Elementary example of energy and momentum of an extended physical system in special relativity

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Abstract

An instructive paradox concerning classical description of energy and momentum of extended physical systems in special relativity theory is explained using an elementary example of two point-like massive bodies rotating on a circle in their center-of-mass frame of reference, connected by an arbitrarily light and infinitesimally thin string. Namely, from the point of view of the inertial observers who move with respect to the rotating system, the sums of energies and momenta of the two bodies oscillate, instead of being constant in time. This result is understood in terms of the mechanism that binds the bodies: the string contributes to the system total energy and momentum no matter how light it is. Its contribution eliminates the unphysical oscillations from the system total four-momentum. Generality of the relativistic approach, applied here to the rotor example, suggests that in every extended physical system its binding mechanism contributes to its total energy and momentum.
I. INTRODUCTION

Relativistic description of extended physical systems is more complicated than the non-relativistic one. Even if a system is assumed known in its center-of-mass frame, description of it in other frames is involved. The reason is that in special relativity one cannot easily separate equations of motion into equations for the relative motion of the system parts and equations for the motion of the system as a whole. Relativity also leads to paradoxes concerning the total energy and momentum of system constituents. We discuss an elementary example of such paradox. Our example is meant to be useful to students of special relativity who try to apply the theory in description of extended objects in motion. The example is also relevant to quantum description of bound states, but we do not discuss quantum theory.

The system we consider consists of two massive points connected via an arbitrarily light and infinitesimally thin string, which may appear negligible as far as the total energy and momentum of the system are concerned. Held by the straight string, the massive points move around the center of mass of the system with constant angular velocity. We shall call this system a rotor, and often refer to the masses at the string ends as endpoint masses, located at the string endpoints.

The issue of proper relativistic description of a rotor is not purely academic. Rotors are of interest in particle physics as models of mesons made of a quark, an anti-quark, and a string of gluons that connects them. The string prototype in a rotor is also useful in developing string theory as a candidate for explaining the nature of particles such as quarks themselves, in the quantum version of the string theory. The need for understanding the rotor is illustrated by the fact that one can imagine that a string between quarks is always straight, while a relativistically acceptable picture of the rotor shows that the string bends when the rotor is in motion. Another field where a relativistic description of a moving rotor may be relevant is the astrophysics of binary systems, especially binary pulsars, although the relativistic effects we talk about are very small in them. They are also likely to be small in the collision of black holes that generates gravitational waves.

The rotor has a feature which simplifies its analysis as a relativistic extended system: there is a clear distinction between its endpoint constituents, i.e., the masses at the string ends, and their binding mechanism – the string. In the Newtonian physics, if we assume that the string is very light, then its contribution to the energy and momentum of the system is
small in every inertial frame of reference. So, in the Newtonian physics, a very light and thin string may appear negligible because it is responsible for no other effect than the circular motion of the endpoint masses.

In the framework of special relativity the situation is different, because the laws of conservation of energy and momentum require a proper treatment of the string. The need for the proper treatment is visible in the case of a moving rotor. Namely, in a frame of reference, in which the rotor moves in the direction perpendicular to its axis of rotation, the total energy and momentum of the endpoint masses oscillate in time. This effect will be explained below. We also show that the string contribution cancels the oscillation, no matter how light the string is. The resulting constant rotor total four-momentum has the proper Lorentz transformation properties.

The reader may find it interesting and encouraging to think about the rotor example, knowing that the Lorentz transformations of the energy and momentum of a moving system had been studied before by Takagi. However, that study is in its nature concerned only with average values of energy and momentum, and the so-called “minimal” addition to energy-momentum of constituents proposed by Takagi is a positive “scalar type volume energy,” which does not contribute to the system three-momentum in any frame of reference. Here, we assume that the laws of conservation of energy and momentum are obeyed in every instant of time in all inertial frames of reference and we introduce a different type of addition – one which contributes to the energy as well as three-momentum.

The great feature of theory of relativity is that it leads to unacceptable results unless the relationships between cause and effect are properly included in description of motion of an extended system. In the rotor case, it turns out that the stresses in a string that connects the endpoint masses contribute a negative amount to the total energy of the rotor in the frames of reference that uniformly move with respect to the rotor. Our reasoning is sufficiently general to say that a relativistic binding mechanism may contribute not only to the energy but also to the momentum of an extended system. Another way to point out this feature of theory of relativity is to say that description of gedanken experiments, the concept introduced by Einstein to imagine what in principle may happen physically, leads to errors unless the events considered in these experiments are physically admissible in the sense that every effect has a cause. Motion of a rotor can be imagined as a gedanken experiment and teaches us about utility of classical relativity theory. For example, one is
forced to correct the description of endpoint masses by including the string. Whatever the
binding mechanism is, its contribution to the energy and momentum of the system must be
included to avoid inconsistency.

It is known that relativistic description of extended systems leads to paradoxes. The
ladder paradox is mentioned quite often. Less popular is Bell’s spaceship paradox over
which disputes continue until today. The right-angle lever paradox was discussed by
Laue. Pauli discussed the Trouton and Noble experiment with a moving condenser.

The rotor paradox discussed in this paper differs from the classic paradoxes mentioned
above. The latter concern extended physical systems looked upon from the frames of re-
ference that uniformly move with respect to these systems, at least momentarily. The rotor
paradox concerns a physical system which cannot ever be considered to be at rest in any
inertial frame of reference; it rotates.

We show on the rotor example that in such circumstances a relativistic description of
an extended system can be obtained using the concept of energy-momentum tensor – the
space-time density of system energy, momentum and stresses – rather than the concept of
force. In relativistic description of extended systems that are more complex than the rotor
of our example, the energy-momentum tensor continues to be a useful tool, worth study and
practice. The rotor example is instructive in this respect.

Textbooks also mention the effect that the interaction that binds constituents, such as
the Coulomb interaction that binds charges, contributes to the system energy, and that the
motion of the bound charges may lead to radiation. For example, see Ref. In the rotor,
the binding is not necessarily electromagnetic. Nevertheless, the rules of special relativity are
powerful enough to determine the form of energy-momentum tensor that provides description
of the binding mechanism in the rotor. The relativity dictum is general enough to imply
that the same treatment of energy-momentum and stresses applies to all kinds of binding
dynamics in extended systems.

For readers interested in further reading about physics of extended systems we ought
to mention that nonrelativistic systems are described in standard textbooks of mechanics.
Classical string theory is instructively addressed in the textbook by Zwiebach. Generaliza-
tion of theory of one-dimensional string-like objects to three dimensional objects (relativistic
elasticity theory) is provided by Kijowski and Magli. The string binding mechanism of an
extended system that is described here can be encoded in a Lagrangian density of the form
that generalizes the one given by Chodos and Thorn. The generalization is required when one allows for the string properties to vary along its length, but the present article only briefly mentions the Lagrangian approach for readers interested in the subject.

This article is organized in the following manner. In Sec. II we describe the paradox concerning the energy and momentum of the rotor when its description does not involve the string. Sec. III is devoted to the construction of a complete energy-momentum tensor of the rotor, including the string, and its analysis. We also present there examples of the rotor energy-momentum tensors including stresses for different choices of the string properties. Summary of our conclusions constitutes Sec. IV. In the Appendix we prove an important theorem that vanishing four-divergence of stress-energy tensor guarantees that four-momentum is conserved and is a four-vector.

II. RELATIVISTIC MOTION OF TWO POINT-LIKE MASSES

One of the most important points of this article is that the sum of four-momenta of the endpoint masses in a rotor cannot be identified in a relativistic theory with a correct energy-momentum of the rotor as a whole. This point is explained in this section.

In the center-of-mass frame of the rotor endpoint masses, which we call the frame $R$, the masses by definition move on a circle. Namely, the position of the first mass is $r\hat{n}(t_R)$ and the position of the second mass is $-r\hat{n}(t_R)$, where $r$ is the rotor radius and $\hat{n}$ is a unit vector dependent on the time $t_R$ in the frame $R$. We assume $\hat{n}$ in the form

$$\hat{n}(t_R) = \begin{bmatrix} \cos \Omega t_R \\ \sin \Omega t_R \\ 0 \end{bmatrix},$$

where $\Omega$ is the angular velocity of rotation. In the frame $R$, the sums of energies and momenta of endpoint masses are conserved: the total energy equals twice the constant energy of one mass moving on a circle, and the total three-momentum is always zero. The observer associated with the frame $R$ is called the observer $R$.

We now ask if the total four-momentum of endpoint masses is also conserved from the point of view of observers whose frames of reference move uniformly with respect to the rotor. In order to check that, one should transform the positions of the endpoints to a moving frame, differentiate them with respect to the time in the moving frame to get the
velocities and thus also energies and momenta, and add respective energies and momenta of the masses at a single instant of time in the moving frame. We describe the resulting total energy-momentum of the endpoint masses in the moving frame and we explain why it is not constant.

The inertial frame of reference of the observer moving with respect to the rotor center of mass, is denoted by $M$. The observer is called the observer $M$. We choose the frame $M$ to move along $x$ axis of the frame $R$, with velocity $-U$, so that the rotor as a whole moves with velocity $+U$ along the $x$ axis of the frame $M$. Time assigned to events in the moving frame, $t_M$, is connected with the time assigned to the same events in the rest frame, $t_R$, via the Lorentz transformation,

$$c t_M = \gamma_U (c t_R + \beta_U x_R),$$

where $c$ is the speed of light in the vacuum, $\gamma_U = (1 - \beta_U^2)^{-1/2}$ is called the Lorentz gamma factor and $\beta_U = U/c$.

Let us assume that the observer $M$, who is at rest at the origin in the moving frame, measures energy-momenta of both endpoint masses at the same time $t_M$. The observer $R$ at rest in the rest frame, can see that $M$’s measurement on the first mass was done at the moment $t_{1R}$, and on the second mass at $t_{2R}$. Substituting $r \cos \Omega t_{1R}$, which is the $x$ component of the position of the first mass, for $x_R$ and $t_{1R}$ for $t_R$ in Eq. (2), one obtains $t_{1R}$ as a function of $t_M$. Similarly, substituting $-r \cos \Omega t_{2R}$, which is the $x$ component of the position of the second mass, for $x_R$ and $t_{2R}$ for $t_R$, one obtains $t_{2R}$ as a function of $t_M$. We observe that if $\beta_U \neq 0$, then $t_{1R}(t_M) \neq t_{2R}(t_M)$, see Fig. 1. Masses’ energies and momenta at an instant of time in $M$ correspond to the masses’ energies and momenta at different instants of time in $R$, and hence their sum depends on time in $M$.

This conclusion can also be arrived at in the following way. The four-momenta of particles seen as separate can be transformed individually from one inertial frame to another using the Lorentz transformation (this is equivalent to saying that they are four-vectors). Geometrically, these four-momenta are assigned to points on the particles’ world-lines. Their sum depends on the choice of these points. Inertial observer’s method of choosing these points is to construct a hyperplane of simultaneity, which is a set of the spacetime points whose coordinates have the same time component, and finding the intersection points of particles’ world-lines with this hyperplane. Different inertial observers choose different hyperplanes, according to the relativity of simultaneity, and the total four-momentum of the endpoint
FIG. 1: Curvy lines represent world lines of the endpoints, dots mark simultaneous measurement of energy-momenta of the masses in the moving frame (represented by \(t_M\) and \(x_M\) axes). In the rest frame (represented by \(t_R\) and \(x_R\) axes) the measurements are not simultaneous. Coordinates \(t_M, x_M\) and \(t_R, x_R\) are related to each other via Lorentz transformation.

masses is time dependent in the frame \(M\) despite that it is constant in the frame \(R\).

The result for energy and momentum of the endpoint masses in the moving frame is,

\[
E_M = 2\gamma_U \gamma_v mc^2 + mU \gamma_U \gamma_v (s_2 - s_1),
\]

\[
\vec{P}_M = 2m \gamma_U \gamma_v \begin{bmatrix} U \\ 0 \\ 0 \end{bmatrix} + m \gamma_v \begin{bmatrix} \gamma_U (s_2 - s_1) \\ c_1 - c_2 \\ 0 \end{bmatrix},
\]

where \(m\) is the mass of the rotor endpoint, \(\Omega\) is the angular velocity of the rotation, and

\[
s_1 = \sin(\Omega t_{1R}(t_M)),
\]

\[
s_2 = \sin(\Omega t_{2R}(t_M)),
\]

\[
c_1 = \cos(\Omega t_{1R}(t_M)),
\]

\[
c_2 = \cos(\Omega t_{2R}(t_M)).
\]

Further, \(\gamma_U\) and \(\gamma_v\) are the Lorentz factors corresponding to, respectively, the speeds \(U\) and \(v = \Omega r\). The latter is the speed of each mass in the frame \(R\). Note that the first terms
on the right-hand sides of Eqs. (3) and (4) are precisely what one would obtain by Lorentz transforming the total four-momentum of the masses from the frame $R$ to the frame $M$. The second terms in these equations oscillate in time $t_M$. An example of these oscillations in total energy and total momentum of the rotor endpoint masses is presented in Fig. 2.

![Graph showing oscillations in total energy and momentum](image)

**FIG. 2:** Oscillations in time $t_M$ of the total energy and total momentum of the rotor endpoint masses in the frame $M$. The total energy is represented by the solid line (black in the online version), marked on the left vertical axis, whereas $x$ and $y$ components of the total momentum are represented by dashed and dotted lines, respectively (blue and green, respectively in the online version), marked on the right vertical axis. Endpoints have masses $m = 1$ kg each, the rotor radius is $r = 1$ m, $v = -0.7c$ and $U = 0.8c$. $T_R$ is the period of rotation in the frame $R$. See also Fig. 4.

### III. THE MECHANISM THAT BINDS THE SYSTEM

The oscillating energy of the rotor endpoint masses indicates that our description of the system misses some ingredient, which is needed in a theory meant to be relativistic. Indeed, in Sec. III we did not include in the description any mechanism that binds the masses and forces them to move on a circle. We can account for the missing ingredient by describing the energy and momentum of the string that connects the masses.
Since we analyze the energy and momentum of an extended body, it is useful to employ energy-momentum tensor $T$. It has sixteen components and depends on the point in space-time, somewhat analogously to the relativistic electromagnetic field tensor $F$ whose components are built from components of the electric and magnetic fields, $\vec{E}$ and $\vec{B}$, that may vary in time and space. $T$ is a symmetric tensor of rank 2, whose $T^{00}$ component is energy density, $T^{0i}$ component, $i = 1, 2, 3$, is the density of $i$-th component of momentum multiplied by the speed of light and $T^{ij}$ components form the stress tensor, i.e., $T^{ij}$ is the density of flux of $j$-th component of momentum through a surface perpendicular to $x^i$ axis.

An introduction to phenomenology of stress-energy tensors can be found in Ref. 24. Descriptions of canonical and symmetric stress-energy tensors are found in Refs. 25, 26. A commonly known example of a stress-energy tensor for extended systems is the one for the perfect fluid,

$$T^{\mu\nu}(x) = \frac{1}{c^2}\epsilon u^\mu u^\nu - p \left( \eta^{\mu\nu} - \frac{u^\mu u^\nu}{c^2} \right), \quad (9)$$

where $\epsilon$ is the local rest frame energy density of the fluid, $p$ is the local rest frame pressure of the fluid, $u^\mu$ is the four-velocity field of the fluid and $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is the inverse of the Minkowski metric matrix. $\epsilon$, $p$ and $u^\mu$ depend on the spacetime position $x$ and “local rest frame” of the fluid (assigned to spacetime point $x$) is an inertial frame in which spatial components of $u^\mu(x)$ are zero. In such frame, $T^{\mu\nu} = \text{diag}(\epsilon, p, p, p)$. Note that $\eta^{\mu\nu} - u^\mu u^\nu/c^2$ is the projection operator on the subspace perpendicular (in four-dimensional sense) to $u^\mu$.

In applications, one usually assumes some relation between $\epsilon$ and $p$ that is characteristic for the medium one wants to describe, e.g., a “gas of photons” in the cosmological models is characterized by $\epsilon = 3p$. An example of similar relation for the string is given below in Eq. (44).

The most important feature of stress-energy tensor for our analysis is that, if the four-divergence of the stress-energy tensor of a system vanishes,

$$\partial_\mu T^{\mu\nu} = 0, \quad (10)$$

then the total four-momentum of the system,

$$P^\mu(t) = \frac{1}{c} \int d^3 x \, T^{0\mu}(t, \vec{x}), \quad (11)$$

is conserved. In other words, Eq. (10) is the locally defined condition that ensures the total energy and momentum conservation laws: a divergenceless tensor $T$ secures that $P$
is a four-vector that does not depend on time. The proof is based on the four-dimensional generalization of the Gauss’s theorem: integration of divergence of a field over a volume in which it is enclosed is equal the flux of the field through the surface of that volume. If the divergence vanishes in the volume, the same amount of field flows in as flows out of that volume. Since the extra fourth dimension is time, and the flux through space describes the total four-momentum of a system, the result that the same flux flows into a space-time volume in the past as flows out of it in the future, means that the four-momentum of the system is conserved. For a complete prove we refer the reader to the Appendix, where we also show that, tensor nature of $T$ guarantees that the total four-momentum is a Lorentz four-vector, even though integration in Eq. (11) is over different spacetime hypersurfaces for different inertial observers.

In Secs. IIIA and IIIB we first build the stress-energy tensor of the string in frame $R$ using physical intuition; then we analyze it in frame $M$ in Secs. IIIC, IIID and IIIE.

A. String energy-momentum tensor, including stresses

We construct the string energy-momentum tensor by using an intuitive idea that the string is a chain of point-like particles. Our first step is to consider the energy-momentum tensor for one point-like particle. One tries first using Eq. (9) with $\epsilon(t, \vec{x}) = mc^2 \delta^{(3)}[\vec{x} - \vec{r}(t)]$ and $p = 0$, but this guess turns out to be wrong, giving wrong energy and momentum of the particle when inserted into Eq. (11). The problem is that the Dirac $\delta$-function needs to be replaced by a properly transforming density. The correct form of stress-energy tensor of one point-like particle is

$$T_{1}^{\mu\nu}(t_R, \vec{x}_R) = \frac{m}{\gamma_u} \delta^{(3)}[\vec{x}_R - \vec{r}(t_R)] u^\mu(t_R) u^\nu(t_R) ,$$

where $m$ is the mass of the particle, $\vec{r}$ is it’s position vector, $u^\mu$ it’s four-velocity, and the Lorentz factor $\gamma_u$ corresponds to the speed

$$u = |\vec{r}| .$$

The Dirac $\delta$-function $\delta^{(3)}(\vec{r} = r_x\hat{x} + r_y\hat{y} + r_z\hat{z})$, where $\hat{x}$, $\hat{y}$ and $\hat{z}$ are three orthonormal basis vectors in space, is a product $\delta(r_x)\delta(r_y)\delta(r_z)$ and it locates the energy and momentum of the particle at the point of it’s position in space in the frame $R$ at the instant of time
The factor $\gamma_u$ needs to be included to describe the Lorentz contraction required in a frame-dependent definition of the infinitesimal spatial volume in which the point-particle is contained. The $u^\mu u^\nu$ term is necessary so that one obtains the proper expressions for the energy and momentum of a particle by integrating $T_1$ in space. Namely, integration of $T_1^{\mu 0}/c$ over $\vec{x}_R$ gives $m u^\mu u^0/\gamma_u c = m u^\mu$, which is the correct value of four-momentum of a point-like particle. Furthermore, our expression for $T_1^{\mu\nu}$ is implicitly Lorentz covariant, which means that the form of Eq. (12) remains the same in any inertial reference frame with Cartesian spatial coordinates.

Discussion of the stress-energy tensors for point-like particles can be found in Refs. 28, 29. We use Eq. (12) to build the stress-energy tensor of the rotor.

1. Part of tensor $T$ due to motion

The first term is the stress-energy tensor of the massive endpoints,

$$T_{\mu\nu}^{mR}(t_R, \vec{x}_R) = \frac{m}{\gamma_u} \delta^{(3)}[\vec{x}_R - \vec{r}_1(t_R)] u_1^\mu(t_R) u_1^\nu(t_R) + \frac{m}{\gamma_u} \delta^{(3)}[\vec{x}_R - \vec{r}_2(t_R)] u_2^\mu(t_R) u_2^\nu(t_R),$$

(14)

where $\vec{r}_1(t_R) = r\vec{n}(t_R)$ and $\vec{r}_2(t_R) = -r\vec{n}(t_R)$ are the positions of the first and the second endpoint mass in the frame $R$, respectively. The four-vectors $u_1^\mu$ and $u_2^\mu$ are their respective four-velocities.

The second part of the stress-energy tensor we construct is the one, which should be present due to the motion of parts of the string in the frame $R$. We introduce

$$T_{\epsilon\nu}^{\mu}(t_R, \vec{x}_R) = \frac{1}{c^2} \int_{-r}^{r} d\sigma \frac{\epsilon_{\sigma}}{\gamma_u} \delta^{(3)}[\vec{x}_R - \sigma\vec{n}(t_R)] u_\mu(\sigma, t_R) u^\nu(\sigma, t_R).$$

(15)

This expression is a sum of stress-energy tensors for point-like particles of masses $\epsilon_{\sigma} d\sigma/c^2$, positions $\sigma\vec{n}(t_R)$ and four-velocities $u_\mu(\sigma, t_R)$, of which the string is meant to be made. The parameter $\sigma$ labels points on the string so that $\sigma = r$ corresponds to the position of the first endpoint mass, while $\sigma = -r$ to the second one. The value of $|\sigma|$ is equal to the distance of a point from the center of the rotor. This means, in particular, that $u_1^\mu(t_R) = u^\mu(r, t_R)$, $u_2^\mu(t_R) = u^\mu(-r, t_R)$.

The symbol $\epsilon_{\sigma}$ denotes the one-dimensional, or linear, energy density of the string at the point labeled by $\sigma$. The linear energy density should not be confused with the energy density $\epsilon$ in Eq. (3), which is a three-dimensional energy density of a medium. Nevertheless,
Eq. (15) gives a term in the string energy-momentum tensor that is analogous to the first term on the right-hand side of Eq. (9).

The symbol $\gamma_{u\sigma}$ is the Lorentz factor corresponding to the speed $u_\sigma = \Omega\sigma$, with which the point labeled by $\sigma$ moves on a circle in the frame $R$. Note that the linear energy density $\epsilon_\sigma$ ought to include the elastic strain energy density at the point labeled by $\sigma$ due to the string tension at that point.

2. Divergence of the part of $T$ due to motion

$T_m$ and $T_\epsilon$ alone are not sufficient to ensure the laws of conservation of energy and momentum, because Eq. (10) is not satisfied for them. The four-divergence of $T_{\epsilon R}$ consists of two terms,

$$\partial_\mu T_{\epsilon \mu} = \frac{1}{c^2} \int_{-\infty}^{\infty} d\sigma \frac{\partial}{\partial ct} \{ \epsilon_\sigma c u' \delta(3) [\vec{x} - \vec{r}(t, \sigma)] \} ,$$

$$\partial_i T_{\epsilon iR} = \frac{1}{c^2} \int_{-\infty}^{\infty} d\sigma \epsilon_\sigma u^\nu \vec{u}_\sigma \cdot \vec{\nabla} \delta(3) [\vec{x} - \vec{r}(t, \sigma)] ,$$

where $\vec{r}(t, \sigma) = \sigma \vec{n}(t)$, $\vec{u}_\sigma = \dot{\vec{r}}$, $\beta_{u_\sigma} = u_\sigma/c$, $u_\sigma = |\vec{u}_\sigma|$, $u^\nu = u(\sigma, t_R)^\nu = [\gamma_{u_\sigma} c, \gamma_{u_\sigma} \vec{u}_\sigma]^\nu$, and $i = 1, 2, 3$. Using

$$\dot{\vec{r}} \cdot \vec{\nabla} \delta(3) [\vec{x} - \vec{r}(t, \sigma)] = -\partial_t \delta(3) [\vec{x} - \vec{r}(t, \sigma)] ,$$

in Eq. (17), and adding Eq. (17) to Eq. (16), we get

$$\partial_\mu T_{\epsilon \mu}^{\nu R} = \frac{1}{c^2} \int_{-\infty}^{\infty} d\sigma \delta(3) [\vec{x} - \vec{r}(t, \sigma)] \frac{\partial}{\partial t} (\epsilon_\sigma u^\nu)$$

$$= \frac{1}{c^2} \int_{-\infty}^{\infty} d\sigma \delta(3) [\vec{x} - \vec{r}(t, \sigma)] \epsilon_\sigma \gamma_{u_\sigma} [0, \sigma \vec{n}]^\nu .$$

Calculation of the four-divergence of $T_m$ in the frame $R$ is done in the same steps as for $T_\epsilon$. The result is

$$\partial_\mu T_{m \mu R} = \delta(3)[\vec{x}_R - \vec{r}_1(t_R)] m \gamma_{u_\sigma} [0, r\vec{n}(t_R)]^\nu + \delta(3)[\vec{x}_R - \vec{r}_2(t_R)] m \gamma_{u_\sigma} [0, -r\vec{n}(t_R)]^\nu .$$

Components zero and $z$ in both Eq. (20) and Eq. (21) are equal zero, but for the components $x$ and $y$ the sum of the two equations does not vanish, irrespective of the shape of $\epsilon_\sigma$. The two Dirac $\delta$-functions in Eq. (21) are located in two different points in space – the two ends
of the string. Their sum cannot give zero. This fact corresponds to the paradox described in Sec. [II]

Addition of Eq. (20) is not sufficient. The points with $\sigma = -r$ and $\sigma = r$ under the integral have measure zero. Hence, they contribute negligible amounts to the result of integration, in comparison to the two Dirac $\delta$-functions in Eq. (21). Furthermore, demanding that $\partial_\mu T_{\mu\nu}^{\epsilon R}$ in Eq. (20) vanishes would imply that $\epsilon_\sigma$ vanishes for every value of $\sigma$, because the Dirac $\delta$-functions under the integral are located in different points in space for different values of $\sigma$.

To make four-divergence of $T_m + T_\epsilon$ vanish, we have to put $m = 0$ and $\epsilon_\sigma = 0$, which would remove the rotor energy-density entirely. This result shows that we need another contribution to the energy-momentum of the rotor, which is not like the contributions we can imagine as built just from moving particles.

3. Parts of tensor $T$ due to stresses

The full stress-energy tensor of the rotor has to include fluxes of four-momentum related to forces present in the system, i.e., the forces responsible for the circular motion of the rotor. Given the interpretation of $T^{ij}$ components, the form of the missing part of the tensor can be guessed on the basis of the fact that the momenta of the system parts always change only along the rotating vector $\vec{n}$. For example, at a moment when the string is placed along the $x$-axis, the tension is also along the $x$-axis. In other words, the flux of momentum which is produced by tension is directed in the $x$ direction. It goes through the surface perpendicular to the $x$-axis. Its only nonzero component is the $x$ component. Therefore, only $T^{xx}$ component of the energy-momentum tensor is nonzero at times when $n_x = 1$ and $n_y = 0$. When the string is placed along $y$-axis, and $n_x = 0$, $n_y = 1$, then only $T^{yy}$ component is nonzero. At other positions, the stress is along $\vec{n}$. Thus, our guess is that $T^{ij} \sim n_i n_j$. Therefore, we write

$$T_{\mu\nu}^{\epsilon R}(t_R, \vec{x}_R) = \int_{-r}^{r} d\sigma \frac{p_\sigma}{\gamma u_\rho} \delta^{(3)} (\vec{x}_R - \sigma \vec{n}(t_R)) [0, \vec{n}(t_R)]^\mu [0, \vec{n}(t_R)]^\nu, \quad (22)$$

where $p_\sigma$ is the tension of the string in the local rest frame of reference of the string piece labeled by $\sigma$, called below the $\sigma$-piece of the string. Note that the linear tension $p_\sigma$ in Eq. (22) differs from the three-dimensional pressure $p$ in Eq. (9), although the two stress-
energy tensors play similar physical roles.

Equation (22) gives the term in the stress-energy tensor of the string analogous to the second term on the right-hand side of Eq. (9). The analogy appears because

\[- [0, \hat{n}(t_R)]^\mu [0, \hat{n}(t_R)]_{\nu}\]

is the projection operator on the spacetime direction of \([0, \hat{n}(t_R)]^\mu\). One can say that the string \(T_p\) in Eq. (22) is similar to the perfect fluid tensor \(T\) of Eq. (9) in the sense that in the string the momentum exchange between its parts happens in one direction, along the string, and in the fluid the momentum is exchanged between elements of the fluid in all spatial directions.

4. Divergenceless energy-momentum tensor

The sum of (14), (15) and (22) is sufficient to define the rotor energy-momentum tensor with vanishing four-divergence. The four-divergence of \(T_p\) is

\[
\partial_\mu T^\mu_{pR}(t, \vec{x}) = \int_{-r}^{r} d\sigma \frac{p_{\sigma}}{\gamma_{u_{\sigma}}} [0, \hat{n}(t)]_{\nu} \vec{r}' \cdot \vec{\nabla} \delta^{(3)} [\vec{x} - \hat{r}(t, \sigma)] ,
\]

where prime denotes differentiation with respect to \(\sigma\) and we replaced \(\vec{n}\) with \(\vec{r}' = (\sigma \vec{n})' = \vec{n}\). Using the identity

\[
\vec{r}' \cdot \vec{\nabla} \delta^{(3)} [\vec{x} - \hat{r}(t, \sigma)] = -\partial_{\sigma} \delta^{(3)} [\vec{x} - \hat{r}(t, \sigma)]
\]

and integrating by parts over \(\sigma\), we remove the differentiation from the Dirac \(\delta\)-function and move it to \(p/\gamma\). The components \(\nu = 0\) and \(\nu = z\) in Eq. (23) vanish while the components \(\nu = x\) and \(\nu = y\) are proportional to \(n_x\) and \(n_y\), respectively. This procedure yields also the boundary terms proportional to three dimensional \(\delta\)-functions \(\delta^{(3)}[\vec{x}_R - \hat{r}_1(t_R)]\) and \(\delta^{(3)}[\vec{x}_R - \hat{r}_2(t_R)]\), corresponding to those present in Eq. (21).

We now recall that \(\partial_\mu T^\mu_{\epsilon R}\) is proportional to \(\vec{\epsilon}\), which is proportional to \(\vec{n}\). Thus, given the forms of \(T_\epsilon\) and \(T_p\), we can adjust \(\epsilon_{\sigma}\) and \(p_{\sigma}\) to make the four-divergence of

\[
T = T_m + T_\epsilon + T_p
\]

vanish. The four-divergence of \(T\) is given by sum of Eqs. (20), (21) and (23). Factors in front of respective Dirac \(\delta\)-functions have to cancel for every \(\sigma\) and in the endpoints separately. Equation (20) and string part of Eq. (23) give

\[
\frac{d}{d\sigma} \frac{p_{\sigma}}{\gamma_{u_{\sigma}}} = \gamma_{u_{\sigma}} c^2 \Omega^2 \sigma ,
\]

14
where $\Omega^2$ comes from $-\ddot{n}$. Equation (21) and boundary terms from Eq. (23) result in the boundary condition

$$\frac{p_r}{\gamma_{ur}} = \frac{p_{-r}}{\gamma_{ur}} = - \gamma_{ur} m \Omega^2 r .$$  \hspace{1cm} (27)

These equations show that if we want the total four-momentum of the rotor to be conserved in every frame of reference, i.e., if we want Eq. (10) to hold, then the energy-momentum tensor of the rotor has to include the part in Eq. (22). This part describes the stresses in the string that are responsible for what we might call the forces that bind the system. The string tension resulting from these stresses causes the endpoint masses to move on a circle in the frame $R$. However, the concept of force is not a natural or intuitive one here because the string may have complex physical properties. They may result in complex solutions to Eq. (26) with boundary conditions of Eq. (27) for the tension $p_\sigma$ and linear energy density $\epsilon_\sigma$ as functions of $\sigma$.

In Sec. III B we analyze Eqs. (26) and (27), which ensure Eq. (10). Subsequently, in Secs. III C, III D and III E we study the contribution of $T_\rho$ to the rotor total four-momentum in the frame $M$, including examples.

\section*{B. Analysis and interpretation of Eqs. (26) and (27)}

Let us first analyze Eq. (27). The factor $m \Omega^2 r = m u_r^2 / r$ on the right hand side, which is equal to the usual, nonrelativistic centripetal force, is needed to make a point mass move on a circle. The relativistic correction, provided by the factor $\gamma_{ur}$, has its origin in the relativistic definition of momentum $\vec{P} = \gamma_v m \vec{v}$ of a particle with mass $m$ and velocity $\vec{v}$. A change in a particle’s momentum is a result of action of a force, which quantitatively is described by equation $\vec{F} = \vec{\dot{P}}$. In uniform circular motion, the magnitude of velocity is constant, $u_r = \Omega r$, and the magnitude of centripetal force is $\gamma_{ur} m u_r^2 / r$. Hence, the right hand side of Eq. (27) is, up to a sign, equal to the centripetal force needed to keep the endpoint masses moving on a circle in the frame $R$. The left-hand side of Eq. (27) comes from the string, the extended system feature which provides the centripetal force that binds the endpoint masses.

We conclude that in the frame $R$, the magnitudes of forces provided by the string that act on the endpoint masses are $p_r / \gamma_{ur}$ and $p_{-r} / \gamma_{ur}$. Furthermore, the tension of the string
at point $\sigma$ as seen in the frame $R$ is $p_\sigma/\gamma_{u_\sigma}$. This conclusion agrees with our interpretation of $p_\sigma$ as the tension of the $\sigma$-element of the string in its local rest frame. If in the local rest frame of $\sigma$-element, say for $\sigma = r$, the second law of dynamics has the form $\vec{F} = d\vec{P}/d\tau$, where $\tau$ is the time parameter in that frame, then in the frame $R$ it will have the form $\vec{F} = \gamma d\vec{P}/d\tau_R$, where $\gamma = dt_R/d\tau$. We assume that $\vec{F}$ and $d\vec{P}$ do not change between these two frames of reference, because the string element moves in $R$ perpendicularly to its axis. So, while $\vec{F}$ is the centripetal force in the local rest frame, in the frame $R$ the centripetal force is equal $\vec{F}/\gamma$.

The content of Eq. (26) can be studied in the nonrelativistic limit ($c \to \infty$), since it simplifies in that limit and the meaning of the gamma factors is already explained in our analysis of Eq. (27). Assuming $\epsilon = \rho c^2$, where $\rho$ is the linear mass density on the string, the nonrelativistic limit of Eq. (26) is

$$\frac{dp}{d\sigma} = \rho \Omega^2 \sigma = \rho u_\sigma^2/\sigma. \quad (28)$$

This equation describes an infinitesimal part of the string, which is located at the point labeled by $\sigma$ and has length $d\sigma$. Thus, $dp$ is the radial component of the force acting on the considered string part. One end is pulled inwards with force $p$, the other end is pulled outwards with force $p + dp$. Components other than radial are zero. The resultant force equals the centripetal force making the element move on a circle, $\rho d\sigma \ u_\sigma^2/\sigma$. This is the content of Eq. (28).

One implication of Eq. (26) and Eq. (27) is that the tension $p$ is negative. It follows from the fact that the boundary tension $p_r$ is negative and $p_\sigma/\gamma_{u_\sigma}$ is decreasing for $\sigma < 0$ and increasing for $\sigma > 0$. This fact can be understood if we imagine the string as a pipe filled with a fluid. A string at rest corresponds to pressure zero. Shortening the pipe results in a positive and extending the pipe results in a negative pressure inside the pipe. A rotating string is stretched to provide the centripetal force to the rotor massive endpoints and every other point of the string. Hence $p$ is negative for the stretched string.

1. **Solution for the tension $p_\sigma$ in Eq. (26)**

We present the solution of Eq. (26) assuming that the string energy density $\epsilon_\sigma$ is given *a priori*, and that as such it satisfies the symmetry condition $\epsilon_{-\sigma} = \epsilon_\sigma$. Equation (26) can be
written in the form
\[
\frac{d}{d\sigma} \gamma_{u_{\sigma}} p_{\sigma} = -\left( \frac{1}{\gamma_{u_{\sigma}}} \right)' \epsilon_{\sigma},
\]  
(29)
where the prime means differentiation with respect to \(\sigma\). This form suggests that integration on the right-hand side could be done by changing the integration variable \(\sigma\) to the variable \(g = 1/\gamma_{u_{\sigma}}\). Consequently, the solution to Eq. (29) can be written in the form
\[
\frac{p_{\sigma}}{\gamma_{u_{\sigma}}} = p_0 + \int_{1/\gamma_{u_{\sigma}}}^{1} \epsilon_{\sigma_g} dg,
\]  
(30)
where \(\sigma_g = \frac{c}{\Omega} \sqrt{1 - g^2}\) and \(p_0\) is the string tension in the middle, i.e., at \(\sigma = 0\). The solution is valid in the whole range of \(\sigma\) from \(-r\) to \(r\).

C. String contribution to the rotor four-momentum in the moving frame

Having secured the energy and momentum conservation via four-dimensionally divergence-less energy-momentum tensor, we can now analyze the contribution of the string to the rotor four-momentum in the moving frame \(M\). The stress-energy tensor of Eq. (25) in that frame is given by
\[
T_{\mu\nu}^M(t_M, x_M) = L_{\alpha}^\mu L_{\beta}^\nu T_{\alpha\beta}^{R} [t_R(t_M, x_M), x_R(t_M, x_M)],
\]  
(31)
where \(L_{\alpha}^\mu\) is the Lorentz transformation matrix for the boost from the frame \(R\) to the frame \(M\), and \(t_R, x_R\) are functions of \(t_M, x_M\) that are determined by the Lorentz transformation \(L^{-1}\). Integrating the zero-component of the tensor density over space in the frame \(M\) according to Eq. (11), one obtains the following example of a rotating extended system four-momentum density in motion in the form
\[
\frac{dP_{M}}{d\sigma} = \frac{1}{c} \epsilon_{\sigma} \gamma_{\sigma} \begin{bmatrix}
\gamma_U \\
\beta_U \gamma_U \\
0 \\
0
\end{bmatrix} + \frac{1}{c} \epsilon_{\sigma} \gamma_{\sigma} \beta_{\sigma} \begin{bmatrix}
-\beta_U \gamma_U s_{\sigma} \\
-\gamma_U s_{\sigma} \\
c_{\sigma} \\
0
\end{bmatrix} - \frac{p_{\sigma}}{c} \frac{\beta_{\sigma}}{c_{\sigma}} \begin{bmatrix}
-\beta_U \gamma_U c_{\sigma} \\
-\gamma_U c_{\sigma} \\
0 \\
0
\end{bmatrix},
\]  
(32)
where \(u_{\sigma} = \Omega \sigma, \beta_{\sigma} = u_{\sigma}/c\) and \(\gamma_{\sigma} = (1 - \beta^2_{\sigma})^{-1/2}\). The first two terms come from the term \(T_{\epsilon}\) and the third one comes from \(T_p\) in Eq. (25), while
\[
s_{\sigma} = \sin[\Omega t_R(t_M, \sigma)],
\]  
(33)
\[
c_{\sigma} = \cos[\Omega t_R(t_M, \sigma)],
\]  
(34)
denote the trigonometric factors. Here, $t_R(t_M, \sigma)$ is a generalization of functions $t_{1R}(t_M) = t_R(t_M, r)$ and $t_{2R}(t_M) = t_R(t_M, -r)$, obtained by inserting $\sigma \cos \Omega t_R$ in place of $x_R$ in Eq. (2), and solving for $t_R$.

The terms originating in the part $T_\epsilon$ hold no surprises: the energy and momentum of a $\sigma$-piece of the string are its Lorentz transformed energy and momentum from the rotor rest frame $R$. Similarly, as observed in Sec. II, the total energy and momentum resulting from the part $T_m$ are not constant in time in the frame $M$. The same holds true for the total energy and momentum that come from integrating the sum $T_m + T_\epsilon$. In these parts, a piece of a string corresponding to some value of $\sigma$ resembles a particle with mass $\epsilon_\sigma d\sigma/c^2$.

In contrast, the part $T_p$ only contributes to the energy-momentum of the rotor in the frame $M$. It does not contribute anything to the rotor energy and momentum in the frame $R$. The tensor structure of $T_p$ in the frame $M$ is given by $w^\mu_M w^\alpha_R = \begin{pmatrix} \beta U^2 \gamma_U c^2 \gamma_U s_\sigma c_\sigma & \beta U^2 \gamma_U c^2 & \beta U^2 \gamma_U s_\sigma c_\sigma & 0 \\ \gamma_U c^2 c_\sigma & \gamma_U c^2 & \gamma_U s_\sigma c_\sigma & 0 \\ \beta U^2 \gamma_U s_\sigma c_\sigma & \gamma_U s_\sigma c_\sigma & s_\sigma^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{\mu\nu}$, so that

$$w^\mu_M = L^\mu_\alpha w^\alpha_R = \begin{pmatrix} \beta U^2 \gamma_U c^2 \gamma_U s_\sigma c_\sigma & \beta U^2 \gamma_U c^2 & \beta U^2 \gamma_U s_\sigma c_\sigma & 0 \\ \gamma_U c^2 c_\sigma & \gamma_U c^2 & \gamma_U s_\sigma c_\sigma & 0 \\ \beta U^2 \gamma_U s_\sigma c_\sigma & \gamma_U s_\sigma c_\sigma & s_\sigma^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{\mu}.$$ (35)

Leaving out the factors in front and the Dirac $\delta$-functions, we can identify the form of “tension” in stress-energy tensor of $\sigma$-element using the matrix,

$$T^\mu_\nu_{pM}(\sigma\text{-element}) \sim p_\sigma \begin{pmatrix} \beta U^2 \gamma_U c^2 \gamma_U s_\sigma c_\sigma & \beta U^2 \gamma_U c^2 & \beta U^2 \gamma_U s_\sigma c_\sigma & 0 \\ \gamma_U c^2 c_\sigma & \gamma_U c^2 & \gamma_U s_\sigma c_\sigma & 0 \\ \beta U^2 \gamma_U s_\sigma c_\sigma & \gamma_U s_\sigma c_\sigma & s_\sigma^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{\mu\nu},$$ (36)

where again $s_\sigma = \sin \Omega t_R(t_M, \sigma)$ and $c_\sigma = \cos \Omega t_R(t_M, \sigma)$.

Since the zero-zero component of the energy-momentum tensor equals the energy density, the energy of a string $\sigma$-element is proportional to $p_\sigma \beta_U^2 \gamma_U^2 \cos^2 \Omega t_R$. Since $p$ is negative, the part of energy of $\sigma$-element that results from the tension is also negative. Similarly, the $x$ component of momentum of a $\sigma$-element is proportional to $p_\sigma \beta_U \gamma_U^2 \cos^2 \Omega t_R$. For $\beta_U$ positive, the rotor as a whole has positive $x$ component of momentum, but the momentum carried by a $\sigma$-element has a negative $x$ component. For $\beta_U$ negative, the rotor velocity is negative and the momentum of a $\sigma$-element is positive in the $x$ direction. Additionally, we have
the $y$ component of the momentum of a $\sigma$-element, which does not have a definite sign. This indicates that the string momentum part resulting from the tension is approximately antiparallel to the velocity of the rotor as a whole. The described features can be observed in the examples presented in Sec. [III D] see Figs. 3 and 4.

The energy and momentum carried by the string have different properties from the ones exhibited by point-like massive particles. For example, a particle carries energy as a zero component of a four-vector and its energy is a non-zero quantity in all frames of reference. In contrast, the string tension contributes to energy as a zero-zero component of a tensor. It is possible that the tension contributes non-zero energy in the moving frame $M$ even if in some frame of reference, such as the rest frame $R$, it contributes no energy. The same happens with momentum.

Our analysis of the rotor indicates that similar intricacies can be expected in relativistic description of dynamics of all extended physical systems in motion, irrespective of the internal mechanisms that bind them. In the case of a rotor, the string contribution to the total energy and momentum in the moving frame, resulting from the string tension, exists regardless of the specific string properties that lead to concrete functions $\epsilon_\sigma$ and $p_\sigma$. Illustrative examples of these functions are presented in Sec. [III D].

D. Examples of the string

We consider two examples of string behavior. Our first, simple model assumes that the string linear energy density is constant as a function of the parameter $\sigma$. Our second model concerns the string that stretches according to the Hooke law.

1. Simple string

In our first example of the string, we assume the linear energy density $\epsilon_\sigma = \rho c^2 = \text{const.}$ Then, using the solution of Eq. (26), given in Eq. (30), we have

$$\frac{p_\sigma}{\gamma_\sigma} = p_0 + \rho c^2 \left(1 - \frac{1}{\gamma_\sigma}\right),$$

(37)
where $\beta_\sigma = \Omega \sigma / c$. From the boundary condition, Eq. (27), we can determine $p_0$, which is the tension of the string in the center of the rotor,

$$p_0 = -\gamma_r m \Omega^2 r - \rho c^2 \left[ 1 - \frac{1}{\gamma_r} \right].$$  \hspace{1cm} (38)

Now, putting this $p_0$ into Eq. (37), we arrive at

$$\frac{p_\sigma}{\gamma_\sigma} = -\gamma_r m \Omega^2 r - \rho c^2 \left[ \frac{1}{\gamma_\sigma} - \frac{1}{\gamma_r} \right],$$  \hspace{1cm} (39)

or, alternatively written,

$$\frac{p_\sigma}{\gamma_\sigma} = -\gamma_r m u_r^2 / r - \rho (u_r^2 - u_\sigma^2) \frac{1}{1/\gamma_r + 1/\gamma_\sigma}.$$  \hspace{1cm} (40)

If $\Omega r$ is small compared to the speed of light, the solution may be approximated by

$$p_\sigma = -\frac{m u_r^2}{r} - \rho (u_r^2 - u_\sigma^2) \frac{1}{2} + O(\beta^2).$$  \hspace{1cm} (41)

This nonrelativistic limit has the following interpretation: the string tension at $\sigma$, denoted by $p_\sigma$, is the sum of centripetal forces needed to hold the endpoint mass and outer part of the string beyond $\sigma$ in their circular motion. The first term, equal $-m \Omega^2 r$, corresponds to the endpoint mass. The second term is the sum of centripetal forces corresponding to all infinitesimal pieces of the outer part of the string or, effectively, of a body of mass $(r - \sigma) \rho$ rotating with frequency $\Omega$ at the distance $(r + \sigma)/2$ from the string center.

The relativistic result of Eq. (40) is more complicated than its nonrelativistic limit. In Sec. III A we argue that the tension as seen in the frame $R$ equals $p_\sigma / \gamma_\sigma$. Since the first term on the right hand side of Eq. (40) is equal to the centripetal force acting on the endpoint mass, including the gamma factor $\gamma_r$, the second term can be interpreted as the centripetal force which holds the outer part of the string or effectively the force needed to hold one body of mass $(r - \sigma) \rho$ at position $(r + \sigma)/2$, with effective gamma factor being the harmonic mean of $\gamma_r$ and $\gamma_\sigma$.

An example of our rotor is presented in Fig. 3. The figure shows momentum carried by different parts of the rotor as observed in the moving frame, assuming the simple model of the string with $\rho = 0.1$ kg/m. The rotor radius $r$ is 1 m and the mass of each endpoint is 1 kg. The momentum density on the string is calculated using Eq. (32). The result is divided into the part proportional to $\epsilon$ (dotted arrows) and the part proportional to $p$ (solid arrows). These parts correspond to the parts $T_\epsilon$ and $T_p$ of the total energy-momentum tensor,
FIG. 3: Momentum carried by different parts of the rotor at four different times in the moving frame. Here, $T_R$ means the period of rotation in the frame $R$. The string is drawn with a solid line (black in the online version). Dashed arrows (red in the online version) indicate momenta of endpoints. Dotted arrows (blue in the online version) indicate the density of momentum of the string which comes from $T_\epsilon$ at four points on the string, for $\sigma/r = -0.75, -0.25, 0.25, 0.75$. Similarly, solid arrows (green in the online version) indicate the density of momentum which comes from $T_p$. The string is assumed light and the rotation of the rotor is not very fast. Therefore, the dotted and solid arrows are magnified 10 and 100 times with respect to the dashed ones, correspondingly. We assume the simple model of the string from Sec. III D 1.
respectively. Momenta carried by the endpoints, which correspond to $T_m$, are indicated by the dashed arrows. One cannot directly compare momentum with momentum density because they have different meanings and dimensions. Therefore, we multiplied the densities by $r/2$, which is one fourth of the total length of the string. Thus, the arrows presented in the figure are approximately equal to momenta carried by pieces of the string of length $r/2$. Furthermore, because the string is much lighter than the endpoint masses, the dotted arrows are magnified 10 times with respect to the dashed ones and, because $p_0/(\rho c^2) \approx -0.11$, the solid arrows are magnified 100 times with respect to the dashed ones. Otherwise, dotted and solid arrows would not be well visible. The frame $M$ moves with respect to the frame $R$ with the speed $U = 0.8c$. The speed of each endpoint in the frame $R$ is $v = -0.1c$, where the minus means that the rotor rotates clockwise when looked at from $z > 0$ half-space. However small, $T_p$ contributes to the total momentum of the rotor and its contribution is different from that of $T_e$. For example, the solid arrows do not have the direction of velocity of the corresponding points of the string, contrary to the dotted arrows that represent the momentum density coming from $T_e$. Another visible aspect of the rotor motion is the apparent bending of the string in the frame $M$. It occurs even though the string is straight at every moment in the frame $R$. This is a consequence of relativity of simultaneity, cf. Fig. 7 in Ref. 30.

2. Hooke’s string

In the previous example, we assume that the string linear energy density $\epsilon$ is constant and we use this constant to derive the tension $p$. But in general, $\epsilon$ will depend on $p$. Both quantities depend on how much the string is stretched. If $p_\sigma$ is uniquely determined by the elongation of the element of the string labeled by $\sigma$, then $\epsilon$ can be considered a function of $p$. Therefore, we do not need to know $\epsilon$ in advance, if we know the law governing the stretching of the string.

To properly account for the effects of stretching of the string, we need to define a quantity which encodes information about its deformation. Assuming that the energy does not depend on whether the string is bent or straight, we need only one number to characterize the degree of stretching locally on the string, and we denote it by $k_\sigma$. We define $k_\sigma$ as the ratio of length of the infinitesimal string element that is relaxed, to the actual length of the same element.
of the string when it is stretched or compressed. If $k_\sigma$ is one, then the piece of the string labeled by $\sigma$ is relaxed. If $k_\sigma < 1$, then the element is stretched. When $k_\sigma > 1$, then the element is compressed. We will call $k_\sigma$ the strain parameter, corresponding to the string parameter $\sigma$.

We now assume that the string obeys Hooke’s law. For a moment, let us ignore the parameter $\sigma$ and let’s assume we have a piece of string at rest, which is stretched from length $l_0$ to $l$. The tension $p$ is

$$p = -\frac{Y}{l_0} (l - l_0) = -Y \left( \frac{1}{k} - 1 \right), \quad (42)$$

where $k = l_0/l$ and $Y$ is a material constant of dimension of force; for a string of a negligibly small but finite thickness it would be the Young modulus times the cross-section area of the string. The energy needed to stretch the string is $\frac{Y}{2l_0} (l - l_0)^2$. If the string has energy $\varepsilon_0 l_0$ before stretching, where $\varepsilon_0$ is the linear energy density of the string in the relaxed state, then after stretching the linear density $\epsilon$ is

$$\epsilon = \frac{\varepsilon_0 l_0 + \frac{Y}{2l_0} (l - l_0)^2}{l} = \left( \varepsilon_0 + \frac{Y}{2} \right) k + \frac{Y}{2k} - Y. \quad (43)$$

Since $p$ is determined by $k$ according to Eq. (42), one can find the inverse function $k(p)$, and put it into $\epsilon(k)$, which yields

$$\epsilon = \frac{1}{1 - \frac{p}{Y}} \left( \varepsilon_0 + \frac{p^2}{2Y} \right). \quad (44)$$

This relation does not depend on the length of a piece of the string and may be considered a local property of the string. It does not depend on the state of motion of the string either, because $\epsilon$ and $p$ are quantities defined point-wise in the local rest frames of the elements of the string. These features are also shared by Eq. (42) and Eq. (43).

Using Eq. (44), we can get rid of $\epsilon$ in Eq. (26), which can then be solved for $p_\sigma$. There is also another possibility, which we pursue: one can insert Eqs. (42) and (43) into Eq. (26), obtaining an equation for $k_\sigma$, which can be solved by method of separation of variables. The result is

$$k_\sigma = \frac{1}{\sqrt{B^2 - \frac{\gamma C}{\gamma}}} \quad (45)$$
where

\[ B^2 = 1 + \frac{2\varepsilon_0}{Y}, \quad (46) \]

\[ C = B^2 - \left[ 1 + \frac{\gamma^2 m u_r^2}{r Y} \right]^2. \quad (47) \]

Using this result, \( p_\sigma \) and \( \epsilon_\sigma \) can be obtained by inserting \( k_\sigma \) of Eq. (45) into Eq. (42) and Eq. (43), respectively.

This solution does not present itself as intuitively understandable, contrary to Eqs. (40) and (41). This is because \( \epsilon_\sigma \) and \( p_\sigma \) and the boundary conditions are mutually related. However, in the limit of an infinitely rigid string, that is, \( Y \to \infty \), \( k \) equals one, \( \epsilon \to \varepsilon_0 \), and \( p_\sigma \) reproduces the solution given by Eq. (40) with \( \rho = \rho_0 := \varepsilon_0/c^2 \). One can also study the nonrelativistic limit of \( c \to \infty \). Tension \( p \) in this limit is

\[ p_\sigma = -Y \left[ \sqrt{\frac{\rho_0(u_r^2 - u_2^2)}{Y}} + \left( 1 + \frac{mu_r^2}{r Y} \right)^2 - 1 \right] + O \left( \frac{1}{c^2} \right). \quad (48) \]

If we further assume that the string is very rigid \( (Y \to \infty) \), then the above equation assumes the form of Eq. (41) with \( \rho = \rho_0 \).

Our conclusion is that the example of Hooke’s string, which is intended to provide more accurate description of the string than the previous example, includes the previous example as a special, limiting, case. In particular, the example of a rotor depicted in Fig. 3 can be reproduced within few percent accuracy with Hooke’s string whose parameters are \( \varepsilon_0 = 88 \text{ kg/m} \cdot \text{c}^2 \) and \( Y = 10^{12} \text{ N} \). The elongation of the string in that case equals about 1000, which is far beyond the elasticity of ordinary strings or ropes.

Another example of the rotor is presented in Fig. 4. It shows momentum carried by the rotor parts in the moving frame when the string obeys Hooke’s law. The momentum density on the string is calculated using Eq. (32), where the string energy density \( \epsilon_\sigma \) and tension \( p_\sigma \) are given by Eqs. (43) and (42), respectively, with \( k \) replaced in these equations by \( k_\sigma \) of Eq. (45).

The total momentum density is divided into the part proportional to \( \epsilon \) (dotted arrows) and the part proportional to \( p \) (solid arrows). This division corresponds to the division of the total energy-momentum tensor into the parts \( T_\epsilon \) and \( T_p \), respectively. Momenta carried by the endpoints, which correspond to \( T_m \), are indicated by dashed arrows. For the purpose of comparing quantities with different dimensions, the densities are multiplied by
Therefore, every dotted and solid arrow might be interpreted as momentum carried by a piece of string of length \( r/2 \), measured in the frame \( R \). The string parameters are \( \varepsilon_0 = 10^5 \text{ kg/m} \cdot c^2 \) and \( Y = 10^{12} \text{ N} \). The frame \( M \) moves with respect to the frame \( R \) with the speed \( U = 0.8c \), the speed of each endpoint in the rotor frame \( R \) is \( v = -0.7c \). The rotor radius \( r \) is 1 m and the mass of each endpoint is 1 kg. These values of parameters imply that the tension is comparable with the energy density, \( p_0/\varepsilon_0 \approx -0.74 \). This means that the presented string is in the regime of ultrarelativistic limit discussed in Sec. [III.E].

Furthermore, the elongation of the string in the middle is \( 1/k_0 \approx 10^5 \). It is extremely large, far beyond what ordinary strings can withstand. In other words, the string has relaxed length of order 10 \( \mu \)m and it is stretched to 1 m. We assumed also that the string is extremely dense in the relaxed state, \( \varepsilon_0 = 10^5 \text{ kg/m} \cdot c^2 \). If the string was much lighter, then we would have \( \epsilon + p < 0 \), which appears unphysical, see Sec. [III.E]. The energy density in the middle of the stretched string is \( \varepsilon_0 \approx 1.55 \text{ kg/m} \cdot c^2 \). Similarly to Fig. 3, solid arrows in Fig. 4 are approximately antiparallel to the dotted ones. The latter have the direction of the velocity of corresponding string points, but this time the arrows are directly comparable. The string also seems to bend in the moving frame, although it is always straight in the rest frame, which can again be compared with Fig. 7 in Ref. 30. The effect of apparent bending of the string is purely kinematic and much more visible here when compared with Fig. 3 because here we assume much faster rotation of the rotor.

### E. Ultrarelativistic limit

Although a truly relativistic rotor cannot be built from real atoms because the interatomic binding forces are too weak to withhold relativistic tension, it is, nevertheless, interesting theoretically what could happen if the angular velocity \( \Omega \) were very large. The issue concerns all kinds of rotors that are bound by any type of forces that resemble a string in their effect of binding.

Let us first analyze our first, simple example. According to Eq. (38), as we increase \( \Omega \) and keep all other quantities constant, the modulus of the tension in the center, \( |p_0| \), also increases. It continues so till at some point \( p_0 \) reaches \( -\rho c^2 \). At that point, \( p_\sigma = \text{const} = p_0 \) and \( \epsilon + p = 0 \) in every point on the string. If we further increase \( \Omega \), then \( \epsilon + p \) becomes negative.
\[ \beta_v = -0.7, \quad \beta_U = 0.8, \quad r = 1.0 \text{ m} \]

FIG. 4: Momentum carried by different parts of the rotor at four different times in the moving frame. \( T_R \) means here the period of rotation in the frame \( R \). The string is drawn with a solid line (black in the online version). Dashed arrows (red in the online version) indicate momenta of endpoints. Dotted arrows (blue in the online version) indicate the density of momentum of the string which comes from \( T_\epsilon \) at four points on the string, for \( \sigma/r = -0.75, -0.25, 0.25, 0.75 \). Similarly, solid arrows (green in the online version) indicate the density of momentum which comes from \( T_p \) in the same four points on the string. We assume that the string obeys Hooke’s law. For more details see Sec. III D 2.
A negative $\epsilon + p$ implies that in some frame of reference the density of energy of the string, given by the $T^{00}$ component of the energy-momentum tensor, is negative, which appears unphysical. Therefore, it seems that the string ought to break before reaching such large $\Omega$. Hooke’s string also admits $\epsilon + p < 0$, which is the case when the constant $C$ is negative.

There exist no ordinary strings that could withstand elongations big enough to get close to the unphysical region of $\epsilon + p < 0$ prior to the breaking. However, there may exist microscopic systems that are close to the case $\epsilon + p = 0$. Such systems are expected to be found in mesons, built as pairs of quarks connected by the gluon string. In such models of mesons, quarks are light and move with the speed close to the speed of light. The string satisfies the condition $\epsilon + p = 0$. This condition is adopted in the mathematical string theory, where the massive endpoints are not included.

F. Remark on Lagrangian approach

Interested readers may ask how the string in our rotor is related to the strings in theories one can find in literature. The latter are typically given in terms of a Lagrangian density, such as in the case of Nambu-Goto\textsuperscript{23} string or Chodos-Thorn\textsuperscript{6} string. As a matter of fact, a general motion of a string is such that each and every infinitesimal bit of the string can be described as rotating around its center-of-mass that is momentarily moving with some velocity. So, our elementary rotor analysis is a necessary ingredient in building a theory of generally moving strings. It turns out that the string dynamics one arrives at using our rotor picture for infinitesimal parts of the string can be described using a Lagrangian. Moreover, the Lagrangians one obtains form a class that includes the Nambu-Goto and Chodos-Thorn Lagrangians. These results will be presented elsewhere.

IV. CONCLUSION

Our relativistic analysis of an elementary example of an extended physical system, a rotor, allows us to draw several conclusions. Energy and momentum of the rotor are not conserved in some inertial frames of reference unless one properly includes the energy and momentum that are carried by the string that binds the endpoint masses. The stresses in
the string contribute to the energy and momentum of the rotor.

The energy contributed by the stresses cancels the oscillations of energy of a moving rotor obtained without proper inclusion of the string. The momentum induced by the stresses is approximately antiparallel to the velocity of the rotor and cancels oscillations in the total momentum of the rotor obtained without account for the momentum due to the stresses.

We wish to stress that the energy-momentum tensor contributions that stabilize the total momentum of the rotor do not manifest themselves in the rest frame of the system. The stresses that provide the rotor binding forces become visibly missing in the frames of reference that move with respect to the rotor center of mass.

Despite the limitation to a specific system, our analysis is general enough to say that, a relativistic description of motion of an extended physical system with bounded internal motion of its parts, necessitates inclusion of the dynamics responsible for the binding. This is true regardless of the binding mechanism – the total energy and momentum of constituents alone oscillate in frames of reference that uniformly move with respect to the system. The energy and momentum carried by the binding mechanism counter these oscillations.

**Appendix A: The four-momentum conservation and its four-vector nature**

The four dimensional Gauss’s theorem applied to tensor $T$, has the form,

$$\int_{\Sigma} dS_\nu T^{\nu\mu} = \int_V d^4x \, \partial_\nu T^{\nu\mu},$$

(A1)

where $V$ is any four-dimensional region in spacetime. $\Sigma$ is it’s boundary, a three-dimensional hypersurface. $dS_\nu$ is a four-vector perpendicular (in a four-dimensional sense) to an infinitesimal element of $\Sigma$, pointing outward of region $V$. Its (four-dimensional) length is a measure of (three-dimensional) volume of that infinitesimal element. In particular, if we express $dS_\nu$ in an inertial frame in which it is perpendicular to a hyperplane of simultaneity, then $dS_\nu = n_\nu d^3x$, where $n_0 = \pm 1$. The sign depends on which direction is the outward one, and $n_i = 0$.

To prove that $P^\mu$ defined in Eq. (11) is conserved in one inertial frame we choose $V$ to be a four-dimensional cylinder whose axis is the time axis and whose bottom and top sides, $\Sigma_1$ and $\Sigma_2$ in Fig. 5 are located on two hyperplanes of constant time, $t = t_1$ and $t = t_2 > t_1$, respectively. The cylinder is made big enough in the spatial directions to contain the whole
FIG. 5: Integration region, which we use to prove that divergenceless energy-momentum tensor $T$ gives conserved four-momentum $P$. The dotted lines represent some spatial extent of a physical system (it does not have to be a rotor).

system. Thus, $T^{\mu\nu}$ is zero on the side surface of the cylinder, as indicated in Fig. 5. On the bottom hypersurface $\Sigma_1$, the normal outward unit vector has the time component $n_0 = -1$, while on the top $\Sigma_2$ it has $n_0 = 1$. The only contributions to the flux integral on the left hand side of Eq. (A1) come from the top and bottom hypersurfaces. These contributions are equal to $-cP^\mu(t_1)$ and $cP^\mu(t_2)$, respectively. They sum to zero, assuming that Eq. (10) holds. Therefore, Eq. (10) implies the four-momentum conservation.

Furthermore, if $T$ is a four-tensor then $P^\mu$ is a four-vector, which means that the extended system’s total four-momenta seen by different inertial observers are connected by a Lorentz transformation. From the physical point of view this fact is not trivial because, as we have seen in Sec. II, hyperplanes of simultaneity are different for different observers. But the Gauss’s theorem is valid no matter how one chooses integration region. We choose $\Sigma_1$ to lie on the hyperplane of simultaneity of observer $R$ and $\Sigma_2$ to lie on the hyperplane of simultaneity of observer $M$. Infinitesimal volume elements are $dS_1$ on $\Sigma_1$ and $dS_2$ on $\Sigma_2$. Again, the side boundary of $V$ is located far enough so that it does not contribute to the left hand side of Eq. (A1), see Fig. 6. Now, all steps that are needed to prove that $P^\mu$ is a
FIG. 6: Integration region, which we use to prove that four-momentum defined in Eq. (11) is a four-vector.

The first equality is a definition of total four-momentum in the frame $M$, using components of $T$ in that frame. This is why they carry the subscript $M$. The second equality makes use of the definition of $dS_2$. The third equality is the Gauss’s theorem. It equates the total momentum flowing through hypersurface $\Sigma_2$ to the total momentum flowing through the surface $\Sigma_1$. Both sides of the Gauss equality are expressed using coordinates and components in frame $M$. In the fourth equality, the index of the four-vector $dS_1$ is raised using the Minkowski metric $\eta$.

The fifth equality expresses the components of $dS_1$ and $T$ in the frame $M$ by their components in the frame $R$ and Lorentz transformation matrix $L$ from frame $R$ to frame $M$. The sixth equality makes use of the identity $L^\alpha_{\kappa}\eta_{\alpha\beta}L^\beta_\lambda = \eta_{\kappa\lambda}$, which is the defining property of the Lorentz transformation matrices. Written without indices, it is $L^T \eta L = \eta$. 

\[
\begin{align*}
P_M^\mu &= \frac{1}{c} \int_{\Sigma_2} d^3x_M T_M^{0\mu} = \frac{1}{c} \int_{\Sigma_2} dS_2 M \nu T_M^{\nu\mu} = -\frac{1}{c} \int_{\Sigma_1} dS_1 M \nu T_M^{\nu\mu} \\
&= -\frac{1}{c} \int_{\Sigma_1} dS_1 M \eta_{\alpha\beta} T_M^{\beta\mu} = -\frac{1}{c} \int_{\Sigma_1} (L^\alpha_{\kappa} dS_1 R) \eta_{\alpha\beta} (L^\beta_{\lambda} L^\mu_{\nu} T_R^{\lambda\nu}) \\
&= -L^\mu_{\nu} \frac{1}{c} \int_{\Sigma_1} dS_1 R \eta_{\kappa\lambda} T_R^{\lambda\nu} = L^\mu_{\nu} \frac{1}{c} \int_{\Sigma_1} d^3x_R T_R^{0\nu} = L^\mu_{\nu} P_R^{\nu} \quad \text{(A2)}
\end{align*}
\]
The seventh equality makes use of the definition of $dS_1$ and the total four-momentum in the frame $R$. As we see, components $P^\mu_M$ and $P^\nu_R$ are connected via Lorentz transformation and, therefore, define a four-vector.

Thus, if $T$ is a tensor in special relativity theory, i.e., if it transforms from one inertial frame to another by applying the Lorentz transformation to both its indices, and if its four-divergence is zero, then total four-momentum defined by such $T$ in Eq. (11) is constant in time and is a Lorentz four-vector.

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