The “optical bars”/“optical lever” topologies of gravitational-wave antennae allow to obtain sensitivity better that the Standard Quantum Limit while keeping the optical pumping energy in the antenna relatively low. Element of the crucial importance in these schemes is the local meter which monitors the local test mirror position. Using cross-correlation of this meter back-action noise and its measurement noise it is possible to further decrease the optical pumping energy. In this case the pumping energy minimal value will be limited by the internal losses in the antenna only. Estimates show that for values of parameters available for contemporary and planned gravitational-wave antennae, sensitivity about one order of magnitude better than the Standard Quantum Limit can be obtained using the pumping energy about one order of magnitude smaller energy than is required in the traditional topology in order to obtain the the Standard Quantum Limit level of sensitivity.

I. INTRODUCTION

First generation large-scale laser interferometric gravitational-wave antennae [1, 2] are being placed into operation nowadays [3]. The second generation of laser gravitational-wave antennae development is under way concurrently [4]. Sensitivity of these second generation antennae will be close to the Standard Quantum Limit (SQL) [5] that is characteristic sensitivity level where the measurement noise (i.e. the shot noise in the laser interferometric devices) and the back-action noise (i.e. the radiation pressure noise) contribute equal parts to the measurement error:

\[ S_{\text{SQL}}(\Omega) = \frac{4\hbar}{M\Omega^2 L^2}. \] (1)

Here \( S_{\text{SQL}}(\Omega) \) is the double-sided spectral density of the equivalent noise for the dimensionless amplitude of the metrics perturbation \( h(t) \), \( \Omega \) is the mean frequency of the gravitational-wave signal, \( M \) is the mass of the interferometer mirrors, and \( L \) is the length of the interferometer arms.

Further improvement of the sensitivity will require to use the Quantum Non-Demolition (QND) measurement methods [6, 7, 8] which allow to eliminate the part produced by the back-action noise from the meter output. Several possible design of QND laser gravitational-wave antennae have been proposed already. They can be divided into three main groups.

The first group is based on the fact that the value of the SQL depends on the nature of the test object. In particular, a harmonic oscillator provides better sensitivity in the narrow band close to its resonance frequency than a free mass does, even if the same SQL-limited
meter is used in both cases [9]. It was shown in articles [10, 11] that in the signal-recycling configuration of the interferometric gravitational-wave antennae an optical rigidity can be created rather easily that will turn test masses into mechanical oscillators. Moreover, this optical rigidity has specific spectral dependence which allows to obtain sensitivity better than the SQL for both free mass and ordinary harmonic oscillator. Using this method it is possible to obtain sensitivity a few times better than the SQL for a free mass in a relatively wide band [10, 11] or “dive” deep below the SQL in a narrow band [12]. It is necessary to note that both regimes require about the same optical pumping energy as standard SQL-limited schemes. It is the author’s opinion that these relatively simple methods could (and should) be implemented already in the second generation of laser gravitational-wave antennae.

The second group of methods requires more substantial modifications of the laser gravitational-wave antenna topology which convert it into a QND device. Examples of this approach are: interferometer with modified input and/or output optics, which implements the spectral variational measurement [13] and different implementations of the quantum speedmeter scheme [14, 15, 16, 17, 18]. In principle, they allow to obtain arbitrarily high sensitivity. In practice, however, they “suffer” from very high optical power circulating in the interferometer arms, which also depends sharply on the required sensitivity.

It can be shown (see [19]) that the optical energy has not to be smaller than the value of

\[ \mathcal{E} = \frac{\mathcal{E}_{\text{SQL}} \xi^2}{2 \xi^2} \]  

(2)

where \( \xi \) is the ratio of the signal amplitude which can be detected to the amplitude corresponding to the Standard Quantum Limit (the smaller is \( \xi \) the better is the sensitivity), and \( \zeta \) is the squeezing factor (\( \zeta = 1 \) for the optical field coherent quantum state and \( \zeta < 1 \) for the squeezed state),

\[ \mathcal{E}_{\text{SQL}} = \frac{ML^2 \Omega^3}{2 \omega_o} \]  

(3)

is the optimal energy for the SQL-limited interferometric antenna, \( \omega_o \) is the pumping frequency, and \( L \) is the length of the interferometer arms [31]. Formula (3) is valid in the wide-band regime where the bandwidth \( \Delta \Omega \sim \Omega \). In the narrow-band regime the energy can be reduced by the factor of \( \sim \Delta \Omega/\Omega \).

If, for example, \( M = 40 \text{ Kg}, L = 4 \text{ Km}, \Omega = 2\pi \times 100 \text{ s}^{-1} \) and \( \omega_o = 2 \times 10^{15} \text{s}^{-1} \) (these values correspond to the proposed gravitation-wave antenna LIGO-II [4]) then \( \mathcal{E}_{\text{SQL}} \approx 40 \text{ J} \). Corresponding circulating optical pumping power is equal to

\[ W_{\text{SQL}} = \frac{c}{4L} \mathcal{E}_{\text{SQL}} \approx 0.75 \text{ MWt} \]  

(4)

It is possible to conclude that the feasibility of these second group methods depends crucially on the experimental progress in preparation of highly squeezed quantum states (with \( \zeta \ll 1 \)).

The third and the most radical approach, intracavity readout scheme, was proposed in the article [20]. It was proposed to measure directly the redistribution of the optical energy inside the antenna in a QND way (without absorption of the optical quanta) instead of monitoring output light beam outside the antenna using photodetectors. In this case the necessary non-classical quantum state of the optical field [factor \( \zeta \) in the formula (2)] is generated automatically and therefore the pumping energy does not depend directly on the required sensitivity.
In the article [21] a possible implementation of the gravitational-wave antenna with intracavity detection, so-called “optical bars” scheme, was proposed (see FIG. 1). In this scheme end mirrors A, B, and an additional local mirror C form two Fabry-Perot cavities coupled by means of the mirror C amplitude transmittance $T$. Relatively weak external pumping (not shown in the picture) is necessary in order to compensate internal losses in the optical elements and to support steady value of the optical energy in the cavities. The optical field acts here as a two rigid springs with one located between the mirrors A and C, while the second one (L-shaped) located between the mirrors B and C. The rigidity of these springs is equal to

$$K = \frac{2\omega_0\mathcal{E}}{L^2\Omega_B}, \quad (5)$$

where

$$\Omega_B = \frac{cT}{L}. \quad (6)$$

is the sloshing frequency of the system of two coupled cavities AC and BC [strictly speaking, $K$ is equal to the asymptotic value of the rigidity at $\Omega \ll \Omega_B$; see formula (A10)].

Due to these springs displacement of the end mirrors A and B caused by the gravitational wave produces displacement of the local mirror C, which can be detected, for example, by a small-scale optical interferometric meter which monitors position of the mirror C relative to reference mass placed outside the optical pumping field.

It was shown in the article [21] that if optical energy exceeds some threshold value of $\mathcal{E}_{\text{thres}}$ then these springs are rigid enough to provide the signal displacement of the local mirror C equal to the displacement of the end mirrors. In this case the sensitivity does not depend on the optical energy and is defined by the sensitivity of the local meter only (compare with formula (2) which contains the factor $\xi^{-2}$). However, it was concluded in the article [21] that the threshold energy have to be rather large and close to $\mathcal{E}_{\text{SQL}}$. 

FIG. 1: The “optical bars” intracavity scheme
In the article [22] an improved version of the “optical bars” scheme was considered. It differs from the original “optical bar” scheme by two additional mirrors A’ and B’ (see FIG. 2) which turn the antenna arms into two Fabry-Perot cavities AA’ and BB’ similar to the standard Fabry-Perot — Michelson topology of the contemporary gravitational-wave antennae. This scheme was called the “optical lever” because it can provide significant gain in the signal displacement of the local mirror similar to the gain which can be obtained using ordinary mechanical lever with unequal arms. The value of this gain is equal to

$$F = \frac{2}{\pi} F,$$  \hspace{1cm} (7)

where $F$ is the finesse of the Fabry-Perot cavities AA’ and BB’. It was shown in the article [22] that in all other aspects the “optical lever” scheme is identical to the “optical bars” one but in the former one the local mirror C transmittance $T$ have to be $F$ times larger, and its mass have to be $F^2$ times smaller. Due to this scaling of mass the gain in the signal displacement by itself does not allow to overcome the SQL as the SQL value increases exactly in the same proportion. But it allows to use less sensitive local position meter and increases the signal-to-noise ratio for miscellaneous noises of non-quantum origin.

No means of reducing the optical power below the level of $E_{SQL}$ were discussed in the article [22].

At the same time different regimes of the “optical bars” scheme were considered in brief in the article [23], and it was mentioned that if a local meter with cross-correlated measurement noise and back-action noise is used then the threshold energy can be substantially lower than the $E_{SQL}$ (see subsection 4.2 of that article). It is evident that this conclusion is also valid for the “optical lever” scheme.

In the current article this cross-correlation regime is analyzed in detail. In the section III the simple mechanical model is considered, which shows how the cross-correlation of the meter noises allows to reduce the pumping energy. In the section IV sensitivity limits are considered and numerical estimates are provided. Most of the calculations are done for the more simple “optical bars” scheme; applicability for the “optical lever” version is considered.
in the subsection III D

II. MECHANICAL MODEL

Consider the mechanical model shown in FIG. 3. Here masses $M$ correspond to the end mirrors of the gravitational-wave antenna (compare with FIG. 1), mass $m$ corresponds to the local mirror, and springs $K/2$ correspond to the optical rigidity. Equal signal forces

$$\frac{F_{\text{grav}}}{2} = \frac{M\ddot{h}}{2}$$

acts on both end masses $M$, and the goal is to detect the signal by monitoring the local mass $m$ state.

Suppose that conditions for the “intermediate case” of the article [23] are fulfilled, namely, the local mass $m$ is small,

$$m\Omega^2 \ll K,$$  \hspace{1cm} (9)

where $\Omega$ is the signal frequency and the rigidity $K$ is also relatively small:

$$K \ll 2M\Omega^2,$$  \hspace{1cm} (10)

Due to condition (9) the signal displacement of the local mass will be equal to the signal displacement of the end ones. However, due to condition (10) the Standard Quantum Limit for the position of the local test mass $\Delta y_{\text{SQL}}$ will be much larger than the Standard Quantum Limit for the positions of the heavy end masses

$$\Delta x_{\text{SQL}} \simeq \sqrt{\frac{\hbar}{M\Omega^2\tau}}.$$  \hspace{1cm} (11)

Really, let $\Delta y_{\text{meas}}$ be the measurement precision provided by the local meter. Due to the uncertainty relation perturbation of the mass $m$ momentum in this case will be equal to

$$\Delta p_{\text{pert}} = \frac{\hbar}{2\Delta y_{\text{meas}} \tau}.$$  \hspace{1cm} (12)

It corresponds to the random force with the uncertainty equal to

$$\Delta F_{\text{pert}} = \frac{\hbar}{2\Delta y_{\text{meas}} \tau},$$  \hspace{1cm} (13)
which produces the additional random displacement of the local test mass

$$\Delta y_{\text{pert}} = \frac{\hbar}{2K\Delta y_{\text{meas}} \tau}. \quad (14)$$

Therefore, the sum error will be equal to

$$\Delta y = \sqrt{(\Delta y_{\text{meas}})^2 + \left(\frac{\hbar}{K\Delta y_{\text{meas}} \tau}\right)^2}. \quad (15)$$

Minimum of this expression is equal to

$$\Delta y_{\text{SQL}} = \sqrt{\frac{\hbar}{2K \tau}}, \quad (16)$$

and this value is $M\Omega^2/K \gg 1$ times larger than the Standard Quantum Limit \(11\). Due to this consideration it was concluded in the article \([21]\) that in order to obtain sensitivity close to the $\Delta x_{\text{SQL}}$ it is necessary to use strong rigidity:

$$K \gtrsim M\Omega^2, \quad (17)$$

and therefore, large pumping energy.

Consider, however, the situation more precisely. If condition \(9\) is fulfilled then equations of motion (in the spectral representation) of the system shown in FIG. 3 looks like:

$$-2M\Omega^2 \hat{x}(\Omega) + K \hat{x}(\Omega) = K \hat{y}(\Omega) + F_{\text{grav}}(\Omega), \quad (18)$$

$$K \hat{y}(\Omega) = K \hat{x}(\Omega) + \hat{F}_{\text{meter}}(\Omega), \quad (19)$$

where $x = (x_1 + x_2)/2$ and $\hat{F}_{\text{meter}}$ is the back-action force of the local meter.

It follows from these equations that the output signal of the meter is equal to:

$$\tilde{y}(\Omega) = \frac{F_{\text{grav}}(\Omega) + \hat{F}_{\text{meter}}(\Omega)}{-2M\Omega^2} + \frac{\hat{F}_{\text{meter}}(\Omega)}{K} + \hat{y}_{\text{meter}}(\Omega), \quad (20)$$

where $\hat{y}_{\text{meter}}$ is the measurement noise which determines the measurement error $\Delta y_{\text{meas}}$.

It is easy to see that if the measurement noise contains the part proportional to the back-action force:

$$\hat{y}_{\text{meter}} = y_{\text{meter}}^{(0)} - \frac{F_{\text{meter}}}{K}, \quad (21)$$

then the main back-action term $\hat{F}_{\text{meter}}/K$ in the equation \(20\) vanishes:

$$\tilde{y}(\Omega) = \frac{F_{\text{grav}}(\Omega) + \hat{F}_{\text{meter}}(\Omega)}{-2M\Omega^2} + y_{\text{meter}}^{(0)}(\Omega). \quad (22)$$

Of course strong perturbation $\hat{F}_{\text{meter}}/K$ still exists in this case but the meter does not “see” it as it is masked by the correlated with $F_{\text{meter}}$ part of $y_{\text{meter}}$.

Equation \(22\) corresponds exactly to the output signal of the position meter with the measurement noise $y_{\text{meter}}^{(0)}$ attached directly to a test mass $2M$. In particular, if an ordinary interferometric position meter with constant pumping power and time- and frequency-independent phase is used, then sensitivity of such a scheme will be limited by the SQL \(11\) even if the rigidities are much smaller than $M\Omega^2$, as long as condition \(9\) is fulfilled.
III. SENSITIVITY LIMITS

We will consider here only two main factors which limit the sensitivity of the scheme being considered: optical losses and noises of the local meter putting aside numerous sensitivity limits which are common for all topologies of the laser gravitational-wave antennae (in particular, miscellaneous internal noises in the mirrors and mirror suspensions). In this case output signal of this scheme normalized as an equivalent gravitational-wave signal can be presented as

\[ \tilde{h} = h + h_{\text{loss}} + h_{\text{meter}} , \]  

where \( h \) is the actual gravitational-wave signal, \( h_{\text{loss}} \) is the noise which arises due to the optical losses, and \( h_{\text{meter}} \) is the one created by the local meter fluctuations. These noises are calculated in appendix B.

A. Optical losses

Taking into account formulae (B4), (A14), and approximations which have been made in the appendix B spectral density of the noise \( h_{\text{loss}} \) can be presented as

\[ S_{h_{\text{loss}}} = \frac{2\hbar L^2 \gamma}{\omega_0 \varepsilon} , \]  

where \( \gamma \) is the Fabry-Perot cavities damping rate. Ratio of this spectral density to the spectral density (1) which corresponds to the SQL is equal to

\[ \xi_{\text{loss}} = \frac{S_{h_{\text{loss}}}}{S_{\text{SQL}}} = \frac{M \Omega^2 \gamma}{2 \omega_0 \varepsilon} = \frac{\varepsilon_{\text{SQL}}}{\varepsilon} \frac{\gamma}{\Omega} . \]  

This formula resembles formula (3) because in both these formulae the larger is ratio \( \varepsilon / \varepsilon_{\text{SQL}} \) the better is the sensitivity. At the same time formula (25) contains the factor \( \gamma / \Omega \) which in principle can be made very small. In particular, modern achievements in fabrication of high-quality mirrors permits to obtain \( \gamma \lesssim 1 \text{s}^{-1} \approx 10^{-3} \Omega \). Therefore, even, say, for \( \varepsilon \simeq 0.1 \varepsilon_{\text{SQL}} \) sensitivity of about one order of magnitude better than the SQL, \( \xi_{\text{loss}} \approx 0.1 \) could be achieved.

B. SQL-limited local meter

Suppose that a SQL-limited position meter with frequency-independent measurement noise \( y_{\text{meter}} \) and back-action noise \( F_{\text{meter}} \) are used as the local meter. For example, a small-scale optical interferometer with the length \( l \ll L \) can be used as such a meter. Suppose also that these noises are cross-correlated [compare with formula (21)]:

\[ \dot{y}_{\text{meter}} = y_{\text{meter}}^{(0)} - \frac{2M}{2M + m} \frac{F_{\text{meter}}}{K} , \]  

This simple frequency-independent cross-correlation can be created rather easily by using a homodine detector with the fixed local oscillator phase [33].
Spectral densities of the measurement noise $S_y$ and the back-action noise $S_F$ satisfy the uncertainty relation

$$S_y S_F = \frac{\hbar^2}{4},$$  \hspace{1cm} (27)

and their ratio depends on the optical pumping energy $\mathcal{E}_{\text{local}}$ in the meter:

$$\frac{S_F}{S_y} = \left( \frac{8Q \mathcal{E}_{\text{local}}}{\ell^2} \right)^2,$$  \hspace{1cm} (28)

where $Q$ is the quality factor of the local meter cavity.

The spectral density $S_{h_{\text{meter}}}^\ast(\Omega)$ of the noise $h_{\text{meter}}$ for this meter has rather sophisticated spectral dependence [see formulae (B8), (B9)] which allows in principle to obtain sensitivity better than the SQL in some narrow band. In this paper, however, wide-band optimization only will be considered.

It is easy to see that if the pumping energy $\mathcal{E}$ is very large ($\mathcal{E} \geq \mathcal{E}_{\text{SQL}}$) then the local meter output is close to the output of the meter attached directly to the test mass $2M + m$ with the signal force $ML\dot{h}$ acting on this mass. Spectral density of the local meter noise $h_{\text{meter}}$ in this case is equal to

$$S_{h_{\text{asympt}}}^\ast(\Omega) = \frac{1}{L^2} \left[ \frac{S_F}{M^2\Omega^4} + \left( \frac{2M + m}{M} \right)^2 S_y \right] \geq \frac{2M + m}{M} \frac{\hbar}{M\Omega^2L^2}. \hspace{1cm} (29)$$
(see curve 1 in FIG. 4). The rightmost part of this formula corresponds to the SQL for the considered scheme which, as well as the SQL (1), is reached at some specific frequency $\Omega = \Omega_{\text{SQL}}$ only. In the case considered here this frequency is equal to

$$\Omega_{\text{SQL}} = \sqrt{\frac{1}{(2M + m)^2}} \frac{S_F}{S_x} = \sqrt{\frac{8Q_{E_{\text{local}}}}{(2M + m)l^2}}. \quad (30)$$

It should be noted that if $m \ll M$ then the SQL (29) corresponds to $\sqrt{2}$ times better sensitivity than the SQL for traditional schemes (1). This gain is obtained because two mirrors $M$ are required for this scheme instead of four ones as in traditional schemes.

At the same time if $E$ is too small then spectral density (B9a) of back-action noise increases sharply at high frequencies; however, some narrow-band gain in sensitivity can be obtained in this regime (see curve 3 in FIG. 4).

Reasonable intermediate value of $E$ can be chosen using, for example, the following algorithm: (i) require that $S_{\text{meter}}(\Omega)$ does not exceeds $S_{\text{asympt}}(\Omega)$ all over the spectral range of interest, up to some given frequency $\Omega_{\text{max}}$ (see curve 2); (ii) with respect to this requirement set $E$ as small as possible; in any case it have to be much smaller than $E_{\text{SQL}}$. In addition condition (A18) which is necessary for the system dynamic stability (see appendix D of the article [21]) is also should be taken into account.

It is easy to show that these requirement can be satisfied only if the sloshing frequency is large,

$$\Omega_B \gg \Omega_{\text{max}}, \quad (31)$$

and in this case the optimal values are equal to

$$m^* \equiv \frac{2Mm}{2M + m} = 12M^2 \frac{\Omega_{\text{max}}^4}{\Omega_B^4}, \quad (32)$$

$$E = \frac{3ML^2\Omega_{\text{max}}^4}{2\omega_0\Omega_B} = 3E_{\text{SQL}} \frac{\Omega_{\text{max}}^4}{\Omega_B}. \quad (33)$$

Suppose, for example, that $T \approx 0.1$, $L = 4$ K$m$, and therefore $\Omega_B \approx 7.5 \times 10^3$ s$^{-1}$. In this case if $\Omega_{\text{max}} = 2\pi \times 10^2$ s$^{-1}$, then $E \approx 0.25E_{\text{SQL}}$ and $m \approx m^* \approx 25$ g.

In principle, more transparent mirror $C$ can be used, which allows further decrease of $E$. However, $E$ depends on $\Omega_B$ as $\Omega_B^{-1}$ only, while $m^*$ depends as $\Omega_B^{-4}$. Therefore, the smaller values of $E$ correspond to very small (sub-gram) values of the mass $m$. It is unclear, if such a small mirror could tolerate tens of the kilowatts of the optical power reflecting from it and hundreds of watts passing through it.

C. QND local meter

The small-scale optical interferometer discussed in the previous subsection can be converted into a QND meter by using, for example, so-called Stroboscopic-Variation Measurement (SVM) technique (see articles [24, 25, 26]). This method permits to filter out the back-action noise by using periodic modulation of the local oscillator phase and/or the pumping energy $E_{\text{local}}$ with frequency which has to be higher than the upper frequency of
the gravitational-wave signal. For small-scale interferometers this modulation can be implemented rather easily.

The residual noise spectral density will be proportional to the spectral density $S_y$ of the meter measurement noise $y_{\text{fluct}}$ and in principle can be made arbitrarily small [see formulae (B9b, C1)] by reducing $S_y$ (i.e. by increasing the local meter sensitivity). However, for technological reasons it is useful to provide the value of the local mirror signal displacement $y_{\text{grav}}$ as large as possible [see formula (B6)].

An optimization algorithm similar to one considered in the previous section can be used here. Start with the simple quasi-static (low-frequency) case, when $m^*\Omega^2 \ll K$. In this case the signal displacement is equal to

$$y_{\text{grav}} = \frac{M^2}{2M + m} L h .$$

and corresponding measurement noise is equal to

$$S_{\text{meas}}^{\text{asympt}} = \frac{1}{L^2} \left( \frac{2M + m}{M} \right)^2 S_y .$$

Require now that $S_{\text{meas}}(\Omega)$ does not exceeds $S_{\text{meas}}^{\text{asympt}}$ for all frequencies within the range $0 \leq \Omega \leq \Omega_{\text{max}}$. It is shown in the appendix C that in this case the pumping energy has to satisfy the following inequality:

$$E \geq k \frac{m^* L^2 \Omega_B \Omega_{\text{max}}^2}{2\omega_o}$$

where $k$ is a numerical factor which varies from 1/8 when $\Omega_B = \Omega_{\text{max}}$ to 1/2 when $\Omega_B \gg \Omega_{\text{max}}$ [see formula (C4)]. This inequality together with the stability condition (A18) can be rewritten as the following condition for the mass $m^*$:

$$\frac{1}{4} \left( \frac{\Omega_{\text{max}}}{\Omega_B} \right)^3 \frac{E}{E_{\text{SQL}}} \leq \frac{m^*}{M} \leq \frac{1}{k} \frac{\Omega_{\text{max}}}{\Omega_B} \frac{E}{E_{\text{SQL}}} .$$

Values of $m^*$ and $\Omega_B$ in formula (36) can vary in wide range and should be chosen considering technological reasons. In principle, heavy local mirror $C$ with low transmittance $T$ (i.e low sloshing frequency) as well as relatively small one with high transmittance $T$ (i.e high sloshing frequency) can be used.

Suppose that $E = 0.1E_{\text{SQL}}$. This value looks like the reasonable one due to limitation caused by the internal losses, see subsection (III A). Suppose also that $M = 40\text{ Kg}$ and $\Omega_{\text{max}} = 2\pi \times 100\text{ s}^{-1}$. In this case typical numerical examples are the following.

Heavy local mirror:

$$T = 0.01,$$

$$\Omega_B \approx 750\text{ s}^{-1},$$

$$10\text{ Kg} \lesssim m^* \lesssim 16\text{ Kg} ;$$

small local mirror:
\[ T = 0.1, \]
\[ \Omega_B \approx 7500 \text{ s}^{-1}, \]
\[ 10 \text{ g} \lesssim m^* \lesssim 700 \text{ g}. \tag{39} \]

Of course all intermediate values between these two examples are also possible.

\textbf{D. The “optical lever” scheme}

In principle, the same idea of the local meter with cross-correlated noise can be used in order to reduce optical pumping energy in the “optical lever” scheme too. However, due to technological limitations in the case of SQL-limited local meter only modest advantages can be obtained. Really, the sloshing frequency in the “optical lever” scheme is equal to

\[ \Omega_B = \frac{cT}{L} < \frac{7.5 \times 10^4 \text{ s}^{-1}}{F}. \tag{40} \]

(it is supposed that \( L = 4Km \)). At the same time it follows from the formula (33) that the sloshing frequency have to be equal to

\[ \Omega_B = \frac{3E_{\text{SQL}}}{E} \Omega_{\text{max}} \approx 2 \times 10^4 \text{ s}^{-1} \frac{E_{\text{SQL}}}{E}. \tag{41} \]

(it is supposed that \( \Omega_{\text{max}} = 2\pi \times 10^2 \text{ s}^{-1} \)). Due to these limitations it is impossible to obtain significant gain \( F \) in the signal displacement using the pumping energy \( E < E_{\text{SQL}} \).

On the other hand, in the case of a QND local meter low sloshing frequency \( \Omega_B \approx \Omega_{\text{max}} \) can be used [see formulae (38)] which makes it possible obtain the gain up to

\[ F = \frac{cT}{L\Omega_{\text{max}}} \approx 10^2. \tag{42} \]

The local mirror mass in the case of the “optical lever” scheme have to be \( F^2 \) time smaller than the figures of formula (38). It follows from the estimates (38) that it has to be equal to about 1 g. This value does not seems unrealistic one as the optical power falling on it will be reduced by the factor of \( F \) and for the case of \( E \approx 0.1E_{\text{SQL}} \) will be equal to about few hundred watts.

\textbf{IV. CONCLUSION}

In the authors opinion, regime of the optical bars/optical lever intracavity topologies considered in this paper looks rather promising for implementing in the third generation of gravitational-wave antennae. It allows to obtain sensitivity better than the SQL and it can do it using rather moderate value of the optical pumping energy: just tens of kilowatts of circulating power, instead of megawatts or tens of megawatts.

At the same time, before the implementation of this method, several issues of a technological origin have to be solved. Some of them are common for all proposed topologies of the laser gravitational-wave antennae: the problem of an internal noises of different origin in the test masses (see, for example, papers [27, 28, 29, 30]) is the most notable one, and some
are specific for the intracavity topologies. It seems that the most important of them is the design of the local meter, and it is evident that this device has to be explored intensively, both experimentally and theoretically before the decisions about the design of the third generation laser gravitational-wave antennae will be made.

Acknowledgments

Author thanks V.B.Braginsky, S.L.Danilishin, M.L.Gorodetsky and S.P.Vyatchanin for useful remarks.

This paper was supported in part by NSF and Caltech grant #PHY0098715, by the Russian Foundation for Basic Research, and by the Russian Ministry of Industry and Science.

APPENDIX A: SYSTEM DYNAMICS

In this appendix we will consider a simplified model of the system similar to one used in the original paper [21]. Replace the Fabry-Perot cavities AC and BC (see FIG. 1) by two coupled single-mode e.m. cavities which eigenfrequencies depend on the positions on the mirrors, see FIG. 5

Equations of motion for this system looks like:

\[
\ddot{q}_1(t) + 2\gamma \dot{q}_1(t) + \omega_o^2 \left(1 + \frac{x_1(t) - y(t)}{L}\right)^2 \dot{q}_1(t) + \omega_o \Omega_B \dot{q}_2(t) = \frac{\omega_o}{\rho} [U_{\text{pump}}(t) + \hat{U}_1(t)], \quad (A1a)
\]

\[
\ddot{q}_2(t) + 2\gamma \dot{q}_2(t) + \omega_o^2 \left(1 - \frac{x_2(t) - y(t)}{L}\right)^2 \dot{q}_2(t) + \omega_o \Omega_B \dot{q}_1(t) = \frac{\omega_o}{\rho} [U_{\text{pump}}(t) + \hat{U}_2(t)], \quad (A1b)
\]

\[
M \ddot{x}_1(t) = -\frac{\omega_o \rho \dot{q}_1^2(t)}{L} + \frac{F_{\text{grav}}(t)}{2}, \quad (A1c)
\]

\[
M \ddot{x}_2(t) = -\frac{\omega_o \rho \dot{q}_2^2(t)}{L} + \frac{F_{\text{grav}}(t)}{2}, \quad (A1d)
\]

\[
m \ddot{y}(t) = -\frac{\omega_o \rho}{L} (\dot{q}_1^2(t) - \dot{q}_2^2(t)) + \hat{F}_{\text{meter}}(t), \quad (A1e)
\]

where \(\dot{q}_{1,2}\) are the generalized coordinates of the cavities, \(U_{\text{pump}}\) is the pumping voltage, \(\Omega_B\) is the sloshing frequency which is proportional to the coupling of the cavities, \(\gamma\) is the damping rate of the cavities, \(\hat{U}_{1,2}\) are the corresponding fluctuational voltages, \(\hat{x}_{1,2}\) are the positions of the masses \(M\), \(\dot{y}\) is the position of the mass \(m\), \(F_{\text{grav}}\) is the signal force, \(\hat{F}_{\text{meter}}\) is the back-action force of the local meter which monitors variable \(y\) (not shown in the picture).
Let introduce new variables then

\[
\dot{q}_\pm(t) = \frac{\dot{q}_1(t) \pm \dot{q}_2(t)}{\sqrt{2}}, \quad \dot{U}_\pm(t) = \frac{\dot{U}_1(t) \pm \dot{U}_2(t)}{\sqrt{2}},
\]

\[
\ddot{x}(t) = \frac{\ddot{x}_1(t) + \ddot{x}_2(t)}{2}, \quad \ddot{X}(t) = \frac{\ddot{x}_1(t) - \ddot{x}_2(t)}{2}.
\]

(A2a, A2b)

For these variables we obtain:

\[
\ddot{q}_+(t) + 2\gamma \dot{q}_+(t) + \omega_+^2 q_+(t) + \frac{2\omega_0^2}{L} \left( \ddot{X}(t) \dot{q}_+(t) + [\dot{x}(t) - \dot{y}(t)] \dot{q}_-(t) \right) = \frac{\omega_0}{\rho} \left[ \sqrt{2} U_{\text{pump}}(t) + \dot{U}_+(t) \right],
\]

(A3a)

\[
\ddot{q}_-(t) + 2\gamma \dot{q}_-(t) + \omega_-^2 q_-(t) + \frac{2\omega_0^2}{L} \left( [\dot{x}(t) - \dot{y}(t)] \dot{q}_+(t) + \ddot{X}(t) \dot{q}_-(t) \right) = \frac{\omega_0}{\rho} \dot{U}_-(t),
\]

(A3b)

\[
2M \ddot{x}(t) = -\frac{2\omega_0\rho}{L} \dot{q}_+(t) \dot{q}_-(t) + F_{\text{grav}}(t),
\]

(A3c)

\[
2M \ddot{X}(t) = -\frac{\omega_0\rho}{L} [\dot{q}_+^2(t) + \dot{q}_-^2(t)],
\]

(A3d)

\[
m\ddot{y}(t) = \frac{2\omega_0\rho}{L} \dot{q}_+(t) \dot{q}_-(t) + \ddot{F}_{\text{meter}}(t),
\]

(A3e)

where \( \omega_\pm = \omega_0 \pm \Omega_B/2 \).

Suppose that the pumping frequency is equal to \( \omega_+ \) and amplitude of the pumping field in the mode “+” is equal to \( q_0 \). Keeping only linear in \( q_0 \) term in the right parts of the equations \( A3 \), these equation can be rewritten as:

\[
q_+(t) = q_0 \cos \omega_+ t,
\]

(A4a)

\[
\ddot{q}_-(t) + 2\gamma \dot{q}_-(t) + \omega_-^2 q_-(t) = \frac{\omega_0}{\rho} \dot{U}_-(t) + \frac{2\omega_0^2 q_0}{L} [\dot{y}(t) - \dot{x}(t)] \cos \omega_+ t,
\]

(A4b)

\[
2M \ddot{x}(t) = -\frac{2\omega_0\rho q_0}{L} \dot{q}_-(t) \cos \omega_+ t + F_{\text{grav}}(t),
\]

(A4c)

\[
m\ddot{y}(t) = \frac{2\omega_0\rho q_0}{L} \dot{q}_-(t) \cos \omega_+ t + \ddot{F}_{\text{meter}}(t).
\]

(A4d)

Using then the rotation polarization approximation:

\[
\dot{q}_-(t) = \dot{q}_c(t) \cos \omega_+ t + \dot{q}_s(t) \sin \omega_+ t,
\]

(A5a)

\[
\dot{q}_-(t) = \omega + (\dot{q}_c(t) \cos \omega_+ t + \dot{q}_s(t) \sin \omega_+ t),
\]

(A5b)

\[
\dot{U}_-(t) = \dot{U}_c(t) \cos \omega_+ t + \dot{U}_s(t) \sin \omega_+ t,
\]

(A5c)

we obtain a simple linear equations set which is convenient for spectral representation:

\[
(i\Omega + \gamma)\dot{q}_c(\Omega) + \Omega_B \dot{q}_s(\Omega) = -\frac{\dot{U}_s(\Omega)}{2\rho},
\]

(A6a)
\[-\Omega_B \dot{q}_c(\Omega) + (i\Omega + \gamma) \ddot{q}_s(\Omega) = \frac{\ddot{U}_c(\Omega)}{2\rho} + \frac{\omega_o q_0}{L} [\dot{y}(\Omega) - \ddot{x}(\Omega)],\]  
\[-2M\Omega^2 \dot{x}(\Omega) = -\frac{\omega_o q_0}{L} \dot{q}_c(t) + F_{\text{grav}}(\Omega),\]  
\[-m\Omega^2 \dot{y}(\Omega) = \frac{\omega_o q_0}{L} \dot{q}_c(\Omega) + \dot{F}_{\text{meter}}(\Omega).\]  

From the first two equations we obtain:

\[\dot{q}_c(\Omega) = -\frac{1}{\mathcal{D}(\Omega)} \left( \frac{\left( i\Omega + \gamma \right) \ddot{U}_s(\Omega) + \Omega_B \ddot{U}_c(\Omega)}{2\rho} + \frac{\omega_o \Omega q_0}{L} [\dot{y}(\Omega) - \ddot{x}(\Omega)] \right),\]  
where

\[\mathcal{D}(\Omega) = (i\Omega + \gamma)^2 + \Omega_B^2.\]

Substitution of this value of \( q_c \) into the last two equations of (A6) gives:

\[-2M\Omega^2 \ddot{x}(\Omega) + \mathcal{K}(\Omega) \dot{x}(\Omega) - \mathcal{K}(\Omega) \ddot{y}(\Omega) = \dot{F}_{\text{loss}}(\Omega) + F_{\text{grav}}(\Omega),\]  
\[-m\Omega^2 \ddot{y}(\Omega) - \mathcal{K}(\Omega) \dot{y}(\Omega) + \mathcal{K}(\Omega) \ddot{x}(\Omega) = -\dot{F}_{\text{loss}}(\Omega) + \dot{F}_{\text{meter}}(\Omega),\]

where

\[\mathcal{K}(\Omega) = \frac{2\omega_o E \Omega B}{L^2 \mathcal{D}(\Omega)} = \frac{K \Omega_B^2}{\mathcal{D}(\Omega)}\]

is the complex pondermotive rigidity,

\[E = \frac{\omega_o q_0^2}{2}\]

is the pumping energy and

\[\dot{F}_{\text{loss}}(\Omega) = \frac{\omega_o q_0}{2L \mathcal{D}(\Omega)} \left( \left( i\Omega + \gamma \right) U_s(\Omega) + \Omega_B U_c(\Omega) \right)\]

is the fluctuational force which arises due to losses in the cavities. Spectral densities of \( \dot{U}_{c,s}(t) \) are equal to

\[S_{U_{c,s}} = 4\hbar \rho \gamma,\]
therefore, spectral density of \( F_{\text{loss}}(t) \) is equal to

\[S_{F_{\text{loss}}} = \frac{2\hbar \omega_o E \gamma \Omega^2}{L^2} \frac{\Omega^2 + \gamma^2 + \Omega_B^2}{|\mathcal{D}(\Omega)|^2}.\]

It follows from the equations (A9), that the position of the local mass is equal to

\[\dot{y}(\Omega) = y_{\text{grav}}(\Omega) + \frac{\mathcal{K}(\Omega) - 2M \Omega^2 \dot{F}_{\text{meter}}(\Omega) + 2M \Omega^2 \dot{F}_{\text{loss}}(\Omega)}{-(2M + m) \Omega^2 |\mathcal{K}(\Omega) - m^* \Omega^2|},\]
where

\[ y_{\text{grav}}(\Omega) = \frac{M}{2M + m} \frac{K(\Omega)}{K(\Omega) - m^*\Omega^2} L h(\Omega) \]  \hspace{1cm} (A16)

is the signal displacement and

\[ m^* = \frac{2Mm}{2M + m}. \]  \hspace{1cm} (A17)

It have to be noted also that as it was shown in the article \[21\] this system is dynamically unstable. If

\[ K \leq \frac{m^*\Omega^2}{4}, \]  \hspace{1cm} (A18)

then this instability is relatively small and can be rather easily suppressed by a feed-back system. In the opposite case, however, very strong asynchronous instability arises (see appendix D of the article \[21\]) which makes the scheme virtually useless. Therefore, condition (A18) has to be considered as necessary one.

**APPENDIX B: THE OUTPUT SIGNAL**

The output signal of the local meter is equal to

\[ \tilde{y}(\Omega) = \hat{y}(\Omega) + \hat{y}_{\text{meter}}(\Omega), \]  \hspace{1cm} (B1)

where \( \hat{y}_{\text{meter}}(\Omega) \) is the measurement noise.

If the meter noises are cross-correlated, see formula (26), then

\[ \tilde{y}(\Omega) = y_{\text{grav}}(\Omega) + \frac{1}{-(2M + m)\Omega^2[K(\Omega) - m^*\Omega^2]} \times \left\{ \left[ K(\Omega) + 2M\Omega^2 \left( \frac{K(\Omega)}{K} - 1 \right) - \frac{2Mm^*\Omega^4}{K} \right] \hat{F}_{\text{meter}}(\Omega) + 2M\Omega^2 \hat{F}_{\text{loss}}(\Omega) \right\} + \hat{y}_{\text{meter}}^{(0)}(\Omega), \]  \hspace{1cm} (B2)

It is convenient to present this expression as follows:

\[ \tilde{y}(\Omega) = \frac{M}{2M + m} \frac{K(\Omega)L}{K(\Omega) - m^*\Omega^2} \left[ h(\Omega) + \hat{h}_{\text{loss}}(\Omega) + \hat{h}_{\text{meter}}(\Omega) \right], \]  \hspace{1cm} (B3)

where

\[ \hat{h}_{\text{loss}}(\Omega) = -\frac{2F_{\text{loss}}(\Omega)}{K(\Omega)L} \]  \hspace{1cm} (B4)

is the equivalent noise produced by the optical losses, and

\[ \hat{h}_{\text{meter}}(\Omega) = \frac{1}{L} \left\{ -\frac{1}{M\Omega^2} - 2 \left( \frac{1}{K} - \frac{1}{K(\Omega)} \right) + \frac{2m^*\Omega^2}{KK(\Omega)} \right\} \hat{F}_{\text{meter}}(\Omega) \]
is the equivalent noise of the meter (both these noises are normalized as an equivalent fluctuational gravitational-wave signals).

Taking into account that in order to obtain \( \xi_\text{loss}^2 < 1 \) [see formula (25)] the optical losses have to be small, \( \gamma \ll \Omega \), expressions (A16), (B5) can be slightly simplified:

\[
y_{\text{grav}}(\Omega) = \frac{M}{2M + m} \frac{Lh(\Omega)}{1 - m^*\Omega^2(\Omega_B^2 - \Omega^2)} \, ,
\]

\[
\hat{h}_{\text{meter}}(\Omega) = \frac{1}{L} \left\{ \left( -\frac{1}{M\Omega^2} - \frac{2\Omega^2}{K\Omega_B^2} + \frac{2m^*\Omega^2(\Omega_B^2 - \Omega^2)}{K^2\Omega_B^2} \right) \hat{F}_{\text{meter}}(\Omega) + \frac{2M + m}{M} \left[ 1 - \frac{m^*\Omega^2(\Omega_B^2 - \Omega^2)}{K\Omega_B^2} \right] \hat{y}_{\text{meter}}^{(0)}(\Omega) \right\}
\]

If a SQL-limited local meter is used, then spectral density of this noise can be presented as a sum

\[
S_{h_{\text{meter}}}^{(\Omega)} = S_{\text{B.A.}}(\Omega) + S_{\text{meas}}(\Omega)
\]

where

\[
S_{\text{B.A.}}(\Omega) = \frac{1}{L^2} \left[ -\frac{1}{M\Omega^2} - \frac{2\Omega^2}{K\Omega_B^2} + \frac{2m^*\Omega^2(\Omega_B^2 - \Omega^2)}{K^2\Omega_B^2} \right]^2 S_F
\]

\[
S_{\text{meas}}(\Omega) = \frac{1}{L^2} \left( \frac{2M + m}{M} \right)^2 \left[ 1 - \frac{m^*\Omega^2(\Omega_B^2 - \Omega^2)}{K\Omega_B^2} \right]^2 S_y
\]

and \( S_F \) and \( S_y \) are spectral densities of the meter back-action noise \( \hat{F}_{\text{meter}} \) and its measurement noise \( \hat{y}_{\text{meter}}(\Omega) \).

**APPENDIX C: NOISE OPTIMIZATION FOR A QND LOCAL METER**

In the case of a QND local meter the back-action noise can be filtered out by some means and sensitivity is limited by the measurement noise only:

\[
S_{h_{\text{meter}}}^{(\Omega)} = S_{\text{meas}}(\Omega)
\]

Require that \( S_{\text{meas}}(\Omega) \) has not to exceed \( S_{\text{meas}}(0) \) for all frequencies \( 0 \leq \Omega \leq \Omega_{\text{max}} \). In this case the following condition has to be fulfilled for all these frequencies:

\[
\left| 1 - \frac{m^*\Omega^2(\Omega_B^2 - \Omega^2)}{K\Omega_B^2} \right| \leq 1
\]

The solution of this inequality can be presented as follows:

\[
K \geq km^*\Omega_B^2_{\text{max}}
\]
where

\[ k = \begin{cases} 
\frac{1}{8} \left( \frac{\Omega_B}{\Omega_{\text{max}}} \right)^2, & \Omega_{\text{max}} \leq \Omega_B \leq \sqrt{2} \Omega_{\text{max}} \\
\frac{1}{2} \left[ 1 - \left( \frac{\Omega_{\text{max}}}{\Omega_B} \right)^2 \right], & \Omega_B > \sqrt{2} \Omega_{\text{max}}.
\end{cases} \] (C4)

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Numerical factors in formulae (11, 12, 30) correspond to the case of the standard Michelson—Fabry-Perot topology with four mirrors having equal masses $M$, which was analyzed in detail in article [13]. Note also the factor $1/2$ in formula (2), which arises for the back-action noise that makes up half of the net noise in the SQL-limited detectors, is eliminated from the output signal of the QND detectors. It allows to obtain sensitivity equal to the SQL ($\xi = 1$) using two times smaller energy or, put it otherwise, to obtain sensitivity $\sqrt{2}$ times better than the SQL ($\xi = 1/\sqrt{2}$) using $E = E_{\text{SQL}}$.

The last term in formula (25) looks a bit misleading because it seems that the smaller is $\Omega$ the worse is sensitivity. It is easy to see that it is not the case because $E_{\text{SQL}}$ also depends on $\Omega$ and the worst case takes place at the highest signal frequency. Therefore, it is for this frequency all estimates should be done.

In principle, the noises $g_{\text{meter}}^{(0)}$ and $F_{\text{meter}}$ can also be made cross-correlated by this method. It can be shown, however, that this additional cross-correlation does not provide any significant advantages so we do not consider this possibility here.