Global existence of solutions to nonlinear Volterra integral equations.

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Abstract

A new method is given for proving the global existence of the solution to nonlinear Volterra integral equations. A bound on the solution is derived. The results are based on a nonlinear inequality proved by the author earlier.

1 Introduction

Consider the equation:

\[ u(t) = f(t) + \int_0^t a(t, s, u(s))ds, \quad t \geq 0. \]  

(1)

The problem is:

Under what assumptions on \( f \) and \( a(t, s, u) \) equation (1) has a solution which is defined on \( \mathbb{R}_+ := [0, \infty) \)?

Many results on the theory of integral equations and many references one can find in [1]. Let us formulate the author’s result basic for our study (see [2], p. 105).

Let

\[ g'(t) \leq -\gamma(t)g(t) + \alpha(t, g(t)) + \beta(t), \quad t \geq 0, \quad g \geq 0; \quad g' = \frac{dg}{dt}, \]  

(2)

where \( \gamma \), \( \beta \) and \( \alpha \) are continuous functions of \( t \in \mathbb{R}_+ \), \( \alpha(t, g) \geq 0 \) is a continuous non-decreasing function of \( g \) on \( \mathbb{R}_+ := [t_0, \infty) \), \( t_0 \geq 0 \).

**Lemma 1.** Assume that there exists a function \( \mu = \mu(t) > 0 \), \( \mu \in C^1([t_0, \infty)) \) such that

\[ \alpha(t, \mu^{-1}(t)) + \beta(t) \leq \mu^{-1}(t) (\gamma(t) - \mu'(t)\mu^{-1}(t)) , \quad \forall t \geq t_0; \]  

(3)

MSC: 45D05, 45G10.
Key words: Nonlinear Volterra integral equations.
and
\[ \mu(t_0)g(t_0) < 1. \]  \hspace{1cm} (4)

Then any solution \( g \geq 0 \) to inequality (2) exists on \( \mathbb{R}_+ \) and
\[ 0 \leq g(t) < \mu^{-1}(t), \quad \forall t \geq t_0. \]  \hspace{1cm} (5)

If \( \mu(t_0)g(t_0) \leq 1 \) then \( 0 \leq g(t) \leq \mu^{-1}(t) \) for all \( t \geq t_0. \)

A proof of Lemma 1 is given in [2], pp. 105-107, see also [3].

A new idea in this paper is to use Lemma 1 with \( \gamma(t) = 0 \). In this case inequality (3) may hold only if \( \mu(t) \) decays as \( t \) grows, and estimate (5) becomes the estimate of the rate of growth of \( u \).

In [3] \( \mu(t) \) was growing to infinity as \( t \to \infty \) and estimate (5) gave results on the stability and long-time behavior of \( g(t) = \|u(t)\| \), where the norm was a Hilbert space norm.

Let us assume that \( t_0 = 0 \) and
\[ |f(t)| + |f'(t)| \leq c_0 e^{-b_0 t}, \quad \forall t \geq 0, \]  \hspace{1cm} (6)

\[ |a(t, t, u)| \leq c_1 e^{-b_1 t}(1 + |u|^{2p}), \quad p > 0, \]  \hspace{1cm} (7)

\[ \int_0^t |a_t(t, s, u(s))|ds \leq c_2 e^{-b t}(1 + |u(t)|^{2p}), \quad a_u(t, s, u) \geq 0, \quad a_t = \frac{\partial a}{\partial t}. \]  \hspace{1cm} (8)

Assume also that \( |a| + a_u \leq c(R_1, R_2) \) for \( t \leq R_1, s \leq R_1 \) and \( |u| \leq R_2, a \) and \( a_t \) are smooth functions of their arguments.

This assumption allows one to use the contraction mapping principle if \( t > 0 \) is sufficiently small and establish the existence and uniqueness of the local solution to equation (1).

Differentiate (11) with respect to \( t \) and get
\[ u' = f' + a(t, t, u(t)) + \int_0^t a_t(t, s, u(s))ds. \]  \hspace{1cm} (9)

**Lemma 2.** Let \( u(t) \in H \), where \( H \) is a Hilbert space, \( \|u\|^2 = (u, u) \), \( \|u\|' = \frac{d\|u\|}{dt} \). If \( u(t) \in C^1(\mathbb{R}_+; H) \) then
\[ \|u\|' \leq \|u\|. \]  \hspace{1cm} (10)

If \( u(t) \in C^1(\mathbb{R}_+) \), then
\[ |u|' \leq |u|. \]  \hspace{1cm} (11)

**Proof of Lemma 2.** One has \( \|u\|^2 = (u, u) \). Thus, \( 2\|u\|'\|u\| = (u', u) + (u, u') \leq 2\|u\|'\|u\|. \)

Since \( \|u\| \geq 0 \), one gets (10).

If \( u(t) \in C^1(\mathbb{R}_+) \), then \( |u(t + h)| - |u(t)| \leq |u(t + h) - u(t)| \). Divide this inequality by \( h > 0 \) and let \( h \to 0 \). This yields (11). \( \square \)

Taking the absolute value of (11), using (10) and setting \( g(t) = |u(t)| \), one obtains
\[ g' \leq c_0 e^{-b_0 t} + c_1 e^{-b_1 t}(1 + g^{2p}(t)) + c_2 e^{-b t}(1 + g^{2p}(t)). \]  \hspace{1cm} (12)
Theorem 1. If (4) and (6)–(8) hold, then the solution to (1) exists on \( \mathbb{R}_+ \), is unique and
\[
|u(t)| \leq ce^{qt}, \quad q > 0,
\]
where \( q > 0 \) is a fixed number and \( c > 0 \) is a sufficiently large constant.

In Section 2 a proof of Theorem 1 is given. From this proof one can get an estimate for the constant \( c \).

2 Proof of Theorem 1

Let us apply to (12) Lemma 1. Choose
\[
\mu = c_3e^{-qt}, \quad q = \text{const} > 0.
\]
Since \( \mu(0) = c_3 \) inequality (4) holds if \( g(0)c_3 < 1 \).

Example 1. Let \( u = 1 + \int_0^tu^2(s)ds \). Then \( u' = u^2 \), \( u(0) = 1 \). A simple integration yields \( u = (1 - t)^{-1} \). So, the solution tends to infinity as \( t \to 1 \).

Remark 1. Without some assumptions on \( f \) and \( a(t, s, u) \) the solution to (1) may not exist globally.

Example 2. Let \( u = 1 + \int_0^tu^2(s)ds \). Then \( u' = u^2 \), \( u(0) = 1 \). A simple integration yields \( u = (1 - t)^{-1} \). So, the solution tends to infinity as \( t \to 1 \).

Remark 2. The method developed in this paper can be used for other decay assumptions, for example, power decay of \( f \) and \( a(t, s, u) \) as \( t \to \infty \).
References

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