Decay $X(3872) \to \pi^0\pi^+\pi^-$ and $S$-wave $D^0\bar{D}^0 \to \pi^+\pi^-$ scattering length

N. N. Achasov and G. N. Shestakov

Laboratory of Theoretical Physics, S. L. Sobolev Institute for Mathematics, 630090 Novosibirsk, Russia

The isospin-breaking decay $X(3872) \to (D^*\bar{D} + \bar{D}^*D) \to \pi^0 D\bar{D} \to \pi^0\pi^+\pi^-$ is discussed. In its atmospheric there is a triangle logarithmic singularity, due to which the dominant contribution to $BR(X(3872) \to \pi^0\pi^+\pi^-)$ comes from the production of the $\pi^+\pi^-$ system in a narrow interval of the invariant mass $m_{\pi^+\pi^-}$ near the value of $2m_{\rho0} \approx 3.73$ GeV. The analysis shows that $BR(X(3872) \to \pi^0\pi^+\pi^-)$ can be expected at the level of $10^{-3} \ldots 10^{-4}$. This estimate includes, in particular, the assumption that the $S$-wave inelastic scattering length $|\alpha_{S,\pi^+\pi^-}| \approx 1/(2m_{\rho0}) \approx 0.25$ GeV$^{-1}$.

I. INTRODUCTION

The state $X(3872)$ (or $\chi_1(3872)$) was first observed in 2003 by the Belle Collaboration in the process $B \to K(X(3872) \to \pi^+\pi^- J/\psi)$ [2]. The $X(3872)$ is a narrow resonance ($\Gamma_X < 1.2$ MeV) [3], its mass consists practically with the $D^0\bar{D}^0$ threshold [1], it has the quantum numbers $I^G(J^{PC}) = 0^+(1^{++})$ [1, 4, 5], in addition to decay into $\pi^+\pi^- J/\psi$ [2, 6, 7], it also decays into $\omega J/\psi$ [8, 11], $D^0\bar{D}^0 + c.c.$ [12, 13], $\gamma J/\psi$ [14, 16], $\gamma\psi(2S)$ [14, 16], and $\pi^0\chi_1(1718)$. The nature of $X(3872)$ remains the subject of much discussion, see, for example, Refs. [14, 33, 33]. Of course, new experiments will allow making a more definite choice between different interpretations.

The search for $X(3872)$ in decay channels that do not contain charm particles or charmion states (i.e., in channels other than $D^0\bar{D}^0 + c.c.$, $D^0\bar{D}^0\pi^0$, $\pi^+\pi^- J/\psi$, $\omega J/\psi$, $\gamma J/\psi$, $\gamma\psi(2S)$, $\pi^+\pi^-\eta(1S)$, $\pi^+\pi^-\chi_1(3872)$, and $\pi^0\chi_1(1718)$) are of great interest [1, 23, 32]. For example, the $c\bar{c} = \chi_1(2P)$ scenario predicts a significant number of various two gluon decays $X(3872) \to (\text{gluon} + \text{gluon}) \to \text{light hadrons}$ [26, 30]. The situation here is qualitatively the same as for the decays $\chi_1(1P) \to (\text{gluon} + \text{gluon}) \to \text{light hadrons}$. On this way, only one channel has been explored so far [1]. Namely, the LHCb Collaboration undertook a search for the decay $X(3872) \to p\bar{p}$, which resulted in the following restriction [34]:

$$BR(B^+ \to X(3872)K^+) \times BR(X(3872) \to p\bar{p}) < 0.25 \times 10^{-2}.$$  \hspace{1cm} (1)

Hence, in view of $BR(B^+ \to J/\psi K^+) \times BR(J/\psi \to p\bar{p}) \approx 2.14 \times 10^{-6}$ [1] and $0.9 \times 10^{-4} < BR(B^+ \to X(3872)K^+) < 2.7 \times 10^{-4}$ [1, 34], it follows that

$$BR(X(3872) \to p\bar{p}) < 0.6 \times 10^{-4}. \hspace{1cm} (2)$$

It is also proposed to investigate the $X(3872)$ coupling to the $p\bar{p}$ channel in the reaction $p\bar{p} \to X(3872) \to \pi^+\pi^- J/\psi$ with the PANDA detector [35].

We propose to obtain an experimental limit on the probability of the decay $X(3872) \to \pi^0\pi^+\pi^-$ and, if lucky, to register this decay. According to our estimate, the branching ratio of the decay $X(3872) \to \pi^0\pi^+\pi^-$ can be expected at the level of $10^{-3} \ldots 10^{-4}$ due to the transition mechanism $X(3872) \to (D^*\bar{D} + \bar{D}^*D) \to \pi^0 D\bar{D} \to \pi^0\pi^+\pi^-$. In this case, the main contribution to $BR(X(3872) \to \pi^0\pi^+\pi^-)$ comes from the production of $\pi^+\pi^-$ pairs in a narrow interval of the invariant mass $m_{\pi^+\pi^-}$ near the value of $2m_{\rho0} \approx 3.73$ GeV.

As for the nature of $X(3872)$, our calculations implicitly imply for him the conventional $c\bar{c}$ nature.

II. ESTIMATE OF $BR(X(3872) \to \pi^0\pi^+\pi^-)$

The decay $X(3872) \to (D^*\bar{D} + \bar{D}^*D) \to \pi^0 D\bar{D}$ (see Fig. 1) is one of the main decay channels of the $X(3872)$ resonance [1]. Due to the final state interaction among $D^0$ and $\bar{D}^0$ mesons, i.e., due to the transition $D^0\bar{D}^0 \to \pi^+\pi^-$, the isospin breaking decay $X(3872) \to (D^0\bar{D}^0 + \bar{D}^0D^0) \to \pi^0 D\bar{D}$ is induced (see Fig. 2).

If the virtual invariant mass squared of the $X(3872)$ resonance $s_1$ falls in the range

$$2(m_{D^0}^2 + m_{D^0}^2) - m_{s_1}^2 = (3.87193 \text{ GeV}^2) > s_1 > (m_{D^{*+}} + m_{D^{*-}})^2 = (3.87168 \text{ GeV}^2), \hspace{1cm} (3)$$

then, in the range of the invariant mass squared of the $\pi^+\pi^-$ system $s_2 = m_{\pi^+\pi^-}^2$

$$\frac{m_{D^0}}{m_{D^{*+}}}(m_{D^{*+}}^2 + m_{D^{*-}}^2 - m_{s_2}^2) + 2m_{D^0}^2 = (3.7299 \text{ GeV}^2), \hspace{1cm} (4)$$

$$> s_2 > 4m_{D^0}^2 = (3.72966 \text{ GeV}^2),$$

[Diagram: $X(3872)$ decay to $\pi^0\pi^+\pi^-$]
the imaginary part of the amplitude of the diagram in Fig. 2 contains the triangle logarithmic singularity. Below, we see that this singularity leads to the resonancelike enhancement in the $\pi^+\pi^-$ mass spectrum at $\sqrt{s_2} = 2m_{\pi^+\pi^-} \approx 2m_{D^*p^0} \approx 3.73$ GeV, i.e., near the $D^0 D^0$ threshold.

The decay $X(3872) \to \pi^0\pi^+\pi^-$ can also be produced via the charged intermediate states, $X(3872) \to (D^{+}\bar{D}^- + D^{-}\bar{D}^+) \to \pi^0 D^0 D^- \to \pi^0\pi^+\pi^-$. From the isotopic symmetry for the coupling constants, it follows that the contributions of the diagrams in Figs. 2 and 3 exactly compensate each other and the isospin breaking decay $X(3872) \to \pi^0\pi^+\pi^-$ is absent, if $m_{D^{*+}} = m_{D^{0+}}$ and $m_{D^{*+}} = m_{D^{0+}}^\rho$. However, the $D^0 D^0$ and $D^0 D^- D^-$ thresholds in the variable $\sqrt{s_1}$ differ by 8.23 MeV ($m_{D^{0+}} + m_{\rho^0} = 3.87168$ GeV, $m_{D_{s+}} + m_{D^-} = 3.87991$ GeV) and the $D^0 D^0$ and $D^0 D^-$ thresholds in the variable $\sqrt{s_2}$ differ by 9.644 MeV ($2m_{D^0} = 3.72966$ GeV, $2m_{D_s} = 3.73930$ GeV). Therefore, in the region of the variables $\sqrt{s_1}$ and $\sqrt{s_2}$ that is significant for the decay $X(3872) \to \pi^0\pi^+\pi^-$ (i.e., for $\sqrt{s_1} \approx m_X \approx m_{D^{0+}} + m_{\rho^0}$, where $m_X$ is the nominal mass of the $X(3872)$ equal to 3.87169 GeV [3], and $\sqrt{s_2} \approx 2m_{\rho^0} \approx 3.73$ GeV), the contributions from the neutral (see Fig. 2) and charged (see Fig. 3) intermediate states weakly compensate each other and the contribution of the diagram in Fig. 2 dominates.

We write the differential probability for the decay of the virtual state $X(3872)$ to $\pi^0\pi^+\pi^-$ in the form

$$d^2B R(X \to \pi^0\pi^+\pi^-; s_1, s_2) = \frac{2\sqrt{s_1}}{\pi} \frac{d\Gamma(X \to \pi^0\pi^+\pi^-; s_1, s_2)}{d\sqrt{s_2}},$$

(5)

where $D_X(s_1)$ is the inverse propagator of the $X(3872)$ resonance [25, 27, 28] which takes into account the couplings of $X(3872)$ with the $D^* D + D^* D$ decay channels, and as well as with all non-$(D^* D + D^* D)$ decay channels; and $d\Gamma(X \to \pi^0\pi^+\pi^-; s_1, s_2)/d\sqrt{s_2}$ is the $X \to \pi^0\pi^+\pi^-$ differential decay width in the variable $\sqrt{s_2} = m_{\pi^+\pi^-}$ caused by the sum of the diagrams in Figs. 2 and 3.

The $X(3872)$ resonance propagator constructed in Refs. [25, 27, 28] has good analytical and unitary properties (as for the case of scalar mesons [41, 42]). The inverse propagator $D_X(s_1)$ has the form

$$D_X(s_1) = m_X^2 - s_1 + \sum_{ab} \left[ \text{Re} \Pi_X^{ab}(m_X^2) - \Pi_X^{ab}(s_1) \right] - im_X\Gamma_{\text{non}},$$

(6)

where $\Gamma_{\text{non}} = \Sigma_i \Gamma_i$ is the total width of the $X(3872)$ decay to all non-$(D^* D + D^* D)$ channels which in the narrow region of the $X(3872)$ peak ($\Gamma_\chi < 1.2$ MeV [43]) is approximated by a constant; $ab = D^0 D^0, D^0 D^0, D^+D^-, D^-D^+$. At $s_1 > (m_a + m_b)^2$,

$$\Pi_X^{ab}(s_1) = \frac{g_A^2}{16\pi} \left[ \frac{m_a (m_a^2 - s_1)}{m_b} \ln \frac{m_b}{m_a} + \rho_{ab}(s_1) \left( i - \frac{1}{\pi} \ln \frac{s_1 - m_a^2 + \sqrt{s_1 - m_a^2} - \sqrt{s_1 - m_a^2}}{s_1 - m_a^2 - \sqrt{s_1 - m_a^2} - \sqrt{s_1 - m_a^2}} \right) \right],$$

(7)

where $\rho_{ab}(s_1) = \sqrt{s_1 - m_a^2} - \sqrt{s_1 - m_a^2} \ln s_1 - m_a^2 / s_1, m_\chi = m_a \pm m_b, m_a > m_b,

$$\text{Im} \Pi_X^{ab}(s_1) = \sqrt{s_1} T_{X \to ab}(s_1) = \frac{g_A^2}{16\pi} \rho_{ab}(s_1),$$

(8)

and $g_A$ is the coupling constant of $X$ with the $D^0\bar{D}^0$ channel. At $m_a^2 < s_1 < m_b^2$,

$$\Pi_X^{ab}(s_1) = \frac{g_A^2}{16\pi} \left[ \frac{m_a (m_a^2 - s_1)}{m_b} \ln \frac{m_b}{m_a} - \rho_{ab}(s_1) \left( 1 - \frac{2}{\pi} \arctan \frac{\sqrt{m_a^2 - s_1}}{s_1 - m_a^2} - \sqrt{s_1 - m_a^2} \right) \right],$$

(9)

where $\rho_{ab}(s_1) = \sqrt{m_a^2 - s_1} - \sqrt{s_1 - m_a^2} / s_1$. If $s_1 \leq m_a^2$, then $\rho_{ab}(s_1) = \sqrt{m_a^2 - s_1} - \sqrt{s_1 - m_a^2} / s_1$. If $s_1 \geq m_b^2$, then $\rho_{ab}(s_1) = 0$.\]
and
\[ \Pi^b_X(s_1) = \frac{g^2 \Gamma_D}{16\pi} \left[ \frac{m_a^+ m_{ab}^-}{\pi s_1} - \frac{m_b^+ m_{ab}^-}{m_a^+} \right] + \rho_{ab}(s_1) \frac{1}{\pi} \ln \frac{\sqrt{m_{ab}^+} - s_1 + \sqrt{m_{ab}^-} - s_1}{\sqrt{m_{ab}^+} - s_1 - \sqrt{m_{ab}^-} - s_1} \right]. \]  

\[ \tag{10} \]

The sum of the probabilities of the X(3872) decay to all modes satisfies the unitarity conditions

\[ BR(X \to (D^* D^0 + c.c.)) + BR(X \to (D^* D^0 + c.c.) + \Sigma_i BR(X \to i) = 1. \]  

\[ \tag{11} \]

The coupling of the X(3872) with the D^*0 D^0 system was introduced in Refs. [23, 28] by means of the Lagrangian

\[ L_{X^{D^* D^0}}(x) = g_A X^{D^0 D^0} + \bar{D}^0 D^0 \]  

\[ \tag{12} \]

and the range of possible values of the coupling constant \( g_A^2 / (16\pi) \) was determined from the analysis of the experimental data [3, 6, 8, 9, 13, 15].

To describe the amplitudes of the D^* \to D\pi^0 decays, we use the expression

\[ V_{D^* D^0} = g_{D^* D^0}(\epsilon_{D^*}, p_{\pi^0} - p_D), \]  

\[ \tag{13} \]

where \( \epsilon_{D^*} \) is the polarization four-vector of the D^* meson, \( p_{\pi^0} \) and \( p_D \) are the four-momenta of \( \pi^0 \) and \( D \), respectively; \( g_{D^* D^0}(\epsilon_{D^*}, p_{\pi^0} - p_D) \).

The effective vertex of the X(3872) \to (D^* D^0 + D^0 D) \to \pi^0 D\bar{D} \to \pi^0 \pi^+\pi^- \) transition corresponding to the sum of the diagrams in Figs. 2 and 3, in which the \( \pi^+\pi^- \) system is produced in the S wave, can be written as

\[ V_{\pi^+\pi^-} = G_{\pi^+\pi^-}((s_1, s_2)(\epsilon_X, p_3 - p_2) = \frac{2}{3} \frac{\bar{g}}{16\pi} [F_0(s_1, s_2) - F_+(s_1, s_2)], \]  

\[ \tag{14} \]

where \( \epsilon_X \) is the polarization four-vector of the X(3872), the amplitudes \( F_0(s_1, s_2) \) and \( F_+(s_1, s_2) \) describe the contributions from the neutral and charged intermediate D^* D states, respectively, and

\[ \bar{g} = g_A g_{D^* D^0} g_{D^0\pi^+\pi^-}. \]  

\[ \tag{15} \]

We assume the S wave amplitudes of the processes \( D^0 D^0 \to \pi^+\pi^- \) and \( D^+ D^- \to \pi^+\pi^- \) (entering in the amplitudes of the diagrams in Figs. 2 and 3) to be equal and approximate them in the region of the D D thresholds by an \( s_2 \)-independent constant \( g_{D^0 D^0}\).\pi^+\pi^-.

Taking into account Eqs. [12, 13], and [14], the amplitude \( F_0(s_1, s_2) \) can be written in the form

\[ F_0(s_1, s_2) = \frac{i}{\pi^3} \epsilon_X \int \frac{d^4k}{(p_1 - k)^2 - m_{\pi^0}^2 + i\varepsilon)} (k^2 - m_{\pi^0}^2 + i\varepsilon) \]  

\[ \times \frac{\left( -g_{\mu\nu} + k_{\mu}k_{\nu} \right)}{m_{\pi^0}^2} \frac{(2p_{3\nu} - k_{\nu})}{(k^2 - m_{\pi^0}^2 + i\varepsilon)} \]  

\[ \tag{16} \]

The four-vector under the integral sign we transform as follows

\[ \left( -g_{\mu\nu} + k_{\mu}k_{\nu} \right) (2p_{3\nu} - k_{\nu}) = -2p_{3\nu} + k_{\nu} \left( m_{\pi^0}^2 - m_{D^0}^2 \right) \]  

\[ -m_{\pi^0}^2 + m_{\pi^0}^2 \right) / m_{\pi^0}^2 - k_{\nu}((k - p_3)^2 - m_{D^0}^2) / m_{\pi^0}^2. \]  

\[ \tag{17} \]

This shows that after reducing the numerator and denominator in Eq. [10] by the factor \((k - p_3)^2 - m_{\pi^0}^2\), the divergent part of the integral is proportional to \( p_{1\mu} \) (i.e., the four-moment of the X(3872) resonance) and does not contribute to \( F_0(s_1, s_2) \) because \( \epsilon_X, p_1 = 0 \). The analysis also shows that the term proportional to \( k_{\nu}((m_{D^0}^2 - m_{\pi^0}^2) / m_{\pi^0}^2 \]  

In Eq. [17] gives a negligible contribution to the integral in Eq. [10] in the vicinity of the \( \sqrt{s_1} \) and \( \sqrt{s_2} \) region under consideration. Thus we get

\[ F_0(s_1, s_2) = -2(\epsilon_X, p_3) \frac{i}{\pi} \int \frac{d^4k}{(k^2 - m_{\pi^0}^2 + i\varepsilon)} \]  

\[ \times \frac{1}{((p_1 - k)^2 - m_{\pi^0}^2 + i\varepsilon)((k - p_3)^2 - m_{D^0}^2 + i\varepsilon)} \]  

\[ \tag{18} \]

The amplitude \( F_+(s_1, s_2) \) is obtained from Eq. [13] by replacing the masses of neutral \( D^* \) and \( D \) mesons by the masses of their charged partners. For the numerical calculation of the amplitudes \( F_0(s_1, s_2) \) and \( F_+(s_1, s_2) \), we use the method developed in Ref. [45].

Using Eqs. [13] we express the differential width \( d\Gamma(X \to \pi^0\pi^+\pi^-; s_1, s_2)/d\sqrt{s_2} \) in terms of the invariant amplitude \( G_{\pi^0\pi^+\pi^-} \)

\[ d\Gamma(X \to \pi^0\pi^+\pi^-; s_1, s_2) \]  

\[ = \frac{2}{3} \frac{G_{\pi^0\pi^+\pi^-}((s_1, s_2))}{4\pi} \rho(s_2) \frac{2}{16\pi} \frac{\sqrt{s_2}}{\pi}, \]  

\[ \tag{19} \]

where

\[ \rho(s_2) = \sqrt{1 - 4m_{\pi^0}^2 / s_2}. \]  

\[ \tag{20} \]

The width of the decay \( X \to \pi^0\pi^+\pi^- \) as a function of \( s_1 \) has the form

\[ \Gamma(X \to \pi^0\pi^+\pi^-; s_1) = \int_{2m_{\pi^0}}^{\sqrt{s_1} - m_{\pi^0}} d\Gamma(X \to \pi^0\pi^+\pi^-; s_1, s_2)/d\sqrt{s_2}, \]  

\[ \tag{22} \]

and the probability of this decay is given by the expression

\[ BR(X \to \pi^0\pi^+\pi^-) = \int_{3m_{\pi^0}}^{\infty} \frac{2}{3\pi} \frac{1}{\sqrt{s_1} \Gamma(X \to \pi^0\pi^+\pi^-; s_1)} d\sqrt{s_1}. \]  

\[ \tag{23} \]
Equations (22) and (23) indicate the kinematically allowable limits of integration. In fact, the main contributions in Eqs. (22) and (23) are concentrated in much smaller intervals.

We now estimate the coupling constants \( g_{D^0 \pi^0} \) and \( g_{D^0 \pi^+ \pi^-} \).

For the total decay width of the \( D^{*0} \) meson, only its upper limit is known so far: \( \Gamma_{D^{*0}} < 2.1 \text{ MeV} \) [1]. On the other hand, the total decay width of the \( D^{*+} \) meson and the branching ratio of the \( D^{*+} \rightarrow (D\pi)^+ \) decay are well known [1]: \( \Gamma_{D^{*+}} \approx 83.6 \text{ keV} \), \( BR(D^{*+} \rightarrow (D\pi)^+) \approx 98.4\% \). Assuming the isotopic symmetry for the coupling constants \( g_{D^* \pi} \), we have

\[
\frac{m_{D^*}^2 \Gamma_{D^{*0} \rightarrow D^0 \pi^0}}{p_{D^0}^3} = \frac{m_{D^*}^2 \Gamma_{D^{*+} \rightarrow (D\pi)^+}}{2p_{D^0}^3 + p_{D^+}^3},
\]

where \( p_{D^0} \) denotes the momentum of the final \( D^0 \) or \( \pi^0 \) meson in the \( D^* \) rest frame. From here we find the decay width \( \Gamma_{D^{*0} \rightarrow D^0 \pi^0} \approx 36 \text{ keV} \) and the coupling constant \( g_{D^{*0} \pi^0} = 3m_{D^*} \Gamma_{D^{*0} \rightarrow D^0 \pi^0}/(2p_{D^0}^3) \approx 2.8 \). Using also the value of \( Br(D^{*0} \rightarrow D^0 \pi^0) \approx 64.7\% \) [1], we get an estimate for the total decay width of the \( D^{*0} \) meson: \( \Gamma_{D^{*0}} \approx 55.6 \text{ keV} \). Here we note in passing the following. As the examples [14, 15] show, the instability of the vector mesons in the intermediate states (i.e., the finiteness of their total widths) is important to take into account when estimating the contributions of logarithmic triangle singularities. In this case, \( \Gamma_{D^{*0}} \) is small. Nevertheless, its accounting in the \( D^{*0} \) propagator (by replacing \( m_{D^{*0}}^2 \rightarrow m_{D^{*0}}^2 - im_{D^{*0}} \Gamma_{D^{*0}} \)) noticeably smoothes the logarithmic singularity in the amplitude of the diagram in Fig. 2 and the computed probability of the \( X (3872) \rightarrow \pi^0 \pi^+ \pi^- \) decay is reduced by approximately 30% as compared to that for \( \Gamma_{D^{*0}} = 0 \).

The constant \( g_{D^{*0} \pi^+ \pi^-} \) is associated with the annihilation cross section \( \sigma_{D^{*0} \pi^+ \pi^-} \) at the \( D^{*0} D^0 \) threshold and with the corresponding inelastic scattering length \( \alpha''_{D^{*0} \pi^+ \pi^-} \) by the relations:

\[
k g_{D^{*0} \pi^+ \pi^-} = |q (m^2_{D^{*0}} - m^{2}_{D^0} + \alpha''_{D^{*0} \pi^+ \pi^-})/8\pi \sqrt{s_2}|^2.
\]

where \( k \) and \( q \) are momenta of the \( D^0 \) and \( \pi^+ \pi^- \) mesons, respectively, in the center-of-mass frame of the reaction \( D^0 D^0 \rightarrow \pi^+ \pi^- \). In the \( D^0 D^0 \) threshold domain of interest to us, \( q/s_2 \approx 1/(4m_{D^0}) \). At present, the values in Eq. (25), which characterizes the S wave \( D^0 D^0 \rightarrow \pi^+ \pi^- \) annihilation at rest, are completely unknown. If we naively put the inelastic scattering length \( |\alpha''_{D^{*0} \pi^+ \pi^-}| \approx 1/(2m_{D^{*0}}) \approx 1/(4 \text{ GeV}) \) (which is in dimensionless units \( m_{\pi^+} + |\alpha''_{D^{*0} \pi^+ \pi^-}| \approx 0.0347 \)), then \( |g_{D^{*0} \pi^+ \pi^-}/(8\pi)|^2 \) is approximately equal to \( \approx 1.8 \). We use this value in further evaluations.

Figure 4 shows an example of the \( \pi^+ \pi^- \) mass spectrum in the decay \( X (3872) \rightarrow \pi^0 \pi^+ \pi^- \), i.e., \( d\Gamma(X \rightarrow \pi^0 \pi^+ \pi^-; s_1, s_2)/d\sqrt{s_2} \) as a function of \( \sqrt{s_2} \), calculated with use of Eq. (19) at \( \sqrt{s_1} = m_X = 3.87169 \text{ GeV} \).

Figure 5 shows the width \( \Gamma(X \rightarrow \pi^0 \pi^+ \pi^-; s_1) \) as a function of \( \sqrt{s_1} \). The constructed example corresponds to \( g_A^2/(16\pi) = 0.25 \text{ GeV}^2 \).
and the coupling constat of $X(3872)$ with the $D^{*0}D^0$ channel $g_A^2/(16\pi) = 0.25$ GeV$^2$ (other possible values for $g_A^2/(16\pi)$ are discussed below). The integration $d\Gamma(X \to \pi^0 \pi^+\pi^-; m_X, s_2)/d\sqrt{s_2}$ over $\sqrt{s_2}$ in the region of 35 MeV wide, i.e., from $m_X - m_{\pi^0} = 0.035$ GeV = 3.70171 GeV to $m_X - m_{\pi^0} = 3.73671$ GeV, results in $\Gamma(X \to \pi^0 \pi^+\pi^-; m_X^2) \approx 3$ keV. However, as can be seen from Fig. 5, this is in fact the maximal value of the $X(3872) \to \pi^0 \pi^+\pi^-$ decay width in the $X(3872)$ resonance region. The width $\Gamma(X \to \pi^0 \pi^+\pi^-; s_1)$ is a sharply changing function of $\sqrt{s_1}$. Two peaks in $\Gamma(X \to \pi^0 \pi^+\pi^-; s_1)$ located near the $D^{*0}D^0$ and $D^+D^-$ thresholds (see Fig. 5) are manifestations of the logarithmic singularities in the amplitudes of the diagrams in Fig. 2 (the left peak) and in Fig. 3 (the right peak). The most important contribution to $BR(X \to \pi^0 \pi^+\pi^-)$ [see Eq. (23)] comes from the left peak. The right peak in $\Gamma(X \to \pi^0 \pi^+\pi^-; s_1)$ practically does not work as it is located far on the right tail of the $X(3872)$ resonance and its contribution to $BR(X \to \pi^0 \pi^+\pi^-)$ is strongly suppressed by the $X(3872)$ propagator module squared.

Table I: $BR(X(3872) \to \pi^0 \pi^+\pi^-)$ in units of $10^{-4}$ for five values of $g_A^2/(16\pi)$ and three values of $\Gamma_{non}$; $m_X = 3.87169$ GeV.

| $g_A^2/(16\pi)$ (in GeV$^2$) | 0.1 | 0.2 | 0.25 | 0.5 | 1.0 |
|-------------------------------|-----|-----|------|-----|-----|
| $\Gamma_{non} = 0.5$ MeV    | 7.42| 8.42| 8.35 | 7.10| 5.19 |
| $\Gamma_{non} = 1$ MeV      | 3.93| 4.99| 5.14 | 4.88| 3.84 |
| $\Gamma_{non} = 2$ MeV      | 1.93| 2.70| 2.89 | 3.07| 2.67 |

We now present numerical estimates for $BR(X \to \pi^0 \pi^+\pi^-)$ using as a guide the values of $g_A$ obtained in Refs. [25, 27, 28]. Figure 6 shows an example of the resonance distribution $2s_1/(\pi|D_X(s_1)|^2)$ calculated at $m_X = 3.87169$ GeV [1], $g_A^2/(16\pi) = 0.25$ GeV$^2$, and $\Gamma_{non} = 1$ MeV. Weighting with this distribution the energy dependent width $\Gamma(X \to \pi^0 \pi^+\pi^-; s_1)$ shown in Fig. 5, we find, according to Eq. (23), that for the above values of the parameters $BR(X \to \pi^0 \pi^+\pi^-) \approx 5 \times 10^{-4}$. Estimates for $BR(X \to \pi^0 \pi^+\pi^-)$ for different values of $g_A^2/(16\pi)$ and $\Gamma_{non}$, which we vary in a fairly wide but reasonable range, are given in Table I at $m_X = 3.87169$ GeV [1].

It is not yet clear whether the mass of the $X(3872)$ state lies slightly above or slightly below the $D^{*0}D^0$ threshold. The $\pm 0.17$ MeV uncertainty that the Particle Data Group [1] indicates allows for both possibilities. Tables II and III show the estimates for $BR(X \to \pi^0 \pi^+\pi^-)$ at the same values of $g_A^2/(16\pi)$ and $\Gamma_{non}$ as in Tab. I but for $m_X = 3.87169 \pm 0.00017$ GeV.

Table II: The same as Tab. I but for $m_X = 3.87169 + 0.00017$ GeV.

| $g_A^2/(16\pi)$ (in GeV$^2$) | 0.1 | 0.2 | 0.25 | 0.5 | 1.0 |
|-------------------------------|-----|-----|------|-----|-----|
| $\Gamma_{non} = 0.5$ MeV    | 6.45| 6.97| 6.82 | 5.63| 3.94 |
| $\Gamma_{non} = 1$ MeV      | 3.76| 4.60| 4.68 | 4.30| 3.27 |
| $\Gamma_{non} = 2$ MeV      | 1.93| 2.64| 2.80 | 2.89| 2.45 |

Table III: The same as Tab. I but for $m_X = 3.87169 - 0.00017$ GeV.

| $g_A^2/(16\pi)$ (in GeV$^2$) | 0.1 | 0.2 | 0.25 | 0.5 | 1.0 |
|-------------------------------|-----|-----|------|-----|-----|
| $\Gamma_{non} = 0.5$ MeV    | 8.04| 11.2| 12.2 | 14.7| 16.3 |
| $\Gamma_{non} = 1$ MeV      | 3.91| 5.57| 6.08 | 7.37| 8.20 |
| $\Gamma_{non} = 2$ MeV      | 1.86| 2.73| 3.01 | 3.70| 4.12 |

III. CONCLUSION

The above analysis shows that $BR(X(3872) \to \pi^0 \pi^+\pi^-)$ can be expected at the level of $10^{-3 \ldots 10^{-4}}$. The dominant contribution to $BR(X(3872) \to \pi^0 \pi^+\pi^-)$ comes from the production of the $\pi^+\pi^-$ system in a narrow (no more than two tens of MeV wide) interval of the invariant mass $m_{\pi^+\pi^-}$ near the value of $2m_{D^0} \approx 3.73$ GeV. The $\pi^+\pi^-$ events with such an invariant mass can serve as a signature of the decay $X(3872) \to (D^{*0}D^0 + D^{*0}D^0) \to \pi^0 D^0 D^0 \to \pi^0 \pi^+\pi^-$. The present work is partially supported by the program No. II.15.1 of fundamental scientific research of the Siberian Branch of the Russian Academy of Sciences, the project No. 0314-2019-0021.
Note that the BESIII experiments on the reactions

\begin{align*}
\pi^+\pi^-\gamma &\rightarrow \omega J/\psi, \\
\pi^+\pi^-\pi^0 &\rightarrow \gamma\omega J/\psi,
\end{align*}

and

\begin{align*}
\rho^+\rho^- &\rightarrow \omega J/\psi
\end{align*}

between the thresholds due to significantly different phases in the amplitude of the decay \(X(3872)\rightarrow \pi^+\pi^-J/\psi\) and the isospin symmetry breaking in the decays \(X(3872)\rightarrow \pi^+\pi^-\gamma\) (17) indicate, apparently, on the resonance which can else contain the destructive or constructive interference with the background.

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[17] A note that the BESIII experiments on the reactions

\begin{align*}
e^+e^- &\rightarrow \gamma\pi^+\pi^-J/\psi, \\
e^+e^- &\rightarrow \gamma\omega(J/\psi)\pi^0,
\end{align*}

and

\begin{align*}
e^+e^- &\rightarrow \gamma\rho(J/\psi)\pi^0(1P)
\end{align*}

indicate, apparently, on the two-gluon production mechanism of \(X(3872)\): 

\begin{align*}
\pi^+\pi^- &\rightarrow \gamma\omega(J/\psi)
\end{align*}

do not appear to be decisive in the question of the nature of the \(X(3872)\) state.

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