π-SYSTEMS OF SYMMETRIZABLE KAC-MOODY ALGEBRAS

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Abstract. As part of his classification of regular semisimple subalgebras of semisimple Lie algebras, Dynkin introduced the notion of a π-system. This is a subset of the roots such that pairwise differences of its elements are not roots. These arise as simple systems of regular semisimple subalgebras. Morita and Naito generalized this notion to all symmetrizable Kac-Moody algebras. In this work, we systematically develop the theory of π-systems of symmetrizable Kac-Moody algebras and establish their fundamental properties. We study the orbits of the Weyl group on π-systems, and completely determine the number of orbits in many cases of interest in physics. In particular, we show that there is a unique π-system of type $HA_1^{(1)}$ (the Feingold-Frenkel algebra) in $E_{10}$ (the rank 10 hyperbolic algebra) up to Weyl group action and negation. For symmetrizable hyperbolic Kac-Moody algebras, we formulate general principles for constructing π-systems and criteria for the non-existence of π-systems of certain types, and use this to determine the set of maximal hyperbolic diagrams in ranks 3-10 relative to the partial order of admitting a π-system.

1. Introduction

1.1. Let $g$ denote a finite dimensional semisimple Lie algebra over $\mathbb{C}$. A semisimple Lie subalgebra of $g$ is said to be regular if it is ad-$h$-invariant for some Cartan subalgebra $h$ of $g$. The classical works of Borel-de Siebenthal [1] and Dynkin [7] contain a complete classification of the possible Cartan-Dynkin types of regular semisimple subalgebras for each simple Lie algebra.

Now fix a Cartan subalgebra $h$ and the corresponding set of roots of $g$. A π-system $\Sigma$ is a subset of the roots satisfying the property that pairwise differences of elements of $\Sigma$ are not roots of $g$. Dynkin showed that linearly independent π-systems arise precisely as simple systems of regular semisimple subalgebras of $g$ (that are ad-invariant under the given Cartan subalgebra).

There is a natural action of the group $\text{Int} g$ of inner automorphisms of $g$ on the set of its regular semisimple subalgebras. The $\text{Int} g$-orbits of regular semisimple subalgebras are in bijection with $W$-orbits of π-systems, where $W$ is the Weyl group of $g$. For each type of π-system, Dynkin computed the number of Weyl group orbits [7, Tables 9-11], and thereby the number of inequivalent regular subalgebras of that type.

Suppose now that $g$ is a symmetrizable Kac-Moody algebra. A π-system $\Sigma$ of $g$ is defined to be a subset of its real roots such that pairwise differences of elements of $\Sigma$ are not roots. This definition is due to Morita [21], who called them fundamental subsets of roots. Naito [22] showed that a linearly independent π-system defines an embedding of some Kac-Moody algebra into $g$. He termed the image of such an embedding a regular subalgebra of $g$; this coincides with Dynkin’s
notion of regular semisimple subalgebras when \( \mathfrak{g} \) is finite-dimensional. Thus regular subalgebras and linearly independent \( \pi \)-systems may be viewed as essentially equivalent notions.

Naito also classified the possible types of \( \pi \)-systems when \( \mathfrak{g} \) is an untwisted affine Kac-Moody algebra \[22, \text{Theorem 4.2}\]. Later work \[12, 25\] extended this to the twisted case as well. Feingold and Nicolai \[11\] constructed families of \( \pi \)-systems when \( \mathfrak{g} \) is an overextended (or Lorentzian) Kac-Moody algebra. Many examples of \( \pi \)-systems of the rank 10 hyperbolic Lie algebra \( E_{10} \) were constructed in \[14, 15\] and studied for their significance in physics.

1.2. Let \( W \) denote the Weyl group of the symmetrizable Kac-Moody algebra \( \mathfrak{g} \). Our primary goal in the first half of this paper is to undertake a systematic study of the Weyl group action on linearly independent \( \pi \)-systems of \( \mathfrak{g} \). Our first key result is Theorem 3.1 which states that every indecomposable linearly independent \( \pi \)-system \( \Sigma \) of \( \mathfrak{g} \) is \( W \)-conjugate to a \( \pi \)-system supported on an affine subdiagram of the Dynkin diagram of \( \mathfrak{g} \). In particular, \( \mathfrak{g} \) admits a \( \pi \)-system of affine type if and only if its Dynkin diagram has a subdiagram of affine type. We further establish in this case that the number of Weyl group orbits of \( \pi \)-systems of affine type is necessarily infinite (see Theorem 4.2 for the precise statements).

The natural next step is to consider \( \pi \)-systems \( \Sigma \) of indefinite type. An important low rank example of an indefinite type Kac-Moody algebra is the Feingold-Frenkel Lie algebra whose Dynkin diagram \( HA_1^{(1)} = A^{++}_1 \) is obtained by overextending the diagram \( A_1 \) of \( \mathfrak{sl}_2 \), i.e., by adding a further vertex to the affinization of \( A_1 \) (figure 5.1).

Let \( \mathfrak{g} \) now denote a Kac-Moody algebra with symmetric generalized Cartan matrix (GCM). We prove (Theorem 6.2) that \( \mathfrak{g} \) has a \( \pi \)-system of type \( A^{++}_1 \) if and only if the Dynkin diagram of \( \mathfrak{g} \) has a subdiagram of overextended type, i.e., one obtained by overextending some finite type Dynkin diagram (figure 5.1). Remarkably, there is a one-to-one correspondence between such subdiagrams of \( \mathfrak{g} \) and \( (W \times \mathbb{Z}_2) \)-orbits of \( \pi \)-systems of type \( A^{++}_1 \) in \( \mathfrak{g} \), where \( W \) is the Weyl group of \( \mathfrak{g} \) and \( \mathbb{Z}_2 \) acts on \( \pi \)-systems by \( \Sigma \mapsto \pm \Sigma \). As a corollary, we obtain that there is a unique \( \pi \)-system of type \( A^{++}_1 \) in the rank 10 hyperbolic Kac-Moody algebra \( E_{10} \), up to conjugation by the Weyl group of \( E_{10} \) and negation. This is a case of particular interest in physics (see § 1.4 below) and was one of the main initial motivations for this paper. More generally, this uniqueness result holds when \( E_{10} \) is replaced by any diagram which has a unique overextended subdiagram. This includes all hyperbolic overextensions (corollary 6.6) and diagrams such as \( E_n \), \( n \geq 11 \) [29].

Strengthening these arguments enables us to prove a general theorem (Theorem 7.1) concerning \( \pi \)-systems of overextended type in \( \mathfrak{g} \). For a fixed overextended Dynkin diagram \( K \), we show that the problem of counting the number of \( (W \times \mathbb{Z}_2) \)-inequivalent \( \pi \)-systems of type \( K \) in \( \mathfrak{g} \) can be reduced to one in which \( K \) is replaced by its underlying finite type diagram \( K^0 \) and \( \mathfrak{g} \) is replaced by certain finite-dimensional simple Lie subalgebras of \( \mathfrak{g} \) (equation (7.1)). Since the number of Weyl group orbits of \( \pi \)-systems is completely known when the ambient Lie algebra is finite-dimensional [7], this explicitly determines the number of \( (W \times \mathbb{Z}_2) \)-orbits for each type of overextended \( \pi \)-system in \( \mathfrak{g} \). In particular, this number is always finite, in contrast to the affine case mentioned above. If further, \( \mathfrak{g} \) is hyperbolic overextended (corollary 7.2) it can be directly read off from Dynkin’s tables [7, Tables 9-11]; the number of \( (W \times \mathbb{Z}_2) \)-orbits turns out to be 1 in most cases, with a few exceptions where
it is 2. Again, this encompasses many examples of interest in physics, such as \( \pi \)-systems of types \( DE_{10} = D_8^{++} \) and \( AE_n = A_n^{++} \) in \( E_{10} \).

1.3. In the second half of the paper, we study the binary relation associated to the notion of admitting a \( \pi \)-system, first introduced by Morita in [21]. Given symmetrizable GCMs \( A, B \), we say \( B \preceq A \) if there is a linearly independent \( \pi \)-system of type \( B \) in the Kac-Moody algebra \( g(A) \). We establish (§8) that \( \preceq \) defines a partial order on the set of symmetrizable hyperbolic GCMs.

It was shown in [28] that the set of symmetric hyperbolic GCMs has the unique maximal element \( E_{10} \) for this partial order. We consider the 142 symmetrizable hyperbolic diagrams in ranks 3-10, and explicitly determine the maximal elements relative to \( \preceq \) (Tables 1-10). To carry out our program, we formulate some widely-applicable principles (§9.1) for the existence and non-existence of \( \pi \)-systems of certain types, which may be of independent interest. We show that there are precisely 22 maximal diagrams, with 5, 9, 5, 3 diagrams occurring in ranks 3, 4, 6 and 10 respectively. Another approach to this poset in terms of Weyl groups is contained in [13,27].

We note that if \( A \) is a symmetrizable hyperbolic GCM, then so is its transpose \( A^T \). The corresponding Kac-Moody algebra \( g(A^T) \) is the dual Lie algebra, whose Dynkin diagram is obtained by reversing all arrows in the diagram of \( A \). We find that the set of maximal diagrams is not closed under transposes, i.e., there is a hyperbolic GCM \( B \) which is maximal, but \( B^T \) is not (§12.5). This leads to the rather surprising conclusion that the dualization map \( A \mapsto A^T \) is not an automorphism of the partial order \( \preceq \). This is in sharp contrast to the subdiagram partial order \( B \subseteq A \) (§4.1) defined by the stronger requirement that the Dynkin diagram of \( A \) have a subdiagram of type \( B \); simply reversing all arrows in the diagram of \( A \) gives \( B^T \subseteq A^T \).

1.4. The physics of regular embeddings. We make some remarks on the motivations from physics that led to this work. It is known that \( E_{10} \) symmetry appears in 11 dimensional supergravity in several ways. In dimensionally reduced supergravity theories, there is a Lie group \( G \) and subgroup \( K \) such that the coset space \( G/K \) fibers over spacetime \( M \) and the scalar fields are maps \( \Sigma : M \to G/K \). Under dimensional reduction to dimension \( D = 1 \), \( G = E_{10}(\mathbb{R}) \) and \( K \) is the subgroup fixed by the Cartan involution [3,16]. The real roots and the representations of \( E_{10} \) have been shown to correspond to the fields of 11 dimensional supergravity at low levels [3].

There is a similar description of the symmetries of Einstein gravity in \( D = 4 \) spacetime dimensions in terms of the rank 3 Feingold-Frenkel hyperbolic Kac-Moody algebra \( A_1^{++,} \), also denoted \( AE_3 \) in the physics literature. Under dimensional reduction to \( D = 1 \) spacetime dimensions, \( AE_3 \) is conjectured to be a symmetry of the dimensionally reduced Lagrangian [16]. The real roots and the representations of \( AE_3 \) correspond to the fields of gravity at low levels [3].

The Lie algebra \( E_{10} \) contains a regular subalgebra isomorphic to \( AE_3 \) (by the results of [28] or Theorem 6.2). This reflects the inclusion of Einstein gravity into 11-dimensional supergravity. Theorem 6.2 states that all \( \pi \)-systems of type \( AE_3 \) in \( E_{10} \) are conjugate (up to negation) under the Weyl group of \( E_{10} \), indicating that there is a ‘canonical’ inclusion of Einstein gravity into 11-dimensional supergravity. We remark that there is also a physics-suggested way to embed \( AE_3 \) into \( E_{10} \) via the ‘gravity truncation’ method of [4, §4].

Weyl orbits of \( \pi \)-systems of affine type occur in [9, §3], where the authors describe a family of embeddings \( A_1^{(1)} \preceq E_8^{(1)} \subseteq E_{10} \) and give a brane interpretation of these.
Finally, we mention that rank 2 hyperbolic Kac-Moody algebras, such as \( H(3) \) with GCM \( (\begin{smallmatrix} 2 & -3 \\ -3 & 2 \end{smallmatrix}) \) also occur as regular subalgebras of \( AE_3 \) and \( E_{10} \). While a supergravity/brane interpretation of these embeddings would be of considerable interest, it has remained elusive thus far.

1.5. Further historical remarks. In Dynkin and Morita’s original definitions, a \( \pi \)-system was required to be linearly independent. Dynkin does however mention \( \pi \)-systems of finite-dimensional simple Lie algebras with this condition relaxed \([7\text{, Table 7}]\) (see also Onischchik-Vinberg \([23\text{, Chap 4, §2, exercises 29-38}]\)). Oshima \([24]\) and Dynkin-Minchenko \([8]\) obtained extensions and variations of the results of \([7]\).

In the symmetrizable Kac-Moody context, Morita \([21]\) and Naito \([22]\) obtained the key initial results. A decade later, Feingold-Nicolai \([11]\) rediscovered the definition of \( \pi \)-systems, but as was pointed out by Henneaux et al \([14, \S 4.3]\), their main theorem on embeddings arising out of \( \pi \)-systems is false unless this condition is imposed. Our Theorem \([2.3]\) is the corrected statement, in the more general setting of \( \pi \)-systems that are not necessarily subsets of the positive real roots. Our Theorem \([3.1]\) serves as a link between the definitions of Morita and Feingold-Nicolai.

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2. \( \pi \)-systems

2.1. An integer matrix \( A = (a_{ij}) \) of size \( n \times n \), where \( n \) is a positive integer, is called a generalized Cartan matrix, GCM for short, if the following conditions are satisfied:

1. \( a_{ii} = 2 \) for all \( 1 \leq i \leq n \)
2. \( a_{ij} \leq 0 \) whenever \( 1 \leq i \neq j \leq n \)
3. \( a_{ij} = 0 \) if \( a_{ji} = 0 \) for \( 1 \leq i, j \leq n \)

Given a GCM \( A \) of size \( n \), we let \( g(A) \) denote the Kac-Moody Lie algebra associated to \( A \) \([18\text{, §1.3}]\), with Cartan subalgebra \( \mathfrak{h}(A) \) and Chevalley generators \( e_i, f_i \) for \( 1 \leq i \leq n \). Let \( g'(A) \) denote the derived subalgebra \([g(A), g(A)]\) of \( g(A) \). Let \( \alpha_i(A), 1 \leq i \leq n \) denote the simple roots of \( g(A) \) and let \( Q(A) \) be its root lattice, i.e., the free abelian group generated by the \( \alpha_i(A) \). Both \( g(A) \) and \( g'(A) \) are \( Q(A) \)-graded Lie algebras, with \( \deg e_i = \alpha_i(A) = -\deg f_i \) and \( \deg \mathfrak{h} = 0 \) for all \( h \in \mathfrak{h}(A) \) \([18\text{, Chapter 1}]\). We let \( \Delta, \Delta^{re}, \Delta^{im} \) denote the sets of roots, real roots and imaginary roots respectively. For a root \( \alpha \), we let \( g(A)_\alpha \) denote the corresponding root space. Each real root \( \alpha \) defines a reflection \( s_\alpha \) of \( \mathfrak{h}^* \) by \( s_\alpha(\lambda) = \lambda - \langle \alpha^\vee, \lambda \rangle \alpha \) where \( \alpha^\vee \in \mathfrak{h}(A) \) is the coroot corresponding to \( \alpha \). The Weyl group \( W(A) \) is the subgroup of \( GL(\mathfrak{h}^*) \) generated by the \( s_\alpha, \alpha \in \Delta^{re} \). We use terminology and notation as in the early chapters of \([18]\) without any further comment.

2.2. Multisets of real roots. Let \( A \) be a GCM, and let \( \Sigma = \{\beta_1, \beta_2, \ldots, \beta_m\} \) be a collection of real roots of \( g(A) \) (possibly with repetitions). We define the \( m \times m \) matrix

\[
M(\Sigma) := \left[\langle \beta_i^\vee, \beta_j^\vee \rangle\right]_{i,j=1}^m
\]

We note that this is not a GCM in general. We let \( \Sigma^\vee := \{\beta_1^\vee, \beta_2^\vee, \ldots, \beta_m^\vee\} \) be the corresponding multiset of coroots. Viewing these as real roots of \( g(A^T) \), we observe \( M(\Sigma^\vee) = M(\Sigma)^T \).
A reordering of the elements of \( \Sigma \) corresponds to a simultaneous permutation of the rows and columns of the matrix: \( M(\Sigma) \to PM(\Sigma)P^T \) for some \( m \times m \) permutation matrix \( P \). We will most often identify two such matrices without explicit mention.

2.3. \( \pi \)-systems.

**Definition 2.1.** Let \( A \) be a GCM. A \( \pi \)-system in \( A \) is a finite collection of distinct real roots \( \{\beta_i\}_{i=1}^m \) of \( \mathfrak{g}(A) \) such that \( \beta_i - \beta_j \) is not a root for any \( 1 \leq i \neq j \leq m \).

This definition is essentially due to Dynkin [7] (for \( A \) of finite type) and Morita [21] (in general), both of whom require that the \( \{\beta_i\}_{i=1}^m \) be linearly independent; Morita calls such sets fundamental subsets of roots. The following proposition is stated in Morita (for the linearly independent case) without proof (see also Naito [22]). We supply the easy details.

**Proposition 2.2.** Let \( A \) be a GCM, and \( \Sigma = \{\beta_i\}_{i=1}^m \) be a \( \pi \)-system in \( A \). Then the matrix \( M(\Sigma) \) is a GCM.

**Proof:** For any real root \( \beta \) we have \( \langle \beta^\vee, \beta \rangle = 2 \). Indeed, letting \( \beta = w\alpha \) for a simple root \( \alpha \) and \( w \) an element of the Weyl group, we have \( \beta^\vee = w(\alpha^\vee) \), and \( \langle \beta^\vee, \beta \rangle = \langle w(\alpha^\vee), w\alpha \rangle = \langle \alpha^\vee, \alpha \rangle = 2 \). Suppose \( \beta \) and \( \gamma \) are distinct real roots such that \( \gamma - \beta \) is not a root. Consider \( \{\gamma - p\beta, \ldots, \gamma + q\beta\} \) the \( \beta \)-string through \( \gamma \) ([18, Prop. 5.1]). Clearly \( p = 0 \) and \( \langle \beta^\vee, \gamma \rangle = p - q \leq 0 \).

With \( \beta \) and \( \gamma \) as in the previous paragraph, if \( \langle \beta^\vee, \gamma \rangle = 0 \), then \( q = 0 \), so that \( \beta + \gamma \) is not a root, so the \( \gamma \)-string \( \{\beta - p'\gamma, \ldots, \beta + q'\gamma\} \) through \( \beta \) consists only of \( \beta \), and so \( \langle \gamma^\vee, \beta \rangle = p' - q' = 0 \). □

We call \( B := M(\Sigma) \) the type of \( \Sigma \), and refer to \( \Sigma \) as a \( \pi \)-system of type \( B \) in \( A \).

2.4. Symmetrizable GCMs and \( \pi \)-systems. An \( n \times n \) GCM \( A \) is symmetrizable if there exists a diagonal \( n \times n \) matrix \( D \) with positive rational diagonal entries such that \( DA \) is symmetric. Let \( \Sigma = \{\beta_i : 1 \leq i \leq m\} \) be a \( \pi \)-system of type \( B \) in \( A \). We note that if \( A \) is a symmetrizable GCM, then so is \( B \). Fix a choice of diagonal matrix \( D \) which symmetrizes \( A \), and let \( (\cdot | \cdot) \) denote the corresponding symmetric bilinear form on \( Q(A) \otimes \mathbb{C} \), defined by:

\[
\langle \alpha_i(A) | \alpha_j(A) \rangle = D_{ii} a_{ij}
\]

Since the \( \beta_i \) are real roots of \( \mathfrak{g}(A) \), we know by [Kac, Chapter 5] that:

\[
b_{ij} = \langle \beta_i^\vee, \beta_j \rangle = \frac{2\langle \beta_i | \beta_j \rangle}{\langle \beta_i | \beta_i \rangle}
\]

Thus, \( D' = \text{diag}(\langle \beta_i | \beta_i \rangle / 2) \) is a diagonal matrix with positive rational entries that symmetrizes \( B \). This choice of symmetrization defines a symmetric bilinear form on \( Q(B) \otimes \mathbb{C} \). As in equation (2.1) above, this is given by \( (\alpha_i(B) | \alpha_j(B)) = D'_{ii} b_{ij} = \langle \beta_i | \beta_j \rangle \). In other words, given the compatible choices of symmetrizations \( (D, \, D') \) as above, the \( \mathbb{C} \)-linear map

\[
q_\Sigma : Q(B) \otimes \mathbb{C} \to Q(A) \otimes \mathbb{C}, \quad \alpha_i(B) \mapsto \beta_i \text{ for } 1 \leq i \leq m
\]

is form preserving. Given \( \alpha \in Q(A) \otimes \mathbb{C} \) with \( (\alpha | \alpha) \neq 0 \), the corresponding reflection \( s_\alpha \) is given by:

\[
s_\alpha(\gamma) = \gamma - \frac{2\langle \gamma | \alpha \rangle}{(\alpha | \alpha)} \alpha
\]

for \( \gamma \in Q(A) \otimes \mathbb{C} \). We note that \( q_\Sigma(s_\alpha(\beta)) = s_{\alpha'}(\beta') \) where \( \alpha, \beta \in Q(B) \otimes \mathbb{C} \) and \( \alpha', \beta' \) are their images under \( q_\Sigma \).

5
Theorem 2.3. Let $A$ be an $n \times n$ symmetrizable GCM and $\Sigma = \{\beta_i\}_{i=1}^m$ a $\pi$-system of type $B$ in $A$. Let $e_{\beta_i}$, $e_{-\beta_i}$ be non-zero elements in the root spaces $\mathfrak{g}(A)_{\beta_i}$ and $\mathfrak{g}(A)_{-\beta_i}$ respectively, such that $[e_{\beta_i}, e_{-\beta_i}] = \beta_i^\vee$. Then there exists a unique Lie algebra homomorphism $i_\Sigma : \mathfrak{g}'(B) \to \mathfrak{g}'(A)$ such that $e_i \mapsto e_{\beta_i}$, $f_i \mapsto e_{-\beta_i}$, $h_i \mapsto \beta_i^\vee$.

Proof: Since $A$ is symmetrizable, so is $B$, and $\mathfrak{g}'(B)$ is generated by $e_i$, $f_i$, $h_i$, $1 \leq i \leq m$ subject to the relations [18, Theorem 9.11]:

\[ \begin{align*}
(2.3) & \quad [h_i, e_j] = b_{ij} e_j \quad [h_i, f_j] = -b_{ij} f_j \\
(2.4) & \quad [h_i, h_j] = 0 \\
(2.5) & \quad [e_i, f_j] = \delta_{ij} h_i \quad \text{and} \\
(2.6) & \quad (\text{ad } e_i)^{1-b_{ij}} e_j = (\text{ad } f_i)^{1-b_{ij}} f_j = 0
\end{align*} \]

Any Lie algebra homomorphism from $\mathfrak{g}'(B)$ is thus determined by the images of $e_i$, $f_i$ and $h_i$ ($1 \leq i \leq m$). Thus there is at most one Lie algebra homomorphism with the requisite properties.

To show that there exists such a homomorphism, we need only verify that the relations in (2.3) through (2.6) are satisfied. Relations (2.3) and (2.4) are clearly satisfied. As for (2.5) we consider two cases: if $j = i$, then it follows since $[e_{\beta_i}, e_{-\beta_i}] = \beta_i^\vee$; if $j \neq i$, then it follows since $\beta_i - \beta_j$ is not a root of $\mathfrak{g}(A)$ by the definition of $\pi$-system. As for (2.6), it follows from the fact [15, Prop. 5.1] that the $\beta_i$-string through $\beta_j$ consists of $\beta_j$, $\beta_j + \beta_i$, $\ldots$, $\beta_j + k \beta_i$, where $k = \langle \beta_i^\vee, \beta_j \rangle$.

The following proposition is equivalent to that of Naito [22, Theorem 3.6], though his proof is different (without using the Serre relations). In the interest of completeness, we give a (slightly simpler) argument.

Proposition 2.4. With notation as in the above theorem, if $\Sigma$ is linearly independent (in $Q(A) \otimes \mathbb{Z} \otimes \mathbb{C}$), one can extend the map $i_\Sigma$ to a map from $\mathfrak{g}'(B)$ to $\mathfrak{g}(A)$. Further, this map is injective.

Proof. Suppose that $\{\mathfrak{h}; \alpha_1^\vee, \ldots, \alpha_n^\vee; \alpha_1, \ldots, \alpha_n \}$ is a realization of $A$ [18, Chapter 1]. Let $\mathfrak{k}$ be any subspace of $\mathfrak{h}$ of smallest possible dimension such that (i) $\mathfrak{k}$ contains $\beta_1^\vee, \ldots, \beta_m^\vee$, and (ii) the restrictions of $\beta_1, \ldots, \beta_m$ to $\mathfrak{k}$ are linearly independent as elements of $\mathfrak{k}^\ast$ (this is possible since we are given that the $\beta_i$ are linearly independent). Then

1. $(\mathfrak{k}, \beta_1^\vee, \ldots, \beta_m^\vee, \beta_1|, \ldots, \beta_m|\mathfrak{k})$ is a realization of $B$.
2. rank $B \geq$ rank $A - 2(n - m)$.

Assertion (1) follows easily from the definition of realization. As for assertion (2), observe that \{\beta_i^\vee\}_{i=1}^m is in the span of \{\alpha_i^\vee\}_{i=1}^n: this follows from the definition of $\beta^\vee$ for a real root $\beta$ as $w(\alpha_i^\vee)$ where $w$ is an element of the Weyl group such that $\beta = w(\alpha_i)$. We have $B = YAX$, where $X = (x_{ij})$ is the $n \times n$ matrix such that $\beta_j = \sum_{i=1}^n x_{ij} \alpha_i$ and $Y = (y_{ij})$ is the $m \times n$ matrix such that $\beta_j^\vee = \sum_{i=1}^n y_{ij} \alpha_i^\vee$. The matrices $X$ and $Y$ are both of rank $m$. The assertion now follows easily from elementary linear algebra.

Now, $\mathfrak{g}'(B)$ is generated by $\mathfrak{k}$, $e_i, f_i$ subject to the relations specified in the proof of Theorem 2.3 together with the following:

\[ [k, e_i] = \beta_i(k) e_i \quad [k, f_i] = -\beta_i(k) f_i \quad [k_1, k_2] = 0 \quad \text{for } k, k_1, k_2 \in \mathfrak{k} \]

We map $\mathfrak{k}$ to $\mathfrak{h}$ via the natural inclusion; $e_i$, $f_i$ are mapped to $e_{\beta_i}$, $e_{-\beta_i}$ as before. We only need to check that the additional relations above hold. But these are obvious.

6
Finally, we show that the homomorphism is an embedding. The kernel of the homomorphism being an ideal of $\mathfrak{g}(B)$, it either contains the derived algebra $\mathfrak{g}'(B)$ or is contained in the center [18 §1.7(b)]. Since $\epsilon_i \mapsto \epsilon_{i \beta_i}$ (and $\epsilon_i$ is contained in $\mathfrak{g}'(B)$ by (2.3)) the first possibility is ruled out. Thus the kernel is contained in the center. But the center is contained in the subspace $\mathfrak{k}$ (18 Prop. 1.6) and on $\mathfrak{k}$ the homomorphism is an inclusion. Thus the kernel is zero.

\begin{proof}
\end{proof}

\begin{remark}
\end{remark}

Remark 2.5. The following easy observations are often useful:

(i) Let $\Sigma$ is linearly independent, then $\pi(\Sigma)$ is an injection.

(ii) If $\det B \neq 0$, then $\Sigma$ is linearly independent.

\begin{example}
\end{example}

(i) Let $A$ be a GCM of finite type. Dynkin [7] showed that if $\mathfrak{m}$ is a regular semisimple subalgebra of $\mathfrak{g}(A)$, then there exists a GCM $B$ of finite type and a $\pi$-system $\Sigma$ of type $B$ in $A$ such that $\mathfrak{m} = i_{\mathfrak{g}}(\mathfrak{g}(B))$.

(ii) Let us take $A = [2]$, so that $\mathfrak{g}(A) = \mathfrak{g}'(A) = \mathfrak{sl}_2 \mathbb{C}$. Let $\Sigma = \{\alpha_1, -\alpha_1\} = \Delta(A)$. This is clearly a $\pi$-system in $A$, of type $B = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$. The corresponding Kac-Moody algebra $\mathfrak{g}(B)$ is the affine Lie algebra $\hat{\mathfrak{sl}}_2 \mathbb{C}$. We then have [18, Chapter 7], $\mathfrak{g}'(B) = \mathfrak{sl}_2 \mathbb{C} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$, the universal central extension of the loop algebra of $\mathfrak{sl}_2$. The generators of $\mathfrak{g}'(B)$ are $e_1 = X, f_1 = Y, e_2 = Y \otimes t, f_2 = X \otimes t^{-1}$, where $X = (0 1)$ and $Y = (0 0)$. The standard generators of $\mathfrak{sl}_2 \mathbb{C}$.

The map defined in Theorem 2.3 is thus:

$$e_1 \mapsto X, f_1 \mapsto Y, e_2 \mapsto Y, f_2 \mapsto X$$

(iii) More generally, let $A$ be any finite type GCM and $\mathfrak{g}(A)$ the corresponding finite dimensional simple Lie algebra, with highest root $\theta$. Consider the $\pi$-system $\Sigma$ consisting of the simple roots of $\mathfrak{g}(A)$ together with $-\theta$. This has type $B$, the GCM of the untwisted affinization of $\mathfrak{g}(A)$. The map defined by Theorem 2.3 coincides with the evaluation map at $t = 1$:

$$\mathfrak{g}'(B) = \mathfrak{g}(A) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \rightarrow \mathfrak{g}(A)$$

$$c \mapsto 0 \text{ and } \zeta \otimes f(t) \mapsto f(1) \zeta \text{ for all } \zeta \in \mathfrak{g}(A), f \in \mathbb{C}[t, t^{-1}]$$

\begin{lemma}
\end{lemma}

Lemma 2.7. Let $A$ be an $n \times n$ GCM. Let $I$ be an ideal of $\mathfrak{g}'(A)$ that does not contain any simple root vectors, i.e., $e_i, f_i \notin I$ for all $i$. Then $I$ does not contain any root vectors, i.e., $\mathfrak{g}'(A) \cap I = (0)$ for all roots $\alpha$.

\begin{proof}
Suppose $\alpha$ is a positive, non-simple root. Assume $e_\alpha \in I$ for some nonzero $e_\alpha \in \mathfrak{g}'(A)$. By [Kac, Lemma 1.5], there exists $i_1$ such that $[f_{i_1}, e_\alpha] \neq 0$. If $\alpha - \alpha_i$ is not a simple root, find $i_2$ such that $[f_{i_2}, [f_{i_1}, e_\alpha]] \neq 0$. Proceeding this way, after finitely many steps we get $[f_{i_k}, \cdots f_{i_2}, [f_{i_1}, e_\alpha]] \cdots = e_i \in I$, which contradicts the hypothesis on $I$. If $\alpha$ were a negative root to begin with, the proof is analogous.

\end{proof}

\begin{remark}
\end{remark}

Remark 2.8. (1) Let $I$ be an ideal of $\mathfrak{g}'(A)$. We observe that if $I$ contains one of $e_i, f_i, \alpha_i^\vee$, then it contains all three.

7
(2) If $A$ is an indecomposable GCM, then any proper ideal of $\mathfrak{g}'(A)$ satisfies the hypothesis of lemma 2.7. To see this, suppose $e_i$ is in $I$. Then, so are $f_i$ and $\alpha_i^\vee$. Since $A$ is indecomposable, for each fixed $j$, there exist $i_1,i_2,\ldots,i_s$ such that $a_{i_1i_2}\ldots a_{i_j} \neq 0$. Since $[\alpha_i^\vee,e_{i_1}] = a_{i_1i_1}e_{i_1}$, we conclude $I$ contains $e_{i_1}$, and hence also $f_{i_1},\alpha_{i_1}^\vee$. Proceeding in this manner, we get $e_j,f_j,\alpha_j^\vee \in I$. Since this holds for all $j$, we obtain $I = \mathfrak{g}'(A)$, a contradiction.

While the map $i_\Sigma$ of Theorem 2.3 need not be injective when $\Sigma$ is linearly dependent, we nevertheless have the following useful result which states that it is injective on each root space.

**Corollary 2.9.** The map $i_\Sigma : \mathfrak{g}'(B) \to \mathfrak{g}'(A)$ defined in Theorem 2.3 is injective when restricted to $\mathfrak{g}'(B)_{\alpha}$ for $\alpha \in \Delta(B)$. Further, the image of $\mathfrak{g}'(B)_{\alpha}$ is contained in $\mathfrak{g}'(A)_{q_\Sigma(\alpha)}$.

**Corollary 2.10.**

1. $q_\Sigma(\Delta^{re}(B)) \subset \Delta^{re}(A)$ and $q_\Sigma(\Delta^{im}(B)) \subset \Delta^{im}(A) \cup \{0\}$.

2. If further $\Sigma$ is linearly independent, then $q_\Sigma(\Delta^{im}(B)) \subset \Delta^{im}(A)$.

**Proof.** Corollary 2.9 implies that if $\alpha$ is a root of $\mathfrak{g}'(B)$, then $q_\Sigma(\alpha)$ is either 0 or a root of $\mathfrak{g}'(A)$. Further, since $(\alpha \mid \alpha) = (q_\Sigma(\alpha) \mid q_\Sigma(\alpha))$, real roots map to real roots and imaginary roots to imaginary roots or 0; since real roots are precisely those roots of positive norm. The second part is obvious from the linear independence assumption, since an imaginary root is nonzero, it cannot map to zero.

The above corollary, for linearly independent $\Sigma$ was first obtained by Naito [22, Theorem 3.8]. Next, we have the converse to Theorem 2.3.

**Proposition 2.11.** Let $A_{n \times n},B_{m \times m}$ be symmetrizable GCMs. Suppose $\phi : \mathfrak{g}'(B) \to \mathfrak{g}'(A)$ is a Lie algebra homomorphism satisfying $0 \neq \phi(e_i) \in \mathfrak{g}'(A)_{\beta_i}$, $0 \neq \phi(f_i) \in \mathfrak{g}'(A)_{-\beta_i}$ for all $1 \leq i \leq m$, for some real roots $\{\beta_i\}_{i=1}^m$ of $\mathfrak{g}'(A)$. Then, the set $\Sigma = \{\beta_i\}_{i=1}^m$ is a $\pi$-system of type $B$ in $A$.

**Proof.** Given a real root $\beta$ and any root $\gamma$ of $\mathfrak{g}'(A)$, it follows from elementary $\mathfrak{sl}_2$ theory (applied to the $\beta$-string through $\gamma$) that

\[(2.7) [\mathfrak{g}'(A)_\beta, \mathfrak{g}'(A)_\gamma] \neq 0 \text{ iff } \beta + \gamma \text{ is a root of } \mathfrak{g}'(A)\]

Now, since $[e_i,f_j] = 0$ for $1 \leq i \neq j \leq m$, we apply $\phi$ to conclude that $[\mathfrak{g}'(A)_{\beta_i}, \mathfrak{g}'(A)_{-\beta_j}] = 0$. Hence $\beta_i - \beta_j$ is not a root of $\mathfrak{g}'(A)$, and $\Sigma$ is thus a $\pi$-system.

Next, we show that the type of this $\pi$-system is exactly $B$. Note that $|\langle \beta_i^\vee, \beta_j \rangle|$ is the largest integer $k$ for which $\beta_j + k'\beta_i$ is a root of $\mathfrak{g}'(A)$ for $0 \leq k' \leq k$. Let $\alpha_i(B)$ denote the simple roots of $\mathfrak{g}'(B)$; their images under $q_\Sigma$ are the $\beta_i$. We have $\ell = |b_{ij}|$ is the largest integer for which $\alpha_j(B) + \ell'\alpha_i(B)$ is a root of $\mathfrak{g}'(B)$ for $0 \leq \ell' \leq \ell$. In fact $\gamma = \alpha_j(B) + \ell\alpha_i(B) \in \Delta^{re}(B)$, and by corollary 2.10, $q_\Sigma(\gamma) \in \Delta^{re}(A)$. Thus, $k \geq \ell$.

By (2.7) above, $[\mathfrak{g}'(B)_{\alpha_i(B)}, \mathfrak{g}'(B)_\gamma] = 0$, and since these two real root spaces map isomorphically to the corresponding real root spaces of $\mathfrak{g}'(A)$, we conclude $[\mathfrak{g}'(A)_{\beta_i}, \mathfrak{g}'(B)_{q_\Sigma(\gamma)}] = 0$. By (2.7) again, $\beta_i + q_\Sigma(\gamma) = \beta_j + (\ell + 1)\beta_i$ is not a root of $\mathfrak{g}'(A)$. Hence $k \leq \ell$, and we obtain $\langle \beta_i^\vee, \beta_j \rangle = b_{ij}$ as required.

**Corollary 2.12.** If $A$ has a $\pi$-system of type $B$ and $B$ has a $\pi$-system of type $C$, then $A$ has a $\pi$-system of type $C$.

**Proof:** Theorem 2.3 gives us Lie algebra morphisms $\mathfrak{g}'(C) \to \mathfrak{g}'(B) \to \mathfrak{g}'(A)$. By corollary 2.9, both these maps are injective on real root spaces. The generators $e_i,f_i$ of $\mathfrak{g}'(C)$ map to real root
vectors of $g'(B)$. Thus, under the composition of these two morphisms, $e_i, f_i$ map to non-zero real root vectors of $g'(A)$. The corresponding roots are clearly negatives of each other. Proposition 2.11 now completes the proof.

If $\Sigma_1, \Sigma_2$ denote the $\pi$-systems of the above corollary, of types $B$ and $C$ respectively, then the $\pi$-system of type $C$ in $A$ that one obtains from the proof above is just $q_{\Sigma_1}(\Sigma_2)$.

2.5. As mentioned in the introduction, $\pi$-systems were first defined by Dynkin in his study of regular semisimple subalgebras of semisimple Lie algebras. In this setting, any set of simple roots of a closed subroot system of the root system (of a semisimple Lie algebra) is a $\pi$-system. The converse is also true, as can be seen from theorem 2.3.

In the infinite dimensional setting, Naito [22] defined a regular subalgebra of a Kac-Moody algebra $g(A)$ to be any subalgebra of the form $i\Sigma(g(B))$ for $\Sigma$ a linearly independent $\pi$-system of type $B$ in $A$, where $B$ varies over all GCMs (cf. Proposition 2.4).

3. Weyl group action on $\pi$-systems

3.1. Let $A$ be a symmetrizable GCM. Let $W(A)$ denote the Weyl group of $A$. It acts on the set of roots of $A$, preserving each of the subsets of real and imaginary roots. Further this action preserves the bilinear invariant form. Thus, there is an induced action of $W(A)$ on the set of all $\pi$-systems in $A$ of a given type $B$.

3.2. When $A$ is of finite type, it is easy to see that every linearly independent $\pi$-system in $A$ is $W(A)$-conjugate to a $\pi$-system contained in the set of positive roots of $A$. To see this, take an element $\gamma \in h^*_R$ which has positive inner product with the elements of the $\pi$-system. The element $w \in W(A)$ which maps $\gamma$ into the dominant Weyl chamber will clearly also map the $\pi$-system to a subset of the positive roots.

This proof fails in the general case; such $w$ does not exist unless $\gamma$ is in the Tits cone. For instance, the negative simple roots of $A$ form a $\pi$-system of type $A$ in $A$. This set cannot be $W(A)$-conjugated to a subset of positive roots if $A$ is not of finite type; this can be seen using for instance [18, Theorem 3.12c]. The next theorem shows that this is essentially the only obstruction.

Theorem 3.1. Let $A, B$ be symmetrizable GCMs and $\Sigma$ a linearly independent $\pi$-system of type $B$ in $A$. If $B$ is indecomposable, then:

1. There exists $w \in W(A)$ such that $w\Sigma \subset \Delta^+_\text{re}(A)$ or $w\Sigma \subset \Delta^\text{re}(A)$.
2. There exist $w_1, w_2 \in W(A)$ such that $w_1\Sigma \subset \Delta^+_\text{re}(A)$ and $w_2\Sigma \subset \Delta^\text{re}(A)$ if and only if $B$ is of finite type.

The proof occupies the next two subsections.

3.3. The proof of theorem 3.1 closely follows that of [18, Proposition 5.9]. The first part of this theorem, in the special case $|\Sigma| = 2$ was proved by Naito in [22]. We first recall some relevant facts about the roots of a Kac-Moody algebra. Let $B$ be an indecomposable GCM, and let $g(B)$ denote the corresponding Kac-Moody algebra. Let $Q(B)$ denote its root lattice. We use the notation introduced already for the sets of roots, real roots, positive roots etc. Let $\mathbb{R}_+$ denote the set of non-negative reals. Define:

$$C^\text{im} = \bigcup_{\alpha \in \Delta^\text{im}(B)} \mathbb{R}_+\alpha, \quad C^\text{re} = \bigcup_{\alpha \in \Delta^\text{re}(B)} \mathbb{R}_+\alpha.$$
We then have the following result due to Kac [17, Proposition 1.8], [18, §5.8]:

**Proposition 3.2.** (Kac) In the metric topology on the real span of $Q(B)$, $C^\text{im}$ is the convex hull of the set of limit points of $C^\text{re}$. In particular, it is a convex cone.

Now suppose $Q(B) \subset E$ for some real vector space $E$. Let $\{\epsilon_i\}_{i=1}^n$ be a basis of $E$. Define $E_+$ to be the $\mathbb{R}_+$ span of the $\epsilon_i$, and let $E_- = -E_+.$

**Lemma 3.3.** If $\Delta(B) \subset E_+ \cup E_-$, then $\Delta^\text{im}_+(B) \subset E_+$ or $\Delta^\text{im}_+(B) \subset E_-.$

*Proof.* Consider the set $C^\text{im}$; it has the following properties: (i) It is convex, by Proposition 3.2. (ii) It is contained in $E_+ \cup E_-$, by the given hypothesis. (iii) It does not contain a line (i.e., for nonzero $x \in E$, both $x$ and $-x$ cannot belong to this set), because $C^\text{im} \subset \mathbb{R}_+(\Delta_+(B))$.

It is easy to see that these properties imply that $C^\text{im}$ must be entirely contained either in $E_+$ or in $E_-$. □

Under the same hypothesis as lemma 3.3 we have:

**Lemma 3.4.** If $\Delta^\text{im}_+(B) \subset E_+$, then all but finitely many real roots of $B$ lie in $E_+$.

*Proof.* First, we define an inner product on $E$ by requiring the $\epsilon_i$ to be an orthonormal basis. This defines the standard metric topology on $E$, and thereby on the $\mathbb{R}$-span of $Q(B)$.

Let $M := \Delta^\text{re}_+(B) \cap E_-$, and $\hat{M} := \{\alpha/\|\alpha\| : \alpha \in M\}$. Here, the norm is that of the Euclidean space $E$. Observe that $\hat{M}$ is a subset of $C^\text{re} \cap E_- \cap S$, where $S$ is the unit sphere in $E$. If $\hat{M}$ is an infinite set, then, it has a limit point, say $\zeta$. Now $\zeta \in E_- \cap S$, and by Proposition 3.2, $\zeta \in C^\text{im}$. But $C^\text{im} \subset E_+$ by hypothesis. This contradiction establishes the lemma. □

**Proposition 3.5.** Let $\Delta(B) \subset E_+ \cup E_-$. There exists $w \in W(B)$ such that $w\Delta_+(B) \subset E_+$ or $w\Delta_+(B) \subset E_-.$

*Proof.* By lemma 3.3, the positive imaginary roots are all contained in $E_+$ or in $E_-; we may suppose (replacing the $\epsilon_i$ with their negatives if need be) that $\Delta^\text{im}_+(B) \subset E_+.$ Consider $F := \Delta^\text{re}_+(B) \cap E_-; this is finite by lemma 3.4. If this set is non-empty, it contains some simple root $\alpha$ of $g(B).$ Since the simple reflection $s_\alpha$ defines a bijective self-map of $\Delta^\text{re}_+(B) \setminus \{\alpha\}$, it is clear that $F' := s_\alpha (\Delta^\text{re}_+(B)) \cap E_- contains one fewer element than $F.$ Iterating this procedure, we can find $w$, a product of simple reflections, such that $w\Delta^\text{re}_+(B) \cap E_-$ is empty, as required. □

3.4. Finally, we are in a position to prove theorem 3.1: With notation as in the theorem, observe that the linear independence of $\Sigma$ implies that $q_\Sigma : Q(B) \rightarrow Q(A)$ is injective. By corollary 2.10, $q_\Sigma (\Delta(B)) \subset \Delta(A) = \Delta_+(A) \cup \Delta_-(A).$ We define $E$ to be the $\mathbb{R}$-span of $\Delta(A)$ and take $\{\epsilon_i\}$ to be the basis of simple roots of $g(A).$ Then, clearly, $q_\Sigma (\Delta(B)) \subset E_+ \cup E_-.$ Identifying $\Delta(B)$ with its image under $q_\Sigma$, and appealing to proposition 3.5 completes the proof of part 1.

To prove part (2), since $w_1\Sigma \subset \Delta^\text{re}_+(A),$ we have $w_1(q_\Sigma (\Delta_+(B))) \subset \Delta_+(A).$ Consider the set $R := q_\Sigma (\Delta^\text{im}_+(B)).$ We have (i) $R \subset \Delta^\text{im}_+(A)$, by corollary 2.10, and (ii) $w_1 R \subset \Delta_+(A).$ Since the sets $\Delta^\text{im}_+(A)$ are both $W(A)$-invariant, this implies $R \subset \Delta^\text{im}_+(A).$ Similarly, from $w_2\Sigma \subset \Delta^\text{re}_+(A),$ we conclude $R \subset \Delta^\text{im}_+(A).$ This means $R$ is empty, or in other words, that $B$ is of finite type.

Conversely, if $B$ is of finite type, then $\Delta_+(B)$ is finite. Hence its intersections with $\Delta_+(A)$ and $\Delta_-(A)$ are both finite sets. The proof of Proposition 3.5 shows that there exist elements of $W(A)$ which map $\Delta_+(B)$ to subsets of $\Delta_+(A).$ □
Lemma 4.1. Let \( S \) denote the set of vertices of \( \Sigma \). Let \( \text{supp} \) denote the null root of \( \Sigma \). We write \( Y \) to be the set \( \{ \alpha \in \Delta^m(A) \mid (\alpha | \delta_y) = 0 \} \) for every \( \alpha \in \Delta^m(B) \). Theorem 4.2. Let \( A, B \) be a symmetrizable GCM and \( B \) is indecomposable and not of finite type, then \( \Sigma \) is either positive or negative. We record below a simple criterion to determine the sign that was obtained in the course of the proof of theorem 3.1.

Proposition 3.6. Let \( A, B \) be symmetrizable GCMs, with \( B \) indecomposable and not of finite type. Let \( \Sigma \) be a linearly independent \( \pi \)-system of type \( B \) in \( A \). Then the following are equivalent:

1. \( \Sigma \) is positive (resp. negative).
2. \( q_\Sigma(\alpha) \in \Delta^+_{\text{im}}(A) \) (resp. \( \Delta^-_{\text{im}}(A) \)) for every \( \alpha \in \Delta^+_{\text{im}}(B) \).
3. \( q_\Sigma(\alpha) \in \Delta^+_{\text{im}}(A) \) (resp. \( \Delta^-_{\text{im}}(A) \)) for some \( \alpha \in \Delta^+_{\text{im}}(B) \).

\( \square \)

3.7. Let \( m(B, A) \) denote the number of \( W(A) \)-orbits of \( \pi \)-systems of type \( B \) in \( A \) (this could be infinity in general). When \( A, B \) are of finite type, Borel-de Siebenthal and Dynkin determined the pairs for which \( m(B, A) > 0 \). Dynkin went further, and also determined the values of \( m(B, A) \); these turn out to be 1 for almost all cases, except for a few where it is 2 [7, Tables 9-11]

4. \( \pi \)-SYSTEMS OF AFFINE TYPE

4.1. Let \( S(A) \) denote the Dynkin diagram associated to the GCM \( A \) [18]. Any subset of the vertices of \( S(A) \) together with the edges between them will be called a subdiagram of \( S(A) \) (and we will use \( \subseteq \) to denote the relation of being a subdiagram). Given \( \alpha = \sum_{i=1}^n c_i \alpha_i \), we define \( \text{supp} \alpha \) to be the set \( \{ i : c_i \neq 0 \} \) and view it as a subset of the vertices of \( S(A) \). Given a subdiagram \( Y \) of \( S(A) \), we say \( \alpha \) is supported in \( Y \) if \( \text{supp} \alpha \) is contained in the set of vertices of \( Y \). We also let \( Y^\perp \) denote the set of vertices of \( S(A) \) that are not connected by an edge to any vertex of \( Y \).

Lemma 4.1. Let \( A \) be a symmetrizable GCM and \( Y \) a subdiagram of \( S(A) \) of affine type. Let \( \delta_Y \) denote the null root of \( Y \). If \( \beta \in \Delta(A) \) is such that \( (\beta | \delta_Y) = 0 \), then \( \text{supp} \beta \subseteq Y \cup Y^\perp \).

Proof. We write \( \beta = \sum_{p \in S(A)} c_p \alpha_p \), where all the coefficients are non-negative, or all non-positive. Let \( \text{supp} \beta \) denote the set of \( p \) for which \( c_p \) is nonzero. Now, \( (\alpha_p | \delta_Y) \) is 0 for \( p \in Y \), and \( \leq 0 \) when \( p \notin Y \). Since all coefficients are of the same sign, every \( p \in \text{supp} \beta \) must be either in \( Y \) or in \( Y^\perp \).

\( \square \)

Theorem 4.2. Let \( A \) be a symmetrizable GCM and \( B \) be a GCM of affine type. Suppose \( \Sigma \) is a linearly independent \( \pi \)-system of type \( B \) in \( A \). Then,

1. There exists an affine subdiagram \( Y \) of \( S(A) \) and \( w \in W(A) \) such that every element of \( w\Sigma \) is supported in \( Y \).
2. Suppose \( \langle Y', w' \rangle \) is another such pair, i.e., with \( Y' \) a subdiagram of affine type, \( w' \in W(A) \) such that \( w'\Sigma \) is supported in \( Y' \). Then \( Y = Y' \) and \( w'w^{-1} \in W(Y \cup Y^\perp) \).
3. \( m(B, A) = \infty \).

Proof. Let \( \Sigma = \{ \beta_i \}_1^n \). Let \( \{ \alpha_i(B) \}_1^n \) denote the simple roots of \( \mathfrak{g}(B) \) and let \( \delta_B \) denote its null root. Let \( \delta_\Sigma = q_\Sigma(\delta_B) \). By corollary 2.10(2) and the fact that \( q_\Sigma \) preserves forms, we obtain that \( \delta_\Sigma \) is an isotropic root of \( \mathfrak{g}(A) \). By [18 Proposition 5.7], there exists \( w \in W(A) \) such that \( w(\delta_\Sigma) \) is
supported on an affine subdiagram $Y$ of $S(A)$ and $w(\delta_\Sigma) = k\delta_Y$ for some nonzero integer $k$, where $\delta_Y$ is the null root of $Y$.

Now, $0 = (\alpha_i(B) \mid \delta_B) = (\beta_i \mid \delta_\Sigma) = k (w\beta_i \mid \delta_Y)$ for all $i = 1, \ldots, m$. We conclude $\text{supp } w\beta_i \subseteq Y \sqcup Y^\perp$, by lemma 4.1. Since $w\beta_i$ is a root, its support is connected, and hence contained entirely in $Y$ or entirely in $Y^\perp$. However, $w\Sigma$ is a $\pi$-system of type $B$, an indecomposable GCM. So, $w\Sigma$ cannot be written as a disjoint union of two mutually orthogonal subsets. This means that either $\text{supp } w\beta_i \subseteq Y$ for all $i$, or $\text{supp } w\beta_i \subseteq Y^\perp$ for all $i$. The latter is impossible since $k\delta_Y = w(\delta_\Sigma)$ is a positive integral combination of the $w\beta_i$. This proves part (1).

Now, if $(Y', w')$ is another such pair, then since the only isotropic roots of $g(A)$ supported on subsets of $Y'$ are the multiples of $\delta_{Y'}$, we obtain $w'(\delta_\Sigma) = k'\delta_{Y'}$, for $k' \neq 0$. Define $\sigma = w'w^{-1}$, so $\sigma(k\delta_Y) = k'\delta_{Y'}$. Since $\delta_Y$ is a positive imaginary root of $g(A)$, so is $\sigma\delta_Y$; thus $k$ and $k'$ have the same sign. We may suppose $k, k' > 0$. Now $k\delta_Y$ and $k'\delta_{Y'}$ are antidominant weights (i.e., their negatives are dominant weights) of $g(A)$, which are $W(A)$-conjugate. By [18, Proposition 5.2b], we get $k\delta_Y = k'\delta_{Y'}$. Thus, $Y = Y'$, $k = k'$ and $\sigma\delta_Y = \delta_Y$.

Since $\delta_Y$ is antidominant, the simple reflections that fix $\delta_Y$ generate the stabilizer of $\delta_Y$. By lemma 4.1, this stabilizer is just $W(Y \sqcup Y^\perp)$. Thus $\sigma \in W(Y \sqcup Y^\perp)$, proving part (2).

Finally, let $\Sigma = w\Sigma$ denote the $\pi$-system of part (1). Now $Y$ is of affine type, untwisted or twisted. In either case, from the description of the real roots of an affine Kac-Moody algebra [18, Chap 6], the following holds: $\Delta^{re}(Y) + 6p\delta_Y \subseteq \Delta^{re}(Y)$ for all $p \in \mathbb{Z}$. Consider

$$\Sigma_p := \{ \alpha + 6p\delta_Y : \alpha \in \Sigma \} \text{ for } p \in \mathbb{Z}.$$ Since $\delta_Y$ is orthogonal to every root of $g(Y)$, it is clear that $\Sigma_p$ is a linearly independent $\pi$-system of type $B$ in $A$, supported in $Y$. From the proof of part (1), we know $q_\Sigma(\delta_B) = k\delta_Y$ for some nonzero integer $k$. From the definition of $\Sigma_p$, we obtain

$$q_{\Sigma_p}(\delta_B) = (k + 6ph)\delta_Y$$

where $h$ is the Coxeter number of the affine Kac-Moody algebra $g(B)$. We claim that the $\Sigma_p$ are pairwise $W(X)$-inequivalent. Suppose $\Sigma_m$ and $\Sigma_n$ are in the same $W(X)$-orbit. Then, from part (2), we obtain $\Sigma_m = \sigma(\Sigma_n)$ for some $\sigma \in W(Y \sqcup Y^\perp)$. In particular, this means $q_{\Sigma_m}(\delta_B) = \sigma(q_{\Sigma_n}(\delta_B))$. Since $\sigma$ fixes $\delta_Y$, equation 4.1 implies $m = n$. This completes the proof of part (3).

Corollary 4.3. Let $A$ be a symmetrizable GCM such that $S(A)$ has no subdiagrams of affine type. Then $A$ contains no linearly independent $\pi$-systems of affine type.

This follows immediately from the proposition. We remark that Figure 5.2 contains examples of such $S(A)$.

Remark 4.4. (1) The conclusion of theorem 4.2 is false without the linear independence assumption, as in Example 2.6(ii), (iii).
(2) Let $A, B$ be symmetrizable GCMs, with $B$ of affine type. Suppose $A$ contains a linearly independent $\pi$-system of type $B$. Theorem 4.2 implies that some affine type subdiagram $Y$ of $S(A)$ also contains a linearly independent $\pi$-system of type $B$. This allows us to determine the possible set of such $B$ in two steps: (i) find all affine subdiagrams $Y$ of $S(A)$, and (ii) for each such $Y$, list out all the $B$’s which occur as GCMs of linearly independent $\pi$-systems of $Y$. 

12
We note that step (ii) above can in-principle be carried out using the results of [25] (see also [6,12,22]).

5. Hyperbolics and Overextensions

5.1. Let $A$ be a symmetrizable GCM and $X = S(A)$ be its Dynkin diagram. If $A$ is symmetric, we will call $X$ simply-laced.

**Definition 5.1.** Let $Z$ be a simply-laced Dynkin diagram. We say that $Z$ is an overextension or of Ext type if there exists a vertex $p$ in $Z$ such that the subdiagram $Y = Z \setminus \{p\}$ is of affine type and $(\delta_Y \mid \alpha_p) = -1$.

We let Ext denote the set of overextensions. It is easy to see that the following is the complete list of overextensions, up to isomorphism:

$A_1^{++}$

$A_n^{++}$ ($n \geq 2$)

$D_n^{++}$ ($n \geq 4$)

$E_6^{++}$

$E_7^{++}$

$E_8^{++}$

(see Figure 5.1). Here, $X_n^{++}$ has $n + 2$ vertices. We remark that the corresponding GCMs are all nonsingular; hence a $\pi$-system of Ext type is necessarily linearly independent.

5.2. From figure 5.1 one makes the important observation (via case-by-case check) that if $Z$ is an overextension, then the vertex $p$ satisfying the condition in definition 5.1 is unique. This vertex is marked by a dashed circle in figure 5.1. We will call $p$ the overextended vertex of $Z$, and $Y$ the affine part of $Z$.

We had $(\delta_Y \mid \alpha_p) = -1$. Let $\delta_Y = \sum_{q \in Y} c_q \alpha_q$ with $c_q \in \mathbb{Z}_+$ for all $q$. Observing that $c_q (\alpha_q \mid \alpha_p) \leq 0$ for all $q$, it follows that: (i) There is a unique vertex $q$ of $Y$ such that $(\alpha_q \mid \alpha_p) \neq 0$, (ii) For this vertex, we have $c_q = 1$ and $(\alpha_q \mid \alpha_p) = -1$, (iii) In particular, this means $q$ is a special vertex of the affine diagram $Y$ (in the terminology of Kac, Chapter 6). Let $Z^o$ denote the finite type diagram obtained from $Y$ by deleting $q$. We will call it the finite part of $Z$. We note that:

$$\delta_Y = \alpha_q + \theta_{Z^o}$$
where $\theta_{Z^o}$ denotes the highest root of $Z^o$. It will be convenient to denote $Y$ by $\hat{Z}$.

5.3. The following trivial observation is useful: let $X$ be a simply-laced Dynkin diagram and $Z$ a diagram of Ext type. Suppose there exists $\pi$, a $\pi$-system of type $Z$ in $X$; we let $\pi^o, \hat{\pi}^o$ denote the subsets of $\pi$ corresponding to the finite and affine parts of $Z$ respectively. For any $w \in W(X)$, $w\pi^o$ is a $\pi$-system of type $Z$ in $X$ and $(w\pi^o) = w(\pi^o), w\hat{\pi}^o = w(\hat{\pi}^o)$.

5.4. Hyperbolics. We recall that an indecomposable, symmetrizable GCM $A$ is said to be of Hyperbolic type if it is not of finite or affine type and every proper principal submatrix of $A$ is a direct sum of finite or affine type GCMs.

There are finitely many GCMs of hyperbolic type in ranks 3-10 and infinitely many in rank 2. The former were enumerated, to varying degrees of completeness and detail, in [5,20,26]. More recently, this list was organized and independently verified in [2]. We will use this latter reference as our primary source for the Dynkin diagrams of hyperbolic type. Note that [2] does not require symmetrizability in the definition of a hyperbolic type GCM, so it contains 142 symmetrizable and 96 non-symmetrizable ones. We let Hyp denote the set of all symmetrizable GCMs of hyperbolic type of rank $\geq 3$.

5.5. We recall from §4.1 the subdiagram partial order on the set of symmetrizable GCMs. We write $B \subseteq A$ if the Dynkin diagram $S(B)$ is a subdiagram of $S(A)$; equivalently $B$ is a principal submatrix of $A$, possibly after a simultaneous permutation of its rows and columns. This is clearly a partial order, once we identify the matrices $\{PAP^T : P$ is a permutation matrix$\}$ with each other.

5.6. We now isolate the symmetric GCMs of hyperbolic type. By checking the classification case-by-case (see for instance [28, Tables 1,2] or [2]), one finds that these are either (i) of Ext type:

(5.1) $A^{++}_n, (1 \leq n \leq 7), D^{++}_n, (4 \leq n \leq 8), E^{++}_n, (6 \leq n \leq 8)$

or (ii) one of the diagrams in Figure 5.2 or (iii) one of the rank 2 symmetric GCMs $\begin{bmatrix} 2 & -a \\ -a & 2 \end{bmatrix}$ for $a \geq 3$. We observe by inspection of figure 5.1 that the diagrams in (ii) and (iii) do not contain a subdiagram of Ext type.

Figure 5.2. Simply-laced hyperbolics (ranks 3-10) that are not of Ext type.

5.7. The next lemma underscores the special role played by the hyperbolic overextensions. These are precisely the minimal elements of the set of overextensions relative to the partial order $\subseteq$.

**Lemma 5.2.**  
$\min(\text{Ext, } \subseteq) = \text{Ext} \cap \text{Hyp}$
Proof. Observe that \( E_7^{++} \subset A_n^{++} \) for \( n \geq 8 \) and \( E_8^{++} \subset D_n^{++} \) for \( n \geq 9 \). We are thus left with the diagrams of equation (5.1) as possible candidates for minimal elements. Now, each of these diagrams except \( D_8^{++} \) contains a unique subdiagram of affine type, obtained by removing a single vertex. So these diagrams cannot contain a proper subdiagram of Ext type. As for the diagram \( Z = D_8^{++} \), it contains two subdiagrams of affine type, \( Y_1 = E_8(1) \) and \( Y_2 = D_8(1) \), obtained by deleting appropriate vertices \( p_1, p_2 \), but only the former satisfies \( (\delta_Y \mid \alpha_p) = -1 \) (this is \(-2\) for the latter). Thus, \( D_8^{++} \) is also minimal. \( \square \)

6. Weyl group orbits of \( \pi \)-systems of type \( A_1^{++} \)

In this section, we focus on the diagram \( A_1^{++} \). The corresponding Kac-Moody algebra was first studied by Feingold and Frenkel [10].

6.1. We consider the problem of determining \( m(A_1^{++}, X) \) for a simply-laced Dynkin diagram \( X \). This is an important special case of the more general result of the next section. The latter result will be obtained by arguments similar to the ones used here, albeit with more notational complexity.

6.2. We begin with the following lemma which asserts that every Dynkin diagram of Ext type has a “canonical” \( \pi \)-system of type \( A_1^{++} \).

**Lemma 6.1.** Given a Dynkin diagram \( Z \) of Ext type, define:

\[
\pi(Z) := \{ \theta_{Z^o}, \delta_Y - \theta_{Z^o}, \alpha_p \}
\]

(notations \( Z^o, Y, p, \theta_{Z^o} \) are as defined in (5.2)). Then \( \pi(Z) \) is a linearly independent, positive \( \pi \)-system of type \( A_1^{++} \).

**Proof.** We only need to show that the type of \( \pi(Z) \) is \( A_1^{++} \), the other assertions following from the observation that the three roots in \( \pi(Z) \) are real, positive and have disjoint supports (cf. [5.2]). Since \( Z \) is simply-laced, we normalize the form such that all real roots have norm 2. Thus

\[
(\theta_{Z^o} \mid \delta_Y - \theta_{Z^o}) = - (\theta_{Z^o} \mid \theta_{Z^o}) = -2
\]

It is clear from [5.2] that \( (\theta_{Z^o} \mid \alpha_p) = 0 \) and \( (\delta_Y \mid \alpha_p) = -1 \). This completes the verification. \( \square \)

**Theorem 6.2.** Let \( X \) be a simply-laced Dynkin diagram. Then:

1. \( X \) has a \( \pi \)-system of type \( A_1^{++} \) if and only if it contains a subdiagram of Ext type.
2. The number of \( W(X) \)-orbits of \( \pi \)-systems of type \( A_1^{++} \) in \( X \) is twice the number of such subdiagrams (and is, in particular, finite).

**Proof.** In light of Theorem 3.1 any \( \pi \)-system of type \( A_1^{++} \) in \( X \) is \( W(X) \)-equivalent to a positive or a negative \( \pi \)-system, but not both. Thus, to prove the above theorem, it is sufficient to construct a bijection from the set of Ext type subdiagrams of \( X \) to \( W(X) \)-equivalence classes of positive \( \pi \)-systems of type \( A_1^{++} \) in \( X \). We claim that the following map defines such a bijection:

\[
Z \mapsto [\pi(Z)]
\]

We will first establish the injectivity. Suppose \( Z_1, Z_2 \) are Ext type subdiagrams of \( X \), with affine parts \( Y_1, Y_2 \) and overextended vertices \( p_1, p_2 \) respectively. Suppose \( \pi(Z_1) \sim \pi(Z_2) \) i.e., there exists \( \sigma \in W(X) \) such that \( \sigma(\pi(Z_1)) = \pi(Z_2) \). Consider the \( \pi \)-systems:

\[
\pi_j = \{ \theta_{Z_j^o}, \delta_{Y_j} - \theta_{Z_j^o} \}, \quad j = 1, 2.
\]

We note that:
(1) \( \pi_j \) is of type \( A_1^{(1)} \).
(2) \( \pi_j \) is supported in the affine subdiagram \( Y_j \) of \( X \).
(3) \( \sigma(\pi_1) = \pi_2 \).

Now, it follows from part (2) of theorem \[4.2\] that \( Y_1 = Y_2 \) and \( \sigma \in W(Y_1 \sqcup Y_1^\perp) \). Since \( p_1 \notin Y_1 \sqcup Y_1^\perp \), we can only have \( \sigma\alpha_{p_1} = \alpha_{p_2} \) if \( p_1 = p_2 \). Thus, \( Z_1 = Z_2 \) as required.

Next, we turn to the surjectivity of this map. Let \( \{\beta_{-1}, \beta_0, \beta_1\} \) be a positive \( \pi \)-system of \( X \) of type \( A_1^{++} \). Since \( \{\beta_0, \beta_1\} \) form a \( \pi \)-system of type \( A_1^{(1)} \), which is affine, it follows from theorem \[4.2\] that there is a unique affine type subdiagram \( Y \) of \( X \) and an element \( w \in W(X) \) such that \( w\beta_i \) is supported in \( Y \) for \( i = 0, 1 \). Further (as in the proof of theorem \[4.2\]), since \( w(\beta_0 + \beta_1) \) is an isotropic root of \( g(X) \), we must have \( w(\beta_0 + \beta_1) = k\delta_Y \) for some nonzero integer \( k \). Since \( (\beta_0 + \beta_1 | \beta_{-1}) = -1 \), we conclude \( k = \pm 1 \). But \( \beta_0 + \beta_1 \in Q_+(X) \) by proposition \[3.6\], and \( w^{-1}(\delta_Y) \in \Delta^{\text{int}}_1 \) since \( \delta_Y \) is a positive imaginary root. This implies \( k = 1 \).

Let \( \beta'_i = w\beta_i \); thus \( \beta'_0, \beta'_1 \) are supported in \( Y \), their sum equals \( \delta_Y \) and \( (\delta_Y | \beta'_{-1}) = -1 \). We now need the following lemma:

**Lemma 6.3.** Let \( X \) be a simply-laced Dynkin diagram, \( Y \) an affine subdiagram of \( X \) and \( \beta \) a real root of \( X \) satisfying \( (\delta_Y | \beta) = -1 \). Then there exists \( \sigma \in W(Y \sqcup Y^\perp) \) such that \( \sigma\beta \) is a simple root of \( X \).

We defer the proof of this lemma to the next subsection. Here, we use it to complete the proof of Theorem \[6.2\] We take \( \beta = \beta'_{-1} \) in lemma \[6.3\]. We obtain \( \sigma \in W(Y \sqcup Y^\perp) \) such that \( \sigma\beta'_{-1} = \alpha_p \) for some vertex \( p \) of \( X \). Define \( Z := Y \sqcup \{p\} \). Since \( \sigma \) stabilizes \( \delta_Y \), we have \( (\delta_Y | \alpha_p) = -1 \); thus \( Z \) is of Ext type.

Since \( \beta'_0, \beta'_1 \) are supported in \( Y \), so are \( \sigma\beta'_0, \sigma\beta'_1 \); further \( \sigma\beta'_0 + \sigma\beta'_1 = \delta_Y \). Now

\[
(\sigma\beta'_1, \alpha_p) = (\sigma\beta'_1, \sigma\beta'_{-1}) = 0.
\]

This implies that \( \sigma\beta'_1 \) is supported in \( Z^\circ \). Since \( Z^\circ \) is a simply-laced finite type diagram, all its real roots are conjugate under its Weyl group. Thus, there exists \( \tau \in W(Z^\circ) \) such that \( \tau\sigma\beta'_1 = \theta_{Z^\circ} \). Since \( \tau \) stabilizes both \( \delta_Y \) and \( \alpha_p \), we conclude that \( \{\tau\sigma\beta'_i : i = -1, 0, 1\} = \pi(Z) \), as required.

6.3. We now turn to the proof of Lemma \[6.3\] We use the notations of the lemma. Since \( \delta_Y \) is an antidominant weight of \( X \), \( \beta \) must be a positive root. Further it is clear from \( (\delta_Y | \beta) = -1 \) that \( \beta \) must have the form:

\[
(6.1) \quad \beta = \alpha_p + \sum_{q \in Y \sqcup Y^\perp} c_q(\beta)\alpha_q
\]

where \( p \) is a vertex of \( X \) such that \( (\delta_Y | \alpha_p) = -1 \), and \( c_q(\beta) \) are non-negative integers. Consider the \( W(Y \sqcup Y^\perp) \)-orbit of \( \beta \). Since the coefficient of \( \alpha_p \) remains the same, any element \( \gamma \) of this orbit is a positive root that has the same form as the right hand side of \( (6.1) \) for some non-negative coefficients \( c_q(\gamma) \). Let \( \gamma \) be a minimal height element of this orbit, i.e., one for which \( \sum q c_q(\gamma) \) is minimal. Then, we have: (i) \( (\gamma | \alpha_q) \leq 0 \) for all \( q \in Y \sqcup Y^\perp \), since otherwise \( s_\gamma \gamma \) would have strictly smaller height, (ii) \( (\gamma | \gamma) = (\alpha_p | \alpha_p) \) since all real roots have the same norm \( (X \text{ is simply-laced}) \). We compute:

\[
0 = (\gamma + \alpha_p | \gamma - \alpha_p) = \sum_{q \in Y \sqcup Y^\perp} c_q(\gamma) (\gamma + \alpha_p | \alpha_q)
\]
Let 6.5. We now have the following corollary of Theorem 6.2.

Remark 6.5. (1) If \( W \) is a subdiagram of \( X \) which have the form of equation (6.1) forms a single orbit under the standard parabolic subgroup \( W(Y \cup Y^\perp) \) of \( W \). In fact, those very same arguments prove a strengthened assertion. We formulate this below.

Given a Dynkin diagram \( X \) with simple roots \( \alpha_i \) and given any \( \alpha \) in its root lattice, we define the coefficients \( c_i(\alpha) \) by:

\[
\alpha = \sum_{i \in X} c_i(\alpha) \alpha_i
\]

If \( J \) is a subdiagram of \( X \), we define \( \alpha_J = \sum_{i \in J} c_i(\alpha) \alpha_i \) and \( \alpha^+_J = \sum_{i \not\in J} c_i(\alpha) \alpha_i \).

**Proposition 6.4.** Let \( X \) be a symmetrizable Dynkin diagram with invariant bilinear form \((\cdot | \cdot)\) and simple roots \( \alpha_i \). Let \( J \) be a subdiagram of \( X \), and fix a nonzero element \( \zeta = \sum_{i \not\in J} b_i \alpha_i \) of the root lattice of \( X \setminus J \). Consider the set

\[
O = \{ \beta \in \Delta^+_e(X) : \beta^+_J = \zeta \text{ and } (\beta | \beta) = (\zeta | \zeta) \}
\]

Then:

(1) If \( \zeta \) is a root of \( g(X \setminus J) \), then \( O = W_J \zeta \) where \( W_J \) is the standard parabolic subgroup \((s_j : j \in J)\) of \( W \).

(2) If \( \zeta \) is not a root of \( g(X \setminus J) \), then \( O \) is empty.

**Proof.** Suppose \( O \) is non-empty, then \( \zeta \) or \(-\zeta \) lies in \( Q_+(X \setminus J) \). We may assume the former case holds, so in fact \( O \subset \Delta^+_e(X) \). Since \( O \) is \( W_J \)-stable, it decomposes into \( W_J \)-orbits. Let \( O' \) denote one such orbit. Let \( \beta \) denote an element of minimal height in \( O' \); as in the proof of Lemma 6.3, this implies \( (\beta | \alpha_j) \leq 0 \) for all \( j \in J \); hence \( (\beta | \alpha) \leq 0 \) for all elements \( \alpha \in Q_+(J) \). We now have

\[
0 = (\beta + \zeta | \beta - \zeta) = (\beta + \beta^+_J | \beta_J).
\]

But as observed already, \( (\beta | \beta_J) \leq 0 \); further \( (\beta^+_J | \beta_J) \leq 0 \) since these elements have disjoint supports. This implies \( (\beta | \beta_J) = (\beta^+_J | \beta_J) = 0 \). Suppose \( \beta_J \) is nonzero, the latter implies that \( \beta = \beta_J + \beta^+_J \) has disconnected support. Hence it cannot be a root. This contradiction shows \( \beta_J = 0 \), i.e., \( \beta = \beta^+_J = \zeta \). In particular, \( \zeta \) is a root, and belongs to any \( W_J \) orbit in \( O \). Hence \( O = W_J \zeta \). \( \square \)

**Remark 6.5.**

(1) If \( X \) is simply-laced and \( J \) is a singleton, say \( J = \{p\} \), and \( \zeta = \alpha_p \), then \( O \) consists precisely of those real roots \( \beta \) of \( X \) which have the form of equation (6.1).

(2) If \( X \) is of finite type and \( \zeta \) is a root of \( X \setminus J \), then Proposition 6.4 is a consequence of Oshima’s lemma [24, Lemma 4.3], [6, Lemma 1.2].

6.5. We now have the following corollary of Theorem 6.2.

**Corollary 6.6.** Let \( X \) be a Dynkin diagram of Ext type. Then:

(1) If \( X \in \text{Hyp} \), then there are exactly two \( \pi \)-systems of type \( A_1^{++} \) in \( X \), up to \( W(X) \)-equivalence. In other words:

\[
m(A_1^{++}, X) = 2 \text{ for } X = A_1^{++} (1 \leq n \leq 7), \ D_4^{++} (4 \leq n \leq 8), \ E_6^{++} (n = 6, 7, 8).
\]

(2) \( m(A_1^{++}, A_8^{++}) = 6, \ m(A_1^{++}, A_9^{++}) = 10 \) for \( n \geq 9 \).
(9) \( m(A_1^{++}, D_9^{++}) = 6, m(A_1^{++}, D_n^{++}) = 4 \) for \( n \geq 10 \).

**Proof:** The first part follows from Lemma 5.2 and Theorem 6.2. For parts (2), (3), we need to count the number of subdiagrams of the ambient diagram which are of Ext type. We list these out in each case, leaving the easy verification to the reader.

1. \( A_{8}^{++} \): one subdiagram of type \( A_{8}^{++} \) and two of type \( E_{7}^{++} \).
2. \( A_{n}^{++} \) (\( n \geq 9 \)): one subdiagram of type \( A_{n}^{++} \) and two each of types \( E_{7}^{++} \) and \( E_{8}^{++} \).
3. \( D_{9}^{++} \): one subdiagram of type \( D_{9}^{++} \) and two of type \( E_{8}^{++} \).
4. \( D_{n}^{++} \) (\( n \geq 10 \)): one subdiagram of type \( D_{n}^{++} \) and one of type \( E_{8}^{++} \).

We also have the following result concerning the simply-laced hyperbolic diagrams not included in the previous corollary.

**Corollary 6.7.** Let \( X \) be a simply-laced hyperbolic Dynkin diagram. If \( X \not\in \text{Ext} \), then \( X \) does not contain a \( \pi \)-system of type \( A_1^{++} \).

**Proof:** This follows from the observation made in §5.6 that such diagrams do not contain subdiagrams of Ext type.

Finally, we remark that Theorem 6.2 can be applied just as easily even when \( X \) is neither in Ext nor Hyp. For example, the diagram \( X = E_{11} \), obtained by further extension of \( E_{8}^{++} \) contains a unique subdiagram of Ext type, namely \( E_{8}^{++} \). Thus, \( m(A_1^{++}, E_{11}) = 2 \).

### 7. The General Case

**Theorem 7.1.** Let \( X \) be a simply-laced Dynkin diagram and let \( K \) be a diagram of Ext type. Then:

1. There exists a \( \pi \)-system in \( X \) of type \( K \) if and only if there exists an Ext type subdiagram \( Z \) of \( X \) such that \( Z^o \) has a \( \pi \)-system of type \( K^o \).
2. The number of \( W(X) \) orbits of \( \pi \)-systems of type \( K \) in \( X \) is given by:

\[
(7.1) \quad m(K, X) = 2 \sum_{Z^o \in X, Z^o \in \text{Ext}} m(K^o, Z^o)
\]

where \( K^o, Z^o \) denote their finite parts.

We remark that equation (7.1) reduces the computation of the multiplicity of \( K \) in \( X \) to a sum of multiplicities involving only finite type diagrams. The latter, as mentioned earlier, are completely known [7]. Observe also that for \( K = A_1^{++}, K^o \) is of type \( A_1 \). Since any \( Z^o \) occurring on the right hand side of (7.1) is simply-laced, we have \( m(K^o, Z^o) = 1 \). So this reduces exactly to Theorem 6.2 in this case.

**Corollary 7.2.** Let \( K \) be a Dynkin diagram of Ext type. Then,

1. \( m(K, X) \) is finite for all simply-laced diagrams \( X \).
2. \( m(K, X) = 2 m(K^o, X^o) \) for all \( X \in \text{Hyp} \cap \text{Ext} \).

We now prove theorem 7.1.
Proof. It is enough to prove the second part of the theorem. Now, by Theorem 3.1 any $\pi$-system in $X$ of type $K$ is either positive or negative, but not both. Consider the sets:

- $\mathcal{A}$: the set of $W(X)$-orbits of positive $\pi$-systems of type $K$ in $X$;
- $\hat{\mathcal{B}}$: the set of all pairs $(Z, \Sigma)$ where $Z$ is an Ext type subdiagram of $X$ and $\Sigma$ is a positive $\pi$-system of type $K^\circ$ in $Z^\circ$.
- $\mathcal{B} = \hat{\mathcal{B}}/\sim$, the equivalence classes of $\hat{\mathcal{B}}$ under the equivalence relation defined by: $(Z, \Sigma) \sim (Z', \Sigma')$ if and only if $Z = Z'$ and $\Sigma'$ is in the $W(Z^\circ)$-orbit of $\Sigma$.

Since $2|\mathcal{A}|$ and $2|\mathcal{B}|$ are the two sides of equation (7.1), it is sufficient to construct a bijection from the set $\mathcal{B}$ to $\mathcal{A}$. We first define a map from $\hat{\mathcal{B}}$ to $\mathcal{A}$. Let $(Z, \Sigma) \in \hat{\mathcal{B}}$. Let $Z^\circ$ and $\hat{Z}_1^\circ$ denote the finite and affine parts of $Z$, and let $p$ denote its overextended vertex. Since $\Sigma$ is a $\pi$-system of type $K^\circ$ in $Z^\circ$, we identify $\Delta(K^\circ)$ with a subset of $\Delta(Z^\circ)$ via corollary 2.10. Let $\theta_\Sigma$ denote the highest root in $\Delta(K^\circ)$ (identified with its image in $\Delta(Z^\circ) \subset Q(Z)$). Consider the set

$$\pi(Z, \Sigma) = \{\alpha_p, \delta_{\hat{Z}_2^\circ} - \theta_\Sigma\} \cup \Sigma$$

It is straightforward to see that this is a $\pi$-system. Further, it is of type $K$.

We now claim that the map: $\hat{\mathcal{B}} \to \mathcal{A}$, $(Z, \Sigma) \mapsto [\pi(Z, \Sigma)]$ factors through $\mathcal{B}$ and defines a bijection between $\mathcal{B}$ and $\mathcal{A}$.

Firstly, suppose $(Z, \Sigma) \sim (Z', \Sigma')$, i.e., $w \Sigma = \Sigma'$ for some $w \in W(Z^\circ)$. Since clearly $w\alpha_p = \alpha_p$, $w\delta_{\hat{Z}_2^\circ} = \delta_{\hat{Z}_2^\circ}$ and $w\theta_\Sigma = w\theta_\Sigma$, we conclude that $\pi(Z, \Sigma') = w \pi(Z, \Sigma)$. So the map does indeed factor through $\hat{\mathcal{B}}$. We will now show it is an injection.

Suppose $(Z_i, \Sigma_i) \in \hat{\mathcal{B}}$, $i = 1, 2$ are such that $[\pi(Z_1, \Sigma_1)] = [\pi(Z_2, \Sigma_2)]$, i.e., there exists $\sigma \in W(X)$ such that $\sigma(\pi(Z_1, \Sigma_1)) = \pi(Z_2, \Sigma_2)$. Let $p_i$ denote the overextended vertex of $Z_i$.

Consider the $\pi$-systems:

$$\pi_j = \{\delta_{\hat{Z}_j^\circ} - \theta_{\Sigma_j}\} \cup \Sigma_j, \quad j = 1, 2.$$ 

We note that: (i) $\pi_j$ is of type $\hat{K}^\circ$, (ii) $\pi_j$ is supported in the affine subdiagram $\hat{Z}_j^\circ$ of $X$, and (iii) $\sigma(\pi_1) = \pi_2$.

Now, it follows from part (2) of theorem 4.2 that $\hat{Z}_1^\circ = \hat{Z}_2^\circ$ and $\sigma \in W(\hat{Z}_1^\circ \cup \hat{Z}_1^\circ^\perp)$. Since $p_1 \notin \hat{Z}_1^\circ \cup \hat{Z}_1^\circ^\perp$, we can only have $\sigma \alpha_{p_1} = \alpha_{p_2}$ if $p_1 = p_2$. Thus, $Z_1 = Z_2$. We write $\sigma = \tau \tau'$ with $\tau \in W(\hat{Z}_1^\circ)$ and $\tau' \in W(\hat{Z}_1^\circ^\perp)$.

Since $\sigma p_1 = p_2$, we obtain $\tau \Sigma_1 = \Sigma_2$ (in fact, $\tau p_1 = p_2$) since $\tau'$ fixes each element of $p_1$ point-wise. Further, $\sigma \alpha_{p_1} = \alpha_{p_1}$ implies that $\sigma \in W(\{p_1\}^\perp)$. In particular, $\tau \in W(\hat{Z}_1^\circ) \cap W(\{p_1\}^\perp) = W(\hat{Z}_1^\circ)$. Hence we obtain $(Z_1, \Sigma_1) \sim (Z_2, \Sigma_2)$, in other words, the map defined above is injective on $\mathcal{B}$.

Next, we show surjectivity of the map. Let $\pi$ be a positive $\pi$-system in $X$ of type $K$; we will show that $[\pi]$ is in the image of the map. Let $\pi^\circ, \hat{\pi}^\circ$ be the subsets of $\pi$ corresponding to the finite and affine parts of $K$ respectively. Now, $\hat{\pi}^\circ$ is a positive $\pi$-system of type $\hat{K}^\circ$ in $X$. By Theorem 4.2, there is an affine type subdiagram $Y$ of $X$, and an element $w \in W(X)$ such that every element of (the positive $\pi$-system) $w(\hat{\pi}^\circ) = (w\pi)^\circ$ is supported in $Y$. Since $[\pi] = [w\pi]$, let us replace $\pi$ with $w\pi$ in what follows. Thus, $\pi$ is a positive $\pi$-system of type $K$ such that $\hat{\pi}^\circ$ is supported in $Y$. Let $\beta \in \pi$ correspond to the overextended vertex of $K$, and let $\delta_{\hat{\pi}^\circ}$ denote the null root of $\hat{\pi}^\circ$, identified with its image in $\Delta(\hat{\pi}^\circ) \subset \Delta(X)$. Thus $\delta_{\hat{\pi}^\circ}$ (i) is a positive imaginary root of $X$ (by corollary 2.10),
(ii) is supported in $Y$, and (iii) satisfies $(\delta_{\pi_x} | \beta) = -1$. The first two conditions imply $\delta_{\pi_x} = r\delta_Y$ for some $r \geq 1$, while the third implies $r = 1$.

As in the proof of Theorem 6.2 we now appeal to Lemma 6.3 to find an element $\sigma \in W(Y \cup Y')$ such that $\sigma \beta = \alpha_p$ for some vertex $p$ of $X$. Define $Z = Y \cup \{p\}$; this is clearly an Ext type subdiagram of $X$. Consider the positive $\pi$-system $\xi = \sigma\pi$ of type $K$. We have:

(a) $\alpha_p \in \xi$, (b) $\hat{\xi}$ is supported in $Y$ and (c) $\delta_{\hat{\xi}} = \delta_Y$.

Further, $(\alpha | \beta) = 0$ for all $\alpha \in \pi^o$ gives us $(\sigma\alpha | \sigma\beta) = 0$, i.e., $(\alpha' | \alpha_p) = 0$ for all $\alpha' \in \xi^o$. This in turn implies that: (d) $\xi^o$ is supported in $Z^o$.

From (a), (c) and (d) we conclude $\xi = \pi(Z,\xi^o)$. Since $[\pi] = [\xi]$ and $\xi^o$ is of type $K^o$, the proof is complete.

$\square$

8. The partial order $\preceq$

Let $A, B$ be GCMs. We define $B \preceq A$ if there is a linearly independent $\pi$-system of type $B$ in $A$. We now show that $\preceq$ defines a partial order on the set of symmetrizable hyperbolic GCMs (where we identify two GCMs that differ only by a simultaneous permutation of rows and columns). Clearly this relation is reflexive. By corollary 2.12 this relation is transitive. We now prove that this relation is anti-symmetric.

Lemma 8.1. Let $A$ be an $n \times n$ GCM (not necessarily symmetrizable). Let $\{\alpha_i\}_{i=1}^n$ be the simple roots of $g(A)$. Let $\{\beta_i\}_{i=1}^n$ be any set of real roots of $g(A)$. Let $\alpha_i^\vee, \beta_i^\vee$ denote the corresponding coroots. Consider the integer matrix: $B = [\langle \beta_i^\vee, \beta_j \rangle]_{i,j}$. Then:

1. $\det A$ divides $\det B$.

2. Further if $A, B$ are invertible with $|\det A| = |\det B|$, then $\{\beta_i\}_{i=1}^n$ and $\{\beta_i^\vee\}_{i=1}^n$ form $\mathbb{Z}$-bases of $Q(A)$ and $Q^\vee(A)$ respectively.

Proof: We write:

$$\beta_i^\vee = \sum_{k=1}^n u_{ik} \alpha_k^\vee$$

$$\beta_j = \sum_{\ell=1}^n v_{j\ell} \alpha_\ell$$

where $u_{ik}, v_{j\ell}$ are integers. Using the equations above, we compute:

$$B = U A V^T$$

where $U = [u_{ij}]$ and $V = [v_{ij}]$ are integer matrices. Taking determinants, we obtain $\det B = \det U \det V \det A$, proving the first assertion. For the second assertion, the given condition implies $|\det U| = |\det V| = 1$, i.e., $U$ and $V$ are in $\GL_n(\mathbb{Z})$. This is clearly equivalent to what needs to be shown. $\square$

Proposition 8.2. Let $A, B$ be $n \times n$ symmetrizable GCMs of hyperbolic type, with $\det A = \det B$. Suppose $\Sigma = \{\beta_i\}_{i=1}^n$ is a $\pi$-system of type $B$ in $A$. Then $\Sigma$ is $W(A)$-conjugate to $\Pi(A)$ or $-\Pi(A)$, where $\Pi(A)$ is the set of simple roots of $g(A)$. In particular, $A$ and $B$ are equal up to a simultaneous permutation of rows and columns.
PROOF: Consider the map \( q_\Sigma : Q(B) \to Q(A) \) of equation (2.2), defined by \( \alpha_j(B) \mapsto \beta_i \) for all \( i \), where \( \Pi(B) = \{ \alpha_j(B) : 1 \leq i \leq n \} \) is the set of simple roots of \( g(B) \). We assume for convenience that the symmetric bilinear forms on \( Q(A) \) and \( Q(B) \) are chosen compatibly as in §2.4 so that \( q_\Sigma \) is form preserving (the arguments below will still work for any choices of standard invariant forms, since they only differ by scaling by positive rationals).

Using the given hypothesis and the fact that hyperbolic GCMs are necessarily invertible, we obtain from the second part of lemma 8.1 that: (i) \( \Sigma \) is a \( \mathbb{Z} \)-basis of \( Q(A) \) and (ii) \( \Sigma^\vee = \{ \beta^\vee_j \}_{j=1}^n \) is a \( \mathbb{Z} \)-basis of \( Q^\vee(A) \).

We observe from (i) above that \( q_\Sigma \) is a form preserving lattice isomorphism of \( Q(B) \) onto \( Q(A) \). We now claim that \( q_\Sigma(\Delta(B)) = \Delta(A) \). Corollary 2.10 implies that \( q_\Sigma(\Delta(B)) \subset \Delta(A) \). We only need to prove the reverse inclusion. Towards this end, we recall the following description of the set of roots of a symmetrizable Kac-Moody algebra \( g(C) \) of Finite, Affine or Hyperbolic type [18 Prop 5.10]:

\[
\Delta^{re}(C) = \{ \alpha = \sum_j k_j \alpha_j(C) \in Q(C) : |\alpha|^2 > 0 \text{ and } k_j |\alpha_j(C)|^2/|\alpha|^2 \in \mathbb{Z} \text{ for all } j \}
\]

\[
\Delta^{im}(C) = \{ \alpha \in Q(C) \setminus \{0\} : |\alpha|^2 \leq 0 \}
\]

forms where \( \alpha_j(C) \) are the simple roots, \( Q(C) \) is the root lattice, and we fix any standard invariant form on \( g(C) \). We apply this when \( C = A, B \) below.

Since \( |q_\Sigma(\alpha)|^2 = |\alpha|^2 \) for all \( \alpha \in Q(B) \), it is clear from equation (8.2) that \( q_\Sigma(\Delta^{re}(B)) = \Delta^{re}(A) \).

Now let \( \beta \in \Delta^{re}(A) \) and define \( \alpha = q_\Sigma^{-1}(\beta) \). We need to prove that \( \alpha \in \Delta^{re}(B) \). Let \( \beta = \sum_j k_j \beta_j \) for some integers \( k_j \); thus \( \alpha = \sum_j k_j \alpha_j(B) \). Since \( \beta \) is a real root, \( |\alpha|^2 = |\beta|^2 > 0 \). Define

\[
c_j = k_j |\alpha_j(B)|^2/|\alpha|^2 = k_j |\beta_j|^2/|\beta|^2
\]

Equation (8.1) implies that \( \alpha \) is a real root of \( g(B) \) if and only if \( c_j \in \mathbb{Z} \) for all \( j \). Consider \( \beta^\vee \in Q^\vee(A) \); by (ii) above, we know that \( \Sigma^\vee \) forms a \( \mathbb{Z} \)-basis of the coroot lattice \( Q^\vee(A) \). Now \( \gamma^\vee = 2n^{-1}(\gamma)/|\gamma|^2 \) for any real root \( \gamma \) of \( g(A) \) [18 Prop. 5.1], where \( n \) is the linear isomorphism from the Cartan subalgebra of \( g(A) \) to its dual induced by the form. A simple computation now shows:

\[
\beta^\vee = \sum_j c_j \beta_j^\vee
\]

This proves the integrality of the \( c_j \), and hence our claim.

Thus, \( q_\Sigma(\Delta(B)) = \Delta(A) \). Since \( q_\Sigma(\Pi(B)) = \Sigma \), this means that \( \Sigma \) is a root basis of \( \Delta(A) \) [18 §5.9], i.e., \( \Sigma \) is a \( \mathbb{Z} \)-basis of \( Q(A) \) such that every element of \( \Delta(A) \) can be expressed as an integral linear combination of \( \Sigma \) with all coefficients of the same sign. By [18 Proposition 5.9], we conclude that \( \Sigma \) is \( W(A) \)-conjugate to \( \pm \Pi(A) \).

Finally, since \( \Pi(A) \) is a \( \pi \)-system of type \( A \) in \( A \), we conclude that \( A = B \), up to a simultaneous permutation of rows and columns. \( \square \)

**Proposition 8.3.** Let \( A_{n \times n} \) and \( B_{m \times m} \) be symmetrizable GCMs of hyperbolic type such that \( A \preceq B \) and \( B \preceq A \). Then, \( m = n \) and there exists a permutation matrix \( P \) such that \( P A P^T = B \).

**PROOF:** Since \( A \preceq B \), there exists a linearly independent \( \pi \)-system of type \( A \) in \( B \); in particular, this implies \( n \leq m \). Similarly, \( m \leq n \), so we obtain \( m = n \). Applying lemma 8.1 we conclude that \( \det A | \det B \) and \( \det B | \det A \), so in fact \( \det A = \pm \det B \). Since hyperbolic GCMs have strictly
negative determinant, we must have \( \det A = \det B \). Proposition 8.2 completes the proof.

In other words, \( \preceq \) is a partial order on the set of equivalence classes of hyperbolic GCMs, where we identify GCMs that differ by a simultaneous reordering of rows and columns. We restrict ourselves to the set Hyp comprising hyperbolic GCMs of rank \( \geq 3 \). In the rest of the paper, we will determine the maximal elements of Hyp with respect to this partial order (up to equivalence).

9. Construction of \( \pi \)-systems

In this section we will develop some principles for constructing \( \pi \)-systems in a given Dynkin diagram. These are generalizations of the principles developed in [28] for simply-laced diagrams.

9.1. In fact, all our principles below are instances of the following simple, but powerful method of constructing \( \pi \)-systems.

**General principle:** Let \( X \) be the Dynkin diagram of a symmetrizable GCM. Let \( \Lambda \) denote a proper subdiagram of \( X \) and let \( \Lambda' \) be the subdiagram formed by the vertices not in \( \Lambda \). Let \( \Sigma, \Sigma' \) be \( \pi \)-systems in \( \Lambda \), \( \Lambda' \) respectively, consisting of positive real roots. Then \( \Sigma \cup \Sigma' \) is a \( \pi \)-system in \( X \).

This principle follows from the observations that (i) the (real) roots of a subdiagram are precisely the (real) roots of the ambient diagram that are supported on the subdiagram, (ii) the difference of two positive roots with disjoint supports will have coefficients of mixed sign, and can therefore not be a root. In all our applications below, we will always take \( \Sigma' \) to consist of the set of all simple roots of \( \Lambda' \).

Observe that the GCM of \( \Sigma \cup \Sigma' \) is of the form

\[
\begin{bmatrix}
B & * \\
* & B'
\end{bmatrix}
\]

where \( B, B' \) are the GCMs of \( \Sigma, \Sigma' \) respectively. The terms denoted * are of the form \( 2(\beta_1 | \beta_2)/(\beta_2 | \beta_2) \) where \( \beta_1 \in \Lambda, \beta_2 \in \Lambda' \) or vice versa. We now isolate some special instances of this general principle, which will be used repeatedly in the sequel.

9.2. For principles A, B below, we assume \( Y \) to be an affine Dynkin diagram, twisted or untwisted, but \( Y \neq A_{2l}^{(2)} \). Let \( \{\alpha_0, \ldots, \alpha_n\} \) denote the simple roots of \( Y \). Let \( \overline{Y} \) denote the underlying finite type diagram, obtained from \( Y \) by deleting the node corresponding to \( \alpha_0 \).

Let \( X \) be the diagram obtained by adding an extra vertex to \( \overline{Y} \), which is connected only to \( \alpha_0 \), and by a single edge. Since \( Y \) is symmetrizable, so is \( X \). We denote the simple root corresponding to this vertex \( \alpha_{-1} \). Let \( A = (a_{ij}) \) denote the GCM of \( X \); thus \( a_{ij} = 2(\alpha_i | \alpha_j)/(\alpha_i | \alpha_i) \) for \(-1 \leq i, j \leq n\).

We note in passing that when \( Y \) is simply-laced, \( X \) is of Ext type. Let \( \delta_Y \) denote the null root of \( Y \), so \( \delta_Y = \sum_{i=0}^{n} a_i \alpha_i \) with \( a_i \in \mathbb{N} \). We let \( s_i \) denote the reflection corresponding to the simple root \( \alpha_i \).

**Principle A:** Since \( Y \) is an affine diagram other than \( A_{2l}^{(2)} \), we have \( a_0 = 1 \) [18 Chapter 4, Tables Aff 1-3]. In the general principle, we take the subdiagram \( \Lambda = Y \) and \( \Lambda' \) to be the singleton set containing the vertex \((-1)\). Define \( \Sigma \) to be the \( \pi \)-system in \( Y \) of type \( Y \) comprising of the roots \( \{s_0 \gamma_i : 0 \leq i \leq n\} \) where the \( \gamma_i \) are given by:

\[ \gamma_0 = \alpha_0 + \delta_Y, \quad \gamma_j = \alpha_j (j \geq 1) \]

Define \( \Sigma' = \{\alpha_{-1}\} \); this is clearly of finite type \( A_1 \).
We let $\Sigma \cup \Sigma' = \{\beta_i : -1 \leq i \leq n\}$ with $\beta_{-1} = \alpha_{-1}$ and $\beta_i = s_0 \gamma_i$ for $i \geq 1$. All the hypotheses of the general principle are satisfied. As observed in equation (9.1), to find the type of $\Sigma \cup \Sigma'$, it only remains to compute the numbers $b_{ij} = 2(\beta_i | \beta_j)/(\beta_i | \beta_i)$ where $i = -1, j \geq 0$ or vice-versa.

Now: (i) $(\beta_{-1} | \beta_j) = (s_0 \beta_{-1} | \gamma_j) = (\alpha_0 | \alpha_j)$ for $j \geq 1$, since $s_0 \alpha_{-1} = \alpha_0 + \alpha_{-1}$ and $\alpha_{-1}$ is orthogonal to all roots of $\Sigma$.

(ii) If $\theta$ is the highest long (respectively short) root of $\Sigma$ if $\Sigma$ is untwisted (respectively twisted). But $(\alpha_{-1} | \theta) = 0$ since as before $\alpha_{-1}$ is orthogonal to all roots of $\Sigma$.

Finally, we compute: $(\beta_{-1} | \beta_0) = (\alpha_{-1} | s_0(\alpha_0 + \delta_Y))$. But $s_0(\alpha_0 + \delta_Y) = -\alpha_0 + \delta_Y = \theta$, where $\delta_Y$ is the highest long (respectively short) root of $\Sigma$ if $\Sigma$ is untwisted (respectively twisted).

Now: (i) $(\beta_{-1} | \beta_0) = (\alpha_{-1} | s_0(\alpha_0 + \delta_Y))$. But $s_0(\alpha_0 + \delta_Y) = -\alpha_0 + \delta_Y = \theta$, where $\delta_Y$ is the highest long (respectively short) root of $\Sigma$ if $\Sigma$ is untwisted (respectively twisted). But $(\alpha_{-1} | \theta) = 0$ since as before $\alpha_{-1}$ is orthogonal to all roots of $\Sigma$.

The Dynkin diagram $S(B)$ is thus obtained from $X = S(A)$ by removing the edge between vertices 0 and $-1$, and instead connecting the vertex $-1$ to every neighbour of 0 with the same edge labels, i.e., such that $b_{j,-1} = a_{j0}$ and $b_{-1,j} = a_{0j}$.

### Principle B:

(i) First let us suppose that $Y$ is untwisted. In the general principle, we choose $\Lambda = Y$. Fix $1 \leq p \leq n$ and let $\Sigma = \{\beta_i\}_{i=0}^n$ be the following $\pi$-system of type $Y$ in $Y$:

$\beta_i = \alpha_i$, for $0 \leq i \leq n, i \neq p$, and $\beta_p = \alpha_p + \delta_Y$.

Let $\Sigma' = \{\alpha_{-1}\}$. Define $\beta_{-1} = \alpha_{-1}$.

We have $\Sigma \cup \Sigma'$ is a $\pi$-system. Let $B = (b_{ij})_{i,j=-1}^n$ denote its type. As above, we only need to compute $b_{ij}$ for $i = -1, j \geq 0$ or vice-versa. Now, clearly $b_{-1,j} = a_{-1,j}$ and $b_{j,-1} = a_{j,-1}$ for $j \geq 0$, $j \neq p$. Further,

$$(\beta_p, \beta_{-1}) = (\alpha_p, \alpha_{-1}) + (\delta_Y, \alpha_{-1}) = (\alpha_0, \alpha_{-1}) = -|\alpha_0|^2$$

Since $|\beta_i|^2 = |\alpha_i|^2$ for all $i$, we conclude that $b_{-1,p} = -|\alpha_0|^2/|\alpha_{-1}|^2 = -1$ and $b_{p,-1} = -|\alpha_0|^2/|\alpha_p|^2$.

Now since $\alpha_0$ is a long root of $Y$, we obtain

$$b_{p,-1} = \begin{cases} -1 & \text{if } \alpha_p \text{ is a long root of } Y \\ -2 & \text{if } Y \neq C_2^{(1)} \text{, and } \alpha_p \text{ is a short root of } Y \\ -3 & \text{if } Y = C_2^{(1)} \text{, and } \alpha_p \text{ is a short root of } Y \end{cases}$$

In terms of Dynkin diagrams, the diagram $S(B)$ coincides with $S(A)$ except that there is a single, double or triple edge joining vertices $-1$ and $p$ (with an arrow pointing towards $p$) depending on the three cases above.

(ii) If $Y$ is twisted, say $Y = X_n^{(r)}$ for $r = 2, 3$ and $Y \neq A_2^{(2)}$. Choose $\Lambda = Y$ as before. Fix a vertex $1 \leq p \leq n$, and define the $\pi$-system $\Sigma = \{\beta_i : 0 \leq i \leq n\}$ of type $Y$ in $Y$ by:

$$\beta_i = \alpha_i$$

and

$$\beta_p = \begin{cases} \alpha_p + \delta_Y & \text{if } \alpha_p \text{ is a short root} \\ \alpha_p + r\delta_Y & \text{if } \alpha_p \text{ is a long root} \end{cases}$$

Let $\beta_{-1} = \alpha_{-1}$ and define $\Sigma' = \{\beta_{-1}\}$. We now have $|\beta_i|^2 = |\alpha_i|^2$ for all $i$. Letting $B$ denote the GCM of $\Sigma \cup \Sigma'$ and reasoning as in (i) above, we obtain: (a) $b_{ij} = a_{ij}$ for $i, j \neq p$, (b) $b_{ij} = a_{ij}$ for $i, j \neq -1$, (c) $b_{p,-1} = -1$ and (d) $b_{p,-1} = -|\alpha_0|^2/|\alpha_p|^2$. Since $\alpha_0$ is a short root of $Y$, we have:
\[
\begin{align*}
    b_{-1, p} = \begin{cases} 
-1 & \text{if } \alpha_p \text{ is a short root of } Y \\
-2 & \text{if } Y \neq D_4^{(3)}, \text{ and } \alpha_p \text{ is a long root of } Y \\
-3 & \text{if } Y = D_4^{(3)}, \text{ and } \alpha_p \text{ is a long root of } Y
\end{cases}
\end{align*}
\]

As before, this implies that the diagram \( S(B) \) coincides with \( S(A) \) except that there is a single, double or triple edge joining vertices \(-1\) and \( p \) (with an arrow pointing away from \( p \)) depending on the three cases above.\(^1\)

(iii) If instead of \( 1 \leq p \leq n \), we choose the vertex \( p = 0 \) in (i) or (ii) above, we obtain \( b_{0, -1} = b_{-1, 0} = -2 \), and \( b_{ij} = a_{ij} \) for all other pairs \( (i, j) \). In the Dynkin diagram \( S(B) \), this would be denoted by a double edge between vertices 0 and \(-1\), marked with two arrows, one pointing toward each vertex.

9.3. For principles \( C, D, E \), we let \( X \) denote the Dynkin diagram of any symmetrizable GCM.

**Principle C:** (Shrinking) Suppose \( I \) is a subset of the vertices of \( X \) such that \( I \) forms a (connected) subdiagram of Finite type. It is well known that \( \beta_\ast = \sum_{i \in I} \alpha_i \) is a root of \( g(I) \). Since \( I \) is of finite type, this root is real. In the general principle, we choose the subset \( \Lambda = I \) and the \( \pi \)-system \( \Sigma = \{ \beta_\ast \} \). Let \( \Sigma' = \{ \alpha_j : j \notin I \} \). Let \( B \) denote the GCM of \( \Sigma \cup \Sigma' \). We have for \( j \notin I \),

\[
\frac{(\beta_\ast, \alpha_j)}{(\alpha_j, \alpha_j)} = \sum_{i \in I} \frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}
\]

Further, letting \( k_i = |\alpha_i|^2/|\beta_\ast|^2 \) for \( i \in I \), we have

\[
\frac{(\beta_\ast, \alpha_j)}{(\beta_\ast, \beta_\ast)} = \sum_{i \in I} k_i \frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}
\]

Thus,

\[\begin{align*}
    b_{\ast j} &= \sum_{i \in I} a_{ji}, \\
    b_{j \ast} &= \sum_{i \in I} k_i a_{ij}
\end{align*}\]

We note that \( k_i \) is the ratio of root lengths in a finite type diagram, and is therefore one of \( 1, \frac{1}{2}, \frac{1}{3}, 1, 2, 3 \). If no two vertices of \( I \) have a common neighbour \( j \notin I \), then the Dynkin diagram \( S(B) \) may be thought of as being obtained from \( X \) by contracting the vertices of \( I \) to a single “fat” vertex \( \bullet \). The edges in \( X \) between \( i \in I \) and \( j \notin I \) are now drawn between \( \bullet \) and \( j \) in \( S(B) \) (with possibly new edge weights). The rest of the diagram \( X \) is carried over unchanged.

**Principle D:** (Deletion) If we delete any subset of vertices from the vertex set of \( X \) and define \( \Sigma \) to be the set of remaining \( \{ \alpha_i \} \), then \( \Sigma \) is a \( \pi \)-system in \( X \). Its Dynkin diagram is clearly a subdiagram of \( X \).

**Principle E:**

(i) Let the vertices of \( X \) be labelled \( 1, 2, \ldots, n \). Suppose \( X \) contains a subdiagram of finite type \( B_2 \), i.e., there are vertices \( p, q \) in \( X \) joined by a double bond directed (say) towards \( p \). In other words, \( a_{pq} = -2, a_{qp} = -1 \). In the general principle, we take \( \Lambda \) to be this subdiagram of type \( B_2 \) and define \( \Sigma = \{ \beta_p, \beta_q \} \) to be the \( \pi \)-system of type \( A_1 \times A_1 \) in \( \Lambda \) given by:

\[
\beta_p = s_p(\alpha_q) = \alpha_q + 2\alpha_p, \quad \beta_q = \alpha_q.
\]

\(^1\)If \( Y \) is of type \( A_2^{(2)} \), analogous results hold for an appropriate modification of \( \Sigma \). We do not dwell on this case, since it is not used in the sequel.
Define \( \beta_j = \alpha_j \) for \( 1 \leq j \leq n, j \neq p, q \) and let \( \Sigma' \) be the set of these \( \beta_j \). Let \( B \) denote the GCM of \( \Sigma \cup \Sigma' = \{ \beta_i : 1 \leq i \leq n \} \); clearly \( b_{ij} = a_{ij} \) for \( i, j \neq p \). Now,

\[
\frac{\langle \beta_p, \beta_j \rangle}{\langle \beta_j, \beta_j \rangle} = \frac{\langle \alpha_q, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} + 2 \frac{\langle \alpha_p, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}, \quad \text{i.e., } b_{jp} = a_{jq} + 2a_{jp}
\]

Since \( |\alpha_q|^2 = 2|\alpha_p|^2 \), we have

\[
\frac{\langle \beta_p, \beta_j \rangle}{\langle \beta_p, \beta_p \rangle} = \frac{\langle \alpha_q, \alpha_j \rangle}{\langle \alpha_q, \alpha_q \rangle} + 2 \frac{\langle \alpha_p, \alpha_j \rangle}{\langle \alpha_p, \alpha_p \rangle} \quad \text{i.e., } b_{pj} = a_{qj} + a_{pj}
\]

Note in particular that since \( \Sigma \) has type \( A \times A \), we have \( b_{pq} = b_{qp} = 0 \), i.e., the double edge between \( p, q \) in \( X \) has been removed in \( S(B) \).

(ii) Now suppose the Dynkin diagram \( X \) has a subdiagram of finite type \( G \), i.e., there are vertices \( p, q \) in \( X \) joined by a triple bond directed towards \( p \). As above, choose \( \Lambda \) to be this subdiagram of type \( G \) and define \( \Sigma = \{ \beta_p, \beta_q \} \) to be the \( \pi \)-system of type \( A \) in \( \Lambda \) given by:

\[
\beta_p = s_p(\alpha_q) = \alpha_q + 3a_p, \quad \beta_q = \alpha_q.
\]

Choose \( \Sigma' \) as above, to consist of all the simple roots \( \alpha_i \) of \( X \) other than \( i = p, q \). A similar computation establishes that \( b_{jp} = a_{jq} + 3a_{jp}, b_{pj} = a_{qj} + a_{pj} \) and \( b_{ij} = a_{ij} \) for all other pairs \( (i, j) \).

Note in particular that since \( \Sigma \) is of type \( A \), one has \( b_{pq} = b_{qp} = -1 \), i.e., the triple edge between \( p, q \) in \( X \) has now been replaced by a single edge in \( S(B) \).

10. Non-Maximal Hyperbolic Diagrams

10.1. In Tables 1–10, we have listed the 142 symmetrizable hyperbolic Dynkin diagrams in ranks 3-10. We will denote by \( \Gamma_k \) the hyperbolic Dynkin diagram occurring with serial number \( k \) in these tables. These diagrams are taken from Tables 1–23 of [2] which contain the full list of 238 hyperbolic diagrams without the assumption of symmetrizability. The diagram \( \Gamma_k \) occurs as item number \( k \) in Tables 1–23 of [2]. Since we only consider the 142 symmetrizable hyperbolic diagrams rather than all 238 of them, there are “gaps” in the serial numbers that occur in our tables.

The entries in our tables contain the following information: for each serial number \( k \), the second column is the corresponding Dynkin diagram, the third column is another serial number, say \( \ell \) such that \( \Gamma_k \preceq \Gamma_\ell \) and the fourth column indicates the principle(s) used to construct a \( \pi \)-system of type \( \Gamma_k \) in \( \Gamma_\ell \). We note that \( \ell \) is not unique in general, but since our primary goal is to identify the maximal diagrams relative to \( \preceq \), we will be content with finding one value of \( \ell \).

The diagrams \( \Gamma_k \) for which we are unable to find a suitable \( \ell \) using any of our principles are candidates for maximal elements. We show in Table 12 that each of these diagrams is indeed maximal. The entries corresponding to these diagrams are indicated by ‘Max’ in the third column while the fourth column contains the value of the determinant of the GCM of the diagram.

In this section we give a few examples to illustrate the Principles A-E developed in the previous section. The other entries of the table may be verified by similar arguments.

**Principle A:** Taking \( X = \Gamma_{219} \) and \( Y = F_4^{(1)} \) in principle A, we obtain a \( \pi \)-system of type \( \Gamma_{207} \) in \( \Gamma_{219} \). Similarly, choosing \( X = \Gamma_{159} \) and \( Y = G_2^{(1)} \), we obtain \( \Gamma_{150} \preceq \Gamma_{159} \).

**Principle B:** Let \( X = \Gamma_{159}, Y = G_2^{(1)} \) and \( \alpha_p \) be the long simple root of \( G_2 \). Applying principle \( B \) allows us to construct a \( \pi \)-system of type \( \Gamma_{129} \) in \( \Gamma_{159} \). Similarly, taking \( X = \Gamma_{160}, Y \) to be the twisted affine diagram \( D_4^{(3)} \) and \( \alpha_p \) to be the short simple root of \( G_2 \), we conclude that \( \Gamma_{130} \preceq \Gamma_{160} \).
**Principle C:** Principle C allows us to shrink diagrams in a specified manner. For instance, one readily obtains from this principle that: $\Gamma_{222} \preceq \Gamma_{226} \preceq \Gamma_{231} \preceq \Gamma_{236}$.

**Principle D:** Typically the deletion principle D is used in conjunction with one of the other principles. For instance, first applying principle B to $X = \Gamma_{163}$, $Y = D_3^{(2)}(2)$ and $p = 0$ (i.e., the affine simple root of $Y$) one obtains the rank 4 diagram obtained from $\Gamma_{163}$ by replacing its single edge by the two-way double edge $\Leftrightarrow$. Now applying principle D to delete the node at the other end gives us $\Gamma_{106}$.

**Principle E:** This principle only applies when the ambient diagram has a double or triple edge. It typically has the effect of shifting the double or triple edge to an adjacent pair of vertices. For example, an application of this principle shows there exists a $\pi$-system of type $\Gamma_{220}$ in $\Gamma_{218}$ and one of type $\Gamma_{161}$ in $\Gamma_{160}$.

10.2. **Principle (*).** As mentioned above, for each non-maximal diagram $\Gamma_k$, the above principles can typically be used to exhibit a diagram $\Gamma_\ell$ such that $\Gamma_k \preceq \Gamma_\ell$. However, the following relation $\Gamma_{172} \preceq \Gamma_{160}$ is not amenable to any of these principles. We give below a special construction for this case.

Consider the Dynkin diagram $\Gamma_{160}$. Let the simple roots be $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$, where $\alpha_4$ is the unique long simple root.

Fixing the normalization
\[
|\alpha_1|^2 = |\alpha_2|^2 = |\alpha_3|^2 = 1 \quad \text{and} \quad |\alpha_4|^2 = 3
\]
we obtain
\[
(\alpha_1, \alpha_2) = (\alpha_3, \alpha_2) = -1/2 \quad \text{and} \quad (\alpha_3, \alpha_4) = -3/2,
\]
the other inner products being zero.

Now consider the following set of roots of $\Gamma_{160}$:
\[
\Sigma = \{\alpha_1 + \alpha_2 + \alpha_3, \alpha_4, \alpha_4 + 3\alpha_3, \alpha_2\}
\]
One readily verifies that $\Sigma$ is a $\pi$-system. Further, using (10.1) and (10.2), we obtain that the type of $\Sigma$ is $\Gamma_{172}$:

11. **Non-existence of $\pi$-systems**

In this section, we give a few simple criteria that can be used to demonstrate the non-existence of $\pi$-systems of certain types in an ambient Lie algebra.

11.1. The following is an immediate corollary of the discussion of §2.4, together with the fact that a real root is Weyl conjugate to some simple root, and therefore has the same length.

**Lemma 11.1.** (Root length criterion) Let $A, B$ be indecomposable symmetrizable GCMs such that $B \preceq A$. For each pair of simple roots of $B$, the ratio of their lengths equals that of some pair of simple roots of $A$ (with respect to any choices of standard invariant forms on $g(A)$ and $g(B)$).
For instance, this implies that there doesn’t exist a \( \pi \)-system of type \( G_2 \) in any other finite type GCM.

The next result follows directly from lemma 8.1, proposition 8.2 and the fact that hyperbolic GCMs have strictly negative determinant. It has been extracted here as a separate statement on account of its wider applicability.

**Lemma 11.2.** (Determinant criterion) Let \( A, B \) be symmetrizable hyperbolic GCMs of the same size. If \( B \preceq A \) and \( B \neq A \) (up to simultaneous reordering of rows and columns), then \( \det B = k \det A \) for some \( k \geq 2 \).

11.2. Let \( X \) be the Dynkin diagram of a symmetrizable Kac-Moody algebra and let \( W \) denote its Weyl group. We define \( X_{\text{short}} \) to be the subdiagram formed by the simple roots of shortest length, i.e,

\[
X_{\text{short}} = \{ p \in X : |\alpha_p| = \min_{i \in X} |\alpha_i| \}
\]

Similarly \( X_{\text{long}} \) is the subdiagram formed by the simple roots of longest length. We also let

\[
\Delta_{\text{short}}^\text{re}(X) = \{ \alpha \in \Delta^\text{re}(X) : |\alpha| = \min_{i \in X} |\alpha_i| \} = W \cdot X_{\text{short}}
\]

and \( \Delta_{\text{long}}^\text{re}(X) = W \cdot X_{\text{long}} \).

We say \( X \) is **doubly-laced** if \( X \) contains only single or double edges (with arrows) and **triply-laced** if it contains only single and triple edges (with arrows). The next lemma is a direct consequence of these definitions.

**Lemma 11.3.** Let \( X \) be a doubly- or triply-laced Dynkin diagram (we set \( d = 2 \) in the former case, \( d = 3 \) in the latter). Then:

1. \( d \mid \langle \alpha_i^\vee, \alpha_j \rangle \) for all \( i \in X_{\text{short}}, j \in X\setminus X_{\text{short}} \).
2. \( d \mid \langle \alpha_j^\vee, \alpha_i \rangle \) for all \( i \in X_{\text{long}}, j \in X\setminus X_{\text{long}} \).

Now consider \( \pi \)-systems \( \Sigma \) in \( X \) such that \( \Sigma \subseteq \Delta_{\text{short}}^\text{re} \) or \( \Sigma \subseteq \Delta_{\text{long}}^\text{re} \). We seek to understand the possible types of such \( \Sigma \). The proposition of the next subsection is the important result that will enable us to answer this question. This proposition is vastly more general and can be applied to a wide variety of settings.

11.3. Let \( X \) be the Dynkin diagram of a symmetrizable Kac-Moody algebra and let \( Y \) be a subdiagram of \( X \). We let \( \Delta(Y) \) denote the set of roots of \( Y \), and identify it with \( Q(Y) \cap \Delta(X) \) where \( Q(Y) = \bigoplus_{i \in Y} \mathbb{Z} \alpha_i \). Let \( W \) denote the Weyl group of \( X \). The following proposition concerns multisubsets \( \Sigma \) of the set \( W \cdot \Delta^\text{re}(Y) = \bigcup_{p \in Y} W\alpha_p \). We recall also the notation \( M(\Sigma) \) from §2.2.

**Proposition 11.4.** Let \( X \) be the Dynkin diagram of a symmetrizable Kac-Moody algebra, \( Y \) a subdiagram of \( X \) and \( d \geq 2 \) an integer. Suppose that either:

1. \( d \mid \langle \alpha_i^\vee, \alpha_j \rangle \) for all \( i \in Y, j \in X\setminus Y \), or
2. \( d \mid \langle \alpha_j^\vee, \alpha_i \rangle \) for all \( i \in Y, j \in X\setminus Y \).

Let \( \Sigma = \{ \beta_i : 1 \leq i \leq m \} \) be a multiset with \( \beta_i \in W \cdot \Delta^\text{re}(Y) \). Then, there exists a multiset \( \Sigma = \{ \overline{\beta}_i : 1 \leq i \leq m \} \) with \( \overline{\beta}_i \in \Delta^\text{re}(Y) \) such that

\[
M(\Sigma) \equiv M(\overline{\Sigma}) \pmod{d}
\]
PROOF: Let $s_i$ denote the simple reflection corresponding to the vertex $i \in X$ and let $W(Y)$ be the (standard parabolic) subgroup of $W$ generated by the $\{s_i : i \in Y\}$. The given hypothesis implies by [18 Prop 3.13] that for each $i \in Y, j \in X \setminus Y$, $(s_is_j)^{m_{ij}} = 1$ where $m_{ij} = 2, 4, 6$ or $\infty$. Since these are even (or $\infty$), it follows that the map $W \rightarrow W(Y)$ defined on the generators by:

$$s_i \mapsto \begin{cases} s_i & i \in Y \\ 1 & i \in X \setminus Y \end{cases}$$

extends to a group homomorphism. We denote it $w \mapsto \overline{w}$.

Let $Q(X), Q^\vee(X)$ denote the root and coroot lattices of $X$. We define sublattices $R, R^\vee$ as follows. If (11.1) holds, then $R := dQ(X)$, and

$$R^\vee := dQ^\vee(Y) \oplus Q^\vee(X \setminus Y) = \bigoplus_{i \in Y} \mathbb{Z}(d\alpha_i^\vee) \oplus \bigoplus_{j \notin Y} \mathbb{Z}\alpha_j^\vee$$

If (11.2) holds, then

$$R := dQ(Y) \oplus Q(X \setminus Y) \quad \text{and} \quad R^\vee = dQ^\vee(X)$$

The given hypotheses readily imply that $R$ and $R^\vee$ are $W$-invariant. We now make the following important observation:

(11.3) Given $(w, \alpha) \in W \times \Delta^{re}(Y)$, we have $w\alpha \in \overline{w}\alpha + R$ and $w(\alpha^\vee) \in \overline{w}(\alpha^\vee) + R^\vee$

It is enough to prove this on the generators $w = s_k$ of $W$. This is obvious when $k \in Y$ and follows from equations (11.1), (11.2) when $k \in X \setminus Y$.

Now, given $\beta \in W \cdot \Delta^{re}(Y)$, say $\beta = \sigma\alpha$ for some $(\sigma, \alpha) \in W \times \Delta^{re}(Y)$, we define $\overline{\beta} := \sigma\overline{\alpha}$. This is a real root of $Y$, and in view of (11.3) above, the association $\beta \mapsto \overline{\beta}$ is well-defined modulo $R$. Further, if $\gamma = \tau\alpha'$ is another root in the $W$-orbit of $\Delta^{re}(Y)$, then

$$\langle \overline{\beta^\vee}, \overline{\gamma} \rangle = \langle \overline{\sigma(\alpha^\vee)}, \overline{\tau\alpha'} \rangle \equiv \langle \sigma(\alpha^\vee), \tau\alpha' \rangle \pmod{d}$$

The congruence modulo $d$ in this equation is an easy consequence of equation (11.3), together with the observations that

$$\langle Q^\vee(Y), R \rangle \equiv \langle R^\vee, Q(Y) \rangle \equiv 0 \pmod{d} \quad \text{if equation (11.1) holds.}$$

$$\langle R^\vee, Q(X) \rangle \equiv \langle Q^\vee(Y), R \rangle \equiv 0 \pmod{d} \quad \text{if equation (11.2) holds.}$$

Finally, if $\Sigma = \{\beta_i : 1 \leq i \leq m\}$ is a multi-subset of $W \cdot \Delta^{re}(Y)$, define $\Sigma = \{\overline{\beta_i} : 1 \leq i \leq m\}$. Equation (11.4) now implies $M(\Sigma) \equiv M(\Sigma) \pmod{d}$ as required. \hfill \square

We obtain several useful corollaries.

**Corollary 11.5.** Let $X$ be a doubly-laced Dynkin diagram. Suppose that $X_{short}$ (respectively $X_{long}$) is of type $A_1$, i.e., is a single vertex, then there is no $\pi$-system of type $A_2$ in $X$ contained wholly in $\Delta^{re}_{short}(X)$ (respectively $\Delta^{re}_{long}(X)$).

**Corollary 11.6.** Let $X$ be a doubly-laced Dynkin diagram. Suppose that $X_{short}$ (respectively $X_{long}$) is of type $A_2$, then there is no $\pi$-system of type $A_2 \times A_1$ in $X$ contained wholly in $\Delta^{re}_{short}(X)$ (respectively $\Delta^{re}_{long}(X)$).

**Corollary 11.7.** Let $X$ be a triply-laced Dynkin diagram. Suppose that $X_{short}$ (respectively $X_{long}$) is of type $A_1$, then there is no $\pi$-system of type $A_1 \times A_1$ in $X$ contained wholly in $\Delta^{re}_{short}(X)$ (respectively $\Delta^{re}_{long}(X)$).
We indicate how to prove Corollary 11.6, the others being similar. Lemma 11.3 allows us to apply Proposition 11.4 with \( Y = X_{\text{short}} \) (or \( X_{\text{long}} \)) and \( d = 2 \). The set of shortest (or longest) real roots of \( X \) is nothing but \( W \cdot \Delta^{re}(Y) \). Given any \( \pi \)-system (in fact any multiset of real roots) \( \Sigma \) of \( X \) contained wholly in the Weyl group orbit of \( \Delta^{re}(Y) \), we obtain the multisubset \( \Sigma \) of \( \Delta^{re}(Y) \) such that \( M(\Sigma) \) coincides with \( M(\Sigma) \) modulo \( d = 2 \). For \( Y \) of type \( A_2 \), it only remains to verify that no such multisubset exists if we take \( M(\Sigma) \) to be the GCM of type \( A_2 \times A_1 \), i.e., the matrix

\[
M = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}
\]

So let \( \Sigma = \{ \beta_1, \beta_2, \beta_3 \} \) be such that \( M(\Sigma) \) is congruent to \( M \) mod 2. We observe that the root system of type \( A_2 \) has the property that given two (real) roots \( \alpha, \beta \), we have \( \langle \alpha^\vee, \beta \rangle \) is even if and only if \( \beta = \pm \alpha \). Since the third row and column of \( M \) is zero mod 2, we conclude that \( \beta_1 \) and \( \beta_2 \) must both be of the form \( \pm \beta_3 \). But this would imply \( \langle \beta_1^\vee, \beta_2 \rangle \) is also even, which is a contradiction. \( \square \)

11.4. The following two lemmas are more restrictive in scope, in that they only apply when the ambient Lie algebra is of finite, affine or (symmetrizable) hyperbolic type, and only to the case of shortest roots.

**Lemma 11.8.** Suppose \( X \) is a triply-laced Dynkin diagram of finite, affine or hyperbolic type. Suppose \( X_{\text{short}} \) is of type \( A_1 \), then there is no \( \pi \)-system of type \( A_2 \) in \( X \) contained wholly in \( \Delta^{re}_{\text{short}}(X) \).

**Proof.** Let \( p \) denote the vertex of \( X \) such that \( X_{\text{short}} = \{ p \} \). We normalize the standard invariant form on \( X \) such that \( |\alpha_p|^2 = \min_{j \in X} |\alpha_j|^2 = 1 \). Since \( X \) is triply-laced, \( |\alpha_j|^2 \) is a nonzero power of 3 for all \( j \neq p \). Now suppose \( \Sigma = \{ \beta_1, \beta_2 \} \) is a \( \pi \)-system of type \( A_2 \) in \( X \) such that \( \Sigma \subset \Delta^{re}_{\text{short}}(X) = W\alpha_p \) (the Weyl group orbit of \( \alpha_p \)). Applying an element of \( W \) if necessary, we can assume \( \beta_1 = \alpha_p \). By the arguments used in the proof of Proposition 11.4 specifically equation (11.3), we obtain:

\[
\beta_2 = \pm \alpha_p + \gamma
\]

for some \( \gamma \in R \), where \( R = \mathbb{Z}(3\alpha_p) \oplus \bigoplus_{j \neq p} \mathbb{Z}\alpha_j \). Thus

\[
\langle \beta_2, \beta_1^\vee \rangle = \pm \langle \alpha_p, \alpha_p^\vee \rangle + \langle \gamma, \alpha_p^\vee \rangle \in \pm 2 + 3 \mathbb{Z}
\]

since \( \langle \alpha_j, \alpha_p^\vee \rangle = 0 \) or \(-3 \) for all \( j \neq p \). Now \( \Sigma \) has type \( A_2 \), so \( \langle \beta_2, \beta_1^\vee \rangle = -1 \). We must have \( \beta_2 = \alpha_p + \gamma \), with \( \langle \gamma, \alpha_p^\vee \rangle = -3 \). We compute:

\[
|\beta_2|^2 = |\alpha_p|^2 + |\gamma|^2 + 2 (\langle \alpha_p, \gamma \rangle) = |\alpha_p|^2 + |\gamma|^2 + \langle \gamma, \alpha_p^\vee \rangle
\]

since \( |\alpha_p|^2 = 1 \). Since \( \beta_2 \) is \( W \)-conjugate to \( \alpha_p \), their norms coincide, and we obtain \( |\gamma|^2 = -\langle \gamma, \alpha_p^\vee \rangle = 3 \). We write

\[
\gamma = 3k_p \alpha_p + \sum_{j \neq p} k_j \alpha_j
\]

where the \( k_j \) are integers. We observe that \( \frac{3k_p |\alpha_p|^2}{|\gamma|^2} = k_p \in \mathbb{Z} \). For \( j \neq p \), \( \frac{k_j |\alpha_j|^2}{|\gamma|^2} = \frac{k_j |\alpha_j|^2}{3} \in \mathbb{Z} \) since 3 divides \( |\alpha_j|^2 \). Since \( X \) is of finite, affine or hyperbolic type, we use equation (8.1) to conclude that \( \gamma \) is a real root of \( X \). But \( \gamma = \beta_2 - \beta_1 \), which contradicts the fact that \( \Sigma \) is a \( \pi \)-system. \( \square \)
Lemma 11.9. Suppose \( X \) is a doubly-laced Dynkin diagram of finite, affine or hyperbolic type. Suppose \( X_{\text{short}} \) is of type \( A_2 \), then there is no \( \pi \)-system of type \( A_1 \times A_1 \) in \( X \) contained wholly in \( \Delta^r_{\text{short}}(X) \).

**Proof.** Let \( X_{\text{short}} = \{p, q\} \) and let \( \{\beta_1, \beta_2\} \) be two elements in the \( W \)-orbit of \( \{\alpha_p, \alpha_q\} \) which form a \( \pi \)-system of type \( A_1 \times A_1 \). Applying an element of \( W \) and interchanging \( p, q \) if necessary, we can assume \( \beta_1 = \alpha_p \). By the arguments used in the proof of Proposition 11.4 we obtain:

\[ \beta_2 = \alpha + \gamma \]

for some \( \alpha \in \Delta^r(X_{\text{short}}) \) and \( \gamma \in R \) where \( R = 2Q(X_{\text{short}}) \oplus Q(X \setminus X_{\text{short}}) \). We have

\[ 0 = \langle \beta_2, \beta_1^\vee \rangle = \langle \alpha, \alpha_p^\vee \rangle + \langle \gamma, \alpha_p^\vee \rangle \in \langle \alpha, \alpha_p^\vee \rangle + 2Z \]

As in the proof of Corollary 11.6, we note that \( \langle \alpha, \alpha_p^\vee \rangle \) is even if and only if \( \alpha = \pm \alpha_p \). Since \( \alpha_p \equiv -\alpha_p \mod R \), we may assume \( \beta_2 = \alpha_p + \gamma \). We conclude \( \langle \gamma, \alpha_p^\vee \rangle = -2 \). Normalizing the standard invariant form such that \( |\alpha_p|^2 = |\alpha_q|^2 = 1 \), we compute: \( |\beta_2|^2 = |\alpha_p|^2 + |\gamma|^2 + \langle \gamma, \alpha_p^\vee \rangle \).

As before, this implies \( |\gamma|^2 = -\langle \gamma, \alpha_p^\vee \rangle = 2 \). Letting:

\[ \gamma = 2k_p \alpha_p + 2k_q \alpha_q + \sum_{j \neq p, q} k_j \alpha_j \]

we obtain: (i) \( 2k_p |\alpha_p|^2 = k_p \in Z \), (ii) \( 2k_q |\alpha_q|^2 = k_q \in Z \), and (iii) \( k_j |\alpha_j|^2 = k_j |\alpha_j|^2 \in Z \) for each \( j \neq p, q \), since in this case \( |\alpha_j|^2 \) is a nonzero power of 2. Equation (8.1) implies \( \gamma \) is a real root of \( X \), contradicting the fact that \( \{\beta_1, \beta_2\} \) was a \( \pi \)-system to begin with. \( \square \)

11.5. We note that both the above lemmas do not hold if ‘short’ is replaced by ‘long’. For example:

1. If \( X = G_2 \), then \( X_{\text{long}} \) is of type \( A_1 \). But the set of all long roots forms a closed subroot system isomorphic to \( A_2 \); a \( \pi \)-system of type \( A_2 \) in \( G_2 \) consisting entirely of long roots is \( \{\alpha_1, \alpha_1 + 3\alpha_2\} \) where \( \alpha_1, \alpha_2 \) are respectively the long and short simple roots of \( G_2 \).
2. If \( X = B_3 \), then \( X_{\text{long}} = \{p, q\} \) (say) is of type \( A_2 \). Consider \( \Sigma = \{-\theta\} \cup \{\alpha_p, \alpha_q\} \) where \( \theta \) is the highest root of \( X \). This forms a \( \pi \)-system consisting entirely of long roots; it has type \( A_3 \), and hence contains a subsystem of type \( A_1 \times A_1 \).

### 12. Maximal Hyperbolic diagrams

In this section, we consider the 22 symmetrizable hyperbolic diagrams \( \Gamma_k \) which cannot be exhibited as \( \pi \)-systems of other diagrams using Principles A-E. Such diagrams only exist in ranks 3, 4, 6 and 10 and there are 5, 9, 5 and 3 such diagrams (respectively) in those ranks. We will prove that these are all in fact maximal diagrams relative to the partial order \( \preceq \). As mentioned in §10, the entries corresponding to these diagrams are labelled ‘Max’ in the third column and contain the determinant of their GCMs in the fourth.

#### 12.1. Rank 10

Since \( \det \Gamma_{238} = -1 \), it is maximal by the determinant criterion (lemma 11.2). The same lemma shows that \( \Gamma_{236} \) and \( \Gamma_{237} \) are not \( \preceq \) comparable. Both these latter diagrams have two root lengths, while \( \Gamma_{238} \) has only one, so the root length criterion (lemma 11.1) shows that neither of them can be \( \preceq \Gamma_{238} \). Thus all three are maximal diagrams of rank 10.
12.2. Rank 6. Since $\Gamma_{218}$ and $\Gamma_{219}$ have determinant $-1$, they are both maximal among rank 6 diagrams by the determinant criterion. The root length criterion ensures that neither of these is $\preceq \Gamma_{238}$, so to show maximality of these two diagrams, it only remains to prove that neither of them can be realized as $\pi$-systems of $\Gamma_{236}$ or $\Gamma_{237}$. But this follows readily from corollary 11.5.

Diagrams $\Gamma_{216}$ and $\Gamma_{217}$ have three root lengths. By the root length criterion they cannot be realized as $\pi$-systems of any of the rank 10 maximal diagrams or of the other candidate diagrams $\Gamma_k$ ($k = 215, 218, 219$) in rank 6. Since each of these two diagrams have determinant $-2$, they are mutually incomparable by the determinant criterion. This establishes maximality of $\Gamma_{216}$ and $\Gamma_{217}$.

Finally to show maximality of $\Gamma_{215}$, we observe that it cannot be realized as a $\pi$-system of: (i) $\Gamma_k$ for $k = 236, 237$ by corollary 11.5 (ii) $\Gamma_{238}$ by the root length criterion (iii) $\Gamma_k$ for $k = 216, 217$ by the determinant criterion (iv) $\Gamma_{218}$ by corollary 11.6 (v) $\Gamma_{219}$ by lemma 11.9.

12.3. Rank 4. Since $\det \Gamma_{159} = \det \Gamma_{160} = -1$, they are maximal amongst rank 4 diagrams. Since both these diagrams are triply laced, they contain a pair of simple roots $\alpha_i, \alpha_j$ such that $|\alpha_i|^2/|\alpha_j|^2 = 3$. However none of the maximal diagrams in rank 6 or 10 have triple edges, so the root length criterion ensures that neither of $\Gamma_{159}, \Gamma_{160}$ occur as $\pi$-systems of those diagrams. Hence $\Gamma_{159}$ and $\Gamma_{160}$ are maximal.

The root length criterion shows that $\Gamma_{173}$ is maximal since it contains 4 root lengths. It also shows that none of the $\Gamma_k$ for $166 \leq k \leq 170$ can be realized as $\pi$-systems of $\Gamma_{159}$ or $\Gamma_{160}$ or of any of the maximal diagrams of ranks 6 or 10. Since $\det \Gamma_k = -2$ or $-3$ for $166 \leq k \leq 170$, the determinant criterion implies they are pairwise incomparable. This establishes their maximality.

Finally to show maximality of $\Gamma_{171}$, we observe that it cannot be realized as a $\pi$-system of: (i) any of the maximal diagrams of rank 6 or 10, by the root length criterion (ii) $\Gamma_k$ for $166 \leq k \leq 170$, by the determinant criterion (iii) $\Gamma_{160}$ by corollary 11.7 (iv) $\Gamma_{159}$ by lemma 11.8.

12.4. Rank 3. The determinant criterion ensures that $\Gamma_k$, $117 \leq k \leq 121$ are pairwise incomparable. By the root length criterion, these diagrams cannot be realized as $\pi$-systems of any diagram of rank $\geq 4$. Thus, they are all maximal.

12.5. Remarks. This completes the verification that all 22 candidate diagrams in ranks 3-10 are in fact maximal. We make the following interesting observation:

$\Gamma$ is a maximal hyperbolic diagram $\not\Rightarrow \Gamma^T$ is maximal

where $\Gamma^T$ is the dual diagram, obtained by reversing all the arrows in $\Gamma$ (corresponds to taking the transpose of the GCM). Examples (in fact the only ones) of such diagrams are:

(1) $\Gamma = \Gamma_{215}$ is maximal, while $\Gamma^T = \Gamma_{214} \preceq \Gamma_{218}$.
(2) $\Gamma = \Gamma_{171}$ is maximal, while $\Gamma^T = \Gamma_{172} \preceq \Gamma_{160}$.

We note that the proof of maximality of these two diagrams involves lemmas 11.8 and 11.9, neither of which holds when “dualized” (as remarked in §11.5). In particular, the above examples show that the operation of taking duals is not an automorphism of the partial order $\preceq$, i.e., if $A, B$ are GCMs such that $B \preceq A$, then it is not necessarily true that $B^T \preceq A^T$.

31
| S. No | Dynkin Diagram | Principle |
|-------|----------------|-----------|
| 3     | ![Diagram](image) | 134 C     |
| 4     | ![Diagram](image) | 135 C     |
| 10    | ![Diagram](image) | 140 C     |
| 11    | ![Diagram](image) | 140 C     |
| 25    | ![Diagram](image) | 166 C     |
| 26    | ![Diagram](image) | 167 C     |
| 27    | ![Diagram](image) | 168 C     |
| 28    | ![Diagram](image) | 169 C     |
| 29    | ![Diagram](image) | 170 C     |
| 30    | ![Diagram](image) | 171 C     |
| 31    | ![Diagram](image) | 172 C     |
| 32    | ![Diagram](image) | 103 B     |
| 40    | ![Diagram](image) | 103 B     |
| 49    | ![Diagram](image) | 164 B, C  |
| 50    | ![Diagram](image) | 165 B, C  |

| S. No | Dynkin Diagram | Principle |
|-------|----------------|-----------|
| 54    | ![Diagram](image) | 157 B     |
| 55    | ![Diagram](image) | 162 B, C  |
| 56    | ![Diagram](image) | 163 B, C  |
| 80.   | ![Diagram](image) | 103 B     |
| 83    | ![Diagram](image) | 113 B     |
| 84.   | ![Diagram](image) | 114 B     |
| 90    | ![Diagram](image) | 123 E     |
| 91    | ![Diagram](image) | 157 B     |
| 103   | ![Diagram](image) | 126 B, D  |
| 104   | ![Diagram](image) | 164 C     |
| 105   | ![Diagram](image) | 165 C     |
| 106   | ![Diagram](image) | 163 B, D  |
| 107   | ![Diagram](image) | 162 B, D  |
| 108   | ![Diagram](image) | 173 C     |
| 109   | ![Diagram](image) | 173 C     |
**Table 2. Rank 3 diagrams (continued)**

| S.No | Dynkin Diagram | `≤` | Principle |
|------|----------------|-----|-----------|
| 110  | ![Dynkin Diagram](image) | 174 | C         |
| 111  | ![Dynkin Diagram](image) | 175 | C         |
| 112  | ![Dynkin Diagram](image) | 103 | B         |
| 113  | ![Dynkin Diagram](image) | 159 | B, C      |
| 114  | ![Dynkin Diagram](image) | 160 | B, C      |
| 115  | ![Dynkin Diagram](image) | 158 | E         |
| 116  | ![Dynkin Diagram](image) | 157 | C         |

**Table 3. Rank 4 diagrams**

| S. No | Dynkin Diagram | `≤` | Principle |
|-------|----------------|-----|-----------|
| 124   | ![Dynkin Diagram](image) | 126 | B         |
| 125   | ![Dynkin Diagram](image) | 126 | B         |
| 126   | ![Dynkin Diagram](image) | 177 | C         |
| 127   | ![Dynkin Diagram](image) | 178 | C         |
| 128   | ![Dynkin Diagram](image) | 179 | C         |
| 129   | ![Dynkin Diagram](image) | 159 | B         |
| 130   | ![Dynkin Diagram](image) | 160 | B         |
| 134   | ![Dynkin Diagram](image) | 162 | B         |
| 135   | ![Dynkin Diagram](image) | 163 | B         |
| 136   | ![Dynkin Diagram](image) | 180 | C         |
| 140   | ![Dynkin Diagram](image) | 171 | B         |
| 146   | ![Dynkin Diagram](image) | 174 | B         |
Table 4. Rank 4 diagrams (continued)

| S. No | Dynkin Diagram | Principle |
|-------|----------------|-----------|
| 148   | ![Diagram](image1.png) | 176 B     |
| 150   | ![Diagram](image2.png) | 159 A     |
| 151   | ![Diagram](image3.png) | 160 A     |
| 152   | ![Diagram](image4.png) | 191 C     |
| 153   | ![Diagram](image5.png) | 189 C     |
| 154   | ![Diagram](image6.png) | 190 C     |
| 155   | ![Diagram](image7.png) | 163 A     |
| 156   | ![Diagram](image8.png) | 162 A     |
| 157   | ![Diagram](image9.png) | 173 E     |
| 158   | ![Diagram](image10.png) | 191 C, E  |
| 159   | ![Diagram](image11.png) | Max det=-1 |
| 160   | ![Diagram](image12.png) | Max det=-1 |
| 161   | ![Diagram](image13.png) | 160 E     |
| 162   | ![Diagram](image14.png) | 197 C     |
| 163   | ![Diagram](image15.png) | 198 C     |
| 164   | ![Diagram](image16.png) | 195 C     |
| 165   | ![Diagram](image17.png) | 196 C     |
| 166   | ![Diagram](image18.png) | Max det=-2 |
| 167   | ![Diagram](image19.png) | Max det=-2 |
| 168   | ![Diagram](image20.png) | Max det=-2 |
| 169   | ![Diagram](image21.png) | Max det=-2 |
| 170   | ![Diagram](image22.png) | Max det=-3 |
| 171   | ![Diagram](image23.png) | Max det=-3 |
| 172   | ![Diagram](image24.png) | 160 *     |
| 173   | ![Diagram](image25.png) | Max det=-4 |
| 174   | ![Diagram](image26.png) | 217 B, D  |
| 175   | ![Diagram](image27.png) | 216 B, D  |
| 176   | ![Diagram](image28.png) | 214 B, C, D |

34
Table 5. Rank 5 diagrams

| S. No | Dynkin Diagram | Principle |
|-------|----------------|-----------|
| 177   |                |           |
| 178   |                |           |
| 179   |                |           |
| 180   |                |           |
| 181   |                |           |
| 184   |                |           |
| 185   |                |           |
| 186   |                |           |
| 187   |                |           |

| S. No | Dynkin Diagram | Principle |
|-------|----------------|-----------|
| 188   |                |           |
| 189   |                |           |
| 190   |                |           |
| 191   |                |           |
| 192   |                |           |
| 193   |                |           |
| 194   |                |           |
| 195   |                |           |
| 196   |                |           |
| 197   |                |           |
| 198   |                |           |
Table 6. Rank 6 diagrams

| S.No | Dynkin Diagram | Principle |
|------|----------------|-----------|
| 199  | ![Dynkin Diagram](image1) | 224 C |
| 200  | ![Dynkin Diagram](image2) | 214 B |
| 203  | ![Dynkin Diagram](image3) | 221 A, C |
| 204  | ![Dynkin Diagram](image4) | 221 C |
| 205  | ![Dynkin Diagram](image5) | 210 A |
| 206  | ![Dynkin Diagram](image6) | 209 A |
| 207  | ![Dynkin Diagram](image7) | 219 A |
| 208  | ![Dynkin Diagram](image8) | 218 A |
| 209  | ![Dynkin Diagram](image9) | 223 C |

| S.No | Dynkin Diagram | Principle |
|------|----------------|-----------|
| 210  | ![Dynkin Diagram](image10) | 222 C |
| 211  | ![Dynkin Diagram](image11) | 214 E |
| 212  | ![Dynkin Diagram](image12) | 237 B, C, E |
| 213  | ![Dynkin Diagram](image13) | 216 E |
| 214  | ![Dynkin Diagram](image14) | 218 E |
| 215  | ![Dynkin Diagram](image15) | Max det=-2 |
| 216  | ![Dynkin Diagram](image16) | Max det=-2 |
| 217  | ![Dynkin Diagram](image17) | Max det=-2 |
| 218  | ![Dynkin Diagram](image18) | Max det=-1 |
| 219  | ![Dynkin Diagram](image19) | Max det=-1 |
| 220  | ![Dynkin Diagram](image20) | 218 E |
### Table 7. Rank 7 diagrams

| S. No | Dynkin Diagram | Principle |
|-------|----------------|-----------|
| 221   |                |           |
| 222   |                | 226 C     |
| 223   |                |           |
| 224   |                |           |

### Table 8. Rank 8 diagrams

| S. No | Dynkin Diagram | Principle |
|-------|----------------|-----------|
| 225   |                |           |
| 226   |                | 231 C     |
| 227   |                | 232 C     |
| 228   |                |           |
| 229   |                |           |
| 230   |                | 230 C     |
| 231   |                |           |
| 232   |                |           |
| 233   |                |           |
| 234   |                |           |

### Table 9. Rank 9 diagrams

| S. No | Dynkin Diagram | Principle |
|-------|----------------|-----------|
| 230   |                |           |
| 231   |                | 236 C     |
| 232   |                | 237 C     |
| 233   |                |           |
| 234   |                |           |
| 235   |                | 235 C     |
| 236   |                |           |
| 237   |                |           |
| 238   |                |           |
### Table 10. Rank 10 diagrams

| S.No | Dynkin Diagram | S.No | Principle |
|------|----------------|------|-----------|
| 235  | ![Dynkin Diagram](image) | 238  | A         |
| 236  | ![Dynkin Diagram](image) | Max  | det=-2    |
| 237  | ![Dynkin Diagram](image) | Max  | det=-2    |
| 238  | ![Dynkin Diagram](image) | Max  | det=-1    |

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39