The master equation for the reduced open-system dynamics, including a Lindbladian description of finite-duration measurement

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Abstract

We consider the problem of the measurement of a system occurring during a finite time interval, while environmentally-induced noise decreases the system-state coherence. We assume a Markovian measuring device and, therefore, use a Lindbladian description for the measurement dynamics. For studying the case of noise produced by a non-Markovian environment, whose definition does not include the measuring apparatus, we use the Redfield approach to the interaction between system and environment. In the present hybrid theory, to trace out the environmental degrees of freedom, we introduce an analytic method based on superoperator algebra and Nakajima-Zwanzig projectors. The resulting master equation, describing the reduced system dynamics, is illustrated in the case of a qubit under phase noise during a finite-time measurement.

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I. INTRODUCTION

Quantum dynamical semigroups [1] constitute a powerful mathematical formalism for the treatment of non-unitary processes in quantum mechanics, which leads to many applications in the field of open systems, such as analysis of dissipation, decoherence and quantum measurement theory [2]. It has been emphasized [5, 6] that such phenomena can not have an adequate description by the conventional formalism of quantum mechanics, based solely on the Liouville-von Neumann equation.

The most general form for the generator of quantum dynamical semigroups is the Lindblad [7, 8], with several applications to Markovian processes [2] and, more recently, to non-Markovian situations as well [9, 10]. Indeed, as demonstrated in [2], even in the case of the quantum Brownian motion it is possible to transform the Caldeira-Leggett master equation [11] into a Lindblad equation, with the addition of a term that becomes small in the high-temperature limit. Moreover, in its stochastic form, the Lindblad equation can be used in numerical methods, such as in the quantum-state diffusion approach [12–14].

Thus, the Lindblad equation is widely used to describe irreversible quantum phenomena, such as the theoretical representation of finite-duration measurements. If we assume that the environment of an open quantum system under measurement does not include the measuring device, the measurement can, therefore, be approached using a Lindblad equation involving a set of Lindbladian operators taken as the measured observables, but with the environmental degrees of freedom also included in the global description. The next step to be taken in such a model is to trace the environment variables out of the formulation, leaving only a master equation describing both, the system and the set of measuring observables. The tracing procedure is usually very complicated due to the non-commutativity of several terms in the total Hamiltonian comprising the system under scrutiny, its environment, and the observables to be measured.

Moreover, in the finite-time measurement, we can consider a Markovian interaction between the system to be measured and the measuring apparatus. In that case, the relevant specification of the apparatus is effectively contained in the Lindbladian and the problem can be treated with the usual tools of that context. However, if we consider that the interaction between the system and its environment is non-Markovian, the action of the environmental degrees of freedom will be described by the Liouvillian only.
In this case, the entanglement between the relevant system and its environment complicates the description of the reduced time evolution of the system. One way of taking into account the correlations between the system and its environment involves the use of a projector superoperator that transforms the total density operator $\hat{\rho}$ into its reduced counterpart, $\hat{\rho}_S$, calculated at the instant under consideration, tensorially multiplied by the reduced density matrix of the environment, $\hat{\rho}_B$, calculated at the initial time, that is, $\hat{\rho}(t) \mapsto \hat{\rho}_S(t) \otimes \hat{\rho}_B(0)$, where the most popular approach is the one by Nakajima and Zwanzig [15, 16], who were motivated by the work of van Hove, Prigogine, and Resibois [17–20]. Even so, however, it is still necessary to trace out the interaction to obtain the temporal equation for the relevant part of the system. Accomplish this task using directly the exponentials of each term to suppress them makes the calculations extremely difficult and prohibitively expensive. The procedure so hard, giving rise to many errors.

In this paper, we will analyze finite-time measurement for a case of Markovian interaction between the principal system and the measurement and non-Markovian interaction with the system and the environment. We present a method to obtain the master equation in the Born-Markov approximation, by tracing out the environmental degrees of freedom using the time-independent thermodynamic projectors and superoperators for each Hamiltonian (system, environment and interaction), including the Lindbladian term. The method is easily demonstrated and the superoperator properties substantially simplify the calculations, rendering the resulting master equation compact and very simple.

Although we use the Born-Markov approximation and time-independent thermodynamic projector superoperators, we believe it to be possible to generalize the present approach to higher orders of the perturbation series, as in [21, 22], non-Markovian cases [23–26], or time-dependent thermodynamic projectors, as in [27, 28].

The paper is structured as follows: the general problem is formulated in the Sec. 2, with the definition of superoperators and thermodynamic projectors; in Sec. 3, the necessary properties of these superoperators are demonstrated; and, in Sec. 4, the master equation is obtained. Finally, in Se. 5, we illustrate the method in the case of a single qubit, subject to phase errors induced by the environment, when a finite-duration measurement is performed on an observable that commute with the unperturbed qubit Hamiltonian.
II. DEFINITIONS

The most general form for a master equation, according to the quantum dynamical semigroups approach, is the Lindblad equation \[1, 2, 7, 8\],

\[
\frac{d}{dt} \hat{\rho}_{SB} = -\frac{i}{\hbar} \left[ \hat{H}, \hat{\rho}_{SB} \right] + \sum_j \left( \hat{L}_j \hat{\rho}_{SB} \hat{L}_j^\dagger - \frac{1}{2} \left\{ \hat{L}_j^\dagger \hat{L}_j, \hat{\rho}_{SB} \right\} \right),
\] (1)

where $\hat{H}$ is the total Hamiltonian and the $\hat{L}_j$ are the Lindblads (Hermitian operators for measurement description, not-Hermitian for dissipation description) that represent a Markovian interaction. Here we will consider the study of finite-time measurement.

It is important to point out that the Lindbladian operator of Eq. (1) appears as an effective device to emulate a measurement on the system. Thus, the Lindbladian evolution plays the role of the measuring-device action on the system. In fact, had we started from an ab initio microscopic description of a measuring apparatus, we would have to distinguish between its microscopic and macroscopic variables. For each of the possible eigenvalues of the observable being measured, there must correspond a single value of a macroscopic variable. However, such a variable alone cannot completely describe the state of the apparatus, for there is a large number of its microscopic states that correspond to the same value of the macroscopic variable. Since, in a measurement, the only variable that matters to the observer is the macroscopic one, we can trace out the microscopic degrees of freedom of the apparatus. This procedure results in a master equation which, under the Born-Markov approximation \[2\], assumes the Lindblad form, which we adopt here \[3, 4\].

Let us consider that the Hamiltonian $\hat{H}$ can be written as

\[
\hat{H} = \hat{H}_B + \hat{H}_{SB} + \hat{H}_S,
\]

where $\hat{H}_B$ describes the environment, $\hat{H}_S$ describes the relevant system, and $\hat{H}_{SB}$ is the Hamiltonian for the system-environment non-Markovian interaction. Here we assume, as usual, that the environment-system interaction is factorizable, i.e.,

\[
\hat{H}_{SB} = \sum_k \hat{S}_k \hat{B}_k,
\] (2)

where, for each $k$, $\hat{S}_k$ operates only on the system and $\hat{B}_k$, only on the environment. The form of the interaction, Eq. \[2\], is general, satisfied by both amplitude and phase damping.
models. By hypothesis, the Lindblads $\hat{L}_j$ will operate only on the system $S$. Our aim is to obtain an equation for the time evolution of the system reduced density matrix, $\hat{\rho}_S$,

$$\hat{\rho}_S (t) = \text{tr}_B \{ \hat{\rho}_{SB} (t) \}.$$ 

Let us begin by defining some superoperators. For any density-matrix operator $\hat{X}$,

$$B\hat{X} = -\frac{i}{\hbar} [\hat{H}_B, \hat{X}], \quad (3)$$

$$S\hat{X} = -\frac{i}{\hbar} [\hat{H}_S, \hat{X}] + \sum_j \left( \hat{L}_j \hat{X} \hat{L}_j^\dagger - \frac{1}{2} \{ \hat{L}_j^\dagger \hat{L}_j, \hat{X} \} \right), \quad (4)$$

and

$$F\hat{X} = -\frac{i}{\hbar} [\hat{H}_{SB}, \hat{X}]. \quad (5)$$

We will also use the Nakajima-Zwanzig thermodynamic projectors $P$ and $Q$ [2, 15, 16]. $P$ is such that its action is defined, for any density operator $\hat{\rho}_B (t_0)$, by

$$P\hat{X} (t) = \hat{\rho}_B (t_0) \otimes \text{Tr}_B \{ \hat{X} (t) \}. \quad (6)$$

It is easy to check that

$$P^2 = P.$$ 

We also define

$$Q = I - P, \quad (7)$$

where $I$ is the identity superoperator ($I\hat{X} (t) = \hat{X} (t)$). It follows from that definition that $Q$ is also a projector, i.e.,

$$Q^2 = Q.$$ 

III. THE SUPEROPERATOR PROPERTIES OF $B$, $S$ AND $F$

As a consequence of the definitions [3] and [4], $B$ acts only on the environment and $S$, only on the system. Hence,

$$BS = SB.$$
It follows from that commutation relation that
\[ \exp (S t + B t) = \exp (S t) \exp (B t) = \exp (B t) \exp (S t). \]

For simplicity, let us consider the initial time as zero \( (t_0 = 0) \) and that the global density operator is initially factorized:
\[ \hat{\rho}_{SB} (0) = \hat{\rho}_S (0) \otimes \hat{\rho}_B (0). \]

For the sake of simplicity, let us ignore the \( \otimes \) symbol henceforth and write
\[ \hat{\rho}_{SB} (0) = \hat{\rho}_S (0) \hat{\rho}_B (0) = \hat{\rho}_B (0) \hat{\rho}_S (0). \]

The partial trace, over the environmental degrees of freedom, of the resulting action of \( \exp (-B t) \) on the global density operator, can be written as
\[
\text{Tr}_B \{ \exp (-B t) \hat{\rho}_{SB} (t) \} = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \text{Tr}_B \{ \mathcal{B}^n \hat{\rho}_{SB} (t) \}
= \text{Tr}_B \{ \hat{\rho}_{SB} (t) \} + \sum_{n=1}^{\infty} \frac{(-t)^n}{n!} \text{Tr}_B \{ \mathcal{B}^n \hat{\rho}_{SB} (t) \}.
\]
But,
\[
\text{Tr}_B \{ \mathcal{B}^n \hat{\rho}_{SB} (t) \} = \text{Tr}_B \{ \mathcal{B} \mathcal{B}^{n-1} \hat{\rho}_{SB} (t) \}
= -\frac{i}{\hbar} \text{Tr}_B \{ [\hat{H}_B, \mathcal{B}^{n-1} \hat{\rho}_{SB} (t)] \}. \tag{8}
\]
The trace is basis independent and, therefore, it is convenient, for the treatment of Eq. (8), to use the environmental basis \( \{|k\} \) of eigenstates of \( \hat{H}_B \), i.e.,
\[ \hat{H}_B |k\rangle = E^B_k |k\rangle, \]
for all \( k \) (where the eigenstates can be degenerate or not). With this consideration,
\[
\text{Tr}_B \{ \mathcal{B}^n \hat{\rho}_{SB} (t) \}
= -\frac{i}{\hbar} \sum_k \langle k | [\hat{H}_B, \mathcal{B}^{n-1} \hat{\rho}_{SB} (t)] |k\rangle
= -\frac{i}{\hbar} \langle k | \hat{H}_B \mathcal{B}^{n-1} \hat{\rho}_{SB} (t) |k\rangle - \langle k | \mathcal{B}^{n-1} \hat{\rho}_{SB} (t) \hat{H}_B |k\rangle
= -\frac{i}{\hbar} \langle k | E^B_k \mathcal{B}^{n-1} \hat{\rho}_{SB} (t) |k\rangle - \langle k | \mathcal{B}^{n-1} \hat{\rho}_{SB} (t) E^B_k |k\rangle
= -\frac{i}{\hbar} [E^B_k \langle k | \mathcal{B}^{n-1} \hat{\rho}_{SB} (t) |k\rangle - E^B_k \langle k | \mathcal{B}^{n-1} \hat{\rho}_{SB} (t) |k\rangle]
= 0, \text{ for all } k.
\]
Therefore,
\[
\text{Tr}_B \{ \exp(-Bt) \hat{\rho}_{SB}(t) \} = \text{Tr}_B \{ \hat{\rho}_{SB}(t) \} + \sum_{n=1}^{\infty} \frac{(-t)^n}{n!} \text{Tr}_B \{ B^n \hat{\rho}_{SB}(t) \} = \text{Tr}_B \{ \hat{\rho}_{SB}(t) \} + 0 = \text{Tr}_B \{ \hat{\rho}_{SB}(t) \}.
\]
(9)

From this property, it follows that
\[
\exp(-St) \hat{\rho}_{S}(t) = \exp(-St) \text{Tr}_B \{ \hat{\rho}_{SB}(t) \} = \exp(-St) \text{Tr}_B \{ \exp(-Bt) \hat{\rho}_{SB}(t) \} = \text{Tr}_B \{ \exp(-St) \exp(-Bt) \hat{\rho}_{SB}(t) \} = \text{Tr}_B \{ \exp(-St - Bt) \hat{\rho}_{SB}(t) \}.
\]
(10)

For the interaction superoperator of Eq. (5), we have, from the hypothesis,
\[
\mathcal{F} \hat{X} = -\frac{i}{\hbar} \left[ \sum_k \hat{S}_k \hat{B}_k, \hat{X} \right] = -\frac{i}{\hbar} \sum_k \left[ \hat{S}_k \hat{B}_k, \hat{X} \right]
\]
for any density operator \( \hat{X} \).

**IV. TRACING OUT THE ENVIRONMENTAL DEGREES OF FREEDOM**

The equation that we want to solve is
\[
\frac{d}{dt} \hat{\rho}_{SB}(t) = -\frac{i}{\hbar} \left[ \hat{H}_S + \hat{H}_B + \hat{H}_{SB}, \hat{\rho}_{SB}(t) \right] + \sum_j \left( \hat{L}_j \hat{\rho}_{SB} \hat{L}_j^\dagger - \frac{1}{2} \left\{ \hat{L}_j^\dagger \hat{L}_j, \hat{\rho}_{SB} \right\} \right).
\]

In terms of the superoperators of Eqs. (3), (4), and (5), the equation becomes
\[
\frac{d}{dt} \hat{\rho}_{SB}(t) = (S + B + \mathcal{F}) \hat{\rho}_{SB}(t).
\]
(11)

With the properties expressed by Eqs. (9) and (10), let us define the operator
\[
\hat{\alpha}(t) = \exp(-St - Bt) \hat{\rho}_{SB}(t).
\]
(12)

Then,
\[
\hat{\rho}_{S}(t) = \exp(St) \text{Tr}_B \{ \hat{\alpha}(t) \}
\]
(13)
and the equation satisfied by $\hat{\alpha}(t)$ is written as

$$
\frac{d}{dt} \hat{\alpha}(t) = \frac{d}{dt} [\exp (-St - Bt) \hat{\rho}_{SB}(t)]
= \frac{d}{dt} \exp (-St - Bt) \hat{\rho}_{SB}(t) + \exp (-St - Bt) \frac{d}{dt} \hat{\rho}_{SB}(t)
= - (S + B) \exp (-St - Bt) \hat{\rho}_{SB}(t) + \exp (-St - Bt) \frac{d}{dt} \hat{\rho}_{SB}(t).
$$

(14)

We can insert Eq. (11) into Eq. (14) and the result is

$$
\frac{d}{dt} \hat{\alpha}(t) = \exp (-St - Bt) \mathcal{F} \exp (St + Bt) \hat{\alpha}(t).
$$

Let us define, then, the superoperator:

$$
\mathcal{G}(t) = \exp (-St - Bt) \mathcal{F} \exp (St + Bt).
$$

(15)

We note that $\mathcal{G}(t)$ is of the order of magnitude of the interaction $\hat{H}_{SB}$, because it is proportional to the super-operator $\mathcal{F}$ that, in turn, is of the order of magnitude of $\hat{H}_{SB}$. Thus, we have

$$
\frac{d}{dt} \hat{\alpha}(t) = \mathcal{G}(t) \hat{\alpha}(t).
$$

(16)

Up to this moment, we have used superoperators based on each of the Hamiltonian and Lindbladian terms ($\mathcal{B}$, $\mathcal{S}$ e $\mathcal{F}$) and obtained Eq. (16), which involves the global density operator $\hat{\rho}_{SB}(t)$. Now, let us use the Nakajima-Zwanzig thermodynamic projectors, defined by Eqs. (6) and (7), in Eq. (16):

$$
\mathcal{P} \frac{d}{dt} \hat{\alpha}(t) = \mathcal{P} \mathcal{G}(t) \hat{\alpha}(t)
= \mathcal{P} \mathcal{G}(t) (\mathcal{P} + \mathcal{Q}) \hat{\alpha}(t)
= \mathcal{P} \mathcal{G}(t) \mathcal{P} \hat{\alpha}(t) + \mathcal{P} \mathcal{G}(t) \mathcal{Q} \hat{\alpha}(t)
= \mathcal{P} \mathcal{G}(t) \mathcal{P}^2 \hat{\alpha}(t) + \mathcal{P} \mathcal{G}(t) \mathcal{Q}^2 \hat{\alpha}(t)
$$

and

$$
\mathcal{Q} \frac{d}{dt} \hat{\alpha}(t) = \mathcal{Q} \mathcal{G}(t) \hat{\alpha}(t)
= \mathcal{Q} \mathcal{G}(t) (\mathcal{P} + \mathcal{Q}) \hat{\alpha}(t)
= \mathcal{Q} \mathcal{G}(t) \mathcal{P} \hat{\alpha}(t) + \mathcal{Q} \mathcal{G}(t) \mathcal{Q} \hat{\alpha}(t)
= \mathcal{Q} \mathcal{G}(t) \mathcal{P}^2 \hat{\alpha}(t) + \mathcal{Q} \mathcal{G}(t) \mathcal{Q}^2 \hat{\alpha}(t).
$$
Since those projectors are time-independent, it follows that

\[
\begin{align*}
\frac{d}{dt} [\mathcal{P} \alpha (t)] &= [\mathcal{P} \mathcal{G} (t) \mathcal{P}] [\mathcal{P} \alpha (t)] + [\mathcal{P} \mathcal{G} (t) \mathcal{Q}] [\mathcal{Q} \alpha (t)] , \\
\frac{d}{dt} [\mathcal{Q} \alpha (t)] &= [\mathcal{Q} \mathcal{G} (t) \mathcal{P}] [\mathcal{P} \alpha (t)] + [\mathcal{Q} \mathcal{G} (t) \mathcal{Q}] [\mathcal{Q} \alpha (t)] .
\end{align*}
\]

Let us formally integrate the second of the Eqs. (17):

\[
\hat{Q} \alpha (t) = \hat{Q} \alpha (0) + \int_0^t dt' [\mathcal{Q} \mathcal{G} (t') \mathcal{P}] [\mathcal{P} \hat{\alpha} (t')] + \int_0^t dt' [\mathcal{Q} \mathcal{G} (t') \mathcal{Q}] [\mathcal{Q} \hat{\alpha} (t')] .
\]

From the definition of \( \hat{\alpha} (t) \), Eq. (12), we know that

\[
\hat{\alpha} (0) = \hat{\rho}_{SB} (0) = \hat{\rho}_{S} (0) \hat{\rho}_{B} (0)
\]

and, therefore,

\[
\hat{Q} \alpha (0) = (\mathcal{I} - \mathcal{P}) \hat{\rho}_{S} (0) \hat{\rho}_{B} (0)
\]

\[
= \hat{\rho}_{S} (0) \hat{\rho}_{B} (0) - \mathcal{P} \hat{\rho}_{S} (0) \hat{\rho}_{B} (0)
\]

\[
= \hat{\rho}_{S} (0) \hat{\rho}_{B} (0) - \hat{\rho}_{B} (0) \hat{\rho}_{S} (0) \text{Tr}_B \{ \hat{\rho}_{S} (0) \hat{\rho}_{B} (0) \}
\]

\[
= \hat{\rho}_{S} (0) \hat{\rho}_{B} (0) - \hat{\rho}_{B} (0) \hat{\rho}_{S} (0) \text{Tr}_B \{ \hat{\rho}_{B} (0) \}
\]

\[
= \hat{\rho}_{S} (0) \hat{\rho}_{B} (0) - \hat{\rho}_{B} (0) \hat{\rho}_{S} (0) 
\]

\[
= 0.
\]

Hence,

\[
\hat{Q} \alpha (t) = \int_0^t dt' [\mathcal{Q} \mathcal{G} (t') \mathcal{P}] [\mathcal{P} \hat{\alpha} (t')] + \int_0^t dt' [\mathcal{Q} \mathcal{G} (t') \mathcal{Q}] [\mathcal{Q} \hat{\alpha} (t')] ,
\]

showing that \( \hat{Q} \alpha (t) \) is of the order of magnitude of \( \hat{H}_{SB} \). Here we use the Born approximation and only keep terms up to the second order of \( \hat{H}_{SB} \). Accordingly, the second-order iteration of Eq. (18) gives

\[
\hat{Q} \alpha (t) = \int_0^t dt' [\mathcal{Q} \mathcal{G} (t') \mathcal{P}] [\mathcal{P} \hat{\alpha} (t')] \\
+ \int_0^t dt' [\mathcal{Q} \mathcal{G} (t') \mathcal{Q}] \left[ \int_0^{t'} dt'' [\mathcal{Q} \mathcal{G} (t'') \mathcal{P}] [\mathcal{P} \hat{\alpha} (t'')] \right] .
\]

Let us formally integrate the first of the Eqs. (17):

\[
\hat{P} \alpha (t) = \hat{P} \alpha (0) + \int_0^t dt' [\mathcal{P} \mathcal{G} (t') \mathcal{P}] [\mathcal{P} \hat{\alpha} (t')] + \int_0^t dt' [\mathcal{P} \mathcal{G} (t') \mathcal{Q}] [\mathcal{Q} \hat{\alpha} (t')] .
\]
Substituing Eq. (20) in the first of the Eqs. (17), we obtain

\[
\frac{d}{dt} [\mathcal{P} \hat{\alpha} (t)] = [\mathcal{P} \mathcal{G} (t) \mathcal{P}] [\mathcal{P} \hat{\alpha} (0)] + [\mathcal{P} \mathcal{G} (t) \mathcal{P}] \int_0^t dt' [\mathcal{P} \mathcal{G} (t') \mathcal{P}] [\mathcal{P} \hat{\alpha} (t')]
+ [\mathcal{P} \mathcal{G} (t) \mathcal{Q}] \int_0^t dt' [\mathcal{Q} \mathcal{G} (t') \mathcal{P}] [\mathcal{P} \hat{\alpha} (t')].
\] (21)

From Eq. (19) we see that the third term on the right-hand side of Eq. (21) is of the third order in \( \hat{H}_{SB} \), and, therefore, we neglect it. Substituing Eq. (19) in the fo rth term of right-hand side of Eq. (21) and keeping only contributions up to second order in \( \hat{H}_{SB} \), we obtain

\[
\frac{d}{dt} [\mathcal{P} \hat{\alpha} (t)] = [\mathcal{P} \mathcal{G} (t) \mathcal{P}] [\mathcal{P} \hat{\alpha} (0)] + [\mathcal{P} \mathcal{G} (t) \mathcal{P}] \int_0^t dt' [\mathcal{P} \mathcal{G} (t') \mathcal{P}] [\mathcal{P} \hat{\alpha} (t')]
+ [\mathcal{P} \mathcal{G} (t) \mathcal{Q}] \int_0^t dt' [\mathcal{Q} \mathcal{G} (t') \mathcal{P}] [\mathcal{P} \hat{\alpha} (t')]
= [\mathcal{P} \mathcal{G} (t) \mathcal{P}] [\mathcal{P} \hat{\alpha} (0)] + \int_0^t dt' [\mathcal{P} \mathcal{G} (t) \mathcal{P}] [\mathcal{P} \mathcal{G} (t') \mathcal{P}] [\mathcal{P} \hat{\alpha} (t')]
+ \int_0^t dt' [\mathcal{P} \mathcal{G} (t) \mathcal{Q} \mathcal{G} (t') \mathcal{P}] [\mathcal{P} \hat{\alpha} (t')]
= [\mathcal{P} \mathcal{G} (t) \mathcal{P}] [\mathcal{P} \hat{\alpha} (0)] + \int_0^t dt' [\mathcal{P} \mathcal{G} (t) \mathcal{G} (t') \mathcal{P}] [\mathcal{P} \hat{\alpha} (t')]
+ \int_0^t dt' [\mathcal{P} \mathcal{G} (t) \mathcal{Q} \mathcal{G} (t') \mathcal{P}] [\mathcal{P} \hat{\alpha} (t')]
= [\mathcal{P} \mathcal{G} (t) \mathcal{P}] [\mathcal{P} \hat{\alpha} (0)] + \int_0^t dt' [\mathcal{P} \mathcal{G} (t) \mathcal{G} (t') \mathcal{P}] [\mathcal{P} \hat{\alpha} (t')]
= [\mathcal{P} \mathcal{G} (t) \mathcal{P}] [\mathcal{P} \hat{\alpha} (0)]
+ \int_0^t dt' [\mathcal{P} \mathcal{G} (t) \mathcal{G} (t') \mathcal{P}] [\mathcal{P} \hat{\alpha} (t')]
+ \int_0^t dt' [\mathcal{P} \mathcal{G} (t) (\mathcal{P} + \mathcal{Q}) \mathcal{G} (t') \mathcal{P}] [\mathcal{P} \hat{\alpha} (t')],
\]

i.e.,

\[
\frac{d}{dt} [\mathcal{P} \hat{\alpha} (t)] = [\mathcal{P} \mathcal{G} (t) \mathcal{P}] [\mathcal{P} \hat{\alpha} (0)] + \int_0^t dt' [\mathcal{P} \mathcal{G} (t) \mathcal{G} (t') \mathcal{P}] [\mathcal{P} \hat{\alpha} (t')].
\]
The first term of this equation can be written as

\[
[\mathcal{P} \mathcal{G} \left( t \right) \mathcal{P}] [\mathcal{P} \dot{\alpha} \left( 0 \right)] = \mathcal{P} \mathcal{G} \left( t \right) \mathcal{P} \mathcal{P} \dot{\alpha} \left( 0 \right) \\
= \mathcal{P} \mathcal{G} \left( t \right) \mathcal{P} \dot{\alpha} \left( 0 \right) \\
= \mathcal{P} \mathcal{G} \left( t \right) \mathcal{P} \dot{\rho}_S \left( 0 \right) \dot{\rho}_B \left( 0 \right) \\
= \mathcal{P} \mathcal{G} \left( t \right) \dot{\rho}_B \left( 0 \right) \text{Tr}_B \{ \dot{\rho}_S \left( 0 \right) \dot{\rho}_B \left( 0 \right) \} \\
= \mathcal{P} \mathcal{G} \left( t \right) \dot{\rho}_B \left( 0 \right) \dot{\rho}_S \left( 0 \right) \\
= \dot{\rho}_B \left( 0 \right) \text{Tr}_B \{ \mathcal{G} \left( t \right) \dot{\rho}_B \left( 0 \right) \dot{\rho}_S \left( 0 \right) \} \\
= \dot{\rho}_B \left( 0 \right) \text{Tr}_B \{ \mathcal{G} \left( t \right) \dot{\rho}_B \left( 0 \right) \} \dot{\rho}_S \left( 0 \right). 
\]

But, following from Eq. (15),

\[
\text{Tr}_B \{ \mathcal{G} \left( t \right) \dot{\rho}_B \left( 0 \right) \} \dot{\rho}_S \left( 0 \right) = \text{Tr}_B \{ \exp \left( -S t \right) \mathcal{F} \exp \left( S t + B t \right) \dot{\rho}_B \left( 0 \right) \} \dot{\rho}_S \left( 0 \right) \\
= \text{exp} \left( -S t \right) \text{Tr}_B \{ \exp \left( -B t \right) \mathcal{F} \exp \left( B t \right) \dot{\rho}_B \left( 0 \right) \} \exp \left( S t \right) \dot{\rho}_S \left( 0 \right),
\]

i.e.,

\[
[\mathcal{P} \mathcal{G} \left( t \right) \mathcal{P}] [\mathcal{P} \dot{\alpha} \left( 0 \right)] = \dot{\rho}_B \left( 0 \right) \exp \left( -S t \right) \text{Tr}_B \{ \exp \left( -B t \right) \mathcal{F} \exp \left( B t \right) \dot{\rho}_B \left( 0 \right) \} \exp \left( S t \right) \dot{\rho}_S \left( 0 \right). 
\]

Now, let us use Eq. (2):

\[
\text{Tr}_B \{ \exp \left( -B t \right) \mathcal{F} \exp \left( B t \right) \dot{\rho}_B \left( 0 \right) \} = -\frac{i}{\hbar} \sum_k \text{Tr}_B \{ \exp \left( -B t \right) [S_k B_k, \exp \left( B t \right) \dot{\rho}_B \left( 0 \right)] \} \\
= -\frac{i}{\hbar} \sum_k \text{Tr}_B \{ \exp \left( -B t \right) S_k [B_k, \exp \left( B t \right) \dot{\rho}_B \left( 0 \right)] \} \\
= -\frac{i}{\hbar} \sum_k S_k \text{Tr}_B \{ \exp \left( -B t \right) [B_k, \exp \left( B t \right) \dot{\rho}_B \left( 0 \right)] \}. 
\]

Analogously to the calculation leading to Eq. (9), we obtain

\[
\text{Tr}_B \left\{ \exp \left( -B t \right) \left[ \dot{B}_k, \exp \left( B t \right) \dot{\rho}_B \left( 0 \right) \right] \right\} = \text{Tr}_B \left\{ \left[ \dot{B}_k, \exp \left( B t \right) \dot{\rho}_B \left( 0 \right) \right] \right\} \\
= \text{Tr}_B \left\{ \dot{B}_k \left[ \exp \left( B t \right) \dot{\rho}_B \left( 0 \right) \right] \right\} \\
- \text{Tr}_B \left\{ \left[ \exp \left( B t \right) \dot{\rho}_B \left( 0 \right) \right] \dot{B}_k \right\} \\
= 0,
\]
which implies that

$$[ \mathcal{P} \mathcal{G}(t) \mathcal{P} ] [ \mathcal{P} \hat{\alpha} (0) ] = 0$$

and

$$\frac{d}{dt} [ \mathcal{P} \hat{\alpha} (t) ] = \int_0^t dt' [ \mathcal{P} \mathcal{G}(t) \mathcal{G}(t') \mathcal{P} ] [ \mathcal{P} \hat{\alpha} (t') ]$$

$$= \int_0^t dt' [ \mathcal{P} \mathcal{G}(t) \mathcal{G}(t') \mathcal{P} \mathcal{P} \hat{\alpha} (t') ]$$

$$= \int_0^t dt' [ \mathcal{P} \mathcal{G}(t) \mathcal{G}(t') \mathcal{P} \hat{\alpha} (t') ] . \quad (22)$$

Integrating Eq. (22) between $t'$ and $t$ yields

$$\mathcal{P} \hat{\alpha} (t) - \mathcal{P} \hat{\alpha} (t') = \int_{t'}^t dt'' \int_{t'}^{t''} dt''' [ \mathcal{P} \mathcal{G}(t) \mathcal{G}(t'') \mathcal{P} \hat{\alpha} (t''') ] ,$$

which shows that the difference between $\mathcal{P} \hat{\alpha} (t)$ and $\mathcal{P} \hat{\alpha} (t')$ is of the second order of magnitude in $\hat{\mathcal{H}}_{SB}$ and, therefore, we can write $\mathcal{P} \hat{\alpha} (t)$ instead of $\mathcal{P} \hat{\alpha} (t')$ in the integrand of Eq. (22), obtaining an equation that obeys the Markov approximation, without violating the Born approximation. Thus, the master equation that we finally obtain, is written

$$\frac{d}{dt} [ \mathcal{P} \hat{\alpha} (t) ] = \int_0^t dt' [ \mathcal{P} \mathcal{G}(t) \mathcal{G}(t') \mathcal{P} \hat{\alpha} (t') ] . \quad (23)$$

In the Born approximation, to obtain the reduced density operator of Eq. (13), we must solve Eq. (23) using the initial condition

$$\text{Tr}_B \{ \hat{\alpha} (0) \} = \text{Tr}_B \{ \hat{\rho}_S (0) \hat{\rho}_B (0) \}$$

$$= \hat{\rho}_S (0) \text{Tr}_B \{ \hat{\rho}_B (0) \}$$

$$= \hat{\rho}_S (0) .$$

V. EXAMPLE

As a simple example of our method, let us consider a two-level system (a spin $\frac{1}{2}$ particle pointing along the $z$ direction) in contact with a thermal bath of quantum harmonic oscillators:

$$\hat{\mathcal{H}}_S = \hbar \omega_0 \hat{\sigma}_z$$
and
\[ \hat{H}_B = \hbar \sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k. \]

We take the interaction to be the phase damping where, in reference to Eq. (2),
\begin{align*}
\hat{S}_k &= \hbar \hat{\sigma}_z, \\
\hat{B}_k &= g_k \hat{b}_k^\dagger + g_k^* \hat{b}_k,
\end{align*}
giving
\[ \hat{H}_{SB} = \hbar \sum_k \hat{\sigma}_z \left( g_k \hat{b}_k^\dagger + g_k^* \hat{b}_k \right). \tag{24} \]

For the initial state of the thermal bath, let us consider the vacuum state \((T = 0)\):
\[ \hat{\rho}_B = (|0\rangle \langle 0| \ldots) \otimes (|0\rangle \langle 0| \ldots). \tag{25} \]

Now we study the case of a measurement of the \(z\) component, using a single Lindblad:
\[ \hat{L} = \lambda \hat{\sigma}_z, \]
where \(\lambda\) is a real number. Then, Eq. (4) gives
\[ S \hat{X} = -i \omega_0 \left[ \hat{\sigma}_z, \hat{X} \right] + \lambda^2 \left( \hat{\sigma}_z \hat{X} \hat{\sigma}_z - \hat{X} \right). \]

To simplify the notation, let us define the following quantities:
\[ \hat{R}(t) \equiv \exp(-St) \hat{\rho}_S(t) \tag{26} \]
and
\[ \mathcal{P} \hat{a}(t) = \hat{R}(t) \hat{\rho}_B. \]

The action of \(\exp(St)\) and \(\exp(Bt)\) can be calculated in the following way. For an arbitrary density operator \(\hat{X}(0)\), let us define
\[ \hat{X} (t) = \exp \left( \mathcal{S} t \right) \hat{X} (0). \]

Hence,

\[ \frac{d}{dt} \hat{X} (t) = \mathcal{S} \exp \left( \mathcal{S} t \right) \hat{X} (0) = \mathcal{S} \hat{X} (t), \quad (27) \]

that is,

\[ \frac{d}{dt} \hat{X} (t) = -i\omega_0 \left[ \hat{\sigma}_z, \hat{X} \right] + \lambda^2 \left( \hat{\sigma}_z \hat{X} \hat{\sigma}_z - \hat{X} \right). \quad (28) \]

The solution of Eq. (28) can be easily determined [13]:

\[
\begin{align*}
X_{11} (t) & = X_{11} (0), \\
X_{12} (t) & = X_{12} (0) e^{-2\lambda^2 t} \left[ \cos \left( 2\omega_0 t \right) - i \sin \left( 2\omega_0 t \right) \right].
\end{align*}
\quad (29)
\]

Analogously to the case of \( \exp \left( \mathcal{S} t \right) \), for an arbitrary density operator \( \hat{X} (0) \), let us define

\[ \hat{X} (t) = \exp \left( \mathcal{B} t \right) \hat{X} (0). \]

From Eq. (3), it follows that

\[ \frac{d}{dt} \hat{X} (t) = -i \hbar \left[ \hat{H}_B, \hat{X} (t) \right], \]

whose solution is, simply,

\[ \hat{X} (t) = e^{-i \frac{\hat{H}_B t}{\hbar}} \hat{X} (0) e^{i \frac{\hat{H}_B t}{\hbar}}. \quad (30) \]

From the above explicit actions of \( \exp \left( \mathcal{S} t \right) \) and \( \exp(\mathcal{B} t) \) it easily follows that

\[
\begin{align*}
\exp \left( \mathcal{S} t \right) \exp \left( \mathcal{S} t' \right) & = \exp \left[ \mathcal{S} \left( t + t' \right) \right], \\
\exp \left( \mathcal{B} t \right) \exp \left( \mathcal{B} t' \right) & = \exp \left[ \mathcal{B} \left( t + t' \right) \right].
\end{align*}
\]

With Eqs. (30) and (29) we are able to solve Eq. (23). Separating the system and environment terms, we obtain, in terms of Eq. (26):
\[ \mathcal{P} \mathcal{G} (t) \mathcal{G} (t') \mathcal{P} \hat{\alpha} (t) = \]
\[ = e^{-St} \hat{\sigma}_z \left\{ e^{S(t-t')} \left[ (e^{St} \hat{R} (t)) \hat{\sigma}_z \right] \right\} \text{tr}_B \left\{ e^{-Bt} \sum_k \hat{B}_k \left[ e^{B(t-t')} \left( e^{Bt} \hat{\rho}_B \right) \sum_{k'} \hat{B}_{k'} \right] \right\} \otimes \hat{\rho}_B + \]
\[ -e^{-St} \left\{ e^{S(t-t')} \left[ (e^{St} \hat{R} (t)) \hat{\sigma}_z \right] \hat{\sigma}_z \text{tr}_B \left\{ e^{-Bt} \left[ e^{B(t-t')} \left( e^{Bt} \hat{\rho}_B \right) \sum_k \hat{B}_k \right] \sum_{k'} \hat{B}_{k'} \right\} \right\} \otimes \hat{\rho}_B + \]
\[ -e^{-St} \hat{\sigma}_z \left\{ e^{S(t-t')} \left[ (e^{St} \hat{R} (t)) \hat{\sigma}_z \right] \right\} \text{tr}_B \left\{ e^{-Bt} \sum_k \hat{B}_k \left[ e^{B(t-t')} \left( e^{Bt} \hat{\rho}_B \right) \sum_{k'} \hat{B}_{k'} \right] \hat{\sigma}_z \right\} \otimes \hat{\rho}_B + \]
\[ + e^{-St} \left\{ e^{S(t-t')} \left[ (e^{St} \hat{R} (t)) \hat{\sigma}_z \right] \hat{\sigma}_z \text{tr}_B \left\{ e^{-Bt} \left[ e^{B(t-t')} \left( e^{Bt} \hat{\rho}_B \right) \sum_k \hat{B}_k \right] \sum_{k'} \hat{B}_{k'} \right\} \right\} \otimes \hat{\rho}_B. \]

Expanding the environmental superoperators using Eq. (30) and grouping the similar terms, we have:

\[ \mathcal{P} \mathcal{G} (t) \mathcal{G} (t') \mathcal{P} \hat{\alpha} (t) = \]
\[ = \left\{ e^{-St} \hat{\sigma}_z \left\{ e^{S(t-t')} \left[ (e^{St} \hat{R} (t)) \hat{\sigma}_z \right] \right\} - e^{-St} \left\{ e^{S(t-t')} \left[ (e^{St} \hat{R} (t)) \hat{\sigma}_z \right] \hat{\sigma}_z \right\} \right\} \otimes \hat{\rho}_B \times \]
\[ \times \text{tr}_B \left\{ e^{iBt} \sum_k \hat{B}_k e^{-i\hat{B}_k t} \hat{\sigma}_z \left. \sum_{k'} \hat{B}_{k'} e^{-i\hat{B}_{k'} t} \right\} \right\} + \]
\[ + \left\{ e^{-St} \left\{ e^{S(t-t')} \left[ \hat{\sigma}_z \left( e^{St} \hat{R} (t) \right) \hat{\sigma}_z \right] \hat{\sigma}_z \right\} \right\} \otimes \hat{\rho}_B \times \]
\[ \times \text{tr}_B \left\{ e^{iBt} \sum_k \hat{B}_k e^{-i\hat{B}_k t} \hat{\sigma}_z \left. \sum_{k'} \hat{B}_{k'} e^{-i\hat{B}_{k'} t} \right\} \right\} . \]

Using the initial state of the environment, Eq. (25), and the interaction between the system and the environment, Eq. (24), we trace out the environmental degrees of freedom and write:

\[ \mathcal{P} \mathcal{G} (t) \mathcal{G} (t') \mathcal{P} \hat{\alpha} (t) = \]
\[ = \left\{ e^{-St} \hat{\sigma}_z \left\{ e^{S(t-t')} \left[ (e^{St} \hat{R} (t)) \hat{\sigma}_z \right] \right\} - e^{-St} \left\{ e^{S(t-t')} \left[ (e^{St} \hat{R} (t)) \hat{\sigma}_z \right] \hat{\sigma}_z \right\} \right\} \otimes \hat{\rho}_B \times \]
\[ \times \sum_k |g_k|^2 \left\{ \cos [\omega_k (t - t')] + i \sin [\omega_k (t - t')] \right\} + \]
\[ + \left\{ e^{-St} \left\{ e^{S(t-t')} \left[ \hat{\sigma}_z \left( e^{St} \hat{R} (t) \right) \hat{\sigma}_z \right] \hat{\sigma}_z \right\} \right\} \otimes \hat{\rho}_B \times \]
\[ \times \sum_k |g_k|^2 \left\{ \cos [\omega_k (t - t')] - i \sin [\omega_k (t - t')] \right\} . \]

Now we rewrite the system superoperators using Eq. (29). Thus, in terms of \( \hat{R} (t) \), we
have

$$\mathcal{P} \mathcal{G} (t) \mathcal{G} (t') \mathcal{P} \hat{\alpha} (t) =$$

$$= -4 \begin{pmatrix} 0 & R_{12} \\ R_{21} & 0 \end{pmatrix} \sum_k |g_k|^2 \cos [\omega_k (t - t')] ,$$

which, in accordance with Eq. (23), gives

$$\frac{d}{dt} \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = -4 \begin{pmatrix} 0 & R_{12} \\ R_{21} & 0 \end{pmatrix} \int_0^t dt' \sum_k |g_k|^2 \cos [\omega_k (t - t')] . \quad (32)$$

Now we make the continuum transformation. Let us define the density of states as

$$J (\omega) = \sum_k |g_k|^2 \delta (\omega - \omega_k) ,$$

which allows us to rewrite Eq. (32) as

$$\frac{d}{dt} \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = -4 \begin{pmatrix} 0 & R_{12} \\ R_{21} & 0 \end{pmatrix} \int_0^t dt' \int_0^\infty d\omega J (\omega) \cos [\omega (t - t')]$$

and, with the change of variable $\tau = t - t'$, we obtain

$$\frac{d}{dt} \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = -4 \begin{pmatrix} 0 & R_{12} \\ R_{21} & 0 \end{pmatrix} \int_0^t d\tau \int_0^\infty d\omega J (\omega) \cos (\omega \tau) . \quad (33)$$

To obtain the solution for the diagonal terms of $\hat{R} (t)$, we do not need further consideration:

$$\begin{cases} R_{11} (t) = R_{11} (0) , \\
R_{22} (t) = R_{22} (0) . \end{cases}$$

However, for the non-diagonal terms, it is necessary to specify $J (\omega)$. Let us use the Ohmic density of states:

$$J (\omega) = \eta \omega e^{-\frac{\omega}{\Omega}} ,$$

where $\eta$ and $\Omega$ are real and positive constants. Hence, Eq. (33) becomes

$$\frac{d}{dt} \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = -4\eta \begin{pmatrix} 0 & R_{12} \\ R_{21} & 0 \end{pmatrix} \int_0^t d\tau \int_0^\infty d\omega \omega e^{-\frac{\omega}{\Omega}} \cos (\omega \tau) .$$
The double integral gives
\[
\int_0^t d\tau \int_0^\infty d\omega \omega e^{-\frac{\omega}{\Omega}} \cos(\omega \tau) = \frac{\Omega^2 t}{1 + (\Omega t)^2},
\]
i.e.,
\[
\begin{aligned}
\frac{d}{dt} R_{12} &= -4\eta \frac{\Omega^2 t}{1 + (\Omega t)^2} R_{12}, \\
\frac{d}{dt} R_{21} &= -4\eta \frac{\Omega^2 t}{1 + (\Omega t)^2} R_{21}.
\end{aligned}
\]
The solution for this system is
\[
\begin{aligned}
R_{12}(t) &= \frac{R_{12}(0)}{[1 + (\Omega t)^2]^{2\eta}}, \\
R_{21}(t) &= \frac{R_{21}(0)}{[1 + (\Omega t)^2]^{2\eta}}.
\end{aligned}
\]
At last, now we calculate the density-operator elements by inverting Eq. (26) and we obtain
\[
\begin{aligned}
\rho_{11}(t) &= \rho_{11}(0), \\
\rho_{12}(t) &= \rho_{12}(0) \frac{e^{-2\lambda^2 t}}{[1 + (\Omega t)^2]^{2\eta}} \left[ \cos(2\omega_0 t) - is\sin(2\omega_0 t) \right],
\end{aligned}
\]
remembering that \(\rho_{22}(t) = 1 - \rho_{11}(t)\) and \(\rho_{21}(t) = \rho_{12}^\ast(t)\).

VI. CONCLUSION

In summary, here we present a new method to describe the dynamics of measurements that occur during a finite time interval, while the system being measured interacts with the rest of the universe and, due to the consequent environmentally-induced noise, undergoes decoherence. We use a Lindbladian description of the measuring apparatus, whose interaction with the system we assume as Markovian. To treat the noise introduced by the fact that, during the finite-duration measurement, the system is perturbed by the environment, we use a Redfield approach to the dynamical description of the interaction between the system and its environment, assumed non-Markovian. The resulting unprecedented hybrid description is shown to be capable of substantially simplifying the tracing procedure, which is usually very complicated due to the non-commutativity of several terms in the total Hamiltonian, comprising the system under scrutiny, its environment, and the observables to be measured.
The superoperators defined in Sec. 2 introduce simplifications of the calculations leading to Eq. (23), that is compact and can be solved in terms of the unperturbed solutions (in Sec. 5, for example, we used Eq. (29)). Moreover, for the Born-Markov approximation, regardless of the model chosen for the environment, the reduction of the density operator becomes evident, as can be verified in Eq. (31). The simple phase-damping-interaction example of Sec. 5 (see Eq. (24)) at zero temperature already provides an important and expected result in the quantum mechanics of open systems: the intensification of the environmentally-induced decoherence, as indicated in the denominator of the second of Eqs. (34), evidencing the power and convenience of the present approach.

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