Retrocausal model for quantum measurement consistent with macroscopic realism

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We analyze the measurement problem by simulating the dynamics of amplitudes associated with backward and forward propagating stochastic equations in a realistic, objective model. By deriving a theorem based on conditional probabilities at the boundaries, we obtain trajectories equivalent to quantum mechanics. The joint densities of complementary variables give the correct quantum probability distribution. We model a measurement on a single-mode system via parametric amplification, showing how a system prepared in a superposition of eigenstates evolves to produce distinct macroscopic outcomes consistent with Born’s rule. The amplified variable corresponds to a backward propagating trajectory. Sampling is carried out according to a future boundary condition determined by the measurement setting. A distinctive feature is the existence of vacuum noise associated with an eigenstate. This noise remains constant at the level of the quantum vacuum throughout the dynamics and is not macroscopically measurable. The precise fluctuations are specified retrocausally, and originate from past and future boundary conditions. Where the separation of eigenstates greatly exceeds the vacuum, we argue consistency with macroscopic realism and causality: the macroscopic outcome of the measurement can be considered determined prior to the measurement. This leads to hybrid macro-causal and micro-retrocausal relations. The states inferred for coupled trajectories conditioned on a measured outcome are not quantum states. They are defined more precisely than allowed by the uncertainty principle, although they approach eigenstates with amplification in the limit of a macroscopic superposition. A full collapse into an eigenstate is simulated by coupling to a second mode, and occurs with loss of information. The model permits Einstein-Podolsky-Rosen correlations and Bell non-locality.

I. INTRODUCTION

Following from Bohr [1] who considered delayed-choice experiments [2] in quantum mechanics, Wheeler speculated that retrocausality may explain some of the quantum paradoxes [3, 4]. Retrocausal dynamics was studied in classical physics even earlier, especially by Dirac, Feynman and Wheeler in their work on electrodynamics [5-6]. Einstein-Rosen-Podolsky (EPR) demonstrated an inconsistency between the completeness of quantum mechanics and the assumptions of local realism [7], while Bell went further to prove incompatibility between local causality and the predictions of quantum mechanics [8-10]. Pegg demonstrated how violation of Bell inequalities can arise from classical fields, using retrocausal solutions from absorber theory [11, 12]. This motivated the question of whether superluminal disturbances, or communications, were possible, leading to no-signaling theorems [13]. These issues have inspired many analyses [2, 14-11] of retrocausal behavior in quantum physics.

Closely connected is the problem of measurement. For a system in a superposition of eigenstates of the measurement operator, the state after the measurement collapses to just one of these states. How does this occur? In the EPR paradox for entangled systems, one can predict the outcome of the measurement on a system with certainty, from a space-like separated measurement event [14, 15]. Then, if we reject realism — that the system was in a state with a well-defined outcome prior to the measurement — it appears as though there is an instantaneous collapse induced by some sort of action at a distance. Different resolutions have been proposed, including Bohmian models which allow nonlocality [10]; many worlds [17]; and decoherence collapse models based on mechanisms additional to quantum mechanics [18].

More recently, these problems have inspired research into studies of quantum correlations and causal order [49-64]. While it is possible to account for Bell violations using classical causal models, this will entail solutions based on, for instance, superluminal causal influences [46, 65-67], retrocausality [11, 22, 11, 68], or super-determinism [69]. However, if classical causality applies, it is claimed that these mechanisms will require fine tuning of causal parameters to explain an observed statistical independence of variables [51, 52, 70].

The overarching mystery remains that effects such as retrocausality, superluminal communication or super-determinism are not observed at a macroscopic level. Experiments proposed by Chaves, Lemos and Pienaar appear to confirm retrocausal effects, but only if one constrains theories to those with a limited dimensionality [52, 56, 59]. Quantum causal models that are based on conditional density operators rather than conditional probabilities have been developed to circumvent some of these issues [54, 56, 57, 71]. However, it is not clear how to present a unified treatment combining realism with causality.

In this paper, we analyze measurement problems using an objective field model motivated by Q function dynamics [72, 74]. This approach provides an explanation of quantum measurement based on a realistic model involving retrocausality. Our model allows for cyclic causation, and is immune to fine-tuning arguments in which directed acyclic graph theory excludes retrocausality [51, 52, 70].
The potential for the theory is that it arises naturally from quantum mechanics, by inclusion of microscopic noise sources generated by interactions. An important feature that distinguishes the theory from previous approaches is the existence of stochastic noise of order $\hbar$ associated with the eigenstates of $\hat{x}$. This noise is determined retrocausally from boundary conditions in the future that depend on the measurement setting.

We show in this paper that this feature allows for a positive probability distribution for stochastic variables associated with a quantum state, and permits explanations allowing for macroscopic realism (that is, macroscopic causality), while also allowing for the EPR paradox and Bell correlations.

Recent work based on this model demonstrates that the dynamics of a measurement $\hat{x}$ on a quantum eigenstate can be represented in terms of stochastic trajectories propagating forward and backward in time [72]. The measurement of $\hat{x}$ is modeled by a direct parametric amplification interaction $H_A$ which amplifies the $\hat{x}$ observable. In this case a decoupling of the measured variable $x$ from its complementary observable $p$ simplifies the solutions. Mathematically, the amplified trajectories that correspond to the final measurement outcome include stochastic noise from a future boundary condition, which is determined by the measurement setting. The complementary variable $p$ deamplifies and corresponds to a forward propagating trajectory.

In this paper, we analyze the measurement problem by considering a superposition $|\psi_{sup}\rangle$ of eigenstates $|x_j\rangle$ of the quadrature $\hat{x}$ for a single mode field. We derive a theorem for the equivalence for the $Q(x,p)$ function distribution and stochastic dynamics of classical-like amplitudes $x$ and $p$. The theorem uses a conditional boundary value to link trajectories associated with complementary observables $x$ and $p$ at any given time in the dynamics. This has the feature that the combination of trajectories that are sampled can be averaged and confirmed to correspond precisely to the $Q(x,p)$ function at the time $t$. Moreover, certain sets of $x$ and $p$ trajectories are coupled according to the boundary value, in a way that has not previously been calculable from quantum mechanics. This provides a description of a state not constrained by the uncertainty principle, giving a model “more complete” than quantum mechanics.

After correcting for growth due to dynamics, the inferred density of the measured amplitudes for $x$ is shown to correspond to the probability $P(x) = |\langle x|\psi\rangle|^2$ of detecting the result $x$, as predicted by Born’s rule for quantum mechanics. This motivates a realistic interpretation — objective field realism — that the amplitudes $x$ and $p$ correspond to a state for the system, with a probability $Q$ [72, 73]. The amplitudes cannot be directly measured, however. This is not due to any lack of reality, but is caused by the fact that even the best measuring instruments have vacuum noise, and interact with the system being measured. We will see that while the part of objective field that represents the eigenvalue for the eigenstate $|x\rangle$ is amplified, the stochastic noise $\xi(t)$ that contributes to the value of $x$ by an amount of order $\hbar$ is not amplified, consistent with the fact that the noise values at the time $t_0$ arise retrocausally from the future boundary condition at time $t_f$.

This leads us to also consider other models of realism, closer to the macroscopic realism understood by an observer. In particular, we propose a hybrid macroscopic-realism model which attributes to the macroscopic superposition state $|\psi_{sup}\rangle \sim |x_1\rangle + |x_2\rangle$ ($|x_1 - x_2| \gg 1$) at time $t = 0$ (prior to the measurement) a definite value for the (amplified) outcome for $\hat{x}$. The causal structure is developed from the tracking of the trajectories of $x$ and $p$ (Figure 1). The model has a hybrid form: given as causal macroscopically, but retrocausal microscopically. We show how this model does not satisfy Bell’s local hidden variable assumptions. The results are therefore not in conflict with the violation of Bell [8] or Leggett-Garg inequalities [75], even those that have been reported at a
Our work examines superpositions $|\psi_{\text{sup}}\rangle$ of two eigenstates $|x_1\rangle$ and $|x_2\rangle$ and also examines more general superposition states, including a superposition $|\text{cat}\rangle$ of two coherent states. The Schrödinger cat paradox is explained in the hybrid macroscopic-realism model not as a failure of macroscopic realism, but as a failure of the completeness of quantum mechanics. We illustrate this by noting how the system prepared in $|\psi_{\text{sup}}\rangle$ at time $t_0$ is amplified by the measurement $H_A$, giving a time $t_g$ at which macroscopic realism is realized according to the hybrid macro-realistic model. The stochastic $Q$ method allows evaluation of inferred states for $x$ and $p$ at the initial time $t_0$, post-selected on the final outcome $x_j$. The inferred states are not quantum states, because they violate the uncertainty relation. For macroscopic superposition states where $|x_1 - x_2| \gg 1$, however, the inferred state approaches the eigenstate $|x_j\rangle$. This gives the origin of the collapse of the wave function as arising from amplification. While the amplification based on the unitary evolution $H_A$ can be reversed, the irreversibility of the collapse is explained by a coupling to a second system.

Finally, we also analyze the measurement of a correlated entangled pair of modes, where one mode acts as a meter by which to measure the signal mode. We evaluate the inferred state of the signal mode, post-selected on the final outcome $x_j$ of the meter. The evaluation involves calculation of the marginal of the two-mode $Q$ function, which implies loss of information. In this way, we see that the “reduction” or “collapse” of the signal system into the eigenstate $|x_j\rangle$ associated with the measured eigenvalue $x_j$ arises both from the objective nature of the $Q$ model (in which the system has a definite set of values for the variables $x$ and $p$ prior to the measurement) and also from the loss of information occurring from the coupling to the second system.

The layout of the paper is as follows. In Section II, we summarize the main results and our approach to the measurement problem, explaining the different models for realism relevant to this paper. In Section III we give details of the stochastic method and of the stochastic boundary Theorem. In Section IV, we solve the equations modeling a measurement $\hat{x}$ of a single mode system in a superposition of eigenstates of $\hat{x}$, using the amplification model $H_A$. Our results are sufficiently general that we consider superpositions of squeezed and coherent states, and also the measurement of the complementary observable $p$. The natural realization of Born’s rule is explained in this section. The coupling of the $x$ and $p$ trajectories in accordance with the Theorem is explained in Section V, giving a mechanism for the collapse of the wave function. We give details of the realism models, the causal relations and the cat paradox in Section VI. The model for measurement based on entangling the system with a meter is given in Section VII. In Section VIII we explain how the retrocausal model will allow EPR entanglement and a violation of Bell and Leggett-Garg inequalities. A conclusion is given in Section IX.

II. MEASUREMENT PROBLEM AND MODELS OF REALISM

A. Measurement problem and a stochastic interpretation

In this paper, we use the interpretation that at all times the system dynamics is described by a unified stochastic model motivated by the $Q$ function [80]. We will investigate the interpretation of the measurement postulate for this stochastic model, which we refer to as the objective field model [13, 72].

Our examples here all use bosonic fields. The approach used here also requires the measurement process to be included in the dynamics. For this purpose, we use a particular model of an optimal measuring device that produces macroscopic outputs, namely the parametric amplifier. These are not fundamental physical restrictions. Fermionic $Q$-functions exist also [81, 82], and have similar dynamical properties and equations. We expect other models of measurement to have similar behavior.
Figure 3. Measurement of \( \hat{p} \) for a system prepared at time \( t_0 \) in a superposition \( |\psi_{sup}\rangle \) of two eigenstates \( |x_1 \rangle \) and \( |x_2 \rangle \), with \( x_1 = 3 \) and \( r = 2 \). The top plot shows trajectories for \( p \) propagating according to \( H_A \) with \( g < 0 \). The final values of \( p \) at \( t_f \) are amplified. In a deterministic-contextual-realism model, the mapping \( p_{j\text{out}} \rightarrow p_j \) at time \( t_f \) is a causal relation. However, the trajectories for \( p \) propagate backwards, from the future boundary at \( t_f \), subject to noise with variance \( \sigma_p \). The lower plot shows forward propagating trajectories for the complementary observable \( x \), which deamplify to the vacuum noise level \( \sigma_x = 1 \).

In the measurement problem a quantum state is expressed as a linear combination of eigenstates \( |x_j\rangle \) of \( \hat{x} \)

\[
|\psi_{sup}\rangle = \sum_j c_j |x_j\rangle
\]

(1)

where \( c_j \) are probability amplitudes. The measurement postulate asserts that for such a state, the set of possible outcomes of the measurement \( \hat{x} \) is the set \( \{x_j\} \) of eigenvalues of \( \hat{x} \). The probability of a given outcome \( x_j \) is \( P_j = |c_j|^2 \). After the measurement, the system “collapses” into the eigenstate \( |x_j\rangle \) associated with that outcome \( x_j \). The measurement problem is to understand the transition from the state \( |\psi\rangle \) to the final state \( |x_j\rangle \). To treat physically realizable states, we replace the eigenstates \( |x_j\rangle \) with squeezed states \( |x_j, r\rangle \) that have a finite precision, so that eigenstates are obtained for \( r \rightarrow \infty \).

Here we outline different models for realism associated with the objective field model, and explain the links with other approaches. To do this, we first summarize the essential results of our approach. Restricting our analysis to a single mode for clarity, we define the \( \hat{x} \) and \( \hat{p} \) observables \( \hat{x} = \hat{a} + \hat{a}^\dagger \) and \( \hat{p} = (\hat{a} - \hat{a}^\dagger)/\sqrt{2} \) where \( \hat{a} \) is the boson annihilation operator \( \hat{a} \). The coherent state \( |\alpha\rangle \) satisfies \( \hat{a}|\alpha\rangle = \alpha|\alpha\rangle \) where \( \alpha = (x + ip)/\sqrt{2} \). The single-mode Q function \( Q(x, p) = \frac{1}{4\pi a^2} |\langle \alpha | \psi \rangle|^2 \) defines the quantum state \( |\psi\rangle \) uniquely as a positive probability distribution \( \psi \).

Without losing any essential features, we consider \( |\psi_{sup}\rangle \) with two states \( |\pm x_1, r\rangle \), where \( r \) indicates the level of squeezing, and let \( c_1 \) real and \( c_2 = i|c_2| \). This gives

\[
Q_{sup}(x, p) = \frac{e^{-x^2/2\sigma_x^2}}{2\pi \sigma_x \sigma_p} \left\{ |c_1|^2 e^{-(x-x_1)^2/2\sigma_x^2} \right. \\
+ |c_2|^2 e^{-(x+x_1)^2/2\sigma_x^2} \\
- 2c_1^* c_2 e^{-(x^2+x_1^2)/2\sigma_x^2} \sin \left[ px_1/\sigma_p^2 \right] \left\}.
\]

(2)

where \( \sigma_x^2 = 1 + e^{-2r} \) and \( \sigma_p^2 = 1 + e^{2r} \). In the limit of \( r \rightarrow \infty \), the \( Q \) function has two Gaussian peaks with fixed variance \( \sigma_x^2 = 1 \), centered at the eigenvalues \( x_1 \) and \( x_2 \), along with a central peak centered at \( x = 0 \) whose amplitude has a fringe pattern which is damped by a term \( e^{-x_1^2/2} \).

We will take a simple model of measurement. To mea-
We will see that for place by amplifying boundary condition, implying retrocausality (Figures 1, 2, 3, lower plots). The complementary observable possible level of the vacuum (Figures 1, 2, 3, lower plots). The means correspond to the eigenvalues \( x_j \), which correspond to the means of the Gaussians in (2) are amplified according to

\[
 x_j \rightarrow X_j = e^{\delta t} x_j
\]

and are therefore ultimately measured (Figure 1, top plots). The means correspond to the eigenvalues \( x_j \), which are therefore the results of the measurement. In the amplification process \( H_A \), the interference term of Eq. (2) is attenuated because of the proportionality to terms \( e^{-x_j^2} \) which decay on amplification of \( x_j \) where \( x_j \rightarrow e^{\delta t} x_j \). The measurement of \( \hat{x} \) amplifies the \( x_j \) variables, but attenuates the \( p \) variables to the minimum possible level of the vacuum (Figures 1, 2, 3, lower plots). The complementary observable \( \hat{p} \) therefore does not appear in the final probability. The amplified \( x_j \) at the time \( t_f \) therefore have a final probability distribution

\[
P(x_j) = Q(x_j, t_f) = |\langle x_j | \psi \rangle|^2
\]

in agreement with the quantum prediction.

However, while the eigenvalues \( x_j \) are amplified, we note that the noise about these values is not amplified. Hence, the \( x_\delta = x - x_j \) is not measurable. We will see that these values for the trajectories are determined by a future boundary condition, implying retrocausality (Figures 1, 2, 3, upper plots).

It is also necessary to consider the measurement of the complementary observable \( \hat{p} \). The \( \hat{p} \) measurement takes place by amplifying \( p_j \), using a different \( H_A \) with \( g < 0 \). We will see that for \( |\psi_{sup}\rangle \) the probability for observing \( p \) at the time \( t_f \) after evolution is

\[
 Q(p, t_f) \rightarrow 1 - \sin[pe^{-|g|/2}(x_1 - x_2)]
\]

which gives a fringe pattern in the amplified variable \( Gp \) (Figure 1, lower right plot), in agreement with the quantum prediction

\[
P(p_j) = Q(p_j, t_f) = |\langle p_j | \psi_{sup} \rangle|^2
\]

for the state \( |\psi_{sup}\rangle \). Here, \( |p\rangle \) is the eigenstate for \( \hat{p} \). The details of the calculations are given in the later sections.

### B. Models for realism

The \( Q \) function is a joint probability distribution for the objective fields \( x \) and \( p \), but the values \( x \) and \( p \) do not directly correspond to the measurable outcomes until one adds a model for the meter itself. As explained in Section VII, this cannot be interpreted as a Bell local hidden variable theory. This is because of the boundary conditions imposed in the future in our implementation of the meter trajectory interpretation. Our model therefore is a contextual model in terms of the outcomes for measurements of \( \hat{x} \) and \( \hat{p} \). These results lead us to consider additional macroscopic models for realism, which we examine in this paper.

#### 1. Objective field model

The first model specifies hidden variables, allowing for a broader conception of realism [43, 42, 73]. The single-mode \( Q \) function gives a probability for amplitudes \( x \) and
$p$ at a given time. These amplitudes determine the future measurement outcomes, subject to boundary conditions both in the past and future, depending on the measurement setting, that is, whether $\hat{x}$ or $\hat{p}$ is measured. In this model, it is postulated that the amplitudes $x$ and $p$ provide an objective realistic picture for the measurement process.

The joint density of values for $x$ and $p$ at the time $t$ corresponds to the $Q(x, p, t)$ function. This gives the correct result for the quantum prediction $P(x)$ for a measurement of $\hat{x}$, after taking into account the dynamics of the measurement process which amplifies the $\hat{x}$. The interpretation of this model for realism is simply that the system at any time $t_0$ is in a state specified by the values $x$ and $p$ with probability $Q(x, p, t_0)$.

This is the simplest interpretation, providing objective trajectories which are time-symmetric. Such an objective model, like the fundamental physical laws, has no preference for past or future. Additionally, it makes no distinction between measurement or any other physical process, and does not require a human agency to define reality.

If human agency is invoked, a common way to define realism is via experimental preparation and measurement. Therefore, we also consider such conventional definitions of macroscopic realism. These provide an epistemological “ladder” to understand the objective, microscopic field picture. Anthropomorphic interpretations like this may be boldly cast aside later, in the sense of Wittgenstein [34].

2. Deterministic contextual realism

The second model (DCR) adopts the more conventional definition of realism that the outcome for a future measurement $\hat{x}$ on the state given by $|\psi_{sup}\rangle$ or its $Q$ function at time $t_0$ is determined at the time $t_0$. The outcome can be described by a variable $\lambda$. Here, realism is linked with causality, that the future is affected by the present and past (not vice versa). In such a model, a given system at time $t_0$ is in a state with a definite value $\lambda$ for the measurement of $\hat{x}$. The definite value is one of the values $\{x_j\}$ that are the eigenvalues of the measurement. Hence, the set of $\lambda$ is the set $\{x_j\}$. The model does not imply that the state is the eigenstate $|x_j\rangle$, but allows a different formulation for the state. In such a model, the probability that the system is in a state giving outcome $x_j$ is indeed $|c_j|^2$.

This model needs qualification. In its simplest form, the model would apparently contradict the knowncontextuality of quantum mechanics [30]. The wave function $|\psi_{sup}\rangle$ can be written as a linear combination of eigenstates of either $\hat{x}$ or $\hat{p}$. The simplest form of the model would then imply simultaneous specification (at time $t_0$) of the outcomes for both $\hat{x}$ and $\hat{p}$, which can be negated.

The $Q$ function is a positive distribution function of variables $x$ and $p$, but it does not give a positive joint probability for outcomes $x = x_j$ and $p = p_j$ at time $t_0$. Hence, the DCR model is specified to account for contextuality. The deterministic contextual realism model specifies that the interpretation of realism can be given for the final amplified value $Gx_j$ (or $Gp_j$), but only in a context where the measurement choice and setting is specified. Such models have been discussed elsewhere [37, 39]. A unitary interaction $U(\theta)$ is required as the first stage of the measurement, in order to specify the measurement setting, for instance, as in the selection of a polarizer angle $\theta$ in a Bell-inequality experiment which measures a spin $S_\theta$. It can then be regarded that the system satisfies deterministic realism for that measurement $S_\theta$ after this interaction $U(\theta)$ has occurred. The solutions of this paper are consistent with the DCR model of realism.

3. Macroscopic realism model for the pointer measurement

We give a third model for realism linked with macroscopic realism, which builds from the model of deterministic contextual realism in a way that naturally takes into account the importance of the measurement setting. We consider a macroscopic superposition

$$|\psi_m\rangle \sim c_1|x_1\rangle + c_2|x_2\rangle$$

of two eigenstates, the term macroscopic meaning that $|x_1 - x_2| \gg 1$. Here units are such that $\hbar = 1$. The outcome of the measurement $\hat{x}$ is either $x_1$ or $x_2$. The macroscopic realism model postulates that the system in the macroscopic state $|\psi_m\rangle$ can be ascribed a definite value for the final outcome of the amplified measurement, which we call the pointer measurement [12, 79, 90, 91]. The value of $\lambda$ is then either $x_1$ or $x_2$, indicating the two final possible amplified outcomes. We call this a macroscopic realism (MR) model. This model is macroscopically causal in that the value of $\lambda$ at time $t_0$ determines the measurement outcome at the later time $t_f$.

The measurement process is modeled as the a dynamical interaction $H_A$. Once it begins, at some time $t_0$ there is an amplification of the system to give a state such as $|\psi_1\rangle$. This is the point made in the Schrodinger cat paradox [22]. Hence the adoption of the macroscopic realism model is consistent with the adoption of the deterministic contextual realism model, because the amplification of $x$ (as opposed to $p$) amounts to the specification of measurement setting. The solutions given in this paper support the MR model.

4. Hybrid retrocausal - macroscopic realism model

The macroscopic realism model implies the outcome of the measurement $\hat{x}$ is specified at time $t_0$ i.e. the causal relation is that the outcome at time $t_f$ depends on the state at $t_0$. This would seem to counter the claim of this paper of a retrocausal model of measurement.
Here, the simulations of the trajectories given in this paper will provide insight. The trajectories for the amplified variable $x$ propagate backward in time. This means that the starting point for each trajectory is at the time $t_f$. The initial values $x_f$ are sampled based on the final marginal $Q(x, t_f)$ for $x$. This simple approach is possible because the measurement Hamiltonian $H_A$ allows a decoupling of the amplified and attenuated variables, $x$ and $p$. The approach is to select the value $x_f$ with the given probability. This may indeed correspond to a model in which the macroscopic values $Gx_1$ or $Gx_2$ are not determined at $t_0$.

As we will see, the trajectories beginning from the amplified $x_f \sim Gx_f$ correspond to the initial Gaussian peak for $x$ with a mean of $x_f$ (Figure 1, upper graph). Hence, we argue the causal interpretation of the macroscopic realism model is possible. In this model, for the macroscopic superposition state $|\psi_m\rangle$ of Eq. (9), the value for the outcome, whether $x_1$ or $x_2$, is determined for a given prepared state at the time $t_0$. The sampling for the trajectories is done the same way in such a model. The initial values $Gx_1$ and $Gx_2$ are selected with probabilities $|c_1|^2$ and $|c_2|^2$, as for a system that at time $t_0$ is in a statistical mixture of the two states $x_1$ and $x_2$.

On the other hand, we cannot argue that the selection of the precise value $x_f - Gx_1$ is causal because this value originates at the time $t_f$ in the simulation. The constraint on the selection of this noise term that comes from the earlier time $t_0$ is only that of the variance $\sigma_f^2 = 1$ (in units where $\hbar = 1$). We therefore argue that there is genuine retrocausality but at the microscopic level of $\hbar$.

C. Links with other approaches

It is often thought that retrocausal models are not likely to be correct because retrocausality is not observed in everyday life, that is, at a macroscopic level. The solutions we give show how retro-causation can exist microscopically, while macroscopic realism and macroscopic causality hold macroscopically. A second criticism is that classical causal models allowing retro-causation require fine-tuning of the parameters of the causal model in order for the model to be consistent with the observed independences of variables e.g. no-signaling [31]. Proofs are however restricted to acyclic causal models, and do not take into account the micro-macro hybrid nature of the trajectories, and our use of conditional boundaries. In Section VI, we provide a diagram of the causal relations associated with a deterministic contextual model.

III. OBJECTIVE Q MODEL FOR DYNAMICS

A. Stochastic model for dynamics

We now analyze the measurement using a $Q$ function model. The $Q$ function probability density $Q(\lambda, t)$ for a phase-space coordinate $\lambda$ is defined with respect to a non-orthogonal operator basis $\hat{\Lambda}(\lambda)$ as

$$Q(\lambda, t) \equiv \text{Tr} \left( \hat{\Lambda}(\lambda) \hat{\rho}(t) \right)$$

(10)

where $\rho$ is the density operator of the system. As $Q(\lambda, t)$ is normalized to unity, it is necessary to normalize the basis so it integrates to unity, and the normalization condition is that $\int \hat{\Lambda}(\lambda) d\lambda = 1$. The basis satisfies $\hat{\Lambda}^2(\lambda) = N(\lambda) \hat{\Lambda}(\lambda)$, different to the usual condition for projectors that $\hat{P}^2 = \hat{P}$, because it is a continuous non-orthogonal basis. From the Schrödinger equation, the dynamics of the probability distribution is obtained from the usual equation $i\hbar \frac{d\hat{P}}{dt} = [\hat{H}, \hat{P}]$. As a result, one obtains an equation for the $Q$-function time-evolution:

$$\frac{dQ(\lambda, t)}{dt} = \frac{i}{\hbar} \text{Tr} \left\{ [\hat{H}, \hat{\Lambda}(\lambda)] \hat{\rho}(t) \right\}$$

(11)

This is equivalent to a zero-trace diffusion equation for the variables $\lambda$, of form: $iQ(\lambda, t) = L(\lambda) Q(\lambda, t)$ where $L(\lambda)$ is the differential operator for $Q$-function dynamics. While the examples given here are bosonic, such distributions can be extended to include fermions as well [31, 32].

We can calculate directly how the $Q$-function evolves in time for a Hamiltonian $H$. To solve for single-mode evolution, we use the bosonic function [30]

$$Q(\alpha) = \frac{1}{4\pi} \langle \alpha | \rho(t) | \alpha \rangle$$

(12)

defined with respect to the nonorthogonal basis of coherent states $|\alpha\rangle$, where the phase-space coordinates $\lambda$ are the real coordinates $x$ and $p$ defined by $\alpha = (x + ip)/2$, and normalized with respect to integration over the $x,p$ variables. Here, $\hat{\alpha}|\alpha\rangle = \alpha|\alpha\rangle$ and we may write $Q(\alpha) \equiv Q(x, p)$. The moments evaluated from the $Q$ function distribution are anti-normally ordered operator moments.

We regard the $Q$ function amplitudes $\lambda (x$ and $p$ in this paper) as representing a reality for the superposition state at the time $t_0$, immediately prior to the measurement, as well during and after the measurement. The evolution of the $Q$ function is given by equation (11). There is an equivalent time-symmetric stochastic action principle for $Q(\lambda, t)$, leading to probabilistic path integrals. These have sample trajectories $\lambda$ that define the realistic path for all times. The important step is to solve the dynamical $Q$-function equations for trajectories of the amplitudes $\lambda (x$ and $p$) with time as the measurement process evolves. For this purpose, we have derived Theorems I and II below, allowing us to determine the stochastic dynamics.
B. Conditional path integral equivalence theorem

1. Definitions and notation

The generalized Fokker-Planck equation (GFPE) satisfied by a Q-function is:

\[ \dot{Q}(\lambda, t) = \mathcal{L}(\lambda) Q(\lambda, t). \]  

(13)

We consider an \( \mathbf{x} \) measurement Hamiltonian. The \( \mathbf{p} \) measurement case is obtained by swapping \( \mathbf{x} \leftrightarrow \mathbf{p} \). The differential operator \( \mathcal{L}(\lambda) \) has both forward and backward components,

\[ \mathcal{L}(\lambda) = \mathcal{L}_p(\mathbf{p}) - \mathcal{L}_x(\mathbf{x}). \]  

(14)

Here \( \mathcal{L}_{p,x} \) are defined as:

\[ \mathcal{L}_p = \sum_j \left[ -\partial_p^j a^j_p + \frac{1}{2} \partial_x^j \partial_p^j d_{p}^{jk} \right], \]  

(15)

where \( \mathcal{L}_x \) is identical except for the substitution of \( x \) for \( p \).

For \( M \) physical modes, the phase-space vector \( \lambda \) is a 2\( M \)-dimensional real vector, and \( \mathbf{p}, \mathbf{x} \) are \( M \)-dimensional real vectors with \( \lambda = (\mathbf{p}, \mathbf{x}) \) and \( \partial^p_j \equiv \partial/\partial p^j, \partial^x_j \equiv \partial/\partial x^j \). Marginals have the notation of \( P(\mathbf{x}, t), P(\mathbf{p}, t) \), where the dimensions that are integrated over removed from the arguments. For the cases in this paper, we only require terms where \( a^p_p = a_p^\prime(\mathbf{p}) \) and \( a^x_x = a_x(\mathbf{x}) \), and we take the diagonal case of \( d_{p}^{jk} = d_{p}^{kj} \).

Define a total propagator as:

\[ G(\lambda, t | \lambda', t') = \lim_{\epsilon \to 0} \int e^{-S_{\text{tot}}_n} \mathcal{D}_n[\lambda], \]  

(16)

where \( \lambda \equiv \lambda_n, t = t_n = t' + n \epsilon \), the total action is:

\[ S_{\text{tot}}_n = \sum_{j=1}^{n} S_{j-1,j} \].

The action for the \( j \)-th step is \( S_{j-1,j} = S_{j-1,j}^{p,x} + S_{j-1,j}^{x} \), with

\[ S_{j-1,j}^{p,x} = S_{j-1,j}^{p,x}(\lambda_{j-1}, \lambda_j) = \frac{\epsilon}{2d_{p,x}^j} |v_{j}^{p,x}|^2, \]  

(17)

where:

\[ v_{j}^{p} = \frac{1}{\epsilon} (p_j - p_{j-1}) - a^p(p_{j-1}) \]

\[ v_{j}^{x} = \frac{1}{\epsilon} (x_j - x_{j-1}) - a^x(x_j). \]  

(18)

The path integral measure over all the coordinates is

\[ \mathcal{D}_n[\lambda] \equiv \frac{1}{(2\pi \epsilon)^{Mn}} \prod_{k=1}^{n-1} d\lambda_k. \]  

(19)

2. Theorem I:

The Q-function solution is given by integrating the product of the conditional, the total propagator and the final marginal probability \( P(\mathbf{x}, t) \) of the field at \( \mathbf{x} \):

\[ Q(\lambda, t) = \int G(\lambda, t | \lambda', t') C(\lambda', t') P(\mathbf{x}, t) \, d\lambda'. \]  

(20)

Here \( C(\lambda, t) \equiv C(p | \mathbf{x}, t) \) is the conditional probability of \( p \) once \( \mathbf{x} \) is known:

\[ C(p | \mathbf{x}, t) = \frac{P(\lambda, t)}{P(\mathbf{x}, t)}. \]  

(21)

Proof:

The total propagator can be written in a factorized form as:

\[ G(\lambda, t | \lambda', t') = G_p(p, t | p', t') G_x(\mathbf{x}, t' | \mathbf{x}, t). \]  

(22)

From now on, we omit the conditional arguments if there is no ambiguity, so \( G_p(p, t) = G_p(p, t | p', t') \), and \( G_x(\mathbf{x}, t') = G_x(\mathbf{x}, t' | \mathbf{x}, t) \). Since they are each constructed as a path integral solution of Fokker-Planck equations in reciprocal time directions, each Green’s functions \( G_{p,x} \) satisfies well-known results for Fokker-Planck equation Green’s functions [93, 95]:

\[ \frac{\partial}{\partial t} G_p(p, t) = \mathcal{L}_p (p) G_p(p, t) \]  

(23)

\[ \frac{\partial}{\partial t} G_x(\mathbf{x}, t') = \mathcal{L}_x (\mathbf{x}) G_x(\mathbf{x}, t'). \]  

To prove the required result, we will now show that \( Q(\lambda, t) \) as defined in Eq. 20 satisfies the GFPE, and also has the correct initial condition at \( t = t' \). Differentiating \( Q(\lambda, t) \) with respect to \( t \) gives that:

\[ \dot{Q} = \int \left[ \frac{\partial}{\partial t} G_p(p, t) \right] G_x(\mathbf{x}', t') C(\mathbf{x}', t') P(\mathbf{x}, t) \, d\lambda' \]

\[ + \int G_p(p, t) \frac{\partial}{\partial t} [G_x(\mathbf{x}', t') P(\mathbf{x}, t)] C(\mathbf{x}', t') \, d\lambda'. \]  

(24)

From Eq. 23, \( \partial G_p(p, t) / \partial t = \mathcal{L}_p (p) G_p(p, t) \). Clearly, \( G_x(\mathbf{x}, t' | \mathbf{x}, t) \) is a Green’s function, and therefore must satisfy a backward Kolmogorov equation in \( \mathbf{x}, t \) [93]. It follows that \( G_x(\mathbf{x}, t' | \mathbf{x}, t) P(\mathbf{x}, t) \) corresponds to the solution of an FPE with an initial value of \( P(\mathbf{x}, t) \) for \( t' < t \). Hence, this product must satisfy the negative-time version of the usual FPE equation in \( \mathbf{x}, t \), in terms of its argument \( \mathbf{x} \), which is that:

\[ \frac{\partial}{\partial t} [G_x(\mathbf{x}', t') P(\mathbf{x}, t)] = -\mathcal{L}_x (\mathbf{x}) G_x(\mathbf{x}', t' | \mathbf{x}, t) P(\mathbf{x}, t) \]
As a result, one immediately obtains that the path integral construction obeys the required GFPE:

\[
\dot{Q} = \left( L_p (p) - L_x (x) \right) Q
\]

Finally, by construction,

\[
\lim_{t \to t'} G (\lambda, t | \lambda', t') = \delta (\lambda - \lambda')
\]

hence, as \( t \to t' \), one has that:

\[
\lim_{t \to t'} Q (\lambda, t) = \int \delta (\lambda - \lambda') C (\lambda', t') P (x, t) d\lambda',
\]

\[
= C (p | x, t') P (x, t').
\]

From the definition of the conditional probability in Eq (21), this is the required initial condition for the Q-function.

3. Theorem II:

The path-integral solution corresponds to a forward-backwards stochastic equation with a specified conditional initial distribution in \( p \) and final marginal distribution in \( x \):

\[
p^f (t) = p^i (t_0) + \int_0^t a^p_p (t') dt' + \sqrt{d_p} \int_0^t dw^p_p
\]

\[
x^f (t) = x^i (t_f) - \int_t^{t_f} a^x_x (t') dt' - \sqrt{d_x} \int_t^{t_f} dw^x_x
\]

(25)

Here \( x (t_f) \) is distributed as \( P (x, t_f) \), \( p (t_0) \) is distributed conditionally according to \( C (p | x, t_0) \), and \( dw^p_p, dw^x_x \) are independent real Gaussian noises such that if \( \mu = 1, 2 \):\n
\[
\langle dw^\mu dw^{\mu'} \rangle = \delta^{\mu \mu'} dt.
\]

**Proof:**

The proof follows immediately using the factorized form of the Green’s function solution combined with the use of standard identities for stochastic equations and Fokker-Planck equations [43, 46, 77], evolving in opposite time directions.

\[
\lambda
\]

\[
x
\]

\[
p
\]

\[
\text{Figure 6. Schematic of the procedure for solving the backward/forward equations using the conditional distribution at the boundary. As for Figure 3 the conditional distribution } C(p|x) \text{ at the boundary } t = t_0 = 0 \text{ determines whether there is a causal relation between } \delta x \text{ and } \delta p \text{ at } t_0. \text{ Depicted is the causal relationship for the superposition } |\psi_{sup}(r)\rangle. \text{ For the mixture } \rho_{mix}, \delta x \text{ and } \delta p \text{ are independent at the boundary and the trajectories for } x \text{ and } p \text{ completely decouple.}
\]

In order to verify the quantitative accuracy of the numerically sampled distributions, they were statistically tested using \( \chi^{(2)} \) methods [98, 99] to compare them with the analytic solutions given here. The tested simulations used up to \( 2 \times 10^6 \) trajectories to obtain good statistics for the distributions. Such verification requires binning on a three-dimensional \((t, x, p)\) grid, to obtain numerical estimates of the probability distributions, combined with numerical integration of the analytic \( Q \) distribution over each bin, using Simpson’s rule for accuracy.

An example for the case described in Fig (9) below used time-steps of \( dt = 0.1 \), combined with a midpoint stochastic integration method for improved accuracy [100]. No significant improvements were found with smaller time-steps. Comparisons were made between the analytic and numerically sampled distributions with \( dx = 0.02 \) and \( dp = 0.05 \). This gave an average of \( \sim 55,000 \) comparison grid-points at each time step, after discarding bins with non-significant populations of \( N < 10 \), following standard practice [101].

In this typical test, the time-averaged statistical error was \( \chi^2 = 55.2 \times 10^3 \). There was an average of \( k = 55.1 \times 10^3 \) significant points per time-step. The statistic is within the expected range of \( \langle \chi^2 \rangle = k \pm \sqrt{2k} \). As anticipated, there is good agreement between quantum theory and stochastic fields. This uses time-averaging in order to cover the complete dynamics of the Q-function, and \( 1.7 \times 10^6 \) comparisons were made in total. Fluctuations are somewhat reduced by time-averaging, but individual tests at each time-point are in general accord with expectations.

C. Numerical tests

The analytic theorems obtained above were tested numerically in multiple cases described in this paper. The trajectory samples plotted in this paper use 40 sample trajectories, to allow improved visibility. Such plots provide an intuitive demonstration of how the trajectories behave.
IV. AMPLIFICATION MODEL FOR A MEASUREMENT OF $\hat{x}$

A. Model Hamiltonian: parametric amplification

We consider a single mode field. In a rotating frame, complementary quadrature phase amplitude operators are defined as $\hat{x} = \hat{a} + \hat{a}^\dagger$ and $\hat{p} = (\hat{a} - \hat{a}^\dagger)/i$ where $\hat{a}$, $\hat{a}^\dagger$ are annihilation and creation operators for the boson field. This implies $\Delta \hat{x} \Delta \hat{p} \geq 1$.

First, we consider the simplest measurement procedure that takes place — that of direct amplification. We model this by the parametric Hamiltonian $H_A = \frac{\theta t}{2}[\hat{a}^2 - \hat{a}^\dagger 2]$ given by Eq. (3) where $g > 0$ is real, which gives a amplification of $\hat{x}$ [34]. For $g > 0$, it is known that the dynamics of $H_A$ gives solutions that amplify the “position” $\hat{x}$ but attenuate the orthogonal “momentum” quadrature $\hat{p} = (\hat{a} - \hat{a}^\dagger)/2i$. This is clear from the standard operator Heisenberg equations which give the solutions

$$\hat{x}(t) = \hat{x}(0) e^{gt}$$
$$\hat{p}(t) = \hat{p}(0) e^{-gt}.$$  (27)

The Hamiltonian $H_A$ is equivalent to the Hamiltonian required to induce squeezing in $\hat{p}$. The solutions give for the means $\bar{x}$ and variances

$$\langle \{\Delta \hat{x}^2\} \rangle = \langle \hat{x}^2(t) - \bar{x}^2 \rangle = e^{2gt}$$
$$\langle \{\Delta \hat{p}^2\} \rangle = \langle \hat{p}^2(t) - \bar{p}^2 \rangle = e^{-2gt}$$
$$\bar{x} = \langle \hat{x} \rangle = \langle \hat{x}(0) \rangle e^{gt}$$
$$\bar{p} = \langle \hat{p} \rangle = \langle \hat{p}(0) \rangle e^{-gt}.$$  (28)

It will be useful to write the variances in terms of anti-normally ordered products [102]. The anti-normal ordering of $\hat{x}^2$ is given by

$$\langle \{\hat{x}^2\} \rangle = \langle \hat{x}^2 + 1 \rangle = e^{2gt} \langle \hat{x}(0) \rangle + 1.$$  (29)

Hence, if $\sigma_x^2(t) = \langle \{\hat{x}^2\} \rangle - \bar{x}^2$ is the anti-normally ordered variance, then

$$\sigma_x^2(t) = 1 + e^{2gt} \langle \hat{x}^2(0) \rangle = 1 + e^{2gt} (\sigma_x^2(0) - 1),$$  (30)

and similarly,

$$\sigma_p^2(t) = \langle \{\hat{p}^2\} \rangle - \bar{p}^2 = 1 - e^{-2gt} (\sigma_p^2(0) - 1).$$  (31)

The anti-normally ordered variances $\sigma_x^2(t)$ and $\sigma_p^2(t)$ are precisely the variances of $x$ and $p$ as defined by the Q function $Q(x, p, t)$.

B. The superposition state

We will consider a measurement on the system prepared at time $t_0$ in the superposition $|\psi_{sup}\rangle = \sum_i c_i |x_i\rangle$, where $|x_i\rangle$ is an eigenstate of $\hat{x}$ of the “position” quadrature with eigenvalue $x_j$. Applying the definition [12] and using the overlap function [39]

$$\langle x_j | \alpha \rangle = \frac{1}{\sqrt{\pi \lambda^4}} e^{i \frac{(x_j - x)^2}{4} + \frac{i}{2} \lambda (x_j - 2x)}$$  (32)

the Q function for this state is given by

$$Q(x, p) = \frac{1}{\pi \sqrt{\pi}} \sum_j c_j e^{-\frac{1}{4} (x - x_j)^2} e^{\frac{i}{2} \lambda (x_j - 2x)}.$$  (33)

which is a sum of Gaussians centered at each eigenvalue $x_j$ along with additional interference cross-terms. For the sake of simplicity without losing the essential features, we examine the simple superposition

$$|\psi_{sup}\rangle = c_1 |x_1\rangle + c_2 |x_2\rangle - x_1$$  (34)

We have also taken without loss of essential features $x_2 = -x_1$. Here the $c_j$ are complex amplitudes satisfying $|c_1|^2 + |c_2|^2 = 1$. The Q function simplifies to

$$Q_{sup}(x, p) \sim |c_1|^2 e^{-(x_1 - x)^2/2} + |c_2|^2 e^{-(x_1 + x)^2/2}$$
$$-2c_1 c_2 e^{x_1 x_2} e^{-(x_1 + x_2)^2/2} \sin(x_1 p)$$  (35)

where we take $c_1$ as real and $c_2 = i |c_2|$. The different choice of the phase of $c_2$ introduces an unimportant phase shift in the sinusoidal term. The Q function differs from that of the mixture $\rho_{mix} = \frac{1}{2} \{ |x_1\rangle\langle x_1 | + |x_2\rangle\langle x_2| \}$ by the addition of the third term which gives fringes. Without a momentum cutoff of $|p| \leq p_m \gg 1$, the distribution is not normalizable, as usual with pure position eigenstates in quantum mechanics. Since the limiting two-dimensional Q function cannot be normalized due to the infinite variance in $\hat{p}$, and in the last line we have written the normalized projection along the $x$ axis for a given $p$.

To obtain a more physical solution for [34], we model the position eigenstates as highly squeezed states in $\hat{x}$. The squeezed state is defined by [34]

$$|\psi(\beta, z)\rangle_{sq} = D(\beta) S(z) |\rangle$$  (36)

Here, $|\rangle$ is the vacuum state satisfying $\hat{a}|\rangle = 0$, and $D(\beta) = e^{\hat{a}^\dagger \beta - \beta^* \hat{a}}$ and $S(z) = e^{\frac{1}{2} (z^* \hat{x}^2 - z \hat{x}^2)}$ are the displacement and squeezing operators respectively, where $z$ and $\beta$ are complex numbers. For the state with squeezed fluctuations in $\hat{x}$, $z = r$ is a real positive number that determines the amount of squeezing in $\hat{x}$. Defining $\hat{x} = \langle \hat{x} \rangle$, $\bar{p} = \langle \hat{p} \rangle$, we find $\langle \Delta \hat{x}^2 \rangle = \langle \hat{x}^2 - \bar{x}^2 \rangle = e^{-2r}$, $\langle \{\Delta \hat{p}^2\} \rangle = \langle \hat{p}^2 - \bar{p}^2 \rangle = e^{2r}$, and $\langle \hat{a} \rangle = (\hat{x} + \bar{p})/2 = \beta \hat{a}$.

We write $\beta_j = (x_j + ip_j)/2$ where $x_j$ and $p_j$ are real. A position eigenstate $|x_j\rangle$ is thus a squeezed state with $r \to \infty$. We take $x_1$ and $x_2$ real, so that with $p_j = 0$. The superposition [34] then becomes the superposition of two squeezed states

$$|\psi_{sup}(r)\rangle = c_1 |\psi(x_1, r)\rangle_{sq} + i |c_2| |\psi(-x_1, r)\rangle_{sq}$$  (37)

We select $c_2 = i |c_2|$ for convenience so that the normalization procedure gives the above form for all values of $r$ and $x_1$. Otherwise, the normalization involves an extra term which vanishes in the limit where the two states forming the superposition are orthogonal. Here, this requires large $r$ i.e. $r \to \infty$. 

The $Q$ function for the squeezed state $|\tilde{\alpha}\rangle$ is

$$Q(x,p) = \frac{1}{2\pi\sigma_x\sigma_p}e^{-(x-x_0)^2/2\sigma_x^2}e^{-(p-p_0)^2/2\sigma_p^2},$$

(38)

where $\sigma_x^2 = 2(1 + \tanh r)^{-1}$ and $\sigma_p^2 = 2(1 - \tanh r)^{-1}$. Here, $\sigma_x^2$ and $\sigma_p^2$ are the variances of $x$ and $p$ for the $Q$ function distribution, which are given as

$$\sigma_p^2 = 1 + e^{2r},$$
$$\sigma_x^2 = 1 + e^{-2r}.$$  

(39)

This is in agreement with the different variances $\langle (\Delta \hat{x})^2 \rangle = e^{-2r}$ and $\langle (\Delta \hat{p})^2 \rangle = e^{2r}$ for the squeezed state $|\tilde{\alpha}\rangle$, once anti-normal ordering is accounted for. The full $Q$ function for $|\tilde{\alpha}\rangle$ is

$$Q_{sup}(x,p) = \frac{e^{-p^2/2\sigma_p^2}}{4\pi\sigma_x\sigma_p} \left\{ |c_1|^2 e^{-(x-x_1)^2/2\sigma_x^2} + |c_2|^2 e^{-(x-x_2)^2/2\sigma_x^2} - 2|c_1c_2|^2 e^{-(x^2+x_1^2)/2\sigma_x^2} \sin(px_1/\sigma_x^2) \right\},$$

(40)

which agrees with Eq. (35) for the idealized superposition $|\tilde{\alpha}\rangle$ in the limit of large squeezing $r$. This function is plotted in Figure 4.

C. $Q$ function stochastic equations

For the system evolving according to the Hamiltonian $H_A = \frac{\hbar \omega}{2} [\hat{a}^{\dagger 2} - \hat{a}^2]$ given by Eq. (3), a dynamical equation for the $Q$ function can be derived [72]. Applying the correspondence rules to transform operators into differential operators, one obtains a generalized Fokker-Planck type equation in terms of complex coherent state variables $\alpha$:

$$\frac{dQ^\alpha}{dt} = -\left[ g \frac{\partial}{\partial \alpha} \alpha^* + g \frac{\partial^2}{2 \partial \alpha^2} + h.c. \right] Q^\alpha$$

(41)

Using the quadrature definitions where the vacuum has unit noise, one has $\alpha = (x + ip)/2$, or $\hat{x} = \hat{a} + \hat{a}^\dagger$. We obtain

$$\frac{dQ}{dt} = \left[ \partial_p(gp) - \partial_x(gx) + g \left( \partial^2_p - \partial^2_x \right) \right] Q$$

(42)

This demonstrates a diffusion matrix which is traceless and equally divided into positive and negative definite parts, and a drift matrix:

$$A = \begin{bmatrix} \hat{x} \\ \hat{p} \end{bmatrix} = \begin{bmatrix} gx \\ -gp \end{bmatrix}$$

(43)

D. Forward-backward stochastic equations

Using the above equivalence Theorem, we solve the measurement dynamics for the dynamics of $H_A = \frac{\hbar \omega}{2} [\hat{a}^{\dagger 2} - \hat{a}^2]$. Here, the $x$ and $p$ dynamics decouple. However, to obtain a mathematically tractable equation for the traceless noise matrix, we follow [72] and the sign of $t$ is reversed in the amplified dynamics of $x$. We arrive at [72, 73]

$$\frac{dp}{dt} = -gp + \xi_p$$

(44)

with a boundary condition in the past, and

$$\frac{dx}{dt} = -gx + \xi_x$$

(45)

with a boundary condition in the future, where $t_- = -t$. Defining $\xi = (\xi_p, \xi_x) = [\xi^\mu]$, the Gaussian random noises $\xi^\mu(t)$ satisfy: $\langle \xi^\mu(t) \xi^\nu(t') \rangle = 2g\delta^\mu_\nu \delta(t-t')$. Thus, there is a forward-backwards stochastic differential equation (FBSDE), for individual trajectories. This describes two individual stochastic trajectories, such that the average of the dynamical trajectories equals the $Q$-function averages. The trajectories are decoupled dynamically, with decay and stochastic noise occurring in each of the time directions. One propagates forward, and one backwards in time.

The corresponding stochastic equations for $g > 0$ are:

$$p(t) = p(t_0) - \int_{t_0}^{t} gp dt' + \sqrt{2g} \int_{t_0}^{t} dw_p$$
$$x(t) = x(t_f) - \int_{t}^{t_f} gx dt' + \sqrt{2g} \int_{t}^{t_f} dw_x$$

(46)

where $\langle dw^\mu dw^\nu \rangle = \delta^\mu_\nu dt$. This leads us to consider a case where the trajectory in $x$ has a future marginal, $P(x,t_f)$. Also, the trajectory in $p$ has a past conditional distribution, $P(p,t_0|x)$, which depends on $x$ in the future (Figure 6), giving a type of acyclic causal behavior.

E. Solutions of the backward and forward trajectories: future boundary condition

To solve the trajectories for the field amplitudes, we stochastically sample according to the noise terms $\xi^\mu(t)$. From Eq. (45), we see that the $x$ solution at $t$ depends on the boundary condition for $x$ imposed in the future time $t_f$, after the measurement has been completed. We solve this equation first, by propagating the trajectories for $x$ backwards in time, starting from the boundary condition at $t_f$. We refer to these trajectories as backward trajectories.

For sufficiently amplified fields $x$, we justify modeling the final stage of measurement as a direct detection of the amplified amplitude $x$ at $t_f$. This model is justified because the amplitudes for $x$ at the final time have a distribution given by the final $Q$ function, $Q(x,t_f)$, which one
can show is precisely that corresponding to the quantum prediction $Q_{\text{sup}}(x,t_f) \rightarrow P(x) = |\langle x|\psi_{\text{sup}}\rangle|^2$. We proved this for a superposition of $\hat{x}$ eigenstates in Section II, and illustrate further by examples below and in Section IV.F.

To evaluate the sampling distribution for the future boundary at time $t_f$, we evaluate the evolving Q function $Q_{\text{sup}}(x,p,t)$ at the time $t = t_f$, from the Hamiltonian $H_A$ of Eq. (4) that amplifies $\hat{x}$. The original Q function in the present time $t_0$ is given by (40). We evaluate the Q function for the amplified system with respect to $x$, after the measurement $H$ at a time $t$. This calculation can be done in two ways. The state formed after the unitary evolution $H_A = e^{-iHt/h}$ for which the Q function can be evaluated directly as

$$Q_{\text{sup}}(x,p,t) = \frac{e^{-p^2/2\sigma_x^2(t)}}{4\pi\sigma_x(t)\sigma_p(t)} \left\{ |c_1|^2 e^{-\{x-G(t)x_1\}^2/2\sigma_x^2(t)} + |c_2|^2 e^{-\{x+G(t)x_1\}^2/2\sigma_x^2(t)} - 2|c_1||c_2| e^{-\{x^2+G^2(t)x_1^2\}/2\sigma_x^2(t)} \times \sin\left(\frac{pG(t)x_1}{\sigma_x^2(t)}\right) \right\},$$

(47)

where the amplification factor is $G(t) = e^{gt}$ and now we define

$$\sigma_{x/p}^2(t) = 2(1 + \tanh(r - gt))^{-1} = 1 + e^{\pm 2(gt-r)}.$$

The solutions can also be found from the dynamical equation (11) for the Q function. We denote the Q function at the future time $t = t_f$ as $Q_{\text{sup}}(x,p,t_f)$.

Due to the separation of variables $x$ and $p$ in Eqs. (44)-[15], the marginal distribution $Q_{\text{sup}}(x,t) = \int Q_{\text{sup}}(x,p,t)dp$ for $x$ at the future time $t = t_f$ determines the sampling distribution for the backward trajectories. This is depicted by the forward causal relation given by the solid blue line in Figures 5 and 6. The marginal for $x$ at the time $t \geq t_0$ is

$$Q_{\text{sup}}(x,t) = \frac{1}{\sqrt{2\pi}\sigma_x(t)} \left\{ |c_1|^2 e^{-\{x-G(t)x_1\}^2/2\sigma_x^2(t)} + |c_2|^2 e^{-\{x+G(t)x_1\}^2/2\sigma_x^2(t)} \right\}.$$ 

(49)

We note the variance can be written as $\sigma_x^2(t) = 1 + G^2(t) \{\sigma_x^2(0) - 1\}$ where $G(t) = e^{gt}$. It is worth adding that the marginal and hence resulting trajectories are identical to those given if the system were initially in the mixture $\rho_{\text{mix}}$ of the two position eigenstates, although this is not true for the corresponding momenta.

The backward trajectories for $r = 2$ are shown in Figure 7 and are depicted by the backward relation shown by the dashed red arrow in Figure 5. The key result is that regardless of the separation $\sim x_1$ between the eigenvalues associated with the two eigenstates $|x_1\rangle$ and $|-x_1\rangle$, the eigenstates are always distinguishable upon measurement for large $gt$, in the asymptotic limit of very large squeezing. The result arises because the fundamental noise given in Eq. (39) by $\sigma_x = 1$ (which is at the vacuum level) for the Q function eigenstate (38) as $r \to \infty$ is not amplified by $H_A$. This feature ensures the natural realization of the measurement postulate and Born’s rule, as we explain further in Section IV.H.

The forward trajectories for the attenuated variable $p$ are shown in Figure 8. These are depicted in Figure 3 by the forward-going red dashed arrows. The trajectories are calculated by solving Eq. (44) using the boundary condition at time $t_0 = 0$, which we refer to as the past or present time. The forward and backward equations decouple, and hence it is the marginal $Q_{\text{sup}}(p,0) = \int Q_{\text{sup}}(x,p)dx$ for $p$ at time $t = 0$ that is

Figure 7. Measurement of $\hat{x}$. Plot of backward propagating trajectories for the variable $x$ versus time $t_f$, for a superposition Eq. (37) of two position eigenstates $|x_1\rangle$ and $|-x_1\rangle$, with $r = 2$ and $|c_1| = |c_2|$. The trajectories originating from the boundary at $t_f$ are sampled from a distribution that is a 50/50 mixture of two Gaussians, with mean $e^{gt}x_1$ and $-e^{gt}x_1$. The top plot shows $x_1 = 8$ where the eigenstates are well separated. The final time is $t_f = 2/g$. The lower plot shows $x_1 = 1$ where the peaks are overlapping, with $t_f = 3/g$. At large $r$, the two eigenstates are always distinguishable at the final measurement $t_f$. 

...
relevant to the sampling. We find

\[ Q_{\text{sup}}(p, 0) = \frac{e^{-p^2/2\sigma_p^2}}{\sqrt{2\pi}\sigma_p} \left\{ 1 - e^{-x_1^2/2\sigma_p^2} \sin(px_1/\sigma_p^2) \right\}. \]

(50)

From this solution, we see that with increasing separation of the two eigenstates \(|x_1\rangle\) and \(|-x_1\rangle\) (so that \(x_1\) increases), the fringes become less prominent. The evolution of \(Q_{\text{sup}}(p, t)\) is shown in Figure 8.

Figure 8. Measurement of \(\dot{x}\). Plot of forward propagating trajectories for the attenuated variable \(p\) versus \(gt\) for the system prepared in a superposition \(|\psi_{\text{sup}}\rangle\) (Eq. (37)) of two position eigenstates, with \(r = 2\), and \(x_1 = 1\). The top plot shows trajectories with \(t_f = 3/g\). The second plot shows the reduction in the variance \(\sigma_p^2 = \langle(\Delta p)^2\rangle\) to a value of 1. The last plot shows the marginal \(Q_{\text{sup}}(p, 0)\) for \(p\) at the initial time \(t = t_0 = 0\), where fringes are evident.

\[ Q_{\text{sup}}(p,t) = \frac{e^{-p^2/2\sigma_p^2}}{\sqrt{2\pi}\sigma_p} \left\{ 1 - e^{-x_1^2/2\sigma_p^2} \sin(px_1/\sigma_p^2) \right\}. \]

(51)

which for large \(\alpha_0\) is the “cat state” [103, 104]. We take \(\alpha_0\) real, which corresponds to \(r = 0\) in the expression Eq. (37), with \(x_1 = 2\alpha_0\). The \(Q\) function is given by (37) with \(r = 0\) and \(x_1 = 2\alpha_0\).

The amplification process acts similarly to that of the superposition of two position eigenstates except it is noticeable that the quantum noise of the \(Q\) function has two contributions: The first noise contribution is that which exists for the eigenstate itself and is not amplified. The second noise term is that for the coherent state itself, corresponding to a measured noise level of \(\langle(\Delta \hat{x})^2\rangle = 1\). Necessarily, since it is measurable, this noise level is amplified with \(g\). This is evident from the Figure 10. The noise levels associated with the initial and final amplified peaks in \(x\) are increased when compared with those of the position eigenstates (Figure 7), due to the noise level \(\langle(\Delta \hat{x})^2\rangle = 1\) in \(\hat{x}\) of the coherent state. This noise level is amplified at time \(t_f\) consistently with the measurement \(H_A\) given by Eq. (3).

We can demonstrate the effectiveness of the model \(H_A\) for the measurement of \(\hat{x}\) by evaluating the final distribution \(Q_{\text{sup}}(x, t_f)\). The final \(Q\) function \(Q_{\text{sup}}(x, t_f)\) in the large amplification limit is given by (51), which corresponds to the \(P(x) = |\langle x |\text{cat}\rangle|^2\) predicted by quantum mechanics. Here, \(|x\rangle\) is the eigenstate for \(\hat{x}\). The equivalence with \(P(x)\) predicted by quantum mechanics is shown in Figure 11 for \(\alpha_0 = 2\). The marginal for \(x\) at large amplification where \(gt \to \infty\) can be written in
Figure 10. Measurement of $\hat{x}$ on a cat state. Plot of backward propagating trajectories for the amplified variable $x$ versus time $t$ in units of $g$, for the system prepared in $|\text{cat}\rangle$ (Eq. (51)). The plots show that the noise $\langle (\Delta \hat{x})^2 \rangle = 1$ associated with a coherent state is amplified and therefore measurable. The top plot shows $x_1 = 10$ where the coherent states are well separated. The second plot shows $x_1 = 2$. Here, the additional noise $\sigma_2^2 = 1$ associated with the $Q$ function ensures that the two peaks in the $Q$ function associated with the two coherent states overlap at $t = 0$, but there is no overlap in the amplified distribution at time $t_f$. For all plots, $t_f = 3/g$.

terms of the scaled variable $\hat{x} = x/e^{gt}$ as

$$Q_x(\hat{x}, t) = \frac{1}{2\sqrt{2\pi}} \left\{ e^{-(\hat{x}-x_1)^2/2} + e^{-(\hat{x}+x_1)^2/2} \right\},$$

(52)

where we use the result that $\sigma_2^2(t) \rightarrow e^{2gt}$. This is in agreement with the $P(x) = |\langle x|\text{cat}\rangle|^2$ predicted by quantum mechanics, as evaluated in [14] using $x_1 = 2\alpha_0$.

From Figure 12, we see that that the trajectories for $p$ for the cat state when $\hat{x}$ is measured are attenuated. The effect is less pronounced when compared to that of the superposition of position eigenstates (Figure 8), because there is a reduced noise in $p$ at the initial time. The measurement of $\hat{x}$ as given by $H_A$ amplifies $\hat{x}$ and squeezes $\hat{p}$. In fact, the noise levels for the initial cat state are approximately at the vacuum level $\langle (\Delta \hat{p})^2 \rangle \sim 1$, and the measurement Hamiltonian has the effect of squeezing the fluctuations in $\hat{p}$, as shown by the plot of the variance in $p$ in Figure 12.

Figure 11. Measurement of $\hat{x}$ on a cat state. Plot of backward propagating trajectories for the amplified variable $x$ versus time $t$ in units of $g$, for the system prepared in a cat state $|\text{cat}\rangle$ (Eq. (51)) with $x_1 = 4$ corresponding to $\alpha_0 = 2$. Also plotted is the distribution for $Q(x, t_f)$ after the amplification. This plot agrees with the distribution $P(x) = |\langle x|\text{cat}\rangle|^2$ given by quantum mechanics for the cat state. Here, $t_f = 4/g$.

G. Measurement of $\hat{p}$

So far, we have considered measurement of $\hat{x}$. We now consider the measurement of $\hat{p}$. This involves amplification of $p$, using a negative $g$ in the Hamiltonian $H_A$. We use

$$H_A = \frac{i\hbar g}{2} [\hat{a}^{\dagger 2} - \hat{a}^2]$$

(53)

where $g$ is real and $g < 0$. The dynamics from the standard operator Heisenberg equations gives the solutions

$$\dot{x} (t) = \dot{x} (0) e^{-|g|t}$$

$$\dot{p} (t) = \dot{p} (0) e^{-|g|t}$$

(54)

and we see that $\dot{p}$ is amplified. The solutions for the dynamics of the $x$ and $p$ variables of the $Q$ function are given as above, except that the $x$ and $p$ exchange roles. The trajectories for $p$ are amplified and propagate back in time. Those for $x$ are attenuated and propagate forward in time.

If we measure $\hat{p}$ by amplifying the $\hat{p}$ quadrature so that $g < 0$, then the full state at the later time is evaluated.
We write the solution as

\[ Q(p) = \frac{1}{2(\sigma^2_x(t))^{1/4}} \left\{ 1 - \sin(pG(t)x_1) \right\}, \]

(57)

using the scaled variable \( \tilde{\rho} = p/\sigma(\rho) = p/\sigma_p(t) \) and noting that \( \sigma_p(t) = e^{\rho/2} e^{\rho r} \) for large \( |\rho| \), with \( g < 0 \).

We can compare with the quantum prediction for the distribution \( P(p) \) for the outcome \( p \) upon measurement of \( \tilde{P} \), given as \( P(p) = |\langle p|\psi^{\text{sup}} \rangle|^2 \) where \( |p\rangle \) is the eigenstate of \( \rho \). We first compare for the cat state (51) with a real amplitude \( \alpha_0 \), where \( r = 0 \). In fact, the solution [83]

\[ P(\rho) = \frac{e^{-\rho^2/2}}{\sqrt{2\pi}} \{ 1 - \sin(2\alpha_0\rho) \} \]

(58)
in agreement with [57], upon noting that with the choice of quadrature scaling, \( x_1 = 2\alpha_0 \). Figure 13 shows the future marginal and the trajectories for \( p \), for large \( |\rho| \). As expected, the fringes are prominent. The comparison is done for the superposition of two eigenstates in Figure 11, giving exact agreement with the quantum prediction.
H. Realization of Born’s rule for measurements \( \hat{x} \) and \( \hat{p} \)

As the amplification \( G \) increases to a macroscopic level, the probability distribution evaluated by sampling over the trajectories for \( x \) becomes that of \( P(x) \), given by quantum mechanics. Here, \( P(x) = |\langle x | \psi \rangle|^2 \) is the probability for detecting the value \( x \) on measurement of \( \hat{x} \) for the system prepared in the state \( |\psi\rangle \). This mechanism was explained in Section II and is evident in the Figures which display the probability for the relative outcomes \( x_1 \) or \( -x_1 \) for the simple superposition Eq. \((37)\). The amplified \( Q \) function is

\[
Q_{sup}(x, t) = \frac{1}{2\sqrt{2\pi}\sigma_x(t)} \left\{ |c_1|^2 e^{-(x-G(t)x_1)^2/2\sigma_x^2(t)} + |c_2|^2 e^{-(x+G(t)x_1)^2/2\sigma_x^2(t)} \right\},
\]

where \( G = e^{gt} \) (\( g > 0 \)) and \( \sigma_x^2(t) = 1 + e^{2gt-2r} \). Defining the scaled variable \( \tilde{x} = x/e^{gt} \), this becomes

\[
Q_{sup}(\tilde{x}, t) = \frac{1}{2\sqrt{2\pi}\sigma_{\tilde{x}}(t)} \left\{ |c_1|^2 e^{-(\tilde{x}-x_1)^2/2(e^{2gt}+e^{-2r})} + |c_2|^2 e^{-(\tilde{x}+x_1)^2/2(e^{2gt}+e^{-2r})} \right\},
\]

On taking \( gt \) large, this gives

\[
Q_{sup}(\tilde{x}, t) = \frac{1}{2\sqrt{2\pi}} \left\{ |c_1|^2 e^{-(\tilde{x}-x_1)^2/(2e^{-2r})} + |c_2|^2 e^{-(\tilde{x}+x_1)^2/(2e^{-2r})} \right\},
\]

The eigenstates of \( \hat{x} \) correspond to \( r \to \infty \) where the Gaussian peaks become infinitely narrow. The value of \( x_1 \) which corresponds to the separation between the peaks is much larger than the width of the Gaussian distributions.

The relative probability for observing \( x_1 \) or \( -x_1 \) is given by the relative weighting of the peaks, \( |c_1|^2 \) and \( |c_2|^2 \), in accordance with Born’s rule for the measurement postulate. The amplified \( Q \) function becomes \( P(x) \) because in this limit of large amplification the vacuum noise term given by \( 1 \) in the expression Eq. \((59)\) for \( \sigma_x^2 \) is not amplified, whereas \( x_1 \) and any extra noise above \( 1 \) is amplified. The analysis can be extended for an arbitrary expansion \( |\psi\rangle = \sum_i c_i |x_i\rangle \) where \( |x_i\rangle \) are eigenstates of \( \hat{x} \), to validate Born’s rule for \( \hat{x} \). The example of validation of \( P(x) = |\langle x | \psi_{sup}\rangle|^2 \) where there is a continuum of states \( |x_i\rangle \) is given by the example of the cat state, Figure 11.

We similarly demonstrate Born’s rule for measurement \( \hat{p} \). We expand in eigenstates \( |p\rangle \) of \( \hat{p} \) as

\[
|\psi_{sup}\rangle_p = \sum_j d_j |p_j\rangle,
\]

where \( d_j \) are probability amplitudes. Using the overlap function

\[
\langle p_j | \alpha \rangle = \frac{1}{\pi^{1/4}} \exp\left(-\frac{(p-p_j)^2}{4} - \frac{i}{2\sigma_r(p_j-2p)}\right),
\]

the \( Q \) function for this state is given by

\[
Q(x, p)_p = \frac{1}{\pi \sigma_r} \sum_j d_j e^{-\frac{1}{2}(p-p_j)^2 - \frac{i}{2\sigma_r(p_j-2p)}}. \tag{64}
\]

This is a sum of Gaussians combined with interference cross terms. By analogy with the analysis of Section II for \( \hat{x} \), we see that on amplification of \( \hat{p} \) the final distribution \( Q(x, p, tf) \) will become a set of narrow Gaussians in \( p \) centered at \( p_j \) with weighting \( d_j^2 \). The interference terms vanish in the limit of large amplification. The realization of Born’s rule for \( \hat{p} \) is evident in the examples of Section IV.G (Figure 13).

Born’s rule arises naturally if one can interpret that at the time \( t_0 \) the system is in a state with a definite outcome \( x_j \) with probability \( |c_j|^2 \). This requires that the amplified outcome \( gx_j \) be determined by the state, at the time \( t_0 \). From Section II, we see that this is the case for the contextual deterministic realism and macroscopic-realism models. Born’s rule for \( \hat{x} \) (and \( \hat{p} \)) arises naturally from that interpretation at that time \( t_g \).

V. COUPLING THE X AND P TRAJECTORIES: MODELING THE COLLAPSE

How then do the results for the trajectories differ from that of the mixture \( \rho_{mix} \) of the eigenstates \( |x_j\rangle \) and how do the results elucidate the meaning of “collapse”? We examine the coupling between the \( x \) and \( p \) trajectories, where one measures \( \hat{x} \).

At time \( t = t_0 = 0 \), the \( Q \) function Eq. \((35)\) and \((40)\) for the superposition \( |\psi_{sup}\rangle \) gives correlations between \( x \) and \( p \). We note this is not the case for the mixture \( \rho_{mix} \), because the \( Q \) function for the eigenstate \( |x_1\rangle \) is separable between \( x \) and \( p \). The \( x \) and \( p \) trajectories for the superposition (but not for a mixture \( \rho_{mix} \)) are therefore coupled. The causal relations are depicted in Figure 6.
A. Conditional distribution at the boundary

We see that a given \( x_f \) from the future time \( t_f \) propagates for each trajectory to a single \( x_p \) in the present time, at \( t_0 \). For each \( x_p \), there is a set of \( p_p \) at the present time. This set is given by the conditional distribution \( P(p_p|x_p) = Q_{\text{sup}}(x_p,p_p)/P(x_p) \) evaluated from the \( Q \) function in the present, where \( t = t_0 = 0 \). Here, \( P(x_p) \) is given by the marginal Eq. \((49)\) at time \( t_0: \)

\[
P(x_p) = Q(x_p, 0). \]

We find

\[
Q(p_p|x_p) = Q_{\text{sup}}(x_p, p_p)/Q_{\text{sup}}(x, 0)
\]

\[
= e^{-r_p^2/2\sigma_p^2} \{ 1 - \sin(p_p x_1/\sigma_p^2) \cos(h x_1/\sigma_p^2) \},
\]

which becomes \(~ 1 - \sin(p_p x_1) \text{sech}(x_1 x_1)\) for \( r \) large. Looking at this function, fringes are evident, these becoming finer for large \( x_1 \), and also increasingly damped, provided \( x_p \neq 0 \). For smaller \( x_1 \), the fringes will be more prominent, regardless of \( G \). The conditional distribution implies that the trajectories for \( x \) and \( p \) are coupled i.e. correlated. For a set of values of \( x_f \) at the time \( t_f \), we can match the set with a set of \( p \) trajectories by propagating each given \( p_p \) from the sample generated by \( Q(p_p|x_p) \), back to the future \( t_f \). We then have sets of variables \( \{ x_f, x_p, p_p, p_f \} \) and all intermediate values on the trajectories. Such sets of trajectories are plotted in Figure 14.

The coupling of the \( p \) trajectories with those for \( x \) is determined by \( Q(p_p|x_p) \). We ask how does this depend on which state \( |x_1\rangle \) or \(-x_1\rangle \) the system is measured to be in? The function \( Q(p_p|x_p) \) is independent of the sign of \( x_p \) and hence, is not sensitive to which state is measured. This is evident in Figure 14 which plots the distribution for the trajectories of \( p \) conditioned on a positive final outcome i.e. \( x_f > 0 \). In the causal model for the deterministic-contextual-realism (or for large \( x_1 \), the macroscopic realism models), this implies that the conditional relation giving \( \delta p \) from \( \delta x \) is not dependent on \( \lambda \) (Figure 5). It is important to note that the conditional distribution \( Q(p_p|x_p) \) for the mixture \( \rho_{\text{mix}} \) of the two eigenstates is given by

\[
Q(p_p|x_p) = e^{-r_p^2/2\sigma_p^2} \sigma_p/\sqrt{2\pi},
\]

which implies complete independence of the trajectories for \( x \) and \( p \) (Figure 5). It is the coupling of the trajectories that distinguishes those of the superposition \( |\psi_{\text{sup}}\rangle \) from the mixture \( \rho_{\text{mix}} \). The trajectories for \( x \) and \( p \) themselves follow identically from the marginals \( Q(x, t_f) \) and \( Q(p, t_0) \) in each case and are indistinguishable.

Next, we ask what happens if we post-select on the final outcome of the measurement for \( \hat{x} \) being \( x_1 \), that is, \( x_f > 0 \)? What can we say about the state at the initial time \( t_p \) in the present? We evaluate the sets \( \{ x_f, x_p, p_p, p_f \} \) conditioned on \( x_f > 0 \), for large amplification \( G \), which corresponds to a strong measurement. In Figure 14 we depict such sets for \( x_f > 0 \), and the forward trajectories in \( p \) for the set \( x_f > 0 \), and also the related conditional distribution for \( p \).

B. Inferred initial state for \( x \) and \( p \) on post-selection of future outcomes

On measurement, one would be inferring the state \( |x_1\rangle \) or \(-x_1\rangle \) of original distribution, at \( t = 0 \). Relevant is
that we can bin the final outcomes into positive and negative $x_f$, which would allow the final observer to infer either $x_1$ or $-x_1$ as the “result of the measurement”, in the limit of large amplification $gt$. In the reality models, one can infer the state corresponding to the coupled trajectories.

In this section, we evaluate the joint distribution $Q_+(x,p,t_0)$ at $t_0 = 0$ corresponding to the sets of coupled trajectories based on the post-selected outcome of $x_1$ for the measurement $\hat{x}$. As explained in the last section, these are evaluated by way of the conditional boundary at time $t_0 = 0$. This distribution $Q_+(x,p,0)$ has the interpretation in the deterministic-contextual and macroscopic-realism models of being the “state” inferred at time $t_0 = 0$, if one measures that the outcome $x_1$ for $\hat{x}$. Here, we evaluate the variances $\sigma_{x,+}^2$ and $\sigma_{p,+}^2$ of the inferred distribution $Q_+(x,p,0)$, and show that they correspond to variances below those possible for the Heisenberg uncertainty principle.

We proceed as follows. For sufficiently large $gt$, each $x_f$ is either positive or negative, associated with the outcome $x_1$ or $-x_1$ which we denote by $+$ or $-$. We can trace the trajectories in $x$ back to the time $t_0 = 0$ given the post-selection of $x_f > 0$, and construct the distribution of $x$ at time $t_0 = 0$ for all such trajectories, as explained in the last sections. At the boundary in the present time $t_0 = 0$, each value of $x$ is coupled to a set of trajectories in $p$, according to the conditional distribution Eq. (65). We can thus construct a joint distribution $Q_+(x,p,t_0) = 0$ (Figure 15). We then may determine the variances $\sigma_{x,+}^2 \equiv \sigma_{x,+}^2(0)$ and $\sigma_{p,+}^2 \equiv \sigma_{p,+}^2(0)$ for $x$ and $p$ for this distribution, and define the associated observed variance for the present $x_p$ and $p_p$ once anti-normal ordering is accounted for:

$$\Delta(x_p|+)^2 = \sigma_{x,+}^2(0) - 1 \quad \Delta(p_p|+)^2 = \sigma_{p,+}^2(0) - 1.$$  (67)

Similar variances $\sigma_{x,-}^2$, $\sigma_{p,-}^2$, $[\Delta(x_p|-)\Delta(x_p|-)]^2$ and $[\Delta(p_p|-)\Delta(p_p|-)]^2$ could be determined for the trajectories post-selected on the $x_f < 0$ corresponding to the outcome $-x_1$. This tells us what we infer about the original state (in the reality model) at time $t = 0$ based on the measurement outcome given by the sign of $x_f$, whether $+$ or $-$. Here, by subtracting the vacuum term 1 associated with the anti-normal ordering operators, we evaluate the variances that would be associated with a measurement of $\hat{x}$.

The variances are given in Figure 16 versus $x_1$ which gives the separation between the states of the superposition, for a large value of $gt$. We also define the uncertainty product for the inferred initial state:

$$\epsilon = \Delta(x_p|+)\Delta(p_p|+).$$  (68)

From the Figures 16 see that $\epsilon < 1$ for all $\alpha$, although $\epsilon \rightarrow 1$ as $x_1 \rightarrow \infty$.

The figures show what happens if we post-select on the positive outcome, $x_1$, for a measurement of $\hat{x}$. This result is highly sensitive to the initial separation (given by $x_1$) of the eigenstates (or of the coherent states $|\alpha_0\rangle$ and $|-\alpha_0\rangle$). For the cat state where $\epsilon = 0$ and $x_1 = 2\alpha_0$ is large, the $x$-variance $[\Delta(x_p|+)\Delta(x_p|-)]^2$ is reduced almost to the vacuum level of 1, as would be expected. This is explained as follows. The overall variance in $x$ at the time $t_p = 0$ is large due to there being two states comprising the superposition, but the final amplified outcome of either $x_1 = 2\alpha_0$ or $-x_1 = -2\alpha_0$ (Figure 11 links the trajectory back to only one of these states, $|\alpha_0\rangle$ or $|-\alpha_0\rangle$, which have a variance in $x$ of 1. In fact, a further reduction for the variance in $x_p$ is observed because of the truncation that occurs with the Gaussian function at $t_f$. The conditioning is done for $x_f > 0$, which does not take account of the negative values in $x_f$ associated with the Gaussian centered at $x_f = \mathcal{G}x_1$. This effect becomes negligible for a superposition of true eigenstates of $\hat{x}$ ($r \rightarrow \infty$) in the limit of a true measurement corresponding to $gt \rightarrow \infty$, as observed when comparing the results of Figure 16. However, the variance $[\Delta(p_p|+)\Delta(p_p|-)]^2$ in $p$ occurs from the distribution for $p$ that has interference fringes. This is more pronounced with smaller separation $x_1$. The fringe pattern leads to a reduction in the variance, as compared to the simple Gaussian in $p$ [105, 107].

This effect is stable for $gt$ and $r$, although at greater $r$ we see that the optimal dip in the variance occurs at smaller separations $x_1$ of eigenstates.

The overall result is that the Heisenberg uncertainty principle is not satisfied for the coupled $x$ and $p$ trajectories i.e., the post-selected Q-function distribution $Q_+(x,p,0)$ does not reflect a “true” quantum state $\psi$ in the standard sense. Due to the fringes becoming finer as
VI. MACROSCOPIC REALISM, CAUSAL MODELS AND FRINGES

We now re-examine the results so far in order to discuss the models for realism presented in Section II.

A. Macroscopic realism and the pointer measurement

We ask what is the interpretation for the state \( |\psi_{\text{sup}}\rangle \) of eigenstates of \( \hat{x} \) at the time \( t_0 = 0 \), prior to the measurement of \( \hat{x} \). This is a superposition

\[
|\psi_{\text{sup}}(r)\rangle = \frac{1}{\sqrt{2}} \{ |\psi(x_1, r)\rangle_{sq} + i |\psi(-x_1, r)\rangle_{sq} \}
\]

of two squeezed states \( |\psi(x_1, r)\rangle_{sq} \) and \( |\psi(-x_1, r)\rangle_{sq} \) as defined by Eq. (36). The squeezed states are eigenstates \( |x_1\rangle \) and \( |-x_1\rangle \) of \( \hat{x} \) with eigenvalues \( x_1 \) and \( -x_1 \) respectively, in the limit of large \( r \). The \( Q \) function \( Q_{\text{sup}}(x, p) \) for this state prepared at time \( t = t_0 = 0 \) is given by Eq. (40) and is depicted by way of the density of \( x \) in Figure 1. The full \( Q \) function is depicted in Figure 4. Regardless of separation, we see from Figure 7 that the amplified values at \( t_f \) can be binned into two categories + and −, which the experimentalist designates as measuring either \( x_1 \), or \( -x_1 \). That the final results become dichotomic is due to the property that the noise \( \sigma_x^2(0) = 1 \) associated with each Gaussian peak in the \( Q \) function at \( t = 0 \) is not amplified, while the amplitude \( x_i \) itself is, thus enhancing the signal to noise ratio.

For macroscopic superpositions, the separation \( \sim 2x_1 \) between the peaks associated with the states \( |\psi(x_1, r)\rangle_{sq} \) and \( |\psi(x_1, r)\rangle_{sq} \) is much greater than the peak variance \( \sigma_x^2(0) = 1 \). Here we see that, essentially, the trajectories stemming from a positive (negative) \( X \) in the future always link to a positive (negative) \( X \) in the present time \( t \). There is a one-to-one correspondence between the initial and final outcomes + and −. Therefore for a measurement of \( \hat{x} \), the inferred state for the present time \( t = 0 \), conditioned on a measurement outcome \( x_f > 0 \) which indicates \( |x_1\rangle \), is exactly the state with positive \( x \) values.

We thus argue that there is consistency with the macroscopic-realism models of Section II. The system at time \( t = 0 \) is in a state that will give the definite result, either \( x_1 \) or \( -x_1 \), for a future measurement of \( \hat{x} \). One can then assign a hidden variable \( \lambda_M \) to the state (69) as prepared at the time \( t_0 = 0 \), where the value \( \lambda_M = 1 \) implies the outcome \( x_1 \) for \( \hat{x} \), and the value \( \lambda_M = -1 \) implies the outcome \( -x_1 \) for \( \hat{x} \).

Where overlap of the peaks in the initial distribution for the \( Q \) function occurs \( \langle x_1 < 1 \rangle \), such an interpretation is no longer justifiable from the point of view of macroscopic realism, but the stochastic process indicated by the trajectories for \( x \) can nonetheless be viewed as statistical. We note this view is supported from the observation that the \( x \) trajectories representing the measured observable are indistinguishable from those of the

\[
\begin{align*}
&\Delta(x_p|p_f > 0), \Delta(p_p|p_f > 0) \text{ and the uncertainty product} \quad \\
&\epsilon = \Delta(x_p|p_f > 0) \Delta(p_p|p_f > 0) \quad \text{conditioned on a positive outcome} \quad x_1 \quad \text{for} \quad \hat{x}. \quad \text{using Eq. (36)} \quad \text{The upper dashed-dotted line} \quad \text{is for a superposition} \quad |\psi_{\text{sup}}\rangle \quad \text{of two eigenstates of} \quad \hat{x} \quad \text{with} \quad |x_1\rangle \quad \text{and} \quad |-x_1\rangle \quad \text{respectively,} \quad \text{in the limit of large} \quad r. \quad \text{The} \quad Q \quad \text{function} \quad Q_{\text{sup}}(x, p) \quad \text{for this state prepared at time} \quad t = t_0 = 0 \quad \text{is given by Eq. (40) and is depicted by way of the density of} \quad x \quad \text{in Figure 1. The full} \quad Q \quad \text{function is depicted in Figure 4. Regardless of separation, we see from Figure 7 that the amplified values at} \quad t_f \quad \text{can be binned into two categories} \quad + \quad \text{and} \quad −. \quad \text{Therefore for a measurement of} \quad \hat{x}, \quad \text{the inferred state for the present time} \quad t = 0, \quad \text{conditioned on a measurement outcome} \quad x_f > 0 \quad \text{which indicates} \quad |x_1\rangle, \quad \text{is exactly the state with positive} \quad x \quad \text{values. We thus argue that there is consistency with the macroscopic-realism models of Section II. The system at time} \quad t = 0 \quad \text{is in a state that will give the definite result, either} \quad x_1 \quad \text{or} \quad -x_1, \quad \text{for a future measurement of} \quad \hat{x}. \quad \text{One can then assign a hidden variable} \quad \lambda_M \quad \text{to the state (69) as prepared at the time} \quad t_0 = 0, \quad \text{where the value} \quad \lambda_M = 1 \quad \text{implies the outcome} \quad x_1 \quad \text{for} \quad \hat{x}, \quad \text{and the value} \quad \lambda_M = -1 \quad \text{implies the outcome} \quad -x_1 \quad \text{for} \quad \hat{x}. \quad \text{Where overlap of the peaks in the initial distribution for the} \quad Q \quad \text{function occurs} \quad \langle x_1 < 1 \rangle, \quad \text{such an interpretation is no longer justifiable from the point of view of macroscopic realism, but the stochastic process indicated by the trajectories for} \quad x \quad \text{can nonetheless be viewed as statistical. We note this view is supported from the observation that the} \quad x \quad \text{trajectories representing the measured observable are indistinguishable from those of the}
\end{align*}
\]
mixed state $\rho_{\text{mix}}$, for which the statistical interpretation is valid. This gives support for the deterministic contextual realism model, which postulates that at the time $t_0 = 0$, the final amplified outcome $x_1$ or $-x_1$ is determined, once the measurement setting is specified. This implies that the system can be assigned a hidden variable $\lambda_M$, as above.

**B. Causal relations**

A question might be "by how much can noise from the future boundary affect the present reality?" For measurements where the separation of outcomes $x_1$ and $-x_1$ is beyond 1 (in units where $h = 1$), the answer is zero. However, microscopic differences for the trajectories are apparent.

We consider the superposition $|\psi_{\text{sup}}\rangle$. Where a hidden variable $\lambda_M$ can be assigned to the state at time $t = 0$ and this determines the outcome $x_1$ or $-x_1$ for $\hat{x}$, we argue that there is a causal relation. This is postulated by both the deterministic contextual realism (DCR) (for all $x_1$) and macroscopic-realism (MR) models (for $x_1$ large), and is consistent with the results of this paper. However, it is a different story for the microscopic fluctuations. The sampling from the future boundary at $t = t_f$ ensures a retrocausality. This is at the level of the quantum vacuum noise.

The causal relations are depicted in Figure 17. The model is classical in its form. The model parameters are the measurement setting $\theta$ (in this case, $\hat{x}$ or $\hat{p}$), and the variables that describe the state of the system, at the times at or just prior to $t_0 = 0$, and at time $t_f$. The $Q$ function specifies the system variables at time $t_0$ (or just prior) to be $x$ and $p$, with joint probability $Q(x, p, t_0)$. We present the causal model associated with the deterministic contextual realism model. An individual system is then also specified by $\lambda$, whose value is one of the set of eigenvalues $\{x_j\}$ (or $\{p_j\}$) of the measurement $\hat{x}$ (or $\hat{p}$). We define the conditional

$$P(\lambda|\theta),$$

where $P(\lambda = x_j) = |c_j|^2$ if $\theta \equiv \hat{x}$ and $P(\lambda = p_j) = |d_j|^2$ if $\theta \equiv \hat{p}$. Here, $|c_j|$ and $|d_j|$ are fully determined by the $Q$ function $Q(x, p)$, from the marginals $Q(x)$ (or $Q(p)$) found by integrating over the complementary variable ($p$ or $x$), but are given quantum mechanically by $c_j = \langle x | \psi \rangle$ and $d_j = \langle p | \psi \rangle$. We note we do not define an underlying joint distribution $P(\lambda_x, \lambda_p)$ for variables that correspond to $\lambda = \lambda_x$ if $\theta \equiv \hat{x}$, and $\lambda = \lambda_p$ if $\hat{p}$.

The final measured outcome value will be $X$, which is the amplified value of $\lambda$. In the model, $X$ is also a system variable at the time $t_f$. The relation between $X$ and $\lambda$ (and $\theta$) is deterministic, ie.

$$X = e^{i\theta t} \lambda,$$

where $\lambda = x_j$ for some $j$, if $\theta \equiv \hat{x}$; and $\lambda = p_j$ for some $j$, if $\theta \equiv \hat{p}$. With the choice of $\theta$, there is a value $\lambda$ selected, with a certain probability.

The system at time $t_f$ includes the fluctuations $\delta q$ that are not measured by amplification. The value of $\delta q$ is then sampled using a random Gaussian function $P_G$ with mean 0 and variance $\sigma^2_x = 1$ at time $t_f$. Hence

$$P(\delta q(t_f)) \equiv P_G(0, \sigma_x).$$

The value of the system variable $q$ is $q = q_j + \delta q(t_f)$, where here $q$ is either $x$ or $p$ (and $q_j$ is either $x_j e^{i\theta t_f}$ or $p_j e^{i\theta t_f}$), depending on the value of $\theta$. The backward trajectory is given by

$$\frac{dq}{dt} = -gq + \xi_1(t),$$

as in (74), with the initial condition being the value of $q$ at time $t_f$. The Gaussian random noise $\xi_1(t)$ satisfies $\langle \xi_1(t) \xi_1(t') \rangle = 2g^2 \delta(t - t')$ and decouples from the complementary variable $q_0$ (which is either $p$ or $x$). We note however we can solve (74) by substituting $q = q_0 + \delta q$, where $q_0$ satisfies

$$\frac{dq_0}{dt} = -gq_0$$

with initial condition $q_0(t_f) = q_j$, and

$$\frac{d(\delta q)}{dt} = -g\delta q + \xi_1,$$
with initial condition \( \delta q = \delta q(t_f) \). Clearly, the solution for \( q_0 \) is the deterministic function \( q_0 = q_0 e^{-|g| t - } \), which is either \( q_0 = x_j e^{i|g| (t_f - t - )} \) or \( q_0 = p_j e^{i|g| (t_f - t - )} \) depending on the value of \( \theta \), the measurement setting. This decaying solution with respect to \( t - \) is evident in the trajectories for \( x \) plotted in Figures 1 and 17. The trajectory for \( \delta q \) has a stochastic solution involving the noise \( \xi_1 \). This noise does not depend on the value of \( q_0 \) or \( q \), but rather has constant size \( g \). The consequence is that the \( \delta q \) has the same initial condition and trajectory equations, regardless of whether \( q \) is \( x \) or \( p \). The trajectory for \( \delta q \) propagates back to \( t_0 \) with a value \( \delta q(t_0) = q_p \) (in the present time). Hence, we write

\[
P(\delta q(t_0)) \equiv P(\delta q(t_0)|\delta q(t_f)),
\]

noting that the noise value \( \delta q \) is independent of the value of \( X \) (that is, \( x_j \) or \( p_j \)) and, perhaps surprisingly, \( \theta \). This defines the backward causal relation marked in the Figure 17.

However, \( q \) is either \( x \) or \( p \) - which is determined at the present time \( t_0 \), from \( \theta \). The value of the complementary variable that is not measured is denoted \( q_c \). Here, \( q_c \) is either \( p \) or \( x \). The value \( \delta q_c(t_0) \) can be specified according to the conditional \( Q(q_c|q) \) at time \( t_0 \). The \( \delta q(t_0) \) defines the value of \( q \) at time \( t_0 \): \( q = \lambda + \delta q(t_0) \). This determines \( q_c \) at time \( t_0 \). Denoting the values of \( x \) and \( p \) at time \( t_0 \) as \( x(t_0) = x_p \) and \( p(t_0) = p_p \) (meaning \( x \) and \( p \) in the present time), we write if \( q = x \) and \( q = p \) that

\[
P(p(t_0)|x(t_0)) = Q(p(t_0)|x(t_0)) \equiv Q(p_p|x_p).
\]

and if \( q = p \) and \( q = x \) that

\[
P(x(t_0)|p(t_0)) = Q(x(t_0)|p(t_0)) \equiv Q(x_p|p_p).
\]

The value of \( q_c(t_0) \) determines the starting point of the trajectory that results in a value \( q_c(t_f) \) for the complementary variable \( q_c \) at time \( t_f \). The probability

\[
P(q_c(t_f)|q_c(t_0))
\]

is well defined stochastically by the forward dynamics given by equation (44), which involves the noise function \( \xi_2 \). This relation is causal and is marked on Figure 17 by the forward arrow.

C. Schrodinger’s cat paradox: macroscopic realism and the incompleteness of quantum mechanics

The cat paradox seeks to interpret the reality of the system in a macroscopic superposition [21]. Such a state is prepared as part of the measurement process. In the Figure 15, we see this in the model for measurement given by amplification \( H_A \). The state formed at time \( t_g \) in the Figure is (taking \( r \) large)

\[
e^{-iH_A t/\hbar}|\psi_{sup}\rangle = \frac{1}{\sqrt{2}} \{ e^{-iH_A t/\hbar}|x_1\rangle + ie^{-iH_A t/\hbar}|-x_1\rangle \}.
\]

For sufficient amplification \( G = e^{gt/\hbar} \) after a time \( t_g \), this state becomes a macroscopic superposition state as represented by the trajectories of Figures 14 and 17. The macroscopic realism model interprets the state at that time \( t_g \) as satisfying macroscopic realism.

This model allows the interpretation of macroscopic realism for the cat in the Schrodinger cat paradox [22], without needing to consider decoherence. The realism of the “cat”, that it be “dead” or “alive”, is achieved with the amplification \( H_A \). For the amplified state (80), at time \( t_g \), the value for the measurement \( \hat{x} \) is determined to be either \( x_1 \) or \( -x_1 \). Macroscopic realism is satisfied, in that the system is in one or other state that will yield an outcome of either \( x_1 \) or \( -x_1 \).

However, a simple analysis shows that the “states” comprising the amplified system at time \( t_g \) cannot be quantum states. We follow the argument in Refs. 77, 108. Let us examine the macroscopic superposition \( |\psi_{sup}\rangle \) given by (69) with \( x_1 \) large. In the model, we assume this system satisfies macroscopic realism, and is in one of two “states” giving either \( x_1 \) or \( -x_1 \). If these states are indeed quantum states, then this would mean the state \( |\psi_{sup}\rangle \) can be expressed as a probabilistic mixture of type

\[
\rho_{mix} = P_+ \rho_+ + P_- \rho_-,
\]

where \( \rho_\pm \) are quantum states giving an outcome \( \pm x_1 \) for the measurement \( \hat{x} \), and \( P_\pm \) are probabilities. However, the expression can be falsified for the predictions of the superposition state \( |\psi_{sup}\rangle \). The uncertainty relation \( \Delta \hat{x} \Delta \hat{p} \geq 1 \) holds for each \( \rho_\pm \), and therefore for the mixture. Therefore, if the \( P_\pm \) distributions associated with the positive and negative \( x_1 \) are measured as Gaussians with variance \( \Delta x \leq 1 \) (as in Figure 11), it would be necessary that the distributions for \( P(\delta q) \) sat-
isfy $P(p) \geq 1$. However, the fringe distribution of Figure 13 gives $\Delta \tilde{p} < 1$. There is the conclusion that if macroscopic realism (MR) holds, then quantum mechanics is incomplete.

The contradiction between the completeness of quantum mechanics and macroscopic realism is supported by the evaluation of the coupled trajectories in Section V. The macroscopic system is described by the coupled values of $x$ and $p$, which have a macroscopically deterministic value of $x$, but which microscopically do not correspond to quantum states, because the uncertainty product $\Delta \hat{x} \Delta \tilde{p}$ of the post-selected state at time $t = 0$ reduces below 1.

The amplification creating the macroscopic superposition state as in Figure 18 at the time $t_g$ is reversible. One may return to the original state $|\psi_{sup}\rangle$ at a time $t_{2g}$ by attenuating $x$, applying the Hamiltonian $H_A = \frac{\hbar g}{2} [\hat{a}^2 - \hat{\tilde{a}}^2]$ of Eq. (3) with $g < 0$. This creates the forward trajectories for $x$. Although the initial state is returned, we note that because the process is stochastic, the individual trajectories do not return to the original value precisely. It is understood that the complete process of the “collapse” to the eigenstate occurs when the process is not reversible. This would occur when there is loss of information from the system due to coupling to a second system, which motivates the next section.

### D. Fringes

Fringes are observed in the conditional distribution $P(p|x)$ for $p$ evaluated at the boundary corresponding to time $t = t_0 = 0$. The conditional distribution is given by Eq. (18). We see from Figure 14 that these fringes do not vanish if one considers the inferred distribution for $p$ post-selected on the outcome of $\hat{x}$ e.g. for $x_f > 0$. It is this feature that gives the reduced variance in $\tilde{p}$, leading to the conclusion that the post-selected state cannot be a quantum state.

Intuitively, it may be thought that the fringes vanish once it is known “which state the system is in”, in analogy to the two-slit experiment. This motivates the next section, which examines precisely a set-up where one may indeed infer “which state the system is in” and simultaneously perform a measurement $\hat{p}$ on the system $A$.

### VII. MEASUREMENT OF AN ENTANGLLED SYSTEM AND METER

We now address a standard model of measurement, where one couples the system to a second macroscopic system $B$, a meter. The measurement on the meter is then used to infer a result for a measurement on the first system, which we refer to as the signal.

#### A. Correlated state for the system and meter

First, we consider a system $A$ prepared in a superposition state $|\psi_{sup}(r)\rangle = \frac{1}{\sqrt{2}}(|\psi(x_1,r)\rangle_{sq} + i|\psi(-x_1,r)\rangle_{sq})$ given by Eq. (17). In the limit of large squeezing $r$, this becomes $|\psi_{sup}\rangle = \frac{1}{\sqrt{2}}(|x_1 + i| - x_1\rangle)$ where $|x_1\rangle$ and $|-x_1\rangle$ are eigenstates of $\hat{x}_4$. Suppose a measurement is made on system $A$ to infer “which of the two states the system $A$ is in”, $|x_1\rangle$ or $|-x_1\rangle$. Such a measurement is made by coupling the system $A$ to a meter $B$. A prototype for the state after such a coupling is the entangled two-mode state

$$|\psi_{ent}\rangle_{sq} = \frac{1}{\sqrt{2}}\{|\psi(x_1,r)\rangle_{sq}|\beta_0\rangle + i|\psi(x_1,r)\rangle_{sq}|-\beta_0\rangle\},$$

which becomes in the limit of large $r$

$$|\psi_{ent}\rangle = \frac{1}{\sqrt{2}}\{|x_1\rangle|\beta_0\rangle + i|x_1\rangle|-\beta_0\rangle\}. \quad (83)$$

We take $x_1$ and $\beta_0$ to be real, and $|\beta_0\rangle$ and $|-\beta_0\rangle$ are coherent states for mode $B$. Normally, it is understood that for an effective measurement, $\beta_0$ would become large. The measurement of the quadrature phase amplitude $\hat{x}_B = \hat{b} + \hat{b}^{\dagger}$ of mode $B$ would indicate “whether the system is in the state $|\beta_0\rangle$ or $|-\beta_0\rangle$”, and hence also be a measurement to indicate the state of the system $A$, “whether $|x_1\rangle$ or $|-x_1\rangle$”. Here, $\hat{b}$ is the destruction operator for field mode $B$. When $r = 0$, the system is a two-mode entangled cat state

$$|\text{cat}_2\rangle = \frac{1}{\sqrt{2}}\{|\alpha_0\rangle|\beta_0\rangle + i|\alpha_0\rangle|-\beta_0\rangle\}, \quad (84)$$

where $\alpha_0 = x_1/2$, and $|\alpha_0\rangle$, $|\beta_0\rangle$, and $|-\alpha_0\rangle$ are coherent states for mode $A$. The measurement on $B$ is then intended to infer whether system $A$ “is in state $|\alpha_0\rangle$ or $|-\alpha_0\rangle$”.

Motivated by this model, we seek to examine the nature of the “collapse” of the system $A$ based on the direct measurement $\hat{x}_B$ on system $B$. The $Q$ function of the entangled cat system (83) is

$$Q_{ent}(\lambda, t_0) = \frac{e^{-(p_A^2 + p_B^2) / 4}}{32\pi^2} \left\{ \begin{array}{l} e^{-(x_A - 2\alpha_0)^2/4 - (x_B - 2\beta_0)^2 / 4} \\
+ e^{-(x_A + 2\alpha_0)^2/4 - (x_B + 2\beta_0)^2 / 4} \\
- 2e^{-(x_A^2 + 2\alpha_0^2 + 4\beta_0^2) / 4} \sin(\alpha_0 p_A + \beta_0 p_B) \end{array} \right\}, \quad (85)$$

where $\lambda = (x_A, x_B, p_A, p_B)$. This can be written succinctly for the more general state (82) as

$$Q(x, p) = N \left[ \cosh \left( \frac{x \cdot x_0}{\sigma_x^2} \right) - \sin \left( \frac{p \cdot x_0}{\sigma_x^2} \right) \right] \quad (86)$$

with the prefactor

$$N = \frac{1}{4\pi^2} \prod_i [\sigma_{p,i}\sigma_{x,i}] e^{-\left(p^2/2\sigma_p^2 + x^2/2\sigma_x^2 + x_0^2/2\sigma_x^2\right)}. \quad (87)$$
Here we use the vector notation \( \mathbf{x}_0 = (x_1, 2\beta_0) \equiv (x_{01}, x_{02}) \), \( \mathbf{x} = (x_A, x_B) \equiv (x_1, x_2) \) and \( \mathbf{p} = (p_A, p_B) \equiv (p_{1}, p_{2}) \) and define \( \mathbf{x} \cdot \mathbf{x}_0/2\sigma^2 \equiv \sum x_i x_{0i}/2\sigma^2_i \), \( \mathbf{p} \cdot \mathbf{p}_0/2\sigma^2 \equiv \sum p_i p_{0i}/2\sigma^2_i \), \( x^2/2\sigma^2_x \equiv \sum x_i^2/2\sigma^2_i \) and \( x_{0i}^2/2\sigma^2_i \). Also, \( \sigma^2_{x1} \) and \( \sigma^2_{x2} \) are the variances for \( x_A \) and \( x_B \) respectively.

For the general state \( |\psi\rangle \), the variances are

\[
\sigma^2_{x,i} = 1 + e^{-2r_i},
\]

\[
\sigma^2_{p,i} = 1 + e^{2r_i},
\]

\[
x_{0i} = x_i, \quad x_{02} = 2\beta_0, \tag{88}
\]

where \( r_i \) is the squeezing parameter \( r \) defined for the squeezed state for each mode. We have taken in Eq. (82) that \( r_1 = r \) and \( r_2 = 0 \).

**B. Measurement on the meter and system**

Following Section IV, the measurement of \( \hat{x}_B \) of the meter field \( B \) is modeled by the local interaction

\[
H_B = \frac{i\hbar g_2}{2} \left[ \hat{b}^2 - \hat{\beta}^2 \right],
\]

where \( g_2 \) is real and \( g_2 > 0 \). It is also possible to measure \( \hat{x}_A \) of system \( A \) via the local Hamiltonian \( H_A = \frac{i\hbar g_1}{2} \left[ \hat{\alpha}^2 - \hat{\beta}^2 \right] \) where \( g_1 > 0 \), as described in Section IV. With \( g_1 < 0 \), the local interaction \( H_A \) describes the measurement of \( \hat{p}_A \). The solution for the amplified \( \hat{Q} \) function of the two-mode system after local interactions \( H_A \) and \( H_B \) for a time \( t \) at site \( A \) and \( B \) respectively is solved using the approach outlined in Section IV. The solution is given by the \( \hat{Q} \) function Eq. (86) with the means and variances becoming

\[
\sigma^2_{x,i} = 1 + e^{2(\mu - r_i)}
\]

\[
\sigma^2_{p,i} = 1 + e^{-2(\mu + r_i)}
\]

\[
x_{01} = x_{1e^{\mu t}}, \quad x_{02} = 2\beta_0 e^{\mu t}, \tag{90}
\]

where we take \( r_1 = r \) and \( r_2 = 0 \).

We first consider joint measurements of \( \hat{x}_A \) and \( \hat{x}_B \). As before, we begin with trajectories starting with an \( x_B^0 \) and an \( x_A^0 \) sampled according to the marginal distribution for \( x_B \) and \( x_A \) of the \( \hat{Q} \) function set in the future at time \( t_f \). Following the procedure to derive equations (45) and (46), we see the evolution for \( x_A \) decouples from that of \( p_A, x_B \) and \( p_B \). The equations for the measurement \( \hat{x}_B \) on the meter at \( B \) are

\[
\frac{dx_B}{dt_-} = -g x_B + \xi_{B2}, \tag{91}
\]

where \( t_- = -t \) with a boundary condition at time \( t_f \), and

\[
\frac{dp_B}{dt} = -g p_B + \xi_{B1}, \tag{92}
\]

Figure 19. A diagram showing the causal relations for trajectories associated with the measurement of \( \hat{x}_A \) using a meter. The system \( A \) has previously been coupled to a second system \( B \) and the correlated state with \( \hat{Q} \) function \( Q(x_A, p_A, x_B, p_B) \) given by Eq. (80) created. When the amplitude \( \beta_0 \) of the meter is large, measurement of \( \hat{x}_B \) of the meter by amplification gives an outcome \( X_B \) which is perfectly correlated with the outcome \( X_A \) for \( \hat{x}_A \) at \( A \), if amplified. In this reality model, the systems \( A \) and \( B \) at time \( t_0 \) possess correlated hidden variables \( \lambda_A \) and \( \lambda_B \) for the amplified outcomes \( X_A \) and \( X_B \) of \( \hat{x}_A \) and \( \hat{x}_B \). This is a causal relation given by the solid blue lines. System \( A \) is therefore in a state of definite outcome \( X_A \) at \( t_0 \) in this model. The \( x_A, p_B, x_B \) and \( p_B \) are not amplified and are not measurable. The associated forward and backward trajectories are indicated by the red dashed lines. The trajectories are local, with the noise terms \( \xi_{A1} \) and \( \xi_{B1} \), being independent. The entanglement between the meter and system \( A \) results in the inseparability of \( Q \) with respect to the variables of \( A \) and \( B \), leading to correlation between trajectories.

with a boundary condition at time \( t_0 \). The Gaussian random noises \( \xi_{\mu}(t) \) satisfy \( \langle \xi_B(t) \xi_B(t') \rangle = 2g_2\delta_{\mu\mu}\delta(t-t') \). Similarly, the trajectories for \( x_A \) decouple from those of \( p_A, x_B \) and \( p_B \). The equations for the measurement \( \hat{x}_A \) at \( A \) are

\[
\frac{dx_A}{dt_-} = -g x_A + \xi_{A2}, \tag{93}
\]

where \( t_- = -t \) with a boundary condition at time \( t_f \), and

\[
\frac{dp_A}{dt} = -g p_A + \xi_{A1}, \tag{94}
\]

with a boundary condition at time \( t_0 \). The Gaussian random noises \( \xi_{\mu}(t) \) satisfy \( \langle \xi_A(t) \xi_A(t') \rangle = 2g_1\delta_{\mu\mu}\delta(t-t') \). While the stochastic evolution of the trajectories at the sites \( A \) and \( B \) are independent, the initial state at the future boundary is correlated, the correlations given by the \( \hat{Q} \) function (80) using (90). Since the \( p \) decouples from the \( x \), the relevant boundary condition for the trajectories \( x_A \) and \( x_B \) is determined by the future marginal for \( x_A \) and \( x_B \). The marginal for \( x \) at time \( t \) can be written on integrating over \( p \), eliminating the second term, and therefore giving two two-mode
ever these are attenuated and are not of direct interest around $2\beta$ trajectory proceeds. As expected, the trajectory for large $A$ system is prepared in the state (82), then we expect the state of system

$$\text{for large } r \text{ where } |\psi(x_1, r)\rangle_{sq} \text{ becomes an eigenstate of } x_1.$$  

\section*{C. Inferred state for the system at time $t_0$ given an outcome for the meter}

What can be inferred from the measurement at $B$ about the state at $A$ as it exists at the initial time $t_0 = 0$? We ask what is inferred for the state of system $A$ in the $Q$ realism model if the outcome at $B$ for $x_B$ is positive $\beta_0$. We make the distinction that we are inferring the state of system $A$ prior to its direct measurement, with respect to the reality model. This has a different meaning to evaluating the $Q$ function for system $A$ conditioned on the outcome $\beta_0$, which can be calculated from standard quantum mechanics by projection. For the system prepared in the two-mode cat state \((84)\) for example, where $\beta_0 = \infty$, the state at $A$ conditioned on the outcome $\beta_0$ for $x_B$ is $|\alpha_0\rangle$.

The trajectories for $x_B$ alone can be evaluated by integrating the $Q$ function \((95)\) over $x_A$ to evaluate the marginal for $x_B$ at time $t_f$. The marginal for $x_B$ is found to be

$$Q_{\text{inf}}(x_A, p_A|x_B) = \frac{e^{-p_{A}^{2}/4}}{8\pi \cosh(\beta_{0}x_{B})} \left\{ e^{-(x_{A}-2\beta_{0})^{2}/2\sigma_{x_{A}}^{2}} \right.$$  

$$+ e^{-\left(\frac{A}{\sigma_{x_{A}}^{2}}\right)^{2}/2\sigma_{x_{A}}^{2}} \right\},$$  

\noindent \textit{where } $G(t) = e^{\sigma_{x_{A}}^{2}t}$ \textit{and } $\sigma_{x_{A}}^{2}$ \textit{is given by \((90)\) evaluated at } $t = t_f$. This provides the distribution from which the sampling for the initial value of the backward trajectory proceeds. As expected, the trajectory for large $\beta_0$ connects back to the positive initial values centered around $2\beta_0$, at time $t_0 = 0$. The $Q$ function of the state in the present time $t_0$ allows evaluation of the distribution $Q_{\text{cond}}$ for $x_A$, $p_A$ and $p_B$ conditioned on $x_B$ in the present, similar to the method given in Section V. However, these are attenuated and are not of direct interest to us.

Our interest is the evaluation of the state at $A$ at the initial time $t_0 = 0$, post-selected on the measurement outcome for the meter $B$. We take the measurement outcome for the meter $B$ to be positive i.e. $x_{f}^{B} > 0$. If the system is prepared in the state \((82)\), then we expect the system $A$ would be found to be in the state $|\psi(x_1, r)\rangle_{sq}$. For large $r$, this state is the eigenstate $|x_1\rangle$; for $r = 0$, this state is the coherent $|\alpha_0\rangle$. Integrating over the variable $p_B$, we can arrive at the inferred $Q$ function $Q(A)_{+\text{inf}}$ for the system $A$, based on a future value $x_{f}^{B}$. This is evaluated as follows. There is a set of backward trajectories emanating from the set $x_{f}^{B} > 0$. For each such trajectory, there is a single $x_B = x_{f}^{B}$ at the time $t_0 = 0$ in the present. We evaluate the distribution $Q_{\text{inf}}$ for system $A$ conditioned on the value $x_B$ at the boundary $t_0 = 0$. For $r = 0$, corresponding to the two-mode cat state, we find after integrating over $p_B$ that

$$Q_{\text{inf}}(x_A, p_A|x_B) = \frac{e^{-p_{A}^{2}/4}}{8\pi \cosh(\beta_{0}x_{B})} \left\{ e^{-\left(\frac{A}{\sigma_{x_{A}}^{2}}\right)^{2}/2\sigma_{x_{A}}^{2}} \right.$$  

$$+ e^{-(x_{A}+2\beta_{0})^{2}/4} e^{-\beta_{0}x_{B}}$$  

$$- 2e^{-(x_{A}^{2}+4\beta_{0}^{2}+4\beta_{0}^{2})/4} \sin(\alpha_0 p_A) \right\},$$  

\noindent \textit{where } $\alpha_0 \gg \beta_0$. We also consider that $x_B$ is justified to be positive, based on the scatter plots of the trajectories for $x_B$ that emanate from $x_{f}^{B} > 0$ for large $x_{f}^{B}$. Then we see that the fringe term is damped and the conditional $Q_{\text{inf}}$ becomes

$$Q_{\text{inf}}(x_A, p_A|x_B) \rightarrow \frac{e^{-p_{A}^{2}/4} e^{-\left(\frac{A}{\sigma_{x_{A}}^{2}}\right)^{2}/2\sigma_{x_{A}}^{2}}}{4\pi},$$  

\noindent \textit{which is indeed that of the coherent state } $|\alpha_0\rangle$. This is based on the amplification due to $\beta_0 \rightarrow \infty$ and is true regardless of the size of $\alpha_0$. The distribution $Q(A)_{+\text{inf}}$ can be evaluated after averaging over all positive future $x_{f}^{B}$, but in the limit corresponding to a measurement where $\beta_0$ is large we expect

$$Q(A)_{+\text{inf}} = \frac{e^{-p_{A}^{2}/4} e^{-\left(\frac{A}{\sigma_{x_{A}}^{2}}\right)^{2}/2\sigma_{x_{A}}^{2}}}{4\pi}.$$  

\noindent \textit{The inferred state of system $A$ is } $|\alpha_0\rangle$ \textit{which is in agreement with the state projected from \((84)\), using standard quantum mechanics. The limiting inferred state does not violate the uncertainty principle for large $\beta_0$, which is similar to the result for the limiting inferred state of Section V where $x_1$ and $\alpha_0$ are large. The inferred distribution function $Q(A)_{+\text{inf}}$ for the state of system $A$ given the measurement at $B$ can be fully calculated from the trajectories in $x_B$, even for small $\alpha_0$. We conclude that in the reality model, the state of system $A$ at time $t_0$ conditioned on the outcome $\beta_0$ for $\hat{x}_B$ corresponds to the "collapsed" or "projected" state $|\alpha_0\rangle$, as predicted by the measurement postulate. The conditional state for system $A$ can be measured directly in an experiment, and is in agreement with the quantum prediction: The reality model gives the extra interpretation about when the system $A$ "collapsed" to the state $|\alpha_0\rangle$. This does not happen with the final measurement at $B$, when $x_{f}^{B}$ is detected. Rather, the system $A$ is in one...}
or other states with amplitudes $x_A$ giving with the final outcome $2|α₀|₀$ or $-2|α₀|₀$ for $x_A$ but correlated with the outcome $2|β₀|₀$ or $-2|β₀|₀$ for $x_B$ of system $B$, at the time $t₀$. This we see from the two-mode $Q$ function which for $β₀ → ∞$ becomes two two-mode Gaussians with correlated means $(2|α₀|₀, 2|β₀|₀)$ and $(-2|α₀|₀, -2|β₀|₀)$. The system *collapses* to the final state $|α₀|₀$ or $|-α₀|₀$ in a limiting sense for $β₀$ large, at the time $t₀$. The collapse was created by a prior interaction $H_C$ which coupled the system $A$ to the macroscopic meter system $B$.

The collapse is not complete, in the sense that the interaction $H_C$ is unitary and can in principle be reversed. This is possible because the fringe terms although small do not completely vanish, no matter how large $β₀$ is. If the system $A$ is *decoupled* from $B$, then reversibility is not possible. The decoupling amounts to a loss of information of the combined systems. The $Q$ function for $A$ in this case is found by integrating over both $x_B$ and $p_B$. The resulting $Q$ function is $Q_{mix}$ with $G(t_f) = 1$, which comprises two two-mode Gaussians, being that of the statistical mixture $p_{mix}$ of the two states $|α₀|₀|β₀|₀$ and $|-α₀|₀|β₀|₀$. The fringe terms vanish completely. At this stage, the system $A$ is precisely in one or other states $|α₀|₀$ or $|-α₀|₀$ and the collapse is completed.

### D. Fringes and which-way information

We now examine the question about how or when the fringes disappear. Here it is possible to measure both $x_A$ and $p_A$ simultaneously, because the outcome for $x_A$ is inferred by the measurement $x_B$ on the meter. Which-way information is gained when $β₀$ is large, and this corresponds directly to the decay of the fringes $|α₀|₀|β₀|₀$ and $|-α₀|₀|β₀|₀$.

We see from Eq. (97) that fringes are present in the state inferred for $A$ given the coupling to the meter at time $t₀$, but are damped by the factor $e^{-β₀}$. In effect, the fringes vanish (but not irreversibly) once the system $A$ has been entangled with the macroscopic meter. Here, macroscopic means $β₀$ is large, which is necessary to ensure effective measurement of $x_A$ by the meter. The coupling to the meter means that the measurement of $x_A$ can be made at any future time, by measuring $x_B$ of system $B$. The conclusion is that (within the interpretation given by reality model) the measurement of system $A$ has taken place once the system is entangled with the macroscopic meter, at the time $t₀$.

The decay of the fringes can be observed experimentally by making joint measurements of $p_A$ on system $A$ and $x_B$ on system $B$. For $β₀$ large, the value of $x_B$ is inferred to correspond to that of $x_A$, based on the correlation. Where $β$ is large, the post-election conditions on the outcome for $x_B$ being $β₀$ or $-β₀$ which implies “which way” information. The distribution shows no observable fringes. For smaller $β₀$ fringes appear, but this does not correspond to an effective measurement outcome for $x_A$, so that which way information is lost.

### VIII. EPR and Bell Correlations

A challenge for any model of measurement that postulates objectivity is to explain the known violation of Bell inequalities. Here, we give a brief explanation of how our model explains measurement of continuous variable Einstein-Podolsky-Rosen (EPR) entanglement. We then identify how the model differs from the local hidden variable (or local causal) models considered by Bell. We explain why the predictions of this objective model will not be constrained by Bell inequalities.

#### A. Einstein-Podolsky-Rosen entanglement

Consider the measurement of EPR correlations between two modes $A$ and $B$. The EPR correlations can be created from the two-mode squeezed state $|45, 111⟩$, $|α₀|₀|β₀|₀$ or $|-α₀|₀|β₀|₀$. The variances of $x_A$ and $p_A$ for which the $Q_{mix}$ function is

\[ Q_{mix}(λ, 0) = \frac{(1 - T^2)}{16π^2} e^{-\frac{1}{2}(x_A - x_B)^2(1 + T)} e^{-\frac{1}{2}(p_A + p_B)^2(1 + T)} \times e^{-\frac{1}{2}(x_A + x_B)^2(1 - T)} e^{-\frac{1}{2}(p_A - p_B)^2(1 - T)} \]

\[ \rightarrow \frac{(1 - T^2)}{16π^2} e^{-\frac{1}{2}(x_A - x_B)^2/4} e^{-\frac{1}{2}(p_A + p_B)^2/4}. \]

Here $λ = (x_A, x_B, p_A, p_B)$. The last step shows the limit of $r → ∞$, where the state becomes a simultaneous eigen-state of $x_− = x_A - x_B$ and $p_− = p_A - p_B$. The variances of $x_± = x_A ± x_B$ and $p_± = p_A ± p_B$ are

\[ ⟨[Δ(x_A ± x_B)]^2⟩ = σ_±^2(0) = 2(1 + e^{±2r}) \]

\[ ⟨[Δ(p_A ± p_B)]^2⟩ = σ_±^2(0) = 2(1 + e^{±2r}). \]

The corresponding measured variances of the operators $x_± = x_A ± x_B$ and $p_± = p_A ± p_B$ are

\[ ⟨[Δ(x_A ± x_B)]^2⟩ = 2e^{±2r} \]

\[ ⟨[Δ(p_A ± p_B)]^2⟩ = 2e^{±2r}. \]

The argument of EPR is based on local realism (LR). If one can predict with certainty the outcome of a measurement of $x$ (or $p$) at a site $A$ without disturbing that system, then the premise of EPR-realism implies there exists an “element of reality”, or hidden variable $λ_A^A(λ_B^B)$, that determines the result for that measurement should it be performed. For spatially separated sites $A$ and $B$, locality implies that the outcome for $x$ (or $p$) is predicted with certainty, by performing a measurement on system $B$. This follows from the correlations given by the $|45, 111⟩$. EPR therefore argued that simultaneous hidden variables $λ_A^A(λ_B^B)$ exist to describe the system A.

EPR then argued that quantum mechanics is incomplete, since such variables ascribed to system $A$ at the time $t_0$ violate the Heisenberg uncertainty relation.
We ask what does the $Q$ model predict for EPR’s argument? For the EPR case, we first consider two-sided measurements in $\hat{x}$, where the fields $A$ and $B$ are spatially separated and measurements of $\hat{x}_A$ and $\hat{x}_B$ are made at the respective sites. The measurements at the sites are local and given by $H_A = \frac{i\hbar g}{2}[\hat{a}^2 - \hat{\sigma}^2]$ and $H_B = \frac{i\hbar g}{2}[\hat{b}^2 - \hat{\sigma}^2]$ where $g$ is real and $g > 0$. This results in an independent amplification of $\hat{x}_A$ and $\hat{x}_B$ at each site. We can understand what is expected for the trajectories from the previous analyses of measurement of $\hat{x}$ on a single system. The trajectories for $x$ are governed by backward propagating equations. The measurement of $\hat{x}_-$ at $A$ and $B$ is such that the amplified outcomes $gx_A - gx_B$ have zero relative noise for large enough $G$. The systems are correlated in $\hat{x}$ so that the measurement of $\hat{x}$ at $B$ implies the result at $A$ for any $x_1$ and $x_2$ no matter how small the difference $|x_1 - x_2|$, despite the existence of the quantum noise in the function $Q$ at time $t_0$. Thus in the continuous variable description given by (100), the outcomes for the $\hat{x}_A$ and $\hat{x}_B$ are precisely correlated, as are those of $\hat{p}_A$ and $\hat{p}_B$.

Similarly to the interpretation of the single mode eigenstate $|x_j\rangle$ that we have given in the previous sections of this paper, one can postulate models of realism for the two-mode EPR eigenstates $|x_1 - x_2\rangle$, $|p_1 + p_2\rangle$ and examine whether they are justified by the retrocausal model using the amplitudes $\lambda$. First, we see that the amplitudes $x_A$ and $p_A$ of system $A$ as defined for the $Q$ function $Q(x_A, p_A, x_B, p_B)$ of the EPR state are not themselves perfectly correlated with those of system $B$. This is due to the fundamental noise $\sigma_x = 1$ and $\sigma_p = 1$ for $x_-$ and $p_+$ that appear in the initial $Q$ function. However, it is clear in this retrocausal model that the final amplified values $gx_A$ and $gx_B$ are perfectly correlated, for sufficient $g$. Similarly, the amplified values $|g|p_A$ and $|g|p_B$ are perfectly anti-correlated after the amplification of $\hat{p}$ at each site ($g < 0$). The $Q$ realization deterministic and hybrid models imply that for the initial EPR state defined at the time $t_0$, the amplified values of $x_A$ and the amplified values of $p_A$ at a later time $t_2$ are (locally) determined (for sufficient $|g|\rangle$. This is not itself a violation of the uncertainty relation, since the $x$ and $p$ are not amplified by the same Hamiltonian.

In summary, in the objective field model, there is no nonlocal effect or action-at-a-distance. The retrocausal trajectory model supports that the system is in a state with a definite future value for the outcomes of $\hat{x}_A$ or $\hat{p}_A$, once the measurement setting is selected. We note that since for two systems $A$ and $B$, measurement settings for $\hat{x}_A$ (at $A$) and $\hat{p}_A$ (by choosing the setting for $\hat{p}_B$ at $B$) can be simultaneously selected, the $Q$ realization model implies that it is possible for the outcomes of both $\hat{x}_A$ and $\hat{p}_A$ to be simultaneously determined. However, the measurement of $\hat{p}_A$ at $A$ would require a further local unitary transformation.

In this respect, the objective realism model gives a more complete picture of quantum mechanics in a way that is consistent with EPR’s local realism, if appropriately qualified. The EPR local realism premise that asserts “If one can predict with certainty the outcome of a measurement of $\hat{x}$ (or $\hat{p}$) at a site $A$ without disturbing that system” is qualified to apply to the system at the time $t_0$, when the measurement setting at $B$ has been established. This is a more restrictive meaning than Bell’s local realism, which considers predetermination of three measurement settings in a bipartite system $[3]$. The causal diagram is similar to that of Figure [19].

B. Quantum nonlocality: EPR and Bell’s premises

This leaves us to understand how (or whether) Bell nonlocality may arise from our model. At first glance, the model appears to be a local realistic theory, and hence it may seem that it could not explain violation of a Bell inequality. In this section, while we do not give an explicit illustration of a Bell violation, we show how Bell’s assumption of local realistic theories breaks down for this model.

First, it has been explained that for noncyclic cause and effect, a classical-causal model one cannot induce nonlocality. In the objective field model, this criticism is overcome because the unitary dynamics associated with the measurement allows for causal loops which bypass this restriction.

Second, the above analysis for the EPR system shows consistency between the model and EPR’s local realism (if appropriately qualified), which also seems counterintuitive to allowing violation of a Bell inequality. However, the qualified form of EPR’s local realism is more restrictive than that assumed in Bell’s theorem. The local realism utilized by Bell (in the first proof $[8]$) is that the final outcome of $\hat{x}$ or $\hat{p}$ for system $A$ can be specified with certainty $in\ principle$, at a time $t$, be later choosing an appropriate measurement setting at $B$. This allowed Bell to assert (for the bipartite system) predetermined outcomes (i.e. hidden variables) for three measurement settings (in the spin case), an assumption which is then falsified by Bell’s theorem. Violations of Bell inequalities have been obtained for observables $\hat{x}_\theta$ which are linear combinations of $\hat{x}$ and $\hat{p}$ $[12, 13]$, but here also the violations required three or more different measurement settings $\theta$. This suggests that the dynamics associated with the choice of measurement setting is important in giving rise to Bell nonlocality.

To observe nonlocality as a violation of a Bell inequality, measurements are performed on both $A$ and $B$. Measurement is made of the joint probability $P(X_A, X_B)$ for outcomes $X_A$ and $X_B$ at the two sites $A$ and $B$ respectively. The $\theta$ and $\phi$ denote the choice of measurement setting at each site. Bell’s most general theorem is based on the premise that hidden variables $\{\lambda\}$ describe the correlated systems, according to a distribution $\rho(\lambda)$ which is independent of measurement settings.

The assumption of locality is also made — that the
probability \(p_A(X_\theta|\lambda)\) for an outcome \(X\) at \(A\), given the system is specified by the set of variables \(\{\lambda\}\), is independent of the choice of measurement \(\phi\) at the location \(B\). Mathematically, Bell’s assumptions imply that the joint probability for outcomes \(X_\theta\) and \(X_\phi\) at \(A\) and \(B\) respectively, given the system is specified by \(\{\lambda\}\), is of the form \([8–10, 114, 115]\)

\[ p(X_\theta, X_\phi|\theta, \phi) = \int \rho(\lambda)d\lambda \, p_A(X_\theta|\lambda, \theta)p_B(X_\phi|\lambda, \phi) \]  

(104)

Here \(p_A(X_\theta|\lambda, \theta)\) is the probability of an outcome \(X_\theta\) at \(A\) given the hidden variables \(\{\lambda\}\) and the measurement setting \(\theta\) at \(A\), which is assumed independent of the measurement setting \(\phi\) at \(B\), \(p_B(X_\phi|\lambda, \phi)\) is defined similarly. The assumption implies that Bell inequalities must hold. In the model presented in this paper, the hidden variables are the phase space amplitudes denoted \(\lambda \equiv \{x_A, p_A, x_B, p_B\}\).

How would this phase-space model violate the condition (104)? The trajectories given by the model are based on the independent amplification of either \(x\) or \(p\) at each site. The measurement setting is the choice of the sign of \(g\) in the Hamiltonians \(H_A\) and \(H_B\), thus determining whether \(\hat{x}\) or \(\hat{p}\) is measured. In the sense that the choice of measurement setting is independent at each site, there is no nonlocal mechanism. Here, we consider \(\theta\) and \(\phi\) to be either \(\hat{x}\) or \(\hat{p}\).

Suppose one considers measurement of \(\hat{x}\) (or \(\hat{p}\)) by amplification. For a given trajectory starting at \(t_f\), there is a single outcome at \(t_0\). At any time \(t\) such that \(t_0 \leq t \leq t_f\), we can then identify a trajectory value \(x\) (or \(p\)). It would seem that there is a local realistic representation of the dynamical development of the correlations from an initial positive distribution \(Q(\lambda, 0)\) for the amplitudes. Regardless, the Bell assumption (104) breaks down for the trajectories. The trajectories for the measurable observable originate from a boundary condition in the future. This boundary condition specifies the correlations between the trajectories for \(A\) and \(B\) according to the \(Q\) function at the final time \(t_f\) i.e. as seen from the \(Q\) function for the amplified variables. The final \(Q\) function is determined by the choice of measurement at both locations, since these are the quantities that are amplified. The amplified \(Q\) function that determines the correlations between the trajectories at \(A\) and \(B\) cannot in general take the role of a \(\rho(\lambda)\) in (104), because in the Bell assumption the \(\rho(\lambda)\) is independent of measurement settings \(\theta\) and \(\phi\). If we consider where \(Q(\lambda, 0)\) is an entangled state, the factorization condition within the integrand is not generally satisfied for the trajectories, because of the correlation between \(A\) and \(B\) evident in the \(Q\) function at the future time \(t_f\) (refer Eq. [55]).

We note two important features that enable the breakdown of the Bell assumption (104). If the \(Q\) function \(Q(\lambda, 0)\) at the time \(t_0\) is that of an entangled state, then the function cannot be expressed as a mixture of functions factorisable with respect to systems \(A\) and \(B\) (refer Eq. [55]). The entanglement is preserved upon the amplification according to \(H_A\) and \(H_B\), and it is this that prevents the assumption of the factorization of the integrand in the Bell assumption (104). For a non-entangled state, the \(Q\) function can be expressed as a mixture of factorizable functions (refer Eq. [55] without the third term), and the Bell assumption (104) holds.

The second feature concerns the existence of the non-amplifiable quantum noise, with variance \(\sigma_x = 1\), in the \(Q\) function. These fluctuations are not amplified by the measurements \(H_A\) and \(H_B\). It is this feature that results in a different form for the amplified \(Q\) functions at time \(t_f\), which determines the final correlations between \(A\) and \(B\) at the future boundary.

**IX. CONCLUSION**

We have presented a model for a quantum measurement which provides a description involving hidden variables. We mostly limit ourselves, for simplicity, to a measurement of \(\hat{x}\) or \(\hat{p}\) on a single field mode and consider the dynamics of the measurement interaction \(H_A\) which amplifies the \(\hat{x}\) or \(\hat{p}\). The model is based on the \(Q\) function \(Q(x, p)\) which uniquely represents the quantum state, and on the equivalence of the dynamical equation for the \(Q\) distribution function with forward-backward stochastic trajectories for amplitudes \(x\) and \(p\). The equivalence is proved in the Theorems, and introduces a technique whereby the forward and backward trajectories are linked by a boundary condition in the present time \(t_0\). We find the solutions for \(x\) and \(p\) involve both forward and backward trajectories. These require a conditional boundary value at time \(t_0\) prior to the measurement interaction, and also a marginal boundary value at the time \(t_f\) after the interaction \(H_A\). The measured variable is amplified, and this is the feature that requires a future boundary condition. In this way, we see that the solutions are dependent on a future boundary condition determined by the choice of measurement setting – whether \(\hat{x}\) or \(\hat{p}\) is measured.

We solve the dynamical trajectories for several examples, illustrating the equivalence to the \(Q\) function throughout the dynamics. This provides a check of the Theorems, verified through \(\chi^2\) statistical tests. Measurement of both \(\hat{x}\) and \(\hat{p}\) on a superposition of eigenstates of \(\hat{x}\) is examined, with Born’s rule validated in each case. The analysis focuses on the superposition of two eigenstates. Regardless of the separation between the eigenstates, the final outcomes for the measurement clearly distinguish between the eigenstates. The analysis is extended to cat states.

The trajectories provide a description for reality that is more complete than quantum mechanics. This was shown for the superposition of two eigenstates for \(\hat{x}\) by analyzing the linked sets of trajectories for \(x\) and \(p\) and then evaluating the variances associated with those distributions post-selected on the final outcome for the measurement of \(\hat{x}\). The variance product reduces below that
allowed by the Heisenberg uncertainty relation, implying that the linked trajectories associated with the single outcome cannot be modeled by a quantum state $|\psi\rangle$.

Different models of realism are discussed and analyzed for compatibility with our model based on these trajectories. The analysis supports the validity of macroscopic realism and of macroscopic causality, despite the fact that retrocausality is embodied in the model. The interpretation is that macroscopic causality holds, and a system prepared in a superposition of two macroscopically distinct eigenstates of $\hat{x}$ at the time $t_0$ has a well defined outcome for the result of the pointer measurement $\hat{x}$: i.e., the system is in a state that will give one or other macroscopically distinct outcome. The final result of the measurement depends on the state at the time $t_0$. However, there is microscopic noise at the level of $\hbar$ which is present in the final outcomes at the time $t_f$. This noise is also present in the microscopic system and the values are determined retrocausally.

The hybrid model provides an interpretation of the Schrödinger cat paradox. Macroscopic realism holds for the macroscopic superposition state, but the state of the system at the microscopic level is such that it cannot be described by quantum mechanics. The $Q$ model provides a more complete description of the state of the “cat” than can be given by standard quantum mechanics. This is because the microscopic fluctuations rely on retrocausality. The incompleteness of quantum mechanics to describe the state of the cat system is shown by the reduced variances in $x$ and $p$ for the trajectories conditioned on the final measurement outcome. The objective field model therefore provides a resolution of Einstein-Podolsky-Rosen-type paradoxes that argue for the completion of quantum mechanics, based on the validity of macroscopic realism $^{42, 79, 108}$. A resolution of similar related paradoxes may also be possible. We also note that the hybrid model does not conflict with the known violations of macrorealism and of macroscopic local realism (as shown through violations of Leggett-Garg $^{76}$ and macroscopic Bell inequalities $^{76, 78, 79}$ because in those cases, extra assumptions are involved in the premises that are falsified. The subtle distinction between weak macroscopic realism and deterministic macroscopic realism is explained in Refs. $^{42, 79, 108, 116}$.

A similar model for realism that is supported by this model is the contextual deterministic model for realism, which postulates that the system in a superposition of two states is in a state with well-defined outcome for measurement once the measurement setting is determined. This means that a suitable unitary interaction has taken place to determine the measurement setting. The hybrid macroscopic-realism model is an example of this model, since the amplification $H_A$ determines the measurement setting. Other unitary interactions however are possible and the meaning of this form of realism is hence more general than that of the hybrid macroscopic-realism model. This model is consistent with contextual explanations of Bell violations $^{87}$.

Einstein-Podolsky-Rosen (EPR) correlations between two field modes $A$ and $B$ are possible for quantum states which are simultaneously eigenstates of both $\hat{x}_A - \hat{x}_B$ and $p_A + p_B$. Here, EPR argued that quantum mechanics was incomplete, based on the validity of the premise of local realism. We have examined how the $Q$ model explains EPR correlations, giving a model consistent with a qualified form of EPR’s local realism that is indeed more complete than quantum mechanics. However, the retrocausal aspect of the model means that the assumption of Bell’s local hidden variables does not hold. Hence one can expect the model to explain Bell violations.

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