Abstract

The second central extension of the planar Galilei group has been alleged to have its origin in the spin variable. This idea is explored here by considering local Galilean covariant field theory for free fields of arbitrary spin. It is shown that such systems generally display only a trivial realization of the second central extension. While it is possible to realize any desired value of the extension parameter by suitable redefinition of the boost operator, such an approach has no necessary connection to the spin of the basic underlying field.

1 Introduction

It is well known that the Galilei group in two spatial dimensions has a two dimensional central extension. Specifically, this means that one can expand the set of seven Galilean operators $P_i, K_i, H, J,$ and $M$ appropriate to a one dimensional central extension satisfying the structure relations

\begin{align*}
[J, K_j] &= i\epsilon_{ij} K_j, [K_i, H] = 0, \\
[J, P_i] &= i\epsilon_{ij} P_j, [K_i, K_j] = 0 \\
[J, H] &= [J, M] = 0, [K_i, P_j] = i\delta_{ij} M \\
[P_i, P_j] &= [P_i, H] = [H, M] = [K_i, M] = [P_i, M] = 0
\end{align*}

by the inclusion of an operator $\kappa$ which appears only in the modified commutator

\begin{equation}
[K_i, K_j] = i\epsilon_{ij} \kappa
\end{equation}

and commutes with all other operators of the theory. The operators $M$ and $\kappa$ thus provide a two dimensional central extension.
While the operator $M$ has a well understood role in Galilean invariant theories, the physical origin of $\kappa$ is less clear. Duval and Horváthy [1] have recently considered this issue as have Jackiw and Nair [2]. Although the latter claim to have established its identification with the spin of the relevant particle, there are a number of issues which need to be raised concerning their derivation. In particular their Eq.(5) has the form

$$J_a = \epsilon_{abc} x^b p^c + \frac{sc^2 p_a}{(p^2)^{\frac{3}{2}}} ,$$

an equation which implies that the spin has the dimension of $1/c^2$ rather than the dimensionless form expected in units in which $\hbar = 1$. This is also the basis for their claim that in the limit of $c \to \infty$ the second term on the rhs of the above equation dominates. Physically, this is an odd result since it seems to assert that the spin angular momentum necessarily dominates over orbital angular momentum in the large $c$ limit. Both of these issues clearly have their origin in the occurrence of seemingly anomalous factors of $c$ in the derivation of [2].

One of the most forthright approaches to an objective assessment of the validity of the Jackiw-Nair claim is to examine this issue in the context of specific models. To this end one seeks (preferably simple) examples in which the spin degree of freedom does indeed give rise to the second central extension parameter $\kappa$. This is certainly feasible within the context of local Galilean covariant field theory describing particles of nonzero spin. Clearly, if one can construct such theories corresponding to a $\kappa$ which either vanishes or is unrelated to spin, the question will be unambiguously settled. In the following section the Galilean field theory of a free spin one-half particle is constructed and shown to have a vanishing second central extension coefficient. In 3 this is extended to the case of arbitrary spin, with similar results being obtained.

## 2 Galilean Spin One-Half

The Galilean covariant wave equation for a spin one-half particle was constructed by Lévy-Leblond [3]. His result in three spatial dimensions requires a four-component spinor just as in the case of the corresponding Dirac equation. Another feature which it shares with the Dirac equation is the prediction of $g = 2$ for the $g$-factor. In the case of two spatial dimensions the four-component spinor equation separates into two two-component equations. Denoting the two spin components $\pm \frac{1}{2}$ by the spin parameter $s$ with $s = \pm 1$ one can write
the Lagrangian for a spin one-half particle as

\[ \mathcal{L} = \psi^\dagger \left[ \frac{1}{2}(1 + \sigma_3)\frac{\partial}{\partial t} - i \sigma \cdot \nabla + m(1 - \sigma_3) \right] \psi \]

where \( \sigma = (\sigma_1, \sigma_2) \) and \( \sigma_i, i = 1, 2, 3 \) denote the usual Pauli matrices. Upon making the identification \( \psi_1 = \phi, \psi_2 = \chi \), one obtains the equation of motion for the independent component \( \phi \)

\[ E\phi + p_\pm \chi = 0 \quad (1) \]

with the dependent component \( \chi \) given by

\[ 2m\chi + p_\pm \phi = 0 \quad (2) \]

where \( E = i \frac{\partial}{\partial t}, p_\pm = -i(\nabla_1 \pm is\nabla_2) \).

The invariance of the Lagrangian under rotations by an angle \( \varphi \) readily follows from the transformation law

\[ \psi'(x') = \exp[i\frac{s}{2}\sigma_3\varphi]\psi(x) \quad (3) \]

appropriate to a particle of spin \( s/2 \). The corresponding result for Galilean boosts with boost parameters \( v_i \) is readily verified using the spinor transformation prescription

\[ \psi'(x', t') = [1 - \frac{1}{4}(\sigma_1 - i\sigma_2)v_\pm]\exp[imv \cdot x + (i/2)mv^2t]\psi(x, t). \quad (4) \]

Corresponding to this statement of Galilean covariance is the existence of the local conservation law

\[ \nabla_j \left[ \psi^\dagger[-mx_i\sigma_j - it\frac{1}{2}\sigma_j\nabla_i - \frac{s}{4}\epsilon_{ij}(1 + \sigma_3)]\psi + it\frac{1}{2}\nabla_i\psi^\dagger\sigma_j\psi \right] \\
+ \frac{\partial}{\partial t} \left[ \psi^\dagger\frac{1}{2}(1 + \sigma_3)(mx_i + it\frac{1}{2}\nabla_i)\psi - it\frac{1}{2}\nabla_i\psi^\dagger\frac{1}{2}(1 + \sigma_3)\psi \right] = 0 \]

and the generator of Galilean boosts

\[ K_i = m \int x_i d^2x \phi^\dagger \phi - tP_i \quad (5) \]

where \( P_i \) is the momentum operator

\[ P_i = -i \int d^2x \phi^\dagger \nabla_i \phi. \quad (6) \]
From this explicit form for the Galilean boost operators and the equal time anticommutation relation

$$\{\phi(x, \phi^+(x')) = \delta(x - x') \tag{7}$$

it is readily inferred that $[K_i, K_j] = 0$. This simple model thus demonstrates that it is indeed possible to construct a nonzero spin field theory which yields a trivial result for the second extension parameter.

### 3 Extension to Arbitrary Spin

There is a standard tool available by which one can extend the results of the preceding section to the case of arbitrary integral or half-integral spin. This is the multispinor approach in which one describes a particle of spin $Ns/2$ by means of a totally symmetric multispinor of rank $N$. This approach has been successfully implemented [4] in the calculation of the $g$-factor of a Galilean particle in three spatial dimensions, and is applied even more easily to the case at hand. To this end one introduces the multispinor $\psi_{a_1a_2...a_N}$ which transforms according to Eqs.(3) and (4) in each index. Upon noting that the matrix $\Gamma \equiv \frac{1}{2}(1 + \sigma_3)$ is invariant under such transformations one infers that the appropriate Lagrangian is

$$\mathcal{L} = \frac{1}{N} \sum_{i=1}^{N} \Gamma_{a_1a'_1}...\Gamma_{a_{i-1}a'_{i-1}} G_{a_ia'_i} \Gamma_{a_{i+1}a'_{i+1}}...\Gamma_{a_Na'_N} \psi_{a_1...a_N} \tag{8}$$

where

$$G \equiv i\Gamma E + \sigma \cdot p + m(1 - \sigma_3).$$

Particularly in the case of two spatial dimensions there is a great deal of redundancy in the equations implied by the Lagrangian (8). Specifically, because of the fact that $\Gamma$ is only nonvanishing between upper components (i.e., $a_i = 1$), there are only two distinct equations implied by (8). These equations are easily seen to involve only the components $\psi_{11...1}$ and $\psi_{21...1}$. By renaming them $\phi$ and $\chi$ respectively, one readily obtains Eqs. (1) and (2) as well as the generalization of Eq. (3)

$$\psi'(x') = \exp[i(N - 1 + \sigma_3)s/2]\psi(x) \tag{9}$$

which are thus seen to describe Galilean particles of spin $Ns/2$ [5]. It immediately follows from this notational change that the expressions for the Galilean boost operators $K_i$ for spin $Ns/2$ are identical to those given by Eqs. (5) and
This together with the anticommutator (7) suffices to establish that the boost operators commute with each other, thereby demonstrating that only trivial realizations of the second central extension are realized in this model.

While the multispinor approach allows only integral \( N \) (i.e., integral and half-integral values of the spin), it is not difficult to prescribe an extension of the two component spinor model for spin \( Ns/2 \) to arbitrary spin merely by changing the definition of the angular momentum operator. In particular this can be achieved by the replacement

\[
J \rightarrow J + \lambda M/m
\]

where \( \lambda \) is any real number and \( M \) is the mass operator

\[
M = m \int d^2x \phi \dagger \phi.
\]

This allows one to assign any value to the spin while leaving intact the structure relations of the Galilei group. The second central extension parameter is again zero for this extended model.

\section{Conclusion}

In this work it has been shown that in a very wide class of arbitrary spin Galilean field theories there is no connection between the spin and the second central extension of the Galilei group. While the extension to arbitrary spin is significant, it is perhaps true that the most relevant result is that for spin one-half. This is because in 2+1 dimensions the Dirac equation goes over unambiguously to the Lévy-Leblond spin one-half equation leaving no possibility for a misidentification of the spin variable as one passes from the special relativistic case to the Galilean relativistic one.

On the other hand the fact that spin does not seem capable of yielding a nonzero \( \kappa \) certainly does not mean that such theories cannot be constructed. It has in fact been noted [6] that one can always proceed from a nonzero \( \kappa \) theory to one with \( \kappa = 0 \) merely by defining a new Galilean boost operator \( K'_i \) by the prescription

\[
K'_i \equiv K_i + \frac{1}{2} \kappa M^{-1} \epsilon_{ij} P_j.
\]  

Such a redefinition takes one to a new set of Galilean operators with a trivial second central extension. Naturally this operation can be carried out in either direction, and one thus easily obtains a \( \kappa \neq 0 \) theory from the class of \( \kappa = 0 \)
models considered in this work merely by implementing the inverse of Eq. (10). The second central extension of the two-dimensional Galilei is thus more closely related to the possibility of including translations in the boost operators than it is to the particle spin.

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