An extension of the Burau representation to a mapping class group associated to Thompson’s group $T$

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Abstract

We study some aspects of the geometric representation theory of the Thompson and Neretin groups, suggested by their analogies with the diffeomorphism groups of the circle. We prove that the Burau representation of the Artin braid groups extends to a mapping class group $A_T$ related to Thompson’s group $T$ by a short exact sequence $B_\infty \hookrightarrow A_T \to T$, where $B_\infty$ is the infinite braid group. This non-commutative extension abelianises to a central extension $0 \to \mathbb{Z} \to A_T / [B_\infty, B_\infty] \to T \to 1$ detecting the discrete version $\overline{g_v}$ of the Bott-Virasoro-Godbillon-Vey class. A morphism from the above non-commutative extension to a reduced Pressley-Segal extension is then constructed, and the class $\overline{g_v}$ is realised as a pull-back of the reduced Pressley-Segal class. A similar program is carried out for an extension of the Neretin group related to the combinatorial version of the Bott-Virasoro-Godbillon-Vey class.

1 Introduction

The purpose of this work is to study some aspects of the geometric representation theory of the Thompson and Neretin groups. Strikingly enough, this turns out to be linked with the Burau representation of the classical braid groups.

Recall that Thompson’s group $T$ (cf. [3]) is the group of piecewise linear dyadic homeomorphisms of the circle; it is finitely presented and simple. Another way to look at its elements is as germs near the boundary of planar partial automorphisms of the binary tree (a planar partial automorphism is defined outside a finite subtree and preserves the cyclic order at each vertex). As for Neretin’s group $N$ (cf. [19]), it is defined as the the group of germs of all the partial automorphisms of the binary tree; it is an uncountable simple group (cf. [12]). Both groups $T$ and $N$ have been observed to present analogies with the group $\text{Diff}(S^1)$ of orientation-preserving diffeomorphisms of the circle. Thus, $T$ looks, up to some extent, as a “lattice” in $\text{Diff}(S^1)$, while $N$ as a $p$-adic (and combinatorial) analogue.

A remarkable point of this analogy concerns the Bott-Virasoro-Godbillon-Vey class. This is a two dimensional (continuous) cohomology class $g_v$ belonging to $H^2(\text{Diff}(S^1) ; \mathbb{R})$, whose derivative – a cohomology class of the Lie algebra $\text{Vect}(S^1)$ of vector fields on the circle – was introduced by Gelfand and Fuks; it corresponds to the Virasoro algebra, the universal central extension of
A relative class $\overline{\mathfrak{g}v}$ for the group $T$ was introduced and studied in [8] by E. Ghys and the second author. Note that it is a $\mathbb{Z}$-valued cohomology class. Moreover, a $\mathbb{Z}/2\mathbb{Z}$-valued class $\overline{GV}$ has been defined for $N$ by the first author ([13]).

A basic aspect of the Godbillon-Vey class is its relation with representation theory. Thus, Pressley and Segal ([24], Chapter 6) introduced representations of $\text{Diff}(S^1)$ in the restricted linear group $GL_{\text{res}}(L^2(S^1))$ (where $L^2(S^1)$ is the Hilbert space of square integrable functions of the circle). The class $e^{i\mathfrak{g}v}$ is then essentially the pull-back of a class in $H^2(GL_{\text{res}}(L^2(S^1)); \mathbb{C}^*)$, which can be described as follows: there is an extension

$$(NC)_{PS} 1 \rightarrow \mathfrak{T} \rightarrow \mathfrak{E} \rightarrow GL_{\text{res}}(L^2(S^1)) \rightarrow 1,$$

where $\mathfrak{T}$ is the group of determinant operators on the Hardy subspace $L^2(S^1)_+$. After dividing by the kernel of the determinant morphism from $\mathfrak{T}$ to $\mathbb{C}^*$, one obtains a central extension detecting the desired cohomology class.

By the machinery of the second quantization, the group $GL_{\text{res}}(\mathcal{H})$ of a Hilbert space $\mathcal{H}$ admits a projective representation in the fermionic Fock space $\Lambda(\mathcal{H})$: the spinor representation. Composed with the Pressley-Segal representations, it provides projective representations of $\text{Diff}(S^1)$. At the infinitesimal level, one obtains unitarisable highest weight modules of the Virasoro algebra (see [20] Chapter 7).

The first aim of this work is to construct similar representations for $T$ and $N$, inducing the classes $\overline{\mathfrak{g}v}$ and $\overline{GV}$ respectively. These classes have non-commutative versions introduced in [9] and [13] by P. Greenberg and the present authors; these are extensions

$$(NC)_T 1 \rightarrow B_\infty \rightarrow A_T \rightarrow T \rightarrow 1,$$

$$(NC)_N 1 \rightarrow \mathfrak{S}_\infty \rightarrow A_N \rightarrow N \rightarrow 1.$$

When divided by the commutators of $B_\infty$ and $\mathfrak{S}_\infty$, they produce central extensions which correspond to $\overline{\mathfrak{g}v}$ and $\overline{GV}$ respectively.

A further aim of this paper is to investigate the link of the above non-commutative extensions with the Pressley-Segal extension $(NC)_{PS}$. By non-commutative extensions we mean extensions of groups which provide central extensions after an abelianisation process. Before stating our main result, we emphasize that such non-commutative extensions appeared in various contexts (see [10], [15], [29] and [14]), and are related to interesting central extensions:

1. In [10], the authors study an extension of $\widetilde{PSL}_2(\mathbb{R})$ (the universal cover of $PSL_2(\mathbb{R})$) by a group of piecewise $PSL_2(\mathbb{R})$ homeomorphisms of $\mathbb{R}$. By abelianisation, they get the Steinberg extension from $K$-theory.

2. Tsuboi ([29]) considers an extension of $\text{Diff}(S^1)$ by a group of area preserving homeomorphisms of the unit disk $D^2$. He proves that after abelianisation one obtains the Euler class of $\text{Diff}(S^1)$.

3. In [14] one describes an action of the Neretin group $N$ on a tower of moduli spaces of real stable curves. Lifting this action to the universal cover provides a non-commutative extension of $N$ by the fundamental group of the tower. Its abelianisation is a non-trivial central extension of $N$ by $\mathbb{Z}/2\mathbb{Z}$.

4. Morita ([15]) looks at the modular group $\Gamma_g$ as an extension of the Torelli group by the Siegel modular group $Sp_{2g}(\mathbb{Z})$. The (non-central!) extension obtained by abelianisation captures the tautological classes of Morita and Mumford.

We stress that all these extensions have a geometrically defined middle term as well.
The construction of \((NC)_T\), as performed in [9], amounts to an embedding \(T \to \text{Out}(B_{\infty})\), which uses a version of \(B_{\infty}\) in which the points to braid are the vertices of an (extended) binary tree. This idea is further developed for \(N\), where the group \(A_N\) is defined in a more direct way [13]. In §3, we shall give a transparent geometric description for the middle term of the extension \((NC)_T\) as well. This makes use of the surface \(S_{\infty}\) which is the complement of a Cantor set in the sphere \(S^2\). We first prove (Theorem 3.3) that \(T\) embeds into the mapping class group of this surface. The new geometric version of \((NC)_T\) is then described in terms of mapping class groups (Definition-Proposition 3.7).

We now give a statement of our main result (Theorem 4.8), which also concerns the Burau representation of the braid groups (see [30] for an up-to-date overview including recent results on the faithfulness question):

**Main Theorem** – The Burau representation of the Artin braid groups (depending on a parameter \(t \in \mathbb{C}^*\)) extends to the mapping class group \(A_T\). More precisely, there exists a Hilbert space \(\mathcal{H}\) and a representation \(\rho^t : A_T \to \text{GL}(\mathcal{H})\) (in the group of bounded operators) such that \(B_{\infty}\) is represented in the subgroup \(\mathfrak{T}\) of determinant operators. This gives a morphism of non-commutative extensions

\[
\begin{array}{cccccc}
(NC)_T & 1 & B_{\infty} & A_T & T & 1 \\
(\mathfrak{T})_{ps} & 1 & \mathfrak{T} & GL(\mathcal{H}) & GL(\mathcal{H})/\mathbb{Z} & 1 \\
\end{array}
\]

inducing a morphism of central extensions

\[
\begin{array}{cccccc}
1 & \mathbb{Z} & A_T/\text{det} & T & 1 \\
1 & \mathbb{C}^* & GL(\mathcal{H})/\mathbb{Z} & GL(\mathcal{H})/\mathbb{Z} & 1 \\
\end{array}
\]

where \(\mathfrak{T}_1 \subset \mathfrak{T}\) is the kernel of the determinant morphism.

The Pressley-Segal extension \((NC)_{PS}\) is a pull-back of the non-commutative extension \((NC)_{ps}\) (see Definition-Proposition 4.4), which will be called the reduced Pressley-Segal extension in our work. We also obtain a quite similar theorem linking the extension \((NC)_N\) to the reduced Pressley-Segal extension \((NC)_{ps}\) (Theorem 4.5).

We would like to mention that several interesting connections between the Thompson (Neretin) groups and infinite dimensional groups and algebras have been recently developed.

A. Reznikov [25] (see also A. Navas [17]) showed that the group \(T\) has not the Kazhdan property. His argument uses in an essential way a 1-cocycle on the group \(\text{Diff}^{1+\alpha}(S^1)\) (into which \(T\) embeds, see [8]) with values in the Hilbert space of Hilbert-Schmidt operators, coming from the representation of \(\text{Diff}^{1+\alpha}(S^1)\) in \(GL_{\text{res}}(L^2(S^1))\) when \(\alpha > \frac{1}{2}\).

D. Farley [6] later proved that the Thompson groups \(T\) are a-T-menable (in other words, have the Haagerup property). This led to a proof of the Baum-Connes conjecture for these groups as well as to a new argument for Reznikov’s result.

Interesting representations of \(N\) appear in [19] and [21], and embeddings of the Thompson groups in the Cuntz-Pimsner algebra have been obtained by V. Nekrashevich [18] and J.-C. Birget [1]. Moreover, a \(C^*\)-algebra version of the class \(\mathfrak{f}\) has been studied by C. Oikonomides [22].

The article is organised as follows: Section 2 presents a convenient way the background concerning the Thompson and Neretin groups. Section 3 is mainly devoted to the construction of the group \(A_T\) as a mapping class group of a surface with infinitely many ends, but begins with a more...
elementary construction describing $T$ itself as a mapping class group (Theorem 3.3). In section 4, we recall the definition of the Pressley-Segal extension, introduce its reduced version, and prove the main theorems (Theorems 4.5, 4.8).

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2 Germ groups of inverse monoids: the Thompson and Nere
tin groups

2.1. The inverse monoid of Fredholm tree automorphisms. Let $T$ be a locally finite tree, $\partial T$ be its boundary at infinity with its usual topology, cf. e.g. [27]. A cofinite domain $D$ is the complement of a finite subgraph of $T$. It has finitely many components, which are subtrees of $T$. A Fredholm automorphism of $T$, or partial tree automorphism of $T$, is a bijection between two cofinite domains $D$ and $D'$ of $T$, which induces a tree isomorphism on each connected component of $D$. The set $Fred(T)$ of Fredholm automorphisms forms an inverse monoid (cf. [23]) with the obvious structure: if $g$ and $h$ in $Fred(T)$ are defined on $D_g$ and $D_h$ respectively, $g \cdot h$ is the Fredholm automorphism on the cofinite domain $D_{g \cdot h} := h^{-1}(h(D_h) \cap D_g)$ defined by restriction of $g \circ h$. The inverse $g^{-1}$ is defined on the cofinite domain $g(D_g)$.

For $g \in Fred(T)$, we call $D_g$ the source, and $g(D_g)$ the target.

The index of a Fredholm automorphism $g$ defined on the cofinite domain $D_g$ is the integer

$$ind g = \text{card}(\text{Vert}(T \setminus D_g)) - \text{card}(\text{Vert}(T \setminus g(D_g)))$$

where $\text{Vert}(G)$ denotes the set of vertices of a graph $G$.

The subset $Fred^0(T)$ of Fredholm automorphisms with null index is an inverse submonoid of $Fred(T)$.

2.2. The group germification. Since the source and target of a Fredholm automorphism $g$ are cofinite, their boundaries at infinity coincide with that of the tree $T$. Since $g$ acts by tree isomorphisms outside a finite subgraph, it induces a homeomorphism $\partial g$ of $\partial T$. The morphism of inverse monoids between $Fred(T)$ and the homeomorphism group of the boundary $\text{Homeo}(\partial T)$ of $T$ (considered as an inverse monoid)

$$g \in Fred(T) \mapsto \partial g \in \text{Homeo}(\partial T)$$

is the group germification map. Its image, denoted $\mathcal{G}(T)$, is the germ group of the inverse monoid $Fred(T)$. Similarly, we shall denote $\mathcal{G}^0(T)$ the image of $Fred^0(T)$.

2.3. Fundamental examples: the Neretin and Thompson groups.

2.3.1. Let $T_2$ be the regular tree whose vertices are all 3-valent ($T_2$ is called the dyadic tree, as its boundary may be identified with the projective line on the field of dyadic numbers $\mathbb{Q}_2$). Clearly, $Fred(T_2) = Fred^0(T_2)$, and the associated germ group is Neretin’s spheromorphism group $N$ defined in [19] (cf. also [11], [12], [13]).

2.3.2. Let $T$ be the planar rooted tree, whose root is 2-valent, while the other vertices are 3-valent. Each vertex inherits a local orientation from the orientation of the plane: the root being at the top of the tree, each vertex $v$ has two descendants (or sons), the left one $\alpha_l(v)$ and the right one $\alpha_r(v)$.

We shall say that $T$ is a finite dyadic rooted subtree of $T$ (f.d.r.s.t. for short) if it is a finite subtree of $T$, rooted in the root of $T$, such that its vertices are 3-valent or 1-valent (except the root, which
is 2-valent). The 1-valent vertices are called the leaves of $T$. We denote the set of leaves of $T$ by $\mathcal{L}(T)$ and say that the type of $T$ is $k$ if it has $k$ leaves.

The canonical labelling of a f.d.r.s.t whose type is $k \in \mathbb{N}^+$ is the list of its leaves $v_0, \ldots, v_{k-1}$, enumerated leftmost first, and reading from left to right (cf. Figure 1a).

**Definition 2.1 (Thompson’s group $T$, see also [3]).** Let $Fred^+(T)$ be the inverse submonoid of $Fred(T) = Fred^d(T)$ consisting of planar partial automorphisms, that is, $g$ belongs to $Fred^+(T)$ if there exist two f.d.r.s.t. $T_0$ and $T_1$ of $T$ such that

1. $D_g := (T \setminus T_0) \cup \mathcal{L}(T_0)$ is the source of $g$;
2. $g((T \setminus T_0) \cup \mathcal{L}(T_0)) = (T \setminus T_1) \cup \mathcal{L}(T_1)$;
3. $g$ induces a cyclic bijection $\mathcal{L}(T_0) \rightarrow \mathcal{L}(T_1)$ in the following sense: if $(v_0, \ldots, v_{k-1})$ and $(w_0, \ldots, w_{k-1})$ are the canonical labellings of the leaves of $T_0$ and $T_1$ respectively, there exists a cyclic permutation $\sigma$ of the set $\{0, \ldots, k-1\}$ identified with $\mathbb{Z}/k\mathbb{Z}$ such that $g(v_i) = w_{\sigma(i)}$ for $i = 0, \ldots, k-1$. The cyclicity of $\sigma$ means that there exists some $i_0 \in \mathbb{Z}/k\mathbb{Z}$ such that for all $i$, $\sigma(i) = i + i_0 \mod k$;
4. for each vertex $v$ of $D_g$, $g(\alpha_i(v)) = \alpha_i(g(v))$, $i = l, r$.

Thompson’s group $T$ is then defined as the germ group of $Fred^+(T)$.

**Symbols.** Each $g \in Fred^+(T)$ may be uniquely represented by a symbol $(T_1, T_0, \sigma)$, with the notations of the above definition. We denote by $[T_1, T_0, \sigma]$ the element of $T$ which is the germ of $g$. We have the following composition rule in $T$:

$$[T_2, T_1, \sigma][T_1, T_0, \tau] = [T_2, T_0, \sigma \circ \tau].$$

**Example 2.2.** Figure 1a represents a symbol coupled with a cyclic permutation $\sigma$.

**Definition 2.3 (Thompson’s group $F$, see also [3]).** Thompson’s group $F$ is the germ group of the inverse submonoid of $Fred^+(T)$ whose elements are represented by symbols $(T_1, T_0, \sigma)$, where $\sigma$ is the identity. It is a subgroup of $T$.

**2.4. Piecewise tree automorphisms.** If $U$ is a null index Fredholm operator of a Hilbert space $H$, we may find a compact operator $K$ such that $U + K$ is an invertible operator. Analogous considerations in the present context lead to the following:

**Definition-Proposition 2.4.** Let $Bij(T^0)$ be the group of bijections on the set of vertices $T^0$ of the tree $T$. A piecewise tree automorphism of $T$ is a bijection $g \in Bij(T^0)$ which induces a Fredholm automorphism outside a finite subset of $T^0$. The set of piecewise tree automorphisms of $T$ forms a subgroup of $Bij(T^0)$, denoted $PAut(T)$.

For each $g \in Fred^0(T)$, there exists $h \in PAut(T)$ inducing the same germ as $g$ in the boundary $\partial T$.
Denote by $\mathcal{S}(T^0)$ the group of finitely supported permutations on the set of vertices $T^0$. The fundamental objects resulting from the preceding considerations are a “non-commutative” extension of the germ group $G^0(T)$ together with an associated central extension:

**Proposition 2.5.** There is a short exact sequence of groups

$$1 \to \mathcal{S}(T^0) \to PAut(T) \to G^0(T) \to 1$$

called the non-commutative extension of the germ group $G^0(T)$. Denote by $\mathfrak{A}(T^0)$ the alternating subgroup of $\mathcal{S}(T^0)$. After dividing by $\mathfrak{A}(T^0)$, the non-commutative extension provides a central extension of $\hat{G}^0(T)$:

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \hat{G}^0(T) \to G^0(T) \to 1,$$

where $\hat{G}^0(T)$ is the quotient group $PAut(T)/\mathfrak{A}(T^0)$.

**Proof.** The verification that $\mathfrak{A}(T^0)$ is normal in $PAut(T)$ is easy. Note that any extension by $\mathbb{Z}/2\mathbb{Z}$ is central.

2.5. Discrete and combinatorial analogues of the Bott-Virasoro-Godbillon-Vey class.

The successful application of the preceding proposition to the Neretin group $N = G^0(T_2)$ (cf. §2.3.1) is the following

**Theorem 2.6.** (Combinatorial (or dyadic) analogue of the Bott-Virasoro-Godbillon-Vey class for the Neretin group, cf. [13]) The central extension of the Neretin spheromorphism group is non-trivial and defines a class $GV$ in $H^2(N, \mathbb{Z}/2\mathbb{Z})$.

However, the analogous construction for Thompson’s group $T$ is trivial. Indeed, at the price of slightly modifying the tree $T$ to a tree $\tilde{T}$ (cf. Remark 2.9 below), we prove that the projection $PAut(\tilde{T}) \to \hat{G}(\tilde{T})$ splits over $T$. The construction of the splitting $T \to PAut(\tilde{T})$ relies on the interpretation of $T$ as a piecewise affine homeomorphism group of the circle that we now recall:

**Thompson’s group $T$ as a group of piecewise dyadic affine homeomorphisms of the circle, cf. e.g. [3].** Label inductively the edges of the tree $T$ by all the dyadic intervals of $[0, 1]$ in the following way: label the left descending edge of the root by $[0, \frac{1}{2}]$, the right one by $[\frac{1}{2}, 1]$. If an edge labelled by $[\frac{k}{2^n}, \frac{k+1}{2^n}]$ gives birth to two descending edges, the left one will be labelled by the first half $[\frac{k}{2^n}, \frac{2k+1}{2^{n+1}}]$, and the right one by the second half $[\frac{2k+1}{2^{n+1}}, \frac{k+1}{2^n}]$.

Label also the vertices by the dyadic rationals of $[0, 1]$ inductively: the root corresponds to $1/2$, and if an edge is labelled by $[\frac{k}{2^n}, \frac{k+1}{2^n}]$, label its bottom vertex by the middle of the interval, namely $\frac{2k+1}{2^{n+1}}$, cf. Figure 2.
Definition-Proposition 2.7. Let \( g \in T \) be given by a symbol \((T_1, T_0, \sigma)\), with \( \sigma \) a cyclic bijection of \( \mathbb{Z}/k\mathbb{Z} \) when the trees \( T_0 \) and \( T_1 \) have \( k \) leaves. Denote by \( I_0, \ldots, I_{k-1} \) (resp. \( J_0, \ldots, J_{k-1} \)) the dyadic intervals corresponding to the terminal edges of \( T_0 \) (resp. \( T_1 \)). The unique piecewise affine map of \([0, 1]\) applying affinely and increasingly \( I_i \) onto \( J_{n(i)} \) for all \( i = 0, \ldots, k-1 \), only depends on \( g \), not on the symbol. It induces an orientation-preserving homeomorphism \( \phi_g \) of the circle, viewed as \([0, 1]/0 \sim 1\). The correspondence \( g \mapsto \phi_g \in \text{Homeo}^+(S^1) \) is a morphism, which embeds \( T \) into the homeomorphism group of the circle.

Remark 2.8. On each dyadic interval \( I_i \) of the subdivision, \( \phi_g \) is the restriction of an affine map of the form \( x \mapsto 2^{n_i}x + \frac{p_i}{2^{n_i}} \), \( n_i \in \mathbb{Z}, p_i \in \mathbb{Z}, q_i \in \mathbb{N} \).

Remark 2.9. Add an edge \( e_0 \) to the tree \( T \), linked to the root of \( T \), and label its terminal vertex by \( 0 \sim 1 \). Denote by \( \tilde{T} \) the resulting tree. Thus, the vertices of \( \tilde{T} \) are in bijection with the dyadic rationals of \([0, 1]/0 \sim 1\). Each \( g \in T \) induces a bijection \( \hat{g} \) on the set of vertices \( \tilde{T}^0 \) via the action of \( \phi_g \) on the set of dyadic rationals. The bijection \( \hat{g} \) is a piecewise tree automorphism, whose germ coincides with \( g \), and the correspondence \( g \in T \mapsto \hat{g} \in \text{PAut}(\tilde{T}) \) is a morphism.

Example 2.10. Figure 1b represents \( \phi_g \) for the element \( g \) of \( T \) defined on Figure 1a.

In [8], the second cohomology group \( H^2(T, \mathbb{Z}) \) is proved to be free abelian on two generators \( \chi \) and \( \alpha \), where \( \chi \) is the Euler class of \( T \), and \( \alpha \) corresponds to the Godbillon-Vey class. In order to recall the formula of the cocycle associated with \( \alpha \), we need to define three functions on \( S^1 = [0, 1]/0 \sim 1 \) associated with an element \( g \in T \):

Definition 2.11. Let \( g \) be an element of \( T \). For \( x \in S^1 = [0, 1]/0 \sim 1 \), \( g'_r(x) \) (resp. \( g'_l(x) \)) is the left (resp. right) derivative number of the affine bijection \( \phi_g \) (it is an integral power of 2), and \( \Delta \log_2 g'_r(x) \) is the integer \( \log_2 g'_r(x) - \log_2 g'_l(x) \).

Theorem 2.12. (Discrete analogue of the Bott-Thurston cocycle for Thompson’s group \( T \), cf. [8]) The function \( \overline{gT} : T \times T \to \mathbb{Z} \) defined by

\[
\overline{gT}(g, h) = \sum_{x \in S^1} \begin{vmatrix} \log_2 h'_r & \log_2 (g \circ h)'_r \\ \Delta \log_2 h'_r & \Delta \log_2 (g \circ h)'_r \end{vmatrix}(x)
\]

is a cocycle whose cohomology class equals \( 2\alpha \). Here \( \begin{vmatrix} a & b \\ c & d \end{vmatrix} \) is the determinant \( ad - bc \).

Note that the sum in the above formula is finite.

In the next section, we shall build geometrically a non-commutative extension of Thompson’s group \( T \) (by a braid group) which abelianises to a central extension by \( \mathbb{Z} \) detecting the class \( \alpha \). This will be a braided version of the non-commutative extension of Neretin’s group (Proposition 2.5 and §2.5).

3 Geometric group extensions related to the genus zero infinite surface

3.1. The infinite surface. Let \( S_\infty \) be the genus zero infinite surface, constructed as an inductive limit of finite subsurfaces \( S_n \): \( S_1 \) is a compact cylinder, with two boundary components, and \( S_{n+1} \) is obtained from \( S_n \) by gluing a copy of a “pair of pants” (that is, a compact surface of genus zero with three boundary components) along each boundary component of \( S_n \) (homeomorphic to a circle). It follows that for each \( n \geq 1 \), \( S_n \) is a \( 2^n \)-holed sphere, and \( S_\infty = \lim_{n \to \infty} S_n \). The surface...
\(\bar{S}_{\infty}\) is oriented, and a homeomorphism of the surface will always be supposed to be orientation-preserving, unless the opposite is explicitly stated.

**Pants decomposition and rigid structure:** By this construction, \(\bar{S}_{\infty}\) is naturally equipped with a pants decomposition, which will be referred to in the sequel as the *canonical decomposition*. We introduce a *rigid structure* on \(\bar{S}_{\infty}\), consisting of three disjoint *seams* on each pair of pants, and two disjoint seams on the cylinder \(\bar{S}_1\), as indicated on Figure 3: a seam (represented by a dotted line on Figure 3) is homeomorphic to a segment, and connects two boundary circles of the pair of pants or the cylinder; each pair of boundary circles of a pair of pants is connected by a unique seam; each seam extends continuously to the seams of the adjacent pairs of pants or cylinder.

We fix also a marked point, or puncture, on each pair of pants as well as on the cylinder. All marked pairs of pants are homeomorphic, by homeomorphisms which respect the seams and the marked points. We call *rigid* such homeomorphisms.

We shall denote by \(S_{\infty}\) the surface \(\bar{S}_{\infty}\) decorated with the punctures.

3.2. **Associativity homeomorphisms.** We shall suppose that \(S_{\infty}\) is lying on an oriented plane, in such a way that the punctures are drawn on the “visible side” of the surface, and the seams separate the visible side from the hidden side. In other words, the visible and hidden sides are the connected components of the complement in \(S_{\infty}\) of the union of the seams.

The *tree of the surface* \(\bar{S}_{\infty}\) (or \(S_{\infty}\)) is the rooted dyadic tree drawn on the visible side of the surface, whose vertices are the punctures of \(\bar{S}_{\infty}\), and edges are transverse to the circles of the canonical pants decomposition of \(\bar{S}_{\infty}\). It will be identified with the rooted dyadic planar tree \(T\) of §2.3.2.

By a *finite subsurface* of \(S_{\infty}\) we shall always mean a connected finite union of pair of pants of the infinite surface, together with the cylinder. To each finite subsurface \(S\) corresponds a f.d.r.s.t. \(T_S\) (cf. §2.3.2), which is the subtree of \(T\) whose internal vertices are the punctures of \(S\). We call it the *tree of the surface* \(S\).

The canonical labelling of the leaves of the tree of a finite subsurface \(S\) (cf. §2.3.2) provides a canonical labelling of the set of boundary components of \(S\). We say that \(S\) has *type* \(k \in \mathbb{N}^*\) if it has \(k\) boundary components. Equivalently, the type of \(S\) is the type of its tree (cf. §2.3.2).

**Definition-Proposition 3.1 (associativity homeomorphisms).** Let \(S_0\) and \(S_1\) be two finite subsurfaces of \(S_{\infty}\) having the same type \(k \in \mathbb{N}^*\), and \(i\) an integer such that \(0 \leq i \leq k - 1\). Up to isotopy, there exists a unique homeomorphism \(\gamma_{1,0}^i : S_0 \to S_1\) mapping the visible side of \(S_0\) onto the visible side of \(S_1\), and the 0\textsuperscript{th} boundary component of \(S_0\) onto the \(i\textsuperscript{th}\) boundary component of \(S_1\).

**Proof.** Cutting \(S_0\) along the seams, one gets two \(2k\)-gons, which are the visible side \(S_0^v\) and the hid-
den side $S_0^h$. Among the sides of the $2k$-gon $S_0^v$, $k$ of them correspond to the boundary components of $S_0$, from which they inherit the same labelling. Call them the distinguished sides of $S_0^v$. Do the same with $S_1$. Since $\gamma_{1,0}^i : S_0 \to S_1$ must preserve the visible and hidden sides, we need to define its restrictions $\gamma_{1,0}^{i,v} : S_0^v \to S_1^v$ and $\gamma_{1,0}^{i,h} : S_0^h \to S_1^h$. But clearly, there is a unique $\gamma_{1,0}^{i,v} : S_0^v \to S_1^v$ which is orientation-preserving, and maps the distinguished side 0 of $S_0^v$ onto the distinguished side 0 of $S_1^v$. This forces the side $j$ to be mapped on the side $j + i \mod k$ (cf. the example of Figure 4: $k = 3$, $i = 2$). In the same way, define $\gamma_{1,0}^{i,h} : S_0^h \to S_1^h$ compatible with $\gamma_{1,0}^{i,v}$ along the boundaries of the $2k$-gons. Thus, $\gamma_{1,0}^{i,v}$ and $\gamma_{1,0}^{i,h}$ agree to induce the expected homeomorphism $\gamma_{1,0} : S_0 \to S_1$. Its unicity up to isotopy is clear.

3.3. Thompson’s group $T$ is a mapping class group.

**Definition 3.2.** A homeomorphism $\gamma$ of the surface $\bar{S}_\infty$ is asymptotically rigid if there exist two finite subsurfaces $S_0$ and $S_1$ of $\bar{S}_\infty$, having the same type, such that the restriction $\gamma : \bar{S}_\infty \setminus S_0 \to \bar{S}_\infty \setminus S_1$ is rigid, that is, maps rigidly a pair of pants onto a pair of pants.

**Theorem 3.3.** Thompson’s group $T$ embeds into the mapping class group of the surface $\bar{S}_\infty$ as the group of isotopy classes of those homeomorphisms representable by asymptotically rigid homeomorphisms which preserve the visible side of $\bar{S}_\infty$.

**Proof.** Let $(T_1, T_0, \sigma)$ be a symbol defining an element $g \in T$: $T_0$ and $T_1$ are $k$-ary trees, and $\sigma \in \mathbb{Z}/k\mathbb{Z}$ prescribes a cyclic bijection from the set of leaves of $T_0$ to the set of leaves of $T_1$. Let $S_0$ and $S_1$ be the finite subsurfaces of $\bar{S}_\infty$ whose associated trees are respectively $T_0$ and $T_1$. If $i = \sigma(0)$, let $\gamma_{1,0}^i : S_0 \to S_1$ be the associativity homeomorphism defined in §3.2. Extend it to the unique homeomorphism $\gamma_g : \bar{S}_\infty \to \bar{S}_\infty$ which is rigid outside $S_0$. We claim that $\gamma_g$ only depends on $g$, not on the choice of the symbol. Indeed, since two symbols $(T_1, T_0, \sigma)$ and $(T_1', T_0', \sigma')$ defining the same $g \in T$ always possess a common refining symbol $(T_1'', T_0'', \sigma'')$ (that is, such that the dyadic subdivision defined by $T_1''$ is a common refinement of the subdivisions defined by $T_0$ and $T_0'$ respectively), it is sufficient to suppose that $(T_1, T_0, \sigma')$ is a refinement of $(T_1, T_0, \sigma)$, and by induction, that $(T_1', T_0', \sigma')$ is a simple refinement of $(T_1, T_0, \sigma)$ (that is, $T_0'$ has just one more leaf than $T_0$). Thus, the surface $S_0'$ associated with $T_0'$ is the connected sum of $S_0$ with a pair of pants glued at some $j$th boundary component, while $S_1'$ associated with $T_1'$ is the connected sum of $S_1$ with a pair of pants glued at its $(j + i)$th (mod $k$) boundary component. Denote by $\gamma' : \bar{S}_\infty \to \bar{S}_\infty$ the asymptotically rigid homeomorphism which rigidly extends the associativity homeomorphism $\gamma_{1,0}^i : S_0' \to S_1'$, where $i' = \sigma'(0)$. At the price of replacing a homeomorphism by an isotopically equivalent one, we may suppose that $\gamma_g|_{S_0'} = \gamma_{1,0}^i$, hence $\gamma_g|_{S_0} = \gamma_g|_{S_0'}$. Since by rigidity $\gamma_g'$ and $\gamma_g$ coincide outside $S_0'$, they coincide everywhere.

The unicity of $\gamma_{1,0}$ implies that the correspondence $g \in T \mapsto \gamma_g \in MC(\bar{S}_\infty)$ (where $MC(\bar{S}_\infty)$ denotes the mapping class group of the surface $\bar{S}_\infty$) is a morphism: if $g, g'$ belong to $T$, we may represent $g$ by the symbol $(T', T, \sigma)$ and $g'$ by the symbol $(T'', T', \sigma')$. It follows that $(T'', T, \sigma' \circ \sigma)$ is a symbol for $g'g$. But clearly, $\gamma_{g'} \circ \gamma_g$ restricted to the finite subsurface $S_T$ induces an associativity homeomorphism associated with the symbol $(T'', T, \sigma' \circ \sigma)$, and it follows that $\gamma_{g'} \circ \gamma_g = \gamma_{g'g}$. 

---

**Figure 4:** Construction of an associativity homeomorphism
Remark 3.4. The authors of [5] prove a similar theorem for Thompson’s group $F$.

3.4. Extension of Thompson’s group $T$ by an infinite braid group.

3.4.1. The infinite surface with tubes. We want to build a new surface $S_{\infty,t}$ by gluing some infinite tubes on $\overline{S}_\infty$. Recall that $\overline{S}_\infty$ has a visible and a hidden sides. In other words, there is an involutive homeomorphism $j$ of $\overline{S}_\infty$, reversing the orientation, stabilizing the circles of the canonical pants decomposition, whose set of fixed points (the union of seams) bound two components of $\overline{S}_\infty$ (the visible and hidden sides).

Let $P$ be a pair of pants of $\overline{S}_\infty$. Consider the seam $s_P$ of $P$ connecting the two circles of its boundary which belong to the boundary of the minimal finite subsurface containing both $P$ and the cylinder $\overline{S}_1$. Cut a small $j$-invariant disk on $P$ overlapping the seam $s_P$, and along the boundary of the resulting hole, glue an infinite tubular surface. Cut also two small $j$-invariant disks on the cylinder $\overline{S}_1$, each overlapping one of its two seams, and glue similarly two tubes along the resulting holes.

Each tube is viewed as a connected countable union of compact cylinders. We add a puncture on each boundary component of those cylinders, in such a way that the punctures are aligned along the tube (see Figure 5). If $P$ is a pair of pants of the surface $\overline{S}_\infty$ (resp. the cylinder $\overline{S}_1$), denote by $t_P$ (resp. $t_P$ and $t_0$) the tubular surface (resp. surfaces) glued on $P$. The puncture (resp. punctures) on the basis of $t_P$ (resp. $t_P$ and $t_0$) is (resp. are) denoted by $v_P$ (resp. $v_P$ and $v_0$). We call the line passing by all the punctures the fibre of the tube, and denote it by $f_{v_P}$ ($f_{v_0}$). We denote by $S_{\infty,t}$ the infinite surface with tubes and punctures (see Figure 6).

The circles bounding the pants or the subcylinders of the tubes are called the circles of the pants-with-tubes decomposition of $S_{\infty,t}$.

Finally we may extend $j$ to an involutive homeomorphism $j_t$ of $S_{\infty,t}$ (reversing the orientation and stabilizing the circles of the pants-with-tubes decomposition of $S_{\infty,t}$) and define the visible side of $S_{\infty,t}$ as one of the two components bounded by the set of fixed points of $j_t$. In particular, each tube has a visible side, mapped by $j_t$ onto its hidden side, and we assume that all the fibres belong to the visible side.

Tree of the surface $S_{\infty,t}$. We draw the tree $\tilde{T}$ (see Remark 2.9) on the visible side of $S_{\infty,t}$, with vertices the punctures lying on the bases of the tubes of the surface. The vertices of the added edge $e_0$ defined in Remark 2.9 are the punctures of the bases of the two tubes glued on $\overline{S}_1$. Thus, $v_0$ is the 1-valent vertex of $e_0$ in $\tilde{T}$.

The union of $\tilde{T}$ with all the fibres of the tubes is a tree $T_t$ embedded in the visible side of the surface, whose vertices are the punctures of $S_{\infty,t}$ (see Figure 6). We call $T_t$ the tree of the surface $S_{\infty,t}$ and denote by $d$ its combinatorial metric.

Paths $\delta_v$. Choose a base point $*$ on the surface $S_{\infty,t}$, say, at the middle of the edge $e_0$. For each
puncture \(v\) of \(S_{\infty,t}\), that is, each vertex of the tree \(T_t\), denote by \(\delta_v\) the geodesic path from \(*\) to \(v\) contained in \(T_t\) (for its metric \(d\)).

3.4.2. Asymptotically rigid homeomorphisms of the surface \(S_{\infty,t}\).

By a finite subsurface of \(S_{\infty,t}\) we shall mean a connected compact subsurface of \(S_{\infty,t}\) containing the cylinder \(S_1\) and bounded by circles of the pants-with-tube decomposition. The bi-type of a finite subsurface is the couple of integers \((k,l)\), where \(k\) is the number of circles of the boundary components of \(S\) which are boundaries of pants, and \(l\) the number of punctures on \(S\). We check easily that \(k\) is also the number of boundary components which are circles of tubes, so that \(S\) has exactly \(2k\) boundary components.

If \(S\) is a connected subsurface of \(S_{\infty,t}\) (finite or not), the tree of \(S\) is the trace of the tree \(T_t\) on \(S\). A homeomorphism \(a\) of \(S_{\infty,t}\) which permutes the punctures of \(S_{\infty,t}\) is called asymptotically rigid if it verifies the following condition: there exist two finite subsurfaces \(S_0\) and \(S_1\) with the same bi-type, such that \(a\) restricts to each connected component \(C\) of \(S_{\infty,t} \setminus S_0\) to a rigid homeomorphism on its image, that is, a homeomorphism mapping the visible side onto the visible side, circles (of the canonical decomposition) onto circles, and the tree of \(C\) onto the tree of its image.

Germ of an asymptotically rigid homeomorphism. It follows from the definition that an asymptotically rigid homeomorphism \(a\) induces a piecewise tree automorphism of \(T_t\) (cf. §2.4). This in turn induces a homeomorphism of the boundary of the tree \(T_t\), called the germ of \(a\), and denoted by \(\partial a\).

3.4.3. The embedding \(T \hookrightarrow \text{Homeo}(\partial T_t)\). Let \(g\) be an element of \(T\); as explained in section 2, Remark 2.3, \(g\) acts on the set of vertices of \(\tilde{T}\), identified with the set of dyadic rationals of \([0,1]/0 \sim 1\). For any vertex \(v\) of \(\tilde{T}\) labelled by the dyadic rational \(x\), denote the integer \(\Delta \log_2 g'_v(x)\) simply by \(g''(v)\). Note that for almost all \(x \in [0,1]/0 \sim 1\), \(\Delta \log_2 g'_v(x) = 0\), and we may check
that $\sum x \Delta \log_2 g'_r(x) = 0$, i.e. $\sum g''(v) = 0$.

**Proposition 3.5.** There is an embedding $T \hookrightarrow \text{Homeo}(\partial T_i)$, $g \mapsto \partial_{T_i} g$, such that each $\partial_{T_i} g$ is induced by a Fredholm automorphism with null index of $T_i$.

**Proof.** If $g$ belongs to $T$, it is possible to extend the bijection $\hat{g}$ induced by $g$ on the set of vertices of $\hat{T}$ (cf. Remark 3.4) to a Fredholm tree automorphism $\hat{g}_{T_i}$ of $T_i$ in the following way:

Let $N = \max_{v \in \hat{T}_0} |g''(v)| + 1$, and for each vertex $v$ of $\hat{T}$, denote by $f^N_v$ the subfibre of $f_v$ whose vertices $s$ verify $d(v, s) \geq N$.

1. If $g''(v) = 0$, define $\hat{g}_{T_i}$ on $f_v$ as the unique isometric bijection from $f_v$ to $f_{\hat{g}(v)}$:

2. if $g''(v) \neq 0$, define $\hat{g}_{T_i}$ on $f^N_v$ as the unique isometric bijection from $f^N_v$ to $f^N_{\hat{g}(v)}$.

Thus defined, $\hat{g}_{T_i}$ is a Fredholm automorphism of $T_i$, and its index is null, thanks to the identity $\sum_{v \in \hat{T}_0} g''(v) = 0$. The germ of $\hat{g}_{T_i}$, acting on the boundary of $T_i$, only depends on $g$, and is denoted $\partial_{T_i} g$. The correspondence $g \in T \mapsto \partial_{T_i} g \in \text{Homeo}(\partial T_i)$ is easily seen to be a group morphism, thanks to the chain rule $(g \circ h)''(v) = g''(h(v)) + h''(v)$.

**Remark 3.6.** If $T_0$ and $T_1$ are finite subtrees of $T_i$ such that $\hat{g}_{T_i}$ restricts to a bijection $T_i \setminus T_0 \to T_i \setminus T_1$, then $\hat{g}_{T_i}$ induces a cyclic bijection $\mathcal{L}(T_0) \to \mathcal{L}(T_1)$ between the set of leaves of $T_0$ and $T_1$: the notion of cyclic bijection is quite similar to that given in Definition 2.3., since we may view $T_i$ (embedded in the visible side of the infinite surface) as a planar tree.

**3.4.4. The mapping class group $A_T$.** Denote by $\text{Homeo}^+(\mathcal{S}_{\infty, t})$ the group of orientation-preserving homeomorphisms of the surface $\mathcal{S}_{\infty, t}$ which permute the punctures.

**Definition-Proposition 3.7 (Mapping class group $A_T$).** Let $A_T$ be the set of isometry classes of homeomorphisms in $\text{Homeo}^+(\mathcal{S}_{\infty, t})$ which:

1. preserve the visible side of $\mathcal{S}_{\infty, t}$;

2. are asymptotically rigid in the sense of §3.4.2;

3. induce partial tree automorphisms of the tree $T_i$, whose germs belong to $T \hookrightarrow \text{Homeo}(\partial T_i)$, cf. §3.4.3.

The set $A_T$ is a group, and there is a short exact sequence

$$1 \to B_\infty[\mathcal{T}_i] \longrightarrow A_T \longrightarrow T \to 1,$$

where $B_\infty[\mathcal{T}_i]$ denotes the mapping class group generated by the isometry classes of half-twists along the edges of $T_i$, that is, the braid group on the (planar) set of punctures of $\mathcal{S}_{\infty, t}$.

**Remark 3.8.** The tree $T_i$ being planar, a presentation of $B_\infty[\mathcal{T}_i]$ may be deduced from [20], with generators the half-twists between all pairs of consecutive vertices of the tree $T_i$, along the edge which connects them.

**Remark 3.9.** The group $A_T$ is defined in [23] in a combinatorial way, by construction of a morphism $T \to \text{Out}(B_\infty[\mathcal{T}_i])$. The main theorem of [23] asserts that the group $A_{F'}$, (where $F'$ is the derived subgroup of $F$), which results from the restriction of the extension to $F'$, is acyclic.
Proof. The set $\mathcal{A}_T$ is clearly a group and the natural map $\mathcal{A}_T \to T$ a morphism. It remains to prove its surjectivity: let $g$ be in $T$, $\hat{g}_T$ the Fredholm automorphism of $T_t$ which induces $g$ in the boundary of $T_t$. Since $\hat{g}_T$ has null index, we may find two finite subsurfaces $S_0$ and $S_1$ with the same bi-type $(k,l)$, and a rigid homeomorphism $g_{0,1} : S_{\infty,t} \setminus S_0 \to S_{\infty,t} \setminus S_1$ (thus, preserving the visible sides), which induces $g_T$. We need to extend $g_{0,1}$ to a $S_0$, such that it maps its visible side onto the visible side of $S_1$.

For $i = 0$ or $1$, denote by $T_i$ the tree of the surface $S_i$, and by $\mathcal{L}(T_i)$ its set of leaves. The $2k$ boundary circles of $S_i$ are labelled by the leaves of $T_i$. The point is that $g_{0,1}$ maps the circles of $\partial S_0$ onto the circles of $\partial S_1$ by the prescription of the bijection $\mathcal{L}(T_0) \to \mathcal{L}(T_1)$ induced by $\hat{g}_T$. Since the latter is cyclic, we may proceed as in the proof of Proposition 4.1 to construct the extension of $g_{0,1}$ respecting the visible sides. The isotopy class of the resulting global homeomorphism of $S_{\infty,t}$ belongs to $\mathcal{A}_T$ and its germ is $g \in T$.

An element of the kernel of the projection $\mathcal{A}_T \to T$ may be represented by a homeomorphism supported in a finite subsurface, preserving its visible side and fixing pointwise its boundary circles. Thus, we may suppose that its compact support is contained in the visible side. Since the homeomorphism can only permute the punctures of the support, its isotopy class may be identified with an Artin braid on the punctures of the support. It follows that the kernel is the union (or inductive limit) of the Artin braid groups defined for any such compact support. \qed

4 Pressley-Segal type representations of the geometric group extensions

4.1 The Pressley-Segal extension

4.1.1. Restricted linear group of a polarised Hilbert space. Let $\mathcal{H}$ be a polarised separable Hilbert space, that is, the orthogonal sum of two isometric separable Hilbert spaces $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Let $J$ be the operator $J = id_{\mathcal{H}_+} \oplus -id_{\mathcal{H}_-}$. The restricted algebra of bounded operators of $\mathcal{H}$ is

$$\mathcal{L}_{\text{res}}(\mathcal{H}) = \{ A \in \mathcal{L}(\mathcal{H}) \mid [J,A] \in \mathcal{L}_2(\mathcal{H}) \}.$$ 

Here $\mathcal{L}(\mathcal{H})$ is the algebra of bounded operators of $\mathcal{H}$, $\mathcal{L}_2(\mathcal{H})$ the ideal of Hilbert-Schmidt operators, endowed with the usual Hilbert-Schmidt norm $\| \cdot \|_2$. The restricted algebra is a Banach algebra for the norm $\| A \| = \| A \| + \| [J,A] \|_2$.

Definition 4.1 (cf. [24]). The restricted linear group $GL_{\text{res}}(\mathcal{H})$ is the group of units of the algebra $\mathcal{L}_{\text{res}}(\mathcal{H})$. It is a Lie-Banach group.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the block decomposition of $A \in \mathcal{L}(\mathcal{H})$ relative to the direct sum $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. The operator $[J,A]$ is Hilbert-Schmidt if and only if $b$ and $c$ are Hilbert-Schmidt operators. If $A$ belongs to $GL_{\text{res}}(\mathcal{H})$, the invertibility of $A$ implies that $a$ is Fredholm in $\mathcal{H}_+$, and has an index $\text{ind}(a) \in \mathbb{Z}$. It is easy to check that $\text{ind} : GL_{\text{res}}(\mathcal{H}) \to \mathbb{Z}$, $A \mapsto \text{ind}(a)$ is a morphism. It can be proved that it induces an isomorphism $\pi_0(GL_{\text{res}}(\mathcal{H})) \cong \mathbb{Z}$.

Let $\mathcal{H}_+$ be the algebra of bounded operators of $\mathcal{H}_+$. If $A$ belongs to $GL_{\text{res}}(\mathcal{H})$, the invertibility of $A$ implies that $a$ is Fredholm in $\mathcal{H}_+$, and has an index $\text{ind}(a) \in \mathbb{Z}$. It is easy to check that $\text{ind} : GL_{\text{res}}(\mathcal{H}) \to \mathbb{Z}$, $A \mapsto \text{ind}(a)$ is a morphism. It can be proved that it induces an isomorphism $\pi_0(GL_{\text{res}}(\mathcal{H})) \cong \mathbb{Z}$.

4.1.2. Pressley-Segal extension of the restricted linear group. Let $L_1(\mathcal{H}_+)$ denote the ideal of trace-class operators of $\mathcal{H}_+$. It is a Banach algebra for the norm $\| b \|_1 = Tr(\sqrt{b^*b})$, where $Tr$ denotes the trace form. We say that an invertible operator $q$ of $\mathcal{H}_+$ has a determinant if $q - id_{\mathcal{H}_+} = t$ is trace-class. Its determinant is the complex number $\det(q) = \sum_{i=0}^{\infty} Tr(\wedge^i t)$, where $\wedge^i t$ is the operator of the Hilbert space $\wedge^i \mathcal{H}_+$ induced by $t$ (cf. [28]).

Denote by $\Xi$ the subgroup of $GL(\mathcal{H}_+)$ consisting of operators which have a determinant, and by $\Xi_1$ the kernel of the morphism $\det : \Xi \to \mathbb{C}^\ast$.
Let $\mathfrak{A}$ be the subalgebra of $L_{res}(\mathcal{H}) \times L(\mathcal{H}_+)$ consisting of couples $(A, q)$ such that $a - q$ is a trace-class operator of $\mathcal{H}_+$. It is a Banach algebra for the norm $\| (A, q) \| = \| A \| + \| a - q \|_1$. Let $\mathfrak{E}$ be the subgroup of $GL_{res}(\mathcal{H}) \times GL(\mathcal{H}_+)$ consisting of the units of $\mathfrak{A}$. If $(A, q)$ belongs to $\mathfrak{E}$, then $\text{ind}(a) = \text{ind}(q + (a - q)) = \text{ind}(q) = 0$, so that $A$ belongs to $GL_{res}^0(\mathcal{H})$, and there is a short exact sequence
\[
(\mathfrak{N}C)_{PS} \quad 1 \to \mathfrak{T} \overset{i}{\longrightarrow} \mathfrak{E} \overset{p}{\longrightarrow} GL_{res}^0(\mathcal{H}) \to 1.
\]

Here $p(A, q) = A$, and $i(q) = (id_\mathcal{H}, q)$.

**Remark 4.2.** The construction of $(\mathfrak{N}C)_{PS}$ can be extended to a short exact sequence $1 \to \mathfrak{T} \longrightarrow \mathfrak{E} \longrightarrow GL_{res}(\mathcal{H}) \to 1$ (cf. [24]). Note, however, that if a simple group is represented in $GL_{res}(\mathcal{H})$, its image is contained in $GL_{res}^0(\mathcal{H})$.

**Definition 4.3 (cf. [24]).** The (non-central) extension $1 \to \mathfrak{T} \longrightarrow \mathfrak{E} \longrightarrow GL_{res}^0(\mathcal{H}) \to 1$ will be called Pressley-Segal’s extension. It induces the central extension
\[
1 \to \mathfrak{T} \overset{i}{\longrightarrow} \mathfrak{E} \cong \mathbb{C}^* \overset{p}{\longrightarrow} GL_{res}^0(\mathcal{H}) \to 1.
\]

The corresponding cohomology class in $H^2(GL_{res}^0(\mathcal{H}); \mathbb{C}^*)$ is denoted by $PS$, and called the Pressley-Segal class of the restricted linear group.

### 4.1.3. Godbillon-Vey class and Pressley-Segal class.

Denote by $\text{Diff}(S^1)$ the group of orientation-preserving diffeomorphisms of the circle, and by $L^2(S^1)$ the Hilbert space of complex valued functions on the circle which are square integrable. Then $L^2(S^1)$ admits a polarisation as $L^2(S^1) = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_+$ is densely generated by $\{e^{in\theta}, n \in \mathbb{Z}, n \geq 0\}$ and $\mathcal{H}_-$ by $\{e^{in\theta}, n \in \mathbb{Z}, n \leq -1\}$. Consider the series of representations, parametrised by $s \in \mathbb{R}$, $\pi_s : \text{Diff}(S^1) \to GL_{res}^0(L^2(S^1))$, $\pi_s(\phi)(f) = f \circ \psi \phi^s$, where $\psi = \phi^{-1}$. When $s = \frac{1}{2}$, the representation is unitary. From computations of [19], the pull-back of the Pressley-Segal class by $\pi_s$ is
\[
\pi_s^*(PS) = e^{2i\pi s \frac{\theta}{2}(\frac{6}{e} - 6 + g v - \frac{1}{2} + \chi)},
\]
where $gv \in H^2_{cont}(\text{Diff}(S^1); \mathbb{R})$ is the Bott-Virasoro-Godbillon-Vey class in the continuous cohomology of $\text{Diff}(S^1)$, and $\chi \in H^2_{cont}(\text{Diff}(S^1); \mathbb{R})$ is the continuous Euler class of $\text{Diff}(S^1)$, that is, the smooth version of the discrete class of the extension $0 \to \mathbb{Z} \longrightarrow \text{Diff}(S^1) \longrightarrow \text{Diff}(S^1) \to 1$, where $\text{Diff}(S^1)$ is the universal cover of $\text{Diff}(S^1)$. Note that in the case $s = \frac{1}{2}$, this result is consistent with [24], Proposition 6.8.5.

### 4.1.4. Reduced Pressley-Segal extension.

The projection $p : A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in L_{res}(\mathcal{H}) \to a \text{ mod } L_1(\mathcal{H}_+) \in \frac{L(\mathcal{H}_+)}{L_1(\mathcal{H}_+)}$ is a morphism of algebras (essentially because the product of two Hilbert-Schmidt operators is trace-class). It follows that the algebra $\mathfrak{A}$ is obtained by the fibre-product
\[
\begin{array}{ccc}
\mathfrak{A} & \longrightarrow & L_{res}(\mathcal{H}) \\
\downarrow & & \downarrow p \\
L(\mathcal{H}_+) & \longrightarrow & \frac{L(\mathcal{H}_+)}{L_1(\mathcal{H}_+)}
\end{array}
\]

Passing to the units, $p$ induces a group morphism $GL_{res}(\mathcal{H}) \to \left(\frac{L(\mathcal{H}_+)}{L_1(\mathcal{H}_+)}\right)^*$. Denote by $\left(\frac{L(\mathcal{H}_+)}{L_1(\mathcal{H}_+)}\right)^*\in GL_{res}(\mathcal{H})$ in $\left(\frac{L(\mathcal{H}_+)}{L_1(\mathcal{H}_+)}\right)^*$.
Definition-Proposition 4.4 (Reduced Pressley-Segal extension). There is a morphism of non-commutative extensions

\[
\begin{array}{ccccccccc}
1 & \overset{\varphi}{\longrightarrow} & \mathbb{T} & \overset{\varepsilon}{\longrightarrow} & GL\mathcal{L}H & GLH & \overset{p}{\longrightarrow} & GL\mathcal{L}H^0 & \overset{\varphi}{\longrightarrow} & 1 \\
1 & \overset{\varphi}{\longrightarrow} & \mathbb{T} & \overset{\varepsilon}{\longrightarrow} & GLH & \overset{p}{\longrightarrow} & GL\mathcal{L}H^0 & \overset{\varphi}{\longrightarrow} & 1 \\
\end{array}
\]

which induces a morphism of central extensions

\[
\begin{array}{ccccccccc}
1 & \overset{\varphi}{\longrightarrow} & \mathbb{T}/\mathbb{T}_1 & \overset{\varepsilon}{\longrightarrow} & GL\mathcal{L}H & GLH & \overset{p}{\longrightarrow} & GL\mathcal{L}H^0 & \overset{\varphi}{\longrightarrow} & 1 \\
1 & \overset{\varphi}{\longrightarrow} & \mathbb{T}/\mathbb{T}_1 & \overset{\varepsilon}{\longrightarrow} & GLH & \overset{p}{\longrightarrow} & GL\mathcal{L}H^0 & \overset{\varphi}{\longrightarrow} & 1 \\
\end{array}
\]

We call the non-commutative extension of \( GL\mathcal{L}H^0 \) the Reduced Pressley-Segal extension. The cohomology class in \( H^2(\mathcal{L}(H^+), \mathbb{C}) \) of the associated central extension is denoted by \( ps \), and called the Reduced Pressley-Segal class. It follows that the class \( PS \) is the pull-back of the reduced class \( ps \) by the projection \( p \).

The proof is straightforward. Since the ideal \( \mathcal{L}_1(H^+) \) is not closed in \( \mathcal{L}(H^+) \), there is no separate topology on the quotient \( \frac{\mathcal{L}(H^+)}{\mathcal{L}_1(H^+)} \). This forces us to treat the group \( \frac{\mathcal{L}(H^+)}{\mathcal{L}_1(H^+)} \), from a cohomological point of view, as a discrete group.

4.2 Pressley-Segal type representations for the non-commutative and central extensions of Neretin’s group

In the next theorem, we refer to the material introduced in \( \S 2.3.1, \S 2.4 \) and \( \S 2.5, \) Theorem 2.6.

Theorem 4.5. Let \( H \) denote the Hilbert space \( \ell^2(T^0_2) \) on the set of vertices of the dyadic complete tree. There exists a unitary representation \( \rho : PAut(T^0_2) \rightarrow GL(H) \) which induces a morphism of non-commutative extensions

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{S}(T^0_2) & \longrightarrow & PAut(T^0_2) & \longrightarrow & N & \longrightarrow & 1 \\
1 & \longrightarrow & \mathbb{T} & \longrightarrow & GLH & \longrightarrow & \left( \frac{\mathcal{L}(H)}{\mathcal{L}_1(H)} \right)^* & \longrightarrow & 1 \\
\end{array}
\]

This, in turn, induces a morphism of central extensions

\[
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & PAut(T^0_2) & \longrightarrow & N & \longrightarrow & 1 \\
1 & \longrightarrow & \mathbb{C} & \longrightarrow & GLH & \longrightarrow & \left( \frac{\mathcal{L}(H)}{\mathcal{L}_1(H)} \right)^* & \longrightarrow & 1 \\
\end{array}
\]

Proof. Let \( \{x_v\}_{v \in T^0_2} \) be the hilbertian basis of \( H \) on the set of vertices of the tree \( T^0_2 \). If \( g \) belongs to \( PAut(T^0_2) \), define the operator \( \rho(g) \) on the basis by \( \rho(g)(x_v) = x_{g(v)} \). This clearly defines a unitary operator of \( H \). If \( g \) is finitely supported, that is, \( g \) belongs to \( \mathcal{S}(T^0_2) \), then \( \rho(g) \) belongs to an inductive limit of finite dimensional linear groups \( GL(n, \mathbb{C}) \), and in particular, to the group \( \mathbb{T} \). The commutativity of the second diagram essentially relies on the remark that the signature of \( g \in \mathcal{S}(T^0_2) \) coincides with the determinant of the operator \( \rho(g) \).

\[ \square \]
4.3 A Pressley-Segal type representation of the mapping class group $A_T$

4.2.1. The Magnus and Burau representations.

**Geometric definition of the Burau representation:** Let $D_n$ be the unit disk with $n$ marked points $q_1, \ldots, q_n$. Define the index morphism by $ind : \pi_1(D_n) \to \mathbb{Z}$, $\gamma_i \mapsto 1$, where $\gamma_1, \ldots, \gamma_n$ is a basis of the free group $\pi_1(D_n)$ (each $\gamma_i$ is the homotopy class of a loop encircling $q_i$, based at a chosen point $*$ on the boundary of $D_n$). The kernel of this epimorphism is the fundamental group of a covering space $\tilde{D}_n$ over $D_n$. The transformation group of the covering $\tilde{D}_n \to D_n$ is isomorphic to $\mathbb{Z}$. If $t$ denotes a generator of this transformation group, viewed as a multiplicative group, then $H_1(\tilde{D}_n)$ is a free module of rank $n - 1$ over $\mathbb{Z}[t, t^{-1}]$.

Choose a point $\hat{*}$ in $\tilde{D}_n$ over $*$. Since each $b \in \text{Homeo}^+(D_n, \partial D_n)$ – the group of orientation-preserving homeomorphisms fixing pointwise the boundary of the disk – acts on $\pi_1(D_n)$ preserving the index, it has a unique lift $\tilde{b}$ to $\tilde{D}_n$ that fixes $\hat{*}$. Hence there is a group morphism $\text{Homeo}^+(D_n, \partial D_n) \to \text{GL}_{n-1}(\mathbb{Z}[t, t^{-1}])$, $b \mapsto \rho(b)$, where $\rho(b)$ is the action induced by $\tilde{b}$ on the homology group $H_1(D_n)$. Since $\rho(b)$ only depends on the isotopy class of $b$, one gets an induced morphism

$$B_n \to \text{GL}_{n-1}(\mathbb{Z}[t, t^{-1}]),$$

where $B_n = \pi_0(\text{Homeo}^+(D_n, \partial D_n))$ is the Artin braid group.

**Algebraic definition of the Burau representation:** using the Magnus representation of $\text{Aut}(F_n)^{\text{ind}}$ (cf. [2]):

Let $F_n$ be the free group on $\gamma_1, \ldots, \gamma_n$, and $\text{Aut}(F_n)^{\text{ind}}$ be the subgroup of $\text{Aut}(F_n)$ consisting of index-preserving automorphisms of $F_n$. There exists a representation $\mathcal{M}$ of $\text{Aut}(F_n)^{\text{ind}}$ in the linear group $\text{GL}_n(\mathbb{Z}[t, t^{-1}])$, called the Magnus representation (cf. [2]). On the other hand, consider the natural embedding of the braid group $B_n$ in $\text{Aut}(F_n)^{\text{ind}}$, deduced from the action of $\text{Homeo}^+(D_n, \partial D_n)$ on $\pi_1(D_n) = F_n$. It happens that the Burau representation is algebraically defined as the composition of the natural morphism $B_n \to \text{Aut}(F_n)^{\text{ind}}$ with the Magnus representation. The resulting representation splits into a one-dimensional trivial representation and an $(n - 1)$-dimensional irreducible part; the latter is the one we have geometrically described.

Explicitly, the action of $B_n$ on $\pi_1(D_n) = F_n$ is defined through the following formula (as usual, $\sigma_1, \ldots, \sigma_{n-1}$ are the standard generators of $B_n$):

$$(\sigma_i)_* : \gamma_i \mapsto \gamma_i \gamma_{i+1} \gamma_i^{-1},$$

$$\gamma_{i+1} \mapsto \gamma_i,$$

$$\gamma_j \mapsto \gamma_j, \quad j \neq i, i + 1.$$  

The Magnus representation of $\text{Aut}(F_n)^{\text{ind}}$ may be elegantly introduced using R. H. Fox’s free differential calculus. To avoid unuseful length, we content ourselves to give the result of the composition of the Magnus representation with the embedding $B_n \hookrightarrow \text{Aut}(F_n)^{\text{ind}}$: introduce a basis of the free module of rank $n$ over $\mathbb{Z}[t, t^{-1}]$, $x_1, \ldots, x_n$, (which may be thought of as the homology classes of the lifts in $\tilde{D}_n$, based at $\hat{*}$, of $\gamma_1, \ldots, \gamma_n$ respectively). It follows that

$$\mathcal{M}((\sigma_i)_*)(x_i) = (1 - t)x_i + tx_{i+1},$$

$$\mathcal{M}((\sigma_i)_*)(x_{i+1}) = x_i,$$

$$\mathcal{M}((\sigma_i)_*)(x_j) = x_j, \quad j \neq i, i + 1.$$  

Finally, $\rho(\sigma_i) \in \text{GL}_{n-1}(\mathbb{Z}[t, t^{-1}])$ is obtained by restricting $\mathcal{M}((\sigma_i)_*)$ to the submodule generated by $x_1 - x_2, \ldots, x_{n-1} - x_n$.
4.2.2. Extension of the Burau representation to the infinite braid group $B_{\infty}$.

For each puncture $q$ on the surface $S_{\infty,t}$, we define a loop $\gamma_q$ based at $*$ in the following way:

- draw a small loop $c_q$ surrounding the puncture $q$;
- slightly deform the edge-path $\delta_q$ to a path $\delta_q$, such that the trace of $\hat{\delta}_q$ contains no punctures of $S_{\infty,t}$, and $\hat{\delta}_q$ connects $*$ to the base point of the loop $c_q$. More precisely, we choose $\hat{\delta}_q$ such that it avoids the punctures by passing on their right (see Figure 7b, where $\hat{\delta}_q'$ passes on the right of $q$), and say that $\hat{\delta}_q$ has the regularity property.
- define $\gamma_q$ as the loop $\hat{\delta}_q . c_q . \hat{\delta}_q^{-1}$ based at $*$. Our convention is to compose the paths from left to right, so that $\hat{\delta}_q . c_q . \hat{\delta}_q^{-1}$ means $(\hat{\delta}_q . c_q) . \hat{\delta}_q^{-1}$.

Let $F_\infty$ be the subgroup of $\pi_1(S_{\infty,t},*)$ generated by the homotopy classes of loops $\gamma_q$, denoted $\gamma_q$. It is a free group on the set of punctures. Recall (see Remark 3.8) that $B_{\infty}[T_t]$ is generated the half-twists between the pairs of consecutive vertices of the tree $T_t$, along the edge which connects them. So, denote by $\sigma_{q,q'}$ the half-twist between two such consecutive vertices. The automorphism of $\pi_1(S_{\infty,t},*)$ induced by $\sigma_{q,q'}$ restricts to an automorphism $(\sigma_{q,q'})_*$ of the free subgroup $F_\infty$, that we now describe:

- $(\sigma_{q,q'})_*(\gamma_q) = \gamma_{q'}$ (Figure 7c),
- $(\sigma_{q,q'})_*(\gamma_q) = \gamma_{q'} \gamma_q \gamma_{q'}^{-1}$ (Figure 7d),
- if $p \neq q, q'$ and $\gamma_p$ does not intersect the edge $qq'$, $(\sigma_{q,q'})_*(\gamma_p) = \gamma_p$,
- if $p \neq q, q'$ but $\gamma_p$ does intersect the edge $qq'$, $(\sigma_{q,q'})_*(\gamma_p) = \gamma_q \gamma_{q'}^{-1} \gamma_p \gamma_q \gamma_{q'}^{-1}$ (Figures 7e, 7f).

From the preceding formula we immediately deduce the

**Lemma 4.6.** Define the index morphism $\text{ind} : F_\infty \to \mathbb{Z}$ by $\text{ind}(\gamma_q) = 1$ for each puncture $q$ of $S_{\infty,t}$. Each $\sigma \in B_{\infty}[T_t]$ induces an automorphism $\sigma_*$ of $F_\infty$ which is index-preserving.

**Proposition 4.7.** Let $\mathcal{H}$ be the Hilbert space $l^2(T_t^0)$ on the set of punctures of the surface $S_{\infty,t}$, that is, the vertices of the tree $T_t$. Let $\mathfrak{T}$ be the group of determinant-operators of $\mathcal{H}$. For each complex number $t \in \mathbb{C}^*$, the Burau representation of the Artin braid groups extends to a representation $\rho_\infty^B : B_{\infty}[T_t] \to \mathfrak{T}$. The composition with the determinant morphism induces an embedding $H_1(B_{\infty}[T_t]) \to \mathbb{C}^*$ if and only if $t$ is not a root of unity.
Proof. Let \( \{x_q\}_q \) be a basis of the free \( 
abla[t, t^{-1}] \)-module on the set of punctures of \( S_{\infty,t} \). Each \( \sigma \in B_{\infty}[T] \) induces \( \sigma_* \) in \( Aut(F_{\infty}) \) (the subgroup of index-preserving automorphisms of \( F_{\infty} \)). Since there is no difficulty to define the Magnus representation \( \mathcal{M}_{\infty} \) in the infinite case, we define \( \rho_\infty(\sigma) \) in \( Aut(\nabla[t, t^{-1}][T_{\mathbb{T}}]) \) by \( \rho_\infty(\sigma) = \mathcal{M}_{\infty}(\sigma_*), \) and we get the formula:

- if \( p = q, \rho_\infty(\sigma)(x_q) = x_{q'}, \)
- if \( p = q', \rho_\infty(\sigma)(x_{q'}) = (1-t)x_{q'} + tx_q, \)
- if \( \gamma_p, p \neq q, q', \) intersects the edge between \( q \) and \( q', \rho_\infty(\sigma)(x_p) = (1-t)(x_{q'} - x_q) + x_p, \)
- in the other cases, \( \rho_\infty(\sigma)(x_p) = x_p. \)

It follows that \( \rho_\infty(\sigma) \) differs from the identity by a finite rank operator. Note however that \( \rho_\infty(\sigma) \) does not belong to an inductive limit \( GL_\infty(\nabla[t, t^{-1}]) = \lim_{n \to \infty} GL_n(\nabla[t, t^{-1}]), \) because an edge between \( q \) and \( q' \) intersects infinitely many loops \( \gamma_p. \)

By evaluation of \( t \) on an invertible scalar \( t \in \mathbb{C}^\ast, \) we get a representation in the Hilbert space \( \mathcal{H}, \)

and satisfy the conditions 1–3 of Definition-Proposition 3.7. The isotopies are assumed to fix the base point. Thus, there is a well-defined representation \( A_1 \to Aut(F_{\infty}). \) One has an exact sequence \( 1 \to B_{\infty,1} \to A_1 \to T \to 1, \) where \( B_{\infty,1} \) is generated by \( B_{\infty} \) and a pure braid \( \tau, \) which can be chosen as follows: let \( v_\ast \) be one of the vertices of the distinguished edge \( e_0, \) and \( \sigma_{v_\ast} \) be the half-twist along the half-edge which joins \( \ast \) to \( v_\ast. \) Define \( \tau \) as the isotopy class of \( \sigma_{v_\ast} \) in \( A_1. \)

Next, consider the commutative diagram of short exact sequences:

4.2.3. Pressley-Segal type representation of the mapping class group \( A_T. \) We are now ready to prove the main result of our paper:

Theorem 4.8. For each \( t \in \mathbb{C}^\ast, \) the Burau representation \( \rho_\infty^k : B_{\infty} := B_{\infty}[T] \to \mathfrak{T} \) extends to a representation \( \rho_\infty^k \) of the mapping class group \( A_T \) in the Hilbert space \( \mathcal{H} = l^2(T^0) \) on the set of punctures of the surface \( S_{\infty,t}. \)

There is a morphism of non-commutative extensions

\[
\begin{array}{ccc}
1 & \longrightarrow & B_{\infty} & \longrightarrow & A_T & \longrightarrow & T & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \mathcal{T} & \longrightarrow & GL(\mathcal{H}) & \longrightarrow & \left( \frac{L(\mathcal{H})}{L_1(\mathcal{H})} \right)_0 & \longrightarrow & 1 \\
\end{array}
\]

inducing a morphism of central extensions

\[
\begin{array}{ccc}
1 & \longrightarrow & H_1(B_{\infty}) & \longrightarrow & \mathcal{A} & \longrightarrow & T & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \mathbb{C}^\ast & \longrightarrow & \frac{GL(\mathcal{H})}{\mathfrak{T}_1} & \longrightarrow & \left( \frac{L(\mathcal{H})}{L_1(\mathcal{H})} \right)_0 & \longrightarrow & 1 \\
\end{array}
\]

where \( \mathfrak{T}_1 \subset \mathfrak{T} \) is the kernel of the determinant morphism.

The vertical arrows are all injective whenever \( t \in \mathbb{C}^\ast \) is not a root of unity.

Proof. It is subdivided into several lemmas.

Lemma 4.9. There exists a representation \( A_T \to Aut(F_{\infty}). \)

Proof. Let \( A_1 \) be the group of isotopy classes of homeomorphisms of \( S_{\infty,t} \) which fix the base point, and satisfy the conditions 1–3 of Definition-Proposition 3.7. The isotopies are assumed to fix the base point. Thus, there is a well-defined representation \( A_1 \to Aut(F_{\infty}). \) One has an exact sequence \( 1 \to B_{\infty,1} \to A_1 \to T \to 1, \) where \( B_{\infty,1} \) is generated by \( B_{\infty} \) and a pure braid \( \tau, \) which can be chosen as follows: let \( v_\ast \) be one of the vertices of the distinguished edge \( e_0, \) and \( \sigma_{v_\ast} \) be the half-twist along the half-edge which joins \( \ast \) to \( v_\ast. \) Define \( \tau \) as the isotopy class of \( \sigma_{v_\ast} \) in \( A_1. \)
The vertical arrows $A_1 \rightarrow A$ and $B_{\infty,1} \rightarrow B_{\infty}$ are induced by forgetting the base point, and $K$, the kernel of those forgetting morphisms, is the normal subgroup of $B_{\infty,1}$ generated by $\tau$. But it is easy to check that $\tau$ induces the identity in $\text{Aut}(F_{\infty})$. It follows that the representation $A_1 \rightarrow \text{Aut}(F_{\infty})$ descends to $A_1/K = \mathcal{A}_T \rightarrow \text{Aut}(F_{\infty})$.

Let $[a]$ be in $\mathcal{A}_T$, and choose any representative $a$ which fixes the base point. There exist two finite subsurfaces $S_0$ and $S_1$ with the same bi-type, such that $a|_{S_{\infty,t}\setminus S_0} : S_{\infty,t} \setminus S_0 \rightarrow S_{\infty,t} \setminus S_1$ is rigid, the action on $T_t \cap (S_{\infty,t} \setminus S_0)$ being prescribed by a partial tree automorphism of $T_t$. Label by $q_1, \ldots, q_n$ the punctures lying on the boundary components of $S_0$, and $C_1, \ldots, C_n$ the corresponding connected components of $S_{\infty,t} \setminus S_0$ (note that $n = 2k$ if $(k,l)$ is the bi-type of $S_0$).

**Lemma 4.10.** For $i = 1, \ldots, n$, there exists $\lambda_i$ in $F_{\infty}$ such that if $q$ is a puncture on the connected component $C_i$, then $a_* (\gamma_q) = \lambda_i \gamma_{a(q)} \lambda_i^{-1}$. Furthermore, the homotopy class $\lambda_i$ belongs to the free subgroup of $F_{\infty}$ generated by the $\gamma_q$'s, for $q \in T_t \cap S_1$.

**Proof.** Compare the homotopy classes, with fixed extremities, of $a(\hat{\delta}_{q_i})$ and $\hat{\delta}_{a(q_i)}^{-1}$. The loop $\lambda_i = a(\hat{\delta}_{q_i}) \hat{\delta}_{a(q_i)}^{-1}$ defines an element $\lambda_i$ of $F_{\infty}$.

For each $q$ on $C_i$, decompose the path $\hat{\delta}_q$ as $\hat{\delta}_{q_i} \hat{\delta}_{a(q_i)}$, where $\hat{\delta}_{q_i}$ connects $q_i$ to $q$. Thus, $\gamma_q = \hat{\delta}_{q_i} \hat{\delta}_{a(q_i)} \hat{\delta}_{a(q_i)}^{-1} \hat{\delta}_{a(q_i)}$. Since the homeomorphism $a$ is rigid on $C_i$, it preserves the regularity property of $\hat{\delta}_{q_i}$ (see §4.2.2), so that $a(\hat{\delta}_{q_i}) = \hat{\delta}_{a(q_i)} a(\hat{q})$. It follows easily that $a_* (\gamma_q) = \lambda_i \gamma_{a(q)} \lambda_i^{-1}$.

**Lemma 4.11.** For each of the finitely many punctures $q$ of $S_0$, there exists $\lambda_q$ in $F_{\infty}$ such that $a_* (\gamma_q) = \lambda_q \gamma_{a(q)} \lambda_q^{-1}$.

**Proof.** The arguments are similar to those of the proof of the previous lemma.

It follows from the preceding three lemmas that each $a \in \mathcal{A}_T$ induces an index-preserving automorphism of the free group $F_{\infty}$. Thus, we extend the Burau representation to $\mathcal{A}_T$ by setting $p_{\infty}^\text{ind}(a) = \mathcal{M}(a_*)$, where $\mathcal{M}$ is, as before, the Magnus representation of $\text{Aut}(F_{\infty})^\text{ind}$. It remains to check that the induced operator $p_{\infty}^\text{ind}(a)$ is indeed continuous:

For $i = 1, \ldots, n$ and $\epsilon = 0$ or $1$, introduce the subbasis $B^i_0 = \{ x_{q_i}, q \in T_t \cap C_i \}$, where $C_i \seteq C_i$, and $C_i := a(C_i)$, as well as the finite subbasis $B_{n+1}^\epsilon = \{ x_{q_i}, q \in T_t \cap S_1 \}$.

If $\lambda_i = \gamma_{q_1}^{\epsilon_1} \cdots \gamma_{q_r}^{\epsilon_r}$, for some punctures $q_{j_1}, \ldots, q_j$ of $S_1$, and $\epsilon_1, \ldots, \epsilon_r \in \{-1, 1\}$, then for any puncture $q$ lying on $C_i$:

$$p_{\infty}(a) (x_q) = P_i (t, t^{-1}) + \epsilon_i t^{m_i} x_{a(q)},$$

where $P_i (t, t^{-1})$ is a vector-valued polynomial in $t, t^{-1}, \epsilon_i \in \{-1, +1\}$, and $m_i \in \mathbb{Z}$, which only depend on $i$. 

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Similarly, for each of the finitely many \( q \)'s on \( S_0 \), there exists a vector-valued polynomial \( P_q(t, t^{-1}) \), an integer \( m_q \in \mathbb{Z} \) and \( \varepsilon_q \in \{-1, +1\} \) such that

\[
\rho_\infty(a)(x_q) = P_q(t, t^{-1}) + \varepsilon_q t^{m_q} x_{a(q)}.
\]

Notice that the vectorial coefficients of the polynomials \( P_i \)'s and \( P_q \)'s belong to the finite dimensional vector space spanned by the basis \( B_{n+1}^1 \). It follows that the matrix of \( \rho^*_\infty(a) \) relative to the bases \( B^0 = (B_1^0, \ldots, B_{n+1}^0) \) and \( B^1 = (B_1^1, \ldots, B_{n+1}^1) \) may be written

\[
\begin{bmatrix}
\pm t^{m_1} & 0 & 0 & 0 \\
0 & \pm t^{m_2} & 0 & 0 \\
0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \pm t^{m_n} \\
* & * & * & * \\
\end{bmatrix}
\]

Each transformation \( \ell^2(B_i^0) \to \ell^2(B_i^1), x_q \mapsto \pm t^{m_i} x_{a(q)}, \) \( i = 1, \ldots, n \), is obviously continuous. Since the \( n + 1 \) blocks \([*]\) define finite rank, hence continuous operators, it follows that \( \rho^*_\infty(a) \) is continuous.

The commutativity of the diagrams involved in the statement of Theorem 4.8 is easy to prove.

### 4.4 Concluding remarks

It would be interesting to know if Theorem 4.8 has a version involving the “non-reduced” Pressley-Segal extension. On the other hand, we do not know if there is an extension of \( \text{Diff}(S^1) \) which is the counterpart of our geometric extension of \( T \) by \( B_\infty \). What could be the kernel of it?

More generally, can one develop a representation theory of \( T \)? Finally, we point out that the results of this paper do not shed any light on Thompson’s group \( V \), acting on the Cantor set. However, geometric group extensions of \( V \) were recently studied by L. Funar and the first author (7), and by M. Brin and J. Meier.

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