R–MATRIX OF THE ORBIFOLD GROMOV–WITTEN THEORY OF AN ELLIPTIC ORBIFOLD

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ABSTRACT. In this paper we present the particular Givental’s action recovering the Gromov–Witten theory of the orbifold $\mathbb{P}^1_{4,4,2}$ via the product of the Gromov–Witten theories of a point. This is done by employing mirror symmetry and certain results in FJRW theory. In particular we present the particular Givental’s action giving the CY/LG correspondence between the Gromov–Witten theory of the orbifold $\mathbb{P}^1_{4,4,2}$ and FJRW theory the pair $(\tilde{E}_7, G_{\text{max}})$. Using Givental’s action again we recover this FJRW via the product of the Gromov–Witten theories of a point. In this way we get the result in “pure” Gromov–Witten theory with the help of modern mirror symmetry conjectures.

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1. INTRODUCTION

Inspired by physicists, cohomological field theories (CohFT for brevity) have been under intensive investigation of mathematicians since the early 90s. From the point of view of physics there was a particular interest in their relation with mirror symmetry, where the Saito–Givental CohFT of an isolated singularity $\tilde{W}$ gives the B–model and Gromov–Witten CohFT of a Calabi–Yau variety $X$ gives the A–model. Another type of A–model CohFT was conjectured in
physics and later constructed by mathematicians. This was the so-called FJRW CohFT of a pair \((W, G_{max})\), where \(W\) is a potential defining an isolated singularity and \(G_{max}\) is a symmetry group of \(W\).

Of particular interest to mathematicians was the classification of all CohFTs, understood axiomatically in a general context (see [KM]). From this point of view, the CohFTs mentioned above are some very special points in the space of all CohFTs. In this general context it’s convenient to work with a CohFT in terms of a certain generating functions for the CohFT, known as the *partition function*. An important tool when working with the CohFTs is the group action of Givental. In \([G]\) two groups were introduced, acting on the space of all partition functions of CohFTs. These groups are now called *upper–triangular* and *lower–triangular* groups of Givental.

Givental’s action is applied to the classificational problem as follows. For an arbitrary CohFT with the partition function \(Z\) one tries to find the upper–triangular group element \(R\) and a lower–triangular group element \(S\), such that

\[
Z = \hat{S} \cdot \hat{R} \cdot Z^{\text{basic}},
\]

where \(\hat{S}\) and \(\hat{R}\) denote the action of \(S\) and \(R\) respectively, \(Z^{\text{basic}}\) is the partition function of some “basic” CohFT. Canonical choice of the “basic” CohFT is given by the product of the Gromov–Witten theories of a point. In this case the upper–triangular group element as above is called a *R–matrix of the CohFT*.

Another phenomenon closely related to mirror symmetry is the *Calabi–Yau/Landau–Ginzburg correspondence* (CY/LG for brevity). Here Givental’s action also has an application. The CY/LG correspondence is a conjecture which in this context states that for two different A–model partition functions \(Z^{GW}\) and \(Z^{FJRW}\), being mirrors to the same B–model, there are an upper–triangular group element \(R\) and a lower–triangular group element \(S\), such that

\[
Z^{FJRW} = \hat{S} \cdot \hat{R} \cdot Z^{GW}.
\]

In this paper we address the two problems outlined above. First we show the CY/LG correspondence for one particular pair \((\tilde{E}_7, G_{max})\) and a particular CY orbifold \(\mathbb{P}^1_{4,4,2}\) by giving the elements \(R\) and \(S\) explicitly. Second we give the Givental’s action formula, expressing the partition function of FJRW theory of \((\tilde{E}_7, G_{max})\) via the partition function of the so–called “untwisted theory”. This step is connected to the work of \([CR, PS, CZ]\). The partition function of the “untwisted theory” differs from the partition function of the product of the GW
theories of a point just by a linear change of the variables, hence we obtain the $R$–matrix of the FJRW theory in this step.

Combining these two results we get the $R$–matrix of the GW theory of the orbifold $\mathbb{P}^1_{4,4,2}$. In this way we get the result in Gromov–Witten theory by using mirror symmetry and modern approach to singularity theory, namely FJRW theory.

However comparing with the formula above our result reads:

$$Z_{\mathbb{P}^1_{4,4,2}} = \lim_{\lambda \to 0} \hat{S} \cdot \hat{R} \cdot Z^{\text{basic}},$$

where the limit on the RHS is the so–called non–equivariant limit. Interestingly, the partition function on the RHS depends on more variables than the partition functions on the LHS. A similar result was obtained in [B1], where it was shown that the Frobenius manifold of the GW theory of the orbifold $\mathbb{P}^1_{2,2,2,2}$ is a submanifold of a certain higher–dimensional Frobenius manifold.

The proof of the CY/LG correspondence is interesting by itself, since it uses the theory of modular forms, but gives a result in terms of Givental’s action. This part of the current article is closely related to the independent work of Shen and Zhou [SZ2]. Their result is more systematic from the point of view of the theory of modular forms, however they don’t give the particular Givental’s action. Furthermore, when requiring some explicit data to be compared Shen–Zhou consider the solutions to certain PDEs fixed by the initial conditions, while we use particular values of the modular forms. This difference is also related to the different approaches to the primitive form change on the B–side. Our approach also shows the holomorphicity of the FJRW theory in question.

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After having proved the CY/LG correspondence in the particular case of $(\tilde{E}_7, G_{\text{max}})$ by our methods, we were informed that Shen and Zhou had independently developed a systematic proof for the CY/LG correspondence for all simple elliptic singularities via the use of modular forms ([SZ2]). We are grateful to Shen and Zhou for the email conversations and also for the sharing the draft versions of our respective texts between us.

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Organization of the paper: In Section 2 we review FJRW theory for a pair \((W, G)\) and give a system of axioms by which one can compute the basic correlators. We don’t give a full definition of the virtual class of Fan–Jarvis–Ruan, but rather restrict ourselves to the situation of this paper in order to avoid some complicated formulas unnecessary to this work. In Section 3 we recall the definition of a cohomological field theory and give many details on the particular cases of the FJRW theory of \((E_7, G_{\text{max}})\) and GW theory of \(\mathbb{P}^1_{4,4,2}\). In Section 4 we show the CY/LG correspondence by using modularity property of the GW theory of \(\mathbb{P}^1_{4,4,2}\) — this is exactly the place, where we have an intersection with [SZ2]. Section 5 is devoted to the Givental’s action. We give there a particular action, which yields the CY/LG correspondence of the A–models discussed. In Section 6 so–called “twisted” correlators are introduced. These give us a partition function, depending on additional parameters, which generalize genus zero FJRW theory. In fact, we recover the FJRW partition function in the limit. In Section 7 we show that the twisted correlators also recover a basic theory, as discussed above, which we call the “untwisted” theory. We show how to recover FJRW theory from the untwisted theory using an upper–triangular Givental action, and use this result to give the \(R\)–matrix of the Gromov–Witten theory of \(\mathbb{P}_{4,4,2}\). In Appendix A we have given a closed formula for \(F_{0,2}^{\mathbb{P}_{4,4,2}}\).

2. Definition of FJRW Theory

We first introduce FJRW theory in some generality, describing the state space and the moduli space of \(W\)–structures together with its virtual class.

2.1. State Space. The Landau–Ginzburg model is provided by FJRW theory. The input is a pair \((W, G)\) of a quasihomogeneous polynomial and a group, which we now describe.

Let \(W \in \mathbb{C}[x_1, \ldots, x_N]\) be a quasihomogeneous polynomial of degree \(d\) with integer weights \(w_1, \ldots, w_N\) such that \(\gcd(w_1, \ldots, w_N) = 1\). For each \(1 \leq k \leq N\), let \(q_k = \frac{w_k}{d}\). The central charge of \(W\) is defined to be

\[
\hat{c} := \sum_{k=1}^{N} (1 - 2q_k).
\]

A polynomial is nondegenerate if

(i) the weights \(q_k\) are uniquely determined by \(W\), and
(ii) the hypersurface defined by \(W\) is non–singular in projective space.
The maximal group of diagonal symmetries is defined as

$$G_{\text{max}} := \left\{ (\Theta_1, \ldots, \Theta_N) \subseteq (\mathbb{Q}/\mathbb{Z})^N \mid W(e^{2\pi i \Theta_1} x_1, \ldots, e^{2\pi i \Theta_N} x_N) = W(x_1, \ldots, x_N) \right\}$$

Note that $G_{\text{max}}$ always contains the exponential grading element $j := (q_1, \ldots, q_N)$. If $W$ is nondegenerate, $G_{\text{max}}$ is finite.

**Remark 2.1.** One can define FJRW theory more generally for admissible subgroups $G \subset G_{\text{max}}$ (see [FJR]), but in the current work we consider only $G = G_{\text{max}}$.

FJRW theory defines a state space and a moduli space of $W$–curves, from which one obtains certain numbers—called correlators—as integrals over the moduli space. Let us first fix some notation. For $h \in G$, let $\text{Fix}(h)$ denote the fixed locus of $\mathbb{C}^N$ with respect to $h$, let $N_h$ denote the dimension of $\text{Fix}(h)$ and let $W_h$ denote $W|_{\text{Fix}(h)}$. Let $W_h^{+\infty} := (\text{Re } W_h)^{-1}(\rho, \infty)$, for $\rho \gg 0$, be the so–called Milnor fiber of $W_h$.

Define

$$\mathcal{H}_h := H^{N_h}(\text{Fix}(h), W_h^{+\infty}; \mathbb{C})^G,$$

that is, $G$–invariant elements of the the middle dimensional relative cohomology of $\text{Fix}(h)$. The state space is the direct sum of the “sectors” $\mathcal{H}_h$, i.e.

$$\mathcal{H}_{W,G} := \bigoplus_{h \in G} \mathcal{H}_h.$$

Let $G^{\text{nar}} = \{ h \in G \mid N_h = 0 \}$. These summands $\mathcal{H}_h$ for $h \in G^{\text{nar}}$ are the so–called narrow sectors.

$\mathcal{H}_{W,G}$ is $\mathbb{Q}$–graded by the $W$–degree. To define this grading, first note that each element $h \in G$ can be uniquely expressed as a tuple of rational numbers

$$h = (\Theta^h_1, \ldots, \Theta^h_N)$$

with $0 \leq \Theta^h_k < 1$.

We first define the degree–shifting number

$$\iota(h) := \sum_{k=1}^N (\Theta^h_k - q_k).$$

For $\alpha_h \in \mathcal{H}_h$, the (real) $W$–degree of $\alpha_h$ is defined by

$$\deg_W(\alpha_h) := N_h + 2\iota(h). \quad (2.1.1)$$

Let $\phi_h$ be the fundamental class in $\mathcal{H}_h$. In the sector indexed by $j$, we have $\deg_W(\phi_j) = 0$. This sector is unique with this property.
Because $\text{Fix}(h) = \text{Fix}(h^{-1})$ there is a non-degenerate pairing
\[ \langle -,- \rangle : \mathcal{H}_h \times \mathcal{H}_{h^{-1}} \to \mathbb{C}, \]
the residue pairing of $W_h$, which induces a symmetric non-degenerate pairing
\[ \langle -,- \rangle : \mathcal{H}_{W,G} \times \mathcal{H}_{W,G} \to \mathbb{C}. \]

2.2. Moduli of $W$–curves. Recall that an $n$–pointed orbifold curve is a stack of Deligne–Mumford type with at worst nodal singularities with orbifold structure only at the marked points and the nodes. We require the nodes to be balanced, in the sense that the action of the generator of the stabilizer group $\mathbb{Z}_k$ be given by
\[ (x,y) \mapsto (e^{2\pi i/k}x, e^{-2\pi i/k}y). \]

Given such a curve $C$, let $\omega$ be its dualizing sheaf. The log–canonical bundle is
\[ \omega_{\log} := \omega(p_1 + \cdots + p_n) \]

In what follows, we will assume $d$, the degree of $W$, is also the exponent of $G_{\text{max}}$, i.e. for each $h \in G_{\text{max}}, h^d = \text{id}$. This is not the case in general, but it simplifies the exposition, while still giving a general enough picture.

The FJRW correlators were first defined in [FJR], but we will follow a slightly different treatment as given in [CR], where it is also shown that the two definitions agree. The reason for our choice, is that [CR] allows us to use Givental’s formalism to determine the FJRW correlators.

A $d$–stable curve is a proper connected orbifold curve $C$ of genus $g$ with $n$ distinct smooth markings $p_1, \ldots, p_n$ such that
(i) the $n$–pointed underlying coarse curve is stable, and
(ii) all the stabilizers at nodes and markings have order $d$.

The moduli stack $\overline{\mathcal{M}}_{g,n,d}$ parametrizing such curves is proper, smooth and has dimension $3g - 3 + n$. It differs from the moduli space of curves only because of the stabilizers over the normal crossings (see [CR]).

We can write $W$ as a sum of monomials $W = W_1 + \cdots + W_s$, where $W_i = c_i \prod_{k=1}^{N} x_k^{a_{ik}}$ with $a_{ik} \in \mathbb{N}$ and $c_i \in \mathbb{C}$. Given line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_N$ on the $d$–stable curve $C$, we define the line bundle
\[ W_i(\mathcal{L}_1, \ldots, \mathcal{L}_N) := \bigotimes_{k=1}^{N} \mathcal{L}_k^{\otimes a_{ik}}. \]
Definition 2.2. A W–structure is comprised of the data
\((C, p_1, \ldots, p_n, L_1, \ldots, L_N, \varphi_1, \ldots, \varphi_N)\),
where \(C\) is an \(n\)–pointed \(d\)–stable curve, the \(L_k\) are line bundles on \(C\) satisfying
\[ W_i(L_1, \ldots, L_N) \cong \omega_{\log}, \]
and for each \(k\), \(\varphi_k : L_k^\otimes d \rightarrow \omega_{\log}^{w_k} \) is an isomorphism of line bundles.

There exists a moduli stack of W–structures, denoted by \(W_{g,n}\) (see [FJR2], [CR] for the construction).

Proposition 2.3 ([CR]). The stack \(W_{g,n}\) is nonempty if and only if \(n > 0\) or \(2g - 2\) is a positive multiple of \(d\). It is a proper, smooth Deligne–Mumford stack of dimension \(3g - 3 + n\). It is etale over \(\overline{M}_{g,n,d}\) of degree \(|G_{\text{max}}|^2g - 1 + n / dN\).

Let \(h = (h_1, \ldots, h_n)\), with \(h_i = (\Theta_{i,1}, \ldots, \Theta_{i,N})\). Define \(W_{g,n}(h)\) to be the stack of \(n\)–pointed, genus \(g\) W–curves for which the generator of the isotropy group at \(p_j\) acts on \(L_k\) by multiplication by \(e^{2\pi i \Theta_{k,j}/d}\). The following proposition describes a decomposition of \(W_{g,n}\) in terms of multiplicities:

Proposition 2.4 ([CR] [FJR]). The stack \(W_{g,n}\) can be expressed as the disjoint union
\[ W_{g,n} = \coprod W_{g,n}(h) \]
with each \(W_{g,n}(h)\) an open and closed substack of \(W_{g,n}\). Furthermore, \(W_{g,n}(h)\) is non–empty if and only if
\[ h_i \in G_{\text{max}}, \quad i = 1, \ldots, n \]
\[ q_k(2g - 2 + n) - \sum_{i=1}^n \Theta_{k,i} \in \mathbb{Z}, \quad k = 1, \ldots, N. \]

The second condition comes from the pushforward of \(L_k\) to the course underlying curve, which must have integer degree.

2.3. Axioms of FJRW theory. For each substack \(W_{g,n}(h)\), one may define a virtual cycle (see [FJR] [FJR2])
\[ [W_{g,n}(h)]^{\text{vir}} \in H_*(\mathcal{W}_{g,n}(h), \mathbb{Q}) \otimes \prod_{i=1}^n H_{N_{h_i}}(\text{Fix}(h_i), W_{h_i}^+; \mathbb{C})^G \]
which satisfies the following axioms:
FJR 1 (Degree). The virtual cycle has degree
\[ 2 \left( (\ell - 3)(1 - g) + n - \sum_{i=1}^{n} \iota(h_i) \right). \]
In particular, if this number is not an integer, then \([W_{g,n}(h)]^{\text{vir}} = 0.\]

FJR 2 (Line bundle degree). The degree of the pushforward \(|L|_{q_k}\)
\[ q_k(2g - 2 + n) - \sum_{i=1}^{n} \Theta^h_{i_k} \]
must be an integer (as in Proposition 2.4), otherwise \([W_{g,n}(h)]^{\text{vir}} = 0.\]

FJR 3 (Symmetric Group invariance). For any \(\sigma \in S_n\), we have
\[ [W_{g,n}(h_1, \ldots, h_n)]^{\text{vir}} = [W_{g,n}(h_{\sigma(1)}, \ldots, h_{\sigma(n)})]^{\text{vir}}. \]

FJR 4 (Deformation invariance). Let \(W_t \in \mathbb{C}[x_1, \ldots, x_N]\) be a family
of nondegenerate quasihomogeneous polynomials depending smoothly on
a real parameter \(t \in [a, b]\). Suppose that \(G\) is the common isomorphism
group of \(W_t\). The corresponding moduli of \(W\)-structures are naturally
isomorphic, and the virtual cycle \([W_{g,n}(h_1, \ldots, h_n)]^{\text{vir}}\) associated to \((W_t, G)\)
is independent of \(t\).

FJR 5 (G\(_{\text{max}}\)-invariance). There is a natural \(G_{\text{max}}\) action on
\(H_\ast(W_{g,n}(h), \mathbb{Q})\) and \(H_{N_{h_i}}(\text{Fix}(h_i), W_{h_i}^{\infty}, \mathbb{C})^G\). The virtual cycle \([W_{g,n}(h)]^{\text{vir}}\) is invariant
under the induced \(G_{\text{max}}\) action on the tensor product.

FJR 6 (Concavity). Suppose that \(h_i \in G^{\text{nar}}\) for all \(i\). If \(\pi_\ast \left( \bigoplus_{k=1}^{N} L_k \right) = 0\), then the virtual class is given by
\[ [W_{g,n}(h)]^{\text{vir}} = c_{\text{top}} \left( R^1 \pi_\ast \bigoplus_{k=1}^{N} L_k \right) \cap [W_{g,n}(h)] \]
and the substack \(W_{g,n}(h)\) is called concave.

Remark 2.5. This last axiom can also be modified to take into account
the restriction to boundary components on \(W_{g,n}\), i.e. \(W\)-curves with
reducible underlying curve (cf. [FJR]).

Remark 2.6. Some authors also include the “Index Zero” axiom, but
in full generality, both concavity and index zero are actually a part of
a larger axiom involving the topological Euler class and the Witten
map, but we will not need the full statement here. We also do not
include the sums of singularities axiom.
There are a few other axioms that are satisfied by \([W_{g,n}(\mathbf{h})]^{\text{vir}}\) that are more complicated to state (cf. \[\text{FJR}\]), so we will not list them here. They show, for example, that the virtual class behaves well with respect to cutting along nodes, ensuring that FJRW theory defines a cohomological field theory, as we will see.

The stacks \(W_{g,n}\) are also equipped with \(\psi\)-classes, which are pulled back from the course underlying curve.

3. COHOMOLOGICAL FIELD THEORIES ON \(\overline{M}_{g,n}\)

We briefly recall some basic facts about cohomological field theories as introduced in \[\text{KM}\].

3.1. Cohomological Field Theory axioms. Let \((V, \eta)\) be a finite–dimensional vector space with a non–degenerate pairing. Consider a system of linear maps

\[
\Lambda_{g,n} : V^\otimes n \to H^* (\overline{M}_{g,n}),
\]

defined for all \(g, n\) such that \(\overline{M}_{g,n}\) exists and is non–empty. The set \(\Lambda_{g,n}\) is called a cohomological field theory on \((V, \eta)\), or CohFT, if it satisfies the following axioms.

**CFT 1** \((S_n\text{ invariance})\). \(\Lambda_{g,n}\) is equivariant with respect to the \(S_n\)-action permuting the factors in the tensor product and the numbering of marked points in \(\overline{M}_{g,n}\).

**CFT 2** \((\text{Cutting trees})\). For the gluing morphism \(\rho: \overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g_1+g_2,n_1+n_2}\) we have:

\[
\rho^* \Lambda_{g_1+g_2,n_1+n_2} = (\Lambda_{g_1,n_1+1} \cdot \Lambda_{g_2,n_2+1}, \eta^{-1}),
\]

where we contract with \(\eta^{-1}\) the factors of \(V\) that correspond to the node in the preimage of \(\rho\).

**CFT 3** \((\text{Cutting loops})\). For the gluing morphism \(\sigma: \overline{M}_{g,n+2} \to \overline{M}_{g+1,n}\) we have:

\[
\sigma^* \Lambda_{g+1,n} = (\Lambda_{g,n+2}, \eta^{-1}),
\]

where we contract with \(\eta^{-1}\) the factors of \(V\) that correspond to the node in the preimage of \(\sigma\).

In this paper we further assume that the CohFT \(\Lambda_{g,n}\) is unital — i.e. there is a fixed vector \(1 \in V\) called the unit such that the following axioms are satisfied.

**U 1.** For every \(\alpha_1, \alpha_2 \in V\) we have: \(\eta(\alpha_1, \alpha_2) = \Lambda_{0,3}(1 \otimes \alpha_1 \otimes \alpha_2)\).
Let $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ be the map forgetting the last marking, then:

$$
\pi^* \Lambda_{g,n}(\alpha_1 \otimes \cdots \otimes \alpha_n) = \Lambda_{g,n+1}(\alpha_1 \otimes \cdots \otimes \alpha_n \otimes 1).
$$

Another important property of CohFTs is the notion of quasi-homogeneity. A CohFT $\Lambda_{g,n}$ on $(V, \eta)$ is called quasi-homogeneous if the vector space $V$ is graded by $\deg : V \to \mathbb{Q}$ and there is a number $\hat{c}$, such that for any $\alpha_1, \ldots, \alpha_n \in V$

$$
\sum_{i=1}^{n} \deg(\alpha_i) = \hat{c} + n + g - 3
$$

whenever $\langle \alpha_1, \ldots, \alpha_n \rangle_{g,n} \neq 0$. The number $\hat{c}$ is called the central charge.

The space of all quasi-homogeneous CohFTs is discrete in the space of all CohFTs. However, these CohFTs possess several properties that make them easier to work with. The CohFTs we are going to work with in this text are quasi-homogeneous. In FJRW theory the state space is graded by $W$-degree, and in GW theory, the state space is graded simply by the cohomological degree.

In what follows we will denote the CohFT just by $\Lambda$ rather than $\Lambda_{g,n}$ when there is no ambiguity.

Let $\psi_i \in H^*(\overline{M}_{g,n})$, $1 \leq i \leq n$ be the so-called $\psi$-classes. The genus $g,n$-point correlators of the CohFT are the following numbers:

$$
\langle \tau_{a_1}(e_{a_1}) \cdots \tau_{a_n}(e_{a_n}) \rangle_{g,n} := \int_{\overline{M}_{g,n}} \Lambda_{g,n}(e_{a_1} \otimes \cdots \otimes e_{a_n}) \psi_{a_1}^1 \cdots \psi_{a_n}^n.
$$

Denote by $F_g$ the generating function of the genus $g$ correlators, called genus $g$ potential of the CohFT:

$$
F_g := \sum \frac{\langle \tau_{a_1}(e_{a_1}) \cdots \tau_{a_n}(e_{a_n}) \rangle_{g,n}^{\Lambda}}{\text{Aut}(\{\alpha, a\})} t_{a_1,a_1}^1 \cdots t_{a_n,a_n}^n.
$$

It is useful to assemble the correlators into a generating function called partition function of the CohFT:

$$
Z := \exp \left( \sum_{g \geq 0} h^g \frac{F_g}{g!} \right).
$$

We will also make use of the so-called primary genus $g$ potential that is a function of the variables $t^a := t^{0,a}$ defined as follows:

$$
F_g := F_g \mid_{t^a = t^{0,a}, t^{0,a} = 0, \forall t \geq 1}
$$

what is also sometimes called a restriction to the small phase space.
### 3.2. CohFT of FJRW theory and Gromov–Witten theory

In the space of all cohomological field theories there are certain special theories, called also sometimes “geometric” since they correspond to some geometry. These include FJRW theory and GW theory.

Consider the FJRW–theory of a pair \((W, G)\). Its moduli space of \(W\)–structures has a good virtual cycle \([W_{g,n}(h)]^{\text{vir}}\) as it was explained in Section 2.3.

However, we can also push forward to \(\overline{\mathcal{M}}_{g,n}\) via the map \(\iota_{s} : \mathcal{M}_{g,n}(h) \to \overline{\mathcal{M}}_{g,n}\). Let \(\alpha_i \in \mathcal{H}_{h_i}\), and \(\alpha = (\alpha_1, \ldots, \alpha_n)\). We define

\[
\Lambda_{FJRW}^{g,n}(\alpha) = \frac{|G|^g}{\deg s} PD_s \left( [W_{g,n}(h)]^{\text{vir}} \cap \prod_{i=1}^{n} \iota_{s}^{*}(\alpha_i) \right).
\]

Here \(PD\) denotes the Poincare dual.

**Theorem 3.1** (Theorem 4.2.2 in [FJR]). For any admissible pair \((W, G)\) the system of maps \(\Lambda_{FJRW}^{g,n}\) defines a unital CohFT on the vector space \(\mathcal{H}_{W,G}\).

In what follows we denote simply by \(Z^{(W,G)}\) and \(F_{g}^{(W,G)}\) the partition function and genus \(g\)–potential of the CohFT above for a fixed admissible pair \((W, G)\). As a consequence of the properties of the virtual cycle \([W_{g,n}]^{\text{vir}}\), these functions also satisfy certain additional properties in addition to those common to all CohFTs.

The second important class of the CohFTs is given by the Gromov–Witten theories. We recall very briefly the definition and refer to [A] for a full exposition. Let \(\mathcal{X}\) be an orbifold and \(\beta \in H_2(\mathcal{X}, \mathbb{Z})\). There is the moduli stack \(\overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta)\) of degree \(\beta\) stable orbifold maps from the genus \(g\) curve with \(n\) marked points to \(\mathcal{X}\). The orbifold cohomology \(H_{\text{orb}}^*(\mathcal{X})\), with the non–degenerate pairing, serves as a state space in this theory. Similarly to the FJRW theory there is a good virtual cycle \([\overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta)]^{\text{vir}}\) so that one could consider the GW–invariants given by the intersection theory on \(\overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta)\).

Again, by considering a push forward \(s : \overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta) \to \overline{\mathcal{M}}_{g,n}\) we get a CohFT associated to \(\mathcal{X}\) with the fixed \(\beta\).

\[
\Lambda_{GW}^{g,n,\beta} := \frac{1}{\deg s} PD_s \left( [\overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta)]^{\text{vir}} \cap \prod_{i=1}^{n} ev_i^{*}(\alpha_i) \right).
\]

The fact that this map defines a CohFT follows from a more general statement and could be found for example in [A].

As with FJRW theory, the CohFT obtained satisfies some additional properties. One of the most important for us is the so–called...
When $H_2(X,\mathbb{Z})$ is one–dimensional, as is the case in this work, it allows us to sum over all classes $\beta$ and obtain a CohFT, $\Lambda^{GW}_{g,n}$ depending on $X$ only. We denote by $\mathcal{Z}^X$ and $\mathcal{F}^X_g$ the partition function and genus $g$–potential of the CohFT $\Lambda^{GW}_{g,n}$.

### 3.3. Reconstruction in genus zero

It is often useful to be able to express all correlators of a given CohFT from some finite list. This is usually referred to as reconstruction.

Due to the topology of the space $\mathcal{M}_{0,n}$ the small phase space potential of a CohFT on $(V,\eta)$ satisfies the so–called WDVV equations. For any four fixed $0 \leq i,j,k,l \leq \dim(V) - 1$ it reads:

$$\sum_{p,q} \frac{\partial^3 F_0}{\partial t^i \partial t^j} \eta^{p,q} \eta_{p,q} \frac{\partial^3 F_0}{\partial t^k \partial t^l} = \sum_{p,q} \frac{\partial^3 F_0}{\partial t^i \partial t^k} \eta^{p,q} \eta_{p,q} \frac{\partial^3 F_0}{\partial t^j \partial t^l}.$$ 

It follows from here that the genus zero three–point correlators endow $V$ with the structure of an associative and commutative algebra by setting $e_i \circ e_j := \sum_{p,k} \langle e_i, e_j, e_p \rangle_{0,3} \cdot \eta^{p,k} \cdot e_k$.

**Definition 3.2.** The vector $\gamma \in V$ is called primitive if there is no $\gamma_1, \gamma_2 \in V$, such that $\gamma = \gamma_1 \circ \gamma_2$ and $0 < \deg(\gamma_1) \leq \deg(\gamma_2) < \deg(\gamma)$. We call a correlator $\langle \ldots \rangle_{0,n}$ basic if it involves at most two non–primitive insertions.

The following lemma will be used later on.

**Lemma 3.3** (Lemma 6.2.8 in [FJR]). Fix a quasihomogeneous CohFT on $(V,\eta)$.

If $\deg(\alpha) \leq \hat{c}$ for all vectors $\alpha \in V$ then all the genus zero correlators are uniquely determined by $\eta$ and the $n$–point genus zero correlators with:

$$n \leq 2 + \frac{1 + \hat{c}}{1 - P}, \quad P := \max_{v \in V} \deg(v).$$

The proof of this lemma is based on the analysis of WDVV equation of a quasihomogeneous CohFT.

### 3.4. FJRW Correlators for $\tilde{E}_7$

In this article, we consider the polynomial $W = x^4 + y^4 + z^2$ defining the $\tilde{E}_7$ singularity. The polynomial $W$ is quasihomogeneous with weights $q_1 = \frac{1}{4}, q_2 = \frac{1}{4}$ and $q_3 = \frac{1}{2}$. The group $G = G_{max}$ is generated by the elements $\rho_1 := (q_1, 0, 0)$, $\rho_2 := (0, q_2, 0)$, $\rho_3 := (0, 0, q_3)$, and $j_W = \rho_1 \rho_2 \rho_3$.

Working through the definition, one sees that in this case only narrow group elements contribute to the state space. With $\phi_h$ defined as
above, the set \( \{ \phi_h \}_{h \in G^{\text{nor}}} \) defines a basis of \( \mathcal{H}_{W,G} \), i.e. we have

\[
\mathcal{H}_{W,G} := \bigoplus_{1 \leq a \leq 3} \bigoplus_{1 \leq b \leq 3} C \cdot \phi_{a1}^a \phi_{b2}^b \phi_{3}^3.
\]

The pairing is determined by the following values on this basis:

\[
\langle \phi_{h_1}, \phi_{h_2} \rangle := \begin{cases} 
1 & \text{if } h_1 = (h_2)^{-1} \\
0 & \text{otherwise.}
\end{cases}
\]

In the following lemma, we show that the entire FJRW theory is concave, namely all substacks \( W_{g,n}(h) \) satisfy the concavity condition. So by the concavity axiom, we can replace the virtual class by the fundamental class capped with the top chern class of a line bundle.

**Proposition 3.4.** The genus zero FJRW theory for \((\tilde{E}_7, G_{\max})\) is concave.

**Proof.** The proof has been given in several places, including [CR, ChIR, PS], so we will not give it in detail here. It consists of checking that over any geometric point \((C, p_1, \ldots, p_n, L_1, L_2, L_3, \varphi_1, \varphi_2, \varphi_3)\) in the moduli space, \(\bigoplus_{k=1}^3 H^0(C, L_k) = 0\). This is done by checking the degree of the line bundle satisfies for each connected component \(C_v\) of \(C\)

\[
\deg(|L_k|_{C_v}) \leq q_k(\#\text{nodes}(C_v) - 2) < \#\text{nodes}(C_v) - 1.
\]

\[\square\]

The genus 0 potential of the FJRW theory is written in the variables \(\tilde{t}_{ab}^r\) for \(1 \leq a \leq 3\) and \(1 \leq b \leq 3\), corresponding to the vectors \(\phi_{a1}^a \phi_{b2}^b \phi_{3}^3\). It is always the case that \(\phi_{1}^1\) is the unit. Thus the variable \(\tilde{t}_{11}^1\) corresponds to the unit of the CohFT for \((\tilde{E}_7, G_{\max})\).

Using axioms FJR [1-FJR [6] (the data is also in [MS1, Section 3.3]), we get the following expression for the genus zero small phase space potential of \((\tilde{E}_7, G_{\max})\):

\[
F_{\tilde{E}_7,G_{\max}}^0 = \tilde{t}_{11}^1 \tilde{t}_{33}^3 + \tilde{t}_{11}^1 \left( \tilde{t}_{21}^2 \tilde{t}_{23}^3 + \tilde{t}_{12}^1 \tilde{t}_{32}^3 + \tilde{t}_{13}^1 \tilde{t}_{31}^3 + \frac{\tilde{t}_{22}^2}{2} \right) + \tilde{t}_{12}^1 \tilde{t}_{21}^2 \tilde{t}_{22}^2
\]

\[
+ \frac{\tilde{t}_{12}^2 \tilde{t}_{31}^3}{2} + \frac{\tilde{t}_{21}^2 \tilde{t}_{13}^3}{2} - \tilde{t}_{33}^3 \left( \frac{\tilde{t}_{21}^2 \tilde{t}_{31}^3}{8} + \frac{\tilde{t}_{12}^2 \tilde{t}_{13}^3}{8} \right) + O(\tilde{t}_{+}^4, \tilde{t}_{33}^3),
\]

where \(\tilde{t}_{+}\) is the set of all coordinates except \(\tilde{t}_{33}\).

We can rephrase Lemma 3.3 for this case in the following lemma.
Lemma 3.5 (Lemma 3.6 in [MSI] and Theorem 3.4 in [KS]). Using the WDVV equation, all genus 0 primary correlators of FJRW theory \((\tilde{E}_7, G_{\text{max}})\) are uniquely determined by the FJRW–algebra and the basic 4–point correlators that have exactly one insertion of \(\rho_1^3 \rho_2^3 \rho_3^3\).

Remark 3.6. It follows immediately from Lemma 3.5 and proof of Lemma 3.3 that the genus 0 potential \(F_{0}^{\tilde{E}_7, G_{\text{max}}} \in \mathbb{Q}[[t]]\) because all the “primary” data is rational and the WDVV equation doesn’t involve anything non–rational.

3.5. Gromov–Witten theory of elliptic orbifolds. Let the ordered set \((a_1, a_2, a_3)\) be either \((3, 3, 3)\), \((4, 4, 2)\) or \((6, 3, 2)\). Consider \(\mathcal{X} := \mathbb{P}^1_{a_1, a_2, a_3}\). These are the so–called elliptic orbifolds. In this case we have

\[
\dim(H^*_\text{orb}(\mathbb{P}^1_{a_1, a_2, a_3})) = 2 + \sum_{i=1}^{3} (a_i - 1).
\]

The space \(H^*_\text{orb}(\mathbb{P}^1_{a_1, a_2, a_3})\) has the generators:

\[
\Delta_0, \Delta_{-1}, \Delta_{i,j}, \quad 1 \leq i \leq 3, \quad 1 \leq j \leq a_i - 1,
\]

so that \(H^*_\text{orb}(\mathbb{P}^1_{a_1, a_2, a_3}) \simeq \mathbb{Q} \Delta_0 \oplus \mathbb{Q} \Delta_{-1} \oplus \bigoplus_{i=1}^{3} \bigoplus_{j=1}^{a_i - 1} \mathbb{Q} \Delta_{i,j}\), and \(H^0(\mathbb{P}^1_{a_1, a_2, a_3}, \mathbb{Q}) \simeq \mathbb{Q} \Delta_0, \quad H^2(\mathbb{P}^1_{a_1, a_2, a_3}, \mathbb{Q}) \simeq \mathbb{Q} \Delta_{-1}, \quad \).

The pairing is given by:

\[
\eta(\Delta_0, \Delta_{-1}) = 1, \quad \eta(\Delta_{i,j}, \Delta_{k,l}) = \frac{1}{a_i} \delta_{i,k} \delta_{j+l,a_i}.
\]

The potential of this CohFT is then written in the coordinates \(t_0, t_{-1}\) and \(t_{i,j}\), corresponding to the classes \(\Delta_0, \Delta_{-1}\) and \(\Delta_{i,j}\) respectively.

As with FJRW theory, it turns out that one needs to know only certain finite list of the correlators in order to compute all the correlators of these GW–theories. Such correlators were found explicitly by [SZ1] and used independently by the first author to write down the genus 0 potentials explicitly. In what follows we will be particularly interested in the GW–theory of \(\mathbb{P}^1_{4,4,2}\). We give explicitly the genus 0 potential of this orbifold in the appendix.

3.6. GW theory of \(\mathbb{P}^1_{4,4,2}\). Let \(\vartheta_2(q), \vartheta_3(q)\) and \(\vartheta_4(q)\) be the following infinite series in a formal variable \(q\):

\[
\vartheta_2(q) = 2 \sum_{k=0}^{\infty} q^{\frac{k}{2}(k+1)^2}, \quad \vartheta_3(q) = 1 + 2 \sum_{k=1}^{\infty} q^{\frac{k^2}{2}},
\]

\[
\vartheta_4(q) = 1 + 2 \sum_{k=1}^{\infty} (-1)^k q^{\frac{k^2}{2}}.
\]
and also
\[ f(q) := 1 - 24 \sum_{k=1}^{\infty} \frac{kq^k}{1 - q^k}. \]

These series are the \( q \)-expansions of the Jacobi theta constants and the second Eisenstein series respectively. However at the moment we consider them only as the formal series in the variable \( q \).

Consider the functions \( x(q), y(q), z(q), w(q) \) defined as follows:
\[
\begin{align*}
x(q) &:= \left( \theta_3(q^8) \right)^2, \\
y(q) &:= \left( \theta_2(q^8) \right)^2, \\
z(q) &:= \left( \theta_2(q^4) \right)^2, \\
w(q) &:= \frac{1}{3} \left( f(q^4) - 2f(q^8) + 4f(q^{16}) \right).
\end{align*}
\]

In what follows the function \( z(q) \) will be sometimes skipped because the following identity holds:
\[ z(q)^2 = 4x(q)y(q). \]

**Proposition 3.7.** The potential \( F_{0}^{\mathbb{P}^1_{4,4,2}} \) has an explicit form via the functions defined above. Namely:
\[
F_{0}^{\mathbb{P}^1_{4,4,2}} \in \mathbb{Q} \left[ t_0, t_{-1}, t_{i,j}, x, y, z, w \right],
\]
where \( x = x(q), y = y(q), z = z(q) \) and \( w = w(q) \) as above. Moreover it satisfies the following homogeneity property:
\[
F_{0}^{\mathbb{P}^1_{4,4,2}} \left( t_0, t_{-1}, t_{i,j}, x, y, z, w \right) = \alpha^{-2} F_{0}^{\mathbb{P}^1_{4,4,2}} \left( t_0, t_{-1}, \alpha \cdot t_{i,j}, \frac{x}{\alpha}, \frac{y}{\alpha}, \frac{z}{\alpha^2}, \frac{w}{\alpha^2} \right),
\]
for any \( \alpha \in \mathbb{C}^* \).

**Proof.** This is clear from the explicit form of the potential — see Appendix [A]. \( \square \)

Up to the 4-th order terms in \( t_{i,k} \) we have:
\[
\begin{align*}
F_{0}^{\mathbb{P}^1_{4,4,2}} &= \frac{1}{2} t_0^2 t_{-1} + t_0 \left( \frac{1}{4} t_{1,1} t_{1,3} + \frac{1}{8} t_{1,2}^2 + \frac{1}{4} t_{2,1} t_{2,3} + \frac{1}{8} t_{2,2}^2 + \frac{1}{4} t_{3,1}^2 \right) \\
&\quad + \frac{1}{8} x(q) \left( t_{1,1}^2 t_{1,2} + t_{2,1}^2 t_{2,2} \right) + \frac{1}{8} y(q) \left( t_{1,2} t_{2,1} + t_{3,1} t_{1,2} \right) \\
&\quad + \frac{1}{4} z(q) t_{1,1} t_{2,1} t_{3,1} + O(t_{i,k}^4, t_{-1}),
\end{align*}
\]
where \( q = \exp(t_{-1}) \).

Considering also the change of the variables \( t_{-1} = t_{-1}(\tau) = \frac{2\pi i}{\tau} \), we can consider the functions \( x(q(\tau)), y(q(\tau)), z(q(\tau)) \) as modular...
forms and \(w(q(\tau))\) as a quasi–modular form. This means in particular that these functions have a large domain of holomorphicity and satisfy certain modularity condition. The first property holds also by the primary potential \(F^{P_{4,4,2}}_0\), however the second—modularity—is slightly more complicated. It was shown in \([B3]\) that primary potentials of all elliptic orbifolds satisfy the modularity property, too.

4. CY/LG CORRESPONDENCE VIA MODULARITY

Consider a unital CohFT \(\Lambda\) on \((V, \eta)\). Let \(\{e_0, \ldots, e_m\}\) be the basis of \(V\), such that \(\eta_{0,k} = \delta_{k,m}\). Define the coordinates \(t_0, \ldots, t_m\) corresponding to this basis. Due to Axiom (U1) of a unital CohFT, the primary genus zero potential of \(\Lambda\) reads in coordinates:

\[
F_0(t_0, \ldots, t_m) = \frac{t_0^2t_m}{2} + t_0 \sum_{0 < \alpha \leq \beta < m} \eta_{\alpha, \beta} \frac{t_\alpha t_\beta}{|\text{Aut}(\alpha, \beta)|} + H(t_1, \ldots, t_m),
\]

where \(H\) is a function, not depending on \(t_0\).

For any \(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C})\) consider another function \(F^A_0 = F^A_0(t_0, \ldots, t_m)\) defined by:

\[
F^A_0 := \frac{t_0^2t_m}{2} + t_0 \sum_{0 < \alpha \leq \beta < m} \eta_{\alpha, \beta} \frac{t_\alpha t_\beta}{|\text{Aut}(\alpha, \beta)|} + \frac{c \left( \sum_{0 < \alpha \leq \beta < m} \eta_{\alpha, \beta} \frac{t_\alpha t_\beta}{|\text{Aut}(\alpha, \beta)|} \right)^2}{2(ct_m + d)}
\]

\[
+ (ct_m + d)^2 H \left( \frac{t_1}{ct_m + d}, \cdots, \frac{t_{m-1}}{ct_m + d}, \frac{at_m + b}{ct_m + d} \right). \tag{4.0.1}
\]

It is not hard to see that \(F^A_0\) is solution to WDVV equation. One could also give a CohFT, whose genus 0 primary potential it is (see \([B2]\) for details). We also write \(A \cdot F_0 := (F_0)^A\). Consider the action of a particular matrix \(A^{\text{CY/LG}}\):

\[
A^{\text{CY/LG}} := \begin{pmatrix} 1 & \pi \Theta \\ 2 & 2 \Theta \\ \pi \Theta & \Theta \end{pmatrix}, \quad \Theta := \frac{\sqrt{2\pi}}{\Gamma \left( \frac{3}{4} \right)}.
\]

Main statement of this section is the following theorem.

**Theorem 4.1.** Let \(F^{(E_7, G_{\text{max}})}_0\) and \(F^{P_{4,4,2}}_0\) be the primary genus 0 potentials of the FJRW theory of \((E_7, G_{\text{max}})\) and GW theory of \(P_{4,4,2}^1\) respectively.
Then we have:

$$F_0^{(\tilde{E}_7, G_{\text{max}})}(\tilde{t}) = A_{\text{CY/LG}} \cdot F_0^{\mathbb{P}^{1,4,2}}(t),$$

where $\tilde{t} = \tilde{t}(t)$ is the following linear change of the variables:

$$t_{1,1} = i\sqrt{2}(\tilde{t}_{12} - \tilde{t}_{21}), t_{1,2} = -\tilde{t}_{13} + \sqrt{2}\tilde{t}_{22} - \tilde{t}_{31}, t_{1,3} = i\sqrt{2}(\tilde{t}_{23} - \tilde{t}_{32}),$$

$$t_{2,1} = \sqrt{2}(\tilde{t}_{12} + \tilde{t}_{21}), t_{2,2} = \tilde{t}_{13} + \sqrt{2}\tilde{t}_{22} + \tilde{t}_{31}, t_{2,3} = \sqrt{2}(\tilde{t}_{23} + \tilde{t}_{32}),$$

$$t_{3,1} = i(\tilde{t}_{13} - \tilde{t}_{31}),$$

$$t_0 = \tilde{t}_{11}, t_{-1} = \tilde{t}_{33}.$$

Moreover the primary potential $F_0^{(\tilde{E}_7, G_{\text{max}})}(\tilde{t})$ is holomorphic in

$$\mathbb{C}^9 \times \{ \tilde{t}_{33} \in \mathbb{C} | |\tilde{t}_{33}| < |\pi \Theta^2| \}$$

and has an expansion with a rational coefficients.

Together with the explicit formulae for the genus 0 small phase space potentials of the GW–theories of the elliptic orbifolds announced in [B2] this theorem gives an explicit closed formula for the FJR potential of $(\tilde{E}_7, G_{\text{max}})$. For example the following expansion holds:

$$F_0^{\tilde{E}_7, G_{\text{max}}} = \frac{1}{2} \tilde{t}_{11} \tilde{t}_{33} + \tilde{t}_{11} \left( \frac{\tilde{t}_{22}}{2} + \tilde{t}_{21} \tilde{t}_{23} + \tilde{t}_{13} \tilde{t}_{31} + \tilde{t}_{12} \tilde{t}_{32} \right) - \tilde{t}_{12} \tilde{t}_{13} \left( \frac{\tilde{t}_{33}}{8} + \frac{\tilde{t}_{53}}{61440} \right)$$

$$+ \tilde{t}_{21} \tilde{t}_{31} \left( -\frac{\tilde{t}_{33}}{8} - \frac{\tilde{t}_{53}}{61440} \right) + \tilde{t}_{13} \tilde{t}_{21} \left( \frac{1}{2} + \frac{\tilde{t}_{43}}{3072} + \frac{\tilde{t}_{83}}{330301440} \right)$$

$$+ \tilde{t}_{12} \tilde{t}_{31} \left( \frac{1}{2} + \frac{\tilde{t}_{43}}{3072} + \frac{\tilde{t}_{83}}{330301440} \right)$$

$$+ \tilde{t}_{12} \tilde{t}_{21} \tilde{t}_{22} \left( 1 + \frac{\tilde{t}_{33}}{32} + \frac{\tilde{t}_{43}}{6144} + \frac{\tilde{t}_{63}}{327680} + \frac{289\tilde{t}_{83}}{2642411520} \right) + O(\tilde{t}_{33}^9, \tilde{t}_{4+}^9)$$

for $\tilde{t}_+ = t\setminus \tilde{t}_{33}$.

4.1. Group action in the formal variable. We make a few preparations, before we prove Theorem [4.1]. The following Proposition appeared first in [SZ1] in a slightly different notation.

**Proposition 4.2** (Section 3.2.3 of [SZ1]). Consider $x(q), y(q)$ and $w(q)$ as the functions of $t = t_{-1}$ by taking $q = \exp(t_{-1})$. The WDVV equation
on $F_0^{\mathbb{P}^{1,2}}$ is equivalent to the following system of equations:
\begin{align*}
\frac{\partial}{\partial t} x(t) &= x(t) \left(2y(t)^2 - x(t)^2 + w(t)\right), \\
\frac{\partial}{\partial t} y(t) &= y(t) \left(2x(t)^2 - y(t)^2 + w(t)\right), \\
\frac{\partial}{\partial t} w(t) &= w(t)^2 - x(t)^4.
\end{align*}

(4.1.1)

The following proposition explains the $\text{SL}(2, \mathbb{C})$–action we consider.

**Proposition 4.3.** Consider $x(q)$, $y(q)$ and $w(q)$ as the functions of $t = t_{-1}$ by taking $q = \exp(t_{-1})$. We have:

(i) for any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C})$ the functions $x^A(t)$, $y^A(t)$ and $w^A(t)$ defined by:
\begin{align*}
x^A(t) &:= \frac{1}{(ct + d)x} \left(\frac{at + b}{ct + d}\right), \\
y^A(t) &:= \frac{1}{(ct + d)y} \left(\frac{at + b}{ct + d}\right), \\
w^A(t) &:= \frac{1}{(ct + d)^2w} \left(\frac{at + b}{ct + d}\right) - \frac{c}{ct + d},
\end{align*}

give solution to (4.1.1).

(ii) The potential $A \cdot F_{0}^{\mathbb{P}^{1,2}}$ is obtained from $F_{0}^{\mathbb{P}^{1,2}}$ by substituting:
\begin{align*}
\{x(t_{-1}), y(t_{-1}), z(t_{-1})\} &\rightarrow \{x^A(t_{-1}), y^A(t_{-1}), z^A(t_{-1})\}.
\end{align*}

**Proof.** Part (i) is easy by using Proposition 4.2 and part (ii) follows immediately from the definition of the $\text{SL}(2, \mathbb{C})$–action on the primary potential, explicit form of $F_{0}^{\mathbb{P}^{1,2}}$ and Proposition 3.7. \[\square\]

The following proposition will be used later.

**Proposition 4.4.** For any $\alpha \in \mathbb{C}^*$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C})$ we have
\begin{align*}
(\alpha x(\alpha^2 t))^A &= x^A(t), \\
(\alpha y(\alpha^2 t))^A &= y^A(t), \\
(\alpha^2 w(\alpha^2 t))^A &= w^A(t),
\end{align*}
where $A' = \begin{pmatrix} a \cdot \alpha & b \cdot \alpha \\ c/\alpha & d/\alpha \end{pmatrix} \in \text{SL}(2, \mathbb{C})$.\[\square\]
Proof. First of all note that if $x(t), y(t)$ and $w(t)$ give a solution to \((4.1.1)\), then $\hat{x}(t) := \alpha x(\alpha^2 t)$, $\hat{y}(t) := \alpha y(\alpha^2 t)$, $\hat{w}(t) := \alpha^2 w(\alpha^2 t)$ is also a solution to \((4.1.1)\) so we can consider the action of $A$ from Proposition 4.3.

Indeed,

\[
\left(\alpha x(\alpha^2 t)\right)^A = \frac{\alpha}{(ct+d)} x \left(\frac{a \cdot at + b}{ct+d}\right) \\
= \frac{1}{(\frac{ct}{\alpha} + \frac{d}{\alpha})} x \left(\frac{aat + ab}{\frac{ct}{\alpha} + \frac{d}{\alpha}}\right) = x'(t).
\]

The same computations are easy to perform for the remaining functions. □

In what follows we are going to consider the explicit values of the functions $\vartheta_k$ and make use of their holomorphicity. For such purposes it’s convenient to write them not as the $q$–expansions, but as the holomorphic functions on $H$. The formal variable $t^{-1}$ is not suitable for these purposes. So we consider the changes of the variables $q = \exp\left(\frac{2\pi i \tau}{4}\right)$. This is equivalent to applying the change of variables $t^{-1} = 2\pi i \tau/4$ mentioned earlier. Applying it to the potential $F_{0142}^{4,4,0}$ will change the terms defining the pairing. Because of this we give a special treatment to this change of the variables.

4.2. Group action via the modular forms. For $p \in \{2, 3, 4\}$ consider the following functions, holomorphic on $H$:

\[
\vartheta_p(\tau) := \vartheta_p(q(\tau)), \quad X_p^{\infty}(\tau) := 2 \frac{\partial}{\partial \tau} \log \vartheta_p(\tau).
\]

Fixing some branch of the square root, denote $\kappa := \sqrt{2\pi i/4}$. We now introduce the new functions

\[
x^\infty(\tau) := \kappa \cdot x(q(\tau)), \quad y^\infty(\tau) := \kappa \cdot y(q(\tau)), \\
z^\infty(\tau) := \kappa \cdot z(q(\tau)), \quad w^\infty(\tau) := \kappa^2 \cdot w(q(\tau)),
\]

For any $\tau_0 \in H$ and $\omega_0 \in \mathbb{C}^*$ consider the functions $x^{(\tau_0, \omega_0)}$, $y^{(\tau_0, \omega_0)}$ and $z^{(\tau_0, \omega_0)}$:

\[
x^{(\tau_0, \omega_0)}(\tau) := \frac{2i \omega_0 \text{Im}(\tau_0)}{(2i \omega_0 \text{Im}(\tau_0) - \tau)} x^\infty \left(\frac{2i \omega_0^2 \text{Im}(\tau_0) - \tau_0 \tau}{2i \omega_0^2 \text{Im}(\tau_0) - \tau}\right),
\]
with $y^{(\tau_0, \omega_0)}$, $z^{(\tau_0, \omega_0)}$ defined similarly, and also

$$w^{(\tau_0, \omega_0)}(\tau) := \frac{(2i\omega_0 \text{Im}(\tau_0))^2}{(2i\omega_0^2 \text{Im}(\tau_0) - \tau)^2} \omega^\infty \left( \frac{2i\omega_0^2 \tau_0 \text{Im}(\tau_0) - \bar{\tau}_0 \tau}{2i\omega_0^2 \text{Im}(\tau_0) - \tau} \right) - \frac{1}{(2i\omega_0^2 \text{Im}(\tau_0) - \tau)}.$$

**Remark 4.5.** The functions introduced make sense from the point of view of the modular forms — these are just expansions of the (quasi)–modular forms $x(\tau)$, $y(\tau)$ and $w(\tau)$ at the point $\tau = \tau_0$ (see [Z]).

**Proposition 4.6.** Fix some $\tau_0 \in \mathbb{H}$ and $\omega_0 \in \mathbb{C}^\ast$. We have:

(i) The functions $x^{(\tau_0, \omega_0)}$, $y^{(\tau_0, \omega_0)}$, $w^{(\tau_0, \omega_0)}$ give a solution to (4.1.1).

(ii) The functions $x^{(\tau_0, \omega_0)}(\tau)$, $y^{(\tau_0, \omega_0)}(\tau)$, $z^{(\tau_0, \omega_0)}(\tau)$, $w^{(\tau_0, \omega_0)}(\tau)$ are holomorphic on:

$$D^{(\tau_0, \omega_0)} := \{ \tau \in \mathbb{C} \mid |\tau| < |2\omega_0^2 \text{Im}(\tau_0)| \}.$$

(iii) Consider the SL–action on $x(t_{-1})$ as in Proposition 4.3. We have:

$$(x(\tau))^{(\tau_0, \omega_0)} = (x(t_{-1}))^A,$$

where

$$A = \begin{pmatrix} i\kappa \tau_0 & \kappa \omega_0 \tau_0 \\ 2\omega_0 \text{Im}(\tau_0) & \omega_0 \kappa \tau_0 \end{pmatrix}.$$  

**Proof.** Part (i) is easily checked by the explicit differentiation and definition of the functions $x^{(\tau_0, \omega_0)}(\tau)$, $y^{(\tau_0, \omega_0)}(\tau)$, $w^{(\tau_0, \omega_0)}(\tau)$. Part (ii) follows from the fact that the theta constants are holomorphic functions on $\mathbb{H}$.

For part (iii) note first that in principle the action of Proposition 4.3 is more general. It can be applied to any solution of (4.1.1). The rest follows from Proposition 4.4. \qed

**Remark 4.7.** The action $x^\infty \to x^{(\tau_0, \omega_0)}$ and so on, has its own meaning. This is, in fact, the action changing the primitive form of the B–model. Having applied this action on the B–side we get the CohFT of a simple elliptic singularity $\tilde{E}_7$ with the primitive form “at $\tau_0$” (see [BT, MR, B2]). This is a coordinate form of the Cayley transform of [SZ2].
4.3. **Proof of Theorem 4.1.** First of all note that the change of variables of Theorem 4.1 identifies the two pairings. This is clear also that the action of any $A \in \text{SL}(2, \mathbb{C})$ doesn’t change the correlators involving insertion of the unit vector of a CohFT.

Applying the linear change of the variables $\tilde{t} = \tilde{t}(t)$ given in the theorem to $A \cdot F_{0}^{4,4,2}$ we get:

$$A \cdot F_{0}^{4,4,2}(\tilde{t}) = \frac{1}{2} f_{11} \tilde{t}_{33} + f_{11} \left( f_{21} \tilde{t}_{23} + f_{13} \tilde{t}_{31} + f_{12} \tilde{t}_{32} + \frac{f_{22}^{2}}{2} \right)$$

$$+ \frac{1}{2} f_{21} (C_{1}(\tilde{t}_{33}) \cdot \tilde{t}_{13} + C_{2}(\tilde{t}_{33}) \cdot \tilde{t}_{31}) + \frac{1}{2} f_{12} (C_{2}(\tilde{t}_{33}) \cdot \tilde{t}_{13} + C_{1}(\tilde{t}_{33}) \cdot \tilde{t}_{31})$$

$$+ \sqrt{2} \left( x^{A}(\tilde{t}_{33}) + y^{A}(\tilde{t}_{33}) \right) \tilde{t}_{12} \tilde{t}_{21} \tilde{t}_{22} + O(\tilde{t}_{i,k}^{4}, \tilde{t}_{11})$$

for $C_{1}(\tilde{t}_{33}) := x^{A}(\tilde{t}_{33}) - y^{A}(\tilde{t}_{33}) + z^{A}(\tilde{t}_{33})$ and $C_{2}(\tilde{t}_{33}) := x^{A}(\tilde{t}_{33}) - y^{A}(\tilde{t}_{33}) - z^{A}(\tilde{t}_{33})$.

**Lemma 4.8.** The equality of the formal series $A \cdot F_{0}^{4,4,2}(\tilde{t}) = F_{0}^{\tilde{E}_{7}, G_{\max}}(\tilde{t})$ is satisfied if and only if

$$A \cdot F_{0}^{4,4,2}(\tilde{t}) - F_{0}^{\tilde{E}_{7}, G_{\max}}(\tilde{t}) \in O(\tilde{t}_{i,k}^{4}, \tilde{t}_{11}).$$

**Proof.** One direction is straightforward and we concentrate on the opposite one.

First of all note that the potential $A \cdot F_{0}^{4,4,2}(\tilde{t})$ satisfies the same quasihomogeneity property as the potential $F_{0}^{4,4,2}(\tilde{t})$. Next one sees easily that the change of the variables $\tilde{t}(t)$ preserves the quasihomogeneity property.

Recall the genus zero reconstruction lemma of Subsection 3.3. The equality above assures also that the algebra structure at the origin coincides on the both sides. Hence the notion of the primitive vectors coincides on the both sides.

Hence the conditions of Lemma 3.3 coincide for the both potentials. For the FJRW theory these conditions were written in Lemma 3.5 to be exactly those as described in the proposition.

However in order to prove the theorem we need to use explicit values of the functions and therefore work with the “modular” variable $\tau \in \mathbb{H}$.

**Lemma 4.9.** Let $\tau_{0} = i$ and $\omega_{0} := \kappa \sqrt{2\pi \tau} / (\Gamma(\frac{3}{2}))^{2}$. The equation

$$A^{\text{CY/\text{LG}}} \cdot F_{0}^{4,4,2}(\tilde{t}) = F_{0}^{\tilde{E}_{7}, G_{\max}}(\tilde{t})$$

...
hold if and only if:

\[ x(t_0, \omega_0)(\tau) - y(t_0, \omega_0)(\tau) + z(t_0, \omega_0)(\tau) = 1 + O(\tau^2), \]
\[ x(t_0, \omega_0)(\tau) - y(t_0, \omega_0)(\tau) - z(t_0, \omega_0)(\tau) = -\frac{\tau}{4} + O(\tau^2), \]
\[ x(t_0, \omega_0)(\tau) + y(t_0, \omega_0)(\tau) = \frac{1}{\sqrt{2}} + O(\tau^2). \]  

(4.3.1)

Proof. By using the lemma above and reconstruction Lemma [3.3] we see that it’s enough to compare the potentials \( F_0^{E_7, G_{\text{max}}} \) and \( A \cdot F_0^{[4,4,2]}(\tilde{t}) \) up to \( O(t_4, t_{ij}, t_{11}) \).

Recall part (iii) of Proposition [4.6]. Note that for the \( t_0 \) and \( \omega_0 \) as in the statement of the Lemma, the matrix \( A' \) coincides with the matrix \( A_{CY/LG} \).

The equalities above are obtained by comparing the coefficients of \( F_0^{E_7, G_{\text{max}}} \) and \( A \cdot F_0^{[4,4,2]}(\tilde{t}) \). The RHS of them are taken from the explicit form of \( F_0^{E_7, G_{\text{max}}} \) (recall Section [3.4]).

It follows from Lemma [3.5] that it’s enough to check these equalities in order for the whole potentials to coincide. \( \square \)

In the remainder of this section we show that (4.3.1) is satisfied by the functions \( x(t_0, \omega_0)(\tau) \), \( y(t_0, \omega_0)(\tau) \), \( z(t_0, \omega_0)(\tau) \) for \( t_0 \) and \( \omega_0 \) as in Lemma [4.9].

Denote by \( \tilde{x}, \tilde{y}, \tilde{z} \) the expansion of the function \( x, y, z \) with the change of variables \( \tau \to A(t_0, \omega_0)\tau \) applied, i.e.

\[ \tilde{x}(\tau) := x^\infty \left( \frac{2i\omega_0^2 t_0 \text{Im}(t_0) - \tau_0 t}{2i\omega_0^2 \text{Im}(t_0) - \tau} \right), \]

and similar for \( \tilde{y}, \tilde{z} \). Define the numbers \( x_0, x_1, y_0, y_1 \) and \( z_0, z_1 \) as the coefficients of the series expansions at \( \tau = 0 \):

\[ \tilde{x} = x_0 + x_1 \tau + O(\tau^2), \quad \tilde{y} = y_0 + y_1 \tau + O(\tau^2), \quad \tilde{z} = z_0 + z_1 \tau + O(\tau^2). \]

The functions \( x(t_0, \omega_0), y(t_0, \omega_0), z(t_0, \omega_0) \) satisfy

\[ x(t_0, \omega_0)(\tau) = \frac{x_0}{\omega_0} + \tau \left( \frac{x_1}{\omega_0} + \frac{x_0}{2i\omega_0^2 \text{Im}(t_0)} \right) + O(\tau^2). \]

To find the coefficients explicitly we use the following derivation formula.
Lemma 4.10. The derivatives of the functions $x, y, z$ satisfy:
\[
\frac{\partial}{\partial \tau} \tilde{x}(\tau) \big|_{\tau=0} = \frac{\kappa}{2\omega_0^2} \left( \vartheta_3(\tau_0)^2 X_3^\infty(\tau_0) + \vartheta_4(\tau_0)^2 X_4^\infty(\tau_0) \right),
\]
\[
\frac{\partial}{\partial \tau} \tilde{y}(\tau) \big|_{\tau=0} = \frac{\kappa}{2\omega_0^2} \left( \vartheta_3(\tau_0)^2 X_3^\infty(\tau_0) - \vartheta_4(\tau_0)^2 X_4^\infty(\tau_0) \right),
\]
\[
\frac{\partial}{\partial \tau} \tilde{z}(\tau) \big|_{\tau=0} = \frac{\kappa}{\omega_0^2} \vartheta_2(\tau_0)^2 X_2^\infty(\tau_0).
\]

Proof. By using the double argument formulae of the Jacobi theta constants we see:
\[
2x(q) = \vartheta_3(q^4)^2 + \vartheta_4(q^4)^2,
\]
\[
2y(q) = \vartheta_3(q^4)^2 - \vartheta_4(q^4)^2,
\]
and all functions $x(q), y(q), z(q)$ are written via $q^4$. Directly from the definition of $X_k^\infty(\tau)$ and the rescaling we get:
\[
\frac{\partial}{\partial \tau} x_\infty(\tau) = \frac{\kappa}{2} \left( \vartheta_3(\tau)^2 X_3^\infty(\tau) + \vartheta_4(\tau)^2 X_4^\infty(\tau) \right),
\]
\[
\frac{\partial}{\partial \tau} y_\infty(\tau) = \frac{\kappa}{2} \left( \vartheta_3(\tau)^2 X_3^\infty(\tau) - \vartheta_4(\tau)^2 X_4^\infty(\tau) \right),
\]
\[
\frac{\partial}{\partial \tau} z_\infty(\tau) = \kappa \vartheta_2(\tau)^2 X_2^\infty(\tau).
\]

The rest follows from the chain rule and the definition of $\tilde{x}, \tilde{y}, \tilde{z}$. □

The values of the theta constants and their logarithmic derivatives at the point $\tau = i$ are known to be:
\[
\vartheta_2(i) = \frac{\pi^{1/4}}{2^{1/4}\Gamma \left( \frac{3}{4} \right)}, \quad \vartheta_3(i) = \frac{\pi^{1/4}}{\Gamma \left( \frac{3}{4} \right)}, \quad \vartheta_4(i) = \frac{\pi^{1/4}}{2^{1/4}\Gamma \left( \frac{3}{4} \right)},
\]
\[
X_2^\infty(i) = \frac{i\pi^2}{4 \left( \Gamma \left( \frac{3}{4} \right) \right)^4} + \frac{i}{2}, \quad X_3^\infty(i) = \frac{i}{2},
\]
\[
X_4^\infty(i) = -\frac{i\pi^2}{4 \left( \Gamma \left( \frac{3}{4} \right) \right)^4} + \frac{i}{2}.
\]

For $K = \frac{\pi^{1/2}}{2^{1/2}(\Gamma(\frac{3}{4}))^2}$ using the lemma above we get:
\[
\kappa^{-1} \tilde{x}(\tau) = \frac{K}{2} \left( \sqrt{2} + 1 \right) + \tau \frac{K_i}{2\omega_0^2} \left( \sqrt{2} + \left( \frac{1}{2} - \frac{K^2\pi}{2} \right) \right) + O(\tau^2),
\]
$$\kappa^{-1} \ ý (\tau) = \frac{K}{2} \left( \sqrt{2} - 1 \right) + \tau \frac{K}{2\omega_0^2} \left( \frac{\sqrt{2}}{2} - \left( \frac{1}{2} - \frac{K^2\pi}{2} \right) \right) + O(\tau^2),$$

$$\kappa^{-1} \ z (\tau) = K + \tau \frac{K}{2\omega_0^2} \left( \frac{1}{2} + \frac{\pi K^2}{2} \right) + O(\tau^2).$$

Hence

$$\kappa^{-1} \left( x^{(i\omega_0)}(\tau) - y^{(i\omega_0)}(\tau) + z^{(i\omega_0)}(\tau) \right)
= \frac{x_0 - y_0 + z_0}{\omega_0} + \tau \left( \frac{x_1 - y_1 + z_1}{\omega_0} + \frac{x_0 - y_0 + z_0}{2i\omega_0^3} \right) + O(\tau^2)
= 2 \frac{K}{\omega_0} + O(\tau^2),$$

$$\kappa^{-1} \left( x^{(i\omega_0)}(\tau) - y^{(i\omega_0)}(\tau) - z^{(i\omega_0)}(\tau) \right)
= \frac{x_0 - y_0 - z_0}{\omega_0} + \tau \left( \frac{x_1 - y_1 - z_1}{\omega_0} + \frac{x_0 - y_0 - z_0}{2i\omega_0^3} \right) + O(\tau^2)
= - \frac{\tau \cdot \pi K^3}{\omega_0^2} + O(\tau^2),$$

$$\kappa^{-1} \left( x^{(i\omega_0)}(\tau) + y^{(i\omega_0)}(\tau) \right)
= \frac{x_0 + y_0}{\omega_0} + \tau \left( \frac{x_1 + y_1}{\omega_0} + \frac{x_0 + y_0}{2i\omega_0^3} \right) + O(\tau^2)
= \sqrt{2} \frac{K}{\omega_0} + O(\tau^2).$$

Fixing $\omega_0 = 2K\kappa$ we get exactly the expansions as in (4.3.1). This completes proof of Theorem 4.1.

5. **Givental’s action and CY/LG correspondence**

In this section we formulate the CY/LG correspondence via the group action on the space of cohomological field theories and give the particular action, connecting $\mathcal{F}_{\mathbb{P}^1,4,2}^1$ and $\mathcal{F}_{\mathbb{E}_7,G_{\text{max}}}^0$.

5.1. **Infinitesimal version of Givental’s action.** In this subsection we introduce Givental’s group action on the partition function of a CohFT via the infinitesimal action computed in [1]. Let $\Lambda_{g,n}$ be a unital CohFT on $(V, \eta)$ with the unit $e_0 \in V$. 
The upper–triangular group consists of all elements \( R = \exp(\sum_{i=1} r_i z^i) \), such that
\[
r(z) = \sum_{i \geq 1} r_i z^i \in \text{Hom}(V, V) \otimes \mathbb{C}[z],
\]
and \( r(z) + r(-z)^* = 0 \) (where the star means dual with respect to \( \eta \)).

Following Givental, we define the quantization of \( R \):
\[
\hat{R} := \exp(\sum_{i=1} \hat{r}_i z^i),
\]
where for \((r_1)^{\alpha, \beta} = (r_1)^{\sigma \eta, \sigma, \eta} \) we have:
\[
\hat{r}_i z^i := - (r_1)^{\alpha} \frac{\partial}{\partial t_i^{l+1, \alpha}} + \sum_{d=0}^{\infty} t^{d, \beta} (r_1)^{\alpha, \beta} \frac{\partial}{\partial t_i^{d, \alpha}} \frac{\partial^2}{\partial t_i^{d, \alpha} t_i^{d, \beta}},
\]

The lower–triangular group consists of all elements \( S = \exp(\sum_{i=1} s_i z^{-i}) \), such that
\[
s(z) = \sum_{i \geq 1} s_i z^{-i} \in \text{Hom}(V, V) \otimes \mathbb{C}[z^{-1}]
\]
and \( s(z) + s(-z)^* = 0 \). Following Givental, we define the quantization of \( S \):
\[
\hat{S} := \exp(\sum_{i=1}^{\infty} (s_i z^{-i}))^*,
\]
where
\[
\sum_{i=1}^{\infty} (s_i z^{-i})^* = -(s_1)^{\alpha} \frac{\partial}{\partial t_0, \alpha} + \frac{1}{\hbar} \sum_{d=0}^{\infty} (s_{d+2})_{1, \alpha} t_i^{d, \alpha} \]
\[
+ \sum_{d=0}^{\infty} (s_1)^{d, \beta} t_i^{d+1, \beta} \frac{\partial}{\partial t_i^{d, \alpha}} + \frac{1}{2\hbar} \sum_{d_1, d_2} (-1)^{d_1} (s_{d_1+d_2+1})_{a_1, a_2} t_i^{d_1, a_1} t_i^{d_2, a_2}.
\]

The following theorem is essentially due to Givental.

**Theorem 5.1** ([G]). The differential operators \( \hat{R} \) and \( \hat{S} \) act on the space of partition functions of CohFTs.

The action of the differential operators \( \hat{R} \) and \( \hat{S} \) on the partition function of the CohFT is called Givental’s \( R \)– and \( S \)–action or upper–triangular and lower–triangular Givental’s group action respectively.

The action of the upper–triangular group can be also formulated on the CohFT itself — not just on its partition function. However it is
more subtle for the lower–triangular group, whose generic element acts on the partition function only (cf. [PPZ]).

5.2. **R–matrix of a CohFT.** Fix a unital CohFT $\Lambda$ on $(V, \eta)$ with unit $e_0$ and $m + 1 = \dim V$. For every fixed indices $i, j, k \in 0, \ldots, m$ define:

$$c^k_{ij}(t) := \sum_{p=0}^{m} \frac{\partial^3 F_0}{\partial t_i \partial t_j \partial t_p} \eta^{pk}.$$  

Because $F_0$ is a solution to WDVV equation (see Section 3.3) the functions $c^k_{ij}(t)$ are structure constants of an associative and commutative algebra. Moreover this algebra turns out to be a Frobenius algebra with respect to the pairing $\eta$.

The CohFT $\Lambda$ is called *semisimple* if the algebra, defined by $c^k_{ij}(t)$ above is semisimple. In this case there are new coordinates $u^0(t), \ldots, u^m(t)$, such that $c^k_{ij}(u) = \delta_{i,j} \delta_{i,k} \Delta_i$ for some functions $\Delta_i = \Delta_i(u)$. Let $\Psi$ be the transformation matrix from the frame $\langle \partial/\partial t^0, \ldots, \partial/\partial t^m \rangle$ to the frame $\langle \partial/\partial u^0, \ldots, \partial/\partial u^m \rangle$.

Consider the partition function $Z^{pt}$ of the GW–theory of a point. This is a partition function of a CohFT on a one–dimensional space, and can be therefore written in coordinates $\{u^\ell \}_{\ell \geq 0}$. In the next formula take the product of $m + 1$ such partition functions indexing however the variables.

$$T^{(m+1)} = \prod_{k=0}^{m} Z^{pt} \left( \{u^\ell \}_{\ell \geq 0} \right). \quad \text{(5.2.1)}$$

The following theorem was conjectured by Givental and later proved by Teleman.

**Theorem 5.2 (Theorem 1 in [1]).** For every quasihomogenous semisimple unital CohFT $\Lambda$ on an $m + 1$–dimensional vector space $V$ there are unique upper–triangular and lower–triangular group elements $R$ and $S$, such that

$$Z^\Lambda = \hat{S} \cdot \hat{R} \cdot \hat{\Psi} \cdot T^{(m+1)},$$

where $\hat{\Psi}$ acts by the change of the variables $u = u(t)$.

Because the $S$–action is more like a change of the coordinates, the most important part of the formula above is located in the action or $R$. This motivates the following definition.

**Definition 5.3.** The upper–triangular group element $R$ as above is called the **$R$–matrix** of the CohFT $\Lambda$. 
In order to find such an R–matrix explicitly one would normally use the recursive procedure described by Givental. After writing \( R = \text{Id} + \sum_{k \geq 1} R_k z^k \) every matrix \( R_k \) is uniquely determined by the preceding matrices. However, it is difficult to perform this procedure to the end to have a closed formula for \( R = R(z) \). Up to now the only explicitly written R–matrix is for the theory of 2–spin curves, which is two–dimensional (cf. [PPZ]).

Furthermore, it could still happen that the formula of the theorem above holds for a non–semisimple CohFT. In this case one doesn’t know if the R–matrix is unique and the recursive procedure above can no longer be be applied. We will return to this question in Section 5.3, where we present an R–matrix for the Gromov–Witten theory of \( \mathbb{P}^1_{4,4,2} \) without the use of the recursive procedure described above.

5.3. Mirror symmetry and CY/LG correspondence. The CY/LG correspondence is best understood using mirror symmetry via the B–model.

Given a hypersurface singularity \( \tilde{W} : \mathbb{C}^N \to \mathbb{C} \) one can construct the so–called Saito–Givental CohFT, which depends non–trivially on the certain special choice of primitive form \( \zeta \), known as the primitive form of Saito. Let \( Z_{\tilde{W}, \zeta} \) be the partition function of this CohFT.

CY–LG mirror symmetry conjectures that the partition function \( Z_{\tilde{W}, \zeta_\infty} \) with the special choice of the primitive form \( \zeta = \zeta_\infty \) coincides with the partition function of the GW theory of some Calabi–Yau variety \( \mathcal{X} \) up to probably a linear change of the variables.

LG–LG mirror symmetry conjectures that the partition function \( Z_{\tilde{W}, \zeta_0} \) with the another special choice of the primitive form \( \zeta = \zeta_0 \) coincides with the partition function of the FJRW theory of some pair \((W, G_{\text{max}})\) up to probably a linear change of the variables, where \( W : \mathbb{C}^N \to \mathbb{C} \) is some hypersurface singularity (generally different from \( \tilde{W} \)).

One says than that the GW theory of \( \mathcal{X} \) and FJRW theory of \((W, G_{\text{max}})\) constitute two mirror A–models of the one B–model of \( \tilde{W} \), taken in the different phases — \( \zeta_\infty \) and \( \zeta_0 \). This lead to the following conjecture.

**Conjecture 5.4.** Let GW theory of \( \mathcal{X} \) and FJRW theory of \((W, G_{\text{max}})\) be two mirror A–models of the same B–model. Then there is an upper–triangular Givental’s action \( R = R(z) \), such that

\[
\hat{R} \cdot Z^{\mathcal{X}}(t) = Z^{(W, G_{\text{max}})}(\tilde{t}(t)),
\]
where $\tilde{f} = f(t)$ is a linear change of the variables.

When two mirror symmetry conjectures of type CY–LG and type LG–LG hold, Conjecture 5.4 is an A–side analogue of the following B–side conjecture:

**Conjecture 5.5.** there is an upper–triangular group element of Givental $R = R(z)$, such that up to a linear change of variables the following equation holds:

$$\hat{R} \cdot Z_{W, \tilde{c}^\infty} = Z_{W, \tilde{c}^0}.$$ 

In the case of simple elliptic singularities this sort of action was investigated in [MR, BL, B2]. In particular, it was shown in [B2] that the $\text{SL}(2, \mathbb{C})$–action of Section 4 has at the same time the meaning of the primitive form change on the B–side and can be written via the certain $R$–action of Givental. In other words, (4.0.1) can be realized as the restriction of the certain action of Givental to the small phase space.

5.4. **CY/LG correspondence via Givental’s action.** For any $\tau, \sigma \in \mathbb{C}$ consider the lower–triangular group element

$$S^\tau(z) = \exp \left( \begin{pmatrix} 0 & \ldots & 0 \\
 & \ddots & 0 \\
 & & \tau \end{pmatrix} z^{-1} \right),$$

and the upper–triangular group element $R^\sigma$:

$$R^\sigma(z) = \exp \left( \begin{pmatrix} 0 & \ldots & \sigma \\
 & \ddots & 0 \\
 & & 0 \end{pmatrix} z \right).$$

For any $c \in \mathbb{C}$, we also define the matrix

$$S^c_0 := \begin{pmatrix} 1 & \ldots & 0 \\
 & \ddots & \vdots \\
 & & c \cdot I_{n-2} \\
 0 & \ldots & c^2 \end{pmatrix},$$

together with an action of $S^c_0$ on $Z(t)$ (which we denote by $\hat{S}^c_0$) defined by

$$t^{\ell, \alpha} \rightarrow (S^c_0)^{\alpha}_{\beta} t^{\ell, \beta} \quad \text{and} \quad h \rightarrow c^2 h.$$ 

Let $\Theta = \frac{\sqrt{2\pi} \pi}{(\Gamma(\frac{3}{4}))^2}$ as in Theorem 4.1 define:

$$\tau := -\frac{\pi}{2}, \quad \sigma := -\frac{1}{\pi \Theta^2}, \quad c := \frac{1}{\Theta}.$$
We give now the Givental’s action form of the CY/LG correspondence in genus zero.

Theorem 5.6. Consider the partition functions $Z(\tilde{E}_7, G_{\text{max}})$ and $Z_{\mathbb{P}^1, 4, 2}$. We have:

$$F_0^{(\tilde{E}_7, G_{\text{max}})} = \text{res}_R \ln \left( \hat{R}^\sigma \cdot \hat{S}_c^0 \cdot \hat{S}^\tau \cdot Z_{\mathbb{P}^1, 4, 2} \right)$$

with the Givental’s element $S^\tau$, $S_c^0$ and $R^\sigma$ defined above.

Proof. One can check (cf. [B2]) that the action of the theorem induces the action of Theorem 4.1 on the primary genus 0 potentials. And in Theorem 4.1 and Corollary 5.1 in [B2] we see that the theorem holds for the primary potentials. We only have to take care of the psi–classes insertions.

However in genus zero all correlators are unambiguously reconstructed from the small phase space correlators by using the topological recursion relation. Hence we can reconstruct these correlators on the LHS from the small phase space. □

6. Extended FJRW Correlators

In this section, we reformulate FJRW theory in order to obtain the genus zero potential $F_0^{(\tilde{E}_7, G_{\text{max}})}$ from a basic CohFT. This method is also used in [CR, ChIR, PS], so we will be brief. The details can be found in these other articles. In this section and the section following, we fix $W = x^4 + y^4 + z^2$, and $G = G_{\text{max}}$.

6.1. $r$–spin theory. Let $(A_r)_{g,n}$ denote the moduli space of genus $g$, $n$–marked $A_r$–curves corresponding to the polynomial $A_r = x^{r+1}$. Such $W$–structures are often referred to as $r$–spin curves. Let $(A_W)_{g,n}$ denote the fiber product

$$(A_W)_{g,n} := (A_3)_{g,n} \times_{\mathcal{M}_{g,n}} (A_3)_{g,n} \times_{\mathcal{M}_{g,n}} (A_1)_{g,n}$$

Proposition 6.1 ([CR]). There is a surjective map

$$s : (A_W)_{g,n} \to W_{g,n}$$

which is a bijection at the level of a point.

Each factor of $(A_r)_{g,n}$ in the product above is equipped with a universal $A_r$–structure. Abusing notation, we denote the universal line bundle over the $k$th factor of $(A_W)_{g,n}$ also by $\mathbb{L}_k$. By the universal properties of the $W$–structure on $W_{g,n}$, we have $s^* \mathbb{L}_k \cong \mathbb{L}_k$ for $1 \leq k \leq 3$. 
Given \( h = (h_1, \ldots, h_n) \), let us denote
\[
A_W(h)_{g,n} := (A_3)_{g,n} (\Theta_1^{h_1}, \ldots, \Theta_1^{h_n}) \times \mathcal{M}_{g,n,A} \times \cdots \times \mathcal{M}_{g,n,A} (\Theta_3^{h_1}, \ldots, \Theta_3^{h_n}).
\]

By the projection formula, we can pull back to \((A_W)_{0,n}\), and obtain the following expression for the genus 0 correlators:
\[
\langle \tau_{a_1}(\phi_{h_1}), \ldots, \tau_{a_n}(\phi_{h_n}) \rangle_{(\tilde{E}_7, G_{\text{max}})}^{0, n} = 32 \int_{A_W(h)_{0,n}} \prod_{i=1}^{n} \psi_i \cap c_{\text{top}} \left( R^1 \tau_* \left( \bigoplus_{i=1}^{3} L_i \right) \right)
\]

From this description, we see that the FJRW theory in this case is a so-called twisted theory. Thus we can use Givental’s formalism to give an expression for the generating function of these correlators.

### 6.2. Twisted theory

We will construct a twisted FJRW theory whose correlators coincide with those of \((\tilde{E}_7, G_{\text{max}})\) in genus zero. We first extend the state space
\[
\mathcal{H}^\text{ext}_{W,G} := \mathcal{H}_{W,G} \oplus \bigoplus_{h \in G \setminus G_{\text{nar}}} C \cdot \phi_h.
\]

Any point \( t \in \mathcal{H}^\text{ext}_{W,G} \) can be written as \( t = \sum_{h \in G} t^h \phi_h \). Let \( i_k(h) := \langle \Theta_k^h - q_k \rangle \), where \( \langle - \rangle \) denotes the fractional part. Notice \( i_k(h) = 1 - q_k \) exactly when \( \Theta_k^h = 0 \). Set
\[
\deg_W(\phi_h) := 2 \sum_{k=1}^{3} i_k(h).
\]

For \( h \in G_{\text{nar}} \), this definition matches the \( W \)-degree defined in (2.1.1).

We extend the definition of our FJRW correlators to include insertions \( \phi_h \) in \( \mathcal{H}^\text{ext}_{W,G} \). Namely, set
\[
\langle \tau_{a_1}(\phi_{h_1}), \ldots, \tau_{a_n}(\phi_{h_n}) \rangle_{(W,G)}^{0, n} = 0
\]
if \( h_i \in G \setminus G_{\text{nar}} \) for some \( i \).

We would like to unify our definition of the extended FJRW correlators, by re-expressing them as integrals over \((A_W)_{0,n}\), a slight variation of \((A_W)_{0,n}\). We will make use of the following lemma.

**Lemma 6.2 ([CR]).** Let \( C \) be a d–stable curve with coarse underlying curve \( C \), and let \( M \) be a line bundle pulled back from \( C \). If \( l | d \), there is an equivalence between two categories of \( l \)th roots \( L \) on d–stable curves:
\[
\left\{ \mathcal{L} | \mathcal{L}^\otimes l \cong M \right\} \leftrightarrow \bigcup_{0 \leq E < \sum l D_i} \left\{ \mathcal{L} | \mathcal{L}^\otimes l \cong M(-E) \text{ mult}_{p_i}(\mathcal{L}) = 0 \right\}.
\]
where the union is taken over divisors $E$ which are linear combinations of integer divisors $D_i$ corresponding to the marked points $p_i$.

**Proof.** Let $p$ denote the map which forgets stabilizers along the markings. The correspondence is simply $L \mapsto p^*p_*(L)$. □

**Definition 6.3.** For $m_1, \ldots, m_n \in \{1, 1, 3, 2, 2, 1\}$, consider the stack $\tilde{A}_3(m_1, \ldots, m_n)_{g,n}$ classifying genus $g$, $n$–pointed, 4–stable curves equipped with fourth roots:

$$\tilde{A}_3(m_1, \ldots, m_n)_{g,n} := \left\{ (C, p_1, \ldots, p_n, L, \phi) \mid \phi : L^{\otimes 4} \sim \omega_{\log} \left( - \sum_{i=1}^{n} 4m_i D_i \right), \text{ mult}_p(L) = 0 \right\},$$

where the integer divisors $D_i$ correspond to the markings $p_i$.

The moduli space $\tilde{A}_3(m_1, \ldots, m_n)_{g,n}$ also has a universal curve $\mathcal{C} \to \tilde{A}_3$ and a universal line bundle $\bar{L}$. We can define everything similarly for $A_1$ and replace it with $\tilde{A}_1$.

We now define an analogue of $(A_W)_{g,n}$, replacing $(A_3)_{g,n}$ with $(\tilde{A}_3)_{g,n}$ in the first two factors, and $(A_1)_{g,n}$ with $(\tilde{A}_1)_{g,n}$. For $1 \leq i \leq n$, let $m_i = (m_{1i}, \ldots, m_{3i})$ be a tuple of fractions satisfying $m_{1i}, m_{2i} \in \{1, 1, 3, 2, 2, 1\}$, and $m_{3i} \in \{2, 1\}$. Let $\mathbf{m}$ denote the $3 \times n$ matrix $(\mathbf{m})_{ki} = m_{ki}$.

Define

$$\tilde{A}_W(\mathbf{m})_{g,n} := \tilde{A}_3(m_{11}, \ldots, m_{1n})_{g,n} \times \mathcal{M}_{g,n,4} \times \mathcal{M}_{g,n,4} \tilde{A}_1(m_{31}, \ldots, m_{3n})_{g,n}.$$ 

$\tilde{A}_W(\mathbf{m})_{g,n}$ carries three universal line bundles $\tilde{L}_1, \tilde{L}_2, \tilde{L}_3$ satisfying

$$(\tilde{L}_k)^{\otimes 4} \cong \omega_{\log} \left( - \sum_{i=1}^{n} 4m_{ki} D_i \right).$$

The above moduli space yields a uniform way of defining the extended FJRW correlators for $(\tilde{E}_7, G_{max})$. Given $\phi_{h_1}, \ldots, \phi_{h_n} \in \mathcal{H}_{W,G}^{ext}$, we define the $3 \times n$ matrix

$$I(\mathbf{h}) = \begin{pmatrix} i_1(h_1) + \frac{1}{4} & \cdots & i_1(h_n) + \frac{1}{4} \\ i_2(h_1) + \frac{1}{3} & \cdots & i_2(h_n) + \frac{1}{3} \\ i_3(h_1) + \frac{1}{2} & \cdots & i_3(h_n) + \frac{1}{2} \end{pmatrix}.$$ 

Consider the following proposition.
Proposition 6.4. On $\tilde{A}_W(I(h))_{0,n}$, $\pi_*(\bigoplus_{k=1}^3 \mathbb{L}_k)$ vanishes and $R^1\pi_*(\bigoplus_{k=1}^3 \mathbb{L}_k)$ is locally free. Furthermore,

$$\langle \tau_{d_1}(\phi_{h_1}), \ldots, \tau_{d_n}(\phi_{h_n}) \rangle_{\tilde{A}_W(I(h))_{0,n},T\mathcal{G}_{\text{max}}} = 32 \int_{\tilde{A}_W(I(h))_{0,n}} \prod_{i} \psi_i^{d_i} \cup c_{\text{top}} \left( R^1\pi_* \left( \bigoplus_{k=1}^3 \mathbb{L}_k \right)^\vee \right).$$

Proof. This proof is given in [CR], and [PS], so we only give an outline. Comparing $A_3$ and $\tilde{A}_3$, we see that if $m_{ki} \in \left\{ \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \right\}$ for all $k, i$, then $h_1, \ldots, h_n \in G^{\text{nar}}$. In this case, we can identify $\tilde{A}_4^4(m)_{g,n}$ with $A_4^4(m)_{g,n}$ via Lemma 6.2 and $R^k\pi_*(\mathbb{L}_k) = R^k\pi_*(\mathbb{L}_k)$.

We must consider the case where $h_i \in G \setminus G^{\text{nar}}$ for some $i$. In this case $(I(h))_{ki} = 1$ for some $k$. Thus it suffices to prove that if $m_{ki} = 1$ for some $i$ and $k$, then $\pi_*(\bigoplus_{k=1}^3 \mathbb{L}_k) = 0$ and $c_{\text{top}} \left( R^1\pi_* \left( \bigoplus_{k=1}^3 \mathbb{L}_k \right)^\vee \right) = 0$.

To do this assume $m_{k1} = 1$, and consider the integer divisor $D_1$ on $\tilde{A}_W(m)_{0,n}$ corresponding to the first marked point. We get the long exact sequence

$$0 \to \pi_*(\mathbb{L}_k) \to \pi_*(\tilde{L}_k(D_1)) \to \pi_*(\tilde{L}_k(D_1)|_{D_1})$$

$$\to R^1\pi_*(\mathbb{L}_k) \to R^1\pi_*(\tilde{L}_k(D_1)) \to R^1\pi_*(\tilde{L}_k(D_1)|_{D_1}) \to 0.$$

As in Lemma 3.4 the first two terms are 0.

With $\pi_*(\tilde{L}_k(D_1))$, there is one alteration. If $C$ is reducible, and $\nu'$ corresponds to the irreducible component carrying the first marked point, then $\deg \tilde{L}_k(D_1)|_{\nu'} < \#\text{nodes}(\nu')$. But any section of $\tilde{L}_k(D_1)$ must still vanish on all other components of $C$, and by degree considerations it must therefore vanish on $\nu'$.

$R^1\pi_*(\tilde{L}_k(D_1)|_{D_1})$ also vanishes, so we have

$$0 \to \pi_*(\tilde{L}_k(D_1)|_{D_1}) \to R^1\pi_*(\tilde{L}_k(D_1)|_{D_1}) \to 0.$$

and

$$c_{\text{top}}(R^1\pi_*\mathbb{L}_k) = c_{\text{top}}(\pi_*(\tilde{L}_k(D_1)|_{D_1}) \cdot c_{\text{top}}(R^1\pi_*\tilde{L}_k(D_1)).$$

But $c_{\text{top}}(\pi_*(\tilde{L}_k(D_1)|_{D_1}) = 0$, as $\tilde{L}_k(D_1)|_{D_1} \cong \mathbb{L}_k|_{D_1}$ is a root of $\omega_{\log}|_{D_1}$ which is trivial. Thus $c_{\text{top}}(R^1\pi_*\tilde{L}_k(D_1)) = 0$ as well.

We may define a $C^*$-equivariant generalization of the above theory. This will allow us to compute correlators which, in the non-equivariant limit coincide with the genus zero FJRW correlators above. Given a point $(C, p_1, \ldots, p_n, \tilde{C}_1, \tilde{L}_1, \tilde{C}_3)$ in $(\tilde{A}_W)_{g,n}$, let $C^*$ act on the
total space of $\bigoplus_{k=1}^{3} \tilde{L}_k$ by multiplication of the fiber. This induces an action on $(\tilde{\mathcal{A}}_W)_{g,n}$.

Set $R = H^*_{\mathbb{C}}(pt, \mathbb{C})[[s_0, s_1, \ldots]]$, the ring of power series in the variables $s_0, s_1, \ldots$ with coefficients in the equivariant cohomology of a point, $H^*_{\mathbb{C}}(pt, \mathbb{C}) = \mathbb{C}[\lambda]$. Define a multiplicative characteristic class $c$ taking values in $R$, by

$$c(E) := \exp\left(\sum_{\ell} s_{\ell} \text{ch}_\ell(E)\right)$$

for $E \in K^*((\tilde{\mathcal{A}}_W)_{g,n})$.

Define the twisted state space

$$\mathcal{H}^t := \mathcal{H}^\text{ext}_{W,G} \otimes R \cong \bigoplus_{h \in G} R \cdot \phi_h$$

and extend the pairing by

$$\langle \phi_{h_1}, \phi_{h_2} \rangle := \begin{cases} 
\prod_{\{k \mid \Theta_{h_1}^k = 0\}} \exp(-s_0) & \text{if } h_1 = (h_2)^{-1} \\
0 & \text{otherwise}.
\end{cases}$$

In this definition, the empty product—i.e. when $h_1 \in G^{nar}$—is understood to be 1.

We may also define twisted correlators as follows. Given $\phi_{h_1}, \ldots, \phi_{h_n}$ basis elements in $\mathcal{H}^t$, define the invariant

$$\langle \tau_{a_1}(\phi_{h_1}), \ldots, \tau_{a_n}(\phi_{h_n}) \rangle^t_{g,n} := 32 \int_{A_W(I(h))_{g,n}} \prod_{j} \psi_{a_j} \cup c\left(R \pi_*\left(\bigoplus_{k=1}^{3} \tilde{L}_k\right)\right).$$

taking values in $R$. We can organize these correlators into generating functions $F^t$ and $Z^t$ as in Section 3.1. It is not clear at first glance that these correlators come from a CohFT. We will see in the next section, however, that they do indeed.

6.3. **From twisted theory to FJRW theory.** Specializing to particular values of $s_{\ell}$ yield different twisted correlators. One particularly important specialization is the following. From the partition function $Z^t$, if we set

$$s_{\ell} = \begin{cases} 
-\ln \lambda & \text{if } \ell = 0 \\
\frac{(\ell-1)!}{\lambda^\ell} & \text{otherwise}
\end{cases}$$

we obtain the (extended) FJRW–theory correlators defined above. To see this first consider the following lemma.
Lemma 6.5. [CR] Lemma 4.1.2] With \( s_\ell \) defined as in (6.3.1), the multiplicative class \( c(-V) = e_{\mathcal{C}^*}(V^\vee) \). In particular, the non–equivariant limit \( \lambda \to 0 \) yields the top chern class of \( V^\vee \).

By Proposition 3.4, \( \pi^*(\bigoplus \tilde{L}_k) = 0 \) and \( c(R\pi^*(\tilde{L}_k)) = c(-R^1\pi^*(\tilde{L}_k)) \).

Setting \( s_\ell \) as in (6.3.1) therefore yields
\[
c(R\pi^*(\bigoplus_{k=1}^3 \tilde{L}_k)) = e_{\mathcal{C}^*}(R^1\pi^*(\bigoplus_{k=1}^3 \tilde{L}_k)^\vee).
\]

We have seen in Proposition 6.4 that the FJRW correlators are obtained by the top chern class of \( R^1\pi^*(\bigoplus_{k=1}^3 \tilde{L}_k) \), so we arrive at the following important result

Corollary 6.6. After specializing \( s_\ell \) to the values in (6.3.1),
\[
\lim_{\lambda \to 0} \mathcal{F}^{tw}_0 = \mathcal{F}^{(\tilde{E}_7,G_{\text{max}})}_0.
\]

7. Computing the R–matrix

In this section, we will begin by describing the so–called “untwisted” theory, which we will see is equivalent to the product of GW theories of a point, as in Section 5.2. From there we will show how to go from this basic theory to the FJRW theory of the pair \((\tilde{E}_7,G_{\text{max}})\), and then using Theorem 5.6, we will obtain the R–matrix for the GW theory of \( \mathbb{P}_{4,4,2} \).

7.1. Untwisted theory. In addition to the specialization mentioned at the end of the previous section, if we specialize to \( s_\ell = 0 \) for all \( \ell \), we obtain the “untwisted” theory. Using the projection formula, we can push all calculations down to \( \overline{\mathcal{M}}_{0,n} \) and obtain (cf. [CR])
\[
\langle \tau_{a_1}(e_{a_1}) \ldots \tau_{a_n}(e_{a_n}) \rangle^{un}_{0,n} := 32 \int_{\overline{\mathcal{M}}_{0,n}(h)} \psi_1^{a_1} \ldots \psi_n^{a_n} = \left( \sum_i a_i \right)_{a_1, \ldots, a_n}
\]
whenever the line bundle degree axiom (axiom FJR 2) is satisfied. From the untwisted theory, we obtain a CohFT. We will denote the generating functions of the untwisted theory by \( \mathcal{F}^{un}_g \) and \( \mathcal{Z}^{un} \).

Recall the function \( \mathcal{T}^{(m+1)} \) of (5.2.1). Take \( m + 1 = |G| \). It’s then connected to \( \mathcal{F}^{un}_0 \) as follows.

Proposition 7.1. The genus zero potential \( \mathcal{F}^{un}_0 \) is obtained from \( \text{res}_h \ln \mathcal{T}^{(|G|)} \) by a linear change of the variables.
Proof. It can be checked explicitly that the function $\mathcal{F}^0_{un}$ defines a semisimple algebra. It is also quasihomogeneous and so we can apply Theorem 5.2. We only need to show that the upper–triangular group element of that theorem is trivial.

Both functions $\mathcal{T}^{(\vert G \vert)}$ and $\mathcal{F}^0_{un}$ are generating functions of the products of the CohFTs. It is enough by considering the “factors” on the both sides. In particular we show that the untwisted theories of $A_3$ and $A_1$ are connected by the linear changes of the variables to the genus zero potentials of $\mathcal{T}^{(4)}$ and $\mathcal{T}^{(2)}$ respectively.

Because of the topological recursion relation in genus zero it’s enough to show this on the small phase space only. Namely, on the level of primary potentials. Let $\mathcal{F}^{KdV \otimes 4}_{0}$ and $\mathcal{F}^{un \otimes 3}_{0}$ stand for the primary genus zero potential of $\mathcal{T}^{(4)}$ and $A_r$ respectively. We have:

$$\mathcal{F}^{KdV \otimes 4}_{0} = \frac{u_0^3}{6} + \frac{u_1^3}{6} + \frac{u_2^3}{6} + \frac{u_3^3}{6}.$$

Using the selection rule we have:

$$\mathcal{F}^{un \otimes 3}_{0} = \frac{1}{2} t_0 t_1^2 + \frac{1}{2} t_0^2 t_2 + \frac{1}{2} t_0 t_3 + \frac{1}{6} t_2^3 + t_1 t_2 t_3.$$

One can check that the desired linear change of the variables is given by $u_0 = t_0 - t_1 + t_2 - t_3$, $u_1 = t_0 + t_1 + t_2 + t_3$, $u_2 = -t_0 + t_2 - i(t_1 - t_3)$, $u_3 = -t_0 + t_2 + i(t_1 - t_3)$.

Recall from Corollary 6.6 that we obtain FJRW theory in genus zero as the non–equivariant limit of the twisted theory. In order to obtain the twisted theory, we use the following proposition.

**Proposition 7.2.** Recall the numbers $i_k(h)$ and $q_k$ from Section 6.2. Consider the upper–triangular group element $R^{tw}$ acting diagonally:

$$R^{tw} (\phi_h) := \prod_{k=1}^{3} \exp \left( \sum_{\ell \geq 0} \frac{B_{\ell+1}(i_k(h) + q_k)}{(\ell + 1)!} z^\ell \right) \phi_h.$$

The action of this upper–triangular group element satisfies

$$\hat{R}^{tw} \cdot Z^{un} = Z^{tw}. \quad (7.1.1)$$

**Proof.** Note first that the identity $B_{\ell}(1 - x) = (-1)^{\ell} B_{\ell}(x)$ implies $R^{tw}$ is in the upper–triangular Givental’s group.

Let the partition functions be written in the variables $t^\ell,h = q^\ell,h + \delta$ with $h \in G$ and $\ell \geq 0$. The proof is the same as the proof in
The basic idea is to consider both sides of (7.1.1) as functions in the variables $s_\ell$. Then one shows that both sides satisfy
\[ \frac{\partial \Phi}{\partial s_\ell} = \sum_{k=1}^{3} p_{\ell}^{(k)} \Phi \] (7.1.2)

where
\[ p_{\ell}^{(k)} = \frac{B_{\ell+1}(q_k)}{\ell + 1!} \frac{\partial}{\partial t^h} + \sum_{a \geq 0} \frac{B_{\ell+1}(i_k(h) + q_k)}{(\ell + 1)!} \frac{\partial}{\partial t^a} \]
\[ + \frac{h}{2} \sum_{a \neq h \neq h'} (1)^{\ell - 1} \eta_{h,h'}^{a,h} B_{\ell+1}(i_k(h) + q_k) (\ell + 1) ! \frac{\partial^2}{\partial t^a \partial t^{a'}} \]

and $\eta_{h,h'}$ denote the entries of the matrix inverse to the pairing. Since both $Z^{tw}$ and $\hat R^{tw} \cdot Z^{un}$ satisfy (7.1.2) and both have the same initial condition (when $s_\ell = 0$), they must then be equal.

By the definition of quantization, it is clear that $\hat R^{tw} \cdot Z^{un}$ satisfies (7.1.2).

That $Z^{tw}$ satisfies (7.1.2) was proven by giving an expression for $\text{ch}_{\ell}(R_{\pi}(\tilde L_k))$ using Grothendieck–Riemann–Roch. This was done in [CZ], and generalized to the extended state space in [CR].

Consider $r_{GW} \in \text{Hom}(\mathcal{H}, \mathcal{H})[z]$ for $\mathcal{H} = \mathcal{H}_{E_7,G_{\text{max}}}^{\text{ext}}$ given by:
\[ r_{GW}(\phi_h) := 3 \sum_{k=1}^{3} \sum_{\ell \geq 0} s_\ell \frac{B_{\ell+1}(i_k(h) + q_k)}{(\ell + 1)!} \frac{\phi_h}{\ell} + \frac{1}{2} \left( \Gamma \left( \frac{3}{4} \right) \right) ^4 \delta_{h_1, i_1} \phi_i. \]

The following theorem gives a formula for the genus zero potential of the GW theory of $\mathbb{P}^1_{4,4,2}$.

**Theorem 7.3.** For the upper–triangular group element $R_{GW} := \exp(r_{GW})$ and some lower–triangular group element $S$ the following holds:
\[ F_{0}^{\mathbb{P}^1_{4,4,2}} = \text{res}_h \ln \left( R_{GW} \cdot Z^{un} \right). \]

Due to Proposition 7.1 partition function $Z^{un}$ differs from the partition function $T^{(m+1)}$ by a linear change of the variables and we get indeed the $R$–matrix reconstructing the Gromov–Witten theory of $\mathbb{P}^1_{4,4,2}$ from the product of the GW–theories of a point.
Proof. Let $\hat{R} := \hat{R}^r$ and $\hat{S} := \hat{S}^r \cdot \hat{S}^r$ as in Theorem 5.6. By composing Corollary 6.6 and Proposition 7.2 we get:

$$\mathcal{F}_0(E_7, G_{max}) = \lim_{\lambda \to 0} \mathcal{F}_0^{tw} = \lim_{\lambda \to 0} \text{res}_h \ln (\hat{R}^{tw} \cdot \mathcal{Z}^{un}).$$

From Theorem 5.6 we get:

$$\mathcal{F}_0(E_7, G_{max}) = \text{res}_h \ln (\hat{R} \cdot \hat{S} \cdot \mathcal{Z}_{\mathbb{P}^{4,4,2}}).$$

Note that when considering Givental’s action in genus zero, only the genus zero correlators of a CohFT given contribute to the Givental–transformed CohFT. By comparing the formal power series in $\hbar$ we get:

$$\mathcal{F}_0^{\mathbb{P}_{4,4,2}} = \text{res}_h \ln \left( \hat{S}^{-1} \cdot \hat{R}^{-1} \cdot \lim_{\lambda \to 0} \hat{R}^{tw} \cdot \mathcal{Z}^{un} \right).$$

Consider the extension of $R^r$ to the state space $\mathcal{H}^{tw}$. This can be done because $R^r$ acts non–trivially only on the vector $\phi_{j-1}^r$ belonging both to $\mathcal{H}_W, G$ and $\mathcal{H}^{tw}$.

Slightly abusing the notation we denote by the same letter $r^c$ the operator on $\mathcal{H}^{tw}$, such that $r^c(\phi_{j}) = \sigma \delta_{\hbar,\text{id}} \phi_{j}$. And again $R^r = \exp(r^c z)$. It’s clear that the action of the differential operator $\hat{R}^c$ is not affected by the limit $\lambda \to 0$ and we get:

$$\mathcal{F}_0^{\mathbb{P}_{4,4,2}} = \text{res}_h \ln \left( \hat{S}^{-1} \cdot \lim_{\lambda \to 0} \hat{R}^{-1} \cdot \hat{R}^{tw} \cdot \mathcal{Z}^{un} \right),$$

$$= \text{res}_h \ln \left( \hat{S}^{-1} \cdot \lim_{\lambda \to 0} \left( \hat{R}^{-1} \hat{R}^{tw} \right) \cdot \mathcal{Z}^{un} \right).$$

This completes the proof. \qed

**Appendix A. Gromov–Witten potential of $\mathbb{P}_{4,4,2}^1$**

In order to shorten the formulae let $t_k := t_{1,k}$ for $1 \leq k \leq 3$, $t_l := t_{2,l-3}$ for $4 \leq l \leq 6$, $t_7 := t_{3,1}$. Let also $t_0$ correspond to the unit, $t_8$ to the hyperplane class of the cohomology ring of $\mathbb{P}^1$ and $x = x(q), y = y(q), z = z(q), w = w(q)$ be as in Section 3.6. Then the following expression for the genus zero GW potential of $\mathbb{P}_{4,4,2}^1$ was first announced by the first named author in [B2].

\[
\begin{align*}
F_0^{\mathbb{P}_{4,4,2}} &= -\frac{(x^6 - 5x^4y^2 - 5x^2y^4 + y^6)}{4128768} \left( \frac{t_8}{t_6} + \frac{t_6}{t_8} \right) + \frac{y(x^4 + 14x^2y^2 + y^4)}{294912} \left( \frac{t_8}{t_6} + \frac{t_6}{t_8} \right) + \frac{y(8x^4 + 8y^4 + 19x^2)}{294912} \left( t_7 + t_8 \right) \\
&- \frac{x(x^2 + y^2)^2}{73728} \left( t_2 t_8 + t_8 t_2 \right) + \frac{y(x^2 + y^2)^2}{73728} \left( t_8 t_5 + t_5 t_8 \right) + \frac{5x^2y^2(x^2 + y^2)}{73728} \left( t_7 t_5 + t_5 t_7 \right) - \frac{(x^4 - 6x^2y^2 + y^4)}{30720} \left( t_7 t_5 + t_5 t_7 \right) \\
&- \frac{(x^4 - 3x^2y^2)}{3072} \left( t_7 t_5 + t_5 t_7 \right) + \frac{x(x^2 + y^2)^2}{6144} \left( t_7 t_5 + t_5 t_7 \right) + \frac{y(x^2 + y^2)}{6144} \left( t_7 t_5 + t_5 t_7 \right). 
\end{align*}
\]
\[ x^2 \frac{4x^2 + y^2}{644} t_5 t_5 + t_2 t_2^2 + 2y \frac{x^2 + y^2}{1536} \left( t_2^2 (t_1 t_3 + t_2 t_4) + t_2 t_5 \left( t_2^2 + t_2^2 \right) \right) + \frac{x^2 y^2}{1536} t_5 t_6 \left( t_3^2 t_4 + t_4 t_5 \right) \\
+ x z \frac{4x^2 + y^2}{1536} t_5 t_5 + t_2 t_2^2 + 2y \frac{x^2 + y^2}{1536} t_5 t_5 + t_2 t_2^2 + \frac{y^2}{512} (x^2 + y^2) t_5 t_6 \left( t_2 t_5 + t_2 t_5 \right) + \frac{3x^2 y^2}{384} \left( t_2^2 + t_2^2 \right) t_2^2 \\
+ \frac{x^2 (x^2 + y^2)}{384} \left( t_1 t_5 t_5 + t_4 t_6 t_6 \right) + \frac{y^2 (x^2 + y^2)}{384} \left( t_1 t_5 t_5 + t_4 t_6 t_6 \right) + \frac{x^2 y^2}{384} \left( t_3 t_4 + t_4 t_5 \right) + \frac{x^3}{384} \left( t_3 t_4 + t_4 t_5 \right) \\
+ \frac{y^2}{384} \left( t_3 t_4 + t_4 t_5 \right) - \frac{3w^2}{384} \left( t_3 t_4 + t_4 t_5 \right) + \frac{x^2}{128} t_5 \left( t_2 t_5 + t_2 t_5 \right) + \frac{y^2}{128} t_5 \left( t_2 t_5 + t_2 t_5 \right) + \frac{x^2 y^2}{128} t_5 \left( t_2 t_5 + t_2 t_5 \right) \\
+ \frac{y^2}{128} t_5 \left( t_2 t_5 + t_2 t_5 \right) + \frac{(2x^2 - y^2 - 3w)}{96} t_2^2 + \frac{x^2}{64} t_5 \left( t_2 t_5 + t_2 t_5 \right) + \frac{y^2}{64} t_5 \left( t_2 t_5 + t_2 t_5 \right) \\
+ \frac{2w^2}{64} \left( t_3 t_4 t_6 + t_1 t_5 t_6 + t_4 t_6 t_6 \right) + \frac{x^2 y^2}{32} \left( t_2 t_5 t_5 + t_2 t_5 t_5 + t_2 t_5 t_5 \right) + \frac{y^2}{32} \left( t_2 t_5 t_5 + t_2 t_5 t_5 + t_2 t_5 t_5 \right) \\
+ \frac{w^2}{32} \left( t_1 t_3 t_6 + t_1 t_5 t_5 \right) + \frac{w^2}{32} \left( t_1 t_3 t_6 + t_1 t_5 t_5 \right) + \frac{x^2 y^2}{32} \left( t_2 t_5 t_5 + t_2 t_5 t_5 \right) + \frac{y^2}{16} t_2 t_5 \left( t_2 t_5 + t_2 t_5 \right) + \frac{1}{16} t_5 t_2^2 \\
+ \frac{1}{8} t_0 \left( t_2^2 + t_2^2 + 2t_1 t_6 + 2t_2 t_4 \right) + \frac{1}{2} t_5 t_5. \]  

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