POINCARÉ GAUGE THEORY FOR GRAVITATIONAL FORCES IN (1 + 1) DIMENSIONS *

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ABSTRACT

We discuss in detail how string-inspired lineal gravity can be formulated as a gauge theory based on the centrally extended Poincaré group in \((1 + 1)\) dimensions. Matter couplings are constructed in a gauge invariant fashion, both for point particles and Fermi fields. A covariant tensor notation is developed in which gauge invariance of the formalism is manifest.
I. INTRODUCTION

In this article we elaborate on our two recent Letters\textsuperscript{1,2} which deal with \((1 + 1)\)-dimensional gravity theory. In the first of these, we gave a gauge theoretical formulation for the “string-inspired” model of “dilaton” gravity\textsuperscript{3,4} complementing an analogous discussions based on the de Sitter group for the constant curvature model\textsuperscript{5} that had been introduced earlier. The gauge group for the “string-inspired” model is the Poincaré group\textsuperscript{4,6} but surprisingly a central extension in the algebra is needed for a manifestly invariant description of the cosmological constant\textsuperscript{1,7}. The second Letter was devoted to a gravity-matter interaction that modifies in a fashion specific to \((1 + 1)\) dimensions the usual geodesic equation of motion for point particles without spoiling general covariance. Also we showed that this additional interaction fits neatly into the gauge theoretical formulation based on the extended Poincaré group, the extension being responsible for the new interaction.

Here we give a fuller account of these matters. We begin in Section II by determining how the geodesic equation can be altered while still preserving general covariance. Possible additions describe novel gravitational forces on point particles and novel gravitational interactions for fields. We examine in detail the modifications for point particles and for Fermi fields. We also compute the functional determinant for massless fermions interacting with gravity, when the conventional interaction is supplemented by our additions.

The above development is presented in the geometric formalism, using the metric tensor and/or the Zweibein. We then rederive the equations in a gauge theoretical formalism based on the extended Poincaré group. To this end, general properties of the extended Poincaré group are summarized in Section III, while Section IV is a brief reprise of Section II, but now in group theoretical language, and it is shown that the novel interactions have a natural
setting here. Our approach to the construction of matter interactions that are Poincaré gauge invariant follows Grignani and Nardelli, but we also present a manifestly covariant tensor formalism.

The discussion of matter dynamics in Sections II and IV is carried out without specifying the gravitational action. In Section V, the gravity sector of the theory is described in terms of connections for the extended Poincaré gauge group. Concluding remarks comprise the last Section VI.

Our conventions are the following. Velocity of light is scaled to unity. Lower case Greek and Latin letters refer respectively to space-time and tangent space components. The former are raised and lowered by the metric tensor $g_{\mu\nu}$; the latter are moved by the Minkowski-space metric $h_{ab} = \text{diag}(1, -1)$. The Zweibein $e_\mu^a$ is related to $g_{\mu\nu}$ by

$$g_{\mu\nu} = e_\mu^a e_\nu^b h_{ab} \quad (1.1)$$

Also we use the anti-symmetric tangent space tensor $\epsilon^{ab}$, normalized by $\epsilon^{01} = 1 = -\epsilon_{01}$, as well as the numerical tensor densities $\epsilon^{\mu\nu}$ and $\epsilon_{\mu\nu}$, where $\epsilon^{\mu\nu}/\sqrt{-g}$ is a contravariant anti-symmetric tensor and $\sqrt{-g} \epsilon_{\mu\nu}$ is covariant, with

$$g \equiv \det g_{\mu\nu} \quad (1.2)$$

$$\sqrt{-g} = \det e_\mu^a = -\frac{1}{2} e_\mu^a e_\nu^b \epsilon^{\mu\nu} \epsilon_{ab}$$

The inverse $E_\alpha^\mu$ of the Zweibein is given by

$$E_\alpha^\mu = -\frac{1}{\sqrt{-g}} \epsilon^{\mu\nu} \epsilon_{ab} e_\mu^a e_\nu^b \quad (1.3)$$

We shall need the spin-connection $\omega_\mu$, either as an independent variable or determined by the Zweibein.

$$\omega_\mu = \frac{1}{\sqrt{-g}} e_\alpha^a h_{ab} \partial_\alpha e_\beta^b \quad (1.4)$$
This follows from the torsion-free condition.

\[
\epsilon^{\mu\nu} \left( \partial_\mu e_\nu^a + \epsilon^a_{\ b} \omega_\mu e_\nu^b \right) = 0 \quad (1.5)
\]

We introduce the Christoffel connection \( \Gamma^\alpha_{\mu\nu} \) by

\[
\partial_\mu e_\nu^a + \epsilon^a_{\ b} \omega_\mu e_\nu^b - \Gamma^\alpha_{\mu\nu} e_\alpha^a = 0 \quad (1.6)
\]

and Eq. (1.5) insures that \( \Gamma^\alpha_{\mu\nu} \) is given by the usual formula.

\[
\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left( \partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu} \right) \quad (1.7)
\]

The scalar curvature \( R \) is constructed from the spin connection.

\[
\partial_\mu \omega_\nu - \partial_\nu \omega_\mu = -\frac{1}{2} \sqrt{-g} \epsilon_{\mu\nu} R \quad (1.8)
\]

When dealing with a two-component Fermi field \( \psi \), we use \( 2 \times 2 \) Dirac matrices \( \gamma_a \) satisfying

\[
\{ \gamma_a, \gamma_b \} = 2 \hbar_{ab} \quad (1.9)
\]

Also defined is the Dirac–Hermitian chiral matrix \( \gamma_5 \),

\[
\gamma_5 = i \gamma_0 \gamma_1 \quad , \quad \gamma_5^\dagger = -\gamma_5 \quad (1.10)
\]

which dualizes the gamma matrices.

\[
\gamma_a \gamma_5 = i \epsilon_{ab} \gamma^b \quad (1.11)
\]

A tangent space Lorentz transformation,

\[
\Lambda_a^b \equiv \delta_a^b \cosh \alpha + e_a^b \sinh \alpha \quad (1.12)
\]
where the rapidity $\alpha$ can be a function of the space-time point $x^\mu$ when the transformation is local, acts on tangent space indices. Explicitly,

$$e^a_\mu \rightarrow (\Lambda^{-1})^a_b e^b_\mu$$

(1.13)

the spin connection transforms as a gauge potential,

$$\omega_\mu \rightarrow \omega_\mu + \partial_\mu \alpha$$

(1.14)

and a Fermi field transforms by

$$\psi \rightarrow e^{\frac{i}{2}\gamma_5 \alpha} \psi$$

(1.15)

so that a derivative supplemented by the spin connection transforms covariantly.

$$\left( \partial_\mu - \frac{i}{2} \omega_\mu \gamma_5 \right) \psi \rightarrow e^{\frac{i}{2}\gamma_5 \alpha} \left( \partial_\mu - \frac{i}{2} \omega_\mu \gamma_5 \right) \psi$$

(1.16)

II. MATTER-GRAVITY INTERACTIONS

A. Point Particle

The usual action for a material point particle of mass $m$, moving on the world line $x^\mu(\tau)$, is constructed from the arc length.

$$I_m = -m \int ds = -m \int d\tau \sqrt{\dot{x}^\mu(x(\tau))g_{\mu\nu}(x(\tau))\dot{x}^\nu(\tau)}$$

(2.A1)

The overdot denotes differentiation with respect to $\tau$, which parametrizes the world line in an arbitrary way — $I_m$ is parametrization invariant. However, the $(1 + 1)$-dimensional setting provides additional, dimension specific possibilities for matter-gravity interactions.

Let us observe that a force $F^\mu$ added to the geodesic equation of motion

$$g^{\mu\nu}(x(\tau)) \frac{\delta}{\delta x^\nu(\tau)} I_m = \frac{d}{d\tau} \frac{1}{N} \dot{x}^\mu + \frac{1}{N} \dot{x}^\alpha \Gamma^\mu_{\alpha\beta} \dot{x}^\beta = F^\mu$$

(2.A2)

$$N \equiv \frac{1}{m} \sqrt{\dot{x}^\alpha g_{\alpha\beta} \dot{x}^\beta}$$
must satisfy various consistency conditions. First, to maintain parametrization invariance, \( F^\mu \) must be linear in \( \dot{x}^\mu \). Also the transversality of the left side in (2.A2) to \( \dot{x}^\mu \) enforces that condition on \( F_\mu \).

\[
\dot{x}^\mu F_\mu = 0 \tag{2.A3}
\]

An option we do not take is \( F^\mu \propto \dot{x}^\mu \), for then (2.A3) could only be true for massless particles, and also the force would be dissipative. We are thus led to the formula

\[
F_\mu = F_{\mu\nu} \dot{x}^\nu \tag{2.A4}
\]

where \( F_{\mu\nu} \) is anti-symmetric. It is clear that an externally prescribed \( F_{\mu\nu} \) will result in loss of general covariance. In order to maintain that principle, \( F_{\mu\nu} \) must be constructed from the dynamical variables of the theory, and experience with electromagnetism (which is not included in the above discussion) shows that general covariance is preserved when \( F_{\mu\nu} \) is a second-rank, anti-symmetric tensor. In dimensions greater than two, such a tensor cannot be constructed from particle and/or gravitational variables; it arises when electromagnetic (or other gauge-field) degrees of freedom are dynamically active — but here we do not include such additional variables. However, in two dimensions, gravitational variables allow constructing the required tensor,

\[
F_{\mu\nu} = \sqrt{-g} \epsilon_{\mu\nu} F \tag{2.A5a}
\]

where \( F \) is a scalar, which evidently can be a constant or built from the scalar curvature. We choose the two simplest contributions to \( F \), which will be seen to fit very naturally into our gauge theoretical formulation.

\[
F = -\frac{A}{2} R - B \tag{2.A5b}
\]
Here $\mathcal{A}$ and $\mathcal{B}$ are constants setting the strength of the addition. The covariant, two-dimensional equation of motion, involving matter and metric variables and generalizing the usual geodesic equation reads

$$\frac{d}{d\tau} \frac{1}{N} \dot{x}^\mu + \frac{1}{N} \dot{x}^\alpha \Gamma^\mu_{\alpha\beta} \dot{x}^\beta + \left( \frac{1}{2} \mathcal{A} R + \mathcal{B} \right) g^{\mu\alpha} \sqrt{-g} \epsilon_{\alpha\beta} \dot{x}^\beta = 0 \quad (2.A6)$$

The first addition, involving $\mathcal{A} R$, is non-minimal, vanishing in the absence of curvature. It will be seen that this term produces interaction with curvature familiar from conformal improvements of dynamics. The second addition\(^2\) to the interaction (2.A6) is similar to a constant external electromagnetic field in flat two-dimensional Minkowski space-time and reduces to that in the absence of curvature. In the flat limit, that interaction preserves the Poincaré symmetry of a non-interacting point particle and similarly the covariantly constant field $\sqrt{-g} \epsilon_{\mu\nu} \mathcal{B}$ respects general covariance. Both terms arise naturally in a gauge theoretical formulation; this will be shown in Section IV.

The additional forces can be derived from additional contributions to the matter gravity action, containing suitable vector potentials contracted with $\dot{x}^\mu$, as is clear from the electromagnetic analogy. The action for the force proportional to curvature evidently involves the spin connection, see (1.8). Construction of the action for the covariantly constant force is geometrically subtle.\(^2\) Consider the volume two-form, $vol \equiv -\frac{1}{2} \sqrt{-g} \epsilon_{\mu\nu} \, dx^\mu \, dx^\nu$, which may also be expressed in terms of the Zweibein as $-\frac{1}{2} \epsilon_{ab} \epsilon^a_{\mu} \epsilon^b_{\nu} \, dx^\mu \, dx^\nu$. Since $vol$ is closed, $d(vol) = 0$, it is locally exact.

$$vol = da \quad (2.A7a)$$

Equation (2.A7a) defines a one-form whose components are also seen to satisfy

$$\partial_{\mu} a_{\nu} - \partial_{\nu} a_{\mu} + \sqrt{-g} \epsilon_{\mu\nu} = 0 \quad (2.A7b)$$
Since the right-hand side of (2.A7b) is a tensor, \(a_\mu\) can be taken as a vector. The covariant action, whose variation leads to the additional force in (2.A6), is now constructed from \(\omega_\mu\) and \(a_\mu\).

\[
I_\mathcal{F} = - \int d\tau \dot{x}^\mu(\tau) \left( \mathcal{A} \omega_\mu(x(\tau)) + \mathcal{B} a_\mu(x(\tau)) \right) \tag{2.A8}
\]

Under coordinate redefinition, \(\omega_\mu\), \(a_\mu\) and \(x^\mu(\tau)\) change, and it is straightforwardly verified that \(I_\mathcal{F}\) is a scalar. Under a tangent space Lorentz transformation (1.12), \(\omega_\mu \rightarrow \omega_\mu + \partial_\mu \alpha\), see (1.14), and \(a_\mu\) is unchanged; however, the defining equations (2.A7) leave a gauge ambiguity.

\[
a_\mu \rightarrow a_\mu + \partial_\mu \beta \tag{2.A9}
\]

But it is seen that (2.A8) changes only by end-point contributions under the gauge transformations (1.14), (2.A9), and the equation of motion (2.A6) is gauge invariant. Also it is local, even though local expressions for \(a_\mu\) and \(I_\mathcal{F}\) are not available.

Next we compute the covariantly conserved energy-momentum tensor \(T^{\alpha\beta}\), which is a functional of the matter variable \(x(\tau)\) and a function of the field argument \(x\).

\[
T^{\alpha\beta}(x(\tau)|x) = -\frac{2}{\sqrt{-g(x)}} \frac{\delta}{\delta g_{\alpha\beta}(x)} (I_m + I_\mathcal{F}) \tag{2.A10}
\]

The variation of \(I_m\) produces the conventional free-particle energy-momentum tensor.

\[
-\frac{2}{\sqrt{-g(x)}} \frac{\delta}{\delta g_{\alpha\beta}(x)} I_m = \frac{1}{\sqrt{-g(x)}} \int \frac{d\tau}{N(\tau)} \dot{x}^\alpha(\tau)x^\beta(\tau) \delta^2(x - x(\tau)) \tag{2.A11}
\]

When varying \(\omega_\mu\) in \(I_\mathcal{F}\) with respect to \(g_{\alpha\beta}\), we use (1.4) for the Zweibein-dependence of \(\omega_\mu\), and (1.1) for the relation between the Zweibein and the metric tensor. But evaluating the metric variation of \(a_\mu\), which is also present in \(I_\mathcal{F}\), is problematic in the absence of an explicit
formula for $a_\mu$. However, we can fix this contribution to the energy-momentum tensor by requiring its covariant conservation. In this way $T^{\alpha\beta}$ is found to be

$$T^{\alpha\beta}(x(\tau)|x) = \frac{1}{\sqrt{-g(x)}} \int \frac{d\tau}{N(\tau)} \dot{x}^\alpha(\tau) \dot{x}^\beta(\tau) \delta^2 \left( x - x(\tau) \right)$$

$$+ \frac{A}{2\sqrt{-g(x)}} \left( \epsilon^{\alpha\gamma} D_\gamma j^\beta(x(\tau)|x) + \epsilon^{\beta\gamma} D_\gamma j^\alpha(x(\tau)|x) \right) - \frac{1}{2} \lambda \left( x(\tau)|x \right) g^{\alpha\beta}(x)$$

(2.A12)

Here $D_\gamma$ is the space-time covariant derivative, while $j^\mu$ is the current,

$$j^\mu(x(\tau)|x) = \frac{1}{\sqrt{-g(x)}} \int d\tau \dot{x}^\mu(\tau) \delta^2 \left( x - x(\tau) \right)$$

(2.A13)

which is covariantly conserved.

$$D_\mu j^\mu = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} j^\mu = 0$$

(2.A14)

The last contribution to (2.A12) comes from varying $-B \int d\tau \dot{x}^\mu a_\mu$. In view of (2.A6), $T^{\alpha\beta}$ will be covariantly conserved

$$D_\alpha T^{\alpha\beta} = 0$$

(2.A15)

if $\lambda$ satisfies

$$\frac{1}{2} \frac{\partial \lambda}{\partial x^\mu}(x(\tau)|x) = -B \sqrt{-g(x)} \epsilon_{\mu\nu} j^\nu(x(\tau)|x)$$

(2.A16)

Equation (2.A16) is solved by

$$\lambda(x(\tau)|x) = -2B \int_x^y dy^\mu \sqrt{-g(y)} \epsilon_{\mu\nu} j^\nu(x(\tau)|y)$$

(2.A17)

Because $j^\mu$ is covariantly conserved the expression (2.A17) for $\lambda$ depends locally on $x$, i.e. it is path-independent. This is established by evaluating the line integral over the closed
contour that describes the possible difference between two different evaluations of \( \lambda \) along two different paths. Alternatively in view of (2.A14), we may write,

\[ j^\mu = \frac{1}{\sqrt{\pi}} \frac{\epsilon^{\mu\nu}}{\sqrt{-g}} \partial_\nu \varphi \]  \hspace{1cm} (2.18)

and then (2.A17) shows that \( \lambda = -\frac{2}{\sqrt{\pi}} B \varphi \). When the current has the explicit form (2.A13), the expressions (2.A16), (2.A17) or (2.A18) may be easily evaluated in the parametrization \( x^0(\tau) = \tau \), and one finds

\[ \lambda = -B \varepsilon \left( x^1 - x^1(t) \right) + \lambda_0 = -\frac{2}{\sqrt{\pi}} B \varphi \]  \hspace{1cm} (2.19)

where \( t \equiv x^0 \) and \( \lambda_0 \) is constant. This is a cosmological “constant” that jumps by \( 2B \) as the particle’s trajectory is crossed, a property that is independent of the above parametrization choice.

The final formula for the covariantly conserved energy-momentum tensor that follows from (2.A12), (2.A18) and (2.A19) is

\[ T^{\alpha\beta}(x(\tau)|x) = \frac{1}{\sqrt{-g(x)}} \int d\tau \frac{\dot{x}^\alpha(\tau)\dot{x}^\beta(\tau)}{N(\tau)} \delta^2(x - x(\tau)) \]

\[ + \frac{A}{\sqrt{\pi}} \left( D^\alpha D^\beta - g^{\alpha\beta} D_\mu D^\mu \right) \varphi(x(\tau)|x) + \frac{B}{\sqrt{\pi}} \varphi(x(\tau)|x) g^{\alpha\beta}(x) \]  \hspace{1cm} (2.20)

This is local in \( x \), just like the equation of motion.

The trace of the energy momentum tensor reads

\[ T^\alpha_{\alpha}(x(\tau)|x) = \frac{m^2}{\sqrt{-g(x)}} \int d\tau N(\tau) \delta^2(x - x(\tau)) \]

\[ - \frac{A}{\sqrt{\pi}} D_\alpha D^\alpha \varphi(x(\tau)|x) + \frac{2B}{\sqrt{\pi}} \varphi(x(\tau)|x) \]  \hspace{1cm} (2.21)

In conformally flat coordinates \( g_{\alpha\beta} = e^{\sigma} h_{\alpha\beta} \) and in the parametrization \( x^0(\tau) = \tau \),

\[ \frac{1}{\sqrt{\pi}} D_\alpha D^\alpha \varphi = \frac{1}{2} e^{-\sigma} \Box \varepsilon \left( x^1 - x^1(t) \right) \]

\[ = -e^{-\sigma} \left\{ a \delta \left( x^1 - x^1(t) \right) + (1 - v^2) \delta \left( x^1 - x^1(t) \right) \right\} \]  \hspace{1cm} (2.22)
where \( v = \dot{x}^1(t) \), \( a = \ddot{x}^1(t) \). For massless particles on a line, \( v^2 = 1 \), \( a = 0 \) and \( \frac{1}{\sqrt{\pi}} D_{\alpha} D^{\alpha} \varphi \) vanishes, leaving in the trace only the cosmological term.

\[
T_{\alpha}^\alpha \left( x(\tau) \right) \bigg|_{m=0} = -\lambda = \frac{2}{\sqrt{\pi}} B \varphi = B \varepsilon \left( x^1 - x^1(t) \right) - \lambda_0 \quad (2.A23)
\]

Note that the axial vector current \( j_5^\mu \), dual to the vector current,

\[
j_5^\mu = \frac{1}{\sqrt{-g}} \epsilon^{\mu \nu} j_\nu = g^{\mu \nu} \frac{1}{\sqrt{\pi}} \partial_\nu \varphi \quad (2.A24a)
\]

occurs naturally in several of the above formulas, for example (2.A17). In the massless case, it also is conserved,

\[
D_{\mu} j_5^\mu = \frac{1}{\sqrt{\pi}} D_{\mu} D^{\mu} \varphi = 0 \quad (2.A24b)
\]

as is seen from (2.A22).

For later use, we observe that the total particle action may also be presented in first-order form.

\[
I_{\text{particle}} = \int d\tau \left( p_a \epsilon_{\mu}^{\alpha} \dot{x}^\alpha + \frac{1}{2} N \left( p^2 - m^2 \right) \right) + I_F \quad (2.A25)
\]

Upon varying and eliminating \( p_a = -\epsilon_{\mu}^{\alpha} \dot{x}^\alpha / N \) and \( N = \sqrt{\dot{x}^\mu g_{\mu \nu} \dot{x}^\nu} / m \), (2.A1) and (2.A8) are regained.

In the absence of gravity, \( \epsilon_{\mu}^{\alpha} = \delta_{\mu}^{\alpha} \), \( \omega_{\mu} = 0 \), \( a_{\mu} = \frac{1}{2} \epsilon_{\mu \nu} x^\nu \) and (2.A25) reduces to

\[
I_{\text{particle}}^{\text{flat}} = \int d\tau \left( \left( p_a - \frac{1}{2} B \epsilon_{ab} x^b \right) \dot{x}^a + \frac{1}{2} N \left( p^2 - m^2 \right) \right) \quad (2.A26)
\]

so that \( p_a - \frac{B}{2} \epsilon_{ab} x^b \) and \( x^a \) are canonically conjugate. Moreover, under space-time translations \( \delta x^a = x_0^a \), \( \delta p_a = 0 \), \( \delta N = 0 \), \( I_{\text{particle}}^{\text{flat}} \) changes by end-point contributions and the conserved energy and momentum

\[
P_a = p_a - B \epsilon_{ab} x^b \quad (2.A27)
\]
possess non-vanishing bracket.

\[ [P_a, P_b] = \mathcal{B} \epsilon_{ab} \quad (2.28) \]

Infinitesimal space-time rotations \( \delta x^a = \alpha \epsilon^a_b x^b \), \( \delta p_a = \alpha \epsilon_a^b p_b \), \( \delta N = 0 \) leave \( T_{\text{particle}}^{\text{flat}} \) invariant and lead to the Lorentz generator

\[ J = x^a \epsilon_a^b P_b - \frac{1}{2} \mathcal{B} x^2 = x^a \epsilon_a^b P_b + \frac{1}{2} \mathcal{B} x^2 \quad (2.29) \]

with bracket

\[ [P_a, J] = \epsilon_a^b P_b \quad (2.30) \]

In flat space-time, the energy-momentum tensor (2.20) becomes

\[
T_{\text{flat}}^{\alpha\beta}(x(\tau)|x) = m \int d\tau \frac{\dot{x}^\alpha(\tau)\dot{x}^\beta(\tau)}{\sqrt{\dot{x}^2(\tau)}} \delta^2(x - x(\tau)) \\
+ \frac{A}{\sqrt{\pi}} (\partial^\alpha \partial^\beta - h^{\alpha\beta} \Box) \varphi(x(\tau)|x) + \frac{B}{\sqrt{\pi}} \varphi(x(\tau)|x) h^{\alpha\beta} \quad (2.31)
\]

The next-to-last term corresponds to the curvature-dependent force; even in the absence of curvature it contributes to the energy-momentum tensor an "improvement" term familiar from conformally invariant coupling. Observe further that \( \int_{-\infty}^{\infty} dx^1 T_{\text{flat}}^{00} \) coincides with the energy obtained from (2.27), apart from an infinite constant proportional to \( \lambda \). However, the spatial integral of \( T_{\text{flat}}^{01} \) is not the momentum; \( \int_{-\infty}^{\infty} dx^1 T_{\text{flat}}^{01} \) is not time-independent even though \( T_{\text{flat}}^{\alpha\beta} \) is conserved. This is because \( T_{\text{flat}}^{11} \) remains non-vanishing at \( x^1 = \pm \infty \) owing to the last term in (2.31). To achieve a time-independent quantity and to obtain the correct momentum from the energy-momentum tensor, one must add \( t \int_{-\infty}^{\infty} dx^1 \frac{d}{dx^1} T_{\text{flat}}^{11} \) to \( \int_{-\infty}^{\infty} dx^1 T_{\text{flat}}^{01} \). Similarly, construction of the Lorentz generator from the energy-momentum tensor, begins with \( t \int_{-\infty}^{\infty} dx^1 T_{\text{flat}}^{01} - \int_{-\infty}^{\infty} dx^1 x^1 T_{\text{flat}}^{00} \). As with the momentum, this is not time-independent, even though \( T_{\text{flat}}^{\alpha\beta} \) is conserved and symmetric, owing to slow drop-off of \( T_{\text{flat}}^{11} \). To
remedy this one adds \( \frac{1}{2} t^2 \int_{-\infty}^{\infty} \frac{d}{dx} T_{11} \), and \( J \) of (2.A29) is then reproduced, apart from the finite constant \( A \) coming from the improvement and an infinite constant coming from the last term in (2.A31).

In the subsequent development it becomes convenient to replace \( p_a \) by \( \epsilon_a \epsilon_b \), whereupon (2.A25) is replaced by

\[
I_{\text{particle}} = \int d\tau \left( \frac{1}{2} \int_0^x \frac{d}{dx} T_{11} - \frac{1}{2} N (p^2 + m^2) - (A \omega_\mu + B a_\mu) \dot{x}^\mu \right)
\]  

(2.A32)

### II.B Fermi Fields

The equation for a Fermi field with mass \( m \) propagating in an external gravitational field

\[
\psi = \left( \partial_\mu - i \frac{\omega_\mu \gamma_5}{2} \right) \psi - m \psi = 0
\]

(2.B1)

\( \gamma_\mu \equiv E^a_a \gamma_\mu = \frac{1}{\sqrt{-g}} \epsilon_{\mu \nu \sigma} \epsilon^{\sigma}_{a b} \gamma^b \)

can be obtained by varying with respect to \( \bar{\psi} \) the following Lagrange density.

\[
\mathcal{L} = \sqrt{-g} \left( \frac{i}{2} \bar{\psi} \gamma_\mu \dot{\psi} - \frac{i}{2} \bar{\psi} \gamma_\mu \gamma_\nu \psi - m \bar{\psi} \psi \right)
\]

(2.B2)

[The spin connection, in the form (1.4), enters the equation of motion (2.B1) when upon variation of \( \bar{\psi} \) in \( \mathcal{L} \) the derivative is moved from \( \bar{\psi} \) to \( \sqrt{-g} \gamma_\mu \); see (1.4).] Therefore the fermion action \( I_{\text{Fermi}} \), including interaction with the covariantly constant and curvature dependent forces, is

\[
I_{\text{Fermi}} = \int d^2 x \sqrt{-g} \left\{ \frac{i}{2} \bar{\psi} \gamma_\mu \left( \partial_\mu + i (A \omega_\mu + B a_\mu) \right) \psi - \frac{i}{2} \bar{\psi} \left( \partial_\mu - i (A \omega_\mu + B a_\mu) \right) \gamma_\mu \psi - m \bar{\psi} \psi \right\}
\]

(2.B3)

and the equation of motion reads

\[
i \gamma_\mu \left( \partial_\mu - i \frac{1}{2} \omega_\mu \gamma_5 + i (A \omega_\mu + B a_\mu) \right) \psi - m \psi = 0
\]

(2.B4)
Evidently under a gauge transformations (1.14) and (2.A9) of $\omega_\mu$ and $a_\mu$, $\psi$ must now transform as

$$\psi \rightarrow e^{\frac{i}{2} \alpha \gamma_5 - i(A\alpha + B\beta)} \psi$$  \hspace{1cm} (2.B5)

to maintain gauge invariance, compare (1.15).

The energy-momentum tensor is found by first computing the Lorentz tensor.

$$T^\alpha_a = -\frac{1}{\sqrt{-g}} \frac{\delta I_{Fermi}}{\delta e^a_\alpha}$$  \hspace{1cm} (2.B6)

Then

$$T^{\alpha\beta} = \frac{1}{2} \left( T^\alpha_a E^\beta_b + T^\beta_a E^\alpha_b \right) h^{ab}$$  \hspace{1cm} (2.B7)

Carrying out the indicated variation gives in complete analogy with (2.A12),

$$T^{\alpha\beta} = \frac{i}{4} \bar{\psi} \left( \gamma^\alpha g^{\beta\mu} + \gamma^\beta g^{\alpha\mu} \right) \left( \partial_\mu + i (A\omega_\mu + B a_\mu) \right) \psi$$

$$- \frac{i}{4} \bar{\psi} \left( \partial_\mu - i (A\omega_\mu + B a_\mu) \right) \left( \gamma^\alpha g^{\beta\mu} + \gamma^\beta g^{\alpha\mu} \right) \psi$$

$$+ \frac{A}{2} \left( \frac{1}{\sqrt{-g}} \epsilon^{\alpha\gamma} D_\gamma j^\beta + \frac{1}{\sqrt{-g}} \epsilon^{\beta\gamma} D_\gamma j^\alpha \right) - \frac{1}{2} \lambda g^{\alpha\beta}$$  \hspace{1cm} (2.B8)

where the covariantly conserved current is now

$$j^\mu = \bar{\psi} \gamma^\mu \psi$$  \hspace{1cm} (2.B9)

and $\lambda$ must again satisfy

$$\frac{1}{2} \frac{\partial \lambda}{\partial x^\mu} = -B \sqrt{-g} \epsilon_{\mu\nu\lambda} j^\nu$$  \hspace{1cm} (2.B10)

so that $T^{\alpha\beta}$ be conserved. The solution

$$\lambda = -2B \int^x dy^\mu \sqrt{-g} \epsilon_{\mu\nu\lambda} j^\nu$$  \hspace{1cm} (2.B11)

remains path-independent thanks to the covariant conservation of $j^\mu$, but the integral cannot be further evaluated. However, if we use (2.A18), which here is viewed as a bosonization
formula, \( \lambda \) is again proportional to \( \varphi \) and the energy-momentum tensor is as in (2.A20), except the kinetic term is now constructed from the Fermi fields.

The trace of the energy momentum tensor reads

\[
T^\alpha_\alpha = m\bar{\psi}\psi - \mathcal{A} D_\mu j_5^\mu - \lambda
\]  

(2.B12)

where the axial current, dual to the vector current (2.B9),

\[
j_5^\mu = \bar{\psi}i\gamma_5\gamma^\mu\psi
\]  

(2.B13)

possesses the covariant divergence \( 2m\bar{\psi}\gamma_5\psi \), so that

\[
T^\alpha_\alpha = m\bar{\psi}(1 - 2\mathcal{A}\gamma_5)\psi - \lambda
\]  

(2.B14)

In the massless case one is simply left with \( T^\alpha_\alpha = -\lambda \). Of course the above does not include the trace and chiral anomalies which are quantum effects, see below.

In the massless case we can evaluate the fermion determinant by making use of known results, which are given in Riemannian space. Adjustments in our formulas necessitated by a positive signature metric are: \( h_{ab} \) is now diag (1, 1), while the Zweibein, spin-connection and curvature are related by

\[
\omega_\mu = -\frac{1}{\sqrt{g}}\epsilon^{\alpha\beta}\epsilon^{a}_\mu h_{ab}\partial_\alpha e^{b}_\beta
\]  

(1.14’)

\[
\partial_\mu \omega_\nu - \partial_\nu \omega_\mu = \frac{1}{2}\sqrt{g}\epsilon_{\mu\nu} R
\]  

(1.8’)

The \( \gamma \) matrices fulfill the Euclidean Clifford algebra.

According to Ref. [8], the determinant of the massless, two-dimensional Dirac operator in curved space and in the presence of an external Abelian gauge field \( A_\mu \) is

\[
W(E, A) \equiv \ln \det \left[ i\gamma^a E^\mu_a \left( \partial_\mu - \frac{i}{2}\omega_\mu \gamma_5 + iA_\mu \right) \right]
\]  

(2.B15)

\[
= -\frac{1}{12} W_g(\omega) + W_g(A)
\]
The functional $W_g(v)$ of the vector field $v_\mu$ is defined in terms of the inverse Laplacian $\nabla_g^{-2}$,

$$g^{\mu\nu}(x) D_\mu D_\nu \nabla_g^{-2}(x, y) = \frac{1}{\sqrt{g}} \delta^2(x - y)$$  \hspace{1cm} (2.B16)

$$\nabla_g^{-2}(x, y) = \nabla_g^{-2}(y, x)$$

and is given by

$$W_g(v) = \frac{1}{2\pi} \int d^2 x \, d^2 y \, \epsilon^{\mu\nu} \partial_\mu v_\nu(x) \nabla_g^{-2}(x, y) \epsilon^{\alpha\beta} \partial_\alpha v_\beta(y)$$  \hspace{1cm} (2.B17)

Local terms are adjusted in (2.B15) to ensure invariance against coordinate transformations, and also against Abelian gauge transformations of $\omega_\mu$ and $A_\mu$.

For us, $\epsilon^{\mu\nu} \partial_\mu \omega_\nu = \frac{1}{2} \sqrt{g} R$, and since $A_\mu = A \omega_\mu + B a_\mu$, $\epsilon^{\mu\nu} \partial_\mu A_\nu = \sqrt{g} (\frac{1}{2} R + B)$. Note that $B^2 \int d^2 x \sqrt{g(x)} d^2 y \sqrt{g(y)} \nabla_g^{-2}(x, y)$ may diverge, e.g. in flat space.

The vacuum expectation value of the energy-momentum tensor is obtained from $W(E, A)$ by varying with respect to gravitational variables. One finds

$$\langle T_{\alpha\beta} \rangle = -\frac{1}{48\pi} \left( \partial_\alpha \sigma \partial_\beta \sigma - \frac{1}{2} g_{\alpha\beta} \partial_\mu \sigma \partial^\mu \sigma \right) + \frac{1}{24\pi} \left( D_\alpha D_\beta - g_{\alpha\beta} D_\mu D^\mu \right) \sigma$$

$$- \left( \partial_\alpha \varphi \partial_\beta \varphi - \frac{1}{2} g_{\alpha\beta} \partial_\mu \varphi \partial^\mu \varphi \right) + \frac{A}{\sqrt{\pi}} \left( D_\alpha D_\beta - g_{\alpha\beta} D_\mu D^\mu \right) \varphi$$

$$- \frac{A}{2\sqrt{\pi}} \left( \partial_\alpha \sigma \partial_\beta \varphi + \partial_\beta \sigma \partial_\alpha \varphi - g_{\alpha\beta} \partial_\mu \sigma \partial^\mu \varphi \right) + \frac{B}{\sqrt{\pi}} \varphi g_{\alpha\beta}$$  \hspace{1cm} (2.B18)

In order to present $\langle T_{\alpha\beta} \rangle$ in local form we have introduced the fields

$$\sigma \equiv -\nabla_g^{-2} R$$  \hspace{1cm} (2.B19)

— a field that coincides with the conformal factor in conformal coordinates — and

$$\varphi \equiv \frac{1}{\sqrt{\pi}} \nabla_g^{-2} \frac{1}{\sqrt{g}} \epsilon^{\mu\nu} \partial_\mu A_\nu$$  \hspace{1cm} (2.B20)

This energy-momentum tensor is covariantly conserved but exhibits a trace anomaly.

$$\langle T^{\alpha\alpha} \rangle = \frac{1}{24\pi} \left( 1 - 12 A^2 \right) R - \frac{1}{\pi} A B + \frac{2}{\sqrt{\pi}} B \varphi$$  \hspace{1cm} (2.B21)
The expressions (2.B18) and (2.B21) are similar to (2.A20) and (2.A21). They can be written as (2.A12) or (2.B8), (2.B12) if we introduce the conserved current,

\[ \langle j^\mu \rangle = -\frac{1}{\sqrt{g}} \frac{\delta W_g}{\delta A_\mu} \]
\[ = -\frac{1}{\sqrt{\pi}} \frac{\epsilon^{\mu\nu}}{\sqrt{g}} \partial_\nu \varphi \]  

(2.B22)

and its dual, the axial current.

\[ \langle j_5^\mu \rangle = \frac{1}{\sqrt{g}} \epsilon^{\mu\nu} j_\nu = \frac{1}{\sqrt{\pi}} g^{\mu\nu} \partial_\nu \varphi \]  

(2.B23)

The latter has an anomalous divergence

\[ D_\mu \langle j_5^\mu \rangle = \frac{1}{\pi} \left( \frac{A}{2} R + B \right) \]  

(2.B24)

which appears explicitly in the trace equation (2.B21);

\[ \langle T^\alpha_\alpha \rangle = \frac{1}{24\pi} R - AD_\mu \langle j_5^\mu \rangle - \lambda \]  

(2.B25a)

\[ \frac{1}{2} \partial_\mu \lambda = -B \sqrt{g} \epsilon_{\mu\nu} \langle j^\nu \rangle \]  

(2.B25b)

compare to (2.B12), (2.B10).

We now return to Minkowski signature.

III. EXTENDED POINCARÉ GROUP

The somewhat haphazard introduction of the additional interactions becomes rationalized in a group theoretical description. We now discuss the extended Poincaré group that we employ.

The Poincaré group consists of two translations (parameters \( \theta^a \)) and single (Lorentz) rotation \( \Lambda \) [parameter \( \alpha \), see (1.12)] with composition law

\[ (\theta_1, \Lambda_1) \circ (\theta_2, \Lambda_2) = (\theta_1 + \Lambda_1 \theta_2, \Lambda_1 \Lambda_2) \]  

(3.1)
Thus $(\theta, \Lambda)$ corresponds to $e^{\theta^a P_a} e^{\alpha J}$, where generators $P_a$ effect translations and $J$ the infinitesimal rotation. We postulate that the generator Lie algebra follows (2.A28), (2.A30), i.e., there is an extension. Henceforth we scale $B$ to unity.

\[
[P_a, P_b] = \epsilon_{ab}\mathcal{I}, \quad (3.2)
\]
\[
[P_a, J] = \epsilon^b_a P_b \quad (3.3)
\]

$\mathcal{I}$ is a central element, commuting with $P_a$ and $J$. Note that $J$ may be supplemented by an arbitrary multiple of $\mathcal{I}$. The algebra is solvable. [The algebra is similar to that of the harmonic oscillator group $Os(1)$, but in our realization we do not impose reality/Hermiticity requirements on the generators.]

The effect of the center is to modify the composition law for group elements, represented by exponentiated generators.

\[
(\theta_1, \Lambda_1) \circ (\theta_2, \Lambda_2) = e^{\frac{1}{2} \theta_1^a \epsilon_{ab} (\Lambda_1 \theta_2)^b \mathcal{I}} (\theta_1 + \Lambda_1 \theta_2, \Lambda_1 \Lambda_2) \quad (3.4)
\]

In order to obtain a faithful representation, we extend the Poincaré group with a $U(1)$ factor, generated by $\mathcal{I}$ and parameter $\beta$. Upon defining group elements

\[
U (\theta, \Lambda, \beta) = e^{\theta^a P_a} e^{\alpha J} e^{\beta \mathcal{I}} \quad (3.5)
\]

we verify that

\[
U (\theta_1, \Lambda_1, \beta_1) U (\theta_2, \Lambda_2, \beta_2) = U (\theta_3, \Lambda_3, \beta_3) \quad (3.6a)
\]

with

\[
\theta_3 = \theta_1 + \Lambda_1 \theta_2
\]
\[
\Lambda_3 = \Lambda_1 \Lambda_2 \quad (3.6b)
\]
\[
\beta_3 = \beta_1 + \beta_2 + \frac{1}{2} \theta_1^a \epsilon_{ab} (\Lambda_1 \theta_2)^b
\]
In this way one is led to a four-parameter group — the extended \((1+1)\)-dimensional Poincaré group.

In a covariant notation, we call the generators \(Q_A\), where \(A\) takes the four values \((a, 2, 3)\), \(Q_A = (P_a, J, I)\). The algebra is four-dimensional.

\[
[Q_A, Q_B] = f_{AB}^\ C Q_C \quad (3.7)
\]

Also we verify

\[
U^{-1}Q_AU = Q_B\left(U^{-1}\right)^B\_A \quad (3.8)
\]

with \(U\) the \(4 \times 4\) matrix

\[
U^A\_B = \begin{pmatrix}
\Lambda^a\_b & -\epsilon^a\_c \theta^c & 0 \\
0 & 1 & 0 \\
\theta^c \epsilon_{cd} \Lambda^d\_b & -\frac{1}{2} \theta^c \theta^c & 1
\end{pmatrix} \quad (3.9)
\]

A. Adjoint Representation

\(U\) gives the adjoint representation, where the generators are represented by \(Q_C\),

\[
(Q_C)^A\_B = f_{CB}^A \quad (3.10a)
\]

explicitly

\[
(P_c)^A\_B = \begin{pmatrix}
0 & -\epsilon^a\_c \\
0 & 0 \\
\epsilon_{cb} & 0
\end{pmatrix}, \quad (J)^A\_B = \begin{pmatrix}
\epsilon^a\_b & 0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}, \quad (I)^A\_B = 0 \quad (3.10b)
\]

It is also true that \(U^A\_B\) coincides with \(U\) of (3.5), with the exponential generators taken in the adjoint representation (3.10). [Our convention is that non-script letters denote group and algebra elements in a specific representation.]
In the usual way, we can define contravariant four-vectors $\xi^A$ that transform according to the adjoint transformation,

$$\xi^A \rightarrow (U^{-1})^A_B \xi^B \quad (3.11a)$$

or infinitesimally

$$\delta \xi^A = f_{BC}^A \xi^B \Theta^C \quad (3.11b)$$

where $\Theta^A = (\theta^a, \alpha, \beta)$. Similarly there are covariant four-vectors with coadjoint transformation.

$$\eta_A \rightarrow \eta_B U^B_A \quad (3.12a)$$

$$\delta \eta_A = -f_{AB}^C \Theta^B \eta_C \quad (3.12b)$$

Evidently $\eta_A \xi^A$ is a scalar invariant.

An invariant mixed tensor is provided by the structure constants.

$$f_{AB}^C \rightarrow (U^{-1})^C_{C'} f_{A'B'C'} U^{A'}_{A} U^{B'}_{B} = f_{AB}^C \quad (3.13)$$

Since the algebra is solvable, there also exist invariant four-vectors: upon defining $i^A = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ and $i_A = (0, 1, 0)$, $i_A i^A = 0$, we see that

$$\begin{pmatrix} (U^{-1})^A_B \\ i^A \end{pmatrix} = i_A \quad (3.14a)$$

$$i_B U^B_A = i_A \quad (3.14b)$$

One understands the existence of such invariant vectors because equivalent to (3.14) are the statements

$$i^A f_{AB}^C = 0 \quad (3.15a)$$

$$f_{AB}^C i_C = 0 \quad (3.15b)$$
The first reflects the commutativity of $J$ with all other generators, the second is true because $J$ is never attained by commuting two generators. These invariant vectors also determine the invariant Cartan–Killing metric,

$$\langle Q_A, Q_{A'} \rangle_{C-K} = f_{AB}^C f_{A'B}^B = 2i_A i_{A'}$$  \hspace{1cm} (3.16)

which is however singular, since the group is not semi-simple.

An invariant, non-singular bilinear form can be defined in the adjoint representation. It is given by the tensor

$$h_{AB} = \begin{pmatrix} h_{ab} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \equiv \langle Q_A, Q_B \rangle$$  \hspace{1cm} (3.17)

$h_{AB}$ is used to raise and lower indices, interchanging covariant and contravariant tensors, as for example

$$i^A = h_{AB} i^B$$  \hspace{1cm} (3.18)

This allows defining an invariant length,

$$\xi^A \xi_A = \xi^a \xi_a - 2 \xi_2 \xi_3$$  \hspace{1cm} (3.19)

and a bilinear Casimir invariant can be constructed from the generators.

$$C = Q^A Q_A = \mathcal{P}^a \mathcal{P}_a - 2IJ$$  \hspace{1cm} (3.20)

In the adjoint representation, the Casimir is determined by the Cartan–Killing metric.

$$C^A {}_B = 2i^A i_B$$  \hspace{1cm} (3.21)

When the upper index on the structure constant is lowered,

$$f_{ABC} = f_{AB}^{C'} h_{C'C}$$ \hspace{1cm} (3.22a)
the resulting covariant tensor is antisymmetric in all three indices and vanishes when any
index equals three. The only non-vanishing components are permutations of

\[ f_{ab2} = -\epsilon_{ab} \]  

(3.22b)

and it is true that

\[ f_{ABC} f_{A'B'C'} = -\frac{1}{2} i_A i_{A'} h_{BB'} h_{CC'} + \text{permutations} \]  

(3.22c)

According to (3.19), the length of a vector \( \xi^A \) is invariant. Moreover, the presence of the
invariant vector \( i_A \) shows that the third component of \( \xi^A \) (alternatively the last component
of \( \xi_A \)) also is invariant, because it can be expressed as \( \xi^A i_A = i^A \xi_A \). In the next Section,
where we present a Poincaré gauge invariant and manifestly covariant formulation of our
dynamics, we shall introduce a “Poincaré coordinate” four-vector \( q^A \), with length \(-2A\) and
third component set to 1.

\[
q^A = \begin{pmatrix} q^a \\ 1 \\ \frac{1}{2} q^b q_b + A \end{pmatrix}
\]  

(3.23)

Also we shall use an on-mass-shell momentum four-vector \( p_A \), with vanishing last component.

\[
p_A = (p_a, p_2, 0), \quad p_a p^a = -m^2
\]  

(3.24)

In view of the above remarks, these restrictions are invariant: \( q^A i_A = 1, q^A q_A + 2A = 0; \)
\( i^A p_A = 0, p^A p_A + m^2 = 0 \). It follows from (3.11) and (3.12) that the transformation law for
\( q^a \) and \( p_a \) is

\[
q^a \rightarrow (\Lambda^{-1})^a_b \left( q^b + \epsilon^b_c \theta^c \right)
\]  

(3.25)

\[
p_a \rightarrow p_b \Lambda^b_a
\]  

(3.26)
Note that the specific translation

\[ T(q) = e^{-q^a \epsilon_a^b p_b} \]  \hspace{1cm} (3.27)

taken in the adjoint representation

\[ T(q)^A_B = \begin{pmatrix} \delta^a_b & -q^a \\ 0 & 1 \\ -q_b & \frac{1}{2} q^c q_c \\ \end{pmatrix} \]  \hspace{1cm} (3.28a)

moves \( q^A \) to its “origin” where \( q^a \) vanishes.

\[ T(q)^A_B q^B = \begin{pmatrix} 0 \\ 1 \\ \mathcal{A} \end{pmatrix} \]  \hspace{1cm} (3.28b)

In the four-dimensional adjoint representation, the momenta are realized non-trivially by commuting matrices and the center is realized trivially, see (3.10). Indeed the matrices provide a four-dimensional representation of the non-extended Poincaré group. Of course in the extended algebra (3.2), (3.3), the center can be represented by the identity only in an infinite-dimensional realization. Nevertheless, we now show that it is possible to represent the center by a finite-dimensional identity matrix, in a formalism where the translations have been “neutralized.”  \(^{10}\)

B. Infinite-Dimensional Representation

Consider a quantity \( \Phi \) transforming according to some unspecified representation in which all generators are realized non-trivially.

\[ \Phi \rightarrow U^{-1} \Phi \]  \hspace{1cm} (3.29)

Here \( U \) is as in (3.5), but realized in the above unspecified representation. Consider next

\[ \Psi = T(q) \Phi \]  \hspace{1cm} (3.30)
where $T$ is as in (3.27), with $P_a$ realized in the appropriate representation. Observe that $\Psi$ transforms as

$$\Psi \to T(\Lambda^{-1}(q+\epsilon\theta)) U^{-1} \Phi = T(\Lambda^{-1}(q+\epsilon\theta)) U^{-1} T^{-1}(q) \Psi$$

(3.31a)

Upon combining factors, it follows that

$$\Psi \rightarrow e^{-\alpha J} e^{-\frac{1}{2}q_a \theta^a} \mathcal{I} \Psi$$

(3.31b)

Unlike $\Phi$, $\Psi$ does not transform in a manifestly covariant fashion; nevertheless it enjoys a definite place in the theory: because $P_a$ has disappeared from the transformation law, the remaining generators, which commute, may be represented by finite-dimensional matrices acting on a finite-dimensional quantity, with $\mathcal{I}$ proportional to the identity matrix. We shall show that Fermi fields behave precisely in this fashion under extended Poincaré gauge transformations. Effectively an infinite number of components in $\Psi$ decouples, leaving a two-component Fermi field $\psi$.

(The phenomenon of representing a group without using the generators whose commutator is a $c$-number can also be seen in the Heisenberg algebra, which forms a nilpotent subalgebra of our extended Poincaré algebra. Calling the Heisenberg generators $X$, $P$ and $\mathbb{I}$ [analogous to the above $iP_+$, $iP_-$ and $-i\mathbb{I}$] we can represent a group element $U(\theta^1, \theta^2, \beta)$ by $e^{i(\theta^1 X+\theta^2 P)} e^{i\beta \mathbb{I}}$ and its action on a function of $x$, $\Phi(x)$, is $\Phi(x) \mapsto U^{-1} \Phi(x) = e^{-i\beta} e^{i\frac{1}{2} \theta^1 \theta^2} e^{-i\theta^1 x} \Phi(x-\theta^2)$. In order to remove the action of the generators $X$ and $P$, introduce two more variables $q^1$ and $q^2$, which also transform according to $q^1 \rightarrow q^1 - \theta^1$, $q^2 \rightarrow q^2 + \theta^2$, and define a new set of functions of $x$, $\Psi(x)$, by $\Psi(x) \equiv e^{-i(q^1 X+q^2 P)} \Phi(x) = e^{-iq^1 x} e^{-\frac{1}{2}q^1 q^2} \Phi(x+q^2)$. It then follows that $\Psi(x)$ responds to the transformation $U(\theta^1, \theta^2, \beta)$ without the appearance of $P$ and $X$: $\Psi(x) \mapsto U^{-1} \Psi(x) = e^{-i(\frac{1}{2}(q^1 \theta^2+q^2 \theta^1))} \Psi(x)$.)
In the infinite-dimensional representation, which we shall use below when discussing
Fermi fields before passing to the finite-dimensional $\psi$, the $P_a$ are matrices familiar from
harmonic oscillator theory: in light-cone components $\left[\pm = \frac{1}{\sqrt{2}}(0 \pm 1)\right]$

$$\langle n' | P_+ | n \rangle = \epsilon \sqrt{\nu + \bar{n}} \delta_{n',n-1} \tag{3.32a}$$

$$\langle n' | P_- | n \rangle = \epsilon \sqrt{\nu + n + 1} \delta_{n',n+1} \tag{3.32b}$$

with $\nu$ arbitrary, $0 \leq \Re \nu < 1$, and $n, n' = 0, \pm 1, \pm 2, \ldots$. Equation (3.2) is verified and $I$ is
realized by the (infinite) identity matrix, $\mathbb{I}$, multiplied by an arbitrary constant $\epsilon^2$: $I = \epsilon^2 \mathbb{I}$.
The quadratic Casimir $C$ in (3.20) will commute with the above infinite $P_\pm$ matrices only if it
is proportional to $\mathbb{I}$. Therefore the Lorentz generator is found from (3.20) to be simultaneously
diagonal with $I$, with which it commutes,

$$\langle n' | J | n \rangle = (j + n) \delta_{n',n} \tag{3.33}$$

where $j$ is an arbitrary number determined by $\nu$ and the Casimir.

C. Gauge Fields

Associated with our extended Poincaré Lie algebra, are gauge connections $A^A_{\mu}$. We define
the Lie algebra valued one-form from the vector potentials that are present in the theory.

$$A = A^A_{\mu} dx^\mu Q_A = e^a_{\mu} dx^\mu P_a + \omega_{\mu} dx^\mu J + a_{\mu} dx^\mu I \tag{3.34}$$

Here $\omega_{\mu}$ and $a_{\mu}$ are independent quantities, not satisfying any other formulas like (1.4) and
(2.A7). The gauge curvature two-form

$$F = dA + A^2 \tag{3.35a}$$
has components

\[ F = F^A Q_A = (de^a + e^a_b \omega e^b) \mathcal{P}_a + d\omega J + \left( da + \frac{1}{2} e^a_{ab} e^b \right) I \] (3.35b)

According to (1.5) and (1.8) the gauge curvature along the translation and rotation generators is the torsion density and scalar curvature density, respectively, while the gauge curvature along the central element coincides with the left side of (2. A7a). Evidently \( F^A \) transforms as a contravariant vector, \( i.e. \) as in (3.11), or \( F \rightarrow U^{-1} F U \), as also does \( A^A \) supplemented by a gauge transformation: \( A \rightarrow U^{-1} A \ U + U^{-1} dU \). In components

\[
\begin{align*}
\epsilon^a_{\mu} & \rightarrow (\Lambda^{-1})^a_b \left( e^b_{\mu} + \epsilon^b_c \theta^c \omega_{\mu} + \partial_{\mu} \theta^b \right) \\
\omega_{\mu} & \rightarrow \omega_{\mu} + \partial_{\mu} \alpha \\
a_{\mu} & \rightarrow a_{\mu} - \theta^a \epsilon_{ab} e^b_{\mu} - \frac{1}{2} \theta^a \theta_a \omega_{\mu} + \partial_{\mu} \beta + \frac{1}{2} \partial_{\mu} \theta^a \epsilon_{ab} \theta^b
\end{align*}
\] (3.36a, 3.36b, 3.36c)

These comprise the previously discussed Lorentz transformation (1.13), (1.14), and gauge transformation on \( a_{\mu} \) (2.A9), now further supplemented by a translation, which also produces a local gauge transformation.

Given a covariant quantity \( \Phi \), we define the gauge covariant derivative \( D_\mu \Phi \) by

\[ D_\mu \Phi \equiv \partial_\mu \Phi + A^A_\mu Q_A \Phi \] (3.37)

For the adjoint representation this formula reads

\[
(D_\mu \xi)^A = \partial_\mu \xi^A + f_{BC}^A A^B_\mu \xi^C
\] (3.38a)

or using the Lie algebra-valued quantity

\[
\xi = \xi^A Q_A
\] (3.38b)

we have
\[ D_\mu \xi = \partial_\mu \xi + [A_\mu, \xi] \]  
(3.38c)

Of special interest is the gauge covariant derivative of the Poincaré coordinate \( q^A \) in (3.23).

\[
(D_\mu q)^A = \partial_\mu q^A + f_{BC}^A A^B_\mu q^C = \begin{pmatrix} (D_\mu q)^a \\ 0 \\ q_b (D_\mu q)^b \end{pmatrix}
\]  
(3.39a)

\[
(D_\mu q)^a = \partial_\mu q^a + \epsilon^a_b (q^b \omega_\mu - e^b_\mu)
\]  
(3.39b)

which transforms as

\[
(D_\mu q)^a \rightarrow (\Lambda^{-1})^a_b (D_\mu q)^b
\]  
(3.40)

[because the third component of \((D_\mu q)^A\) vanishes].

D. Comments

We conclude with several observations. The metric (3.17) on the algebra that we have introduced can be generalized: one may add an arbitrary multiple of the Cartan–Killing metric (3.16).

\[
\tilde{h}_{AB} = h_{AB} + c i_A i_B = \begin{pmatrix} h_{ab} & 0 & 0 \\ 0 & c & -1 \\ 0 & -1 & 0 \end{pmatrix}
\]  
(3.41)

Evidently

\[
q_1^A \tilde{h}_{AB} q_2^B = q_1^A h_{AB} q_2^B + c (q_1^A i_A) (i_B q_2^B)
\]  
(3.42)

\[
Q_A \tilde{h}^{AB} Q_B = \mathcal{P}_a h_{ab} \mathcal{P}_b - 2 \mathcal{J} \mathcal{I} - c \mathcal{I}^2
\]

Thus, with the modified metric, the “length” and inner product of vectors is shifted by an arbitrary amount. Equivalently we see that introducing the parameter \( c \) is tantamount to shifting \( \mathcal{J} \) by a multiple of \( \mathcal{I} \): \( \mathcal{J} \rightarrow \mathcal{J} + (c/2) \mathcal{I} \). In our application below, we deal with vectors of arbitrary length and/or vanishing component along \( i_A \), so it does not appear that the one-parameter freedom of (3.41) adds any more generality to our theory — beyond
the freedom that is already contained in the fact that the Poincaré coordinate $q^A$ satisfies $q^A q_A + 2A = 0$, and that $J$ may be shifted at will. Thus we ignore the possibility (3.41) and remain with (3.17). [Note that it was already observed, in the end of Section II.A, that the Lorentz generator (2.A29) differs by a constant from the expression obtained with the energy momentum tensor.]

The metric (3.17) may be diagonalized, and it is then seen that the signature is $(1, -1, 1, -1)$. Consequently our Poincaré group adjoint representation also gives a representation for a subgroup of $SO(2, 2)$. The occurrence of this group in the present context is mysterious. To be sure, $SO(2, 2)$ is the conformal group in $(1 + 1)$ dimensions, and our Poincaré coordinate with $A = 0$ is like Dirac’s projective variable for realizing conformal transformations linearly. However, in general $A \neq 0$, and also we do not deal exclusively with massless particles, so there does not seem to be any actual relevance of the conformal group.

IV. POINCARÉ GAUGE INVARIANCE OF MATTER-GRA VITY DYNAMICS

In this Section we present the actions (2.A32) and (2.B3) for the point particle and the Fermi field, respectively, in a formalism that is invariant against gauge transformations of the extended Poincaré group. The gauge invariant expressions are written both in component form and in a manifestly covariant tensor formalism.

Our strategy for coupling matter to gravity in a gauge invariant manner follows (a modified version of) the approach due to Grignani and Nardelli, which we first describe in general terms.

To achieve a Poincaré gauge invariant description, one begins by presenting the dynamics in terms of the Zweibeine $e^a_\mu$, spin connection $\omega_\mu$ and the additional Abelian connection
$a_\mu$ associated with the center. In this form, the action is invariant against Lorentz gauge
transformations, but not against translations since neither $e^a_\mu$ nor $a_\mu$ are translation covariant,
compare (3.36). To achieve gauge invariance against translations, a Poincaré-coordinate $q^a$
is introduced, which transforms as in (3.25), i.e. as the first two components of a four-vector
$q^A$, whose third component is 1, $q^A i_A = 1$, and whose length, $q^A q_A = -2A$; compare (3.23).
The Poincaré coordinate provides a mapping from Minkowski space to the tangent space, i.e.
it is a function of $x^\mu$: $q^a(x)$; in the point-particle application $q^a$ is evaluated on the path
$x^\mu(\tau)$ and so may be taken as a function of $\tau$: $q^a(x(\tau)) \equiv q^a(\tau)$.

The invariant action is now obtained by replacing $-\epsilon^{ab} e^b_\mu$ by $(D_\mu q)^a$, and $A_\omega + a_\mu$
by $-q^A A^A_\mu + \frac{1}{2} q^a \epsilon_{ab} \partial_\mu q^b$. This renders a Lorentz gauge invariant action also Poincaré gauge
invariant because of the following two facts. First, $(D_\mu q)^a$ is unaffected by translations — it
transforms solely by a Lorentz rotation, see (3.40). Second, $-q^A A^A_\mu + \frac{1}{2} q^a \epsilon_{ab} \partial_\mu q^b$
changes by a total derivative,

$$-q^A A^A_\mu + \frac{1}{2} q^a \epsilon_{ab} \partial_\mu q^b \rightarrow$$

$$-q^A A^A_\mu + \frac{1}{2} q^a \epsilon_{ab} \partial_\mu q^b + \partial_\mu \left( A_\alpha + \beta - \frac{1}{2} q^a \theta^a \right)$$

(4.1)

and this lack of invariance can be rendered innocuous.

The invariant action now depends on an additional variable — the Poincaré coordinate
$q^a$. Dynamical equations are obtained by varying all the variables — the original ones and
$q^a$. However, the invariant content of the dynamics is not affected by the additional variable.
This is established by the following consideration.

Observe from (3.25) that gauge transformations shift $q^a$ by an arbitrary amount, so
that $q^a$ may be set to zero — indeed the $T$ transformation (3.28) accomplishes this. At
$q^a = 0$, $(D_\mu q)^a$ reduces to $-\epsilon^{ab} e^b_\mu$, see (3.39b), while $-q^A A^A_\mu + \frac{1}{2} q^a \epsilon_{ab} \partial_\mu q^b$
becomes $A_\omega + a_\mu$. 
Therefore the equations of motion, obtained by varying dynamical variables other than $q^a$, coincide at $q^a = 0$ with the equations of motion of the Lorentz invariant theory; at the same time $q^a = 0$ represents an attainable gauge choice in the Poincaré invariant theory. It remains to examine the equation obtained by varying $q^a$. Here one sees that at $q^a = 0$ this equation is automatically satisfied, when all the other equations of motions hold. The proof follows from gauge invariance of the action: Upon denoting all variables other than $q^a$ by $\chi$, with infinitesimal gauge transform $\delta \chi$, and with $\delta q^a = -\epsilon^a{}_b q^b \alpha + \epsilon^a_b \theta^b$ being the infinitesimal gauge transform of $q^a$ [see (3.25)], gauge invariance can be stated as the following property of the total action $I_t$

$$0 = \int \frac{\delta I_t}{\delta \chi} \delta \chi + \int \frac{\delta I_t}{\delta q^a} \delta q^a = \int \frac{\delta I_t}{\delta \chi} \delta \chi + \int \frac{\delta I_t}{\delta q^a} (-\epsilon^a{}_b q^b \alpha + \epsilon^a_b \theta^b) \tag{4.2a}$$

The first term in the last equality vanishes when equations of motion for the $\chi$ variables are satisfied, while the last term leaves at $q^a = 0$

$$\int \frac{\delta I_t}{\delta q^a} \bigg|_{q^a=0} \epsilon^a_b \theta^b = 0 \tag{4.2b}$$

Since $\theta^a$ is arbitrary, (4.2) shows that $\delta I_t/\delta q^a|_{q^a=0}$ vanishes when the other equations hold.

[This result is true provided the gravitational variables are dynamical, i.e. they are included in the set $\chi$ and are present in the total action $I_t$. The gravitational action is given in Section V.]

A. Point Particle

We carry out the above-described procedure for the point-particle action, whose first-order and Lorentz invariant form is (2.A32). The Poincaré gauge invariant expression therefore reads

$$I_{\text{particle}} = \int d\tau \left\{ p_a (D_\tau q)^a - \frac{1}{2} N (p^2 + m^2) + q_A A^A_{\mu} \dot{x}^\mu - \frac{1}{2} q^a \epsilon_{ab} \dot{q}^b \right\} \tag{4.A1a}$$
Here the Poincaré coordinate is taken to be a function of \( \tau \), so from (3.39) \((D_\tau q)^a = (\dot{x}^\mu D_\mu q)^a = q^a + e^a_b \left( q^b \omega_\mu - \gamma^b_\mu \right) \dot{x}^\mu\). When \( q_A A^A \) is expressed in components, (4.A1a) becomes

\[
I_{\text{particle}} = \int d\tau \left\{ \left( p_a + \frac{1}{2} \epsilon_{ab} q^b \right) (D_\tau q)^a - \frac{1}{2} N \left( p^2 + m^2 \right) - \left( A \omega_\mu + a_\mu - \frac{1}{2} q_a e^a_\mu \right) \dot{x}^\mu \right\}
\]

(4.A1b)

and reduces to (2.A32) at \( q^a = 0 \). Performing gauge transformations according to (3.25), (3.26), (3.36) and (3.40) shows that the action changes by end-point contributions.

\[
I_{\text{particle}} \rightarrow I_{\text{particle}} - \int d\tau \frac{d}{d\tau} \left( A \alpha + \beta - \frac{1}{2} q_a \theta^a \right)
\]

(4.A2)

Dynamical equations are gauge invariant, but the Lagrangian acquires a total derivative, even when the transformation is global: Owing to the \( \frac{1}{2} \dot{q}_a \epsilon_{ab} q^b \) term, which is like a constant electromagnetic field in the tangent space, the Lagrangian is not invariant against translations of \( q^a \).

Our formulas can be written in terms of tensors that are covariant under transformations of the extended group. The action (4.A1) may be presented as

\[
I_{\text{particle}} = \int d\tau \left\{ p_A (D_\tau q)^A - \frac{1}{2} N \left( p_A p^A + m^2 \right) + \left( q_A A^A \dot{x}^\mu + \frac{1}{2} q^A f_{AB} \dot{q}^B \right) - \lambda_1 \left( q_A q^A + 2A \right) - \lambda_2 \left( i_A q^A - 1 \right) - \lambda_3 \left( p_A i^A \right) \right\}
\]

(4.A3)

The \( \lambda_i \) are Lagrange multipliers enforcing the special properties of \( q^A \) and \( p_A \); viz. \( q_A q^A + 2A = 0 \), \( i_A q^A = 1 \), and \( p_A i^A = 0 \). Non-invariance of the Lagrangian is again seen in the presence of the non-covariant term: \( \frac{1}{2} q^A f_{AB} \dot{q}^B = -\frac{1}{2} q^a \epsilon_{ab} \dot{q}^b \). But since the action changes only by end-point contributions, the equations of motion are gauge invariant; when \( \lambda_i \) are eliminated they read in terms of \( p \equiv p_A Q^A \) and \( q \equiv q^A Q_A \)

\[
D_\tau q = N \left( p + \langle p, q \rangle I \right)
\]

(4.A4a)

\[
D_\tau p + \langle D_\tau p, q \rangle I = [D_\tau q, q]
\]

(4.A4b)
Also the gauge current

\[ J^\mu(x(\tau)|x) = Q^A J^\mu_A(x(\tau)|x) = -Q^A \frac{1}{\sqrt{-g}} \frac{\delta I_{\text{particle}}}{\delta A^A_\mu(x)} \]

is covariantly conserved.

\[ \frac{1}{\sqrt{-g}} D_\mu \sqrt{-g} J^\mu = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} J^\mu + [A_\mu, J^\mu] = 0 \] (4.A6)

Equation (4.A6) effectively entails,

\[ D_\tau ([p, q] - q) = 0 \] (4.A7)

which is readily established from (4.A4). One further equation follows from varying \( x^\mu(\tau) \).

\[ \int d^2 x \langle F_{\mu\nu}(x), J^\nu(x(\tau)|x) \rangle = 0 \] (4.A8)

Here \( F_{\mu\nu} \) is the gauge curvature (3.35) and (4.A4) as well as (4.A7) have been used to simplify (4.A8).

**B. Fermi Fields**

When the steps outlined in the introductory paragraphs are implemented, the Lorentz gauge invariant Fermi field action (2.B3) gives rise to the following Poincaré gauge invariant expression, which reduces to (2.B3) in the gauge \( q^a = 0 \). [Here the Poincaré coordinate is a field: \( q^A(x) \).]

\[
I_{\text{Fermi}} = \int d^2 x e(q) \left\{ \frac{i}{2} \bar{\psi} \gamma^\mu(q) \left( \not\partial - i \left( q_A A^A_\mu - \frac{1}{2} q^a \epsilon_{ab} \partial_\mu q^b \right) \right) \psi \\
- \frac{i}{2} \bar{\psi} \left( \not\partial + i \left( q_A A^A_\mu - \frac{1}{2} q^a \epsilon_{ab} \partial_\mu q^b \right) \right) \gamma^\mu(q) \psi - m \bar{\psi} \psi \right\} 
\] (4.B1)
\begin{align*}
\gamma^\mu(q) & \equiv \frac{1}{e(q)} e^{\nu} (D_\nu q)^a \gamma_a \\
e(q) & \equiv \frac{1}{2} e^{\mu\nu} e_{ab} (D_\mu q)^a (D_\nu q)^b
\end{align*}

where \( e(q) \) is a \( q \)-dependent generalization of \( \det e^a_{\mu} = \sqrt{-g} \), just as \( -e^{a}_{\mu} (D_\mu q)^b \) generalizes \( e^a_{\mu} \). Invariance is complete when the Fermi fields transform as

\begin{align*}
\psi & \rightarrow e^{\frac{i}{2} \alpha \gamma_5 - i(A_\alpha + \beta - \frac{1}{2} q_a \theta^a)} \psi \\
\bar{\psi} & \rightarrow \bar{\psi} e^{-\frac{i}{2} \alpha \gamma_5 + i(A_\alpha + \beta - \frac{1}{2} q_a \theta^a)}
\end{align*}

This transformation, which reduces to (2.5) at \( q^a = 0 \), is needed to compensate for the total derivative that arises when \( q A A^\mu - \frac{1}{2} q^a \epsilon_{ab} \partial_\mu q^b \) is transformed, see (4.1).

The above formulas are not manifestly covariant. The route to a covariant formalism is indicated by the remarks in Section III, Eqs. (3.27) – (3.31): what is needed is a representation of the extended Poincaré group, with \( I \) realized by a multiple of the identity, which is necessarily infinite-dimensional. Then the two-component \( \psi \) field and \( 2 \times 2 \) Dirac matrices that occur in (4.B1) are obtained by projection.

The infinite representation of the extended Poincaré group that we use here is given in (3.32) and (3.33). We also need an infinite-dimensional representation of the Dirac matrices satisfying

\begin{align*}
\{ \Gamma_a, \Gamma_b \} &= 2 h_{ab} I \\
[\Gamma_a, J] &= \epsilon^b_{a} \Gamma_b
\end{align*}

A solution is

\begin{align*}
\langle n' | \Gamma_+ | n \rangle &= \begin{cases} 
0 & n \text{ even} \\
\sqrt{2} \delta_{n', n-1} & n \text{ odd}
\end{cases} \\
\langle n' | \Gamma_- | n \rangle &= \begin{cases} 
\sqrt{2} \delta_{n', n+1} & n \text{ even} \\
0 & n \text{ odd}
\end{cases}
\end{align*}
We shall also need various anti-commutators. In the above representation we find

\[
\langle n' \mid \frac{1}{2i} \{ \Gamma^a, P_a \} \mid n \rangle = M(n) \delta_{n'n}
\]  

(4.B7a)

\[
M(2N) = M(2N + 1) = -i \epsilon \sqrt{2(\nu + 2N + 1)}
\]

(4.B7b)

Also we have

\[
\langle n' \mid \{ \Gamma_a, J \} \mid n \rangle = (2j + n' + n) \langle n' \mid \Gamma_a \mid n \rangle
\]  

(4.B8)

We see that \(J, M\) and \(\Gamma_a\) are \(2 \times 2\) matrices imbedded along the diagonal of an infinite matrix. A set of these \(2 \times 2\) matrices, given by fixing adjacent values of \(n\), corresponds to a diagonal Lorentz generator \(J\) with eigenvalues differing by 1, a diagonal \(M\) matrix with equal eigenvalues that will become identified with the fermion mass \(m\), and a conventional \(2 \times 2\) realization of Dirac matrices. While it is true that the action of the infinite matrices representing \(P_a\) takes one out of the above-mentioned \(2 \times 2\) blocks, we shall be able to disregard this since we shall employ the projection (3.30) to “neutralize” the translation generators.

An invariant action is constructed from infinite component fields \(\Phi\), and \(\bar{\Phi}\), transforming by

\[
\Phi \to U^{-1} \Phi, \quad \bar{\Phi} \to \bar{\Phi} U
\]

(4.B9)

where \(U\) is as in (3.5), with generators given by the above infinite matrices.

\[
I_{\text{Fermi}} = \int d^2 x \ e(q) \left\{ \frac{i}{2} \bar{\Phi} \Gamma^\mu(q) \overset{\rightarrow}{D}_\mu \Phi - \frac{i}{2} \bar{\Phi} \overset{\leftarrow}{D}_\mu \Gamma^\mu(q) \Phi \right\}
\]

\[
\Gamma^\mu(q) = \frac{1}{e(q)} \epsilon^{\mu\nu} (D_\nu q)^a T^{-1}(q) \Gamma_a T(q)
\]

\[
\overset{\rightarrow}{D}_\mu \Phi = \partial_\mu \Phi + A_\mu \Phi
\]

\[
\overset{\leftarrow}{D}_\mu = \partial_\mu \Phi - \Phi A_\mu
\]

(4.B10)

Of course the Lie algebra-valued gauge field and the group element \(T(q)\) are realized in the above infinite-dimensional representation. It is straightforward to verify that (4.B10) is
invariant; one uses the fact that when \( q \) and the connections in \( \Gamma^\mu(q) \) are transformed, the result is an adjoint transformation of that quantity.

\[
\Gamma^\mu(q) \to U^{-1} \Gamma^\mu(q) U \tag{4.B11}
\]

Note that here we do not use the constrained covariant four-vector \( q^A \); rather we remain with the non-covariant two-component Poincaré coordinate \( q^a \). It enters (4.B10) only through \( \Gamma^\mu(q) \), which however is a covariant object, as is seen from (4.B11).

It is interesting that only a gauged kinetic term is present in the action (4.B10). A mass term is not explicitly included; it emerges in the subsequent reduction. Following (3.30) we define

\[
\Phi = T^{-1}(q) \bar{\Psi} \quad \text{and} \quad \bar{\Phi} = \bar{\Psi} T(q) \tag{4.B12}
\]

Substituting (4.B12) into (4.B10) and combining the exponentials results in

\[
I_{\text{Fermi}} = \int d^2 x \, e(q) \left\{ \frac{i}{2} \bar{\Psi} \tilde{\Gamma}^\mu(q) \left( \overleftarrow{\partial}_\mu - I \left( q A A^A_\mu - \frac{1}{2} q^a \epsilon_{ab} \partial_\mu q^b \right) \right) \bar{\Psi} \right. \\
- \frac{i}{2} \bar{\Psi} \left( \overleftarrow{\partial}_\mu + I \left( q A A^A_\mu - \frac{1}{2} q^a \epsilon_{ab} \partial_\mu q^b \right) \right) \tilde{\Gamma}^\mu(q) \bar{\Psi} \\
\left. + \frac{i}{2} \bar{\Psi} \{ \Gamma^a, P_a \} \bar{\Psi} + \frac{i}{2} \omega_\mu \bar{\Psi} \left\{ \tilde{\Gamma}^\mu, J - AI \right\} \bar{\Psi} \right\} \\
\tilde{\Gamma}^\mu(q) = \frac{1}{e(q)} \epsilon^{\mu\nu} (D_\nu q)^a \Gamma_a = T(q) \Gamma^\mu(q) T^{-1}(q) \tag{4.B13}
\]

The translational generator here is present only in \( \{ \Gamma^a, P_a \} \), which according to (4.B7) is diagonal. Hence the infinite matrices in (4.B13) are composed of decoupled \( 2 \times 2 \) blocks, and we can focus on one of these blocks.

To compare (4.B13) with (4.B1) and also (3.31b) with (4.B2), we look at two adjacent components of \( \Psi \) and call them \( \psi \equiv \left( \begin{array}{c} \Psi_{n+1} \\ \Psi_n \end{array} \right) \), where for definiteness \( n \) is chosen even. For
(3.31b) to agree with (4.2), we set $I$ equal to $iI$ i.e., $\epsilon^2 = i$; also on this two-dimensional subspace, we can define the restriction of $J$ as $-\frac{i}{2} \gamma_5 + iA\gamma$, i.e. $j = -n - \frac{1}{2} + iA$, where now $n$ is the fixed number chosen above, and $\gamma_5 = i\sigma_3$. Then with the help of the identities (4.8) and (4.9) the action (4.13) reproduces (4.1): the last term in (4.13) vanishes while the next-to-last gives the mass in (4.1) provided we define

$$m = \sqrt{2} \sqrt{A + i \left( j - \nu + \frac{1}{2} \right)}$$

(4.14)

Therefore the Casimir in this representation is

$$C = (2A - m^2) I.$$  

(4.15)

[Had we not scaled $B$ to unity, it would scale the mass as $m/\sqrt{B}$.]

The equations of motion following from (4.10) by varying $\bar{\Phi}$ and $\Phi$ are

$$i \bar{\Phi} D_{\mu} \Gamma^\mu(q) \Phi + \frac{i}{4} \epsilon^{\mu\nu} f_{AB} C F^A_{\mu\nu} q^B T^{-1}(q) \Gamma_C T(q) \Phi = 0$$

(4.16a)

$$i \bar{\Phi} D_{\mu} \Gamma^\mu(q) + \frac{i}{4} \epsilon^{\mu\nu} f_{AB} C F^A_{\mu\nu} q^B \bar{\Phi} T^{-1}(q) \Gamma_C T(q) = 0$$

(4.16b)

In order to present these equations in an explicitly covariant form, we have introduced four quantities $\Gamma_A$, coinciding with $\Gamma_a$ for $A = 0, 1$, unspecified $\Gamma_2$ and vanishing $\Gamma_3$; the constraint is invariant: $i^A \Gamma_A = 0$. Thus $T^{-1}(q) \Gamma_A T(q)$ behaves in the same way as $p_A$ in the particle case, and one further verifies from (4.11) the transformation law,

$$T^{-1}(q) \Gamma_a T(q) \rightarrow T^{-1} (\Lambda^{-1}(q + e\theta)) \Gamma_a T (\Lambda^{-1}(q + e\theta)) = U^{-1} T^{-1}(q) \Gamma_b T(q) U \Lambda^b_a$$

(4.17)

which again parallels the behavior of $p_a$, see (3.26).

Varying $q^a$ in (4.10) leads to

$$i \frac{\epsilon^{\mu\nu}}{e(q)} \bar{\Phi} D_{\mu} T^{-1}(q) \Gamma_a T(q) D_{\nu} \Phi + \frac{i}{4} \frac{\epsilon^{\mu\nu}}{e(q)} \bar{\Phi} \{ T^{-1}(q) \Gamma_a T(q), F_{\mu\nu} \} \Phi = 0$$

(4.18)
which again is covariant once $\Gamma_a$ is extended to $\Gamma_A$.

The gauge current

$$J_A^\mu = -\frac{1}{\sqrt{-g}} \frac{\delta I_{\text{Fermi}}}{\delta A_A^\mu}$$

$$= -\frac{i}{2} \frac{e(q)}{\sqrt{-g}} \Phi \{ \Gamma^\mu(q), Q_A \} \Phi$$

$$+ \frac{i}{2} f_{ABC} q^B \frac{1}{\sqrt{-g}} \epsilon^{\mu\nu} \left( \Phi T^{-1}(q) \Gamma_C T(q) \overrightarrow{D_\nu} \Phi - \Phi \overleftarrow{D_\nu} T^{-1}(q) \Gamma_C T(q) \Phi \right)$$

(4.B19)

can be shown to be covariantly conserved

$$\frac{1}{\sqrt{-g}} (D_\mu \sqrt{-g} J^\mu)_A = 0$$

(4.B20)

with the help of equations of motion (4.B16), (4.B18).

V. GRAVITATIONAL ACTION

The gravitational variables that are present in our gauge-theoretical matter Lagrangians comprise the conventional Zweibein $e^a_\mu$ and spin connection $\omega_\mu$; also there is the unconventional connection $a_\mu$, associated with the center in our Poincaré algebra. The gravitational action provides equations that determine these quantities. We anticipate that the Zweibein equation will encode the torsion free-condition (1.5), thereby evaluating the spin connection as in (1.4). Also we anticipate that $a_\mu$ satisfies (2.A7b). Furthermore, the action must give a determination of the curvature $R$ in (1.8). Finally, all these conditions should arise in a gauge theoretical formalism based on the extended Poincaré group.

The above requirements are successfully met in a gravitational action that is related to the one for “string inspired, dilaton” gravity, which reads, in terms of variables employed originally,\(^3,4\)

$$I_{\text{dilaton}} = \frac{1}{G} \int d^2 x \sqrt{-g'} e^{-2\phi} \left( R' - 4 \partial_\mu \phi \partial^\mu \phi - \lambda_0 \right)$$

(5.1)
Here, $G$ is the gravitational coupling strength, $\lambda_0$ is a cosmological constant, and temporarily we use a primed metric variable and scalar curvature to distinguish them from the unprimed ones, occurring in our formulation. With the definitions

\[
e^{2\varphi} = \frac{1}{\eta}
\]

\[
g'_{\mu\nu} = \frac{g_{\mu\nu}}{\eta}
\]

the action (5.1) becomes

\[
I_{\text{dilaton}} = \frac{1}{G} \int d^2x \sqrt{-g} \left( \eta R - \lambda_0 \right)
\]

(5.3)

This is similar to the action for the constant curvature de Sitter model,

\[
I_{\text{de Sitter}} = \frac{1}{G} \int d^2x \sqrt{-g} \eta (R - \lambda_0)
\]

(5.4)

but (5.3) differs from (5.4) because $\eta$ does not multiply the cosmological constant in the dilaton model. Nevertheless, apart from a boundary term, (5.3) may be obtained from (5.4) in a singular limit: shift $\eta$ in (5.4) by $\lambda_0'/\lambda_0$ and set $\lambda_0$ to zero.\(^{11}\) [The contribution $\frac{\lambda_0'}{\lambda_0} \frac{1}{G} \int d^2x \sqrt{-g} R$ is a boundary term.]

This singular limit has an algebraic parallel. The de Sitter algebra, which leads to the gauge theoretical formulation of (5.4),\(^5\) reads

\[
[\mathcal{P}_a, \mathcal{P}_b] = -\frac{1}{2} \lambda_0 \epsilon_{ab} \mathcal{J}
\]

(5.5)

\[
[\mathcal{P}_a, \mathcal{J}] = \epsilon_{ab} \mathcal{P}_b
\]

(5.6)

Since $\mathcal{J}$ is a generator of Abelian rotations, it may be shifted by an arbitrary multiple of $\mathcal{I}$. When $\mathcal{J}$ is shifted by $-\frac{2}{\lambda_0} \mathcal{I}$ and $\lambda_0$ is set to zero, the de Sitter algebra (5.5), (5.6) contracts to extended Poincaré algebra (3.2), (3.3).
The gauge theoretical formulation of (5.3) uses the quartet of gauge curvatures \( F^A \), constructed from the gravitational variables and transforming under the adjoint representation, together with a quartet of Lagrange multiplier fields \( \eta_A \), transforming according to the co-adjoint representation. Thus the Poincaré gauge invariant gravitational action is

\[
I_{\text{gravity}} = \frac{1}{G} \int \eta_A F^A
\]

\[
= \frac{1}{G} \int \left( \eta_a \left( de^a + \epsilon^a_{\ b} \omega^b \right) + \eta_2 d\omega + \eta_3 \left( da + \frac{1}{2} \epsilon^a_{\ ab} \epsilon^b \right) \right)
\]

(5.7)

Setting to zero the variation of \( \eta_a \) yields the required torsion free-condition, while varying \( \eta_3 \) equates \( da \) with the volume two-form. Finally, variation of \( \eta_2 \) shows that the curvature vanishes. [Nevertheless, the model remains non-trivial because the “physical” metric in the dilaton-string context is given by \( g_{\mu\nu}/\eta \), see (5.2b).]

The relation between (5.7) and (5.3) is seen in the following manner. First we enforce the torsion-free condition and evaluate \( \omega \) in (5.7) on (1.4). Next we vary \( a_\mu \), and assume that this quantity does not occur in the matter Lagrangian; i.e. matter dynamics does not include our unconventional, dimension-specific force. The equation that results from the variation shows that \( d\eta_3 \) vanishes, i.e. \( \eta_3 \) is constant, which we set equal to \( \lambda_0 \). With these steps and the definition \( \eta_2 \equiv 2\eta \) we see that \( I_{\text{gravity}} \) becomes \( I_{\text{dilaton}} \), apart from the boundary term \( \int \lambda_0 da \).

In the more general case, when our unconventional, dimension-specific force does modify matter dynamics, the equation for \( \eta_3 \) coincides with that for \( \lambda \) [Eqs. (2.A16) or (2.B10)], and \( I_{\text{gravity}} \) still resembles \( I_{\text{dilaton}} \), but the cosmological term is no longer constant.

By varying the Lagrange multiplier \( \eta_A \), the gravitational equations can be presented in gauge invariant form.

\[
F^A_{\mu\nu} = 0
\]

(5.8)
Varying the connections $A^A_\mu$ gives an equation for the $\eta_A$.

$$
\frac{1}{\sqrt{-g}} \epsilon^{\mu\nu} (D_\nu \eta)_A = -GJ^\mu_A
$$

(5.9)

$$(D_\mu \eta)_A = -GJ^5_{\mu A}$$

Here the right side contains the matter gauge current (4.A5) for point particles or (4.B19) for Fermi fields.

We recall\(^2\) that in the absence of matter, the solution for $\eta_A$ produces the “black-hole” metric tensor $g'_{\mu\nu} = 2g_{\mu\nu}/\eta_2$, while the invariant $\eta_A \eta^A$ determines the product of the “black-hole” mass with the cosmological constant $\lambda_0$.

The most direct solution to (5.8)

$$A^A_\mu = 0$$

(5.10)

simplifies drastically the matter equations, while (5.9) reduces to

$$\partial_\mu \eta^0_A = -GJ^5_{\mu A} \bigg|_{A^A_\mu = 0}$$

(5.11)

Of course, this solution is highly singular from the geometrical point of view: since $A^A_\mu$ vanishes, so do the Zweibein, spin-connection and $a_\mu$; no geometry can be described. However, geometric content is easily regained by performing a gauge transformation on (5.10),

$$A_\mu \equiv A^A_\mu Q_A \rightarrow U^{-1} \partial_\mu U$$

(5.12)

where $U$ is arbitrary, but chosen so that geometrical quantities are non-singular. After the gauge transformation $H^0 \equiv \eta^0_A Q^A$ becomes replaced by $H = U^{-1}H^0U$, but the invariant content of a solution, coded e.g. in $\eta_A \eta^A$, remains unchanged. In the absence of matter (5.11) is solved by four constants comprising $\eta^0_A$, which correspond to the cosmological constant, as well as to the black hole location in space-time and its mass.

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VI. CONCLUSIONS

In this work we have given details not included in our two recent Letters\(^1\),\(^2\) about a gauge theory of lineal gravity built on the centrally extended Poincaré group. We have also considered the coupling of matter points and fields, whose gauge invariance is seen explicitly in our manifestly covariant tensor calculus. Within this formalism, additional non-standard and dimension-specific matter gravity interactions are elegantly incorporated. This is accomplished by using the connection \(A^A_\mu\) and the Poincaré four-vector \(q^A\), whose presence in the action in the combination (4.1) gives rise to the additional forces, which are also seen to be correlated with the extension of the Poincaré group. The strength \(A\) of the curvature-dependent force is determined by the length of \(q^A\) and can take any value. Since the quantity (4.1) may be deleted from the action without upsetting gauge invariance, it is recognized that the additional forces need not be included in the matter Lagrangian, even when the extended group is used for the gravity sector. Finally, we call attention to the fact that the cosmological constant in gauged gravity arises dynamically, and is not a parameter in the Lagrangian. Similar mechanisms for generating a cosmological constant have been previously posited in the physical four-dimensional theory,\(^1\),\(^2\) but for us this is dictated by group structure. Moreover, our additional interactions result in space-time variation of the cosmological term.
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