ON THE BONGIORNO’S NOTION OF ABSOLUTE
CONTINUITY

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Abstract. We show that the classes of \( \alpha \)-absolutely continuous functions in the sense of Bongiorno coincide for all \( 0 < \alpha < 1 \).

1. Introduction

The classical Vitali’s definition says that when \( \Omega \subseteq \mathbb{R} \), a function \( f : \Omega \rightarrow \mathbb{R} \) is absolutely continuous if for all \( \varepsilon > 0 \), there exists \( \delta > 0 \) so that for every finite collection of disjoint intervals \( \{[a_i, b_i]\}_{i=1}^k \subset \Omega \) we have (where \( \mathcal{L}^n \) denotes the Lebesgue measure on \( \mathbb{R}^n \))

\[
\sum_{i=1}^k \mathcal{L}^1[a_i, b_i] < \delta \Rightarrow \sum_{i=1}^k |f(a_i) - f(b_i)| < \varepsilon.
\]

(1.1)

The study of the space of absolutely continuous functions on \( [0, 1] \) and their generalizations to domains in \( \mathbb{R}^n \) is connected to the problem of finding regular subclasses of Sobolev spaces which goes back to Cesari and Calderón [6, 5]. There are several natural ways of generalizing the definition of absolute continuity for functions of several variables (cf. [7, 10, 12, 11, 9]).

Recently Bongiorno [2] introduced a generalization of absolute continuity, which is simultaneously similar to Arzelà’s notion of bounded variation for functions on \( \mathbb{R}^2 \), cf. [7], and to Malý’s notion of absolute continuity [11].

Definition 1.1. (Bongiorno [2]) Let \( 0 < \alpha < 1 \). A function \( f : \Omega \rightarrow \mathbb{R}^l \), where \( \Omega \subset \mathbb{R}^n \) is open, is said to be \( \alpha \)-absolutely continuous (denoted \( \alpha \)-AC\(^{(n)}\)(\( \Omega, \mathbb{R}^l \)) or \( \alpha \)-AC) if for all \( \varepsilon > 0 \), there exists \( \delta > 0 \), such that for any finite collection of disjoint \( \alpha \)-regular intervals \( \{[a_i, b_i] \subset \Omega\}_{i=1}^k \) we have

\[
\sum_{i=1}^k \mathcal{L}^n([a_i, b_i]) < \delta \Rightarrow \sum_{i=1}^k |f(a_i) - f(b_i)|^n < \varepsilon.
\]

(1.2)

Here, for \( \mathbf{a} \in \mathbb{R}^l \), \( |\mathbf{a}| \) denotes the Euclidean norm of \( \mathbf{a} \), and we say that an interval \( [\mathbf{a}, \mathbf{b}] \) \( \overset{\text{def}}{=} \{\mathbf{x} = (x_{\nu})_{\nu=1}^n \in \mathbb{R}^n : a_{\nu} \leq x_{\nu} \leq b_{\nu}, \nu = 1, \ldots, n\} \) is
\(\alpha\)-regular if
\[
\frac{L^n([a, b])}{(\max_\nu |a_\nu - b_\nu|)^n} \geq \alpha.
\]

The goal of this paper is to prove that the classes \(\alpha\)-AC coincide for all \(\alpha \in (0, 1)\). We show, however, that the choice of \(\delta\) in (1.2) cannot be made uniformly for all \(\alpha \in (0, 1)\).

Our method uses the class 1-AC which was introduced in [8], see Definition 3.1 below.

2. Preliminaries

In 2002, Hencl [9] introduced the following class of absolutely continuous functions.

**Definition 2.1.** (Hencl [9]) A function \(f : \Omega \to \mathbb{R}^l (\Omega \subset \mathbb{R}^n \text{ open})\) is said to be in \(AC^{(n)}_H(\Omega, \mathbb{R}^l)\) (briefly \(AC_H\)) if there exists \(\lambda \in (0, 1)\) (equivalently, for all \(\lambda \in (0, 1)\)) so that for all \(\varepsilon > 0\), there exists \(\delta > 0\) so that for any finite collection of disjoint closed balls \(\{B(x_i, r_i) \subset \Omega\}_{i=1}^k\) we have

\[
\sum_{i=1}^k L^n(B(x_i, r_i)) < \delta \Rightarrow \sum_{i=1}^k \text{osc}^n(f, B(x_i, \lambda r_i)) < \varepsilon.
\]

Hencl proved that \(AC_H \subset W^{1,n}_{loc}\) and that all functions in \(AC_H\) are differentiable a.e. and satisfy the Luzin (N) property and the change of variables formula.

Bongiorno [3] introduced a modification of the notion of \(\alpha\)-absolute continuity in the spirit of Definition 2.1. To define it, we will use the following notation.

Given interval \([x, y] \subset \mathbb{R}^n\), we denote \(|f([x, y])| = |f(y) - f(x)|\), and for \(0 < \lambda < 1\), we denote by \(\lambda[x, y]\) the interval with center \((x + y)/2\) and sides of length \(\lambda(y_\nu - x_\nu), \nu = 1, \ldots, n\).

**Definition 2.2.** (Bongiorno [3]) A function \(f : \Omega \to \mathbb{R}^l (\Omega \subset \mathbb{R}^n \text{ open})\) is said to be in \(\alpha\)-\(AC^{(n)}_H(\Omega, \mathbb{R}^l)\) (briefly \(\alpha\)-\(AC_H\)) if there exists \(\lambda \in (0, 1)\) (equivalently, for all \(\lambda \in (0, 1)\)) so that for all \(\varepsilon > 0\), there exists \(\delta > 0\) such that for any finite collection of disjoint \(\alpha\)-regular intervals \(\{[a_i, b_i] \subset \Omega\}_{i=1}^k\) we have

\[
(2.1) \quad \sum_{i=1}^k L^n([a_i, b_i]) < \delta \Rightarrow \sum_{i=1}^k |f(\lambda[a_i, b_i])|^n < \varepsilon.
\]

Bongiorno [3] proved that for all \(0 < \alpha < 1\), the classes \(\alpha\)-\(AC_H\) and \(AC_H\) coincide.
Theorem 2.3. ([3] Theorem 3) For every $0 < \alpha < 1$,
\[
\alpha\text{-}AC_H^{(n)}(\Omega, \mathbb{R}^l) = AC_H^{(n)}(\Omega, \mathbb{R}^l).
\]

3. The main result

In [3] we introduced an analog of Bongiorno’s notion for $\alpha = 1$, which
will be an important tool for the main result of this paper.

Definition 3.1. We say that a function $f : \Omega \to \mathbb{R}^l$ ($\Omega \subset \mathbb{R}^n$ open) is $1$-absolutely continuous, denoted $1$-AC$(\Omega, \mathbb{R}^l)$ or $1$-AC, if for every
$\varepsilon > 0$, there exists $\delta > 0$, such that for any finite collection of disjoint
1-regular intervals $\{[a_i, b_i] \subset \Omega\}_{i=1}^k$ we have
\[
\sum_{i=1}^k \mathcal{L}^n([a_i, b_i]) < \delta \Rightarrow \sum_{i=1}^k |f(a_i) - f(b_i)|^n < \varepsilon.
\]

Note that for all $\alpha < 1$, every $\alpha$-regular interval is also 1-regular. Thus for all $\alpha < 1$,
\[
\alpha\text{-}AC \subseteq 1\text{-}AC.
\]

Properties of the class $1$-AC were studied in [3], where it was shown that it contains many pathological functions. In particular functions in $1$-AC don’t need to be continuous and even if they are differentiable a.e. they do not need to belong to the Sobolev space $W^{1,n}$. However, the class $1$-AC is very useful for characterizing the classes $\alpha$-AC.

Theorem 3.2. For all $0 < \alpha < 1$ we have
\[
\alpha\text{-}AC^{(n)}(\Omega, \mathbb{R}^l) = 1\text{-}AC^{(n)}(\Omega, \mathbb{R}^l) \cap AC_H^{(n)}(\Omega, \mathbb{R}^l).
\]

Proof. By definitions, $\alpha\text{-}AC^{(n)}(\Omega, \mathbb{R}^l) \subset 1\text{-}AC^{(n)}(\Omega, \mathbb{R}^l)$, and by [2] Theorem 7], $\alpha\text{-}AC^{(n)}(\Omega, \mathbb{R}^l) \subset AC_H^{(n)}(\Omega, \mathbb{R}^l)$, which proves that
\[
\alpha\text{-}AC^{(n)}(\Omega, \mathbb{R}^l) \subset 1\text{-}AC^{(n)}(\Omega, \mathbb{R}^l) \cap AC_H^{(n)}(\Omega, \mathbb{R}^l).
\]

For the other direction, let $f \in 1\text{-}AC^{(n)}(\Omega, \mathbb{R}^l) \cap AC_H^{(n)}(\Omega, \mathbb{R}^l)$ and $\varepsilon > 0$. Let $\beta = (\frac{\alpha}{2-\alpha})^n$ and $\lambda = 1 - \frac{\alpha}{2}$. By Theorem [2,3], $f \in \beta\text{-}AC_H^{(n)}(\Omega, \mathbb{R}^l)$ and there exists $\delta_1 > 0$ so that for each finite family of disjoint $\beta$-regular intervals $\{[a_i, b_i] \subset \Omega\}$
\[
\sum_i \mathcal{L}^n([a_i, b_i]) < \delta_1 \Rightarrow \sum_i |f(\lambda[a_i, b_i])|^n < \frac{\varepsilon}{3^n+1}.
\]

Since $f \in 1\text{-}AC^{(n)}(\Omega, \mathbb{R}^l)$, there exists $\delta_2 > 0$ so that for each finite family of disjoint 1-regular intervals $\{[a_i, b_i] \subset \Omega\}$
\[
\sum_i \mathcal{L}^n([a_i, b_i]) < \delta_2 \Rightarrow \sum_i |f([a_i, b_i])|^n < \frac{\varepsilon}{3^n+1}.
\]
Let \( \delta = \min\{\delta_1, \delta_2\} \) and \( \{[a_i, b_i] \subset \Omega\} \) be a finite family of disjoint \( \alpha \)-regular intervals with \( \sum_{i} L^n([a_i, b_i]) < \delta \). We will show that \( \sum_{i} |f([a_i, b_i])|^n < \varepsilon \).

For each \([a_i, b_i]\) from this family, let \( \eta_i = \max_{1 \leq \nu \leq n} (b_{i\nu} - a_{i\nu}). \) Since \([a_i, b_i]\) is an \( \alpha \)-regular interval, we get that
\[
\prod_{j=1}^{n} \left( \frac{b_{ij} - a_{ij}}{\eta_i} \right) \geq \alpha.
\]

Thus, for every \( j = 1, \ldots, n, \)
\[
\frac{b_{ij} - a_{ij}}{\eta_i} \geq \alpha.
\]

Let
\[
c_i = \left( a_{ij} + \frac{\alpha}{4} \eta_i \right)^n, \quad d_i = \left( b_{ij} - \frac{\alpha}{4} \eta_i \right)^n.
\]

Then, for every \( j = 1, \ldots, n, \) \( c_{ij} < d_{ij} \), and for every \( i \)
\[
\max_{j} (d_{ij} - c_{ij}) = \max_{j} \left( b_{ij} - a_{ij} - \frac{\alpha}{2} \eta_i \right) = \left( 1 - \frac{\alpha}{2} \right) \eta_i.
\]

Hence
\[
\frac{L^n([c_i, d_i])}{\max_{j} (d_{ij} - c_{ij})^n} = \prod_{j=1}^{n} \left( \frac{b_{ij} - a_{ij} - \frac{\alpha}{2} \eta_i}{(1 - \frac{\alpha}{2}) \eta_i} \right) \geq \prod_{j=1}^{n} \left( \frac{\alpha \eta_i - \frac{\alpha}{2} \eta_i}{(1 - \frac{\alpha}{2}) \eta_i} \right) = \left( \frac{\alpha}{2 - \alpha} \right)^n = \beta.
\]

Thus interval \([c_i, d_i]\) is \( \beta \)-regular.

Let
\[
\overline{c_i} = \left( a_{ij} + b_{ij} - \frac{1}{\lambda} \left( \frac{a_{ij} + b_{ij}}{2} - a_{ij} - \frac{\alpha}{4} \eta_i \right) \right)_{j=1}^{n},
\]
\[
\overline{d_i} = \left( c_{ij} + d_{ij} - \frac{1}{\lambda} \left( \frac{c_{ij} + d_{ij}}{2} - c_{ij} \right) \right)_{j=1}^{n},
\]

Then \([\overline{c_i}, \overline{d_i}] \subset [a_i, b_i] \), and
\[
\lambda^{[\overline{c_i}, \overline{d_i}]} = [c_i, d_i],
\]
\[
|f([a_i, b_i])|^n \leq 3^n \left[ |f([a_i, c_i])|^n + |f(c_i, d_i)|^n + |f(d_i, b_i)|^n \right].
\]
\[ \sum_i (\mathcal{L}^n[a_i, c_i] + \mathcal{L}^n[d_i, b_i]) < \sum_i \mathcal{L}^n[a_i, b_i] < \delta, \]
\[ \sum_i \mathcal{L}^n[a_i, b_i] < \sum_i \mathcal{L}^n[a_i, c_i] < \delta. \]

Since the intervals \([a_i, c_i]\) and \([d_i, b_i]\) are 1-regular, and the intervals \([c_i, d_i]\) are \(\beta\)-regular we get
\[ \sum_i |f(a_i, b_i)|^n \leq 3^n \left[ \sum_i |f([a_i, c_i])|^n + \sum_i |f([c_i, d_i])|^n + \sum_i |f([d_i, b_i])|^n \right] < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \]

As an immediate consequence of Theorem 3.2 we obtain the following corollary.

**Corollary 3.3.** For all \(0 < \alpha, \beta < 1\) and all \(n, l \in \mathbb{N}\) we have
\[ \alpha-AC^{(n)}(\Omega, \mathbb{R}^l) = \beta-AC^{(n)}(\Omega, \mathbb{R}^l). \]

However the choice of \(\delta > 0\) in (1.2) cannot be made uniformly for all \(\alpha \in (0, 1)\).

**Proof.** To prove the final statement, suppose that \(f\) is a function so that for all \(\varepsilon > 0\), there exists \(\delta > 0\), such that for all \(\alpha \in (0, 1)\) and for any finite collection of disjoint \(\alpha\)-regular intervals \(\{[a_i, b_i] \subset \Omega\}_{i=1}^k\), the implication (1.2) holds. Note that every nontrivial interval in \(\mathbb{R}^n\), i.e. an interval \([x, y]\) such that \(x_\nu < y_\nu\) for all \(\nu = 1, \ldots, n\), is \(\alpha\)-regular for
some $\alpha \in (0, 1)$. Thus (1.2) holds for any finite collection of nontrivial intervals. By [8, Theorem 3.1] this implies that $f$ is constant. □

Remark 3.4. In [4] Bongiorno introduced another generalization of absolute continuity, a class $AC^{n}_{\Lambda}(\Omega, \mathbb{R}^{m})$, briefly $AC_{\Lambda}$, which strictly contains the class $AC_{H}$, and such that all functions in $AC_{\Lambda}$ are differentiable a.e. and satisfy the Luzin (N) property, but $AC^{n}_{\Lambda}(\Omega, \mathbb{R}^{l}) \not\subset W^{1,n}_{loc}(\Omega, \mathbb{R}^{l})$. It would be interesting to determine what are the classes $1-AC \cap AC_{\Lambda}$ and $1-AC \cap AC_{\Lambda} \cap W^{1,n}_{loc}$ (since, by [8], $1-AC \not\subset W^{1,n}_{loc}$).

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