Pseudoscalar bosonic excitations in the color-flavor locked phase at moderate densities

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The properties of pseudoscalar bosonic excitations in the color-flavor locked phase at moderate densities are studied within a model of the Nambu–Jona-Lasinio type. Our previous analysis [1] is extended to Goldstone bosons with hidden flavor and to higher-lying modes which stay massive in the chiral limit. The bosons are constructed explicitly by solving the Bethe-Salpeter equation for quark-quark scattering in random phase approximation. The masses and weak decay constants of the Goldstone bosons are found in good agreement with predictions from the low-energy effective theory. In the non-Goldstone sector we find an SU(3) octet which is weakly bound, while the singlet appears to be unbound.

I. INTRODUCTION

The ground state of quark matter at asymptotically high baryon densities is the color-flavor locked (CFL) phase [2]. In this phase up, down, and strange quarks are paired in a particularly symmetric way. As all quark flavors and colors participate in a condensate, all fermionic modes are gapped and do not appear in the low-energy excitation spectrum. Spontaneous breaking of baryon number and chiral symmetry leads to the emergence of one scalar and eight pseudoscalar Goldstone bosons. A ninth pseudoscalar Goldstone boson is expected due to the spontaneous breaking of the $U_A(1)$ symmetry, which is a symmetry of QCD at high density [3, 4]. These Goldstone bosons are the lowest lying excitations and should play an important role for the thermodynamics of strongly interacting matter at high density. In nature, this could have phenomenological consequences for compact star physics [5-8].

The pseudoscalar Goldstone bosons in the CFL phase have been studied extensively within the low-energy effective theory (LEET) which is based on the symmetry breaking pattern $\eta^0, \eta, \eta^\prime$. At very high densities, the corresponding low-energy constants can be calculated from QCD within high density effective theory (HDET) [13, 16, 17, 18, 19]. The Goldstone bosons are the lowest lying excitations and should play an important role for the thermodynamics of strongly interacting matter at high density. In nature, this could have phenomenological consequences for compact star physics [3, 6, 7, 8].

The properties of pseudoscalar bosonic excitations in the CFL phase have been studied extensively within the low-energy effective theory (LEET) which is based on the symmetry breaking pattern $\eta^0, \eta, \eta^\prime$. At very high densities, the corresponding low-energy constants can be calculated from QCD within high density effective theory (HDET) [13, 16, 17, 18, 19]. The bosons form an SU(3) octet and a singlet and there is a one-to-one correspondence to the well-known octet and singlet of pseudoscalar mesons in vacuum. They are therefore often called “mesons” as well and can be identified with pions, kaons, $\eta$, and $\eta^\prime$, according to their quantum numbers. An interesting prediction made within the LEET is that the strange quark mass acts as an effective strangeness chemical potential, eventually leading to kaon condensation in the CFL phase [11, 13, 14].

So far only few authors have investigated the CFL mesons starting from quark degrees of freedom [1, 20, 21, 22, 23]. In Ref. [1] we have studied meson masses and decay constants as well as the onset of kaon condensation within an NJL-type model at moderate densities. In the numerical part, we have restricted ourselves to open-flavor Goldstone bosons, i.e., kaons and charged pions, which are technically easier to describe. Moreover, this is the sector where meson condensation can occur. Our results are consistent with the LEET analysis, but we found quantitative differences from the weak-coupling limit of QCD.

One focus of the present article is to extend these studies to the hidden-flavor sector, i.e., $\pi^0$, $\eta$, and $\eta^\prime$. Again, we will compare our results with the predictions of the LEET.

In spite of the formal correspondence of the pseudoscalar Goldstone bosons in the CFL phase and in vacuum, their physical nature is very different. In fact the “mesons” in the CFL phase are mainly superpositions of diquark and di-hole states, rather than quark-antiquark states. This is possible because baryon number is not a good quantum number in the CFL phase. In Ref. [1] we argued that there is a second octet and a second singlet of pseudoscalar bosons, which come about as superpositions of diquark and di-hole states orthogonal to the Goldstone modes. These higher-lying excitations have been investigated in Ref. [20], but only in the chiral limit. In the present paper, we study their properties for equal and non-equal quark masses.

This article is organized as follows. In Sec. II we introduce our model and briefly summarize the formalism developed in Ref. [1]. After that we present our results for the Goldstone bosons in Sec. III and for the higher-lying modes in Sec. IV. We conclude with a short summary in Sec. V.

II. FORMALISM

In this section we briefly summarize our NJL model for pseudoscalar mesons in the CFL phase. Further details can be found in Ref. [1].

We consider the Lagrangian

$$\mathcal{L} = \bar{q}(i\gamma - \hat{m})q + \mathcal{L}_{qq},$$

where $q$ is a quark field with three flavor and three color degrees of freedom, $\hat{m} = \text{diag}(m_u, m_d, m_s)$ is the mass
matrix, and

\[
\mathcal{L}_{qq} = H \sum_{A,A'=2,5,7} \left[ (\bar{q}i\gamma_5\tau_A\lambda_A'Cq^T)(q^T C i\gamma_5\tau_A\lambda_A'q) + (\bar{q}\tau_A\lambda_A'Cq^T)(q^T C\tau_A\lambda_A'q) \right] 
\]

(2)
describes an SU(3)\text{\textsubscript{color}} × U(3)\text{\textsubscript{L}} × U(3)\text{\textsubscript{R}} symmetric four-point interaction with a dimensionful coupling constant \( C = i\gamma^0 \) is the matrix of charge conjugation, and \( \tau \) and \( \lambda \) denote Gell-Mann matrices acting in flavor and color space, respectively. In this article, we follow the convention that the indices \( A \) and \( A' \) are used for the antisymmetric Gell-Mann matrices only, i.e., \( A, A' \in \{2, 5, 7\} \).

Working in Nambu-Gorkov formalism, the interaction Lagrangian gives rise to the quark-quark scattering kernel

\[
\hat{K} = \Gamma_i K_{ij} \Gamma_j , \quad K_{ij} = 4H \delta_{ij} , 
\]

(3)
where \( \Gamma_i = \gamma_0 \Gamma_i \gamma_0 \) with 18 scalar operators

\[
\Gamma_{AA'}^{1} = \begin{pmatrix} 0 & i\gamma_5\tau_A\lambda_A' \\ 0 & 0 \end{pmatrix} , \quad \Gamma_{AA'}^{\dagger} = \begin{pmatrix} 0 & \tau_A\lambda_A' \\ 0 & 0 \end{pmatrix} , 
\]

(4)
and 18 pseudoscalar operators

\[
\Gamma_{AA'}^{p} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} , \quad \Gamma_{AA'}^{p\dagger} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} . 
\]

(5)
In the CFL phase, the inverse dressed quark propagator in Nambu-Gorkov space is given by

\[
S^{-1}(p) = \left( \begin{array}{cc} \bar{\psi} + \mu \gamma^0 - \bar{m} & \sum_{A=2,5,7} \Delta^*_A \gamma_5\tau_A\lambda_A \\ -\sum_{A=2,5,7} \Delta_A \gamma_5\tau_A\lambda_A & \bar{\psi} - \mu \gamma^0 - \bar{m} \end{array} \right) . 
\]

(6)
The gap parameters \( \Delta_A \), which enter the anomalous components, are obtained through minimizing the mean-field thermodynamic potential \( \Omega \), leading to three gap equations, \( \frac{\partial \Omega}{\partial \Delta_A} = 0 \). In addition, we require electric and color neutrality, which fixes the electric and color chemical potentials at given temperature and quark number chemical potential.

In the following we only consider the (fully gapped) CFL phase at zero temperature in the isospin symmetric limit, \( m_u = m_d \). In this case, we only need a nonzero color chemical potential \( \mu \) to ensure neutrality.

We calculate the mesonic excitations by solving the RPA equation for the \( T \)-matrix in Nambu-Gorkov space,

\[
T(q) = K + KJ(q)T(q) = [1 - KJ(q)]^{-1}K , 
\]

(7)
where the matrix \( K \) is related to the scattering kernel, see Eq. (4). The elements of the one-loop polarization matrix \( J(q) \) are given by

\[
J_{ij}(q) = i \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left[ \Gamma_i S(k + q) \Gamma_j S(k) \right] , 
\]

(8)
where we have introduced a “vacuum-like” notation for brevity. In medium the zero components of \( q \) and \( k \) should be replaced by bosonic and fermionic Matsubara frequencies, respectively, and we should replace \( i \int \frac{dk^0}{2\pi} \) by the Matsubara sum \(-T \sum_n\). The remaining three-momentum integral is divergent and will be regularized by a sharp cutoff \( \Lambda \).

The matrices \( T \) and \( J \) are 36 × 36 matrices in the space of the operators given in Eq. (4) and (5). They are block diagonal because scalar and pseudoscalar operators do not mix with each other. The two \( 18 \times 18 \) blocks can be decomposed further, each into six \( 2 \times 2 \) blocks and one \( 6 \times 6 \) block. These blocks can be attributed to different flavor quantum numbers: The \( 2 \times 2 \) blocks correspond to the open-flavor mesons (i.e., in the pseudoscalar sector, \( \pi^+, \pi^-, K^+, K^-, K^0, \bar{K}^0 \)), while the \( 6 \times 6 \) block contains the hidden-flavor mesons (\( \pi^0, \eta_8, \eta_0 \)). Here \( \eta_8 \) and \( \eta_0 \) are the SU(3) octet and singlet states. As in vacuum they mix to the physical states \( \eta \) and \( \eta' \) when flavor SU(3) is broken explicitly.

By diagonalizing the blocks we get one Goldstone mode and one massive excitation for each meson. In the vicinity of their poles, we can parameterize each mode as a free boson with mass \( m_M \) in the presence of a boson chemical potential \( \mu_M \). In the following, we restrict ourselves to the meson rest frame, \( q = (q_0, 0) \). Then the parameterization reads

\[
T^{(M)}(q_0) \approx -\frac{g_M^2}{(q_0 + \mu_M)^2 - m_M^2} . 
\]

(9)
The constant \( g_M \) is a wave function renormalization constant, which can be interpreted as a coupling constant of the boson to an external quark.

The (time-like) weak decay constant \( f_M \) of the pseudoscalar meson \( M \) is given by the expression

\[
f_M = \frac{1}{q_0} \int \frac{d^4p}{(2\pi)^4} \frac{1}{2} \text{Tr} \left[ \tilde{A}_M S(p + q) g_M \Gamma_M S(p) \right] |_{q_0 = m_M} . 
\]

(10)
Here \( \Gamma_M \) denotes the meson-quark-quark vertex of the eigenmode \( M \) resulting from the diagonalization discussed above. \( \tilde{A}_M^0 \) is the vertex of an external axial current in the corresponding flavor channel.

In this paper we evaluate the decay constants in the flavor-SU(3) limit only. We then only need the singlet channel, i.e., the \( \eta^0 \) and one representative of the octet, e.g., the \( \pi^0 \). The corresponding vertices of the external axial currents read

\[
\tilde{A}_{\eta^0}^0 = \gamma^0 \gamma_5 \frac{\tau_0}{4} \mathbb{I}_{NG} 
\]

(11)
and

\[
\tilde{A}_{\pi^0}^0 = \gamma^0 \gamma_5 \frac{\tau_1}{2} \mathbb{I}_{NG} 
\]

(12)
where \( \mathbb{I}_{NG} \) indicates a unit matrix in Nambu-Gorkov space. \( \tau_0 = \sqrt{\frac{3}{4}} \mathbb{I}_f \) is proportional to the unit matrix in
flavor space.¹

III. GOLDSTONE BOSONS

In this section we discuss our numerical results for the pseudoscalar Goldstone bosons. The results for the higher-lying pseudoscalar excitations will be presented in Sec. IV.

We restrict ourselves to $T = 0$ and a fixed quark number chemical potential $\mu = 500$ MeV. We choose a three-momentum cutoff $\Lambda = 600$ MeV and, if not otherwise stated, a diquark coupling $H = 1.4 \Lambda^{-2}$. For these parameters we have $\Delta = 79.1$ MeV in the chiral limit and we are in the fully gapped CFL phase for all values of $m_s$ we consider.

A. Weak decay constants in the chiral limit

We begin with the weak decay constants in the chiral limit, $m_u = m_d = m_s = 0$. In the chiral limit all nine pseudoscalar Goldstone bosons are exactly massless. However, since the CFL ground state is symmetric under $SU(3)$ symmetry, but not under $U(1)$, we have to distinguish between the decay constants in the octet (“$f_\pi$”) and in the singlet (“$f_\eta$”).

Our numerical results are shown in Fig. 1. For technical reasons the calculations have been performed with $m_u = m_d = m_s = 0.1$ MeV, but the difference to the exact chiral limit is negligible. In the figure, $f_\pi$ and $f_\eta$ are displayed as functions of the gap parameter $\Delta$, which was varied by varying the coupling constant $H$. The numerical points are indicated by the points, which have been connected by straight lines to guide the eye. The curve for the octet decay constant $f_\pi$ was already shown in Ref. [1]. There we restricted ourselves to mesons with open flavor, but the decay constants of the hidden-flavor octet mesons, $\pi^0$ and $\eta$, are of course the same. The decay constant in the singlet channel, $f_\eta$, was not calculated in Ref. [1]. As one can see in the figure, it is a few percent larger than $f_\pi$. This is well understood in the weak-coupling limit, where $f_\pi$ and $f_\eta$ are given by

$$f_\pi^2 = \frac{21 - 8 \ln(2)}{18} \left( \frac{\mu^2}{2\pi^2} \right) \quad \text{and} \quad f_\eta^2 = \frac{3}{4} \left( \frac{\mu^2}{2\pi^2} \right).$$

These limiting values are marked by the thin horizontal lines in Fig. 1. Indeed, for $\Delta \to 0$ our results approach the weak-coupling limit, both, in the singlet and in the octet channel. For $f_\eta$, this was confirmed in a more careful analysis in Ref. [12], where a semi-analytical formula for the decay constant was derived. An analogous analysis could be done for $f_\eta$ as well.

For larger values of $\Delta$, we find significant deviations from Eq. (13). This is not surprising as the weak-coupling assumption is not valid in that region.

B. Meson masses

Next, we discuss the masses of the (pseudo) Goldstone bosons for non-vanishing quark masses. As pointed out earlier, our focus is on the hidden-flavor sector. To obtain a complete picture, however, we will also include our results for the open-flavor mesons, which have been discussed already in great detail in Ref. [1].

We begin with the case of non-zero, but equal quark masses $m_u = m_d = m_s \equiv m_q$. Our results are shown in Fig. 2, where the numerical calculations are indicated by the points. Since the $SU(3)$ symmetry is still intact, the octet remains degenerate. This means, $\pi^0$ and $\eta \equiv \eta_8$ have the same mass $m_q$ as the flavored mesons discussed in Ref. [1]. The $\eta' \equiv \eta_0$, on the other hand, forms a singlet and has a slightly higher mass for $m_q \neq 0$.

To very high accuracy, both, the octet and the singlet masses, grow linearly with the quark mass. This is expected from the LEET, where, to leading order in the quark mass, the octet $[13]$ and the singlet $[24]$ masses are given by

$$m_\pi = \sqrt{\frac{8A}{f_\pi^2} m_q} \quad \text{and} \quad m_\eta' = \sqrt{\frac{8A}{f_\eta^2} m_q}, \quad (14)$$

respectively. Whereas this form is universal, the constant $A$, which determines the coefficient, depends on the interaction. For QCD in the weak-coupling limit it has

¹ The exact normalization of the generators in flavor space is a matter of convention. The normalization conventions in Eqs. (11) and (12) have been chosen to be consistent with the standard definitions of the weak decay constants in the LEET approach. Note that the generator in the singlet channel, $t_0 = \frac{A}{2}$, is a factor of two smaller than introduced in our previous paper [1].
be calculated in Ref. [9]. In Ref. [1] we derived an analytical expression for $A$ for the present NJL model (see Eq. (E1) in [1]). Inserting that formula into Eq. (14) and using the chiral limit values of $f_\pi$ and $f_\eta'$ from the previous subsection, we obtain the results which are indicated by the lines in Fig. 2. Obviously, they are in excellent agreement with the numerical calculations.

Next, we study the effect of unequal quark masses. We choose a fixed mass of 30 MeV for $m_u$ and $m_d$ and vary the strange quark mass. The results are shown in Fig. 3. Since the hidden-flavor mesons are not sensitive to the effective strangeness chemical potential induced by $m_s - m_u$, their masses are directly given by the poles of the $T$-matrix. For comparison we also display the kaon mass in the figure, which has already been shown in Ref. [1].

Our numerical results are indicated by the points. Because of the explicit breaking of the $SU(3)$-flavor symmetry, the octet is no longer degenerate. On the other hand, since we kept $m_u = m_d = m_q$, isospin is still an exact symmetry of the Lagrangian. Therefore, the $\pi^0$ has the same mass as the charged pions already discussed in Ref. [1].

The masses of $\pi$, $K$ and $\eta$ exhibit the “inverse ordering” already predicted in Ref. [9], with the pion being the heaviest and the $\eta$ being the lightest among these excitations. In fact, the mass of the $\eta$ meson barely changes with increasing strange quark mass. The $\eta'$, on the other hand, drops out of the inverse ordering scheme and is slightly heavier than the pion.

To understand this behavior, we again compare our results with the LEET predictions. For $\pi$ and $K$ these are given by [9, 13, 14, 24]

$$m_\pi^2 = \frac{8A}{f_\pi^2}m_q$$
$$m_K^2 = \frac{4A}{f_\pi^2}m_q(m_q + m_s).$$  

(15)

For $\eta$ and $\eta'$ the situation is more complicated because of the mixing of the octet with the singlet for $m_s \neq m_q$. To that end we have to diagonalize the mass matrix

$$M^2 = \begin{pmatrix} m_{\eta_0}^2 & m_{\pi_0}^2 \\ m_{\pi_0}^2 & m_{\eta_0}^2 \end{pmatrix},$$

(16)

where

$$m_{\eta_0}^2 = \frac{8A}{3f_{\eta'}^2}m_q(2m_q + m_s), \quad m_{\eta_0}^2 = \frac{8A}{3f_{\eta'}^2}m_q(m_s + 2m_q),$$

(17)

are the singlet and octet masses, respectively, and

$$m_{mix}^2 = \frac{8\sqrt{2}A}{3f_{\eta'}f_{\pi}}m_q(m_s - m_q)$$

(18)

describes the mixing [24].

Inserting the values for $f_{\pi}$, $f_{\eta'}$, and $A$ used in the equal mass case into Eqs. (15), (17), and (18), and diagonalizing Eq. (16), we obtain the mass eigenvalues which are indicated by the lines in Fig. 3. Obviously, the agreement with the numerical NJL results is very good, with small deviations only at higher values of $m_s$.

To get more compact expressions for the $\eta$ and $\eta'$ masses, we can make use of the fact that the difference between $f_{\pi}$ and $f_{\eta'}$ is small. We may thus write

$$f_{\eta'}^2 = f_{\pi}^2 - \delta f^2$$

(19)

and expand the eigenvalues of the mass matrix Eq. (16) until the order $\delta f^2$. One finds

$$m_{\eta_0}^2 = \frac{8A}{f_{\pi}^2}(1 + \frac{1}{3}\delta f^2) m_q^2$$

(20)

and

$$m_{\eta_0}^2 = \frac{8A}{f_{\pi}^2}(1 + \frac{2}{3}\delta f^2) m_q m_s = (1 + \frac{2}{3}\delta f^2) m_s^2.$$

(21)

This explains why the $\eta$ mass stays (approximately) constant with $m_s$, whereas the $\eta'$ behaves very similar to the pion, but scaled by a constant factor.
IV. HIGHER-LYING EXCITATIONS

As explained in Sec. II, besides the Goldstone modes, there also exist an octet and a singlet of higher-lying pseudoscalar excitations, which stay massive even in the chiral limit. In this section, we analyze the masses and decay constants of these mesons.

A. Masses

In Ref. [20] the higher-lying pseudoscalar modes in the CFL phase have been studied in the chiral limit within a similar model. The authors did not find any singlet solutions, whereas for the octet they report the existence of resonance states above $2\Delta$, i.e., the threshold for decay into two quasiparticles. These solutions were identified as poles on the second Riemann sheet in the complex energy plane.

The model of Ref. [20] is practically the same as ours. It was thus to our surprise that we found bound-state solutions in the higher-lying octet. This is shown in Fig. 4. For a direct comparison with Ref. [20], we used the same parameters ($\Lambda = 602.3$ MeV and $H = 1.73925 \Lambda^{-2}$) and calculated the octet mass as a function of the quark number chemical potential $\mu$ (solid line). In the chosen interval, $360$ MeV < $\mu$ < $500$ MeV, the masses of the excitations vary between $189$ and $223$ MeV. We also show the decay-threshold $2\Delta$ (dotted line). As one can see, the meson masses closely follow this line, but always stay below. This means that the octet modes are bound in the entire interval.

![FIG. 4: The higher-lying pseudoscalar octet mode in the chiral limit as a function of the quark chemical potential $\mu$. The parameters are $H = 1.73925 \Lambda^{-2}$, $\Lambda = 602.3$ MeV [21]. The dotted line indicates the decay threshold $2\Delta$.](image)

For technical reasons, the chiral limit was again approximated in our code by taking very small quark masses, $m_u = m_d = m_s = 0.1$ MeV. On the other hand, the authors of Ref. [20] restricted themselves to the chiral limit from the beginning. As a consequence they were able to derive a somewhat simpler equation for the meson masses, see Eq. (33) in [20]. We checked that this equation has indeed bound-state solutions, which are in excellent agreement with the results shown in Fig. 4. We are therefore convinced that our results (as well as Eq. (33) in [20]) are correct.

We would like to point out that it is not excluded that there are several branches of solutions. This means, we cannot exclude that the resonance-state solutions found in Ref. [20] are correct as well. Unfortunately, the extension of our method to evaluate the polarization integral Eq. (8) above threshold would require additional effort, which is beyond the scope of the present paper. Our analysis is therefore restricted to the regime below threshold. For the same reason we could not determine the mass of the higher-lying excitation in the singlet channel, which is always above threshold.

We now return to our standard parameters ($\mu = 500$ MeV, $\Lambda = 600$ MeV, $H = 1.4 \Lambda^{-2}$) and discuss the effect of finite quark masses. As before, we begin with the simplified case of equal quark masses, $m_u = m_d = m_s \equiv m_q$. In Fig. 5 we show the masses of the two octets (Goldstone bosons and higher-lying excitations) as functions of $m_q$. While the masses of the Goldstone

![FIG. 5: Masses of the pseudoscalar octet excitations as functions of a common quark mass $m_q$: higher-lying modes in comparison with the Goldstone modes. The dotted line indicates the decay threshold $2\Delta$.](image)

2 The Lagrangian used in Ref. [20] contains an additional quark-antiquark interaction. However, in the CFL phase in the chiral limit this term does not contribute to the modes we discuss here.

3 The behavior of the $T$-matrix below threshold seems to indicate that the mass of the singlet meson is rather close to the threshold as well. However, because of threshold effects a quantitative estimate is difficult.
bosons increase linearly with a sizeable slope, the masses of the higher-lying excitations stay nearly constant. In fact, they stay again very close to the decay threshold $2\Delta$ (dotted line). With increasing quark mass they approach this threshold and the mesons become eventually unbound at $m_q \approx 190$ MeV. At this point our curve terminates, because, as mentioned above, our method does not allow to find solutions above threshold.

Next, we study the effect of an explicit breaking of the SU(3)-flavor symmetry on the higher-lying excitations. Again, we choose $m_u = m_d = 30$ MeV and vary the strange quark mass $m_s$.

As in the case of equal quark masses, we only find higher-lying excitations for the octet modes. The positions of the poles are plotted in Fig. 6. The poles of the pions and the $\eta$ stay nearly constant around 155 MeV whereas the pole of the antikaons ($\bar{K}^-$ and $\bar{K}^0$) grows from 156 MeV for a quark mass of 30 MeV to 177 MeV for a quark mass of 140 MeV. For the kaons ($K^+$ and $K^0$) the situation is reversed, the position of their pole moves from 156 MeV to 136 MeV.

It should be noted that in the case of unequal quark masses the threshold for decay into two quasiparticles is no longer equal to $2\Delta$. In fact, in this case the fermionic excitation spectrum contains five different particle branches (two singlets, two doublets and one triplet) with different excitation gaps, see, e.g. Ref. [25]. Thus, the decay threshold for the different meson modes depends in a complicated way on their respective quasiparticle composition and rises with $m_s$ in some channels, while it drops in others. It turns out that the splitting of the various mesonic modes closely follows the splitting of the decay thresholds.

In Fig. 6 we only show the pole positions $\omega^+_q$ at positive values of $q_0$. In addition, each mode has another pole $\omega^-_q$ at negative energies, which is not shown in the figure. Applying Eq. (9) we can calculate the masses and effective chemical potential for each meson as

$$m_M = \frac{1}{2}(\omega^+_M - \omega^-_M), \quad \mu_M = -\frac{1}{2}(\omega^+_M + \omega^-_M).$$

The results are plotted in Fig. 7. In the upper part of the figure the masses are shown. They first slightly increase and then decrease with $m_s$. However, this happens on a very small scale from 156.4 to 155.2 MeV, i.e., unlike the Goldstone bosons, they stay nearly constant. On this scale we can also see that the $\eta$ and the pions are not degenerate. (This was already the case for the poles shown in Fig. 4 but hardly visible.)

In the lower part of Fig. 7 we show the effective meson chemical potentials. Our numerical results are indicated by the points. The chemical potentials vanish identically in the pion and $\eta$ channels, whereas they are negative for kaons and positive for antikaons, with equal absolute values.

Of course, this was to be expected: Since the “effective meson chemical potentials” are induced by the mass differences of different quark flavors, and since we kept $m_u = m_d$, there is only an effective strangeness chemical potential, but no isospin chemical potential. Hence, pions and $\eta$ remain unaffected, while kaons and antikaons feel opposite chemical potentials. In fact, the effective meson chemical potentials should only depend on the flavor content of the mesons. This means, there should be no difference between Goldstone bosons and higher-lying modes. Therefore, we can compare our numerical results with the LEET predictions for the Goldstone bosons [14],

$$\mu_{\pi^\pm} = 0, \quad \mu_{K^\pm} = \mu_{K^0,\bar{K}^0} = \pm \frac{m_s^2 - m_q^2}{2\mu}.$$  

These functions are indicated by the lines in Fig. 7. Obviously, they nicely fit the numerical results.

### B. Decay constants

For completeness, we briefly discuss the decay constants of the higher-lying octet modes. Utilizing chiral Ward-Takahashi identities one can show that either the masses or the decay constants of the pseudoscalar excitations vanish in the chiral limit (see Ref. [1] for details in the context of the present model).

In Fig. 8 we have plotted the decay constant of the higher-lying octet modes as a function of the squared common quark mass $m_q$. We find that the numerical results (points) are very well described by a straight line, meaning that $f_\pi$ behaves like $m_q^2$. This implies that it
FIG. 7: Masses (upper panel) and effective meson chemical potentials (lower panel) of the higher-lying excitations as functions of the strange quark mass $m_s$ for $m_u = m_d = 30$ MeV. The various points indicate the numerical calculations using Eq. (22). In the upper panel, these points have been connected by straight lines to guide the eye. The lines in the lower part correspond to Eq. (23).

V. SUMMARY & CONCLUSIONS

We studied the properties of the pseudoscalar bosonic excitations in the color-flavor locked phase at moderate densities within an NJL-type model. Extending our previous analysis [1], our focus was on the hidden-flavor Goldstone bosons and on the higher-lying pseudoscalar modes. Our results are consistent with the model independent predictions of the low-energy effective theory and with predictions from axial Ward-Takahashi identities.

First, we discussed the weak decay constants of the Goldstone bosons in the chiral limit. Since the Goldstone bosons form an $SU(3)$ octet and a singlet, there are two different decay constants, correspondingly. For the hidden-flavor octet mesons $\pi^0$ and $\eta$ we confirmed our previous results obtained in the flavored sector [1]. In addition we calculated the decay constant $f_{\eta'}$ of the singlet. We found that the weak-coupling limit of $f_{\eta'}$ is correctly reproduced, whereas for stronger couplings, i.e., for higher values of $\Delta$, we found deviations from this limit.

Next, we investigated the masses of the Goldstone bosons. For the $SU(3)$-symmetric case (equal quark masses), the octet remains degenerate and the hidden-flavor mesons $\pi^0$ and $\eta$ have the same mass as the flavored mesons. The $\eta'$, however, has a slightly higher mass. We found a linear dependence on the quark mass in both cases. Our results are in good agreement with the low-energy effective theory predictions if the NJL-model values for the decay constants and the coefficient $A$ are used. This remains true in the case of explicit $SU(3)$ breaking via a larger strange quark mass.

In the last part, we studied the higher-lying pseudoscalar modes, which appear naturally in the diagonalization procedure of the $T$-matrix for quark-quark scattering. We found that the octet states are bound in most cases and are always very close to their respective threshold for decay into two quasiparticles. The singlet mode, on the other hand, is unbound.

Finally, we calculated the decay constant of the higher-lying octet modes for equal quark masses. In agreement with axial Ward-Takahashi identities it is very small and vanishes in the chiral limit.
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