On the generating function of the Pearcey process

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Abstract

The Pearcey process is a universal point process in random matrix theory. In this paper, we study the generating function of the Pearcey process on any number \( m \) of intervals. We derive an integral representation for it in terms of a Hamiltonian that is related to a system of \( 6m + 2 \) coupled nonlinear equations. We also obtain asymptotics for the generating function as the size of the intervals get large, up to and including the constant term. This work generalizes some recent results of Dai, Xu and Zhang, which correspond to \( m = 1 \).

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1 Introduction and statement of results

In random matrix theory, the universality conjecture asserts that the microscopic behavior of the eigenvalues of large random matrices is similar for many different models. More precisely, it is expected that the local eigenvalue statistics only depend on the symmetry class of the matrix ensemble and on the nature of the point around which these statistics are considered [28, 30, 38]. The Pearcey process is one of the canonical point processes from the theory of random matrices: it models the asymptotic behavior of the eigenvalues near the points of the spectrum where the density of states admits a cusp-like singularity. This process appears in Gaussian random matrices with an external source [12, 13, 8], Hermitian random matrices with independent, not necessarily identically distributed entries [27], and in general Wishart matrices with correlated entries [32]. The Pearcey process is also universal in a sense that goes beyond random matrix theory: it appears in certain models of skew plane partitions [39] and of Brownian motions [1, 2, 8, 46, 31].

The Pearcey process is the determinantal point process on \( \mathbb{R} \) associated with the kernel

\[
K_\rho(x, y) = \frac{\mathcal{P}(x)\mathcal{Q}(y) - \mathcal{P}'(x)\mathcal{Q}'(y) + \mathcal{P}''(x)\mathcal{Q}(y) - \rho\mathcal{P}(x)\mathcal{Q}(y)}{x - y},
\]  

where \( \rho \in \mathbb{R} \),

\[
\mathcal{P}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{\pi}{4}t^4 - \frac{\rho}{2}t^2 + itx} dt,
\quad
\mathcal{Q}(y) = \frac{1}{2\pi} \int_{\Sigma} e^{\frac{\pi}{4}t^4 + \frac{\rho}{2}t^2 + ity} dt,
\]

and \( \Sigma = (e^{\pi i/4}, 0) \cup (0, e^{\pi i/4}) \cup (e^{-\pi i/4}, 0) \cup (0, e^{-\pi i/4}) \). As \( \rho \) increases, the point configurations with few points near 0 are increasingly likely to occur, and when \( \rho \) tends to +\( \infty \), the Pearcey process “factorizes” (in the sense of gap probabilities) into two independent Airy processes [6]. Important progress on the large gap asymptotics for any fixed \( \rho \in \mathbb{R} \) have only recently been obtained in [23].

This paper is inspired by the work [24] of Dai, Xu and Zhang, and is concerned with the moment generating function of the Pearcey process. Let \( X \) be a locally finite random point configuration distributed according to the Pearcey process, and let \( N(x) := \# \{ \xi \in X : \xi \in (-x, x) \} \) be the associated counting function. We are interested in the \( m \)-point generating function

\[
F(r, u) := \mathbb{E} \left[ \prod_{j=1}^{m} e^{u_j N(r_j)} \right],
\]  

where \( r \) and \( u \) are vectors of real numbers.
where \( r > 0, \ m \in \mathbb{N}_{>0}, \ \vec{u} = (u_1, \ldots, u_m) \in \mathbb{R}^m, \ \vec{x} = (x_1, \ldots, x_m) \in \mathbb{R}_{\text{ord}}^{+m}, \)

and \( \mathbb{R}_{\text{ord}}^{+m} := \{(x_1, \ldots, x_m) : 0 < x_1 < \ldots < x_m < +\infty\}. \)

Our main results are stated in Theorems 1.3 and 1.4 below and can be summarized as follows:

- Theorem 1.3 establishes an integral representation for \( F(r \vec{x}, \vec{u}) \) in terms of a Hamiltonian related to a system of \( 6m + 2 \) coupled differential equations. This system of equations admits at least one solution, which we derive from the Lax pair of a Riemann-Hilbert (RH) problem. The asymptotic properties of this solution are stated in Theorem 1.1.

- Theorem 1.4 is concerned with the asymptotic properties of the generating function as the size of the intervals get large. Specifically, Theorem 1.4 gives a precise asymptotic formula, up to and including the constant term, for \( F(r \vec{x}, \vec{u}) \) as \( r \to +\infty. \)

For \( m = 1, \) Theorems 1.1, 1.3 and 1.4 have previously been obtained in [24]. The general case \( m \geq 2 \) allows to capture the correlation structure of the Pearcey process, see Corollary 1.5, and to analyze the joint fluctuations of the counting function, see Corollary 1.6. We also expect the case \( m = 2 \) of Theorem 1.4 to play an important role in future studies of the rigidity of the Pearcey process (we comment more on that at the end of this section).

The relevant system of \( 6m + 2 \) coupled differential equations depends on unknown functions which are denoted

\[
p_0(r), \ q_0(r), \ p_{j,k}(r), \ q_{j,k}(r), \quad k = 1, 2, 3, \ j = 1, \ldots, m, \tag{1.3}
\]

and is as follows:

\[
\begin{align*}
p'_0(r) &= -\sqrt{2} \sum_{j=1}^{m} x_j p_{j,3}(r) q_{j,2}(r), \\
p'_0(r) &= \sqrt{2} \sum_{j=1}^{m} x_j p_{j,2}(r) q_{j,1}(r), \\
q'_{j,1}(r) &= \frac{2}{r} S_{11}(r) q_{j,1}(r) + x_j q_{j,2}(r) + \frac{2}{r} S_{31}(r) q_{j,3}(r), \\
q'_{j,2}(r) &= \sqrt{2} p_0(r) x_j q_{j,1}(r) + \frac{2}{r} S_{32}(r) q_{j,2}(r) + x_j q_{j,3}(r), \\
q'_{j,3}(r) &= (r x_j^2 + \frac{2}{r} S_{13}(r)) q_{j,1}(r) + \sqrt{2} q_0(r) x_j q_{j,2}(r) + \frac{2}{r} S_{33}(r) q_{j,3}(r), \\
p'_{j,1}(r) &= -\frac{2}{r} S_{11}(r) p_{j,1}(r) - \sqrt{2} p_0(r) x_j p_{j,2}(r) - (r x_j^2 + \frac{2}{r} S_{13}(r)) p_{j,3}(r), \\
p'_{j,2}(r) &= -r x_j p_{j,1}(r) - \frac{2}{r} S_{22}(r) p_{j,2}(r) - \sqrt{2} q_0(r) x_j p_{j,3}(r), \\
p'_{j,3}(r) &= -\frac{2}{r} S_{31}(r) p_{j,1}(r) - x_j p_{j,2}(r) - \frac{2}{r} S_{33}(r) p_{j,3}(r),
\end{align*}
\tag{1.4}
\]

where \( j = 1, \ldots, m, \) and

\[
S_{kl}(r) = \sum_{j=1}^{m} p_{j,k}(r) q_{j,l}(r), \quad k, l = 1, 2, 3.
\]

Furthermore, we require the functions (1.3) to satisfy the following \( m \) relations

\[
\sum_{k=1}^{3} p_{j,k}(r) q_{j,k}(r) = 0, \quad j = 1, \ldots, m. \tag{1.5}
\]

Let

\[
H(r) = H(r; p_0, q_0, \{p_{j,1}, q_{j,1}, p_{j,2}, q_{j,2}, p_{j,3}, q_{j,3}\}_{j=1}^{m})
\]

be defined by

\[
H(r) = \sqrt{2} p_0(r) \sum_{j=1}^{m} x_j p_{j,2}(r) q_{j,1}(r) + \sqrt{2} q_0(r) \sum_{j=1}^{m} x_j p_{j,3}(r) q_{j,2}(r)
\]

\[
+ \sum_{j=1}^{m} x_j p_{j,1}(r) q_{j,2}(r) + \sum_{j=1}^{m} x_j p_{j,2}(r) q_{j,3}(r) + \sum_{j=1}^{m} r x_j^2 p_{j,3}(r) q_{j,1}(r)
\]
There exists at least one solution $H$ satisfying the following asymptotics. As $r \to 0$, we have

$$p_0(r) = \frac{\sqrt{3}}{2\sqrt{2\pi}} \sum_{k=1}^{m} u_k x_0^{\frac{2}{3}} r^{\frac{3}{2}} + \frac{1}{\sqrt{2}} \left( \frac{\rho_2 + \rho_2^3}{2} \right) + O(r^{-\frac{3}{2}}),$$  

where

$$\theta_3(r) = \frac{3}{2} r^2 + \frac{\rho_2}{2} r^2,$$

$$\mathcal{A}_j = |\Gamma(1 - \frac{u_j}{2\pi})| \exp \left( -\frac{u_j}{3} - \frac{m}{3} \frac{u_k}{2} - \sum_{k=k+1}^{m} \frac{u_k}{2\pi} \arctan \frac{\sqrt{3} x_k^{2/3}}{x_j^{2/3} + 2 x_k^{2/3}} \right),$$

and therefore $H$ is a Hamiltonian for the system (1.4)–(1.5).

**Theorem 1.1.** Let $\rho \in \mathbb{R}$, $r > 0$, $m \in \mathbb{N}_{>0}$, $\vec{u} = (u_1, \ldots, u_m) \in \mathbb{R}^m$, and $\vec{x} = (x_1, \ldots, x_m) \in \mathbb{R}_{ord}^m$.

There exists at least one solution $(p_0, q_0)_{j=1}^m$ to the system of equations (1.4) and (1.5) satisfying the following asymptotics. As $r \to +\infty$, we have

$$p_0(r) = \frac{\sqrt{3}}{2\sqrt{2\pi}} \sum_{k=1}^{m} u_k x_0^{\frac{2}{3}} r^{\frac{3}{2}} + \frac{1}{\sqrt{2}} \left( \frac{\rho_2 + \rho_2^3}{2} \right) + O(r^{-\frac{3}{2}}),$$

$$q_0(r) = -\frac{3}{2\sqrt{2\pi}} \sum_{k=1}^{m} u_k x_0^{\frac{2}{3}} r^{\frac{3}{2}} + \frac{1}{\sqrt{2}} \left( -\frac{\rho_2 + \rho_2^3}{2} \right) + O(r^{-\frac{3}{2}}),$$

where $\theta_2(r) = \frac{3}{2} r^2 + \frac{\rho_2}{2} r^2$,

$$\mathcal{A}_j = |\Gamma(1 - \frac{u_j}{2\pi})| \exp \left( -\frac{u_j}{3} - \frac{m}{3} \frac{u_k}{2} - \sum_{k=k+1}^{m} \frac{u_k}{2\pi} \arctan \frac{\sqrt{3} x_k^{2/3}}{x_j^{2/3} + 2 x_k^{2/3}} \right),$$

and $\Gamma$ is Euler’s Gamma function. As $r \to 0$, we have

$$p_0 = \frac{1}{\sqrt{2}} \left( \frac{\rho_2}{2} + \frac{\rho_2^3}{54} \right) + O(r), \quad q_0 = \frac{1}{\sqrt{2}} \left( -\frac{\rho_2}{2} + \frac{\rho_2^3}{54} \right) + O(r),$$

$$p_{j,1}(r) = O(r), \quad p_{j,2}(r) = O(1), \quad p_{j,3}(r) = O(r), \quad j = 1, \ldots, m,$$

$$q_{j,1}(r) = O(1), \quad q_{j,2}(r) = O(r), \quad q_{j,3}(r) = O(1), \quad j = 1, \ldots, m.$$
The following relation holds
\[ K_m \rightarrow \text{const} \]

Furthermore, the asymptotics of \( F \) with \( H_0 \) given by (1.6), and where \( (p_0,q_0,\{p_{1,1},q_{1,1},p_{1,2},q_{1,2},p_{1,3},q_{1,3}\}_{j=1}^m) \) is a solution to the system of equations (1.4) and (1.5) which satisfies the asymptotic formulas (1.8) and (1.11). Furthermore,
\[ H(r) = O(1), \quad \text{as} \ r \to 0, \quad (1.13) \]
and as \( r \to +\infty, \)
\[ H(r) = \sum_{j=1}^m \left( \frac{\sqrt{3}}{2\pi} u_j x_j^\frac{3}{2} r^\frac{1}{2} - \frac{\rho}{2\sqrt{3}\pi} u_j x_j^\frac{3}{2} r^{-\frac{1}{2}} + \frac{u_j^2}{3\pi^2 r} - \frac{u_j}{3\sqrt{3}\pi r} \cos(2\theta_j(r)) \right) + O(r^{-\frac{3}{2}}), \quad (1.14) \]
where \( \theta_j(r) \) is defined in (1.10).

Since the functions \( (p_0,q_0,\{p_{j,1},q_{j,1},p_{j,2},q_{j,2},p_{j,3},q_{j,3}\}_{j=1}^m) \) appearing in the integral representation (1.12) are rather complicated objects, it is natural to try to approximate \( F(r\vec{x},\vec{u}) \) for small and large values of \( r \) with some explicit asymptotic formulas. Because \( F(r\vec{x},\vec{u}) \) is a Fredholm determinant (see (2.1) below), the asymptotics of \( F(r\vec{x},\vec{u}) \) as \( r \to 0 \) can be easily obtained from an analysis of the kernel \( \mathcal{K}(x,y) \) near \((x,y) = (0,0)\). A much more complicated question is to approximate \( F(r\vec{x},\vec{u}) \) for large values of \( r \). In the case of the sine, Airy and Bessel processes, the asymptotics for the \( m \)-point generating functions are known, see [5, 15] for sine, [10, 17] for Airy, [11, 14] for Bessel and [20] for the transition between Bessel and Airy. The asymptotics for the 1-point generating function of the Pearcey process, up to and including the notoriously difficult constant term, have recently been established in [24, Theorem 2.3]. We provide here the generalization of [24, Theorem 2.3] to an arbitrary \( m \).

**Theorem 1.3.** Let \( \rho \in \mathbb{R}, \)
\[ r > 0, \quad m \in \mathbb{N}_{>0}, \quad \vec{u} = (u_1, \ldots, u_m) \in \mathbb{R}^m, \quad \text{and} \quad \vec{x} = (x_1, \ldots, x_m) \in \mathbb{R}^\text{ord}_m. \]
The following relation holds
\[ F(r\vec{x},\vec{u}) = \exp \left( 2 \int_0^\rho H(\tau) d\tau \right), \quad (1.12) \]
with \( H \) given by (1.6), and where \( (p_0,q_0,\{p_{j,1},q_{j,1},p_{j,2},q_{j,2},p_{j,3},q_{j,3}\}_{j=1}^m) \) is a solution to the system of equations (1.4) and (1.5) which satisfies the asymptotic formulas (1.8) and (1.11). Furthermore,
\[ H(r) = O(1), \quad \text{as} \ r \to 0, \quad (1.13) \]
and as \( r \to +\infty, \)
\[ H(r) = \sum_{j=1}^m \left( \frac{\sqrt{3}}{2\pi} u_j x_j^\frac{3}{2} r^\frac{1}{2} - \frac{\rho}{2\sqrt{3}\pi} u_j x_j^\frac{3}{2} r^{-\frac{1}{2}} + \frac{u_j^2}{3\pi^2 r} - \frac{u_j}{3\sqrt{3}\pi r} \cos(2\theta_j(r)) \right) + O(r^{-\frac{3}{2}}), \quad (1.14) \]
where \( \theta_j(r) \) is defined in (1.10).

Since the functions \( (p_0,q_0,\{p_{j,1},q_{j,1},p_{j,2},q_{j,2},p_{j,3},q_{j,3}\}_{j=1}^m) \) appearing in the integral representation (1.12) are rather complicated objects, it is natural to try to approximate \( F(r\vec{x},\vec{u}) \) for small and large values of \( r \) with some explicit asymptotic formulas. Because \( F(r\vec{x},\vec{u}) \) is a Fredholm determinant (see (2.1) below), the asymptotics of \( F(r\vec{x},\vec{u}) \) as \( r \to 0 \) can be easily obtained from an analysis of the kernel \( \mathcal{K}(x,y) \) near \((x,y) = (0,0)\). A much more complicated question is to approximate \( F(r\vec{x},\vec{u}) \) for large values of \( r \). In the case of the sine, Airy and Bessel processes, the asymptotics for the \( m \)-point generating functions are known, see [5, 15] for sine, [10, 17] for Airy, [11, 14] for Bessel and [20] for the transition between Bessel and Airy. The asymptotics for the 1-point generating function of the Pearcey process, up to and including the notoriously difficult constant term, have recently been established in [24, Theorem 2.3]. We provide here the generalization of [24, Theorem 2.3] to an arbitrary \( m \).

**Theorem 1.4.** Let
\[ \rho \in \mathbb{R}, \quad m \in \mathbb{N}_{>0}, \quad \vec{u} = (u_1, \ldots, u_m) \in \mathbb{R}^m, \quad \text{and} \quad \vec{x} = (x_1, \ldots, x_m) \in \mathbb{R}^\text{ord}_m. \]
As \( r \to +\infty, \) we have
\[ F(r\vec{x},\vec{u}) = \exp \left( \sum_{j=1}^m u_j \mu_\rho(x_j) + \sum_{j=1}^m \frac{u_j^2}{2} \sigma^2(r x_j) + \sum_{1 \leq j < k \leq m} u_j u_k \Sigma(x_k, x_j) \right. \]
\[ + \left. \sum_{j=1}^m 2 \log \left( G(1 - \frac{u_j}{2\pi}) G(1 + \frac{u_j}{2\pi}) \right) \right) + O(r^{-\frac{3}{2}}), \quad (1.15) \]
where \( G \) is Barnes’ \( G \)-function, and \( \mu_\rho, \sigma^2 \) and \( \Sigma \) are given by
\[ \mu_\rho(x) = \frac{3\sqrt{3}}{4\pi} x^\frac{3}{2} - \frac{\sqrt{3}\rho}{2\pi} x^\frac{5}{2}, \]
\[ \sigma^2(x) = \frac{4}{3\pi^2} \log x + \frac{1}{\pi^2} \log \frac{9}{2}, \]

\[ \Sigma(x_k, x_j) = \frac{1}{\pi^2} \log \left| \frac{x_j^{2/3} - \omega x_k^{2/3}}{x_j^{2/3} - x_k^{2/3}} \right|, \]

where \( \omega = e^{\frac{2i\pi}{3}} \). Furthermore, (1.15) holds uniformly for \( r \) in compact subsets of \( \mathbb{R} \), for \( u \) in compact subsets of \( \mathbb{R}^m \), and for \( \tilde{x} \) in compact subsets of \( \mathbb{R}^n \). The asymptotic formula (1.15) can also be differentiated any number of times with respect to \( u_1, \ldots, u_m \) at the expense of increasing the error term in the following way. Let \( F(x, u) \) be the right-hand side of (1.15) without the error term, and denote the error term by \( E = \log F(x, u) - \log F(x, \tilde{u}) \). For any \( k_1, \ldots, k_m \in \mathbb{N}_{\geq 0} \), we have

\[ \partial_{u_1}^{k_1} \cdots \partial_{u_m}^{k_m} E = O \left( \frac{(\log r)^{k_1+\cdots+k_m}}{r^{2/3}} \right), \quad \text{as } r \to +\infty. \quad (1.16) \]

We end this section by providing several new applications of Theorem 1.4, and we also discuss its relevance in future studies of the rigidity of the Pearcey process.

**Applications of Theorem 1.4.** Using (1.15) with \( m = 1 \), (1.16),

\[ \partial_u \log F(r, u)|_{u=0} = E[N(r)] \quad \text{and} \quad \partial_u^2 \log F(r, u)|_{u=0} = \mathrm{Var}[N(r)], \]

it readily follows that

\[ E[N(r)] = \mu_r(r) + O \left( \frac{\log r}{r^{2/3}} \right), \quad \text{as } r \to +\infty, \quad (1.17) \]

\[ \mathrm{Var}[N(r)] = \sigma^2(r) + \frac{1 + \gamma_E}{\pi^2} + O \left( \frac{(\log r)^2}{r^{2/3}} \right), \quad \text{as } r \to +\infty, \quad (1.18) \]

where \( \gamma_E \) is Euler’s gamma constant. These asymptotic formulas for the expectation and variance of \( N(r) \) are not new and were already obtained in [24, equations (2.30) and (2.31)]. We see from (1.18) that the large \( r \) asymptotics of \( \mathrm{Var}[N(r)] \) are of the form \( c_1 \log r + c_2 + o(1) \) for some explicit constants \( c_1 \) and \( c_2 \). The asymptotics for the variance of the counting functions of other classical point processes such as the sine, Airy and Bessel point processes are also of the same form [42, 15, 17, 14] (with different values for \( c_1 \) and \( c_2 \)). This phenomena is expected to be universal, in the sense that it is expected to hold for many point processes in random matrix theory and other related fields, see the very general predictions [40, 41]. We emphasize that the proof of (1.17) and (1.18) only relies on Theorem 1.4 with \( m = 1 \). Using Theorem 1.4 with \( m = 2 \) allows to obtain new results on the correlation structure of the Pearcey process. More precisely, using (1.15) with \( m = 2 \), (1.16), and

\[ \partial_u^2 \log \left( \frac{F(x_1, x_2, u)}{F(x_1, u)F(x_2, u)} \right)|_{u=0} = 2 \mathrm{Cov}(N(x_1), N(x_2)), \]

we directly obtain the following result.

**Corollary 1.5.** Let \( x_2 > x_1 > 0 \) be fixed. As \( r \to +\infty \),

\[ \mathrm{Cov}(N(r;x_1), N(r;x_2)) = \Sigma(x_k, x_j) + O \left( \frac{(\log r)^2}{r^{2/3}} \right). \]

In [24, Corollary 2.4], the authors also proved that the random variable \( (N(r) - \mu_r(r))/\sqrt{\sigma^2(r)} \) converges in distribution as \( r \to +\infty \) to a normal random variable with mean 0 and variance 1. Using Theorem 1.4, we obtain the following generalization of this result.

**Corollary 1.6.** Let \( 0 < x_1 < \ldots < x_m < +\infty \) be fixed and consider the random variables \( N_j^{(r)} \) defined by

\[ N_j^{(r)} = \frac{N(r;x_j) - \mu_r(r;x_j)}{\sqrt{\sigma^2(r;x_j)}}, \quad j = 1, \ldots, m. \]

As \( r \to +\infty \), we have

\[ (N_1^{(r)}, N_2^{(r)}, \ldots, N_m^{(r)}) \xrightarrow{d} \mathcal{N}(\vec{0}, I_m), \quad (1.19) \]

where \( I_m \) is the \( m \times m \) identity matrix, and \( \mathcal{N}(\vec{0}, I_m) \) is a multivariate normal random variable of mean \( \vec{0} = (0, \ldots, 0) \) and covariance matrix \( I_m \).
Proof. Let $a_1, \ldots, a_m \in \mathbb{R}$ be arbitrary and fixed (i.e. independent of $r$). It directly follows from (1.2) and (1.15) with $u_j = \frac{a_j}{\sqrt{3\pi} \log r}$, $j = 1, \ldots, m$, that
\[
\mathbb{E} \left[ \prod_{j=1}^{m} e^{a_j N_j(r)} \right] = \exp \left( \frac{\sum_{j=1}^{m} a_j^2}{2} + O \left( \frac{1}{\sqrt{\log r}} \right) \right), \quad \text{as } r \to +\infty.
\]
In other words, the moment generating function of $(N_1^{(r)}, N_2^{(r)}, \ldots, N_m^{(r)})$ converges as $r \to +\infty$ pointwise in $\mathbb{R}^m$ to the moment generating function of $N(0, I_m)$. This implies the convergence in distribution (1.19) by standard probability theorems, see e.g. [7, Corollary of Theorem 25.10].

Possible future applications of Theorem 1.4. Corollary 1.6 gives information about the joint fluctuations of the counting function at $m$ well-separated points $rx_1, \ldots, rx_m$. A more difficult question is to understand the global rigidity of the Pearcey process, that is, to understand the maximum fluctuation of the counting function. In recent years in random matrix theory, there has been a lot of progress in the study of rigidity of various point processes, see [29, 4] for important early works, [34] for the sine process, and [18] for the Airy and Bessel point processes. Of particular interest for us is the following result on the rigidity of the Pearcey process, which was proved in [16] (by combining results from [18] and [24]): for any $\epsilon > 0$, the probability that
\[
\mu_r(x) - \left( \frac{4\sqrt{2}}{3\pi} + \epsilon \right) \log x \leq N(x) \leq \mu_r(x) + \left( \frac{4\sqrt{2}}{3\pi} + \epsilon \right) \log x \quad \text{for all } x > r
\]
tends to 1 as $r \to +\infty$. Roughly speaking, this means that with high probability and for all large $x$, $N(x)$ lies in a tube centered at $\mu_r(x)$ and of width $(\frac{4\sqrt{2}}{3\pi} + 2\epsilon) \log x$, see Figure 1 (left). Equivalently, (1.20) can be rewritten for the normalized counting function as follows
\[
\lim_{r \to \infty} \mathbb{P} \left( \sup_{x > r} \left| \frac{N(x) - \mu_r(x)}{\log x} \right| \leq \frac{4\sqrt{2}}{3\pi} + \epsilon \right) = 1,
\]
see Figure 1 (right). It has also been conjectured in [16] that the upper bound (1.21) is sharp, in the sense that the following complementary lower bound is expected to hold: for any $\epsilon > 0$,
\[
\lim_{r \to \infty} \mathbb{P} \left( \sup_{x > r} \left| \frac{N(x) - \mu_r(x)}{\log x} \right| \geq \frac{4\sqrt{2}}{3\pi} - \epsilon \right) = 1,
\]
and this is also supported by Figure 1 (right). Such lower bounds are notoriously difficult to prove.

Figure 1: Rigidity of the Pearcey process (the pictures are taken from [16]). Left: the smooth blue lines correspond to the upper and lower bounds in (1.20) with $\epsilon = 0.05$, and $N(x)$ is the discontinuous blue line. Right: the blue line is $\frac{N(x) - \mu_r(x)}{\log x}$, and the four orange lines correspond to the constants $\pm \frac{4\sqrt{2}}{3\pi} + \epsilon$, $\pm \frac{4\sqrt{2}}{3\pi} - \epsilon$ with $\epsilon = 0.05$.

By analogy with the method developed in [22], we expect that Theorem 1.4 with $m = 2$ will be useful to establish (1.22). However, we also expect that proving (1.22) will also require other estimates that are not provided in this paper, such as the large $r$ asymptotics for $\mathbb{E}[e^{u_1 N(r x_1) + u_2 N(r x_2)}]$ when simultaneously $|x_1 - x_2| \to 0$. This regime requires a completely different analysis than the one of Theorem 1.4, and we shall not pursue this here.
The main result of this section is a differential identity which expresses \( \frac{\partial_r \log F(r \vec{x}, \vec{u})}{\partial r} \) in terms of the solution \( \Phi \) to a 3 \times 3 RH problem. In Section 3, we derive the system of equations (1.4) and establish the formula \( \frac{\partial_r \log F(r \vec{x}, \vec{u})}{\partial r} = 2H(r) \) by analyzing a natural Lax pair associated to \( \Phi \). In Section 4, we use the Deift–Zhou [26] steepest descent method to obtain the large \( r \) asymptotics of \( \Phi \). A main technical challenge here is to analyze the behavior of the global parametrix at certain points, which becomes particularly delicate for \( m \geq 2 \), see e.g. the asymptotic formulas of Subsection 4.3.2. The steepest descent analysis of \( \Phi \) for small \( r \) is simpler and is performed in Section 5. In Section 6, we use the small and large \( r \) asymptotics of \( \Phi \) together with some remarkable identities for the Hamiltonian to complete the proofs of Theorems 1.1, 1.3 and 1.4.

2 Differential identity

The main result of this section is a differential identity which expresses \( \frac{\partial_r \log F(r \vec{x}, \vec{u})}{\partial r} \) in terms of the solution \( \Phi \) to a 3 \times 3 RH problem.

It is well-known, see e.g. [43, Theorem 2], that the moment generating function (1.2) is equal to the Fredholm determinant

\[
F(r \vec{x}, \vec{u}) = \det (1 - K_{\rho}^{\text{Pe}}), \quad K_{\rho}^{\text{Pe}} := \sum_{j=1}^{m} (1 - s_j)K_{\rho}^{\text{Pe}}|_{rA_j},
\]

where \( s_j := e^{\mu_j + \cdots + \mu_m} \in (0, +\infty) \), \( j = 1, \ldots, m \),

\[
A_1 = (-x_1, x_1), \quad A_j = (-x_{j-1}, -x_j) \cup (x_{j-1}, x_j), \quad j = 2, \ldots, m,
\]

and \( K_{\rho}^{\text{Pe}}|_{rA_j} \) is the trace-class operator acting on \( L^2(rA_j) \) whose kernel is \( K_{\rho}^{\text{Pe}} \). We now recall a formula from [8] which expresses \( K_{\rho}^{\text{Pe}} \) in terms of the solution \( \Psi \) of a 3 \times 3 RH problem.

2.1 Background from [8]

**RH problem for \( \Psi \)**

(a) \( \Psi : \mathbb{C} \setminus \bigcup_{j=0}^5 \Sigma_j \cup \{0\} \rightarrow \mathbb{C}^{3 \times 3} \) is analytic, where

\[
\Sigma_0 = (0, +\infty), \quad \Sigma_1 = e^{2 \pi i} (0, +\infty), \quad \Sigma_2 = e^{-2 \pi i} (+\infty, 0),
\]

\[
\Sigma_3 = (-\infty, 0), \quad \Sigma_4 = e^{-2 \pi i} (+\infty, 0), \quad \Sigma_5 = e^{2 \pi i} (0, +\infty).
\]

(b) For \( z \in \bigcup_{j=0}^5 \Sigma_j \), we denote \( \Psi_+(z) \) (resp. \( \Psi_-(z) \)) for the limit of \( \Psi(s) \) as \( s \rightarrow z \) from the left (resp. right) of \( \bigcup_{j=0}^5 \Sigma_j \) (here “left” and “right” refer to the orientation of \( \bigcup_{j=0}^5 \Sigma_j \) as indicated in (2.2)).

For \( z \in \Sigma_j \), we have \( \Psi_+(z) = \Psi_-(z)J_j \), \( j = 0, \ldots, 5 \), where \( J_0, J_1, J_2, J_3, J_4 \) and \( J_5 \) are respectively given by

\[
\left(\begin{array}{ccc}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{array}\right), \quad 
\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 1 & 1
\end{array}\right), \quad 
\left(\begin{array}{ccc}
0 & 0 & 1 \\
0 & 1 & 0 \\
-1 & 0 & 0
\end{array}\right), \quad 
\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & -1 & 1
\end{array}\right), \quad 
\left(\begin{array}{ccc}
1 & 0 & 0 \\
1 & 1 & -1 \\
0 & 0 & 1
\end{array}\right).
\]

(c) As \( z \rightarrow \infty, \pm i \text{Im} z > 0 \),

\[
\Psi(z) = \sqrt{\frac{2\pi}{3}} e^\frac{\pi^2}{3} i \Psi_0 \left( I + \frac{\Psi_1}{z} + O(z^{-2}) \right) \text{diag}(z^{-\frac{i}{2}}, 1, z^\frac{i}{2}) L_3 e^{\theta(z)},
\]

where

\[
\Psi_0 = \left(\begin{array}{ccc}
1 & 0 & 0 \\
\kappa_3(\rho) + \frac{\rho^2}{2} & 0 & 0 \\
0 & 1 & 0
\end{array}\right), \quad 
\Psi_1 = \left(\begin{array}{ccc}
0 & \kappa_3(\rho) & 0 \\
\kappa_0(\rho) & 0 & \kappa_3(\rho) \pm \frac{\rho^2}{2} \\
0 & \kappa_0(\rho) & 0
\end{array}\right),
\]

\[
\kappa_3(\rho) = \frac{\rho^3}{54} - \frac{\rho}{6}, \quad \kappa_0(\rho) = \frac{\rho^6}{5832} - \frac{\rho^4}{162} + \frac{\rho^2}{72} + \frac{7}{32},
\]
\[ \tilde{\kappa}_0(\rho) = \kappa_0(\rho) + \frac{1}{3} \kappa_0(\rho) - \frac{1}{3}, \quad \kappa_0(\rho) = \kappa_0(\rho) - \kappa_3(\rho)^2 + \frac{\rho^2}{9} - \frac{1}{3}. \]

\[ L_+ = \begin{pmatrix} -\omega & \omega^2 & 1 \\ -1 & 1 & 1 \\ -\omega & \omega & 1 \end{pmatrix}, \quad L_- = \begin{pmatrix} \omega^2 & \omega & 1 \\ 1 & 1 & 1 \\ \omega & \omega^2 & 1 \end{pmatrix}, \]

\[ \Theta(z) = \begin{cases} \text{diag} (\theta_1(z), \theta_2(z), \theta_3(z)), & \text{Im } z > 0, \\ \text{diag} (\theta_2(z), \theta_1(z), \theta_3(z)), & \text{Im } z < 0, \end{cases} \quad \theta_k(z) = \frac{3}{4} \omega^{2k} z^4 + \frac{\rho}{2} \omega^k z^2, \quad k = 1, 2, 3, \quad (2.6) \]

and \( \omega = e^{2\pi i}. \)

(d) \( \Psi(z) \) remains bounded as \( z \to 0. \)

Consider the following functions

\[ \mathcal{P}_j(z) = \int_{\Gamma_j} e^{-\frac{4}{9}t^4 - \frac{8}{9}t^2 + utz} dt, \quad j = 0, 1, \ldots, 5, \quad (2.7) \]

where

\[ \Gamma_0 = (-\infty, \infty), \quad \Gamma_1 = (i\infty, 0] \cup [0, \infty), \quad \Gamma_2 = (i\infty, 0] \cup [0, -\infty), \]
\[ \Gamma_3 = (-i\infty, 0] \cup [0, -\infty), \quad \Gamma_4 = (-i\infty, 0) \cup [0, +\infty), \quad \Gamma_5 = (-i\infty, i\infty), \]

and define

\[ \tilde{\Psi}(z) = \begin{pmatrix} P_0(z) & P_1(z) & P_4(z) \\ P_0'(z) & P_1'(z) & P_4'(z) \end{pmatrix}, \quad z \in \mathbb{C}. \quad (2.8) \]

It was shown in [8, Section 8.1] that the RH problem for \( \Psi \) admits a unique solution which can be explicitly written in terms of \( \mathcal{P}_j, \quad j = 0, \ldots, 5. \) For example, for \( \arg z \in \left( \frac{\pi}{4}, \frac{3\pi}{4} \right), \) we have \( \Psi(z) = \tilde{\Psi}(z). \) The explicit expression of \( \Psi \) in the other sectors is not needed for us so we do not write it down, but we refer the interested reader to [8, equations (8.12)–(8.17)]. The Pearcey kernel can be written as follows (see [8, equation (10.19)]):

\[ K_{\rho \text{Pe}}(x, y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \tilde{\Psi}(y)^{-1} \tilde{\Psi}(x) \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^t, \quad x, y \in \mathbb{R}, \quad (2.9) \]

where \((\cdot)^t\) denotes the transpose operation. Let \( \tilde{K}_{\rho \text{Pe}} \) be the kernel of the operator \( \tilde{K}_{\rho \text{Pe}} \) appearing in (2.1):

\[ \tilde{K}_{\rho \text{Pe}}(x, y) := \sum_{j=1}^{m} (1 - s_j) K_{\rho \text{Pe}}(x, y) \chi_{rA_j}(y), \]

where \( \chi_{A_j} \) denotes the characteristic function of \( A_j, \) i.e. \( \chi_{A_j}(x) = 1 \) if \( x \in A_j \) and 0 otherwise. From (2.9), it is easy to see that \( \tilde{K}_{\rho \text{Pe}} \) can be written as

\[ \tilde{K}_{\rho \text{Pe}}(x, y) = \frac{f(x)^t h(y)}{x-y}, \quad (2.10) \]

where

\[ f(x) = \tilde{\Psi}(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad h(y) = \sum_{j=1}^{m} (1 - s_j) \chi_{rA_j}(y) \tilde{\Psi}(y)^{-t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \sum_{j=1}^{m} a_j \chi_{rB_j}(y) \tilde{\Psi}(y)^{-t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \quad (2.11) \]

with \( a_j := \frac{s_{j+1} - s_j}{2\pi i} \) and \( B_j = (-x_j, x_j), \) \( j = 1, \ldots, m. \) Since \( \tilde{u} \in \mathbb{R}^m, \) it follows from (1.2) that \( F(r\bar{x}, \bar{u}) \in (0, +\infty). \) Thus, by (2.1), we have \( \det(1 - \tilde{K}_{\rho \text{Pe}}) > 0 \) and in particular \( 1 - \tilde{K}_{\rho \text{Pe}} \) is invertible. Using now standard identities for trace-class operators, we obtain

\[ \partial_r \log F(r\bar{x}, \bar{u}) = \partial_r \log \det(1 - \tilde{K}_{\rho \text{Pe}}) = -\text{Tr} \left( (1 - \tilde{K}_{\rho \text{Pe}})^{-1} \tilde{\kappa}_0(\rho) \right) \]
\begin{align}
- \sum_{j=1}^{m} x_j \left( \lim_{v \searrow -r x_j} R(v,v) + \lim_{v \nearrow +r x_j} R(v,v) \right) + \sum_{j=1}^{m-1} x_j \left( \lim_{v \searrow -r x_j} R(v,v) + \lim_{v \nearrow +r x_j} R(v,v) \right),
\end{align}

where \( R \) is the kernel of the resolvent operator \((1 - \tilde{K}_\rho^{Pe})^{-1} \tilde{K}_\rho^{Pe} \). Formula (2.10) shows in particular that \( \tilde{K}_\rho^{Pe} \) is integrable (of size 3) in the sense of \([35]\). Hence, by \([25, \text{Lemma 2.12}]\), we have

\begin{align}
R(u,v) = F(u)H(v), \quad u,v \in \mathbb{R},
\end{align}

where

\begin{align}
F(u) = (1 - \tilde{K}_\rho^{Pe})^{-1} f(u), \quad H(v) = Y^{-1} h(v), \quad u,v \in \mathbb{R},
\end{align}

and \( Y \) is given by

\begin{align}
Y(z) = I - \int_{-r x_m}^{r x_m} \frac{F(w)h(w)}{w-z} dw.
\end{align}

Furthermore, \( Y \) is the unique solution to the following RH problem.

**RH problem for \( Y \)**

(a) \( Y : \mathbb{C} \setminus [-r x_m, r x_m] \to \mathbb{C}^{3 \times 3} \) is analytic.

(b) \( Y \) satisfies the jumps

\begin{align}
Y_+(x) = Y_-(x) (I - 2\pi i f(x) h(x) \psi), \quad x \in (-r x_m, r x_m) \setminus \bigcup_{j=1}^{m-1} \{-r x_j, r x_j\}.
\end{align}

(c) As \( z \to \infty \), \( Y(z) = I + \frac{Y_0}{z} + O(z^{-2}) \).

(d) As \( z \to z_0 \in \bigcup_{j=1}^{m-1} \{-r x_j, r x_j\} \), we have \( Y(z) = O(\log(z - z_0)) \).

Now, we apply a transformation which changes \( Y \) into another function \( \Phi \) whose jump matrices are piecewise constant. Following \([24, \text{eq (3.24)}]\), we define \( \Phi(z) = \Phi(z;r) \) as

\begin{align}
\Phi(z) = \begin{cases}
Y(z) \psi(z), & z \in I \cup III \cup IV \cup VI, \\
Y(z) \tilde{\psi}(z), & z \in II,
\end{cases}
\end{align}

where the regions I, II, III, IV, V and VI are shown in Figure 2. It is easily verified from the RH problems for \( Y \) and \( \psi \) that \( \Phi \) satisfies the following RH problem.
RH problem for $\Phi$

(a) $\Phi : \mathbb{C} \setminus \{ \cup_{j=0}^{m} \Sigma_j^{(r)} \cup \{ -rx_m, rx_m \} \} \to \mathbb{C}^{3 \times 3}$ is analytic, where

$$
\Sigma_0^{(r)} = (rx_m, +\infty), \quad \Sigma_1^{(r)} = rx_m + e^{\frac{\pi i}{3}}(0, +\infty), \quad \Sigma_2^{(r)} = -rx_m + e^{\frac{2\pi i}{3}}(0, +\infty),$$
$$
\Sigma_3^{(r)} = (-\infty, -rx_m), \quad \Sigma_4^{(r)} = -rx_m + e^{-\frac{2\pi i}{3}}(0, +\infty), \quad \Sigma_5^{(r)} = rx_m + e^{-\frac{2\pi i}{3}}(0, +\infty),
$$

and $\Sigma_6^{(r)} = (-rx_m, rx_m)$, see also Figure 2.

(b) For $z \in \Sigma_j^{(r)}$, we have $\Phi_j(z) = \Phi_j(z)J_j$, $j = 0, \ldots, 5$, where $J_0, J_1, J_2, J_3, J_4$ and $J_5$ are given by (2.3). For $z \in (-rx_m, rx_m) \setminus \cup_{j=1}^{m-1} \{ -rx_j, rx_j \}$, we have $\Phi_j(z) = \Phi_j(z)J_0(z)$, where

$$
J_0(z) = \begin{pmatrix} 1 & s_j & s_j \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in rA_j, \quad j = 1, \ldots, m.
$$

(c) As $z \to \infty$, $\pm \text{Im} z > 0$, we have

$$
\Phi(z) = \left( I + \frac{\Phi_1}{z} + \frac{\Phi_2}{z^2} + \mathcal{O}(z^{-3}) \right) \text{diag} \left( z^{-\frac{1}{4}}, 1, z^\frac{1}{2} \right) L_\lambda e^{\Theta(z)},
$$

where $\Phi_1, \Phi_2$ are independent of $z$ and $\Phi_1 = \Psi_1 + \Psi_0^{-1} Y_1 \Psi_0$.

(d) As $z \to rx_j$, $j = 1, \ldots, m$, we have

$$
\Phi(z) = \tilde{\Phi}_j(z) \begin{pmatrix} 1 & -s_j \log(z - rx_j) & -s_j \log(z - rx_j) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I, 1 - s_{j+1} & -s_{j+1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in \Pi,
$$

where we recall that $s_j := \frac{s_{j+1} - s_j}{2\pi i}$. The matrix $\tilde{\Phi}_j$ is analytic at $rx_j$ and satisfies

$$
\tilde{\Phi}_j(z) = \Phi_j^{(0)}(r) \left( I + \Phi_j^{(1)}(r)(z - rx_j) + \mathcal{O}((z - rx_j)^2) \right), \quad z \to rx_j,
$$

for some matrices $\Phi_j^{(0)}(r)$ and $\Phi_j^{(1)}(r)$.

(e) $\Phi$ satisfies the symmetry

$$
\Phi(z) = -\text{diag}(1, -1, 1) \Phi(-z) \text{B}, \quad \text{B} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.
$$

Proposition 2.1. We have

$$
\partial_t \log F(\vec{x}, \vec{u}) = -\sum_{j=1}^{m} 2s_j x_j \left[ \left( \Phi_j^{(1)}(r) \right)_{21} + \left( \Phi_j^{(1)}(r) \right)_{31} \right].
$$

Proof. The proof is a minor adaptation of [24, Proposition 3.5]. For $v \in \mathbb{R}$, by (2.11), (2.14) and (2.17), we have

$$
F(v) = \sqrt{\frac{2\pi}{3}} e^{\frac{\pi^2}{3}} i \Psi_0 \Phi_+ (v) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad H(v) = \sum_{j=1}^{m} s_j \chi_{rB_j}(v) \frac{-\Psi_0^{-t}}{\sqrt{2\pi} e^{\frac{\pi^2}{3}}} \Phi_+(v)^{-t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.
$$

Using (2.13), (2.17), (2.22) and (2.24), we find

$$
R(v, v) = \sum_{j=1}^{m} s_j \chi_{rB_j}(v) \left[ \left( \Phi_+(v)^{-1} \Phi_+(v) \right)_{21} + \left( \Phi_+(v)^{-1} \Phi_+(v) \right)_{31} \right] = R(-v, -v), \quad v \in \mathbb{R},
$$
which allows us to rewrite (2.12) as

$$\partial_t \log F(r \vec{x}, \vec{u}) = -2 \sum_{j=1}^{m} x_j \lim_{v \to r x_j} R(v, v) + 2 \sum_{j=1}^{m-1} x_j \lim_{v \to r x_j} R(v, v)$$

$$= -2 \sum_{j=1}^{m} x_j \mathfrak{g}_j \left( [\Phi_+(x_j)^{-1} \Phi'_+(x_j)]_{21} + [\Phi_+(x_j)^{-1} \Phi'_+(x_j)]_{31} \right).$$

By (2.20) and (2.21), $[\Phi_+(x_j)^{-1} \Phi'_+(x_j)]_{k1} = (\Phi_j^{(k)}(r))_{k1}$, for all $j = 1, \ldots, m$ and $k = 2, 3$, which finishes the proof.

3 Lax pair

In this section,

- we find an explicit solution to (1.4)–(1.5) in terms of $\Phi$,
- we prove the relation $\partial_t \log F(r \vec{x}, \vec{u}) = 2H(r)$,
- we derive some further identities for $H$ which will be useful in Section 6.

These results generalize part of the content of [24, Section 4] to an arbitrary $m$.

**Proposition 3.1.** The functions $(p_0, q_0, \{p_{j,1}, q_{j,1}, p_{j,2}, q_{j,2}, p_{j,3}, q_{j,3}\}_{j=1}^{m})$ defined by

$$p_0(r) := \frac{1}{\sqrt{2}}(\rho/3 + \Phi_{1,23}), \quad q_0(r) := \frac{1}{\sqrt{2}}(\rho/3 - \Phi_{1,12}),$$

$$(p_{j,1}(r), p_{j,2}(r), p_{j,3}(r)) := -\mathfrak{g}_j \Phi_j^{(0)}(r)^{-t} \left[ \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right], \quad (q_{j,1}(r), q_{j,2}(r), q_{j,3}(r)) := \Phi_j^{(0)}(r) \left[ \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right], \quad j = 1, \ldots, m,$$

satisfy (1.5) and the system of coupled equations (1.4).

**Remark 3.2.** Since $\Phi$ exists by (2.14), (2.15) and (2.17), Proposition 3.1 implies that there exists at least one solution to (1.4)–(1.5).

**Proof.** Following [24], we will proceed by analyzing the following Lax pair

$$L(z; r) := \partial_z \Phi(z; r) \cdot \Phi(z; r)^{-1}, \quad U(z; r) := \partial_r \Phi(z; r) \cdot \Phi(z; r)^{-1}.$$

Since the jump matrices of $\Phi$ are independent of $z$ and $r$, $L(z) = L(z; r)$ and $U(z) = U(z; r)$ are analytic in $\mathbb{C} \setminus \{-rx_m, \ldots, -rx_1, 0, rx_1, \ldots, rx_m\}$. Furthermore, using (2.22), we infer that they satisfy

$$L(z) = -\text{diag}(1, -1, 1)L(-z)\text{diag}(1, -1, 1), \quad U(z) = \text{diag}(1, -1, 1)U(-z)\text{diag}(1, -1, 1).$$

By (2.19), (3.1) and (3.3), we find

$$L(z) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} z + A_0(r) + \frac{L_1}{z} + O(z^{-2}), \quad \text{as } z \to \infty,$$

where

$$A_0(r) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 0 & 0 \end{pmatrix} + \Phi_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$L_1 = \begin{pmatrix} -1/3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} + \Phi_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \left[ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right] \Phi_1, \Phi_1 \right] + \left[ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right] \Phi_1, \Phi_1 \right] + \left[ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right] \Phi_1, \Phi_1 \right],$$

and where we have used the notation $[B_1, B_2] := B_1 B_2 - B_2 B_1$. Also, by (2.20)–(2.21) and (3.2), we have

$$L(z) = \frac{A_j(r)}{z - rx_j} + O(1), \quad \text{as } z \to rx_j, \quad j = 1, \ldots, m.$$
with
\[ A_j(r) = -s_j \Phi_j^{(0)}(r) \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Phi_j^{(0)}(r)^{-1} = \begin{pmatrix} q_{j,1} \\ q_{j,2} \\ q_{j,3} \end{pmatrix} \begin{pmatrix} p_{j,1} & p_{j,2} & p_{j,3} \end{pmatrix}. \] (3.7)

Since \( \det \Phi(z) \) is constant, we have \( \text{Tr}L(z) = \text{Tr}A_j(r) = \sum_{k=1}^{3} p_{j,k}(r)q_{j,k}(r) = 0 \), which already proves (1.5). Combining (3.3), (3.4) and (3.6), we have shown that
\[ L(z) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ z & 0 & 0 \end{pmatrix} + A_0(r) + \sum_{j=1}^{m} \begin{pmatrix} A_j(r) \\ z - rx_j \\ z - rx_j \end{pmatrix}, \] (3.8)

where
\[ A_{−j}(r) = \text{diag}(1, −1, 1)A_j(r)\text{diag}(1, −1, 1) = \begin{pmatrix} q_{j,1} \\ -q_{j,2} \\ q_{j,3} \end{pmatrix} \begin{pmatrix} p_{j,1} & -p_{j,2} & p_{j,3} \end{pmatrix}. \] (3.9)

For the computation of \( U \), we use (2.19) and (2.20)–(2.21) to obtain
\[ U(z) = O(z^{-1}), \quad \text{as } z \to \infty, \quad U(z) = -x_j \frac{A_j(r)}{z - rx_j} + O(1), \quad \text{as } z \to rx_j. \]

Using also (3.3), we conclude that
\[ U(z) = \sum_{j=1}^{m} \left( -x_j \frac{A_j(r)}{z - rx_j} + x_j \frac{A_{−j}(r)}{z + rx_j} \right). \] (3.10)

It remains to show that the functions \((p_0, q_0, \{p_{j,1}, q_{j,1}, p_{j,2}, q_{j,2}, p_{j,3}, q_{j,3}\}_{j=1}^{m})\) satisfy the system of equations (1.4). For this, we note that the compatibility condition \( \partial_2 \partial_3 \Phi(z) = \partial_1 \partial_3 \Phi(z) \) is equivalent to the relation
\[ \partial_1 L(z) - \partial_2 U(z) = [U(z), L(z)]. \] (3.11)

On the other hand, by (3.8) and (3.10), we have
\[ \partial_1 L(z) - \partial_2 U(z) = A_0'(r) + \sum_{j=1}^{m} \left( \frac{A_j'(r)}{z - rx_j} + \frac{A_{−j}(r)}{z + rx_j} \right). \] (3.12)

Substituting (3.8) and (3.10) in the above two equations, and then taking \( z \to \infty \), we get
\[ A_0'(r) = \sum_{j=1}^{m} x_j \begin{pmatrix} A_{−j}(r) - A_j(r) \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ -2 \sum_{j=1}^{m} x_j p_{j,3}(r)q_{j,2}(r) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right), \]
which yields the first two equations in (1.4). We now prove the last six equations of (1.4). A direct computation using (2.20) and (2.21) shows that
\[ x_j L(z) + U(z) = (x_j \partial_2 \Phi(z) + \partial_2 \Phi(z)) \Phi(z)^{-1} = \partial_1 \Phi_j^{(0)}(r) \cdot \Phi_j^{(0)}(r)^{-1} + o(1), \quad \text{as } z \to rx_j, \] (3.13)
and using (3.8) and (3.10), we get
\[ x_j L(z) + U(z) = M_j(r) - \frac{1}{r} A_j(r) + o(1), \quad \text{as } z \to rx_j, \] (3.14)

with
\[ M_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ rx_j^2 & 0 & 0 \end{pmatrix} + x_j A_0 + \frac{1}{r} \sum_{k=m}^{m} A_k = \begin{pmatrix} 0 & 0 & 0 \\ \frac{\sqrt{2} S_{11}}{r} & \frac{x_j}{r} & \frac{\sqrt{2} S_{31}}{r} \\ \frac{\sqrt{2} S_{22}}{r} & \frac{x_j}{r} & \frac{\sqrt{2} S_{33}}{r} \end{pmatrix}. \] (3.15)
In (3.15), we have omitted the \( r \)-dependence of various functions for notational convenience. Combining (3.13) and (3.14) yields
\[
\partial_r \Phi_j^{(0)}(r) = \left( M_j(r) - \frac{1}{r} A_j(r) \right) \Phi_j^{(0)}(r).
\]
Taking the first column of the above equation and using (1.5), (3.5) and (3.2), we get
\[
(q'_{j,1}(r) \ q'_{j,2}(r) \ q'_{j,3}(r))^t = M(r) (q_{j,1}(r) \ q_{j,2}(r) \ q_{j,3}(r))^t,
\]
and it is a direct computation to verify that (3.16) is equivalent to the third, fourth and fifth equations of (1.4). Finally, using (3.8), (3.10), (3.11) and (3.12), and letting \( z \to rx_j \), we get
\[
A'_j(r) = -[A_j(r), M_j(r)].
\]
Combining (3.17) with (3.5) and (3.16), we get
\[
(p'_j,1(r) \ p'_j,2(r) \ p'_j,3(r)) = - (p_{j,1}(r) \ p_{j,2}(r) \ p_{j,3}(r)) M(r),
\]
which yields the last three equations in (1.4).

For later use, we also note that by taking \( z \to \infty \) in (3.4) and then by reading the \( z^{-1} \) term of the (1.3) entry, we get
\[
\sum_{j=1}^{m} (A_{j,13}(r) + A_{-j,13}(r)) = 2S_{31}(r) = \frac{\rho}{3} + \Phi_{1,12}(r) - \Phi_{1,23}(r) = \rho - \sqrt{2}(p_0(r) + q_0(r)),
\]
where we have also used (3.8) and (3.1). In the rest of this section, we prove some identities for \( H \) which will be useful in Section 6.

**Proposition 3.3.** Let \( H \) be the Hamiltonian given in (1.6) with \( \{p_0, q_0, \{p_{j,1}, q_{j,1}, p_{j,2}, q_{j,2}, p_{j,3}, q_{j,3}\}_{j=1}^{m}\} \) defined as in (3.1)–(3.2). We have
\[
\partial_r \log F(rx, \tilde{u}) = 2H(r).
\]

**Proof.** By (2.23), the claim (3.19) is equivalent to
\[
H(r) = - \sum_{j=1}^{m} \bar{s}_j x_j \text{Tr} \left( \Phi_j^{(1)}(r) \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) = - \sum_{j=1}^{m} \bar{s}_j x_j \left[ \Phi_j^{(1)}(r) \right]_{21} + \left[ \Phi_j^{(1)}(r) \right]_{31}.
\]
Let \( \mathcal{H}(r) \) be the right-hand side of (3.20). We must show that \( H(r) = \mathcal{H}(r) \). By reading the \( \mathcal{O}(1) \) term in the expansion of \( \partial_r \Phi(z) = L(z) \Phi(z) \) as \( z \to rx_j \) (using (3.8) and (2.20)), we obtain
\[
\Phi_j^{(1)}(r) = \bar{s}_j \left[ \Phi_j^{(1)}(r), \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right]
+ \Phi_j^{(0)}(r)^{-1} \left( \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ rx_j & 0 & 0 \end{pmatrix} + A_0(r) + \sum_{\ell=1}^{m} \frac{A_{\ell}(r)}{rx_j - rr_{x_{\ell}}} + \frac{A_{-\ell}(r)}{rr_{x_{\ell}} + rx_{x_{\ell}}} + \frac{A_{-j}(r)}{2rx_j} \right) \Phi_j^{(0)}(r).
\]
Substituting (3.21) in the right-hand side of (3.20) leads to
\[
\mathcal{H}(r) = - \sum_{j=1}^{m} \bar{s}_j x_j \left( \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ rx_j & 0 & 0 \end{pmatrix} + A_0(r) + \sum_{\ell=1}^{m} \frac{A_{\ell}(r)}{rx_j - rr_{x_{\ell}}} + \frac{A_{-\ell}(r)}{rr_{x_{\ell}} + rx_{x_{\ell}}} + \frac{A_{-j}(r)}{2rx_j} \right) \Phi_j^{(0)}(r) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.
\]

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Inserting (3.2) and (3.5) in (3.22), we get
\[ \mathcal{H}(r) = \sum_{j=1}^{m} x_j \left( \frac{p_{j,1}}{p_{j,2}} \right)^t \begin{pmatrix} 0 & 1 & 0 \\ \sqrt{2p_0} & 0 & 1 \\ r x_j & \sqrt{2q_0} & 0 \end{pmatrix} \sum_{\ell=1}^{m} \left( A_{\ell} \right) \left( \frac{A_{\ell} - A_{\ell - \ell}}{r x_j + r x_{\ell}} \right) + \frac{A_{(j-1)} - A_{j}}{2r x_j} \right) \left( q_{j,1} \right) \left( q_{j,2} \right) \left( q_{j,3} \right). \]

The double sum in the above expression can be simplified as
\[ \sum_{j=1}^{m} x_j \left( \frac{p_{j,1}}{p_{j,2}} \right)^t \sum_{\ell=1}^{m} \left( A_{\ell} \right) \left( \frac{A_{\ell} - A_{\ell - \ell}}{r x_j + r x_{\ell}} \right) \left( q_{j,1} \right) \left( q_{j,2} \right) \left( q_{j,3} \right) = \frac{1}{2} \sum_{j=1}^{m} \left( p_{j,1} \right) \left( q_{j,1} \right) \left( q_{j,2} \right) \left( q_{j,3} \right). \]

Using (3.7) and (3.9), it is now a direct computation to check that indeed $\mathcal{H}(r) = H(r)$, which concludes the proof.

**Proposition 3.4.** Let $H$ be the Hamiltonian given in (1.6) with $\{p_0, q_0, \{p_{j,1}, q_{j,1}, p_{j,2}, q_{j,2}, p_{j,3}, q_{j,3}\}_j=1\}$ defined as in (3.1)–(3.2). We have
\[ p_0(r)q_0(r) + \sum_{j=1}^{m} \sum_{k=1}^{m} p_{j,k}(r)q_{j,k}(r) - H(r) \]
\[ = H(r) + \frac{1}{4} \frac{d}{dr} \left( 2p_0(r)q_0(r) + \sum_{j=1}^{m} \left[ p_{j,2}(r)q_{j,2}(r) + 2p_{j,3}(r)q_{j,3}(r) \right] - 3r H(r) \right). \]  

Furthermore,
\[ \partial_\gamma \left( p_0(r)q_0(r) + \sum_{j=1}^{m} \sum_{k=1}^{m} p_{j,k}(r)q_{j,k}(r) - H(r) \right) = \frac{d}{dr} \left( \sum_{j=1}^{m} \sum_{k=1}^{m} p_{j,k}(r) \partial_\gamma q_{j,k}(r) + p_0(r) \partial_\gamma q_0(r) \right) \]  

where $\gamma$ is any parameter among $u_1, \ldots, u_m$.

**Proof.** Formula (3.23) follows directly from (1.4) and (1.5), and formula (3.24) follows from (1.7) together with
\[ \partial_\gamma H(r) = \frac{\partial H}{\partial p_0}(r) \partial_\gamma p_0(r) + \frac{\partial H}{\partial q_0}(r) \partial_\gamma q_0(r) + \sum_{j=1}^{m} \sum_{k=1}^{m} \left( \frac{\partial H}{\partial p_{j,k}}(r) \partial_\gamma p_{j,k}(r) + \frac{\partial H}{\partial q_{j,k}}(r) \partial_\gamma q_{j,k}(r) \right). \]

\[ \square \]

4  Asymptotic analysis of $\Phi(z; r)$ as $r \to +\infty$

In this section, we perform a Deift-Zhou steepest descent analysis to obtain the large $r$ asymptotics of $\Phi$. The case $m = 1$ of this analysis was previously done in [24].

4.1 First transformation: $\Phi \to T$

Define
\[ T(z) = \text{diag} \left( r^{\frac{3}{4}}, 1, r^{-\frac{3}{4}} \right) \Phi(rz; r)e^{-\Theta(rz)}. \]  

The jumps for $T$ on $(-x_m, x_m)$ are given by
\[ T_+(z) = T_-(z) \begin{pmatrix} e^{\theta_2(z) - \theta_1(z)} & s_j e^{\theta_2(z) - \theta_3(z)} & s_j e^{\theta_2(z) - \theta_3(z)} \\ 0 & e^{\theta_1(z) - \theta_2(z)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad z \in (x_{j-1}, x_j), \]
\[ T_+(z) = T_-(z) \begin{pmatrix} e^{\theta_3(z) - \theta_1(z)} & s_j e^{\theta_2(z) - \theta_3(z)} & s_j e^{\theta_2(z) - \theta_3(z)} \\ 0 & 1 & 0 \\ 0 & 0 & e^{\theta_3(z) - \theta_2(z)} \end{pmatrix}, \quad z \in (-x_j, -x_{j-1}), \]
The next transformation is defined by $\gamma x$ respectively, starting at $T_{4.2}$. Second transformation:

where $T_{RH}$ problem for $S$ satisfies the following RH problem.

As $z \to \infty$, $\pm \Im z > 0$, we have

$$T(z) = \left( I + \frac{T_1}{z} + O(z^{-2}) \right) \mathrm{diag} \left( z^{-\frac{1}{2}}, 1, z^{\frac{1}{2}} \right) \Phi_1$$

where $T_1 = \frac{1}{z} \mathrm{diag} \left( r^\frac{1}{2}, 1, r^{-\frac{1}{2}} \right) \Phi_1 \mathrm{diag} \left( r^{-\frac{1}{2}}, 1, r^\frac{1}{2} \right)$.  

4.2 Second transformation: $T \to S$

For each $j = 1, \ldots, m$, let $\gamma_{j+}$ and $\gamma_{j-}$ be open curves, lying in the upper and lower half planes respectively, starting at $x_{j-1}$ and ending at $x_j$. We also orient the open curves $\gamma_{j+} := -\gamma_{j-}$ and $\gamma_{j-} := -\gamma_{j+}$ from $-x_j$ to $-x_{j-1}$. Let

$$\gamma_{m+1} := \Sigma_{m+1}^{(1)}, \quad \gamma_{m+1,-} := \Sigma_{m+1}^{(1)}, \quad \gamma_{m-1,+} := \Sigma_{m-1}^{(1)}, \quad \gamma_{m-1,-} := \Sigma_{m-1}^{(1)},$$

$s_{m+1} := 1, x_{m+1} := +\infty$ and for $j = 1, \ldots, m+1$, define

$$J_{\gamma_{j+},-}(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -e^{\theta_1(rz) - \theta_3(rz)} \\ 0 & 0 & 1 \end{pmatrix}, \quad J_{\gamma_{j+},+}(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{\theta_2(rz) - \theta_3(rz)} \\ 0 & 0 & 1 \end{pmatrix},$$

$$J_{\gamma_{j-},+}(z) = \begin{pmatrix} 1 & 0 & 0 \\ e^{\theta_3(rz) - \theta_1(rz)} & 1 & 0 \\ e^{\theta_3(rz) - \theta_2(rz)} & 0 & 1 \end{pmatrix}, \quad J_{\gamma_{j-},-}(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{\theta_3(rz) - \theta_1(rz)} \\ 0 & 0 & 1 \end{pmatrix}.$$  

The next transformation is defined by

$$S(z) = T(z) \begin{cases} J_{\gamma_{j+},-}(z), & \text{Im } z < 0 \text{ and } z \text{ above } \gamma_{j-}, \quad j \in \{1, \ldots, m\}, \\ J_{\gamma_{j+},+}(z)^{-1}, & \text{Im } z > 0 \text{ and } z \text{ below } \gamma_{j+}, \quad j \in \{1, \ldots, m\}, \\ J_{\gamma_{j-},+}(z)^{-1}, & \text{Im } z > 0 \text{ and } z \text{ below } \gamma_{j-}, \quad j \in \{1, \ldots, m\}, \\ J_{\gamma_{j-},-}(z), & \text{Im } z < 0 \text{ and } z \text{ above } \gamma_{j-}, \quad j \in \{1, \ldots, m\}, \\ I, & \text{otherwise}. \end{cases}$$

$S$ satisfies the following RH problem.

**RH problem for $S$**

(a) $S : \mathbb{C} \setminus \Sigma_S \to \mathbb{C}^{2\times 3}$ is analytic, where $\Sigma_S := (-\infty, +\infty) \cup \bigcup_{j=1}^{m+1} \left( \gamma_{j-} \cup \gamma_{j+} \cup \gamma_{j-} \cup \gamma_{j-}\right)$.  

(b) For $z \in \Sigma_S \setminus \bigcup_{j=0}^{m} \{-x_j, x_j\}$, $S_+(z) = S_-(z)J_S(z)$, where

$$J_S(z) = J_{\gamma_{j+},-}(z), \quad z \in \gamma_{j-}, \quad J_S(z) = J_{\gamma_{j+},+}(z), \quad z \in \gamma_{j+},$$
Inspired by \[24\], we consider three functions $d \in \bigcup \theta$

Using the definitions of 4.3 Global parametrix

\[ J_S(z) = \begin{cases} 0 & 0 \quad s_j \quad 0, \\ -s_j^{-1} & 0 \quad 0 \quad 0, \\ 0 & 0 \quad 0 \quad 1, \end{cases} \quad z \in \gamma_{j-1}, \quad J_S(z) = \begin{cases} 0 & 0 \quad s_j \quad 0, \\ 0 & 1 \quad 0 \quad 0, \\ -s_j^{-1} & 0 \quad 0 \quad 0, \end{cases} \quad z \in (-x_j, -x_{j-1}), \]

where $j = 1, \ldots, m, m + 1$.

(c) As $z \to \infty$, $\pm \text{Im} \ z > 0$, we have

\[ S(z) = \left( I + \frac{T_j}{z} + \mathcal{O}(z^{-2}) \right) \text{diag} \left( z^{-\frac{1}{2}}, 1, z^{\frac{1}{2}} \right) L_{\pm}. \]

(d) As $z \to z_+ \in \bigcup_{j=1}^m \{ -x_j, x_j \}$, we have $S(z) = \mathcal{O}(\log(z - z_+))$.

As $z \to 0$, $S(z) = \mathcal{O}(1)$.

(e) $S$ satisfies the symmetry $S(z) = -\text{diag}(1, -1, 1)S(-z)B$, where $B$ is defined in (2.22).

### 4.3 Global parametrix

Using the definitions of $\theta_1, \theta_2, \theta_3$ given in (2.6), it is easily checked that $J_S(z) \to I$ as $r \to \infty$ for each $z \in \bigcup_{j=1}^m \gamma_{j-1} \cup \gamma_{j+1} \cup \gamma_{j-2} \cup \gamma_{j+2}$. The following RH problem, whose solution is denoted $N$ and called the global parametrix, has the same jump conditions on $(-\infty, +\infty)$ than the RH problem for $S$, and no other jumps. We will show in Subsection 4.7 that $N$ is a good approximation to $S$ outside small neighborhoods of $\bigcup_{j=0}^m \{ -x_j, x_j \}$. The RH problem for $N$ is as follows.

**RH problem for $N$**

(a) $N : \mathbb{C} \setminus (-\infty, +\infty) \to \mathbb{C}^{3 \times 3}$ is analytic.

(b) $N$ satisfies the following jump relations:

\[ N_+(z) = N_-(z) \begin{pmatrix} 0 & 0 & s_j \\ 0 & 1 & 0 \\ -s_j^{-1} & 0 & 0 \end{pmatrix}, \quad z \in (-x_j, -x_{j-1}), \quad j = 1, \ldots, m, m + 1 \]

\[ N_+(z) = N_-(z) \begin{pmatrix} 0 & s_j & 0 \\ 0 & 0 & 0 \\ -s_j^{-1} & 0 & 0 \end{pmatrix}, \quad z \in (x_{j-1}, x_j), \quad j = 1, \ldots, m, m + 1. \]

(c) As $z \to \infty$, $\pm \text{Im} \ z > 0$, we have

\[ N(z) = \left( I + \frac{1}{z} N_1 + \mathcal{O}(z^{-2}) \right) \text{diag} \left( z^{-\frac{1}{2}}, 1, z^{\frac{1}{2}} \right) L_{\pm}, \quad (4.3) \]

for a certain matrix $N_1$.

(d) As $z \to z_+ \in \bigcup_{j=1}^m \{ -x_j, x_j \}$, $N(z) = \mathcal{O}(1)$.

As $z \to 0$, $N(z) = \mathcal{O}(1)\text{diag} \left( z^{-\frac{1}{2}}, 1, z^{\frac{1}{2}} \right)\mathcal{O}(1)$.

(e) $N$ satisfies the symmetry $N(z) = -\text{diag}(1, -1, 1)N(-z)B$.

For $m = 1$, the above RH problem was solved explicitly in [24, Section 5.3]. Let us define

\[ \beta_j := \frac{1}{2\pi i} u_j = \frac{1}{2\pi i} \log \frac{s_j}{s_{j+1}}, \quad j = 1, \ldots, m. \quad (4.4) \]

Inspired by [24], we consider three functions $d_1, d_2, d_3$ defined by

\[ d_1(z) = \begin{cases} \lambda(z^{\frac{1}{2}}), & \text{Im} \ z > 0, \\ \lambda(\omega^{-1}z^{\frac{1}{2}}), & \text{Im} \ z < 0, \end{cases} \]

\[ d_2(z) = \begin{cases} \lambda(\omega z^{\frac{1}{2}}), & \text{Im} \ z > 0, \\ \lambda(z^{\frac{1}{2}}), & \text{Im} \ z < 0, \end{cases} \]

\[ d_3(z) = \lambda(\omega z^{\frac{1}{2}}), \]
As $z \to \infty$, we have

$$N(z) = \left( I + \frac{1}{z} N_1 + O(z^{-2}) \right) \text{diag}(z^{-\frac{1}{4}}, 1, z^{\frac{1}{4}}) L_k \text{diag}(d_1(z), d_2(z), d_3(z)), \quad \pm \text{Im} z > 0,$$

where $N_1$ is of the form

$$N_1 = \begin{pmatrix} 0 & i\sqrt{3} \sum_{j=1}^{m} \beta_j x_j^{2/3} & 0 \\ 0 & 0 & i\sqrt{3} \sum_{j=1}^{m} \beta_j x_j^{2/3} \\ 0 & * & 0 \end{pmatrix}.$$

The entries $(N_1)_{21}$ and $(N_1)_{32}$ can also be computed explicitly, but their expressions are longer and not important for us.

### 4.3.2 Asymptotics of $N(z)$ as $z \to x_j$, $j = 1,\ldots, m$

As $z \to x_j$, $\text{Im} z > 0$, $j = 1,\ldots, m$,

$$d_1(z) = d_{1,x_j}^{(0)}(z - x_j)^{-\beta_j} (1 + d_{1,x_j}^{(1)}(z - x_j) + O((z - x_j)^2)),$$

$$d_{1,x_j}^{(0)} = \left( \frac{3\sqrt{3} x_j}{2} \right)^{\beta_j} e^{-\frac{\pi i}{3} \sum_{k \neq j} m \frac{(x_j^{2/3} - \omega x_k^{2/3})^{\beta_k}}{(x_j^{2/3} - x_k^{2/3})^{\beta_k}},$$

$$d_{1,x_j}^{(1)} = \frac{(\omega - 5)\beta_j}{6(\omega - 1)x_j} + \sum_{k \neq j} m \frac{2(1 - \omega) x_k^{2/3}}{3x_j^{1/3}(x_j^{2/3} - x_k^{2/3})(x_j^{2/3} - \omega x_k^{2/3})},$$

$$d_2(z) = d_{2,x_j}^{(0)}(z - x_j)^{\beta_j} (1 + d_{2,x_j}^{(1)}(z - x_j) + O((z - x_j)^2)),$$

$$d_{2,x_j}^{(0)} = \left( \frac{2}{3\sqrt{3} x_j} \right)^{\beta_j} e^{-\frac{\pi i}{3} \sum_{k \neq j} m \frac{(x_j^{2/3} - x_k^{2/3})^{\beta_k}}{(x_j^{2/3} - x_k^{2/3})^{\beta_k}},$$

$$d_{2,x_j}^{(1)} = \frac{(1 - 5\omega)\beta_j}{6(\omega - 1)x_j} + \sum_{k \neq j} m \frac{2(1 - \omega) x_k^{2/3}}{3x_j^{1/3}(x_j^{2/3} - x_k^{2/3})(x_j^{2/3} - \omega x_k^{2/3})},$$

### 4.3.1 Asymptotics of $N(z)$ as $z \to \infty$

As $z \to \infty$, $\pm \text{Im} z > 0$, we have

$$N(z) = \left( I + \frac{1}{z} N_1 + O(z^{-2}) \right) \text{diag}(z^{-\frac{1}{4}}, 1, z^{\frac{1}{4}}) L_k \text{diag}(d_1(z), d_2(z), d_3(z)), \quad \pm \text{Im} z > 0,$$

where $N_1$ is of the form

$$N_1 = \begin{pmatrix} 0 & i\sqrt{3} \sum_{j=1}^{m} \beta_j x_j^{2/3} & 0 \\ 0 & 0 & i\sqrt{3} \sum_{j=1}^{m} \beta_j x_j^{2/3} \\ 0 & * & 0 \end{pmatrix}.$$
where  

\[ d_3(z) = d_{3,x_j}^{(0)}(1 + d_{3,x_j}^{(1)}(z - x_j) + O((z - x_j)^2)), \]

\[ d_{3,x_j}^{(0)} = e^{-\pi i} \prod_{k = 1, k \neq j}^{m} \frac{(x_j^{2/3} - \omega x_k^{2/3})^{\beta_k}}{(x_j^{2/3} - \omega^{2/3})^{\beta_k}}, \]

\[ d_{3,x_j}^{(1)} = \frac{2(\omega + 1)\beta_j}{3(\omega - 1)x_j} \sum_{k = 1, k \neq j}^{m} \frac{2(\omega^2 - \omega)x_k^{2/3} \beta_k}{3x_j^{2/3}(x_j^{2/3} - \omega x_k^{2/3})(x_k^{2/3} - \omega^2 x_j^{2/3})}. \]

The above asymptotic expansions hold as \( z \to x_j \), \( \Im z > 0 \), and the jumps are given by

\[ |m \prod_{k = 1, k \neq j}^{m} \frac{(x_j^{2/3} - \omega x_k^{2/3})^{\beta_k}}{(x_j^{2/3} - \omega^{2/3})^{\beta_k}}| = \exp \left( -\sum_{k = 1, k \neq j}^{m} i\beta_k \arctan \frac{\sqrt{3} x_k^{2/3}}{x_j^{2/3} + 2x_j^{2/3}} - \sum_{k = j + 1}^{m} \pi i \beta_k \right), \]

\[ \arg \prod_{k = 1, k \neq j}^{m} \frac{(x_j^{2/3} - \omega x_k^{2/3})^{\beta_k}}{(x_j^{2/3} - \omega^{2/3})^{\beta_k}} = -\sum_{k = 1, k \neq j}^{m} i\beta_k \log \left| \frac{x_j^{2/3} - \omega x_k^{2/3}}{x_j^{2/3} - \omega^{2/3}} \right| \mod 2\pi. \]

Hence, as \( z \to x_j \), \( \Im z > 0 \), \( j = 1, \ldots, m \),

\[ N(z) = (N_{x_j}^{(0)} + (z - x_j)N_{x_j}^{(1)} + O((z - x_j)^2))(z - x_j)^{-\beta_j \sigma_{3,1}}, \quad (4.11) \]

where \( \sigma_{3,1} = \text{diag} (1, -1, 0) \) and

\[ N_{x_j}^{(0)} = C_N \text{diag} (x_j^{-1/3}, 1, x_j^{1/3}) L_+ \text{diag} (d_{1,x_j}^{(0)}, d_{2,x_j}^{(0)}, d_{3,x_j}^{(0)}), \]

\[ N_{x_j}^{(1)} = C_N \left( \text{diag} (x_j^{-1/3}, 1, x_j^{1/3}) L_+ \text{diag} (d_{1,x_j}^{(0)}, d_{1,x_j}^{(1)}, d_{2,x_j}^{(0)}, d_{2,x_j}^{(1)}, d_{3,x_j}^{(0)}, d_{3,x_j}^{(1)}) + \text{diag} (-\frac{1}{3}x_j^{-2/3}, 0, \frac{1}{3}x_j^{-2/3}) L_+ \text{diag} (d_{1,x_j}^{(0)}, d_{2,x_j}^{(0)}, d_{3,x_j}^{(0)}) \right). \]

### 4.3.3 Asymptotics of \( N(z) \) as \( z \to 0 \)

As \( z \to 0, \Im z > 0 \),

\[ d_{\ell}(z) = d_{\ell,0}^{(0)}(1 + d_{\ell,0}^{(1)}(z - x_j) + O(z^2)), \quad \ell = 1, 2, 3, \]

\[ d_{1,0}^{(0)} = e^{-\pi i} \sum_{k = 1}^{m} \beta_k (-\omega) \sum_{k = 1}^{m} \beta_k = s_1^{-\frac{1}{3}}, \quad d_{2,0}^{(0)} = d_{3,0}^{(0)} = \omega \sum_{k = 1}^{m} \beta_k = s_1^{-\frac{1}{3}}, \]

\[ d_{1,0}^{(1)} = (1 - \omega^2) \sum_{k = 1}^{m} \beta_k x_k^{-\frac{2}{3}}, \quad d_{2,0}^{(1)} = (\omega - 1) \sum_{k = 1}^{m} \beta_k x_k^{-\frac{2}{3}}, \quad d_{1,1}^{(0)} = (\omega^2 - \omega) \sum_{k = 1}^{m} \beta_k x_k^{-\frac{2}{3}}, \]

where the branches are the principal ones. As \( z \to 0, \Im z > 0 \),

\[ N(z) = C_N \text{diag} (-z^{-\frac{1}{3}}, 1, z^{\frac{1}{3}}) L_+ \text{diag} (d_{1,0}^{(0)}, d_{1,0}^{(1)}, d_{2,0}^{(0)}, d_{2,0}^{(1)}, d_{3,0}^{(0)}, d_{3,0}^{(1)}) \left( I + z^{\frac{2}{3}} \text{diag} (d_{1,0}^{(1)}, d_{2,0}^{(1)}, d_{3,0}^{(1)}) + O(z^{\frac{5}{3}}) \right). \quad (4.13) \]

#### Local parametrices.

For each \( p \in \{-x_m, \ldots, -x_1, 0, x_1, \ldots, x_m\} \), let \( \mathcal{D}_p \) be a small open disk centered at \( p \). The local parametrix \( P^{(p)} \) is defined inside \( \mathcal{D}_p \), has the same jumps as \( S \) in \( \mathcal{D}_p \), and satisfies \( S(z)P^{(p)}(z)^{-1} = O(1) \) as \( z \to p \). Furthermore, we require \( P^{(p)} \) to satisfy the following matching condition with \( P^{(\infty)} \) on \( \partial \mathcal{D}_p \):

\[ P^{(p)}(z) = (I + o(1))P^{(\infty)}(z), \quad \text{as } r \to +\infty, \quad (4.14) \]

uniformly for \( z \in \partial \mathcal{D}_p \).

### 4.4 Local parametrices near \( x_j, \ j = 1, \ldots, m \)

Since the \((1,2)\) and \((2,1)\) entries of \( J_3(z) \) have each a discontinuity at \( z = x_j \), we can follow [36] and build \( P^{(x_j)} \) using the model RH problem \( \Phi_{\mathcal{HC}} \) which is presented in Appendix A. We also refer to [24, Section
5.5] for more details about this construction. The local parametrix $P^{(x_j)}$ is of the form

$$P^{(x_j)}(z) = E_{x_j}(z) \begin{pmatrix} \Phi_{HG,11}(r^{\frac{3}{4}}f_{x_j}(z); \beta_j) & \Phi_{HG,12}(r^{\frac{3}{4}}f_{x_j}(z); \beta_j) \\ \Phi_{HG,21}(r^{\frac{3}{4}}f_{x_j}(z); \beta_j) & \Phi_{HG,22}(r^{\frac{3}{4}}f_{x_j}(z); \beta_j) \end{pmatrix} \times (s_j s_{j+1})^{-\sigma_3,1} e^{\frac{2}{3}i(\theta_2(r_{x_j}) - \theta_1(r_{x_j})) \sigma_3,1} A_{x_j}(z),$$

where $\sigma_3,1 = \text{diag}(1, -1, 0)$, $\pm$ stands for $\pm \text{Im } z > 0$, $E_{x_j}$ is analytic in $\mathcal{D}_{x_j}$ and given by

$$E_{x_j}(z) = N(z)(s_j s_{j+1})^{-\sigma_3,1} \begin{pmatrix} \frac{x_j - x_{j+1}}{s_j} \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{Im } z > 0$$

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{Im } z < 0$$

and $f_{x_j}$ is given by

$$f_{x_j}(z) = r^{-\frac{3}{4}} [(\theta_2(r_{x_j}) - \theta_1(r_{x_j})) - (\theta_2(r_{x_j}) - \theta_1(r_{x_j}))] = \frac{i \sqrt{3}}{4} \left(3(z^\frac{3}{2} - x_j^\frac{3}{2}) - \frac{2\rho}{r^2}(z^\frac{3}{2} - x_j^\frac{3}{2})\right).$$

It is easily checked that $A_{x_j}(z)$ is exponentially small as $r \to +\infty$ uniformly for $z \in \mathcal{D}_{x_j}$, and that

$$f_{x_j}(x_j) = 0, \quad f_{x_j}'(x_j) = i \left(\sqrt{3}x_j^{1/3} - \frac{\rho}{\sqrt{3}x_j^{1/3} r^{2/3}}\right), \quad f_{x_j}''(x_j) = i \left(\frac{1}{\sqrt{3}x_j^{2/3}} + \frac{\rho}{3\sqrt{3}x_j^{4/3} r^{2/3}}\right).$$

Using (4.11), we obtain

$$E_{x_j}(x_j) = N(z)\frac{1}{\sqrt{s_j + 1}} e^{-\frac{1}{2}(\theta_2(r_{x_j}) - \theta_1(r_{x_j})) \sigma_3,1} (r^{\frac{3}{4}} |f'(x_j)|)^{\beta_j} \sigma_3,1,$$

$$E_{x_j}(x_j)^{-1} E_{x_j}'(x_j) = \begin{pmatrix} f_{x_j}'(x_j) & \frac{d_0^{(1)}}{d_{x_j}^{(2)}} \\ \frac{f_{x_j}''(x_j)}{f_{x_j}'(x_j)} & \frac{d_0^{(2)}}{d_{x_j}^{(2)}} \end{pmatrix}, \quad E_{x_j}^{-1}(z) = \begin{pmatrix} \frac{d_0^{(2)}}{d_{x_j}^{(2)}} & -\frac{d_0^{(1)}}{d_{x_j}^{(2)}} \\ \frac{d_0^{(2)}}{d_{x_j}^{(2)}} & \frac{d_0^{(2)}}{d_{x_j}^{(2)}} \end{pmatrix}, \quad \sigma_j = \sqrt{s_j + 1} e^{-\frac{1}{2}(\theta_2(r_{x_j}) - \theta_1(r_{x_j}))} (r^{\frac{3}{4}} |f'(x_j)|)^{\beta_j}.$$

Using (A.2), we obtain

$$P^{(x_j)}(z)N(z)^{-1} = I + \frac{1}{r^{3/4} f_{x_j}(z)} E_{x_j}(z) \begin{pmatrix} \Phi_{HG,1}(\beta_j)_{11} & \Phi_{HG,1}(\beta_j)_{12} \\ \Phi_{HG,1}(\beta_j)_{21} & \Phi_{HG,1}(\beta_j)_{22} \end{pmatrix} 0 1 E_{x_j}(z)^{-1} + O(r^{-\frac{3}{4}}),$$

as $r \to +\infty$ uniformly for $z \in \partial \mathcal{D}_{x_j}$.

4.5 Local parametrix near $-x_j$, $j = 1, \ldots, m$

$P^{(-x_j)}$ can be constructed in terms of $\Phi_{HG}$ in a similar way as $P^{(x_j)}$. Alternatively, we can use the symmetry stated in condition (e) of the RH problem for $S$. This observation saves us some effort and allows us to see immediately that

$$P^{(-x_j)}(z) = -\text{diag}(1, -1, 1) P^{(x_j)}(z) B, \quad z \in \mathcal{D}_{-x_j}.$$
4.6 Local parametrix near 0

For the local parametrix $P^{(0)}$, we need to use the model RH problem for $\Psi$ from [8] (and which is recalled in Subsection 2.1 for the convenience of the reader). Define

$$ P^{(0)}(z) = E_0(z)^{-1} P_0(z) e^{-\Theta(rz)} \text{diag}(s_{1}^{-\frac{2}{3}}, s_{1}^{-\frac{1}{3}}, s_{1}^{\frac{1}{3}}), $$

where $E_0$ is analytic inside $D_0$ and given by

$$ E_0(z) = -\sqrt{\frac{3}{2\pi}} \sqrt{\frac{\gamma}{\pi}} \tau N(z) \text{diag}(s_{1}^{\frac{2}{3}}, s_{1}^{-\frac{1}{3}}, s_{1}^{-\frac{1}{3}}) L^{-1}\text{diag}((rz)^{\frac{1}{2}}, 1, (rz)^{-\frac{1}{2}}) \Psi^{-1}, \quad \pm \text{Im} z > 0, $$

and

$$ \Psi_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \kappa_3 + \frac{\rho^3}{\rho^3} & 0 & 1 \end{pmatrix}, \quad \kappa_3 = \frac{\rho^3}{54} - \frac{\rho}{6}. $$

In a similar way as in [24, Proposition 5.13], we verify that $P^{(0)}$ has the same jumps as $S$ inside $D_0$. It is also clear from condition (d) of the RH problem for $\Psi$ that $P^{(0)}(z)$ remains bounded as $z \to 0$. Finally, using (4.22) and (2.4), we obtain

$$ P^{(0)}(z) N(z)^{-1} = I + \frac{1}{r.z} \hat{E}_0(z) \tilde{\Psi}_1 \hat{E}_0(z)^{-1} + O(r^{-\frac{2}{3}}), \quad \tilde{\Psi}_1 = \begin{pmatrix} 0 & \kappa_3 & 0 \\ 0 & 0 & \kappa_3 + \frac{\rho^3}{\rho^3} \\ 0 & 0 & 0 \end{pmatrix}, $$

as $r \to +\infty$ uniformly for $z \in \partial D_0$, where

$$ \hat{E}_0(z) := N(z) \text{diag}(s_{1}^{\frac{2}{3}}, s_{1}^{-\frac{1}{3}}, s_{1}^{-\frac{1}{3}}) L^{-1}\text{diag}(z^{\frac{1}{2}}, 1, z^{-\frac{1}{2}}). $$

For future reference, using (4.13) we note that $\hat{E}_0(0) = C_N$.

4.7 Final transformation

Define

$$ R(z) = \begin{cases} S(z) N(z)^{-1}, & z \in \mathbb{C} \setminus \left( \cup_{j=1}^{m} (D_{x_j} \cup D_{-x_j}) \cup D_0 \right), \\ S(z) P^{(0)}(z)^{-1} & z \in D_p, \quad p \in \{-x_m, \ldots, -x_1, 0, x_1, \ldots, x_m\}. \end{cases} $$

(4.24)

Using the analysis of Subsections 4.4–4.6, we conclude that $R$ is analytic in $\cup_{j=1}^{m} (D_{x_j} \cup D_{-x_j}) \cup D_0$, and therefore $R$ is analytic in $\mathbb{C} \setminus \Sigma_R$, where

$$ \Sigma_R := \cup_{j=1}^{m} (\partial D_{x_j} \cup \partial D_{-x_j}) \cup \partial D_0 \cup \Sigma_S \setminus \left( \cup_{j=1}^{m} (D_{x_j} \cup D_{-x_j}) \cup D_0 \cup \mathbb{R} \right). $$

For convenience, we orient the boundaries of the $2m+1$ disks in the clockwise direction, and for $z \in \Sigma_R$, we define $J_{R}(z) := R_{-1}(z) R_{+}(z)$. Using the definitions (2.6) of $\theta_1, \theta_2, \theta_3$, condition (b) of the RH problem for $S$ and (4.24), we verify that

$$ J_{R}(z) = I + O(c^{-\alpha}), \quad \text{as } r \to +\infty \text{ uniformly for } z \in \Sigma_R \setminus \left( \cup_{j=1}^{m} (D_{x_j} \cup D_{-x_j}) \cup D_0 \right) $$

for a certain $c > 0$. Also, by (4.20), (4.21) and (4.23), we have

$$ J_{R}(z) = I + O(r^{-\frac{1}{3}}), \quad \text{as } r \to +\infty \text{ uniformly for } z \in \cup_{j=1}^{m} (\partial D_{x_j} \cup \partial D_{-x_j}), $$

$$ J_{R}(z) = I + \frac{J^{(1)}(z)}{r^{\frac{1}{3}}} + O(r^{-\frac{1}{3}}), \quad \text{as } r \to +\infty \text{ uniformly for } z \in \partial D_0, $$

where $J^{(1)}(z) = \frac{1}{z} \hat{E}_0(z) \tilde{\Psi}_1 \hat{E}_0(z)^{-1}$. Hence, $R$ satisfies a small norm RH problem [26], and we have

$$ R(z) = I + R^{(1)}(z) r^{-\frac{1}{3}} + O(r^{-\frac{1}{3}}), \quad \text{as } r \to +\infty, $$

(4.25)

uniformly for $z \in \mathbb{C} \setminus \Sigma_R$, with

$$ R^{(1)}(z) = \frac{1}{2\pi i} \int_{\partial D_0} \frac{J^{(1)}(x)}{x - z} dx = \begin{cases} \frac{C_N \hat{\Psi}_1 C_N^{-1}}{z}, & z \notin D_0, \\ \frac{C_N \hat{\Psi}_1 C_N^{-1}}{z} - J^{(1)}(z), & z \in D_0, \end{cases} $$

(4.26)
and where we recall that $\partial \mathcal{D}_0$ is oriented in the clockwise direction. (the method of [26] also implies that $R$ exists for all sufficiently large $r$; note however that here we already know from (2.15) and from \( \det(1 - \mathcal{K}^p) > 0 \) that $Y$ exists for all $r > 0$, which implies by the transformations $Y \to \Phi \to T \to S \to R$ that $R$ also exists for all $r > 0$). Finally, the same analysis as in [20, Section 3.5] shows that for any $k_1, \ldots, k_m \in \mathbb{N}_{\geq 0}$ with $k_1 + \ldots + k_m \geq 1$, we have

$$
\partial_{u_1} \ldots \partial_{u_m} R(z) = \partial_{u_1} \ldots \partial_{u_m} R^{(1)}(z) r^{-\frac{j}{4}} + \mathcal{O}((\log r)^{k_1+\ldots+k_m} r^{-\frac{j}{4}}), \quad \text{as } r \to +\infty,
$$

(4.27) uniformly for $z \in \mathbb{C} \setminus \Sigma_R$.

5 Asymptotic analysis of $\Phi$ as $r \to 0$

The analysis of $\Phi$ as $r \to 0$ is much simpler than the large $r$ analysis of Section 4. Here we generalize [24, Section 6] to an arbitrary $m$. Let $\delta > 0$ be fixed. Define

$$
\widetilde{N}(z) := -\sqrt{\frac{2\pi}{2\pi}} e^{-\frac{\pi^2}{2\pi} i} \Phi_0^{-1} \Psi(z) \times \begin{cases}
J_1, & \text{for } \arg z < \frac{\pi}{4} \text{ and } \arg(z - rx_m) > \frac{\pi}{4}, \\
J_2, & \text{for } \arg z > \frac{3\pi}{4} \text{ and } \arg(z + rx_m) < \frac{3\pi}{4}, \\
J_1^{-1}, & \text{for } \arg z < -\frac{3\pi}{4} \text{ and } \arg(z + rx_m) > -\frac{3\pi}{4}, \\
J_2^{-1}, & \text{for } \arg z > -\frac{3\pi}{4} \text{ and } \arg(z - rx_m) < -\frac{3\pi}{4},
\end{cases}
$$

(5.1)

and for $|z| < \delta$, define

$$
\widetilde{R}^{(0)}(z) := -\sqrt{\frac{2\pi}{2\pi}} e^{-\frac{\pi^2}{2\pi} i} \Phi_0^{-1} \Psi(z)
$$

$$
\times \left( I + \sum_{j=1}^{m} \frac{s_j - s_j+1}{2\pi i} \left( \log (z - rx_j) - \log (z + rx_j) \right) \frac{1}{0 1 1 0 0 0 0 0} \right)
$$

(5.2)

where the principal branches are chosen for the log, and we recall that $s_{m+1} := 1$, the regions $I, II, III, IV, V$ and VI are shown in Figure 2, $\Psi$ is defined in (2.8), and the matrices $J_j$, $j = 0, \ldots, 5$ are defined in (2.3). Let

$$
\widetilde{R}(z) := \begin{cases}
\Phi(z) \widetilde{N}(z)^{-1}, & |z| > \delta, \\
\Phi(z) \widetilde{R}^{(0)}(z)^{-1}, & |z| < \delta.
\end{cases}
$$

(5.3)

The definitions (5.1) and (5.2) also ensure that $\widetilde{R}$ has no jumps on $\cup_{j=0}^{5} S^{(r)}_j$. Since

$$
\widetilde{R}^{(0)}(z)^{-1} \widetilde{R}^{(0)}(z) = J_5 J_0 J_1 \begin{pmatrix}
1 & s_j - 1 & s_j - 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & s_j & s_j \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} = \Phi_-(z)^{-1} \Phi_+(z)
$$

(5.4)

holds for all $z \in (-rx_j, -rx_{j-1}) \cup (rx_{j-1}, rx_j)$, it follows that $\widetilde{R}(z)$ is also analytic in $(-rx_m, rx_m) \setminus \cup_{j=1}^{m-1} (-rx_j, rx_j)$. Furthermore, from a direct inspection of (5.2) and (5.3), we see that the singularities of $\widetilde{R}(z)$ at $z = -rx_m, \ldots, -rx_1, 0, rx_1, \ldots, rx_m$ are removable. Hence, $\widetilde{R}(z)$ is analytic for $z \in \mathbb{C} \setminus \{z : |z| = \delta\}$. Let us orient the circle $|z| = \delta$ in the clockwise direction, and define $J_R := -\mathcal{R}^{-1}\mathcal{R}$. By (5.3), $\mathcal{J}_R(z) = \mathcal{R}^{(0)}(z) \mathcal{N}(z)^{-1}$, and by (2.4), (2.19) and (5.3), $\mathcal{R}(z) \to I + \mathcal{O}(z^{-1})$ as $z \to \infty$. We also check using (5.1) and (5.2) that $J_R(z) = I + \mathcal{O}(r)$ as $r \to 0$ uniformly for $|z| = \delta$. Thus

$$
\mathcal{R}(z) \to I + \mathcal{O}(r), \quad \text{as } r \to 0 \text{ uniformly for } z \in \mathbb{C} \setminus \{z : |z| = \delta\}.
$$

(5.5)

6 Proof of the main results

6.1 Proof of Theorem 1.1

We already proved in Section 3 that the functions $(p_0, q_0, \{p_{j_1}, q_{j_1}, \ldots, p_{j_k}, q_{j_k} \}_{j=1}^{m_1})$ defined by (3.1)–(3.2) exist and satisfy (1.4)–(1.5). In this subsection we complete the proof of Theorem 1.1 by obtaining the asymptotic formulas (1.8) and (1.11).
Asymptotics of $p_0$, $q_0$, $p_{j,k}$ and $q_{j,k}$ as $r \to +\infty$

We first compute the asymptotics of $p_0$ and $q_0$ using (3.1). Using (4.25) and (4.26), we see that $R(z) = I + \frac{R_0}{z} + O(z^{-2})$ as $z \to \infty$, where

$$R_1 = C_N \Phi_0 C_N^{-1} r^{-\frac{\rho}{3}} + O(r^{-\frac{\rho}{2}}), \quad \text{as } r \to +\infty. \quad (6.1)$$

Inverting the transformations $\Phi \mapsto T \mapsto S \mapsto R$ in the region outside the disks using (4.1), (4.2) and (4.24), we get

$$\Phi(rz) = \text{diag} (r^{-\frac{\rho}{3}}, 1, r^{\frac{\rho}{6}}) R(z) N(z) e^{\Theta(rz)}, \quad z \in \mathbb{C} \setminus \bigcup_{j=1}^{m} (D_{x_j} \cup D_{-x_j} \cup D_0).$$

From this expression, (2.19) and (4.9), we deduce that

$$\Phi_1 = r \text{diag} (r^{-\frac{\rho}{3}}, 1, r^{\frac{\rho}{6}})(R_1 + N_1) \text{diag} (r^{\frac{\rho}{6}}, 1, r^{-\frac{\rho}{3}}),$$

which implies by (4.10) and (6.1) that

$$\Phi_{1,12} = \frac{i}{\sqrt{3}} \rho \sum_{j=1}^{m} \beta_j x_j + \frac{\rho^3}{54} - \frac{\rho}{6} + O(r^{-\frac{\rho}{2}}), \quad \Phi_{1,23} = \frac{i}{\sqrt{3}} \rho \sum_{j=1}^{m} \beta_j x_j + \frac{\rho^3}{54} - \frac{\rho}{6} + O(r^{-\frac{\rho}{2}}),$$

as $r \to +\infty$. Substituting these asymptotics in (3.1) gives the large $r$ asymptotics of $p_0$ and $q_0$ stated in (1.8a) and (1.8e).

We now compute the large $r$ asymptotics of $p_{j,k}$, $j = 1, \ldots, m$, $k = 1, 2, 3$ using the definitions (3.2). It follows from (2.20) and (2.21) that

$$\Phi_j^{(0)}(r) = \lim_{z \to \infty} \Phi_j(r) \begin{pmatrix} 1 & s_j \log(rz - rx_j) & s_j \log(rz - rx_j) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (6.2)$$

For $z \in D_{x_j}$ and $z$ outside the lenses, by (4.1), (4.2) and (4.24) we have

$$\Phi(rz) = \text{diag} (r^{-\frac{\rho}{3}}, 1, r^{\frac{\rho}{6}}) R(z) P_{j}(x_j)(z)e^{\Theta(rz)}. \quad (6.3)$$

Hence, using also (4.15), we get

$$\Phi_j^{(0)}(r) = \text{diag} (r^{-\frac{\rho}{3}}, 1, r^{\frac{\rho}{6}}) R(x_j) E_{x_j}(x_j) \lim_{z \to \infty} \begin{pmatrix} \Phi_{HG,11}(r^{\frac{\rho}{6}} f_{x_j}(z); \beta_j) & \Phi_{HG,12}(r^{\frac{\rho}{6}} f_{x_j}(z); \beta_j) & 0 \\ \Phi_{HG,21}(r^{\frac{\rho}{6}} f_{x_j}(z); \beta_j) & \Phi_{HG,22}(r^{\frac{\rho}{6}} f_{x_j}(z); \beta_j) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\times (s_j s_{j+1})^{-\frac{\rho}{6 \pi}} \widetilde{\Theta}(z) \begin{pmatrix} 1 & s_j \log(rz - rx_j) & s_j \log(rz - rx_j) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (6.4)$$

where

$$\widetilde{\Theta}(z) = \frac{1}{e^{\frac{1}{2} (\theta_2(rz) - \theta_1(rz)) \sigma_1}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{\theta_2(rz) - \theta_1(rz)} \\ 0 & 0 & 1 \end{pmatrix} e^{\Theta(rz)} = \frac{1}{e^{\frac{1}{2} \theta_1(rz)}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\frac{1}{2} \theta_1(rz)} \end{pmatrix}. \quad (6.5)$$

Using (4.4), we note that

$$\frac{\sin(\pi \beta_j)}{\pi} = \frac{1}{\Gamma(\beta_j) \Gamma(1 - \beta_j)} = \frac{s_j}{\sqrt{s_j s_{j+1}}}. \quad (6.6)$$

Therefore, using (A.4), we can rewrite (6.4) as

$$\Phi_j^{(0)}(r) = e^{-\frac{1}{2} \theta_1(rz)} \begin{pmatrix} 1 & s_j (\log r - \log(\frac{1}{2} f'_{x_j}(x_j)|i)|i) & \frac{1}{2}\theta_1(rz) \\ 0 & 1 & 0 \\ 0 & 0 & e^{\frac{1}{2} \theta_1(rz)} \end{pmatrix}, \quad (6.6)$$

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where \( \gamma \) is Euler’s gamma constant. Combining (6.6) with (3.2), we get

\[
(q_j,1(r) \ q_j,2(r) \ q_j,3(r))^t = e^{s_j r x \ j} \text{diag}(r^{-\frac{1}{2}}, 1, r^{-\frac{1}{2}}) R(x_j) E_{x_j}(x_j) \left( \gamma_j^{(0)} \ \gamma_j^{(0)} \right)^t,
\]

\[
(p_j,1(r) \ p_j,2(r) \ p_j,3(r))^t = -e^{s_j r x \ j} \text{diag}(r^{\frac{1}{2}}, 1, r^{\frac{1}{2}}) R(x_j)^{-t} E_{x_j}(x_j)^{-t} \text{diag}(\gamma_j^{(0)} - 1, 1) (0 1 0)^t.
\]

We then obtain (1.8b), (1.8c), (1.8d), (1.8f), (1.8g) and (1.8h) after a long computation. We omit the details.

**6.2 Proof of Theorem 1.3**

Using (5.3) and (5.5), we have, for \(|z| > \delta\),

\[
\Phi(z) = \left(\frac{\sqrt{2\pi}}{\sqrt{3} e^{\frac{1}{3} i}}\right)^{-1} (I + O(r)) \Psi_0^{-1} \Psi(z), \quad \text{as } r \to 0.
\]

Using also (2.4) and (3.1), we obtain (1.11). We now compute the asymptotics of \( p_{j,k} \) and \( q_{j,k} \) as \( r \to 0 \).

\[
\Phi(rz) = R(rz) \left(\frac{\sqrt{2\pi}}{\sqrt{3} e^{\frac{1}{3} i}}\right)^{-1} \Psi(rz) \begin{bmatrix} I - \sum_{j=1}^{m} a_j \log(rz - r x_j) - \log(rz + r x_j) \\ 0 1 1 \\ 0 0 0 \end{bmatrix}.
\]

It then follows from (6.2) that

\[
\Phi_j^{(0)}(r) = R(r x_j) \left(\frac{\sqrt{2\pi}}{\sqrt{3} e^{\frac{1}{3} i}}\right)^{-1} \Psi(r x_j) \begin{bmatrix} I + \sum_{j=1}^{m} a_j \log(2r x_j) \\ 0 1 1 \\ 0 0 0 \end{bmatrix},
\]

and then by (2.8) and (5.5), we get

\[
\Phi_j^{(0)}(r) = \left(\frac{\sqrt{2\pi}}{\sqrt{3} e^{\frac{1}{3} i}}\right)^{-1} \Psi(0) + O(r) \begin{bmatrix} I + \sum_{j=1}^{m} a_j \log(2r x_j) \\ 0 1 1 \\ 0 0 0 \end{bmatrix}.
\]

On the other hand, by [24, equation (7.22)], we have

\[
\Psi(0) (1 0 0)^t = (P_0(0) 0 P_0'(0))^t, \quad \Psi(0)^{-t} (0 1 1)^t = \left(0 1 (P_0(0))^t \right)^t,
\]

where \( P_0(0) \neq 0 \). The asymptotics of (1.11b) and (1.11c) now directly follows from (3.2).

**6.2 Proof of Theorem 1.3**

The asymptotics of \( H(r) \) as \( r \to 0 \) given by (1.13) are directly obtained from (1.6) and (1.11a)–(1.11c).

Since \( F(0) = a = 1 \) by (1.2), the integral representation (1.12) follows by integrating (2.19) from 0 to an arbitrary \( r > 0 \).

To compute the asymptotics of \( H(r) \) as \( r \to +\infty \), we follow the method of [24] and rely on (3.20). Using (2.20) and (2.21), we obtain

\[
\Phi_j^{(1)}(r) = \frac{1}{r} \Phi_j^{(0)}(r)^{-1} \lim_{x \to x_j} \left[ \Phi(rz) \begin{bmatrix} 1 & 0 & 1 \\ a_j \log(rz - r x_j) & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right]^t,
\]

where \( ' \) denotes the derivative with respect to \( z \). We see from (6.6) that

\[
(0 1 1) \Phi_j^{(0)}(r)^{-1} = e^{\frac{1}{2} s_j r x \ j} (0 1 0) \text{diag}((\gamma_j^{(0)})^{-1}, 1) E_{x_j}(x_j)^{-1} R(x_j)^{-1} \text{diag}(r^{\frac{1}{2}}, 1, r^{\frac{1}{2}}).
\]
Also, by (6.3) and (4.15), we have
\[
\lim_{z \to z_j} \Phi(z) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]
\[
\times \lim_{z \to z_j} R(z) E(z) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]
where \( \Theta \) is defined in (6.5). A direct computation using (A.4) shows that this last limit is given by
\[
e^{-\frac{\beta_j}{\sqrt{s_j}} \text{diag} (r^{-\beta_j}, 1, r^{-\beta_j})} (r^{1/2} f_{jx}(x) R(z) E_j(x)) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (1 \ 0 \ 0)^t
\]
where \( \gamma_{j,21} = \frac{\beta_j}{\sqrt{s_j}} \text{sm}(\pi \beta_j) \). Combining the above equations with (3.20), we then find
\[
H(r) = -\frac{1}{\pi} \sum_{j=1}^{m} \beta_j x_j \left[ \text{diag} ((\gamma_j^{(0)})^{-1}, 1) E_j(x) \right] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = O(r^{-\frac{2}{3}}), \quad \text{as} \ r \to +\infty.
\]
For the second part of (6.10), we use (6.7) and get
\[
\begin{align*}
\left[ \text{diag} ((\gamma_j^{(0)})^{-1}, 1) E_j(x) \right] & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{\beta_j}{3 x_j} + 2i \beta_j \text{Im}(d_j^{(1)}), \\
\text{and} \quad \text{as} \ r \to +\infty \quad \text{we obtain}
\end{align*}
\]
\[
\begin{align*}
\frac{\beta_j}{3 x_j} + 2i \beta_j \text{Im}(d_j^{(1)}) = O(r^{-\frac{2}{3}}), \quad \text{as} \ r \to +\infty.
\end{align*}
\]
Substituting the above formulas in (6.10), and noting the remarkable simplification
\[
\begin{align*}
\sum_{j=1}^{m} 2i \beta_j x_j \text{Im}(d_j^{(1)}) & = \sum_{j=1}^{m} 2i \beta_j x_j \text{Im} \left( \frac{\beta_j}{6(\omega - 1)x_j} \right) + \sum_{1 \leq j < k \leq m} \text{Re} \left( \frac{4(\omega - 1)x_k^3 x_j^3}{3(x_j^2 - x_k^2)} \right) \\
& = \sum_{j=1}^{m} 2i \beta_j x_j \text{Im} \left( \frac{\beta_j}{6(\omega - 1)x_j} \right) = \sum_{j=1}^{m} \beta_j^2,
\end{align*}
\]
we obtain (1.14). This finishes the proof of Theorem 1.3.
6.3 Proof of Theorem 1.4

Integrating (3.23) from 0 to an arbitrary \( r > 0 \), we get

\[
\int_0^r H(\tau) d\tau = \int_0^r \left( p_0(\tau)q_0(\tau) + \sum_{j=1}^3 \sum_{k=1}^3 p_{j,k}(\tau)q'_{j,k}(\tau) - H(\tau) \right) d\tau - \frac{1}{4} \left[ 2p_0(\tau)q_0(\tau) + \sum_{j=1}^3 \left( p_{j,2}(\tau)q_{j,2}(\tau) + 2p_{j,3}(\tau)q_{j,3}(\tau) \right) - 3\tau H(\tau) \right]_{\tau=0}^r .
\]

(6.11)

For convenience, we write \( \vec{0} = (0, \ldots, 0) \in \mathbb{R}^m \),

\[
\vec{\beta} := (\beta_1, \ldots, \beta_m), \quad \vec{\beta}_\ell := (\beta_1, \ldots, \beta_\ell, 0, \ldots, 0), \quad \vec{\beta}'_\ell := (\beta_1, \ldots, \beta_{\ell-1}, \beta'_\ell, 0, \ldots, 0).
\]

We also write explicitly the dependence of \( p_{j,k}, q_{j,k}, p_0, q_0 \) and \( H \) in \( \beta_1, \ldots, \beta_m \) using the notation \( p_{j,k}(r; \vec{\beta}), q_{j,k}(r; \vec{\beta}), p_0(r; \vec{\beta}), q_0(r; \vec{\beta}) \) and \( H(r; \vec{\beta}) \). By (4.4), the parameter \( \gamma \) in (3.24) can also be chosen to be any parameter among \( \beta_1, \ldots, \beta_m \). Let \( \ell \in \{1, \ldots, m\} \). Using (3.24) with \( \vec{\beta} = \vec{\beta}_{\ell-1} \) and \( \gamma = \beta_\ell \), and integrating in \( r \) from 0 to \( r \) and integrating in \( \beta_\ell \) from 0 to \( \vec{\beta}_\ell \), we get

\[
\int_0^r \left( p_0(\tau; \vec{\beta})q_0(\tau; \vec{\beta}_\ell) + \sum_{j=1}^3 \sum_{k=1}^3 p_{j,k}(\tau; \vec{\beta}_\ell)q'_{j,k}(\tau; \vec{\beta}_\ell) - H(\tau; \vec{\beta}_\ell) \right) d\tau
\]

\[
- \int_0^r \left( p_0(\tau; \vec{\beta}_{\ell-1})q_0(\tau; \vec{\beta}_{\ell-1}) + \sum_{j=1}^3 \sum_{k=1}^3 p_{j,k}(\tau; \vec{\beta}_{\ell-1})q'_{j,k}(\tau; \vec{\beta}_{\ell-1}) - H(\tau; \vec{\beta}_{\ell-1}) \right) d\tau
\]

\[
= \int_0^{\vec{\beta}_\ell} \left( \sum_{k=1}^3 \sum_{j=1}^3 p_{j,k}(r; \vec{\beta}_\ell)\partial_{\vec{\beta}'} q_{j,k}(r; \vec{\beta}_\ell') + p_0(r; \vec{\beta}_\ell)\partial_{\vec{\beta}'} q_0(r; \vec{\beta}_\ell') \right) d\vec{\beta}'_\ell .
\]

(6.12)

where we have used (1.11a)–(1.11c) to conclude that

\[
\int_0^{\vec{\beta}_\ell} \left( \sum_{k=1}^3 \sum_{j=1}^3 p_{j,k}(0; \vec{\beta}_\ell)\partial_{\vec{\beta}'} q_{j,k}(0; \vec{\beta}_\ell') + p_0(0; \vec{\beta}_\ell)\partial_{\vec{\beta}'} q_0(0; \vec{\beta}_\ell') \right) d\vec{\beta}'_\ell = 0.
\]

Summing (6.12) over \( \ell = 1, \ldots, m \), we then obtain

\[
\int_0^r \left( p_0(\tau; \vec{\beta})q_0(\tau; \vec{\beta}) + \sum_{j=1}^3 \sum_{k=1}^3 p_{j,k}(\tau; \vec{\beta})q'_{j,k}(\tau; \vec{\beta}) - H(\tau; \vec{\beta}) \right) d\tau
\]

\[
- \int_0^r \left( p_0(\tau; 0)q_0(\tau; 0) + \sum_{j=1}^3 \sum_{k=1}^3 p_{j,k}(\tau; 0)q'_{j,k}(\tau; 0) - H(\tau; 0) \right) d\tau
\]

\[
= \sum_{\ell=1}^m \int_0^{\vec{\beta}_\ell} \left( \sum_{k=1}^3 \sum_{j=1}^3 p_{j,k}(r; \vec{\beta}_\ell)\partial_{\vec{\beta}'} q_{j,k}(r; \vec{\beta}_\ell') + p_0(r; \vec{\beta}_\ell)\partial_{\vec{\beta}'} q_0(r; \vec{\beta}_\ell') \right) d\vec{\beta}'_\ell .
\]

(6.13)

If \( \beta_\ell = 0 \) (or equivalently, if \( s_\ell = s_{\ell+1} \)), it follows from (3.2) that \( p_{1,1}(r) = p_{1,2}(r) = p_{1,3}(r) = 0 \) for all \( r > 0 \). Also, by (3.20), we have \( H(r; 0) = 0 \), and by (2.11) and (2.16), we have \( \Phi'(s_{\ell-1}) \equiv I \). Hence, by (3.1) and the relations \( \Phi_1 = \Psi_1 + \Psi_0^{-1}Y_1\Psi_0 \) and (2.5), we have \( p_0(r; 0) = \frac{1}{\sqrt{2}} \left( \frac{\rho^2}{\rho_1^2} + \frac{p_0^2}{2} \right) \) and \( q_0(r; 0) = \frac{1}{\sqrt{2}} \left( -\rho^2 + \frac{p_0^2}{2} \right) \). Thus,

\[
\int_0^r \left( p_0(\tau; 0)q_0(\tau; 0) + \sum_{j=1}^3 \sum_{k=1}^3 p_{j,k}(\tau; 0)q'_{j,k}(\tau; 0) - H(\tau; 0) \right) d\tau = 0
\]

and (6.13) can be simplified as

\[
\int_0^r \left( p_0(\tau; \vec{\beta})q_0(\tau; \vec{\beta}) + \sum_{j=1}^3 \sum_{k=1}^3 p_{j,k}(\tau; \vec{\beta})q'_{j,k}(\tau; \vec{\beta}) - H(\tau; \vec{\beta}) \right) d\tau
\]
Combining the asymptotics (6.16)–(6.18), we get

\[ = \sum_{k=1}^{\ell} \int_{0}^{\beta_{r}} \left( \sum_{k=1}^{\ell} p_{j,k}(r; \beta_{r}) \partial_{j,\beta_{r}} q_{j,k}(r; \beta_{r}) + p_{0}(r; \beta_{r}) \partial_{j,\beta_{r}} q_{0}(r; \beta_{r}) \right) d\beta_{r}. \]  

(6.14)

Substituting (6.14) in (6.11), we obtain

\[
\int_{0}^{r} H(r; \beta) d\tau = \sum_{k=1}^{\ell} \int_{0}^{\beta_{r}} \left( \sum_{k=1}^{\ell} p_{j,k}(r; \beta_{r}) \partial_{j,\beta_{r}} q_{j,k}(r; \beta_{r}) + p_{0}(r; \beta_{r}) \partial_{j,\beta_{r}} q_{0}(r; \beta_{r}) \right) d\beta_{r} \]

\[
- \frac{1}{4} \left( 2p_{0}(r; \beta) q_{0}(r; \beta) - \sum_{j=1}^{\ell} \left( 2p_{j,1}(r; \beta) q_{j,1}(r; \beta) + p_{j,2}(r; \beta) q_{j,2}(r; \beta) \right) - 3r H(r; \beta) \right) + \frac{1}{2} p_{0}(0; \beta) q_{0}(0; \beta),
\]

where we have also used (1.5). Using (1.8b)–(1.8d), (1.8f)–(1.8h) and (4.27), we get

\[
p_{j,1}(r; \beta_{r}) \partial_{\beta_{r}} q_{j,1}(r; \beta_{r}) = p_{j,1}(r; \beta_{r}) q_{j,1}(r; \beta_{r}) \partial_{\beta_{r}} \log q_{j,1}(r; \beta_{r})
\]

\[
= - \frac{2i \beta}{3} \left( \sin(2\theta_{j}(r; \beta_{r}) - \frac{2\pi}{3}) \right. + \left. \left( \sin(2\theta_{j}(r; \beta_{r})) - \frac{\sqrt{3}}{2} \right) \sum_{n=1}^{\ell} \sqrt{3} i \beta \frac{x_{n}^{2/3}}{x_{j}^{2/3}} \right) \]

\[
\times \left( \partial_{\beta_{r}} \log \mathcal{A}_{j}(\beta_{r}) + \cot(\theta_{j}(r; \beta_{r}) - \frac{\pi}{3}) \partial_{\beta_{r}} \theta_{j} \right) \left( 1 + O\left( \frac{\log r}{r^{2/3}} \right) \right), \quad \text{as } r \to +\infty,
\]

(6.16)

where we have explicitly written the dependence of \( \theta_{j} \) and \( \mathcal{A}_{j} \) on \( r \) and \( \beta_{r} \). For \( p_{j,2}(r; \beta_{r}) \partial_{\beta_{r}} q_{j,2}(r; \beta_{r}) \) and \( p_{j,3}(r; \beta_{r}) \partial_{\beta_{r}} q_{j,3}(r; \beta_{r}) \), we obtain after another computation (using again (1.8b)–(1.8d), (1.8f)–(1.8h) and (4.27)) that

\[
p_{j,2}(r; \beta_{r}) \partial_{\beta_{r}} q_{j,2}(r; \beta_{r}) = - \frac{2i \beta}{3} \sin(2\theta_{j}) \left( \partial_{\beta_{r}} \log \mathcal{A}_{j}(\beta_{r}) + \cot(\theta_{j}) \partial_{\beta_{r}} \theta_{j} \right) \left( 1 + O\left( \frac{\log r}{r^{2/3}} \right) \right),
\]

(6.17)

\[
p_{j,3}(r; \beta_{r}) \partial_{\beta_{r}} q_{j,3}(r; \beta_{r}) = \frac{-2i \beta}{3} \sin(2\theta_{j} - \frac{2\pi}{3}) \left( \partial_{\beta_{r}} \log \mathcal{A}_{j}(\beta_{r}) + \cot(\theta_{j} + \frac{\pi}{3}) \partial_{\beta_{r}} \theta_{j} \right)
\]

\[
+ \frac{2i \beta}{3} \left( \sin(2\theta_{j} - \frac{\sqrt{3}}{2}) \right) \left( \partial_{\beta_{r}} \log \mathcal{A}_{j}(\beta_{r}) + \cot(\theta_{j} - \frac{\pi}{3}) \partial_{\beta_{r}} \theta_{j} \right) \sum_{n=1}^{\ell} \sqrt{3} i \beta \frac{x_{n}^{2/3}}{x_{j}^{2/3}} + \sqrt{3} \frac{x_{n}^{2/3}}{x_{j}^{2/3}} \right) \]

\[
\times \left( 1 + O\left( \frac{\log r}{r^{2/3}} \right) \right),
\]

(6.18)

as \( r \to +\infty \), where \( \theta_{j} = \theta_{j}(r; \beta_{r}) \). Furthermore, using (1.9) and (4.4), we find

\[
\partial_{\beta_{r}} \log \mathcal{A}_{j}(\beta_{r}) = \begin{cases} 
\frac{-2\pi}{3} + \partial_{\beta_{r}} \log |\Gamma(1 - \beta_{r})|, & \text{if } j = \ell, \\
\log \left( \frac{x_{j}^{2/3} - \omega x_{n}^{2/3}}{|x_{j}^{2/3} - x_{n}^{2/3}|} \right), & \text{if } j < \ell.
\end{cases}
\]

Combining the asymptotics (6.16)–(6.18), we get

\[
\sum_{k=1}^{\ell} \sum_{j=1}^{\ell} p_{j,k}(r; \beta_{r}) \partial_{j,\beta_{r}} q_{j,k}(r; \beta_{r}) = \sum_{j=1}^{\ell} \partial_{\beta_{r}} \left( \frac{x_{j}^{2/3}}{x_{j}^{2/3}} - \frac{2}{\sqrt{3}} \sin(2\theta_{j}) \frac{x_{j}^{2/3}}{x_{j}^{2/3}} - 2i \partial_{\beta_{r}} \partial_{j} \right) + O\left( \frac{\log r}{r^{2/3}} \right)
\]

as \( r \to +\infty \), where again \( \theta_{j} = \theta_{j}(r; \beta_{r}) \). Using (3.18) and the fact that \( p_{j,1}(r) = p_{j,2}(r) = p_{j,3}(r) = 0 \) if \( \beta_{j} = 0 \), we note that

\[
p_{0}(r; \beta_{r}) \partial_{\beta_{r}} q_{0}(r; \beta_{r}) = -\sqrt{2} \sum_{j=1}^{\ell} p_{j,1}(r; \beta_{r}) q_{j,1}(r; \beta_{r}) \partial_{\beta_{r}} q_{0}(r; \beta_{r}) - \left( q_{0}(r; \beta_{r}) - \frac{\rho}{\sqrt{2}} \right) \partial_{\beta_{r}} q_{0}(r; \beta_{r}).
\]

Integrating this identity in \( \beta_{r} \) from 0 to an arbitrary \( \beta_{r} \in i\mathbb{R} \), we get

\[
\int_{0}^{\beta_{r}} p_{0}(r; \beta_{r}) \partial_{\beta_{r}} q_{0}(r; \beta_{r}) d\beta_{r} = -\sqrt{2} \sum_{j=1}^{\ell} \int_{0}^{\beta_{r}} p_{j,1}(r; \beta_{r}) q_{j,1}(r; \beta_{r}) \partial_{\beta_{r}} q_{0}(r; \beta_{r}) d\beta_{r}
\]

\[
- \left[ \frac{1}{2} q_{0}(r; \beta_{r})^{2} - \frac{\rho}{\sqrt{2}} q_{0}(r; \beta_{r}) \right]^{\beta_{r}}_{\beta_{r}=0}.
\]

(6.19)
Using (1.8d), (1.8e), (1.8f) and (4.27), as \( r \to +\infty \) we get
\[
- \sqrt{2} \sum_{j=1}^{\ell} p_{j,3}(r; \beta_{j})q_{j,1}(r; \beta_{j})q_{j,2}(r; \beta_{j}) = \sum_{j=1}^{\ell} \frac{x_{j}^{2/3}}{x_{j}} \beta_{j} \left( \frac{2}{\sqrt{3}} \sin(2 \theta_{j}(r; \beta_{j})) - 1 \right) + O\left( \frac{\log r}{r^{2/3}} \right). \tag{6.20}
\]
Hence, substituting (6.16)–(6.20) in (6.15), we obtain
\[
\int_{r}^{1} H(r; \beta) \, dr = -2 \sum_{l=1}^{m} \int_{0}^{\beta_{l}} \left\{ \sum_{j=1}^{\ell} i \beta_{j} \partial_{\beta_{j}} \psi_{j}(r; \beta_{j}) + i \beta_{l} \partial_{\beta_{l}} \psi_{l}(r; \beta_{l}) \right\} d\beta_{l}
- \frac{1}{2} q_{0}(r; \beta \bar{\beta}) - \frac{\rho}{\sqrt{2}} q_{0}(r; \beta \bar{\beta}) + 1 \frac{1}{2} p_{0}(0; \beta \bar{\beta}) q_{0}(0; \beta \bar{\beta})
- \frac{1}{4} \left( 2 p_{0}(r; \beta \bar{\beta}) q_{0}(r; \beta \bar{\beta}) - \sum_{j=1}^{m} \left( 2 p_{j,1}(r; \beta \bar{\beta}) q_{j,1}(r; \beta \bar{\beta}) + p_{j,2}(r; \beta \bar{\beta}) q_{j,2}(r; \beta \bar{\beta}) - 3 r H(r; \beta \bar{\beta}) \right) \right)
+ O\left( \frac{\log r}{r^{2/3}} \right), \quad \text{as } r \to +\infty. \tag{6.21}
\]
Since \( p_{0}(r; \bar{\beta}) = \frac{1}{\sqrt{2}} (\frac{\rho}{\sqrt{2}} + \frac{\rho}{\sqrt{2}}) \), \( q_{0}(r; \bar{\beta}) = \frac{1}{\sqrt{2}} (\frac{\rho}{\sqrt{2}} + \frac{\rho}{\sqrt{2}}) = q_{0}(0; \beta \bar{\beta}) \), we see that
\[
\left( \frac{1}{2} q_{0}(r; \bar{\beta}) - \frac{\rho}{\sqrt{2}} q_{0}(r; \bar{\beta}) \right) + \frac{1}{2} p_{0}(0; \beta \bar{\beta}) q_{0}(0; \beta \bar{\beta}) = - \frac{\rho}{2 \sqrt{2}} q_{0}(0; \beta \bar{\beta}) = \frac{\rho^{4}}{216} - \frac{\rho^{2}}{8}. \tag{6.22}
\]
Also, using (1.8), (1.14) and (3.18), we obtain
\[
- \left( \frac{1}{2} q_{0}(r; \bar{\beta}) - \frac{\rho}{\sqrt{2}} q_{0}(r; \bar{\beta}) \right) = \frac{1}{4} \left( 2 p_{0}(r) q_{0}(r) - \sum_{j=1}^{m} \left( 2 p_{j,1}(r) q_{j,1}(r) + p_{j,2}(r) q_{j,2}(r) \right) - 3 r H(r) \right)
= \frac{q_{0}(r)}{\sqrt{2}} \left( \frac{\rho}{2} + \sum_{j=1}^{m} p_{j,3}(r) q_{j,1}(r) \right) + \frac{3}{4} r H(r) + \frac{1}{4} \sum_{j=1}^{m} \left( 2 p_{j,1}(r) q_{j,1}(r) + p_{j,2}(r) q_{j,2}(r) \right)
= \sum_{j=1}^{m} \left( 3 \sqrt{3} i \beta_{j} (r x_{j})^{\frac{3}{2}} - \sqrt{3} \frac{\rho}{2} i \beta_{j} (r x_{j})^{\frac{3}{2}} - \beta_{j}^{2} \right) - \frac{\rho^{4}}{216} + \frac{\rho^{2}}{8} + O(r^{-\frac{3}{2}}), \quad \text{as } r \to +\infty. \tag{6.23}
\]
It follows from [24, equation (7.51)] that
\[
-2 \sum_{j=1}^{m} \int_{0}^{\beta_{j}} i \beta_{j} \partial_{\beta_{j}} \psi_{j}(r; \beta_{j}) d\beta_{j} = \sum_{j=1}^{m} \log \left( G(1 + \beta_{j}) G(1 - \beta_{j}) \right) + \sum_{j=1}^{m} \beta_{j}^{2} \left( 1 - 4 \frac{3}{4} \log(r x_{j}) - \log \left( \frac{9}{2} \right) \right). \tag{6.24}
\]
For \( m \geq 2 \), we also need the relation
\[
-2 \sum_{\ell=1}^{m} \int_{0}^{\beta_{\ell}} \sum_{j=1}^{\ell} i \beta_{\ell} \partial_{\beta_{\ell}} \psi_{j}(r; \beta_{\ell}) d\beta_{\ell} = -2 \sum_{\ell=1}^{m} \sum_{j=1}^{\ell} \beta_{\ell} \beta_{j} \log \frac{|x_{j}^{2/3} - \omega x_{j}^{2/3}|}{|x_{j}^{2/3} - x_{\ell}^{2/3}|}, \tag{6.25}
\]
which can be proved directly from (1.10) and a direct computation. The asymptotic formula (1.15) (without the error term) now follows after substituting (6.22) and (6.23) in (6.21) and performing a rather long calculation which uses (6.24) and (6.25). The fact that the error term in (1.15) is \( O(r^{-\frac{3}{2}}) \) and not \( O(r^{-\frac{3}{4}} \log r) \) follows directly from (1.12) and (1.14), and (1.16) follows from (4.27). This finishes the proof of Theorem 1.4.

## A Confluent hypergeometric model RH problem

In this appendix we recall a well-known model RH problem, whose solution depends on a parameter \( \beta \in i \mathbb{R} \) and is denoted \( \Phi_{HG}(\cdot) = \Phi_{HG}(\cdot; \beta) \).

(a) \( \Phi_{HG} : \mathbb{C} \setminus \Sigma_{HG} \to \mathbb{C}^{2 \times 2} \) is analytic, where \( \Sigma_{HG} = e^{\frac{\pi i}{4}} (-\infty, \infty) \cup e^{\frac{3\pi i}{4}} (-\infty, \infty) \cup e^{\frac{2\pi i}{3}} (-\infty, \infty) \).
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(b) $\Phi_{HG}$ satisfies the jump relations

$$
\Phi_{HG,+}(z) = \Phi_{HG,-}(z) \tilde{J}_k, \quad z \in \Gamma_{k}^{HG}, \quad k = 1, \ldots, 6,
$$

where

$$
\Gamma_1^{HG} = (0, i\infty), \quad \Gamma_2^{HG} = (0, e^{\frac{3\pi}{4}} \infty), \quad \Gamma_3^{HG} = (e^{-\frac{3\pi}{4}} \infty, 0),
$$

$$
\Gamma_4^{HG} = (-i\infty, 0), \quad \Gamma_5^{HG} = (e^{-\frac{\pi}{2}} \infty, 0), \quad \Gamma_6^{HG} = (0, e^{\frac{\pi}{4}} \infty),
$$

and

$$
\tilde{J}_1 = \begin{pmatrix} 0 & e^{-i\pi \beta} \\ -e^{-i\pi \beta} & 0 \end{pmatrix}, \quad \tilde{J}_2 = \begin{pmatrix} 0 & e^{i\pi \beta} \\ -e^{i\pi \beta} & 0 \end{pmatrix}, \quad \tilde{J}_3 = \tilde{J}_5 = \begin{pmatrix} 1 & 0 \\ e^{-i\pi \beta} & 1 \end{pmatrix}.
$$

(c) As $z \to \infty$, $z \notin \Sigma_{HG}$, we have

$$
\Phi_{HG}(z) = \left( I + \frac{\Phi_{HG,1}(\beta)}{z} + \mathcal{O}(z^{-2}) \right) z^{-\beta \sigma z} e^{-\frac{i}{2} z^2} \begin{cases} 0, & -\frac{\pi}{2} < \arg z < \frac{\pi}{2}, \\ \frac{\pi}{2}, & -\frac{\pi}{2} < \arg z < \frac{\pi}{2} \end{cases},
$$

where $z^2 = |z|^2 e^{\beta \alpha z}$ with $\arg z \in (-\frac{\pi}{2}, \frac{3\pi}{2})$ and

$$
\Phi_{HG,1}(\beta) = \beta^2 \begin{pmatrix} 0 & 1 \\ -1 & \tau(-\beta) \end{pmatrix}, \quad \tau(\beta) = \frac{-\Gamma(-\beta)}{\Gamma(\beta + 1)},
$$

(d) $\Phi_{HG}(z) = \mathcal{O}(\log z)$ as $z \to 0$.

This model RH problem can be solved explicitly using confluent hypergeometric functions [36]. By a computation similar to [15, equation (A.9)] and [24, equation (A.10)], we have

$$
\Phi_{HG}(z) = \Upsilon^{(0)}(I + \Upsilon^{(1)} z + \mathcal{O}(z^2)) \begin{pmatrix} 1 & \sin(\pi \beta) \\ 0 & \pi \log z \end{pmatrix}, \quad \text{as } z \to 0, \quad \arg z \in (\frac{3\pi}{4}, \frac{5\pi}{4}),
$$

where

$$
\Upsilon^{(0)} = \begin{pmatrix} \Gamma(1 - \beta) & -1 \\ \Gamma(1 + \beta) & -1 \end{pmatrix} \begin{pmatrix} \Gamma'(1 - \beta) + 2\gamma_E - i\pi \\ \Gamma'(1 + \beta) + 2\gamma_E - i\pi \end{pmatrix}, \quad \Upsilon^{(1)}_{21} = \frac{\beta \pi}{\sin(\pi \beta)},
$$

$\gamma_E$ is Euler’s gamma constant and

$$
\log z = \log |z| + i \arg z, \quad \arg z \in \left( -\frac{\pi}{2}, \frac{3\pi}{2} \right),
$$

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