The blow up analysis of solutions of the elliptic sinh-Gordon equation

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Abstract In this paper, using a geometric method we show that the blow-up values of the elliptic sinh-Gordon equation are multiples of $8\pi$.

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1 Introduction

The aim of this paper is to explore the relationship between the analytic aspects of the elliptic sinh-Gordon equation

$$u_{zz} + \lambda \sinh u = 0,$$

in particular the blow-up analysis, on a two-dimensional surface $(\Sigma, g)$ and its geometric interpretation in terms of constant mean curvature surfaces and harmonic maps. We therefore hope that our work will provide a better understanding of solutions of the sinh-Gordon equation.
equation. In fact, this equation plays a very important role in the study of the construction of constant mean curvature surfaces initiated by Wente. See [41] or the next section.

Wente’s work [40] on the existence problem for constant mean curvature surfaces and the simultaneous work of Sacks–Uhlenbeck [31] on two-dimensional harmonic map started the investigation of blow-up phenomena for variational problems that possess a noncompact invariance group and represent limiting cases where the Palais-Smale condition just fails. Wente’s seminal work then lead to the subsequent work of Steffen [35], Struwe [36] and Brezis–Coron [5] which completed the understanding of the blow-up for constant mean curvature surfaces from a geometric point of view (the analogous analysis for harmonic maps was achieved by Brezis–Coron [6] and Jost [14].)

Spruck [34] introduced an analytic point of view into the study of the sinh-Gordon equation (1). When \( \Sigma \) is a rectangle in \( \mathbb{R}^2 \), he studied the behavior of nonnegative solutions of (1) with a Dirichlet boundary condition as \( \lambda \) tends to zero. In particular, he proved that a sequence of nonnegative and nontrivial solutions \((\lambda_k, u_k)\) for the Dirichlet problem of (1) tends to the Green function \(-\log |G(z)|^2\) as \( \lambda_k \to 0 \) in a suitable sense. Here \( G(z) \) is the conformal map of \( \Sigma \) onto the unit disk.

Equation (1) arises also from many mathematical and physical problems. See for instance [9, 18, 21, 23–25, 28, 42], and the references therein. In this paper, with the help of differential geometry, we shall investigate the blow up analysis of solutions to (1) when \( \Sigma \) is a Riemann surface or a bounded smooth domain in \( \mathbb{R}^2 \), and we shall give a more precise asymptotic behavior when the sequence of solutions blows up as \( \lambda_n \to \lambda \). Let \( v_n \) be a sequence of solutions of (1), i.e. \( v_n \) satisfies

\[
-\Delta v_n = \lambda_n (e^{v_n} - e^{-v_n}) \quad \text{in } \Sigma
\]

with the condition

\[
\int_{\Sigma} \lambda_n (e^{v_n} + e^{-v_n}) dv_g \leq C < \infty
\]

and \( \lim_{n \to \infty} \lambda_n = \lambda \). In order to state our main result, we define the blow-up set of the sequence \( \{v_n\} \) by

\[
S_1 = \{ x \in \Sigma \mid \exists x_n \to x \text{ such that } v_n(x_n) \to \infty \},
\]

\[
S_2 = \{ x \in \Sigma \mid \exists x_n \to x \text{ such that } -v_n(x_n) \to \infty \}.
\]

Up to subsequences, it is not difficult to show (see Lemma 3.2) that \( S_1 \) and \( S_2 \) are finite sets. For \( p \in S_1 \cup S_2 \), set

\[
m_1(p) = \lim_{r \to 0} \lim_{n \to \infty} \int_{B_r(p)} \lambda_n e^{v_n} dv_g \quad \text{and} \quad m_2(p) = \lim_{r \to 0} \lim_{n \to \infty} \int_{B_r(p)} \lambda_n e^{-v_n} dv_g.
\]

These are two different types of blow-up. Our main theorem is

**Theorem 1.1** The blow-up values \( m_1 \) and \( m_2 \) are multiples of \( 8\pi \).

This is an analogue of the result of Li and Shafrir [19] for the Liouville equation

\[
-\Delta u = \lambda e^u.
\]

Analytically, the blow-up analysis of the Liouville equation can be seen as a special case \((S_2 = \emptyset)\) of that for the sinh-Gordon equation.
In view of a relationship established in [25],
\[ [m_1(p) - m_2(p)]^2 = 8\pi [m_1(p) + m_2(p)], \] (5)
a direct consequence of Theorem 1.1 is

**Corollary 1.2** The blow-up values of the sinh-Gordon equation (2) can only be
\[ (m_1(p), m_2(p)) = 8\pi \left( \frac{\ell(\ell - 1)}{2}, \frac{\ell(\ell + 1)}{2} \right) \] or
\[ 8\pi \left( \frac{\ell(\ell + 1)}{2}, \frac{\ell(\ell - 1)}{2} \right) \] (6)
for some integer \( \ell > 0 \).

This result was in fact conjectured in [25].

The proof of Theorem 1.1 is more geometric. We shall use differential geometry of surfaces of constant mean curvature to transfer our problem into a blow-up phenomenon for harmonic maps. Then we apply a result about no loss of energy during bubbling off for a sequence of harmonic maps, which was proved in [15,26]. See also [11] and [20]. A direct analytic proof of Theorem 1.1 is an interesting problem. However, it seems to be subtle, at least the method presented in [19] seems difficult to generalize to the sinh-Gordon equation. There are many examples in partial differential equations in which the geometric structure of equations plays a crucial role. Good examples are Wente’s inequality [39] and Helein’s proof [12] of the regularity of weak harmonic maps. See also the recent work [30]. For the blow-up analysis of the Toda system see [16,17].

**Problem 1** Is the constant \( \ell \) in Corollary 1.2 one?

Without any boundary constraint, we believe that there are examples with \( \ell > 1 \). However it is not easy to give an example (cf. [8] for the Liouville equation). We shall consider this problem elsewhere. With a suitable boundary condition, for examples that \( (\max_{\partial \Sigma} v_n - \min_{\partial \Sigma} v_n) \) is uniformly bounded, we believe that \( \ell = 1 \).

## 2 The sinh-Gordon equation and constant mean curvature surfaces

Let \( f : \Sigma \to \mathbb{R}^3 \) be a conformal immersion of a surface in \( \mathbb{R}^3 \). Its first fundamental form is
\[ I = e^{2u}|dz|^2 \]
and the second fundamental form is
\[ II = \frac{Q}{2}dz^2 + He^{2u}dzd\bar{z} + \frac{\bar{Q}}{2}d\bar{z}^2. \]
Here \( H \) is the mean curvature and \( Qdz^2 \) is the Hopf differential. The Gauss-Codazzi equation gives
\[ -\Delta u = H^2e^{2u} - |Q|^2e^{-2u} \quad \text{and} \quad Q\bar{z} = e^{2u}H\bar{z}, \] (7)
where \( Q \bar{z} = \partial Q/\partial \bar{z} \) and \( H\bar{z} = \partial H/\partial \bar{z} \). When this conformal immersion has constant mean curvature (denoted by CMC), i.e. \( H = \text{constant} \), then the Hopf differential \( Q \) is holomorphic. When \( Q = 0 \) (for instance in the case where \( \Sigma \) is a sphere), equation (7) gives the Liouville equation (after a rescaling)
\[ -\Delta u = e^{2u}. \] (8)
When \( Q = c \neq 0 \) (for instance in the case where \( \Sigma \) is a torus), we have (after a rescaling and transformation as \( u \mapsto u + \sigma \)) the sinh-Gordon equation

\[-\Delta u = e^{2u} - e^{-2u}.\]  

(9)

Hence from a CMC surface, we have a solution of the sinh-Gordon equation (9). Vice versa, from a solution of (9), one can get a CMC surface if \( \Sigma \) is a simply connected domain. This plays a very important role in the construction of constant mean curvature immersions, which was initiated by Wente (see also \([1,3,27]\)).

Let \( \nu \) be the unit outer normal vector of the immersion \( f \). One can check that \( \nu \), as a map from \( \Sigma \to S^2 \), the Gauss map, satisfies

\[-\Delta \nu = 2H \bar{f}_z \bar{f} - 2Q \bar{e}^{-2u} \bar{f} + (H^2 e^{2u} + |Q|^2 e^{-2u}) \nu.\]  

(10)

Thus \( f \) has constant mean curvature if and only if \( -\Delta \nu \) is a multiple of \( \nu \), which is equivalent to \( \nu \) being a harmonic map from \( \Sigma \to S^2 \), a well known result, see e.g. \([4]\) or Lemma 4.2 below. In this case, we have

\[|\nabla \nu|^2 = H^2 e^{2u} + |Q|^2 e^{-2u}.\]  

(11)

Hence the condition (3) is equivalent to the Gauss map having finite energy. Therefore as mentioned, the blow-up analysis of the sinh-Gordon equation becomes equivalent to the blow-up analysis of CMC surfaces.

3 Preliminary results

We assume first that \( \Sigma = \Omega \) is a smooth bounded domain in \( \mathbb{R}^2 \). We consider the blow-up analysis of the following equation

\[
\begin{aligned}
-\Delta v_n &= \lambda_n (e^{v_n} - e^{-v_n}) \quad \text{in } \Omega \\
v_n &= 0 \quad \text{on } \partial \Omega 
\end{aligned}
\]  

(12)

under a finite energy condition

\[\int_{\Omega} \lambda_n (e^{v_n} + e^{-v_n}) \, dx \leq C < \infty.\]  

(13)

We have then

Lemma 3.1 Let \( v_n \) be a family of \( C^2 \) solutions of (12) verifying (13). For \( B_r(x) \subset \Omega \), if we have

\[
\lim_{n \to \infty} \int_{B_r(x)} \lambda_n e^{v_n} \, dx < 8\pi \quad \left( \text{resp. } \lim_{n \to \infty} \int_{B_r(x)} \lambda_n e^{-v_n} \, dx < 8\pi \right).
\]

then there is a constant \( C > 0 \) such that

\[
\max_{B_{r/2}(x)} \lambda_n e^{v_n} \leq C \quad \left( \text{resp. } \max_{B_{r/2}(x)} \lambda_n e^{-v_n} \leq C \right).
\]
Proof We use the following inequality in [32]: If \( p \) is a positive \( C^2 \) solution satisfying 
\[-\Delta \log p \leq p \text{ in } B_r(x_0) \subset \mathbb{R}^2,\]
then
\[
\log p(x_0) \leq \frac{1}{\pi r^2} \int_{B_r(x_0)} \log p(y)dy - 2 \log \left(1 - \frac{1}{8\pi} \int_{B_r(x_0)} p(y)dy\right).
\]  \( \text{(14)} \)
By our assumption,
\[
\int_{B_r(x)} \lambda_n e^{v_n} dx \leq C_0 < 8\pi
\]
for sufficiently large \( n \). Taking \( p = \lambda_n e^{v_n} \), we get \(-\Delta \log p \leq p \) in \( \Omega \), so that for \( n \) large enough and any \( z \in B_{r/2}(x) \),
\[
v_n(z) \leq \frac{4}{\pi r^2} \int_{B_{r/2}(z)} v_n(y)dy - 2 \log \left(1 - \frac{C_0}{8\pi}\right).
\]
Moreover, by elliptic theory, \( v_n \) is uniformly bounded in \( W^{1,q}_0(\Omega) \) for any \( q < 2 \). The above inequality means that \( v_n \) is uniformly upper bounded in \( B_{r/2}(x) \). We can do the same for \( \lambda_n e^{-v_n} \).

Set \( e(v_n) = \lambda_n (e^{v_n} + e^{-v_n}) \) and
\[
S = \left\{ x \in \Omega \mid \lim_{\delta \to 0} \limsup_{n \to \infty} \int_{B_\delta(x)} e(v_n)dx \geq 8\pi \right\}.
\]

Lemma 3.2 We have \( S = S_1 \cup S_2 \) and \( v_n \) is uniformly bounded in any compact \( K \subset \Omega \setminus S \).

From this Lemma, it is easy to see that \( S \) consists of a finite number of points. Another immediate consequence of Lemma 3.2 is

Proposition 3.3 Let \( v_n \) be a sequence of solutions to (12) satisfying \( \lim_{n \to \infty} \|v_n\|_\infty = \infty \) and \( \lim_{n \to \infty} \lambda_n = 0 \). Up to a subsequence, there exists a finite, non empty set \( S = S_1 \cup S_2 \) in \( \Omega \) such that
\[
\lambda_n e^{v_n} dx \to \sum_{x_0 \in S_1} m_1(x_0)\delta_{x_0} \quad \text{and} \quad \lambda_n e^{-v_n} dx \to \sum_{y_0 \in S_2} m_2(y_0)\delta_{y_0}
\]
Moreover, \( v_n \) converges to \( G \) in \( C^\infty_{loc}(\Omega \setminus S) \) and in \( W^{1,q}_0(\Omega) \) for any \( q < 2 \). Here, \( G \) is the Green function defined by
\[
\begin{cases} 
-\Delta G = \sum_{p \in S} [m_1(p) - m_2(p)]\delta_p & \text{in } \Omega \\
G = 0 & \text{on } \partial \Omega
\end{cases}
\]
where \( m_i(p) \geq 8\pi \) if \( p \in S_i \) and we define \( m_i(p) = 0 \) if \( p \in \Omega \setminus S_i \) (\( i = 1, 2 \)).

When \( \lim_{n \to \infty} \lambda_n > 0 \), we get the asymptotic behavior of \( v_n \) as follows.

Proposition 3.4 Let \( v_n \) be a sequence of solutions to (12) verifying (13) with \( \lim_{n \to \infty} \lambda_n = \lambda > 0 \). Up to a subsequence, we have
- either \( v_n \) is bounded in \( L^\infty_{loc}(\Omega) \);
Proof Without loss of generality, we assume that 

\[ \frac{\lambda}{\Omega_1} \]

uniformly on any open set of \( u \). Similar results hold, for example, for the following equation

\[ u \]

The dichotomy result is an immediate consequence of Theorem 4.2 in [22] with 

\[ u_{1,n} = v_n \] and \( u_{2,n} = -v_n \). We need only to remark that it is impossible to have 

\[ u_{1,n} \to -\infty \]

uniformly on any open set of \( \Omega \), since if it is the case, the condition in (13) is false by 

\[ u_{1,n} + u_{2,n} = 0 \]

in \( \Omega \).

\[ \square \]

**Problem 2** Does the second case of Proposition 3.4 really occur?

Next we want to characterize the blow-up value at blow up points in \( \Omega \) for solutions of 

(2), which is essentially achieved in [25] by using a symmetrization method. Here we follow 

the arguments of [7] and use the Pohozaev identity.

**Lemma 3.5** With the notations in Proposition 3.3 or 3.4, for any \( p \in S \), we have

\[ [m_1(p) - m_2(p)]^2 = 8\pi [m_1(p) + m_2(p)]. \]

**Proof** Without loss of generality, we assume that \( p = 0 \) and for sufficiently small \( r_0 > 0 \), 

\[ B_{r_0}(0) \cap S = \{0\} \]. Multiplying \( x \cdot \nabla v_n \) to (12) and integrating in 

\( B = B_r(0) \) with \( r \in (0, r_0) \), we get the following Pohozaev identity

\[ r \int_{\partial B_r} \left( \frac{\partial v_n}{\partial v} \right)^2 d\sigma - \frac{r}{2} \int_{\partial B_r} |\nabla v_n|^2 d\sigma \]

\[ = 2 \int_{B_r} \lambda_n (e^{v_n} + e^{-v_n}) dx - r \int_{\partial B_r} \lambda_n (e^{v_n} + e^{-v_n}) d\sigma. \]

Since in both cases \( \lambda = 0 \) or \( \lambda > 0 \), we have always

\[ v_n(x) \to -\frac{m_1(0) - m_2(0)}{2\pi} \log |x| + H(x) \quad \text{in} \quad C^\infty_{\text{loc}}(B \setminus \{0\}) \]

with some \( H \in C^\infty(B \setminus \{0\}) \), letting first \( n \to \infty \), then \( r \to 0 \), we obtain

\[ [m_1(0) - m_2(0)]^2 = 8\pi [m_1(0) + m_2(0)]. \]

The proof is completed.

\[ \square \]

From our analysis, we can see that it is not necessary to have the same coefficients in front

of \( e^{v_n} \) and \( e^{-v_n} \), and we can also combine the study of the cases \( \lambda > 0 \) and \( \lambda = 0 \). Therefore similar results hold, for example, for the following equation

\[
\begin{aligned}
-\Delta v_n &= \frac{\lambda_1 e^{v_n}}{\int_{\Omega} e^{v_n} dx} - \frac{\lambda_2 e^{-v_n}}{\int_{\Omega} e^{-v_n} dx} \quad \text{in} \ \Omega \\
&

v_n = 0 \quad \int_{\Omega} e^{v_n} dx \quad \text{on} \ \partial \Omega.
\end{aligned}
\]

We leave the details for interested readers.
Consider now the equations (12) or (15) on a surface $\Sigma$ with boundary, instead of a bounded domain $\Omega \subset \mathbb{R}^2$. Using local isothermal charts, we can remark that by changing everywhere $8\pi$ to a suitable positive constant $\varepsilon_0$, all the arguments for Lemmas 3.1, 3.2 continue to work, so we still have a finite set of singularities $S$. Using again a local analysis, we can repeat the proof of Propositions 3.3, 3.4 and Lemma 3.5 by just replacing the Green function $G$ by the one of $\Sigma$. See for example [25].

By the same argument, similar results hold for solutions of

$$-\Delta v_n = \frac{\lambda e^{v_n}}{\int_{\Sigma} e^{v_n} dx} - \frac{\lambda e^{-v_n}}{\int_{\Sigma} e^{-v_n} dx} \quad \text{in } \Sigma \text{ and } \int_{\Sigma} v_n dx = 0,$$

with a closed surface $\Sigma$.

4 Proof of the main theorem

In this section we prove the main theorem, Theorem 1.1. First we follow closely the paper of Bobenko [4] to construct a harmonic map from a solution of a sinh-Gordon equation. Let $\Omega$ be a simply connected domain in $\mathbb{R}^2$ and let $u$ be a solution of

$$-\Delta u = 2\lambda_1 e^u - 2\lambda_2 e^{-u} \quad \text{in } \Omega,$$

for two positive constants $\lambda_1$ and $\lambda_2$. Define two matrices

$$U = \frac{1}{2} \left( \begin{array}{cc} u_x & -\sqrt{\lambda_2} e^{-\frac{u}{2}} \\ \sqrt{\lambda_1} e^{\frac{u}{2}} & 0 \end{array} \right), \quad V = \frac{1}{2} \left( \begin{array}{cc} 0 & -\sqrt{\lambda_1} e^{\frac{u}{2}} \\ \sqrt{\lambda_2} e^{-\frac{u}{2}} & u_x \end{array} \right).$$

(18)

It is easy to check the following equivalence.

**Lemma 4.1** $u$ is a solution of (17) if and only if $U$ and $V$ satisfy

$$U \bar{z} + [U, V] = 0.$$  

(19)

Let $\mathbb{H}$ denote the algebra of quaternions and $\{1, i, j, k\}$ be the standard basis of $\mathbb{H}$ with $ij = k, jk = i$ and $ki = j$. It is convenient to express quaternions using matrices. In order to do so, we only need to identify the Pauli matrices as follows

$$\sigma_1 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) = i, \quad \sigma_2 = \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right) = ij,$$

$$\sigma_3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) = ik, \quad \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) = 1.$$

We can also identify $\mathbb{R}^3$ with the space of imaginary quaternions $\text{Im}\mathbb{H}$ by

$$A = -i \sum_{i=1}^{3} a_i \sigma_i \in \text{Im}\mathbb{H} \iff A = (a_1, a_2, a_3) \in \mathbb{R}^3.$$

Let $\mathbb{H}_u = \mathbb{H}\setminus\{0\}$ be the multiplicative quaternion group. Any element $\phi$ in $\mathbb{H}_u$ can be expressed by

$$\phi = \left( \begin{array}{cc} a & b \\ -\bar{b} & \bar{a} \end{array} \right) \quad \text{with } |a|^2 + |b|^2 \neq 0.$$
Define a one-form $\alpha$ valued in the algebra $H$ by

$$\alpha = Ud\bar{z} + Vd\bar{z}.$$  

We also point out that $d + \alpha$ is a flat connection if and only if $U$, $V$ satisfy (19).

Consider the following equation

$$d\phi \cdot \phi^{-1} = \alpha,$$  \hspace{1cm} (20)

for $\phi : \Omega \rightarrow \mathbb{H}_s$. Equation (20) is equivalent to

$$\phi_z = U\phi \quad \text{and} \quad \phi_{\bar{z}} = V\phi.$$  \hspace{1cm} (21)

The compatibility condition for (21) is just (19), which is equivalent to (17) by Lemma 4.1. Therefore for any solution $u$ of (17), by the Frobenius theorem, on a simply connected domain $\Omega$, equation (20) has a solution $\phi : \Omega \rightarrow H^*$. Let $x_0 \in \Omega$ be a fixed point. With the following normalization

$$\phi(x_0) = e^{u(0)/4}1$$  \hspace{1cm} (22)

the solution of (21) is unique. First we claim that

$$\det \phi = e^{u/2}.$$  

Note that $\det \phi \neq 0$ and set $h = \log (\det \phi)$. In view of (21) it is easy to verify that

$$h_z = \text{tr}(\phi^{-1}\phi_z) = \text{tr}(U) = u_z/2,$$

$$h_{\bar{z}} = \text{tr}(\phi^{-1}\phi_{\bar{z}}) = \text{tr}(V) = u_{\bar{z}}/2.$$  

Hence $h - \frac{u}{2}$ is a constant over $\Omega$. By the normalization (22), we have $\det \phi = e^{u/2}$.

From the map $\phi$, we define a map $N : B_1 \rightarrow S^2$ by

$$N = \phi^{-1}k\phi.$$  

It is clear that

$$\phi^{-1} = \frac{1}{\det \phi} \begin{pmatrix} \bar{a} & -b \\ b & a \end{pmatrix}$$  

and

$$N = \frac{-i}{\det \phi} \begin{pmatrix} |a|^2 - |b|^2 \\ 2\bar{a}b \\ 2|b|^2 - |a|^2 \end{pmatrix}.$$  

It follows that $\|N\|^2 = 1$ and $N : \Omega \rightarrow \mathbb{R}^3 = \text{Im} \mathbb{H}$. It is easy now to check that

**Lemma 4.2** The map $N$ satisfies the harmonic map equation

$$- \Delta N = |\nabla N|^2 N.$$  \hspace{1cm} (23)

**Proof** In fact, $N$ is the Gauss map of a CMC surface and hence is a harmonic map. This is a well-known fact, see for instance [4]. Here for convenience of readers, we give a direct computation. We set

$$F_z = -ie^{u/2}\phi^{-1}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \phi, \quad F_{\bar{z}} = -ie^{u/2}\phi^{-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \phi.$$  

By [4], $F : \Omega \rightarrow \mathbb{R}^3 = \text{Im} \mathbb{H}$ is also well-defined and is a CMC surface, but here we will not discuss the surface $F$.  

$\square$ Springer
As \( (\phi^{-1})_z = -\phi^{-1} \mathcal{U} \) and \( (\phi^{-1})_{\bar{z}} = -\phi^{-1} \mathcal{V} \), a direct computation gives
\[
F_{zz} = F_{\bar{z} \bar{z}} = \frac{\sqrt{\lambda_1} e^u}{2} N
\] (24)
and
\[
F_{z\bar{z}} = u_z F_{\bar{z}} + \frac{\sqrt{\lambda_2}}{2} N, \quad F_{\bar{z}z} = u_{\bar{z}} F_{z} + \frac{\sqrt{\lambda_2}}{2} N.
\] (25)
We get also
\[
N_z = -\phi^{-1} \mathcal{U} k \phi + \phi^{-1} \mathcal{U} \mathcal{V} \phi = -\sqrt{\lambda_1} F_{\bar{z}} - \sqrt{\lambda_2} e^{-u} F_{\bar{z}}
\]
and \( N_{\bar{z}} = \phi^{-1} [k, V] \phi = -\sqrt{\lambda_2} e^{-u} F_{z} - \sqrt{\lambda_1} F_{z} \). Since \( \|F_z\|^2 = \|F_{\bar{z}}\|^2 = e^u / 2 \) and \( \langle F_z, F_{\bar{z}} \rangle = 0 \), we get readily
\[
|\nabla N|^2 = 2\lambda_1 e^u + 2\lambda_2 e^{-u}.
\] (26)
Using (24) and (25), we have
\[
-N_{z\bar{z}} = \sqrt{\lambda_1} F_{\bar{z}z} + \sqrt{\lambda_2} e^{-u} (F_{\bar{z}z} - u_{\bar{z}} F_{z}) = \frac{\lambda_1 e^u + \lambda_2 e^{-u}}{2} N.
\]
This finishes the proof of the Lemma. \( \square \)

Hence \( N \) is a harmonic map and its Hopf differential is
\[
\langle N_z, N_{\bar{z}} \rangle = \sqrt{\lambda_1 \lambda_2}.
\]
Therefore, \( N \) is conformal if and only if one of \( \lambda_1, \lambda_2 \) is zero. In this case equation (17) is just the Liouville equation.

Now we want to consider the holomorphic and anti-holomorphic part of the energy density \( |\nabla N|^2 \). The standard complex structure of \( S^2 \) is given by
\[
J : T_N S^2 \to T_N S^2, \quad JX = N \wedge X,
\]
if we consider \( N \in S^2 \subset \mathbb{R}^3 \) as a point in \( \mathbb{R}^3 \). In our expression of \( \mathbb{R}^3 \), the standard complex structure is, in terms of matrices,
\[
J : T_N S^2 \to T_N S^2, \quad JX = \frac{1}{2} [N, X],
\]
the Lie bracket. The \( \partial \)-energy density and \( \bar{\partial} \)-energy density are
\[
e_\partial(N) = \frac{1}{4} \left| \frac{1}{2} [N, N_z] + N_{\bar{z}} \right|^2, \quad e_{\bar{\partial}}(N) = \frac{1}{4} \left| \frac{1}{2} [N, N_{\bar{z}}] - N_z \right|^2
\]

Lemma 4.3 We have
\[
e_\partial(N) = \lambda_1 e^u \quad \text{and} \quad e_{\bar{\partial}}(N) = \lambda_2 e^{-u}.
\] (27)

Proof We can check that \( [N, F_z] = 2i F_{\bar{z}} \) and \( [N, F_{\bar{z}}] = -2i F_z \) and then we use the expansion of \( N_z \) and \( N_{\bar{z}} \) by \( F_z \) and \( F_{\bar{z}} \). \( \square \)
Theorem 4.4  Let \( \{v_n\} \) be a sequence of solutions of
\[- \Delta v_n = 2 \lambda_1,ne^{v_n} - 2 \lambda_2,ne^{-v_n} \quad \text{in} \quad B_r(p) \subset \Omega. \tag{28}\]
Assume that \( p \) is the unique blow-up point in \( B_r(p) \) and set
\[ m_1(p) = \lim_{r \to 0} \lim_{n \to \infty} \int_{B_r(p)} 2 \lambda_1,ne^{v_n}dx \quad \text{and} \quad m_2(p) = \lim_{r \to 0} \lim_{n \to \infty} \int_{B_r(p)} 2 \lambda_2,ne^{-v_n}dx. \]

Then
\[ m_1(p) \in 8\pi \mathbb{N} \quad \text{and} \quad m_2(p) \in 8\pi \mathbb{N}. \]

Proof  For each \( v_n \), we find a unique harmonic map \( N_n : B_r(p) \to \mathbb{S}^2 \) as above with the normalization (22). In view of (26) the energy density of \( N_n \) is
\[ e(N_n) = \lambda_1,ne^{v_n} + \lambda_2,ne^{-v_n}. \]

By the assumption that \( p = 0 \) is the only blow-up point, the sequence of harmonic maps \( N_n \) blows up only at 0. Now the result in [15] and [26] (see also [11]) and [20]) tells us that there are harmonic spheres \( u_j : \mathbb{S}^2 \to \mathbb{S}^2 \) \((i = \{1, \cdots, k\})\) such that (up to subsequence)
\[ \lim_{r \to 0} \lim_{n \to \infty} \int_{B_r(p)} e(N_n)dx = \sum_{j=1}^{k} E(u_j). \]

It is well-known that any harmonic sphere from \( \mathbb{S}^2 \) to \( \mathbb{S}^2 \) is holomorphic or anti-holomorphic of degree \( d \) with energy \( 4\pi |d| \). The holomorphic energy and anti-holomorphic energy also satisfy a similar identity. In view of Lemma 4.3, we have
\[ m_1(p) = \lim_{r \to 0} \lim_{n \to \infty} \int_{B_r(p)} 2 \lambda_1,ne^{v_n}dx = \lim_{r \to 0} \lim_{n \to \infty} 2 \int_{B_r(p)} e^\partial(N_n)dx \in 8\pi \mathbb{N}. \]
Similarly \( m_2(p) \in 8\pi \mathbb{N} \). This finishes the proof. \( \square \)

Theorem 1.1 and Corollary 1.2 follow from Theorem 4.4.

Remark 4.5  Theorem 4.4, and hence Theorem 1.1 and Corollary 1.2 hold for
\[- \Delta v_n = 2 \lambda_1,ne^{v_n} - 2 \lambda_2,ne^{-v_n} \quad \text{in} \quad B_r(p) \subset \Omega \tag{29}\]
with a positive \( C^2 \) function \( V \). In this case
\[ m_1(p) = \lim_{r \to 0} \lim_{n \to \infty} \int_{B_r(p)} 2 \lambda_1,ne^{v_n}dx, \quad m_2(p) = \lim_{r \to 0} \lim_{n \to \infty} \int_{B_r(p)} 2 \lambda_2,ne^{-v_n}dx. \]
This is because all the analysis in Sect. 3 holds also on a surface \( \Sigma \). Therefore, the argument utilizing harmonic maps is completely the same, and we can transfer equation (29) to (28) with a new metric \( V(dx^2 + dy^2) \).
5 Existence result

As an application, we give an existence result for the equation (15). The functional associated to (15) is

\[ J_{\lambda_1, \lambda_2}(u) = \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx - \lambda_1 \log \left( \int_\Omega e^v \, dx \right) - \lambda_2 \log \left( \int_\Omega e^{-v} \, dx \right). \]

It was showed in [25] and [33] that if \( \lambda_1 \leq 8\pi \) and \( \lambda_2 \leq 8\pi \) then there exists a positive constant \( C \) such that

\[ J_{\lambda_1, \lambda_2}(u) \geq -C, \quad \text{for} \quad u \in H^1_0(\Omega). \] (30)

Inequality (30) is a generalization of the Moser-Trudinger inequality.

When \( \lambda_1 < 8\pi \) and \( \lambda_2 < 8\pi \), the existence of a solution of (15) was studied in [32] and [25]. For other existence results, see [42] and [29]. With the help of Theorem 1.1, we can consider existence of (30) for \((\lambda_1, \lambda_2) \in (8\pi, 16\pi) \times (0, 8\pi)\) or \((\lambda_1, \lambda_2) \in (0, 8\pi) \times (8\pi, 16\pi)\). Now we assume that \( \Omega \) is a non simply-connected domain as in [10] for the Liouville equation.

**Theorem 5.1** Let \( \Omega \) be a non simply-connected domain in \( \mathbb{R}^2 \). If \( \lambda_1 \in (8\pi, 16\pi) \) and \( \lambda_2 \in (0, 8\pi) \), then equation (15) admits a solution.

**Proof** The argument follows from a trick given by Struwe [37] and the blow-up analysis presented above. Since the method now becomes rather well-known (see for instance [10, 38]), here we just give a sketch. \( \square \)

**Step 1** We first define the center of mass of a function \( v \in H^1_0(\Omega) \) by

\[ m_c(v) = \left( \int_\Omega e^v \, dx \right)^{-1} \left( \int_\Omega x e^v \, dx \right). \]

Assume for simplicity that \( \partial \Omega = \Gamma_+ \cup \Gamma_- \) has only two disjoint components. Define a family of functions

\[ \gamma : \mathbb{R} \to H^1_0(\Omega) \times H^1_0(\Omega) \] (31)

satisfying

\[ J_{\lambda_1, \lambda_2}(\gamma(t)) \to -\infty \quad \text{as} \quad t \to -\infty \] (32)

and

\[ m_c(\gamma) \to \Gamma_{\pm} \quad \text{as} \quad t \to \pm \infty. \] (33)

The existence of such a family is guaranteed by \( \lambda_1 > 8\pi \). Define a minimax value

\[ \alpha := \inf_{\gamma \in \mathcal{X}} \sup_{t \in \mathbb{R}} J_{\lambda_1, \lambda_2}(\Gamma(t)), \] (34)

where \( \mathcal{X} \) is the set of all such families \( \gamma \).

**Step 2** The minimax value \( \alpha > -\infty \).

Inequality (30) can be improved under a condition introduced by Aubin [2].
Lemma 5.2 Let $\Omega_1$ and $\Omega_2$ be two subsets of $\overline{\Omega}$ satisfying $\text{dist}(\Omega_1, \Omega_2) \geq \delta_0 > 0$ and $\delta \in (0, 1/2)$. For any $\epsilon > 0$, there exists a constant $c = c(\epsilon, \delta_0, \delta) > 0$ such that

$$J_{(16\pi - \epsilon, 8\pi - \epsilon)}(u) \geq -c$$

holds for all $u \in H^1_0(\Omega)$ satisfying

$$\int_{\Omega_1} e^u dx \geq \delta \int_{\Omega_1} e^u dx \quad \text{and} \quad \int_{\Omega_2} e^u dx \geq \delta \int_{\Omega_2} e^u dx.$$

Let $\Gamma_0 \subset \Omega$ be a closed curve enclosing the inner boundary of $\Omega$. Each curve $\gamma$ starting from $\Gamma_-$ and ending at $\Gamma_+$ intersects with $\Gamma_0$. By (35), for $(\lambda_1, \lambda_2) \in (8\pi, 16\pi) \times (0, 8\pi)$ we can show that

$$J_{\lambda_1, \lambda_2}(u) > -c,$$

for any $u \in H^1_0(\Omega)$ with center of mass $m_c(u) \in \Gamma_0$. See the argument in [10]. Hence $\alpha > -\infty$.

Step 3 Now fix $\lambda_2$ and apply the trick of Struwe to obtain a dense subset $\Lambda$ of $(8\pi, 16\pi)$ such that for any pair $(\lambda, \lambda_2)$ with $\lambda \in \Lambda$, $\alpha$ is achieved by a function $v$. It is clear that $v$ is a solution of equation (15) with the pair $(\lambda, \lambda_2)$. This is an important step. See [10,37] and [38].

Step 4 Now for any $\lambda_1 \in (8\pi, 16\pi)$, there is a sequence $\lambda^k \in \Lambda$ with $\lambda^k \to \lambda_1$ as $k \to \infty$, since $\Lambda$ is dense. Applying Step 3, we have a solution $v_k$ of (15) for $(\lambda^k, \lambda_2)$. Now we use the blow-up analysis established above to show that $v_k$ converges to $v$, which will be a solution of (15) for $(\lambda_1, \lambda_2)$. Assume by contradiction that $v_k$ does not converge. As discussed above, there might be two types of blow-up. We first exclude that such two types of blow-up occur at the same point $p$. Set

$$m_1(p) = \lim_{r \to 0} \lim_{n \to \infty} \lambda^k \left( \int_{B_r(p)} e^{v_k} dx \right)^{-1} \int_{B_r(p)} e^{v_k} dx$$

and

$$m_2(p) = \lim_{r \to 0} \lim_{n \to \infty} \lambda_2 \left( \int_{B_r(p)} e^{-v_k} dx \right)^{-1} \int_{B_r(p)} e^{-v_k} dx.$$

By Corollary 1.2 we have

$$(m_1(p), m_2(p)) = 8\pi \left( \frac{\ell(\ell - 1)}{2}, \frac{\ell(\ell + 1)}{2} \right) \quad \text{or} \quad 8\pi \left( \frac{\ell(\ell + 1)}{2}, \frac{\ell(\ell - 1)}{2} \right).$$

It is trivial to see that $m_1(p) < 16\pi$ and $m_2(p) < 8\pi$. Therefore $\ell \leq 1$. Hence one of $m_i(p)$ must be zero, which means that the two types of blow-up cannot occur at the same point.

Since the residual term $r_1 = 0$ if $S_1 \setminus (S_1 \cap S_2) \neq \emptyset$ (see for instance [25]), we get

$$\lim_{k \to \infty} \lambda_k = \sum_p m_1(p) \quad \text{or} \quad \lambda_2 = \sum_p m_2(p),$$

if blow-up happens. Here the summation is taken over the set of all blow-up points. Thus, $\lambda_1 = 8\pi n_1$ or $\lambda_2 = 8\pi n_2$ for some integers $n_i$. This is a contradiction. Therefore $v_k$ converges to $v$ which is a solution of (15) for $(\lambda_1, \lambda_2)$.
Remark 5.3  Formula (36) might not be true (see Proposition 3.4), if there are points at which both of these two types of blow-up occur.

Remark 5.4  Theorem 5.1 is also true for (29).

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