The prismatic finite element with the interpolation procedure vector form for the engineering structures’ strength calculations

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Abstract. The technique of forming a stiffness matrix of a volume prismatic finite element with a triangular base and with six nodes located at the vertices of the prism is presented. The discretization element is formed on the basis of the interpolation procedure vector form with consideration as the interpolation object of the displacement vector of an arbitrary point of the engineering structure. It is proposed to improve the compatibility of the prismatic discretization element at the boundaries of the docking of the bases by using Lagrange multipliers as additional nodal unknowns, which are introduced in additional nodes located in the middle of the prismatic discretization element bases’ sides. The presented vector form of interpolation allows one to obtain the correct finite element solutions in the problems of determining the stress-strain state of engineering structures using curvilinear coordinate systems.

1. Introduction
The modern development of the construction industry requires improving the methods of strength analysis of building structures and architectural objects. The fast-paced digitalization of the economy highlights the need for the development of numerical methods of calculation [1-5] based on the advanced computer technologies. Among all numerical methods, the finite element method (FEM) [6–9] should be considered as a priority, combining universality, the ability to fully algorithmize the computational process, and indifference to the types of external load and boundary conditions. Among the vast family of discretization elements developed to date, the researchers prefer volumetric finite elements. Therefore, the task of developing and improving three-dimensional discretization elements remains quite relevant. In addition, special attention should be paid to the fact that the geometric forms of structures and architectural forms are becoming more complicated, a large number of objects of this kind have a curved configuration. In this regard, the use of curved coordinate systems: cylindrical, spherical, toroidal, becomes most convenient in computational algorithms. However, the use of curvilinear coordinate systems leads to the problem of taking into account the displacement of the finite element as an absolutely rigid body. A correct solution to this problem becomes possible through the use of the vector form of the interpolation procedure, based on the displacement vector consideration itself as interpolation objects, rather than its individual components [10-11].
In the present work, the method for the formation of a stiffness matrix of a bulk prismatic finite element formed on the basis of the vector form of the interpolation procedure using Lagrange multipliers used to improve compatibility at the boundaries of the bases’ junction of a sampling element of this type has been presented.

2. Materials and methods

2.1. Geometric relationships

The radius vector of the reference surface of a building structure or architectural form with a curvilinear generatrix can be given by the following formula

\[ \vec{R}^0 = x(\beta_1, \beta_2)\hat{i} + y(\beta_1, \beta_2)\hat{j} + z(\beta_1, \beta_2)\hat{k}, \]  

(1)

where \( \beta_1, \beta_2 \) are the curved coordinates of the reference surface; \( x, y, z \) and \( \vec{i}, \vec{j}, \vec{k} \) are the Cartesian coordinates and their unit vectors.

As a result of differentiation (1) by \( \beta_1, \beta_2 \) we can obtain the expressions for the vectors of the local basis located in the plane that is in contact with the reference surface

\[ \vec{a}_1^0 = \vec{R}_{\beta_1}^0; \quad \vec{a}_2^0 = \vec{R}_{\beta_2}^0, \]  

(2)

where the comma indicates the differentiation in the corresponding coordinate.

The unit vector normal to the reference surface can be obtained by the vector product of the vectors in (2)

\[ \vec{a}^0 = \vec{a}_1^0 \times \vec{a}_2^0 / \sqrt{\vec{a}_1^0 \cdot \vec{a}_1^0}, \]  

(3)

Between the point basis vectors \( M^0 \) of the reference surface and the orts of the Cartesian coordinate system based on (2) and (3) the following matrix dependencies can be arranged

\[ \begin{bmatrix} \vec{a}^0 \end{bmatrix} = \begin{bmatrix} \vec{a}_1^0 \cdot \vec{a}_2^0 \vec{a}_2^0 \cdot \vec{a}_2^0 \end{bmatrix} - \begin{bmatrix} \vec{a}_1^0 \cdot \vec{a}_1^0 \vec{a}_2^0 \cdot \vec{a}_2^0 \end{bmatrix}, \]

(4)

where \[ \begin{bmatrix} \vec{a}^0 \end{bmatrix} = \begin{bmatrix} \vec{a}_1^0 \vec{a}_2^0 \vec{a}_3^0 \end{bmatrix}, \quad \begin{bmatrix} \vec{a}_1^0 \end{bmatrix} = \begin{bmatrix} \vec{i} \end{bmatrix}, \quad \begin{bmatrix} \vec{a}_2^0 \end{bmatrix} = \begin{bmatrix} \vec{j} \end{bmatrix}, \quad \begin{bmatrix} \vec{a}_3^0 \end{bmatrix} = \begin{bmatrix} \vec{k} \end{bmatrix}. \]

If to the reference surface at the point \( M^0 \) a tangent plane and restore the perpendicular at the given point is drawn, then the position of the point \( M^{0h} \), located on this perpendicular at a distance \( h \) from point \( M^0 \), can be defined by a radius vector

\[ \vec{R}^{0h} = \vec{R}^0 + \vec{a}^0 h. \]

(5)

In the process of deformation of an engineering structure under the action of an applied load, the point \( M^{0h} \) will move to a point \( M^h \). The vector of a given displacement and its first-order partial derivatives can be represented by the components of the basis of a point \( M^0 \)

\[ \vec{v} = v^0\vec{a}_0^0 + v\vec{a}^0; \quad \vec{v}_{\beta_1} = t^0_{\beta_1}\vec{a}_0^0 + t_{\beta_1}\vec{a}^0; \quad \vec{v}_{\beta_2} = t^0_{\beta_2}\vec{a}_0^0 + t_{\beta_2}\vec{a}^0, \]

(6)

where \( t^0_{\beta_1}, t_{\beta_1}, t^0_{\beta_2}, t_{\beta_2} \) are the polynomials containing components of the displacement vector and their partial derivatives of the first order; \( \beta = 1, 2 \).

Point position \( M^h \) is defined by a radius vector

\[ \vec{R}^h = \vec{R}^{0h} + \vec{v}. \]

(7)
Differentiating (5) and (7), we can obtain the expressions for the basis vectors of the point $M^{0h}$ and $M^h$

$$
\bar{g}^0_p = \bar{R}^{0h}_\beta \bar{g}^0 \ ; \ \bar{g}_h = \bar{R}^{0h}_\beta \ ; \ \bar{g}_h = \bar{R}^{0h} = \left( \bar{R}^{0h} + \bar{v} \right)_\beta \ ; \ \bar{g}^0_h = \bar{R}^{0h} = \left( \bar{R}^{0h} + \bar{v} \right)_h.
$$

(8)

Performing the scalar multiplication operation over (8), we can write the relations for the covariant components of the metric tensor at the points $M^{0h}$ and $M^h$

$$
\bar{g}_p^0 = \bar{g}_{p}^0 \cdot \bar{g}_{\tau}^0 \ ; \ \bar{g}_p = \left( \bar{R}^{0h} + \bar{v} \right)_{\beta} \left( \bar{R}^{0h} + \bar{v} \right)_\beta \ ; \ \bar{g}_p^0 = \bar{g}_{p}^0 \cdot \bar{g}_{\tau}^0 \ ; \ \bar{g}_p = \left( \bar{R}^{0h} + \bar{v} \right)_{\beta} \left( \bar{R}^{0h} + \bar{v} \right)_h.
$$

(9)

Deformations at a point $M^{0h}$ can be obtained on the continuum mechanics classical formula basis [12]

$$
\varepsilon_{st} = \left( g_{st}^0 - g_{st}^0 \right) / 2 \ ; \ (s, t = 1, 2, 3)
$$

(10)

2.2 Prismatic discretization element with vector form of interpolation procedure

The sampling element is a prism with a triangular base and with six nodes located at the vertices of the prism. Columns of the sought values of the prismatic discretization element in the local $\xi, \eta, \zeta$ and global $\beta_1, \beta_2, h$ coordinate system were presented as

$$
U^L = \begin{bmatrix}
\{ u_{i,j}^L \}_{i=1,j=24} \\
\{ v_{i,j}^L \}_{i=1,j=24} \\
\{ v_{i,j}^L \}_{i=1,j=24}
\end{bmatrix};
$$

(11)

$$
U^G = \begin{bmatrix}
\{ u_{i,j}^G \}_{i=1,j=24} \\
\{ v_{i,j}^G \}_{i=1,j=24} \\
\{ v_{i,j}^G \}_{i=1,j=24}
\end{bmatrix},
$$

(12)

where $\{ q^L \}_{i=1,j=24} = \{ q^i q^j q^k q^m q^n q_\xi q_\eta q_\zeta \}$; $\{ q^G \}_{i=1,j=24} = \{ q^i q^j q^k q^m q^n q_\beta_1 q_\beta_2 q_\zeta \}$; $q$ is the displacement vector component $v^1, v^2$ or $v^3$; $i, j, k$ and $l, m, n$ are the nodes located at the vertices of the lower and upper bases of the prismatic element, respectively.

The standard interpolation procedure [6-10] in the prismatic discretization element corresponds to the following interpolation dependence

$$
q = \begin{bmatrix}
\{ u^L \}_{i=1,j=24} \\
\{ v^L \}_{i=1,j=24} \\
\{ v^L \}_{i=1,j=24}
\end{bmatrix},
$$

(13)

where $\{ u_{i,j}^L \}_{i=1,j=24}$ is the form function column.

The use of formula (13) in algorithms for the strength calculation of engineering structures with curved outlines when using curved coordinate systems can lead to a very significant calculation error due to disregarding the displacements of the prismatic discretization element as an absolutely rigid body. The solution to this problem becomes possible using the vector form of the interpolation procedure. In this case, the interpolation dependence will have the following form

$$
\bar{v} = \begin{bmatrix}
\{ \bar{v}_{i,j}^L \}_{i=1,j=24} \\
\{ \bar{v}_{i,j}^L \}_{i=1,j=24} \\
\{ \bar{v}_{i,j}^L \}_{i=1,j=24}
\end{bmatrix} = [L] \begin{bmatrix}
\{ \bar{u}^L_{i,j} \}_{i=1,j=24} \\
\{ \bar{u}^L_{i,j} \}_{i=1,j=24} \\
\{ \bar{u}^L_{i,j} \}_{i=1,j=24}
\end{bmatrix},
$$

(14)

where $\{ \bar{u}^L_{i,j} \}_{i=1,j=24} = \{ \bar{u}^i \bar{u}^j \bar{u}^k \bar{u}^m \bar{u}^n \bar{u}^\zeta \}$; $\bar{v}_{i,j}^L$ are the nodal displacements of the prismatic discretization element. The use of interpolation functions involving vector arguments with a minimal number of additional calculations allows for the creation of algorithms for the strength calculation of engineering structures with curved outlines when using curved coordinate systems.
Given (19), the matrix  \( \{ G \} \) taking into account (6), can be represented as the result of the product of matrices

\[
\{ G \} = [A] \{ b \},
\]

(15)

where \( [A] \) is a matrix whose structural elements are row submatrices containing nodal vectors of the local basis \( \{ a_{i} \} \); superscript indicates the nodes of the prismatic sampling element \( i, j, k, l, m, n \).

Column in (15) \( \{ b \} \) has the following form

\[
\{ b \} = \begin{pmatrix} v^{1} \\ v^{2} \\ \vdots \\ v^{n} \end{pmatrix} = \begin{pmatrix} v_{i}^{1} \\ v_{j}^{1} \\ \vdots \\ v_{n}^{1} \end{pmatrix},
\]

(16)

The interpolation expression (14), taking into account (15) and (16), can be transformed to the form

\[
\tilde{v} = [y]^{T} \begin{pmatrix} A \\ G \end{pmatrix} [T] [U^{G}],
\]

(17)

where the matrix \([G]\) is compiled on the basis of equality

\[
[L] [A] = [A] [G],
\]

(18)

and the matrix \([T]\) is a column transition matrix \( \{ b \} \) to the column \( \{ U \} \).

The matrices included in the structure \([A]\) underarm \( \{ a \} \) taking into account (4), can be expressed by the matrix which elements are the vectors of the local basis of an arbitrary point of the prismatic discretization element

\[
\{ a_{0} \} = [b_{0}] [T]^{-1} [a_{0}],
\]

(19)

Given (19), the matrix \([A]\) can be replaced with a matrix sum

\[
[A] = a_{0}^{T} [A_{0}] + a_{1}^{T} [A_{1}] + a_{2}^{T} [A_{2}] + \ldots + a_{m}^{T} [A_{m}],
\]

(20)

Taking into account (6) and (20), expression (17) can be written in the following form

\[
v^{1} \tilde{a}_{0} + v^{2} \tilde{a}_{0} + v \tilde{a}_{0} = [y]^{T} [a_{0}^{T} [A_{0}^{T}] + a_{1}^{T} [A_{1}^{T}] + a_{2}^{T} [A_{2}^{T}] + \ldots + a_{m}^{T} [A_{m}^{T}] [Z]],
\]

(21)

where \( [Z] = [G] [T] [U^{G}] \).

From (21) it is possible to obtain the interpolation dependences for the displacement vector components corresponding to the vector form of the interpolation procedure

\[
v^{1} = [y]^{T} [A_{0}] [Z], \quad v^{2} = [y]^{T} [A_{1}] [Z], \quad v = [y]^{T} [A] [Z].
\]

(22)
The compatibility of the proposed prismatic discretization element with the vector form of the interpolation procedure can be improved by using the Lagrange multipliers introduced as auxiliary unknowns in additional nodes located in the middle of the sides of the lower (1, 2, 3) and the upper (4, 5, 6) bases triangular prism [13-14]. It is proposed to ensure the equality of the derivatives of the normal components of the displacement vectors along the normals to the sides of the bases at additional nodes with the help of Lagrange correction factors

\[ \lambda_a \left( \frac{\partial \tilde{v}_a}{\partial \tilde{n}_a} - \frac{\partial v'_a}{\partial n'_a} \right) = 0, \]  

(23)

where \( \tilde{n}_a \) is the normal to the base side of the prismatic discretization element in the additional node \( \alpha (\alpha = 1, 2, 3, 4, 5, 6) \); the primes denote the values of the adjacent sampling element.

Considering a separate prismatic discretization element, for its lower and upper bases we can write the following equalities

\[ \lambda_1 \left( \frac{\partial \tilde{v}_1}{\partial \tilde{n}_1} + \frac{\partial \tilde{v}_2}{\partial \tilde{n}_2} \right) + \lambda_3 \left( \frac{\partial \tilde{v}_3}{\partial \tilde{n}_3} \right) = 0; \]

\[ \lambda_4 \left( \frac{\partial \tilde{v}_4}{\partial \tilde{n}_4} + \frac{\partial \tilde{v}_5}{\partial \tilde{n}_5} \right) + \lambda_6 \left( \frac{\partial \tilde{v}_6}{\partial \tilde{n}_6} \right) = 0. \]

(24)

The partial derivatives of the normal components of the displacement vectors along the normals to the sides of the bases at additional nodes can be represented by the sum

\[ \tilde{v}_a = \tilde{v}_a / \tilde{\beta}_1 \cdot \cos \theta_1 + \tilde{v}_a / \tilde{\beta}_2 \cdot \cos \theta_2, \]

(25)

where \( \theta_1, \theta_2 \) are the angles between the normal vector \( \tilde{n}_a \) to the sides of the bases and basis vectors \( \tilde{a}_1^\alpha, \tilde{a}_2^\alpha \) in additional nodes \( \alpha \).

The derivatives \( \tilde{v}_a / \tilde{\beta}_1 \) and \( \tilde{v}_a / \tilde{\beta}_2 \) when using the vector form of the interpolation procedure, are determined by the differentiation (17)

\[ \tilde{v}_a = \left( \psi_{\xi} \cdot \tilde{\xi}_{\beta_p} + \psi_{\eta} \cdot \tilde{\eta}_{\beta_p} + \psi_{\zeta} \cdot \tilde{\zeta}_{\beta_p} \right) \left[ \tilde{A} \right] \left[ G \right] \left[ T \right] \left[ U^G \right], \]

(26)

or taking into account (6) and (20)

\[ t_{\beta_p} \tilde{a}_1^\alpha + t_{\beta_p} \tilde{a}_2^\alpha = \left( \psi_{\xi} \cdot \tilde{\xi}_{\beta_p} + \psi_{\eta} \cdot \tilde{\eta}_{\beta_p} + \psi_{\zeta} \cdot \tilde{\zeta}_{\beta_p} \right) \left[ \tilde{A} \right] + \tilde{a}_1^\alpha \left[ \tilde{A} \right] + \tilde{a}_2^\alpha \left[ \tilde{A} \right] \left[ Z_y \right]. \]

(27)

From (27), we can obtain the interpolation expressions for the partial derivatives of the first-order displacement vector along global curvilinear coordinates, for example,

\[ t_{\beta_p} = \left( \psi_{\xi} \cdot \tilde{\xi}_{\beta_p} + \psi_{\eta} \cdot \tilde{\eta}_{\beta_p} + \psi_{\zeta} \cdot \tilde{\zeta}_{\beta_p} \right) \left[ A \right] \left[ Z_y \right], \]

(28)

where \( t_{\beta_p} \) is a polynomial whose terms are the partial derivatives of the first-order normal component along global curvilinear coordinates \( \beta_1 \) or \( \beta_2 \).

Equalities (24), taking into account (28), can be transformed to the form

\[ \{ \lambda_{11} \} \left[ D_{11} \right] \left[ U^G \right] = 0; \quad \{ \lambda_{11} \} \left[ D_{11} \right] \left[ U^G \right] = 0, \]

(29)

where \( \{ \lambda_{11} \} = \{ \lambda_1, \lambda_2, \lambda_3 \}; \quad \{ \lambda_{11} \} = \{ \lambda_4, \lambda_5, \lambda_6 \}. \)

The stiffness matrix and the column of nodal forces of the prismatic discretization element with the vector form of the interpolation procedure can be obtained by minimizing the conditional Lagrange functional with the inclusion of additional terms (29)

\[ \Phi_x = \int_V \{ \varepsilon_{uu} \} \sigma^u \text{d}V - \int_V \{ U \} \{ P \} \text{d}V + \int_V \{ \lambda_{11} \} \left[ D_{11} \right] \left[ U^G \right] + \int_V \{ \lambda_{11} \} \left[ D_{11} \right] \left[ U^G \right], \]

(30)
where \( \{ \varepsilon_{st} \}^T = \{ e_{11} 2 e_{12} 2 e_{13} e_{22} 2 e_{23} e_{33} \} \); \( \{ \sigma_{st} \}^T = \{ \sigma_{11} \sigma_{12} \sigma_{13} \sigma_{22} \sigma_{23} \sigma_{33} \} \) are the deformations and stresses at an arbitrary point of the structure, spaced from the reference surface at a distance \( \bar{h} \); \( \{ \mathbf{U} \}^T = [v^1 v^2 v^3] \) – matrix row components of the displacement vector; \( \{ \mathbf{P} \} = \{ p^1 p^2 p^3 \} \) – matrix is a row of external load.

The inbox (30)\( \{ \varepsilon_{st} \} \), \( \{ \sigma_{st} \} \), and \( \{ \mathbf{U} \}^T \) taking into account (6), (23) and (27) and Hooke’s law [12], can be replaced by the matrix products
\[
\{ \varepsilon_{st} \} = [\mathbf{B}] \{ \mathbf{U} \}^G; \quad \{ \sigma_{st} \} = [\mathbf{C}] \{ \varepsilon_{st} \}; \quad \{ \mathbf{U} \} = [\mathbf{A}] \{ \mathbf{U} \}^G.
\]
(31)

Performing this and taking into account (31), the minimization procedure (30) by \( \{ \mathbf{U} \}^G \), \( \{ \mathbf{S} \}^T \) and \( \{ \mathbf{P} \}^T \), we can obtain the following system of algebraic equations
\[
\begin{align*}
\frac{\partial \Phi_L}{\partial \{ \mathbf{U} \}^G} &= \int \left[ [\mathbf{B}]^T [\mathbf{C}] [\mathbf{B}] \mathbf{h} \mathbf{V} \{ \mathbf{U} \}^G \right] - \int \left[ [\mathbf{A}]^T \{ \mathbf{P} \} \mathbf{d} \mathbf{V} + [\mathbf{D}_1]^T \{ \mathbf{S}_1 \} + [\mathbf{D}_2]^T \{ \mathbf{S}_2 \} \right] = 0; \\
\frac{\partial \Phi_L}{\partial \{ \mathbf{S} \}^T} &= [\mathbf{D}_1]^T \{ \mathbf{U} \}^G = 0; \\
\frac{\partial \Phi_L}{\partial \{ \mathbf{P} \}^T} &= [\mathbf{D}_2]^T \{ \mathbf{U} \}^G = 0,
\end{align*}
\]
(32)

which can be represented in compressed matrix form
\[
\begin{bmatrix} [\mathbf{K}] \{ \mathbf{V} \}^G \end{bmatrix} = \{ \mathbf{F} \},
\]
(33)

where \( [\mathbf{K}] \) is the stiffness matrix of the prismatic discretization element; \( \{ \mathbf{V} \}^G = \{ \mathbf{U} \}^G \{ \mathbf{S} \}^T \{ \mathbf{P} \}^T \) is a column of nodal forces of a prismatic discretization element.

3. Results
Analyzing the interpolation dependences (23), (28) obtained in the article, we can note their fundamental differences from the standard interpolation procedure (13), according to which the component of the displacement vector \( \mathbf{q} \) interpolated through the nodal values of the same component in isolation from two other components. When using the vector form of the interpolation procedure according to (23), (28), the component of the displacement vector is interpolated through the column \( \{ \mathbf{U} \}^G \), which structure includes the nodal values of all three components of the displacement vector and their partial derivatives of the first order.

4. Summary
Increasing the rank of the interpolated object tensor from zero to one (from a scalar to a vector) makes it possible to obtain the correct finite-element solutions in the problems of determining the stress-strain state of engineering structures using curvilinear coordinate systems.

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