Quadratic Mean-Field Reflected BSDEs

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Abstract

In this paper, we analyze mean-field reflected backward stochastic differential equations when the driver has quadratic growth in the second unknown $z$. Using linearization technique and BMO martingale theory, we first apply fixed point argument to establish uniqueness and existence result for the case with bounded terminal condition and obstacle. Then, with the help of a $\theta$-method, we develop a successive approximation procedure to remove the boundedness condition on the terminal condition and obstacle when the generator is concave (or convex) with respect to the 2nd unknown $z$.

Key words: mean-field, reflected BSDEs, linearization technique, $\theta$-method

MSC-classification: 60H10, 60H30

1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a given complete probability space under which $B$ is a $d$-dimensional standard Brownian motion. Suppose $(\mathcal{F}_t)_{0 \leq t \leq T}$ is the natural filtration generated by $B$ augmented by the $P$-null sets and $\mathcal{P}$ the corresponding sigma algebra of progressive sets of $\Omega \times [0,T]$. This paper is devoted to the study of the following mean-field type reflected backward stochastic differential equations (BSDEs):

$$
\begin{aligned}
Y_t &= \xi + \int_t^T f(s, Y_s, P_{Y_s}, Z_s)ds - \int_t^T Z_s dB_s + K_T - K_t, \quad 0 \leq t \leq T, \\
Y_t &\geq h(t, Y_t, P_{Y_t}), \quad \forall t \in [0, T] \text{ and } \int_0^T (Y_t - h(t, Y_t, P_{Y_t}))dK_t = 0,
\end{aligned}
$$

where $P_{Y_t}$ is the marginal probability distribution of the process $Y$ at time $t$, the terminal condition $\xi$ is a scalar-valued $\mathcal{F}_T$-measurable random variable, the driver $f : \Omega \times [0, T] \times \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \times \mathbb{R}^d \to \mathbb{R}$ and the constraint $h : \Omega \times [0, T] \times \mathcal{P}_1(\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ are progressively measurable maps with respect to $\mathcal{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathcal{P}_1(\mathbb{R})) \times \mathcal{B}(\mathbb{R}^d)$ and $\mathcal{P} \times \mathcal{B}(\mathcal{P}_1(\mathbb{R})) \times \mathcal{B}(\mathbb{R})$ respectively.

It is well known that El Karoui et al. [18] introduced the following reflected BSDE

$$
\begin{aligned}
Y_t &= \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s + K_T - K_t, \quad 0 \leq t \leq T, \\
Y_t &\geq L_t, \quad \forall t \in [0, T] \text{ and } \int_0^T (Y_t - L_t)dK_t = 0,
\end{aligned}
$$

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in which the obstacle $L$ is a given stochastic process. When the terminal condition is square-integrable and the driver is Lipschitz in the unknowns $(y, z)$, the authors of [18] obtained the existence and uniqueness of solution to reflected BSDE (2) both by a fixed point argument and by penalization method. Great progress has since then been made in this field, as it has rich connections with obstacle problems of partial differential equations, American option pricing, zero-sum games, switching problems and many others, e.g., see [4, 14, 19, 21, 24, 26] and the references therein for more details on this topic. In particular, the term $Y$ can be seen as a solution of an optimal stopping problem

\[ Y_t = \operatorname{ess} \sup_{\tau \text{ stopping time } \geq t} \mathbb{E}_t \left[ \eta 1_{\{\tau = T}\} + L_{\tau} 1_{\{\tau < T\}} + \int_t^\tau f(s, Y_s, Z_s)ds \right], \quad \forall t \leq T. \quad (3) \]

Recently, in order to study partial hedging of financial derivatives, various mean-field type reflected BSDEs were introduced, in which the driver $f$ and the obstacle $h$ may depend on the law of the term $Y$. For example, Briand, Elie and Hu [8] considered BSDEs with mean reflection to study the super-hedging problem under running risk management constraint. We refer the reader to [4, 6, 12, 16, 32] and the references therein for some other important contributions.

In particular, motivated by applications in pricing life insurance contracts with surrender options, Djehiche, Elie and Hamadène formulated in [15] mean-field reflected BSDEs of the form (1). Under the Lipschitz hypothesis on the driver, they used a fixed point method to prove the existence and uniqueness result for mean-field reflected BSDEs (1) via the Snell envelope representation (3):

\[ \Gamma(U)_t = \operatorname{ess} \sup_{\tau \text{ stopping time } \geq t} \mathbb{E}_t \left[ \xi 1_{\{\tau = T\}} + h(\tau, U_\tau, (P_{U_s})_{s = \tau}) 1_{\{\tau < T\}} + \int_\tau^T f(s, U_s, P_{U_s})ds \right], \quad \forall t \leq T, \quad (4) \]

in which the driver $f$ is independent of the second unknown $z$. Indeed, any solution $Y$ to (1) is a fixed point of the solution map $\Gamma(U)$. Note that the comparison principle for mean-filed BSDE is quite restricted, which involves some additional monotone hypothesis on the driver (see [11]). Thus, under some additional assumptions, they applied a penalization method to obtain the existence of a solution when the driver $f$ also depends on the second unknown $z$. More precisely, they used a global domination condition in the $z$ component and assumed that

\[ f(s, Y_s, P_{Y_s}, Z_s) = F(s, Y_s, \mathbb{E}[Y_s], Z_s), \quad h(t, Y_t, P_{Y_t}) = H(t, Y_t, \mathbb{E}[Y_t]), \]

where $F$ and $H$ are non-decreasing with respect to $\mathbb{E}[Y_s]$.

Recently, Djehiche, Dumitrescu and Zeng [17] studied the mean-field reflected BSDEs with jumps and right-continuous and left-limited obstacle. In the Lipschitz driver case, they introduced a novel fixed point argument to establish the existence as well as the uniqueness of the solution without these additional assumptions. The main idea is based on the following nonlinear Snell envelope representation for the reflected BSDE (2):

\[ Y_t = \operatorname{ess} \sup_{\tau \text{ stopping time } \geq t} \mathcal{E}^\eta_{t, \tau}[\eta 1_{\{\tau = T\}} + L_{\tau} 1_{\{\tau < T\}}], \quad \forall t \leq T, \quad (5) \]

where $\mathcal{E}^\eta_{t, \tau}[\eta 1_{\{\tau = T\}} + L_{\tau} 1_{\{\tau < T\}}] := y^\tau_\eta$ is the solution to the following standard BSDE:

\[ y^\tau_\eta = \eta 1_{\{\tau = T\}} + L_{\tau} 1_{\{\tau < T\}} + \int_\tau^T f(s, y^\tau_s, z^\tau_s)ds - \int_\tau^T z^\tau_s dB_s, \quad (6) \]

which does not explicitly involve the term $Z$ and allows to construct a solution map $\Gamma$ when the driver $f$ depends on the second unknown $z$. Our aim is to establish the existence and uniqueness of the solution to the mean-field reflected BSDE (1) with quadratic driver, i.e., the driver $f$ is allowed to have quadratic growth in the second unknown $z$.
In the BSDEs theory, the research of quadratic case is significantly more difficult than that of Lipschitz case. Based on the monotone convergence method and PDE-based approximation technique, Kobylanski [29] established the solvability of real-valued quadratic BSDEs with bounded terminal condition. Then, using monotone convergence method and localization stopping times, Briand and Hu [9] extended the existence result of real-valued quadratic BSDEs to the case that the terminal condition can have exponential moment of certain order. Under the additional assumption that the driver \( f \) is concave (or convex) with respect to the 2nd unknown \( z \), Briand and Hu [10] used a \( \theta \)-method to obtain the uniqueness result.

With the help of the afore-mentioned results, some generalizations were obtained for quadratic reflected BSDEs. Indeed, Kobylanski et al. [30] and Bayraktar and Yao [2] made a counterpart study for the case of bounded terminal condition and obstacle and of unbounded terminal condition and obstacle, respectively. In particular, they also established the corresponding nonlinear Snell envelope representation (5), which allows us to construct a iteration map \( U \to \Gamma(U) \) as that of Lipschitz case.

However, the monotone convergence method for mean-field BSDE is quite restricted. Thus, we have to develop an alternative approximation approach to obtain a fixed point of the quadratic solution map \( \Gamma \). Note that Tevzadze [36] proposed a fixed point method for real-valued quadratic BSDEs with bounded terminal condition through BMO martingale theory. However, when the terminal condition is unbounded, the fixed point argument fails to work since the 2nd unknown \( Z \) may be unbounded in the BMO space. Recently, it was found in Fan et al. [21] that \( \theta \)-method also provides an approximation procedure for (multi-dimensional) quadratic BSDEs when the driver is concave (or convex) and terminal value has exponential moments of arbitrary order. For more research on this field, we refer the reader to [1, 7, 13, 25, 27, 33, 37] and the references therein.

Thanks to these results, we could show that the quadratic solution map \( \Gamma \) admits a unique fixed point by contraction map argument and the \( \theta \)-method. When the terminal condition and obstacle are bounded, we combine nonlinear Snell envelope representation and BMO martingale theory to show the quadratic solution map is a contraction. In comparison to that of [17], we use linearization technique to estimate the difference of two solutions instead of Itô’s formula. As a byproduct, our argument removes a domination condition [17, Assumption 2.1 (ii)(b)] for the Lipschitz case.

In the unbounded case, we first apply nonlinear Snell envelope representation to introduce an approximation procedure through a sequence of quadratic reflected BSDEs with unbounded terminal condition and obstacle. Then, utilizing quadratic BSDEs theory and the \( \theta \)-method, we show the convergence of the approximating sequences by some delicate and involved technique computations. In particular, the corresponding limit is the unique solution to the quadratic mean-field reflected BSDE (11) with the concave driver and the terminal value of exponential moments of arbitrary order. In conclusion, we develop quadratic mean-field reflected BSDEs theory which gives some extension of the result from [13] and [17] to the quadratic case.

The paper is organized as follows. In section 2, we start with some technical lemmas and a revisit to Lipschitz case to illustrate the main idea. Section 3 is devoted to the quadratic case with bounded terminal condition and obstacle, while Section 4 removes the boundedness condition using convexity on the driver.

Notation.

For each Euclidian space, we denote by \( \langle \cdot, \cdot \rangle \) and \( | \cdot | \) its scalar product and the associated norm, respectively. Then, for each \( p \geq 1 \), we consider the following collections:

- \( L^p \) is the collection of real-valued \( \mathcal{F}_T \)-measurable random variables \( \xi \) satisfying

\[
\| \xi \|_{L^p} = \mathbb{E}[|\xi|^p]^{\frac{1}{p}} < \infty;
\]
• $\mathcal{L}^{\infty}$ is the collection of real-valued $\mathcal{F}_T$-measurable random variables $\xi$ satisfying
  \[ \|\xi\|_{\mathcal{L}^{\infty}} = \text{ess sup}_{\omega \in \Omega} |\xi(\omega)| < \infty; \]

• $H^{p,d}$ is the collection of $\mathbb{R}^d$-valued $\mathcal{F}$-progressively measurable processes $(z_t)_{0 \leq t \leq T}$ satisfying
  \[ \|z\|_{H^{p,d}} = \mathbb{E}\left[ \left( \int_0^T |z_t|^2 dt \right)^{\frac{p}{2}} \right] < \infty; \]

• $S^{p}$ is the collection of real-valued $\mathcal{F}$-adapted continuous processes $(y_t)_{0 \leq t \leq T}$ satisfying
  \[ \|y\|_{S^{p}} = \mathbb{E}\left[ \sup_{t \in [0,T]} |y_t|^p \right]^{\frac{1}{p}} < \infty; \]

• $S^{\infty}$ is the collection of real-valued $\mathcal{F}$-adapted continuous processes $(y_t)_{0 \leq t \leq T}$ satisfying
  \[ \|y\|_{S^{\infty}} = \text{ess sup}_{(t,\omega) \in [0,T] \times \Omega} |y(t,\omega)| < \infty; \]

• $A^{p}$ is the collection of continuous non-decreasing processes $(K_t)_{0 \leq t \leq T} \in S^{p}$ with $K_0 = 0$;

• $P_p(\mathbb{R})$ is the collection of all probability measures over $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ with finite $p^{th}$ moment, endowed with the $p$-Wasserstein distance $W_p$;

• $L^p$ is the collection of real-valued $\mathcal{F}_T$-measurable random variables $\xi$ satisfying $\mathbb{E}[e_p|\xi|] < \infty$;

• $S^p$ is the collection of all stochastic processes $Y$ such that $e^Y \in S^p$;

• $L$ is the collection of all random variables $\xi \in L^p$ for any $p \geq 1$, and $H^{d}$, $A$ and $S$ are defined in a similar way;

• $T_\tau$ is the collection of $[0,T]$-valued $\mathcal{F}$-stopping times $\tau$ such that $\tau \geq t$ $\mathbb{P}$-a.s.;

• $BMO$ is the collection of $\mathbb{R}^d$-valued progressively measurable processes $(z_t)_{0 \leq t \leq T}$ such that
  \[ \|z\|_{BMO} := \sup_{\tau \in \tau_0} \text{ess sup}_{\omega \in \Omega} \mathbb{E}_\tau \left[ \int_\tau^T |z_s|^2 ds \right]^{\frac{1}{2}} < \infty. \]

Denote by $\ell_{[a,b]}$ the corresponding collections for the stochastic processes with time indexes on $[a,b]$ for $\ell = H^{p,d}, S^p, S^{\infty}$ and so on. For each $Z \in BMO$, we set
\[ \delta(\cdot;B)_0 = \exp \left( \int_0^t Z_s dB_s - \frac{1}{2} \int_0^t |Z_s|^2 ds \right), \]
which is a martingale by $2\mathbb{N}$. Thus it follows from Girsanov’s theorem that $(B_t - \int_0^t Z_s ds)_{0 \leq t \leq T}$ is a Brownian motion under the equivalent probability measure $\delta(Z;B)_0 d\mathbb{P}$. 

4
2 A reminder in the Lipschitz case

2.1 Preliminaries

Let us start by giving the definition of a solution and some technical results, which will be frequently used in our subsequent discussions.

**Definition 2.1** By a solution to (1), we mean a triple of progressively measurable processes \((Y, Z, K)\) such that (1) holds.

For each \(\mathcal{F}\)-stopping time \(\tau\) taking values in \([0, T]\) and for every \(\mathcal{F}_\tau\)-measurable function \(\eta \in L^p\) for some \(p > 1\), we first define the following \(g\)-evaluation (see [35]):

\[
\mathcal{E}_{t, \tau}^g[\eta] := y^\tau_t, \quad \forall t \in [0, T],
\]

where \(y^\tau\) is the solution to the following BSDE on the random time horizon \([0, \tau]\)

\[
y^\tau_t = \eta + \int_t^\tau g(s, y^\tau_s, z^\tau_s)ds - \int_t^\tau z^\tau_s dB_s.
\] (7)

If the BSDE (7) admits a unique solution \((Y, Z) \in S^p \times H^p, d\), then it is easy to check that

\[
y^\tau_t = y^\tau_{t \wedge \tau}, z^\tau_t = z^\tau_{t \wedge \tau} \mathbf{1}_{[t, \tau]}(t), \quad \forall t \in [0, T].
\]

Next, we introduce the following reflected BSDE:

\[
\begin{aligned}
Y_t &= \eta + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s + K_T - K_t, \quad 0 \leq t \leq T, \\
Y_t &\geq L_t, \quad \forall t \in [0, T] \quad \text{and} \quad \int_0^T (Y_t - L_t) dK_t = 0,
\end{aligned}
\] (8)

where \(\eta \in L^p\), \(L \in S^p\) with \(L_T \leq \eta\). We assume that the driver \(g\) satisfies the following comparison principle.

**Comparison principle:** For \(i = 1, 2\), let \((y^i, z^i, v^i) \in S^p \times H^p, d \times A^p\) be a solution to the following BSDE

\[
y^i_t = \xi^i + \int_t^T g(s, y^i_s, z^i_s)ds + v^i_T - v^i_t - \int_t^T z^i_s dB_s.
\]

If \(\xi^1 \geq \xi^2\) and \(v^2 \equiv 0\), then \(y^1_t \geq y^2_t\) for every \(t \in [0, T]\).

We have the following nonlinear Snell envelope representation for the solution of the reflected BSDE (8).

**Lemma 2.2** Let \((Y, Z, K) \in S^p \times H^p, d \times A^p\) be a solution to the reflected BSDE (8). If \(g\) satisfies the comparison principle, then

\[
Y_t = \text{ess sup}_{\tau \in \mathcal{T}_t} \mathcal{E}_{t, \tau}^g[\eta \mathbf{1}_{\{\tau = T\}} + L_T \mathbf{1}_{\{\tau < T\}}].
\]

In particular, the reflected BSDE (8) has at most one solution.

**Proof.** For any \(\tau \in \mathcal{T}_t\), we have

\[
Y_s = Y_\tau + \int_s^\tau g(r, Y_r, Z_r)dr - \int_s^\tau Z_r dB_r + K_\tau - K_r, \quad \forall s \in [t, \tau].
\]
Note that $Y_r \geq \eta 1_{\{r=T\}} + L_r 1_{\{r<T\}}$ and $K$ is a non-decreasing process. It follows from the comparison principle that

$$Y_t \geq \mathcal{E}_{t,T}^g[\eta 1_{\{\tau=T\}} + L_{\tau} 1_{\{\tau<T\}}], \quad \forall \tau \in \mathcal{T}_t.$$ 

On the other hand, we define the stopping time $\tau^* = \inf\{r \in [t,T] : Y_r = L_r \} \wedge T$. Since $Y_r \geq L_r$ and $\int_t^T (Y_r - L_r) dK_r = 0$, we conclude that $K_{\tau^*} = K_t$, which indicates that

$$Y_s = Y_{\tau^*} + \int_{\tau^*}^s g(r, Y_r, Z_r) dr - \int_{\tau^*}^s Z_r dB_r, \quad \forall s \in [t, \tau^*].$$

Note that $Y_{\tau^*} = \xi 1_{\{\tau^*=T\}} + L_{\tau^*} 1_{\{\tau^*<T\}}$ by the definition of $\tau^*$. It follows that

$$Y_t = \mathcal{E}_{t,\tau^*}^g[\eta 1_{\{\tau^*=T\}} + L_{\tau^*} 1_{\{\tau^*<T\}}],$$

which completes the proof. \[\square\]

**Remark 2.3** It is obvious that the comparison principle and the nonlinear Snell envelope representation hold in the Lipschitz case. We refer to \cite{2,3} for some sufficient conditions under which the results hold for the quadratic case.

**Remark 2.4** It follows from Lemma 2.2 that any solution $Y$ to the mean-field reflected BSDE \cite{1} is a fixed point of the following map $\Gamma$:

$$\Gamma(U)_t := \text{ess sup}_{\tau \in \mathcal{T}_t} \mathcal{E}_{t,\tau}^U [\xi 1_{\{\tau=T\}} + h(\tau, U_\tau, (P_{U_\tau})_s=\tau) 1_{\{\tau<T\}}], \quad \forall t \in [0,T],$$

where the driver $f^U$ is given by $f^U(t, z) := f(t, U_t, P_{U_t}, z)$.

The representation given by Lemma 2.2 is an important tool for our main results in the following sections. With the help of the representation, we will combine a linearization technique, a fixed point argument and the $\theta$-method to study quadratic mean-field reflected BSDEs. The authors of \cite{17} applied the representation result and a fixed point argument to prove existence and uniqueness of a solution for mean-field reflected BSDEs with jumps when the driver is Lipschitz. In order to illustrate our main idea and present some preliminaries involved, we first deal with the Lipschitz case via a linearization technique and a fixed point method. As a byproduct, our argument removes a domination condition \cite{17}, Assumption 2.1 (ii)(b)].

### 2.2 Revisit to the Lipschitz case

In what follows, we make use of the following conditions on the terminal condition $\xi$, the driver $f$ and the constraint $h$.

- **(B1)** There exists a constant $p > 1$ such that $\xi \in \mathcal{L}^p$ with $\xi \geq h(T, \xi, P_\xi)$.

- **(B2)** The process $f(t, 0, \delta_0, 0)$ belongs to $\mathcal{H}^{p,1}$ and there exists a constant $\lambda > 0$ such that for any $t \in [0,T]$, $y_1, y_2 \in \mathbb{R}$, $v_1, v_2 \in \mathcal{P}_1(\mathbb{R})$ and $z_1, z_2 \in \mathbb{R}^d$

$$|f(t, y_1, v_1, z_1) - f(t, y_2, v_2, z_2)| \leq \lambda (|y_1 - y_2| + W_1(v_1, v_2) + |z_1 - z_2|).$$

- **(B3)** The process $h(t, y, v)$ belongs to $\mathcal{S}^p$ for any $y \in \mathbb{R}$, $v \in \mathcal{P}_1(\mathbb{R})$ and there exist two constants $\gamma_1, \gamma_2 > 0$ such that for any $t \in [0,T]$, $y_1, y_2 \in \mathbb{R}$, $v_1, v_2 \in \mathcal{P}_1(\mathbb{R})$

$$|h(t, y_1, v_1) - h(t, y_2, v_2)| \leq \gamma_1 |y_1 - y_2| + \gamma_2 W_1(v_1, v_2).$$
We are now ready to state the main result of this section.

**Theorem 2.5** Assume that (B1)-(B3) are satisfied. If \( \gamma_1 \) and \( \gamma_2 \) satisfy
\[
(\gamma_1 + \gamma_2) \left( \frac{p}{p-1} \right)^{\frac{p}{p-1}} < 1,
\]
then the mean-field reflected BSDE (11) admits a unique solution \((Y, Z, K) \in S^p \times H^{p,d} \times A^p\).

**Remark 2.6** We remove the additional domination condition from [17, Assumption 2.1 (ii)(b)], that requires:

\[
\sup_{(y,v) \in \mathbb{R} \times P_1(\mathbb{R})} |h(t, y, v)| \text{ belongs to } S^p.
\]

**Remark 2.7** Note that the enhanced sufficient condition (9) is the same as in [15, Theorem 3.1]. The authors of [17] proved that mean-field reflected BSDEs with jumps admit a unique solution under the following enhanced sufficient condition
\[
2^{p-1}(\gamma_1^p + \gamma_2^p) \leq 1.
\]

When \( \gamma_1 = 0 \), it reduces to the condition \( \gamma_2 \leq 2^{\frac{1}{p-1}} \), whereas (9) reduces to the condition \( \gamma_2 \leq 1 \). On the other hand, it is easy to check that
\[
(\gamma_1 + \gamma_2)^{p-1} \left( \frac{p}{p-1} \right)^{\frac{p}{p-1}} \geq 2^{p-1}(\gamma_1^p + \gamma_2^p)
\]
when \( \gamma_1 = \gamma_2 \).

We are now ready to prove Theorem 2.5. More precisely, we first state the existence and uniqueness of the solution on a small time interval \([T-h, T]\), in which \( h \) is to be determined later. Then, we stitch the local solutions to build the global solution.

According to Assumption (B3), it is obvious that \((h(s, U_s, P_{U_s}))_{s \in [T-h, T]} \in S^p_{[T-h, T]}\) for any \( U \in S^p_{[T-h, T]}\). It follows from Lemma 2.2 and [23, Theorem 3] that \( \Gamma(U) \) is the \( S^p \)-solution to the reflected BSDE (8) with data \((\eta, g, L) = (\xi, f^U, h, (U, P_U))\). Thus for any \( h \in (0, T] \), we have
\[
\Gamma \left( S^p_{[T-h, T]} \right) \subset S^p_{[T-h, T]}.
\]

Let us now show uniqueness and existence of the local solution for the mean-field reflected BSDE (11).

**Lemma 2.8** Assume that (B1)-(B3) hold. If \( \gamma_1 \) and \( \gamma_2 \) satisfy (9), then there exists a constant \( \delta > 0 \) depending only on \( p, \lambda, \gamma_1 \) and \( \gamma_2 \) such that for any \( h \in (0, \delta] \), the mean-field reflected BSDE (11) admits a unique solution \((Y, Z, K) \in S^p_{[T-h, T]} \times H^{p,d}_{[T-h, T]} \times A^p_{[T-h, T]}\) on the time interval \([T-h, T]\).

**Proof.** The proof will be divided into three steps.

**Step 1 (A priori estimate).** Let \( U^i \in S^p \), \( i = 1, 2 \). It follows from Lemma 2.2 that
\[
\Gamma(U^i)_{t} \coloneqq \text{ess sup}_{\tau \in T_t} y^{i, \tau}_t, \quad \forall t \in [0, T],
\]
in which \( y^{i, \tau}_t \) is the solution of the following BSDE
\[
y^{i, \tau}_t = \xi 1_{\{\tau = T\}} + h(\tau, U^i_{\tau}, (P_{U^i})_{s=\tau}) 1_{\{\tau < T\}} + \int_t^\tau f(s, U^i_s, P_{U^i_s}, z^{i, \tau}_s)ds - \int_t^\tau z^{i, \tau}_s dB_s. \quad (11)
\]
For each \( t \in [0, T] \), denote by
\[
\beta_t = \frac{f^{U_1}(t, z^{1, \tau}_t) - f^{U_1}(t, z^{2, \tau}_t)}{|z^{1, \tau}_t - z^{2, \tau}_t|^2}(z^{1, \tau}_t - z^{2, \tau}_t)1_{\{|z^{1, \tau}_t - z^{2, \tau}_t| \neq 0\}}.
\]
Then, the pair of processes \((y^{1, \tau}_t, y^{2, \tau}_t, z^{1, \tau}_t, z^{2, \tau}_t)\) solves the following BSDE:
\[
y^{1, \tau}_t - y^{2, \tau}_t = h(\tau, U^1_t, (P_{U^1})_{s=\tau})1_{\{\tau < T\}} - h(\tau, U^2_t, (P_{U^2})_{s=\tau})1_{\{\tau < T\}} - \int_t^T 1_{[0, \tau]}(s)(z^{1, \tau}_s - z^{2, \tau}_s)dB_s + \int_t^T \left( \beta_s(z^{1, \tau}_s - z^{2, \tau}_s)^T + f^{U_1}(s, z^{2, \tau}_s) - f^{U_2}(s, z^{2, \tau}_s) \right)1_{[0, \tau]}(s)ds.
\]
Note that \( \tilde{B}_t := B_t - \int_0^t \beta_s^T 1_{[0, \tau]}(s)ds \), defines a Brownian motion under the equivalent probability measure \( \tilde{P} \) given by \( d\tilde{P} := \delta(\beta 1_{[0, \tau]} \cdot B)^T dP \). It follows that for every \( t \in [0, T] \)
\[
y^{1, \tau}_t - y^{2, \tau}_t = \mathbb{E}_t\left[ h(\tau, U^1_t, (P_{U^1})_{s=\tau})1_{\{\tau < T\}} - h(\tau, U^2_t, (P_{U^2})_{s=\tau})1_{\{\tau < T\}} \right]
+ \int_t^T \left( f^{U_1}(s, z^{2, \tau}_s) - f^{U_2}(s, z^{2, \tau}_s) \right)ds.
\]
Noting that \( |\beta_t| \leq \lambda \) and by a standard computation, we have that for any \( q \geq 1 \),
\[
\mathbb{E}_t \left[ |\beta 1_{[0, \tau]} \cdot B|^q \right] \leq \exp \left( \frac{\lambda^2}{2} (q^2 - q)(T - t) \right).
\]
In view of Hölder’s inequality, we have for any \( \mu \in (1, p) \) and any \( t \in [T - h, T] \),
\[
|y^{1, \tau}_t - y^{2, \tau}_t|^\mu \leq \exp \left( \frac{\lambda^2 h}{2(\mu - 1)} \right) \mathbb{E}_t\left[ \left( \gamma_1 + \lambda h \sup_{s \in [T-h, T]} |U^1_s - U^2_s| + \gamma_2 + \lambda h \sup_{s \in [T-h, T]} \mathbb{E}[|U^1_s - U^2_s|] \right)^\mu \right]^{\frac{1}{\mu}},
\]
which together with (10) implies the following, for any \( t \in [T - h, T] \)
\[
|\Gamma(U^1)_t - \Gamma(U^2)_t|^p \leq \exp \left( \frac{p \lambda^2 h}{2(\mu - 1)} \right) \mathbb{E}_t\left[ \left( \gamma_1 + \lambda h \sup_{s \in [T-h, T]} |U^1_s - U^2_s| + \gamma_2 + \lambda h \sup_{s \in [T-h, T]} \mathbb{E}[|U^1_s - U^2_s|] \right)^\mu \right]^{\frac{1}{\mu}}, \tag{12}
\]
Step 2 (The contraction). The convexity inequality \((ax + by)^\rho \leq (a + b)^{\rho - 1}(ax^\rho + by^\rho)\) holds for any non-negative constants \( a, b, x, y \) and \( \rho \geq 1 \). It follows that
\[
\mathbb{E}_t\left[ \left( \gamma_1 + \lambda h \sup_{s \in [T-h, T]} |U^1_s - U^2_s| + \gamma_2 + \lambda h \sup_{s \in [T-h, T]} \mathbb{E}[|U^1_s - U^2_s|] \right)^\mu \right]^{\frac{1}{\mu}} \leq (\gamma_1 + \gamma_2 + 2\lambda h)^{\frac{p-1}{\rho}} \mathbb{E}_t\left[ \left( \gamma_1 + \lambda h \sup_{s \in [T-h, T]} |U^1_s - U^2_s|^\rho \right)^\mu \right]^{\frac{1}{\mu}} \leq (\gamma_1 + \gamma_2 + 2\lambda h)^{\rho - 1} \mathbb{E}_t\left[ \left( \gamma_1 + \lambda h \sup_{s \in [T-h, T]} |U^1_s - U^2_s|^\rho \right)^\mu \right]^{\frac{1}{\mu}}.
\]
Recalling (12) and applying Doob’s maximal inequality, we derive

\[
E \left[ \sup_{t \in [T-h, T]} |\Gamma(U^1)_t - \Gamma(U^2)_t|^p \right] \leq \exp \left( \frac{p\lambda^2 h}{2(\mu - 1)} \right) (\gamma_1 + \gamma_2 + 2\lambda h)^{p-1}
\]

\[
\times \left( (\gamma_1 + \lambda h) \left( \frac{p}{p - \mu} \right)^{\frac{p}{2}} E \left[ \sup_{s \in [T-h, T]} |U^1_s - U^2_s|^p \right] + (\gamma_2 + \lambda h) \sup_{s \in [T-h, T]} E[|U^1_s - U^2_s|^p] \right).
\]

Consequently, for any \( \mu \in (1, p) \) and \( h \in (0, (\mu - 1)^2] \), we have

\[
E \left[ \sup_{t \in [T-h, T]} |\Gamma(U^1)_t - \Gamma(U^2)_t|^p \right] \leq \Lambda(\mu) E \left[ \sup_{s \in [T-h, T]} |U^1_s - U^2_s|^p \right]^{\frac{1}{p}}
\]

with

\[
\Lambda(\mu) = \exp \left( \frac{\lambda^2(\mu - 1)}{2} \right) (\gamma_1 + \gamma_2 + 2\lambda(\mu - 1)^2) \left( \frac{p}{p - \mu} \right)^{\frac{p}{2}} + (\gamma_2 + \lambda(\mu - 1)^2) \right)^{\frac{1}{p}}.
\]

Under Assumption (9), we can then find a small enough constant \( \mu^* \in (1, p) \) depending only on \( p, \lambda, \gamma_1 \) and \( \gamma_2 \) such that \( \Lambda(\mu^*) < 1 \). Let us define

\[
\delta := (\mu^* - 1)^2.
\]

(13)

It is now obvious that \( \Gamma \) is a contraction map on the time interval \([T-h, T]\) for any \( h \in (0, \delta) \).

**Step 3 (Uniqueness and existence).** Note that any solution \( Y \) to the mean-field reflected BSDE (1) is a fixed point of the map \( \Gamma \). For any \( h \in (0, \delta] \), \( \Gamma \) has a unique fixed point \( Y \in S^p_{[T-h, T]} \), so that

\[
Y_t = \text{ess sup}_{\tau \in T} E^\mathcal{F}_{t, \tau} \left[ 1_{\{\tau = T\}} + h(\tau, Y_{\tau}, (P_{Y_{\tau}}, k_{\tau}))1_{\{\tau < T\}} \right], \quad \forall t \in [T-h, T].
\]

On the other hand, the reflected BSDE (8) with data \((\eta, g, L) = (\xi, f^Y, h(\cdot, Y, P_Y))\) admits a unique solution

\[
(Y, Z, K) \in S^p_{[T-h, T]} \times H^p_{[T-h, T]} \times A^p_{[T-h, T]}.
\]

It follows from Lemma 2.2 that \( \bar{Y} = \Gamma(Y) = Y \), which implies that \((Y, Z, K)\) is a solution to the mean-field reflected BSDE (1) on the time interval \([T-h, T]\).

Let us now turn to the proof of uniqueness. Suppose \((Y', Z', K')\) is also a solution to the mean-field reflected BSDE (1) on the time interval \([T-h, T]\). In the spirit of Lemma 2.2, \( Y' \) is the fixed point of the map \( \Gamma \), which indicates that \( Y = Y' \). Applying Itô’s formula to \(|Y - Y'|^2\) yields that \( Z = Z' \) and then \( K = K' \). This completes the proof.

Now we are in a position to complete the proof of the main result.

**Proof of Theorem 2.5.** The uniqueness of the global solution on \([0, T]\) is inherited from the uniqueness of the local solution on each small time interval. It suffices to prove the existence.

By Lemma 2.8 there exists a constant \( \delta > 0 \) depending only on \( p, \lambda, \gamma_1 \) and \( \gamma_2 \), such that the mean-field reflected BSDE (1) admits a unique solution

\[
(Y^1, Z^1, K^1) \in S^p_{[T-\delta, T]} \times H^p_{[T-\delta, T]} \times A^p_{[T-\delta, T]}.
\]
on the time interval \([T - \delta, T]\). Next, taking \(T - \delta\) as the terminal time and applying Lemma \(2.8\) again, the mean-field reflected BSDE (1) admits a unique solution

\[
(Y^2, Z^2, K^2) \in S^p_{[T - 2\delta, T - \delta]} \times H^p_{[T - 2\delta, T - \delta]} \times A^p_{[T - 2\delta, T - \delta]}
\]
on the time interval \([T - 2\delta, T - \delta]\). Denote by

\[
Y_t = \sum_{i=1}^{2} Y^i_t 1_{[T - i\delta, T - (i-1)\delta]} + Y^1_T 1_{T}, \quad Z_t = \sum_{i=1}^{2} Z^i_t 1_{[T - i\delta, T - (i-1)\delta]} + Z^1_T 1_{T},
\]

\[
K_t = K^2_T 1_{[T - i\delta, T - (i-1)\delta]} + (K^2_T + K^1_T) 1_{[T - \delta, T]}.
\]

It is easy to check that \((Y, Z, K) \in S^p_{[T - 2\delta, T]} \times H^p_{[T - 2\delta, T]} \times A^p_{[T - 2\delta, T]}\) is a solution to the mean-field reflected BSDE (1). Repeating this procedure, we get a global solution \((Y, Z, K) \in S^p \times H^p \times A^p\).

The proof of the theorem is complete.

\(\square\)

3 Bounded terminal condition and obstacle

In this section, we will use a linearization technique and a fixed point argument to investigate the quadratic case for the mean-field reflected BSDE (1) with bounded terminal condition and obstacle. In comparison to the Lipschitz case, the BMO martingale theory plays a key role here.

In what follows, we make use of the following conditions on the terminal condition \(\xi\), the driver \(f\) and the constraint \(h\).

(H1) The terminal condition \(\xi \in L^\infty\) with \(\xi \geq h(T, \xi, P_\xi)\).

(H2) There exist three positive constants \(\alpha, \beta\) and \(\gamma\) such that for any \(t \in [0, T], y \in \mathbb{R}, v \in P_1(\mathbb{R})\) and \(z \in \mathbb{R}^d\)

\[
|f(t, y, v, z)| \leq \alpha + \beta(|y| + W_1(v, \delta_0)) + \frac{\gamma}{2}|z|^2.
\]

(H3) The process \(h(\cdot, y, v) \in S^\infty\) is uniformly bounded with respect to \((t, \omega, y, v)\).

(H4) There exist two constants \(\gamma_1, \gamma_2 > 0\) such that for any \(t \in [0, T], y_1, y_2 \in \mathbb{R}, v_1, v_2 \in P_1(\mathbb{R})\)

\[
|h(t, y_1, v_1) - h(t, y_2, v_2)| \leq \gamma_1|y_1 - y_2| + \gamma_2 W_1(v_1, v_2).
\]

(H5) There exists a constant \(\kappa\) such that for each \(t \in [0, T], y_1, y_2 \in \mathbb{R}, v_1, v_2 \in P_1(\mathbb{R})\) and \(z_1, z_2 \in \mathbb{R}^d\)

\[
|f(t, y_1, v_1, z_1) - f(t, y_2, v_2, z_2)| \leq \beta(|y_1 - y_2| + W_1(v_1, v_2)) + \kappa(1 + |z_1| + |z_2|)|z_1 - z_2|.
\]

We are now ready to state the main result of this section.

Theorem 3.1 Assume that (H1)-(H5) are satisfied. If \(\gamma_1\) and \(\gamma_2\) satisfy

\[
\gamma_1 + \gamma_2 < 1,
\]

then the quadratic mean-field reflected BSDE (1) admits a unique solution \((Y, Z, K) \in S^\infty \times BMO \times A\).

In order to prove Theorem 3.1, we need to analyze the quadratic solution map \(\Gamma\).
Lemma 3.2 Assume that (H2) and (H5) are satisfied and that \( \eta \in \mathcal{L}^\infty, U \in \mathcal{S}^\infty \). Then, the following quadratic BSDE:

\[
y^\tau_t = \eta + \int_t^\tau f(s, U_s, P_{U_s}, Z_s) ds - \int_t^\tau z^\tau_s dB_s,
\]

admits a unique solution \((y^\tau, z^\tau) \in \mathcal{S}^\infty \times \text{BMO}\).

**Proof.** The result is an immediate consequence of \([38, \text{Theorem 7.3.3}]\). ■

Lemma 3.3 Assume that (H1)-(H5) are satisfied and \( U \in \mathcal{S}^\infty \) with \( U_T = \xi \). Then, the following quadratic reflected BSDE:

\[
\left\{
\begin{array}{ll}
Y_t = \xi + \int_t^T f(s, U_s, P_{U_s}, Z_s) ds - \int_t^T Z_s dB_s + K_T - K_t, & 0 \leq t \leq T, \\
Y_t \geq h(t, U_t, P_{U_t}), & \forall t \in [0, T] \quad \text{and} \quad \int_0^T (Y_t - h(t, U_t, P_{U_t})) dK_t = 0,
\end{array}
\right.
\]

(15)

admits a unique solution \((Y, Z, K) \in \mathcal{S}^\infty \times \mathcal{H}^{d,2} \times \mathcal{A}^2\).

**Proof.** It follows from \([38, \text{Theorem 7.3.1}]\) that the comparison principle holds for the driver \((t, z) \mapsto f(t, U_t, P_{U_t}, z)\) under Assumptions (H2) and (H5). This, together with Lemma 3.2, implies that \( Y = \Gamma(U) \) for any solution \( Y \) to the quadratic reflected BSDE (15). Thus, it suffices to prove the existence. From Assumptions (H1)-(H4), it is easy to check that the driver \((t, z) \mapsto f(t, U_t, P_{U_t}, z)\) and the obstacle \( t \mapsto h(t, U_t, P_{U_t})\) satisfy \([30, \text{Conditions (H1)-(H3)}]\). Applying \([30, \text{Theorem 1}]\), the reflected BSDE (15) has a solution \((Y, Z, K) \in \mathcal{S}^\infty \times \mathcal{H}^{d,2} \times \mathcal{A}^2\). Using Lemma A.1 in Appendix, we derive that \((Z, K) \in \text{BMO} \times \mathcal{A}\). This completes the proof. ■

We are now ready to complete the proof of the main result of this section.

**Proof of Theorem 3.1** Let \( U^i \in \mathcal{S}^\infty, i = 1, 2 \). It follows from Lemma 3.2 that

\[
\Gamma(U^i)_t := \text{ess sup}_{\tau \in \mathcal{T}_t} y^{1,\tau}_t, \quad \forall t \in [0, T],
\]

(16)

in which \( y^{1,\tau}_t \) is the solution to the BSDE (11). Following the proof of Lemma 3.2 step by step (noting that \( \beta_t \in \text{BMO} \) in this case), we have for every \( t \in [0, T] \)

\[
y^{1,\tau}_t - y^{2,\tau}_t
\]

\[
= \mathbb{E}^P_t \left[ h(\tau, U^1_t, (P_{U^1_t})_{s=\tau}) \mathbf{1}_{\{\tau < T\}} - h(\tau, U^2_t, (P_{U^2_t})_{s=\tau}) \mathbf{1}_{\{\tau < T\}} + \int_t^\tau \left( f^{U^1}(s, z^2_s) - f^{U^2}(s, z^2_s) \right) ds \right],
\]

which together with assumptions (H4) and (H5) implies that for any \( t \in [T - h, T] \),

\[
|y^{1,\tau}_t - y^{2,\tau}_t| \leq (\gamma_1 + \beta h)\|U^1 - U^2\|s_1^\infty_{[T - h, T]} + (\gamma_2 + \beta h) \sup_{s \in [T - h, T]} \mathbb{E}[\|U^1_s - U^2_s\|].
\]

The above inequality combined with (16) implies the following,

\[
\|\Gamma(U^1) - \Gamma(U^2)\|s_1^\infty_{[T - h, T]} \leq (\gamma_1 + \gamma_2 + 2\beta h)\|U^1 - U^2\|s_1^\infty_{[T - h, T]}.
\]

Under assumption (14), we can then find a small enough constant \( h \) depending only on \( \beta, \gamma_1 \) and \( \gamma_2 \) such that \( \gamma_1 + \gamma_2 + 2\beta h < 1 \). It is now obvious that \( \Gamma \) defines a contraction map on the time interval \([T - h, T]\). Finally, proceeding exactly as in Theorem 2.5 and Lemma 2.8, we complete the proof. ■

**Remark 3.4** When the terminal condition is unbounded, the process \((\beta_t)\) may be unbounded in the BMO space, so that \( \widetilde{P} \) is not well-defined. Thus, the conventional fixed point argument fails to work in the unbounded terminal condition case.
4 Unbounded terminal condition and obstacle

In this section, we will use the θ-method to deal with quadratic mean-field reflected BSDEs taking the form (11) with unbounded terminal condition and obstacle. For this purpose, we need to assume the driver is concave or convex with respect to the second unknown \( z \). In what follows, we make use of the following conditions on the terminal condition \( \xi \), the driver \( f \) and the constraint \( h \).

(H1') The terminal condition \( \xi \in L \) with \( \xi \geq h(T, \xi, P_\xi) \).

(H3') For any \( y \in \mathbb{R}, v \in \mathcal{P}_1(\mathbb{R}) \), the process \( h(t, y, v) \) belongs to \( S \).

(H5') For each \( t \in [0, T], y_1, y_2 \in \mathbb{R}, v_1, v_2 \in \mathcal{P}_1(\mathbb{R}) \) and \( z \in \mathbb{R}^d \)
\[
|f(t, y_1, v_1, z) - f(t, y_2, v_2, z)| \leq \beta (|y_1 - y_2| + W_1(v_1, v_2)).
\]

(H6) For each \( (t, \omega, y, v) \in [0, T] \times \Omega \times \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \), \( f(t, y, v, \cdot) \) is concave or convex.

We are now ready to state the main result of this section.

**Theorem 4.1** Assume that (H1'), (H2), (H3'), (H4), (H5') and (H6) hold. If \( \gamma_1 \) and \( \gamma_2 \) satisfy
\[
4(\gamma_1 + \gamma_2) < 1,
\]
then the quadratic mean-field reflected BSDE (11) admits a unique solution \((Y, Z, K) \in S \times H^d \times \mathcal{A} \).

In order to prove Theorem 4.1, we need to recall some technical results on the representation of solutions of quadratic BSDEs. First, we introduce some general conditions on the generator.

(H7) There exists a positive progressively measurable process \((\alpha_t)_{0 \leq t \leq T} \) with \( \int_0^T \alpha_t \, dt \in L \) such that for each \((t, \omega, z) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \), \( |g(t, y, z)| \leq \alpha_t + \beta |y| + \frac{\beta^2}{2} |z|^2 \).

(H8) For each \((t, \omega, y, v) \in [0, T] \times \Omega \times \mathbb{R}^d \), \( g(t, y, z) \leq \alpha_t + \beta |y| + \frac{\beta^2}{2} |z|^2 \).

**Remark 4.2** Suppose that \( g \) satisfies assumptions (H5'), (H6) and (H7). It follows from [10, Corollary 6] that the quadratic BSDE (11) with \( \eta \in L \) admits a unique solution \((y^\tau, z^\tau) \in S \times H^d \). In particular, the comparison principle also holds by [2, Proposition 5.1] or [11, Theorem 5] (up to a slight modification).

Under assumptions (H2), (H5') and (H6), the quadratic reflected BSDE (8) with \( L \in S \) admits a unique solution \((Y, Z, K) \in S \times H^d \times \mathcal{A} \) by [2, Theorems 3.2 and 4.1].

The following result plays a key role in our subsequent calculus, and can be derived from [20, Proposition 1].

**Lemma 4.3** Assume that \((y^\tau, z^\tau) \in S^2 \times H^{d, d}\) is a solution to (11). Suppose that there is a constant \( p \geq 1 \) such that
\[
\mathbb{E} \left[ \exp \left\{ 2p\gamma e^{\beta T} \sup_{t \in [0, T]} |y^\tau_t| + 2p\gamma \int_0^T c_t e^{\beta t} \, dt \right\} \right] < \infty.
\]

Then, we have

(i) Let Assumption (H7) hold. Then, for each \( t \in [0, T], \) Then for each \( t \in [0, T] \) and \( p \geq 1, \)
\[
\exp \{ p\gamma |y^\tau_t| \} \leq \mathbb{E}_t \left[ \exp \left\{ p\gamma e^{\beta(t-T)} |\eta| + p\gamma \int_t^T c_s e^{\beta(s-t)} \, ds \right\} \right].
\]
(ii) Let Assumption (H8) hold. Then, for each \( t \in [0, T] \),
\[
\exp \left\{ \gamma_i y_i^{t, \tau} \right\} \leq \mathbb{E} \left[ \exp \left\{ \gamma_i e^{\beta(T-t)\eta^+} + \gamma_i \int_t^T \alpha_s e^{\beta(s-t)} ds \right\} \right].
\]

We are now ready to prove Theorem 4.1. Indeed, we will make use of the \( \theta \)-method to prove existence and uniqueness of the solution of the quadratic mean-field reflected BSDE (I).

**Lemma 4.4** Assume that all the conditions of Theorem 4.1 hold. Then, the quadratic mean-field reflected BSDE (I) has at most one solution \( (Y, Z, K) \in \mathbb{S} \times \mathcal{H}^d \times \mathcal{A} \).

**Proof.** For \( i = 1, 2 \), let \((Y^i, Z^i, K^i)\) be a \( \mathbb{S} \times \mathcal{H}^d \times \mathcal{A}\)-solution to the quadratic mean-field reflected BSDE (I). From Lemma 2.2 and Remark 4.2 we have
\[
Y_i := \text{ess sup}_{\tau \in \mathcal{T}_i} y_i^{1, \tau}, \quad \forall t \in [0, T],
\]
in which \( y_i^{1, \tau} \) is the solution of the following quadratic BSDE
\[
y_i^{1, \tau} = \xi 1_{\{\tau = T\}} + h(\tau, Y_i^{1, \tau}, (P_{Y_i^{1}})_{s=\tau}) 1_{\{\tau < T\}} + \int_t^\tau f(s, Y_i^1, Z_i^1, \tau, Y_i^{1, \tau}, (P_{Y_i^{1}})_{s=\tau}) ds - \int_t^\tau z_{i,\tau} dB_s.
\]

Assume without loss of generality that \( f(t, y, v, \cdot) \) is concave (see Remark 4.5), for each \( \theta \in (0, 1) \), denote by
\[
\delta_\theta = \frac{\theta \ell_1 - \ell^2}{1 - \theta}, \quad \delta_\theta = \frac{\theta \ell^2 - \ell_1}{1 - \theta} \quad \text{and} \quad \delta_\theta Y := |\delta_\theta Y| + |\delta_\theta Y|
\]
for \( \ell = Y, y^r \) and \( z^r \). Then, the pair of processes \( (\delta_\theta y_i^{1, \tau}, \delta_\theta z_i^{1, \tau}) \) satisfies the following BSDE:
\[
\delta_\theta y_i^{1, \tau} = \delta_\theta \eta + \int_t^\tau (\delta_\theta f(s, \delta_\theta z_i^{1, \tau}) + \delta_\theta f_0(s)) ds - \int_t^\tau \delta_\theta z_i^{1, \tau} dB_s,
\]
where the terminal condition and generator are given by
\[
\delta_\theta \eta = -\xi 1_{\{\tau = T\}} + \frac{\theta h(\tau, Y_i^{1, \tau}, (P_{Y_i^{1}})_{s=\tau}) - h(\tau, Y_i^{2, \tau}, (P_{Y_i^{2}})_{s=\tau})}{1 - \theta} 1_{\{\tau < T\}},
\]
\[
\delta_\theta f_0(t) = \frac{1}{1 - \theta} \left( f(t, Y_i^{1}, P_{Y_i^{1}}, z_i^{1, \tau}) - f(t, Y_i^{2}, P_{Y_i^{2}}, z_i^{2, \tau}) \right),
\]
\[
\delta_\theta f(t, z) = \frac{1}{1 - \theta} \left( \theta f(t, Y_i^{1}, P_{Y_i^{1}}, z_i^{1, \tau}) - f(t, Y_i^{1}, P_{Y_i^{1}}, -z) \right).
\]

Recalling assumptions (H2), (H4), (H5') and (H6), we have that
\[
\delta_\theta \eta \leq |\xi| + |h(\tau, 0, \delta_0)| + |\gamma_1(2|Y_i^1| + |\delta_0 Y_\tau|) + |\gamma_2(2E[|Y_i^1|]_{s=\tau} + E[|\delta_0 Y_s|]_{s=\tau})|
\]
\[
\delta_\theta f_0(t) \leq \beta(|Y_i^1| + |\delta_0 Y_{1, s} + E[|Y_i^1|] + E[|\delta_0 Y_{s}])|
\]
\[
\delta_\theta f(t, z) \leq -f(t, Y_i^{1}, P_{Y_i^{1}}, -z) \leq \alpha + \beta(|Y_i^1| + E[|Y_i^1|] + \frac{\gamma}{2} |z|^2).
\]

Set \( C_1 := \sup_{i \in \{1, 2\}} \sup_{s \in [0, T]} |Y_i^1| \) and
\[
\chi = \alpha T + \sup_{s \in [0, T]} |h(s, 0, \delta_0)| + 2(\gamma_1 + \beta T) \left( \sup_{s \in [0, T]} |Y_i^1| + \sup_{s \in [0, T]} |Y_i^2| \right) + 2(\gamma_1 + \beta T) C_1,
\]
\[
\tilde{\chi} = \alpha T + \sup_{s \in [0, T]} |h(s, 0, \delta_0)| + 2(1 + \gamma_1 + \beta T) \left( \sup_{s \in [0, T]} |Y_i^1| + \sup_{s \in [0, T]} |Y_i^2| \right) + 2(\gamma_2 + \beta T) C_1.
\]
Using assertion (ii) of Lemma 4.3 to (18), we derive that for any \( p \geq 1 \)

\[
\exp \left\{ p \gamma (\delta_0 y_t^+) \right\} \leq E_t \left[ \exp \left\{ p \gamma \left( |\xi| + \chi + (\gamma_1 + \beta(T-t)) \sup_{s \in [t,T]} |\delta_0 Y_s| + (\gamma_2 + \beta(T-t)) \sup_{s \in [t,T]} E[|\delta_0 Y_s|] \right) \right\},
\]

which indicates that

\[
\exp \left\{ p \gamma (\delta_0 Y_t^+) \right\} \leq \text{ess sup}_{t \in T} \exp \left\{ p \gamma (\delta_0 y_t^+) \right\}
\]

\[
\leq E_t \left[ \exp \left\{ p \gamma \left( |\xi| + \chi + (\gamma_1 + \beta(T-t)) \sup_{s \in [t,T]} |\delta_0 Y_s| + (\gamma_2 + \beta(T-t)) \sup_{s \in [t,T]} E[|\delta_0 Y_s|] \right) \right\}. \tag{19}
\]

Using a similar method, we derive that

\[
\exp \left\{ p \gamma (\delta_0 \tilde{Y}_t^+) \right\} \leq E_t \left[ \exp \left\{ p \gamma \left( |\xi| + \chi + (\gamma_1 + \beta(T-t)) \sup_{s \in [t,T]} |\delta_0 \tilde{Y}_s| + (\gamma_2 + \beta(T-t)) \sup_{s \in [t,T]} E[|\delta_0 \tilde{Y}_s|] \right) \right\}. \tag{20}
\]

In view of the fact that

\[
(\delta_0 Y)_- \leq (\delta_0 \tilde{Y})^+ + 2|Y^2| \quad \text{and} \quad (\delta_0 \tilde{Y})_- \leq (\delta_0 Y)^+ + 2|Y^1|,
\]

and recalling (19) and (20), we have

\[
\exp \left\{ p \gamma |\delta_0 Y_t| \right\} \lor \exp \left\{ p \gamma |\delta_0 \tilde{Y}_t| \right\} \leq \exp \left\{ p \gamma \left( (\delta_0 Y_t^+) + (\delta_0 \tilde{Y}_t^+) + 2|Y^1| + 2|Y^2| \right) \right\} \leq E_t \left[ \exp \left\{ p \gamma \left( |\xi| + \chi + (\gamma_1 + \beta(T-t)) \sup_{s \in [t,T]} \delta_0 \tilde{Y}_s + (\gamma_2 + \beta(T-t)) \sup_{s \in [t,T]} E[\delta_0 \tilde{Y}_s] \right) \right\} \right]^2.
\]

Applying Doob’s maximal inequality and Hölder’s inequality, we get that for each \( p \geq 1 \) and \( t \in [0,T] \)

\[
E \left[ \exp \left\{ p \gamma \sup_{s \in [t,T]} \delta_0 \tilde{Y}_s \right\} \right] \leq E \left[ \exp \left\{ p \gamma \sup_{s \in [t,T]} |\delta_0 Y_s| \right\} \exp \left\{ p \gamma \sup_{s \in [t,T]} |\delta_0 \tilde{Y}_s| \right\} \right] \leq 4E \left[ \exp \left\{ 4p \gamma \left( |\xi| + \chi + (\gamma_1 + \beta(T-t)) \sup_{s \in [t,T]} \delta_0 \tilde{Y}_s + (\gamma_2 + \beta(T-t)) \sup_{s \in [t,T]} E[\delta_0 \tilde{Y}_s] \right) \right\} \right] \leq 4E \left[ \exp \left\{ 4p \gamma \left( |\xi| + \chi + (\gamma_1 + \beta(T-t)) \sup_{s \in [t,T]} \delta_0 \tilde{Y}_s \right) \right\} \right] E \left[ \exp \left\{ 4p \gamma (\gamma_2 + \beta(T-t)) \sup_{s \in [t,T]} \delta_0 Y_s \right\} \right] \tag{21}
\]

where we used Jensen’s inequality in the last inequality.

Under assumption (17), there exist two constants \( h \in (0, T] \) and \( \nu > 1 \) depending only on \( \beta, \gamma_1 \) and \( \gamma_2 \) such that

\[
4(\gamma_1 + \gamma_2 + 2\beta h) < 1 \quad \text{and} \quad 4\nu(\gamma_1 + \beta h) < 1.
\]
In the spirit of Hölder’s inequality, we derive that for any $p \geq 1$

$$\mathbb{E}\left[ \exp\left\{ \frac{p\gamma}{\nu} \sup_{s \in [T-h,T]} \delta_{\theta} Y_{s} \right\} \right] \leq 4\mathbb{E}\left[ \exp\left\{ \frac{4p\gamma}{\nu-1} (|\xi| + \bar{\chi}) \right\} \right]^{\frac{1}{\gamma}} \mathbb{E}\left[ \exp\left\{ \frac{4p\gamma}{\nu} \sup_{s \in [t,T]} \delta_{\theta} Y_{s} \right\} \right]^{\frac{1}{\gamma}} \mathbb{E}\left[ \exp\left\{ \frac{p\gamma}{\nu} \sup_{s \in [t,T]} \delta_{\theta} Y_{s} \right\} \right]^{4(\gamma_2 + \beta h)} \leq 4\mathbb{E}\left[ \exp\left\{ \frac{4p\gamma}{\nu} \sup_{s \in [T-h,T]} \delta_{\theta} Y_{s} \right\} \right]^{\frac{1}{\gamma}} \mathbb{E}\left[ \exp\left\{ \frac{4p\gamma}{\nu-1} (|\xi| + \bar{\chi}) \right\} \right]^{\frac{1}{\gamma}} \mathbb{E}\left[ \exp\left\{ \frac{p\gamma}{\nu} \sup_{s \in [T-h,T]} \delta_{\theta} Y_{s} \right\} \right]^{4(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5).}

which together with the fact that $4(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5) < 1$ implies that for any $p \geq 1$ and $\theta \in (0,1)$

$$\mathbb{E}\left[ \exp\left\{ \frac{p\gamma}{\nu} \sup_{s \in [T-h,T]} \delta_{\theta} Y_{s} \right\} \right] \leq 4\mathbb{E}\left[ \exp\left\{ \frac{4p\gamma}{\nu} \sup_{s \in [T-h,T]} \delta_{\theta} Y_{s} \right\} \right]^{\frac{1}{\gamma}} \mathbb{E}\left[ \exp\left\{ \frac{4p\gamma}{\nu-1} (|\xi| + \bar{\chi}) \right\} \right]^{\frac{1}{\gamma}} \mathbb{E}\left[ \exp\left\{ \frac{p\gamma}{\nu} \sup_{s \in [T-h,T]} \delta_{\theta} Y_{s} \right\} \right]^{4(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5)} < \infty.

Note that $Y^1 - Y^2 = (1 - \theta)(\delta_{\theta} Y + Y^1)$. It follows that

$$\mathbb{E}\left[ \sup_{t \in [T-h,T]} |Y^1_t - Y^2_t| \right] \leq (1 - \theta) \left( \frac{1}{\gamma} \mathbb{E}\left[ \exp\left\{ \frac{4p\gamma}{\nu-1} (|\xi| + \bar{\chi}) \right\} \right]^{\frac{1}{\gamma}} \mathbb{E}\left[ \exp\left\{ \frac{p\gamma}{\nu} \sup_{s \in [T-h,T]} \delta_{\theta} Y_{s} \right\} \right]^{4(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5)} + \mathbb{E}\left[ \sup_{t \in [0,T]} |Y^1_t| \right] \right) \mathbb{E}\left[ \exp\left\{ \frac{4p\gamma}{\nu} \sup_{s \in [T-h,T]} \delta_{\theta} Y_{s} \right\} \right]^{\frac{1}{\gamma}} \mathbb{E}\left[ \exp\left\{ \frac{p\gamma}{\nu} \sup_{s \in [T-h,T]} \delta_{\theta} Y_{s} \right\} \right]^{4(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5)} \mathbb{E}\left[ \exp\left\{ \frac{p\gamma}{\nu} \sup_{s \in [T-h,T]} \delta_{\theta} Y_{s} \right\} \right]^{4(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5).}

Letting $\theta \to 1$ yields that $Y^1 = Y^2$ and then $(Z^1, K^1) = (Z^2, K^2)$ on $[T-h,T]$. Repeating iteratively this procedure a finite number of times, we get the uniqueness on the given interval $[0,T]$. The proof is complete.

**Remark 4.5** In the convex case, one should use $\ell^1 - \ell^2$ and $\ell^2 - \ell^1$ instead of $\ell^1 - \ell^2$ and $\ell^2 - \ell^1$ in the definition of $\delta_{\theta} \ell$ and $\delta_{\theta} \ell$, respectively. Then the terminal condition and generator of BSDE (18) satisfies

$$\delta_{\theta} f_{\ell} \leq |\xi| + h(r, 0, \delta_{\theta}) + \gamma_{1}(2|Y_{t}^2| + |\delta_{\theta} Y_{t}|) + \gamma_{2}(2\mathbb{E}[|Y_{t}^2|]_{s=\tau} + \mathbb{E}[|\delta_{\theta} Y_{s}|]_{s=\tau})),
$$

$$\delta_{\theta} f_{\ell} \leq \beta(|Y_{t}^2| + |\delta_{\theta} Y_{t}| + \mathbb{E}[|Y_{t}^2|] + \mathbb{E}[|\delta_{\theta} Y_{t}|]),
$$

$$\delta_{\theta} f_{\ell}(t, z) \leq f(t, Y_{t}^2, \mathbb{E}[Y_{t}^2], z) \leq \alpha + \beta(|Y_{t}^2| + \mathbb{E}[|Y_{t}^2|] + \frac{\gamma}{2}|z|^2) + \frac{\gamma}{2}|z|^2.
$$

By a similar analysis, one can check that (19), (20) and (21) still hold.

**Remark 4.6** Note that we do not obtain directly a uniform estimate for $(\delta_{\theta} Y_{t})^{+}$ in (13), which involves the term $|\delta_{\theta} Y_{t}|$. Otherwise, the condition (17) could reduce to the condition (14).

Let us now turn to the proof of existence.

**Lemma 4.7** Assume that all the conditions of Theorem 4.1 hold and $U \in \mathbb{S}$ with $U_{T} = \xi$. Then, the following quadratic reflected BSDE:

$$\begin{cases}
Y_{t} = \xi + \int_{t}^{T} f(s, Y_{s}, \mathbb{P}_{U_{s}}, Z_{s}) ds - \int_{t}^{T} Z_{s} dB_{s} + K_{T} - K_{t}, & 0 \leq t \leq T, \\
Y_{t} \geq h(t, U_{t}, \mathbb{P}_{U_{t}}), \quad \forall t \in [0,T] & \text{and} \int_{0}^{T} (Y_{t} - h(t, U_{t}, \mathbb{P}_{U_{t}})) dK_{t} = 0,
\end{cases}
$$

admits a unique solution $(Y, Z, K) \in \mathbb{S} \times \mathcal{H}^{d} \times \mathcal{A}$.

**Proof.** From Assumptions (H2), (H4) and (H6), we have

$$|f(t, y, P_{U_{t}}, z)| \leq \alpha + \beta(y + \mathbb{E}[|U_{t}|]) + \frac{\gamma}{2}|z|^2, \quad h(t, U_{t}, \mathbb{P}_{U_{t}}) \leq h(t, 0, \delta_{0}) + \gamma_{1}|U_{t}| + \gamma_{2}\mathbb{E}[|U_{t}|]. \quad (22)
$$

It follows from [2, Proposition 5.1] or [10, Theorem 5] that the driver $(t, y, z) \mapsto f(t, y, P_{U_{t}}, z)$ satisfies the comparison principle. Then, the above quadratic reflected BSDE has at most one $\mathbb{S} \times \mathcal{H}^{d} \times \mathcal{A}$-solution by Lemma 2.2. In particular, $(t, y, z) \mapsto f(t, y, P_{U_{t}}, z)$ satisfies [2, Conditions (H1)] and $(t, U_{t}, P_{U_{t}}) \in \mathbb{S}$. Consequently, applying [2, Theorem 3.2], we get the desired result. □
Remark 4.8 For a process $U \in \mathcal{S}$, the driver $(t, z) \mapsto f(t, U_t, P_{U_t}, z)$ may not satisfy Condition (H1)). We use the driver $(t, y, z) \mapsto f(t, y, P_{U_t}, z)$ instead in Lemma 4.7.

Lemma 4.9 Assume that $\xi \in \mathcal{L}$. Then, the process $(E_t[\xi])_{0 \leq t \leq T} \in \mathcal{S}$.

Proof. Using Jensen’s inequality yields that

$$\exp \left\{ |E_t[\xi]| \right\} \leq E_t \left[ \exp \left\{ |\xi| \right\} \right],$$

which together with Doob’s maximal inequality indicates that

$$E \left[ \exp \left\{ p \sup_{t \in [0, T]} |E_t[\xi]| \right\} \right] \leq \left( \frac{p}{p - 1} \right)^p E_t \left[ \exp \left\{ p|\xi| \right\} \right] < \infty, \forall p > 1.$$

The proof is complete. ■

Then, based on Lemmas 4.7 and 4.9, we could define recursively a sequence of stochastic processes $(Y^{(m)})_{m=1}^{\infty}$ through the following quadratic reflected BSDE:

$$\begin{cases} Y_t^{(m)} = \xi + \int_t^T f(s, Y_s^{(m)}, P_{Y_s^{(m-1)}}, Z_s^{(m)}) ds - \int_t^T Z_s^{(m)} dB_s + K_T^{(m)} - K_t^{(m)}, & 0 \leq t \leq T, \\ Y_t^{(m)} \geq h(t, Y_t^{(m-1)}, P_{Y_t^{(m-1)}}, Y_t^{(m)}), & \forall t \in [0, T] \text{ and } Y_t^{(0)} = \xi \text{ for } t \in [0, T]. \end{cases}$$

where $Y^{(0)} = \xi$ for $t \in [0, T]$. In particular, we have that $(Y^{(m)}, Z^{(m)}, K^{(m)}) \in \mathcal{S} \times \mathcal{H}^d \times \mathcal{A}$. Next, we use a $\theta$-method to prove that the limit of $Y^{(m)}$ is a desired solution. The following uniform estimates are crucial for our main result.

Lemma 4.10 Assume that the conditions of Theorem 4.1 are fulfilled. Then, for any $p \geq 1$, we have

$$\sup_{m \geq 0} \left[ E \left\{ \exp \left( p \gamma \sup_{s \in [0, T]} |Y_s^{(m)}| \right) \right\} + \left( \int_0^T |Z_t^{(m)}|^2 \, dt \right)^p + |K_T^{(m)}|^p \right] < \infty.$$

Proof. The proof will be given in Appendix. ■

Lemma 4.11 Assume that all the conditions of Theorem 4.1 hold. Then, for any $p \geq 1$, we have

$$\Pi(p) := \sup_{\theta \in (0, 1)} \lim_{m \to \infty} \sup_{q \geq 1} \left[ E \left\{ \exp \left( p \gamma \sup_{s \in [0, T]} \delta_{\theta Y_s^{(m,q)}} \right) \right\} \right] < \infty,$$

where we use the following notations

$$\delta_{\theta} Y^{(m,q)} = \frac{\theta Y^{(m+q)} - Y^{(m)}}{1 - \theta}, \quad \delta_{\theta} Y^{(m,q)} = \frac{\theta Y^{(m)} - Y^{(m+q)}}{1 - \theta} \quad \text{and} \quad \delta_{\theta} \theta := |\delta_{\theta} Y^{(m,q)}| + |\delta_{\theta} Y^{(m,q)}|.$$

Proof. The proof will be given in Appendix. ■

We are now in a position to complete the proof of the main result.

Proof of Theorem 4.1 It is enough to prove the existence, the uniqueness was dealt with in Lemma 4.4. Note that for any integer $p \geq 1$ and $\theta \in (0, 1),$

$$\lim_{m \to \infty} \sup_{q \geq 1} \sup_{t \in [0, T]} E \left[ |Y_t^{(m+q)} - Y_t^{(m)}| \right] \leq 2^{p-1} (1 - \theta)^p \left( \frac{\Pi(1)p!}{\gamma^p} + \sup_{m \geq 1} E \left[ \sup_{t \in [0, T]} |Y_t^{(m)}|^p \right] \right).$$
Sending \( \theta \to 1 \) and recalling Lemmas 4.10 and 4.11 we could find a continuous process \( Y \in \mathbb{S} \) such that

\[
\lim_{m \to \infty} \mathbb{E} \left[ \sup_{t \in [0,T]} |Y_t^{(m)} - Y_t|^{p} \right] = 0, \quad \forall p \geq 1. \tag{24}
\]

Applying Itô’s formula to \( |Y_t^{(m+q)} - Y_t^{(m)}|^2 \) and by a standard calculus, we have

\[
\mathbb{E} \left[ \int_0^T |Z_t^{(m+q)} - Z_t^{(m)}|^2 dt \right] \leq \mathbb{E} \left[ \sup_{t \in [0,T]} |Y_t^{(m+q)} - Y_t^{(m)}|^2 + \sup_{t \in [0,T]} |Y_t^{(m+q)} - Y_t^{(m)}| |\Delta^{(m,q)}| \right]
\]

\[
\leq \mathbb{E} \left[ \sup_{t \in [0,T]} |Y_t^{(m+q)} - Y_t^{(m)}|^2 \right] + \mathbb{E} \left[ |\Delta^{(m,q)}|^2 \right] \frac{1}{2} \mathbb{E} \left[ \sup_{t \in [0,T]} |Y_t^{(m+q)} - Y_t^{(m)}|^2 \right] \frac{1}{2}
\]

with

\[
\Delta^{(m,q)} := \int_0^T \left| f(t, Y_t^{(m+q)}, P_{Y_t^{(m+q)-1}}, Z_t^{(m+q)}) - f(t, Y_t^{(m)}, P_{Y_t^{(m)-1}}, Z_t^{(m)}) \right| dt + |K_T^{(m+q)}| + |K_T^{(m)}|.
\]

which together with Lemma 4.10 [24] and dominated convergence theorem indicates that there exists a process \( Z \in \mathcal{H}^d \) so that

\[
\lim_{m \to \infty} \mathbb{E} \left[ \left( \int_0^T |Z_t^{(m)} - Z_t|^2 dt \right)^p \right] = 0, \quad \forall p \geq 1. \tag{25}
\]

Set

\[
K_t = Y_t - Y_0 + \int_0^t f(s, Y_s, P_{Y_s}, Z_s) ds - \int_0^t Z_s dB_s.
\]

Applying dominated convergence theorem again yields that for each \( p \geq 1 \),

\[
\lim_{m \to \infty} \mathbb{E} \left[ \left( \int_0^T |f(t, Y_t^{(m)}, P_{Y_t^{(m)-1}}, Z_t^{(m)}) - f(t, Y_t, P_{Y_t}, Z_t)| dt \right)^p \right] = 0,
\]

which implies that \( \mathbb{E} \left[ \sup_{t \in [0,T]} |K_t - K_t^{(m)}|^p \right] \to 0 \) as \( m \to \infty \) for each \( p \geq 1 \) and that \( K \) is a non-decreasing process. Note that

\[
\lim_{m \to \infty} \mathbb{E} \left[ \sup_{t \in [0,T]} |h(t, Y_t^{(m-1)}, P_{Y_t^{(m-1)}}) - h(t, Y_t, P_{Y_t})| \right] \leq (\gamma_1 + \gamma_2) \lim_{m \to \infty} \mathbb{E} \left[ \sup_{t \in [0,T]} |Y_t^{(m-1)} - Y_t| \right] = 0.
\]

Then it is obvious that \( Y_t \geq h(t, Y_t, P_{Y_t}) \). Moreover, recalling [3, Lemma 13], we have

\[
\int_0^T (Y_t - h(t, Y_t, P_{Y_t})) dK_t = \lim_{m \to \infty} \int_0^T (Y_t^{(m)} - h(t, Y_t^{(m-1)}, P_{Y_t^{(m-1)}})) dK_t^{(m)} = 0,
\]

which implies that \( (Y, Z, K) \in \mathbb{S} \times \mathcal{H}^d \times \mathcal{A} \) is a solution to quadratic mean-field reflected [1]. The proof is complete. \( \blacksquare \)

Appendix

A.1

**Lemma A.1** Let \((Y, Z, K)\) be a \( \mathcal{S}^\infty \times \mathcal{H}^2 \times \mathcal{A}^2 \)-solution to the reflected BSDEs [3]. Assume the driver \( g \) satisfies Assumption (H2). Then, \((Z, K) \in \text{BMO} \times \mathcal{A} \).
Proof. Applying Itô’s formula to $e^{-2\gamma Y_t}$ yields that for any $\tau \in \mathcal{T}_0$

$$2\gamma^2 \int_\tau^T e^{-2\gamma Y_s} |Z_s|^2 ds \leq e^{-2\gamma \eta} - 2\gamma \int_\tau^T e^{-2\gamma Y_s} g(s, Y_s, Z_s) ds + 2\gamma \int_\tau^T e^{-2\gamma Y_s} (Z_s dB_s - dK_s)$$

$$\leq e^{-2\gamma \eta} + 2\gamma \int_\tau^T e^{-2\gamma Y_s} \left( \alpha + \beta |Y_s| + \frac{\gamma}{2} |Z_s|^2 \right) ds + 2\gamma \int_\tau^T e^{-2\gamma Y_s} Z_s dB_s,$$

where we used the fact that $K$ is a non-decreasing process in the last inequality. Thus, we could derive that

$$\gamma^2 e^{-2\gamma \|Y\|_{\mathcal{S}}^\infty} \int_\tau^T |Z_s|^2 ds \leq \gamma^2 \int_\tau^T e^{-2\gamma Y_s} |Z_s|^2 ds$$

$$\leq (1 + 2\gamma T \alpha + \beta \|Y\|_{\mathcal{S}}^\infty) e^{2\gamma \|Y\|_{\mathcal{S}}^\infty} + \gamma \int_\tau^T e^{-2\gamma Y_s} Z_s dB_s,$$

which implies that $Z \in \mathcal{BMO}$. In particular $Z \in \mathcal{H}^d$. Then, by a standard calculus, we could get that $K \in \mathcal{A}$, which ends the proof. \[\blacksquare\]

Let us now turn to the proofs of Lemma 4.10 and Lemma 4.11. The main idea is the same as in Lemma 4.4 (see also that of [21, Theorem 2.8]). For the reader’s convenience, we shall give the sketch of these proofs.

### A.2 Proof of Lemma 4.10

It follows from Lemma 4.2 and Remark 4.2 that for any $m \geq 1$

$$Y^{(m)}_t := \text{ess sup}_{\tau \in \mathcal{T}_t} y^{(m),\tau}_t, \quad \forall t \in [0, T],$$

(26)

in which $y^{(m),\tau}_t$ is the solution of the following quadratic BSDE

$$y^{(m),\tau}_t = \xi 1\{\tau = T\} + h(\tau, Y^{(m-1)}_\tau, (P_{y^{(m-1)}_s})_{s < \tau}) 1\{\tau < T\} + \int_\tau^T f(s, y^{(m),\tau}_s, P_{y^{(m-1)}_s}, z^{(m),\tau}_s) ds - \int_\tau^T z^{(m),\tau}_s dB_s.$$

Thanks to assertion (i) of Lemma 4.3 (taking $\alpha_t = \alpha + \beta E[|Y^{(m-1)}_t|]$) and in view of (22), we get for any $t \in [0, T]$,

$$\exp \left\{ \gamma |y^{(m),\tau}_t| \right\}$$

$$\leq E_t \left[ \exp \left\{ \gamma e^{\beta(T-t)} \left( |\xi| + \eta + \gamma_1 \sup_{s \in [t,T]} |Y^{(m-1)}_s| + (\gamma_2 + \beta(T-t)) \sup_{s \in [t,T]} E[|Y^{(m-1)}_s|] \right) \right\} \right],$$

(27)

in which $\eta = \alpha T + \sup\limits_{s \in [0,T]} |h(s, 0, \delta_0)|$. Recalling (26) and applying Doob’s maximal inequality and Jensen’s inequality, we get that for each $m \geq 1, p \geq 2$ and $t \in [0, T]$

$$E \left[ \exp \left\{ p\gamma \sup_{s \in [t,T]} |Y^{(m)}_s| \right\} \right] \leq 4 E \left[ \exp \left\{ p\gamma e^{\beta(T-t)} \left( |\xi| + \eta + \gamma_1 \sup_{s \in [t,T]} |Y^{(m-1)}_s| \right) \right\} \right]$$

$$\times E \left[ \exp \left\{ p\gamma e^{\beta(T-t)} (\gamma_2 + \beta(T-t)) \sup_{s \in [t,T]} |Y^{(m-1)}_s| \right\} \right].$$
Under assumption (17), we can then find three constants $h \in (0, T]$ and $\nu, \tilde{\nu} > 1$ depending only on $\beta, \gamma_1$ and $\gamma_2$ such that

$$4e^{\beta h} \tilde{\nu} (\gamma_1 + \gamma_2 + \beta h) < 1 \quad \text{and} \quad 4e^{\beta h} \nu \tilde{\nu} \gamma_1 < 1.$$  \hfill (28)

In the spirit of Hölder’s inequality, we derive that for any $p \geq 2$

$$\mathbf{E} \left[ \exp \left\{ p \gamma \sup_{s \in [T-h, T]} |Y^{(m)}_s| \right\} \right] \leq 4 \mathbf{E} \left[ \exp \left\{ \frac{p \nu \gamma}{\nu - 1} e^{\beta h} (|\xi| + \eta) \right\} \right] \leq 4 \mathbf{E} \left[ \exp \left\{ \frac{2p \nu \gamma}{\nu - 1} e^{\beta h} \eta \right\} \right] e^{\beta h} (\gamma_1 + \gamma_2 + \beta h).$$

Define $\rho = \frac{1}{1 - e^{\beta h} (\gamma_1 + \gamma_2 + \beta h)}$ and

$$\mu := \begin{cases} \frac{T}{h}, & \text{if } \frac{T}{h} \text{ is an integer;} \\ \left\lfloor \frac{T}{h} \right\rfloor + 1, & \text{otherwise.} \end{cases}$$

If $\mu = 1$, it follows from the previous inequality that for each $p \geq 2$ and $m \geq 1$

$$\mathbf{E} \left[ \exp \left\{ p \gamma \sup_{s \in [0, T]} |Y^{(m)}_s| \right\} \right] \leq 4 \mathbf{E} \left[ \exp \left\{ \frac{2p \nu \gamma}{\nu - 1} e^{\beta h} \eta \right\} \right] \mu \mathbf{E} \left[ \exp \left\{ \frac{p \nu \gamma}{\nu - 1} e^{\beta h} \eta \right\} \right] e^{\beta h} (\gamma_1 + \gamma_2 + \beta h).$$

Iterating the above procedure $m$ times, we get,

$$\mathbf{E} \left[ \exp \left\{ p \gamma \sup_{s \in [0, T]} |Y^{(m)}_s| \right\} \right] \leq 4^m \mathbf{E} \left[ \exp \left\{ \frac{2p \nu \gamma}{\nu - 1} e^{\beta h} \eta \right\} \right] \mu^m \mathbf{E} \left[ \exp \left\{ p \gamma \sup_{s \in [0, T]} |Y^{(0)}_s| \right\} \right] e^{\beta h} (\gamma_1 + \gamma_2 + \beta h)^m, \quad (29)$$

which is uniformly bounded with respect to $m$. If $\mu = 2$, proceeding identically as in the above, we have for any $p \geq 2$,

$$\mathbf{E} \left[ \exp \left\{ p \gamma \sup_{s \in [T-h, T]} |Y^{(m)}_s| \right\} \right] \leq 4^m \mathbf{E} \left[ \exp \left\{ \frac{2p \nu \gamma}{\nu - 1} e^{\beta h} \eta \right\} \right] \mu^m \mathbf{E} \left[ \exp \left\{ p \gamma \sup_{s \in [T-h, T]} |Y^{(0)}_s| \right\} \right] e^{\beta h} (\gamma_1 + \gamma_2 + \beta h)^m, \quad (30)$$

Then, consider the following quadratic reflected BSDEs on time interval $[0, T - h]$:

$$\begin{cases} Y^{(m)}_t = Y^{(m)}_{T-h} + \int_t^{T-h} f(s, Y^{(m-1)}_s, \mathbf{P}^{(m-1)}_s, Z^{(m)}_s) ds - \int_t^{T-h} Z^{(m)}_s dB_s + K^{(m)}_T, & 0 \leq t \leq T - h, \\
Y^{(m)}_t \geq h(t, Y^{(m-1)}_t, \mathbf{P}^{(m-1)}_t), & \forall t \in [0, T - h] \quad \text{and} \quad \int_0^{T-h} (Y^{(m)}_t - h(t, Y^{(m-1)}_t, \mathbf{P}^{(m-1)}_t)) dB_t = 0. \end{cases}$$
In view of the derivation of (29), we deduce that

\[
\mathbb{E} \left[ \exp \left\{ p_{\gamma} \sup_{s \in [0,T-h]} \left| Y_s^{(m)} \right| \right\} \right] \\
\leq 4^p \mathbb{E} \left[ \exp \left\{ \frac{2p_{\gamma} E_{\beta_h}}{\nu - 1} \left| Y_{T-h}^{(m)} \right| \right\} \right]^\frac{1}{2} \mathbb{E} \left[ \exp \left\{ \frac{2p_{\gamma} E_{\beta_h}}{\nu - 1} \left| Y_T^{(m)} \right| \right\} \right]^\frac{1}{2} \mathbb{E} \left[ \exp \left\{ p_{\gamma} \sup_{s \in [0,T]} \left| Y_s^{(0)} \right| \right\} \right]^{e^{m \beta_h (\gamma_1 + \gamma_2 + \beta h)^m}} \\
\leq 4^p \mathbb{E} \left[ \exp \left\{ \frac{2p_{\gamma} E_{\beta_h}}{\nu - 1} \left| Y_{T-h}^{(m)} \right| \right\} \right]^\frac{1}{2} \mathbb{E} \left[ \exp \left\{ \frac{2p_{\gamma} E_{\beta_h}}{\nu - 1} \left| Y_T^{(m)} \right| \right\} \right]^\frac{1}{2} \mathbb{E} \left[ \exp \left\{ p_{\gamma} \sup_{s \in [0,T]} \left| Y_s^{(0)} \right| \right\} \right]^{e^{m \beta_h (\gamma_1 + \gamma_2 + \beta h)^m}} \\
\times \mathbb{E} \left[ \exp \left\{ \frac{2p_{\gamma} E_{\beta_h}}{\nu - 1} \sup_{s \in [0,T]} \left| Y_s^{(0)} \right| \right\} \right]^{\frac{1}{2}} e^{m \beta_h (\gamma_1 + \gamma_2 + \beta h)^m},
\]

where we used (30) in the last inequality. Putting the above inequalities together and applying Hölder’s inequality again yields that for any \( p \geq 2 \)

\[
\mathbb{E} \left[ \exp \left\{ p_{\gamma} \sup_{s \in [0,T]} \left| Y_s^{(m)} \right| \right\} \right] \leq \mathbb{E} \left[ \exp \left\{ 2p_{\gamma} \sup_{s \in [0,T-h]} \left| Y_s^{(m)} \right| \right\} \right] \mathbb{E} \left[ \exp \left\{ 2p_{\gamma} \sup_{s \in [0,T]} \left| Y_s^{(m)} \right| \right\} \right] \mathbb{E} \left[ \exp \left\{ p_{\gamma} \sup_{s \in [0,T-h]} \left| Y_s^{(0)} \right| \right\} \right]^{e^{m \beta_h (\gamma_1 + \gamma_2 + \beta h)^m}} \\
\leq 4^p + \mathbb{E} \left[ \exp \left\{ \frac{2p_{\gamma} E_{\beta_h}}{\nu - 1} \left| Y_{T-h}^{(m)} \right| \right\} \right]^\frac{2+p}{2} \mathbb{E} \left[ \exp \left\{ \frac{2p_{\gamma} E_{\beta_h}}{\nu - 1} \left| Y_T^{(m)} \right| \right\} \right]^\frac{2+p}{2} \mathbb{E} \left[ \exp \left\{ p_{\gamma} \sup_{s \in [0,T]} \left| Y_s^{(0)} \right| \right\} \right]^{1+ e^{m \beta_h (\gamma_1 + \gamma_2 + \beta h)^m}} \\
\times \mathbb{E} \left[ \exp \left\{ \frac{4p_{\gamma} E_{\beta_h}}{\nu - 1} \sup_{s \in [0,T]} \left| Y_s^{(0)} \right| \right\} \right],
\]

which is uniformly bounded with respect to \( m \). Iterating the above procedure \( \mu \) times in the general case and recalling (2. Theorem 3.2), we eventually get

\[
\sup_{m \geq 0} \mathbb{E} \left[ \exp \left\{ p_{\gamma} \sup_{s \in [0,T]} \left| Y_s^{(m)} \right| \right\} \right] + \left( \int_0^T |Z_t^{(m)}|^2 dt \right)^p + |K_T^{(m)}|^p < \infty, \ \forall p \geq 1,
\]

which concludes the proof.

A.3 Proof of Lemma 4.11

Without loss of generality, assume \( f(t, y, v, \cdot) \) is concave, since the other case can be proved in a similar way, see Remark 4.5. For each fixed \( m, q \geq 1 \) and \( \theta \in (0, 1) \), we can define similarly \( \delta_{\theta} \ell^{(m, q)} \) and \( \delta_{\theta} \bar{\ell}^{(m, q)} \) for \( y^\tau, z^\tau \). Then, the pair of processes \( (\delta_{\theta} y_t^{(m, q), \tau}, \delta_{\theta} z_t^{(m, q), \tau}) \) satisfies the following BSDE:

\[
\delta_{\theta} y_t^{(m, q), \tau} = \delta_{\theta} \eta_t^{(m, q), \tau} + \int_t^T \left( \delta_{\theta} f^{(m, q)}(s, \delta_{\theta} y_s^{(m, q), \tau}, \delta_{\theta} z_s^{(m, q), \tau}) + \delta_{\theta} f_0^{(m, q)}(s) \right) ds - \int_t^T \delta_{\theta} z_s^{(m, q), \tau} dB_s, \tag{32}
\]

where the terminal condition and generator are given by

\[
\delta_{\theta} \eta_T^{(m, q)} = -\xi 1_{\{\tau = T\}}, \quad \frac{\theta h(\tau, Y_T^{(m+q-1)}, (P_{Y_{T-h}^{(m+q-1)}, z_T^{(m+q-1)}})_{s=T}) - h(\tau, Y_T^{(m-1)}, (P_{Y_{T-h}^{(m-1)}, z_T^{(m-1)}})_{s=T})}{1 - \theta} 1_{\{\tau < T\}}, \\
\delta_{\theta} f_0^{(m, q)}(t) = \frac{1}{1 - \theta} \left( f(t, y_t^{(m, \tau), \tau}, P_{Y_t^{(m+q-1)}, z_t^{(m+q-1)}, \tau}) - f(t, y_t^{(m, \tau), \tau}, P_{Y_t^{(m-1), z_t^{(m-1), \tau}}}) \right), \\
\delta_{\theta} f^{(m, q)}(t, y, z) = \frac{1}{1 - \theta} \left( \theta f(t, y_t^{(m+q, \tau), \tau}, P_{Y_t^{(m+q-1), z_t^{(m+q-1), \tau})}) - \theta f(t, y_t^{(m+q, \tau), \tau}, P_{Y_t^{(m-1), z_t^{(m-1), \tau}}}) \right) + h(t, (1 - \theta)y + \theta y_t^{(m+q, \tau), \tau}, (1 - \theta)z + \theta z_t^{(m+q, \tau), \tau}).
\]
Recalling Assumptions (H2), (H4), (H5') and (H6), we have

\[
\delta_{\theta} f^{(m,q)}(t) \leq \beta \left( E[|Y_t^{(m+q-1)}|] + E[|\delta_{\theta} Y_t^{(m-1,q)}|] \right),
\]

\[
\delta_{\theta} f^{(m,q)}(t, y, z) \leq \beta |y| + \beta |Y_t^{(m+q-1)}| - f(t, Y_t^{(m+q-1)}, \mathcal{P}_Y^{(m+q-1)}, -z)
\]

\[
\leq \alpha + 2\beta |Y_t^{(m+q-1)}| + \beta E[|Y_t^{(m+q-1)}|] + \beta |y| + \frac{2}{2} |z|^2.
\]

For any \(m, q \geq 1\), set \(C_2 := \sup m \mathbb{E}[\sup_{s \in [0,T]} |Y_s^{(m)|} < \infty\) and

\[
\zeta^{(m,q)} = e^{\beta T} \left( |\xi| + \eta + \gamma_1 \left( \sup_{s \in [0,T]} |Y_s^{(m-1)}| + \sup_{s \in [0,T]} |Y_s^{(m+q-1)}| \right) \right) (\gamma_2 + \beta T) C_2,
\]

\[
\chi^{(m,q)} := \eta + 2\gamma_1 \left( \sup_{s \in [0,T]} |Y_s^{(m+q-1)}| + \sup_{s \in [0,T]} |Y_s^{(m-1)}| \right) + 2(\gamma_2 + \beta T) C_2,
\]

\[
\chi^{(m,q)} := \chi^{(m,q)} + \sup_{s \in [0,T]} |Y_s^{(m+q)}| + \sup_{s \in [0,T]} |Y_s^{(m))}|
\]

Using assertion (ii) of Lemma 4.3 to (32) and Hölder’s inequality, we derive that for any \(p \geq 1\)

\[
\exp \left\{ p \gamma (\delta_{\theta} y_t^{(m,q)}, r)^+ \right\} \leq \mathbb{E}_t \left[ \exp \left\{ p \gamma e^{\beta (T-t)} \left( |\xi| + \chi^{(m,q)} + 2\beta T \sup_{s \in [t,T]} |y_s^{(m+q), r}| + \gamma_1 \sup_{s \in [t,T]} |\delta_{\theta} Y_s^{(m-1,q)}| \right) \right. \\
+ (\gamma_2 + \beta (T-t)) \sup_{s \in [t,T]} E[|\delta_{\theta} Y_s^{(m+q-1)}|] \right\}
\]

\[
\leq \mathbb{E}_t \left[ \exp \left\{ 2p \gamma e^{\beta (T-t)} \left( |\xi| + \chi^{(m,q)} + \gamma_1 \sup_{s \in [t,T]} |\delta_{\theta} Y_s^{(m-1,q)}| \right) \right. \\
+ (\gamma_2 + \beta (T-t)) \sup_{s \in [t,T]} E[|\delta_{\theta} Y_s^{(m+q-1)}|] \right\}\right]^\frac{1}{2} \\
\times \mathbb{E}_t \left[ \exp \left\{ 4p \gamma e^{2\beta T} \zeta^{(m,q)} \right\}\right]^{\frac{1}{2}}.
\]

Recalling (27) and using Doob’s maximal inequality, we conclude that for each \(p \geq 2\) and \(t \in [0, T]\)

\[
\mathbb{E}_t \left[ \exp \left\{ p \gamma \sup_{s \in [t,T]} |y_s^{(m,q), r}| \right\} \right] \wedge \mathbb{E}_t \left[ \exp \left\{ p \gamma \sup_{s \in [t,T]} |y_s^{(m+q), r}| \right\} \right] \leq 4\mathbb{E}_t \left[ \exp \{ p \gamma \zeta^{(m,q)} \} \right], \forall m, q \geq 1.
\]

It follows from (26) that

\[
\exp \left\{ p \gamma (\delta_{\theta} Y_t^{(m,q)})^+ \right\} \leq \text{ess sup}_{t \in [0,T]} \exp \left\{ p \gamma (\delta_{\theta} y_t^{(m,q), r})^+ \right\}
\]

\[
\leq 4\mathbb{E}_t \left[ \exp \left\{ 2p \gamma e^{\beta (T-t)} \left( |\xi| + \chi^{(m,q)} + \gamma_1 \sup_{s \in [t,T]} |\delta_{\theta} Y_s^{(m-1,q)}| + (\gamma_2 + \beta (T-t)) \sup_{s \in [t,T]} E[|\delta_{\theta} Y_s^{(m+q-1)}|] \right) \right\}\right]\right]^\frac{1}{2} \\
\times \mathbb{E}_t \left[ \exp \left\{ 4p \gamma e^{2\beta T} \zeta^{(m,q)} \right\}\right]^{\frac{1}{2}}.
Using a similar method, we derive that
\[
\exp \left\{ p\gamma (\delta_y Y_t^{(m,q)})^+ \right\} \\
\leq 4E \left[ \exp \left\{ 2p\gamma e^{\beta(T-t)} \left[ |\xi| + \chi^{(m,q)} + \gamma_1 \sup_{s \in [t,T]} |\delta_y Y_s^{(m,q)}| + (\gamma_2 + \beta(T-t)) \sup_{s \in [t,T]} E[|\delta_y Y_s^{(m,q)}|] \right] \right\} \right]^{\frac{1}{2}}
\times E \left[ \exp \left\{ 4p\gamma e^{2\beta T} \zeta^{(m,q)} \right\} \right]^{\frac{1}{4}}.
\]

According to the fact that
\[
\left( \delta_y Y^{(m,q)} \right)^- \leq \left( \delta_y Y^{(m,q)} \right)^+ + 2|Y^{(m)}| \text{ and } \left( \delta_y Y^{(m,q)} \right)^- \leq \left( \delta_y Y^{(m,q)} \right)^+ + 2|Y^{(m+q)}|,
\]
we deduce that
\[
\exp \left\{ p\gamma |\delta_y Y_t^{(m,q)}| \right\} \vee \exp \left\{ p\gamma (|\delta_y Y_t^{(m,q)}| + (\delta_y Y_t^{(m,q)})^+ + 2|Y^{(m)}| + 2|Y^{(m+q)}|) \right\} \\
\leq 4^\gamma E \left[ \exp \left\{ 2p\gamma e^{\beta(T-t)} \left[ |\xi| + \chi^{(m,q)} + \gamma_1 \sup_{s \in [t,T]} |\delta_y Y_s^{(m,q)}| + (\gamma_2 + \beta(T-t)) \sup_{s \in [t,T]} E[|\delta_y Y_s^{(m,q)}|] \right] \right\} \right]^{\frac{1}{2}}
\times E \left[ \exp \left\{ 4p\gamma e^{2\beta T} \zeta^{(m,q)} \right\} \right]^{\frac{1}{4}}.
\]

Applying Doob’s maximal inequality and Hölder’s inequality, we get that for each \( p > 1 \) and \( t \in [0,T] \)
\[
E \left[ \exp \left\{ p\gamma \sup_{s \in [t-h,T]} |\delta_y Y_s^{(m,q)}| \right\} \right]
\leq 4^\gamma E \left[ \exp \left\{ \frac{8p\gamma}{\nu} e^{2\beta h} \left[ \sup_{s \in [T-h,T]} \delta_y Y_s^{(m,q)} \right] \left[ \sup_{s \in [T-h,T]} \zeta^{(m,q)} \right] \right\} \right]^{\frac{1}{2}}
\times E \left[ \exp \left\{ \frac{8p\gamma}{\nu} e^{2\beta T} \zeta^{(m,q)} \right\} \right]^{\frac{1}{4}}
\] (33)

Recalling the definitions of \( h, \nu \) and \( \nu \) in (28), we have
\[
E \left[ \exp \left\{ p\gamma \sup_{s \in [T-h,T]} |\delta_y Y_s^{(m,q)}| \right\} \right]
\leq 4^\gamma E \left[ \exp \left\{ \frac{8p\gamma}{\nu} e^{2\beta h} \left[ \sup_{s \in [T-h,T]} \delta_y Y_s^{(m,q)} \right] \left[ \sup_{s \in [T-h,T]} \zeta^{(m,q)} \right] \right\} \right]^{\frac{1}{2}}
\times E \left[ \exp \left\{ \frac{8p\gamma}{\nu} e^{2\beta T} \zeta^{(m,q)} \right\} \right]^{\frac{1}{4}}
\]

Set \( \bar{\nu} = \frac{1}{1 - 4e^{m\gamma_1(1+\gamma_2+\beta h)}} \). If \( \mu = 1 \), it follows from (33) that for each \( p \geq 1 \) and \( m, q \geq 1 \)
\[
E \left[ \exp \left\{ p\gamma \sup_{s \in [0,T]} |\delta_y Y_s^{(m,q)}| \right\} \right]
\leq 4^\gamma E \left[ \exp \left\{ \frac{8p\gamma}{\nu} e^{2\beta h} \left[ \sup_{m,q \geq 1} \delta_y Y_s^{(m,q)} \right] \left[ \sup_{m,q \geq 1} \zeta^{(m,q)} \right] \right\} \right]^{\frac{1}{2}}
\times E \left[ \exp \left\{ \frac{8p\gamma}{\nu} e^{2\beta T} \zeta^{(m,q)} \right\} \right]^{\frac{1}{4}}
\]

\[
\times E \left[ \exp \left\{ p\gamma \sup_{s \in [0,T]} |\delta_y Y_s^{(1,q)}| \right\} \right]^{e^{(m-1)\beta h}(4q_1+4q_2+4\beta h)^{-m-1}}.
\]
Applying Lemma 4.10, we have for any $\theta \in (0, 1)$
\[
\lim_{m \to \infty} \sup_{q \geq 1} \mathbb{E} \left[ \exp \left\{ p \gamma \sup_{s \in [0, T]} \delta_\theta Y_s^{(1, q)} \right\} \right] e^{(m-1)\beta h (4\gamma_1+4\gamma_2+4\beta h)^{m-1}} = 1.
\]

It follows that
\[
\sup_{\theta \in (0, 1)} \lim_{m \to \infty} \sup_{q \geq 1} \mathbb{E} \left[ \exp \left\{ p \gamma \sup_{s \in [0, T]} \delta_\theta Y_s^{(m, q)} \right\} \right]
\leq 4^{\tilde{\beta}} \mathbb{E} \left[ \exp \left\{ \frac{8\nu \nu' e^{\beta h}}{\nu - 1} \chi^{(m, q)} \right\} \right]^{\frac{\tilde{\beta}}{4}} \times \sup_{m, q \geq 1} \mathbb{E} \left[ \exp \left\{ \frac{4\nu \nu' e^{2\beta T} \zeta^{(m, q)}}{\nu - 1} \right\} \right]^{\frac{\tilde{\beta}}{4}} < \infty.
\]

If $\mu = 2$, proceeding identically as to derive (31), we have for any $p \geq 1$
\[
\mathbb{E} \left[ \exp \left\{ p \gamma \sup_{s \in [0, T]} \delta_\theta Y_s^{(m, q)} \right\} \right]^{\frac{\tilde{\beta}}{4} + \frac{\tilde{\beta}^2}{4}} \leq 4^{\tilde{\beta} + \frac{\tilde{\beta}^2}{4}} \mathbb{E} \left[ \exp \left\{ \frac{8\nu \nu' e^{\beta h}}{\nu - 1} \right\} \right]^{\frac{\tilde{\beta}}{4} + \frac{\tilde{\beta}^2}{4}} \times \sup_{m, q \geq 1} \mathbb{E} \left[ \exp \left\{ \frac{8\nu \nu' e^{2\beta T} \zeta^{(m, q)}}{\nu - 1} \right\} \right]^{\frac{\tilde{\beta}}{4} + \frac{\tilde{\beta}^2}{4}} \times \mathbb{E} \left[ \exp \left\{ \frac{32\nu \nu' e^{\beta h}}{\nu - 1} \right\} \right]^{\frac{\tilde{\beta}}{4} + \frac{\tilde{\beta}^2}{4}} \times \mathbb{E} \left[ \exp \left\{ \frac{8\nu \nu' e^{\beta h}}{\nu - 1} \right\} \right]^{\frac{\tilde{\beta}}{4} + \frac{\tilde{\beta}^2}{4}} \times \sup_{m, q \geq 1} \mathbb{E} \left[ \exp \left\{ \frac{8\nu \nu' e^{2\beta T} \zeta^{(m, q)}}{\nu - 1} \right\} \right]^{\frac{\tilde{\beta}}{4} + \frac{\tilde{\beta}^2}{4}} \times \mathbb{E} \left[ \exp \left\{ \frac{8\nu \nu' e^{\beta h}}{\nu - 1} \right\} \right]^{\frac{\tilde{\beta}}{4} + \frac{\tilde{\beta}^2}{4}} \times \mathbb{E} \left[ \exp \left\{ \frac{8\nu \nu' e^{2\beta T} \zeta^{(m, q)}}{\nu - 1} \right\} \right]^{\frac{\tilde{\beta}}{4} + \frac{\tilde{\beta}^2}{4}}.
\]

which also implies the desired assertion in this case. Iterating the above procedure $\mu$ times in the general case, we get the desired result.

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