Bose-Einstein condensation of finite number of confined particles

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Abstract

The partition function and specific heat of a system consisting of a finite number of bosons confined in an external potential are calculated in different spatial dimensions. Using the grand partition function as the generating function of the partition function, an iterative scheme is established for the calculation of the partition function of system with an arbitrary number of particles. The scheme is applied to finite number of bosons confined in isotropic and anisotropic parabolic traps and in rigid boxes. The specific heat as a function of temperature is studied in detail for different number of particles, different degrees of anisotropy, and different spatial dimensions. The cusp in the specific heat is taken as an indication of Bose-Einstein condensation (BEC). It is found that the results corresponding to a large number of particles are approached quite rapidly as the number of bosons in the system increases. For large number of particles, results obtained within our iterative scheme are consistent with those of the semiclassical theory of BEC in an external potential based on the grand canonical treatment.
I. INTRODUCTION

Following the foundation laid down by Einstein\textsuperscript{4}, discussions on the phenomena of Bose-Einstein Condensation (BEC) have been of constant interest to physicists and can be found in standard textbooks on statistical physics.\textsuperscript{2} The conventional treatment is usually based on the grand canonical ensemble within the context of an ideal Bose gas and the thermodynamic limit. The recent beautiful experiments\textsuperscript{3} on BEC in atomic gases, however, lead to the need of a deeper understanding of the problem of BEC in confined systems with finite number of bosons. The fact that the number of bosons is finite and relatively small in the trap created experimentally and that the ultracold atomic gas does not exchange particles with reservoirs implies that the canonical ensemble is more appropriate.\textsuperscript{4,5}

It is interesting to study how the thermodynamical quantities such as the specific heat evolves as the number of particles in the system increases. In Fermion systems, the differences among results obtained within the different ensembles have been studied.\textsuperscript{6} Recently, there have been some work on finite boson systems in a parabolic confinement within the context of micro-canonical and canonical ensembles.\textsuperscript{4,5} In the present work, using the grand partition function as the generating function of the partition function, an iterative scheme is established for the calculation of the partition function of system with an arbitrary number of particles. The scheme is applied to bosons confined in isotropic and anisotropic parabolic potentials and in rigid boxes in arbitrary spatial dimensions. In particular, we aim out studying the effects of dimensionality, anisotropy, and finite number of particles. Although there exist many interesting approaches to the problem in the literature,\textsuperscript{7} we concentrate on comparing our results, wherever possible, with those in the conventional theory and in the semiclassical theory of Bagnato \textit{et al.},\textsuperscript{8} which is basically an extension of the conventional theory to the case with confining potentials.

The plan of the paper is as follows. In Sec. II, the general formalism and the recursive scheme are given. Results for bosons confined in isotropic and anisotropic parabolic potentials are given in Sec. III. The corresponding results for confinement in rigid boxes are
given in Sec. IV together with a discussion on the validity of the integral approximation commonly used. A brief summary is given in Sec. V.

II. FORMALISM

We consider a system of \( N \) non-interacting bosons. The partition function \( Z_N(v, T) \) is related to the \( N \)-th derivative with respect to the fugacity \( z \) of the grand partition function \( Q(z, v, T) \) through

\[
Z_N(v, T) = \frac{1}{N!} \frac{\partial^N Q(z, v, T)}{\partial z^N} \bigg|_{z=0},
\]

where \( T \) is the temperature and \( z = \exp(\mu/kT) \) is the fugacity. Here \( v \) is the variable representing the volume of the system for confinement in a rigid box and the confining strength in the case of a parabolic confining potential. In general, the grand partition function for an ideal bose gas can be written as

\[
Q(z, v, T) = \prod_p \frac{1}{1 - ze^{-\beta \epsilon_p}},
\]

where the product is over all the single particle states labelled by \( p \) with energies \( \epsilon_p \), and \( \beta = 1/kT \) with \( k \) being the Boltzmann constant. The zeros of the chemical potential and the single particle energy are set to be the same. Combining Eqs.(1) and (2) gives a relation

\[
Z_N = \frac{1}{N} \sum_{j=1}^{N} B_j Z_{N-j}
\]

between the partition function \( Z_N \) of a \( N \)-particle system and the partition functions \( Z_{N'} \), with \( N' = 1, \ldots, N - 1 \), for systems with less than \( N \) particles. The factor \( B_j \) is defined as \( B_j = \sum_p e^{-j \beta \epsilon_p} \). For simplicity, the arguments \( v \) and \( T \) are left out in the partition functions in Eq.(3). The internal energy \( E \) and the heat capacity \( C_v \) can then be calculated using the standard formulae \( E = -\partial \ln Z_N/\partial \beta \) and \( C_v = \partial E/\partial T \). These expressions involve the first and second derivatives of the partition function with respect to temperature.

The first and second derivatives of the partition function of a \( N \)-particle system can also be related to quantities in systems with less particles. Let \( b \) be a dimensionless parameter representing the temperature. From Eq.(3), we have
\[
\frac{\partial Z_N}{\partial b} = \frac{1}{N} \sum_{j=1}^{N} \left[ B_j \frac{\partial Z_{N-j}}{\partial b} + \frac{\partial B_j}{\partial b} Z_{N-j} \right],
\]
and
\[
\frac{\partial^2 Z_N}{\partial b^2} = \frac{1}{N} \sum_{j=1}^{N} \left[ B_j \frac{\partial^2 Z_{N-j}}{\partial b^2} + 2 \frac{\partial B_j}{\partial b} \frac{\partial Z_{N-j}}{\partial b} + \frac{\partial^2 B_j}{\partial b^2} Z_{N-j} \right].
\]
Equations (3)-(5) imply that the partition function and other thermodynamical quantities of a $N$-particle system can be calculated from the partition functions and their derivatives of systems with less particles iteratively by using the conditions $Z_0 = 1$, $\partial Z_0/\partial b = 0$, and $\partial^2 Z_0/\partial b^2 = 0$. These conditions, when substituted into Eqs.(3)-(5), give the correct single particle partition function $Z_1$ and its derivatives. It should be pointed out that these conditions are independent of the form of the confining potential and spatial dimension. The effects of the confinement are reflected in the energy spectrum and the explicit form of Eqs.(3)-(5) when they are applied to specific problems. In what follows, we apply Eqs.(3)-(5) to systems with finite number of non-interacting bosons confined either in a parabolic potential or in a rigid box. In particular, the position of the cusp in the heat capacity as a function of temperature is taken to be an indication of Bose-Einstein condensation in the system. We study the dependence of the position of the cusp on the number of particles in the system.

III. PARABOLIC CONFINEMENT

A. Isotropic confinement in 1, 2, and 3 dimensions

For $M$-dimensional isotropic parabolic confinements characterized by the energy $\hbar \omega$, the energies of the single particle states are given by
\[
\varepsilon_{n_i} = \sum_{i=1}^{M} n_i \hbar \omega,
\]
where $n_i \ (i = 1, \ldots, M)$ takes on non-negative integers and the zero-point energy has been absorbed in the definition of the zero of energy. Identifying the dimensionless parameter $b$
as \( b \equiv \exp(-1/\tau) \), where \( \tau = kT/\hbar\omega \) is the effective temperature, and using Eqs.(3)-(5), we have, in \( M \) dimensions, the relations

\[
Z_N = \frac{1}{N} \sum_{j=1}^{N} \frac{Z_{N-j}}{(1-b^M)}, \quad (7)
\]

\[
\frac{\partial Z_N}{\partial b} = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{(1-b^M)} \left[ \frac{\partial Z_{N-j}}{\partial b} + \frac{Mj^2b^{j-1}}{1-b^j}Z_{N-j} \right], \quad (8)
\]

\[
\frac{\partial^2 Z_N}{\partial b^2} = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{(1-b^M)} \left[ \frac{\partial^2 Z_{N-j}}{\partial b^2} + \frac{2Mj^2b^{j-1}}{1-b^j} \frac{\partial Z_{N-j}}{\partial b} + Mj^2b^{j-2} \frac{j}{(1-b^j)^2}Z_{N-j} \right]. \quad (9)
\]

Equations (7)-(9), together with the conditions on \( Z_0 \) and its derivatives, can be used iteratively to obtain \( Z_1, \ldots, Z_N \) and their derivatives. These equations can be easily implemented numerically for any spatial dimensions. The heat capacity can be evaluated using

\[
C_v = k \frac{1}{\tau^2} \left[ \frac{b}{Z_N} \frac{\partial Z_N}{\partial b} \left( 1 - b \frac{\partial Z_N}{\partial b} \right) + b \frac{\partial^2 Z_N}{\partial b^2} \right]. \quad (10)
\]

It should be pointed out that, although exact expression for \( Z_N \) may not be possible for general dimensions, exact expressions exist for \( Z_N \) and \( C_v \) in one dimension (1D) and they read

\[
Z_N^{(1D)} = \frac{1}{\prod_{\ell=1}^{N} (1-b^\ell)}, \quad (11)
\]

and

\[
C_v^{(1D)} = k \left( \frac{\hbar\omega}{kT} \right)^2 \sum_{\ell=1}^{N} \frac{\ell^2b^\ell}{(1-b^\ell)^2}. \quad (12)
\]

Slightly different expression for \( Z_N^{(1D)} \) has been obtained by Brosens et al.\(^4\) The difference comes from our choice of setting the zero of energy at the ground state energy of a \( m \)-dimensional harmonic oscillator. This choice, of course, does not affect the result of the heat capacity. Figure 1 shows the results of the specific heat \( C_v/Nk \) calculated using Eq.(10) as a function of \( kT/(\hbar\omega N^{1/M}) \) for systems in \( M = 1, 2, \) and 3 dimensions (1D, 2D, 3D) with different number of bosons. In Fig. 1(a), it is obvious that the results obtained numerically
using Eqs.(7)-(10) agree well with the results obtained from the exact expression given by Eq.(12). This, in turn, indicates that our iterative approach is reliable. In 1D, the specific heat approaches its large $N$ limit very rapidly in that the curves corresponding to systems with $N = 10$ and $N = 10^4$ particles are very close to each other. As temperature increases, $C_v$ approaches its limiting value of $Nk$. In contrast to 2D and 3D systems, $C_v$ varies smoothly with temperature and there is no cusp. This implies that there is no BEC in 1D system with parabolic confinement. This result is consistent with that of the semiclassical treatment of Bagnato and Kleepner. These authors studied BEC in traps within the grand canonical ensemble and the thermodynamic limit so that the notion of a density of states is valid. They found that 1D ideal Bose gas will display BEC only if the external potential is more confining than the parabolic form.

Figure 1(b) shows that cusp appears in the specific heat when the number of particles in the system is large enough in 2D isotropic parabolic confinement. The position of the cusp shifts to higher temperature as the number of particles increases. As there will be, strictly speaking, no sharp transition in a system with finite number of particles, the appearance of a cusp in the specific heat is taken to be an indication of BEC. For system with $N = 10^4$ particles, the cusp is at a temperature $kT/(\hbar\omega N^{1/2}) \approx 0.75$; while the semiclassical treatment gives a transition temperature $T_c$ at $kT_c/(\hbar\omega N^{1/2}) = \sqrt{6}/\pi \approx 0.7797$ for $N \to \infty$. Thus, our results are consistent with the semiclassical results and we expect the position of the cusp in $C_v$ to approach a value slightly larger than $kT/(\hbar\omega N^{1/2}) \approx 0.75$ as $N$ increases.

In 3D isotropic parabolic confinement, the specific heat shows a cusp as a function of temperature for systems with large enough $N$ with the position of the cusp shifts up in temperature as $N$ increases and approaches a limit (see Fig.1(c)). However, the behaviour is quite different from that in 2D. As $N$ increases, the cusp becomes sharper and $C_v$ decreases rapidly across the cusp and eventually leads to a discontinuity in $C_v$. For systems with $N = 10^4$ particles, the position of the cusp is at $kT/(\hbar\omega N^{1/3}) \approx 0.9$ and the discontinuity in $C_v$ is about $-6.5Nk$. According to the semiclassical treatment for BEC in 3D harmonic traps, the transition temperature is at $kT_c/(\hbar\omega N^{1/3}) \approx 0.912$ for $N \to \infty$, and there will
be a discontinuity in $C_v$ of the amount given by $[C_v(T_c^+) - C_v(T_c^-)]/Nk \approx -6.57$. Therefore, our results for large $N$ are consistent with that of semiclassical theory. In contrast, the specific heat in $C_v$ is continuous across the cusp in 2D.

**B. Anisotropic confinement in 2 dimensions**

We choose to illustrate the effects of anisotropy by considering two-dimensional systems. The energies of the single particle states in a 2D anisotropic harmonic confinement are

$$
\varepsilon_{n_1,n_2} = \hbar \omega_1 (n_1 + \alpha n_2), \quad n_1, n_2 = 0, 1, 2, \ldots,
$$

where $\alpha = \omega_2/\omega_1$ is the ratio of the frequencies of the harmonic potentials in the two directions. In this case, the factor $B_j$ is defined to be $B_j(b) = 1/(1 - b^j)(1 - b^{\alpha j})$, where $b = \exp(1/\tau)$ with $\tau = \frac{kT}{\hbar \omega_1}$. Equations (3)-(5) can then be applied iteratively to obtain the heat capacity. Figure 2 shows the dependence of the specific heat on temperature by plotting $C_v/Nk$ as a function of $\frac{\tau}{(\alpha N)^{1/2}} = \frac{kT}{\hbar (\omega_1 \omega_2)^{1/2} N^{1/2}}$ for different values of $\alpha$ with different number of particles in the system. Figure 2(a) shows the case with fixed degree of anisotropy $\alpha = 80$. For $N = 10^4$ particles in such an anisotropic trap, the cusp appears at $\frac{\tau}{(\alpha N)^{1/2}} \approx 0.71$; while semiclassical theory gives a value of 0.7797 for $N \to \infty$. For small number of particles, say $N = 10$, $C_v$ appears to approach the 1D limit of $Nk$ at low temperature. However, as temperature increases, $C_v$ increases again and eventually approaches the 2D limit of $2Nk$ at high temperature. This behaviour can be understood in that the anisotropy leads to different energy spectra in the two directions. At low temperature, only the levels corresponding to the less confining direction are populated and thus the system appears to behave in a one-dimensional way. At higher temperature, the levels corresponding to the larger confining direction start to become populated and the system takes on two-dimensional character. Figure 2(b) shows the effects of anisotropy for a fixed number of particles $N = 10^4$ in the system. For small $\alpha$, there is no cusp in $C_v$ - a character of a 1D system. However, it should be pointed out that even for small values of $\alpha$, $C_v$ approaches the 2D limit of $2Nk$ at high temperatures.
IV. RIGID BOX

The iterative approach can also be used to calculate the thermodynamical quantities of a system of \(N\) non-interacting Bosons confined in a \(M\)-dimensional rigid box of volume \(v = L^M\). The energies of the single-particle states for a particle of mass \(m\) in such a box are given by

\[
\varepsilon_{n_i} = \sum_{i=1}^{M} \frac{n_i^2 h^2}{8mL^2},
\]

where the quantum numbers \(n_i (i = 1, 2, \cdots, M)\) takes on positive integers. The factor \(B_j\) in \(M\) dimensions is given by

\[
B_j(b) = [\Theta_j(b)]^M
\]

with \(\Theta_j(b)\) being the factor \(B_j\) in 1D defined by

\[
\Theta_j(b) \equiv \sum_{n=1}^{\infty} b^{jn^2}.
\]

The dimensionless parameter \(b\) is taken to be \(b \equiv \exp(-1/\tau)\), where \(\tau\) is the effective temperature given by

\[
\tau \equiv \frac{8mkT L^2}{h^2}.
\]

In \(m\) dimensions, Eqs. (3)-(5) become

\[
Z_N = \frac{1}{N} \sum_{j=1}^{N} \Theta_j^M Z_{N-j},
\]

\[
\frac{\partial Z_N}{\partial b} = \frac{1}{N} \sum_{j=1}^{N} \Theta_j^M \left[ \frac{\partial Z_{N-j}}{\partial b} + \frac{M}{\Theta_j} \frac{\partial \Theta_j}{\partial b} Z_{N-j} \right],
\]

\[
\frac{\partial^2 Z_N}{\partial b^2} = \frac{1}{N} \sum_{j=1}^{N} \Theta_j^M \left\{ \frac{\partial^2 Z_{N-j}}{\partial b^2} + \frac{2M}{\Theta_j^M} \frac{\partial \Theta_j}{\partial b} \frac{\partial Z_{N-j}}{\partial b} \right. + \left. M \left[ \frac{M - 1}{\Theta_j^2} \left( \frac{\partial \Theta_j}{\partial b} \right)^2 + \frac{1}{\Theta_j} \frac{\partial^2 \Theta_j}{\partial b^2} \right] Z_{N-j} \right\}.
\]

The function \(\Theta_j(b)\), which is related to the Jacobi’s Theta-function, and its derivatives can be evaluated numerically by summing the corresponding infinite series to convergence. The heat capacity can then be obtained using Eq. (10). Figure 3 shows the specific heat for different number of bosons in a rigid box in 1D, 2D, and 3D. Figure 3(a) shows that in 1D rigid box the results for systems with a small number of particles approach that of a larger number of particles quite rapidly. This feature is similar to that in a 1D parabolic trap. Another feature is that the high temperature limit of \(C_v = 0.5k\) is approached quite slowly. For example, \(C_v/Nk\) increases from approximately 0.478 to 0.489 as \(\tau/N^2\) increases from 20 to 80.
Figure 3(b) shows that in 2D the large $N$ behaviour is approached very rapidly. The high temperature limit is approached more rapidly than in the 1D case. There is no cusp in the specific heat for both 1D and 2D. This is consistent with the well-known result that BEC for ideal bose gas in the absence of external confining potential does not exist in dimensions lower than three. Figure 3(c) gives the results in 3D. The position of the cusp shifts to approach the large $N$ limit from above as the number of particles increases. For $N = 10^4$, our numerical results give the maximum specific heat at the cusp the value $C_v/N k = 2.087$ at the corresponding temperature $8mkT_L^2/h^2N^{2/3} \approx 0.7556$. It should be contrasted with the standard ideal Bose gas results in 3D in the thermodynamic limit that the maximum specific heat is $1.925Nk$ at the transition temperature given by $8mkT_cL^2/h^2N^{2/3} \approx 0.6713$. In contrast to the results corresponding to a 3D parabolic potential, the specific heat is continuous across the cusp.

The infinite series in $\Theta_j(b)$ is sometimes approximated by an integral which yields $\Theta_j \approx (\pi \tau/4j)^{1/2}$. With this approximation, a suitable dimensionless parameter is $b \equiv (\pi \tau/4)^{M/2}$. An approximate expression of the the factor $B_j(b)$ in $M$ dimensions is

$$B_j(b) \approx j^{-M/2}b.$$  

(18)

From the identity

$$\sum_{n=1}^{\infty} e^{-jn^2/\tau} = \frac{1}{2} \left( \frac{\pi \tau}{j} \right)^{1/2} + \left( \frac{\pi \tau}{j} \right)^{1/2} \sum_{q=1}^{\infty} e^{-\pi^2 q^2 \tau/j} - \frac{1}{2},$$  

(19)

the integral approximation amounts to retaining only the first term. Within this approximation, Eqs.(15)-(17) read

$$Z_N = \frac{1}{N} \sum_{j=1}^{N} j^{-M/2}bZ_{N-j},$$  

(20)

$$\frac{\partial Z_N}{\partial b} = \frac{1}{N} \sum_{j=1}^{N} j^{-M/2} \left[ b \frac{\partial Z_{N-j}}{\partial b} + Z_{N-j} \right],$$  

(21)

$$\frac{\partial^2 Z_N}{\partial b^2} = \frac{1}{N} \sum_{j=1}^{N} j^{-M/2} \left[ b \frac{\partial^2 Z_{N-j}}{\partial b^2} + 2 \frac{\partial Z_{N-j}}{\partial b} \right].$$  

(22)

Within this approximation, the formula for heat capacity is
\[ C_v(b) = \frac{Mk}{2} \left[ \left( \frac{M}{2} + 1 \right) \frac{b}{Z_N} \frac{\partial Z_N}{\partial b} - \frac{M}{2} \left( \frac{b}{Z_N} \frac{\partial Z_N}{\partial b} \right)^2 + \frac{M}{2} \frac{b^2}{Z_N} \frac{\partial^2 Z_N}{\partial b^2} \right] , \] (23)

and the calculated results are shown in Fig.4. The general features are quite similar to that of the exact results shown in Fig.3. It can be shown that the integral approximation gives reliable results except for systems with only a few particles, in those cases the approximation may lead to erroneous results.

V. CONCLUSIONS

We have studied the BEC of finite number of confined bosons. An iterative scheme has been used to obtain the specific heat for systems with different number of particles confined in an external potential. As emphasized by van Hove\textsuperscript{10}, no singularities would appear in the partition function and the thermodynamical quantities of a finite system. Therefore the cusp of specific heat curve was taken as an indication of BEC. Numerical results show that the specific heat approaches its large \( N \) limit rapidly as the number of particles in the system increases. Results in different spatial dimensions are obtained. Results for systems with large \( N \), typically \( N \approx 10^4 \) in our calculations, are consistent with those in the semiclassical theory of Bagnato \textit{et al.}\textsuperscript{8} As the early experiments have typically \( 10^4 \text{-} 10^5 \) particles in the trap\textsuperscript{3} we expect that treatments within the canonical and grand canonical ensembles would give at least qualitatively consistent results. Although the iterative scheme is applied here to Bose systems, it can equally be applied to systems with finite number of \textit{Fermions} confined in an external potential. In this case, the variable \( B_j \) should be modified to be \( B_j = \sum_p (\begin{pmatrix} j \end{pmatrix} e^{-j\beta\varepsilon_p} \).
ACKNOWLEDGMENTS

One of us (PMH) acknowledges the support of a grant from the British Council under the UK-HK Joint Research Scheme. WD is supported by a Postdoctoral Fellowship of the Chinese University of Hong Kong.
REFERENCES

1 A. Einstein, Ber. Berl. Akad. 261 (1924); ibid. 3 (1925).

2 See, for example, K. Huang, *Statistical Mechanics*, (Wiley, New York, 1963).

3 M.H. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman, and E.A. Cornell, Science 269, 198 (1995); C.C. Bradley, C.A. Sackett, J.J. Tollett, and R.G. Hulet, Phys. Rev. Lett. 75, 1687 (1995); K.B. Davis, M.-O. Mewes, M.R. Andrews, N.J. van Druten, D.S. Durfee, D.M. Kurn, and W. Ketterle, Phys. Rev. Lett. 75, 3969 (1995).

4 F. Brosens, J.T. Devreese, and L. F. Lemmens, Solid State Commun. 100, 123 (1996).

5 S. Grossmann, and M. Holthaus, Phys. Rev. E 54, 3495 (1996).

6 See, for example, R. Denton, B. Muhlschlegel, and D. J. Scalapino, Phys. Rev. B 7, 3589 (1973); H. P. Chen and H. F. Cheung, J. Phys.: Condens. Matter 7, 6707 (1995) and references therein.

7 S. R. de Groot, G.J. Hooyman, and C.A. ten Seldam Proc. R. Soc. A 203, 266 (1950); D. L. Mills, Phys. Rev. 134, A 306 (1964); P. K. Pathria, Can. J. Phys. 61, 228 (1983); K. Kirsten, and D.J. Toms, Phys. Rev. A 54, 4188 (1996); Phys. Lett. A 222, 148 (1996); Phys. Lett. B 368, 119 (1995); W. Ketterle and N. J. van Druten, Phys. Rev. A 54, 656 (1996); S. Grossmann and M. Holthaus, Phys. Lett. A 208, 188 (1995); Z. Phys. B 97, 319 (1995); Z. Naturforsch. a50, 921 (1995).

8 V. Bagnato, D. E. Pritchard, and D. Kleppner, Phys. Rev. A 35, 4354 (1987); V. Bagnato and D. Kleppner, Phys. Rev. A 44, 7439 (1991).

9 See, for example, E. T. Whittaker and G. N. Watson, *A course of modern analysis*, 4th edition, Chapters 20-22, (Cambridge Univ. Press, Cambridge, England, 1952).

10 L. van Hove, Physica, 15, 951 (1949).
FIG. 1. The specific heat of a finite number $N$ of bosons confined in an isotropic parabolic potential as a function of the dimensionless temperature $\tau/N^{1/M} = kT/(\hbar\omega N^{1/M})$, where $M$ is the spatial dimension for (a) $M = 1$, (b) $M = 2$, and (c) $M = 3$. The lines are results of the iterative scheme for different values of $N$. The symbols in (a) are the results obtained using the exact expression Eq.(12).
FIG. 2. The specific heat of $N$ bosons confined in a two-dimensional anisotropic parabolic potential as a function of the dimensionless temperature $\tau/(\alpha N)^{1/2} = kT/h(\omega_1\omega_2)^{1/2}N^{1/2}$. (a) The degree of anisotropy $\alpha$ is taken to be 80 and the number of particles takes on four different values of $N = 10, 10^2, 10^3$, and $10^4$. (b) The number of particle is fixed at $N = 10^4$, and the degree of anisotropy $\alpha$ takes on $\alpha = 10^{-4}, 10^{-3},$ and $10^{-2}$. 
FIG. 3. The specific heat of $N$ bosons confined in $M$-dimensional rigid box as a function of the dimensionless temperature $\tau/N^{2/M} = 8mkTL^2/h^2N^{2/M}$ for (a) $M = 1$, (b) $M = 2$, and (c) $M = 3$. Results are obtained using Eqs.(15)-(17) by summing up the infinite series for $\Theta_j(b)$ to convergence.
FIG. 4. The specific heat of $N$ bosons confined in $M$-dimensional rigid box as a function of the dimensionless temperature $\tau / N^{2/M} = 8\hbar k T L^2 / \hbar^2 N^{2/M}$ for (a) $M = 1$, (b) $M = 2$, and (c) $M = 3$. Results are obtained using Eqs.(20)-(22) within the integral approximation.