THE COMPLETE LIST OF PRIME KNOTS WHOSE FLAT PLUMBING BASKET NUMBERS ARE 6 OR LESS

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Abstract. Flat plumbing basket surfaces of links were introduced to study the geometry of the complement of the links. These flat plumbing basket surface can be presented by a sequential presentation known as flat plumbing basket code first found by Furihata, Hirasawa and Kobayashi. The minimum number of flat plumbings to obtain a flat plumbing basket surfaces of a link is defined to be the flat plumbing basket number of the given link. In present article, we use these sequential presentations to find the complete classification theorem of prime knots whose flat plumbing basket number 6 or less. As applications, this result improves the work of Hirose and Nakashima which finds the flat plumbing basket number of prime knots up to 9 crossings.

1. Introduction

Orientable surfaces whose boundary is the given link, known as Seifert surfaces have been studied for many interesting invariants of links such as Seifert pairings, Alexander polynomials, signatures and etc. A plumbing surface obtained from a 2-dimensional disc by plumbings annuli found by Rudolph [16] used to study extensively for the fibreness of links and surfaces [1–3, 6, 13, 15, 18]. In particular, if we only use flat annuli plumbings, the resulting surface is called a flat plumbing surface. The main focus of the present article is flat plumbing basket surfaces, a precise definition can be found in Definition 2.1. A flat plumbing basket surface can be regarded as a flat plumbing surface, but not vice versa. There exists a Seifert surface which is obtained from a disk by successively plumbing flat annuli, but which is not isotopic to any flat plumbing basket surface [1].

The third author’s first preprint about these plumbing surfaces from a canonical Seifert surface had a critical mistake. In the process of resolving this mistake, the third author, Kwon and Lee proved the existence of banded surfaces and flat banded surfaces [12] by weakening some conditions of plumbings. The third author also proved that every link L is the boundary of an oriented surface which is obtained from a graph embedding of a dipole graph, this surface is also known as a braidzel surface [14], and a complete bipartite graph $K_{2,n}$, where all voltage assignments on the edges of dipole graph and $K_{2,n}$ are 0 [10]. The mistake was finally fixed in [9].

The present work is one of articles in this series of results presenting links as a boundary of the surface obtained in a embedding of certain graphs as described in [5].

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One might consider these plumbing surfaces as special embeddings of the bouquets of circles [4].

A sequence of articles by Hayashi and Wada [8], Furihata, Hirasawa and Kobayashi [1] and the third author [9] proved the existence of a flat plumbing basket surface of a given link $L$. We can define the flat plumbing basket number of $L$, denoted by $fpbk(L)$, to be the minimal number of flat annuli to obtain a flat plumbing basket surface of the link $L$. However, finding the flat plumbing basket number of a link $L$ is very difficult and has been beyond the reach for more than 30 years because it is defined to be the minimum over all possible flat plumbing basket surfaces whose boundaries is the given link $L$.

The work of Furihata et al. [1] provided not only the existence theorem using a very tangible alternating definition of the flat plumbing basket surface but also a coding algorithm, the resulting code is called flat plumbing basket code, to present links as the boundaries of flat plumbing basket surfaces from a special closed braid presentation of the link.

In present article, we use these sequential codes to find all prime knots of the flat plumbing basket number 6 by applying DT-notation and a computer program “knotfinder” of Knotscape [19].

**Theorem 3.6.** The prime knot $K$ has the flat plumbing basket number 6 if and only if it is either $5_1$, $5_2$, $6_1$, $6_2$, $6_3$, $7_6$, $7_7$, $8_1$, $8_3$, $8_{12}$, $8_{20}$, $8_{21}$, $9_{42}$, $9_{44}$, $9_{46}$, $9_{48}$, $10_{132}$, $10_{136}$, $10_{137}$, $10_{140}$, $11_{n38}$, $12n_{462}$, $13n_{973}$, $14n_{17954}$, $15n_{45460}$ or $16n_{246032}$.

When the third author first presented this work at the TAPU conference in 2013 summer, Carter pointed out that this flat plumbing basket code of a link can be very useful to calculate Alexander polynomials because all components in Seifert matrix can be found directly from the presentation and they are either 0 or $\pm 1$. A very recent work by Hirose and Nakashima [7] found a theorem which provide two lower bounds of the flat plumbing basket number using Alexander polynomials and genera of links. Using these lower bounds, they succeed to find the flat plumbing basket number of all prime knots up to 9 crossings except 24 knots.

As an application of our classification theorem, we find the flat plumbing basket number of five knots out of 24 knots and sharpens the range of the flat plumbing basket number of three knots.

The outline of this paper is as follows. We first provide some preliminary definitions and results in Section 2. We provided an explicit coding algorithm to find the flat plumbing basket presentation of a link from its braid presentation and canonical Seifert surface. Also we provide two classification theorems of the flat plumbing basket number of 4 and 6 with a explanation how we find DT-notation and use the computer program “knotfinder” of Knotscape in Section 3. We conclude with a remark on further research in Section 4.

### 2. Preliminaries

A compact orientable surface $\mathcal{F}$ is called a Seifert surface of a link $L$ if the boundary of $\mathcal{F}$ is isotopic to the given link $L$. The existence of such a surface was first proven
by Seifert using an algorithm on a diagram of \( L \), this algorithm was named after him as Seifert’s algorithm \([17]\). A Seifert surface \( \mathcal{F}_L \) of an oriented link \( L \) produced by applying Seifert’s algorithm to a link diagram is called a canonical Seifert surface.

The main topic of the article is the flat plumbing basket surfaces. Rudolph first defined the top plumbing as follows. Let \( \alpha \) be a proper arc on a Seifert surface \( S \). Let \( \beta_\alpha \) be a 3-cell on top of \( S \) along a tubular neighborhood \( C_\alpha \) of \( \alpha \) on \( S \). Let \( A_n \subset \beta_\alpha \) be an \( n \) times full twisted annulus such that \( A_n \cap \partial \beta_\alpha = C_\alpha \). The top plumbing on \( S \) along a path \( \alpha \) is the new surface \( S' = S \cup C_\alpha \) where \( A_n, \beta_\alpha, C_\alpha \) satisfy the previous conditions as depicted in Figure 1. Thus, two consecutive plumbings are non-commutative in general. Rudolph found a few interesting results with regards to the top and bottom plumbings in \([16]\). For the rest of article, all plumbings are top plumbing unless state differently.

**Definition 2.1.** A Seifert surface \( \mathcal{F} \) is a flat plumbing basket surface if \( \mathcal{F} = D^2 \) or if \( \mathcal{F} = \mathcal{F}_0 *_\alpha A_0 \) which can be constructed by plumbing \( A_0 \) to a basket \( \mathcal{F}_0 \) along a proper arc \( \alpha \subset D^2 \subset \mathcal{F}_0 \). We say that a link \( L \) admits a flat plumbing basket representation if there exists a flat plumbing basket surface \( \mathcal{F} \) such that \( \partial \mathcal{F} \) is equivalent to \( L \).

An alternative definition of the flat plumbing basket surfaces is given in \([1]\) and it is very easy to follow. The trivial open book decomposition of \( \mathbb{R}^3 \) is a decomposition of \( \mathbb{R}^3 \) into the half planes in the following form. In a cylindrical coordinate, it can be presented

\[
\mathbb{R}^3 = \bigcup_{\theta \in [0,2\pi)} \{(r, \theta, z)| r \geq 0, z \in \mathbb{R}\}
\]

where \( \{(r, \theta, z)| r \geq 0, z \in \mathbb{R}\} \) is called a page for \( \theta \in [0,2\pi) \). Let \( \mathcal{O} \) be the trivial open book decomposition of the 3-sphere \( S^3 \) which is obtained from the trivial open book decomposition of \( \mathbb{R}^3 \) by the one point compactification. A Seifert surface is said to be a flat plumbing basket surface if it consists of a single page of \( \mathcal{O} \) as a 2-disc \( D^2 \) and finitely many bands which are embedded in distinct pages \([1]\). Flat plumbing basket surfaces of (\( i \)) the trefoil knot and (\( ii \)) the figure eight knot in the trivial open book decomposition are depicted in Figure 2 where \( D^2 \) is presented as a
Figure 2. (a) A flat 4-banded surface of the trefoil knot and (b) a flat 4-banded surface of the figure eight knot.

shaded rectangular region and the top horizontal line of the rectangle is in the \(z\)-axis and the top hemi-spherical annuli are contained in different pages.

Using this definition, for a given link \(L\), Furihata et al. [1] found an algorithm to find a flat plumbing basket surface from a closed braid \(\beta = L\). So we can define the flat plumbing basket number of \(L\), denoted by \(fpbk(L)\), to be the minimal number of flat annuli to obtain a flat plumbing basket surface of \(L\).

**Theorem 2.2.** ( [1]) Let \(L\) be an oriented link which is a closed \(n\)-braid with a braid word \(\sigma_{n-1}\sigma_{n-2}\ldots\sigma_1W\) where the length of \(W\) is \(m\) and \(W\) has \(p\) positive letters, then there exists a flat plumbing basket surface \(S\) with \(m + 2p\) bands such that \(\partial S\) is isotopic to \(L\), i.e., \(fpbk(L) \leq m + 2p\).

This upper bound has been improved by the third author [9] where the link is prime but not splittable.

**Theorem 2.3.** ( [9]) Let \(L\) be an oriented link which is a closed \(n\)-braid with a braid word \(\beta\) whose length is \(m\) and let \(ps(\sigma_i^{\pm 1})\) be the power sum of \(\sigma_i^{\pm 1}\) in \(\beta\) for all \(i = 1, 2, \ldots, n - 1\). Let \(\gamma\) be the cardinality of the set
\[
\Omega = \{i|1 \leq i \leq n - 1, \sigma_i \text{ and } \sigma_i^{-1} \text{ both appear in } \beta\}.
\]

Let
\[
\epsilon_i = \begin{cases} 
1 & \text{if } 1 \leq ps(\sigma_i^1) \leq ps(\sigma_i^{-1}) \text{ or } ps(\sigma_i^{-1}) = 0, \\
-1 & \text{if } 1 \leq ps(\sigma_i^{-1}) \leq ps(\sigma_i^1) \text{ or } ps(\sigma_i^1) = 0.
\end{cases}
\]

Then the flat plumbing basket number of \(L\) is bounded by \(m + n - 1 - 4\gamma + 2\sum_{i=1}^{n-1} ps(\sigma_i^{\epsilon_i})\), i.e.,
\[
fpbk(L) \leq m + n - 1 - 4\gamma + 2\sum_{i=1}^{n-1} ps(\sigma_i^{\epsilon_i}).
\]

The third author proved that every link \(L\) admits a flat plumbing basket representation from a canonical Seifert surface \(F_L\) of \(L\) to have a property that the Seifert graph \(\Gamma(D_L)\) has a co-tree edge alternating spanning tree \(T\) [9].

**Theorem 2.4.** ( [9]) Let \(\Gamma\) be an Seifert graph of canonical Seifert surface \(S\) of a link \(L\) with \(|V(\Gamma)| = n\), \(|E(\Gamma)| = m\) and the sign labeling \(\phi\). Let \(G(\Gamma)\) be the Seifert
Let $T$ be a co-tree edge alternating spanning tree of $\Gamma$ and $\mu$ a labeling on $T$ chosen in [9, Theorem 3.3]. Let $\delta(T)$ be the cardinality of the set
$$\Psi(T) = \{ e \in E(T) | \mu(e) \neq \phi(\overline{e}) \text{ for all } \overline{e} \in \Gamma(e) \},$$
and let $\zeta(T)$ be the cardinality of the set
$$\Upsilon(T) = \{ \overline{e} \in E(\Gamma(T)) \mid \mu(e) = \phi(\overline{e}), \ e \in E(T) - \Psi(T) \}.$$ 
and let $\eta(T)$ be the cardinality of the set
$$\Phi(T) = \{ \overline{e} \in E(\Gamma) - E(\Gamma(T)) \mid \mu(\overline{e}) = \nu(e) \}$$
where $\nu(e) = +(-, \text{ resp.})$ if there is one extra positive(negative, respectively) sign in the path $P_e$ joining end vertices of the edge $e$ in $T$. Then the flat plumbing basket number of $L$ is bounded by $m - 3(n - 1) + 2(2\delta(T) + \zeta(T) + \eta(T))$, i.e.,
$$\text{fpbk}(L) \leq m - 3(n - 1) + 2(2\delta(T) + \zeta(T) + \eta(T)) \text{.}$$

3. The flat plumbing basket codes and results

For a given link $L$, Furihata et al. [1] found an algorithm to find a flat plumbing basket surface from a closed braid $\beta = L$ as follows.

**Algorithm 1** [1]

- **Step 1.** For a given link $L$, we find its braid representation $\beta$, the closed braid $\beta = L$.
- **Step 2.** Apply the method in [1] to obtain a flat plumbing basket surface $F$ which is obtained from a disc by successively plumbing flat annuli.

This algorithm can be demonstrated in the following Example 3.1.

**Example 3.1.** A flat plumbing basket code of the knot $5_2$ is $(1, 2, 3, 4, 5, 6, 4, 5, 1, 2, 3, 6)$.

**Proof.** For the knot $5_2$, we first present it as a closed braid $\sigma_2\sigma_1^{-1}(\sigma_2)^{-3}\sigma_1^{-1}$ on three strings as illustrated in Figure 3(a). Although theorem in [1] stated differently, one can choose any two generators of the Artin’s braid group $B_3$ as stated in Theorem 2.3. We choose the first $\sigma_2\sigma_1^{-1}$ to have a disc $D$ which is the union of three discs, bounded by three Seifert circles, joined by two half twisted bands presented by $\sigma_2\sigma_1^{-1}$ as indicated by the dashed purple line in Figure 3(a). Since the rest word $(\sigma_2)^{-3}\sigma_1^{-1}$ has the length 4 and $(\sigma_2)^{-3}$ has the different sign to $\sigma_2$, we need three flat plumbings. However $\sigma_1^{-1}$ has the same sign to $\sigma_1^{-1}$, we first change the sign of half twisted band by adding two flat annuli as shown in Figure 3(b). Now we pick as starting point as indicated as a red dot in Figure 3(b). Then, we read the flat bands along the disc in the direction as given in Figure 3(b). By isotoping the original disc to a standard disc, we obtain a diagram in Figure 3(c). By rewriting labels in the set $\{7, 8, \ldots, 12\}$ by one in the set $\{1, 2, \ldots, 6\}$ depend on how $1, 2, \ldots, 6$ are connected to $7, 8, \ldots, 12$, and by the rule that for annulus presented by $i$ is in front of the annulus presented by $j$ whenever $i > j$, we obtain the first 12-tuple $(1, 2, 3, 4, 5, 6, 4, 5, 1, 2, 3, 6)$. \hfill \Box
Let us remark that two groups of annuli presented by \{1, 2, 3\} and \{4, 5\} do not involve each other, one may obtain different code 12-tuple \((1, 2, 6, 1, 2, 3, 4, 5, 6, 3, 4, 5)\).

One may see that the direction and the initial page of the open book decomposition was not fixed. By moving the initial page from 1 to 2, the effect on the flat plumbing basket code is an action by the permutation \(\sigma = \begin{pmatrix} 1 & 2 & \ldots & n-1 & n \\ 2 & 3 & \ldots & n & 1 \end{pmatrix}\). By reversing the direction of pages in the open book decomposition, the effect on the flat plumbing...

**Figure 3.** (a) The knot 5\(_2\) as a closed braid, (b) Seifert surface of 5\(_2\) to apply algorithm in [1], (c) a flat plumbing basket surface of 5\(_2\).
KNOTS OF THE FLAT PLUMBING BASKET NUMBER 6

basket code is an action by the permutation \( \tau = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix} \). By summarizing these observations, we find the following theorem.

**Theorem 3.2.** For a positive integer \( n \),

1. The number of the \( 2n \)-tuple flat plumbing basket codes presenting the same link are divisible by \( n \).
2. The number of the \( 2n \)-tuple flat plumbing basket codes presenting the same link are divisible by \( 2 \).

Now we will provide some classification theorems of knots and links by the flat plumbing basket numbers. Without using a computer program, the third author was able to prove the following classification theorem of links of the flat plumbing basket number 0, 1, 2, 3 and 4 by using slightly different presentations of the flat plumbing basket surfaces, known as permutation presentations.

**Theorem 3.3.** \( [11] \)

1. A link \( L \) has the flat plumbing basket number 0 if and only if \( L \) is the trivial knot.
2. A link \( L \) has the flat plumbing basket number 1 if and only if \( L \) is the trivial link of two components.
3. A link \( L \) has the flat plumbing basket number 2 if and only if \( L \) is the trivial link of three components.
4. A link \( L \) has the flat plumbing basket number 3 if and only if \( L \) is either the trivial link of four components or the Hopf link which is denoted by \( L_{2a1} \).
5. A link \( L \) has the flat plumbing basket number 4 if and only if \( K \) is either the trefoil knot, the figure eight knot, \( L_{2a1} \sqcup O \), \( L_{2a1} \# L_{2a1} \), \( L_{6a5} \), or the trivial link of five components.

where \( \sqcup \) presents the disjoint union.

It is fairly easy to see that the number of components of the link whose flat plumbing basket number \( n \) is always congruent to \( n+1 \) modulo 2. Thus, the flat plumbing basket number of a prime knot has to be an even integer. From Theorem 3.3 one can easily obtain the following corollary which addresses the classification of all prime knots of the flat plumbing basket number 0, 2 and 4.

**Corollary 3.4.**

1. A prime knot \( K \) has the flat plumbing basket number 0 if and only if \( L \) is the trivial knot.
2. There does not exist a prime knot \( K \) whose flat plumbing basket number 2.
3. A knot \( K \) has the flat plumbing basket number 4 if and only if \( K \) is either the trefoil knot or the figure eight knot.

**Example 3.5.**

1. The flat plumbing basket number of the link \( 4_{1}^{2} \) is 5.
2. The flat plumbing basket number of the knots \( 5_{2} \) is 6.

**Proof.** Flat plumbing basket surfaces of the link \( 4_{1}^{2} \) with five annuli are depicted in Figure 4. By Theorem 3.3, the link \( 4_{1}^{2} \) can not have the flat plumbing basket number less than 4. The remaining possibility is 4. But, the number of components in a
flat plumbing basket surface of 4 bands is odd. Therefore, the flat plumbing basket number of the link $4_2^1$ must be 5.

A flat plumbing basket surface of the link $5_2^1$ with six annuli are depicted in Figure 3 and its flat plumbing basket code is given $(1, 2, 3, 4, 5, 6, 4, 5, 1, 2, 3, 6)$ in Example 3.1. By Corollary 3.4, the knot $5_2^1$ can not have the flat plumbing basket number less than 6. $\square$

**Theorem 3.6.** The prime knot $K$ has the flat plumbing basket number 6 if and only if it is either $5_1^1$, $5_2^1$, $6_1^1$, $6_2^1$, $6_3$, $7_6$, $7_7$, $8_1$, $8_3$, $8_{12}$, $8_{20}$, $8_{21}$, $9_{42}$, $9_{44}$, $9_{46}$, $9_{48}$, $10_{132}$, $10_{136}$, $10_{137}$, $10_{140}$, $11n_{38}$, $12n_{462}$, $13n_{973}$, $14n_{17954}$, $15n_{45460}$ or $16n_{246032}$.

**Proof.** To consider all flat plumbing basket surface with 6 annuli, we first count all possible flat plumbing basket codes of $\{1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6\}$. Since these are presented as circular shapes, we can fix the first elements to be 1. Thus, there are $\frac{11!}{2^6} = 7, 484, 400$ many such flat plumbing basket codes. Among these flat plumbing basket codes, 6, 415, 200 of them present links and 1, 069, 200 present knots. But some sequences admit a Type I move as depicted in Figure 5 (for the case that the distance between the same number is 2) so that these are not flat plumbing basket surfaces of the flat plumbing basket number 6.

Among 1, 069, 200 flat plumbing basket codes presenting knots, there are 874, 080 codes which admit a Type I move. It leaves us 195, 120 flat plumbing basket codes. 105, 162 codes presents the unknot and 2, 268 codes present composite knots.
The second author has written C+ programs which determine whether a given flat plumbing basket code produces a knot or a link with more than 1 components and finds a DT-notation of the given flat plumbing basket code. All remaining 87,690 DT-notations of the flat plumbing basket codes are identified the computer program “knotfinder” of Knotscape identified as in Table 1. □

While using “knotfinder” of Knotscape, some of DT-notations of the given flat plumbing basket code were not identified since the program itself may run infinitely since it repeatedly uses Reidemeister moves on DT-notation and aborts by the time limit set up by the program. However, we are lucky enough that at least one of DT-notations of the flat plumbing basket codes which produce the same knot by the action of $\sigma = \begin{pmatrix} 1 & 2 & \ldots & n-1 & n \\ 2 & 3 & \ldots & n & 1 \end{pmatrix}$ as stated in Theorem 3.2 (1). One can also observe that the number of the flat plumbing basket codes for a knot in Table 1 are all divisible by 12 except 31, 41 and 61 because some of flat plumbing basket codes are invariant by the action of $\sigma$ or $\tau$.

Since the trefoil knot and the figure eight knot have the flat plumbing basket number 4, there are exactly 26 prime knots whose flat plumbing basket numbers are exactly 6. The results in [7] found the flat plumbing basket number of prime knots up to 9 crossings except 24 knots. Using Theorem 3.6 we find the following corollary.

**Corollary 3.7.** (1) The flat plumbing basket number of knots 72, 74 and 945 is 8.

(2) The flat plumbing basket number of knots 81, 944 is 6.

(3) The flat plumbing basket number of knots 92, 95 and 935 are either 8 or 10.
4. Conclusion

Authors already have all required DT-notations for the flat plumbing basket codes of the flat plumbing basket number 8 and 10. But we are not able to find a complete table like Table I. There are two difficulties arise for these cases 1) the computer program “knotfinder” of Knotscape does not have a complete list of DT-notations for prime knots of more than 16 crossings, 2) the number of DT-notations we have to deal with increases exponentially. The text file contains all DT-notations for the flat plumbing basket codes of the flat plumbing basket number 10 is more than 2-gigabytes already.

To solve these problems, authors are trying to write a new computer program which identify whether two DT-notations are the same and revise “knotfinder” of Knotscape.

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References

[1] R. Furihata, M. Hirasawa and T. Kobayashi, Seifert surfaces in open books, and a new coding algorithm for links, Bull. London Math. Soc. 40(3) (2008), 405–414.
[2] D. Gabai, The Murasugi sum is a natural geometric operation, in: Low-Dimensional Topology (San Francisco, CA, USA, 1981), Amer. Math. Soc., Providence, RI, 1983, 131–143.
[3] D. Gabai, The Murasugi sum is a natural geometric operation II, in: Combinatorial Methods in Topology and Algebraic Geometry (Rochester, NY, USA, 1982), Amer. Math. Soc., Providence, RI, 1985, 93–100.
[4] J. Gross, D. Robbins and T. Tucker, Genus distribution for bouquets of circles, J. Combin. Theory B. Soc. 47 (3) (1989) 292–306.
[5] J. Gross and T. Tucker, Topological graph theory, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley & Sons, New York, 1987.
[6] J. Harer, How to construct all fibered knots and links, Topology 21 (3) (1982) 263–280.
[7] S. Hirose and Y. Nakashima, Seifert surfaces in open books, and pass moves on links, arXiv:1311.3383.
[8] C. Hayashi and M. Wada, Constructing links by plumbing flat annuli, J. Knot Theory Ramifications 2 (1993), 427–429.
[9] D. Kim, Basket, flat plumbing and flat plumbing basket surfaces derived from induced graphs, preprint, arXiv:1108.1455.
[10] D. Kim, The boundaries of dipole graphs and the complete bipartite graphs $K_{2,n}$, preprint, arXiv:1302.3829.
[11] D. Kim, A classification of links of the flat plumbing basket numbers 4 or less, Korean J. of Math. 22(2) (2014), 253–264.
[12] D. Kim, Y. S. Kwon and J. Lee, Banded surfaces, banded links, band indices and genera of links, J. Knot Theory Ramifications 22(7) 1350035 (2013), 1–18, arXiv:1105.0059.
[13] T. Nakamura, On canonical genus of fibered knot, J. Knot Theory Ramifications 11 (2002), 341–352.
[14] T. Nakamura, Notes on braidzel surfaces for links, Proc. of AMS 135(2) (2007), 559–567.
[15] L. Rudolph, Quasipositive annuli (Constructions of quasipositive knots and links IV.), J. Knot Theory Ramifications 1 (4) (1992) 451–466.
[16] L. Rudolph, *Hopf plumbing, arborescent Seifert surfaces, baskets, espaliers, and homogeneous braids*, Topology Appl. 116 (2001), 255–277.

[17] H. Seifert, *Über das Geschlecht von Knoten*, Math. Ann. 110 (1934), 571–592.

[18] J. Stallings, *Constructions of fibred knots and links*, in: Algebraic and Geometric Topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, CA, 1976), Part 2, Amer. Math. Soc., Providence, RI, 1978, pp. 55–60.

[19] M. Thistlethwaite, *Knotscape*, available at [http://www.math.utk.edu/~morwen/knotscape.html](http://www.math.utk.edu/~morwen/knotscape.html).

[20] T. Van Zandt. PSTricks: PostScript macros for generic TEX. Available at [ftp://ftp.princeton.edu/pub/tvz/](ftp://ftp.princeton.edu/pub/tvz/).

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