RESTRICTED SUMMABILITY OF THE MULTI-DIMENSIONAL CESÁRO MEANS OF WALSH-KACZMARZ-FOURIER SERIES

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ABSTRACT. The properties of the maximal operator of the \((C, \alpha)\)-means \((\alpha = (\alpha_1, \ldots, \alpha_d))\) of the multi-dimensional Walsh-Kaczmarz-Fourier series are discussed, where the set of indices is inside a cone-like set. We prove that the maximal operator is bounded from \(H^p_{\gamma_0} \) to \(L_p \) for \( p < p_0 \) \((p_0 = \max 1/(1 + \alpha_k) : k = 1, \ldots, d)\) and is of weak type \((1, 1)\). As a corollary we get the theorem of Simon \[15\] on the a.e. convergence of cone-restricted two-dimensional Fejér means of integrable functions. At the endpoint case \( p = p_0 \), we show that the maximal operator \( \sigma_{L, \alpha, \alpha}^* \) is not bounded from the Hardy space \( H^p_{\gamma_0} \) to the space \( L^p_{\gamma_0} \).

Key words and phrases: Walsh-Kaczmarz system, maximal operator, multi-dimensional system, restricted summability, a.e. convergence, Cesàro means.

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1. Definitions and notation

Now, we give a brief introduction to the theory of dyadic analysis (for more details see \[11, 13\]). Let \( \mathbb{P} \) denote the set of positive integers, \( \mathbb{N} := \mathbb{P} \cup \{0\} \). Denote \( \mathbb{Z}_2 \) the discrete cyclic group of order 2, that is \( \mathbb{Z}_2 \) has two elements 0 and 1, the group operation is the modulo 2 addition. The topology is given by that every subset is open. The Haar measure on \( \mathbb{Z}_2 \) is given such that the measure of a singleton is 1/2, that is, \( \mu(\{0\}) = \mu(\{1\}) = 1/2 \). Let \( G \) be the complete direct product of countable infinite copies of the compact groups \( \mathbb{Z}_2 \). The elements of \( G \) are sequences of the form \( x = (x_0, x_1, \ldots, x_k, \ldots) \) with coordinates \( x_k \in \{0, 1\} (k \in \mathbb{N}) \). The group operation on \( G \) is the coordinate-wise addition, the measure (denoted by \( \mu \)) is the product measure and the topology is the product topology. The compact Abelian group \( G \) is called the Walsh group. A base for the neighbourhoods of \( G \) can be given by

\[
I_0(x) := G, \quad I_n(x) := I_n(x_0, \ldots, x_{n-1}) := \{y \in G : y = (x_0, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots)\},
\]

\((x \in G, n \in \mathbb{N})\), \( I_n(x) \) are called dyadic intervals. Let \( 0 = (0 : i \in \mathbb{N}) \in G \) denote the null element of \( G \), and for the simplicity we write \( I_n := I_n(0) \quad (n \in \mathbb{N}) \). Set \( e_n := (0, \ldots, 0, 1, 0, \ldots) \in G \), the \( n \)-th coordinate of which is 1 and the rest are zeros \((n \in \mathbb{N})\).

Let \( r_k \) denote the \( k \)-th Rademacher function, it is defined by

\[
r_k(x) := (-1)^{x_k}
\]

\((k \in \mathbb{N}, x \in G)\).

The Walsh-Paley system is defined as the product system of Rademacher functions. Now, we give more details. If \( n \in \mathbb{N} \), then \( n \) can be expressed in the form \( n = \sum_{i=0}^{\infty} n_i 2^i \), where
\( n_i \in \{0, 1\} \quad (i \in \mathbb{N}) \). Define the order of a natural number \( n \) by \( |n| := \max\{j \in \mathbb{N} : n_j \neq 0\} \), that is \( 2^{|n|} \leq n < 2^{|n|+1} \).

The Walsh-Paley system is

\[
\ w_n(x) := \prod_{k=0}^{\infty} \left( r_k(x) \right)^{n_k} = r_{|n|}(x) (-1)^{\sum_{k=0}^{\lfloor n \rfloor - 1} n_k x_k} \quad (x \in G, n \in \mathbb{P}).
\]

The Walsh-Kaczmarz functions are defined by \( \kappa_0 = 1 \) and for \( n \geq 1 \)

\[
\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{\lfloor n \rfloor - 1} \left( r_{n-1-k}(x) \right)^{n_k} = r_{|n|}(x) (-1)^{\sum_{k=0}^{\lfloor n \rfloor - 1} n_k x_{|n|-1-k}}.
\]

The set of Walsh-Kaczmarz functions and the set of Walsh-Paley functions are equal in dyadic blocks. Namely,

\[
\{ \kappa_n : 2^k \leq n < 2^{k+1} \} = \{ w_n : 2^k \leq n < 2^{k+1} \}
\]

for all \( k \in \mathbb{P} \). Moreover, \( \kappa_0 = w_0 \).

V.A. Skvortsov (see [18]) gave a relation between the Walsh-Kaczmarz functions and the Walsh-Paley functions. Namely, he defined a transformation \( \tau_A : G \rightarrow G \)

\[
\tau_A(x) := (x_{A-1}, x_{A-2}, \ldots, x_1, x_0, x_A, x_{A+1}, \ldots)
\]

for \( A \in \mathbb{N} \). By the definition of \( \tau_A \), we have the following connection

\[
\kappa_n(x) = r_{|n|}(x) w_{n-2^{\left\lfloor \log_2 n \right\rfloor - \alpha}}(\tau_{|n|}(x)) \quad (n \in \mathbb{N}, x \in G).
\]

Let \( 0 < \alpha \) and let

\[
A_j^\alpha := \binom{j + \alpha}{j} = \frac{(\alpha + 1)(\alpha + 2)\ldots(\alpha + j)}{j!} \quad (j \in \mathbb{N}; \alpha \neq -1, -2, \ldots).
\]

It is known that \( A_j^\alpha \sim O(j^\alpha) \) \( (j \in \mathbb{N}) \) (see Zygmund [29]). The one-dimensional Dirichlet kernels and Cesáro kernels are defined by

\[
D_n^\psi := \sum_{k=0}^{n-1} \psi_k, \quad K_n^{\psi, \alpha}(x) := \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_n^{\alpha-1-k} D_k^\psi(x),
\]

for \( \psi_n = w_n \) \( (n \in \mathbb{P}) \) or \( \psi_n = \kappa_n \) \( (n \in \mathbb{P}) \), \( D_0^\psi := 0 \).

Choosing \( \alpha = 1 \) we defined the \( n \)th Fejér mean, as special case. For Walsh-Paley-Fejér kernel functions we have \( K_n(x) \rightarrow 0 \), while \( n \rightarrow \infty \) for every \( x \neq 0 \). However, they can take negative values which is a different situation from the trigonometric case. On the other hand, for Walsh-Kaczmarz-Fejér kernel functions we have \( K_n(x) \rightarrow \infty \), while \( n \rightarrow \infty \) at every dyadic rational. The last fact shows that the behavior of Walsh-Kaczmarz system worse than the behavior of Walsh-Paley system in this special sense. See later inequality [2], as well.

It is well-known that the \( 2^m \)th Dirichlet kernels have a closed form (see e.g. [13])

\[
D_n^{2^m}(x) = D_{2n}^{2^m}(x) = D_{2^m}^{2^n}(x) = \begin{cases} 0, & \text{if } x \notin I_n, \\ 2^n, & \text{if } x \in I_n. \end{cases}
\]
The Kronecker product \((\psi_n : n \in \mathbb{N}^d)\) of \(d\) Walsh-(Kaczmarz) system is said to be the \(d\)-dimensional (multi-dimensional) Walsh-(Kaczmarz) system. That is,

\[
\psi_n(x) = \psi_{n_1}(x^1) \cdots \psi_{n_d}(x^d),
\]

where \(n = (n_1, \ldots, n_d)\) and \(x = (x^1, \ldots, x^d)\).

If \(f \in L^1(G^d)\), then the number \(\hat{f}(n) := \int_{G^d} f \psi_n \quad (n \in \mathbb{N}^d)\) is said to be the \(n\)th Walsh-(Kaczmarz)-Fourier coefficient of \(f\). We can extend this definition to martingales in the usual way (see Weisz [23, 24]).

The \(d\)-dimensional \((C, \alpha) (\alpha = (\alpha_{n_1}, \ldots, \alpha_{n_d}))\) or Cesàro mean of a martingale is defined by

\[
\sigma_n^{\psi, \alpha} f(x) := \frac{1}{\prod_{i=1}^d A_{\alpha_i}^n} \sum_{j=1}^d \prod_{i=1}^d A_{\alpha_i}^{n_j - k_i} \psi_k(f; x).
\]

It is known that

\[
K_n^{\psi, \alpha}(x) = K_{n_1}^{\psi, \alpha_1}(x^1) \cdots K_{n_d}^{\psi, \alpha_d}(x^d).
\]

In 1948 Šneider [19] introduced the Walsh-Kaczmarz system and showed that the inequality

\[
(2) \quad \limsup_{n \to \infty} \frac{D_n(x)}{\log n} \geq C > 0
\]

holds a.e. This inequality shows a big difference between the two arrangements of Walsh system.

In 1974 Schipp [14] and Young [22] proved that the Walsh-Kaczmarz system is a convergence system. Skvortsov in 1981 [18] showed that the Fejér means with respect to the Walsh-Kaczmarz system converge uniformly to \(f\) for any continuous functions \(f\). Gát [3] proved for any integrable functions, that the Fejér means with respect to the Walsh-Kaczmarz system converge almost everywhere to the function. Moreover, he showed that the maximal operator \(\sigma^{\psi, \alpha}\) of Walsh-Kaczmarz-Fejér means is weak type \((1, 1)\) and of type \((p, p)\) for all \(1 < p \leq \infty\). Gát’s result was generalized by Simon [16], who showed that the maximal operator \(\sigma^{\psi, \alpha}\) is of type \((H_p, L_p)\) for \(p > 1/2\). In the endpoint case \(p = 1/2\) Goginava [7] proved that the maximal operator \(\sigma^{\psi, \alpha}\) is not of type \((H_{1/2}, L_{1/2})\) and Weisz [26] showed that the maximal operator is of weak type \((H_{1/2}, L_{1/2})\). Recently, the rate of the deviant behaviour in the endpoint \(p = 1/2\) was discussed by Goginava and the author [9]. The case \(0 < p < 1/2\) can be found in the papers of Tephnadze [20, 21].

In 2004, the boundedness of the maximal operator of the Cesàro means \((0 < \alpha \leq 1)\) was investigated by Simon [17]. It was showed that the maximal operator is bounded from the Hardy space \(H_p\) to the space \(L_p\) for \(p > p_0 := 1/(1 + \alpha)\) and of weak type \((1, 1)\). Moreover, he showed the inequality

\[
(3) \quad \sup_n \|K_n^{\psi, \alpha}\|_1 < \infty \quad \text{for all} \ 0 < \alpha \leq 1.
\]

In the endpoint case Goginava showed that the endpoint \(p_0 := 1/(1 + \alpha)\) is essential. That is, he proved that the maximal operator of Cesàro means is not bounded from the Hardy space \(H_{p_0}\) to the space \(L_{p_0}\) [8].

For \(x = (x^1, x^2, \ldots, x^d) \in G^d\) and \(n = (n_1, n_2, \ldots, n_d) \in \mathbb{N}^d\) the \(d\)-dimensional rectangles are defined by \(I_n(x) := I_{n_1}(x^1) \times \cdots \times I_{n_d}(x^d)\). For \(n \in \mathbb{N}^d\) the \(\sigma\)-algebra generated by the
rectangles \( \{I_n(x), x \in G^d\} \) is denoted by \( \mathcal{F}_n \). The conditional expectation operators relative to \( \mathcal{F}_n \) are denoted by \( E_n \).

Suppose that for all \( j = 2, \ldots, d \) the functions \( \gamma_j : [1, +\infty) \to [1, +\infty) \) are strictly monotone increasing continuous functions with properties \( \lim_{+\infty} \gamma_j = +\infty \) and \( \gamma_j(1) = 1 \). Moreover, suppose that there exist \( \zeta, c_{j,1}, c_{j,2} > 1 \) such that the inequality

\[
\tag{4}
c_{j,1} \gamma_j(x) \leq \gamma_j(\zeta x) \leq c_{j,2} \gamma_j(x)
\]

holds for each \( x \geq 1 \). In this case the functions \( \gamma_j \) are called CRF (cone-like restriction functions). Let \( \gamma := (\gamma_2, \ldots, \gamma_d) \) and \( \beta_j \geq 1 \) be fixed \( (j = 2, \ldots, d) \). We will investigate the maximal operator of the multi-dimensional \( (C, \alpha) \) means and the convergence over a cone-like set \( L \) (with respect to the first dimension), where

\[
L := \{ n \in \mathbb{N}^d : \beta_j^{-1} \gamma_j(n_1) \leq n_j \leq \beta_j \gamma_j(n_1), j = 2, \ldots, d \}.
\]

If each \( \gamma_j \) is the identical function then we get a cone. The cone-like sets were introduced by Gát in dimension two [4]. The condition (4) on the function \( \gamma \) is natural, because Gát [4] proved that to each cone-like set with respect to the first dimension there exists a larger cone-like set with respect to the second dimension and reversely, if and only if the inequality (4) holds.

Weisz defined a new type martingale Hardy space depending on the function \( \gamma \) (see [25]). For a given \( n_1 \in \mathbb{N} \) set \( n_j := |\gamma_j(2^{n_1})| \ (j = 2, \ldots, d) \), that is, \( n_j \) is the order of \( \gamma_j(2^{n_1}) \) (this means that \( 2^n \leq \gamma_j(2^{n_1}) < 2^{n+1} \) for \( j = 2, \ldots, d \)). Let \( \overline{n}_1 := (n_1, \ldots, n_d) \). Since, the functions \( \gamma_j \) are increasing, the sequence \( (\overline{n}_1, n_1 \in \mathbb{N}) \) is increasing, too. It is given a class of one-parameter martingales \( f = (f_{\overline{n}_1}, n_1 \in \mathbb{N}) \) with respect to the \( \sigma \)-algebras \( (\mathcal{F}_{\overline{n}_1}, n_1 \in \mathbb{N}) \).

The maximal function of a martingale \( f \) is defined by \( f^* = \sup_{n_1 \in \mathbb{N}} |f_{\overline{n}_1}| \). For \( 0 < p < \infty \) the martingale Hardy space \( H^\gamma_p(G^d) \) consists of all martingales for which \( \|f\|_{H^\gamma_p} := \|f^*\|_p < \infty \).

It is known (see [24]) that \( H^\gamma_p \sim L_p \) for \( 1 < p < \infty \), where \( \sim \) denotes the equivalence of the norms and spaces.

If \( f \in L_1(G^d) \) then it is easy to show that the sequence \( (S_{2^{\alpha_1}, \ldots, 2^{\alpha_d}}(f) : \overline{n}_1 = (n_1, \ldots, n_d), n_1 \in \mathbb{N}) \) is a one-parameter martingale with respect to the \( \sigma \)-algebras \( (\mathcal{F}_{\overline{n}_1}, n_1 \in \mathbb{N}) \). In this case the maximal function can also be given by

\[
f^*(x) = \sup_{n_1 \in \mathbb{N}} \frac{1}{\text{mes}(I_{\overline{n}_1}(x))} \left| \int_{I_{\overline{n}_1}(x)} f(u) d\mu(u) \right| = \sup_{n_1 \in \mathbb{N}} |S_{2^{\alpha_1}, \ldots, 2^{\alpha_d}}(f, x)|
\]

for \( x \in G^d \).

We define the maximal operator \( \sigma^\kappa_{L}^{\alpha, \gamma, \alpha} \) by

\[
\sigma^\kappa_{L}^{\alpha, \gamma, \alpha} f(x) := \sup_{n \in \mathcal{L}} |\sigma_n^{\kappa, \alpha} f(x)|.
\]

For double Walsh-Paley-Fourier series, Móricz, Schipp and Wade [11] proved that \( \sigma_n f \) converge to \( f \) a.e. in the Pringsheim sense (that is, no restriction on the indices other than \( \min(n_1, n_2) \to \infty \)) for all functions \( f \in L \log^+ L \). The a.e. convergence of Fejér means \( \sigma_n f \) of integrable functions, where the set of indices is inside a positive cone around the identical function, that is \( \beta^{-1} \leq n_1/n_2 \leq \beta \) is provided with some fixed parameter \( \beta \geq 1 \), was proved by Gát [5] and Weisz [27]. A common generalization of results of Móricz, Schipp, Wade [11] and Gát [5], Weisz [27] for cone-like set was given by the first author and Gát in [6]. Namely,
a necessary and sufficient condition for cone-like sets in order to preserve the convergence property, was given. The trigonometric case was treated by Gát [4].

Connecting to the original paper [4] on trigonometric system, Gát asked the following. What could we state for other systems for example Walsh-Paley, Walsh-Kaczmarz and Vilenkin systems and for other means for example logarithmic means, Riesz means, \((C,\alpha)\) means? Some parts of Gát’s question was answered by Weisz [25], Blahota and the first author [2, 12] and naturally in paper [6].

In 2011, the properties of the maximal operator of the \((C,\alpha)\) and Riesz means of a multi-dimensional Vilenkin-Fourier series provided that the supremum in the maximal operator is taken over a cone-like set, was discussed by Weisz [25]. Namely, it was proved that the maximal operator is bounded from dyadic Hardy space \(H_p\) to the space \(L_p\) for \(p_0 < p \leq \infty\) \((p_0 := \max\{1/(1 + \alpha_k) : k = 1, \ldots, d\})\) and is of weak type \((1,1)\). Recently, it was showed that the index \(p_0\) is sharp. Namely, it was proven that the maximal operator is not bounded from the dyadic Hardy space \(H_{p_0}\) to the space \(L_{p_0}\) [2]. Detailed list of the reached results for one- and several dimensional Walsh-like systems can be found in [28].

For the two-dimensional Walsh-Kaczmarz-Fourier series Simon proved [15] that the cone-restricted maximal operator of the Fejér means is bounded from the Hardy space \(H_{1/2}\) to the space \(L_{1/2}\) for all \(1/2 < p\) (here the set of indices is inside a positive cone around the identical function). That is, the a.e. convergence of cone-restricted Fejér means holds for the Walsh-Kaczmarz system, as well. Moreover, it was proved that \(p = 1/2\) is essential. In 2007, Goginava and the first author proved that the cone-restricted maximal operator is not bounded from the Hardy space \(H_{1/2}\) to the space weak-\(L_{1/2}\) [10]. The cone-like restricted two-dimensional maximal operator of Fejér means was discussed in [12].

Motivated by the works of Weisz [25], Simon [15] and the above mentioned question of Gát we prove that the maximal operator \(\sigma_{L}^{\kappa,\alpha,*}\) is bounded from the dyadic Hardy space \(H_{p}^{\gamma}\) to the space \(L_{p}\) for \(p_0 < p \leq \infty\) and is of weak type \((1,1)\). As a corollary we get the theorem of Simon [15] on the a.e. convergence of cone-restricted Fejér means. At the endpoint \(p = p_0\), we show that the maximal operator \(\sigma_{L}^{\kappa,\alpha,*}\) is not bounded from the Hardy space \(H_{p_0}^{\gamma}\) to the space \(L_{p_0}\).

In dimension 2, the case \(\alpha_1 = \alpha_2 = 1\) was discussed in [12]. Unfortunately, the counterexample martingale and the method is not suitable for case \(0 < \alpha_i < 1\) \((i = 1, \ldots, d)\).

2. Auxiliary propositions and main results

First, we formulate our main theorems.

**Theorem 1.** Let \(\gamma\) be CRF. The maximal operator \(\sigma_{L}^{\kappa,\alpha,*}\) is bounded from the Hardy space \(H_{p}^{\gamma}\) to the space \(L_{p}\) for \(p_0 < p \leq 1\) \((p_0 := \max\{1/(1 + \alpha_i) ; i = 1, \ldots, d\})\).

By standard argument we have that, if \(1 < p \leq \infty\) then \(\sigma_{L}^{\kappa,\alpha,*}\) is of type \((p,p)\) and of weak type \((1,1)\).

**Theorem 2.** Let \(\gamma\) be CRF. Then for any \(f \in L^1\)

\[
\lim_{n \to \infty} \sigma_n^{\kappa,\alpha} f = f
\]

holds almost everywhere.

We immediately have the theorem of Simon [15] as a corollary.
Corollary 1 (Simon \cite{15}). Let $f \in L^1$ and $\beta \geq 1$ be a fixed parameter. Then
\[
\lim_{\beta^{-1} \leq n_1/n_2 \leq \beta} \sigma_n f = f
\]
holds a.e.

Theorem 3. Let $\gamma$ be CRF and $\alpha_1 \leq \ldots \leq \alpha_d$. The maximal operator $\sigma_{L,\alpha,\gamma}$ is not bounded from the Hardy space $H_{\frac{\gamma}{p_0}}$ to the space $L_{p_0}$.

To prove our Theorem 1, 2, 3 we need the following Lemma of Weisz \cite{24}, the concept of the atoms (for more details see \cite{25}) and a Lemma of Goginava \cite{8}.

A bounded measurable function $a$ is a $p$-atom, if there exists a dyadic $d$-dimensional rectangle $I \in \mathcal{F}_{\pi_1}$, such that
\begin{itemize}
  \item[a)] $\text{supp } a \subseteq I$,
  \item[b)] $\|a\|_{\infty} \leq \mu(I)^{-1/p}$,
  \item[c)] $\int_I \text{ad} \mu = 0$.
\end{itemize}

Lemma 1 (Weisz \cite{24}). Suppose that the operator $T$ is $\sigma$-sublinear and $p$-quasilocal for any $0 < p < 1$. If $T$ is bounded from $L_\infty$ to $L_\infty$, then
\[
\|T f\|_p \leq c_p \|f\|_{H_p} \text{ for all } f \in H_p.
\]

Lemma 2 (Goginava \cite{8}). Let $n \in \mathbb{N}$ and $0 < \alpha \leq 1$. Then
\[
\int \max_{G, 1 \leq N < 2^n} (A_{N-1}^n |K_N^n(x)|)^{1/2} d\mu(x) \geq c(\alpha) \frac{n}{\log(n + 2)}.
\]

3. Proofs of the theorems

Now, we prove our main Theorems.

Proof of Theorem 1. Using Lemma 1 of Weisz we have to prove that the operator $\sigma_{L,\alpha,\gamma}$ is bounded from the space $L_\infty$ to the space $L_\infty$. It immediately follows from the inequality (3).

Let $a$ be a $p$-atom, with support $I$. We can assume that $I = I_{N_1} \times \ldots \times I_{N_d}$ (with $2^{N_j} < 2^{N_j+1}$, $j = 2, \ldots, d$), $\|a\|_{\infty} \leq 2^{(N_1+\ldots+N_d)/p}$ and $\int_I \text{ad} \mu = 0$.

Set $\delta := \max\{\zeta_{\log c_{j,1} 2^{\beta_j+1}} \colon j = 2, \ldots, d\}$. If $n_1 \leq 2^{N_1}/\delta$, then
\[
\begin{align*}
n_j & \leq \beta_j \gamma_j(n_1) \leq \beta_j \gamma_j(2^{N_1}\zeta_{-\log c_{j,1} 2^{\beta_j-1}}) \leq \frac{1}{\beta_j c_{j,2}} \
& \leq \frac{1}{\beta_j c_{j,2}} \gamma_j(2^{N_1}) \leq \frac{\gamma_j(2^{N_1})}{2} \leq 2^{N_j}.
\end{align*}
\]

Let $\zeta, c_{j,1}, c_{j,2} > 1$, $\beta_j \geq 1$ imply $n_1 < 2^{N_1}$ and $n_j \leq \gamma_j(2^{N_1})/2 < 2^{N_j}$ ($j = 2, \ldots, d$). In this case the $(m_1, \ldots, m_d)$-th Fourier coefficients are zeros for $m_1 \leq n_1, \ldots, m_d \leq n_d$. This gives $\sigma_n f = 0$ ($n = (n_1, \ldots, n_d)$).

That is, we could suppose that $n_1 > 2^{N_1}/\delta$. This yields that
\[
\begin{align*}
n_j \geq \frac{\gamma_j(n_1)}{\beta_j} \geq \frac{\gamma_j(2^{N_1}/\delta)}{\beta_j} \geq \frac{1}{\max\{\log c_{j,1} 2^{\beta_j+1} \colon j = 2, \ldots, d\}} \gamma_j(2^{N_1}) \geq \frac{\gamma_j(2^{N_1})}{\delta_j} \geq \frac{2^{N_j}}{\delta}
\end{align*}
\]
with $\delta' := \max_{j=2,\ldots,d} \delta_j^j$ for all $j = 2, \ldots, d$. $\delta' > 1$ can be assumed.

The proof will be complete, if we show that the maximal operator $\sigma_{L,\alpha}^\kappa$ is $p$-quasilocal for $p_0 < p \leq 1$. That is, there exists a constant $c_p$ such that the inequality

$$\int_\mathcal{D} |\sigma_{L,\alpha}^\kappa a|^p d\mu \leq c_p < \infty$$

holds for all atom $a$ in $H_p^\gamma$ with support $I = I_{N_1} \times \ldots \times I_{N_d}$ (with $N_1 = (N_1, \ldots, N_d)$).

It is well known that the concept of $p$-quasi-locality of the maximal operator $\sigma_{L,\alpha}^\kappa$ can be modified as follows [13]: there exists $r = 0, 1, \ldots,$ such that

$$(5) \quad \int_{\mathcal{D}} |\sigma_{L,\alpha}^\kappa a|^p d\mu \leq c_p < \infty,$$

where $I^r := I_{N_1}^r \times \ldots \times I_{N_d}^r := I_{N_1-r} \times \ldots \times I_{N_d-r} (N_j - r \geq 0$ for all $j = 1, \ldots, d)$. We will give the value of $r$ later.

Let us set $x = (x^1, \ldots, x^d) \in \mathcal{T}$.

$$|\sigma_n^\kappa a(x)| \leq 2^{(N_1+\ldots+N_d)/p} \int_{I_{N_1}} |K_{n_1}^{\kappa,\alpha_1}((x^1 + t^1) \ldots K_{n_d}^{\kappa,\alpha_d}(x^d + t^d)) d\mu(t)| \ldots \int_{I_{N_d}} |K_{n_d}^{\kappa,\alpha_d}(x^d + t^d)) d\mu(t|^d)$

Now, we decompose the set $\mathcal{T} = \mathcal{T}_{N_1}^r \times \ldots \times \mathcal{T}_{N_d}^r$ as the following disjoint union

$$\mathcal{T} = (\mathcal{T}_{N_1}^r \times \ldots \times \mathcal{T}_{N_d}^r) \cup \cup (\mathcal{T}_{N_1}^r \times \mathcal{T}_{N_2}^r \times \ldots \times \mathcal{T}_{N_d}^r) \cup \ldots \cup (\mathcal{T}_{N_1}^r \times \ldots \times \mathcal{T}_{N_{d-1}}^r \times \mathcal{T}_{N_d}^r) \cup \ldots \cup (\mathcal{T}_{N_1}^r \times \ldots \times \mathcal{T}_{N_{d-1}}^r \times \mathcal{T}_{N_d}^r)$.

Let us set $\delta'' := \max\{\delta, \delta', \delta''\}$, set $r \in \mathbb{P}$ such that $2^{-r} \leq 1/\delta'' \leq 2^{-r+1}$, and $L^{r,l} := I_{N_1}^r \times \ldots \times I_{N_{l+1}}^r \times \ldots \times I_{N_d}^r$ for $l = 0, \ldots, d$. We define

$$J_i := \int\mathcal{T}_{N_i} \left( \sup_{n_i \geq 2^{N_i} / \delta''} \int_{I_{N_i}} |K_{n_i}^{\kappa,\alpha_i}(x^i + t^i)) d\mu(t^i) \right)^p \mu(x^i), \quad i = 1, \ldots, l,$$

$$\mathcal{J}_j := \int\mathcal{T}_{N_j} \left( \sup_{n_j \geq 2^{N_j} / \delta''} \int_{I_{N_j}} |K_{n_j}^{\kappa,\alpha_j}(x^j + t^j)) d\mu(t^j) \right)^p \mu(x^j), \quad j = l+1, \ldots, d.$$

Now, we get

$$(6) \quad \int_{L^{r,l}} |\sigma_{L,\alpha}^\kappa a|^p d\mu \leq 2^{N_1+\ldots+N_d} J_1 \ldots J_l \cdot \mathcal{J}_{l+1} \ldots \mathcal{J}_d.$$
Second, we discuss the integrals $\mathcal{T}_j$ ($j = l+1, \ldots, d$). In the paper [17, 48-59 pages], Simon showed that
\begin{equation}
\left( \sup_{n \geq 2^N} \int_{I_N} |K_n^{\kappa,\alpha}(x+t)|d\mu(t) \right)^p \mu(x) \leq c_p 2^{-N} \quad \text{if } p > \frac{1}{1 + \alpha} \quad (0 < \alpha \leq 1).
\end{equation}
Using this we write
\begin{equation}
\mathcal{T}_j \leq \left( \sup_{n_j \geq 2^{N_j-r}} \int_{I_{N_j}} |K_n^{\kappa,\alpha_j}(x^j + t^j)|d\mu(t^j) \right)^p \mu(x^j) \leq c_p 2^{-N_j} \quad \text{if } p > \frac{1}{1 + \alpha_j}.
\end{equation}
Inequalities (6), (8) yield
\[ \int_{L^r} |\sigma_L^{\kappa,\alpha,*}a|^pd\mu \leq c_p \quad \text{for all } l \text{ if } p > p_0. \]
The decomposition of $\mathcal{T}r$ gives
\[ \int_{\mathcal{T}r} |\sigma_L^{\kappa,\alpha,*}a|^pd\mu \leq c_{p,d} \quad \text{if } p > p_0. \]
This completes the proof of Theorem 1 \[ \Box \]

The weak type $(1,1)$ inequality follows by interpolation. The set of Walsh-Kaczmarz polynomials is dense in $L_1$. The weak-type $(1,1)$ inequality and the usual density argument imply Theorem 2.

**Proof of Theorem 3** In the present proof we use a counterexample martingale. Let us set
\[ f_{m_1}(x) := (D_{2^{n_1+1}}(x^1) - D_{2^{n_1}}(x^1)) \prod_{j=2}^d w_{2^n_{j-1}}(x^j), \]
where $n_2, \ldots, n_d$ is defined to $n_1$, earlier.

Now, we calculate $S_j^x(f_{m_1}, x)$. Since, $\omega_{2^n_{j-1}}(x) = r_{n-1}(x)(-1)^{\sum_{i=0}^{n-2} x_i} = \kappa_{2^n_{j-1}}(x)$, we have
\[ \hat{f}^x_{m_1}(k) = \begin{cases} 1, & \text{if } k_1 = 2^{n_1}, \ldots, 2^{n_1+1} - 1, \text{ and } k_j = 2^n - 1 \text{ for all } j = 2, \ldots, d; \\ 0, & \text{otherwise}. \end{cases} \]
\begin{align}
S_j^x(f_{m_1}, x) &= \sum_{\nu=0}^{j-1} \hat{f}^x_{m_1}(\nu, 2^{n_2} - 1, \ldots, 2^{n_d} - 1) \omega_\nu(x) \prod_{l=2}^d w_{2^{n_l-1}}(x^l) \\
&= \begin{cases} (D_j(x^1) - D_{2^{n_1}}(x^1)) \prod_{l=2}^d w_{2^{n_l-1}}(x^l) & \text{if } j_1 = 2^{n_1} + 1, \ldots, 2^{n_1+1} - 1, \text{ and } j_1 \geq 2^n \text{ for all } l = 2, \ldots, d; \\
0 & \text{if } j_1 \geq 2^{n+1} \text{ and } j_l \geq 2^n \text{ for all } l = 2, \ldots, d; \\
f_{m_1}(x) & \text{otherwise}. \end{cases}
\end{align}

We immediately have that
\[ f_{m_1}(x) = \sup_{m_1 \in \mathbb{N}} |S_{2^{m_1}, \ldots, 2^m}(f_{m_1}, x)| = |f_{m_1}(x)|, \]
where $\overline{m_1} = (m_1, \ldots, m_d)$.
\begin{equation}
\|f_{m_1}\|_{H^p_{\gamma_0}} = \|f_{m_1}\|_{p_0} = \|D_{2^{n_1}}\|_{p_0} = 2^{(1-1/p_0)n_1} < \infty.
\end{equation}
That is $f_{\pi} \in H_{p_0}^\gamma$.

We can write the $n$th Dirichlet kernel with respect to the Walsh-Kaczmarz system in the following form:

$$D_n^\kappa(x) = D_{2^{[n]}}(x) + \sum_{k=2^{[n]}}^{n-1} r_{[k]}(x) w_{k-2^{[n]}}(\tau_{[k]}(x))$$

(11)

$$= D_{2^{[n]}}(x) + r_{[n]}(x) D_{n-2^{[n]}}^w(\tau_{[n]}(x))$$

We set $L_1^N := 2^{n_1} + N$, where $0 < N < 2^{n_1}$ and $L_j^N := [\gamma_j(2^{n_1} + N)]$ for $j = 2, \ldots, d$ (where $[x]$ denotes the integer part of $x$). In this case $L_i^N := (L_i^N, \ldots, L_d^N) \in L$. Let us calculate $\sigma_{L,N}^\kappa f_{\pi}$.

By inequalities (11) and (9) we get

$$|\sigma_{L,N}^\kappa f_{\pi}(x)| = \frac{1}{\prod_{j=1}^d A_{L_j^N}^{\alpha_j}} \left| \sum_{j=1}^d \sum_{k_j=0}^{L_j^N} \prod_{i=1}^d A_{L_i^N-k_i}^{\alpha_i-1} S_k^\kappa(f_{\pi}; x) \right|$$

$$= \frac{1}{\prod_{j=1}^d A_{L_j^N}^{\alpha_j}} \left| \sum_{j=2}^d \sum_{k_j=2^{n_j}}^{L_j^N} \sum_{k_1=0}^{L_1^N} \prod_{i=2}^d A_{L_i^N-k_i}^{\alpha_i-1} \prod_{i=2}^d w_{2^{n_i}-1}(x^i)(D_{k_1}(x^i) - D_{2^{n_i}}(x^i)) \right|$$

$$= \frac{1}{\prod_{j=1}^d A_{L_j^N}^{\alpha_j}} \left| \sum_{j=2}^d \sum_{k_j=2^{n_j}}^{L_j^N} \sum_{k_1=0}^{L_1^N} \prod_{i=2}^d A_{L_i^N-k_i}^{\alpha_i-1} \prod_{i=2}^d A_{L_i^N-2^{n_i}-k_i}^{\alpha_i} \prod_{i=2}^d A_{L_i^N-2^{n_i}-k_i}^{\alpha_i} \prod_{i=2}^d D_{k_1}(x^i) \right|$$

$$\geq \frac{c(\alpha)}{2^{n_1+1}} A_{N}^{\alpha_1} |K_N^{w,\alpha_1}(\tau_{n_1}(x^1))|.$$
This completes the proof of Theorem 3. □

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