Simultaneously identify right-hand side and lowest coefficient in a parabolic equation

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Abstract. The theory and practice of inverse problem of partial differential equations play an important role in science and engineering applications. This paper deals with the numerical solutions of the inverse problem to simultaneously identify the right-hand and lowest coefficient that dependent on time only in a two dimensional parabolic equation. Backward difference in time and finite element procedure in space are constructed for solving this kind of inverse problem. Based on a special decomposition, the algorithm transforms the original problem into three standard elliptic problems at the new time level. Numerical examples are given to demonstrate the ability of the proposed computational algorithm for solving such coefficient inverse problem.

1. Introduction

Inverse problems, corresponding to direct problems, arise in many mathematical modelling in the fields of mathematical physics and engineering. Based on the unknowns, there are several types of inverse problems, such as coefficient inverse problem, boundary inverse problem and time inverse problems, etc. More information on the classification of inverse problems can be seen in [1] and references therein.

According to Hadamard [2], inverse problem of partial differential equations is classified as a non-classical problem, usually an ill-posed or conditionally well-posed problem, a small disturbance in the input data can lead to large errors in the results. In addition, inverse problem is always nonlinear, which makes solving them more difficult, large amount of calculation is also a problem in solving an inverse problem. Therefore, the study of numerical solution to inverse problems is of great significance.

In this paper, we consider the numerical solutions of right-hand side and lowest coefficient in parabolic equation, which is a coefficient inverse problem. In recent years, the inverse problem of coefficient has attracted more attention. The study of existence and uniqueness of the solution and the well-posedness in various functional classes were given in [3-5]. Numerical solution of coefficient inverse problem were considered in many works, Erdogan and Ashyralyev in their work [6] considered on second order implicit difference schemes for the problem of identification of right-hand side, the method similar to bordering method for reconstructing a distributed right-hand problem was discussed in [7], Vabishchevich considered coefficient inverse problem of finding an unknown lower coefficient in two dimensional elliptic equation in [8], a low-cost method based on fast Fourier
transform was studied in [9], the inverse problem to identify the time-dependent diffusion coefficient from integral condition was given in [10], etc.

In the present work, finite element method combined with a special decomposition is used to determine only time dependent right-hand and lowest coefficient in a parabolic equation. The construction of this paper is as follows, section 2 gives an introduction of coefficient inverse problem, which will be considered. In section 3, we talk about the computational algorithm and apply it on the inverse problem. Numerical example is given in section 4 to illustrate the ability of the computational method. Section 5 gives a brief conclusion and some references are listed in the end.

2. Mathematical formulation

We will consider the following second order parabolic equation (1) with time dependent coefficient \( p(t) \) and right-hand side \( q(t) \) on a bounded domain \( \Omega \subseteq \mathbb{R}^2 \) with a smooth boundary \( \partial \Omega \), where \( t \in [0, T] \),

\[
\frac{\partial u}{\partial t} - \text{div}(k(x)\text{grad} u) + p(t)u = q(t)g(x,t),
\]

with the initial conditions,

\[
u(x,0) = u_0(x), \quad x \in \Omega,
\]

and Dirichlet boundary conditions,

\[
u(x,t) = h(t), \quad x \in \partial \Omega,
\]

where \( 0 < k_1 \leq k(x) \leq k_2, \ x = (x, y) \) and \( u_0(x), \ h(t) \) are known functions.

In the coefficient inverse problem, we will find the solution \( u(x,t) \) together with the only time dependent coefficient \( p(t) \) and the right-hand side \( q(t) \). As we all know, when consider an inverse problem of partial differential equation, some additional conditions are needed. And for a coefficient inverse problem, the additional conditions are always formulated as some interior solutions of the master equation or as the average values integrated over the whole domain. In our work, we take the additional conditions as two specifications of the solution at two interior points, which are given as follows,

\[
u(x^+,t) = \Phi(t).
\]

\[
u(x^-,t) = \Psi(t).
\]

Many works have been done for the research of determination of time dependent coefficients in parabolic equations. Some theoretical studies, including existence, uniqueness and the condition well-posedness are given in the works [5, 11, 12]. There are also a lot of published articles give a research of the numerical solutions of the coefficient inverse problems. In [13], the determination of time dependent coefficient in heat equation was considered, the work supplied unique solvability and regularization method for solving this inverse problem. Finite difference method were used to identify the right-hand and lowest coefficient in our works [14, 15], a finite difference method combined with the MATLAB toolbox routine \textit{lsqnonlin} was studied in [16]. More information about inverse problem please refer to the books [17, 18] and the references therein.

3. Computational algorithm

To solve the inverse problem (1)–(4), we define the uniform time grid,
\[
\mathcal{O}_\tau = \{ I^n \mid t^n = n\tau, \; n = 1, \cdots, N, \; N\tau = T\},
\]

using the backward difference scheme for approximation in time and the new notation \( u^n(x) = u(x, t^n) \),

\[
\frac{\partial u}{\partial t}(x, t^{n+1}) \approx \frac{u^{n+1}(x) - u^n(x)}{\tau}, \quad (5)
\]

and employing finite element approximation in space. In the domain \( \Omega \), triangulation is performed and a finite dimensional space \( V \subseteq H^1_e(\Omega) \) of finite element is introduced for computational grid.

We use the test function \( \forall \; \nu \in V \), which vanishes on the boundary of \( \Omega \), and the fully implicit scheme, together with the initial conditions, we obtain the following variational problem.

\[
\int \frac{u^{n+1} - u^n}{\tau} \nu \, dx + \int k(x) \text{grad } u^{n+1} \text{grad } \nu \, dx + p^{n+1} \int u^{n+1} \nu \, dx = \int g^{n+1} \nu \, dx,
\]

\[
\int u(x, 0) \nu \, dx = \int u_0 \nu \, dx,
\]

with the boundary conditions,

\[
u^{n+1}(x) = h^{n+1}, \quad x \in \partial \Omega,
\]

and the additional conditions,

\[
u^{n+1}(x^*) = \Phi^{n+1}, \quad \nu^{n+1}(x^**) = \Psi^{n+1}.
\]

For simple evaluate the inverse problem, we use the following equation instead (6),

\[
\int \frac{u^{n+1} - u^n}{\tau} \nu \, dx + \int k(x) \text{grad } u^{n+1} \text{grad } \nu \, dx + p^{n+1} \int u^n \nu \, dx = \int g^{n+1} \nu \, dx,
\]

\[
\int u(x, 0) \nu \, dx = \int u_0 \nu \, dx,
\]

introducing the following decomposition for solution \( u^{n+1}(x) \) at new time level,

\[
u^{n+1}(x) = \mu^{n+1}(x) + p^{n+1} w_1^{n+1}(x) + q^{n+1} w_2^{n+1}(x).
\]

To find \( \mu^{n+1}(x) \), substituting the above decomposition into (9), we can obtain the following equation,

\[
\int \frac{\mu^{n+1}}{\tau} \nu \, dx + \int k(x) \text{grad } \mu^{n+1} \text{grad } \nu \, dx = \int \frac{u^n}{\tau} \nu \, dx,
\]

with the boundary conditions \( \mu^{n+1}(x) = h^{n+1} \). And the auxiliary functions \( w_1^{n+1}(x) \), \( w_2^{n+1}(x) \) are determined from two equations, which are given as follows,

\[
\int \frac{w_1^{n+1}}{\tau} \nu \, dx + \int k(x) \text{grad } w_1^{n+1} \text{grad } \nu \, dx = -\int u^n \nu \, dx,
\]

\[
\int \frac{w_2^{n+1}}{\tau} \nu \, dx + \int k(x) \text{grad } w_2^{n+1} \text{grad } \nu \, dx = \int g^n \nu \, dx,
\]
on the boundary has \( w_1^{n+1}(x) = 0 \) and \( w_2^{n+1}(x) = 0 \).

To evaluate the solutions \( p^{n+1} \) and \( q^{n+1} \) the additional conditions (8) are used, substituting the decomposition of solution \( u^{n+1}(x) \) into (8), we can obtain the following equations

\[
\begin{align*}
p^{n+1} w_1^{n+1}(x^*) + q^{n+1} w_2^{n+1}(x^*) &= \Phi^{n+1} - \mu^{n+1}(x^*), \\
p^{n+1} w_1^{n+1}(x^{**}) + q^{n+1} w_2^{n+1}(x^{**}) &= \Psi^{n+1} - \mu^{n+1}(x^{**}),
\end{align*}
\]

(13)

in order to determine \( p^{n+1} \) and \( q^{n+1} \), we assume that,

\[
w_1^{n+1}(x^*)w_2^{n+1}(x^{**}) - w_1^{n+1}(x^{**})w_2^{n+1}(x^*) \neq 0.
\]

(14)

Thus, the inverse problem (1)–(4) is solved based on the linearized scheme (7)–(9) combined with the decomposition of \( u^{n+1}(x) \) by using the auxiliary functions \( \mu^{n+1} \), which is obtained from (10) , and \( w_1^{n+1} \), which is obtained from (11), \( w_2^{n+1} \) is got from (12), the parabolic problem is transformed into three elliptic equations. The further evaluation of \( p^{n+1} \) and \( q^{n+1} \) from the above linear system and the solutions of \( u^{n+1} \) can be identified by bringing back to the decomposition.

4. Numerical examples

To demonstrate the ability of the algorithm for numerical solution of the only time dependent right-hand and lowest coefficient in a parabolic equation, numerical example is given in this section.

In this example we take the known conditions as,

\[
k(x) = 1, \quad g(x, t) = -6(x + y),
\]

(15)

with the initial conditions,

\[
u_0(x) = x^3 + y^3.
\]

(16)

The coefficient \( p(t) \) and right-hand \( q(t) \) take as the forms,

\[
p(t) = -\frac{2\zeta \exp(-\zeta^2)}{2 - \exp(-\zeta^2)}, \quad q(t) = 2 - \exp(-\zeta^2),
\]

(17)

where \( \zeta \) is a constant not less than zero. The figures of \( p(t) \) and \( q(t) \) with different \( \zeta \) are given in figure 1.

In the computation of this example we take \( T = 1 \) and on the triangular mesh dolfin-2.xml.gz distributed with DOLFIN, which can be seen in figure 2.

The numerical solution with \( \zeta = 10 \) at the final time is shown in figure 3.

We considered the inverse problem use the triangular grid, which is presented in figure 2 and time step \( \tau = 0.002 \), the two observation points are chosen as \( x^* = (0.4, 0.75) \) and \( x^{**} = (0.35, 0.3) \). Figure 4 shows the pictures of exact and numerical solutions of \( p(t) \) and \( q(t) \) with different \( \zeta \). It can be seen that the numerical solutions are well overlapped with exact one.
We use two errors, Maxer and RMSE, which are defined as follows, to consider the impact of different time step \( \tau \) on the results,

\[
\text{Maxer} = \max_{1 \leq n \leq N} \left| p(t^n) - \tilde{p}(t^n) \right|, \\
\text{RMSE} = \sqrt{\frac{1}{N} \sum_{n=1}^{N} \left| p(t^n) - \tilde{p}(t^n) \right|^2},
\]

where \( \tilde{p}(t^n) \) is the numerical solution at time \( t^n \), while \( p(t^n) \) is the exact one. For \( q(t) \) are defined by the same definition. Table 1 and Table 2 present the two errors of \( p(t) \) and \( q(t) \) with different \( \zeta \), respectively.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
& \( \tau \) & 0.01 & 0.005 & 0.002 \\
\hline
\( \zeta = 2 \) & Maxer & 1.92e-2 & 8.99e-3 & 3.29e-3 \\
& RMSE & 8.87e-3 & 4.11e-3 & 1.01e-3 \\
\hline
\end{tabular}
\caption{Two errors of \( p(t) \) with different time steps \( \tau \)}
\end{table}
| ζ = 5 | Maxer | 4.55e-2 | 2.10e-3 | 5.75e-3 |
|-------|-------|---------|---------|---------|
| RMSE  | 1.81e-2 | 8.37e-3 | 2.03e-3 |
| ζ = 10| Maxer  | 8.87e-2 | 4.07e-2 | 9.86e-3 |
| RMSE  | 3.03e-2 | 1.40e-2 | 3.41e-3 |

Table 2. Two errors of \( q(t) \) with different time steps \( \tau \).

| \( \tau \) | 0.01 | 0.005 | 0.002 |
|------------|------|-------|-------|
| \( \zeta = 2 \) Maxer | 2.41e-4 | 1.36e-4 | 2.41e-4 |
| RMSE       | 3.38e-5 | 1.65e-5 | 1.27e-5 |
| \( \zeta = 5 \) Maxer | 2.39e-4 | 1.34e-4 | 2.38e-4 |
| RMSE       | 3.37e-5 | 1.64e-5 | 1.26e-5 |
| \( \zeta = 10 \) Maxer | 2.35e-4 | 1.31e-4 | 2.36e-4 |
| RMSE       | 3.35e-5 | 1.65e-5 | 1.25e-5 |

We consider the results with different observation points \( x^* = (x, 0.75) \) and \( x^{**} = (0.35, y) \). Tables 3, 4 are presented with different \( x \) and \( y \) with time step \( \tau = 0.002 \). From the Tables we can see that the choice of observation points also has some impact on the results.

Table 3. Two errors of \( p(t) \) with different observation points.

| \((x, y)\) | \((0.2, 0.1)\) | \((0.4, 0.3)\) | \((0.6, 0.9)\) |
|-----------|----------------|----------------|----------------|
| \( \zeta = 2 \) Maxer | 1.68e-2 | 3.29e-3 | 1.19e-2 |
| RMSE      | 1.21e-3 | 1.01e-3 | 1.28e-3 |
| \( \zeta = 5 \) Maxer | 1.93e-2 | 5.57e-3 | 9.96e-3 |
| RMSE      | 2.16e-3 | 2.03e-3 | 1.98e-3 |
| \( \zeta = 10 \) Maxer | 2.34e-2 | 9.86e-3 | 6.61e-3 |
| RMSE      | 3.51e-3 | 3.41e-3 | 3.18e-3 |

Table 4. Two errors of \( q(t) \) with different observation points.

| \((x, y)\) | \((0.2, 0.1)\) | \((0.4, 0.3)\) | \((0.6, 0.9)\) |
|-----------|----------------|----------------|----------------|
| \( \zeta = 2 \) Maxer | 7.23e-4 | 2.41e-4 | 1.23e-3 |
| RMSE      | 3.17e-5 | 1.27e-5 | 9.67e-4 |
| \( \zeta = 5 \) Maxer | 7.23e-4 | 2.38e-4 | 1.27e-3 |
| RMSE      | 3.18e-5 | 1.26e-5 | 9.81e-4 |
| \( \zeta = 10 \) Maxer | 7.24e-4 | 2.36e-4 | 1.34e-3 |
| RMSE      | 3.18e-5 | 1.25e-5 | 1.01e-4 |

When solving an inverse problem numerically, due to measurement error, the input data are always given with noisy, much attention is paid to the numerical algorithm for approximation of solution in an inverse problem with the input data is given with some errors. In our inverse problem, the input data are the observation solutions at interior points. We will consider the influence of inaccuracies in the two functions \( \Phi(t) \) and \( \Psi(t) \) on the numerical results.

To give an illustration of the algorithm with errors of the input data, we use \( \Phi_\delta \) and \( \Psi_\delta \), defined as follows,

\[
\Phi_\delta = \Phi + \delta(2^\gamma - 1), \quad \Psi_\delta = \Psi + \delta(2^\gamma - 1)
\]  \( (19) \)
where $\delta$ is tolerated noise level, $\gamma$ is a random variable uniformly distributed on the interval $[0, 1]$. 

Figure 5. The solution of $p(t)$ and $q(t)$ with $\delta = 10^{-3}$.

In this computation we use $\tau = 0.005$ and the two points $x^* = (0.45, 0.8), x^{**} = (0.7, 0.95)$. To smooth $\Phi_\delta$ and $\Psi_\delta$, the B-splines method is used. Figure 5 presents the solutions of $p(t)$ and $q(t)$ with different $\zeta$ and $\delta = 10^{-3}$. The figures of error RMSE of $p(t)$ and $q(t)$ with different noisy level are present in figure 6.

From figure 5 and 6 can be seen that the numerical method in solving the coefficient inverse problem of simultaneous identification the lowest coefficient and the right hand side in parabolic equation is effective and capable, even when there is a small error in the input data.

Figure 6. The RMSE of $p(t)$ and $q(t)$ with different $\delta$.

5. Conclusions
In this paper we consider the inverse problem to simultaneously determine the only time dependent lowest coefficient and the right hand side in a parabolic equation. Based on a special decomposition combined with the backward difference scheme in time, we transform the parabolic equation into three elliptic equations and finite element method in space is used to solve the problems. Numerical examples demonstrate the effectiveness and accuracy of this algorithm.

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