Computation of Stability Criterion for Fractional Ocean Circulation Box Model Using Optimal Routh–Hurwitz Conditions

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Abstract. Atlantic ocean thermohaline circulation is a deep ocean circulation occur in the Atlantic ocean which shows mixed of salt and freshwater transportation. The ocean circulation box model is defined to cover the large-scale behavior of the thermohaline circulation. On the other hand, fractional order dynamical systems are more flexible and realistic for real-life problems if compare with integer order dynamical systems. Hence, research on the stability for fractional dynamical systems is still infant and more difficult to analyze analytically. In this paper, we will extend the ocean circulation 3-box model into fractional order and investigate stability criterion for this fractional model by applying fractional Routh-Hurwitz conditions. Routh-Hurwitz conditions allow us to find the range of adjustable control parameter $F_1$ which can detect the stability criterion for the fractional ocean circulation model.

Keywords: Fractional ocean circulation box model, optimal Routh–Hurwitz conditions, stability criterion.

1. Introduction
The thermohaline circulation of the Atlantic ocean is a deep ocean circulation transport atmospheric freshwater from low- to high-latitudes in Atlantic [1]. The physical processes in the ocean involved the salt content and freshwater transport, and the mixed water transports northward. The thermohaline circulation of the Atlantic ocean is a significance research area since the abrupt change of the circulation will effects on the ecosystems and our society [2]. Recently, the effect of the Atlantic thermohaline circulation on climate change had been studied by several researchers [2–5]. In addition, meridional overturning circulation is a major component and is categorized as thermohaline circulation [2]. Both of them show the impacts on ecosystems in climate change. Hence, the warming climate shows its impacts on a weakened Atlantic meridional overturning circulation had been studied by researchers [6].

In this research direction, the paradigmatic ocean circulation 4-box model was introduced in [1], and was derived to become more simple 3-box model. The 3-box dynamical model was set up and proposed in [7]. There are 3 boxes occured and represent the South and North Atlantic deep water forms in box 1 and box 2 respectively, and the upper layer of the tropics (box 3). The homoclinic bifurcation of integer order dynamical equations has been done and presented in [7]. In this paper, we extend the problem to fractional order since the fractional order model is more...
realistic for real-life problems. Applications of Routh-Hurwitz conditions enable us to perform stability analysis of fractional order differential equations. In [8–10], they find the negative real parts of zeros of the cubic order characteristic polynomial. In [11], they extended the generalized differential equation systems with fractional order $0 \leq \alpha < 1$ to order $0 \leq \alpha < 2$. Meanwhile, we will find the stability criterion for the quadratic order characteristic polynomial in this paper. We emphasize the computation of stability criterion for the fractional order ocean circulation 3-box model

$$\begin{align*}
\frac{C_0}{\alpha}D^{\alpha}_t S_1(t) &= \frac{1}{V}S_{\text{ref}}F_1 + \frac{1}{V}m(S_2(t) - S_1(t)) \\
\frac{C_0}{\alpha}D^{\alpha}_t S_2(t) &= -\frac{1}{V}S_{\text{ref}}F_2 + \frac{1}{V}m(S_3(t) - S_2(t))
\end{align*}$$

(1)

where the variables $S_i$ are the salt content for each box, $F_1$ is the freshwater flux transports from box 1 to box 3, and $F_2$ transports from box 3 to box 2, $V$ is the volume same for each box, $S_{\text{ref}}$ is a reference salinity which converts the freshwater fluxes into salt fluxes, and overturning rate $m$ which is proportional to the density difference between North and South Atlantic and depends linearly on temperature and salinity:

$$m = k(\rho_{\text{North}} - \rho_{\text{South}})$$

$$= k(\beta(S_2 - S_1) - \sigma T^*)$$

(2)

where

$$T^* = T_{2,\text{const}} - T_{1,\text{const}}.$$  

(3)

The constants $\sigma$ and $\beta$ are expansion coefficients for temperature and salinity respectively. The parameter $k$ is a hydraulic constant and is tunable. The temperature $T_i$, $i = 1, 2, 3$ are set to be constant for each box. The total salt content, $S_c$ of the three boxes is always conserved, that is

$$S_c = S_1 + S_2 + S_3.$$  

(4)

The model (1) is a general conceptual model for ocean circulation, and the water characteristics of Atlantic Ocean are considered by [7]. In regard to this, we perform the stability analysis without changing this ocean model’s qualitative behavior. Here, we use the following parameter values: $F_2 = 0.17 \text{ Sv}$ (Sverdrup is a common unit in oceanography: $1 \text{ Sv} = 1 \times 10^6 \text{ m}^3\text{s}^{-1}$), $S_{\text{ref}} = 35 \text{ psu}$ (practical salinity units), $\sigma = 0.00017$, $\beta = 0.0008$, $k = 23 \times 10^{17} \text{ m}^3\text{yr}^{-1}$, $V = 10^{17} \text{ m}^3$, $T^* = -2 \text{ K}$, $S_c = 99.5 \text{ psu}$.

2. Preliminaries

$\frac{C_0}{\alpha}D^{\alpha}_t$ stated in (1) is the Caputo derivative operator defined as follows:

**Definition 1** [9] Let $\alpha > 0, n \in \mathbb{N}, n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$, then the left Caputo fractional derivative of a function $f$ of order $\alpha$, is denoted by

$$\frac{C_0}{\alpha}D^{\alpha}_t f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau.$$  

(5)

3. Description for Ocean Circulation Box Model

Throughout this paper, we assume the freshwater flux $F_1$ as the adjustable control parameter here. Freshwater fluxes are the important bifurcation parameters represent the effect of global warming on thermohaline circulation. On the other hand, we notice that the stability analysis
will not depend on the parameter $F_2$ if applying the Routh-Hurwitz conditions. Thus, $F_1$ is considered as the bifurcation parameter in our analysis.

If $F_1 > 0$, then there are two equilibria appear, which are

$$E^\pm = \left( \frac{1}{3} S_c - \frac{T^* \sigma (2F_1 - F_2)}{3\beta F_1} + \frac{S_{ref}(F_1 + F_2)(2F_1 - F_2)}{3\beta F_1 k(3R - S_0)}, R \right),$$  \hspace{1cm} (6)$$

with

$$R = \frac{1}{3} S_c + \frac{F_1 + F_2}{6\beta F_1 k} \left( T^* \sigma k \pm \sqrt{k(T^* \sigma^2 k - 4\beta F_1 S_{ref})} \right).$$  \hspace{1cm} (7)$$

Given standard parameter values in Section 1, the equilibria $E^\pm$ with unknown parameter $F_1$ as follows:

$$E^\pm = (\bar{S}_1, \bar{S}_2)$$

\begin{align*}
\bar{S}_1 &= \left[ \sqrt{2.373928572 \cdot 10^{12} - F_1 \left( 4.61340129 \cdot 10^{-6} F_1^2 + 7.809566623 \cdot 10^{-7} F_1 ight. \right. \\
&\left. - 5.646646872 \cdot 10^{-10} \right) \pm \left( 1.268115942 \cdot 10^{-14} F_1^3 - 7.108124 F_1^2 - 1.203263542 F_1 \\
&\left. + 0.0008700104167 \right) \right] \div \left[ F_1 \left( 1.379193211 \cdot 10^{-7} F_1 + 2.344628458 \cdot 10^{-8} \right) \times \right. \\
&\left. \sqrt{2.373928572 \cdot 10^{12} - F_1 \pm (-0.21249999 F_1 - 0.036125) \right] \\
\bar{S}_2 &= \frac{1}{F_1} \left( 33.09583334 F_1 - 0.01204166667 \times 4.597310702 \cdot 10^{-10} (100 F_1 + 17) \times \right. \\
&\left. \sqrt{2.373928572 \cdot 10^{12} - F_1} \right) .
\end{align*}  \hspace{1cm} (8)$$

Then, we obtain the general characteristic equation of the system (1) from the linearization along (6)

$$\lambda^2 + a\lambda + b = 0,$$  \hspace{1cm} (9)$$

where

$$a = -\frac{k}{V} \left( 3T^* \sigma + \beta (3S_1 - 6S_2 + S_0) \right)$$

$$b = \frac{3k^2}{V^2} \left( T^* \sigma + 2\beta (S_1 - S_2) \right) \left( T^* \sigma + \beta (S_1 - S_2) \right).$$

By substituting the standard parameter values into (9), we have the coefficients of characteristic equation in term of $S_1, S_2$ as follows:

$$a = 0.1104 S_2 - 0.0552 S_1 - 1.80734$$

$$b = 0.00203136 S_1^2 - 0.000406272 S_1 S_2 + 0.00203136 S_2^2 - 0.001294992 S_1$$

$$+ 0.001294992 S_2 + 0.0001834572$$  \hspace{1cm} (10)$$

If substituting the equilibria into characteristic equation, we will get two different pairs of polynomial coefficients $a$ and $b$. Hence, in this paper, we will only use the larger equilibrium $E^+$ to investigate the stability analysis. Nevertheless, stability criterion for smaller equilibrium $E^-$ can be developed in the same manner as we present in Section 5 for $E^+$. Putting $E^+$ (8) into (10), the coefficients of characteristic equation become:
4. Routh-Hurwitz Conditions for Quadratic Polynomials

We consider the classical quadratic polynomial, \( P(\lambda) \)

\[
P(\lambda; a, b) = \lambda^2 + a\lambda + b
\]

with real coefficients \( a \) and \( b \).

Refer to \([11, 12]\), applying the fractional Routh-Hurwitz conditions to (12), all the zeros \( \lambda_i \) of (12) \((i = 1, 2)\) satisfy the Matignon stability sector \( |\arg(\lambda_i)| > \frac{\alpha\pi}{2} \), if and only if the following conditions holds:

\[
b > 0, a + 2\sqrt{b}\cos\left(\frac{\alpha\pi}{2}\right) > 0, 0 < \alpha < 2. \tag{13}
\]

The extension of Matignon stability sector for the fractional order \( \alpha \in (1, 2) \) is proved by \([13]\).

5. Stability Analysis of the Ocean Circulation Box Model

Applying the fractional Routh-Hurwitz conditions, we can analyze the local asymptotic stability of the equilibria (6) of the ocean circulation box model with respect to the adjustable control parameter \( F_1 \). The stability analysis of the ocean circulation box model can be computed if all the conditions (13) in Section 4 are satisfied.

First, we analyze the first inequality of (13). \( b > 0 \) is satisfied when \( F_1 \) range:

\[
0.000002016053879 < F_1 < 2.373928572 \cdot 10^{12}. \tag{14}
\]

For the second inequality \( a + 2\sqrt{b}\cos\left(\frac{\alpha\pi}{2}\right) > 0 \), where \( 0 < \alpha < 2 \), substituting \( a \) and \( b \) in (11), we get the inequality

\[
A + B\sqrt{C} > -2\sqrt{D + E\sqrt{C}\cos\left(\frac{\alpha\pi}{2}\right)} \tag{15}
\]

with
\[ A \equiv A(F_1) = 0.782F_1^2 + 0.06647F_1 - 0.0112999 \]
\[ B \equiv B(F_1) = 8 \cdot 10^{-8}F_1^2 + 1.7 \cdot 10^{-8}F_1 + 5.780000002 \cdot 10^{-10} \]
\[ C \equiv C(F_1) = 3.822025 \cdot 10^{14} - 161F_1 \]
\[ D \equiv D(F_1) = -3.864000002 \cdot 10^{-13}F_1^5 + 0.9172780004F_1^4 + 0.31187572F_1^3 \]
\[ + 0.026591962F_1^2 + 7.64405 \cdot 10^{-13}F_1 - 2 \cdot 10^{-14} \]
\[ E \equiv E(F_1) = 4.692000201 \cdot 10^{-8}F_1^4 + 1.59528 \cdot 10^{-8}F_1^3 + 1.355988127 \cdot 10^{-9}F_1^2 - 1.10^{-20}. \]

(16)

We assume the left-hand side of (15) by \( f(F_1) \) and right-hand side by \( g(F_1;\alpha) \), the inequality (15) shows
\[ f(F_1) > g(F_1;\alpha). \]

A direct calculation shows that the left-hand side, \( f(F_1) > 0 \) for any \( 0 < F_1 < 2.373928572 \cdot 10^{12} \). Meanwhile for the right-hand side, \( g(F_1;\alpha) \leq 0 \) only when \( 0 < \alpha \leq 1 \) and the \( F_1 \) range same as (14). Therefore, inequality (17) is satisfied for any \( 0.000002016053879 < F_1 < 2.373928572 \cdot 10^{12} \) and \( 0 < \alpha \leq \alpha_0 = 1 \). Thus, we continue our analysis for (17) is satisfied when \( \alpha_0 < \alpha < 2 \).

Using the notions of concavity and convexity of a function, we investigate the sign of \( \frac{d}{dF_1} f(F_1), \frac{d^2}{dF_1^2} f(F_1), \frac{d}{dF_1} g(F_1;\alpha), \) and \( \frac{d^2}{dF_1^2} g(F_1;\alpha) \). For any fixed \( 1 < \alpha < 2 \), \( g(F_1;\alpha) \) is positive and increasing, convex in \( 0.000007019786692 < F_1 < 1.303511098 \cdot 10^{12} \) and is changing to concave in \( 1.303511098 \cdot 10^{12} < F_1 < 2.048658102 \cdot 10^{12} \). Next for \( 2.048658102 \cdot 10^{12} < F_1 < 2.373928572 \cdot 10^{12} \), \( g(F_1;\alpha) \) is positive, decreasing and concave. Besides, \( f(F_1) \) has the same trend of curve as \( g(F_1) \). \( f(F_1) \) has an inflection point at \( F_1 = 1.427254586 \cdot 10^{12} \) and turns from ascent to decline at \( F_1 = 2.174421666 \cdot 10^{12} \). Since the curves \( f \) and \( g \) are the same in variation trend but different in inflection point and turning point, there is possible for \( f \) and \( g \) to have more than one intersections for \( F_1 < 2.373928572 \cdot 10^{12} \) and any fixed \( 1 < \alpha < 2 \).

Furthermore, we make some analysis to know how many points where the functions \( f(F_1) \) and \( g(F_1;\alpha) \) intersect. At the maximum \( \alpha \) value, \( g(F_1,2) = 2\sqrt{D(F_1) + E(F_1)\sqrt{C(F_1)}} \) intersects with \( f(F_1) \) at \( F_1^* = 0.000004032116766 \) and \( F_1^{**} = 2.055856245 \cdot 10^{12} \). Thus, (15) is satisfied when \( F_1 < F_1^* \) and \( F_1 > F_1^{**} \). (See Figure 1 and 2). When \( \alpha \) is monotonically decreasing from \( \alpha_{\text{max}} = 2 \) to \( \alpha_0 = 1 \), \( f(F_1) \) did not change since it did not depends on \( \alpha \) value, but \( g(F_1;\alpha) \) is moving down towards \( F_1 \)-axis until \( g(F_1,1) = 0 \). In regards to the observations, we can conclude that there exists two intersection points of functions \( f \) and \( g \) for \( \alpha < 2 \) and arrive at a critical value \( \alpha_{\text{cr}} \) which only exists an intersection point for \( f(F_1) = g(F_1;\alpha_{\text{cr}}) \). In addition, there is no intersection between functions \( f \) and \( g \) for any fixed \( \alpha_0 < \alpha < \alpha_{\text{cr}} \).

Now, we need to determine the unique solution where \( f \) and \( g \) intersect at the only one point when \( 0.000002016053879 < F_1 < 2.373928572 \cdot 10^{12} \) and \( 1 < \alpha < 2 \). To find this point, we assume (15) to become equality \( f = g \) at critical value \( \alpha_{\text{cr}} \) as follows:
\[ A(F_1) + B(F_1)\sqrt{C(F_1)} = -2\sqrt{D(F_1) + E(F_1)\sqrt{C(F_1)}} \cos \left( \frac{\alpha\pi}{2} \right) \]
and compute its tangent line which enables us to achieve the \( \alpha \) critical value:
\[ \frac{d}{dF_1} \left( A(F_1) + B(F_1)\sqrt{C(F_1)} \right) = \frac{d}{dF_1} \left( -2\sqrt{D(F_1) + E(F_1)\sqrt{C(F_1)}} \cos \left( \frac{\alpha\pi}{2} \right) \right). \]

(19)
Figure 1: Functions $f(F_1)$ and $g(F_1; \alpha)$ at $\alpha = 2$.

Figure 2: Functions $f(F_1)$ and $g(F_1; 2)$ for $F_1 < F_1^*$.

Solving the equation (18) and (19) numerically by using Maplesoft command \texttt{fsolve}, and obtain the critical values for $\alpha$ and $F_1$

$$\alpha_{cr} \approx 1.666668746, (F_1)_{cr} \approx 0.4004372045.$$ (20)

Finally, to conclude our observations for stability region, we have the following theorems:

\textbf{Theorem 1} For $0 < \alpha < \alpha_{cr}$, the equilibrium $E^+$ of system (1) is locally asymptotically stable for all $0.000002016053879 < F_1 < 2.373928572 \cdot 10^{12}$.

\textbf{Theorem 2} For $\alpha_{cr} < \alpha < 2$, the equilibrium $E^+$ of the system (1) is locally asymptotically stable for either $F_1 < F_1^*$ or $F_1 > F_1^{**}$, where the couple of values $F_1^* < F_1^{**}$ are uniquely determined in the interval $0.000002016053879 < F_1 < 2.373928572 \cdot 10^{12}$ via equation (18).

6. Numerical Result

In this section, we illustrate the simulation of the stability analysis for the fractional order ocean circulation 3-box model (1). We have chosen the generalized Adams-Bashford-Moulton type predictor-corrector scheme for solving fractional differential equations [14].

In this case, we assume the fractional differential equation problem as

$$^{C}D_2^\alpha y(x) = l(x, y(x))$$

with initial conditions

$$y^{(k)}(0) = y_0^{(k)}, \quad k = 0, 1, ..., m - 1$$ (22)

Consider the function $l$ exists on interval $[0, T]$, $t_n = nh, n = 0, 1, ..., N$ with positive integer $N$ (number of time-steps), step-size $h = T/N$, then we have the predictor-corrector algorithm. The corrector formula is given by

$$y_h(t_{n+1}) = \sum_{k=0}^{[\alpha]-1} \frac{t_n^{k+1}}{k!} y_0^{(k)} + \frac{h^\alpha}{\Gamma(\alpha + 2)} l(t_{n+1}, y_h(t_{n+1})) + \frac{h^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^{n} a_j n + 1 l(t_j, y_h(t_j))$$ (23)
Applying the predictor-corrector algorithm, we have the analysis to the equilibrium $E^+_+$ in (8) of the system (1). We consider $\alpha = 0.9$, $F_1 = 0.5$, then the numerical solution shows that equilibrium $E^+_+$ achieves stable condition (see Figure 3). The explicit result from Section 5, equilibrium $E^+_+$ is locally asymptotically stable for $\alpha = 0.9$ ($< \alpha_{cr}$) and $F_1 = 0.5$. Hence, the numerical and explicit solutions are equivalent to each other.

7. Conclusions

In this paper, we have investigate the stability criterion for a fractional ocean circulation box model by using optimal Routh-Hurwitz conditions. We indicate freshwater flux $F_1$ as the control parameter. We present the way analysis and observations to achieve the stability criterion for a system with quadratic order polynomial. We found that for the larger equilibrium $E^+_+$ of this ocean circulation box model, when $0 < \alpha < \alpha_{cr}$, both the zeros $\lambda_i$ of (9) are located inside the stability sector if $0.000002016053879 < F_1 < 2.373928572 \cdot 10^{12}$. When $\alpha_{cr} < \alpha < 2$, there exists a couple of bifurcation values $F_1^*$ and $F_1^{**}$, where the equilibrium changing from stable to unstable and vice versa. The same method can be applied to the smaller equilibrium $E^-$, but the stability region is much smaller regards to the first inequality $b > 0$ of Routh-Hurwitz conditions. For equilibrium $E^-$, the coefficient of polynomial $b > 0$ only for $0 < F_1 < 0.0006746578905$. Thus, we only present the stability analysis solution for equilibrium $E^+_+$. Besides, the stable result was shown in Section 6 by predictor-corrector scheme for fractional systems.
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References
[1] Rahmstorf S 1996 On the freshwater forcing and transport of the Atlantic thermohaline circulation *Climate Dynamics* 12(12) 799-811.
[2] Kessler A 2020 Atlantic thermohaline changes and its implications on the carbon cycle during the Last Interglacial.
[3] Wrzesiński D, Marsz A A, Styszyńska A, and Sobkowiak L 2019 Effect of the North Atlantic Thermohaline circulation on changes in climatic conditions and river flow in Poland *Water* 11(8) 1622.
[4] Velasco J A, Estrada F, Calderón-Bustamante O, Swingedouw D, Ureta C, Gay C, and Defrance D 2021 Synergistic impacts of global warming and thermohaline circulation collapse on amphibians *Communications biology* 4(1) 1-7.
[5] Bagatinsky V A, and Diansky N A 2021 Variability of the North Atlantic Thermohaline Circulation in Different Phases of the Atlantic Multidecadal Oscillation from Ocean Objective Analyses and Reanalyses *Izvestiya, Atmospheric and Oceanic Physics* 57(2) 208-219.
[6] Liu W, Fedorov A V, Xie S P, and Hu S 2020 Climate impacts of a weakened Atlantic Meridional Overturning Circulation in a warming climate *Science advances* 6(26) eaaz4876.
[7] Titz S, Kuhlbrodt T and Feudel U 2002 Homoclinic bifurcation in an ocean circulation box model *International Journal of Bifurcation and Chaos* 12(04) 869-875.
[8] Čermák J and Nechvátal I 2017 The Routh–Hurwitz conditions of fractional type in stability analysis of the Lorenz dynamical system *Nonlinear Dynamics* 87(2) 939-954.
[9] Ng Y X and Phang C 2019 Computation of Stability Criterion for Fractional Shimizu–Morioka System Using Optimal Routh–Hurwitz Conditions *Computation* 7(2) 23.
[10] Čermák J and Nechvátal I 2019 Stability and chaos in the fractional Chen system *Chaos, Solitons & Fractals* 125 24-33.
[11] Bourafa S, Abdelouahab M S and Moussaoui A 2020 On some extended Routh–Hurwitz conditions for fractional-order autonomous systems of order $\alpha \in (0, 2)$ and their applications to some population dynamic models *Chaos, Solitons & Fractals* 133 109623.
[12] Čermák J and Kisela T 2015 Stability properties of two-term fractional differential equations *Nonlinear Dynamics* 80(4) 1673-1684.
[13] Moze M, Sabatier J and Oustaloup A 2005 LMI tools for stability analysis of fractional systems *In International Design Engineering Technical Conferences and Computers and Information in Engineering Conference* 47438 1611-1619.
[14] Toh Y T, Phang C and Loh J R 2019 New predictor-corrector scheme for solving nonlinear differential equations with Caputo-Fabrizio operator *Mathematical Methods in the Applied Sciences* 42(1) 175-185.