Spectral subspaces of spectra of Abelian lattice-ordered groups in size aleph one

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Abstract. It is well known that the lattice \( \text{Id}_c G \) of all principal \( \ell \)-ideals of any Abelian \( \ell \)-group \( G \) is a completely normal distributive 0-lattice; yet not every completely normal distributive 0-lattice is a homomorphic image of some \( \text{Id}_c G \), via a counterexample of cardinality \( \aleph_2 \). We prove that every completely normal distributive 0-lattice with at most \( \aleph_1 \) elements is a homomorphic image of some \( \text{Id}_c G \). By Stone duality, this means that every completely normal generalized spectral space with at most \( \aleph_1 \) compact open sets is homeomorphic to a spectral subspace of the \( \ell \)-spectrum of some Abelian \( \ell \)-group.

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1. Introduction

A subset \( I \) in a lattice-ordered group (in short \( \ell \)-group) \( G \) is an \( \ell \)-ideal if it is an order-convex normal subgroup closed under the lattice operations. An \( \ell \)-ideal \( I \) of \( G \) is prime if \( I \neq G \) and \( x \land y \in I \) implies that \( \{x, y\} \cap I \neq \emptyset \) whenever \( x, y \in G \). In case \( G \) is Abelian, the \( \ell \)-spectrum of \( G \) is defined as the set \( \text{Spec} G \) of all prime \( \ell \)-ideals of \( G \), endowed with the topology whose closed subsets are the \( \{P \in \text{Spec} G \mid X \subseteq P\} \) for \( X \subseteq G \) (often called the hull-kernel topology). Denote by \( \mathcal{G} \) the class of all Abelian \( \ell \)-groups.

The problem of the description of \( \ell \)-spectra of all Abelian \( \ell \)-groups (say the \( \ell \)-spectrum problem) is stated, in the language of MV-algebras, in Mundici [10, Problem 2]. Now under Stone duality (cf. Grätzer [6, § II.5], Johnstone [8, § II.3], Rump and Yang [13] for the case without top element, and Wehrung [18, § 2.2] for a summary), for any \( G \in \mathcal{G} \), \( \text{Spec} G \) corresponds to the lattice \( \text{Id}_c G \) of all principal \( \ell \)-ideals of \( G \); that is, \( \text{Id}_c G = \)
\{\langle a \rangle \mid a \in G^+ \} \text{ where each } \langle a \rangle \overset{\text{def}}{=} \{x \in G \mid (\exists n \in \mathbb{N})(|x| \leq na)\}. \text{ This enables us to restate the } \ell\text{-spectrum problem as the description problem of the class } \Id_c \mathcal{G} \overset{\text{def}}{=} \{D \mid (\exists G \in \mathcal{G})(D \cong \Id_c G)\}. \text{ All such lattices are clearly distributive with smallest element (usually denoted by 0). They are also completely normal (cf. Bigard, Keimel, and Wolfenstein [3, Ch. 10]), that is, they satisfy the statement}
\[(\forall a, b)(\exists x, y)(a \lor b = a \lor y = x \lor b \text{ and } x \land y = 0).
\]
Delzell and Madden observed in [4, Theorem 2], \textit{via} a counterexample of cardinality \(\aleph_1\), that those properties are not sufficient to characterize \(\Id_c \mathcal{G}\). On the other hand, the second author proved in [15] that every countable completely normal distributive 0-lattice belongs to \(\Id_c \mathcal{G}\). The categorical concept of \textit{condensate}, initiated in the second author’s work [5] with Pierre Gillibert, together with the main result of [16], enabled the second author to prove in [17] that \(\Id_c \mathcal{G}\) is not the class of models of any class of \(\mathcal{L}_{\infty \lambda}\) sentences of lattice theory, for any infinite cardinal \(\lambda\). Using further tools from infinitary logic, the second author extended those results in [19] by proving that \(\Id \mathcal{G}\) is not the \textit{complement of a projective class} over \(\mathcal{L}_{\infty \infty}\), thus verifying in particular that the additional property of all lattices \(\Id_c G\) coined by the first author in his proof of [11, Theorem 2.1] is still not sufficient to characterize \(\Id_c \mathcal{G}\).

As observed in the above-cited references, all those results extend to the class of all (lattice) homomorphic images of lattices \(\Id_c G\). On the other hand, not every homomorphic image of a lattice of the form \(\Id_c G\) belongs to \(\Id_c \mathcal{G}\) (cf. Wehrung [15, Example 10.6]). Recast in terms of spectra, \textit{via} Stone duality, this means that \textit{not every spectral subspace of an }\ell\text{-spectrum is an }\ell\text{-spectrum}.

Moreover, not every completely normal bounded distributive lattice is a homomorphic image of some \(\Id_c G\): a counterexample of cardinality \(\aleph_2\) is constructed in Wehrung [16].

In this paper we complete the picture above, by establishing that every completely normal distributive 0-lattice \(D\) with at most \(\aleph_1\) elements is a homomorphic image of \(\Id_c G\) for some Abelian \(\ell\)-group \(G\). This also strengthens the first author’s result, obtained in [11], that \(D\) is \textit{Cevian}. In fact, we verify the slightly more general statement that \(G\) may be taken a vector lattice over any given countable totally ordered division ring \(k\) (cf. Theorem 4), modulo the obvious change in the definition of an \(\ell\)-ideal (i.e., \(\ell\)-ideals need to be closed under scalar multiplication by elements of \(k\); see Wehrung [18, §2.3] for more detail). Due to the results of [18, §9], the countability assumption on \(k\) cannot be dispensed with.

Our argument will roughly follow the one from Wehrung [15], with the “Main Extension Lemma” [15, Lemma 4.2] strengthened from finite lattices to certain infinite lattices, and streamlined \textit{via} the introduction of \textit{consonance kernels} (cf. Definition 1), as Lemma 10. The proof of the “closure step” [15, Lemma 7.2] fails in that more general context, so we get only “homomorphic image” as opposed to “isomorphic copy”, of \(\Id_c G\). This will also require a few known additional properties of finite distributive lattices and their homomorphisms, \textit{via} Birkhoff duality (see in particular Lemma 4). Our final argument,
given a completely normal distributive 0-lattice $L$, will start by expressing $L$ as a directed union of an ascending $\omega_1$-sequence $\bar{L} = (L_\xi \mid \xi < \omega_1)$ of countable completely normal distributive 0-lattices, and then, with the help of Lemma 10, iteratively lift all subdiagrams $(L_\xi \mid \xi < \alpha)$ with $\alpha < \omega_1$, with respect to the functor $\text{Id}_c$. That part of our argument turns out to be valid not only for the chain $\omega_1$ but for any tree in which every element has countable height (cf. Theorem 3).

2. Basic concepts

2.1. Sets, posets, lattices

For any set $X$, $\text{Pow} X$ denotes the powerset algebra of $X$. By “countable” we will mean “at most countable”. For an element $a$ in a partially ordered set (from now on poset) $P$, we set $P \downarrow a \overset{\text{def}}{=} \{p \in P \mid p \leq a\}$ (or $\downarrow a$ if $P$ is understood). A subset $A$ of $P$ is a lower subset of $P$ if $P \downarrow a \subseteq A$ whenever $a \in A$. A poset $P$ with bottom element is a tree if $P \downarrow a$ is well-ordered under the induced order whenever $a \in P$.

For a subset $P$ in a poset $Q$ and for $x \in Q$, $x^P$ (resp., $x_P$) denotes the least $y \in P$ such that $x \leq y$ (resp., the largest $y \in P$ such that $y \leq x$) if it exists. We say that $P$ is relatively complete in $Q$ if $x^P$ and $x_P$ both exist for all $x \in P$. If $P$ is a subalgebra of a Boolean algebra $Q$, it suffices to verify that $x^P$ exists whenever $x \in Q$ (resp., $x_P$ exists whenever $x \in Q$).

Relative completeness has been used in a description of projective Boolean algebras. For the proof of the following (easy) assertion see Heindorf and Shapiro [7, Lemma 1.2.7].

**Lemma 1.** Let $A$, $A'$ be subalgebras of a Boolean algebra $B$ with $A'$ finitely generated over $A$. If $A$ is relatively complete in $B$, then so is $A'$.

For posets $P$ and $Q$ with respective top elements $\top_P$ and $\top_Q$, a map $f: P \to Q$ is top-faithful if $f^{-1}\{\top_Q\} = \{\top_P\}$. For any poset $P$, $P^{\uparrow \infty}$ denotes the poset obtained by adding an extra element, usually denoted by $\infty$, atop of $P$. For any map $f: P \to Q$, we denote by $f^{\uparrow \infty}: P^{\uparrow \infty} \to Q^{\uparrow \infty}$ the unique extension of $f$ sending $\infty$ to $\infty$. Such maps are exactly the top-faithful maps from $P^{\uparrow \infty}$ to $Q^{\uparrow \infty}$.

We denote by $\text{Ji}_L$ (resp., $\text{Mi}_L$) the set of all join-irreducible (resp., meet-irreducible) elements in a lattice $L$, endowed with the induced ordering. For any join-irreducible element $p$ in a finite distributive lattice $D$, we denote by $p^*$ the unique lower cover of $p$ in $D$, and by $p^\dagger$ the largest element of $D$ not above $p$; so $p^* = p \land p^\dagger$. The assignment $p \mapsto p^\dagger$ defines an order-isomorphism from $\text{Ji}_D$ onto $\text{Mi}_D$.

As in Wehrung [15,18], two elements $a$ and $b$ in a 0-lattice (i.e., lattice with a bottom element) $D$ are consonant if there exist $u, v \in D$ such that $a \leq u \vee b$, $b \leq a \vee v$, and $u \wedge v = 0$. A subset $X$ of $D$ is consonant if any pair of elements in $X$ is consonant. The lattice $D$ is completely normal if it is consonant within itself.
The assignment \( D \mapsto \text{Ji}D \) is part of Birkhoff duality between finite distributive lattices with 0,1-lattice homomorphisms and finite posets with isotope maps (cf. Grätzer [6, § II.1.3]). The Birkhoff dual of a 0,1-lattice homomorphism \( \varphi: D \rightarrow E \) is the map \( \text{Ji}E \rightarrow \text{Ji}D \), \( q \mapsto q^\varphi \overset{\text{def}}{=} \min \{ x \in D \mid q \leq \varphi(x) \} \).

For any distributive 0-lattice \( D \), we denote by \( \text{BR}(D) \) the generalized Boolean algebra \( R\)-generated by \( D \) in the sense of Grätzer [6, § II.4] (aka the Boolean envelope of \( D \)). Equivalently, \( \text{BR}(D) \) is the universal generalized Boolean algebra of \( D \). Up to isomorphism, \( \text{BR}(D) \) is the unique generalized Boolean algebra generated by \( D \) as a 0-sublattice. The assignment \( D \mapsto \text{BR}(D) \) canonically extends to a functor, which turns 0-lattice embeddings to embeddings of generalized Boolean algebras. For a 0-sublattice \( D \) of a distributive lattice \( E \) with 0, we will thus identify \( \text{BR}(D) \) with its canonical image in \( \text{BR}(E) \). If \( D \) is a finite distributive lattice and \( P \overset{\text{def}}{=} \text{Ji}D \), then the assignment \( x \mapsto P \downarrow x \) defines an isomorphism from \( D \) onto the lattice \( \text{Down}P \) of all lower subsets of \( P \). Since the universal Boolean algebra of \( \text{Down}P \) is the powerset lattice of \( P \), with each \( \{ p \} = (\downarrow p) \setminus (\uparrow p)_* \), it follows that the atoms of \( \text{BR}(D) \) are exactly the \( p \wedge \neg p_* \) for \( p \in \text{Ji}D \).

**Lemma 2.** The following statements hold, for any distributive 0-lattice \( D \):

1. For all \( a_1, a_2, b_1, b_2 \in D \), \( a_1 \wedge \neg b_1 \leq a_2 \wedge \neg b_2 \) within \( \text{BR}(D) \) iff \( a_1 \leq a_2 \lor b_1 \) and \( a_1 \wedge b_2 \leq b_1 \) within \( D \). (This does not require \( D \) be finite.)

2. If \( D \) is finite, then \( a \wedge \neg b = \vee \{ p \wedge \neg p_* \mid p \in \text{Ji}D, p \leq a, p \not\leq b \} \) within \( \text{BR}(D) \), whenever \( a, b \in D \).

**Lemma 3.** Let \( D \) and \( L \) be distributive 0-lattices with \( D \) finite, let \( \varphi: D \rightarrow L \) be a 0-lattice homomorphism, let \( a, b \in D \), and let \( c \in L \). Then \( \varphi(a) \leq \varphi(b) \lor c \) iff \( \varphi(p) \leq \varphi(p_*) \lor c \) whenever \( p \in \text{Ji}D \) with \( p \leq a \) and \( p \not\leq b \).

**Proof.** \( \varphi(a) \leq \varphi(b) \lor c \) iff \( \text{BR}(\varphi)(a \wedge \neg b) \leq c \), iff \( \text{BR}(\varphi)(p \wedge \neg p_*) \leq c \) whenever \( p \in \text{Ji}D \) such that \( p \leq a \) and \( p \not\leq b \) (we apply Lemma 2(2)). Now \( \text{BR}(\varphi)(p \wedge \neg p_*) \leq c \) iff \( \varphi(p) \leq \varphi(p_*) \lor c \). \( \square \)

For any elements \( x \) and \( y \) in a lattice \( E \) let \( x \rightarrow_E y \) denote the largest \( z \in E \), if it exists, such that \( x \wedge z \leq y \) (it is also called the pseudocomplement of \( x \) relative to \( y \)); so \( \rightarrow_E \) is the Heyting implication on \( E \). If \( \rightarrow_E \) is defined on every pair of elements then we say that \( E \) is a generalized Heyting algebra.

If, in addition, \( E \) has a bottom element, then we say that \( E \) is a Heyting algebra. Every Heyting algebra is a bounded distributive lattice, and every finite distributive lattice is a Heyting algebra.\(^1\)

Dually, we denote by \( x \wedge_E y \) the least \( z \in E \), if it exists, such that \( x \leq y \lor z \). It is the dual pseudocomplement of \( y \) relative to \( x \).

A lattice homomorphism \( \varphi: D \rightarrow E \) is closed if whenever \( a_0, a_1 \in D \) and \( b \in E \), if \( \varphi(a_0) \leq \varphi(a_1) \lor b \), then there exists \( x \in D \) such that \( a_0 \leq a_1 \lor x \).

\(^1\) Strictly speaking we should set the Heyting implication \( \rightarrow \) apart from the lattice signature, thus for example stating that “every finite distributive lattice expands to a unique Heyting algebra”. The shorter formulation, which we shall keep for the sake of simplicity, reflects a standard abuse of terminology that should create no confusion here.
and \( \varphi(x) \leq b \). If \( \varphi \) is an inclusion map we will say that \( D \) is a closed sublattice of \( E \).

The following folklore lemma, whose easy proof we leave to the reader as an exercise, enables to read, on the Birkhoff dual, whether a given homomorphism, between finite distributive lattices, is a homomorphism of Heyting algebras or a closed homomorphism, respectively.

Lemma 4. The following statements hold, for any finite distributive lattices \( D \) and \( E \) and any 0, 1-lattice homomorphism \( \varphi: D \to E \):

1. \( \varphi \) is a homomorphism of Heyting algebras iff for all \( p \in \text{J}_iD \) and all \( q \in \text{J}_iE \), if \( p \leq q^{\varphi} \), then there exists \( x \in \text{J}_iE \) such that \( x \leq q \) and \( x^{\varphi} = p \).
2. \( \varphi \) is closed iff for all \( p \in \text{J}_iD \) and all \( q \in \text{J}_iE \), if \( q^{\varphi} \leq p \), then there exists \( x \in \text{J}_iE \) such that \( q \leq x \) and \( x^{\varphi} = p \).

2.2. The lattices \( \text{Bool}(\mathcal{F}, \Omega) \), \( \text{Op}(\mathcal{F}, \Omega) \), and \( \text{Op}^- (\mathcal{F}, \Omega) \)

For more detail on this subsection we refer the reader to Wehrung [15,18]. For a right vector space \( E \) over a totally ordered division ring \( k \), a map \( f: E \to k \) is an affine functional if \( f - f(0) \) is a linear functional. Note that the affine functionals on \( E \) form a left vector space over \( k \).

For functions \( f \) and \( g \) with common domain \( \Omega \) and values in a poset \( T \), we set \( [f \leq g] \stackrel{\text{def}}{=} \{ x \in \Omega \mid f(x) \leq g(x) \} \); and similarly for \( [f < g] \), \( [f = g] \), \( [f \neq g] \), and so on. Throughout this paper, \( f \) and \( g \) will always be restrictions, to a convex set \( \Omega \), of continuous affine functionals on a topological vector space \( E \) over a totally ordered division ring \( k \). For a set \( \mathcal{F} \) of maps from \( \Omega \) to \( k \), we will denote by \( \text{Bool}(\mathcal{F}, \Omega) \) the Boolean subalgebra of the powerset of \( \Omega \) generated by all subsets \( [f > 0] \) and \( [f < 0] \) for \( f \in \mathcal{F} \). As in [18], we will also denote by \( \text{Op}^- (\mathcal{F}, \Omega) \) the 0-sublattice of \( \text{Bool}(\mathcal{F}, \Omega) \) generated by all \( [f > 0] \) and \( [f < 0] \) where \( f \in \mathcal{F} \), and then set \( \text{Op}(\mathcal{F}, \Omega) \stackrel{\text{def}}{=} \text{Op}^- (\mathcal{F}, \Omega) \cup \{ \Omega \} \). Evidently, \( \text{Bool}(\mathcal{F}, \Omega) \) is generated, as a Boolean algebra, by its 0-sublattice \( \text{Op}(\mathcal{F}, \Omega) \); so \( \text{Bool}(\mathcal{F}, \Omega) = \text{BR}(\text{Op}(\mathcal{F}, \Omega)) \).

For any set \( I \) and any totally ordered division ring \( k \), we will denote by \( k^{(I)} \) the collection of all \( I \)-indexed families of elements in \( k \) that vanish outside some finite subset of \( I \). We will occasionally identify every element \( a = (a_i \mid i \in I) \in k^{(I)} \) with the corresponding (continuous) linear functional \( \sum_{i \in I} a_i \delta_i \) (where \( \delta_i \) denotes the \( i \)th projection), thus justifying such notations as \( \text{Bool}(k^{(I)}, k^{(I)}) \) and \( \text{Op}(k^{(I)}, k^{(I)}) \); observe that in those notations, the first (resp., second) occurrence of \( k^{(I)} \) is endowed with its structure of left (resp., right) vector space over \( k \). Moreover, in its second occurrence, \( k^{(I)} \) is endowed with the coarsest topology making all canonical projections \( \delta_i \) continuous.

Denote by \( F_\ell(I, k) \) the free left \( k \)-vector lattice on a set \( I \). As observed in Baker [1], Bernau [2], Madden [9, Ch. III] (see also Wehrung [18, page 13] for a summary), \( F_\ell(I, k) \) canonically embeds into \( k^{k^{(I)}} \). We sum up a few related facts.

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2 “Right” and “left” appear to have been unfortunately mixed up at various places in [18], particularly on pages 12 and 13. Since this is mostly a matter of choosing sides, that paper’s results are unaffected. We nonetheless attempt to fix this here.
Lemma 5. (Folklore)

(1) $F_\ell(I, k)$ is isomorphic to the sublattice of $k^{k(I)}$ generated by all linear functionals $\sum_{i \in I} a_i \delta_i$ associated to elements $a \in k^{(I)}$, via the assignment $i \mapsto \delta_i$.

(2) The assignment $(x) \mapsto [x \neq 0]$ defines an isomorphism from the lattice $\text{Id}_c F_\ell(I, k)$, of all principal $\ell$-ideals of the left $k$-vector lattice $F_\ell(I, k)$, onto $\text{Op}^-(k^{(I)}, k^{(I)})$.

3. Consonance kernels

In this section we introduce a tool, the consonance kernels, expressing the consonance of the image a lattice homomorphism via its behavior on join-irreducible elements.

Definition 1. Let $D$ and $L$ be distributive lattices, with $D$ finite and $L$ with a zero element, and let $f : D \to L$ be a join-homomorphism. Set $P \overset{\text{def}}{=} \text{Ji} D$. A consonance kernel for $f$ is a family $(e_p \mid p \in P)$ of elements of $L$ such that

$$
(3.1) \quad f(p) = f(p_\ast) \lor e_p, \quad \text{whenever } p \in P; \\
(3.2) \quad e_p \land e_q = 0, \quad \text{whenever } p, q \in P \text{ are incomparable}.
$$

We then set $x \otimes_y \overset{\text{def}}{=} \bigvee \{e_p \mid p \in (P \downarrow x) \setminus (P \downarrow y)\}$, whenever $x, y \in D$.

Lemma 6. In the context of Definition 1, $f(x) = f(x \land y) \lor (x \otimes y)$ whenever $x, y \in D$. Moreover, $f$ is a lattice homomorphism.

Proof. Setting $c \overset{\text{def}}{=} f(x \land y) \lor (x \otimes y)$, it is obvious that $c \leq f(x)$. In order to prove that $f(x) \leq c$, it suffices to prove that $f(p) \leq c$ whenever $p \in P \downarrow x$. By way of contradiction, let $p$ be a minimal element of $P \downarrow x$ with $f(p) \not\leq c$. Since $p \leq y$ implies $f(p) \leq f(x \land y) \leq c$, we get $p \in (P \downarrow x) \setminus (P \downarrow y)$, so $f(p) = f(p_\ast) \lor e_p$. Since $e_p \leq c$, we get $f(p_\ast) \not\leq c$. The case $p_\ast = 0$ is impossible, because $f(0) \leq f(x \land y) \leq c$. Since $f$ is a join-homomorphism, we get $f(p_\ast) = \bigvee \{f(q) \mid q \in P \downarrow p_\ast\}$. By the minimality assumption on $p$, we get $f(q) \leq c$ for every $q \in P \downarrow p_\ast$, hence $f(p_\ast) \leq c$, a contradiction.

Now let $x, y \in D$. By the result of the paragraph above, $f(x) = f(x \land y) \lor (x \otimes y)$ and $f(y) = f(x \land y) \lor (y \otimes x)$. Due to (3.2), $(x \otimes y) \land (y \otimes x) = 0$; whence $f(x) \land f(y) = f(x \land y)$. \qed

Lemma 7. Let $D$ and $L$ be distributive lattices, with $D$ finite and $L$ with a zero element. Then a lattice homomorphism $f : D \to L$ has a consonance kernel iff the range of $f$ is consonant in $L$.

Proof. Suppose first that the range of $f$ is consonant in $L$. Since $D$ is finite, there exists a finite 0-sublattice $K$ of $L$, containing $f[D]$, such that the range of $f$ is consonant in $K$. Setting $e_p \overset{\text{def}}{=} f(p) \setminus_K f(p_\ast)$ for each $p \in \text{Ji} D$, Condition (3.1) is obviously satisfied. Let $p, q \in \text{Ji} D$ be incomparable. From $p \land q \leq p_\ast$ we get

$$
e_p = f(p) \setminus_K f(p_\ast) \leq f(p) \setminus_K f(p \land q) = f(p) \setminus_K (f(p)$$

\cite{Birkhäuser}
\[ \wedge f(q) = f(p) \setminus_K f(q), \]

and, similarly, \( e_q \leq f(q) \setminus_K f(p) \). Since \( f(p) \) and \( f(q) \) are consonant within \( K \), we get \( (f(p) \setminus_K f(q)) \wedge (f(q) \setminus_K f(p)) = 0 \); whence \( e_p \wedge e_q = 0 \).

Let, conversely, \( (e_p \mid p \in \text{Ji} D) \) be a consonance kernel for \( f \) and set \( P \defeq \text{Ji} D \). Let \( x, y \in D \), set \( u \defeq x \otimes_E y \) and \( v \defeq y \otimes_E x \). It follows from Lemma 6 that \( f(x) \leq f(y) \vee u \) and \( f(y) \leq f(x) \vee v \). Moreover, for all \( p \in (P \downarrow x) \setminus (P \downarrow y) \) and \( q \in (P \downarrow y) \setminus (P \downarrow x) \), \( p \) and \( q \) are incomparable, thus \( e_p \wedge e_q = 0 \); whence \( u \wedge v = 0 \). Therefore, the pair \((u, v)\) witnesses the consonance of \( f(x) \) and \( f(y) \) in \( L \).

\[ \square \]

4. An extension lemma for infinite distributive lattices

This section’s main result, Lemma 10, states conditions under which a homomorphism \( f : D \to L \) of distributive lattices can be extended to a homomorphism \( f : E \to L \) in case \( E \) is generated over \( D \) by two disjoint elements \( a \) and \( b \). One of its main improvements, over the original [15, Lemma 4.2] it stems from, is the possibility of \( D \) being infinite.

**Definition 2.** A 0, 1-sublattice \( D \) of a bounded distributive lattice \( E \) is a semi-Heyting sublattice if for all \( x, y \in D \), \( x \to_D y \) and \( x \to_E y \) both exist and are equal.

In particular, every semi-Heyting sublattice of \( E \) is a Heyting algebra (\( E \) itself may not be a Heyting algebra).

**Notation 1.** Let \( D \) be a finite 0, 1-sublattice of a bounded distributive lattice \( E \) and let \( f : D \to L \) be a 0-lattice homomorphism. We set

\[ f_{\vec{e}}(a) \defeq \bigvee \{ e_p \mid p \in \text{Ji} D, \ p \leq p_* \vee a \} \]

for every consonance kernel \( \vec{e} \) of \( f \) and every \( a \in E \).

The following lemma arises from Wehrung [18, Remark 4.6]. We include a proof for convenience.

**Lemma 8.** Let \( D \) be a finite semi-Heyting sublattice of a bounded distributive lattice \( E \), let \( f : D \to L \) be a 0-lattice homomorphism, and let \( a, b \in E \) such that \( a \wedge b = 0 \). Then any join-irreducible elements \( p \) and \( q \) in \( D \) such that \( p \leq p_* \vee a \) and \( q \leq q_* \vee b \) are incomparable. In particular, \( f_{\vec{e}}(a) \wedge f_{\vec{e}}(b) = 0 \) for any consonance kernel \( \vec{e} \) for \( f \).

**Proof.** Suppose otherwise, say \( p \leq q \); thus \( p^\ddagger \leq q^\ddagger \). From \( a \wedge b = 0 \) we get \( p \wedge b \leq (p_* \vee a) \wedge b = p_* \wedge b \leq p_* \), thus, by assumption, \( b \leq p \to_E p_* = p \to_D p_* = p^\ddagger \). Since \( p^\ddagger \leq q^\ddagger \), we get \( b \leq q^\ddagger \), so \( q \leq q_* \vee b \leq q^\ddagger \), a contradiction.

\[ \square \]

The following lemma ought to be well known, however we could not locate any reference for it. We include a proof for convenience.
Lemma 9. Let $D$ and $L$ be lattices, with $L$ distributive, and let $X$ a generating subset of $D$. Then a map $f: X \to L$ can be extended to a (necessarily unique) lattice homomorphism $g: D \to L$ if and only if
\[
\bigwedge_{i=1}^{m} x_i \leq \bigvee_{j=1}^{n} y_j \implies \bigwedge_{i=1}^{m} f(x_i) \leq \bigvee_{j=1}^{n} f(y_j)
\] (4.1)
for all integers $m, n > 0$ and all $x_1, \ldots, x_m, y_1, \ldots, y_n \in X$.

Proof. Let us express (4.1) by stating that the map $f$ is consistent (on its domain $X$). Clearly, it suffices to prove that $f$ extends to a consistent map defined on the whole lattice $D$.

Set $\mathcal{C} \overset{\text{def}}{=} \{ Y \subseteq D \mid X \subseteq Y \text{ and } f \text{ extends to } Y \to L \}$. Trivially, $X \in \mathcal{C}$. Let $Y \in \mathcal{C}$ with unique consistent extension $g: Y \to L$ of $f$ and let $U \subseteq Y$ be nonempty finite; set $v \overset{\text{def}}{=} \bigwedge U$. We claim that $Y \cup \{ v \} \in \mathcal{C}$. The case where $v \in Y$ being trivial, we may assume that $v \notin Y$. Denote by $h: Y \cup \{ v \} \to L$ the unique extension of $g$ such that $h(v) = \bigwedge g[U]$. We need to prove that $h$ is the unique consistent extension of $f$ to $Y$. Since $g$ witnesses that $Y$ belongs to $\mathcal{C}$, any consistent extension $h'$ of $f$ to $Y \cup \{ v \}$ extends $g$, so it also satisfies $h'(v) = \bigwedge g[U]$ (consider the inequality $\bigwedge U \leq v$ together with the monotonicity of $h'$), and so it extends $h$. The uniqueness statement on $h$ follows. It thus remains to verify that $h$ is consistent. Several cases need to be considered, among which the only nontrivial one arises from an inequality $\bigwedge_{i=1}^{m} x_i \leq \bigvee_{j=1}^{n} y_j \vee v$ where $m > 0$, $n \geq 0$, and all $x_i, y_j \in Y$. For each $u \in U$, it follows from the inequality $\bigwedge_{i=1}^{m} x_i \leq \bigvee_{j=1}^{n} y_j \vee u$ that $\bigwedge_{i=1}^{m} g(x_i) \leq \bigvee_{j=1}^{n} g(y_j) \vee g(u)$; whence, meeting over $u \in U$ and applying the distributivity of $L$, $\bigwedge_{i=1}^{m} g(x_i) \leq \bigvee_{j=1}^{n} g(y_j) \vee h(v)$, as required for our claim. By iterating that process and since $\mathcal{C}$ is obviously closed under directed unions, it follows that the closure $X^\wedge$ of $X$ under finite meets belongs to $\mathcal{C}$. Dually, the closure $X^\vee$ of $X$ under finite joins also belongs to $\mathcal{C}$. Since $\mathcal{C}$ is closed under directed unions and $X$ generates $D$, it follows that $D = \bigcup \{ X^{(\wedge \vee)^n} \mid n < \omega \}$ belongs to $\mathcal{C}$. 

We are now reaching this section’s main goal.

Lemma 10 (Main Extension Lemma). Let $D$ be a semi-Heyting sublattice of a bounded distributive lattice $E$ and let $a, b \in E$. Setting $B \overset{\text{def}}{=} \text{BR}(D)$, we assume the following:

1. $E$ is generated, as a lattice, by $D \cup \{ a, b \}$.
2. $a \land b = 0$.
3. All elements $a_B, b_B, (a \lor b)_B, a^B$, and $b^B$ are defined.
4. $(a \lor b)_B = a_B \lor b_B$.

Then
\[
c^B \in D \text{ whenever } c \in \{ a, b, a \lor b \}.
\] (4.2)
Further, for every 0-lattice homomorphism $f: D \to L$ and all $\alpha, \beta \in L$, the following conditions are equivalent.

\[ c^B \in D \quad \text{whenever} \quad c \in \{ a, b, a \lor b \}. \]
(i) \((\alpha, \beta) = (g(a), g(b))\) for some lattice homomorphism \(g: E \to L\) extending \(f\);
(ii) \(\alpha \leq f(a^B), \beta \leq f(b^B), \alpha \wedge \beta = 0\), \(\text{BR}(f)(a_B) \leq \alpha\), and \(\text{BR}(f)(b_B) \leq \beta\).

Moreover, for any finite semi-Heyting sublattice \(D'\) of \(D\) such that \(\{a_B, b_B\} \subseteq \text{BR}(D')\) and \(\{a^B, b^B\} \subseteq D'\), and any consonance kernel \(\bar{c}\) of \(f' \overset{\text{def}}{=} f|_{D'}\), the pair \((f'_\alpha(a), f'_\beta(b))\) satisfies (ii).

Note. By the same token as the one used in the proof of Lemma 3, the condition that \(\text{BR}(f)(a_B) \leq \alpha\) is equivalent to saying that for all \(x, y \in D\), \(x \leq y \lor a \Rightarrow f(x) \leq f(y) \lor \alpha\). By Lemma 3, if \(D\) is finite, then it suffices to restrict ourselves to the case where \(x = p \in J_1 D\) and \(y = p^*\). Note that \(\text{BR}(f)(a_B)\) is an element of \(\text{BR}(L)\), usually not in \(L\), so it cannot be taken as the lowest possible value of \(\alpha\) a priori.

Proof. We start by proving (4.2). By (3), there is an expression of the form \(c^E = \bigwedge_{i<n}(\neg u_i \lor v_i)\) (within \(B\)) where \(n < \omega\) and all \(u_i, v_i \in D\). For each \(i < n, c \leq \neg u_i \lor v_i\) within \(\text{BR}(E)\), thus \(u_i \land c \leq v_i\), and thus, since \(D\) is a semi-Heyting sublattice of \(E\), \(c \leq u_i \to_E v_i = u_i \to_D v_i\); whence, setting \(w = \bigwedge_{i<n}(u_i \to_D v_i)\), we get \(c \leq w\). For each \(i < n, w \leq u_i \to_D v_i\) with \(w \in D\), thus \(u_i \land w \leq v_i\), so \(w \leq \neg u_i \lor v_i\) within \(B\), and so \(w \leq c^B\). Since \(w \in D\), it follows that \(w = c^B = c^D\).

Now it is obvious that for every lattice homomorphism \(g: E \to L\) extending \(f\), the pair \((\alpha, \beta) \overset{\text{def}}{=} (g(a), g(b))\) satisfies \(\alpha \leq f(a^B), \beta \leq f(b^B), \alpha \wedge \beta = 0\), \(\text{BR}(f)(a_B) \leq \alpha\), and \(\text{BR}(f)(b_B) \leq \beta\). Let, conversely, \((\alpha, \beta)\) be such a pair.

By virtue of Lemma 9, we need to verify the implication (4.1) for \(x_i, y_j \in D \cup \{a, b\}\), at the unique extension of \(f\) to \(D \cup \{a, b\}\) sending \(a\) to \(\alpha\) and \(b\) to \(\beta\) (note that \(a \in D \Rightarrow \alpha = f(a)\) and \(b \in D \Rightarrow \beta = f(b)\)). Since \(f\) is a lattice homomorphism, we may assume that exactly one \(x_i\) and exactly one \(y_j\) belong to \(D\). Since \(a \land b = 0\), the inequality \(x \land a \leq y \lor b\) is equivalent to \(x \land a \leq y\). Hence (4.1) boils down to the equation \(\alpha \land \beta = 0\) (which is assumed) and the following implications:

\[
\begin{align*}
x \leq y \lor a & \Rightarrow f(x) \leq f(y) \lor \alpha; \quad (4.3) \\
x \leq y \lor b & \Rightarrow f(x) \leq f(y) \lor \beta; \quad (4.4) \\
x \leq y \lor a \lor b & \Rightarrow f(x) \leq f(y) \lor \alpha \lor \beta; \quad (4.5) \\
x \land a \leq y & \Rightarrow f(x) \land \alpha \leq f(y); \quad (4.6) \\
x \land b \leq y & \Rightarrow f(x) \land \beta \leq f(y). \quad (4.7)
\end{align*}
\]

The implications (4.3) and (4.4) follow from \(\text{BR}(f)(a_B) \leq \alpha\) and \(\text{BR}(f)(b_B) \leq \beta\). Owing to Condition (4), the implication (4.5) follows from the inequalities

\[
\text{BR}(f)((a \lor b)_B) = \text{BR}(f)(a_B \lor b_B) = \text{BR}(f)(a_B) \lor \text{BR}(f)(b_B) \leq \alpha \lor \beta.
\]

Suppose that \(x \land a \leq y\). Since \(D\) is a semi-Heyting sublattice of \(E\), it follows that \(a \leq x \to_E y = x \to_D y\), thus, using (4.2), \(a^D = a^B \leq x \to_D y\). It follows that \(\alpha \leq f(a^B) \leq f(x \to_D y)\), thus \(f(x) \land \alpha \leq f(x) \land f(x \to_D y) \leq f(y)\). The implication (4.6) follows. The proof of (4.7) is similar.
For the remainder of the proof, let $D'$ be a finite semi-Heyting sublattice of $D$ such that $\{a^B, b^B\} \subseteq BR(D')$ and $\{a^B, b^B\} \subseteq D'$ (cf. Fig. 1), and let $\vec{c}$ be a consonance kernel of $f' \overset{\text{def}}{=} f|_{D'}$. Set $(\alpha, \beta) \overset{\text{def}}{=} (f'_\vec{c}(a), f'_\vec{c}(b))$.

For every $p \in Ji(D')$, $p \leq p^* \lor a$ (within $E$) implies that $p \leq p^* \lor a^B$ (within $D'$), thus, since $p \in Ji(D')$, we get $p \leq a^B$, whence $e_p \leq f(p) \leq f(a^B)$. This proves that $\alpha \leq f(a^B)$. Similarly, $\beta \leq f(b^B)$. Further, the equation $\alpha \land \beta = 0$ follows from Lemma 8.

Let $c \in \{a, b\}$ and let $x, y \in D$ such that $x \leq y \lor c$, we need to prove that $f(x) \leq f(y) \lor f'_\vec{c}(c)$. From $x \land \neg y \leq c$ (within BR($E$)) it follows that $x \land \neg y \leq c_B$ (within BR($D$)). Set $X \overset{\text{def}}{=} \{p \in Ji(D') \mid p \land \neg p^* \leq c_B\} = \{p \in Ji(D') \mid p \leq p^* \lor c\}$. By (3) and since BR($D'$) is a finite Boolean algebra with atoms $p \land \neg p^*$ for $p \in Ji(D')$,

$$c_B = \bigvee \{p \land \neg p^* \mid p \in X\} \text{ within } B. \quad (4.8)$$

By the definition of $X$,

$$f(p) = f(p^*) \lor e_p \leq f(p^*) \lor f'_\vec{c}(c) \quad \text{whenever } p \in X,$$

so $f(p) \land \neg f(p^*) \leq f'_\vec{c}(c)$ within BR($L$), whenever $p \in X$; whence, using (4.8),

$$\text{BR}(f)(c_B) = \bigvee \{f(p) \land \neg f(p^*) \mid p \in X\} \leq f'_\vec{c}(c) \quad \text{within } \text{BR}(L). \quad (4.9)$$

Using (4.9), we get

$$f(x) \land \neg f(y) = \text{BR}(f)(x \land \neg y) \leq \text{BR}(f)(c_B) \leq f'_\vec{c}(c),$$

so $f(x) \leq f(y) \lor f'_\vec{c}(c)$. \hfill \qed

5. Adjunctions between lattices $\text{Bool}(\mathcal{F}, k(I))$

Throughout this section $k$ will be a totally ordered division ring. We shall state a few properties of Boolean algebras of the form $\text{Bool}(\mathcal{F}, \Omega)$, mostly related to relative completeness between such algebras.

The following observation is contained in the proof of Wehrung [15, Lemma 6.6].
Lemma 11. Let $\Omega$ be a convex subset in a right vector space $\mathbb{E}$ over $k$ and let $F \cup \{a\}$ be a set of affine functionals on $\mathbb{E}$. Set $A^+ \overset{\text{def}}{=} [a > 0]$ and $A^- \overset{\text{def}}{=} [a < 0]$. Then for every $U \in \text{Bool}(F, \Omega)$, if $U \subseteq A^+ \cup A^-$, then there are $U^+, U^- \in \text{Bool}(F, \Omega)$ such that $U = U^+ \cup U^-$ whereas $U^+ \subseteq A^+$ and $U^- \subseteq A^-$. Proof. Since $U$ is the union of finitely many cells, each of which being the intersection of finitely many sets of the form either $[±f > 0]$ or $[±f \geq 0]$ where $f \in F$, it suffices to consider the case where $U$ is such a cell. If $U$ meets both $A^+$ and $A^-$, pick $x \in U \cap A^+$ and $y \in U \cap A^-$; so $a(x) > 0$ and $a(y) < 0$. Then $x \overset{\text{def}}{=} (a(y) - a(x))^{-1} a(y)$ belongs to the open interval $]0, 1[$ and $a(x\lambda + y(1-\lambda)) = 0$, that is, $x\lambda + y(1-\lambda) \notin A^+ \cup A^-$. On the other hand, since $U$ is convex, $x\lambda + y(1-\lambda) \in U$; a contradiction since $U \subseteq A^+ \cup A^-$. Therefore, $U$ is disjoint either from $A^+$ or from $A^-$, thus it is contained either in $A^+$ or in $A^-$. Corollary 1. In the context of Lemma 1, $(A^+ \cup A^-)_{\text{Bool}(F, \Omega)}$ exists iff both $(A^+)_{\text{Bool}(F, \Omega)}$ and $(A^-)_{\text{Bool}(F, \Omega)}$ exist, and then $(A^+ \cup A^-)_{\text{Bool}(F, \Omega)} = (A^+)_{\text{Bool}(F, \Omega)} \cup (A^-)_{\text{Bool}(F, \Omega)}$. In what follows we will identify every element $f \in k(I)$ with the associated linear functional on $k(I)$, that is, $x \mapsto \sum_{i \in I} f_i x_i$. Moreover, whenever $I \subseteq J$, we will identify $k(I)$ with the subset of $k(J)$ consisting of all vectors with support contained in $I$. Notation 2. For $I \subseteq J$, we define mappings
\[
\varepsilon_{I,J} : \text{Pow } k(I) \rightarrow \text{Pow } k(J), \quad X \mapsto \{y \in k(J) \mid y\upharpoonright_I \in X\},
\]
\[
\rho_{I,J} : \text{Pow } k(J) \rightarrow \text{Pow } k(I), \quad Y \mapsto \{y\upharpoonright_I \mid y \in Y\}.
\]
It is straightforward to verify that $\varepsilon_{I,J}$ is an embedding of $\text{Bool}$ algebras with left adjoint $\rho_{J,I}$ (i.e., $\rho_{J,I}(Y) \subseteq X$ iff $Y \subseteq \varepsilon_{I,J}(X)$), whenever $X \in \text{Pow } k(I)$ and $Y \subseteq k(J)$. In particular, $\rho_{I,J}$ is a $(\vee, 0)$-homomorphism and $\rho_{J,I}\varepsilon_{I,J}$ is the identity map on $\text{Pow } k(I)$. Lemma 12. Let $I$ and $J$ be sets with $I \subseteq J$. Then $\rho_{J,I}[\text{Bool}(k(J), k(J))] = \text{Bool}(k(I), k(J))$. Proof. For every $X \in \text{Bool}(k(I), k(I))$, $X = \rho_{J,I}\varepsilon_{I,J}(X)$ with $\varepsilon_{I,J}(X) \in \text{Bool}(k(J), k(J))$, thus $X \in \rho_{J,I}[\text{Bool}(k(J), k(J))]$; whence $\rho_{J,I}[\text{Bool}(k(J), k(J))]$ contains $\text{Bool}(k(I), k(I))$. Let us establish the converse containment. Since $\rho_{J,I}$ is a $(\vee, 0)$-homomorphism, it suffices to prove that $\rho_{J,I}(Y) \in \text{Bool}(k(I), k(I))$ whenever $Y$ is a set of the form $\bigcap_{i < m}[a_i \geq 0] \cap \bigcap_{j < n}[b_j > 0]$ where $m, n < \omega$ and all $a_i, b_j \in k(J)$. Set $a_i' \overset{\text{def}}{=} a_i\upharpoonright_I$ and $a_i'' \overset{\text{def}}{=} a_i\upharpoonright_{J \setminus I}$, for all $i < m$, and define similarly $b_j'$ and $b_j''$ for $j < n$. An element $x \in k(I)$ belongs to $\rho_{J,I}(Y)$ iff
there exists $z \in \k^{(J \setminus I)}$ such that each $a_i'(x) + a_i''(z) \geq 0$ and each $b'_j(x) + b''_j(z) > 0$. The set $V$ of all $(m + n)$-tuples of elements of $\k$ of the form $(a'_0(z), \ldots, a'_{m-1}(z), b'_0(z), \ldots, b''_{n-1}(z))$ is a vector subspace of $\k^{m+n}$. Hence, an element $x \in \k^{(I)}$ belongs to $\rho_{I,1}(Y)$ if there exists $u \in V$ such that $a'_i(x) + u_i \geq 0$ whenever $i < m$ and $b'_j(x) + u_{m+j} > 0$ whenever $j < n$. Since membership in $V$, of any $(m+n)$-tuple of elements of $\k$, can be expressed by a finite set of linear equations, the statement that a given $x \in \k^{(I)}$ belongs to $\rho_{I,1}(Y)$ can be expressed by a sentence, over the first-order language $\mathcal{L} \overset{\text{def}}{=} \{<, 0, -, +\} \cup \{\cdot, \lambda \mid \lambda \in \k\}$ of ordered Abelian groups augmented with right scalar multiplications by elements of $\k$, in $(a'_0(x), \ldots, a'_{m-1}(x), b'_0(x), \ldots, b''_{n-1}(x))$. Now every $\mathcal{L}$-sentence is equivalent, over all nonzero totally ordered right $\k$-vector spaces, to a quantifier-free $\mathcal{L}$-sentence (cf. van den Dries [14, Corollary I.7.8]). Therefore, $\rho_{I,1}(Y)$ belongs to $\text{Bool}(\mathcal{F}, \k^{(I)})$ for a finite set $\mathcal{F}$ of linear combinations of the $a'_i$ and the $b'_j$. □

**Proposition 1.** Let $I$ and $J$ be sets with $I \subseteq J$ and let $D$ be a finite subset of $\k^{(J)}$. Then $\text{Bool}(\k^{(I)} \cup D, \k^{(J)})$ is relatively complete in $\text{Bool}(\k^{(J)}, \k^{(J)})$.

**Proof.** We first prove that $\text{Bool}(\k^{(I)}, \k^{(J)})$ is relatively complete in $\text{Bool}(\k^{(J)}, \k^{(J)})$. Let $Y \in \text{Bool}(\k^{(I)}, \k^{(J)})$. Then $Y \subseteq Z \in \text{Bool}(\k^{(I)}, \k^{(J)})$ implies that $\varepsilon_{I,1,\rho_{I,1}}(Y) \subseteq \varepsilon_{I,1,\rho_{I,1}}(Z) = Z$. Thus, $Y^{\text{Bool}(\k^{(I)}, \k^{(J)})} = \varepsilon_{I,1,\rho_{I,1}}(Y)$ which, by Lemma 12, belongs to $\text{Bool}(\k^{(I)}, \k^{(J)})$.

Since $\text{Bool}(\k^{(I)} \cup D, \k^{(J)})$ is finitely generated over $\text{Bool}(\k^{(I)}, \k^{(J)})$, via the additional generators $[d > 0]$ and $[d < 0]$ for $d \in D$, the desired conclusion follows from Lemma 1. □

### 6. Extending a top-faithful map

In Lemmas 13 and 14 we fix a totally ordered division ring $\k$. The following lemma takes care of the “domain step” required in the proof of Theorem 4.

**Lemma 13.** Let $I$ and $J$ be sets, let $L$ be a completely normal distributive $0$-lattice, let $D$ be a finite subset of $\k^{(J)}$, and let $e \in \k^{(J)}$. Then every top-faithful $0$-lattice homomorphism $f: \text{Op}(\k^{(I)} \cup D, \k^{(J)}) \to L^{\ominus\infty}$ extends to a top-faithful lattice homomorphism $g: \text{Op}(\k^{(I)} \cup D \cup \{e\}, \k^{(J)}) \to L^{\ominus\infty}$ (cf. Fig. 2).

**Proof.** Set $E \overset{\text{def}}{=} D \cup \{e\}$, $D \overset{\text{def}}{=} \text{Op}(\k^{(I)} \cup D, \k^{(J)})$, $E \overset{\text{def}}{=} \text{Op}(\k^{(I)} \cup E, \k^{(J)})$, $B \overset{\text{def}}{=} \text{BR}(D) = \text{Bool}(\k^{(I)} \cup D, \k^{(J)})$, and $C \overset{\text{def}}{=} \text{Bool}(\k^{(I)} \cup E, \k^{(J)})$. By Proposition 1, $B$ is relatively complete in $C$. In particular, setting $a \overset{\text{def}}{=} [e > 0]$ and $L^{\ominus\infty} \xrightarrow{g} \text{Op}(\k^{(I)} \cup D \cup \{e\}, \k^{(J)})$.

**Diagram 1.** A commutative triangle for Lemma 13.
b \equiv [e < 0]$, the elements $a^B, b^B, a_B, b_B$, and $(a \lor b)_B$ are all defined. By Corollary 1, $(a \lor b)_B = a_B \lor b_B$. Let $\mathcal{D}'$ be a finite subset of $\mathbb{k}^I \cup \mathcal{D}$ such that $a^B, b^B, a_B$, and $b_B$ all belong to $B' \equiv \text{Bool}(\mathcal{D}', \mathbb{k}^{\mathcal{J}})$. By Wehrung [15, Lemma 5.4] (see also Wehrung [18, Lemma 4.1] for the more general form of that statement), $D$ is a Heyting subalgebra of $E$ and $\mathcal{D}' \equiv \text{Op}(\mathcal{D}', \mathbb{k}^{\mathcal{J}})$ is a Heyting subalgebra of $D$. Since $L$ is completely normal and $f[\mathcal{D}']$ is finite, it follows from Lemma 7 that $f' \equiv f|_{\mathcal{D}'}$ has a consonance kernel $(e_P \mid P \in \text{Ji} \mathcal{D}')$. By Lemma 10, $f$ extends to a unique lattice homomorphism $g: \mathcal{D} \rightarrow L$ such that $g(x) = f'_e(x)$ whenever $x \in \{a, b\}$. For any $P \in \text{Ji} \mathcal{D}'$ such that $P \subseteq P_x \cup x$, $0 \notin P_x \cup x$, thus $0 \notin P$, that is, $P$ is not the top element of $\text{Op}(\mathbb{k}^I, \mathbb{k}^{\mathcal{J}})$. Since $f$ is top-faithful, it follows that $e_P \leq f(P) < \infty$; whence $f'_e(x) < \infty$. It follows that $g$ is top-faithful.

The “surjectivity step” is much more easily taken care of:

**Lemma 14.** Let $I$ and $J$ be sets with $I \subset J$ and $J \setminus I$ infinite, let $L$ be a distributive 0-lattice, let $\mathcal{D}$ be a finite subset of $\mathbb{k}^I$, and let $c \in L$. Then for every top-faithful 0-lattice homomorphism $f: \text{Op}(\mathbb{k}^I \cup \mathcal{D}, \mathbb{k}^{\mathcal{J}}) \rightarrow L^{\cup \infty}$, there are $e \in \mathbb{k}^I$ and a top-faithful lattice homomorphism $g: \text{Op}(\mathbb{k}^I \cup \mathcal{D} \cup \{e\}, \mathbb{k}^{\mathcal{J}}) \rightarrow L^{\cup \infty}$ extending $f$ such that $g(e) = c$.

**Proof.** Since $\mathcal{D}$ is finite and $J \setminus I$ is infinite, there exists $j \in J \setminus I$ not in the support of any element of $\mathcal{D}$. Take $e \equiv \delta_j$, the $j$th canonical projection $\mathbb{k}^{\mathcal{J}} \rightarrow \mathbb{k}$. By the argument of Wehrung [15, Lemma 8.3], $\text{Op}(\mathbb{k}^I \cup \mathcal{D} \cup \{\delta_j\}, \mathbb{k}^{\mathcal{J}})$ is the (internal) free amalgamated sum of $\text{Op}(\mathbb{k}^I \cup \mathcal{D}, \mathbb{k}^{\mathcal{J}})$ and $\{\emptyset, [\delta_j > 0], [\delta_j < 0], [\delta_j \neq 0], \mathbb{k}^{\mathcal{J}}\}$ within the category of bounded distributive lattices. Hence $f$ extends to a unique lattice homomorphism $g: \text{Op}(\mathbb{k}^I \cup \mathcal{D} \cup \{\delta_j\}, \mathbb{k}^{\mathcal{J}}) \rightarrow L$ such that $g([\delta_j > 0]) = c$ and $g([\delta_j < 0]) = 0$. Since $c < \infty$ and $f$ is top-faithful, it follows that $g$ is also top-faithful. 

\section{Representing trees of countable lattices}

In this section we will reach the paper’s main goal, Theorem 4, which states that if $\mathbb{k}$ is countable, then every completely normal distributive 0-lattice with at most $\aleph_1$ elements is a homomorphic image of some $\text{Id}_c F$ for some $\mathbb{k}$-vector lattice $F$. In order to reach that result we will in fact prove (cf. Theorem 3) the apparently stronger statement that every diagram of countable completely normal distributive 0-lattices, indexed by a tree in which every element has countable height, can be represented in that fashion.

Towards that goal, our main technical tool is the following “one-step extension” theorem, which relies on the results of Sect. 6, together with the observation that for $\mathcal{F} \subseteq \mathbb{k}^I$, $\text{Op}(\mathcal{F}, \mathbb{k}^I) = \text{Op}^-(\mathcal{F}, \mathbb{k}^I) \cup \{\infty\}$ (where $\infty$ denotes here the full space $\mathbb{k}^I$; so the top-faithful maps $\text{Op}(\mathcal{F}, \mathbb{k}^I) \rightarrow L^{\cup \infty}$ are exactly the $g^{\cup \infty}$ where $g: \text{Op}^-(\mathcal{F}, \mathbb{k}^I) \rightarrow L$).

**Theorem 1.** Let $\mathbb{k}$ be a countable totally ordered division ring, let $I$ and $J$ be countable sets with $I \subset J$ and $J \setminus I$ infinite, let $K$ and $L$ be distributive 0-lattices with $L$ countable and completely normal, let $\varphi: K \rightarrow L$ be a 0-lattice.
homomorphism, and let \( f : \text{Op}^{-}(k^{(I)}, k^{(I)}) \to K \) be a 0-lattice homomorphism. Then there exists a surjective lattice homomorphism \( g : \text{Op}^{-}(k^{(J)}, k^{(J)}) \to L \) such that \( g \circ \varepsilon_{I,J} = \varphi \circ f \).

The settings for Theorem 1 can be read in Fig. 3. Its proof can be followed in Fig. 4.

**Proof.** An iterative application of Lemmas 13 and 14, similar to the proof of Wehrung [15, Theorem 9.1] but easier since we do not need any analogue of the "closure step" [15, Lemma 7.1]. Let \( k^{(J)} = \{ v_n \mid n < \omega \} \) and \( L = \{ c_n \mid n < \omega \} \). Given an extension \( g_n : \text{Op}^{-}(k^{(I)} \cup D_n, k^{(J)}) \to L \) of \( g_0 \overset{\text{def}}{=} \varphi \circ f \), where \( D_n \subseteq k^{(J)} \) is finite, we extend the top-faithful extension \( g_n^\cup : \text{Op}(k^{(I)} \cup D_n, k^{(J)}) \to L^\cup \) of \( g_n \) to a top-faithful lattice homomorphism \( g_{n+1}^\cup : \text{Op}(k^{(I)} \cup D_{n+1}, k^{(J)}) \to L^\cup \), with \( D_n \subseteq D_{n+1}, v_{|n/2|} \in D_{n+1} \) if \( n \) is even (via Lemma 13), and \( c_{|n/2|} \in \text{rng } g_{n+1} \) if \( n \) is odd (via Lemma 14). The common extension \( g \) of all \( g_n \) is as required. \( \square \)

By virtue of Lemma 5, Theorem 1 can be recast in terms of \( \ell \)-ideal lattices of free vector lattices over \( k \), as follows.

**Theorem 2.** Let \( k \) be a countable totally ordered division ring, let \( I \) and \( J \) be countable sets with \( I \subset J \) and \( J \setminus I \) infinite, let \( K \) and \( L \) be distributive 0-lattices with \( L \) countable and completely normal, let \( \varphi : K \to L \) be a 0-lattice homomorphism, and let \( f : \text{Id}_c F_\ell I, k \to K \) be a 0-lattice homomorphism. Denote by \( \eta_{I,J} : \text{Id}_c F_\ell I, k \to \text{Id}_c F_\ell J, k \) the canonical embedding. Then there exists a surjective lattice homomorphism \( g : \text{Id}_c F_\ell J, k \to L \) such that \( g \circ \eta_{I,J} = \varphi \circ f \).

By using the functoriality of the assignment \( I \mapsto \text{Id}_c F_\ell I, k \), Theorem 2 can further be extended to diagrams indexed by trees, as follows.

**Theorem 3.** Let \( k \) be a countable totally ordered division ring, let \( T \) be a tree in which every element has countable height, and let \( L \overset{\text{def}}{=} (L_s, \varphi_{s,t} \mid s \leq t \text{ in } T) \) be a commutative \( T \)-indexed diagram of distributive 0-lattices such that \( L_t \) is countable and completely normal whenever \( t \in T \setminus \{ \bot \} \). Let \( I_{\bot} \subseteq \{ \bot \} \times \omega \) and
set $I_t \overset{\mathrm{def}}{=} (T \downarrow t) \times \omega$ whenever $t \in T \setminus \{ \bot \}$. Set $\bar{I} \overset{\mathrm{def}}{=} (I_s, \eta_{I_s, I_t} \mid s \leq t \text{ in } T)$. Then every 0-lattice homomorphism $\chi_\bot : \operatorname{Id}_c F_\ell(I_\bot, k) \to L_\bot$ extends to a natural transformation $\bar{\chi} : \operatorname{Id}_c F_\ell(\bar{I}, k) \to \bar{L}$ such that $\chi_t$ is a surjective lattice homomorphism whenever $t \in T \setminus \{ \bot \}$.

**Proof.** The proof can be partly followed on Fig. 5.

By Zorn’s Lemma, there exists a maximal lower subset $T'$ of $T$, containing $\{ \bot \}$, on which the conclusion of Theorem 3 holds. Suppose, by way of contradiction, that $T' \neq T$ and let $t$ be a minimal element of $T \setminus T'$; so $T' \cup \{ t \}$ is also a lower subset of $T$. Since the height of $t$ is countable, so are the lattice $L_{< t} \overset{\mathrm{def}}{=} \lim_{s < t} L_s$ (with transition maps $\varphi_{s, s'}$ where $s \leq s' < t$ and limiting maps $\varphi_{s, < t} : L_s \to L_{< t}$ for $s < t$) and the set $I_{< t} \overset{\mathrm{def}}{=} \bigcup \{ I_s \mid s < t \}$. The universal property of the colimit ensures the existence of unique 0-lattice homomorphisms

$$\eta_{I_{< t}, I_t} : \operatorname{Id}_c F_\ell(I_{< t}, k) = \lim_{s < t} \operatorname{Id}_c F_\ell(I_s, k) \to \operatorname{Id}_c F_\ell(I_t, k)$$

and $\varphi_{< t, t} : L_{< t} \to L_t$, such that $\eta_{I_{< t}, I_t} \circ \eta_{I_s, I_{< t}} = \eta_{I_s, I_t}$ and $\varphi_{< t, t} \circ \varphi_{s, < t} = \varphi_{s, t}$ whenever $s < t$. Further, the natural transformation $(\chi_s \mid s < t)$ induces a unique 0-lattice homomorphism

$$\chi_t : \operatorname{Id}_c F_\ell(I_{< t}, k) \to L_{< t}$$

such that $\chi_t \circ \eta_{I_s, I_{< t}} = \varphi_{s, t} \circ \chi_s$ whenever $s < t$. By Theorem 2, there exists a surjective lattice homomorphism $\chi_t : \operatorname{Id}_c F_\ell(I_t, k) \to L_t$ such that $\chi_t \circ \eta_{I_{< t}, I_t} = \varphi_{< t, t} \circ \chi_t$. Therefore, for each $s < t$,

$$\chi_t \circ \eta_{I_s, I_{< t}} = \chi_t \circ \eta_{I_{< t}, I_t} \circ \eta_{I_s, I_{< t}} = \varphi_{< t, t} \circ \chi_t \circ \eta_{I_s, I_{< t}} = \varphi_{< t, t} \circ \varphi_{s, < t} \circ \chi_s = \varphi_{s, t} \circ \chi_s.$$

This shows that our conclusion holds at $T' \cup \{ t \}$, in contradiction with the maximality assumption on $T'$.

□

This leads us to the following positive solution of the problem stated at the end of Wehrung [20].

**Theorem 4.** Let $k$ be a countable totally ordered division ring. Then every completely normal distributive 0-lattice $L$ with at most $\aleph_1$ elements is a surjective homomorphic image of $\operatorname{Id}_c F$ for some vector lattice $F'$ over $k$.
Proof. Write $L$ as the directed union of an ascending $\omega_1$-sequence $\bar{L} = (L_\xi \mid \xi < \omega_1)$ of countable completely normal distributive 0-lattices, with $L_0 = \{0\}$. Theorem 3, applied to the well-ordered chain $\omega_1$, yields an $\omega_1$-indexed commutative diagram $\bar{F} = (F_\xi, f_{\xi, \eta} \mid \xi \leq \eta < \omega_1)$ of $k$-vector lattices together with a natural transformation $\bar{\chi} : \text{Id}_c \bar{F} \to \bar{L}$ all of whose components are surjective lattice homomorphisms. Letting $F \overset{\text{def}}{=} \lim_{\rightarrow} \bar{F}$, the universal property of the colimit yields a surjective homomorphism from $\text{Id}_c F$ onto $L$.

Due to Wehrung [18, Corollary 9.5], Theorem 4 cannot be generalized to uncountable totally ordered division rings $k$. On the other hand, setting $k$ as any countable Archimedean totally ordered field (for example the rationals), $\text{Id}_c F$ is identical to the $\ell$-ideal lattice of the underlying $\ell$-group of $F$. Hence,

**Corollary 2.** Every completely normal distributive 0-lattice $L$ with at most $\aleph_1$ elements is a surjective homomorphic image of $\text{Id}_c F$ for some Abelian $\ell$-group $F$.

By applying Stone duality for distributive 0-lattices, we obtain the following formulation in terms of spectra.

**Corollary 3.** Every completely normal generalized spectral space with at most $\aleph_1$ compact open sets embeds, as a spectral subspace, into the $\ell$-spectrum of an Abelian $\ell$-group.

Corollary 2 also strengthens Ploščica [11, Theorem 3.2], which states that every completely normal distributive 0-lattice of cardinality at most $\aleph_1$ is Cevian; that is, it carries a binary operation $(x, y) \mapsto x \setminus y$ such that $x \leq y \lor (x \setminus y) \land (y \setminus x) = 0$, and $x \setminus z \leq (x \setminus y) \lor (y \setminus z)$ for all elements $x, y, z$. Indeed, $\text{Id}_c G$ is Cevian for any Abelian $\ell$-group $G$, and any homomorphic image of a Cevian lattice is Cevian (cf. Wehrung [16, § 5]).

As in the second author’s paper [15], say that a distributive 0-lattice has countably based differences if for all $a, b \in D$ there exists a countable subset $\{c_n \mid n < \omega\}$ of $D$ such that for all $x \in D$, $a \leq b \lor x$ if and only if there exists $n < \omega$ such that $c_n \leq x$. As observed in the second author’s paper [15], the lattice $\text{Id}_c G$ has countably based differences whenever $G$ is an Abelian $\ell$-group. The question thus arises whether every completely normal distributive 0-lattice $D$ with countably based differences is isomorphic to $\text{Id}_c G$ for some Abelian $\ell$-group $G$. The cases where $\text{card} D \leq \aleph_0$ and $\text{card} D \geq \aleph_2$ are settled in Wehrung [15, 16], in the positive and the negative, respectively; the counterexample constructed in [16] is not even Cevian, thus it is not a homomorphic image of any $\text{Id}_c G$. A Cevian counterexample (of size continuum plus) is constructed in Ploščica [11].

The authors’ recent preprint [12] solves the case where $\text{card} D = \aleph_1$, stated as an open problem at the end of [11]: a distributive 0-lattice $D$ of cardinality $\aleph_1$ is isomorphic to $\text{Id}_c G$ for some Abelian $\ell$-group $G$ iff it is completely normal and has countably based differences.

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Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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