Absence of a critical nematic phase in the vicinity of the SU(3) ferromagnetic point for the one-dimensional spin-1 bilinear-biquadratic model

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The absence of a critical nematic phase in the vicinity of the SU(3) ferromagnetic point for the one-dimensional spin-1 bilinear-biquadratic model is demonstrated by means of the tensor network algorithms. As it turns out, the phase transition from the ferromagnetic phase to the dimerized phase at the SU(3) ferromagnetic point is direct, but not of the first-order. The transition point features highly degenerate ground states, which are scale but not conformally invariant, with the fractal dimension being equal to 2. The conceptual developments in effective field theories - the fractal dimension and the counting rule of the Goldstone modes - play a pivotal role in clarifying the numerical artifacts arising from the finiteness of the bond dimension in the tensor network simulations, which are attributed to a proximity effect to a highly entangled scale or conformally invariant ground state.

PACS numbers:

Introduction.- The spin-1 bilinear-biquadratic model has been investigated extensively, either analytically or numerically [1–13]. It is described by the Hamiltonian with the nearest-neighbor interactions

$$H = \sum_{j} \left[ \cos \phi (S_{j}S_{j+1}) + \sin \phi (S_{j}S_{j+1})^{2} \right], \quad (1)$$

where $S_{j} = (S_{j}^{x}, S_{j}^{y}, S_{j}^{z})$ denote the spin-1 operators acting on the $j$-th site. The model exhibits rich physics, with the ground-state phase diagram being shown in Fig. 1. A remarkable example is the Haldane gap in the spin-1 Heisenberg antiferromagnet [14] at $\phi = 0$, which is adiabatically connected to the Affleck-Kennedy-Lieb-Tasaki point at $\phi = \arctan (1/3)$ [15], with its ground state being the valence-bond-solid state. This exactly solved point provides compelling evidence for the existence of the gapped Haldane phase with hidden topological order [14, 16] for $-\pi/4 < \phi < \pi/4$. The importance of the model [1] is also embodied in the fact that it is exactly solvable by means of the Bethe ansatz at a few selected points, including $\phi = \pi/4$ [17], $\phi = -\pi/2$ [18], and $\phi = -\pi/4$ [19]. In addition, the model exhibits the SU(3) symmetry [20, 22] at $\phi = \pm \pi/2, \pi/4,$ and $-3\pi/4$ (cf. also Sec. SI in the Supplementary Material (SM)). However, whether or not the phase transition from the ferromagnetic phase to the dimerized phase is direct remains to be controversial up to present.

The controversy dates back to a pioneering work by Chubukov [1], who suggested that a gapped nematic phase exists between the dimerized phase and the ferromagnetic phase. In contrast, Fáth and Sólyom [2] advocated that the phase transition from the ferromagnetic phase to the dimerized phase is direct and of the first-order, thus ruling out the existence of the gapped nematic phase. They also argued that an exponentially decaying small gap opens once away from the SU(3) ferromagnetic point. However, no mechanism is offered to account for the opening of such an exponentially decaying small gap. Subsequently, an extensive investigation has been done [3–10], in attempt to settle this controversial issue. Buchta et al [5] showed that the dimerized phase prevails down to the SU(3) point, although a non-dimerized phase is possible in a narrow range close to the SU(3) ferromagnetic point. In Refs. [7, 10], a critical nematic phase is suggested as a possible non-dimerized phase between the ferromagnetic phase and the dimerized phase. Meanwhile, Rizzi et al [6] concluded that there is no intermediate nematic phase, but a tendency is enhanced towards the nematic order. Instead, Läuchli, Schmid, and Trebst [7] suggested that an unconventional crossover might lie at the heart of the complications concerning the presence or absence of a critical nematic phase. In a recent work [10], the SU(2) symmetry is implemented in a finite-size matrix product state (MPS) algorithm under the periodic boundary conditions, which is sufficient to confirm the absence of a critical nematic phase for $\phi/\pi > -0.72$, but leaving it open the possibility for the existence of a critical nematic phase in the vicinity of the SU(3) ferromagnetic point.

FIG. 1: (color online) A sketch of the ground state phase diagram for the one-dimensional spin-1 bilinear-biquadratic model. Here, $\phi$ ranges from 0 to $2\pi$, but we focus on a region in the vicinity of the SU(3) ferromagnetic point at $\phi = 5/4\pi$. The question mark indicates a possible critical nematic phase, which turns out to be absent.
chosen to be at the SU(3) ferromagnetic point. Thus, the data is fitted in the form:

\[ \chi \rightarrow \infty \]

Here, the termination point is chosen to be at the SU(3) ferromagnetic point. Thus, the data is fitted in the form: \( \phi, \chi \rightarrow 5/4\pi + p\chi^q \), where \( p = -2.54 \) and \( q = -0.0064 \).

In this work, we adopt a completely different strategy to address this controversy. In our opinion, the current state-of-the-art algorithms in the context of the tensor network representations are powerful enough to efficiently simulate the model [1], including the infinite time evolving block decimation (iTEBD) [23], the U(1) infinite density matrix renormalization group (iDMRG) algorithm, and the SU(2) iDMRG algorithm [24]. For our purpose, we focus on the region \( 5\pi/4 < \phi < 3\pi/2 \). As it turns out, different types of numerical artifacts arise from the finiteness of the bond dimension in the tensor network simulations, if different symmetries are implemented. The artifacts may be characterized in terms of the fractal dimension [25] and the counting rule of the Goldstone modes (GMs) [26] - recent conceptual developments in effective field theories. As a consequence, there are two distinct types of the artifacts, which may be attributed to a proximity effect to two types of highly entangled ground states: one is conformally invariant and the other is scale invariant, thus leading to a pseudo critical regime and a pseudo fractal regime. Here, by “pseudo” we mean an artifact arising from the finiteness of the bond dimension in the tensor network simulations. It is the presence of the numerical artifacts that makes it very hard to extract the underlying physics behind the model [1].

The iTEBD simulation. The iTEBD simulation is performed in the region (1.25π, 1.5π]. In Fig. 2, the entanglement entropy \( S(\phi, \chi) \), which quantifies the entanglement of a bipartite system [27], is plotted as a function of \( \phi \), with the bond dimension \( \chi = 20, 40, 80, 100, 200 \) and 300. Here, we note that in the infinite MPS representation, the entanglement entropy \( S(\phi, \chi) \) for the semi-infinite chain may be written as

\[ S(\phi, \chi) = -\sum_{\alpha, \beta} \lambda^2_{\alpha \beta}(\phi, \chi) \ln \lambda^2_{\alpha \beta}(\phi, \chi), \]

with \( \lambda^2_{\alpha \beta}(\phi, \chi) \) being the Schmidt decomposition coefficients. The entanglement entropy \( S(\phi, \chi) \) exhibits a singular peak, with each singular point being indicative of a pseudo critical point \( \phi_c(\chi) \). Hence, there is a pseudo critical point \( \phi_c(\chi) \) for a given value of the bond dimension \( \chi \), separating the whole region into a pseudo critical regime and a dimerized regime. As an illustrative example, for the bond dimension \( \chi = 300 \), the pseudo critical point \( \phi_c(\chi) \) is located at \( \phi_c(\chi) = 1.298\pi \).

In order to characterize the pseudo critical regime, we need to extract the central charge \( c \) from the finite-entanglement scaling [28,29]:

\[ S(\phi, \chi) = \frac{c}{6} \ln \xi(\phi, \chi) + a(\phi), \]

where \( \xi(\phi, \chi) \approx \chi^{\kappa(\phi)} \), with \( \kappa(\phi) \) being the finite entanglement scaling exponent, and \( a(\phi) \) is an additive constant. The best linear fit is performed for \( \phi = 1.26\pi, 1.27\pi, 1.28\pi \) and 1.29\pi, with the bond dimension ranging from 16 to 300, respectively. The central charge \( c \) is estimated to be \( c = 2 \), with a relative error being less than 2%, in the entire pseudo critical regime (for details, see Sec. SII in SM). Meanwhile, the central charge \( c \) is also extracted from the pseudo critical points \( \phi_c(\chi) \), yielding \( c = 1.86 \), with a relative error being less than 7%, compared to the expected exact value \( c = 2 \) (cf. Sec. SII in SM). We attribute a higher relative error in this case to the fact that the accuracies of the pseudo critical points \( \phi_c(\chi) \) are relatively lower, due to high computational costs.

Physically, the fact that the central charge \( c = 2 \) in the pseudo critical regime may be explained in terms of the counting rule of the (pseudo) GMs adapted to numerical artifacts. Actually, this arises from the pseudo spontaneous symmetry breaking (SSB) [30] from SU(2) to U(1), in which two generators are spontaneously broken, thus resulting in two pseudo GMs of type-A, with the central charge \( c \) being the number of the pseudo GMs, given that the central charge \( c \) measures the number of the gapless excitations [31]. Here, we note that the infinite MPS algorithm naturally leads to infinitely degenerate ground states in a pseudo critical regime, due to the finiteness of the bond dimension. In fact, if the bond dimension \( \chi \) tends to infinity, then a pseudo local order parameter \( \langle O \rangle \) arising from pseudo SSB tends to vanish, as required to keep consistency with the Mermin-Wagner theorem [32]. In our case, the pseudo local order parameter \( \langle O \rangle \) is defined as follows:

\[ \langle O \rangle = \sum_{\alpha, \beta = 1}^8 g_{\alpha \beta}^{op}(K_{\alpha}^j)(K_{\beta}^j), \]

where \( g_{\alpha \beta}^{op} \) is a metric tensor: \( g_{\alpha \beta}^{op} = \sum_{\gamma, \delta, \epsilon} \gamma_{\gamma \delta \epsilon} \gamma_{\alpha \beta \epsilon} \), with \( \gamma_{\alpha \beta \epsilon} \) being the structural constants of the SU(3) group, and \( K_{\alpha}^j \) is the local components of the SU(3) generators at the \( j \)-th site. Since the iTEBD simulation is performed for a randomly chosen initial state, it yields a ground state wave function in the infinite MPS representation, with different expectation values of the generators \( K_{\alpha}^j \). However, the pseudo local order operator \( \langle O \rangle \) does not depend on an initial state, within accuracies. It is found that the same set of the pseudo critical points \( \phi_c(\chi) \) are detected in terms of the pseudo local order parameter \( \langle O \rangle(\chi) \), consistent with those.
from the entanglement entropy $S(\phi, \chi)$ and the pseudo local order parameter ($O(\chi)$) is scaled down to zero (for details, cf Sec. SIV in SM).

In addition, the dimerized phase is characterized in terms of the local order parameter ($D_{j,j+1} = \langle S_j S_{j+1} - S_{j+1} S_j \rangle$), which tends to be saturated as the bond dimension $\chi$ increases.

A crucial question concerns whether or not the pseudo critical points $\phi_1(\chi)$ terminate at the SU(3) ferromagnetic point, if the bond dimension $\chi$ tends to infinity. As shown in Fig. 2, it is possible for the pseudo critical points $\phi_1(\chi)$ to terminate at the SU(3) ferromagnetic point, where the termination point is chosen to be at the SU(3) ferromagnetic point. Thus, the fitting function takes the form: $\phi(\chi) = 5/4\pi + p\chi^q$, where $p = -2.54$ and $q = -0.0064$. However, such an extrapolation also works within the accuracies, if we make any other choice of the termination point, as long as it is close enough to the SU(3) ferromagnetic point. As a consequence, two possible scenarios arise: (i) the pseudo critical points $\phi_1(\chi)$ terminate at the SU(3) ferromagnetic point, when the bond dimension $\chi$ tends to infinity; (ii) the pseudo critical points $\phi_1(\chi)$ terminate at a point $\phi_1$ away from the SU(3) ferromagnetic point, when the bond dimension $\chi$ tends to infinity. In the first scenario, there is no critical nematic phase between the SU(3) ferromagnetic point and the dimerized phases. In the second scenario, there is a critical nematic phase between the SU(3) ferromagnetic point and the dimerized phases, with the central charge $c$ being 2 (for details, cf Sec. SIV in SM).

The U(1) iDMRG simulation.- The U(1) iDMRG algorithm with two-site translational invariance, is exploited, targeting at a ground state, with the z-component of the total spin being zero, to simulate the model (1) in the same region. In Fig. 3 we plot the entanglement entropy $S(\phi, \chi)$ as a function of $\phi$, with the bond dimension $\chi = 30, 200, 300, 500, 1000$ and 2000, from the U(1) iDMRG simulation. One pseudo first-order transition point $\phi_{f1}(\chi)$ and one pseudo critical point $\phi_{c1}(\chi)$ are detected as a (discontinuous) dip and a peak for each value of the bond dimension $\chi$, respectively. Inset: an extrapolation of the pseudo critical points $\phi_{c1}(\chi)$, as the bond dimension $\chi$ tends to infinity, assuming that the pseudo critical points $\phi_{c1}(\chi)$ terminate at the SU(3) ferromagnetic point. Thus, the fitting function takes the form: $\phi(\chi) = 5/4\pi + p\chi^q$, where $p = -2.587$ and $q = -0.0026$.

To characterize the pseudo fractal regime, we resort to the finite block-size scaling of the entanglement entropy $S(n)$. As it turns out, it scales logarithmically with the block size $n$ [25]:

$$S(n) = \frac{d_f}{2} \ln n + b.$$  \hspace{1cm} (4)

Here, $d_f$ is the fractal dimension, and $b$ is an additive constant. For three different values of $\phi$: $\phi = 1.26\pi, 1.254\pi$ and $1.252\pi$ in the pseudo fractal regime, we perform the best linear fit, yielding that $d_f = 1$, with a relative error being less than $4\%$, consistent with a general statement that the fractal dimension $d_f$ is equal to the number of the (pseudo) GMs of type-B [25](also cf. Sec. SVII in SM).

From the finite-entanglement scaling (11), the central charge $c$ is extracted for $\phi = 1.26\pi, 1.27\pi, 1.28\pi$ and $1.29\pi$ in the pseudo critical regime, yielding $c = 2$, with a relative error being less than $4\%$. This is in agreement with that from the iTEBD simulation. Meanwhile, the central charge $c$ is extracted from a pseudo critical point $\phi_{c1}(\chi)$ between the pseudo critical regime and the dimerized regime, which yields $c = 1.833$, with a relative error being less than $8.4\%$, compared to the exact value $c = 2$.

Both the pseudo fractal regime and the pseudo critical regime may be characterized in terms of the pseudo order parameter ($O_j$). From the U(1) iDMRG simulation, one finds that only $K'_j = I - 3/2(S_j^z)^2$ yields a non-zero expectation value. Therefore, the pseudo local order parameter ($O_j$) may be replaced by $\langle (S_j^z)^2 \rangle$. Hence, the pseudo first-order phase transition point $\phi_{f1}(\chi)$ and the pseudo critical points $\phi_{c1}(\chi)$ are determined from the pseudo local order parameter $\langle (S_j^z)^2 \rangle$, consistent with those from the entanglement entropy $S(\phi, \chi)$.

We stress that the two regimes may be distinguished by means of the pseudo local order parameter $\langle (S_j^z)^2 \rangle$, with pseudo SSB from SU(2) to U(1) in two distinct ways. In the pseudo fractal regime, we have $\langle (S_j^z)^2 \rangle > 2/3$: SU(2) is spontaneously broken to U(1), with two broken generators corresponding to one pseudo GM of type-B. In contrast, in the pseudo critical regime, we have $\langle (S_j^z)^2 \rangle < 2/3$: SU(2) is spontaneously broken to U(1), with two broken generators corresponding to two pseudo GMs of type-A. Note that $\langle (S_j^z)^2 \rangle$ approaches 2/3 to
recover the SU(2) invariance, as the bond dimension $\chi$ goes to infinity.

The presence of the pseudo first-order phase transition points $\phi_{\text{f}}(\chi)$ from the U(1) iDMRG simulation makes it possible to refine the two scenarios from the iTEBD. In particular, this entails an important implication for the second scenario. Suppose the pseudo critical points $\phi_{\text{c}}(\chi)$ terminate at a critical point $\phi_c$, as the bond dimension $\chi$ tends to infinity. Then, the critical point $\phi_c$ must be located at a value of $\phi$ greater than $\phi_f$. Here, $\phi_f$ denotes an onset point where the pseudo fractal regime sets in, which reflects ferromagnetic fluctuations from the SU(3) ferromagnetic point. Physically, this amounts to stating that the pseudo fractal regime is not allowed to extend to a point beyond the critical point $\phi_c$, if it exists. For the first scenario, both $\phi_{\text{f}}(\chi)$ and $\phi_{\text{c}}(\chi)$ approach the SU(3) ferromagnetic point, as the bond dimension $\chi$ tends to infinity.

The SU(2) iDMRG simulation.- We exploit the SU(2) iDMRG algorithm to simulate the model (1) in the region $(1.25\pi, 1.33\pi)$. In the SU(2) iDMRG simulation, note that no phase transition is detected in the region $(1.262\pi, 1.33\pi)$, in which the simulation results are reliable.

FIG. 4: The entanglement entropy $S(\phi, \chi)$ as a function of $\phi$, with the bond dimension $\chi = 30, 90, 240, 480, 600$ and $5000$, from the SU(2) iDMRG simulation. Note that no phase transition is detected in the region $(1.262\pi, 1.33\pi)$, in which the simulation results are reliable.
ent symmetries are implemented. There are two types of the artifacts, attributed to a proximity effect to a highly entangled ground state, either a conformally invariant critical regime or a scale invariant fractal regime. The artifacts may be characterized in terms of the fractal dimension $d_{fract}$ and the counting rule of the pseudo GMs $2^{d_{fract}}$.

A few closing remarks are in order. First, the suggestion made by Chubukov [1], though it has been ruled out, is not far from being true, in the sense that the two critical nematic phases with the central charge $c=1$ are asymptotically close to the SU(3) ferromagnetic point, though no SU(2) WZW regime, as the bond dimension increases. It is the proximity effect to the two critical nematic phases that provides a mechanism responsible for the opening of an exponentially decaying small gap away from the SU(3) ferromagnetic point. Accordingly, an essential singularity results from a catastrophe point, in a similar way to the Kosterlitz-Thouless transitions [20]. Second, the phase transition from the ferromagnetic phase to the dimerized phase is direct, consistent with Ref. [2]. The transition point, located at the SU(3) ferromagnetic point, features scale invariant ground states, with the fractal dimension being equal to 2. Indeed, the SU(2) ferromagnetic states are smoothly embedded into the SU(3) ferromagnetic states from the ferromagnetic phase. In contrast, the essential singularity appears if the SU(3) ferromagnetic point is approached from the dimerized phase. As a consequence, the phase transition is not of the first-order, in contrast to the claim in Ref. [2]. Third, instead of being a crossover [2], the artifacts involve the pseudo first-order phase transition and the pseudo critical points, thus much more complicated than anticipated.

Acknowledgements.- We thank Murray Batchelor, Sam Young Cho, John Ove Fjærestad, Javier Rodríguez-Laguna, Silvia N. Santalla, and Germán Sierra for enlightening discussions. The work is supported by the National Natural Science Foundation of China (Grant No. 11805285).

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Supplementary Material for “absence of a critical nematic phase in the vicinity of the SU(3) ferromagnetic point for the one-dimensional spin-1 bilinear-biquadratic model”

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PACS numbers:

SI. The two SU(3) symmetries

The Hamiltonian for the one-dimensional spin-1 bilinear-biquadratic model (1) possesses the SU(2) symmetry, with the generators \( \sum_j S^x_j \), \( \sum_j S^y_j \) and \( \sum_j S^z_j \), for a generic value of the parameter \( \phi \). However, its symmetry is enlarged to SU(3) at four points \( \phi = \pi/2, 3\pi/2, \phi = \pi/4, \) and \( \phi = 5\pi/4 \): one is staggered at \( \phi = \pi/2 \) and \( \phi = 3\pi/2 \), and the other is uniform at \( \phi = \pi/4 \) and \( \phi = 5\pi/4 \).

At \( \phi = \pi/2 \) and \( \phi = 3\pi/2 \), the staggered SU(3) symmetry is realized in terms of the spin-1 operators: \( J_\alpha = \sum_j J^\alpha_j \) (\( \alpha = 1, 2, \cdots, 8 \)), with \( J_1 = 1/2 \sum_j S^z_j \), \( J_2 = 1/2 \sum_j S^y_j \), \( J_3 = 1/2 \sum_j S^x_j \), \( J_4 = 1 - 3/2 \sum_j (-1)^{j+1} (S^z_j)^2, \) \( J_5 = 1/2 \sum_j (-1)^{j+1} (S^z_j S^x_j + S^x_j S^z_j), \) \( J_6 = 1/2 \sum_j (-1)^{j+1} (S^y_j S^z_j + S^z_j S^y_j), \) \( J_7 = 1/2 \sum_j (-1)^{j+1} (S^x_j S^y_j + S^y_j S^x_j) \) and \( J_8 = 1/2 \sum_j (-1)^{j+1} (S^y_j S^x_j + S^x_j S^y_j) \).

At \( \phi = \pi/4 \) and \( \phi = 5\pi/4 \), the uniform SU(3) symmetry is realized in terms of the spin-1 operators: \( K_\alpha = \sum_j K^\alpha_j \) (\( \alpha = 1, 2, \cdots, 8 \)), with \( K_1 = 1/2 \sum_j S^x_j \), \( K_2 = 1/2 \sum_j S^y_j \), \( K_3 = 1/2 \sum_j S^z_j \), \( K_4 = 1 - 3/2 \sum_j (S^y_j)^2 \), \( K_5 = 1/2 \sum_j (S^x_j)^2 - (S^y_j)^2 \), \( K_6 = 1/2 \sum_j S^x_j (S^y_j + S^z_j), \) \( K_7 = 1/2 \sum_j (S^x_j + S^y_j) (S^y_j + S^z_j) \) and \( K_8 = 1/2 \sum_j S^x_j (S^y_j + S^z_j) \).

SII. The central charge \( c \) in the pseudo critical regime and at the pseudo critical points

The entanglement entropy is defined as \( S = -Tr \rho \ln \rho \), which may be exploited to quantify the bipartite entanglement \([S4]\). In our case, the density matrix \( \rho = \vert \phi \rangle \langle \phi \vert \) is generated from the ground state wave function \( \langle \phi \rangle \), in the infinite MPS representation from the tensor network simulations. For our purpose, we consider the entanglement entropy for a semi-infinite chain, which may be rewritten as

\[
S(\phi, \chi) = -\sum_{\mu} \lambda_{\mu}^2(\phi, \chi) \ln \lambda_{\mu}^2(\phi, \chi),
\]

where \( \lambda_{\mu}^2(\phi, \chi) \) is the Schmidt decomposition coefficients. The finite-entanglement scaling (2) \([S5, S6]\) is performed for the \( S(\phi, \chi) \) to extract the central charge \( c \) in the pseudo critical regime as well as from the pseudo critical points \( \phi_c(\chi) \). Note that the correlation length \( \xi \) is defined in terms of the ratio between the largest eigenvalue \( \epsilon_0 \) and the second largest eigenvalue \( \epsilon_1 \) of the transfer matrix: \( \xi = 1/\ln[\epsilon_0/\epsilon_1] \).

For our purpose, we focus on the region \((1.25\pi, 1.5\pi)\) of the parameter \( \phi \) and exploit the iTEBD to simulate the model (1). As a result, a pseudo critical point \( \phi_c(\chi) \) is detected from the iTEBD simulation for each value of the bond dimension \( \chi \). Therefore, the region is separated into the pseudo critical regime and the dimerized regime.

In order to characterize the pseudo critical regime, the finite-entanglement scaling for the entanglement entropy \( S(\phi, \chi) \) is performed for various values of \( \phi \) in the pseudo critical regime, with the bond dimension \( \chi \) ranging from 16 to 300. Here, we have randomly chosen \( \phi = 1.26\pi, 1.27\pi, 1.28\pi \) and \( 1.29\pi \), which are located in the pseudo critical regime. In Fig. \([S1]\) the best linear fit is exploited to estimate the central charge \( c \), which is listed in Table \([S1]\). The iTEBD simulation yields that the central charge \( c = 2 \) with a relative error being less than 2% in the pseudo critical regime.

In addition, the central charge \( c \) is extracted by performing...
the finite-entanglement scaling from the pseudo critical points \( \phi_c(\chi) \). In Fig. S2 we plot the entanglement entropy \( S(\phi, \chi) \) versus \( \ln \xi(\chi) \), with the bond dimension \( \chi \) ranging from 10 to 200. The best linear fit is exploited to estimate the central charge \( c = 1.8624 \), with a relative error being less than 7\%, compared to the expected value \( c = 2 \). Here, we stress that the less accurate central charge \( c \) extracted from the pseudo critical points results from an error in determining the locations of the pseudo critical points \( \phi_c(\chi) \), due to the high computational costs involved in the iTEBD simulation, which may be significantly improved if more trials are implemented.

The finite-entanglement scaling is performed for the entanglement entropy \( S(\phi, \chi) \), with \( \phi = 1.265\pi, 1.27\pi, 1.28\pi \) and \( 1.29\pi \), which are located in the pseudo critical regime from the U(1) iDMRG simulation. In Fig S3 we plot the entanglement entropy \( S(\phi, \chi) \) versus \( \ln \xi(\chi) \), with the bond dimension \( \chi \) ranging from 30 to 1000. The best linear fit is exploited to estimate the central charge \( c \), which is listed in Table S2. The U(1) iDMRG simulation yields that the central charge is \( c = 2 \), with a relative error being less than 4\% in the pseudo critical regime, consistent with the iTEBD simulation.

The central charge \( c \) is also extracted by performing the finite-entanglement scaling from the pseudo critical points \( \phi_c(\chi) \) between the pseudo critical regime and the dimerized regime, determined from the U(1) iDMRG simulation. In Fig S4 the best linear fit is exploited to estimate the central charge \( c = 1.833 \), with a relative error being less than 8.4\%. Similar to the iTEBD simulation, this may be significantly improved if more trials are implemented.

2. The central charge \( c \) from the U(1) iDMRG simulation

The U(1) iDMRG simulation is performed for the model (1) in the region \([1.25\pi, 1.4\pi] \). Two pseudo phase transition points are detected from the U(1) iDMRG simulation for each value of the bond dimension \( \chi \). One is a pseudo first-order phase transition point \( \phi_{f}(\chi) \), and the other is a pseudo critical point \( \phi_{c}(\chi) \). The latter also occurs in the iTEBD simulation, though the former does not. The two pseudo phase transition points separate the region into three regimes: the pseudo fractal regime, the pseudo critical regime and the dimerized regime.

![FIG. S2: (color online) The iTEBD simulation for the finite-entanglement scaling: \( S(\phi, \chi) \) versus \( \ln \xi(\chi) \), at the pseudo critical points \( \phi(\chi) \). The bond dimension \( \chi \) ranges from 10 to 200. The central charge \( c \) is extracted to be \( c = 1.8624 \), close to the expected value \( c = 2 \).](image1)

![FIG. S3: (color online) The U(1) iDMRG simulation for the finite-entanglement scaling: \( S(\phi, \chi) \) versus \( \ln \xi(\chi) \), with the bond dimension \( \chi \) ranging from 10 to 200. The central charge \( c \) is extracted to be \( c = 2 \) in the pseudo critical regime.](image2)

![FIG. S4: (color online) The U(1) iDMRG simulation for the finite-entanglement scaling: \( S(\phi, \chi) \) versus \( \ln \xi(\chi) \), at the pseudo critical points \( \phi(\chi) \). The bond dimension \( \chi \) ranges from 60 to 600. The central charge \( c \) is extracted to be \( c = 1.833 \), with a relative error being less than 8.4\%, compared to the expected value \( c = 2 \).](image3)

| \( \phi \) | 1.265\pi | 1.27\pi | 1.28\pi | 1.29\pi |
|---|---|---|---|---|
| \( c \) | 2.0478 | 2.0664 | 2.0478 | 2.055 |

Table S2: The central charge \( c \) is extracted to be \( c = 2 \) in the pseudo critical regime from the U(1) iDMRG simulation. Here, \( \phi = 1.265\pi, 1.27\pi, 1.28\pi \) and \( 1.29\pi \).
SIII. Pseudo SSB in the pseudo critical regime: the iTEBD and the U(1) iDMRG simulations

To set the stage, we introduce a few notations. For a directional vector \( \vec{n} = (\cos \zeta \cos \eta, \cos \zeta \sin \eta, \sin \zeta) \), the spin component \( S_{j}^{z} \) along the direction of \( \vec{n} \) takes the form: \( S_{j}^{z} = \cos \zeta \cos \eta S_{j}^{x} + \cos \zeta \sin \eta S_{j}^{y} + \sin \zeta S_{j}^{z} \). Our task is to determine the directional vector \( \vec{n} \), along which the global spin \( \vec{S} \) is spontaneously polarized for a given ground state wave function generated from the iTEBD and the U(1) iDMRG algorithms. For this purpose, a subroutine is developed to determine the directional vector \( \vec{n} \) for spontaneous polarization as a result of pseudo SSB in the pseudo critical regime, with flow chart being shown in Fig. S5.

For a given \( \zeta \in [0, \pi] \) and \( \eta \in [0, 2\pi] \), we compute the ground state fidelity per lattice site \( d \) [S12] (also cf. Eq. S2 below in Sec. S5) between \( |\psi\rangle \) and \( |\phi\rangle \). Here, \( |\psi\rangle \) is a ground state wave function generated from the iTEBD and the U(1) iDMRG algorithms, and \( U = \exp(i\omega S^{\vec{n}}) \), with \( \omega \) being a real number. The idea is to optimize the ground state fidelity per lattice site \( d \), when \( \zeta \) and \( \eta \) are varied.

The procedure is as follows. (i) Input an initial value of \( \zeta \) and \( \eta \), \( \zeta \) is varied from 0 to \( \pi \), and \( \eta \) is varied from 0 to \( 2\pi \), respectively, with the step size being 0.01. (ii) Compute the ground state fidelity per lattice site \( d \). (iii) Compare the ground state fidelity per lattice site \( d \) to 1, to see if \( |d - 1| > \varepsilon \), with \( \varepsilon \) being a preset error. If \( |d - 1| > \varepsilon \), then return to (i). If \( |d - 1| < \varepsilon \), then exit and save the values of \( \zeta \) and \( \eta \). In our computation, \( \omega \) takes an arbitrary value, and the preset error \( \varepsilon \) is set to be \( \varepsilon = 10^{-6} \). Note that the final outcome should not depend on the value of \( \omega \).

1. Pseudo SSB in the pseudo critical regime: the iTEBD simulation

The iTEBD algorithm is exploited to generate ground state wave functions for a few selected values of \( \phi \): \( \phi = 1.26\pi, 1.27\pi, 1.28\pi \) and \( 1.29\pi \), in the pseudo critical region. Here, we choose the bond dimension \( \chi = 30 \). The directional vector \( \vec{n} \) is determined for spontaneous polarization as a result of pseudo SSB. The results for \( \zeta \) and \( \eta \) thus yielded are listed in Table S3. This indicates that \( \vec{n} \) is random, meaning that spontaneous polarization in any direction is possible, consistent with pseudo SSB from SU(2) to U(1). Here, U(1) is generated from \( \vec{S}^{\vec{n}} \).

\[
\begin{array}{cccc}
\phi & 1.26\pi & 1.27\pi & 1.28\pi & 1.29\pi \\
\zeta & 4.7124 & 4.7124 & 4.7124 & 4.7124 \\
\eta & 0 & 0 & 0 & 0
\end{array}
\]

2. Pseudo SSB in the pseudo critical regime: the U(1) iDMRG simulation

The U(1) iDMRG algorithm is exploited to generate ground state wave functions for a few selected values of \( \phi \): \( \phi = 1.26\pi, 1.27\pi, 1.28\pi \) and \( 1.29\pi \) in the pseudo critical regime, with the bond dimension \( \chi = 40 \).

The directional vector \( \vec{n} \) is determined for spontaneous polarization as a result of pseudo SSB. The results for \( \zeta \) and \( \eta \) thus yielded are listed in Table S4, meaning that \( \vec{n} \) is always along the \( z \) axis, as anticipated. This is consistent with pseudo SSB from SU(2) to U(1). Here, U(1) is generated from \( \vec{S}^{\vec{n}} \).

\[
\begin{array}{cccc}
\phi & 1.26\pi & 1.27\pi & 1.28\pi & 1.29\pi \\
\zeta & 0 & 0 & 0 & 0 \\
\eta & 0 & 0 & 0 & 0
\end{array}
\]

SIV. The pseudo local order parameter \( \langle O_{\chi}(\vec{n}) \rangle \) and the dimerized local order parameter \( \langle D_{\chi,1}(\vec{n}) \rangle \)

As noted in Ref. S7, the tensor network algorithms in the infinite MPS representation naturally lead to infinitely degenerate ground states in a critical regime, due to the finiteness of the bond dimension. This results in the so-called pseudo continuous SSB [S7], with the pseudo symmetry-breaking order being quantified in terms of a pseudo local order parameter \( \langle O_{\chi}(\vec{n}) \rangle \). Here, we note that the pseudo GMs involved are of type-A, if we adapt the classification of the GMs in
effective field theories [S8] to the pseudo GMs arising from the finiteness of the bond dimension in the tensor network simulations. In order to keep consistency with the Mermin-Wagner theorem [S9], \( \langle O(\phi) \rangle \) must be scaled down to zero, when the bond dimension \( \chi \) tends to infinity. Here, by the Mermin-Wagner theorem we mean a statement that no continuous symmetry group is spontaneously broken in one dimensional quantum many-body systems, if the GMs involved are of type-A. As a consequence, the Mermin-Wagner theorem does not rule out the possibility for continuous SSB in one dimensional many-body quantum systems, if the GMs involved are of type-B [S8, S10].

Physically, the finiteness of the bond dimension \( \chi \) in the implementation of the tensor network algorithms in the infinite MPS representation amounts to introducing extra long-range interactions into the model Hamiltonian, which turns a quantum many-body system in one spatial dimension into one in higher than one spatial dimensions, if one insists to consider only the short-range interactions. This makes it possible for continuous SSB with the pseudo GMs of type-A to occur in numerical simulations of one dimensional quantum many-body systems, which constitutes the origin of a pseudo critical regime.

For the model (1), a pseudo critical regime with the central charge \( c = 2 \) does occur, as a result of pseudo SSB from \( SU(2) \) to \( U(1) \), which has been confirmed in Sec. SIIII. Given the directional vector \( \mathbf{h} \) for spontaneous polarization is random, the three spin components \( S^+, S^z \), and \( S^- \) are broken, but leaving \( S^\theta \) invariant. That is, \( \langle S^+ \rangle = 0, \langle S^z \rangle = 0, \) and \( \langle S^- \rangle = 0 \) in the iTEBD simulation. As it turns out, the remaining five local components of the eight generators, \( K^\alpha_\ell \) \((\alpha = 1, \ldots, 8)\), of the \( SU(3) \) group, discussed in Sec. SI, yield non-zero expectation values, thus constituting a vector which acts as a pseudo local order parameter. However, it is convenient to choose a pseudo local order parameter, which does not depend on an initial state. For this purpose, we adopt \( \langle O_j(\phi) \rangle \), defined in (3), as the pseudo local order parameter. Note that \( \langle K^\alpha_i \rangle \), \( \langle K^\alpha_2 \rangle \), and \( \langle K^\gamma \rangle \), which are essentially \( \langle S^\alpha \rangle \), \( \langle S^\beta \rangle \), and \( \langle S^\gamma \rangle \), are included in (3), since they are all zero. Indeed, the (Cartan) metric tensor \( g_{\alpha\beta} \) is defined as \( g_{\alpha\beta} = \sum_{\ell=1}^{8} \gamma_{\alpha\beta\ell} \gamma_{\alpha\beta\ell} \), with \( \gamma_{\alpha\beta\ell} \) being the structural constants of the \( SU(3) \) group: \( [K_\alpha, K_\beta] = i \sum_{\ell=1}^{8} \gamma_{\alpha\beta\ell} K_\ell \). Alternatively, the metric tensor \( g_{\alpha\beta} \) may be defined through the Killing form as \( g_{\alpha\beta} = B(K_\alpha, K_\beta) \) [S11]. The independence of \( \langle O_j(\phi) \rangle \) on an initial state is confirmed.

In addition, the dimerized local order parameter \( \langle D_{j,j+1}(\phi) \rangle \) is introduced to characterize the dimerized phase. The dimerized local order parameter \( \langle D_{j,j+1}(\phi) \rangle \), together with the pseudo local order parameter \( \langle O_j(\phi) \rangle \), provide other means to confirm the pseudo phase transition points detected from the entanglement entropy \( S(\phi, \chi) \).

### 1. The pseudo local order parameter \( \langle O_j(\phi) \rangle \) and the dimerized local order parameter \( \langle D_{j,j+1}(\phi) \rangle \): the iTEBD simulation

In Fig. S6 we plot the pseudo local order parameter \( \langle O_j(\phi) \rangle \) as a function of \( \phi \). From this we see the same pseudo critical points \( \phi_c(\chi) \) as those detected from the entanglement entropy \( S(\phi, \chi) \). Specifically, the pseudo critical points \( \phi_c(\chi) \) between the pseudo critical regime and the dimerized regime are located at \( \phi_c(\chi) = 1.33\pi, 1.306\pi, 1.299\pi, \) and \( 1.298\pi \) for the bond dimension \( \chi = 30, 100, 200, \) and \( 300, \) respectively.

We remark that the pseudo local order parameter \( \langle O_j(\phi) \rangle \) is nonzero in the pseudo critical regime, due to the finiteness of the bond dimension \( \chi \), which is scaled down to zero, as the bond dimension \( \chi \) tends to infinity. This is shown in Fig. S6 as required to keep consistency with the Mermin-Wagner theorem [S9].

We plot the dimerized local order parameter \( \langle D_{j,j+1}(\phi) \rangle \) in Fig. S7 as a function of \( \phi \) in the region \( (1.25\pi, 1.5\pi) \), with the bond dimension \( \chi = 30, 80, 100, 200 \), and \( 300, \) Note that the dimerized local order parameter \( \langle D_{j,j+1}(\phi) \rangle \) tends to be saturated, as the bond dimension \( \chi \) increases. The same pseudo
1. The pseudo local order parameter \( \langle O_j(\chi) \rangle \) and the entanglement entropy \( S(\phi, \chi) \)

\[
\langle O_j(\chi) \rangle = \frac{3 - 9/2(\langle S_j^z \rangle)^2}{1 - 3/2(\langle S_j^z \rangle)^2}
\]

In the U(1) iDMRG simulation, both the pseudo fractal regime and the pseudo critical regime may be characterized in terms of the pseudo order parameter \( \langle O_j(\chi) \rangle \). Note that, since the \( z \)-component of the total spin is preserved to be zero, only \( K_f^j = 1 - 3/2(\langle S_j^z \rangle)^2 \) yields a non-zero expectation value. Therefore, the pseudo local order parameter \( \langle O_j(\chi) \rangle \), as defined in (3), is reduced to \( \langle O_j \rangle = 3 - 9/2(\langle S_j^z \rangle)^2 \). Therefore, \( \langle O_j(\chi) \rangle \) may be replaced by the pseudo local order parameter \( \langle (S_j^z)^2 \rangle \). In Fig. S8 we plot the pseudo local order parameter \( \langle (S_j^z)^2 \rangle \) as a function of \( \phi \), for the bond dimension \( \chi = 30, 200, 500, 1000 \) and 2000, respectively. From this we observe that two pseudo phase transition points are detected from the pseudo local order parameter \( \langle (S_j^z)^2 \rangle \), consistent with those from the entanglement entropy \( S(\phi, \chi) \).

2. The pseudo local order parameter \( \langle O_j(\chi) \rangle \) and the dimerized local order parameter \( \langle D_{j,j+1}(\chi) \rangle \): the U(1) iDMRG simulation

The SU(2) iDMRG simulation is performed for the model (1) in the region \( (1.251 \pi, 1.335 \pi) \). In Fig. S9 we plot the dimerized local order parameters \( \langle D_{j,j+1}(\chi) \rangle \) as a function of \( \phi \), with the bond dimension \( \chi = 480, 600 \) and 5000.

3. The dimerized local order parameter \( \langle D_{j,j+1}(\chi) \rangle \) from the SU(2) iDMRG simulation

The SU(2) iDMRG simulation is performed for the model (1) in the region \( (1.251 \pi, 1.335 \pi) \). In Fig. S10 we plot the dimerized local order parameters \( \langle D_{j,j+1}(\chi) \rangle \) as a function of \( \phi \), with the bond dimension \( \chi = 480, 600 \) and 5000.
the bond dimension $\chi = 480,600$ and 5000, in the region $(1.251\pi, 1.28\pi]$. Fig. [S10] indicates that the dimerized order parameter ($D_{j,j+1}(\phi)$) decreases, as $\phi = 5\pi/4$ is approached. However, we remark that the accuracies get worse, and so the simulation results become less reliable, when $\phi$ gets close to $5\pi/4$.

**SV. Ground state fidelity per lattice site $d(\phi_1, \phi_2)$ from the SU(2) iDMRG simulation**

For two given ground states $|\psi(\phi_1)\rangle$ and $|\psi(\phi_2)\rangle$, the ground state fidelity $F = \langle \psi(\phi_1)|\psi(\phi_2)\rangle$ asymptotically scales as $F(\phi_1, \phi_2) \sim d(\phi_1, \phi_2)^{\chi}$, with $L$ being the system size. Here, $d(\phi_1, \phi_2)$ is the ground state fidelity per lattice site, which is well-defined in the thermodynamic limit [S12]:

$$\ln d(\phi_1, \phi_2) \equiv \lim_{L \to \infty} \ln \frac{F(\phi_1, \phi_2)}{L}. \quad (S2)$$

As a convention, we choose $|\psi(\phi_1)\rangle$ as a reference state, and regard $d(\phi_1, \phi_2)$ as a function of $\phi_2$ for fixed $\phi_1$.

In Fig [S11] we plot the ground state fidelity per lattice site $d(\phi_1, \phi_2)$ as a function of $\phi_2$, with $\chi = 5000$, in the region $(1.251\pi, 1.28\pi]$. Here, the reference state $|\psi(\phi_1)\rangle$ has been chosen to be located at $\phi_1 = 1.252\pi$, $1.274\pi$ and $1.278\pi$, respectively. No pinch point is observed, indicating that there is no phase transition in the region $(1.251\pi, 1.28\pi]$.

**SVI. Two scenarios arising from the tensor network simulations**

For each value of the bond dimension $\chi$, the iTEBD simulation yields a pseudo critical point $\phi_c(\chi)$, which may be detected by means of the entanglement entropy $S(\phi, \chi)$, the pseudo local order parameter (O$_\chi$($\chi$)) and the dimerized local order parameter ($D_{j,j+1}(\chi)$), as shown in Fig. 2, Fig. [S6] and Fig. [S7] respectively. The central charge $c$ is estimated to be $c = 2$, with a reasonable relative error, in the pseudo critical regime. One may perform an extrapolation of the pseudo critical point $\phi_c(\chi)$ to see if they terminate at the SU(3) ferromagnetic point. In Fig. 2, we have shown that such an extrapolation is possible if one requires the pseudo critical points $\phi_c(\chi)$ to terminate at the SU(3) ferromagnetic point. However, the same extrapolation works well if the pseudo critical points $\phi_c(\chi)$ terminate somewhere next to the SU(3) ferromagnetic point, within the accuracies achievable. Therefore, two scenarios arise: (i) the pseudo critical points $\phi_c(\chi)$ terminate at the SU(3) ferromagnetic point when the bond dimension $\chi$ tends to infinity; (ii) the pseudo critical points $\phi_c(\chi)$ terminate at a point $\phi_c$ away from the SU(3) ferromagnetic point. Here, $\phi_c$ is defined to be a point, to which $\phi_c(\chi)$ approaches, when the bond dimension $\chi$ tends to infinity.

The two scenarios from the iTEBD simulation have been sketched in Fig [S12]. If the first scenario is valid, then the phase transition from the ferromagnetic phase to the dimerized phase is direct, with the phase transition point located at the SU(3) ferromagnetic point. If the second scenario is valid, then there are two phase transitions from the ferromagnetic phase to the dimerized phase: one is located at the SU(3) ferromagnetic point, and the other is located at the point $\phi_c$. Hence, there is a critical nematic phase between the ferromagnetic phase and the dimerized phase.

In the U(1) iDMRG simulation, we target at a ground state with the $z$-component of the total spin being zero. As shown in Fig. 3, Fig. [S8] and Fig. [S9] from the entanglement entropy $S(\phi, \chi)$, the pseudo local order parameter (O$_\chi$($\chi$)) and the dimerized local order parameter ($D_{j,j+1}(\chi)$), two phase transitions are detected: one is the pseudo first-order phase transition points $\phi_f(\chi)$, and the other is the pseudo critical points $\phi_c(\chi)$, which are captured by the iTEBD simulation. The occurrence of the pseudo first-order phase transition points $\phi_f(\chi)$ makes it possible to refine the two scenarios.

For the first scenario, the situation is simple: the pseudo critical regime from the iTEBD simulation is now divided into a pseudo fractal regime and a pseudo critical regime, with the pseudo first-order phase transition separating the two regimes. For the second scenario, one may argue that the pseudo fractal regime should not extend up to a point beyond $\phi_c$. That is, $\phi_c$ must always be in the pseudo critical regime for any value of the bond dimension $\chi$. Actually, performing a truncation by means of the bond dimension $\chi$ amounts to introducing an energy scale so that a gapped ground state becomes gapless, as long as the gap is small enough, compared to the energy scale thus introduced. With this in mind, we may expect that a pseudo critical point appears as a shift of the critical point $\phi_c$, but both are always in the pseudo critical regime for any value of the bond dimension $\chi$. In addition, the pseudo fractal regime reflects fluctuations from the SU(3) ferromagnetic point. Hence, it has to terminate at a point $\phi_c$, which represents the onset of the ferromagnetic fluctuations from the SU(3) ferromagnetic point. Therefore, we have $\phi_c < \phi_f$. This is in accord with our U(1) iDMRG simulation results. In practice, $\phi_f$ may be replaced by the pseudo first-order phase transition point $\phi_f(\chi)$ corresponding to the smallest value of the bond dimension $\chi$ achievable. The two scenarios from the
The fractal dimension $d_f$ in the pseudo fractal regime: the spin-1/2 XXZ model and the spin-1 bilinear-biquadratic model

According to the finite block-size scaling (4) of the entanglement entropy $S(n)$, we are able to extract the fractal dimension $d_f$ from the tensor network simulations for the spin-1 bilinear-biquadratic model in the pseudo fractal regime. We emphasize that such a fractal regime only occurs in the U(1) iDMRG simulation.

For an illustrative purpose, we include our simulation results for the spin-1/2 quantum XXZ model. We remark that it is necessary to develop a subroutine to efficiently calculate the block entanglement entropy $S(n)$. As it turns out, the fractal dimension $d_f$ is equal to the number of the pseudo GMs of type-B, which may be regarded as an adaptation of a general statement [S10] about the connection between the fractal dimension and the number of the GMs of type-B to the numerical artifacts.

1. The fractal dimension $d_f$ for the spin-1/2 XXZ model in the vicinity of the ferromagnetic point at $\Delta = -1$

The Hamiltonian for the spin-1/2 anisotropic XXZ model takes the form:

$$H_{XXZ} = \sum_j \left[ S^x_j S^x_{j+1} + S^y_j S^y_{j+1} + \Delta S^z_j S^z_{j+1} \right],$$

where $\Delta$ denotes the anisotropic coupling parameter. At $\Delta = -1$, the model possesses the SU(2) symmetry, generated by $\sum_j (-1)^j S^z_j$, $\sum_j (-1)^j S^y_j$, and $\sum_j S^z_j$, which are staggered.

The ground state phase diagram for the spin-1/2 XXZ model involves three distinct phases in the entire parameter region: the ferromagnetic phase ($\Delta < -1$), the critical XY phase ($-1 < \Delta \leq 1$) with the central charge $c = 1$, and the Néel phase ($\Delta > 1$). In the Néel phase, the ground state wave functions are two-fold degenerate, due to the spin-spin interaction.
one-site translational invariance. In the ferromagnetic phase, the ground state wave functions are two-fold degenerate and fully polarized along the z axis. The model is special at the point $\Delta = -1$. In fact, it possesses highly degenerate ground states and is not conformally invariant. A proper description requires to introduce the fractal dimension $d_f$ \([S13]\). Indeed, this description is also necessary to understand the simulation results from the U(1) iDMRG algorithm, if one targets at a ground state with the z-component of the total spin being zero.

In Fig.\([S14]\) the finite-entanglement scaling of the entanglement entropy $S(n)$ is performed for a few chosen values of $\Delta$. An interesting observation is that, if $\Delta$ is far away from the ferromagnetic point at $\Delta = -1$, then all the data falls on a straight line, as the correlation length $\xi(\chi)$ increases with $\chi$. If $\Delta$ gets close enough to $\Delta = -1$, the data starts to fall on two segments with a different slope asymptotically (see also Ref. \([S14]\)). Physically, this implies a crossover from the pseudo fractal regime to the pseudo critical regime in the vicinity of $\Delta = -1$, as the bond dimension $\chi$ increases.

In the critical XY regime, the central charge $c$ may be extracted from the finite block-size scaling $S(n) \sim c/3 \ln n$ \([S13]\), which is listed in Table \([S5]\). In the fractal regime, the fractal dimension $d_f$ may be extracted from the finite block-size scaling of the entanglement entropy $S(n)$ \([4]\), which is listed in Table \([S6]\).

A conclusion one may draw from the simulation results is that, as $\Delta = -1$ is approached, the fractal dimension $d_f$ tends to 1, as anticipated. We remark that the U(1) iDMRG algorithm works well even when $\Delta$ is so close to $-1$, given that it still captures the crossover behavior from the pseudo fractal regime to the pseudo critical regime, as the bond dimension $\chi$ increases.

### Table S5: The central charge $c$ in the critical XY regime is extracted from the U(1) iDMRG algorithm for the spin-1/2 XXZ model, with the block size $n$ being from 6 to 24. Here, $\Delta = -0.2$ and $-0.5$.

| $\chi$ | 20  | 40  | 60  | 50  | 60  |
|---|---|---|---|---|---|
| $\Delta = -0.2$ | $c$ | 1.0143 | 1.0032 | 1.0026 | 1.002 |
| $\Delta = -0.5$ | $c$ | 1.0395 | 1.0086 | 1.0056 | 1.0038 |

### Table S6: The fractal dimension $d_f$ in the pseudo fractal regime is extracted from the U(1) iDMRG algorithm for the spin-1/2 XXZ model, with the block size $n$ being from 6 to 24. When $\Delta = -1$ is approached, $d_f$ tends to 1.

| $\chi$ | 20  | 40  | 60  | 70  | 78  |
|---|---|---|---|---|---|
| $\Delta = -0.9999$ | $d_f$ | 0.799 | 0.897 | 0.9406 | 0.9504 | 0.953 |
| $\Delta = -1$ | $d_f$ | 0.7998 | 0.8992 | 0.9486 | 0.97 | 0.972 |

![Figure S15](image.png)

FIG. S15: (color online) The finite block-size scaling of the entanglement entropy $S(n)$ in the pseudo fractal regime for the spin-1 bilinear-biquadratic model, with the bond dimension $\chi = 60, 80, 100, 120, 140$ and 150, for (a) $\phi = 1.26\pi$; (b) $\phi = 1.254\pi$; and (c) $\phi = 1.252\pi$.

### Table S7: The fractal dimension $d_f$ in the pseudo fractal regime is extracted from the U(1) iDMRG simulation for the spin-1 bilinear-biquadratic model, with the block size $n$ being from 6 to 16. Here, $\phi = 1.26\pi, 1.254\pi$ and $1.252\pi$.

| $\chi$ | 60  | 80  | 100 | 120 | 140 | 150 |
|---|---|---|---|---|---|---|
| $\phi = 1.26\pi$ | $d_f$ | 0.9802 | 1.0024 | 1.0122 | 1.0384 | 1.035 | 1.0364 |
| $\phi = 1.254\pi$ | $d_f$ | 0.9604 | 0.9554 | 1.0178 | 1.0148 | 1.0368 | 1.0418 |
| $\phi = 1.252\pi$ | $d_f$ | 0.9874 | 0.9794 | 0.9898 | 1.0018 | 1.0314 | 1.0404 |

2. The fractal dimension $d_f$ for the spin-1 bilinear-biquadratic model in the vicinity of the SU(3) ferromagnetic point

The fractal dimension $d_f$ is 2 for the model (1) at the SU(3) ferromagnetic point \([S10]\). We need to extract the fractal dimension $d_f$ from the finite block-size scaling in the pseudo fractal regime in the vicinity of the SU(3) ferromagnetic point for the spin-1 bilinear-biquadratic model. In Fig.\([S15]\) the finite block-size scaling of the entanglement entropy $S(n)$ is performed, with the bond dimension $\chi = 60, 80, 100, 120, 140$ and 150, for (a) $\phi = 1.26\pi$; (b) $\phi = 1.254\pi$; and (c) $\phi = 1.252\pi$, which are located in the pseudo fractal regime. The best linear fit yields that the fractal dimension $d_f = 1$, with $n$ ranging from 6 to 16, with a relative error being less than 5%, as shown in Table \([S7]\). Note that the fractal dimension $d_f$ tends to saturation, as the bond dimension $\chi$ increases.
is adapted from Ref. [S17] undergoing a quadratic Zeeman effect. Here, \( U(1) \) is generated from charge. Therefore, two generators are spontaneously broken, yielding two pseudo GMs of type-A. Therefore, the central charge \( c \) takes \( c = 2 \), given that the central charge \( c \) measures the number of gapless low-lying excitations [S16]. The same argument also applies to a ground state wave function generated from the U(1) iDMRG simulation, with the only difference being that spontaneous polarization is always along the \( z \)-axis in the spin space. This is in sharp contrast to the SU(2) iDMRG simulation, in which the full SU(2) symmetry is implemented so that no pseudo SSB is allowed to occur. Thus, the pseudo critical regime only occurs in the iTEBD simulation and the U(1) iDMRG simulation, but not in the SU(2) iDMRG simulation.

As for the pseudo fractal regime, it only occurs in the U(1) iDMRG simulation, since a ground state with the \( z \)-component of the total spin being zero is targeted, ensuring that it is highly entangled so that a scale invariant ground state, characterized in terms of the fractal dimension \( d_f \) is equal to 2. Here, we note that the fractal dimension \( d_f \) measures the number of low-lying excitations [S10] (also cf. Sec. SVII in SM). As a consequence, one may conclude that the pseudo fractal regime originates from the proximity effect to a highly entangled scale invariant ground state at the SU(3) ferromagnetic point.

The final question we need to address concerns the origin of the pseudo critical regime. Given there is only one phase transition point from the ferromagnetic phase to the dimerized phase located at the SU(3) ferromagnetic point, the model (1) itself does not accommodate any critical nematic phase close to the transition point. Therefore, one may expect that a critical nematic phase exists in the vicinity of the SU(3) ferromagnetic point, if extra interactions are introduced into the model (1). Since \( \langle S^z_j \rangle = 0 \), \( \langle S^y_j \rangle = 0 \), and \( \langle S^z_j \rangle = 0 \), in both the iTEBD simulation and the U(1) simulation, the linear Zeeman effect is not relevant. Instead, we turn to a quadratic Zeeman effect, described by the Hamiltonian:

\[
H_{BBXY} = J \sum_j \left[ (\cos \phi(S^z_j S^z_{j+1}) + \sin \phi(S^y_j S^y_{j+1})^2) \right] + D_x \sum_j (S^x_j)^2 + D_y \sum_j (S^y_j)^2 + D_z \sum_j (S^z_j)^2. \tag{S4}
\]

Here, \( D_x \), \( D_y \), and \( D_z \) are coupling constants describing a quadratic Zeeman effect. A special case with \( D_x = 0 \) and \( D_z = 0 \) has been investigated in Ref. [S17]. The ground state phase diagram is sketched in Fig. S16 close to the SU(3) ferromagnetic point. It is found that the XY nematic phase, with the central charge \( c = 1 \), extends up to the SU(3) ferromagnetic point. In addition, the phase transition from the XY nematic phase to the dimerized phase is of the Kosterlitz-Thouless type. Therefore, for the model (S4), there should be the XY and ZX critical nematic phases, with the central charge \( c = 1 \), in the vicinity of the SU(3) ferromagnetic point.

We have demonstrated in Sec. SIII that, in the pseudo critical regime, pseudo SSB from SU(2) symmetry to U(1) occurs. Here, U(1) is generated from \( S^z \), with the directional vector \( \vec{r} \) being random and depending on an initial state for a ground state wave function generated from the iTEBD simulation. Therefore, two generators are spontaneously broken, yielding two pseudo GMs of type-A. Therefore, the central charge \( c \) takes \( c = 2 \), given that the central charge \( c \) measures the number of gapless low-lying excitations [S16]. The same argument also applies to a ground state wave function generated from the U(1) iDMRG simulation, with the only difference being that spontaneous polarization is always along the \( z \)-axis in the spin space. This is in sharp contrast to the SU(2) iDMRG simulation, in which the full SU(2) symmetry is implemented so that no pseudo SSB is allowed to occur. Thus, the pseudo critical regime only occurs in the iTEBD simulation and the U(1) iDMRG simulation, but not in the SU(2) iDMRG simulation.

As for the pseudo fractal regime, it only occurs in the U(1) iDMRG simulation, since a ground state with the \( z \)-component of the total spin being zero is targeted, ensuring that it is highly entangled so that a scale invariant ground state, characterized in terms of the fractal dimension \( d_f \) is equal to 1 corresponding to one pseudo GM of type-B, is able to compete with a ground state, characterized in terms of the central charge \( c = 2 \) corresponding to two pseudo GMs of type-A. In contrast, the \( z \)-component of the total spin is not preserved in the iTEBD simulation. Thus, it is impossible to produce such a highly entangled scale invariant ground state. We remark that the fractal dimension \( d_f \) is equal to 1 in the pseudo fractal regime, since there is one pseudo GM of type-B as a result of pseudo SSB from SU(2) to U(1). This pseudo GM of type-B evolves into one of the two GMs at the SU(3) ferromagnetic point. Indeed, at this point, two GMs of type-B emerge, as a consequence of SSB from SU(3) to SU(2) \( \otimes \) U(1), implying that the fractal dimension \( d_f \) is equal to 2. Here, we note that the fractal dimension \( d_f \) measures the number of low-lying excitations [S10] (also cf. Sec. SVII in SM). As a consequence, one may conclude that the pseudo fractal regime originates from the proximity effect to a highly entangled scale invariant ground state at the SU(3) ferromagnetic point.
due to the local constraints: \((S^x_j)^2 + (S^y_j)^2 + (S^z_j)^2 = 2\), if one chooses \(D_0\) to be a fixed constant. The two critical nematic phases are sketched in Fig. [S1] which are symmetric under the unitary transformation: \(S^x_j \leftrightarrow S^z_j\) and \(S^y_j \leftrightarrow -S^z_j\). We emphasize that no other critical regime with the central charge \(c = 1\) exists in the vicinity of the SU(3) ferromagnetic point, since the model \([S4]\) possesses a U(1) symmetry only when \(D_y = D_x = D_z\), which is necessary for a critical nematic phase to emerge. This explains why two pseudo GMs of type-A emerge in the iTEBD and U(1) iDMRG simulations. However, as long as it is close enough to the SU(3) ferromagnetic point, in other words, it is the proximity effect to the two critical nematic phases with \(c = 1\), which are infinitesimally close to each other in the vicinity of the SU(3) ferromagnetic point, that constitutes the origin of the pseudo critical regime with \(c = 2\) in the iTEBD and U(1) iDMRG simulations. However, the two critical nematic phases with \(c = 1\) do not meet each other at the SU(3) ferromagnetic point, though asymptotically close, implying that no SU(2) WZW model with level \(k = 4\) exists as a limit of the pseudo critical regime, as the bond dimension \(\chi\) tends to infinity. Finally, the proximity effect to the two critical nematic phases with \(c = 1\) offers a mechanism to account for the opening of an exponentially decaying small gap, when the SU(3) ferromagnetic point is approached from the dimerized phase. This is due to the facts that the phase transition from the XY/ZX nematic phase to the dimerized phase is of the Kosterlitz-Thouless type, meaning an essential singularity arising from a marginally relevant perturbation at the Kosterlitz-Thouless transition points, and that the two lines of the critical points themselves get close to each other in the vicinity of the SU(3) ferromagnetic point.

We thank Murray Batchelor, Sam Young Cho, John Ove Fjærestad, Javier Rodríguez-Laguna, Silvia N. Santalla, and Germán Sierra for enlightening discussions. The work is supported by the National Natural Science Foundation of China (Grant No. 11805285).

\[\text{[S1]} \text{ I. Affleck, J. Phys.: Condens. Matter 2, 405 (1990).}\]