2. $p$-primary part of the Milnor $K$-groups and
Galois cohomologies of fields of characteristic $p$

Oleg Izhboldin

2.0. Introduction

Let $F$ be a field and $F^{\text{sep}}$ be the separable closure of $F$. Let $F^{\text{ab}}$ be the maximal abelian extension of $F$. Clearly the Galois group $G^{\text{ab}} = \text{Gal}(F^{\text{ab}}/F)$ is canonically isomorphic to the quotient of the absolute Galois group $G = \text{Gal}(F^{\text{sep}}/F)$ modulo the closure of its commutant. By Pontryagin duality, a description of $G^{\text{ab}}$ is equivalent to a description of

$$\text{Hom}_{\text{cont}}(G^{\text{ab}}, \mathbb{Z}/m) = \text{Hom}_{\text{cont}}(G, \mathbb{Z}/m) = H^1(F, \mathbb{Z}/m),$$

where $m$ runs over all positive integers. Clearly, it suffices to consider the case where $m$ is a power of a prime, say $m = p^i$. The main cohomological tool to compute the group $H^1(F, \mathbb{Z}/m)$ is a pairing

$$(\cdot, \cdot)_m : H^1(F, \mathbb{Z}/m) \otimes K_n(F)/m \to H^{n+1}_m(F)$$

where the right hand side is a certain cohomological group discussed below.

Here $K_n(F)$ for a field $F$ is the $n$-th Milnor $K$-group $K_n(F) = K_n^M(F)$ defined as

$$(F^*)^n / J$$

where $J$ is the subgroup generated by the elements of the form $a_1 \otimes \ldots \otimes a_n$ such that $a_i + a_j = 1$ for some $i \neq j$. We denote by $\{a_1, \ldots, a_n\}$ the class of $a_1 \otimes \ldots \otimes a_n$. Namely, $K_n(F)$ is the abelian group defined by the following generators: symbols $\{a_1, \ldots, a_n\}$ with $a_1, \ldots, a_n \in F^*$ and relations:

$$\{a_1, \ldots, a_i a'_i, \ldots, a_n\} = \{a_1, \ldots, a_i, \ldots, a_n\} + \{a_1, \ldots, a'_i, \ldots, a_n\}$$

$$\{a_1, \ldots, a_n\} = 0 \quad \text{if} \ a_i + a_j = 1 \text{ for some } i \text{ and } j \text{ with } i \neq j.$$

We write the group law additively.

Published 10 December 2000: © Geometry & Topology Publications
Consider the following example (definitions of the groups will be given later).

**Example.** Let \( F \) be a field and let \( p \) be a prime integer. Assume that there is an integer \( n \) with the following properties:

(i) the group \( H_{n+1}^p(F) \) is isomorphic to \( \mathbb{Z}/p \),

(ii) the pairing

\[
( , )_p : H^1(F, \mathbb{Z}/p) \otimes K_n(F)/p \to H_{n+1}^p(F) \simeq \mathbb{Z}/p
\]

is non-degenerate in a certain sense.

Then the \( \mathbb{Z}/p \)-linear space \( H^1(F, \mathbb{Z}/p) \) is obviously dual to the \( \mathbb{Z}/p \)-linear space \( K_n(F)/p \). On the other hand, \( H^1(F, \mathbb{Z}/p) \) is dual to the \( \mathbb{Z}/p \)-space \( G_{ab}/(G_{ab})^p \).

Therefore there is an isomorphism

\[
\Psi_{F,p} : K_n(F)/p \simeq G_{ab}/(G_{ab})^p.
\]

It turns out that this example can be applied to computations of the group \( G_{ab}/(G_{ab})^p \) for multidimensional local fields. Moreover, it is possible to show that the homomorphism \( \Psi_{F,p} \) can be naturally extended to a homomorphism \( \Psi_F : K_n(F) \to G_{ab} \) (the so called reciprocity map). Since \( G_{ab} \) is a profinite group, it follows that the homomorphism \( \Psi_F : K_n(F) \to G_{ab} \) factors through the homomorphism \( K_n(F)/DK_n(F) \to G_{ab} \) where the group \( DK_n(F) \) consists of all divisible elements:

\[
DK_n(F) := \bigcap_{m \geq 1} mK_n(F).
\]

This observation makes natural the following notation:

**Definition** (cf. section 6 of Part I). For a field \( F \) and integer \( n \geq 0 \) set

\[
K_n^+(F) := K_n(F)/DK_n(F),
\]

where \( DK_n(F) := \bigcap_{m \geq 1} mK_n(F) \).

The group \( K_n^+(F) \) for a higher local field \( F \) endowed with a certain topology (cf. section 6 of this part of the volume) is called a topological Milnor \( K \)-group \( K_{top}(F) \) of \( F \).

The example shows that computing the group \( G_{ab} \) is closely related to computing the groups \( K_n(F), \ K_n^+(F), \) and \( H_{n+1}^m(F) \). The main purpose of this section is to explain some basic properties of these groups and discuss several classical conjectures.

Among the problems, we point out the following:

- discuss \( p \)-torsion and cotorsion of the groups \( K_n(F) \) and \( K_n^+(F) \),
- study an analogue of Satz 90 for the groups \( K_n(F) \) and \( K_n^+(F) \),
- compute the group \( H_{n+1}^m(F) \) in two "classical" cases where \( F \) is either the rational function field in one variable \( F = k(t) \) or the formal power series \( F = k((t)) \).

We shall consider in detail the case (so called "non-classical case") of a field \( F \) of characteristic \( p \) and \( m = p \).
2.1. Definition of $H^{n+1}_m(F)$ and pairing $(\cdot, \cdot)_m$

To define the group $H^{n+1}_m(F)$ we consider three cases depending on the characteristic of the field $F$.

**Case 1 (Classical).** Either $\text{char}(F) = 0$ or $\text{char}(F) = p$ is prime to $m$.

In this case we set

$$H^{n+1}_m(F) := H^{n+1}(F, \mu_m^\otimes n).$$

The Kummer theory gives rise to the well known natural isomorphism $F^*/F^*_m \to H^1(F, \mu_m^\otimes n)$. Denote the image of an element $a \in F^*$ under this isomorphism by $(a)$. The cup product gives the homomorphism

$$F^n \otimes \cdots \otimes F^n \to H^n(F, \mu_m^\otimes n), \quad a_1 \otimes \cdots \otimes a_n \mapsto (a_1, \ldots, a_n)$$

where $(a_1, \ldots, a_n) := (a_1) \cup \cdots \cup (a_n)$. It is well known that the element $(a_1, \ldots, a_n)$ is zero if $a_i + a_j = 1$ for some $i \neq j$. From the definition of the Milnor $K$-group we get the homomorphism

$$\eta_m: K^M_n(F)/m \to H^n(F, \mu_m^\otimes n), \quad \{a_1, \ldots, a_n\} \mapsto (a_1, \ldots, a_n).$$

Now, we define the pairing $(\cdot, \cdot)_m$ as the following composite

$$H^1(F, \mathbb{Z}/m) \otimes K_n(F)/m \xrightarrow{\text{id} \otimes \eta_m} H^1(F, \mathbb{Z}/m) \otimes H^n(F, \mu_m^\otimes n) \xrightarrow{\cup} H^{n+1}_m(F).$$

**Case 2.** $\text{char}(F) = p \neq 0$ and $m$ is a power of $p$.

To simplify the exposition we start with the case $m = p$. Set

$$H^{n+1}_p(F) = \text{coker}(\Omega^n_F \to \Omega^n_F / d \Omega^{n-1}_F)$$

where

$$d(a db_2 \wedge \cdots \wedge db_n) = da \wedge db_2 \wedge \cdots \wedge db_n,$$

$$\varphi(a \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n}) = (a^p - a) \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n} + d \Omega^{n-1}_F$$

($\varphi = C^{-1} - 1$ where $C^{-1}$ is the inverse Cartier operator defined in subsection 4.2). The pairing $(\cdot, \cdot)_p$ is defined as follows:

$$(\cdot, \cdot)_p: F/\varphi(F) \times K_n(F)/p \to H^{n+1}_p(F),$$

$$(a, \{b_1, \ldots, b_n\}) \mapsto a \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n}$$

where $F/\varphi(F)$ is identified with $H^1(F, \mathbb{Z}/p)$ via Artin–Schreier theory.
To define the group $H_{p_i}^{n+1}(F)$ for an arbitrary $i \geq 1$ we note that the group $H_{p_i}^{n+1}(F)$ is the quotient group of $\Omega_p^n$. In particular, generators of the group $H_{p_i}^{n+1}(F)$ can be written in the form $adb_1 \wedge \cdots \wedge db_n$. Clearly, the natural homomorphism

$$F \otimes F^* \otimes \cdots \otimes F^* \to H_{p_i}^{n+1}(F), \quad a \otimes b_1 \otimes \cdots \otimes b_n \mapsto \frac{adb_1}{b_1} \wedge \cdots \wedge \frac{db_n}{b_n}$$

is surjective. Therefore the group $H_{p_i}^{n+1}(F)$ is naturally identified with the quotient group $F \otimes F^* \otimes \cdots \otimes F^*/J$. It is not difficult to show that the subgroup $J$ is generated by the following elements:

- $(a^p - a) \otimes b_1 \otimes \cdots \otimes b_n$,
- $a \otimes a \otimes b_2 \otimes \cdots \otimes b_n$,
- $a \otimes b_1 \otimes \cdots \otimes b_n$, where $b_i = b_j$ for some $i \neq j$.

This description of the group $H_{p_i}^{n+1}(F)$ can be easily generalized to define $H_{p_i}^{n+1}(F)$ for an arbitrary $i \geq 1$. Namely, we define the group $H_{p_i}^{n+1}(F)$ as the quotient group

$$W_i(F) \otimes \underbrace{F^* \otimes \cdots \otimes F^*}_{n} / J$$

where $W_i(F)$ is the group of Witt vectors of length $i$ and $J$ is the subgroup of $W_i(F) \otimes F^* \otimes \cdots \otimes F^*$ generated by the following elements:

- $(F(w) - w) \otimes b_1 \otimes \cdots \otimes b_n$,
- $(a, 0, \ldots, 0) \otimes a \otimes b_2 \otimes \cdots \otimes b_n$,
- $w \otimes b_1 \otimes \cdots \otimes b_n$, where $b_i = b_j$ for some $i \neq j$.

The pairing $(\ , )_{p_i}$ is defined as follows:

$$(\ , )_{p_i}: W_i(F)/\wp(W_i(F)) \times K_n(F)/p^i \to H_{p_i}^{n+1}(F),$$

$$(w, \{b_1, \ldots, b_n\}) \mapsto w \otimes b_1 \otimes \cdots \otimes b_n$$

where $\wp = F - \text{id}: W_i(F) \to W_i(F)$ and the group $W_i(F)/\wp(W_i(F))$ is identified with $H^1(F, \mathbb{Z}/p^i)$ via Witt theory. This completes definitions in Case 2.

Case 3. $\text{char}(F) = p \neq 0$ and $m = m'p^i$ where $m' > 1$ is an integer prime to $p$ and $i \geq 1$.

The groups $H_{m}^{n+1}(F)$ and $H_{p_i}^{n+1}(F)$ are already defined (see Cases 1 and 2). We define the group $H_{m}^{n+1}(F)$ by the following formula:

$$H_{m}^{n+1}(F) := H_{m}^{n+1}(F) \oplus H_{p_i}^{n+1}(F)$$

Since $H^1(F, \mathbb{Z}/m) \simeq H^1(F, \mathbb{Z}/m') \oplus H^1(F, \mathbb{Z}/p^i)$ and $K_n(F)/m \simeq K_n(F)/m' \oplus K_n(F)/p^i$, we can define the pairing $(\ , )_m$ as the direct sum of the pairings $(\ , )_{m'}$ and $(\ , )_{p_i}$. This completes the definition of the group $H_{m}^{n+1}(F)$ and of the pairing $(\ , )_m$.
Part I. Section 2. $K$-groups and Galois cohomologies of fields of characteristic $p$

Remark 1. In the case $n = 1$ or $n = 2$ the group $H_n^m(F)$ can be determined as follows:

$$H_1^m(F) \cong H^1(F, \mathbb{Z}/m) \quad \text{and} \quad H_2^m(F) \cong m \text{Br}(F).$$

Remark 2. The group $H_n^{n+1}(F)$ is often denoted by $H^{n+1}(F, \mathbb{Z}/m(n))$.

2.2. The group $H_n^{n+1}(F)$

In the previous subsection we defined the group $H_n^{n+1}(F)$ and the pairing $(\ , \ )_m$ for an arbitrary $m$. Now, let $m$ and $m'$ be positive integers such that $m'$ is divisible by $m$. In this case there exists a canonical homomorphism

$$i_{m,m'}: H_n^{n+1}(F) \to H_n^{n+1}(F).$$

To define the homomorphism $i_{m,m'}$ it suffices to consider the following two cases:

**Case 1.** Either $\text{char}(F) = 0$ or $\text{char}(F) = p$ is prime to $m$ and $m'$.

This case corresponds to Case 1 in the definition of the group $H_n^{n+1}(F)$ (see subsection 2.1). We identify the homomorphism $i_{m,m'}$ with the homomorphism

$$H_n^{n+1}(F, \mu \otimes \mathbb{Z}/m(n)) \to H_n^{n+1}(F, \mu \otimes \mathbb{Z}/m(n))$$

induced by the natural embedding $\mu_m \subset \mu_{m'}$.

**Case 2.** $m$ and $m'$ are powers of $p = \text{char}(F)$.

We can assume that $m = p^i$ and $m' = p^{i'}$ with $i \leq i'$. This case corresponds to Case 2 in the definition of the group $H_n^{n+1}(F)$. We define $i_{m,m'}$ as the homomorphism induced by

$$W_i(F) \otimes F^* \otimes \ldots F^* \rightarrow W_{i'}(F) \otimes F^* \otimes \ldots F^*,$$

$$(a_1, \ldots, a_i) \otimes b_1 \otimes \ldots \otimes b_n \mapsto (0, \ldots, 0, a_1, \ldots, a_i) \otimes b_1 \otimes \ldots \otimes b_n.$$

The maps $i_{m,m'}$ (where $m$ and $m'$ run over all integers such that $m'$ is divisible by $m$) determine the inductive system of the groups.

**Definition.** For a field $F$ and an integer $n$ set

$$H_n^{n+1}(F) = \lim_{\rightarrow m} H_n^{n+1}(F).$$

**Conjecture 1.** The natural homomorphism $H_n^{n+1}(F) \rightarrow H_n^{n+1}(F)$ is injective and the image of this homomorphism coincides with the $m$-torsion part of the group $H_n^{n+1}(F)$.
This conjecture follows easily from the Milnor–Bloch–Kato conjecture (see subsection 4.1) in degree \( n \). In particular, it is proved for \( n \leq 2 \). For fields of characteristic \( p \) we have the following theorem.

**Theorem 1.** Conjecture 1 is true if \( \text{char}(F) = p \) and \( m = p^i \).

### 2.3. Computing the group \( H^{n+1}_m(F) \) for some fields

We start with the following well known result.

**Theorem 2** (classical). Let \( F \) be a perfect field. Suppose that \( \text{char}(F) = 0 \) or \( \text{char}(F) \) is prime to \( m \). Then

\[
H^{n+1}_m(F((t))) \simeq H^{n+1}_m(F) \oplus H^n_m(F)
\]

\[
H^{n+1}_m(F(t)) \simeq H^{n+1}_m(F) \oplus \bigoplus_{\text{monic irred } f(t)} H^n_m(F[t]/f(t)).
\]

It is known that we cannot omit the conditions on \( F \) and \( m \) in the statement of Theorem 2. To generalize the theorem to the arbitrary case we need the following notation. For a complete discrete valuation field \( K \) and its maximal unramified extension \( K_{ur} \) define the groups \( H^n_{m,ur}(K) \) and \( \tilde{H}^n_m(K) \) as follows:

\[
H^n_{m,ur}(K) = \ker(H^n_m(K) \to H^n_m(K_{ur})) \quad \text{and} \quad \tilde{H}^n_m(K) = H^n_m(K)/H^n_{m,ur}(K).
\]

Note that for a field \( K = F((t)) \) we obviously have \( K_{ur} = F_{\text{sep}}((t)) \). We also note that under the hypotheses of Theorem 2 we have \( H^n(K) = H^n_{m,ur}(K) \) and \( H^n(K) = 0 \). The following theorem is due to Kato.

**Theorem 3** (Kato, [K1, Th. 3 §0]). Let \( K \) be a complete discrete valuation field with residue field \( k \). Then

\[
H^{n+1}_{m,ur}(K) \simeq H^{n+1}_{m,ur}(k) \oplus H^n_m(k).
\]

In particular, \( H^{n+1}_{m,ur}(F((t))) \simeq H^{n+1}_m(F) \oplus H^n_m(F) \).

This theorem plays a key role in Kato’s approach to class field theory of multidimensional local fields (see section 5 of this part).

To generalize the second isomorphism of Theorem 2 we need the following notation. Set

\[
\begin{align*}
H^{n+1}_{m,\text{sep}}(F(t)) &= \ker(H^{n+1}_m(F(t)) \to H^{n+1}_m(F_{\text{sep}}(t))) \quad \text{and} \\
\tilde{H}^{n+1}_m(F(t)) &= H^{n+1}_m(F(t))/H^{n+1}_{m,\text{sep}}(F(t)).
\end{align*}
\]

If the field \( F \) satisfies the hypotheses of Theorem 2, we have \( H^{n+1}_{m,\text{sep}}(F(t)) = H^{n+1}_m(F(t)) \) and \( \tilde{H}^{n+1}_m(F(t)) = 0 \).

In the general case we have the following statement.
Theorem 4 (Izhboldin, [12, Introduction]).

\[ H_{n+1}^{m, \text{sep}}(F(t)) \simeq H_{n+1}^m(F) \oplus \prod_{\text{monic irred } f(t)} H_{m}^n(F[t]/f(t)), \]

\[ \tilde{H}_{n+1}^m(F(t)) \simeq \bigoplus_v \tilde{H}_{n+1}^m(F(t)_v) \]

where \( v \) runs over all normalized discrete valuations of the field \( F(t) \) and \( F(t)_v \) denotes the \( v \)-completion of \( F(t) \).

2.4. On the group \( K_n(F) \)

In this subsection we discuss the structure of the torsion and cotorsion in Milnor \( K \)-theory. For simplicity, we consider the case of prime \( m = p \). We start with the following fundamental theorem concerning the quotient group \( K_n(F)/p \) for fields of characteristic \( p \).

Theorem 5 (Bloch–Kato–Gabber, [BK, Th. 2.1]). Let \( F \) be a field of characteristic \( p \). Then the differential symbol

\[ d_F : K_n(F)/p \to \Omega_F^n, \quad \{a_1, \ldots, a_n\} \mapsto \frac{da_1}{a_1} \wedge \cdots \wedge \frac{da_n}{a_n} \]

is injective and its image coincides with the kernel \( \nu_n(F) \) of the homomorphism \( \varphi \) (for the definition see Case 2 of 2.1). In other words, the sequence

\[ 0 \to K_n(F)/p \xrightarrow{d_F} \Omega_F^n \xrightarrow{\varphi} \Omega_F^n / d\Omega_F^{n-1} \]

is exact.

This theorem relates the Milnor \( K \)-group modulo \( p \) of a field of characteristic \( p \) with a submodule of the differential module whose structure is easier to understand. The theorem is important for Kato’s approach to higher local class field theory. For a sketch of its proof see subsection A2 in the appendix to this section.

There exists a natural generalization of the above theorem for the quotient groups \( K_n(F)/p^i \) by using De Rham–Witt complex ([BK, Cor. 2.8]).

Now, we recall well known Tate’s conjecture concerning the torsion subgroup of the Milnor \( K \)-groups.

Conjecture 2 (Tate). Let \( F \) be a field and \( p \) be a prime integer.

(i) If \( \text{char } (F) \neq p \) and \( \zeta_p \in F \), then \( pK_n(F) = \{\zeta_p\} \cdot K_{n-1}(F) \).

(ii) If \( \text{char } (F) = p \) then \( pK_n(F) = 0 \).

This conjecture is trivial in the case where \( n \leq 1 \). In the other cases we have the following theorem.
Theorem 6. Let \( F \) be a field and \( n \) be a positive integer.

1. Tate’s Conjecture holds if \( n \leq 2 \) (Suslin, [S]).
2. Part (ii) of Tate’s Conjecture holds for all \( n \) (Izhboldin, [11]).

The proof of this theorem is closely related to the proof of Satz 90 for \( K \)-groups. Let us recall two basic conjectures on this subject.

Conjecture 3 (Satz 90 for \( K_n \)). If \( L/F \) is a cyclic extension of degree \( p \) with the Galois group \( G = \langle \sigma \rangle \) then the sequence

\[
K_n(L) \xrightarrow{1-\sigma} K_n(L) \xrightarrow{N_{L/F}} K_n(F)
\]

is exact.

There is an analogue of the above conjecture for the quotient group \( K_n(F)/p \). Fix the following notation till the end of this section:

Definition. For a field \( F \) set

\[
k_n(F) = K_n(F)/p.
\]

Conjecture 4 (Small Satz 90 for \( k_n \)). If \( L/F \) is a cyclic extension of degree \( p \) with the Galois group \( G = \langle \sigma \rangle \), then the sequence

\[
k_n(F) \oplus k_n(L) \xrightarrow{i_{F/L} \oplus (1-\sigma)} k_n(L) \xrightarrow{N_{L/F}} k_n(F)
\]

is exact.

The conjectures 2,3 and 4 are not independent:

**Lemma** (Suslin). Fix a prime integer \( p \) and integer \( n \). Then in the category of all fields (of a given characteristic) we have

\((\text{Small Satz } 90 \text{ for } k_n) + (\text{Tate conjecture for } pK_n) \iff (\text{Satz } 90 \text{ for } K_n)\).

Moreover, for a given field \( F \) we have

\((\text{Small Satz } 90 \text{ for } k_n) + (\text{Tate conjecture for } pK_n) \Rightarrow (\text{Satz } 90 \text{ for } K_n)\)

and

\((\text{Satz } 90 \text{ for } K_n) \Rightarrow (\text{small Satz } 90 \text{ for } k_n)\).

Satz 90 conjectures are proved for \( n \leq 2 \) (Merkurev-Suslin, [MS1]). If \( p = 2 \), \( n = 3 \), and \( \text{char } (F) \neq 2 \), the conjectures were proved by Merkurev and Suslin [MS] and Rost. For \( p = 2 \) the conjectures follow from recent results of Voevodsky. For fields of characteristic \( p \) the conjectures are proved for all \( n \):
Part I. Section 2. \(K\)-groups and Galois cohomologies of fields of characteristic \(p\)

**Theorem 7** (Izhboldin, [I1]). Let \(F\) be a field of characteristic \(p\) and \(L/F\) be a cyclic extension of degree \(p\). Then the following sequence is exact:

\[
0 \rightarrow K_n(F) \rightarrow K_n(L) \xrightarrow{1-\sigma} K_n(L) \xrightarrow{N_{L/F}} K_n(F) \rightarrow H_{n+1}^p(F) \rightarrow H_{n+1}^p(L)
\]

### 2.5. On the group \(K^t_n(F)\)

In this subsection we discuss the same issues, as in the previous subsection, for the group \(K^t_n(F)\).

**Definition.** Let \(F\) be a field and \(p\) be a prime integer. We set

\[
DK_n(F) = \bigcap_{m \geq 1} mK_n(F) \quad \text{and} \quad D_pK_n(F) = \bigcap_{i \geq 0} p^iK_n(F).
\]

We define the group \(K^t_n(F)\) as the quotient group:

\[
K^t_n(F) = K_n(F)/DK_n(F) = K_n(F)/\bigcap_{m \geq 1} mK_n(F).
\]

The group \(K^t_n(F)\) is of special interest for higher class field theory (see sections 6, 7 and 10). We have the following evident isomorphism (see also 2.0):

\[
K^t_n(F) \simeq \text{im} \left( K_n(F) \rightarrow \lim_{\leftarrow m} K_n(F)/m \right).
\]

The quotient group \(K^t_n(F)/m\) is obviously isomorphic to the group \(K_n(F)/m\). As for the torsion subgroup of \(K^t_n(F)\), it is quite natural to state the same questions as for the group \(K_n(F)\).

**Question 1.** Are the \(K^t\)-analogue of Tate’s conjecture and Satz 90 Conjecture true for the group \(K^t_n(F)\)?

If we know the (positive) answer to the corresponding question for the group \(K_n(F)\), then the previous question is equivalent to the following:

**Question 2.** Is the group \(DK_n(F)\) divisible?

At first sight this question looks trivial because the group \(DK_n(F)\) consists of all divisible elements of \(K_n(F)\). However, the following theorem shows that the group \(DK_n(F)\) is not necessarily a divisible group!

**Theorem 8** (Izhboldin, [I3]). For every \(n \geq 2\) and prime \(p\) there is a field \(F\) such that \(\text{char}(F) \neq p\), \(\zeta_p \in F\) and

1. The group \(DK_n(F)\) is not divisible, and the group \(D_pK_2(F)\) is not \(p\)-divisible.

Geometry & Topology Monographs, Volume 3 (2000) – Invitation to higher local fields
(2) The $K^t$-analogue of Tate’s conjecture is false for $K^t_{n}$:
\[ pK^t_{n}(F) \neq \{ \zeta_p \} \cdot K^t_{n-1}(F). \]

(3) The $K^t$-analogue of Hilbert 90 conjecture is false for group $K^t_{n}(F)$.

**Remark 1.** The field $F$ satisfying the conditions of Theorem 8 can be constructed as the function field of some infinite dimensional variety over any field of characteristic zero whose group of roots of unity is finite.

Quite a different construction for irregular prime numbers $p$ and $F = \mathbb{Q}(\mu_p)$ follows from works of G. Banaszak [B].

**Remark 2.** If $F$ is a field of characteristic $p$ then the groups $D_pK^t_{n}(F)$ and $DK^t_{n}(F)$ are $p$-divisible. This easily implies that $pK^t_{n}(F) = 0$. Moreover, Satz 90 theorem holds for $K^t_{n}$ in the case of cyclic $p$-extensions.

**Remark 3.** If $F$ is a multidimensional local fields then the group $K^t_{n}(F)$ is studied in section 6 of this volume. In particular, Fesenko (see subsections 6.3–6.8 of section 6) gives positive answers to Questions 1 and 2 for multidimensional local fields.

**References**

[B] G. Banaszak, Generalization of the Moore exact sequence and the wild kernel for higher $K$-groups, Compos. Math., 86(1993), 281–305.

[BK] S. Bloch and K. Kato, $p$-adic étale cohomology, Inst. Hautes Études Sci. Publ. Math. 63, (1986), 107–152.

[F] I. Fesenko, Topological Milnor $K$-groups of higher local fields, section 6 of this volume.

[I1] O. Izhboldin, On $p$-torsion in $K^M_{s}$ for fields of characteristic $p$, Adv. Soviet Math., vol. 4, 129–144, Amer. Math. Soc., Providence RI, 1991

[I2] O. Izhboldin, On the cohomology groups of the field of rational functions, Mathematics in St.Petersburg, 21–44, Amer. Math. Soc. Transl. Ser. 2, vol. 174, Amer. Math. Soc., Providence, RI, 1996.

[I3] O. Izhboldin, On the quotient group of $K_{2}(F)$, preprint, www.maths.nott.ac.uk/personal/ibf/stqk.ps

[K1] K. Kato, Galois cohomology of complete discrete valuation fields, In Algebraic $K$-theory, Lect. Notes in Math. 967, Springer-Verlag, Berlin, 1982, 215–238.

[K2] K. Kato, Symmetric bilinear forms, quadratic forms and Milnor $K$-theory in characteristic two, Invent. Math. 66(1982), 493–510.

[MS1] A. S. Merkur’ev and A. A. Suslin, $K$-cohomology of Severi-Brauer varieties and the norm residue homomorphism, Izv. Akad. Nauk SSSR Ser. Mat. 46(1982); English translation in Math. USSR Izv. 21(1983), 307–340.
[MS2] A. S. Merkur’ev and A. A. Suslin, The norm residue homomorphism of degree three, Izv. Akad. Nauk SSSR Ser. Mat. 54(1990); English translation in Math. USSR Izv. 36(1991), 349–367.

[MS3] A. S. Merkur’ev and A. A. Suslin, The group $K_3$ for a field, Izv. Akad. Nauk SSSR Ser. Mat. 54(1990); English translation in Math. USSR Izv. 36(1991), 541–565.

[S] A. A. Suslin, Torsion in $K_2$ of fields, K-theory 1(1987), 5–29.