COLORFUL COVERINGS OF POLYTOPES AND PIERCING NUMBERS OF COLORFUL $d$-INTERVALS

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We prove a common strengthening of Bárány’s colorful Carathéodory theorem and the KKMS theorem. In fact, our main result is a colorful polytopal KKMS theorem, which extends a colorful KKMS theorem due to Shih and Lee [Math. Ann. 296 (1993), no. 1, 35–61] as well as a polytopal KKMS theorem due to Komiya [Econ. Theory 4 (1994), no. 3, 463–466]. The (seemingly unrelated) colorful Carathéodory theorem is a special case as well. We apply our theorem to establish an upper bound on the piercing number of colorful $d$-interval hypergraphs, extending earlier results of Tardos [Combinatorica 15 (1995), no. 1, 123–134] and Kaiser [Discrete Comput. Geom. 18 (1997), no. 2, 195–203].

1. Introduction

The KKM theorem of Knaster, Kuratowski, and Mazurkiewicz [11] is a set covering variant of Brouwer’s fixed point theorem. It states that for any covering of the $k$-simplex $\Delta_k$ on vertex set $[k+1]$ with closed sets $A_1, \ldots, A_{k+1}$ such that the face spanned by vertices in $S$ is contained in $\bigcup_{i \in S} A_i$ for every $S \subset [k+1]$, the intersection $\bigcap_{i \in [k+1]} A_i$ is nonempty.

The KKM theorem has inspired many extensions and variants, some of which we will briefly survey in Section 2. Important strengthenings include a colorful extension of the KKM theorem due to Gale [9] that deals with $k+1$ possibly distinct coverings of the $k$-simplex and the KKMS theorem of Shapley [16], where the sets in the covering are associated to faces of the $k$-simplex instead of its vertices. Further generalizations of the KKMS theorem are a polytopal version due to Komiya [12] and the colorful KKMS theorem of Shih and Lee [17].

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In this note we prove a colorful polytopal KKMS theorem, extending all results above. This result is finally sufficiently general to also specialize to Bárány’s celebrated colorful Carathéodory theorem [5] from 1982, which asserts that if $X_1, \ldots, X_{k+1}$ are subsets of $\mathbb{R}^k$ with $0 \in \text{conv} X_i$ for every $i \in [k+1]$, then there exists a choice of points $x_1 \in X_1, \ldots, x_{k+1} \in X_{k+1}$ such that $0 \in \text{conv}\{x_1, \ldots, x_{k+1}\}$. Carathéodory’s classical result is the case $X_1 = X_2 = \cdots = X_{k+1}$. We deduce the colorful Carathéodory theorem from our main result in Section 3.

For a set $\sigma \subset \mathbb{R}^k$ we denote by $C_{\sigma}$ the cone of $\sigma$, that is, the union of all rays emanating from the origin that intersect $\sigma$. Our main result is the following:

**Theorem 1.1.** Let $P$ be a $k$-dimensional polytope with $0 \in P$. Suppose for every nonempty, proper face $\sigma$ of $P$ we are given $k+1$ points $y^{(1)}_{\sigma}, \ldots, y^{(k+1)}_{\sigma} \in C_{\sigma}$ and $k+1$ closed sets $A^{(1)}_{\sigma}, \ldots, A^{(k+1)}_{\sigma} \subset P$. If $\sigma \subset \bigcup_{\tau \subset \sigma} A^{(j)}_{\tau}$ for every face $\sigma$ of $P$ and every $j \in [k+1]$, then there exist faces $\sigma_1, \ldots, \sigma_{k+1}$ of $P$ such that $0 \in \text{conv}\{y^{(1)}_{\sigma_1}, \ldots, y^{(k+1)}_{\sigma_{k+1}}\}$ and $\bigcap_{i=1}^{k+1} A^{(i)}_{\sigma_i} \neq \emptyset$.

Our proof of this result relies on a topological mapping degree argument. As such, it is entirely different from Bárány’s proof of the colorful Carathéodory theorem, and thus provides a new topological route to prove this theorem. Our argument is also less involved than the topological proof given recently by Meunier, Mulzer, Sarrabezolles, and Stein [14] to show that algorithmically finding the configuration whose existence is guaranteed by the colorful Carathéodory theorem is in PPAD (that is, informally speaking, it can be found by a path-following algorithm). Our method, however, involves a limiting argument and thus does not have immediate algorithmic consequences. Finally, our proof of Theorem 1.1 exhibits a surprisingly simple way to prove KKMS-type results and their polytopal and colorful extensions.

As an application of Theorem 1.1 we prove a bound on the piercing numbers of colorful $d$-interval hypergraphs. A $d$-interval is a union of at most $d$ disjoint closed intervals on $\mathbb{R}$. A $d$-interval $h$ is separated if it consists of $d$ disjoint interval components $h = h^1 \cup \cdots \cup h^d$ with $h^{i+1} \subset (i, i+1)$ for $i \in \{0, \ldots, d-1\}$. A hypergraph of (separated) $d$-intervals is a hypergraph $H$ whose vertex set is $\mathbb{R}$ and whose edge set is a finite family of (separated) $d$-intervals.

A matching in a hypergraph $H = (V, E)$ with vertex set $V$ and edge set $E$ is a set of disjoint edges. A cover is a subset of $V$ intersecting all edges. The matching number $\nu(H)$ is the maximal size of a matching, and the covering number (or piercing number) $\tau(H)$ is the minimal size of a cover. Tardos
[19] and Kaiser [10] proved the following bound on the covering number in hypergraphs of $d$-intervals:

**Theorem 1.2 (Tardos [19], Kaiser [10]).** In every hypergraph $H$ of $d$-intervals we have $\tau(H) \leq (d^2 - d + 1)\nu(H)$. Moreover, if $H$ is a hypergraph of separated $d$-intervals, then $\tau(H) \leq (d^2 - d)\nu(H)$.

Matoušek [13] showed that this bound is not far from the truth: There are examples of hypergraphs of $d$-intervals in which $\tau = \Omega(d^2 \log d)$. Aharoni, Kaiser and Zerbib [1] gave a proof of Theorem 1.2 that used the KKMS theorem and Komiya’s polytopal extension, Theorem 2.1. Using Theorem 1.1 we prove here a colorful generalization of Theorem 1.2:

**Theorem 1.3.** Let $F_i$, $i \in [k+1]$, be $k+1$ hypergraphs of $d$-intervals and let $F = \bigcup_{i=1}^{k+1} F_i$.

1. If $\tau(F_i) > k$ for all $i \in [k+1]$, then there exists a collection $\mathcal{M}$ of pairwise disjoint $d$-intervals in $F$ of size $|\mathcal{M}| \geq \frac{k+1}{d^2 - d + 1}$, with $|\mathcal{M} \cap F_i| \leq 1$.
2. If $F_i$ consists of separated $d$-intervals and $\tau(F_i) > kd$ for all $i \in [k+1]$, then there exists a collection $\mathcal{M}$ of pairwise disjoint separated $d$-intervals in $F$ of size $|\mathcal{M}| \geq \frac{k+1}{d^2 - d + 1}$, with $|\mathcal{M} \cap F_i| \leq 1$.

Note that Theorem 1.2 is the case where all the hypergraphs $F_i$ are the same. In Section 2 we introduce some notation and, as an introduction to our methods, provide a new simple proof of Komiya’s theorem. Then, in Section 3, we prove Theorem 1.1 and use it to derive Bárány’s colorful Carathéodory theorem. Section 4 is devoted to the proof of Theorem 1.3.

## 2. Coverings of polytopes and Komiya’s theorem

Let $\Delta_k$ be the $k$-dimensional simplex with vertex set $[k+1]$ realized in $\mathbb{R}^{k+1}$ as \( \{ x \in \mathbb{R}_{\geq 0}^{k+1} : \sum_{i=1}^{k+1} x_i = 0 \} \). For every $S \subset [k+1]$ let $\Delta^S$ be the face of $\Delta_k$ spanned by the vertices in $S$. Recall that the KKM theorem asserts that if $A_1, \ldots, A_{k+1}$ are closed sets covering $\Delta_k$ so that $\Delta^S \subset \bigcup_{i \in S} A_i$ for every $S \subset [k+1]$, then the intersection of all the sets $A_i$ is non-empty. We will refer to covers $A_1, \ldots, A_{k+1}$ as above as KKM covers.

A generalization of this result, known as the KKMS theorem, was proven by Shapley [16] in 1973. Now we have a cover of $\Delta_k$ by closed sets $A_T$, $T \subset [k+1]$, so that $\Delta^S \subset \bigcup_{T \subset S} A_T$ for every $S \subset [k+1]$. Such a collection of sets $A_T$ is called a KKMS cover. The conclusion of the KKMS theorem is that there exists a balanced collection $T_1, \ldots, T_m$ of subsets of $[k+1]$ for which
Here \( T_1, \ldots, T_m \) form a balanced collection if the barycenters of the corresponding faces \( \Delta_{T_1}, \ldots, \Delta_{T_m} \) contain the barycenter of \( \Delta_k \) in their convex hull.

A different generalization of the KKM theorem is a colorful version due to Gale [9]. It states that given \( k+1 \) KKM covers \( A^{(i)}_1, \ldots, A^{(i)}_{k+1}, i \in [k+1], \) of the \( k \)-simplex \( \Delta_k \), there is a permutation \( \pi \) of \( [k+1] \) such that \( \bigcap_{i \in [k+1]} A^{(i)}_{\pi(i)} \) is nonempty. This theorem is colorful in the sense that we think of each KKM cover as having a different color; the theorem then asserts that there is an intersection of \( k+1 \) sets of pairwise distinct colors associated to pairwise distinct vertices. Asada et al. [2] showed that one can additionally prescribe \( \pi(1) \).

In 1993 Shih and Lee [17] proved a common generalization of the KKMS theorem and Gale’s colorful KKM theorem: Given \( k+1 \) KKMS covers \( A^i_1, T \subset [k+1], i \in [k+1], \) of \( \Delta_k \), there exists a balanced collection \( T_1, \ldots, T_{k+1} \) of subsets of \( [k+1] \) for which we have \( \bigcap_{i=1}^m A^i_{\pi(i)} \neq \emptyset \).

Another far reaching extension of the KKMS theorem to general polytopes is due to Komiya [12] from 1994. Komiya proved that the simplex \( \Delta_k \) in the KKMS theorem can be replaced by any \( k \)-dimensional polytope \( P \), and that the barycenters of the faces can be replaced by any points \( y_\sigma \) in the face \( \sigma \):

**Theorem 2.1 (Komiya’s theorem [12]).** Let \( P \) be a polytope, and for every nonempty face \( \sigma \) of \( P \) choose a point \( y_\sigma \in \sigma \) and a closed set \( A_\sigma \subset P \). If \( \sigma \subset \bigcup_{T \subset \sigma} A_T \) for every face \( \sigma \) of \( P \), then there are faces \( \sigma_1, \ldots, \sigma_m \) of \( P \) such that \( y_P \in \text{conv}\{y_{\sigma_1}, \ldots, y_{\sigma_m}\} \) and \( \bigcap_{i=1}^m A_{\pi(i)} \neq \emptyset \).

This specializes to the KKMS theorem if \( P \) is the simplex and each point \( y_\sigma \) is the barycenter of the face \( \sigma \). Moreover, there are quantitative versions of the KKMS theorem due to De Loera, Peterson, and Su [6] as well as Asada et al. [2] and KKM theorems for general pairs of spaces due to Musin [15].

To set the stage we will first present a simple proof of Komiya’s theorem. Recall that the KKM theorem can be easily deduced from Sperner’s lemma on vertex labelings of triangulations of a simplex. Our proof of Komiya’s theorem – just as Shapley’s original proof of the KKMS theorem – first establishes an equivalent Sperner-type version. A **Sperner–Shapley labeling** of a triangulation \( T \) of a polytope \( P \) is a map \( f: V(T) \longrightarrow \{\sigma: \sigma \text{ a nonempty face of } P\} \) from the vertex set \( V(T) \) of \( T \) to the set of nonempty faces of \( P \) such that \( f(v) \subset \text{supp}(v) \), where \( \text{supp}(v) \) is the minimal face of \( P \) containing \( v \). We prove the following polytopal Sperner–Shapley theorem that will imply Theorem 2.1 by a limiting and compactness argument:
Theorem 2.2. Let $T$ be a triangulation of the polytope $P \subset \mathbb{R}^k$, and let $f: V(T) \rightarrow \{\sigma: \sigma$ a nonempty face of $P\}$ be a Sperner–Shapley labeling of $T$. For every nonempty face $\sigma$ of $P$ choose a point $y_\sigma \in \sigma$. Then there is a face $\tau$ of $T$ such that $y_P \in \text{conv}\{y_{f(v)}: v \text{ vertex of } \tau\}$.

Proof. The Sperner–Shapley labeling $f$ maps vertices of the triangulation $T$ of $P$ to faces of $P$; thus mapping a vertex $v$ to the chosen point $y_{f(v)}$ in the face $f(v)$ and extending linearly onto faces of $T$ defines a continuous map $F: P \rightarrow P$. By the Sperner–Shapley condition for every face $\sigma$ of $P$ we have that $F(\sigma) \subset \sigma$. This implies that $F$ is homotopic to the identity on $\partial P$, and thus $F|_{\partial P}$ has degree one. Then $F$ is surjective and we can find a point $x \in P$ such that $F(x) = y_P$. Let $\tau$ be the smallest face of $T$ containing $x$. By the definition of $F$ the image $F(\tau)$ is equal to the convex hull $\text{conv}\{y_{f(v)}: v \text{ vertex of } \tau\}$.

Proof of Theorem 2.1 Let $\varepsilon > 0$, and let $T$ be a triangulation of $P$ such that every face of $T$ has diameter at most $\varepsilon$. Given a cover $\{A_\sigma: \sigma$ a nonempty face of $P\}$ that satisfies the covering condition of the theorem we define a Sperner–Shapley labeling in the following way: For a vertex $v$ of $T$, label $v$ by a face $\sigma \subset \text{supp}(v)$ such that $v \in A_\sigma$. Such a face $\sigma$ exists since $v \in \text{supp}(v) \subset \bigcup_{\sigma \subset \text{supp}(v)} A_\sigma$. Thus by Theorem 2.2 there is a face $\tau$ of $T$ whose vertices are labeled by faces $\sigma_1, \ldots, \sigma_m$ of $P$ such that $y_P \in \text{conv}\{y_{\sigma_1}, \ldots, y_{\sigma_m}\}$. In particular, the $\varepsilon$-neighborhoods of the sets $A_{\sigma_i}$, $i \in [m]$, intersect. Now let $\varepsilon$ tend to zero. As there are only finitely many collections of faces of $P$, one collection $\sigma_1, \ldots, \sigma_m$ must appear infinitely many times. By compactness of $P$ the sets $A_{\sigma_i}$, $i \in [m]$, then all intersect since they are closed.

Note that Theorem 2.1 is true also if all the sets $A_\sigma$ are open in $P$. Indeed, given an open cover $\{A_\sigma: \sigma$ a nonempty face of $P\}$ of $P$ as in Theorem 2.1, we can find closed sets $B_\sigma \subset A_\sigma$ that have the same nerve as $A_\sigma$ (namely, any collection of sets $\{B_{\sigma_i}: i \in I\}$ intersects if and only if the corresponding collection $\{A_{\sigma_i}: i \in I\}$ intersects) and still satisfy $\sigma \subset \bigcup_{\tau \subset \sigma} B_\tau$ for every face $\sigma$ of $P$.

3. A colorful Komiya theorem

Recall that the colorful KKMS theorem of Shih and Lee [17] states the following: If for every $i \in [k+1]$ the collection $\{A_\sigma^i: \sigma$ a nonempty face of $\Delta_k\}$ forms a KKMS cover of $\Delta_k$, then there exists a balanced collection of faces $\sigma_1, \ldots, \sigma_k+1$ so that $\bigcap_{i=1}^{k+1} A_{\sigma_i}^i \neq \emptyset$. Theorem 1.1, proved in this section, is a
colorful extension of Theorem 2.1, and thus generalizes the colorful KKMS theorem to any polytope.

Let $P$ be a $k$-dimensional polytope. Suppose that for every nonempty face $\sigma$ of $P$ we choose $k + 1$ points $y^{(1)}_{\sigma}, \ldots, y^{(k+1)}_{\sigma} \in \sigma$ and $k + 1$ closed sets $A^{(1)}_{\sigma}, \ldots, A^{(k+1)}_{\sigma} \subset P$, so that $\sigma \subseteq \bigcup_{\tau \subseteq \sigma} A^{(j)}_{\tau}$ for every face $\sigma$ of $P$ and every $j \in [k + 1]$. Theorem 2.1 now guarantees that for every fixed $j \in [k + 1]$ there are faces $\sigma^{(j)}_1, \ldots, \sigma^{(j)}_{m_j}$ of $P$ such that $y^{(j)}_P \in \text{conv}\{y^{(j)}_{\sigma_1}, \ldots, y^{(j)}_{\sigma_{m_j}}\}$ and $\bigcap_{i=1}^{m_j} A^{(j)}_{\sigma_i}$ is nonempty. Now let us choose $y^{(1)}_P = y^{(2)}_P = \cdots = y^{(k+1)}_P$ and denote this point by $y_P$. The colorful Carathéodory theorem implies the existence of points $z_j \in \{y^{(j)}_{\sigma_1}, \ldots, y^{(j)}_{\sigma_{m_j}}\}$, $j \in [k + 1]$, such that $y_P \in \text{conv}\{z_1, \ldots, z_{k+1}\}$.

Theorem 1.1 shows that this implication can be realized simultaneously with the existence of sets $B_j \in \{A^{(j)}_{\sigma_1}, \ldots, A^{(j)}_{\sigma_{m_j}}\}$, $j \in [k + 1]$, such that $\bigcap_{j=1}^{k+1} B_j$ is nonempty. We prove Theorem 1.1 by applying the Sperner–Shapley version of Komiya’s theorem – Theorem 2.2 – to a labeling of the barycentric subdivision of a triangulation of $P$. The same idea was used by Su [18] to prove a colorful Sperner’s lemma. For related Sperner-type results for multiple Sperner labelings see Babson [3], Bapat [4], and Frick, Houston-Edwards, and Meunier [7].

**Proof of Theorem 1.1** Let $\varepsilon > 0$, and let $T$ be a triangulation of $P$ such that every face of $T$ has diameter at most $\varepsilon$. We will also assume that the chosen points $y^{(1)}_{\sigma}, \ldots, y^{(k+1)}_{\sigma}$ are contained in $\sigma$. This assumption does not restrict the generality of our proof since $0 \in \text{conv}\{x_1, \ldots, x_{k+1}\}$ for vectors $x_1, \ldots, x_{k+1} \in \mathbb{R}^k$ if and only if $0 \in \text{conv}\{\alpha_1 x_1, \ldots, \alpha_{k+1} x_{k+1}\}$ with arbitrary coefficients $\alpha_i > 0$. Denote by $T'$ the barycentric subdivision of $T$. We now define a Sperner–Shapley labeling of the vertices of $T'$: For $v \in V(T')$ let $\sigma_v$ be the face of $T$ so that $v$ lies at the barycenter of $\sigma_v$, let $\ell = \dim \sigma_v$, and let $\tau$ be the minimal supporting face of $P$ containing $\sigma_v$. By the conditions of the theorem, $v$ is contained in a set $A^{(\ell+1)}_{\tau}$ where $\tau \subseteq \sigma$. We label $v$ by $\tau$. Thus by Theorem 2.2 there exists a face $\tau$ of $T'$ (without loss of generality $\tau$ is a facet) whose vertices are labeled by faces $\sigma_1, \ldots, \sigma_{k+1}$ of $P$ such that $0 \in \text{conv}\{y^{(1)}_{\sigma_1}, \ldots, y^{(k+1)}_{\sigma_{k+1}}\}$. In particular, the $\varepsilon$-neighborhoods of the sets $A^{(i)}_{\sigma_i}$, $i \in [k + 1]$, intersect. Now use a limiting argument as before.

Note that by the same argument as before, Theorem 1.1 is true also if all the sets $A^{(i)}_{\sigma_i}$ are open.

For a point $x \neq 0$ in $\mathbb{R}^k$ let $H(x) = \{y \in \mathbb{R}^k : \langle x, y \rangle = 0\}$ be the hyperplane perpendicular to $x$ and let $H^+(x) = \{y \in \mathbb{R}^k : \langle x, y \rangle \geq 0\}$ be the closed halfspace with boundary $H(x)$ containing $x$. Let us now show that Bárány’s colorful Carathéodory theorem is a special case of Theorem 1.1.
Theorem 3.1 (Colorful Carathéodory theorem, Bárány [5]). Let $X_1,\ldots,X_{k+1}$ be finite subsets of $\mathbb{R}^k$ with $0 \in \text{conv} X_i$ for every $i \in [k+1]$. Then there are $x_1 \in X_1,\ldots,x_{k+1} \in X_{k+1}$ such that $0 \in \text{conv}\{x_1,\ldots,x_{k+1}\}$.

Proof. We will assume that $0$ is not contained in any of the sets $X_1,\ldots,X_{k+1}$, for otherwise we are done. Let $P \subset \mathbb{R}^k$ be a polytope containing $0$ in its interior, such that if points $x$ and $y$ belong to the same face of $P$, then $\langle x,y \rangle \geq 0$. For example, a sufficiently fine subdivision of any polytope that contains $0$ in its interior (slightly perturbed to be a strictly convex polytope) satisfies this condition. We can assume that any ray emanating from the origin intersects each $X_i$ in at most one point by arbitrarily deleting any additional points from $X_i$. This will not affect the property that $0 \in \text{conv} X_i$. Furthermore, we can choose $P$ in such a way that for each face $\sigma$ and $i \in [k+1]$ the intersection $C_\sigma \cap X_i$ contains at most one point.

Now for each nonempty, proper face $\sigma$ of $P$ choose points $y_{\sigma}^{(i)}$ and sets $A_{\sigma}^{(i)}$ in the following way: If there exists $x \in C_\sigma \cap X_i$, then let $y_{\sigma}^{(i)} = x$ and $A_{\sigma}^{(i)} = \{y \in P : \langle y,x \rangle \geq 0\} = P \cap H^+(x)$; otherwise let $y_{\sigma}^{(i)}$ be some point in $\sigma$ and let $A_{\sigma}^{(i)} = \sigma$.

Suppose the statement of the theorem was incorrect; then in particular, we can slightly perturb the vertices of $P$ and those points $y_{\sigma}^{(i)}$ that were chosen arbitrarily in $\sigma$, to make sure that for any collection of points $y_{\sigma_1}^{(1)},\ldots,y_{\sigma_{k+1}}^{(k+1)}$ and any subset $S$ of this collection of size at most $k$, $0 \notin \text{conv} S$.

Let us now check that with these definitions the conditions of Theorem 1.1 hold. Clearly, all the sets $A_{\sigma}^{(i)}$ are closed. The fact that $P$ is covered by the sets $A_{\sigma}^{(i)}$ for every fixed $i$ follows from the condition $0 \in \text{conv} X_i$. Indeed, this condition implies that for every $p \in P$ there exists a point $x \in X_i$ with $\langle p,x \rangle \geq 0$, and therefore, for the face $\sigma$ of $P$ for which $x \in C_\sigma$ we have $p \in A_{\sigma}^{(i)}$.

Now fix a proper face $\sigma$ of $P$. We claim that $\sigma \subset A_{\sigma}^{(i)}$ for every $i$. Indeed, either $X_i \cap C_\sigma = \emptyset$ in which case $A_{\sigma}^{(i)} = \sigma$, or otherwise, pick $x \in X_i \cap C_\sigma$ and let $\lambda > 0$ such that $\lambda x \in \sigma$; then for every $p \in \sigma$ we have $\langle p,\lambda x \rangle \geq 0$ by our assumption on $P$, and thus $\langle p,x \rangle \geq 0$, or equivalently $p \in A_{\sigma}^{(i)}$.

Thus by Theorem 1.1 there exist faces $\sigma_1,\ldots,\sigma_{k+1}$ of $P$ such that $\bigcap_{i=1}^{k+1} A_{\sigma_i}^{(i)} \neq \emptyset$ and $0 \in \text{conv}\{y_{\sigma_1}^{(1)},\ldots,y_{\sigma_{k+1}}^{(k+1)}\}$. We claim that $\bigcap_{i=1}^{k+1} A_{\sigma_i}^{(i)}$ can contain only the origin. Indeed, suppose that $0 \neq x_0 \in \bigcap_{i=1}^{k+1} A_{\sigma_i}^{(i)}$. Fix $i \in [k+1]$. If $y_{\sigma_i}^{(i)} \in C_{\sigma_i} \cap X_i$, then since $x_0 \in A_{\sigma_i}^{(i)}$ we have $y_{\sigma_i}^{(i)} \in H^+(x_0)$ by definition. Otherwise $x_0 \in A_{\sigma_i}^{(i)} = \sigma_i$ and $y_{\sigma_i}^{(i)} \in \sigma_i$, so by our choice of $P$ we obtain again that $y_{\sigma_i}^{(i)} \in H^+(x_0)$. Thus all the points $y_{\sigma_1}^{(1)},\ldots,y_{\sigma_{k+1}}^{(k+1)}$ are in $H^+(x_0)$. But
since $0 \in \text{conv}\{y^{(1)}_{\sigma_1}, \ldots, y^{(k+1)}_{\sigma_{k+1}}\}$ this implies that the convex hull of the points in $\{y^{(1)}_{\sigma_1}, \ldots, y^{(k+1)}_{\sigma_{k+1}}\} \cap H(x_0)$ contains the origin. Now, the dimension of $H(x_0)$ is $k-1$, and thus by Carathéodory’s theorem there exists a set $S$ of at most $k$ of the points in $y^{(1)}_{\sigma_1}, \ldots, y^{(k+1)}_{\sigma_{k+1}}$ with $0 \in \text{conv} S$, in contradiction to our general position assumption.

We have shown that $\bigcap_{i=1}^{k+1} A^{(i)}_{(\sigma_i)} = \{0\}$, and thus in particular, $A^{(i)}_{(\sigma_i)} \neq \sigma_i$ for all $i$. By our definitions, this implies $y^{(i)}_{\sigma_i} \in X_i$ for all $i$, concluding the proof of the theorem.

Remark 3.2. Note that we could have avoided the usage of Carathéodory’s theorem in the proof of Theorem 3.1 by taking a more restrictive assumption on the polytope $P$, namely, that $\langle x, y \rangle > 0$ whenever the points $x$ and $y$ belong to the same face of $P$. Therefore, in particular, Theorem 3.1 specializes to Carathéodory’s theorem in the case where all the sets $X_i$ are the same.

4. A colorful $d$-interval theorem

Recall that a fractional matching in a hypergraph $H = (V, E)$ is a function $f : E \rightarrow \mathbb{R}_{\geq 0}$ satisfying $\sum_{e : v \in e} f(e) \leq 1$ for all $v \in V$. A fractional cover is a function $g : V \rightarrow \mathbb{R}_{\geq 0}$ satisfying $\sum_{v : e \cap v} g(v) \geq 1$ for all $e \in E$. The fractional matching number $\nu^*(H)$ is the maximum of $\sum_{e \in E} f(e)$ over all fractional matchings $f$ of $H$, and the fractional covering number $\tau^*(H)$ is the minimum of $\sum_{v \in V} g(v)$ over all fractional covers $g$. By linear programming duality, $\nu \leq \nu^* = \tau^* \leq \tau$. A perfect fractional matching in $H$ is a fractional matching $f$ in which $\sum_{e : v \in e} f(e) = 1$ for every $v \in V$. It is a simple observation that a collection of sets $I \subset 2^{[k+1]}$ is balanced if and only if the hypergraph $H = ([k+1], I)$ has a perfect fractional matching (see e.g., [1]). The rank of a hypergraph $H = (V, E)$ is the maximal size of an edge in $H$. $H$ is $d$-partite if there exists a partition $V_1, \ldots, V_d$ of $V$ such that $|e \cap V_i| = 1$ for every $e \in E$ and $i \in [d]$.

For the proof of Theorem 1.3 we will use the following theorem by Füredi.

Theorem 4.1 (Füredi [8]). If $H$ is a hypergraph of rank $d \geq 2$, then $\nu(H) \geq \frac{\nu^*(H)}{d-1+\frac{1}{d}}$. If, in addition, $H$ is $d$-partite, then $\nu(H) \geq \frac{\nu^*(H)}{d-1}$.

We will also need the following simple counting argument.

Lemma 4.2. If a hypergraph $H = (V, E)$ of rank $d$ has a perfect fractional matching, then $\nu^*(H) \geq \frac{|V|}{d}$.
Proof. Let \( f : E \to \mathbb{R}_{\geq 0} \) be a perfect fractional matching of \( H \). Then \( \sum_{v \in V} \sum_{e : v \in e} f(e) = \sum_{v \in V} 1 = |V| \). Since \( f(e) \) was counted \( |e| \leq d \) times in this equation for every edge \( e \in E \), we have that \( \nu^*(H) \geq \sum_{e \in E} f(e) \geq \frac{|V|}{d} \). ■

We are now ready to prove Theorem 1.3. The proof is an adaption of the methods in [1]. For the first part we need the simplectic version of Theorem 1.1, which was already proven by Shih and Lee [17], while the second part requires our more general polytopal extension.

**Proof of Theorem 1.3.** For a point \( \vec{x} = (x_1, \ldots, x_{k+1}) \in \Delta_k \) let \( p_{\vec{x}}(j) = \sum_{i=1}^j x_t [0,1] \). Since \( \mathcal{F} \) is finite, by rescaling \( \mathbb{R} \) we may assume that \( \mathcal{F} \subset (0,1) \). For every \( T \subset [k+1] \) let \( A_T^i \) be the set consisting of all \( \vec{x} \in \Delta_k \) for which there exists a \( d \)-interval \( f \in \mathcal{F}_i \) satisfying:

(a) \( f \subset \bigcup_{j \in T} (p_{\vec{x}}(j-1), p_{\vec{x}}(j)) \), and

(b) \( f \cap (p_{\vec{x}}(j-1), p_{\vec{x}}(j)) \neq \emptyset \) for each \( j \in T \).

Note that \( A_T^i = \emptyset \) whenever \( |T| > d \).

Clearly, the sets \( A_T^i \) are open. The assumption \( \tau(\mathcal{F}_i) > k \) implies that for every \( \vec{x} = (x_1, \ldots, x_{k+1}) \in \Delta_k \), the set \( P(\vec{x}) = \{ p_{\vec{x}}(j) : j \in [k] \} \) is not a cover of \( \mathcal{F}_i \), meaning that there exists \( f \in \mathcal{F}_i \) not containing any \( p_{\vec{x}}(j) \). This, in turn, means that \( \vec{x} \in A_T^i \) for some \( T \subset [k+1] \), and thus the sets \( A_T^i \) form a cover of \( \Delta_k \) for every \( i \in [k+1] \).

To show that this is a KKMS cover, let \( \Delta^S \) be a face of \( \Delta_k \) for some \( S \subset [k+1] \). If \( \vec{x} \in \Delta^S \) then \( (p_{\vec{x}}(j-1), p_{\vec{x}}(j)) = \emptyset \) for \( j \notin S \), and hence it is impossible to have \( f \cap (p_{\vec{x}}(j-1), p_{\vec{x}}(j)) \neq \emptyset \). Thus \( \vec{x} \in A_T^i \) for some \( T \subset S \). This proves that \( \Delta^S \subset \bigcup_{T \subset S, i} A_T^i \) for all \( i \in [k+1] \).

By Theorem 1.1 there exists a balanced collection of sets \( \mathcal{T} = \{ T_1, \ldots, T_{k+1} \} \) of subsets of \( [k+1] \), satisfying \( \bigcap_{i=1}^{k+1} A_{T_i}^i \neq \emptyset \). In particular, \( |T_i| \leq d \) for all \( i \). (Recall that we think of a collection of sets \( \mathcal{T} \subset 2^{[k+1]} \) as faces of the \( k \)-dimensional simplex to apply the earlier geometric definition of balancedness.) Then by the observation mentioned above, the hypergraph \( H = ([k+1], \mathcal{T}) \) of rank \( d \) has a perfect fractional matching, and thus by Lemma 4.2 we have \( \nu^*(H) \geq \frac{k+1}{d^{2-d+1}} \). Therefore, by Theorem 4.1, \( \nu(H) \geq \frac{\nu^*(H)}{d^{-1} + \frac{1}{d}} \geq \frac{k+1}{d^{2-d+1}} \).

Let \( M \) be a matching in \( H \) of size \( m \geq \frac{k+1}{d^{2-d+1}} \). Let \( \vec{x} \in \bigcap_{i=1}^{k+1} A_{T_i}^i \). For every \( i \in [k+1] \) let \( f(T_i) \) be the \( d \)-interval of \( \mathcal{F}_i \) witnessing the fact that \( \vec{x} \in A_{T_i}^i \). Then the set \( \mathcal{M} = \{ f(T_i) : T_i \in \mathcal{M} \} \) is a matching of size \( m \) in \( \mathcal{F} \) with \( |\mathcal{M} \cap \mathcal{F}_i| \leq 1 \). This proves the first assertion of the theorem.

Now suppose that \( \mathcal{F}_i \) is a hypergraph of separated \( d \)-intervals for all \( i \in [k+1] \). For \( f \in \mathcal{F} \) let \( f^t \subset (t-1,t) \) be the \( t \)-th interval component of \( f \). We
can assume without loss of generality that \( f^t \) is nonempty. Let \( P = (\Delta_k)^d \).

For a \( d \)-tuple \( T = (j_1, \ldots, j_d) \subset [k+1]^d \) let \( A^i_T \) consist of all \( \vec{X} = \vec{x}^1 \times \cdots \times \vec{x}^d \in P \) for which there exists \( f \in F_i \) satisfying \( f^t \subset (t-1+p_{\vec{x}^j}(j_t-1), t-1+p_{\vec{x}^j}(j_t)) \) for all \( t \in [d] \).

Since \( \tau(F) > kd \), the points \( t-1+p_{\vec{x}^j}(j) \), \( t \in [d], j \in [k], \) do not form a cover of \( F \). Therefore, as before, the sets \( A^i_T \) are open and satisfy the covering condition of Theorem 1.1. Thus, by Theorem 1.1, there exists a set \( \mathcal{T} = \{T_1, \ldots, T_{k+1}\} \) of \( d \)-tuples in \([k+1]^d\) containing the point \( (\frac{1}{k+1}, \ldots, \frac{1}{k+1}) \times \cdots \times (\frac{1}{k+1}, \ldots, \frac{1}{k+1}) \in P \) in its convex hull and satisfying \( \bigcap_{i \in [k+1]} A^i_{T_i} \neq \emptyset \). Then the \( d \)-partite hypergraph \( H = (\bigcup_{i=1}^d V_i, \mathcal{T}) \), where \( V_i = [k+1] \) for all \( i \), has a perfect fractional matching, and hence by Lemma 4.2 we have \( \nu^*(H) \geq k+1 \). By Theorem 4.1, this implies \( \nu(H) \geq \frac{\nu^*(H)}{d-1} \geq \frac{k+1}{d-1} \).

Now, by the same argument as before, by taking \( \vec{X} \in \bigcap_{i \in [k+1]} A^i_{T_i} \) we obtain a matching in \( F \) of the same size as a maximal matching in \( H \), concluding the proof of the theorem.

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