Energy-efficient gas pipeline transportation

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Gas transportation via long pipelines is considered. Distributed parameter, dynamic modelling with series and shunt energy dissipation and gas stream, equivalent capacitance and inductance effects are proposed. Hybrid analysis techniques, wherein both the distributed and the lumped, concentrated elements of the pipeline system are included in the overall model, are advocated. Constrained optimisation procedures, with the introduction of the Hamiltonian function to minimise the pipeline, inflow–outflow difference, are invoked thereby promoting impedance matching and the energy-efficient transportation of the specified, gas volume flow rate. Illustrative application studies are outlined thereby validating the analytical methods employed and the determination of the optimum, pipeline exit resistance.

Keywords: gas; pipeline; transportation; modelling; optimisation

Nomenclature

\[ C_j \] capacitance of the \( j \)th airway per unit length scalar

\[ L_j \] inductance of the \( j \)th airway per unit length scalar

\[ r_j \] series resistance of the \( j \)th airway per unit length scalar

\[ g_j \] shunt conductance (admittance) of the \( j \)th airway per unit length scalar

\[ p_j(s) \] pressure at inlet to the \( j \)th airway per unit length function

\[ p_{j+1}(s) \] pressure at outlet from the \( j \)th airway per unit length function

\[ q_j(s) \] volume flow at inlet to the \( j \)th airway per unit length function

\[ q_{j+1}(s) \] volume flow at outlet from the \( j \)th airway per unit length function

\[ \zeta_j(s) \] characteristics impedance of the \( j \)th airway function

\[ \Gamma_j(s) \] propagation function for the \( j \)th airway function

\[ T \] phase lead time constant scalar

\[ \tau \] phase lag time constant scalar

\[ R \] frictional resistance at pipeline exit scalar

\[ \rho \] gas density scalar

1. Introduction

The transportation of gas over long distances by pipelines will be considered in this contribution. This method of supply is a relatively safe, reliable and a cost-effective form of conveying natural gas and oil which is universally employed.

Constructing and the installation of gas or oil pipelines is an expensive, labour-intensive and politically sensitive operation. These networks often span remote regions, cross-national boundaries and ecologically protected areas resulting in international agreements following delicate, protracted negotiations.

Beyond this, the running cost associated with gas pipelines is substantial. Owing to the frictional energy dissipation arising from the internal pipeline roughness, welds, joints, bends and discontinuities, there is a continuous reduction in the gas stream pressure and, hence, volume gas steam flow rate. To counter this effect, compressors and heat exchangers are installed, at strategic locations along the pipeline thereby restoring the pressure loss and gas volume flow rate.

Due to the length of gas pipelines and the proportional pressure loss, the compressors and heat exchangers employed operate continuously. Consequently, the running cost, maintenance and refit charges associated with this requirement are substantial.

This problem is exacerbated by the remote locations, monitoring and operation of the compressor drive systems and gas coolers. These active devices must also respond to varying load demands with the requirement for constant delivery pressure and supply rates.

As with all continuously operated, long-duration, duty cycle systems, any operational economy is translated into significant savings owing to the accumulating reduction in running, maintenance and delivery costs. To assess
demands under which maximum economy can be achieved whilst satisfying specified delivery conditions, an accurate model for long pipeline systems is required. Once this has been established, optimum operating conditions can be investigated.

Unfortunately, the classical theory for spatially dispersed pipeline systems results in irrational, multivariable, input–output models which are incomplete in the Laplace transform variable; see, for example, Schwartz & Friedland (1965) and Takahashi (1970). Technically, it is possible to obtain the predicted system responses from these models. However, the procedures involved do not provide simple, usable results which can be incorporated into design, analysis or optimisation investigations.

Finite element techniques may be used to assess the pipeline dynamics. With this procedure, large matrix models arise from the modelling procedure thereby attracting computational errors. Considerable speculation surrounding the computed pipeline performance may also be encountered in that the number and composition of the elements employed are unspecified in finite element modelling, as discussed in Section 2.

In view of this, the focus of this contribution will be on deriving a pipeline modelling method which includes all of the salient features of classical analysis whilst avoiding the above complications arising from multiple, lumped parameter methods. Once established, this provision invites the use of constrained optimisation and the employment of impedance matching to confine operational energy consumption.

Ultimately, the delivery of specified, regulated gas supplies is the principal attraction. This, when considering pipelines with varying, distributed frictional characteristics, will be investigated using the calculus of variations following the derivation of an accurate, unambiguous model, for the pipeline system.

2. Modelling methods

In the transportation of gases via long pipelines and compressor networks, geographical dispersion is a significant feature. Owing to this, obtaining estimates for the volume flow of gas using conventional, lumped parameter theory is inappropriate.

It may be considered that using multiple lumping with the application of finite element techniques would be sufficient. However, this is not so, since the dynamics of spatially dispersed systems comprise a combination of travelling, stationary and reflected pressure and volume flow waveforms. Hence, there is no equivalent lumped, cascaded model counterpart since travelling and reflected transient components cannot materialise in lumped representations, in that all spatial dispersion effects are absent from these realisations.

In any case when considering pipelines of >10 km in length, the matrix models, derived from finite element methods, would be dimensionally very large (Esahanian & Behbani-Nejad, 2002). Consequently, in addition to the analytical disadvantages cited above, numerical computational errors which would further contaminate the results would be encountered (Bradie, 2006).

Alternatively, with the employment of hybrid, distributed–lumped modelling an accurate modelling method is available (Whalley, 1988; Whalley & Abdul-Ameer, 2009). This procedure allows pipeline elements, which are clearly distributed, to be modelled using distributed parameter methods. Otherwise, relatively point-wise components and sub-assemblies such as valves, compressors, bends and restrictions may be represented using lumped analysis methods without too much loss of accuracy. This allows engineering judgement to be exercised in selecting the appropriate modelling method for each sub-system element.

Importantly, the many boundary conditions and complexities arising from the use of distributed parameter methods universally are avoided with this approach, whilst including simple point-wise representations for components which form the pipeline connecting elements and series or parallel branches.

Following the modelling of the individual elements of pipelines, distributed–lumped analysis allows the construction of an overall hybrid matrix configuration for the system. This final model provides a general, component-identifiable, accurate, impedance admittance realisation enabling dynamic simulation, analysis and regulator design.

3. Series and parallel representations

Elements comprising an overall pipeline system may be assembled series, parallel or in series–parallel form where in each case the steady-state volume flow would be inversely proportional to the pipeline input impedance.

In the analysis herein, lumped parameter components are represented by simple passive impedances, \( G_{ij}(s) \). However, any analogous two-port network representation could be employed to model these units, as shown in Whalley (1990).

4. The distributed parameter modelling of pipelines

Although the flow of gas in pipeline systems is usually turbulent, three-dimensional and nonlinear, there are compelling reasons for the formulation of simple, usable models for perturbed flow changes, relative to the given steady-state conditions. Essentially, this type of model would enable the analysis of complex, interconnected applications. Ideally this would be via general solutions for the spatially distributed pressure–flow relationships following input or disturbance changes. Moreover, simulation studies could be easily accommodated using this form of representation with the advantage that regulator design exercises could now be embarked upon using existing theoretical techniques and algorithms.
In fact, pioneering work, as detailed in Iberall (1960), Nichols (1961) and Brown (1962), showed that theoretically derived, first-order, perturbed, acoustic, one-dimensional approximations for the Navier–Stokes equations were available. These modelling restrictions included zero bulk modulus and radiant heat transfer effects. A continuous, homogeneous medium was also assumed with no radial or axial heat transfer effects.

Further work within this framework was undertaken in Bartlett and Whalley (1998) where a general discrete, distributed–lumped parameter representation for linear systems was presented. Low-temperature application studies using this approach were proposed in Bartlett and Whalley (1995) where all of these representations related to the perturbed pressure-variation dynamics relative to steady-state equilibrium conditions.

Extending this work, this contribution focuses on the distributed parameter system model element shown in Appendix 1, Figure A1 where \( L_j, C_j, r_j \) and \( g_j \) are the pipeline system distributed inductance, capacitance and series (longitudinal) and shunt (radial) flow resistance and conductance effects per metre length of pipeline, respectively. The governing equations for this type of element for the \( j \)th pipeline section are

\[
\frac{\partial p_j}{\partial x}(t,x) = -L_j \frac{\partial q_j}{\partial t} - r_j q_j(t,x) \quad (1)
\]

and

\[
\frac{\partial q_j}{\partial x}(t,x) = -C_j \frac{\partial p_j}{\partial t} - g_j p_j(t,x) \quad \text{as shown in Figure 1.} \quad (2)
\]

Following Laplace transformation with zero initial conditions, Equations (1) and (2) yield the solution for the \( j \)th distributed parameter model of a system of \( m \) elements, as shown in Appendix 1:

\[
\begin{bmatrix}
P_j(s, l_j) \\
P_{j+1}(s, l_{j+1})
\end{bmatrix}
= \begin{bmatrix}
\zeta_j(s)(s)w_j(s) & -\zeta_j(s)(w_j^2(s) - 1)^{1/2} \\
\zeta_j(s)(w_j^2(s) - 1)^{1/2} & -\zeta_j(s)w_j(s)
\end{bmatrix}
\times \begin{bmatrix}
Q_j(s, l_j) \\
Q_{j+1}(s, l_{j+1})
\end{bmatrix},
\]

where \( j = 2k + 1, k = 0, 1, \ldots, m - 1 \) (m is the number of distributed parameter elements),

\[
\zeta_j(s) = \left( \frac{(L_j s + r_j)}{(C_j s + g_j)} \right)^{1/2},
\]

\[
w_j(s) = \frac{(e^{s_{\tau_j} r_j} + 1)}{(e^{s_{\tau_j} r_j} - 1)}
\]

and

\[
\Gamma_j(s) = \left( \frac{(L_j s + r_j)}{(C_j s + g_j)} \right)^{1/2}.
\]

Consequently, even with all of the constraints mentioned earlier the input–output relationship for a typical distributed parameter gas pipeline network model is multivariable, irrational and is incomplete in the Laplace variable \( s \). This difficulty effectively masks any correspondence between the actual system performance and the governing equations so that extracting dynamic information from this representation is markedly impaired.

Although in Equation (3), for example, \( \Gamma_j(s) \) and \( \zeta_j(s) \) have branch points in the complex frequency plane, inversion with the use of Bromwich’s contour and the Laplace error function, \( \text{Erf}(t) = \left( 2/\pi \right) \int_0^t e^{-u^2} \, du \), see, for example, Spiegel (1965), is possible yielding the time–response characteristics following input pressure changes.

Figure 1. Distributed–lumped parameter model of pipeline, compressor and motor.
However, this does little to aid the recovery of the original aim of producing a “user friendly” distributed parameter description for large-scale, pipeline system modelling, analysis, simulation and regulator design.

In this regard, of interest here is the nature of the series impedance and shunt admittance of the infinitesimal pipeline element, shown in Figure A1. The series impedance frictional drag \( r_j \), for example, represents the effect of the gas flow on the pressure gradient arising from shear action at the pipeline wall boundary layer. Contrasting this, the shunt admittance or conductance arises from compressibility effects, as shown in Robertson and Crowe (1990) where the frictional drag arises from varying gas path compliance, owing to cross-flow turbulence and molecular heat transfer.

In pipeline systems, the flow impedance is principally due the entrance/exit losses and the \( r_j \) and \( g_j \), distributed pipeline frictional resistance effects, where in general:

\[
\frac{1}{g_j} \neq r_j, \quad (4)
\]

In dimensionally “long” pipelines both the series \( r_j \) and the shunt \( g_j \) frictional factors contribute to the overall pressure drop and diminishing volume flow characteristics. The analysis herein also confirms that the inclusion of both these dissipation mechanisms is mandatory.

Moreover, the per unit length energy storage parameters, as shown in Eckman (1958) and Palm (2005), for a circular pipeline, diameter \( 2a_j \), are the gas path capacitance and inductance of

\[
C_j = \frac{\pi a_j^2}{\gamma R_g g_j} \quad (5)
\]

and

\[
L_j = \frac{1}{\pi a_j^2}, \quad (6)
\]

respectively, where

\[
L_j \gg C_j \quad (7)
\]

for engineering applications.

In view of the inequalities of Equations (4) and (7) rationality may be recovered by equating

\[
\left( \frac{C_j s}{g_j} + 1 \right) \prod_{k=1}^{N} \left( T_{jk}s + 1 \right)^2 \simeq \left( \frac{L_j s}{r_j} + 1 \right) \prod_{k=1}^{N} \left( \tau_{jk}s + 1 \right)^2. \quad (8)
\]

It should be noted that with an appropriate choice of \( T_{jk} \) and \( \tau_{jk} \) the approximation of Equation (8), with \( s = j\omega \), is accurate at

low frequencies \( \omega < \frac{r_j}{L_j} \).

high frequencies \( \omega > \frac{g_j}{C_j} \)

and at \((2N - 1)\) intermediate frequencies \( \omega^* \) where

\[
\frac{r_j}{L_j} < \omega^* < \frac{g_j}{C_j}
\]

as shown by the Bode plot of Figure 2, where \( N = 1 \).

Then from Equation (A16) since

\[
\Gamma_j(s) = \left[ r_j \left( \frac{L_j s}{r_j} + 1 \right) \left( \frac{C_j s}{g_j} + 1 \right) g_j \right]^{1/2}. \quad (9)
\]

Equation (9) becomes, following the substitution shown in Equation (8):

\[
\Gamma_j(s) = \alpha_j \left( \frac{T_{j1}s + 1}{(T_{j1}s + 1)} \right) \left( \frac{T_{j2}s + 1}{(T_{j2}s + 1)} \right) \left( \frac{C_j}{g_j} + 1 \right), \quad (10)
\]

where \( \alpha_j = \sqrt{r_j g_j} \).

Equally, following Equations (A13) and (A14) since

\[
\zeta_j(s) = \sqrt{\frac{(L_j s + r_j)}{(C_j s + g_j)}} \quad (11)
\]

then Equation (11), with the substitution of Equation (8), continuing with \( N = 2 \) for illustration purposes, is

\[
\zeta_j(s) = \tilde{\alpha}_j \left( \frac{T_{j1}s + 1}{(T_{j1}s + 1)} \right) \left( \frac{T_{j2}s + 1}{(T_{j2}s + 1)} \right) \text{ where } \tilde{\alpha}_j = \sqrt{\frac{r_j}{g_j}}. \quad (12)
\]

The remaining important function of Equation (2) is \( w_j(s) \). Since

\[
w_j(s) = \frac{(e^{\gamma_j s})^2 + 1}{(e^{\gamma_j s})^2 - 1}. \quad (13)
\]
then with $\Gamma_j(s)$ from Equation (10):

$$w_j(s) = \frac{(e^{2\sqrt{\text{Re}_k}s} + 1)}{(e^{\sqrt{\text{Re}_k}s} - 1)}. \quad (14)$$

Consequently, upon substituting for Equation (14):

$$(w_j^2(s) - 1)^{1/2} = \frac{2e^{\sqrt{\text{Re}_k}s}}{(e^{\sqrt{\text{Re}_k}s} - 1)}$$

where in Equation (15):

$$\chi_j(s) = \left(\frac{T_{1j}s + 1}{\tau_{1j}s + 1}\right) \left(\frac{T_{2j}s + 1}{\tau_{2j}s + 1}\right) \left(\frac{C_j s + 1}{g_j} + 1\right). \quad (16)$$

From Equation (16), following division, the truncated series evaluation is always a simple $P+D$ term, given by

$$\chi_j \equiv a_j s + b_j, \quad (17)$$

where from Equation (17), expanding $\chi(s)$ for high frequencies gives

$$a_j = \left[\frac{C_j}{g_j} + (T_{1j} + T_{2j}) - (\tau_{1j} + \tau_{2j})\right]$$

and $b_j = 1$.

It is evident from Equations (13)–(17) that the distributed parameter model is now in an attractive form. The functions comprising Equation (3) are free from origin branch point problems with each component $\zeta_j(s)$, $w_j(s)$ and $(w_j^2(s) - 1)^{1/2}$ being single valued and complete, in the Laplace variable $s$ with simple steady-state values of

$$\zeta_j(0) = a_j, \quad w_j(0) = \frac{e^{2\sqrt{\text{Re}_k}s}(0) + 1}{e^{\sqrt{\text{Re}_k}s}(0) - 1} \quad \text{and} \quad (w_j^2(0) - 1)^{1/2} = \frac{2e^{\sqrt{\text{Re}_k}s}(0)}{e^{2\sqrt{\text{Re}_k}s}(0) - 1}.$$

Although values for the equivalent of the energy storage via the distributed capacitance and inductance effects per unit length in gas pipelines can be obtained with accuracy from Equations (5) and (6), the distributed, per unit length, series and shunt resistance values are more difficult to ascertain. From the theory of fluid dynamics, the pressure drop due to friction is inversely proportional to the Reynolds number, $Re < 2500$, for a given gas flow velocity, gas density, ducting length and cross-sectional area. For higher Reynolds numbers, owing to the elevated velocities in gas pipelines the empirical law of Blasius could be used to obtain estimates of the frictional flow coefficients, as discussed in Rogers and Mayhew (1970). These estimates and those obtained from Moody diagrams (Gautam, 2009) however are based on steady flow conditions, in a pipeline of constant cross-sectional area. Consequently, as in all engineering system problems, the frictional coefficient values are known with least confidence. Empirically based results, derived from direct measurement, may be used if the system exists. Otherwise, upper and lower bounded values for $r_j$ and $g_j$ may be employed, with the system response characteristics reflecting these estimates.

5. Lumped parameter elements

The assumption here is that pipeline restrictions can be modelled using elements having the ladder network structure shown in Langill (1965). In the overall configuration, presented in Equation (18), $g_{j+k+1}(s)$, $g_{j+k+2}(s)$ and $g_{j+k+3}(s)$ are the lumped impedances for the $j$th lumped element where $j = 2k$, $k = 1, \ldots, m$. Models of greater sophistication could be employed if so desired. However, for many applications the arrangement shown provides an adequate, equivalent analogue representation for sub-assemblies such as compressors, bends and valves for analysis purposes. The generic multivariable representation given in Rosenbrock (1974) may be adopted here, if required, to replicate impedance models of greater complexity.

6. Integrated hybrid model

An overall model structure must now be derived enabling the assembly of the system matrix. For purposes of illustration, if a distributed–lumped configuration is assumed, then an appropriate system matrix can be constructed by adding consecutively distributed or lumped system descriptions and in so doing eliminate all intermediate variables.

In this case, the system model for a total of $m$ distributed–lumped interconnected sections results in the system equation:

$$(P_1(s), 0, \ldots, 0)^T = \Omega(s)(Q_1(s), Q_2(s), \ldots, Q_m(s))^T,$$

where

$$\Omega(s) = \begin{bmatrix}
\zeta_1(s)w_1(s) & -\zeta_1(s)w_2(s) - 1^{1/2} \\
-\zeta_1(s)w_1(s) - 1^{1/2} & \zeta_1(s)w_2(s) - 1^{1/2} \\
0 & 0 \\
\vdots & \ddots \\
0 & 0 & \ddots \\
-g_{1,12}(s) & 0 & \ddots \\
\vdots & \ddots & \ddots \\
0 & 0 & \ddots \\
-g_{1,22}(s) & \zeta_2(s)w_2(s) & \zeta_2(s)w_2(s) - 1^{1/2} \\
-g_{2,12}(s) & \zeta_2(s)w_2(s) - 1^{1/2} & \zeta_2(s)w_2(s) - 1^{1/2} \\
0 & 0 & 0 \\
0 & 0 & \ddots \\
0 & 0 & \ddots \\
0 & 0 & \ddots \\
\end{bmatrix}.$$

This impedance matrix representation is in respect of the distributed–lumped–distributed–lumped system topology with $\bar{g}_{1,11}(s), \bar{g}_{1,22}, \bar{g}_{1,11}$, etc. being the diagonal elements of the termination matrix. However, there is no restriction on assembling alternative matrix descriptions for distributed–distributed–distributed realisations, for example, when modelling a series of purely distributed parameter pipeline elements of varying dimensions.
As Equation (18) shows, $\Omega(s)$ is a skew, symmetric tridiagonal matrix enabling simple recursive procedures to be employed in the computation of the determinant, as shown in Barnett (1992).

The theoretical basis outlined here is sufficient for the analysis procedures required. With these methods the dynamic models for complex series–parallel pipeline systems can be constructed in a simple, systematic and scientific manner. Moreover, the admittance functions, following the inversion of Equation (18), are easily realisable in terms of rational and irrational Laplace functions which can be used for the analysis or simulation purposes for the complete hybrid distributed–lumped system model of Equation (18), as illustrated in the following application study.

### 7. Pipeline and compressor model

In this application a model representing a single, long pipeline and a compressor will be considered. The arrangement for analysis purposes is shown in Figure 1.

From the theory of Section 6, the system Equation (18) is relevant, since there is only a single distributed parameter section and a single termination, lumped resistance element.

Hence

$$
\begin{bmatrix}
    P_1(s) \\
    P_2(s)
\end{bmatrix}
= \begin{bmatrix}
    \zeta(s)w(s) & -\zeta(s)(w^2(s) - 1)^{1/2} \\
    \zeta(s)(w^2(s) - 1)^{1/2} & -\zeta(s)w(s)
\end{bmatrix}
\begin{bmatrix}
    Q_1(s) \\
    Q_2(s)
\end{bmatrix},
$$

where in Equation (19), the termination relationship between the transformed pressure change $P_2(s)$ and the transformed airflow change $Q_2(s)$ is simply

$$P_2(s) = RQ_2(s).$$

Consequently, following inversion Equation (19) becomes

$$
\begin{bmatrix}
    \zeta(s)w(s) + R & -\zeta(s)(w^2(s) - 1)^{1/2} \\
    \zeta(s)(w^2(s) - 1)^{1/2} & -\zeta(s)w(s)
\end{bmatrix}
\begin{bmatrix}
    P_1(s) \\
    0
\end{bmatrix}
= \begin{bmatrix}
    Q_1(s) \\
    Q_2(s)
\end{bmatrix}.
$$

If the compressor unit is assumed to be relatively lumped, in comparison with the pipeline, comprising rotors and bearings, see, for example, Hodder (2008) then

$$P_1(s) = \frac{k_r U(s)}{(\tau_f(s) + 1)},$$

where $k_r$ is the gain and $\tau_f$ is the fan time constant and $U(s)$ is the applied voltage for the electrical drive. For this particular application, the parameters are as follows:

- $a$ pipeline radius 0.5 m
- $\theta$ absolute gas temperature 313 K
- $R$ exit resistance 1.0 N s/m$^3$
- $R_g$ characteristic gas constant 287 J/kg K
- $l$ pipeline length 1000, 5000 and 10,000 m
- $\tau_c$ compressor time constant 5.0 s
- $k_c$ compressor gain 10 kg s/m$^3$v
- $\gamma$ adiabatic index 1.31
- $g$ shunt conduction per metre 10$^{-4}$ m$^5$/N s
- $r$ series resistance per metre 0.6 $\times$ 10$^{-4}$ N s/m$^5$

From Equations (5) and (6), the gas capacitance and inductance effects per metre length of pipeline are

$$C = \frac{\pi a^2}{\gamma R_g \theta} = 0.643 \times 10^{-4} \text{ m}^2$$

and

$$L = \frac{1}{\pi a^2} = 1.2835 \text{ m}^2,$$

respectively.

From Equations (10) and (12), and (11), respectively, we have:

$$\tilde{\alpha} = \sqrt{\frac{\tau_f}{g}} = 0.7745,$$

$$\alpha = \sqrt{\frac{\tau_f}{g}} = 0.7745 \times 10^{-4}$$

and

$$\zeta(s) = \tilde{\alpha} \left( \frac{T_1s + 1}{\tau_1s + 1} \right) \left( \frac{T_2s + 1}{\tau_2s + 1} \right).$$

Also Equation (8) requires that for a single pipeline section where for illustration purposes, with $N = 2$ and where the double subscripts have been dropped for purposes of clarity:

$$\frac{(T_1s + 1)^2 (T_2s + 1)}{(\tau_1s + 1)^2 (\tau_2s + 1)} \approx \frac{(L/r)s + 1}{(C/g)s + 1},$$

where $L/r = 2.1392 \times 10^4$ s, $C/g = 0.0643$s, $T_1 = 3750$, $T_2 = 3.0$, $\tau_1 = 110.1$ and $\tau_2 = 0.175$ s.

The break frequencies selected for the Bode characteristics for $1/(L/r), 1/(C/g), 1/T_1, 1/T_2, 1/\tau_1$ and $1/\tau_2$ are $0.4674 \times 10^{-4}, 15.552, 6 \times 10^{-4}, 0.4, 0.05$ and $6.02$ rad/s, respectively.
where in delay form and and that in a series form, as shown in Figure 4, be computed from Equation (19), are fully defined and the outputs may now be evaluated for the values given that \( w(s) \) and \( \zeta(s) \), in Equation (19), are fully defined and the outputs may now be computed from

\[
Q_1(s) = \frac{(\zeta(s)w(s) + R)}{(\zeta(s)w(s)R + \zeta(s))}
\]

and

\[
Q_2(s) = \frac{(w^2(s) - 1)^{1/2}}{(w(s)R + \zeta(s))},
\]

where in delay form

\[
w(s) = (1 + e^{-2lax(s)})/(1 - e^{-2lax(s)})
\]

and

\[
\hat{w}(s) = (w^2(s) - 1)^{1/2} = \frac{2e^{-lax(s)}}{(1 - e^{-2lax(s)})}.
\]

Alternatively, commensurate with the pipeline system topology, Equations (22) and (23) may be written as

\[
Q_1(s) = \frac{(\zeta(s)w(s) + R)}{(\zeta(s)w(s)R + \zeta(s))}
\]

and

\[
Q_2(s) = \frac{\zeta(s)(w^2(s) - 1)^{1/2}}{(\zeta(s)w(s) + R)}.
\]

The block diagram for the representation given by Equations (24) and (25) is in a series form, as shown in Figure 4, and whereas Equations (22) and (23) provide the parallel, equivalent realisation, for this system model.

To simplify the simulation process it would be prudent to construct sub-system blocks for \( w(s) \) and \( \hat{w}(s) \). From Equation (17)

\[
\chi(s) = as + b,
\]

where the low-frequency approximation is

\[
a = \frac{C}{g} + (T_1 + T_2) - (\tau_1 + \tau_2) \quad \text{and} \quad b = 1
\]

so that

\[
w(s) = \frac{(1 + e^{-2lax(s)})}{(1 - e^{-2lax(s)})}.
\]

Following a step input change on the \( w(s) \) sub-system, it is easy to show that

\[
\frac{w(s)}{s} = \frac{1}{s} \left( 1 + 2 \sum_{n=1}^{\infty} e^{-2nla(s+b)} \right)
\]

so that the output following any arbitrary finite input change would be stable since

\[
b > 0.
\]

From the geometry of the approximation given by Equation (8), given in Figure 1, evidently

\[
(T_1 + T_2) > (\tau_1 + \tau_2)
\]

so that

\[
a > 0,
\]

resulting in the finite time delay \( e^{-2lax} \).
Also, \( \dot{w}(s) \) in delay form is
\[
\dot{w}(s) = \frac{2e^{-la(s+a)}}{1 - e^{-2la(s+a)}} \tag{28}
\]
then following a unit step change on this sub-system
\[
\frac{\dot{w}(s)}{s} = \sum_{n=1}^{\infty} e^{-nlal(s+a)}
\]
and this produces a stable output response since again \( b > 0 \) and \( a > 0 \) gives a finite time delay \( e^{-nlal} \) and attenuation \( e^{-nlalb} \), respectively.

In this case, when
\[
r = 0.6 \times 10^{-4} \text{ Nsec/m}^5
\]
then following division of the low-frequency approximation from Equation (26) is
\[
\chi(s) = 1 + 1648.9s \quad \text{and} \quad \zeta(s) = \tilde{\alpha} \left( \frac{166.6s + 1}{20s + 1} \right) \left( \frac{2.5s + 1}{0.166s + 1} \right).
\]

As the pipeline length varies, the characteristic impedance \( \zeta(s) \), the exit resistance \( R \) and the compressor model remain invariant. Consequently, only \( w(s) \) and \( \dot{w}(s) \) need to be adjusted in the simulation model to obtain the gas flow characteristics for any length of pipeline with the same diameter and per unit length resistance values. In this regard, substituting for \( \chi(s) \) in the equations for \( w(s) \) and \( \dot{w}(s) \) are given by Equations (27) and (28).

Hence for the 1000 m pipeline
\[
w(s) = \frac{1 + 0.8555e^{-77.0627s}}{1 - 0.8555e^{-77.0627s}},
\]
\[
\dot{w}(s) = \frac{1.8508e^{-38.5813s}}{1 - 0.8555e^{-77.0627s}},
\]

for the 5000 m pipeline
\[
w(s) = \frac{1 + 0.4609e^{-385.3s}}{1 - 0.4609e^{-385.3s}},
\]
\[
\dot{w}(s) = \frac{1.3578e^{-192.5794s}}{1 - 0.4609e^{-385.3s}},
\]

and for the 10,000 m pipeline
\[
w(s) = \frac{1 + 0.2125e^{-770.62s}}{1 - 0.2125e^{-770.62s}},
\]
\[
\dot{w}(s) = \frac{0.9218e^{-385.15s}}{1 - 0.2125e^{-770.62s}}.
\]

The distributed–lumped parameter block representation for the series configuration including the compressor unit can be given by
\[
\frac{P_1(s)}{v(s)} = \frac{k_c}{\tau_c s + 1} \quad \text{is shown in Figure 4.}
\]

Following unit step changes on the compressor motor voltage input, the changes in the volume flow at \( Q_1(t) \) and \( Q_2(t) \) are shown in Figure 5, in dotted and bold lines, respectively. This figure is initially for a 1000 m long, 1.0 m diameter, distributed parameter pipeline model, with \( r = 0.6 \times 10^{-4} \text{ Nsec/m}^5 \) and \( g = 10^{-4} \text{ m}^5/\text{N s} \), with all the
remaining parameters given earlier. Upon increasing the pipeline length to 5000 m and then 10,000 m the responses, following a 1% change in the compressor motor voltage, are also shown in Figure 5. Once again dotted and bold traces for the inlet and exit volume flow transients, respectively, are employed.

8. Gas flow transportation and energy requirement

The steady-state energy required to transport the input gas flow to the pipeline exit is proportional to \( Q_1^2(0) - Q_2^2(0) \). This energy difference between the input and output gas flow rate increases with pipeline roughness and discontinuities in the form of connections, bends and branches and with the gas, volume flow delivery rate. These aspects are also instrumental in increasing turbulence, noise, uneven gas flow and, hence, pipeline erosion.

These damaging prospects are exacerbated in that a larger capacity compressor and electrical drives are then required to maintain constant gas flows dissipating thereby increasing electrical drive and energy consumption. Consequently, frictional losses are minimised by the use of smooth bore pipelines which may be lined.

To further reduce the pipeline energy losses, of interest here is the relationship between the pipeline characteristic impedance and the exit resistance \( R \). With appropriate “matching” between \( R \) and \( \zeta(0) \) the possibility of minimising the input–output energy dissipation difference remains a tempting proposition.

Since the direct application of minimisation analysis would result in negative and/or infinite exit resistances, the problem requires the employment of constrained minimisation (Barnett, 1992). To achieve this, the use of Hamiltonian function and Lagrange multiplier for the pipeline model is proposed.

9. Constrained steady-state optimisation

From Equation (3) the squared, steady-state difference in the volume gas flow is given by

\[
\frac{(\zeta(0)w(0) + R)^2 - \zeta^2(0)\dot{w}^2(0)}{(\zeta(0) + w(0)R)^2\zeta^2(0)} = Q_1^2(0) - Q_2^2(0) = J,
\]

where in Equation (29)

\[
w(0) = \frac{e^{2ax(0)}}{e^{2ax(0)} - 1} \quad \text{and} \quad \dot{w}(0) = \frac{2e^{2ax(0)}}{(e^{2ax(0)} - 1)},
\]

and the performance index \( J > 0 \), since \( \zeta(0), R, w(0) \) and \( \dot{w}(0) > 0 \) and \( w(0) > \dot{w}(0) \).

Minimising the squared, steady-state volume gas flow difference would also minimise the squared, steady-state input and exit gas velocity difference which is proportional to the energy dissipation in the delivery process. Minimising this difference also minimises the continuous, input–exit volume gas stream expansion or compression effects. Consequently, these changes would be instrumental in minimising erosion, vibration, noise and turbulence within the pipeline system.

Essentially, the principal energy dissipation effects in this horizontal, constant-diameter gas pipeline transportation system arise from the pipeline series and shunt frictional resistance and conductance, \( r \) and \( g \), respectively, and from the exit resistance \( R \). Since \( r \) and \( g \) arise from the pipeline characteristics, the constraint adopted here relates...
to the pipeline roughness parameter $\zeta(0)$ and exit resistance $R$, via the simple constraint equation

$$R = k\zeta(0).$$  \hfill (30)

Consequently, if the squared volume flow difference between the pipeline input and exit is minimised with respect to $\zeta(0)$, then a suitable value for $k$ and hence $R$ could be selected to achieve this constrained minimisation.

From Equations (29) and (30) the Hamiltonian for this system can be formed, so that in accordance with (Hodder, 2008) the supremum value for $\zeta(0)$ and $R$ could be determined from

$$\frac{\partial H}{\partial \zeta(0)} = 0 \quad \hfill (31)$$

and

$$\frac{\partial H}{\partial R} = 0, \quad \hfill (32)$$

where in Equations (31) and (32), the Hamiltonian function $H$ is represented by

$$H = \frac{(\zeta(0)w(0) + R)^2 - \zeta^2(0)\dot{w}^2(0)}{(\zeta(0) + w(0)R)^2\zeta^2(0)} + (R - k\zeta(0))\lambda,$$

where $\lambda$ is the Lagrange multiplier associated with the constraint, given by Equation (30). Upon evaluating Equations (31) and (32)

$$\frac{\partial H}{\partial \zeta(0)} = \frac{2[(\zeta(0)w(0) + R)w(0) - \zeta(0)\dot{w}(0)]}{(\zeta(0) + w(0)R)\zeta^2(0)} - \frac{2[(\zeta(0)w(0) + R)^2 - \zeta^2(0)\dot{w}^2(0)]}{(\zeta(0) + w(0)R)^2\zeta^2(0)} - \frac{2[(\zeta(0)w(0) + R)^2w(0) - \zeta(0)\dot{w}^2(0)]}{(\zeta(0) + w(0)R)^3\zeta^2(0)} - \lambda k = 0, \quad \hfill (33)$$

$$\frac{\partial H}{\partial R} = \frac{2[(\zeta(0)w(0) + R)]}{(\zeta(0) + w(0)R)^2\zeta^2(0)} - \frac{2w(0)[(\zeta(0)w(0) + R)^2 - \zeta^2(0)\dot{w}^2(0)]}{(\zeta(0) + w(0)R)^3\zeta^2(0)} + \lambda = 0. \quad \hfill (34)$$

In Equations (33) and (34) if

$$x = \frac{2[(\zeta(0)w(0) + R)]}{(\zeta(0) + w(0)R)^2\zeta^2(0)}$$

and

$$y = \frac{2[(\zeta(0)w(0) + R)^2 - \zeta^2(0)\dot{w}^2(0)]}{(\zeta(0) + w(0)R)^3\zeta^2(0)}$$

then Equations (33) and (34) are simply

$$xw(0) - y - y(\frac{\zeta(0) + w(0)R}{\zeta(0)}) = k\lambda, \quad \hfill (35)$$

and

$$x - yw(0) = -\lambda. \quad \hfill (36)$$

Hence, from Equations (35) and (36) following the elimination of $\lambda$

$$x = \frac{[(\zeta(0)w(0) + R)(\zeta(0) + w(0)R)]}{(\zeta(0) + w(0)R)^2 - \zeta^2(0)\dot{w}^2(0)}.$$

Since, $R = k\zeta(0)$ Equation (37) becomes

$$\frac{[(\zeta^2(0)w(0) + k)(1 + w(0)k)]}{(w(0) + k)^2\zeta^2(0) - \zeta^2(0)\dot{w}^2(0)} \Rightarrow \frac{2\zeta(0)w(0)\dot{w}(0)k + w(0)\dot{w}(0)k}{\zeta(0)w(0) + k}. \hfill (38)$$

Finally Equation (38) yields the simple optimum, least volume flow difference relationships of

$$k = \sqrt{2}\dot{w}(0) - w(0). \hfill (39)$$

From Equation (39), the optimum value of $R$ is given by

$$R = (\sqrt{2}\dot{w}(0) - w(0))\zeta(0) \hfill (40)$$

guaranteeing $R > 0$.

10. Steady-state volume flow difference

If from Equations (22) and (23), the final values of $Q_1(0)$ and $Q_2(0)$, for a step change on $P_1(s)$, $(P_1(s) = 1/s, s \to 0)$, are given by

$$Q_1(0) = \frac{(\zeta(0)w(0) + R)}{\zeta(0)(\zeta(0) + w(0)R)} \hfill (41)$$

and

$$Q_2(0) = \frac{\dot{w}(0)}{(\zeta(0) + w(0)R)} \hfill (42)$$

then the steady-state volume flow difference constrained minimum for $(Q_1^2(0) - Q_2^2(0))$ would occur with

$$R = (\sqrt{2}\dot{w}(0) - w(0))\zeta(0)$$

and from Equations (41) and (42)

$$Q_1(0) = \frac{\sqrt{2}\dot{w}(0)}{[1 - w^2(0)] + \sqrt{2}w(0)\dot{w}(0)} \zeta(0) \hfill (43)$$

and

$$Q_2(0) = \frac{\dot{w}(0)}{[1 - w^2(0)] + \sqrt{2}w(0)\dot{w}(0)} \zeta(0) \hfill (44)$$

Consequently

$$Q_1(0) - Q_2(0) = \frac{(\sqrt{2} - 1)\dot{w}(0)}{[1 - w^2(0)] + \sqrt{2}w(0)\dot{w}(0)} \zeta(0). \hfill (45)$$

Then, from Equations (43) and (44)

$$Q_2(0) = \frac{1}{\sqrt{2}} Q_1(0). \hfill (46)$$

Hence, with this output resistance, $R = (\sqrt{2}\dot{w}(0) - w(0))\zeta(0)$, the steady-state volume output flow, following
input changes on \( P_1(s) \), always increases to 0.7071 of the input volume flow, irrespective of the pipeline roughness characteristics, \( r \) and \( g \). This is an important result.

From Equations (43) and (44) it is also evident that

\[
Q_1^2(0) - Q_2^2(0) = \frac{\dot{w}(0)}{\xi^2(0)[(1 - w^2(0)) + \sqrt{2}w(0)\dot{w}(0)]}
\]

so that \( Q_1^2(0) = 2Q_2^2(0) \).

Hence, the efficiency of the delivery with \( Q_2(0) = \frac{1}{\sqrt{2}} Q_1(0) \), and \( R = (\sqrt{2}\dot{w}(0) - w(0))\zeta(0) \), with the constraint that \( R = k\zeta(0), k > 0 \), rises to 50%.

Higher delivery efficiencies than these are achievable with increasing values of \( R \). However, lower steady-state rates of volume flow, for \( Q_2(0) \), would be experienced with \( Q_2(0) \) falling to zero, as \( R \) is increased to large values.

11. Application study

For the 1000 m long, 1.0 m diameter pipeline considered earlier, determine the optimum value of the pipeline exit resistance \( R \) which would deliver a volume flow of \( Q_2(0) = 0.7071Q_1(0) \), minimising the input–output volume flow difference, with the constraint \( R = k\zeta(0), k > 0 \).

The pipeline series and shunt resistance and conductance values here, as stated earlier, are \( r = 0.6 \times 10^{-4} \) Ns/m² and \( g = 10^{-4} \) m²/Ns, respectively. Consequently

\[
\tilde{a} = \zeta(0) = \sqrt{\frac{r}{g}} = 0.7745
\]

and

\[
\alpha = \sqrt{rg} = 0.7745 \times 10^{-4},
\]

\[
w(0) = \frac{(1 + e^{-2\alpha})}{(1 - e^{-2\alpha})} = 12.9374
\]

and

\[
\dot{w}(0) = \frac{2e^{-\alpha}}{1 - e^{-2\alpha}} = 12.8987.
\]

Hence, from Equation (39) the optimum output resistance is

\[
R = (\sqrt{2}\dot{w}(0) - w(0))\zeta(0) = 4.0817
\]

and from the constraint, given by Equation (30)

\[
k = \frac{R}{\zeta(0)} = 5.2701 \quad \text{and} \quad \tan^{-1} k \cong 80^\circ.
\]

This result can be validated graphically by plotting the performance index \( J \) against \( \zeta(0) \). The least value of the \( J \) contours which contact the constraint line, angle 80°, gives the minimum input–output volume flow which in this case is \( J = 0.0147 \). Figure 6 shows the constant \( J \) contours which occupy the first quadrant of the \( R - \zeta \) plane, \( (R, \zeta) > 0 \), with the \( J = \infty \) contour mapping into the origin. The 80° constraint line is tangential to \( J = 0.0147 \), as predicted and this is the least value of \( J \) commensurate with the constraint requirement, implied by Equation (30).

12. Energy consumption efficiency

The block diagram representation for this system is shown in Figure 4. The transient response for the system following a 1% change on the compressor motor voltage is as shown earlier, for the 1000 m pipeline with \( R = 1 \), with \( Q_1(t) \) and \( Q_2(t) \) increasing to their maximum steady-state values in approximately 6000 s.

It is very easy to obtain the proportional steady-state energy dissipation curves from this simulation, for a range

![Figure 6. Constant performance index J contours against characteristic impedance ζ(0) and exit resistance R.](image-url)
of exit resistances $R$. Figure 7 shows the steady-state performance $Q_1(t), Q_2(t)$ and the steady-state delivery curves for this application.

The steady-state energy consumption efficiency is proportional to $[(Q_1^2(0) - Q_2^2(0)) / Q_1^2(0)]$. This equation indicates that either a large volume gas flow, $Q_2(0)$, can be delivered with low energy efficiency or small volume gas flow can be delivered with high energy efficiency. From Section 10, the squared flow difference $(Q_1^2(0) - Q_2^2(0))$ with the constraint $R = k\zeta(0)$ ensures that this can always be achieved with an efficiency of 50% whilst guaranteeing a high delivery load of $Q_2(0) = 0.7071Q_1(0)$.

This delivery load, volume flow, is independent of the pipeline roughness.

The compact volume flow response functions derived invite the optimisation analysis of Sections 8–11. This advancement considered the constrained minimum energy dissipation problem via impedance matching. By selecting the constraint $R = k\zeta(0)$, the analysis restricts the identification of the output valve exit resistance $R$ to practical realisable values.

To do this, the Hamiltonian function for the system and a Lagrange multiplier were employed. This led to the sample elegant solution relating the exit resistance to the pipeline characteristics with the final result showing that

$$Q_2(0) = 0.7071Q_1(0)$$

with the inclusion of the optimum exit resistance of

$$R = (\sqrt{2}\hat{w}(0) - w(0))\zeta(0).$$

Importantly, the absolute pipeline roughness values would not be required to evaluate this resistance.

This exit resistance for the 1000 m pipeline under consideration would give the smoothest exit flow with the least energy dissipation for this steady-state exit volume flow rate.

13. Conclusion

In this contribution, the theory for modelling the gas flow in long pipelines was presented. It was shown that accurate, unambiguous, simple models could be easily constructed for pipeline–compressor configurations with the model-simulation block diagram requiring no more than four basic sub-system models, which would include the compressor model and for the pipeline functions of $w(s), (w^2(s) - 1)^{1/2}$ and $\zeta(s)$ for the complete system representation.

The distributed parameter theory involved was also uncomplicated with the incorporation of the continuous series and shunt resistances and the gas stream equivalent, capacitance and inductance, energy storage effects. Consequently, the perturbed transient volume flow responses following any input variation are readily available from the theoretical development and simulation process established herein.

The compact volume flow response functions derived invite the optimisation analysis of Sections 8–11. This advancement considered the constrained minimum energy dissipation problem via impedance matching. By selecting the constraint $R = k\zeta(0)$, the analysis restricts the identification of the output valve exit resistance $R$ to practical realisable values.

To do this, the Hamiltonian function for the system and a Lagrange multiplier were employed. This led to the sample elegant solution relating the exit resistance to the pipeline characteristics with the final result showing that

$$Q_2(0) = 0.7071Q_1(0)$$

with the inclusion of the optimum exit resistance of

$$R = (\sqrt{2}\hat{w}(0) - w(0))\zeta(0).$$

Importantly, the absolute pipeline roughness values would not be required to evaluate this resistance.

This exit resistance for the 1000 m pipeline under consideration would give the smoothest exit flow with the least energy dissipation for this steady-state exit volume flow rate.

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Bartlett, H., & Whalley, R. (1995). Gas flow in pipes and tunnels. *Proceedings of the Institution of Mechanical Engineers, Part I, 209* (6), 41–52.
Bartlett, H., & Whalley, R. (1998). Analogue solution to the modelling and simulation of distributed-lumped parameter systems. *Proceedings of the Institution of Mechanical Engineers, Part I, 212* (12), 99–114.
Appendix 1. Derivation of the input–output matrix for a distributed parameter element

Acoustic approximations for first-order, small-amplitude Navier–Stokes equations with zero bulk viscosity and neglecting radiant heat transfer were derived in Iberall (1960) and Nichols (1961).

If in addition to these constraints axial heat transfer is negligible and the pressure over any cross-section is assumed to be uniform, then in a continuous medium for small-amplitude disturbances with rigid, constant temperature walls the longitudinal dynamics for long pipelines may be considered via the elemental, infinitesimal, per unit length model shown in Figure A1. These constraints are adopted and employed in Brown (1962), Bartlett and Whalley (1999), and Bartlett and Whalley (1995). This enables analysis procedures based on partial differential calculus, differential equations, and Laplace transforms to be applied.

Taking Laplace transformations with respect to time with zero initial conditions allows Equations (A7) and (A8) to be written as

\[
\frac{d^2Q}{dx^2} = (Cs + g) \frac{dP}{dx} \quad \text{and} \quad \frac{d^2P}{dx^2} = -(Ls + r) \frac{dQ}{dx}.
\]

Hence

\[
\frac{d^2Q}{dx^2} - (Ls + r)(Cs + g)Q = 0 \quad (A7)
\]

and

\[
\frac{d^2P}{dx^2} = -(Ls + r) \frac{dQ}{dx} = (Ls + r)(Cs + g)P. \quad (A8)
\]

Consequently, as Equations (A7) and (A8) show, the homogeneous output and input equations are of the same form and have identical, general solutions. If now a propagation function

\[
\begin{align*}
&\frac{d^2Q}{dx^2} - (Ls + r)(Cs + g)Q = 0 \quad \text{and} \quad \frac{d^2P}{dx^2} = -(Ls + r)Q + (Ls + r)(Cs + g)P.
\end{align*}
\]

By dividing by \(dx\) and taking the limit as \(dx \to 0\) Equations (A1) and (A2) become

\[
\frac{\partial p}{\partial x}(t,x) = -L\frac{\partial q}{\partial t}(t,x) - rq(t,x) \quad (A3)
\]

and

\[
\frac{\partial q}{\partial x}(t,x) = -C\frac{\partial p}{\partial t}(t,x) - gp(t,x). \quad (A4)
\]

where for the element shown in Figure A1.

\[
p(t,x + dx) - p(t,x) = -\left( L\frac{\partial q}{\partial t}(t,x) + rq(t,x) \right) \, dx. \quad (A1)
\]

Also

\[
q(t,x + dx) - q(t,x) = -\left( C\frac{\partial p}{\partial t}(t,x) + gp(t,x) \right) \, dx. \quad (A2)
\]
is defined as
\[ \Gamma(s) = [(Ls + r)(C_s + g)]^{1/2} \]
then the solutions required are
\[ P(s, x) = A \cosh \Gamma(s)x + B \sinh \Gamma(s)x, \quad (A9) \]
\[ Q(s, x) = \tilde{C} \sinh \Gamma(s)x + D \cosh \Gamma(s)x \quad (A10) \]
then when \( x = 0 \)
\[ A = p(s, 0), \]
\[ D = q(s, 0). \]
Differentiating Equations (A9) and (A10) with respect to \( x \) and equating to (A5) and (A6) gives
\[-(Ls + r)Q(s, x) = A\Gamma(s) \sinh \Gamma(s)x + B\Gamma(s) \cosh \Gamma(s)x, \quad (A11)\]
\[-(Cs + g)P(s, x) = \tilde{C}\Gamma(s) \cosh \Gamma(s)x + D\Gamma(s) \sinh \Gamma(s)x. \quad (A12)\]
Again by setting \( x = 0 \) from Equation (A11)
\[ B = -(Ls + r)_{\Gamma(s)} Q(s, 0) = -\sqrt{\frac{(Ls + r)}{(Cs + g)}} Q(s, 0) \quad (A13)\]
and from Equation (A12)
\[ \tilde{C} = -(Cs + g)_{\Gamma(s)} P(s, 0) = -\sqrt{\frac{(Cs + g)}{(Ls + r)}} P(s, 0) \quad (A14)\]
or, from Equations (A13) and (A14):
\[ B = -\zeta(s) Q(s, 0) \quad \text{and} \quad \tilde{C} = -\zeta^{-1}(s) P(s, 0), \]
where \( \zeta(s) = \sqrt{\frac{(Ls + r)}{(Cs + g)}} \) is the characteristic impedance.

With this notation Equations (A9) and (A10) become
\[ P(s, x) = \cosh \Gamma(s) x P(s, 0) - \zeta(s) \sinh \Gamma(s) x Q(s, 0), \]
\[ Q(s, x) = -\zeta^{-1}(s) \sinh \Gamma(s) x P(s, 0) + \cosh \Gamma(s) x Q(s, 0). \]
Consequently, the output at \( l \) is given by
\[ \begin{bmatrix} P(s, l) \\ Q(s, l) \end{bmatrix} = \begin{bmatrix} \cosh \Gamma(s) l & -\zeta(s) \sinh \Gamma(s) l \\ -\zeta^{-1}(s) \sinh \Gamma(s) l & \cosh \Gamma(s) l \end{bmatrix} \begin{bmatrix} P(s, 0) \\ Q(s, 0) \end{bmatrix}. \quad (A15)\]

Equation (A15) can be arranged in impedance form with:
\[ \text{etnh} \Gamma(s) l = \frac{(e^{2\Gamma(s) l} + 1)}{(e^{2\Gamma(s) l} - 1)} = w(s) \]
and \( \text{csch} \Gamma(s) l = (w^2(s) - 1)^{1/2} \).

With this notation Equation (A15) becomes for the \( j \)th distributed section of pipeline
\[ \begin{bmatrix} P_j(s, l_j) \\ Q_{j+1}(s, l_{j+1}) \end{bmatrix} = \begin{bmatrix} \zeta(s) w_j(s) & -\zeta(s)(w_j^2(s) - 1)^{1/2} \\ \zeta(s)(w_j^2(s) - 1)^{1/2} & -\zeta(s) w_j(s) \end{bmatrix} \]
\[ \times \begin{bmatrix} P_j(s, l_j) \\ Q_{j+1}(s, l_{j+1}) \end{bmatrix} \quad (A16) \]
which completes the analysis.