On nonequilibrium states
in QFT model with boundary interaction

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Abstract

We prove that certain nonequilibrium expectation values in the boundary sine-Gordon model coincide with associated equilibrium-state expectation values in the systems which differ from the boundary sine-Gordon in that certain extra boundary degrees of freedom ($q$-oscillators) are added. Applications of this result to actual calculation of nonequilibrium characteristics of the boundary sine-Gordon model are also discussed.

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1. Introduction

In this paper we study the so called boundary sine-Gordon model with zero bulk mass (referred below as BSG). Its action is

\[ A_{BSG} = \frac{1}{4\pi g} \int_{-\infty}^{\infty} dt \int_{-\infty}^{0} dx \left( \Phi_t^2 - \Phi_x^2 \right) + \frac{\kappa}{g} \int_{-\infty}^{\infty} dt \cos (\Phi_B + Vt) \]  

(1.1)

Here \( \Phi = \Phi(x,t) \) is a scalar field defined on a half line \( -\infty < x \leq 0 \), \( \Phi_B \equiv \Phi(0,t) \) is its boundary value, \( \Phi_t = \partial_t \Phi \), \( \Phi_x = \partial_x \Phi \) and \( g \), \( \kappa \) and \( V \) are parameters. The way \( g \) enters (1.1) allows one to interpret it as a quantum parameter, because it always appears in the combination \( gh \); in what follows we set \( \hbar = 1 \). In the quantum theory the parameter \( \kappa \) carries the dimension \( \text{[mass]}^{-1} \). The boundary interaction in (1.1) contains explicit time dependence through the term \( Vt \). Although this time dependence can be eliminated by a simple change of the field variables \( \Phi \to \Phi - Vt \) (which instead brings in the term linear in \( \Phi_t \)), the above form (1.1) is more convenient for our analysis. Below we almost always assume that \( V > 0 \), more generally we will consider complex \( V \) with \( \Re e V > 0 \). For future references let us write down the Hamiltonian corresponding to (1.1),

\[ H_{BSG} = \frac{1}{4\pi g} \int_{-\infty}^{0} dx \left( \Pi^2 + \Phi_x^2 \right) - \frac{\kappa}{g} \cos (\Phi_B + Vt) \]  

(1.2)

Here \( \Phi(x), \Pi(x) \) are field operators obeying canonical commutation relations

\[ [\Pi(x), \Phi(x')] = -2\pi ig \delta(x - x') \]  

(1.3)

and again \( \Phi_B \equiv \Phi(0) \).

At a nonzero \( V \) and a temperature \( T \) the system (1.2) develops a stationary nonequilibrium state which can be thought of as the result of an infinite time evolution of the equilibrium state of the corresponding “free” system, with the interaction term (the last term in (1.2)) adiabatically switched on. We will denote \( \langle A \rangle_{BSG} \) the expectation value of an observable \( A \) over this nonequilibrium stationary state.

Besides being an interesting model of Quantum Field Theory on its own, the theory finds important applications in other branches of physics. As explained in [1], the model (1.1) describes a quantum particle with the coordinate \( X = \Phi_B + Vt \) and the potential

\[ 1^{\text{We assume here that the normalization of the boundary field } \cos(\Phi_B + Vt) \text{ is fixed by the condition } \langle \cos(\Phi_B(t) + Vt) \cos(\Phi_B(t') + Vt') \rangle_{BSG} \to 2^{-1} \left( i(t - t') + 0 \right)^{-2g} \text{ as } 0 < t - t' \to 0.} \]
energy $-2\pi \kappa \cos(X)$, interacting with dissipative environment, the bulk part of the field $\Phi$ playing the role of the latter (see also \[3,4\]). In this case $V$ is interpreted as an external driving force. The model (1.1) is also believed to describe an electric current through a point contact in the quantum Hall system \[5\]; in this case $V$ is proportional to the voltage drop across the contact. In all cases the quantities of interest are the correlation functions of the boundary fields

$$V_+(t) = e^{i\Phi_B(t)} e^{iVt}, \quad V_-(t) = e^{-i\Phi_B(t)} e^{-iVt}.$$ (1.4)

Here $\Phi_B(t) = S^{-1}(t) \Phi_B S(t)$, and $S(t)$ is the time evolution operator corresponding to the Hamiltonian (1.2). We will be particularly interested in the expectation values $\langle V_\pm \rangle_{BSG}$.

The main goal of this paper is to show that the expectation values $\langle V_\pm \rangle_{BSG}$ (and indeed some more general correlation functions of (1.1)) coincide with equilibrium expectation values of certain operators in a system which differs from (1.2) in that it involves certain additional boundary degree of freedom. Namely, let us define a Hamiltonian

$$H_+ = \frac{1}{4\pi g} \int_{-\infty}^{0} dx \left( \Pi_x^2 + \Phi_x^2 \right) - Vh - \frac{\kappa}{2g} \left( a_- e^{i\Phi_B} + a_+ e^{-i\Phi_B} \right),$$ (1.5)

where $\Phi(x)$, $\Pi(x)$ are again the Bose field operators obeying the same commutation relations (1.3) as in (1.2), and the operators $h$, $a_+$, $a_-$ commute with $\Phi(x)$, $\Pi(x)$ and form among themselves the so called "$q$-oscillator algebra", i.e.

$$[h, a_\pm] = \pm a_\pm; \quad q a_+ a_- - q^{-1} a_- a_+ = q - q^{-1}$$ (1.6)

with

$$q = e^{i\pi g}.$$ (1.7)

Let $\rho_+$ be some representation of (1.6) such that the spectrum of $\rho_+(h)$ is real and bounded from above. The Hamiltonian (1.3) acts in the space

$$\mathcal{H}_+ = \mathcal{F} \otimes \rho_+,$$ (1.8)

where $\mathcal{F}$ is the space of states of the Bose field representing the commutation relations (1.3), and for $V > 0$ this Hamiltonian is bounded from below. Then, for $V > 0$ the system (1.3) has a thermal equilibrium state described by the standard density matrix

$$\mathbf{P}_+ = Z_+^{-1}(\kappa, V) e^{-R\mathbf{H}_+},$$ (1.9)
where $R$ is proportional to the inverse temperature,

$$R = g/T,$$

and $Z_+(\kappa, V) = \text{Tr}_{\mathcal{H}_+}[e^{-R\mathcal{H}_+}]$ is the corresponding partition function. Here and below we treat $g$ as a constant, and therefore we do not include it in the list of arguments of $Z_+$. Let us denote $\langle A \rangle_+$ the expectation value of an observable $A$ over this thermal equilibrium state. We will show that

$$\langle V_+ \rangle_{BSG} = \langle W_+ \rangle_+ = \langle W_- \rangle_+, \quad (1.10)$$

where

$$W_+ = a_+ e^{i\Phi_B}; \quad W_- = a_- e^{-i\Phi_B}. \quad (1.11)$$

While the second equality in (1.10) is a simple property of the equilibrium state (1.9) (see Sect.3), the relation between the nonequilibrium and equilibrium expectation values in (1.10) looks rather unusual and suggestive. In fact, the Eq.(1.10) is a particular case of a more general relation. Namely, the correlation functions in (1.2) involving any number of the Heisenberg operators $V_+(t)$ (but not $V_-(t)$) coincide with the corresponding equilibrium-state correlation functions of the operators $W_+(t)$ in (1.5).

Similarly, one can relate the correlation functions which involve the operators $V_-(t)$ (but not $V_+(t)$) of (1.2) to certain equilibrium-state correlation functions. Let $\rho_-$ be any representation of (1.6) such that the spectrum of $\rho_-(\mathcal{H})$ is bounded from below. Consider the Hamiltonian

$$\mathcal{H}_- = \frac{1}{4\pi g} \int_{-\infty}^{0} dx \left( \Pi^2 + \Phi_x^2 \right) + V \mathcal{H} - \frac{\kappa}{2g} \left( a_+ e^{i\Phi_B} + a_- e^{-i\Phi_B} \right) \quad (1.12)$$

acting in $\mathcal{F} \otimes \rho_-$, and associated equilibrium state density matrix

$$\mathcal{P}_- = Z_-^{-1}(\kappa, V) e^{-R\mathcal{H}_-}, \quad (1.13)$$

which is well defined for $\Re V > 0$. Then

$$\langle V_- \rangle_{BSG} = \langle \tilde{W}_- \rangle_- = \langle \tilde{W}_+ \rangle_-, \quad (1.14)$$

where $\langle A \rangle_-$ stands for the expectation value of $A$ over the equilibrium state (1.13), and

$$\tilde{W}_+ = a_+ e^{i\Phi_B}; \quad \tilde{W}_- = a_- e^{-i\Phi_B}. \quad (1.15)$$
Although the Hamiltonian (1.12) is related to (1.5) through a simple change \( \Phi \rightarrow -\Phi, V \rightarrow -V \), we treat it as a distinct one because we always imply \( \Re V > 0 \).

Besides being interesting by themselves, the relations (1.10) and (1.14) provide efficient tool for actual computation of the nonequilibrium expectation values \( \langle V_{\pm} \rangle_{BSG} \). Using (1.10), (1.14) one can easily show that

\[
\langle V_{\pm} \rangle_{BSG} = T \partial_\kappa \log Z_{\pm}(\kappa, V),
\]

(1.16)

where \( Z_{\pm} \) and \( Z_{-} \) are the partition functions in (1.9) and (1.13), respectively. On the other hand, the above partition functions are related in a simple way to the vacuum eigenvalues \( Q_{\pm}(\lambda, p) \) of the CFT analogs of Baxter’s operators \( Q_{\pm}(\lambda) \) studied in [6,7] (see Sect.4 for some details),

\[
Z_{\pm}(\kappa, V)/Z_{\pm}(0, V) = \lambda^{\mp 2\pi i p/\beta^2} Q_{\pm}(\lambda, p).
\]

(1.17)

where \( Z_{\pm}(0, V) \) are the partition functions for the “free” systems defined by (1.5) and (1.12) with \( \kappa = 0 \). The parameters \( \lambda, p \) and \( \beta \) used in [6,7] are defined as

\[
\lambda = i\kappa \frac{\sin(\pi g)}{g} \left( \frac{g}{2\pi T} \right)^{1-g}, \quad p = -iV \frac{g}{4\pi T}, \quad \beta^2 = g.
\]

(1.18)

Many exact results for these operators exist [6,7]. These include high and low temperature expansions, but most importantly, the vacuum eigenvalues \( Q_{\pm}(\lambda, p) \) are shown in [6] to satisfy closed integral equations – the Destri-de Vega equations – which allow for their evaluation to an arbitrary degree of accuracy. In view of (1.10), (1.14) and (1.16) all these results directly apply to the non-equilibrium expectation values of the operators (1.4).

In applications, the quantity of a particular interest is

\[
J = V + \langle \Phi_x(0,t) \rangle_{BSG},
\]

(1.19)

which is interpreted as the current through the point contact in the quantum Hall systems, or as the drift velocity in dissipative quantum mechanics. It was the main subject of interest

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2 The voltage \( V \) and current \( J \) (1.19) differs in normalization from the real voltage \( V^{(phys)} \) and current \( J^{(phys)} \) in the Hall system

\[
V^{(phys)} = e^{-1} V, \quad J^{(phys)} = \frac{e}{2\pi \hbar} g J,
\]

where \( e \) and \( \hbar \) are the electron charge and Plank’s constant. Also, \( g \) coincides with the fractional filling of the Luttinger state in a Hall bar and the temperature \( T \) is measured in energy units.
in many recent papers \([5,8,9,10,11,6]\). In particular, for \(g = \frac{1}{3}\) in (1.1), Bethe-Ansatz computation of this current was given in \([9]\), under the assumption that the Boltzmann equation for the backscattering electrons in the Hall contact holds exactly. Further, it was conjectured in \([11]\) that for all values of this parameter

\[
J = V + i\pi T \mu \partial_{\mu} \log \left( \frac{Z_{2p}(\mu)}{Z_{-2p}(\mu)} \right),
\]  

(1.20)

with

\[
\mu = \kappa \frac{\pi}{g} \left( \frac{g}{2\pi T} \right)^{1-g}, \quad p = -iV \frac{g}{4\pi T}.
\]  

(1.21)

The “partition function” \(Z_{2p}\) was defined in Ref.\([11]\) as a power series

\[
Z_{2p}(\mu) = 1 + \sum_{n=1}^{\infty} \mu^{2n} I_{2n}(p)
\]  

(1.22)

with the coefficients \(I_{2n}\) given by multiple infinite sums over all ordered sets \(\mathbf{m} = (m_1, m_2, \ldots, m_n)\) of non-negative integers \(m_1 \geq m_2 \geq \cdots \geq m_n \geq 0\)

\[
I_{2n}(p) = \frac{1}{\Gamma^{2n}(g)} \sum_{\mathbf{m}} \prod_{i=1}^{n} \frac{\Gamma(m_i + g(n - i + 1)) \Gamma(2p + m_i + g(n - i + 1))}{\Gamma(m_i + g(n - i) + 1) \Gamma(2p + m_i + g(n - i) + 1)}.
\]  

(1.23)

Similarly looking conjecture

\[
J = V + i\pi T \kappa \partial_{\kappa} \log \left( \frac{Z_{+}(\kappa, V)}{Z_{-}(\kappa, V)} \right),
\]  

(1.24)

where \(Z_{\pm}(\kappa, V)\) are just the partition functions in \([1.9]\), \([1.13]\) and \([1.17]\), was proposed independently in \([8]\). The (plausible) equivalence between \((1.20)\) and \((1.24)\) is not yet established. However, it is easy to see that \((1.24)\) is a simple consequence of \((1.16)\) and the equation

\[
\Phi_x(0, t) = i\pi \kappa \left( \mathbf{V}_+(t) - \mathbf{V}_(t) \right),
\]  

(1.25)

which is the boundary condition corresponding to \((1.1)\), i.e. our result \((1.10), (1.14)\) actually proves the conjecture \((1.24)\).
2. Expectation values in boundary sine-Gordon model

If $V > 0$ the system (1.1) evolves towards a stationary state which is characterized by nonzero expectation values $\langle \Phi_t \rangle_{BSG}$, $\langle \Phi_x \rangle_{BSG}$. This is not a thermodynamic equilibrium state and no simple explicit expression for its density matrix is known. In this section we describe the definition of this state in terms of real-time perturbation theory and discuss some properties of corresponding expectation values.

Let us split the total Hamiltonian (1.2) into the free and interaction parts
\[
H_0 = \frac{1}{4\pi g} \int_{-\infty}^{0} dx \left( \Pi^2 + \Phi_x^2 \right), \quad H_1 = -\frac{\kappa}{g} \cos(\Phi_B + Vt).
\]  

(2.1)

Using corresponding interaction representation one can write the density matrix $P(t)$ as
\[
P(t) = e^{-iH_0t} S(t, -\infty) P_0 S(-\infty, t) e^{iH_0t},
\]

(2.2)

where
\[
S(t, t_0) = \mathcal{T} \exp \left\{ -i \int_{t_0}^{t} d\tau \ H_1^{(int)}(\tau) \right\} = 1 + \sum_{k=1}^{\infty} (-i)^k \int_{t_0}^{t} D_k(\{\tau\}) \ H_1^{(int)}(\tau_1) H_1^{(int)}(\tau_2) \cdots H_1^{(int)}(\tau_n)
\]

(2.3)

and
\[
H_1^{(int)}(t) = e^{iH_0t} H_1 e^{-iH_0t}.
\]

(2.4)

In (2.3) and below the shorthand notation for the multiple ordered integrals
\[
\int_{t_0}^{t} D_k(\{\tau\}) = \int_{t_0}^{t} d\tau_1 \int_{t_0}^{\tau_1} d\tau_2 \cdots \int_{t_0}^{\tau_{k-1}} d\tau_k
\]

(2.5)

is used. In writing (2.2) we have assumed that the interaction has been adiabatically switched on in the infinite past ($t = -\infty$) when the system was in thermodynamic equilibrium at the temperature $T$, i.e.
\[
P_0 = Z_0^{-1} e^{-RH_0}, \quad R = g/T.
\]

(2.6)

Then, the expectation value of an arbitrary operator $A$ can be written as
\[
\text{Tr}_F \left[ P(t) A \right] = \text{Tr}_F \left[ P_0 S(-\infty, t) A^{(int)}(t) S(t, -\infty) \right] = \langle S(-\infty, t) A^{(int)}(t) S(t, -\infty) \rangle_0,
\]

(2.7)
where \( \langle \ldots \rangle_0 \) denotes the expectation value over the equilibrium state (2.6) of the free
system, and the superscript “(int)” means that this operator is taken in the interaction
representation, i.e. \( \mathbf{A}^{\text{(int)}}(t) = e^{i\mathbf{H}_0 t} \mathbf{A} e^{-i\mathbf{H}_0 t} \). Equivalently, one may write the above
expectation value as
\[
\langle \mathbf{A} \rangle_{\text{BSG}} = \text{Tr}_{\mathcal{F}} \left[ \mathbf{P} \mathbf{A}(t) \right],
\]
where \( \mathbf{P} \) stands for density matrix of the system at \( t = 0 \), i.e. \( \mathbf{P} = \mathbf{P}(0) \), and \( \mathbf{A}(t) \) is the
full Heisenberg operator
\[
\mathbf{A}(t) = \mathbf{S}(0, t) \mathbf{A}^{\text{(int)}}(t) \mathbf{S}(t, 0).
\]

All above formulae are very well known (see e.g. [12]); we included them here to fix
the notations. In this paper we are interested in the expectation values
\( \langle \mathbf{V}_\pm \rangle_{\text{BSG}} \) of the Heisenberg operators
\[
\mathbf{V}_\pm(t) = \mathbf{S}(0, t) \mathbf{V}_\pm^{\text{(int)}}(t) \mathbf{S}(t, 0).
\]
where
\[
\mathbf{V}_\pm^{\text{(int)}}(t) = \exp\{ \pm i \Phi_B^{\text{(int)}}(t) \pm iVt \},
\]
and \( \Phi_B^{\text{(int)}}(t) \) is the boundary field \( \Phi_B \) in the interaction representation. It is convenient
to introduce also auxiliary operators
\[
\mathbf{V}_\pm(t, t_0) = \mathbf{S}(t_0, t) \mathbf{V}_\pm^{\text{(int)}}(t) \mathbf{S}(t, t_0),
\]
where \( t_0 \) is a parameter. For \( t_0 = 0 \) (2.12) coincide with the Heisenberg operators (2.10),
and according to (2.7) the expectation values of (2.10) can be expressed through (2.12) as follows
\[
\langle \mathbf{V}_\sigma(t) \rangle_{\text{BSG}} = \lim_{t_0 \to -\infty} \langle \mathbf{V}_\sigma(t, t_0) \rangle_0,
\]
where \( \sigma = \pm 1 \). Our nearest goal is to prove the following useful representation for the operators (2.12)
\[
\mathbf{V}_\sigma(t, t_0) = \mathbf{V}_\sigma^{\text{(int)}}(t) \times
\left\{ 1 + \sum_{k=1}^{\infty} \sum_{\sigma_1, \ldots, \sigma_k = \pm 1} C_k(\sigma | \sigma_1, \ldots, \sigma_k) \int_{t_0}^{t} D_k(\{t\}) \mathbf{V}_\sigma^{\text{(int)}}(t_1) \cdots \mathbf{V}_\sigma^{\text{(int)}}(t_k) \right\},
\]
where the sum is taken over all arrangements of the “charges” \( \sigma_1, \ldots, \sigma_k = \pm 1 \), and the
coefficients \( C_k(\sigma | \sigma_1, \ldots, \sigma_k) \) have the following explicit form
\[
C_k(\sigma | \sigma_1, \ldots, \sigma_k) = \left( -\frac{k}{g} \right)^k \sigma_1 \cdots \sigma_k \eta_k \sum_{j=1}^{k} \sigma_j \eta_j \prod_{j=1}^{k} \sin(\pi g \eta_j),
\]
\[ \eta_j = \sigma + \sum_{s=1}^{j-1} \sigma_s . \] (2.16)

The fact that this operator can be written as the sum (2.14) with some coefficients \( C_k(\sigma|\sigma_1, \ldots, \sigma_k) \) is obvious from its definition (2.12). Indeed, the series expansions (2.3) for the evolution operators \( S \) in (2.12) allow one to represent (2.12) as a series of multiple integrals of certain products of \( V^{(\text{int})}_\pm \). Then, using the commutation relations

\[ V^{(\text{int})}_{\sigma_1}(t_1) V^{(\text{int})}_{\sigma_2}(t_2) = q^{2\sigma_1 \sigma_2} V^{(\text{int})}_{\sigma_2}(t_2) V^{(\text{int})}_{\sigma_1}(t_1), \quad t_1 > t_2, \quad q = e^{i\pi g} , \] (2.17)

where \( \sigma_1, \sigma_2 = \pm 1 \), one can always rewrite each of these integrals as a combination of the time-ordered integrals as in (2.14). The easiest way to obtain the coefficients \( C_k \) in (2.14) is to notice that in view of (2.3) the operators (2.12) satisfy the differential equation

\[ i \frac{\partial}{\partial t_0} V_\sigma(t, t_0) = [H_1^{(\text{int})}(t_0), V_\sigma(t, t_0)] \] (2.18)

with the initial condition

\[ V_\sigma(t, t_0) \big|_{t_0=t} = V^{(\text{int})}_\sigma(t) . \] (2.19)

It is easy to check that the expansion (2.14) satisfies (2.18) provided the coefficients \( C_k \) solve the recurrence relations

\[ C_k(\sigma|\sigma_1, \ldots, \sigma_k) = \frac{i \kappa}{2g} (1 - q^{-2\sigma_k \eta_k}) C_{k-1}(\sigma|\sigma_1, \ldots, \sigma_{k-1}) , \] (2.20)

where the notation (2.16) is used. With the initial condition \( C_0 = 1 \), which follows from (2.19), these relations lead to (2.15).

A simple consequence of (2.14) is the infinite series representation for the expectation value (2.13),

\[ \langle V_\sigma \rangle_{\text{BSG}} = -\frac{2\pi \sigma T}{\kappa \sin(\pi g)} \sum_{n=1}^{\infty} \lambda^{2n} \sum_{\sigma_1, \ldots, \sigma_{2n-1}} \left( \prod_{j=1}^{2n-1} \frac{\sin(\pi g \eta_j)}{\sin(\pi g)} \right) J(\sigma, \sigma_1, \ldots, \sigma_{2n-1}|p) , \] (2.21)

where the sum is taken over all arrangements of \( \sigma_1, \ldots, \sigma_{2n-1} = \pm 1 \) with zero total charge \( \sigma + \sum_{s=1}^{2n-1} \sigma_s = 0 \), and

\[ J_n(\sigma_0, \sigma_1, \ldots, \sigma_{2n-1}|p) = \int_{-\infty}^{0} D_{2n-1}(\{\tau\}) e^{-2p \sum_{j=1}^{2n-1} \sigma_j \tau_j} \prod_{0 \leq j < l \leq 2n-1} \left( 2 \sinh \left( \frac{\tau_j - \tau_l}{2} \right) \right)^{2g \sigma_j \sigma_l} \] (2.22)
with \( \tau_0 \equiv 0 \). Here the parameters \( \lambda \) and \( p \) are defined by (1.18). This representation (more precisely, the corresponding representation for the current (1.19)) was previously obtained in Ref.[10] by a combinatorial method. The above consideration provides an alternative derivation of this result. It also promotes the formula (2.21) to the level of the operator relation (2.14) which can be used in evaluating the multitime correlation functions.

Let us emphasis here an important feature of the expression (2.14) and (2.21). The sums over \( \sigma_1, \sigma_2, \ldots, \sigma_{2n-1} \) there exclude configurations where any of the “cumulative charges” \( \eta_j, j = 1, \ldots, 2n-1 \) defined by (2.16) vanish, because one of the factors \( \sin(\pi g \eta_j) \) in (2.21) (as well as in (2.15)) then turn to zero. As a result all integrals (2.22) appearing in (2.21) converge at the lower limit, and that is why the limit \( t_0 \to -\infty \) in (2.13) poses no difficulty. And of course RHS of (2.13), (2.21) does not actually depend on \( t \) as a consequence of the invariance of the expectation values upon an overall time shift.

3. The system with \( q \)-oscillator

Now we turn to the systems (1.5) and (1.12) which involve the boundary \( q \)-oscillators, with the aim to derive our main relations (1.10) and (1.14). These two systems are quite similar and in what follows only (1.5) is studied explicitly, and only (1.10) is actually derived; the relation (1.14) can be obtained by obvious modifications of the arguments presented below.

Like in (2.1), let us split (1.5) as \( \mathbf{H}_+ = \mathbf{H}_0 + \mathbf{H}_1 \),

\[
\mathbf{H}_0 = \mathbf{H}_0 - \mathbf{V} \mathbf{h}, \quad \mathbf{H}_1 = -\frac{\kappa}{2g} \left( a_- e^{i\Phi_+} + a_+ e^{-i\Phi_+} \right),
\]

where \( \mathbf{H}_0 \) is given by (2.1) and the operators \( \mathbf{h}, a_\pm \) are defined in (1.6). In the Matsubara representation the equilibrium density matrix (1.9) reads

\[
\mathbf{P}_+ = Z_+^{-1}(\kappa, V) e^{-R\mathbf{H}_0} \mathbf{S}(-iR, 0),
\]

where the Matsubara operator \( \mathbf{S}(-iR, 0) \) is the “imaginary time” version of the time-evolution operator in the corresponding interaction representation,

\[
\mathbf{S}(t_2, t_1) = \mathcal{T}\exp\left\{-i \int_{t_1}^{t_2} dt \, \mathbf{H}_1^{(int)}(t) \right\}, \quad \mathbf{H}_1^{(int)}(t) = e^{i\mathbf{H}_0 t} \mathbf{H}_1 e^{-i\mathbf{H}_0 t}.
\]

Note that in view of the commutation relations (1.3)

\[
\mathbf{H}_1^{(int)}(t) = -\frac{\kappa}{2\beta^2} \left( a_- \mathbf{V}_+^{(int)}(t) + a_+ \mathbf{V}_-^{(int)}(t) \right),
\]

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where $V_{\pm}^{(int)}(t)$ are exactly the operators (2.11).

We are interested in the expectation values of the operators (1.11), which can be written as

$$\langle W_{\pm}\rangle_+ = Z_+^{-1} \text{Tr}_{H_+} \left[ e^{-R \bar{H}_0} \bar{S}(t) \right] .$$

(3.5)

The fact that $\langle W_+\rangle_+ = \langle W_-\rangle_+$ (as stated in (1.10)) is a particular manifestation of the detailed balance principle for the equilibrium system (1.5); on formal level it is easily established if one notices that the commutation relations are invariant with respect to the transformation $a_+ \rightarrow \Lambda a_+,$ $a_- \rightarrow \Lambda^{-1} a_-$, where $\Lambda$ is a constant. Therefore

$$\langle W_{\pm}\rangle_+ = T \partial_\kappa \log Z_+(\kappa, V) .$$

(3.6)

It is convenient for our purposes to introduce again an auxiliary time $t_0$ and rewrite (3.3) as

$$\langle W_+\rangle_+ = Z_+^{-1} \text{Tr}_{H_+} \left[ e^{-R \bar{H}_0} \bar{S}(t_0 - iR, t_0) W_+(0, t_0) \right] ,$$

(3.7)

where

$$W_+(t, t_0) = \bar{S}(t_0, t) e^{i\bar{H}_0 t} W_+ e^{-i\bar{H}_0 t} \bar{S}(t, t_0) .$$

(3.8)

Our proof of (1.10) will be based on remarkably simple representation for the operator $W_+(t, t_0)$ similar to (2.14). Using perturbative expansions for the operators $\bar{S}$ in (3.3) with the explicit form (3.4) of the interaction Hamiltonian, and then applying the commutation relations (2.17) to achieve the full time ordering for the operators $V_{\pm}^{(int)}$, one can bring (3.3) to the form analogous to (2.14), i.e.

$$W_+(t, t_0) = a_- V_+^{(int)}(t) +$$

$$V_+^{(int)}(t) \sum_{k=1}^{\infty} \sum_{\sigma_1, \ldots, \sigma_k = \pm 1} \tilde{C}_k(\sigma_1, \ldots, \sigma_k) \int_{t_0}^{t} D_k(\{t\}) V^{(int)}_{\sigma_1}(t_1) \cdots V^{(int)}_{\sigma_k}(t_k) ,$$

(3.9)

where this time the coefficients $\tilde{C}_k(\sigma_1, \ldots, \sigma_k)$ are not $c$-numbers, but some operators acting in $\rho_+$. In general, from this analysis one would expect these coefficients to be complicated polynomials in the operators $a_+$ and $a_-$. It turns out, however, that these coefficients contain only $a_-$ but not $a_+$, and moreover

$$\tilde{C}_k(\sigma_1, \ldots, \sigma_k) = (a_-)^{\eta_k+1} C_k(\sigma_1, \ldots, \sigma_k) ,$$

(3.10)

where $\eta_k$ is given by (2.16) with $\sigma = +1$, and $C_k(\sigma | \sigma_1, \ldots, \sigma_k)$ are exactly the same numerical coefficients (2.13) as in (2.14). The rest of the notations in (3.9) is the same as
in (2.14). The derivation of (3.10) is very similar to our proof of (2.15) in the previous section. By its definition, the operator (3.8) must satisfy the differential equation
\[ i \frac{\partial}{\partial t_0} W_+(t, t_0) = \left[ \bar{H}_1^{(int)}(t_0), W_+(t, t_0) \right], \]
with the initial condition
\[ W_+(t, t_0)\big|_{t_0=t} = a_- V_+^{(int)}(t). \]
Substituting (3.9) with yet unknown operators \( \bar{C}_k \), one obtains the recurrence relations
\[ \bar{C}_k(\sigma_1, \ldots, \sigma_k) = \frac{i}{2g} \left( \bar{C}_{k-1}(\sigma_1, \ldots, \sigma_{k-1}) a_{-\sigma_k} - q^{-2\sigma_k \eta_k} a_{-\sigma_k} \bar{C}_{k-1}(\sigma_1, \ldots, \sigma_{k-1}) \right) \]
with the initial condition \( \bar{C}_0 = a_- \). It is straightforward to check that (3.10) satisfy (3.13). Note that because of (3.10) and (2.15), the representation (3.9) enjoys the same remarkable property as (2.14), namely the series (3.9) contains only the terms where no of the “cumulative charges” \( \eta_j \), \( j = 1, \ldots, 2k - 1 \) vanish. We want to stress also that this simple form of the coefficients \( \bar{C}_k \) is a very special property of the operator \( W_+(t, t_0) \). Had we taken, say, \( W_- \) instead of \( W_+ \) in (3.8), no such simplification in (3.9) would occur, i.e. the resulting coefficients \( \bar{C}_k \) would indeed come out to be complicated polynomials of both \( a_- \) and \( a_+ \), and in particular the above “no zero cumulative charges” property would not hold.

Our proof of (1.10) is based on the representation (3.9). Consider the expression (3.7) for \( \langle W_+ \rangle \) and take the limit \( t_0 \to -\infty \). Due to the above “no zero cumulative charges” property of (3.9) the trace in (3.7) in this limit factorizes as
\[ \text{Tr}_{\mathcal{H}_+} \left[ e^{-R\bar{H}_0} \bar{S}(t_0 - iR, t_0)^{-1} W_+(t, t_0) \right] \rightarrow \]
\[ \frac{\text{Tr}_{\mathcal{H}_+} \left[ e^{-R\bar{H}_0} W_+(t, t_0) \right]}{\text{Tr}_{\mathcal{H}_+} \left[ e^{-R\bar{H}_0} \right]} \text{Tr}_{\mathcal{H}_+} \left[ e^{-R\bar{H}_0} \bar{S}(t_0 - iR, t_0) \right], \]
and we obtain
\[ \langle W_+ \rangle_+ = \lim_{t_0 \to -\infty} \frac{\text{Tr}_{\mathcal{H}_+} \left[ e^{-R\bar{H}_0} W_+(0, t_0) \right]}{\text{Tr}_{\mathcal{H}_+} \left[ e^{-R\bar{H}_0} \right]} . \]
The trace in the denominator is the partition function of (1.5) with \( \kappa = 0 \). Now, using here the representation (3.9) with (3.10) for the operator \( W_+(0, t_0) \), and taking advantage of the fact that
\[ \text{Tr}_{\mathcal{H}_+} \left[ e^{-R\bar{H}_0} (a_-)^n \right] = \delta_{n,0} \text{Tr}_{\mathcal{H}_+} \left[ e^{-R\bar{H}_0} \right] , \]

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one arrives exactly at the series in (2.21) for the expectation value (3.15). This proves the relation (1.10). Let us also note that using the same arguments based on the properties of (2.14), (3.9) and on (3.16), one can establish more general relation between the multitime correlation functions of the systems (1.2) and (1.5), namely

\[ \langle V_+(t_1) \cdots V_+(t_n) \rangle_{BSG} = \langle W_+(t_1) \cdots W_+(t_n) \rangle_+ . \] (3.17)

Here \( W_+(t) = W_+(t, 0) \) are the Heisenberg operators associated with \( W_+ \).

4. Discussion

The relations (1.10) and (1.14) between nonequilibrium expectation values in (1.1) and equilibrium expectation values in (1.3) and (1.12) is the main result of this paper. We want to stress here that these relations are actually derived (in Sect. 2 and 3) from first principles, and in particular this result proves the validity of the expression (1.24) for the current (1.19) conjectured earlier in [6]. General significance of this result depends on whether it can be extended to wider class of systems with nonequilibrium dynamics. At this time we feel reluctant to enter any speculations on this matter leaving that to when better understanding of the situation is achieved. Instead, in this section we discuss how these relations can be applied towards actual computation of the nonequilibrium expectations \( \langle V_\pm \rangle_{BSG} \) and of the current (1.19).

According to (1.10) and (1.24), computation of these expectation values reduces to finding the partition functions \( Z_\pm(\kappa, V) \) of the equilibrium systems (1.3) and (1.12). By itself this does not help much in solving the problem. Of course one can use Matsubara theory to write down a perturbative series in \( \kappa^2 \) for \( Z_\pm(\kappa, V) \). However, \( n \)-th order coefficients of this series are expressed in terms of \( 2n - 1 \) fold integrals (written down explicitly in [3]) and it is not clear how to evaluate or simplify these integrals in general. In this respect this approach does not have any significant advantages over the series (2.21); in fact, (2.21) looks more compact. The same remark seems to apply to the approach based on the representation (1.20) conjectured in [11]. Even if (1.20) is proven, evaluation of all terms of the series (1.22) requires calculation of the sums (1.23) of ever growing degree of complexity. Each of these representations — (1.24), (2.21) and (1.20) — provides more or less efficient way to calculate few first terms of the \( \kappa^2 \) expansion of (1.19) but neither seem to give a full solution to the problem. In fact, if one is concerned with the actual evaluation of the coefficients of these \( \kappa^2 \) expansions, even more efficient technique can be obtained.
by combining certain analyticity properties and functional relations for $Q$ operators (see below); we describe this technique in the Appendix.

What makes the relations (1.1), (1.14) a useful computational tool is remarkable relation (1.17) between the partition functions $Z_{\pm}(\kappa, V)$ and certain eigenvalues of so-called $Q_{\pm}$-operators. These operators were introduced in [6] as CFT analogue of Baxter’s $Q$-matrix [13]. The operators $Q_{\pm}(\lambda)$ of [6] act in the Fock space of free Bose field with spatial coordinate compactified on a circle. The Fock vacua $|p\rangle$ are parameterized by the value $p$ of the zero-mode momentum. These vacua are eigenstates of the operators $Q_{\pm}(\lambda)$, and we denote $Q_{\pm}(\lambda, p)$ the corresponding eigenvalues, i.e.

$$Q_{\pm}(\lambda, p) = \langle p | Q_{\pm}(\lambda) | p \rangle . \quad (4.1)$$

We will also use the notations

$$A_{\pm}(\lambda, p) = \lambda^{\mp 2\pi ip/\beta^2} Q_{\pm}(\lambda, p) . \quad (4.2)$$

The identity (1.17), i.e.

$$Z_{\pm}(\kappa, V)/Z_{\pm}(0, V) = A_{\pm}(\lambda, p) , \quad (4.3)$$

(where the parameters $\kappa, V, g$ and $\lambda, p, \beta^2$ are related by (1.18)) can be verified by direct comparison of the definition of $Z_{\pm}(\kappa, V)$ in Sect.1 and the definition of the $Q$-operators in [6]. At the same time the above operators $Q_{\pm}(\lambda)$ exhibit some very useful properties. First, the operators $\lambda^{\mp 2\pi ip/\beta^2} Q_{\pm}(\lambda)$, and hence the functions (1.1), are entire functions of $\lambda^2$, with known asymptotics at $\lambda^2 \to -\infty$ along the real axis [3]. Second, these operators obey certain functional equations (one is utilized in Appendix, see (A.5)), notably the famous Baxter’s $T-Q$ relation. These analytic characteristics and functional equations lead to closed nonlinear integral equation for the functions (1.3) (explicitly written down in [3]), known as Destri-de Vega equation [14,15]. This equation has been used in [3] to analyze low-temperature expansions for the current (1.19) and to confirm notable duality relation for this current first suggested in [4] (some relevant results concerning this duality were obtained in [3,3,16]). Detailed study of the solutions to this equation lies beyond the scope of this work.

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3 In the form written down in [6] the Destri-de Vega equation applies only to the case of real $2p > -g$, i.e. pure imaginary $V$. For generic complex values of $p$ the equation must be modified, namely the $\lambda$ integral over the real axis must be replaced by contour integral with the contour encircling all zeroes of $A_{\pm}(\lambda, p)$ in the $\lambda$ plane, which become complex at complex $p$. 

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Appendix A.

In this section we describe a technique of calculating the perturbative expansions in $\kappa^2$ ("high temperature expansions") of the expectation values (1.10), (1.14) and (1.19). The technique is based on the functional relations for the $Q$-operators derived in [6,7]. Unlike the main text of the paper which was meant to be more or less self-contained, this Appendix relies heavily on the results of Refs. [6,7], so some familiarity with these works will be useful in reading it.

Let us remind some properties of the functions (4.2) which follow from the analysis in [6,7].

(i) The functions $A_\pm(\lambda, p)$ are entire functions of $\lambda^2$, and $A_\pm(0, p) = 1$. The power series

$$\log A_+(\lambda, p) = -\sum_{n=1}^{\infty} a_n(p) \lambda^{2n}$$

has finite radius of convergence which is determined by position of closest to origin zero of $A_+(\lambda, p)$. For real $2p > -g$ this zero is a real and positive. Let us mention here that according to (1.16), (2.21) and (4.3), the coefficient $a_n(p)$ can be expressed in terms of the integrals (2.22),

$$a_n(p) = \frac{\pi}{n \sin(\pi g)} \sum_{\sigma_1, \ldots, \sigma_{2n-1} = -1} \left( \prod_{j=1}^{2n-1} \frac{\sin(\pi g \eta_j)}{\sin(\pi g)} \right) J(+1, \sigma_1, \ldots, \sigma_{2n-1}|p),$$

(A.2)

The coefficients $a_n(p)$ here differ in normalization from $H_n^{(vac)}(p)$ used in [6], namely

$$a_n(p) = g^{-2n} \Gamma^{2n}(1 - g) H_n^{(vac)}(p).$$
with $\sigma_s = \pm 1$, $\eta_j = 1 + \sum_{s=1}^{j-1} \sigma_s$. Note that this relation holds as is only when $0 < g < \frac{1}{2}$ and $\Re(2p) > -g$, since otherwise the integrals (2.22) generally diverge; analytic continuation in these parameters is required outside this domain.

(ii) The coefficients $a_n(p)$ are meromorphic functions of $p$. Moreover, they are analytic in the half plane $\Re(2p) > -g$. The last statement follows directly from Eqs.(2.22) and (A.2).

(iii) The following asymptotics,
\[ a_n(p) \to \frac{\Gamma(n-g) \Gamma\left(1 - \frac{1}{2} + n(1-g)\right)}{2 \sqrt{\pi n!}} \Gamma^{2n}(1 - g) \ p^{1 - 2n + 2ng}, \quad \text{as } p \to \infty, \quad (A.3) \]
hold in the half-plane $\Re(2p) > -g$.

(iv) $A_-(\lambda, p)$ is related to $A_+(\lambda, p)$ as
\[ A_-(\lambda, p) = A_+(\lambda, -p) = \exp\left\{-\sum_{n=1}^{\infty} a_n(-p) \lambda^{2n}\right\}. \quad (A.4) \]

(v) The functions $A_\pm(\lambda, p)$ obey so called "quantum Wronskian" condition,
\[ e^{2\pi ip} A_+(q^{\frac{1}{2}}\lambda, p) A_-(q^{-\frac{1}{2}}\lambda, p) - e^{-2\pi ip} A_+(q^{-\frac{1}{2}}\lambda, p) A_-(q^{\frac{1}{2}}\lambda, p) = 2i \sin(2\pi p), \quad (A.5) \]
where $q = e^{i\pi g}$.

The Eqs.(A.4) and (A.5), supplemented with the analyticity (ii) and asymptotic conditions (A.3) constitute a Riemann-Hilbert problem which defines the functions $A_\pm(\lambda, p)$ completely. Indeed, substituting the expansions (A.1), (A.4) into the (A.5), one obtains relations of the form
\[ \sin(\pi ng + 2\pi p) \ a_n(p) - \sin(\pi ng - 2\pi p) \ a_n(-p) = R_n(p), \quad n = 1, 2, \ldots, \quad (A.6) \]
where the functions $R_n(p)$ are expressed through $a_k(p)$ with $k = 1, \ldots n - 1$ only. For example,
\[ R_1(p) = 0, \]
\[ R_2(p) = -\left(q a_1(p) + q^{-1} a_1(-p)\right)^2 e^{4\pi ip} \sin(2\pi p)/2. \quad (A.7) \]
Since $a_n(p)$ are analytic in the half plane $\Re(2p) > -g$, and $a_n(p) \to \text{const} \ p^{1 - 2n + 2ng}$ as $p \to \infty$ there, one can solve the Eq.(A.6) with respect to $a_n(p)$,
\[ a_1(p) = \frac{\pi \Gamma(1 - 2g)}{\sin(\pi g)} \frac{\Gamma(g + 2p)}{\Gamma(1 - g + 2p)}, \]
\[ a_n(p) = (-1)^n \frac{i}{\pi} \frac{\Gamma(2 - n + ng + 2p)}{\Gamma(n - 1 - ng + 2p)} \times \]
\[ \int_{-\infty}^{+\infty} dx \ \frac{R_n(ix)}{2\pi} \frac{\Gamma(n - 1 - ng + 2i x) \Gamma(n - 1 - ng - 2i x)}{x + ip}, \quad (A.8) \]
where \( n = 2, 3, \ldots \). In writing (A.8), we assume that \( 0 < g < \frac{1}{2} \) and \( \Re p > 0 \). As the functions \( R_n(p) \) are uniquely expressed (by the use of (A.7)) through “lower” \( a_k(p) \) with \( k = 1, 2, \ldots, n - 1 \), the Eq.(A.8) provides a recursion for evaluation of \( a_n(p) \). It allows one to represent \( a_n(p) \) in terms of \((n - 1)\)-fold integral. Let us present explicit formulae for \( a_2 \) and \( a_3 \) \((0 < g < \frac{1}{2} \) and \( \Re p > 0 \)),

\[
a_2(p) = 2^{1 - 4g} \frac{\Gamma^2(1 - g)}{\Gamma^2(\frac{1}{2} + g)} \frac{\Gamma(2g + 2p)}{\Gamma(1 - 2g + 2p)} \int_{-\infty}^{+\infty} \frac{dx}{2\pi} \frac{S_1(x)}{x + ip},
\]

\[
a_3(p) = 2^{2 - 6g} \frac{\sqrt{\pi} \Gamma^3(1 - g)}{\Gamma^3(\frac{1}{2} + g)} \frac{\Gamma(3g - 1 + 2p)}{\Gamma(2 - 3g + 2p)} \times \left\{ -\frac{\sin(4\pi g)}{\pi^2} \int_{-\infty}^{+\infty} \frac{dy}{2\pi} \int_{-\infty}^{+\infty} \frac{dx}{2\pi} \frac{S_2(x, y)}{(y + ip)(x - y - i0)} + \frac{1}{3} \int_{-\infty}^{+\infty} \frac{dx}{2\pi} S_3(x) \right\},
\]

where the functions \( S \) are

\[
S_1(x) = \sinh(2\pi x) \Gamma(1 - 2g + 2ix) \Gamma(1 - 2g - 2ix) \left( \Gamma(g + 2ix) \Gamma(g - 2ix) \right)^2,
\]

\[
S_2(x, y) = \sinh(2\pi y) \sinh(2\pi x) \Gamma(g + 2iy) \Gamma(g - 2iy) \Gamma(2g + 2iy) \Gamma(2g - 2iy) \times \Gamma(2 - 3g + 2iy) \Gamma(2 - 3g - 2iy) \Gamma(1 - 2g + 2ix) \Gamma(1 - 2g - 2ix) \times (\Gamma(g + 2ix) \Gamma(g - 2ix))^2,
\]

\[
S_3(x) = \sinh(2\pi x) \Gamma(2 - 3g + 2ix) \Gamma(2 - 3g - 2ix) \left( \Gamma(g + 2ix) \Gamma(g - 2ix) \right)^3 \times \frac{\sin(4i\pi x + 2\pi g) - 2\sin(2\pi g)}{\sin(2i\pi x + 2\pi g)}.
\]

Eqs.(A.8), (A.9) have obvious computational advantages over (A.2). First, the number of integrations is greatly reduced - while (A.2) requires evaluation of \( 2n - 1 \)-fold integral (2.22) to compute \( a_n(p) \), the recursion relation (A.8) leads to \( n - 1 \) fold integral for this quantity. Second, it is very convenient in numerical calculations because the integral in (A.8) converges very fast at infinity. It is important to note also that analytic continuation of the expressions (A.8), (A.9) outside the domain \( 0 < g < \frac{1}{2} \) and \( \Re p > 0 \) is rather straightforward — it is done by appropriate deformation of integration contours. For example, the analytical continuation of \( a_2(p) \) into the domain \( \frac{1}{2} < g < 1 \) and \( \Re p > 0 \) reads,

\[
a_2(p) = 2^{1 - 4g} \frac{\Gamma^2(1 - g)}{\Gamma^2(\frac{1}{2} + g)} \frac{\Gamma(2g + 2p)}{\Gamma(1 - 2g + 2p)} \left\{ \int_{-\infty}^{+\infty} \frac{dx}{2\pi} S_1(x) \right\} - \frac{\sin(2\pi g) \Gamma(3 - 4g) \Gamma^2(1 - g) \Gamma^2(3g - 1)}{(2p + 1 - 2g)(2p - 1 + 2g)},
\]

where the function \( S_1(x) \) is the same as in (A.10).
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