Rings and Balls

Lee Brekke\textsuperscript{a}

Shane J. Hughes\textsuperscript{b}

and

Tom D. Imbo\textsuperscript{a}

We examine the various linkings in space-time of “ball-like” and “ring-like” topological solitons in certain nonlinear sigma models in 2+1 and 3+1 dimensions. By going to theories where soliton overlaps are forbidden, these linkings become homotopically nontrivial and can be studied by investigating the topology of the corresponding configuration spaces. Our analysis reveals the existence of topological terms which give the linking number of the world-tubes of distinct species of ball solitons in 2+1 dimensions, or which in 3+1 dimensions count the number of times a ball or ring soliton threads through the center of a ring of a different species. We explicitly construct these terms for our models, and generalize them to cases where soliton overlaps are no longer strictly forbidden so the terms are no longer purely topological. One of the (3+1)-dimensional theories we consider also has topological solitons which consist of two rings (of distinct species) linked in space.

\textsuperscript{a}Department of Physics, University of Illinois at Chicago, Chicago, IL 60607-7059

\textsuperscript{b}Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138
1. Introduction

The existence of linking phenomena in 3 dimensions is familiar from everyday experience. The importance of linking (or “braiding”) for the quantum mechanics of point particles in 2+1 dimensions has also been well explored. For instance, the linking of the world lines of identical particles leads to the possibility of fractional statistics [1], while the linking of particles of distinct species leads to possible additional phases in the quantum theory which can give rise to interesting statistical behavior for composite particles [2]. In 3+1 dimensions linking of the world lines of point particles is no longer possible, but there are new types of linking if one adds fundamental string loops to the model [3]. In particular, loops can be linked with each other in space, and point particles or loops can thread through other string loops as a function of time. These latter time-dependent processes allow the introduction of additional phases in the quantum mechanics of these objects.

In this paper, we look for analogous linking behavior in certain nonlinear sigma models by studying the topology of the corresponding configuration spaces. The role of the point particles in the quantum mechanical case is played by topological solitons possessing a “ball-like” structure. The particular examples we explore use the $O(3)$-invariant sigma model (with target space $S^2$) in 2+1 dimensions [4][5] and the $O(4)$-invariant model (or “2-flavor Skyrme model” with target space $S^3$) in 3+1 dimensions [6]. Both of these models have ball-like topological solitons. In 3+1 dimensions the $O(3)$-invariant sigma model has topological solitons that are expected to have a “ring-like” structure [7], so they can play the role of the string loops in quantum mechanics. By taking cross products of these target spaces one can get theories with both ring and ball solitons, or with more than one species of ring and/or ball solitons.

One can now examine the linking of rings and balls in these field theories. Any such linking, however, is topologically trivial because the solitons in these models can pass through each other (since the target space is a cross product). This problem can be solved by changing the form of the target space from $A \times B$ to $A \vee B$ — the wedge (or one-point union) of $A$ and $B$, that is, $A$ and $B$ joined at a single point [8]. This may be realized by starting with an $A \times B$ model in which spatial infinity gets mapped to the point $(a_0, b_0)$ in the target space. Then, introduce a potential that gives an infinite amount of energy to any configuration not of the form $(a_0, b(\vec{x}))$ or $(a(\vec{x}), b_0)$. In such wedge models, solitons

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1 In all of the above models, in order to obtain nonsingular solutions to the equations of motion (whether of ring or ball type), stabilizing terms must be added to the lagrangians.
associated with the space $A$ cannot pass through those associated with $B$ so that a study of the topology of configuration space should detect any linking between them. (More generally, there can be no overlap between $A$ and $B$ energy densities.) Furthermore, there should exist topological terms which can be added to the action of these models and which supply the possible phases seen in the quantum mechanical case. We explicitly construct these terms in specific models, and also their (nontopological) generalizations to the corresponding cross product models.

The paper is organized as follows. In section two we review the $O(3)$-invariant sigma model in both 2+1 and 3+1 dimensions, using the $CP^1$ formulation. We recall that the (2+1)-dimensional solitons may be anyonic and that such quantizations can be implemented by using a Hopf term. We also discuss the ring-like structure of the solitons in 3+1 dimensions. In section three, we consider the (3+1)-dimensional sigma models with target spaces $S^2 \times S^3$ and $S^2 \lor S^3$, which contain both ball and ring solitons. We examine the time-dependent configurations where a ball soliton passes through the center of a ring and construct a term that counts the number of such linkings. In section four, we examine the $S^2 \lor S^2$ sigma model in both 2+1 and 3+1 dimensions. In 2+1 dimensions, we find noncontractible loops in configuration space that correspond to linking the “world-tubes” of the two distinct species of ball solitons that exist. In 3+1 dimensions we find a rich structure of ring solitons and examine the various homotopically distinct loops in configuration space. In both cases we construct topological terms that count the number of relevant linkings. We also extend these terms to the $S^2 \times S^2$ model.

2. The $S^2$ Model in 2+1 and 3+1 Dimensions

In the (2+1)-dimensional $O(3)$-invariant nonlinear sigma model, the scalar field $n^a$, $a = 1, 2, 3$, is constrained to lie on a sphere $S^2$ ($n^a n^a = 1$). Thus, at any fixed time the system is described by a map from $\mathbb{R}^2$ to $S^2$. To ensure that configurations have finite energy, all points at spatial infinity must be mapped to the same point on $S^2$. So the configuration space $Q$ of the system is equal to $Map_*(S^2, S^2)$ — the set of base point preserving maps from compactified space $S^2$ to the target space $S^2$. Since $\pi_0(Q) = \pi_2(S^2) = \mathbb{Z}$, this model has topological solitons and we may write $Q = \bigcup_{N=-\infty}^{\infty} Q_{(N)}$, where the $N$-soliton sector $Q_{(N)}$ consists of the maps of degree $N$ from $S^2$ to $S^2$. The degree $N$ of a configuration may be calculated from the conserved topological current

$$j^\mu = \frac{1}{8\pi} \epsilon^{\mu\nu\lambda} \epsilon^{abc} n^a \partial_\nu n^b \partial_\lambda n^c,$$  \hspace{1cm} (2.1)
through $N = \int d^2 x j^0$. A convenient description of these solitons is afforded by the so-called \( CP^1 \) formulation of the model where the fact that $S^2 = SU(2)/U(1)$ is used to describe the theory in terms of an $SU(2)$-valued field together with a $U(1)$ gauge symmetry which eliminates the extra degrees of freedom. More explicitly, $SU(2)$ may be parametrized by $\zeta = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2$ with $|z_1|^2 + |z_2|^2 = 1$ via

$$U = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}.$$  \hspace{1cm} (2.2)

The $S^2$ field $n^a(x)$ corresponding to $\zeta(x)$ is

$$n^a = \zeta^\dagger \sigma^a \zeta,$$  \hspace{1cm} (2.3)

where $\sigma^a$ are the Pauli matrices. Under the local $U(1)$ symmetry in the model, $\zeta$ is multiplied by a phase, $\zeta \rightarrow e^{i\chi(x)} \zeta$, while the $n^a$ remain fixed. The corresponding gauge field has no kinetic term in the Lagrangian – it is an auxiliary field which can be written as $A_\mu = i \zeta^\dagger \partial_\mu \zeta$. The topological current in this language is

$$j^\mu = \frac{1}{4\pi} \epsilon^{\mu\nu\lambda} F_{\nu\lambda},$$  \hspace{1cm} (2.4)

where $F_{\nu\lambda}$ is the $U(1)$ field strength. Thus,

$$N = \int d^2 x j^0 = \frac{1}{2\pi} \int_C \vec{A} \cdot \vec{dl},$$  \hspace{1cm} (2.5)

where $C$ is the contour at spatial infinity. Note that an $N$-soliton configuration looks like a $U(1)$ vortex with $N$ units of “magnetic flux”. This vortex interpretation of the solitons is also suggested by the long exact homotopy sequence for the Hopf fibration

$$U(1) \hookrightarrow SU(2) \downarrow \pi_2(S^2) = \pi_1(U(1)) = \mathbb{Z}.$$  \hspace{1cm} (2.6)

This shows that the topological stability of the solitons has its origins in nontrivial windings in the $U(1)$ sector of the theory.
Some physics also resides in the fact that the fundamental group of the zero soliton sector $Q_{(0)}$ is nontrivial:

$$\pi_1(Q_{(0)}) = \pi_3(S^2) = \pi_3(SU(2)) = \mathbb{Z}. \quad (2.7)$$

This group is generated by (the homotopy class of) a time-dependent configuration where, out of the vacuum, a soliton of degree one and its corresponding anti-soliton are created, the soliton undergoes a $2\pi$ rotation and then the soliton anti-soliton pair mutually annihilate.\footnote{One can also show that $\pi_1(Q_{(N)}) = \mathbb{Z}$ for any $N$. The following interpretations of the generator also hold for any $N$. Analogous statements apply to the various configuration spaces considered in this paper.}

(More generally, if the above $2\pi$ rotation is assigned the integer 1 in $\pi_1(Q_{(0)})$, then a rotation by $2\pi n$ of an $N$-soliton corresponds to $nN^2 \in \pi_1(Q_{(0)})$.) In the quantum theory, homotopically nontrivial processes such as this may be weighted by distinct phases, where the only requirement on these phases is that they form a representation of $\pi_1(Q_{(0)})$. The one-dimensional unitary representations of $\mathbb{Z}$ are labelled by an arbitrary angle $\theta$. They are given by $m :\rightarrow e^{im\theta}$ for any $m \in \mathbb{Z}$. Quantizing using such a representation, we see that the above $2\pi$ rotation picks up the phase $e^{i\theta}$ showing that the solitons may possess fractional spin. The above generator of $\pi_1(Q_{(0)})$ may also be represented by the time-dependent configuration where two soliton anti-soliton pairs of unit degree are created out of the vacuum, the two solitons interchange position (counterclockwise), and the “new” soliton anti-soliton pairs annihilate. Hence this process also picks up the phase $e^{i\theta}$ in the above quantum theory and we see that solitons exhibiting fractional spin also exhibit the corresponding fractional statistics in keeping with the spin-statistics relation \footnote{Throughout this paper, all rotations will be taken as counterclockwise.}.

To realize a quantization with “statistical angle” $\theta$, the Hopf term $H$ may be added to the action $S$ \footnote{One can also show that $\pi_1(Q_{(N)}) = \mathbb{Z}$ for any $N$. The following interpretations of the generator also hold for any $N$. Analogous statements apply to the various configuration spaces considered in this paper.}.

$$S \rightarrow S + \theta H. \quad (2.8)$$

This term can be written as

$$H = \frac{1}{2\pi} \int d^3 x A_\mu j^\mu = \frac{1}{8\pi^2} \int d^3 x \epsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda}. \quad (2.9)$$

Evaluated on a time-dependent configuration in which an $N$-soliton is rotated by $2\pi n$, $H$ is equal to $nN^2$.\footnote{Throughout this paper, all rotations will be taken as counterclockwise.}
A construction similar in spirit to the one above may be used to show that the $O(3)$-invariant nonlinear sigma model in 3+1 dimensions also has solitons, but of a different nature than their (2+1)-dimensional counterparts. Now the configuration space of the theory is $Q = \text{Map}_*(S^3, S^2)$ and the soliton’s existence is revealed by

$$\pi_0(Q) = \pi_3(S^2) = \mathbb{Z}. \quad (2.10)$$

Again, the configuration space may be decomposed into sectors of varying soliton number, $Q = \bigcup_{N=-\infty}^{\infty} Q(N)$. However, the structure of the solitons is expected to be that of a ring; such a homotopically nontrivial configuration being identical to that of an $O(3)$ soliton creation, rotation and annihilation in 2+1 dimensions, only now the role of time is played by the “extra” spatial dimension. The (3+1)-dimensional time plays no role. Thus the $N = 1$ soliton in the (3+1)-dimensional theory can be viewed as a ring of energy with one unit of $U(1)$ flux running through it and also having a nontrivial $2\pi$-twist providing topological stability [7]. A ring with $n_f$ units of flux and $n_t$ $2\pi$-twists has soliton number $n_t n_f^2$. In other words, the Hopf term $H$ in 2+1 dimensions now becomes the topological charge in the (3+1)-dimensional model. More precisely, $H = \int d^3x h^0$ where

$$h^\mu = \frac{1}{8\pi^2} \epsilon^{\mu\rho\sigma} A_\rho F_{\rho\sigma} \quad (2.11)$$

is the full conserved topological current. Note that under the gauge transformation $A_\mu \rightarrow A_\mu - \partial_\mu \chi$, the current $h^\mu$ changes by the total derivative $-\frac{1}{8\pi^2} \partial_\nu \epsilon^{\mu\nu\rho\sigma} \chi F_{\rho\sigma}$. However, the topological charge $H$ is gauge invariant. To check for the possibility of nontrivial spin, we calculate

$$\pi_1(Q_{(0)}) = \pi_4(S^2) = \mathbb{Z}_2. \quad (2.12)$$

The generator of this $\mathbb{Z}_2$ corresponds to creating a 1-soliton anti-soliton pair out of the vacuum, rotating the soliton by $2\pi$ and then annihilating the pair. (This is done in (3+1)-dimensional time.) Thus we only have the option of quantizing the ring solitons as integer or half-integer spin objects.

3. The $S^2 \times S^3$ and $S^2 \vee S^3$ Models

We now consider a (3+1)-dimensional sigma model with target space $S^2 \times S^3$. This theory has configuration space $Q = \text{Map}_*(S^3, S^2 \times S^3)$ and

$$\pi_0(Q) = \pi_3(S^2) \times \pi_3(S^3) = \mathbb{Z} \times \mathbb{Z}. \quad (3.1)$$
There are two types of solitons; ring solitons from the $S^2$ part of the theory and ordinary ball solitons from the $S^3$ part (as in the 2-flavor Skyrme model) \[^4\]. We decompose $Q$ into $\bigcup_{N=-\infty}^{\infty} \bigcup_{M=-\infty}^{\infty} Q(N,M)$, where $N$ counts the $S^2$ soliton number and $M$ that of $S^3$. Thus, there are two conserved topological currents in this model; the “Hopf current” $h^\mu$ discussed above for the $S^2$ fields, and a new current $J^\mu$ for the $S^3$ fields. Since $S^3$ is the group manifold of $SU(2)$, we may view the $S^3$ fields as $SU(2)$ matrices $U(x)$. If we define

$$R_\mu = U \partial_\mu U^\dagger,$$

then we have

$$J^\mu = \frac{1}{24\pi^2} \epsilon^{\mu
u\rho\sigma} \text{tr}(R_\nu R_\rho R_\sigma).$$

We also have

$$\pi_1(Q_{(0,0)}) = \pi_4(S^2) \times \pi_4(S^3) = \mathbb{Z}_2 \times \mathbb{Z}_2,$$

and the nontrivial generator of $\pi_4(S^3)$ is represented by the creation, rotation and annihilation of an $S^3$ 1-soliton anti-soliton pair. Thus we may independently quantize the ring 1-soliton and the ball 1-soliton as integer or half-integer spin objects.\[^6\]

In this theory, there is the interesting possibility of sliding a ball soliton through the center of a ring. The linking number $\gamma$ of the world-line of a point and the world-sheet of an infinitely thin closed string is given by \[^5\]

$$\gamma = \frac{1}{4\pi^2} \int d^2 \Sigma_{\mu\nu}(X) \int dY_\lambda \epsilon^{\mu\nu\lambda\sigma} \frac{(X - Y)_\sigma}{|X - Y|^4}.$$  

Here $d^2 \Sigma_{\mu\nu} = d\sigma d\tau \epsilon^{\alpha\beta} \partial_\alpha X_\mu \partial_\beta X_\nu$ is the infinitesimal area element of the string’s world-sheet, and $dY_\lambda$ the infinitesimal element of the point’s world-line. It is possible to add a term to the action which, in the limit that the ball soliton becomes a point and the ring becomes a one-dimensional string, reduces to this term. To see this, take $J^\mu_{\mu}$ to be the topological current of the $S^3$ soliton and $A^\mu$ to be the auxiliary gauge field of the $S^2$ ring. Then, generalize the above area and line elements as

$$dY_\lambda \to J_\lambda(y) d^4 y$$  

\[^4\] Surprisingly, it has been found that the baryon number 2 solutions in the 2-flavor Skyrme model (or equivalently, the $(0,2)$ solution in the above model with an appropriate stabilizing term) has a ring-like structure \[^9\]. However, unlike the ring solitons we consider, one cannot treat these objects as having a $U(1)$ flux running through them since $\pi_2(S^3)$ is trivial. In what follows, we will treat all $S^3$ solitons as ball-like.
and
\[ d^2 \Sigma_{\mu\nu}(x) e^{\mu\nu\lambda\sigma} \rightarrow \frac{1}{\pi} F^{\lambda\sigma}(x) d^4 x, \] (3.7)
where \( F^{\lambda\sigma} \) is the field strength associated with \( A^\mu \). Modifying the action by the resulting term
\[ \Delta S = \frac{\theta}{4\pi^3} \int d^4 x \int d^4 y F^{\lambda\sigma} J_\lambda \frac{(x - y)\sigma}{|x - y|^4}, \] (3.8)
we pick up the phase \( \exp(in_b n_f \gamma \theta) \) every time a ball soliton (of topological charge \( n_b \)) passes \( \gamma \) times through the center of a ring (with \( n_f \) units of flux), provided that there is no overlap of the energy density of the ball with that of the ring during the process. Note that this phase is independent of the number of \( 2\pi \)-twists \( n_t \) in the ring. Further, upon integrating by parts and noting that
\[ \partial^\sigma \left( \frac{(x - y)\sigma}{|x - y|^4} \right) = -\frac{1}{2} \partial^\sigma \partial_\sigma \left( \frac{1}{|x - y|^2} \right) = -2\pi^2 \delta^{(4)}(x - y), \] (3.9)
we see that the extra term in the action is local,
\[ \Delta S = \frac{\theta}{2\pi} \int d^4 x A^\mu(x) J_\mu(x). \] (3.10)
In this form, \( \Delta S \) can be thought of as giving the \((0, n_b)\) ball soliton a \( U(1) \) electric charge \( \theta n_b/2\pi \). The factor \( \exp(in_b n_f \theta) \) acquired upon threading a ball through the center of a ring can now be seen as the usual Aharonov-Bohm phase \( \exp(iq\Phi) \) which results from bringing a charge \( q = \theta n_b/2\pi \) around a flux \( \Phi = 2\pi n_f \). As noted above, \( \Delta S \) has its topological interpretation as \( \theta n_b n_f \gamma \) only when there is no overlap between the ball and the ring. However, the \( S^2 \) and \( S^3 \) energy densities are allowed to overlap and our term alters the equations of motion. This reveals why there is no suggestion of the existence of this term from the topology of configuration space. However, if we consider a model in which these overlaps are forbidden, such a term should be suggested by topological arguments. As indicated in the introduction, this can be accomplished by choosing the target space to be \( S^2 \vee S^3 \). We now turn to a discussion of this model.

The configuration space of the \( S^2 \vee S^3 \) model is \( Q = Map_*(S^3, S^2 \vee S^3) \). Solitons analogous to the ring and ball solitons of the \( S^2 \times S^3 \) model exist, as revealed by
\[ \pi_0(Q) = \pi_3(S^2 \vee S^3) = \pi_3(S^2) \times \pi_3(S^3) = \mathbb{Z} \times \mathbb{Z}, \] (3.11)

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5 The homotopy groups of a wedge of spheres have a very rich structure. It can be shown that \( \pi_p(S^n \vee S^m) = \pi_p(S^n) \times \pi_p(S^m) \times K \) for \( n, m > 1 \), where the abelian group \( K \) is trivial if \( p < n + m - 1 \) and is isomorphic to \( \mathbb{Z} \) if \( p = n + m - 1 \). If \( p > n + m - 1 \), then \( K \) can be computed using the Hilton-Milnor Theorem [8].
and we can write \( Q = \bigcup_{N=-\infty}^{\infty} \bigcup_{M=-\infty}^{\infty} Q_{(N,M)} \). Since at no time can the \( S^2 \) and \( S^3 \) energy densities overlap, we expect any time-varying configuration where a ball soliton passes through the center of a ring to provide a nontrivial homotopy class. Indeed the calculation of \( \pi_1(Q_{(0,0)}) \) bears this out,

\[
\pi_1(Q_{(0,0)}) = \pi_4(S^2 \vee S^3) = \pi_4(S^2) \times \pi_4(S^3) \times \mathbb{Z} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}. \tag{3.12}
\]

The first two factors of \( \mathbb{Z}_2 \) are the familiar spin factors for the two species of solitons. The factor of \( \mathbb{Z} \) can be generated by a time-dependent configuration in which a \((0,1)\) soliton anti-soliton pair and a single unit flux, zero twist ring are created out of the vacuum, and the \((0,1)\) ball goes through the center of the ring before the pair of ball solitons annihilate and the ring shrinks away. Thus, in this model the term (3.10) should rise to the status of a topological term with no effect on the classical dynamics. In fact, in the \( S^2 \vee S^3 \) model the integrand in \( \Delta S \) is locally a total derivative because whenever \( J_\mu \) is nonzero, \( A_\mu \) must be pure gauge. So as expected, the addition of \( \Delta S \) does not change the equations of motion.

4. The \( S^2 \vee S^2 \) and \( S^2 \times S^2 \) Models

Another interesting class of sigma models we may consider are those with target space \( S^2 \vee S^2 \). In 2+1 dimensions the configuration space is \( Q = \text{Map}_*(S^2, S^2 \vee S^2) \). We then have

\[
\pi_0(Q) = \pi_2(S^2 \vee S^2) = \mathbb{Z} \times \mathbb{Z}, \tag{4.1}
\]

and \( Q \) can be written as \( \bigcup_{N=-\infty}^{\infty} \bigcup_{M=-\infty}^{\infty} Q_{(N,M)} \). The solitons corresponding to the \((N,0)\) and \((0, M)\) sectors of \( Q \) are the corresponding \( O(3) \)-invariant sigma model solitons associated with the first and second target \( S^2 \) respectively. These two species of solitons are distinct, so it may be useful to assign colors to them — say red and black — to denote which sphere in the target space they arise from. There are two conserved topological currents \( j_{(R)}^\mu \) and \( j_{(B)}^\mu \) of the form (2.4), one for each color. Further, we find

\[
\pi_1(Q_{(0,0)}) = \pi_3(S^2 \vee S^2) = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}. \tag{4.2}
\]

The processes consisting of the creation, rotation and annihilation of the red \((1,0)\) and black \((0,1)\) solitons represent generators of two of these factors of \( \mathbb{Z} \). Thus, these solitons may be quantized as anyons with independent statistical angles. Any such choice may be
realized by adding to the action both “red” and “black” Hopf terms $H^{(R)}$ and $H^{(B)}$ (as in (2.3)) with appropriate coefficients. The third factor of $\mathbb{Z}$ in (1.2) can be generated by the creation of $(1, 0)$ and $(0, 1)$ soliton-antisoliton pairs, taking (say) the red soliton once counterclockwise around the black soliton (and no other), and then annihilating both pairs. The $2\pi$-rotation of a composite $(1, 1)$ soliton represents the element of $\pi_1(Q_{(0,0)})$ which is the product of the above three generators. Since each of these generators may be represented by an arbitrary phase, we see that the spin of this composite is not uniquely determined by that of the constituents. Now consider adding to the action the topological term $\theta L$ where

$$L = \frac{1}{4\pi} \int d^3x \left( A^{(R)}_\mu j^{(B)}_\mu + A^{(B)}_\mu j^{(R)}_\mu \right) = \frac{1}{16\pi^2} \int d^3x e^{\mu\nu\sigma} \left( A^{(R)}_\mu F^{(B)}_{\nu\sigma} + A^{(B)}_\mu F^{(R)}_{\nu\sigma} \right).$$

(4.3)

Here $A^{(R)}_\mu$ (respectively, $A^{(B)}_\mu$) is the auxiliary gauge field for the red (respectively, black) $S^2$ and $F^{(R)}_{\nu\sigma}$ (respectively, $F^{(B)}_{\nu\sigma}$) its corresponding field strength. This “mixed” Hopf term can be thought of as providing an $(N, 0)$ (respectively, $(0, N)$) soliton with a black (respectively, red) electric charge $\theta N/4\pi$. Evaluated on a time-dependent configuration in which an $(N, 0)$ soliton is brought $n$ times counterclockwise around a $(0, M)$ soliton we obtain $L = nNM$. Thus, $L$ can also be viewed as the generalization to our sigma model of the linking number $n$ of two closed curves $C_1$ and $C_2$ in $\mathbb{R}^3$, which can be written as $[10]

$$n = \frac{1}{4\pi} \epsilon^{ijk} \int_{C_1} dX_i \int_{C_2} dY_j \frac{(X - Y)_k}{|X - Y|^3}. \quad (4.4)$$

If we add the term

$$\Delta S = \theta^{(R)} H^{(R)} + \theta^{(B)} H^{(B)} + \theta L \quad (4.5)$$

to the action, then for a rotation by $2\pi n$ of an $(N, M)$ soliton we have $\Delta S = nN^2\theta^{(R)} + nM^2\theta^{(B)} + nNM\theta$. The mixed Hopf term may also be added to the $S^2 \times S^2$ model, where it can be given the same interpretation. However, unlike here, it will no longer be locally a total derivative. Similar behavior for two species of identical particles in (2+1)-dimensional quantum mechanics has been found in [2].

In 3+1 dimensions the $S^2 \lor S^2$ model configuration space is $Q = Map_*(S^3, S^2 \lor S^2)$ and we have

$$\pi_0(Q) = \pi_3(S^2 \lor S^2) = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}. \quad (4.6)$$

So $Q$ can be written as $\cup_{N=-\infty}^\infty \cup_{M=-\infty}^\infty \cup_{L=-\infty}^\infty Q_{(N,M,L)}$. The solitons corresponding to the red $(N, 0, 0)$ and black $(0, M, 0)$ sectors of $Q$ are the usual twisted rings associated with
the first and second target $S^2$ respectively. The solitons of type $(0, 0, 1)$ have the structure of two untwisted unit-flux rings, one red and one black, which link each other once. Since the target space is $S^2 \vee S^2$, red and black flux tubes may not overlap and these linked configurations are topologically stable. A configuration where a ring having $n_1$ units of red flux and $n_t 2\pi$-twists has linking number $n$ with a ring having $n_2$ units of black flux and $m_t 2\pi$-twists, belongs to the $(n_t n_1^2, m_t n_2^2, n n_1 n_2)$-th sector of $Q$. This model has three conserved topological currents; one Hopf current of the form $(2.11)$ for each target $S^2$, as well as an additional “mixed” Hopf current

$$\ell^\mu = \frac{1}{16\pi^2} \epsilon^{\mu\nu\sigma\rho} (A^{(R)}_\nu F^{(B)}_{\sigma\rho} + A^{(B)}_\nu F^{(R)}_{\sigma\rho}).$$

(4.7)

Evaluated on the general configuration described above containing one black and one red ring, the quantity $\int d^3 x \ell^0$ is equal to $n n_1 n_2$. This topological charge is gauge invariant although $\ell^\mu$ itself changes by a total derivative under both red and black gauge transformations.

In checking for nontrivial spin and other phases we calculate

$$\pi_1(Q_{(0,0,0)}) = \pi_4(S^2 \vee S^2) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z}.$$  

(4.8)

The creation, rotation and annihilation processes for the $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ solitons represent the generators of the three factors of $\mathbb{Z}_2$, and hence we see that we may quantize these solitons as either integer or half-integer spin objects. One of the two factors of $\mathbb{Z}$ can be generated by the process of creating from the vacuum a $(1,0,0)$ (that is, a twisted red) soliton anti-soliton pair and a unit flux, zero twist black ring, pulling the black ring around the red ring and then annihilating the pair of red solitons as well as the black ring. The other factor of $\mathbb{Z}$ comes from passing a $(0,1,0)$ black ring through the center of a unit flux red ring. We can introduce an arbitrary phase into the quantum theory for each $\mathbb{Z}$. A term in the action which supplies one such phase is

$$\Delta S = \frac{\theta}{2\pi} \int d^4 x h^\mu \xi^{(R)} A^{(B)}_\mu,$$

(4.9)

where $h^\mu$ is the red Hopf current. Here $\Delta S$ provides a red $(N,0,0)$ soliton with a black electric charge $\theta N/2\pi$. This term is locally a total derivative since $A^{(B)}_\mu$ must be pure gauge whenever $h^\mu$ is not zero. $\Delta S$ is not manifestly gauge invariant, however. Under a red gauge transformation $A^{(R)}_\mu \to A^{(R)}_\mu - \partial_\mu \chi$ we have

$$\Delta S \to \Delta S - \frac{\theta}{32\pi^3} \int d^4 x \epsilon^{\mu\nu\rho\sigma} \xi^{(R)} F^{(R)}_{\mu\nu} F^{(B)}_{\rho\sigma}.$$  

(4.10)
Since in the $S^2 \vee S^2$ model red and black fluxes are never allowed to overlap, the field strengths $F_{\mu\nu}^{(R)}$ and $F_{\rho\sigma}^{(B)}$ are never nonzero at the same space-time point. Thus $\Delta S$ is invariant under the $U(1)$ gauge transformations associated with the red $S^2$ part of the sigma model. Under the black gauge transformation $A_{\mu}^{(B)} \rightarrow A_{\mu}^{(B)} - \partial_{\mu}\lambda$ we obtain

$$\Delta S \rightarrow \Delta S + \frac{\theta}{2\pi} \int d^4x \partial_{\mu} h_{(R)}^{\mu} \lambda.$$  (4.11)

Since the current $h_{(R)}^{\mu}$ is conserved, we see that $\Delta S$ is also invariant under this transformation and is indeed a well-defined term in the $S^2 \vee S^2$ sigma model. The process where an $(N,0,0)$ soliton passes through the center of a black ring with flux $m$ (and any number of twists) gets a phase $e^{iNm\theta}$ from this term. Similarly we may add an analogous term to the action with the roles of the red and the black fields interchanged, which we may use to assign an arbitrary phase to the process where a $(0,M,0)$ soliton goes through the center of a red ring of flux $n$.

In the previous wedge models we considered, the topological terms (3.10) and (4.3) were still well-defined in the corresponding cross product models, only in the latter these terms were no longer of a topological nature. We may similarly ask about the fate of (4.9) in the $(3+1)$-dimensional sigma model with target space $S^2 \times S^2$. This model has configuration space $Q = \text{Map}^*_+(S^3, S^2 \times S^2)$ and has ring solitons associated with each target $S^2$, which will also be referred to as red and black (these and all the other relevant conventions used in the $S^2 \vee S^2$ model will be followed here). That there are these solitons and no others is shown by

$$\pi_0(Q) = \pi_3(S^2) \times \pi_3(S^2) = \mathbb{Z} \times \mathbb{Z},$$  (4.12)

and thus $Q = \bigcup_{N=-\infty}^{\infty} \bigcup_{M=-\infty}^{\infty} Q_{(N,M)}$, where $N$ counts the number of red solitons and $M$ counts the number of black solitons. Here, of course, the linked solitons of the $S^2 \vee S^2$ model are no longer topologically stable because the red and black fluxes can simply pass through each other. Similarly, the topology of the configuration space does not suggest any topological terms that yield phases when, for instance, a red soliton threads through the hole in a black ring. More specifically,

$$\pi_1(Q_{(0,0)}) = \pi_4(S^2) \times \pi_4(S^2) = \mathbb{Z}_2 \times \mathbb{Z}_2,$$  (4.13)

where the two $\mathbb{Z}_2$’s indicate that the $(1,0)$ and $(0,1)$ solitons may be independently quantized as integer or half-integer spin objects. It is straightforward to see that $\Delta S$ in (4.9) is
not even well-defined for the $S^2 \times S^2$ model. While invariant under black gauge transformations for the same reasons as in the $S^2 \vee S^2$ case, the fact that the red and black fluxes are now allowed to overlap means that it is not invariant under red gauge transformations. However, all is not lost. We may add a term to $\Delta S$ that makes the total term well-defined (but not topological) and supplies an arbitrary phase to the process of sliding a twisted red ring through the center of a black ring. Indeed write $\zeta \in \Phi^2$, where $\zeta$ parametrizes the red $S^2$, as

$$\zeta = \left( e^{i\omega_1 \cos(\phi)} e^{i\omega_2 \sin(\phi)} \right)$$

(4.14)

and define $\omega = (\omega_1 + \omega_2)/2$. Then under the red gauge transformation $A^{(R)}_\mu \rightarrow A^{(R)}_\mu - \partial_\mu \chi$ we have $\omega \rightarrow \omega + \chi$. Note that $\omega$ is only defined modulo $\pi$, and is completely ambiguous if either $\cos(\phi)$ or $\sin(\phi)$ equals zero. However, since $F^{(R)}_{\mu\nu}$ vanishes in this latter case and

$$\int d^4x e^{\mu\rho\sigma} F^{(R)}_{\mu\nu} F^{(B)}_{\rho\sigma} = 0,$$

(4.15)

we see that

$$\Delta S' = \frac{\theta}{2\pi} \int d^4x (h^{(R)}_\mu A^{(B)}_\mu + \frac{\omega}{32\pi^3} e^{\mu\rho\sigma} F^{(R)}_{\mu\nu} F^{(B)}_{\rho\sigma})$$

(4.16)

is well-defined. Moreover, by (4.10), $\Delta S'$ is invariant under red gauge transformations. It remains invariant under black gauge transformations since only the black field strength is involved in the modification. Also, the modification is nonzero only when the field strengths of the red and black fluxes overlap, so we see that this term still supplies the phase $e^{i N m \theta}$ to the process where a red $(N,0)$ ring passes through the center of a black ring with flux $m$. A similar term, again with red and black interchanged, supplies a phase to the passing of a twisted black ring through the center of a red ring.

5. Conclusion

In various nonlinear sigma models, we have explicitly constructed terms which give the linking number of the world-tubes of distinct species of ball solitons in 2+1 dimensions or which detect the threading of balls or loops through other loops (of distinct species) in 3+1 dimensions. We also found that the (3+1)-dimensional $S^2 \vee S^2$ model has solitons consisting of two flux rings of distinct species linked in space. The $A \vee B$ type models which appear in our analysis may be considered as theories in their own right by adding additional fields to smooth out the target space singularity at the point where $A$ and $B$
are joined (which can be done without essentially altering the topology). Alternatively, they may be considered as limits of cross product models as discussed earlier. In any case, they should be generally useful for thinking about linking effects in field theories since they reduce certain questions about such phenomena to questions about topology.

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