ON THE ULTRAVIOLET DIVERGENCE IN QED

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Abstract

The well-known physical equivalence drawn from hole theory is applied in this article. The author suggests to replace, in the part of Feynman diagram which cannot be fixed by experiments, each fermion field operator, and hence fermion propagator, by pairs of equivalent fermion field operators and propagators. The formulation of this article thus yields additional terms which reveal characteristic effects that have not been explored previously; such characteristic effects lead to the appearance of logarithmic running terms and that finite radiative corrections are directly obtained in calculations.

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About the Author

Ji Sun (1921-1997), one of the influential physicists in China, has been dedicated in research and education of physics for half of a century. He graduated from the Department of Physics, Shanghai Jiaotong University, Shanghai, China in 1947. Since then, he had engaged in research and education in quantum machenics and particle physics in high educational institutions of China like Nankai University, Tsinghua University and Peking University. He was one of the founder and leading researcher of the Department of Technical Physics in Peking University. In the last ten years of his life, he had struggled with prostate cancer. The tremendous pain and even the high level paralysis caused by the disease did not prevent him from pursuing scientific truth until the last minute of his life. His dedication to physics and scientific world would be remembered by generations to come.
1 Introduction

Ultraviolet divergence is an important problem in QED. Investigation on this problem, to expose more characteristics of the divergence, may be helpful in the developments of the quantum field theory [1]. This article is an attempt on this problem. In this article, we start from the well-known and well-established physical equivalence drawn from Dirac’s hole theory [2], which gives also pairs of physically equivalent fermion propagators. In general, as measurements or observations fix only one of a physically equivalent pair, the physical equivalence from the hole theory thus gives nothing new, so such discussion is superfluous. However, there are certain parts of Feynman diagrams in QED processes in which the fermion fields or fermion propagators cannot be fixed experimentally (e.g. fermion propagators in a self energy loop). Both of the two equivalent propagators are equally probable to happen; thus the physical equivalence might lead to additional term or terms. It will be shown in section 3 that the new additional terms coming from physical equivalence really reveal new characteristic physical effects which are closely related with ultraviolet divergences and can yield finite radiative corrections and hence finite results in direct calculations of QED processes.

It might be quite interesting to note that one of the striking characteristics of the ultraviolet divergence, i.e. the appearance of, for example, logarithmic running of QED coupling constant with scale, which has been verified in precise electroweak measurements, is also given by the formulation of this article, as will be shown in Sec. 3; the logarithmic running terms really appear in this article; however, there are, meanwhile, really more such logarithmic running terms with different charges. Such logarithmic terms will combine to give finite radiative corrections.

As the first paper of this work, main focus is given to the fundamental assumption and formulation, together with their foundations. Here as illustration, one QED process, the vertex, is calculated; other QED processes
will be given in subsequent papers.

2 Fundamental Assumption and the Foundation of the Formulation

2.1 Preliminary discussions on hole theory

In order to propose the fundamental assumption, we need first to investigate the hole theory \[2\]. Before the investigation on hole theory, we review first the characteristics of tensors formed by fermion and boson field operators under the transformation \(x \rightarrow -x\), given by refs. \[4\] and \[5\]; their relevant results are rearranged in the form:

\[ \sum kU^+ = \sum U^-, \quad \sum kU^- = \sum U^+ \tag{1P} \]

There are altogether two possible substitutions keeping Eq. (1P) invariant:

\[ k_i \rightarrow -k_i(k_i = -i \frac{\partial}{\partial x_l}, l = 1, 2, 3, 4), U^+ \rightarrow U^+, U^- \rightarrow -U^- \tag{2P} \]

(4)

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(4)

As the propagation vector \(k_i\) belongs to \(U^-\) class, so only (2P) among the two substitutions (2P), (2'P) is consistent; which may also be considered as: when \(k_i \rightarrow -k_i\), if Eq. (1P) is required to remain invariant, then \(U^+ \rightarrow U^+, U^- \rightarrow -U^-\); and hence \(T \rightarrow T, S \rightarrow -S\). As the invariance of field equation (1P) is a fundamental requirement, so we may always write: If \(x_\mu \rightarrow -x_\mu\), then \(T \rightarrow T, S \rightarrow -S\), which is (1b). Similarly for (1a).
Under the transformation \( x_\mu \rightarrow -x_\mu \): 

\[
\text{fermions fields: } T \rightarrow -T, \ S \rightarrow S. \tag{1a}
\]

\[
\text{bosons fields: } T \rightarrow T, \ S \rightarrow -S. \tag{1b}
\]

\( T, S \) in (1a), ( in (1b) ) are tensors formed by fermion (boson) field operators. \( T \) represents tensors of even rank, including scalars, skew symmetric and symmetric tensors of second rank such as energy momentum tensor, etc. \( S \) represents tensors of odd rank, including vectors such as the charge current density vector, etc.

Now, as a preliminary to the fundamental assumption, we discuss the physical equivalences drawn from hole theory.

We begin our discussion by reexpressing the well known equivalences drawn from the hole theory:

\[
b_{-\vec{p}, r=3, 4} \equiv d_{\vec{p}, r=1, 2}; \quad b_{-\vec{p}, r=3, 4} \equiv d_{\vec{p}, r=1, 2};
\]

\[
b_{\vec{p}, r=1, 2} \equiv d_{\vec{p}, r=3, 4}; \quad b_{\vec{p}, r=1, 2} \equiv d_{\vec{p}, r=3, 4} \tag{2}
\]

The equivalences may also be written as

\[
\sum_{\vec{p}, r=3, 4} b_{-\vec{p}, r} u^{(r)}(-\vec{p}) = \sum_{\vec{p}, r=1, 2} d_{\vec{p}, r} v^{(r)}(\vec{p})
\]

etc. Here \( b (d) \) refer to \(-e (+e)\) fermions, \( \equiv \) denotes physical equivalence, the both sides of which express the same physical entity or process.

Now we combine the equivalences (2) with (1a). As from (1a) that for fermions, \( p(\vec{p}, E) \) reverses its sign on reversing \( x(\vec{x}, t) \), it is reasonable to associate \( p(\vec{p}, E) \) with \( x(\vec{x}, t) \), (and hence \(-p(-\vec{p}, -E)\) with \(-x(-\vec{x}, -t)\))

\(^1\)Fermion fields here mean free fermion fields; non-free fields may include a boson term, e.g. \( p + e/c A \). The fermion field \( p \) and the boson field term \( A \) obeys (1a) and (1b) respectively.
to agree with experimental facts. We obtain immediately a time dependent expression of the equivalences drawn from hole theory. The example \( b_{-\vec{p}, r=3, 4} \equiv d_{-\vec{p}, r=1, 2}^\dagger \) is now expressed as the equivalence between the processes (a) (b) in Fig. 1; namely, there are two equivalent processes: (a) an electron with charge \(-e\), momentum \(-\vec{p}\), energy \(-E\) propagating in \(-t\) sense, denoted here and here after as \((-e, -\vec{p}, -E, -t)\), is annihilated at \(-x(-\vec{x}, -t)\). (-\(x(-\vec{x}, -t)\) is just the space-time point \(x(\vec{x}, t)\) viewed in the frame \(-x\).) (b) an electron with \((+e, \vec{p}, E, t)\) is created at \(x(\vec{x}, t)\). (a) and (b) are two expressions of a same physical process. All other equivalences in (2) can be reexpressed similarly.

The equivalences drawn from hole theory also lead directly to pairs of equivalent electron propagators as depicted in Fig. 2, provided the equivalences hold at both \(x_1\) and \(x_2\). Thus there are two physically equivalent propagators in Fig. 2: Fig. 2 (a) is an electron with \(-e, \vec{p}, E\), propagating from \(x_1\) to \(x_2\) in \(+t\) sense, denoted as \((-e, \vec{p}, E, t)\); while Fig. 2 (b) is an electron with \(+e, -\vec{p}, -E\) propagating from \(x_2' = -x_2\) to \(x_1' = -x_1\), in \(-t\) sense., denoted as \((+e, -\vec{p}, -E, -t)\). Mathematical forms of Fig. 2 (a), (b) will be given below.

There are thus equivalences between \((-e, \vec{p}, E, t)\) and \((+e, -\vec{p}, -E, -t)\) processes. Relations between the equivalent fermion pairs (a), (b). both of Fig. 1 and of Fig. 2 are: (a) \(\rightarrow\) (b) and (b) \(\rightarrow\) (a) will occur under the simultaneous reflections \((x_\mu \rightarrow -x_\mu)\) and \((Q \rightarrow -Q)\). Note that \(p_\mu \rightarrow -p_\mu\) is just a consequence of \(x_\mu \rightarrow -x_\mu\) by (1a); and \((Q \rightarrow -Q)\) is a consequence of transpose, which interchanges the initial and final states (see below, 2.2.1 and 2).

The pair (a),(b) of Fig. 2, for example, may be considered as one fermion propagator, which is (a) or \((\vec{p}, E; \vec{x}, t)\), if it is regarded as \(-e\) propagator; while it is (b) or \((-\vec{p}, -E; -\vec{x}, -t)\), if it is regarded as \(+e\) propagator. Thus it follows that the axes of reference frames of physical equivalent \(-e\) and \(+e\) fermion fields (particle and antiparticle) are opposite
Figure 1: Equivalent pair of fermion field operators: (a), annihilation of $(-e, -\vec{p}, -E)$; (b), creation of $(+e, \vec{p}, E)$. $x$ ($x'$) being reference frame of coordinate of $-e$ (+$e$) electron; $(x' = -x)$.
Figure 2: Equivalent pair of fermion propagators: (a), $(-e, \vec{p}, E, \vec{x}, t)$; (b), $(+e, -\vec{p}, -E, -\vec{x}, -t)$. $x$ ($x'$) being reference frame of coordinate of $-e$ ($+e$) electron; ($x' = -x$).
in sense.

2.2 Physically equivalent pairs of fermion field operators

1) The definition

The well known equivalences drawn from Dirac’s hole theory can be formulated or generalized as: ”To each fermion field operator $\Psi(x)$ there is a $\Psi^R(x)$, which is physically equivalent to $\Psi(x)$, defined as

$$\Psi^R(x) = R\Psi(x)R^{-1} = \Psi^T(x)$$

(3)

where $R = r tr$, $r$ being the reflection $(x \rightarrow -x)$ operator, which bring $\Psi(x)$ into $\Psi'(x) = O\Psi(-x)$, $O$ being a matrix, $tr$ being the transpose operator, which bring $\Psi(x)$ into $\Psi^T(x)$, with the initial and final states interchanged. ”

2) Several notes:

i) The operator $O$ given here is a matrix, which is proved, under the requirement of invariance of Dirac equation, etc, as $O = \xi \gamma_5$. ($\xi = \pm 1, \pm i$) [5]. Thus,

$$\Psi^R(x) = \xi \gamma_5 \Psi^T(-x)$$

(3a)

which shows that the transformation of $R$ is just the same as that of joint operation $CPT$.

ii) Eq. (3a) shows that the charge of $\Psi^R(x)$ should be $-Q$ if that of $\Psi(x)$ is $Q$, since they are connected by the transformation $CPT$. Eq. (3) gives directly that $\Psi^R(x)$ is the fermion proceeding in the sense of time $-t$, if $\Psi(x)$ is that proceeding in the sense of time $+t$; the opposite sense of time gives the interchange of creation and annihilation operators. Thus, e.g., the creation of $-e$ charge of $\Psi(x)$ is turned into annihilation of $-e$, or creation of $+e$ charge of $\Psi^R(x)$; therefore $\Psi^R(x)$ and $\Psi(x)$ are opposite in charge. This is just the physical equivalence drawn from hole theory.

9
The transformation (3) or (3a) is called "strong reflection" by Pauli \[3\] (see also \[3\]).

iii) Thus, if we take \(\Psi(x)\) as a \(Q = -e\) fermion with \(+p(+E)\) proceeding in the sense of \(+t\), i.e., \(\Psi_{-e,p,+E,+t}\), \(\Psi^R(x)\) will then be \(\Psi^R_{+e,-p,-E,-t}\) (see also Eq. (2)). The physical equivalence between \(\Psi_{-e,p,+E,+t}\) and \(\Psi^R_{+e,-p,-E,-t}\) is just the equivalences in Eq. (2) (e.g. \(b^\dagger_{\mathbf{p},r=1,2} \equiv d_{-\mathbf{p},r=3,4}\) etc., \(b, d\) being operators of \(-e, +e\) respectively, remember that the transpose makes \(\Psi^R\) to proceed in the sense \(-t\)).

iv) The fact that \(-E, -t\) fermions are closely related to the \(+E, +t\) fermions with opposite charge was already given by Feynman \[3\]. However, the problem is treated here in a somewhat different point of view; in this article, all processes of fermion field operators and of products of them always proceed in the order of increasing time in its "own frame of reference" (namely the frame that the \(t\)-axis is directed in \(+t\) \((-t)\) sense for the fermion \(\Psi_{+t} (\Psi_{-t})\) ); \(-t\) arises only when a process is viewed in other frame of reference. This explains also why "a particle travelling from \(x_1\) to \(x_2\) is the same as an antiparticle travelling from \(x_2\) to \(x_1\)." (see also Fig.2).

The transpose operation in (3) and (3a) keeps that all processes proceed in the order of increasing of time.

3) It is significant to note that it has been proved that the transformation of \(R\) (Eq. (3)) leaves both field equations and commutation relations invariant \[7\]. Also it has been proved that, if the electromagnetic fields \(A_\mu\) is transformed by \(R\) simultaneously, i.e.,

\[
A^R_\mu(x) = RA_\mu(x)R^{-1} = A^T_\mu(x) \tag{3'}
\]

the transformation of \(R\) leaves both field equations and commutation relations of fermion and boson fields invariant. Moreover, the fundamental equations of quantum electrodynamics are invariant under the transformation of \(R\) \[7\].

Therefore the transformation of \(R\) is consistently defined for full interacting quantum field theory.
2.3 The Fundamental Assumption

1) Before proposing the fundamental assumption, it might be significant to notice that QED processes consist of two possible cases (or parts of Feynman diagram): (a). observable case (or parts), which can be directly observed or determined by measurements, (e.g. external fermion or boson lines). (b). unobservable case (or parts), which cannot be directly determined by measurements or observations (e.g. fermion propagators in the loops of self energy processes).

In case (a), the observable case, although there are equivalent expressions \( \Psi \) and \( \Psi^R \), a fermion field or a physical process is always uniquely expressed by only one expression fixed by measurement (e.g. \( \Psi \), if it is expressed with \( +E, +t \) which agrees with measurement condition). Everything is as usual, nothing new can be given by the physical equivalence defined in Eq. (3).

In case (b), the unobservable case, a fermion field operator or physical process cannot be determined by measurements or observations. *No measurement can determine which expression should be taken.* The two equivalent expressions, e.g. \( \Psi \) and \( \Psi^R \), are then equally probable.

It should be noted that: the case (a) is the ordinary case, in which each \( C, P, T \) reflection can be operated separately; while in case (b), although the transformation \( R \) satisfies CPT theorem, single reflections \( P, T, C \) cannot be performed separately in general, since no observation or measurement can be performed in case (b); hence a single reflection, e.g., \( T \), cannot be defined. Only case (a) has been considered nowadays; while case (b) has never been investigated so far (case (b) appears only in some intermediate steps.).

In case (b), the physically equivalent fermion field \( \Psi(x) \) and \( \Psi^R(x) \), which satisfy the same quantum field equation with e.m. interaction and commutation relation, are observed under different conditions (e.g. \( \Psi(x) \) is measured or viewed in the frame of reference \( +x(+\vec{e}, +t), Q = -e \); while
Ψ^R(x) is viewed in the frame \(-x(-\vec{x}, -t), Q = +e\). In each measurement, we can only measure either \(\Psi(x)\) or \(\Psi^R(x)\), but not both; so it is suitable to apply, as in quantum mechanics, the superposition principle. As no measurement can determine which of \(\Psi\) and \(\Psi^R\) is more probable than the other; so \(\Psi\) and \(\Psi^R\) should appear with equal probability; we arrive thus naturally at the fundamental assumption.

2) Fundamental assumption:

In the unobservable case (case (b)), a fermion field operator, which is \(\Psi(x)\) in the observable case (case (a)), is expressed as

\[
\frac{1}{\sqrt{2}} \left[ \Psi(x) + \Psi^R(x) \right] (4)
\]

where \(\Psi^R(x) = R\Psi(x)R^{-1}\) is defined in Eq. (3) and the statements in that paragraph. This is the fundamental assumption of this article.

It should be noted that, in the case (a), \(\Psi^R(x)\) is automatically turned into \(\Psi(x)\) which is expressed in the frame of reference \(+x(\vec{x}, +t)\). This is due to that \(\Psi(x)\), \(\Psi^R(x)\) are just two expressions of the same \(-e\) fermion field viewed from frame of reference \(+x(\vec{x}, +t), Q = -e\) and \(-x(-\vec{x}, -t), Q = +e\), respectively; in other words, \(\Psi^R(x)\) is just \(\Psi(x)\) viewed in the frame \(-x(-\vec{x}, -t), Q = +e\). The case (a) is, e.g., to observe \(\Psi^R(x)\), which should be transformed to the frame \(+x(\vec{x}, t)\), i.e. into \(\Psi(x)\), as

\[
R\Psi^R(x)R^{-1} |O\rangle = \Psi(x) |O\rangle (5)
\]

\(|O\rangle\) is the observable state which is fixed in the frame \(+x(\vec{x}, +t)\).

The transformation (5) between two expressions of the same field leaves the fermion field unchanged. Transformation (5) is just twice operation of transformation \(R\) on \(\Psi(x)\).

As \(\Psi(x)\) and \(\Psi^R(x)\) in expression (4) should be measured in two different measurements, if we pass from case (b) to case (a), the norm becomes, by (3) and (4),

\[
\frac{1}{2} \left[ |\Psi(x)|^2 + |\Psi^R(x)|^2 \right] = |\Psi(x)|^2.
\]

Thus, each fermion field operator is well connected between unobservable and observable parts of
a Feynman diagram; this guarantees the conservation of fermion number and everything of fermion fields.

In the case (b), as \(\Psi(x)\) and \(\Psi^R(x)\) are equally probable, the complete set of fermion field operators should be extended to that given by (A1) and (A1') in Appendix A.

### 2.4 The S-matrix formulation

The formulation of this article is the same as the S-matrix formulation of conventional QED, i.e.

\[
S = \sum_n S^{(n)}
\]

\[
S^{(n)} = \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} d^4x_n \int_{-\infty}^{\infty} d^4x_{n-1} \cdots \int_{-\infty}^{\infty} d^4x_1 
T \left[ H_I(x_n)H_I(x_{n-1}) \cdots H_I(x_1) \right] 
\]

except only that in the unobservable case (case (b)), \(\Psi(x)\) in each \(H_I(x) = -ie\nabla(x)\gamma_\mu \Psi(x)A_\mu(x)\) is replaced by \(\frac{1}{\sqrt{2}} [\Psi(x) + \Psi^R(x)]\) given in (4). \(A_\mu(x)\) multiplied with \(\Psi^R(x)\) is correspondingly transformed by \(R\), (3'), since \(\Psi\) interacts with \(A\) at same space-time points (while in case (a) nothing is changed).

In (3), \(H_I(x)\) denote interaction Hamiltonian densities. The chronological operator \(T\) is defined as usual. (i.e. all factors in the bracket after \(T\) are arranged so that the time of the factors are increasing from right to left; a factor \(\delta_p\) is multiplied where \(p\) is the number of permutations of fermion field operators to bring them into chronological order).

In case (b), due to the replacement of \(\Psi(x)\) by \(\frac{1}{\sqrt{2}} [\Psi(x) + \Psi^R(x)]\), each \(H_I(x)\) in Eq. (3) is then replaced by, \(\frac{1}{2} [H_I(x) + H^R_I(x)], H^R_I(x) = -ie\nabla^R(x)\gamma_\mu \Psi^R(x)A^R_\mu(x)\).

Actually \(H_I(x)\) becomes \(\frac{ie}{2} [\nabla(x)\gamma_\mu \Psi(x)A_\mu(x) + \nabla^R(x)\gamma_\mu \Psi^R(x)A^R_\mu(x)]\) on replacing \(\Psi(x)\) by \(\frac{1}{\sqrt{2}} [\Psi(x) + \Psi^R(x)]\); all other terms that might appear in
the direct replacement are excluded by the requirement that every term in $H_I(x)$ should be an electromagnetic interaction term which is a product of various fields at the same space-time point. Therefore, in case (b), $H_I(x)$ is replaced by $\frac{1}{\sqrt{2}} [H_I(x) + H^R_I(x)]$. Also, if we pass from case (b) to case (a), as each $\Psi^R(x)$ is automatically turned into $\Psi(x)$, $\frac{1}{\sqrt{2}} [H_I(x) + H^R_I(x)]$ is turned into $H_I(x)$ automatically.

The product of $\frac{1}{\sqrt{2}} [H_I(x) + H^R_I(x)]$ is arranged by the chronological operator $T$ into two products (the product of $H_I(x)$ and $H^R_I(x)$). It may be interesting to note that the order of factors in the product of $H_I(x)$ is automatically reversed.

### 2.5 Summarization of the formulation of this article

I). The formulation of this article differs from conventional QED only in one point, namely: *In the unobservable case, (or parts of a Feynman diagram; the case (b)), each fermion field operator $\Psi(x)$ is replaced by $\frac{1}{\sqrt{2}} [\Psi(x) + \Psi^R(x)]$, (the fundamental assumption).*

*Everything except this point is the same as conventional QED and hence need not be qualified or discussed.*

II). The critical point of the replacement stated in I) is the transformation of $R$ drawn from hole theory ([3] and [3']).

1). It has been proved that the fundamental equations of quantum electrodynamics are invariant under the transformation $R$ [4]. *Therefore the transformation of $R$ of this article leaves the fundamental equations of QED invariant, and hence is consistently defined for full interacting quantum field theory.*

2). *Causality is satisfied*, which can be seen directly by the chronological operator $T$ in [4]. Also, *all processes defined above proceed in the sense of increasing time; $-t$ arises only when a process is viewed in another frame of reference.* Furthermore, if only observable states defined above, such as $\Psi(x)$, are considered, there appears only $+t$. 14
3) *Unitarity is satisfied.*

i) The transformation $R$ defined in this article is itself unitary. Actually, the transformation $R$ satisfies $CPT$ theorem. Therefore the formulation of this article satisfies unitarity.

ii) The fermion(s) (and fermion propagator(s)) in each $S^{(n)}$ in (3) are the same as in $S^{(0)}$ of the corresponding conventional $S$-matrix theory, throughout each process except that $\Psi(x)$ is replaced by $\frac{1}{\sqrt{2}} \left[ \Psi(x) + \Psi^R(x) \right]$ in the observable case (case (b)).

a) In the expression $\frac{1}{\sqrt{2}} \left[ \Psi(x) + \Psi^R(x) \right]$ the sum of probabilities of $\Psi(x)$ and $\Psi^R(x)$, which cannot be measured simultaneously, is equal to unity.

b) The two expressions $\Psi(x)$ and $\Psi^R(x)$ of the same fermion field in Eq. (4) can only be measured in two different measurements; so if we pass from case (b) to case (a), $\Psi^R(x)$ should be transformed into $\Psi(x)$, as given by Eq. (5). So the norm of $\frac{1}{\sqrt{2}} \left[ \Psi(x) + \Psi^R(x) \right]$ is then $\frac{1}{2} \left[ |\Psi(x)|^2 + |\Psi^R(x)|^2 \right] = |\Psi(x)|^2$ which is just that of $\Psi(x)$ in the case (a). Therefore, each fermion field operator is well connected between unobservable and observable parts of a Feynman diagram. This guarantees the unitarity and conservation of quantum number and everything of fermion fields.

4) The transformation $R$ is also called ”strong reflection” by Pauli [3] (see also [5]), which satisfies $CPT$ theorem. However, it should be noted that separate $P$, $T$, ..., reflections, which is defined only in observable case (case (a)), is not defined in the unobservable case (case (b)).

### 3 Consequences

The fundamental assumption and the formulation given in Sec. 2 lead directly to the following consequences:

1. *Two equivalent pairs of fermion propagators.*

   We have already given in Sec. 2 that the occurrence of two physically
equivalent expressions of each fermion field operator leads to the occurrence of:

(1) two equivalent expressions of each fermion propagator. This is a direct consequence of the physical equivalence drawn from hole theory, as shown in Fig. 2. There are in general two equivalent propagators for each fermion propagator in Feynman diagram. In the case when a propagator can be determined by measurement (the observable case (a)), the two equivalent propagators will automatically become identical.

(2) two equivalent pairs of fermion propagators.

If we include the unobservable case (b), the appearance of pairs of physically equivalent fermion field operators extends the complete set of fermion field operators to (A1), (A1’) given in Appendix A. Such extended complete set of fermion field operators can constitute four fermion propagators, i.e. two equivalent pairs of fermion propagators. Namely, the propagator (A) formed by (A1a), (A1’a) and its equivalent (B) by (A1b), (A1’b); and similarly (C) and (D) formed respectively by (A1c), (A1’c) and (A1d), (A1’d), as given in Appendix A. The results of Appendix A, the propagators (A), (B), (C), (D), are written here:

\[
\frac{i}{(2\pi)^4} \int d^4 p \frac{-i\hat{p} + m}{p^2 + m^2} e^{ip(x_2-x_1)}
\] (A)

\[
\frac{i}{(2\pi)^4} \int d^4 p \frac{i\hat{p} - m}{p^2 + m^2} e^{ip(x_2-x_1)}
\] (B)

\[
\frac{i}{(2\pi)^4} \int d^4 p \frac{-i\hat{p} - m}{p^2 + m^2} e^{ip(x_2-x_1)}
\] (C)

\[
\frac{i}{(2\pi)^4} \int d^4 p \frac{i\hat{p} + m}{p^2 + m^2} e^{ip(x_2-x_1)}
\] (D)

2. Occurrence of characteristic effects in the calculations of typical divergence problems in QED.

From the statement in 1, in the unobservable case (b), the formulation Eq. (6) gives directly, for each fermion propagator in the Feynman diagram of a QED process:
(1) a sum of two propagators of an equivalent pair each multiplied with its interacting e.m. field operator.

(2) two equivalent pairs of fermion propagators [A], [B] and [X], [Y], each pair occurs with equal probability.

(1) and (2) give two significant characteristic effects which will be discussed in and after the illustrative examples of QED process given below. In order to illustrate the fundamental assumption and the formulation of the article, typical divergence processes have been calculated. As the calculations are somewhat lengthy, we choose only one of them, the vertex, here to show the ability of the formulation of this article; other processes will be given in subsequent papers.

**The vertex**

The $S$-matrix element of the vertex of an electron incoming at $x_1$ with $(-e, \vec{p}_1, E_1)$ outgoing at $x_2$ with $(-e, \vec{p}_2, E_2)$ and interacting with the external field $A^e_\mu(x_3) = A^e_\mu e^{iQx_3}$ at $x_3$, is $\langle f | S^{(3)} | i \rangle$. The $S^{(3)}$ in Eq. (3) has been rewritten, in Appendix B., as

$$S^{(3)} = -\frac{e^3}{8} \int_{-\infty}^{\infty} d^4 x_2 \int_{-\infty}^{\infty} d^4 x_3 \int_{-\infty}^{\infty} d^4 x_1 T \{(\Psi(x_2) \hat{A}(x_2) \Psi(x_2) \Psi(x_3) + \overline{\Psi}^R(x_2) \hat{A}^R(x_2) \Psi^R(x_3)) \hat{A}^e_\mu(x_3) \}
$$

The terms which are zero in the matrix element $\langle f | S^{(3)} | i \rangle$ have not been written, where $|i\rangle = b_{p_1, r_1}|0\rangle$, $|f\rangle = b_{p_2, r_2}|0\rangle$; the factor $\frac{1}{3!}$ is omitted since only one of the $3!$ possible figures is taken. The space-time variables in (7) are only $x_2, x_3, x_1$; $x'_3, x'_1$ in (7) are just $\pm x_3, \pm x_1$, the sign is $+$ or $-$ according to the fermion propagators taken (see Appendix A.2).

The external field $\hat{A}^e_\mu(x)$ in (7) is equally well be multiplied to the second square bracket; namely written as $\hat{A}^e_\mu(x'_3)$ instead of $\hat{A}^e_\mu(x_3)$.

As pairs of fermion propagators in the two square brackets in (7) should be taken over all possible pairs, (A),(B) and (C),(D), there are thus four cases:
I). (A), (B) in both square brackets.
II). (A), (B) in the first, (C), (D) in the second square bracket.
III). (C), (D) in the first, (A), (B) in the second square bracket.
IV). (C), (D) in both square brackets.

As it can be easily shown that (III), (IV) give the same value as (I), (II); so it is only necessary to calculate (I), (II) with the result times 2. So Eq. (7) can be written as

\[ S^{(3)} = -\frac{e^3}{4} \int_{-\infty}^{\infty} d^4x_2 \int_{-\infty}^{\infty} d^4x_3 \int_{-\infty}^{\infty} d^4x_1 T\{\{I\} + \{II\}\} \quad (8) \]

Where

\[ I = \left\{ \left[ (\Psi(x_2)\hat{A}^c(x_2)A_{x_2,x_3} + \overline{\Psi}^R(x_2)\hat{A}^{Rc}(x_2)(B)_{x_2,x_3}) \right] \hat{A}_\mu(x_3) \right. \]

\[ \left. \left[ ((A)_{x_3,x_1}^c \hat{A}(x_1')\Psi(x_1') + (B)_{x_3,x_1}^c \hat{A}^{Rc}(x_1')\Psi^R(x_1')) \right] \right\} \quad (8'1a) \]

\[ II = \left\{ \left[ (\Psi(x_2)\hat{A}^c(x_2)A_{x_2,x_3} + \overline{\Psi}^R(x_2)\hat{A}^{Rc}(x_2)(B)_{x_2,x_3}) \right] \hat{A}_\mu(x_3) \right. \]

\[ \left. \left[ (((C)_{x_3,x_1}^c \hat{A}(x_1')\Psi(x_1') + (D)_{x_3,x_1}^c \hat{A}^{Rc}(x_1')\Psi^R(x_1')) \right] \right\} \quad (8'2a) \]

\( (A)_{x_2,x_3} \) being \( \Psi(x_2)\overline{\Psi}(x_3) \) with propagator (A); similarly for propagators (B), (C), (D).

I and II are equally written as:

\[ I = \left\{ \left[ (\Psi(x_2)\hat{A}^c(x_2)A_{x_2,x_3} + \overline{\Psi}^R(x_2)\hat{A}^{Rc}(x_2)(B)_{x_2,x_3}) \right] \overline{\hat{A}}_\mu(x_3') \right. \]

\[ \left. \left[ ((A)_{x_3,x_1}^c \hat{A}(x_1')\Psi(x_1') + (B)_{x_3,x_1}^c \hat{A}^{Rc}(x_1')\Psi^R(x_1')) \right] \right\} \quad (8'1b) \]

\[ II = \left\{ \left[ (\Psi(x_2)\hat{A}^c(x_2)A_{x_2,x_3} + \overline{\Psi}^R(x_2)\hat{A}^{Rc}(x_2)(B)_{x_2,x_3}) \right] \overline{\hat{A}}_\mu(x_3') \right. \]

\[ \left. \left[ (((C)_{x_3,x_1}^c \hat{A}(x_1')\Psi(x_1') + (D)_{x_3,x_1}^c \hat{A}^{Rc}(x_1')\Psi^R(x_1')) \right] \right\} \quad (8'2b) \]
III and IV are defined similarly, hence,

\[ \langle f \mid S^{(3)} \mid i \rangle = \frac{-ie^3}{4} \pi^{(r_2)}(\vec{p}_2) \int \frac{d^4k}{k^2 + \lambda^2} \gamma_\nu \]

\[ \left\{ \frac{-i(\hat{p}_2 - \hat{k}_2) + m}{(p_2 - k)^2 + m^2} \gamma_\mu \frac{-i(\hat{p}_1 - \hat{k}_2) + m}{(p_1 - k)^2 + m^2} \right\} \delta^4(p_1 - p_2 + q) \]

\[ + \left\{ \frac{i(\hat{p}_2 + \hat{k}_2) - m}{(p_2 + k)^2 + m^2} \gamma_\mu \frac{i(\hat{p}_1 + \hat{k}_2) - m}{(p_1 + k)^2 + m^2} \right\} \delta^4(p_1 - p_2) \]

\[ \gamma_\nu a_\mu(q)u^{(r_1)}(\vec{p}_1) \]

(8)

The factor \( \delta^4(p_1 - p_2 + q) \) in the first term of (8), is different from the corresponding factor \( \delta^4(p_1 - p_2) \) in the second.

In the first term of (8), which is the integral of I, (8'1a) or (8'1b), in which the propagators in both square brackets are all (A), (B); so \( x'_3 = x_3 \). Hence (8'1a) = (8'1b) as required.

While the second term of (8), which is the integral of (II), (8'2a) or (8'2b), in which the propagators in the first square bracket are (A), (B); while those in the second square bracket are (C), (D). So \( x'_3 = -x_3 \) (see Appendix A). However, it is required that (8'2a) has to be equal to (8'2b) for all values of \( x_3 \); namely it is required that \( \hat{A}_\mu(x_3) = \hat{A}_\mu(x'_3) \), i.e., it is required that \( e^{iqx_3} = e^{-iqx_3} \); as \( x_3 \) runs over all space-time points, so it is required \( q = 0 \).}

\[ 19 \]

\[ ^\dagger \]Actually, \( q = 0 \) is a significant physical condition; since as boson fields behaves differently from fermion fields under the reflection \( x \rightarrow -x \) as given by (1a), (1b), the factor \( \cos x \) of a fermion propagator with \( \mu' = p - e/c A \) is unchanged under \( x \rightarrow -x \) if \( A = 0 \) is assumed, since \( e^{i(-b)(-x)} = e^{inx} \) by (1a).
The calculation of (8) is given in Appendix B. The final is

$$A^{(2)}_{\mu f}(p_1, p_2; q) = -\frac{\alpha}{\pi} \left\{ \left( \frac{q^2}{3m^2} \ln \frac{m}{\lambda} - \frac{q^2}{8m^2} \right) \gamma_\mu - \frac{i}{8m} (\gamma_\mu \hat{q} - \hat{q} \gamma_\mu) \right\}. \quad (9)$$

to order $q^2$, which is the same as that obtained in the renormalization

**About the characteristic effects**

The two terms in the square bracket of (8) means that we should take the average of I and II which appear with equal probability. In order to take an insight into Eq. (8) consider for a moment the fictitious processes in which there is only I or II alone, i.e.

$$S^{(3)} = -\frac{e^3}{4} \int_{-\infty}^{\infty} d^4 x_2 \int_{-\infty}^{\infty} d^4 x_3 \int_{-\infty}^{\infty} d^4 x_1 T\{I\}$$

$$S^{(3)} = -\frac{e^3}{4} \int_{-\infty}^{\infty} d^4 x_2 \int_{-\infty}^{\infty} d^4 x_3 \int_{-\infty}^{\infty} d^4 x_1 T\{II\}$$

We have, from (A4), (A4I), (A4II) in Appendix B,

$$(\langle f | S^{(3)} | i \rangle)_I = -\frac{ie^3}{2} \pi^{(r_2)}(\vec{p}_2) \int d^4 k \int_0^1 dx \int_0^x dy \left\{ \frac{(2 - 2x - x^2)m^2 - \frac{k^2}{2} + (1 - x + y)(1 - y)q^2}{k^2 + m^2 x^2 + q^2 y(x - y) + \lambda^2(1 - x)} \gamma_\mu \right\} a_\mu(q)u^{(r_1)}(p_1)$$

$$(\langle f | S^{(3)} | i \rangle)_I = -\frac{ie^3}{2} \pi^{(r_2)}(\vec{p}_2) \int d^4 k \int_0^1 dx \int_0^x dy \left\{ \frac{\{-(2 - 2x + x^2)m^2 + \frac{k^2}{2}\} \gamma_\mu}{k^2 + m^2 x^2 + \lambda^2(1 - x)} \right\} a_\mu(q)u^{(r_1)}(p_1)$$

We see at once that there are logarithmic running terms in both fictitious processes $(\langle f | S^{(3)} | i \rangle)_I$ and $(\langle f | S^{(3)} | i \rangle)_II$ (due to the $\pm \frac{k^2}{2}$ terms in the numerators of both integrands.). This shows why the conventional
formulation of QED, which includes \((\langle f | S^{(3)} | i \rangle)_I\) only, gives logarithmic
running terms. While in the formulation of this article, although there
appear also logarithmic terms, the \(\pm \frac{k^2}{2}\) terms in the numerators of \((A_{4I})\),
and \((A_{4II})\) concealed each other before integration over \(d^4k\); thus there is
no divergence in all processes of this article; the finite radiative corrections
are obtained directly.

We see that the first pair of propagators in \(\{I\}\) and \(\{II\}\) are the
same, they differ only in the second pair, (A), (B) in \(\{I\}\), while (C),
(D) in \(\{II\}\). As (A),(B) and (C),(D) are propagators \((-e, \vec{p}, E, \vec{x}, t)\),
\((+e, -\vec{p}, -E, -\vec{x}, -t)\) and \((-e, -\vec{p}, -E, -\vec{x}, -t)\), \((+e, \vec{p}, E, \vec{x}, t)\) respectively,
so, e.g., the propagators \((\vec{p}, E, \vec{x}, t)\) in (A),(B) and in (C),(D) are respec-
tively (A) and (D) which are opposite in charge. Similarly, the two propa-
gators \((-\vec{p}, -E, -\vec{x}, -t)\) are also opposite in charge. So the charge content
of the pair (A),(B) is different from that of (C),(D).

Therefore the terms \(\{I\}\) and \(\{II\}\) in (8) are really coexisting physical
processes with different charges. So it is not surprising that we can measure
logarithmic running of , e.g., coupling constant , while the two coexisting
processes \(\{I\}\) and \(\{II\}\) have the effect of concellation with each other at
very large \(k\) or very small distance (see the \(\pm \frac{k^2}{2}\) terms in the numerators
of \((A_{4I})\) and \((A_{4II})\)) which renders the final results finite.

So far we have discussed, in the process of the vertex, the characteris-
tic effect coming from the averaging of coexisting \(\{I\}\) and \(\{II\}\) processes.
There is another characteristic effect coming from replacing a fermion prop-
gagator interacting with e.m. (photon) field by two propagators of an equi-
vant pair each multiplied with its interacting e.m. field operator. This
effect will cancel the logarithmic divergence in the self energy of free elec-
trons; while for non-free electrons, we obtain, on combining this effect with
the averaging of coexisting processes \(\{I\}\) and \(\{II\}\) discussed above, finite
radiative corrections. These will be given by the processes in subsequent
papers.
4 Conclusion and Discussions

1. Based on the fundamental assumption, \( \Psi(x) \) is replaced by

\[
\frac{1}{\sqrt{2}} \left[ \Psi(x) + \Psi^R(x) \right]
\]

in case (b), the formulation of this article leads, in the case (b), to that:

1) every propagator in Feynman diagram is replaced by two propagators of an equivalent pair, each multiplied with its interacting e.m. field operator.

2) there occur two pairs of fermion propagators, (A), (B) and (C), (D), each with equal probability.

Each of 1), 2) gives a characteristic effect which is closely related to the ultraviolet divergence.

2. As given above, we have obtained, on one hand, the appearance of logarithmic running terms; while on the other, the finite results of calculations. This is due to that, there appear, in this article, logarithmic running terms with different charges (see the statements near the end of Sec. 3), so the cancellation of logarithmic terms at very large values of \( k \), or very small distance, is a very natural physical effect. The cancellation of logarithmic terms takes place before the final integration over \( k \); so no ultraviolet divergences appear finally in our calculations; finite radiative corrections are obtained in direct calculations. It might be possible that, according to the author, the appearance of ultraviolet divergence might be due to the negligence of the case (b).

3. Since the calculations of each process are somewhat lengthy, only one of them can be given here. Actually the author have calculated the three typical divergence problems, i.e., the self energy of electron, the self energy of photon (vavuum polarization), and the vertex. Preliminary results show that all the radiative corrections are the same as those obtained in conventional renormalization treatments. Such works will be given in subsequent papers.
The three typical divergences are the source of all ultraviolet divergences in QED; so it might be expected that in the treatment of this article, the results of all higher order diagrams may be all the same as those given by the conventional renormalization treatment. However, such works, which are somewhat lengthy, can only be given in a series of subsequent papers.

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A Appendix A

1. The complete set of fermion field operators

Since the unobservable case (b) is included in this article, the expansion of $\Psi$ and of $\overline{\Psi}$ should be written, to include explicitly physically equivalent fermion field operators, as

$$\Psi = \frac{1}{\sqrt{2}} (\Psi_{-e,p,x} + \Psi_{+e,-p,-x} + \Psi_{-e,-p,-x} + \Psi_{+e,p,x}) \quad (A1)$$

where,

$$\Psi_{-e,p,x} = \sum_{p,r=1,2} b_r (p) u^{(r)} (p) e^{ipx} \quad (A1a)$$

$$\Psi_{+e,-p,-x} = \sum_{p,r=3,4} d_r^+ (-p) v^{(r)} (-p) e^{ipx} \quad (A1b)$$

$$\Psi_{-e,-p,-x} = \sum_{p,r=3,4} b_r (-p) u^{(r)} (-p) e^{ipx} \quad (A1c)$$

$$\Psi_{+e,p,x} = \sum_{p,r=1,2} d_r^+ (p) v^{(r)} (p) e^{ipx} \quad (A1d)$$
\[ \Psi = \frac{1}{\sqrt{2}}(\Psi_{-e,p,x} + \Psi_{+e,-p,-x} + \Psi_{-e,-p,-x} + \Psi_{+e,p,x}) \quad (A1') \]

where,

\[ \Psi_{-e,p,x} = \sum_{p,r=1,2} b_r^+(\vec{p})\varphi^{(r)}(\vec{p})e^{-ipx} \quad (A1'a) \]

\[ \Psi_{+e,-p,-x} = \sum_{p,r=3,4} d_r(-\vec{p})\varphi^{(r)}(-\vec{p})e^{-ipx} \quad (A1'b) \]

\[ \Psi_{-e,-p,-x} = \sum_{p,r=3,4} b_r^+(-\vec{p})\varphi^{(r)}(-\vec{p})e^{-ipx} \quad (A1'c) \]

\[ \Psi_{+e,p,x} = \sum_{p,r=1,2} d_r(\vec{p})\varphi^{(r)}(\vec{p})e^{-ipx} \quad (A1'd) \]

2. Fermion propagators.
There can be formed, from the complete set of fermion field operators, (A1) and (A1'), two equivalent pairs of fermion propagators, i.e. the propagators formed by (A1a), (A1’a) and by (A1c), (A1’c) together with their respective equivalents formed by (A1b), (A1’b) and by (A1d), (A1’d). Now we derive the mathematical forms of the four fermion propagators:

We derive first the mathematical form of the fermion propagator formed by (A1a), (A1’a) and its equivalent (by (A1b), (A1’b)), i.e. Fig. 2 (a) and (b). The propagator (a), the electron propagator with \((-e, \vec{p}, E, \vec{x}, t)\) has already been given in the usual way as the matrix element of \(T(\Psi(x_2)\Psi(x_1))\) between vacuum states, where \(\Psi(x) = \sum_{p,r=1,2} b_{p,r} \varphi^{(r)}(\vec{p})\exp(ipx)\), \((x_{02} > x_{01})\), as

\[ 0 \langle | T(\Psi(x_2)\Psi(x_1)) | \rangle_0 = \frac{1}{(2\pi)^3} \int d^3p \frac{-i\hat{p} + m}{2E} e^{ip(x_2-x_1)} \quad (A2a) \]

(A2a) is the propagator \((-e, \vec{p}, E, \vec{x}, t)\) in its own frame of reference, called the frame of \(-e\); it is also the propagator \((+e, \vec{p}, E, \vec{x}, t)\) in the frame of \(+e\), since (A2a) is independent of sign of \(Q\). To obtain the propagator
(b) of Fig. 2, we may write first the propagator \((+e, -\vec{p}, -E, -\vec{x}, -t)\) in the frame of \(+e\) by operating \(r (x_\mu \to -x_\mu)\) on (A2a), as

\[
0 \langle | T \left( \Psi (x_2)^R \Psi (x_1) \right) | \rangle_0 = -0 \langle | T \left( \Psi (x_1)^R \Psi (x_2) \right) | \rangle_0 = \frac{1}{(2\pi)^3} \int d^3p \frac{-i\hat{p} - m}{2E} e^{ip(x_2 - x_1)} \quad (A2b')
\]

Mathematical forms of an equivalent pair of propagators (i.e. an equivalent pair of expressions of one propagator) should be written in same frame of reference since they have to be calculated together. So we have to write the propagator (b) as \((+e, -\vec{p}, -E, -\vec{x}, -t)\) in the same frame as (a), i.e. in the frame of \(-e\). As the axes of frames of reference of physically equivalent \(-e\) and \(+e\) fermion fields are opposite in sense, as stated at the end of Sec. 2.1, we have only to replace \(x\) in (A2b') by \(-x\), and hence \(p\) by \(-\hat{p}\) according to (1a), which leads to the required propagator \((+e, -\hat{p}, -E, -\vec{x}, -t)\) in the frame of \(-e\), as

\[
0 \langle | T \left( \Psi (x_2)^R \Psi (x_1) \right) | \rangle_0 = \frac{1}{(2\pi)^3} \int d^3p \frac{-i\hat{p} + m}{2E} e^{ip(x_2 - x_1)} \quad (A2b)
\]

Note that for a fermion field operator, e.g., the equivalence field \(\Psi^R(x)\) of \(\Psi(x)\) is just \(\Psi(x)\) when viewed from \(-e\) frame \((+x, +t)\); however, the equivalent propagator (A2b') is not exactly (A2a) when viewed in the frame of \(-e\); but differ in a minus sign. This is due to that a propagator consists of two operators, one production and one annihilation operator. If the frame of reference is changed, the order of the two operators should also be changed, and hence a minus sign is brought in. (Notice that there are now two operations, the transpose operation and the operation of chronological operator \(T\).) The pair of equivalent propagators (A2a), (A2b) should, as in usual field theory, be written in four dimensional form, denoted as (A),(B),

\[
\frac{i}{(2\pi)^4} \int d^4p \frac{-i\hat{p} + m}{p^2 + m^2} e^{ip(x_2 - x_1)}
\]
We proceed next to propagator formed by (A1c),(A1’c) and its equivalent (by (A1d),(A1’d)). This equivalent pair of fermion propagators, called (C),(D), can be more quickly obtained from (A),(B) through $x_\mu \to -x_\mu$ as:

$$\frac{i}{(2\pi)^4} \int d^4p \frac{i\hat{p} - m}{p^2 + m^2} e^{i\hat{p}(x_2 - x_1)} \quad (C)$$

$$\frac{i}{(2\pi)^4} \int d^4p \frac{-i\hat{p} + m}{p^2 + m^2} e^{i\hat{p}(x_2 - x_1)} \quad (D)$$

Each of (A),(B),(C),(D) includes, in the usual way, two three-dimensional propagators; e.g. (A) includes: $(-e, \vec{p}, E)$ for $x_{02} > x_{01}$, and $(+e, \vec{p}, E)$ for $x_{01} > x_{02}$. Similarly for (B),(C),(D). The space-time variable of propagator (A2a) is chosen as $x$, which is also that of (A). In this way the space-time variables of (A),(B),(C),(D) are $x$, $-x$, $-x$, $x$ respectively.

(C),(D) should be considered as independent propagators, although they could be obtained from (A),(B) through $(x_\mu \to -x_\mu)$, since we are not permitted to make the transformation $(x_\mu \to -x_\mu)$, e.g., on (A) alone, which is only a part of Feynman diagram of a whole QED process. There are altogether two equivalent pairs of propagators (A),(B) and (C),(D). The four propagators (A),(B),(C),(D) given here are all written in the frame of $-e$.

**B Appendix B**

1. The step from $S^{(3)}$ in (3) to (4).

We write first, by (3)

$$S^{(3)} = \frac{-e^3}{2\sqrt{2}} \int_{-\infty}^{\infty} d^4x_2 \int_{-\infty}^{\infty} d^4x_3 \int_{-\infty}^{\infty} d^4x_1 T\{ \left[ \bar{\Psi}(x_2)\gamma_\mu \Psi(x_2)A_\mu(x_2) + \bar{\Psi}^R(x_2)\gamma_\mu \Psi^R(x_2)A_\mu^R(x_2) \right] \}$$
\[ \left[ \bar{\Psi} (x_3) \gamma_\mu \Psi^R (x_3) A^e_\mu (x_3) + \bar{\Psi} (x_3) \gamma_\mu \Psi^R (x_3) A^{eR}_\mu (x_3) \right] \\
\left[ \bar{\Psi}^R (x_1) \gamma_\mu \Psi (x_1) A^e_\mu (x_1) + \bar{\Psi}^R (x_1) \gamma_\mu \Psi^R (x_1) A^{eR}_\mu (x_1) \right] \} (A3) \]

The external field \( A^e_\mu (x) \) is a classical e.m. field, which can be observed or measured directly; there are no physical equivalence, no creation or annihilation of photons; so \( A^{eR}_\mu (x) \) is just \( A^e_\mu (x) \), and hence can be extracted from the square bracket. \((A3)\) is then written as \((\bar{A})\). (If \(CPT\) operations are applied, it can be seen that the twice minus sign cancelled out, leaving \( A^{eR}_\mu (x) = A^e_\mu (x) \).)

We have written the space-time variable as in the first square bracket in \((8)\), while as \(x\) in the second. This is due to that the pair of propagator in each square bracket may be either taken as \((A),(B)\) or as \((C),(D)\). For example, if \((A),(B)\) are in both square brackets (case I), \(x' = x\); if \((A),(B)\) in the first while \((C),(D)\) in the second (case II) \(x' = -x\).

2. The steps from \((8)\) to \((9)\).

Eq. \((8)\) is

\[
\langle f | S^{(3)} | i \rangle = \frac{-ie^3}{4} \pi^{(r2)}(\vec{p}_2) \int \frac{d^4k}{k^2 + \lambda^2} \gamma_\nu \\
\left[ \frac{-i(\hat{p}_2 - \hat{k}_2) + m}{(p_2 - k)^2 + m^2} \gamma_\mu \frac{-i(\hat{p}_1 - \hat{k}_2) + m}{(p_1 - k)^2 + m^2} \\
+ \frac{i(\hat{p}_2 + \hat{k}_2) - m}{(p_2 + k)^2 + m^2} \gamma_\mu \frac{i(\hat{p}_1 + \hat{k}_2) - m}{(p_1 + k)^2 + m^2} \right] \delta^4(p_1 - p_2 + q) \\
+ \left\{ \frac{-i(\hat{p}_2 - \hat{k}_2) + m}{(p_2 - k)^2 + m^2} \gamma_\mu \frac{i(\hat{p}_1 - \hat{k}_2) + m}{(p_1 - k)^2 + m^2} \\
+ \frac{i(\hat{p}_2 + \hat{k}_2) - m}{(p_2 + k)^2 + m^2} \gamma_\mu \frac{-i(\hat{p}_1 + \hat{k}_2) - m}{(p_1 + k)^2 + m^2} \right\} \delta^4(p_1 - p_2) \gamma_\nu a_\mu (q) u^{(r_1)}(\vec{p}_1) \]
\]

(8)

Start now from \((8)\). We need only to calculate the first terms in each curly bracket in \((8)\), called \((8)_a\); the second terms in each curly bracket can be obtained from \((8)_a\) through \(k \rightarrow -k\). It is readily proved that the
second terms give the same results as the first, i.e. \((8)_a\); so the result of \((8)\) is twice that of \((8)_a\).

As the numerator and the denominator of the first term in the first curly bracket

\[
\frac{-i(p_2 - \hat{k}_2) + m}{(p_2 - k)^2 + m^2} = \frac{-i(p_1 - \hat{k}_2) + m}{(p_1 - k)^2 + m^2} \gamma^{\mu} \delta^4(p_1 - p_2 + q)
\]

are respectively \(\{2m^2 \gamma_\mu - \hat{q} \gamma_\mu \hat{q} + \hat{k} \gamma_\mu \hat{k} - \hat{k} \gamma_\mu \hat{q} + 2imk_\mu\}\) and

\[
\frac{1}{(k^2 + \lambda^2)(k^2 - 2p_1k)(k^2 - 2p_2k)}
\]

Similarly, those in the second curly bracket are respectively \(\{-2m^2 \gamma_\mu - \hat{k} \gamma_\mu \hat{k} + im(\gamma_{m\mu} \hat{k} - \hat{k} \gamma_{m\mu}\}\) and \((k^2 + \lambda^2)(p_2 - k)^2 + m^2\)(\(p_1 - k)^2 + m^2\).

We may write \((8)_a\) as

\[
\langle f | S^{(3)} | i \rangle_a = \frac{-ie^3}{2} d^{(r_2)}(\vec{p}_2) \int d^4k [(I)_a + (II)_a] a_\mu(q) u^{(r_1)}(p_1)
\]

where

\[
(I)_a = \frac{2(2m^2 \gamma_\mu - \hat{q} \gamma_\mu \hat{q} + \hat{k} \gamma_\mu \hat{k} - \hat{k} \gamma_\mu \hat{q} + 2imk_\mu)}{(k^2 + \lambda^2)(p_2 - k)^2 + m^2}(p_1 - p_2 + q) \delta^4(p_1 - p_2 + q)
\]

\[
= 4 \int_0^1 dx \int_0^1 dy \frac{(2 - 2x - x^2)m^2 - k^2}{k^2 + m^2 + q^2} \gamma_\mu + im\gamma_\mu(1 + x)q_\mu + mx(1 - x)\sigma_{\mu\nu} q_\nu
\]

by usual calculation; here, \(\sigma_{\mu\nu}\)

\[
(II)_a = \frac{2(-2m^2 \gamma_\mu - \hat{k} \gamma_\mu \hat{k} + im(\gamma_{m\mu} \hat{k} - \hat{k} \gamma_{m\mu}\))}{(k^2 + \lambda^2)(p_2 - k)^2 + m^2}(p_1 - p_2)
\]

\[
= 4 \int_0^1 dx \int_0^1 dy \frac{(2 - 2x - x^2)m^2 + k^2}{(k^2 + m^2 + x^2 + \lambda^2)(1 - x)^3} \gamma_\mu
\]

Here, in the case (II), \(x_1' = -x_1\), the corresponding \(p_1\) should be turned into \(-p_1\), (see (14)), Dirac Eq. for \(-p(-p', -E)\) case should be \((-i\hat{p} + m)u = 0\) in place of \((i\hat{p} + m)u = 0\) for \(p(\vec{p}, E)\) case.
\[
\begin{align*}
\langle 8 \rangle_a &= -ie^3\bar{u}(r_2)(\vec{p}_2) \int d^4k \int_0^1 dx \int_0^1 dy \\
&\left\{ \gamma_\mu(1-x+y)(1-y)q^2 + mx(1-x)\sigma_{\mu\nu}q_\nu \right\} + \gamma_\mu[(2-2x-x^2)m^2 \frac{k^2}{2}] \\
&\left[ \frac{1}{(k^2+l^2)^3} - \frac{1}{(k^2+l_0^2)^3} \right] - \frac{2m^2x^2}{(k^2+l^2)^3} \gamma_\mu \right\} 
\end{align*}
\]

where, \( l^2 = m^2x^2 + q^2y(x-y) + \lambda^2(1-x) \), \( l_0^2 = m^2x^2 + \lambda^2(1-x) \), the Lorentz condition \( q_\mu a_\mu(q) = 0 \) has been used.

Here in \((A6)\), there are only the first terms in each curly bracket in \((8)\), so for the whole \((8)\), we have

\[
\langle 8 \rangle = -2ie^3\bar{u}(r_2)(\vec{p}_2) \int d^4k \int_0^1 dx \int_0^1 dy \\
\left\{ \gamma_\mu(1-x+y)(1-y)q^2 + mx(1-x)\sigma_{\mu\nu}q_\nu \right\} + \gamma_\mu[(2-2x-x^2)m^2 \frac{k^2}{2}] \\
\left[ \frac{1}{(k^2+l^2)^3} - \frac{1}{(k^2+l_0^2)^3} \right] - \frac{2m^2x^2}{(k^2+l^2)^3} \gamma_\mu \right\} 
\]

The last term is just \(-\pi^2i\gamma_\mu\), a first order vertex. The other terms in \((A7)\), denoted as \((8')\), can be, in the case of \( q \ll m \), expanded in power series in \( q \). We have then, to the order of \( q^2 \),

\[
\langle 8 \rangle \approx \pi e^3 \int_0^1 xdx \left\{ \frac{q^2x^2\gamma_\mu}{6l_0^2} - \frac{q^2m^2x^2}{6l_0^2} (2-2x-x^2)\gamma_\mu \\
+ \frac{q^2}{l_0^2}(1-x+x^2)\gamma_\mu + \frac{m}{l_0^2}x(1-x)\sigma_{\mu\nu}q_\nu \right\} 
\]

On using the well known integrals \( \int_0^1 \frac{dx}{l^2}, \int_0^1 \frac{x dx}{l^2}, \int_0^1 \frac{x^2 dx}{l^2}, \int_0^1 \frac{dx}{l^2}, \int_0^1 \frac{xdx}{l^2} \), \( \int_0^1 \frac{x^{n+2} dx}{l^2} \) and taking \( \lambda \to 0 \), we have, by ordinary calculations,

\[
\Lambda_{\mu\nu}^{(2)}(p_1, p_2; q) = -\frac{\alpha}{\pi} \left\{ \left( \frac{q^2}{3m^2} \ln \frac{m}{\lambda} - \frac{q^2}{8m^2} \right) \gamma_\mu - \frac{i}{8m} (\gamma_\mu \hat{q} - \hat{q} \gamma_\mu) \right\} 
\]

to the order \( q^2 \), which is just Eq. \((3)\).
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