Abstract. The prevalence of convolution in applications within signal processing, deep neural networks, and numerical solvers has motivated the development of numerous fast convolution algorithms. In many of these problems, convolution is performed on terabytes or petabytes of data, so even constant factors of improvement can significantly reduce the computation time. We leverage the formalism of bilinear algorithms to describe and analyze all of the most popular approaches. This unified lens permits us to study the relationship between different variants of convolution as well as to derive error bounds and analyze the cost of the various algorithms. We provide new derivations, which predominantly leverage matrix and tensor algebra, to describe the Winograd family of convolution algorithms as well as reductions between 1D and multidimensional convolution. We provide cost and error bounds as well as experimental numerical studies. Our experiments for two of these algorithms, the overlap-add approach and Winograd convolution algorithm with polynomials of degree greater than one, show that fast convolution algorithms can rival the accuracy of the fast Fourier transform (FFT) without using complex arithmetic. These algorithms can be used for convolution problems with multidimensional inputs or for filters larger than size of four, extending the state-of-the-art in Winograd-based convolution algorithms.

Key words. convolution, bilinear algorithms, Winograd convolution, convolutional neural networks

AMS subject classifications. 65F99, 68W01

1. Introduction. Discrete convolution is a bilinear function that combines two sequences of data to produce a third. Problems such as multiplication \cite{35, 77, 2}, signal processing \cite{14, 13, 54, 38}, statistics \cite{66, 57}, acoustics \cite{1, 73}, image processing \cite{55, 72}, and numerical solvers for partial differential equations within physics and chemistry \cite{90, 92, 46} use convolution. Consequently, fast methods for convolution can reduce the computation time for various problems. Given two inputs of size \( n \), a direct computation of convolution performs at most \( n(n-1) \) additions and \( n^2 \) multiplications.

Over the years, fast algorithms have been studied and used to compute convolution. Most fast algorithms operate in three steps: compute the linear combinations of both inputs, calculate the element-wise product of those linear combinations, and then recover the result by computing the linear combinations of the products. The first known fast algorithm is Karatsuba’s algorithm \cite{44}, which achieves a complexity of \( O(n \log_2(3)) \). The most prominent fast algorithm employs the discrete Fourier transform (DFT) to obtain suitable linear combinations. This fast algorithm leverages the fast Fourier transform (FFT) to obtain the linear combinations in \( O(n \log_2(n)) \) time \cite{35, 34}. The FFT-approach reduces the number of bilinear products necessary to \( O(n) \), yielding an algorithm with an overall cost of \( O(n \log_2(n)) \) instead of the \( O(n^2) \) cost incurred by the direct method.

For a convolution with two \( n \)-dimensional vectors, the cost and stability of the FFT make it the method of choice. However, in many scenarios, including signal processing and convolutional neural networks (CNN), a small filter of size \( r \) is convolved with a large vector of size \( n \). A naive application of the FFT requires \( O(n \log_2(n)) \) cost, which is worse than the \( O(nr) \) cost of the direct method when \( r < \log_2(n) \). The use of \( n/r \) FFTs of size \( O(r) \) yields a lower cost of \( O(n \log_2(r)) \). Furthermore, when \( r \) is small, the constant factors incurred by FFT, due in part to the use of com-
plex arithmetic, can make it uncompetitive [25]. Given a direct implementation of complex arithmetic, an FFT-based convolution with $n$-dimensional vectors requires $18n \log_2(2n) + O(n)$ real additions and $12n \log_2(2n) + O(n)$ real multiplications. For sufficiently small dimensions, the direct approach requires less work than the FFT. While the direct approach is efficient in such cases, other fast algorithms can obtain yet lower constant factors, yielding practical benefits. Consider the use of convolution in CNNs. The convolutional layer of the CNN architecture, AlexNet [53], takes approximately three hours, about ninety percent of the CNN’s overall runtime, to convolve 256 images when running on a single-threaded CPU [17]. Even a constant factor improvement over the direct method can save minutes to hours for this type of problem. Fast algorithms present a variety of methods with lower cost complexities.

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Beyond adaptation for small filters, another remaining challenge is the development of efficient methods for multidimensional (especially, 2D and 3D) convolution algorithms. Efficient algorithms for 2D and 3D convolution are important for applications within scientific computing and CNNs. The FFT-based approach is well-suited for the former domain, but the use of small filters in CNNs again leaves room for further innovation. To the best of our knowledge, the main algorithms for computing convolution in CNNs are either matrix-multiplication [16], the FFT [27], or a few variants of Winograd’s convolution algorithm [91, 55, 6]. We propose other variants of the general Winograd formulation that are suitable for higher dimensions, such as 2D, 3D, and 4D convolutions.

1.1. Previous surveys and key related work. Convolution has been studied and surveyed before in signal processing [67, 39, 62, 8, 12, 79]. Some of these methods have been presented as bilinear algorithms, which provide a framework to define new algorithms for larger convolutions via a matrix nesting by the Kronecker product and embedding using the Winograd convolution algorithm. In addition to using the formalism of bilinear algorithms to define these methods, we provide explicit formulations on how to generate the matrices for the various convolution algorithms. We provide new, simple derivations of many of the key methods, and specially address multidimensional and small-filter convolution scenarios.

An important consideration for bilinear algorithms is the number of additions and element-wise multiplications required to compute the algorithm. As the problem size grows, the overhead of applying linear combinations increases quadratically. Variations of bilinear algorithms for convolution can trade-off the number of linear combinations and element-wise multiplications needed [8], which has subsequently been studied and optimized for various implementations of convolution algorithms [10]. We provide similar tables as well as supplementary material for readers to generate the matrices themselves.

With the advent of parallel computing, the scalability of convolution algorithms is crucial for building highly efficient algorithms. The parallelization of convolution with Sobel or Gaussian filters has been studied [70, 29]. Sobel filters are used for edge-detection [70, 43] and Gaussian filters are used for reducing the noise in a signal [29, 22]. However, a more general study of the parallel efficiency of convolution algorithms may be useful as convolution filters are not restricted to the Sobel and Gaussian variants. In the context of CNNs, a direct computation of discrete convolution is fairly straightforward to parallelize [52]. Convolution can be reduced to a matrix-multiplication and fast Fourier transform problem, both of which can leverage

1https://github.com/jucaleb4/Bilinear-Algorithms-for-Convolution
efficient library packages, such as cuDNN [16] for direct convolution on GPUs and FFTW [27] for FFT on shared-memory machines (but also many other for GPUs, shared-memory, and distributed-memory architectures). The family of fast convolution algorithms from signal processing (aside from the FFT) has been largely unused for CNNs prior to the paper by Lavin and Gray [55]. They propose a method based on Winograd’s formulation of convolution algorithms, although it is later noted to be a variant of the Toom-Cook (interpolation) method [5]. The Winograd-based algorithm [55] divides the convolution into three steps, each step performing a sequence of matrix multiplications. Experiments on GPUs suggest that the Winograd-based algorithm can be highly scalable for small-filter convolution problems [55]. In general, both the FFT and Winograd-based method achieve comparable execution times. When executed on GPUs, the FFT and Winograd method achieve speed-ups of up to 4\times over the direct (matrix-multiplication-based) approach for AlexNet [48]. Additionally, parallel implementations [24, 42] and specialized hardware designs [74, 14, 71] have been shown to improve the speed of convolution. We focus on the sequential arithmetic complexity and stability of fast algorithms for convolution.

The use of linear combinations in fast algorithms leverages cancellation of sums to reduce the number of element-wise multiplications, however it may also introduce significant error for certain inputs. Since the Winograd-based convolution [55], a modified Toom-Cook method, relies on Vandermonde matrix, the algorithm can quickly become inaccurate for inputs of size greater than four [55, 91]. The absolute error of the Toom-Cook method is proportional to norm of the inputs and the three matrices that compose the bilinear algorithm [5]. Better nodes and the use of the Winograd convolution algorithm with polynomials of degree greater than one have shown promising results in reducing error [5, 6]. In addition to summarizing these results, we show that decomposing a large convolution into nested smaller convolutions can result in more stable algorithms.

Given the wide array of work for convolution in signal processing, image processing, and more recently CNNs, our main contribution is to provide a comprehensive guide for various convolution algorithms and derive simple constructions of the algorithms. To do so, we leverage the general bilinear algorithm formulation, which enables derivation and analysis of fast algorithms using basic algebraic transformations of matrices and tensors.

1.2. Convolution and its variations. The convolution of two continuous functions \( u \) and \( v \) is defined as

\[
(u * v)(t) = \int_{-\infty}^{\infty} u(\rho)v(t - \rho)d\rho.
\]

Given the input vectors \( f \in \mathbb{R}^r \) and \( g \in \mathbb{R}^n \) (assume \( n \geq r \)), the discrete convolution between \( f \) and \( g \) is defined as

\[
y_k = \sum_i f_i g_{k-i}.
\]

An explicit definition of the summation bounds for equation (1.2) produces different types of discrete convolution, detailed in Table 1. The linear convolution, \( y = f * g \), is equivalent to equation (1.2) and using bounds that keep the indices within the range of input and output vector dimensions. Cyclic convolution wraps the vectors by evaluating the indices modulo \( n \). Equivalently, cyclic convolution is the linear convolution of a periodic signal \( g \). When we only want the subset of elements from
linear convolution, where every element of the filter is multiplied by an element of \( g \), we can use correlation algorithms, as introduced by Winograd [91]. We can see these are the middle \( n - r + 1 \) elements from a discrete convolution. Given a filter \( f \in \mathbb{R}^r \) and input \( g \in \mathbb{R}^{m+r-1} \), correlation algorithms compute \( m \) outputs.

Table 1: Convolution variations: different formulae for the output element \( y_i \) as well as the whole vector \( y \).

| Linear          | Cyclic | Correlation |
|-----------------|--------|-------------|
| \( y_k = \sum_{i=\max(0,k+1)}^{\min(k,r-1)} f_i g_{k+i} \) | \( y = T_{(f,n)} g \) | \( y = T_{(g,r)} f \) |
| \( y = T_{(f,n)} g \) | \( y = T_{(g,r)} f \) | \( y = T_{(x,k)} \) |

Each of the three convolution algorithms can be used to solve the other two. Using the Matrix Interchange Theorem (Theorem 3.2), we can derive correlation algorithms from linear convolution algorithms and vice versa. Cyclic convolution can be computed with linear convolution by appending inputs \( f \) and \( g \) with copies of themselves. To solve cyclic convolution with linear convolution, the inputs \( f \) and \( g \) are appended with zeros until they are each of size \( n + r - 1 \). Convolution can also be cast as a matrix–vector product, where the matrix has a Toeplitz or Hankel structure. For linear convolution, either the filter \( f \) or input \( g \) can be embedded as a lower-trapezoidal Toeplitz matrix. The linear convolution of the vector \( x \in \mathbb{R}^m \) with an arbitrary \( k \)-dimensional vector can computed with \( T_{(x,k)} \in \mathbb{R}^{(m+k-1)\times k} \), so that

\[
T_{(x,k)} = \begin{bmatrix}
    x_1 \\
    \vdots \\
    x_m \\
    \vdots \\
    x_1 \\
    \vdots \\
    x_m
\end{bmatrix}.
\]

The multiplication \( y = T_{(f,n)} g \) and \( y = T_{(g,r)} f \) both compute linear convolution. Other variants may be computed using other Toeplitz matrices as shown in Table 1.

Multidimensional convolution corresponds to convolving along each mode of the inputs. Given a 2D filter \( F \in \mathbb{R}^{r\times r} \) and input \( G \in \mathbb{R}^{n\times n} \), linear convolution is computed by

\[
y_{lm} = \sum_{i=\max(0,l-n+1)}^{\min(l,r-1)} \sum_{j=\max(0,m-n+1)}^{\min(m,r-1)} f_{ij} \cdot g_{l-i,m-j}.
\]

(1.3)
1.3. Paper overview. We survey different applications of convolution in section 2. We formulate the convolution algorithm as a bilinear algorithm in section 3, following Pan’s formalism for matrix-multiplication [68]. Then, we present specific implementations of fast algorithms in section 4, section 5, section 6, and section 7. We leverage our general formulation of fast convolution algorithms to quantify their cost complexity in section 8. We derive bounds on the numerical stability of algorithms (providing a simplified summary of previous results from [5]) for the Toom-Cook method and provide solutions to reduce the error in section 9. We provide experimental results on the stability of a variety of 1D and multidimensional convolution algorithms in section 10. Finally, we present open questions in section 11.

2. Problems and applications of convolution. Convolution is a key component for many scientific and engineering problems, such as signal processing, partial differential equations, and image processing. In the following section, we examine how linear discrete convolution and correlation convolution algorithms are used in a variety of fields.

2.1. Signal processing. One of the most important tasks in digital signal processing is the filtering of a long signal, represented by a sequence of real and complex numbers. The filtering of the signal is calculated by a digital filter [8], which produces a new signal called an output sequence. FIR filters, or finite-impulse-response filters, is a digital filter that captures the strength of the incoming signal for only a finite period of time [54]. The computation of the output sequence from an FIR filter can be synthesized by discrete convolution [30]. The ubiquity of FIR filters within domains such as noise removal in EKGs [13], image processing for texture mapping [38], and mobile communications [75] have consequently led to the development of highly efficient algorithms for 1D discrete convolution, such as new nesting schemes [14, 78], the Fermat number transform [2], and the Good-Thomas PFA.

2.2. Integer multiplication. Let \( a \) and \( b \) be two \( n \)-digit integers. The value of the two integers can be rewritten as, \( a = \sum_{i=0}^{n-1} a_i \cdot 10^i \) and \( b = \sum_{i=0}^{n-1} b_i \cdot 10^i \), where \( a_i \) and \( b_i \) are the individual digits of integers \( a \) and \( b \) respectively. A direct computation of the product \( ab \) can be formulated as

\[
(2.1) \quad ab = \sum_{k=0}^{2n-2} \min(k,n-1) \sum_{i=\max(0,k-n+1)}^{n-1} (a_i \cdot b_{k-i})10^k.
\]

The similarity of equation (2.1) to equation (1.2) allows integer multiplication to be viewed as discrete convolution and vice versa.

2.3. Numerical methods for differential equations. Within physics, chemistry, and engineering, many numerical PDE solvers are based on determining the solution to continuous convolution equations. For example, integral equations for initial and boundary value problems of linear differential equations seek to describe the solution \( v \), which arises in \( u \ast v \) with the integration domain described by boundary conditions, where \( u \) is the Green’s function of the differential operator [51]. These problems are sometimes reduced to discrete convolution, especially when regular grids are used for discretization, often yielding 3D discrete convolution problems. Iterative methods for solutions to convolution equations leverage repeated application of the convolution operator, yielding a series of discrete convolutions. Techniques for fast convolution algorithms, such as the discrete Fourier transform, also provide a way of
solving convolution equations directly. Such regular-grid-based solvers are prevalent across a variety of major numerical PDE applications in scientific computing. For example, they are used for acoustic scattering problems [11], for long-range interactions in molecular dynamics (particle-mesh Ewald method) [20], and within quantum chemistry for electronic structure calculations [33, 47, 45, 46] and dynamics [90].

2.4. Image processing. An important step in image recognition is feature extraction. Given an input image and a feature, both of which can be represented as matrices, the goal is compute a feature map for the image. A feature map assigns scores, or activations, to different areas of the original image, which can be computed via a 2D convolution. By connecting a series of 2D convolutions, a CNN can identify global features from localized changes in values, such as inferring shapes from curvature, which enables identification of objects in images [81, 60].

The first successful CNN was LeNet5, developed by LeCun et al. in 1994 [56]. Research in CNNs remained relatively modest until the success of AlexNet [53] in the 2012 ImageNet competition. Since then, there has been a myriad of CNN designs [94, 86, 36, 41]. In many contexts, CNNs can be very large in size, making their training very expensive and their performance for inference to be contingent on the efficiency of convolution [15]. A variety of avenues have been pursued for decreasing memory footprint and runtime of CNNs, one example being the reduction in the number of parameters [15, 32, 59, 61].

Fast algorithms for 2D convolution provide a direct approach to speed-up both the training and inference of CNNs. The state of practice of 2D convolution algorithms for CNNs consists of either matrix-multiplications [16], the FFT [27], and an algorithm [55] based on Winograd’s minimal filtering method [91].

We now formally define how the series of 2D convolution are performed, leveraging the correlation form of convolution. A CNN is associated with a set of $K$ filters of size $S \times R$ stored in the tensor $F$. An input to a CNN will be a set of $N$ images stored in the tensor $G$. Each filter and image has $H$ channels, such as the RGB channels for color images. The convolutions are summed over the channels and stored in $Y$,

\[
y_{ikxy} = \sum_{c=1}^{H} \sum_{v=1}^{R} \sum_{u=1}^{S} f_{kcuv} \cdot g_{i,c,x+u,y+v}.\tag{2.2}
\]

In equation (2.2), the variable $i$ is the index for which of the $N$ images we are convolving, and the variable $k$ denotes which of the $K$ filters is being used. The variables $l, m$ represent the index of the summed feature maps.

2.5. Cosmological data analysis. One example of large-scale data analysis via CNNs arises in cosmology. By studying the distribution of matter across various galaxies, key cosmological parameters, such as halo mass, metallicity, environment, and age can be predicted [23]. However, the study of large cosmological systems such as the Universe are hindered by the size (terabytes to petabytes) and complexity (3D and 4D) of the datasets that describe them.

While previous attempts to identify the parameters rely on hand-crafted statistical parameters [23], recent work has explored use of learning models such as variational autoencoders [49] and generative adversarial networks [31]. The use of CNNs to identify the parameters have been able to outperform the results of hand-crafted statistical measurements [64, 63] and have shown the potential to handle noise in the data [76].
Although CNNs have currently brought accurate results, designing efficient methods to process the large datasets is still an ongoing research question. Due to the size and dimensionality of the data, previous works that relied on CNNs can require up to twenty days of runtime when running on the TensorFlow framework [63]. For applications of convolution that act on large amounts of multidimensional data, such as in image processing and cosmology, there is a need for highly efficient and scalable convolution algorithms.

3. Bilinear algorithm representation for convolution. A direct computation of an 1D linear convolution requires $O(nr)$ additions and multiplications. Faster algorithms can generally be represented using the framework of bilinear algorithms [68]. The linear convolution of 1D vectors $f \in \mathbb{R}^r$ and $g \in \mathbb{R}^n$ can be defined by a bilinear function,

\[ y_k = \mathcal{F}_T(f, g), \]

where $y_k = \sum_{i,j} t_{ijk} f_i g_j$, with $t_{ijk} = \begin{cases} 1 : i + j - k = 0 \\ 0 : \text{otherwise} \end{cases}$.

A CP decomposition [50] of tensor $T$, given by matrices $A \in \mathbb{C}^{r \times R}$, $B \in \mathbb{C}^{n \times R}$, and $C \in \mathbb{C}^{(n+r-1) \times R}$ via

\[ t_{ijk} = \sum_{l=0}^{R-1} a_{il} b_{jl} c_{kl}, \]

specifies a bilinear algorithm [68] for computing $\mathcal{F}_T$,

\[ y_k = \sum_{l=0}^{R-1} c_{kl} \left( \sum_{i=0}^{r-1} a_{il} f_i \right) \left( \sum_{j=0}^{n-1} b_{jl} g_j \right), \]

i.e., $y = C \left[ (A^T f) \odot (B^T g) \right]$.

The value $R$ is the bilinear rank of the algorithm. Matrices $A$ and $B$ specify linear combinations for inputs $f$ and $g$ respectively, which serve as respective inputs to a set of $R$ products of the two sets of linear combinations. The matrix $C$ takes linear combinations of these products to obtain each entry of the output $y$. We refer to the multiplication by matrices $A$ and $B$ as encoding and the multiplication by matrix $C$ as decoding. When an algorithm is applied recursively many times, the bilinear rank $R$ plays a key role, since the rank determines the number of recursive calls needed. The asymptotic complexity of the recursive bilinear algorithm usually depends on $R$ and not on the particular structure of the matrices $(A, B, C)$.

Given a filter of size $r$ and input of size $n$, a direct computation of linear convolution has a rank of $R = nr$. For filter size $r$ and output size $m$, the correlation algorithm has bilinear rank of $R = mr$ when directly computed. However, algorithms with bilinear rank of $m + r - 1$ exist. The optimality of this bilinear rank has been proven by Winograd [91].

**Theorem 3.1.** The minimum rank of a correlation convolution algorithm with filter of size $r$ and output of size $m$ is $m + r - 1$.

A proof of the above theorem is presented in [91]. Winograd also shows that by casting the bilinear algorithm to a trilinear algorithm, linear convolution algorithms can be derived from correlation algorithms by swapping variables. Blahut and Barabasz et al. derive a similar result by using Matrix Interchange [5, 8]. We provide an alternative proof by simply swapping the indices of the tensor $T$. 

Theorem 3.2 (Matrix Interchange). Let the bilinear algorithm for linear convolution $f$ and $g$ be defined as $C((A^T f) \odot (B^T g))$. The correlation algorithm with output size $m = n$ is

\begin{equation}
B((A^T f) \odot (C^T g)).
\end{equation}

Proof. From equation (3.1), the tensor $T$ in $\sum_{ij} t_{ijk} f_i g_j$ satisfies $t_{ijk} = 1$ if $i + j - k = 0$ and otherwise $t_{ijk} = 0$. The bilinear function computing correlation can be expressed via tensor $T_{\text{corr}}$ as

\begin{equation}
y_k = \sum_{ij} t^\text{corr}_{ijk} f_i g_j = \sum_{i=0}^{r-1} f_i g_{k+i}
\end{equation}

with $t^\text{corr}_{ijk} = 1$ if $i - j + k = 0$, and consequently,

\begin{equation}
t^\text{corr}_{ijk} = t^c_{ikj}.
\end{equation}

Therefore, given a bilinear algorithm $(A, B, C)$ to compute convolution, we obtain a bilinear algorithm $(A, C, B)$ for the correlation algorithm, since

\begin{equation}
t^\text{corr}_{ijk} = t^c_{ikj} = \sum_{l=0}^{R-1} a_i b_{kl} c_{jl}.
\end{equation}

The conversion between correlation and linear convolution algorithm preserves the number of element-wise multiplications, and subsequently the rank as well.

Corollary 3.3. The minimum rank on convolution with an input of size $n$ and filter of size $r$ is $R = n + r - 1$.

We now present various bilinear algorithms that achieve the minimal rank.

4. Convolution using polynomial interpolation. Given a discrete set of points $x$ with corresponding values $y$, interpolation derives the polynomial $v$ that fits the values of $y$ as accurately as possible. Given $n$ points, a unique $n-1$ degree polynomial $v$ exists that satisfies $v(x_i) = y_i$ for all $i$ [37].

Recall from subsection 2.2 that polynomial multiplication is equivalent to linear convolution. Let the vectors $f$ and $g$ be the coefficients for a degree $r-1$ polynomial $p$ and degree $n-1$ polynomial $q$ respectively. The linear convolution of $f$ and $g$ is equivalent to the coefficients of the polynomial product $v = pq$. By viewing linear convolution as polynomial multiplication, we can apply a family of fast algorithms to convolution, one of which is based on interpolation. The intuition is that we first multiply the values of $p$ and $q$ at $n + r - 1$ discrete nodes. These products are equivalent to $v$ at those same $n + r - 1$ points. We then interpolate on these values to compute the coefficients for polynomial $v$. By carefully selecting the nodes and the basis for interpolation, we can derive algorithms that are both stable and compute linear convolution in asymptotically less time.

Let the matrix $V \in \mathbb{C}^{R \times R}$ be the Vandermonde matrix with $R = n + r - 1$ distinct nodes. The bilinear algorithm’s encoding matrices $A \in \mathbb{C}^{r \times R}$ and $B \in \mathbb{C}^{n \times R}$ are defined by keeping the first $r$ and $n$ rows of $V^T$, respectively [91, 8, 10]. The decoding matrix $C \in \mathbb{C}^{R \times R}$ is then given by $V^{-1}$. This construction of the matrices $A$, $B$, and $C$ creates a bilinear algorithm with rank $R = n + r - 1$. 
4.1. Karatsuba’s Algorithm. In the late 1950s, Kolmogorov conjectured that integer multiplication (2.1) had a cost complexity of $\Omega(n^2)$. Karatsuba refuted the conjecture by developing an algorithm running in $O(n^{\log_2(3)})$ time [44]. Karatsuba’s algorithm reuses the previous element-wise multiplications to compute the middle term of a two-digit integer multiplication problem,

$$a \times b = \sum_{k=0}^{2} \sum_{i = \max(0, k-1)}^{\min(k, 1)} (a_i \cdot b_{k-i}) 10^k$$

$$= (a_1 \cdot b_1)10^2 + (a_1 \cdot b_0 + a_0 \cdot b_1)10 + (a_0 \cdot b_0)$$

$$= (a_1 \cdot b_1)10^2 + ((a_1 \cdot b_1 + a_0 \cdot b_0) - (a_0 - a_1)(b_0 - b_1))10 + (a_0 \cdot b_0).$$

With the reformulation, the multiplication now only requires three unique element-wise multiplications instead of four. When the input is larger than two, equation (4.1) can be applied by breaking the integer into two smaller integers and recursively computing each element-wise multiplication. By reducing the problem by a factor of two and making three recursive calls, the asymptotic cost of this algorithm is $T(n) = 3T(n/2) + O(n) = O(n^{\log_2(3)})$.

Karatsuba’s algorithm operates in three distinct steps: take linear combinations of the input, compute the element-wise multiplications, and compute the linear combinations of the products. The combination of these three steps is captured by the bilinear algorithm,

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}.$$  

This bilinear algorithm can be viewed as an interpolation-evaluation problem using the nodes 0, 1, and $\infty$. The use of the node $\infty$ will be explained in the next section on Toom-Cook algorithms. Toom-Cook algorithms encompass a family of fast algorithms, such as Karatsuba’s algorithm, that operate using a more general bilinear algorithm formulation.

4.2. The Toom-Cook method. Soon after the publication of Karatsuba’s algorithm, Toom developed a generalized algorithm for any input size $k$ [88]. Cook’s Ph.D. thesis formalized Toom’s algorithm into what is now known as the Toom-Cook method [18], which is an explicit definition of the interpolation approach from the beginning of section 4.

The designer of the Toom-Cook method can freely choose the basis and nodes. Regardless of the basis, both the input and output must be represented in the monomial basis, since convolution is equivalent to polynomial multiplication only in this basis. The Toom-Cook algorithm can be defined by the Lagrangian basis [9, 93, 62, 8]. Using this basis, the polynomial multiplication, $v = pq$, is computed by the summation,

$$v(x) = \sum_{j=0}^{r+n-2} \prod_{i=0, i \neq j}^{r+n-2} p(x_j) \cdot q(x_j) \frac{(x - x_i)}{(x_j - x_i)},$$

where $x_0, \ldots, x_{r+n-2}$ are the set of $r + n - 1$ unique nodes. Equation (4.2) can be rewritten as the multiplication by the inverse Vandermonde matrix,

$$y = V^{-1} \begin{bmatrix} p(x_0) \cdot q(x_0) \\ \vdots \\ p(x_{r+n-2}) \cdot q(x_{r+n-2}) \end{bmatrix}. $$
By defining the matrices $A$ and $B$ by the truncated Vandermonde matrix and matrix $C$ by the inverse Vandermonde matrix, as explained in the beginning of section 4, the bilinear algorithm $(A, B, C)$ computes the Toom-Cook algorithm. A common choice of nodes are small integer values, such as $0, 1, -1, 2, -2, \ldots$. Small integers can limit the magnitude of the scalars in the Vandermonde matrix.

As the number of nodes increases, the number of nonzeros in the Vandermonde matrix grows quadratically. The number of nonzeros in the $A$, $B$, and $C$ matrices can be reduced by selecting $\infty$ as a node $[62, 10]$. The $\infty$ node computes the product between the leading terms of inputs $f$ and $g$. To use the $\infty$ node, the last row for each of the decoding matrices, $A$ and $B$, is set to all zeros except for the last entry, which is set to 1. Similarly, the decoding matrix is set to $C = \tilde{V}^{-1}$, where $\tilde{V}$ is the original Vandermonde matrix with the last row set to all zeros, and the last entry is set to 1. The Karatsuba algorithm (4.1) is a Toom-Cook algorithm with the nodes 0, 1, and $\infty$.

4.3. Discrete Fourier transform. The use of integer nodes creates Vandermonde matrices that are ill-conditioned, limiting the Toom-Cook method to small linear convolutions. Instead, the set of nodes can be defined by the first $n$ nonnegative powers of the primitive $n$th primitive root of unity, $\omega_{(n)} = \exp(-2\pi i/n)$. The use of the powers of $\omega_{(n)}$ as nodes generates a Vandermonde matrix that is equivalent to the discrete Fourier matrix, $D^{(n)} \in \mathbb{C}^{n \times n}$ with $d_{mk}^{(n)} = \omega_{(n)}^{mk}$. The conditioning of this matrix, which is bounded by a constant value, permits us to extend the use of the Toom-Cook method beyond sizes of four. The inverse discrete Fourier matrix is simply $D^{(n)^{-1}} = (1/n)D^{(n)^*}$. The use of the discrete Fourier matrix and its inverse also defines bilinear algorithms for cyclic convolution $[8]$.

**Theorem 4.1 (Discrete cyclic convolution theorem).** The bilinear algorithm $(D^{(n)^T}, D^{(n)^T}, D^{(n)^{-1}})$ computes cyclic convolution.

**Proof.** By expanding the bilinear algorithm, $y = D^{(n)^{-1}}((D^{(n)} f) \odot (D^{(n)} g))$, we have the summation,

$$y_k = \frac{1}{n} \sum_{i=0}^{n-1} \omega_{(n)}^{-ki} \left( \sum_{j=0}^{n-1} \omega_{(n)}^{ij} f_j \right) \left( \sum_{t=0}^{n-1} \omega_{(n)}^{gt} g_t \right) = \frac{1}{n} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{t=0}^{n-1} \omega_{(n)}^{(j+t-k)i} f_j g_t.$$

It suffices to observe that for any fixed $u = j + t - k \neq 0$ or $\neq n$, the outer summation yields a zero result, since the geometric sum simplifies to

$$\sum_{i=0}^{n-1} \omega_{(n)}^{ui} = (1 - \omega_{(n)}^u)/(1 - \omega_{(n)}^u) = 0.$$

Therefore the only non-zero values in the summation are $f_j g_{k-j \pmod n}$, yielding cyclic convolution. \[\square\]

Other transformations to compute cyclic convolution may be defined based on roots of unity in other finite fields. One example is the Fermat number transform (FNT) $[2]$. The FNT leverages roots of unity in the ring of integers modulo the Fermat number, $F_{(n)} = 2^{2^t} + 1$ for some nonnegative integer $t$. The roots of unity can then be selected as powers of 2, yielding a transformation that requires only $O(n \log_2(n))$ integer or bitmask additions and bit-shifts.
4.4. Fast Fourier transform. Applying the DFT using the fast Fourier transform (FFT) can reduce the complexity of this algorithm from $O(n^2)$ to $O(n \log_2(n))$. The FFT applies a divide-and-conquer structure to the DFT, which can be seen by breaking the indices into even and odd components,

$$y_k = \sum_{i=0}^{n-1} x_i \omega_n^{ik} = \sum_{i=0}^{n/2-1} x_{2i} \omega_n^{ik} + \omega_n^k \sum_{i=0}^{n/2-1} x_{2i+1} \omega_n^{ik/2}.$$  

Computing both terms in equation (4.4) recursively gives the split-radix-2 variant of the Cooley-Tukey algorithm. In general, this division can be extended to larger parities. For example, consider breaking an $n = n_1 n_2$-length FFT into $n_1$ FFTs of size $n_2$,

$$y_{(kn_1+t)} = \sum_{s=0}^{n_1-1} \omega_{n_1}^s \left[ \omega_{n_2}^{sk} \sum_{i=0}^{n_2-1} x_{(in_1+s)i} \omega_{n_2}^{ik} \right].$$

This decomposition produces a split-radix-$n_1$ FFT algorithm, which uses $n_1$ FFTs of size $n_2$ followed by $n_2$ FFTs of size $n_1$. Both approaches yield an $O(n \log_2(n))$ cost.

4.5. Discrete trigonometric transform. A disadvantage of the DFT is its reliance of complex arithmetic. Complex additions require two real additions, and a direct computation of complex multiplication yields 4 real multiplications and 1 real addition. While this overhead does not affect the algorithm’s asymptotic cost, the additional work can reduce the efficiency of the algorithm in practice. An alternative is to apply an FFT-like transform with real values, which can be achieved with either the discrete sine or cosine transform.

Instead of using both the sine and cosine values, which necessitates complex values, the use of only the cosine leads to the discrete cosine transform (DCT), which can be used for deriving cyclic convolution algorithms \[7\]. This construct has the following bilinear algorithm,

$$f * g = C_N^{-1} \left( (C_N^T f) \odot (C_N^T g) \right),$$

where $C_N^T$ is the matrix derived from the DCT. Similar to the fast Fourier transform, the DCT also has a fast transform with a cost complexity of $O(n \log_2(n))$. However, while DCT avoids complex arithmetic, it requires computation of $O(n \log_2(n))$ cosine functions. The cosine transform is orthogonal, so the DCT algorithm, like the FT approach, is stable \[84\].

5. Convolution using modular polynomial arithmetic. Winograd presents a more general family of convolution algorithms \[91\] based on modular arithmetic over polynomials. Consider evaluating the remainder of the product $v = pq$, expressed as $r = v \mod M$. When $\text{deg}(M) > \text{deg}(v)$, where we denote the degree of a polynomial by $\text{deg}()$, then $r = v$. If instead $\text{deg}(M) \leq \text{deg}(v)$, then $r \neq v$, as the remainder of $v/M$ will produce a polynomial $r$ of degree at most $\text{deg}(M) - 1$. Winograd shows that computing remainders of $v$ (evaluating $p$ and $q$) with well-chosen polynomial divisors will produce new fast and stable linear convolution algorithms. We first present Winograd’s algorithm for recovering $v$ with $\text{deg}(M) > \text{deg}(v)$.

5.1. Winograd’s convolution method. In interpolation, each polynomial is evaluated at a set of discrete points. In Winograd’s convolution algorithm, the remainder of the product $v = pq$ is computed using $k$ distinct polynomial divisors, $m_i$. 
The $k$ polynomial divisors $m^{(1)}, m^{(2)}, \ldots , m^{(k)}$, must be coprime, or share no common roots. Together, the product of the $k$ polynomials define the larger polynomial divisor, $M = \prod_i m^{(i)}$. After computing the remainders with each the $k$ polynomial divisors, $m^{(i)}$, the remainder $r = v \mod M$ is recovered via the Chinese remainder theorem.

The Chinese remainder theorem for polynomials provides a specification for recovering $v = pq$ from the remainders of the $k$ polynomial remainders,

$$w^{(i)} \equiv r \equiv v \pmod{m^{(i)}}.$$  

so long as $\deg(v) < \deg(M)$. The bound on degree, in combination with the fact that $m^{(1)}, \ldots , m^{(k)}$ are coprime, ensures that the remainder polynomials $w^{(i)}, \ldots , u^{(k)}$ uniquely specify $v$. Consequently, defining $M^{(i)} = M/m^{(i)}$, Bézout’s identity implies that there exists polynomials $n^{(i)}$ and $N^{(i)}$ such that

$$M^{(i)} N^{(i)} + m^{(i)} n^{(i)} = 1.$$  

A set of such polynomials $N^{(1)}, \ldots , N^{(k)}$ can be computed by the extended Euclidean algorithm. The desired polynomial $v$ satisfying the set of equivalences (5.1) can be recovered as

$$v = \left( \sum_{i=1}^k u^{(i)} M^{(i)} N^{(i)} \right) \mod M,$$

since $u^{(i)} M^{(i)} N^{(i)} \equiv 0 \pmod{m^{(j)}}$ for $i \neq j$, while

$$u^{(i)} M^{(i)} N^{(i)} = u^{(i)} (1 - m^{(i)} n^{(i)}) \equiv u^{(i)} \pmod{m^{(i)}}.$$  

Interpolation is a particular instance of a Winograd’s convolution algorithm. By selecting the polynomial divisors $m^{(i)}$ to be the polynomial $x - \rho_i$, where $\rho_i$ are nodes, Winograd’s algorithm is equivalent to the Toom-Cook method using Lagrangian interpolation [8]. The DFT algorithm for linear convolution may be obtained by the polynomial $M(x) = x^k - 1$ with $k = n + r - 1$, whose roots are equally spaced on the unit circle on the complex plane [62]. With the choice $M(x) = x^k - 1$ for $k = n = r$, we obtain cyclic convolution [67], since the remainder polynomial $r$ has the right coefficients as

$$\sum_{i=0}^{2n-1} v_i x^i \equiv \sum_{i=0}^{n-1} (v_{n+i} + v_i) x^i \mod x^n - 1.$$  

Polynomial divisors can also be chosen to be of degree $d > 1$ (superlinear polynomials) [6]. Different degree choices for the polynomial divisors will yield trade-offs between the bilinear rank and the number of additions necessary. A few examples of this trade-off are shown in Table 2 [8, Table 5.2]. The degree choices also affect numerical stability.

5.2. Bilinear algorithm for Winograd’s convolution method. We now present a formulation of the bilinear algorithm for Winograd’s convolution algorithm. As before, we denote the coefficients of an arbitrary polynomial $p$ as $p$. Let $X_{(m,d)} \in \mathbb{C}^{\deg(m) \times (d+1)}$ be a matrix that can act on the coefficients of any degree $d$ polynomial $p$ to compute the coefficients of $r = p \pmod{m}$ as $r = X_{(m,d)} p$. Tolimieri provides
Table 2: Number of additions for Winograd’s convolution algorithm with different bilinear ranks

| n | b | Rank | Adds |
|---|---|------|------|
| 2 | 2 | 3 | 3 |
| 2 | 2 | 4 | 7 |
| 3 | 3 | 5 | 20 |
| 3 | 3 | 6 | 10 |
| 3 | 3 | 9 | 4 |
| 4 | 4 | 7 | 41 |
| 4 | 4 | 9 | 15 |

examples of this matrix [62], which he refers to as the matrix $E$. We provide a succinct algebraic construction of this linear operator,

\begin{equation}
X_{(m,d)} = \begin{bmatrix} I & -LU^{-1} \end{bmatrix},
\end{equation}

where $I$ is an identity matrix of size $\deg(m)$, $L$ contains the top $\deg(m)$ rows of $T_{(m,d-\deg(m)+1)}$, and $U$ contains the bottom $d+1$ rows of $T_{(m,d-\deg(m)+1)}$.

**Theorem 5.1.** Let $r = p \pmod{m}$, with $d = \deg(p)$, then $r = X_{(m,d)}p$.

**Proof.** Let $q = p/m$, so that $r = p - qm$. As $\deg(r) < \deg(p)$, then $\deg(p) = d - \deg(m)$. Defining $w = qm$, let

\[ p = \begin{bmatrix} p_{\text{upper}} \\ p_{\text{lower}} \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} w_{\text{upper}} \\ w_{\text{lower}} \end{bmatrix}, \]

where $p_{\text{upper}}, w_{\text{upper}} \in \mathbb{C}^{\deg(m)}$, so $p_{\text{upper}} = \begin{bmatrix} I & O \end{bmatrix} p$. Then we have that $r = p_{\text{upper}} - w_{\text{upper}}$. Furthermore, observing that $w = T_{(m,\deg(p)+1)}q$ and separating $T_{(m,\deg(p)+1)} = \begin{bmatrix} L \\ U \end{bmatrix}$, where $L \in \mathbb{C}^{\deg(m) \times (d-\deg(m)+1)}$ is lower-triangular and $U \in \mathbb{C}^{(d-\deg(m)+1) \times (d-\deg(m)+1)}$ is upper-triangular, we have

\[ w_{\text{upper}} = Lq. \]

Further, since $w = \begin{bmatrix} p - r \\ 0 \end{bmatrix}$, we have that $p_{\text{lower}} = w_{\text{lower}} = Uq$, and so $q = Up_{\text{lower}}$. Therefore, we obtain

\[ r = p_{\text{upper}} - LU^{-1}p_{\text{lower}} = X_{(m,d)}p. \]

Using this linear operator, we can now construct an operator for modular polynomial multiplication. Since,

\[ pq \mod m = (p \mod m)(q \mod m) \mod m, \]

we have that

\[ X_{(m,\deg(p)+\deg(q)-1)}(p \ast q) = X_{(m,2\deg(m)-1)}((X_{(m,\deg(p)})p) \ast (X_{(m,\deg(q)})q)). \]
Further, given a bilinear algorithm \((A,B,C)\) to compute linear convolution of two \(m\)-dimensional vectors, we can obtain an algorithm to compute \(r = pq \mod m\),

\[
r = X_{(m,2\deg(m)-1)} C((A^T X_{(m,\deg(p))} p) \odot (B^T X_{(m,\deg(q))} q)).
\]

To implement the Winograd’s convolution algorithm, we need to compute \(pq \mod m_i\) for \(i \in \{1, \ldots, k\}\). After obtaining these remainders \(u^{(1)}, \ldots, u^{(k)}\), it suffices to compute (5.3) by multiplying each \(u^{(i)}\) with the matrix,

\[
X_{(M,\deg(M)+\deg(m^{(i)})-2)} T_e^{(i)} X_{(m^{(i)},2\deg(m^{(i)})-1)},
\]

where \(e^{(i)} = M^{(i)} N^{(i)} \mod M\). Consequently, we can interpret Winograd’s convolution algorithm as a prescription for building a new bilinear algorithm for convolution from a set of \(k\) bilinear algorithms that compute the linear convolution between two sequences of vectors with dimension \(\deg(m^{(1)}), \ldots, \deg(m^{(k)})\).

**Definition 5.2 (Winograd’s Convolution Algorithm).** Given \(M = \prod_{i=1}^k m^{(i)}\) where \(\deg(M) = n + r - 1\) and \(m^{(1)}, \ldots, m^{(k)}\) are coprime, as well as \((A^{(i)}, B^{(i)}, C^{(i)})\) for \(i \in \{1, \ldots, k\}\), where \((A^{(i)}, B^{(i)}, C^{(i)})\) is a bilinear algorithm for linear convolution of vectors of dimension \(\deg(m^{(i)})\). Winograd’s convolution algorithm yields a bilinear algorithm \((A,B,C)\) for computing linear convolution with vectors of dimension \(r\) and \(n\), where

\[
A = \begin{bmatrix} X^T_{(m^{(1)},r-1)} A^{(1)} & \cdots & X^T_{(m^{(k)},r-1)} A^{(k)} \end{bmatrix},
\]

\[
B = \begin{bmatrix} X^T_{(m^{(1)},n-1)} B^{(1)} & \cdots & X^T_{(m^{(k)},n-1)} B^{(k)} \end{bmatrix}, \quad \text{and}
\]

\[
C = \begin{bmatrix} \tilde{C}^{(1)} & \cdots & \tilde{C}^{(k)} \end{bmatrix},
\]

with \(\tilde{C}^{(i)} = X_{(M,\deg(M)+\deg(m^{(i)})-2)} T_e^{(i)} X_{(m^{(i)},2\deg(m^{(i)})-1)} C^{(i)}\) and polynomial \(e^{(i)} = M^{(i)} N^{(i)} \mod M\).

To automatically generate Winograd’s convolution algorithm, it suffices to have a prescription to obtain \(e^{(i)} = M^{(i)} N^{(i)} \mod M\). Below, we give, to the best our knowledge, the first approach that computes the coefficients of \(N^{(i)}\) using only linear algebra.

**Definition 5.3.** Given coprime polynomials \(M\) and \(m\), the coefficients of polynomials \(N\) and \(n\) satisfying \(MN + mn = 1\) are

\[
\begin{bmatrix} N \\ n \end{bmatrix} = \begin{bmatrix} T_{(M,\deg(m)-1)} & T_{(m,\deg(M)-1)} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.
\]

**Proof.** The polynomials degrees of \(N\) and \(n\) are at most \(\deg(N) \leq \deg(m) - 1\) and \(\deg(n) \leq \deg(M) - 1\) [4]. Therefore, we can rewrite the equivalence \(MN + mn = 1\) as

\[
\begin{bmatrix} T_{(M,\deg(m)-1)} & T_{(m,\deg(M)-1)} \end{bmatrix} \begin{bmatrix} N \\ n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.
\]
Therefore, the cost of the fast symmetric algorithm is $O(l)$ of an symmetric matrix and a vector \[82\]. These algorithms compute the multiplication

Consequently, we can leverage fast nested bilinear algorithms to compute the product

We can embed (for simplicity) $H_{(f,n)}$ within a square Hankel matrix, $H_{(2n-1)}(x) \in \mathbb{R}^{(2n-1)\times(2n-1)}$, by appending $n-1$ zero columns to $H_{(f,n)}$ (using $x = [0^T \ f^T \ 0^T]^T$ to define each anti-diagonal of the matrix), so that $y = H_{(2n-1)}(x) \begin{bmatrix} \hat{g} \\ 0 \end{bmatrix}$. Now, we can observe that this square Hankel matrix is symmetric, and further that this type of matrix can be subdivided recursively into Hankel matrices,

Consequently, we can leverage fast nested bilinear algorithms to compute the product of a symmetric matrix and a vector \[82\]. These algorithms compute the multiplication of an $l \times l$ symmetric matrix with a vector using $l(l+1)/2$ multiplications. The choice of $l = 2$, requires 3 multiplications, and yields the fastest asymptotic complexity (same as Karatsuba’s algorithm $O(n \log_2(3))$). This variant of the algorithm performs the Hankel matrix–vector product $y = H_{(2k)}(x)z$ using the transformation,

The new form can be computed with 3 Hankel–vector products of half the dimension. The addition of the Hankel submatrices can be computed with $O(k)$ additions. Therefore, the cost of the fast symmetric algorithm is $T(n) = 3T(n/2) + O(n) = O(n \log_2(3/2))$ by directly computing the convolution once $n \approx r$.  

6. Other fast algorithms for convolution. We now discuss two other techniques for fast convolution, which are not based on polynomial algebra.

6.1. Fast symmetric multiplication. Recall that convolution can be solved by a Toeplitz matrix–vector product, $y = T(f,n)g$. Consider, for simplicity, the scenario when $n = r$ is the dimension of both $f$ and $g$. This problem can be converted to a Hankel matrix-vector product by reversing the order of the elements in the vector $g$ with $y = H_{(f,n)}\hat{g}$, where

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$$y_1 = \frac{H_{(k)}(x_1)z_1 + H_{(k)}(x_2)z_2}{H_{(k)}(x_2)z_1 + H_{(k)}(x_3)z_2}$$

$$y_2 = \frac{(H_{(k)}(x_1) - H_{(k)}(x_2))z_1 + H_{(k)}(x_2)(z_1 + z_2)}{H_{(k)}(x_2)(z_1 + z_2) + (H_{(k)}(x_3) - H_{(k)}(x_2))z_2}.$$ 

The new form can be computed with 3 Hankel–vector products of half the dimension. The addition of the Hankel submatrices can be computed with $O(k)$ additions. Therefore, the cost of the fast symmetric algorithm is $T(n) = 3T(n/2) + O(n) = O(n \log_2(3/2))$ by directly computing the convolution once $n \approx r$.  

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Consequently, we can leverage fast nested bilinear algorithms to compute the product of a symmetric matrix and a vector \[82\]. These algorithms compute the multiplication of an $l \times l$ symmetric matrix with a vector using $l(l+1)/2$ multiplications. The choice of $l = 2$, requires 3 multiplications, and yields the fastest asymptotic complexity (same as Karatsuba’s algorithm $O(n \log_2(3))$). This variant of the algorithm performs the Hankel matrix–vector product $y = H_{(2k)}(x)z$ using the transformation,
6.2. Minimizing scalar products. There remain other bilinear algorithms for convolution not covered by the techniques in the previous sections. For example, a bilinear algorithm for linear convolution of 3-dimensional vectors can be derived by the factorization [8],

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 1 & 0 & 0 \\
-1 & 1 & -1 & 0 & 1 & 0 \\
0 & -1 & -1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{bmatrix}
\odot
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
f_0 \\
f_1 \\
f_2
\end{bmatrix}
\odot
\begin{bmatrix}
g_0 \\
g_1 \\
g_2
\end{bmatrix}.
\]

While this bilinear algorithm does not achieve the minimal rank, the cost of encoding and decoding is lower than for the bilinear algorithm of the optimal rank since \((A, B, C)\) are sparse and require only additions or subtractions to apply.

7. Adaptations of convolution algorithms. All fast algorithms described so far can be adapted to efficiently perform multidimensional convolution, as well as to handle convolution when the filter size is small, i.e., \(r \ll n\). We describe adaptations to these scenarios in a general way, using the bilinear algorithm representation.

7.1. Multidimensional convolution via 1D convolution. For problems in image processing and scientific computing, where the inputs are 2D, 3D, or 4D, we need methods for multidimensional convolution. We provide a way to construct 2D convolution algorithms from 1D convolution algorithms, which extends in a natural way to higher-dimensional convolutions. Given \(F \in \mathbb{R}^{r \times r}\) and \(G \in \mathbb{R}^{n \times n}\), the 2D linear convolution \(Y = F \ast G\) with \(Y \in \mathbb{R}^{(n+r-1) \times (n+r-1)}\) gives

\[
y_{ab} = \min(a,r-1) \sum_{i=\max(0,a-n+1)}^{\min(b,r-1)} \sum_{j=\max(0,b-n+1)}^{\min(a,r-1)} f_{ij} g_{a-i,b-j}.
\]

A 2D convolution can be broken into a convolution of convolutions. That is, each row is individually convolved and then the rows are convolved amongst each other. Given a bilinear algorithm for a linear 1D convolution, \((A, B, C)\), the bilinear algorithm for a linear 2D convolution [55] is

\[
Y = C \left[ (A^T FA) \odot (B^T GB) \right] C^T.
\]

Correctness of this algorithm can be shown by defining the 2D convolution tensor, \(T^{(2D)} = T \otimes T\), so that

\[
t^{(2D)}_{i+n,j+a,r+v,a(n+r-1)+b} = t_{iua} t_{jvb}.
\]

This tensor computes 2D convolution as \(\hat{y}_k = \sum_{i,j} t^{(2D)}_{ijk} \hat{f}_i \hat{g}_j\), where \(\hat{y} = \text{vec}(Y)\), \(\hat{f} = \text{vec}(F)\), and \(\hat{g} = \text{vec}(G)\), since,

\[
y_{ab} = \sum_{i=0}^{r} \sum_{j=0}^{r} \sum_{u=0}^{n} \sum_{v=0}^{n} t_{iua} t_{jvb} f_{ij} g_{uv}.
\]

A rank \(R^2\) decomposition of \(T^{(2D)}\) can be constructed from a rank \(R\) decomposition of \(T\) as \((A \otimes A, B \otimes B, C \otimes C)\). The bilinear algorithm

\[
\hat{y} = (C \otimes C) \left[ ((A \otimes A)^T \hat{f}) \odot ((B \otimes B)^T \hat{g}) \right]
\]

is algebraically equivalent to (7.2).
7.2. Linear 1D convolution via multidimensional linear convolution.
We can also compute a long 1D linear convolution with multidimensional convolution using the technique called overlap-add [65, 58]. For simplicity, we assume both the filter \( f \) and input \( g \) are \( n \)-dimensional vectors. Suppose we want to decompose the \( n \)-length linear convolution, where \( n = \gamma \eta \), into \( \gamma \) linear convolutions for \( \eta \)-dimensional vectors. We represent overlap-add by the recomposition matrix \( Q^{(\gamma,\eta)} \in \mathbb{R}^{2n-1 \times (2\gamma-1)(2\eta-1)} \), defined by

\[
q_{ij}^{(\gamma,\eta)} = \begin{cases} 
1 : & \text{if } i = j - (\eta - 1)\left\lfloor \frac{j}{2\eta-1} \right\rfloor \\
0 : & \text{otherwise}
\end{cases}, \quad \text{with block structure}
\]

\[
Q^{(\gamma,\eta)} = \begin{bmatrix}
I_{\eta-1} & & & \\
 & I_{\eta-1} & I_{\eta-1} & \\
& & & & \ddots & \\
& & & & & I_{\eta-1} & I_{\eta-1} & \\
& & & & & & & I_{\eta-1}
\end{bmatrix}
\]

Theorem 7.1. Let \( \tilde{Y} = \tilde{F} \ast \tilde{G} \), where \( \tilde{F}, \tilde{G} \in \mathbb{R}^{\gamma \times \eta} \). Then if \( f = \text{vec}(\tilde{F}) \), \( g = \text{vec}(\tilde{G}) \), \( f \ast g = \text{vec}(Q^{(\gamma,\eta)} \tilde{Y}) \).

Proof. It suffices to show that multiplication along the last mode of \( T^{(2D)} = T^{(\gamma)} \otimes T^{(\eta)} \) with \( Q^{(\gamma,\eta)} \) gives \( T^{(\gamma\eta)} \), where we denote the linear convolution tensor for \( n \)-dimensional vectors by \( T^{(n)} \). Using (7.3), we can express \( Q^{(\gamma,\eta)} \) as

\[
q_{a\eta+b,c(2\eta-1)+d}^{(\gamma,\eta)} = \delta(a\eta + b, c(2\eta - 1) + d - (\eta - 1)c) = \delta(a\eta + b, c\eta + d),
\]

where \( b < \eta, d < 2\eta - 1 \), and \( \delta(i, j) \) is the Kronecker delta. Then the product of \( Q^{(\gamma,\eta)} \) and \( T^{(2D)} \) gives

\[
\sum_{c=0}^{2\gamma-2} \sum_{d=0}^{2\eta-2} f^{(2D)}_{i\eta+j, u\eta+v, c(2\eta-1)+d} \delta(a\eta + b, c\eta + d) = \sum_{c=0}^{2\gamma-2} \sum_{d=0}^{2\eta-2} f^{(\gamma)}_{i, j+c} f^{(\eta)}_{c, (\gamma) u, v, a\eta + b, c\eta + d} = f^{(\gamma\eta)}_{i\eta+j, u\eta+v, a\eta+b}.
\]

7.3. Cyclic 1D convolution via multidimensional cyclic convolution.
While the overlap-add approach decomposes a linear convolution, an \( n \)-length cyclic convolution can be broken into an \( n_1 \times n_2 \)-length nested cyclic convolution, where \( n = n_1n_2 \) and \( n_1 \) and \( n_2 \) are coprime, using the Agarwal-Cooley Algorithm [3]. The Agarwal-Cooley algorithm uses the Chinese remainder theorem to decompose the indices of cyclic convolution. To denote modulo arithmetic, let the notation \((x)_{\zeta}\) be
equivalent to \( x \mod z \). We start with the cyclic convolution between vectors \( f \) and \( g \),

\[
y_k = \sum_{i=0}^{n-1} f_i g_{k-i} n.
\]

We seek to decompose the 1D variables \( k \) and \( i \) into 2D variables. Define the following variables, \( k_1 = (k)_{n_1}, k_2 = (k)_{n_2}, i_1 = (i)_{n_1} \), and \( i_2 = (i)_{n_2} \). The Chinese remainder theorem asserts there is a unique bijection between the remainders of \( k_1, k_2 \) (and similarly for \( i_1, i_2 \)) to the original index \( k \) (and similarly \( i \)), through the mapping,

\[
k = (k_1 e_2 + k_2 e_1) n \quad \text{and} \quad i = (i_1 e_2 + i_2 e_1) n,
\]

where \( e_1 = n_2 m_2, e_2 = n_1 m_1, \) and \( m_1 \) and \( m_2 \) are integers that satisfy Bézout’s identity \((5.2)\),

\[
n_1 m_1 + n_2 m_2 = 1.
\]

Therefore, \((7.5)\) can be rewritten as

\[
y_{(e_1 k_1+e_2 k_2) n} = \sum_{i_1=0}^{n_1-1} \sum_{i_2=0}^{n_2-1} f_{i_1} g_{(e_1 i_1+e_2 i_2) n} e_{i_1} e_{i_2} k_1 \tilde{g}_{k_1-i_1, k_2-i_2}.
\]

Consequently, the 2D cyclic convolution of \( \tilde{F} \) and \( \tilde{G} \) gives \( \tilde{Y} \). Further, there exists a permutation matrix \( P \) such that \( p_{ij} = 1 \) if \( i = i_1 n_2 + j_2 \) and \( j = (e_1 i_1 + e_2 i_2) n \), so that \( \text{vec}(\tilde{Y}) = Py, \text{vec}(\tilde{F}) = Pf, \) and \( \text{vec}(\tilde{G}) = Pg \). This permutation allows us to obtain a bilinear algorithm for cyclic convolution by nesting two cyclic convolution algorithms, \((A(n_1), B(n_1), C(n_1))\) and \((A(n_2), B(n_2), C(n_2))\), as

\[
y = P^T (C(n_1) \otimes C(n_2)) (A(n_1) \otimes A(n_2)) (P f) \otimes (B(n_1) \otimes B(n_2)) (P g).
\]

### 7.4. Convolution with small filters.

Many popular CNN architectures today use filters (referred to as kernels in CNNs) that are small in size. The 2D filter’s size ranges from \( 11 \times 11 \) down to \( 3 \times 3 \) [94, 86, 36, 41, 53], whereas the images are of dimension \( 256 \times 256 \) and larger [53]. While fast algorithms, such as the interpolation approach, can produce efficient convolution algorithms for any input size, these algorithms are subject to large errors when the dimensions are larger than four [55, 91].

This error can be reduced by breaking a long convolution into smaller parts. We have shown how to decompose a convolution between two equally sized inputs into a series of smaller convolutions using the overlap-add method and the Agarwal-Cooley algorithm. Another strategy is to divide the inputs into subvectors. Let the Toeplitz matrix–vector multiplication, \( T(f, n) g \), compute the 1D convolution problem \( f * g \), with the block form,

\[
T(f, n) g = \begin{bmatrix} A & B & \cdots & \cdots & B & A \end{bmatrix} \begin{bmatrix} g_{n/r-1} \\ \vdots \\ \vdots \\ g_0 \end{bmatrix}.
\]

The block Toeplitz matrix \( T(f, n) \) can be written using Kronecker products [80], \( T(f, n) = (I \otimes A) + (I' \otimes B) \), where \( I' \) is a matrix with a sub-diagonal of ones.
Let \( X \in \mathbb{R}^{s \times (n/r)} \) be the matrix where \( \text{vec}(X) = x \). We can rewrite the Toeplitz matrix–vector multiplication problem as

\[
T_{(f,n)}g = (I \otimes A)\text{vec}(G) + (I' \otimes B)\text{vec}(G)
= \text{vec}(AG) + \text{vec}(BGIT').
\]

Given a fast convolution algorithm with a cost of \( T(r) \), the asymptotic complexity of computing this entire convolution is \( O(nr \cdot T(r)) \). When \( n \gg r \), this formulation can reduce the cost of the fast Fourier transform from \( O(n \log_2(n)) \) to \( O(n \log_2(r)) \).

8. Fast algorithm cost comparison. The bilinear rank of a convolution algorithm is most important for understanding its asymptotic complexity, especially when the algorithm is used in a nested manner. However, the number of additions required for computing linear combinations is nevertheless important and typically controls the constant-factor on the leading order term in the algorithmic cost. The composition of the bilinear algorithm, especially for larger convolution problems, can significantly affect the number of additions and scalar multiplications required to apply the linear combinations. Many bilinear algorithms exhibit an inverse relationship between the bilinear rank and the number of flops needed for applying the linear combination [8]. Different decomposition of the same convolution can lead to varied amounts of additions in the encoding and decoding step and bilinear ranks [10]. In this section, we build upon previous examinations on the number of flops required for various compositions of bilinear algorithms [8, 10] by analyzing the number of element-wise multiplications as well as flops from linear combinations. To do so, we pay particular attention to the structure of the matrices of the bilinear algorithms.

8.1. Cost bounds for general bilinear algorithms. For bilinear algorithms without structure, as in some variants of the Toom-Cook and Winograd’s convolution algorithm, a direct computation is needed. To bound this cost, we will study the structure of the matrices from certain Toom-Cook and Winograd’s algorithms by counting the number of non-zeros as \( \text{nnz}(A,B,C) \).

For applying a matrix–vector product \( Ax \), we can bound the number of additions \( a(A) \) and multiplications \( m(A) \) as

\[
a(A) \leq (\text{nnz}(A) - \#\text{row}(A)) \quad \text{and} \quad m(A) \leq \text{nnz}(A).
\]

We use an upper bound since the number of non-zeros does not necessarily correspond to additions, since some of these can be reused for later computation. The same bound can be applied for matrices \( B \) and \( C \).

We represent a bilinear algorithm \( F \) by its encoding and decoding matrices, \( F = (A, B, C) \). In general, the rank \( R \) of a bilinear algorithm \( F \) is implicit in the number of rows within matrices \( A \) and \( B \). With this notation, we can count the number of flops needed for any non-nested bilinear algorithm as

\[
a(F) \leq a(A) + a(B) + a(C) \quad \text{and} \quad m(F) \leq m(A) + m(B) + m(C) + R.
\]

8.2. Costs of fast transform algorithms. For a bilinear algorithm where the matrices \( (A, B, C) \) have an inherent recursive structure, a divide-and-conquer approach, such as the FFT and DCT, can yield asymptotically fast algorithms. For the radix-2 FFT algorithm, the cost in terms of complex additions \( \bar{a}(n) \) and multiplies \( \bar{m}(n) \), \( T(n) = (\bar{a}(n), \bar{m}(n)) \) is

\[
T(n) = 2T(n/2) + (n/2, n/2) \quad \text{with} \quad T(2) = (0, 2), \quad \text{so} \quad T(n) = (n(\log_2(n) - 1)/2, n \log_2(n)/2).
\]
8.3. Costs of multidimensional methods. Given a bilinear algorithm \( F_1 = (A_1, B_1, C_1) \) and \( F_2 = (A_2, B_2, C_2) \), let the Kronecker product of these bilinear algorithms be \( F = F_1 \otimes F_2 = (A_1 \otimes A_2, B_1 \otimes B_2, C_1 \otimes C_2) \). To bound the cost of the decoding matrix \( A \), we can use [10, Theorem 22],
\[
a(A) = a(A_{(1)}) \cdot \#\text{col}(A_{(2)}) + \#\text{row}(A_{(1)}) \cdot a(A_{(2)}) \quad \text{and} \\
m(A) = m(A_{(1)}) \cdot \#\text{col}(A_{(2)}) + \#\text{row}(A_{(1)}) \cdot m(A_{(2)}).
\]
This bound also applies to matrices \( B \) and \( C \). The rank of the new bilinear algorithm \( F \) is the product of the two smaller ranks, \( R = R_{(1)} R_{(2)} \). This nesting of bilinear algorithms can be extended to higher dimensions as well. Consider a set of nested bilinear algorithms \( F_1, \ldots, F_k \). We bound the cost of applying the nested linear combinations, similar to the 2D case in (8.3).

**Claim 8.1.** Given a nested bilinear algorithm \( F_1 \otimes \cdots \otimes F_k \), the cost for encoding with matrix \( A = A_{(1)} \otimes \cdots \otimes A_{(k)} \), where we define cost as \( T(A) = (a(A), m(A)) \), is
\[
(8.3) \quad T(A) = \sum_{i=1}^{k} \left( T(A_{(i)}) \cdot \prod_{j=1}^{i-1} \#\text{row}(A_{(j)}) \cdot \prod_{j=i+1}^{k} \#\text{col}(A_{(j)}) \right).
\]
The same cost can be applied for encoding the matrix \( B = B_{(1)} \otimes \cdots \otimes B_{(k)} \) and for the decoding with matrix \( C = C_{(1)} \otimes \cdots \otimes C_{(k)} \). For matrices where there are more rows than columns, such as for the encoding matrices \( A \) and \( B \) of linear convolution, the bottom-most level \( i = k \) dominates. If there are more columns than rows, then the first level dominates the cost.

8.4. Fast CNN algorithm costs. Both the training and inferencing of CNNs, which rely on a series of 2D convolutions, are computation heavy. As noted in the introduction, the convolutional layer can be the most expensive step. To better understand this cost, we will extend our cost model for bilinear algorithms to bound the costs of the convolutional layer in CNNs. In (2.2), a CNN performs many convolutions and adds them over multiple channels [55],
\[
(8.4) \quad Y^{(i,k,\hat{x},\hat{y})} = C \left[ \sum_{c=1}^{H} (A^T F^{(k,c)} A) \otimes (B^T G^{(i,c,\hat{x},\hat{y})} B) \right] C^T,
\]
where the indices \( \hat{x}, \hat{y} \) represents the different partitions of a 2D slice \( G \) to be convolved with a 2D slice of \( F \). Unlike in signal processing, convolutions in CNNs are associated with:
1. filters (kernels) that are often much smaller than the image,
2. filters that are reused in many convolutions,
3. separate convolutions over multiple channels are added altogether.
To reduce the cost of the filter transformations, the transformations of the filters, stored on the tensor \( U \), can be separated from the image transformations, which are stored on the tensor \( V \) [55]. The element-wise multiplications are then computed by
\[
(8.5) \quad M^{(i,k,\hat{x},\hat{y})} = \left[ \sum_{c=1}^{H} U^{(k,c)} \otimes V^{(c,\hat{y})} \right].
\]
The tensor \( M \) stores the resultant matrices from the convolutions between every combination of the \( N \) images, \( K \) filters, and \( P = D_H D_W / m^2 \) partitions of \( \hat{x} \) and \( \hat{y} \),
where \( D_W \times D_H \) is the dimension of the inputs images and \( m \) is the output size of each correlation convolution. Within the bracket of (8.5), each of the \( H \) channels needs to perform a convolution. Given a bilinear algorithm \( F = (A, B, C) \) to compute the convolution of (8.5), the cost of the convolutional layer in the CNN is the sum of \( T(D) \) (cost of image transformations), \( T(F) \) (cost of filter transformations), \( T(I) \) (cost of inverse transformations), and \( T(M) \) (cost of the bilinear multiplications), where

\[
\begin{align*}
T(F) &= KH \cdot T(A), \\
T(D) &= PNH \cdot T(B), \\
T(M) &= PKHN \cdot R^2, \text{ and} \\
T(I) &= PKN \cdot T(C).
\end{align*}
\]

Our cost is identical to the cost model proposed by Lavin and Gray in [55, Equation 23], which is of the form,

\[
\alpha' (1 + \beta'/K + \gamma'/P + \delta'/H)ND_H D_W HK,
\]

where \( \alpha' = R^2/m^2 \), \( \beta' = T(B)/R^2 \), \( \gamma' = T(A)/R^2 \), and \( \delta' = T(C)/R^2 \).

Lavin and Gray note that instead of directly computing the \( PKHN R^2 \) element-wise multiplications, (8.5) can be transformed to a matrix–matrix multiplication between a \( K \times H \) and \( H \times PN \) matrix. By applying a fast matrix-multiplication algorithm such as Strassen’s algorithm [85], the bilinear rank of this algorithm can be asymptotically smaller than a direct computation.

8.5. Generating fast algorithms for CNNs. The algorithm analyzed in subsection 8.4 is one of a handful of approaches to compute the convolutional layer. The prominent libraries for convolution employ an optimized direct computation, matrix-matrix multiplications [16], the FFT [27], or Lavin and Gray’s Winograd-based algorithm [55]. It is not immediately clear which algorithm has the most optimal performance. Experimental results from [48, 95] highlight the mixed performances of each approach. It appears that both the problem size and computer architecture can significantly affect the performance for each of the convolution algorithms.

To quantify the costs for each convolution algorithm, it is important to uncover both the structure of the algorithms \( (A, B, C) \) and the rank of the bilinear algorithm. These values determine the overhead of the linear convolutions as well as the asymptotic complexity of the algorithm. To do so, we first analyze the structure of the Toom-Cook matrices from section 4 and Winograd’s convolution algorithms from section 5. We calculate the structure of the matrices for a linear convolution where \( n = r \), \( (A, B, C) \), of the Toom-Cook method in Table 3 and nested Toom-Cook in Table 4. When the filter \( f \) is the same as size as \( g \), the encoding matrices are equivalent, so we only list the structure for the matrix \( A \).

The use of Winograd’s algorithm with a single superlinear polynomial divisor has been shown to reduce error [6], and it also can make the matrices \( (A, B, C) \) sparser at the expense of a larger bilinear rank. To study this more closely, we calculate the structure of these algorithms in Table 5 and the polynomial divisors that define the bilinear algorithm\(^2\). For a convolution of \( n \)-dimensional vectors, we reused the polynomials from the convolution of \( n - 1 \)-dimension vectors, i.e. the polynomials for \( n = 4 \) uses polynomials \( x + 2 \) and \( x - 2 \) and the polynomials from \( n = 3 \).

\(^2\)Matrices generated using gen_bilinear.py from https://github.com/jucaleb4/Bilinear-Algorithms-for-Convolution
Table 3: (nnz, adds, mults) of Toom-Cook-based linear convolution algorithms

| n | A       | C       | Rank |
|---|---------|---------|------|
| 2 | (4, 1, 4) | (5, 2, 5) | 3    |
| 3 | (11, 6, 11) | (16, 11, 16) | 5    |
| 4 | (22, 15, 22) | (36, 29, 36) | 7    |
| 5 | (37, 28, 37) | (65, 56, 65) | 9    |
| 6 | (56, 45, 56) | (101, 90, 101) | 11   |
| 7 | (79, 66, 79) | (145, 132, 145) | 13   |
| 8 | (106, 91, 106) | (197, 182, 197) | 15   |
| 9 | (137, 120, 137) | (257, 240, 257) | 17   |

Table 4: (nnz, adds, mults) of nested Toom-Cook-based linear convolution algorithms

| n | Nesting | A       | C       | Rank |
|---|---------|---------|---------|------|
| 4 | 2 × 2   | (16, 7, 16) | (25, 18, 25) | 9    |
| 6 | 2 × 3   | (44, 29, 44) | (76, 65, 76) | 15   |
| 8 | 2 × 4   | (88, 67, 88) | (162, 147, 162) | 21   |
| 8 | 2 × 2 × 2 | (64, 37, 64) | (125, 110, 125) | 27   |
| 9 | 3 × 3   | (121, 96, 121) | (228, 211, 228) | 25   |

We find that choosing a combination of polynomial divisors in the form \( x + \rho \) and \( x^2 + \rho \) produce matrices \((A, B, C)\) with a good balance between sparsity, rank, and conditioning. Similar to choosing nodes for interpolation [5], we choose the scalar \( \rho \) to be small powers of two as well as its reciprocal, its negative, and its negative reciprocal. We observe that choosing more linear polynomials than superlinear polynomials often produces algorithms with the lowest bilinear rank and that are still highly accurate.

9. Fast algorithm accuracy comparison. Although fast bilinear algorithms compute convolution in asymptotically less time than the direct approach, the use of linear combinations can introduce considerable error from floating-point arithmetic, especially when the multiplicative constants are large. For example, algorithms that use the Vandermonde matrix directly for its encoding and decoding step may incur too large of an error to be used in practice. The Toom-Cook method is consequently rarely used for inputs larger than four [91, 55, 89]. Although a CNN can still perform well under substantial error [19], other applications of convolution, such as in cosmology and physics, must have highly accurate convolutions.

There exist pairs of vectors of any magnitude that yield an exact output of zero for both linear and circular convolution. Consequently, convolution is an ill-posed problem, when considering the full space of nonzero inputs, meaning the relative error due to an input perturbation can be unbounded for any algorithm. To quantify the magnitude of the absolute error, Barabasz et al. derive an upper bound for the absolute error of the Toom-Cook algorithm [5], which is bounded by the product between the Frobenius norms of the matrices \((A, B, C)\) and the 2-norm for inputs \( f \) and \( g \). As with matrix products [40], the use of mixed norms yields a constant factor proportional to the input size. Below, we give a simple bound for general bilinear
Theorem 9.1 (1D bilinear algorithm convolution error). Given inputs \( f \in \mathbb{R}^r \) and \( g \in \mathbb{R}^n \), and perturbations \( \delta f, \delta g \) such that \( \|\delta f\| \leq \epsilon \|f\| \) and \( \|\delta g\| \leq \epsilon \|g\| \), the absolute error of the bilinear algorithm \((A, B, C)\) is

\[
\|\delta y\| \leq 2(\|C\| \cdot \|A\| \cdot \|B\| \cdot \|f\| \cdot \|g\|) \epsilon + O(\epsilon^2),
\]

where \( \cdot \parallel \) is the 2-norm.

Proof. We have that

\[
\delta y = C^T ((A\delta f) \odot (Bg) + (Af) \odot (B\delta g) + (A\delta f) \odot (B\delta g))
\]

Now, since

\[
\|x \odot y\|^2 = \sum_i |x_i y_i|^2 \leq (\sum_i |x_i|^2)(\sum_i |y_i|^2) = \|x\|^2 \cdot \|y\|^2,
\]

we have \(\|x \odot y\| \leq \|x\| \cdot \|y\|\). Therefore,

\[
\|\delta y\| \leq \|C\| \cdot (\|A\delta f\| \cdot \|Bg\| + \|Af\| \cdot \|B\delta g\|) + O(\epsilon^2),
\]

and the bound in the theorem follows by basic matrix and vector norm inequalities. \(\square\)

This error bound can be extended to higher dimensions as well [5].

Corollary 9.2. The convolution between order \( d \) inputs \( \mathcal{F} \in \mathbb{R}^{r \times \cdots \times r} \) and \( \mathcal{G} \in \mathbb{R}^{n \times \cdots \times n} \) yielding the tensor \( \mathcal{Y} = \mathcal{F} \ast \mathcal{G} \) using the nested 1D algorithm \((A, B, C)\) has an error of

\[
\|\delta \mathcal{Y}\| \leq 2(\|C\|^d \cdot \|A\|^d \cdot \|B\|^d \cdot \|\text{vec}(\mathcal{F})\| \cdot \|\text{vec}(\mathcal{G})\|) \epsilon + O(\epsilon^2).
\]

Corollary 9.2 shows that the error is proportional to the norm of the bilinear algorithm’s matrices \((A, B, C)\) and exponential to the dimension of the problem. For algorithms like the Toom-Cook method subsection 4.2, \(A\) and \(B\) are submatrices of a Vandermonde matrix and \(C\) is its inverse. Consequently, the absolute error of Toom-Cook convolution scales with the condition number of the Vandermonde matrix. When the nodes of the Vandermonde matrix are restricted to real values, Pan
proves that its condition number will be exponential in its dimension [69]. Therefore, the Toom-Cook method with real interpolation nodes will produce encoding matrices whose norm is exponential to the problem size. Selecting complex nodes can fix the ill-conditioning (e.g., via DFT subsection 4.3), but smarter selections of real nodes can also somewhat improve the conditioning.

9.1. Improved accuracy by nodes and scaling. Chebyshev nodes can improve the conditioning of the Vandermonde matrix without requiring additional costs [28]. While these nodes produce matrices with smaller condition numbers than that of integer nodes, the condition number still grows exponentially with respect to the dimension of the inputs. Empirical experiments based on exhaustive search show that choosing nodes with few significant mantissa bits and “symmetric” nodes, or nodes that are the negative, reciprocal, and negative reciprocal of previously chosen nodes, will yield better conditioned matrices [5]. For instance, the norm of the Vandermonde matrix can remain relatively low when selecting the points $2, -2, 1/2$, and $-1/2$. Finding a “widely accepted strategy for selecting [good] points” without using the complex domain is an open question [5].

Another technique to improve accuracy is diagonal scaling [89, 87]. Diagonal scaling introduces a diagonal matrix multiplication to each of the matrices $(A, B, C)$ while preserving the correct convolution output. This scaling reduces the magnitude of the entries in the matrices, which can improve the condition number of the matrices. By empirically identifying the best weights for the diagonal matrix, diagonal scaling reduces the maximum relative error of convolution. Experimental results of AlexNet show that for a correlation convolution algorithm with filter of size $r = 5$ and output of size $m = 9$ and using well-chosen nodes, diagonal scaling can reduce the maximum relative error from $7.53 \times 10^{-2}$ to $5.49 \times 10^{-4}$ [89]. By comparison, a direct computation of the correlation algorithm achieves a maximum relative error of $2.81 \times 10^{-6}$.

9.2. Improved accuracy by small nested convolutions. When highly accurate convolution algorithms are needed, well-chosen nodes may not offer enough norm reductions to significantly reduce the error. Instead, another strategy, proposed in signal processing, is to break a long convolution into a series of smaller ones. To illustrate why this works, recall that the condition number of an $n \times n$ Vandermonde matrix is $\Omega(n^n)$ [69]. Instead, if the bilinear algorithm is decomposed from an $n = n_1 n_2$-length convolution into a sequence of $n_1$-sized convolution nested with $n_2$-sized convolution, the condition number of the nested Vandermonde matrix by a Kronecker product is $\Omega((n_1 + n_2)^{n_1 + n_2})$. By repeating this decomposition, the accuracy of fast convolution algorithms that rely on the Vandermonde matrix can be greatly improved.

In order to devise such nested algorithms, we can employ the overlap-add approach for linear convolution and the Agarwal-Cooley algorithm for cyclic convolution from subsection 7.1. The error bound of using the Agarwal-Cooley algorithm is identical to Corollary 9.2, as the additional permutation matrices $P$ have a norm of 1. For the overlap-add approach, the recomposition matrix $Q^{(r/n)}$ introduces some floating-point error due to its additions at the end of each nested convolution. This error is relatively small as long as the dimension size is not too large, as shown by the following bound.

**Theorem 9.3.** Given inputs $f \in \mathbb{R}^{n_1}$ and $g \in \mathbb{R}^{n_2}$ and perturbations $\delta f$, $\delta g$ such that $\|\delta f\| \leq \varepsilon\|f\|$ and $\|\delta g\| \leq \varepsilon\|g\|$, the convolution of $n$-dimensional vectors based on the 1D linear convolution bilinear algorithm $(A, B, C)$ nested using the overlap-add
method has an error of

$$\|\delta y\| \leq 2^{d/2+1} \cdot \|C\|^d \cdot \|A\|^d \cdot \|B\|^d \cdot \|f\| \cdot \|g\| \cdot \varepsilon + O(\varepsilon^2).$$

Proof. Let $Q^{(i)} = Q^{(i)}_{\eta^{(i)}}, \gamma^{(i)}$ be the overlap-add matrix for the $i$th level of the nested bilinear algorithm. Using Corollary 9.2, we have that to first order in $\varepsilon$,

$$\|\delta Y\| \leq 2\|Q^{(d)} \otimes \cdots \otimes Q^{(1)} C\| \cdot \|A \otimes \cdots \otimes A\| \cdot \|B \otimes \cdots \otimes B\| \cdot \|f\| \cdot \|g\| \cdot \varepsilon.$$

Notice that $\|Q^{(i)}\| \leq \sqrt{2}$ since each row has at most two ones. Simplifying leads to the bound in the theorem.

9.3. Orthogonal polynomials as a basis. Decomposing a long convolution into a series of small nested convolutions can help us achieve highly accurate convolution. For cases where we need very accurate convolutions, one approach is to simply use the DFT. The discrete Fourier matrix has bounded conditioning, making it the ideal choice when accuracy is imperative. For cases where we want the same accuracy without use of complex arithmetic, we can instead use orthogonal polynomials.

By using orthogonal polynomials to define the encoding and decoding matrices $(A, B, C)$, the resulting matrices are generally well-conditioned. The trade-off is that the input must be converted to and from its monomial basis to the orthogonal basis. For certain orthogonal polynomials, this conversion introduces large multiplicative scalars [26], thereby negating the accuracy of the orthogonal polynomials. To avoid this conversion, the use of the orthogonal basis as defined by the Chebyshev polynomials can provide a workaround. Convolution can be computed with the Chebyshev polynomials without any conversion between the two bases. To achieve this, zeros may be appended to both inputs $f$ and $g$ until they are of size $n + r - 1$, and then a Chebyshev generalized Vandermonde matrix is used. This approach corresponds to an extension of the DCT transform algorithm described in subsection 4.5 to linear convolution.

10. Numerical experiments. We provide experimental results on the numerical accuracy of the following bilinear algorithms for linear convolution: Toom-Cook with integer nodes, Toom-Cook with Chebyshev nodes, Winograd convolution algorithm with superlinear polynomial divisors, and the nested Toom-Cook method. All the code is written in Python with NumPy. The inputs are composed of randomly chosen real numbers from the set $[0, 1)$. We use NumPy’s seed() function with a seed of 1 to ensure these results are reproducible. To compute the relative error, we compute a convolution from a bilinear algorithm using compute_bilinear_algorithm() and compare it with the convolution from a direct computation using direct_conv(). The relative error of each algorithm is averaged over ten trials.

10.1. Accuracy of the Toom-Cook method. Figures 1a to 1d show the relative error of the Toom-Cook method using small integer and Chebyshev nodes. Both methods incur substantial errors when the input size exceeds size six, especially as the dimension of the problem increases. For multidimensional problems, the use of Chebyshev nodes can significantly reduce the relative error from using integer nodes. However, we observe that once the algorithm is used for 3D or 4D convolution with inputs greater than size seven, the use of Chebyshev nodes still leads to high errors.

3Methods available in test.py from https://github.com/jucaleb4/Bilinear-Algorithms-for-Convolution
10.2. Accuracy of the Winograd convolution algorithm. We implement Winograd’s convolution algorithm based on the formulation from section 5 using the list of polynomial divisors \( m^{(i)} \) from Table 5 and plot the average relative errors in Figures 1a to 1d. We observe that Winograd’s convolution algorithm with just one superlinear polynomial divisor (increasing rank by 1 with respect to optimum), as is the case for convolutions of size up to 6, can significantly reduce the relative error compared to the Toom-Cook method with Chebyshev nodes. Furthermore, the number of flops required is only marginally larger than that of the Toom-Cook method. For a 5-dimensional convolution, the number of additions, multiplications, and rank of Winograd’s convolution algorithm Table 5 and the Toom-Cook method Table 3 suggests that Winograd’s convolution algorithm can achieve highly accurate results without a significant increase in arithmetic or use of complex arithmetic as with the DFT.

10.3. Accuracy of the nested Toom-Cook method. We show the accuracy of the nested Toom-Cook method in Figures 1a to 1d. Like the Winograd convolution algorithm, this algorithm can significantly reduce the error of the Toom-Cook method. A downside to the nested Toom-Cook approach is that the bilinear rank is greatly increased as compared to Winograd’s convolution algorithm. For example, our implementation of a Winograd’s convolution algorithm for 8-dimensional vectors has a rank of 17, whereas the nested Toom-Cook algorithm has a bilinear rank of 27.

However, a benefit of the nested Toom-Cook method is that the number of nonzeros in both the encoding and decoding matrices are lower than Winograd’s convolution algorithm, as shown seen in Table 4 and Table 5. For certain decompositions, the magnitude of the constants in the matrices never exceeds 1, as these matrices are built from very small Toom-Cook methods. A nested Toom-Cook algorithm for \( n = 8 \) can
be created by a triply nested Toom-Cook algorithm for \( n = 2 \), whose matrices are composed of zeros and the scalars 1 and \( 1/2 \) as well as their negatives. For convolutions where the overhead of applying the encoding and decoding is computationally expensive (such as near the leaves of the recursion tree), the nested Toom-Cook approach offers an accurate and efficient approach.

11. Future work. While novel strategies to reduce error from section 9, such as better node points and diagonal scaling, have the potential to improve the accuracy of the convolution algorithm by a constant factor [5, 89], it remains an open question on how much this strategy can improve accuracy. Deriving a lower bound on how much the error can be reduced, as well as an algorithm that determines the set of nodes and diagonal scaling with the optimal norm, could extend the use of Toom-Cook algorithms to larger filter sizes. Similarly, there remains space for a more comprehensive search of divisor polynomials to produce encoding and decoding matrices with an optimal balance between sparsity, rank, and conditioning.

Another area for future study is the design and analysis of parallel faster convolution algorithms. Performance of convolution algorithms in the parallel setting is generally dominated by communication costs, which is the amount of data movement required to compute the algorithm. As the size of the dataset grows, one can see that more communication is needed. Demmel and Dinh examined optimal communication for a direct approach [21]. Their work assumes a seven-nested layer algorithm, which computes convolution directly. An open question is determining how much communication is needed for fast bilinear algorithms such as the Toom-Cook algorithm, Winograd algorithm, and Toom-Cook with overlap-add. General approaches for deriving communication lower bounds of bilinear algorithms [83] may provide one avenue towards understanding communication costs in fast convolution algorithms.

Another question is whether the interpolation and Winograd techniques covered in this paper encompass all possible fast bilinear algorithms for convolution. If these techniques do cover all possible fast bilinear algorithms, this knowledge can narrow the search for optimal bilinear algorithms in accuracy and number of flops.

12. Conclusion. We present the various convolution algorithms as a bilinear algorithm. Using the formalism of bilinear algorithms, we present different variants of convolution, such as a polynomial interpolation, modular polynomial arithmetic, and an approach based on products of symmetric matrices and vectors. We derive simple formulations for generating these bilinear algorithms. We use these explicit formulations to quantify the cost of the various algorithms as well as simplify previous error bounds. Our analysis and experiments show that the nested convolution via overlap-add and Winograd’s convolution algorithm with superlinear polynomials can extend the use case of convolution for large and multidimensional filters and inputs. We propose some open questions about the computational efficiency and lower bounds for the error of these fast bilinear algorithms. With the simplified construction of these convolution algorithms in the language of linear algebra, we hope researchers in scientific computing, applied mathematics, and machine learning can discover new uses and methods for fast convolution.

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