Universality of scaling of correlations across probability distributions

Vaibhav Wasnik
Indian Institute of Technology, Goa

Scale invariance and the resulting power law behaviours are seen in diverse systems. In this work we consider translation, rotational and scale invariant systems defined on a lattice, such that the variables defining the state at every lattice site take on the same range of finite values, with these values collectively picked up from probability distribution that can be arbitrary. We show that the exponent that describes the scaling of the two point correlation function in these systems will match the scaling exponent of an equilibrium statistical mechanical model described by a Boltzmannian distribution at criticality. This work therefore extends the concept of universality in statistical mechanics to probability distributions that do not have a Boltzmannian form.

Introduction

Scale invariance expressed in terms of power laws is seen in varied systems. Be it in natural and artificial collective systems [1], in percolation [2], in vocalization sequences [3], scale invariance in the repeating fast radio burst [4], in examples of turbulence [5], in river run offs [6], in stock market crashes [7], criticality in biological systems [8] etc.

That power law behaviours are a feature of independent systems, begs reasons for their similarities. In equilibrium statistical mechanics, different systems in thermal equilibrium show similar behaviour at the critical point. At thermal equilibrium, the probability distribution of finding the system in a particular state is proportional to $e^{-H/k_BT}$, where $H$ is the Hamiltonian describing the system state. For a system defined on a lattice $H$ could have the form

$$H = \sum_i J_i S_i + \sum_{ij} J_{ij} S_i S_j + \sum_{ijk} J_{ijk} S_i S_j S_k...$$

(1)

Here $S_i$ is a variable describing the state at lattice site $i$. At the critical point, the critical exponents of the system end up being independent of microscopic details of the Hamiltonian describing the system and are instead given by parameters such as its symmetry, dimensions in which the system exists etc [9] . For example value of $\eta$ in the correlation function

$$\langle S_i S_j \rangle_B - \langle S_i \rangle_B \langle S_j \rangle_B \sim \frac{1}{|I-J|^{d+\eta-2}},$$

(2)

is independent of the form of the Hamiltonian, instead determined by things such as symmetry etc. Here

$$\langle \phi \rangle_B = \frac{\sum_{\{S_i\}} e^{-H/k_BT} \phi}{\sum_{\{S_i\}} e^{-H/k_BT}}.$$ 

(3)

In this write up, $|I-J|$ denotes the distance between lattice points labeled by $I$ and $J$.

Continuous systems that are scale invariant can be defined by operators that scale as $\phi_i(\lambda x) = \lambda^\sigma \phi_i(x)$. If one improves this symmetry to conformal invariance (with restrictions on the form of $\lambda(x)$ depending on the dimensions of space), the operators scale as $\phi_i(\lambda(x)x) = \lambda(x)^\sigma \phi_i(x)$. It is known that if the scaling dimensions of operators, their spin and three point correlations are known, then it is possible to evaluate all possible correlation functions in the theory, without knowing the Hamiltonian [10, 11]. The Hamiltonian appears in the probability distribution as $e^{-H/k_BT}$, but the natural question to ask is whether the functional form of the probability distribution is of relevance, since the correlation functions can be evaluated without knowledge of the Hamiltonian.

*Electronic address: wasnik@iitgoa.ac.in
In this work, we consider translational, rotational and scale invariant systems defined on a lattice, such that the variable that defines the state at a lattice site takes on the same finite range of values. These variables are collectively picked up from an arbitrary probability distribution. We show that the scaling of the two point correlation function is similar to the scaling of statistical mechanical models at equilibrium described by Boltzmannian distribution at criticality. This therefore extends the concept of universality beyond Boltzmannian probability distributions of statistical mechanics. We would like to explicitly state that by a Boltzmannian probability distribution we mean a probability distribution that goes as $e^{H}$ where $H$ is a functional of the degrees of freedom in the problem. For example if we say that the variable $S_{i}$ represents the state at lattice site $i$, $H$ will have a form like $H\{S_{i}\} = a + \sum_{i}a_{i}S_{i} + \sum_{ij}a_{ij}S_{i}S_{j} + \sum_{ijk}a_{ijk}S_{i}S_{j}S_{k} + ...$ where $a, a_{ij}, a_{ijk}...$ etc are constants. Hence an arbitrary probability distribution $P\{S_{i}\}$ cannot be written as $e^{\ln P(S_{i})}$ and then labelled as a Boltzmannian probability distribution. This is because $\ln P\{S_{i}\}$ is a non-analytic function of $\{S_{i}\}$, because of the presence of branch cuts.

We summarize the arguments of the proof for the interest of the reader. Proposition 1, starts with the attesting that if all possible measured observables are translationally invariant, the probability distribution describing the system should be translationally invariant. Proposition 2, states that translation invariance and discrete rotational invariance automatically makes the two point correlation function fully rotational invariant and hence functionally dependent on distance between the two points on the lattice. Scale invariance then sets the correlation function $C(||a-b||) = \frac{C}{|a-b|^{\alpha}}$. In proposition 3, we claim that translation invariance of the probability distribution implies that the probability distribution can be written as a function of translation invariant terms, that cannot be expressed in terms of each other. This information allows us to write the probability distribution as a Fourier transform as in Eq[10]. We can hence write the two point correlation function $C(||a-b||)$ for the system in terms of correlation functions $C(a, b)_{i_{1}, i_{2}}...$ evaluated using probability weights which have an exponential form described in Eq[13]. Rotational invariance of $C(a, b)_{i_{1}, i_{2}}... = C(||a-b||)_{i_{1}, i_{2}}...$ is emphasised. Proposition 4, then considers the case when distance between points $a$ and $b$ is orders of magnitude larger than the lattice spacing to show that in that limit $C(||a-b||)_{i_{1}, i_{2}, i_{3}}... \sim \frac{e^{-\lambda_{i_{1}, i_{2}, i_{3}}...|a-b|}}{|a-b|^{\alpha_{i_{1}, i_{2}, i_{3}}...}}$. Proposition 6, still looks in the limit when the distance between points $a$ and $b$ is orders of magnitude larger than the lattice spacing to show that wherever $C(||a-b||)_{i_{1}, i_{2}, i_{3}}...$ makes a non-zero contribution to $C(||a-b||)$, $\alpha_{i_{1}, i_{2}, i_{3}}...$ has to equal $\alpha$. Finally in Proposition 7, we show that at least one of the $\lambda_{i_{1}, i_{2}, i_{3}}...$ has to equal zero, implying that $\alpha$ also equals the scaling of a correlation function of a system at criticality described by a Boltzmannian probability distribution, completing our proof. Some of the propositions stated may have been known earlier but we offer a proof for them for the sake of completeness. However all the propositions build up to prove the main assertion of the paper which is a new addition to literature, that the scaling coefficient of the two point function for a translational, rotational and scale invariant system described by an arbitrary probability distribution is similar to that of a translational, rotational and scale invariant statistical mechanical system at equilibrium.

Conventions

Consider a system defined on a lattice. Let us label lattice sites by vectorial indices $i$. Let $S_{i}$ be the variable defining the state at lattice site $i$. If the lattice is one dimensional $i$ is an integer. If the lattice is two dimensional $i = (x_{i}, y_{i})$ where $x_{i}, y_{i}$ are integers. If the lattice is three dimensional $i = (x_{i}, y_{i}, z_{i})$ where $x_{i}, y_{i}, z_{i}$ are integers. The lattice in question is constructed by primitive translation vectors and hence possesses a discrete rotational symmetry.

We also note that $i$ appearing as a part of a subscript is a label and not to be confused with $i = \sqrt{-1}$ that appears in exponentials of a Fourier transform in this paper.

Define translation as the following operation: Consider any variable $\{Z_{i}\}$ defined on the lattice. Assign the value of the variable for lattice site $i+1$ the value of the variable of lattice site $i$, with $i$ running over all lattice sites. Denote this operation by $Z_{i} \rightarrow Z_{i+1}$. A quantity being translation invariant then would imply that the quantity is invariant under translations.

In the above when we say lattice site $i+1$ we are implying the number $i+1$ if the system is one dimensional, lattice site $(x_{i}+1, y_{i})$ along with $(x_{i}, y_{i}+1)$ if $i = (x_{i}, y_{i})$ for a two dimensional lattice or implying the lattice site $(x_{i}+1, y_{i}, z_{i})$ along with lattice sites $(x_{i}, y_{i}+1, z_{i})$ and $(x_{i}, y_{i}, z_{i}+1)$, if $i = (x_{i}, y_{i}, z_{i})$ for a three dimensional lattice. $(1,0), (0,1)$ and $(1,0,0), (0,1,0), (0,0,1)$ are the primitive translation vectors of the lattice in two and three dimensions respectively. Nowhere it should be assumed that the primitive translation vectors have to be perpendicular to each other in space.

To elaborate, let us consider the system in two dimensions. The operation $Z_{i} \rightarrow Z_{i+1}$ implies assigning the value of variable on lattice site $(x_{i}, y_{i})$ to the variable on lattice site $(x_{i}+1, y_{i})$ for all values of $(x_{i}, y_{i})$ in one realization
of the translation and then assigning the value of variable on lattice site \((x_i, y_i)\) to the variable on the lattice site \((x_i, y_i + 1)\), for all values of \((x_i, y_i)\) in another realization of the translation.

Define

\[
\sum_i S_{i+m}S_{i+n}S_{i+p} = \sum_{k=-\infty}^{k=\infty} S_{m+k}S_{n+k}S_{p+k}
\]

(4)
in case of one dimension.

\[
\sum_{i} S_{i+m}S_{i+n}S_{i+p} = \sum_{k,l=-\infty}^{k,l=\infty} S_{m+k,y_m+l}S_{n+k,y_n+l}S_{p+k,y_p+l}
\]

(5)
in case of two dimension and similarly for three dimensions. Note that the above terms are invariant under translation as defined above.

Proof

The probability for a configuration \(\{S_i\}\) is denoted by \(f(\{S_i\})\). The ensemble average of any quantity which is a function of \(\{S_i\}\), \(O(\{S_i\})\), is defined as

\[
\langle O(\{S_i\}) \rangle = \sum_{\{S_i\}} O(\{S_i\}) f(\{S_i\}).
\]

(6)

Here \(\sum_{\{S_i\}}\) refers to summing over all possible configurations.

Observed translational invariance of the system implies the translational invariance of the correlation functions. With this in mind, we make the following proposition,

**Proposition 1:**

*Since correlation functions are translation invariant, the probability distribution \(f(\{S_i\})\) has to be translation invariant.*

If

\[
\langle S_a \rangle = \sum_{\{S_i\}} S_a \ f(\{S_i\}).
\]

\[
\langle S_a S_b \rangle = \sum_{\{S_i\}} S_a S_b \ f(\{S_i\}).
\]

\[
\langle S_a S_b S_c \rangle = \sum_{\{S_i\}} S_a S_b S_c \ f(\{S_i\}).
\]

...  

(7)

are all translation invariant, we have to have that \(f(\{S_i\})\) is invariant under \(S_i \rightarrow S_{i+1}\).

**Proposition 2:**

*Translation invariance coupled with discrete rotational invariance implies full rotational invariance of the two point correlation function.*
Consider a two-dimensional case. Let two points on the lattice be \( i = (m_1, n_1) \) and \( j = (m_2, n_2) \). Translation invariance implies the two-point correlation function has to have the functional form \( C(i, j) = g(m_1 - m_2, n_1 - n_2) \). A discrete rotation is done by a matrix

\[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\]

where \( a, b, c, d \) are constants. The discrete rotation symmetry which will determine the \( a, b, c, d \), would be a subgroup of the symmetry group of the lattice. Under a discrete rotation we have

\[
\begin{bmatrix}
  m_1' - m_2' \\
  n_1' - n_2'
\end{bmatrix} = \begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix} \begin{bmatrix}
  m_1 - m_2 \\
  n_1 - n_2
\end{bmatrix}
\]

In order to have that \( g(m_1 - m_2, n_1 - n_2) = g(m_1' - m_2', n_1' - n_2') \), for any lattice points \( i = (m_1, n_1) \) and \( j = (m_2, n_2) \), we have to have that, \( g(m_1 - m_2, n_1 - n_2) \) is a functional of \( \sqrt{(m_1 - m_2)^2 + (n_1 - n_2)^2} \), implying full rotational invariance of the two-point correlation function. Since, the systems we are interested in are scale invariant, the correlation function should have the functional form \( C(|a - b|) = \frac{C}{|a - b|^\nu} \).

**Proposition 3:**

Translation invariance of \( f(\{S_i\}) \), implies that generically \( f(\{S_i\}) \) is a function of translationally invariant terms that cannot be expressed in terms of each other. Some examples of these terms are \( \sum_i S_i, \sum_i S_i + n S_i + m, \sum_i S_i + p S_i + m S_i + n, \) etc.

Next, because none of the \( \sum_i S_i, \sum_i S_i + n S_i + m, \sum_i S_i + p S_i + m S_i + n, \) etc can be broken up into a product of two terms, it implies they cannot be written in terms of each other.

We can hence express as a Fourier transform

\[
f(\{S_i\}) = \int \Pi dJ_i \quad a(J_1, J_2, J_3...) \quad e^{iJ_1 \sum_i S_i + i \sum_{m,n} J_2^{m,n} \sum_i S_i + n + i \sum_{m,n,p} J_3^{m,n,p} \sum_i S_i + p S_i + m S_i + n} \ldots + (*),
\]

\((*)\) in this text refers to complex conjugation of all terms to the left. The ... in the exponential above includes all other translationally invariant terms that cannot be written in terms of each other. Each of them multiplied by a corresponding \( J_i \). In the integration we do not explicitly write the \( m, n, p, \) etc labels on the \( J_i \)'s, but their presence is understood as we are integrating over all possible \( J_i \)'s. To regularize terms that may be ill defined because of integrating over the \( J_i \)'s, we use the epsilon prescription.

\[
f(\{S_i\}) = \int \Pi dJ_i \quad a(J_1, J_2, J_3...) \quad e^{iJ_1 (1 + i\epsilon\Gamma) \sum_i S_i + i \sum_{m,n} J_2^{m,n} (1 + i\epsilon\Gamma) \sum_i S_i + n + i \sum_{m,n,p} J_3^{m,n,p} (1 + i\epsilon\Gamma) \sum_i S_i + p S_i + m S_i + n} \ldots + (*),
\]

\((*)\) is added to regularize the exponentials. We take \( \epsilon \to 0^+ \) in the end of any calculation. This is inspired by the method of regularizing a path integral by taking the time co-ordinate to have a slight imaginary component.

Now,
\[
C(|a - b|) = \langle S_a S_b \rangle - \langle S_a \rangle \langle S_b \rangle \\
= \int \sum_{\{S_i\}} \Pi_j dJ_j \ a(J_1, J_2, ...) S_a S_b \\
\times e^{iJ_1 (1 + i\epsilon \Gamma) \sum S_i + i \sum_m J_2^{m,n} (1 + i\epsilon \Gamma) \sum S_i + S_i + S_{i+m} + i \sum_{m,n} J_3^{m,n} (1 + i\epsilon \Gamma) \sum S_i + S_i + S_{i+m} S_{i+n}} + (\ast) - \langle S_a \rangle \langle S_b \rangle \\
= \int \sum_{\{S_i\}} \Pi_j dJ_j \ a(J_1, J_2, ...) Z(J_1, J_2, ...) \langle S_a S_b \rangle J_1, J_2, ... + (\ast) - \langle S_a \rangle \langle S_b \rangle \\
= \int \Pi_j dJ_j a(J_1, J_2, ...) Z(J_1, J_2, ...) C(a, b) J_1, J_2, J_3, \ldots \\
\text{function of position's } a \text{ and } b \\
\text{+ } \int \Pi_j dJ_j Z(J_1, J_2, ...) a(J_1, J_2, ...) \langle S_a \rangle J_1, J_2, J_3, \ldots \langle S_b \rangle J_1, J_2, J_3, \ldots + (\ast) \\
- \langle S_a \rangle \langle S_b \rangle, \\
\text{(13)}
\]

Where, we have defined

\[
Z(J_1, J_2, \ldots) = \sum_{\{S_i\}} e^{iJ_1 (1 + i\epsilon \Gamma) \sum S_i + i \sum_m J_2^{m,n} (1 + i\epsilon \Gamma) \sum S_i + S_i + S_{i+m} + i \sum_{m,n} J_3^{m,n} (1 + i\epsilon \Gamma) \sum S_i + S_i + S_{i+m} S_{i+n}} \\
\langle O(\{S_i\}) \rangle_{J_1, J_2, J_3, \ldots} = \frac{\sum_{\{S_i\}} O(\{S_i\}) e^{iJ_1 (1 + i\epsilon \Gamma) \sum S_i + i \sum_m J_2^{m,n} (1 + i\epsilon \Gamma) \sum S_i + S_i + S_{i+m} + i \sum_{m,n} J_3^{m,n} (1 + i\epsilon \Gamma) \sum S_i + S_i + S_{i+m} S_{i+n}}}{Z(J_1, J_2, \ldots)} \\
C(a, b) J_1, J_2, J_3, \ldots = \langle S_a S_b \rangle J_1, J_2, J_3, \ldots - \langle S_a \rangle J_1, J_2, J_3, \ldots \langle S_b \rangle J_1, J_2, J_3, \ldots, \\
\text{(14)}
\]

and \(O(\{S_i\})\) is any functional of the \(S_i\) configuration on the lattice. Since \(\langle S_a \rangle \), \(\langle S_a \rangle J_1, J_2, J_3, \ldots\), \(\langle S_b \rangle J_1, J_2, J_3, \ldots\) are independent of lattice sites \(a, b\) we have

\[
\int \Pi_j dJ_j a(J_1, J_2, ...) Z(J_1, J_2, ...) \langle S_a \rangle J_1, J_2, J_3, \ldots \langle S_b \rangle J_1, J_2, J_3, \ldots + (\ast) - \langle S_a \rangle \langle S_b \rangle = \text{Ind}(a, b), \\
\text{(15)}
\]

where, \(\text{Ind}(a, b)\) does not depend on \(a\) or \(b\).

Since \(C(|a - b|)\) is rotationally invariant, it is not possible for \(C(a, b) J_1, J_2, J_3, \ldots\) to be not rotationally invariant for all possible combinations of \(a\) and \(b\). The power of this statement cannot be understated. Rotational invariance of \(C(|a - b|)\), sets only those \(a(J_1, J_2, \ldots)\)’s to be non zero for which \(C(a, b) J_1, J_2, J_3, \ldots\) is rotationally invariant. Because of rotational invariance, we can write \(C(a, b) J_1, J_2, J_3, \ldots = C(|a - b|) J_1, J_2, J_3, \ldots\).

**Proposition 4:** If \(|a - b|\) is order’s of magnitude greater than the lattice spacing

\[
C(|a - b|) J_1, J_2, J_3, \ldots \sim \frac{e^{-\lambda_{J_1, J_2, J_3, \ldots} |a - b|}}{|a - b|^{\lambda_{J_1, J_2, J_3, \ldots}}}, \\
\text{(16)}
\]

with \(\lambda_{J_1, J_2, J_3, \ldots} \geq 0\).

Consider

\[
1 = N \int \Pi_j D\psi_j e^{-iJ_1 \sum_i \psi_i \sum_{i} J_2^{m,n} \sum_{i} \psi_i \psi_{i+m} - i \sum_{m,n} J_3^{m,n} \sum_{i} \psi_i \psi_{i+m} \psi_{i+n}} \\
\text{(17)}
\]
ψ ∈ [−∞, ∞] during integration. N is an normalization constant. Finiteness of the integral is obtained by an epsilon prescription as was done in Eq.11 and Eq.12. The J’s above have to be read as being multiplied by (1 + iα) (we do not write this multiplication explicitly to prevent the equations below from appearing too messy). Redefining ψ_i → ψ_i + S_i, we get

\[ 1 = N \int \Pi_i D\psi_i e^{-iJ_1 \sum_i \psi_i - iJ_2 \sum_i S_i + i \sum_{m,n} J_3^{m,n} \sum_i \psi_i \psi_{i+n} - i \sum_{m,n} J_3^{m,n} \sum_i \psi_i \psi_{i+n} - i \sum_{m,n} J_3^{m,n} \sum_i S_i S_{i+m} S_{i+n} \ldots } \]

\[ \times e^{-2i \sum_m J_2^{m} \sum_i S_i \psi_{i+m}} \times e^{-3i \sum_{m,n} J_3^{m,n} \sum_i \psi_i \psi_{i+n} - 3i \sum_{m,n} J_3^{m,n} \sum_i S_i S_{i+m} \psi_{i+n} \ldots } \]

or,

\[ e^{iJ_1 \sum_i S_i + i \sum_m J_2^{m} \sum_i S_i S_{i+m} + i \sum_{m,n} J_3^{m,n} \sum_i S_i S_{i+m} S_{i+n} \ldots } = N \int \Pi_i D\psi_i e^{-iJ_1 \sum_i \psi_i - i \sum_m J_2^{m} \sum_i \psi_i \psi_{i+m} - i \sum_{m,n} J_3^{m,n} \sum_i \psi_i \psi_{i+n} - i \sum_{m,n} J_3^{m,n} \sum_i S_i S_{i+m} \psi_{i+n} \ldots } \]

\[ \times e^{-2i \sum_m J_2^{m} \sum_i S_i \psi_{i+m}} \times e^{-3i \sum_{m,n} J_3^{m,n} \sum_i \psi_i \psi_{i+n} - 3i \sum_{m,n} J_3^{m,n} \sum_i S_i S_{i+m} \psi_{i+n} \ldots } \]

So,

\[ Z = \Pi_i(S_i) e^{iJ_1 \sum_i S_i + i \sum_m J_2^{m} \sum_i S_i S_{i+m} + i \sum_{m,n} J_3^{m,n} \sum_i S_i S_{i+m} S_{i+n} \ldots } = N \Pi_i(S_i) \int \Pi_i D\psi_i e^{-iJ_1 \sum_i \psi_i - i \sum_m J_2^{m} \sum_i \psi_i \psi_{i+m} - i \sum_{m,n} J_3^{m,n} \sum_i \psi_i \psi_{i+n} - i \sum_{m,n} J_3^{m,n} \sum_i S_i S_{i+m} \psi_{i+n} \ldots } \]

\[ \times e^{-2i \sum_m J_2^{m} \sum_i S_i \psi_{i+m}} \times e^{-3i \sum_{m,n} J_3^{m,n} \sum_i \psi_i \psi_{i+n} - 3i \sum_{m,n} J_3^{m,n} \sum_i S_i S_{i+m} \psi_{i+n} \ldots } \]

(20)

The RHS in the second equation after summing over all S_i’s can be written as \( e^{-iS(\psi, \psi_{i+1}, \ldots)} \), where S(\psi_1, \psi_{i+1}, \ldots) is a function of all values of \( \psi_i \)'s. Because every term in every exponent in the integral above is translation invariant S(\psi_1, \psi_{i+1}, \ldots) is translation invariant, in the continuum limit (considering to correspond to dimensions much larger than the lattice spacing, where the lattice appears as a continuum) we have

\[ S = \int d^d\psi [c_1 \nabla \psi \cdot \nabla \psi + c_2 \psi^2 + \sum_{m,n,p} c_{mnp} \psi^m (\nabla \psi \cdot \nabla \psi)^n \nabla^2 \psi], \]

(21)

The c’s are functions of the J’s. (Eq. 15 to Eq. 19 are inspired from Eq. 5.5 to Eq 5.6 from [12], where the transformation are done for partition functions). Rotational invariance of the expression above is necessary in ensure that C(|a − b| J_1, J_2, J_3) is rotationally invariant. The above form for S is set by including all possible rotationally invariant terms, which should be allowed in the continuum limit. It is well known that for a system described by a field theory, the correlation function at distances r orders of magnitude larger than the lattice spacing scales as

\[ \sim e^{-\lambda J_1, J_2, J_3 \cdot r} \]

(22)

The reason behind this is for large distances (infrared limit), higher derivative terms and higher powers of \( \psi \) contribute minimiscularly in any effective field theory [13]. Hence, the correlation function is just one that is obtained from an action \( \int d^d\psi [c_1 \nabla \psi \cdot \nabla \psi + c_2 \psi^2 + \ldots] \), where the ... are the relevant and marginal operators. The correlation function has a form as above.

This furnishes proof of \( C(|a − b| J_1, J_2, J_3) \sim e^{-\lambda J_1, J_2, J_3 \cdot |a−b|} \) at large |a − b|.

**Proposition 5:** \( Ind(a, b) = 0 \)
Proof:
Since our system is scale invariant \(C(|a - b|) = \frac{C}{|a-b|^{\alpha}}, \) where \(C\) is a constant and \(|a - b|\) is quite large

\[
\frac{C}{|a-b|^{\alpha}} = \int \Pi_j dJ_j a(J_1, J_2...) Z(J_1, J_2...) e^{-\lambda_{j_1, j_2, j_3, |a-b|}} |a-b|^{\alpha_{j_1, j_2, j_3}} + (*) + \text{Ind}(a, b).
\]

(23)

In the above we have absorbed the constant of proportionality in \(C(|a - b|)J_1, J_2, J_3... \sim e^{-\lambda_{j_1, j_2, j_3, |a-b|}}\) into \(a(J_1, J_2...)\).

Now, take \(|a - b| \to \infty\). Then, since first term on RHS is zero and LHS is zero, it implies \(\text{Ind}(a, b) = 0\) since \(\text{Ind}(a, b)\) does not depend on either \(a\) or \(b\).

**Proposition 6:**
\(\alpha_{j_1, j_2, j_3} = \alpha, \) if \(a(J_1, J_2...) \neq 0\).

Proof: Differentiate both sides of Eq. (23) with respect to \(a\). This would give

\[
-\alpha C = \int \Pi_j dJ_j a(J_1, J_2...) Z(J_1, J_2...) \frac{-\alpha_{J_1, J_2, J_3} e^{-\lambda_{j_1, j_2, j_3, |a-b|}}}{|a-b|^{\alpha_{j_1, j_2, j_3} + 1}} |a-b|^{\alpha_{j_1, j_2, j_3}} + (*)
\]

- \(\frac{\alpha}{|a-b|^{\alpha + 1}}\)

(24)

To see the last step, equalize the RHS of the first and second line of the above equation to the RHS of Eq. (23) after multiplying on both sides by \(\frac{\alpha}{|a-b|}\). Consistency for all possible values of \(|a - b|\) in the last equation (which are still large enough), requires \(\int \Pi_j dJ_j a(...J_1, J_{i+1}...) Z(J_1, J_2...) \frac{\lambda_{j_1, j_2, j_3} e^{-\lambda_{j_1, j_2, j_3, |a-b|}}}{|a-b|^{\alpha_{j_1, j_2, j_3} + 1}} + (*) = 0\), as it dominates over \(\int \Pi_j dJ_j a(J_1, J_2...) Z(J_1, J_2...) \frac{-\alpha_{J_1, J_2, J_3} e^{-\lambda_{j_1, j_2, j_3, |a-b|}}}{|a-b|^{\alpha_{j_1, j_2, j_3} + 1}} + (*)\).

Hence we get

\[
-\alpha C = \int \Pi_j dJ_j a(J_1, J_2...) Z(J_1, J_2...) \frac{-\alpha_{J_1, J_2, J_3} e^{-\lambda_{j_1, j_2, j_3, |a-b|}}}{|a-b|^{\alpha_{j_1, j_2, j_3} + 1}} + (*)
\]

(25)

Differentiating again with respect to \(a\) similarly gives

\[
\frac{\alpha(\alpha + 1)}{|a-b|^{\alpha + 2}} C = \int \Pi_j dJ_j a(J_1, J_2...) Z(J_1, J_2...) \frac{\alpha_{J_1, J_2, J_3} e^{-\lambda_{j_1, j_2, j_3, |a-b|}}}{|a-b|^{\alpha_{j_1, j_2, j_3} + 2}}
\]

+ \(\int \Pi_j dJ_j a(J_1, J_2...) Z(J_1, J_2...) \frac{\lambda_{j_1, j_2, j_3} e^{-\lambda_{j_1, j_2, j_3, |a-b|}}}{|a-b|^{\alpha_{j_1, j_2, j_3} + 1}} + (*)\)

\[
= \int \Pi_j dJ_j a(J_1, J_2...) Z(J_1, J_2...) \frac{\alpha_{J_1, J_2, J_3} e^{-\lambda_{j_1, j_2, j_3, |a-b|}}}{|a-b|^{\alpha_{j_1, j_2, j_3} + 2}},
\]

(26)

where last line is because of multiplying Eq. (23) by \(\alpha(\alpha + 1)\). We hence get by arguments similar to ones below Eq. (24).
\[ \int \Pi_J dJ a(J_1, J_2...) Z(J_1, J_2...) \frac{\alpha_{J_1, J_2, J_3} (\alpha_{J_1, J_2, J_3} + 1) e^{-\lambda_{J_1, J_2, J_3} |a-b|}}{|a-b|^{\alpha_{J_1, J_2, J_3} + 1}} = \int \Pi_J dJ a(J_1, J_2...) Z(J_1, J_2...) \frac{\alpha (\alpha + 1) e^{-\lambda_{J_1, J_2, J_3} |a-b|}}{|a-b|^{\alpha_{J_1, J_2, J_3} + 1}} + (*) \]

One can keep taking derivatives to get

\[ \frac{P(\alpha, n)C}{|a-b|^{\alpha + n}} = \int \Pi_J dJ a(J_1, J_2...) Z(J_1, J_2...) \frac{P(\alpha_{J_1, J_2, J_3}, n) e^{-\lambda_{J_1, J_2, J_3} |a-b|}}{|a-b|^{\alpha_{J_1, J_2, J_3} + n}}
+ \int \Pi_J dJ a(...J_1, J_{1+1}...) Z(J_1, J_2...) \frac{\lambda_{J_1, J_2, J_3} P(\alpha, n-1) e^{-\lambda_{J_1, J_2, J_3} |a-b|}}{|a-b|^{\alpha_{J_1, J_2, J_3} + n-1}} + (*) \]

\[ = \int \Pi_J dJ a(J_1, J_2...) Z(J_1, J_2...) \frac{P(\alpha_{J_1, J_2, J_3}, n) e^{-\lambda_{J_1, J_2, J_3} |a-b|}}{|a-b|^{\alpha_{J_1, J_2, J_3} + n}} + (*) \]

and hence

\[ \int \Pi_J dJ a(J_1, J_2...) Z(J_1, J_2...) \frac{P(\alpha_{J_1, J_2, J_3}, n) e^{-\lambda_{J_1, J_2, J_3} |a-b|}}{|a-b|^{\alpha_{J_1, J_2, J_3} + n}} + (*) \]

where \(P(x, n) = x(x + 1)(x + n - 1)\) for \(n > 0\) and \(P(x, n = 0) = 1\) and \(P(x, n = -1) = 0\). Only way the above equation is possible for all large values of \(|a-b|\) and all \(n > 0\) is if \(\alpha_{J_1, J_2, J_3} = \alpha\) for \(a(...J_1, J_{1+1}...) \neq 0\).

**Proposition 7:**

Atleast one \(\lambda_{J_1, J_2, J_3} = 0\), for \(a(...J_1, J_{1+1}...) \neq 0\).

**Proof:** From Proposition 6, we get that

\[ \int \Pi_J dJ a(J_1, J_2...) Z(J_1, J_2...) e^{-\lambda_{J_1, J_2, J_3} |a-b|} + (*) = C \]

where \(C\) is constant for all values of \(|a-b|\). If \(|a-b| \to \infty\), this constant is equal to zero in case all of the \(\lambda_{J_1, J_2, J_3} \neq 0\) as all \(\lambda_{J_1, J_2, J_3} > 0\). Hence atleast one \(\lambda_{J_1, J_2, J_3} = 0\), for \(a(...J_1, J_{1+1}...) \neq 0\), if we have to have a non zero value for \(C\). Having a particular \(\lambda_{J_1, J_2, J_3} = 0\), implies we are really talking about the critical point of a field theory with an action of the form Eq[21]. Hence, the critical exponent \(\alpha\) is the scaling exponent for such a field theory. At criticality such models fall into universality classes, with models in each universality class having the same critical exponent. This completes our proof that the scaling critical exponent in the lattice model described by an arbitrary probability distribution will match the scaling exponent of a statistical mechanical model in a universality class. Implicit in this statement is the observation that the scaling exponent of the correlation function in a quantum field theory is same as in a corresponding statistical field theory at the fixed point.

**Discussion**

We note that what is said above is only confirmed for systems defined on a lattice, where the variable defining the state of the system at a lattice site takes the same possible values on every lattice site. These values are also finite. The
result that the way the correlations decay in systems having long range order described by any arbitrary probability distributions, are similar to statistical models on a lattice at criticality is a new addition to the understanding of universality classes in physics. Critical nature has been hypothesized in non statistical mechanical systems going from retinal neurons [8], to natural images [14], which could be represented as systems on a lattice. Our result above says that irrespective of the actual probability distributions describing the statistics of these systems, the scaling exponents if measured would fall into universality classes of statistical mechanical models. The result in the paper could also aid in the working out analytically the critical exponents of of statistical mechanical models at criticality, by considering appropriate alternative probability distributions, instead of the Boltzmann distribution in which the calculation is tedious.

**Additional Information**

There are no competing interests.

**Data Availability**

Data sharing not applicable – no new data generated.

[1] Khaluf, Yara, et al. "Scale invariance in natural and artificial collective systems: a review." Journal of the royal society interface 14.136 (2017): 20170662.
[2] Schröder, Malte, Wei Chen, and Jan Nagler. "Discrete scale invariance in supercritical percolation." New Journal of Physics 18.1 (2016): 013042.
[3] Khatami, Fatemeh, et al. "Origins of scale invariance in vocalization sequences and speech." PLoS computational biology 14.4 (2018): e1005996.
[4] Lin, Hai-Nan, and Yu Sang. "Scale-invariance in the repeating fast radio burst 121102." Monthly Notices of the Royal Astronomical Society 491.2 (2020): 2156-2161.
[5] Djenidi, L., R. A. Antonia, and S. L. Tang. "Scale invariance in finite Reynolds number homogeneous isotropic turbulence." Journal of Fluid Mechanics 864 (2019): 244-272.
[6] Coutandin, Thomas, and Carlo Albert. "Scaling Laws in River Runoff." EGUGA (2018): 7707.
[7] Jhun, Jennifer, Patricia Palacios, and James Owen Weatherall. "Market crashes as critical phenomena? Explanation, idealization, and universality in econophysics." Synthese 195.10 (2018): 4477-4505.
[8] Mora, T., & Bialek, W. (2011). Are biological systems poised at criticality?. Journal of Statistical Physics, 144(2), 268-302.
[9] Chaikin, Paul M., Tom C. Lubensky, and Thomas A. Witten. Principles of condensed matter physics. Vol. 10. Cambridge: Cambridge university press, 1995.
[10] Cardy, J. L. (1988). Conformal invariance and statistical mechanics. Les Houches.
[11] Poland, David, and David Simmons-Duffin. "The conformal bootstrap." Nature Physics 12.6 (2016): 535-539.
[12] Altland, A., Simons, B. D. (2010). Condensed matter field theory. Cambridge university press pg 196-199.
[13] Burgess, Cliff P. "An introduction to effective field theory." Annu. Rev. Nucl. Part. Sci. 57 (2007): 329-362.
[14] Stephens G J, Mora T, Tkacik G and Bialek W, 2008 [arXiv:0806.2694 [q-bio.NC]]