Phase-locked and phase drift solutions of phase oscillators with asymmetric coupling strengths

YAMADA Hiroyasu *†

Abstract
Phase-locked solutions of coupled oscillators are studied with asymmetric coupling strengths or inhomogeneous natural frequencies. The solutions show remarkable profiles of phase lags from the pacemaker corresponding to the ratio of upward and downward coupling strengths. By means of the existence condition of phase-locked solutions, the transition points from phase-locked to phase drift states are estimated. The application of the existence condition to the case of the linear gradient of natural frequency illustrates some scaling properties in the frequency diagrams.

1 Introduction
Coupled oscillators have been dealt with in wide and diverse areas associated with oscillation and synchronization phenomena. The phenomenon of collective synchronization of phases or frequencies attracts attentions in not only biology but also in physics and engineering [1,2]. Theoretical studies have been deeply carried out for the case of global coupling, and clarified characters of phase transition of entrained oscillators [3–7]. The case of local coupling has also been studied in various contexts motivated by rhythmic phenomena in biological systems. They include animal gaits [8], fish swimming [9,10], and the peristaltic movement of gastrointestinal tracts [1,11,12]. In recent years, researches of coupled oscillators are extended to the case of non-local (but not global) coupling [13] and more complicated network systems [14].

Since such complicated networks have, in general, intricate boundaries and inhomogeneous couplings, we have been found little clues to analyses of dynamical behaviour of them. We can, however, analyse some phase-locked solutions for more simple and symmetric trees of couplings. In the case of $m$-ary trees with the height $n$ and $m^n$ leaves, for instance, the system of coupled oscillators has some particular solutions with identical phases in each level, and these solutions are reduced to the phase-locked solutions for chains of oscillators with asymmetric coupling strengths [15]. As well as complicated network structures, intricate dynamical processes appear in biological systems, of which it is impossible to drag the inherent or isolated parts out. It is one of the ways to study such systems that we construct models from the phenomenological evidence and

* A visiting researcher at R-laboratory, Department of Physics, Nagoya University, Nagoya 464–8602, Japan
† A visiting researcher at Bio-mimetic Control Research Centre, The Institute of Physical and Chemical Research (RIKEN), Nagoya 463–0003, Japan
draw qualitative features of them. When we compare the models with data ob-
served in real systems such as efficiency of movement and the transport volume,
we will need to know the dynamical behaviour and solutions of the models in
detail.

In this paper, we study chains of coupled phase oscillators. It is the remark-
able point that we can derive phase-locked solutions of the systems by algebraic
calculations, even if the coupling strengths are asymmetric or the natural fre-
quencies are inhomogeneous. Moreover, the analytical form of the solutions is
not much cumbersome, and thus the profile of solutions are easy to see. From
the existence conditions of the phase-locked solutions, we can estimate transition
points of phase-locked to phase drift states.

We present coupled phase oscillators with asymmetric coupling strengths in
the following section, and derive the phase-locked solutions in section 3. The
profile of the phase-locked solutions and bifurcation phenomena are studied
for the cases of asymmetric coupling strengths (section 4) and of a linear fre-
quency gradient (section 5). In section 6, we discuss the difference analogue of
the continuous system corresponding to the oscillator systems with asymmetric
coupling strengths.

2 Coupled phase oscillators

A linear chain of phase oscillators with asymmetric couplings is governed by the
following equations of phases \( \theta_j \) (\( j = 0, \ldots, n \)):

\[
\begin{align*}
\dot{\theta}_0 &= \omega_0 + a_u^* h(\theta_1 - \theta_0), \\
\dot{\theta}_j &= \omega_j + a_d h(\theta_{j-1} - \theta_j) + a_u h(\theta_{j+1} - \theta_j), \quad (j = 1, \ldots, n - 1), \\
\dot{\theta}_n &= \omega_n + a_u^* h(\theta_{n-1} - \theta_n),
\end{align*}
\]

where \( \omega_j \) is the natural frequency, \( a_u \) and \( a_d \) are coupling coefficients, and \( h \) is a
periodic function. For the 0th and nth oscillators, coupling coefficients, \( a_u^* \) and
are set according to the boundary conditions.

Introducing new variables defined by neighbouring sites,

\[
\psi_j := \theta_j - \theta_{j-1}, \quad d_j := \omega_j - \omega_{j-1},
\]

we obtain \( n \) equations for the phase differences:

\[
\dot{\psi} = d + A_u h_+ - A_d h_-, \tag{2}
\]

\[
\begin{bmatrix}
\psi_1 \\
\psi_2 \\
\vdots \\
\psi_{n-1} \\
\psi_n
\end{bmatrix}, \quad
\begin{bmatrix}
d_1 \\
d_2 \\
\vdots \\
d_{n-1} \\
d_n
\end{bmatrix}, \quad
\begin{bmatrix}
h(\pm \psi_1) \\
h(\pm \psi_2) \\
\vdots \\
h(\pm \psi_{n-1}) \\
h(\pm \psi_n)
\end{bmatrix}, \quad
\begin{bmatrix}
-a_u^* & a_u & 0 & \cdots & 0 \\
-a_u & a_u & -a_u & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & -a_u & a_u & -a_u
\end{bmatrix}
\]
In the analysis for phase locked/drift solutions shown below, we assume $h$ is an odd function, $h(-\psi) = -h(\psi)$, and thereby $h_- = -h_+$.  

3 Phase lags

Phase-locked solutions of eq. (2) mean all the oscillators have the same period and the phase difference is constant, $\dot{\psi}_j = 0$. Thus they are solutions of

$$A h + d = 0,$$

where $A := A_u + A_d$ and $h := h_+$. We solve formal solution of the above equations under the free boundary conditions (mutual entrainment), and one-sided coupling conditions (forced entrainment).

Fast, we study the case of free boundary conditions, $a_u^* = a_u, a_d^* = a_d$, where the boundary oscillators and their neighbours are coupled in the bi-directional manner. Under these conditions, all of the oscillators have the same period

$$\dot{\theta}_j = \Omega_n / \Lambda_n,$$

where $\Omega_n$ and $\Lambda_n$ are defined below. From eq. (3) we get the solution $h_j (j = 1, \ldots, n)$, the $j$th element of $h$, in the form of

$$a_d h_j = \Delta_{n-j} - (\Delta_n / \Lambda_n) \Lambda_{n-j},$$

or

$$a_u h_j = \tilde{\Delta}_{j-1} - (\tilde{\Delta}_n / \tilde{\Lambda}_n) \tilde{\Lambda}_{j-1},$$

where

$$\begin{align*}
\Omega_j &:= \sum_{k=0}^{j} \omega_k \left( \frac{a_u}{a_d} \right)^k, \\
\Lambda_j &:= \sum_{k=0}^{j} \left( \frac{a_u}{a_d} \right)^k, \\
\tilde{\Lambda}_j &:= \sum_{k=0}^{j} \left( \frac{a_d}{a_u} \right)^k, \\
\Delta_j &:= \sum_{k=0}^{j} (\omega_{j+k} - \omega_0) \left( \frac{a_u}{a_d} \right)^k, \\
\tilde{\Delta}_j &:= \sum_{k=0}^{j} (\omega_{n-k} - \omega_n) \left( \frac{a_d}{a_u} \right)^k.
\end{align*}$$

The phase difference, $\psi_j$, is obtained by inverting eq. (5). When the coupling function $h$ is continuous and bounded, solutions exist only if $\min h < h_j < \max h$ for all $j$. The linear-stability conditions of solutions are obtained from the eigenvalue problem of the linearized matrix, $Ah'$, where the $j$th elements of $h'$ is $h'(\psi_j)$.

We comment the cases of forced oscillations, that is, one of the terminals is a forced oscillator. Suppose that the 0th oscillator is one-sided coupling to its neighbour, $a_u^* = 0$, but the opposite end satisfies the free boundary condition, $a_d^* = a_d$. Under these conditions, a solution of eq. (3) is $h_j = \Delta_{n-j} / a_d$. If the boundary conditions are exchanged, that is, the $n$th oscillator is a forced oscillator, $a_u^* = 0$, and the 0th oscillator satisfies the free condition, $a_d^* = a_u$, a solution of eq. (3) is obtained in the form of $h_j = \tilde{\Delta}_{j-1} / a_u$.  

3
4 Asymmetry couplings

We study the mutual entrainment to a pacemaker with asymmetry couplings in the present section. We assume that all the oscillators have the same natural frequency but only the 0th oscillator, the pacemaker, has the high frequency. If we set \( \omega_0 > 0 \) and \( \omega_1 = \cdots = \omega_n = 0 \), eq. (5) becomes
\[
h_j = -\frac{\omega_0 \Lambda_{n-j}}{a_d \Lambda_n}.
\]
The entrained frequency is obtained from eq. (4),
\[
\bar{\omega}_n = \begin{cases} 
\frac{\omega_0}{1 + \cdots + \lambda^n}, & (a_u > a_d), \\
\frac{\omega_0 \lambda^n}{1 + \cdots + \lambda^n}, & (a_u < a_d), 
\end{cases}
\]
where \( \lambda := \max(a_u, a_d)/\min(a_u, a_d) \). We set \( a_u, a_d, \) and \( h'(0) \) are positive below. For numerical calculations shown in figures, we chose the coupling function as \( h(\psi) = \sin \psi + 0.1 \sin 2\psi - 0.03 \sin 3\psi \) to obtain generic results.

When the coupling strengths are asymmetric, \( a_u \neq a_d \), the profile of \( h_j \) given in eq. (7) has the exponential dependence on \( j \). In fig. 1, we illustrate some features of phase-locked solutions: the profile of coupling interaction, phase difference and phase lag from the pacemaker. Calculations of these quantities are carried out for the system of 20 oscillators \( n = 19 \) and for three ratios of couplings, \( a_u/a_d = 2, 1, \) and \( 1/2 \). If the coupling function \( h(\psi) \) is approximated to the linear function for small \( \psi \), then the dependence of the phase lag, \( \theta_j - \theta_0 \), on \( j \) is almost (i) exponential for \( a_u > a_d \), (ii) parabolic for \( a_u = a_d \), and (iii) linear for \( a_u < a_d \). Such dependence is inherited for more complicated network systems on trees [15].

For small coupling strengths, the phase-locked state is broken and the phase of each oscillator drifts from the phase of the pacemaker. Such a phase drift solution has two regions of the frequency-entrainment since the pacemaker and other oscillators have different natural frequencies. The phase drift occurs between these two regions. Fig. 2 shows the transition between phase-locked and phase drift solutions for the three ratios of \( a_u \) and \( a_d \). Each system has 10 oscillators \( n = 9 \). The averaged frequency of the \( j \)th oscillator and the coupling strength are defined by
\[
\langle \omega_j \rangle := \lim_{t \to \infty} (\theta_j(t) - \theta_j(0))/t, \quad \epsilon := \min(a_u, a_d) \max h.
\]
In fig. 2 \( \langle \omega_j \rangle \) is obtained from numerical integrations of eq. (1) with the step width in \( \epsilon, \Delta \epsilon = 10^{-3} \times \max h \). We can estimate the transition point of phase-locked and phase drift solutions from the necessary condition for the existence of the phase-locked solutions,
\[
\epsilon > \epsilon_n := \frac{1 + \cdots + \lambda^{n-1}}{1 + \cdots + \lambda^n}.
\]
The critical point \( (\epsilon_n, \bar{\omega}_n) \) is plotted by the open circle in fig. 2.

Numerical calculations show that phase drift states always consist of two regions: the pacemaker and the other oscillators. It implies that if the pacemaker can entrain its neighbouring oscillator, all oscillators will be entrained to the
pacemaker, but if not so, no oscillators entrained to the pacemaker. It is also possible to estimate this entrainment feature from the existence condition of the phase-locked solutions. We assume that the pacemaker is entraining \( j \) oscillators. Then we divide the system into two sets of coupling oscillators: (i) upper \((j + 1)\) oscillators including the pacemaker, and (ii) lower \((n - j)\) oscillators. For the upper set (i), the phase-locked solution exist only if

\[
\frac{\omega_0}{1 + \ldots + \lambda^{j-1}} < \epsilon. \tag{10}
\]

For the lower set (ii), we add one oscillator with the entrainment frequency of the set (i) to the upper side as the pacemaker. If the coupling strength \( \epsilon \) satisfies

\[
\epsilon < \frac{\bar{\omega}_j}{1 + \ldots + \lambda^{n-j-1}}, \tag{11}
\]

then the phase drift solution exists for the set (ii) with the pacemaker. Since no value of the coupling strength \( \epsilon \) satisfies both of conditions (10) and (11), oscillators entraining to the pacemaker are all or none.

5 Linear frequency gradient

Daido [12] has studied chains of coupled phase oscillators with a linear gradient of natural frequencies by numerical calculations. He has shown some scaling properties in the frequency diagram, a plot of averaged frequencies against the coupling strength. In the present section, we attempt to derive some of them from eq. (5).

We assume that the natural frequency of the \( j \)th oscillator is \( \omega_j := j/n \) and coupling strengths are symmetric \( a := a_u = a_d \). Then a phase-locked solution is obtained from eq. (5) in the form of

\[
h_j = \frac{\delta j(n+1-j)}{a}, \tag{12}
\]

where \( \delta := 1/n \) is the difference of natural frequency between neighboring oscillators. The entrained frequency of this solution is

\[
\bar{\omega}_n = \frac{\delta n}{2} \left( = \frac{1}{2} \right). \tag{13}
\]

We set \( a \) and \( b'(0) \) are positive, and chose the coupling function as the previous section for numerical calculations shown in figures below.

We note that \( h_j \) takes the maximal value at the centre of the chain of entrained oscillators from eq. (12). If the phase difference \( \psi_j \) is small enough to approximate that the coupling function \( h(\psi) \) has the linear dependence on \( \psi \), \( \psi_j \) has the convex dependence on \( j \) as well as \( h_j \). Then the phase lag (\( \theta_j - \theta_0 \)) has an inflection point at the centre of the chain. Figure (3) shows the profile of phase lags and other variables.

When the coupling strength is diminished, the entrained region is divided into some clusters of frequency-entrained oscillators, called frequency plateaus [1,11,12]. In the limiting case that the coupling strength is vanishing, entrained domains of oscillators will split into each oscillator.
Although the frequency diagram shows fine and complicated bifurcation structure, we can estimate the approximate arrangement of transition points by means of the necessary conditions for existence of phase-locked solutions. To illustrate the frequency diagram, we define the averaged frequency of the \( j \)th oscillator and the coupling strength as eq. (9). We pick out the frequency plateau with \( m \) oscillators from the \((n + 1)\)-oscillator system, and consider these oscillators as the isolated system with the free boundary conditions in both sides. Phase-locked solutions of the \( m \)-oscillator system exist only if

\[
\epsilon > \epsilon_m := \begin{cases} 
(\delta/8)m^2, & (m \text{ : even}), \\
(\delta/8)(m^2 - 1), & (m \text{ : odd}).
\end{cases}
\]  

(14)

The entrained frequency is the average of natural frequencies,

\[
\bar{\omega}_{k,k+m-1} = (\omega_k + \cdots + \omega_{k+m-1})/m \\
= \delta(m + 2k - 1)/2, \quad (k = 0, \ldots, n - m).
\]  

(15)

In fig. 4, we put the approximate critical point \((\bar{\omega}_{k,k+m-1}, \epsilon_m)\) on the frequency diagrams for the system of 10 oscillators \((n = 9)\). Some of approximate points are located near the transition points, but others have no transition points around them. The latter points indicate that it is impossible for the system to have some states such as small plateaus located near the boundaries or extremely asymmetric arrangements of plateaus. We obtain the scaling behaviour of \(\bar{\omega}_{0,m-1}\) to \(\epsilon_m/\epsilon_{n+1}\) from eqs. (14) and (15) as

\[
\bar{\omega}_{0,m-1} \sim \frac{1}{2} \left( \frac{\epsilon_m}{\epsilon_{n+1}} \right)^{1/2}
\]  

(16)

in the limit of \(n \to \infty\).

6 Discussion

We can consider the phase-difference system is derived from some continuous partial differential equation by means of the difference analogue. If the coupling function \( h \) is assumed to be the odd function, the continuous system corresponding to eq. (2) becomes

\[
\partial_t \psi + \Gamma \partial_x h(\psi) = d(x) + D \partial_x^2 h(\psi),
\]  

(17)

where \(x\) is the spatial coordinate. Coefficients \(\Gamma\) and \(D\) are defined by

\[
\Gamma := (a_d - a_u)\Delta x, \quad D := \frac{a_d + a_u}{2} (\Delta x)^2
\]

for the difference interval \(\Delta x\). Here we took the central difference for the differential of first order. In the continuous system \([17]\) entrained solutions are obtained from the ordinary differential equation in the form of

\[
D \frac{d^2 h}{dx^2} - \Gamma \frac{dh}{dx} + d(x) = 0.
\]  

(18)
When all of natural frequencies are identical, \( d(x) \equiv 0 \), solutions of eq. (18) are

\[
h(x) = \begin{cases} 
  c + b \exp(\Gamma x/D), & (\Gamma \neq 0), \\
  c + bx, & (\Gamma = 0),
\end{cases}
\]

where \( b \) and \( c \) are constants defined by boundary conditions. These solutions are consistent with eq. (7) and fig. 1. In another case corresponding to the linear frequency gradient, \( d(x) = \delta \) (const.), a solution of eq. (18) are obtained in the form of

\[
h(x) = c + bx - (\delta/2D)x^2.
\]

This is also consistent with eq. (12) and fig. 3, if we set arbitrary constants \( b \) and \( c \) in proper values.

Here we give more examples. When the both cases mentioned above are combined, that is, the natural frequency has the linear gradient in space, \( d(x) = \delta \), and coupling strengths are asymmetric, \( \Gamma \neq 0 \), a solution of eq. (18) is

\[
h(x) = c + (\Gamma \delta/D)x + b \exp(\Gamma x/D).
\]

Even if \( \Gamma = 0 \), we can obtain the similar dependence of \( h \) on \( x \) from the system in which the natural frequency has the exponential gradient in space as \( d(x) = \gamma \exp(\gamma(x - L)) \) (L is the system size). A solution of this system is

\[
h(x) = c + bx - (1/D\gamma) \exp(\gamma(x - L)).
\]

In the last, we comment on the case that the coupling functions is divided into the odd and even parts. In general, the division of a function \( h(x) \) is carried out in the form of

\[
h(x) = h_{\text{odd}}(x) + h_{\text{even}}(x),
\]

\[
h_{\text{odd}}(x) := \frac{h(x) - h(-x)}{2}, \quad h_{\text{even}}(x) := \frac{h(x) + h(-x)}{2}.
\]

The system of coupled phase oscillators (2) is derived from the following continuous system by the discretization of space

\[
\partial_t \psi + \Gamma_{\text{odd}} \partial_x h_{\text{odd}}(\psi) - \Gamma_{\text{even}} \partial_x h_{\text{even}}(\psi) = d(x) + D_{\text{odd}} \partial_x^2 h_{\text{odd}}(\psi) - D_{\text{even}} \partial_x^2 h_{\text{even}}(\psi), \tag{19}
\]

where coefficients are defined as

\[
\Gamma_{\text{odd}} := (a_d - a_u)\Delta x, \quad \Gamma_{\text{even}} := (a_d + a_u)\Delta x \\
D_{\text{odd}} := \frac{a_d + a_u}{2}(\Delta x)^2, \quad D_{\text{even}} := \frac{a_d - a_u}{2}(\Delta x)^2.
\]

We note that the coefficients of advection and diffusion terms are almost exchanged between the odd and even parts,

\[
D_{\text{odd}} = (\Delta x/2)\Gamma_{\text{even}}, \quad D_{\text{even}} = (\Delta x/2)\Gamma_{\text{odd}}.
\]

It is also possible to obtain the phase-locked solutions from eq. (19) as above, but we do not mention them here.
7 Conclusions

We obtained the phase-locked solutions of coupled phase oscillators with asymmetric coupling strengths or inhomogeneous natural frequencies. These solutions show the different profiles of phase lags between the three types of ratios of the upward to downward coupling strength. The phase differences obtained in analytical form, must be applicable to consider the efficiencies of biological motions and transportations based on the collective oscillation. Moreover, inherent structures of the real systems, the asymmetry of couplings, for example, are suggested by the spatio-temporal patterns observed in the systems.

From the existence conditions of phase-locked solutions, we derived transition points from phase-locked to phase drift states. We also obtained that the length of entrained region to the pacemaker is approximately zero or the system size. However, for the discrete time-dependent Ginzburg-Landau equations, the number of entraining oscillators to the pacemaker depends on the coupling strength, and the sequence of bifurcations is observed numerically [15].

Next, we showed the profiles of phase-locked solutions for the linear frequency gradient. The phase differences take the maximal absolute value around the centre of the chain system. This means that the phase drift tends to occur around the centre when the coupling strengths are diminished. As well as the case of the asymmetric coupling strengths, we estimate the approximate transition points in the frequency diagram from the existence conditions of phase-locked solutions, and derive some scaling properties.

References

[1] A. T. Winfree, *The Geometry of Biological Time* (Springer, New York, 1980)

[2] Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence* (Springer, Berlin, 1984)

[3] Y. Kuramoto, in *Proceedings of the International Symposium on Mathematical Problems in Theoretical Physics*, ed. H. Araki (Springer, New York, 1975) p. 420.

[4] S. H. Strogatz and R. E. Mirollo, J. Stat. Phys. 63 (1991) 613.

[5] H. Daido, Prog. Theor. Phys. 88 (1992) 1213; 89 (1993) 929; Phys. Rev. Lett. 73 (1994) 760; Physica D 91 (1996) 24.

[6] H. D. Crawford, Phys. Rev. Lett. 74 (1995) 4341.

[7] S. H. Strogatz, Physica D 143 (2000) 1.

[8] J. J. Collins and I. N. Stewart, J. Nonlinear Sci. 3 (1993) 349; Biol. Cybern. 68 (1993) 187.

[9] A. H. Cohen, P. J. Holmes, and R. R. Rand, J. Math. Biol. 13 (1982) 345.

[10] J. D. Murray, *Mathematical Biology*, 2nd Corrected Ed. (Springer, Berlin, 1993)
[11] G. B. Ermentrout and N. Kopell, SIAM J. Math. Anal. 15 (1984) 215; N. Kopell and G. B. Ermentrout, Commun. Pure Appl. Math. 39 (1986) 623.

[12] H. Daido, Prog. Theor. Phys. 102 (1999) 197; 1055(E).

[13] Y. Kuramoto, Prog. Theor. Phys. 94 (1995) 321; Y. Kuramoto and H. Nakao, Phys. Rev. Lett. 76 (1996) 4352.

[14] S. H. Strogatz, Nature 410 (2001) 268.

[15] H. Yamada and T. Nakagaki: Oscillation patterns in cytoplasmic networks of the Physarum plasmodium, in Traffic and Granular Flow ’01, October 15–71, 2001, Nagoya University, Japan, p. 49.

Figures

Figure 1: The profiles of phase-locked solutions for phase oscillators with asymmetric couplings: the coupling interaction, $h_j$, phase difference, $\psi_j$, and phase lag from the pacemaker, $(\theta_j - \theta_0)$, where $\psi_j$ and $(\theta_j - \theta_0)$ are plotted with $\pi$ units. The left, centre, and right columns are the cases of (i) $a_u/a_d = 2$ ($a_u = 1.0, a_d = 0.5$), (ii) $a_u/a_d = 1$ ($a_u = 1.5, a_d = 1.5$), and (iii) $a_u/a_d = 1/2$ ($a_u = 1.5, a_d = 3.0$), respectively. Solid lines are connecting the points derived eq. (7), while open circles are obtained from numerical calculations of the phase equation (1).
Figure 2: The frequency diagrams for phase oscillators with the asymmetric couplings. The left, middle, and right diagrams are the cases of (i) $a_u/a_d = 2$, (ii) $a_u/a_d = 1$, and (iii) $a_u/a_d = 1/2$, respectively. The averaged frequencies $\langle \omega_j \rangle$ are plotted against the coupling strength $\epsilon$. The dots are obtained from numerically calculations of eq. (1). The open circles denote critical points of the existence condition for phase-locked solutions.

Figure 3: The profiles of phase-locked solutions for phase oscillators with linear frequency gradient: the coupling interaction, $h_j$, phase difference, $\psi_j$, and phase lag from the 0th oscillator, $(\theta_j - \theta_0)$. The upper left shows the natural frequency of each oscillator.
Figure 4: The frequency diagram for phase oscillators with the linear gradient of natural frequencies. The diagram is plotted the averaged frequencies $\langle \omega_j \rangle$ against the coupling strength $\epsilon$. Averaged frequencies plotted by dots are numerically calculated. The open circles denote critical points for the existence condition of phase-locked solutions with $m$ oscillators ($m = 1, \ldots, n + 1$).