Dynamical Determination of the Innermost Stable Circular Orbit of Binary Neutron Stars

Pedro Marronetti, Matthew D. Duez, and Stuart L. Shapiro
Department of Physics, University of Illinois at Urbana-Champaign, Urbana, IL 61801

Thomas W. Baumgarte
Department of Physics and Astronomy, Bowdoin College, Brunswick, ME 04011

We determine the innermost stable circular orbit (ISCO) of binary neutron stars (BNSs) by performing dynamical simulations in full general relativity. Evolving quasiequilibrium (QE) binaries that begin at different separations, we bracket the location of the ISCO by distinguishing stable circular orbits from unstable plunges. We study $\Gamma = 2$ polytropes of varying compactions in both corotational and irrotational equal-mass binaries. For corotational binaries we find an ISCO orbital angular frequency somewhat smaller than that determined by applying turning-point methods to QE initial data. For the irrotational binaries the initial data sequences terminate before reaching a turning point, but we find that the ISCO frequency is reached prior to the termination point. Our findings suggest that the ISCO frequency varies with compactness but does not depend strongly on the stellar spin. Since the observed gravitational wave signal undergoes a transition from a nearly periodic “chirp” to a burst at roughly twice the ISCO frequency, the measurement of its value by laser interferometers (e.g. LIGO) will be important for determining some of the physical properties of the underlying stars.

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The emission of gravitational radiation drives the slow inspiral of neutron star and black hole binaries towards their coalescence and merger. Fully general-relativistic numerical simulations are required for the accurate description of the late inspiral and plunge epochs of the binary evolution (see, e.g., [1] for a recent review). While complete orbits of binary black holes have not been numerically simulated yet, simulations of binary neutron star (BNS) mergers are now becoming sufficiently mature to provide results of astrophysical interest (e.g. [2]).

One piece of information of great astrophysical interest is the location of the innermost stable circular orbit (ISCO). The evolution of a binary system occurs in three distinct phases [3]: (1) A slow, adiabatic inspiral phase that is driven by gravitational radiation reaction forces and can be approximated as a sequence of quasi-circular orbits; (2) a brief transition phase, where the inward radial motion increases and the orbital motion changes from slow inspiral to rapid plunge; (3) a plunge phase, terminating in the merger of the objects. The ISCO resides within the transition region; its identification is complicated by the fact that it is not arbitrarily sharp and cannot be localized precisely. The gravitational wave quasi-periodic “chirp” signal of the slow binary inspiral ends at about the twice the orbital angular frequency of the ISCO, where it changes its form to a wave signal characteristic of a burst (compare [4]).

Within the framework of Newtonian and post-Newtonian gravity, the ISCO has been determined by different methods (see, e.g., the reviews [5, 6] and references therein). Much less is known for fully relativistic binaries. For corotational binaries, a turning point on a curve of the binding energy vs. separation for quasiequilibrium (QE) models along a sequence of constant rest mass marks the onset of secular instability [7, 8]. In the following we will refer to this point as the QE-ISCO. No such theorem exists for irrotational binaries or for the onset of dynamical instability [9]. Locating the ISCO dynamical instability therefore requires dynamical evolution simulations of the full set of Einstein’s equations for the gravitational field, coupled to relativistic hydrodynamics in the case of BNSs.

In this paper we present the first attempt to dynamically locate this ISCO. We identify BNS configurations that correspond to stable and unstable circular orbits by evolving binary initial data sets for different separations. The objective is to bracket the location of the ISCO by distinguishing configurations that can maintain quasi-circular motion for more than one orbital period and systems that plunge and coalesce in a fraction of that time.

We adopt the QE initial data presented by Marronetti and Shapiro [10], describing two identical neutron stars in quasi-circular orbit. These data have been constructed using the conformal thin-sandwich (CTS) decomposition of the constraint equations, together with maximal slicing and spatial conformal flatness. The formalism introduced in [10] allows for a free specification of the spin of the stars in an approximate fashion. In this paper we consider corotational binaries as well as “irrotational” binaries [11] with zero (equatorial) fluid circulation, which are believed to be more realistic astrophysically [12]. Sequences of corotating binaries feature a minimum in the binding energy at the QE-ISCO. Irrotational binaries,
however, typically terminate before reaching this minimum (see Figs. 7 and 8 in [10]; also [13, 14] as well as [1] for more references).

We adopt a $\Gamma = 2$ polytropic equation of state (EOS) for which the maximum rest mass (gravitational mass) of a star in isolation in nondimensional units [17] is $m_0 = 0.180$ ($m = 0.164$) with a compaction ratio of $(m/R)_\infty = 0.216$. We study models that have two different compaction ratios in isolation: a moderate value $(m/R)_\infty = 0.142$ (both corotational and irrotational binaries; cases A and B) and a high value $(m/R)_\infty = 0.195$ (only irrotational binaries; case C). These compaction ratios correspond to individual stars with rest masses $m_0 = 0.1469$ and $m_0 = 0.1767$, respectively. The fully general relativistic hydrodynamical code employed for this study has been introduced in Duez et al. [10]. We evolve the gravitational fields using the BSSN formalism [17] with a Courant factor of 0.46. We approximate maximal slicing with a “K-driver” and use a “Gamma-driver” shift condition that keeps $\bar{\Gamma} = \bar{\gamma}^{lm} \bar{\Gamma}_lm_0$ approximately constant. The simulations were performed on uniform Cartesian grids in a reference frame that rotates with the binary, which improves conservation of angular momentum [12, 18] and reduces the spurious eccentricities of the stable orbits. In all cases presented here the spatial volume covered by the grids is completely enclosed by the light cylinder (defined by coordinate radius $R_L = 1/\Omega_{\text{orb}}$). A more detailed description of the evolution of the gravitational and hydrodynamical fields, the boundary conditions and their numerical implementation can be found in Duez et al. [10].

**Results** Figures [1, 2] and [3] show the evolution of the coordinate separation $d$ between maximum baryonic density points in each star for cases A, B and C [10]. The filled circles mark the points of surface contact for those runs that result in a merger. In each plot we present results for different grid resolutions and bounding box sizes, which are listed in Table II. We use the highest quality results to bracket the ISCO; these are labeled as *Stable* and *Merger* in the plots. Results obtained on smaller computational grids agree with these brackets fairly well. Note that all three merger cases experience surface contact (and the related mass interchange) well after the start of the inspiral plunge. The ISCO parameters for each of the three cases are estimated as the average of the parameters corresponding to the bracketing orbits labeled *Stable* and *Merger* on each figure, while the “errors” span the difference. We note that these “errors” are partly due to numerical errors (see below), and partly by the conceptual difficulty of defining a sharp “ISCO”. The results are included in Table I. For the corotating sequence A we also compare with the QE-ISCO at the QE turning-point. The irrotational sequences considered in this paper terminate before reaching a turning point (compare [13, 14]). The termination of a sequence indicates that equilibrium models do not exist at smaller separations, but since the numerical determination of this point relies on the breakdown of a numerical (equilibrium) code, its accuracy is somewhat uncertain. For both sequences we find that the ISCO is reached before the sequence terminates. For the irrotational sequence C one can also compare with the first-order post-Newtonian ellipsoidal results of Lombardi et al. [20], who find an angular velocity of $\Omega m_0 = 0.0226$ for $(m/R)_\infty = 0.2$ binaries. We consistently find that orbits become unstable before the onset of instability as determined by QE methods, meaning at a smaller orbital frequency. This is not

![Figure 1](image1.png)

**FIG. 1:** Coordinate separation vs. time for sequences A. The separation $d$ is the coordinate distance between points of maximum rest mass density and is given in units of the total rest mass of the binary $M_0$. The curves show runs with different grid sizes and resolutions which are detailed in Table II. The filled circles mark the time of surface contact for the merger orbits. The empty square on the $y$ axis marks the QE estimation of the ISCO separation.

![Figure 2](image2.png)

**FIG. 2:** Coordinate separation vs. time for sequences B.
and irrotational values for the fairly gradual \[3\]. The similarity between the corotating binaries. This result is also consistent with earlier sug-
surprising, since the orbital decay becomes fairly rapid
just outside the QE-ISCO (compare, e.g., Fig. 2 in \[4\]),
so that our criterion for merger orbits applies to those
binaries. This result is also consistent with earlier sug-
gestions that the transition through the ISCO may be
fairly gradual \[4\]. The similarity between the corotating
and irrotational values for the \((m/R)_{\infty} = 0.142\) orbits
suggests that the dependence of the ISCO parameters on
the stellar spins is not strong.

During all simulations we monitored the Hamiltonian
and Momentum constraints as well as the conservation of
the total ADM mass \(M\) and angular momentum \(J\) \[21\]
(the rest mass \(M_0\) is conserved identically in our evolution
scheme). An example of these for case B is shown in
Fig. 4. In all our simulations all quantities are conserved
very well up to merger, after which hydrodynamical ef-
fects including shocks and shear are handled only crudely
by our artificial viscosity scheme. Stable runs ultimately
break down due to accumulation of numerical error. We
find that the latter is sometimes dominated by hydrody-
namical effects, leading to deviations in the angular mo-
momentum, and sometimes by gravitational effects, leading
to violations of the constraint equations. These effects
are improved by increasing the grid resolution and the
separation to the outer boundaries, as well as using a co-
ordinate system that rotates with the binary as closely as possible.

**Summary** We present prototype simulations to de-
termine dynamically the ISCO of BNSs. Evolving QE
initial data at different separations, we bracket the ISCO
by distinguishing stable orbits, that remain in approxi-
mately circular orbit for well over a period, from unstable
ones, which decay within a period. The uncertainty in
our results is caused both by numerical error and the
conceptual difficulty in defining a sharp ISCO. Consis-
tent with earlier results \[3, 4\] we find that binary orbits
start to plunge somewhat outside of the “QE-ISCO” as
determined by turning-point methods applied to QE ini-
tial data, resulting in a cut-off in gravitational “chirp”
signals at somewhat smaller frequency. Our preliminary
results also seem to indicate that the dependence of the
ISCO parameters on the stellar spins is not very strong.

One source of error is our assumption of zero radial
velocity in our binary initial data. More realistic ini-
tial data at finite binary separation would incorporate
a radial velocity that corresponds to the non-zero rate of inspiral at that separation. Miller \[23\] indicates that
this approximation may lead to non-negligible error, es-
pecially for black hole binaries. However, for the neutron
star binaries and separations we consider here, the ra-
dial velocities would be at most \(1 - 3\%\) of the tangential
velocity \([4, 23]\). Consequently, the error introduced by
neglecting this component is likely to be smaller than
the error bars already provided in Table II.

It was recently pointed out that in dynamical evolu-
tions of CTS initial data describing corotating BNSs, the
four-velocity \(u^\alpha\) quickly deviates from being proportional
to an exact helicoidal Killing vector, as assumed in con-
structing the initial models \[24\]. We confirm this result
but find that this deviation arises from a small readjust-
ment of the gravitational fields \[25\], and not from appreci-
able changes in the density and velocity profiles. We
find no evidence of a significant breakdown of quasequi-
librium for stable orbits and any spurious orbital eccen-
cricities sharply decrease with increasing grid size.

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| Case Fig. | Curves | Res. | \(B/M_0\) | Grid points | Res. | Box |
|-----------|--------|------|-----------|-------------|------|-----|
| A         | 1      | dashed | 20 | 18.7 | \(64^2 \times 128\) | Low | Large |
| A         | 2      | solid  | 40 | 18.7 | \(128^2 \times 256\) | High | Large |
| B         | 1      | dashed | 20 | 18.7 | \(64^2 \times 128\) | Low | Large |
| B         | 2      | solid  | 40 | 18.7 | \(128^2 \times 256\) | High | Large |
| C         | 3      | dashed | 40 | 11.5 | \(128^2 \times 256\) | High | Small |
| C         | 3      | dotted | 30 | 15.2 | \(128^2 \times 256\) | Low | Large |
| C         | 3      | solid  | 40 | 15.5 | \(172^2 \times 344\) | High | Large |

TABLE I: Grid sizes and resolutions. The resolution (Res.) is
given in number of grid points across the stellar diameter. The
bounding box length \(B\) gives the extent of the physical space
covered in each direction in units of total rest mass \(M_0\) (i.e.;
the numerical grid spans from \([-B, 0, 0]\) to \([B, B, B]\) since
we make use of the equatorial and \(\pi\) symmetries of the
systems.)
The wave frequency at the ISCO is $f_{\text{ISCO}}$.

TABLE II: Summary of our binary cases A, B, and C and results for their ISCO. For each sequence, we show the rotation state, the rest mass of the individual stars $m_0$, the compaction in isolation $(m/R)_\infty$, the total initial ADM mass $M_i$ and angular momentum $J_i/M_i^2$, as well as the binary coordinate separation at the ISCO $d/M_0$ (where $M_0$ is the total rest mass $2m_0$) and the corresponding orbital angular velocity $\Omega_{\text{ISCO}}$. The frequency at the ISCO is $f_{GW}$ and $m_{1.4}$ is the stellar gravitational mass in units of 1.4 $M_\odot$.

![Quality control for the $(m/R)_\infty = 0.142$ irrotational runs. We show from top to bottom the total gravitational mass, angular momentum, the Hamiltonian constraint, and the total initial ADM mass for binaries at different separation.](image)

For simplicity, in the remainder of the paper we refer to the zero circulation models as irrotational. See [10] for comparisons between these models and irrotational binaries.

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Fortner Fellow

† Department of Astronomy & NCSA, University of Illinois at Urbana-Champaign, Urbana, IL 61801

‡ Department of Physics, University of Illinois at Urbana-Champaign, Urbana, IL 61801

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The time coordinate has been transformed into fractions of the initial orbital period. Note that the periods differ for binaries at different separation.

[10] J. C. Lombardi, F. A. Rasio and S. L. Shapiro, Phys. Rev. D 56, 3416 (1997).

[21] The total angular momentum like the gravitational mass is not strictly conserved due to the emission of gravitational waves. However, the loss rate of angular momentum for a typical orbit at these separations is of the order of 1% to 2%. Deviations exceeding these values therefore indicate numerical error.

[22] This figure presents the $L_2$ norm of the constraint residuals. The details of these plots as well as the normalizations used here can be found in [10]. In this paper the momentum constraint curve shows the average of the three spatial components.

[23] M. Miller, gr-qc/0305024

[24] M. Miller and W. M. Suen, gr-qc/0301112 (2003), and...
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[25] E.g., the loss of a small amount of spurious radiation present in the initial data, which leaves \( M \) and \( J \) nearly unchanged.