A quasi–regular Sasakian structure on a manifold \( L \) is equivalent to writing \( L \) as the unit circle subbundle of a holomorphic Seifert \( \mathbb{C}^* \)-bundle over a complex algebraic orbifold \( (X, \Delta = \sum (1 - \frac{1}{m_i})D_i) \). The Sasakian structure is called positive if the orbifold first Chern class \( c_1(X) - \Delta = -(K_X + \Delta) \) is positive. We are especially interested in the case when the Riemannian metric part of the Sasakian structure is Einstein. By a result of [Kob63, BG00], this happens iff

1. \(-(K_X + \Delta)\) is positive,
2. the first Chern class of the Seifert bundle \( c_1(L/X) \) is a rational multiple of \(-(K_X + \Delta)\), and
3. there is an orbifold Kähler–Einstein metric on \((X, \Delta)\).

If the first two conditions hold, we say that \( f: L \to (X, \Delta) \) is a pre-SE (or pre–Sasakian–Einstein) Seifert bundle.

Note that if \( H_2(L, \mathbb{Q}) = 0 \) then \( H_2(S, \mathbb{Q}) \cong \mathbb{Q} \) and so the second condition is automatic.

While it is not true that all pre-SE Seifert bundles carry a Sasakian–Einstein structure, all known topological obstructions to the existence of a Sasakian–Einstein structure in dimension 5 are consequences of the pre-SE condition.

A 2–dimensional orbifold \((S, \Delta)\) such that \(-(K_S+\Delta)\) is positive is also called a log Del Pezzo surface. For any such, \( S \) is a rational surface with quotient singularities. By [Kol05, 2.4], Seifert \( \mathbb{C}^* \)-bundles over \((S, \sum (1 - \frac{1}{m_i})D_i)\) are uniquely classified by a homology class \( B \in H_2(S, \mathbb{Z}) \) and integers \( 0 < b_i < m_i \) with \((b_i, m_i) = 1\). We denote the corresponding Seifert \( \mathbb{C}^* \)-bundle (resp. \( S^1 \)-bundle) by \( Y(S, B, \sum b_i/m_i D_i) \) (resp. \( L(S, B, \sum b_i/m_i D_i) \)).

1 (Classification problems). Following [Kol05], we want to address 3 problems.

1. Given \((S, \Delta)\) and \( B \), determine the 5–manifold \( L(S, B, \sum b_i/m_i D_i) \).
2. Given a 5–manifold \( L \), decide if it can be written as \( L(S, B, \sum b_i/m_i D_i) \) for some \((S, \Delta)\) and \( B \).
3. Given a 5–manifold \( L \), describe all of its representations as \( L(S, B, \sum b_i/m_i D_i) \) for some \((S, \Delta)\) and \( B \).

The case when \( L \) is simply connected was studied in [Kol05]. Problem (1) was solved in [Kol05, 5.7]. As to Problem (2), note that by a result of [Sma62], such manifolds are determined by the second homology group \( H_2(L, \mathbb{Z}) \) if \( w_2 = 0 \). (See [Bar65] for the \( w_2 \neq 0 \) case.) The torsion part of \( H_2(L, \mathbb{Z}) \) can be completely described as follows. (There are partial results about the rank of the free part.)

**Theorem 2.** [Kol05, 1.4] Let \( L \) be a compact 5–manifold such that \( H_1(L, \mathbb{Z}) = 0 \). Assume that \( L \) has a positive Sasakian structure \( f: L \to (S, \Delta) \). Then the torsion subgroup of the second homology, \( \text{tors} H_2(L, \mathbb{Z}) \), is determined by \((S, \Delta)\) and it is one of the following.
Note that the theorem gives restrictions on $L$ but does not say anything about $(S, \Delta)$. While it seems hopelessly complicated to describe all Seifert bundle structures on $S^5$, there are few positive Sasakian structures on complicated 5–manifolds. In particular, Problem (3) was settled for $\text{tors} H_2(L, \mathbb{Z}) \cong (\mathbb{Z}/5)^4$ [Kol05, 1.8.2].

The aim of this note is fourfold. First, we extend several of the results to 5–manifolds which are not simply connected. Second, I would like to correct the mistake in the classification of the case (2.1) which was brought to my attention by K. Galicki. Third, Problem (3) is solved in the $(\mathbb{Z}/4)^4$ and $(\mathbb{Z}/3)^{8}$ cases. Finally, we show that in many cases, all positive Sasakian structures are given by hypersurfaces in $\mathbb{C}^4$ and also give explicit equations for these.

Toward the first goal, we show that the groups listed in (2.1–4) give large subgroups of all possible $\text{tors} H_2(L, \mathbb{Z})$.

**Theorem 3.** Let $L$ be a compact 5–manifold with a positive Sasakian structure. Then $\text{tors} H_2(L, \mathbb{Z})$ has a subgroup $G$ as in (2.1–4) such that the quotient group $\text{tors} H_2(L, \mathbb{Z})/G$ is $\mathbb{Z}/r$, $\mathbb{Z}/r + \mathbb{Z}/2$ or $\mathbb{Z}/r + \mathbb{Z}/3$ for some $r$.

Our next aim is to prove that in most cases the existence of the subgroup $G$ controls $L$ and the Sasakian structure very tightly.

**Theorem 4.** Let $f : L \to (S, \Delta)$ be a compact 5–manifold with a positive Sasakian structure. Assume that $H_2(L, \mathbb{Z}) \supset (\mathbb{Z}/m)^2$ where $m \geq 28$ or $m \geq 12$ and $(m, 6) = 1$. Then

1. $S$ is a Del Pezzo surface with cyclic Du Val singularities only and $\Delta = (1 - \frac{1}{m'})C$ where $C \in |-K_S|$ is a smooth elliptic curve and $m'$ is a multiple of $m$.
2. there are 132 families of such $(S, C)$ up to deformations that preserve the singularities,
3. for every such $S$, $\pi_1(S^0)$ is abelian of order $\leq 9$, where $S^0 := S \setminus \text{Sing } S$ denotes the set of smooth points. $\pi_1^{\text{orb}}(S, (1 - \frac{1}{m'})C)$ is also abelian of order $\leq 9$, save 3 cases which are nonabelian of order $2^3, 2^4$ or $3^3$. The precise values are in Table 2.
4. The fundamental group of $L$ is given by a central extension

$$0 \to \mathbb{Z}/d \to \pi_1(L) \to \pi_1^{\text{orb}}(S, (1 - \frac{1}{m'})C) \to 1,$$

for some $d$.
5. The torsion of the second homology group of $L$ sits in an exact sequence

$$0 \to (\mathbb{Z}/m')^2 \to \text{tors} H_2(L, \mathbb{Z}) \to \pi_1(S^0).$$

It is quite likely that the above theorem holds for all $m \geq 12$. However, for $m < 12$, there are many counter examples, see [Kol05] 6.6–7.

5 (Sasakian–Einstein structures). In each of the 132 cases the existence of positive Sasakian and of pre–Sasakian–Einstein structures over $(S, \Delta)$ is effectively decidable, but some of the computations may be lengthy.
There are 93 cases of \((S, C)\) when \(\pi_1(S^0) = 1\). In [Kol05, 1.8.1] I claimed, incorrectly, that they all admit a Seifert bundle with Sasakian–Einstein structure. We see in (20) that these all admit Seifert bundles with a positive Sasakian structure, but only 19 of them admit a Seifert bundle with pre–SE structure.

For the remaining 39 cases with \(\pi_1(S^0) \neq 1\) my computations are not yet complete. There are some cases that do not have any smooth Seifert bundles over them (26), and in many other cases Sasakian–Einstein structures exist.

6 (Hypersurface examples). Concrete examples of Sasakian structures can be obtained using \(\mathbb{C}^*\)-actions on algebraic hypersurfaces. A linear \(\mathbb{C}^*\)-action on \(\mathbb{C}^n\) can be diagonalized and so given by weights \((w_1, \ldots, w_n)\). We consider only actions which are effective (thus the weights are relatively prime) and the origin is an attracting fixed point (thus \(w_i > 0\) for every \(i\)). Let \(Y \subset \mathbb{C}^n\) be an algebraic variety, smooth away from the origin, which is invariant under the \(\mathbb{C}^*\)-action. Then its link \(L := Y \cap S^{2n-1}(1)\) inherits a natural quasi–regular Sasakian structure and every quasi–regular Sasakian structure arises this way.

This description is especially simple when \(Y\) is a hypersurface. In this case everything is described by the weights \((w_1, \ldots, w_n)\) and a weighted homogeneous polynomial \(f(x_1, \ldots, x_n)\) which defines \(Y\).

Thus, up to deformations, we need to specify the weights \((w_1, \ldots, w_n)\) and the weighted degree \(d\) of \(f\). I write \(L^*(w_1, \ldots, w_n; d)\) for any such quasi–regular Sasakian manifold. (The * is to remind one that using weights is in some sense dual to the notation in [BGK05].)

The simplest examples where \(\pi_1(L) \neq 1\) are created by taking a quotient by a subgroup of \(\mathbb{C}^*\). These are all cyclic. Thus the symbol

\[
L^*(w_1, \ldots, w_n; d)/(\mathbb{Z}/m) \quad \text{or} \quad L^*(w_1, \ldots, w_n; d)/\mathbb{Z}/m(w_1, \ldots, w_n)
\]

stands for any Sasakian manifold obtained as \(L/(\mathbb{Z}/m) \subset Y/(\mathbb{Z}/m)\) where \(Y := (f = 0)\) is the zero set of a degree \(d\) weighted homogeneous polynomial such that \(Y\) is smooth outside the origin and we take the quotient by the \(\mathbb{Z}/m\)-action generated by

\[
(x_1, \ldots, x_n) \mapsto (\epsilon^{w_1} x_1, \ldots, \epsilon^{w_n} x_n) \quad \text{where} \quad \epsilon = e^{2\pi i/m}.
\]

We call these the obvious quotients.

One can also take quotients by groups that are not contained in \(\mathbb{C}^*\). For instance, one can take quotients

\[
L^*(w_1, \ldots, w_n; d)/\left(\frac{1}{r}(a_1, \ldots, a_n) \times \frac{1}{m}(w_1, \ldots, w_n)\right)
\]

where the first factor of \(\mathbb{Z}/r \times \mathbb{Z}/m\) acts via

\[
(x_1, \ldots, x_n) \mapsto (\eta^{a_1} x_1, \ldots, \eta^{a_n} x_n) \quad \text{where} \quad \eta = e^{2\pi i/r},
\]

and the second factor acts as above. One should keep in mind that not all hypersurfaces \(L^*(w_1, \ldots, w_n; d)\) admit such a \(\mathbb{Z}/r\)-action. In using this notation, we assume that we consider a case when the action exists.

We are now ready to state that classification theorem for those cases when \(H_2(L,\mathbb{Z})\) contains a large torsion subgroup.

**Theorem 7.** Let \(L\) be a compact 5–manifold with a positive Sasakian structure.
(1) If $H_2(L, \mathbb{Z}) \supset (\mathbb{Z}/5)^4$ then $H_2(L, \mathbb{Z}) = (\mathbb{Z}/5)^4$ and $L = L^{*}(5, 5, 15, 6; 30)$ or an obvious quotient by $\mathbb{Z}/m$ for $(m, 15) = 1$. The simplest equation is 
\[(x^6 + y^6 + z^2 + t^5 = 0)/\frac{1}{m}(5, 5, 15, 6).\]

The moduli space of Sasakian–Einstein structures is naturally parametrized by the moduli space of genus 2 curves.

(2) If $H_2(L, \mathbb{Z}) \supset (\mathbb{Z}/3)^8$ then $H_2(L, \mathbb{Z}) = (\mathbb{Z}/3)^8$ and $L = L^{*}(3, 3, 15, 10; 30)$ or an obvious quotient by $\mathbb{Z}/m$ for $(m, 15) = 1$. The simplest equation is 
\[(x^{10} + y^{10} + z^2 + t^3 = 0)/\frac{1}{m}(3, 3, 15, 10).\]

The moduli space of Sasakian–Einstein structures is naturally parametrized by the moduli space of hyperelliptic curves of genus 4.

(3) If $H_2(L, \mathbb{Z}) \supset (\mathbb{Z}/4)^4$ then either $H_2(L, \mathbb{Z}) = (\mathbb{Z}/4)^4$ or $H_2(L, \mathbb{Z}) = \mathbb{Z} + (\mathbb{Z}/4)^4$. In the first case, there are 5 families:

(a) $L = L^{*}(4, 4, 8, 5; 20)$ or an obvious quotient by $\mathbb{Z}/m$ for $(m, 2) = 1$, with sample equation
\[(x^5 + y^5 + yz^2 + t^4 = 0)/\frac{1}{m}(4, 4, 8, 5),\]

(b) $Y = L^{*}(2, 2, 6, 3; 12)/\frac{1}{2}(1, 1, 1, e)$ where $e \in \{0, 1\}$ or an obvious quotient by $\mathbb{Z}/m$ for $(m, 6) = 1$. The simplest equation is 
\[(x^6 + y^6 + z^2 + t^4 = 0)/\frac{1}{m}(2, 2, 6, 3).\]

(c) $L = L^{*}(4, 8, 20, 9; 40)$ or an obvious quotient by $\mathbb{Z}/m$ for $(m, 6) = 1$. The simplest equation is
\[(x^{10} + y^5 + z^2 + xt^4 = 0)/\frac{1}{m}(4, 8, 20, 9).\]

(d) $L = L^{*}(2, 4, 10, 5; 20)/\frac{1}{2}(1, 0, 1, 1)$ or an obvious quotient by $\mathbb{Z}/m$ for $(m, 10) = 1$. The simplest equation is
\[(x^{10} + y^5 + z^2 + t^4 = 0)/\frac{1}{2}(1, 0, 1, 1) \times \frac{1}{m}(2, 4, 10, 5).\]

In the second case, there are infinitely many positive Sasakian structures with the same $\pi_1$ but only one which is pre–SE. These are:

(e) $L = L^{*}(4, 4, 12, 5; 24)$ or an obvious quotient by $\mathbb{Z}/m$ for $(m, 10) = 1$. The simplest equation is
\[(x^6 + y^6 + z^2 + xt^4 = 0)/\frac{1}{m}(4, 4, 12, 5).\]

The moduli space of Sasakian–Einstein structures is naturally parametrized by the moduli space of pairs $(C, p)$ where $\pi : C \to P^1$ is a genus 2 curve and $p \in C$ is not a branch point in cases (a),(b),(e) and $p \in C$ is a branch point in cases (c),(d).

The above parametrizations of the moduli spaces of Sasakian–Einstein structures are one–to–one. However, if we consider the moduli spaces of the resulting Einstein metrics, we get a two–to–one parametrization since a curve and its complex conjugate give the same Einstein metric, see [BGK05] 18–21.

I see no a priori reason why all these cases should be realizable as hypersurface quotients. K. Galicki told me that he found the example $(x^6 + y^6 + z^2 + t^5 = 0)$ with $H_2(L, \mathbb{Z}) = (\mathbb{Z}/5)^4$. This lead me to realize that in fact all examples with $H_2(L, \mathbb{Z}) = (\mathbb{Z}/5)^4$ are hypersurfaces. The equations are also easy to see in the $(\mathbb{Z}/3)^8$ case, but are rather mysterious from the point of view of my proof for several of the $(\mathbb{Z}/4)^4$ cases.
1. Reduction to algebraic geometry

8. As in [BGK05, Kol05], to \( L = Y \cap S^{2n-1}(1) \) we associate a projective algebraic variety \( X = Y \setminus \{0\}/\mathbb{C}^+ \) with cyclic quotient singularities. There is also a natural \( \mathbb{Q} \)-divisor \( \Delta = \sum (1 - \frac{1}{m_i}) D_i \) on \( X \) where \( D_i \subset X \) is a divisor such that the stabilizer of the \( \mathbb{C}^* \)-action has order \( m_i \) over the points of \( D_i \). The positivity of the Sasakian structure is equivalent to the log Fano condition:

\[
-(K_X + \sum (1 - \frac{1}{m_i}) D_i) \text{ is ample.}
\]

Thus if \( \dim L = 5 \), we are looking for pairs \((S, \Delta)\) where

1. \( S \) is a projective surface with cyclic quotient singularities,
2. \( \Delta = \sum (1 - \frac{1}{m_i}) D_i \) is a \( \mathbb{Q} \)-divisor, and
3. \(-(K_S + \Delta)\) is ample.

There are a few more conditions which we do not need for now.

A rather easy result [Kol05, 6.3] on log Del Pezzo surfaces gives the following restriction on the \( D_i \):

4. there is at most one index \( i \) such that \( g(D_i) > 0 \).

Because of the special role of \( D_i \), we frequently set \( D := D_i \), \( r := m_i \) and use the notation \((S, (1 - \frac{1}{r}) D + \Delta)\).

It turns out that the ampleness of \(-(K_S + (1 - \frac{1}{r}) D + \Delta)\) forces \( g(D) \) or \( r \) to be quite small and this accounts for the restrictions in Theorem 2.

This is relevant to our purposes since [Kol05, 5.7] relates the torsion of \( H_2(L, \mathbb{Z}) \) to the divisor \( \sum (1 - \frac{1}{m_i}) D_i \):

5. If \( H_1(L, \mathbb{Z}) = 0 \) then tors \( H_2(L, \mathbb{Z}) = \sum (\mathbb{Z}/m_i)^{2g(D_i)} \). Thus, in the log Del Pezzo case, tors \( H_2(L, \mathbb{Z}) \cong (\mathbb{Z}/r)^{2g(D)} \).

If \( H_1(L, \mathbb{Z}) \neq 0 \), the spectral sequence in [Kol05, 5.10] becomes quite messy, but in the log Del Pezzo case only one nonzero map involves \( H^3(L, \mathbb{Z}) \). The same proof gives the following weaker result when the first ordinary homology of the base is trivial.

**Proposition 9.** Let \( f : L^5 \to (S, \Delta = \sum (1 - \frac{1}{m_i}) D_i) \) be a smooth Seifert bundle over a projective surface with quotient singularities. Assume that \( H_1(S, \mathbb{Z}) = 0 \) and let \( s = \text{rank } H^2(S, \mathbb{Q}) \). Then there is an exact sequence

\[
H_1(S^0, \mathbb{Z}) \cong H^3(S, \mathbb{Z}) \to H^3(L, \mathbb{Z}) \to \mathbb{Z}^{s-1} + \sum (\mathbb{Z}/m_i)^{2g(D_i)} \to 0. \quad (\text{4.1})
\]

**Proof.** The argument very closely follows [Kol05, 5.9–10]. As there, we have exact sequences

\[
0 \to R^1 f_* \mathcal{L}_S \to \mathcal{L}_S \to Q \to 0 \quad (\text{4.2})
\]

and

\[
0 \to \sum_i \mathbb{Z}/n_j \to Q \to \sum_i \mathbb{Z}/m_i \to 0, \quad (\text{4.3})
\]

where \( P_j \in S \) are the singular points. This implies that \( H^1(S, \mathbb{Q}) = \sum_i H^1(D_i, \mathbb{Z}/m_i) \) for \( i \geq 1 \). The key piece of the long cohomology sequence of (4.2) is

\[
H^1(S, \mathbb{Z}) \to \sum H^1(D_i, \mathbb{Z}/m_i) \to H^2(S, R^1 f_* \mathcal{L}_S) \to H^2(S, \mathbb{Z}) \to \sum H^2(D_i, \mathbb{Z}/m_i)
\]
Here $H^1(S, \mathbb{Z}) = 0$ and $H^2(S, \mathbb{Z}) \cong \mathbb{Z}^s$ by assumption. The right hand group is torsion, hence there is a noncanonical isomorphism

$$H^2(S, R^1 f_* \mathbb{Z}_L) \cong \mathbb{Z}^s + \sum_i H^1(D_i, \mathbb{Z}/m_i).$$

Therefore, in the Leray spectral sequence $H^i(S, R^j f_* \mathbb{Z}_L) \Rightarrow H^{i+j}(L, \mathbb{Z})$ the $E_2$ term is

$$
\begin{array}{cccc}
\mathbb{Z} & \mathbb{Z}^s + \sum_i (\mathbb{Z}/m_i)^{2g(D_i)} & \mathbb{Z}^s & \mathbb{Z} \\
\mathbb{Z} & 0 & \mathbb{Z}^s & H^3(S, \mathbb{Z}) & \mathbb{Z} \\
\end{array}
$$

Since $H^3(S, \mathbb{Z}) \cong H_1(S^0, \mathbb{Z})$ by [Kol05, 4.2], we get the exact sequence

$$H_1(S^0, \mathbb{Z}) \to \text{tors} H^3(L, \mathbb{Z}) \to \sum_i H^1(D_i, \mathbb{Z}/m_i) \to 0. \quad \square$$

**Corollary 10.** Notation and assumptions as in (2). Let $p_j \in S$ be the singular points and $n_j$ the order of the local fundamental group.

1. If any two of the numbers $m_i, n_j$ are relatively prime then the sequence (11) is left exact.
2. If, in addition $|H_1(S^0, \mathbb{Z})|$ is relatively prime to $\prod m_i$ then the sequence (11) splits.

Proof. If any two of the numbers $m_i, n_j$ are relatively prime, the sequence in (2) splits and $H^0(S, Q) \cong \mathbb{Z}/(\prod n_j \cdot \prod m_i)$. This implies that $H^1(S, R^1 f_* \mathbb{Z}_L) = 0$. The $E_2$ term of the Leray spectral sequence (11) is

$$
\begin{array}{cccc}
\mathbb{Z} & 0 & \mathbb{Z}^s + \sum_i (\mathbb{Z}/m_i)^{2g(D_i)} & \mathbb{Z} \\
\mathbb{Z} & 0 & \mathbb{Z}^s & H^3(S, \mathbb{Z}) & \mathbb{Z} \\
\end{array}
$$

Thus $H^3(L, \mathbb{Z})$ sits in an exact sequence

$$0 \to H^3(S, \mathbb{Z}) \to H^3(L, \mathbb{Z}) \to \mathbb{Z}^s + \sum_i (\mathbb{Z}/m_i)^{2g(D_i)} \to \mathbb{Z}$$

As before, $H^3(S, \mathbb{Z}) \cong H_1(S^0, \mathbb{Z})$ and the extension of the torsion part splits if $|H_1(S^0, \mathbb{Z})|$ is relatively prime to $\prod m_i$.

This result turns out to be quite useful, since the groups $H_1(S^0, \mathbb{Z})$ are rather special when $(S, \Delta)$ is a log Del Pezzo surface. Thus $S$ is a rational surface and so $H_1(S, \mathbb{Z}) = 0$. A project of C. Xu aims to give a complete determination of the possible fundamental groups $\pi_1(S^0)$ where $S$ is a rational surface with quotient singularities. While this is not easy, the first homology $H_1(S^0, \mathbb{Z})$ is not hard to control.

The proof of (3) is obtained by combining (9) and (11).

**Lemma 11.** Let $S$ be a log Del Pezzo surface with quotient singularities. Then $H_1(S^0, \mathbb{Z})$ is either $\mathbb{Z}/m$, $\mathbb{Z}/m + \mathbb{Z}/2$ or $\mathbb{Z}/m + \mathbb{Z}/3$ for some $m \geq 1$.

Proof. Let $S' \to S$ be the corresponding Galois cover with Galois group $G$. The stabilizers of points are subgroups of the local fundamental groups at the singularities, and hence cyclic. Moreover, no $1 \neq g \in G$ fixes a curve pointwise. Thus it is enough to prove the following.

**Lemma 12.** Let $G$ be an Abelian group acting on a rational surface with cyclic stabilizers. Assume also that no element fixes a curve of genus $\geq 1$ pointwise. Then $G$ is either $\mathbb{Z}/m$, $\mathbb{Z}/m + \mathbb{Z}/2$ or $\mathbb{Z}/m + \mathbb{Z}/3$. 


Proof. We can take a $G$-equivariant resolution and then pass to the $G$-minimal model $T$. (Note that if a smooth point $p \in T$ is fixed by an Abelian group $H$ then $H$ also has a fixed point on the blow up $B_p T$, so a fixed point on one model gives a fixed point on any other model.)

Our aim is to find a $G$-invariant subset $Z \subset T$ such that either $Z \cong \mathbb{P}^1$ or $Z$ is at most 3 points. In the first case, $G$ acts on $\mathbb{P}^1$ hence it either has a fixed point, or it acts through $\langle \mathbb{Z}/2 \rangle^2$ and an index 2 subgroup has a fixed point. In the second case a subgroup of index \leq 3 has a fixed point. Such a subgroup is cyclic and so $G$ is $\mathbb{Z}/m, \mathbb{Z}/m + \mathbb{Z}/2$ or $\mathbb{Z}/m + \mathbb{Z}/3$ as required.

Consider first the case when there is a $G$-equivariant ruling $f : T \to \mathbb{P}^1$. We are done if $f_* : G \to \text{Aut}(\mathbb{P}^1)$ is injective. Otherwise, any $g \in \ker f_*$ fixes 2 points in each fiber of $f$, thus $g$ fixes either 1 or 2 irreducible curves pointwise and their union is $G$-invariant. We are done if there is a $G$-invariant curve.

Otherwise, every $g \in \ker f_*$ fixes 2 disjoint curves $E_1, E_2$ pointwise and $G$ permutes these curves. By the Hodge index theorem $E_i^2 \leq 0$, thus they generate the “other” extremal ray of the cone of curves. In particular, either $T = \mathbb{P}^1 \times \mathbb{P}^1$ or these curves are unique and every element of $\ker f_*$ fixes the same curves. If $G$ has a fixed point $p \in \mathbb{P}^1$ then $Z := f^{-1}(p) \cap (E_1 \cup E_2)$ is a 2–element set fixed by $G$. If there is no such $p$ then $G$ acts on $\mathbb{P}^1$ and also on $E_1 \cup E_2$ through $\langle \mathbb{Z}/2 \rangle^2$.

Furthermore, there is an index 2 subgroup $H \subset G$ which fixes each $E_i$ and has a fixed point on each $E_i$. Thus $H$ is cyclic and so $G \cong \mathbb{Z}/m$ or $G \cong \mathbb{Z}/m + \mathbb{Z}/2$.

The $\mathbb{P}^1 \times \mathbb{P}^1$ case is left to the end.

Otherwise $T$ is a Del Pezzo surface of $G$-Picard number 1.

Assume that some $g \in G$ pointwise fixes some curves $C_i$. The $C_i$ are smooth, disjoint, rational curves. By the adjunction formula, $C_i^2 \in \{0, -1\}$ or $T \cong \mathbb{P}^2$. Their sum $\sum_i C_i$ is $G$-invariant and not ample, but this contradicts $G$-minimality. Thus either $T \cong \mathbb{P}^2$ or every element of $G$ acts with isolated fixed points.

Assume that there is $H < G$ with $H \cong \langle \mathbb{Z}/2 \rangle^2$. Then $T/H$ is a Del Pezzo surface with $A_1$-singularities only. From Table 2 in [21] we see that $\deg T/H = 2$ and $T \cong \mathbb{P}^1 \times \mathbb{P}^1$. Thus, aside from this case, the 2–part of $G$ is cyclic.

If $T = \mathbb{P}^2$, take any $g \in G$. It has either 3 fixed points or a fixed point and a fixed line (and the second case must happen if $g^2 = 1$). In the second case, the fixed point is $G$-fixed (and so $G$ is cyclic) and in the former case either all 3 points are $G$-fixed or $G$ permutes them cyclically.

If $\deg T \in \{1, 2, 3, 4, 5\}$ then $\text{Aut}(T)$ acts faithfully on $H^2(T, \mathbb{Z})$. Indeed, if $g$ acts trivially on $H^2(T, \mathbb{Z})$ then it descends to an automorphism of $\mathbb{P}^2$ with $\geq 4$ fixed points in general position, hence $g = 1$. Thus, $G$ is a subgroup of the Weyl group of $E_8, E_7, E_6, D_5, A_4$ (cf. [Man86, Sec.25]).

If $\deg T = 1$ then $|K_T|$ has a unique base point which is fixed by $\text{Aut}(T)$, so $G$ is cyclic.

If $\deg T = 2$ then there is a unique degree two morphism $T \to \mathbb{P}^2$. If $|G|$ is odd and $H < G$ then any $H$-fixed point on $\mathbb{P}^2$ is dominated by (one or two) $H$-fixed point(s) on $T$. If $G$ is even, then a $G$-fixed point on $\mathbb{P}^2$ is dominated by two points and each is fixed by an index 2 subgroup of $G$.

If $\deg T \in \{3, 4, 5\}$ then by looking at the order of the Weyl groups, we see that the odd part $G_{\text{odd}} \subset \mathbb{Z}/15$, except possibly when $\deg T = 3$ where the 3–part could be bigger. Any action of a group on a cubic surface $T$ induces an action on $\mathbb{P}^3 \cong |K_T|$. For odd order groups this action lifts to $\mathbb{C}^4$ since the kernel of
$SL_4 \to PSL_4$ is $\mathbb{Z}/4$. Thus we get an eigenvector on $\mathbb{C}^4$ and a fixed point on the cubic $T$.

If $\deg T = 6$ then only $\mathbb{Z}/3$ can act on the $(-1)$-curves nontrivially, and even for the $\mathbb{Z}/3$-action there is a $\mathbb{Z}/3$-invariant set of 3 disjoint $(-1)$-curves. Thus the action descends to $\mathbb{P}^2$.

There are no $G$-minimal Del Pezzo surfaces of degree 7 and in degree 8 we get $\mathbb{P}^1 \times \mathbb{P}^1$.

Thus we are left with $G$ acting on $\mathbb{P}^1 \times \mathbb{P}^1$. If any $g \in G$ interchanges the two factors, then $g$ has 2 fixed points or a fixed rational curve from the Lefschetz fixed point formula. In both cases an index 2 subgroup of $G$ has a fixed point.

The last case is when $G$ preserves the coordinate projections. The image of $G$ in each $\text{Aut}(\mathbb{P}^1)$ is either cyclic or $(\mathbb{Z}/2)^2$. If the first case happens at least once, then an index 2 subgroup has a fixed point. Finally we have to deal with subgroups of $G \subset (\mathbb{Z}/2)^4$. If the order is 8 or 16 then there are $g_1, g_2 \in G$ which act trivially on the first (resp. second) factor. Thus $\langle g_1, g_2 \rangle$ is a noncyclic subgroup with a fixed point, a contradiction. □

Putting these together, we obtain the following. Note that the torsion in $H_2$ is dual to the torsion in $H^3$, thus a quotient of $H^3$ becomes a subgroup of $H_2$.

**Corollary 13.** Let $L$ be a compact 5–manifold with a positive Sasakian structure. Then the torsion subgroup of the second homology, $\text{tors} H_2(L, \mathbb{Z})$ has a subgroup $G$ such that

1. $G$ is $(\mathbb{Z}/m)^2$, $(\mathbb{Z}/5)^4$, $(\mathbb{Z}/4)^4$, $(\mathbb{Z}/3)^2n$ for $n \in \{2, 3, 4\}$ or $(\mathbb{Z}/2)^2n$ and
2. $\text{tors} H_2(L, \mathbb{Z})/G$ is $\mathbb{Z}/r$, $\mathbb{Z}/r + \mathbb{Z}/2$ or $\mathbb{Z}/r + \mathbb{Z}/3$ for some $r$. □

**Remark 14.** Most Abelian groups can not be written in the above form. If $G$ exists, then it is almost always unique up to isomorphism. The only ambiguity is with the 6-torsion part.

To see this, we can consider the $p$-parts separately. The main cases are

1. $A_p = (\mathbb{Z}/p^a)^2 + \mathbb{Z}/p^b$ and $A_p/G \cong \mathbb{Z}/p^b$,
2. $A_p = \mathbb{Z}/p^a + \mathbb{Z}/p^b$ and $A_p/G \cong \mathbb{Z}/p^b-a$.

These are the only possibilities for $p \geq 7$, with a few more cases for $p = 2, 3, 5$.

**15** (Plan of the proofs of [11] and [7]). We follow the approach in [Kol05 6.8]. Start with $(S, (1 - \frac{1}{r})D + \Delta)$ satisfying the conditions (S1–3). Let $g : S' \to S$ be the minimal resolution of $S$ and $h : S' \to S''$ a minimal model of $S'$. In the sequence of blow ups leading from $S$ to $S'$ and the subsequent blow downs leading from $S'$ to $S''$ every intermediate surface $T$ satisfies the following condition:

$(*)$ We can write $-K_T = (1 - \frac{1}{r})D_T + \Delta_T + H_T$ where $D_T$ denotes the birational transform of $D$ on $T$, $\Delta_T$ is an effective linear combination of rational curves (coming from $\Delta$ and the exceptional curves of $g$) and $H_T$ is nef and big (this is a general divisor numerically equivalent to the pull back/push forward of $-(K_S + (1 - \frac{1}{r})D + \Delta)$).

This turns out to be very restrictive in many cases.

It is easy to see that $S$ is a rational surface, so $S''$ is either $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or a minimal ruled surface $\mathbb{F}_n$ for some $n \geq 2$. For the latter, let $E \subset \mathbb{F}_n$ denote the negative section and $F$ a fiber. By an easy case analysis (cf. [Kol05 6.8]) we get the following.
The plan is, in each case, to start with $(S, (1 - \frac{1}{r})D^m + \Delta^m)$ and to write down all possible $(S', (1 - \frac{1}{r})D' + \Delta')$. It is then easy to get a complete list of all $(S, (1 - \frac{1}{r})D + \Delta)$. Finally we need to check the additional conditions, especially [Kol05 3.6 and 4.8].

2. The exceptional cases

16 (The $(\mathbb{Z}/5)^4$ and $(\mathbb{Z}/3)^8$ cases). Assume that $H_2(L, \mathbb{Z})$ contains a subgroup isomorphic to $(\mathbb{Z}/5)^4$ or $(\mathbb{Z}/3)^8$. Let $G \subset H_2(L, \mathbb{Z})$ be the subgroup given in [13]. Since the odd torsion in $H_2(L, \mathbb{Z})/G$ is either cyclic or $(\mathbb{Z}/3)^2$, we conclude that either

(1) $G = (\mathbb{Z}/5)^4$,
(2) $G = (\mathbb{Z}/3)^8$, or
(3) $G = (\mathbb{Z}/3)^6$ and $H_2(L, \mathbb{Z})/G = (\mathbb{Z}/3)^2$.

Thus $g(D) = 2$ in the first case and $g(D) \geq 3$ in the second and third cases.

This implies that $S^0$ is simply connected. Indeed, any nontrivial cover $\pi : S' \to S$ would give another log Del Pezzo surface where $D' := \pi^{-1}(D)$ has genus $\geq 3$ in the first case and genus $\geq 5$ in the other cases. By the list in [15] there are no such surfaces. In particular, the case $G = (\mathbb{Z}/3)^6$ and $H_2(L, \mathbb{Z})/G = (\mathbb{Z}/3)^2$ does not happen.

Next, as in [Kol05 9.9], we show that $S' = S^m$ and $S$ is obtained by contracting the negative section $E$. Indeed, any one point blow up of $F_m$ maps either to $F_{m-1}$ (if we blow up a point not on $E$) or $F_{m+1}$ (if we blow up a point on $E$). Since $S^m$ is unique in each case by the list in [14], $S' \to S^m$ is an isomorphism.

If $S = S^m$ then

$$\Delta^m + H^m \equiv \sum (1 - \frac{1}{r_i})D_i + H$$

where the $D_i$ are rational and $H$ is ample. In the $(\mathbb{Z}/5)^4$ case, this would give

$$\frac{3}{5}E + \frac{1}{2}F \equiv \sum (1 - \frac{1}{r_i})D_i + H.$$

The intersection number of the left hand side with $F$ is $\frac{3}{5} < \frac{1}{2}$, so any $D_i$ is a fiber. Now intersecting with $E$ shows that there can not be any $D_i$. This is impossible since the left hand side is not nef and big.

Similarly, in the $(\mathbb{Z}/3)^8$ case we would need to solve

$$\frac{2}{3}E + \frac{1}{3}F \equiv \sum (1 - \frac{1}{r_i})D_i + H,$$

which is also impossible.

This proves [7], except for the equations.

There is a systematic way to obtain $Y$ and the $\mathbb{C}^*$-action from $(S, \Delta)$ (cf. [Kol05 2.5]), but I found it much easier to do this by guessing. Note that by contracting the negative section in $F_n$, we get the weighted projective plane $\mathbb{P}(1, 1, n)$ and $D \in |2E + 2nF|$ is a curve given by a degree $2n$ equation. After completing the square, these are of the form $z^2 + f_{2n}(x, y)$. The explicit relationship between $\Delta$
and the weights exhibited in [BGK05, 6] now leads to \( f = t^5 + z^2 + f_6(x, y) \) in case \( r = 5, S^m = \mathbb{F}_3 \) and to \( f = t^5 + z^2 + f_{10}(x, y) \) in case \( r = 3, S^m = \mathbb{F}_5 \).

The case analysis is harder for \((\mathbb{Z}/4)^4\) and we need some way to see how to get \( S' \) from \( S^m \).

**17 (Blow up criterion).** Assume that \( \pi : T_1 \to T \) is the inverse of the blowing up of \( p \in T \) with exceptional curve \( E_1 \subset T_1 \) and that \( (T_1, (1 - \frac{1}{r})D_1 + \Delta_1 + H_1) \) satisfies \((15.\ast)\). That is, \(-K_{T_1} \equiv (1 - \frac{1}{r})D_1 + \Delta_1 + H_1 \) is effective and \( H_1 \) is nef and big.

Set \( D := \pi_*D_1, \Delta := \pi_*\Delta_1 \) and \( H := \pi_*H_1 \). Then \((T, (1 - \frac{1}{r})D + \Delta + H)\) also satisfies \((15.\ast)\). Since \((K_{T_1} \cdot E_1) = -1\), we conclude that
\[
((1 - \frac{1}{r})D_1 + \Delta_1 + H_1) \cdot E_1 = 1,
\]
hence
\[
\text{mult}_p ((1 - \frac{1}{r})D + \Delta + H) \geq 1.
\]
In practice we know \((T, (1 - \frac{1}{r})D)\) and we would like to find \( p \). This is possible if the numerical class of \( \Delta + H \) is small, but we get many possibilities if \( \Delta + H \) is bigger.

**18 (The \( (\mathbb{Z}/4)^4 \) case).** Here there are two possibilities for \((S^m, D^m)\) and \( S' \to S^m \) need not be an isomorphism.

Let us start with \( S^m = \mathbb{F}_2 \) and \( D^m \in |2E + 5F| \).

**Case 1.** If \( S' = S^m \) then, as in \((16)\), from \((E \cdot (\Delta^m + H^m)) < 0\) we see that \( E \) must be contracted. Thus \( S \) is the weighted projective plane \( \mathbb{P}(1, 1, 2) \) and \( D \) is given by a weighted degree 5 equation \( f_5(x, y, z) = 0 \) where \( w(z) = 2 \). (Alternatively, \( S \) is the quadric cone in \( \mathbb{P}^3 \) and \( D \) is a curve of degree 5 passing through its vertex.) Thus we get the equation \((t^4 + f_5(x, y, z) = 0) \) for \( Y \). This gives us \((17.3.a)\).

**Case 2.** If we perform at least one blow up in going from \( S^m \) to \( S' \), then as before, we could have ended up with \( S^m = \mathbb{F}_1 \) or \( S^m = \mathbb{F}_3 \) instead. The first case is impossible from the list of \((16)\) and the second one we consider next.

**Case 3.** Thus assume that \( S^m = \mathbb{F}_3 \) and \( C \in |2E + 6F| \) is a smooth curve. Thus \( \Delta^m + H^m \equiv \frac{1}{2}E + \frac{1}{2}F \).

If \( S' = S^m \) then, as before, we see that \( E \) must be contracted. This leads to the surface \( S = \mathbb{P}(1, 1, 3) \) and \( D \in |O_S(6)| \) a smooth curve.

This never leads to simply connected 5–manifolds by [Kol05, 4.8]. All Seifert bundles are of the form \( Y(S, B, \frac{1}{4}D) \) where \( B = aF \) for some \( a \in \mathbb{Z} \) and \( b \in \{1, 3\} \). The corresponding Chern class is \((a + \frac{4b}{2})F\), always a half integer. Thus we get the basic cases with \( c_1(Y/S) = \frac{1}{2} \) and their obvious quotients.

Since \( D \in |O_S(6)| \) and 6 is even, \( H^1 orb(S, \frac{1}{4}D) = \mathbb{Z}/2 \) and there are two basic cases: \( a = -1, b = 1 \) and \( a = -4, b = 3 \).

The double cover of \((S, \frac{1}{4}D)\) is given by
\[
S_6 = (f_6(x, y, z) + T^2 = 0) \subset \mathbb{P}(1, 1, 3, 3),
\]
and the involution is \((x : y : z : T) \leftrightarrow (x : y : z : -T)\). Keep in mind that these are projective coordinates, so the same involution can be given as \((x : y : z : T) \leftrightarrow (-x : -y : -z : T)\). The basic Seifert bundle pulls back to \( Y = L^4(2, 2, 6, 3; 12) \) with typical equation
\[
x^6 + y^6 + z^2 + t^4 = 0.
\]
There are 2 ways to lift the involution to a fixed point free involution on $Y$. These are $(x, y, z, t) \mapsto (-x, -y, -z, -t)$ and $(x, y, z, t) \mapsto (-x, -y, z, t)$. These give the 2 families listed in (7.3.b).

**Case 4.** Next we blow up at least 1 point on $S^m$. This point can not be on $E$ since the resulting surface would also dominate $F_4$. Write
\[
\frac{1}{2}E + \frac{1}{2}F = \Delta^m + H^m = \sum a_i F_i + bE + R^m,
\]
where the $F_i$ are distinct fibers and $R^m$ has no irreducible component which is $E$ or a fiber. By intersecting with $F$ we see that $b \leq \frac{1}{2}$. Intersecting with $E$ gives
\[
0 \leq (E \cdot R^m) = 3b - 1 - \sum a_i,
\]
in particular $b \geq \frac{1}{3}$ and $\frac{1}{2} - b \leq \frac{1}{3}(\frac{1}{2} - \sum a_i)$, thus $\sum a_i \leq \frac{1}{2}$. If $p$ lies on $F_1$ then
\[
\text{mult}_p(\Delta^m + H^m) \leq a_1 + (F \cdot R^m) = a_1 + (\frac{1}{2} - b) \leq \frac{1}{6} + \frac{1}{2}a_1 - \frac{1}{3} \sum a_i \leq \frac{1}{7}.
\]
The condition (15+) is
\[
\text{mult}_p(\frac{4}{3}D + \Delta^m + H^m) \geq 1,
\]
which is only possible if $p \in C$ and $a_1 \geq 1/8$. Furthermore, if $a_1 \leq a_2$ then we get $\frac{1}{8} \leq a_1 \leq a_2$. But $\sum a_i \leq \frac{1}{2}$, and this would lead to $R^m = (\frac{1}{8} - b)E$ which is impossible. Thus we conclude:

**Claim.** There is a unique fiber $F$ such that $S'$ is obtained from $S^m$ by blowing up points in or above $F \cap D$.

The general case is when $F \cap D$ consists of 2 points and the special case is when $F \cap D$ is a single point where $F$ and $D$ are tangent.

**Case 5.** Let us deal first with the general case and blow up $p \in F \cap D$. We get $S_1 \to S^m$ with exceptional curve $G_1$. Let $F_1$ and $D_1$ be the birational transforms of $F$ and $D$. Write
\[
\Delta_1 + H_1 = aF_1 + bE_1 + cG_1 + R_1.
\]
From (17) we see that
\[
c \leq \text{mult}_p(\frac{4}{3}D + \Delta^m + H^m) - 1 \leq \frac{3}{4} a + \frac{1}{3}(\frac{1}{2} - a) - 1 = \frac{2}{3}a - \frac{1}{12}.
\]
So at every point $q \in G_1 \setminus (D_1 + F_1)$ the multiplicity of $\Delta_1 + H_1$ is at most
\[
c + \text{mult}_q R_1 \leq c + \text{mult}_p R^m \leq \frac{2}{3}a - \frac{1}{12} + \frac{1}{3}a = \frac{1}{12} + \frac{4}{3}a < 1,
\]
thus we can not blow these up. At $G_1 \cap F_1$ the multiplicity is at most
\[
a + c + \text{mult}_q R_1 \leq a + \frac{1}{12} + \frac{1}{3}a < 1.
\]
Finally, at $G_1 \cap D_1$ the multiplicity is at most
\[
\frac{2}{3} + \frac{1}{12} + \frac{2}{3}a = \frac{5}{6} + \frac{1}{3}a < 1,
\]
extcept when $a = \frac{1}{6}$. But this again would lead to $R^m = (\frac{1}{6} - b)E$ which is impossible.

Thus we conclude that we can blow up one or both of the points $F \cap D$ and then we get $S'$.

If we blow up only one point, we have to contract $F_1$ and we are in the already considered case when $S' = F_2$. 
Case 6. If we blow up both points of $F \cap D$, we have to contract $E_1 \cup F_1$. We get a surface with Picard number 2 and a single cyclic quotient singularity of the form $\mathbb{C}^2/\mathbb{Z}_3(1, 2)$.

Claim. The surface $S$ is isomorphic to a quasi-smooth hypersurface $S_0 \subset \mathbb{P}(1, 1, 3, 5)$ and $D$ is the complete intersection of $S_0$ with $(t = 0)$.

I found this isomorphism by computing the quotient by the hyperelliptic involution of $D$ which acts on $S^n, S'$ and also on $S$. Once the isomorphism is guessed, it is easier to verify by working backwards from the surface $S_0 \subset \mathbb{P}(1, 1, 3, 5)$. Its equation can be written, after coordinate changes, as

$$S_0 = (z^2 + f_6(x, y) + \ell_1(x, y)t = 0) \subset \mathbb{P}(1, 1, 3, 5).$$

Notice that its intersection with $(\ell_1 = 0)$ is the reducible curve $(z^2 + f_6(x, \alpha x) = 0) \subset \mathbb{P}(1, 3, 5)$ for some $\alpha$. The two irreducible components correspond to the two exceptional curves of $S' \to S^n$. The rest is a straightforward computation. The surface is specified by choosing 6 points in $\mathbb{P}^1$ (given by $f_6 = 0$) plus one more corresponding to the choice of $\ell_1$.

This leads to the case (7.3.e).

Case 7. Next we deal with the special case when $F \cap D$ is a single point where $F$ and $D$ are tangent. Computations as above yield that in this case we can blow up the point $p$ on $D$ at most 3–times.

Case 8. If $S'$ is obtained by 1 blow up, we factor through $\mathbb{P}_2$ as before. If we do 2 blow ups, we get $S = \mathbb{P}(1, 2, 5)$ and $D \in |O(10)|$. As in Case 3, $H^0_{\text{orb}}(S, \frac{4}{3}D) = \mathbb{Z}/2$. Set $L := (x = 0)$. There are two basic cases, corresponding to $Y(S, -2L, \frac{4}{3}D)$ and $Y(S, -7L, \frac{4}{3}D)$. The index 2 point $(0 : 1 : 0) \in S$ adds a further complication since the first of these does not satisfy the smoothness condition [Kol05, 4.8.1].

The orbifold double cover of $(S, \frac{4}{3}D)$ is

$$S_{10} := (f_{10}(x, y, z) + T^2 = 0) \subset \mathbb{P}(1, 2, 5, 5)$$

and the involution is $(x : y : z : T) \mapsto (x : y : z : -T)$. Since these are weighted projective coordinates, this is the same as $(x : y : z : T) \mapsto (-x : y : z : T)$. (Note that $-1$ acts by sending a coordinate $u$ to $(-1)^{w(u)}u$, hence the + sign in front of $y$.)

This leads to the basic examples $L = L^*(2, 4, 10, 5; 20)$ with sample equation $(x^{10} + y^5 + z^2 + t^4 = 0)$, The lifting of the involution is either $(x, y, z, t) \mapsto (-x, y, -z, -t)$ or $(x, y, z, t) \mapsto (-x, y, z, t)$. Here the second action has fixed point, this is consistent with our earlier considerations. Thus we get only one family, as in (7.3.d).

Case 9. Finally, if we blow up 3–times, we can contract the birational transforms of $E, F$ and of the first 2 exceptional curves. This gives a surface with a single singular point of the form $\mathbb{C}^2/\mathbb{Z}_3(2, 5)$.

Claim. The surface $S$ is isomorphic to a quasi-smooth hypersurface $S_{10} \subset \mathbb{P}(1, 2, 5, 9)$ and $D$ is the complete intersection of $S_{10}$ with $(t = 0)$.

Once again, it is easier to verify this by working backwards. The equation of $S_{10}$ is

$$(f_5(x^2, y) + z^2 + xt = 0) \subset \mathbb{P}(1, 2, 5, 9).$$
Its intersection with \((x = 0)\) is the curve \((ay^5 + bz^2 = 0)\) \(\subset\mathbb{P}(2, 5, 9)\). This is a smooth (but not quasi-smooth) rational curve, and it corresponds to the last exceptional curve of \(S' \to S^m\). The surface is specified by choosing 6 points in \(\mathbb{P}^1\) (given by \(xf_5 = 0\)), that is a genus 3 hyperelliptic curve plus a specified branch point. The rest is a straightforward computation.

Thus we obtain (7.3.c). \(\Box\)

19 (Other examples). It is easy to write down infinitely many positive Sasakian structures on certain simply connected 5–manifolds \(L\) either by hand or by consulting the (partially unpublished) lists of Boyer and Galicki. Below I write the orbifolds \((S, \Delta)\) and the simplest equation. (Here \(C_k\) denotes a general curve of degree \(k\) in the weighted projective plane \(\mathbb{P}(a, b, c)\) with coordinates \(x, y, z\) and \(\ell := (x = 0)\).)

1. \(H_2(L, \mathbb{Z}) = (\mathbb{Z}/3)^2\). For any \((k, 6) = 1\), take \((\mathbb{P}(1, 1, 2), (1 - \frac{1}{k})C_4 + (1 - \frac{1}{k})\ell)\) and \(x^{4k} + y^2 + z^2 + t^3\).
2. \(H_2(L, \mathbb{Z}) = (\mathbb{Z}/5)^2\). For any \((k, 30) = 1\), take \((\mathbb{P}(1, 2, 3), (1 - \frac{1}{k})C_5 + (1 - \frac{1}{k})\ell)\) and \(x^{6k} + y^2 + z^2 + t^5\).
3. \(H_2(L, \mathbb{Z}) = (\mathbb{Z}/3)^2\). For any \((k, 30) = 1\), take \((\mathbb{P}(1, 2, 5), (1 - \frac{1}{k})C_{10} + (1 - \frac{1}{k})\ell)\) and \(x^{10k} + y^5 + z^2 + t^3\).
4. \(H_2(L, \mathbb{Z}) = (\mathbb{Z}/2)^{2n}\). For \(n \geq 0\) and \((k, 2n(2n + 1)) = 1\) take \((\mathbb{P}(1, 1, n), (1 - \frac{1}{k})C_{2n+1} + (1 - \frac{1}{k})\ell)\) and \(x^{k(2n+1)} + y^{2n+1} + yz^2 + t^2\).

Sasakian–Einstein structures are known to exist in the first 3 cases, but for the last one the criterion of [DK01] fails and existence is not known.

The situation is more complicated in the next two cases:
5. \(H_2(L, \mathbb{Z}) = (\mathbb{Z}/4)^2\) or \((\mathbb{Z}/6)^2\). For any \(k\), take \((\mathbb{P}(1, 2, 3), (1 - \frac{1}{k})C_6 + (1 - \frac{1}{k})\ell)\) resp. \((\mathbb{P}(1, 2, 3), (1 - \frac{1}{k})C_6 + (1 - \frac{1}{k})\ell)\).

These are the right examples as log Del Pezzo surfaces, but the condition [Kol05, 4.8.2] fails, so the corresponding Seifert bundles are not simply connected. Since my approach is to rule out everything at the surface level, these cases could be very hard to settle using my methods.

This leaves open the finiteness question for
6. \(H_2(L, \mathbb{Z}) = (\mathbb{Z}/m)^2\) with \(7 \leq m \leq 11\), and
7. \(H_2(L, \mathbb{Z}) = (\mathbb{Z}/3)^6\).

The computations in Section 4 suggest that in these cases there should be only finitely many families of pre-SE structures.

However, as the examples with \(H_2(L, \mathbb{Z}) = (\mathbb{Z}/4)^4\) suggest, the case analysis can be rather tricky and unexpected special configurations may arise.

3. The main series

Let us start with fixing an error in [Kol05].
The theorem asserts that for each \( m \geq 12 \), there are exactly 93 pre–SE Seifert bundles \( f : L \to (S, \Delta) \) on compact 5–manifolds \( L \) satisfying \( H_1(L, \mathbb{Z}) = 0 \) and tors \( H_2(L, \mathbb{Z}) \cong (\mathbb{Z}/m)^2 \).

The construction of the 93 families of pairs \((S, \Delta)\) is correct. Going from \((S, \Delta)\) to the Seifert bundle is, however, done incorrectly since the conditions for a Sasakian structure and for a pre–Sasakian–Einstein structure have been thoroughly mixed up.

The construction of the 93 families given in [Kol05, 7.6] starts with a surface \( T \) which is one of \( \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^2, Q_S, \mathbb{P}(1, 2, 3) \). For each of these write \(-K_T \sim d(T)H\) where \( H \in \text{Weil}(T) \) is a positive generator. We have

\[
d(\mathbb{P}^1 \times \mathbb{P}^1) = 2, d(\mathbb{P}^2) = 3, d(Q) = 4, d(S_5) = 5, d(\mathbb{P}(1, 2, 3)) = 6.
\]

Next we perform some weighted blow ups [Kol05, 7.3] to get \( S = B_{m_1, \ldots, m_k}T \). There are \( k \) exceptional curves \( E_1, \ldots, E_k \). Each \( E_i \) passes through a unique singular point \( p_i \in S \) and \( E_i \) generates the local class group which is \( \mathbb{Z}/m_i \). Set \( d(S) := \gcd(m_1, \ldots, m_k, d(T)) \).

The divisor class group \( \text{Weil}(S) \) is freely generated by \( \pi^*H, E_1, \ldots, E_k \) and \( K_S \sim -d(T)\pi^*H + \sum m_iE_i \).

The main condition that was overlooked is the smoothness criterion for Seifert bundles [Kol05, 3.6].

In our case there is only one curve \( D \) and \( S \) is smooth along \( D \). Thus, by [Kol05, 3.6], the corresponding Seifert bundle \( Y(S, B, \frac{1}{m}D) \) is smooth iff

\[
B \text{ generates the local class group at every point.} \quad (20.1)
\]

**Claim 20.2.** Notation as above. Then

\[
Y(B_{m_1, \ldots, m_k}T, aH + c_1E_1 + \cdots + c_kE_k, \frac{b}{m}D) \quad \text{is smooth}
\]

iff \((m_i, c_i) = 1 \) for \( i = 1, \ldots, k \) and \((a, d(T)) = 1 \) if \( T \) is singular.

As a consequence, we see that all 93 cases correspond to positive Sasakian structures on smooth 5–manifolds.

A Seifert bundle is pre–SE iff its Chern class \( c_1(Y/S) = B + \frac{b}{m}D \) is a rational multiple of \(-K_S + (1 - \frac{1}{m})D\). In our case \( D \sim -K_S \), hence \( B \) itself is a rational multiple of \(-K_S \). Thus

\[
B = r\left( \frac{d(T)}{d(S)} \pi^*H - \sum \frac{m_i}{d(S)} E_i \right)
\]

for some positive integer \( r \). Thus if (20.1) holds then \( d(S) = m_i \) for every \( i \).

Furthermore, in the cases when \( T \) is singular, \( B \) generates the local class group at each singular point of \( T \) iff \( d(T) = d(S) \). Thus we obtain the following.

**Claim 20.3.** Notation as above. Then

\[
Y(B_{m_1, \ldots, m_k}T, aH + c_1E_1 + \cdots + c_kE_k, \frac{b}{m}D) \quad \text{is smooth and pre–SE iff}
\]

(1) \( m_1 = \cdots = m_k = d(S) \),

(2) \( aH + c_1E_1 + \cdots + c_kE_k = r\left( -\frac{d(T)}{d(S)}H + E_1 + \cdots + E_k \right) \) for some \( r \in \mathbb{Z} \), and

(3) if \( T \) is singular, then \((r, d(T)) = 1 \) and \( d(S) = d(T) \).

\[\square\]
These conditions cut down considerably the list given in [Kol05, 7.6] and we get the following 19 cases:

| surfaces                                                                 | $d(S)$ |
|--------------------------------------------------------------------------|--------|
| $B_1 \mathbb{P}^2$, $B_{11} \mathbb{P}^2$, $B_{1111111} \mathbb{P}^2$   | 1      |
| $\mathbb{P}^1 \times \mathbb{P}^1$, $B_2 \mathbb{P}^1 \times \mathbb{P}^1$, $B_{22} \mathbb{P}^1 \times \mathbb{P}^1$, $B_{222} \mathbb{P}^1 \times \mathbb{P}^1$ | 2      |
| $\mathbb{P}^2$, $B_3 \mathbb{P}^2$, $B_{33} \mathbb{P}^2$               | 3      |
| $Q$, $B_4 Q$                                                              | 4      |
| $S_5$                                                                    | 5      |
| $\mathbb{P}(1, 2, 3)$                                                   | 6      |

Table 1

The condition [Kol05, 4.8.2] says that the resulting Seifert bundle $L$ satisfies $H_1(L, \mathbb{Z}) = 0$ iff $H^2(S, \mathbb{Z}) \to H^2(D, \mathbb{Z}) \to \mathbb{Z}/m$ is surjective. If $S = T$, that is, we do no blow ups at all, then by [Kol05, 9.8]

$$\text{im}[H^2(T, \mathbb{Z}) \to H^2(D, \mathbb{Z})] = d(T)H^2(D, \mathbb{Z}).$$

After blow ups, the new curves that come in are the exceptional curves $E_i$. Here $(E_i \cdot D) = 1$ but the $E_i$ pass through the singular points and they are only homology classes. To get a cohomology class (or Cartier divisor), we need to take $m_i E_i$ since $m_i$ is also the index of the singular point. Thus we conclude:

**Claim 20.4.** For $S = B_{m_1, \ldots, m_k} T$, $\text{im}[H^2(S, \mathbb{Z}) \to H^2(D, \mathbb{Z})] = d(S) \cdot H^2(D, \mathbb{Z})$. 

**Corollary 21.** Let $S$ be a projective surface with Du Val singularities such that $\pi_1(S^0) = 1$. Let $D \in |-K_S|$ be a smooth elliptic curve. There is a simply connected Seifert bundle $f : L \to (S, (1 - \frac{1}{m}) D)$ with a pre-SE structure iff

1. $S$ is one of the surfaces in Table 1, and
2. $m$ is relatively prime to $d(S)$.

In these cases, $f : L \to (S, (1 - \frac{1}{m}) D)$ is uniquely determined by $(S, (1 - \frac{1}{m}) D)$ (up to reversing the orientation of the fibers) and $L$ carries a Sasakian–Einstein metric for $m \geq 7$.

**Proof.** We have proved everything, except the claims about the existence of Sasakian–Einstein metrics. This will be established in [33].

22 (Equations). Quite surprisingly, all the singular surfaces on the list can be realized in weighted projective 3–spaces. All of these examples are on the Boyer–Galicki lists. Here also claim the converse: every singular Del Pezzo surface in Table 1 is isomorphic to a corresponding surface below.

1. $B_2 \mathbb{P}^1 \times \mathbb{P}^1$: $S_3 \subset \mathbb{P}(1, 1, 1, 2)$ with simplest equation

$$x^3 + y^3 + z^3 + xt^m = 0 \quad \text{for} \quad (m, 2) = 1.$$

2. $B_{22} \mathbb{P}^1 \times \mathbb{P}^1$: $S_4 \subset \mathbb{P}(1, 1, 2, 2)$ with simplest equation

$$x^4 + y^4 + z^2 + xt^m = 0 \quad \text{for} \quad (m, 2) = 1.$$

3. $B_{222} \mathbb{P}^1 \times \mathbb{P}^1$: $S_6 \subset \mathbb{P}(1, 2, 3, 2)$ with simplest equation

$$x^6 + y^3 + z^2 + t^3m = 0 \quad \text{for} \quad (m, 2) = 1.$$
Remark 23. There are no hypersurface links on the Boyer–Galicki lists giving infinitely many \((S, \Delta)\) where 
\(S \in \{B_1P^2, \ldots, B_{11111}P^2\}\).

I claim that these can not be realized as the link \(L\) of a hypersurface with a \(\mathbb{C}^*\)-action, at least when \(H_2(L, \mathbb{Z}) \supset (\mathbb{Z}/m)^2\) for \(m \geq 12\) and \((S, \Delta)\) is log Del Pezzo. Assume the contrary. Then we get that \(S\) is a hypersurface in a weighted projective space \(\mathbb{P}(a, b, c, d)\) such that \(K_S + (1 - \frac{1}{m})D\) is proportional to \(H|_S\) where \(H\) is the hyperplane class of the weighted projective space. Since \(D \in | - K_S|\), we conclude that \(K_S\) is proportional to \(H\). By the Grothendieck–Lefschetz theorem, this implies that 
\[-K_S = dH\text{ for some } d \in \mathbb{Z}.\]
In our cases \(-K_S\) is not divisible, so \(d = 1\). This implies that 
\(h^0(S, \mathcal{O}_S(-K_S)) = h^0(\mathbb{P}(a, b, c, d), \mathcal{O}_\mathbb{P}(1)) \leq 4,\)
since this dimension is the number of times that 1 occurs among \(a, b, c, d\). In the above cases, however, \(h^0(S, \mathcal{O}_S(-K_S)) \geq 5\).

Galicki told me that the link \(L^+(2, 3, 4, 7; 14)\) realizes \(S = B_{11111}P^2\), but the corresponding \(\Delta\) is not a rational multiple of \(-K_S\).

24 (The cases with nontrivial fundamental group). The classification of minimal Del Pezzo surfaces with Du Val singularities is completed in [Fur86, MZ88, MZ93, Ye02]. We are interested only in those that have cyclic quotient singularities. The 5 cases where \(\pi_1(S^0) = 1\) were considered in [Kol05]. The following table lists the

(4) \(B_3P^2\): \(S_4 \subset \mathbb{P}(1, 1, 2, 3)\) with simplest equation
\[x^4 + y^4 + z^2 + xt^m = 0 \quad \text{for } (m, 3) = 1.\]

(5) \(B_{33}P^2\): \(S_6 \subset \mathbb{P}(1, 2, 3, 3)\) with simplest equation
\[x^6 + y^3 + z^2 + zt^m = 0 \quad \text{for } (m, 3) = 1.\]

(6) \(Q\): \(S_4 \subset \mathbb{P}(1, 1, 2, 4)\) with simplest equation
\[x^4 + y^4 + z^2 + tm = 0 \quad \text{for } (m, 2) = 1.\]

(7) \(B_4Q\): \(S_6 \subset \mathbb{P}(1, 2, 3, 4)\) with simplest equation
\[x^6 + y^3 + z^2 + yt^m = 0 \quad \text{for } (m, 2) = 1.\]

(8) \(S_5\): (sorry for the notation) \(S_6 \subset \mathbb{P}(1, 2, 3, 5)\) with simplest equation
\[x^6 + y^3 + z^2 + zt^m = 0 \quad \text{for } (m, 5) = 1.\]

(9) \(P(1, 2, 3)\): \(S_6 \subset \mathbb{P}(1, 2, 3, 6)\) with simplest equation
\[x^6 + y^3 + z^2 + t^m = 0 \quad \text{for } (m, 6) = 1.\]
Here $G_n$ is a nonabelian group of order $n$.

(1) $G_8$ is the quaternion group,

(2) $G_{16} \subset GL_4$ is generated by $(x, y, z, t) \mapsto (z, t, -x, y)$ and $(x, y, z, t) \mapsto (y, -x, t, -z)$.

(3) $G_{27} \subset GL_3$ is generated by $(x, y, z) \mapsto (y, z, x)$ and $(x, y, z) \mapsto (x, \epsilon y, \epsilon^2 z)$ where $\epsilon^3 = 1$.

The computation of the table: The papers [Fur86] p.13-15, [MZ88] p.71, [MZ93] p.193 and [Ye02] contain tables for the first 4 columns, except Weil / Pic in the 4 cases with Picard number 2. (The fundamental group of $A_7 + A_1$ is listed in [MZ88] p.71 as $(Z/2)^2$, but it is $Z/4$.) The Picard number and the singularities of the universal cover are listed in [MZ88] p.71, from this it is easy to work out where the surface is on the list [Ko05] 7.6.

By [Ye02] 1.2 and 1.6, for each singularity type there is a unique surface, except for $2A_3$ for which there is a 1-parameter family.

Once we have $\tilde{S} \to S$ as the universal cover and $\tilde{D} \subset \tilde{S}$ is the preimage of $D$ then we have an exact sequence

$$\pi_1(\tilde{S}^0 \setminus \tilde{D}) \to \pi_1(S^0 \setminus D) \to \pi_1(S^0) \to 1.$$ 

Each time $\tilde{S}$ is obtained by a blow up of weight 1, the resulting $\mathbb{P}^1 \subset \tilde{S}$ intersects $\tilde{D}$ transversally at 1 point, so $\pi_1(\tilde{S}^0 \setminus \tilde{D}) = 1$. In the remaining cases one needs to write down the action of the $\pi_1(S^0)$ and see how it lifts to the universal cover of $S^0 \setminus D$.

Computing any entry of the table is an elementary task. Some computations are quick but a few are quite tedious. It is unfortunately easy to miss or misdraw a $-1$-curve after performing many blow ups, so anyone wishing to rely on a particular entry is advised to recheck it.

| degree | singularities | $\pi_1(S^0)$ | Weil / Pic | univ. cover | $\pi_1(S^0 \setminus D)$ |
|--------|---------------|--------------|-------------|-------------|-----------------|
| 1      | $A_8$         | $\mathbb{Z}/3$ | $\mathbb{Z}/3$ | $B_{3111}\mathbb{P}^2$ | $\mathbb{Z}/3$ |
| 1      | $A_7 + A_1$   | $\mathbb{Z}/4$ | $\mathbb{Z}/4$ | $B_{2111}\mathbb{P}^2$ | $\mathbb{Z}/4$ |
| 1      | $A_5 + A_2 + A_1$ | $\mathbb{Z}/6$ | $\mathbb{Z}/6$ | $B_{1111}\mathbb{P}^2$ | $\mathbb{Z}/6$ |
| 1      | $4A_2$        | $(\mathbb{Z}/3)^2$ | $(\mathbb{Z}/3)^2$ | $\mathbb{P}^2$ | $G_{27}$ |
| 1      | $2A_3 + 2A_1$ | $\mathbb{Z}/2 + \mathbb{Z}/4$ | $\mathbb{Z}/2 + \mathbb{Z}/4$ | $\mathbb{P}^1 \times \mathbb{P}^1$ | $G_{16}$ |
| 1      | $2A_4$        | $\mathbb{Z}/5$ | $\mathbb{Z}/5$ | $B_{1111}\mathbb{P}^2$ | $\mathbb{Z}/5$ |
| 2      | $A_7$         | $\mathbb{Z}/2$ | $\mathbb{Z}/4$ | $B_{31}\mathbb{P}^2$ | $\mathbb{Z}/2$ |
| 2      | $A_5 + A_2$   | $\mathbb{Z}/3$ | $\mathbb{Z}/6$ | $B_{11}\mathbb{P}^2$ | $\mathbb{Z}/3$ |
| 2      | $2A_3 + A_1$  | $\mathbb{Z}/4$ | $\mathbb{Z}/2 + \mathbb{Z}/4$ | $\mathbb{P}^1 \times \mathbb{P}^1$ | $\mathbb{Z}/2 + \mathbb{Z}/4$ |
| 2      | $6A_1$        | $(\mathbb{Z}/2)^2$ | $(\mathbb{Z}/2)^2$ | $\mathbb{P}^1 \times \mathbb{P}^1$ | $G_8$ |
| 2      | $2A_3$        | $\mathbb{Z}/2$ | $\mathbb{Z}/2 + \mathbb{Z}/4$ | $B_{21}\mathbb{P}^2 \times \mathbb{P}^1$ | $\mathbb{Z}/4$ |
| 3      | $A_5 + A_1$   | $\mathbb{Z}/2$ | $\mathbb{Z}/6$ | $B_{3}\mathbb{P}^2$ | $\mathbb{Z}/6$ |
| 3      | $3A_2$        | $\mathbb{Z}/3$ | $(\mathbb{Z}/3)^2$ | $\mathbb{P}^2$ | $(\mathbb{Z}/3)^2$ |
| 4      | $A_3 + 2A_1$  | $\mathbb{Z}/2$ | $\mathbb{Z}/2 + \mathbb{Z}/4$ | $Q$ | $(\mathbb{Z}/2)^2$ |
| 4      | $4A_1$        | $\mathbb{Z}/2$ | $(\mathbb{Z}/2)^3$ | $\mathbb{P}^1 \times \mathbb{P}^1$ | $(\mathbb{Z}/2)^2$ |

Table 2


In the simply connected case there are many isomorphisms between blow ups, but this does not happen for the general case.

**Lemma 25.** Let $S$ be a Del Pezzo surface with cyclic quotient singularities such that $|\pi_1(S^0)| > 1$. There is a unique line in Table 2 such that $S$ is a weighted blow up of a surface on that line. (We do not claim that the blow up itself is unique.)

**Proof.** The blow ups do not change the fundamental group [Kol05, 7.3] and we create only $A_1$ and $A_2$ singularities since we can only have blow ups $B_{m_1, \ldots, m_r}T$ where $\sum m_i < \deg T - 1 \leq 3$. It turns out that the fundamental group and the collection of $A_i : i \geq 3$ singularities uniquely determine in which line of Table 2 the surface is. The Picard number and the $A_i : i \leq 2$ singularities now determine the number and type of blow ups performed. \hfill \Box

If $\deg T = 1$ (resp. 2, 3) then we get 1 (resp. 2, 4, 7) blown up surfaces, including $T$ itself. Thus we get 39 deformation types of Del Pezzo surfaces with cyclic quotient singularities such that $|\pi_1(S^0)| > 1$. The 93 cases where $\pi_1(S^0) = 1$ were enumerated in [Kol05, 7.6], giving a total of 132 deformation types.

**26 (Existence of smooth Seifert bundles).** Let $S$ be one of the surfaces in Table 2 and $D \in |-K_S|$ a smooth elliptic curve. As in (20), we are considering Seifert bundles $Y(S, B, \frac{D}{m})$ where $B$ is a Weil divisor class on $S$.

By (20.1) $Y(S, B, \frac{D}{m})$ is smooth iff $B$ generates the local class group at every point.

Finding all such $B$ requires a detailed computation of the divisor class group $\text{Weil}(S)$ and the restriction map

$$\text{Weil}(S)/\text{Pic}(S) \to \sum_{p \in \text{Sing } S} \text{Weil}(p, S).$$

On a Del Pezzo surface of degree $\leq 7$, the curves $C$ with $(C \cdot K_S) = -1$ generate $\text{Weil}(S)$. On the minimal desingularization $S' \to S$ these are the $-1$ curves. Thus if we have a description of $S'$ as a blow up of $\mathbb{P}^2$, we see all such curves by looking at lines through 2 blow up points, conics through 5 blow up points, etc. (See [Man86, Sec.26] for the complete list in degrees 2 and 1.) The description given in [Fur86] gives exactly these blow ups. See also [MZ88, Figure 1].

In some cases $\text{Weil}(S)/\text{Pic}(S)$ is too small to get surjection onto some $\text{Weil}(p, S)$, and then there are no smooth Seifert bundles at all. This happens in 3 cases:

$$A_8, A_7 + A_1, A_7.$$  

More surprising is the mildly singular $A_3 + 2A_1$ case which again has no smooth Seifert bundle over it. On the minimal desingularization, the configuration of $-1$ and $-2$ curves is

$$\begin{array}{ccc} & -2 & -2 \\ -1 & -2 & -2 \\ -1 & -2 & -2 \end{array}$$

In order to generate both $\mathbb{Z}/2$ on the ends, we need to take the two $-1$-curves with odd coefficients, but then we get only twice the generator of the $\mathbb{Z}/4$ of the middle singularity.

The more complicated $A_5 + A_1$ case leads to the configuration

$$\begin{array}{ccc} & -2 & -2 \\ -1 & -2 & -2 \\ -1 & -2 & -2 \end{array}$$

plus an extra $-1$-curve. The $-1$-curve shown generates both local class groups, so there are smooth Seifert bundles, even with SE structure.
A glance at the diagrams for $A_5 + A_2 + A_1$ and for $2A_4$ in [MZ88] Figure 1] shows
−1-curves which generate all local class groups.
As another concrete example, the group $G_{27}$ operates freely on $\mathbb{C}^3$ outside the
origin, thus $S^5/G_{27} \to \mathbb{P}^2 / (\mathbb{Z}/3)^2$ is a smooth Seifert bundle.
Just for illustration, let us compute one simple case completely.

**Example 27 (3A_2 case).** We can write this as $S = \mathbb{P}^2 / (\mathbb{Z}/3)$ by the action
\[(x : y : z) \mapsto (x : ey : e^2z) \quad \text{where} \quad e^3 = 1.\]
We can take $D = (x^3 + y^3 + z^3 = 0)$. The universal cover of $\mathbb{P}^2 \setminus D$ is the cubic
\[(x^3 + y^3 + z^3 = t^3) \subset \mathbb{P}^3.\]
We get a $(\mathbb{Z}/3)^2$-action generated by
\[(x : y : z : t) \mapsto (x : ey : e^2z : t) \quad \text{and} \quad (x : y : z : t) \mapsto (x : y : z : et).\]
The 3 coordinate lines give the curves $A, B, C \subset \mathbb{P}^2 / (\mathbb{Z}/3)$. These generate
Weil($S$) subject to the relations $3A = 3B = 3C = A + B + C$. We can thus rewrite
\[\text{Weil}(S) = \mathbb{Z}[A] + \mathbb{Z}/3[A - B].\]
By explicit computations, the class $B_{uv} := uA + v(A - B)$ corresponds to a smooth
Seifert bundle iff none of the numbers $u, v, u + v$ is divisible by 3. Whenever this
holds, there is a smooth pre-SE Seifert bundle $L_{uv} \to (S, B_{uv}, (1 - \frac{1}{m})D)$ for any
$(m, 3) = 1$. It has a Sasakian–Einstein metric for every $m \geq 4$.

**Remark 28.** As a consequence of the classification, [MZ93] Thm, p.184] concludes
that the fundamental group of a Del Pezzo surface with Du Val singularities is
abelian.
This is easy to see directly as follows. Let $g : S' \to S$ be the universal cover.
Pick a smooth elliptic curve $C \in | - K_S|$. Then $C' := g^{-1}(C) \in | - K_{S'}|$, thus it
is also a smooth elliptic curve. Hence the fundamental group is the same as the
kernel of the group homomorphism $C' \to C$, hence an abelian group with at most
2 generators.

4. Klt conditions

While we did not use it in the proof, it is instructive to see the relationship
between finding $S'$ and the klt condition for $(S^m, (1 - \frac{1}{r})D^m + \Delta^m + H^m).
Since $\Delta'$ is a nonnegative linear combination, all exceptional curves of $S' \to S^m$
have nonpositive discrepancy with respect to $(S^m, (1 - \frac{1}{r})D^m + \Delta^m + H^m)$. If the
latter pair is klt, then, by an observation of Shokurov, there are only finitely many
such curves on any birational model of $S^m$ and they can be found explicitly (cf.
[KM98 2.36]).
The problem with using this result is that we do not know $\Delta^m$ and $H^m$, only
the numerical class
\[\Delta^m + H^m = -(K_{S^m} + (1 - \frac{1}{r})D^m).\]
The usual proofs of the above finiteness result start by taking a log resolution of
$(S^m, (1 - \frac{1}{r})D^m + \Delta^m + H^m)$, and in essence we would need to understand all
smooth surfaces dominating $S^m$. 
The key part of the proofs in \([16]\) and \([20]\) is to describe all exceptional curves over \(S^m\) which have nonpositive discrepancy with respect to some \((S^m, (1 - \frac{1}{m})D^m + \Delta^m + H^m)\).

To this end one can use the following result which sharpens the klt conditions used in \([Ko05\text{ Sec.8}]\).

**Proposition 29.** Let \(S\) be a smooth surface, \(C\) a smooth curve on \(S\) and \(D\) an effective \(\mathbb{Q}\)-divisor on \(S\) such that \(C \not\subseteq \text{Supp} \, D\). Let \(p \in S\) be point and \(n\) a natural number. Then \((S, (1 - \frac{1}{m})C + cD)\) is klt at \(p\) for

\[
c < \min \left\{ \frac{1}{(C \cdot D)_p} + \frac{1}{n \cdot \text{mult}_p D}, \frac{1}{\text{mult}_p D} \right\}.
\]

Proof. Choose local coordinates \((x, y)\) such that \(C = (y = 0)\) and let \(f(x, y) = 0\) be an equation of \(mD\) for some \(m\) such that \(mD\) is an integral divisor.

Consider the local degree \(n\) cover \(\pi : T \to S\) given by \(y = z^n\). By \([Ko92\text{ 20.3.2}]\), \((S, (1 - \frac{1}{m})C + cD)\) is klt at \(p\) iff \((T, \frac{1}{m}(f(x, z^n) = 0))\) is klt at \(p\).

We aim to apply a theorem of Varchenko \([Var76]\) which gives a condition for \((T, \gamma g(x, z) = 0))\) to be klt in terms of the Newton polygon of \(g\) in a suitable coordinate system which is achieved after a series of coordinate changes of the form \((x, z) \mapsto (x - \alpha x^i, z)\) or \((x, z) \mapsto (x, z - \alpha x^i)\). The problem is that we can handle only those coordinate changes which are compatible with \(\pi\). That is, those of the form \((x, z) \mapsto (x - \alpha x^i, z)\). Thus we have to look carefully at the proof, not just the final result.

We state the result in the rather artificial form that we need. The reader should consult the proof given in \([KSC04\text{ 6.40}]\), especially pp.172–3.

**Lemma 30.** Write \(g(x, z) = \sum b(i, j)x^iy^j\). Assume that one of the following holds.

1. (Main case.) There are \((i, j)\) and \((i', j')\) such that \(b(i, j) \neq 0, b(i', j') \neq 0\) and the line segment \([[(i, j), (i', j')]]\) contains a point \((\gamma, \gamma)\) with \(\gamma < c^{-1}\).
2. (Degenerate case.) There is \((i, j)\) such that \(b(i, j) \neq 0\) and \(i, j < c^{-1}\).

Then one of the following also holds:

3. (Klt case.) \((T, c\gamma g(x, z) = 0))\) is klt.
4. (Coordinate change.) There are natural numbers \(u, v, w\) and \(e\) such that \((u, v) = 1\), \(e > w/(u + v)\) and

\[
\sum_{ui + vj = uw} b(i, j)x^iy^j \quad \text{is divisible by} \quad (\alpha x^u + \beta y^w)^e.
\]

In this case necessarily \(u = 1\) or \(v = 1\), there is a unique such factor and the new coordinates are \((x, \alpha x^u + \beta y^w)\) if \(u = 1\) and \((\alpha x + \beta y^w, y)\) if \(v = 1\).

Set \(a = (L \cdot D)_p\) and \(d = \text{mult}_p D\). As before, write \(f = \sum b(i, j)x^iy^j\). Then \(b(am, 0) \neq 0\) and \(b(i, j) \neq 0\) for some \(i + j = md\). Thus

\[
f(x, z^n) = \sum b(i, j)x^iy^j.
\]

If \(b(i, j) \neq 0\) for some \(i + j = md\) and \(j \geq md/(n + 1)\) then the main case \(30.1\) applies using the line segment \([[(am, 0), (i, nj)]\]. The maximum value of \(\gamma\) is achieved when \(i = 0\), giving the condition \(c < a^{-1} + (nd)^{-1}\).

Otherwise \(b(i, j) \neq 0\) for some \(i + j = md\) and \(j < md/(n + 1)\) Thus \(i < md\). \(nj < md\) and \(30.2\) applies, giving the condition \(c < d^{-1}\).
Corollary 32. The rest follows from (29). The other two cases are similarly easy. □

Corollary 33. Let $C \subset S$ be a smooth elliptic curve where $S$ is $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_2$. Let $D$ be an effective $\mathbb{Q}$-divisor such that $D \equiv \frac{1}{n}C$. Then $(S, (1-\frac{1}{n})C + D)$ is klt for $n \geq 7$.

Proof. Start with the $S = \mathbb{P}^2$ case. If $H \subset \mathbb{P}^2$ is a line then $C \equiv 3H$ and so $D \equiv \frac{1}{3}H$. Thus $(C \cdot D)_p \leq \frac{2}{3}$ and $\text{mult}_p D \leq (H \cdot D) = \frac{2}{3}$. Since

\[
\frac{2}{3} + \frac{2}{3} > 1 \quad \text{for } n \geq 7,
\]

the rest follows from (29). The other two cases are similarly easy. □

Corollary 31. Let $C \subset \mathbb{P}_4$ be a smooth curve of genus 3 such that $C \equiv 2E + 8F$. Let $D$ be an effective $\mathbb{Q}$-divisor such that $D \equiv -(K + \frac{2}{3}C)$. Then $(\mathbb{P}_4, \frac{2}{3}C + D)$ is klt.

Proof. Here $D \equiv \frac{2}{3}(E + F)$, thus

\[
(C \cdot D)_p \leq \frac{2}{3} \cdot (2E + 8F) \cdot (E + F) = \frac{4}{3},
\]

and $\text{mult}_p D \leq \frac{2}{3}$. Since $\frac{2}{3} + \frac{2}{3} > 1$ the rest follows from (29). □

One can also use (29) to check the existence condition [DK01] for orbifold Kähler–Einstein metrics.

Corollary 33. Let $S$ be a Del Pezzo surface with quotient singularities and $C \in |{-K}_S|$ a smooth elliptic curve. Then $(S, (1-\frac{1}{n})C)$ has an orbifold Kähler–Einstein metric whenever $n > \frac{2}{3}K_S^3$.

Proof. Set $d = K_S^2$. Let $D$ be an effective $\mathbb{Q}$-divisor such that $D \equiv \frac{1}{n}C$. We need to check that $(S, (1-\frac{1}{n})C + \frac{2}{3}D)$ is klt. As in (31), this holds if

\[
\frac{2}{3} < \min\left\{\frac{n}{d}, \frac{2}{n^2d}, \frac{2}{3}\right\} = \frac{2}{3}.
\]

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