A regularity theory for random elliptic operators

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Abstract. Since the seminal results by Avellaneda & Lin it is known that elliptic operators with periodic coefficients enjoy the same regularity theory as the Laplacian on large scales. In a recent inspiring work, Armstrong & Smart proved large-scale Lipschitz estimates for such operators with random coefficients satisfying a finite-range of dependence assumption. In the present contribution, we extend the intrinsic large-scale regularity of Avellaneda & Lin (namely, intrinsic large-scale Schauder and Calderón-Zygmund estimates) to elliptic systems with random coefficients. The scale at which this improved regularity kicks in is characterized by a stationary field $r_\ast$ which we call the minimal radius. This regularity theory is qualitative in the sense that $r_\ast$ is almost surely finite (which yields a new Liouville theorem) under mere ergodicity, and it is quantifiable in the sense that $r_\ast$ has high stochastic integrability provided the coefficients satisfy quantitative mixing assumptions. We illustrate this by establishing optimal moment bounds on $r_\ast$ for a class of coefficient fields satisfying a multiscale functional inequality, and in particular for Gaussian-type coefficient fields with arbitrary slow-decaying correlations.

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1. Introduction

This article is the first of a series that develops a quantitative theory for large-scale properties of random elliptic operators. It presents digested and optimized versions of the proofs of the first complete version of the manuscript (dated August 2015). The series consists of three parts: An intrinsic large-scale regularity theory in the present contribution, applications to quantitative stochastic homogenization in [40], and the characterization of large-scale fluctuations in [27, 28, 29].

The classical theory of homogenization for elliptic systems $-\nabla \cdot a \nabla$ with periodic, uniformly elliptic coefficients $a$ started with contributions of the French, the Italian and the Russian schools (e.g. see [64, 54, 18, 22, 62, 47, 46]). Classical homogenization states that on large scales (i.e., scales much larger than the period of $a$) the resolvent of $-\nabla \cdot a \nabla$ is close to the resolvent of the so-called homogenized operator $-\nabla \cdot a_{\text{hom}} \nabla$, where $a_{\text{hom}}$ are spatially homogeneous coefficients. In the seminal work [9] Avellaneda and Lin observed that
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A regularity theory for random elliptic operators can be used to lift the regularity theory for the homogenized (constant-coefficient) operator $-\nabla \cdot a_{\text{hom}} \nabla$ to the original variable-coefficient operator $-\nabla \cdot a \nabla$. Since elliptic systems with measurable coefficients basically only enjoy $L^2$-regularity theory (as opposed to the maximal regularity of elliptic systems with constant coefficients), Avellaneda and Lin’s results yield a strong improvement of regularity (on large scales). More precisely, in [9, Section 3.1] Avellaneda and Lin derive intrinsic $C^{1,1-}$-a priori estimates on $a$-harmonic functions (where $C^{k,1-}$ means $C^{k,\alpha}$ for all $\alpha < 1$). Here intrinsic refers to the fact that the estimates are formulated not in Euclidean (flat) coordinates, but with help of the so-called harmonic coordinates, which are based on the notion of the corrector – a key object in the theory of homogenization. In the present paper we extend the intrinsic large-scale regularity theory of Avellaneda and Lin to the case of elliptic systems with random (in particular stationary & ergodic) coefficients. This extension from the periodic to the random setting is non-trivial since the original argument of Avellaneda and Lin crucially relies on a compactness argument (related to the compactness of the torus associated with the periodic coefficients).

Qualitative stochastic homogenization of uniformly elliptic equations with random coefficients was first established by Papanicolaou and Varadhan [58] and by Kozlov [48]. The argument of Papanicolaou and Varadhan [58] is based on Tartar’s method of oscillating test-functions, and, in the core of the analysis, extends the notion of corrector to the random setting: Roughly speaking, if $a$ denotes a random coefficient field, and $e$ a fixed unit direction of $\mathbb{R}^d$, then the associated corrector $\phi$ is defined as a sublinearly growing solution of

$$-\nabla \cdot a \nabla \phi = \nabla \cdot ae \quad \text{in } \mathbb{R}^d \quad (1)$$

(see Lemma 1 below for the precise statement). The first example of a large-scale regularity result due to randomness is the higher stochastic integrability of the gradient $\nabla \phi$ of the corrector obtained in [41, 43, 38] (in the course of proving quantitative results in stochastic homogenization). Developing a quantitative theory obviously requires quantitative ergodicity assumptions, which we make in the form of a functional inequality (e.g. a spectral gap estimate), inspired by the unpublished work [56] by Naddaf and Spencer. Among other estimates, we proved that if the random coefficients $a$ satisfy a spectral gap estimate (see Definition 1), then for all $1 \leq p < \infty$

$$\left\langle \left( \int_{B_1(0)} |\nabla \phi|^2 \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}} \lesssim 1 \lesssim \left\langle \left( \int_{B_1(0)} |ae|^2 \right)^{\frac{p}{2}} \right\rangle^{\frac{1}{p}},$$

where $\lesssim$ stands for $\leq C \times$ for a multiplicative constant depending on $p, \lambda$, and the dimension. (Note that the case $p = 2$ follows by an elementary energy estimate.) In view of (1), this result is reminiscent of a Calderón-Zygmund estimate. It is a large-scale regularity result since it involves taking the expectation $\langle \cdot \rangle$, which by Birkhoff’s ergodic theorem turns into the “large-scale” spatial average $\lim_{R \to \infty} \int_{B_R} (\cdot)$. In terms of Hölder regularity, the first large-scale regularity result in the random setting was by Marahrens
and the third author in [53]: For scalar equations and under a strong quantitative ergodicity assumption in form of a Logarithmic Sobolev Inequality (see Definition 1), (large-scale) $C^{0,1}$-estimates for $a$-harmonic functions were established (see also [37]). Common key elements to these works are functional inequalities to quantify ergodicity, a sensitivity calculus to estimate the dependence of a solution, like $\nabla \phi$, on the coefficient field $a$, and input from deterministic regularity theory, which in [41, 43, 38, 53] are encoded in form of Green’s function estimates that crucially use De Giorgi-Nash-Moser regularity theory. The latter restricts the results of these works to scalar equations (see however [15, 16] for systems, and, thanks to [21], also [53] extends to systems). A motivation for the present work is to replace the deterministic regularity theory used in these works by the intrinsic large-scale regularity theory developed in the present paper – this will be addressed in our follow-up work [40].

With another flavor and under the sole assumption of stationarity and ergodicity (as opposed to the strong quantitative ergodicity assumption of [53]), Benjamini, Duminil-Copin, Kozma, and Yadin proved in [20] a Liouville theorem in a very general context which states that strictly sublinear $a$-harmonic functions are constants. Note that Liouville theorems and Schauder theory are intimately connected: E.g. Simon derived Schauder estimates [61, Theorem 1] indirectly from a Liouville result [61, Lemma 1]; while Avellaneda and Lin [10] obtained Liouville theorems of any order for elliptic systems in the periodic setting by appealing to their large-scale regularity theory.

In a recent inspiring work, Armstrong and Smart [5] developed a large-scale regularity theory in the random setting. It is the first result that implements the general strategy [9] of Avellaneda and Lin (of lifting the regularity theory for the homogenized operator to the original operator) in a situation where the above-mentioned compactness argument fails. Roughly speaking, in their approach the compactness argument is replaced by a quantitative estimate of the homogenization error that can be established under quantitative ergodicity. In contrast to our framework based on nonlinear mixing conditions (as developed in [41, 43, 38] and the present paper), Armstrong and Smart quantify ergodicity in terms of linear mixing conditions. In [5] they consider the scalar random case under a finite-range of dependence assumption (the strongest of the linear mixing conditions). On the one hand they reformulate the Campanato iteration of [9] in an abstract functional-analytic form that is oblivious to the PDE (see [5, Lemma 5.1]) and essentially states that if a function is close at all scales (down to unit scale) to functions with improvement of flatness, then that function must itself have an improvement of flatness (down to unit scale). Next, they show that if for Dirichlet problems the homogenization error decays algebraically, then $a$-harmonic functions are indeed close at all scales to functions with improvement of flatness (see [5, Proposition 4.1]), so that they are themselves Lipschitz (from unit scale onwards). On the other hand, they establish the algebraic (although largely suboptimal) decay of the homogenization error (at an $L^2$-level) within their
assumptions (scalar equation, finite-range of dependence, symmetric coefficients) using an ingenious combination of subadditivity methods, duality, and a concentration argument. In contrast to the corrector-based, intrinsic regularity approach of Avellaneda and Lin, their method is Euclidean. The improvement over [53] is twofold: In terms of regularity ($C^{1,0}$ versus $C^{0,1}$) and in terms of stochastic integrability (finite nearly-optimal exponential moment versus finite algebraic moments).

In the first version of the present paper (cf. the arXiv preprint [39] in fall 2014), inspired by the works [9] of Avellaneda and Lin and [5] of Armstrong and Smart, we developed the first intrinsic large-scale regularity theory for (possibly non-symmetric) elliptic systems (including linear elasticity) in the random setting. This large-scale regularity goes beyond Lipschitz estimates and is at the same time qualitative (it applies to merely ergodic coefficients) and quantifiable (in terms of stochastic integrability under quantitative mixing conditions). In place of a finite-range of dependence condition, we quantify ergodicity via a Logarithmic Sobolev Inequality in a variant that is flexible enough to cover coefficients with both weak and strong correlations (which is also new). On the one hand, this regularity theory (since it applies to the general ergodic case) is of interest for the study of the random conductance model in probability theory (see [50, 19] for recent surveys). Quenched invariance principles and heat kernel estimates for degenerate general ergodic conductances indeed received much attention recently — e.g. see [1, 23], and the Liouville theorems obtained by the approach of the present paper in [57] (for the random conductance model) and in [12] (for degenerate elliptic systems). On the other hand, the quantification of ergodicity via functional inequalities (in the form of multiscale functional inequalities as in the present paper) is particularly well-suited for the class of coefficient fields considered in the applied sciences: Most models studied in materials science are indeed generated starting from a (typically hidden) product or Gaussian structure (see [65], a reference textbook on random heterogeneous materials), and therefore not only satisfy linear mixing conditions but also nonlinear mixing conditions that can be captured in form of multiscale functional inequalities (see [25, 26]). As we shall see, such nonlinear mixing conditions are crucial to establish the optimal stochastic integrability for the large-scale regularity theory for these models, see Remark 3 below.

Between the first posted version [39] and the current version of this paper, Armstrong and Mourrat [4], and subsequently Armstrong, Mourrat and Kuusi [6, 7, 8] significantly extended the regularity theory of [5] in several directions: First, using the framework of the Fitzpatrick duality theory, they were able to treat not only convex integral functionals with quadratic growth, but also monotone operators with quadratic growth (recovering the case of non-symmetric systems we studied in [39], albeit with a stronger notion of coercivity). Second, they showed that the subadditivity method of [5] can be pushed forward to treat weaker linear mixing conditions on the coefficients (such as $\alpha$-mixing), lifted the Lipschitz theory to higher-order regularity, and
proved moment bounds on the gradient of the corrector that depend on the alpha-mixing decay rate (algebraic decay rate yields algebraic moments, exponential decay rate yields exponential moments, albeit with an arbitrarily small loss of stochastic integrability). Third, they established optimal growth estimates on the corrector and characterized its large-scale fluctuations (as well as other related quantities) under the finite-range of dependence assumption — their proof cannot does not extend in a straightforward way to the setting of functional inequalities (and therefore Gaussian statistics). Likewise, the results of the first version [39] of this paper have been extended in several directions, see discussion in the next paragraph. Whereas [4, 6, 7] rely on variational arguments, the present work and its extensions rely on PDE analysis.

To conclude this introduction, let us summarize the achievements of the present contribution. We develop a complete intrinsic large-scale regularity theory for random elliptic operators that shows that the regularity theory for constant-coefficients elliptic systems extends to random elliptic operators at large scales. We illustrate this by establishing maximal regularity at the level of $C^{1,1-}$, i.e. Schauder theory, and of $\dot{H}^{1,p}$, i.e. Calderón-Zygmund theory. A key object that we introduce in this contribution is a stationary random field $r^*$, which we call the minimal radius. It characterizes the scale at which the improved intrinsic regularity theory kicks in. In particular, this large-scale regularity theory is

(i) qualitative in the sense that $r^*$ is almost surely finite under the mere qualitative assumption of stationarity and ergodicity. This is crucial to prove a Liouville result for subquadratic $a$-harmonic functions, see Corollary 1;

(ii) and at the same time quantifiable in the sense that $r^*$ can be proved to have stretched exponential moments if the ensemble of coefficient fields satisfies suitable functional inequalities. We prove this on the representative example of a family of Gaussian coefficient fields the covariance of which decays arbitrarily slowly, cf. Theorems 3 & 4, and obtain the optimal stochastic integrability (as opposed to the nearly-optimal stochastic integrability one would get using the approach of [5]).

The definition of $r^*$ is based on an extended corrector $(\phi, \sigma)$, see Lemma 1, which allows to represent the residuum of the homogenization error in divergence form. In addition to the standard corrector $\phi$, the extended corrector involves a skew-symmetric tensor field, which we call $\sigma$. While it has not been used in stochastic homogenization before, it is a standard object in periodic homogenization, see for instance [46, p.27], where it is used to establish quantitative two-scale expansions. The tensor $\sigma$ is related to the flux of the corrector and appears to be as important as the corrector itself. (This does not come as a surprise in view of the very definition of qualitative H-convergence: Weak convergence of the gradient of the solution and weak convergence of the flux.) This notion of flux corrector turns out to be fundamental in stochastic homogenization, and has been taken up by subsequent work: Intrinsic
higher-order regularity and Liouville theorems of all orders [30], in half spaces [32], in degenerate environments [12], quantitative estimates on the corrector and error estimates both for ensembles that satisfy functional inequalities and ensembles of finite range of dependence [40, 44, 45, 13], characterization of fluctuations (not only of the corrector but also of the solution operator) [27, 28, 29], notion of multipoles [14], long-time homogenization of the wave equation [17], results on non-symmetric discrete models [16], quantification of invariance principles for the random conductance in degenerate environments [2], etc.

Our approach to large-scale regularity is mainly inspired by the work of Avellaneda and Lin. In particular, it is close to [9, Section 3.1], see Remark 3. Incidentally, $\sigma$ is not used for that result and only used marginally in that paper [9, p.845], and not capitalizing on its skew-symmetry. Compared to the series [4, 6, 7], our approach has advantages both in terms of stochastic results and large-scale regularity. Compared to the large-scale Lipschitz regularity of [5] (which we call here the mean-value property), our result is based on a mere smallness condition as opposed to an algebraic convergence rate. This has two consequences: First it allows to cover the case of qualitative ergodicity, and second it allows to capture the optimal stochastic integrability of $r_*$. In terms of large-scale regularity, our approach is intrinsic (which allows us to place ourselves at scale 1, and not at mesoscales) and the large-scale Calderón-Zygmund theory we derive here does not come with a loss of integrability and holds for all $1 < p < \infty$.

2. Statement of the main results

2.1. Assumptions and notation

We start by specifying our assumptions on the coefficient fields, and then recall the standard definition of the corrector and the (slightly less standard) definition of the flux corrector.

Assumptions on the ensemble of coefficient fields. Our two assumptions on the space of (admissible) coefficient fields $a$ are pointwise boundedness and uniform ellipticity. Without loss of generality, we may assume that the bound is unity:

$$|a(x)\xi| \leq |\xi| \quad \text{for all } \xi \in \mathbb{R}^d \text{ and } x \in \mathbb{R}^d.$$  (2)

We require uniform ellipticity with constant $\lambda > 0$ only in the integrated form of

$$\int \nabla \zeta \cdot a \nabla \zeta \geq \lambda \int |\nabla \zeta|^2 \quad \text{for all smooth and compactly supported } \zeta.$$  (3)

This form of ellipticity is weaker than pointwise ellipticity for systems, and allows one to consider linear elasticity tensors $a$ that do not necessarily have a sign (so that the duality theory at the basis of [5] does not automatically apply). Throughout this paper, we use scalar notation for convenience. However, we only use arguments that are available in the case of systems, that is,
when \( \mathbb{R} \)-valued functions \( \zeta \) are replaced by fields with values in some finite dimensional Euclidean space \( H \). More precisely, we only use the energy estimate and consequences thereof, like the Caccioppoli estimate and the higher integrability coming from the hole-filling argument. In particular, we do not appeal to De Giorgi’s theory. Clearly, in the case of systems, all constants acquire an additional dependence on \( H \).

We now address the minimal assumptions on the “ensemble” \( \langle \cdot \rangle \), a probability measure on the space of (admissible) coefficient fields as introduced in [58, Section 2], which will be assumed throughout the paper. They are related to the operation of the shift group \( \mathbb{R}^d \) on the space of coefficient fields, that is, for any shift vector \( z \in \mathbb{R}^d \) and any coefficient field \( a \), the shifted field \( a(\cdot + z) : x \mapsto a(x + z) \) is again a coefficient field. The first assumption is stationarity, which means that for any shift \( z \in \mathbb{R}^d \) the random coefficient fields \( a \) and \( a(\cdot + z) \) have the same (joint) distribution. The second assumption is ergodicity, which means that any (integrable) random variable \( F(a) \) that is shift invariant, that is, \( F(a(\cdot + z)) = F(a) \) for all shift vectors \( z \in \mathbb{R}^d \) and \( \langle \cdot \rangle \)-almost coefficient field \( a \), is actually constant, that is \( F(a) = \langle F \rangle \) for \( \langle \cdot \rangle \)-almost every coefficient field \( a \).

Under assumptions (2), (3), stationarity, and ergodicity, homogenization (in the sense of Murat and Tartar’s notion of H-convergence, see [54]) holds (for Dirichlet boundary data in the case of systems due to the weak notion of ellipticity), and the homogenized coefficients also satisfy

\[
\int \nabla \zeta \cdot a_{\text{hom}} \nabla \zeta \geq \lambda \int |\nabla \zeta|^2 \quad \text{for all smooth and compactly supported } \zeta
\]

and \( |a_{\text{hom}} \xi| \leq \frac{(1+\lambda^2)^{\frac{1}{2}}}{\lambda} |\xi| \). In particular, \( a_{\text{hom}} \) is uniformly elliptic in the scalar case,

\[
\xi \cdot a_{\text{hom}} \xi \geq \lambda |\xi|^2 \quad \text{and} \quad |a_{\text{hom}} \xi| \leq \frac{1}{\lambda} |\xi| \quad \text{for all } \xi \in \mathbb{R}^d,
\]

and satisfies the Legendre-Hadamard condition in the case of systems. The proof of these statements is the same as the proof given by Murat and Tartar in the case of a second-order elliptic equation (e.g. see [54, 55]). (In both cases, the elliptic operator \(-\nabla \cdot a_{\text{hom}} \nabla \) enjoys a full regularity theory.)

**Extended corrector.** Throughout this paragraph \( i = 1, \ldots, d \) denotes a coordinate direction. We recall the definition of the extended corrector \( (\phi_i, \sigma_{ijk}) \) in the following lemma, the (rather standard) proof of which is displayed for the reader’s convenience (see in particular [46, Section 7.2]).

**Lemma 1.** Let \( \langle \cdot \rangle \) be stationary and ergodic. Then there exist two random tensor fields \( \{\phi_i\}_{i=1,\ldots,d} \) and \( \{\sigma_{ijk}\}_{i,j,k=1,\ldots,d} \) with the following properties: The gradient fields \( \nabla \phi_i \) and \( \nabla \sigma_{ijk} \) are stationary, by which we understand that for \( \langle \cdot \rangle \)-a.e. \( a \) and any shift vector \( z \in \mathbb{R}^d \) we have \( \nabla \phi_i(a(\cdot + z)) = \nabla \phi_i(a(\cdot + z) ; \cdot) \) and \( \nabla \sigma_{ijk}(a(\cdot + z)) = \nabla \sigma_{ijk}(a(\cdot + z) ; \cdot) \) a.e. in \( \mathbb{R}^d \), and have bounded second
moments and vanishing expectations:

\[
\langle |\nabla \phi_i|^2 \rangle \leq \frac{1}{\lambda^2}, \quad \sum_{j,k=1,\ldots,d} \langle |\nabla \sigma_{ijk}|^2 \rangle \leq 4d \left( \frac{1}{\lambda^2} + 1 \right),
\]

\[
\langle \nabla \phi_i \rangle = \langle \nabla \sigma_{ijk} \rangle = 0.
\]

Moreover, the field \(\sigma\) is skew-symmetric in its last indices, that is,

\[
\sigma_{ijk} = -\sigma_{ikj}.
\]

Finally, we have for \(\langle \cdot \rangle\)-a.e. \(a\) the equations

\[
-\nabla \cdot a(\nabla \phi_i + e_i) = 0,
\]

\[
\nabla \cdot \sigma_i = q_i,
\]

\[
-\Delta \sigma_{ijk} = \partial_j q_{ik} - \partial_k q_{ij},
\]

in the distributional sense on \(\mathbb{R}^d\) with \(\{q_{ij}\}_{i,j=1,\ldots,d}\) given by

\[
q_i := a(\nabla \phi_i + e_i) - a_{\text{hom}} e_i, \quad a_{\text{hom}} e_i := \langle a(\nabla \phi_i + e_i) \rangle,
\]

where the (distributional) divergence of a tensor field is defined as \( (\nabla \cdot \sigma)_j := \sum_{k=1}^d \partial_k \sigma_{ijk} \). □

In the rest of this paper we use the abbreviations \(\phi = (\phi_1, \ldots, \phi_d)\), \(\sigma = (\sigma_{ijk})_{i,j,k=1,\ldots,d}\), and \(\phi_\xi := \sum_{i=1}^d \xi_i \phi_i\) for \(\xi \in \mathbb{R}^d\).

### 2.2. Regularity theory and the minimal radius

In the Euclidean context, the \(C^{1,\alpha}\)-seminorm of a function measures its local deviation from linear functions. As is customary in the \(C^{1,\alpha}\)-theory based on energy estimates, that deviation is measured in the \(L^2\)-sense at the level of gradients, giving rise to Campanato spaces that are equivalent to Hölder spaces. We name this expression “excess”, cf. (11), in (linear) analogy to the quantity in the regularity theory for minimal surfaces introduced by De Giorgi, [24, Teorema 3.3]. In the context of homogenization, it is natural to replace the space of linear functions (which is \(d\)-dimensional once one factors out constants) by the \(d\)-dimensional set of harmonic coordinates, that is, \(\{x \mapsto \xi \cdot x + \phi_\xi(x)\}_{\xi \in \mathbb{R}^d}\). We therefore define for any square-integrable vector field \(g\) and ball \(B\) the excess as

\[
\text{Exc}(g; B) := \inf_{\xi \in \mathbb{R}^d} \int_B |g - (\xi + \nabla \phi_\xi)|^2,
\]

which measures the deviation of \(g\) from \(a\)-linear functions on \(B\). The theorem below shows that for an \(a\)-harmonic function \(u\), \(\text{Exc}(\nabla u; B)\) can be controlled provided the corrector is well-behaved in the sense that \((\phi, \sigma)\) has sufficiently small linear growth in a spatially averaged, but quantitative, sense. To make this precise, we associate to a given constant \(C > 0\) a random variable \(r^*_a\) defined by the expression

\[
r^*_a := \inf \left\{ r > 0 \mid \forall R \geq r : \frac{1}{R^2} \int_{BR} |(\phi, \sigma) - \int_{BR} (\phi, \sigma)|^2 \leq \frac{1}{C} \right\},
\]

where
with the understanding that \( r_* = \infty \) if the set is empty. Since the extended corrector \((\phi, \sigma)\) exists in all directions \( \xi \in \mathbb{R}^d \) almost surely for ergodic coefficients, we implicitly assume above and in the sequel (and in particular for deterministic estimates) that \( a \) belongs to the set of full measure of coefficients for which the extended corrector, or at least its stationary gradient on which \( r_* \) only depends, is well-defined.

In the rest of the article, we write \( C \) for a generic positive constant that may change from line to line in the statements and in the proofs (unless otherwise stated), and display its dependence upon the parameters of the problem in the form of \( C = C(\cdot) \) (e.g. \( C = C(d, \lambda) \) if \( C \) only depends on the dimension \( d \) and the ellipticity ratio \( \lambda \)).

**Theorem 1.** For any Hölder exponent \( \alpha \in (0, 1) \) there exists a constant \( C(d, \lambda, \alpha) < \infty \) with the following properties: Let \( r_* \) (the “minimal radius associated with \( \alpha \)”) be defined by (12) with \( C = C(d, \lambda, \alpha) \). Let \( u \in H^1(B_R) \) with \( R \geq r_* \) denote an \( a \)-harmonic function in \( B_R \), that is,

\[
-\nabla \cdot a \nabla u = 0 \quad \text{in} \quad B_R. \tag{13}
\]

Then we have “excess decay” in the sense of

\[
\forall r \in [r_*, R], \quad \text{Exc}(\nabla u; B_r) \leq C(d, \lambda, \alpha) \left( \frac{r}{R} \right)^{2\alpha} \text{Exc}(\nabla u; B_R). \tag{14}
\]

Moreover the correctors enjoy the following non-degeneracy property

\[
\forall r \geq r_*, \forall \xi \in \mathbb{R}^d, \quad \frac{1}{2} |\xi|^2 \leq \int_{B_r} |\xi + \nabla \phi_\xi|^2 \leq C(d, \lambda, \alpha) |\xi|^2. \tag{15}
\]

Finally, we have the mean-value property (for which \( \alpha > 0 \) can be fixed, say, \( \alpha = \frac{1}{2} \))

\[
\forall r \in [r_*, R], \quad \int_{B_r} |\nabla u|^2 \leq C(d, \lambda) \int_{B_R} |\nabla u|^2. \tag{16}
\]

The regularity result provided by Theorem 1 is “quenched”, that is, entirely deterministic in the sense that the smallness condition is expressed in terms of the given “realization” of \((\phi, \sigma)\). (In case of thermal randomness, one would speak of a “pathwise result”.) On the one hand, qualitative ergodicity implies that for almost every coefficient field the “minimal radius” \( r_* \) is finite, see (105) in the proof of Corollary 1. On the other hand, mild quantitative ergodicity conditions imply that \( r_* \) is a random variable with stretched exponential moments. The latter is the content of our second main result, see Theorem 2 below.

**Remark 1.** Let us compare Theorems 1 and 2 (see below) to the results of Armstrong & Smart in [5]. The random field \( Y \) of [5, Theorem 1.2], which plays a similar role as \( r_* \) in terms of “Lipschitz-regularity” (cf. (16) in Theorem 1), is defined there as the smallest radius from which on an algebraic decay (arbitrarily small, yet fixed) holds, cf. [5, Lemma 5.1]. First, our weaker quantitative smallness condition (12) allows one to treat the borderline case
of qualitative ergodicity (for which no algebraic decay is available in general) and to avoid any loss of stochastic integrability. In particular, for coefficients that satisfy a standard Logarithmic Sobolev Inequality, [5, Theorem 1.2] would essentially take the form \( \langle \exp(\mathcal{Y}(\varepsilon)) \rangle < \infty \) for any \( \varepsilon > 0 \) whereas Theorem 2 yields \( \langle \exp(\frac{C}{\varepsilon}r^d) \rangle \leq 2 \) for some \( 0 < C < \infty \) large enough. Second, [5, Theorem 1.2] is not formulated using harmonic coordinates, and the estimate that essentially corresponds to (14) only holds on “mesoscales” and for exponents \( 0 \leq \alpha < \beta \) (where \( \beta \) depends on \( \lambda \) and \( d \)), that is typically for \( r \in [R^{\alpha}, R] \) for some \( 0 < c < 1 \) instead of \([r_*, R] \).

A fairly easy consequence of Theorem 1 in form of (14) is the Liouville property for subquadratic functions stated in Corollary 1. This partially answers to the affirmative a specific version of a question raised in [20, Question 5, p.33] on whether the dimension of the space of a-harmonic functions of a given growth exponent agrees with the dimension in the Euclidean case. The answer is partial, because only subquadratic growth is treated, and deals with a special case, because only the case of uniformly elliptic coefficient fields is treated. In the even more special case of periodic coefficient fields the answer is affirmative for all growth rates [10]. Our qualitative result holds, as it should, under the purely qualitative condition of ergodicity.

**Corollary 1.** Let \( \langle \cdot \rangle \) be stationary and ergodic. Then for \( \langle \cdot \rangle \)-a.e. coefficient field \( a \), the following Liouville property holds: Suppose that \( u \) is a-harmonic, that is \( -\nabla \cdot a \nabla u = 0 \) in all \( \mathbb{R}^d \), and that it grows subquadratically in the sense that there exists an exponent \( \alpha < 1 \) such that

\[
\lim_{R \uparrow \infty} R^{-2(1+\alpha)} \int_{B_R} u^2 = 0. \tag{17}
\]

Then \( u \) is a-linear in the sense that there exists \( (c, \xi) \in \mathbb{R} \times \mathbb{R}^d \) such that

\[
u(x) = c + \xi \cdot x + \phi_\xi(x) \quad \text{for Lebesgue-a. e. } x \in \mathbb{R}^d. \tag{18}
\]

The next corollary establishes a \( C^{1,1} \)-a priori estimate for a-harmonic functions similar to the one for plain harmonic functions. There are two restrictions: As expected from Theorem 1, such an estimate only holds on scales that are large with respect to the minimal radius \( r_* \), see (19). Moreover, it only holds for an effective gradient which is the projection of the microscopic gradient onto a-linear functions, a projection localized at the level of the minimal radius \( r_* \), cf. (20).

**Corollary 2.** Let \( \alpha \in (0, 1) \) be given and let \( r_* \) denote the associated minimal radius (see Theorem 1). We denote by \( r_*(a, x) := r_*(a(\cdot + x)) \) the stationary extension of the minimal radius, so that

\[
\forall x \in \mathbb{R}^d, \forall r \geq r_*(x) : \frac{1}{r^2} \int_{B_r(x)} |(\phi, \sigma) - \int_{B_r(x)} (\phi, \sigma)|^2 \leq \frac{1}{C(d, \lambda, \alpha)}. \tag{19}
\]
For any $a$-harmonic function $u$ in a ball $B_R$, cf. (13), consider the vectors $\xi_+$ and $\xi_-$ characterized by
\[ \int_{B_{r^*}(\pm x)} |\nabla u - (\xi_+ + \nabla \phi_{\xi_+})|^2 = \text{Exc}(\nabla u; B_{r^*}(\pm x)), \]
which we think of as the effective gradient of $u$ in $x$ and $-x$ at scale $r^*$, respectively. Then we have, provided $R \geq 8 \max\{|x|, r^*(\pm x)|$,
\[ |\xi_+ - \xi_-|^2 \leq C(d, \lambda, \alpha) \left( \frac{\max\{|x|, r^*(\pm x)|}{R} \right)^2 \text{Exc}(\nabla u; B_R). \]
\[ \square \]

Another application of Theorem 1 are intrinsic Schauder estimates for elliptic systems in divergence form.

**Corollary 3 (Large-scale Schauder estimates).** Let a Hölder exponent $\alpha \in (0, 1)$ be given and let $r^*$ denote the minimal radius (12) associated with the constant $C = C(d, \lambda, \alpha')$ of Theorem 1 for some fixed $\alpha' \in (\alpha, 1)$. Below the notation $\lesssim$ stands for $\leq$ for a generic multiplicative constant $C$ that only depends on $d$, $\lambda$, and $\alpha$. Let the function $u$ (with square-integrable gradient) and the (square-integrable) vector fields $g$ and $h$ on $B_R$ be related by
\[ -\nabla \cdot a(\nabla u + g) = \nabla \cdot h \quad \text{in } B_R. \]

Then we have
\[ \sup_{r \in [r^*, R]} \frac{1}{r^{2\alpha}} \text{Exc}(\nabla u + g; B_r) \lesssim \frac{1}{R^{2\alpha}} \text{Exc}(\nabla u + g; B_R) \]
\[ + \sup_{r \in [r^*, R]} \frac{1}{r^{2\alpha}} \int_{B_r} (|g - \int_{B_r} g|^2 + |h - \int_{B_r} h|^2). \]

If $R = +\infty$, we obtain
\[ \sup_{r \geq r^*} \frac{1}{r^{2\alpha}} \text{Exc}(\nabla u + g; B_r) \lesssim \sup_{r \geq r^*} \frac{1}{r^{2\alpha}} \int_{B_r} (|g - \int_{B_r} g|^2 + |h - \int_{B_r} h|^2) \]
\[ (\text{with the understanding that } \nabla u + g \text{ is square-integrable on } \mathbb{R}^d). \]

For later purpose, we also state the following extension of (16)
\[ \sup_{r \in [r^*, R]} \int_{B_r} (|\nabla u + g|^2 \lesssim \int_{B_R} (|\nabla u + g|^2 + \sup_{r \in [r^*, R]} \left( \frac{R}{r} \right)^{2\alpha} \int_{B_r} (|g - \int_{B_r} g|^2 + |h - \int_{B_r} h|^2). \]
\[ \square \]

A last application of Theorem 1 are the following large-scale Calderón-Zygmund estimates, which are an improved form of a result proved in the first version of [27].
Corollary 4 (Large-scale Calderón-Zygmund estimates).

Let $C_0 = C(d, \lambda, \frac{1}{2}) < \infty$ be the constant associated with $\alpha = \frac{1}{2}$ in Theorem 1. There exists a $\frac{1}{8}$-Lipschitz stationary field $r_\ast$ that satisfies $r_\ast(C_0) \leq r_\ast \leq r_\ast(3^{d+2}C_0)$ (where $r_\ast(C)$ is the minimal radius associated with the constant $C$ as defined in (12)) and such that for any suitably decaying scalar field $u$ and vector field $g$ related in $\mathbb{R}^d$ by

$$-\nabla \cdot a \nabla u = \nabla \cdot g, \quad (26)$$

and any exponent $1 < p < \infty$, we have

$$\left( \int \left( \int_{B_\ast(x)} |\nabla u|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}} \lesssim \left( \int \left( \int_{B_\ast(x)} |g|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}, \quad (27)$$

where $B_\ast(x) := B_{r_\ast}(x)$, and $\lesssim$ means $\leq C$ for a constant only depending on $d$, $\lambda$, $p$. □

There is nothing particular in the definition of $C_0 = C(d, \lambda, \alpha)$, and we may choose $C_0 = C(d, \lambda, \alpha)$ for any $\alpha > 0$ so that large-scale $C^{1,\alpha}$-regularity also holds on scales $R \geq r_\ast$ (which we use in the proof of Corollary 4 only in the form of the mean-value property). Note that a weaker (and unweighted) version of Corollary 4 is proved in [3]: However, there is a loss in the exponent in the LHS (and only $p \geq 2$ is addressed).

For applications in [40] and [28], we shall need weighted versions of these large-scale Calderón-Zygmund estimates — which we prove for a restricted class of weights (although for these specific weights, the condition $0 \leq \gamma < d(p-1)$ below is equivalent to the condition $\omega \in A_p$, the correct Muckenhoupt class, cf. [63, Chapter V, Section 3]).

Corollary 5 (Large-scale weighted Calderón-Zygmund estimates).

Let $2 \leq p < \infty$, $\gamma < d(p-1)$, and $\omega = \omega(r) > 0$ satisfy

$$\omega(r) \leq \omega(R) \leq (\frac{R}{r})^\gamma \omega(r) \quad \text{for all } r \leq R. \quad (28)$$

In the notation of Corollary 4,

$$\left( \int \omega(|x| + r_\ast(0)) \left( \int_{B_\ast(x)} |\nabla u|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}} \lesssim \left( \int \omega(|x| + r_\ast(0)) \left( \int_{B_\ast(x)} |g|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}, \quad (29)$$

where $\lesssim$ means $\leq C$ with $C$ only depending on $d$, $\lambda$, $p$, and $\gamma$. □

Loosely speaking, Corollaries 3 & 4 state that from the minimal radius $r_\ast$ onwards, one is in the regime of $C^{1,\alpha}$- and $L^p$-maximal regularity, respectively. For higher-order regularity, we refer the reader to the subsequent work [30, 31] by Fischer and the third author.
Remark 2. At the price of including the adjoint of the extended corrector (that is, the extended corrector associated with the pointwise transpose coefficient field \( a^* \)) in the definition of \( r_* \), one may w.l.o.g. assume that the above regularity theory holds for both the operators \(-\nabla \cdot a\nabla\) and \(-\nabla \cdot a^*\nabla\). The estimates of \( r_* \) obtained in this article remain unchanged since pointwise transposition is a local linear operation that does not change statistical properties. This will be used in the proofs of (27) (for the range \( 1 < p \leq 2 \)) and Corollary 5 for which we argue by duality. □

We close this section by stating the main ingredient for Theorem 1; namely Proposition 1. It relies on the following observation: Harmonic functions \( u \) have the property that for all radii \( r \leq R \) there exists a \( \xi \in \mathbb{R}^d \) (in fact, \( \xi = \nabla u(0) \) or \( \xi = \nabla u \) will do) such that

\[
\int_{B_r} |\nabla u - \xi|^2 \leq C(d)(\frac{r}{R})^2 \int_{B_R} |\nabla u|^2. \tag{30}
\]

Proposition 1 establishes a perturbation of (30) for \( a \)-harmonic functions, provided the affine function \( x \mapsto \xi \cdot x \) is replaced by its \( a \)-harmonic version \( x \mapsto \xi \cdot x + \phi_\xi(x) \), and where the perturbation is controlled by the amount of linear growth of the corrector \((\phi, \sigma)\).

**Proposition 1.** There exists an exponent \( \varepsilon = \varepsilon(d, \lambda) > 0 \) with the following property. Let \( u \) be an \( a \)-harmonic function in a ball \( B_R \) with radius \( R > 0 \), cf. (13). Then for all \( r \leq R \), there exists a vector \( \xi \in \mathbb{R}^d \) such that

\[
\int_{B_r} |\nabla u - (\xi + \nabla \phi_\xi)|^2 \leq C(d, \lambda) \left( \left( \frac{r}{R} \right)^2 + \delta^2 \left( \frac{R}{r} \right)^{d+2} \right) \int_{B_R} |\nabla u|^2, \tag{31}
\]

where we have set for abbreviation

\[
\delta := \frac{1}{R} \left( \int_{B_R} |(\phi, \sigma) - \int_{B_R} (\phi, \sigma)|^2 \right)^{\frac{1}{2}}. \tag{32}
\]

Moreover, we have the following non-degeneracy property

\[
\frac{1}{C(d, \lambda)}(1 - C(d, \delta))|\xi| \leq \left( \int_{B_{\frac{R}{2}}} |\xi + \nabla \phi_\xi|^2 \right)^{\frac{1}{2}} \leq C(d, \lambda)(1 + \delta)|\xi| \tag{33}
\]

for all \( \xi \in \mathbb{R}^d \). □

**Remark 3.** Theorem 1 and its main ingredient, Proposition 1, should be compared to the work of Avellaneda & Lin, more precisely, to [9, Section 3.1]: Like in (31), the distance between \( \nabla u \) and \( \xi + \nabla \phi_\xi \) for a suitable \( \xi \) (there, it is given by the spatial average of \( \nabla u \)) is monitored — however, on an \( L^\infty \) instead of an \( H^1 \)-level, see [9, Lemma 14], which is the analogue of Proposition 1. Like for Proposition 1, [9, Lemma 14] is a perturbation of an estimate for the (constant-coefficients) homogenized operator. In fact, [9, Lemma 14] does not use periodicity in an explicit way, but only H-convergence of the elliptic operator \(-\nabla \cdot a\nabla\) (see [54, 55]), in its scaled-down version, to the homogenized limit \(-\nabla \cdot a_{\text{hom}}\nabla\). More precisely, it uses an upgraded version
of H-convergence, where the solutions converge in $L^\infty$, an upgrade which in case of scalar equations may be obtained by appealing to the uniform Hölder regularity of $a$-harmonic functions (De Giorgi’s result) and which in [9, Section 2.2] is obtained in the system’s case by first deriving a $C^{0,\alpha}$-estimate by a similar strategy to the $C^{1,\alpha}$-estimate. Incidentally, [9, Lemma 14] also uses implicitly the sublinear growth of the corrector $\phi$. The main new ingredient in Proposition 1 is a quantification of H-convergence (which is a purely qualitative concept) in terms of the sublinear growth of $\phi$ and $\sigma$. This also requires a suitable cut-off argument since we want to use the whole-space corrector $(\phi, \sigma)$ and thus need to introduce a boundary layer. The passage from Proposition 1 to Theorem 1 mimics the passage from [9, Lemma 14] to [9, Lemma 15]. Note that, in contrast to our work, [9] assumes smoothness of $a$ which helps handle the small scales.

In view of Theorem 1 and Corollaries 3, 4, and 5, it is clearly of interest to control the size of the stationary random field $r_*$, which is almost surely finite under mere ergodicity of the coefficients, cf. the proof of Corollary 1. In order to obtain a quantitative control, one needs to make quantitative assumptions.

2.3. Control of the minimal radius

In this section $\alpha \in (0,1)$ is fixed and we denote by $r_*$ the associated minimal radius (see Theorem 1), so that the mean-value property (16) for gradients of $a$-harmonic functions $u$ on $B_R$ holds for balls centered at the origin and of radius larger than $r_*$:

$$\forall r \in [r_*, R], \int_{B_r} |\nabla u|^2 \leq C(d, \lambda, \alpha) \int_{B_R} |\nabla u|^2.$$

(34)

We shall also use the $C^{1,\alpha}$-Schauder estimate from Corollary 3.

We consider two extreme situations on the statistics of $\langle \cdot \rangle$:

- Strong decay of correlation that leads to the best integrability of $r_*$ one can expect, cf. Theorem 2;
- Arbitrarily slow decay of correlation that leads to weaker (typically stretched exponential or algebraic) integrability of $r_*$, cf. Theorems 3 and 4.

These results are proved using a mixing condition in the form of functional inequalities, which ensure strong nonlinear concentration properties (typically stronger than other more standard mixing conditions). We split the rest of this section into two parts. In the first part we specialize to standard functional inequalities, state Theorem 2, and describe the general structure of the proof in that setting. In the second part, we address more general fields based on multiscale functional inequalities (as first introduced using non-uniform partitions coarsening away from the origin in the first version of this article, and recently reformulated and extensively studied in the form of the multiscale inequalities we use here in [25, 26]), which allows us to treat most coefficient fields considered in the applied sciences (cf. e.g. [65]). The structure of the proof is similar for Theorems 2, 3, and 4. As a general rule,
in the actual proofs of these results, we first focus on standard functional inequalities so that the core argument appears as clear as possible, and only later on address the multiscale case.

2.3.1. Control of the minimal radius using standard functional inequalities.

By definition, controlling the minimal radius $r_*$ means controlling the sublinear growth of the corrector $(\phi, \sigma)$. The sublinear growth of the corrector (a key element to most homogenization results) is a result of the cancellations coming from $\langle \nabla (\phi, \sigma) \rangle = 0$, which due to ergodicity translates into $\lim_{r \to \infty} \int_{B_r} \nabla (\phi, \sigma) = 0$, cf. the proof of Corollary 1. The quantification of this relies on two distinct ingredients:

- On the one hand, one needs good locality properties of $\frac{1}{r^2} \int_{B_r} |(\phi, \sigma) - \langle \phi, \sigma \rangle|^2$, and thus of $(\phi, \sigma)$. By this it is meant that the solution $(\phi, \sigma)$ of the elliptic system (7) & (9) at some point $x$ depends only weakly on the coefficient field $a$ far away from that point. To establish this locality, we shall use the modified extended corrector $(\phi_T, \sigma_T)$, cf. (37)–(39) below, and relate the sublinear growth of $(\phi, \sigma)$ to that of $(\phi_T, \sigma_T)$, cf. Proposition 2.

- On the other hand, one needs good mixing properties of the ensemble $\langle \cdot \rangle$ of random coefficient fields $a$. By this it is meant that the random value of $a$ at some point $x$ statistically depends only weakly on its values far away. For that purpose we appeal to the multiscale functional inequalities (MFI) introduced by Duerinckx and the first author in [25].

We start by recalling the standard logarithmic Sobolev inequality, cf. [51, 52]. In what follows, sup is a shorthand notation for the essential supremum.

**Definition 1.** [25, Section 2] For all $\ell \geq 0$ and $x \in \mathbb{R}^d$, denote by $B_{\ell+1}(x)$ the ball centered at $x \in \mathbb{R}^d$ and of radius $\ell + 1$. We consider two types of derivative for a function $F$ on the space of coefficient fields $a$:

- $|\partial^\text{fct}_{x, \ell+1} F|$ denotes the $L^1(B_{\ell+1}(x))$-norm of the functional (or Malliavin) derivative of $F$ with respect to $a$, that is
  $$|\partial^\text{fct}_{x, \ell+1} F| := \int_{B_{\ell+1}(x)} |\frac{\partial F}{\partial a(z)}|dz.$$  

- $|\partial^\text{osc}_{x, \ell+1} F|$ denotes the (essential) oscillation\footnote{To make this quantity measurable, the supremum and infimum in the oscillation have to be slightly modified, see [25] based on [11]. This is not essential in this article since we shall bound such oscillations by measurable quantities.} of $F$ with respect to the restriction of $a$ on $B_{\ell+1}(x)$, that is,
  $$|\partial^\text{osc}_{x, \ell+1} F(a)| := \sup \{ F(a') - F(a'') : a' = a'' = a \text{ in } \mathbb{R}^d \setminus B_{\ell+1}(x) \}.$$
We say that \( \langle \cdot \rangle \) satisfies a standard Logarithmic Sobolev Inequality (LSI) if there exists \( \kappa > 0 \) such that for all random variables \( F \) we have
\[
\forall L \geq 1, \quad \text{Ent}_L(F) := \langle F^2 \log F^2 \rangle_L - \langle F^2 \rangle_L \langle \log F^2 \rangle_L \leq \frac{1}{\kappa} \int_{\mathbb{R}^d} |\partial_{x,1}^\text{fct/osc} F|^2 dx_L,
\]
where \( \langle \cdot \rangle_L \) denotes the ensemble scaled by \( L \), i.e. for rescaled coefficient fields \( a(Lr) \). □

**Remark 4.** Imposing (35) for all \( L \geq 1 \) (instead of just \( L = 1 \)) is not restrictive: It is automatically met in the constructive approach of [26], and essentially follows from the fact that mixing properties improve under such rescaling — see [26, Remark 2.2] for details. □

Under assumption (35), we have the following result on the integrability of \( r_* \).

**Theorem 2.** Assume that \( \langle \cdot \rangle \) is stationary and satisfies the standard Logarithmic Sobolev Inequality (35). For all \( \alpha \in (0, 1) \), consider \( r_* \) defined in (12) with constant \( C(d, \lambda, \alpha) \). Then there exists a positive constant \( C = C(d, \lambda, \kappa, \alpha) \) such that
\[
\langle \exp \left( \frac{1}{C} r_*^d \right) \rangle \leq 2.
\]
(36) □

The following remark shows that (36) is the best stochastic integrability one can hope for.

**Remark 5.** Consider the case of discrete stationarity in the form of a Bernoulli random checkerboard and a scalar equation. On a square of side length \( R \), the probability to approximate the coefficients the classical counter example to (large-scale) De Giorgi’s regularity (e.g. the quasi-conformal mapping of [35, Section 12.1] in dimension 2) is at least \( \left( \frac{1}{2} \right)^{R^d} \), in which case \( r_* \) has to be larger than \( R \). As already observed by Armstrong and Smart in [5], this directly implies that \( \langle \exp \left( \frac{1}{C} g(r_*^d) \right) \rangle = \infty \) if \( g \geq 0 \) is such that
\[
\liminf_{r \to \infty} \frac{g(r)}{r^d} = \infty.\]
On the other hand, the fact that (36) holds owes to the large constant \( 0 < C < \infty \). Indeed, this constant quantifies the fraction of the subset of coefficients that do not satisfy the Lipschitz regularity at size \( R \), which is best seen by writing (36) in form of \( \langle I(r_* \geq R) \rangle \leq 2 \exp(-\frac{1}{C} R^d) \). In the case of the random checkerboard, this quantifies the ratio between the number of “bad” coefficients on a square of size \( R \) and the total number \( 2^{R^d} \) of realizations: It does not exceed \( 2 \left( \exp(-\frac{1}{C}) \right)^{R^d} \). □

Let us now describe the main ingredients to the proof of Theorem 2, which relies on the one hand on five deterministic results, and on the other hand on (nonlinear) concentration properties. We start with the five deterministic ingredients to the proof of Theorem 2, namely
• Proposition 2, which states that the sublinear growth (in a locally square-averaged form) of the extended corrector is controlled by the sublinear growth of the \textit{modified} extended corrector \((\phi_T, \sigma_T)\) introduced below;

• Proposition 3, which shows that the modified extended corrector can be controlled by the \textit{average} of the (squared) \(H^{-1}\)-norm of field and flux of a \textit{more localized} modified extended corrector \((\phi_t, \sigma_t)\);

• Proposition 4, which shows that this squared \(H^{-1}\)-norm is indeed approximately local – a property that we use to control fluctuations of its spatial averages by concentration of measure (cf. Lemma 4);

• Lemma 2, which allows to control the \(H^{-1}\)-norm of field and flux by their spatial averages, on the level of second stochastic moments;

• Lemma 3, which establishes a deterministic sensitivity estimate for these spatial averages.

The modified extended corrector \((\phi_T, \sigma_T)\) associated with a fixed direction \(e\) and a cut-off scale \(\sqrt{T} \geq 1\) is defined as the unique solution to

\[
\frac{1}{T} \phi_T - \nabla \cdot a(\nabla \phi_T + e) = 0, \quad (37)
\]

\[
q_T := a(\nabla \phi_T + e), \quad (38)
\]

\[
\frac{1}{T} \sigma_T - \Delta \sigma_T = \nabla \times q_T, \quad (39)
\]

(in the distributional sense on \(\mathbb{R}^d\)) in the class

\[
\{ v \in H^1_{\text{loc}}(\mathbb{R}^d) : \sup_{x \in \mathbb{R}^d} \int_{B_1(x)} |v|^2 + |\nabla v|^2 < \infty \},
\]

where \(B_1(x)\) denotes the unit ball centered at \(x\). In (39) and in the rest of this paper, we use the notation \(\nabla \times\) inspired by the case of \(d = 3\) for the differential operator in the RHS of (9). Note that the “massive” term \(\frac{1}{T} \phi_T\) introduces a length scale \(R = \sqrt{T}\) that plays a role in the sequel. These whole-space problems are well-posed for all coefficient fields on a purely deterministic level (see e.g. [43, Appendix A.1]). Moreover, as it is well-known in homogenization, if \(\langle \cdot \rangle\) is stationary and ergodic, then \(\langle \cdot \rangle\)-almost surely \((\nabla \phi, \nabla \sigma)\) can be recovered as the weak limit as \(T \uparrow \infty\) in \(L^2_{\text{loc}}(\mathbb{R}^d)\) of \((\nabla \phi_T, \nabla \sigma_T)\), while \(\frac{1}{T} \phi_T\) and \(\frac{1}{T} \sigma_T\) converge to zero in \(L^2_{\text{loc}}(\mathbb{R}^d)\) (see for instance Step 1 in the proof of Proposition 2).

Proposition 2 below states that the sublinear growth of the extended corrector \((\phi, \sigma)\) and thus \(r_\ast\) is controlled by the sublinear growth of the modified extended corrector \(\{(\phi_T, \sigma_T)\}_T\), provided both are slightly quantified.

**Proposition 2.** Suppose that for some exponent \(\nu > 0\) and radius \(r_{**}\), we have for all dyadic \(R = 2^k\)

\[
\int_{B_R} \frac{1}{T} \| (\phi_T, \sigma_T) \|^2 \leq \left( \frac{r_{**}}{R} \right)^{2\nu} \quad \text{for all } R \geq r_{**} \text{ and } T = R^2. \quad (40)
\]
Then there exists a constant $C = C(d, \lambda, \alpha, \nu)$ such that
\[
\frac{1}{R^2} \int_{B_R} |(\phi, \sigma) - \int_{B_R} (\phi, \sigma)|^2 \leq C\left(\frac{R^{**}}{R}\right)^{2\nu} \quad \text{for all } R \geq r_{**}; \tag{41}
\]
furthermore,
\[
r_* \leq Cr_{**}. \tag{42}
\]

The following Proposition 3 relates the sublinear growth of the modified extended corrector $(\phi_T, \sigma_T)$ to the smallness of a negative norm of the corresponding field $\nabla \phi_T$ and flux $q_T = a(\nabla \phi_T + e)$. More precisely, the negative norm monitors the fluctuations $q_T - \langle q_T \rangle$ (note that $\nabla \phi_T - \langle \nabla \phi_T \rangle = \nabla \phi_T$). Whereas the (homogeneous) $H^{-1}$-norm of $\nabla \phi_T$ is conveniently given by the $L^2$-norm of $\phi_T$ itself, for the negative norm of $q_T - \langle q_T \rangle$ we introduce the vector field
\[
\frac{1}{T} g_T - \Delta g_T = \frac{1}{\sqrt{T}}(q_T - \langle q_T \rangle) \tag{43}
\]
and take the $L^2$-norm of $(g_T, \sqrt{T}\nabla g_T)$ as a version of the $H^{-1}$-norm of $q_T - \langle q_T \rangle$ with a cut-off for scales $\gtrsim \sqrt{T}$ (the latter being important for the locality property). The normalization in (43) is chosen such that $g_T$ and $\phi_T$ live on the same footing. The point of Proposition 3 is that the sublinear growth of $(\phi_T, \sigma_T)$ is controlled by negative norms of $(\nabla \phi_T, q_T - \langle q_T \rangle)$ for any $t \leq T$. It is convenient to introduce a scale of exponential averaging functions
\[
\omega_T(x) := \frac{1}{|\partial B_1|^{(d-1)!}} \frac{1}{(C\sqrt{T})^d} \exp\left(-\frac{|x|}{C\sqrt{T}}\right), \tag{44}
\]
with a constant $C = C(d, \lambda)$ chosen such that the localized energy estimate for $\frac{1}{T} - \nabla \cdot a\nabla$ holds, see (169) below — for some estimates, we shall further need to increase the constant $C$ without changing notation.

**Proposition 3.** For all $0 < t \leq T$ we have
\[
\int \omega_T \frac{1}{T} |(\phi_T, \sigma_T)|^2 \leq C(d, \lambda) \int \omega_T \left(\frac{1}{t} \phi_t^2 + \frac{1}{t} |g_t|^2 + |\nabla g_t|^2\right). \tag{45}
\]

Proposition 4 establishes the locality of the RHS integrand $F_t$ in (45). More precisely, it considers local averages $F_t$ on scale $\sqrt{t}$ of the integrand, cf. (46). By locality of such a random variable, i.e. a function(al) $F_t = F_t(a)$ of the coefficient field $a$, we understand that it essentially does not depend on $a$ at distances $\gg \sqrt{t}$. Proposition 4 establishes that this is true up to an exponential error, see (48).

**Proposition 4.** For all $t > 0$ consider the function $F_t = F_t(a)$ given by
\[
F_t := \int \omega_t \left(\frac{1}{t} \phi_t^2 + \frac{1}{t} |g_t|^2 + |\nabla g_t|^2\right). \tag{46}
\]
Then for every $\lambda$-uniformly elliptic coefficient field $a$ we have

$$|F_t(a)| \leq C(d, \lambda),$$

and $F_t$ is approximately $\sqrt{t}$-local in the sense that for all balls $B_R$ and all $\lambda$-uniformly elliptic coefficient fields $a, a'$ we have

$$a = a' \text{ in } B_R \Rightarrow |F_t(a) - F_t(a')| \leq C(d, \lambda) \exp\left(-\frac{1}{C(d, \lambda)} \frac{R}{\sqrt{t}}\right).$$

□

It remains to provide the deterministic elements for the estimate of the expectation of $F_t$ defined in (46), which by stationarity is given by $\langle \frac{1}{T}\phi_T^2 + \frac{1}{T}|g_T|^2 + |\nabla g_T|^2 \rangle$. The following lemma shows that this truncated version of the $H^{-1}$-norm of the field/flux pair $(\nabla \phi_T, q_T - \langle q_T \rangle)$ can be estimated by spatial averages of $(\nabla \phi_T, q_T - \langle q_T \rangle)$, where for a field $f$ we denote by $f_\ast$ the convolution by a Gaussian of variance $t$ (and thus length-scale $\sqrt{t}$). Because of the semi-group property, it is indeed convenient to take spatial averages by convolving with Gaussians.

**Lemma 2.** For all $T > 0$ we have

$$\langle \frac{1}{T}\phi_T^2 + \frac{1}{T}|g_T|^2 + |\nabla g_T|^2 \rangle \leq C(d) \frac{1}{T} \int_0^T \langle |(\nabla \phi_T)_t|^2 + |(q_T)_t - \langle q_T \rangle|^2 \rangle dt.$$  

□

The next result is a (suboptimal) sensitivity estimate for the RHS of (49).

**Lemma 3.** There exist an exponent $\varepsilon = \varepsilon(d, \lambda) > 0$ (coming from hole-filling) and a constant $C = C(d, \lambda)$ such that for all $1 \leq t \leq T$ we have

$$(\ell + 1)^{-d} \int_{\mathbb{R}^d} \left| \partial^\text{fct/osc}_{x, \ell + 1} (\nabla \phi_T, q_T)_t \right|^2 dx \leq C \left( \frac{\sqrt{T}}{\sqrt{t}} \right)^d (\ell + 1)^\varepsilon d.$$  

□

We conclude with the stochastic arguments. Equipped with Lemmas 2 and 3, we obtain using LSI the following control of the expectation of $F_t$:

**Corollary 6.** Let $\langle \cdot \rangle$ be stationary and satisfy the standard Logarithmic Sobolev Inequality (35). Then there exists a constant $C = C(d, \lambda, \kappa)$ such that for all $T > 0$ we have

$$\langle \frac{1}{T}\phi_T^2 + \frac{1}{T}|g_T|^2 + |\nabla g_T|^2 \rangle \leq CT^{-\varepsilon},$$  

where $\varepsilon = \varepsilon(d, \lambda) > 0$ is defined in Lemma 3. □

We finally recall a result to control fluctuations of random variables that behave like simple averages (cf. [25], see also [51, 52]).
Lemma 4 (Concentration for averages). [25, Proposition 4.3] Assume that $\langle \cdot \rangle$ is stationary and satisfies the standard Logarithmic Sobolev Inequality (35). Let $t \geq 1$ and let $F_t$ denote a bounded random variable that is approximately $\sqrt{t}$-local in the sense of (47) and (48). Then there exists a positive constant $C = C(d, \lambda, \kappa)$ such that we have for all $\delta > 0$ and $T \geq t$

\[
\left\langle I \left( \left| \int \omega_T(x)F_t(a(\cdot + x)) \, dx - \langle F_t \rangle \right| > \delta \right) \right\rangle \leq \exp \left( -\frac{\delta^2}{C} \left( \frac{\sqrt{T}}{\sqrt{t}} \right)^d \right). \tag{52}
\]

□

The proof of Theorem 2 is now as follows. By Proposition 2, $r_*$ is controlled by $r_{**}$, a minimal radius based on the modified corrector. Averages of the modified corrector in turn can be controlled by averages of an even more localized quantity, cf. Proposition 3. On the one hand, by Proposition 4, this quantity is local enough to apply Lemma 4 to control the size of its fluctuations. On the other hand, its expectation is controlled by Corollary 6 using Lemma 2. By a union bound argument, this yields moment bounds for $r_{**}$, and therefore for $r_*$. 2.3.2. Control of the minimal radius using multiscale functional inequalities. In this paragraph we extend Theorem 2 to more general situations where the coefficient field is more strongly correlated. We shall prove two results (Theorems 3 and 4) that address the general case of multiscale functional inequalities with functional derivative and oscillation. We start with the definitions of these functional inequalities.

Definition 2. [25, Definition 2.2] Let $\pi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be an integrable function. For $L \geq 1$ we denote by $\langle \cdot \rangle_L$ the scaled ensemble.

- We say that $\langle \cdot \rangle$ satisfies a multiscale Logarithmic Sobolev Inequality (MLSI) for the weight $\pi$ with the functional derivative/oscillation if for all random variables $F$ we have

\[
\forall L \geq 1, \ \text{Ent}_L(F) \leq \left\langle \int_0^\infty \pi(\ell)(\ell + 1)^{-d} \int_{\mathbb{R}^d} |\partial_{x,\ell+1}^{\text{osc}} F|^2 \, dx d\ell \right\rangle_L. \tag{53}
\]

- We say that $\langle \cdot \rangle$ satisfies a multiscale covariance inequality (MCI) for the weight $\pi$ with the oscillation if for all random variables $F, G$ we have

\[
\forall L \geq 1, \ \text{cov}_L [F; G] := \langle (F - \langle F \rangle_L)(G - \langle G \rangle_L) \rangle_L 
\leq \int_0^\infty \pi(\ell)(\ell + 1)^{-d} \int_{\mathbb{R}^d} \left\langle |\partial_{x,\ell+1}^{\text{osc}} F|^2 \right\rangle_L^{1/2} \left\langle |\partial_{x,\ell+1}^{\text{osc}} G|^2 \right\rangle_L^{1/2} \, dx d\ell. \tag{54}
\]

- We say that $\langle \cdot \rangle$ satisfies a multiscale spectral gap (MSG) for the weight $\pi$ with the oscillation if for all random variables $F$ we have

\[
\forall L \geq 1, \ \text{var}_L [F] := \langle F^2 \rangle_L - \langle F \rangle_L^2 
\leq \left\langle \int_0^\infty \pi(\ell)(\ell + 1)^{-d} \int_{\mathbb{R}^d} |\partial_{x,\ell+1}^{\text{osc}} F|^2 \, dx d\ell \right\rangle_L. \tag{55}
\]
Remark 6. Note that both (53) (for the oscillation) and (54) imply (55). If an ensemble satisfies MLSI with a compactly supported weight, then it satisfies the standard LSI (after suitable rescaling of space).

The weight $\pi$ typically is related to the covariance function of $\langle \cdot \rangle$ (see in particular (58) below for details). In order to state our main result on the integrability of $r_*$, we need to introduce some further quantities. Given a weight $\pi$, we consider the antiderivative $\gamma$ of $-\pi$ defined by

$$
\gamma(\ell) := \int_{\ell}^{\infty} \pi(s) \, ds,
$$

and assume that it decays at least algebraically in the sense that there exist $0 < \beta \ll 1$ and $C(\beta)$ (both depending only on $\pi$) such that

$$
\gamma(\ell) \leq C(\beta)(\ell + 1)^{-\beta} \quad \text{and} \quad \liminf_{\ell \to \infty} \frac{\gamma(\ell \theta)}{\gamma(\ell)} \geq \theta^{-\beta} \quad \text{for all } \theta \in (0, 1).
$$

We finally introduce the function

$$
\pi_*(r) := \left( \int_{B_r} \gamma(|x|) \, dx \right)^{-1},
$$

which, as the following result shows, captures the stochastic integrability of $r_*$ for MLSI with functional derivative.

**Theorem 3.** Assume that $\langle \cdot \rangle$ is stationary and satisfies the MLSI (53) for the functional derivative with integrable weight $\pi$, where $\pi$ satisfies (56a) & (56b), and let $\pi_*$ be as in (56c). For all $\alpha \in (0, 1)$, consider $r_*$ defined in (12) with constant $C(d, \lambda, \alpha)$. Then there exists a positive constant $C = C(d, \lambda, \pi, \alpha)$ such that

$$
\langle \exp \left( \frac{1}{C} \pi_*(r_*) \right) \rangle \leq 2.
$$

Before we turn to MFI with the oscillation, let us present the prototypical example of Gaussian coefficient fields. Apply a local pointwise Lipschitz nonlinear transform to possibly several independent copies of a stationary Gaussian scalar field with covariance function $|c(x)| \leq \gamma(|x|)$ where $\gamma$ is non-increasing and decays at least algebraically at infinity. Then, relying on the Brascamp-Lieb inequality, one can prove ([26, Section 3.1]) that the ensemble $\langle \cdot \rangle$ is stationary and satisfies MLSI (53) with weight

$$
\pi(\ell) \sim -\gamma'(\ell).
$$

We then abusively say that $\langle \cdot \rangle$ is Gaussian with correlation function $\gamma$. The application of Theorem 3 to this example takes the following form.
Corollary 7. Let $\langle \cdot \rangle$ be Gaussian with correlation function $\gamma(\ell) = (\ell + 1)^{-\beta}$ for some $\beta > 0$. Then
\[
\pi_*(r) \sim \begin{cases} 
(r+1)^\beta & \beta < d, \\
(r+1)^d \log^{-1}(r+2) & \beta = d, \\
(r+1)^d & \beta > d,
\end{cases}
\tag{59}
\]
and (57) thus takes the form

- subcritical case $\beta > d$: $\langle \exp\left(\frac{1}{C} r_\gamma^d\right)\rangle \leq 2$;
- critical case $\beta = d$: $\langle \exp\left(\frac{1}{C} \frac{r_\gamma^d}{\log(r_\gamma+1)}\right)\rangle \leq 2$;
- supercritical case $0 < \beta < d$: $\langle \exp\left(\frac{1}{C} r_\gamma^\beta\beta^{-d}\right)\rangle \leq 2$,

for some positive constant $C = C(d, \lambda, \beta, \alpha)$. □

This corollary directly follows from Theorem 3 provided we prove (59), which is itself a consequence of the definition (56c) of $\pi_*$ and of the elementary calculation
\[
\int_{B_r} \gamma(|x|) \, dx = r^{-d} \int_0^r (\ell + 1)^{-\beta} \ell^{d-1} d\ell \sim \begin{cases} 
(r+1)^{-\beta} & \beta < d, \\
(r+1)^{-d} \log(r+2) & \beta = d, \\
(r+1)^{-d} & \beta > d.
\end{cases}
\]

Note that a stronger version of Corollary 7 was obtained by Fischer and the third author in [31] in the range $0 < \beta \ll 1$. There, by a more direct use of the Brascamp-Lieb inequality, the same stochastic integrability (up to the optimal iterated logarithm) as in Corollary 7 is established for a minimal radius $r_{\ast\ast}$ that satisfies (41) and (42) for $\nu = 1$.

For MFI with the oscillation, the control of the moments of $r_\ast$ is weaker than for MFI with the functional derivative. In terms of stochastic integrability of $r_\ast$, MLSI yields stronger results than MCI which in turn yields stronger results than MSG (for the same weight).

Theorem 4. Assume that $\langle \cdot \rangle$ is stationary. Let $\pi$ be an integrable weight and let $\gamma$ be as in (56a). For all $\alpha \in (0, 1)$, consider $r_\ast$ defined in (12) with constant $C(d, \lambda, \pi, \alpha)$. Then there exists a positive constant $C = C(d, \lambda, \pi, \alpha)$ such that

- if $\gamma$ has algebraic decay at infinity, and $\langle \cdot \rangle$ satisfies the MSG (55), then
  \[
  \langle \frac{1}{C} \gamma(r_\ast^{-1}) \rangle \leq 2; \tag{60}
  \]
- if $\gamma(\ell) \leq C_\pi \exp\left(-\frac{1}{C_\pi} \ell^\beta\right)$ for some $\beta > 0$, and $\langle \cdot \rangle$ satisfies the MSG (55), then
  \[
  \langle \exp\left(\frac{1}{C} r_\ast^{\beta \vee \frac{d}{2}}\right)\rangle \leq 2; \tag{61}
  \]
- if $\gamma(\ell) \leq C_\pi \exp\left(-\frac{1}{C_\pi} \ell^\beta\right)$ for some $\beta > 0$, and $\langle \cdot \rangle$ satisfies the MCI (54), then
  \[
  \langle \exp\left(\frac{1}{C} r_\ast^{(\beta \vee \frac{d}{2}) \vee \frac{d}{\pi \gamma d}}\right)\rangle \leq 2; \tag{62}
  \]
• if $\gamma(\ell) \leq C_\pi \exp(-\frac{1}{C_\pi} \ell^\beta)$ for some $\beta > 0$, and $\langle \cdot \rangle$ satisfies the MLSI (53) for the oscillation, then

$$\langle \exp \left( \frac{1}{C} r_\ast^{\beta/d} \right) \rangle \leq 2. \quad (63)$$

□

Before we turn to the ingredients to the proof of Theorems 3 and 4, let us present prototypical examples of coefficient fields satisfying MLSI, MCI or MSG with the oscillation (borrowed from [26, Section 3]). The first example considers random tessellations (RT) and is given by

$$a(x) = \sum_i \lambda_i \text{Id}(x \in V_i), \quad (64)$$

where $I$ is the indicator function, $\lambda_i$ are i.i.d. random variables that take values in $[\lambda, 1]$, and $V_i$ are the Voronoi cells associated with a Poisson point process (RT-PPP) of fixed intensity or with the random parking measure (RT-RPM), see [59]. In the case of RT-PPP (resp. RT-RPM), by [26, Proposition 3.2] (resp. [26, Proposition 3.3]), $\langle \cdot \rangle$ satisfies MLSI, cf. (53), with $\pi(\ell) \sim \exp(-\frac{\ell^d}{C_\pi})$ (resp. $\pi(\ell) \sim \exp(-\frac{\ell^d}{C})$), which yields $\gamma(\ell) \sim \exp(-\frac{\ell^d}{C_\pi})$ (resp. $\gamma(\ell) \sim \exp(-\frac{\ell^d}{C})$). The second example considers random inclusions (RI) and consists of a constant background coefficient field perturbed by random inclusions centered at the points of a Poisson point process of fixed intensity with i.i.d. random radii. More precisely, if $\{z_k\}_{k}$ denotes the Poisson points, and $r_k$ denotes the radius of the inclusion $B_{r_k}(z_k)$ centered at $z_k$, we consider the inclusion set $I := \bigcup_k B_{r_k}(z_k)$, and might for instance define the coefficient field as

$$a(x) = \lambda \text{Id} + (1 - \lambda)\text{Id} I(x \in I). \quad (65)$$

Let $\Gamma(\ell) := \langle I(\ell - \frac{1}{2} \leq r \leq \ell + \frac{1}{2}) \rangle$ for $\ell \geq 0$ and $\Gamma(\ell) := 0$ for $\ell < 0$ be the distribution function of the radii on $\mathbb{R}$. Then, by [26, Proposition 3.4], there is some $C = C(d) > 0$ such that $\langle \cdot \rangle$ satisfies the MCI, cf. (54), with weight

$$\pi(\ell) \sim \sup_{|\ell'| \leq \frac{1}{2}} (\ell + \ell')^d \Gamma(\ell + \ell').$$

We refer to this example as (RI-PPP). If instead of the Poisson point process we consider inclusions centered at the points of the random parking measure (RI-RPM), then, by [26, Proposition 3.4], $\langle \cdot \rangle$ satisfies the MCI with weight

$$\pi(\ell) \sim \sup_{|\ell'| \leq \frac{1}{2}} (\ell + \ell')^d \Gamma(\ell + \ell') + \exp(-\frac{1}{C} \ell),$$

where the additional term comes from the correlations of the points (of the RPM) themselves. Finally, if $\Gamma$ has compact support (that is, if the radii are uniformly bounded), then $\langle \cdot \rangle$ satisfies MLSI with the same weights as above. Corollary 8 below applies Theorem 4 to these examples (the proof, which is obvious, is omitted). Next to these results, we display what can be proved using linear concentration argument based on the associated $\alpha$-mixing. In most of these examples the stochastic integrability of $r_\ast$ implied by the MFI
are strictly stronger than what would follow from the associated \( \alpha \)-mixing conditions (cf. [25, Remark 4.6] for the spatial averages of the coefficient field itself).

**Corollary 8.**

(i) For RT-PPP and RI-PPP with uniformly bounded radii, (63) in Theorem 4 takes the form

\[
\left\langle \exp\left(\frac{1}{C}r_*^d\right) \right\rangle \leq 2, \quad \text{while } \alpha \text{-mixing yields } \left\langle \exp\left(\frac{1}{C}r_*^\frac{d}{2}\right) \right\rangle \leq 2.
\]

For RT-RPM, (61) in Theorem 4 takes the form

\[
\left\langle \exp\left(\frac{1}{C}r_*\right) \right\rangle \leq 2, \quad \text{while } \alpha \text{-mixing yields } \left\langle \exp\left(\frac{1}{C}r_*^{\frac{d}{d+1}}\right) \right\rangle \leq 2.
\]

(ii) For the example of random inclusions (65) with random radii and distribution function \( \Gamma : \mathbb{R}_+ \rightarrow [0,1] \), Theorem 4 takes the form

(a) If \( \Gamma(\ell) \leq 1 - e^{-\beta(\ell + 1)^{-1}} \) for some \( \beta > 0 \), then we have \( \Gamma(\ell) \sim (\ell + 1)^{-1}, \pi(\ell) \sim (\ell + 1)^{-1}, \gamma(\ell) \sim (\ell + 1)^{-1} \), and (60) yields

\[
\left\langle \frac{1}{C}r_*^\beta \right\rangle \leq 2 \quad \text{(same with } \alpha \text{-mixing).}
\]

(b) If \( \Gamma(\ell) \leq 1 - e^{-\beta(\ell)^\beta} \) for some \( \beta, C > 0 \), then \( \Gamma(\ell) \sim e^{-\beta(\ell)^\beta} \), and (62) yields in the case of RI-PPP

\[
\left\langle \exp\left(\frac{1}{C}r_*^{(\beta \wedge 1)^d + \frac{d}{d+1}}\right) \right\rangle \leq 2,
\]

while \( \alpha \text{-mixing yields } \left\langle \exp\left(\frac{1}{C}r_*^{\frac{d}{d+1}}\right) \right\rangle \leq 2 \);

and in the case of RI-RPM

\[
\left\langle \exp\left(\frac{1}{C}r_*^{(\beta \wedge 1)^d + \frac{d}{d+1}}\right) \right\rangle \leq 2,
\]

while \( \alpha \text{-mixing yields } \left\langle \exp\left(\frac{1}{C}r_*^{(\beta \wedge 1)^d + 1}\right) \right\rangle \leq 2. \]

\( \square \)

We finally turn to the proof of Theorems 3 and 4. The general structure is similar to that of Theorem 2. Two ingredients need to be refined: The concentration result of Lemma 4 and the control of the second moment of the extended corrector in Corollary 6. We start with the former.

**Lemma 5 (Concentration for averages).** [25, Propositions 4.3 & 4.5, and Remark 4.4] Assume that \( \langle \cdot \rangle \) is stationary. Let \( \pi \) denote an integrable weight satisfying (56b), where \( \gamma \) and \( \pi_* \) are defined in (56a) and (56c). Let \( t \geq 1 \) and let \( F_t \) denote a bounded random variable that is approximately \( \sqrt{t} \)-local in the sense of (48). For all \( T \geq t \), set \( (F_t)_T(a) := \int \omega_T(x)F_t(a(\cdot + x)) \, dx. \)
Then there exists a positive constant $C = C(d, \lambda, \pi)$ such that for all $\delta > 0$ and all $T \geq t$:

(i) if $\langle \cdot \rangle$ satisfies the MSG (55) with weight $\pi(\ell) \leq C_\pi(\ell + 1)^{-\beta - 1}$ for some $\beta > 0$, then

$$\langle I\left((F_t)_T - \langle F_t \rangle \geq \delta\right) \rangle \leq C e^{-\delta (1 + \delta^{-2 \frac{d}{2}} |\log \delta|)(\sqrt{t}^{-1})^{-\beta}}; \tag{66}$$

(ii) if $\langle \cdot \rangle$ satisfies the MSG (55) with weight $\pi(\ell) \leq C_\pi \exp(-\frac{1}{C_\pi} \ell^\beta)$ for some $\beta > 0$, then

$$\langle I\left((F_t)_T - \langle F_t \rangle \geq \delta\right) \rangle \leq \exp \left(- \frac{\delta^2}{C} \left(\frac{\sqrt{t}}{\sqrt{T}}\right)^{\beta \wedge \frac{d}{2}}\right); \tag{67}$$

(iii) if $\langle \cdot \rangle$ satisfies the MCI (54) with weight $\pi(\ell) \leq C_\pi \exp(-\frac{1}{C_\pi} \ell^\beta)$ for some $\beta > 0$, then

$$\langle I\left((F_t)_T - \langle F_t \rangle \geq \delta\right) \rangle \leq C \exp \left(- \frac{\delta^2}{C} \left(\frac{\sqrt{t}}{\sqrt{T}}\right)^{\beta \wedge \frac{d}{2}}\right); \tag{68}$$

(iv) if $\langle \cdot \rangle$ satisfies the MLSI (53) for the oscillation with weight $\pi(\ell) \leq C_\pi \exp(-\frac{1}{C_\pi} \ell^\beta)$ for some $\beta > 0$, then

$$\langle I\left((F_t)_T - \langle F_t \rangle \geq \delta\right) \rangle \leq \exp \left(- \frac{\delta^2}{C} \left(\frac{\sqrt{t}}{\sqrt{T}}\right)^{\beta \wedge \frac{d}{2}}\right); \tag{69}$$

(v) if $\langle \cdot \rangle$ satisfies the MLSI (53) with weight $\pi$ for the functional derivative, then

$$\langle I\left((F_t)_T - \langle F_t \rangle \geq \delta\right) \rangle \leq \exp \left(- \frac{\delta^2}{C} \pi^*(\sqrt{t})\right). \tag{69}$$

\[\square\]

Remark 7. For $t = 1$, the proof of Lemma 5 adapts the Herbst argument taking into account the specific properties of averages. For $t > 1$ the general statement reduces to the statement for $t = 1$ by rescaling and using (53), (54), or (55) with $L = \sqrt{t}$. \[\square\]

We conclude with the extension of Corollary 6.

Corollary 9. Assume that $\langle \cdot \rangle$ is stationary and satisfies the MLSI (53) with the functional derivative or the MSG (55) with the oscillation for an integrable weight $\pi$ that satisfies (56a) & (56b). Then there exist an exponent $\varepsilon = \varepsilon(d, \lambda, \pi) > 0$ (depending on the hole-filling exponent) and a constant $C = C(d, \lambda, \pi)$ such that for all $T > 0$ we have

$$\langle \frac{1}{T} \phi_T^2 + \frac{1}{T} |g_T|^2 + |\nabla g_T|^2 \rangle \leq CT^{-\varepsilon}. \tag{70}$$

\[\square\]
Remark 8. Corollaries 6 and 9 are the only places in our strategy of proof where the functional inequality is applied to a truly nonlinear random variable. We refer the reader to [44] for a similar result that does not rely on functional inequalities.

3. Proof of the regularity theory

3.1. Proof of Lemma 1: Construction of correctors

This proof is standard and essentially based on [46, Section 7.2]. We could also argue by massive approximation, cf. (37)–(39). The argument below relies on the Lax-Milgram theorem in the space of potential fields in probability. Following [58] we define the “horizontal derivatives” $D_j$ as the generators of the $d$ (shift) $L^2(\Omega)$-semigroups. More explicitly, we set

$$H^1(\Omega) := \{ \zeta \in L^2(\Omega) \mid \lim_{h \to 0} \frac{1}{h} (\zeta(a(\cdot + he_j)) - \zeta(a)) \text{ exists as a limit in } L^2(\Omega) \text{ for all } j \},$$

$$D_j : H^1(\Omega) \to L^2(\Omega), \quad D_j \zeta := \lim_{h \to 0} \frac{1}{h} (\zeta(a(\cdot + he_j)) - \zeta(a)).$$

In the following argument we suppress the index $i$ (which is fixed throughout the proof) in our notation for the tensor fields $\phi_i$, $\sigma_{ijk}$, and $q_{ij}$.

Step 1. Construction of a potential field $g$ (playing the role of $\nabla \phi$) and of a solenoidal field $q$.

Consider the space of curl-free vector fields with vanishing expectation

$$X := \{ g \in L^2(\Omega, \mathbb{R}^d) \mid D_j g_k = D_k g_j \text{ distributionally}, \langle g_j \rangle = 0 \}.$$ 

This is a closed subspace of $L^2(\Omega, \mathbb{R}^d)$. Because of stationarity, $-D_j$ is the (formal) adjoint of $D_j$. By stationarity, ergodicity, and the density of $\{D\phi \mid \phi \in H^1(\Omega)\} \subset X$ in $X$, (3) translates into

$$\forall g \in X \quad \langle g \cdot a(0) g \rangle \geq \lambda \langle |g|^2 \rangle. \quad (71)$$

By the Lax-Milgram theorem, there thus exists a unique

$$g \in X \text{ such that } \forall \tilde{g} \in X \quad \langle \tilde{g} \cdot a(0) (g + e) \rangle = 0. \quad (72)$$

With help of (2) and (71), we see that it satisfies the bound

$$\langle |g|^2 \rangle \leq \frac{1}{\lambda^2}. \quad (73)$$

Since $\{D\phi \mid \phi \in H^1(\Omega)\} \subset X$, (72) implies in particular

$$D \cdot a(0)(g + e) = 0 \quad (74)$$

in a weak sense. We define the homogenized coefficients $a_{\text{hom}}$ in direction $e$ as

$$a_{\text{hom}} e = \langle a(0)(g + e) \rangle. \quad (75)$$

In particular, the random vector $q \in L^2(\Omega, \mathbb{R}^d)$

$$q := a(0)(g + e) - a_{\text{hom}} e = a(0)(g + e) - \langle a(0)(g + e) \rangle, \quad (76)$$

satisfies

$$D \cdot a(h \cdot e) = 0 \quad (77)$$

and

$$\langle |q|^2 \rangle \leq \frac{1}{\lambda^2}. \quad (78)$$
which we may think of as a flux correction, satisfies
\[ \langle |q|^2 \rangle \leq \frac{1}{\lambda^2} + 1, \quad \langle q \rangle = 0, \quad D \cdot q = 0, \] (77)

(the bound is seen as follows \( \langle |q|^2 \rangle \leq \langle |a(0)(g + e)|^2 \rangle \leq \langle |g + e|^2 \rangle = \langle |g|^2 \rangle + 1 \leq \frac{1}{\lambda^2} + 1 \), the +1 is the price to pay for knowing (71) only for \( g \)'s with \( \langle g \rangle = 0 \) which mimics the properties of the field correction, namely
\[ \langle |g|^2 \rangle \leq \frac{1}{\lambda^2}, \quad \langle g \rangle = 0, \quad D_j g_k = D_k g_j. \]

Step 2. Construction of a curl-free matrix field \( b \).

For the construction of \( \sigma_{jk} \) we first introduce an auxiliary third-order tensor field \( b = b_{jkl} \). In this step, and throughout the article unless explicitly stated, we use the Einstein summation convention on repeated indices. Let \( \mathbb{R}^{d \times d}_{\text{sym}} \) denote the space of symmetric matrices and consider the space of curl-free symmetric matrix fields of vanishing expectation
\[ Y := \{ \tilde{b} \in L^2(\Omega, \mathbb{R}^{d \times d}_{\text{sym}}) \mid D_k b_{lm} = D_m b_{lk} \text{ distributionally, } \langle \tilde{b}_{kl} \rangle = 0 \}, \] (78)
which is a closed subspace of \( L^2(\Omega, \mathbb{R}^{d \times d}_{\text{sym}}) \). We denote by \( b_{j} \in Y \) the \( L^2(\Omega, \mathbb{R}^{d \times d}_{\text{sym}}) \)-orthogonal projection of the tensor field \( q_{j} \text{Id} \) onto \( Y \), where \( \text{Id} \) denotes the identity matrix in \( \mathbb{R}^{d \times d} \). As a projection, \( b_{j} \) satisfies the estimate
\[ \langle |b_{j}|^2 \rangle \leq \langle |q_{j} \text{Id}|^2 \rangle = d \langle q_{j}^2 \rangle. \] (79)

We claim that the third order tensor \( b = b_{jkl} \) satisfies
\[ b_{jkk} = q_{j}, \] (80)
\[ b_{kkj} = 0. \] (81)

We first prove (80). Define \( H^2(\Omega) \) as the set of \( H^1(\Omega) \) functions \( \zeta \) such that \( D\zeta \in H^1(\Omega, \mathbb{R}^d) \). Since \( \{ D^2 \zeta | \zeta \in H^2(\Omega) \} \subset Y \), we have by orthogonality and the curl-free condition in the definition (78) of \( Y \) in form of \( D_m b_{jkl} = D_k b_{jml} \) (that holds by symmetry and that we use for \( m = l \)),
\[ 0 = \langle D^2 \zeta \cdot (b_{j} - q_{j} \text{Id}) \rangle = \langle D_k D_l \zeta (b_{jlk} - q_{j} \delta_{lk}) \rangle = -\langle D_k \zeta D_l b_{jkl} \rangle - \langle D_k D_k \zeta q_{j} \rangle = \langle D_k D_k \zeta q_{j} \rangle = \langle D_k D_k \zeta (b_{jll} - q_{j}) \rangle. \]

This implies (80), since by ergodicity the range of \( \{ \sum_{k=1}^{d} D_k D_k \zeta | \zeta \in H^2(\Omega) \} \) is dense in \( \{ \zeta \in L^2(\Omega) | \langle \zeta \rangle = 0 \} \) and both \( b_{j} \) and \( q_{j} \) have vanishing expectation.

The remaining identity (81) follows from \( D \cdot q = 0 \), cf. (77). Indeed, by the curl-free and symmetry conditions
\[ D_l D_l b_{kkj} = D_l D_l b_{kkkl} = D_l D_j b_{kkjl} = D_j D_k b_{kkl} \quad \text{for all } j, k, \ell \]
in a distributional sense. Hence, for all \( j \) we have
\[ \langle D_l D_l \zeta b_{kkj} \rangle = \langle D_j D_k \zeta b_{kkl} \rangle \overset{(80)}{=} \langle D D_j \zeta \cdot q \rangle \overset{(77)}{=} 0, \]
and (81) follows from ergodicity.

Step 3. Construction of potential scalar and vector fields for $g$ and $b$.

By construction of $g$ and $b$, these fields are horizontally curl-free in a distributional sense:

$$D_j g_k = D_k g_j \quad \text{and} \quad D_l b_{jkm} = D_m b_{jkl}.$$ 

We extend the random variables $g$, $q$, and $b$ to stationary fields according to $g(a; x) = g(a(\cdot + x))$, however keeping the same symbol so that in particular (76) is consistent with (10). By definition of the horizontal derivative, spatial and horizontal derivatives are then related by $(\partial_j g)(a; x) = (D_j g)(a(\cdot + x))$, so that we obtain in particular

$$\partial_j g_k = \partial_k g_j \quad \text{and} \quad \partial_l b_{jkm} = \partial_m b_{jkl}.$$ 

Therefore, there exist fields $\phi = \phi(a; x)$ and $\sigma_{jk} = \sigma_{jk}(a; x)$ with the property that

$$g_j = \partial_j \phi, \quad b_{jkl} - b_{kjl} = \partial_l \sigma_{jk}.$$  \hspace{1cm} (82)

The fields $\phi$ and $\sigma_{jk}$ are uniquely determined by (82) up to a random additive constant in $x$, which we may fix by requiring that their average on the unit ball centered at the origin vanishes, e.g. $\int_B \phi = \int_B \sigma_{jk} = 0$. This makes the fields (generically) non-stationary and ensures that $\{\sigma_{jk}\}_{jk}$ inherits the build-in skew-symmetry of $\{b_{jkl} - b_{kjl}\}_{jk}$, and thus (6) follows. Clearly, the build-in vanishing expectation properties of $g$ and $b$ translate into those in (5). Moreover, the bounds stated in (5) follow from the moment bounds on $g$ and $q$, cf. (73), (77), and (79).

We note that by definition (82) and (74), the latter rewritten in terms of spatial instead of horizontal derivatives as $\nabla \cdot a(g + e) = 0$, we obtain (7).

For (8), we note that

$$\partial_l \sigma_{jl} \overset{(82)}{=} b_{jll} - b_{ljl} = b_{jll} - b_{ljl} \overset{(80),(81)}{=} q_j.$$ 

Finally (9) can be seen as follows

$$\partial_l \partial_l \sigma_{jk} \overset{(82)}{=} \partial_l b_{jkl} - \partial_l b_{kjl}$$

$$= \partial_l b_{ijk} - \partial_l b_{kij} \quad \text{by symmetry of} \ b$$

$$= \partial_k b_{jll} - \partial_j b_{kll} \quad \text{by curl-freeness of} \ b$$

$$\overset{(80)}{=} \partial_k q_j - \partial_j q_k.$$  

3.2. Proof of Proposition 1: Large-scale regularity by perturbation

Following [9], we recover the improvement (31) for $a$-harmonic functions as a perturbation of a result for $a_{\text{hom}}$-harmonic functions. By a scaling argument, we may assume $R = 1$. To ease notation, we also assume $\int_{B_1}(\phi, \sigma) = 0$. 

Step 1. Two PDE ingredients. 
On the one hand, we claim that there exists an exponent $\varepsilon = \varepsilon(d, \lambda) > 0$ such that if $w$ and $g$ satisfy

$$-\nabla \cdot a \nabla w = \nabla \cdot g \quad \text{in } B_1, \quad w = 0 \quad \text{on } \partial B_1,$$

(83)

then we have the weighted energy estimate

$$\int_{B_1} (1 - |x|)^\varepsilon |\nabla w|^2 \lesssim \int_{B_1} (1 - |x|)^\varepsilon |g|^2.$$

(84)

On the other hand, for any function $u_{\text{hom}}$ with

$$-\nabla \cdot a_{\text{hom}} \nabla u_{\text{hom}} = 0 \quad \text{in } B_1,$$

(85)

we claim the inner regularity estimate

$$\sup_{B_1 - \nu} \left( \rho |\nabla^2 u_{\text{hom}}| + |\nabla u_{\text{hom}}| \right) \lesssim \left( \frac{1}{\rho^{d/2}} \int_{B_1} |\nabla u_{\text{hom}}|^2 \right)^{1/2},$$

(86)

for any boundary layer width $\rho \leq 1$.

We first address (84). By Caccioppoli’s estimate we have for any cut-off function $\eta$ in $B_1$

$$\int_{B_1} \eta^2 |\nabla^2 w|^2 \lesssim \int_{B_1} \eta^2 |g|^2 + \int_{B_1} |\nabla \eta|^2 w^2,$$

(87)

see (205) in the proof of Lemma 6 in Appendix A. Choosing $\eta^2 = (1 - |x|)^\varepsilon$ for some $\varepsilon \in (0, 1)$ to be fixed later this turns into

$$\int_{B_1} (1 - |x|)^\varepsilon |\nabla w|^2 \lesssim \int_{B_1} (1 - |x|)^\varepsilon |g|^2 + \varepsilon^2 \int_{B_1} (1 - |x|)^{\varepsilon - 2} w^2,$$

where $\lesssim$ stands for a constant that depends on $d$ and $\lambda$ but not on $\varepsilon$. In order to absorb the second RHS term for $\varepsilon \ll 1$, we appeal to Hardy’s inequality

$$\int_{B_1} (1 - |x|)^{\varepsilon - 2} w^2 \lesssim \int_{B_1} (1 - |x|)^\varepsilon |\nabla w|^2.$$

(88)

For the convenience of the reader, we display the standard argument. We first argue that it is enough to prove (88) for functions $w$ that are compactly supported in $B_1$. Indeed, given a general $w \in H^1_{01}(B_1)$, which we trivially extend by zero on $\mathbb{R}^d \setminus B_1$ as a function in $H^1(\mathbb{R}^d)$, we define for all $0 < \kappa < 1$ the function $w_\kappa : B_1 \to \mathbb{R}, x \mapsto \kappa^{-\frac{d}{2}} w(x/\kappa)$. So defined, $w_\kappa$ is compactly supported in $B_1$, and (88) takes the form after a change of variables

$$\int_{B_1} (1 - \kappa |x|)^{\varepsilon - 2} w^2 = \int_{B_1} (1 - |x|)^{\varepsilon - 2} w_\kappa^2$$

$$\lesssim \int_{B_1} (1 - |x|)^\varepsilon |\nabla w_\kappa|^2 = \frac{1}{\kappa^2} \int_{B_1} (1 - \kappa |x|)^\varepsilon |\nabla w|^2.$$

We then take the limit as $\kappa \uparrow 1$ of this inequality, and conclude by the monotone convergence theorem that (88) holds true for $w$. 

We may thus assume that $w$ is compactly supported in $B_1$. By polar coordinates, it is enough to establish
\[
\int_0^1 (1-r)^{\varepsilon-2} w^2 r^{d-1} dr \leq \frac{4}{(1-\varepsilon)^2} \int_0^1 (1-r)^{\varepsilon} (\partial_r w)^2 r^{d-1} dr.
\]
(89)

To this purpose we note $(1-r)^{\varepsilon-2} r^{d-1} \leq \frac{d}{dr} \left( \frac{1}{1-\varepsilon} (1-r)^{\varepsilon-1} r^{d-1} \right)$ so that
\[
(1-r)^{\varepsilon-2} w^2 r^{d-1} \leq \frac{d}{dr} \left( \frac{1}{1-\varepsilon} (1-r)^{\varepsilon-1} w^2 r^{d-1} \right) - \frac{2}{1-\varepsilon} (1-r)^{\varepsilon-1} w \partial_r w r^{d-1}.
\]

When integrating this inequality over $r \in (0, 1)$, we note that the boundary contribution from $r = 0$ is non-positive, while the one from $r = 1$ vanishes identically by the support condition on $w$. By Cauchy-Schwarz’ inequality we thus obtain
\[
\int_0^1 (1-r)^{\varepsilon-2} w^2 r^{d-1} dr \leq \frac{2}{1-\varepsilon} \left( \int_0^1 (1-r)^{\varepsilon-2} w^2 r^{d-1} dr \right)^{\frac{1}{2}} \int_0^1 (1-r)^{\varepsilon} (\partial_r w)^2 r^{d-1} dr \right)^{\frac{1}{2}},
\]
which implies (89) since the LHS is finite by the support condition on $w$.

We now turn to (86) which is a consequence of the estimate
\[
|\rho |\nabla^2 u_{\text{hom}}(z)| + |\nabla u_{\text{hom}}(z)| \lesssim \left( \int_{B_\rho(z)} |\nabla u_{\text{hom}}|^2 \right)^{\frac{1}{2}} \quad \text{for all } z \in B_{1-\rho}.
\]
By translation and rescaling the latter follows from the inner regularity estimate
\[
\sup_{B_1} |\nabla^2 v|^2 + \sup_{B_1} |\nabla v|^2 \lesssim \int_{B_2} |\nabla v|^2,
\]
for any $a_{\text{hom}}$-harmonic function $v$ on $B_2$. For the sake of brevity we focus on the first estimate, that is
\[
\sup_{B_1} |\nabla^2 v|^2 \lesssim \int_{B_2} |\nabla v|^2.
\]
Since the coefficients $a_{\text{hom}}$ are constant, to the effect that also the components of $\nabla v$ are harmonic, this amounts to showing
\[
\sup_{B_1} |\nabla v|^2 \lesssim \int_{B_2} |v|^2.
\]
By Sobolev’s embedding, it is enough to show for some integer $k$ with $k > \frac{d}{2} + 1$ that
\[
\int_{B_1} (|\nabla^k v|^2 + |\nabla^{k-1} v|^2 + \cdots + |\nabla v|^2) \lesssim \int_{B_2} |v|^2.
\]
Again, since the components of the tensor $\nabla^m v$, $m = 0, \cdots, k - 1$, are $a_{\text{hom}}$-harmonic, this follows from a $k$-fold application of the Caccioppoli estimate (203) in Appendix A, where the radius decreases at every step by the amount of $\frac{1}{k}$. 
Step 2. The harmonic approximation. We consider the Lax-Milgram solution $u_{\text{hom}}$ to
\[ -\nabla \cdot a_{\text{hom}} \nabla u_{\text{hom}} = 0 \text{ in } B_1, \quad u_{\text{hom}} = u \text{ on } \partial B_1. \tag{90} \]
We claim that
\[ \int_{B_1} |\nabla u_{\text{hom}}|^2 \lesssim \int_{B_1} |\nabla u|^2. \tag{91} \]
Indeed, we rewrite (90) as $-\nabla \cdot a_{\text{hom}} \nabla (u_{\text{hom}} - u) = \nabla \cdot a_{\text{hom}} \nabla u$ in $B_1$ with $u_{\text{hom}} - u = 0$ on $\partial B_1$, so that by testing with $u_{\text{hom}} - u$ we obtain from the ellipticity of $a_{\text{hom}}$ that $\int_{B_1} |\nabla (u_{\text{hom}} - u)|^2 \lesssim \int_{B_1} |\nabla u|^2$. Now (91) follows by the triangle inequality.

Step 3. Representation formula in conservative form. For a given boundary layer thickness $\rho \in (0, \frac{1}{2}]$ we select a cut-off function $\eta$ with
\[ \eta = 1 \text{ in } B_{1-2\rho}, \quad \eta = 0 \text{ outside of } B_{1-\rho}, \quad |\nabla \eta| \lesssim \frac{1}{\rho}, \tag{92} \]
and consider the error in the two-scale expansion
\[ w := u - (1 + \eta \phi_i \partial_i) u_{\text{hom}}, \tag{93} \]
which thanks to $\eta$ vanishes on $\partial B_1$. We claim that we have (83) with RHS
\[ g := (1 - \eta)(a - a_{\text{hom}}) \nabla u_{\text{hom}} + (\phi_i a - \sigma_i) \nabla (\eta \partial_i u_{\text{hom}}). \tag{94} \]
Indeed, applying the gradient to (93), we obtain by Leibniz’s rule
\[ \nabla w = \nabla u - (\nabla u_{\text{hom}} + \eta \partial_i u_{\text{hom}} \nabla \phi_i + \phi_i \nabla (\eta \partial_i u_{\text{hom}})). \tag{95} \]
Applying $-\nabla \cdot a$, this yields because of (13)
\[ -\nabla \cdot a \nabla w \]
\[ = \nabla \cdot (a \nabla u_{\text{hom}} + \eta \partial_i u_{\text{hom}} a \nabla \phi_i + \phi_i a \nabla (\eta \partial_i u_{\text{hom}})) \]
\[ = \nabla \cdot ((1 - \eta) a \nabla u_{\text{hom}} + \eta \partial_i u_{\text{hom}} a \nabla \phi_i + \phi_i a \nabla (\eta \partial_i u_{\text{hom}})) \]
\[ = \nabla \cdot ((1 - \eta) a \nabla u_{\text{hom}} + \phi_i a \nabla (\eta \partial_i u_{\text{hom}})) + \nabla (\eta \partial_i u_{\text{hom}}) \cdot a (\nabla \phi_i + e_i). \tag{7} \]
Writing $\nabla (\eta \partial_i u_{\text{hom}}) \cdot a_{\text{hom}} e_i = \nabla \cdot (\eta \partial_i u_{\text{hom}} a_{\text{hom}} e_i) = \nabla \cdot (\eta a_{\text{hom}} \nabla u_{\text{hom}})$, and appealing to (90) in form of $\nabla \cdot (\eta a_{\text{hom}} \nabla u_{\text{hom}}) = -\nabla \cdot ((1 - \eta) a_{\text{hom}} \nabla u_{\text{hom}})$, we see that the above turns into
\[ -\nabla \cdot a \nabla w = \nabla \cdot ((1 - \eta)(a - a_{\text{hom}}) \nabla u_{\text{hom}}) + \phi_i a \nabla (\eta \partial_i u_{\text{hom}}) \]
\[ + \nabla (\eta \partial_i u_{\text{hom}}) \cdot (a (\nabla \phi_i + e_i) - a_{\text{hom}} e_i). \]
Using $\nabla \cdot \sigma_i = q_i = a(\nabla \phi_i + e_i) - a_{\text{hom}} e_i$, cf. (8), and the skew-symmetry of $\sigma_i$, cf. (6), in form of
\[ \nabla \zeta \cdot (\nabla \cdot \sigma_i) = \partial_j \zeta \partial_k \sigma_{ijk} \tag{6} \]
\[ \overset{\partial_k (\partial_j \zeta \sigma_{ijk}) \tag{6}}{=} -\nabla \cdot (\sigma_i \nabla \zeta), \]
we obtain (83) with $g$ defined as in (94).

Step 4. Estimate of $g$. 

We claim that
\[
\int_{B_1} (1 - |x|)^\varepsilon |g|^2 \lesssim (\rho^\varepsilon + \rho^{-d-2} \int_{B_1} |(\phi, \sigma)|^2) \int_{B_1} |\nabla u|^2.
\] (96)
Indeed, by definition (94) of $g$ and definition (92) of $\eta$ we have
\[
\int_{B_1} (1 - |x|)^\varepsilon |g|^2 \lesssim \int_{B_1 \setminus B_{1-2\rho}} (1 - |x|)^\varepsilon |\nabla u_{\text{hom}}|^2 \\
+ \sup_{B_{1-\rho}} (|\nabla^2 u_{\text{hom}}|^2 + \rho^{-2} |\nabla u_{\text{hom}}|^2) \int_{B_1} |(\phi, \sigma)|^2,
\]
so that by (86)
\[
\int_{B_1} (1 - |x|)^\varepsilon |g|^2 \lesssim (\rho^\varepsilon + \rho^{-d-2} \int_{B_1} |(\phi, \sigma)|^2) \int_{B_1} |\nabla u_{\text{hom}}|^2.
\]
Inserting (91) yields (96).

Step 5. Estimate by $w$.

We claim that for any $r \leq \frac{1}{4}$ we have
\[
\int_{B_r} |\nabla u - \partial_i u_{\text{hom}}(0)(e_i + \nabla \phi_i)|^2 \\
\lesssim (r^2 + r^{-d} \int_{B_1} |\phi|^2) \int_{B_1} |\nabla u|^2 + r^{-d-2} \int_{B_1} (1 - |x|)^\varepsilon |\nabla w|^2.
\] (97)
Indeed, since $u - (u_{\text{hom}}(0) + \partial_i u_{\text{hom}}(0)(x_i + \phi_i))$ is an $a$-harmonic function, we have by Caccioppoli’s estimate
\[
\int_{B_r} |\nabla u - \partial_i u_{\text{hom}}(0)(e_i + \nabla \phi_i)|^2 \\
\lesssim r^{-2} \int_{B_{2r}} \left( u - (u_{\text{hom}}(0) + \partial_i u_{\text{hom}}(0)(x_i + \phi_i)) \right)^2.
\] (98)
Using that $w = u - (1 + \phi_i \partial_i) u_{\text{hom}}$ on $B_{2r}$, cf. (92) & (93), we obtain by the triangle inequality in $L^2$
\[
\int_{B_{2r}} \left( u - (u_{\text{hom}}(0) + \partial_i u_{\text{hom}}(0)(x_i + \phi_i)) \right)^2 \\
\lesssim \int_{B_{2r}} w^2 + \sup_{B_{2r}} \left( u_{\text{hom}} - (u_{\text{hom}}(0) + \partial_i u_{\text{hom}}(0)x_i) \right)^2 \\
+ \sup_{B_{2r}} (\partial_i u_{\text{hom}} - \partial_i u_{\text{hom}}(0))^2 \int_{B_{2r}} \phi_i^2.
\]
Combining Taylor’s formula, (86) with $\rho = \frac{1}{2}$, and (91), yields
\[
\sup_{B_{2r}} \left( u_{\text{hom}} - (u_{\text{hom}}(0) + \partial_i u_{\text{hom}}(0)x_i) \right)^2 + r^2 \sup_{B_{2r}} (\partial_i u_{\text{hom}} - \partial_i u_{\text{hom}}(0))^2 \\
\lesssim r^4 \sup_{B_{\frac{1}{2}}} |\nabla^2 u_{\text{hom}}|^2 \lesssim r^4 \int_{B_1} |\nabla u_{\text{hom}}|^2 \lesssim r^4 \int_{B_1} |\nabla u|^2,
\]
and thus
\[
\int_{B_{2r}} \left( u - (u_{\text{hom}}(0) + \partial_i u_{\text{hom}}(0)(x_i + \phi_i)) \right)^2 \\
\lesssim \int_{B_{2r}} w^2 + (r^4 + r^{2-d} \int_{B_1} |\phi|^2) \int_{B_1} |\nabla u|^2.
\]

We combine this with (88)
\[
\int_{B_{2r}} w^2 \lesssim r^{-d} \int_{B_1} (1 - |x|)^{\epsilon-2} w^2 \lesssim r^{-d} \int_{B_1} (1 - |x|)^{\epsilon} |\nabla w|^2.
\]
The combination with (98) yields (97).

\textbf{Step 6. Proof of (31).}

Recall that by scaling we may assume \( R = 1 \) so that \( \delta = (f_{B_1} |(\phi, \sigma)|^2)^{\frac{1}{2}} \).

Inserting (84) and (96) into (97) gives for \( 0 < r, \rho \leq \frac{1}{4} \)
\[
\int_{B_r} |\nabla u - \partial_i u_{\text{hom}}(0)(e_i + \nabla \phi_i)|^2 \\
\lesssim \left( r^2 + r^{-d-2} (\rho^{\epsilon} + \rho^{-d-2} \int_{B_1} |(\phi, \sigma)|^2) \right) \int_{B_1} |\nabla u|^2.
\]

Provided \( f_{B_1} |(\phi, \sigma)|^2 \ll 1 \), we may choose \( \rho = (f_{B_1} |(\phi, \sigma)|^2)^{\frac{\epsilon}{d+2+\epsilon}} \) and obtain (31) with \( \frac{\epsilon}{d+2+\epsilon} \) playing the role of \( \epsilon \) and \( \nabla u_{\text{hom}}(0) \) the role of \( \xi \). If \( f_{B_1} |(\phi, \sigma)|^2 \gtrsim 1 \) or \( r \geq \frac{1}{4} \) we may choose \( \xi = 0 \) and thus trivially obtain (31).

\textbf{Step 7. Proof of (33).}

The upper bound follows from Caccioppoli’s estimate, cf. (87), applied to the \( a \)-harmonic function \( \xi \cdot x + \phi_\xi \) in form of
\[
\left( \int_{B_{\frac{1}{2}}} |\xi + \nabla \phi_\xi|^2 \right)^{\frac{1}{2}} \lesssim \left( \int_{B_1} (\xi \cdot x + \phi_\xi)^2 \right)^{\frac{1}{2}},
\]
followed by the triangle inequality in \( L^2 \). The lower bound follows from Poincaré’s inequality (with mean-value zero) and the triangle inequality in \( L^2 \)
\[
\left( \int_{B_{\frac{1}{2}}} |\xi + \nabla \phi_\xi|^2 \right)^{\frac{1}{2}} \gtrsim \left( \int_{B_{\frac{1}{2}}} (\xi \cdot x + \phi_\xi)^2 \right)^{\frac{1}{2}} \\
\geq \left( \int_{B_{\frac{1}{2}}} (\xi \cdot x)^2 \right)^{\frac{1}{2}} - \left( \int_{B_{\frac{1}{2}}} (\phi_\xi - \phi)^2 \right)^{\frac{1}{2}} \\
\geq \left( \int_{B_{\frac{1}{2}}} (\xi \cdot x)^2 \right)^{\frac{1}{2}} - \left( \int_{B_{\frac{1}{2}}} \phi_\xi^2 \right)^{\frac{1}{2}} \\
\geq C(d) |\xi| - \frac{1}{C(d)} \left( \int_{B_1} \phi_\xi^2 \right)^{\frac{1}{2}}.
\]
3.3. Proof of Theorem 1: Excess-decay and the minimal radius

We split the proof into two steps, and make use of the short-hand notation $\text{Exc}(r) := \text{Exc}(\nabla u; B_r)$.

**Step 1.** Proof of (14) and (15).

Given a $\delta \leq 1$ to be fixed in the sequel as a function of $d$, $\lambda > 0$, and $\alpha < 1$, we define $r_*$ in line with (12) as

$$r_* = \inf \left\{ r > 0 \mid \forall \rho \geq r \, \rho^2 \int_{B_\rho} |(\phi, \sigma) - \int_{B_\rho} (\phi, \sigma)|^2 \leq \delta \right\}.$$

Let $R \geq r_*$. With this notation, (31) in Proposition 1 assumes the form

$$\text{Exc}(r) \leq C_1 \left( (\frac{r}{R'})^2 + \delta^{2\varepsilon} \left( \frac{R'}{r} \right)^{d+2} \right) \int_{B_{R'}} |\nabla u|^2$$

for all $a$-harmonic functions $u$ in $B_R$ and all radii $r_* \leq r \leq R' \leq R$, where $C_1$ denotes some constant only depending on $d$, $\lambda$, and $\alpha$ the value of which we retain momentarily. Replacing the $a$-harmonic function $u$ by the $a$-harmonic function $x \mapsto u(x) - (\xi \cdot x + \phi \xi(x))$, cf. (7), and optimizing in $\xi$, (99) yields

$$\text{Exc}(r) \leq C_1 \left( (\frac{r}{R'})^2 + \delta^{2\varepsilon} \left( \frac{R'}{r} \right)^{d+2} \right) \text{Exc}(R').$$

(100)

We now first choose $\theta \leq 1$, which is a placeholder for the ratio $\frac{r}{R'}$, so small that $C_1 \theta^2 \leq \frac{1}{2} \theta^{2\alpha}$. Since $\alpha < 1$, this can be done and $\theta$ just depends on $d$, $\lambda$, and $\alpha$. We then choose $\delta \leq 1$ so small that $C_1 \delta^{2\varepsilon} \left( \frac{R'}{r} \right)^{d+2} \leq \frac{1}{2} \theta^{2\alpha}$; again, this $\delta$ just depends on $d$, $\lambda$, and $\alpha$. With these choices, (100) assumes the form

$$\text{Exc}(\theta R') \leq \theta^{2\alpha} \text{Exc}(R')$$

for all radii $R' \leq R$ with $R' \geq r_*$. It is this form that may be iterated to yield

$$\text{Exc}(\theta^n R) \leq (\theta^n)^{2\alpha} \text{Exc}(R)$$

for all $n \in \mathbb{N}$ with $\theta^{n-1} R \geq r_*$. For $r_* \leq r \leq R$, choose now $n$ such that $\theta^{n+1} R < r \leq \theta^n R$ and thus on the one hand $\theta^n \leq \theta^{-1} \frac{r}{R}$ while on the other hand $\text{Exc}(r) \leq \theta^{-d} \text{Exc}(\theta^n R)$. This implies the desired estimate (14)

$$\text{Exc}(r) \leq \theta^{-(d+2\alpha)} \left( \frac{r}{R} \right)^{2\alpha} \text{Exc}(R).$$

Clearly, (15) is an immediate consequence of (33), possibly further reducing the constant in (12).

**Step 2.** Proof of (16).

In view of the non-degeneracy condition (15), for any $r_* \leq \rho \leq R$, there exists a unique $\xi_{\rho} \in \mathbb{R}^d$ such that

$$\int_{B_{\rho}} |\nabla u - (\xi_{\rho} + \nabla \phi \xi_{\rho})|^2 = \text{Exc}(\rho),$$

(101)
so that $\xi_\rho$ can be interpreted as an effective gradient of $u$ on scale $\rho$. We claim that the dependence of $\xi_\rho$ on the scale $\rho$ is well-controlled by the excess in the sense that for all $R \geq R' \geq r \geq r_*$

$$|\xi_r - \xi_{R'}|^2 \lesssim \text{Exc}(R'),$$  \hfill (102)

here and below $\lesssim$ denotes $\leq$ up to a generic constant that only depends on $d$ and $\alpha > 0$. By a dyadic argument which we will sketch presently, it is enough to consider two radii $\rho$ and $R'$ that are close in the sense of $\rho \leq R' \leq 2\rho$ and to show

$$|\xi_\rho - \xi_{R'}|^2 \lesssim \text{Exc}(R').$$  \hfill (103)

Here comes the dyadic argument: Let $N$ be the non-negative integer such that $2^{-(N+1)}R' < \rho \leq 2^{-N}R'$. By (103) for $n = 0, \ldots, N - 1$ we have

$$|\xi_\rho - \xi_{2^{-n}R'}|^2 \lesssim \text{Exc}(2^{-n}R'), \quad |\xi_{2^{-(n+1)}R'} - \xi_{2^{-n}R'}|^2 \lesssim \text{Exc}(2^{-n}R'),$$

and thus by the triangle inequality and since $\alpha > 0$, we obtain (102):

$$|\xi_\rho - \xi_{R'}|^2 \lesssim \left( \sum_{n=0}^{N} \sqrt{\text{Exc}(2^{-n}R')} \right)^2 \lesssim \left( \sum_{n=0}^{N} (2^{-n})^\alpha \sqrt{\text{Exc}(R')} \right)^2 \lesssim \text{Exc}(R').$$

We now turn to the argument for (103): By the non-degeneracy condition (15) on scale $\rho$ applied to $\xi_\rho - \xi_{R'}$, we have

$$|\xi_\rho - \xi_{R'}|^2 \lesssim \int_{B_\rho} |(\xi_\rho - \xi_{R'}) + \nabla\phi_{\xi_\rho - \xi_{R'}}|^2,$$

which by linearity we may rewrite as

$$|\xi_\rho - \xi_{R'}|^2 \lesssim \int_{B_\rho} |(\xi_\rho + \nabla\phi_\rho) - (\xi_{R'} + \nabla\phi_{\xi_{R'}})|^2,$$

so that by the triangle inequality in $L^2(B_\rho)$, and using $\rho \sim R'$, we obtain

$$|\xi_\rho - \xi_{R'}|^2 \lesssim \int_{B_\rho} |\nabla u - (\xi_\rho + \nabla\phi_\rho)|^2 + \int_{B_{R'}} |\nabla u - (\xi_{R'} + \nabla\phi_{\xi_{R'}})|^2.$$

By definition (101), and using once more $\rho \sim R'$ this turns as desired into

$$|\xi_\rho - \xi_{R'}|^2 \lesssim \text{Exc}(\rho) + \text{Exc}(R') \lesssim \text{Exc}(R').$$

We now may conclude the argument for (16). By the triangle inequality in $L^2$, the definition (11) of the excess, and the non-degeneracy condition (15), we get the two estimates

$$\int_{B_r} |\nabla u|^2 \lesssim \text{Exc}(r) + |\xi_r|^2,$$

$$\text{Exc}(R) + |\xi_R|^2 \lesssim \int_{B_R} |\nabla u|^2,$$  \hfill (104)

which combined with (14) in form of $\text{Exc}(r) \lesssim \text{Exc}(R) \leq \int_{B_R} |\nabla u|^2$ and (102) (with $R' = R$) yields (16) as desired.
3.4. Proof of Corollary 1: Almost-sure Liouville property

The Liouville property is a fairly simple consequence of Theorem 1 and the following sublinear growth property

\[ \lim_{r \to \infty} \frac{1}{r^2} \int_{B_r} |(\phi, \sigma) - \int_{B_r} (\phi, \sigma)|^2 = 0 \quad \text{for a.e. } a. \quad (105) \]

For \( \phi \) this statement (in a more involved form) is a key ingredient for the quenched invariance principle and can be established based on ergodicity and stationarity, see [60]. We argue in Step 1 that the same argument can be used to establish this property for \( \sigma \).

**Step 1. Proof of (105).**

To keep notation lean, we just focus on \( \sigma \) and consider only one of the components \( \sigma_{ijk} \) of the tensor field \( \sigma \). We drop the indices. The key property of the random, typically non-stationary field \( \sigma(a, x) \) is that \( \nabla \sigma \) is stationary and of zero expectation and finite variance, see (5) in the statement of Lemma 1. For all \( r > 0 \), define the rescaled tensor field \( \sigma_r(x) := r^{-1} \left( \sigma(rx) - \int_{B_1} \sigma(ry) dy \right) \). On the one hand, by the pointwise ergodic theorem,

\[ \nabla \sigma_r = (\nabla \sigma)(r \cdot) \quad \overset{r \uparrow \infty}{\to} \quad 0 \]

weakly in \( L^2(B_1) \) almost surely, so that \( \int_{B_1} |\nabla \sigma_r|^2 \) is a bounded sequence almost surely. On the other hand, by Poincaré’s inequality on \( \hat{B}_1 \) with mean value zero

\[ \int_{B_1} |\sigma_r|^2 \lesssim \int_{B_1} |\nabla \sigma_r|^2 \leq \sup_{\rho \geq 1} \int_{B_1} |\nabla \sigma_\rho|^2 < \infty. \quad (107) \]

By (106), (107), and the Rellich theorem, \( \sigma_r \) thus converges strongly to 0 in \( L^2(B_1) \) as \( r \uparrow \infty \) almost surely. Rescaling back, this yields (105).

**Step 2. Conclusion.**

We now give the argument for the almost-sure Liouville property. Recall our short-hand notation \( \text{Exc}(r) := \text{Exc}(\nabla u; B_r) \). By (105), we may restrict ourselves to those coefficient fields for which \( \lim_{r \uparrow \infty} \frac{1}{r^2} \int_{B_r} |(\phi, \sigma) - \int_{B_r} (\phi, \sigma)|^2 = 0 \). Hence there exists a radius \( r < \infty \) such that (12) holds for \( C(\alpha, d, \lambda) \). Now we are given an \( a \)-harmonic function \( u \) with (17). By Caccioppoli’s estimate (203) (cf. Lemma 6 in Appendix A), this can be upgraded to

\[ \lim_{R \uparrow \infty} \frac{1}{R^{2\alpha}} \int_{B_R} |\nabla u|^2 = 0, \]

which in turn trivially yields

\[ \lim_{R \uparrow \infty} \frac{1}{R^{2\alpha}} \text{Exc}(R) = 0. \]

(109)

By (14) this implies for all \( \rho \geq r \)

\[ \inf_{\xi \in \mathbb{R}^d} \int_{B_\rho} |\nabla u - (\xi + \nabla \phi)\xi|^2 = \text{Exc}(\rho) = 0, \]

(110)
that is
\[ \forall \rho < \infty \ \exists \xi \in \mathbb{R}^d \ \text{s.t.} \ \nabla u = \xi + \nabla \phi \xi \ \text{a.e. in } B_\rho, \]  
(111)
which upgrades to
\[ \exists \xi \in \mathbb{R}^d \ \text{s.t.} \ \nabla u = \xi + \nabla \phi \xi \ \text{a.e. in } \mathbb{R}^d, \]  
(112)
and thus in turn implies (18).

3.5. Proof of Corollary 2: Intrinsic large-scale $C^{1,1-}$-regularity

In this proof, we use the short-hand notation $\text{Exc}(D) := \text{Exc}(\nabla u; D)$ for any domain $D$. In view of the non-degeneracy condition (15), for any $\rho \geq r_*(\pm x)$, there exists a unique $\xi_{\rho, \pm} \in \mathbb{R}^d$ such that
\[ \int_{B_\rho(x_\pm)} |\nabla u - (\xi_{\rho, \pm} + \nabla \phi \xi_{\rho, \pm})|^2 = \text{Exc}(B_\rho(x_\pm)), \]  
(113)
so that $\xi_{\rho, \pm}$ can be interpreted as an effective gradient of $u$ at $\pm x$ on scale $\rho$. Recall that we use the shorthand notation $\xi_{\pm} = \xi_{r_*, \pm}$. As in (102) in the proof of Theorem 1 we have that the dependence of $\xi_{\rho, \pm}$ on the scale $\rho$ is well-controlled by the excess in the sense that we have for all $\rho \geq r_*(\pm x)$
\[ |\xi_{\pm} - \xi_{r_*, \pm}|^2 \lesssim \text{Exc}(B_\rho(\pm x)), \]  
(114)
where here and in the remainder of the proof, $\lesssim$ denotes $\leq$ up to a generic constant that only depends on $d$, $\lambda$, and $\alpha$.

We set for abbreviation
\[ r := \max\{4|x|, 2r_*(x), r_*(-x)\}, \]  
(115)
so that $\frac{r}{4} \geq |x|$, $\frac{r}{2} \geq r_*(x)$, $r \geq r_*(-x)$.

We now claim that on this scale $r$ (which up to the cut-off $r_*$ is essentially the distance between the points $x$ and $-x$), the difference of the corresponding effective gradients $\xi_{r, +}$ and $\xi_{r, -}$ is well-controlled by the excess on that scale in the sense of
\[ |\xi_{r, +} - \xi_{r, -}|^2 \lesssim \text{Exc}(B_r(x)) + \text{Exc}(B_r(-x)). \]  
(116)
Indeed, by the non-degeneracy condition (15) and thanks to (115), we have
\[ |\xi_{r, +} - \xi_{r, -}|^2 \lesssim \int_{B_{\frac{r}{2}}(x)} |(\xi_{r, +} - \xi_{r, -}) + \nabla \phi \xi_{r, +} - \xi_{r, -}|^2. \]  
By linearity of $\nabla \phi \xi$ in $\xi$, the triangle inequality, and $B_{\frac{r}{2}}(x) \subset B_r(\pm x)$, this yields
\[ |\xi_{r, +} - \xi_{r, -}|^2 \lesssim \int_{B_r(x)} |\nabla u - (\xi_{r, +} + \nabla \phi \xi_{r, +})|^2 + \int_{B_r(-x)} |\nabla u - (\xi_{r, -} + \nabla \phi \xi_{r, -})|^2, \]
which turns into (116) by definition of $\xi_r$ and of the excess.

By the triangle inequality, estimates (114) and (116) combine to
\[ |\xi_+ - \xi_-|^2 \lesssim \text{Exc}(B_r(x)) + \text{Exc}(B_r(-x)). \]  
(117)
Since by (115) we have \( r \geq r_*(\pm x) \), and by assumption on \( R \) we have \( r \leq R \), we may apply Theorem 1 to the effect of

\[
\text{Exc}(B_r(x)) + \text{Exc}(B_r(-x)) \lesssim \left( \frac{r}{R} \right)^{2\alpha} \left( \text{Exc}(B_{\frac{R}{2}}(-x)) + \text{Exc}(B_{\frac{R}{2}}(x)) \right).
\]

Since by assumption \( R \geq 4|x| \) we have in particular \( B_{\frac{R}{2}}(\pm x) \subset B_R \) so that trivially by definition of the excess,

\[
\text{Exc}(B_{\frac{R}{2}}(x)) + \text{Exc}(B_{\frac{R}{2}}(-x)) \lesssim \text{Exc}(B_R).
\]

The combination of the three last estimates turns into (21).

### 3.6. Proof of Corollary 3: Intrinsic large-scale Schauder-estimates

We select \( \alpha' \in (\alpha, 1) \), say \( \alpha' := \frac{1+\alpha}{2} \), and choose \( C \) in the definition (12) of \( r_* \) so small that Theorem 1 holds with \( \alpha' \) playing the role of \( \alpha \).

**Step 1. Proof of (23) and (24).**

Let \( r_* \leq r \leq \rho \leq R \). We first argue that for some constant \( C_1 = C_1(d, \lambda, \alpha) \) we have

\[
\text{Exc}(\nabla u + g; B_r) \leq C_1 \left( \frac{r}{\rho} \right)^{2\alpha'} \text{Exc}(\nabla u + g; B_{\rho}) + \left( \frac{\rho}{r} \right)^d \int_{B_{\rho}} (|g - \int_{B_{\rho}} g|^2 + |h - \int_{B_{\rho}} h|^2) \right). \tag{118}
\]

To this end for \( \xi := \int_{B_{\rho}} g \) we consider the Lax-Milgram solution \( w \) of

\[
\begin{align*}
-\nabla \cdot a \nabla w &= \nabla \cdot (a(g - \xi) + h) & \text{in } B_{\rho}, \\
\quad w &= 0 & \text{on } \partial B_{\rho},
\end{align*}
\]

which is made such that on the one hand, by (22) \( x \mapsto u + \xi \cdot x - w \) is \( a \)-harmonic in \( B_{\rho} \), so that by (14),

\[
\text{Exc}(\nabla u + \xi - \nabla w; B_r) \lesssim \left( \frac{r}{\rho} \right)^{2\alpha'} \text{Exc}(\nabla u + \xi - \nabla w; B_{\rho}),
\]

and on the other hand, one has the energy estimate

\[
\int_{B_{\rho}} |\nabla w|^2 \lesssim \int_{B_{\rho}} (|g - \int_{B_{\rho}} g|^2 + |h - \int_{B_{\rho}} h|^2).
\]

By the triangle inequality in \( L^2 \) and \( \int_{B_r} \leq \left( \frac{\rho}{r} \right)^d \int_{B_{\rho}} \), the combination of these implies (118).

We then argue in favor of (23) based on (118), which we rewrite in terms of \( \theta = \frac{r}{\rho} \).

\[
\text{Exc}(\nabla u + g; B_{\theta \rho}) \leq C_1 (\theta^{2\alpha'} \text{Exc}(\nabla u + g; B_{\rho}) + \theta^{-d} \int_{B_{\rho}} (|g - \int_{B_{\rho}} g|^2 + |h - \int_{B_{\rho}} h|^2)),
\]
divide by $(\theta \rho)^{2\alpha}$, and take the supremum over $\rho \in [\frac{r_*}{R}, R]$:

$$
\sup_{r \in [r_*, \theta R]} \frac{1}{r^{2\alpha}} \text{Exc}(\nabla u + g; B_r)
$$

$$
\leq C_1 \theta^{2(\alpha' - \alpha)} \sup_{r \in [r_*, \theta R]} \frac{1}{r^{2\alpha}} \text{Exc}(\nabla u + g; B_r)
$$

$$
+ \theta^{-d-2\alpha} \sup_{r \in [r_*, \theta R]} \frac{1}{r^{2\alpha}} \int_{B_r} (|g - \int_{B_r} g|^2 + |h - \int_{B_r} h|^2).
$$

We now choose $\theta = \theta(d, \lambda, \alpha) \leq 1$ so small that $C_1 \theta^{2(\alpha' - \alpha)} \leq \frac{1}{2}$; which yields

$$
\sup_{r \in [r_*, \theta R]} \frac{1}{r^{2\alpha}} \text{Exc}(\nabla u + g; B_r)
\lesssim \sup_{r \in [\theta R, R]} \frac{1}{r^{2\alpha}} \text{Exc}(\nabla u + g; B_r)
$$

$$
+ \sup_{r \in [r_*, \theta R]} \frac{1}{r^{2\alpha}} \int_{B_r} (|g - \int_{B_r} g|^2 + |h - \int_{B_r} h|^2).
$$

Since $\sup_{r \in [\theta R, R]} \frac{1}{r^{2\alpha}} \text{Exc}(\nabla u + g; B_r) \lesssim \frac{1}{R^{2\alpha}} \text{Exc}(\nabla u + g; B_R)$, this yields (23) in case of $R < \infty$. In case of $R = \infty$ we obtain (24) from (23) in the limit $R \uparrow \infty$ by the square integrability of $\nabla u + g$ on $\mathbb{R}^d$ in form of

$$
\text{Exc}(\nabla u + g, B_R) \leq \int_{B_R} |\nabla u + g|^{2} \downarrow 0.
$$

**Step 2. Proof of (25).**

Starting point is (23), which also holds in the more general form of: For all $r \geq r_*$,

$$
\sup_{\rho \in [r, R]} \left( \frac{R}{\rho} \right)^{2\alpha} \text{Exc}(\nabla u + g; B_\rho) \lesssim \text{Exc}(\nabla u + g; B_R)
$$

$$
+ \sup_{\rho \in [r, R]} \left( \frac{R}{\rho} \right)^{2\alpha} \int_{B_\rho} (|g - \int_{B_\rho} g|^2 + |h - \int_{B_\rho} h|^2)
$$

since we may increase $r_*$ at our pleasure. As in Step 2 of the proof of Theorem 1 we denote by $\xi_r$ the optimal $\xi$ in the definition of $\text{Exc}(\nabla u + g; B_r)$. An inspection of the proof of (102) in that step shows that we have

$$
|\xi_r - \xi_R|^2 \lesssim \sup_{\rho \in [r, R]} \left( \frac{R}{\rho} \right)^{2\alpha} \text{Exc}(\nabla u + g; B_\rho)
$$

as soon as $\alpha > 0$. Giving away some and using the triangle inequality in $\mathbb{R}^d$, the two last estimates combine to

$$
|\xi_r|^2 + \text{Exc}(\nabla u + g; B_r)
\lesssim |\xi_R|^2 + \text{Exc}(\nabla u + g; B_R)
$$

$$
+ \sup_{\rho \in [r, R]} \left( \frac{R}{\rho} \right)^{2\alpha} \int_{B_\rho} (|g - \int_{B_\rho} g|^2 + |h - \int_{B_\rho} h|^2).
$$
Combined with the triangle inequality in $L^2$, the definition of the excess, and the non-degeneracy property in the form of
\[
\int_{B_r} |\nabla u + g|^2 \lesssim |\xi_r|^2 + \text{Exc}(\nabla u + g; B_r),
\]
\[
|\xi_R|^2 + \text{Exc}(\nabla u + g; B_R) \lesssim \int_{B_R} |\nabla u + g|^2,
\]
cf. (104), we may pass from (119) to (25).

3.7. Proof of Corollary 4: Large-scale Calderón-Zygmund estimates

We follow the standard approach to Calderón-Zygmund in the constant-coefficients case that passes via a BMO-estimate (see for instance [34, Section 7.1.1]). The main ingredient is excess decay for solutions of the homogeneous equation (cf. the excess-decay estimate (23) in Corollary 3 in our variable-coefficients case). There are two main differences with respect to [34, Section 7.1.1]: First, the excess decay is limited to the scale $r_*$, and second, we work on $L^2$-based quantities rather than $L^1$-based quantities. In Step 1, we choose a suitable minimal radius $r_*$ for the estimate (which we then simply call $r_*$ in the rest of the proof). In Step 2, we show that we control the energy by the intrinsic excess on dyadic cubes. In Step 3, we turn to the control of sub-level sets, which yields control of the $L^p$-norm in Step 4. In Step 5 we prove the equivalence of discrete and continuous norms, which we use in Step 6 to prove (27) in the range $2 \leq p < \infty$. In Step 7 we argue by duality to derive (27) in the remaining range of exponents $1 < p \leq 2$.

**Step 1. Choice of $r_*$.**

Let $C_0 > 0$ be such that one has the mean-value property for all $R \geq r_*(C_0)$, and therefore for all $R \geq r_*(C)$ with $C \geq C_0$. We shall define $r_*$ to be the largest function with Lipschitz constant $L$ below $r_*(C)$ where $C$ and $L$ will be chosen below, that is
\[
r_*(x) = \inf_{y \in \mathbb{R}} (r_*(y; C) + L|x - y|). \tag{121}
\]

With this definition, (12) survives with $r_*$ replaced by $r_*$ in form of
\[
\frac{1}{R} \left( \int_{B_R(x)} |(\phi, \sigma) - \int_{B_R(x)} (\phi, \sigma)|^2 \right)^{\frac{1}{2}} \leq \left( \frac{1}{L} + 1 \right)^{\frac{d}{2} + 1} \frac{1}{C}
\]
for all $R \geq r_*(x)$ and $x \in \mathbb{R}^d$.

Indeed, for a point $x \in \mathbb{R}^d$ and a radius $R < \infty$ with $r_*(x) < R$, by definition (121), there exists $y \in \mathbb{R}$ such that $|x - y| \leq \frac{R}{2}$ and $r_*(y) \leq R$. The former implies $B_R(x) \subset B_{\bar{R}}(y)$ where $\bar{R} := (\frac{1}{2} + 1)R$ so that
\[
\frac{1}{R} \left( \int_{B_R(x)} |(\phi, \sigma) - \int_{B_R(x)} (\phi, \sigma)|^2 \right)^{\frac{1}{2}} 
\leq \left( \frac{\bar{R}}{R} \right)^{\frac{d}{2} + 1} \frac{1}{R} \left( \int_{B_{\bar{R}}(y)} |(\phi, \sigma) - \int_{B_{\bar{R}}(y)} (\phi, \sigma)|^2 \right)^{\frac{1}{2}}.
\]
The latter implies
\[
\frac{1}{R} \left( \int_{B_R(x)} |(\phi, \sigma) - \int_{B_R(x)} (\phi, \sigma)|^2 \right)^{\frac{1}{2}} \leq \left( \frac{1}{L} + 1 \right)^{\frac{2}{2} + 1} \frac{1}{C}.
\]
Choosing \( L = \frac{1}{8} \) and \( C = 3^{d+2}C_0 \), we thus have that \( r_* \) is \( \frac{1}{8} \)-Lipschitz and satisfies \( r_*(C_0) \leq r_* \leq r_*(3^{d+2}C_0) \), as claimed. In particular, we have the mean-value property for all \( R \geq r_* \). In the rest of the proof, we use the short-hand notation \( r_* \) for \( r_*(x) \).

**Step 2. Control of the energy via the intrinsic excess.**

In the standard approach to Calderón-Zygmund for constant coefficients, see for instance [34, Proposition 6.31], the main part consists in controlling the (standard) energy density \( \rho_{stan} \) by the (standard) excess density \( e_{stan} \) (the “sharp function” of \( \nabla u \), see for instance [34, Section 6.3.4]) where
\[
\rho_{stan} := |\nabla u|^2 \quad \text{and} \quad e_{stan}(x) := \sup_r \left( \inf_{\xi} \int_{B_r(x)} |\nabla u - \xi|^2 \right),
\]
which amounts to a result by Fefferman & Stein, see for instance [34, Theorem 6.30]. More precisely, the next simple step in the standard approach consists in deriving the weak-strong estimate
\[
|B \cap \{ \rho_{stan} \leq 1 \}| \sim |B| \quad \text{if} \quad M \gg 1, \quad \int_B e_{stan} \lesssim |B|
\]
for any ball \( B \subset \mathbb{R}^d \). In the standard theory, this is clearly based on the fact the \( \xi \) in the definition of the excess is constant; whereas in our case it is replaced by \( \xi_i(e_i + \nabla \phi_i) \) and thus is only approximately constant on scales \( \gtrsim r_* \). In fact, we shall appeal to (15) in Theorem 1, namely
\[
\int_{B_R(x)} |\xi_i(e_i + \nabla \phi_i)|^2 \sim |\xi|^2 \quad \text{provided} \quad R \geq r_*(x).
\]
In what follows, it will be convenient to have a partition \( \mathcal{P} \) of \( \mathbb{R}^d \) into dyadic cubes \( Q \in \{2^k(\mathbb{Z}^d + [-\frac{1}{2}, \frac{1}{2})^d), k \in \mathbb{Z} \} \) such that
\[
r_* \sim \text{diam} (Q) \quad \text{on} \quad Q.
\]
Such a partition can be constructed like a Calderón-Zygmund decomposition: \( Q \) consists of those dyadic cubes \( Q \) which are such that \( Q \) and all its ancestors \( Q' \) (that is, \( Q' \in \{2^k(\mathbb{Z}^d + [-\frac{1}{2}, \frac{1}{2})^d), k \in \mathbb{Z} \} \) and \( Q \subset Q' \)) satisfy
\[
\int_Q r_* \geq \text{diam} (Q) \quad \text{and} \quad \int_{Q'} r_* < \text{diam} (Q').
\]
The second inequality implies
\[
\int_Q r_* \leq 2^{d+1} \text{diam} (Q).
\]
Since \( r_* \) is \( \frac{1}{8} \)-Lipschitz-continuous and since we may w.l.o.g. assume that \( r_* \) is bounded away from zero, \( Q \) defines indeed a (countable) partition of \( \mathbb{R}^d \).
Since \( r_* \) is \( \frac{1}{8} \)-Lipschitz continuous, the last inequalities imply (125). We now may pass from balls to cubes, more precisely, from (124) to

\[
\int_Q |\xi_i(e_i + \nabla \phi_i)|^2 \sim |\xi|^2 \quad \text{for all } Q \in \mathcal{Q}.
\]  

(126)

Indeed, select an \( x \in Q \); by (125) we have \( B_r(x) \subset Q \subset B_R(x) \) for two radii \( r, R \sim r_*(x) \), so that (124) translates into (126).

The price to pay for the lower scale \( r_* \) is a softening of the standard version \( \rho_{stan} \) of the energy density as follows

\[
\rho := \int_Q |\nabla u|^2 \quad \text{on } Q \quad \text{for all } Q \in \mathcal{P}.
\]  

(127)

Equipped with this modification, we recover (123) for our objects: We will argue that for any dyadic \( D \)

\[
|D \cap \{ \rho \leq 1 \}| \sim |D| \quad \text{if} \quad M \gg 1, \quad e(D) \lesssim |D| \quad \implies \quad |D \cap \{ \rho \geq M \}| \lesssim \frac{1}{M} e(D),
\]  

(128)

where we have set for abbreviation

\[
e(D) := \inf_\xi \int_D |\nabla u - \xi_i(e_i + \nabla \phi_i)|^2.
\]  

(129)

Note that because of the fact that \( \rho \) is piecewise constant on \( \mathcal{P} \), cf. (127), (128) is trivially satisfied for a \( D \) that is contained in \( \mathcal{P} \) or finer. We thus consider a dyadic \( D \) coarser than elements of \( \mathcal{P} \), and let \( Q \in \mathcal{P} \) be an arbitrary cube contained in \( D \). On the one hand, we have by the upper bound in (126) and the definition (127) of \( \rho \):

\[
\rho = \int_Q |\nabla u|^2 \lesssim \int_Q |\nabla u - \xi_i(e_i + \nabla \phi_i)|^2 + |\xi|^2 \quad \text{on } Q.
\]  

(130)

On the other hand, we have by the lower bound in (126)

\[
|\xi|^2 \lesssim \int_Q |\nabla u - \xi_i(e_i + \nabla \phi_i)|^2 + \rho \quad \text{on } Q.
\]

Summing the integral of the latter over \( Q \) for all \( Q \in \mathcal{P} \) such that \( Q \subset D \) and \( \rho|Q \leq 1 \), we obtain

\[
|D \cap \{ \rho \leq 1 \}| |\xi|^2 \lesssim \int_D |\nabla u - \xi_i(e_i + \nabla \phi_i)|^2 + |D|.
\]

Choosing \( \xi \) to be the minimizer in (129), this turns into \( |D \cap \{ \rho \leq 1 \}| |\xi|^2 \lesssim e(D) + |D| \), so that by the assumptions in (128) we have \( |\xi|^2 \lesssim 1 \). The combination of this with (130) yields because of \( M \gg 1 \)

\[
\rho \geq M \quad \text{on } Q \quad \implies \quad \int_Q |\nabla u - \xi_i(e_i + \nabla \phi_i)|^2 \gtrsim M,
\]

which we rewrite as (recall that \( \rho \) is constant on \( Q \))

\[
|Q \cap \{ \rho \geq M \}| \lesssim \frac{1}{M} \int_Q |\nabla u - \xi_i(e_i + \nabla \phi_i)|^2.
\]
Summing over all $Q \subset D$ with $Q \in \mathcal{P}$ yields the RHS of (128).

Step 3. Control of the sub-level sets of the energy by the sub-level sets of the intrinsic excess.

The next step in the standard theory starts from (123) and establishes control of the global measure of sub-level sets of $\rho_{\text{stan}}$ by the one of sub-level sets of $e_{\text{stan}}$. More precisely, it consists in passing from (123) to

$$\{|\rho_{\text{stan}}| \geq M\| \lesssim \{|e_{\text{stan}}| \geq \theta\| + \frac{\theta}{M} \{|\rho_{\text{stan}}| \geq 1\| \text{ for } \theta \ll 1 \ll M, \quad (131)$$

see for instance [34, Proposition 6.31]. This holds verbatim also in our case

$$\{|\rho| \geq M\| \lesssim \{|e| \geq \theta\| + \frac{\theta}{M} \{|\rho| \geq 1\| \text{ for } \theta \ll 1 \ll M, \quad (132)$$

where we even may relax the definition of the standard version $e_{\text{stan}}$ of the excess density, cf. (122), which could also be defined with help of the family of dyadic cubes $D$ as $e_{\text{stan}}(x) := \sup_{D \ni x} \inf_{\xi} \left( f_D |\nabla u - \xi|^2 \right)$ by restricting the supremum over dyadic cubes to those that are ancestors of cubes in the decomposition $\mathcal{P}$:

$$e(x) := \sup_{D \ni x} \left\{ \frac{e(D)}{|D|} \left| Q \subset D \text{ for some } Q \in \mathcal{P} \right\}, \quad (133)$$

where $e(D)$ is defined in (129). The argument in passing from (128) to (132) is identical to the argument for passing from (123) to (131). By successive divisions we construct a Calderón-Zygmund partition $\mathcal{D}$ based on the characteristic function of $\{|\rho| \geq 1\}$. In other words, $\mathcal{D}$ consists of those dyadic cubes $D$ such that it and all its ancestors $D' \supset D$ satisfy

$$|D \cap \{|\rho| \geq 1\}| > \frac{1}{2d+1} |D| \quad \text{and} \quad |D' \cap \{|\rho| \geq 1\}| \leq \frac{1}{2d+1} |D'|. \quad (134)$$

This yields a disjoint decomposition of $\mathbb{R}^d$ into $\{D\}_{D \in \mathcal{D}}$ and a set where $\rho < 1$. Hence for (132) it is enough to show for every cube $D \in \mathcal{D}$:

$$|D \cap \{|\rho| \geq M\| \lesssim |D \cap \{|e| \geq \theta\| + \frac{\theta}{M} |D \cap \{|\rho| \geq 1\| \text{ for } \theta \ll 1 \ll M. \quad (135)$$

Note that the first and second properties in (134) imply in particular

$$\frac{1}{2d+1} |D| \leq |D \cap \{|\rho| \geq 1\}| \leq \frac{1}{2} |D|, \quad (136)$$

and therefore one of the LHS conditions in (128). In addition, this yields that $\rho$ is not constant on $D$. By the definition of the dyadic decomposition $\mathcal{P}$ and by definition (127) of $\rho$ this implies that there is a $Q \in \mathcal{P}$ such that $Q \subset D$ (strictly, in fact). Hence in view of the definition (133),

$$|D| e(x) \geq e(D) \quad \text{for all } x \in Q. \quad (137)$$

To recover the other LHS condition in (128), we first consider the case $e(D) < \theta |D| \leq |D|$ in which case the RHS of (128) assumes the form $|D \cap \{|\rho| \geq M\| \leq \frac{\theta}{M} |D|$, which together with (136) yields (135). It remains to consider the case
\( e(D) \geq \theta |D| \), which by (137) implies \( D \cap \{ e \geq \theta \} = D \) so that (135) is automatically met.

**Step 4.** Conversion of the control of the sub-level sets to an \( L^p \)-estimate.

The previous-to-last step in the standard argument is to convert (131) into the \( L^p \)-estimate \( \int \rho^p_{stan} \lesssim \int e^p_{stan} \) for any \( 1 < p < \infty \), see for instance [34, Theorem 6.30]. By the same argument we may pass from (132) to

\[
\int \rho^p \lesssim \int e^p. \tag{138}
\]

Indeed, by a scaling argument in form of \( u = t \hat{u} \) we may upgrade (132) to

\[
|\{ \rho \geq Mt \}| \lesssim |\{ e \geq \theta t \}| + \frac{\theta}{M} |\{ \rho \geq t \}| \quad \text{for } \theta \ll 1 \ll M \text{ and all } t > 0.
\]

Integrating against \( t^{p-1} \) we obtain

\[
\frac{1}{M^p} \int \rho^p \lesssim \frac{1}{\theta^p} \int e^p + \frac{\theta}{M} \int \rho^p \quad \text{for } \theta \ll 1 \ll M.
\]

This yields (138) by first fixing an \( M \sim 1 \) sufficiently large for which this inequality holds, and then choosing \( \theta \sim 1 \) sufficiently small so that the last term may be absorbed.

**Step 5.** Equivalence of discrete and continuous norms.

In this step we prove that for all \( 1 < p < \infty \) and non-negative functions \( h \) we have

\[
\left( \int \left( \int_{B_*(x)} h \right)^{\frac{2}{p}} dx \right)^{\frac{p}{2}} \sim \left( \sum_{Q \in \mathcal{P}} |Q| \left( \int_{Q} h \right)^{\frac{2}{p}} \right)^{\frac{p}{2}}, \tag{139}
\]

where and \( A \sim B \) means \( \frac{1}{C} A \leq B \leq CA \) for a generic constant \( C \) depending only on \( d \) (and not on \( p \)). In particular, for \( p = 2 \), this takes the form

\[
\int \int_{B_*(x)} h \, dx \sim \int h. \tag{140}
\]

We split the rest of this step into two parts.

**Substep 5.1.** Proof that for all \( 1 \leq p < \infty \),

\[
\left( \int \left( \int_{B_*(x)} h \right)^{\frac{2}{p}} dx \right)^{\frac{p}{2}} \lesssim \left( \sum_{Q \in \mathcal{P}} |Q| \left( \int_{Q} h \right)^{\frac{2}{p}} \right)^{\frac{p}{2}}. \tag{141}
\]

For all \( Q, Q' \in \mathcal{P} \), we write \( Q' \sim Q \) if there exists \( x \in Q \) such that \( B_*(x) \cap Q' \neq \emptyset \), and first claim that if \( Q' \sim Q \) then

\[
\text{diam} (Q') \sim \text{diam} (Q), \tag{142}
\]

\[
\text{dist} (Q, Q') \lesssim \text{diam} (Q'). \tag{143}
\]

We first note that (142) implies (143) in the form \( \text{dist}(Q, Q') \leq r_* (x) \) \((125)\)

\[
\text{diam} (Q) \sim \text{diam} (Q'). \tag{142}
\]

We then prove (142), and let \( y \in Q' \) be such that \( y \in B_*(x) \). By the Lipschitz continuity of \( r_* \) we have

\[
|r_* (y) - r_* (x)| \leq \frac{1}{8} |x - y| \leq \frac{1}{8} r_* (x),
\]
so that \( \text{diam}(Q') \sim r_*(y) \sim r_*(x) \sim \text{diam}(Q) \), that is, (142). We now argue that (142) and (143) imply that

\[
\sup_{Q' \in \mathcal{P}} \# \{ Q \in \mathcal{P} \mid Q' \sim Q \} \lesssim 1, \quad \sup_{Q' \in \mathcal{P}} \# \{ Q' \in \mathcal{P} \mid Q' \sim Q \} \lesssim 1. \tag{144}
\]

We only prove the first estimate: From (143) we learn that \( \cup_{Q' \sim Q} Q \subseteq B_{C \text{diam}(Q')} \) for some generic \( C = C(d) < \infty \) (which may change from line to line in the estimates below), whereas from (142) we learn that \( |Q| \gtrsim \text{diam}(Q')^d \) for all \( Q \) with \( Q' \sim Q \). The combination of these properties yields (144).

We are in the position to conclude the proof of (141). Let \( Q \in \mathcal{P} \). For all \( x \in Q \), we have

\[
\int_{B_*(x)} h \leq \sum_{Q' : Q' \sim Q} |Q'| \int_{B_*(x)} h \overset{(142),(125)}{\lesssim} \sum_{Q' : Q' \sim Q} \int_{Q'} h, 
\]

so that

\[
\int \left( \int_{B_*(x)} h \right)^{\frac{p}{2}} dx \leq C^p \sum_{Q \in \mathcal{P}} |Q| \left( \sum_{Q' \sim Q} \int_{Q'} h \right)^{\frac{p}{2}} \overset{(144)}{\leq} C^p \sum_{Q' \in \mathcal{P}} \sum_{Q : Q' \sim Q} |Q| \left( \int_{Q'} h \right)^{\frac{p}{2}} \overset{(144),(142)}{\leq} C^p \sum_{Q' \in \mathcal{P}} |Q'| \left( \int_{Q'} h \right)^{\frac{p}{2}},
\]

as claimed.

**Substep 5.2.** Proof that for all \( 1 \leq p < \infty \),

\[
\left( \int \left( \int_{B_*(x)} h \right)^{\frac{p}{2}} dx \right)^{\frac{2}{p}} \gtrsim \left( \sum_{Q \in \mathcal{P}} |Q| \left( \int_{Q} h \right)^{\frac{p}{2}} \right)^{\frac{2}{p}}. \tag{146}
\]

Let \( Q \in \mathcal{P}, \ell = \text{diam} (Q) \), and set \( Q_\varepsilon (x) := x + [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]^d \). Since \( r_* \sim \ell \) on \( Q \), for \( 0 < \varepsilon \ll 1 \) small enough, we have

\[
\int_{Q} \left( \int_{B_*(x)} h \right)^{\frac{p}{2}} dx \geq C^{-p \varepsilon d \frac{p}{2}} \int_{Q} \left( \int_{Q_{\varepsilon \ell}(x)} h \right)^{\frac{p}{2}} dx, \tag{147}
\]

where \( C \) denotes a constant depending only on \( d \) (and that may change from line to line like in Substep 5.1). W.l.o.g. we may assume that \( \varepsilon \) is chosen such
that \( \{Q_{\frac{1}{2}\varepsilon}^z(z)\}_{z \in Z} \), with \( Z := \{ z \in \frac{1}{2}\varepsilon \mathbb{Z}^d : Q_{\frac{1}{2}\varepsilon}^z(z) \subset Q \} \) is a partition of \( Q \) into disjoint cubes. Thus, (147) turns into
\[
\int_Q \left( \int_{B_r(x)} h \right)^{\frac{p}{2}} dx \geq C^{-p\varepsilon^d} \varepsilon^d \sum_{z \in Z} \int_Q \left( \int_{Q_{\frac{1}{2}\varepsilon}^z(x)} h \right)^{\frac{p}{2}} dx.
\]
Since
\[
x \in Q_{\frac{1}{2}\varepsilon}^z(z) \implies \left( \int_{Q_{\frac{1}{2}\varepsilon}^z(x)} h \right)^{\frac{p}{2}} \geq 2^{-d^p} \left( \int_{Q_{\frac{1}{2}\varepsilon}^z(x)} h \right)^{\frac{p}{2}},
\]
we conclude that
\[
\int_Q \left( \int_{B_r(x)} h \right)^{\frac{p}{2}} dx \geq C^{-p\varepsilon^d} \varepsilon^d \sum_{z \in Z} \left( \int_Q \left( \int_{Q_{\frac{1}{2}\varepsilon}^z(z)} h \right)^{\frac{p}{2}} \right).
\]
For \( 2 \leq p < \infty \) Jensen’s inequality yields
\[
\int_Q \left( \int_{B_r(x)} h \right)^{\frac{p}{2}} \geq C^{-p\varepsilon^d} \varepsilon^d \left( \sum_{z \in Z} \int_Q \left( \int_{Q_{\frac{1}{2}\varepsilon}^z(z)} h \right)^{\frac{p}{2}} \right) \geq C^{-p\varepsilon^d} \left( \int_Q \left( \int_{B_r(x)} h \right)^{\frac{p}{2}} \right),
\]
while for \( 1 \leq p < 2 \) the discrete estimate \( \| \cdot \|_{\ell^p} \leq \| \cdot \|_{\ell^1} \) yields
\[
\int_Q \left( \int_{B_r(x)} h \right)^{\frac{p}{2}} \geq C^{-p\varepsilon^d} \varepsilon^d \left( \sum_{z \in Z} \int_Q \left( \int_{Q_{\frac{1}{2}\varepsilon}^z(z)} h \right)^{\frac{p}{2}} \right) \geq C^{-p\varepsilon^d} \left( \int_Q h \right)^{\frac{p}{2}}.
\]
Since \( \varepsilon \) can be chosen only depending on \( d \), (146) follows.

**Step 6.** Proof of (27) for \( p \geq 2 \).

We now return from cubes to balls and start with the excess. Based on the Lipschitz continuity of \( r_* \) we claim that for any point \( x \)
\[
e(x) \lesssim \sup_{R > r_*(x)} \inf_{\xi} \left( \int_{B_R(x)} |\nabla u - \xi(e_i + \nabla \phi_i)|^2 \right).
\]
The definition (133) prompts us to prove that for any dyadic cube \( D \) with \( x \in D \) and such that there exists a \( Q \in \mathcal{P} \) with \( Q \subset D \), we have
\[
e(D) |D| \lesssim \frac{e(B_R(x))}{|B_R|}
\]
for some \( R \geq r_*(x) \).

Since \( Q \subset D \) for some \( Q \in \mathcal{P} \), we have \( \inf_D r_* \lesssim \text{diam}(Q) \leq \text{diam}(D) \) in view of (125). By the Lipschitz continuity of \( r_* \), \( x \in D \) then yields \( r_*(x) \lesssim \text{diam}(D) \), which implies \( D \subset B_R(x) \) for some \( r_*(x) \leq R \sim \text{diam}(D) \). This gives both \( e(D) \leq e(B_R(x)) \), cf. (129), and \( |D| \gtrsim |B_R| \), so that the claim (149) follows.

We continue to revert back to balls from cubes and look at the energy density. By (139) for \( h = |\nabla u|^2 \), we have
\[
\int_Q \left( \int_{B_r(x)} |\nabla u|^2 \right)^{\frac{p}{2}} dx \lesssim \sum_{Q \in \mathcal{P}} |Q| \left( \int_Q |\nabla u|^2 \right)^{\frac{p}{2}} \overset{(127)}{=} \int \rho^p.
\]
We may now conclude: We use (23) with $\alpha = 0$ and combine it with (149) in the form

$$e(x) \lesssim \sup_{R > r_*(x)} \left( \int_{B_R(x)} |g|^2 \right).$$

By (139) for $p = 2$, we have

$$\sup_{R > r_*(x)} \int_{B_R(x)} |g|^2 \lesssim \sup_{R > 0} \int_{B_R(x)} |g|^2,$$

where $g_*(x) := \left( \int_{B_*(x)} |g|^2 \right)^{\frac{1}{2}}$. We now combine (138) with the Maximal Function estimate applied to $|g_*|^2$ with exponent $\frac{p}{2} > 1$ to obtain for all $p > 2$

$$\int \rho^p \lesssim \int \left( \int_{B_+(y)} |g|^2 \right)^{\frac{p}{2}} dy \lesssim \int |g|^p,$$

which in combination with (150) yields the claim for $p > 2$. For $p = 2$, (150) takes the simpler form

$$\int \left( \int_{B_+(x)} |\nabla u|^2 \right) dx \sim \sum_{Q \in \mathcal{P}} \int_Q |\nabla u|^2 = \int |\nabla u|^2,$$

so that the result is a consequence of the simple energy estimate for (26). The claim then follows by the Riesz-Thorin interpolation theorem.

**Step 7.** Proof of (27) for $1 < p < 2$.

By (139), it is enough to prove the claim by replacing the integral of averages on $\mathbb{R}^d$ by the sum of averages on the partition $\mathcal{P}$. We argue by a standard duality argument that appeals to (27) for the dual problem (cf. Remark 2): By Step 6, for any exponent $2 \leq q < \infty$ and any decaying $v, h$ related through $-\nabla \cdot a^* \nabla v = \nabla \cdot h$, we have

$$\sum_{Q \in \mathcal{P}} |Q| \left( \int_Q |\nabla v|^2 \right)^{\frac{2}{q}} \lesssim \sum_{Q \in \mathcal{P}} |Q| \left( \int_Q |h|^2 \right)^{\frac{2}{q}}. \quad (151)$$

Now, by discrete duality, we have for our solution $u$ and all $1 < p \leq 2$ and $q = \frac{p}{p-1}$,

$$\left( \sum_{Q \in \mathcal{P}} |Q| \left( \int_Q |\nabla u|^2 \right)^{\frac{2}{q}} \right)^{\frac{q}{2}} = \sup_{h \neq 0} \left( \frac{\sum_{Q \in \mathcal{P}} |Q| \int_Q \nabla u \cdot h}{\left( \sum_{Q \in \mathcal{P}} |Q| \left( \int_Q |h|^2 \right)^{\frac{2}{q}} \right)^{\frac{q}{2}}} \right). \quad (152)$$

Let $h$ be a test function. We then consider the solution $v$ of $-\nabla \cdot a^* \nabla v = \nabla \cdot h$ and compute using the defining equations for $u$ and $v$, and Hölder’s inequality.
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with exponents \((p, q)\)

\[
\left| \sum_{Q \in \mathcal{P}} |Q| \int_{Q} \nabla u \cdot h \right| \\
= \left| \int \nabla u \cdot h \right| = \left| \int \nabla u \cdot a^* \nabla v \right| = \left| \int \nabla v \cdot g \right| \\
\leq \sum_{Q \in \mathcal{P}} |Q| \left( \int_{Q} |\nabla v|^2 \right)^{\frac{1}{2}} \left( \int_{Q} |g|^2 \right)^{\frac{1}{2}} \\
\leq \left( \sum_{Q \in \mathcal{P}} |Q| \left( \int_{Q} |\nabla v|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{q}} \left( \sum_{Q \in \mathcal{P}} |Q| \left( \int_{Q} |g|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{p}} \\
\leq \left( \sum_{Q \in \mathcal{P}} |Q| \left( \int_{Q} |h|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{q}} \left( \sum_{Q \in \mathcal{P}} |Q| \left( \int_{Q} |g|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{p}},
\]

from which (27) on the level of \(\mathcal{P}\) follows by (152) and the arbitrariness of \(h\).

### 3.8. Proof of Corollary 5: Large-scale weighted Calderón-Zygmund estimates

We split the proof into four steps.

**Step 1.** Decay property in \(L^2\).

Suppose in addition that \(\text{supp} g \subset B_r\) for some \(r \geq r_\ast(0)\). Then we claim for all \(R \geq r\)

\[
\left( \frac{1}{R^d} \int_{|x| > R} |\nabla u|^2 \right)^{\frac{1}{2}} \lesssim \left( \frac{r}{R} \right)^d \left( \frac{1}{r^d} \int_{|x| < r} |g|^2 \right)^{\frac{1}{2}}.
\]  

(153)

We argue by duality: Given a square-integrable vector field \(h\) supported in \(\{|x| > R\}\) we denote by \(v\) the Lax-Milgram solution of \(-\nabla \cdot a^* \nabla v = \nabla \cdot h\), so that we have the identity \(\int h \cdot \nabla u = \int g \cdot \nabla v\), which by the support condition on \(g\) implies

\[
\int h \cdot \nabla u \leq \left( \int_{|x| < r} |g|^2 \right)^{\frac{1}{2}} \left( \int_{|x| < r} |\nabla v|^2 \right)^{\frac{1}{2}}.
\]

Since by the support condition on \(h\), \(v\) is \(a^*\)-harmonic in \(\{|x| < R\}\), we may apply (16) (recall \(r \geq r_\ast(0) \geq r_\ast(0)\)) to the effect of \(\int_{|x| < r} |\nabla v|^2 \lesssim \int_{|x| < R} |\nabla v|^2\). Combined with the energy estimate, this yields

\[
\int_{|x| < r} |\nabla v|^2 \lesssim \left( \frac{r}{R} \right)^d \int |h|^2.
\]

Choosing \(h\) to be the restriction of \(\nabla u\) on \(\{|x| > R\}\), we obtain (153).

**Step 2.** Decay property in \(L^p\).
As in Step 1 suppose that \( \text{supp} g \subset B_r \) for some \( r \geq r_\ast(0) \). Then we claim for all \( R \geq 4r \)
\[
\frac{1}{R^d} \int_{|x| > R} \left( \int_{B_*,(x)} |\nabla u|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}} \leq \left( \frac{r}{R} \right)^d \left( \frac{1}{r^d} \int_{|x| < 4r} \left( \int_{B_*,(x)} |g|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}. \tag{154}
\]
By a decomposition of \( \{|x| > R\} \) in dyadic annuli, it is enough to show for \( R \geq 4r \)
\[
\sup_{R < |x| < 2R} \int_{B_*,(x)} |\nabla u|^2 \leq \sup_{R < |x| < 2R} \int_{B_{\frac{R}{2}}(x)} |\nabla u|^2 \leq \int_{\frac{R}{2} < |y| < 4R} |\nabla u|^2.
\]
We note that for \( x \) with \( R < |x| < 2R \) we have \( r_\ast(x) \leq r_\ast(0) + \frac{1}{8} |x| \leq R \), so that \( B_*(x) \subset B_{\frac{R}{2}}(x) \subset \{\frac{R}{2} < |y| < 4R\} \), which in turn is contained in \( \{|y| > r\} \). Hence by our support assumption on \( g, u \) is \( a \)-harmonic so that by (16)
\[
\sup_{R < |x| < 2R} \int_{B_*,(x)} |\nabla u|^2 \leq \sup_{R < |x| < 2R} \int_{B_{\frac{R}{2}}(x)} |\nabla u|^2 \leq \int_{\frac{R}{2} < |y| < 4R} |\nabla u|^2. \]
According to (153) in the previous step (applied to three neighboring dyadic annuli) we obtain
\[
\int_{\frac{R}{2} < |y| < 4R} |\nabla u|^2 \leq \left( \frac{r}{R} \right)^{2d} \int_{|x| < r} |g|^2.
\]
Finally by (140), c.f. Step 5 in the proof of Corollary 4, in conjunction with \( B_*(x) \cap \{|y| < r\} = \emptyset \) provided \( |x| > 4r \) (which in turn relies on \( r_\ast(x) \leq r + \frac{1}{8} |x| \)) , we have
\[
\int_{|x| < r} |g|^2 \leq \int_{|x| < 4r} \int_{B_*,(x)} |g|^2 \leq \left( \int_{|x| < 4r} \left( \int_{B_*,(x)} |g|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}.
\]
Step 3. Under the assumptions of the corollary we claim for \( R \geq 8r_\ast(0) \)
\[
\left( \int_{|x| > R} \left( \int_{B_*,(x)} |\nabla u|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}} \leq \left( \int_{|x| > R} \left( \int_{B_*,(x)} |g|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}} + \left( \int_{|x| < R} \frac{|x|}{R}^\gamma \left( \int_{B_*,(x)} |g|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}. \tag{155}
\]
W.l.o.g. we may assume that \( R \) is a dyadic multiple of \( r_\ast(0) \); also \( r \) below runs over dyadic multiple of \( r_\ast(0) \). We decompose \( g \) into
\[
g_R := I(|x| > \frac{R}{4}) g,
\]
\[
g_r := I(\frac{R}{2} < |x| < r) g \quad \text{for} \quad r_\ast(0) < r \leq \frac{R}{4}, \quad g_{r_\ast(0)} := I(|x| < r_\ast(0)) g.
\]
Let $u_r$ denote the corresponding Lax-Milgram solutions (to $-\nabla \cdot a \nabla u_r = \nabla \cdot g_r$) so that we have by the triangle inequality
\[
\left( \int_{|x|>R} \left( \int_{B_*(x)} |\nabla u_r|^2 \right)^{\frac{p}{2}} \, dx \right)^{\frac{1}{p}} \leq \left( \int_{|x|>R} \left( \int_{B_*(x)} |\nabla u_r|^2 \right)^{\frac{p}{2}} \, dx \right)^{\frac{1}{p}} + \sum_{r_* (0) \leq r \leq \frac{R}{4}} \left( \int_{|x|>R} \left( \int_{B_*(x)} |\nabla u_r|^2 \right)^{\frac{p}{2}} \, dx \right)^{\frac{1}{p}}. \tag{157}
\]
We start with the RHS term in line (156): Replacing the integration over the set $\{|x|>R\}$ by the integration over the whole space and appealing to (27) in Corollary 4 we have
\[
\left( \int_{|x|>R} \left( \int_{B_*(x)} |\nabla u_r|^2 \right)^{\frac{p}{2}} \, dx \right)^{\frac{1}{p}} \lesssim \left( \int_{|x|<\frac{R}{4}} \left( \int_{B_*(x)} |g_r|^2 \right)^{\frac{p}{2}} \, dx \right)^{\frac{1}{p}}. \tag{158}
\]
Using that $B_*(x) \cap \{|y|>\frac{R}{4}\} = \emptyset$ for $|x|<\frac{R}{8}$, we see that this term is indeed contained in (both terms of) the RHS of (155). We now turn to the terms in line (157). Since for $r \leq \frac{R}{4}$, $g_r$ is supported in $\{|x|<r\}$ we apply (154) in the previous step to the couple $(u_r, g_r)$:
\[
\left( \int_{|x|>R} \left( \int_{B_*(x)} |\nabla u_r|^2 \right)^{\frac{p}{2}} \, dx \right)^{\frac{1}{p}} \lesssim \left( \frac{r}{R} \right)^d \left( \frac{R}{r} \right)^d \int_{|x|<4r} \left( \int_{B_*(x)} |g_r|^2 \right)^{\frac{p}{2}} \, dx \right)^{\frac{1}{p}} \leq \left( \frac{r}{R} \right)^d \left( \frac{R}{r} \right)^{(d+\gamma)} \left( \frac{|x|}{R} \right)^\gamma \left( \int_{B_*(x)} |g|^2 \right)^{\frac{p}{2}} \, dx \right)^{\frac{1}{p}}.
\]
By our assumption on $\gamma$, the exponent is positive, so that the sum over dyadic $r < R$ converges, to the effect that also the contribution from (157) is estimated by the (second term on the) RHS of (155).

**Step 4. Conclusion.**

We set for abbreviation $U := (\int_{B_*(x)} |\nabla u|^2)^{\frac{p}{2}}$, $G := (\int_{B_*(x)} |g|^2)^{\frac{p}{2}}$, and $r := 8r_*(0)$; we fix a $\gamma' \in (\gamma, d(p-1))$. By (155) in the previous step and by (27) in Corollary 4 we have
\[
\int_{|x|>R} U \lesssim \int_{|x|>R} G + \int_{|x|<R} \left( \frac{|x|}{R} \right)^{\gamma'} G \text{ for } R \geq r, \quad \int U \lesssim \int G. \tag{158}
\]
From the properties (28) of $\omega$ we infer by division into dyadic annuli
\[
\int_{|x|>r} \omega(|x|) U \sim \omega(r) \int_{|x|>r} U + \sum_{R>r} (\omega(R) - \omega(\frac{R}{2})) \int_{|x|>R} U,
\]
which we also apply with $U$ replaced by $G$, so that by (158) we get
\[
\int_{|x|>r} \omega(|x|)U \lesssim \int_{|x|>r} \omega(|x|)G + \omega(r) \int_{|x|<r} \left(\frac{|x|}{r}\right)^\gamma G
\]
\[
+ \sum_{R>r} \left(\omega(R) - \omega\left(\frac{R}{2}\right)\right) \int_{|x|<R} \left(\frac{|x|}{R}\right)^\gamma G.
\]
Appealing to (28) in form of $\omega(r)I(r>|x|)(\frac{|x|}{r})^\gamma \leq \omega(|x|)$ (where we need $\gamma \leq \gamma'$) and in form of $\sum_{R>r} \omega(R)I(R>|x|)(\frac{|x|}{R})^\gamma \lesssim \omega(|x|)$ (where we need $\gamma < \gamma'$), and recalling $r = 8r_*(0)$, this turns into
\[
\int_{|x|>8r_*(0)} \omega(|x|)U \lesssim \int \omega(|x|)G.
\]
Combining the latter with the last estimate in (158) and appealing to $\omega(|x| + r_*(0)) \lesssim I(|x| > 8r_*(0))\omega(|x| + \omega(r_*(0))$ and $\omega(|x| + \omega(r_*(0)) \lesssim \omega(|x| + r_*(0))$ (which both follow from (28)), we obtain (29) in form of $\int \omega(|x| + r_*(0))U \lesssim \int \omega(|x| + r_*(0))G$.

4. Optimal stochastic integrability of $r_*$: Proof of deterministic results

4.1. Proof of Proposition 2: From modified corrector to corrector
We split the proof into three steps and start with a reduction argument.

Step 1. Reduction.
For notational ease, we replace $\tilde{f}_{BR} |f - \tilde{f}_{BR} f|^2$ by $\inf_c \tilde{f}_{BR} |f - c|^2$. We start with a couple of reductions: We first claim that it is enough to establish (41) under the additional condition $R \geq \max\{r_*, r_{**}\}$. Indeed, if $C_0 = C_0(d, \lambda, \alpha)$ denotes the constant in (12), then, by definition of $r_{**}$, there exists a constant $C_1 = C_1(d, \lambda, \alpha, \nu)$ such that
\[
\frac{1}{R^2} \inf_c \int_B |(\phi, \sigma) - c|^2 \leq \frac{1}{2d+2} \frac{1}{C_0} \text{ for all } R \geq \max\{r_*, C_1 r_{**}\}.
\]
Because of the elementary inequality $\frac{1}{R^2} \int_B |(\phi, \sigma) - c|^2 \leq \frac{1}{R^2} \int_B |f - c|^2$ this implies in turn
\[
\frac{1}{R^2} \inf_c \int_B |(\phi, \sigma) - c|^2 \leq \frac{1}{C_0} \text{ for all } R \geq \frac{1}{2} \max\{r_*, C_1 r_{**}\}.
\]
Since we may take $r_*$ to be the smallest radius with (12), we obtain $r_* \leq \frac{1}{2} \max\{r_*, C_1 r_{**}\}$ and thus $r_* \leq C_1 r_{**}$. This yields both (42) and the fact that (41) holds for all $R \geq C_1 r_{**}$, and thus, at the expense of a worse constant, for all $R \geq r_{**}$.

Moreover, using the elementary inequality $\frac{1}{R^2} \int_B |f - c|^2 \leq \frac{1}{R^2} \int_B |f|^2$ again, we find that it suffices to establish (41) for dyadic radii $R$. 

\[
\int_{|x|>r} \omega(|x|)U \lesssim \int_{|x|>r} \omega(|x|)G + \omega(r) \int_{|x|<r} \left(\frac{|x|}{r}\right)^\gamma G
\]
\[
+ \sum_{R>r} \left(\omega(R) - \omega\left(\frac{R}{2}\right)\right) \int_{|x|<R} \left(\frac{|x|}{R}\right)^\gamma G.
\]
We now turn to the last reduction argument. By qualitative homogenization (see for instance [36, Theorem 1]), we have
\[ \langle |\nabla (\phi_T, \sigma_T) - \nabla (\phi, \sigma)|^2 \rangle_T \uparrow \infty \rightarrow 0, \]
and thus by stationarity and Poincaré’s inequality,
\[ \langle \frac{1}{R^2} \inf_c \int_{B_R} |(\phi_T - \phi, \sigma_T - \sigma) - c|^2 \rangle \lesssim \left( \int_{B_R} |\nabla (\phi_T, \sigma_T) - \nabla (\phi, \sigma)|^2 \right) T \uparrow \infty \rightarrow 0. \]
Hence, by a diagonal extraction, there exists a sequence \( \sqrt{T} \) (that is, a subsequence of \((2^m)_{m \in \mathbb{N}}\)) such that for all dyadic radii \( R \) we have
\[ \lim_{k \rightarrow \infty} \frac{1}{R^2} \inf_c \int_{B_R} |(\phi_{T_k} - \phi, \sigma_{T_k} - \sigma) - c|^2 = 0 \quad \langle \cdot \rangle \text{-almost surely}. \]
We may thus conclude that it suffices to establish (41) in the modified form:

For all \( T_0 \) with \( \sqrt{T_0} \) dyadic we have
\[ \frac{1}{R^2} \inf_c \int_{B_R} |(\phi_{T_0} - \phi, \sigma_{T_0} - \sigma)|^2 \lesssim (\frac{r^*}{R})^{2\nu} \]
for \( \max\{r_*, r^{**}\} \leq R \leq \sqrt{T_0}, \quad (159) \]
where \( \lesssim \) stands for \( \leq C(d, \lambda, \alpha, \nu) \).

**Step 2.** Proof of (159) by a Campanato-iteration argument.
We shall prove (159) based on the one-step yet iterable estimate
\[ \frac{1}{r^2} \left( \inf_c \int_{B_r} (\phi_{T_0} - c)^2 + (\frac{r}{R})^d \inf_c \int_{B_r} |\sigma_{T_0} - c|^2 \right) \lesssim \left( \frac{r^*}{R} \right)^{2\nu} \]
\[ + \left( \frac{R}{r} \right)^{d+2} \int_{B_R} \frac{1}{T} |(\phi_T, \sigma_T)|^2 \]
for \( \max\{r_*, r^{**}\} \leq r \leq R \leq \sqrt{T_0} \) with \( T = R^2 \). \quad (160) \]
Let us argue how to pass from (160) to (159). For some ratio \( M = \frac{R}{r} \) to be fixed later we introduce the abbreviation \( E(r) := \frac{1}{T_0^2} \inf_c \int_{B_r} (\phi_{T_0} - c)^2 \]
\[ + M^{-d} \frac{1}{r^2} \inf_c \int_{B_r} |\sigma_{T_0} - c|^2, \]
so that (160) turns into
\[ E(r) \lesssim (1 + \frac{M^{d+2}r^2}{T_0^2}) E(Mr) + M^{d+2} (\frac{r^{**}}{Mr})^{2\nu} \]
for \( \max\{r_*, r^{**}\} \leq r \leq \sqrt{T_0}/M, \)
where we used (40) to estimate the last term in (160). Restricting the range of \( r \)'s a bit, this simplifies to

\[
E(r) \lesssim E(Mr) + M^{d+2} \left( \frac{r_{**}}{Mr} \right)^{2\nu} \quad \text{for} \quad \max \{ r_*, r_{**} \} \leq r \leq \left( \frac{1}{M} \right)^{\frac{d}{4} + 1} \sqrt{T_0},
\]

which we multiply with \( r^{2\nu} \) and make the constant \( C = C(d, \lambda, \alpha, \nu) \) explicit:

\[
r^{2\nu} E(r) \leq C \left( M^{-2\nu} (Mr)^{2\nu} E(Mr) + M^{d+2} \left( \frac{T_{**}}{M} \right)^{2\nu} \right).
\]

We now choose \( M = M(d, \lambda, \alpha, \nu) \) so large that \( CM^{-2\nu} = \frac{1}{2} \), which is possible because of \( \nu > 0 \), and so obtain

\[
\sup_{\max \{ r_*, r_{**} \} \leq r \leq \left( \frac{1}{M} \right)^{\frac{d}{2}} \sqrt{T_0}} r^{2\nu} E(r) \leq \left( \frac{1}{M} \right)^{\frac{d}{2} + 1} \sqrt{T_0} \sup_{R \leq \sqrt{T_0}} R^{2\nu} E(R) + r_{**}^{2\nu},
\]

which yields

\[
\sup_{\max \{ r_*, r_{**} \} \leq r \leq \sqrt{T_0}} r^{2\nu} E(r) \lesssim \sup_{\frac{1}{M} \sqrt{T_0} \leq R \leq \sqrt{T_0}} R^{2\nu} E(R) + r_{**}^{2\nu},
\]

after adding \( \sup_{\left( \frac{1}{M} \right)^{\frac{d}{2} + 1} \sqrt{T_0} \leq R \leq \sqrt{T_0}} R^{2\nu} E(R) \) to both sides of the inequality and absorbing then \( \frac{1}{2} \sup_{\max \{ r_*, r_{**} \} \leq r \leq \left( \frac{1}{M} \right)^{\frac{d}{2}} \sqrt{T_0}} r^{2\nu} E(r) \) in the LHS. It is here that we use \( T_0 < \infty \), so that we only have to take the supremum over a finite range of radii and may therefore absorb the RHS into the LHS without any a priori assumption of finiteness. In view of the definition of \( E(r) \), which in particular yields using that \( R \geq r_{**} \)

\[
\sup_{\left( \frac{1}{M} \right)^{\frac{d}{2} + 1} \sqrt{T_0} \leq R \leq \sqrt{T_0}} R^{2\nu} E(R) \lesssim \sqrt{T_0}^{2\nu} \int_{B_{\sqrt{T_0}}} \frac{1}{T_0} |(\phi_{T_0}, \sigma_{T_0})|^2 \lesssim r_{**}^{2\nu},
\]

this last estimate turns into (159).

**Step 3.** Proof of (160)

By the triangle inequality in \( L^2 \) and \( f_{Br} \leq (\frac{R}{r})^d f_{BR} \), it is enough to show

\[
\frac{1}{r^2} \left( \inf_{c} \int_{B_r} (\phi_{T_0} - \phi_T - c)^2 \right) \lesssim \left( \frac{1}{R^2} + \left( \frac{R}{r} \right)^d \frac{1}{T_0} \right) \left( \inf_{c} \int_{B_{Rr}} (\phi_{T_0} - \phi_T - c)^2 \right) + \left( \frac{R}{r} \right)^d \frac{1}{T} |(\phi_T, \sigma_T)|^2 \text{ for } r_* \leq r \leq \frac{1}{4} R \leq \sqrt{T_0} \text{ with } T = R^2.
\]
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(The exponent \(d + 2\) in the last RHS term of (160) comes from the control of \(\frac{1}{r^2} \int_{B_r} \phi_T^2\).) By Poincaré’s inequality the latter follows from

\[
\int_{B_r} |\nabla (\phi_{T_0} - \phi_T)|^2 + \left(\frac{r}{R}\right)^d \int_{B_r} |\nabla (\sigma_{T_0} - \sigma_T)|^2 \\
\lesssim \left(\frac{1}{R^2} + \left(\frac{R}{r}\right)^d \frac{1}{T_0}\right) \left(\inf_c \int_{B_R} (\phi_{T_0} - \phi_T - c)^2 + \left(\frac{r}{R}\right)^d \inf_c \int_{B_R} |\sigma_{T_0} - \sigma_T - c|^2\right) \\
+ \left(\frac{R}{r}\right)^d \int_{B_R} \frac{1}{T}|(\phi_T, \sigma_T)|^2,
\]

always in the same range. We note that we may split (161) into

\[
\int_{B_r} |\nabla (\phi_{T_0} - \phi_T)|^2 \\
\lesssim \left(\frac{1}{R^2} + \left(\frac{R}{r}\right)^d \frac{1}{T_0}\right) \left(\inf_c \int_{B_R} (\phi_{T_0} - \phi_T - c)^2 + \left(\frac{R}{r}\right)^d \int_{B_R} \frac{1}{T}\phi_T^2\right),
\]

and

\[
\int_{B_r} |\nabla (\sigma_{T_0} - \sigma_T)|^2 \lesssim \left(\frac{1}{R^2} + \left(\frac{R}{r}\right)^d \frac{1}{T_0}\right) \left(\inf_c \int_{B_R} |\sigma_{T_0} - \sigma_T - c|^2\right) \\
+ \left(\frac{R}{r}\right)^d \left(\int_{B_{\frac{R}{2}}} \frac{1}{T}|\sigma_T|^2 + \int_{B_{\frac{R}{2}}} |\nabla (\phi_{T_0} - \phi_T)|^2\right).
\]

Indeed, we first use (162) for \(r = \frac{R}{2}\), insert the result into (163) and multiply with \(\left(\frac{R}{r}\right)^d\); we then add (162) to it to obtain (161).

We claim that both (162) and (163) may be inferred from the following a priori estimate: Suppose the functions \(u, f\), and the vector field \(g\) are related by

\[
\frac{1}{T_0} u - \nabla \cdot a \nabla u = f + \nabla \times g,
\]

then we have

\[
\int_{B_r} |\nabla u|^2 \lesssim \left(\frac{1}{R^2} + \left(\frac{R}{r}\right)^d \frac{1}{T_0}\right) \left(\inf_c \int_{B_R} (u - c)^2\right) \\
+ \left(\frac{R}{r}\right)^d \left(R^2 \int_{B_R} f^2 + \int_{B_R} |g|^2\right) \quad \text{for} \quad r_* \leq r \leq \frac{R}{2}.
\]

In order to obtain (162), we apply this to \(u = \phi_{T_0} - \phi_T\), which by (37) satisfies (164) with \(g = 0\) and \(f = \left(\frac{1}{T} - \frac{1}{T_0}\right)\phi_T\), recalling that \(T = R^2\). For (163), we apply (165) to \(u = \sigma_{T_0} - \sigma_T\) and the identity matrix playing the role of \(a\) (so that trivially \(r_* = 0\)) and \(R\) replaced by \(\frac{R}{2}\). Indeed, by (39) we have (164) with \(g = a \nabla (\phi_{T_0} - \phi_T)\) and \(f = \left(\frac{1}{T} - \frac{1}{T_0}\right)\sigma_T\).

**Step 4.** Argument for the a priori estimate (165).

We start by applying Caccioppoli estimate (205) to (164). Because of the presence of the massive term, the Caccioppoli estimate is slightly more subtle: We test (164) with \(\eta^2(u - \bar{u})\), where \(\eta\) is a smooth cut-off for \(B_{\frac{R}{2}}\) in \(B_R\)
and where \( \bar{u} := \frac{\eta^2 u}{\int \eta^2} \) is the corresponding spatial average, to the effect of \( \int \eta^2 (u - \bar{u})u = \int \eta^2 (u - \bar{u})^2 \). Hence we obtain
\[
\int_{B_R} \left( \frac{1}{T_0} \eta^2 (u - \bar{u})^2 + |\nabla (\eta(u - \bar{u}))|^2 \right) \\
\leq \int_{B_R} \left( \frac{1}{R^2} (u - \bar{u})^2 + R^2 f^2 + |g|^2 \right) \\
\leq \inf_{c} \int_{B_R} \left( \frac{1}{R^2} (u - c)^2 + R^2 f^2 + |g|^2 \right).
\]
From testing (164) with \( \eta^2 \) and rewriting the elliptic term like \( \nabla \eta^2 \cdot a \nabla u = 2 \nabla \eta \cdot a \nabla (\eta(u - \bar{u})) - 2(u - \bar{u}) \nabla \eta \cdot a \nabla \eta \) we obtain
\[
R^2 \left( \frac{1}{T_0} \bar{u} \right)^2 \leq \int_{B_R} \left( \frac{1}{R^2} (u - \bar{u})^2 + R^2 f^2 + |g|^2 + |\nabla (\eta(u - \bar{u}))|^2 \right) \\
\leq \inf_{c} \int_{B_R} \left( \frac{1}{R^2} (u - c)^2 + R^2 f^2 + |g|^2 + |\nabla (\eta(u - \bar{u}))|^2 \right).
\]
The combination of these two estimates yields
\[
R^2 \left( \frac{1}{T_0} \bar{u} \right)^2 + \int_{B_{\frac{R}{2}}} \left( \frac{1}{T_0} (u - \bar{u})^2 + |\nabla u|^2 \right) \leq \inf_{c} \int_{B_{\frac{R}{2}}} \left( \frac{1}{R^2} (u - c)^2 + R^2 f^2 + |g|^2 \right). \tag{166}
\]
We now split \( u \) on \( B_{\frac{R}{2}} \) into two functions \( v \) and \( w \) defined through the auxiliary boundary value problems
\[
- \nabla \cdot a \nabla v = - \frac{1}{T_0} \bar{u} \text{ in } B_{\frac{R}{2}}, \quad v = u \text{ on } \partial B_{\frac{R}{2}}, \\
- \nabla \cdot a \nabla w = - \frac{1}{T_0} (u - \bar{u}) + f + \nabla \times g \text{ in } B_{\frac{R}{2}}, \quad w = 0 \text{ on } \partial B_{\frac{R}{2}}.
\]
By the energy estimate, combined with Poincaré’s estimate with vanishing boundary conditions, we have
\[
\int_{B_{\frac{R}{2}}} |\nabla v|^2 \leq R^2 \left( \frac{1}{T_0} \bar{u} \right)^2 + \int_{B_{\frac{R}{2}}} |\nabla u|^2, \tag{167}
\]
\[
\int_{B_{\frac{R}{2}}} |\nabla w|^2 \leq \int_{B_{\frac{R}{2}}} \left( \frac{R^2}{T_0} (u - \bar{u})^2 + R^2 f^2 + |g|^2 \right). \tag{168}
\]
In order to apply the Schauder theory from Corollary 3 to \( v \), we note that its RHS may be rewritten as \( - \nabla \cdot a \nabla v = \nabla \cdot h \) with \( h := \frac{1}{T_0} \bar{u} \). Because of
\[
\sup_{1 \leq \rho \leq \frac{R}{2}} \left( \frac{R}{\rho} \right)^{2\alpha} \inf_{\xi \in \mathbb{R}^d} \int_{B_{\rho}} |h - \xi|^2 \alpha^{\leq 1} \leq R^2 \left( \frac{1}{T_0} \bar{u} \right)^2,
\]
we obtain from (25) that for \( r_* \leq r \leq \frac{R}{2} \)
\[
\int_{B_r} |\nabla v|^2 \leq R^2 \left( \frac{1}{T_0} \bar{u} \right)^2 + \int_{B_{\frac{R}{2}}} |\nabla v|^2.
\]
Hence (167) & (168) turn into
\[
\int_{B_r} |\nabla v|^2 \lesssim R^2 (\frac{1}{T_0} \bar{u})^2 + \int_{B_{\frac{r}{2}}} |\nabla u|^2;
\]
\[
\int_{B_r} |\nabla w|^2 \lesssim (\frac{R}{r})^d \int_{B_{\frac{r}{2}}} \left( \frac{R^2}{T_0} (u - \bar{u})^2 + R^2 f^2 + |g|^2 \right).
\]

Inserting (166) into these estimates, we obtain
\[
\int_{B_r} |\nabla v|^2 \lesssim \inf_c \int_{B_R} \left( \frac{1}{R^2} (u - c)^2 + R^2 f^2 + |g|^2 \right),
\]
\[
\int_{B_r} |\nabla w|^2 \lesssim (\frac{R}{r})^d \left( \inf_c \int_{B_R} \left( \frac{1}{T_0} (u - c)^2 + R^2 f^2 + |g|^2 \right) \right).
\]

By the triangle inequality in $L^2$, this yields (165).

4.2. Proof of Proposition 3: Localization of averages of the modified corrector

The main building block is the following localized energy estimate. Suppose the function $u$ is of the class sup $y \int_{B_1(y)} (u^2 + |\nabla u|^2) < \infty$ and satisfies $\frac{1}{T} u - \nabla \cdot a \nabla u = \frac{1}{T} f + \nabla \cdot h$ for some scalar field $f$ and some vector field $h$. Then we have
\[
\int \omega_T (\frac{1}{T} u^2 + |\nabla u|^2) \lesssim \int \omega_T (\frac{1}{T} f^2 + |h|^2).
\]

Indeed, this follows from testing the equation with $\eta^2 u$ and arguing like in case of the Caccioppoli estimate, cf. Appendix A, that $\int (\frac{1}{T} (\eta u)^2 + |\nabla (\eta u)|^2) \lesssim \int (\frac{1}{T} f^2 + \eta^2 |h|^2 + |\nabla \eta|^2 u^2)$. We then use this for $\eta = \sqrt{T} \omega_T$ and note that $\eta^2 \sim \omega_T$ and $|\nabla \eta|^2 \ll \frac{1}{T} \omega_T$, provided the constant $C = C(d, \lambda)$ in (44) is chosen large enough.

We now seek to apply (169) to $u = \phi_T - \phi_t$, which by (37) satisfies $\frac{1}{T} u - \nabla \cdot a \nabla u = (\frac{1}{t} - \frac{1}{T}) \phi_t$, but want to bring $\frac{1}{t} \phi_t$ in divergence-form. We denote the convolution of a function $f$ with the (centered) Gaussian $G_t(z) = (\frac{1}{\sqrt{2\pi t}})^d \exp(-\frac{|z|^2}{2t})$ of variance $t$ by $f_{*t}$. Since then $\partial_t f_{*t} = \frac{1}{2} \Delta f_{*t}$ we have $\frac{1}{t} \phi_t = \frac{1}{t} (\phi_t)_{*t} - \frac{1}{2t} \int_0^t \Delta (\phi_t)_{*t} d\tau$, so that using (37) and (38) we obtain
\[
(\frac{1}{t} - \frac{1}{T}) \phi_t = \nabla \cdot h \quad \text{with} \quad h := (1 - \frac{t}{T}) ((q_t - \langle q_t \rangle)_{*t} - \frac{1}{2t} \int_0^t (\nabla \phi_t)_{*t} d\tau). \]
We now may apply (169) to \( u = \phi_T - \phi_t \) and the above \( h \) so that by the triangle inequality in \( L^2(\omega_T) \) and by the definition of \( g_t \), cf. (43),
\[
\int \omega_T \left( \frac{1}{T}(\phi_T - \phi_t)^2 + |\nabla(\phi_T - \phi_t)|^2 \right)
\lesssim \int \omega_T|(g_t - \langle g_t \rangle_\ast t)|^2 + \left( \frac{1}{t} \int_0^t \int \omega_T|(\nabla \phi_t)_{\ast t}|^2 \right)^{\frac{3}{2}} d\tau.
\]

Using (43),
\[
\int \omega_T|(g_t)_{\ast t}|^2 + t \int \omega_T|(\Delta g_t)_{\ast t}|^2
+ \left( \frac{1}{t} \int_0^t \int \omega_T|(\nabla \phi_t)_{\ast t}|^2 \right)^{\frac{3}{2}} d\tau.
\tag{170}
\]

In order to conclude we appeal to a couple of properties of the convolution operator. We note that because of Jensen’s inequality in form of \((f^2)_{\ast t} \leq |f|^2_{\ast t}\) and the dominance of Gaussians by exponentials in form of \((\omega_T)_{\ast t} \lesssim \omega_T\) for \( t \leq T \) (up to increasing the constant \( C \) in (44) which we implicitly assume without changing notation) we have
\[
\int \omega_T (f_{\ast t})^2 \lesssim \int \omega_T f^2 \quad \text{for} \quad t \leq T.
\]

Furthermore, since for our Gaussian we have \( \nabla G_t(z) = -\frac{z}{t} G_t(z) \) and \( \Delta G_t = \left( \frac{|z|^2}{t^2} - \frac{d}{t} \right) G_t \), and thus \( |\nabla G_t| \lesssim \frac{1}{\sqrt{t}} G_{\frac{1}{2} t} \) and \( |\Delta G_t| \lesssim \frac{1}{t} G_{\frac{1}{2} t} \), we obtain
\[
|\nabla f_{\ast t}| \lesssim \frac{1}{\sqrt{t}} |f|_{\ast t} \quad \text{and} \quad |\Delta f_{\ast t}| \lesssim \frac{1}{t} |f|_{\ast t}.
\]

Equipped with these auxiliary statements, we see that (170) turns into
\[
\int \omega_T \left( \frac{1}{T}(\phi_T - \phi_t)^2 + |\nabla(\phi_T - \phi_t)|^2 \right) \lesssim \int \omega_T \frac{1}{t} |g_t|^2 + \phi_t^2. \tag{171}
\]

We now turn to the \( \sigma \)-part. Note that by an application of the differential operator \( \sqrt{T} \nabla \times \) to (43) we recover (39), and thus
\[
\sigma_t = \sqrt{T} \nabla \times g_t. \tag{172}
\]

Hence by (38) and once more by (39) and (43) we get
\[
\frac{1}{T}(\sigma_T - \sigma_t) - \Delta(\sigma_T - \sigma_t) = \left( 1 - \frac{t}{T} \right) \nabla \times \left( \frac{1}{\sqrt{t}} g_t \right) + \nabla \times a \nabla(\phi_T - \phi_t),
\]

so that by (169) we have in particular
\[
\int \omega_T \frac{1}{T} |\sigma_T - \sigma_t|^2 \lesssim \int \omega_T \left( \frac{1}{t} |g_t|^2 + |\nabla(\phi_T - \phi_t)|^2 \right).
\]

The combination of this with (171) yields
\[
\int \omega_T \frac{1}{T} |(\phi_T, \sigma_T) - (\phi_t, \sigma_t)|^2 \lesssim \int \omega_T \frac{1}{t} |(\phi_t, g_t)|^2.
\]

By the triangle inequality in \( L^2 \), this clearly implies (45) in conjunction with
\[
\int \omega_T \frac{1}{T} |\sigma_t|^2 \lesssim \int \omega_T |\nabla g_t|^2,
\]

which follows from (172).
4.3. Proof of Proposition 4: Locality of the modified corrector

We start by noting that (47) is an easy consequence of (169) (with $T$ replaced by $t$): We first apply it to $u = \phi_t$ and thus $h = ac$ and $f = 0$, see (37), to the effect of $\int \omega_t (\frac{1}{t} \phi_t^2 + |\nabla \phi_t|^2) \lesssim 1$; for later reference we note

$$\int \omega_t |e + \nabla \phi_t|^2 \lesssim 1. \quad \text{(173)}$$

In view of (38) and stationarity in form of $|\langle q_t \rangle|^2 = \langle |q_t|^2 \rangle \leq \langle \int \omega_t |q_t|^2 \rangle$ this yields in particular $\int \omega_t |q_t - \langle q_t \rangle|^2 \lesssim 1$. We then apply (169) (with $\text{Id}$ playing the role of $a$) to $u = g_t$ and thus $f = \sqrt{t}(q_t - \langle q_t \rangle)$ and $h = 0$, see (43), which yields $\int \omega_t (\frac{1}{t} |g_t|^2 + |\nabla g_t|^2) \lesssim \int \omega_t |q_t - \langle q_t \rangle|^2$. The combination gives (47).

We now turn to (48) and write for abbreviation $(\phi_t', q_t', q_t') = (\phi_t, q_t, q_t)(a')$ in order to reserve $(\phi_t, q_t, q_t)$ for $(\phi_t, q_t, q_t)(a)$. We first apply (169) (always with $T$ replaced by $t$) to $u = \phi_t' - \phi_t$ so that $h = (a' - a)(e + \nabla \phi_t)$, see (37), and with $a'$ playing the role of $a$, to obtain

$$\int \omega_t (\frac{1}{t} (\phi_t' - \phi_t)^2 + |\nabla (\phi_t' - \phi_t)|^2) \lesssim \int \omega_t |(a' - a)(e + \nabla \phi_t)|^2. \quad \text{(174)}$$

We then apply (169) to $u = g_t' - g_t$ and $f = \sqrt{t}(q_t' - q_t)$, see (43), to the effect of

$$\int \omega_t (\frac{1}{t} |g_t' - g_t|^2 + |\nabla (g_t' - g_t)|^2) \lesssim \int \omega_t |q_t' - q_t|^2. \quad \text{(175)}$$

In view of (38) and the triangle inequality in $L^2(\omega_t)$ we obviously have

$$\int \omega_t |q_t' - q_t|^2 \lesssim \int \omega_t |\nabla (\phi_t' - \phi_t)|^2 + \int \omega_t |(a' - a)(e + \nabla \phi_t)|^2. \quad \text{(176)}$$

These three estimates combine to

$$\int \omega_t (\frac{1}{t} (\phi_t' - \phi_t)^2 + \frac{1}{t} |g_t' - g_t|^2 + |\nabla (g_t' - g_t)|^2) \lesssim \int \omega_t |(a' - a)(e + \nabla \phi_t)|^2. \quad \text{(177)}$$

Using our assumption that $a' - a$ vanishes on $B_R$, appealing to the elementary estimate $\omega_t \lesssim \exp(-\frac{|x|}{2C\sqrt{t}})\omega_{\frac{1}{2}}$, see (44), and using that $\int \omega_{\frac{1}{2}} |e + \nabla \phi_t|^2 \lesssim 1$ (which follows from (173) since by shift-covariance the latter yields $\sup_y \int_{B_{\sqrt{t}}(y)} |e + \nabla \phi_t|^2 \lesssim 1$), we have

$$\int \omega_t |(a' - a)(e + \nabla \phi_t)|^2 \lesssim \exp(-\frac{R}{2C\sqrt{t}}).$$

By the triangle inequality in $L^2(\omega_t)$ and by (47), the two last estimates yield (48).

4.4. Proof of Lemma 2: Control by averages

We first argue that

$$\langle \phi_T^2 \rangle \lesssim \int_0^T \langle |(\nabla \phi_T)_{st}|^2 + |(q_T - \langle q_T \rangle)_{st}|^2 \rangle dt. \quad \text{(178)}$$
By definition of $f_{st}$ as the convolution of $f$ with the Gaussian of variance $t$ we have $\partial_t f_{st} = \frac{1}{2} \Delta f_{st}$ and thus $\partial_t (f_{st})^2 = -|\nabla f_{st}|^2 + \nabla \cdot (f_{st} \nabla f_{st})$. Applied to the stationary $f = \phi_T$ we thus obtain

$$\frac{d}{dt} (\langle (\phi_T)_{st} \rangle^2) = -\langle |(\nabla \phi_T)_{st}|^2 \rangle. \tag{175}$$

From (37) and (38) in form of $\phi_T = T \nabla \cdot (q_T - \langle q_T \rangle)$ and the semi-group property of convolution with Gaussians in form of $(\cdot)_{st} = (\cdot)_{\frac{t}{2}} \ast \frac{t}{2}$ we obtain $\langle (\phi_T)_{st} \rangle = T \langle \nabla G_{\frac{t}{2}} \ast (q_T - \langle q_T \rangle) \rangle_{\frac{t}{2}}$, where $G_{\frac{t}{2}}$ denotes the Gaussian of variance $\frac{t}{2}$, so that by Jensen’s inequality

$$\langle (\langle (\phi_T)_{st} \rangle^2 \rangle \leq T^2 \langle \langle |\nabla G_{\frac{t}{2}}|^2 \rangle \langle |(q_T - \langle q_T \rangle)_{\frac{t}{2}}|^2 \rangle \rangle \tag{176} \leq \frac{T^2}{t} \langle \langle |(q_T - \langle q_T \rangle)_{\frac{t}{2}}|^2 \rangle \rangle.$$

By appealing to the elementary inequality for the function $[0, T] \ni t \mapsto \langle (\langle (\phi_T)_{st} \rangle^2 \rangle$,

$$\langle \phi_T^2 \rangle \leq \int_0^T \left| \frac{d}{dt} \langle (\langle (\phi_T)_{st} \rangle)^2 \rangle \right| dt + \frac{1}{T} \int_{\frac{t}{2}}^T \langle \langle (\langle (\phi_T)_{st} \rangle)^2 \rangle \rangle dt,$$

into which we insert (175) & (176), we obtain (174).

We now argue that

$$\langle |g_T|^2 + T |\nabla g_T|^2 \rangle \leq \int_0^T \langle \langle (q_T - \langle q_T \rangle)_{\ast t} \rangle^2 \rangle dt. \tag{177}$$

Indeed, from $\partial_t f_{st} = \frac{1}{2} \Delta f_{st}$ it is easy to check that $u = \int_0^\infty \exp(-\frac{t}{2T}) f_{st} dt$ provides the solution of $\frac{1}{T} u - \Delta u = f$, so that from (43) we obtain

$$g_T = \frac{1}{\sqrt{T}} \int_0^\infty \exp(-\frac{t}{2T}) (q_T - \langle q_T \rangle)_{\ast t} dt.$$

Testing (43) with $g_T$, using the stationarity of $g_T$ and the above representation, we obtain

$$\langle \frac{1}{T} |g_T|^2 + |\nabla g_T|^2 \rangle = \frac{1}{T} \int_0^\infty \exp(-\frac{t}{2T}) \langle \langle (q_T - \langle q_T \rangle)_{\ast \frac{t}{2}} \rangle^2 \rangle dt,$$

where on the RHS we used the semi-group property and symmetry of $(\cdot)_{st}$ in form of $\langle \langle (q_T - \langle q_T \rangle)_{\ast \frac{t}{2}} \rangle \cdot (q_T - \langle q_T \rangle) \rangle = \langle \langle (q_T - \langle q_T \rangle)_{\ast \frac{t}{2}} \rangle^2 \rangle$. After the change of variables $\frac{t}{2} = t'$, splitting the integral into $\int_0^T dt' + \int_T^\infty dt'$ and using that by Jensen’s inequality $\langle \langle (q_T - \langle q_T \rangle)_{\ast t'} \rangle^2 \rangle \leq \langle \langle (q_T - \langle q_T \rangle)_{\ast t''} \rangle^2 \rangle$ for $t' \geq t''$ we obtain (177).

4.5. Proof of Lemma 3: Deterministic sensitivity estimate

We split the proof into five steps. In Step 1 we reformulate the carré du champ via partition norms, which enables us to argue by duality. In Step 2 we provide a deterministic sensitivity estimate for $(\nabla \phi_T, q_T)$. It relies on the hole-filling argument provided in Step 5. In Step 3 we treat the case of the functional derivative and in Step 4 the case of the oscillation.
Step 1. Reformulation by duality.
Set \( P := \{ x + Q_0 : x \in \mathbb{Z}^d \} \), where \( Q_0 = [-\frac{1}{2}, \frac{1}{2})^d \) denotes the unit cube. Let \( F = F(a) \) and \( \ell \geq 0 \). On the one hand we argue that for the functional derivative we have the implication
\[
\sup_{a' \neq a} \frac{|F(a') - F(a)|}{\|a - a'|_{\ell+1,*}} \leq 1 \quad \Rightarrow \quad (\ell + 1)^{-d} \sup_a \int |\partial_{x,\ell+1}^\text{fct} F|^2 dx \lesssim 1, \quad (178)
\]
where \( \|a - a'|_{\ell+1,*} := \sum_{Q \in (\ell+1)P} \sup_{x \in Q} \|a - a'|^2 \). On the other hand, for the oscillation, we have the corresponding implication
\[
\sup_{z \in \mathbb{R}^d} \sup_{a} \sum_{Q \in z + 2(\ell+1)P} |\partial_{x,\ell+1}^\text{osc} F|^2 \leq 1 \quad \Rightarrow \quad (\ell + 1)^{-d} \sup_a \int |\partial_{x,\ell+1}^\text{osc} F|^2 dx \lesssim 1, \quad (179)
\]
where in line with Definition 1
\[
|\partial_{x,\ell+1}^\text{osc} F(a)| := \sup \{ F(a') - F(a'') : a' = a'' = a \text{ in } \mathbb{R}^d \setminus Q \}.
\]
We start with the proof of (178). By Definition 1, we obviously have for \( x \in Q \)
\[
|\partial_{x,\ell+1}^\text{fct} F| \leq \int_{B_{\ell+1}(Q)} \left| \frac{\partial F}{\partial a(z)} \right| dz,
\]
where \( B_{\ell+1}(Q) := \{ x : \text{dist}(x, Q) < \ell + 1 \} \). Combined with the additivity of the functional derivative with respect to sets, this yields
\[
\int |\partial_{x,\ell+1}^\text{fct} F|^2 dx = \sum_{Q \in (\ell+1)P} \int_Q |\partial_{x,\ell+1}^\text{fct} F|^2 dx \leq \sum_{Q \in (\ell+1)P} (\ell + 1)^d \left( \int_{B_{\ell+1}(Q)} \left| \frac{\partial F}{\partial a(z)} \right| dz \right)^2 \leq 3^2d(\ell + 1)^d \sum_{Q \in (\ell+1)P} \left( \int_Q \left| \frac{\partial F}{\partial a(z)} \right| dz \right)^2.
\]
Since \((\ell+1)P\) is a partition and the norm \( \| \cdot \|_{\ell+1,*} \) is adapted to this partition, we have by duality
\[
\left( \sum_{Q \in (\ell+1)P} \left( \int_Q \left| \frac{\partial F}{\partial a(z)} \right| dz \right)^2 \right)^{\frac{1}{2}} = \sup_{\| \delta a \|_{\ell+1,*} = 1} \int \frac{\partial F}{\partial a(x)} \cdot \delta a(x) dx,
\]
and the desired estimate (178) follows by bounding the functional derivative by the Lipschitz norm.

We turn now to the proof of (179). For all \( x \in \mathbb{R}^d \), by Definition 1,
\[
|\partial_{x,\ell+1}^\text{osc} F(a)| = \sup \{ F(a') - F(a'') : a' = a'' = a \text{ in } \mathbb{R}^d \setminus B_{\ell+1}(x) \} \leq |\partial_{x+2(\ell+1)Q_0}^\text{osc} F(a)|,
\]
so that

\[
\int |\partial_{x,\ell+1}^{\text{osc}} F|^2\,dx \leq (2(\ell + 1))^d \sum_{z \in (\ell+1)^d} \int_{z+2(\ell+1)Q_0} |\partial_{x,\ell+1}^{\text{osc}} F|^2\,dx
\]

\[
= (2(\ell + 1))^d \int_{2(\ell+1)Q_0} \sum_{Q \in z+2(\ell+1)P} |\partial_{Q}^{\text{osc}} F|^2\,dz,
\]

and (179) follows.

**Step 2.** Deterministic sensitivity estimate using hole-filling.

Fix two \(\lambda\)-uniformly elliptic coefficient fields \(a, a'\) and set for abbreviation

\[
\delta F := (\nabla \phi_T(a'), q_T(a')) - (\nabla \phi_T(a), q_T(a)).
\]

Then there exists an exponent \(\varepsilon = \varepsilon(d, \lambda) > 0\) (coming from hole-filling) such that

\[
\int \omega_T |\delta F|^2 \lesssim \|a - a'\|_{\ell+1,*}^2 (\frac{\ell + 1}{\sqrt{T}})^{\varepsilon d}.
\]  

(180)

For the argument set \(\delta a := a' - a, \delta \phi_T := \phi_T(a') - \phi_T(a)\) and \(\delta q_T := q_T(a') - q_T(a)\) and note that

\[
\left(\frac{1}{T} - \nabla \cdot a \nabla\right) \delta \phi_T = \nabla \cdot \delta a (\nabla \phi_T(a') + e),
\]

(181)

\[
\delta q_T = \delta a (\nabla \phi_T(a') + e) + a \nabla \delta \phi_T.
\]

(182)

The energy estimate (169) applied to (181) yields

\[
\int \omega_T \left(\frac{1}{T} \delta \phi_T^2 + |\nabla \delta \phi_T|^2\right) \lesssim \int \omega_T |\delta a (\nabla \phi_T(a') + e)|^2,
\]

and thus by (182)

\[
\int \omega_T |\delta F|^2 \lesssim \int \omega_T |\delta a (\nabla \phi_T(a') + e)|^2
\]

\[
\leq \sum_{Q \in (\ell+1)P} \sup_{Q} |a' - a|^2 \int_{Q} \omega_T |\nabla \phi_T(a') + e|^2
\]

\[
\leq \|a' - a\|_{\ell+1,*}^2 \sup_{Q \in (\ell+1)P} \int_{Q} \omega_T |\nabla \phi_T(a') + e|^2.
\]

Hence for (180), it suffices to prove

\[
\sup_{Q \in (\ell+1)P} \int_{Q} \omega_T |\nabla \phi_T(a') + e|^2 \lesssim \left(\frac{\ell + 1}{\sqrt{T}}\right)^{\varepsilon d}
\]  

(183)

which we postpone to the last step.

**Step 3.** Proof of (50) for the functional derivative.

Set \(F_{st} := (\nabla \phi_T, q_T)_{st}\). In view of Step 1, cf. (178), it suffices to show that for any pair of coefficient fields \(a, a'\) we have

\[
|F_{st}(a') - F_{st}(a)|^2 \lesssim \left(\frac{\sqrt{T}}{\sqrt{t}}\right)^d \left(\frac{\ell + 1}{\sqrt{T}}\right)^{\varepsilon d} \|a' - a\|_{\ell+1,*}^2.
\]  

(184)
Using Jensen’s inequality for the Gaussian measure, followed by the relation $G_t \lesssim \left( \frac{\sqrt{T}}{\sqrt{t}} \right)^d \omega_T$ (since $T \geq t$) between the Gaussian and exponential weights $G_t$ and $\omega_T$, we obtain with $\delta F$ defined as in Step 2,

$$|F_t(a') - F_t(a)|^2 \leq (|\delta F|^2)_{st} \lesssim \left( \frac{\sqrt{T}}{\sqrt{t}} \right)^d \int \omega_T |\delta S|^2.$$  

The desired estimate (184) then follows from (180) in Step 2.

**Step 4. Proof of (50) for the oscillation.**

Set again $F_t := (\nabla \phi_T, q_T)_{st}$. In view of Step 1, cf. (179), using

$$|\partial_{osc} Q F_t(a)| \leq 2 \sup \{|F_t(a_Q) - F_t(a)| \mid a_Q = a \text{ in } \mathbb{R}^d \setminus Q\},$$

and by discrete duality, it suffices to show that for any coefficient field $a$, any shift $z \in \mathbb{R}^d$, any family $\{a_Q\}_{Q \in z + (\ell + 1)P}$ of coefficient fields $a_Q$ with $a = a_Q$ outside of $Q$, and any real sequence $\omega = \{\omega_Q\}_{Q \in z + (\ell + 1)P}$, we have

$$\left( \sum_{Q \in z + (\ell + 1)P} \omega_Q \left( F_t(a_Q) - F_t(a) \right) \right)^2 \lesssim \left( \frac{\sqrt{T}}{\sqrt{t}} \right)^d \left( 1 + \frac{1}{\sqrt{T}} \right)^{\varepsilon d} \sum_{Q \in z + (\ell + 1)P} \omega_Q^2. \quad (185)$$

For notational convenience we replace $z$ by 0. For any $Q \in (\ell + 1)P$ set

$$\delta_Q a := a_Q - a, \quad \delta_Q \phi_T := \phi_T(a_Q) - \phi_T(a), \quad \delta_Q q_T := q_T(a_Q) - q_T(a),$$

and

$$\delta F := \sum_{Q \in (\ell + 1)P} \omega_Q (\nabla \delta_Q \phi_T, \delta_Q q_T).$$

Then as in Step 3 we have

$$\left( \sum_{Q \in (\ell + 1)P} \omega_Q \left( F_t(a_Q) - F_t(a) \right) \right)^2 \leq (|\delta F|^2)_{st} \lesssim \left( \frac{\sqrt{T}}{\sqrt{t}} \right)^d \int \omega_T |\delta S|^2.$$

Moreover, $(\delta_Q \phi_T, \delta_Q q_T)$ are decaying solutions of

$$\left( \frac{1}{T} - \nabla \cdot a \nabla \right) \delta_Q \phi_T = \nabla \cdot \delta_Q a (\nabla \phi_T(a_Q) + e),$$

$$\delta_Q q_T = \delta_Q a (\nabla \phi_T(a_Q) + e) + a \nabla \delta_Q \phi_T.$$

Multiplying with $\omega_Q$ and summing over $Q$, the energy estimate (169) yields as in Step 2

$$\int \omega_T |\delta F|^2 \lesssim \int \omega_T \left| \sum_{Q \in (\ell + 1)P} \omega_Q \delta_Q a (\nabla \phi_T(a_Q) + e) \right|^2.$$
Since $\text{supp} \delta_Q a \subset Q$ this yields
\[
\int \omega_T |\delta F|^2 \lesssim \sum_{Q \in (\ell+1)P} (\sup_Q |a_Q - a|)^2 \omega_Q \int_Q \omega_T |\nabla \phi_T(a_Q) + e|^2 \\
\lesssim \left( \sum_{Q \in (\ell+1)P} \omega_Q^2 \right) \sup_{Q \in (\ell+1)P} \int_Q \omega_T |\nabla \phi_T(a_Q) + e|^2,
\]
so that also (185) follows from (183).

**Step 5. Proof of (183)**

In view of the energy estimate (169), the LHS of (183) is $\lesssim 1$, so that it is enough to consider the case of $\ell + 1 \leq \sqrt{T}$, for which (183) assumes the form
\[
\int_Q |\nabla \phi_T + e|^2 \lesssim \left( \frac{\sqrt{T}}{\ell + 1} \right)^{d(1-\varepsilon)}.
\]
This estimate follows from
\[
\int_Q \frac{1}{T} \phi_T^2 + |\nabla \phi_T + e|^2 \overset{(169)}{\lesssim} \left( \frac{\sqrt{T}}{\ell + 1} \right)^d
\]
by Widman’s hole-filling argument, see for instance [33, p.81]. Since we could not find the variant with a massive term in the literature, we give the argument for $(187) \Rightarrow (186)$ presently. W.l.o.g. we may assume that $Q$ is centered at 0 so that $Q \subset B_R$ for $R \sim \ell + 1$. In view of (187), it is enough to show
\[
\int_{B_R} 1 + \frac{1}{T} \phi_T^2 + |\nabla \phi_T + e|^2 \lesssim (R \sqrt{T})^\varepsilon d
\]
for all (dyadic) radii $R \leq \sqrt{T}$. With $\varepsilon = \frac{\ln 1/\theta}{\ln 2}$, this estimate is obtained via iteration from
\[
\int_{B_R} 1 + \frac{1}{T} \phi_T^2 + |\nabla \phi_T + e|^2 \leq \theta \int_{B_{2R}} 1 + \frac{1}{T} \phi_T^2 + |\nabla \phi_T + e|^2
\]
for some $\theta = \theta(d, \lambda) < 1$. In order to derive the latter with $\theta = \frac{C_0}{C_0 + 1}$, it is enough to establish
\[
\int_{B_R} 1 + \frac{1}{T} \phi_T^2 + |\nabla \phi_T + e|^2 \leq C_0 \int_{B_{2R} \setminus B_R} 1 + \frac{1}{T} \phi_T^2 + |\nabla \phi_T + e|^2
\]
for some $C_0 = C_0(d, \lambda) < \infty$ – this is the origin of the name “hole-filling”. The last estimate is obtained from the following Caccioppoli estimate: Test (37) with $\eta^2(\phi_T - c)$, where $\eta$ is a cut-off for $B_R$ in $B_{2R}$ and $c = \int_{B_{2R} \setminus B_R} \phi_T$ to the effect of
\[
\int \eta^2 \frac{1}{T} \phi_T(\phi_T - c) + \eta^2 \nabla \phi_T \cdot a(\nabla \phi_T + e) = -2 \int \eta(\phi_T - c) \nabla \eta \cdot a(\nabla \phi_T + e),
\]
which by uniform ellipticity implies
\[
\int \eta^2 \left( \frac{1}{T} \phi_T^2 + |\nabla \phi_T + e|^2 \right) \lesssim \int \eta^2 \frac{1}{T} |c||\phi_T| + (\eta^2 + \eta|\phi_T - c||\nabla \eta|)|\nabla \phi_T + e|.
\]
By Young’s inequality, this yields
\[
\int \eta^2 \left( \frac{1}{T} \phi_T^2 + |\nabla \phi_T + e|^2 \right) \lesssim \int \eta^2 \left( \frac{1}{T} |c|^2 + 1 \right) + (\phi_T - c)^2 |\nabla \eta|, 
\]
and thus by the choice of the cut-off \( \eta \)
\[
\int_{B_R} \frac{1}{T} \phi_T^2 + |\nabla \phi_T + e|^2 \lesssim R^d \left( \frac{1}{T} |c|^2 + 1 \right) + \frac{1}{R^2} \int_{B_{2R} \setminus B_R} (\phi_T - c)^2. 
\]
Because of the choice of \( c \) we obtain by Jensen’s inequality for the first RHS term and by the Poincaré estimate with mean value zero on the annulus \( B_{2R} \setminus B_R \) for the last one
\[
\int_{B_R} \frac{1}{T} \phi_T^2 + |\nabla \phi_T + e|^2 \lesssim R^d \left( \frac{1}{T} |c|^2 + 1 \right) + \hat{B}_{2R \setminus B_R} |\nabla \phi_T|^2, 
\]
which can be rewritten in the iterable form (188).

5. Optimal stochastic integrability of \( r_* \): Proof of stochastic results

5.1. Proof of Theorems 2, 3, and 4: Optimal stochastic integrability of \( r_* \)

We only prove Theorems 2 and 3. The argument for Theorem 4 is simpler (we need not prove properties of \( \pi_* \) in that case). Unless stated otherwise we use \( \lesssim \) for up to a multiplicative constant only depending on \( d, \lambda \) and \( \pi \).

Step 1. Reformulation.

In this proof, for all \( \nu > 0 \) we define \( r_{**} \) (implicitly depending on \( \nu \)) as the smallest (possibly infinite) random variable that satisfies
\[
\int \omega_T \frac{1}{T} |(\phi_T, \sigma_T)|^2 \leq \left( \frac{T_{**}}{R} \right)^{2\nu} 
\]
for all dyadic \( R \geq r_{**} \) and \( T = R^2, \) (189)
which, in view of (44), is a slightly stronger form of (40) in Proposition 2. Proposition 2 then implies that \( r_* \leq Cr_{**} \) for some constant \( C \). Since the function \( \pi_* \) is increasing and satisfies the scaling relation (56b), the above implies \( \frac{1}{C'} \pi_*(r_*) \leq \pi_*(r_{**}) \) for some \( C' \) depending only on \( C \) and \( \pi_* \). Hence, the claim of the theorem follows if we prove that \( r_{**} \) satisfies the moment bound (36) (resp. (57)) for a suitable range of \( \nu > 0 \) depending on \( \pi \) (which also encodes the dependence on \( \kappa \) for the standard LSI). In this first step of the proof, we focus on super-level sets of \( r_{**} \), and shall show that there exist some dyadic threshold \( r_0 = r_0(d, \lambda, \pi) \) and some (generic) constants \( C(d, \lambda, \pi), \tilde{C}(d, \lambda, \pi) \) such that for all \( 0 < \nu \leq \varepsilon \) (with \( \varepsilon = \varepsilon(d, \lambda, \pi) \) given in Corollaries 6 and 9), the random variable \( r_{**} \) associated with the exponent \( \nu \) (through (189)) satisfies
\[
\forall \text{ dyadic } r \geq r_0 : 
\]
\[
\langle I(r_{**} \geq r) \rangle \leq \sum_{R \geq r \text{ dyadic}} \left\langle I \left( \int \omega_T (F_t - \langle F_t \rangle) > \frac{1}{C} \left( \frac{r}{R} \right)^{2\nu} \right) \right\rangle, 
\]
(190)
where \( F_t := \int \omega_t \left( \frac{1}{t} \phi_t^2 + \frac{1}{t} |g_t|^2 + |\nabla g_t|^2 \right) \) and for \( R, T, r, \) and \( t \) related via

\[
\sqrt{T} = R \quad \text{and} \quad t = \tilde{C}(\frac{R}{r})^{2\nu} \leq \frac{T}{2}. \tag{191}
\]

Here comes the argument. By contraposition of the definition (189) of \( r_{**} \) and a union bound, we have for all dyadic \( r \)

\[
\langle I(r_{**} \geq r) \rangle \leq \sum_{R \geq r \text{ dyadic}} \left( I \left( \int \frac{1}{T} |(\phi_T, \sigma_T)|^2 > (\frac{r}{R})^{2\nu} \right) \right). \tag{192}
\]

Since for all functions \( h \geq 0 \) and all \( t \leq \frac{T}{2} \) we have \( \int \omega_T h \leq \int \omega_T (h * \omega_t) \), Proposition 3 yields for some \( C_1 = C_1(d, \lambda) \) and all \( t \leq T \),

\[
\int \omega_T \frac{1}{T} |(\phi_T, \sigma_T)|^2 \leq C_1 \int \omega_T F_t. \tag{193}
\]

In turn, Corollary 6 (resp. Corollary 9) yields for all \( T, t \geq 1 \)

\[
\langle F_t \rangle \leq C_2 t^{-\varepsilon} \tag{194}
\]

for some constant \( C_2 = C_2(d, \lambda, \pi) \). We choose now a dyadic \( r_0 = r_0(d, \lambda, \pi) \) so large that

\[
(2C_1C_2)^\frac{1}{2} \frac{1}{r_0^2} \leq \frac{1}{2}. \tag{195}
\]

Given \( 0 < \nu \leq \varepsilon \), and dyadic \( R \geq r \geq r_0 \) we specify \( t \) in line with (191) as

\[
t := (2C_1C_2)^\frac{1}{2} \left( \frac{R}{r} \right)^{2\nu}, \tag{196}
\]

and note that our choice (194) of \( r_0 \) ensures that

\[
t \leq \frac{1}{2} \left( 2C_1C_2 \right)^\frac{1}{2} \left( \frac{R}{r} \right)^2 = \left( 2C_1C_2 \right)^\frac{1}{2} \frac{1}{R^2} \frac{r \\geq r_0}{r_0^2} \leq T(2C_1C_2)^\frac{1}{2} \frac{1}{R^2} \leq \frac{T}{2}. \tag{197}
\]

The combination of (192) and (193) thus yields for all \( 0 < \nu \leq \varepsilon \) and all dyadic \( R \geq r \geq r_0 \), \( \sqrt{T} = R \), and \( t \) given by (195)

\[
\int \omega_T \frac{1}{T} |(\phi_T, \sigma_T)|^2 > (\frac{r}{R})^{2\nu} \Rightarrow \int \omega_T F_t > \frac{1}{C_1} (\frac{r}{R})^{2\nu} \tag{198}
\]

\[
\Rightarrow \int \omega_T (F_t - \langle F_t \rangle) > \frac{1}{2C_1} (\frac{r}{R})^{2\nu}. \tag{199}
\]

This implies (190) in the regime of parameters (191) for some \( C, \tilde{C} \) depending only on \( C_1, C_2, \) and \( \varepsilon \).

**Step 2.** Proof for the standard LSI.

In this case, Proposition 4 ensures that \( F_t \) satisfies the assumptions of Lemma 4, which yields the existence of a positive constant \( C' = C'(d, \lambda, \kappa) \) such that for all \( 0 < \nu \leq \varepsilon \), all dyadic \( \sqrt{T} = R \geq r \geq r_0 \) and \( t \) given by (191),

\[
\left\langle I \left( \int \omega_T (F_t - \langle F_t \rangle) > \frac{1}{C} (\frac{r}{R})^{2\nu} \right) \right\rangle \leq \exp \left( - \frac{1}{C'} (\frac{r}{R})^{4\nu} (\frac{\sqrt{T}}{\sqrt{t}})^{d} \right). \tag{195}
\]

\[
\left\langle I \left( \int \omega_T (F_t - \langle F_t \rangle) > \frac{1}{C} (\frac{r}{R})^{2\nu} \right) \right\rangle \leq \exp \left( - \frac{1}{C'} (\frac{r}{R})^{4\nu} (\frac{\sqrt{T}}{\sqrt{t}})^{d} \right). \tag{196}
\]
It remains to choose $\nu$. By (191),
\[
\left(\frac{\sqrt{T}}{\sqrt{t}}\right)^d \gtrsim r^d \left(\frac{R}{r}\right)^d (1 - \frac{\nu}{4})
\]
and thus \( \left(\frac{r}{R}\right)^{4\nu} \left(\frac{\sqrt{T}}{\sqrt{t}}\right)^d \gtrsim r^d \left(\frac{R}{r}\right)^d - 4\nu (1 + \frac{\nu}{4}) \),

so that provided \(0 < \nu < \frac{d\varepsilon}{4\varepsilon + \beta}\) the RHS of (196) is summable w.r.t. \(R \geq r\) dyadic, with the contribution from \(R = r\) being dominant. Fixing such a \(\nu = \nu(d, \lambda)\), we conclude with help of (190) that for some positive constant \(C'' = C''(d, \lambda, \kappa)\) and for all dyadic \(r \geq r_0\),
\[
\langle I(r_{**} \geq r) \rangle \leq C''\exp\left(-\frac{1}{C''} \frac{r}{R}\right).
\]
By summation over dyadic \(r \geq r_0\), this yields the desired (stretched) exponential moment bound on \(r_{**}\), and therefore on \(r_*\) by Step 1.

**Step 3.** Proof for MLSI.
As in the previous step, the combination of Proposition 4 with Lemma 5 (v) yields for some positive constant \(C' = C'(d, \lambda, \pi)\), for all \(0 < \nu \leq \varepsilon\), all dyadic \(\sqrt{T} = R \geq r \geq r_0\) and \(t\) given by (191),
\[
\langle I \left( \int \omega_t (F_t - \langle F_t \rangle) > \frac{1}{C} \left(\frac{r}{R}\right)^{2\nu} \right) \rangle \leq \exp\left(-\frac{1}{C'} \left(\frac{r}{R}\right)^{4\nu} \pi_\ast \left(\frac{\sqrt{T}}{\sqrt{t}}\right)\right).
\]
In Step 4 below, we shall argue that for any \(0 < \tilde{\beta} < \beta\), and in particular for \(\tilde{\beta} = \frac{\beta}{2}\), there exists \(\ell_0 = \ell_0(\pi) \gg 1\) such that
\[
\forall K \geq 1 \text{ and } \ell \geq \ell_0 : K^{\tilde{\beta}} \pi_\ast(\ell) \leq \pi_\ast(K\ell).
\]
Combined with the definition (191) of \(t\) and the first estimate in (197), this implies
\[
\left(\frac{r}{R}\right)^{4\nu} \pi_\ast \left(\frac{\sqrt{T}}{\sqrt{t}}\right) \gtrsim \left(\frac{R}{r}\right)^{\tilde{\beta}(1 - \frac{\nu}{2}) - 4\nu} \pi_\ast(r).
\]
Choosing \(\nu = \nu(d, \lambda, \pi)\) in the range \(0 < \nu < \frac{\beta\varepsilon}{4\varepsilon + \beta}\), we continue to argue as in Step 2.

**Step 4.** Proof of (198).
Recall that \(\frac{1}{\pi_\ast(\ell)} = \int_{B_\ell} \gamma(|x|)dx\). Using spherical coordinates, we have
\[
\frac{1}{\pi_\ast(\ell)\gamma(\ell)} = d \int_0^1 \theta^{d-1} \frac{\gamma(\theta \ell)}{\gamma(\ell)} d\theta.
\]
By appealing to (56b) and Fatou’s lemma we deduce that
\[
\liminf_{\ell \to \infty} \frac{1}{\pi_\ast(\ell)\gamma(\ell)} \geq d \int_0^1 \theta^{d-1} \liminf_{\ell \to \infty} \frac{\gamma(\theta \ell)}{\gamma(\ell)} d\theta \geq d \int_0^1 \theta^{d-1 - \beta} d\theta = \frac{d}{d - \beta}.
\]
Hence, for any \(0 < \tilde{\beta} < \beta\) we can find \(\ell_0 \gg 1\) such that
\[
\frac{1}{\pi_\ast(\ell)\gamma(\ell)} \geq \frac{d}{d - \beta} \quad \text{for all } \ell \geq \ell_0.
\]
In addition, we have \((\frac{1}{\pi_\ast})'(\ell) = d \int_0^1 \theta^d \gamma'(\theta \ell) d\theta = -d \frac{1}{\ell} \pi_\ast(\ell)(1 - \gamma(\ell) \pi_\ast(\ell))\), so that for \(K \geq 1\)
\[
\log \pi_\ast(\ell) - \log \pi_\ast(K\ell) = -\int_\ell^K \frac{d}{r} (1 - \gamma(r) \pi_\ast(r)) dr.
\]
Combined with (199), this yields for \(\ell \geq \ell_0\)
\[
\log \pi_\ast(\ell) - \log \pi_\ast(K\ell) \leq -\int_\ell^K \frac{1}{s} ds = \log(K^{-\bar{\beta}}),
\]
which we rewrite as the desired estimate \(K^{\bar{\beta}} \pi_\ast(\ell) \leq \pi_\ast(K\ell)\).

5.2. Proof of Corollaries 6 and 9: Control of the expectation

By Lemma 2, it is enough to control the quantity
\[
\frac{1}{T} \int_0^T \langle |(\nabla \phi_T)_{st}|^2 + |(q_T)_{st} - \langle q_T \rangle|^2 \rangle dt,
\]
which is the integral of a variance. For \(t \leq T^{1-\varepsilon}\), by Jensen’s inequality and stationarity, we have
\[
\frac{1}{T} \int_0^{T-\varepsilon} \langle |(\nabla \phi_T)_{st}|^2 + |(q_T)_{st} - \langle q_T \rangle|^2 \rangle dt \lesssim T^{-\varepsilon} \langle |\nabla \phi_T|^2 + 1 \rangle \lesssim T^{-\varepsilon}.
\]  
(200)

For \(t \geq T^{1-\varepsilon}\), we appeal MSG (which follows from MLSI) in the form
\[
\langle |(\nabla \phi_T)_{st}|^2 + |(q_T)_{st} - \langle q_T \rangle|^2 \rangle \leq \left( \int_0^\infty \pi(\ell)(\ell + 1)^{-d} \int_{\mathbb{R}^d} |\partial_{x,\ell+1}^{\text{fct/osc}} F_{st}|^2 dx d\ell \right)
\]  
(201)

with \(F_{st} := (\nabla \phi_T, q_T)_{st}\). (For the standard LSI, replace the integral over \(\ell\) by the integrand for \(\ell = 0\).) We split the rest of the proof into two steps. In the first step, we prove the claim for the standard LSI, and conclude with the general case in Step 2.

Step 1. Proof of (51).

By Lemma 3 (applied with \(\ell = 0\)),
\[
\int_{\mathbb{R}^d} |\partial_{x,1}^{\text{fct/osc}} F_{st}|^2 dx \lesssim \left( \frac{\sqrt{T}}{\sqrt{t}} \right)^d \sqrt{T}^{-\varepsilon d},
\]
so that
\[
\frac{1}{T} \int_{T^{1-\varepsilon}} T |\partial_{x,1}^{\text{fct/osc}} F_{st}|^2 dx dt \lesssim \sqrt{T}^{-2-\varepsilon d} \int_{T^{1-\varepsilon}} t^{-d} dt \lesssim T^{-\varepsilon},
\]
where in dimension \(d = 2\), we slightly reduced \(\varepsilon > 0\) to absorb the logarithm. Combined with (200) and (201), the desired result follows.

Step 2. Proof of (70).

Starting from a general partition \((\ell + 1)\mathcal{P}\) for \(\ell \geq 0\), Lemma 3 yields
\[
(\ell + 1)^{-d} \int_{\mathbb{R}^d} |\partial_{x,\ell+1}^{\text{fct/osc}} F_{st}|^2 dx \lesssim \left( \frac{\sqrt{T}}{\sqrt{t}} \right)^d \left( \frac{\ell + 1}{\sqrt{T}} \wedge 1 \right)^\varepsilon d.
\]
In particular, this implies
\[
\int_0^\infty \pi(\ell)(\ell + 1)^{-d} \int_{\mathbb{R}^d} |\partial_{x,\ell+1}^{\text{fct/osc}} F_{st}|^2 \, dx \, d\ell
\lesssim \left( \frac{\sqrt{T}}{\sqrt{t}} \right)^d \int_0^\infty (\ell + 1)^{-\varepsilon d} \pi(\ell) \, d\ell
\]
\[
\lesssim \left( \frac{\sqrt{T}}{\sqrt{t}} \right)^d \int_0^{\sqrt{T}} \pi(\ell)(\ell + 1)^{-\varepsilon d} \, d\ell + \int_{\sqrt{T}}^\infty \pi(\ell) \, d\ell.
\]
By (56a) & (56b), and an integration by parts,
\[
\int_0^\infty \pi(\ell)(\ell + 1)^{-d} \, d\ell \overset{(56a)}{\lesssim} 1 + \int_0^\infty \gamma(\ell)(\ell + 1)^{-\varepsilon d - 1} \, d\ell
\]
\[
\overset{(56b)}{\lesssim} 1 + \int_0^\infty (\ell + 1)^{-\varepsilon d - 1 - \beta} \, d\ell \lesssim 1
\]
provided $\varepsilon d - \beta < 0$ (which we may assume w.l.o.g. by reducing the hole-filling exponent $\varepsilon$). By (56a), $\gamma$ is non-increasing, so that for all $R > 0$ we have using (56c)
\[
\int_R^\infty \pi(\ell) \, d\ell \overset{(56a)}{=} \gamma(R) \leq \int_{B_R} \gamma(|x|) \, dx \overset{(56c)}{=} \pi_*(R)^{-1}.
\]
Hence,
\[
\int_0^\infty \pi(\ell)(\ell + 1)^{-d} \int_{\mathbb{R}^d} |\partial_{x,\ell+1}^{\text{fct/osc}} F_{st}|^2 \, dx \, d\ell \lesssim \left( \frac{\sqrt{T}}{\sqrt{t}} \right)^d (\sqrt{T}^{-\varepsilon d} + \pi_*(\sqrt{T})^{-1}).
\]
Combined with (56b) in form of $\pi_*(r)^{-1} \lesssim r^{-\beta}$ and the relation $\varepsilon d - \beta < 0$, this turns into
\[
\int_0^\infty \pi(\ell)(\ell + 1)^{-d} \int_{\mathbb{R}^d} |\partial_{x,\ell+1}^{\text{fct/osc}} F_{st}|^2 \, dx \, d\ell \lesssim \left( \frac{\sqrt{T}}{\sqrt{t}} \right)^d \sqrt{T}^{-\varepsilon d}.
\]
As in Step 1, this yields
\[
\frac{1}{T} \int_T^{T^{1-\varepsilon}} \int_0^\infty \pi(\ell)(\ell + 1)^{-d} \int_{\mathbb{R}^d} |\partial_{x,\ell+1}^{\text{fct/osc}} F|^2 \, dx \, d\ell \, dt \lesssim T^{-\varepsilon},
\]
and therefore proves the desired estimate in combination with (200) and (201).

**Appendix A. Caccioppoli’s inequality**

In the proofs, we shall make intensive use of the classical Caccioppoli argument, which we state for future reference, and prove for the reader’s convenience.

**Lemma 6.** Let $R \geq 1$. Consider $u, g$ related in a distributional sense by
\[
-\nabla \cdot a \nabla u = -\nabla \cdot g \quad \text{in } B_R.
\]
There exists a constant $C = C(d, \lambda) > 0$ such that for any constant $c$ we have
\[
\forall 0 < \rho < R : \int_{B_{R-\rho}} |\nabla u|^2 \leq C \left( \int_{B_R} |g|^2 + \frac{1}{\rho^2} \int_{B_R \setminus B_{R-\rho}} (u - c)^2 \right). \tag{203}
\]

\begin{proof}
For the convenience of the reader, we recall the standard argument under the weak ellipticity assumption (3), which thanks to the homogeneity of the coefficients could be weakened further, see [33, Proposition 2.1]. By scaling, we may w. l. o. g. assume that $R = 1$ and by adding a constant, $c = 0$, so that it remains to show
\[
\int_{B_{1-\rho}} |\nabla u|^2 \lesssim \int_{B_1} |g|^2 + \frac{1}{\rho^2} \int_{B_1 \setminus B_{1-\rho}} u^2. \tag{204}
\]
where here and below $\lesssim$ stands for $\leq$ up to a constant that depends on $d$ and $\lambda$. To this purpose we test $-\nabla \cdot a \nabla u = 0$ with $\eta^2 u$, where $\eta$ is a cut-off for $B_{1-\rho}$ in $B_1$; using Leibniz’s rule in form of
\[
\nabla (\eta^2 u) \cdot a \nabla u = \nabla (\eta u) \cdot a \nabla (\eta u) + u \nabla \eta \cdot a \nabla (\eta u) - u \nabla (\eta u) \cdot a \nabla \eta - u^2 \nabla \eta \cdot a \nabla \eta,
\]
we obtain with (202) the identity
\[
\int \nabla (\eta u) \cdot a \nabla (\eta u) = \int g \cdot \nabla (\eta^2 u) + \int (-u \nabla \eta \cdot a \nabla (\eta u) + u \nabla (\eta u) \cdot a \nabla \eta + u^2 \nabla \eta \cdot a \nabla \eta).
\]
With $|\nabla (\eta^2 u)| \leq |u| |\nabla \eta| + \eta |\nabla (u \eta)|$ and by uniform ellipticity and boundedness of $a$, cf. (3) and (2), this yields
\[
\lambda \int |\nabla (\eta u)|^2 \leq \int |g| |\nabla (\eta u)| + |\nabla \eta| |u| + \int (2 |u| |\nabla \eta| |\nabla (\eta u)| + u^2 |\nabla \eta|^2).
\]
By Young’s inequality this entails
\[
\int |\nabla (\eta u)|^2 \lesssim \int |g|^2 \eta^2 + \int u^2 |\nabla \eta|^2,
\]
so that by the properties of the cut-off function we obtain (204). Note that this also yields
\[
\int |\nabla u|^2 \eta^2 \lesssim \int |g|^2 \eta^2 + \int u^2 |\nabla \eta|^2. \tag{205}
\]
\end{proof}

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