Supersymmetric gradient flow in $\mathcal{N} = 1$ SYM

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Abstract The gradient flow equation is derived in four-dimensional $\mathcal{N} = 1$ supersymmetric Yang–Mills theory in terms of the component field of the Wess–Zumino gauge. We show that the flow-time derivative and supersymmetry transformation that is naively extended to 4+1 dimensions by replacing the four-dimensional fields with the corresponding flowed fields commute with each other up to a gauge transformation. In this sense, the obtained flow is supersymmetric in the Wess–Zumino gauge. We also discuss more about the symmetry of the flow equation.

1 Introduction

The gradient flow [1,2] has been applied to various studies in lattice QCD [3–13]. These applications are based on the UV-finiteness of correlators of flowed fields and the smoothing effects obtained by the flow [3]. It also has great potential for supersymmetric Yang–Mills (SYM) flow because we have not only the same applications as in QCD [14,15] but also ones specific to SYM such as the supercurrent needed to construct the correct continuum limit [16,17].

There are mainly two possibilities in defining a gradient flow equation in SYM. One way is to use the non SUSY flow as introduced in QCD [1,2,4], while the other way is to use a SUSY flow associated with the gradient of the SYM action. In the former case, the gaugino is treated as the adjoint matter coupled to the gauge field. In this sense, the flow is irrelevant to SUSY and the flowed fermions receive extra renormalizations. In the latter case, one can expect that no such extra renormalizations exist thanks to SUSY.

The SYM-gradient flow has already been given in terms of vector superfield [18], which manifestly respects super and extended gauge transformations.1,2 It is, however, not straightforward to derive a flow equation in terms of component fields taking the WZ gauge. In Ref. [18], a term by which the extended gauge symmetry is fixed is introduced in the SYM-flow equation to take the Wess–Zumino gauge, but the obtained equation is not invariant under the ordinary gauge transformation. It is not clear whether or not the flow is compatible with supersymmetry in the WZ gauge [21,22].

The SYM flow at the WZ gauge will be useful for the lattice calculation performed with the gauge [14,15] and for knowing the clear difference from the non-SUSY flow. When we try to compare some analytic calculations performed in the component fields directly to the flow theory, the flow given by the component fields will be useful.

In this paper, we propose a natural way to derive a supersymmetric gradient flow equation in the Wess–Zumino gauge in $\mathcal{N} = 1$ SYM and discuss the symmetry properties of the equation. Taking a gauge fixing term that satisfies a condition milder than that of [18], we are free to choose any gauge for the ordinary gauge symmetry, in other words, the gauge symmetry is not fixed in the flow. We also show that the flow equation is compatible to supersymmetry in the sense that the commutator of the flow-time derivative and supersymmetry transformation that is naturally extended to $d + 1$-dimensions vanishes up to a gauge transformation.

This paper is organized as follows. Starting from the definition of non-SUSY flow (the Yang–Mills flow) in Sect. 2, we review the SYM-gradient flow given in Ref. [18] in Sect. 3. The superfield formalism is given in the Euclidean space in Sect. 3.1 and the original derivation of the SYM-gradient flow in terms of vector superfield is shown in Sect. 3.2. We discuss

1 In the context of Langevin equation, a similar equation has been introduced by Nakazawa [19,20].
2 The extended gauge transformation is a gauge transformation generated by a chiral superfield, which is defined in eq. (36) in Sect. 3. The ordinary gauge transformation $A_μ → A_μ − δ_μ ω$ is included in it as a partial transformation.
the issue of the original flow equation in the Wess–Zumino gauge in Sect. 3.3. In Sect. 4, we derive a supersymmetric gradient flow equation in the Wess–Zumino gauge taking a new gauge fixing term and show that it is compatible with supersymmetry in the Wess–Zumino gauge. The conclusion and outlook are shown in Sect. 5.

2 Yang–Mills gradient flow

We review the gradient flow in four-dimensional Yang–Mills theory, while fixing the notations used in this paper. The gauge group is $SU(N)$ and the group generators $T^a$ $(a = 1, \ldots, N^2 - 1)$ are hermitean matrices that satisfy the standard relations given in Appendix 1A. The Einstein summation convention is used throughout this paper.

The action of Yang–Mills theory in four dimensional Euclidean space whose coordinate is expressed as $x_\mu$ $(\mu = 0, 1, 2, 3)$ is given by

$$S = \frac{1}{2g^2} \int d^4x \, \text{tr} \left\{ F_{\mu \nu}^2(x) \right\},$$

(1)

where

$$F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i [A_\mu, A_\nu].$$

(2)

The gauge field $A_\mu^a(x)$ is now expressed as a matrix-valued field as $A_\mu(x) = \sum_{a=1}^{N^2-1} A_\mu^a(x) T^a$.

To define the gradient flow equation, we introduce a flow-time $t (\geq 0)$ and assume that $A_\mu(x)$ depends on the flow time as $A_\mu(x) \rightarrow B_\mu(t, x)$ which satisfies a boundary condition,

$$B_\mu(t, x)|_{t=0} = A_\mu(x).$$

(3)

Then the gradient flow equation is formally defined as the gradient of the Yang–Mills action:

$$\partial_t B_\mu^a(t, x) = -g^2 \left. \frac{\delta S_{\text{SYM}}}{\delta A_\mu^a(x)} \right|_{A_\mu^a(x) \rightarrow B_\mu^a(t, x)}$$

(4)

where $\frac{\delta}{\delta A_\mu^a(x)}$ in the right hand side is the functional derivative with respect to $A_\mu^a(x)$. We thus have

$$\partial_t B_\mu(t, x) = D_\nu G_{\nu \mu}(t, x),$$

(5)

where

$$D_\mu \varphi = \partial_\mu \varphi + i [B_\mu, \varphi],$$

$$G_{\mu \nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + i [B_\mu, B_\nu].$$

(6)

Throughout this paper, $D_\mu$ that acts on any $t$-dependent field $\varphi(t, x)$ is defined by (6).

The flow equation is manifestly covariant under a time independent gauge transformation. It is known that the gauge covariance becomes an obstacle in the perturbative analyses of the flow equation. We introduce a gauge fixing term when performing the perturbative calculations as

$$\partial_t B_\mu = D_\nu G_{\nu \mu} + \alpha D_\mu \partial_\nu B_\nu$$

(7)

with a positive parameter $\alpha$.

The solution of the gauge fixed flow equation (7) with any $\alpha$ is related with the solution with $\alpha = 0$ through the time dependent gauge transformation:

$$B_\mu = \Lambda (B_\mu|_{t=0} - i \partial_\mu) \Lambda^{-1},$$

(8)

where

$$\partial_t \Lambda = -i \alpha \partial_\mu B_\mu \Lambda.$$  

(9)

This fact means that for any gauge invariant operator at $t = 0$ the two Eqs. (5) and (7) are equivalent.

The flow equation (7) itself has a kind of gauge covariance with a time dependent gauge transformation:

$$B_\mu^a = B_\mu - D_\nu \omega,$$

(10)

where $\omega(t, x)$ obeys

$$\partial_t \omega = \alpha D_\mu \partial_\nu \omega.$$  

(11)

The boundary condition on $\omega$ at $t = 0$ can be chosen arbitrarily and the gauge symmetry at $t = 0$ remains unfixed.

A surprising result of the Yang–Mills gradient flow is that any correlation function of the flowed gauge field is UV finite in all order of perturbation theory once the boundary four-dimensional theory is renormalized in the standard way [3]. The time dependent gauge symmetry in (10) and (11) then play an essential role in the proof of the finiteness of correlation functions.

3 Gradient flow equation in $\mathcal{N} = 1$ SYM

We basically follow Ref. [23] with the Wick rotation to the Euclidean space. The convention of superfields and differential operators after the Wick rotation are summarized in Appendix 1B.

3.1 $\mathcal{N} = 1$ SYM action and superfield formalism

The Euclidean action of $\mathcal{N} = 1$ SYM is given by

$$S = \frac{1}{g^2} \int d^4x \, \text{tr} \left\{ \frac{1}{2} F_{\mu \nu}^2 + 2i \bar{\chi} \gamma_\mu D_\mu \chi + D^2 \right\}(x)$$

(12)

where $\chi(x)$ and $\bar{\chi}(x)$ with $\alpha = 1, 2$ are two-component spinors, $A_\mu(x)$ is the gauge field, $D(x)$ is an auxiliary field ($D^a(x) \in \mathbb{R}$). Four dimensional sigma matrices are defined as $\sigma_\mu = (-i I, \sigma^i)$ and $\bar{\sigma}_\mu = (-i I, -\sigma^i)$ with the standard Pauli matrices $\sigma^i$. The gauge field tensor $F_{\mu \nu}(x)$ is given by (2) and the covariant derivative $D_\mu$ is defined by

$$D_\mu \varphi = \partial_\mu \varphi + i [B_\mu, \varphi].$$
\[ D_\mu \phi(x) = \delta_\mu \phi(x) + i[A_\mu(x), \phi(x)], \]
for any field \( \phi(x) \) in the adjoint representation of the gauge group. The action is invariant under the infinitesimal transformation,
\[
\delta_\alpha A_\mu(x) = -D_\mu \omega(x) \quad \delta_\alpha \phi(x) = i[\alpha(x), \phi(x)].
\]
The supersymmetry transformation is defined by
\[
\begin{align*}
\delta_\xi A_\mu(x) &= i \xi \sigma_\mu \bar{\lambda}(x) + i \bar{\xi} \bar{\sigma}_\mu \lambda(x) \\
\delta_\xi \lambda(x) &= \sigma_\mu \mu(x) \xi F_{\mu
u}(x) - \xi D(x) \\
\delta_\xi \bar{\lambda}(x) &= \bar{\sigma}_\mu \mu(x) \xi F_{\mu
u}(x) + \bar{\xi} D(x) \\
\delta_\xi D(x) &= i \xi \sigma_\mu \bar{\lambda}(x) - i \bar{\xi} \bar{\sigma}_\mu \lambda(x),
\end{align*}
\]
where \( \xi \) and \( \bar{\xi} \) are two-component Grassmann parameters that do not depend on \( x \). We can show that the action is also invariant under this transformation using several identities given in Appendix 1.

We now define \( \mathcal{N} = 1 \) SYM in the superfield formalism. The supersymmetry transformation of any superfield \( F(x, \theta, \bar{\theta}) \) is given by
\[
\delta_\xi F(x, \theta, \bar{\theta}) = (\xi Q + i \bar{\xi} \bar{Q}) F(x, \theta, \bar{\theta}),
\]
where
\[
Q_\alpha = \frac{\partial}{\partial \theta^\alpha} - i(\sigma_\mu)_{\alpha\beta} \bar{\theta}^\beta \partial_\mu \\
\bar{Q}_\dot{\alpha} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + i \theta^{\dot{\alpha}} (\sigma_\mu)_{\alpha\dot{\beta}} \partial_\mu .
\]

The transformation of each component field can be read from (16). Note that (16) is a linear transformation although (15) is non-linear since \( Q_\alpha \) and \( \bar{Q}_{\dot{\alpha}} \) are linear derivative operators.

The vector superfield \( V(x, \theta, \bar{\theta}) \) is defined as
\[
V(x, \theta, \bar{\theta}) = C(x) + i \theta \eta(x) - i \bar{\theta} \bar{\eta}(x) + \frac{i}{2} \theta \eta (M(x) + i N(x)) - \frac{i}{2} \bar{\theta} \bar{\eta} (M(x) - i N(x)) - \theta \bar{\sigma}_\mu \bar{A}_\mu(x) + i \bar{\theta} \theta \bar{\sigma}_\mu \lambda(x)
\]
\[
+ i \theta \bar{\theta} \left( \bar{\lambda}(x) + i \frac{1}{2} \bar{\sigma}_\mu \partial_\mu \eta(x) \right) \\
- i \theta \bar{\theta} \left( \lambda(x) + i \frac{1}{2} \sigma_\mu \partial_\mu \bar{\eta}(x) \right) \\
+ \frac{i}{2} \theta \bar{\theta} \left( D(x) + 1/2 \Box C(x) \right),
\]
where \( C, D, M, N \) and \( A_\mu \) are real fields, and \( \eta, \bar{\eta}, \lambda, \bar{\lambda} \) are spinor fields. The superfields \( \Phi(x, \theta, \bar{\theta}) \) and \( \bar{\Phi}(x, \theta, \bar{\theta}) \) that satisfy \( D_\alpha \Phi = \bar{D}_{\alpha} \Phi = 0 \) are called chiral superfields where
\[
D_\alpha = \frac{\partial}{\partial \theta^\alpha} + i(\sigma_\mu)_{\alpha\beta} \bar{\theta}^\beta \partial_\mu \\
\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i \theta^{\dot{\alpha}} (\sigma_\mu)_{\alpha\dot{\beta}} \partial_\mu
\]
which are covariant derivatives because they commute with \( Q_\alpha \) and \( \bar{Q}_{\dot{\alpha}} \).

The \( \mathcal{N} = 1 \) SYM action is then given by
\[
S_{\text{SYM}} = -\int d^4 \times \frac{1}{2g^2} \text{tr} \left( W^\alpha W_\alpha \big|_{\delta \theta \delta \bar{\theta}} + \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \big|_{\delta \theta \delta \bar{\theta}} \right),
\]
where
\[
W_\alpha = -\frac{1}{8} \bar{D} e^{-2V} D_\alpha e^{2V} \\
\bar{W}_{\dot{\alpha}} = \frac{1}{8} D e^{2V} \bar{D}_{\dot{\alpha}} e^{-2V}.
\]
\( W_\alpha \) and \( \bar{W}_{\dot{\alpha}} \) are chiral superfields since \( D_\alpha \bar{W}_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}} W_\alpha = 0 \). The integrals of \( \theta \theta \) and \( \bar{\theta} \bar{\theta} \) components in (20) are invariant under linear supersymmetry transformation (16).

The symmetry of the superfield action (20) is higher than that of (12), (20) is indeed invariant under an extended gauge transformation,
\[
e^{2V'} = e^{-\bar{\Lambda}} e^{2V} e^{i \Lambda} \\
e^{-2V'} = e^{-\bar{\Lambda}} e^{-2V} e^{-i \Lambda},
\]
where \( \bar{D}_\alpha \Lambda = D_\alpha \bar{\Lambda} = 0 \). As the \( \theta \) and \( \bar{\theta} \) expansions, we have
\[
\Lambda(y, \theta) = A(y) + \sqrt{2} \theta \psi(y) + i \theta \theta F(y) \\
\bar{\Lambda}(\bar{y}, \bar{\theta}) = \bar{A}(\bar{y}) + \sqrt{2} \bar{\theta} \bar{\psi}(\bar{y}) + i \bar{\theta} \bar{\theta} \bar{F}(\bar{y})
\]
where \( y_\mu = x_\mu + i \theta \sigma_\mu \bar{\theta} \) and \( \bar{y}_\mu = \bar{x}_\mu - i \sigma_\mu \theta \). We should note that (22) is a non-linear transformation.

We can set
\[
C = \eta = \bar{\eta} = M = N = 0,
\]
choosing the component fields of \( \Lambda \) and \( \bar{\Lambda} \) by hand. This is so-called Wess–Zumino gauge fixing. The vector superfield is then given only by the field variables in (12):
\[
V_{\text{WZ}}(x, \theta, \bar{\theta}) = -i \theta \sigma_\mu \bar{\theta} A_\mu(x) + i \theta \bar{\theta} \bar{\sigma}_\mu \lambda(x) - i \theta \bar{\theta} \theta \lambda(x) + \frac{i}{2} \theta \bar{\theta} \theta \bar{\theta} D(x).
\]

Consequently,
\[
W_\alpha(y, \theta) = -i \lambda_\alpha(y) + i \theta \sigma_\mu \theta A_\mu(y) - i (\sigma_\mu \theta)_{\alpha} F_{\mu\nu}(y) + \theta \theta (\sigma_\mu \bar{D}_\mu \lambda)(y) \\
\bar{W}_{\dot{\alpha}}(\bar{y}, \bar{\theta}) = i \bar{\lambda}^{\dot{\alpha}}(\bar{y}) + i \bar{\theta} \bar{\theta} D(\bar{y}) + i (\bar{\sigma}_\mu \bar{\theta})^{\dot{\alpha}} F_{\mu\nu}(\bar{y}) - \bar{\theta} \bar{\theta} (\bar{\sigma}_\mu D_\mu \lambda)^{\dot{\alpha}}(\bar{y})
\]
and the superfield action (20) coincides with (12).

The original linear supersymmetry transformation \( \delta_\xi \) breaks the Wess–Zumino gauge. Another supersymmetry transformation that keeps the gauge is defined by adding an infinitesimal extended gauge transformation \( \delta_{\text{gauge}}^{\Lambda} \) to \( \delta_\xi \) [21]:
\[
\delta_\xi = \delta_\xi^{\text{gauge}}.
\]
3.2 Original derivation of the SYM flow equation

We use $t \geq 0$ as a flow time and define $t$-dependent vector superfield $V(t, x, \theta, \bar{\theta})$ as

$$V(t, x, \theta, \bar{\theta}) = V(x, \theta, \bar{\theta}) \bigg|_{field \ replacement}$$

with the field replacements,

$$C(x) \rightarrow C'(t, x)$$
$$\eta(x) \rightarrow \eta'(t, x)$$
$$\tilde{\eta}(x) \rightarrow \tilde{\eta}'(t, x)$$
$$M(x) \rightarrow M'(t, x)$$
$$N(x) \rightarrow N'(t, x)$$
$$A_{\mu}(x) \rightarrow B_{\mu}(t, x)$$
$$\lambda(x) \rightarrow \chi(t, x)$$
$$\tilde{\lambda}(x) \rightarrow \tilde{\chi}(t, x)$$
$$D(x) \rightarrow H(t, x)$$

and the boundary condition $V(t, x, \theta, \bar{\theta})|_{t=0} = V(x, \theta, \bar{\theta})$.

In order to define a gradient flow equation such that it is covariant under linear supersymmetry and the extended gauge transformations, we define the metric that is invariant under these two transformations:

$$||\delta V||^2 = \int d^8 z \frac{1}{2} \text{tr} \left( e^{-2V} \delta e^{2V} \cdot e^{-2V} \delta e^{2V} \right),$$

where $z = (x, \theta, \bar{\theta})$ and $d^8 z = d^4 x d^2 \theta d^2 \bar{\theta}$. The metric $g_{ab}$ is read from (30) as $||\delta V||^2 = \int d^8 z g_{ab}(V) \delta V^a \delta V^b$. With $g^{ac} g_{cb} = \delta_{ab}$, we thus have

$$g^{ab}(V) = 4 \text{tr} \left\{ T^a \frac{L^2_V}{\cosh(2L^2_V)} - T^b \right\},$$

where $L_{A B} \equiv [A, B]$.

The gradient flow equation for $V$ is given in terms of superfield as

$$\partial_t V^a(t, z) = \frac{1}{2} g^{ab} \frac{\delta S_{SYM}}{\delta V^b} (V_{(z)} \rightarrow V(t, z))$$

where $\delta \delta V^b(x, \theta, \bar{\theta})$ is a functional derivative. After a short calculation, (32) is also expressed as

$$\partial_t V = \frac{L_V}{1 - e^{-2L_V}} \left( \frac{1}{2} D^a W_a \right)$$
$$- \frac{L_V}{1 - e^{-2L_V}} \left( \frac{1}{2} \tilde{D}_{\alpha} \tilde{W}^\alpha \right) \bigg|_{V \rightarrow V}$$

with

$$D^a W_a = D^a W_a + \{ e^{-2V} (D^a e^{2V}), W_a \},$$
$$\tilde{D}_{\alpha} \tilde{W}^\alpha = \tilde{D}_{\alpha} \tilde{W}^\alpha + \{ e^{2V} (\tilde{D}_{\alpha} e^{-2V}), \tilde{W}^\alpha \},$$

which are covariant under super and gauge transformations.

With and without the metric, (32) is covariant under $t$-independent supersymmetry transformation which is obtained by replacing $F(z)$ in (16) with $V(t, z)$ since the both sides are given in terms of superfield. However the extended gauge transformation is broken without the metric since it is non-linear transformation. With the metric, (32) is also covariant for the extended gauge transformation as in the case of general covariance.

3.3 The issue of SYM flow equation in the Wess–Zumino gauge

The Wess–Zumino gauge fixing (24) is broken by the time evolution because the breaking term appear in R.H.S of (32). The authors of Ref. [18] have tried to construct the flow equation in the gauge adding a fixing term given by an extended gauge transformation to (32):

$$\partial_t V^a(t, z) = \frac{1}{2} g^{ab} \frac{\delta S_{SYM}}{\delta V^b} + \delta_{\Lambda}^{\text{gauge}} V^a \bigg|_{V(z) \rightarrow V(t, z)},$$

where

$$\delta_{\Lambda}^{\text{gauge}} V = \frac{L_V}{1 - e^{-2L_V}} (i \Lambda) + \frac{L_V}{1 - e^{2L_V}} (i \bar{\Lambda}),$$

which is the infinitesimal form of (22).

In Ref. [18], the gauge parameters $\Lambda$ and $\bar{\Lambda}$ are chosen such that the following three conditions hold:

(i) $\Lambda$ and $\bar{\Lambda}$ are functions of $V$.
(ii) The flow equation is invariant under $\delta_\xi$.
(iii) The WZ gauge is kept at any flow time. The unique solution of these three conditions is given by

$$i \Lambda = \frac{1}{8} \delta^2 (D^2 V + 2[D^2 V, V])$$
$$i \bar{\Lambda} = \frac{1}{8} \delta^2 (\bar{D}^2 V - 2[\bar{D}^2 V, V]).$$

Substituting (37) into (35) and setting $V$ at $t = 0$ to $V_{WZ}$, the flow equation is defined in terms of component fields.

We make a few comments on the flow equation in the Wess–Zumino gauge with (37). In the case of the Yang–Mills flow (7), $\alpha$ is a free parameter and the flow equation is gauge covariant if we set $\alpha = 0$. However, such gauge covariance is broken in the present case due to the choice (37). With the time dependent gauge transformation (8), the solution with any non-zero $\alpha$ is equivalent to that without the gauge fixing term ($\alpha = 0$) in the Yang–Mills flow. The corresponding supersymmetric relation is, however, unknown. Furthermore, it is not clear whether or not the flow equation is consistent.
with SUSY transformation in the Wess–Zumino gauge \((15)\) [22].

We will reconstruct the flow equation in the Wess–Zumino gauge and study the symmetry properties in the next section.

4 New derivation of gradient flow equation in the Wess–Zumino gauge

We propose a method of defining a SYM-gradient flow in the Wess–Zumino gauge and show that the constructed flow is supersymmetric in the sense that the five-dimensional supersymmetry and the flow-time derivative commute with each other up to a gauge transformation.

4.1 Supersymmetric gradient flow

In the prescription of the Wess–Zumino gauge fixing, the linear supersymmetry transformation \(\delta^E_x\) defined in \((27)\) is broken. The component fields of chiral superfields, which are parameters of extended gauge transformation, are chosen by hand to keep the gauge. So we can say that \((i)\) and \((ii)\) in Sect. 3.2 are too strong conditions which are unnecessary.

We impose only the condition \((iii)\) and choose the component fields of \(A\) and \(\tilde{A}\) by hand to keep the Wess–Zumino gauge at any flow time:

\[
A = D - \theta \\
\tilde{A} = -D - \theta \\
\psi = \sqrt{2i} \sigma_\mu D_\mu \lambda \\
\tilde{\psi} = \sqrt{2i} \tilde{\sigma}_\mu D_\mu \lambda \\
F = \tilde{F} = 0, \tag{38}
\]

where \(\theta^a(t, x)\) is any real function. We thus obtain a flow equation in the Wess–Zumino gauge:

\[
\partial_t C' = \partial_t \eta' = \partial_t \tilde{\eta}' = \partial_t M' = \partial_t N' = 0, \tag{39}
\]

and

\[
\partial_t B_\mu = D_\mu G_{\nu \mu} - (\tilde{\chi} \bar{\sigma}_\mu \chi + \chi \sigma_\mu \tilde{\chi}) - D_\mu \theta \\
\partial_t \chi = -\sigma_\mu \bar{\sigma}_\nu D_\mu \chi - i[\chi, H] + i[\theta, \chi] \\
\partial_t \tilde{\chi} = -\tilde{\sigma}_\mu \sigma_\nu D_\mu \tilde{\chi} + i[\tilde{\chi}, H] + i[\tilde{\theta}, \tilde{\chi}] \\
\partial_t H = D_\mu D_\mu H \\
+ (D_\mu \chi \sigma_\nu D_\mu \chi + \tilde{\chi} \bar{\sigma}_\mu D_\mu \tilde{\chi} - D_\mu \bar{\chi} \sigma_\mu \tilde{\chi} - \chi \sigma_\mu D_\mu \chi) \\
+ i[\theta, H]. \tag{40}
\]

The Wess–Zumino gauge fixing maintains at any flow time as shown in \((39)\). Equation \((38)\) leads to non-linear couplings of fields in \((40)\). The flow equation for the fermion is given as \(D^2 \psi\) instead of \(D_\mu D_\mu \psi\) which is proposed in Ref. [4]. The second terms of R.H.S. of \(\partial_t B_\mu\) and \(\partial_t H\) is actually proportional to the group generator noting that \(\tilde{\chi}^a \bar{\sigma}_\mu \chi^b = -\chi^b \sigma_\mu \tilde{\chi}^a\).

We do not have any constraint on \(\theta\) that is the real part of \(A\) in the Minkowski space, which corresponds to the degree of freedom of the ordinary gauge transformation \((14)\). Thus the Wess–Zumino gauge of SYM-flow equation is independently taken from the ordinary gauge transformation. Indeed, if we set \(\theta = 0\), the flow equation is manifestly gauge covariant for \(t\)-independent gauge transformations. As in the case of Yang–Mills flow, we can choose any gauge fixing term such as \(\theta = -\alpha \partial_\mu B_\mu\) with any parameter \(\alpha\) although the flow equation of Ref. [18] is given with a specific gauge fixing term \(\theta = -\partial_\mu B_\mu\) \((\alpha = 1)\).

4.2 Symmetry of the SYM flow equation

Let us study the symmetry properties of the flow equation.

As a natural extension, five dimensional supersymmetry can be defined replacing \(A_\mu, \lambda, \tilde{\lambda}, D\) in \((15)\) with corresponding flow fields:

\[
\delta_x B_\mu = i\xi \sigma_\mu \tilde{\chi} + i\bar{\xi} \tilde{\sigma}_\mu \chi \\
\delta_x \chi = \sigma_\mu \xi G_{\mu \nu} - \xi H \\
\delta_x \tilde{\chi} = \tilde{\sigma}_\mu \bar{\xi} G_{\mu \nu} + \bar{\xi} H \\
\delta_x H = i\xi \sigma_\mu D_\mu \chi - i\bar{\xi} \tilde{\sigma}_\mu D_\mu \chi \tag{41}
\]

One can naively expect that \(\partial_t \delta_x = \delta_x \partial_t\) if supersymmetry and the flow are consistent with each other.

After some calculations, we find that the commutation relation does not hold for \(V_{WZ}\) and we actually have

\[
(\partial_t \delta_x - \delta_x \partial_t) V_{WZ} = \delta^E_x V_{WZ}, \tag{42}
\]

where \(\delta^E_x\) represents the ordinary gauge transformation with the gauge transformation function,

\[
\omega = i D_\mu (\xi \sigma_\mu \tilde{\chi} + \bar{\xi} \tilde{\sigma}_\mu \chi) + \xi \theta, \tag{43}
\]

where the gauge transformation of flowed fields are replacing \(\omega(x)\) in \((14)\) by \(\omega(t, x)\). The relation \((42)\) implies that the flow and supersymmetry are consistent with each other for any gauge invariant operator since R.H.S. vanishes. In this sense, the constructed flow equation is supersymmetric.

The gauge covariance \((8)\) with \((9)\) also holds for the SUSY flow equation \((40)\). We can show that two solutions of the flow equations with and without \(\theta\) are related as

\[
B_\mu = \Lambda \left( B_\mu |_{\theta = 0} - i \partial_\mu \right) \Lambda^{-1} \\
\psi = \Lambda \psi |_{\theta = 0} \Lambda^{-1}, \tag{44}
\]

where \(\partial_t \Lambda = i \theta \Lambda\) and \(\psi = \chi, \tilde{\chi}, H\). The existence of this transformation suggests us that we can take any \(\theta\) in perturbative analyses of the flow equation. Without fixing the WZ gauge, this kind of relation also holds for \((35)\):

\[
ed^2 V = e^{-i \tilde{\xi} D_\mu \psi} e^{2V} \bigg|_{\Lambda = 0} e^{i \tilde{\xi} D_\mu \psi}, \tag{45}
\]
where $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ obey
\[ \partial_t e^{i\mathcal{Z}} = i e^{i\mathcal{Z}} \Lambda, \quad \partial_t e^{-i\tilde{\mathcal{Z}}} = -i \tilde{\Lambda} e^{-i\tilde{\mathcal{Z}}}. \] (46)

Note that (44) is also derived from (45) and (46) setting the WZ gauge.

We find another relation which is a supersymmetric version of (10) and (11). The SYM flow equation (40) is covariant under a time dependent gauge transformation,
\[ B_\mu^\omega = B_\mu - D_\mu \omega, \]
\[ \varphi^\omega = \varphi + i[\omega, \varphi], \] (47)

where $\omega(t, x)$ obeys $\partial_t \omega = \delta_\omega^\phi \theta + i[\theta, \omega]$. Here $\theta$ is assumed to be a function of the component fields, such as $\theta = -\alpha \partial_\mu B_\mu$. For the superfields, it is easily shown that $\mathcal{V}_\mathcal{Z}$ defined by
\[ e^{2\mathcal{V}_\mathcal{Z}} = e^{2\mathcal{V}} - i \tilde{\mathcal{Z}} e^{2\mathcal{V}} + i e^{2\mathcal{V}} \mathcal{Z}, \] (48)

is another solution of the fixed Eq. (35) for infinitesimal transformation parameters $\mathcal{Z}$, $\tilde{\mathcal{Z}}$ satisfying
\[ \partial_t \mathcal{Z} = i[\mathcal{Z}, A] + \delta_\mathcal{Z} \Lambda, \]
\[ \partial_t \tilde{\mathcal{Z}} = i[\tilde{\mathcal{Z}}, \tilde{A}] + \delta_{\tilde{\mathcal{Z}}} \tilde{\Lambda}, \] (49)

where $\delta_\mathcal{Z} \Lambda = \Lambda(\mathcal{V}_\mathcal{Z}) - \Lambda(\mathcal{V})$. Note again that (47) is derived from (48) and (49) setting the WZ gauge.

5 Summary

We have defined a gradient flow equation in $\mathcal{N} = 1$ supersymmetric Yang–Mills theory in terms of the component fields of the Wess–Zumino gauge. We have shown that the obtained flow is consistent with supersymmetry (de Wit-Freedman transformation) such that the commutator of time-derivative and five-dimensional SUSY transformation vanishes up to a gauge transformation. In this sense, one can say that the flowed gauge multiplet in the Wess–Zumino gauge remains as a supermultiplet at non-zero flow time.

Although the flowed adjoint fermion and the auxiliary field receive extra renormalizations for the non-SUSY flow, one can expect that there are no such extra renormalizations in the supersymmetric flow. This is the direct consequence that the renormalization property of the gauge field is extended to $\mathcal{N} = 1$ gauge multiplet thanks to five-dimensional supersymmetry. One can also expect that the energy-momentum tensor, supercurrent and R-current form a five-dimensional Ferrara-Zumino multiplet. The SUSY flow is thus very attractive, but the actual renormalization property is still not clear and further studies are needed.

The SUSY gradient flow of the same kind can be defined in various SUSY models. The method presented in this paper could be very useful to build the theory of SUSY flow in $\mathcal{N} = 1$ SQCD and $\mathcal{N} = 2$ theories. But it would not be straightforward for $\mathcal{N} = 4$ SYM since $\mathcal{N} = 4$ theory has no off-shell formulation which is important in showing that the flow is supersymmetric. The SUSY flow can also be given for the low-dimensional SUSY theories and the Wess–Zumino model. We can say that the SUSY flow is a systematic and promising approach and will be widely used in the actual lattice computations in the near future.

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B Supersymmetry in four dimensional Euclidean space

We basically follow the convention of Ref. [23] with the Wick rotation to four-dimensional Euclidean space. The Minkowski time \( t \) is replaced as \( t \rightarrow -it \) where \( t = x^0 \) and \( \partial_t \rightarrow i\partial_t \). The gauge field \( A_0 \) and the auxiliary fields \( X \) are then replaced as \( A_0 \rightarrow iA_0, X \rightarrow iX \). After these replacements we identify the Minkowski action \( S(M) \) as the Euclidean action \( iS(E) \).

B.1 Convention and spinor algebra in Euclidean space

In the Euclidean space \( \psi_\alpha \) and \( \bar{\psi}_\bar{\alpha} \) (\( \alpha = 1, 2 \)) transform as independent spinors under \( SU(2)_R \) and \( SU(2)_L \) group and they are not related with each other under the complex conjugate. We define the invariant tensors of \( SU(2)_R \) and \( SU(2)_L \) as

\[
\epsilon_{21} = \epsilon^{12} = \epsilon_{\hat{2}\hat{1}} = \epsilon^{\hat{2}\hat{1}} = 1, \quad \epsilon_{12} = \epsilon^{21} = \epsilon_{\hat{1}\hat{2}} = \epsilon^{\hat{1}\hat{2}} = -1
\]

(54)

with the others are zero. Note that \( \epsilon_{\alpha\beta}\epsilon_{\gamma\gamma} = \delta_{\alpha\gamma} \) and \( \epsilon_{\hat{\alpha}\hat{\beta}}\epsilon_{\hat{\gamma}\hat{\gamma}} = \delta_{\hat{\alpha}\hat{\gamma}} \). Then

\[
\psi_\chi \equiv \psi_\alpha \chi_\alpha, \quad \bar{\psi}_\bar{\chi} \equiv \bar{\psi}_{\bar{\alpha}} \bar{\chi}_{\bar{\alpha}}
\]

are Lorenz scalars. We use a totally anti-symmetric tensor \( \epsilon_{\mu\nu\rho\sigma} \) with \( \epsilon_{0123} = -1 \).

The four dimensional sigma matrices in the Euclidean space \( (\sigma_\mu)_\alpha^\beta \) and \( (\bar{\sigma}_\mu)^\alpha_\beta \) are defined as

\[
\sigma_0 = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

(55)

\[
(\bar{\sigma}_\mu)^\alpha_\beta = \epsilon^{\dot{\beta}\dot{\alpha}}(\sigma_\mu)^{\dot{\alpha}}_{\dot{\beta}}, \quad \bar{\sigma}_0 = \sigma_0, \quad \bar{\sigma}_i = -\sigma_i \quad (i = 1, 2, 3).
\]

Note that the matrix elements of \( \sigma_\mu \) are different from those of Minkowski’s \( \sigma_\mu \).\(^3\) As with (55), the contractions of \( \sigma_\mu \bar{\psi} \) and \( \bar{\sigma}_\mu \psi \) mean \( \chi^\alpha(\sigma_\mu)_\alpha^\beta \bar{\psi}_\beta \) and \( \bar{\chi}_\alpha(\bar{\sigma}_\mu)^\alpha_\beta \psi_\beta \), respectively.

We present the useful identities for sigma matrices,

\[
\begin{align*}
\text{Tr}(\sigma_\mu \bar{\sigma}_\nu) &= -2\delta_{\mu\nu}, \\
(\sigma_\mu)_\alpha^\beta (\bar{\sigma}_\nu)^\beta_\alpha &= -2\delta_{\alpha\beta} \delta_{\mu\nu}, \\
(\sigma_\mu \bar{\sigma}_\nu + \bar{\sigma}_\mu \sigma_\nu)_\alpha^\beta &= -2\delta_{\mu\nu} \delta_{\alpha\beta}, \\
(\bar{\sigma}_\mu \sigma_\nu + \sigma_\mu \bar{\sigma}_\nu)^\alpha_\beta &= -2\delta_{\mu\nu} \delta_{\alpha\beta}.
\end{align*}
\]

(56)

\[^{3}\] They are related as \( \sigma_0^{(E)} = -i\sigma_0^{(M)}, \bar{\sigma}_0^{(E)} = -i\bar{\sigma}_0^{(M)} \), and \( \sigma_j^{(E)} = \sigma_j^{(M)}, \bar{\sigma}_j^{(E)} = \bar{\sigma}_j^{(M)} \) for \( j = 1, 2, 3 \).

and the generators of \( SO(4) \) in the spinor representation,

\[
\begin{align*}
(\sigma_{\mu\nu})_\alpha^\beta &= \frac{1}{4}((\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu)_\alpha^\beta, \\
(\bar{\sigma}_{\mu\nu})^\alpha_\beta &= \frac{1}{4}((\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu)^\alpha_\beta.
\end{align*}
\]

(59)

The identities for \( \sigma_{\mu\nu} \) and two and three \( \sigma \) are

\[
\begin{align*}
(\sigma_{\mu\nu})_\alpha^\beta &= (\sigma_{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}}, \\
(\sigma_{\mu\nu})^\alpha_\beta &= (\sigma_{\mu\nu})_{\alpha}^{\dot{\beta}} = 0, \\
(\sigma_{\mu\nu})^\alpha_\beta \epsilon^\gamma_\mu &= (\sigma_{\mu\nu})_{\alpha}^{\dot{\gamma}} \epsilon^\dot{\gamma}_\mu = (\sigma_{\mu\nu})_{\alpha}^{\dot{\gamma}} \epsilon^\dot{\gamma}_\mu. \\
(\sigma_{\mu\nu})_{\alpha}^{\dot{\gamma}} &= 2\sigma_{\mu\nu} \delta_{\alpha\gamma} - 2\epsilon_{\mu\nu\rho\sigma} \sigma_{\rho\sigma} = -2\sigma_{\mu\nu} \delta_{\alpha\gamma}, \\
(\sigma_{\mu\nu})_{\alpha}^{\dot{\gamma}} &= 2(\sigma_{\mu\nu})_{\alpha}^{\dot{\gamma}} - \delta_{\mu\nu} \delta_{\alpha\gamma} \\
(\sigma_{\mu\nu})_{\alpha}^{\dot{\gamma}} &= 2(\sigma_{\mu\nu})_{\alpha}^{\dot{\gamma}} - \delta_{\mu\nu} \delta_{\alpha\gamma} \\
\epsilon_{\mu\nu\rho\sigma} \sigma_{\rho\sigma} &= -2\epsilon_{\mu\nu\rho\sigma} \\
\epsilon_{\mu\nu\rho\sigma} \sigma_{\rho\sigma} &= -2\epsilon_{\mu\nu\rho\sigma}; \\
\epsilon_{\mu\nu\rho\sigma} \sigma_{\rho\sigma} &= -2\epsilon_{\mu\nu\rho\sigma} \\
\epsilon_{\mu\nu\rho\sigma} \sigma_{\rho\sigma} &= -2\epsilon_{\mu\nu\rho\sigma}.
\end{align*}
\]

(60)

The results of spinor algebra are given as follows:

\[
\begin{align*}
\theta^\alpha \bar{\theta}^\beta &= -\frac{1}{2} \epsilon^{\alpha\beta} \theta \bar{\theta}, \\
\theta^\alpha \bar{\theta}^\beta &= \frac{1}{2} \epsilon_{\alpha\beta} \theta \bar{\theta}, \\
\bar{\theta}^\alpha \bar{\theta}^\beta &= \frac{1}{2} \epsilon^{\alpha\beta} \theta \bar{\theta}, \\
\theta^\alpha \bar{\theta}^\beta &= \frac{1}{2} \epsilon_{\alpha\beta} \theta \bar{\theta}. \\
\theta^\alpha \bar{\theta}^\beta(\theta \bar{\theta}) &= -\frac{1}{2} \theta \bar{\theta} \theta \bar{\theta} \delta_{\mu\nu}, \\
\theta^\alpha \bar{\theta}^\beta &= -\frac{1}{2} \theta \bar{\theta} \theta \bar{\theta} \delta_{\mu\nu}. \\
\theta^\alpha \bar{\theta}^\beta &= -\frac{1}{2} \theta \bar{\theta} \theta \bar{\theta} \delta_{\mu\nu}.
\end{align*}
\]

(61)

\[
\begin{align*}
\epsilon_{\mu\nu\rho\sigma} \bar{\theta}^\alpha \bar{\theta}^\beta &= \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \theta \bar{\theta}\theta \bar{\theta} \delta_{\alpha\beta}, \\
\epsilon_{\mu\nu\rho\sigma} \theta \bar{\theta}^\alpha \bar{\theta}^\beta &= \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \theta \bar{\theta}\theta \bar{\theta} \delta_{\alpha\beta}. \\
\theta^\alpha \bar{\theta}^\beta(\theta \bar{\theta}) &= -\frac{1}{2} \theta \bar{\theta} \theta \bar{\theta} \delta_{\mu\nu}, \\
\theta^\alpha \bar{\theta}^\beta &= \frac{1}{2} \theta \bar{\theta} \theta \bar{\theta} \delta_{\mu\nu}. \\
\theta^\alpha \bar{\theta}^\beta &= \frac{1}{2} \theta \bar{\theta} \theta \bar{\theta} \delta_{\mu\nu}.
\end{align*}
\]

(62)

For anti-commuting spinors \( \phi, \psi, \bar{\phi}, \bar{\psi} \), we have

\[
(\theta \phi)(\theta \psi) = -\frac{1}{2} (\theta \theta)(\phi \psi).
\]

\[^{3}\] Springer
B.2 Chiral and vector superfields

The superfield $F$ is defined as a Lorentz covariant function of $x, \theta, \bar{\theta}$ with supersymmetry transformation,

$$\delta_{\xi} F(x, \theta, \bar{\theta}) = (\xi Q + \bar{\xi} \bar{Q}) F(x, \theta, \bar{\theta}),$$

where $\xi_x$ and $\bar{\xi}_\bar{x}$ are two component Grassmann global parameters and the difference operators $Q_a, \bar{Q}_a$ are defined by

$$Q_a = \frac{\partial}{\partial \theta^a} - i(\sigma_\mu)_{a\bar{a}} \bar{\theta}^\bar{a} \partial_\mu,$$

$$\bar{Q}_a = -\frac{\partial}{\partial \bar{\theta}^a} + i \theta^a(\sigma_\mu)_{a\bar{a}} \partial_\mu.$$

The associated super covariant derivatives which commute with $Q_a$ and $\bar{Q}_a$ are defined as

$$D_a = \frac{\partial}{\partial \theta^a} + i(\sigma_\mu)_{a\bar{a}} \bar{\theta}^\bar{a} \partial_\mu,$$

$$\bar{D}_{\bar{a}} = -\frac{\partial}{\partial \bar{\theta}^\bar{a}} - i \theta^\bar{a}(\sigma_\mu)_{a\bar{a}} \partial_\mu.$$

These difference operators obey

$$\{Q_a, \bar{Q}_b\} = 2i(\sigma_\mu)_{a\bar{a}} \bar{\theta}^\bar{a} \partial_\mu,$$

$$\{D_a, \bar{D}_{\bar{b}}\} = -2i(\sigma_\mu)_{a\bar{a}} \partial_\mu,$$

and the other anti-commutation relations vanish.

The supersymmetry transformation of each component field is obtained by expanding $F(x, \theta, \bar{\theta})$ with respect to $\theta$ and $\bar{\theta}$ and comparing the coefficients of the $\theta$-expansion between the both sides of (84).

The chiral and anti-chiral superfields $\Phi(x, \theta, \bar{\theta})$ and $\bar{\Phi}(x, \theta, \bar{\theta})$ are defined as superfields that satisfy $\bar{D}_a \Phi = 0$ and $D_a \bar{\Phi} = 0$, respectively. They are expanded in $\theta$ and $\bar{\theta}$ as

$$\Phi(x, \theta, \bar{\theta}) = A(x) + i \sigma_\mu \bar{\theta} \partial_\mu A(x) + \frac{1}{4} \theta \bar{\theta} \partial^2 A(x)$$

$$+ \sqrt{2} i \theta \bar{\theta} \psi(x) \sigma_\mu \bar{\theta} + i \theta \bar{\theta} F(x),$$

$$\bar{\Phi}(x, \theta, \bar{\theta}) = \bar{A}(x) - i \sigma_\mu \theta \partial_\mu \bar{A}(x) + \frac{1}{4} \theta \bar{\theta} \partial^2 \bar{A}(x)$$

$$+ \sqrt{2} i \theta \bar{\theta} \bar{\psi}(x) + i \theta \bar{\theta} \bar{F}(x),$$

where $A, \bar{A}, F$ and $\bar{F}$ are independent complex bosonic fields, and $\psi, \bar{\psi}$ are two component fermions. Although $\bar{\Phi}$ is complex conjugate of $\Phi$ in the Minkowski space, that relation is broken in the Euclidean space.
Introducing new coordinate \((y, \theta, \bar{\theta})\) with \(y_\mu = x_\mu + i \partial \sigma_\mu \bar{\theta}\), the derivative operators and \(\Phi\) are expressed as

\[
Q_a = \frac{\partial}{\partial \theta^a}, \quad Q_a = -\frac{\partial}{\partial \bar{\theta}^a} + 2i \theta^a (\sigma_\mu)_{ab} \bar{\theta}^b \partial_\mu \\
D_a = \frac{\partial}{\partial \theta^a} + 2i (\sigma_\mu)_{ab} \bar{\theta}^b \partial_\mu \\
\bar{D}_a = -\frac{\partial}{\partial \bar{\theta}^a} - 2i \theta^a (\sigma_\mu)_{ab} \bar{\theta}^b \partial_\mu
\]

(90)

while in \((\bar{y}, \theta, \bar{\theta})\) with \(\bar{y}_\mu = x_\mu - i \theta \sigma_\mu \bar{\theta}\),

\[
\bar{Q}_a = \frac{\partial}{\partial \bar{\theta}^a}, \quad \bar{Q}_a = -\frac{\partial}{\partial \theta^a} + 2i \bar{\theta}^a (\sigma_\mu)_{ab} \theta^b \partial_\mu \\
\bar{D}_a = \frac{\partial}{\partial \bar{\theta}^a} + 2i \bar{\theta}^a (\sigma_\mu)_{ab} \theta^b \partial_\mu \\
\bar{D}_a = -\frac{\partial}{\partial \theta^a} + 2i \theta^a (\sigma_\mu)_{ab} \bar{\theta}^b \partial_\mu
\]

(91)

Note that \(\bar{y}\) is not a complex conjugate of \(y\) in the Euclidean space.

The vector superfield is defined as

\[
V(x, \theta, \bar{\theta}) = C(x) + i \theta \eta(x) - i \bar{\theta} \bar{\eta}(x)
\]

\[
\quad + \frac{i}{2} \theta \eta (M(x) + i N(x)) - \frac{i}{2} \bar{\theta} \bar{\eta} (M(x) - i N(x))
\]

\[
- \theta \sigma_\mu \bar{\theta} A_\mu(x)
\]

\[
+ i \bar{\theta} \bar{\theta} \left( \bar{\lambda}(x) + \frac{i}{2} \sigma_\mu \partial_\mu \eta(x) \right)
\]

\[
- i \theta \theta \left( \lambda(x) + \frac{i}{2} \sigma_\mu \partial_\mu \bar{\eta}(x) \right)
\]

\[
+ \frac{i}{2} \theta \bar{\theta} \theta \bar{\theta} \left( i D(x) + \frac{1}{2} \square C(x) \right)
\]

(92)

where \(C, D, M, N\) and \(A_\mu\) are real bosonic fields, and \(\eta, \bar{\eta}, \lambda, \bar{\lambda}\) are spinor fields. The Euclidean vector superfield (92) is obtained by the Wick rotation of vector superfield in the Minkowski space \(V_M\) which satisfies \(V_M^\dagger = V_M\).

B.3 \(\mathcal{N} = 1\) super Yang–Mills in four dimensional Euclidean space

In the gauge theory, the vector superfield is a matrix-valued field as

\[
V = V^a T^a
\]

(93)

The \(N = 1\) SYM action is given in terms of vector superfield:

\[
S_{\text{SYM}} = - \int d^4x \frac{1}{2g^2} \text{tr} \left( W^a W_a \big|_{\partial_\theta} + \tilde{W}_a \tilde{W}_a \big|_{\bar{\partial}_\bar{\theta}} \right)
\]

(94)

where

\[
W_a = - \frac{1}{8} \bar{D} e^{-2V} D_a e^{2V}
\]

\[
\tilde{W}_a = \frac{1}{8} D e^{2V} \bar{D}_a e^{-2V}
\]

(95)

which are chiral superfields since \(\bar{D}_a W_\beta = D_a \tilde{W}_\beta = 0\).

The extended gauge transformation is defined by

\[
e^{2V} \rightarrow e^{2V'} = e^{-i \tilde{A}} e^{2V} e^{i \Lambda}
\]

\[
e^{-2V} \rightarrow e^{-2V'} = e^{-i \Lambda} e^{-2V} e^{i \tilde{A}}
\]

(96)

where

\[
\bar{D}_a A = D_a \tilde{A} = 0
\]

(97)

with \(A = A^a T^a\) and \(\tilde{A} = \tilde{A}^a T^a\). The chiral superfields \(W_a\) and \(\tilde{W}_a\) transform in a covariant manner under the extended gauge transformation (96).

\[
W_a \rightarrow W_a' = e^{-i \Lambda} W_a e^{i \tilde{A}}
\]

\[
\tilde{W}_a \rightarrow \tilde{W}_a' = e^{-i \tilde{A}} \tilde{W}_a e^{i \Lambda}
\]

(98)

(99)

The infinitesimal extended gauge transformation is expressed as

\[
\delta V = V' - V
\]

\[
= \frac{i}{2} \mathcal{L}_V \cdot [(\Lambda + \tilde{A}) + \coth(\mathcal{L}_V) \cdot (\Lambda - \tilde{A})]
\]

(100)

(101)

where the Lie derivative \(\mathcal{L}_V \cdot \Lambda = [V, \Lambda]\).

We can eliminates \(C, M, N, \eta, \bar{\eta}\) fields in (92) by choosing the component fields of \(\Lambda, \tilde{\Lambda}\) as the Wess–Zumino gauge fixing. The vector superfield in the Wess–Zumino gauge is given by

\[
V_{\text{WZ}}(x, \theta, \bar{\theta}) = - \theta \sigma_\mu \bar{\theta} A_\mu(x) + i \theta \bar{\theta} \bar{\theta} \lambda(x) - i \bar{\theta} \theta \lambda(x)
\]

\[
+ \frac{i}{2} \theta \bar{\theta} \theta \bar{\theta} \left( i D(x) + \frac{1}{2} \square C(x) \right)
\]

(102)

Then,

\[
W_a(y, \theta) = - i \lambda_\alpha(y) + i \partial_\alpha D(y) - i (\sigma_\mu \theta_\alpha) F_{\mu\nu}(y)
\]

\[
+ \bar{\theta} \theta (\sigma_\mu D_\mu \lambda_\alpha)(y)
\]

\[
\tilde{W}_a(y, \bar{\theta}) = i \bar{\lambda}^\alpha (y) + i \bar{\theta} \bar{\theta} D(y) + i (\bar{\sigma}_\mu \bar{\theta} \bar{\lambda}^\alpha) F_{\mu\nu}(y)
\]

\[
- \bar{\theta} \theta (\bar{\sigma}_\mu D_\mu \lambda_\alpha)(y),
\]

(103)

where

\[
D_\mu \lambda = \partial_\mu \lambda + i [A_\mu, \lambda],
\]

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i [A_\mu, A_\nu].
\]

\[
\text{Springer}
\]
The $\mathcal{N} = 1$ super Yang–Mills action in the Wess–Zumino gauge is given in terms of the component fields,

$$S_{\text{SYM}} = \frac{1}{g^2} \int d^4x \text{ tr} \left\{ \frac{1}{2} F_{\mu\nu}^2 + 2i\bar{\lambda} \gamma_{\mu} D_\mu \lambda + D^2 \right\}(x).$$

(106)

The action is invariant under supersymmetry transformation [21],

$$\delta \xi A_\mu = i \xi \sigma_\mu \bar{\lambda} + i \bar{\xi} \bar{\sigma}_\mu \lambda,$$

$$\delta \xi \lambda = \sigma_{\mu\nu} \xi F_{\mu\nu} - \xi D,$$

$$\delta \xi \bar{\lambda} = \bar{\sigma}_{\mu\nu} \bar{\xi} F_{\mu\nu} + \bar{\xi} D,$$

$$\delta \xi D = i \xi \sigma_\mu D_\mu \bar{\lambda} - i \bar{\xi} \bar{\sigma}_\mu D_\mu \lambda.$$

(107)

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