34) for $k = 1$. On the other hand, if assume $t > s$ with out loss of generality, then
\[
\lim_{x \to (-\alpha)^+} \Lambda_{s,t}(x) = \lim_{x \to (-s)^+} \Lambda_{s,t}(x) \\
= \lim_{u \to 0^+} \frac{2u(u + t - s)}{2u + t - s} \left[ \frac{\psi(u + t - s) - \psi(u)}{t - s} - \frac{1}{2u(u + t - s)} \right]
\]
is proved. Consequently, the necessities for the function \( H_{s,t;\lambda}(x) \) to be LCM on \((-\alpha, \infty)\) are verified.

The left proofs are similar and so omitted. Theorem 2 is proved. \(\square\)

**Proof of Theorem 3.** In [42], the following asymptotic relation was obtained:

\[
\lim_{x \to \infty} \frac{\Gamma(x+s)}{x^s \Gamma(x)} = 1 \quad (44)
\]

for real \( s \) and \( x \) holds. This implies that

\[
H_{s,t;\lambda}(x) = \frac{x + s}{(x + s)(x + t)^{1/2}} \left[ \frac{\Gamma(x + t)}{(x + s)^{t-s} \Gamma(x + s)} \right]^{1/(t-s)} \frac{(x + t)^{1/2(t-s)}}{(x + s)^{1/2(t-s)}}
\]

(45)

as \( x \to \infty \) for \( s \neq t \). As a result, from the fact that the function \( H_{s,t;1}(x) \) on \((-\alpha, \infty)\) is increasing for \( 0 < |t-s| < 1 \) and decreasing for \( |t-s| > 1 \), it is deduced that the inequality

\[
\left[ \frac{\Gamma(x + t)}{\Gamma(x + s)} \right]^{1/(t-s)} < \frac{(x + s)^{1/(t-s)+1}/2}{(x + t)^{1/(t-s)-1/2}}
\]

(46)

on \((-\alpha, \infty)\) holds for \( 0 < |t-s| < 1 \) and reverses for \( |t-s| > 1 \). Letting \( x + s = a \) and \( x + t = b \) in the above inequality gives

\[
\left[ \frac{\Gamma(b)}{\Gamma(a)} \right]^{1/(b-a)} < \left( \frac{a^{1/(b-a)+1}/2}{b^{1/(b-a)-1/2}} \right)
\]

which is equivalent to (24).

By definition of LCM function and the fact [37, p. 82] that a completely monotonic function which is non-identically zero cannot vanish at any point on \((0, \infty)\), it is easy to see that when \( 0 < |t-s| < 1 \), the inequality

\[
(-1)^k \left[ \ln H_{s,t;\lambda}(x) \right]^{(k)} = (-1)^k \left[ \theta_{s,t;\lambda}(x) \right]^{(k-1)} > 0
\]

(46)

for \( k \in \mathbb{N} \) holds if and only if \( \lambda \geq \frac{1}{|t-s|} \) and reverses if and only if \( \lambda \geq 1 \). The inequality (46) may be rewritten as

\[
(-1)^{k-1} \left[ \psi^{(k-1)}(x+t) - \psi^{(k-1)}(x+s) \right] \frac{t-s}{2} < \frac{(k-1)!}{(x+t)^{k}} \left[ \frac{1}{t-s} + \lambda \right] \frac{1}{(x+s)^{k}} + \left( \lambda - \frac{1}{t-s} \right) \frac{1}{(x+t)^{k}}
\]

(47)

Consequently, utilizing the complete monotonicity of \( \theta_{s,t;\lambda}(x) \) or the logarithmically complete monotonicity of \( H_{s,t;\lambda}(x) \) concludes the double inequality.
\[
\frac{(k-1)!}{2} \left[ \left( \frac{1}{t-s} + \beta \right) \frac{1}{(x+s)^k} + \left( \frac{\beta - 1}{t-s} \right) \frac{1}{(x+t)^k} \right] \\
< \frac{(-1)^{k-1}}{t-s} [\psi^{(k-1)}(x+t) - \psi^{(k-1)}(x+s)] \\
< \frac{(k-1)!}{2} \left[ \left( \frac{1}{t-s} + \gamma \right) \frac{1}{(x+s)^k} + \left( \gamma - \frac{1}{t-s} \right) \frac{1}{(x+t)^k} \right]
\]

on \((-\alpha, \infty)\) holds either for \(0 < |t-s| < 1\) if and only if \(\gamma \geq \frac{1}{|t-s|}\) and \(\beta \leq 1\) or for \(|t-s| > 1\) if and only if \(\beta \leq \frac{1}{|t-s|}\) and \(\gamma \geq 1\). Replacing \(x + s\) and \(x + t\) by \(a\) and \(b\) respectively in (48) leads to (25). The proof of Theorem 3 is complete. \(\square\)

3. Remarks

Remark 1. Taking \(\lambda = s-t > 0\) in Theorem 1 produces that the function \(\Gamma(x+s)\) on \((-t, \infty)\) is increasingly convex for \(s-t > 1\) and increasingly concave for \(0 < s-t < 1\). For detailed information, please refer to [25, Remark 4.2 and Remark 4.5] on the paper [20].

Remark 2. From the proofs of Theorem 1 and Theorem 2, the following conclusions may be summarized: For real numbers \(s, t\), \(\alpha = \min\{s, t\}\) and \(\lambda\), the function

\[
\theta_{s,t,\lambda}(x) = \begin{cases} \\
\psi(x+t) - \psi(x+s) - \frac{1 + \lambda(2x+s+t)}{2(x+s)(x+t)}, & s \neq t \\
\psi'(x+s) - \frac{1 + 2\lambda(x+s)}{2(x+s)^2}, & s = t
\end{cases}
\]

on \((-\alpha, \infty)\) has the following completely monotonic properties:

(1) For \(0 < |t-s| < 1\),
   (a) the function \(\theta_{s,t,\lambda}(x)\) is CM if and only if \(\lambda \leq 1\),
   (b) the function \(-\theta_{s,t,\lambda}(x)\) is CM if and only if \(\lambda \geq \frac{1}{|t-s|}\);

(2) For \(|t-s| > 1\),
   (a) the function \(\theta_{s,t,\lambda}(x)\) is CM if and only if \(\lambda \leq \frac{1}{|t-s|}\),
   (b) the function \(-\theta_{s,t,\lambda}(x)\) is CM if and only if \(\lambda \geq 1\);

(3) For \(s = t\), the function \(\theta_{s,s,\lambda}(x)\) is CM if and only if \(\lambda \leq 1\);

(4) For \(|t-s| = 1\),
   (a) the function \(\theta_{s,t,\lambda}(x)\) is CM if and only if \(\lambda < 1\),
   (b) so is the function \(-\theta_{s,t,\lambda}(x)\) if and only if \(\lambda > 1\),
   (c) and \(\theta_{s,t,1}(x) \equiv 0\).

Remark 3. In (24), taking \(b = x + 1\) and \(a = x + \frac{1}{2}\) yields

\[
\left[ \frac{\Gamma(x+1)}{\Gamma(x+1/2)} \right]^2 < \left( x + \frac{1}{2} \right) \sqrt{\frac{x+1/2}{x+1}}
\]

for \(x > -\frac{1}{2}\), which is a refinement of the inequality

\[
\left[ \frac{\Gamma(x+1)}{\Gamma(x+1/2)} \right]^2 - x < \frac{1}{2}
\]

for \(x > -\frac{1}{2}\), obtained in [41].
Remark 4. In [10, Lemma 1.2] and [11, Lemma 1.2], it was discovered that if $a \leq -\ln 2$ and $b \geq 0$, then
\[ a - \ln(e^{1/x} - 1) < \psi(x) < b - \ln(e^{1/x} - 1) \] (52)
holds for $x > 0$.

In [9, Theorem 2.8], the inequality (52) was sharpened as $a \leq -\gamma$ and $b \geq 0$.

In [4], the function $\phi(x)$ defined by (13) was proved to be strictly increasing on $(0, \infty)$ and
\[ \lim_{x \to \infty} \phi(x) = 0. \] (53)

In [35], among other things, the function $\phi(x)$ was proved to be both strictly increasing and concave on $(0, \infty)$, with $\lim_{x \to 0^+} \phi(x) = -\gamma$ and the limit (53).

It is not difficult to see that all these results extend, refine and generalize the one-side inequality (12) or the increasing property of $\phi(x)$.

Remark 5. After the monotonic and convex properties of the function (18) were perfectly procured in [16, Theorem 1], several alternative proofs were supplied in [14, 23, 26, 30, 39, 40]. The investigation of the function (18) has a long history, see [20, 21, 41] or the survey articles [24, 25] and related references therein.

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