FFT, DA, and Mori-Tanaka approximation to determine the elastic moduli of three-phase composites with the random inclusions

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Abstract. In this work, some solutions such as Mori-Tanaka approximation (MTA), Differential approximations (DA), and Fast Fourier transformation method (FFT) were applied to estimate the elastic bulk and shear modulus of three-phase composites in 2D. In which two different sizes of circular inclusions are arranged randomly non-overlapping in a continuous matrix. The numerical solutions using FFT analysis were compared with DA, MTA, and Hashin-Strikman’s bounds. The MTA and DA reasonably agreeable solution with the FFT solution shows the effectiveness of the approximation methods, which makes MTA, DA useful with simplicity and ease of application.

Keywords: Elastic modulus / Fast Fourier transformation method (FFT) / Mori-Tanaka approximation / differential approximations / composite materials

1 Introduction

Investigations on the macroscopic properties of multiphase materials and media have been started with the classical works of Maxwell, Voigt, Reuss, Einstein [1], and continued with the bounds of Hill [2], and Hashin and Shtrikman [3], Pham & Nguyen [4, 5]. Further studies aim to construct better estimates by including more detailed information about the microstructure of the materials. Approximation schemes have been constructed based on microscopic models. Based on the classical work of Eshelby [6] on the elastic ellipsoidal inclusion problem, effective medium approximations have been developed to predict the overall behavior of the general non-dilute composites such as approximation methods (Christensen [7], Mori and Tanaka [8], and the recent Polarization approximation [9]), in which construction of approximate formulae that can describe the macroscopic property accurately over ranges of volume proportions of component materials for engineering uses. Strong numerical methods such as the Finite Element and Fast Fourier ones have been developed and used effectively. Numerical homogenization techniques determining the effective properties give reliable results but challenge engineers by computational costs, especially in the case of complex microstructure. The Fast Fourier Transform has been used to compute the effective properties of periodic composites by Michel et al. [10], Moulinec [11], Bonnet [12] in the context of elasticity. However, most of these studies were applied to two-phase material composites with simple periodic structures such as square or hexagonal models. In this work, some solutions are proposed to determine the macro-elastic modulus for 3-phase composite materials in 2D (or transverse isotropic unidirectional fiber-reinforced composite) with a more complex structure than previous studies [13]. In Section 2, the Mori-Tanaka approximation for elastic moduli is introduced with the explicit expressions of the estimates for the three-phase composite with random two different phase inclusions. In Section 3, the DA approximation for elastic bulk and shear moduli are developed, HS bounds are presented in Section 4. In Section 5, the FFT method is developed to directly calculate the effective moduli of three-phase composites with the random distribution of fiber inclusions. After that (Sect. 6), numerical examples will be presented to compare the results of MTA, DA, and Hashin-Shtrikman bounds and the FFT method.

2 Mori-Tanaka approximation (MTA) for elastic modulus

Consider n-component transversely isotropic unidirectional elastic composites of randomly oriented inclusions of type α (α = 2, ..., n). The matrix phase has the volume fraction \( v_M \) and the α-inclusion has the volume fraction \( v_{\alpha} \). The bulk...
modulus and shear modulus of the matrix are $K_M$ and $\mu_M$, respectively, those of the $\alpha$ inclusion phases are $K_{I\alpha}$ and $\mu_{I\alpha}$. The MTA, derived as an approximate solution to the field equations for the composite to compute the elastic bulk modulus $K_{MTA}$ and shear modulus $\mu_{MTA}$ has the expressions [1,8]

$$K_{MTA} = \frac{v_M K_M + \sum_{a=2}^{n} v_{Ia} K_{Ia} D_{Ka}(K_{Ia}, \mu_{Ia}, K_M, \mu_M)}{v_M + \sum_{a=2}^{n} v_{Ia} D_{Ka}(K_{Ia}, \mu_{Ia}, K_M, \mu_M)}$$

$$\mu_{MTA} = \frac{v_M \mu_M + \sum_{a=2}^{n} v_{Ia} \mu_{Ia} D_{Ka}(K_{Ia}, \mu_{Ia}, K_M, \mu_M)}{v_M + \sum_{a=2}^{n} v_{Ia} D_{Ka}(K_{Ia}, \mu_{Ia}, K_M, \mu_M)}$$

where

$$D_{Ka}(K_{Ia}, \mu_{Ia}, K_M, \mu_M) = \frac{K_M + \mu_M}{K_{Ia} + \mu_{Ia}}$$

$$D_{\mu a}(K_{Ia}, \mu_{Ia}, K_M, \mu_M) = \frac{\mu_M + K_{Ia} + \mu_{Ia}}{\mu_M + K_{Ia} + \mu_{Ia}}$$

The three-component composite that are two circular inclusions having elastic bulk modulus $K_1$, $K_2$ shear modulus $\mu_1$, $\mu_2$ and volume fraction $v_1$, $v_2$ in a matrix having the elastic moduli $K_M$, $\mu_M$ and volume fraction $v_M$. In the case of three-component matrix composites, the bulk modulus $K$ and the elastic shear modulus $\mu$ formula of Mori-Tanaka approximation can be written as:

$$K_{MTA} = \frac{v_M K_M + v_1 K_1 D_{K1}(K_1, \mu_1, K_M, \mu_M) + v_2 K_2 D_{K2}(K_2, \mu_2, K_M, \mu_M)}{v_M + v_1 D_{K1}(K_1, \mu_1, K_M, \mu_M) + v_2 D_{K2}(K_2, \mu_2, K_M, \mu_M)}$$

$$\mu_{MTA} = \frac{v_M \mu_M + v_1 \mu_1 D_{\mu 1}(K_1, \mu_1, K_M, \mu_M) + v_2 \mu_2 D_{\mu 2}(K_2, \mu_2, K_M, \mu_M)}{v_M + v_1 D_{\mu 1}(K_1, \mu_1, K_M, \mu_M) + v_2 D_{\mu 2}(K_2, \mu_2, K_M, \mu_M)}$$

3 Differential approximations (DA)

Consider $n$-phase suspension of randomly oriented inclusions of type $\alpha$ ($\alpha = 2, ..., n$), with elastic bulk modulus $K_{I\alpha}$, shear modulus $\mu_{I\alpha}$ (volume proportion $v_{I\alpha}$) in a matrix of elastic moduli $K_M$, $\mu_M$ (volume proportion $v_M$). The differential scheme construction process for the suspension starts with the base matrix phase $M$. At each step of the procedure, we add proportionally infinitesimal volume amounts $v_{I\alpha} \Delta t$ ($\Delta t \ll 1$, $\alpha = 2, ..., n$) of randomly oriented inclusions into already constructed composite of the previous step, which contains volume fractions $v_{I\alpha} t$ of the inclusion phases (the parameter $t$ increases from 0 to 1, as the differential scheme proceeds). The newly added particles will see an effective continuum, owing to their relative sizes, and the new composite can be considered as a dilute suspension of particles from phases $\alpha$, of volume fractions

$$\frac{v_{I\alpha} \Delta t}{1 + \sum_{a=2}^{n} v_{I\alpha} \Delta t} = \frac{v_{I\alpha} \Delta t}{1 + v_{I}\Delta t}$$

where $v_I = \sum_{a=2}^{n} v_{I\alpha}$, in a matrix of elastic bulk modulus $K$, shear modulus $\mu$ ($v_I$ is the total volume fractions of the included phases). The elastic modulus of the new composite is

$$K + dK = K + \sum_{a=2}^{n} \frac{v_{I\alpha} \Delta t}{1 + v_{I}\Delta t} (K_{I\alpha} - K) D_{Ka}(K_{Ia}, \mu_{Ia}, K, \mu)$$

$$\mu + d\mu = \mu + \sum_{a=2}^{n} \frac{v_{I\alpha} \Delta t}{1 + v_{I}\Delta t} (\mu_{I\alpha} - \mu) D_{\mu a}(K_{Ia}, \mu_{Ia}, K, \mu)$$

where the dilute suspension expression $D_{Ka}$, $D_{\mu a}$ for an inclusion $\alpha$ has been defined in (3), (4). Since the volume fraction of the included phase $\alpha$ increases by

$$v_{I\alpha} dt = \frac{v_{I\alpha} t + v_{I\alpha} \Delta t}{1 + v_{I}\Delta t} - v_{I\alpha} t = \frac{v_{I\alpha} \Delta t}{1 + v_{I}\Delta t} (1 - v_I t)$$

See equations (5)–(6) below

where the dilute suspension expression $D_{Ka}$, $D_{\mu a}$ for an inclusion $\alpha$ has been defined in (3), (4). Equations (5) and (6) will be used to determine the elastic moduli of three-phase composites.
we obtain the following differential equation for the elastic bulk modulus $K$, shear modulus $\mu$ of the composite

$$\frac{dK}{dt} = \frac{1}{1 - v_f \Delta t} \sum_{a=2}^n v_{fa}(K_{fa} - K)D_{Kfa}(K_{fa}, \mu_{fa}, K, \mu)$$  \hspace{1cm} (11)$$

$$\frac{d\mu}{dt} = \frac{1}{1 - v_f \Delta t} \sum_{a=2}^n v_{fa}(\mu_{fa} - K)D_{\mu fa}(K_{fa}, \mu_{fa}, K, \mu)$$  \hspace{1cm} (12)$$

$$K(0) = K_M, \mu(0) = \mu_M, 0 \leq t \leq 1. \hspace{1cm} (13)$$

Differential equations (11) and (12) will be used to determine the elastic moduli of three-phase composites. Though the above construction process of differential scheme corresponds to certain idealistic hierarchical models formed on widely-separated scales, the approximation aims at usual multiphase suspensions of inclusions in a matrix.

4 Hashin-Strikman bounds

HS bounds on the effective elastic moduli of isotropic $d$-dimensional composites can be presented as [3]

- Elastic bulk modulus

$$P_k\left(\frac{2(d-1)}{d} \mu_{min}\right) \leq K^{eff} \leq P_k\left(\frac{2(d-1)}{d} \mu_{max}\right)$$  \hspace{1cm} (14)$$

where

See equation (15) below.

- Shear modulus

$$P_\mu(\mu^*(K_{min}, \mu_{min})) \leq \mu^{eff} \leq P_\mu(\mu^*(K_{max}, \mu_{max}))$$  \hspace{1cm} (16)$$

where

See equation (17) below.

5 FFT simulation for three-phase composites

The FFT method uses the classical expansion along with the Neuman series of the solution of the periodic elastic problem in Fourier space, based on the Green’s tensor and exact expressions of the shape factors in Fourier space [11,12,14,15]. In this section, the FFT method is briefly presented for calculating the effective elastic moduli of three-component materials in 2D.

Behavior of the component materials is described by Hooke’s law:

$$\sigma(x) = C(x \cdot \varepsilon(x)$$  \hspace{1cm} (18)$$

where $\sigma(x)$ and $\varepsilon(x)$ are respectively the local stress and strain fields, the stress field satisfies the equilibrium condition

$$\nabla \cdot \sigma(x) = 0. \hspace{1cm} (19)$$

Let $x$ denote the position of a point in the unit cell. $C(x)$ is the fourth order local elastic tensor of the heterogeneous medium, one is given by

$$C(x) = \sum_\alpha C_\alpha I_\alpha(x), \quad I_\alpha(x) = \begin{cases} 1, & x \in V_\alpha \\ 0, & x \notin V_\alpha \end{cases}$$  \hspace{1cm} (20)$$

$\alpha$ designates the phase ($\alpha = I_1; I_2$ or $M$).

We shall denote the Fourier transform of a $V$-periodic function $F(x)$ of cartesian $x(x_1, x_2, x_3)$ as $\hat{F}(\xi)$

$$F(x) = \sum_\xi \hat{F}(\xi)e^{i\xi \cdot x}, \quad \hat{F}(\xi) = \langle F(x)e^{-i\xi \cdot x} \rangle$$

$$= \frac{1}{V} \int_V F(x)e^{-i\xi \cdot x}dx$$  \hspace{1cm} (21)$$

with $\xi(\xi_1, \xi_2, \xi_3)$ being the wave vector, the symbol $\langle \cdot \rangle$ designates the product of convolution $\xi = \xi_k e_k, \xi_j = \frac{n\pi}{a_j}$, $(n_j = 0, \pm 1, \pm 2, ..., \pm \infty)$, $j = 1, 2, 3$.
The Fourier transformation of elastic tensor is
\[ C(\xi) = \overline{\int C(x) e^{-i\xi \cdot x} dx} = \sum_a C_a I_a(\xi) \] (22)
where \( I_a(\xi) \) are the shape functions, defined by
\[ I_a(\xi) = \frac{1}{V_a} \int e^{i\xi \cdot x} dV. \] (23)

In the case of circle-inclusion, the function \( I_a(\xi) \) is given by Nemat-Nasser [16]
\[ I_a(\xi) = 2 S_a \frac{J_1(\eta)}{\eta} e^{i\xi \cdot x(a)} \] (24)
where \( \eta = R(\xi_1^2 + \xi_2^2)^{1/2} \), \( S_a = \pi R^2 \), \( R \) is the radii of circle-inclusion, \( x(a) \) is the vector position of the center of the inclusion \( \alpha \); and \( \xi_1, \xi_2 \) are the components of \( \xi \); \( J_1 \) is the Bessel function of first kind and first order. \( I_a(\xi) \) can be derived from relation
\[ \sum_a I_a(\xi) = 0, \quad \forall \xi \neq 0. \] (25)

For \( \xi = 0 \), one have \( I_a(0) = V_a/V \).

Substituting the Fourier transformation of the local stress, strain fields into equation (19), the problem in a unit cell is solved by explicit recurrence process in Fourier space. That can be rewritten in the form [10, 11]
\[
\begin{cases}
\hat{\varepsilon}^{i+1} = \hat{\varepsilon}^{i} - \hat{\Gamma}(\xi) \cdot \sum_a C_a I_a(\xi) \hat{\varepsilon}^{i} \quad \xi \neq 0 \\
\hat{\varepsilon}^{i+1} = \mathbf{E}^{0}, \quad \xi = 0
\end{cases}
\] (26)
in which \( \hat{\sigma}(\xi) \) and \( \hat{\varepsilon}(\xi) \) are respectively Fourier transformation of \( \sigma(x) \) and \( \varepsilon(x) \). \( \hat{\Gamma}(\xi) \) is the Greens's tensor, the symbol \( \ast \) designates the product of convolution. The value of the Green's tensor is given for an isotropic reference medium by Mura [1].

\[ \hat{\Gamma}_{iii} = 4A \frac{\xi_i^2}{\xi_i^2} - B \frac{\xi_i^2}{\xi_i^2} \] (27)
\[ \hat{\Gamma}_{iij} = -4B \frac{\xi_i^2 \xi_j^2}{\xi_i^2} \] (28)
\[ \hat{\Gamma}_{ijj} = 2A \frac{\xi_i \xi_j}{\xi_i^2} - B \frac{\xi_i \xi_j}{\xi_i^2} \] (29)

\[
\begin{cases}
\hat{\varepsilon}^{i+1}(\xi) = \hat{\varepsilon}^{i}(\xi) - \hat{\Gamma}(\xi) \cdot \sum_a C_a I_a(\xi) \hat{\varepsilon}^{i}(\xi), \quad \xi \neq 0 \\
\hat{\varepsilon}^{i+1} = \mathbf{E}^{0}, \quad \xi = 0
\end{cases}
\] (30)
where \( A = \frac{1}{2} \mu, B = \frac{E}{(2\mu+\lambda)\mu} ; \lambda \) and \( \mu \) are Lamé coefficients. Relationship between \( \hat{\sigma}(\xi) \) and \( \hat{\varepsilon}(\xi) \) is described by expression
\[ \hat{\sigma}(\xi) = C(\xi) \ast \hat{\varepsilon}(\xi). \] (31)

For the three-component medium considered, the expression (26) can be written as
\[ \frac{\|\hat{\sigma}(\xi)\| - \|\hat{\varepsilon}(\xi)\|}{\|\hat{\varepsilon}(\xi)\|} < \varepsilon, \] where \( \varepsilon \) is a prescribed value (\( \varepsilon = 10^{-3} \)).

6 Applications

In this section, we use the FFT method, MTA approximation, and DA approximation to estimate the effective elastic moduli of the elastically-isotropic composites 2D.

We consider 3 examples, with
(A): \( K_M = 2, \mu_M = 1, K_N = 100, \mu_N = 60, K_T = 20, \mu_T = 10 \)
(B): \( K_M = 30, \mu_M = 16, K_N = 10, \mu_N = 6, K_T = 1, \mu_T = 0.5 \)
(C): \( K_M = 30, \mu_M = 16, K_N = 4, \mu_N = 2, K_T = 100, \mu_T = 60 \)

We consider three-phase composites, in which two different size balls are arranged randomly non-overlapping (Fig. 1). For numerical FFT illustrations, 60 circles inclusion were planted randomly in a unit cell by Matlab program such that there is no circle overlapping, in which a unit cell having the dimension \( L = 1 \) along each space direction containing inclusion (Fig. 1, left), the minimum
Fig. 1. Unit cell containing 60 circle-inclusion randomly placed (left), three phase model of composite with two different circle inclusions (right).

Fig. 2. Elastic bulk (left) and shear modulus (right) of three-phase composites with the case (A).

Fig. 3. Elastic bulk (left) and shear modulus (right) of three-phase composites with the case (B).
distance of is 0.01. In our calculations, a grid 128 × 128 is considered. The FFT result simulation is obtained from the algorithm in Section 4. The FFT results compared with Differential approximation, Mori-Tanaka approximation, and Hashin–Shtrikman bounds over ranges of \( v_I = v_{I1} + v_{I2}, v_{I1} = 2v_{I2} \) (all the inclusions in one phase have the same size, the dimensionless radius \( R_{I1} \) varies from 0.01 to 0.063 and \( R_{I2} \) varies from 0.0071 to 0.0445) are reported in Figures 2–4 and Tables 1–6. Numerical FFT results and MTA, DA approximations are quite close and converge at small volume fractions of inclusion phases and diverge at large proportions of suspended particles, all the results fall inside the Hashin-Shtrikman’ bounds, as expected. The DA, MTA approximations are asymptotically exact at dilute suspensions of included particles, but become inevitably less so good at higher proportions of included phases. At large proportions of included phases, the details of particles’ interactions of particular microstructures should be accounted for more accurate estimations.

### Table 1. Comparison of results (\( K^{eff} \)) of FFT, MTA, DA, and HS bound for the case (A), \( R_1 = \sqrt{2}R_2 \).

| \( R_1 \) | \( v_I = v_{I1} + v_{I2} \) | HS_Lower | FFT | MTA | AD | HS_Upper |
|---|---|---|---|---|---|---|
| 0.01 | 0.0141 | 2.0401 | 2.0019 | 2.0401 | 2.0402 | 2.4266 |
| 0.02 | 0.0565 | 2.1670 | 2.1905 | 2.1670 | 2.1690 | 3.7422 |
| 0.03 | 0.1272 | 2.4039 | 2.4446 | 2.4039 | 2.4158 | 6.0626 |
| 0.04 | 0.2262 | 2.8020 | 2.8722 | 2.8020 | 2.8481 | 9.6103 |
| 0.05 | 0.3534 | 3.4749 | 3.6133 | 3.4749 | 3.6235 | 14.7730 |
| 0.055 | 0.4276 | 3.9002 | 4.2142 | 3.9002 | 4.2501 | 18.1543 |
| 0.06 | 0.5089 | 4.7102 | 5.1462 | 4.7102 | 5.1663 | 22.2267 |
| 0.063 | 0.5611 | 5.2929 | 6.0566 | 5.2929 | 5.9368 | 25.0714 |

### Table 2. Comparison of results (\( \mu^{eff} \)) of FFT, MTA, DA, and HS bounds for the case (A), \( R_1 = \sqrt{2}R_2 \).

| \( R_1 \) | \( v_I = v_{I1} + v_{I2} \) | HS_Lower | FFT | MTA | AD | HS_Upper |
|---|---|---|---|---|---|---|
| 0.01 | 0.0141 | 1.0201 | 1.0009 | 1.0201 | 1.0202 | 1.2139 |
| 0.02 | 0.0565 | 1.0838 | 1.0957 | 1.0838 | 1.0848 | 1.8755 |
| 0.03 | 0.1272 | 1.2028 | 1.2251 | 1.2028 | 1.2087 | 3.0493 |
| 0.04 | 0.2262 | 1.4028 | 1.4485 | 1.4028 | 1.4260 | 4.8609 |
| 0.05 | 0.3534 | 1.7413 | 1.7413 | 1.7413 | 1.8163 | 7.5346 |
| 0.055 | 0.4276 | 2.0009 | 2.1967 | 2.0009 | 2.1323 | 9.3102 |
| 0.06 | 0.5089 | 2.3642 | 2.7370 | 2.3642 | 2.5952 | 11.4750 |
| 0.063 | 0.5611 | 2.6568 | 3.2650 | 2.6568 | 2.9853 | 13.0046 |
### Table 3. Comparison of results ($K^{eff}$) of FFT, MTA, DA, and HS bounds for the case (B), $R_1 = \sqrt{2}R_2$.

| $R_1$ | $v_I = v_{I1} + v_{I2}$ | HS_Lower | FFT | MTA | AD | HS_Upper |
|-------|-----------------|----------|-----|-----|----|----------|
| 0.01  | 0.0141          | 27.0008  | 29.3346 | 29.3073 | 29.3045 | 29.3073   |
| 0.02  | 0.0565          | 20.7361  | 27.3577 | 27.3490 | 27.3013 | 27.3490   |
| 0.03  | 0.1272          | 14.8922  | 24.4704 | 24.4361 | 24.2275 | 24.4361   |
| 0.04  | 0.2262          | 10.6114  | 20.9567 | 20.9591 | 20.4373 | 20.9591   |
| 0.05  | 0.3534          | 7.6847   | 17.1532 | 17.2799 | 16.3414 | 17.2799   |
| 0.055 | 0.4276          | 6.5947   | 15.1940 | 15.4534 | 14.3044 | 15.4534   |
| 0.06  | 0.5089          | 5.6915   | 13.1736 | 13.6700 | 12.3320 | 13.6700   |
| 0.063 | 0.5611          | 5.2239   | 11.8714 | 12.6282 | 11.9374 | 12.6282   |

### Table 4. Comparison of results ($\mu^{eff}$) of FFT, MTA, DA, and HS bound for the case (B), $R_1 = \sqrt{2}R_2$.

| $R_1$ | $v_I = v_{I1} + v_{I2}$ | HS_Lower | FFT | MTA | AD | HS_Upper |
|-------|-----------------|----------|-----|-----|----|----------|
| 0.01  | 0.0141          | 14.3571  | 15.6492 | 15.6325 | 15.6310 | 15.6325   |
| 0.02  | 0.0565          | 10.9579  | 14.6187 | 14.5954 | 14.5693 | 14.5954   |
| 0.03  | 0.1272          | 7.8257   | 13.0759 | 13.0572 | 12.9430 | 13.0572   |
| 0.04  | 0.2262          | 5.5546   | 11.1903 | 11.2284 | 10.9405 | 11.2284   |
| 0.05  | 0.3534          | 4.0132   | 9.0875  | 9.3016  | 8.7766  | 9.3016    |
| 0.055 | 0.4276          | 3.4413   | 7.9843  | 8.3483  | 7.6984  | 8.3483    |
| 0.06  | 0.5089          | 2.9685   | 6.8710  | 7.4195  | 6.6516  | 7.4195    |
| 0.063 | 0.5611          | 2.7241   | 6.1943  | 6.8779  | 6.0456  | 6.8779    |

### Table 5. Comparison of results ($K^{eff}$) of FFT, MTA, DA, and HS bound for the case (C), $R_1 = \sqrt{2}R_2$.

| $R_1$ | $v_I = v_{I1} + v_{I2}$ | HS_Lower | FFT | MTA | AP | HS_Upper |
|-------|-----------------|----------|-----|-----|----|----------|
| 0.01  | 0.0141          | 28.8402  | 29.6093 | 29.5712 | 29.5695 | 29.8412   |
| 0.02  | 0.0565          | 25.8158  | 28.4026 | 28.3316 | 28.3002 | 29.3683   |
| 0.03  | 0.1272          | 21.9081  | 26.5124 | 26.4089 | 26.2583 | 28.5910   |
| 0.04  | 0.2262          | 17.9787  | 24.1064 | 23.9813 | 23.5522 | 27.5253   |
| 0.05  | 0.3534          | 14.4934  | 21.3444 | 21.2405 | 20.3287 | 26.1921   |
| 0.055 | 0.4276          | 12.9700  | 19.8680 | 19.8086 | 18.5774 | 25.4331   |
| 0.06  | 0.5089          | 11.5948  | 18.3266 | 18.3615 | 16.7648 | 24.6169   |
| 0.063 | 0.5611          | 10.8379  | 17.3536 | 17.4929 | 15.6576 | 24.1013   |

### Table 6. Comparison of results ($\mu^{eff}$) of FFT, MTA, DA, and HS bound for the case (C), $R_1 = \sqrt{2}R_2$.

| $R_1$ | $v_I = v_{I1} + v_{I2}$ | HS_Lower | FFT | MTA | AP | HS_Upper |
|-------|-----------------|----------|-----|-----|----|----------|
| 0.01  | 0.0141          | 15.3370  | 15.7757 | 15.7512 | 15.7527 | 15.9080   |
| 0.02  | 0.0565          | 13.6258  | 15.0802 | 15.0445 | 15.0269 | 15.6341   |
| 0.03  | 0.1272          | 11.4519  | 13.9902 | 13.9531 | 13.8694 | 15.1854   |
| 0.04  | 0.2262          | 9.3072   | 12.5996 | 12.5897 | 12.3542 | 14.5728   |
| 0.05  | 0.3534          | 7.4385   | 11.0038 | 11.0697 | 10.5768 | 13.8106   |
| 0.055 | 0.4276          | 6.6314   | 10.1583 | 10.2837 | 9.6234  | 13.3787   |
| 0.06  | 0.5089          | 5.9078   | 9.2987  | 9.4948  | 8.6453  | 12.9159   |
| 0.063 | 0.5611          | 5.5115   | 8.7828  | 9.0239  | 8.0523  | 12.6245   |
7 Conclusions

There have been many previous studies on the elastic moduli of two-phase material composites or three-phase with periodic structures such as square or hexagonal models. In practical materials, the structure of materials is often randomly distributed and has multi- phases. In this work, with different asymptotic solutions, the paper has solved the problem of the elastic moduli for a three-phase material model in 2D.

FFT algorithm is developed to calculate the effective elastic moduli of some complex material models such as three-phase composites with an arbitrary distribution of two inclusion phases. The numerical results fall inside the Hashin-Shtrikman bounds.

DA, MTA approaches are to solve for the three-phase material model of two different sizes of circular inclusions. FFT, MTA, and DA give quite close results. All results satisfy HS bounds over all the volume proportions of the components.

MTA and DA have explicit algebraic expressions, so that easy to apply the estimates for the effective elastic moduli, hence might be more useful for engineers as first estimates of the effective elastic moduli of the composites.

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