Local sampling of the quantum phase-space distribution of a continuous-wave optical beam

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Abstract. It is shown how the quantum phase-space distribution of a continuous-wave (cw) optical beam can be obtained independently at each point in phase space by a combination of unbalanced homodyne and balanced-heterodyne techniques. The unbalanced homodyning allows for the local sampling of phase space, whereas the heterodyne part, although introducing a loss of detection efficiency, provides a highly efficient reduction of 1/f noise that is required for the sampling of cw fields. The method complements well-known techniques for light pulses by providing a robust method for cw optical beams.

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1. Introduction

Balanced optical homodyning has been successfully used in the past to measure phase-dependent quadrature statistics and to show reductions of the quadrature fluctuations below the standard quantum limit, which can then be attributed to incident squeezed light. Taking a closer look at the first observations of squeezed light [1–4], it becomes clear that beat signals were recorded. The continuous-wave (cw) squeezed light was superimposed by a frequency shifted local-oscillator (LO) field in order to be able to measure the beat signal by a spectrum analyser (SA). In this way $1/f$ excess noise can be efficiently suppressed in the measured signal. Similar ideas have been exploited when squeezing was measured on frequency-modulated side bands [5, 6]. As these methods are based on the measurement of a beat signal, they could equally well be denoted as heterodyne measurements.

Another way to suppress the influence of excess noise is by the use of light pulses instead of cw fields. Triggering the measurements with the pulses allows for an efficient suppression of noise at frequencies below the inverse pulse duration. This method has been used to demonstrate the first quantum-state reconstruction from quadrature statistics measured in balanced optical homodyne tomography [7], whereas for cw light heterodyne-type methods were used [6].

Alternatives to balanced-homodyne tomography [8], a method for the local sampling of the phase-space distribution of the light field, have been proposed in the past [9, 10]. This method is based on displacing the signal light field in phase space by the use of an unbalanced beam splitter and an LO field. Subsequently, the photon statistics in the output channel of maximum transmittance of the signal field is measured. In this way, the quantum phase-space distribution can be sampled locally at each point in phase space. The method is optimal for obtaining the phase-space distribution according to a recent theoretical work [11].

This idea has been implemented for cw input states that generate very low count rates with an avalanche photo diode operating in the Geiger regime [12], and for pulsed light [13, 14]. However, for general input states, other implementations must be searched for that combine a sufficient photon-number resolution together with a broad range of detectable photon numbers, such as, for example, time-multiplexing detectors as realized in [15] for pulsed light. A solution...
to this problem has been proposed by replacing the photo detector (PD) by a phase-randomized balanced-homodyne detector [16]. The resulting setup is a cascaded homodyne scheme that includes an unbalanced and a balanced stage [17]. A modified version of it has been proposed later for obtaining the phase-space distribution of an intra-cavity field [18].

If balanced, unbalanced or cascaded homodyne setups are applied to cw light fields, the electronic signals generated by the amplified PDs are directly processed without spectral filtering or temporal triggering. As their full spectrum from dc up to the full bandwidth of the detector is recorded, practical problems emerge due to unavoidable technical noise sources, such as electronic noise, etc. These noise sources provide a noise background in the measured signal that typically increases with decreasing frequency, for which it is commonly denoted as $1/f$ noise [19]. A tomographic reconstruction of the phase-space distribution of the cw light field is then strongly compromised by the presence of this typically dominant technical noise. On the other hand, for pulsed light or heralded photons the signal may be spectrally analysed at the repetition rate of trigger events, which typically lies well above the frequencies where $1/f$ noise is dominant. In this way $1/f$ noise can be largely suppressed.

In this paper, we show how the proposal of cascaded homodyning can be modified to be experimentally feasible for the local sampling of the quantum phase space of an incident cw light field. We show how this can be reached by combining unbalanced homodyne and balanced-heterodyne techniques. The unbalanced homodyning allows for the local sampling of phase space, which represents an advantage over non-local reconstruction methods such as inverse Radon transforms. And the heterodyne part, although introducing a loss of detection efficiency, provides a highly efficient reduction of $1/f$ excess noise that is necessary in the case of cw detection.

2. Local sampling of phase space

2.1. Unbalanced homodyning

A local sampling of the quasi-probability distribution in the phase space of an optical mode can be obtained by unbalanced homodyning [9, 10]. The outline of the setup is shown in figure 1. Both signal (S) and LO fields have a fixed difference phase within the time of data acquisition, and the relative phase $\phi$ and amplitude $A$ of the LO field are adjustable. Both fields are superimposed on a beam splitter with very low reflectivity, $|r| \ll 1$, so that almost all of the signal light is transmitted into the detector. The effect of the LO light is to add a coherent amplitude $rAe^{i\phi}$ to the field incident on the PD. The PD is ideally measuring the photo-electron statistics related to the incoming photon flux integrated over the detection response time $\tau_{\text{det}}$, which must be smaller than the coherence time of the light, $\tau_{\text{det}} < \tau_{\text{coh}}$, in order to properly map information on the photon statistics into the photo-electron statistics.

Given the quantum efficiency $\eta_{\text{PD}}$ of the PD, the overall detection efficiency is $\eta = \eta_{\text{PD}}|t|^2$, where $t$ is the transmittance of the beam splitter, and the PD ‘sees’ a signal field displaced by $\alpha = (r/t)Ae^{i\phi}$. It can be shown that the photo-electron statistics $P_n(\alpha; \eta)$ measured in this way contains all the information necessary for obtaining the value of the $s$-ordered quasi-probability distribution $W$ at the phase-space point $\alpha$, i.e.

$$W(\alpha; s) = \frac{2}{\pi(1-s)} \sum_{n=0}^{\infty} [-\xi(s; \eta)]^n P_n(\alpha; \eta),$$

(1)
Figure 1. The unbalanced optical homodyne setup. The signal field $S$ is superimposed with a local oscillator field LO on a beam splitter with high transmittance $|t|^2 \sim 1$ and the photo-electron statistics of the mixed and transmitted signal fields is measured with the PD. The point in phase space where the quasi-probability of the signal field is adapted by changes the amplitude and phase of the LO.

where

$$\xi(s; \eta) = \frac{2 - \eta(1 - s)}{\eta(1 - s)}.$$

The main advantage of this setup over optical balanced-homodyne tomography is that the phase-space distribution is obtained at each point in phase space independently. Thus, a local sampling of the phase-space distribution is obtained. Unfortunately, the setup in figure 1 requires a highly efficient photon-number resolving detector, which is currently available only in the form of avalanche photo diodes in the Geiger regime. As such diodes may only resolve between zero or some photons, it must be guaranteed by other means that the incident photon flux integrated over the detector response time is not larger than 1. Practically, this means that only phase-space distributions can be measured that are sufficiently narrow. Under these conditions, an experimental implementation was shown in [12].

2.2. Cascaded homodyning

To overcome the lack of highly efficient PDs with output signals that resolve the actual photon number, it has been proposed in [17] to substitute the PD by a phase-randomized balanced-homodyne detector [16]. This has the advantages that highly efficient linearly responding photo diodes can be used and that the resolution of photon numbers is easily obtained for sufficiently large intensities of the additional second LO field. Such a cascaded homodyne scheme has the outline shown in figure 2. The displaced signal field is now superimposed by LO2 with the 50 : 50 beam splitter BS2. The two output intensities are measured with photo diodes PD1 and PD2, and their photo currents are then subtracted to obtain the statistics of quadratures. If the phase of LO2 is integrated over a $2\pi$ interval or if this integral is approximated by...
randomization, the photon statistics can be obtained from the phase-integrated quadrature statistics \( p(q; \alpha, \eta) \) as [16]

\[
P_n(\alpha, \eta) = \int dq f_{nn}(q)p(q; \alpha, \eta),
\]

where \( f_{nn}(x) \) are pattern functions that can be obtained from the regular and irregular eigenfunctions of the Schrödinger equation of the harmonic oscillator [20, 21]. Inserting equation (3) into (1), together with equation (2), the \( s \)-ordered phase-space distribution at the point \( \alpha \) determined by LO1 is obtained directly as

\[
W(\alpha; s) = \int dq S(q; s, \eta)p(q; \alpha, \eta),
\]

where the unique sampling function is given as [17]

\[
S(q; s, \eta) = \frac{\eta}{\pi[\eta(1-s)-1]} f_{00}\left(\frac{q}{\sqrt{\eta(1-s)-1}}\right),
\]

with the pattern function being determined by the Dawson integral \( F(x) \) [22] as \( f_{00}(x) = 2 - 4x F(x) \).

This setup, while resolving the issue of high efficiency combined with photon-number resolution, is still problematic for cw light. As the measured observable is the difference of two photo currents that are generated in the photo diodes by integrating the incoming photon flux over the response time \( \tau_{det} \), no spectral filtering or selection may be performed. That is,
all frequencies from dc up to the bandwidth of the PD are measured. As a consequence, the obtained histogram of quadratures is strongly affected by $1/f$ excess noise that may be dominant at low frequencies. Thus, for cw light, the implementation of the cascaded homodyne setup is very demanding due to the need to beat the excess noise by ultra-low noise detectors and electronics [27].

2.3. Cascaded homodyne–heterodyne setup

If LO2 was operated at a slightly different frequency than the light coming from BS1, the balanced-homodyne part would become a heterodyne one and PD1 and PD2 would detect the beat signal at the difference frequency. If this difference frequency is sufficiently high, the $1/f$ excess noise could be efficiently suppressed, similar to the squeezing experiments [1–4]. To implement the integration or randomization of the phase of LO2, one may now use a second independent laser source for LO2. As there will be no phase relation between signal, LO1 and LO2, the phase will be effectively randomized on the time scale of the coherence time. In this way no phase control of LO2 is needed, as shown in the setup of this homodyne–heterodyne scheme, shown in figure 3.

This setup combines now the advantages of the previous ones, resolving their respective caveats. Firstly, the method is a local one, meaning that the phase-space distribution is obtained in each point in phase space independently. Thus no inverse Radon transform of the measured data is required. The measured data are simply integrated with the unique sampling function (5). Secondly, the PDs should have high quantum efficiency but need not be photon-number...
Figure 4. Spectral analysis of the heterodyne signal. The photo detectors PD1 and PD2 generate the photo currents $I_1$ and $I_2$ from the incident photon flux $\Phi_1$ and $\Phi_2$, respectively. The currents are transformed into voltages $U_1$ and $U_2$ by the trans-impedance amplifiers TIA1 and TIA2, respectively, both with trans-impedance $R$ and bandwidth $\Delta\omega_{\text{det}}$. The difference voltage is processed in the SA, which consists in a bandpass filter (BP) with resolution bandwidth $\Delta\omega_{\text{rbw}}$ and mid-frequency $\omega_0$, followed by an envelope detector (ED).

resolving. The resolution of photon numbers is obtained by a sufficiently intense LO2. Thirdly, excess noise is efficiently suppressed due to the heterodyne stage, so that the measured data truly reflect the quantum statistics of the signal field. Finally, the use of an independent laser source for LO2 leads to a natural phase randomization so that a phase control of LO2 is not required. All this, however, comes with a well-known disadvantage that is due to the spectral filtering of the heterodyne beat signal: the quantum efficiency of the heterodyne stage as an effective PD is half the efficiency of the employed photo diodes [25]. That means that the local sampling can be performed only for $s$-ordered quasi-probability phase-space distributions with $s \leq -1$, i.e. for the Husimi $Q$ function or Gaussian convolutions of it. However, this phase-space distribution still contains the complete information on the quantum state of the signal light. Therefore, one may guess that by means of maximum-likelihood methods [23, 24] a corresponding deconvolution to, e.g., the Wigner function may be performed.

3. Cascaded homodyne–heterodyne setup

In the following, a detailed analysis of the proposed cascaded homodyne–heterodyne setup is performed in order to obtain experimentally relevant parameters and to show its experimental feasibility.

3.1. Spectrally analysed heterodyne signal

To show the electronic processing of the heterodyne signal, as shown in figure 4, we assume for the moment a unit quantum efficiency of the photo diodes. Imperfect detection will then be treated in the following section.

The incident photon flux coming from the output channels of BS2, cf figure 3,

$$\Phi_d(t) = \frac{c\varepsilon_0}{\hbar V_0} \int d S \hat{E}^{(d)(-)}(\vec{r}, t) \hat{E}^{(d)(+)}(\vec{r}, t) \quad (d = 1, 2),$$

(6)
are transformed by PD1 and PD2 into the short-circuit photo-electron currents \( \hat{I}_d(t) = e \Phi(t) \). These currents are amplified with two identical trans-impedance amplifiers TIA1 and TIA2 with bandwidth \( \delta \omega_{\text{det}} \) to obtain the output voltages
\[
\hat{U}_d(t) = R \int dt' f_{\text{det}}(t-t') \hat{I}_d(t'),
\]
which are convolutions of the currents with the time response of the amplifiers \( f_{\text{det}}(t) \) with \( \int dt f_{\text{det}}(t) = 1 \). The dc trans-impedance \( R \) determines the conversion of the photo currents to measurable voltages. The voltages are then subtracted to obtain
\[
\Delta \hat{U}(t) = \hat{U}_1(t) - \hat{U}_2(t),
\]
in terms of the incident photon flux reads
\[
\Delta \hat{U}(t) = eR \int dt' f_{\text{det}}(t-t')[\hat{\Phi}_1(t') - \hat{\Phi}_2(t')].
\]
The photo diodes, together with their trans-impedance amplifiers, determine the smallest time scale of measurable signal fluctuations, given by the inverse detection bandwidth \( \tau_{\text{det}} \sim 2\pi/\delta \omega_{\text{det}} \). As the amplifiers’ low-pass characteristics \( f_{\text{det}}(t) \) can be well approximated by a time-integrating circuit, the difference photon flux is basically integrated over the detector response time interval \( \tau_{\text{det}} \), i.e.
\[
\Delta \hat{U}(t) \approx U_{\text{det}} \Delta \hat{N}(t),
\]
where \( U_{\text{det}} = eR/\tau_{\text{det}} \) is the voltage generated by a single photo-electron within the response time of the detector. Here \( \Delta \hat{N}(t) = \hat{N}_1(t) - \hat{N}_2(t) \), where the incident photon numbers accumulated during \( \tau_{\text{det}} \) at the detectors are
\[
\hat{N}_d(t) = \int_{t-\tau_{\text{det}}}^{t} dt' \hat{\Phi}_d(t').
\]
The difference voltage (9) is then given on the SA, which consists of the band pass (BP) filter followed by the envelope detector (ED). The BP filter is employed to select the part of the signal that oscillates with the beat frequency \( \Delta = \omega_1 - \omega_0 \) of the superimposed light fields, i.e. its mid-frequency \( \omega_0 \) is set to the difference frequency of the light fields \( \omega_0 \approx \Delta \). The spectral width of this filter, \( \delta \omega_{\text{rbw}} \), known as the resolution bandwidth (RBW), is chosen so as to let pass through the beating of all spectral components of the optical field. That is, \( \delta \omega_{\text{rbw}} > \delta \omega_1 \), where \( \delta \omega_1 \) is the spectral bandwidth of the light.

The filtered signal is still oscillating at the chosen mid-frequency \( \omega_0 \) of the BP filter. The second stage of the SA is detecting the amplitude of this oscillation by the use of an ED, which measures the peak value within one period of the oscillation. In this way a non-oscillating signal is generated that is still random due to the randomness of the signal injected into the SA\(^2\).

Mathematically, the effect of the BP filter is a further temporal convolution
\[
\hat{U}_{\text{bp}}(t) = \int dt' f_{\text{bp}}(t-t') \Delta \hat{U}(t'),
\]
where the time response of the BP filter is
\[
f_{\text{bp}}(t) = g_{\text{bp}}(t) e^{i\omega_0 t} + \text{c.c.}
\]
\(^2\) As the final stage the SA would allow us to smooth this randomness by a further low-pass filter with bandwidth denoted as video bandwidth (VBW). However, for our purpose such a smoothing should be avoided by setting VBW to its maximum value.
The function \( g_{bp}(t) \) satisfies \( \int dt g_{bp}(t) = 1 \) and has a temporal width \( \tau_{rbw} = 2\pi/\delta \omega_{rbw} \) with \( \delta \omega_{rbw} \) and \( \omega_0 \) being the above-mentioned RBW and mid-frequency of the SA. We may approximate the filter function as \( g_{bp}(t) \approx \tau_{rbw}^{-1} \) for \( t \in [-\tau_{rbw}, 0] \) and \( g_{bp}(t) = 0 \) elsewhere, to obtain from equation (11)

\[
\hat{U}_{bp}(t) \approx U_{det} \tau_{rbw}^{-1} \int_{t-\tau_{rbw}}^{t} dt' e^{i \omega_0(t-t')} \Delta \hat{N}(t') + c.c.
\]  

To obtain the envelope of this fast oscillation, the peak value within the period \( 2\pi/\omega_0 \) is measured by the ED. The result becomes the SA signal

\[
\hat{U}_{sa}(t) \approx 2 U_{det} \tau_{rbw}^{-1} \int_{t-\tau_{rbw}}^{t} dt' \cos(\phi_{ed} - \omega_0 t') \Delta \hat{N}(t'),
\]

where \( \phi_{ed} \) is the phase of the corresponding peak value of the fast oscillation.

### 3.2. Quantum statistics of the spectrally analysed signal

The photo diodes typically have non-unit quantum efficiency \( \eta \), so that the measured photo-electron statistics is different from the statistics of incident photons. To take this into account, we employ the quantum theory of photo detection (see e.g. [26]) to obtain the statistics of the output voltage of the spectrum analyser \( \hat{U}_{sa} \).

The heterodyne signal is measured with PDs that integrate the incident photon flux over the time interval \( \tau_{det} \). Therefore, in the spectrally analysed signal (14), we may replace the time integration over the interval \( \tau_{rbw} \) by a sum over discrete time intervals \( \tau_{det} \),

\[
\hat{U}_{sa}(t) \approx U_{rbw} \sum_{k=0}^{K-1} \xi_k \Delta \hat{N}(t_k)
\]

with \( t_k = t - k \tau_{det} \) and \( K = [\delta \omega_{det}/\delta \omega_{rbw}] \), where \( K \gg 1 \) because the highest detectable frequency is determined by the detector, i.e. \( \delta \omega_{rbw}, \omega_0 \ll \delta \omega_{det} \). Here \( U_{rbw} = eR/\tau_{rbw} \) is the voltage generated by a single photo-electron within the temporal resolution of the spectral analyser \( \tau_{rbw} \) and

\[
\xi_k = 2 \cos(\phi_{ed} - \omega_0 t_k).
\]

Using the discretized representation (15), the probability density to measure at time \( t \) the spectrally analysed difference voltage \( U \) with overall quantum efficiency \( \eta \), given a phase-space displacement \( \alpha \), can now be formulated as

\[
p(U, t; \alpha, \eta) = \sum_{\{n_{1,k}, n_{2,k}\}} P_{\{n_{1,k}, n_{2,k}\}}(\{t_k\}; \alpha, \eta) \delta \left[ U - U_{rbw} \sum_{k=0}^{K-1} \xi_k (n_{1,k} - n_{2,k}) \right].
\]

Here the joint probability for detecting during the time intervals \( \{[t_k - \tau_{det}, t_k] \} \) the accumulated photon numbers \( \{n_{1,k}, n_{2,k}\} \) with quantum efficiency \( \eta \) reads as [26]

\[
P_{\{n_{1,k}, n_{2,k}\}}(\{t_k\}; \alpha, \eta) = \left\langle \prod_{k=0}^{K-1} \hat{P}_{n_{1,k} n_{2,k}}(t_k; \eta) \right\rangle_\alpha,
\]

where the operators are defined as

\[
\hat{P}_{n_{1},n_{2}}(t; \eta) = \frac{[\eta \hat{N}_1(t)]^{n_1} [\eta \hat{N}_2(t)]^{n_2}}{n_1! n_2!} e^{-\eta [\hat{N}_1(t) + \hat{N}_2(t)]},
\]
and \( \langle \cdots \rangle_a \) denotes the expectation value in the quantum state of the signal field, coherently displaced by the amplitude \( \alpha \).

Using the Fourier representation of the Dirac delta function, equation (17) can be rewritten as

\[
p(U, t; \alpha, \eta) = \frac{1}{2\pi U_{\text{rbw}}} \int dx \left\langle \prod_{k=0}^{K-1} \hat{C}(x\xi_k, t_k; \eta) \right\rangle \exp \left( i x \frac{U}{U_{\text{rbw}}} \right),
\]

(20)

with the operator-valued characteristic function for the difference counts \( m = n_1 - n_2 \) being defined as

\[
\hat{C}(x, t; \eta) = \sum_m \hat{P}_m(t; \eta) e^{-ixm},
\]

(21)

where the difference-count statistics is defined by the operator

\[
\hat{P}_m(t; \eta) = \sum_{k} \hat{P}_{k+m,k}(t; \eta).
\]

(22)

Given LO2 in a coherent state with mean photon flux \( \Phi_{\lambda_0} \), the accumulated mean photon number within \( \tau_{\text{det}} \) is \( N_{\lambda_0, \text{det}} = \Phi_{\lambda_0} \tau_{\text{det}} \). Supposing that this number is much larger than that of the signal field, \( N_{\lambda_0, \text{det}} \gg \langle \hat{N}(t) \rangle \), the operator (22) can be approximated as a function of the quasi-continuous variable \( m \), \( \hat{P}_m(t; \eta) \approx \hat{P}(m, t; \eta, N_{\lambda_0, \text{det}}) \) with the normalized Gaussian operator

\[
\hat{P}(m, t; \eta, \mathcal{N}) = (2\pi \eta\mathcal{N})^{-1/2} \exp \left\{ -\frac{1}{2} \left[ \frac{m - \eta \Delta \hat{N}(t)}{\sqrt{\eta \mathcal{N}}} \right]^2 \right\}.
\]

(23)

Using this approximation, in equation (21) the sum can be replaced by an integral. The scaled argument \( x\xi_k \) in equation (20) can then be incorporated by substitution of the integration variable \( m \rightarrow m' = m \cdot \xi_k \), which together with the relation \( \hat{P}(m/\xi, t; \eta, \mathcal{N}) = \xi \hat{P}(m, t; \xi \eta, \xi \mathcal{N}) \), cf equation (23), results as

\[
\hat{C}(x\xi_k, t_k; \eta) = \int dm \hat{P}(m, t_k; \xi k \eta, \xi k N_{\lambda_0, \text{det}}) e^{-ixm}.
\]

(24)

Furthermore, using equations (23) and (24), equation (20) can be seen to be a convolution of Gaussians of the form (23),

\[
p(U, t; \alpha, \eta) = U_{\text{rbw}}^{-1} \int dm_{K-1} \ldots \int dm_1 \left\langle \prod_{k=0}^{K-1} \hat{P}(m_{k+1} - m_k, t_k; \xi_k \eta, \xi_k N_{\lambda_0, \text{det}}) \right\rangle \alpha,
\]

(25)

with \( m_0 = 0 \) and \( m_K = U / U_{\text{rbw}} \). The result is again Gaussian and reads as

\[
p(U, t; \alpha, \eta) = \frac{1}{\sqrt{2\pi} \delta U} \left\langle \exp \left\{ -\frac{1}{2} \left[ \frac{U - \eta \dot{U}_{\lambda\alpha}(t)}{\delta U} \right]^2 \right\} \right\rangle \alpha,
\]

(26)

where

\[
\delta U = U_{\text{rbw}} \sqrt{\eta N_{\lambda_0, \text{det}} \xi^2}.
\]

(27)
Here we made use of equation (15) and the width of the Gaussian (26) is modified by the time-dependent function
\[
\overline{\xi^2}(t) = \sum_{k=0}^{K-1} \xi_k^2(t).
\]  
(28)

This function can be obtained for \(\omega_0 \gg \delta \omega_{\text{rbw}}\) as the constant \(\overline{\xi^2}(t) \approx 2\tau_{\text{rbw}}/\tau_{\text{det}}\). In turn this leads to the root mean square voltage spread
\[
\delta U = U_{\text{rbw}} \sqrt{2\eta N_{\text{lo,rbw}}},
\]  
(29)

where \(N_{\text{lo,rbw}} = \Phi_0 \tau_{\text{rbw}}\) is now the accumulated photon number of the LO field within the integration time \(\tau_{\text{rbw}}\) determined by the RBW of the SA. The additional factor \(\sqrt{2}\) is well known to appear due to the use of a heterodyne instead of a homodyne setup [25]. It is due to the fact that the spectrum of the real-valued beat signal has symmetric components at frequencies \(\pm \Delta\), whereas the SA selects only the positive-frequency part and thus discards half of the signal. This translates into the additional factor \(1/2\) in the exponent of the Gaussian (26), which finally leads to a reduction of the overall quantum efficiency by 50%.

3.3. Phase-randomized quadrature distribution

As \(\Delta \hat{N}(t)\) is a heterodyne signal composed of two signals with mid-frequencies \(\omega_S\) (signal field) and \(\omega_L\) (LO2 field), for perfect balanced detection it oscillates with the beat frequency \(\Delta = \omega_S - \omega_L\) as
\[
\Delta \hat{N}(t) = \Delta \hat{N}^{(+)}(t) e^{i\Delta t} + \Delta \hat{N}^{(-)}(t) e^{-i\Delta t},
\]  
(30)

where for perfect mode matching the slowly varying operators read as
\[
\Delta \hat{N}^{(+)}(t) = \Phi_0 \tau_{\text{det}} \sqrt{\langle \hat{n}_{\text{lo}} \rangle} \hat{a}_s e^{-i\phi_{\text{lo}}(t)}, \quad \Delta \hat{N}^{(-)}(t) = [\Delta \hat{N}^{(+)}(t)]^*,
\]  
(31)

with \(\hat{n}_{\text{lo}} = \hat{a}_{\text{lo}}^\dagger \hat{a}_{\text{lo}}\). Here \(\phi_{\text{lo}}(t)\) is the phase of the LO2 field and the single-photon flux is defined as
\[
\Phi_0 = \frac{c \varepsilon_0 \bar{\hbar} \nu_0}{\hbar V_0} \int dS |E(\vec{r})|^2,
\]  
(32)

with \(E(\vec{r})\) being the transverse mode of the beams. In this way, the LO2 photon flux becomes \(\Phi_{\text{lo}} = \Phi_0 \langle \hat{a}_{\text{lo}}^\dagger \hat{a}_{\text{lo}} \rangle\).

Given that the BP mid-frequency is tuned to the beat frequency \(\omega_0 \approx \Delta\) and that \(\omega_0 \gg \delta \omega_{\text{rbw}}\), the integration time \(\sim \tau_{\text{rbw}}\) in equation (14) is sufficiently long to suppress the counter-rotating terms. As a consequence, the two remaining resonant terms can be written as
\[
\hat{U}_{\text{sa}}(t) \approx U_{\text{rbw}} \Delta N_{\text{lo,rbw}} \sqrt{2} \hat{q} (\phi_{\text{lo}}(t) - \phi_{\text{ed}}),
\]  
(33)

where the signal quadrature is defined as
\[
\hat{q}(\phi) = \frac{1}{\sqrt{2}} [\hat{a}_s e^{-i\phi} + \text{c.c.}].
\]  
(34)

The variance of the fluctuations of the accumulated LO2 photon number within \(\tau_{\text{rbw}}\) is
\[
\Delta N_{\text{lo,rbw}} = \Phi_0 \Delta n_{\text{lo}} \tau_{\text{rbw}},
\]  
(35)
where Poissonian fluctuations of the LO have been used,

\[ \Delta n_{lo} = \sqrt{\langle (\Delta \hat{n}_{lo})^2 \rangle} = \sqrt{\langle \hat{n}_{lo} \rangle}. \]  

As a consequence, these fluctuations can be equivalently written as \( \Delta N_{lo,rbw} = \sqrt{N_{lo,rbw}}, \) so that we obtain

\[ \hat{U}_{sa}(t) = U_{rbw}\sqrt{2N_{lo,rbw}} \hat{q} \left( \phi_{lo}(t) - \phi_{ed} \right). \]  

The LO phase \( \phi_{lo}(t) \) is a random variable that changes on the time scale given by the coherence time of the LO field \( \tau_{coh} \sim 2\pi/\delta\omega_l \), where \( \delta\omega_l \) is the spectral bandwidth of the laser light. That is, given that \( \tau_{rbw} \ll \tau_{coh} \) (i.e. \( \delta\omega_l \ll \delta\omega_{rbw} \)), equation (26) represents the probability density for the quadrature with a well-defined LO phase. Performing now a time average of the measured voltage statistics over a time \( T \gg \tau_{coh} \) reveals automatically the phase-randomized statistics \( p(U; \alpha, \eta) = \tilde{p}(U; \alpha, \eta) \) with

\[ p(U; \alpha, \eta) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \frac{1}{\sqrt{2\pi}\delta U} \left\{ \text{exp} \left\{ -\frac{1}{2} \left[ \frac{U - \sqrt{\eta}\delta U \hat{q}(\phi)}{\delta U} \right]^2 \right\} \right\}_{\alpha} , \]  

where equations (26) and (37) have been used.

From this equation, the linear relationship between the measured voltage and quadrature can be seen to be

\[ U = \sqrt{\eta}\delta U q, \]  

so that the probability density \( p(q; \alpha, \eta) \) for the phase-randomized quadrature can be obtained from \( p(U; \alpha, \eta) \) \( dU = p(q; \alpha, \eta) dq \) as \( p(q; \alpha, \eta) = p(\sqrt{\eta}\delta U q; \alpha, \eta)\sqrt{\eta}\delta U \), which becomes

\[ p(q; \alpha, \eta) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \frac{1}{\sqrt{2\pi}\delta q} \left\{ \text{exp} \left\{ -\frac{1}{2} \left[ \frac{q - \hat{q}(\phi)}{\delta q} \right]^2 \right\} \right\}_{\alpha} , \]  

with \( \delta q = 1/\sqrt{\eta} \), which again is \( \sqrt{2} \) times larger than in the case of a homodyne measurement.

### 3.4. Experimental requirements

For a proper functioning of the heterodyne stage, specific frequency ranges of the laser fields and adjustments of the SA are required. Suppose that the light sources are external-cavity diode lasers with line widths of the order of \( \Delta\omega_l/2\pi \sim 1 \text{ MHz} \) and coherence times \( \tau_{coh} \sim 1 \mu\text{s} \). In order that the photo diodes generate photo currents that contain information on the photon statistics, the detection time should be much smaller \( \tau_{det} \ll \tau_{coh} \), which may be satisfied by \( \delta\omega_{det}/2\pi = 100 \text{ MHz} \) trans-impedance amplifiers, corresponding to \( \tau_{det} \sim 10 \text{ ns} \). Furthermore, we have seen that the BP-filter bandwidth of the SA should be \( \delta\omega_{det} \ll \delta\omega_{rbw} < \delta\omega_{det} \) and that \( \delta\omega_{rbw} < \omega_0 < \delta\omega_{det} \), so that the time scales are ordered as

\[ \delta\omega_{det} < \delta\omega_{rbw} < \omega_0 < \delta\omega_{det}. \]  

These conditions can be satisfied by choosing the bandwidth \( \delta\omega_{rbw}/2\pi \sim 10 \text{ MHz} \) and the filter mid-frequency as \( \omega_0/2\pi \sim 20 \text{ MHz} \), the latter being matched by the laser detuning \( \Delta = \omega_0 \).

The overall quantum efficiency of the cascaded homodyne–heterodyne setup is

\[ \eta = |t|^2 \frac{\eta_{det}}{2}. \]
Figure 5. Sampling function for an overall quantum efficiency of $\eta = 0.4$ and $s = s_{\text{max}} - \Delta s$ with $s_{\text{max}} = -1.5$. Values for $\Delta s$ are: 0.5 (blue curve), 0.1 (green curve) and 0.05 (red curve).

As a consequence, a convergent sampling function (5) is possible only for the sampling of a phase-space quasi-probability with ordering parameter $s < s_{\text{max}}$ where the maximum value is

$$s_{\text{max}} = 1 - \frac{2}{|U|^2 |\eta|_{\text{det}}}.$$  \hspace{1cm} (43)

From the phase-integrated quadrature distribution (40) the phase-space quasi-probability at the point $\alpha$ is obtained by integration over the sampling function (5) as given in equation (3). In terms of the measured voltage statistics this can be rewritten using (39) as

$$W(\alpha; s) = \int \text{d}U \tilde{S}(U; s, \eta) p(U; \alpha, \eta),$$  \hspace{1cm} (44)

where the scaled sampling function is

$$\tilde{S}(U; s, \eta) = S \left( \frac{U}{\sqrt{\eta} \delta U}; s, \eta \right).$$  \hspace{1cm} (45)

The original sampling function for the quadrature is a well-located non-zero function with an approximate range $|q| \lesssim 1$. As a consequence, the sampling function for the voltage has a non-zero range of $|U| \lesssim \sqrt{\eta} \delta U = \eta U_{rbw} \sqrt{2N_{lo,rbw}}$. One may also observe from figure 5 that within this range the sampling function attains one maximum and two minima, so that roughly the distance between these extrema may be approximated by $\Delta U \sim \sqrt{\eta} \delta U / 3$. On the other hand, the spectrally analysed voltages are measured with a certain maximum error $\Delta U_e$ that should be considerably smaller than $\Delta U$ in order to provide a voltage distribution with sufficient information content. That is, we require that the measurement error is

$$\Delta U_e \ll \frac{\sqrt{\eta} \delta U}{3} \approx 0.5 \eta U_{rbw} \sqrt{N_{lo,rbw}}.$$  \hspace{1cm} (46)
For a given measurement error, this condition can be typically satisfied either by increasing the amplifier trans-impedance $R$ or by increasing the intensity of LO2, i.e. $N_{\text{lo,rbw}}$. Strictly speaking, the measurement error generates a convolution of the voltage distribution (38). However, as we require condition (46) for a proper sampling, it is guaranteed that $\Delta U_e \ll \delta U$, so that the modified spread of the operator function in equation (38) can be estimated as being unaffected.

Typical values for condition (46) are obtained for an LO2 power 10 mW at 800 nm, trans-impedance $R \sim 100 \, \text{k}\Omega$ and $\delta \omega_{\text{rbw}}/2\pi \sim 10 \, \text{MHz}$ as $\delta U_{\text{rbw}} \sim 1.6 \, \text{nV}$ and $N_{\text{lo,rbw}} \sim 4 \times 10^{12}$ so that the measurement error should be $\Delta U_e \ll 1 \, \text{mV}$, which is feasible in experiment.

A further source of error is in the adjustment of the point in phase space $\alpha = r \exp(i \phi)$ where the quasi-probability is sampled. Typically, the intensity and phase of LO1 are controlled independently. Suppose, therefore, that we may fix amplitude and phase with uncertainties $\Delta r_e$ and $\Delta \phi_e$. Whereas the radial error may be supposed to be relative to the actual radius, $\Delta r_e/r = \varepsilon_r$, we assume that the phase error $\Delta \phi_e$ is constant. The chosen phase-space point is therefore precise within an area $\Delta S_e = r^2 \varepsilon_r \Delta \phi_e$. As a reference we may use the Gaussian phase-space distribution of a coherent state $|\alpha_0\rangle$, which reads as

$$W(\alpha; s) = \frac{2}{\pi(1-s)} \exp \left( -\frac{2|\alpha - \alpha_0|^2}{1-s} \right).$$

(47)

The size of the uncertainty area of equation (47) is $\delta S_{\text{coh}} = (1-s)/2$, so that for resolving the details of this Gaussian, the precision of the adjustment of the phase-space point by LO1 should be

$$\Delta S_e \ll \delta S_{\text{coh}}.$$

(48)

The maximum value $s_{\text{max}}$, cf equation (43), corresponds to the smallest uncertainty area of the coherent state, $\delta S_{\text{coh}} \geq (1 - s_{\text{max}})/2$, which together with equation (42) leads to

$$r^2 \varepsilon_r \Delta \phi_e \ll 1.$$

(49)

This condition becomes more and more stringent for sampling at increasing distances from the origin of phase space. For the phase-space distribution (47) we may define the maximum radius $r_{\text{max}} \sim \sqrt{1-s}$ of the area that is required to be sampled$^3$. Assuming that one would try to come close to the maximum value $s_{\text{max}}$, this results in the simple relation

$$\varepsilon_r \Delta \phi_e \ll \eta \quad \text{(coherent state).}$$

(50)

This requirement on the precision of the adjustment of amplitude and phase of LO1 is easily satisfied in experiment.

4. Summary and conclusions

In summary, we have shown that a setup consisting of an unbalanced optical homodyne stage followed by a balanced optical heterodyne stage can be employed to locally sample the phase space of the incident signal field. The local sampling means that for each choice of amplitude and phase of the first LO a specific point in the phase space of the signal field is addressed. Integrating the measured quadrature histograms with a unique sampling function leads then to

$^3$ If the phase-space distribution is not centred at the origin of phase space it may be convenient to superimpose onto LO1 a constant coherent, phase-locked field that may recentre the distribution around the origin.
the value of the quasi-probability distribution of the quantum state of the signal field at this particular point in phase space. The heterodyne stage has been shown to provide for an efficient suppression of excess noise by allowing for a spectral analysis of the measured beat signal. Furthermore, as the second LO is an independent laser source the necessary phase integration is naturally obtained due to the lack of correlation of signal and LO phase.

Whereas the heterodyne technique, on the one hand, reduces the overall quantum efficiency of the detection by 50%, on the other hand, it provides for a very efficient suppression of excess noise in the measured data. Furthermore, as the quasi-probability distribution is obtained locally by scanning over the area of interest in phase space, the remaining noise in the data is not amplified as it possibly would be in an inverse Radon transform. Thus, we may expect rather noise-free phase-space distributions, which may then be used in maximum-likelihood methods to estimate the corresponding quasi-probability distribution with a larger value of the operator-ordering parameter.

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