Abstract

Robustness is an important property of complex networks. Up to now, there are plentiful researches focusing on the network’s robustness containing error and attack tolerance of network’s connectivity and the shortest path. In this paper, the error and attack tolerance of network’s community structure are studies through randomly and purposely disturbing interaction of networks. Two purposely perturbation methods are designed, that one methods is based on cluster coefficient and the other is attacking triangle. Dissimilarity function \( D \) is used to quantify the changes of community structure and modularity \( Q \) is used to quantify the significance of community structure. The numerical results show that after perturbation, network’s community structure is damaged to be more unclear. It is also discovered that purposely attacking damages more to the community structure than randomly attacking.

1 Introduction

In recent years, more and more systems in many fields are depicted as complex networks. Robustness, as an important function of many systems, is studies under framework of complex networks recently. The past researches mainly focus on the the topological aspects of robustness and are usually done either by removal of nodes, random failure of nodes or targeted attack, or by replacing a sector of edges or adding new edges\(^\text{\[1\]}\). Network with special structure, for instance scale-free network, is found to be less robust to targeted attack and more robust to random failure. The robustness is usually measured by either the size of the largest connected component or the length of the shortest path between pairs of nodes. Some of the investigations try to find effective way to improve robustness of real networks against attack.

\*Author for correspondence: yfan@bnu.edu.cn
During the studies on networks, lots of evidences show that there are communities in social networks, metabolic networks, economic networks, and so on. Community structure is an important character to understand the functional properties of complex networks. For instance, in social networks, communities can formed depending on careers or ages. In food web, communities may reveal the subsystem of ecosystem. In biochemical or neural networks, communities may correspond to functional groups. In the world wide web, the community analysis have found thematic groups. Email network can be divided into departmental groups whose work is distinct and the communities reveal organization structures. Moreover, the study of dynamics in complex networks shows that vertices belonged to the same community reach synchronization easily. Thus deep understanding on community structures will make us comprehend and analyze the characteristics of systems better. Most research about the community structure mainly focus on the algorithm of detecting the community structure in networks.

Robustness of community structure hasn’t been paid enough attention. Recently, Brian Karrer et al. proposed a random perturbation method with rewiring edges randomly, and studies the robustness of community structure, which is an error tolerance study. In [15], it is proposed that robustness study can be another method of measuring the significance of community structure. The attack tolerance of community structure hasn’t been studied yet, which is mainly focused in this paper. Based on the idea in [15], we propose two targeted methods and investigated their effect on community structure in networks. This paper is organized as following. In section 2, we introduce three network perturbation ways, including one random disturbing way. In section 3 random and attack tolerance of community structure are investigated through dissimilarity functions $D$. We disturb networks using targeted attack and random attack method on artificial and real network and numerical results are analyzed. Section 4 attack’s effect on the significance of community structure and network’s topology character is explored through modularity $Q$’s and average cluster coefficient $C$. It is found that targeted attack cause serious damage to network’s structure. The attack is simulated both on artificial and real network and numerical results are analyzed. Section 5 is a conclusion.

2 Attack Methods on network

In network, triangles as an basic unit plays important role in communities structure. There are divisive algorithms basing on the triangle structure to find community structure in networks, such as the algorithm invited by Filippo Radicchi, et al. and good results are obtained. It is found that edges connecting nodes in different communities are included in few or no triangles. Edge-clustering coefficient is introduced first for the edge-connecting node $i$ to node $j$:

$$C_{ij}^{3} = \frac{z_{ij}^{3}}{\min[(k_{i} - 1), (k_{j} - 1)]}$$

(1)

where $z_{ij}^{3}$ is the number of triangles built on that edge and $\min[(k_{i} - 1), (k_{j} - 1)]$ is the minimal possible number of them. Edges connecting nodes within communities are included in more triangles and tend to have large values of $C_{ij}^{3}$. Our first targeted attack method is attacking edges with the largest $C_{ij}^{3}$, and second targeted attack method is attacking edges...
which is included in the largest triangles. Initially a network with \(N\) nodes and \(M\) edges is given.

The targeted attack method based on edge-clustering coefficient (EC method in short) is as followings.

1. Compute \(C_{ij}^3\) on every edge and ascending the edge according to the cluster coefficient.
2. Remove the former \(\alpha M\) edges from the network.
3. Go through every pair nodes \((i, j)\) and rewire \(\alpha M\) edges randomly selected from the network.

It can be known that no edge is moved as \(\alpha = 0\). As \(\alpha = 1\), all edges are moved. The number of nodes and edges keeps constant ever if moved.

The targeted attack method based on triangles (T method in short), is as followings.

1. Compute \(z_{ij}^3\) on every edge and ascending the edge according to \(z_{ij}\).
2. calculate the total number of \(t = z_{ij}^3 \neq 0\) and remove the former \(\alpha_1 * t\) edges from the network.
3. Randomly add \(\alpha_1 * t\) edges to the network.

In T method, \(t\) represents the total number of edges included in triangles. It can be known that no edge is moved as \(\alpha_1 = 0\). As \(\alpha_1 = 1\), \(t\) edges are moved. For comparison to other perturbation method, here we still use \(\alpha\) to represent the ratio of number of edges actually moved to the total number of edges. \(\alpha = \frac{\alpha_1 * t}{M}\). The number of nodes and edges keeps constant ever if moved.

The random attack methods (R method in short) is designed here for comparison to the targeted attack.

1. remove \(\alpha * M\) edges from the network randomly.
2. Randomly add \(\alpha * M\) edges to the network.

The main difference between above two targeted attack methods is that the second methods disturbing the only edges contained in triangles purposely rather than all of the edges in the network. Thus the radices of the perturbation are changed. In the second methods if \(\alpha = 0\), no edges are rewired. However, if \(\alpha = 1\), all of the edges that consist of triangles are rewired, rather than all of the edges in the network. During the perturbation the average degree and size \(N\) of the network keep constant. The attack methods which is different from the method in [15], is that our perturbation scheme generates networks in which the expected degrees of vertices aren’t the same as the original degrees.
3 Robustness of Community Structure

3.1 Quantification of variation of Community Structure

Quantifying difference of community structures is not a new problem\[17, 18, 19\]. The community structure divided by algorithm is usually compared with the actual communities if known. When analyzing the precision of a certain algorithm, the comparison of different community structures is also needed. Thus several methods for measuring similarities or differences between divisions of community structure have been designed. Function $D$\[20\] is a simple and efficient measurement to the difference between two community division and is chosen here as an measurement of robustness of community structure.

In this section, based on the three perturbation methods proposed above, we measure the variation of robustness of community structure after perturbation using dissimilarity function $D$. While in Karrer’s study, he analyzed the robustness using function $V$ based information entropy \[15\]. The reason we choose $D$ is that $D$ reflects the same character of variation of information of community structure as $V$ does, and $D$ can be normalized to 1. When doing the computer experiments, it is found that $D$ is more sensitive than $V$.

The idea of dissimilarity function $D$ is introduced by \[20\]. Discuss the similarity and dissimilarity of two sets $A$ and $B$ that defined as the subsets of $\Omega$. Similarity is expressed by $A \cap B$, and dissimilarity corresponds to be $(A \cap \overline{B}) \cup (\overline{A} \cap B)$. The normalized similarity and dissimilarity can be represented as

$$
\begin{align*}
    s &= \frac{|A \cap B|}{|A \cup B|} \\
    d &= \frac{|(A \cap \overline{B}) \cup (\overline{A} \cap B)|}{|A \cup B|}
\end{align*}
$$

(2)

Discussing two particular divisions of a network, each of them have many communities, and we assuming that both of them have $k$ communities. First, construct the correspondence between communities that from different sets, which makes them have biggest similarity. Second, calculate the dissimilarity of each pair of the subsets. And then using the dissimilarities of all subsets to calculate the integer sets’ dissimilarity.

$$
D = \sum_{i=1}^{k} \frac{d_{X_iY_i}}{k}
$$

(3)

However, in most cases, two community structures do not have the same number of communities, which means not every subset has correspondence subset. To solve this problem, the subset $X_i$ that has no correspondence, correspond with $\Phi$. The $k$ equal to the larger number of communities. Under this definition, the maximum value of $D$ is 1 and minimum value of $D$ is 0, where (0, 1) means no and largest differences respectively.

3.2 Numerical results of Robustness of Community Structure

For a given network, now we have all the components to analyze the robustness. First, we get the community structure $C$ of this network by any algorithm existing now. Here we
use EO algorithm to detect the community structure for its good character \cite{21}. Then, we disturb the original network with the methods proposed in above section separately, and get the new community structure $C'$. And then, measure the varieties of community structure by function $D$ which have introduced in section 2. Since the disturbing methods including some choosing by random, we will repeat the second step for some times, and get the average value of variation to make sure it not be impacted by special cases. And the whole process is done many times concomitantly with the change of parameter $\alpha$ from 0 to 1.

To test the efficiency of targeted attack methods proposed here, three attack methods are simulated respectively on artificial networks generated by GN benchmark and LFR benchmark, and real network including Karate network, football network, and econophysicist collaboration network.

As first, we apply the methods on benchmarks of Girvan-Newman (GN) proposed in \cite{22}. In GN benchmark, homogeneous networks are generated and used widely in the evaluation of community detection algorithms. These networks consist of 128 vertices divided into 4 communities of 32 nodes each. Every node is connected on average with $\langle k_{in} \rangle$ nodes of its own group and $\langle k_{out} \rangle$ of the rest of the network. The total degree of each node is equal to $k = k_{in} + k_{out}$ and always kept constant to 16. As the average number $k_{out}$ of between-group connections per vertex is increased from zero, the community structure in the network, stark at first, becomes gradually obscured until, at the point where between- and within-group edges are equally likely, the network becomes a standard Poisson random graph with no community structure at all. Here, two cases that $k_{out}= 2$ and $k_{out}= 10$ are used.
Fig.1(a,b) shows the results of the application of our analysis method to graphs of this type. The figure shows the value of the variation of community structure $D$ as a function of the parameter $\alpha$ that measures the amount of perturbation. For small value of $k_{out}$ that $k_{out} = 2$ the variation of community structure increases faster under two targeted attacks than under random attack as a function of $\alpha$ as shown in Fig.1(a). As we can see, the variation of community structure $D$ starts at zero when $\alpha = 0$, as we would expect for an unperturbed network, rises rapidly, then levels off as $\alpha$ approaches its maximum value of 1.

The curves of targeted methods depart significantly from that of random method, indicating that the community structure discovered by the algorithm is less robust against targeted perturbation. Furthermore, the curve of ET method depart significantly from that of T method, indicating that the community structure discovered by the algorithm is relatively fragile against the ET method.

We can find that the curves represent the three methods don’t have the same length. The reason is the total edges that are changed in different methods are not the same. For R method ET method, all of the edges in the network can be moved. For T method disturbing towards triangles, the edges that can be disturbed are the edges that forming triangles, number of which may be much fewer than the total number of the edges.

Large values of $k_{out}$ in GN benchmark that generates network with obscure community structure, and $k_{out} = 10$ here. As shown in Fig.1(b), $D$ keeps a high value and almost unchanged as $\alpha$ changes from zero to one under three attack methods, indicating that the attack tolerance and error tolerance of obscure community structure is closed to each other. It is necessary to point out that $D$’s minimal value is not always exact zero when $\alpha = 0$, which is determined by the EO algorithm.

Then, on benchmarks of Lancichinetti et al. (LFR) in [23], the methods is applied to disturb the network generated. LFR is a generalization of the GN benchmark to heterogeneous group sizes and graph degree distribution. Groups are also a priori fixed with the degrees and the community sizes following a power-like distribution. As before, nodes have $k_{in}$ connections within its own group and $k_{out}$ edges linking elsewhere. For investigation of robustness of community structure, networks with significant community structure is needed here, and the parameters of LFR benchmark is set as following. The average degree $\langle k \rangle = 10$, and size is 500. The degree distribution follows a power-law $P(k) \sim k^\gamma$, with $\gamma = 2.5$; and the community sizes distribution follows a power-law $P(k) \sim k^\beta$, with $\beta = 2$. The mixing parameter $\mu = k_{out}/k$ indicates the "strength" of the communities, and is set to be 0.3 here. The maximal community size is 80, and the minimal is 20. Under such parameters, the network with clear community structure can be generated.

For the LFR benchmark, as shown in Fig.1(c), the curves of targeted methods also depart significantly from that of random method, indicating that the community structure discovered by the algorithm is fragile against targeted perturbation. With moving a small amount of edges with high $C$ or high number of triangles, the community structure changes more than with random moving edges. The results on the three artificial networks suggest that targeted methods is more efficiently destroying the community structure than random attack.

Turning now to real-world networks, we have tested our method on examples mainly including social networks. A selection of results are shown in Fig.2.

Fig.2(a) shows the curve of variation of community structure as a function of $\alpha$ for one of the best studied examples of community structure in a social network, the karate club.
network of Zachary [24]. The vertices in this network represent members of a karate club at a US university in the 1970s and the edges represent friendship between members based on independent observations by the experimenter. The network is widely believed to show strong community structure and repeated studies have upheld this view.

The black(square) and blue(triangle) points in the figure show the variation of community structure under targeted attacks while the red points show the results under random attack. It is clear in this case that the community structure is essentially the same robust against random perturbation with targeted perturbation.

Then we apply the methods to the football network [25] are shown in Fig.2(b). In the network of American college football teams. is a representation of the schedule of Division I games for the 2000 season: vertices in the graph represent teams (identified by their college names) and edges represent regular-season games between the two teams they connect. The network contains 115 nodes, 613 edges and is proved to have significant community structure. The curves of targeted methods also depart significantly from that of random method, indicating that the community structure discovered by the algorithm is relatively less robust against targeted perturbation. And for a certain possibility that edges have been disturbed, the variation of community structure caused by the methods we introduced are larger than variation of community structure caused by the random disturbing.

The result of robustness of econophysicists collaboration network [26] are shown in Fig.2(c). In the econophysicists collaboration network, notes represent econophysicists, the edges represent their collaboration relationship. And we analyze the largest component of it, which contains 271 nodes. For the same reason that different methods disturbing different edges, the curves in Fig.2(c) are not in the same length. The curves of ET method departs significantly from that of R method, indicating that the community structure discovered by the algorithm is relatively fragile robust against ET perturbation.

Comparing the analysis in the three real networks, we can find that the disarrange methods we proposed are always more efficiency than random disturbing. The results shows that under the same perturbation strength, targeted attack methods is more efficient than random attack method. Targeted methods always cause more damage to the original community structure. These results indicate that when investigating robustness of community structure, targeted attacks based on edge-clustering coefficient or triangles can be more efficient methods.

4 Modularity and Edge-clustering coefficient

In above sections, the robustness of community structure is discussed through targeted and perturbation on network, and it is found that the community structure is fragile to targeted attack. In this paper, we mainly focus on the perturbation’s effect on the topology character of network, such as modularity function and edge-clustering coefficient, which are related to community structure.

4.1 Modularity function

Modularity function $Q$ [27] now is implemented widely to measure the significance of network’s community structure.
\[ Q = \frac{1}{2m} \sum_{ij} [A_{ij} - \frac{k_i k_j}{2m}] \delta(c_i, c_j) \tag{4} \]

where \( A_{ij} \) is an element of the adjacency matrix of the network, \( A_{ij} = 1 \) if there is an edge between node \( i \) and node \( j \), otherwise \( A_{ij} = 0 \). \( k_i \) represents the degree of the vertex \( i \), which is defined to be the number of edges connected to node \( i \), \( m = \frac{1}{2} \sum_{ij} A_{ij} \) is the number of edges in the whole graph, and \( c_i \) shows that vertex \( i \) belongs to community \( c_i \). If the community structure is divided properly, the fraction of edges within communities should be large than the expect for the randomized network. The larger value of the function \( Q \) means the better community structure. For a given process people can calculate \( Q \) for each split of a network into communities, and there are only one or two local peaks. The position of these peaks usually correspond closely to the expected divisions.

When using EO algorithm, modularity analysis is necessary to find which partition is the best one. The optimal modularity for each networks, including original one and the disturbed networks, can be calculated. By doing this, we can measure whether the best community structure partition for the disturbed network is clear. Fig.3 and Fig.4 show the modularity changing on above three artificial and three real networks, caused by the three different methods introduced above. We can find that modularity become smaller with the disturbing strengthen going up except the GN benchmark when \( k_{out} = 10 \). The modularity of this artificial network is low and keeps almost unchanged under targeted attack and random attack. Meanwhile, the curves of targeted methods depart significantly from that of random method, indicating that targeted attacks make the community structure discovered by the algorithm more obscure than random attack.

![Figure 3: Value of Modularity Q versus α. Three attack methods are simulated on the arbitrary network benchmark.(a)k_{out} = 2 on GN benchmark;(b)k_{out} = 10 on GN benchmark;(c) LFR benchmark, N = 500, k=10, p(k) \sim k^{-\gamma}, \gamma = 2.5, p(s) \sim k^{-\lambda}, \lambda = 2.0, mixing parameter \mu = 0.3.](image)

### 4.2 Edge-clustering coefficient

In section 2, it has been mentioned that edge-clustering coefficient is a measurement of the strength of the network’s connection, and network with significant community structure usually has large edge-clustering coefficient value. In this part, we mainly investigate the variation of average edge-clustering coefficient \( C \) to perturbation strength \( \alpha \). It is found that in most cases \( C \) decreases fast under targeted methods with \( \alpha \)’s increasing from zero as shown in Fig.5 and Fig.6. Meanwhile, interesting phenomena appears that on GN benchmark when \( k_{out} =10 \) and football network, that \( C \) decreases at the beginning and increases later.
Figure 4: Value of Modularity $Q$ versus $\alpha$. Three attack methods are simulated on real network, $Q$ decreases with $\alpha$, which means perturbation makes community structure unclear. (a) Three attack methods are simulated on Karate club network. (b) Three attack methods are simulated on football network. (c) Three attack methods are simulated on econo-physicist network.

In this part, through modularity and edge-clustering coefficient, the results suggest that targeted attack method caused more damage to topology of networks, which is relating to community structure.

Figure 5: Value of edge-clustering coefficient $C$ versus $\alpha$. Three attack methods are simulated on the arbitrary network benchmark. (a)$k_{out} = 2$ on GN benchmark;(b)$k_{out} = 10$ on GN benchmark;(c) LFR benchmark, $N = 500$, $k=10$, $p(k) \sim k^{-\gamma}$, $\gamma = 2.5$, $p(s) \sim k^{-\lambda}$, $\lambda = 2.0$, mixing parameter $\mu = 0.3$.

5 conclusion

In the conclusion, we propose targeted methods to disturb the network, both of which are different from random disturbing. And then we use the random disturbing method and the our two methods to disturbing the original network and using function $D$ to compare the variation of community structure, which is more convenient than $V$. The results show that targeted attack methods based on edge-clustering coefficient and triangles damage more to community structure through analysis of $D$, $Q$, $C$ than random disturbing. These facts indicate that community structure is fragile against targeted attack and imply that triangle is important and deserves more attention in the study of community structure.

Acknowledgement

This work is supported by NSFC under the grant No.70771011.
Figure 6: Value of edge-clustering coefficient $C$ versus $\alpha$. Three attack methods are simulated on real network. (a) Three attack methods are simulated on Karate club network. (b) Three attack methods are simulated on football network. (c) Three attack methods are simulated on econophysicist network.

References

[1] Reka Albert, Hawoong Jeong, Albert-László Barabási, Error and attack tolerance of complex networks, Nature (London) 2000, 406.
[2] M. Girvan, M. E. J. Newman, Proc. Natl. Acad. Sci. USA 99 (2002) 7821-7826.
[3] M. Boss, H. Elsinger, M. Summer and S. Thurner, arXiv:cond-mat/0309582 (2003).
[4] E. Ravasz, A. L. Somera, D. A. Mongru, Z. N. Olvai and A. L. Barabási, Science 297 (2002) 1551.
[5] R. Guimerà, L. A. N. Amaral, Nature, 433 (2005) 895-900.
[6] P. Holme, M. Huss and H. Jeong, Bioinformatics 19 (2003) 532.
[7] P. Gleiser, L. Danon, Advances in Complex Systems, 6 (2003) 565-573.
[8] R. J. Williams, N. D. Martinez, Nature 404 (2000) 180-183.
[9] J.P. Eckmann and E. Moses, Proc. Natl. Acad. Sci. 99 (2002) 5825.
[10] H. Zhou and R. Lipowsky, Lecture Notes Comput. Sci. 3038 (2004) 1062-1069.
[11] J. R. Tyler, D. M. Wilkinson, B. A. Huberman, arXiv:cond-mat/0303264 (2003).
[12] R. Guimerà, L. Danon, A. Díaz-Guilera, F. Giralt, A. Arenas, Phys. Rev. E 68 (2003) 065103.
[13] Alex Arenas, Albert Díaz-Guilera, Conrad J. Pérez-Vicente, Phys. Rev. Lett 96 2006 114102.
[14] Filippo Radicchi, Claudio Castellano, Federico Cecconi, Vittorio Loreto, Domenico Parisi, Defining and identifying communities in networks, PNAS, 2004, vol.101, no.9, 2658-2663.
[15] Brian Karrer, Elizaveta Levina, M. E. J. Newman, Phys. Rev.E. 77 (2008): 046119.
[16] F. Radicchi, C. Castellano, F. Cecconi, V. Loreto, C. Parisi, Proc. Natl. Acad. Sci. USA 101 2004: 2658 - 2663.

[17] Danon L., Guilera A., Duch J, Comparing community structure identification[J], J. Stat. Mech,2005, P09008.

[18] Kuncheva L I, Hadjitodorov S T, Using diversity in cluster ensembles[C]. IEEE International Conference Systems, Man and Cybernetics, 2004, 2: 12141219.

[19] Fred A LN, Jain A K, Robust data clustering [C]. Proceeding of IEEE Computer Society Conference on Computer Vision and Pattern Recognition, Madison. USA, IEEE, 2003, 2: 128 - 133.

[20] Zhang Peng, Menghui Li, Jinshan Wu, Zengru Di, Ying Fan, Physica A 367 (2006): 577 - 583.

[21] J. Duch and A. Arenas, Phys Rev E. 72 (2005).027104.

[22] M. Girvan and M. E. J. Newman, Proc. Natl. Acad. Sci. U.S.A., 2002 99, 7821.

[23] Lancichinetti A, Fortunato S, Radicchi F, Benchmark graphs for testing community detection algorithms, Phys. Rev. E. 78(2008): 046110.

[24] W. W. Zachary, J. Anthropol. Res. 1977,33, 452.

[25] M. Girvan, M. E. J. Newman, Community structure in social and biological networks, Proc. Natl. Acad. Sci., 99(2002), 12: 7821 - 7826.

[26] Menghui Li, Ying Fan, Jiawei Chen et al, Physica A, 350( 2005) 643-656.

[27] Aaron Clauset, M. E. J. Newman, Cristopher Moore, Phys. Rev. E 70 2004 066111.