Leech Constellations of Construction-A Lattices
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Abstract—The problem of communicating over the additive white Gaussian noise (AWGN) channel with lattice codes is addressed in this paper. Theoretically, Voronoi constellations have proved to yield very powerful lattice codes when the fine/coding lattice is AWGN-good and the coarse/shaping lattice has an optimal shaping gain. However, achieving Shannon capacity with these premises and practically implementable encoding algorithms is in general not an easy task. In this paper, a new way to encode and demap Construction-A Voronoi lattice codes is presented. As a meaningful application of this scheme, the second part of the paper is focused on Leech constellations of low-density Construction-A (LDA) lattices: LDA Voronoi lattice codes are presented whose numerically measured waterfall region is situated at less than 0.8 dB from Shannon capacity. These LDA lattice codes are based on dual-diagonal nonbinary low-density parity-check codes. With this choice, encoding, iterative decoding, and demapping have all linear complexity in the block length.

Index Terms—Construction A, dual-diagonal LDPC codes, LDA lattices, leech lattice, Voronoi constellations.

I. INTRODUCTION

URING the last forty years, the problem of transmitting digital information via lattice constellations has been extensively studied, mainly for the interest that lattices arise when dealing with continuous channels [1], [2]; interest that is lasting with time, since lattice codes may play a role in physical-layer network coding in communication networks under standardization [3].

We can divide the results on Euclidean lattice codes into two main groups: the information-theoretical ones, aimed to analytically prove the capacity-achieving properties of lattice codes [4]–[12]; and coding results, in which authors design lattice families particularly adapted to fast and efficient decoding with satisfactory performance [13]–[19]. At the price of some technical challenges - whose solution is not always straightforward - this second group focuses on translating to the Euclidean space the techniques used for designing effective iteratively decodable error-correcting codes over finite fields, like turbo codes, low-density parity-check (LDPC) codes, and polar codes. Information-theoretical analyses exist for some lattice families in the second group, aiming at establishing a unified theory that surpasses this dichotomy [18], [20], [21].

Most of the available practical results treat the properties of lattices as infinite constellations, comparing performance with the theoretical limits established in [5] and [10]. When we move our attention to finite constellations, the theory tells that a winning strategy to achieve capacity is to carve Voronoi constellations [22] out of Polyhrev-limit-achieving infinite constellations, using shaping lattices that are good for quantization [8], [9], [21]. However, in practice this approach is hard to realize, mainly due to the complexity of quantization algorithms for general lattices. Erez and ten Brink proposed a scheme which employs trellis shaping to design constellations with good shaping gain and close-to-capacity performance [23]. Other implementable shaping schemes have been proposed [24] and mainly applied to low-density lattice codes (LDLC) [25]–[27]. Very recently, the problem of encoding constellations of nested lattices with an application-oriented approach was treated by Kurkoski [28].

In this work, we focus on Voronoi constellations for the additive white Gaussian noise (AWGN) channel where the fine lattices are nonbinary Construction-A lattices. We have chosen Construction A because it constitutes a powerful tool to build both Polyhrev-limit-achieving lattices and optimal or near-optimal shaping regions [6], [8], [21], [29]. Furthermore, Construction A yields integer lattices, whose encoding and decoding algorithms are more easily implemented with respect to noninteger lattice constellations.

The construction of a Voronoi lattice code is divided into two main steps: first, finding a complete set of representatives of the quotient group defined by the coding lattice modulo the shaping lattice; second, reducing the representatives modulo the shaping lattice (using lattice quantization) to find the ones with the smallest norm. These points lie by construction in the Voronoi region of the shaping lattice and form the Voronoi lattice code.

The main theoretical novelty of this paper is Lemma 1, which deals with the first of the two steps above. It characterizes a specific set of representatives of the quotient group, when the coding lattice is Construction-A and the shaping lattice is contained in pZ^n. We apply Lemma 1 to describe a new scheme to encode and demap Construction-A lattice codes.

In the second part of the paper, we show how our method works when the shaping lattice consists of the direct sum of scaled copies of the Leech lattice. For this reason, we call the resulting lattice codes Leech constellations. We learn from [30, Ch. 12] or [31, Th. 4.1] that the Leech lattice is the unique (up to isomorphism) even unimodular lattice
in $\mathbb{R}^{24}$ without vectors of Euclidean square norm 2. The Leech lattice has numerous interesting properties and has been extensively studied [30]. Its very peculiar structure allows a deep and complete algebraic investigation. In dimension 24, it is the best known quantizer [30, p. 61] and it corresponds to the densest sphere-packing, among both lattice and nonlattice constructions [32].

Using direct sums of small-dimensional lattices to build the shaping lattice for Voronoi constellations is a well-known technique [24], [28]. In this way, the high-dimensional shaping lattice inherits the same shaping gain of the small-dimensional lattices, while the complexity of the quantization operation needed for shaping remains algorithmically manageable. The scope of this paper is not to focus on the choice of the shaping lattice, but rather on the encoding and demapping scheme. Therefore, in Leech constellations we fix the small-dimensional lattices used for shaping to be (properly scaled) copies of the Leech lattice, whose shaping gain is 1.03 dB, only 0.50 dB away from the optimal shaping gain of a spherical infinite-dimensional shaping region. The Leech lattice can be substituted with any other integer lattice without changing the theoretical description of our scheme.

To build efficiently encodable and decodable Leech constellations in very high dimensions, we cut them out of dual-diagonal low-density Construction-A (LDA) lattices. To the best of our knowledge, dual-diagonal LDPC codes were never employed in the nonbinary case and as base element for Construction A. We have chosen LDA lattices for our Leech constellations because of the decoding performance that they showed in our previous work, from both the theoretical and practical point of view [16], [21]. This performance is confirmed by the infinite-constellation simulations shown in Section VI on the unconstrained AWGN channel. Moreover and very importantly, the family of dual-diagonal LDA lattices is designed on purpose to allow fast encoding, which can be performed via the chain in the parity-check matrix. Experimentally, we reach low error rates at a distance of only 0.8 dB from Shannon capacity (cf. Section VI). To the best of our knowledge, it is the first time that numerical results of decoding finite nonhypercubic constellations of LDA lattices are published.

The paper is structured as follows. Section II recalls some knowledge about lattices and lattice codes for the AWGN channel. In Section III, we give the general description of our new encoding and demapping scheme for Construction-A lattices, based on Lemma 1. Section IV defines Leech constellations and Section V introduces nonbinary dual-diagonal LDA lattices. The latter are defined by the means of their parity-check matrix and are used in Section VI for numerical simulations. The paper finishes with Section VII, which summarizes our achievements and contains some conclusive remarks.

A. Notation

In this paper we use the bold type for vectors in row convention: $\mathbf{x} = (x_1, \ldots, x_n)$; a general coordinate of a vector is indicated by $x_i$. Capital bold letters are used for matrices and their entries are written in most cases in lower case with double index; e.g., $G = \{g_{i,j}\}$. Calligraphic capital letters indicate sets: $\mathcal{B}$. The notation $O(f(n))$ indicates a function whose absolute value is upper bounded by $af(n)$ for some positive constant $a$ and for all $n$ big enough.

II. LATTICES FOR THE AWGN CHANNEL

The scope of this section is to recall the definitions that we will need throughout the paper and fix some notation. For more details on lattices, we refer the reader to [2], [30], [31].

Lattices are $\mathbb{Z}$-modules in the Euclidean space $\mathbb{R}^n$ or, equivalently, discrete additive subgroups of $\mathbb{R}^n$. These two definitions correspond to the following: let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k\}$ be a basis over $\mathbb{R}$ of $\mathbb{R}^n$ as a vector space, then a lattice $\Lambda$ is the set of all possible linear combinations of the $\mathbf{b}_i$’s with integer coefficients; if $G$ is the $n \times n$ matrix whose rows are the $\mathbf{b}_i$’s, then $\Lambda = \{\mathbf{x} \in \mathbb{Z}^n : \mathbf{x} = \mathbf{zG}, \exists \mathbf{z} \in \mathbb{Z}^n\}$. In this setting, $\mathcal{B}$ is called a basis of the lattice, $G$ a generator matrix, and the quantity $\text{Vol}(\Lambda) = |\det(G)|$ is called the volume of the lattice. It can be shown that $\text{Vol}(\Lambda) = \text{Vol}(\mathcal{V}(\Lambda))$, where $\mathcal{V}(\Lambda)$ is the Voronoi region of the lattice:

$$\mathcal{V}(\Lambda) = \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{y}\| \leq \|\mathbf{y} - \mathbf{x}\|, \forall \mathbf{x} \in \Lambda \smallsetminus \{\mathbf{0}\}\}.$$ It is very well known that although a single lattice has (infinitely) many different bases, its volume is a characterizing invariant. Notice that we have restricted our definition to full-rank lattices, i.e., those which have $n$ independent generator vectors in an $n$-dimensional space. We do not need to treat lower-rank lattices for the purposes of this work.

A famous and useful way to construct lattices is Construction A [30]; this method consists in embedding into $\mathbb{R}^n$ an infinite number of copies of a linear code over a finite field, in a way that preserves linearity. More precisely, let $C = C[n, k]_p \subseteq \mathbb{F}_p^n$ be a linear code over the prime field $\mathbb{F}_p$ of dimension $k$ and length $n$. Identifying $\mathbb{F}_p^n$ with its image via an embedding $\mathbb{F}_p^n \hookrightarrow \mathbb{Z}^n$, we define the lattice obtained by Construction A from $C$ as $\Lambda = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{c} \bmod p, \exists \mathbf{c} \in C\}$. Equivalently, we can write that

$$\Lambda = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{c} + p\mathbf{z}, \exists \mathbf{c} \in C, \exists \mathbf{z} \in \mathbb{Z}^n\} = C + p\mathbb{Z}^n \subseteq \mathbb{Z}^n.$$ It is known that [2, Prop. 2.5.1(d)]:

$$\text{Vol}(\Lambda) = p^{n-k}. \quad (1)$$

In this paper, we are interested in lattices as constellations of points for the transmission of information over the AWGN channel; this channel has lattice points $\mathbf{x}$ as inputs and returns $\mathbf{y} = \mathbf{x} + \mathbf{n}$, where the $n_i$ are i.i.d. Gaussian random variables of mean 0 and variance $\sigma^2$. If we do not suppose any limitation for the energy of the input $\mathbf{x}$, we say that we are considering the unconstrained AWGN channel. A seminal theorem by Poltyrev states what follows [5]:

**Theorem 1 (Poltyrev):** Over the unconstrained AWGN channel and for every $\varepsilon > 0$, there exists a lattice $\Lambda \subseteq \mathbb{R}^n$ (in dimension $n$ big enough) that can be decoded with error probability less than $\varepsilon$ if and only if $\text{Vol}(\Lambda) > (\sqrt{2\pi e}\sigma^2)^n.$
The condition above is often written as $VNR > 1$, where VNR indicates the so called \textit{volume-to-noise ratio} of $\Lambda$ [33]:

$$\text{VNR} = \frac{\text{Vol}(\Lambda)^{\frac{2}{n}}}{2\pi e \sigma^2}.$$  \hfill (2)

\textbf{Corollary 1:} In the set of all lattices $\Lambda$ with fixed normalized volume $\text{Vol}(\Lambda)^{\frac{2}{n}} = \nu$, there exists a lattice that can be decoded with vanishing error probability over the unconstrained AWGN channel only if the noise variance satisfies $\sigma^2 < \frac{\text{Vol}(\Lambda)}{2\pi e \sigma^2} = \sigma_{\text{max}}^2$.

This corollary does not add anything new to Theorem 1, but it is interesting for an operational reason: the quantity $\sigma^2$ can be interpreted as the \textit{maximum tolerable noise variance} for lattices with normalized volume $\nu$ and is often called \textit{Poltyrev limit} or \textit{Poltyrev capacity}. Whenever we work with a specific family of lattices over the unconstrained AWGN channel, an important task is to show how the decoding probability of those lattices behaves for noise variances close to $\sigma_{\text{max}}^2$. Families of lattices that have vanishing error probability for every $\sigma^2 < \sigma_{\text{max}}^2$ are said to be \textit{good} for coding or \textit{AWGN-good} or \textit{Poltyrev-limit-achieving}. As an example, Construction-A lattices were shown to be AWGN-good [6], [12], [29]; the same holds for some ensembles of LDA lattices [20], [34] and generalized low-density (GLD) lattices [35].

If we impose to the AWGN channel input $\mathbf{x} = (x_1, \ldots, x_n)$ the power condition $\mathbb{E}[x_i^2] \leq P$, for some $P > 0$, then we call the \textit{signal-to-noise ratio} of the channel the quantity

$$\text{SNR} = \frac{P}{\sigma^2}.$$  \hfill (3)

It is known that the capacity of the AWGN channel is $\frac{1}{2} \log_2 (1 + \text{SNR})$ bits per dimension [36, p. 365]. AWGN-good lattices are essential ingredients in lattice constructions that achieve capacity of the (constrained) AWGN channel [8], [18], [21].

A typical efficient way of building finite - hence power-constrained - sets of lattice points for the AWGN channel, called \textit{lattice codes}, is to use pairs of nested lattices to build Voronoi constellations: we say that two lattices $\Lambda$ and $\Lambda_f$ are \textit{nested} if one is included in the other, $\Lambda \subseteq \Lambda_f$. The bigger lattice (as a set) is sometimes called the \textit{fine} lattice, hence the index $f$; its sublattice $\Lambda$ is the \textit{coarse} lattice. $\Lambda$ is a subgroup of $\Lambda_f$, therefore we can consider the quotient group:

$$\Lambda_f/\Lambda = \{ x + \Lambda : x \in \Lambda_f \}.$$  

The sets $x + \Lambda = \{ x + z : z \in \Lambda \}$ are called the \textit{cosets} of $\Lambda$ in $\Lambda_f$ [33]. In this notation, $\mathbf{x}$ is called the \textit{leader} of the coset. The group structure is such that the coset of $\mathbf{x} + \mathbf{y}$ is equal to the coset of $\mathbf{x}$ plus the coset of $\mathbf{y}$, i.e., $(x + y) + \Lambda = (x + \Lambda) + (y + \Lambda)$. Notice that with a little abuse of notation which does not lead to confusion, the “+” symbols in the previous formula represent both the addition in $\Lambda_f$ and the addition in the quotient group $\Lambda_f/\Lambda$. It is known that the cardinality of $\Lambda_f/\Lambda$ is $M = \text{Vol}(\Lambda_f)/\text{Vol}(\Lambda_f)$, i.e., there exist exactly $M$ different cosets.

The lattice code $C$ given by the coset leaders of $\Lambda$ in $\Lambda_f$ with smallest Euclidean norm is called the \textit{Voronoi constellation (or code)} of $\Lambda_f$ with \textit{shaping lattice} $\Lambda$ [22], [37]. Equivalently, $C = \Lambda_f \cap \mathcal{V}(\Lambda)$. In this context, the fine lattice $\Lambda_f$ is also called the \textit{coding lattice}. From now on, we will always assume that $\Lambda \subseteq \Lambda_f$, and $\Lambda$ and $\Lambda_f$ will be for us the standard notation for respectively the coarse/shaping and the fine/coding lattice.

From an operational point of view, building a Voronoi constellation (i.e., encoding our lattice code) consists of two main steps:

1) Given the coding and the shaping lattice, being able to construct all the $M$ cosets. This means to characterize a set of $M$ different coset leaders.

2) Given the cosets, find for each one of them the coset leader of minimum Euclidean norm, which is a point of the Voronoi constellation. This is done by the \textit{quantization} operation: the \textit{quantizer} associated with the Voronoi region of $\Lambda$ is the function

$$Q_{\mathcal{V}(\Lambda)} : \mathbb{R}^n \to \Lambda$$

$$\mathbf{y} \mapsto \arg \min_{\mathbf{z} \in \Lambda} \| \mathbf{z} - \mathbf{y} \|.$$  \hfill (4)

Hence, if $\mathbf{x}$ is a point of a given coset of $\Lambda$ in $\Lambda_f$, its coset leader with minimum norm is $\mathbf{x} - Q_{\mathcal{V}(\Lambda)}(\mathbf{x}) \in \Lambda_f$.

Much attention has to be paid to the fact we are using a quantizer (or nearest-neighbor decoder) of $\Lambda$ for the procedure of encoding points of $\Lambda_f$ into a Voronoi constellation [22]. Therefore, it is important to have efficient quantization algorithms for the shaping lattice. We do not only mean optimal, mathematically well-defined, or “numerically precise”; we mainly mean of “manageable” complexity. Many nice theoretical Voronoi constructions, including the capacity-achieving ones [8], [21], cannot be implemented in high dimensions because of the complexity of the associated quantizer.

In order to achieve capacity over the AWGN channel with Voronoi constellations, optimization of both the shaping lattice and the coding lattice has to be performed [38]. Namely, the coding lattice needs to be Poltyrev-limit-achieving and the shaping lattice needs a “spherical” Voronoi region, that is, its shaping gain has to be optimal: we call \textit{shaping gain} of $\Lambda$ the shaping gain $\gamma_s(\Lambda) = \gamma_s(\mathcal{V}(\Lambda))$ of its Voronoi region [37]:

$$\gamma_s(\Lambda) = \frac{n \text{Vol}(\Lambda)^{\frac{1}{2} + \frac{2}{n}}}{12 \int_{\mathcal{V}(\Lambda)} \| x \|^2 \, d x}^{-1}.$$  \hfill (5)

It is an established result that $\gamma_s(\Lambda) \leq \frac{n}{12} \approx 1.53$ dB for every lattice $\Lambda$. A family of lattices has an optimal shaping gain if it tends to $\frac{n}{12}$ when $n$ tends to infinity. A family with this property is called \textit{good for quantization} [29]. The capacity results obtained with Construction A and LDA lattices in [8] and [21] are based on shaping lattices which are good for quantization and coding lattices which are Poltyrev-limit-achieving.

\section*{III. Encoding and Demapping Construction-A Lattice Codes}

In this section we show a way to perform step 1) above when the coding lattice $\Lambda_f$ is built with Construction A and the shaping lattice $\Lambda$ is contained in $\rho \mathbb{Z}^n$. Notice that under
these premises \( \Lambda \) and \( \Lambda_f \) are nested because \( \Lambda \subseteq p\mathbb{Z}^n \subseteq \Lambda_f \).
The following lemma characterizes the quotient group \( \Lambda_f/\Lambda \) by producing an explicit set of coset leaders. After the lemma, we will describe how to map and demap information to and from lattice codewords.

**Lemma 1:** Let \( \Gamma \subseteq \mathbb{Z}^n \) be any integer lattice and let us define \( \Lambda = p\Gamma \subseteq p\mathbb{Z}^n \). Let us call \( \mathbf{T} \) a lower triangular generator matrix of \( \Gamma \) with \( t_{ij} > 0 \) for every \( i > j \):

\[
\mathbf{T} = \begin{pmatrix}
t_{11} & 0 & \cdots & 0 \\
t_{12} & t_{22} & \cdots & \vdots \\
& \ddots & \ddots & \ddots \\
t_{1n} & \cdots & t_{n,n-1} & t_{nn}
\end{pmatrix} \in \mathbb{Z}^{n \times n}.
\]

Let us call \( S \) the set \( S = \{0, \ldots, t_{1,1}-1\} \times \{0, \ldots, t_{2,2}-1\} \times \cdots \times \{0, \ldots, t_{n,n}-1\} \).

Let \( \Lambda_f = C + p\mathbb{Z}^n \) be a Construction-A lattice; we consider the usual embedding of \( C \) in \( \mathbb{Z}^n \) via the coordinate-wise morphism \( \mathbb{F}_p \hookrightarrow \{0, 1, \ldots, p-1\} \subseteq \mathbb{Z} \), hence \( C \subseteq \{0, 1, \ldots, p-1\}^n \).

Then \( C + pS = \{c + ps \in \mathbb{Z}^n : c \in C, s \in S\} \) is a complete set of coset leaders of \( \Lambda_f/\Lambda \).

**Proof:** In principle \( |C + pS| \leq |C||S| \), though it is easy to show that the equality is achieved: suppose that \( c + ps = d + pt \) for some \( c, d \in C \subseteq \{0, 1, \ldots, p-1\}^n \) and \( s, t \in S \); then \( c \equiv d \) \( \pmod{p} \). This means that \( c = d \), because all the \( c_i \)'s and \( d_i \)'s are in \( \{0, 1, \ldots, p-1\} \). This in turn implies that \( s = t \). Hence every pair \( (c, s) \in C \times S \) generates a different point in \( C + pS \) and \( |C||S| = |C + pS| \).

Now, by the triangularity of \( \mathbf{T} \) and the definition of \( S \), we have that

\[
\text{Vol}(\Lambda) = \text{Vol}(p\Gamma) = p^n \prod_{i=1}^{n} t_{ii} = p^n|S|.
\]

If we also take into account (1) and that the cardinality of \( \Lambda_f/\Lambda \) is

\[
M = \frac{\text{Vol}(\Lambda)}{\text{Vol}(\Lambda_f)} = \frac{\text{Vol}(\Lambda)}{p^n-k}.
\]

then we easily obtain that

\[
|C + pS| = |C||S| = p^k \frac{\text{Vol}(\Lambda)}{p^n} = M.
\]

We have just proved that \( C + pS \) and \( \Lambda_f/\Lambda \) have the same cardinality. At this point, to conclude the proof of the lemma, it is sufficient to show that any two elements of \( C + pS \) belong to different cosets. Equivalently, we will prove the following: if \( x, y \in C + pS \) belong to the same coset, then \( x = y \).

Now, \( x = c + ps \) and \( y = d + pt \) are in the same coset if and only if \( x - y = c - d + p(s - t) \) is in \( \Lambda \subseteq p\mathbb{Z}^n \). This holds only if \( c - d \in p\mathbb{Z}^n \) and consequently only if \( c_i - d_i = 0 \) for every \( i \), because 0 is the only element of \( p\mathbb{Z}^n \) that can be obtained by subtracting two numbers of \( \{0, 1, \ldots, p-1\} \). Thus \( x \) and \( y \) are in the same coset only if \( c = d \) and \( x - y = p(s - t) \in \Lambda = p\Gamma \). This implies that \( s - t \in \Gamma \), i.e., \( s - t = z\mathbf{T} \) for some \( z \in \mathbb{Z}^n \).

\[\text{Let } \mathbf{U} = [u_{i,j}] \text{ be the inverse of } \mathbf{T}: \text{ it is lower triangular and } u_{i,i} = t_{i,i}^{-1} \text{ for every } i. \text{ The relation } (s - t)\mathbf{U} = z \text{ implies that}
\[
\sum_{j=1}^{n} (s_j - t_j)u_{j,i} \in \mathbb{Z} \text{ for every } i.
\]

When \( i = n, \) the condition is simply \( (s_n - t_n)t_{n,1}^{-1} \in \mathbb{Z} \), which implies that \( s_n - t_n = 0 \) because of definition of \( S \) we have \( |s_n - t_n| \leq t_{n,n} - 1 < t_{n,n} \). Using the equality \( s_n = t_n \) in the case \( i = n - 1 \), we obtain that \( s_{n-1} - t_{n-1} = 0 \). Moving recursively backwards to \( i = n - 2, n - 3, \ldots, 1 \) and using each time the new equalities, we conclude that \( s_i = t_i \) for every \( i \), i.e., \( s = t \) and, as wanted, \( x = y \).

\[\]
encoding of \( C \), this approach will allow us in Section V to build Voronoi constellations whose encoding complexity is linear in \( n \), whereas, in general, encoding via the lattice generator matrix has complexity proportional to \( n^2 \).

B. Demapping

In the process of communication, besides encoding and decoding a constellation point, it is necessary to specify how demapping works, i.e., how to reobtain the information vector \( \mathbf{x} \). Namely, we apply the following steps:

1) \( Q_{\psi(A)}(\mathbf{y}) \in \Lambda = p\Gamma \subseteq p\mathbb{Z}^n \) for every \( \mathbf{y} \in \mathbb{R}^n \), hence we can obtain \( \mathbf{c} \) simply by reducing modulo \( p \) the point \( \mathbf{x} = \mathbf{c} + ps - Q_{\psi(A)}(\mathbf{x}) \).

2) How to derive \( \mathbf{u} \) from \( \mathbf{c} \). Namely, we apply the following steps:

   a) \( \mathbf{c} = \text{enc}(\mathbf{u}) = (\mathbf{u}, \mathbf{s}) \) for some parity symbols \( \mathbf{c} = (\mathbf{c}) \) and some \( \mathbf{s} \) automatically given by the information symbols of \( \mathbf{c} \).

   b) We now need to compute \( \mathbf{s} \).

   c) At this point, we know both \( \mathbf{x} \) and \( \mathbf{c} \). We can recover
      \[
      \mathbf{r} = \frac{(\mathbf{x} - \mathbf{c})}{p} = \mathbf{s} - \frac{1}{p} Q_{\psi(A)}(\mathbf{x}) = \mathbf{s} - \mathbf{q} \in \mathbb{Z}^n, \tag{6}
      \]
      for some \( \mathbf{q} \in \Gamma = p^{-1}\Lambda \). In particular, if \( \mathbf{T} \) is the triangular generator matrix of \( \Gamma \) as in (10), \( \mathbf{r} = \mathbf{s} - \mathbf{zT} \) for some unknown \( \mathbf{z} \in \mathbb{Z}^n \).

   d) By triangularity of \( \mathbf{T} \), the \( i \)-th coordinate of the previous equality is:
      \[
      r_i = s_i - z_i t_{i,j} - \sum_{j=i+1}^{n} z_j t_{i,j}, \tag{7}
      \]
      For \( i = n \), the rightmost term of (7) is absent and we have
      \[
      s_n = r_n + z_n t_{n,n}. \tag{8}
      \]
      \( r_n \) is known from (6) and we aim to find \( s_n \). This is easy, because the two following conditions uniquely identify it:

      - \( s_n \equiv r_n \mod t_{n,n} \);  
      - \( 0 \leq s_n \leq t_{n,n} - 1 \) (by definition of \( S \)).

   e) Furthermore, after having found \( s_n \), we can compute \( z_n \) from (8).

   f) For \( i = n - 1 \), (7) yields
      \[
      s_{n-1} = r_{n-1} + z_{n-1} t_{n-1,n-1} + z_n t_{n,n-1}. \tag{9}
      \]

5) The two unknowns here are \( s_{n-1} \) and \( z_{n-1} \). In a similar way as before, \( s_{n-1} \) can be explicitly computed because:

      - \( s_{n-1} \equiv r_{n-1} + z_n t_{n,n-1} \mod t_{n-1,n-1} \);  
      - \( 0 \leq s_{n-1} \leq t_{n-1,n-1} - 1 \) (by definition of \( S \)).

Once we have \( s_{n-1} \), it is easy to obtain \( z_{n-1} \) from (9).

6) Going on with the same strategy, using recursively at the \( i \)-th step the values of \( s_j \) and \( z_j \) already computed for \( j = i + 1, i + 2, \ldots, n \), we obtain \( s_i \) for the remaining \( i = n - 2, n - 3, \ldots, 1 \). This concludes the demapping procedure.

IV. LEECH CONSTELLATIONS

From now on, we will put into practice the coding scheme described in Section III with the goal of designing lattice codes with encoding and demapping complexity linear in \( n \). In this section, we choose a standard solution to simplify the algorithmic problem of quantization for shaping. We will focus on Voronoi constellations in which the shaping lattice \( \Lambda \) is the direct sum of low-dimensional lattices: \( \Lambda = \Lambda_{24}^\oplus \), for some \( \ell \) proportional to \( n \) and some lattice \( \Lambda_\ell \subseteq \mathbb{R}^n_\ell \). This is a standard approach, which yields a coarse lattice with the same shaping gain of \( \Lambda_\ell \). Kurkoski [28] considers this construction when the fine lattice is built via Construction A and the authors of [24] use it with low-density lattice codes (LDLC).

The choice of taking \( \Lambda = \Lambda_{24}^\oplus \) results in a low-complexity quantizer of the shaping lattice. In particular, for every \( \mathbf{y} \in \mathbb{R}^n \), we have:

\[
Q_{\psi(A)}(\mathbf{y}) = \left( Q_{\psi(A)}(\mathbf{y}_1) \mid Q_{\psi(A)}(\mathbf{y}_2) \mid \cdots \mid Q_{\psi(A)}(\mathbf{y}_\ell) \right),
\]

where \( Q_{\psi(A)}(\cdot) \) is the \( n/\ell \)-dimensional Voronoi quantizer of \( \Lambda_\ell \), and for every \( i = 1, 2, \ldots, \ell \),

\[
y_i = (y_{1+n(i-1)/\ell}, y_{2+n(i-1)/\ell}, \ldots, y_{n/i}).
\]

Hence, applying \( Q_{\psi(A)}(\cdot) \) is equivalent to apply \( \ell \) independent quantizers \( Q_{\psi(A)}(\cdot) \). When \( \Lambda_\ell \) has constant dimension in \( n \), the complexity of the quantization operation is \( O(\ell) = O(n) \).

The scope of this paper is not to introduce any fundamental novelty concerning the construction of the shaping lattice. For this reason, we choose to fix it once for all: from now on, \( \Lambda_\ell \) will be (a scaled copy of) the Leech lattice. It is known that we need shaping lattices with a high shaping gain to obtain Voronoi constellations with decoding performance close to capacity. The Leech lattice is the best-known quantizer in dimension 24 [30, p. 61] and has a shaping gain of about 1.03 dB [40]. This corresponds to a difference of around 0.50 dB from the optimal shaping gain. If we work with AWGN-good fine lattices, our experimental target is to achieve numerically measured decoding error probabilities with a waterfall region situated at around 0.50 dB from Shannon capacity. We will show in Section VI how close we can get to this result.

Now, let \( \Lambda_f = C[n,k]_p + p\mathbb{Z}^n \) be the \( n \)-dimensional Construction-A fine lattice and let us suppose from now on that \( n = 24\ell \) for some integer \( \ell \). Let \( \mathbf{G}_{24} \) be the lower triangular generator matrix of the Leech lattice proposed by Conway and Sloane in [30, p. 133], but with all the coordinates multiplied by \( \sqrt{2} \). In particular, we are considering an integer version of the Leech lattice: \( \mathbf{G}_{24} \in \mathbb{Z}^{24\times24} \). In spite of scaling, we can still call it without confusion the Leech lattice and we denote it \( \Lambda_{24} \). Also, one can check that

\[
\det(\mathbf{G}_{24}) = \text{Vol}(\Lambda_{24}) = (\sqrt{2})^{24} = 2^{36}.
\]
Now, consider the lattice given by the following direct sum of \( \ell \) copies of the Leech lattice:
\[
\Lambda_{24}^{\oplus \ell} = \{ \mathbf{x} = (x_1 | x_2 | \cdots | x_l) \in \mathbb{R}^n : x_i \in \Lambda_{24}, \forall i \} \subseteq \mathbb{Z}^n
\]
and let us call \( \Gamma = a \Lambda_{24}^{\oplus \ell} \), for \( a \in \mathbb{N} \setminus \{0\} \). The generator matrix of \( \Gamma \) is the \( n \times n \) diagonal matrix obtained by diagonally juxtaposing \( \ell \) copies of \( \Gamma_{24} \) multiplied by \( a \) (and filling with zeroes all the other entries):
\[
T = \begin{pmatrix}
\alpha \mathbf{G}_{24} & 0 & \cdots & 0 \\
0 & \alpha \mathbf{G}_{24} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix} \in \mathbb{Z}^{n \times n}. \tag{10}
\]

If we denote \( V = (\sqrt{2})^{24} \) the volume of \( \Lambda_{24} \), it is easy to compute that \( \text{Vol}(\Gamma) = a^n V^{\ell} \). In what follows, the shaping lattice will always be \( \Lambda = p \Gamma = pa \Lambda_{24}^{\oplus \ell} \).

Definition 1: We call Leech constellation of a Construction-A lattice its Voronoi constellation when the shaping lattice is a direct sum of (conveniently scaled) copies of the Leech lattice: the fine lattice is \( \Lambda_f = C + p\mathbb{Z}^n \) and the shaping lattice is \( \Lambda = pa \Lambda_{24}^{\oplus \ell} \).

Leech constellations are well defined because
\[
\Lambda = p \Gamma = pa \Lambda_{24}^{\oplus \ell} \subseteq p\mathbb{Z}^n \subseteq C + p\mathbb{Z}^n = \Lambda_f.
\]

If we call \( g_1, g_2, \ldots, g_{24} \) the diagonal elements of \( \mathbf{G}_{24} \), then the set \( S \) of Lemma 1 becomes:
\[
S = \{0, 1, \ldots, a g_1 - 1\} \times \cdots \times \{0, 1, \ldots, a g_{24} - 1\}\times^{\ell}.
\]

We can easily compute the cardinality \( M \) of the Leeuc constellation: if \( R = k/n \) is the rate of the code \( C \),
\[
M = |\Lambda_f \cap \varphi(\Lambda)| = |\Lambda_f / \Lambda| = \frac{\text{Vol}(\Lambda)}{\text{Vol}(\Lambda_f)} = \frac{p^n a^n V^{\ell}}{p^k a^n V^{\ell}} = \left( p^k \frac{a V^{1/24}}{a V^{1/24}} \right)^n.
\]
Consequently, the information rate of the Leeuc constellation \( C \) is
\[
R_C = \frac{\log_2 M}{n} = R \log_2 p + \log_2 a + \frac{3}{2} \text{ bits/dim },
\]
because \( V = (\sqrt{2})^{24} \). By tuning the parameters \( p, R, \) and \( a \), we can fix different information rates. As an example, the values \( a = 1, \ p = 13, \) and \( R = 1/3 \) that are used in the simulations of Section VI, yield a rate of \( R_C \approx 2.73 \) bits per dimension.

Independently from the decoder used for \( \Lambda_f \), if we apply the coding scheme of Section III to Leeuc constellations, we can observe the following:

- The complexity of the encoding algorithm resides in steps 2) and 4) of Section III-A. Because of what we pointed out at the beginning of this section, step 4) (quantization) has practical complexity, linear in \( n \). The linearity constant is a power of 24, due to the complexity of the Leech quantizer, polynomial in its dimension. Numerical simulations like the ones of Section VI tell us that the complexity of the Leeuc-constellation encoder is manageable and we are capable of simulating encoding and decoding up to dimension \( n = 10^6 + 8 \) (the addition of 8 to the round number \( 10^6 \) is needed to make \( n \) divisible by 24). Notice that these simulations use the codes that we will design in Section V, for which also step 2) of Section III-A (encoding of \( C \)) is \( O(n) \).
- In general, the complexity of demapping resides in computing \( s_i \) from (7) (as in (9)) for \( i = n - 1 \). Nevertheless, in the case of Leeuc constellations, for every given \( i \), all but at most 24 of the \( t_{j,i} \)'s are equal to zero. Therefore, each step from 1) to 6) of Section III-B requires a constant \( (in n) \) number of operations for every \( i \) and the complexity of demapping is \( O(n) \) too. More generally, when using copies of a lattice of dimension \( d \) in the direct sum that produces the shaping lattice, the complexity of demapping is \( O(dn) \).

V. Dual-Diagonal LDA Lattices

In Section II, we mentioned that we need fine lattices with good performance (ideally Poltyrev-limit-achieving) for the construction of strong Voronoi constellations; furthermore, through Section III-A and IV we established that linear-complexity encoding of Leeuc constellations is possible if the encoding of the underlying \( p \)-ary code \( C \) is linear in \( n \) too. In this section, we propose the algebraic construction of a lattice family which possesses both qualities: good performance over the unconstrained AWGN channel and fast encoding. As fine lattices, we choose a particular family of LDA lattices:

Definition 2: We call a low-density Construction-A (LDA) lattice a lattice built with Construction A when the underlying code \( C \) is a low-density parity-check (LDPC) code.

LDPC codes were invented by Gallager [41], have had a huge success, and do not need further introduction. LDA lattices were proposed by the authors of this work a few years ago [16]; they are endowed with an iterative low-complexity decoder which allows fast decoding with satisfactory performance. Well-defined ensembles of LDA lattices were proved to be Poltyrev-limit-achieving first [34], then also Shannon-capacity-achieving [21] and good for other communication-related problems [20].

Definition 3: A square matrix \( A = (a_{i,j}) \) is said dual-diagonal if all its entries are equal to 0 except for the \( a_{i,i} \)'s and the \( a_{i,i-1}'s \).

We look for LDA lattices that can be rapidly encoded. Our solution is to use Construction A with LDPC codes whose parity-check matrix \( H \) has a dual-diagonal submatrix. By extension, we call them dual-diagonal LDPC codes and their associated LDA lattices dual-diagonal LDA lattices. Recall that a parity-check matrix of a code \( C \) is a matrix \( H \) which defines the code as: \( C = \{ \mathbf{x} \in \mathbb{F}_p^n : \mathbf{Hx}^T \equiv \mathbf{0} \mod p \} \). For our construction, we impose that \( H \) has the following structure:
\[
H = (L | R); \tag{11}
\]

2More precisely, with our choice of \( G_{24} \), there are in average 5.625 nonzero \( t_{j,i} \)'s per column; the minimum is 1 and the maximum is 21.
\( H \) has \( n-k \) rows, \( n \) columns, and its right submatrix \( R \) is the following square dual-diagonal matrix:

\[
\begin{pmatrix}
  h_{1,1} & 0 & \cdots & 0 \\
  h_{2,1} & h_{2,2} & \cdots & \vdots \\
  0 & \cdots & h_{n-k-1,n-2} & h_{n-k-1,n-1} \\
  \vdots & \cdots & \vdots & \ddots & \vdots \\
  0 & \cdots & 0 & h_{n-k,n}
\end{pmatrix},
\]

with \( h_{i,i+k} \neq 0 \) for every \( i = 1, 2, \ldots, n-k \) and \( h_{i,k+i-1} \neq 0 \) for every \( i = 2, 3, \ldots, n-k \). Moreover, to build LDA lattices, we need \( H \) to be sparse, hence its left submatrix \( L \) (of size \((n-k) \times k \)) has to be sparse too. In particular, we choose it also to be regular: it has a fixed constant number of nonzero entries in every column and row. We call these numbers respectively the column degree \( d_c \) and the row degree \( d_r \) of \( L \). Notice that, a priori, \( L \) could be taken irregular and its degree distribution could be optimized, but the standard techniques used for binary LDPC codes cannot be applied in this nonbinary context. Fine-tuning the degree distribution of \( L \) goes beyond the scope of this paper.

By construction, \( H \) is full-rank, hence the rate of the LDPC code that it identifies is \( R = k/n \). Furthermore, all the rows of \( H \) have degree \( d_r + 2 \), except for the first row, that has degree \( d_r + 1 \). If we count the number of nonzero entries at first column by column and then row by row, we relate the degrees and the code parameters via the following equality:

\[ d_c k + 2(n - k - 1) + 1 = (d_r + 2)(n - k - 1) + d_r + 1. \]

By simplifying the previous formula, we can easily derive that

\[ R = \frac{k}{n} = \frac{d_r}{d_r + d_c}. \tag{12} \]

This kind of dual-diagonal parity-check matrix has been used for several practical applications of binary LDPC and repeat-accumulate codes [42, Sec. 6.5], but to the best of our knowledge it was never applied to nonbinary constructions or lattice constructions. The main advantage of using nonbinary LDPC codes is that their design has an additional degree of freedom: when building \( H \), once we fix the degrees, the only freedom that we have in the binary case concerns the choice of the positions of the 1’s in \( H \); in the nonbinary case, instead, we also have to fix the values of the nonzero entries among \{1, 2, \ldots, p - 1\}. This choice plays a nontrivial role.

### A. Encoding Dual-Diagonal LDPC Codes

The particular shape of the parity-check matrix allows to use it for encoding; given \( H \) as in (11) and an information vector \( u \in \mathbb{F}_p^k \), the codeword \( c \in \mathbb{F}_p^n \) associated with \( u \) is obtained in the following way:

1. \( c_i = u_j \), for every \( i = 1, 2, \ldots, k \).
2. The first parity-check equation of \( C \), defined by the first equation of \( H \), is:

\[ \sum_{j=1}^{k} h_{1,j} c_j + h_{1,k+1} c_{k+1} \equiv 0 \text{ mod } p. \]

The only unknown is \( c_{k+1} \in \{0, 1, \ldots, p - 1\} \), that can therefore be computed easily.

3. For \( i = 2, 3, \ldots, n-k \), the \( i \)-th parity-check equation is:

\[ \sum_{j=1}^{k} h_{i,j} c_j + h_{i,k+i-1} c_{k+i-1} + h_{i,k+i} c_{k+i} \equiv 0 \text{ mod } p. \]

A priori, the unknowns in the previous congruence are \( c_{k+i-1} \) and \( c_{k+i} \). Yet, for \( i = 2 \), the only unknown is \( c_{k+2} \), because we computed \( c_{k+1} \) in step 2). Thus, \( c_{k+2} \) can be obtained too.

4. In turn, this means that we can compute \( c_{k+3} \) from the third parity-check equation, and so on so forth, we recursively obtain all the \( c_i \)’s for \( i = k + 1, k + 2, \ldots, n \).

The key observation here is that, because of the sparsity of \( L \), most of the \( h_{i,j} \)’s are equal to 0 for \( i = 1, 2, \ldots, n - k \) and \( j = 1, 2, \ldots, k \). More precisely, exactly \( d_r \) of them are nonzero for every fixed \( i \). Therefore, the number of operations required to compute \( c_{k+i} \) in steps 2)-4) is constant in \( n \) for every fixed \( i \) and the whole encoding procedure of the LDPC code has complexity \( O(d_r(n-k)) = O(\frac{d_r d_c}{d_r + d_c}) \).

We would have a higher complexity if we encoded via the generator matrix of the code, which is in general not sparse. As a consequence of the comments made in Section III-A and Section IV, the whole encoding algorithm of a Leech constellation with underlying dual-diagonal LDPC code has complexity linear in \( n \).

Finally, notice that when we apply the steps 1)-4) above, the codeword \( c \) is built systematically, in the sense that the information vector \( u \) coincides with the first \( k \) coordinates of \( c \). This guarantees that step 2) of Section III-B has no computational complexity.

### VI. Simulation Results

The purpose of this section is to show some numerical decoding performance of Leech constellations of dual-diagonal LDA lattices in very high dimensions. How close to capacity will we be able to get? In [2, Ch. 9], Zamir considers Voronoi constellations of lattices for which the transmission scheme includes uniform dithering at the channel input and lattice decoding. Under these premises, it is shown that for high SNR, the gap to capacity equals the sum of the shaping loss of the coarse lattice and the coding loss of the fine lattice (where by shaping loss we mean the gap to optimal shaping and by coding loss we mean the gap to Poltyrev limit). Our coding scheme differs from the one considered by Zamir because we are not using dithering and in simulations we apply an iterative decoder instead of a lattice decoder. Nonetheless, we will see that the gap to capacity in our numerical results is consistent with the theoretical results stated in [2, Ch. 9].

Now, let us start by describing the parameters of the LDA family with which we are experimenting; this corresponds to making explicit all the choices that characterize the construction of the underlying parity-check matrix \( H \):

- As already mentioned, \( H \) is as in (11).
- We fix \( p = 13 \). This \( p \) is “big enough” in a sense that will be clear later.
For the construction of $L$, we fix $d_c = 2$ and $d_r = 1$. This is the simplest choice for a regular $L$ and has the advantage of speeding the decoding procedure, because smaller degrees correspond to less edges in the associated Tanner graph and therefore to a faster iterative decoding algorithm [42]. According to (12), the associated LDPC codes have rate $R = 1/3$. The resulting $H$ is almost regular: only the first row and the last column have different degrees from the other rows and columns.

- $L$ has size $2k \times k$. We take

$$L = \begin{pmatrix} \Pi_1 \\ \Pi_2 \end{pmatrix},$$

where $\Pi_1$ and $\Pi_2$ are two permutation matrices of size $k \times k$ chosen at random among all permutations that do not create 4-cycles in the Tanner graph associated with $H$ (hence the girth of this graph is at least 6).

- The nonzero entries of $H$ are optimized with the same strategy used in [16]: for every fixed row of $H$, its nonzero entries $(d_r + 2 = 3$ in general or $d_r + 1 = 2$ for the first row) are chosen at random among all the triples (or couples for the first row) of coefficients that guarantee that the minimum Euclidean distance of the single-parity-check code defined by the corresponding parity-check equation is bigger than $\sqrt{2}$. The reader can find in [16, Sec. V-A] a more detailed explanation of this technique, which has experimentally proved to yield better performance than a completely random choice of the nonzero entries of $H$.

### A. Infinite Constellations of Dual-Diagonal LDA Lattices

The first feature to investigate is the performance of the infinite constellations of dual-diagonal LDA lattices. Fig. 1 provides some numerical evaluations of it for values of $n$ up to $10^6 - 1$ (notice that with our choice of the parameters $n$ has to be divisible by 3). The decoder that we used is the same iterative belief-propagation decoder used in [16], whose complexity is $O(p^2 n)$. Fig. 1 shows the symbol-error-rate (SER) as a function of the VNR. Notice that by (1),

$$\text{rate (SER) as a function of the VNR. Notice that by (1),}$$

the normalized volume of $\Lambda_f$ is $\text{Vol}(\Lambda_f)^{\frac{1}{2}} = p^{2(1-R)}$. It is clear from (2) that fixing values of VNR bigger than 1 = 0 dB is the same as fixing noise variances less than the Poltyrev limit $\sigma_{\text{max}}^2$ defined in Corollary 1. The waterfall region of our family of dual-diagonal LDA lattices is situated only at less than 0.3 dB from this limit. This tells that this family is “AWGN-good enough” and its elements are good candidates for being the fine lattices in Leech constellations.

Fig. 1 also shows a lower bound for the SER: since $p\mathbb{Z}^n \subseteq \Lambda_f$, the decoding performance of our LDA lattices are bounded below by those of $p\mathbb{Z}^n$. Therefore, the decoding error probability per coordinate is bounded as follows:

$$p_c(x_i) \geq p_e(p\mathbb{Z}) = 2Q \left( \frac{p}{2\sigma} \right) = 2Q \left( \sqrt{\frac{\pi e p^{2R} \text{VNR}}{2}} \right),$$

where we obtain the last equality using (2) and (1). It is interesting to notice how the SER “touches” the bound for $n = 10^6 - 1$. The choice of $p = 13$ is made on purpose to let the bound be less than $10^{-6}$ at 0.3 dB from Poltyrev limit. This is the reason why we said before that $p$ is “big enough.” For smaller $p$, the bound would be higher in the waterfall region of Fig. 1 and would not allow to fully appreciate the decoding potential of our family in the closest regions to Poltyrev limit (VNR = 0 dB).

### B. Leech Constellations of Dual-Diagonal LDA Lattices

In the previous subsection we established that our family of dual-diagonal LDA lattices has good infinite-constellation performance. Now it is time to validate the goodness of its Leech constellations too. As anticipated at the end of Section IV, with our choice of the parameters and fixing $\alpha = 1$, the information rate of the constellations is $R_C \approx 2.73$ bits per dimension. In Fig. 2, we can see the SER numerically measured as a function of $E_b/N_0$ in dimensions up to $n = 10^6 + 8$ (recall that in this case $n$ has to be divisible by 24). In this scenario, Shannon capacity corresponds to $E_b/N_0 = 8.98$ dB, where,
as usual, $\sigma^2 = N_0/2$ and $E_b = P / R_C$ is the average energy per bit of our constellation.

The experimental gap to Shannon capacity shown in Fig. 2 equals 0.8 dB. As announced at the beginning of the section, it corresponds to the sum of the gap to Poltyrev limit measured in Fig. 1 (around 0.3 dB) plus the gap to optimal shaping due to our choice of shaping the constellation with a direct sum of copies of the Leech lattice (around 0.5 dB).

The Leech-lattice quantizer that we use to perform step 4) of encoding (cf. Section III-A) is the sphere decoder by Viterbo and Boutros [43]; an alternative can be the ML Leech decoder of [44]. The fine dual-diagonal LDA lattices are decoded with the same iterative belief-propagation decoder employed for the infinite constellations of Fig. 1. However, before decoding, in this case we multiply the channel output $y = x + n$ by the Wiener coefficient:

$$w = \frac{\text{SNR}}{1 + \text{SNR}},$$

with SNR as in (3) and $P = E_b R_C$. In other words, the iterative decoder input is $wy$ instead of $y$. The multiplication by $w$ is known as minimum mean squared error (MMSE) scaling [2]. The importance of MMSE scaling to achieve capacity with lattices over the AWGN channel and an optimal lattice decoder is explained in [8], [21], and [45]. Its usefulness for decoding lattice codes with belief-propagation iterative decoders is explained in [46]. Fig. 3 shows an example of the difference in decoding the same Leech constellation with or without MMSE scaling. The constellation that we have taken into account is the same whose performance is plotted in Fig. 2 for $n = 10^4 + 8$. MMSE scaling before decoding allows to gain up to almost 0.1 dB at low $E_b/N_0$. Notice that this is very close to the theoretical performance gain of $10 \log_10(1/w)$ dB predicted by the scheme of [8]. This gain is not negligible when we work close to capacity.

VII. Conclusion

This paper contains two main results: the description of a novel encoding and demapping scheme for Construction-A lattices and its application to design Voronoi constellations whose encoding, decoding, and demapping complexities are all linear in the lattice dimension $n$. The latter result is the combination of several factors:

1) The use of dual-diagonal LDPC codes for Construction A. Their encoding algorithm exploits the particular shape of the parity-check matrix and has linear complexity in $n$.

2) The application of an encoding scheme for Construction-A lattices which does not require multiplication by the lattice generator matrix. This scheme is based on the new characterization of a set of coset leaders of $\Lambda_f / \Lambda$ given in Lemma 1.

3) The use of a direct sum of copies of the Leech lattice as a shaping lattice. This choice makes demapping linear in $n$ and contributes to the linearity of the encoding complexity.

4) The use of a low-complexity iterative decoding algorithm.

The effectiveness of our construction is confirmed by the numerical simulations of the previous section. To obtain that performance, we selected a very specific family of LDA lattices, optimized in many little but nontrivial senses. The result of these choices is satisfying and invites to further investigate these kinds of constructions. Notice that the encoding, decoding, and demapping procedures are independent from the choice of the shaping lattice and using the Leech lattice for shaping is not mandatory. With the same principles applied in this paper, we can build Voronoi constellations using any Construction-A fine lattice and any integer lattice $\Gamma$ to build the coarse lattice $p \Gamma = \Lambda$. In particular, lattices in smaller dimensions can substitute the Leech lattice to improve the encoding complexity; others with better shaping gain can be used to improve the decoding performance.

REFERENCES

[1] J. G. Forney, R. G. Gallager, G. Lang, F. M. Longstaff, and S. U. Qureshi, "Efficient modulation for band-limited channels," IEEE J. Sel. Areas Commun., vol. SAC-2, no. 5, pp. 632–647, Sep. 1984.

[2] R. Zamir and B. Nazet, Lattice Coding for Signals and Networks: A Structured Coding Approach to Quantization, Modulation and Multiuser Information Theory. Cambridge, U.K.: Cambridge Univ. Press, 2014.

[3] Z. Ma, Z. Zhang, Z. Ding, P. Fan, and H. Li, “Key techniques for 5G wireless communications: Network architecture, physical layer, and MAC layer perspectives,” Sci. China Inf. Sci., vol. 58, no. 4, pp. 1–20, Apr. 2015.

[4] R. de Buda, “Some optimal codes have structure,” IEEE J. Sel. Areas Commun., vol. 7, no. 6, pp. 893–899, Aug. 1989.

[5] G. Poltyrev, “On coding without restrictions for the AWGN channel,” IEEE Trans. Inf. Theory, vol. 40, no. 2, pp. 409–417, Mar. 1994.

[6] H.-A. Loeliger, “Averaging bounds for lattices and linear codes,” IEEE Trans. Inf. Theory, vol. 43, no. 6, pp. 1767–1773, Nov. 1997.

[7] R. Urbanke and B. Rimoldi, “Lattice codes can achieve capacity on the AWGN channel,” IEEE Trans. Inf. Theory, vol. 44, no. 1, pp. 273–278, Jan. 1998.

[8] U. Erez and R. Zamir, “Achieving $1/2 \log(1+\text{SNR})$ on the AWGN channel with lattice encoding and decoding,” IEEE Trans. Inf. Theory, vol. 50, no. 10, pp. 2293–2314, Oct. 2004.
