FUSION RULES ON A PARAMETRIZED SERIES OF GRAPHS

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Abstract. A series of pairs of graphs \((\Gamma_k, \Gamma'_k), k = 0, 1, 2, \ldots\) has been considered as candidates for dual pairs of principal graphs of subfactors of small Jones index above 4 and it has recently been proved that the pair \((\Gamma_k, \Gamma'_k)\) comes from a subfactor if and only if \(k = 0\) or \(k = 1\). We show that nevertheless there exists a unique fusion system compatible with this pair of graphs for all non-negative integers \(k\).

1. Introduction

A subfactor \(N \subset M\) with finite index and finite depth generates finitely many isomorphism classes of bimodules with four different combinations of left and right coefficients. They form a bi-graded fusion category. Its Grothendieck ring form a fusion ring or a fusion hypergroup, namely a bi-graded \(\mathbb{Z}\)-algebra \(\mathcal{A}\) with following properties:

- it has a basis given by finitely many irreducible bimodules of four different kinds \(\mathcal{X} = N\mathcal{X}_N \sqcup N\mathcal{X}_M \sqcup M\mathcal{X}_N \sqcup M\mathcal{X}_M\) (we call the labels \(N, M\) right or left coefficients, depending on the position),
- an involution \(X \in P\mathcal{X}_Q \to \overline{X} \in Q\mathcal{X}_P\) is defined, where \(P, Q \in \{N, M\}\).
- a product is defined for a pair of bimodules with “matching” coefficient, namely, for a pair \((X, Y) \in \mathcal{X} \times \mathcal{X}\) such that the right coefficient of \(X\) and the left coefficient of \(Y\) match, \(XY\) is defined. It decomposes as follows:

\[
XY = \sum N_{X,Y}^Z Z,
\]

where the sum is taken over those \(Z \in \mathcal{X}\) that have the same left (resp. right) coefficient as \(X\) (resp. \(Y\)), and \(N_{X,Y}^Z \in \mathbb{N}_0\), moreover Frobenius reciprocity holds:

\[
N_{X,Y}^Z = N_{Z,Y}^X = N_{Y,Z}^X = N_{Y,Z}^X = N_{Y,Z}^X = N_{Y,Z}^X.
\]

- There are identity objects \(1_N \in N\mathcal{X}_N, 1_M \in M\mathcal{X}_M\) that act as identity with respect to the product, whenever it is defined.

The involution extends linearly to define an involution on \(\mathcal{A}\). For a fusion ring \(\mathcal{A}\), there is a unique weight function \(\mu : \mathcal{A} \to \mathbb{R}_{\geq}\) satisfying

\[
\mu(1_N) = \mu(1_M) = 1,
\]

\[
\mu(XY) = \mu(X)\mu(Y),
\]

where \(X, Y, Z \in \mathcal{X}\) are with suitable coefficients for each equality, so that \(XY\) and \(X + Z\) are defined. The (dual) principal graph of the subfactor encode partial information of the fusion

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algebra: namely, the (dual) principal graph has the vertices corresponding to \( N\mathcal{X}_N \sqcup N\mathcal{X}_M \) (resp. \( M\mathcal{X}_N \sqcup M\mathcal{X}_M \)), with the number of the edges between vertices \( N\mathcal{X}_N \) and \( N\mathcal{Y}_M \) (resp. \( M\mathcal{X}_M \) and \( M\mathcal{Y}_N \)) given by \( N\mathcal{Y}_N \mathcal{X}_M \) (resp. \( N\mathcal{Y}_M \mathcal{X}_N \)).

On the other hand, one may start with a pair of graphs, and may consider if there is a fusion algebra compatible with the fusion constraints determined by the graphs. Such investigation may be used to exclude graphs as (dual) principal graphs of subfactors. For example, type \( E_7 \) and \( D_{2n+1} \) Dynkin diagrams are proved not to be (dual) principal graphs of subfactors, by showing that the fusion constraints given by the graphs give rise to inconsistency in fusion rules \([8], [9]\)). Note that the existence of a fusion algebra compatible with a given pair of graphs do not imply the existence of a subfactor with given graphs as (dual) principal graphs.

In this paper, we deal with the following series of pairs of graphs:

\[
\Gamma_k : \begin{array}{cccccc}
\alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_n \\
\end{array}
\quad \beta_1 \quad \beta_2 \quad \beta_3
\]

\[
\Gamma'_k : \begin{array}{cccccc}
\alpha'_0 & \alpha'_1 & \alpha'_2 & \cdots & \alpha'_{n-1} & \alpha_n \\
\gamma_1 & \gamma_2 & \gamma_3
\end{array}
\quad \beta_1 \quad \beta_2
\]

where \( n = 4k + 3, k = 0, 1, \ldots \). Let These graphs are a part of the list of the graphs that were candidates for (dual) principal graphs of a subfactor with indices between 4 and \( 3 + \sqrt{3} \) given by the second author \([6]\)). Note that the notation used here is somewhat different from the one used in \([6]\). It has been already proved that, for \( k = 0, 1 \), \( \Gamma_k \) (resp. \( \Gamma'_k \)) are (dual) principal graphs of a subfactors \([2], [4]\)), and for \( k > 1 \), they are not realized as (dual) principal graphs \([3]\)). In this paper, we prove that, despite that \( \Gamma_k \) (resp. \( \Gamma'_k \)) are not principal graphs for \( k > 1 \), there are still fusion algebras consistent with the graphs, and moreover such fusion algebras are unique for each \( k \). Namely we prove the following:

**Theorem 1.1.** Let \( V_{11} := \{ \text{even vertices of } \Gamma_k \} \), \( V_{12} := \{ \text{odd vertices of } \Gamma_k \} \), \( V_{21} := \{ \text{odd vertices of } \Gamma'_k \} \), \( V_{22} := \{ \text{even vertices of } \Gamma'_k \} \), and \( V := V_{11} \sqcup V_{12} \sqcup V_{21} \sqcup V_{22} \). For each \( k \), there is a unique fusion algebra \( \mathcal{A} = \mathbb{Z}\mathcal{X} \), where

\[
\mathcal{X} = N\mathcal{X}_N \sqcup N\mathcal{X}_M \sqcup M\mathcal{X}_N \sqcup M\mathcal{X}_M
\]

compatible with the graphs \( \Gamma_k, \Gamma'_k \). Namely

\[
N\mathcal{X}_N = V_{11},
N\mathcal{X}_M = V_{12},
M\mathcal{X}_N = V_{21},
\]
$M \mathcal{X}_M = V_{22}$

as sets, and

$$N^Y_{X, \alpha_1} (\text{resp. } N^Y_{X, \gamma_1}) = \begin{cases} 1 & \text{if } X \text{ and } Y \text{ are connected by an edge} \\ 0 & \text{else}, \end{cases}$$

$$N^Y_{X, 1} = \delta_{X,Y}$$

holds, where $X, Y \in \mathcal{X}$, and 1 denotes identity objects $1_N = \alpha_0 \in N \mathcal{X}_N$ or $1_M = \alpha_0' \in M \mathcal{X}_M$.

The content of this paper is as follows. In Section 2 we show that if there is a fusion system compatible with the graphs $\Gamma_k, \Gamma'_k$, it must be unique. In Section 3 we show the existence of such a fusion system.

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2. Uniqueness, positivity, and integrality of the fusion rules

In this section we prove that, if there is a fusion algebra compatible with the graphs, it is unique. Positivity and integrality of fusion coefficients is derived: we do not impose them in showing uniqueness of the fusion rules.

2.1. Fusion rules for the even vertices. In this subsection we show that there is a unique fusion algebra structure on $\mathcal{A}_1 = \mathbb{Z} N \mathcal{X}_N$ compatible with the graph $\Gamma_k$. The main issue is to determine the fusion rule among $\beta_1, \beta_3, \gamma_1, \gamma_3$. The rest will follow easily from this.

In the following we assume that there is a fusion algebra compatible with $(\Gamma_k, \Gamma'_k)$. The involution $\gamma \in V \to \overline{\gamma} \in V$ extends linear to a map on $\mathbb{R} V$. For simplicity, we refer to the objects in $\mathcal{X}$ by corresponding vertices in $V$. For $X := \sum N^X_Z Z \in \mathbb{R} V$ and $Y \in V$, we denote

$$< X, Y >=< Y, X > := N^Y_X.$$

Observe that $< \cdot, \cdot >$ expended linearly to define a bilinear form on $\mathbb{R} V$, and

$$< XY, Z >=< X, ZY > =< Y, XZ >$$

holds by Frobenius reciprocity. The graph $\Gamma_k$ encodes s the decomposition of $X \alpha_1$ for $X$ in $V_{11}$ into a direct sum of vertices from $V_{12}$ and the decomposition of $Y \overline{\alpha}_1$ into a direct sum of vertices from $V_{11}$. Let $G$ be the adjacency matrix for $(V_{11}, V_{12})$, namely

$$G = (G_{X,Y})_{X \in V_{11}, Y \in V_{12}},$$

where $G_{X,Y} = (\text{the number of the edges connecting } X \text{ and } Y)$

$$= < X \alpha_1, Y >$$

$$= < Y \overline{\alpha}_1, X >,$$
which is written as the following \((\frac{n+1}{2} + 4) \times (\frac{n+1}{2} + 2)\)-matrix:

\[
G = \begin{pmatrix}
\beta_2 & \gamma_2 & \alpha_n & \alpha_{n-2} & \cdots & \alpha_1 \\
\beta_3 & 1 & 0 & 0 & 0 & \cdots & 0 \\
\beta_1 & 1 & 0 & 1 & 0 & \cdots & 0 \\
\gamma_3 & 0 & 1 & 0 & 0 & \cdots & 0 \\
\gamma_1 & 0 & 1 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\alpha_{n-1} & 0 & 1 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\alpha_2 & 0 & 0 & \cdots & 0 & 1 & 1 \\
\alpha_0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 1
\end{pmatrix},
\]

(1)

Let

\[
\Delta := \begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix},
\]

then

\[
\Delta^2 = \begin{pmatrix} GG^t & 0 \\ 0 & G'^t G \end{pmatrix}.
\]

We put \(\mathbb{D} := GG^t\), which acts on \(\overline{A}_1 := \mathbb{R}V_{11}\). We utilize certain eigen vectors of \(\mathbb{D}\) to determine the fusion structure of \(A_1\).

Observe from the graph that

\[
\Delta \beta_1 = \alpha_n + \beta_2, \quad \Delta \gamma_1 = \alpha_n + \gamma_2,
\]
\[
\Delta \beta_2 = \beta_1 + \beta_3, \quad \Delta \gamma_2 = \gamma_1 + \gamma_2,
\]
\[
\Delta \beta_3 = \beta_2, \quad \Delta \gamma_3 = \gamma_2.
\]

Put

\[
\xi = (\beta_1 - \gamma_1) + (\beta_3 - \gamma_3)
\]
\[
\eta = (\beta_1 - \gamma_1) - (\beta_3 - \gamma_3).
\]

Then

\[
\mathbb{D}\xi = \Delta^2 \xi = \Delta(2\beta_2 - 2\gamma_2) = 2\xi,
\]
\[
\mathbb{D}\eta = \Delta^2 \eta = 0.
\]

Let \(E(\mathbb{D}, c), c \in \mathbb{R}\) be the eigenspace of the eigenvalue \(c\) for \(\mathbb{D}\) in \(\mathbb{R}(V_{11})\).

Lemma 2.1.

\[
\dim E(\mathbb{D}, 2) = E(\mathbb{D}, 0) = 2
\]

Proof
Since the matrix $G$ is a symmetric matrix, it follows that the dimensions of the eigenspaces for $G$ are equal to $2\times2$ matrix. Hence $0$ and $2$ are roots of multiplicity $2$ in $\rho_k$. Hence $0$ and $2$ are eigenvalues of $G$.

Recall that $n = 4k + 3$. Let $\rho_k(x) := \det(tI - D)$ be the characteristic polynomial of $D = GG^t$. It was proved in [1] that the characteristic polynomial of $G^tG$ is equal to $(t - 2)^2q_k(t)$, where the polynomials $q_k(t), k \geq 0$, can be defined recursively by

\[
q_0(t) = t^2 - 5t + 3
\]

\[
q_1(t) = (t - 1)(t^3 - 8t^2 + 17t - 5)
\]

\[
q_k(t) = (t^2 - 4t + 2)q_{k-1}(t) - q_{k-2}(t), \quad k \geq 2
\]

Since the matrix $G$ has $2k + 6$ rows and $2k + 4$ columns, $GG^t$ is a unitary conjugate of $G^tG \oplus O_2$, where $O_2$ is the zero $2 \times 2$ matrix. Hence

\[
\rho_k(t) = t^2 \det(tI - G^tG)
= t^2(t - 2)^2q_k(t).
\]

Using the recursion formula for $q_k(t)$, one gets $q_0(0) = 2k + 3$ and $q_k(2) = (-1)^k(k + 1)(2k + 3)$.

In particular neither $0$ nor $2$ is a root of $q_k$. Hence $0$ and $2$ are roots of multiplicity $2$ in $\rho_k$. Since $D = GG^t$ is a symmetric matrix, it follows that the dimensions of the eigenspaces for $D$ for the eigenvalues $0$ and $2$ are both equal to two.

Bases of $E(D, 2)$, $E(D, 0)$ may be taken as follows:

$E(D, 2) := \text{span}\{x_1, x_2\}$

$E(D, 0) := \text{span}\{y_1, y_2\}$,

where

\[
x_1 := 2(\alpha_0 + \alpha_2) - 2(\alpha_4 + \alpha_6) + \cdots + (-1)^k2(\alpha_{4k} + \alpha_{4k+2})
+ (-1)^{k+1}(\beta_1 + \gamma_1 + \beta_3 + \gamma_3)
\]

\[
x_2 := \xi = (\beta_1 - \gamma_1) + (\beta_3 - \gamma_3)
\]

\[
y_1 := 2\alpha_0 - 2\alpha_2 + \cdots + 2\alpha_{4k} - 2\alpha_{4k+2} + (\beta_1 + \gamma_1) - (\beta_3 + \gamma_3)
\]

\[
y_2 := \eta = (\beta_1 - \gamma_1) - (\beta_3 - \gamma_3)
\]
Assume that we have a fusion algebra compatible with the pair of the graphs \((\Gamma_k, \Gamma'_k)\), and let \(\pi\) and \(\pi'\) be the conjugate maps \(\gamma \mapsto \overline{\gamma}\) on \(V_{11}\) and \(V_{22}\). By the argument used in [6, pp28-31], \(\pi'\) fixes every element of \(V_{22}\). For \(\pi\), there are only two possibilities:

**Case 1** (= case (b) in [6, p31])

\[ \overline{\beta_1} = \beta_1, \overline{\gamma_1} = \gamma_1, \overline{\beta_3} = \gamma_3 (\iff \overline{\gamma_3} = \beta_3), \]

**Case 2** (= case (a) in [6, p31]. To be eliminated.)

\[ \overline{\beta_1} = \gamma_1 (\iff \overline{\gamma_1} = \beta_1), \overline{\beta_3} = \beta_3, \overline{\gamma_3} = \gamma_3. \]

In both cases, \(\overline{\alpha}_{2j} = \alpha_{2j}\) for \(j = 0, 1, \ldots, 2k + 1\). Note that \(\pi\) extends linearly to \(A_1\) and \(\overline{A_1} = \mathbb{R}V_{11}\). Let \(E(\mathbb{D}, c)_{sc} := E(\mathbb{D}, c)_{\pi}\). Observe that

\[ c_1 \overline{\beta_1} + c_2 \overline{\gamma_2} = c_1 x_1 + c_2 x_2, \quad c_1, c_2 \in \mathbb{R} \]

holds if and only if \(c_2 = 0\) in both cases 1 and 2, and similarly

\[ c_1 \overline{\beta_1} + c_2 \overline{\gamma_2} = c_1 y_1 + c_2 y_2, \quad c_1, c_2 \in \mathbb{R} \]

if and only if \(c_2 = 0\) in both cases. Therefore

\[ E(\mathbb{D}, 2)_{sc} = \mathbb{R}x_1 \]
\[ E(\mathbb{D}, 0)_{sc} = \mathbb{R}y_1 \]

By the definition of principal graphs, the matrix \(\mathbb{D} : \mathbb{R}V_{11} \to \mathbb{R}V_{11}\) corresponds to the fusion rule of the right tensor product by \(\alpha \overline{\alpha}\), where \(\alpha = \alpha_1\). Therefore

\[ \mathbb{D}(\overline{\xi}\xi) = \overline{\xi}\mathbb{D}(\xi) = 2\overline{\xi}\xi \]
\[ \mathbb{D}(\overline{\eta}\eta) = \overline{\eta}\mathbb{D}(\eta) = 0. \]

Hence

\[ \overline{\xi}\xi \in E(\mathbb{D}, 2)_{sc} = \mathbb{R}x_1, \]
\[ \overline{\eta}\eta \in E(\mathbb{D}, 0)_{sc} = \mathbb{R}y_1. \]

Thus

\[ \langle \overline{\xi}\xi, \alpha_0 \rangle = \langle \xi, \xi_0 \rangle = \langle \xi, \xi \rangle = 4. \]

Hence the coefficient of \(\overline{\xi}\xi\) at \(\alpha_0\) is 4. Since \(\overline{\xi}\xi \in \mathbb{R}x_1\), we have \(\overline{\xi}\xi = 2x_1\). Likewise we obtain \(\overline{\eta}\eta = 2y_1\). Noting that

\[ \overline{\xi} = \begin{cases} \eta & \text{for Case 1} \\ -\eta & \text{for Case 2} \end{cases} \]

we have

- In Case 1: \(\xi\eta = 2y_1, \eta\xi = 2x_1\),
- In Case 2: \(\xi\eta = -2y_1, \eta\xi = -2x_1\).

**Lemma 2.2.**

\[ \xi^2 = 0, \eta^2 = 0. \]
Proof. Since $\mathbb{D}(\xi^2) = \xi \mathbb{D}(\xi) = 2\xi^2$, $\xi^2 = c_1 x_1 + c_2 x_2$ for some $c_1, c_2 \in \mathbb{R}$. Moreover, since $<\xi, \eta> = 0$, we have

$$<\xi^2, \alpha_0> = <\xi, \xi \alpha_0> = \pm <\xi, \eta> = 0$$

Together with $<c_1 x_1 + c_2 x_2, \alpha_0> = 2c_1$, $c_1, c_2 \in \mathbb{R}$, we obtain

$$\xi^2 = c_2 x_2 = c_2 \xi.$$  

We show that $c_2 = 0$:

$$4c_2 = <c_2 \xi, c_2 \xi> = <\xi^2, \xi> = 4 <x_1, y_1> = (2 - 2) - (2 - 2) + \cdots (-1)^k (2 - 2) + (1 + 1 - 1 - 1) = 0.$$  

We used that $\xi = 2x_1$, $\xi \xi = 2y_1$ for both cases. Thus $\xi^2 = 0$. Then $\xi^2 = \eta^2 = 0$ for both cases.

Remark 2.3. For $k$ even (i.e. $n = 3 \pmod{8}$) and $k = 2l$,

$$\frac{1}{2} (x_1 + y_1) = 2(\alpha_0 - \alpha_6 + \alpha_8 - \alpha_{14} + \alpha_{16} - \cdots + \alpha_{8l}) - (\beta_3 + \gamma_3)$$

and for $k$ odd (i.e. $n = 7 \pmod{8}$) and $k = 2l + 1$,

$$\frac{1}{2} (x_1 + y_1)$$

$$= 2(\alpha_0 - \alpha_6 + \alpha_8 - \alpha_{14} + \alpha_{16} - \cdots + \alpha_{8l} - \alpha_{8l+6}) + (\beta_1 + \gamma_1)$$

Consider next the sequence of polynomials $R_n$ given recursively by

$$R_0(t) = 1,$$

$$R_1(t) = t,$$

$$R_m(t) = t R_{m-1}(t) - R_{m-2}(t), n \geq 2.$$  

as in [6] p33–34. Note that $R_m(t) = U_m(\frac{t}{2})$, where $U_m$ is the $m$-th Chebyshev polynomial of second kind [5]. Moreover,

$$R_m(2 \cos \theta) = \frac{\sin(m + 1)\theta}{\sin \theta}, \quad 0 < \theta < \pi.$$
By the recursion formula for $R_n$, it follows that
\[
R_j(\Delta)\alpha_0 = \alpha_j, \quad 0 \leq j \leq n,
\]
\[
R_{n+1}(\Delta)\alpha_0 = \beta_1 + \gamma_1,
\]
\[
R_{n+2}(\Delta)\alpha_0 = \alpha_n + \beta_2 + \gamma_2,
\]
\[
R_{n+3}(\Delta)\alpha_0 = \alpha_{n-1} + \beta_1 + \gamma_1 + \beta_3 + \gamma_3.
\]

Hence
\[
\beta_3 + \gamma_3 = (R_{n+3}(\Delta) - R_{n+1}(\Delta) - R_{n-1}(\Delta))\alpha_0
\]
\[
= (R_{4k+6}(\Delta) - R_{4k+4}(\Delta) - R_{4k+2}(\Delta))\alpha_0
\]

For $m$ even, $R_m(t)$ is an even polynomial in $t$, thus there is are unique polynomials $(Q_j)_{j=0,1,2,...}$ with $\deg(Q_l) = l$, such that
\[
Q_j(t^2) = R_{2j}(t), \quad t \in \mathbb{R}, \quad j = 0, 1, 2, \ldots.
\]

With this notation, we have
\[
\beta_3 + \gamma_3 = (Q_{2k+3}(\Delta) - Q_{2k+2}(\Delta) - Q_{2k+1}(\Delta))\alpha_0
\]
\[
= (Q_{2k+3} - Q_{2k+2} - Q_{2k+1})(\alpha_0)\alpha_0.
\]

Therefore
\[
(\beta_3 - \gamma_3)(\beta_3 + \gamma_3) = (Q_{2k+3} - Q_{2k+2} - Q_{2k+1})(\beta_3 - \gamma_3)
\]
\[
= \frac{1}{2}(Q_{2k+3} - Q_{2k+2} - Q_{2k+1})(\xi - \eta)
\]

Since $\Re \xi = 2\xi$ and
\[
Q_m(2) = R_{2j}(\sqrt{2}) = \frac{\sin(2j + 1)\pi/4}{\sin \pi/4}
\]
\[
= \begin{cases} 
1, & j = 0, 1 \pmod{4} \\
-1, & j = 2, 3 \pmod{4},
\end{cases}
\]

we have
\[
Q_j(\Re)\xi = \begin{cases} 
\xi, & j = 0, 1 \pmod{4} \\
-\xi, & j = 2, 3 \pmod{4},
\end{cases}
\]

Similarly, since $\Re \eta = 0$ and
\[
Q_j(0) = R_{2j}(0) = \frac{\sin(2j + 1)\pi/2}{\sin \pi/2} = (-1)^j,
\]

we have
\[
Q_j(\Re)\eta = (-1)^j\eta, \quad j = 0, 1, 2, \ldots.
\]

Therefore we have
\[
(Q_{2k+3}(\Re) - Q_{2k+2}(\Re) - Q_{2k+1}(\Re))\xi
\]
\[
= \begin{cases} 
(Q_{4l+3}(\Re) - Q_{4l+2}(\Re) - Q_{4l+1}(\Re))\xi = -\xi & \text{for } k = 2l, \ l \in \mathbb{N}_0 \\
(Q_{4l+5}(\Re) - Q_{4l+4}(\Re) - Q_{4l+3}(\Re))\xi = \xi & \text{for } k = 2l + 1, \ l \in \mathbb{N}_0,
\end{cases}
\]
and in both cases
\[ (Q_{2k+3}(D) - Q_{2k+2}(D) - Q_{2k+1}(D))\eta = -\eta. \]

Hence
\[
(\beta - \gamma_3)(\beta + \gamma_3) = \frac{1}{2}(Q_{2k+3} - Q_{2k+2} - Q_{2k+1})(D)(\xi - \eta)
\]
\[
= \begin{cases} 
\frac{1}{2}(-\xi + \eta) = \gamma_3 - \beta_3, & k \text{ even,} \\
\frac{1}{2}(\xi + \eta) = \beta_1 - \gamma_1, & k \text{ odd.}
\end{cases}
\]

Using the contragradient map we get

**For Case 1:**

\[
(\beta_3 + \gamma_3)(\beta_3 - \gamma_3) = (\gamma_3 + \beta_3)(\gamma_3 - \beta_3)
\]
\[
= - (\beta_3 - \gamma_3)(\beta_3 + \gamma_3)
\]
\[
= \begin{cases} 
- (\gamma_3 - \beta_3) = -(\beta_3 - \gamma_3), & k \text{ even,} \\
- (\beta_1 - \gamma_1) = -(\beta_1 - \gamma_1), & k \text{ odd,}
\end{cases}
\]

**For Case 2** (to be eliminated):

\[
(\beta_3 + \gamma_3)(\beta_3 - \gamma_3) = (\gamma_3 + \beta_3)(\gamma_3 - \beta_3)
\]
\[
= (\gamma_3 - \beta_3)(\beta_3 + \gamma_3)
\]
\[
= \begin{cases} 
\gamma_3 - \beta_3, & k \text{ even,} \\
\beta_1 - \gamma_1, & k \text{ odd.}
\end{cases}
\]

Thus in both cases
\[
(\beta_3 + \gamma_3)(\beta_3 - \gamma_3) = \begin{cases} 
\gamma_3 - \beta_3, & k \text{ even,} \\
\gamma_1 - \beta_1, & k \text{ odd.}
\end{cases}
\]

So far, we have obtained the following three formulae:

**[A]**

\[
(\beta_3 - \gamma_3)^2 = \begin{cases} 
- \frac{1}{2}(x_1 - y_1) \text{ in Case 1} \\
\frac{1}{2}(x_1 - y_1) \text{ in Case 2}
\end{cases}
\]

**[B]**

\[
(\beta_3 - \gamma_3)(\beta_3 + \gamma_3) = \begin{cases} 
\frac{1}{2}(-\xi + \eta) = \gamma_3 - \beta_3, & k \text{ even,} \\
\frac{1}{2}(\xi + \eta) = \beta_1 - \gamma_1, & k \text{ odd.}
\end{cases}
\]

**[C]**

\[
(\beta_3 + \gamma_3)(\beta_3 - \gamma_3) = \begin{cases} 
\gamma_3 - \beta_3, & k \text{ even,} \\
\gamma_1 - \beta_1, & k \text{ odd.}
\end{cases}
\]

Next we compute \((\beta_3 + \gamma_3)^2\), in order to find \(\beta_3^2, \gamma_3^2, \beta_3\gamma_3\) and \(\gamma_3\beta_3\).

**Claim 2.4.**
Our strategy of the proof is as follows: first we find a sequence of polynomials \( (S_j) \) such that 
\[
S_j = c_j + \gamma_j \quad (j = 1, 2, \ldots, n)
\]
where \( c_j \)'s are defined by \( c_0 = 0, c_1 = 1, c_2 = 0 \) and \( c_j = c_{j-1} + c_{j-2} + c_{j-3} \) for \( j \geq 3 \).

**Proof**
Recall that
\[
(\beta_3 + \gamma_3)^2 = (Q_{2k+3} - Q_{2k+2} - Q_{2k+1})(\mathbb{D})\alpha_0
\]
\[
= (R_{4k+6}(\Delta) - R_{4k+4}(\Delta) - R_{4k+2}(\Delta))\alpha_0,
\]
thus
\[
(\beta_3 + \gamma_3)^2 = (R_{4k+6}(\Delta) - R_{4k+4}(\Delta) - R_{4k+2}(\Delta))(\beta_3 + \gamma_3) \quad \cdots (\ast)
\]
Our strategy of the proof is as follows: first we find a sequence of polynomials \( (S_j) \) such that 
\( S_j(\Delta)(\beta_3 + \gamma_3) \) is given by a simple formula. Next we rewrite the right hand side of \( (\ast) \) using \( (S_j) \)'s.
Observe that we obtain from the graph the following:
\[
R_0(\Delta)(\beta_3 + \gamma_3) = (\beta_3 + \gamma_3),
R_1(\Delta)(\beta_3 + \gamma_3) = (\beta_2 + \gamma_2),
R_2(\Delta)(\beta_3 + \gamma_3) = \Delta(\beta_2 + \gamma_2) - (\beta_3 + \gamma_3) = \beta_1 + \gamma_1,
R_3(\Delta)(\beta_3 + \gamma_3) = \Delta(\beta_1 + \gamma_1) - (\beta_2 + \gamma_2) = 2\alpha_n,
R_4(\Delta)(\beta_3 + \gamma_3) = 2\Delta\alpha_n - (\beta_1 + \gamma_1) = 2\alpha_{n-1} + \beta_1 + \gamma_1,
\]
We define the polynomials \( (S_j(t))_{j\geq3} \) by the following recursive formula:
\[
S_3(t) = R_3(t),
S_4(t) = R_4(t) - R_2(t),
S_j(t) = tS_{j-1}(t) - S_{j-2}(t), \quad j \geq 5.
\]
By definition \( S_3(\Delta)(\beta_3 + \gamma_3) = 2\alpha_n, \quad S_4(\Delta)(\beta_3 + \gamma_3) = 2\alpha_{n-1} \). Since \( \alpha_{l-1} = \Delta\alpha_l - \alpha_{l+1} \) for \( l = 1, 2, \ldots, n - 1 \), we easily obtain
\[
S_j(\Delta)(\beta_3 + \gamma_3) = 2\alpha_{n-j+3}
\]
for \( j = 3, 4, \ldots, n + 3 \). Next we express \( R_j \)'s in terms of \( S_j \)'s.

**Lemma 2.5.** For \( j \geq 2 \),
\[
R_{2j-1} = d_0S_{2j-1} + d_1S_{2j-3} + \cdots + d_{j-2}S_3 + (d_{j-1} - d_{j-2})R_1,
R_{2j} = d_0S_{2j} + d_1S_{2j-2} + \cdots + d_{j-2}S_4 + d_{j-1}R_2 + d_{j-3}R_0,
\]
where \( d_j \)'s satisfy
\[
d_j = d_{j-1} + d_{j-2} + d_{j-3}
\]
\[
d_{-1} = 0, \quad d_0 = d_1 = 1,
\]

**Proof of Lemma:**
For \( j = 2 \) it is obvious by the definition of \( S_j \)'s. We proceed with induction. Assume that it is
true for \( j \geq 2 \). Using the recursion formulae for \( R_j \)'s and \( S_j \)'s, we have
\[
R_{2j+1}(t) = tR_{2j}(t) - R_{2j-1}(t)
\]
\[
= t(d_0S_{2j} + d_1S_{2j-2} + \cdots + d_{j-2}S_4 + d_{j-1}R_2 + d_{j-3})
\]
\[
- (d_0S_{2j-1} + d_1S_{2j-3} + \cdots + d_{j-2}S_3 + (d_{j-1} - d_{j-2})R_1)
\]
\[
= d_0S_{2j+1} + d_1S_{2j-1} + \cdots + d_{j-2}S_5 + t(d_{j-1}R_2 + d_{j-3}) - (d_{j-1} - d_{j-2})R_1
\]
\[
= d_0S_{2j+1} + d_1S_{2j-1} + \cdots + d_{j-2}S_5 + d_{j-1}(tR_2 - R_1) + td_{j-3} - d_{j-2}R_1
\]
\[
= d_0S_{2j+1} + d_1S_{2j-1} + \cdots + d_{j-2}S_5 + d_{j-1}S_3 + (d_{j-3} - d_{j-2})R_1.
\]
The last equality was obtained using \( S_3 = R_3, \) \( R_1 = t, \) and \( d_{j-2} + d_{j-3} = d_j - d_{j-1}. \) Likewise we have
\[
R_{2j+2}(t) = tR_{2j+1}(t) - R_{2j}(t)
\]
\[
= d_0S_{2j+2} + d_1S_{2j} + \cdots + d_{j-2}S_6
\]
\[
+ t(d_{j-1}S_3 + (d_j - d_{j-1})R_1) - (d_{j-1}R_2 + d_{j-3}R_0)
\]
\[
= d_0S_{2j+2} + d_1S_{2j} + \cdots + d_{j-2}S_6 + d_{j-1}R_4
\]
\[
+ (d_j - d_{j-1})(R_2 + R_0) - d_{j-3}R_0
\]
\[
= d_0S_{2j+2} + d_1S_{2j} + \cdots + d_{j-2}S_6 + d_{j-1}S_4 +
\]
\[
+ d_{j}R_2 + (d_j - d_{j-1} - d_{j-3})R_0
\]
\[
= d_0S_{2j+2} + d_1S_{2j} + \cdots + d_{j-2}S_6 + d_{j-1}S_4 + d_jR_2 + d_{j-2}R_0.
\]

\[ \square \]

Let us go back to (\( \text{4} \)). Using Lemma 2.3
\[
R_{4k+6} - R_{4k+4} - R_{4k+2}
\]
\[
= d_0S_{4k+6} + (d_1 - d_0)S_{4k+4} + d_{-1}S_{4k+2} + d_0S_{4k} + d_1S_{4k-2} + \cdots
\]
\[
+ d_{2k-2}S_4 + d_{2k-1}R_2 + d_{2k-3}R_0
\]
\[
= S_{4k+6} + d_0S_{4k} + d_1S_{4k-2} + \cdots
\]
\[
+ d_{2k-2}S_4 + d_{2k-1}R_2 + d_{2k-3}R_0
\]

Recall
\[
S_j(\Delta)\beta_3 + \gamma_3 = 2\alpha_{n-j+3}
\]
\[
R_2\beta_3 + \gamma_3 = \beta_1 + \gamma_1.
\]

Letting \( c_0 := 1, c_1 = c_2 = 0, c_j := d_{j-3} \) for \( j \geq 3 \), we obtain the formula [D]. This concludes the proof for Claim 2.4. \[ \square \]

Thus far we obtained the formulae for \((\beta_3 - \gamma_3)^2, (\beta_3 - \gamma_3)(\beta_3 + \gamma_3), (\beta_3 + \gamma_3)(\beta_3 - \gamma_3)\) and \((\beta_3 + \gamma_3)^2\) as in [A], [B], [C], [D]. This enable us to understand the fusion rules among \( \beta_3, \gamma_3 \) and their conjugates. We obtain the following:

**Proposition 2.6.** The Case 2 does not occur. Namely \( \beta_1, \gamma_1 \) are self conjugate and \( \overline{\beta_3} = \gamma_3 \) if there is a fusion algebra compatible with the graphs \( \Gamma_k, \Gamma_k' \).

**Proof.**

First observe that, by the definition \((c_j)_{j \geq 0}\) used in Claim 2.4 it follows that \( c_j \pmod{4} \) is periodic in \( j \) with period 8. The values are given in the following Table 1.
Table 1.

| j (mod 8) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----------|---|---|---|---|---|---|---|---|
| c_j (mod 4) | 1 | 0 | 0 | 1 | 1 | 2 | 0 | 0 |

In particular,

\[
\begin{cases} 
  c_{2j} = 1 \pmod{4} & \text{for } j \text{ even}, \\
  c_{2j} = 0 \pmod{4} & \text{for } j \text{ odd}.
\end{cases}
\]

In the following we assume Case 2 and derive contradiction.

• for k even:
  By [B] and [C], we have
  \[(\beta_3 - \gamma_3)(\beta_3 + \gamma_3) = (\beta_3 + \gamma_3)(\beta_3 - \gamma_3),\]
  hence
  \[
  \beta_3\gamma_3 = \gamma_3\beta_3 = \frac{1}{2}(\beta_3\gamma_3 + \gamma_3\beta_3)
  = \frac{1}{4}((\beta_3 + \gamma_3)^2 - (\beta_3 - \gamma_3)^2).
  \]
  From [A] (Case 2), [D] and Remark 2.3 it follows that the coefficient of \(\beta_3\) in the expansion of \(\beta_3\gamma_3\) in irreducible objects is equal to
  \[
  \frac{c_{2k} + 1}{4}.
  \]
  Since k is even, \(c_{2k} = 1 \pmod{4}\) by (⋆), \((c_{2k} + 1)/4\) is not an integer. This implies that Case 2 does not occur if k is even.

• for k odd:
  From [B], [C], we get
  \[(\beta_3 - \gamma_3)(\beta_3 + \gamma_3) = -(\beta_3 + \gamma_3)(\beta_3 - \gamma_3).\]
  Hence
  \[
  \beta_3^2 = \gamma_3^2 = \frac{1}{2}(\beta_3^2 + \gamma_3^2)
  = \frac{1}{4}((\beta_3 + \gamma_3)^2 + (\beta_3 - \gamma_3)^2).
  \]
  From [A] (Case 2), [D] and Remark 2.3 it follows that the coefficient of \(\beta_1\) in the expansion of \(\beta_3^2\) in irreducible objects is equal to
  \[
  \frac{c_{2k+2} + 1}{4}.
  \]
  Since k is odd, \(c_{2k+2} = 1 \pmod{4}\) by (⋆), \((c_{2k+2} + 1)/4\) is not an integer. This excludes Case 2 for k odd as well.

In the following we determine all the irreducible decompositions for the products of any two objects in V, and show that the coefficients are non-negative integers. Since we excluded Case 2, we rewrite the formula [A]:

[A*] For \(k = 2l, l = 0, 1, 2, \ldots,\)

\[(\beta_3 - \gamma_3)^2 = -2(\alpha_0 - \alpha_6 + \alpha_8 - \alpha_{14} + \alpha_{16} - \cdots + \alpha_{8l}) - (\beta_3 + \gamma_3),\]

and for \(k = 2l + 1, l = 0, 1, 2, \ldots,\)

\[(\beta_3 - \gamma_3)^2 = -2(\alpha_0 - \alpha_6 + \alpha_8 - \alpha_{14} + \alpha_{16} - \cdots + \alpha_{8l} - \alpha_{8l+6}) + (\beta_1 + \gamma_1).\]
Put
\[ A := (\beta_3 - \gamma_3)^2 \]
\[ B := (\beta_3 - \gamma_3)(\beta_3 + \gamma_3) \]
\[ C := (\beta_3 + \gamma_3)(\beta_3 - \gamma_3) \]
\[ D := (\beta_3 + \gamma_3)^2. \]

Then
\[ \beta_3 \gamma_3 = \frac{(D - A) + (B - C)}{4} \]
\[ \gamma_3 \beta_3 = \frac{(D - A) - (B - C)}{4} \]
\[ \beta_3^2 = \frac{(D + A) + (B + C)}{4} \]
\[ \gamma_3^2 = \frac{(D + A) - (B + C)}{4} \]

We introduce new constants \((f_j)_{j\geq 0}, (g_j)_{j\geq 0}\) by
\[
\begin{cases}
  f_j = \frac{1}{2}(c_j + 1), & g_j = \frac{1}{2}(c_j - 1) \text{ when } j = 0(\text{mod } 4), \\
  f_j = \frac{1}{2}(c_j - 1), & g_j = \frac{1}{2}(c_j + 1) \text{ when } j = 3(\text{mod } 4), \\
  f_j = g_j = \frac{1}{2}c_j \text{ when } j = 1, 2(\text{mod } 4).
\end{cases}
\]

Note that \(f_j + g_j = c_j\) for all \(j\). Furthermore, from Table 1, observe that \(f_j, g_j\)'s are non-negative integers for all \(j \geq 0\). The list of some values for \(f_j\)'s and \(g_j\)'s are given in Table 2.

**Table 2.**

| \(j\) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-------|---|---|---|---|---|---|---|---|---|---|----|----|----|
| \(f_j\) | 1 | 0 | 0 | 0 | 1 | 1 | 2 | 3 | 7 | 12 | 22 | 40 | 75 |
| \(g_j\) | 0 | 0 | 0 | 1 | 0 | 1 | 2 | 4 | 6 | 12 | 22 | 41 | 74 |

For \(k\) even, using the formulae \([A'], [B], [C], [D]\), we have
\[
\frac{D - A}{4} = f_0 \alpha_0 + f_1 \alpha_2 + \cdots + f_{2k+1} \alpha_{4k+2} + \frac{1}{4} c_{2k+2}(\beta_1 + \gamma_1) + \frac{1}{4} (c_{2k} - 1)(\beta_3 + \gamma_3),
\]
\[
\frac{D + A}{4} = g_0 \alpha_0 + g_1 \alpha_2 + \cdots + g_{2k+1} \alpha_{4k+2} + \frac{1}{4} c_{2k+2}(\beta_1 + \gamma_1) + \frac{1}{4} (c_{2k} + 1)(\beta_3 + \gamma_3),
\]
\[
\frac{B - C}{4} = 0,
\]
\[
\frac{B + C}{4} = \frac{1}{2}(\gamma_3 - \beta_3).
\]

Since \(k\) is even, \(c_{2k+2} = 2f_{2k+2} = 2g_{2k+2}, c_{2k} + 1 = 2f_{2k}, c_{2k} - 1 = 2g_{2k}\). Hence we obtain the following theorem:
Theorem 2.7. For $k$ even,

$$
\beta_3 \gamma_3 = \gamma_3 \beta_3 = f_0 \alpha_0 + f_1 \alpha_2 + \cdots + f_{2k+1} \alpha_{4k+2}
+ \frac{1}{2} f_{2k+2} (\beta_1 + \gamma_1) + \frac{1}{2} (f_{2k} - 1) (\beta_3 + \gamma_3),
$$

$$
\beta_3^2 = g_0 \alpha_0 + g_1 \alpha_2 + \cdots + g_{2k+1} \alpha_{4k+2}
+ \frac{1}{2} g_{2k+2} (\beta_1 + \gamma_1) + \frac{1}{2} g_{2k} \beta_3 + \frac{1}{2} (g_{2k} + 2) \gamma_3,
$$

$$
\gamma_3^2 = g_0 \alpha_0 + g_1 \alpha_2 + \cdots + g_{2k+1} \alpha_{4k+2}
+ \frac{1}{2} g_{2k+2} (\beta_1 + \gamma_1) + \frac{1}{2} (g_{2k} + 2) \beta_3 + \frac{1}{2} g_{2k} \gamma_3.
$$

All the coefficients of irreducible elements are non-negative integers.

**Proof.** The only remaining thing to prove is that $f_{2k+2}$ is even, $f_{2k}$ is odd, $g_{2j}$ is even for any $j$. Since $k$ is even, $c_{2k+2} = 0 \pmod{4}$. Thus $f_{2k+2} = \frac{1}{2} c_{2k+2}$ is even. Likewise $c_{2k} = 1 \pmod{4}$, thus $f_{2k} = \frac{1}{2} (c_{2k} + 1)$ is odd.

$$
g_{2j} = \begin{cases} 
\frac{1}{2} (c_{2j} - 1) & \text{for } j \text{ even}, \\
\frac{1}{2} c_{2j} & \text{for } j \text{ odd}
\end{cases}
$$

Since $c_{2j} - 1 = 0 \pmod{4}$ for $j$ even, $c_{2j} = 0 \pmod{4}$ for $j$ odd, $g_{2j}$ is even for any $j$. □

In the same way, we get for $k$ odd:

$$
\frac{D - A}{4} = f_0 \alpha_0 + f_1 \alpha_2 + \cdots + f_{2k+1} \alpha_{4k+2}
+ \frac{1}{4} (c_{2k+2} + 1) (\beta_1 + \gamma_1) + \frac{1}{4} c_{2k} (\beta_3 + \gamma_3),
$$

$$
\frac{D + A}{4} = g_0 \alpha_0 + g_1 \alpha_2 + \cdots + g_{2k+1} \alpha_{4k+2}
+ \frac{1}{4} (c_{2k+2} - 1) (\beta_1 + \gamma_1) + \frac{1}{4} c_{2k} (\beta_3 + \gamma_3),
$$

$$
\frac{B - C}{4} = \frac{1}{2} (\beta_1 - \gamma_1),
$$

$$
\frac{B + C}{4} = 0.
$$

Since $k$ is odd, $c_{2k+2} + 1 = 2 f_{2k+2}$, $c_{2k+2} - 1 = 2 g_{2k+2}$, $c_{2k} = 2 f_{2k} = 2 g_{2k}$. Hence we get:
**Theorem 2.8.** For $k$ odd,

$$
\beta_3 \gamma_3 = f_0 \alpha_0 + f_1 \alpha_2 + \cdots + f_{2k+1} \alpha_{4k+2}
+ \frac{1}{2} (f_{2k+2} + 1) \beta_1 + \frac{1}{2} (f_{2k+2} - 1) \gamma_1 + \frac{1}{2} f_{2k} (\beta_3 + \gamma_3),
$$

$$
\gamma_3 \beta_3 = f_0 \alpha_0 + f_1 \alpha_2 + \cdots + f_{2k+1} \alpha_{4k+2}
+ \frac{1}{2} (f_{2k+2} + 1) \beta_1 + \frac{1}{2} (f_{2k+2} - 1) \gamma_1 + \frac{1}{2} f_{2k} (\beta_3 + \gamma_3),
$$

$$
\beta_3^2 = \gamma_3^2 = g_0 \alpha_0 + g_1 \alpha_2 + \cdots + g_{2k+1} \alpha_{4k+2}
+ \frac{1}{2} g_{2k+2} (\beta_1 + \gamma_1) + \frac{1}{2} g_{2k} (\beta_3 + \gamma_3).
$$

All the coefficients of irreducible elements are non-negative integers.

**Proof.**

It remains to show that $f_{2k+2}$ is odd, $f_{2k}$ is even. In the proof of Theorem 2.7, it has been already proved that $g_{2j}$ is even for any $j$.

Since $k$ is odd, $c_{2k+2} = 1 \pmod{4}$. Thus $f_{2k+2} - 1 = \frac{1}{2} (c_{2k+2} - 1)$ is even, i.e. $f_{2k+2}$ is odd. Likewise $c_{2k} = 0 \pmod{4}$, thus $f_{2k} = \frac{1}{2} c_{2k}$ is even.

Thus far we determined that $\beta_1$ and $\gamma_1$ are self-conjugate, and computed full irreducible decomposition of $\beta_3, \gamma_3$, in particular $\beta_3^2 = \gamma_3^2$. This determines the rest of the fusion rule. Note that the conjugate map $\pi$ on $\mathbb{Z}V_{11}$ is now determined.

First, for $\alpha_{2j}$, $j = 0, 1, \ldots, 2k + 1$, the right and left multiplication of $\alpha_{2j}$ on any other object from $V_{11}$ is represented by the matrices $Q_j(\mathbb{D})$ and $Q_j(\pi \mathbb{D} \pi)$ respectively.

**Claim 2.9.** The entries of the matrices $R_i(\Delta)$ for $i = 0, 1, \ldots, 4k + 3$ are non-negative integers. In particular, the entries of the matrices $Q_j(\mathbb{D})$ for $j = 0, 1, \ldots, 2k + 1$ are non-negative integers.

**Proof.**

Immediate from the result in [1], which states that, when $\Delta$ is an adjacency matrix of a graph with norm greater than 2, then $R_i(\Delta)$ has non-negative integer entries for any $i$.

It remains to determine the decomposition of tensor product of $\beta_1, \gamma_1$ with themselves and $\beta_3, \gamma_3$.

Since by the graph $\beta_1 = \beta_3 \alpha_2, \gamma_1 = \gamma_3 \alpha_2$, the fusion among $\beta_3$ and $\gamma_3$, together with the fusion of $\alpha_2$ with all the objects determine $\beta_3 \gamma_1, \beta_3 \gamma_1, \beta_3 \beta_3$ by imposing associativity. Taking the conjugate, we obtain $\beta_1 \gamma_3, \beta_1 \gamma_3, \beta_1 \beta_3$ as well. $\beta_1 = \beta_1 \gamma_3 \alpha_2, \gamma_1 = \gamma_1 \gamma_3 \alpha_2, \beta_1 \gamma_1 = \beta_1 \gamma_3 \alpha_2, \gamma_1 \beta_1 = \gamma_1 \beta_3 \alpha_2$ are thus all determined. Since there is no division, subtraction of objects are involved in the process of determining each desired fusion rule, the coefficients are all non-negative integers.

### 2.2. Fusion rules on $\mathcal{X}_N \times \mathcal{X}_M$.

We identify $\mathcal{X}_N$ with $V_{11}$, $\mathcal{X}_M$ with $V_{12}$. From Claim 2.9 $\alpha_i Y$ for $i$ even and any $Y \in V_{12}$ are determined, so are $X \alpha_j$ for $X \in V_{11}$ and $i$ odd. Thus it remains to obtain $\beta_i Y$ and $\gamma_i Y$, where $i = 1, 3, Y = \beta_2$ or $\gamma_2$. They are easily determined, since $\beta_2 = \beta_3 \alpha_1, \gamma_2 = \gamma_3 \alpha_1$, and the fusion among $\beta_i, \gamma_j, i, j = 1, 3$ are already determined. Here we imposed associativity again. Since the fusion coefficients among $\beta_i$’s and $\gamma_j$’s are non-negative integers and product of $\alpha_1$ from the right gives fusion with non-negative integers, the fusion coefficients of $\beta_i Y$ and $\gamma_i Y$ are non-negative integers as well.
2.3. **Fusion rules on** $N\mathcal{X}_M \times M\mathcal{X}_N$. Let $X \in N\mathcal{X}_M$. Then for $j$ odd,

$$X\overline{\alpha}_j = R_j(\Delta)X.$$ 

From Claim 2.9 $R_j(\Delta)X$ is a linear combination of the objects in $N\mathcal{X}_N$ with non-negative integer coefficients. It remains to show that $\beta_2\overline{\beta}_2$, $\gamma_2\overline{\gamma}_2$, and $\gamma_2\overline{\alpha}$ also have this property. It is immediate, since $\beta_2 = \alpha_1 + \beta_3$, $\gamma_2 = \alpha_1 + \gamma_3$, and all the fusion rules involved have decompositions into simple objects with $\mathbb{Z}_{\geq 0}$-coefficients.

2.4. **Fusion rules on** $M\mathcal{X}_M \times M\mathcal{X}_M$ and $M\mathcal{X}_M \times M\mathcal{X}_N$. Recall that we have identification $M\mathcal{X}_M = V_{22}$ and $M\mathcal{X}_N = V_{21}$. Let $\Delta'$ be the adjacency matrix for $\Gamma'$. Then the fusion rules of the tensor products of $\alpha'_j$’s for $j = 0, 2, \ldots, n-1$, as well as $\alpha_k$’s for $k = 1, 3, \ldots, n-1$ with any objects in $V_{21} \sqcup V_{22}$ are given by the matrices $R_l(\Delta')$, where $l = 0, 1, \ldots, n$. Similarly to Claim 2.9, the entries of $R_l(\Delta')$ are all non-negative integers. Furthermore, using Frobenius reciprocity, this also takes care of the coefficients of $\alpha'_j$’s and $\overline{\alpha}_k$’s in the tensor product of two bimodules.

2.5. **Fusion rules on** $M\mathcal{X}_M \times M\mathcal{X}_M$. The remaining issue is to determine the fusion rule among $f$ and $g$. By observing the Perron-Frobenius weights, $\overline{f} = f$, $\overline{g} = g$. Since for $j$ even, all the $\alpha'_j$’s are self-conjugate as well, we have $fg = gf$.

**Theorem 2.10.**

$$< f^2, f > = d_{2k-1}, \quad < fg, f >= d_{2k},$$

$$< fg, g > = d_{2k+1}, \quad < g^2, g >= d_{2k+2},$$

where $d_k$’s are as in the proof of Claim 2.4, namely defined by $d_j = d_{j-1} + d_{j-2} + d_{j-3}$, $d_{-1} = 0$, $d_0 = d_1 = 1$.

**Lemma 2.11.**

$$< f^2, f > - < fg, g > = d_{2k-1} - d_{2k+1},$$

$$< fg, f > - < g^2, g > = d_{2k} - d_{2k+2},$$

$$< fg, g > - < g^2, g > = d_{2k+1} - d_{2k+2}.$$ 

**Proof of Lemma 2.11** We use the similar strategy as in Claim 2.4. Let $G'$ be the adjacency matrix for $(V_{22}, V_{21})$ corresponding to the graph $\Gamma'_k$, and let

$$\Delta' := \begin{pmatrix} 0 & G' \\ G'^t & 0 \end{pmatrix}.$$ 

\[ \Gamma'_k: \]

\[ \alpha'_0 \overline{\alpha}_1 \alpha'_2 \ldots \alpha'_{n-1} \overline{\alpha}_n \]

\[ f \rightarrow \beta_2 \]

\[ g \rightarrow \overline{\gamma}_2 \]
Observe

\[
R_0(\Delta')(g - f) = (g - f),
\]
\[
R_1(\Delta')(g - f) = \gamma_2 + \beta_2,
\]
\[
R_2(\Delta')(g - f) = g + f,
\]
\[
R_3(\Delta')(g - f) = 2\alpha'_n,
\]
\[
R_4(\Delta')(g - f) = 2\alpha'_{n-1} + f + g,
\]

where \(\alpha'_j = \pi_j\) for \(j\) odd. Then we have

\[
S_j(\Delta')(g - f) = 2\alpha'_{n-j+3}
\]

for \(j = 3, 4, \ldots, n + 3\), where the polynomials \(S_j\)'s are as defined in the proof of Claim 2.4. On the other hand,

\[
g + f = R_{n+1}(\mathbb{D}')\alpha'_0
\]

\[
= R_{4k+4}(\mathbb{D}')\alpha'_0 = Q_{2k+2}(\mathbb{D}_1)\alpha_0.
\]

Using Lemma 2.5

\[
(g + f)(g - f)
\]

\[
= (d_0S_2(2k+2) + d_1S_2(2k+1) + \cdots + d_{2k}S_4 + d_{2k+1}R_2 + d_{2k-1}R_0)(\Delta')(g - f)
\]

\[
= (\text{linear combination of } \alpha'_j\text{'s}) + d_{2k+1}(g + f) + d_{2k-1}(g - f)
\]

\[
= (\text{linear combination of } \alpha'_j\text{'s}) + (d_{2k+1} + d_{2k-1})g + (d_{2k+1} - d_{2k-1})f.
\]

Therefore we have

\[
< (g - f)(g + f), g > = < g^2, g > - < f^2, g >
\]

\[
= d_{2k+1} + d_{2k-1} = d_{2k+2} - d_{2k},
\]

\[
< (g - f)(g + f), f > = < g^2, f > - < f^2, f > = d_{2k+1} - d_{2k-1}. \tag{b1}
\]

We obtain further information by investigating \(R_2(\Delta')(g + f)(g - f)\). Note that \(R_2(\Delta')(g + f) = 2\alpha'_{n-1} + f + 3g\). Therefore

\[
R_2(\Delta')(g + f)(g - f)
\]

\[
= (2\alpha'_{n-1} + f + 3g)(g - f)
\]

\[
= 2\alpha'_{n-1}(g - f) + 3g^2 - f^2 - 2fg
\]

\[
= (\alpha'_j\text{'s}) + 2(d_{2k}(g + f) + d_{2k-2}(g - f)) + 3g^2 - f^2 - 2fg
\]

\[
= (\alpha'_j\text{'s}) + 2(d_{2k} + d_{2k-2})g + 2(d_{2k} - d_{2k-2})f + 3g^2 - f^2 - 2fg \quad \text{(#1)}
\]
On the other hand,
\[
R_2(\Delta')(g + f)(g - f) = R_2(\Delta')(2(d_0\alpha'_2 + d_1\alpha'_4 + \cdots + d_{2k}\alpha'_{4k+2})) + (d_{2k+1} + d_{2k-1})R_2(\Delta')g \\
+ (d_{2k+1} - d_{2k-1})R_2(\Delta')f \\
= (\alpha'_s) + 2d_{2k}(f + g) + (d_{2k+1} + d_{2k-1})(\alpha'_{n-1} + f + 2g) \\
+ (d_{2k+1} - d_{2k-1})(\alpha'_{n-1} + g) \\
= (\alpha'_s) + (2d_{2k} + d_{2k+1} + d_{2k-1})f + (2d_{2k} + 3d_{2k+1} + d_{2k-1})g. \tag{\#2}
\]

Comparing (\#1) and (\#2) we obtain
\[
3 < g^2, g > - < f^2, g > - 2 < fg, g > = 3d_{2k+1} + d_{2k-1} - 2d_{2k-2}, \\
3 < g^2, f > - < f^2, f > - 2 < fg, f > = d_{2k+1} + d_{2k-2} + 2d_{2k-2} \tag{\#2}
\]

Combining the equations (\#1) and (\#2) we obtain the statement of the Lemma. Note that we use Frobenius reciprocity such as \(< fg, f > = < f^2, g >\) etc. \(\square\)

**Lemma 2.12.**

\(< g^2, g > = d_{2k+2},\)

which implies, together with Lemma 2.11, Theorem 2.10.

**Proof** Since \(g = \overline{\beta}_2\alpha_1 = \overline{\gamma}_2\alpha_1,\)

\[
2g = (\overline{\beta}_2 + \overline{\gamma}_2)\alpha_1 = (\overline{\beta}_3 + \overline{\gamma}_3)\alpha_1 = \overline{\alpha}_1(\beta_3 + \gamma_3)\alpha_1.
\]

Also note \(\overline{\gamma}_2 = \overline{\gamma}_3\alpha_1 = \overline{\alpha}_1\beta_3.\) Therefore

\[
4 < g^2, g > = \overline{\alpha}_1(\beta_3 + \gamma_3)\overline{\alpha}_1(\beta_3 + \gamma_3)\alpha_1, \overline{\alpha}_1\beta_3\alpha_1 > \\
= \overline{\alpha}_1\overline{\alpha}_1(\beta_3 + \gamma_3)\overline{\alpha}_1(\beta_3 + \gamma_3)\alpha_1, \overline{\alpha}_1\beta_3\alpha_1 > \\
= (\beta_3 + \gamma_3)^2(\overline{\alpha}_1\overline{\alpha}_1)^3, \beta_3 > \\
= (\beta_3 + \gamma_3)^2, \beta_3(\overline{\alpha}_1\overline{\alpha}_1)^3 >,
\]

where we used \(\alpha_1\overline{\alpha}_1(\beta_3 + \gamma_3) = \beta_1 + \beta_3 + \gamma_1 + \gamma_3 = \overline{\beta}_1 + \overline{\beta}_3 + \overline{\gamma}_1 + \overline{\gamma}_3 = (\overline{\beta}_3 + \overline{\gamma}_3)\alpha_1\overline{\alpha}_1 = (\beta_3 + \gamma_3)\alpha_1\overline{\alpha}_1 .\) By computation using the graph \(\Gamma_k,\) one obtains

\[
\beta_3(\alpha_1\overline{\alpha}_1)^3 = 5\beta_3 + 10\beta_1 + 6\alpha_{n-1} + 6\gamma_1 + \alpha_{n-3} + \gamma_3.
\]

The formula for \((\beta_3 + \gamma_3)^2\) is given in Claim 2.24. Using it we obtain

\[
< (\beta_3 + \gamma_3)^2, \beta_3(\alpha_1\overline{\alpha}_1)^3 > \\
= 8c_{2k} + 12c_{2k+1} + 16c_{2k+2} \\
= 4c_{2k+1} + 8c_{2k+2} + 8c_{2k+3} \\
= 4c_{2k+2} + 4c_{2k+3} + 4c_{2k+4} = 4c_{2k+5} = 4d_{2k+2}.
\]

Therefore \(< g^2, g > = d_{2k+2}.\)
2.6. Fusion rules on $M \times M \times N$. The remaining problem is to determine the fusion rule on $\{f, g\} \times \{\beta, \gamma\}$.

$$< f \beta_2, \beta_2 > = < f, \beta_2 > = < f, \alpha \beta_2 > = < \alpha f, \beta_2 > = < \alpha_n, \beta_2 > = < \beta_3, \beta_1 + \beta_2, \gamma_1 > + < \beta_3, \alpha_n > .$$

Using Theorems 2.7 and 2.8

$$< f \beta_2, \beta_2 > = g_{2k+2} + g_{2k+1} .$$

Both values are non-negative integers. Similarly we obtain

$$< f \gamma_2, \gamma_2 > = < f \beta_2 > = f_{2k+2} + f_{2k+1} ,$$

$$< f \gamma_2, \gamma_2 > = f_{2k+2} + f_{2k+1} .$$

$$< f \beta_2, \beta_2 > = < f \beta_2 > = < f, \alpha \beta_2 > = < \alpha f, \beta_2 > = < \alpha_n, \beta_2 > = < \beta_3, \beta_1 + \beta_2, \beta_2 > = < \alpha_1 \alpha \beta_3, \beta_3 > = < \alpha_1 \alpha \beta_3, \beta_3 > = < \alpha_1 \alpha \beta_3 > .$$

Thus, using Theorems 2.7 and 2.8 we obtain

$$< g \beta_2, \beta_2 > = \begin{cases} f_{2k+1} + 2f_{2k+2} + f_{2k} - 1 & \text{if } k \text{ even} \\ f_{2k+1} + 2f_{2k+2} + f_{2k} & \text{if } k \text{ odd} \end{cases}$$

Similarly,

$$< g \beta_2, \beta_2 > = < g \beta_2 > = \begin{cases} g_{2k+1} + 2g_{2k+2} + g_{2k} + 2 & \text{if } k \text{ even} \\ g_{2k+1} + 2g_{2k+2} + g_{2k} & \text{if } k \text{ odd} \end{cases}$$

$$< g \gamma_2, \gamma_2 > = < g \gamma_2 > = \begin{cases} g_{2k+1} + 2g_{2k+2} + g_{2k} + 2 & \text{if } k \text{ even} \\ g_{2k+1} + 2g_{2k+2} + g_{2k} & \text{if } k \text{ odd} \end{cases} .$$

3. Existence of the fusion algebra

Let $k \in \mathbb{N}_0$, and put $n = 4k + 3$ as before. In this section we will reserve the symbols

$$(\alpha_j)_{0 \leq j \leq n}, \ (\beta_j)_{1 \leq j \leq 3}, \ (\gamma_j)_{1 \leq j \leq 3}$$
for elements in a certain bi-graded \(\mathbb{Z}\)-algebra \(A\) which we define later. Therefore we relabel the vertices of the graph \(\Gamma_k\) in the following way:

\[
\begin{array}{cccccc}
& & & b_2 & & b_3 \\
& a_0 & a_1 & a_2 & a_{n-1} & a_n \\
& & & b_1 \\
& c_1 & c_2 & c_3 & & \\
\end{array}
\]

As in Section 2.1, we let \(G\) be the adjacency matrix for \((\Gamma_k^{\text{even}}, \Gamma_k^{\text{odd}})\), where

\[
\Gamma_k^{\text{even}} = \{a_0, a_2, \ldots, a_{n-1}, b_1, c_1, b_3, c_3\},
\]
\[
\Gamma_k^{\text{odd}} = \{a_1, a_3, \ldots, a_n, b_2, c_2\},
\]
we set \(D = GG^t\), and

\[
\Delta := \begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix}.
\]
We set \((q_k)_{k=0}^\infty\) be the sequence of polynomials defined

\[
q_0(t) = t^2 - 5t + 3,
\]
\[
q_1(t) = (t - 1)(t^3 - 8t^2 + 17t - 5),
\]
\[
q_k(t) = (t^2 - 4t + 2)q_{k-1}(t) - q_{k-2}(t), \quad k \geq 2
\]
as in Section 2.1. Then the characteristic polynomial for \(D\) is

\[
\chi_k(t) = t^2(t - 2)^2 q_k(t).
\]
(\text{cf. Section 2.1.}) Moreover \(q_k(t)\) is a polynomial of degree \(2k + 2\) with \(2k + 2\) distinct roots, because by [3], either \(q_k(t)\) or \(q_k(t)/(t - 1)\) is an irreducible polynomial. From the recursion formula for the \(q_k\)-polynomials, one obtains

\[
q_k(0) = 2k + 3,
\]
\[
q_k(2) = (-1)^{k+1}(2k + 3),
\]
In particular, 0 and 2 are not roots of \(q_k\). Let \(k \in \mathbb{N}_0\) be now fixed. From the above, we knot that \(\chi_k(t)\) has exactly \(2k + 4\) distinct roots \((t_j)_{j=1}^{2k+4}\), where \(t_1 = 0\), \(t_2 = 2\), and \(t_3, \ldots, t_{2k+4}\) are the roots of \(q_k(t)\). Since \(D = GG^t\) is a positive operator, \(t_j \geq 0\) for \(1 \leq j \leq 2k + 4\).

**Lemma 3.1.** Let \(E_j\) be the orthogonal projection on the eigenspace of \(D\) corresponding to the eigenvalue \(t_j\) \((1 \leq j \leq 2k + 4)\) and put

\[
\mu_j = \langle E_j a_0, a_0 \rangle,
\]

where \(\langle \cdot, \cdot \rangle\) is the inner product in \(L^2(\Gamma_k^{\text{even}})\). Then
(a) $\Sigma_{j=1}^{2k+4} \mu_j = 1$,

(b) $\mu_j > 0$ for $1 \leq j \leq 2k + 4$,

(c) $\mu_1 = \mu_2 = \frac{1}{2k+3}$.

**Proof.**

(a): Since $\mathbb{D}$ is a symmetric matrix, $\Sigma_{j=1}^{2k+4} E_j = I$, which proves (a).

(b): From Section 2.1, we have

\[
Q_j(\mathbb{D})a_0 = R_{2j}(\Delta)a_0 = a_{2j} \quad (0 \leq j \leq 2k + 1),
\]

\[
Q_{2k+2}(\mathbb{D})a_0 = R_{4k+4}(\Delta)a_0 = b_1 + c_1,
\]

\[
Q_{2k+3}(\mathbb{D})a_0 = R_{4k+6}(\Delta)a_0 = b_1 + c_1 + b_3 + c_3.
\]

Since \{a_0, a_2, \ldots, a_{4k+2}, b_1 + c_1, b_1 + c_1 + b_3 + c_3\} is a set of $2k + 4$ linear independent vectors in $l^2(\Gamma_k^\text{even})$, and since $(Q_j)_{0 \leq j \leq 2k+3}$ spans the set of polynomials of degree less or equal to $2k + 3$, we have

\[
P(\mathbb{D})a_0 \neq 0
\]

for every non-zero polynomial $P \in \mathbb{R}[x]$ with $\deg(P) \leq 2k + 3$. On the other hand, since $\mathbb{D}$ is diagonalisable with with eigenvalues $(t_j)_{j=1}^{2k+4}$, we have

\[
E_j = P_j(\mathbb{D}),
\]

where

\[
P_j(t) = \Pi_{i \neq j} \frac{t - t_i}{t_j - t_i}, \quad (t \in \mathbb{R})
\]

is a polynomial of degree $2k + 3$. Hence

\[
\mu_j = \langle E_j a_0, a_0 \rangle = ||E_j a_0||^2 > 0 \quad (1 \leq j \leq 2k + 4).
\]

(c): From Section 2.1 we have

\[
\text{range}(E_1) = E(\mathbb{D}, 0) = \text{span}\{y_1, y_2\},
\]

\[
\text{range}(E_2) = E(\mathbb{D}, 2) = \text{span}\{x_1, x_2\},
\]

where

\[
x_1 := 2(a_0 + a_2) - 2(a_4 + a_6) + \cdots + (-1)^{k+1}(a_{4k} + a_{4k+2})
\]

\[
+ (-1)^{k+1}(b_1 + c_1 + b_3 + c_3)
\]

\[
x_2 := (b_1 - c_1) + (b_3 - c_3)
\]

\[
y_1 := 2a_0 - 2a_2 + \cdots + 2a_{4k} - 2a_{4k+2} + (b_1 + c_1) - (b_3 + c_3)
\]

\[
y_2 := (b_1 - c_1) - (b_3 - c_3)
\]

Since $y_1 \perp y_2$ and $y_2 \perp a_0$, we get

\[
\mu_1 = \langle E_1 a_0, a_0 \rangle = \frac{\langle y_1, a_0 \rangle^2}{||y_1||^2} = \frac{1}{2k+3}
\]

and similarly

\[
\mu_2 = \langle E_2 a_0, a_0 \rangle = \frac{\langle x_1, a_0 \rangle^2}{||x_1||^2} = \frac{1}{2k+3}.
\]

□
Corollary 3.2. Let \((e_{ij})_{i,j=1}^{2k+4}\) be the matrix units of \(M_{2k+4}(\mathbb{R})\). Put
\[
B = \text{span}_\mathbb{R}\{e_{11}, e_{12}, e_{21}, e_{22}, e_{33}, \ldots, e_{2k+4,2k+4}\}
\cong M_2(\mathbb{R}) \oplus l^\infty(\{3, 4, \ldots, 2k + 4\}, \mathbb{R})
\]
Then \(B\) is a finite dimensional real \(C^*\)-algebra and \(\mu : B \to \mathbb{R}\) given by
\[
\mu(b) := \sum_{j=1}^{2k+4} \mu_j b_{jj}, \quad b = (b_{ij})_{i,j=1}^{2k+4} \in B
\]
is a faithful trace state on \(B\).

Proof: it is clear from (a) and (b) in Lemma 3.1 that \(\mu\) is a faithful state on \(B\) and the trace property
\[
\mu(bc) = \mu(cb), \quad b, c \in B
\]
follows from (c) in Lemma 3.1. \(\square\)

Lemma 3.3. Let \(k \in \mathbb{N}_0\) be fixed and let \(\mu : B \to \mathbb{R}\) be the trace defined above, and put
\[
A := \text{diag}(0, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{t_{2k+4}}),
\]
where \(t_3, \ldots, t_{2k+4}\) are the roots of \(q_k\). Then
(a) For every even polynomial \(P \in \mathbb{R}[x]\)
\[
\mu(P(A)) = < P(\Delta) a_0, a_0 >.
\]
(b) Let \(P, Q \in \mathbb{R}[x]\) be two polynomials, which are either both even or both odd. Then
\[
\mu(P(A)Q(A)) = < P(\Delta) a_0, Q(\Delta) a_0 >.
\]
(c) Let \(n = 4k + 3\) (as usual), then
\[
R_{n+4}(A) - R_{n+2}(A) - R_n(A) - R_{n-2}(A) = 0.
\]

Proof
(a): Let \(Q \in \mathbb{R}[x]\) be so that \(P(t) = Q(t^2)\). Then
\[
< P(\Delta) a_0, a_0 > = < Q(\mathbb{D}) a_0, a_0 >.
\]
Let \(E_j\) denote the spectral projection of \(\mathbb{D}\) corresponding to the eigenvalue \(t_j\) \(1 \leq j \leq 2k + 4\) as before, where \(t_1 = 0\) and \(t_2 = 0\). Then
\[
Q(\mathbb{D}) = \sum_{j=1}^{2k+4} Q(t_j) E_j.
\]
Hence
\[
< Q(\mathbb{D}) a_0, a_0 > = \sum_{j=1}^{2k+4} Q(t_j) < E_j a_0, a_0 >
\]
\[
= \sum_{j=1}^{2k+4} \mu_j Q(t_j)
\]
\[
= \mu(Q(A^2)) = \mu(P(A)).
\]
(b): Under the assumption on $P$ and $Q$, the product $PQ$ is an even polynomial. Hence by (a) we have

$$\mu(P(A)Q(A)) = \langle P(\Delta)Q(\Delta)a_0, a_0 \rangle = \langle P(\Delta)a_0, Q(\Delta)a_0 \rangle.$$ 

(c): Put $P = Q = R_{n+4} - R_{n+2} - R_n - R_{n-2}$, which is an odd polynomial. By (b),

$$\mu(P(A)^2) = ||P(\Delta)a_0||^2_2.$$ 

From the recursive formula for the polynomials $R_j$ one has

\[
\begin{align*}
R_{n-2}(\Delta)a_0 &= a_{n-2}, \\
R_n(\Delta)a_0 &= a_n, \\
R_{n+2}(\Delta)a_0 &= a_n + b_2 + c_2, \\
R_{n+4}(\Delta)a_0 &= a_{n-2} + 2a_n + b_2 + c_2 \\
&= (R_{n+2}(A) + R_n(A) + R_{n-2}(A))a_0.
\end{align*}
\]

Hence $\mu(P(A)^2) = ||P(\Delta)a_0||^2_2 = 0$, and since $\mu$ is a faithful trace on $B$, we have $P(A) = 0$. □

**Remark 3.4.** Since $P = R_{n+4} - R_{n+2} - R_n - R_{n-2}$ is an odd polynomial and $P(A) = 0$, we know that $P(t)$ has at least $n + 4 = 4k + 7$ roots

$$0, \pm \sqrt{2}, \pm \sqrt{3}, \ldots, \sqrt{t_{2k+4}},$$

which are exactly the distinct roots of $t(t^2 - 2)q_k(t^2)$. Since $P$ and $t(t^2 - 2)q_k(t^2)$ are both monic polynomial of degree $4k + 7$, it follows that

$$(R_{n+4} - R_{n+2} - R_n - R_{n-2})(t) = t(t^2 - 2)q_k(t^2).$$

It is not hard to prove this identity directly by using the recursion formulas for the polynomials $\{q_k\}$'s and $\{R_j\}$'s.

**Definition 3.5.** Let $k \in \mathbb{N}_0$, $n = 4k + 3$, and let $(B, \mu)$ and $A = \text{diag}(\sqrt{1}, \sqrt{2}, \ldots, \sqrt{t_{2k+4}}) \in B$ be as before. Let $(f_{ij})^2_{i,j=1}$ be the matrix units in $M_2(\mathbb{R})$, and put

$$V := V_{11} \sqcup V_{12} \sqcup V_{21} \sqcup V_{22},$$

where $V_{ij} \subset B \otimes f_{ij}$ $(i, j = 1, 2)$ are described as below:

1. $V_{11} = \{\alpha_0, \alpha_2, \alpha_4, \ldots, \alpha_{4k+2}, \beta_1, \gamma_1, \beta_3, \gamma_3\}$, where

   - $\alpha_{2j} = R_{2j}(A) \otimes f_{11}, \ 0 \leq j \leq 2k + 1,$
   - $\beta_1 = \frac{1}{2}(R_{n+1}(A) + \sqrt{2k + 3(e_{12} + e_{21})}) \otimes f_{11},$
   - $\gamma_1 = \frac{1}{2}(R_{n+1}(A) - \sqrt{2k + 3(e_{12} + e_{21})}) \otimes f_{11},$
   - $\beta_3 = \frac{1}{2}((R_{n+3} - R_{n+1} - R_{n-1})(A) + \sqrt{2k + 3(e_{12} - e_{21})}) \otimes f_{11},$
   - $\gamma_3 = \frac{1}{2}((R_{n+3} - R_{n+1} - R_{n-1})(A) - \sqrt{2k + 3(e_{12} - e_{21})}) \otimes f_{11}$
b) \( V_{12} = \{\alpha_1, \alpha_3, \alpha_5, \ldots, \alpha_{4k+3}, \beta_2, \gamma_2\} \) where
\[
\begin{align*}
\alpha_{2j+1} &= R_{2j+1}(A) \otimes f_{12}, \quad 0 \leq j \leq 2k + 1, \\
\beta_2 &= \frac{1}{2}((R_{n+2} - R_n)(A) + \sqrt{2(2k+3)}e_{12}) \otimes f_{12}, \\
\gamma_2 &= \frac{1}{2}((R_{n+2} - R_n)(A) - \sqrt{2(2k+3)}e_{12}) \otimes f_{12},
\end{align*}
\]

c) \( V_{21} = \{\overline{\alpha_1}, \overline{\alpha_3}, \overline{\alpha_5}, \ldots, \overline{\alpha_{4k+3}}, \overline{\beta_2}, \overline{\gamma_2}\} \) where
\[
\begin{align*}
\overline{\alpha_{2j+1}} &= R_{2j+1}(A) \otimes f_{21}, \quad 0 \leq j \leq 2k + 1, \\
\overline{\beta_2} &= \frac{1}{2}((R_{n+2} - R_n)(A) + \sqrt{2(2k+3)}e_{21}) \otimes f_{21}, \\
\overline{\gamma_2} &= \frac{1}{2}((R_{n+2} - R_n)(A) - \sqrt{2(2k+3)}e_{21}) \otimes f_{21},
\end{align*}
\]
d) \( V_{22} = \{\alpha'_0, \alpha'_2, \ldots, \alpha'_{4k+2}, f, g\} \) where
\[
\begin{align*}
\alpha'_j &= R_{2j}(A) \otimes f_{22}, \quad 0 \leq j \leq 2k + 1, \\
f &= \frac{1}{2}(R_{n-1} + 2R_{n+1} - R_{n+3})(A) \otimes f_{22}, \\
g &= \frac{1}{2}(R_{n+3} - R_{n-1})(A) \otimes f_{22}.
\end{align*}
\]
e) The conjugation map \( V_{12} \to V_{21} \) and \( V_{21} \to V_{12} \) is already defined earlier. For \( V_{11}, V_{22} \) all the elements are defined to be self-conjugate except \( \beta_3 \) and \( \gamma_3 \) which are defined to be conjugate of each other. Note that for every \( X \in V_{ij} \), the conjugate \( \overline{X} \) is equal to \( X^* \) (or \( X^t \), since all the matrices here are real).
f) We will equip \( \mathbb{R}V_{ij} \subset B \otimes f_{ij} \) with inner products given by
\[
\langle b \otimes f_{ij}, c \otimes f_{ij} \rangle_\mu := \mu(c^i b) = \mu(bc^i)
\]
for every \( b, c \in \mathbb{R}V_{ij} \) \((i, j = 1, 2)\).

Lemma 3.6. Let \( i, j \in \{1, 2\} \). For \( X, Y \in V_{ij} \),
\[
\langle X, Y \rangle_\mu = \begin{cases} 1 & \text{if } X = Y, \\
0 & \text{if } X \neq Y. \end{cases}
\]

Proof
Let \( (b, c)_\mu := \mu(c^i b) = \mu(bc^i), b, c \in B \) be the inner product on \( B \) given by \( \mu \), and put \( \|b\|_\mu, (b, b)_\mu^{1/2}, b \in B \).
a) Case \((i, j) = (1, 1)\). It suffices to show that
\[
S_1 := \{R_0(A), R_2(A), \ldots, R_{n+1}(A), (R_{n+3} - R_{n+1} - R_{n-1})(A), e_{12} + e_{21}, e_{12} - e_{21}\}
\]
is an orthogonal set in \( B \) and that
\[
\begin{align*}
\|R_{2j}(A)\|_\mu^2 &= 1, \quad 0 \leq j \leq \frac{n-1}{2}, \\
\|R_{n+1}(A)\|_\mu^2 &= 2, \\
\|(R_{n+3} - R_{n+1} - R_{n-1})(A)\|_\mu^2 &= 2, \\
\|e_{12} + e_{21}\|_\mu^2 &= \|e_{12} - e_{21}\|_\mu^2 = \frac{2}{2k+3}.
\end{align*}
\]
By the definition of $\mu$ in Corollary 3.2, it is clear that $e_{12} + e_{21}$ and $e_{12} - e_{21}$ are $\mu$-orthogonal to the remaining matrices in $S_1$, because $R_j(A)$ is a diagonal matrix for all $j \in \mathbb{N}_0$. Moreover, by Lemma 3.1
\[
< e_{12} + e_{21}, e_{12} - e_{21} >_\mu = \mu(e_{11} - e_{22}) = \mu_1 - \mu_2 = 0,
\]
\[
||e_{12} + e_{21}||_\mu^2 = ||e_{12} - e_{21}||_\mu^2 = \mu(e_{11} + e_{22}) = \mu_1 + \mu_2 = \frac{2}{2k + 3}.
\]
By Lemma 3.3 (b), the remaining part of the proof in the $V_{11}$-case reduces to show that
\[
T_1 := \{R_0(\Delta)a_0, R_2(\Delta)a_0, \ldots, R_{n+1}(\Delta)a_0, (R_{n+3}(\Delta) - R_{n+1}(\Delta) - R_{n-1}(\Delta))a_0\}
\]
is an orthogonal set in $l^2(\Gamma_k)$ with
\[
||R_{2j}(\Delta)a_0||^2 = 1, \ 0 \leq j \leq n - 1,
\]
\[
||R_{n+1}(\Delta)a_0||^2 = 2, 
\]
\[
||(R_{n+3} - R_{n+1} - R_{n-1})(\Delta)a_0||^2 = 2.
\]
This follows from the fact that
\[
T_1 = \{a_0, a_2, \ldots, a_{n-1}, b_1 + c_1, b_3 + c_3\}.
\]
b) cases $(i, j) = (1, 2)$ and $(i, j) = (2, 1)$. It suffices to show that
\[
S_2 := \{R_1(A), R_3(A), \ldots R_n(A), (R_{n+2} - R_n)(A), e_{12}\}
\]
is an orthonormal set in $B$ and that
\[
||R_{2j+1}(A)||^2_\mu = 1, \ 0 \leq j \leq \frac{n - 1}{2},
\]
\[
||(R_{n+2} - R_n)(A)||^2_\mu = 2, 
\]
\[
||e_{12}||^2_\mu = \frac{1}{2k + 3}.
\]
It is easy to check that $e_{12}$ is orthogonal to the remaining elements of $S_2$ and that $||e_{12}||^2_\mu = (2k + 3)^{-1}$ by Lemma 3.3 (b). The remaining statement about the set $S_2$ follow from the fact that
\[
T_2 = \{R_1(\Delta)a_0, R_3(\Delta)a_0, \ldots, R_n(\Delta)a_0, (R_{n+2} - R_n)(\Delta)a_0\}
\]
\[
= \{a_1, a_3, \ldots, a_n, b_2 + c_2\}
\]
is an orthonormal set in $l^2(\Gamma_k)$, and that
\[
||a_{2j+1}||^2 = 1, \ 0 \leq j \leq \frac{n - 1}{2},
\]
\[
||b_2 + c_2||^2 = 2.
\]
c) Case $(i, j) = (2, 2)$. The statement follows in this case if we can show that
\[
S_3 := \{R_0(A), R_2(A), \ldots, R_{n-1}(A), \frac{1}{2}(R_{n-1} + 2R_{n+1} - R_{n+3})(A), \frac{1}{2}(R_{n+3} - R_{n-1})(A)\}
\]
is a $\mu$-orthogonal set in $B$. By Lemma 3.3 (b) this reduces to showing that
\[
T_3 := \{a_0, a_2, \ldots, a_{n-1}, \frac{1}{2}(b_1 + c_1 + b_3 + c_3), \frac{1}{2}(b_1 + c_1 - b_3 - c_3)\}
\]
is an orthogonal set in $l^2(\Gamma_k)$, which is obvious. □
Theorem 3.7. Let $V = V_{11} \sqcup V_{12} \sqcup V_{21} \sqcup V_{22}$ as in Definition 3.5. Then $ZV \subset M_2(B)$ form a fusion ring, with coefficients given by

$$N^Z_{XY} = < XY, Z >_\mu,$$

where $X \in V_{ij}, Y \in V_{jk}, Z \in V_{ik}, (i, j, k) \in \{1, 2\}^3$, and with units $\alpha_0 \in V_{11}$ and $\alpha'_0 \in V_{22}$. Moreover the graph with vertices $V_{11} \sqcup V_{12}$ obtained by right multiplication by $\alpha = \alpha_1$ is $\Gamma_k$ and the graph with vertices $V_{21} \sqcup V_{22}$ obtained by right multiplication $\overline{\alpha}$ is $\Gamma'_k$.

**Proof.**

Note that by Lemma 3.6, $V_{ij}$ is a linear independent set in $B \otimes f_{ij}$ for all $i, j \in \{1, 2\}$. Hence

$$\text{dim}(R_{V_{11}}) = |V_{11}| = 2k + 6$$

and

$$\text{dim}(R_{V_{12}}) = \text{dim}(R_{V_{21}}) = \text{dim}(R_{V_{22}}) = 2k + 4.$$ 

This implies that

$$
\begin{align*}
R_{V_{11}} & = B \otimes f_{11}, \\
R_{V_{12}} & = \text{span}\{e_{12}, e_{22}, e_{33}, \ldots, e_{2k+4,2k+4}\} \otimes f_{12}, \\
R_{V_{21}} & = \text{span}\{e_{21}, e_{22}, e_{33}, \ldots, e_{2k+4,2k+4}\} \otimes f_{21}, \\
R_{V_{22}} & = \text{span}\{e_{11}, e_{22}, e_{33}, \ldots, e_{2k+4,2k+4}\} \otimes f_{22},
\end{align*}
$$

because the four inclusions $\subset$ are obvious, and the right hand sides have dimensions $2k + 6$ (resp. $2k + 4, 2k + 4, 2k + 4$). Therefore

$$RV = R_{V_{11}} \oplus R_{V_{12}} \oplus R_{V_{21}} \oplus R_{V_{22}}$$

form a bi-graded $\mathbb{R}$-algebra, and the conjugation $X \rightarrow X^t$ extends by linearity to all of $RV$ and it is given by transposition of matrices. Moreover, for $X \in V_{ij}, Y \in V_{jk}, (i, j, k \in \{1, 2\})$, we have a unique decomposition

$$XY = \sum_{Z \in V_{ik}} N^Z_{XY} Z,$$

where by Lemma 3.6

$$N^Z_{XY} = < XY, Z >_\mu \in \mathbb{R}.$$ 

The identities

$$N^Z_{XY} = N^X_{Z,Y} = N^Y_{X,Z} = N^{Y^t}_{Z,X} = N^{X^t}_{Y,Z}$$

is now a simple consequence of the fact that $\mu$ is a trace state on the real $C^*$-algebra $B$, so in particular

$$\mu(b) = \mu(b^t), b \in B,$$

$$\mu(bc) = \mu(cb), b, c \in B.$$ 

It remains to be proved that $N^Z_{X,Y} \in \mathbb{N}_0$ and that multiplication from the right by $\alpha = \alpha_1$ (resp $\overline{\alpha}$) on $V_{11}$ (resp $V_{22}$) generates the graph $\Gamma_k$ (resp. $\Gamma'_k$).

**Lemma 3.8.** Let $\alpha = \alpha_1$.

a) For $X \in V_{11}, Y \in V_{12}$,

$$< X\alpha, Y >_\mu = < X, Y\overline{\alpha} >_\mu \in \mathbb{N}_0,$$
and \((< X\alpha, Y >_\mu)_X\in V_{11}, Y\in V_{12}\) is the adjacency matrix \(G_k\) for \(\Gamma_k\).

b) For \(X \in V_{22}, Y \in V_{21}\),
\[< X\bar{\alpha}, Y >_\mu = < X, Y \alpha >_\mu \in \mathbb{N}_0,\]
and \((< X\bar{\alpha}, Y >_\mu)_X\in V_{22}, Y\in V_{21}\) is the adjacency matrix \(G'_k\) for \(\Gamma'_k\).

**Proof**

This follows from simple computations using Definition 3.5, Lemma 3.6, the recursion formula
\[tR_n(t) = R_{n+1}(t) + R_{n-1}(t), n \geq 1\]
and the identity from Lemma 3.3(c)
\[R_{n+4}(A) - R_{n+2}(A) - R_n(A) - R_{n-2}(A) = 0:\]
a) It follows immediately from \((\star)\) that for \(1 \leq j \leq 2k + 1,\)
\[\alpha_{2j} \alpha = \alpha_{2j+1} + \alpha_{2j-1}\]
which shows that \(\alpha_{2j} \in V_{11}\) is connected to \(\alpha_{2j+1}\) and \(\alpha_{2j-1}\) in \(V_{12}\) (with simple edges) and not connected to any other \(Y \in V_{12}\). To prove that we recover the graph \(\Gamma_k\) this way we just have to check that \(\alpha_0 \alpha = \alpha_1\), which is obvious, and that \(\beta_1 \alpha = \alpha_n + \beta_2, \beta_3 \alpha = \beta_2\). The last one follows from
\[
\beta_3 \alpha = \frac{1}{2}((R_{n+3} - R_{n+1} - R_{n-1})(A) + \sqrt{2k+3}(e_{12} + e_{21}))A) \otimes f_{12}
\]
\[
= \frac{1}{2}(R_{n+4} - 2R_n - R_{n-2})(A) + \sqrt{2(2k+3)e_{12}} \otimes f_{12}
\]
\[
= \frac{1}{2}((R_{n+2} - R_n)(A) + \sqrt{2(2k+3)e_{12}}) \otimes f_{12}
\]
\[
= \beta_2,
\]
where we have used \((\star)\) and \((\star\star)\) and the fact that \(e_{12}A = \sqrt{2}e_{12}, e_{21}A = 0\). The proof of \(\beta_1 \alpha = \alpha_n + \beta_2\) is similar.

b) To recover the graph \(\Gamma'_k\) from \(V_{22} \sqcup V_{21}\), it suffices to prove that
\[
\alpha'_0 \bar{\alpha} = \bar{\alpha}_1,
\]
\[
\alpha'_{2j} \bar{\alpha} = \bar{\alpha}_{2j+1} + \bar{\alpha}_{2j-1} (1 \leq j \leq 2k + 1)
\]
\[
f \bar{\alpha} = \bar{\alpha}_n
\]
\[
g \bar{\alpha} = \bar{\alpha}_n + \bar{\beta}_2 + \gamma_2
\]
The first two are obvious. Let us prove \(f \bar{\alpha} = \bar{\alpha}_n\). The formula for \(g \bar{\alpha}\) is obtained in the same way
\[
f \bar{\alpha} = \frac{1}{2}((R_{n-1}(A) + 2R_{n+1}(A) - R_{n+3}(A))A) \otimes f_{21}
\]
\[
= \frac{1}{2}(R_{n-2} + 3R_n + R_{n+2} - R_{n+4})(A) \otimes f_{21}
\]
\[
= \frac{1}{2} \cdot 2R_n(A) \otimes f_{21}
\]
\[
= \frac{1}{2}(A) \otimes f_{21}
\]
where we again have used \((\star)\) and \((\star\star)\).
Lemma 3.9. Put
\[ \xi := (\beta_1 - \gamma_1) + (\beta_3 - \gamma_3). \]
Then
\[ \bar{\xi} := (\beta_1 - \gamma_1) - (\beta_3 - \gamma_3), \]
and
\[
\begin{align*}
\frac{1}{2} \xi \xi &= 2\alpha_0 - 2\alpha_2 + \cdots + 2\alpha_{4k} - 2\alpha_{4k+2} + (\beta_1 + \gamma_1) - (\beta_3 + \gamma_3) \\
\frac{1}{2} \bar{\xi} \bar{\xi} &= 2(\alpha_0 + \alpha_2) - 2(\alpha_4 + \alpha_6) + \cdots + (-1)^k2(\alpha_{4k} + \alpha_{4k+2}) \\
&\quad + (-1)^{k+1}(\beta_1 + \gamma_1 + \beta_3 + \gamma_3)
\end{align*}
\]

Proof.
Clearly \( \bar{\xi} = (\beta_1 - \gamma_1) - (\beta_3 - \gamma_3). \) By Lemma 3.8, we know that the linear maps
\[
R_\alpha : \mathbb{R}V_{11} \rightarrow \mathbb{R}V_{12} \\
R_\tau : \mathbb{R}V_{12} \rightarrow \mathbb{R}V_{11}
\]
obtained by right multiplication by \( \alpha \) (resp. \( \bar{\alpha} \)) have the matrices \( G^t \) (resp. \( G \)) expressed with respect to bases \( V_{11} \) for \( \mathbb{R}V_{11} \) and \( V_{11} \) for \( \mathbb{R}V_{12} \). Hence
\[
R_{\alpha \bar{\alpha}} := R_\tau R_\alpha : \mathbb{R}V_{11} \rightarrow \mathbb{R}V_{12}
\]
has the matrix \( D = GG^t \) with respect to the basis \( V_{11} \) for \( \mathbb{R}V_{11} \). We can now argue exactly as in Case 1 of Section 2.1 to get
\[
\xi \xi \in E(D, 0)_{sc} = \mathbb{R}y_1, \\
\bar{\xi} \bar{\xi} \in E(D, 2)_{sc} = \mathbb{R}x_1,
\]
where
\[
\begin{align*}
y_1 &= 2\alpha_0 - 2\alpha_2 + \cdots + 2\alpha_{4k} - 2\alpha_{4k+2} + (\beta_1 + \gamma_1) - (\beta_3 + \gamma_3) \\
x_1 &= 2(\alpha_0 + \alpha_2) - 2(\alpha_4 + \alpha_6) + \cdots + (-1)^k2(\alpha_{4k} + \alpha_{4k+2}) \\
&\quad + (-1)^{k+1}(\beta_1 + \gamma_1 + \beta_3 + \gamma_3).
\end{align*}
\]
Since \( \langle \xi \xi, \alpha_0 \rangle = \langle \bar{\xi} \bar{\xi}, \alpha_0 \rangle = \langle \xi, \xi \rangle = \mu = 4 \) and \( \langle y_1, \alpha_0 \rangle = \langle x_1, \alpha_0 \rangle = \mu = 2 \), it follows that
\[
\xi \xi = 2y_1 \text{ and } \bar{\xi} \bar{\xi} = 2x_1.
\]

End of proof of Theorem 3.7
It remains to be proved that \( N_{X,Y}^Z \in \mathbb{N}_0 \) for all \( X \in V_{ij}, Y \in V_{jk} \) and \( Z \in V_{ik}, (i, j, k \in \{1, 2, 3\}) \). Having established the formulas for \( \xi \xi \) and \( \bar{\xi} \bar{\xi} \) in Lemma 3.8, the proof of \( N_{X,Y}^Z \in \mathbb{N}_0 \) can be obtained from Section 2. Using that
\[
N_{X,Y}^Z = N_{Z,Y}^X = N_{Y,X}^Z,
\]
one gets that if \( X, Y \) or \( Z \) is one of the elements \( (\alpha_j)_{0 \leq j \leq n}, (\alpha'_j)_{0 \leq j \leq n} \) (where \( \alpha'_{2k+1} = \bar{\alpha}_{2k+1} \)), then \( N_{X,Y}^Z \) is an entry of the matrix \( R_j(\Delta) \) or \( R_j(\Delta') \), which by \( 7 \) is a non-negative integer. In the remaining cases, \( X, Y, Z \) are compatible and comes from the list
\[
\beta_1, \gamma_1, \beta_3, \gamma_3, \beta_2, \gamma_2, \bar{\beta}_2, \bar{\gamma}_2, f, g.
\]
For \(X, Y, Z \in \{\beta_1, \gamma_1, \beta_3, \gamma_3\}\), we have \(N^\infty_{X, Y} \in \mathbb{N}_0\) by Theorem 2.7, 2.8 and the remark at the end of Section 2.1. The case \(X, Y, Z \in \{f, g\}\) is treated in Theorem 2.10 and the remaining cases can easily be reduced to these two cases by using \(\beta_2 = \beta_3\alpha\) and \(\gamma_2 = \gamma_3\alpha\) (c.f. Sections 2.2 and 2.6).

**Remark 3.10.** From Definition 3.5, we have
\[
\xi = (\beta_1 - \gamma_1) + (\beta_3 - \gamma_3) = 2\sqrt{2k + 3}e_1 \otimes f_{11},
\]
\[
\overline{\xi} = (\beta_1 - \gamma_1) - (\beta_3 - \gamma_3) = 2\sqrt{2k + 3}e_2 \otimes f_{11}.
\]
Thus
\[
\xi \overline{\xi} = 4(2k + 3)e_{11} \otimes f_{11},
\]
\[
\overline{\xi} \xi = 4(2k + 3)e_{22} \otimes f_{11}.
\]
Since \(A = \text{diag}(0, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{2k+4})\), where \(t_3, \ldots, t_{2k+4}\) are the distinct roots of \(q_k(t)\), and since \(0, 2 \notin \{t_3, \ldots, t_{2k+4}\}\), \(e_{11}\) and \(e_{22}\) are the projections on the eigenspaces for \(A\) with eigenvalues 0 and 2 respectively. Using \(q_k(0) = 2k + 3\) and \(q_k(2) = (-1)^{k+1}(2k + 3)\), one gets
\[
(2 - A^2)q_k(A^2) = 2(2k + 3)e_{11},
\]
\[
A^2q_k(A^2) = (-1)^{k+1}(2k + 3)e_{22},
\]
because the polynomial \((2 - t)q_k(t)\) vanishes at \(t = 2\) and \(t = t_j, 3 \leq j \leq 2k + 4\) and has the value \(2(2k + 3)\) at \(t = 0\). Similarly \(tq_k(t)\) vanishes at \(t = 0\) and \(t = t_j, 3 \leq j \leq 2k + 4\) and has the value \((-1)^{k+1}(2k + 3)\) at \(t = 2\). Hence the following two identities hold:
\[
\xi \overline{\xi} = 2(2 - A^2)q_k(A^2) \otimes f_{11} = 2(1_N - \alpha \overline{\alpha})q_k(\alpha \overline{\alpha}),
\]
\[
\overline{\xi} \xi = (-1)^{k+2}2A^2q_k(A^2) \otimes f_{11} = (-1)^{k+2}2\alpha \overline{\alpha}q_k(\alpha \overline{\alpha}),
\]
where \(1_N = \alpha_0\) and \(\alpha = \alpha_1\). Let \(Q_j\) denote as usual the polynomial for which \(R_{2j}(t) = Q_j(t^2), t \in \mathbb{R}\). Then by Definition 3.5,
\[
\alpha_{2j} = Q_j(\alpha \overline{\alpha}),
\]
\[
\beta_1 + \gamma_1 = Q_{2k+2}(\alpha \overline{\alpha}),
\]
\[
\beta_3 + \gamma_3 = (Q_{2k+3} - Q_{2k+2} - Q_{2k+1})(\alpha \overline{\alpha}).
\]

Hence a more direct proof of Lemma 3.8 can be obtained if the two polynomial identities (i) and (ii) below holds: Put
\[
r_k(t) = (2 - t)q_k(t), \ s_k(t) = (-1)^{k+1}tq_k(t).
\]
Then
\[
(i) \quad r_k = (2Q_0 - 2Q_1 + \cdots + 2Q_{2k} - 2Q_{2k+1})
\]
\[
\quad + (Q_{2k+1} + 2Q_{2k+2} - Q_{2k+3})
\]
\[
(ii) \quad s_k = 2(Q_0 + Q_2) - 2(Q_2 + Q_4) + \cdots + (-1)^k2(Q_{2k} + Q_{2k+1})
\]
\[
\quad + (-1)^{k+1}(Q_{2k+3} - Q_{2k+1}).
\]
These two polynomials identities are actually true, and they can be proved by using the recursion formulas for \((q_k)_{k=0}^\infty\) and \((R_j)_{j=0}^\infty\). □
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