A Bounded mean oscillation (BMO) theorem for small distorted diffeomorphisms from $\mathbb{R}^D$ to $\mathbb{R}^D$ and PDE.

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Abstract

This announcement considers the following problem. We produce a bounded mean oscillation theorem for small distorted diffeomorphisms from $\mathbb{R}^D$ to $\mathbb{R}^D$. A revision of this announcement is in the memoir preprint: arXiv: 2103.09748, [1], submitted for consideration for publication.

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1 Notation and Table of Contents.

We will work throughout in Euclidean space $\mathbb{R}^D$. Here and throughout, $D \geq 2$ is fixed. $|.|$ will denote the Euclidean norm on $\mathbb{R}^D$ and $|.|_{L_2(\cdot)}$ the $L_2$ norm on $\mathbb{R}^D$ when suitably defined. The letters $C, C', c, c_1, ..$ will be used for positive constants depending on $D$ and possibly other quantities. The later will be indicated when necessary. $B := B(z,r)$ will denote a ball in $\mathbb{R}^D$ with center $z$ and radius $r > 0$. The $D$ dimensional volume of a ball $B$ will be denoted by $\text{Vol}(B)$. $\varepsilon$ will be reserved for a small enough positive constant depending on $D$ and fixed. A map $f : \mathbb{R}^D \to \mathbb{R}^D$ is distorted with small $c$ if $(1 + c)^{-1}|x - y| \leq |f(x) - f(y)| \leq |x - y|(1 + c), \ x, y \in \mathbb{R}^D$.

$\Phi : \mathbb{R}^D \to \mathbb{R}^D$ is a special map which is a small distorted diffeomorphism. Euclidean motions from $\mathbb{R}^D$ to $\mathbb{R}^D$ will be denoted by $A, A_1$ given by their action $x \to T(x) + x_0$, $x_0 \in \mathbb{R}^D$ fixed and $T \in O(D)$. $O(D)$ is the orthogonal group. All other functions will be denoted by $f$. For a $D \times D$ matrix, $M = (M_{ij})$, we write $|M|$ to denote the Hilbert-Schmidt norm

$$|M| = \left(\sum_{ij} |M_{ij}|^2\right)^{1/2}.$$  

We note moving forward that if $M$ is real and symmetric and if

$$(1 - \lambda)I \leq M \leq (1 + \lambda)I$$

we have

$$\text{tr}(M) = \text{tr}(\frac{1}{2}(1 + \lambda)I) = D(1 + \lambda)$$

and

$$\text{tr}(M) = \text{tr}(\frac{1}{2}(1 - \lambda)I) = D(1 - \lambda).$$

**Notes:**

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as matrices, where $0 < \lambda < 1$, then

$$|M - I| \leq C\lambda. \quad (1.1)$$

This follows from working in an orthonormal basis for which $M$ is diagonal. $I$ is the usual identity matrix. All constants, maps, matrices, sets may be different in different occurrences. All other notation will be defined in context.

2 The breakdown of the paper is as follows

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3 Main Results

This paper studies BMO estimates for maps which are small distorted diffeomorphisms from $\mathbb{R}^D$ to $\mathbb{R}^D$. It centers around three objects and a connection between them, each interesting independently:

- Small distorted diffeomorphisms from $\mathbb{R}^D$ to $\mathbb{R}^D$.
- Functions of bounded mean oscillation (BMO).
- The John Nirenberg inequality.

There is overlap between what follows and [3].

Let us provide our framework for our main results by speaking to each of the above:

Thus we begin with:
3.1 $\varepsilon$-distorted diffeomorphisms from $\mathbb{R}^D$ to $\mathbb{R}^D$.

In this subsection we look at the above and define a special map $\Phi : \mathbb{R}^D \to \mathbb{R}^D$ which is a small distorted diffeomorphism. We recall that we fix a dimension $D \geq 2$ and a $\varepsilon > 0$ small enough.

An $\varepsilon$-distorted diffeomorphism $\Phi : \mathbb{R}^D \to \mathbb{R}^D$ is a diffeomorphism (hence one to one and onto) map with
\[
(1 + \varepsilon)^{-1} I \leq (\Phi'(x))^T (\Phi'(x)) \leq (1 + \varepsilon) I
\]
as matrices. Such maps $\Phi$ satisfy
\[
\left| (\Phi'(x))^T (\Phi'(x)) - I \right| \leq C\varepsilon.
\] (3.1)

An easy application of Bochner’s theorem then tells us that if $\Phi$ is an $\varepsilon$-distorted diffeomorphism then
\[
(1 + \varepsilon)^{-1} |x - y| \leq |\Phi(x) - \Phi(y)| \leq (1 + \varepsilon) |x - y|,
\]
$x,y \in \mathbb{R}^D$.

3.2 Functions of bounded mean oscillation (BMO).

In this subsection recall the definition of the space of functions $f : \mathbb{R}^D \to \mathbb{R}^D$ of bounded mean oscillation (BMO).

The mean oscillation of a locally integrable function $f : \mathbb{R}^D \to \mathbb{R}$ over a ball $B \subset \mathbb{R}^D$ is defined as the value of the following integral:
\[
\frac{1}{\text{Vol}(B)} \int_B |f(y) - C_B| dy
\]
where the real number $C_B := C(B, f) := \frac{1}{\text{Vol}(B)} \int_B f(y) dy$ is the average value of $f$ over the ball $B$. $f$ is of bounded mean oscillation (BMO) if the supremum of the mean oscillation taken over all balls $B \subset \mathbb{R}^D$ is finite. State another way: That is, there exists $K \geq 0$ such that, for every ball $B \subset \mathbb{R}^D$,
\[
\frac{1}{\text{Vol}B} \int_B |f(y) - C_B| dy \leq K.
\]
The least such $K$ is denoted by $||f||_{\text{BMO}}$. BMO functions were studied first by Fritz John and Louis Nirenberg who introduced and studied them for the first time in the context of elasticity. See [5, 6]. (The BMO function space is often called a John-Nirenberg space). They are more easily understood in a probabilistic framework as is apparent from their definition.

The space BMO appears in two classical results. The first due to Charles Fefferman who proved [4] that as Banach spaces, the space BMO and the Hardy space $H_1$ are dual to each other.

The second classical result is called:
3.3 The John-Nirenberg inequality.

In this subsection, we recall the John-Nirenberg inequality. The John-Nirenberg inequality asserts the following: Let \( f \in \text{BMO} \) and let \( B \subset \mathbb{R}^D \) be a ball. Then there exists a real number \( C_B := C(B, f) \) such that

\[
\text{vol} \left\{ x \in B : |f(x) - C_B| > C\lambda \|f\|_{\text{BMO}} \right\} \leq \exp(-\lambda)\text{Vol} B, \, \lambda \geq 1.
\] (3.2)

We are now ready for our main results:

3.4 Main results

We are ready to state the main results of this paper. We make one quick comment before we do: The definition of BMO, the notion of the BMO norm and the John-Nirenburg inequality carry through to the case of functions with domain \( \mathbb{R}^D \) and which take their values in the space of \( D \times D \) matrices. To see this, take the constant \( C_B := C(B, f) \) to be a \( D \times D \) matrix for the function \( f \).

Moving forward, we now state the main results of this paper:

The first is given by:

**BMO Theorem 1.** Let \( \Phi : \mathbb{R}^D \to \mathbb{R}^D \) be an \( \varepsilon \) distorted diffeomorphism. Let \( B \subset \mathbb{R}^D \) be a ball. Then, there exists \( T_B \in O(D) \) such that for every \( \lambda \geq 1 \),

\[
\text{vol} \left\{ x \in B : |\Phi'(x) - T_B| > C\lambda \varepsilon \right\} \leq \exp(-\lambda)\text{vol} (B).
\] (3.3)

Moreover, Theorem 1 is sharp in the sense of small volume if one takes a slow twist defined as follows: For \( x \in \mathbb{R}^D \), let \( S_x \) be the block-diagonal matrix

\[
\begin{pmatrix}
D_1(x) & 0 & 0 & 0 & 0 \\
0 & D_2(x) & 0 & 0 & 0 \\
0 & 0 & . & 0 & 0 \\
0 & 0 & 0 & . & 0 \\
0 & 0 & 0 & 0 & D_r(x)
\end{pmatrix}
\]

where, for each \( i \), either \( D_i(x) \) is the \( 1 \times 1 \) identity matrix or else

\[
D_i(x) = \begin{pmatrix}
\cos f_i(|x|) & \sin f_i(|x|) \\
-\sin f_i(|x|) & \cos f_i(|x|)
\end{pmatrix}
\]

for a function \( f_i \) of one variable. Now define for each \( x \in \mathbb{R}^D \), \( \Phi(x) = \Theta^T S_x(\Theta x) \) where \( \Theta \) is any fixed matrix in \( SO(D) \). One checks that \( \Phi \) is \( \varepsilon \)-distorted, provided for each \( i \), \( t|f_i'(t)| < c\varepsilon \) for all \( t \in [0, \infty) \).

**Proof** The proof of the theorem follows from (3.2) and (3.3) below. \( \square \).
The second is given by:

**BMO Theorem 2.** Let $\Phi : \mathbb{R}^D \to \mathbb{R}^D$ be an $\varepsilon$ diffeomorphism and let $B \subset \mathbb{R}^D$ be a ball. Then, there exists $T_B \in O(D)$ such that

$$
\frac{1}{\text{vol } B} \int_B |\Phi'(x) - T_B| \, dx \leq c_1 \varepsilon^{1/2}.
$$

(3.4)

The third is given as a refinement of Theorem 1:

**BMO Theorem 3.** Let $\Phi : \mathbb{R}^D \to \mathbb{R}^D$ be an $\varepsilon$ diffeomorphism and let $B \in \mathbb{R}^D$ be a ball. Then, there exists $T_B \in O(D)$ such that

$$
\frac{1}{\text{vol } B} \int_B |\Phi'(x) - T_B| \, dx \leq C \varepsilon.
$$

(3.5)

The fourth is given as a PDE theorem for overdetermined systems:

**Theorem 4** Let $\Omega_1, ..., \Omega_D$ and $f_{ij}, i, j = 1, ..., D$ be smooth functions on $\mathbb{R}^D$. Assume that the following holds:

(1)

$$
\frac{\partial \Omega_i}{\partial x_j} + \frac{\partial \Omega_j}{\partial x_i} = f_{ij}, i, j = 1, ..., D
$$

on $\mathbb{R}^D$. Here, $\Omega_i$ and $f_{ij}$ are $C^\infty$ functions on $\mathbb{R}^D$.

(2)

$$
\|f_{ij}\|_{L^2(B(0,1))} \leq 1.
$$

Then, there exist real numbers $\Delta_{ij}, i, j = 1, ..., D$ such that

(3)

$$
\Delta_{ij} + \Delta_{ji} = 0, \forall i, j.
$$

(4)

$$
\left\| \frac{\partial \Omega_i}{\partial x_j} - \Delta_{ij} \right\|_{L^2(B(0,1))} \leq C.
$$
4 Proof of Theorem 4.

In this section, we prove Theorem 4. Henceforth will need some potential theory. If \( f \) is a smooth function of compact support in \( \mathbb{R}^D \), then we can write \( \Delta^{-1} f \) to denote the convolution of \( f \) with the Newtonian potential. Thus, \( \Delta^{-1} f \) is smooth and \( \Delta(\Delta^{-1} f) = f \) on \( \mathbb{R}^D \).

We will use the estimate:

\[
\left\| \frac{\partial}{\partial x_i} \Delta^{-1} \frac{\partial}{\partial x_j} f \right\|_{L^2(\mathbb{R}^D)} \leq C \| f \|_{L^2(\mathbb{R}^D)}, \quad i, j = 1, \ldots, D
\]

valid for any smooth function \( f \) with compact support. Estimate (4) follows by applying the Fourier transform.

**Proof of Theorem 4.** From (1), we see at once that

\[
\frac{\partial \Omega_i}{\partial x_i} = \frac{1}{2} f_{ii},
\]

for each \( i \). Now, by differentiating (1) with respect to \( x_j \) and then summing on \( j \), we see that

\[
\Delta \Omega_i + \frac{1}{2} \frac{\partial}{\partial x_j} \left( \sum_j f_{jj} \right) = \sum_j \frac{\partial f_{ij}}{\partial x_j},
\]

for each \( i \). Therefore, we may write

\[
\Delta \Omega_i = \sum_j \frac{\partial}{\partial x_j} g_{ij},
\]

for smooth functions \( g_{ij} \) with

\[
\| g_{ij} \|_{L^2(B(0,4))} \leq C.
\]

This holds for each \( i \). Let \( \chi \) be a \( C^\infty \) cutoff function on \( \mathbb{R}^D \) equal to 1 on \( B(0,2) \) vanishing outside \( B(0,4) \) and satisfying \( 0 \leq \chi \leq 1 \) everywhere. Now let

\[
\Omega_i^{err} = \Delta^{-1} \sum_j \frac{\partial}{\partial x_j} (\chi g_{ji})
\]

and let

\[
\Omega_i^* = \Omega_i - \Omega_i^{err}.
\]

Then,

\[
(5) \quad \Omega_i = \Omega_i^* + \Omega_i^{err}.
\]

each \( i \).
\( \Omega_i^* \)

is harmonic on \( B(0,2) \) and

\[ \| \nabla \Omega_i^{\text{err}} \|_{L^2(B(0,2))} \leq C \]

thanks to \((\cdot)\). By \((1), (2), (5), (7))\), we can write

\[ \frac{\partial \Omega_i^*}{\partial x_j} + \frac{\partial \Omega_i^*}{\partial x_i} = f_{ij}^*, \ i, j = 1, \ldots, D. \]

on \( B(0,2) \) and with

\[ \| f_{ij}^* \|_{L^2(B(0,2))} \leq C. \]

From \((6)\) and \((8)\), we see that each \( f_{ij}^* \) is a harmonic function on \( B(0,2) \). Consequently, \((9)\) implies

\[ \sup_{B(0,1)} \| \nabla f_{ij}^* \| \leq C. \]

From \((8)\), we have for each \( i, j, k \),

\[ \frac{\partial^2 \Omega_i^*}{\partial x_j \partial x_k} + \frac{\partial^2 \Omega_k^*}{\partial x_i \partial x_j} = \frac{\partial f_{ik}^*}{\partial x_j} + \frac{\partial f_{ik}^*}{\partial x_i} + \frac{\partial^2 \Omega_j^*}{\partial x_i \partial x_k} + \frac{\partial^2 \Omega_j^*}{\partial x_i \partial x_k} = \frac{\partial f_{ij}^*}{\partial x_k}. \]

Now adding the equations \((11)\) and subtracting \((12)\), we obtain:

\[ 2 \frac{\partial^2 \Omega_i^*}{\partial x_j \partial x_k} = \frac{\partial f_{ik}^*}{\partial x_j} + \frac{\partial f_{ij}^*}{\partial x_k} - \frac{\partial f_{jk}^*}{\partial x_i} \]

on \( B(0,1) \). Now from \((10)\) and \((13)\), we obtain the estimate

\[ \left| \frac{\partial^2 \Omega_i^*}{\partial x_j \partial x_k} \right| \leq C. \]

on \( B(0,1) \) for each \( i, j, k \). Now for each \( i, j \), let

\[ \Delta_{ij}^* = \frac{\partial \Omega_i^*}{\partial x_j}(0). \]

By \((14)\), we have
(16)\[
\left| \frac{\partial \Omega_i^*}{\partial x_j} - \Delta_{ij}^* \right| \leq C
\]
on $B(0,1)$ for each $i,j$. Recalling (5) and (7), we see that (16) implies that
(17)\[
\left\| \frac{\partial \Omega_i}{\partial x_j} - \Delta_{ij}^* \right\|_{L^2(B(0,1))} \leq C.
\]
Unfortunately, the $\Delta_{ij}^*$ need not satisfy (3). However, (1), (2) and (17) imply the estimate
\[
|\Delta_{ij}^* + \Delta_{ji}^*| \leq C.
\]
for each $i,j$. Hence, there exist real numbers $\Delta_{ij}$, $(i,j = 1,\ldots,D)$ such that
(18)\[
\Delta_{ij} + \Delta_{ji} = 0.
\]
and
(19)\[
|\Delta_{ij}^* - \Delta_{ij}| \leq C.
\]
for each $i,j$. From (17) and (19), we see that
(20)\[
\left\| \frac{\partial \Omega_i}{\partial x_j} - \Delta_{ij} \right\|_{L^2(B(0,1))} \leq C.
\]
for each $i$ and $j$.

Thus (18) and (20) are the desired conclusions of Theorem 4. □

5 Proof of Theorem 2.

In this section, we prove Theorem 2.

Proof \[5.1\] is preserved by translations and dilations. Hence we may assume without loss of
generality that
\[
B = B(0,1).
\]
From [2], there exists an Euclidean motion $A : \mathbb{R}^D \rightarrow \mathbb{R}^D$ such that
\[
|\Phi(x) - A(x)| \leq C\varepsilon
\]
for $x \in B(0,10)$.

Our desired conclusion (1) holds for $\Phi$ iff it holds for $A^{-1} \circ \Phi$ (with a different $T$). Hence, without
loss of generality, we may assume that $A = I$. Thus, (4.3) becomes
\[
|\Phi(x) - x| \leq C\varepsilon, x \in B(0,10).
\]
We set up some notation: We write the diffeomorphism $\Phi$ in coordinates by setting:

$$\Phi(x_1, \ldots, x_D) = (y_1, \ldots, y_D)$$  \hspace{1cm} (5.4)

where for each $i$, $1 \leq i \leq D$,

$$y_i = \psi_i(x_1, \ldots, x_D).$$  \hspace{1cm} (5.5)

First claim: For each $i = 1, \ldots, D$,

$$\int_{B(0,1)} \left| \frac{\partial \psi_i(x)}{\partial x_i} - 1 \right| \leq C\varepsilon. \hspace{1cm} (5.6)$$

For this, for fixed $(x_2, \ldots, x_D) \in B'$, we apply (4.4) to the points $x^+ = (1, \ldots, x_D)$ and $x^- = (1, \ldots, x_D)$. We have

$$|\psi_1(x^+) - 1| \leq C\varepsilon$$

and

$$|\psi_1(x^-) + 1| \leq C\varepsilon.$$

Consequently,

$$\int_{-1}^{1} \frac{\partial \psi_1}{\partial x_1}(x_1, \ldots, x_D) dx_1 \geq 2 - C\varepsilon. \hspace{1cm} (5.7)$$

On the other hand, since,

$$\left(\psi'(x)\right)^T \left(\psi'(x)\right) \leq (1 + \varepsilon)I,$$

we have the inequality for each $i = 1, \ldots, D$,

$$\left(\frac{\partial \psi_i}{\partial x_i}\right)^2 \leq 1 + \varepsilon.$$

Therefore,

$$\left| \frac{\partial \psi_1}{\partial x_i} \right| - 1 \leq \sqrt{1 + \varepsilon} - 1 \leq \varepsilon. \hspace{1cm} (5.8)$$

Set

$$I^+ = \left\{ x_1 \in [-1,1] : \frac{\partial \psi_1}{\partial x_1}(x_1, \ldots, x_D) - 1 \leq 0 \right\},$$

$$I^- = \left\{ x_1 \in [-1,1] : \frac{\partial \psi_1}{\partial x_1}(x_1, \ldots, x_D) - 1 \geq 0 \right\},$$

$$\Delta^+ = \int_{I^+} \left( \frac{\partial \psi_1}{\partial x_1}(x_1, \ldots, x_D) - 1 \right) dx_1$$

and

$$\Delta^- = \int_{I^-} \left( \frac{\partial \psi_1}{\partial x_1}(x_1, \ldots, x_D) - 1 \right) dx_1.$$

The inequality (4.8) implies that $-\Delta^- \leq \varepsilon$ and the inequality (4.9) implies that $\frac{\partial \psi_1}{\partial x_1} - 1 \leq \varepsilon$. 

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Integrating the last inequality over $I^+$, we obtain $\Delta^+ \leq C\varepsilon$. Consequently,

$$
\int_{-1}^{1} \left| \frac{\partial \psi}{\partial x_1}(x_1, ..., x_D) - 1 \right| \, dx_1 = \Delta^+ - \Delta^- \leq C\varepsilon. \quad (5.9)
$$

Integrating this last equation over $(x_2, ..., x_D) \in B'$ and noting that $B(0, 1) \subset [-1, 1] \times B'$, we conclude that

$$
\int_{B(0, 1)} \left| \frac{\partial \psi}{\partial x_1}(x_1, ..., x_D) - 1 \right| \, dx \leq C\varepsilon.
$$

Similarly, for each $i = 1, ..., D$, we obtain (4.7).

Second claim: For each $i, j = 1, ..., D, i \neq j$,

$$
\int_{B(0, 1)} \left| \frac{\partial \psi_i(x)}{\partial x_j} \right| \, dx \leq C\sqrt{\varepsilon}. \quad (5.10)
$$

Since

$$(1 - \varepsilon)I \leq (\Phi'(x))^T(\Phi'(x)) \leq (1 + \varepsilon)I,$$

we have

$$
\sum_{i,j=1}^{D} \left( \frac{\partial \psi_i}{\partial x_j} \right)^2 \leq (1 + C\varepsilon)D. \quad (5.11)
$$

Therefore,

$$
\sum_{i \neq j} \left( \frac{\partial \psi_i}{\partial x_j} \right)^2 \leq C\varepsilon + \sum_{i=1}^{D} \left( 1 - \frac{\partial \psi_i}{\partial x_i} \right) \left( 1 + \frac{\partial \psi_i}{\partial x_i} \right).
$$

Using (4.9) for $i$, we have $\left| \frac{\partial \psi_i}{\partial x_i} \right| + 1 \leq C$. Therefore,

$$
\sum_{i \neq j} \left( \frac{\partial \psi_i}{\partial x_j} \right)^2 \leq C\varepsilon + C \left| \frac{\partial \psi_i}{\partial x_i} - 1 \right|.
$$

Now integrating the last inequality over the unit ball and using (4.7), we find that

$$
\int_{B(0, 1)} \sum_{i \neq j} \left( \frac{\partial \psi_i}{\partial x_j} \right)^2 \, dx \leq C\varepsilon + \int_{B(0, 1)} \left| \frac{\partial \psi_i}{\partial x_i} - 1 \right| \, dx \leq C\varepsilon. \quad (5.12)
$$

Consequently, by the Cauchy-Schwartz inequality, we have

$$
\int_{B(0, 1)} \sum_{i \neq j} \left| \frac{\partial \psi_i}{\partial x_j} \right| \, dx \leq C\sqrt{\varepsilon}.
$$
Third claim:

$$\int_{B(0,1)} \left| \frac{\partial \psi_i}{\partial x_i} \right| \, dx \leq C \sqrt{\varepsilon}. \quad (5.13)$$

Since,

$$\int_{B(0,1)} \left( \frac{\partial \psi_i}{\partial x_i} - 1 \right)^2 \, dx \leq \int_{B(0,1)} \left| \frac{\partial \psi_i}{\partial x_i} - 1 \right| \left| \frac{\partial \psi_i}{\partial x_i} + 1 \right| \, dx,$$

using (4.7) and $|\frac{\partial \psi_i}{\partial x_i}| \leq 1 + C\varepsilon$, we obtain

$$\int_{B(0,1)} \left( \frac{\partial \psi_i}{\partial x_i} \right)^2 \, dx \leq C\varepsilon.$$

Thus, an application of Cauchy Schwartz, yields (4.15).

Final claim: By the Hilbert Schmidt definition, we have

$$\int_{B(0,1)} \left| \Phi'(x) - I \right| \, dx = \int_{B(0,1)} \left( \sum_{i,j=1}^{D} \left( \frac{\partial \psi_i}{\partial x_j} - \delta_{ij} \right)^2 \right)^{1/2} \leq \int_{B(0,1)} \sum_{i,j=1}^{D} \left| \frac{\partial \psi_i}{\partial x_j} - \delta_{ij} \right| \, dx.$$

The estimate (4.11) combined with (4.15) yields:

$$\int_{B(0,1)} \left| \Phi'(x) - I \right| \, dx \leq C\varepsilon^{1/2}.$$

Thus we have proved (4.1) with $T = I$. The proof of the BMO Theorem 1 is complete. □

**Corollary** Let $\Phi : \mathbb{R}^D \to \mathbb{R}^D$ be an $\varepsilon$-distorted diffeomorphism. For each, ball $B \subset \mathbb{R}^D$, there exists $T_B \in O(D)$, such that

$$\left( \frac{1}{\text{vol} B} \int_{B} \left| \Phi'(x) - T \right|^4 \, dx \right)^{1/4} \leq C\varepsilon^{1/2}.$$

The proof follows from the first BMO Theorem just proved and the John Nirenberg inequality. (See (2.4). □.

### 6 A Refined BMO Theorem

We prove:
BMO Theorem 2 Let $\Phi : \mathbb{R}^D \to \mathbb{R}^D$ be an $\varepsilon$ diffeomorphism and let $B \in \mathbb{R}^D$ be a ball. Then, there exists $T_B \in O(D)$ such that
\[
\frac{1}{\text{vol } B} \int_B |\Phi'(x) - T_B| \, dx \leq C\varepsilon. \tag{6.1}
\]

Proof We may assume without loss of generality that
\[
B = B(0, 1). \tag{6.2}
\]
We know that there exists $T^*_B \in O(D)$ such that
\[
\left( \int_B |\Phi'(x) - T^*_B| \, dx \right)^{1/4} \leq C\varepsilon^{1/2}. \tag{6.3}
\]
Our desired conclusion holds for $\Phi$ iff it holds for $(T^*_B)^{-1} \circ \Phi$. Hence without loss of generality, we may assume that $T^*_B = I$. Thus we have
\[
\left( \int_B |\Phi'(x) - I| \, dx \right)^{1/4} \leq C\varepsilon^{1/2}. \tag{6.4}
\]
Let
\[
\Omega(x) = (\Omega_1(x), \Omega_2(x), ..., \Omega_D(x)) = \Phi(x) - x, \ x \in \mathbb{R}^D. \tag{6.5}
\]
Thus (5.3) asserts that
\[
\left( \int_{B(0,1)} |\nabla \Omega(x)| \, dx \right)^{1/4} \leq C\varepsilon^{1/2}. \tag{6.6}
\]
We know that
\[
|\langle \Phi'(x) \rangle^T \Phi'(x) - I| \leq C\varepsilon, \ x \in \mathbb{R}^D. \tag{6.7}
\]
In coordinates, $\Phi'(x)$ is the matrix $\left( \delta_{ij} + \frac{\partial \Omega_i(x)}{\partial x_j} \right)$, hence $\Phi'(x)^T \Phi'(x)$ is the matrix whose $ij$th entry is
\[
\delta_{ij} + \frac{\partial \Omega_j(x)}{\partial x_i} + \frac{\partial \Omega_i(x)}{\partial x_j} + \sum_l \frac{\partial \Omega_l(x)}{\partial x_i} \frac{\partial \Omega_l(x)}{\partial x_j}.
\]
Thus (6.5) says that
\[
\left| \frac{\partial \Omega_j(x)}{\partial x_i} + \frac{\partial \Omega_i(x)}{\partial x_j} + \sum_l \frac{\partial \Omega_l(x)}{\partial x_i} \frac{\partial \Omega_l(x)}{\partial x_j} \right| \leq C\varepsilon
\]
on $\mathbb{R}^D$, $i,j = 1, ..., D$. Thus, we have from (5.5), (5.7) and the Cauchy Schwartz inequality the estimate
\[
\left\| \frac{\partial \Omega_i}{\partial x_j} + \frac{\partial \Omega_j}{\partial x_i} \right\|_{L^2(B(0,1))} \leq C\varepsilon.
\]
By the PDE Theorem, there exists, for each $i,j$, an antisymmetric matrix $S = (S)_{ij}$, such that
\[
\left\| \frac{\partial \Omega_i}{\partial x_j} - S \right\|_{L^2(B(0,1))} \leq C\varepsilon. \tag{6.8}
\]
Recalling (5.4), this is equivalent to
\[ \| \Phi' - (I + S) \|_{L^2(B(0,1))} \leq C \varepsilon. \] (6.9)

Note that (5.5) and (5.8) show that
\[ |S| \leq C \varepsilon^{1/2} \]
and thus,
\[ |\exp(S) - (I + S)| \leq C \varepsilon. \]

Hence, (5.9) implies via Cauchy Schwartz.
\[ \int_{B(0,1)} |\Phi'(x) - \exp(S)(x)| \, dx \leq C \varepsilon^{1/2}. \] (6.10)

This implies the result because \( S \) is antisymmetric which means that \( \exp(S) \in O(D) \). \( \square \).

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