SQUARES IN PIATETSKI–SHAPIRO SEQUENCES

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ABSTRACT. We study the distribution of squares in a Piatetski-Shapiro sequence \((\lfloor n^c \rfloor)_{n \in \mathbb{N}}\) with \(c > 1\) and \(c \notin \mathbb{N}\). We also study more general equations \(\lfloor n^c \rfloor = sm^2\), \(n, m \in \mathbb{N}\), \(1 \leq n \leq N\) for an integer \(s\) and obtain several bounds on the number of solutions for a fixed \(s\) and on average over \(s\) in an interval. These results are based on various techniques chosen depending on the range of the parameters.

1. Introduction

1.1. Motivation and formulation of the problem. Piatetski-Shapiro sequences (PS-sequences), that is, sequences of the form
\[
\mathbb{N}^c = (\lfloor n^c \rfloor)_{n \in \mathbb{N}} \quad (c > 1, \ c \notin \mathbb{N}),
\]
where \(\lfloor z \rfloor\) is the integer part of a real \(z\), have been extensively studied by many authors since their introduction by Piatetski-Shapiro [19], see [1, 2, 3, 4, 5, 6, 10, 21] and the references therein.

Here we consider the distribution of perfect squares in PS-sequences, which seems to be a new, yet natural question to study. More precisely, for a real \(c > 1\) and positive integers \(N\) and \(s\), we denote by \(Q_c(s; N)\) the number of solutions to the equation
\[
\lfloor n^c \rfloor = sm^2, \quad 1 \leq n \leq N, \ m, n \in \mathbb{Z}.
\]
Clearly, we have the following trivial bound
\[
(1.1) \quad Q_c(s; N) \leq \min \{N, s^{-1/2}N^{c/2}\}.
\]
Here we use a variety of different techniques to obtain asymptotic formulas, or upper bounds improving (1.1). We also study \(Q_c(s; N)\) on average over positive square-free integers \(s \leq S\), that is, the quantity
\[
\Omega_c(S, N) = \sum_{s \leq S} Q_c(s; N).
\]

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We remark that only the case $S \leq N^c$ is meaningful, hence we always assume this. Having nontrivial upper bounds on $Q_c(S, N)$ immediately implies a lower bound on the number of distinct square-free parts of the integers $[n^c]$, $1 \leq n \leq N$. In turn, this can be reformulated as a lower bound on the number of distinct quadratic fields in the sequence of fields $\mathbb{Q}\left(\sqrt{[n^c]}\right)$, $1 \leq n \leq N$.

1.2. Notation. Among other methods, our results are also based on the square sieve of Heath-Brown [14] coupled with a bound of character sums due to Baker and Banks [3]. We also employ the method of exponent pairs, we refer to [9, Chapter 3], [16, Sections 7.3 and 17.4], [17, Chapter 8] and [18, Chapter 3] for an exact definition, properties and examples of exponent pairs.

Throughout the paper, as usual $U \ll V$ and $U = O(V)$ are both equivalent to the inequality $|U| \leq BV$ with some constant $B > 0$, which maybe depend on the parameter $c$ (and sometimes, where obvious, on the some other auxiliary parameters), however it is always uniform with respect to our main parameters $N$, $s$ and $S$.

For two quantities $U$ and $V$, which among other parameters also depend on $N$, we use $U \ll V$ to denote that $U \leq VN^{o(1)}$ as $N \to \infty$.

We also write $u \sim U$ to denote that $U < u \leq 2U$.

The letters $\ell$ and $p$, with or without subscripts, always denote prime numbers.

As usual $(k/q)$ denotes the Jacobi symbol modulo $q$, which we use only for prime $q$, when it is called the Legendre symbol, or for products of two primes).

We also use $\Box$ to denote a nonspecified integer square, that is, $n = \Box$ is equivalent to the statement that $n$ is a perfect square and thus we can write

$$Q_c(s; N) = \sum_{\substack{n \leq N \cr [n^c] = s\Box}} 1.$$ 

2. Main Results

2.1. Results for a fixed $s$. We start with an asymptotic formula for $Q_c(s; N)$ for the values of $c$ close to 1. We refer to [9, Chapter 3], [16, Sections 7.3 and 17.4], [17, Chapter 8] and [18, Chapter 3] for a background on exponent pairs.
Theorem 2.1. For any $c > 1$, $c \notin \mathbb{N}$ and any exponent pair $(\kappa, \lambda)$ we have an asymptotic formula

$$Q_c(s; N) = \gamma(2\gamma - 1)^{-1}s^{-1/2}N^{1-c/2}$$

$$+ O \left( s^{-\rho_1(\kappa, \lambda)}N^{\vartheta_1(c, \kappa, \lambda)+o(1)} + s^{-\rho_2(\kappa, \lambda)}N^{\vartheta_2(c, \kappa, \lambda)+o(1)} \right)$$

as $N \to \infty$, where $\gamma = 1/c$,

$$\rho_1(\kappa, \lambda) = \frac{\lambda}{2(\kappa + 1)}, \quad \vartheta_1(c, \kappa, \lambda) = \frac{2\kappa + c\lambda}{2(1 + \kappa)}$$

and

$$\rho_2(\kappa, \lambda) = \frac{(\lambda - \kappa)}{2}, \quad \vartheta_2(c, \kappa, \lambda) = \frac{2\kappa + c(\lambda - \kappa)}{2}.$$

For example, taking $(\kappa, \lambda) = (9/56, 37/56)$ (see [9, Chapter 7]), we have

$$Q_c(s; N) = \gamma(2\gamma - 1)^{-1}s^{-1/2}N^{1-c/2}$$

$$+ O \left( s^{-\frac{37}{130}}N^{(18+37c)/130+o(1)} + s^{-\frac{1}{4}}N^{(9+14c)/56+o(1)} \right),$$

which gives an asymptotic formula for $s = 1$ and $1 < c < 56/51 \approx 1.09804$.

Furthermore, taking $(\kappa, \lambda) = (1/2, 1/2)$ (see [9, Chapter 3]), we obtain

$$Q_c(s; N) = \gamma(2\gamma - 1)^{-1}s^{-1/2}N^{1-c/2}$$

$$+ O \left( s^{-1/6}N^{(2+c)/6+o(1)} + N^{1/2+o(1)} \right)$$

$$\ll s^{-1/6}N^{(2+c)/6} + N^{1/2}$$

for $c > 1$, $c \notin \mathbb{N}$ (which is nontrivial for $s = 1$ and $1 < c < 4$, $c \notin \mathbb{N}$).

For larger values of $c$ we have a less explicit bound, which is nontrivial for any $c > 2$. This bound depends on an absolute constant $\beta(c) > 0$, depending only on $c$ such that for any positive integers $N$ and $q$, for characters sums

$$T_{c, \chi}(q; N) = \sum_{N/2 < n \leq N} \chi([n^c]),$$

with a primitive Dirichlet character $\chi$ modulo $q$ (see [17, Chapter 3] for a background on characters), we have

$$T_{c, \chi}(q; N) \ll q^{1/2}N^{1-\beta(c)}.$$ 

The existence of such $\beta(c)$ for any $c > 2$ of the form

$$\beta(c) = \beta/c^2$$

with an absolute constant $\beta > 0$ is essentially a result of Baker and Banks [3, Theorem 1.6], which we also present as Lemma 3.6 below.
Theorem 2.2. For any $c > 2$, $c \not\in \mathbb{N}$ and $\beta(c)$ satisfying (2.2), for $N \to \infty$ we have

$$Q_c(s;N) \ll N^{1-\beta(c)/2+o(1)}.$$  

We note that the proof of Theorem 2.2 is based on the square sieve method of Heath-Brown [14] which seems to be the first application of this method in the context of PS-sequences for large $c$, where usually the method of exponential sum is used for small $c$. This has become possible because of the recent results of Baker and Banks [3].

2.2. Results on average over $s$. Here we show that using a result of Fouvry and Iwaniec [7, Theorem 3], when $c$ is near to 1, we can take advantage of averaging over $s$ and estimate the sum $\Omega_c(S,N)$ better than via a direct applications of Theorem 2.1.

Our result, as it is natural to expect, depends on the following function

$$\Phi(S) = \sum_{s \leq S, s \text{ square-free}} s^{-1/2}.$$  

Using the well known result, see [11, Theorem 333],

$$\sum_{s \leq t, s \text{ square-free}} 1 = \frac{6}{\pi^2} t + O(\sqrt{t})$$

and partial summation, we easily derive

$$\Phi(S) = \frac{12}{\pi^2} S^{1/2} + O(\log S),$$

which we can use together with the bound of Theorem 2.1.

Theorem 2.3. For any $c > 1$, $c \not\in \mathbb{N}$, for $N \to \infty$ we have

$$\Omega_c(S,N) - \frac{12\gamma}{\pi^2(2\gamma - 1)} S^{1/2} N^{1-c/2} \ll S^{1/5} N^{(1+2c)/5} + S^{5/8} N^{3c/8} + S^{1/8} N^{(2+3c)/8} + SN^{1-c}$$

with $\gamma = 1/c$.

In particular, we have:

Corollary 2.4. For any $c > 1$, $c \not\in \mathbb{N}$, any $\varepsilon > 0$ and $S \leq N^{\tau(c)-\varepsilon}$, where

$$\tau(c) = \begin{cases} (8 - 3c)/5, & \text{for } 1 < c \leq 12/7, \\ 2(2-c), & \text{for } c > 12/7, \end{cases}$$

we have

$$\Omega_c(S,N) = o(N)$$

as $N \to \infty$.  

Clearly Corollary 2.4 is nontrivial only for \( c < 2, \ c \not\in \mathbb{N} \).

**Remark 2.1.** Applying Theorem 2.1 with \((\kappa, \lambda) = (1/2, 1/2)\) and trivial estimate, we may take \( \tau(c) = (4-c)/5 \), which is nontrivial for \( c < 4, \ c \not\in \mathbb{N} \). But for large \( c \), we need refer to the square sieve again to get a positive value for \( \tau(c) \).

**Theorem 2.5.** For any \( c > 2, \ c \not\in \mathbb{N} \) and \( \beta(c) \) satisfying (2.2), for \( N \to \infty \) we have

\[
Q_c(S, N) \ll SN^{1-\beta(c)} + S^{3/4}N^{1-\beta(c)/2}.
\]

In particular, we have:

**Corollary 2.6.** For any \( c > 2, \ c \not\in \mathbb{N}, \ \beta(c) \) satisfying (2.2), and any \( \varepsilon > 0 \), for \( S \leq N^{2\beta(c)/3 - \varepsilon} \), we have

\[
Q_c(S, N) = o(N)
\]

as \( N \to \infty \).

Corollary 2.4 and Corollary 2.6 cover the full range \( c > 1, \ c \not\in \mathbb{N} \), provided (2.2) holds. Hence, combining this with (2.3), which we have by Lemma 3.6 below, we obtain:

**Corollary 2.7.** For any \( c > 1, \ c \not\in \mathbb{N} \), there exists a constant \( \vartheta(c) > 0 \) such that the square-free parts of almost all integers of the type \([n^c], n \leq N\) are larger than \( N^{\vartheta(c)}\).

3. **Preparations**

3.1. **Some general statements.** As usual, we define the function \( \psi(u) = u - \lfloor u \rfloor - 1/2 \). We use the following result of Vaaler [22], see also [9, Theorem A.6].

**Lemma 3.1.** Let \( H \geq 1 \). There are functions \( a(h) \) and \( b(h) \), such that for \( 1 \leq |h| \leq H \) we have

\[
a(h) \ll \frac{1}{|h|}, \quad b(h) \ll \frac{1}{H},
\]

and

\[
\left| \psi(t) - \sum_{1 \leq |h| \leq H} a(h)e(ht) \right| \leq \sum_{|h| \leq H} b(h)e(ht).
\]

Note that we can take explicitly,

\[
a(h) = (2\pi ih)^{-1}F \left( \frac{h}{H+1} \right) \quad \text{and} \quad b(h) = \frac{1}{2H+2} \left( 1 - \frac{|h|}{H+1} \right),
\]

where

\[
F(y) = \frac{1}{y} - \log(1+y).
\]
with $F(u) = \pi u (1 - |u|) \cot(\pi u)$ in Lemma 3.1. We also remark the right hand side of Lemma 3.1 is a real nonnegative number, so now absolute value symbol is necessary. It is also important to notice that the summation on the right hand side also includes $h = 0$.

We also need the following technical result, see [9, Lemma 2.4].

**Lemma 3.2.** Let

$$L(Z) = \sum_{i=1}^{u} A_i Z^{a_i} + \sum_{j=1}^{v} B_j Z^{-b_j},$$

where $A_i, B_j, a_i$ and $b_j$ are positive. Let $0 \leq Z_1 \leq Z_2$. Then there is some $Z \in (Z_1, Z_2]$ with

$$L(Z) \ll \sum_{i=1}^{u} \left( \sum_{j=1}^{v} A_i B_j^{a_i/ (a_i + b_j)} \right)^{1/2} + \sum_{i=1}^{u} A_i Z_1^{a_i} + \sum_{j=1}^{v} B_j Z_2^{-b_j},$$

where the implied constant depends only on $u$ and $v$.

The following is a form of the square sieve of Heath-Brown [14] which is given by Friedlander and Iwaniec [8, Proposition 3.1] (combined with the trivial observation that if for some integers $r$ and $s$ we have $r = s \square$ then $rs = \square$).

**Lemma 3.3.** Let $a_r$, $r = 1, \ldots, R$, be an arbitrary finite sequence non-negative real numbers and let $P \geq 2$. Then we have

$$\sum_{r=1}^{R} a_r \leq 10 P^{-2} \sum_{r=1}^{R} a_r \left( \left( \sum_{P < p \leq 2P} \left( \frac{sr}{p} \log p \right) \right)^2 + (\log sr)^2 \right).$$

One can apply Lemma 3.3 directly to $Q_c(s; N)$ but it is technically easier to work with dyadic intervals, so we define

$$Q^*_c(s; N) = \sum_{N/2 < n \leq N \atop [n^c] = s \square} 1.$$  \hspace{1cm} (3.1)

Taking $a_r$ to be the characteristic function of the event $r = [n^c]$ for some positive integer $N/2 < n \leq N$ we obtain:

**Corollary 3.4.** For any positive integers $N$, $P \geq 2$ and $s$, we have

$$Q^*_c(s; N) \ll P^{-2} \sum_{N/2 < n \leq N} \left( \sum_{P < p \leq 2P} \left( \frac{s \lfloor n^c \rfloor}{p} \log p \right) \right)^2 + P^{-2} N \log^2 N.$$  

We need the following mean value estimate for real character sums, which is Theorem 1 of [15] (see also [17, Theorem 7.20]).
Lemma 3.5. For any integers $M, N \geq 1$ and complex numbers $a_n$, $n = 1, \ldots, N$ we have

$$\sum_{\substack{m \leq M \text{ square-free}}} a_n \left(\frac{n}{m}\right)^2 \leq (MN)^{o(1)}(M + N) \sum_{n \leq N} |a_n|^2,$$

as $MN \to \infty$.

3.2. Character sums with PS-sequences. We now recall the following bound on the sums $T_{c, \chi}(q; x)$ defined by (2.1), given by Baker and Banks [3, Theorem 1.6] (used with $y = x = N/2$), which is nontrivial for any $c > 2$ (provided that $N$ is sufficiently large compared to $q$).

Lemma 3.6. Let $N \geq 2$ and $q \geq 3$. Then for $c > 2$, $c \not\in \mathbb{N}$, there exists an absolute constant $\beta > 0$ such that

$$T_{c, \chi}(q; N) \ll q^{1/2}N^{1-\beta/c^2}.$$

In particular, Lemma 3.6 shows that (2.3) is satisfied for some $\beta > 0$ and thus the assumption (2.2) is not void.

There is no doubt that the value of $\beta$ in Lemma 3.6 can be explicitly evaluated.

3.3. Exponential sums with monomials. We need the following bound due to Fouvry and Iwaniec [7, Theorem 3]. We remark that the more recent bound of Robert and Sargos [20, Theorem 1] does not bring any improvement to our results (as the bounds of [7, Theorem 3] and [20, Theorem 1] have some common terms and these are exactly the terms that dominate in our applications).

Lemma 3.7. Let $\alpha, \alpha_1, \alpha_2$ be real constants such that

$$\alpha \neq 1 \quad \text{and} \quad \alpha \alpha_1 \alpha_2 \neq 0.$$

Let $M, M_1, M_2, x \geq 1$ and let

$$\Phi = (\varphi_m)_{m \sim M} \quad \text{and} \quad \Psi = (\psi_{m_1, m_2})_{m_1 \sim M_1, m_2 \sim M_2}$$

be two sequences of complex numbers supported on $m \sim M$, $m_1 \sim M_1$ and $m_2 \sim M_2$ with $|\varphi_m| \leq 1$ and $|\psi_{m_1, m_2}| \leq 1$. Then for the sum

$$S_{\Phi, \Psi}(x; M, M_1, M_2) = \sum_{m \sim M} \sum_{m_1 \sim M_1} \sum_{m_2 \sim M_2} \varphi_m \psi_{m_1, m_2} e \left( x \frac{m^\alpha m_1^{\alpha_1} m_2^{\alpha_2}}{M^\alpha M_1^{\alpha_1} M_2^{\alpha_2}} \right),$$
we have
\[ S_{\Phi, \Psi}(x; M_1, M_2) \ll (x^{1/4}M^{1/2}(M_1M_2)^{3/4} + M^{7/10}M_1M_2 + M(M_1M_2)^{3/4} \]
\[ + x^{-1/4}M^{11/10}M_1M_2) \log^2(2M M_1 M_2). \]

4. Proofs of main results

4.1. Proof of Theorem 2.1. For \( c > 1, c \notin \mathbb{N} \), let \( \gamma = 1/c \). It is easy to see that \([n^c] = s \square \) if and only if
\[
(s \square)^\gamma \leq n < (s \square + 1)^\gamma.
\]
We also set \( M = s^{-1/2}N^{c/2} \). Using a similar argument as that in Heath-Brown [13] (and in many other works on PS-sequences), we have
\[
Q_c(s; N) = \sum_{m \leq M} \left( \lfloor -s^\gamma m^{2\gamma} \rfloor \right) + O(1).
\]
Let
\[
\psi(x) = x - \lfloor x \rfloor - 1/2.
\]
Then we obtain
\[
Q_c(s; N) = \sum_{m \leq M} \left( (sm^2 + 1)^\gamma - s^\gamma m^{2\gamma} \right) - \sum_{m \leq M} \psi(-s^\gamma m^{2\gamma})
\]
\[
+ \sum_{m \leq M} \psi \left( - (sm^2 + 1)^\gamma \right) + O(1).
\]

The first term in the right side is
\[
\sum_{m \leq M} \left( (sm^2 + 1)^\gamma - s^\gamma m^{2\gamma} \right)
\]
\[
= \sum_{m \leq M} s^\gamma m^{2\gamma} \left( \gamma m^{-2}s^{-1} + O\left( m^{-4}s^{-2} \right) \right)
\]
\[
= \gamma(2\gamma - 1)^{-1} s^{\gamma - 1}(N^{c/2}s^{-1/2})^{2\gamma - 1} + O(1)
\]
\[
= \gamma(2\gamma - 1)^{-1} N^{1-c/2}s^{-1/2} + O(1),
\]
which gives the desired main term.

Now we need to estimate the other two sums with the \( \psi \)-functions. We only estimate the first sum with \( \psi(-s^\gamma m^{2\gamma}) \) and the other sum with \( \psi \left( - (sm^2 + 1)^\gamma \right) \) can be treated similarly and admits the same upper bound.
We now fix some parameter $H \geq 1$ and using Lemma 3.1, we obtain

\[ \sum_{m \leq M} \psi(-s^\gamma m^{2\gamma}) \ll |E_1(N, H, c)| + |E_2(N, H, c)| + MH^{-1}, \]

where

\[ E_1(N, H, c) = \sum_{m \leq M} \sum_{0 < |h| \leq H} a(h)e(-hs^\gamma m^{2\gamma}) \]

and

\[ E_2(N, H, c) = \sum_{m \leq M} \sum_{0 < |h| \leq H} b(h)e(-hs^\gamma m^{2\gamma}) \]

(the term $MH^{-1}$ corresponds to the choice $h = 0$ in the summation on the right hand side of Lemma 3.1). We deal with $E_1(N, H, c)$ first.

Switching the summation, we get

\[ E_1(N, H, c) \ll \sum_{0 < |h| \leq H} \frac{1}{h} \left| \sum_{m \leq M} e(hs^\gamma m^{2\gamma}) \right|. \]

For the inner sum over $m$, we have

\[ \sum_{m \leq M} e(hs^\gamma m^{2\gamma}) \ll \log N \max_{1 \leq L \leq M} \left| \sum_{L < m \leq 2L} e(hs^\gamma m^{2\gamma}) \right|. \]

Using an exponent pair $(\kappa, \lambda)$, see [9, 16, 17, 18], we obtain

\[ \sum_{L < m \leq 2L} e(hs^\gamma m^{2\gamma}) \ll (hs^\gamma L^{2\gamma-1})^\kappa L^\lambda. \]

Then

\[ \sum_{m \leq M} e(hs^\gamma m^{2\gamma}) \ll h^\kappa s^{\kappa-\lambda}/2 N^{c(2\gamma\kappa-\kappa+\lambda)/2}, \]

which yields

\[ H = s^{(\lambda-\kappa-1)/(2+2\kappa)} N^{c(1+\kappa-2\gamma\kappa-\lambda)/(2+2\kappa)} \]

\[ E_1(N, H, c) \ll H^\kappa s^{(\kappa-\lambda)/2} N^{c(2\gamma\kappa-\kappa+\lambda)/2}. \]

By a similar argument, we can also get

\[ E_2(N, H, c) \ll H^\kappa s^{(\kappa-\lambda)/2} N^{c(2\gamma\kappa-\kappa+\lambda)/2}. \]

Applying Lemma 3.2 to the bounds on terms in (4.3), we obtain

\[ \sum_{m \leq s^{-1/2} N^{\kappa/2}} \psi(-s^\gamma m^{2\gamma}) \ll s^{-\rho_1(\kappa, \lambda)} N^{\theta_1(\kappa, \kappa, \lambda)} + s^{-\rho_2(\kappa, \lambda)} N^{\theta_2(\kappa, \kappa, \lambda)}. \]

Now the result follows from (4.1) and (4.2).
4.2. Proof of Theorem 2.2 and 2.5. We fix some integer $P$ with $2 \leq P \leq N$ (to be optimised later). It is also clear that it is enough to obtain the desired bounds for $Q^*_c(s; N)$, defined by (3.1).

Using Corollary 3.4 and then opening the square, changing the order of summation and separating the diagonal terms (with the total contribution at most $NP^{1+o(1)}$), we obtain

$$Q^*_c(s; N) \ll P^{-2} \sum_{P < \ell, p \leq 2P} \log \ell \log p \left| \sum_{N/2 < n \leq N} \left( \frac{|n^c|}{\ell p} \right) \right| + P^{-1} N^{1+o(1)}$$

We remark that $s$ is no present anymore in the expression on the right hand side, and thus the estimates below are uniform in $s$.

Note that the Jacobi symbols here are primitive characters thus (2.2) applies and yields

$$Q^*_c(s; N) \ll N^{1-\beta(c)} P + NP^{-1}.$$ 

Taking $P = N^{\beta(c)/2}$, we get

$$Q^*_c(s; N) \ll N^{1-\beta(c)/2},$$

which concludes the proof Theorem 2.2.

To prove Theorem 2.5 and we only need to consider

$$\Omega^*_c(S, N) = \sum_{s \leq S} Q^*_c(s; N).$$

By Corollary 3.4 again, we have

$$\Omega^*_c(S, N) \ll P^{-2} \sum_{P < \ell, p \leq 2P} \left| \sum_{s \leq S} \left( \frac{s}{\ell p} \right) \sum_{N/2 < n \leq N} \left( \frac{|n^c|}{\ell p} \right) \right| + P^{-1} SN.$$ 

Applying (2.2), we see that

$$\Omega^*_c(S, N) \ll P^{-1} N^{1-\beta(c)} \sum_{r \leq 4P^2} \left| \sum_{s \leq S} \left( \frac{s}{r} \right) \right| + P^{-1} SN$$

$$\ll P^{-1} N^{1-\beta(c)} \sum_{r \leq 4P^2} \left| \sum_{s \leq S} \left( \frac{s}{r} \right) \right| + P^{-1} SN.$$
Now by the Cauchy inequality, Lemma 3.5 and choosing an optimal $P$, we get

$$\mathcal{S}_c(S, N) \ll (PS^{1/2} + S)N^{1-\beta(c)} + P^{-1}SN \ll SN^{1-\beta(c)} + S^{3/4}N^{1-\beta(c)/2},$$

which yields Theorem 2.5.

4.3. Proof of Theorem 2.3.

4.3.1. Preliminaries. We proceed as in the proof of Theorem 2.1, then we have

\begin{equation}
\mathcal{Q}_c(S, N) = S_0 - E_1 + E_2 + O(1),
\end{equation}

where

$$S_0 = \sum_{\substack{sm^2 \leq N^c \\ s \leq S}} ((sm^2 + 1)\gamma - s^\gamma m^{2\gamma}),$$

contributes to the main term and

$$E_1 = \sum_{\substack{sm^2 \leq N^c \\ s \leq S}} \psi(-s^\gamma m^{2\gamma}) \quad \text{and} \quad E_2 = \sum_{\substack{sm^2 \leq N^c \\ s \leq S}} \psi(-(sm^2 + 1)\gamma).$$

contribute to the error term.

4.3.2. Evaluation of the main term $S_0$. Using (4.2), we compute $S_0$ directly as follows:

\begin{equation}
S_0 = \gamma(2\gamma - 1)^{-1}N^{1-c/2}\Phi(S) + O(SN^{1-c}).
\end{equation}

4.3.3. Reductions in the error terms $E_1$ and $E_2$. By Lemma 3.1, we obtain the following analogue of (4.3):

\begin{equation}
E_1 \ll |E_{11}| + |E_{12}| + H^{-1}N^{c/2}S^{1/2},
\end{equation}

where

$$E_{11} = \sum_{\substack{sm^2 \leq N^c \\ s \leq S}} \sum_{0 < |h| \leq H} a(h)e(-hs^\gamma m^{2\gamma}),$$

and

$$E_{12} = \sum_{\substack{sm^2 \leq N^c \\ s \leq S}} \sum_{0 < |h| \leq H} b(h)e(-hs^\gamma m^{2\gamma})$$

for some $H \geq 2$. Using the same $H$, we also have

\begin{equation}
E_2 \ll |E_{21}| + |E_{22}| + H^{-1}N^{c/2}S^{1/2},
\end{equation}
where
\[ E_{21} = \sum_{\substack{sm^2 \leq N^c \\ s \text{ square-free}}} \sum_{0 < |h| \leq H} a(h)e\left(-h(sm^2 + 1)\gamma\right) \]
and
\[ |E_{22}| = \sum_{\substack{sm^2 \leq N^c \\ s \leq S \\ s \text{ square-free}}} \sum_{0 < |h| \leq H} b(h)e\left(-h(sm^2 + 1)\gamma\right) \]

As usual, the sums \( E_{12} \) and \( E_{22} \) can be estimated similarly as \( E_{11} \) and \( E_{21} \), respectively, and by partial summation, \( E_{21} \) can be converted to exponential sums which is similar to \( E_{11} \) (see [13, Section 2] for details). In particular, we obtain same upper bounds for \( E_{11}, E_{12}, E_{21} \) and \( E_{22} \). Hence we only concentrate on the sum \( E_{11} \).

### 4.3.4. Estimating \( E_{11} \)

Using \( \mu^2(s) = \sum_{s=rd^2} \mu(d) \),
we can write
\[ E_{11} = \sum_{0 < |h| \leq H} a(h) \sum_{rd^2m^2 \leq N^c \quad rd^2 \leq S} \mu(d)e(-hr\gamma d^2 m^2 \gamma). \]

Then, splitting the ranges of variables into dyadic ranges, for some real positive parameters \( R, D \) and \( M \), satisfying
\[ RD^2M^2 \ll N^c \quad \text{and} \quad RD^2 \ll S, \]
we obtain
\[ E_{11} \ll \sum_{0 < |h| \leq H} h^{-1} |S(R, D, M; h)|, \]

where
\[ S(R, D, M; h) = \sum_{r \sim R, d \sim D, m \sim M, \quad rd^2m^2 \leq N^c \quad rd^2 \leq S} \mu(d)e(-hr\gamma d^2 m^2 \gamma). \]

Now we estimate \( S(R, D, M; h) \). Clearly we can assume that \( 1 < c < 2 \) as for \( c > 2 \) the result is trivial (due to the presence of the term \( S^{1/5}N^{(1+2c)/5} > N^{(1+2c)/5} \geq N \) for \( c > 2 \)). We can remove the restrictive conditions
\[ rd^2m^2 \leq N^c \quad \text{and} \quad rd^2 \leq S \]
at the cost of a small factor \((SN)^{o(1)}\) in a standard way (see, for example, [12, Sections 2.3 and 3.2]), which yields

\[
S(R, D, M; h) \ll \left| \sum_{r \sim R, d \sim D, m \sim M} \alpha_1(R)\alpha_2(D)\alpha_3(M)e(hr^\gamma d^{2\gamma}m^{2\gamma}) \right| + M
\]

with some coefficients \(|\alpha_i(n)| \leq 1\) for \(n \in \mathbb{N}\) and \(i = 1, 2, 3\). Applying Lemma 3.7 to the right hand side of the above formula, we get

\[
S(R, D, M; h) \ll (hR^\gamma D^{2\gamma}M^{2\gamma})^{1/4}R^{1/2}(DM)^{3/4} + R^{7/10}DM + R(DM)^{3/4} + (hR^\gamma D^{2\gamma}M^{2\gamma})^{-1/4}R^{11/10}DM.
\]

Noting \(\gamma > 1/2\), it is easy to check that the fourth term can be absorbed by the third term on the right side. Thus by conditions (4.8), we have

\[
(4.10) \quad E_{11} \ll H^{1/4}S^{1/8}N^{1/4+3c/8} + S^{1/5}N^{c/2} + S^{5/8}N^{3c/8}.
\]

4.3.5. **Concluding the proof.** Bound (4.10) together with (4.6) and (4.7) yields

\[
|E_1| + |E_2| \ll H^{1/4}S^{1/8}N^{1/4+3c/8} + H^{-1}N^{c/2}S^{1/2} + S^{1/5}N^{c/2} + S^{5/8}N^{3c/8}.
\]

Now Lemma 3.2 gives

\[
(4.11) \quad |E_1| + |E_2| \ll S^{1/5}N^{(1+2c)/5} + S^{5/8}N^{3c/8} + S^{1/8}N^{1/4+3c/8},
\]

where the term \(S^{1/5}N^{c/2}\) is absorbed by \(S^{1/5}N^{(1+2c)/5}\), since we suppose \(1 < c < 2\). Using the bound (4.11) together with (2.4), (4.4) and (4.5), and noting the contribution of \(O(\log S)\) in (2.4) can also be absorbed by \(S^{1/5}N^{(1+2c)/5}\), we obtain the desired result.

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