Baryonic sphere: a spherical domain wall carrying baryon number

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Abstract

We construct a spherical domain wall which has baryon charge distributed on a sphere of finite radius in a Skyrme model with a sixth order derivative term and a modified mass term. Its distribution of energy density likewise takes the form of a sphere. In order to localize the domain wall at a finite radius we need a negative coefficient in front of the Skyrme term and a positive coefficient of the sixth order derivative term to stabilize the soliton. Increasing the pion mass pronounces the shell-like structure of the configuration.
1 Introduction

Domain walls appear in many different theories if they possess degenerate and discrete vacua. They appear in field theory \cite{1} including supersymmetric field theory \cite{2, 3}, quark matter \cite{4}, cosmology \cite{5}, and various condensed matter systems \cite{6}.

By the scaling argument known as Derrick’s theorem \cite{7}, a localized, finite-energy scalar configuration with only a standard kinetic term and a potential can only exist in $1+1$ dimensions. This can be side-stepped by making the scalar field dependent on only 1 spatial coordinate. If we now contemplate compactifying this configuration to a 3-sphere (in $3+1$ dimensions), the above scaling argument tells us that it will shrink to a point. This can be avoided by adding higher derivative terms, like for instance the Skyrme term. The configuration just described is simply a Skyrmion \cite{8}, but with a modified mass term – first introduced in the baby Skyrme model in $d=2+1$ dimensions \cite{9} and slightly later in the Skyrme model \cite{10} (see also \cite{11}) – namely a mass term for the pion fields with two degenerate discrete vacua.

Let us look at the configuration from the point of view of it being a domain wall (the higher derivative terms are simply there to stabilize its size), and describe it in terms of a four-vector $\mathbf{n}$ and potential $\frac{1}{2} m^2 (1-n_n^2)$. The model admits two discrete degenerate vacua $n_4 = -1$ and $n_4 = +1$ with unbroken $SO(3)$-symmetry, and a domain wall interpolating between them. This domain wall possesses $S^2$ moduli because it breaks the vacuum symmetry down to $SO(2)$ \cite{12, 13}. Our configuration is simply a configuration with (say) $n_4 = -1$ at the origin and $n_4 = +1$ at spatial infinity. If in turn we make the $S^2$ moduli wind along the world volume of the domain wall, which is also $S^2$, then remembering the radial “winding” along the domain wall, it carries a topological charge $\pi_3(S^3)$, namely a Skyrmion or baryon charge.

In fact, what we have just described is simply the traditional Skyrmion with the addition of a modified potential for the pions. Now topologically speaking everything checks out, but intuitively or physically, the so-called vacuum inside the soliton is point-like and furthermore the energy density does not vanish at said point. In this note we modify the model by including a higher derivative term than the Skyrme term (i.e. a sixth order derivative term) and show by choosing a negative sign for the Skyrmie term, keeping the coefficient of the sixth order term positive, that the domain wall with winding moduli can be physically pushed out from the origin and thus really resembling a domain wall. Interestingly, the latter description is a model proposed by Jackson et. al. \cite{15} while their motivation was to make the interaction of what is interpreted as the $\omega$-meson and using the Skyrme term to make scalar exchange attractive (whereas in the original Skyrme model the central potential of nucleon-nucleon interaction is all repulsive).

A hand-waving explanation of how our domain wall works is as follows. We choose not to touch the sign of the standard kinetic term and by Derrick’s theorem, the highest derivative term needs to have a positive coefficient. Let us further consider the situation in which the pion mass is very large compared to other scales in the system. If all the derivative terms have positive coefficients, the cheapest way energy-wise, to interpolate the two vacua is for the chiral angle function to make a steep descent just at the origin. This makes the standard solutions have the energy peak at the center. Considering two, four and six derivative terms with a negative sign for only the fourth order term, the cancellation between the terms allows for the transition between the vacua to be moved to a higher radius.

The spherical domain wall in our model is a $3+1$ dimensional generalization of a $2+1$ dimensional model; an $O(3)$ nonlinear sigma model admitting two discrete degenerate vacua and a domain wall with a $U(1)$ modulus interpolating between these vacua \cite{16, 17}. If one makes a closed domain wall with the $S^1$ modulus twisted along the $S^1$ world-volume, it is a lump \cite{17} or baby Skyrmion \cite{18} with a topological charge \cite{19}.

\footnote{The Skyrme model admits also exact domain wall solutions of non-topological nature, which we do not consider here.}
charge of $\pi_2(S^2)$. In the next section we will start by reviewing the Skyrme model with a modified mass term allowing for two discrete and degenerate vacua.

## 2 The Skyrme model with a modified mass term

Let us consider the Skyrme model \[8\]

$$
\mathcal{L} = \frac{f_\pi^2}{16} \text{Tr} \left( \partial_\mu U^\dagger \partial^\mu U \right) + \frac{1}{32e^2} \text{Tr} \left( \left[ U^\dagger \partial_\mu U, U^\dagger \partial_\nu U \right]^2 \right) - V(U),
$$

where $f_\pi$ is the pion decay constant, $e$ is a coupling constant, $U$ is an $SU(2)$-valued matrix field and $\mu = 0, 1, 2, 3$ runs over 4-dimensional spacetime indices. Instead of the usual mass term, $\propto \text{Tr} [2U - U - (U^\dagger)]$, with only one vacuum, we consider a modified mass term which allows for domain walls, as two vacua are present \[10\]

$$
V(U) = \frac{m^2 e^2 f_\pi^4}{256} \text{Tr} \left[ (21_2 - U - U^\dagger)(21_2 + U + U^\dagger) \right],
$$

where $m$ is the (here dimensionless) pion mass and $1_2$ is the two-by-two unit matrix. Introducing a field $n$ such that

$$
U = in_\alpha \sigma^\alpha + n_4 1_2 \equiv n \cdot t,
$$

where $\alpha = 1, 2, 3$ is summed over, $\sigma^\alpha$ are the Pauli matrices and $U^\dagger U = 1_2$ is equivalent to $n \cdot n = 1$, we obtain the $O(4)$ sigma model with the Skyrme term

$$
\mathcal{L} = \frac{1}{2} \partial_\mu n \cdot \partial^\mu n + \frac{1}{4} (\partial_\mu n \cdot \partial_\nu n) (\partial^\nu n \cdot \partial^\sigma n) - \frac{1}{4} (\partial_\mu n \cdot \partial^\nu n)^2 - V(n),
$$

$$
V(n) = \frac{1}{2} m^2 (1 - n_4^2),
$$

where we have rescaled the coordinates $x^\mu \rightarrow \frac{2}{\sqrt{f_\pi}} x^\mu$ and the energy is given in units of $f_\pi/(2e)$. The Skyrmion number is given by the 3rd homotopy group of the 3-sphere and reads

$$
B = -\frac{1}{24\pi^2} \int d^3 x \epsilon_{ijk} \text{Tr} \left( U^\dagger \partial_i U U^\dagger \partial_j U U^\dagger \partial_k U \right) = -\frac{1}{12\pi^2} \int d^3 x \epsilon_{ijk} \epsilon^{abcd} \partial_i n^a \partial_j n^b \partial_k n^c n^d.
$$

In order to find a spherical solution we proceed by reviewing the standard Hedgehog Ansatz in the next section.

## 3 The Skyrmion with a modified mass term

The most naive attempt of constructing the spherical domain wall is to simply use the standard Hedgehog Ansatz for the Skyrmion (in the standard way of constructing the Skyrmion solution) and study the solutions and energy densities as functions of the mass parameter $m$, which is the parameter controlling the width of the domain wall. The Hedgehog Ansatz reads

$$
U = \exp \{ i f(r) \hat{x}^i \sigma^i \} = 1_2 \cos f(r) + i \hat{x}^i \sigma^i \sin f(r),
$$

which in terms of $n$ is

$$
n^i = \hat{x}^i \sin f(r), \quad n^4 = \cos f(r),
$$

\[2\]
for which the Lagrangian density reads

\[- \mathcal{L} = \frac{1}{2} f_r^2 + \frac{1}{r^2} \sin^2 f (1 + f_r^2) + \frac{1}{2r^4} \sin^4 f + \frac{1}{2} m^2 \sin^2 f ,\]  

(9)

where \( f_r \equiv \partial_r f \) and the energy (Skyrmion mass) is

\[ E = 2\pi \int dr \left\{ r^2 f_r^2 + 2 \sin^2 f (1 + f_r^2) + \frac{1}{r^2} \sin^4 f + r^2 m^2 \sin^2 f \right\}. \]  

(10)

The equation of motion reads

\[ (r^2 + 2 \sin^2 f) f_{rr} + 2 r f_r + \sin 2f \left( f_r^2 - 1 - \frac{1}{2} r^2 m^2 - \frac{\sin^2 f}{r^2} \right) = 0. \]  

(11)

If we apply the boundary conditions \( f(\infty) = 0 \) and \( f(0) = \pi \), the Skyrmion solution corresponds to a domain wall with vacuum \( n_4 = 1 \) at spatial infinity and vacuum \( n_4 = -1 \) on the inside. The volume of the region with vacuum \( n_4 = -1 \) is expected to be point-like. Let us calculate the Skyrmion number of this configuration

\[ B = -\frac{2}{\pi} \int dr \sin^2(f) f_r = -\frac{1}{2\pi} \int dr \partial_r (2f - \sin 2f) = \frac{f(0) - f(\infty)}{\pi} = 1, \]  

(12)

where the last equality depends on the boundary conditions and is true for the above given ones.

In fig. 1 is shown the spherical domain wall (i.e. the Skyrmion with the modified mass term) for \( m = 0, 1, 2, \ldots, 10 \). The energy density grows with \( m \) near the center of the Skyrmion and the size of the domain wall shrinks as \( m \) increases. The volume of the vacuum of the core of the Skyrmion does not grow and remains point-like for any value of the mass \( m \). The “vacuum” on the inside of the domain wall is not a vacuum in the traditional sense of the word. That is, the energy density does not exhibit a minimum at said point.

In fig. 11 is shown a numerical estimate of the width of the domain wall. We can estimate the scaling in the limit of large \( m \gg 1 \) by rescaling \( r \to \mu r \) in eq. (11), neglecting the terms proportional \( 1/\mu \) (for large \( m \), the terms proportional to \( 1/\mu \) are negligible compared to the mass term which is proportional to \( 1/\mu^3 \)) and then varying with respect to \( \mu \):

\[ \mu = 3^{\frac{3}{4}} \sqrt{m} \left( \frac{\int dr \sin^2 f}{\int dr (2\sin^2(f) f_r^2 + \frac{1}{r^2} \sin^4 f)} \right)^{\frac{1}{4}}, \]  

(13)

giving the naive scaling estimate of the domain wall width \( w \sim 3^{-\frac{3}{4}}/\sqrt{m} \).

In the next section we will attempt to “open up” the inner vacuum such that it is evident that we really are studying a compactified domain wall.

4 Higher-derivatives Skyrmion: a spherical domain wall

In this section we consider a way of “opening up” the inner vacuum, such that it is evident from the energy density that the deformed Skyrmion is really a spherical domain wall, namely we will add even higher derivative terms than the Skyrme term to the Lagrangian, keeping spherical symmetry intact. The simplest possible extension is to add just a sixth-order derivative term to the action. For simplicity, we will only consider the type of higher-derivative terms that gives rise to a second-order equation of motion, along the lines of Marleau [21]. The action can be written as

\[ \mathcal{L} = c_2 \mathcal{L}_2 + c_4 \mathcal{L}_4 + c_6 \mathcal{L}_6 - V, \]  

(14)
Figure 1: (a) Profile function, (b) energy density and (c) baryon number density of the spherical domain wall for \( m = 0, 1, 2, \ldots, 10 \). (d) shows a full-width-half-maximum estimate of the width of the domain wall. The fit shows that the simple scaling estimate, \( w \sim 3^{-1/4}/\sqrt{m} \) is a quite good approximation at large \( m \).
necessary but not sufficient condition for having a finite-sized soliton solution is that the coefficient of the highest derivative term is positive; in this case a soliton (wall) to have a finite size (note that \( \mu \) shrinks the solution.

We thus have

\[
E = c_2 e_2 + \mu c_4 e_4 + \mu^3 c_6 e_6 + \frac{1}{\mu^3} m^2 v ,
\]

and perform a scale transformation, \( x^\mu \to x'^\mu = \mu x^\mu \):

\[
E(\mu) = \frac{c_2}{\mu} e_2 + \mu c_4 e_4 + \mu^3 c_6 e_6 + \frac{1}{\mu^3} m^2 v ,
\]

then according to Derrick’s theorem, \( E'(\mu) = 0 \) must have a real and finite solution for \( \mu \) in order for the soliton (wall) to have a finite size (note that \( \mu \to \infty \) corresponds to the soliton shrinking to a point). A necessary but not sufficient condition for having a finite-sized soliton solution is that the coefficient of the highest derivative term is positive; in this case \( c_6 > 0 \). \( c_4 \) can have either sign. From eq. (20) we can see that \( e_2 \) and \( v \) shrink the soliton and \( e_{4,6} \) make the soliton grow if \( c_4 > 0 \). If \( c_4 < 0 \) then \( e_4 \) also tends to shrink the solution.

Interestingly enough, if we choose the negative sign for \( c_4 \), which means that only the sixth derivative term prevents the soliton from shrinking to a point, the model is basically that proposed by [15] in which the authors choose the sign on phenomenological grounds. Their motivation lies in simulating attractive scalar exchange whereas our motivation is to study the full parameter space of the model and find a region where the energy density is concentrated in a shell-like structure.

We can fix two of the coefficients by fixing the units of the length scale and the energy scale. Let us explicitly scale the energy by sending \( E \to \lambda E \) and fixing \( \mu = \sqrt{c_2/c_4} \) and \( \lambda = \sqrt{c_2/c_4} \), which leaves us with 2 free parameters: \( c_6' = c_2 c_6/c_4^2 \) and \( m' = m\sqrt{|c_4|}/c_2 \) as well as the sign of \( c_4 \) (dropping the primes)

\[
E = e_2 + \epsilon e_4 + c_6 e_6 + m^2 v , \quad \epsilon \equiv \text{sign}(c_4) .
\]

We thus have

\[
E = \frac{1}{2} f_r^2 + \frac{1}{r^2} \sin^2 f + \frac{1}{r^2} \sin^2(f) f_r^2 + \frac{1}{r^2} \sin^4 f + \frac{1}{2r^4} \sin^4(f) f_r^2 + \frac{1}{2} m^2 \sin^2 f ,
\]

giving rise to the equation of motion

\[
f_{rr} + \frac{2}{r} f_r + \frac{2}{r^2} \sin^2(f) f_{rr} - \frac{1}{r^2} \sin 2 f \left[ 1 - \epsilon f_r^2 \right] - \frac{1}{2} m^2 \sin 2 f + \frac{2c_6}{r^3} \sin^4 f \left[ f_{rr} - \frac{2}{r} f_r \right]
\]

\[
+ \frac{1}{r^4} \sin 2 f \sin^2 \left[ -\epsilon f_r^2 + 2c_6 f_r^2 \right] = 0 ,
\]

which we will solve with the boundary conditions \( f(0) = \pi \) and \( f(\infty) = 0 \).

In order to study the “vacuum” near \( r = 0 \), let us expand the chiral angle function \( f \) as

\[
f = \pi + f_1 r + \frac{1}{3!} f_3 r^3 + \frac{1}{5!} f_5 r^5 + O(r^7) ,
\]
where the would-be $f_{2,4,6}$ vanish due to the equation of motion. Plugging this expansion into the energy density yields

$$\mathcal{E} = \frac{1}{2} f_1^2 \left[ 3 + 3\epsilon f_1^2 + 2c_6 f_1^4 \right] + \frac{1}{6} f_1 \left[ 3m^2 f_1 - 2f_1^3 - 4\epsilon f_1^5 - 4c_6 f_1^7 + 5f_3 + 10\epsilon f_1^2 f_3 + 10c_6 f_1^4 f_3 \right] r^2$$

$$+ \frac{1}{360} \left[ -60m^2 f_1^4 + 16f_1^6 + 52\epsilon f_1^8 + 72c_6 f_1^{10} + 60m^2 f_1 f_3 - 80f_1^2 f_3 - 320\epsilon f_1^2 f_3 - 480c_6 f_1^4 f_3 
+ 250\epsilon f_1^2 f_3^2 + 55f_1^3 f_3^2 + 390c_6 f_1^4 f_3^2 + 21f_1 f_3 + 42\epsilon f_1^2 f_5 + 42c_6 f_1^4 f_5 \right] r^4 + \mathcal{O}(r^6),$$

which means that a sufficient condition for the energy density to vanish around $r \to 0$ is that the first derivative vanishes at $r = 0$ (see the equations of motion below). Using the equation of motion at order $\mathcal{O}(r)$ and $\mathcal{O}(r^3)$, we can determine the third and fifth derivative in terms of $f_1$

$$f_3 = \frac{4c_6 f_1^7 - 2\epsilon f_1^5 - 4f_1^3 + 3f_1 m^2}{5 \left( 2c_6 f_1^4 + 2\epsilon f_1^2 + 1 \right)},$$

$$f_5 = \frac{-20m^2 f_1^3 + 8f_1^5 + 16\epsilon f_1^7 - 24c_6 f_1^9 + 5f_3 m^2 - 20f_1^2 f_3 + 30\epsilon f_1^4 f_3 + 140c_6 f_1^6 f_3 - 40\epsilon f_1^2 f_5 - 80c_6 f_1^4 f_5}{7 \left( 1 + 2\epsilon f_1^2 + 2c_6 f_1^4 \right)},$$

which we can insert into the energy density (25). Since we are interested in the expression for small $f_1 \ll 1$, we give the expression to just fourth order in $f_1$:

$$\mathcal{E} = \frac{3}{2} \left( f_1^2 + \epsilon f_1^4 \right) + \left( m^2 f_1^2 - f_1^4 \right) r^2 + \left[ \frac{9}{50} m^4 f_1^2 - \left( \frac{22}{25} m^2 + \frac{17}{50} \epsilon m^4 \right) f_1^4 \right] r^4 + \mathcal{O}(r^6, f_1^6).$$

Unfortunately, we cannot find an analytic expression for $f_1$ as it is not a perturbative object, but encodes information about the soliton as a whole. We can, however, solve the equation of motion numerically and study $f_1$ as function of $m$ and $c_6$ for $\epsilon = \pm$ which we show in fig. 2.

Finally, we can show numerical solutions in the interesting region of parameter space, i.e. for small $c_6$ and relatively large pion mass $m$. In fig. 2 is shown the profile function $f$, the energy density and the baryon number charge density for two values of $c_6 = 1, 0.6$ and various pion masses $m = 0, 1, \ldots, 10$.  

Figure 2: The first derivative of the chiral angle function $f_1$ at $r = 0$ as function of the pion mass $m$ and the sixth order derivative term coefficient $c_6$ for (a) $\epsilon = +$ and (b) $\epsilon = -$. In (b) four black solid lines are shown, representing the iso-curves $f_1 = -0.2, -0.1, -0.05$ and $-0.03$ from above.
Figure 3: Sixth order derivative Skyrmions with negative Skyrme term for $c_6 = 1$ (left) and $c_6 = 0.6$ (right) and various pion masses $m = 0, 1, . . . 10$.
5 Summary and Discussion

We have constructed a spherical domain wall with baryon charge distributed on the surface of a sphere with a finite radius in a Skyrme model with the addition of a modified mass term and a sixth-order derivative term. The width of the domain wall is inversely proportional to (the square root of) the pion mass, in units of the coefficient of the kinetic term. By means of a series expansion near the origin, we have related the energy density at small radii to the first derivative of the chiral angle function at the origin, $f_1$. When $f_1$ is parametrically small, the energy as well as the baryon charge density remain parametrically small near the origin of the soliton. This separation of the domain wall from the origin point reveals that the lump of energy of a Skyrmion in fact is a domain wall in disguise. In order to practically achieve an almost vanishing $f_1$ (the first derivative of $f$) it is necessary to flip the sign of the Skyrme term with respect to the conventional choice. This in turn necessitates a positive sixth order term.

It is possible to obtain a similar behavior by keeping a positive coefficient, $c_4$, of the Skyrme term by having a negative $c_6$ which in turn necessitates a positive (and sufficiently large) $c_8$, for an eighth-order derivative term, e.g. \[ \mathcal{L}_8 = \frac{1}{r^6} \sin^6(f) f_r^2 - \frac{1}{4r^8} \sin^8 f. \] (29)

In this case the balance of forces is three versus two; namely the kinetic term, the sixth order term and the potential tend to shrink the soliton whereas the Skyrme term and the eighth order term tend to make the soliton grow.

In the Skyrme model, there is no primary reason for using the conventional mass term (i.e. $\frac{1}{2}m^2(1-n_4)$) instead of the modified mass term for the pion mass. One can also consider higher derivative terms like a sixth order term – as in our case – for a model of baryons. Therefore, there is a possibility that baryons are spherical. What impact it has in nuclear physics remains a future problem.

Interestingly, a BPS proposal which has been put forward in \[22\], relies on the sixth-order derivative term (only) for saturation of the BPS bound and the near BPS region (which has only small contributions from the second and fourth order terms) is phenomenologically compelling because it gives a low binding energy and an almost linear relation between the baryon number and mass \[23\].

It is an open problem to construct higher winding Skyrmions. In particular, it is to be studied whether the energy distributions of the minimum-energy configurations can be spherical or need to be in separated lumps. For this purpose, the rational map Ansatz \[24\] may be useful as for usual Skyrmions \[2\]. Likewise, the interaction between spherical domain walls as Skyrmions is an important subject. Another interesting topic is a coupling to gravity, resulting in a gravitational Skyrmion or a black hole.

It is interesting to study low-energy modes of a spherical domain wall. For a flat domain wall, the effective theory is a nonlinear sigma model with the target space $\mathbb{R} \times S^2$, describing the fluctuations of the domain wall surface and the $S^2$ Nambu-Goldstone modes \[13\]. The effective field theory of a spherical domain wall may contain a radial fluctuation field which should be light but not massless, and the $S^2$ Nambu-Goldstone modes which are twisted. Low-energy effective field theories on curved soliton world-volumes have not been studied in depth thus far. This model provides a primary example of such.

When we deform the model with an additional mass term $V_2 = -m_2^2n_2$, where $m_3 \ll m$, there appears a domain line inside a domain wall \[14\]. In our case, a circular domain line will appear in a spherical domain wall. If we further deform the model by adding $V_3 = -m_4^2n_4$ with $m_2 \ll m_3 \ll m$, sine-Gordon kinks appear on the domain line \[26\]. In our setting, sine-Gordon kinks will appear on a circular domain

\[2\] For analytic properties of multi-Skyrmions in the standard Skyrme model, see \[25\].

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line in a spherical domain wall. We may construct a domain wall junction on a sphere \( \Sigma \) by properly choosing the potential.

In the introduction, we mentioned that our spherical domain wall is a \( d = 3 + 1 \) dimensional generalization of a circular domain wall being a baby Skyrmion with a topological charge of \( \pi_2(S^2) \) in \( d = 2 + 1 \) dimensions \[19\]. Higher dimensional versions are also possible in an \( O(N+1) \) sigma model in \( d = N + 1 \) dimensions admitting two discrete degenerate vacua and a domain wall with \( S^{N-1} \) moduli interpolating between these vacua \[14\]. This model will admit an \( S^{N-1} \) domain wall with a topological charge \( \pi_N(S^{N}) \simeq \mathbb{Z} \) in \( d = N + 1 \) dimensions.

There is another higher-dimensional generalization. The \( O(3) \) sigma model mentioned above also admits domain walls in a torus shape \( T^2 = S^1 \times S^1 \) in \( d = 3 + 1 \) dimensions, along whose two cycles the \( U(1) \) modulus is twisted \[28\]. This toroidal domain wall carries Hopf charge characterized by \( \pi_3(S^2) \simeq \mathbb{Z} \). Therefore, our domain wall with an \( S^2 \times S^2 \) world-volume is a possibility in \( d = 5 + 1 \) dimensions and it may carry a topological charge of \( \pi_5(S^3) \simeq \mathbb{Z}_2 \).

In this paper, we have studied a spherical domain wall as a twisted soliton, i.e., the \( S^2 \) moduli are twisted along a closed domain wall world-volume, in the nonlinear sigma model. On the other hand, Yang-Mills Higgs theories with certain matter contents have been proposed to admit an \( S^3 \) domain wall, \( S^2 \) vortex sheet, or \( S^1 \) monopole string as a Yang-Mills instanton-particle in \( d = 4 + 1 \) dimensions, if the \( S^3 \), \( S^2 \) or \( S^1 \) moduli are twisted along the world-volume \[29\]. Thus far, we know of several examples of twisted solitons regarded as Skyrmions or instantons. There should exist a general framework for studying which topological charges are carried by twisted solitons.

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