Circular quantum secret sharing

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A circular quantum secret sharing protocol is proposed, which is useful and efficient when one of the parties of secret sharing is remote to the others who are in adjacent, especially the parties are more than three. We describe the process of this protocol and discuss its security when the quantum information carrying is polarized single photons running circularly. It will be shown that entanglement is not necessary for quantum secret sharing. Moreover, the theoretic efficiency is improved to approach 100% as almost all the instances can be used for generating the private key, and each photon can carry one bit of information without quantum storage. It is straightforwardly to utilize this topological structure to complete quantum secret sharing with multi-level two-particle entanglement in high capacity securely.

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I. INTRODUCTION

Secret sharing is a useful tool in classical secure communication [1,2]. It can be used to accomplish a special task. Suppose a president of a bank, Alice wants to send a secret message to her two agents, Bob and Charlie who are at a distant place for carrying out a business on her behalf. Alice cannot determine whether both of them are honest. There may be at most one dishonest agent among Bob and Charlie pair, but Alice does not know who the dishonest one is. She knows that the honest one will keep the dishonest one from destroying the business if they coexist in the process of the business. For the security of the secret message (SA), she will split it into two pieces, SB and SC, and only sends one to Bob and another to Charlie respectively. The two pieces of message can be used to reconstruct the secret message SA when Bob and Charlie collaborate, otherwise none of them can get any useful information about SA.

Encryption can be used to complete the task if Alice created a private key KB (KC) with Bob (Charlie). For example, if Alice wants to send the message SB to Bob securely, she can encrypt the message SB with the private key KB using the one time-pad crypt-system and then send the ciphertext CB = SB ⊕ KB to Bob, where ⊕ means modulo 2 summation. With the key KB, Bob can decrypt CB to read out the message SB, but any one else can not obtain anything about it. Similarly, Alice can encrypt the message SA with KA = KB ⊕ KC, i.e., CA = SA ⊕ KA, and then sends the ciphered text CA to both of Bob and Charlie. They can get the original message SA only when they collaborate. In essence, this is the classical secret sharing whose security depends on the privacy of the key KB and KC. The distribution of a private key between two remote parties or multi-parties is important for secure communication.

Quantum key distribution (QKD) is an important application of quantum mechanics within the field of information, and it provides a secure way for generating a private key between two remote parties since Bennett and Brassard [3,4]. The secret sharing has been generalized to the quantum scenario by using entanglement [2,5], namely quantum secret sharing (QSS). Different from classical secret sharing, the shared information in QSS can be both classical and quantum. In particular, QSS is useful for creating a private key among multi-party of secure communications. There have been many theoretical and experimental interests in QSS [6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22].

A pioneering QSS scheme [2], called HBB99 scheme, was proposed by Hillery, Bužek and Berthiaume in 1999 by using three-particle entangled Greenberger-Horne-Zeilinger (GHZ) states. In this scheme, the bank president, Alice prepares a GHZ triplet state

$$|\psi\rangle_{abc} = \frac{1}{\sqrt{2}}(|000\rangle_{abc} + |111\rangle_{abc}),$$  

(1)

where the state |0⟩ and |1⟩ are two eigenvectors of two-level quantum system, such as the polarization of single photons along the z-direction (σz). Alice sends the particle b and c to Bob and Charlie respectively, and keeps the particle a. They all agree that they choose randomly one of the two measuring bases (MBs), σx and σy, to perform the measurement on their particles. When they all choose σx or one chooses σy and the others choose

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their results are correlated and will be kept for key, otherwise they discard the results. Its intrinsic efficiency for qubits $\epsilon_q$, the ratio of number of valid qubits to the number of transmitted qubits, is about 50% as half of the instances will be abandoned. Subsequently, Karlsson, Koashi and Imoto (KKI) put forward a QSS scheme with two-photon polarization-entangled states, and its efficiency $\epsilon_q$ is also 50%.

There is a common feature in the existing QSS protocols, for instance, in Refs. 2, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21, that the quantum information carrier (QIC) runs only between two parties among the participants, i.e., from Alice to Bob, and from Alice to Charlie in three-party secret sharing, no transmission between Bob and Charlie. In general, Alice is remote to both Bob and Charlie, and Bob and Charlie are likely two adjacent agents. Then they can complete a QSS by making the QIC run a circle, namely, the QIC runs from Alice to Bob, and then from Bob to Charlie, finally back to Alice. Though the change in the process seems small, but it reduces the resource requirement greatly and the intrinsic efficiency is also increased. In this paper, we will present a quantum secret sharing protocol of classical information based on the circular motion idea using polarized single photons. This basic circular transmission idea is not restricted to the use of single photons, but also could be used in other systems and the generalization of the protocol with entangled states is also presented.

II. CIRCULAR QUANTUM SECRET SHARING WITH POLARIZED SINGLE PHOTONS

A. The circular-QSS protocol with single photons

The basic idea of the circular-QSS protocol with polarized single photons is shown in Fig.1. The president, Alice prepares the QICs which are polarized single photons in this QSS protocol, using two sets of measuring basis (MB) into one of the following four states randomly

$$\{|+z\rangle, |−z\rangle, |+x\rangle, |−x\rangle\}, \quad (2)$$

where

$$|+z\rangle = |0\rangle, \quad |−z\rangle = |1\rangle, \quad |+x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |−x\rangle = \frac{1}{\sqrt{2}}(|0\rangle − |1\rangle), \quad (3)$$

respectively. The states $|±z\rangle$ and $|±x\rangle$ are the eigenstates of $\sigma_z$ and $\sigma_x$, respectively. Before the quantum communication, Alice, Bob and Charlie agree that Bob and Charlie choose randomly the control mode or the coding mode for the quantum signal received, similar to the Ping-Pong quantum key distribution scheme in Refs. 23, 24. When they choose the control mode, they perform single-photon measurement on the signal using one of the two MBs, $\sigma_z$ and $\sigma_x$ randomly and record the MBs and outcomes of the measurements, denoted as $R_B$ and $R_C$, respectively. If they choose the coding mode, they perform randomly one of the two unitary operations, $U_0$ and $U_1$ which represent the bits 0 and 1 respectively, on the single photon received,

$$U_0 = |0\rangle \langle 0| + |1\rangle \langle 1|, \quad U_1 = |0\rangle \langle 1| − |1\rangle \langle 0|. \quad (5)$$

The operation $U_0$ is the identity operation and does nothing on the single photon. The nice feature of the $U_1$ operation is that it flips the state in both measuring basis: $U_1$ negates the states in the two conjugate MBs, i.e.,

$$U_1|+z\rangle = −|−z\rangle, \quad U_1|−z\rangle = |+z\rangle, \quad U_1|+x\rangle = −|−x\rangle, \quad U_1|−x\rangle = |+x\rangle. \quad (6)$$

For creating the private key $K_A$, Alice sends the quantum signal to Bob first, and Bob chooses the control mode or the coding mode randomly for each photon. If he chooses the control mode, Bob performs measurement on the photon. Otherwise, he codes the photon with the two unitary operations $U_0$ and $U_1$ chosen randomly and then sends it to Charlie. Charlie performs the operation in a way just like Bob. He sends the photon to Alice after coding with unitary operations if he chooses the coding mode. Alice performs single-photon measurement on the photon received with the same MB as she prepares it. As the two unitary operations $U_0$ and $U_1$ do not change the MB, Alice can get a deterministic outcome for each photon returned, e.g., $U_A = U_B \otimes U_C$, where $U_B$ and $U_C$ are the operations done by Bob and Charlie on the same photon respectively, $U_A$ is the total operation on photon.

In order to prevent any eavesdropper from getting the information about the key $K_A$ which is represented by $U_A$ when the parties confirm that the whole process of quantum communication is secure, Alice, Bob and Charlie should sample instances twice for analyzing the error rates. The first sequence of the samples are those that
have been chosen and measured by Bob or Charlie when they choose the control mode. It can be divided into two parts: one contains those measured by Bob, denoted by \( s_{1b} \); the other contains those measured by Charlie, say \( s_{1c} \). The second sample sequence are randomly chosen from those instances that returned back to Alice, denoted by \( s_{2} \). In this second sample sequence, Bob and Charlie both choose the coding mode, and hence accordingly their coding operations form two coding sequences denoted by \( s_{2b} \) and \( s_{2c} \).

For analyzing the error rate in samples \( s_{1b} \), Bob publishes the information about the samples \( s_{1b} \), including the positions of the measured photons in the photon sequence he receives from Alice, the MBs and the results of the measurements in \( s_{1b} \), and Alice compares it with the information of these photons. In those cases where Bob has chosen the same measuring-basis as Alice’s, Alice can determine the error rate \( \varepsilon_{1s} \).

The error rate of the sample sequence \( s_{1c} \), denoted as \( \varepsilon_{1c} \), can be similarly determined by Alice and Charlie. But here Charlie first announces the positions of \( s_{1c} \) photons, and then Bob is asked to publish his unitary operations he performs on these photons, then Bob publishes the outcome of his measurement and the corresponding measuring-basis.

To analyze the error rate in the second sample sequence \( s_{2} \), Alice asks Bob and Charlie to publish the unitary operations of the sampled photons in the sequence. Since Alice’s result should be the product of unitary operations in the sequence he receives from Alice, the MBs and the results of the error rate \( \varepsilon_{2s} \).

The error rate of the sample sequence \( s_{2c} \) can be similarly determined by Alice and Charlie.

B. Security analysis of the QSS protocol with single photons

Suppose the dishonest one between Bob and Charlie is denoted as Bob*. As discussed in Ref. \([3,27]\), if the dishonest one Bob* can be detected by the other two parties, say Alice and Charlie* when he/she eavesdrops the quantum communication, then any eavesdropper can be found out. We will discuss the security of this circular QSS protocol in two cases with Bob* being Bob, and Charlie respectively.

If Bob* is Bob, the security of this QSS protocol is simplified to prevent Bob from eavesdropping the secret key \( K_A \). In fact, the task of eavesdropping check is to determine whether Bob obtained the information about the unitary operations \( U_C \) which is just the key \( K_C \) under ideal condition. Any other cheat done by Bob in the process of quantum communication will be found out in secret sharing if Bob cannot get the information about the \( U_C \). The fake information that Bob publishes about his operations \( U_B \) on \( s_{1b} \) will be exposed as it was announced before Charlie publishes the MBs and the results for \( s_{1c} \). In this way, the process for security analysis between Alice and Charlie is equal to that in BB84 QKD \([3,27]\), which is proved unconditionally secure, for example, seeing Refs. \([28,29,30]\).

We can calculate the information \( I_B \) that Bob can obtain about the unitary operations \( U_C \) done by Charlie with the probability of being detected \( \varepsilon_B \) as follows, in a way similar to those used in Refs. \([23,31]\).

We discuss the security in the case that any eavesdropper can only make individual attacks. The reason is discussed in Refs. \([32,33,34,35,36,37,38,39]\). As discussed in Ref. \([26]\), the limitation on the error rate introduced by Bob’s eavesdropping is 25% for which Bob intercepts the quantum signal Alice sends to Charlie and sends a fake photon to him. The purpose that Bob eavesdroppers the quantum signal is to learn more information about it and introduces as little error as possible into the results.

The error rate introduced by Bob comes from the wrong MBs chosen, i.e., Alice prepares the quantum signal with \( \sigma_A \) but Bob chooses \( \sigma_z \) for eavesdropping, or vice versa \([32]\). We assume that Alice prepares the quantum states with \( \sigma_z \) and Bob with \( \sigma_x \) for eavesdropping (the condition that Alice chooses \( \sigma_x \) and Bob \( \sigma_z \) is the same for the security analysis). In this way, the information stolen by Bob about the state of the photon coded is equal to that about the operation done by Charlie \([28]\).

The optimal individual attack done by an eavesdropper can be realized by a unitary operation \( U_E \) on the photon \([23, 27, 32, 33, 34, 36, 37, 38, 39]\) with an ancilla whose initial state is \( |0\rangle \).

\[
U_E |0\rangle = |0\rangle |0\rangle, \quad U_E |1\rangle = \cos \phi |1\rangle |0\rangle + \sin \phi |0\rangle |1\rangle, \tag{9}
\]

where \( \phi \in [0, \frac{\pi}{4}] \) characterizes the strength of Eve’s attack \([32]\).

The probability \( \varepsilon_B \) that Bob will be detected is same as the error rate introduced by the eavesdropping \([23, 27]\). As Alice makes the photon in the four states \( \{|+\rangle, \{-\rangle, \{|+\rangle \} \} \) with the same probability, then the error rate is \([20, 33]\)

\[
\varepsilon_B = \frac{1}{2} \sin^2 \phi. \tag{10}
\]

The state of the photon that Alice prepares can be described with a density matrix

\[
\rho_A = \frac{1}{2} |0\rangle \langle 0| + \frac{1}{2} |1\rangle \langle 1|. \tag{11}
\]

After Bob’s eavesdropping, the joint state \( \psi_3 \) of the system \( 3 \) composed of the photon \( A \) and the ancilla \([23, 27, 31]\) \( P \) can be written as

\[
\rho_{AP} = \frac{1}{2} \{ |00\rangle \langle 00| + \cos^2 \phi |10\rangle \langle 10| + \cos \phi \sin \phi |01\rangle \langle 01| + \sin^2 \phi |01\rangle \langle 01| + |0\rangle \langle 0| + |1\rangle \langle 1| \}, \tag{12}
\]
where $|ij⟩ ≡ |i⟩_A |j⟩_B$, $i, j \in \{0, 1\}$. The effect of the unitary operations done by Charlie is just to change the state of the photon $A$ in $\rho_{AP}′$. Suppose the probabilities that Charlie chooses $U_0$ and $U_1$ are $P_{c_0}$ and $P_{c_1}$, respectively. After the coding, the state $ψ_3$ becomes

$$\rho'_{AP} = \frac{1}{2} \left( P_{c_0} |00⟩⟨00| + P_{c_1} |10⟩⟨10| + P_{c_0} |00⟩⟨10| \right) + \sin^2 \phi |11⟩⟨11|$$

i.e.,

$$I_B = S(\rho''_p) = -\sum_{i=0}^{1} \lambda_i \log_2 \lambda_i, \quad (18)$$

where $\lambda_i$ ($i = 0, 1$) are the roots of the characteristic polynomial $\det(\rho''_p - \lambda I)$ [23], yielding the two eigenvalues

$$\lambda_0 = \frac{1}{2} (1 + \cos^2 \phi), \quad (19)$$

$$\lambda_1 = \frac{1}{2} \sin^2 \phi. \quad (20)$$

So we have

$$I_B = 1 - \frac{1}{2} \{(1 + \cos^2 \phi) \log_2(1 + \cos^2 \phi) + \sin^2 \phi \log_2 \sin^2 \phi\} = -\varepsilon_B \log_2 \varepsilon_B - (1 - \varepsilon_B) \log_2 (1 - \varepsilon_B). \quad (21)$$

The relation between $I_B$ and $\varepsilon_B$ is shown in Fig.2. It is shown in the figure that Bob has to face a detection probability $\varepsilon_B > 0$ if he wants to gain information $I_B > 0$. If $I_B$ is not small, Bob will be detected, otherwise Alice and Charlie can distill the key $K_C$ with privacy amplification.

As Bob wants to eavesdrop the quantum communication for creating a private key, he should send the photon coded to Alice and only measure the ancilla, which is different to that for direct communication [23]. We can trace out the state of the photon $A$ from the joint state $ψ_3$ with MB $σ_z (\{|0⟩, |1⟩\})$ to get the state of the ancilla, $ρ''_p$,

$$\rho''_p = \frac{1}{2} \left( 1 + \cos^2 \phi |0⟩⟨0| + \sin^2 \phi |1⟩⟨1| \right), \quad (15)$$

which can be projected to orthogonal measuring basis $\{|0⟩, |1⟩\}$ (it is one of the best measurements for distilling the information from the state) and written as

$$\rho''_p = \frac{1}{2} \left( 1 + \cos^2 \phi \begin{array}{cc} 1 & 0 \\ 0 & \sin^2 \phi \end{array} \right). \quad (16)$$

The information $I_B$ that Bob can obtain is equal to the Von Neumann entropy of the state of the ancilla and the Von Neumann entropy can be calculate as follows [26, 27],

$$I_B = S(\rho''_p) = -\text{Tr}(\rho''_p \log_2 \rho''_p), \quad (17)$$

![FIG. 2: The relation between $I_B$ and $\varepsilon_B$.](image)

The operation $U_B$ done by Bob on the photon does not change the character of the state for Charlie as

$$\rho_{AP''} = \frac{1}{4} \left[ \begin{array}{cccc} 1 + z & \frac{1}{4} + \frac{1}{4} & -z & \frac{1}{4} + x \\ \frac{1}{4} + x & 1 & -z & \frac{1}{4} + x \\ -z & 1 & -z & \frac{1}{4} - x \\ \frac{1}{4} + x & \frac{1}{4} - x & \frac{1}{4} - x & 1 \end{array} \right] \rho_{AP} = \rho_{AP''}, \quad (23)$$

where $P_{b_0}$ and $P_{b_1}$ are the probabilities that Bob chooses the unitary operations $U_0$ and $U_1$, respectively. No matter what quantum signal Charlie eavesdrops, the security analysis is same as BB84-QKD [4]. So this QSS is secure if Bob* is Charlie.

Bob* may cheat in the communication, for instance he publishes a wrong information about his unitary operations $U_{B*}$, or he does not use the right key $K_{B*}$ in
secret sharing. Inevitably, his action can be detected. For example, the wrong information about his unitary operations will be found out when Alice and Charlie compare the results in \( s_{2n} \) in quantum secret sharing. After the key \( K_A = K_B' \oplus K_C' \) is created, the cheat that Bob* does not use \( K_{B*} \) for decrypting the ciphered text \( C_A = S_A \oplus K_A \) can also be detected before the \( C_A \) is transmitted. Alice, Bob and Charlie need only determine whether the key \( K_A' = K_B' \oplus K_C' \) obtained by combining Bob’s key and Charlie’s key when they cooperate is identical to her key \( K_A \) obtained by Alice’s measurement before she sends a secret message to her two remote assistants, Bob and Charlie. The process can be achieved by choosing at random a sufficiently large subset of bits in the key \( K_A' \) to compare the results with those in the key \( K_A \). If the error rate is zero, Alice confirms that there is no dishonest one in Bob and Charlie pair, and she sends the secret message to them after encrypting it with the key \( K_A \); otherwise she has to abort the secret message communication.

The parties encode the photons with unitary operations and each photon can carry one bit of information in \( K_A \) in principle. The efficiency for qubit is improved to approach 100\%. Moreover, they do not exchange the information about the MBs for almost all the instances, and they also need not store the single photons.

### III. CIRCULAR QUANTUM SECRET SHARING WITH MULTI-LEVEL TWO-PARTICLE ENTANGLEMENT

For two-particle quantum system, \( d \)-dimension Bell-basis states in a symmetric channel are \([40, 41, 42, 43, 44]\),

\[
|\Psi_{nm}\rangle = \sum_j e^{2\pi i jn/d} |j\rangle \otimes |j + m \text{ mod } d\rangle /\sqrt{d},
\]

where \( n, m = 0, 1, ..., d - 1 \). The unitary operations

\[
U_{nm} = \sum_j e^{2\pi i jn/d} |j + m \text{ mod } d\rangle \langle j|
\]

can transform the Bell-basis state

\[
|\Psi_{00}\rangle = \sum_j |j\rangle \otimes |j\rangle /\sqrt{d}
\]

into the Bell-basis state \( |\Psi_{nm}\rangle \), i.e., \( U_{nm}|\Psi_{00}\rangle = |\Psi_{nm}\rangle \).

For two-party communication, one particle can carry \( \log_2 dq^2 \) bits of information while running forth and back. In a more generalized case, non-symmetric quantum channel is possible where the two particles of the entangled quantum system have the different dimensions \([44, 45]\), for example, the first particle has \( p \) dimensions and the second one has \( q \) dimensions. Then the capacity is \( \log_2 pq \).

The source coding capacity of this circular QSS can be improved largely with super-dense coding \([40, 43, 44]\) and quantum state storage \([46, 47, 48]\). We will generalize this circular QSS with Einstein-Podolsky-Rosen (EPR) pairs, two-particle maximally entangled states, following the ideas in dense coding \([10]\). The case for other multi-level two-particle entanglement is just the same as it.

An EPR pair can be in one of the four Bell states \([27, 49]\),

\[
|\psi^-\rangle_{HT} = \frac{1}{\sqrt{2}}(|0\rangle_T |1\rangle_H - |1\rangle_T |0\rangle_H), \quad (27)
\]

\[
|\psi^+\rangle_{HT} = \frac{1}{\sqrt{2}}(|0\rangle_T |1\rangle_H + |1\rangle_T |0\rangle_H), \quad (28)
\]

\[
|\phi^-\rangle_{HT} = \frac{1}{\sqrt{2}}(|0\rangle_T |0\rangle_H - |1\rangle_T |1\rangle_H), \quad (29)
\]

\[
|\phi^+\rangle_{HT} = \frac{1}{\sqrt{2}}(|0\rangle_T |0\rangle_H + |1\rangle_T |1\rangle_H). \quad (30)
\]

The four local unitary operations \( U_{Li} \) \( (i = 0, 1, 2, 3) \) can transform the four Bell states into each other.

\[
U_{L0} = I = |0\rangle \langle 0| + |1\rangle \langle 1|, \quad (31)
\]

\[
U_{L1} = i\sigma_y = |0\rangle \langle 1| - |1\rangle \langle 0|, \quad (32)
\]

\[
U_{L2} = \sigma_x = |1\rangle \langle 0| + |0\rangle \langle 1|, \quad (33)
\]

\[
U_{L3} = \sigma_z = |0\rangle \langle 0| - |1\rangle \langle 1|, \quad (34)
\]

i.e.,

\[
I \otimes U_{L0}|\psi^\pm\rangle = |\psi^\pm\rangle, \quad I \otimes U_{L0}|\phi^\pm\rangle = |\phi^\pm\rangle, \quad (35)
\]

\[
I \otimes U_{L1}|\psi^\pm\rangle = |\phi^\mp\rangle, \quad I \otimes U_{L1}|\phi^\pm\rangle = -|\psi^\mp\rangle, \quad (36)
\]

\[
I \otimes U_{L2}|\psi^\pm\rangle = |\phi^\pm\rangle, \quad I \otimes U_{L2}|\phi^\pm\rangle = |\psi^\pm\rangle, \quad (37)
\]

\[ I \otimes U_{L3}|\psi^\mp\rangle = -|\psi^\mp\rangle, \quad I \otimes U_{L3}|\phi^\pm\rangle = |\phi^\mp\rangle. \quad (38)
\]

The process of this QSS with EPR pairs is similar to that with single photons discussed above. The president, Alice prepares the two-particle entangled state \( |\psi^-\rangle_{HT} \), and she keeps the particle \( H \) and sends the particle \( T \) to Bob first, shown in Fig.1. He chooses the control mode and the coding mode randomly. When Bob chooses the coding mode, he performs one of the four unitary operations \( U_{Li} \) \( (i = 0, 1, 2, 3) \) which represent the bits 00, 01, 10 and 11 respectively, on the particle \( T \) randomly. Otherwise, he chooses the two MBs, \( \sigma_z \) and \( \sigma_x \) to measure the particle \( T \), and tells Alice which particle he chooses the control mode. Alice does the correlated measurement on the particle \( H \) in the EPR pair in which Bob measures the particle \( T \), that is, Bob tells Alice the position of the particle and his MB for it, and Alice performs the measurement with the same MB as Bob on the particle \( H \).

As for Charlie, after he receives the particle Bob sends to Charlie after coding with an unitary operation, Charlie chooses randomly the control mode and the coding mode. If he chooses the coding mode, he performs randomly one of the four coding operations and then send the particle to Alice. When Bob chooses the control mode, he measures the particle choosing randomly one of the two MBs \( \sigma_z \) and \( \sigma_x \).
The eavesdropping check can be adapted here straightforwardly. In fact, no matter who the dishonest Bob is, the way for checking the security of quantum communication is same as that for the BBM-QKD [50] which has been proven unconditionally secure for key generation [34, 51]. As pointed out by Bechmann-Pasquinucci and Peres [52], the QKD with multi-level quantum system is more secure than that with two-level one.

IV. DISCUSSION AND CONCLUSION

In general, QSS is accomplished with entanglement, which normally requires more complicated experimental setups. Though big progress has been made for producing and measuring entanglement, the efficiency is still low [33, 34, 51]. QSS with single photons will be more convenient for being implemented in laboratory and practical application. On the other hand, the source coding capacity of QSS can be improved largely with super-dense coding in which entanglement is necessary. With development of technology, it is likely feasible to implement QSS based on entanglement, especially with multi-level entanglement in high capacity.

Certainly, another important function of QSS is to split a secret message into \( n \) pieces and completes the task of an \( m – out – of – n \) quantum secret splitting scheme, or so-called \((m, n)\) threshold scheme [3]. Unfortunately, this circular QSS scheme cannot be used to accomplish the full goal of quantum secret splitting. That is, it can not be used for \( m – out – of – n \) scheme in which any \( m \) parties can reconstruct the secret message when they collaborate. However, it is useful for accomplishing a partial goal, \( n-out-af-n \) scheme. In other words, the circular QSS can be used to reconstruct the secret message when all of the other \( n \) parties cooperate with some classical information published by Alice, the president.

In summary, a circular QSS scheme is proposed. It is useful and efficient when the president Alice is remote to all her agents, Bobs who are in adjacent, especially the parties of secret sharing are more than three. In this scheme, the quantum information carrier, single photons or entangled particles, will run circularly, and the parties choose randomly the control mode or coding mode to operate the QIC. They measure the QIC only when they choose control mode, otherwise, they encode the QIC with some unitary operations. If the QIC is single photon, all the parties of communication including Alice do not need to store the quantum state. If the QIC is entangled quantum system, only Alice is required to possess the technique of quantum storage, others need not. This is convenient for realizing QSS in practical application. Moreover, each QIC can be used to carry information except for the samples for eavesdropping check, and classical information exchanged is reduced largely as the parties need not announce the MBs for the QIC.

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