ON NON-EMPTY CROSS-INTERSECTING FAMILIES

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Let $2^n$ and $\binom{n}{i}$ be the power set and the collection of all $i$-subsets of $\{1,2,\ldots,n\}$, respectively. We call $t$ ($t \geq 2$) families $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_t \subseteq 2^n$ cross-intersecting if $A_i \cap A_j \neq \emptyset$ for any $A_i \in \mathcal{A}_i$ and $A_j \in \mathcal{A}_j$ with $i \neq j$. We show that, for $n \geq k + l, l \geq r \geq 1, c > 0$ and $\mathcal{A} \subseteq \binom{n}{k}, \mathcal{B} \subseteq \binom{n}{l}$, if $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting and $\binom{n-k}{l-r} \leq |\mathcal{B}| \leq \binom{n-1}{l-1}$, then

$$|\mathcal{A}| + c|\mathcal{B}| \leq \max \left\{ \binom{n}{k} - \binom{n-r}{k} + c \binom{n-r}{l-r}, \binom{n-1}{k-1} + c \binom{n-1}{l-1} \right\}.$$  

This implies a result of Tokushige and the second author (Theorem 3.1) and also yields that, for $n \geq 2k$, if $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_t \subseteq \binom{n}{k}$ are non-empty cross-intersecting, then

$$\sum_{i=1}^{t} |\mathcal{A}_i| \leq \max \left\{ \binom{n}{k} - \binom{n-k}{k} + t - 1, t \binom{n-1}{k-1} \right\},$$

which generalizes the corresponding result of Hilton and Milner for $t = 2$. Moreover, the extremal families attaining the two upper bounds above are also characterized.

1. Introduction

For a natural number $n$, we write $[n] = \{1,2,\ldots,n\}$ and denote by $2^n$ the power set of $[n]$. In particular, for an integer $i > 0$ we denote by $\binom{n}{i}$ the collection of all $i$-subsets of $[n]$. Every subset of $2^n$ is called a family. We call a family $\mathcal{A}$ intersecting if $A \cap B \neq \emptyset$ for any $A, B \in \mathcal{A}$, and call $t$ ($t \geq 2$) families $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_t$ cross-intersecting [2,20] if $A_i \cap A_j \neq \emptyset$ for any $A_i \in \mathcal{A}_i$

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and \( A_j \in \mathcal{A} \) with \( i \neq j \). For a set \( A \in 2^{[n]} \), we define its complement as usual by \( \overline{A} = [n] \setminus A \) and, for a family \( \mathcal{A} \subseteq 2^{[n]} \), we denote \( \overline{\mathcal{A}} = \{ \overline{A} : A \in \mathcal{A} \} \).

Extremal intersecting family is one of the main themes in extremal set theory. The following theorem, known as Erdős–Ko–Rado theorem, is a fundamental result on this theme.

**Theorem 1.1 (Erdős–Ko–Rado, [3])**. For two positive integers \( n \) and \( k \), if \( n \geq 2k \) and \( \mathcal{A} \subseteq \binom{[n]}{k} \) is an intersecting family, then

\[
|\mathcal{A}| \leq \binom{n - 1}{k - 1}.
\]

The Erdős–Ko–Rado theorem has a large number of variations and generalizations, see [1,4,9,12,16,18] for examples. A natural direction is to extend the notion of an intersecting family to a class of cross-intersecting families. Notice that if \( \mathcal{A}_1 = \mathcal{A}_2 = \cdots = \mathcal{A}_t \) (\( t \geq 2 \)), then the families \( \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_t \) are cross-intersecting if and only if \( \mathcal{A}_1 \) is intersecting. In this sense, the notion of cross-intersecting for families is indeed a generalization of that of intersecting for a family. The following result was proved by Hilton, a simple proof was given later by Borg [2]. Recently, using the Katona’s Circle Method, the second author gave another simple proof [5].

**Theorem 1.2 (Hilton, [12])**. Let \( n, k \) and \( t \) be positive integers with \( n \geq 2k \) and \( t \geq 2 \). If \( \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_t \subseteq \binom{[n]}{k} \) are cross-intersecting families, then

\[
\sum_{i=1}^{t} |\mathcal{A}_i| \leq \begin{cases} 
\binom{n}{k}, & \text{if } t \leq \frac{n}{k}; \\
t\binom{n-1}{k-1}, & \text{if } t \geq \frac{n}{k}.
\end{cases}
\]

For \( t = 2 \), lots of variations of Theorem 1.2 were also considered in the literature by imposing some particular restrictions on the families, e.g., the Sperner type restriction [21], \( r \)-intersecting restriction [6,20] and non-empty restriction [6,14,20]. For non-empty restriction, Hilton and Milner gave the following result:

**Theorem 1.3 (Hilton and Milner, [14])**. Let \( n \) and \( k \) be two positive integers with \( n \geq 2k \) and \( \mathcal{A}, \mathcal{B} \subseteq \binom{[n]}{k} \). If \( \mathcal{A} \) and \( \mathcal{B} \) are non-empty cross-intersecting, then

\[
|\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{k} - \binom{n-k}{k} + 1.
\]

In this paper we focus on non-empty cross-intersecting families. Inspired by Theorem 1.3, we prove the following generalization of it.
Theorem 1.4. Let $n,k,l,r$ be any integers with $n \geq k+l,l \geq r \geq 1$, $c$ be a positive constant and $\mathcal{A} \subseteq \binom{[n]}{k}$, $\mathcal{B} \subseteq \binom{[n]}{l-r}$. If $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting and $(\binom{n-r}{l-r}) \leq |\mathcal{B}| \leq (\binom{n-1}{l-1})$, then

$$|\mathcal{A}| + c|\mathcal{B}| \leq \max \left\{ \binom{n}{k} - \binom{n-r}{k} + c\binom{n-r}{l-r}, \frac{n-1}{k-1} + c\frac{n-1}{l-1} \right\}$$

and the equality holds if and only if, up to isomorphism, one of the following holds:

(i) $n > k+l$, $\mathcal{A} = \{A \in \binom{[n]}{k} : [s] \cap A \neq \emptyset \}$ and $\mathcal{B} = \{B \in \binom{[n]}{l} : [s] \subseteq B \}$, where $s=1$ if

$$\binom{n}{k} - \binom{n-r}{k} + c\binom{n-r}{l-r} \leq \frac{n-1}{k-1} + c\frac{n-1}{l-1},$$

or $s=r$ if the ‘$\leq$’ in (2) is ‘$\geq$’;

(ii) $n=k+l$, $\mathcal{B} \subseteq \binom{[n]}{l-r}$ with $|\mathcal{B}| = \binom{n-r}{l-r}$ if $c<1$ or $(\binom{n-r}{l-r}) \leq |\mathcal{B}| \leq (\binom{n-1}{l-1})$ if $c=1$ or $|\mathcal{B}| = (\binom{n-1}{l-1})$ if $c>1$, and $\mathcal{A} = \binom{[n]}{k} \setminus \mathcal{B}$.

The following result is a simple, but probably more applicable, consequence of Theorem 1.4. It provides another generalization of Theorem 1.3 by extending two families to arbitrary number of families and it is also sharpening of Theorem 1.2.

Theorem 1.5. Let $n,k$ and $t$ be positive integers with $n \geq 2k$ and $t \geq 2$. If $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_t \subseteq \binom{[n]}{k}$ are non-empty cross-intersecting families, then

$$\sum_{i=1}^{t} |\mathcal{A}_i| \leq \max \left\{ \binom{n}{k} - \binom{n-k}{k} + t-1, \binom{n-1}{k-1} \right\}$$

and the equality holds if and only if, up to isomorphism, one of the following holds:

(i) $n > 2k$, $\mathcal{A}_i = \{A \in \binom{[n]}{k} : i \in A \}$ for every $i \in \{1,2,\ldots,t\}$ if

$$\binom{n}{k} - \binom{n-k}{k} + t-1 \leq \binom{n-1}{k-1},$$

or $\mathcal{A}_i = \{A \in \binom{[n]}{k} : [k] \cap A \neq \emptyset \}$ and $\mathcal{A}_j = \{[k]\}$ for every $j \in \{2,\ldots,t\}$ (up to rearrangment of families) if the ‘$\leq$’ in (4) is ‘$\geq$’;

(ii) $n=2k$, $t=2$, $\mathcal{A}_1 \subseteq \binom{[n]}{k}$ with $0 < |\mathcal{A}_1| < \binom{n}{k}$ and $\mathcal{A}_2 = \binom{[n]}{k} \setminus \mathcal{A}_1$;

(iii) $n=2k$, $t \geq 3$, $\mathcal{A}_1 \subseteq \binom{[n]}{k}$ is intersecting with $|\mathcal{A}_1| = (\binom{n-1}{k-1})$ and $\mathcal{A}_2 = \cdots = \mathcal{A}_t = \mathcal{A}_1$. 
2. Proof of Theorem 1.4 and Theorem 1.5

Proof of Theorem 1.4. Let \( \prec_L \), or \( \prec \) for short, be the lexicographic order on \( \binom{[n]}{i} \) where \( i \in \{1, 2, \ldots, n\} \), that is, for any two sets \( A, B \in \binom{[n]}{i} \), \( A \prec B \) if and only if \( \min\{a: a \in A \setminus B\} < \min\{b: b \in B \setminus A\} \). For a family \( \mathcal{A} \subseteq \binom{[n]}{k} \), let \( \mathcal{A}_L \) denote the family consisting of the first \( |\mathcal{A}| \) k-sets in order \( \prec \), and call \( \mathcal{A} \) \( L \)-initial if \( \mathcal{A}_L = \mathcal{A} \).

In our forthcoming argument, the well-known Kruskal–Katona theorem [15,17] will play a key role, an equivalent formulation of which was given in [7,13] as follows:

**Kruskal theorem.** For \( \mathcal{A} \in \binom{[n]}{k} \) and \( \mathcal{B} \in \binom{[n]}{l} \), if \( \mathcal{A} \) and \( \mathcal{B} \) are cross-intersecting, then \( \mathcal{A}_L \) and \( \mathcal{B}_L \) are cross-intersecting as well.

For any \( i \in \{1, 2, \ldots, k\} \), let
\[
\mathcal{P}_i^{(l)} = \left\{ P \in \binom{[n]}{l} : P \supseteq [i] \right\} \quad \text{and} \quad \mathcal{R}_i^{(k)} = \left\{ R \in \binom{[n]}{k} : R \cap [i] \neq \emptyset \right\}.
\]

**Lemma 2.1.** Let \( n, k, l \) be any integers with \( n \geq k+l \). For any \( i \in \{1, 2, \ldots, k\} \), \( \mathcal{R}_i^{(k)} \) is the largest family that is cross-intersecting with \( \mathcal{P}_i^{(l)} \) and, vice versa. Moreover, \( \mathcal{R}_i^{(k)} \) and \( \mathcal{P}_i^{(l)} \) are both \( L \)-initial.

**Proof.** Assume that \( A \) is a \( k \)-set that intersects every \( l \)-set in \( \mathcal{P}_i^{(l)} \). Choose an arbitrary \( (l-i) \)-set \( B \) from \( \{i+1, i+2, \ldots, n\} \setminus A \) (such \( B \) exists since \( n \geq k+l \)). Then \( [i] \cup B \in \mathcal{P}_i^{(l)} \). Since \( A \cap B = \emptyset \) and \( A \) intersects every \( l \)-set in \( \mathcal{P}_i^{(l)} \), we must have \( A \cap [i] \neq \emptyset \). Hence, \( A \in \mathcal{R}_i^{(k)} \) and thus, \( \mathcal{R}_i^{(k)} \) is largest. The reverse is analogous. Finally, the last part follows directly from the definitions of \( \mathcal{P}_i^{(l)} \) and \( \mathcal{R}_i^{(k)} \).

By the Kruskal–Katona theorem, when investigating the maximum of \( |\mathcal{A}| + c|\mathcal{B}| \), we may assume that both \( \mathcal{A} \) and \( \mathcal{B} \) are \( L \)-initial families. Moreover, for given \( \mathcal{B} \), \( \mathcal{A} \) is the largest family that is cross-intersecting with \( \mathcal{B} \) and vice versa. That is,
\[
\mathcal{A} = \left\{ A \in \binom{[n]}{k} : A \cap B \neq \emptyset \quad \text{for all} \quad B \in \mathcal{B} \right\},
\]
\[
\mathcal{B} = \left\{ B \in \binom{[n]}{l} : A \cap B \neq \emptyset \quad \text{for all} \quad A \in \mathcal{A} \right\}.
\]

We call such a pair \((\mathcal{A}, \mathcal{B})\) a **maximal pair**. Hence, the condition \(|\mathcal{B}| \geq \binom{n-r}{l-r}\) implies \( \mathcal{P}_r^{(l)} \subseteq \mathcal{B} \). Define \( s \) to be the minimal integer such that \( \mathcal{P}_s^{(l)} \subseteq \mathcal{B} \). Therefore, \( 1 \leq s \leq r \).
In the case that \( s = 1 \), we have \( \mathcal{P}^{(l)}_1 = \{ P \in [n]: 1 \in P \} \) and \( \mathcal{R}^{(k)}_1 = \{ R \in \binom{[n]}{k}: 1 \in R \} \). Since \( \mathcal{P}^{(l)}_1 \subseteq \mathcal{B}, \binom{n-1}{l-1} = |\mathcal{P}^{(l)}_1| \leq |\mathcal{B}| \leq \binom{n}{l-1} \). This means that the only possibility is \( \mathcal{B} = \mathcal{P}^{(l)}_1 \). So by Lemma 2.1, \( \mathcal{A} = \mathcal{R}^{(k)}_1 \) and, hence, \( |\mathcal{A}| + c|\mathcal{B}| = \binom{n-1}{l-1} + c\binom{n}{l-1} \). Theorem 1.4 follows in this case.

From now on we assume that \( 2 \leq s \leq r \). By the minimality of \( s \), we have

\[
(5) \quad \mathcal{P}^{(l)}_s \subseteq \mathcal{B} \subseteq \mathcal{P}^{(l)}_{s-1}.
\]

By Lemma 2.1, \( \mathcal{B} \subset \mathcal{P}^{(l)}_{s-1} \) means that \( \mathcal{R}^{(k)}_{s-1} \) is cross-intersecting with \( \mathcal{B} \). Hence, \( \mathcal{R}^{(k)}_{s-1} \subseteq \mathcal{A} \) since \( \mathcal{A} \) is largest. On the other hand, \( \mathcal{P}^{(l)}_s \subseteq \mathcal{B} \) means that \( \mathcal{A} \) is cross-intersecting with \( \mathcal{P}^{(l)}_s \) since \( \mathcal{A} \) is cross-intersecting with \( \mathcal{B} \). So, again by Lemma 2.1, we have \( \mathcal{A} \subseteq \mathcal{R}^{(k)}_s \). In conclusion, (5) implies

\[
\mathcal{R}^{(k)}_{s-1} \subseteq \mathcal{A} \subseteq \mathcal{R}^{(k)}_s.
\]

Consider the cross-intersecting pair \( \mathcal{B}_0 := \mathcal{B} \setminus \mathcal{P}^{(l)}_s \) and \( \mathcal{A}_0 := \mathcal{A} \setminus \mathcal{R}^{(k)}_{s-1} \). By (5), we have \( B \cap [s] = [s-1] \) for any \( B \in \mathcal{B}_0 \), and \( A \cap [s] = \{ s \} \) for any \( A \in \mathcal{A}_0 \). Let

\[
Y = \binom{s+1, n}{l-s+1}, \quad X = \binom{s+1, n}{k-1},
\]

where \( [s+1, n] = \{ s+1, s+2, \ldots, n \} \). Define \( G_s \) to be the bipartite graph with bipartite sets \( X \) and \( Y \), in which \( PQ \) is an edge if and only if \( P \in X, Q \in Y \) and \( P \cap Q = \emptyset \). It is clear that \( G_s \) is biregular; that is, the vertices in the same partite set have the same degree.

**Lemma 2.2.** Let \( G \) be a bipartite biregular graph with partite sets \( P \) and \( Q \), and let \( c \) be a positive real constant. Let \( P_0 \subseteq P \) and \( Q_0 \subseteq Q \). If \( P_0 \cup Q_0 \) is independent, then \( |P_0| + c|Q_0| \leq \max\{|P|, c|Q|\} \). Moreover, if \( G \) is connected, then equality is possible only for \( P_0 \cup Q_0 = P \) or \( Q \).

**Proof.** For a set \( W \) of vertices, we denote by \( N(W) \) the neighbourhood of \( W \). Since \( P_0 \cup Q_0 \) is independent, we have \( N(P_0) \cap Q_0 = \emptyset \) and \( N(Q_0) \cap P_0 = \emptyset \). Further, \( |N(P_0)| \geq |P_0||Q|/|P| \) and \( |N(Q_0)| \geq |Q_0||P|/|Q| \) since \( G \) is biregular. Moreover, if \( G \) is connected, then equality holds only if \( P_0 = P \) or \( Q \) and \( Q_0 = Q \) or \( \emptyset \). Hence, if \( |P| \geq c|Q| \), then we have

\[
|P_0| + c|Q_0| \leq |P_0| + \frac{|P|}{|Q|}|Q_0| \leq |P_0| + \frac{|P|}{|Q|}|Q_0|N(Q_0) \leq |P|.
\]

The discussion for the case that \( |P| \leq c|Q| \) is analogous. \( \blacksquare \)
Let us first consider the case \( n > k + l \). Set \( \mathcal{A}_1 = \{ A\setminus\{s\} : A \in \mathcal{A}_0 \} \) and \( \mathcal{B}_1 = \{ B\setminus\{s\} : B \in \mathcal{B}_0 \} \). Then for any \( A \in \mathcal{A}_1 \) and \( B \in \mathcal{B}_1 \), we have \( A \cap B \neq \emptyset \) since \( A \cup \{s\} \in \mathcal{A}, B \cup \{s-1\} \in \mathcal{B} \) while \( \mathcal{A} \) and \( \mathcal{B} \) are cross-intersecting. This means that \( \mathcal{A}_1 \cup \mathcal{B}_1 \) is independent in \( G_s \). Let us note that \( G_s \) is connected for \( n > k + l \). So by Lemma 2.2,

\[
|\mathcal{A}_0| + c|\mathcal{B}_0| = |\mathcal{A}_1| + c|\mathcal{B}_1| \leq \max\{|X|, c|Y|\}.
\]

Moreover, for \( n > k + l \), equality is possible in (6) only if \( \mathcal{A}_0 = X, \mathcal{B}_0 = \emptyset \) or \( \mathcal{A}_0 = \emptyset, \mathcal{B}_0 = Y \). Consequently, either the maximal pair \( (\mathcal{A}, \mathcal{B}) \) is \( (\mathcal{R}_s(k), \mathcal{P}_s(l)) \) or it is \( (\mathcal{R}_s(k) - 1, \mathcal{P}_s(l) - 1) \). Hence, we have

\[
\max\{|\mathcal{A}| + c|\mathcal{B}|\} = |\mathcal{R}_i^{(k)}| + c|\mathcal{P}_i^{(l)}|.
\]

for some \( i \in \{1, 2, \ldots, r\} \).

We claim that the maximum in (7) is achieved for \( i = 1 \) or \( i = r \). To prove this, it is sufficient to prove that there is no \( i \) with \( 2 \leq i < r \) satisfying both

\[
|\mathcal{R}_i^{(k)}| + c|\mathcal{P}_i^{(l)}| \geq |\mathcal{R}_{i-1}^{(k)}| + c|\mathcal{P}_{i-1}^{(l)}|,
\]

\[
|\mathcal{R}_i^{(k)}| + c|\mathcal{P}_i^{(l)}| \geq |\mathcal{R}_{i+1}^{(k)}| + c|\mathcal{P}_{i+1}^{(l)}|.
\]

Equivalently,

\[
\binom{n-i}{k-1} \geq c\binom{n-i+1}{l-i+1} - c\binom{n-i}{l-i} = c\binom{n-i}{l-i+1},
\]

\[
c\binom{n-i-1}{l-i} \geq \binom{n-i-1}{k-1}.
\]

Multiplying the two inequalities yields

\[
c\binom{n-i}{k-1}\binom{n-i-1}{l-i} \geq c\binom{n-i}{l-i+1}\binom{n-i-1}{k-1}
\]

or equivalently,

\[
\binom{n-i}{k-1}/\binom{n-i-1}{k-1} \geq \binom{n-i}{l-i+1}/\binom{n-i-1}{l-i}.
\]

Hence,

\[
\frac{1}{n-i-k+1} \geq \frac{1}{l-i+1}.
\]

This contradicts the assumption that \( n > k + l \). Our claim follows.
Thus, we have proved that the only maximal pairs are \((\mathcal{P}_1^{(k)}, \mathcal{P}_1^{(l)})\) or \((\mathcal{P}_r^{(k)}, \mathcal{P}_r^{(l)})\). This concludes the proof of (1). The uniqueness for initial families follows as well.

To extend uniqueness to general families, we will apply a result proved independently by Füredi, Griggs and Mörs. To state it we need a definition. For two integers \(i, j\) with \(n \geq i + j\) and a family \(\mathcal{F} \subset \binom{[n]}{i}\), let us define

\[
\mathcal{D}_j(\mathcal{F}) = \left\{ D \in \binom{[n]}{j} : \exists F \in \mathcal{F}, D \cap F = \emptyset \right\}.
\]

With this terminology, \(\mathcal{A}\) and \(\mathcal{B}\) are cross-intersecting if and only if \(\mathcal{A} \cap \mathcal{D}_k(\mathcal{B}) = \emptyset\) or equivalently \(\mathcal{B} \cap \mathcal{D}_k(\mathcal{A}) = \emptyset\). They form a maximal pair if and only if \(\mathcal{A} = \binom{[n]}{k} \setminus \mathcal{D}_k(\mathcal{B})\) and \(\mathcal{B} = \binom{[n]}{l} \setminus \mathcal{D}_l(\mathcal{A})\).

**Proposition 2.3** (Füredi, Griggs \[10\], Mörs \[19\]). Suppose that \(n > k + l\), \(\mathcal{B} \subset \binom{[n]}{i}\), \(|\mathcal{B}| = \binom{n-r}{l-r}\) for some \(r\) with \(1 \leq r \leq l\). Then

\[
|\mathcal{D}_k(\mathcal{B})| \geq \binom{n-r}{k}
\]

with strict inequality unless for some \(R \in \binom{[n]}{r}\), \(\mathcal{B} = \{B \in \binom{[n]}{l} : R \subset B\}\).

We should note that (8) follows from the Kruskal–Katona theorem, the contribution of \[10\] and \[19\] is the uniqueness part. Actually, they proved analogous results for a much wider range but we only need this special case.

Let us continue with the proof of the uniqueness in the case \(n > k + l\), \(|\mathcal{A}| = \binom{n}{k} - \binom{n-r}{k}, |\mathcal{B}| = \binom{n-r}{l-r}\). From Proposition 2.3 and \(|\mathcal{A}| = \binom{n}{k} - |\mathcal{D}_k(\mathcal{B})|\), we infer \(|\mathcal{D}_k(\mathcal{B})| = \binom{n-r}{k}\). Hence, for some \(R \in \binom{[n]}{r}\), \(\mathcal{B} = \{B \in \binom{[n]}{l} : R \subset B\}\) and \(\mathcal{A} = \binom{[n]}{k} \setminus \mathcal{D}_k(\mathcal{B}) = \{A \in \binom{[n]}{k} : A \cap R \neq \emptyset\}\).

Let us next consider the case \(n = k + l\). First note that every \(k\)-set (resp., \(l\)-set) \(F\) is disjoint to only one \(l\)-set (resp., \(k\)-set), that is, its complement \(\overline{F} = [n] \setminus F\). Consequently, for a family \(\mathcal{B} \subset \binom{[n]}{l}\), \(\mathcal{D}_k(\mathcal{B}) = \overline{\mathcal{B}} = \{\overline{B} : B \in \mathcal{B}\}\). Hence, for any maximal pair \((\mathcal{A}, \mathcal{B})\), \(\mathcal{A} = \binom{[n]}{k} \setminus \overline{\mathcal{B}}\) and \(|\mathcal{A}| + |\mathcal{B}| = \binom{n}{k} - \binom{n-r}{k} + \binom{n-r}{l-r} = \binom{n}{k}\) since \(n = k + l\). This shows that for \(c = 1\), \(|\mathcal{A}| + |\mathcal{B}| = \binom{n}{k}\) holds if and only if \(\mathcal{B}\) is an arbitrary family with \(\binom{n-r}{l-r} \leq \mathcal{B} \leq \binom{n-1}{l-1}\) and \(\mathcal{A} = \binom{[n]}{k} \setminus \overline{\mathcal{B}}\).

For \(c > 1\), the maximum in Theorem 1.4 is \(\binom{n-1}{k-1} + c\binom{n-1}{l-1}\). It is realized by any pair \((\mathcal{A}, \mathcal{B})\) with \(|\mathcal{B}| = \binom{n-1}{l-1}, \mathcal{A} = \binom{[n]}{k} \setminus \overline{\mathcal{B}}\).

For \(c < 1\), the maximum is \(\binom{n}{k} - \binom{n-r}{k} + c\binom{n-r}{l-r}\). To realize it we can choose an arbitrary \(\mathcal{B} \subset \binom{[n]}{l}\) satisfying \(|\mathcal{B}| = \binom{n-r}{l-r}\) and set \(\mathcal{A} = \binom{[n]}{k} \setminus \overline{\mathcal{B}}\). This completes the proof of Theorem 1.4.
Proof of Theorem 1.5. Without loss of generality we assume that $|\mathcal{A}_1| \leq |\mathcal{A}_2| \leq \cdots \leq |\mathcal{A}_t|$. For $i \in \{1, 2, \ldots, t\}$, write $\mathcal{B}_i = (\mathcal{A}_i)_L$. Then, $\mathcal{B}_1 \supseteq \mathcal{B}_2 \supseteq \cdots \supseteq \mathcal{B}_t$ and $\sum_{i=1}^{t} |\mathcal{B}_i| = \sum_{i=1}^{t} |\mathcal{A}_i|$. Further, by the Kruskal–Katona Theorem, $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_t$ are cross-intersecting and, therefore, $\mathcal{B}_i$ is intersecting for $i \geq 2$ as $\mathcal{B}_1 \supseteq \mathcal{B}_i$. So by the Erdős–Ko–Rado theorem, we have $|\mathcal{A}_i| = |\mathcal{B}_i| \leq \binom{n-1}{k-1}$ for $i \geq 2$.

We notice that $\mathcal{A}_1$ and $\mathcal{A}_i$ are cross-intersecting for each $i \in \{2, \ldots, t\}$. Setting $c = t - 1$ and $r = k = l$ in Theorem 1.4, we have

$$\mathcal{A}_1 + (t-1)\mathcal{A}_i \leq \max \left\{ \binom{n}{k} - \binom{n-k}{k} + t - 1, \ t \binom{n-1}{k-1} \right\},$$

where $i = 2, \ldots, t$. Summing the above $t-1$ inequalities up, we obtain

$$\sum_{i=1}^{t} |\mathcal{A}_i| = \frac{1}{t-1} \sum_{i=2}^{t} (\mathcal{A}_1 + (t-1)\mathcal{A}_i)$$

$$\leq \max \left\{ \binom{n}{k} - \binom{n-k}{k} + t - 1, \ t \binom{n-1}{k-1} \right\}$$

and, hence, (3) follows.

Finally, let us consider the upper bound in (3). By Theorem 1.4, if $\mathcal{A}_1$ and $\mathcal{A}_i$ attain the upper bound in (9), then $\mathcal{A}_i$ is uniquely determined by $\mathcal{A}_1$. This implies that if the upper bound in (10) is attained, then $\mathcal{A}_2 = \cdots = \mathcal{A}_t$ and, hence, $\mathcal{A}_i$ is intersecting for each $i \in \{2, \ldots, t\}$ by the cross-intersecting property. Therefore, (i) follows directly from Theorem 1.4 (i). If $n = 2k$ and $t = 2$, then $c = 1$. So by Theorem 1.4 (ii), $1 \leq |\mathcal{A}_2| \leq \binom{n-1}{k-1} = \frac{1}{2} \binom{n}{k}$ and, hence, (ii) follows by symmetry. If $n = 2k$ and $t > 2$, then $c > 1$. Again by Theorem 1.4 (ii), $|\mathcal{A}_i| = \binom{n-1}{k-1}$ and $\mathcal{A}_1 = \binom{n}{k} \setminus \mathcal{A}_i = \mathcal{A}_i$. (iii) thereby follows. This completes our proof.

3. An application of Theorem 1.4

Let us recall a related result.

Theorem 3.1 (Frankl and Tokushige, [8]). Let $\mathcal{A} \subset \binom{[n]}{k}$ and $\mathcal{B} \subset \binom{[n]}{l}$ be non-empty cross-intersecting families with $n \geq k + l$ and $k \geq l$. Then

$$|\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{k} - \binom{n-l}{k} + 1.$$

We should mention that this result had found several applications, in particular in [8] it used to provide a simple proof of the following important result.
Theorem 3.2 (Hilton–Milner stability Theorem [14]). Suppose that 
\( \mathcal{F} \subset \binom{[n]}{k} \) is intersecting, \( \bigcap_{F \in \mathcal{F}} F = \emptyset \) and \( n > 2k \). Then
\[
|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.
\]

Let us derive now (11) from Theorem 1.4. Without loss of generality, we may assume that \( \mathcal{A} = \mathcal{A}_L, \mathcal{B} = \mathcal{B}_L \), i.e., both families are initial. We distinguish two cases.

(a) \( |\mathcal{A}| > \binom{n-1}{k-1} \).

Since \( \mathcal{A} \) is initial, \( \mathcal{A} \supset \mathcal{P}^{(k)}_1 = \mathcal{B}^{(k)}_1 \) follows. Now the cross-intersecting property implies \( \mathcal{B} \subset \mathcal{P}^{(l)}_1 \), in particular, \( |\mathcal{B}| \leq \binom{n-1}{l-1} \).

Applying Theorem 1.4 with \( c=1 \) and \( r=l \) yields
\[
(12) \quad |\mathcal{A}| + |\mathcal{B}| \leq \max \left\{ \binom{n}{k} - \binom{n-l}{k} + 1, \binom{n-1}{k} + \binom{n-1}{l-1} \right\}.
\]

(b) \( |\mathcal{A}| \leq \binom{n-1}{k-1} \).

Since \( |\mathcal{A}| \geq \binom{n-k}{k-1} = 1 \), we may apply Theorem 1.4 with the role of \( \mathcal{A} \) and \( \mathcal{B} \) interchanged, \( r=k, c=1 \), for obtaining
\[
(13) \quad |\mathcal{B}| + |\mathcal{A}| \leq \max \left\{ \binom{n}{l} - \binom{n-k}{l} + 1, \binom{n-1}{l-1} + \binom{n-1}{k-1} \right\}.
\]

Comparing (12) and (13) with (11), to conclude the proof we must show the following two inequalities:

\[
(14) \quad \binom{n-1}{k-1} + \binom{n-1}{l-1} \leq \binom{n}{k} - \binom{n-l}{k} + 1,
\]
\[
(15) \quad \binom{n}{l} - \binom{n-k}{l} \leq \binom{n}{k} - \binom{n-l}{k}.
\]

Using the formulae
\[
\binom{n-1}{l-1} = \binom{n-2}{l-1} + \binom{n-3}{l-2} + \cdots + \binom{n-l-1}{0}
\]
and
\[
\binom{n}{k} - \binom{n-l}{k} = \binom{n-1}{k-1} + \cdots + \binom{n-l}{k-1},
\]
(14) is equivalent to
\[
\binom{n-2}{l-1} + \cdots + \binom{n-l}{1} \leq \binom{n-2}{k-1} + \cdots + \binom{n-l}{k-1}.
\]
This inequality follows by termwise comparison

\[
\binom{n-i}{l+1-i} \leq \binom{n-i}{k-1}, \quad 2 \leq i \leq l.
\]

Since for \(i \geq 2, l+1-i \leq k-1\) and \((l+1-i)+(k-1) = k+l-i \leq n-i\), (16) and thereby (14) hold.

To prove (15) is not hard either. If \(n = k+l\), then we have equality. Let us apply induction on \(n\), supposing that (15) holds for all triples \((\tilde{n}, \tilde{k}, \tilde{l})\) with \(\tilde{n} \geq k+l, \tilde{k} \geq \tilde{l} \geq 1\). Note that the case \(k = l\) in (15) is trivial. So we may assume \(k \geq l+1\). Then by the induction hypothesis, (15) holds for \((n-1, k-1, l)\) and \((n-1, k, l-1)\), i.e.,

\[
\binom{n-1}{l} - \binom{(n-1)-(k-1)}{l} \leq \binom{n-1}{k-1} - \binom{(n-1)-(l-1)}{k-1}
\]

and

\[
\binom{n-1}{l-1} - \binom{(n-1)-k}{l-1} \leq \binom{n-1}{k} - \binom{(n-1)-(l-1)}{k}.
\]

Further, since \(k \geq l\), it is clear that

\[
\binom{n-1-k}{l-1} \leq \binom{n-1-l}{k-1}.
\]

Summing the above three inequalities up yields (15) and concludes the new proof of Theorem 3.1.

### 4. Remark and open problems

Let us recall that two families \(\mathcal{A}, \mathcal{B}\) are called cross-\(q\)-intersecting if \(|A \cap B| \geq q\) for all \(A \in \mathcal{A}, B \in \mathcal{B}\). The following are two related results concerning cross-\(q\)-intersecting families.

**Theorem 4.1 (Frankl and Kupavskii, [6]).** Let \(\mathcal{A}, \mathcal{B} \subset \binom{[n]}{k}\) be non-empty cross-\(q\)-intersecting families with \(k > q \geq 1\) and \(n > 2k-q\). Then

\[
|\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{k} - \sum_{i=0}^{q-1} \binom{k}{i} \binom{n-k}{k-i} + 1.
\]
Theorem 4.2 (Wang and Zhang, [20]). Let \( n \geq 4, k, l \geq 2, q < \min\{k, l\}, n > k + l - q, (n, q) \neq (k + l, 1), \binom{n}{k} \leq \binom{n}{l} \). Then for any non-empty cross-q-intersecting families \( \mathcal{A} \subset \binom{[n]}{k} \) and \( \mathcal{B} \subset \binom{[n]}{l} \),

\[
|\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{k} - \sum_{i=0}^{q-1} \binom{k}{i} \binom{n-k}{l-i} + 1.
\]

Following the notion of cross-intersecting, we extend the notion of cross-q-intersecting to arbitrary number of families, that is, \( \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_t \) are cross-q-intersecting if \( |A_i \cap A_j| \geq q \) for any \( A_i \in \mathcal{A}_i \) and \( A_j \in \mathcal{A}_j \) with \( i \neq j \). Based on the two theorems above, the following three problems are inspired naturally by Theorem 1.5:

Problem 4.3. Let \( \mathcal{A}_1 \subset \binom{[n]}{k_1}, \mathcal{A}_2 \subset \binom{[n]}{k_2}, \ldots, \mathcal{A}_t \subset \binom{[n]}{k_t} \) be non-empty cross-intersecting families with \( k_1 \geq k_2 \geq \cdots \geq k_t, n > k_1 + k_2 - 1 \) and \( t \geq 2 \). Is it true that

\[
\sum_{i=1}^{t} |\mathcal{A}_i| \leq \max \left\{ \binom{n}{k_1} - \binom{n-k_1}{k_1} + \sum_{i=2}^{t} \binom{n-k_i}{k_i-k_i} \cdot \sum_{i=1}^{t} \binom{n-1}{k_i-1} \right\}.
\]

We note that if we set \( c = t - 1 \) in Theorem 1.4, then we obtain a positive answer to Problem 4.3 for the special case that \( k_2 = \cdots = k_t \).

Problem 4.4. Let \( \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_t \subset \binom{[n]}{k} \) be non-empty cross-q-intersecting families with \( k > q \geq 1, n > 2k - q \) and \( t \geq 2 \). Is it true that

\[
\sum_{i=1}^{t} |\mathcal{A}_i| \leq \max \left\{ \binom{n}{k} - \sum_{i=0}^{q-1} \binom{k}{i} \binom{n-k}{k-i} + t - 1, t \binom{n-q}{k-q} \right\}.
\]

Problem 4.5. Let \( \mathcal{A}_1 \subset \binom{[n]}{k_1}, \mathcal{A}_2 \subset \binom{[n]}{k_2}, \ldots, \mathcal{A}_t \subset \binom{[n]}{k_t} \) be non-empty cross-q-intersecting families with \( k_1 \geq k_2 \geq \cdots \geq k_t > q \geq 1, n > k_1 + k_2 - q \) and \( t \geq 2 \). Is it true that

\[
\sum_{i=1}^{t} |\mathcal{A}_i| \leq \max \left\{ \binom{n}{k_1} - \sum_{i=0}^{q-1} \binom{k_t}{i} \binom{n-k_t}{k_1-i} + \sum_{i=2}^{t} \binom{n-k_t}{k_i-k_t} \cdot \sum_{i=1}^{t} \binom{n-q}{k_i-q} \right\}.
\]

We note that a positive answer to Problem 4.5 would imply that to Problem 4.3 and Problem 4.4. Moreover, the upper bound in Problem 4.5 is
attained by setting $\mathcal{A}_1 = \{A \in \binom{[n]}{k_1} : |A \cap [k_t]| \geq q\}$ and $\mathcal{A}_i = \{A \in \binom{[n]}{k_i} : A \supseteq [k_t]\}$ for $i \in \{2, 3, \ldots, t\}$ if

\[
\binom{n}{k_1} - \sum_{i=0}^{q-1} \binom{k_t}{i} \binom{n-k_t}{k_1-i} + \sum_{i=2}^{t} \binom{n-k_t}{k_i-k_t} \geq \sum_{i=1}^{t} \binom{n-q}{k_i-q},
\]

or setting $\mathcal{A}_i = \{A \in \binom{[n]}{k_i} : A \supseteq [q]\}$ for all $i \in \{1, 2, \ldots, t\}$ if the ‘$\geq$’ in (18) is ‘$\leq$’. Recently, Gupta, Mogge, Piga and Schülke [11] gave a proof of (17) under the condition that $n \geq 2k_1 + k_{t-1} - q$ and $|\bigcap_{i=1}^{t} A_i| \geq q$ for any $A_i \in \mathcal{A}_i, i = 1, 2, \ldots, t$.

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