UP-DOWN ORDERED CHINESE RESTAURANT PROCESSES WITH TWO-SIDED IMMIGRATION, EMIGRATION AND DIFFUSION LIMITS

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We introduce a three-parameter family of up-down ordered Chinese restaurant processes PCRP(α)(θ₁, θ₂), α ∈ (0, 1), θ₁, θ₂ ≥ 0, generalising the two-parameter family of Rogers and Winkel. Our main result establishes self-similar diffusion limits,SSIP(α)(θ₁, θ₂)-evolutions generalising existing families of interval partition evolutions. We use the scaling limit approach to extend stationarity results to the full three-parameter family, identifying an extended family of Poisson–Dirichlet interval partitions. Their ranked sequence of interval lengths has Poisson–Dirichlet distribution with parameters α ∈ (0, 1) and θ := θ₁ + θ₂ − α ≥ −α, including for the first time the usual range of θ > −α rather than being restricted to θ ≥ 0. This has applications to Fleming–Viot processes, nested interval partition evolutions and tree-valued Markov processes, notably relying on the extended parameter range.

1. Introduction. The primary purpose of this work is to study the weak convergence of a family of properly rescaled continuous-time Markov chains on integer compositions [22] and the limiting diffusions. Our results should be compared with the scaling limits of natural up-down Markov chains on branching graphs, which have received substantial focus in the literature [8, 35, 36]. In this language, our models take place on the branching graph of integer compositions and on its boundary, which was represented in [22] as a space of interval partitions. This paper establishes a proper scaling limit connection between discrete models [44] and their continuum analogues [14, 15, 18] in the generality of [47].

We consider a class of ordered Chinese restaurant processes with departures, parametrised by α ∈ (0, 1) and θ₁, θ₂ ≥ 0. This model is a continuous-time Markov chain (C(t), t ≥ 0) on vectors of positive integers, describing customer numbers of occupied tables arranged in a row. At time zero, say that there are k ∈ N = {1, 2, ...} occupied tables and for each i ≤ k the i-th occupied table enumerated from left to right has nᵢ ∈ N customers, then the initial state is C(0) = (n₁, ..., nₖ). New customers arrive as time proceeds, either taking a seat at an existing table or starting a new table, according to the following rule, illustrated in Figure 1:

• for each occupied table, say there are m ∈ N customers, a new customer comes to join this table at rate m − α;
• at rate θ₁, a new customer enters to start a new table to the left of the leftmost table;
• at rate θ₂, a new customer begins a new table to the right of the rightmost table;
• between each pair of two neighbouring occupied tables, a new customer enters and begins a new table there at rate α.

We refer to the arrival of a customer as an up-step. Furthermore, each customer leaves at rate 1 (a down-step). By convention, the chain jumps from the null vector ∅ to state (1) at rate θ := θ₁ + θ₂ − α if θ > 0, and ∅ is an absorbing state if θ ≤ 0. At every time t ≥ 0, let C(t)

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be the vector of customer numbers at occupied tables, listed from left to right. In this way we have defined a continuous-time Markov chain \((C(t), t \geq 0)\). This process is referred to as a Poissonised up-down ordered Chinese restaurant process (PCRP) with parameters \(\alpha, \theta_1, \theta_2\), denoted by \(\text{PCRP}^{(\alpha)}(\theta_1, \theta_2)\).

This family of Markov chains is closely related to the well-known Chinese restaurant processes due to Dubins and Pitman (see e.g. [38]) and their ordered variants studied in [26, 39]. When \(\theta_2 = \alpha\), a \(\text{PCRP}^{(\alpha)}(\theta_1, \alpha)\) is studied in [44]. Notably, our generalisation includes cases \(\theta = \theta_1 + \theta_2 - \alpha \in (-\alpha, 0)\), which did not arise in [26, 39, 44]. Though we focus on the range \(\alpha \in (0, 1)\) in this paper, our model is clearly well-defined for \(\alpha = 0\) and it is straightforward to deal with this case; we include a discussion in Section 4.7.

To state our first main result, we represent PCRPs in a space of interval partitions. For \(M \geq 0\), an interval partition \(\beta = \{U_i, i \in I\}\) of \([0, M]\) is a (finite or countably infinite) collection of disjoint open intervals \(U_i = (a_i, b_i) \subseteq (0, M)\), such that the (compact) set of partition points \(G(\beta) := [0, M] \setminus \bigcup_{i \in I} U_i\) has zero Lebesgue measure. We refer to the intervals \(U \in \beta\) as blocks, to their lengths \(\text{Leb}(U)\) as their masses. We similarly refer to \(\|\beta\| := \sum_{U \in \beta} \text{Leb}(U)\) as the total mass of \(\beta\). We denote by \(\mathcal{I}_H\) the set of all interval partitions of \([0, M]\) for all \(M \geq 0\). This space is equipped with the metric \(d_H\) obtained by applying the Hausdorff metric to the sets of partition points: for every \(\gamma, \gamma' \in \mathcal{I}_H\),

\[
d_H(\gamma, \gamma') := \inf \left\{ r \geq 0 : G(\gamma) \subseteq \bigcup_{x \in G(\gamma')} (x-r, x+r), \ G(\gamma') \subseteq \bigcup_{x \in G(\gamma)} (x-r, x+r) \right\}.
\]

Although \((\mathcal{I}_H, d_H)\) is not complete, the induced topological space is Polish [16, Theorem 2.3]. For \(c > 0\) and \(\beta \in \mathcal{I}_H\), we define a scaling map by

\[
c\beta := \{(ca, cb) : (a, b) \in \beta\}.
\]

We shall regard a PCRP \(\text{PCRPs}(C(t), t \geq 0)\) as a càdlàg process in \((\mathcal{I}_H, d_H)\), by identifying any vector of positive integers \((n_1, \ldots, n_k)\) with an interval partition of \([0, n_1 + \cdots + n_k]\):

\[
(n_1, \ldots, n_k) \longleftrightarrow \{(s_{i-1}, s_i), 1 \leq i \leq k\} \quad \text{where } s_i = n_1 + \cdots + n_i.
\]

We are now ready to state our main result, which is a limit theorem in distribution in the space of \(\mathcal{I}_H\)-valued càdlàg functions \(\mathcal{D}(\mathbb{R}_+, \mathcal{I}_H)\) with \(\mathbb{R}_+ := [0, \infty)\), endowed with the \(J_1\)-Skorokhod topology (see e.g. [6] for background).

**Theorem 1.1.** Let \(\alpha \in (0, 1)\) and \(\theta_1, \theta_2 \geq 0\). For \(n \in \mathbb{N}\), let \(\text{PCRPs}^{(n)}(C^{(n)}(t), t \geq 0)\) be a PCRPs\((\alpha)(\theta_1, \theta_2)\) starting from \(C^{(n)}(0) = \gamma^{(n)}\). Suppose that the initial interval partitions \(\frac{1}{n} \gamma^{(n)}\) converge in distribution to \(\gamma \in \mathcal{I}_H\) as \(n \to \infty\), under \(d_H\). Then there exists an \(\mathcal{I}_H\)-valued path-continuous Hunt process \((\beta(t), t \geq 0)\) starting from \(\beta(0) = \gamma\), such that

\[
(1) \quad \frac{1}{n} C^{(n)}(2nt), t \geq 0 \xrightarrow{n \to \infty} (\beta(t), t \geq 0), \quad \text{in distribution in } \mathcal{D}(\mathbb{R}_+, \mathcal{I}_H).
\]

Moreover, set \(\zeta^{(n)} = \inf\{t \geq 0 : C^{(n)}(t) = \emptyset\}\) and \(\zeta = \inf\{t \geq 0 : \beta(t) = \emptyset\}\) to be the respective first hitting times of \(\emptyset\). If \(\gamma \neq \emptyset\), then (1) holds jointly with \(\zeta^{(n)} / 2n \to \zeta\), in distribution.
We call the limiting diffusion \((\beta(t), t \geq 0)\) on \(I_H\) an \((\alpha, \theta_1, \theta_2)\)-self-similar interval partition evolution, denoted by SSIP\((\alpha)(\theta_1, \theta_2)\). These processes are indeed self-similar with index 1, in the language of self-similar Markov processes [31], see also [29, Chapter 13]: if \((\beta(t), t \geq 0)\) is an SSIP\((\alpha)(\theta_1, \theta_2)\)-evolution, then \((c\beta(c^{-1}t), t \geq 0)\) is an SSIP\((\alpha)(\theta_1, \theta_2)\)-evolution starting from \(c\beta(0)\), for any \(c > 0\).

While most SSIP-evolutions have been constructed before [14, 15, 18, 47], in increasing generality, Theorem 1.1 is the first scaling limit result with an SSIP-evolution as its limit. In special cases, this was conjectured in [44]. In the following, we state some consequences and further developments. We relate to the literature in more detail in Section 1.1. We refer to Section 1.3 for applications particularly of the generality of the three-parameter family.

Our interval partition evolutions have multiple connections to squared Bessel processes. More precisely, a squared Bessel process \(Z = (Z(t), t \geq 0)\) starting from \(Z(0) = m \geq 0\) and with “dimension” parameter \(\delta \in \mathbb{R}\) is the unique strong solution of the following equation:

\[
Z(t) = m + \delta t + 2 \int_0^t \sqrt{Z(s)} dB(s),
\]

where \((B(t), t \geq 0)\) is a standard Brownian motion. We refer to [23] for general properties of squared Bessel processes. Let \(\zeta(Z) := \inf\{t \geq 0 : Z(t) = 0\}\) be the first hitting time of zero. To allow \(Z\) to re-enter \((0, \infty)\) where possible after hitting 0, we define the lifetime of \(Z\) by

\[
\zeta(Z) := \begin{cases} 
\infty, & \text{if } \delta > 0, \\
\zeta(Z), & \text{if } \delta \leq 0.
\end{cases}
\]

We write \(\text{BESQ}_m(\delta)\) for the law of a squared Bessel process \(Z\) with dimension \(\delta\) starting from \(m\), in the case \(\delta \leq 0\) absorbed in \(\emptyset\) at the end of its (finite) lifetime \(\zeta(Z)\). When \(\delta \leq 0\), by our convention \(\text{BESQ}_0(\delta)\) is the law of the constant zero process.

In an SSIP\((\alpha)(\theta_1, \theta_2)\)-evolution, informally speaking, each block evolves as \(\text{BESQ}(-2\alpha)\), independently of other blocks [14, 15]. Meanwhile, there is always immigration of rate \(2\alpha\) between “adjacent blocks”, rate \(2\theta_1\) on the left [18] and rate \(2\theta_2\) on the right [47]. Moreover, the total mass process \((||\beta(t)||, t \geq 0)\) of any SSIP\((\alpha)(\theta_1, \theta_2)\)-evolution \((\beta(t), t \geq 0)\) is \(\text{BESQ}_{||\beta(0)||}(2\theta)\) with \(\theta := \theta_1 + \theta_2 - \alpha\). We discuss this more precisely in Section 4.3. We refer to \(2\theta\) as the total immigration rate if \(\theta > 0\), and as the total emigration rate if \(\theta < 0\).

There are pseudo-stationary SSIP\((\alpha)(\theta_1, \theta_2)\)-evolutions, that have fluctuating total mass but stationary interval length proportions, in the sense [15] of the following proposition, for a family \(\text{PDIP}(\alpha)(\theta_1, \theta_2), \alpha \in (0, 1), \theta_1, \theta_2 \geq 0\), of Poisson–Dirichlet interval partitions on the space \(\mathcal{I}_{H,1} \subset \mathcal{I}_H\) of partitions of the unit interval. This family notably extends the subfamilies of [21, 39, 47], whose ranked sequence of interval lengths in the Kingman simplex

\[
\nabla_{\infty} := \left\{ (x_1, x_2, \ldots) : x_1 \geq x_2 \geq \cdots \geq 0, \sum_{i \geq 1} x_i = 1 \right\}
\]

are members of the two-parameter family \(\text{PD}(\alpha)(\theta), \alpha \in (0, 1), \theta \geq 0\) of Poisson–Dirichlet distributions. Here, we include new cases of interval partitions, for which \(\theta \in (-\alpha, 0)\), completing the usual range of the two-parameter family of \(\text{PD}(\alpha)(\theta)\) with \(\alpha \in (0, 1)\) of [38, Definition 3.3]. The further case \(\theta_1 = \theta_2 = 0\), is degenerate, with \(\text{PDIP}(\alpha)(0, 0) = \delta_{((0,1))}\).

**Proposition 1.2** (Pseudo-stationarity). For \(\alpha \in (0,1)\) and \(\theta_1, \theta_2 \geq 0\), consider independently \(\tilde{\gamma} \sim \text{PDIP}(\alpha)(\theta_1, \theta_2)\) and a \(\text{BESQ}(2\theta)\)-process \((Z(t), t \geq 0)\) with any initial distribution and parameter \(\theta = \theta_1 + \theta_2 - \alpha\). Let \((\beta(t), t \geq 0)\) be an SSIP\((\alpha)(\theta_1, \theta_2)\)-evolution starting from \(\beta(0) = Z(0)\tilde{\gamma}\). Fix any \(t \geq 0\), then \(\beta(t)\) has the same distribution as \(Z(t)\tilde{\gamma}\).
We refer to Definition 2.6 for a description of the probability distribution $\text{PDIP}^{(\alpha)}(\theta_1, \theta_2)$ on unit interval partitions. We prove in Proposition 2.9 that $\text{PDIP}^{(\alpha)}(\theta_1, \theta_2)$ gives the limiting block sizes in their left-to-right order of a new three-parameter family of composition structures in the sense of [22]. For the special case $\theta_2 = \alpha$, which was introduced in [21, 39], we also write $\text{PDIP}^{(\alpha)}(\theta_1, \alpha)$ and recall a construction in Definition 2.4.

As in the case $\theta_2 = \alpha$ studied in [15, 18], we define an associated family of $\mathcal{I}_{H,1}$-valued evolutions via time-change and renormalisation (“de-Poissonisation”).

**Definition 1.3 (De-Poissonisation and $\text{IP}^{(\alpha)}(\theta_1, \theta_2)$-evolutions).** Consider $\gamma \in \mathcal{I}_{H,1}$, let $\beta := (\beta(t), t \geq 0)$ be an SSIP$^{(\alpha)}(\theta_1, \theta_2)$-evolution starting from $\gamma$ and define a time-change function $\tau_{\beta}$ by

$$
\tau_{\beta}(u) := \inf \left\{ t \geq 0 : \int_0^t \| \beta(s) \|^{-1} ds > u \right\}, \quad u \geq 0.
$$

Then the process on $\mathcal{I}_{H,1}$ obtained from $\beta$ via the following de-Poissonisation

$$
\overline{\beta}(u) := \| \beta(\tau_{\beta}(u)) \|^{-1} \beta(\tau_{\beta}(u)), \quad u \geq 0,
$$

is called a Poisson–Dirichlet ($\alpha, \theta_1, \theta_2$)-interval partition evolution starting from $\gamma$, abbreviated as $\text{IP}^{(\alpha)}(\theta_1, \theta_2)$-evolution.

**Theorem 1.4.** Let $\alpha \in (0, 1)$, $\theta_1, \theta_2 \geq 0$. An $\text{IP}^{(\alpha)}(\theta_1, \theta_2)$-evolution is a path-continuous Hunt process on $(\mathcal{I}_{H,1}, d_H)$, is continuous in the initial state and has a stationary distribution $\text{PDIP}^{(\alpha)}(\theta_1, \theta_2)$.

Define $\mathcal{H}$ to be the commutative unital algebra of functions on $\nabla_{\infty}$ generated by $q_k(x) = \sum_{i \geq 1} x_i^{k+1}$, $k \geq 1$, and $q_0(x) = 1$. For every $\alpha \in (0, 1)$ and $\theta > -\alpha$, define an operator $B_{\alpha, \theta} : \mathcal{H} \to \mathcal{H}$ by

$$
B_{\alpha, \theta} := \sum_{i \geq 1} x_i \frac{\partial^2}{\partial x_i^2} - \sum_{i,j \geq 1} \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i \geq 1} (\theta x_i + \alpha) \frac{\partial}{\partial x_i}.
$$

It has been proved in [35] that there is a Markov process on $\nabla_{\infty}$ whose (pre-)generator on $\mathcal{H}$ is $B_{\alpha, \theta}$, which shall be referred to as the Ethier–Kurtz–Petrov diffusion with parameter $(\alpha, \theta)$, for short $\text{EKP}(\alpha, \theta)$-diffusion; moreover, $\text{PD}^{(\alpha)}(\theta)$ is the unique invariant probability measure for $\text{EKP}(\alpha, \theta)$. In [17], the following connection will be established.

- Let $\alpha \in (0, 1)$, $\theta_1, \theta_2 \geq 0$ with $\theta_1 + \theta_2 > 0$. For an $\text{IP}^{(\alpha)}(\theta_1, \theta_2)$-evolution $(\overline{\beta}(u), u \geq 0)$, list the lengths of intervals of $\overline{\beta}(u)$ in decreasing order in a sequence $W(u) \in \nabla_{\infty}$. Then the process $(W(u/2), u \geq 0)$ is an $\text{EKP}(\alpha, \theta)$-diffusion with $\theta := \theta_1 + \theta_2 - \alpha > -\alpha$.

1.1. **Connections with interval partition evolutions in the literature.** The family of SSIP$^{(\alpha)}(\theta_1, \theta_2)$-evolutions generalises the two-parameter model of [18]:

- for $\alpha \in (0, 1)$ and $\theta_1 > 0$, an SSIP$^{(\alpha)}(\theta_1, \alpha)$-evolution is an $(\alpha, \theta_1)$-self-similar interval partition evolution in the sense of [18], which we will also refer to as an SSIP$^{(\alpha)}(\theta_1)$-evolution. The properties of these limiting processes have been proved in [18].

For this smaller class with $\theta_2 = \alpha$, [44] provides a study of the family $\text{PCRP}^{(\alpha)}(\theta_1, \alpha)$ and conjectures the existence of diffusion limits, which is thus confirmed by our Theorem 1.1. We conjecture that the convergence in Theorem 1.1 can be extended to the case where $G^{(\overline{\gamma}(n))}$ converges in distribution, with respect to the Hausdorff metric, to a compact set of positive
Lebesgue measure; then the limiting process is a generalized interval partition evolution in the sense of [18, Section 4].

For the two-parameter case ($\theta_2 = \alpha$), [43] obtains the scaling limits of a closely related family of discrete-time up-down ordered Chinese restaurant processes, in which at each time exactly one customer arrives according to probabilities proportional to the up-rates of $\text{PCRP}^{(\alpha)}(\beta_1, \alpha)$, see Definition 2.1, and then one customer leaves uniformly, such that the number of customers remains constant. The method in [43] is by analysing the generator of the Markov processes, which is quite different from this work, and neither limit theorem implies the other. It is conjectured that the limits of [43] are IP$^{(\alpha)}(\theta_1, \alpha)$-evolutions, and we further conjecture that this extends to the three-parameter setting of Definition 1.3.

An SSIP$^{(\alpha)}(\theta_1, \theta_2)$-evolution $\beta = (\beta(t), t \geq 0)$ killed at its first hitting time $\zeta(\beta)$ of $\emptyset$ has been constructed in [47]. We denote this killed process by SSIP$^\dagger_{\emptyset}(\theta_1, \theta_2)$. For the SSIP$^{(\alpha)}(\theta_1, \theta_2)$-evolution itself, there are three different phases, according to the parameter $\theta := \theta_1 + \theta_2 - \alpha \geq -\alpha$.

- $\theta \geq 1$: $\zeta(\beta) = \infty$ a.s.. In this case, SSIP$^{(\alpha)}(\theta_1, \theta_2)$-evolutions have been constructed in [47], including a proof of Proposition 1.2 that is the key to the proof of Theorem 1.4.
- $\theta \in (0, 1)$: $\zeta(\beta)$ is a.s. finite with $\zeta(\beta) \overset{d}{=} \|\beta(0)\|/2G$, where $G \sim \Gamma(1 - \theta, 1)$, the Gamma distribution with shape parameter $1 - \theta$ and rate parameter 1. The construction in [47] does not cover this case in full. We will construct in Section 4.5 an SSIP$^\dagger_{\emptyset}(\theta_1, \theta_2)$-evolution as a recurrent extension of SSIP$^\dagger_{\emptyset}(\theta_1, \theta_2)$-evolutions and study its properties.
- $\theta \in [-\alpha, 0]$; $\emptyset$ is an absorbing state, and hence an SSIP$^\dagger_{\emptyset}(\theta_1, \theta_2)$-evolution coincides with an SSIP$^{(\alpha)}(\theta_1, \theta_2)$-evolution. In [47], we were unable to establish the pseudo-stationarity stated in Proposition 1.2 for this case. Our proof of Proposition 1.2 relies crucially on the convergence in Theorem 1.1.

Note that the law of $\zeta(\beta)$ and the phase transitions can be observed directly from the boundary behaviour at zero of the total mass process $\text{BESQ}(\theta)$, see e.g. [23, Equation (13)].

1.2. SSIP$^{(\alpha)}(\theta_1, \theta_2)$-excursions. When $\theta = \theta_1 + \theta_2 - \alpha \in (0, 1)$, a PCRP$^{(\alpha)}(\theta_1, \theta_2)$ is reflected at $\emptyset$. When $\theta = \theta_1 + \theta_2 - \alpha \leq 0$, a PCRP$^{(\alpha)}(\theta_1, \theta_2)$ is absorbed at $\emptyset$, and if the initial interval partitions $\frac{1}{n}\gamma^{(n)}$ converge in distribution to $\emptyset \in \mathcal{I}_H$ as $n \to \infty$ under $d_H$, then the limiting process in Theorem 1.1 is the constant process that stays in $\emptyset$. In both cases we refine the discussion and establish the convergence of rescaled PCRP excursions to a non-trivial limit in the following sense.

**Theorem 1.5.** Let $\alpha \in (0, 1)$, $\theta_1, \theta_2 \geq 0$ and suppose that $\theta = \theta_1 + \theta_2 - \alpha \in (-\alpha, 1)$. Let $(C(t), t \geq 0)$ be a PCRP$^{(\alpha)}(\theta_1, \theta_2)$ starting from state 1 and denote by $P^{(n)}$ the law of the process $(C^{(n)}(t)) := C(2nt \wedge \zeta(C)), t \geq 0$, where $\zeta(C) := \inf\{t \geq 0 : C(t) = \emptyset\}$. Then the following convergence holds vaguely under the Skorokhod topology:

$$
\frac{\Gamma(1 + \theta)}{1 - \theta} n^{1 - \theta} P^{(n)} \rightarrow \Theta,
$$

where the limit $\Theta$ is a $\sigma$-finite measure on the space of continuous excursions on $\mathcal{I}_H$.

A description of the limit $\Theta$ is given in Section 4.4. We refer to $\Theta$ as the excursion measure of an SSIP$^{(\alpha)}(\theta_1, \theta_2)$-evolution, which plays a crucial role in the construction of recurrent extensions mentioned above (when $\theta \in (0, 1)$), as well as in the study of nested interval partition evolutions (when $\theta \in (-\alpha, 0)$) in Section 5.3.
1.3. Further applications. A remarkable feature of the three-parameter family is that it includes the emigration case with $\theta < 0$; this cannot happen in the two-parameter case with $\theta_2 = \alpha$ where $\theta = \theta_1 \geq 0$, but it is naturally included by Petrov [35] in the unordered setting. The discrete approximation method developed in this work in particular permits us to understand pseudo-stationarity and SSIP-excursions in the emigration case, which has further interesting applications. We discuss a few in this paper, listed as follows.

1.3.1. Measure-valued processes with $\theta \in [-\alpha, 0)$. In [19], we introduced a family of Fleming–Viot superprocesses parametrised by $\alpha \in (0, 1)$, $\theta \geq 0$, taking values on the space $(\mathcal{M}_1^\alpha, d_{\mathcal{M}})$ of all purely atomic probability measures on $[0, 1]$ endowed with the Prokhorov distance. Our construction in [19] is closely related to our construction of interval partition evolutions. We can now extend this model to the case $\theta \in [-\alpha, 0)$ and identify the desired stationary distribution, the two-parameter Pitman–Yor process, here exploiting the connection with an SSIP(\alpha)(\theta + \alpha, 0)-evolution. This is discussed in more detail in Section 5.1.

1.3.2. Nested interval partition evolutions. Let us recall a well-known identity [38, (5.24)] involving the two-parameter family PDIP(\alpha)(\theta) and associated fragmentation operators. For $0 \leq \bar{\alpha} \leq \alpha < 1$ and $\theta > 0$, let $(A_i, i \geq 1)$ be a random variable on $\nabla_{\infty}$ with distribution PDIP(\alpha)(\theta), and let $(A'_{i,j}, i \geq 1)$, $i \geq 1$, be an i.i.d. sequence of PDIP(\alpha)(-\bar{\alpha}) further independent of $(A_i, i \geq 1)$. Then the decreasing rearrangement of the collection $A_i A'_{i,j}$, $i, j \geq 1$, counted with multiplicities, has distribution PDIP(\alpha)(\bar{\theta})$. In other words, a PDIP(\alpha)(\bar{\theta})$ fragmented by PDIP(\alpha)(-\bar{\alpha}) has distribution PDIP(\alpha)(\bar{\theta})$. In Section 5.2, we extend this to the setting of our three-parameter family PDIP(\alpha)(\theta_1, \theta_2) of interval partitions.

In Sections 5.3, we study nested interval partition evolutions $(\beta_c, \beta_f)$, such that at every time $t \geq 0$, the endpoints of the intervals in $\beta_f(t)$ form a subset of those in $\beta_c(t)$. A particular case of our results establishes a dynamical and ordered version of this identity. Informally speaking, for $0 < \bar{\alpha} \leq \alpha < 1$ and $\theta, \theta_1, \theta_2 \geq 0$ with $\theta_1 + \theta_2 - \alpha = -\bar{\alpha} < 0$, we can find a coupling of stationary IP(\alpha)(\theta)- and IP(\alpha)(\bar{\theta})-evolutions $\beta_c$ and $\beta_f$, such that at each time $u \geq 0$, $\beta_f(u) \sim$ PDIP(\alpha)(\bar{\theta})$ can be regarded as fragmenting each interval of $\beta_c(u) \sim$ PDIP(\alpha)(\theta)$ according to PDIP(\alpha)(\theta_1, \theta_2).

Finally, Section 5.4 extends Theorem 1.1 to the setting of nested PCRP and nested interval partitions.

1.3.3. Connections with random trees. An initial motivation of this work was from studies of diffusions on a space of continuum trees. Aldous [1] introduced a Markov chain on the space of binary trees with $n$ leaves, by removing a leaf uniformly and reattaching it to a random edge. This Markov chain has the uniform distribution as its stationary distribution. As $n \to \infty$, Aldous conjectured that the limit of this Markov chain is a diffusion on continuum trees with stationary distribution given by the Brownian continuum random tree (CRT), i.e. the universal scaling limit of random discrete trees with finite vertex degree variance. Among different approaches [32, 13] investigating this problem, [13] describes the evolution via a consistent system of spines endowed with lengths and subtree masses, which relies crucially on interval partition evolutions.

This motivates us to construct stable Aldous diffusions, with stationary distributions given by stable Lévy trees with parameter $\rho \in (1, 2)$ [12, 11], which are the infinite variance analogues of the Brownian CRT. The related Markov chains on discrete trees have been studied in [49]. A major obstacle for the study in the continuum is that the approaches in [32, 13] cannot be obviously extended from binary to multifurcating trees with unbounded degrees. The current work provides tools towards overcoming this difficulty; with these techniques
one could further consider more general classes of continuum fragmentation trees, including the alpha-gamma models [9] or a two-parameter Poisson–Dirichlet family [24].

To demonstrate a connection of our work to continuum trees, recall the spinal decompositions developed in [24]. In a stable Lévy tree with parameter \( \rho \in (1, 2) \), there is a natural diffuse “uniform” probability measure on the leaves of the tree. Sample a leaf uniformly at random and consider its path to the root, called the spine. To decompose along this spine, we say that two vertices \( x, y \) of the tree are equivalent, if the paths from \( x \) and \( y \) to the root first meet the spine at the same point. Then equivalence classes are bushes of spinal subtrees rooted at the same spinal branch point; by deleting the branch point on the spine, the subtrees in an equivalence class form smaller connected components. The collection of equivalence classes is called the coarse spinal decomposition, and the collection of all subtrees is called the fine spinal decomposition. With \( \bar{\alpha} = 1 - 1/\rho \) and \( \alpha = 1/\rho \), it is known [24, Corollary 10] that the decreasing rearrangement of masses of the coarse spinal decomposition has distribution \( \text{PD}^{(\alpha)}(\bar{\alpha}) \); moreover, the mass sequence of the fine spinal decomposition is a \( \text{PD}^{(\alpha)}(\alpha - 1) \)-fragmentation of the coarse one and has \( \text{PD}^{(\alpha)}(\bar{\alpha}) \) distribution.

Some variants of our aforementioned nested interval partition evolutions can be used to represent the mass evolution of certain spinal decompositions in a conjectured stable Aldous diffusion. The order structure provided by the interval partition evolutions also plays a crucial role: at the coarse level, the equivalence classes of spinal bushes are naturally ordered by the distance of the spinal branch points to the root; at the fine level, a total order of the subtrees in the same equivalence class aligns with the semi-planar structure introduced in [49], which is used to explore the evolutions of sizes of subtrees in a bush at a branch point.

In Section 5.5, we give a broader example of a Markov chain related to random trees that converges to our nested SSIP-evolutions. The study of stable Aldous diffusions is, however, beyond the scope of the current paper and will be further investigated in future work.

1.4. Organisation of the paper. In Section 2 we generalise the two-parameter ordered Chinese Restaurant Processes to a three-parameter model and establish their connections to interval partitions and composition structures. In Section 3, we prove Theorem 1.1 in the two-parameter setting, building on [14, 15, 18, 44]. In Section 4, we study the three-parameter setting, both for processes absorbed in \( \emptyset \) and for recurrent extensions, which we obtain by constructing excursion measures of \( \text{SSIP}^{(\alpha)}(\theta_1, \theta_2) \)-evolutions. This section concludes with proofs of all results stated in this introduction. We finally turn to applications in Section 5.

2. Poisson–Dirichlet interval partitions. Throughout the rest of the paper, we fix a parameter \( \alpha \in (0, 1) \). In this section we will introduce the three-parameter family of random interval partitions \( \text{PDIP}^{(\alpha)}(\theta_1, \theta_2) \), with \( \theta_1, \theta_2 \geq 0 \), as the limit of a family of discrete-time ordered Chinese restaurant processes (without customers leaving).

2.1. Ordered Chinese restaurant processes in discrete time. For \( n \in \mathbb{N} \), let \( [n] := \{1, 2, \ldots, n\} \). Let \( C := \{(n_1, n_2, \ldots, n_k) \in \mathbb{N}^k, k \geq 1\} \cup \{\emptyset\} \). We view \( C \) as a space of integer compositions: for any \( n \geq 0 \), the subset \( C_n := \{(n_1, \ldots, n_k) \in C : n_1 + \cdots + n_k = n\} \) is the set of compositions of \( n \). Recall that we view \( C \) as a subset of the space \( \mathcal{I}_H \) of interval partitions introduced in the introduction. We still consider the metric \( d_H \), and all operations and functions defined on \( \mathcal{I}_H \) shall be inherited by \( C \). We also introduce the concatenation of a family of interval partitions \( (\beta_a)_{a \in A} \), indexed by a totally ordered set \( (A, \leq) \):

\[
\bigstar_{a \in A} \beta_a := \{(x + S_\beta(a-), y + S_\beta(a-)) : a \in A, (x, y) \in \beta_a\}, \quad \text{where} \quad S_\beta(a-) := \sum_{b \prec a} \|\beta_b\|.
\]

When \( A = \{1, 2\} \), we denote this by \( \beta_1 \bigstar \beta_2 \). Then each composition \( (n_1, n_2, \ldots, n_k) \in C \) is identified with the interval partition \( \bigstar_{i=1}^k \{(0, n_i)\} \in \mathcal{I}_H \).
Moreover, as Q. SHI AND M. WINKEL

At each step \(n\) with \(\theta\) in \([21, \text{Section 8}]\) and \([39]\). In particular, they show that an oCRP partition of \(N\) (table, then this gives rise to the well-known (unordered) \(C\) of \(N\) compositions \(\alpha, \theta_1\) and \(\theta_2\), or oCRP\((\alpha)(\theta_1, \theta_2)\). We also denote the distribution of \(N\) by \(\text{oCRP}_n(\alpha)(\theta_1, \theta_2)\).

In the degenerate case \(\theta_1 = \theta_2 = 0\), an oCRP\((\alpha)(0, 0)\) is simply a deterministic process \(N = (n), n \in \mathbb{N}\). If we do not distinguish the location of the new tables, but build the partition of \(N\) that has \(i\) and \(j\) in the same block if the \(i\)-th and \(j\)-th customer sit at the same table, then this gives rise to the well-known (unordered) \((\alpha, \theta)\)-Chinese restaurant process with \(\theta := \theta_1 + \theta_2 - \alpha\); see e.g. [38, Chapter 3].

When \(\theta_1 = \alpha\), this model encompasses the family of composition structures studied in [21, Section 8] and [39]. In particular, they show that an oCRP\((\alpha)(\alpha, \theta_2)\) is a regenerative composition structure in the following sense.

DEFINITION 2.2 (Composition structures [21, 22]). A Markovian sequence of random compositions \((C(n), n \in \mathbb{N})\) with \(C(n) \in \mathcal{C}_n\) is a composition structure, if the following property is satisfied:

- Weak sampling consistency: for each \(n \in \mathbb{N}\), if we first distribute \(n + 1\) identical balls into an ordered series of boxes according to \(C(n + 1)\) and then remove one ball uniformly at random (deleting an empty box if one is created), then the resulting composition \(\widetilde{C}(n)\) has the same distribution as \(C(n)\).

Moreover, a composition structure is called regenerative, if it further satisfies

- Regeneration: for every \(n \geq m\), conditionally on the first block of \(C(n)\) having size \(m\), the remainder is a composition in \(\mathcal{C}_{n - m}\), with the same distribution as \(C(n - m)\).

For \(n \geq m\), let \(r(n, m)\) be the probability that the first block of \(C(n)\) has size \(m\). Then \((r(n, m), 1 \leq m \leq n)\) is called the decrement matrix of \((C(n), n \in \mathbb{N})\).

LEMMA 2.3 ([39, Proposition 6]). For \(\theta_2 \geq 0\), an oCRP\((\alpha)(\alpha, \theta_2)\) \((C(n), n \in \mathbb{N})\) starting from \((1) \in \mathcal{C}\) is a regenerative composition structure with decrement matrix

\[
\begin{bmatrix}
(n - m)\alpha + m\theta_2 & \Gamma(m - \alpha)\Gamma(n - m + \theta_2)
\end{bmatrix}n \Gamma(1 - \alpha)\Gamma(n + \theta_2), \quad 1 \leq m \leq n.
\]

For every \((n_1, n_2, \ldots, n_k) \in \mathcal{C}_n\), we have

\[
\mathbb{P}(C(n) = (n_1, n_2, \ldots, n_k)) = \prod_{i=1}^{k} r_{\theta_2}(N_{i,k}, n_i), \quad \text{where} \quad N_{i,k} := \sum_{j=i}^{k} n_j.
\]

Moreover, \(\frac{1}{n}C(n)\) converges a.s. to a random interval partition \(\tilde{\gamma} \in \mathcal{I}_H\), under the metric \(d_H\), as \(n \to \infty\).
The limit $\hat{\gamma}$ is called a regenerative $(\alpha, \theta_2)$ interval partition in [21] and [39]. For $\beta \in \mathcal{I}_D$, the left-right reversal of $\beta$ is

$$
\text{rev}(\beta) := \{(\|\beta\| - b, \|\beta\| - a) : (a, b) \in \beta\} \in \mathcal{I}_D.
$$

DEFINITION 2.4 (PDIP$^{(\alpha)}(\theta_1)$). For $\theta_1 \geq 0$, let $\hat{\gamma}$ be a regenerative $(\alpha, \theta_1)$ interval partition. Then the left-right reversal $\text{rev}(\hat{\gamma})$ is called a Poisson–Dirichlet$(\alpha, \theta_1)$ interval partition, whose law on $\mathcal{I}_D$ is denoted by PDIP$^{(\alpha)}(\theta_1)$.

By the left-right reversal, it follows clearly from Lemma 2.3 that PDIP$^{(\alpha)}(\theta_1)$ also describes the limiting proportions of customers at tables in an oCRP$(\alpha, \theta_2)$. We record from [15, Proposition 2.2(iv)] a decomposition for future usage: with independent $B \sim \text{Beta}(\alpha, 1-\alpha)$ and $\bar{\gamma} \sim \text{PDIP}^{(\alpha)}(\alpha)$, we have

$$
\{(0,1-B)\} \star B\bar{\gamma} \sim \text{PDIP}^{(\alpha)}(0).
$$

To understand the distribution of $(C(n), n \in \mathbb{N}) \sim \text{oCRP}^{(\alpha)}(\theta_1, \theta_2)$, let us present a decomposition as follows. Recall that there is an initial table with one customer at time 1. Let us distinguish this initial table from other tables. At time $n \in \mathbb{N}$, we record the size of the initial table by $N^{(n)}_0$, the composition of the table sizes to the left of the initial table by $C^{(n)}_1$, and the composition to the right of the initial table by $C^{(n)}_2$. Then there is the identity $C(n) = C^{(n)}_1 \star \{(0, N^{(n)}_0)\} \star C^{(n)}_2$.

Let $(N^{(n)}_1, N^{(n)}_2) := (\|C^{(n)}_1\|, \|C^{(n)}_2\|)$. Then $(N^{(n)}_1, N^{(n)}_0, N^{(n)}_2)$ can be described as a Pólya urn model with three colours. More precisely, when the current numbers of balls of the three colours are $(n_1, n_0, n_2)$, we next add a ball whose colour is chosen according to probabilities proportional to $n_1 + \theta_1, n_0 - \alpha$ and $n_2 + \theta_2$. Starting from the initial state $(0, 1, 0)$, we get $(N^{(n)}_1, N^{(n)}_0, N^{(n)}_2)$ after adding $n-1$ balls. In other words, the vector $(N^{(n)}_1, N^{(n)}_0, N^{(n)}_2)$ has Dirichlet-multinomial distribution with parameters $n-1$ and $(\theta_1, 1-\alpha, \theta_2)$; i.e. for $n_1, n_0, n_2 \in \mathbb{N}$ with $n_0 \neq 0$ and $n_1 + n_0 + n_2 = n$,

$$
p_n(n_1, n_0, n_2) := \mathbb{P}\left((N^{(n)}_1, N^{(n)}_0, N^{(n)}_2) = (n_1, n_0, n_2)\right)
=$$  
$$
\frac{\Gamma(1-\alpha + \theta_1 + \theta_2) (n-1)! \Gamma(n_0 - \alpha) \Gamma(n_1 + \theta_1) \Gamma(n_2 + \theta_2)}{\Gamma(1-\alpha) \Gamma(\theta_1) \Gamma(\theta_2) \Gamma(n - \alpha + \theta_1 + \theta_2) (n_0 - 1)! n_1! n_2!}.
$$

Furthermore, conditionally given $(N^{(n)}_1, N^{(n)}_0, N^{(n)}_2)$, the compositions $C^{(n)}_1$ and $C^{(n)}_2$ are independent with distribution oCRP$^{(\alpha)}_{\gamma^{(n)}_1}(\theta_1, \alpha)$ and oCRP$^{(\alpha)}_{\gamma^{(n)}_2}(\theta_2, \alpha)$ respectively, for which there is an explicit description in Lemma 2.3, up to an elementary left-right reversal.

PROPOSITION 2.5. Let $\theta_1, \theta_2 \geq 0$ and $(C(n), n \in \mathbb{N})$ an oCRP$^{(\alpha)}(\theta_1, \theta_2)$. Then for every $(n_1, n_2, \ldots, n_k) \in \mathcal{C}$ with $n = n_1 + n_2 + \cdots + n_k$, we have

$$
\mathbb{P}(C(n) = (n_1, \ldots, n_k)) = \sum_{i=1}^{k} p_n(N_{1:i-1}, n_i, N_{i+1:k}) \prod_{j=1}^{i-1} r_{\theta_1}(N_{1:j}, n_j) \prod_{j=i+1}^{k} r_{\theta_2}(N_{j:k}, n_j),
$$

where $N_{i:j} = n_i + \cdots + n_j$, $p_n$ is given by (6) and $r_{\theta_1}, r_{\theta_2}$ are as in Lemma 2.3. Furthermore, $(C(n), n \in \mathbb{N})$ is a composition structure in the sense that it is weakly sampling consistent.

PROOF. The distribution of $C(n)$ is an immediate consequence of the decomposition explained above. To prove the weak sampling consistency, let us consider the decomposition
$C(n + 1) = C_1^{(n+1)} \ast \{(0, N_0^{(n+1)})\} \ast C_2^{(n+1)}$. By removing one customer uniformly at random (a down-step), we obtain in the obvious way a triple $(\widetilde{C}_1^{(n)}, \widetilde{N}_0^{(n)}, \widetilde{C}_2^{(n)})$, with the exception for the case when $N_0^{(n+1)} = 1$ and this customer is removed by the down-step: in the latter situation, to make sure that $\widetilde{N}_0^{(n)}$ is strictly positive, we choose the new marked table to be the nearest to the left with probability proportional to $\|C_1^{(n+1)}\|$, and the nearest to the right with the remaining probability, proportional to $\|C_2^{(n+1)}\|$, and we further decompose according to this new middle table to define $(\widetilde{C}_1^{(n)}, \widetilde{N}_0^{(n)}, \widetilde{C}_2^{(n)})$.

Therefore, for $n_1, n_0, n_2 \in \mathbb{N}_0$ with $n_0 \neq 0$ and $n_1 + n_0 + n_2 = n$, the probability of the event $\{ (\|\widetilde{C}_1^{(n)}\|, \widetilde{N}_0^{(n)}, \|\widetilde{C}_2^{(n)}\|) = (n_1, n_0, n_2) \}$ is

$$p_{n+1}(n_1+1, n_0, n_2) \frac{n_1+1}{n+1} + p_{n+1}(n_1, n_0, n_2+1) \frac{n_2+1}{n+1} + p_{n+1}(n_1, n_0+1, n_2) \frac{n_0+1}{n+1} + \frac{n_1+n_0}{n(n+1)} \frac{n+1}{n+1} \tau_{\theta_1}(n_1+n_0, n_0) + p_{n+1}(n_1, n_0+1, n_2) \frac{n_0+n_2}{n(n+1)} \tau_{\theta_2}(n_0+n_2, n_2),$$

where the meaning of each term should be clear. Straightforward calculation shows the sum is equal to $p_n(n_1, n_0, n_2)$.

The triple description above shows that, conditionally on $\|C_1^{(n+1)}\| = n_1, C_1^{(n+1)}$ has distribution $\text{oCRP}_{n_1}^{(\alpha)}(\theta_1, \alpha)$. Conditionally on $(\|\widetilde{C}_1^{(n)}\|, \widetilde{N}_0^{(n)}, \|\widetilde{C}_2^{(n)}\|) = (n_1, n_0, n_2)$, we still have that $\widetilde{C}_1^{(n)} \sim \text{oCRP}_{n_1}^{(\alpha)}(\theta_1, \alpha)$. This could be checked by looking at each situation: in the down-step, if the customer is removed from the marked table (with size $\geq 2$) or the right part, then $\widetilde{C}_1^{(n)} = C_1^{(n+1)}$, which has distribution $\text{oCRP}_{n_1}^{(\alpha)}(\theta_1, \alpha)$; if the customer is removed from the left part, then this is a consequence of the weak sampling consistency of $C_1^{(n)}$ given by Lemma 2.3; if the marked table has one customer and she is removed, then the claim holds because Lemma 2.3 yields that $C_1^{(n+1)}$ is regenerative. By similar arguments we have $\widetilde{C}_2^{(n)} \sim \text{oCRP}_{n_2}^{(\alpha)}(\theta_2)$. Summarising, we find that $(\widetilde{C}_1^{(n)}, \widetilde{N}_0^{(n)}, \widetilde{C}_2^{(n)})$ has the same distribution as $(C_1^{(n)}, N_0^{(n)}, C_2^{(n)})$. This completes the proof.

\[\square\]

2.2. The three-parameter family PDIP^{(\alpha)}(\theta_1, \theta_2). Our next aim is to study the asymptotics of an $\text{oCRP}_{n}^{(\alpha)}(\theta_1, \theta_2)$, as $n \to \infty$. Recall that Lemma 2.3 gives the result for the case $\theta_1 = \alpha$ with the limit distributions forming a two-parameter family, whose left-right reversals give the corresponding result for $\theta_2 = \alpha$. The latter limits were denoted by PDIP^{(\alpha)}(\theta_1), $\theta_1 \geq 0$. Let us now introduce a new three-parameter family of random interval partitions, generalising PDIP^{(\alpha)}(\theta_1). To this end, we extend the parameters of Dirichlet distributions to every $\alpha_1, \ldots, \alpha_m \geq 0$ with $m \in \mathbb{N}$: say $\alpha_i, \ldots, \alpha_{i_k} > 0$ and $\alpha_j = 0$ for any other $j \leq m$, let $(B_1, \ldots, B_i) \sim \text{Dir}(\alpha_1, \ldots, \alpha_k)$ and $B_j := 0$ for all other $j$. Then we define $\text{Dir}(\alpha_1, \ldots, \alpha_m)$ to be the law of $(B_1, \ldots, B_m)$. By convention $\text{Dir}(\alpha) = \delta_1$ for any $\alpha > 0$.

**Definition 2.6 (PDIP^{(\alpha)}(\theta_1, \theta_2)).** For $\theta_1, \theta_2 \geq 0$, let $(B_1, B_0, B_2) \sim \text{Dir}(\theta_1, 1-\alpha, \theta_2)$, $\gamma_1 \sim \text{PDIP}^{(\alpha)}(\theta_1)$, and $\gamma_2 \sim \text{PDIP}^{(\alpha)}(\theta_2)$, independent of each other. Let $\gamma = B_1 \gamma_1 \ast \{(0, B_0)\} \ast \text{rev}(B_2 \gamma_2)$. Then we call $\gamma$ an $(\alpha, \theta_1, \theta_2)$-Poisson–Dirichlet interval partition with distribution PDIP^{(\alpha)}(\theta_1, \theta_2).

The case $\theta_1 = \theta_2 = 0$ is degenerate with PDIP^{(\alpha)}(0, 0) = $\delta_{\{(0,1)\}}$. When $\theta_2 = \alpha$, we see e.g. from [39, Corollary 8] that PDIP^{(\alpha)}(\theta_1, \theta_2) coincides with PDIP^{(\alpha)}(\theta_1) in Definition 2.4.

**Lemma 2.7.** Let $(C(n), n \in \mathbb{N}) \sim \text{oCRP}^{(\alpha)}(\theta_1, \theta_2)$. Then $\frac{1}{n} C(n)$ converges a.s. to a random interval partition $\gamma \sim \text{PDIP}^{(\alpha)}(\theta_1, \theta_2)$, under the metric $d_H$, as $n \to \infty$. 

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PROOF. We use the triple-description of $C(n)$ in the proof of Proposition 2.5. Consider independent $(R_1(n), n \in \mathbb{N}) \sim \text{oCRP}^{(\alpha)}(\theta_1, \alpha)$, $(R_2(n), n \in \mathbb{N}) \sim \text{oCRP}^{(\alpha)}(\alpha, \theta_2)$, and a Pólya urn model with three colours $(N_1^{(n)}, N_0^{(n)}, N_2^{(n)})$, $n \in \mathbb{N}$. Then we can write $C(n) = R_1(N_1^{(n)}) \ast \{(0, N_0^{(n)})\} \ast R_2(N_2^{(n)})$.

The asymptotics of a Pólya urn yield that $\frac{1}{n} (N_1^{(n)}, N_0^{(n)}, N_2^{(n)})$ converges a.s. to some $(B_1, B_0, B_2) \sim \text{Dir}(\theta_1, 1 - \alpha, \theta_2)$. By Lemma 2.3, there exist $\gamma_1 \sim \text{PDIP}^{(\alpha)}(\theta_1)$ and $\gamma_2 \sim \text{PDIP}^{(\alpha)}(\theta_2)$, independent of each other, such that $\frac{1}{n} R_1(n) \rightarrow \gamma_1$ and $\frac{1}{n} R_2(n) \rightarrow \text{rev}(\gamma_2)$; both convergences hold a.s. as $n \rightarrow \infty$. Therefore, we conclude that $\frac{1}{n} C(n)$ converges a.s. to $(B_1 \gamma_1) \ast \{(0, B_0)\} \ast (B_2 \text{rev}(\gamma_2))$, which has distribution $\text{PDIP}^{(\alpha)}(\theta_1, \theta_2)$ by definition. 

We now give some decompositions of $\text{PDIP}^{(\alpha)}(\theta_1, \theta_2)$, extending [39, Corollary 8] to the three-parameter case. With independent $B \sim \text{Beta}(1 - \alpha + \theta_1, \theta_2)$, $\gamma \sim \text{PDIP}^{(\alpha)}(\theta_1, 0)$, and $\beta \sim \text{PDIP}^{(\alpha)}(\alpha, \theta_2)$, it follows readily from Definition 2.6 that

$$B \gamma \ast (1 - B) \beta \sim \text{PDIP}^{(\alpha)}(\theta_1, \theta_2).$$

When $\theta_1 \geq \alpha$, we also have a different decomposition as follows.

**Corollary 2.8.** Suppose that $\theta_1 \geq \alpha$. With independent $B' \sim \text{Beta}(\theta_1 - \alpha, \theta_2)$, $\gamma \sim \text{PDIP}^{(\alpha)}(\theta_1, 0)$, and $\beta \sim \text{PDIP}^{(\alpha)}(\alpha, \theta_2)$, we have

$$B' \gamma \ast (1 - B') \beta \sim \text{PDIP}^{(\alpha)}(\theta_1, \theta_2),$$

and $\gamma \overset{d}{=} V' \gamma_1 \ast \{(0, 1 - V')\}$ for independent $V' \sim \text{Beta}(\theta_1, 1 - \alpha)$ and $\gamma_1 \sim \text{PDIP}^{(\alpha)}(\theta_1)$.

**Proof.** Consider an $\text{oCRP}^{(\alpha)}(\theta_1, \theta_2)$. For the initial table, we colour it in red with probability $\theta_1 - \alpha)/(\theta_1 - \alpha + \theta_2)$. If it is not coloured in red, then each time a new table arrives to the left of the initial table, we flip an unfair coin with success probability $1 - \alpha/\theta_1$ and colour the new table in red at the first success. In this way, we separate the composition at every step into two parts: the tables to the left of the red table (with the red table included), and everything to the right of the red table. It is easy to see that the sizes of the two parts follow a Pólya urn such that the asymptotic proportions follow $\text{Dir}(\theta_1 - \alpha, \theta_2)$. Moreover, conditionally on the sizes of the two parts, they are independent $\text{oCRP}^{(\alpha)}(\theta_1, 0)$ and $\text{oCRP}^{(\alpha)}(\alpha, \theta_2)$ respectively. Now the claim follows from Lemma 2.7 and Definition 2.6.

An ordered version [37] of Kingman’s paintbox processes is described as follows. Let $\gamma \in \mathcal{I}_{H,1}$ and $(Z_i, i \in \mathbb{N})$ be i.i.d. uniform random variables on $[0, 1]$. Then customers $i$ and $j$ sit at the same table, if and only if $Z_i$ and $Z_j$ fall in the same block of $\gamma$. Moreover, the tables are ordered by their corresponding intervals. For any $n \in \mathbb{N}$, the first $n$ variables $(Z_i, i \in [n])$ give rise to a composition of the set $[n]$, i.e. an ordered family of disjoint subsets of $[n]$:

$$C^*_n(\gamma) = \{B_U(n) : B_U(n) \neq \emptyset, U \in \gamma\},$$

where $B_U(n) := \{j \leq n : Z_j \in (\inf U, \sup U)\}$.

Let $P_{n, \gamma}$ be the distribution of the random composition of $n$ induced by $C^*_n(\gamma)$.

The following statement shows that the composition structure induced by an ordered CRP is a mixture of ordered Kingman paintbox processes.

**Proposition 2.9.** The probability measure $\text{PDIP}^{(\alpha)}(\theta_1, \theta_2)$ is the unique probability measure on $\mathcal{I}_H$, such that there is the identity

$$\text{oCRP}^{(\alpha)}_n(\theta_1, \theta_2)(A) = \int_{\mathcal{I}_H} P_{n, \gamma}(A) \text{PDIP}^{(\alpha)}(\theta_1, \theta_2)(d\gamma), \quad \forall n \in \mathbb{N}, \forall A \subseteq \mathcal{C}_n.$$
Proof. Since an $oCRP^{(\alpha)}(\theta_1, \theta_2)$ is a composition structure by Proposition 2.5 and since renormalised $oCRP^{(\alpha)}_n(\theta_1, \theta_2)$ converges weakly to $PDIP^{(\alpha)}(\theta_1, \theta_2)$ as $n \to \infty$ by Lemma 2.7, the statement follows from [22, Corollary 12].

Remark. If we label the customers by $\mathbb{N}$ in an $oCRP^{(\alpha)}(\theta_1, \theta_2)$ defined in Definition 2.1, then we also naturally obtain a composition of the set $[n]$ when $n$ customers have arrived. However, it does not have the same law as the $C^\ast(n)$ obtained from the paintbox in (8) with $\tilde{\gamma} \sim PDIP^{(\alpha)}(\theta_1, \theta_2)$, though we know from Proposition 2.9 that their induced integer compositions of $n$ have the same law. Indeed, $C^\ast(n)$ obtained by the paintbox is exchangeable [22], but it is easy to check that an $oCRP^{(\alpha)}(\theta_1, \theta_2)$ with general parameters is not, the only exceptions being for $\theta_1 = \theta_2 = \alpha$.

Recall that $PD^{(\alpha)}(\theta)$ denotes the Poisson–Dirichlet distribution on the Kingman simplex $\nabla_{\infty}$.

Proposition 2.10. Let $\theta_1, \theta_2 \geq 0$ with $\theta_1 + \theta_2 > 0$. The ranked interval lengths of a $PDIP^{(\alpha)}(\theta_1, \theta_2)$ have $PD^{(\alpha)}(\theta)$ distribution on $\nabla_{\infty}$ with $\theta := \theta_1 + \theta_2 - \alpha$.

Proof. Let $C^{(n)} \sim oCRP^{(\alpha)}_n(\theta_1, \theta_2)$, then it follows immediately from its construction that $C^{(n)}$ ranked in decreasing order is an unordered $(\alpha, \theta)$-Chinese restaurant process with $\theta := \theta_1 + \theta_2 - \alpha > -\alpha$. As a consequence of Lemma 2.7, the ranked interval lengths of a $PDIP^{(\alpha)}(\theta_1, \theta_2)$ have the same distribution as the limit of $(\alpha, \theta)$-Chinese restaurant processes, which is known to be $PD^{(\alpha)}(\theta)$.

3. Proof of Theorem 1.1 when $\theta_2 = \alpha$. We first recall in Section 3.1 the construction and some basic properties of the two-parameter family of $SSIP^{(\alpha)}(\theta_1)$-evolutions from [14, 15, 18], and then prove that they are the diffusion limits of the corresponding $PCR^{(\alpha)}(\theta_1, \alpha)$, in Sections 3.4 and 3.5 for $\theta_1 = 0$ and for $\theta_1 \geq 0$ in general, respectively, thus proving Theorem 1.1 for the case $\theta_2 = \alpha$. The proofs rely on a representation of $PCR^{(\alpha)}(\theta_1, \alpha)$ by Rogers and Winkel [44] that we recall in Section 3.2 and an in-depth investigation of a positive-integer-valued Markov chain in Section 3.3.

3.1. Preliminaries: $SSIP^{(\alpha)}(\theta_1)$-evolutions. In this section, we recall the scaffolding-and-spindles construction and some basic properties of an $(\alpha, \theta_1)$ self-similar interval partition evolution, $SSIP^{(\alpha)}(\theta_1)$. The material is collected from [14, 15, 18].

Let $E$ be the space of non-negative càdlàg excursions away from zero. Then for any $f \in E$, we have $\zeta(f) := \inf\{t > 0: f(t) = 0\} = \sup\{t \geq 0: f(t) > 0\}$. We will present the construction of $SSIP$-evolutions via the following skewer map introduced in [14].

Definition 3.1 (Skewer). Let $N = \sum_{i \in I} \delta(t_i, f_i)$ be a point measure on $\mathbb{R}_+ \times E$ and $X$ a càdlàg process such that

$$\sum_{\Delta X(t) > 0} \delta(t, \Delta X(t)) = \sum_{i \in I} \delta(t, \zeta(f_i)).$$

The skewer of the pair $(N, X)$ at level $y$ is (when well-defined) the interval partition

$$\text{SKWER}(y, N, X) := \{(M^y(t^-), M^y(t)) : M^y(t^-) < M^y(t), t \geq 0\},$$

where $M^y(t) = \int_{[0,t] \times E} f(y - X(s^-)) N(ds, df)$. Denote the process by

$$\text{SKWER}(N, X) := (\text{SKWER}(y, N, X), y \geq 0).$$
Let $\theta \in (-1, 1)$. We know from [23] that $\text{BESQ}(2\theta)$ has an exit boundary at zero. Pitman and Yor [40, Section 3] construct a $\sigma$-finite excursion measure $\Lambda^{(2\theta)}_{\text{BESQ}}$ associated with $\text{BESQ}(2\theta)$ on the space $\mathcal{E}$, such that

$$\Lambda^{(2\theta)}_{\text{BESQ}}(\zeta > y) := \Lambda^{(2\theta)}_{\text{BESQ}} \{ f \in \mathcal{E} : \zeta(f) > y \} = \frac{2^{\theta-1}}{\Gamma(2-\theta)} y^{-1+\theta}, \quad y > 0,$$

and under $\Lambda^{(2\theta)}_{\text{BESQ}}$, conditional on $\{\zeta = y\}$ for $0 < y < \infty$, the excursion is a squared Bessel bridge from 0 to 0 of length $y$, see [42, Section 11.3]. [40, Section 3] offers several other equivalent descriptions of $\Lambda^{(2\theta)}_{\text{BESQ}}$; see also [14, Section 2.3].

For $\alpha \in (0, 1)$, let $N$ be a Poisson random measure on $\mathbb{R}^+ \times \mathcal{E}$ with intensity $c_0 \text{Leb} \otimes \Lambda_{\text{BESQ}}^{(-2\alpha)}$, denoted by $\text{PRM}(c_0 \text{Leb} \otimes \Lambda_{\text{BESQ}}^{(-2\alpha)})$, where

$$c_0 := 2\alpha/(1+\alpha)/\Gamma(1-\alpha).$$

Each atom of $N$, which is an excursion function in $\mathcal{E}$, shall be referred to as a spindle, in view of illustration of $N$ as in Figure 2. We pair $N$ with a scaffolding function $\xi_N := (\xi_N(t), t \geq 0)$ defined by

$$\xi_N(t) := \lim_{z \downarrow 0} \left( \int_{[0,t] \times \{ g \in \mathcal{E} : \zeta(g) > z \}} \zeta(f) N(ds, df) - \frac{(1+\alpha)t}{2}\Gamma(1+\alpha) \Gamma(1-\alpha) \right).$$

This is a spectrally positive stable Lévy process of index $(1+\alpha)$, with Lévy measure $c_0 \Lambda^{(-2\alpha)}_{\text{BESQ}}(\zeta \in dy)$ and Laplace exponent $(2^{1-\alpha}q^{1+\alpha}/\Gamma(1+\alpha), q \geq 0)$.

For $x > 0$, let $f \sim \text{BESQ}_x(-2\alpha)$, independent of $N$. Write $\text{Clade}_x(\alpha)$ for the law of a clade of initial mass $x$, which is a random point measure on $\mathbb{R}^+ \times \mathcal{E}$ defined by

$$\text{Clade}(f, N) := \delta(0, f) + N_{\lfloor 0, T_{-\zeta(\eta_N)} \rfloor \times \mathcal{E}}, \quad \text{where} \quad T_{-y}(\xi_N) := \inf \{ t \geq 0 : \xi_N(t) = -y \}.$$

**Definition 3.2 (SSIP$^{(\alpha)}$)** (0-evolution). For $\gamma \in \mathcal{I}_H$, let $(N_U, U \in \gamma)$ be a family of independent clades, with each $N_U \sim \text{Clade}_{\text{Leb}(U)}(\alpha)$. An SSIP$^{(\alpha)}$ (0)-evolution starting from $\gamma \in \mathcal{I}_H$ is a process distributed as $\beta = (\beta(y), y \geq 0)$ defined by

$$\beta(y) := \bigotimes_{U \in \gamma} \text{SKEWER}(y, N_U, \xi_N(U)), \quad y \geq 0.$$
For any \( y \geq 0 \), let
\[
T_{\alpha}^{-y} := \inf\{t \geq 0 : X_\alpha(t) = -y\} = \inf\{t \geq 0 : \ell(t) \geq y\}.
\]
Notice that \( \inf_{u \leq t} X_{\theta_1}(u) = -(\alpha/\theta_1) \ell(t) \), then we have the identity
\[
T_{\theta_1}^{-y} := \inf\{t \geq 0 : X_{\theta_1}(t) = -y\} = T_{\alpha}^{-\theta_1/\alpha}y.
\]
For each \( j \in \mathbb{N} \), define an interval-partition-valued process
\[
\beta_j(y) := \text{skewer}(y, N|_{[0, T_{\alpha_1}]}, j + X_{\theta_1}|_{[0, T_{\alpha_1}]}, y \in [0, j].
\]
For any \( z > 0 \), the shifted process \((z + X_{\alpha}(T_{\alpha}^{-z} + t), t \geq 0)\) has the same distribution as \( X_{\alpha} \), by the strong Markov property of \( X_\alpha \). As a consequence, \((-z + \ell(t + T_{\alpha}^{-z}), t \geq 0)\) has the same law as \( \ell \). Combing this and (14), we deduce that, for any \( k \geq j \), the following two pairs have the same law:
\[
\left( N \circ L_{T_{\alpha_1}^{-k}} \bigl|_{[0, T_{\alpha_1}^{-k} - T_{\alpha_1}^{-j} - k]}, k + X_{\theta_1} \circ L_{T_{\alpha_1}^{-k}} \bigl|_{[0, T_{\alpha_1}^{-k} - T_{\alpha_1}^{-j} - k]} \right) \overset{d}{=} \left( N \big|_{[0, T_{\alpha_1}^{-j}]}, j + X_{\theta_1} \big|_{[0, T_{\alpha_1}^{-j}]}, \right)
\]
where \( L \) stands for the shift operator and we have also used the Poisson property of \( N \). This leads to \((\beta_j(y), y \in [0, j]) \overset{d}{=} (\tilde{\beta}_k(y), y \in [0, j])\). Thus, by Kolmogorov’s extension theorem, there exists a process \((\tilde{\beta}(y), y \geq 0)\) such that
\[
(\beta_j(y), y \in [0, j]) \overset{d}{=} (\tilde{\beta}_j(y), y \in [0, j]) \quad \text{for every } j \in \mathbb{N}.
\]
**Definition 3.3 (SSIP\((\alpha)/(\theta_1)\)-evolution).** For \( \theta_1 > 0 \), let \((\tilde{\beta}(y), y \geq 0)\) be as in (15) and \((\beta_j(y), y \geq 0)\) an independent SSIP\((\alpha)/(0)\)-evolution starting from \( \gamma \in \mathcal{I}_H \). Then \((\beta(y) = \tilde{\beta}(y) \ast \beta_j(y), y \geq 0)\) is called an SSIP\((\alpha)/(\theta_1)\)-evolution starting from \( \gamma \).

In [18], an SSIP\((\alpha)/(\theta_1)\)-evolution is defined in a slightly different way that more explicitly handles the Poisson random measure of excursions of \( X_\alpha \) above the minimum. Indeed, the passage from \( \alpha \) to \( \theta_1 \) in [18] is by changing the intensity by a factor of \( \theta_1/\alpha \). The current correspondence can be easily seen to have the same effect.

**Proposition 3.4 ([18, Theorem 1.4 and Proposition 3.4]).** For \( \theta_1 \geq 0 \), an SSIP\((\alpha)/(\theta_1)\)-evolution is a path-continuous Hunt process and its total mass process is a BESQ\((2\theta_1)\).

We refer to [18] for the transition kernel of an SSIP\((\alpha)/(\theta_1)\)-evolution.

**3.2. Poissonised ordered up-down Chinese restaurant processes.** For \( \theta > -1 \), let \( Z := (Z(t), t \geq 0) \) be a continuous-time Markov chain on \( \mathbb{N}_0 \), whose non-zero transition rates are
\[
Q_{i,j}(\theta) = \begin{cases} 
\theta + 1, & i \geq 1, j = i + 1; \\
i, & i \geq 1, j = i - 1; \\
\theta \lor 0, & i = 0, j = 1.
\end{cases}
\]
In particular, 0 is an absorbing state when \( \theta \leq 0 \). For \( k \in \mathbb{N}_0 \), we define
\[
\pi_k(\theta) : \text{the law of the process } Z \text{ starting from } Z(0) = k.
\]
Let \( \zeta(Z) := \inf\{t > 0 : Z(t) = 0\} \) be its first hitting time of zero.

Let \( \alpha \in (0, 1) \) and \( \theta_1, \theta_2 \geq 0 \). Recall from the introduction that a Poissonised ordered up-down Chinese restaurant process (PCRP) with parameters \( \alpha, \theta_1 \) and \( \theta_2 \), starting from \( C \in \mathcal{C} \), is denoted it by \( \text{PCRP}_{\mathcal{C}}(\alpha, \theta_1, \theta_2) \).
When $\theta_2 = \alpha$, a PCRP$(\alpha)(\theta_1, \alpha)$ is well-studied by Rogers and Winkel [44]. They develop a representation of a PCRP by using scaffolding and spindles, in a similar way to the construction of an SSIP$(\alpha)(\theta_1)$-evolution. Their approach draws on connections with splitting trees and results of the latter object developed in [20, 30].

Let $D \sim \text{PM}(\alpha \cdot \text{Leb} \otimes \pi_1(-\alpha))$ and define its scaffolding function by

$$J_D(t) := -t + \int_{[0,t] \times E} \zeta(f) D(ds, df), \quad t \geq 0.$$  

Let $Z \sim \pi_m(-\alpha)$ with $m \in \mathbb{N}$, independent of $D$. Then a discrete clade with initial mass $m$ is a random point measure on $\mathbb{R}_+ \times E$ defined by

$$\text{CLADE}_Z^D(Z, D) := \delta(0, Z) + D\left([0,T_{-\zeta(x)}(J_D)] \times E\right),$$

where $T_{-\zeta}(J_D) = \inf \{t \geq 0 : J_D(t) = -y\}$. Write $\text{CLADE}_Z^D(\alpha)$ for the law of $\text{CLADE}_Z^D(Z, D)$.

**Lemma 3.5 ([44, Theorem 1.2]).** For $\gamma \in C$, let $(D_U, U \in \gamma)$ be an independent family with each $D_U \sim \text{CLADE}_Z^D(\alpha)_{\text{Leb(U)}}(\alpha)$. Then the process $\left( \bigstar_{U \in \gamma} \text{SKWER}(y, D_U, J_{D_U}), y \geq 0 \right)$ is a PCRP$(\alpha)$ $(0, \alpha)$.

To construct a PCRP$(\alpha)(\theta_1, \alpha)$ with $\theta_1 > 0$, define for $t \geq 0$

$$J_{\theta_1, D}(t) := J_D(t) + \left(1 - \frac{\alpha}{\theta_1}\right) \ell(t), \quad \text{where } \ell(t) := -\inf_{u \leq t} J_D(u).$$

Then $\inf \{t \geq 0 : J_D(t) = -z\} = \inf \{t \geq 0 : J_{\theta_1, D}(t) = -(\alpha/\theta_1) z\} = T_{\theta_1}^{-\alpha/\theta_1} z$ for $z \geq 0$. Set

$$\tilde{C}_j(y) := \text{SKWER}\left(y, D|_{[0,T_{\theta_1}^{-1}]}, J_{\theta_1, D}|_{[0,T_{\theta_1}^{-1}]}, y \in [0,j], \quad j \in \mathbb{N}.\right.$$ 

Then, for any $k > j$, we have

$$\left(D|_{[T_{\theta_1}^{-1}(j-k), T_{\theta_1}^{-1}]}, D_{[T_{\theta_1}^{-1}(j-k), T_{\theta_1}^{-1}]}, J_{\theta_1, D}|_{[0,T_{\theta_1}^{-1}]}, j + J_{\theta_1, D}|_{[0,T_{\theta_1}^{-1}]}) \right) \cong \left(D|_{[0,T_{\theta_1}^{-1}]}, J_{\theta_1, D}|_{[0,T_{\theta_1}^{-1}]} \right).$$

As a result, $(\tilde{C}_k(y), y \in [0,j]) \cong (\tilde{C}_j(y), y \in [0,j])$. Then by Kolmogorov's extension theorem there exists a càdlàg process $(\tilde{C}(y), y \geq 0)$ such that

$$(\tilde{C}(y), y \in [0,j]) \cong (\tilde{C}_j(y), y \in [0,j]) \quad \text{for all } j \in \mathbb{N}.\right.$$

**Theorem 3.6 ([44, Theorem 2.5]).** For $\theta_1 > 0$, let $(\tilde{C}(y), y \geq 0)$ be the process defined in (20). For $\gamma \in C$, let $(\tilde{C}(y), y \geq 0)$ be a PCRP$(\alpha)(0, \alpha)$ starting from $\gamma$. Then the $\mathbb{C}$-valued process $(C(y) := \tilde{C}(y) \star \tilde{C}(y), y \geq 0)$ is a PCRP$(\alpha)(\theta_1, \alpha)$ starting from $\gamma$.

### 3.3. Study of the up-down chain on positive integers.

For $\theta > -1$ and $n, k \in \mathbb{N}$, define a probability measure:

$$\pi^{(n)}_k(\theta)$$

is the law of the process $(n^{-1} Z(2ny), y \geq 0)$, where $Z \sim \pi_k(\theta)$ as in (16).

In preparation of proving Theorem 1.1, we present the following convergence concerning scaffoldings and spindles.
For $n \in \mathbb{N}$, let $N^{(n)}$ be a Poisson random measure on $\mathbb{R}_+ \times \mathcal{E}$ with intensity $\text{Leb} \otimes (2\alpha n^{1+\alpha} \cdot \pi_1^{(n)}(-\alpha))$, and define its scaffolding $\xi^{(n)} := (\xi^{(n)}(t))_{t \geq 0}$, where

$$
\xi^{(n)}(t) := J_{N^{(n)}}(t) := -n^\alpha t + \int_{[0,t] \times \mathcal{E}} \zeta(f) N^{(n)}(ds, df), \quad t \geq 0.
$$

Write $\ell^{(n)} := (\ell^{(n)}(t) := -\inf_{s \in [0,t]} \xi^{(n)}(s), t \geq 0)$. Let $N \sim \text{PRM}(c_\alpha \cdot \text{Leb} \otimes \Lambda_{\text{BESQ}}^{(-2\alpha)})$, where $\Lambda_{\text{BESQ}}^{(-2\alpha)}$ is the excursion measure associated with $\text{BESQ}(-2\alpha)$ introduced in Section 3.1 and $c_\alpha = 2\alpha(1+\alpha)/\Gamma(1-\alpha)$ as in (11). Define its scaffolding $\xi_N$ as in (12), and $\ell_N(t) = -\inf_{s \in [0,t]} \xi_N(s), t \geq 0$. Then the joint distribution of the triple $(N^{(n)}, \xi^{(n)}, \ell^{(n)})$ converges to $(N, \xi_N, \ell_N)$ in distribution, under the product of vague and Skorokhod topologies.

Note that, for $n \in \mathbb{N}$, we can obtain $N^{(n)}$ from $D \sim \text{PRM}(\text{Leb} \otimes \pi_1(-\alpha))$, such that for each atom $A \in N^{(n)}$, $D$ has atom $\delta(\theta^{-1}A, f(2n \cdot))$. This suggests a close relation between $N^{(n)}$ and a rescaled PCRP, which will be specified later on.

The up-down chain defined in (21) plays a central role in the proof of Proposition 3.7. Let us first record a convergence result obtained in [44, Theorem 1.3–1.4]. Similar convergence in a general context of discrete-time Markov chains converging to positive self-similar Markov processes has been established in [5].

**Lemma 3.8 ([44, Theorem 1.3–1.4]).** Fix $a > 0$ and $\theta > -1$. For every $n \in \mathbb{N}$, let $Z^{(n)} \sim \pi_{[na]}^{(n)}(\theta)$. Then the following convergence holds in the space $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$ of càdlàg functions endowed with the Skorokhod topology:

$$
Z^{(n)} \xrightarrow{n \to \infty} Z \sim \text{BESQ}_a(2\theta) \text{ in distribution.}
$$

Moreover, if $\theta \in (-1, 0]$, then the convergence holds jointly with the convergence of first hitting times of 0.

For our purposes, we study this up-down chain in more depth and obtain the following two convergence results.

**Lemma 3.9.** In Lemma 3.8, the joint convergence of first hitting time of 0 also holds when $\theta \in (0, 1)$.

Recall that for $\theta \geq 1$, the first hitting time of 0 by $\text{BESQ}(2\theta)$ is infinite.

**Proof.** We adapt the proof of [5, Theorem 3(i)], which establishes such convergence of hitting times in a general context of discrete-time Markov chains converging to positive self-similar Markov processes. This relies on Lamperti’s representation for $Z \sim \text{BESQ}_a(2\theta)$

$$
Z(t) = \exp(\xi(\sigma(t))), \quad \text{where } \sigma(t) = \inf \left\{ s \geq 0: \int_0^s e^{\xi(r)} dr > t \right\},
$$

for a Brownian motion with drift $\xi(t) = \log(a) + 2B(t) - 2(1-\theta)t$, and corresponding representations

$$
Z_n(t) = \exp(\xi_n(\sigma_n(t))), \quad \text{where } \sigma_n(t) = \inf \left\{ s \geq 0: \int_0^s e^{\xi_n(r)} dr > t \right\},
$$

for continuous-time Markov chains $\xi_n, n \geq 1, \text{ with increment kernels}

$$
L^n(x, dy) = 2n e^x \left( ne^x + \theta e^{\log(1+1/ne^x)}(dy) + ne^x \delta_{\log(1-1/ne^x)}(dy) \right), \quad x \geq -\log n.
$$
We easily check that for all $x \in \mathbb{R}$

$$
\int_{y \in \mathbb{R}} y L_n(x, dy) \to -2(1-\theta) \quad \text{and} \quad \int_{y \in \mathbb{R}} y^2 L_n(x, dy) \to 4,
$$
as well as

$$
\sup_{x: |x| \leq x} \int_{y \in \mathbb{R}} y^2 1_{\{|y| \geq \varepsilon\}} L_n(x, dy) = 0 \quad \text{for } n \text{ sufficiently large}.
$$

To apply [25, Theorem IX.4.21], we further note that all convergences are locally uniform, and we extend the increment kernel $\tilde{L}(x, dy) := L(x, dy)$, $x \geq \log(2) - \log(n)$, by setting

$$
\tilde{L}^n(x, dy) := L^n(\log(2) - \log(n), dy), \quad x < \log(2) - \log(n),
$$
to be definite. With this extension of the increment kernel, we obtain $\tilde{\xi}_n \to \xi$ in distribution on $\mathbb{D}([0, \infty), \mathbb{R})$. This implies $\xi_n \to \xi$ in distribution also for the process $\xi_n$ that jumps from $-\log(n)$ to $-\infty$, but only if we stop the processes the first time they exceed any fixed negative level.

Turning to extinction times $\tau_n$ of $Z_n$, we use Skorokhod’s representation $\xi_n \to \xi$ almost surely. Then we want to show that also

$$
\tau_n = \int_0^\infty e^{\xi_n(s)} ds \to \tau = \int_0^\infty e^{\xi(s)} ds \quad \text{in probability}.
$$

We first establish some uniform bounds on the extinction times when $Z_n$, $n \geq 1$, are started from sufficiently small initial states. To achieve this, we consider the generator $L^n$ of $Z_n$ and note that for $g(x) = x^\beta$, we have

$$
L^n g(x) = 2n((nx + \theta)g(x + 1/n) + nxg(x - 1/n) - (2nx + \theta)g(x)) \leq -C/x
$$
for all $n \geq 1$ and $x \geq K/n$ if and only if

$$
\frac{g(1+h) - 2g(1) + g(1-h)}{h^2} + \theta \frac{g(1+h) - g(1)}{h} \leq -C/2 \quad \text{for all } h \leq 1/K.
$$

But since $g''(1) + \theta g'(1) = \beta(\beta - 1 + \theta) < 0$ for $\beta \in (0, 1 - \theta)$, the function $g$ is a Foster-Lyapunov function, and [34, Corollary 2.7], applied with $q = p/2 = \beta/(1 + \beta)$ and $f(x) = x^{(1+\beta)/2}$, yields

$$
\exists C' > 0 \forall n \geq 1, \forall x \geq K/n \quad \mathbb{E}_x((\tau_n^K)^{\eta}) < C' x^\beta,
$$
where $\tau_n^K = \inf\{t \geq 0: M_n(t) \leq K/n\}$. An application of Markov’s inequality yields

$$
\mathbb{P}_x(\tau_n^K > t) \leq C' x^\beta t^{-\beta/(1+\beta)}.
$$

In particular,

$$
\forall \varepsilon > 0 \forall t > 0 \exists \eta > 0 \forall n \geq 1 \forall K + 1 \leq i \leq \eta n \quad \mathbb{P}_{i/n}(\tau_n^K > t/6) \leq \varepsilon/8.
$$

Furthermore, there is $n_0$ such that for $n \geq n_0$, the probability that $M_n$ starting from $K/n$ takes more than time $t/6$ to get from $K/n$ to 0 is smaller than $\varepsilon/8$. Now choose $R$ large enough so that

$$
\mathbb{P}(\exp(\xi(R)) < \eta/2) \geq 1 - \varepsilon/8 \quad \text{and} \quad \mathbb{P}\left(\int_R^\infty e^{\xi(s)} ds > t/3\right) \leq \varepsilon/4.
$$

We can also take $n_1 \geq n_0$ large enough so that

$$
\mathbb{P}(|\exp(\xi_n(R)) - \exp(\xi(R))| < \eta/2) \geq 1 - \varepsilon/8 \quad \text{for all } n \geq n_1.
Then, considering \( \exp(\xi_n(R)) \) and applying the Markov property at time \( R \),
\[
\mathbb{P}(\exp(\xi_n(R)) > \eta) < \varepsilon/4 \quad \text{and} \quad \mathbb{P}\left( \int_0^\infty e^{-\xi_n(s)} ds > t/3 \right) \leq \varepsilon/2, \quad \text{for all } n \geq n_1.
\]
But since \( \xi_n \to \xi \) almost surely, uniformly on compact sets, we already have
\[
\int_0^R e^{\xi_n(s)} ds \to \int_0^R e^{\xi(s)} ds \quad \text{almost surely.}
\]
Hence, we can find \( n_2 \geq n_1 \) so that for all \( n \geq n_2 \)
\[
\mathbb{P}\left( \left| \int_0^R e^{\xi_n(s)} ds - \int_0^R e^{\xi(s)} ds \right| > t/3 \right) < \varepsilon/4.
\]
We conclude that, for any given \( t > 0 \) and any given \( \varepsilon \), we found \( n_2 \geq 1 \) such that for all \( n \geq n_2 \)
\[
\mathbb{P}\left( \left| \int_0^\infty e^{\xi_n(s)} ds - \int_0^\infty e^{\xi(s)} ds \right| > t \right) < \varepsilon,
\]
as required.

**Proposition 3.10.** Let \( \theta \in (-1,1) \) and \( Z \sim \pi_1(\theta) \). Denote by \( \tilde{\pi}_1^{(n)}(\theta) \) the distribution of \( \left( \frac{1}{n} R(2nt \wedge \zeta(Z)) \right), t \geq 0 \). Then the following convergence holds vaguely
\[
\frac{\Gamma(1+\theta)}{1-\theta} n^{1-\theta} \cdot \tilde{\pi}_1^{(n)}(\theta) \to \Lambda_{\text{BESQ}}^{(2\theta)}
\]
on the space of càdlàg excursions equipped with the Skorokhod topology.

**Proof.** Denote by \( A(f) = \sup |f| \) the supremum of a càdlàg excursion \( f \). In using the term "vague convergence" on spaces that are not locally compact, but are bounded away from a point (here bounded on \( \{A > a\} \) for all \( a > 0 \)), we follow Kallenberg [28, Section 4.1]. Specifically, it follows from his Lemma 4.1 that it suffices to show for all \( a > 0 \)

1. \( \Lambda_{\text{BESQ}}^{(2\theta)}(A = a) = 0 \),
2. \( (\Gamma(1+\theta)/(1-\theta))n^{1-\theta} \cdot \tilde{\pi}_1^{(n)}(\theta)(A > a) \to \Lambda_{\text{BESQ}}^{(2\theta)}(A > a) \),
3. \( \tilde{\pi}_1^{(n)}(\theta)(\cdot | A > a) \to \Lambda_{\text{BESQ}}^{(2\theta)}(\cdot | A > a) \) weakly.

See also [10, Proposition A2.6.II].

1. is well-known. Indeed, we have chosen to normalise \( \Lambda_{\text{BESQ}}^{(2\theta)} \) so that \( \Lambda_{\text{BESQ}}^{(2\theta)}(A > a) = a^{1-\theta} \).
See e.g. [40, Section 3]. Cf. [14, Lemma 2.8], where a different normalisation was chosen.

2. can be proved using scale functions. Let \( s \) be a scale function for the birth-death chain with up-rates \( i + \theta \) and down-rates \( i \) from state \( i \geq 1 \). Set \( s(0) = 0 \) and \( s(1) = 1 \). For \( s \) to be a scale function, we need
\[
(k + \theta)(s(k + 1) - s(k)) + k(s(k) - s(k)) = 0 \quad \text{for all } k \geq 1.
\]
Let \( d(k) = s(k) - s(k-1), k \geq 1 \). Then
\[
d(k+1) = \frac{k}{k + \theta} d(k) = \frac{\Gamma(k+1)\Gamma(1+\theta)}{\Gamma(k+1+\theta)} \sim \Gamma(1+\theta) k^{-\theta} \quad \text{as } k \to \infty,
\]
and therefore
\[
s(k) = \sum_{i=1}^k d(i) = \sum_{i=1}^k \frac{\Gamma(i+1)\Gamma(1+\theta)}{\Gamma(i+1+\theta)} \sim \frac{\Gamma(1+\theta)}{1-\theta} k^{1-\theta}.
\]
Then the scale function applied to the birth-death chain is a martingale. Now let \( p(k) \) be the probability of hitting \( k \) before absorption in 0, when starting from 1. Applying the optional stopping theorem at the first hitting time of \( \{0, k\} \), we find \( p(k)s(k) = 1 \), and hence

\[
\frac{\Gamma(1+\theta)}{1-\theta} n^{1-\theta} p([na]) = \frac{\Gamma(1+\theta)n^{1-\theta}}{(1-\theta)s([na])} \to a^{1-\theta},
\]
as required.

3. can be proved by using the First Description of \( \Lambda^{(2\theta)}_{\text{BESQ}} \) given in [40, (3.1)], which states, in particular, that the excursion under \( \Lambda^{(2\theta)}_{\text{BESQ}}(\cdot | A > a) \) is a concatenation of two independent processes, an \( \uparrow \)-diffusion starting from 0 and stopped when reaching \( a \) followed by a 0-diffusion starting from \( a \) and run until absorption in 0. In our case, the 0-diffusion is \( \text{BESQ}(2\theta) \), while [40, (3.5)] identifies the \( \uparrow \)-diffusion as \( \text{BESQ}(4 - 2\theta) \). Specifically, straightforward Skorokhod topology arguments adjusting the time-changes around the concatenation times, see [25, VI.1.15], imply that it suffices to show:

(a) The birth-death chain starting from 1 and conditioned to reach \( [na] \) before 0, rescaled, converges to a \( \text{BESQ}(4 - 2\theta) \) stopped at \( a \), jointly with the hitting times of \( [na] \).
(b) The birth-death chain starting from \( [na] \) run until first hitting 0, rescaled, converges to a \( \text{BESQ}(2\theta) \) starting from \( a \) stopped when first hitting 0, jointly with the hitting times.

(b) was shown in [44, Theorem 1.3]. See also Lemmas 3.8–3.9 here, completing the convergence of the hitting time. For (a), we adapt that proof. But first we need to identify the conditioned birth-death process. Note that the holding rates are not affected by the conditioning. An elementary argument based purely on the jump chain shows that the conditioned jump chain is Markovian, and its transition probabilities are adjusted by factors \( s(i \pm 1)/s(i) \) so that the conditioned birth-death process has up-rates \( (i+\theta)s(i+1)/s(i) \) and down-rates \( is(i-1)/s(i) \) from state \( i \geq 1 \). Rescaling, our processes are instances of \( \mathbb{R} \)-valued pure-jump Markov process with jump intensity kernels \( \tilde{K}^n(x, dy) = 0 \) for \( x \leq 0 \) and, for \( x > 0 \),

\[
\tilde{K}^n(x, dy) = 2n\left([nx] + \theta\right) s\left([nx+1]\right) s\left([nx]\right) \delta_{/n}(dy) + [nx] s\left([nx-1]\right) s\left([nx]\right) \delta_{-1/n}(dy).
\]

We now check the drift, diffusion and jump criteria of [25, Theorem IX.4.21]: for \( x > 0 \)

\[
\begin{align*}
\int_{\mathbb{R}} y\tilde{K}^n(x, dy) &= 2\left[nx\right] s\left([nx+1]\right) - s\left([nx-1]\right) + 2\theta s\left([nx+1]\right) s\left([nx]\right) \\
&\to 4 - 4\theta + 2\theta = 4 - 2\theta,
\end{align*}
\]

\[
\begin{align*}
\int_{\mathbb{R}} y^2\tilde{K}^n(x, dy) &= 2\left[nx\right] s\left([nx+1]\right) + s\left([nx-1]\right) + 2\theta s\left([nx+1]\right) \left(s\left([nx]\right)\right) \\
&\to 4x + 0 = 4x,
\end{align*}
\]

\[
\int_{\mathbb{R}} y^2 \mathbb{1}_{\{|y| \geq \varepsilon\}}\tilde{K}^n(x, dy) = 0 \quad \text{for } n \text{ sufficiently large},
\]

all uniformly in \( x \in (0, \infty) \), as required for the limiting \((0, \infty)\)-valued \( \text{BESQ}(4 - 2\theta) \) diffusion with infinitesimal drift \( 4 - 2\theta \) and diffusion coefficient \( 4x \). The convergence of hitting times of \([na]\), which is the first passage time above level \([na]\), follows from the regularity of the limiting diffusion after the first passage time above level \( a \). See e.g. [44, Lemma 3.3].

**Proof of Proposition 3.7.** Proposition 3.10 shows that the intensity measure of the Poisson random measure \( N^{(n)} \) converges vaguely as \( n \to \infty \). Then the weak convergence of
\( N^{(n)} \), under the vague topology, follows from [28, Theorem 4.11]. The weak convergence \( \xi^{(n)} \rightarrow \xi_N \) has already been proved by Rogers and Winkel [44, Theorem 1.5].

Therefore, both sequences \((N^{(n)}, n \in \mathbb{N})\) and \((\xi^{(n)}, n \in \mathbb{N})\) are tight (see e.g. [25, VI 3.9]), and the latter implies the tightness of \((\xi^{(n)}, n \in \mathbb{N})\). We hence deduce immediately the tightness of the triple-valued sequence \(((N^{(n)}, \xi^{(n)}, \ell^{(n)}), n \in \mathbb{N})\). As a result, we only need to prove that, for any subsequence \((N^{(n)}, \xi^{(n)}, \ell^{(n)})\) that converges in law, the limiting distribution is the same as \((N, \xi_N, \ell_N)\). By Skorokhod representation, we may assume that \((N^{(n)}, \xi^{(n)}, \ell^{(n)})\) converges a.s. to \((N, \xi, \ell)\), and it remains to prove that \(\xi = \xi_N\) and \(\ell = \ell_N\) a.s..

For any \(\varepsilon > 0\), since a.s. \(N\) has no spindle of length equal to \(\varepsilon\), the vague convergence of \(N^{(n)}\) implies that, a.s. for any \(t \geq 0\), we have the following weak convergence of finite point measures:

\[
\sum_{s \leq t} 1\{|\Delta \xi^{(n)}(s)| > \varepsilon\} \delta \left(s, \Delta \xi^{(n)}(s)\right) = \sum_{s \leq t} 1\{|\Delta \xi_N(s)| > \varepsilon\} \delta \left(s, \Delta \xi_N(s)\right).
\]

The subsequence above also converges a.s. to \(\sum_{s \leq t} 1\{|\Delta \xi^*(s)| > \varepsilon\} \delta \left(s, \Delta \xi^*(s)\right)\), since \(\xi^{(n)} \rightarrow \xi^*\) in \(D([R_+, R])\). By the Lévy–Itô decomposition, this is enough to conclude that \(\xi = \xi_N\) a.s.

For any \(t \geq 0\), since \(\xi^{(n)} \rightarrow \xi_N\) a.s. and \(\xi_N\) is a.s. continuous at \(t\), we have \((\xi^{(n)}(s), s \in [0, t]) \rightarrow (\xi_N(s), s \in [0, t])\) in \(D([0, t], R)\) a.s. Then \(\inf_{s \in [0, t]} \xi^{(n)}(s) \rightarrow \inf_{s \in [0, t]} \xi_N(s)\) a.s., because it is a continuous functional (w.r.t. the Skorokhod topology). In other words, \(\tilde{\ell}(t) = \ell_N(t)\) a.s.. By the continuity of the process \(\ell_N\) we have \(\tilde{\ell} = \ell_N\) a.s., completing the proof.

**Lemma 3.11** (First passage over a negative level). Suppose that \((N^{(n)}, \xi^{(n)})\) as in Proposition 3.7 converges a.s. to \((N, \xi_N)\) as \(n \rightarrow \infty\). Define

\[
T^{(n)}_{-y} := T_{-y}(\xi^{(n)}) := \inf \{t \geq 0 : \xi^{(n)}(t) = -y\}, \quad y > 0,
\]

and similarly \(T_{-y} := T_{-y}(\xi_N)\). Let \((h^{(n)})_{n \in \mathbb{N}}\) be a sequence of positive numbers with \(\lim_{n \rightarrow \infty} h^{(n)} = h > 0\). Then \(T^{(n)}_{-h^{(n)}}\) converges to \(T_{-h}\) a.s..

**Proof.** Since the process \(\xi^{(n)}\) is a spectrally positive Lévy process with some Laplace exponent \(\Phi^{(n)}\), we know from [3, Theorem VII.1] that \((T^{(n)}_{(-y)+}, y \geq 0)\) is a subordinator with Laplace exponent \((\Phi^{(n)})^{-1}\). On the one hand, the convergence of \(\Phi^{(n)}\) leads to \((T^{(n)}_{(-y)+}, y \geq 0) \rightarrow (T_{(-y)+}, y \geq 0)\) in distribution under the Skorokhod topology. Since \(\xi_N\) is a.s. continuous at \(T_{-h}\), we have \(T^{(n)}_{-h^{(n)}} \rightarrow T_{-h}\) in distribution.

On the other hand, we deduce from the convergence of the process \(\xi^{(n)}\) in \(D([R_+, R])\) that, for any \(\varepsilon > 0\), a.s. there exists \(N_1 \in \mathbb{N}\) such that for all \(n > N_1\),

\[
|\xi^{(n)}(T_{-h}) - (-h)| < \varepsilon.
\]

We may assume that \(|h^{(n)} - h| < \varepsilon\) for all \(n \in \mathbb{N}\). As a result, a.s. for any \(n > N_1\) and \(y' < h^{(n)} - 2\varepsilon\), we have \(T^{(n)}_{-y'} < T_{-h}\). Hence, by the arbitrariness of \(\varepsilon\) and the left-continuity of \(T_{-y}\) with respect to \(y\), we have \(\limsup_{n \rightarrow \infty} T^{(n)}_{-h^{(n)}} \leq T_{-h}\) a.s.. Recall that \(T^{(n)}_{-h^{(n)}} \rightarrow T_{-h}\) in distribution, it follows that \(T^{(n)}_{-h^{(n)}} \rightarrow T_{-h}\) a.s..
3.4. The scaling limit of a PCRP\(^{(\alpha)}(0, \alpha)\).

**Theorem 3.12** (Convergence of PCRP\(^{(\alpha)}(0, \alpha)\)). For \(n \in \mathbb{N}\), let \((C^{(n)}(y), y \geq 0)\) be a PCRP\(^{(\alpha)}(0, \alpha)\) starting from \(C^{(n)}(0) \in C\) and \((\beta(y), y \geq 0)\) be an SSIP\(^{(\alpha)}(0)\)-evolution starting from \(\beta(0) \in \mathcal{I}_H\). Suppose that the interval partition \(\frac{1}{n}C^{(n)}(0)\) converges in distribution to \(\beta(0)\) as \(n \to \infty\), under \(d_H\). Then the rescaled process \(\{\frac{1}{n}C^{(n)}(2ny), y \geq 0\}\) converges in distribution to \((\beta(y), y \geq 0)\) as \(n \to \infty\) in the Skorokhod sense and hence uniformly.

We start with the simplest case.

**Lemma 3.13.** The statement of Theorem 3.12 holds, if \(\beta(0) = \{(0, b)\}\) and \(C^{(n)}(0) = \{(0, b^{(n)})\}\), where \(\lim_{n \to \infty} n^{-1}b^{(n)} = b > 0\).

To prove this lemma, we first give a representation of the rescaled PCRP. Let \((N^{(n)}, \xi^{(n)})\) be as in Proposition 3.7. For each \(n \in \mathbb{N}\), we define a random point measure on \(\mathbb{R}_+ \times \mathcal{E}\) by

\[
N^{(n)}_{\text{cld}} := \delta(0, f^{(n)}) + N^{(n)}\big|_{\{0, T_{\xi^{(n)}}(\xi^{(n)})\} \times \mathcal{E}},
\]

where \(f^{(n)} \sim \pi^{(n)}_{\text{cld}}(-\alpha)\), independent of \(N^{(n)}\), and \(T_{\xi^{(n)}}(\xi^{(n)}) := \inf\{t \geq 0 : \xi^{(n)}(t) = -y\}\). Then we may assume that \(N^{(n)}_{\text{cld}}\) is obtained from \(D^{(n)} \sim \text{Clade}_{b^{(n)}(\alpha)}\) defined in (18) such that for each atom \(\delta(s, f)\) of \(D^{(n)}\), \(N^{(n)}_{\text{cld}}\) has an atom \(\delta(n^{-1+\alpha}s, n^{-1}f(2n\cdot))\). Let \(\xi^{(n)}_{\text{cld}}\) be the scaffolding associated with \(N^{(n)}_{\text{cld}}\) as in (17). As a consequence, we have the identity

\[
\beta^{(n)}(y) := \text{SKEWER}\left(y, N^{(n)}_{\text{cld}}, \xi^{(n)}_{\text{cld}}\right) = \frac{1}{n} \text{SKEWER}\left(2ny, D^{(n)}, \xi_{\text{D}^{(n)}}\right), \quad y \geq 0,
\]

where \(\xi_{\text{D}^{(n)}}\) is defined as in (17). By Lemma 3.5, we may assume that \(\beta^{(n)}(y) = \frac{1}{n}C^{(n)}(2ny)\) with \(C^{(n)}\) a PCRP\(^{(\alpha)}(0, \alpha)\) starting from \(C^{(n)}(0) = \{(0, b^{(n)})\}\).

**Proof of Lemma 3.13.** With notation as above, we shall prove that the rescaled process \(\beta^{(n)} := (\beta^{(n)}(y), y \geq 0)\) converges to an SSIP\(^{(\alpha)}(0)\)-evolution \(\beta := (\beta(y), y \geq 0)\) starting from \(\{(0, b)\}\). By Definition 3.2 we can write \(\beta = \text{SKEWER}(N_{\text{cld}}, \xi_{\text{cld}})\), with \(N_{\text{cld}} = \text{CLADE}(f, \mathcal{N})\) and \(\xi_{\text{cld}}\) its associated scaffolding, where \(f \sim \text{BESQ}_{b^{(2\alpha)}}\) and \(\mathcal{N}\) is a Poisson random measure on \([0, \infty) \times \mathcal{E}\) with intensity \(c_\alpha \text{Leb} \otimes \Lambda^{(2\alpha)}_{\text{BESQ}}\), independent of \(f\). Using Proposition 3.7 and Lemma 3.8, we have \((N^{(n)}, \xi^{(n)}) \to (N, \xi)\) and \((f^{(n)}, \zeta(f^{(n)})) \to (f, \zeta(f))\) in distribution, independently. Then it follows from Lemma 3.11 that this convergence also holds jointly with \(T_{-\xi(f^{(n)})}(\xi^{(n)}) \to T_{-\xi(f)}(\xi)\). As a consequence, we have \((N^{(n)}_{\text{cld}}, \xi^{(n)}_{\text{cld}}) \to (N_{\text{cld}}, \xi_{\text{cld}})\) in distribution.

Fix any subsequence \((m_j)_{j \in \mathbb{N}}\). With notation as above, consider the subsequence of triple-valued processes \((N^{(m_j)}, \xi^{(m_j)}, \|\beta^{(m_j)}\|)\) \(j \in \mathbb{N}\). For each element in the triple, we know its tightness from Proposition 3.7 and Lemma 3.8, then the triple-valued subsequence is tight. Therefore, we can extract a further subsequence \(\left(N^{(n_i)}, \xi^{(n_i)}_{\text{cld}}, \|\beta^{(n_i)}\|\right)\), that converges in distribution to a limit process \((N_{\text{cld}}, \xi_{\text{cld}}, \widetilde{M})\). Using Skorokhod representation, we may assume that this convergence holds a.s.. We shall prove that \(\beta^{(n_i)}\) converges to \(\beta\) a.s., from which the lemma follows.

We stress that the limit \(\widetilde{M}\) has the same law as the total mass process \(\|\beta\|\), but at this stage it is not clear if they are indeed equal. We will prove that \(\widetilde{M} = \|\beta\|\) a.s..
To this end, let us consider the contribution of the spindles with lifetime longer than $\rho > 0$. On the space $\mathbb{R}_+ \times \{ f \in \mathcal{E} : \zeta(f) > \rho \}$, $N_{cld}$ has a.s. a finite number of atoms, say enumerated in chronological order by $(t_j, f_j)_{j \leq K}$ with $K \in \mathbb{N}$. Since $N_{cld}$ has no spindle of length exactly equal to $\rho$, by the a.s. convergence $N_{cld}^{(n)} \to N_{cld}$, we may assume that each $N_{cld}^{(n)}$ also has $K$ atoms $(t_j^{(n)}, f_j^{(n)})_{j \leq K}$ on $\mathbb{R}_+ \times \{ f \in \mathcal{E} : \zeta(f) > \rho \}$, and, for every $j \leq K$, that

$$\lim_{n \to \infty} t_j^{(n)} = t_j, \quad \limsup_{i \to \infty} \left| f_j^{(n)}(t) - f_j(t) \right| = 0, \quad \text{and} \quad \lim_{i \to \infty} \zeta(f_j^{(n)}) = \zeta(f_j) \quad \text{a.s..}$$

Note that $\zeta(f_j^{(n)}) = \Delta \zeta_{cld}^{(n)}(t_j^{(n)})$. Since $\zeta_{cld}^{(n)} \to \zeta_{cld}$ in $\mathbb{D}(\mathbb{R}_+, \mathbb{R})$, we deduce that

$$\lim_{i \to \infty} \zeta_{cld}^{(n)}(t_j^{(n)}) = \zeta_{cld}(t_j) \quad \text{a.s..}$$

By deleting all spindles whose lifetimes are smaller than $\rho$, we obtain from $\beta^{(n)}$ an interval partition evolution

$$\beta_{\rho}^{(n)}(y) := \left\{ \begin{array}{l} M_{k-1}^{(n)}(y, \rho), M_k^{(n)}(y, \rho), \\ 1 \leq k \leq K \end{array} \right\}, \quad y \geq 0,$$

where $M_k^{(n)}(y, \rho) = \sum_{j \in [k]} f_j^{(n)} \left( y - \xi_{cld}^{(n)}(t_j^{(n)}) \right)$. Define similarly $M_k(y, \rho)$ and $\beta_{\rho}(y)$ from $\beta$. By (25) and (24), for all $k \leq K$,

$$\lim_{n \to \infty} \sup_{y \geq 0} \left| M_k^{(n)}(y, \rho) - M_k(y, \rho) \right| = 0 \quad \text{a.s..}$$

It follows that

$$\lim_{i \to \infty} \sup_{y \geq 0} d_H \left( \beta_{\rho}^{(n)}(y), \beta_{\rho}(y) \right) = 0 \quad \text{a.s..}$$

In particular, for all $y, \rho > 0$,

$$\tilde{M}(y) = \lim_{i \to \infty} \left\| \beta^{(n)}(y) \right\| \geq \lim_{i \to \infty} \left\| \beta_{\rho}^{(n)}(y) \right\| = \left\| \beta_{\rho}(y) \right\|, \quad \text{a.s..}$$

Then monotone convergence leads to, for all $y > 0$,

$$\tilde{M}(y) \geq \lim_{\rho \downarrow 0} \left\| \beta_{\rho}(y) \right\| = \left\| \beta(y) \right\|, \quad \text{a.s..}$$

Moreover, since $\tilde{M}$ and $\left\| \beta \right\|$ also have the same law, we conclude that $\tilde{M}$ and $\left\| \beta \right\|$ are indistinguishable.

Next, we shall show that $\beta_{\rho}$ approximates arbitrarily closely to $\beta$ as $\rho \to 0$. Write $M_{\leq \rho} := \left\| \beta \right\| - \left\| \beta_{\rho} \right\|$. For any $\varepsilon > 0$, we can find a certain $\rho > 0$ such that $\sup_{y \geq 0} M_{\leq \rho}(y) < \varepsilon$. Indeed, suppose by contradiction that this is not true, then there exist two sequences $\rho_i \downarrow 0$ and $y_i \geq 0$, such that $M_{\leq \rho_i}(y_i) \geq \varepsilon$ for each $i \in \mathbb{N}$. Since the extinction time of a clade is a.s. finite, we may assume that (by extracting a subsequence) $\lim_{i \to \infty} y_i = y \geq 0$. By the continuity of $\beta$, this yields that $\liminf_{i \to \infty} M_{\leq \rho_i}(y) \geq \varepsilon$. This is absurd, since we know by monotone convergence that $\lim_{i \to \infty} M_{\leq \rho_i}(y) = 0$. With this specified $\rho$, since $d_H(\beta(y), \beta_{\rho}(y)) \leq M_{\leq \rho}(y)$ for each $y \geq 0$, we have

$$\sup_{y \geq 0} d_H(\beta(y), \beta_{\rho}(y)) < \varepsilon.$$ 

Using (26) and the uniform convergence $\left\| \beta^{(n)}(y) \right\| \to \tilde{M} = \left\| \beta \right\|$, we deduce that the process $M_{\leq \rho}^{(n)} := \left\| \beta^{(n)}(y) \right\| - \left\| \beta_{\rho}^{(n)}(y) \right\|$ converges to $M_{\leq \rho}$ uniformly. Then, for all $n$ large enough, we also have

$$\sup_{y \geq 0} d_H \left( \beta^{(n)}(y), \beta_{\rho}^{(n)}(y) \right) \leq \sup_{y \geq 0} M_{\leq \rho}^{(n)}(y) < 2\varepsilon.$$
Combining this inequality with (26) and (27), we deduce that
\[
\limsup_{i \to \infty} \sup_{y \geq 0} d_H \left( \beta^{(n_i)}(y), \beta(y) \right) \leq 3\varepsilon.
\]
As \(\varepsilon\) is arbitrary, we conclude that \(\beta^{(n_i)}\) converges to \(\beta\) a.s. under the uniform topology, completing the proof.

To extend to a general initial state, let us record the following result that characterises the convergence under \(d_H\).

\textbf{Lemma 3.14 ([18, Lemma 4.3]).} Let \(\beta, \beta_n \in \mathcal{I}_H, n \geq 1\). Then \(d_H(\beta_n, \beta) \to 0\) as \(n \to \infty\) if and only if
\[
\forall (a,b) \in \beta \exists n_0 \geq 1 \forall n \geq n_0 \exists (a_n, b_n) \in \beta_n \ a_n \to a \text{ and } b_n \to b
\]
and
\[
\forall (n_k)_{k=1}^\infty n_k \to \infty \forall (c_k,d_k) \in \beta_{n_k}, k \geq 1: d_k \to d \in (0,\infty], c_k \to c \neq d \ (c,d) \in \beta.
\]

\textbf{Proof of Theorem 3.12.} For the case \(\beta(0) = \emptyset\), by convention \(\beta(y) = \emptyset\) for every \(y \geq 0\). Then the claim is a simple consequence of the convergence of the total mass processes.

So we may assume that \(\beta(0) \neq \emptyset\). By Definition 3.2, we can write \(\beta = \bigstar_{U \in \beta(0)} \beta_U\), where each process \(\beta_U := (\beta_U(y), y \geq 0) \sim \text{Clade}_{\text{Leb}(U)}(\alpha)\), independent of the others. For any \(\varepsilon > 0\), a.s. we can find at most a finite number of intervals, say \(U_1, U_2, \ldots, U_k \in \beta(0)\), listed from left to right, such that

\[
\sup_{y \geq 0} R_k(y) < \varepsilon, \quad \text{where } R_k(y) := \|\beta(y)\| - \sum_{i=1}^k \|\beta_{U_i}(y)\|.
\]

Indeed, since the process \(\|\beta\| \sim \text{BES}_0(\|\beta(0)\|)(0), \|\beta_{U_i}\| \sim \text{BES}_0(\text{Leb}(U_i))(0)\) for each \(i \in [k]\), and \(R_k\) is independent of the family \(\{\beta_{U_i}, i \in [k]\}\), we deduce from Proposition 3.4 that \(R_k \sim \text{BES}_0 r_k(0)\) with \(r_k := \|\beta(0)\| - \sum_{i=1}^k \text{Leb}(U_i)\). Hence, by letting \(r_k\) be small enough, we have (30).

For each \(n \in \mathbb{N}\), we similarly assume that \(C^{(n)}(y) = \bigstar_{U \in C^{(n)}(0)} C_{U}^{(n)}(y), y \geq 0\), where each process \((C_{U}^{(n)}(y), y \geq 0) \sim \text{PCRP}_{\text{Leb}(U)}^{(\alpha)}(0, \alpha)\), independent of the others.

Due to the convergence of the initial state \(\pi^{(n)}(0) \to \beta(0)\) and by Lemma 3.14 we can find for each \(i \leq k\) a sequence \(U^{(n)}_i = (a_i^{(n)}, b_i^{(n)}) \in C^{(n)}(0), n \in \mathbb{N}\), such that \(a_i^{(n)} / n \to \inf U_i\) and \(b_i^{(n)} / n \to \sup U_i\). In particular, we have \(\text{Leb}(U_i^{(n)}) / n \to \text{Leb}(U_i)\) for every \(i \leq k\). Then we may assume by Lemma 3.13 that, for all \(i \leq k\),
\[
\limsup_{n \to \infty} d_H \left( \frac{1}{n} C_{U_i^{(n)}}^{(n)}(2ny), \beta_U(y) \right) = 0 \quad \text{a.s.}
\]

Moreover, it is easy to see that the total mass of a \(\text{PCRP}^{(\alpha)}(0, \alpha)\) is a Markov chain described by \(\pi(0)\) in (16). By independence, the rescaled process
\[
R^{(n)}_k(y) := \frac{1}{n} \left\| C^{(n)}(2ny) \right\| - \frac{1}{n} \sum_{i=1}^k \left\| C_{U_i^{(n)}}^{(n)}(2ny) \right\|, \quad y \geq 0,
\]
has the law of \(\pi^{(n)}_k(0)\) as in (21), where \(r^{(n)}_k := \left\| C^{(n)}(0) \right\| - \sum_{i=1}^k \text{Leb}(U_i^{(n)})\). By Lemma 3.8 and Skorokhod representation, we may also assume \(\sup_{y \geq 0} |R^{(n)}_k(y) - R_k(y)| \to 0\) a.s.
An easy estimate shows that
\[ d_H \left( \frac{1}{n} C^{(n)}(2ny), \beta(y) \right) \leq 2R_k^{(n)}(y) + 2R_k(y) + \sum_{i=1}^k d_H \left( \frac{1}{n} C^{(n)}_{U_i}, (2ny), \beta_{U_i}(y) \right). \]

As a result, combining (30) and (31), we have
\[
\limsup_{n \to \infty} \sup_{y \geq 0} d_H \left( \frac{1}{n} C^{(n)}(2ny), \beta(y) \right) \leq 4\varepsilon \quad \text{a.s.}
\]
By the arbitrariness of $\varepsilon$ we deduce the claim.  

3.5. The scaling limit of a PCRP$^{(\alpha)}(\theta_1, \alpha)$.

**Proposition 3.15** (Convergence of a PCRP$^{(\alpha)}(\theta_1, \alpha)$). Let $\theta_1 \geq 0$. For $n \in \mathbb{N}$, let $(C^{(n)}(y), y \geq 0)$ be a PCRP$^{(\alpha)}(\theta_1, \alpha)$ starting from $C^{(n)}(0) \in C$. Suppose that the interval partition $\frac{1}{n} C^{(n)}(0)$ converges in distribution to $\beta(0) \in \mathcal{I}_H$ as $n \to \infty$, under $d_H$. Then the process $(\frac{1}{n} C^{(n)}(2ny), y \geq 0)$ converges in distribution to an SSIP$^{(\alpha)}(\theta_1)$-evolution starting from $\beta(0)$, as $n \to \infty$, in $\mathcal{D}(\mathbb{R}_+, \mathcal{I}_H)$ under the Skorokhod topology.

**Proof.** We only need to prove the case when $\theta_1 > 0$ and $C^{(n)}(0) = \emptyset$ for every $n \in \mathbb{N}$; then combining this special case and Theorem 3.12 leads to the general result. The arguments are very similar to those in the proof of Lemma 3.13; we only sketch the strategy here and omit the details.

Fix $j \in \mathbb{N}$. Let $(N^{(n)}, \xi^{(n)}, \ell^{(n)})_{n \in \mathbb{N}}$ be the sequence given in Proposition 3.7. For each $n \in \mathbb{N}$, by using Theorem 3.6, we may write
\[ \beta^{(n)}(y) := \frac{1}{n} C^{(n)}(2ny) = \text{SKWER} \left( y, N^{(n)} \big|_{[0,T^{(n)}_j/\alpha]}, j + \xi^{(n)} \big|_{[0,T^{(n)}_j/\alpha]} \right), \quad y \in [0, j], \]
where $\xi^{(n)}_{\theta_1} := \xi^{(n)} + (1 - \alpha/\theta_1) \ell^{(n)}$ and $T^{(n)}_{j/\theta_1/\alpha} := T_{j/\theta_1/\alpha}(\xi^{(n)}) = T_{j}(\xi^{(n)}_{\theta_1})$. By Proposition 3.7 and Skorokhod representation, we may assume that $(N^{(n)}, \xi^{(n)}, \ell^{(n)}_{\theta_1})$ converges a.s. to $(N, X_\alpha, X_{\theta_1})$. Then it follows from Lemma 3.11 that $T^{(n)}_{j/\theta_1/\alpha} \to T_{j/\theta_1/\alpha}(X_\alpha) = T_{j}(X_{\theta_1})$, cf. (14). Next, in the same way as in the proof of Lemma 3.13, we consider for any $\rho > 0$ the interval partition evolution $\beta^{(n)}_{>\rho}$ associated with the spindles of $\beta^{(n)}$ with lifetime longer than $\rho$. By proving that for any $\rho > 0$, $(\beta^{(n)}_{>\rho}(y), y \in [0, j]) \to (\beta_{>\rho}(y), y \in [0, j])$ as $n \to \infty$, and that $\|\beta^{(n)} - \beta_{>\rho}\| \to 0$ as $\rho \downarrow 0$ uniformly for all $n \in \mathbb{N}$, we deduce the convergence of $(\beta^{(n)}(y), y \in [0, j])$. This leads to the desired statement.

4. Convergence of the three-parameter family. In this section we consider the general three-parameter family PCRP$^{(\alpha)}(\theta_1, \theta_2, \alpha)$ with $\theta_1, \theta_2 \geq 0$. In Section 4.1 we establish a related convergence result, Theorem 4.3, for the processes killed upon hitting $\emptyset$, with the limiting diffusion being an SSIP$^\dagger$-evolution introduced in [47]. Using Theorem 4.3, we obtain a pseudo-stationary distribution for an SSIP$^\dagger$-evolution in Proposition 4.4, which enables us to introduce an excursion measure and thereby construct an SSIP$^\dagger$-evolution from excursions, for suitable parameters, in Sections 4.4 and 4.5 respectively. In Section 4.6, we finally complete the proofs of Theorem 1.1 and the other results stated in the introduction.
4.1. Convergence when $\emptyset$ is absorbing. If we choose any table in a PCRP\textsuperscript{\textit{(a)}}($\theta_1, \theta_2$), then its size evolves as a $\pi(-\alpha)$-process until the first hitting time of zero; before the deletion of this table, the tables to its left form a PCRP\textsuperscript{\textit{(a)}}($\theta_1, \alpha$) and the tables to its right a PCRP\textsuperscript{\textit{(a)}}($\alpha, \theta_2$). This observation suggests us to make such decompositions and to use the convergence results obtained in the previous section. A similar idea has been used in [47] for the construction of an SSIP-evolution with absorption in $\emptyset$, abbreviated as SSIP\textsubscript{\textit{I}}-evolution, which we shall now recall. Specifically, define a function $\phi: I_H \to (I_H \times (0, \infty) \times I_H) \cup \{(0, 0, \emptyset)\}$ by setting $\phi(\emptyset) := (\emptyset, 0, \emptyset)$ and, for $\beta \neq \emptyset$,

\begin{equation}
\phi(\beta) := \begin{cases} (\beta \cap (0, \inf U), \text{Leb}(U), \beta \cap (\sup U, ||\beta||) - \sup U), \quad \text{where } U \text{ is the longest interval in } \beta; \end{cases}
\end{equation}

Moreover, we refer to the convergence results obtained in the previous section. A similar idea has been used in [47] for the construction of an SSIP-evolution with absorption in $\emptyset$, abbreviated as SSIP\textsubscript{\textit{I}}-evolution, which we shall now recall. Specifically, define a function $\phi: I_H \to (I_H \times (0, \infty) \times I_H) \cup \{(0, 0, \emptyset)\}$ by setting $\phi(\emptyset) := (\emptyset, 0, \emptyset)$ and, for $\beta \neq \emptyset$,

\begin{equation}
\phi(\beta) := \begin{cases} (\beta \cap (0, \inf U), \text{Leb}(U), \beta \cap (\sup U, ||\beta||) - \sup U), \quad \text{where } U \text{ is the longest interval in } \beta; \end{cases}
\end{equation}

We refer to $(T_k)_{k \geq 1}$ as the renaissance levels at $\emptyset$. Set $T_0 := 0$ and $\beta(0) := \emptyset$. For $k \geq 0$, suppose by induction that we have obtained $(\beta(t), t \leq T_k)$.

- If $\beta(T_k) = \emptyset$, then we stop and set $T_i := T_k$ for every $i \geq k$ and $\beta(t) := \emptyset$ for $t \geq T_k$.
- If $\beta(T_k) \neq \emptyset$, denote $(\beta_i^{(k)}, m_i^{(k)}):= \phi(\beta(T_k))$. Conditionally on the history, let $f^{(k)} \sim \text{BESQ}_{m_i^{(k)}}(-2\alpha)$ and $\gamma_i^{(k)} = (\gamma_i^{(k)}(s), s \geq 0)$ be independent. Set $T_{k+1} := T_k + \zeta(f^{(k)})$. We define $\beta(t) := \gamma_i^{(k)}(t-T_k) * \left\{ (0, f^{(k)}(t-T_k)) \right\} * \text{rev}(\gamma_i^{(k)}(t-T_k))$, $t \in (T_k, T_{k+1}]$.

We refer to $(T_k)_{k \geq 1}$ as the renaissance levels and $T_\infty := \sup_{k \geq 1} T_k \in [0, \infty]$ as the degeneration level. If $T_\infty < \infty$, then by convention we set $\beta(t) := \emptyset$ for every $t \geq T_\infty$. Then the process $\beta := (\beta(t), t \geq 0)$ is called an SSIP\textsubscript{\textit{I}}\textsuperscript{(a)}($\theta_1, \theta_2$)-evolution starting from $\emptyset$.

Note that $\emptyset$ is an absorbing state of an SSIP\textsubscript{\textit{I}}\textsuperscript{(a)}($\theta_1, \theta_2$)-evolution by construction. Let us summarise a few results obtained in [47, Theorem 1.4, Corollary 3.7].

**Theorem 4.2 ([47]).** For $\theta_1, \theta_2 \geq 0$, let $(\beta(t), t \geq 0)$ be an SSIP\textsubscript{\textit{I}}\textsuperscript{(a)}($\theta_1, \theta_2$)-evolution, with renaissance levels $(T_k, k \geq 0)$ and degeneration level $T_\infty$. Set $\theta = \theta_1 + \theta_2 - \alpha$.

(i) (Hunt property) $(\beta(t), t \geq 0)$ is a Hunt process with continuous paths in $(I_H, d_H)$.

(ii) (Total-mass) $||\beta(t)||, t \geq 0$ is a BESQ\textsubscript{\textit{I}}$(2\alpha)$ killed at its first hitting time of zero.

(iii) (Degeneration level) If $\theta > 1$ and $\beta(0) \neq \emptyset$, then a.s. $T_\infty = \infty$ and $\beta(t) \neq \emptyset$ for every $t \geq 0$; if $\theta < 1$, then a.s. $T_\infty < \infty$ and $\lim_{t \to T_\infty} d_H(\beta(t), \emptyset) = 0$.

(iv) (Self-similarity) For any $c > 0$, the process $(c\beta(c^{-1}t), t \geq 0)$ is an SSIP\textsubscript{\textit{I}}\textsuperscript{(a)}($\theta_1, \theta_2$)-evolution starting from $c\beta(0)$.

(v) When $\theta_2 = \alpha$, the SSIP\textsubscript{\textit{I}}\textsuperscript{(a)}($\theta_1, \alpha$)-evolution $(\beta(t), t \geq 0)$ is an SSIP\textsuperscript{(a)}($\theta_1$)-evolution killed at its first hitting time at $\emptyset$.

**Theorem 4.3.** Let $\theta_1, \theta_2 \geq 0$ and $\theta = \theta_1 + \theta_2 - \alpha$. For $n \in \mathbb{N}$, let $(C^{(n)}(t), t \geq 0)$ be a PCRP\textsuperscript{\textit{(a)}}($\theta_1, \theta_2$) starting from $C^{(n)}(0) = \gamma^{(n)}$ and killed at $\zeta^{(n)} = \inf \{t \geq 0: C^{(n)}(t) = \emptyset\}$.

Let $(\beta(t), t \geq 0)$ be an SSIP\textsubscript{\textit{I}}\textsuperscript{(a)}($\theta_1, \theta_2$)-evolution starting from $\gamma$ and $\zeta = \inf \{t \geq 0: \beta(t) = \emptyset\}$. Suppose that $\frac{1}{n} \gamma^{(n)}$ converges in distribution to $\gamma$ as $n \to \infty$, under $d_H$, then the following convergence holds in $\mathbb{D}([0, \infty], I_H)$:

$$
\frac{1}{n} C^{(n)}((2nt) \wedge \zeta^{(n)}), t \geq 0 \quad \text{a.s.}
$$

Moreover, $\frac{1}{2n} \zeta^{(n)}$ converges to $\zeta$ in distribution.


Proof of Theorem 4.3. We shall construct a sequence of \( PCRP^{(\alpha)}(\theta_1, \theta_2) \) on a sufficiently large probability space by using \( PCRP^{(\alpha)}(\theta_1, \alpha) \), \( PCRP^{(\alpha)}(\alpha, \theta_2) \) and up-down chains of law \( \pi_k(-\alpha) \) defined in (16); the idea is similar to Definition 4.1. Then the convergences obtained in Proposition 3.15 and Lemmas 3.8–3.9 permit us to conclude.

By assumption, \( \frac{1}{n}C^{(n)}(0) \) converges in distribution to \( \gamma \in \mathcal{I}_H \) under the metric \( d_H \). Except for some degenerate cases when \( \gamma = 0 \), we may use Skorokhod representation and Lemma 3.14 to find \( \left( C^{(n)}_1(0), m^{(n)}(0), C^{(n)}_2(0) \right) \) for all \( n \) sufficiently large, with \( m^{(n)}(0) \geq 1 \) and \( C^{(n)}_1(0), C^{(n)}_2(0) \in \mathcal{C} \), such that \( C^{(n)}_1(0) \ast \{(0, m^{(n)}(0))\} \ast C^{(n)}_2(0) = C^{(n)}(0) \), and that, as \( n \to \infty \),

\[
\left( \frac{1}{n}C^{(n)}_1(0), \frac{1}{n}m^{(n)}(0), \frac{1}{n}C^{(n)}_2(0) \right) \rightarrow (\gamma_1, m, \gamma_2) := \phi(\gamma), \quad \text{a.s.,}
\]

where \( \phi \) is the function defined by (32).

For every \( n \in \mathbb{N} \), let \( f^{(n,0)} \sim \pi_m^{(n,0)}(-\alpha) \) be as in (16), \( \gamma^{(n,0)}_1 \) a \( PCRP^{(\alpha)}(\theta_1, \alpha) \) starting from \( C^{(n)}_1(0) \) and \( \gamma^{(n,0)}_2 \) a \( PCRP^{(\alpha)}(\alpha, \theta_2) \) starting from \( C^{(n)}_2(0) \); the three processes \( \gamma^{(n,0)}_1, f^{(n,0)} \) and \( \gamma^{(n,0)}_2 \) are independent. By Proposition 3.15, Lemma 3.8 and Skorokhod representation, we may assume that a.s.

\[
\left( \frac{1}{n}\gamma^{(n,0)}_1(2n \cdot), \frac{1}{n}f^{(n,0)}(2n \cdot), \frac{1}{n}\gamma^{(n,0)}_2(2n \cdot) \right) \rightarrow \left( \gamma^{(0)}_1, f^{(0)}, \gamma^{(0)}_2 \right).
\]

The limiting triple process \( (\gamma^{(0)}_1, f^{(0)}, \gamma^{(0)}_2) \) starting from \( (\gamma_1, m, \gamma_2) \) can serve as that in the construction of \( \beta \) in Definition 4.1. Write \( T_1 = \zeta(f^{(0)}) \) and \( T_{n,1} = \zeta(f^{(n,0)}) \), then

\[
\frac{1}{n}\gamma^{(n,0)}_1(T_{n,1}) \ast \frac{1}{n}\gamma^{(n,0)}_2(T_{n,1}) \rightarrow \gamma^{(0)}_1(T_1) \ast \gamma^{(0)}_2(T_1) =: \beta(T_1), \quad \text{a.s.}
\]

With \( \phi \) the function defined in (32), set

\[
(C^{(n,1)}_1, m^{(n,1)}, C^{(n,1)}_2) := \phi \left( \gamma^{(0)}_1(T_{n,1}) \ast \gamma^{(0)}_2(T_{n,1}) \right).
\]

Since \( T_1 \) is independent of \( (\gamma^{(0)}_1, \gamma^{(0)}_2) \), \( \beta(T_1) \) a.s. has a unique largest block. By this observation and (34) we have \( \frac{1}{n}(C^{(n,1)}_1, m^{(n,1)}, C^{(n,1)}_2) \rightarrow \phi(\beta(T_1)) \), since \( \phi \) is continuous at any interval partition whose longest block is unique.

For each \( n \geq 1 \), if \( (C^{(n,1)}_1, m^{(n,1)}, C^{(n,1)}_2) = (0, 0, 0) \), then for every \( i \geq 1 \), we set \( T_{n,i} := T_{n,1} \) and \( \left( \gamma^{(n,i)}_1, f^{(n,i)}, \gamma^{(n,i)}_2 \right) := (0, 0, 0) \). If \( (C^{(n,1)}_1, m^{(n,1)}, C^{(n,1)}_2) \neq (0, 0, 0) \), then conditionally on the history, let \( f^{(n,i)} \sim \pi_m^{(n,i)}(-\alpha) \), and consider \( \gamma^{(n,1)}_1 \), a \( PCRP^{(\alpha)}(\theta_1, \alpha) \) starting from \( C^{(n,1)}_1 \) and \( \gamma^{(n,1)}_2 \), a \( PCRP^{(\alpha)}(\alpha, \theta_2) \) starting from \( C^{(n,1)}_2 \), independent of each other. Set \( T_{n,2} = T_{n,1} + \zeta(f^{(n,1)}) \). Again, by Proposition 3.15, Lemma 3.8 and Skorokhod representation, we may assume that a similar a.s. convergence as in (34) holds for \( (\gamma^{(n,1)}_1, f^{(n,1)}, \zeta(f^{(n,1)}), \gamma^{(n,1)}_2) \).

By iterating arguments above, we finally obtain for every \( n \geq 1 \) a sequence of processes \( (\gamma^{(n,i)}_1, f^{(n,i)}, \gamma^{(n,i)}_2) \geq 0 \) with renaissance levels \( (T_{n,i})_{i \geq 0} \), such that, inductively, for every \( k \geq 0 \), the following a.s. convergence holds:

\[
\left( \frac{1}{n}\gamma^{(n,k)}_1(2n \cdot), \frac{1}{n}f^{(n,k)}(2n \cdot), \frac{1}{n}\gamma^{(n,k)}_2(2n \cdot) \right) \rightarrow \left( \gamma^{(k)}_1, f^{(k)}, \zeta(f^{(k)}), \gamma^{(k)}_2 \right).
\]

Using the limiting processes \( (\gamma^{(k)}_1, f^{(k)}, \gamma^{(k)}_2) \) \( k \geq 0 \), we build according to Definition 4.1 an SSIP^{(\alpha)}(\theta_1, \theta_2)-evolution \( \beta = (\beta(t), t \geq 0) \), starting from \( \gamma \), with renaissance levels \( T_k = \sum_{i=0}^{k-1} \zeta(f^{(i)}) \) and \( T_\infty = \lim_{k \to \infty} T_k \).
Then for every $t \geq 0$ and $k \in \mathbb{N}$, on the event $\{T_k > t\}$ we have by (35) the a.s. convergence of the process $(\frac{1}{n}C(n)(2ns), s \leq t) \to (\beta(s), s \leq t)$ when $\theta \geq 1$, since the event $\{T_\infty = \infty\} = \bigcap_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} \{T_k > t\}$ has probability one by Theorem 4.2, the convergence in Theorem 4.3 holds a.s.

We now turn to the case $\theta < 1$, where we have by Theorem 4.2 that $T_\infty < \infty$ a.s. and that, for any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that

$$\mathbb{P}\left(\sup_{t \geq T_k} \|\beta(t)\| > \varepsilon\right) < \varepsilon \quad \text{and} \quad \mathbb{P}(T_\infty > T_K + \varepsilon) < \varepsilon. \tag{36}$$

For each $n \in \mathbb{N}$, consider the concatenation

$$C(n)(t) = \begin{cases} \gamma^{(n,i)}_1(t-T_{n,i}) \ast \{(0, t^{(n,i)}(t-T_{n,i}))\} \ast \gamma^{(n,i)}_2(t-T_{n,i}), & t \in [T_{n,i}, T_{n,i+1}), i \leq K-1, \\ C(n)(t-T_{n,K}), & t \geq T_{n,K}, \end{cases}$$

where $\tilde{C}(n)$ is a PCRP($\alpha$)($\theta_1, \theta_2$) starting from $C(n)(T_{n,K}^-)$ and killed at $\emptyset$, independent of the history. Then $C(n)$ is a PCRP($\alpha$)($\theta_1, \theta_2$) starting from $C(n)(0)$. We shall next prove that its rescaled process converges to $(\beta(t), t \geq 0)$ in probability, which completes the proof.

By Lemmas 3.8–3.9, under the locally uniform topology

$$\left(\frac{1}{n}\|\tilde{C}(n)(2n\cdot)\|, \frac{1}{2n} \zeta(\tilde{C}(n)) \right) \Rightarrow \left(\|\beta(\cdot + T_K)\|, \zeta(\beta(\cdot + T_K)) \right)$$

in distribution.

By the convergence (35), there exists $N \in \mathbb{N}$ such that for every $n > N$, we have

$$\mathbb{P}\left(\sup_{s \in [0,T_K]} d_H\left(\frac{1}{n}C(n)(2ns), \beta(s)\right) > \varepsilon\right) < \varepsilon \quad \text{and} \quad \mathbb{P}\left(\frac{1}{2n}(T_{n,K} - T_K) > \varepsilon\right) < \varepsilon. \tag{37}$$

Furthermore, by the convergence of $\frac{1}{n}\|\tilde{C}(n)\|$, there exists $\tilde{N} \in \mathbb{N}$ such that for every $n > \tilde{N},$

$$\mathbb{P}\left(\sup_{s \geq 0} \frac{1}{n}\|\tilde{C}(n)(s)\| > \varepsilon\right) < \varepsilon \quad \text{and} \quad \mathbb{P}\left(\frac{1}{2n}\zeta(\tilde{C}(n)) - \zeta(\beta(\cdot + T_K)) > \varepsilon\right) < \varepsilon. \tag{38}$$

Summarising (36) and (38), for every $n > \tilde{N}$, we have

$$\mathbb{P}\left(\sup_{s \in [0,\infty)} d_H\left(\frac{1}{n}\tilde{C}(n)(2ns), \beta(s + T_K)\right) > 3\varepsilon\right) \leq 3\varepsilon.$$

Together with (37), this leads to the desired convergence in probability. \hfill \Box

4.2. Pseudo-stationarity of SSIP$_1$-evolutions.

Proposition 4.4 (Pseudo-stationary distribution of an SSIP$_1^{(\alpha)}(\theta_1, \theta_2)$-evolution). For $\theta_1, \theta_2 \geq 0$ and $\theta := \theta_1 + \theta_2 - \alpha$, let $(Z(t), t \geq 0)$ be a BESQ($2\theta$) killed at zero with $Z(0) > 0$, independent of $\gamma \sim \text{PDP}^{(\alpha)}(\theta_1, \theta_2)$. Let $(\beta(t), t \geq 0)$ be an SSIP$_1^{(\alpha)}(\theta_1, \theta_2)$-evolution starting from $Z(0)\tau$. Fix any $t \geq 0$, then $\beta(t)$ has the same distribution as $Z(t)\tau$.

Analogous results for SSIP$(\alpha)(\theta_1)$-evolutions have been obtained in [14, 18], however, the strategy used in their proofs does not easily apply to our three-parameter model. We shall use a completely different method, which crucially relies on the discrete approximation by PCRP$(\alpha)(\theta_1, \theta_2)$ in Theorem 4.3. It is easy to see that the total mass of a PCRP$(\alpha)(\theta_1, \theta_2)$ evolves according to a Markov chain defined by $\pi(\theta)$ as in (16), with $\theta = \theta_1 + \theta_2 - \alpha$. Conversely, given any $C(0) \in C$ and $Z \sim \pi_{\|C(0)\|}(\theta)$, we can embed a
process \((C(t), t \geq 0) \sim \text{PCRP}^{(\alpha)}(\theta_1, \theta_2)\), starting from \(C(0)\), such that its total-mass evolution is \(Z\). More precisely, in the language of the Chinese restaurant process, at each jump time when \(Z\) increases by one, add a customer according to the seating rule in Definition 2.1; and whenever \(Z\) decreases by one, perform a down-step, i.e., one uniformly chosen customer leaves. It is easy to check that this process indeed satisfies the definition of \(\text{PCRP}^{(\alpha)}(\theta_1, \theta_2)\) in the introduction. Recall the probability law \(\text{oCRP}^{(\alpha)}(\theta_1, \theta_2)\) from Definition 2.1.

Lemma 4.5 (Marginal distribution of a \(\text{PCRP}^{(\alpha)}(\theta_1, \theta_2)\)). Consider a \(\text{PCRP}^{(\alpha)}(\theta_1, \theta_2)\) \((C(t), t \geq 0)\) starting from \(C(0) \sim \text{oCRP}^{(\alpha)}_m(\theta_1, \theta_2)\) with \(m \in \mathbb{N}_0\). Then, at any time \(t \geq 0\), \(C(t)\) has a mixture distribution \(\text{oCRP}^{(\alpha)}_\|C(t)\|(\theta_1, \theta_2)\), where the total number of customers has distribution \((\|C(t)\|, t \geq 0) \sim \pi_m(\theta)\) with \(\theta := \theta_1 + \theta_2 - \alpha\).

Proof. Let \(Z \sim \pi_{\|C(0)\|}(\theta)\) and we consider \((C(t), t \geq 0) \sim \text{PCRP}^{(\alpha)}(\theta_1, \theta_2)\), starting from \(C(0)\), as a process embedded in \(Z \sim \pi_{\|C(0)\|}(\theta)\), in the way we just explained as above. Before the first jump time \(J_1\) of \(Z\), \(C(t) = C(0) \sim \text{oCRP}^{(\alpha)}_m(\theta_1, \theta_2)\) by assumption. At the first jump time \(J_1\) of \(Z\), it follows from Proposition 2.5 that, given \(Z(J_1)\), \(C(J_1)\) has conditional distribution \(\text{oCRP}^{(\alpha)}_{\|Z(J_1)\|}(\theta_1, \theta_2)\). The proof is completed by induction.

Proof of Proposition 4.4. For \(n \in \mathbb{N}\), consider a process \(C^{(n)}(0) \sim \text{PCRP}^{(\alpha)}(\theta_1, \theta_2)\), starting from \(C^{(n)}(0) \sim \text{oCRP}^{(\alpha)}_{[n\pi(0)]}(\theta_1, \theta_2)\) and killed at 0. It follows from Lemma 4.5 that, for every \(t \geq 0\), \(C^{(n)}(t)\) has the mixture distribution \(\text{oCRP}^{(\alpha)}_{\|C(t)\|_{\pi(0)}}(\theta_1, \theta_2)\) with \((\|C(t)\|, t \geq 0) \sim \pi_{\|C(0)\|_{\pi(0)}}(\theta)\). By Lemma 2.7, \(\frac{1}{n} C^{(n)}(0)\) converges in distribution to \(Z(0)^{\gamma}\) under \(d_H\). For any fixed \(t \geq 0\), it follows from Theorem 4.3 that \(\frac{1}{n} C^{(n)}(2nt)\) converges in distribution to \(\beta(t)\). Using Lemmas 3.8–3.9 and 2.7 leads to the desired statement.

4.3. SSIP-evolutions. Let \(\alpha \in (0, 1)\) and \(\theta_1, \theta_2 \geq 0\). Recall that the state 0 has been defined to be a trap of an SSIP\(^{(\alpha)}_1(\theta_1, \theta_2)\)-evolution. In this section, we will show that, for certain cases, depending on the value of \(\theta := \theta_1 + \theta_2 - \alpha\), we can include 0 as an initial state such that it leaves 0 continuously. More precisely, consider independent \((Z(t), t \geq 0) \sim \text{BESQ}_0(2\theta)\) and \(\gamma \sim \text{PDIP}^{(\alpha)}(\theta_1, \theta_2)\). Define for every \(t \geq 0\) a probability kernel \(K_t\) on \(\mathcal{I}_H\) for \(\beta_0 \in \mathcal{I}_H\) and measurable \(A \subseteq \mathcal{I}_H\).

\[
K_t(\beta_0, A) = \mathbb{P}(\beta(t) \in A, t < \zeta(\beta)) + \int_0^t \mathbb{P}(Z(t-r)^\gamma \in A) \mathbb{P}(\zeta(\beta) \in dr),
\]

where \(\beta = (\beta(t), t \geq 0)\) is an SSIP\(^{(\alpha)}_1(\theta_1, \theta_2)\)-evolution starting from \(\beta_0\), and \(\zeta(\beta)\) is the first hitting time of 0 by \(\beta\). Note that [42, Corollary XI.(1.4)] yields for fixed \(s \geq 0\), that

\[
(Z(t), t \geq 0) \sim \text{BESQ}_0(2\theta), \quad \theta > 0, \quad \Rightarrow \quad Z(s) \sim \text{Gamma}(\theta, 1/2s).
\]

When \(\beta_0 = 0\), we have by convention \(\zeta(\beta) = 0\) and the first term in (39) vanishes.

Theorem 4.6. Let \(\theta_1, \theta_2 \geq 0\). The family \((K_t, t \geq 0)\) defined in (39) is the transition semigroup of a path-continuous Hunt process on the Polish space \(\mathcal{I}_H\).

Definition 4.7 (SSIP\(^{(\alpha)}_1(\theta_1, \theta_2)\)-evolutions). For \(\theta_1, \theta_2 \geq 0\), a path-continuous Markov process with transition semigroup \((K_t, t \geq 0)\) is called an SSIP\(^{(\alpha)}_1(\theta_1, \theta_2)\)-evolution.

Proposition 4.8. For \(\theta_1, \theta_2 \geq 0\), let \((\beta(t), t \geq 0)\) be a Markov process with transition semigroup \((K_t, t \geq 0)\). Then the total mass \((\|\beta(t)\|, t \geq 0)\) is a BESQ\((2\theta)\) with \(\theta = \theta_1 + \theta_2 - \alpha\).
PROOF. We know from Theorem 4.2 that the total mass of an SSIP$_i^{(\alpha)}(\theta_1, \theta_2)$-evolution evolves according to a BESQ(2\theta) killed at zero. Therefore, the description in (39) implies that \((\|\beta(t)\|, t \geq 0)\) has the semigroup of BESQ(2\theta).

The proof of Theorem 4.6 is postponed to Section 4.5. We distinguish three phases:

• \(\theta \in [-\alpha, 0)\): by convention, \(Z \sim \text{BESQ}_0(2\theta)\) is the constant zero process and thus the second term in (39) vanishes; then \((K_t, t \geq 0)\) is just the semigroup of an SSIP$_i^{(\alpha)}(\theta_1, \theta_2)$-evolution. In this case Theorem 4.6 is encompassed by Theorem 4.2.

• \(\theta \in (0, 1)\): by Theorem 4.2 (ii) and [23, Equation (13)] we deduce that \(\zeta(\beta)\) is a.s. finite in (39), with \(\zeta(\beta) \overset{d}{=} \|\beta(0)\|/2G\), where \(G \sim \text{Gamma}(1-\theta, 1)\). In this case, we will construct an SSIP$_i^{(\alpha)}(\theta_1, \theta_2)$-evolution as a recurrent extension of SSIP$_i^{(\alpha)}(\theta_1, \theta_2)$-evolutions, by using an excursion measure that will be introduced in Section 4.4.

• \(\theta \geq 1\): since \(\zeta(\beta) = \infty\) a.s., the second term in (39) vanishes unless \(\beta(0) = 0\) and \(0\) is an entrance boundary with an entrance law \(K_\theta(0, \cdot) = \mathbb{P}(Z(t) \bar{\gamma} \in \cdot)\), by Proposition 4.4. See also [47, Proposition 5.11], where this was shown using a different construction and a different formulation of the entrance law, which is seen to be equivalent to (39) by writing \(\bar{\gamma} = B'(V'\bar{\gamma}_1 \ast \{(0, 1-V')\}) \ast (1-B') \beta \sim \text{PDIP}^{(\alpha)}(\theta_1, \theta_2)\) as in Corollary 2.8.

4.4. The excursion measure of an SSIP-evolution when \(\theta \in (-\alpha, 1)\). In this section, we fix \(\theta_1, \theta_2 \geq 0\) and suppose that \(-\alpha/2 = \theta_1 + \theta_2 - \alpha < 1\). We shall construct an SSIP$_i^{(\alpha)}(\theta_1, \theta_2)$ excursion measure \(\Theta := \Theta^{(\alpha)}(\theta_1, \theta_2)\), which is a \(\sigma\)-finite measure on the space \(\mathbb{C}([0, \infty), \mathcal{I}_H)\) of continuous functions in \((\mathcal{I}_H, \mathcal{d}_H)\), endowed with the uniform metric and the Borel \(\sigma\)-algebra. Our construction is in line with Pitman and Yor [40, (3.2)], by the following steps.

• For each \(t > 0\), define a measure \(N_t\) on \(\mathcal{I}_H\) by

\[
N_t(A) := \mathbb{E}[(Z(t))^{\theta-1}1_A(Z(t)\bar{\gamma})], \quad \text{measurable } A \subseteq \mathcal{I}_H \setminus \{\emptyset\},
\]

\[
N_t(\emptyset) := \infty,
\]

where \(Z = (Z(t), t \geq 0) \sim \text{BESQ}_0(4-2\theta)\) and \(\bar{\gamma} \sim \text{PDIP}^{(\alpha)}(\theta_1, \theta_2)\) are independent. As \(4-2\theta > 2\), the process \(Z\) never hits zero. We have \(N_t(\mathcal{I}_H \setminus \{\emptyset\}) = t^{\theta-1}/2^{1-\theta}(2-\theta)\).

Then \((N_t, t \geq 0)\) is an entrance law for an SSIP$_i^{(\alpha)}(\theta_1, \theta_2)$-evolution \((\beta(t), t \geq 0)\). Indeed, with notation above, we have by Proposition 4.4 that, for every \(s, t \geq 0\) and \(f\) non-negative measurable,

\[
\int \mathbb{E}[f(\beta(s)) | \beta(0) = \gamma] N_t(d\gamma) = \mathbb{E}[(Z(t))^{\theta-1}\mathbb{E}_{Z(t)\bar{\gamma}}[f(\beta(s))]]
\]

\[
= \mathbb{E}[(Z'(0))^{\theta-1}\mathbb{E}_{Z'(0)}[f(Z'(s)\bar{\gamma})]],
\]

where \((Z'(s), s \geq 0)\) is a BESQ(2\theta) killed at zero with \(Z'(0) = Z(t)\). Since we know from the duality property of BESQ(2\theta), see e.g. [40, (3.b) and (3.5)], that

\[
(Z'(0))^{\theta-1}\mathbb{E}_{Z'(0)}[g(Z'(s))] = \mathbb{E}_{Z'(0)}[g(\bar{Z}(s))(\bar{Z}(s))^{\theta-1}], \quad \forall s > 0,
\]

where \(\bar{Z} \sim \text{BESQ}(4-2\theta)\) starting from \(Z'(0)\), it follows from the Markov property that

\[
\mathbb{E}[(Z'(0))^{\theta-1}\mathbb{E}_{Z'(0)}[f(Z'(s)\bar{\gamma})]] = \mathbb{E}[[\bar{Z}(s)]^{\theta-1}f(\bar{Z}(s)\bar{\gamma})]
\]

\[
= \mathbb{E}[(Z(t+s))^{\theta-1}f(Z(t+s)\bar{\gamma})] = \int f(\gamma)N_{t+s}(d\gamma).
\]
We conclude that
\[ \int \mathbb{E}[f(\beta(s)) \mid \beta(0) = \gamma] N_t(d\gamma) = \int f(\gamma) N_{t+s}(d\gamma), \quad \forall s, t \geq 0. \]

- As a consequence, there exists a unique \( \sigma \)-finite measure \( \Theta \) on \( \mathbb{C}([0, \infty), \mathcal{I}_H) \) such that for all \( t > 0 \) and \( F \) bounded measurable functional, we have the identity
\[ (42) \quad \Theta[F \circ L_t] = \int \mathbb{E}[F(\beta(s), s \geq 0) \mid \beta(0) = \gamma] N_t(d\gamma), \]
where \( (\beta(s), s \geq 0) \) is an SSIP\((\alpha)(\theta_1, \theta_2)\)-evolution and \( L_t \) stands for the shift operator. See [45, VL48] for details. In particular, for each \( t > 0 \) and measurable \( A \subseteq \mathcal{I}_H \setminus \{0\} \), we have the identity \( \Theta\{(\beta(s), s > 0) \in \mathbb{C}((0, \infty), \mathcal{I}_H) : \beta(t) \in A\} = N_t(A) \). In particular,
\[ (43) \quad \Theta(\zeta > t) = \Theta\{(\beta(s), s > 0) \in \mathbb{C}((0, \infty), \mathcal{I}_H) : \beta(t) \neq \emptyset\} = \theta^{t-1/2}1-\theta(t - \theta). \]
- The image of \( \Theta \) by the mapping \( (\beta(t), t > 0) \mapsto (\|\beta(t)\|, t > 0) \) is equal to the pushforward of \( \Lambda_{\text{BESQ}}^{(2\theta)} \) from \( \mathbb{C}((0, \infty), \mathcal{I}_H) \) to \( \mathbb{C}((0, \infty), \mathcal{I}_H) \) under the restriction map, where \( \Lambda_{\text{BESQ}}^{(2\theta)} \) is the excursion measure of BESQ\((2\theta)\) as in (10). In particular, we have for \( \Theta \)-almost every \( (\beta(t), t > 0) \in \mathbb{C}((0, \infty), \mathcal{I}_H) \)
\[ (44) \quad \limsup_{t \downarrow 0} \parallel \beta(t) \parallel = 0 \quad \implies \quad \lim_{t \downarrow 0} d_H(\beta(t), \emptyset) = 0. \]
Therefore, we can “extend” \( \Theta \) to \( \mathbb{C}([0, \infty), \mathcal{I}_H) \), by defining \( \Theta(0) = \emptyset \) for \( \Theta \)-almost every \( (\beta(t), t > 0) \in \mathbb{C}((0, \infty), \mathcal{I}_H) \), and we also set
\[ (45) \quad \Theta\{\beta \in \mathbb{C}([0, \infty), \mathcal{I}_H) : \beta \equiv \emptyset\} = 0. \]

Summarising, we have the following statement.

**Proposition 4.9.** Let \( \theta_1, \theta_2 \geq 0 \) and suppose that \(-\alpha < \theta = \theta_1 + \theta_2 - \alpha < 1\). Then there is a unique \( \sigma \)-finite measure \( \Theta = \Theta^{(\alpha)}(\theta_1, \theta_2) \) on \( \mathbb{C}([0, \infty), \mathcal{I}_H) \) that satisfies (42) and (45). Moreover, the image of \( \Theta \) by the mapping \( (\beta(t), t \geq 0) \mapsto (\|\beta(t)\|, t \geq 0) \) is \( \Lambda_{\text{BESQ}}^{(2\theta)} \), the excursion measure of BESQ\((2\theta)\). 

For the case \( \theta_1 = \theta_2 = 0 \), the law PDIP\((\alpha)\) coincides with the Dirac mass \( \delta_{(0,1)} \). As a consequence, the SSIP\((\alpha)\) \((0, 0)\) excursion measure is just the pushforward of \( \Lambda_{\text{BESQ}}^{(2\theta)} \) by the map \( x \mapsto \{(0, x)\} \) from \([0, \infty)\) to \( \mathcal{I}_H \). When \( \theta_1 = 0 \) and \( \theta_2 = \alpha \), it is easy to check using [18, Proposition 2.12(i), Lemma 3.5(ii), Corollary 3.9] that \( 2\alpha \Theta^{(\alpha)}(0, \alpha) \) is the push-forward via the mapping \( \text{SKEWER} \) in Definition 3.1 of the measure \( \nu_{\text{lcl}}^{(\alpha)} \) studied in [18, Section 2.3].

### 4.5. Recurrent extension when \( \theta \in (0, 1) \)

Consider the SSIP\((\alpha)\) \((\theta_1, \theta_2)\) excursion measure \( \Theta := \Theta^{(\alpha)}(\theta_1, \theta_2) \) and suppose that \( \theta = \theta_1 + \theta_2 - \alpha \in (0, 1) \). It is well-known [46] in the theory of Markov processes that excursion measures such as \( \Theta \) can be used to construct a recurrent extension of a Markov process. To this end, let \( G \sim \text{PRM}(\text{Leb} \otimes b_\theta \Theta) \), where \( b_\theta = 2^{1-\theta} \Gamma(1-\theta)/\Gamma(\theta) \).

For every \( s \geq 0 \), set \( \sigma_s = \int_{[0,s] \times \mathcal{I}_H} \zeta(\gamma) G(dr, d\gamma) \). As the total mass process under \( \Theta \) is the BESQ\((2\theta)\) excursion measure with \( \theta \in (0, 1) \), the process \( (\sigma_s, s \geq 0) \) coincides with the inverse local time of a BESQ\((2\theta)\), which is well-known to be a subordinator. We define
\[ (46) \quad \beta(t) = \bigstar_{\text{points } (s, \sigma_s) \text{ of } G, \sigma_s < t \leq \sigma_s} \gamma_s(t - \sigma_s), \quad t \geq 0. \]
This “concatentation” consists of at most one interval partition since \( (\sigma_s, s \geq 0) \) is increasing.
Proposition 4.10. The process \((\beta(t), t \geq 0)\) of (46) is a path-continuous Hunt process with transition semigroup \((K_t, t \geq 0)\). Its total mass process \(||\beta(t)||, t \geq 0\) is a BESQ(2\(\theta\)).

Proof. We can use [46, Theorem 4.1], since we have the following properties:

- an SSIP\(_+\)\((\theta_1, \theta_2)\)-evolution is a Hunt process;
- \(\Theta\) is concentrated on \(\{\gamma \in \mathbb{C}([0, \infty), \mathcal{H}): 0 < \zeta(\gamma) < \infty, \gamma(t) = \emptyset \text{ for all } t \geq \zeta(\gamma)\}\);
- for any \(\alpha > 0\), we have \(\Theta\{\gamma \in \mathbb{C}([0, \infty), \mathcal{H}): \sup_{t \geq 0} ||\gamma(t)|| \geq \alpha\} < \infty\);
- \(f(1 - e^{-\zeta(\gamma)})b_0(\theta(d\gamma)) = 1\);
- (42) holds;
- \(\Theta\) is infinite and \(\Theta\{\gamma \in \mathbb{C}([0, \infty), \mathcal{H}): \gamma(0) \neq \emptyset\} = 0\).

It follows that \((\beta(t), t \geq 0)\) is a Borel right Markov process with transition semigroup \((K_t, t \geq 0)\). Moreover, the total mass process \(||\beta(t)||, t \geq 0\) evolves according to a BESQ(2\(\theta\)) by Proposition 4.9.

In fact, \((\beta(t), t \geq 0)\) a.s. has continuous paths. Fix any path on the almost sure event that the total mass process \(||\beta(t)||, t \geq 0\) and all excursions \(\gamma_s\) are continuous. For any \(t \geq 0\), if \(\sigma_+ < t < \sigma_\alpha\) for some \(\alpha \geq 0\), then the continuity at \(t\) follows from that of \(\gamma_s\). For any other \(t\), we have \(\beta(t) = \emptyset\) and the continuity at \(t\) follows from the continuity of the BESQ(2\(\theta\)) total mass process. This completes the proof.

We are now ready to give the proof of Theorem 4.6, which claims that \((K_t, t \geq 0)\) defined in (39) is the transition semigroup of a path-continuous Hunt process.

Proof of Theorem 4.6. When \(\theta \in (0, 1)\), this is proved by Proposition 4.10. When \(\theta \leq 0\), the state \(\emptyset\) is absorbing, and an SSIP\(_+\)\((\theta_1, \theta_2)\)-evolution coincides with an SSIP\(_+\)\((\theta_1, \theta_2)\) \_evolution. For \(\theta \geq 1\), the state \(\emptyset\) is inaccessible, but an entrance boundary of the SSIP\(_+\)\((\theta_1, \theta_2)\)-evolution, see also [47, Proposition 5.11]. For these cases, the proof is completed by Theorem 4.2, the only modification is when starting from \(\emptyset\). Specifically, the modified semigroup is still measurable. Right-continuity starting from \(\emptyset\) follows from the continuity of the total mass process, and this entails the strong Markov property by the usual approximation argument.

4.6. Proofs of results in the introduction. We will first prove Theorem 1.1 and identify the limiting diffusion in Theorem 1.1 with an SSIP-evolution as defined in Definition 4.7. Then we complete proofs of the other results in the introduction.

Lemma 4.11. Let \(\alpha \in (0, 1)\), \(\theta_1, \theta_2 \geq 0\) and \(\theta = \theta_1 + \theta_2 - \alpha\). For \(n \in \mathbb{N}\), let \((\PDIP^{(\alpha)}(\theta_1, \theta_2), t \geq 0)\) be a PCRP\(_{+}\)\((\theta_1, \theta_2)\) starting from \(C^{(\alpha)}(0) = \gamma^{(\alpha)}\). If \(\frac{1}{n}\gamma^{(\alpha)}\) converges to \(\emptyset\) under \(d_H\), then for any \(t \geq 0\),

\[
\frac{1}{n}C^{(\alpha)}(2nt) \to Z(t)\tilde{\gamma} \quad \text{in distribution,}
\]

where \((Z(t), t \geq 0) \sim \PDIP_0(2\theta)\) and \(\tilde{\gamma} \sim \PDIP^{(\alpha)}(\theta_1, \theta_2)\) are independent. In particular, this limit is constant \(\emptyset\) when \(\theta \leq 0\).

Proof. We start with the case when \(\theta < 1\). Let \(\zeta^{(\alpha)}\) be the hitting time of \(\emptyset\) for \(C^{(\alpha)}\). For any \(\varepsilon > 0\), for all \(n\) large enough, \(\zeta^{(\alpha)}\) is stochastically dominated by the hitting time of zero of an up-down chain \(Z^{(\alpha)} \sim \pi(2\theta)\) starting from \(\lfloor ne\rfloor\), which by Lemmas 3.8 and 3.9 converges in distribution to \(\varepsilon/2G\) with \(G\) a Gamma variable. Letting \(\varepsilon \to 0\), we conclude that \(\zeta^{(\alpha)}/2n \to 0\) in probability as \(n \to \infty\).
For any \( t > 0 \) and any bounded continuous function \( f \) on \( \mathcal{I}_H \), we have
\[
(47) \quad \mathbb{E} \left[ f \left( \frac{1}{n} C^{(n)}(2nt) \right) \right] = \mathbb{E} \left[ \mathbf{1}\{\zeta^{(n)} \leq 2nt\} f \left( \frac{1}{n} C^{(n)}(2nt - \zeta^{(n)}) \right) \right]
+ \mathbb{E} \left[ \mathbf{1}\{\zeta^{(n)} > 2nt\} f \left( \frac{1}{n} C^{(n)}(2nt) \right) \right],
\]
where \( C^{(n)}(s) = C(n)(s + \zeta^{(n)}) \), \( s \geq 0 \). As \( n \to \infty \), since \( \zeta^{(n)}/2n \to 0 \) in probability, the second term tends to zero. By the strong Markov property and Lemma 4.5, \( C^{(n)}(s) \) has the mixture distribution \( \text{oCRP}_{\|C^{(n)}(s)\|}(\theta_1, \theta_2) \). Since \( \|C^{(n)}(2nt)\|/n \to Z(t) \) in distribution by Lemma 3.8, we deduce by Lemma 2.7 that the first term tends to \( \mathbb{E} [ f(Z(t)\bar{\gamma})] \), as desired.

For \( \theta \geq 1 \), at least one of \( \theta_1 \geq \alpha \) or \( \theta_2 \geq \alpha \). Say, \( \theta \geq \alpha \). We may assume that \( C^{(n)}(t) = C_1^{(n)}(t) \ast C_0^{(n)}(t) \ast C_2^{(n)}(t) \) for independent \( (C_1^{(n)}(t), t \geq 0) \sim \text{PCRP}(\alpha, 0) \) starting from \( \emptyset \), \( (C_0^{(n)}(t), t \geq 0) \sim \text{PCRP}(\alpha, 0) \) starting from \( C^{(n)}(0) \), and \( (C_2^{(n)}(t), t \geq 0) \sim \text{PCRP}(\alpha, 0) \) starting from \( \emptyset \). For the middle term \( C_0^{(n)}(t) \), the \( \theta \leq 0 \) case yields that \( \frac{1}{n} C_0^{(n)}(2nt) \to \emptyset \) in distribution. For the other two, applying (40) and Lemmas 4.5, 3.8 and 2.7 yields \( \frac{1}{n} C_1^{(n)}(2nt) \to Z_1(t)\bar{\gamma}_1 \) in distribution, with \( Z_1(t) \sim \text{Gamma}(\theta_1 - \alpha, 1/2t) \) and \( \bar{\gamma}_1 \sim \text{PDIP}(\alpha, \theta_1, 0) \), and \( \frac{1}{n} C_2^{(n)}(2nt) \to Z_2(t)\bar{\gamma}_2 \) in distribution, with \( Z_2(t) \sim \text{Gamma}(\theta_2, 1/2t) \) and \( \bar{\gamma}_2 \sim \text{PDIP}(\alpha, \theta_2) \). We complete the proof by applying the decomposition (7).

\[\square\]

\textbf{PROOF OF THEOREM 1.1.} When \( \theta \leq 0 \), the state \( \emptyset \) is absorbing, and an SSIP\(^{(\alpha)}\)(\(\theta_1, \theta_2\))-evolution coincides with an SSIP\(_{+}^{(\alpha)}\)(\(\theta_1, \theta_2\))-evolution. For this case the proof is completed by Theorem 4.3.

So we shall only consider \( \theta > 0 \) and prove that the limiting diffusion is given by an SSIP\(^{(\alpha)}\)(\(\theta_1, \theta_2\))-evolution \( \beta = (\beta(t), t \geq 0) \) with \( \zeta(\beta) = \inf\{t \geq 0 : \beta(t) = \emptyset\} \) as defined in Definition 4.7. It suffices to prove the convergence in \( \mathbb{D}([0, T], \mathcal{I}_H) \) for a fixed \( T > 0 \). The convergence of finite-dimensional distributions follows readily from Theorem 4.3, Lemma 4.11 and the description in (39). Specifically, for \( \theta \in (0, 1) \), we proceed as in the proof of Lemma 4.11 and see the first term in (47) converge to \( \mathbb{E}[\mathbf{1}\{\zeta(\beta) \leq t\} f(Z_0(t - \zeta(\beta))\bar{\gamma})] \) where \( Z_0 \sim \text{BESQ}(2\theta) \) and \( \bar{\gamma} \sim \text{PDIP}(\alpha, \theta_1, 0) \) are independent and jointly independent of \( \beta \), while the second term converges to \( \mathbb{E}[\mathbf{1}\{\zeta(\beta) > t\} f(\beta(t))] \). For \( \theta \geq 1 \) and \( (\beta(t) = \emptyset) \), convergence of marginals holds by Lemma 4.11 and (39). Theorem 4.3 then establishes finite-dimensional convergence, also when \( \beta(0) \neq \emptyset \).

It remains to check tightness. Let \( \beta^{(n)}(t) = (\beta^{(n)}(t), t \geq 0) := \frac{1}{n} C^{(n)}(2nt) \). Since we already know from Lemma 3.8 that the sequence of total mass processes \( \|\beta^{(n)}\|, n \geq 1 \), converges in distribution, it is tight. For \( h > 0 \), define the modulus of continuity by
\[
\omega\left(\|\beta^{(n)}\|, h\right) = \sup \left\{ \left|\|\beta^{(n)}(s)\| - \|\beta^{(n)}(t)\|\right| : |s - t| \leq h \right\}.
\]
For any \( \varepsilon > 0 \), the tightness implies that there exists \( \Delta' \) such that for any \( h \leq 2\Delta' \),
\[
\lim_{n \to \infty} \mathbb{E}\left[ \omega\left(\|\beta^{(n)}\|, h\right) \wedge 1 \right] < \varepsilon;
\]
this is an elementary consequence of [25, Proposition VI.1.26]. See also [27, Theorem 16.5].

For \( 1 \leq i \leq [T/\Delta'] \), set \( t_i = i\Delta' \) and let \( \beta^{(n)}(t_i) \) be the process obtained by shifting \( \beta^{(n)}(t) \) to start from \( t_i \), killed at \( \emptyset \). The convergence of the finite-dimensional distributions yields that each \( \beta^{(n)}(t_i) \) converges weakly to \( \beta(t_i) \). Since \( \beta(t_i) \neq \emptyset \) a.s., by Theorem 4.3 each sequence
$\beta_i^{(n)}$ converges in distribution as $n \to \infty$. So the sequence $(\beta_i^{(n)}, n \in \mathbb{N})$ is tight, as the space $(\mathcal{I}_H, d_H)$ is Polish. By tightness there exists $\Delta_i$ such that for any $h < \Delta_i$,

$$\limsup_{n \to \infty} \mathbb{E} \left[ \omega \left( \beta_i^{(n)}, h \right) \wedge 1 \right] < 2^{-i} \varepsilon.$$ 

Now let $\Delta = \min(\Delta', \Delta_0, \Delta_1, \ldots, \Delta_{\lfloor T/\Delta' \rfloor})$. For any $s \leq t \leq T$ with $t - s \leq \Delta$, consider $i$ such that $t_i \leq s < t_{i+1}$, then $t - t_i < \Delta + t - s \leq 2\Delta'$. If $\zeta(\beta_i^{(n)}) \leq t - t_i$, then $\beta_i^{(n)}$ touches $\emptyset$ during the time interval $[t_i, t]$ and thus $\max(\|\beta_i^{(n)}(s)\|, \|\beta_i^{(n)}(t)\|) \leq \omega \left( \|\beta_i^{(n)}\|, 2\Delta' \right)$.

Therefore, we have

$$d_H \left( \beta_i^{(n)}(s), \beta_i^{(n)}(t) \right) \leq d_H \left( \beta_i^{(n)}(s), \beta_i^{(n)}(t) \right) + 2 \omega \left( \|\beta_i^{(n)}\|, 2\Delta' \right).$$

It follows that for $h < \Delta$,

$$\mathbb{E} \left[ \omega \left( \beta_i^{(n)}, h \right) \wedge 1 \right] \leq 2 \mathbb{E} \left[ \omega \left( \|\beta_i^{(n)}\|, 2\Delta' \right) \wedge 1 \right] + \sum_{i=0}^{\lfloor T/\Delta' \rfloor} \mathbb{E} \left[ \omega \left( \beta_i^{(n)}, \Delta_i \right) \wedge 1 \right].$$

So we have $\limsup_{n \to \infty} \mathbb{E} \left[ \omega \left( \beta_i^{(n)}, h \right) \wedge 1 \right] \leq 4\varepsilon$. This leads to the tightness, e.g. via [27, Theorem 16.10].

**Proof of Proposition 1.2.** This follows from Proposition 4.4 and the semigroup description in (39).

**Theorem 4.12.** Let $\alpha \in (0, 1)$, $\theta_1, \theta_2 \geq 0$ and $\gamma_n \in \mathcal{I}_H$ with $\gamma_n \to \gamma \in \mathcal{I}_H$. Let $\beta_n, n \geq 1$, and $\beta$ be SSIP($\alpha$)($\theta_1, \theta_2$)-evolutions starting from $\gamma_n, n \geq 1$, and $\gamma$, respectively. Then $\beta_n \to \beta$ in distribution in $\mathbb{C}(\mathbb{R}_+, \mathcal{I}_H)$ equipped with the locally uniform topology.

**Proof.** It follows easily from Lemma 3.14 that we may assume that $\gamma_n = \beta_{n,1}^{(0)} * \{(0, m_{n}^{(0)}(i)) \} * \beta_{n,2}^{(0)}$ with $m_{n}^{(0)} \to m^{(0)}$, $\beta_{n,1}^{(0)} \to \beta_{1}^{(0)}$, and $\beta_{n,2}^{(0)} \to \beta_{2}^{(0)}$. We will now couple the constructions in Definition 4.1 and use the notation from there.

Given $(\beta_{n,1}^{(k)}, m_{n}^{(k)}, \beta_{n,2}^{(k)}) \to (\beta_{1}^{(k)}, m^{(k)}, \beta_{2}^{(k)})$ a.s., for some $k \geq 0$, the Feller property of [18, Theorem 1.8] allows us to apply [27, Theorem 19.25] and, by Skorokhod representation, we obtain $\gamma_{n,1}^{(k)} \to \gamma_{1}^{(k)}$ a.s. in $\mathbb{C}(\mathbb{R}_+, \mathcal{I}_H)$, $i = 1, 2$, as $n \to \infty$. For $f^{(k)} \sim \text{BE}^{m_{n}^{(k)}}(-2\alpha)$ and $f_{n}^{(k)}(s) := (m_{n}^{(k)}/m^{(k)})f^{(k)}((m_{n}^{(k)}/m^{(k)})s), s \geq 0$, we find $f_{n}^{(k)} \sim \text{BE}^{m_{n}^{(k)}}(-2\alpha)$. As $n \to \infty$, $(f_{n}^{(k)}, \zeta(f_{n}^{(k)})) \to (f^{(k)}, \zeta(f^{(k)}))$ a.s. And as $\gamma_{1}^{(k)}(\zeta(f^{(k)})) * \gamma_{2}^{(k)}(\zeta(f^{(k)}))$ has a.s. a unique longest interval, $\phi(\gamma_{1}^{(k)}(\zeta(f^{(k)})), \gamma_{2}^{(k)}(\zeta(f^{(k)}))) \to \phi(\gamma_{1}^{(k)}(\zeta(f^{(k)})) * \gamma_{2}^{(k)}(\zeta(f^{(k)})))$, a.s.

Inductively, the convergences stated in this proof so far hold a.s. for all $k \geq 0$. When $\theta := \theta_1 + \theta_2 - \alpha \geq 1$, this gives rise to coupled $\beta_n$ and $\beta$. When $\theta < 1$, arguments as at the end of the proof of Theorem 4.3 allow us to prove the convergence until the first hitting time of $\emptyset$ jointly with the convergence of the hitting times. When $\theta \leq 0$, we extend the constructions by absorption in $\emptyset$. When $\theta \in (0, 1)$, we extend by the same SSIP($\alpha$)($\theta_1, \theta_2$) starting from $\emptyset$. In each case, we deduce that $\beta_n \to \beta$ a.s., locally uniformly.

**Proof of Theorem 1.4.** For an SSIP-evolution, we have established the pseudo-stationarity (Proposition 1.2), self-similarity, path-continuity, Hunt property (Theorem 1.1) and the continuity in the initial state (Theorem 4.12). With these properties in hand, we can easily prove this theorem by following the same arguments as in [15, proof of Theorem 1.6]. Details are left to the reader.
We now prove Theorem 1.5, showing that when \( \theta = \theta_1 + \theta_2 - \alpha \in (-\alpha, 1) \), the excursion measure \( \Theta := \Theta(\alpha)/(\theta_1, \theta_2) \) of Section 4.4 is the limit of rescaled PCRP excursion measures.

Recall that the total mass process of \( \text{PCRP}^{(\alpha)}(\theta_1, \theta_2) \) has distribution \( \pi_1(\theta) \). We have already obtained the convergence of the total mass process from Proposition 3.10.

**Proof of Theorem 1.5.** Recall that \( \zeta(\gamma) = \inf\{t > 0: \gamma(t) = 0\} \) denotes the lifetime of an excursion \( \gamma \in \mathcal{D}([0, \infty), \mathcal{I}_H) \). To prove vague convergence, we proceed as in the proof of Proposition 3.10. In the present setting, we work on the space of measures on \( \mathcal{D}([0, \infty), \mathcal{I}_H) \) that are bounded on \( \{\zeta > t\} \) for all \( t > 0 \). We denote by \( \mathbb{P}^{(n)} \) the distribution of \( C^{(n)} \), a killed \( \text{PCRP}^{(\alpha)}(\theta_1, \theta_2) \) starting from \( 1 \). It suffices to prove for fixed \( t > 0 \),

1. \( \Theta(\zeta = 0) = 0 \),
2. \( (\Gamma(1 + \theta)/(1 - \theta)) n^{1-\theta} \cdot \mathbb{P}^{(n)}(\zeta > t) \nrightarrow \Theta(\zeta > t) \),
3. \( \mathbb{P}^{(n)}(\cdot | \zeta > t) \nrightarrow \Theta(\cdot | \zeta > t) \) weakly.

1. This follows from (43).

2. Since the total-mass process \( \|C^{(n)}\| \) is an up-down chain of law \( \pi_1^{(n)}(\theta) \), Proposition 3.10 implies the following weak convergence of finite measures on \( (0, \infty) \):

\[
\frac{\Gamma(1 + \theta)}{1 - \theta} n^{1-\theta} \mathbb{P}^{(n)} \left( \|C^{(n)}(t)\| \in \cdot; \zeta(C^{(n)}) > t \right) \nrightarrow \Lambda^{(2\theta)} \cdot \{f \in \mathcal{C}([0, \infty), \mathcal{I}_H): \|f(t)\| > \cdot; \zeta(f) > t\} = N_\gamma \{\gamma \in \mathcal{I}_H: \|\gamma\| \in \cdot\},
\]

where \( N_\gamma \) is the entrance law of \( \Theta \) given in (41). This implies the desired convergence.

3. For any \( t > 0 \), given \( \{\|C^{(n)}(r)\|, r \leq 2nt\} \), we know from Lemma 4.5 that the conditional distribution of \( C^{(n)}(t) = \frac{1}{n} C^{(2nt)} \) is the law of \( \nu^{(n)}_C \), where \( C^{(n)} \) is \( \text{ocRP}^{(\alpha)}(\theta_1, \theta_2) \) with \( m = \|C^{(2nt)}\| \). By Lemma 2.7, we can strengthen (48) to the following weak convergence on \( \mathcal{I}_H \setminus \{\emptyset\} \):

\[
\mathbb{P} \left( C^{(n)}(t) \in \cdot; \zeta(C^{(n)}) > t \right) \nrightarrow N_t(\cdot | \mathcal{I}_H \setminus \{\emptyset\}).
\]

Next, by the Markov property of a PCRP and the convergence result Theorem 4.3, we deduce that, conditionally on \( \{\zeta(C^{(n)}) > t\} \), the process \( (C^{(n)}(t + s), s \geq 0) \) converges weakly to an SSIP\(^{(\alpha)}(\theta_1, \theta_2) \)-evolution \( \{\beta(s), s \geq 0\} \) starting from \( \beta(0) \sim N_t(\cdot | \mathcal{I}_H \setminus \{\emptyset\}) \). By the description of \( \Theta \) in (42), this implies the convergence of finite-dimensional distributions for times \( t \leq t_1 < \cdots < t_k \). For \( t > t_1 \), this holds under \( \mathbb{P}^{(n)}(\cdot | \zeta > t_1) \) and \( \Theta(\cdot | \zeta > t_1) \) and can be further conditioned on \( \{\zeta > t\} \), by 1. and 2.

It remains to prove tightness. For every \( n \geq 1 \), let \( \tau_n \) be a stopping time with respect to the natural filtration of \( C^{(n)} \) and \( h_n \) a positive constant. Suppose that the sequence \( \tau_n \) is bounded and \( h_n \to 0 \). By Aldous’s criterion [27, Theorem 16.11], it suffices to show that for any \( \delta > 0 \),

\[
\lim_{n \to \infty} \mathbb{P} \left( d_H \left( C^{(n)}(\tau_n + h_n), C^{(n)}(\tau_n) \right) > \delta \left| \zeta(C^{(n)}) > t \right. \right) = 0.
\]

By the total mass convergence in Proposition 3.10, for any \( \varepsilon > 0 \), there exists a constant \( s > 0 \), such that

\[
\limsup_{n \to \infty} \mathbb{P} \left( \sup_{r \leq 2s} \|C^{(n)}(r)\| > \delta/3 \left| \zeta(C^{(n)}) > t \right. \right) \leq \varepsilon.
\]

Moreover, since \( (C^{(n)}(s + z), z \geq 0) \) conditionally on \( \{\zeta(C^{(n)}) > s\} \) converges weakly to a continuous process, by [25, Proposition VI.3.26] we have for any \( u > s \),

\[
\lim_{n \to \infty} \mathbb{P} \left( \sup_{r \in [s, u]} d_H \left( C^{(n)}(r + h_n), C^{(n)}(r) \right) > \delta/3 \left| \zeta(C^{(n)}) > t \right. \right) = 0.
\]

Then (49) follows from (50) and (51). This completes the proof. \( \square \)
4.7. The case $\alpha = 0$. In a PCRP model with $\alpha = 0$, the size of each table evolves according to an up-down chain $\pi(0)$ as in (16), and new tables are only started to the left or to the right, but not between existing tables. We can hence build a PCRP$^{(0)}(\theta_1, \theta_2)$ starting from $(n_1, \ldots, n_k) \in C$ by a Poissonian construction. Specifically, consider independent $f_i \sim \pi_{n_i}(0)$, $i \in [k]$, as size evolutions of the initial tables, $F_1 \sim \text{PRM}(\theta_1 \text{Leb} \otimes \pi_1(0))$ whose atoms describe the birth times and size evolutions of new tables added to the left, and $F_2 \sim \text{PRM}(\theta_2 \text{Leb} \otimes \pi_1(0))$ for new tables added to the right. For $t \geq 0$, set $C_1(t) = \bigstar_{\text{atoms } (s,f) \text{ of } F_1, s \leq t} \{(0, f(t-s))\}$, where $\downarrow$ means that the concatenation is from larger $s$ to smaller, $C_0(t) = \bigstar_{i \in [k]} \{(0, f_i(t))\}$, and $C_2(t) = \bigstar_{\text{atoms } (s,f) \text{ of } F_2, s \leq t} \{(0, f(t-s))\}$. Then $(C(t) = C_1(t) \ast C_0(t) \ast C_2(t), t \geq 0)$ is a PCRP$^{(0)}(\theta_1, \theta_2)$ starting from $(n_1, \ldots, n_k)$.

**Proposition 4.13.** The statement of Theorem 1.1 still holds when $\alpha = 0$.

**Proof.** We only prove this for the case when the initial state is $C^{(n)}(0) = \{(0, b^{(n)})\}$ with $\lim_{n \to \infty} b^{(n)}/n = b > 0$. Then we can extend to a general initial state in the same way as we passed from Lemma 3.13 to Theorem 3.12.

For each $n \geq 1$, we may assume $C^{(n)}$ is associated with $F_1 \sim \text{PRM}(\theta_1 \text{Leb} \otimes \pi_1(0))$, $F_2 \sim \text{PRM}(\theta_2 \text{Leb} \otimes \pi_1(0))$, and $F_0^{(n)} \sim \pi_{\theta b^{(n)}}(0)$. Replacing each atom $\delta(s,f)$ of $F_1$ and $F_2$ by $\delta(s/2n, f/(2n^\alpha)/n)$, we obtain $F_1^{(n)} \sim \text{PRM}(2n\theta_1 \text{Leb} \otimes \pi^{(n)}_1(0))$ and $F_2^{(n)} \sim \text{PRM}(2n\theta_2 \text{Leb} \otimes \pi^{(n)}_1(0))$. Note that $(\frac{1}{n}C^{(n)}(2nt), t \geq 0)$ is associated with $(F_1^{(n)}, F_2^{(n)}, f^{(n)}(2n\cdot)/n)$.

Since Proposition 3.10 shows that $n\pi_1^{(n)}(0) \to \Lambda^{(0)}_{\text{BESQ}}$, by [28, Theorem 4.11], we deduce that $F_1^{(n)}$ and $F_2^{(n)}$ converge in distribution respectively to $F_1^{(\infty)} \sim \text{PRM}(2\theta_1 \text{Leb} \otimes \Lambda^{(0)}_{\text{BESQ}})$ and $F_2^{(\infty)} \sim \text{PRM}(2\theta_2 \text{Leb} \otimes \Lambda^{(0)}_{\text{BESQ}})$. By Lemma 3.8, $f^{(n)}(2n\cdot)/n \to f^{(\infty)} \sim \text{BESQ}_0(0)$ in distribution.

As a result, we can deduce that $(\frac{1}{n}C^{(n)}(2nt), t \geq 0)$ converges to an $I_H$-valued process $(\beta(t), t \geq 0)$ defined by

\begin{equation}
\beta(t) = \left(\bigstar_{\text{atoms } (s,f) \text{ of } F_1^{(\infty)}, s \leq t} \{(0, f(t-s))\}\right) \ast \left(\bigstar_{\text{atoms } (s,f) \text{ of } F_2^{(\infty)}, s \leq t} \{(0, f(t-s))\}\right).
\end{equation}

A rigorous argument can be made as in the proof of Lemma 3.13. □

The limiting process in (52) can be viewed as an SSIP$^{(0)}(\theta_1, \theta_2)$-evolution, which is closely related to the construction of measure-valued processes in [48]. See also [19, Section 7.1].

5. Applications.

5.1. Measure-valued processes. In [19], we introduced a two-parameter family of superprocesses taking values in the space $(M^\nu, d_M)$ of all purely atomic finite measures on a space of allelic types, say $[0,1]$. Here $d_M$ is the Prokhorov distance. Our construction is closely related to that of SSIP-evolutions, here extracting from scaffolding and spindles via the following superskewer mapping. See Figure 2 on page 13 for an illustration.
DEFINITION 5.1 (Superskewer). Let
\[ V = \sum_{i \in I} \delta(t_i, f_i, x_i) \]
be a point measure on \( \mathbb{R} \times \mathcal{E} \times [0, 1] \) and \( X \) a càdlàg process such that
\[ \sum_{\Delta X(t) > 0} \delta(t, \Delta X(t)) = \sum_{i \in I} \delta(t_i, \zeta(f_i)). \]
The superskewer of the pair \((V, X)\) at level \( y \) is the atomic measure
\[ \text{SSKWER}(y, V, X) := \sum_{i \in I : X(t_i) \leq y < X(t_{i+1})} f_i(y - X(t_i)) \delta(x_i). \]

For \( \alpha \in (0, 1) \), \( \theta \geq 0 \), recall the scaffolding-and-spindles construction of an SSIP\(^{(\alpha)}(\theta)\)-evolution starting from \( \gamma \in T_H \); in particular, for each \( U \in \gamma \), there is an initial spindle \( f_U \sim \text{BESQ}_{\text{Leb}(U)}(-2\alpha) \). For any collection \( x_U \in (0, 1) \), \( U \in \gamma \), we can construct a self-similar superprocess \( \text{SSSP}^{(\alpha)}(\theta) \) starting from \( \pi = \sum_{U \in \gamma} \text{Leb}(U) \delta(x_U) \) as follows. We mark each initial spindle \( f_U \) by the allelic type \( x_U \) and all other spindles in the construction by i.i.d. uniform random variables on \([0, 1]\). Then we obtain the desired superprocess by replacing the process \( \text{SSSP}^{(\alpha)}(\theta) \)-evolution in Definitions 3.2–3.3, with skewer replaced by superskewer, and concatenation replaced by addition. We refer to [19] for more details.

We often write \( \pi \in \mathcal{M}^a \) in canonical representation \( \pi = \sum_{i \geq 1} b_i \delta(x_i) \) with \( b_1 \geq b_2 \geq \cdots \) and \( x_i < x_{i+1} \) if \( b_i = b_{i+1} \). We write \( \| \pi \| := \pi([0, 1]) = \sum_{i \geq 1} b_i \) for the total mass of \( \pi \).

DEFINITION 5.2. Let \( \alpha \in (0, 1) \) and \( \theta \in [-\alpha, 0) \). We define a process \((\pi(t), t \geq 0)\) starting from \( \pi(0) \in \mathcal{M}^a \) by the following construction.

1. Set \( T_0 = 0 \). For \( \pi(0) = \sum_{i \geq 1} b_i \delta(x_i) \) in canonical representation, consider \( x^{(0)} := x_1 \) and independent \( \mathbb{f}^{(0)} \sim \text{BESQ}_{b_1}(-2\alpha) \) and \( \lambda^{(0)} \sim \text{SSSP}^{(\alpha)}(\theta + \alpha) \) starting from \( \sum_{i \geq 1} b_i \delta(x_i) \).
2. For \( k \geq 1 \), suppose by induction we have obtained \( (\lambda^{(i)}, \mathbb{f}^{(i)}, x^{(i)}, T_i)_{0 \leq i \leq k-1} \). Then we set \( T_k = T_{k-1} + \zeta(\mathbb{f}^{(k-1)}) \) and
\[ \pi(t) = \lambda^{(k-1)}(t-T_{k-1}) + \mathbb{f}^{(k-1)}(t-T_{k-1}) \delta(x^{(k-1)}), \quad t \in [T_{k-1}, T_k]. \]

Write \( \pi(T_k) = \sum_{i \geq 1} b_i^{(k)} \delta(x_i^{(k)}) \), with \( b_1^{(k)} \geq b_2^{(k)} \geq \cdots \), for its canonical representation.

Conditionally on the history, construct independent \( \lambda^{(k)} \sim \text{SSSP}^{(\alpha)}(\theta + \alpha) \) starting from \( \sum_{i \geq 2} b_i^{(k)} \delta(x_i^{(k)}) \) and \( \mathbb{f}^{(k)} \sim \text{BESQ}_{b_1^{(k)}}(-2\alpha) \). Let \( x^{(k)} = x_1^{(k)} \).

3. Let \( T_\infty = \lim_{k \to \infty} T_k \) and \( \pi(t) = 0 \) for \( t \geq T_\infty \).

The process \( \pi := (\pi(t), t \geq 0) \) is called an \((\alpha, \theta)\) self-similar superprocess, \( \text{SSSP}^{(\alpha)}(\theta) \).

For any \( \pi(0) = \sum_{i \geq 1} b_i \delta(x_i) \in \mathcal{M}^a \) consider \( \beta(0) = \{(s(i-1), s(i)), i \geq 1\} \in T_H \), where \( s(i) = b_1 + \cdots + b_i, i \geq 0 \). Consider an SSIP\(^{(\alpha)}(\theta + \alpha)\)-evolution starting from \( \beta(0) \), built in Definition 4.1 and use notation therein. As illustrated in Figure 2, we may assume that each interval partition evolution is obtained from the skewer of marked spindles. Therefore, we can couple each SSIP\(^{(\alpha)}(\theta + \alpha)\)-evolution \( \gamma_1^{(k)} \) with an \( \lambda_1^{(k)} \sim \text{SSSP}^{(\alpha)}(\theta + \alpha) \), such that the atom sizes of the latter correspond to the interval lengths of the former. Similarly, each SSIP\(^{(\alpha)}(0)\)-evolution \( \gamma_2^{(k)} \) corresponds to a \( \lambda_2^{(k)} \sim \text{SSSP}^{(\alpha)}(0) \). Then \( \lambda^{(k)} = \lambda_1^{(k)} + \lambda_2^{(k)} \) is an SSIP\(^{(\alpha)}(\theta + \alpha)\) by definition. Let \( \mathbb{f}^{(k)} \) be the middle (marked) spindle in Definition 4.1, which is a BESQ\((-2\alpha)\), and \( x^{(k)} \) be its type. In this way, we obtain a sequence \((\lambda^{(k)}, \mathbb{f}^{(k)}, x^{(k)})_{k \geq 0} \) and thus \( \pi = (\pi(t), t \geq 0) \sim \text{SSSP}^{(\alpha)}(\theta) \) as in Definition 5.2. It is coupled with \( \beta = (\beta(t), t \geq 0) \sim \text{SSIP}^{(\alpha)}(\theta + \alpha, 0) \) as in Definition 4.1, such that atom sizes and interval lengths are matched, and the renaissance level \((T_k)_{k \geq 0}\) are exactly the same.

The next theorem extends [19, Theorem 1.2] to \( \theta \in [-\alpha, 0) \).
Theorem 5.3. Let $\alpha \in (0,1)$, $\theta \in [-\alpha,0)$. An SSSP$^{(\alpha)}(\theta)$ is a Hunt process with BESQ$(2\theta)$ total mass, paths that are total-variation continuous, and its finite-dimensional marginals are continuous along sequences of initial states that converge in total variation.

Proof. For any $\pi(0) = \sum_{i \geq 1} b_i \delta(x_i) \in M^\alpha$ consider $(\pi(t), t \geq 0) \sim$ SSSP$^{(\alpha)}(\theta)$ and $(\beta(t), t \geq 0) \sim$ SSSP$^{(\alpha)}(\theta + \alpha, 0)$ coupled, as above. By [47, Theorem 1.4], their (identical) total mass processes are BESQ$(2\theta)$. Moreover, by this coupling and [47, Corollary 3.7],

$$T_\infty < \infty, \text{ and } \lim_{t \to T_\infty} \|\pi(t)\| = 0 \text{ a.s.,}$$

which implies the path-continuity at $T_\infty$. Since an SSSP$^{(\alpha)}(\theta + \alpha)$ has continuous paths [19, Theorem 1.2], we conclude the path-continuity of SSSP$^{(\alpha)}(\theta)$ by the construction in Definition 5.2, both in the Prokhorov sense and in the stronger total variation sense.

To prove the Hunt property, we adapt the proof of [47, Theorem 1.4] and apply Dynkin’s criterion to a richer Markov process that records more information from the construction. Specifically, in the setting of Definition 5.2, let

$$(\lambda(t), f(t), x(t)) := \left(\lambda^{(k-1)}(t-T_{k-1}), f^{(k-1)}(t-T_{k-1}), x^{(k-1)}\right), \quad t \in [T_{k-1}, T_k), k \geq 1.$$ 

and $(\lambda(t), f(t), x(t)) := (0,0,0)$ for $t \geq T_\infty$. We shall refer to this process as a triple-valued SSSP$^{(\alpha)}(\theta)$ with values in $\tilde{J} := (M^\alpha \times (0,\infty) \times [0,1]) \cup \{(0,0,0)\}$. This process induces the $M^\alpha$-valued SSSP$^{(\alpha)}(\theta)$ as $\pi(t) = \lambda(t) + f(t)\delta(x(t))$. Since each $(\lambda^{(k)}, f^{(k)}, x^{(k)})$ is Hunt and is built conditionally on the previous ones according to a probability kernel, then $(\lambda(t), f(t), x(t))$, $t \geq 0$, is a Borel right Markov process by [2, Théorème II 3.18].

To use Dynkin’s criterion to deduce that $(\pi(t), t \geq 0)$ is Borel right Markovian, and hence Hunt by path-continuity, we consider any $(\lambda_1(0), f_1(0), x_1(0)), (\lambda_2(0), f_2(0), x_2(0)) \in \tilde{J}$ with $\lambda_1(0) + f_1(0)\delta(x_1(0)) = \lambda_2(0) + f_2(0)\delta(x_2(0))$. It suffices to couple triple-valued SSSP$^{(\alpha)}(\theta)$ from these two initial states whose induced $M^\alpha$-valued SSSP$^{(\alpha)}(\theta)$ coincide.

First note that (unless they are equal) the initial states are such that for $t = 0$ and $i = 1,2$, 

$$\lambda_1(t) = \mu(t) + f_2(t)\delta(x_2(t)) \quad \text{and} \quad \lambda_2(t) = \mu(t) + f_1(t)\delta(x_1(t))$$

for some $\mu(t) \in M^\alpha$. We follow similar arguments as in the proof of [47, Lemma 3.3], via a quintuple-valued process $(\mu(t), f_1(t), x_1(t), f_2(t), x_2(t))$, $0 \leq t < S_N$, that captures two marked types. Let $S_0 := 0$. For $j \geq 0$, suppose we have constructed the process on $[0,S_j]$.

- Conditionally on the history, consider an SSSP$^{(\alpha)}(\theta+2\alpha)$-evolution $\mu^{(j)}$ starting from $\mu(S_j)$, and $f_1^{(j)} \sim \text{BESQ}_t(S_j)(-2\alpha)$, $i = 1,2$, independent of each other. Let $\Delta_j := \min\{\zeta(f_1^{(j)}), \zeta(f_2^{(j)})\}$ and $S_{j+1} := S_j + \Delta_j$. For $t \in [S_j, S_{j+1})$, define

$$(\mu(t), f_1(t), x_1(t), f_2(t), x_2(t)) := \left(\mu(t)-S_j, f_1^{(j)}(t-S_j), x_1(S_j), f_2^{(j)}(t-S_j), x_2(S_j)\right).$$

- Say $\Delta_j = \zeta(f_1^{(j)})$. If $f_2^{(j)}(\Delta_j)$ exceeds the size of the largest atom in $\mu^{(j)}(\Delta_j)$, let $N = j + 1$. The construction is complete. Otherwise, let $(f_2(S_{j+1}), x_2(S_{j+1})) := (f_2^{(j)}(\Delta_j), x_2(S_j))$ and decompose $\mu^{(j)}(\Delta_j) = \mu(S_{j+1}) + f_1(S_{j+1})\delta(x_1(S_{j+1}))$ by identifying its largest atom, giving rise to the five components. Similar operations apply when $\Delta_j = \zeta(f_2^{(j)})$.

For $t \in [0,S_N)$, define $\lambda_i(t)$, $i = 1,2$, by (55). In general, we may have $N \in \mathbb{N} \cup \{\infty\}$. On the event $\{N < \infty\}$, we further continue with the same triple-valued SSSP$^{(\alpha)}(\theta)$ starting...
from the terminal value \((\mu^N(\Delta_{N-1}),\mathbf{f}^N(\Delta_{N-1}),\mathbf{x}_i(\Delta_N))\), with \(i \in \{1,2\}\) being the index such that \(\mathbf{f}^N_i(\Delta_{N-1}) > 0\).

By [19, Corollary 5.11 and remark below] and the strong Markov property of these processes applied at the stopping times \(S_j\), we obtain two coupled triple-valued SSSP\((\alpha)(\theta)\), which induce the same \(\mathcal{M}^0\)-valued SSSP\((\alpha)(\theta)\), as required. Indeed, the construction of these two processes is clearly complete on \(\{N < \infty\}\). On \(\{N = \infty\}\), by (54) one has \(\{S_\infty < \infty\}\) and the total mass tends to zero as \(t \uparrow b_\infty\), and hence the construction is also finished.

For the continuity in the initial state, suppose that \(\pi_n(0) = \mathbf{f}^{(0)}_n(0)\delta(1) + \lambda^{(0)}_n(0) \rightarrow \pi(0) = \mathbf{f}^{(0)}(0)\delta(1) + \lambda^{(0)}(0)\) in total variation. First note that a slight variation of the proof of [19, Proposition 3.6] allows to couple \(\Lambda^{(0)}_n\) and \(\Lambda^{(0)}\) so that \(\lambda^{(0)}_n(t_n) \rightarrow \lambda^{(0)}(t)\) in total variation a.s., for any fixed sequence \(t_n \rightarrow t\). Also coupling \(\mathbf{f}^{(0)}_n\) and \(\mathbf{f}^{(0)}\), we can apply this on \(\{\zeta(\mathbf{f}^{(0)}_n) > t\}\) to obtain \(\pi_n(t) \rightarrow \pi(t)\) for any fixed \(t\), and on \(\{\zeta(\mathbf{f}^{(0)}_n) < t\}\) to obtain \(\zeta(\mathbf{f}^{(0)}_n) \rightarrow \zeta(\mathbf{f}^{(0)})\) and \(\pi_n(\zeta(\mathbf{f}^{(0)}_n)) \rightarrow \pi(\zeta(\mathbf{f}^{(0)}))\) in total variation a.s.. By induction, this establishes the convergence of one-dimensional marginals on \(\{T_\infty > t\}\), and trivially on \(\{T_\infty < t\} = \{\pi(t) = 0\}\). A further induction extends this to finite-dimensional marginals. \(\square\)

For \(\alpha \in (0,1), \theta \in [-\alpha,0), \) let \((B_1,B_2,\ldots) \sim \text{PD}(\alpha)(\theta)\) be a Poisson–Dirichlet sequence in the Kingman simplex and \((X_i,i \geq 1)\) i.i.d. uniform random variables on \([0,1]\), further independent of \((B_1,B_2,\ldots)\). Define \(\text{PD}^\alpha(\theta)\) to be the distribution of the random probability measure \(\bar{\pi} := \sum_{i \geq 1} B_i \delta(X_i)\) on \(\mathcal{M}_1^\alpha := \{\mu \in \mathcal{M}^\alpha : \|\mu\| = 1\}\). If \(\theta = -\alpha\), then \(\bar{\pi} = \delta(X_1)\).

**Proposition 5.4.** Let \((Z(t),t \geq 0)\) be a BESQ\((2\theta)\) killed at zero with \(Z(0) > 0\), independent of \(\pi \sim \text{PD}^\alpha(\theta)\). Let \((\pi(t),t \geq 0)\) be an SSSP\((\alpha)(\theta)\) starting from \(Z(0)\bar{\pi}\). Fix any \(t \geq 0\), then \(\pi(t)\) has the same distribution as \(Z(t)\bar{\pi}\).

**Proof.** By the coupling between an SSSP\((\alpha)(\theta+\alpha,0)\)-evolution and an SSSP\((\alpha)(\theta)\) mentioned above, the claim follows from Proposition 4.4 and the definition of \(\text{PD}^\alpha(\theta)\). \(\square\)

For \(\theta \geq -\alpha\), let \(\gamma := (\pi(t),t \geq 0)\) be an SSSP\((\alpha)(\theta)\) starting from \(\mu \in \mathcal{M}_1^\alpha\). Define an associated \(\mathcal{M}_1^\alpha\)-valued process via the de-Poissonisation as in Definition 1.3:

\[
\pi(u) := \|\pi(\tau_\pi(u))\|^{-1}\pi(\tau_\pi(u)), \quad u \geq 0,
\]

where \(\tau_\pi(u) := \inf\{t \geq 0 : \int_0^t \|\pi(s)\|^{-1}ds > u\}\). The process \((\pi(u),u \geq 0)\) on \(\mathcal{M}_1^\alpha\) is called a **Fleming–Viot (\alpha,\theta) process** starting from \(\mu\), denoted by \(\text{FV}(\alpha)(\theta)\).

Using Proposition 5.4, we easily deduce the following statement by the same arguments as in the proof of [19, Theorem 1.7], extending [19, Theorem 1.7] to the range \(\theta \in [-\alpha,0)\).

**Theorem 5.5.** Let \(\alpha \in (0,1)\) and \(\theta \geq -\alpha\). A \(\text{FV}(\alpha)(\theta)\) is a total-variation path-continuous Hunt process on \((\mathcal{M}_1^\alpha,d_{\mathcal{M}})\) and has a stationary distribution \(\text{PD}^\alpha(\theta)\).

### 5.2. Fragmenting interval partitions.

We define a fragmentation operator for interval partitions, which is associated with the random interval partition \(\text{PDIP}(\alpha)(\theta_1,\theta_2)\) defined in Section 2. Fragmentation theory has been extensively studied in the literature; see e.g. [4].

**Definition 5.6 (A fragmentation operator).** Let \(\alpha \in (0,1)\) and \(\theta_1,\theta_2 \geq 0\). We define a Markov transition kernel \(\text{Frag} := \text{Frag}(\alpha)(\theta_1,\theta_2)\) on \(\mathcal{I}_H\) as follows. Let \((\gamma_i)_{i \in \mathbb{N}}\) be i.i.d. with distribution \(\text{PDIP}(\alpha)(\theta_1,\theta_2)\). For \(\beta = \{(a_i,b_i) \in \mathbb{N}\} \in \mathcal{I}_H\), with \((a_i,b_i)_{i \in \mathbb{N}}\) enumerated in decreasing order of length, we define \(\text{Frag}(\beta,\cdot)\) to be the law of the interval partition obtained from \(\beta\) by splitting each \((a_i,b_i)\) according to the interval partition \(\gamma_i\), i.e.

\[
\{((a_i + (b_i - a_i))l,a_i + (b_i - a_i)r) : i \in \mathbb{N},(a_i,b_i) \in \beta,(l,r) \in \gamma_i\}.
\]
LEMMA 5.7. For \( \alpha, \beta \in (0, 1) \) and \( \theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2 \geq 0 \), suppose that \( \theta_1 + \theta_2 + \bar{\alpha} = \alpha \).

Let \( \beta_c \sim \text{PDIP}(\alpha)(\bar{\theta}_1, \bar{\theta}_2) \) and \( \beta_f \) a random interval partition whose regular conditional distribution given \( \beta_c \) is \( \text{Frag}(\alpha)(\theta_1, \theta_2) \). Then \( \beta_f \sim \text{PDIP}(\alpha)(\theta_1 + \theta_2, \theta_2 + \bar{\theta}_2) \).

A similar result for the particular case with parameter \( \bar{\alpha} = \bar{\theta}_2 = 0, \theta_1 = 0 \) and \( \theta_2 = \alpha \) is included in [21, Theorem 8.3]. Lemma 5.7 is also an analogous result of [38, Theorem 5.23] for \( \text{PD}(\alpha, \theta) \) on the Kingman simplex.

To prove Lemma 5.7, we now consider a pair of nested ordered Chinese restaurant processes \( (C_c(n), C_f(n))_{n \in \mathbb{N}} \). The coarse one \( C_c \sim \text{oCRP}^\alpha(\theta_1, \theta_2) \) describes the arrangement of customers in a sequence of ordered clusters. We next obtain a composition of each cluster of customers by further seating them at ordered tables, according to the \( (\alpha, \theta_1, \theta_2) \)-seating rule. These compositions are concatenated according to the cluster order, forming the fine process \( C_f \). Then, as illustrated in Figure 3, we can easily check that \( C_f \sim \text{oCRP}^\alpha(\theta_1 + \theta_2, \theta_2 + \bar{\theta}_2) \), due to the identity \( \theta_1 + \theta_2 + \bar{\alpha} = \alpha \). Nested (unordered) Chinese restaurant processes have been widely applied in nonparametric Bayesian analysis of the problem of learning topic hierarchies from data, see e.g. [7].

Lemma 5.7 follows immediately from the following convergence result.

LEMMA 5.8 (Convergence of nested oCRP). Let \( (C_c(n), C_f(n))_{n \in \mathbb{N}} \) be a pair of nested ordered Chinese restaurant processes constructed as above. Then \( (n^{-1}C_c(n), n^{-1}C_f(n)) \) converges a.s. to a limit \( (\beta_c, \beta_f) \) for the metric \( d_H \) as \( n \to \infty \); furthermore, we have \( \beta_c \sim \text{PDIP}(\alpha)(\bar{\theta}_1, \bar{\theta}_2) \), \( \beta_f \sim \text{PDIP}(\alpha)(\theta_1 + \bar{\theta}_1, \theta_2 + \bar{\theta}_2) \), and the regular conditional distribution of \( \beta_f \) given \( \beta_c \) is \( \text{Frag}(\alpha)(\theta_1, \theta_2) \).

PROOF. By Lemma 2.7, we immediately deduce that \( (n^{-1}C_c(n), n^{-1}C_f(n)) \) converges a.s. to a limit \( (\beta_c, \beta_f) \). It remains to determine the joint distribution of the limit.

Consider a sequence of i.i.d. \( \gamma_i \sim \text{PDIP}(\alpha)(\theta_2, \theta_2) \), \( i \geq 1 \) and an independent \( \beta'_c := \{(a_i, b_i), i \in \mathbb{N}\} \sim \text{PDIP}(\alpha)(\bar{\theta}_1, \bar{\theta}_2) \). Let \( \beta'_f \) be obtained from \( \beta'_c \) by splitting each \( (a_i, b_i) \) according to \( \gamma_i \), then the regular conditional distribution of \( \beta'_f \) given \( \beta'_c \) is \( \text{Frag}(\alpha)(\theta_1, \theta_2) \). We will show that \( (\beta_c, \beta_f) \overset{d}{=} (\beta'_c, \beta'_f) \).

To this end, apply the paintbox construction described before Proposition 2.9 to the nested \( \beta'_c \) and \( \beta'_f \), by using the same sequence of i.i.d. uniform random variables \( (Z_j, j \in \mathbb{N}) \) on \([0, 1]\). For each \( n \in \mathbb{N} \), let \( C_c^*(n) \) and \( C_f^*(n) \) be the compositions of the set \([n]\) obtained as in (8), associated with \( \beta'_c \) and \( \beta'_f \) respectively. Write \( (C_c^*(n), C_f^*(n)) \) for the integer compositions associated with \( (C_c(n), C_f(n)) \), then \( (n^{-1}C_c^*(n), n^{-1}C_f^*(n)) \) converges a.s. to \( (\beta'_c, \beta'_f) \) by [22, Theorem 11].
Note that each \((a_i, b_i) \in \beta_c^t\) corresponds to a cluster of customers in \(C^*_c(n)\), which are further divided into ordered tables in \(C^*_f(n)\). This procedure can be understood as a paint-box construction, independent of other clusters, by using \(\gamma_i \sim \text{PDIP}^\alpha(\theta_1, \theta_2)\) and i.i.d. uniform random variables \(\{(Z_j - a_i)/(b_i - a_i) : Z_j \in (a_i, b_i), j \in \mathbb{N}\}\) on \([0, 1]\). By Proposition 2.9, it has the same effect as an oCRP \((\theta_1, \theta_2)\). For each \(n \in \mathbb{N}\), as we readily have \(C^*_c(n) \xrightarrow{d} C_c(n)\) by Proposition 2.9, it follows that \((C^*_c(n), C^*_f(n)) \xrightarrow{d} (C_c(n), C_f(n))\). As a result, we deduce that the limits also have the same law, i.e. \((\beta_c, \beta_f) \xrightarrow{d} (\beta^t_c, \beta^t_f)\).  

5.3. Coarse-fine interval partition evolutions. We consider the space 

\[ T^2_{\text{nest}} := \{(\gamma_c, \gamma_f) \in \mathcal{I}_H \times \mathcal{I}_H : G(\gamma_c) \subseteq G(\gamma_f), \|\gamma_c\| = \|\gamma_f\|\}. \]

In other words, for each element \((\gamma_c, \gamma_f)\) in this space, the interval partition \(\gamma_f\) is a refinement of \(\gamma_c\) such that each interval \(U \in \gamma_c\) is further split into intervals in \(\gamma_f\), forming an interval partition \(\gamma_f|_U\) of \([\inf U, \sup U]\). We also define the shifted interval partition 

\[ \gamma_f|_U := \{(a, b) : (a + \inf U, b + \inf U) \in \gamma_f|_U\} \]

of \([0, \text{Leb}(U)]\) and note that \(\gamma_f|_U \in \mathcal{I}_H\). We equip \(T^2_{\text{nest}}\) with the product metric 

\[ d_H^2((\gamma_c, \gamma_f), (\gamma'_c, \gamma'_f)) = d_H(\gamma_c, \gamma'_c) + d_H(\gamma_f, \gamma'_f). \]

**Lemma 5.9.** For each \(n \geq 1\), let \((\beta_n, \gamma_n) \in T^2_{\text{nest}}\). Suppose that \((\beta_n, \gamma_n)\) converges to \((\beta_\infty, \gamma_\infty)\) under the product metric \(d_H^2\). Then \((\beta_\infty, \gamma_\infty) \in T^2_{\text{nest}}\).

**Proof.** This requires us to prove \(G(\beta_\infty) \subseteq G(\gamma_\infty)\). As \(G(\gamma_\infty)\) is closed, it is equivalent to show that, for any \(x \in G(\beta_\infty)\), the distance \(d(x, G(\gamma_\infty))\) from \(x\) to the set \(G(\gamma_\infty)\) is zero. For any \(y_n \in G(\beta_n) \subseteq G(\gamma_n)\), we have \(d(x, G(\gamma_\infty)) \leq d(x, y_n) + d_H(\gamma_n, \gamma_\infty)\). It follows that 

\[ d(x, G(\gamma_\infty)) \leq \inf_{y_n \in G(\beta_n)} d(x, y_n) + d_H(\gamma_n, \gamma_\infty) \leq d_H(\beta_\infty, \beta_n) + d_H(\gamma_n, \gamma_\infty) \]

as \(n \to \infty\), the right-hand side converges to zero. So we conclude that \(d(x, G(\gamma_\infty)) = 0\) for every \(x \in G(\beta_\infty)\), completing the proof. 

We shall now construct a coarse-fine interval partition evolution in the space \(T^2_{\text{nest}}\). To this end, let us first extend the scaffolding-and-spindles construction in Section 3.1 to the setting where each spindle is an interval-partition-valued excursion. Denote by \(E_T\) the space of continuous \(\mathcal{I}_H\)-valued excursions. Given a point measure \(W\) on \(\mathbb{R}_+ \times E_T\) and a scaffolding function \(X : \mathbb{R}_+ \to \mathbb{R}\), we define the following variables, if they are well-defined. The coarse skewer of \((W, X)\) at level \(y \in \mathbb{R}\) is the interval partition

\[ c\text{SKWER}(y, W, X) := \left\{ \left( M^y_{W,X}(t-), M^y_{W,X}(t) \right) : t \geq 0, M^y_{W,X}(t) = \inf_{t < s \leq t} \gamma(y - X(s)) \right\}, \]

where \(M^y_{W,X}(t) := \int_{[0,t] \times E_T} \gamma(y - X(s)) |W(ds, d\gamma)|, t \geq 0\). Let \(c\text{SKWER}(W, X) := (c\text{SKWER}(y, W, X), y \geq 0)\). The fine skewer of \((W, X)\) at level \(y \in \mathbb{R}\) is the interval partition 

\[ f\text{SKWER}(y, W, X) := \{ \text{points} (t, \gamma_t) \text{ of } W : M^y_{W,X}(t-) < M^y_{W,X}(t) \}. \]

Let \(f\text{SKWER}(W, X) := (f\text{SKWER}(y, W, X), y \geq 0)\).

Let \(\theta_1, \theta_2 \geq 0\). Suppose that \(\theta = \theta_1 + \theta_2 - \alpha \in [-\alpha, 0]\), then we have an SSIP\(^\alpha\)(\(\theta_1, \theta_2\))-excursion measure \(\Theta := \Theta^{\alpha}(\theta_1, \theta_2)\) defined as in Section 4.4. Write \(\bar{\alpha} := -\theta \in (0, \alpha)\) and let \(W\) be a Poisson random measure on \(\mathbb{R}_+ \times E_T\) with intensity \(c_\alpha \text{Leb} \otimes \Theta\), where \(c_\alpha := 2\bar{\alpha}(1 + \bar{\alpha})/\Gamma(1 - \bar{\alpha})\). We pair \(W\) with a (coarse) scaffolding \((\xi_W^{(\bar{\alpha})}(t), t \geq 0)\) defined by

\[ (56) \quad \xi^{(\bar{\alpha})}_W(t) := \lim_{z \downarrow 0} \left( \int_{[0,t] \times (\zeta \in E_T : \zeta(\gamma) > z)} \zeta(\gamma) |W(ds, d\gamma)| - \frac{(1 + \bar{\alpha})t}{(2z)^\bar{\alpha} \Gamma(1 - \bar{\alpha}) \Gamma(1 + \bar{\alpha})} \right). \]
This is a spectrally positive stable Lévy process of index $1 + \alpha$. Let $\beta$ be an SSIP$^{(\alpha)}_1(\theta_1, \theta_2)$-evolution starting from $\gamma_0 \in \mathcal{I}_U$ with first hitting time $\zeta(\beta)$ of $\emptyset$. We define by $Q_{\gamma_0}^{(\alpha)}(\theta_1, \theta_2)$ the law of the following a random point measure on $[0, \infty) \times \mathcal{E}$:

$$(57) \quad \delta(0, \beta) + W|_{\{0, T_{c \alpha}(\beta)\} \times \mathcal{E}} \quad \text{where} \quad T_{-y} := \inf \{ t \geq 0 : \xi^{(\alpha)}_W(t) = -y \}. $$

**Definition 5.10 (Coarse-fine SSIP-evolutions).** Let $\alpha \in (0, 1)$, $\theta_1, \theta_2 \geq 0$ with $\theta_1 + \theta_2 < \alpha$. Let $\bar{\alpha} := \alpha - \theta_1 - \theta_2 \in (0, \alpha]$. For $(\gamma_c, \gamma_f) \in \mathcal{I}_\text{nest}^2$, let $W_U \sim Q^{(\alpha)}_{\gamma_U}(\theta_1, \theta_2)$, $U \in \gamma_c$, be an independent family with scaffolding $\xi^{(\alpha)}_W$ as in (56). Then the pair-valued process

$$(\star) \quad \text{cSK} (y, W_U, \xi^{(\alpha)}_W), \quad \text{fSK} (y, W_U, \xi^{(\alpha)}_W), \quad y \geq 0,$$

is called a coarse-fine $(\alpha, \theta_1, \theta_2, 0)$-self-similar interval partition evolution, starting from $(\gamma_c, \gamma_f)$, abbreviated as cfSSIP$^{(\alpha, \theta_1, \theta_2)}(0)$-evolution.

Roughly speaking, it is a random refinement of an SSIP$^{(\alpha)}(0)$-evolution according to SSIP$^{(\alpha)}(\theta_1, \theta_2)$-excursions. To add immigration to this model, let $W \sim \text{PRM}_c(\text{Leb} \otimes \Theta)$ and consider its coarse scaffolding $\xi^{(\alpha)}_W$ as in (56). For $\bar{\alpha} \geq 0$, as in (13), define the process

$$(58) \quad X_{\bar{\alpha}}(t) := \xi^{(\alpha)}_W(t) + (1 - \alpha/\bar{\theta}) \ell(t), \quad t \geq 0, \quad \text{where} \quad \ell(t) := \inf_{u \leq t} \xi^{(\alpha)}_W(u).$$

For each $j \in \mathbb{N}$, set $T_{\gamma}^{-j} := \inf\{ t \geq 0 : X_{\bar{\alpha}}(t) = -j \}$ and define nested processes

$$(\bar{\beta}_{c,j}(y) := \text{cSK}(y, W|_{[0, T_{\gamma}^{-j}]), j + X_{\bar{\alpha}}|_{[0, T_{\gamma}^{-j}])}, \quad y \in [0, j], $$

$$f_{c,j}(y) := \text{fSK}(y, W|_{[0, T_{\gamma}^{-j}]), j + X_{\bar{\alpha}}|_{[0, T_{\gamma}^{-j}])}, \quad y \in [0, j].$$

As in Section 3.1, we find that $((\bar{\beta}_{c,j}(y), \bar{\beta}_{f,j}(y)), y \in [0, j]) \overset{d}{=} (\bar{\beta}_{c,k}(y), \bar{\beta}_{f,k}(y)), y \in [0, j])$ for all $k \geq j$. Thus, by Kolmogorov’s extension theorem, there exists a process $((\bar{\beta}_{c}, \bar{\beta}_{f})$ such that $((\bar{\beta}_{c}(y), \bar{\beta}_{f}(y), y \in [0, j]) \overset{d}{=} (\bar{\beta}_{c,j}(y), \bar{\beta}_{f,j}(y)), y \in [0, j])$ for every $j \in \mathbb{N}$.

**Definition 5.11 (Coarse-fine SSIP-evolutions with immigration).** Let $\bar{\alpha}, \theta_1, \theta_2 \geq 0, \alpha \in (0, 1), \bar{\alpha} = \alpha - \theta_1 - \theta_2 \in (0, \alpha]$ and $(\gamma_c, \gamma_f) \in \mathcal{I}_\text{nest}^2$. Let $((\bar{\beta}_{c}, \bar{\beta}_{f})$ be defined as above and $((\bar{\beta}_{c}, \bar{\beta}_{f})$ an independent cfSSIP$^{(\alpha, \theta_1, \theta_2)}(0)$-evolution starting from $(\gamma_c, \gamma_f)$. Then we call $((\bar{\beta}_{c}(y), \bar{\beta}_{f}(y)) \overset{d}{=} (\bar{\beta}_{c,j}(y), \bar{\beta}_{f,j}(y)), y \geq 0$, a coarse-fine $(\alpha, \theta_1, \theta_2)$-self-similar interval partition evolution with immigration rate $\alpha$, starting from $(\gamma_c, \gamma_f)$, or a cfSSIP$^{(\alpha, \theta_1, \theta_2)}(0)$-evolution.

By construction, the coarse process of a cfSSIP$^{(\alpha, \theta_1, \theta_2)}(0)$-evolution is an SSIP$^{(\alpha)}(\bar{\alpha})$-evolution. For the special case $\theta_1 = \theta_2 = 0$, the fine process coincides with the coarse one.

**Remark.** By combining Definition 5.11 and Definition 4.1, one can further construct a coarse-fine SSIP-evolution with the coarse process being an SSIP$^{(\alpha)}(\theta_1, \theta_2)$-evolution.

### 5.4. Convergence of nested PCRP$s$. For $\alpha \in (0, 1)$ and $\bar{\alpha} \geq 0$, let $(C_c(t), t \geq 0)$ be a Poissonised Chinese restaurant process PCRP$^{(\alpha)}(\bar{\alpha})$ as in Section 3.2. Recall that for each cluster of $C_c$, the mass evolves according to a Markov chain of law $\pi(-\bar{\alpha})$ as in (16). Let $\alpha \in (\bar{\alpha}, 1)$ and $\theta_1, \theta_2 \geq 0$. Suppose that there is the identity

$$\theta = \theta_1 + \theta_2 - \alpha = -\bar{\alpha} < 0,$$
then the total mass evolution of a \( \text{PCRPR}^{(\alpha)}(\theta_1, \theta_2) \) also has distribution \( \pi(-\bar{\alpha}) \). Therefore, we can fragment each cluster of \( C_c \) into \( \text{PCRPR}^{(\alpha)}(\theta_1, \theta_2) \) as follows. In each cluster, customers are further attributed into an ordered sequence of tables: whenever a customer joins this cluster, they choose an existing table or add a new table according to the \((\alpha, \theta_1, \theta_2)\)-seating rule; whenever the cluster size reduces by one, a customer is chosen uniformly to leave. As a result, we embed a \( \text{PCRPR}^{(\alpha)}(\theta_1, \theta_2) \) into each cluster of \( C_c \), independently of the others. The rates at which customers arrive are illustrated in Figure 3. For each time \( t \geq 0 \), by concatenating the composition of ordered table size configuration of each cluster, from left to right according to the order of clusters, we obtain a composition \( C_f(t) \), representing the numbers of customers at all tables. Then \( C_f(t) \) is a refinement of \( C_c(t) \). One can easily check that \((C_f(t), t \geq 0)\) is a \( \text{PCRPR}^{(\alpha)}(\theta_1 + \bar{\theta}, \theta_2 + \bar{\alpha}) \). We refer to the pair \(((C_c(t), C_f(t)), t \geq 0)\) as a pair of nested PCRPs.

**Theorem 5.12 (Convergence of nested PCRPs).** For each \( n \in \mathbb{N} \), let \((C_c^{(n)}, C_f^{(n)})\) be a pair of nested PCRPs as defined above, starting from \((\gamma^{(n)}, \gamma_f^{(n)})\) in \( \mathcal{I}_c^{2\text{nest}} \). Suppose that \( \frac{1}{n}\langle \gamma^{(n)}, \gamma_f^{(n)} \rangle \) converges to \((\gamma^{\infty}, \gamma_f^{\infty})\) in \( \mathcal{I}_c^{2\text{nest}} \) under the product metric \( d_H^2 \). Then the following convergence holds in distribution in the space of càdlàg functions on \( \mathcal{I}_H \times \mathcal{I}_H \) endowed with the Skorokhod topology,

\[
\left( \frac{1}{n}\left(C_c^{(n)}(2nt), C_f^{(n)}(2nt)\right), t \geq 0 \right) \xrightarrow{n \to \infty} \left( (\beta_c(t), \beta_f(t)), t \geq 0 \right),
\]

where the limit \((\beta_c, \beta_f) = ((\beta_c(t), \beta_f(t)), t \geq 0)\) is a \( \text{cfSSIP}^{(\alpha, \theta_1, \theta_2)}(\bar{\theta}) \).

**Proof.** The arguments are very similar to those in the proof of Theorem 3.12 and Proposition 3.15, with an application of Theorem 1.5, which replaces the role of Proposition 3.10. Let us sketch the main steps:

- Let \( W^{(n)} \) be a Poisson random measure of rescaled excursions of \( \text{PCRPR}^{(\alpha)}(\theta_1, \theta_2) \) with intensity \( 2\alpha n^{1+\bar{\alpha}}P^{(n)} \), where \( P^{(n)} \) is as in Theorem 1.5. Write \( \xi^{(n)} \) for the associated scaffolding of \( W^{(n)} \) defined as in (17) and \( M^{(n)} \) the total mass of the coarse skewer. Since by Theorem 1.5 the intensity measure converges vaguely to the \( \text{SSIP}^{(\alpha)}(\theta_1, \theta_2) \)-excursion measure \( c_\alpha \Theta \), in analogy with Proposition 3.7, the sequence \((W^{(n)}, \xi^{(n)}, M^{(n)})\) can be constructed such that it converges a.s. to \((W, \xi, M)\), where \( \xi \) and \( M \) are the scaffolding defined as in (56) and the coarse skewer total mass of \( W \sim \text{PRM}(c_\alpha \text{Leb} \otimes \Theta) \), respectively.
- Using similar methods as in Theorem 3.12 proves the convergence when \( \bar{\theta} = 0 \). More precisely, using the sequence \( W^{(n)} \) obtained in the previous step, we give a scaffolding-and-spindles construction for each rescaled nested pair \( \left( \frac{1}{n}C_c^{(n)}(2n\cdot), \frac{1}{n}C_f^{(n)}(2n\cdot) \right) \), as in the description below Lemma 3.13 and in Section 3.2. We first study the case when the initial state of the coarse process is a single interval as in Lemma 3.13, and then extend to any initial state by coupling the large clades and controlling the total mass of the remainder.
- When \( \bar{\theta} > 0 \), we proceed as in the proof of Proposition 3.15: we prove that the modified scaffolding converges and then the skewer process also converges.

Summarising, we deduce the convergence of nested PCRPs to the coarse-fine skewer processes, as desired.

Having Theorem 5.12, we can now identify the fine process by Theorem 1.1.

**Proposition 5.13 (Nested SSIP-evolutions).** Let \( \alpha \in (0, 1), \theta_1, \theta_2, \bar{\theta} \geq 0 \) and suppose that \( \theta = \theta_1 + \theta_2 - \alpha < 0 \). In a \( \text{cfSSIP}^{(\alpha, \theta_1, \theta_2)}(\bar{\theta}) \)-evolution, the coarse and fine processes are \( \text{SSIP}^{(\alpha)}(\bar{\theta}) \)- and \( \text{SSIP}^{(\alpha)}(\theta_1 + \bar{\theta}, \theta_2 + \bar{\alpha}) \)-evolutions respectively, where \( \bar{\alpha} = -\alpha \).
PROOF. We may assume this cfSSIP is the limit of a sequence of nested PCRPs. Since the coarse processes form a sequence of PCRP\(^{(α)}(\tilde{θ})\) that converges in its own right, Theorem 1.1 shows that the limit is an SSIP\(^{(α)}(\tilde{θ})\)-evolution. Similarly, since the fine process is the limit of a sequence of PCRP\(^{(α)}(θ_1+\tilde{θ}, θ_2+\tilde{θ})\), it is an SSIP\(^{(α)}(θ_1+\tilde{θ}, θ_2+\tilde{θ})\)-evolution.

**Proposition 5.14 (Pseudo-stationarity).** Let \(α ∈ (0,1)\), \(θ_1, θ_2 ≥ 0\) with \(\tilde{α} := α - θ_1 - θ_2 \in (0, α]\), and \(\tilde{θ} ≥ 0\). Let \(Z \sim BESQ(2\tilde{α})\) and \(\tilde{γ}_c \sim PDIP^{(α)}(\tilde{θ}, \tilde{α})\) be independent and \(\tilde{γ}_f \sim Frag^{(α)}(θ_1, θ_2)(\tilde{γ}_c, \cdot)\). Let \((\tilde{β}_c(t), \tilde{β}_f(t)), t ≥ 0\) be a cfSSIP\(^{(α,θ_1,θ_2)}(\tilde{θ})\)-evolution starting from \((Z(0)\tilde{γ}_c, Z(0)\tilde{γ}_f)\). Then \((\tilde{β}_c(t), \tilde{β}_f(t)) \overset{d}{=} (Z(t)\tilde{γ}_c, Z(t)\tilde{γ}_f)\) for each \(t ≥ 0\).

PROOF. We may assume this cfSSIP-evolution is the limit of a sequence of nested PCRPs \((C_c^{(n)}, C_f^{(n)})\), with \((C_c^{(n)}, C_f^{(n)})\) starting from nested compositions of \([n]\) with distribution as in Lemma 5.8. By similar arguments as in Lemma 4.5, we deduce that, given the total number of customers \(m := ||C_c^{(n)}(t)|| = ||C_f^{(n)}(t)||\) at time \(t ≥ 0\), the conditional distribution of \((C_c^{(n)}(t), C_f^{(n)}(t))\) is given by nested oCRP\(^{(α)}(\tilde{θ}, \tilde{α})\) and oCRP\(^{(α)}(θ_1+\tilde{θ}, θ_2+\tilde{θ})\) described above Lemma 5.8. The claim then follows from Lemma 5.8 and Theorem 5.12.

**Proposition 5.15 (Markov property).** A cfSSIP\(^{(α,θ_1,θ_2)}(\tilde{θ})\)-evolution is a Markov process on \((T_{nest}^2, d_{H}^2)\) with continuous paths.

To prove Proposition 5.15, we first give a property of the excursion measure \(Θ^{(α)}(θ_1, θ_2)\). For any \(Z_H\)-valued process \(γ(γ(y), y ≥ 0)\) and \(α > 0\), let \(H^α(γ) := \inf\{y ≥ 0; ||γ(y)|| > α\}\).

**Lemma 5.16.** For \(α > 0\), let \(β(β(y), y ≥ 0) ∼ \Theta^{(α)}(θ_1, θ_2)(\cdot | H^α < ∞)\). Conditionally on \((β(r), r ≤ H^α(β))\), the process \((β(H^α(β) + z), z ≥ 0)\) is an SSIP\(^{(α)}(θ_1, θ_2)\)-evolution starting from \(β(⟨H^α(β)⟩)\).

PROOF. For \(k ∈ ℕ\), let \(H_k^α := 2^{-k} [2^k H^α] ∧ 2^k\). Then \(H_k^α\) is a stopping time that a.s. only takes a finite number of possible values and eventually decreases to \(H^α\). By (42), the desired property is satisfied by each \(H_k^α\). Then we deduce the result for \(H^α\) by approximation, using the path-continuity and Hunt property of SSIP\(^{(α)}(θ_1, θ_2)\)-evolutions of Theorem 4.6.

For \((γ_c, γ_f) ∈ T_{nest}^2\), let \((W_U, U ∈ γ_c)\) be a family of independent clades, with each \(W_U ∼ Q_{γ_c}^{(α)}(θ_1, θ_2)\). Let \(ξ_U^{(α)}\) be the scaffolding associated with \(W_U\) as in (56) and write \(\text{len}(W_U) := \inf\{s ≥ 0; ξ_U^{(α)}(s) = 0\}\) for its length, which is a.s. finite. Then we define the concatenation of \((W_U, U ∈ γ_c)\) by

\[
\bigstar_{U ∈ γ_c} W_U := \sum_{U ∈ γ_c} \int \delta(g(U) + t, β)W_U(dt, dβ), \quad \text{where } g(U) = \sum_{V ∈ γ_c, \sup \text{len}(W_V) ≤ \inf U} \text{len}(W_V).
\]

Write \(Q_{γ_c,γ_f}^{(α)}(θ_1, θ_2)\) for the law of \(\bigstar_{U ∈ γ_c} W_U\). We next present a Markov-like property for such point measures of interval partition excursions, analogous to [14, Proposition 6.6].

**Lemma 5.17.** For \((γ_c, γ_f) ∈ T_{nest}^2\), let \(W ∼ Q_{γ_c,γ_f}^{(α)}(θ_1, θ_2)\) and \(X = c^{(α)}_{γ_f}\). For \(y ≥ 0\), set \(\text{cutoff}_{W}^{y} = \sum_{\text{points } (t, γ_t) \text{ of } W} 1\{X(t−) ≥ y\}δ(σ^y(t), γ_t) + 1\{y ∈ \{X(t−), X(t)\}\}δ(σ^y(t), γ^y_{t}),\)
where \(σ^y(t) = \text{Leb}\{u ≤ t: X(u) > y\}\) and \(γ^y_{t} = (γ_t - X(t−) + z, z ≥ 0)\). Similarly define \(\text{cutoff}_{W}^{y}\). Given \((β_c(y), β_f(y)) = (\text{cSkewer}(y, W, X), \text{fSkewer}(y, W, X)),\) \(\text{cutoff}_{W}^{y}\) is conditionally independent of \(\text{cutoff}_{W}^{y}\) and has conditional distribution \(Q_{γ_c,γ_f}^{(α)}(β_c(y), β_f(y))(θ_1, θ_2)\).
PROOF. Recall that the construction of the nested processes is a modification of the scaffolding-and-spindles construction of the coarse component, with the same scaffolding and the \( \Lambda_{\operatorname{BESQ}}(-2\alpha) \)-excursions being replaced by the interval-partition excursions under \( \Theta \). In view of this, we can follow the same arguments as in the proof of [14, Proposition 6.6], with an application of Lemma 5.16.

PROOF OF PROPOSITION 5.15. The path-continuity follows directly from that of an SSIP-evolution. As in [14, Corollary 6.7], Lemma 5.17 can be translated to the skewer process under \( Q^{(\alpha)}_{(\gamma_c,\gamma_f)}(\theta_1,\theta_2) \), thus giving the Markov property for \( \operatorname{cfSSIP}^{(\alpha,\theta_1,\theta_2)}(0) \)-evolutions.

When the immigration rate is \( \bar{\theta} > 0 \), we introduce an excursion measure \( \Theta_{\operatorname{nest}} \) on the space of continuous \( \mathcal{I}_{\operatorname{nest}} \)-excursions, such that the coarse excursion is a \( \Theta^{(\alpha)}(0,\bar{\alpha}) \), and each of its \( \operatorname{BESQ}(-2\alpha) \)-excursions is split into a \( \Theta^{(\alpha)}(\theta_1,\theta_2) \)-excursion. More precisely, for \( y > 0 \), it has the following properties:

1. \( \Theta_{\operatorname{nest}}(\zeta > y) = \Theta^{(\alpha)}(0,\bar{\alpha})(\zeta > y) = y^{-1} \).
2. If \( (\beta_c,\beta_f) \sim \Theta_{\operatorname{nest}}(\cdot | \zeta > y) \), then \( (\beta_c(y),\beta_f(y)) \sim \Gamma_{\operatorname{Gamma}}(1-\bar{\alpha},1/2y) \), \((\tilde{\gamma}_c,\tilde{\gamma}_f) \sim \Gamma_{\operatorname{Gamma}}(1-\bar{\alpha},1/2y) \), where \( \tilde{\gamma}_c \) and \( \tilde{\gamma}_f \) are given by the construction in [18, Section 3].

Having obtained the pseudo-stationarity (Proposition 5.14) and the Markov property of \( \operatorname{cfSSIP}^{(\alpha,\theta_1,\theta_2)}(0) \)-evolutions, the construction of \( \Theta_{\operatorname{nest}} \) can be made by a similar approach as in Section 4.4.

Using \( \mathbf{F} \sim \operatorname{PRM}(\bar{\theta} \operatorname{Leb} \otimes \Theta_{\operatorname{nest}}) \), by the construction in [18, Section 3], the following process has the same law as a \( \operatorname{cfSSIP}^{(\alpha,\theta_1,\theta_2)}(0) \)-evolution starting from \((\emptyset,\emptyset)\), for \( y \geq 0 \),

\[
\beta_c(y) = \operatorname{points}(s,\gamma_c,\gamma_f) \in \mathbf{F}: s \in [0,y] \downarrow \gamma_c(y-s), \quad \beta_f(y) = \operatorname{points}(s,\gamma_c,\gamma_f) \in \mathbf{F}: s \in [0,y] \downarrow \gamma_f(y-s).
\]

The Markov property of \( \operatorname{cfSSIP}^{(\alpha,\theta_1,\theta_2)}(0) \)-evolutions is now a consequence of this Poissonian construction and the form of \( \Theta_{\operatorname{nest}} \); see the proof of [18, Lemma 3.10] for details.

THEOREM 5.18. For any \( \theta \geq 0 \) and pairwise nested \( \gamma_\alpha \in \mathcal{I}_H, \alpha \in (0,1) \), there exists a nested family \( (\beta_\alpha,\alpha \in (0,1)) \) of \( \operatorname{SSIP}^{(\alpha)}(0) \)-evolutions, in the following sense:

1. each \( \beta_\alpha \) is an \( \operatorname{SSIP}^{(\alpha)}(0) \)-evolution starting from \( \gamma_\alpha \);
2. for any \( 0 < \bar{\alpha} < \alpha < 1 \), \( \beta_{\bar{\alpha}} \) takes values in \( \mathcal{I}_{\operatorname{nest}}^2 \).

PROOF. For \( 0 < \bar{\alpha} < \alpha < 1 \), let \( \beta_{\bar{\alpha}}(\cdot,\beta_f) = ((\beta_c(\cdot),\beta_f(\cdot)), y \geq 0) \) be a \( \operatorname{cfSSIP}^{(\alpha,0,\alpha-\alpha)}(0) \)-evolution starting from \( \gamma_{\alpha}(\cdot) \in \mathcal{I}_{\operatorname{nest}}^2 \). Then by Proposition 5.13, the coarse process \( \beta_c \) is an \( \operatorname{SSIP}^{(\alpha)}(0) \)-evolution and the fine process \( \beta_f \) is an \( \operatorname{SSIP}^{(\alpha)}(0) \)-evolution. This induces a kernel \( \kappa_{\bar{\alpha}}(\cdot) \) from the coarse process to the fine process. Arguing by approximation as in Theorem 5.12, we can prove that \( \kappa_{\alpha_1,\alpha_2}(\cdot) \leq \kappa_{\alpha_3,\alpha_4}(\cdot) \) for all \( 0 < \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < 1 \). More generally, for any finitely many \( 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < 1 \), we can find nested \( (\beta_{\alpha_i}, 1 \leq i \leq n) \) that are consistently related by these kernels. We can thus construct the full family by using Kolmogorov’s extension theorem.

Let \( \mathcal{I}_{\operatorname{nest},1}^2 := \{(\gamma_c,\gamma_f) \in \mathcal{I}_{\operatorname{nest}}^2 : \|\gamma_c\| = \|\gamma_f\| = 1\} \) be the space of nested partitions of \([0,1]\).
THEOREM 5.19. For any \(\theta \geq 0\) and pairwise nested \(\tilde{\gamma}_\alpha \in \mathcal{I}_{H;1}\), \(\alpha \in (0,1)\), there exists a family of processes \((\tilde{\beta}_\alpha, \alpha \in (0,1))\) on \(\mathcal{I}_{H;1}\), such that

1. each \(\tilde{\beta}_\alpha\) is an \(\text{IP}^{(\alpha)}(\theta)\)-evolution starting from \(\tilde{\gamma}_\alpha\);
2. for any \(0 < \tilde{\alpha} < \alpha < 1\), \((\tilde{\beta}_\alpha, \tilde{\beta}_\alpha)\) takes values in \(\mathcal{I}_{\text{nest};1}\).

PROOF. Build a family of SSIP-evolutions \((\beta_\alpha, \alpha \in (0,1))\) as in Theorem 5.18 on the same probability space. In particular, they have the same total mass process and thus the same de-Poissonisation. So the de-Poissonised family \((\tilde{\beta}_\alpha, \alpha \in (0,1))\) is still nested. \(\square\)

5.5. An application to alpha-gamma trees. For \(n \geq 1\), let \(T_n\) be the space of all (non-planar) trees without degree-2 vertices, a root vertex of degree 1, and exactly \(n\) further degree-1 vertices, leaves labelled by \([n] = \{1, \ldots, n\}\). For \(\alpha \in (0,1)\) and \(\gamma \in [0,\alpha]\), we construct random trees \(T_n\) by using the following \((\alpha, \gamma)\)-growth rule [9]: \(T_1\) and \(T_2\) are the unique elements in \(T_1\) and \(T_2\). Given \(T_k\) with \(k \geq 2\), assign weight \(1-\alpha\) to each of the \(k\) edges adjacent to a leaf, weight \(\gamma\) to each of the other edges, and weight \((d-2)\alpha-\gamma\) to each branch point with degree \(d \geq 3\). To create \(T_{k+1}\) from \(T_k\), choose an edge or a branch point proportional to the weight, and insert the leaf \(k+1\) to the chosen edge or branch point. This generalises Rémy’s algorithm [41] of the uniform tree (when \(\alpha = \gamma = 1/2\)) and Marchal’s recursive construction [33] of \(\rho\)-stable trees with \(\rho \in (1,2)\) (when \(\alpha = 1-1/\rho\) and \(\gamma = 1-\alpha\)).

For each \(T_n\), consider its spinal decomposition as discussed in the introduction, the spine being the path connecting the leaf 1 and the root. Let \(C_c(n)\) be the sizes of bushes at the spinal branch points, ordered from left to right in decreasing order of their distances to the root. Then the \((\alpha, \gamma)\)-growth rule implies that the \((C_c(n), n \in \mathbb{N})\) is an \(\text{oCRP}^{(\gamma)}(1-\alpha, \gamma)\). Similar as the semi-planar \((\alpha, \gamma)\)-growth trees in [49], we further equip each spinal branch point with a left-to-right ordering of its subtrees, such that the sizes of the sub-trees in each bush follow the \((\alpha, 0, \alpha-\gamma)\)-seating rule. By concatenating the sub-tree-configurations of all bushes according to the order of bushes, we obtain the composition \(C_f(n)\) of sizes of subtrees. Then \((C_f(n), n \in \mathbb{N})\) is an \(\text{oCRP}^{(\alpha)}(1-\alpha, \alpha)\) nested to \((C_c(n), n \in \mathbb{N})\), as in Figure 3.

Let us introduce a continuous-time Markov chain \((T(s), s \geq 0)\) on \(T = \bigcup_{n \geq 1} T_n\), the space of labelled rooted trees without degree-2 vertices. Given \(T(s)\), assign weights to its branch points and edges as in the \((\alpha, \gamma)\)-growth model, such that for each branch point or edge, a new leaf arrives and is attaches to this position at the rate given by its weight. Moreover, fix the root and the leaf 1, and delete any other leaf at rate one, together with the edge attached to it; in this operation, if a branching point degree is reduced to two, we also delete it and merge the two edges attached to it.

For each \(n \geq 1\), consider such a continuous-time up-down Markov chain \((T^{(n)}(s), s \geq 0)\) starting from a random tree \(T_n\) built by the \((\alpha, \gamma)\)-growth rule. At each time \(s \geq 0\), with the spine being the path connecting the leaf 1 and the root, we similarly obtain a nested pair \((C_c^{(n)}(s), C_f^{(n)}(s))\), representing the sizes of spinal bushes and subtrees respectively. Then it is clear that \((C_c^{(n)}(s), s \geq 0)\) is a \(\text{PCRP}^{(\gamma)}(1-\alpha)\) and that \((C_f^{(n)}(s), s \geq 0)\) is a \(\text{PCRP}^{(\alpha)}(1-\alpha, \alpha)\) nested within \(C_c^{(n)}\), such that the size evolution of the subtrees in each bush gives a \(\text{PCRP}^{(\alpha)}(0, \alpha-\gamma)\).

PROPOSITION 5.20. For each \(n \geq 1\), let \(((C_c^{(n)}(t), C_f^{(n)}(t)), t \geq 0)\) be a pair of nested \(\text{PCRP}\)s defined as above, associated with a tree-valued process \((T^{(n)}(s), s \geq 0)\) starting from \(T_n\). As \(n \to \infty\), \((T^{(n)}(2nt), C_f^{(n)}(2nt)), t \geq 0)\) converges in distribution to a \(\text{cFSIP}^{(\alpha, 0, \alpha-\gamma)}(1-\alpha)\)-evolution starting from \((\gamma_c, \gamma_f)\), where \(\gamma_c \sim \text{PDIP}^{(\gamma)}(1-\alpha, \gamma)\) and \(\gamma_f \sim \text{Frag}^{(\alpha)}(0, \alpha-\gamma)(\gamma_c, \cdot)\)
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