Witness sets

Gérard Cohen¹, Hugues Randriam¹, and Gilles Zémor²

¹ Ecole Nationale Supérieure des Télécommunications,  
46 rue Barrault,  
75 634 Paris 13, France  
cohen@enst.fr, randriam@enst.fr  
² Institut de Mathématiques de Bordeaux,  
Université de Bordeaux, UMR 5251,  
351 cours de la Libération,  
33405 Talence, France.  
Gilles.Zemor@math.u-bordeaux1.fr

Abstract. Given a set $C$ of binary $n$-tuples and $c \in C$, how many bits of $c$ suffice to distinguish it from the other elements in $C$? We shed new light on this old combinatorial problem and improve on previously known bounds.

1 Introduction

Let $C \subset \{0,1\}^n$ be a set of distinct binary vectors that we will call a code, and denote by $[n] = \{1,2,...,n\}$ the set of coordinate positions. It is standard in coding theory to ask for codes (or sets) $C$ such that every codeword $c \in C$ is as different as possible from all the other codewords. The most usual interpretation of this is that every codeword $c$ has a large Hamming distance to all other codewords, and the associated combinatorial question is to determine the maximum size of a code that has a given minimal Hamming distance $d$. The point of view of the present paper is to consider that “a codeword $c$ is as different as possible from all the other codewords” means that there exists a small subset $W \subset [n]$ of coordinates such that $c$ differs from every other codeword in $W$. Put differently, it is possible to single out $c$ from all the other codewords by focusing attention on a small subset of coordinates. More precisely, for $x \in \{0,1\}^n$, and $W \subset [n]$ let us define the projection $\pi_W$

$$
\pi_W : \{0,1\}^n \to \{0,1\}^W  
x \mapsto (x_i)_{i \in W}
$$

and let us say that $W$ is a witness set (or a witness for short) for $c \in C$ if $\pi_W(c) \neq \pi_W(c')$ for every $c' \in C$, $c \neq c'$. Codes for which every codeword has a small witness set arise in a variety of contexts, in particular in machine learning theory [1, 3, 4] where a witness set is also called a specifying set or a discriminant: see [5, Ch. 12] for a short survey of known results and also [2] and references therein for a more recent discussion of this topic and some variations.
Let us now say that a code has the \( w \)-witness property, or is a \( w \)-witness code, if every one of its codewords has a witness set of size \( w \). Our concern is to study the maximum possible cardinality \( f(n, w) \) of a \( w \)-witness code of length \( n \). We shall give improved upper and lower bounds on \( f(n, w) \) that almost meet.

The paper is organised as follows. Section 2 gives some easy facts for reference. Section 3 is devoted to upper bounds on \( f(n, w) \) and introduces our main result, namely Theorem 2. Section 4 is devoted to constant weight \( w \)-witness codes, and we derive precise values of the cardinality of optimal codes. Section 5 studies mean values for the number of witness sets of a codeword and the number of codewords that have a given witness set. Section 6 is devoted to constructions of large \( w \)-witness codes, sometimes giving improved lower values of \( f(n, w) \). Finally, Section 7 concludes with some open problems.

2 Easy and known facts

Let us start by mentioning two self-evident facts

– If \( C \) is a \( w \)-witness code, so is any translate \( C + x \),
– \( f(n, w) \) is an increasing function of \( n \) and \( w \).

Continue with the following example. Let \( C \) be the set of all \( n \) vectors of length \( n \) and weight 1. Then every codeword of \( C \) has a witness of size 1, namely its support. Note the dramatic change for the slightly different code \( C \cup \{0\} \). Now the all-zero vector 0 has no witness set of size less than \( n \). Bondy [3] shows however that if \( |C| \leq n \), then \( C \) is a \( w \)-witness code with \( w \leq |C| - 1 \) and furthermore \( C \) is a uniform \( w \)-witness code, meaning that there exists a single subset of \( [n] \) of size \( w \) that is a witness set for all codewords.

We clearly have the upper bound \( |C| \leq 2^w \) for uniform \( w \)-witness codes. For ordinary \( w \)-witness codes however, the best known upper bound is, [5, Proposition 12.2],

\[
f(n, w) \leq 2^w \binom{n}{w}.
\] (1)

The proof is simple and consists in applying the pigeon-hole principle. A subset of \( [n] \) can be a witness set for at most \( 2^w \) codewords and there are at most \( \binom{n}{w} \) witness sets.

We also have the following lower bound on \( f(n, w) \), based on a trivial construction of a \( w \)-witness code.

**Proposition 1.** We have: \( f(n, w) \geq \binom{n}{w} \).

**Proof.** Let \( C = \binom{[n]}{w} \) be the set of all vectors of weight \( w \). Notice that for all \( c \in C \), \( W(c) = \text{support}(c) \) is a witness set of \( c \).

Note that the problem is essentially solved for \( w \geq n/2 \); since \( f(n, w) \) is increasing with \( w \), we then have:

\[
2^n \geq f(n, w) \geq f(n, n/2) \geq \binom{n}{n/2} \geq 2^n/(2n)^{1/2}.
\]
We shall therefore focus in the sequel on the case \( w \leq n/2 \).

In the next section we improve the upper bound (1) to a quantity that comes close to the lower bound of Proposition 1.

### 3 An improved upper bound

The key result is the following.

**Theorem 1.** Let \( g(n, w) = f(n, w)/(\binom{n}{w}) \). Then, for fixed \( w \), \( g(n, w) \) is a decreasing function of \( n \). That is:

\[
    n \geq v \geq w \implies g(n, w) \leq g(v, w).
\]

**Proof.** Let \( C \) be a binary code of length \( n \) having the \( w \)-witness property, with maximal cardinality \( |C| = f(n, w) \). Fix a choice function \( \phi : C \rightarrow \binom{[n]}{w} \) such that for any \( c \in C \), \( \phi(c) \) is a witness for \( c \). For any \( V \in \binom{[n]}{v} \), denote by \( C_V \) the subset of \( C \) formed by the \( c \) satisfying \( \phi(c) \subseteq V \). Remark that the projection \( \pi_V \) is injective on \( C_V \), since each element of \( C_V \) has a witness in \( V \). Then \( \pi_V(C_V) \) also has the \( w \)-witness property.

Remark now that if \( V \) is uniformly distributed in \( \binom{[n]}{v} \) and \( W \) is uniformly distributed in \( \binom{[n]}{w} \) and independent from \( V \), then for any function \( \psi : \binom{[n]}{w} \rightarrow \mathbb{R} \) one has

\[
    E_W(\psi(W)) = E_V(E_W(\psi(W) | W \subseteq V)), \tag{2}
\]

where we denote by \( E_W(\psi(W)) \) the mean value (or expectation) of \( \psi(W) \) as \( W \) varies in \( \binom{[n]}{w} \), and so on.

We apply this with \( \psi(W) = |\phi^{-1}(W)| \) to find

\[
    g(n, w) = \binom{n}{w}^{-1} |C| = \binom{n}{w}^{-1} \sum_{W \in \binom{[n]}{w}} |\phi^{-1}(W)|
    = E_W(|\phi^{-1}(W)|)
    = E_V(E_W(|\phi^{-1}(W)| | W \subseteq V))
    = E_V \left( \binom{v}{w}^{-1} \sum_{W \in \binom{[v]}{w}} |\phi^{-1}(W)| \right)
    = E_V \left( \binom{v}{w}^{-1} |C_V| \right)
    = E_V \left( \binom{v}{w}^{-1} |\pi_V(C_V)| \right)
    \leq g(v, w)
\]

the last inequality because \( \pi_V(C_V) \) is a binary code of length \( v \) having the \( w \)-witness property.
Remark: It would be interesting to try to improve Theorem 1 using some unexploited aspects of the above proof, such as the fact that the choice function $\phi$ may be non-unique, or the fact that the last inequality not only holds in mean value, but for all $V$. For instance, suppose there is a codeword $c \in C$ (with $C$ optimal as in the proof) that admits two distinct witnesses $W$ and $W'$, with $W \not\subset W'$. Let $\phi$ be a choice function with $\phi(c) = W$, and let $\phi'$ be the choice function that coincides everywhere with $\phi$, except for $\phi'(c) = W'$. Let $V$ contain $W'$ but not $W$. If we denote by $C'_V$ the subcode obtained as $C_V$ but using $\phi'$ as choice function, then $C'_V = C_V \cup \{c\}$ (disjoint union), so $|\pi_V(C_V)| = |\pi_V(C'_V)| - 1 < f(v, w)$, and $g(n, w) < g(v, w)$.

Theorem 1 has a number of consequences: the following is straightforward.

**Corollary 1.** For fixed $w$, the limit

$$\lim_{n \to \infty} g(n, w) = \frac{f(n, w)}{\binom{n}{w}}$$

exists.

The following theorem gives an improved upper bound on $f(n, w)$.

**Theorem 2.** For $w \leq n/2$, we have the upper bound:

$$f(n, w) \leq 2^w \frac{n}{w}.$$  

*Proof.** Choose $v = 2w$ and use $f(v, w) \leq 2^w$; then $f(n, w) \leq \binom{n}{w} f(2w, w) / \binom{2w}{w}$ and the result follows by Stirling’s approximation.

Set $w = \omega n$ and denote by $h(x)$ the binary entropy function

$$h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x).$$

Theorem 2 together with Proposition 1 yield:

**Corollary 2.** We have

$$\lim_{n \to \infty} \frac{1}{n} \log_2 f(n, \omega n) = h(\omega)$$

for $0 \leq \omega \leq 1/2$ and $1/2 \leq \omega \leq 1$.

### 4 Constant-weight codes

Denote now by $f(n, w, k)$ the maximal size of a $w$-witness code with codewords of weight $k$. The following result is proved using a folklore method usually attributed to Bassalygo and Elias, valid when the required property is invariant under some group operation.
**Proposition 2.** We have:

\[
\max_k f(n, w, k) \leq f(n, w) \leq \min_k \frac{f(n, w, k)2^n}{n \choose k}.
\]

**Proof.** The lower bound is trivial.

For the upper bound, fix \(k\), pick an optimal \(w\)-witness code \(C\) and consider its \(2^n\) translates by all possible vectors. Every \(n\)-tuple, in particular those of weight \(k\), occurs exactly \(|C|\) times in the union of the translates; hence there exists a translate (also an optimal \(w\)-witness code of size \(f(n, w)\) - see the remark at the beginning of Section 2) containing at least the average number \(|C| \times n \choose k\) of vectors of weight \(k\). Since \(k\) was arbitrary, the result follows.

We now deduce from the previous proposition the exact value of the function \(f(n, w, k)\) in some cases.

**Corollary 3.** For constant-weight codes we have:

- If \(k \leq w \leq n/2\) then \(f(n, w, k) = \left(\begin{array}{c} n \\ k \end{array}\right)\) and an optimal code is given by \(S_k(0)\), the Hamming sphere of radius \(k\) centered on \(0\).
- If \(n - k \leq w \leq n/2\), then \(f(n, w, n - k) = \left(\begin{array}{c} n \\ k \end{array}\right)\) and an optimal code is given by the sphere \(S_k(1)\).

**Proof.** If \(k \leq w \leq n/2\), we have the following series of inequalities:

\[
\left(\begin{array}{c} n \\ k \end{array}\right) \leq f(n, k, k) \leq f(n, w, k) \leq \left(\begin{array}{c} n \\ k \end{array}\right).
\]

If \(n - k \leq w \leq n/2\), perform wordwise complementation.

### 5 Some mean values

Let \(C\) be a binary code of length \(n\) (not necessarily having the \(w\)-witness property). Let

\[
\mathcal{W}_{C, w}: C \to 2^{\left(\begin{array}{c} n \\ w \end{array}\right)}, \quad \mathcal{W}_{C, w}(c) = \{W \in \left(\begin{array}{c} n \\ w \end{array}\right): W \text{ is a witness for } c\},
\]

and symmetrically,

\[
\mathcal{C}_{C, w}: \left(\begin{array}{c} n \\ w \end{array}\right) \to 2^C, \quad \mathcal{C}_{C, w}(W) = \{c \in C: W \text{ is a witness for } c\}.
\]

Remark that if \(C' \subset C\) is a subcode, then \(\mathcal{W}_{C', w}(c) \supseteq \mathcal{W}_{C, w}(c)\) for any \(c \in C'\), while \(\mathcal{C}_{C', w}(W) \supseteq (C' \cap \mathcal{C}_{C, w}(W))\) for any \(W \in \left(\begin{array}{c} n \\ w \end{array}\right)\).
Lemma 1. With these notations, the mean values of $|W_{C,w}|$ and $|C_{C,w}|$ are related by

$$|C|E_c(|W_{C,w}(c)|) = \binom{n}{w} E_W(|C_{C,w}(W)|),$$

or equivalently

$$\frac{|C|}{\binom{n}{w}} = \frac{E_W(|C_{C,w}(W)|)}{E_c(|W_{C,w}(c)|)}.$$

Proof. Double count the set $\{(W,c) \in \binom{[n]}{w} \times C : W$ is a witness for $c\}$.

Now let $\gamma(C, w) = E_W(|C_{C,w}(W)|)$ and let $\gamma^+(n, w)$ be the maximum possible value of $\gamma(C, w)$ for $C$ a binary code of length $n$, and $\gamma^{++}(n, w)$ be the maximum possible value of $\gamma(C, w)$ for $C$ a binary code of length $n$ having the $w$-witness property.

Lemma 2. With these notations, one has $\gamma^+(n, w) = \gamma^{++}(n, w)$.

Proof. By construction $\gamma^+(n, w) \geq \gamma^{++}(n, w)$. On the other hand, let $C$ be a binary code of length $n$ with $\gamma(C, w) = \gamma^+(n, w)$, and let then $C'$ be the subcode of $C$ formed by the $c$ having at least one witness of size $w$, i.e. $C' = \bigcup_{W \in \binom{[n]}{w}} C_{C,w}(W)$. Then $C'$ has the $w$-witness property, and

$$\gamma^{++}(n, w) \geq \gamma(C', w) \geq \gamma(C, w) = \gamma^+(n, w).$$

The technique of the proof of Proposition 1 immediately adapts to give:

Proposition 3. With these notations, $w$ being fixed, $\gamma^+(n, w)$ is a decreasing function of $n$. That is:

$$n \geq v \geq w \Rightarrow \gamma^+(n, w) \leq \gamma^+(v, w).$$

Proof. Let $C$ be a binary code of length $n$ with $\gamma(C, w) = \gamma^+(n, w)$. For $V \in \binom{[n]}{v}$, denote by $C_V$ the subset of $C$ formed by the $c$ having at least one witness of size $w$ included in $V$, i.e. $C_V = \bigcup_{W \in \binom{[n]}{w}} C_{C,w}(W)$. Then $C_V$ has the $w$-witness property, $C_{C,w}(W) \subset C_{C_V,w}(W)$ for any $W \subset V$, and $\pi_V$ is injective on $C_V$. Using this and (2), one gets:

$$\gamma^+(n, w) = E_W(|C_{C,w}(W)|)$$
$$= E_V(E_W(|C_{C,w}(W)| \mid W \subset V))$$
$$\leq E_V(E_W(|C_{C_V,w}(W)| \mid W \subset V))$$
$$= E_V(E_W(|C_{\pi_V(C_V),w}(W)| \mid W \subset V))$$
$$= E_V(\gamma(\pi_V(C_V), w))$$
$$\leq \gamma^+(v, w).$$
6 Constructions

6.1 A generic construction

Let \( \mathcal{F} \subset \binom{[n]}{\leq w} \) be a set of subsets of \( \{1, \ldots, n\} \) all having cardinality at most \( w \).

Let \( \mathcal{C}_{\mathcal{F}} \subset \{0, 1\}^n \) be the set of words having support included in one and only one \( W \in \mathcal{F} \). Then:

**Proposition 4.** With these notations, \( \mathcal{C}_{\mathcal{F}} \) has the \( w \)-witness property.

**Proof.** For each \( c \in \mathcal{C}_{\mathcal{F}} \), let \( W_c \) be the unique \( W \in \mathcal{F} \) containing the support of \( c \). Then \( W_c \) is a witness for \( c \).

**Example 1.** For \( \mathcal{F} = \binom{[n]}{w} \) we find \( \mathcal{C}_{\mathcal{F}} = S_w(0) \), and

\[
\begin{align*}
f(n, w) & \geq |\mathcal{C}_{\mathcal{F}}| = \binom{n}{w}.
\end{align*}
\]

**Example 1’.** Suppose \( w \geq n/2 \). Then for \( \mathcal{F} = \binom{[n]}{n/2} \) we find \( \mathcal{C}_{\mathcal{F}} = S_{n/2}(0) \), and

\[
\begin{align*}
f(n, w) & \geq |\mathcal{C}_{\mathcal{F}}| = \binom{n}{n/2}
\end{align*}
\]

(where for ease of notation we write \( n/2 \) instead of \( \lfloor n/2 \rfloor \)).

**Example 2.** For \( \mathcal{F} = \{W\} \) with \( |W| \leq w \) we find \( \mathcal{C}_{\mathcal{F}} = \{0, 1\}^W \) (where we see \( \{0, 1\}^W \) as a subset of \( \{0, 1\}^n \) by extension by 0 on the other coordinates), and

\[
\begin{align*}
f(n, w) & \geq |\mathcal{C}_{\mathcal{F}}| = 2^w.
\end{align*}
\]

**Example 3.** Let \( \mathcal{F} \) be the set of (supports of) words of a code with constant weight \( w \) and minimal distance \( d \) (one can suppose \( d \) even). Then for all distinct \( W, W' \in \mathcal{F} \) one has \( |W \cap W'| \leq w - d/2 \), so for all \( W \in \mathcal{F} \), the code \( \mathcal{C}_{\mathcal{F}} \) contains all words of weight larger than \( w - d/2 \) supported in \( W \). This implies:

**Corollary 4.** For all \( d \) one has

\[
\begin{align*}
f(n, w) & \geq A(n, d, w)B(w, d/2 - 1)
\end{align*}
\]

where:

- \( A(n, d, w) \) is the maximal cardinality of a code of length \( n \) with minimal distance at least \( d \) and constant weight \( w \)
- \( B(w, r) = \sum_{1 \leq i \leq r} \binom{w}{i} \) is the cardinality of the ball of radius \( r \) in \( \{0, 1\}^w \).

For \( d = 2 \), this construction gives the sphere again. For \( d = 4 \), this gives \( f(n, w) \geq (1 + w)A(n, d, w) \). We consider the following special values:

- \( n = 4, d = 4, w = 2: A(4, 4, 2) = 2 \)
- \( n = 8, d = 4, w = 4: A(8, 4, 4) = 14 \)
- \( n = 12, d = 4, w = 6: A(12, 4, 6) = 132 \)
the last two being obtained with $F$ the Steiner system $S(3, 4, 8)$ and $S(5, 6, 12)$ respectively.

The corresponding codes $C_F$ have same cardinality as the sphere $(2 \times 3 = 6, 14 \times 5 = 70$ and $132 \times 7 = 924$ respectively), but they are not translates of a sphere. Indeed, when $C$ is a (translate of a) sphere with $w = n/2$, one has $C_{C,w}(W) = 2$ for any window $W \in \binom{[n]}{w}$. On the other hand, for $C = C_F$ as above, one has by construction $C_{C,w}(W) = w + 1$ for $W \in F$.

6.2 Another construction

Let $D \subset \{0, 1\}^w$ be a binary (non-linear) code of length $w > n/2$ and minimal weight at least $2w - n$.

Let $C_1$ be the code of length $n$ obtained by taking all words of length $w$ that do not belong to $D$, and completing them with 0 on the last $n - w$ coordinates. Thus $|C_1| = 2^w - |D|$.

Let $C_2$ be the code of length $n$ formed by the words $c$ of weight exactly $w$, and such that the projection of $c$ on the first $w$ coordinates belongs to $D$. Thus if $n_k$ is the number of codewords of weight $k$ in $D$, one finds $|C_2| = \sum_k n_k \binom{n - k}{w - k}$.

Now let $C$ be the (disjoint!) union of $C_1$ and $C_2$. Then $C$ has the $w$-witness property. Indeed, let $c \in C$. Then if $c \in C_1$, $c$ admits $[w]$ as witness, while if $c \in C_2$, $c$ admits its support as witness.

As an illustration, let $D$ be the sphere of radius $w - t$ in $\{0, 1\}^w$, for $t \in \{1, \ldots, \frac{n - w}{2}\}$. Then

$$f(n, w) \geq |C| = 2^w + \binom{w}{w - t} \left( \binom{n - w}{t} - 1 \right).$$

If $w$ satisfies $2^w > \binom{n}{n/2}$ but $w < n - 1$, this improves on examples 1, 1’, and 2 of the last subsection, in that one finds then

$$f(n, w) \geq |C| \geq \max \left( \binom{n}{w}, \binom{n}{n/2}, 2^w \right).$$

On the other hand, remark that $C_1 \subset \{0, 1\}^w$ and $C_2 \subset S_w(0)$, so that $|C| \leq 2^w + \binom{n}{w}$.

7 Conclusion and open problems

We have determined the asymptotic size of optimal $w$-witness codes. A few issues remain open in the non-asymptotic case, among which:

- When is the sphere $S_w(0)$ the/an optimal $w$-witness code? Do we have $f(n, w) = \binom{n}{w}$ for $w \leq n/2$? In particular do we have $f(2w, w) = \binom{2w}{w}$?
- For $w > n/2$, do we have $f(n, w) \leq \max \left( \binom{n}{w/2}, 2^w + \binom{n}{w} \right)$?
- Denoting by $f(n, w, \geq d)$ the maximal size of a $w$-witness code with minimum distance $d$, can the asymptotics of Proposition 2 be improved to

$$\frac{1}{n} \log_2 f(n, \omega n, \geq \delta n) < h(\omega) ?$$
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