Effect of Landau damping on ion acoustic solitary waves in a collisionless unmagnetized plasma consisting of nonthermal and isothermal electrons

S Dalui and A Bandyopadhyay*
Department of Mathematics, Jadavpur University, Kolkata 700 032, India

Received: 15 May 2019 / Accepted: 26 November 2019 / Published online: 19 March 2020

Abstract: A Korteweg–de Vries (KdV) equation including the effect of linear Landau damping of electrons is derived to study the propagation of weakly nonlinear and weakly dispersive ion acoustic waves in a collisionless unmagnetized plasma consisting of warm adiabatic ions and two species of electrons at different temperatures. It is found that the coefficient of the nonlinear term of this KdV-like evolution equation vanishes along different family of curves in different parameter planes. In this context, a modified KdV (MKdV) equation including the effect of linear Landau damping of electrons describes the nonlinear behaviour of ion acoustic waves. Again, the coefficients of the nonlinear terms of the KdV and MKdV-like evolution equations are simultaneously equal to zero along a family of curves in the parameter plane. In this situation, we have derived a further modified KdV (FMKdV) equation including the effect of linear Landau damping of electrons. The multiple time scale method has been applied to obtain the solitary wave solution of the evolution equations having the nonlinear term \( \frac{\phi^{(1)}_{o}}{r} \), where \( \phi^{(1)}_{o} \) is the first-order perturbed electrostatic potential and \( r = 1, 2, 3 \). The amplitude of the ion acoustic solitary wave decreases with time for all \( r = 1, 2, 3 \).

Keywords: Nonthermal electrons; Ion acoustic wave; Landau damping; Modified Korteweg–de Vries equation; Solitary wave solution

PACS No.: 52.35.Fp; 52.35.Mw; 52.35.Sb

1. Introduction

The observations of electric field structures by the Freja Satellite [1] in the auroral zone of the upper ionosphere, the FAST [2–6] satellite and the Viking Satellite [7, 8] in the auroral zone indicate the presence of cooler and hotter electron species. The cooler electron species can be modelled by the Maxwell–Boltzmann velocity distribution, whereas the hotter electron species can be described by considering Cairns [9] distributed nonthermal electrons. The existence of different species of electrons at different temperatures has already been reported by Dalui et al. [10]. In the present paper, we have considered the effect of linear Landau damping of electrons on ion acoustic (IA) solitary wave in a collisionless unmagnetized electron–ion plasma consisting of warm adiabatic ions, isothermal and nonthermal electrons.

Several authors [10–29] investigated different linear and nonlinear properties of IA waves in a plasma consisting of one or two ion species and one or two electron species. In the present paper, we have investigated the effect of linear Landau damping of electrons of two different populations at different temperatures on IA solitary waves. But, Yu and Luo [30] reported that for phenomena on long-time scales, one can consider electrons into two different species if the electrons are physically separated in space/time domain of interest. So, Maxwell–Boltzmann distributed electrons and Cairns [9] distributed nonthermal electrons can be considered as two different electron species only when those electron species are physically separated in the phase space by external or self-consistent fields. On the basis of the assumption that the two groups of electrons occupy different regions of phase space, several authors [16, 22, 27]...
considered two populations of electrons at different temperatures.

Longitudinal electron plasma oscillations are damped during the propagation through a collisionless plasma. In particular, Vlasov [31] used the linearized Boltzmann equation to investigate the small amplitude steady-state longitudinal electron plasma oscillations. Shortly afterwards, Landau [32] pointed out that these oscillations are damped. This damping of longitudinal electron plasma waves in a collisionless plasma is known as linear electron Landau damping. For the first time, Ott and Sudan [33] investigated the effect of linear Landau damping of electrons on IA solitary waves in a collisionless plasma. Several authors investigated the effect of Landau damping on IA solitary waves in unmagnetized or magnetized plasmas theoretically [34–41] and experimentally [42]. In particular, Tajiri and Nishihara [36] investigated the effect of Landau damping on finite amplitude IA solitary waves in a collisionless unmagnetized electron–ion plasma consisting of cold ions and two distinct populations of isothermal electrons at different temperatures by considering a KdV-like evolution equation including the effect of Landau damping. Bandyopadhyay and Das [37] derived a Korteweg–de Vries–Zakharov–Kuznetsov (KdV–ZK) and a modified KdV–ZK equations including the effect of linear Landau damping of electrons to investigate the nonlinear behaviour of IA waves in a magnetized plasma consisting of warm adiabatic ions and nonthermal electrons. Recently, Ghai et al. [43] investigated the dust acoustic solitary and shock structures under the influence of Landau damping in a dusty plasma containing two different temperature ion species.

To investigate the effect of linear Landau damping of electrons on IA solitary waves in a collisionless unmagnetized electron–ion plasma consisting of two distinct populations of electrons at different temperatures, we have considered coupled Vlasov–Poisson model for two different electron species along with the fluid model for ions. So, in the present plasma system, the kinetic effects of two different species of electrons at different temperatures have been investigated on IA solitary structures with special emphasis on the following cases:

Case-1: Using the reductive perturbation method, an evolution equation has been derived which describes the nonlinear behaviour of IA waves along with a correction due to the kinetic effects of two different species of electrons. This evolution equation reduces to a well-known Korteweg–de Vries (KdV) equation if electron-to-ion mass ratio is neglected.

Case-2: It is found that a factor \((B_1)\) of the coefficient of the nonlinear term of the evolution equation derived in Case-1 vanishes along different family of curves in different parameter planes. In this situation, i.e. when \(B_1 = 0\), a modified evolution equation including the effect of linear Landau damping of electrons describes the nonlinear behaviour of IA waves and this modified evolution equation becomes a modified KdV (MKdV) equation having the nonlinear term \(\left(\phi^{(1)}\right)^2 \frac{\partial \phi^{(1)}}{\partial t}\) if electron-to-ion mass ratio is neglected, where \(\phi^{(1)}\) is the perturbed electrostatic potential and \(\xi\) is the stretched space variable.

Case-3: It has been observed that a factor \((B_2)\) of the coefficient of the nonlinear term of the evolution equation derived in Case-2 vanishes along a family of curves in the parameter plane. In this context, a further modified evolution equation including the effect of linear Landau damping of electrons can describe the nonlinear behaviour of IA waves when the conditions \(B_1 = 0\) and \(B_2 = 0\) hold simultaneously and this further modified evolution equation reduces to a further modified KdV (FMKdV) equation having nonlinear term \(\left(\phi^{(1)}\right)^3 \frac{\partial \phi^{(1)}}{\partial t}\) if electron-to-ion mass ratio is neglected. For the first time, we have derived a FMKdV equation having nonlinear term \(\left(\phi^{(1)}\right)^3 \frac{\partial \phi^{(1)}}{\partial t}\) including the effect of linear Landau damping of electrons.

Case-4: Using the multiple time scale analysis, we have developed a general method to find the solitary wave solution of the evolution equation having nonlinear term \(\left(\phi^{(1)}\right) \frac{\partial \phi^{(1)}}{\partial \xi}\) including the effect of linear Landau damping of electrons.

Case-5: The amplitudes of the solitary wave solutions of the different evolution equations including the effect of linear Landau damping of electrons have been investigated for \(r = 1, 2, 3\) and it is found that the amplitude of the solitary wave solution decreases with time for all \(r = 1, 2, 3\).

2. Basic equations

In this paper, we have considered the effect of linear Landau damping of electrons on the IA solitary waves. So, to describe the nonlinear behaviour of IA waves including the effect of linear Landau damping of electrons, we take the Vlasov–Poisson model for two different electron species and the fluid model for ions. In this section, we have shown that if we neglect the electron-to-ion mass ratio or if we neglect the inertia of electrons, i.e. if we neglect the effect of linear Landau damping of electrons, then the system of equations reduces to a system of hydrodynamic equations. These hydrodynamic equations can describe the nonlinear behaviour of IA waves and small amplitude IA solitary waves can be described by usual KdV and different modified KdV equations. So, here Vlasov–Poisson model of electron species depends on the inertia of electrons only, i.e. if we neglect the inertia of electrons, then the system of equations reduces to a
system of hydrodynamic equations. Therefore, to study the
effect of linear Landau damping of electrons on IA solitary
waves, we cannot neglect the inertia of electrons. In fact,
considering Vlasov–Poisson model for electrons and the fluid
model for ions, Ott and Sudan [33] derived a KdV equation
along with an extra term responsible for the effect of linear
Landau damping of electrons. In the present paper, we have
considered a fully ionized collisionless unmagnetized plasma
consisting of warm adiabatic ions, isothermal and nonthermal
[9] electrons. So, to describe the effect of linear Landau
damping of electrons on the nonlinear behaviour of IA waves
propagating along x-axis, we consider the Vlasov–Poisson
model for two different electron species and the fluid model
for ions. The Vlasov–Poisson model for two electron species at
different temperatures can be written in the following form:

\[ \sqrt{\frac{m_e}{m}} \frac{\partial f_{ce}}{\partial \tau} + \frac{v_i}{c} \frac{\partial f_{ce}}{\partial x} + \frac{\partial \phi}{\partial x} \frac{\partial f_{ce}}{\partial v_i} = 0, \]

(1)

\[ \sqrt{\frac{m_e}{m}} \frac{\partial f_{se}}{\partial \tau} + \frac{v_i}{c} \frac{\partial f_{se}}{\partial x} + \frac{\partial \phi}{\partial x} \frac{\partial f_{se}}{\partial v_i} = 0, \]

(2)

\[ \frac{e^2 \phi}{c^2} = n_{ce} + n_{se} - n, \]

(3)

where

\[ n_{ce} = \int_{-\infty}^{\infty} f_{ce} dv_i, \quad n_{se} = \int_{-\infty}^{\infty} f_{se} dv_i. \]

(4)

The above equations along with the equation of continuity of
electrons and the equation of motion for ion fluid form a
system of coupled equations. The continuity equation and
the momentum equation for ion fluid can be taken as

\[ \frac{\partial n}{\partial \tau} + \frac{\partial (n u)}{\partial x} = 0, \]

(5)

\[ \frac{\partial u}{\partial \tau} + \frac{u}{c} \frac{\partial u}{\partial x} + \frac{1}{n} \frac{\partial p}{\partial x} = -\frac{\partial \phi}{\partial x}. \]

(6)

In the momentum equation (6), the pressure term has been
included to get the effect of ion temperature. To make a
closed system of equations, we take the following adiabatic
pressure law:

\[ p = n^\gamma, \]

(7)

where we have neglected the effects of viscosity, thermal
conductivity and energy transfer due to collisions.

In Eqs. (1)–(7), \( f_{ce}, f_{se}, n, u, v_i, p, \phi, x \) and \( \tau \) are
the velocity distribution function of nonthermal electrons,
the velocity distribution function of isothermal electrons, the
ion number density, the ion fluid velocity, the velocity of
electrons in phase space, the ion pressure, the electrostatic
potential, the spatial variable and time, respectively, and
these quantities have been normalized by \( n_0 \) (unperturbed
ion number density), \( n_0, n_0, c_s(= \sqrt{K_B T_{ei}/m}), \)

\( V_{ce}(= \sqrt{K_B T_{ei}/m}), \quad n_0 T_{ei}, \quad K_B T_{ei}/e, \quad \lambda_D(= \sqrt{K_B T_{ei}/4\pi n_0 e^2}) \) and \( \omega_{pe}^{-1}(= \sqrt{m/4\pi n_0 e^2}) \), where \( \sigma = T_i/T_{ei} \) and \( \gamma(= 3) \) is the adiabatic index. Again, \( K_B \) is the
Boltzmann constant, \( m \) is the mass of an ion, \( n_e \) is the mass
of an electron, \( -e \) is the charge of an electron, \( T_i \) is the
average ion temperature and \( T_{ei} \) is given by the following
equation [10]:

\[ \frac{n_{ce} + n_{se}}{T_{ei}} = \frac{n_{ce}}{T_{ce}} + \frac{n_{se}}{T_{se}}, \]

(8)

where \( n_{ce}, n_{se}, T_{ce} \) and \( T_{se} \) are, respectively, unperturbed
nonthermal electron number density, unperturbed isother-
mal electron number density, average temperature of
nonthermal electrons and average temperature of isothermal
electrons.

On the basis of the above-mentioned normalization of
the independent and dependent variables, the unperturbed
velocity distribution functions of nonthermal Cairns [9]
distributed electrons and isothermal electrons can be written
in the following form:

\[ f_0 = \frac{\sigma_e}{2\pi} \left( 1 + \frac{\omega_{pe}^2}{1 + 3\omega_{pe}^2} \right) \exp \left[ -\frac{\sigma_e \omega_{pe}^2}{2} \right], \]

(9)

\[ f_0 = \frac{\sigma_e}{2\pi} \exp \left[ -\frac{\sigma_e \omega_{pe}^2}{2} \right], \]

(10)

where \( \omega_{pe} \) is the nonthermal parameter associated with
the Cairns model [9] for electron species and the expressions of \( \frac{n_{ce} n_{se}}{n_{ce}}, \frac{n_{ce}}{n_{se}}, \frac{n_{se}}{n_{ce}}, \frac{T_{ce}}{T_{se}}, \frac{T_{se}}{T_{se}} \) are given by

\[ n_{ce} = \frac{n_{ce}}{n_{ce}}, \quad n_{se} = \frac{n_{ce}}{n_{se}}, \quad \frac{n_{ce}}{n_{se}} = \frac{T_{ce}}{T_{se}}, \quad \frac{n_{se}}{n_{ce}} = \frac{T_{se}}{T_{ce}}. \]

(11)

Using (11), Eq. (8) and the unperturbed charge neutrality
condition \( n_{ce} + n_{se} = n_0 \) can be written as

\[ \frac{n_{ce} \sigma_c + n_{se} \sigma_s}{1 + n_{ce}} = 1, \quad \frac{n_{ce} \sigma_s + n_{se} \sigma_c}{1 + n_{se}} = 1. \]

(12)

Following Dalui et al. [10] and using Eq. (12), we can
write the expressions of \( \frac{n_{ce}}{n_{ce}}, \frac{n_{se}}{n_{ce}}, \frac{n_{ce}}{n_{se}}, \frac{T_{ce}}{T_{se}}, \frac{T_{se}}{T_{se}} \) in the following form:

\[ \frac{n_{ce}}{n_{ce}} = \frac{n_{se}}{n_{ce}}, \quad \frac{n_{ce}}{n_{ce}} = \frac{1}{1 + n_{se}}, \quad \frac{n_{ce}}{n_{se}} = \frac{n_{ce} \sigma_c + n_{se} \sigma_s}{\sigma_c + n_{se}}, \]

(13)

(14)

where \( n_{ce} = \frac{n_{ce}}{n_{ce}} \) and \( \sigma_{ce} = \frac{T_{ce}}{T_{se}} \).

If we neglect the electron-to-ion mass ratio, then (1) and
(2) assume the following form:

\[ \frac{\partial f_{ce}}{\partial \tau} + \frac{\partial \phi}{\partial x} \frac{\partial f_{ce}}{\partial v_i} = 0; \]

(15)
The linearized dispersion relation of the IA wave obtained from a set of Eqs. (5), (6), (7) and the Poisson equation (3) can be written as

$$\frac{\omega}{k} = M_s \sqrt{\frac{(M_s^2 - \gamma \sigma)^{-1} + \frac{\sigma^2}{\beta^2} k^2}{(M_s^2 - \gamma \sigma)^{-1} + k^2}},$$  \hfill (21)

where $\omega$ is the normalized wave frequency and $k$ is the normalized wave number and we have used Eqs. (19) and (20) to describe $n_{ce}$ and $n_{se}$ in Eq. (3). The expression of $M_s$ is given by

$$M_s = \sqrt{\frac{1}{\sigma_c + (1 - \beta_e)\sigma_e \epsilon}}.$$  \hfill (22)

Now, for long-wavelength plane wave perturbation, i.e. for $k \to 0$, from the linear dispersion relation (21), we have

$$\lim_{k \to 0} \frac{\omega}{k} = M_s \text{ and } \lim_{k \to 0} \frac{\partial \omega}{\partial k} = M_s.$$  \hfill (23)

Therefore, for long-wavelength plane wave perturbation (for small value of $k$), the phase of the wave can be written as

$$kx - \omega t = k(x - M_s t) + \left\{ \frac{M_s^2 - \gamma \sigma}{\sqrt{2M_s}} \right\} k^3 t + O(k^5).$$  \hfill (24)

This equation suggests to choose the stretched space coordinate and stretched time as

$$\xi = \epsilon \frac{k}{\epsilon} (x - M_s t), \tau = \epsilon^2 t,$$ \hfill (25)

where $k = \epsilon^2$ and consequently, $\epsilon$ measures the weakness of dispersion. Since, we have considered the weakly nonlinear and weakly dispersive IA wave, then $\epsilon$ also measures the weakness of nonlinearity if we assume that the weakness of nonlinearity is of the same order of weakness of dispersion. Therefore, $\epsilon$ measures the weakness of dispersion as well as the weakness of nonlinearity.

In the present paper, our main aim is to consider the effect of linear Landau damping of electrons on IA solitary waves. Now, if we neglect the electron-to-ion mass ratio, then the nonlinear behaviour of the IA wave can be expressed by a set hydrodynamic equations (5), (6), (7) and (3) along with equations (19) and (20). From these hydrodynamic equations, one can analyse the nonlinear behaviour of the small amplitude IA wave with the help of usual KdV or modified KdV equations. So, to include the kinetic effect of electrons or to study the effect of linear Landau damping of electrons on IA solitary wave, we cannot neglect electron-to-ion mass ratio. But we have assumed that the effect of electron Landau damping on the nonlinear behaviour of IA wave is small and the effect of linear Landau damping of electrons on the nonlinear behaviour of IA wave is of the same order of nonlinearity, i.e. dispersion, nonlinearity and the effect of linear Landau damping of electrons are small but of the same order of magnitude. Therefore, following Ott and Sudan [33], we replace $\sqrt{m_e/m}$ by $\epsilon z_1$ in Eqs. (1) and (2), and consequently these two equations can be written in the following form:

$$x_1 \frac{\partial f_{ce}}{\partial t} + v_{||} \frac{\partial f_{ce}}{\partial x} + \frac{\partial}{\partial x} \left\{ \frac{M_s^2 - \gamma \sigma}{\sqrt{2M_s}} \right\} k^3 t \frac{\partial f_{ce}}{\partial v_{||}} = 0,$$ \hfill (26)

$$x_1 \frac{\partial f_{se}}{\partial t} + v_{||} \frac{\partial f_{se}}{\partial x} + \frac{\partial}{\partial x} \left\{ \frac{M_s^2 - \gamma \sigma}{\sqrt{2M_s}} \right\} k^3 t \frac{\partial f_{se}}{\partial v_{||}} = 0.$$  \hfill (27)

Now using (7), the momentum equation (6) can be written as

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\gamma \sigma n_{se}}{\sqrt{2M_s}} \frac{\partial n_{se}}{\partial x} = -\frac{\partial \phi}{\partial x},$$ \hfill (28)

Again, using (4), the Poisson equation (3) can be written as

$$\frac{\partial^2 \phi}{\partial x^2} = \int_{-\infty}^{\infty} f_{ce} dv_{||} + \int_{-\infty}^{\infty} f_{se} dv_{||} - n.$$ \hfill (29)

Therefore, Eqs. (26), (27), (29), (5) and (28) are the basic equations to derive Korteweg–de Vries (KdV) equation and different modified Korteweg–de Vries equations including the effect of linear Landau damping of electrons. Finally, we have solved the different macroscopic nonlinear evolution equations including the kinetic effect of electrons on IA waves by considering appropriate initial and boundary conditions.

3. Derivation of different evolution equations

To derive different nonlinear evolution equations including the kinetic effect of electrons on IA waves propagating along $x$-axis, we consider the following stretching of the space coordinate and time:
\[ \zeta = e^2 (x - Vt), \tau = e^4 t, \]  
where \( V \) is a constant and \( \epsilon \) is a small parameter.

3.1. KdV equation including the effect Landau damping

To derive the KdV equation including the effect of linear Landau damping of electrons, we take the following perturbation expansions of the dependent variables:

\[ \Lambda = \Lambda^{(0)} + \sum_{i=1}^{\infty} \epsilon^{i} \Lambda^{(i)}(\zeta, \tau), \]  
where \( \Lambda = n, u, \phi, f_{\text{ce}} \) and \( f_{\text{se}} \) with \( (n^{(0)}, u^{(0)}, \phi^{(0)}, f_{\text{ce}}^{(0)}, f_{\text{se}}^{(0)}) = (1, 0, 0, f_{0\text{e}}, f_{0\text{s}}) \).

Substituting (30) and (31) into Eqs. (26), (27), (29), (5) and (28) and collecting the terms of different powers of \( \epsilon \) on both sides of each equation, we get a sequence of equations and from this sequence of equations, we get the following nonlinear evolution equation:

\[ \begin{align*}
\frac{\partial \phi^{(1)}}{\partial \tau} + A B_{1} \phi^{(1)} \frac{\partial \phi^{(1)}}{\partial \zeta} &+ \frac{1}{2} A c_{s}^{3} \frac{\partial^{2} \phi^{(1)}}{\partial \zeta^{2}} \\
&+ \frac{1}{2} A E \varepsilon \int_{-\infty}^{\infty} \frac{\partial \phi^{(1)}}{\partial \zeta} \frac{d \zeta'}{\zeta - \zeta'} = 0,
\end{align*} \]  
where we have used the same procedure of Bandyopadhyay and Das [37] to derive Eq. (32).

The coefficients \( A, B_{1}, E \) are given by

\[ A = \frac{1}{V} (V^{2} - \sigma_{\gamma}^{2}), \]  
\[ B_{1} = \frac{1}{2} \left[ \frac{3V^{2} + \sigma_{\gamma}^{2}(\gamma - 2)}{(V^{2} - \sigma_{\gamma}^{2})^{3}} - \left( \bar{n}_{0\text{e}} \sigma_{c}^{2} + \bar{n}_{0\text{s}} \sigma_{s}^{2} \right) \right], \]  
\[ E = \frac{V}{\sqrt{2\pi}} \left[ \bar{n}_{0\text{e}} \sigma_{c}^{3/2} \left( 1 - \frac{3}{4} \beta_{c} \right) + \bar{n}_{0\text{s}} \sigma_{s}^{3/2} \right]. \]

The constant \( V \) is given by

\[ (V^{2} - \sigma_{\gamma})(1 - \bar{n}_{0\text{e}} \sigma_{c} \beta_{c}) = 1, \]  
where \( \beta_{c} = \frac{\sigma_{c}}{1 + 3 \sigma_{c}} \) and the physically admissible range of \( \beta_{c} \) is \( 0 \leq \beta_{c} \leq \frac{4}{3} \). The physically admissible range of \( \beta_{c} \) is pointed out by Verheest and Pillay [44]. The calculation regarding the physically admissible range of \( \beta_{c} \) has been given by Debnath et al. [45], although, mathematically, \( \beta_{c} \) is restricted by the inequality: \( 0 \leq \beta_{c} < \frac{4}{3} \).

If we neglect electron-to-ion mass ratio, i.e. if we set \( \alpha_{1} = 0 \), then the nonlinear evolution equation (32) simply reduces to the well-known KdV equation.

Equation (32) describes the propagation of weakly nonlinear and weakly dispersive IA solitary waves in a multi-species collisionless unmagnetized plasma consisting of nonthermal and isothermal electrons including the effect of linear Landau damping of electrons.

From Eq. (32), we see that the nonlinearity of the IA wave is only due to the second term of (32), i.e. \( AB_{1} \) is responsible for the nonlinearity of the system. When \( AB_{1} = 0 \), i.e. \( B_{1} = 0 \) (as \( A \neq 0 \) for any set of physically admissible values of the parameters of the system), it is not possible to discuss the nonlinear behaviour of IA waves with the help of the evolution equation (32).

In Fig. 1, \( B_{1} \) is plotted against \( \sigma_{\text{se}} \) for \( \gamma = 3, \sigma = 0.001 \) and for (a) \( n_{\text{se}} = 0.05 \), (b) \( n_{\text{se}} = 0.2 \), (c) \( n_{\text{se}} = 0.3 \) and (d) \( n_{\text{se}} = 0.5 \). Here, red, black, green and blue curves of each figure correspond to \( \beta_{c} = 0, \beta_{c} = 0.2, \beta_{c} = 0.4 \) and \( \beta_{c} = 0.57 \) respectively. From Fig. 1(a), (b) and (c), we see that there exists a value \( \sigma_{\text{se}}^{(c)} \) of \( \sigma_{\text{se}} \) such that \( B_{1} = 0 \) at \( \sigma_{\text{se}} = \sigma_{\text{se}}^{(c)} \), and more specifically, \( B_{1} < 0 \) for \( \sigma_{\text{se}} < \sigma_{\text{se}}^{(c)} \) and \( B_{1} > 0 \) for \( \sigma_{\text{se}} > \sigma_{\text{se}}^{(c)} \). Again, from Fig. 1(d), we see that \( B_{1} > 0 \) for all values of \( \beta_{c} \). From Fig. 1, it is evident that there exists a region \( R_{I} = \{(n_{\text{se}}, \sigma_{\text{se}}, \beta_{c}) \mid B_{1}(n_{\text{se}}, \sigma_{\text{se}}, \beta_{c}) \neq 0 \} \) such that each point of the collection must satisfy equation \( B_{1}(n_{\text{se}}, \sigma_{\text{se}}, \beta_{c}) = 0 \) and consequently for these values of the parameters \( n_{\text{se}}, \sigma_{\text{se}} \) and \( \beta_{c} \) we cannot use the KdV-like evolution equation to investigate the effect of linear Landau damping of electrons on IA solitary waves. To confirm the existence of a region \( R_{II} = \)
}\{(\sigma_{\text{sc}}, \sigma_{\text{sc}}, \beta_e) : B_1(\sigma_{\text{sc}}, \sigma_{\text{sc}}, \beta_e) = 0\}\) in the entire parameter space, we consider the following figures in different parameter planes.

Now, it is simple to check that \(B_1\) is a function of \(n_{\text{sc}}, \sigma_{\text{sc}}\) and \(\beta_e\) for any prescribed value of \(\sigma\) and \(\gamma\), i.e. \(B_1 = B_1(n_{\text{sc}}, \sigma_{\text{sc}}, \beta_e)\). Throughout this paper, we take \(\gamma = 3\) and \(\sigma = 0.001\), then all the coefficients \(A, B_1, E\) can be regarded as functions of \(n_{\text{sc}}, \sigma_{\text{sc}}\) and \(\beta_e\). Therefore, \(B_1\) is a function of \(\sigma_{\text{sc}}\) and \(n_{\text{sc}}\) for any given value of \(\beta_e\), and consequently, \(B_1 = 0\) gives a functional relationship between \(\sigma_{\text{sc}}\) and \(n_{\text{sc}}\). This functional relationship between \(\sigma_{\text{sc}}\) and \(n_{\text{sc}}\) is plotted in Fig. 2 when \(B_1(n_{\text{sc}}, \sigma_{\text{sc}}, \beta_e) = 0\) for different values of \(\beta_e\). Here, red, black, green and blue curves correspond to \(\beta_e = 0, \beta_e = 0.4, \beta_e = 0.5\) and \(\beta_e = 0.57\) respectively. From this figure, we see that the interval of existence of \(\sigma_{\text{sc}}\) increases with increasing \(\beta_e\) when \(B_1(n_{\text{sc}}, \sigma_{\text{sc}}, \beta_e) = 0\).

Again, from the equation \(B_1(n_{\text{sc}}, \sigma_{\text{sc}}, \beta_e) = 0\), we get a functional relationship between \(n_{\text{sc}}\) and \(\beta_e\) for any given value of \(\sigma_{\text{sc}}\). In Fig. 3, \(n_{\text{sc}}\) is plotted against \(\beta_e\) when \(B_1(n_{\text{sc}}, \sigma_{\text{sc}}, \beta_e) = 0\) for (a) \(\sigma_{\text{sc}} = 0.05\), (b) \(\sigma_{\text{sc}} = 0.1\), (c) \(\sigma_{\text{sc}} = 0.2\) and (d) \(\sigma_{\text{sc}} = 0.3\). From this figure, we see that the interval of existence of \(\beta_e\) decreases with increasing \(\sigma_{\text{sc}}\) when \(B_1(n_{\text{sc}}, \sigma_{\text{sc}}, \beta_e) = 0\).

Similarly, when \(B_1(n_{\text{sc}}, \sigma_{\text{sc}}, \beta_e) = 0\), we get a functional relationship between \(\sigma_{\text{sc}}\) and \(\beta_e\) for any fixed value of \(n_{\text{sc}}\). When \(B_1(n_{\text{sc}}, \sigma_{\text{sc}}, \beta_e) = 0\), then the functional relation between \(\sigma_{\text{sc}}\) and \(\beta_e\) is plotted in Fig. 4 for different values of \(n_{\text{sc}}\) with \(\sigma = 0.001\). Red, black, green and blue curves correspond to \(n_{\text{sc}} = 0.1, n_{\text{sc}} = 0.2, n_{\text{sc}} = 0.3\) and \(n_{\text{sc}} = 0.4\) respectively. From this figure, we see that the interval of existence of \(\beta_e\) increases with increasing \(n_{\text{sc}}\) whereas \(\sigma_{\text{sc}}\) decreases with increasing \(n_{\text{sc}}\) for any fixed \(\beta_e\).

So, Figs. 1, 2, 3 and 4 confirm the existence of a region \(R_H = \{(n_{\text{sc}}, \sigma_{\text{sc}}, \beta_e) : B_1(n_{\text{sc}}, \sigma_{\text{sc}}, \beta_e) = 0\}\) in the parameter space such that each point of \(R_H\) satisfies the equation \(B_1(n_{\text{sc}}, \sigma_{\text{sc}}, \beta_e) = 0\). Therefore, for \(B_1(n_{\text{sc}}, \sigma_{\text{sc}}, \beta_e) = 0\) or \((n_{\text{sc}}, \sigma_{\text{sc}}, \beta_e) \in R_H\), it is necessary to modify the KdV-

like evolution equation to investigate the effect of linear Landau damping of electrons on IA solitary waves.

3.2. MKdV equation including the Landau damping effect

When \(B_1 = 0\), we take the following perturbation expansions of the dependent variables:

\[
\Lambda = \Lambda^{(0)} + \sum_{i=1}^{\infty} \epsilon^i \Lambda^{(i)}(\xi, \tau),
\]

(37)

where \(\Lambda = n, u, \phi, f_{sc}\) and \(f_{sc}\) with \((n(0), u(0), \phi(0), f_{sc}(0), f_{sc}(0)) = (1, 0, 0, f_{co}, f_{cd})\).

Substituting (30) and (37) into Eqs. (26), (27), (29), (5) and (28) and collecting the terms of different powers of \(\epsilon\),
we get a sequence of equations and from this sequence of equations, following Bandyopadhyay and Das [37], we get the following nonlinear evolution equation:

$$\frac{\partial \phi(x)}{\partial t} + A B_2 \phi(x) \frac{\partial \phi(x)}{\partial x} + \frac{1}{2} A + \frac{1}{2} A E x_1 P$$

$$\int_{-\infty}^{\infty} \frac{\partial \phi(x)}{\partial x} \frac{\partial \phi(x)}{\partial x} = 0.$$ (38)

Here, it is important to mention that the condition $B_1 = 0$ has been used to eliminate the term $AB_1 \frac{\partial \phi(x)}{\partial x}$ from the final form of (38). The expressions of $A$, $E$, $V$ are given by (33), (35), (36), respectively, and the expression of $B_2$ can be written as

$$B_2 = \frac{1}{4} \left[ H_2 - \left( \mathbf{n}_c \sigma_e^2 (1 + 3 \beta_e) + \mathbf{n}_a \sigma_o^2 \right) \right],$$ (39)

where

$$H_2 = \frac{1}{(V^2 - \sigma_e^2)} \left[ 15 V^4 + \sigma_e^2 (\gamma^2 + 13 \gamma - 18) V^2 \right] + \sigma_o^2 (\gamma^2 - 2(2 \gamma - 3)).$$ (40)

If $x_l = 0$, then the nonlinear evolution equation (38) simply reduces to the well-known MKdV equation.

From Eq. (38), we see that the nonlinearity of the IA wave is only due to the second term of (38). So, Eq. (38) describes the nonlinear dynamics of IA waves when $B_1 = 0$ and $B_2 \neq 0$.

Now, in Fig. 5, $B_2$ is plotted against $\beta_e$ when $B_1 = 0$ for $\gamma = 3$ and $\sigma = 0.001$, and for different values of $n_{sc}$. In fact, for given values of $\gamma$, $\sigma$ and $n_{sc}$, $B_1$ is a function of $\sigma_{sc}$ and $\beta_e$ only and consequently if we solve the equation $B_1 = 0$ with respect to the unknown $\sigma_{sc}$, we get $\sigma_{sc}$ as a function of $\beta_e$. If we put all the values of $\gamma$, $\sigma$, $n_{sc}$ and $\sigma_{sc}$ in the expression of $B_2$, we get $B_2$ as a function of $\beta_e$. This $B_2$ is plotted against $\beta_e$ in Fig. 5. Here, red, black and blue curves correspond to $n_{sc} = 0.02$, $n_{sc} = 0.05$ and $n_{sc} = 0.08$ respectively. This figure clearly shows that there exists a value $\beta_e^{(c)}$ of $\beta_e$ such that $B_2 = 0$ at $\beta_e = \beta_e^{(c)}$ and more specifically, $B_2 < 0$ for $\beta_e < \beta_e^{(c)}$, $B_2 > 0$ for $\beta_e > \beta_e^{(c)}$ and $B_2 = 0$ at $\beta_e = \beta_e^{(c)}$. In particular, for $n_{sc} = 0.05$, the value of $\beta_e^{(c)}$ is approximately equal to 0.2847. Therefore, there exist points $(n_{sc}, \sigma_{sc}, \beta_e)$ in the parameter space such that $B_1 = B_2 = 0$. So, now it is necessary to divide the region $R_{II}$ into two regions $R_{II}^{(a)}$ and $R_{II}^{(b)}$ such that $R_{II}^{(a)} = \{ (n_{sc}, \sigma_{sc}, \beta_e) : B_1 = 0$ and $B_2 \neq 0 \}$ and $R_{II}^{(b)} = \{ (n_{sc}, \sigma_{sc}, \beta_e) : B_1 = B_2 = 0 \}$.

We see that Eq. (38) is free from any nonlinear effect when $B_1 = B_2 = 0$, i.e. if $(n_{sc}, \sigma_{sc}, \beta_e) \in R_{II}^{(b)}$. To explain the existence of the region $R_{II}^{(b)}$, we consider Fig. 6. Now, it is simple to check that $B_1$ and $B_2$ are the functions of $n_{sc}$, $\sigma_{sc}$ and $\beta_e$ for any prescribed value of $\sigma$ and $\gamma$, i.e. $B_1 = B_1(n_{sc}, \sigma_{sc}, \beta_e)$ and $B_2 = B_2(n_{sc}, \sigma_{sc}, \beta_e)$ for any given value of $\sigma$ and $\gamma$. We have mentioned earlier that throughout this paper we take $\gamma = 3$ and $\sigma = 0.001$. Now, for given values of $\beta_e$ and $\sigma_{sc}$, $B_1(n_{sc}, \sigma_{sc}, \beta_e) = 0$ gives an equation for the unknown $n_{sc}$ and consequently, $B_1 = 0$ gives a real solution for $n_{sc}$. Let $n_{sc} = n_{sc}(\beta_e, \sigma_{sc})$ be the physically admissible real solution of the equation $B_1(n_{sc}, \sigma_{sc}, \beta_e) = 0$, i.e. the physically admissible real solution $n_{sc}$ of the equation $B_1(n_{sc}, \sigma_{sc}, \beta_e) = 0$ can be considered as a function of $\beta_e$ and $\sigma_{sc}$. If we put this value of $n_{sc}(\beta_e, \sigma_{sc})$ in the expression of $B_2(n_{sc}, \sigma_{sc}, \beta_e)$ then the function $B_2$ is reduced to a function of $\beta_e$ and $\sigma_{sc}$ only, i.e. $B_2 = B_2(\beta_e, \sigma_{sc})$. Again, $B_2 = B_2(\beta_e, \sigma_{sc}) = 0$ gives a functional relationship between $\sigma_{sc}$ and $\beta_e$. This functional relationship between $\sigma_{sc}$ and $\beta_e$ is plotted in Fig. 6, for fixed values of the other parameters, i.e. in Fig. 6, $\sigma_{sc}$ is plotted against the nonlinear parameter $\beta_e$ when $B_1 = 0$ and $B_2 = 0$. Figure 6 shows a variation of $\sigma_{sc}$ against the
3.3. FMKdV equation including the Landau damping effect

To derive the FMKdV equation including the effect of linear Landau damping of electrons when the conditions $B_1 = 0$ and $B_2 = 0$ hold simultaneously, we take the following perturbation expansions of the dependent variables:

$$\Lambda = \Lambda^{(0)} + \sum_{i=1}^{\infty} \epsilon^i \Lambda^{(i)}(\xi, \tau),$$  \hspace{1cm} (41)

where $\Lambda = n, u, \phi, f_{ce}$ and $f_{se}$ with $(n^{(0)}, u^{(0)}, \phi^{(0)}, f_{ce}^{(0)}, f_{se}^{(0)}) = (1, 0, 0, f_{ce0}, f_{se0})$.

Substituting (30) and (41) into Eqs. (26), (27), (29), (5) and (28) and collecting the terms of different powers of $\epsilon$ on both sides of each equation, we get a sequence of equations.

3.3.1. Equations for ion fluid at the order $\epsilon^{5/6}$

At the order $\epsilon^{5/6}$, solving the equation of continuity and the equation of motion of ion fluid for the unknowns $n^{(1)}$ and $u^{(1)}$, we get

$$n^{(1)} = \frac{1}{V^2 - \sigma^2} \phi^{(1)}, u^{(1)} = \frac{V}{V^2 - \sigma^2} \phi^{(1)}.$$  \hspace{1cm} (42)

3.3.2. Vlasov–Boltzmann equation at the order $\epsilon^{5/6}$

The Vlasov–Boltzmann equation of nonthermal electrons at the order $\epsilon^{5/6}$ is

$$\frac{\partial f_{ce}^{(1)}}{\partial \xi} + \frac{\partial \phi^{(1)}}{\partial \xi} \frac{\partial f_{ce}}{\partial \dot{\xi}^2} = 0.$$  \hspace{1cm} (43)

The above equation does not have a unique solution and consequently to get the unique solution of Eq. (43), we follow the method of Ott and Sudan [33]. This method suggests to add an extra higher-order time derivative term $\epsilon^{17/6} \sigma_1 \phi^{(1)}$ with the Vlasov–Boltzmann equation at the order $\epsilon^{5/6}$. So, Eq. (43) can be written in the following form:

$$\xi \epsilon^2 \frac{\partial f_{ce}^{(1)}}{\partial \xi} + \frac{\partial \phi^{(1)}}{\partial \xi} \frac{\partial f_{ce}}{\partial \dot{\xi}^2} + \frac{\partial \phi^{(1)}}{\partial \xi} \frac{\partial f_{ce0}}{\partial \dot{\xi}^2} + \frac{\partial \phi^{(1)}}{\partial \xi} \frac{\partial f_{se0}}{\partial \dot{\xi}^2} = 0,$$  \hspace{1cm} (44)

where $f_{ce}^{(1)}$ is replaced by $f_{ce}^{(1)}$ and one can get $f_{ce}^{(1)}$ from the solution of the above equation by considering the following relation for $j = 1$.

$$f_{ce}^{(j)} = \lim_{\epsilon \rightarrow 0} f_{ce}, \quad j = 1, 2, 3, \ldots.$$  \hspace{1cm} (45)

To solve (44), we have assumed that the time dependence of any perturbed quantity is of the form $\exp(i \omega t)$ and we can write Eq. (44) as

$$i e \epsilon^2 \frac{\partial f_{ce}^{(1)}}{\partial \xi} + \frac{\partial \phi^{(1)}}{\partial \xi} \frac{\partial f_{ce}}{\partial \dot{\xi}^2} + \frac{\partial \phi^{(1)}}{\partial \xi} \frac{\partial f_{ce0}}{\partial \dot{\xi}^2} + \frac{\partial \phi^{(1)}}{\partial \xi} \frac{\partial f_{se0}}{\partial \dot{\xi}^2} = 0.$$  \hspace{1cm} (46)

Now, taking the Fourier transform of this equation with respect to $\xi$, we get

$$\tilde{f}_{ce}^{(1)} = -2 \frac{\partial f_{ce0}}{\partial \xi} \frac{\sigma \phi^{(1)}}{\sigma \phi^{(1)} + 2 \epsilon \sigma \phi^{(1)}},$$  \hspace{1cm} (47)

where the Fourier transform of $g$ with respect to $\xi$ is defined as

$$\tilde{g} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\xi) e^{-i\epsilon \xi} d\xi.$$  \hspace{1cm} (48)

Again, using the Landau prescription to resolve the singularities involved, Eq. (47) can be written as

$$\tilde{f}_{ce}^{(1)} = -2 \frac{\partial f_{ce0}}{\partial \xi} \left( \frac{1}{\sigma \phi^{(1)} + 2 \epsilon \sigma \phi^{(1)}} + i \pi \sigma \phi^{(1)} \delta(\sigma \phi^{(1)} + 2 \epsilon \sigma \phi^{(1)}) \right) \phi^{(1)}.$$  \hspace{1cm} (49)

Taking limit $\epsilon \rightarrow 0$, we get

$$\tilde{f}_{ce}^{(1)} = -2 \frac{\partial f_{ce0}}{\partial \xi} \left( \frac{1}{\sigma \phi^{(1)}} + i \pi \phi^{(1)} \delta(\phi^{(1)}) \right) \phi^{(1)},$$  \hspace{1cm} (50)

where we have used the relation (45) for $j = 1$.

Now, using the relations $\sigma \phi^{(1)} = 1$ and $\epsilon \phi^{(1)} = 0$, Eq. (50) can be simplified as

$$\tilde{f}_{ce}^{(1)} = -2 \frac{\partial f_{ce0}}{\partial \xi} \phi^{(1)}.$$  \hspace{1cm} (51)

Taking Fourier inversion of the above equation, we get

$$f_{ce}^{(1)} = -2 \frac{\partial f_{ce0}}{\partial \xi} \phi^{(1)}.$$  \hspace{1cm} (52)

Similarly, considering the Vlasov–Boltzmann equation of isothermal electrons at the order $\epsilon^{5/6}$, we get
3.3.5. Vlasov–Boltzmann equation at the order \( \epsilon^7/6 \)

At the order \( \epsilon^7/6 \), the Vlasov–Boltzmann equation for nonthermal and isothermal electrons are

\[
\begin{aligned}
\frac{\partial f_{ee}^{(2)}}{\partial v_{||}} + \frac{\partial f_{e0}^{(2)}}{\partial c_{e}} + \frac{\partial f_{0e}^{(1)}}{\partial v_{||}} &= 0, \\
\frac{\partial f_{ee}^{(2)}}{\partial \zeta} + \frac{\partial f_{e0}^{(2)}}{\partial c_{e}} + \frac{\partial f_{0e}^{(1)}}{\partial \zeta} &= 0.
\end{aligned}
\]  

(58)

(59)

Following exactly the same analysis as given in Sect. 3.3.2, the solutions of (58) and (59) can be written as follows:

\[
\begin{aligned}
f_{ee}^{(2)} &= -2 \frac{\partial f_{e0}}{\partial v_{||}} \phi_{(2)}^{(2)} - 2 \frac{\partial g_{0e}}{\partial v_{||}} \psi_{(2)}, \\
f_{se}^{(2)} &= -2 \frac{\partial f_{e0}}{\partial v_{||}} \phi_{(2)}^{(2)} - 2 \frac{\partial g_{0e}}{\partial v_{||}} \psi_{(2)},
\end{aligned}
\]  

(60)

(61)

where

\[
\psi^{(2)} = - (\phi^{(1)})^2, \quad g_{e0} = \frac{\partial f_{e0}}{\partial v_{||}}, \quad g_{0e} = \frac{\partial f_{0e}}{\partial v_{||}}.
\]  

(62)

3.3.6. Poisson equation at the order \( \epsilon^5 \)

It is simple to check that the Poisson equation at the order \( \epsilon^5 \) is identically satisfied due to the dispersion relation (36) and the condition \( B_1 = 0 \).

3.3.7. Equations for ion fluid at the order \( \epsilon^{9/6} \)

Again, at the order \( \epsilon^{9/6} \), solving the continuity equation of ions and the momentum equation of ions for the unknowns \( n^{(3)} \) and \( u^{(3)} \), we obtain the following equations:

\[
\begin{aligned}
n^{(3)} &= \frac{\phi^{(3)}}{V^2 - \sigma_\gamma} + \frac{3V^2 + \sigma_\gamma(\gamma - 2)}{(V^2 - \sigma_\gamma)^3} \phi^{(1)}(\phi^{(2)} + \frac{1}{6}H_2[\phi^{(1)}]^3, \\
u^{(3)} &= \frac{V\phi^{(3)}}{V^2 - \sigma_\gamma} + \frac{V(V^2 + \sigma_\gamma^2)}{(V^2 - \sigma_\gamma)^3} \phi^{(1)}(\phi^{(2)} + G_2[\phi^{(1)}]^3).
\end{aligned}
\]  

(63)

(64)

where

\[
G_2 = \frac{V}{6} \left[ \frac{3V^4 + \sigma_\gamma^2(\gamma + 7)V^2 + \sigma_\gamma^2\gamma^3(2\gamma - 1)}{(V^2 - \sigma_\gamma)^5} \right].
\]  

(65)

3.3.8. Vlasov–Boltzmann equation at the order \( \epsilon^{9/6} \)

Following exactly the same analysis as given in Sect. 3.3.2, the solutions of the Vlasov–Boltzmann equations of nonthermal and isothermal electrons at the order \( \epsilon^{9/6} \) can be written as follows:

\[
\begin{aligned}
f_{ee}^{(3)} &= -2 \frac{\partial f_{e0}}{\partial v_{||}} \phi_{(3)}^{(3)} - 2 \frac{\partial g_{0e}}{\partial v_{||}} \psi_{(3)}^{(3)} - 2 \frac{\partial h_{e0}}{\partial v_{||}} \chi_{(3)}^{(3)}, \\
f_{se}^{(3)} &= -2 \frac{\partial f_{e0}}{\partial v_{||}} \phi_{(3)}^{(3)} - 2 \frac{\partial g_{0e}}{\partial v_{||}} \psi_{(3)}^{(3)} - 2 \frac{\partial h_{e0}}{\partial v_{||}} \chi_{(3)}^{(3)},
\end{aligned}
\]  

(66)

(67)

where

\[
\begin{aligned}
\psi^{(3)} &= -2\phi^{(1)}(\phi^{(2)}), \\
\chi^{(3)} &= \frac{2}{3} (\phi^{(1)})^3, \\
h_{e0} &= \frac{\partial g_{0e}}{\partial v_{||}}, \\
h_{0e} &= \frac{\partial g_{0e}}{\partial v_{||}}.
\end{aligned}
\]  

(68)

3.3.9. Poisson equation at the order \( \epsilon \)

It is simple to check that the Poisson equation at the order \( \epsilon \) is also identically satisfied due to the dispersion relation (36) and the conditions \( B_1 = 0 \) and \( B_2 = 0 \).

3.3.10. Equations for ion fluid at the order \( \epsilon^{11/6} \)

At the order \( \epsilon^{11/6} \), solving the continuity equation and the momentum equation of ions, \( \frac{\partial n^{(4)}}{\partial v_{||}} \) and \( \frac{\partial u^{(4)}}{\partial v_{||}} \) can be expressed as functions of \( \phi^{(1)}, \phi^{(2)}, \phi^{(3)} \) and \( \phi^{(4)} \) along with their
different derivatives with respect to \( \xi \) and \( \tau \). In particular, \( \frac{\partial n^{(4)}}{\partial \xi} \) can be written as

\[
\frac{\partial n^{(4)}}{\partial \xi} = \frac{1}{(V^2 - \sigma^2)} \frac{\partial (V^2 - \sigma^2)}{\partial \xi} \frac{\partial \phi^{(1)}}{\partial \xi} + \frac{2V}{3} \frac{\partial \phi^{(1)}}{\partial \xi} + \frac{3V^2 + \sigma \gamma (\gamma - 2)}{(V^2 - \sigma^2)^2} \frac{\partial (V^2 - \sigma^2)}{\partial \xi}(\frac{\partial (V^2 - \sigma^2)}{\partial \xi})^2 + \frac{2H_2}{\partial \xi} \left( (\frac{\partial (V^2 - \sigma^2)}{\partial \xi})^2 + \frac{H_5}{6} \frac{(\partial V^2 - \sigma^2)}{\partial \xi} \frac{(\partial (V^2 - \sigma^2))}{\partial \xi} \right),
\]

(69)

where

\[ H_5 = \frac{1}{(V^2 - \sigma^2)} \left( 105V^6 + \sigma \gamma (3^2 + 21\gamma^2 + 161\gamma - 174)V^4 + \sigma^2 \gamma^2(8\gamma^2 + 53\gamma^2 - 162\gamma + 108)V^2 + \sigma^3 \gamma^3(\gamma - 2)(2\gamma^2 - 3\gamma - 4) \right). \]

(70)

### 3.3.11. Vlasov–Boltzmann equation at the order \( \epsilon^{11/6} \)

At the order \( \epsilon^{11/6} \), the Vlasov–Boltzmann equation of nonthermal electrons is

\[
\frac{\partial f_c(e)}{\partial \xi} + \frac{\partial \phi^{(4)}}{\partial \xi} \frac{\partial f_c(e)}{\partial \xi} + \frac{\partial \psi^{(4)}}{\partial \xi} \frac{\partial g_c(e)}{\partial \xi} + \frac{\partial \nu^{(4)}}{\partial \xi} \frac{\partial h_c(e)}{\partial \xi} + \frac{\partial \kappa^{(4)}}{\partial \xi} \frac{\partial \kappa_c(e)}{\partial \xi} + 2\alpha V \frac{\partial f_c(e)}{\partial \xi} \frac{\partial \phi^{(4)}}{\partial \xi} = 0,
\]

(71)

where we have used Eqs. (52), (60) and (66) to get Eq. (71) and in this equation, we have used the following notations:

\[
\psi^{(4)} = -2\phi^{(1)}(3 + (\phi^{(2)})^2), \quad \nu^{(4)} = 2(\phi^{(1)})^2 \phi^{(2)}, \quad \kappa^{(4)} = -\frac{1}{3} (\phi^{(1)})^4, \quad \kappa_c = \frac{\partial h_c(e)}{\partial \xi}.
\]

(72)

Including an extra higher-order time derivative term \( \epsilon^{23/6} \frac{\partial f_c^{(4)}}{\partial \xi} \), Eq. (71) can be written as

\[
\frac{\partial f_c^{(4)}}{\partial \xi} + \frac{\partial \phi^{(4)}}{\partial \xi} \frac{\partial f_c^{(4)}}{\partial \xi} + \frac{\partial \psi^{(4)}}{\partial \xi} \frac{\partial g_c^{(4)}}{\partial \xi} + \frac{\partial \nu^{(4)}}{\partial \xi} \frac{\partial h_c^{(4)}}{\partial \xi} + \frac{\partial \kappa^{(4)}}{\partial \xi} \frac{\partial \kappa_c^{(4)}}{\partial \xi} + 2\alpha_1 V \frac{\partial f_c^{(4)}}{\partial \xi} \frac{\partial \phi^{(4)}}{\partial \xi} = 0,
\]

(73)

where \( f_c^{(4)} \) is replaced by \( f_c^{(4)} \) and \( f_c^{(4)} \) can be obtained from the unique solution of Eq. (73) by considering the relation \( (45) \) for \( j = 4 \).

Now, assuming the \( \tau \) dependence of the perturbed quantities is of the form \( \exp(i\omega \tau) \) and taking the Fourier transform with respect to \( \xi \), we get the following equation from Eq. (73):

\[
\tilde{f}_c^{(4)} = -\frac{2}{\omega} \left[ \frac{\partial \tilde{f}_c^{(4)}}{\partial \xi} \tilde{\phi}_c^{(4)} + \frac{\partial \tilde{g}_c^{(4)}}{\partial \xi} \tilde{\psi}_c^{(4)} + \frac{\partial \tilde{h}_c^{(4)}}{\partial \xi} \tilde{\nu}_c^{(4)} + \frac{\partial \kappa_c^{(4)}}{\partial \xi} \tilde{\kappa}_c^{(4)} \right]
\times \frac{\partial (\tilde{s}^{(4)})}{\partial \xi} + \frac{\partial \tilde{\phi}_c^{(4)}}{\partial \xi} + \frac{\partial \tilde{\psi}_c^{(4)}}{\partial \xi} \tilde{\nu}_c^{(4)} + \frac{\partial \tilde{\kappa}_c^{(4)}}{\partial \xi} \tilde{\nu}_c^{(4)}
\]

(74)

where \( \tilde{\phi}_c^{(4)}, \tilde{\psi}_c^{(4)}, \tilde{\kappa}_c^{(4)}, \tilde{\kappa}_c^{(4)} \) and \( \tilde{\phi}_c^{(4)} \) are, respectively, the Fourier transform of \( \phi^{(4)}, \psi^{(4)}, \kappa^{(4)}, \kappa^{(4)} \) and \( \phi^{(4)} \).

Now, making \( \epsilon \to 0 \) and using the relations \( xP(1/x) = 1, x\delta(x) = 0 \) and \( s\delta(s) = \text{sgn}(s) \delta(s) \), we get the following expression of \( \tilde{f}_c^{(4)} \):

\[
\tilde{f}_c^{(4)} = -2 \left[ \frac{\partial \tilde{f}_c^{(4)}}{\partial \xi} \tilde{\phi}_c^{(4)} + \frac{\partial \tilde{g}_c^{(4)}}{\partial \xi} \tilde{\psi}_c^{(4)} + \frac{\partial \tilde{h}_c^{(4)}}{\partial \xi} \tilde{\nu}_c^{(4)} + \frac{\partial \tilde{\kappa}_c^{(4)}}{\partial \xi} \tilde{\kappa}_c^{(4)} \right]
\times \frac{\partial (\tilde{s}^{(4)})}{\partial \xi} + \text{sgn}(s) \delta(s) \tilde{\phi}_c^{(4)}.
\]

(75)

Integrating (75) over the velocity space, we get

\[
is\int_{-\infty}^{\infty} f_c^{(4)} d\xi = -2 \left[ F_c^0 \tilde{\phi}_c^{(4)} + G_c^0 \tilde{\psi}_c^{(4)} + H_c^0 \tilde{\nu}_c^{(4)} + K_c^0 \tilde{\kappa}_c^{(4)} \right]
\]

(76)

where \( F_c^0, G_c^0, H_c^0, K_c^0, Z_c \) are given in Appendix 1.

Taking Fourier inversion of (76), we get

\[
\frac{\partial}{\partial \xi} \left( \int_{-\infty}^{\infty} f_c^{(4)} d\xi \right) = -2 \left[ F_c^0 \tilde{\phi}_c^{(4)} + G_c^0 \tilde{\psi}_c^{(4)} + H_c^0 \tilde{\nu}_c^{(4)} + K_c^0 \tilde{\kappa}_c^{(4)} \right]
\]

(77)

where we have used the convolution theorem of Fourier transform to find the inverse Fourier transform of \( \tilde{\phi}_c^{(4)} \). Here, \( \frac{\partial \tilde{\phi}_c^{(4)}}{\partial \xi} \) is the value of \( \frac{\partial \tilde{\phi}_c^{(4)}}{\partial \xi} \) at \( \xi = \xi' \).

Similarly, considering the Vlasov–Boltzmann equation of the isothermal electrons at the order \( \epsilon^{11/6} \), we get

\[
\frac{\partial}{\partial \xi} \left( \int_{-\infty}^{\infty} f_c^{(4)} d\xi \right) = -2 \left[ F_c^0 \tilde{\phi}_c^{(4)} + G_c^0 \tilde{\psi}_c^{(4)} + H_c^0 \tilde{\nu}_c^{(4)} + K_c^0 \tilde{\kappa}_c^{(4)} \right]
\]

(78)

where \( F_c^0, G_c^0, H_c^0, K_c^0, Z_c \) are given in Appendix 2.

### 3.3.12. Poisson equation at the order \( \epsilon^{4/3} \)

From the Poisson equation at the order \( \epsilon^{4/3} \), we get
\[ n^{(4)}(\xi) = \int_{-\infty}^{\infty} f^{(4)}(\xi') \, d\xi' + \int_{-\infty}^{\infty} f^{(4)}(\xi') \, d\xi'' \cdot \frac{\partial^2 \phi^{(1)}}{\partial \xi^2}. \quad (79) \]

Differentiating this equation with respect to \( \xi \), using equations (77) and (78) in the resulting equation, we get the following expression of \( \frac{\partial n^{(4)}}{\partial \xi} \) as follows:

\[
\frac{\partial n^{(4)}}{\partial \xi} = -\frac{\partial^3 \phi^{(1)}}{\partial \xi^3} + \frac{1}{2} \left[ \frac{\partial^3 \phi^{(1)}}{\partial \xi^3} + \frac{1}{2} \left( \phi^{(2)} \right)^2 \right] + \frac{1}{2} \left[ \frac{\partial \sigma_0^2}{\partial \xi} (1 + 3\sigma_0^2) + \frac{\partial \sigma_1^2}{\partial \xi} \right] \left( \phi^{(1)} \right)^2 \left( \phi^{(2)} \right) + \frac{1}{6} \left[ \frac{\partial \sigma_0^2}{\partial \xi} (1 + 8\sigma_0^2) + \frac{\partial \sigma_1^2}{\partial \xi} \right] \left( \phi^{(1)} \right)^3 \frac{\partial \phi^{(1)}}{\partial \xi} - \frac{1}{2} \int_{-\infty}^{\infty} \frac{\partial \phi^{(1)}}{\partial \xi} \frac{d\xi''}{\xi - \xi''}. \quad (80) \]

Now, eliminating \( \frac{\partial n^{(4)}}{\partial \xi} \) from Eqs. (69) and (80), we get

\[
\frac{\partial \phi^{(1)}}{\partial \tau} + AB \left[ \frac{\partial \phi^{(1)}}{\partial \xi} + \frac{1}{2} A \frac{\partial^3 \phi^{(1)}}{\partial \xi^3} + \frac{1}{2} A \frac{\partial \phi^{(1)}}{\partial \xi} \right] \int_{-\infty}^{\infty} \frac{d\xi''}{\xi - \xi''} = 0, \quad (81) \]

where we have used the dispersion relation (36), conditions \( B_1 = 0 \) and \( B_2 = 0 \) to eliminate the terms \( \frac{\partial \phi^{(4)}}{\partial \xi} \), \( AB \frac{\partial \left[ \left( \phi^{(1)} \right)^3 + \frac{1}{2} \left( \phi^{(2)} \right)^2 \right] }{\partial \xi} \) and \( AB \frac{\partial \left[ \left( \phi^{(1)} \right)^2 \left( \phi^{(2)} \right) \right] }{\partial \xi} \) respectively, to simplify Eq. (81).

Here, \( B_3 \) is given by

\[ B_3 = \frac{1}{12} \left[ H_3 - \left( \sigma_0 \sigma_0^2 (1 + 8\sigma_0^2) + \sigma_0 \sigma_0^4 \right) \right], \quad (82) \]

where \( H_3 \) is given by Eq. (70).

Therefore, the Poisson equation at the order \( \xi^3 \) gives a FMKdV equation including the effect of Landau damping which describes the nonlinear behaviour of IA waves when \( B_1 = 0, B_2 = 0 \) but \( B_3 \neq 0 \).

4. Solitary wave solution

In more compact form, we can write the KdV equation, MKdV equation and FMKdV equation as

\[
\frac{\partial \phi^{(1)}}{\partial \tau} + AB \frac{\partial \phi^{(1)}}{\partial \xi} + \frac{1}{2} \frac{\partial^3 \phi^{(1)}}{\partial \xi^3} + \frac{1}{2} A \phi^{(1)} \int_{-\infty}^{\infty} \frac{d\xi''}{\xi - \xi''} = 0, \quad (83) \]

where \( r = 1, 2, 3 \).

If we put \( \alpha_1 = 0 \) in Eq. (83), then Eq. (83) reduces to a KdV equation for \( r = 1 \), an MKdV equation for \( r = 2 \) and a FMKdV equation for \( r = 3 \).

For a solitary wave solution of (83) with \( \alpha_1 = 0 \), we consider the following transformation of the independent variables:

\[ X = \xi - U \tau, \tau' = \tau. \quad (84) \]

Under the above transformation of independent variables, Eq. (83) with \( \alpha_1 = 0 \) assumes the following form:

\[
\frac{\partial \phi^{(1)}}{\partial \tau} - U \frac{\partial \phi^{(1)}}{\partial X} + AB \phi^{(1)} \frac{\partial \phi^{(1)}}{\partial X} + \frac{1}{2} A \frac{\partial^3 \phi^{(1)}}{\partial X^3} = 0, \quad (85) \]

where we drop the prime on the independent variable \( \tau \) to simplify the notation.

For the travelling wave solution of (85), we take

\[ \phi^{(1)} = \phi_0(X). \quad (86) \]

Substituting (86) into (85), we get the following ordinary differential equation of \( \phi_0 \):

\[ -U \frac{d \phi_0}{dX} + AB \phi_0 \frac{d \phi_0}{dX} + \frac{1}{2} A \frac{d^3 \phi_0}{dX^3} = 0. \quad (87) \]

To get the solitary wave solution of (87), we use the boundary conditions: \( \phi_0, \frac{\partial \phi_0}{\partial X} \to 0 \) as \( |X| \to \infty \) for \( n = 1, 2, 3, \ldots \) and using these conditions, the solitary wave solution of (87) can be written as

\[ \phi_0 = a \sech \left[ \frac{\xi}{W} \right], \quad (88) \]

where the amplitude \( (a) \) and width \( (W) \) are given by

\[ a' = \frac{(r + 1)(r + 2)U}{2AB} \quad \text{and} \quad W^2 = \frac{r^2 U}{2A}. \quad (89) \]

Now, using (89), Eq. (88) can be written as

\[ \phi_0 = a \sech \left[ \frac{\xi}{W} \right] \int_{-\infty}^{\infty} \frac{d\xi'}{(r + 1)(r + 2)} \left[ \frac{\xi - \frac{2AB \phi_0}{\partial \xi}}{(r + 1)(r + 2)} \right]. \quad (90) \]

Again, multiplying Eq. (83) by \( \phi^{(1)} \) and then integrating the resulting equation with respect to \( \xi \) within the interval \( (\infty, \infty) \), and finally, using the boundary conditions: \( \phi^{(1)}, \frac{\partial \phi^{(1)}}{\partial \xi} \to 0 \) as \( |\xi| \to \infty \) for \( n = 1, 2, 3, \ldots \), we get the following equation:

\[
\frac{\partial}{\partial \xi} \int_{-\infty}^{\infty} \left( \phi^{(1)} \right)^2 d\xi = -A \phi_1 \int_{-\infty}^{\infty} \phi^{(1)} d\xi \quad \text{and} \quad \left[ \int_{-\infty}^{\infty} \frac{d\xi}{\xi} \right] \left[ \int_{-\infty}^{\infty} \frac{d\xi'}{\xi'} \right] = 0. \quad (91) \]
If we neglect the electron-to-ion mass ratio, then Eq. (91) reduces to the following equation: 
\[ \frac{\partial}{\partial \tau} \int_{-\infty}^{\infty} (\phi^{(1)})^2 d\xi = 0. \]
This equation shows that the wave energy is conserved. On the other hand, if \( \alpha \neq 0 \) and if the initial perturbation is of the form (90), then the integral appearing in the right-hand side of (91) is positive for \( r = 1, 2, 3 \) and consequently from (91), we have the inequality: 
\[ \frac{\partial}{\partial \tau} \int_{-\infty}^{\infty} (\phi^{(1)})^2 d\xi < 0 \]
for any values of the parameters of the system, because \( A, E, \alpha \) are all strictly positive. This inequality shows that the initial perturbation of the form (90) will decay to zero. This phenomenon suggests that the amplitude of the solitary wave solution of the form (90) is not a constant but decreases slowly with time.

Now for \( \alpha \neq 0 \), to get a solitary wave solution of Eq. (83), we shall follow the method of Ott and Sudan [33]. So, using the prescription of Ott and Sudan [33], we have introduced the following space coordinate:
\[ X = \sqrt{\frac{r^2 a^2 B_t}{(r + 1)(r + 2)}} \int_{0}^{\tau} a \, d\tau, \]
(92)
where the amplitude \( a \) is a slowly varying function of time. Therefore, considering \( \phi^{(1)} \) as a function of \( X \) and \( \tau \), i.e. \( \phi^{(1)} = \phi^{(1)}(X, \tau) \), Eq. (83) can be written as
\[
\frac{\partial \phi^{(1)}}{\partial \tau} + \left[ -\frac{2AB_t a^2 W}{(r + 1)(r + 2)} + \frac{rX}{2a} \frac{\partial a}{\partial \tau} \right] \frac{\partial \phi^{(1)}}{\partial X} + \frac{AB_t W \phi^{(1)}}{2} \frac{\partial^2 \phi^{(1)}}{\partial X^2} \right] \frac{\partial \phi^{(1)}}{\partial X} + \frac{1}{2} \frac{A^2 W \phi^{(1)}}{2} \frac{\partial^3 \phi^{(1)}}{\partial X^3} \right] \frac{\partial \phi^{(1)}}{\partial X} = 0,
\]
(93)
where \( \frac{\partial \phi^{(1)}}{\partial X} = \frac{\partial \phi^{(1)}}{\partial \tau} \) at \( X = X' \).

To find the solitary wave solution, we follow the procedure of Ott and Sudan [33] and considering two time scales with respect to \( \alpha \) as \( \tau_0 = \tau, \tau_1 = \alpha \tau \). We take the solution of (93) as
\[ \phi^{(1)}(X, \tau) = q^{(0)}(X, \tau_0, \tau_1) + \alpha_1 q^{(1)}(X, \tau_0, \tau_1) + O(\alpha_1^2). \]
(94)
Substituting (94) into (93) and equating the coefficients of order unity \([\alpha_1^0]\) and order \( \alpha_1 \) \([\alpha_1^1]\) on each side of the resulting equation, we get the following equations:
\[
\rho \left[ \frac{\partial q^{(0)}}{\partial \tau_0} + \frac{rX}{2a} \frac{\partial a}{\partial \tau_0} \frac{\partial q^{(0)}}{\partial X} \right] + \frac{L}{\partial q^{(0)}} = 0,
\]
(95)
\[
\rho \left[ \frac{\partial q^{(1)}}{\partial \tau_0} + \frac{rX}{2a} \frac{\partial a}{\partial \tau_0} \frac{\partial q^{(1)}}{\partial X} \right] + \frac{\partial [L q^{(1)}]}{\partial X} = \rho M q^{(0)},
\]
(96)
where
\[
\rho = \frac{2}{A W^3}, L = \frac{\partial^2}{\partial X^2} + \frac{2(r + 1)(r + 2)}{r^2 a^2} q^{(0)} - \frac{4}{r^2}, \]
(97)
\[
-M q^{(0)} = \frac{\partial q^{(0)}}{\partial \tau_1} + \frac{rX}{2a} \frac{\partial a}{\partial \tau_1} \frac{\partial q^{(0)}}{\partial X} + \frac{1}{2} A E W \int_{-\infty}^{\infty} \frac{\partial q^{(0)}}{\partial X} dX. \]
(98)
Now, in view of initial and boundary conditions:
\[ \phi^{(1)}(X, 0) = a_0 \sech \tilde{X} \quad \text{and} \quad \phi^{(1)}(\pm \infty, \tau) = 0, \]
it is simple to check that \( q^{(0)} = a \sech \tilde{X} \) is the soliton solution of (95) if and only if \( \frac{\partial a}{\partial \tau_0} = 0 \) and consequently the solution of (95) can be written in the following form:
\[ q^{(0)} = a(\tau_1) \sech \tilde{X}, \]
where \( a(\tau_1) \) is an arbitrary function of \( \tau_1 \) except for the initial condition \( a(0) = a_0 \). Therefore, Eq. (96) can be written as
\[ \rho \frac{\partial q^{(1)}}{\partial \tau_0} + \frac{\partial [L q^{(1)}]}{\partial \tau_1} = \rho M q^{(0)}. \]
(99)
Now, for the existence of the solution of (99), we have the following consistency condition:
\[ \int_{-\infty}^{\infty} \sech \tilde{X} M q^{(0)} dX = 0. \]
(100)
The above equation states that the right-hand side of (99) is perpendicular to the kernel of adjoint operator of \( \frac{\partial}{\partial \tau} \) \([L] \) and this kernel is \( \sech \tilde{X} \), which satisfies the boundary conditions at \( X = \pm \infty \), i.e. \( \sech \tilde{X} \rightarrow 0 \) as \( X \rightarrow \pm \infty \).

Eq. (100) gives the following differential equation for the solitary wave amplitude:
\[ M_t \left( \frac{\partial}{\partial \tau_0} \left( \frac{a}{a_0} \right) + W_t \left( \frac{a}{a_0} \right)^{\tau_{-1}} \right) = 0, \]
(101)
where \( a_0 \) is the value of \( a \) when \( \tau = 0 \) and
\[ M_t = \int_{-\infty}^{\infty} \sech \tilde{X} (1 + X \tanh X) dX, \]
(102)
\[ W_t = \frac{1}{2} I_t A E W \int_{-\infty}^{\infty} \frac{r^2 a^2 B_t}{(r + 1)(r + 2)} dX, \]
(103)
\[ I_t = P \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sech \tilde{X} \sech \tilde{X} \frac{dX dX'}{X - X'}. \]
(104)
Now, it is simple to check that \( M_1 = 1, \ M_2 = 1, \ M_3 \approx 0.6468 \). In Appendix 3, we have generalized the method of Weiland et al. [46] to find \( I_t \). Using this method and MATHEMATICA [47], we get the following numerical values of \( I_t \) for \( r = 1, 2, 3 : I_1 \approx 2.9231, I_2 \approx 2.7776, I_3 \approx 2.6649 \).
For \( r = 1, 2, 3 \), the solution of (101) can be written as
where $T_r$ is given by the following equation:

$$T_r = \left[ \frac{r - a_b B_r}{4 M_r \sigma_b} \sqrt{r^2 - a_b B_r (r + 1)(r + 2) T_r} \right]^{-1}. $$

Eq. (105) shows that the amplitude of solitary wave solution is proportional to $(1 + \tau)^{-\frac{1}{2}}$ for $r = 1, 2, 3$.

Therefore, the first-order solitary wave solution of the evolution Eq. (83) can be written in the following form: $\phi^{(1)} = a \sech \frac{1}{2} X$ for $r = 1, 2$ and 3, where the amplitude ($a$) of the solitary wave is not a constant but it is a function of time $\tau$ and its functional form is given by Eq. (105).

From Eq. (105), we see that the amplitude of the solitary wave decreases slowly with time $\tau$.

5. Conclusions

We have considered a collisionless unmagnetized electron–ion plasma consisting of warm adiabatic ions and two distinct populations of electrons at different temperatures—a cooler one is isothermally distributed and follows Maxwell–Boltzmann distribution, whereas the hotter one is nonthermally distributed and obeys the distribution function of Cairns et al. [9].

Considering the Vlasov–Poisson model for two different electron species and the fluid model for ions, we have derived a KdV-like evolution equation including the effect of linear Landau damping of electrons. We have studied the propagation of weakly nonlinear and weakly dispersive IA waves using this KdV-like evolution equation.

We have seen that the coefficient of the nonlinear term of the KdV-like evolution equation vanishes along different family of curves in different parameter planes, viz., $\sigma_{sc} - n_{sc}, \beta_e - \sigma_{sc}, \beta_e - n_{sc}$. In this situation, to describe the nonlinear behaviour of IA waves, we have derived an MKdV-like evolution equation including the effect of linear Landau damping of electrons having nonlinear term $\left( \phi^{(1)} \right)^2 \frac{\partial \phi^{(1)}}{\partial \tau}$ but the term responsible for the effect of linear Landau damping of electrons remains the same in all KdV, MKdV and FMKdV-like evolution equations.

The evolution equations can be written in a more compact form by considering the nonlinear term of the form $\left( \phi^{(1)} \right)^2 \frac{\partial \phi^{(1)}}{\partial \tau}$ for $r = 1, 2, 3$. For $r = 1, 2$ and 3, we, respectively, get KdV, MKdV and FMKdV-like evolution equations. Using the multiple time scale analysis with respect to the small parameter $x_1$, we have generalized the method of Ott and Sudan [33] to solve evolution equation (83).

The solitary wave solution of the evolution equation (83) can be simplified as $\phi^{(1)} = a \sech \frac{1}{2} X$, where the amplitude $a$ of the solitary wave solution of (83) is a decreasing function of time and its functional form is given by Eq. (105).

For the first time, we have found the solitary wave solution of FMKdV-like evolution equation and we have seen that the amplitude of solitary wave solution of FMKdV-like evolution equation is proportional to $(1 + \tau)^{-\frac{1}{2}}$, where $T_3$ is given by Eq. (106) for $r = 3$.

For $r = 1$, the amplitude $a$ of the KdV soliton is plotted against $\tau$ in Fig. 7 for $\gamma = 3, \sigma = 0.001, \sigma_{sc} = 0.25$ and $n_{sc} = 0.3$ and for different values of $\beta_e$. Here, red, black and blue curves correspond to $\beta_e = 0, \beta_e = 0.4$ and $\beta_e = 0.57$ respectively. From this figure, we see that the amplitude $a$ of the KdV soliton increases with increasing $\beta_e$ for any fixed $\tau$. This figure also shows that the amplitude decreases with time.

For $r = 2$, the amplitude $a$ of the MKdV soliton is plotted against $\tau$ in Fig. 8 when $B_1 = 0$ for $\gamma = 3, \sigma = 0.001$ and $\sigma_{sc} = 0.25$, and for different values of $\beta_e$. Here,
The amplitude of the FMKdV soliton decreases with increasing time for any fixed value of $\beta_e$ (colour figure online).

For $r = 3$, the amplitude $a$ of the FMKdV soliton is plotted against $\tau$ in Fig. 9 when $B_1 = B_2 = 0$ for $\gamma = 3$ and $\sigma = 0.001$, and for different values of $\beta_c$. Red, black and blue curves correspond to $\beta_c = 0$, $\beta_c = 0.352$ and $\beta_c = 0.42$ respectively. This figure shows that the amplitude decreases with time.

Therefore, from Figs. 7, 8 and 9, we can conclude that the amplitude of the IA soliton decreases with time for all $r = 1, 2, 3$ if the effect of linear Landau damping of electrons is taken into account.

Finally, it is important to note that if we neglect the effect of linear Landau damping of electrons, then Eqs. (1)–(7) reduce to a full set of hydrodynamic equations and simultaneously the nonlinear evolution equation (83) reduces to KdV and different modified KdV equation for different values of $r = 1, 2$ and 3. These equations can describe the small amplitude solitary wave solutions under different circumstances of the present plasma system, viz., the nonlinear evolution equation is a KdV-like equation if $B_1 \neq 0$ or a modified KdV-like equation if $B_1 = 0$ but $B_2 \neq 0$ or a further modified KdV-like equation if $B_1 = B_2 = 0$ but $B_3 \neq 0$. In fact, here Vlasov–Poisson model of electron species depends on the inertia of electrons, i.e. if we neglect the inertia of electrons, then the system of equations reduces to a system of hydrodynamic equations and all the usual nonlinear evolution equations can be obtained from Eq. (83) by neglecting the effect of linear Landau damping of electrons. Therefore, one can assume that the treatment made in this paper is physically consistent when we are going to consider the effect of linear Landau damping of electrons on IA solitary waves. In fact, VanDam and Taniuti [34] clearly stated that Ott and Sudan [33] considered the electron Landau damping only, being based on an approximation in powers of mass ratio, related to the smallness of electron inertia. Hence, it cannot be applied to treat ion Landau damping. Furthermore, Meiss and Morrison [35] considered nonlinear electron Landau damping on IA solitons. They reported that the theory of Ott and Sudan [33] is valid for time much less than the electron bounce time, i.e. nonlinear effects are important for time greater than electron bounce time. It is also important to note that the last terms of left-hand side of Eqs. (32), (38) and (81) are all equal as these terms are responsible for the effect of linear Landau damping of electrons. But, of course, the more realistic physical situation is to consider nonlinear wave modulation along with nonlinear Landau damping.

Acknowledgements The authors are grateful to all reviewers for their constructive comments, without which this paper could not have been written in its present form. The authors are grateful to Prof. Basudev Ghosh, Department of Physics, Jadavpur University, for his helpful suggestions.

Appendix 1

Coefficients of Eq. (76):

$$J_{e0} = \int_{-\infty}^{+\infty} \frac{\partial j_{e0}}{\partial v_\|^2} dv_\|, \quad Z_{e0} = \left. \frac{\partial f_{e0}}{\partial v_\|^2} \right|_{v_\|=0},$$

where $J = F, G, H, K$ for $j = f, g, h, k$, respectively.

Appendix 2

Coefficients of Eq. (78):

$$J_{j0} = \int_{-\infty}^{+\infty} \frac{\partial j_{j0}}{\partial v_\|^2} dv_\|, \quad Z_{j0} = \left. \frac{\partial f_{j0}}{\partial v_\|^2} \right|_{v_\|=0},$$

where $J = F, G, H, K$ for $j = f, g, h, k$, respectively.
Appendix 3

Method of finding $I_r$ associated with Eqs. (103) and (104):

$$I_r = \mathcal{P} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left[ \text{sech} \frac{X'}{2} \right] \frac{\text{sech} \frac{X}{2}}{\partial X} \frac{dXdX'}{X - X'}. \tag{109}$$

Now $I_r$ can be written as

$$I_r = - \int_{-\infty}^{\infty} \left[ \text{sech} \frac{z}{2} \right] dI_r dz; \tag{110}$$

where $X = z'$, $X' = z$ and

$$I_{1r} = \mathcal{P} \int_{-\infty}^{+\infty} \left[ \text{sech} \frac{z}{2} \right] \frac{dz}{z - z'}. \tag{111}$$

Using the following known result

$$\int_{-\infty}^{0} e^{i(z-z')} ds = \pi \delta(z-z') - i\mathcal{P} \frac{1}{z-z'}, \tag{112}$$

form Eq. (112), we get

$$\mathcal{P} \frac{1}{z-z'} = \frac{1}{2i} \int_{-\infty}^{\infty} \frac{s}{s} e^{i(z-z')} ds. \tag{113}$$

Using (113), Eq. (111) can be written as

$$I_{1r} = \frac{1}{2i} \int_{-\infty}^{\infty} s F(s) e^{isz} ds, \tag{114}$$

where

$$F(s) = \int_{-\infty}^{\infty} \frac{s}{s} \text{sech} \frac{z}{2} e^{-isz} dz. \tag{115}$$

Therefore, Eq. (110) can be written as

$$I_r = \int_{0}^{\infty} s[F(s)]^2 ds. \tag{116}$$

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