A NEW CONVERGENCE PROOF OF 
AUGMENTED LAGRANGIAN-BASED METHOD 
WITH FULL JACOBIAN DECOMPOSITION FOR 
STRUCTURED VARIATIONAL INEQUALITIES

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Abstract. In the work, we present a new proof for global convergence of a classical method, augmented Lagrangian-based method with full Jacobian decomposition, for a special class of variational inequality problems with a separable structure. This work can be regarded as an improvement to work [14]. The convergence result of the work is established under more general conditions and proven in a new way.

1. Introduction. We begin the work by introducing a problem concerned, a separable variational inequality problem, which is defined as follows: Find a point \( v^* \in V \) such that

\[
G(v^*)^T (v - v^*) \geq 0, \quad \forall v \in V,
\]

where

\[
v = \begin{pmatrix}
y_1 \\
y_2 \\
\vdots \\
y_m \\
\lambda
\end{pmatrix}, 
G(v) = \begin{pmatrix}
g_1(y_1) - B_1^T \lambda \\
g_2(y_2) - B_2^T \lambda \\
\vdots \\
g_m(y_m) - B_m^T \lambda \\
\sum_{i=1}^m B_i y_i - b
\end{pmatrix},
\]

\( y_i \in Y_i \subseteq R^{n_i}, (i = 1, 2, \ldots, m), \lambda \in R^l, V = \prod_{i=1}^m Y_i \times R^l, \) and \( g_i : Y_i \to R^{n_i}, (i = 1, 2, \ldots, m) \) are mappings. In the work, we assume that \( Y_i, (i = 1, 2, \ldots, m) \) are nonempty, close, and convex sets, the mappings \( g_i, (i = 1, 2, \ldots, m) \) are monotone, and the solution set of (1)-(2) is nonempty.

Clearly, (1)-(2) is called as a specially structured variational inequality problem in that each \( g_i \) is a function of only variable \( y_i \). The separable variational inequality problem arises naturally from many applications such as economics, transportation,

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and engineering [1, 8, 13]. Moreover, the problem plays an important role in optimization, which has been demonstrated in the literature where many optimization problems can be reformulated as separable variational inequality problems. For example, a separable convex minimization problem with linear constraints defined as

\[
\min_{y_1, \ldots, y_m} \left\{ \sum_{i=1}^{m} \phi_i(y_i) \mid \sum_{i=1}^{m} B_i y_i = b, y_i \in \mathcal{Y}_i, i = 1, 2, \ldots, m \right\},
\]

can be reformulated as the variational inequality (1)-(2), where \( g_i \) is corresponding to a subgradient of \( \phi_i \) and \( \lambda \) is a Lagrangian multiplier to the linear constraints \( \sum_{i=1}^{m} B_i y_i = b \).

Due to wide applications, a significant portion of optimization research was dedicated towards designing algorithms for solving the problem (1)-(2), see [2, 3, 5, 11, 10, 9, 12, 7]. While many strategies exist for such problem, the alternating direction method of multipliers has been shown to subsume and dominate them since it was developed by Gabay and Mercier [4]. The process of the alternating direction algorithm is as follows:

For a given \( \lambda^k \), sequentially solve the following subproblems (3) in the order of \( i = 1, 2, \ldots, m \) to obtain \( y_i^{k+1} \), respectively,

\[
(y_i - y_i)T \left\{ g_i(y_i) - B_i^T \left[ \lambda^k - \beta \left( \sum_{j=1}^{i-1} B_j y_j^{k+1} + \sum_{j=i+1}^{m} B_j y_j^{k} - b \right) \right] \right\} \geq 0, \\
\forall y_i' \in \mathcal{Y}_i, i = 1, 2, \ldots, m.
\]

(3)

Then, improve \( \lambda \) by formula (4),

\[
\lambda^{k+1} = \lambda^k - \beta \left( \sum_{i=1}^{m} B_i y_i^{k+1} - b \right).
\]

(4)

The scheme (3)-(4) can be comprehended as an augmented Lagrangian-based method with full Gauss-Seidel decomposition and its global convergence has not been proven until the most recent work by Han and Yuan [6] which shows that the scheme (3)-(4) with involved strongly monotone functions \( g_i, (i = 1, 2, \ldots, m) \) is globally convergent.

Another famous approach is the augmented Lagrangian-based method with full Jacobian decomposition which was exploited by He et al. [9]. The approach has computational attractiveness due to its iterative scheme described as follows:

For a given \( \lambda^k \), simultaneously obtain \( y_i^{k+1} \) such that

\[
(y_i - y_i)T \left\{ g_i(y_i) - B_i^T \left[ \lambda^k - \beta \left( \sum_{j \neq i} B_j y_j^{k} + B_i y_i \right) \right] \right\} \geq 0, \forall y_i' \in \mathcal{Y}_i, i = 1, 2, \ldots, m.
\]

(5)

Then update \( \lambda \) through (4). The global convergence of the augmented Lagrangian-based method with full Jacobian decomposition is stated by Wang et al. [14] where it is required that each involved function \( g_i \) is strongly monotone with modulus \( \mu_i \), and each \( \mu_i \) is larger than a constant. Actually, the requirements on modulus \( \mu_i \) are strict since the coefficient \( \mu_i \) is always small for a general strongly monotone operator and hard to be estimated in practice. To avoid this weakness, we provide a new proof of the global convergence of the scheme (5)-(4) for the separable
variational inequality problem (1)-(2). In brief, the main contribution of the paper is that I provide a new proof of the global convergence of the scheme (5)-(4) for the separable variational inequality problem (1)-(2) by reducing the requirement of each modulus $\mu_{g_i}$.

The rest of the paper is broken down as follows. Next section includes some notations and fundamental concepts concerned in our study. Section 3 describes the process of the augmented Lagrangian-based method with full Jacobian decomposition and provides the convergence result of the concerned algorithm. Finally, Section 4 gives the conclusions of this work.

2. Preliminaries. To present a remarkably reader-friendly convergence analysis, we first explain some basic definitions and notations. In the sequent sections, $\mathbb{R}^n$ means an $n$-dimensional Euclidean space, $^T$ means the transpose, $\|\cdot\|$ means standard Euclidean norm, and $\delta_{\text{max}}(B)$ means the maximum eigenvalue of square matrix $B$. All vectors in the following analysis are column vectors. Hence, let $x^Ty$ be the standard inner product for any two vectors $x, y \in \mathbb{R}^n$. We are interested in analyzing the global convergence of the method in the sense of the matrix norm so we define $M$-norm as $\|y\|_M := \sqrt{y^TMy}$, where $M$ is a symmetric and positive definite matrix. In the present work, we define

$$M = \begin{pmatrix}
\beta B_1^TB_1 & \beta B_1^TB_2 & \cdots & \beta B_1^TB_m \\
\beta B_2^TB_1 & \beta B_2^TB_2 & \cdots & \beta B_2^TB_m \\
\vdots & \vdots & \ddots & \vdots \\
\beta B_m^TB_1 & \beta B_m^TB_2 & \cdots & \beta B_m^TB_m \\
\beta^{-1}I
\end{pmatrix},$$

where $I$ is the identity matrix in $\mathbb{R}^{l\times l}$. Then, it gives access to an important result

$$\|v - v^*\|_M^2 := \beta \left(\|B_1y_1 - B_1y_1^*\|^2 + \cdots + \|B_my_m - B_my_m^*\|^2\right) + \|\lambda - \lambda^*\|_2^2.$$

**Definition 2.1.**

a) A mapping $g : \mathbb{R}^n \to \mathbb{R}^n$ is said to be monotone, if

$$(y_1 - y_2)^T(g(y_1) - g(y_2)) \geq 0, \ \forall y_1, y_2 \in \mathbb{R}^n.$$

b) A mapping $g : \mathbb{R}^n \to \mathbb{R}^n$ is said to be $\mu$-strongly monotone if

$$(y_1 - y_2)^T(g(y_1) - g(y_2)) \geq \mu \|y_1 - y_2\|^2, \ \forall y_1, y_2 \in \mathbb{R}^n.$$

In the present work, it is assumed that each $g_i$ is strongly monotone with modulus $\mu_{g_i}, (i = 1, 2, \cdots, m)$.

3. Convergence analysis. In the section, for the completeness of the paper, firstly, the process of the augmented Lagrangian-based method with Jacobian decomposition for the separable variational inequality problem (1)-(2) is stated. Then, we conduct an analysis on the convergence of the method in a new manner.

**Augmented Lagrangian-based method with full Jacobian decomposition for (1)-(2)**

**S0.** Choose a starting point $v^0 = (y_1^0, \ldots, y_m^0, \lambda^0) \in \mathcal{V}$ and $\beta$. Set $k = 0$. 
Remark 1. In the paper, an easily implementable stopping criterion is given by
\[
\frac{1}{2}y^T \sum_{i, j \neq i} (B_j y_j^k + B_i y_i - b) \geq 0, \quad \forall y_i' \in Y_i, \quad i = 1, 2, \ldots, m. \tag{7}
\]

Lemma 3.1. We assume 
\[
y_i = \sum_{k=1}^m B_i y_i^{k+1} - b \tag{8}
\]
\[
\text{S3. Update } \lambda^{k+1}, \quad \lambda^{k+1} = \lambda^k - \beta \left( \sum_{i=1}^m B_i y_i^{k+1} - b \right).\tag{9}
\]
\[
\text{S4. If the new point } v^{k+1} = (y_1^{k+1}, \ldots, y_m^{k+1}, \lambda^{k+1}) \text{ satisfies some stopping criterion, stop. Otherwise, set } k = k + 1 \text{ and go to Step 1.}
\]

Remark 1. In the paper, an easily implementable stopping criterion is given by
\[
\max \left\{ \max_i \|B_i y_i^k - B_i y_i^{k+1}\|, \|\lambda^k - \lambda^{k+1}\| \right\} \leq \epsilon. \tag{9}
\]

The stopping criterion (9) was justified by Wang et al. [14].

Now, we concentrate on the convergence analysis. In order to get the convergence result, we first present some useful lemmas.

Lemma 3.1. We assume 
\[
v^* = (y_1^*, \ldots, y_m^*, \lambda^*) \in V \text{ to be a solution of the problem (1)-(2). Then, for the sequence } v^k = (y_1^k, \ldots, y_m^k, \lambda^k) \text{ generated by the augmented Lagrangian-based method with Jacobian decomposition, there exists the following result,}
\]
\[
(\lambda^k - \lambda^*)^T \left( \sum_{i=1}^m B_i y_i^{k+1} - b \right) \geq \beta \|\sum_{i=1}^m B_i y_i^{k+1} - b\|^2 + \beta \sum_{i=1}^m \sum_{j \neq i} (B_i y_i^{k+1} - B_i y_j^*)^T (B_j y_j^k - B_j y_j^{k+1}) + \sum_{i=1}^m \mu_i \|y_i^{k+1} - y_i^*\|^2. \tag{10}
\]

Proof. Since 
\[
v^* = (y_1^*, \ldots, y_m^*, \lambda^*) \in V \text{ is a solution of the problem, we have}
\]
\[
G(v^*)^T (v^{k+1} - v^*) \geq 0. \tag{11}
\]
Moreover, set 
\[
y_i' = y_i^* \quad (i = 1, 2, \ldots, m) \text{ in each inequality of (7) such that}
\]
\[
(y_i' - y_i^*)^T \left\{ g_i(y_i^{k+1}) - B_i^T \lambda^k + \beta B_i^T \left( \sum_{j \neq i} B_j y_j^k + B_i y_i^{k+1} - b \right) \right\} \geq 0, \quad i = 1, 2, \ldots, m. \tag{12}
\]
Using the sum taken over all the inequalities included in (11) and (12) and the equation 
\[
\sum_{i=1}^m B_i y_i^* = b, \text{ we derive that}
\]
\[
\sum_{i=1}^m (y_i^{k+1} - y_i^*)^T \left\{ g_i(y_i^*) - g_i(y_i^{k+1}) + B_i^T (\lambda^k - \lambda^*) - \beta B_i^T \left( \sum_{j \neq i} B_j y_j^k + B_i y_i^{k+1} - b \right) \right\} \geq 0.
\]
Noticing the strong monotonicity of all the functions $g_i$ $(i = 1, 2, \ldots, m)$ and the fact $\sum_{i=1}^{m} B_i y_i^* = b$, we rewrite the above inequality as

\[
(\lambda^k - \lambda^*)^T \left( \sum_{i=1}^{m} B_i y_i^{k+1} - b \right) \\
\geq \sum_{i=1}^{m} \beta \left[ (B_i y_i^{k+1} - B_i y_i^*)^T (\sum_{j \neq i} B_j y_j^k + B_i y_i^{k+1} - b) \right] + \sum_{i=1}^{m} \|y_i^{k+1} - y_i^*\|^2.
\]

Note that $b$ can be replaced by $\sum_{i=1}^{m} B_i y_i^*$. Then,

\[
(\lambda^k - \lambda^*)^T \left( \sum_{i=1}^{m} B_i y_i^{k+1} - b \right) \\
\geq \sum_{i=1}^{m} \beta \left[ (B_i y_i^{k+1} - B_i y_i^*)^T (\sum_{j \neq i} B_j y_j^k + B_i y_i^{k+1} - \sum_{j=1}^{m} B_j y_j^*) \right] + \sum_{i=1}^{m} \mu_{g_i} \|y_i^{k+1} - y_i^*\|^2
\]

\[
= \sum_{i=1}^{m} \beta \left[ (B_i y_i^{k+1} - B_i y_i^*)^T (\sum_{j \neq i} (B_j y_j^k - B_j y_j^*) + B_i y_i^{k+1} - B_i y_i^*) \right] + \sum_{i=1}^{m} \mu_{g_i} \|y_i^{k+1} - y_i^*\|^2
\]

\[
= \beta \sum_{i=1}^{m} \|B_i y_i^{k+1} - B_i y_i^*\|^2 + \beta \sum_{i=1}^{m} \sum_{j \neq i} (B_i y_i^{k+1} - B_i y_i^*)^T (B_j y_j^k - B_j y_j^*)
\]

\[
+ \sum_{i=1}^{m} \mu_{g_i} \|y_i^{k+1} - y_i^*\|^2
\]

\[
= \beta \sum_{i=1}^{m} \|B_i y_i^{k+1} - b\|^2 + \beta \sum_{i=1}^{m} \sum_{j \neq i} (B_i y_i^{k+1} - B_i y_i^*)^T (B_j y_j^k - B_j y_j^{k+1})
\]

\[
+ \sum_{i=1}^{m} \mu_{g_i} \|y_i^{k+1} - y_i^*\|^2,
\]

where the last equation of above formula is obtained by the formula for square of sum of $m$ numbers. Hence proved. \qed
Lemma 3.2. Given \( v^k = (y^k_1, ..., y^k_m, \lambda^k) \) at iteration \( k \), the new point \( v^{k+1} = (y^{k+1}_1, ..., y^{k+1}_m, \lambda^{k+1}) \) satisfies

\[
\beta \sum_{i=1}^{m} \| B_i y_i^{k+1} - B_i y_i^* \|^2 + \| \lambda^{k+1} - \lambda^* \|^2_{\beta} - \lambda^* \|_{\beta} ^2 - \lambda^* \|_{\beta} ^2 
\leq \beta \sum_{i=1}^{m} \| B_i y_i^k - B_i y_i^* \|^2 + \| \lambda^k - \lambda^* \|^2_{\beta} - \beta \sum_{i=1}^{m} \| \sum_{j \neq i} B_j y_j^{k+1} + B_i y_i^k - b \|^2.
\]

Proof. Using the result of Lemma 3.1 and the equation (8), we get

\[
\| \lambda^{k+1} - \lambda^* \|^2_{\beta} - \lambda^* \|_{\beta} ^2 - \lambda^* \|_{\beta} ^2 = \| \lambda^k - \lambda^* \|^2_{\beta} - 2(\lambda^k - \lambda^*)^T(\sum_{i=1}^{m} B_i y_i^{k+1} - b) + \beta \| \sum_{i=1}^{m} B_i y_i^{k+1} - b \|^2
\]

\[
\leq \| \lambda^k - \lambda^* \|^2_{\beta} - 2\beta \left\{ \| \sum_{i=1}^{m} B_i y_i^{k+1} - b \|^2 + \sum_{i=1}^{m} \sum_{j \neq i} (B_i y_i^{k+1} - B_i y_i^*)^T(B_j y_j^k - B_j y_j^{k+1}) \right\}
\]

\[
- 2 \sum_{i=1}^{m} \mu_i y_i^{k+1} - y_i^* \|_{\beta} ^2 - \sum_{i=1}^{m} B_i y_i^{k+1} - b \|^2.
\]

Now, we investigate the terms \( \beta \| B_i y_i^{k+1} - B_i y_i^* \|^2, \) \( i = 1, 2, ..., m \).

Since

\[
(B_i y_i^k - B_i y_i^*)^T(B_i y_i^k - B_i y_i^{k+1})
\]

\[
= (B_i y_i^{k+1} - B_i y_i^*)^T(B_i y_i^k - B_i y_i^{k+1}) + \| B_i y_i^k - B_i y_i^{k+1} \|^2, \quad i = 1, 2, ..., m,
\]

it follows that

\[
\beta \| B_i y_i^{k+1} - B_i y_i^* \|^2
\]

\[
= \beta \| B_i y_i^k - B_i y_i^* \|^2 + \beta \| B_i y_i^{k+1} - B_i y_i^k \|^2
\]

\[
- 2\beta \left\{ (B_i y_i^{k+1} - B_i y_i^*)^T(B_i y_i^k - B_i y_i^{k+1}) + \| B_i y_i^k - B_i y_i^{k+1} \|^2 \right\}, \quad i = 1, 2, ..., m.
\]
The sum of all the inequalities and equations in (13) and (14) gives that

\[
\begin{align*}
\beta \sum_{i=1}^{m} ||B_i y_i^{k+1} - B_i y_i^*||^2 + \|\lambda^{k+1} - \lambda^*\|_{\beta_i}^2 \\
\leq \beta \sum_{i=1}^{m} ||B_i y_i^k - B_i y_i^*||^2 + \|\lambda^k - \lambda^*\|_{\beta_i}^2 + \beta \sum_{i=1}^{m} ||B_i y_i^{k+1} - B_i y_i^*||^2 \\
+ \beta \| \sum_{i=1}^{m} B_i y_i^{k+1} - b \|^2 - 2 \beta \left\{ \| \sum_{i=1}^{m} B_i y_i^{k+1} - b \|^2 \right\} \\
+ \sum_{i=1}^{m} \sum_{j \neq i} (B_i y_i^{k+1} - B_i y_i^*)^T (B_j y_j^k - B_j y_j^{k+1}) \\
+ \sum_{i=1}^{m} (B_i y_i^{k+1} - B_i y_i^*)^T (B_i y_i^k - B_i y_i^{k+1}) + \sum_{i=1}^{m} ||B_i y_i^k - B_i y_i^{k+1}||^2 \right\} \\
- 2 \sum_{i=1}^{m} \mu_{y_i} ||y_i^{k+1} - y_i^*||^2 \\
= \beta \sum_{i=1}^{m} ||B_i y_i^k - B_i y_i^*||^2 + \|\lambda^k - \lambda^*\|_{\beta_i}^2 + \beta \sum_{i=1}^{m} ||B_i y_i^{k+1} - b||^2 \\
- \beta \sum_{i=1}^{m} ||B_i y_i^{k+1} - B_i y_i^*||^2 \\
- 2 \beta \left\{ \| \sum_{i=1}^{m} B_i y_i^{k+1} - b \|^2 + \sum_{i=1}^{m} \sum_{j=1}^{m} (B_i y_i^{k+1} - B_i y_i^*)^T (B_j y_j^k - B_j y_j^{k+1}) \right\} \\
- 2 \sum_{i=1}^{m} \mu_{y_i} ||y_i^{k+1} - y_i^*||^2 \\
= \beta \sum_{i=1}^{m} ||B_i y_i^k - B_i y_i^*||^2 + \|\lambda^k - \lambda^*\|_{\beta_i}^2 + \beta \sum_{i=1}^{m} ||B_i y_i^{k+1} - b||^2 \\
- \beta \left\{ \sum_{i=1}^{m} ||B_i y_i^{k+1} - B_i y_i^k||^2 + 2 \sum_{i=1}^{m} ||B_i y_i^{k+1} - b||^2 \\
+ 2 \sum_{j=1}^{m} \sum_{i=1}^{m} (B_i y_i^{k+1} - b)^T (B_j y_j^k - B_j y_j^{k+1}) \right\} \\
- 2 \sum_{i=1}^{m} \mu_{y_i} ||y_i^{k+1} - y_i^*||^2 \\
= \beta \sum_{i=1}^{m} ||B_i y_i^k - B_i y_i^*||^2 + \|\lambda^k - \lambda^*\|_{\beta_i}^2 + \beta (m-1) \sum_{i=1}^{m} ||B_i y_i^{k+1} - b||^2 \\
- \beta \sum_{i=1}^{m} \| \sum_{j \neq i} B_j y_j^{k+1} + B_i y_i^k - b \|^2 - 2 \sum_{i=1}^{m} \mu_{y_i} ||y_i^{k+1} - y_i^*||^2,
\end{align*}
\]
where the last equation of (15) is deduced based on the formula for square of sum of two numbers. Since

\[
\| \sum_{i=1}^{m} B_i y_i^{k+1} - b \|^2 \\
\leq \sum_{i=1}^{m} \| B_i y_i^{k+1} - B_i y_i^* \|^2 \\
\leq \sum_{i=1}^{m} \delta_{\text{max}}(B_i^T B_i) \| y_i^{k+1} - y_i^* \|^2,
\]

from (15), it follows that

\[
\beta \sum_{i=1}^{m} \| B_i y_i^{k+1} - B_i y_i^* \|^2 + \| \lambda^{k+1} - \lambda^* \|^2_{\beta-1} \\
\leq \beta \sum_{i=1}^{m} \| B_i y_i^k - B_i y_i^* \|^2 + \| \lambda^k - \lambda^* \|^2_{\beta-1} \\
- \sum_{i=1}^{m} [2\mu_{g_i} - \beta(m-1)\delta_{\text{max}}(B_i^T B_i)] \| y_i^{k+1} - y_i^* \|^2 \\
- \beta \sum_{i=1}^{m} \| \sum_{j \neq i} B_j y_j^{k+1} + B_i y_i^k - b \|^2.
\]

The proof is completed. \(\Box\)

Lemma 3.2 implies that

\[
\| v^{k+1} - v^* \|^2_M \\
\leq \| v^k - v^* \|^2_M - \sum_{i=1}^{m} [2\mu_{g_i} - \beta(m-1)\delta_{\text{max}}(B_i^T B_i)] \| y_i^{k+1} - y_i^* \|^2 \\
- \beta \sum_{i=1}^{m} \| \sum_{j \neq i} B_j y_j^{k+1} + B_i y_i^k - b \|^2,
\]

where the matrix \(M\) is given by (6).

With the above analysis, the convergence theorem of the method is ready to be presented.

**Theorem 3.3.** With the following hypotheses: For each \(i \in \{1, 2, ..., m\}\), \(g_i(y_i)\) is continuous and \(\mu_{g_i}\)-strongly monotone, for any

\[
0 < \beta \leq \min_i \left\{ \frac{2\mu_{g_i}}{(m-1)\delta_{\text{max}}(B_i^T B_i)} \right\},
\]

the sequence \(\{v^k\}\) generated by the augmented Lagrangian-based method with full Jacobian decomposition converges to a global optimal point of the problem (1)-(2).
Proof. From the inequality (17) deduced by Lemma 3.2, we get
\[
\|v^{k+1} - v^*\|_M^2 - \|v^k - v^*\|_M^2 \\
\leq - \sum_{i=1}^{m} \left[ 2\mu g_i - \beta (m - 1)\delta_{\max}(B_i^T B_i) \right] \|y_i^{k+1} - y_i^*\|^2 \\
- \beta \sum_{j=1}^{m} \| \sum_{j \neq i} B_j y_j^{k+1} + B_i y_i^k - b \|^2,
\]
(18)
where \(2\mu g_i - \beta (m - 1)\delta_{\max}(B_i^T B_i) > 0\).
Thus,
\[
\|v^{k+1} - v^*\|_M^2 \leq \|v^k - v^*\|_M^2 \leq \cdots \leq \|v^0 - v^*\|_M^2 \leq +\infty,
\]
which implies the boundedness of the sequence \(\{v^k\}\) generated by the algorithm. Consequently,
\[
\sum_{k=1}^{+\infty} \sum_{i=1}^{m} \left[ 2\mu g_i - \beta (m - 1)\delta_{\max}(B_i^T B_i) \right] \|y_i^{k+1} - y_i^*\|^2 \\
+ \sum_{k=1}^{+\infty} \sum_{i=1}^{m} \beta \| \sum_{j \neq i} B_j y_j^{k+1} + B_i y_i^k - b \|^2 \\
< \sum_{k=1}^{+\infty} (\|v^k - v^*\|_M^2 - \|v^{k+1} - v^*\|_M^2) < +\infty,
\]
which means that
\[
\lim_{k \to +\infty} \|y_i^{k+1} - y_i^*\| = 0, \quad i = 1, 2, \ldots, m,
\]
\[
\lim_{k \to +\infty} \| \sum_{j \neq i} B_j y_j^{k+1} + B_i y_i^k - b \| = 0, \quad i = 1, 2, \ldots, m.
\]
(19)
Thus,
\[
\sum_{i=1}^{m} B_i y_i^* = b.
\]
(20)
Notice that, for each \(i \in \{1, 2, \ldots, m\}\), \(y_i^{k+1}\) satisfies the following inequality,
\[
(y_i^* - y_i^{k+1})^T \left( g_i(y_i^{k+1}) - B_i^T \lambda^k + \beta B_i^T \left( \sum_{j \neq i} B_j y_j^k + B_i y_i^{k+1} - b \right) \right) \geq 0, \quad \forall y_i^* \in \mathcal{Y}_i,
\]
(21)
Moreover, because \(\{\lambda^k\}\) is bounded, there is a subsequence \(\{\lambda^{k_j}\}\) that converges to a point \(\lambda^*\). Taking limit of (21) along this subsequence for each \(i \in \{1, 2, \cdots, m\}\), we obtain
\[
(y_i^* - y_i^*)^T \left( g_i(y_i^*) - B_i^T \lambda^* \right) \geq 0, \quad \forall y_i^* \in \mathcal{Y}_i, \quad i = 1, 2, \ldots, m.
\]
(22)
Along with (22) and (20), we can conclude that the sequence generated is globally convergent. This completes the proof. \(\square\)
4. **Conclusion.** The augmented Lagrangian-based method with full Jacobian decomposition developed by He et al. [9] is of great interest for solving the separable variational inequality problem (1)-(2). It allows us to get a solution in a parallel perspective. The convergence of the method is ensured with the hypotheses on the strongly monotone modulus $u_g$. In fact, it is practically difficult to satisfy and verify the hypotheses. In this work, a new convergence result is shown by the relaxation of the hypotheses. This result makes a contribution to convergence of the the augmented Lagrangian-based method with full Jacobian decomposition under only strongly monotone assumptions on the involved functions.

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