Exact Solution of a Master Equation Applied to the  
Two Level System of an Atom

Kazuyuki FUJII *
International College of Arts and Sciences
Yokohama City University
Yokohama, 236–0027
Japan

Abstract

In this paper we discuss a master equation applied to the two level system of an atom and derive an exact solution to it in an abstract manner. We also present a problem and a conjecture based on the three level system.

Our results may give a small hint to understand the huge transition from Quantum World to Classical World.

To the best of our knowledge this is the finest method up to the present.

Keywords : quantum mechanics; decoherence theory; two level system; master equation; exact solution.

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*E-mail address : fujii@yokohama-cu.ac.jp
1 Introduction

The target of this paper is to study and solve the time evolution of a quantum state (which is a superposition of two physical states) under decoherence.

In order to set the stage and to introduce proper notation, let us start with a system of principles of Quantum Mechanics (QM in the following for simplicity). See for example [1], [2], [3] and [4]. That is,

System of Principles of QM

1. Superposition Principle
If $|a\rangle$ and $|b\rangle$ are physical states then their superposition $\alpha|a\rangle + \beta|b\rangle$ is also a physical state where $\alpha$ and $\beta$ are complex numbers.

2. Schrödinger Equation and Evolution
Time evolution of a physical state proceeds like

$$|\Psi\rangle \longrightarrow U(t)|\Psi\rangle$$

where $U(t)$ is the unitary evolution operator ($U^\dagger(t)U(t) = U(t)U^\dagger(t) = 1$ and $U(0) = 1$) determined by a Schrödinger Equation.

3. Copenhagen Interpretation

Let $a$ and $b$ be the eigenvalues of an observable $Q$, and $|a\rangle$ and $|b\rangle$ be the normalized eigenstates corresponding to $a$ and $b$. When a state is a superposition $\alpha|a\rangle + \beta|b\rangle$ and we observe the observable $Q$ the state collapses like

$$\alpha|a\rangle + \beta|b\rangle \longrightarrow |a\rangle \text{ or } \alpha|a\rangle + \beta|b\rangle \longrightarrow |b\rangle$$

where their collapsing probabilities are $|\alpha|^2$ and $|\beta|^2$ respectively ($|\alpha|^2 + |\beta|^2 = 1$).

This is called the collapse of the wave function and the probabilistic interpretation.

There are some researchers who are against this terminology, see for example [4]. However, I don’t agree with them because the terminology is nowadays very popular in the world.
4. Many Particle State and Tensor Product

A multiparticle state can be constructed by the superposition of the Kronecker products of one particle states, which are called the tensor products. For example,

\[ \alpha|a\rangle \otimes |a\rangle + \beta|b\rangle \otimes |b\rangle \equiv \alpha|a,a\rangle + \beta|b,b\rangle \]

is a two particle state.

Here is an important comment. Beginners of QM might think that a quantum state created by an experiment would undergo the unitary time evolution (U) forever.

This is nothing but an illusion because the quantum state is in an environment (a kind of heat bath) and the interaction with it will disturb the quantum state. For example, readers should imagine an oscillator on the desk.

In order to understand QM deeply readers should take decoherence (: interaction with environment) into consideration correctly. For this topic see for example [5].

2 Two Level System and Master Equation

In this section we discuss a master equation applied to the two level system of an atom and solve the equation exactly under certain conditions.

For the discussion of the two level system of an atom let us prepare some notations from Quantum Optics. See for example [6], [7].

For the system the target space is \( \mathbb{C}^2 = \text{Vect}_\mathbb{C}(|0\rangle, |1\rangle) \) with bases

\[ |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]

Then Pauli matrices \{\sigma_1, \sigma_2, \sigma_3\} with the identity \(I_2\)

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]
act on the space. By setting
\[\sigma_+ \equiv \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- \equiv \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\]
it is easy to see
\[\sigma_+\sigma_- = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma_-\sigma_+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\].

For the initial time \(t = 0\) we may assume that the Hamiltonian (of the atom) is of a diagonal form
\[H_0 = \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix}\] (1)
where \(E_0\) and \(E_1\) are the two eigenvalues (\(E_0 < E_1\) for simplicity) of the atom. It is easy to see
\[H_0|0\rangle = E_0|0\rangle, \quad H_0|1\rangle = E_1|1\rangle.\]

For \(t > 0\) we consider an interaction of the atom with some laser field. Then the interaction term is included as the non-diagonal terms of the Hamiltonian
\[H = \begin{pmatrix} E_0 & \gamma \\ \bar{\gamma} & E_1 \end{pmatrix}\]. (2)
Here we assume for simplicity that \(\gamma\) is a complex constant.

First, let us calculate the eigenvalues of the interacting Hamiltonian (2):
\[0 = |\lambda_{12} - H| = \left| \begin{array}{cc} \lambda - E_0 & -\gamma \\ -\bar{\gamma} & \lambda - E_1 \end{array} \right| = \lambda^2 - (E_0 + E_1)\lambda + E_0E_1 - |\gamma|^2\]
\[\Rightarrow \lambda_{\pm} = \frac{E_0 + E_1 \pm \sqrt{(E_1 - E_0)^2 + 4|\gamma|^2}}{2}\].
Note the order
\[\lambda_+ > E_1 > E_0 > \lambda_-\].
Next, the eigenvector of $\lambda_-$ is given by

$$ |\lambda_-\rangle = \frac{|\gamma|}{\sqrt{|\gamma|^2 + (E_0 - \lambda_-)^2}} \left( \begin{array}{c} 1 \\ -\frac{E_0 - \lambda_-}{\gamma} \end{array} \right) $$

$$ = \frac{|\gamma|}{\sqrt{|\gamma|^2 + (E_0 - \lambda_-)^2}} |0\rangle - \frac{|\gamma|}{\gamma \sqrt{|\gamma|^2 + (E_0 - \lambda_-)^2}} |1\rangle $$

(we omit the details of $|\lambda_+\rangle$).

This state (having the eigenvalue $\lambda_-$) is just a superposition of $|0\rangle$ and $|1\rangle$. This example shows that a superposition of quantum states can lower the energy level. Maskawa says in [8] that this phenomenon is the essence of superposition in QM.

See the following figure:

From now on we study the time evolution of (2) including decoherence interactions. To treat the decoherence phenomena in a correct manner it is important to adopt the density matrix formulation instead of the pure state formulation discussed on far. The general definition of density matrix $\rho$ is

$$ \rho^\dagger = \rho \text{ and } \text{tr}\rho = 1, $$

so we can write $\rho = \rho(t)$ as

$$ \rho = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \quad (a = \bar{a}, \ d = \bar{d}, \ a + d = 1). \quad (3) $$

---

2 This point is a bit difficult to understand for beginners
Here we have suppressed the \( t \) dependence of the components like \( a = a(t) \), etc for simplicity.

The general form of the master equation ([9], [10] or [11]) is well–known to be

\[
\frac{d}{dt} \rho = -i[H, \rho] + D\rho \quad (\Leftrightarrow \hbar = 1 \text{ for simplicity})
\]

where

\[
D\rho = \mu \left( \sigma_- \rho \sigma_+ - \frac{1}{2} \sigma_+ \sigma_- \rho - \frac{1}{2} \rho \sigma_+ \sigma_- \right) + \nu \left( \sigma_+ \rho \sigma_- - \frac{1}{2} \sigma_- \sigma_+ \rho - \frac{1}{2} \rho \sigma_- \sigma_+ \right)
\]

and \( \mu \) and \( \nu \) are positive constants (\( \mu, \nu > 0 \)) representing phenomenologically the feeble interactions with the environment. Note that \( \mu \) and \( \nu \) are determined by models.

We must solve the equation (4). By use of the transformation

\[
\rho = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \longrightarrow \hat{\rho} = \begin{pmatrix} a \\ b \\ \bar{b} \\ d \end{pmatrix}
\]

the master equation can be rewritten as

\[
\begin{pmatrix} a \\ b \\ \bar{b} \\ d \end{pmatrix} \frac{d}{dt} = \begin{pmatrix} -\mu & i\bar{\gamma} & -i\gamma & \nu \\ i\gamma & i(E_1 - E_0) - \frac{\mu + \nu}{2} & 0 & -i\gamma \\ -i\bar{\gamma} & 0 & -i(E_1 - E_0) - \frac{\mu + \nu}{2} & i\bar{\gamma} \\ \mu & -i\bar{\gamma} & i\gamma & -\nu \end{pmatrix} \begin{pmatrix} a \\ b \\ \bar{b} \\ d \end{pmatrix}.
\]

The derivation is left to readers. For example, refer to [12].

First, we must look for eigenvalues of the matrix \( W \)

\[
W = \begin{pmatrix} -\mu & i\bar{\gamma} & -i\gamma & \nu \\ i\gamma & i(E_1 - E_0) - \frac{\mu + \nu}{2} & 0 & -i\gamma \\ -i\bar{\gamma} & 0 & -i(E_1 - E_0) - \frac{\mu + \nu}{2} & i\bar{\gamma} \\ \mu & -i\bar{\gamma} & i\gamma & -\nu \end{pmatrix};
\]
which is very hard. Since

\[ 0 = |\lambda I_4 - W| \]

\[
\begin{vmatrix}
\lambda + \mu & -i\bar{\gamma} & i\gamma & -\nu \\
-i\gamma & \lambda - i(E_1 - E_0) + \frac{\mu + \nu}{2} & 0 & i\gamma \\
i\bar{\gamma} & 0 & \lambda + i(E_1 - E_0) + \frac{\mu + \nu}{2} & -i\bar{\gamma} \\
-\mu & i\bar{\gamma} & -i\gamma & \lambda + \nu
\end{vmatrix}
\]

\[ = \ldots \]

\[ = \lambda \begin{vmatrix} 1 & 0 & 0 & 0 \\
-i\gamma & \lambda - i(E_1 - E_0) + \frac{\mu + \nu}{2} & 0 & 2i\gamma \\
i\bar{\gamma} & 0 & \lambda + i(E_1 - E_0) + \frac{\mu + \nu}{2} & -2i\bar{\gamma} \\
-\mu & i\bar{\gamma} & -i\gamma & \lambda + \mu + \nu
\end{vmatrix} \]

we obtain one trivial root \( \lambda = 0 \) and a cubic equation

\[
\left\{ \left( \lambda + \frac{\mu + \nu}{2} \right)^2 + (E_1 - E_0)^2 \right\} (\lambda + \mu + \nu) + 2|\gamma|^2(2\lambda + \mu + \nu) = 0.
\]

Let us transform this. By setting

\[ \Lambda = \lambda + \frac{\mu + \nu}{2} \implies \lambda = \Lambda - \frac{\mu + \nu}{2} \]

the cubic equation becomes

\[ \Lambda^3 + \frac{\mu + \nu}{2} \Lambda^2 + \{(E_1 - E_0)^2 + 4|\gamma|^2\} \Lambda + (E_1 - E_0)^2 \frac{\mu + \nu}{2} = 0. \tag{7} \]

Since the equation is cubic we can solve it by use of the Cardano formula formally. See for example [13]. However, the formula does not suit our purpose well.

Here we set

\[ f(\Lambda) = \Lambda^3 + \frac{\mu + \nu}{2} \Lambda^2 + \{(E_1 - E_0)^2 + 4|\gamma|^2\} \Lambda + (E_1 - E_0)^2 \frac{\mu + \nu}{2} \]

\[ 7 \]
and treat its roots in an abstract way. Note that \( f(\Lambda) > 0 \) for \( \Lambda \geq 0 \) because all coefficients are positive. Since
\[
f(0) = (E_1 - E_0)\frac{\mu + \nu}{2} > 0 \quad \text{and} \quad f\left(-\frac{\mu + \nu}{2}\right) = -2|\gamma|^2(\mu + \nu) < 0
\]
there is (at least) one root \(-\frac{\mu + \nu}{2} < \Lambda_0 < 0\) satisfying \( f(\Lambda_0) = 0 \). By denoting
\[
f(\Lambda) = \Lambda^3 + a\Lambda^2 + b\Lambda + c
\]
for simplicity we have a decomposition
\[
f(\Lambda) = (\Lambda - \Lambda_0)(\Lambda^2 + (\Lambda_0 + a)\Lambda + (\Lambda_0^2 + a\Lambda_0 + b)) = 0.
\]
From this we obtain other two roots
\[
\Lambda_\pm = -(\Lambda_0 + a) \pm \sqrt{(\Lambda_0 + a)^2 - 4(\Lambda_0^2 + a\Lambda_0 + b)} \quad \frac{2}{2}
\]
Note that \( \Lambda_0 + a = \Lambda_0 + \frac{\mu + \nu}{2} > 0 \).

If \( \Lambda_0^2 + a\Lambda_0 + b < 0 \) then \( \Lambda_+ > 0 \), which is a contradiction. Therefore, \( \Lambda_0^2 + a\Lambda_0 + b > 0 \).

As a result,
\[
\Lambda_- < \Lambda_+ < 0
\]
if \( (\Lambda_0 + a)^2 - 4(\Lambda_0^2 + a\Lambda_0 + b) > 0 \) (real roots) and
\[
\Lambda_\pm = -(\Lambda_0 + a) \pm i\sqrt{4(\Lambda_0^2 + a\Lambda_0 + b) - (\Lambda_0 + a)^2} \quad \frac{2}{2}
\]
if \( (\Lambda_0 + a)^2 - 4(\Lambda_0^2 + a\Lambda_0 + b) < 0 \) (complex conjugate roots). In this case, the real part is negative
\[
\text{Re} \Lambda_\pm = -\frac{\Lambda_0 + a}{2} < 0.
\]

The solutions of the characteristic polynomial of \( W (= |\lambda_1 - W|) \) are
\[
\lambda_1 = 0, \quad \lambda_2 = \Lambda_0 - \frac{\mu + \nu}{2}, \quad \lambda_3 = \Lambda_+ - \frac{\mu + \nu}{2}, \quad \lambda_4 = \Lambda_- - \frac{\mu + \nu}{2}
\]
and
\[
\lambda_2 < 0, \quad \lambda_3 < 0, \quad \lambda_4 < 0 \quad \text{or} \quad \lambda_2 < 0, \quad \text{Re} \lambda_3 < 0, \quad \text{Re} \lambda_4 < 0
\]
under the conditions stated above.

Next, we look for the eigenvectors corresponding to the eigenvalues. For the purpose let us prepare some notations. We use the convention that the ket vector \(|\lambda\rangle\) is normalized, while the round ket vector \(|\lambda\rangle\) is not normalized (that is, \(\langle\lambda|\lambda\rangle = 1\) and \(|\lambda|\lambda\rangle \neq 1\)).

It is easy to obtain the eigenvectors of \(W^T\) rather than those of \(W\) as shown in the following. Namely,

\[
W^T = \begin{pmatrix}
-\mu & i\gamma & -i\gamma & \mu \\
i\gamma & i(E_1 - E_0) - \frac{\mu + \nu}{2} & 0 & -i\gamma \\
-i\gamma & 0 & -i(E_1 - E_0) - \frac{\mu + \nu}{2} & i\gamma \\
\nu & -i\gamma & i\gamma & -\nu
\end{pmatrix}.
\]  

(10)

Of course, \(W\) and \(W^T\) share the same eigenvalues. Let us list the eigenvectors of \(W^T\):

\[|\lambda_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \]

and we set

\[|\lambda_2\rangle = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix}, \quad |\lambda_3\rangle = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}, \quad |\lambda_4\rangle = \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix}.\]

See the next section why we make such a choice.

**Note.** Let us show how to construct an eigenvector \(|\lambda\rangle\) from the eigenvalue \(\lambda\). In order to avoid complicated expressions (equations) we restrict to the case of \(n = 3\). That is, the equation is

\[
\begin{pmatrix}
a_1 & b_1 & c_1 \\
a_2 & b_2 & c_2 \\
a_3 & b_3 & c_3
\end{pmatrix}
\begin{pmatrix}
x \\ y \\ z
\end{pmatrix} = \lambda \begin{pmatrix}
x \\ y \\ z
\end{pmatrix}.
\]
From the first and second rows we have
\[
\begin{pmatrix}
  a_1 & b_1 \\
  a_2 & b_2
\end{pmatrix}
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
+ \begin{pmatrix}
  c_1 z \\
  c_2 z
\end{pmatrix}
= \lambda \begin{pmatrix}
  x \\
  y
\end{pmatrix}
\]
or
\[
\begin{pmatrix}
  \lambda - a_1 & -b_1 \\
  -a_2 & \lambda - b_2
\end{pmatrix}
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
= z \begin{pmatrix}
  c_1 \\
  c_2
\end{pmatrix}.
\]
If we assume that the determinant is non-zero
\[
\begin{vmatrix}
  \lambda - a_1 & -b_1 \\
  -a_2 & \lambda - b_2
\end{vmatrix}
= (\lambda - a_1)(\lambda - b_2) - a_2 b_1 \neq 0
\]
we have
\[
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
= z \begin{pmatrix}
  \lambda - a_1 & -b_1 \\
  -a_2 & \lambda - b_2
\end{pmatrix}^{-1}
\begin{pmatrix}
  c_1 \\
  c_2
\end{pmatrix}
= \frac{z}{(\lambda - a_1)(\lambda - b_2) - a_2 b_1}
\begin{pmatrix}
  (\lambda - b_2) c_1 + b_1 c_2 \\
  a_2 c_1 + (\lambda - a_1) c_2 \\
  (\lambda - a_1)(\lambda - b_2) - a_2 b_1
\end{pmatrix}.
\]
Therefore, we obtain
\[
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix}
= \frac{z}{(\lambda - a_1)(\lambda - b_2) - a_2 b_1}
\begin{pmatrix}
  (\lambda - b_2) c_1 + b_1 c_2 \\
  a_2 c_1 + (\lambda - a_1) c_2 \\
  (\lambda - a_1)(\lambda - b_2) - a_2 b_1
\end{pmatrix}.
\]
As a result, the eigenvector \(|\lambda\rangle\) is given by
\[
|\lambda\rangle = \begin{pmatrix}
  (\lambda - b_2) c_1 + b_1 c_2 \\
  a_2 c_1 + (\lambda - a_1) c_2 \\
  (\lambda - a_1)(\lambda - b_2) - a_2 b_1
\end{pmatrix}.
\]
If the determinant above is zero then we have only to apply the same procedure to other two rows.

For the readers let us give one exercise:
\[
A = \begin{pmatrix}
  2 & -1 & 0 \\
  -1 & 2 & -1 \\
  0 & -1 & 2
\end{pmatrix}.
\]
If we set

\[
O = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \begin{pmatrix}
1 & x_2 & x_3 & x_4 \\
0 & y_2 & y_3 & y_4 \\
0 & z_2 & z_3 & z_4 \\
1 & w_2 & 1 & 1
\end{pmatrix}
\]  

(11)

we have \(O \in GL(4; \mathbb{C})\) and

\[
O^{-1} = \frac{1}{|O|} \begin{pmatrix}
\hat{O}_{11} & \hat{O}_{12} & \hat{O}_{13} & \hat{O}_{14} \\
* & * & * & * \\
* & * & * & * \\
* & * & * & *
\end{pmatrix}
\]

where * denotes unnecessary terms in the following. Here, the cofactors are

\[
\hat{O}_{11} = \begin{vmatrix}
y_2 & y_3 & y_4 \\
z_2 & z_3 & z_4 \\
w_2 & 1 & 1
\end{vmatrix}, \quad \hat{O}_{12} = - \begin{vmatrix}
x_2 & x_3 & x_4 \\
z_2 & z_3 & z_4 \\
w_2 & 1 & 1
\end{vmatrix}, \quad \hat{O}_{13} = \begin{vmatrix}
x_2 & x_3 & x_4 \\
y_2 & y_3 & y_4 \\
w_2 & 1 & 1
\end{vmatrix}, \quad \hat{O}_{14} = - \begin{vmatrix}
x_2 & x_3 & x_4 \\
y_2 & y_3 & y_4 \\
z_2 & z_3 & z_4
\end{vmatrix}.
\]

We know that each term is very complicated. Note that

\[
|O| = \hat{O}_{11} + \hat{O}_{14} \implies 1 = \frac{\hat{O}_{11}}{|O|} + \frac{\hat{O}_{14}}{|O|}.
\]  

(12)

Now we are in a position to diagonalize \(W\). Since

\[
W^T = OD_WO^{-1}
\]

with \(D_W\) being the diagonal matrix

\[
D_W = \begin{pmatrix}
0 & & & \\
& \lambda_2 & & \\
& & \lambda_3 & \\
& & & \lambda_4
\end{pmatrix}
\]  

(13)

we have

\[
W = (O^T)^{-1}D_WO^T.
\]  

(14)
Here, let us go back to the equation (5). If we set
\[
(\hat{\rho} =) \Psi = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix}
\]
for simplicity, the equation (5) reads
\[
\frac{d}{dt} \Psi = W \Psi
\]
and the general solution is given by (14)
\[
\Psi(t) = e^{tW} \Psi(0) = (O^T)^{-1} e^{tD} O^T \Psi(0).
\]

Since we are interested in the final state \( \Psi(\infty) \) we must look for the asymptotic limit \( \lim_{t \to \infty} e^{tD} \). From (9) and (13) it is easy to see
\[
\lim_{t \to \infty} e^{tD} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = |0\rangle \langle 0|,
\]
so we obtain
\[
\Psi(\infty) = (O^T)^{-1} |0\rangle \langle 0| O^T \Psi(0) = \frac{1}{|O|} \begin{pmatrix} \hat{O}_{11} & 0 & 0 & \hat{O}_{11} \\ \hat{O}_{12} & 0 & 0 & \hat{O}_{12} \\ \hat{O}_{13} & 0 & 0 & \hat{O}_{13} \\ \hat{O}_{14} & 0 & 0 & \hat{O}_{14} \end{pmatrix} \Psi(0).
\]

This equation gives
\[
\Psi(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \implies \Psi(\infty) = \frac{1}{|O|} \begin{pmatrix} \hat{O}_{11} \\ \hat{O}_{12} \\ \hat{O}_{13} \\ \hat{O}_{14} \end{pmatrix}
\]
and it is equivalent to
\[ \rho_0(0) = |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \implies \rho_0(\infty) = \frac{1}{|O|} \begin{pmatrix} \hat{O}_{11} & \hat{O}_{12} \\ \hat{O}_{13} & \hat{O}_{14} \end{pmatrix}. \] (16)

Similarly,
\[ \Psi(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \implies \Psi(\infty) = \frac{1}{|O|} \begin{pmatrix} \hat{O}_{11} \\ \hat{O}_{12} \\ \hat{O}_{13} \\ \hat{O}_{14} \end{pmatrix} \]
is equivalent to
\[ \rho_1(0) = |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \implies \rho_1(\infty) = \frac{1}{|O|} \begin{pmatrix} \hat{O}_{11} & \hat{O}_{12} \\ \hat{O}_{13} & \hat{O}_{14} \end{pmatrix}. \] (17)

Let us state our result once more:
\[ \rho_0(0) = |0\rangle\langle 0|, \quad \rho_1(0) = |1\rangle\langle 1| \implies \rho_0(\infty) = \rho_1(\infty). \] (18)

We would like to interpret the final density matrix as “classical one”.

At the end of this section, let us present an important problem.

**Problem** Generalize the result to the case of \( N \) level system of an atom.

For \( N = 3 \) we conjecture that
\[
\Psi(\infty) = (O^T)^{-1}|0\rangle\langle 0|O^T \Psi(0)
\]
\[
= \frac{1}{|O|} \begin{pmatrix}
\hat{O}_{11} & 0 & 0 & \hat{O}_{11} & 0 & 0 & \hat{O}_{11} \\
\hat{O}_{12} & 0 & 0 & \hat{O}_{12} & 0 & 0 & \hat{O}_{12} \\
\hat{O}_{13} & 0 & 0 & \hat{O}_{13} & 0 & 0 & \hat{O}_{13} \\
\hat{O}_{14} & 0 & 0 & \hat{O}_{14} & 0 & 0 & \hat{O}_{14} \\
\hat{O}_{15} & 0 & 0 & \hat{O}_{15} & 0 & 0 & \hat{O}_{15} \\
\hat{O}_{16} & 0 & 0 & \hat{O}_{16} & 0 & 0 & \hat{O}_{16} \\
\hat{O}_{17} & 0 & 0 & \hat{O}_{17} & 0 & 0 & \hat{O}_{17} \\
\hat{O}_{18} & 0 & 0 & \hat{O}_{18} & 0 & 0 & \hat{O}_{18} \\
\hat{O}_{19} & 0 & 0 & \hat{O}_{19} & 0 & 0 & \hat{O}_{19}
\end{pmatrix} \Psi(0)
\]

and

\[ \rho_0(0) = |0\rangle\langle 0|, \quad \rho_1(0) = |1\rangle\langle 1|, \quad \rho_2(0) = |2\rangle\langle 2| \]

\[ \implies \rho_0(\infty) = \rho_1(\infty) = \rho_2(\infty) = \frac{1}{|\mathcal{O}|} \begin{pmatrix} \hat{O}_{11} & \hat{O}_{12} & \hat{O}_{13} \\ \hat{O}_{14} & \hat{O}_{15} & \hat{O}_{16} \\ \hat{O}_{17} & \hat{O}_{18} & \hat{O}_{19} \end{pmatrix} \]

with some notations changed from \( N = 2 \) to \( N = 3 \).

We expect that young researchers will attack and solve the problem.

### 3 Special Case

The cubic equation is formally solved by the Cardano formula. However, in this case we cannot obtain a compact form of solutions \(^3\), so we assume

\[ E_1 = E_0 \tag{19} \]

in this section. Then

\[ W = \begin{pmatrix} -\mu & i\bar{\gamma} & -i\gamma & \nu \\ i\gamma & -\frac{\mu + \nu}{2} & 0 & -i\gamma \\ -i\bar{\gamma} & 0 & -\frac{\mu + \nu}{2} & i\bar{\gamma} \\ \mu & -i\bar{\gamma} & i\gamma & -\nu \end{pmatrix}. \tag{20} \]

The equation (7) becomes

\[ \Lambda \left\{ \Lambda^2 + \frac{\mu + \nu}{2} - \Lambda + 4|\gamma|^2 \right\} = 0 \]

and the solutions are

\[ \Lambda_0 = 0, \quad \Lambda_\pm = -\frac{\mu + \nu}{4} \pm \frac{1}{2} \sqrt{\left( \frac{\mu + \nu}{2} \right)^2 - 16|\gamma|^2}. \]

\(^3\)One can check this by MATHEMATICA
Therefore, the eigenvalues of $W$ (20) are given by (from $\lambda = \Lambda - \frac{\mu + \nu}{2}$)

$$\lambda_1 = 0, \quad \lambda_2 = -\frac{\mu + \nu}{2},$$

$$\lambda_3 = -\frac{3}{4}(\mu + \nu) + \frac{1}{2}\sqrt{\left(\frac{\mu + \nu}{2}\right)^2 - 16|\gamma|^2},$$

$$\lambda_4 = -\frac{3}{4}(\mu + \nu) - \frac{1}{2}\sqrt{\left(\frac{\mu + \nu}{2}\right)^2 - 16|\gamma|^2}. \quad (21)$$

For

$$W^T = \begin{pmatrix}
-\mu & i\gamma & -i\bar{\gamma} & \mu \\
i\bar{\gamma} & -\frac{\mu + \nu}{2} & 0 & -i\bar{\gamma} \\
-i\gamma & 0 & -\frac{\mu + \nu}{2} & i\gamma \\
\nu & -i\gamma & i\bar{\gamma} & -\nu
\end{pmatrix} \quad (22)$$

the corresponding eigenvectors are given by

$$|\lambda_1\rangle = \begin{pmatrix}1 \\ 0 \\ 0 \\ 1\end{pmatrix}, \quad |\lambda_2\rangle = \begin{pmatrix}0 \\ \bar{\gamma} \\ \gamma \\ 0\end{pmatrix},$$

$$|\lambda_3\rangle = \begin{pmatrix}-1 + \frac{2(\mu - \nu)}{\mu - \nu + \frac{1}{2}\sqrt{(\mu - \nu)^2 - 16|\gamma|^2}} \\
2i\bar{\gamma} + \frac{\mu - \nu}{\mu - \nu + \frac{1}{2}\sqrt{(\mu - \nu)^2 - 16|\gamma|^2}} \\
-1 + \frac{2(\mu - \nu)}{\mu - \nu + \frac{1}{2}\sqrt{(\mu - \nu)^2 - 16|\gamma|^2}} \\
-2i\gamma + \frac{\mu - \nu}{\mu - \nu + \frac{1}{2}\sqrt{(\mu - \nu)^2 - 16|\gamma|^2}}
\end{pmatrix}, \quad |\lambda_4\rangle = \begin{pmatrix}-1 + \frac{2(\mu - \nu)}{\mu - \nu - \frac{1}{2}\sqrt{(\mu + \nu)^2 - 16|\gamma|^2}} \\
2i\gamma - \frac{\mu + \nu}{\mu - \nu - \frac{1}{2}\sqrt{(\mu + \nu)^2 - 16|\gamma|^2}} \\
-1 + \frac{2(\mu - \nu)}{\mu - \nu - \frac{1}{2}\sqrt{(\mu + \nu)^2 - 16|\gamma|^2}} \\
-2i\gamma - \frac{\mu + \nu}{\mu - \nu - \frac{1}{2}\sqrt{(\mu + \nu)^2 - 16|\gamma|^2}}
\end{pmatrix}. \quad (23)$$

Verification of the result is left to readers.
4 Perturbation

Since \( \mu \) and \( \nu \) in (4) are in general small compared to the terms in the Hamiltonian we can apply a perturbation method to the master equation (like [14]) in order to obtain an approximate solution.

Let us decompose \( W \) into two parts:

\[
W = \begin{pmatrix}
-\mu & i\tilde{\gamma} & -i\gamma & \nu \\
i\gamma & i(E_1 - E_0) - \frac{\mu + \nu}{2} & 0 & -i\gamma \\
-\tilde{\gamma} & 0 & -i(E_1 - E_0) - \frac{\mu + \nu}{2} & i\tilde{\gamma} \\
\mu & -i\tilde{\gamma} & i\gamma & -\nu
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0 & i\tilde{\gamma} & -i\gamma & 0 \\
i\gamma & i(E_1 - E_0) & 0 & -i\gamma \\
-\tilde{\gamma} & 0 & -i(E_1 - E_0) & i\tilde{\gamma} \\
0 & -i\tilde{\gamma} & i\gamma & 0
\end{pmatrix}
+ \begin{pmatrix}
-\mu & 0 & 0 & \nu \\
0 & -\frac{\mu + \nu}{2} & 0 & 0 \\
0 & 0 & -\frac{\mu + \nu}{2} & 0 \\
\mu & 0 & 0 & -\nu
\end{pmatrix}
\equiv \hat{H} + \hat{D}.
\]

The general solution of (5) is given by

\[
\Psi(t) = e^{t(\hat{H} + \hat{D})} \Psi(0).
\]  

(24)

However, it is not easy to calculate the term \( e^{t(\hat{H} + \hat{D})} \) exactly, so we use a simple approximation

\[
e^{t(\hat{H} + \hat{D})} = e^{t\hat{D}} e^{t\hat{H}} \approx e^{t\hat{D}} e^{t\hat{H}}.
\]

In general, we must use the Zassenhaus formula (see for example [7], [15]).

**Zassenhaus Formula** For operators (or square matrices) \( A \) and \( B \) we have an expansion

\[
e^{t(A+B)} = \cdots e^{-\frac{t^2}{2}([A,B]+[\{A,B\},A])} e^{\frac{t^2}{2} \{ A,B \}} e^{tB} e^{tA}.
\]  

(25)

The formula is a bit different from that of [15].

From now on we discuss the approximate solution

\[
\Psi(t) \approx e^{t\hat{D}} e^{t\hat{H}} \Psi(0).
\]  

(26)
First, let us calculate $e^{t\hat{D}}$. For the purpose we set
\[
K = \begin{pmatrix}
-\mu & \nu \\
\mu & -\nu
\end{pmatrix}
\]
and calculate $e^{tK}$. The eigenvalues of $K$ are $\{0, -(\mu + \nu)\}$ and corresponding eigenvectors (not normalized) are
\[
0 \leftrightarrow \begin{pmatrix}
\nu \\
\mu
\end{pmatrix}, \quad -(\mu + \nu) \leftrightarrow \begin{pmatrix}
1 \\
-1
\end{pmatrix}.
\]
If we define the matrix
\[
O = \begin{pmatrix}
\nu & 1 \\
\mu & -1
\end{pmatrix} \implies O^{-1} = \frac{1}{\mu + \nu} \begin{pmatrix}
1 & 1 \\
\mu & -\nu
\end{pmatrix}
\]
then it is easy to see
\[
K = O \begin{pmatrix}
0 \\
-(\mu + \nu)
\end{pmatrix} O^{-1}
\]
and
\[
e^{tK} = O \begin{pmatrix}
1 \\
e^{-t(\mu + \nu)}
\end{pmatrix} O^{-1} = \frac{1}{\mu + \nu} \begin{pmatrix}
\nu + \mu e^{-t(\mu + \nu)} & \nu - \nu e^{-t(\mu + \nu)} \\
\mu - \mu e^{-t(\mu + \nu)} & \mu + \nu e^{-t(\mu + \nu)}
\end{pmatrix}.
\]
Therefore, we have
\[
e^{t\hat{D}} = \begin{pmatrix}
\frac{\nu + \mu e^{-t(\mu + \nu)}}{\mu + \nu} & 0 & 0 & \frac{\nu - \nu e^{-t(\mu + \nu)}}{\mu + \nu} \\
0 & e^{-t\frac{\mu + \nu}{2}} & 0 & 0 \\
0 & 0 & e^{-t\frac{\mu + \nu}{2}} & 0 \\
\frac{\mu - \mu e^{-t(\mu + \nu)}}{\mu + \nu} & 0 & 0 & \frac{\mu + \nu e^{-t(\mu + \nu)}}{\mu + \nu}
\end{pmatrix} \approx \begin{pmatrix}
\nu & 0 & 0 & \nu \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\mu & 0 & 0 & \mu
\end{pmatrix}
\] (27)
if $t$ is large enough ($t \gg 1/(\mu + \nu)$).

Next, let us calculate $e^{t\hat{H}}$. Since we need some properties of tensor products in the following see for example [7]. We can express $\hat{H}$ as
\[
\hat{H} = -i \left( H \otimes I_2 - I_2 \otimes H^T \right).
\]
In fact,
\[
\hat{H} = -i \left\{ \left( \begin{array}{cc} E_0 & \gamma \\ \bar{\gamma} & E_1 \end{array} \right) \otimes \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) - \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \otimes \left( \begin{array}{cc} E_0 & \bar{\gamma} \\ \gamma & E_1 \end{array} \right) \right\}
\]
\[
= -i \left\{ \left( \begin{array}{ccc} E_0 & 0 & \gamma \\ 0 & E_0 & 0 \\ \bar{\gamma} & 0 & E_1 \end{array} \right) - \left( \begin{array}{ccc} 0 & E_0 & \gamma \\ \gamma & E_1 & 0 \\ 0 & 0 & \bar{E}_0 \end{array} \right) \right\}
\]
\[
= -i \left( \begin{array}{ccc} 0 & -\bar{\gamma} & \gamma & 0 \\ -\gamma & -(E_1 - E_0) & 0 & \gamma \\ \bar{\gamma} & 0 & E_1 - E_0 & -\bar{\gamma} \\ 0 & \bar{\gamma} & -\gamma & 0 \end{array} \right).
\]

It is well–known that
\[
e^{\hat{H}t} = e^{-it(\hat{H} \otimes 1_2) \otimes H^T} = e^{-itH \otimes H^T} = (e^{-itH} \otimes 1_2) \left( 1_2 \otimes e^{itH^T} \right) = e^{-itH} \otimes e^{itH^T},
\]
so we must calculate
\[
e^{-itH} = \exp \left\{ -it \left( \begin{array}{cc} E_0 & \gamma \\ \bar{\gamma} & E_1 \end{array} \right) \right\}.
\]

Since \( H \) in (2) is expressed as
\[
\left( \begin{array}{cc} E_0 & \gamma \\ \bar{\gamma} & E_1 \end{array} \right) = \left( \frac{E_0 + E_1}{2} \right) + \left( \frac{E_1 - E_0}{2} \right) 
\]
\[
= \Delta_+ 1_2 + \left( \begin{array}{cc} -\Delta_- & \gamma \\ \bar{\gamma} & \Delta_- \end{array} \right) \text{ where } \Delta_\pm = \frac{E_1 \pm E_0}{2}
\]
the calculation is reduced to
\[
e^{-itH} = e^{-it\Delta_+} \exp \left\{ -it \left( \begin{array}{cc} -\Delta_- & \gamma \\ \bar{\gamma} & \Delta_- \end{array} \right) \right\}.
\]

This exponential is well–known, see for example [7]. That is,
\[
\exp \left\{ -it \left( \begin{array}{cc} -\Delta_- & \gamma \\ \bar{\gamma} & \Delta_- \end{array} \right) \right\} = \left( \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right) \quad (28)
\]
where

\begin{align*}
a_{11} &= \cos(t \sqrt{\Delta_2^2 + |\gamma|^2}) + \frac{i \sin(t \sqrt{\Delta_2^2 + |\gamma|^2})}{\sqrt{\Delta_2^2 + |\gamma|^2}} \Delta_-, \\
a_{12} &= -i \frac{\sin(t \sqrt{\Delta_2^2 + |\gamma|^2})}{\sqrt{\Delta_2^2 + |\gamma|^2}} \gamma, \\
a_{21} &= -i \frac{\sin(t \sqrt{\Delta_2^2 + |\gamma|^2})}{\sqrt{\Delta_2^2 + |\gamma|^2}} \bar{\gamma}, \\
a_{22} &= \cos(t \sqrt{\Delta_2^2 + |\gamma|^2}) - \frac{i \sin(t \sqrt{\Delta_2^2 + |\gamma|^2})}{\sqrt{\Delta_2^2 + |\gamma|^2}} \Delta_-. 
\end{align*}

Similarly, we obtain

\[ e^{itH_T} = e^{it\Delta_+} \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix}. \]

Therefore, we arrive at

\[ e^{-itH} \otimes e^{itH_T} = \left( \begin{array}{cccc} a_{11} & a_{12} & & \\
 & a_{21} & a_{22} & \\
 a_{12} & -a_{11} & a_{12}a_{22} & -a_{12}a_{21} \\
 & a_{22} & -a_{12}a_{21} & a_{11}a_{21} & a_{11}a_{22} \\
 & & & \end{array} \right) \]

\[ \equiv \left( \begin{array}{cccc} c_{11} & c_{12} & c_{13} & c_{14} \\
 & c_{42} & c_{43} & c_{44} \\
 & * & * & * \\
 & * & * & * \\
 & * & * & * \\
 & * & * & * \\
 & * & * & * \end{array} \right) \]

(30)

where *’s in the matrix are elements not used in later discussion.

From (30) and (29) it is easy to see

\[ c_{11} + c_{41} = 1, \quad c_{12} + c_{42} = 0, \quad c_{13} + c_{43} = 0, \quad c_{14} + c_{44} = 1. \]

(31)
Therefore, from (26), (27), (30) and (31) we obtain

\[ \Psi(t) \approx \frac{1}{\mu + \nu} \begin{pmatrix} \nu & 0 & 0 & \nu \\ 0 & 0 & 0 & 0 \\ \mu & 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ * & * & * & * \\ * & * & * & * \\ c_{41} & c_{42} & c_{43} & c_{44} \end{pmatrix} \Psi(0) \]

\[
= \frac{1}{\mu + \nu} \begin{pmatrix} \nu & 0 & 0 & \nu \\ 0 & 0 & 0 & 0 \\ \mu & 0 & 0 & \mu \end{pmatrix} \Psi(0) \]

(32)

for \( t \gg 1/(\mu + \nu) \).

From (3)

\[ \rho(t) = \begin{pmatrix} a(t) & b(t) \\ \overline{b}(t) & d(t) \end{pmatrix}, \quad \rho(0) = \begin{pmatrix} a(0) & b(0) \\ \overline{b}(0) & d(0) \end{pmatrix} \]

we have

\[
\rho(\infty) = \frac{1}{\mu + \nu} \begin{pmatrix} \nu (a(0) + d(0)) & 0 \\ 0 & \mu (a(0) + d(0)) \end{pmatrix} = \frac{1}{\mu + \nu} \begin{pmatrix} \nu & 0 \\ 0 & \mu \end{pmatrix} \]

\[ = \frac{\nu}{\mu + \nu} |0\rangle \langle 0| + \frac{\mu}{\mu + \nu} |1\rangle \langle 1| \]

(33)

because \( \text{tr} \rho(0) = a(0) + d(0) = 1 \)

We believe that the result in this section is deeply related to the proof of the Copenhagen interpretation, see [14].

5 Concluding Remarks

In this paper we have derived the solutions to the master equation of the two level system of an atom under decoherence. How do we understand the result from the physical point of view? We would like to interpret the final density matrix as a representation of some classical state.
In general, to solve a master equation exactly is very hard, so we are usually satisfied by solving it approximately. For example, see [14] and [16]. As far as we know our result is the finest one up to the present.

We want to apply the results in the paper to our method of Quantum Computation based on Cavity QED, see [17] and [18]. In the quantum computation we must take decoherence time into consideration, which is an essential point. Some results will be reported in the near future.

In standard textbooks of QM decoherence theory is usually not contained, so it may be hard for beginners (young students) to understand. For example a book [11] or a recent review paper [19] would be very helpful for beginners.

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