String theory and the 4D/3D reduction of Seiberg duality. A Review

Antonio Amariti, Domenico Orlando and Susanne Reffert

Albert Einstein Center for Fundamental Physics
Institute for Theoretical Physics
University of Bern,
Sidlerstrasse 5, ch-3012 Bern, Switzerland
Abstract

We review the reduction of four-dimensional $\mathcal{N} = 1$ Seiberg duality to three dimensions focusing on the D brane engineering approach. We start with an overview of four-dimensional Seiberg duality for theories with various types of gauge groups and matter content both from a field-theoretic and a brane engineering point of view. Then we describe two families of $\mathcal{N} = 2$ three-dimensional dualities, namely Giveon–Kutasov-like and Aharony-like dualities. The last part of our discussion is devoted to the 4D/3D reduction of the dualities studied above. We discuss both the analysis at finite radius, crucial for preserving the duality in the dimensional reduction, and the zero-size limit that must be supported by a real mass flow and a Higgsing, which can differ case by case. We show that this mechanism is reproduced in the brane description by T-duality, supplying a unified picture for all the different cases. As a bonus we show that this analysis provides a brane description for Aharony-like dualities.
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1 Introduction

Dualities have played an extremely important role in theoretical physics in the last decades. In this review, we bring together four-dimensional Seiberg-like dualities, three-dimensional dualities and the reduction from four to three dimensions, with a string-theoretic outlook on the problem, namely brane constructions. These brane constructions give rise to a clear picture that unifies all the reductions of four-dimensional dualities to dualities in three dimensions.

A duality is an exact equivalence of two seemingly different physical systems. Especially so-called strong/weak dualities are an important tool for accessing non-perturbative regimes of quantum field theories. In general, we can approach a physical problem via perturbation theory, where we have a Hamiltonian

\[ H = H_0 + gH_1, \]  

(1.0.1)

where \( H_0 \) is solvable. Observables are given by a perturbation series in the parameter \( g \), which is valid for small values of \( g \). For large \( g \), the perturbative approach loses its applicability, and often we are at a complete loss of how to study the system in the strongly coupled regime. Sometimes, however, the system allows two different descriptions,

\[ H = H_0 + gH_1, \]
\[ = H'_0 + g'H'_1, \]

(1.0.2)

leading to two different perturbative expansions. We speak of a strong/weak duality, when the coupling constants \( g, g' \) are related via

\[ g' = \frac{1}{g}. \]

(1.0.3)

When \( g \) becomes large, the perturbation series in \( g' \) becomes an accurate description of the system, allowing important insights into the theory at a regime which was inaccessible in the first description.

Maybe the first prototype of such a duality in four dimensions is electric/magnetic duality, i.e. the invariance of source-free Maxwell theory under the mapping

\[ \vec{E} \to \vec{B}, \]
\[ \vec{B} \to -\vec{E}, \]

(1.0.4)

which in the quantum theory includes the transformation of the coupling

\[ e \to e' = \frac{2\pi}{e}. \]

(1.0.5)

This duality becomes more interesting when sources are present, i.e. for non-Abelian gauge theories which include monopole solutions. While the electric charges become more strongly coupled and less point-like for larger \( e \), the monopoles appear to become lighter and smaller, suggesting that they could be quantized

\[ \text{For an elementary introduction to the concept of duality, see [2].} \]
as fundamental fields for very large $e$. These observations led to the Montonen–Olive conjecture [2] stating that non-Abelian gauge theories with monopoles are invariant under a strong/weak electric/magnetic duality. It turned however out that without adding supersymmetry, even gathering only circumstantial evidence was impossible. Witten and Olive [3] showed that for $\mathcal{N} = 2$ super Yang–Mills ($\text{sym}$) the calculation of the monopole masses is exact. In $\mathcal{N} = 4$ $\text{sym}$, also the spins of the electric charges and the monopoles can be shown to match [4], and the mapping of the coupling constant gets extended to the full S–duality group $SL(2,\mathbb{Z})$.

In 1994, Seiberg [5] proposed a duality for $\mathcal{N} = 1$ supersymmetric quantum chromodynamics ($\text{sqcd}$), which unlike the versions with higher supersymmetry which are exact for all values of the coupling constant, is a infrared (IR) duality. In this review, we will concentrate on Seiberg-duality and its many generalizations and extensions. These four-dimensional dualities have sparked interest also in their three-dimensional analogs, starting with the work of Aharony [6] in 1997. The question of how the three-dimensional dualities are related to their four-dimensional cousins is obvious. As was discussed at length in [7], a naive dimensional reduction does not yield the correct results.

A large amount of literature and good reviews exist about dualities of four-dimensional supersymmetric gauge theories (see e.g. [8–12]). Also, the literature on three-dimensional dualities is extensive [6,13–26], even though no real reviews exist on the subject. The aim of this review is to bring together the field theoretic material of the above topics and to add as a new ingredient the string-theoretic point of view. We will discuss the brane constructions not only for the relevant gauge theories, but also their reduction. This approach has allowed us to obtain a brane picture for Aharony duality, which had been missing in the literature. The reason for this lack is intrinsic in the brane engineering of a field theory. Via this picture, it is quite easy to capture the classical dynamics, for example the global symmetries and the classical moduli space. Extracting quantum information is however not obvious. To understand which of the global symmetries become anomalous at the quantum level for example requires more information, such as the uplift to M-theory. The key role in capturing the quantum aspects of the reduction of four-dimensional dualities and the brane picture of Aharony duality is played by T-duality in string theory. T–duality provides the missing quantum information, necessary to reconstruct the three-dimensional physics in this picture.

The plan of the review is as follows.

• In Section 2 we review some aspects of four-dimensional $\mathcal{N} = 1$ gauge theories: the algebra, the matter content and the interactions (Sec. 2.1). We also discuss some aspects of the analysis of the moduli space, mass deformations and anomalies. The discussion is focused on the case of $\text{sqcd}$ (Sec. 2.1.2). After a brief introduction, we introduce some basic aspects of Seiberg duality for $\text{sqcd}$ and some generalizations, restricting our analysis to theories with a single gauge group. We consider theories with unitary or real gauge groups, fundamental and tensor matter (Sec. 2.2). We review the main aspects of these dualities, i.e. the dual gauge groups, the matter content of the dual theory and the superpotential. Finally, we discuss the engineering of these systems in type IIA string theory by introducing the realization in terms of NS branes, D branes and orientifold.
O-planes. We discuss the relation between Seiberg duality and the Hanany–Witten [27] (hw) transition, showing how to reproduce the field theory results in the brane language (Sec. 2.3).

• In Section 3, we discuss some known fact about three-dimensional \( \mathcal{N} = 2 \) gauge theories with a single gauge group. We start by reviewing the main aspects of these theories, i.e. their field content, their interactions and the moduli space. In particular, we stress the role of the monopole operators and their relation to the Coulomb branch. We also discuss the Chern–Simons (cs) interactions, and their relation with the real masses that exist in three dimensions (Sec. 3.1.1 and Sec. 3.1.2). After this review, we start discussing two classes of dualities: the one found by Aharony in [6] and the one found by Giveon and Kutasov in [17]. Our focus is on the similarities and the differences between these three-dimensional dualities and four-dimensional Seiberg duality (Sec. 3.2). We also discuss the generalization of these dualities to real gauge groups and tensor matter and we conclude the section by reviewing the brane engineering of the three-dimensional dualities (Sec. 3.3). We discuss in detail the dualities of Giveon–Kutasov [17] (gk) type and we introduce some brane engineerings of more involved configurations that (to our knowledge) have not yet appeared in the literature. We conclude commenting on the absence of a brane engineering for Aharony duality and for its generalizations.

• In Section 4, we discuss the dimensional reduction of four-dimensional Seiberg duality to three dimensions along the lines of [7]. We review the main aspects of this reduction, discussing the problems of a naive reduction on \( S^1 \) and how the standard prescription has to be modified by considering effective dualities at finite radius. We also show that these are related to the more conventional dualities of Aharony and of Giveon–Kutasov via a real mass flow. The cases with unitary and real gauge groups require a very different treatment of the decoupling limit on the field theory side and apparently an ad hoc procedure has to be adopted (Sec. 4.1). In the second part of this section, we show that these reductions can be understood via brane engineering (Sec. 4.2) and a unified picture emerges. All the different cases are obtained by considering a double-scaling limit on the real masses and on the radius of the compactification circle. By defining the circle along the spatial direction \( x_3 \) and considering the three-dimensional gauge theory living at \( x_3 = 0 \), a very important role is played by the mirror point on the circle, \( x_3^\circ = R\pi \), where \( R \) is the radius of \( S^1 \). The double-scaling limit can be performed in the same way in each case because the physics at that point can be described in a unified way in terms of branes.

• In Section 5, we discuss checks of the dualities via localization. We review the derivation of the three-dimensional partition function on the squashed sphere via the circle reduction of the four-dimensional superconformal index (Sec. 5.1); we apply this technique to the reduction of dualities (Sec. 5.2) and we show as an application the reduction of Kutasov–Schwimmer–Seiberg [28] (kss) duality.

We hope to close a gap in the literature with this review by collecting not only the various generalizations of Seiberg duality in four dimensions, but by finally providing an overview over the Seiberg-like dualities in three dimensions and
their relation to the four-dimensional case. Last but not least, we believe that the string theory point of view succeeds in unifying the understanding of these various dualities.
2 The four-dimensional case

In this section, we discuss the four-dimensional case. We first recall the general properties, multiplets and invariant Lagrangians for $\mathcal{N} = 1$ gauge theories. Then we specialize to $\text{SQCD}$ with gauge group $SU(N_c)$, our main example. Then, we discuss the dualities, starting from Seiberg duality for SQCD and discussing its extensions. Finally, we move to the string theory point of view and recreate the dualities via D brane constructions.

2.1 $\mathcal{N} = 1$ supersymmetric gauge theories in four dimensions

In the following, we will summarize the basics of supersymmetric gauge theories in four dimensions, following in presentation and notation Ch. 9 of [29]. Then we will review the specific case of SQCD, which is the most basic example of Seiberg duality.

2.1.1 Generalities

The $\mathcal{N} = 1$ supersymmetry algebra is given by

$$\left\{ Q_{\alpha}, \bar{Q}_{\dot{\beta}} \right\} = 2 \sigma^{\mu}_{\alpha \dot{\beta}} P_{\mu},$$

(2.1.1)

where the supersymmetry generators $Q$, $\bar{Q}$ are two-component spinors with spinor indices $\alpha, \dot{\beta}$, $\sigma^{\mu} = (1, \vec{\sigma})$ are the Pauli matrices, and $P_{\mu}$ is the four-momentum.

The smallest irreducible representations which can describe massless fields include the chiral multiplet which contains as propagating degrees of freedom a complex scalar $\phi$ and a chiral fermion $\psi_{\alpha}$, and the vector multiplet containing a chiral fermion $\lambda_{\alpha}$ and a vector field $A_{\mu}$ which are both in the adjoint representation of the gauge group $G$.

We will be using the superspace formulation, which extends spacetime with coordinates $x^{\mu}$ by two anti-commuting Grassmann variables $\theta_{\alpha}, \bar{\theta}_{\dot{\alpha}}$. The supersymmetry generators $Q, \bar{Q}$ act on functions of the superspace variables $f(x^{\mu}, \theta, \bar{\theta})$ as differential operators:

$$Q_{\alpha} = \frac{\partial}{\partial \theta_{\alpha}} - i \sigma^{\mu}_{\alpha \dot{\beta}} \theta^{* \dot{\beta}} \partial_{\mu}, \quad \bar{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}} + i \theta^{* \alpha} \sigma^{\mu}_{\alpha \dot{\beta}} \partial_{\mu}. \quad (2.1.2)$$

The covariant derivatives are defined by

$$D_{\alpha} = \frac{\partial}{\partial \theta_{\alpha}} + i \sigma^{\mu}_{\alpha \dot{\beta}} \theta^{* \dot{\beta}} \partial_{\mu}, \quad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}} - i \theta^{* \alpha} \sigma^{\mu}_{\alpha \dot{\beta}} \partial_{\mu}. \quad (2.1.3)$$

They anti-commute with the supersymmetry generators and are thus useful for constructing invariant expressions. The condition

$$\bar{D}_{\dot{\alpha}} \Phi = 0 \quad (2.1.4)$$
on a superfield $\Phi$ is invariant under supersymmetry transformations. Fields satisfying this condition are called \textit{chiral superfields}. A chiral superfield $\Phi$ has the expansion in $\theta$, $\bar{\theta}$

$$\Phi = \phi(x) + i \theta \sigma^\mu \bar{\theta} \partial_\mu \phi + \frac{1}{4} \theta^2 \bar{\theta}^2 \partial^2 \phi + \sqrt{2} \theta \phi - \frac{i}{\sqrt{2}} \theta \bar{\theta} \partial_\mu \phi \partial^\mu \phi + \theta^2 F,$$  \hspace{1cm} (2.1.5)

where as mentioned before, $\phi$ is a complex scalar, $\psi_\alpha$ a chiral fermion, and $F$ is a scalar field, which as we will see in the following, does not propagate and is thus called \textit{auxiliary}. The spinor indices are all fully contracted and thus suppressed.

As mentioned, \textit{vector fields} form another irreducible representation of the algebra, they satisfy

$$V = V^\dagger.$$  \hspace{1cm} (2.1.6)

Also this condition is preserved under supersymmetry transformations. The expansion of a vector superfield $V$ in a power series in $\theta$ is given by

$$V = i \chi - i \chi^\dagger - \theta \sigma^\mu \theta^\alpha A_\mu + i \theta^2 \bar{\theta} \lambda - i \bar{\theta}^2 \theta \lambda + \frac{1}{2} \theta^2 \bar{\theta}^2 D,$$  \hspace{1cm} (2.1.7)

where $D$ is a real scalar field, which like $F$ is auxiliary as it does not propagate. The vector field $A_\mu$ hints at an underlying gauge symmetry: let us take $\Phi$ to transform as

$$\Phi \rightarrow e^{-i \Lambda} \Phi$$  \hspace{1cm} (2.1.8)

under gauge transformations, where $\Lambda$ is a matrix-valued chiral superfield. In order for the combination

$$\Phi^\dagger e^V \Phi, \hspace{1cm} V \text{ a matrix valued superfield}$$  \hspace{1cm} (2.1.9)

to be invariant, $V$ must transform as

$$e^V \rightarrow e^{-i \Lambda} e^V e^{i \Lambda}$$  \hspace{1cm} (2.1.10)

under gauge transformations. We can use the gauge freedom to set the $\chi$ component of $V$ to zero, this is known as the Wess–Zumino gauge. The gauge-covariant field strength is given by the expression

$$W_a = -\frac{i}{4} \bar{D}^2 e^{-V} D_a e^V.$$  \hspace{1cm} (2.1.11)

It transforms under gauge transformations like a chiral field in the adjoint representation,

$$W_a \rightarrow e^{-i \Lambda} W_a e^{i \Lambda}.$$  \hspace{1cm} (2.1.12)

\textbf{Supersymmetric Lagrangians.} As we have now collected all the ingredients, we are ready to construct manifestly supersymmetric and gauge invariant Lagrangians, using the fact that $\theta$ has dimension $-1/2$, while chiral superfields have dimension 1 and vector superfields dimension 0.

The most general renormalizable Lagrangian consists of the following pieces:

$$L_{\text{kin}} = \int d^4 \theta \sum_i \Phi_i^\dagger e^V \Phi_i,$$  \hspace{1cm} (2.1.13)
where the sum runs over all the matter multiplets and $V$ is in a representation that is appropriate for the field $\Phi_i$.

$$L_W = \int d^2 \theta W(\Phi_i) + \text{c.c}, \quad (2.1.14)$$

where $W(\Phi_i)$ is a holomorphic function of the $\Phi_i$, the so-called superpotential. For the theory to be renormalizable, the superpotential must take the form

$$W = \frac{1}{2} m_{ij} \Phi_i \Phi_j + \frac{1}{3} \lambda_{ijk} \Phi_i \Phi_j \Phi_k. \quad (2.1.15)$$

The gauge part of the Lagrangian is given by

$$L_{\text{gauge}} = \frac{1}{g^{(i)2}} \int d^2 \theta W^{(i)} W^{\dagger(i)}. \quad (2.1.16)$$

The full Lagrangian density is given by

$$\mathcal{L} = L_{\text{kin}} + L_W + L_{\text{gauge}}. \quad (2.1.17)$$

In case the gauge group $G$ has $U(1)$ factors, we can include a supersymmetric and gauge invariant Fayet–Iliopoulos term for each of them:

$$\xi \int d^4 \theta V, \quad (2.1.18)$$

where $\xi$ is a real Fayet–Iliopoulos parameter.

Thanks to the superspace formalism, the above Lagrangian is very compact, but in order to make the physical meaning of these terms more transparent, we express them in terms of the component fields:

$$L_{\text{kin}} = |\partial_\mu \phi_i|^2 + i\psi_i \partial_\mu \sigma_i \phi_i^* + F_i F_i^* + \text{c.c}, \quad (2.1.19)$$

We see now that the $F_i$ appear without derivatives, which is why they are called auxiliary fields.

$$L_W = \frac{\partial W}{\partial \Phi_i} F_i + \frac{\partial^2 W}{\partial \Phi_i \partial \Phi_j} \psi_i \psi_j \quad (2.1.20)$$

$$= F_i (m_{ij} \Phi_j + \lambda_{ijk} \Phi_j \Phi_k) + (m_{ij} + \lambda_{ijk} \Phi_k) \psi_i \psi_j + \text{c.c},$$

where the second equality holds for the choice of superpotential Eq. (2.1.15). The full Lagrangian in Wess–Zumino gauge in terms of component fields is given by

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu \nu}^a \phi^2 - i \lambda^a \sigma^\mu D_\mu \lambda^a + |D_\mu \phi_i|^2 - i \psi_i^a \sigma^\mu D_\mu \psi_i^a$$

$$\quad + \frac{1}{2g^2} \left( D^a \right)^2 + D^a \sum_i \phi_i^a T^a \phi_i + F_i^a F_i - F_i \frac{\partial W}{\partial \phi_i} + \text{c.c} \quad (2.1.21)$$

$$+ \sum_{i,j} \frac{1}{2} \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \psi_i \psi_j + i \sqrt{2} \sum_{a,j} \phi_j^a T^a \phi_i^a + \text{c.c},$$

In the following we will deal with non-renormalizable superpotentials as well. They will be related to dangerously irrelevant interactions that will lead the flow to strongly coupled fixed points in the IR.
where \( a \) is the gauge index and \( T^a \) are the generators of the Lie algebra associated to \( G \).

We can solve for the auxiliary fields \( F_i \) and \( D^a \) via their equations of motion and find

\[
F_i = \frac{\partial W}{\partial \phi^*_i}, \quad D^a = \sum_i (g^a \phi^*_iT^a \phi_i).
\]

Substituted back into the Lagrangian, we find the scalar potential

\[
V = |F_i|^2 + \frac{1}{2g^2}(D^a)^2.
\]

Allowing also non-renormalizable terms, the most general, globally \( N = 1 \) super-symmetric theory with at most two derivatives has the form

\[
L = \int d^4\theta K(\Phi_i, \Phi^*_i) + \int d^2\theta W(\Phi_i) + c.c. + \int d^2\theta f_a(\Phi_i)(W^{(a)})^2 + c.c.,
\]

where the Kähler potential \( K(\Phi_i, \Phi^*_i) \) gives rise to the kinetic terms, and the superpotential and the gauge kinetic function \( f_a(\Phi_i) \) are functions of the chiral fields only. In \( N = 1 \) supersymmetric theories, non-renormalization theorems state that the superpotential is not corrected in perturbation theory beyond its tree-level value and that \( f_a \) is at most renormalized at one loop.

**R–symmetry.** Supersymmetric theories have global symmetries which rotate the supercharges, so-called R–symmetries. For \( N = 1 \) supersymmetric theories, the R–symmetry group is isomorphic to a global \( U(1)_R \), or a discrete subgroup thereof. The defining feature of a continuous R–symmetry is that \( \theta, \bar{\theta} \) transform as

\[
\theta \rightarrow e^{i\alpha} \theta, \quad \bar{\theta} \rightarrow e^{-i\alpha} \bar{\theta},
\]

with \( \alpha \) the transformation parameter, meaning that they have R–charges 1, \(-1\). The supersymmetry generators transform thus as

\[
Q \rightarrow e^{-i\alpha} Q, \quad \bar{Q} \rightarrow e^{i\alpha} \bar{Q},
\]

i.e. they have R–charges \(-1, +1\) and do not commute with the R–symmetry generator:

\[
[R, Q] = -Q, \quad [R, \bar{Q}] = \bar{Q}.
\]

The different component fields within a superfield carry different R–charges. If the theory is invariant under R–symmetry, each superfield \( S \) can be assigned an R–charge \( r_S \) defined by

\[
S(x^\mu, \theta, \bar{\theta}) \rightarrow e^{ir_S} S(x^\mu, e^{-i\theta}, e^{i\bar{\theta}}).
\]

The R–charge of a product of superfields is the sum of the individual charges of the fields. In Table 2.1 the R–charges of the various quantities are collected.

All Lagrangian terms involving gauge superfields are automatically R–symmetric, including the couplings to the chiral fields. The superpotential, however, must carry R–charge \(+2\) in order to conserve R–symmetry which is not automatic and often not the case.
Table 2.1: \( U(1)_R \) R-charges of the different quantities appearing in \( \mathcal{N} = 1 \) supersymmetric gauge theories.

| \( \theta_a \) | \( \bar{\theta}_k \) | \( Q_a \) | \( d^2 \theta \) | \( D_k \) | \( W_a \) | \( A^\mu \) | \( \lambda_a \) | \( D \) | \( W \) | \( \psi \) | \( \psi_A \) | \( F \) |
|-----------|---------|---------|----------|--------|--------|--------|--------|----|----|-----|-----|----|
| \( r \)   | \( 1 \)  | \( -1 \) | \( -1 \) | \( -2 \) | \( -1 \) | \( 1 \)   | \( 0 \)   | \( 2 \) | \( r_\Phi \) | \( r_\Phi - 1 \) | \( r_\Phi - 2 \) | 

**Moduli space.** Supersymmetric theories in general do not have isolated vacua but exhibit a continuous family of connected vacua. The moduli space of inequivalent vacua is the set of all zero-energy field configurations modulo gauge transformations. The vacuum degeneracy of the classical moduli space \( M_{cl} \) is however not protected by any symmetries and can be lifted in the quantum theory by a dynamically generated effective superpotential. For an \( \mathcal{N} = 1 \) supersymmetric gauge theory with zero tree-level superpotential, the classical squark potential is given by

\[
V_{\text{squark}} = i\sqrt{2}\sum_{i,a} \lambda^a \psi_i T^a \phi_i^\dagger + \text{c.c.} + \frac{1}{2} \sum_a \left( \sum_i \phi_i^\dagger T^a \phi_i \right)^2,
\]

where the last term comes from the \( D \)-term in the scalar potential given in Eq. (2.1.23). The classical moduli space \( M_{cl} \) is the space of squark expectation values \( \langle \phi_i \rangle \) modulo gauge equivalence, along which the potential Eq. (2.1.29) vanishes. \( M_{cl} \) can be given a gauge-invariant description in terms of expectation values of gauge-invariant combinations of the fields (such as e.g. mesons), subject to classical constraints. A more detailed description of the moduli space can be found e.g. in [8].

### 2.1.2 SQCD with \( G = SU(N_c) \)

In this section, we will study supersymmetric quantum chromodynamics with gauge group \( U(N_c) \), where \( N_c \) stands for the number of colors. The gauge field is encoded in the \( \mathcal{N} = 1 \) vector multiplet

\[
V = (\lambda^a, A^a_\mu, D^a)
\]

with gauge index \( a \) as discussed in the last section. The matter content consists of \( N_f \) flavors of quarks, encoded in the chiral multiplets

\[
\tilde{Q}_i^a = (\phi_{iQ_a}, \psi_{iQ_a}, F_{iQ_a}), \quad i = 1, \ldots, N_f,
\]

and \( N_f \) flavors of anti-quarks, encoded in the chiral multiplets

\[
\tilde{\bar{Q}}_i^a = (\psi_{i\bar{Q}_a}, \phi_{i\bar{Q}_a}, F_{i\bar{Q}_a}), \quad i = 1, \ldots, N_f.
\]

This means that in total, there are \( 2N_cN_f \) chiral degrees of freedom. The quarks transform in the fundamental representation \( N_c \) of the gauge group, while the anti-quarks transform in the anti-fundamental \( \bar{N}_c \). There is no perturbative superpotential, \( W = 0 \). We will see in the following that the qualitative features of SQCD change very drastically for different relative values of \( N_f \) and \( N_c \).

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2 Such a superpotential is generated non-perturbatively and does not violate the non-renormalization theorems.

3 For more details, see [5, 8].
Global symmetries. Classically, this theory has the following global symmetries:

- the flavor symmetry $SU(N_f)_L \times SU(N_f)_R$
- the baryonic symmetry $U(1)_B$
- the axial symmetry $U(1)_A$
- the R–symmetry $U(1)_R$.

At the quantum level, only the $U(1)_A$ is anomalous. In Table 2.2, we have collected the transformation properties of the quark multiplets under the global symmetries.

Moduli space. For $N_f < N_c$, the classical moduli space is parameterized by gauge-invariant mesons

$$M^j_i = Q^j_i \tilde{Q}^q_j,$$

which transform in the bifundamental representation $N_f \times \bar{N}_f$ of $SU(N_f)_L \times SU(N_f)_R$. The gauge group $SU(N_c)$ is partially broken to $SU(N_c - N_f)$. It turns out, however, that $M_{cl}$ is lifted completely in this range of $N_f$ by a dynamically generated superpotential, so the quantum theory has no supersymmetric ground state.

The classical moduli space for $N_f \geq N_c$ is parameterized in a gauge-invariant way by

$$M^j_i = Q^j_i \tilde{Q}^q_j,$$

$$B^{i_1 \ldots i_{N_c}} = Q^{i_1}_{i_1} \cdots Q^{i_{N_c}}_{i_N} \epsilon_{a_1 \ldots a_{N_c}},$$

$$\tilde{B}^{i_1 \ldots i_{N_c}} = \tilde{Q}^{a_1}_{i_1} \cdots \tilde{Q}^{a_{N_c}}_{i_{N_c}} \epsilon_{a_1 \ldots a_{N_c}},$$

where $B$ is a baryon and $\tilde{B}$ an anti-baryon. These basic generators are however not independent, but subject to classical constraints. At a generic point of $M_{cl}$, the gauge group is completely broken ("Higgsed"). In the range $N_f \geq N_c$, the vacuum degeneracy cannot be lifted and $M_q$ exists. While $N_f = N_c$, $N_f = N_c + 1$ are special cases (see e.g. the discussion in [8]), for $N_f \geq N_c + 2$, the quantum and classical moduli spaces are identical, $M_q = M_{cl}$.

---

Table 2.2: Transformation properties of the quarks under the global symmetries of $sqcd$.

|       | $SU(N_f)_L$ | $SU(N_f)_R$ | $U(1)_B$ | $U(1)_A$ | $U(1)_R$ |
|-------|-------------|-------------|---------|---------|---------|
| $Q^a_i$ | $N_f$       | 1          | 1       | 1       | $\frac{N_f - N_c}{N_f}$ |
| $\tilde{Q}^a_i$ | 1          | $N_f$     | -1      | 1       | $\frac{N_f - N_c}{N_f}$ |

---

4 At this point, the assignment of $r_Q$ is unmotivated.
Phases of SQCD.  

\[ \beta(g) = -\frac{g^3}{16\pi^2} \frac{3N_c - N_f(1 - \gamma_0)}{1 - \frac{g^2}{8\pi^2}N_c}, \]

(2.1.37)

\[ \gamma_0(g) = -\frac{g^2}{8\pi^2} \frac{N_f^2 - 1}{N_c} + \mathcal{O}(g^4), \]

(2.1.38)

where \( \gamma_0 \) is the anomalous dimension of the mass. We see clearly that \( N_f = 3N_c \) is a special point. In fact, for \( N_f < 3N_c \), the beta-function is negative. If was shown that for \( N_c, N_f \gg 1 \) and \( 3N_c - N_f \ll N_c \), a conformal fixed point exists. This fixed point

\[ g = g_* \approx \sqrt{\frac{(3N_c - N_f)}{3N_c}} \frac{1}{N_c} \]

(2.1.39)

is a stable infrared fixed point (\( c \) is a positive number of order one). We see that \( g_* \) increases for diminishing \( N_f \). Via arguments involving the operator dimensions, it can be shown that the description of the theory must change completely for \( N_f < 3/2N_c \). It has been argued that such a conformal fixed point exists in the whole range \( 3/2N_c < N_f < 3N_c \). In this range, the symmetry is enhanced to the superconformal algebra and we speak of the conformal window of SQCD. In the conformal window, the infrared theory is a non-trivial interacting super-conformal field theory (scft). The elementary quarks and gluons are not confined but appear as interacting massless particles. The potential between external electric sources at distance \( R \) behaves like \( V(R) \sim 1/R \), so we are in a non-Abelian Coulomb phase.

In the range \( N_f \geq 3N_c \), the theory is not asymptotically free: the coupling decreases at large distances due to screening effects. The potential between external electric sources behaves like \( V(R) \sim \frac{1}{R \log(R\Lambda)} \), so we are in a non-Abelian free electric phase (\( g^2 \sim \frac{1}{R \log(R\Lambda)} \)).

In the range \( N_c + 2 \leq N_f \leq 3/2N_c \), the theory becomes infinitely coupled and we need a different description. It is in fact in a non-Abelian free magnetic phase and should be described by a different set of degrees of freedom.

In Figure 2.1, the different properties of SQCD for different values of \( N_f \) are summarized.

2.1.3 SQCD with real gauge groups

So far we have discussed theories with unitary gauge groups. We will also study theories with real gauge groups, namely symplectic \( Sp(2N_c) \) and orthogonal \( SO(N_c) \) gauge theories.

Let us briefly summarize some properties of these theories. We refer to the unitary symplectic group as \( Sp(2N_c) \), i.e. the subgroup of \( SU(2N_c) \) leaving an antisymmetric tensor \( f \) invariant. This group has dimension \( N_c(2N_c + 1) \) and there are \( 2N_f \) quarks in the fundamental representation of \( Sp(2N_c) \). This representation has dimension \( 2N_c \). There is a meson operator of dimension \( N_f(2N_f - 1) \) and,

\[ \text{This applies at the origin of moduli space.} \]
If the gauge group is orthogonal, it corresponds to a model with an \( \text{so}(N_c) \) algebra. Depending on the global properties, there are different choices for the gauge group. These groups can be constructed starting from an \( \text{SO}(N_c) \) gauge group and gauging a discrete \( \mathbb{Z}_2 \) symmetry. There are different choices for this gauging, leading to either an \( \text{O}(N_c)_- \), an \( \text{O}(N_c)_+ \), or a \( \text{Spin}(N_c) \) gauge group. These cases are slightly different, e.g. matter fields in the spinor representation are allowed in the \( \text{Spin}(N_c) \) case. We will ignore this issue in the rest of this review, referring the reader to [32] for details.

We consider \( \text{SO}(N_c) \text{sQCD} \), and in this case the gauge group has dimension \( N_c(N_c - 1)/2 \). We consider \( N_f \) flavors in the vector representation of the gauge group. There is a mesonic operator of dimension \( (N_f + 1)/(2N_f) \), and there are baryons as well.

### 2.2 Seiberg duality in four dimensions and some generalizations

In this section we review some salient features of four-dimensional Seiberg duality relevant for our discussion. First we introduce the notion of Seiberg duality for \( SU(N_c) \text{sQCD} \) with \( N_f \) flavors then we turn the discussion to the generalization of this construction in presence of tensor matter and for real gauge groups.

#### 2.2.1 SQCD with \( G = SU(N_c) \)

In his original paper [5], Seiberg considered \( \text{sQCD} \) as discussed in Section 2.1.2 with superpotential

\[
W = 0
\]

We have seen that this theory has different IR behaviors, depending on the ratio \( x = N_f/N_c \). We will focus on the case of \( 3/2 \leq x \leq 3 \), where the theory is superconformal. One of the main consequences of superconformality is that the \( R \)-current belongs to a supermultiplet, which contains the stress energy tensor as higher component. In this window, the theory can be studied via conformal
perturbation theory. In general, there is a weakly coupled regime for \( x > 2 \) and a strongly coupled regime for \( x < 2 \). In the special case of \( x = 2 \), corresponding to \( N_f = 2 N_c \), one can equivalently describe the physics, i.e. the partition functions and the correlations in terms of a different set of fundamental degrees of freedom, the fundamentals and antifundamentals \( q \) and \( \bar{q} \). In this description, it is necessary to introduce a gauge singlet, that we will refer to as \( M \). This singlet is in the bifundamental representation of the flavor symmetry group and it couples to the fields \( q \) and \( \bar{q} \) through a superpotential coupling,

\[
W = h M q \bar{q}.
\]

The transformation properties of these fields under the global symmetries of SQCD are summarized in Table 2.3.

One can study the properties of the fixed point of these theories in both cases: in the first, “electric”, case there is one coupling involved, the gauge coupling \( \lambda \), while in the second, “magnetic”, case there are two couplings, the gauge coupling \( \bar{\lambda} \) and the superpotential coupling \( h \). The two descriptions are physically equivalent. This equivalence can be reformulated in terms of a duality, by observing that the singlet \( M \) has the same quantum numbers of the gauge invariant operator \( Q \bar{Q} \). One can claim a mapping

\[
M = \frac{1}{\mu} Q \bar{Q}.
\]

between the elementary singlet of the magnetic theory and the composite operator of the electric theory. In equation (2.2.3), \( \mu \) represents the renormalization group (RG) scale at which we consider the theory. This mapping can be extended to the full conformal window. In general, for \( x \neq 2 \), the magnetic theory has gauge group \( SU(N_f - N_c) \). For \( x < 2 \), the magnetic theory is more weakly coupled than the electric theory, while this regime is inverted in the \( x > 2 \) window, see Figure 2.2. This behavior is reminiscent of electric/magnetic duality (see e.g. the case of \( N = 4 \) SYM, where S-duality maps the holomorphic gauge coupling as \( \tau' = -1/\tau \)).

The holomorphic scale \( \Lambda \) is defined in terms of the holomorphic gauge coupling \( \tau = \frac{4 \pi i}{g^2} + \frac{\beta}{2\pi} \) as

\[
\Lambda = \mu e^{\pi i b/\tau},
\]

where \( b \) is the numerator of the 1-loop \( \beta \)-function for the gauge coupling,

\[
b = 3 N_c - N_f
\]

for SQCD, see also Eq. (2.1.37).

| Field | SU(Nf)_L | SU(Nf)_R | U(1)_B | U(1)_A | U(1)_R |
|-------|-----------|-----------|---------|---------|---------|
| M     | Nf        | Nf        | 0       | 2       | 2 \( \frac{N_f - N_c}{N_f} \) |
| q     | Nf        | 1         | \( \frac{N_c}{N_f - N_c} \) | -1 | \( \frac{N_c}{N_f} \) |
| \( \bar{q} \) | 1 | Nf | \( \frac{N_c}{N_f - N_c} \) | -1 | \( \frac{N_c}{N_f} \) |

Table 2.3: Transformation properties of the fields in the magnetic description under the global symmetries of SQCD.
In general, we can associate the holomorphic scale $\Lambda$ of the electric theory to the scale of the magnetic theory via the relation

$$\Lambda^b \tilde{\Lambda}^\tilde{b} = (-1)^{N_f} \mu^{2N_f}. \quad (2.2.6)$$

The duality has been extended beyond the conformal window: for $1 < x < 3/2$, the electric theory is ultraviolet (uv) free and strongly coupled in the ir, while the magnetic theory is ir free. This regime is inverted in the $x > 3$ window.

### 2.2.2 Checks

In the last two decades, many checks of Seiberg duality have been performed. The duality relates a weakly coupled description to a strongly coupled one, which requires the development of non-perturbative techniques in order to perform some of the checks.

One of the first checks examined the structure of the moduli spaces. In some cases, the moduli space of one of the descriptions is classical while in the dual theory, there are quantum effects that have to be taken into account. Another class of tests analyzes the behavior of the duality under mass deformations and Higgsing. We refer the reader to [8] for an exhaustive analysis.

Another powerful check of Seiberg duality consists in matching the 't Hooft anomalies for the global symmetries between the dual descriptions. It has been performed with success on all generalizations that we will discuss below. The basic idea underlying the anomaly matching is that during an $\mathcal{R}$ flow from a uv fixed point to the ir, the value of the chiral anomaly has to be independent of the energy scale. This condition can be equivalently applied to a duality: despite the different degrees of freedom used to describe the model, the chiral anomalies have to match. In other words, the dual theories have to respond in the same manner to an external background gauge perturbation.

The anomalies in the case of $\text{sqcd}$ are summarized in Table 2.4. They have been shown to match in the electric and in the magnetic phases. Also this test has been passed with success by all the other dualities presented in this review. Note that the 't Hooft anomaly matching is a necessary but not sufficient condition for a duality (see [33] for further elaboration on this point).

Other possible checks consist in matching other exact quantities such as the Witten index [34] and the Hilbert series [35].

In the recent past, a new set of checks has been furnished by localization: many tests have been performed by matching the superconformal index of the dual phases
Table 2.4: Anomalies in squcd.

| $SU(N_f)^3$: | $N_c$ | $U(1)_B SU(N_f)^2$: | $\frac{N_f}{2}$ |
|-------------|-------|----------------------|-----------------|
| $U(1)_R SU(N_f)^2$: | $-\frac{N_c^2}{2N_f}$ | $U(1)_B^3$: | 0 |
| $U(1)_B$: | 0 | $U(1)_B U(1)_R^2$: | 0 |
| $U(1)_R$: | $-N_c^2 - 1$ | $U(1)_R^3$: | $-\frac{2N_f^2}{N_f^2} + N_c^2 - 1$ |

(see for example [36]). A detailed analysis of the identities that relate the dualities discussed in this review is given in [37, 38]. Note that in some of the cases, the identities have been proven, while in some other cases only partial results are known.

2.2.3 Extensions

Seiberg duality has been extended in many directions over the years. It has been shown for example that a similar pattern exists for orthogonal and symplectic gauge groups. Another extension of the duality includes the presence of tensor matter in the adjoint, symmetric and antisymmetric representations. Another interesting extension is related to product groups. In the following, we will discuss the former two situations, and briefly comment on the latter.

**Tensor matter**

$SU(N_c)$ with adjoint

This was the first generalization of Seiberg duality with tensor matter, introduced in [28]. The electric theory corresponds to the squcd considered above with an extra field $X$ in the adjoint representation of the $SU(N_c)$ gauge group turned on. There is also a superpotential interaction

$$ W = \text{Tr} \ X^{n+1}, \quad (2.2.7) $$

where $n$ is a positive integer. For $n = 1$, the adjoint can be integrated out and the original squcd is recovered. The case $n > 1$ is more interesting. In this case the singlets, generalizing the meson $M$ of squcd, are

$$ M_j = QX^j\tilde{Q} \quad (2.2.8) $$

and the chiral ring relations (i.e. the F-term of $X$) classically constrain $j < n$.

The dual theory in this case has an $SU(nN_f - N_c)$ gauge group, a dual adjoint $Y$ and dual fundamentals and antifundamentals interacting with the singlets $M_j$. The superpotential for this theory is

$$ W = \text{Tr} \ Y^{n+1} + \sum_{j=0}^{n-1} Y^j q M_{n-j-1}\tilde{q}. \quad (2.2.9) $$

$SU(N_c)$ with symmetric flavor

This duality was introduced in [39]. In this case one can consider the field $S$
in the symmetric representation of the gauge group, and its conjugate \( \tilde{S} \), with an interaction
\[
W = \text{Tr}(SS)^{n+1}.
\]
(2.2.10)

This induces a truncation on the chiral ring, such that the mesons that we have to consider are
\[
M = Q(\tilde{SS})^{j+1}\tilde{Q}, \quad P = Q(\tilde{SS})^{j}\tilde{q}Q, \quad \tilde{P} = \tilde{Q}(SS)^{j}\tilde{q}, \quad j = 0, \ldots, n - 1,
\]
(2.2.11)

where \( M \) is a bifundamental operator while \( P \) and \( \tilde{P} \) are symmetric in the flavor indices.

The dual theory has \( SU((2n + 1)N_f + 4n - N_c) \) gauge group, there are dual fundamentals and antifundamentals \( q \) and \( \tilde{q} \), dual symmetric and conjugate \( s \) and \( \tilde{s} \), and the singlet fields are the mesons \( M \), \( P \) and \( \tilde{P} \). The dual superpotential is
\[
W = (s\tilde{s})^{n+1} + \sum_{j=0}^{n} M_{n-j}q(s\tilde{s})^{j}\tilde{q} + \sum_{j=0}^{n-1} \left( P_{n-j-1}q(s\tilde{s})^{j}\tilde{q} + \tilde{P}_{n-j-1}\tilde{q}s(s\tilde{s})^{j}\tilde{a} \right).
\]
(2.2.12)

**SU\((N_c)\) with antisymmetric flavor**

This duality was introduced in [39]. It has an \( SU(N_c) \) gauge groups, \( N_f \) pairs of fundamentals and antifundamentals \( Q, \tilde{Q}, \) and two tensor matter fields \( A, \tilde{A} \), in the antisymmetric and in the conjugate antisymmetric representations. Note that the presence of the conjugate antisymmetric is required in order for the theory to be anomaly free. There is also a superpotential term
\[
W = \text{Tr}(A\tilde{A})^{n+1}.
\]
(2.2.13)

This induces a truncation on the chiral ring, such that the mesons that we have to consider are
\[
M = Q(\tilde{AA})^{j+1}\tilde{Q}, \quad P = Q(\tilde{AA})^{j}\tilde{q}Q, \quad \tilde{P} = \tilde{Q}(AA)^{j}\tilde{q}, \quad j = 0, \ldots, n - 1,
\]
(2.2.14)

where \( M \) is a bifundamental operator while \( P \) and \( \tilde{P} \) are antisymmetric in the flavor indices.

The dual theory has \( SU((2n + 1)N_f - 4n - N_c) \) gauge group, there are dual fundamentals and antifundamentals, \( q \) and \( \tilde{q} \), dual antisymmetric and conjugate, \( a \) and \( \tilde{a} \) and the singlet fields are the mesons \( M \), \( P \) and \( \tilde{P} \). The dual superpotential is
\[
W = (a\tilde{a})^{n+1} + \sum_{j=0}^{n} M_{n-j}q(a\tilde{a})^{j}\tilde{q} + \sum_{j=0}^{n-1} \left( P_{n-j-1}q(a\tilde{a})^{j}\tilde{q} + \tilde{P}_{n-j-1}\tilde{q}a(a\tilde{a})^{j}\tilde{a} \right).
\]
(2.2.15)

**Orthogonal case**

Another extension of Seiberg duality consists of theories with real gauge group. We will be interested in these realizations of the duality as they can be easily realized in
terms of D branes and orientifold planes, and play a crucial role in understanding the role of the orientifold in the reduction.

We will start by discussing the case of SQCD with orthogonal gauge group and $N_f$ vectors (corresponding to the fundamental and the antifundamental representations in the unitary case). Then we turn to the case of tensor matter. The two most interesting cases are those with symmetric and with antisymmetric (i.e. adjoint) tensors.

**$SO(N_c)$ with $N_f$ vectors**

In this case, the electric theory has gauge group $SO(N_c)$ and $N_f$ vectors $q$. Observe that here, we do not distinguish the $N_c$ even and odd cases. The superpotential is

$$W = 0$$

(2.2.16)

and does not impose constraints on the chiral ring; there is a meson operator of the form

$$M = QQ.$$  

(2.2.17)

The dual theory was worked out in [40] and it corresponds to an $SO(N_f - N_c + 4)$ gauge theory with $N_f$ vectors $q$ and the meson $M$. The dual superpotential is

$$W = Mqq.$$  

(2.2.18)

**$SO(N_c)$ with $N_f$ vectors and an adjoint**

In this case, the electric theory has gauge symmetry $SO(N_c)$, with $N_f$ vectors $Q$ and an adjoint $A$ (in this case the adjoint corresponds to an antisymmetric tensor). The superpotential is

$$W = \text{Tr} A^{2(n+1)},$$

(2.2.19)

and the mesons in the chiral ring are identified as

$$M_j = QA^jQ, \quad j = 0, \ldots, 2n,$$

(2.2.20)

symmetric in the flavor indices for even $j$ and antisymmetric for odd $j$.

The dual theory, obtained in [41], has $SO((2n + 1)N_f - N_c + 4)$ gauge group, $N_f$ vectors $q$, an adjoint $a$ and the mesons $M_j$. The superpotential of this dual theory is

$$W = a^{2(n+1)} + \sum_{j=0}^{2n} M_{2n-j}aq.$$  

(2.2.21)

**$SO(N_c)$ with $N_f$ vectors and a traceless symmetric tensor**

In this case the electric theory has $SO(N_c)$ gauge symmetry, with $N_f$ vectors $Q$ and a traceless symmetric tensor $S$

$$W = \text{Tr} S^{n+1},$$

(2.2.22)

and the mesons in the chiral ring are identified as

$$M_j = QS^j\bar{Q}, \quad j = 0, \ldots, n - 1.$$  

(2.2.23)
The dual theory, obtained in [39], has $SO(n(N_f + 4) - N_c)$ gauge group, $N_f$ vectors $q$, a traceless symmetric tensor $s$ and the mesons $M_j$. The superpotential of this dual theory is

$$W = s^{n+1} + \sum_{j=0}^{n-1} M_{k-j}q^js^j. \quad (2.2.24)$$

**Symplectic case**

Here we discuss the case of symplectic gauge groups, by fixing the convention $Sp(2) = SU(2)$. We first discuss the case of gauge group $Sp(2N_c)$ in presence for $2N_f$ fundamentals. Then we generalize the construction by considering also a symmetric (adjoint) or antisymmetric tensor.

* $Sp(2N_c)$ with $2N_f$ fundamentals

In this case the electric theory has gauge group $Sp(2N_c)$ and there are $2N_f$ fundamental quarks $Q$. The superpotential is

$$W = 0 \quad (2.2.25)$$

and there is a meson

$$M = QQ. \quad (2.2.26)$$

The dual theory, found in [31], has $Sp(2(N_f - N_c - 2))$ gauge group with $2N_f$ fundamentals $q$ and the meson $M$. The superpotential of this dual theory is

$$W = Mqq. \quad (2.2.27)$$

* $Sp(2N_c)$ with $2N_f$ fundamentals and an adjoint

In this case the electric theory has gauge group $Sp(2N_c)$ and there are $2N_f$ fundamental quarks $Q$ and an adjoint $S$ (in this case we use this terminology because the adjoint is a symmetric tensor). The superpotential is

$$W = \text{Tr} S^{2(n+1)}, \quad (2.2.28)$$

and there are mesons

$$M_j = QS^jQ, \quad j = 0, \ldots , 2n, \quad (2.2.29)$$

symmetric in the flavor indices for odd $j$ and antisymmetric for even $j$. The dual theory, obtained in [44], has $Sp(2((2n + 1)N_f - N_c - 2))$ gauge group with $2N_f$ fundamentals $q$, a dual adjoint $s$ and the mesons $M_j$. The superpotential of this dual theory is

$$W = s^{2(n+1)} + \sum_{j=0}^{2n} M_{2n-j}q^js^j. \quad (2.2.30)$$

* $Sp(2N_c)$ with $2N_f$ fundamentals and an antisymmetric tensor

In this case the electric theory has gauge group $Sp(2N_c)$ and there are $2N_f$
fundamental quarks \( Q \) and a traceless antisymmetric tensor \( A \). The superpotential is
\[
W = \text{Tr} A^{n+1}. \tag{2.2.31}
\]
and there are mesons
\[
M_j = QA^j Q, \quad j = 0, \ldots, n - 1. \tag{2.2.32}
\]
The dual theory, obtained in \([39]\), has \( Sp(2(n(N_f - 2) - N_c)) \) gauge group with \( 2N_f \) fundamentals \( q \), a dual antisymmetric \( a \) and the mesons \( M_j \). The superpotential of this dual theory is
\[
W = a^{n+1} + \sum_{j=0}^{n-1} M_{n-j} q a^j q. \tag{2.2.33}
\]

### 2.3 The brane picture in four dimensions

In this section we discuss the realization of four-dimensional Seiberg duality and its generalization in terms of type \( IIA \) D brane configurations. We start by reviewing some general aspects of the brane realization of four-dimensional \( \mathcal{N} = 1 \) theories in Sec. 2.3.1. Then we show how this picture is modified when flavors are added, realizing the electric \( SQCD \) in terms of D branes and NS branes (Sec. 2.3.2). The dual theory is constructed via a \( \text{hw} \) transition that we review in Sec. 2.3.3. One generalization of this duality is obtained by adding an adjoint field with a power-law superpotential, leading to the \( kss \) duality (Sec. 2.3.4). Another generalization involves theories with real gauge group and with tensor matter, realized by introducing orientifold planes in the brane setup. We review some of these dualities explicitly in Sec. 2.3.5 and sketch some further generalizations in Sec. 2.3.6.

#### 2.3.1 Generalities

Here we introduce some basic aspects of brane dynamics necessary to study the realization of Seiberg duality and its extensions in four-dimensional \( \mathcal{N} = 1 \) theories. As this subject has been extensively described in the literature (see e.g. \([3]\)), we will give a general overview as opposed to a description in full detail. We will try nevertheless to be self-consistent, discussing the details necessary to the analysis of the reduction of the four-dimensional dualities to three dimensions.

The D brane realization of four-dimensional \( \mathcal{N} = 1 \) theories requires a ten-dimensional type \( IIA \) description. The ten-dimensional theory is non-chiral: there are two supercharges \( Q_L \) and \( Q_R \), generated respectively by the left and right-moving worldsheet degrees of freedom. The supercharges satisfy the relation
\[
\Gamma^{0123456789} Q_{L/R} = \pm Q_{L/R}, \tag{2.3.1}
\]
where \( \Gamma^{0123456789} = \Gamma^0 \Gamma^1 \cdots \Gamma^9 \).

The theory has two sectors, the Neveu–Schwartz (\( NS \)) and Ramond (\( \kappa \)) sectors, in which we can have different potentials. There are two main branes charged under these potentials that will play a crucial role in the description, namely Neveu–Schwarz (\( NS \)) and Dirichlet-\( p \) (\( D_p \)) branes. Let us discuss their main properties separately.
Table 2.5: Summary of the four dimensional dimensional dualities reviewed in this section. Here $F$ is the fundamental representation, $\tilde{F}$ the antifundamental, $S$ the symmetric, and $\tilde{S}$ its conjugate, $A$ the antisymmetric, $\tilde{A}$ its conjugate, $X$ the adjoint and $V$ the vector. There are also singlets in the spectrum of the dual theories that we do not report in the table, corresponding to the generalized mesons.
Table 2.6: Brane configuration for four-dimensional brane engineering.

|    | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|----|---|---|---|---|---|---|---|---|---|---|
| NS | × | × | × | × | × | × |   |   |   |   |
| NS’| × | × | × | × |   | × |   |   |   |   |
| D4 | × | × | × | × | × | × | × |   |   |   |
| D6 | × | × | × | × | × | × | × | × |   |   |
| D6’| × | × | × | × | × | × | × |   |   |   |

• The NS branes are five-dimensional membranes, magnetically coupled to the two-dimensional antisymmetric $B$ field of the $\text{ns}$ sector. They have tension $(2\pi)^{-5}g_s^{-2}(\alpha')^{-3}$ and do not admit a simple conformal field theory (cft) description because the dilaton blows up close to their core. They preserve some of the original supersymmetry (susy): for an NS brane extended along $x_{12345}$, the preserved supercharge is $\epsilon_L Q_L + \epsilon_R Q_R$, where the spinors satisfy $\epsilon_L/R = \Gamma^{012345}\epsilon_L/R$. Unless explicitly stated, we will denote with NS a brane extended along $x_{12345}$ and with NS’ a brane extended in $x_{12389}$.

• The other necessary ingredient are the $D_p$ branes. They are defined with respect to their charge under the $p+1$ forms of the $R$ sector, where $p = 0, 2, 4, 6, 8$. Their tension is $(2\pi)^{-p}(\alpha')^{-(p+1)/2}$ and they admit a cft description at weak string coupling. We will mostly focus on $D_4$ branes along $x_{01236}$ and on $D_6$ branes along $x_{0123789}$; we will also introduce $D_6'$ branes along $x_{0123457}$ when necessary. The brane directions are summarized in Table 2.6.

The $\epsilon_L Q_L + \epsilon_R Q_R$ supercharges preserved by a $D_4$ brane requires $\epsilon_L = \Gamma^{012345}\epsilon_R$ while for a $D_6$ brane we have $\epsilon_L = \Gamma^{0123789}\epsilon_R$. The worldvolume theory on a $D_p$ brane corresponds to a field theory in $p+1$ dimensions with 16 supercharges. In the spectrum there is a massless $U(1)$ gauge field and $9 - p$ massless scalars, corresponding to the position of the $D_p$ brane in the transverse geometry. This description is valid as long as the brane is infinite, but we will deal with $D_4$ branes extended along a segments in $x_6$. This is possible because at the extremal points we will put an NS or a $D_6$ brane. In this case, we will interpret the theory as effectively four dimensional.

Combining the two types of branes, in the simplest case of a $D_4$ suspended between two NS branes, the resulting theory is $\mathcal{N} = 2$ sym. In fact, in this case the system of branes preserves 8 supercharges. The distance between the NS branes (the length of the $x_6$ segment) is inversely proportional to the gauge coupling:

$$g_s^2 = \frac{g_s(2\pi)^2\sqrt{\alpha'}}{\ell_6} = \text{sym}.$$  \hspace{1cm} (2.3.2)

There are in this case two types of massless fields, a $U(1)$ gauge field (and its supersymmetric completion) and a complex scalar, related to the freedom of moving the $D_4$ segment along the NS branes (i.e. along the directions $x_{45}$), together with its supersymmetric fermionic partner.

We can also consider more than one $D_4$ brane (a stack of $N$ $D_4$ branes). In this case the type IIA description is dressed with Chan–Paton indices to obtain a $U(N)$
Figure 2.3: The two possible configurations of D4–D5–NS–NS'.

theory; now we have a $U(N)$ gauge field and the scalar field becomes an $N \times N$ matrix, transforming in the adjoint representation of the gauge group. This is not the end of the story. In the four-dimensional picture, the $U(1) \subset U(N)$ decouples through a Green–Schwarz mechanism because one component of the massive vector boson becomes massive in the $\text{ir}$. In geometrical terms, the decoupling of the $U(1)$ corresponds to the freedom in the choice of the position of the center of mass of the stack of D4 branes. In this way we are left with an $SU(N)$ gauge theory in the $\text{ir}$.

We can also break half of the supersymmetry by rotating one of the NS branes into an NS' brane (extended in $x_{012389}$). Geometrically we lose the freedom to slide the D4 brane along the NS branes, and on the field theory side this corresponds to integrating out the adjoint field. This configuration corresponds to $\mathcal{N} = 1$ sym with an $SU(N)$ gauge group.

### 2.3.2 Adding flavor: SQCD

In this section we discuss the flavoring of the model discussed above. Flavors are introduced by adding a stack of $N_f$ D6 branes to the setup. This will lead us directly to the discussion of physically equivalent configurations and to the idea of Hanany–Witten [27] transitions, i.e. the brane description of Seiberg duality.

The D6s are considered as heavy objects and they have four dimensions common with the D4 branes (they are extended in $x_{0123789}$). The dynamical degrees of freedom in the four-dimensional picture are $N_f$ fundamental fields of the $SU(N_f)$ gauge group, describing the 4–6 strings (stretching from the D4 branes to the D6 branes) plus the adjoint fields from the 4–4 strings. There are two inequivalent cases: in one case the NS is placed between the D6 and the NS' along the direction $x_6$; in the other, the positions of the two NS branes are interchanged (see Figure 2.3).

In the first case, we cannot move the D4 branes between the NS' and the D6 branes in any direction, as they do not share directions orthogonal to the D4. In the second case, we can actually consider two separate stacks of D4s, one stretched between the stack of D6s and the NS', the other stretched between the NS' and the
NS. The first stack of D4s can move in the $x_{89}$ direction. The relative position in $x_{89}$ of the two stacks of D4s translates into a mass term for the quarks $Q$ and $\tilde{Q}$. This mass term is associated to the presence of a coupling of the fundamentals with a singlet $M$, and moving the branes apart corresponds to assigning a vacuum expectation value (vev) to this singlet. In other words, in the first D brane configuration, we have vanishing superpotential $W = 0$, while in the second case, we have

$$W = MQ\tilde{Q}.$$ (2.3.3)

Let us discuss the global symmetries of the model. There is a $U(1)$ symmetry associated to the decoupling of the center of mass of the stack of D4s, which is the overall $U(1)$ in $U(N_c)$ discussed above. There is also a flavor symmetry $U(N_f)^2$ associated to the stack of $N_f$ D4 branes ending on the D6 branes. The two overall $U(1)$s contained in the flavor groups however are not independent, since only the relative displacement of the two stacks of D4 branes is physically relevant. They form the baryonic $U(1)$ of $\text{sqcd}$ in terms of the gauge theory description.

More interestingly, there are two $U(1)$ symmetries associated to rotations in the $x_{45}$ and $x_{89}$ planes. This $U(1)_{45} \times U(1)_{89}$ is part of the Lorentz group in $9 + 1$ dimensions and rotates the supercharges. It is hence an $R$–symmetry. However, the axial $U(1)$ subgroup leaves the supercharges invariant while it rotates $Q$ and $\tilde{Q}$ in the same way, appearing as axial symmetry $U(1)_A$ in the field theory. While both $U(1)_R$ and $U(1)_A$ are classically present, the latter is anomalous. In the brane construction, this can be seen if one uplifts the system to M-theory, where the NS brane bends away from the D4 branes. This is a classical effect in the M-theory picture and a quantum effect in the type $\text{IIA}$ setup and signals the breakdown of one of the $U(1)$ symmetries discussed above. We obtain a geometrical realization of the axial anomaly in four dimensions. We will come back to this issue later, when discussing the reduction of the duality to three dimensions, showing how this breaking is realized in the T–dual case, when $x_3$ is compact.

In the following section we will show how the two configurations in Figure 2.3 are related via the so-called Hanany–Witten transition, which is the brane realization of Seiberg duality.

### 2.3.3 Hanany–Witten transition as Seiberg duality

In this section we review the realization of Seiberg duality as an equivalence between two brane realizations. As we will see, the two flavored models discussed above (the two sides in Figure 2.3) can be transformed into each other via a brane transition. This transformation corresponds to the duality on the field theory side. The brane transition has been obtained in [27] by Hanany and Witten, and we will refer to it as the hw transition. The main idea behind the transition is the following.

Consider a D6 brane and an NS brane (or equivalently a D6' and an NS' brane). If we move the D6 along the $x_6$ direction to the other side of the NS brane, we obtain an equivalent configuration, in which a D4 brane has been created between the 6

The brane picture only realizes a $U(N_f)$ symmetry. The full flavor symmetry is obtained by superimposing the D6 to the NS as discussed in [17]. The situation will be different in three dimensions, see Sec. 3.3.1.

Here we adapt the discussion to the type $\text{IIA}$ setup, even if the original discussion was done in the type $\text{IIB}$ case.
D6 and the NS brane. The converse is also true: if there was a D4 in the initial configuration between the D6 and the NS brane, after moving the D6 to the left of the NS brane, this brane is destroyed (see Figure 2.4).

Figure 2.4: hw transition. When the D6 brane crosses the NS brane a new D4 brane is created. The two configurations are physically equivalent.

For more general configurations, one has to consider the flux sourced by each brane and study its effect on the action of the other branes. This defines the so called linking number, representing the total magnetic charge of the gauge fields coupled to the worldvolume of the branes [27]. This number has to be preserved in the transition in order to make the brane action well defined. This will be important later, when we will add other characters to the picture, i.e. other extended objects carrying D brane charge. In this discussion, there is always an underlying assumption that can be formulated in terms of a constraint, the so-called s-rule. This principle states that the configurations that preserve supersymmetry have necessarily 0 or 1 D4 branes stretched between a D6 and an NS brane [27].

We are now ready to discuss the Seiberg duality in terms of a hw transition. Let us start considering the setup in Figure 2.3(a). There are $N_c$ D4 branes between an NS and an NS' brane and, on the left side, $N_f$ D4s between the $N_f$ D6s and the NS' brane. Let us now recombine $N_c$ D4 branes, stretching them between the NS and the $N_f$ D6 branes. We can now move the NS branes along $x_6$ and cross the NS' brane (Figure 2.3(b)). After this, we can recombine the D4 branes between the NS' and the D6 brane. In this way, we obtain a configuration with $N_f - N_c$ D4s between the NS and the NS' brane and $N_f$ D4s between the NS' and the D6 branes. This configuration corresponds to the one discussed in the section above with superpotential $W = MQ\tilde{Q}$, i.e. the Seiberg-dual configuration expected from the field theory discussion.
2.3.4 Multiple NS and KSS duality

In this section, we discuss in some detail a generalization of Seiberg duality in presence of a power law superpotential for an adjoint matter field, i.e. the Kutasov–Schwimmer–Seiberg \([28]\) duality.

First we need to understand how to engineer a power-law superpotential for the adjoint field. This is done by considering a stack of \(n\) NS branes (or equivalently NS’ branes). A stack of NS branes corresponds to a singular limit, and the correct physical interpretation is obtained by considering the desingularization of this geometry.

In order to understand the origin of the superpotential, we need to take a step back. As discussed above, the case of \(\mathcal{N} = 2\) SYM is associated to a pair of NS branes and \(N_c\) D4s stretched between them. The mass term of the adjoint breaking supersymmetry to \(\mathcal{N} = 1\) was obtained by rotating the NS into an NS’ brane. In other words, the case \(n = 1\) corresponds to \(W = X^{n+1} = X^2\), where \(X\) is an adjoint field of \(SU(N_c)\). For general \(n\), the claim is then that the superpotential is \(W = X^{n+1}\).

There is a classical way to verify this result. It consists in studying a polynomial deformation
\[
\Delta W = \sum_{i=1}^{k-1} \lambda_i \text{Tr } X^i
\]
in the \(X\), and assigning a vev to \(X\), breaking \(SU(N_c)\) to \(SU(r_1) \times \cdots \times SU(r_n)\) with \(r_1 + \cdots + r_n = N_c\). This corresponds to studying a set of decoupled sqCD sectors, each with \(N_f\) flavors.

On the brane side, this is precisely the desingularization of the geometry, and it is obtained by separating the NS branes along the \(x_{89}\) direction. In this way we obtain \(n\) different sqCD sectors, as expected. This corresponds on the field theory side to considering the superpotential deformation (2.3.4), and breaking the theory into a set of sqCD sectors. There is a subtlety when we consider the flavor branes, though. Having \(N_f\) pairs of fundamentals and antifundamentals in each sector we must consider in total \(nN_f^2\) flavor D4 and D6 branes in this setup even if it does not enhance the global symmetry to \(SU(nN_f)^2\). There are in total \(nN_f^2\) mesonic components, corresponding to
\[
M_j = QX^j \bar{Q}, \quad j = 0, \ldots, n.
\]

Even if it is unclear how to realize this possibility when the \(n\) NS branes are coincident, when the geometry is desingularized, the counting becomes more obvious, because we have a single meson \(M = Q^n Q_{n_l}\), where the subscript \(l\) refers to the fact that we are summing over the gauge index in each sqCD sector, \(a_1 = 1, \ldots, r_l\), for \(l = 1, \ldots, n\).

We can now proceed as above, by performing the HW transition in each sector and recombining the branes at the end. The final configuration has \(nN_f - N_c\) D4 branes stretched between the NS’ and the \(n\) NS branes, and \(N_f\) D4 branes stretched between the NS brane and the \(N_f\) D6 branes. This configuration corresponds to the expected KSS dual theory, and realizes the field theory duality in the brane setup (see Figure 2.5).
2.3.5 Orientifold planes

In this section, we introduce another extended object that plays a crucial role in realizing four-dimensional $\mathcal{N} = 1$ field theories from the brane perspective: the orientifold plane. Orientifolds allow us to construct real orthogonal and symplectic gauge groups and matter fields in the symmetric and antisymmetric representation of the gauge group.

An orientifold can be defined from its action in perturbative string theory. It is the combined action of

- a parity inversion $\sigma$ of the coordinates transverse to the plane
- a world-sheet parity $\Omega$
- $(-1)^{F_L}$, $F_L$ being the left-moving fermion number.

We can have in general $p$-dimensional orientifold planes, denoted as $O_p$ planes, and we can use the definition both in type IIA (even $p$) and in type IIB (odd $p$). In the following, we will restrict our analysis to the case of $p = 4, 6$ in type IIA.

It is important to distinguish the action of the orientifold on the NS and on the R sectors. They are two different $\mathbb{Z}_2$ parities:

- in the NS sector, the $\mathbb{Z}_2$ action corresponds to a perturbative action on the string theory side and we denote it by a $\pm$ sign;
- in the R sector, the $\mathbb{Z}_2$ action is non-perturbative, and we denote it by a tilde ($\sim$) sign.

Figure 2.5: HW transition for the KSS duality.
There are in total four possibilities, $Op^\pm$ and $\tilde{O}p^\pm$ [42][43]. By specifying the $\mathbb{Z}_2$ charges, we completely characterize the action of the orientifold on the gauge theory from the string theory perspective (geometrically we specify the so-called “discrete torsion”).

It is important to note that the orientifolds are charged under the $r$ sector, i.e. they carry D brane charge. This fact will be crucial when studying the duality in terms of $\text{hw}$ transition, as will will see in the following.

**O4 planes**

In this section we add an O4 plane on top of the stack of D4 branes that realizes the sqcd model as discussed above.

There are some subtleties arising in this case:

- The charge of the $Op$ plane in the $r$ sector enters the calculation of the linking number in the $\text{hw}$ transition. These charges are $c_R[O4^\pm] = \pm 1/2$. This forces the number of D4 branes to be even.

- In presence of an $\tilde{O}4^-$, the orientifold charge has an extra 1/2 factor. This forces the number of D4 gauge branes to be odd, $2N_c + 1$.

Putting these things together, we find that for $O4^-$ we have gauge group $SO(2N_c)$, for $O4^+$ we have $Sp(2N_c)$, for $O4^-$ we have $SO(2N_c + 1)$ and for $O4^+$ we have again $Sp(2N_c)$ but with a different non-perturbative sector.

One more subtlety comes from the $n_s$ sector: the $n_s$ charge of the $Op$ plane has to be flipped when crossing an NS brane, so that an O4$^+$ plane on the left of an NS brane is equivalent to an O4$^-$ plane on the right.

We can now proceed with the duality, by performing the $\text{hw}$ transition. When a D6 brane crosses an NS brane, the number of D4 that is created is shifted by $\pm 4$ depending on the $\pm$ orientifold charge. This reproduces the field theory results $Sp(2N_c) \rightarrow Sp(2(N_f - N_c - 2))$ and $SO(N_c) \rightarrow SO(N_f - N_c + 4)$ as in Sec. 2.2.3.

**O6 planes**

Adding O6 branes to the picture proves very useful in the construction of models with tensor matter but requires some modifications to the setup discussed above.

Consider for example a configuration with $2N_c$ D4 branes and two NS branes, that we refer to as $\text{NS}_{\pm \theta}$, rotated by an angle $\theta$ in the $x_{45}$ and in the $x_{89}$ plane. Similarly, we can also add $D6_{\pm \theta}$ branes.

One possibility corresponds to considering the branes as shown in Figure 2.6. In this case, we can place the O6 plane symmetrically as in the Figure. In this case the original theory corresponds to $SU(2N_c)$ sqcd with $2F$ flavors. Depending on the charge of the O6 plane (here $c_R[O6^\pm] = \pm 2$) we obtain a real or a symplectic gauge group. Again we can perform the $\text{hw}$ transition and obtain the expected result for the rank of the dual gauge group.

**2.3.6 Realization of other dualities**

We have now collected all the ingredients needed to represent the models described on the field theory side in terms of their brane realizations.
Figure 2.6: hw transition in the presence of an O6 plane.

Real gauge groups and tensor matter require the presence of NS, D4, D6 and Op planes. The dual model is obtained by performing an hw transition, exchanging the NS branes. We will not discuss in detail all of these examples, and refer the reader to the original literature [44–47] for their explicit embeddings.

Here we will instead concentrate on a single case, which we will use as a toy example also when discussing the reduction of the four-dimensional dualities to three dimensions. It consists of an $Sp(2N_c)$ gauge group with an adjoint and $2N_f$ fundamentals. This is the theory discussed in Section 2.2.3. This example can be engineered with $2N_c$ D4 branes and an O4$^-$ plane stretched between $2n + 1$ NS branes and one NS’ brane. The fundamentals are obtained by considering $2N_f$ D6 branes on the NS branes. The gauge group is broken by a polynomial superpotential for the adjoint into the product

$$Sp(2r_0) \times \prod_{i=1}^{n} U(r_i), \quad \text{with} \quad \sum_{i=0}^{n} r_i = N_c.$$  \hspace{1cm} (2.3.6)

This polynomial deformation is obtained in the brane picture by separating the NS branes along the $x_{45}$ plane (Figure 2.7(a)).

The magnetic theory is obtained by a hw transition in each sector and eventually reconnecting the NS5s, see Figure 2.7. Before reconnecting the branes, the gauge group is broken to

$$Sp(2(N_f - r_0 - 2)) \times \prod_{i=1}^{n} U(N_f - r_i).$$  \hspace{1cm} (2.3.7)

In the final step the dual gauge group becomes $Sp(2((2n + 1)N_f - N_c - 2))$ as expected (Figure 2.7(b)).
Figure 2.7: Electric and magnetic sides of the duality for $Sp(2N_c)$ gauge theories with adjoint matter.
3 The three-dimensional case

In the following, we will repeat the four-dimensional discussion for the three-dimensional case, i.e. we will first discuss the generalities of supersymmetric gauge theories with four supercharges in three dimensions and then specialize to the case of three-dimensional $\mathcal{N}=\text{soCD}$ with gauge group $U(N_c)$, following largely [13, 48]. After this, we discuss the various three-dimensional dualities and go on to describe them in the brane picture.

3.1 $\mathcal{N}=2$ supersymmetric gauge theories in three dimensions

In this section, we will discuss the basics of gauge theories with four supercharges in three dimensions. This is the same number of supercharges as for $\mathcal{N}=1$ in four dimensions, but in three dimensions, it corresponds to $\mathcal{N}=2$. These theories can be obtained by dimensional reduction from four dimensions. While theories with $\mathcal{N}=1$ have no holomorphy properties in three dimensions, for $\mathcal{N}=2$, there are holomorphic objects and non-renormalization theorems.

As the gauge coupling is dimensionful in three dimensions, none of the theories are conformal. Their IR fixed points are in general strongly interacting theories, so when we study IR dualities, we do not have effective IR descriptions. The analysis of the conformal window in $\mathcal{N}=2$ three-dimensional theories is more complicated than for the four-dimensional case. This is mostly due to the fact that in three dimensions, there are strong coupling effects in the IR and many of the techniques applicable in the four-dimensional case are not available. It has been nevertheless possible to gain new insights into the analysis of the scaling dimensions of the fields from localization techniques via the F-maximization principle [49–52]. One of the main issues in this analysis is the presence of accidental symmetries in the IR [53, 54]. We refer the reader to [55, 56] for an analysis of the conformal window and of the accidental symmetries for Aharony duality and its generalizations.

3.1.1 Generalities

A representation of the Clifford algebra in three dimensions with $\eta = (-, +, +)$ is given by

$$\gamma^i_{\alpha\beta} = (i\sigma_2, \sigma_3, \sigma_1), \quad i = 0, 1, 2, \quad (3.1.1)$$

with $\sigma_i$ the Pauli matrices. We see that here, the $\gamma^i$ are all real. The fundamental fermion representation in $2+1$ dimensions is a two-component Majorana fermion $\psi^\alpha_i, \alpha = 1, 2$. When performing the dimensional reduction, the scalars remain scalars, $\phi(x^\mu) \to \phi(x^i)$, where $\mu = 0, \ldots, 3$ and $i = 0, 1, 2$. The vector field reduces as

$$A_\mu(x^\mu) \to A_i(x^i), \quad \sigma(x^i) = A_3(x^i), \quad (3.1.2)$$
and a four-dimensional Weyl spinor $\psi$ with four real components turns into two independent Majorana spinors with two real components each, which can also be combined into a complex spinor.

The dimensional reduction of the four-dimensional $\mathcal{N} = 1$ supersymmetry algebra is given by

$$\{Q_\alpha, \overline{Q}_\beta\} = 2\gamma^i_{\alpha \beta} P_i + 2i \epsilon_{\alpha \beta} Z, \quad (3.1.3)$$

$$\{Q_\alpha, Q_\beta\} = \{\overline{Q}_\alpha, \overline{Q}_\beta\} = 0. \quad (3.1.4)$$

The central term $Z$ is given by the $P_3$ component of the four-dimensional momentum. As in the four-dimensional case, the R–symmetry is isomorphic to a global $U(1)_R$. The operators $Q_\alpha$, $\overline{Q}_\alpha$ and $D_\alpha$, $\overline{D}_\alpha$ are defined from the $\theta$, $\overline{\theta}$ analogously to Eq. (2.1.2) and (2.1.3). In three dimensions, there are chiral superfields $Q$ satisfying $\overline{D}_\alpha Q = 0$, anti-chiral fields $\overline{Q}$ satisfying $D_\alpha \overline{Q} = 0$ and vector superfields $V$ satisfying $V = V^\dagger$. Both chiral and vector multiplets contain two real bosonic degrees of freedom and two Majorana fermionic degrees of freedom. In addition, there can be linear superfields $\Sigma$ which satisfy

$$\epsilon_{\alpha \beta} D_\alpha D_\beta \Sigma = \epsilon_{\alpha \beta} \overline{D}_\alpha \overline{D}_\beta \Sigma = 0. \quad (3.1.5)$$

The lowest component of $\Sigma$ is a real scalar.

The four-dimensional chiral superfield Eq. (2.1.5) reduces straightforwardly as discussed. In the reduction of the four-dimensional vector field Eq. (2.1.7), the real scalar $\sigma$ appears:

$$V = -i\theta \overline{\theta} \sigma - \theta \gamma^i \overline{\theta} A_i + i\theta^2 \overline{\theta} \lambda + i\overline{\theta}^2 \theta \lambda + \frac{1}{2} \theta^2 \overline{\theta}^2 D. \quad (3.1.6)$$

Unlike in the four-dimensional theory, $\sigma$ can acquire a vev that breaks the gauge group to its maximal Abelian torus, leading to a Coulomb branch of the moduli space.

In three dimensions, in a free Abelian theory with gauge coupling $e_3$, the photon $A_i$ can be dualized into a scalar $\gamma$ via

$$\partial_i \gamma = \frac{\pi}{e_3} \epsilon_{ijk} F^{jk}, \quad (3.1.7)$$

where $F^{jk}$ is the field strength of $A_j$. The quantization of the magnetic charge requires that $\gamma$, which is known as the dual photon, is periodic, $\gamma \sim \gamma + 2\pi$. The dual field strength $J = \epsilon_{ijk} F^{jk}$ is a one-form. From Maxwell’s equation, we see that it is divergence-free, $\partial_i J_i = 0$. Three-dimensional theories have therefore a global topological symmetry $U(1)_J$ with conserved current $J$.

It is possible to describe the vector superfield equivalently with a linear superfield $\Sigma$:

$$\Sigma = -\frac{i}{2} \epsilon^{\alpha \beta} \overline{D}_\alpha D_\beta V$$

$$= \sigma + \theta \lambda + \overline{\theta} \lambda + \frac{1}{2} \theta \gamma^i \overline{\theta} J_i + i\theta \overline{\theta} D + \frac{1}{2} \theta^2 \theta \gamma^i \partial_i \lambda - \frac{i}{2} \overline{\theta}^2 \overline{\theta} \gamma^i \partial_i \overline{\lambda} + \frac{1}{2} \theta^2 \overline{\theta}^2 \partial^2 \sigma. \quad (3.1.8)$$

It is furthermore possible to dualize this linear multiplet into a chiral multiplet $\Phi$ with lowest component

$$\Phi = \frac{2\pi}{e_3} \sigma + i\gamma. \quad (3.1.9)$$
Having collected all the necessary ingredients, we can now write down invariant Lagrangians in three dimensions.

**Gauge fields.** For the gauge part, we can again form the gauge invariant field strength given in Eq. (2.11), resulting in

$$\mathcal{L}_{\text{gauge}} = \frac{1}{e^2} \int d^2 \theta W^a W^a + \text{c.c.}$$

(3.1.10)

Equivalently, we can use the description in terms of the linear superfield $\Sigma$:

$$\mathcal{L}_{\text{gauge}} = \frac{1}{e^2} \int d^4 \theta \Sigma^2.$$  

(3.1.11)

Unlike in four dimensions, the above are not the only gauge-invariant combinations of the gauge fields. It is possible to add a Chern–Simons term to the Lagrangian:

$$\mathcal{L}_{CS} = \frac{k}{4\pi} \text{Tr} \left( \varepsilon^{ijk} (A_i \partial_j A_k + \frac{2i}{3} A_i A_j A_k) + 2D\sigma - \lambda^* \lambda \right),$$

(3.1.12)

where $k \in \mathbb{Z}$ is the Chern–Simons level. In the Abelian case, the above can be rewritten very compactly as

$$\mathcal{L}_{CS} = \frac{k}{4\pi} \int d^4 \theta \Sigma V.$$  

(3.1.13)

As in four dimensions, we can add a Fayet–Iliopoulos term which has the same form as Eq. (2.118) whenever the gauge group contains a $U(1)$ factor.

**Matter fields.** The Lagrangian for the chiral superfields $Q$ is given by

$$\mathcal{L}_{\text{matter}} = \int d^4 \theta K(Q, Q^\dagger) + \int d^2 \theta (W(Q) + \text{c.c.}).$$

(3.1.14)

For supersymmetric theories, the Kähler potential takes the usual form

$$K = Q e^V Q^\dagger,$$

(3.1.15)

so in terms of component fields, we have

$$\mathcal{L}_{\text{matter}} = |D_i \phi_Q|^2 + \phi_Q^* \sigma^2 \phi_Q + i \phi_Q^* D \phi_Q + i \psi^* \gamma^i D_i \psi +$$

$$- i \psi^* \sigma \psi + i \phi_Q^* \lambda^* \psi - i \psi^* \lambda \phi_Q + |F|^2.$$  

(3.1.16)

We see that for $\langle \sigma \rangle \neq 0$, a mass term is induced for $Q$.  

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**Real and complex mass terms.** In three dimensions, different types of mass terms can be written for a chiral superfield $Q$. In the vector-like theory, we can add the term

$$W_{mc} = m_C Q \bar{Q}$$

(3.1.17)

to the superpotential. Since $m_C$ is complex, this is called a complex mass term. It is the analog of the usual mass term in four dimensions.

Alternatively, we can induce a mass term via $\langle \sigma \rangle \neq 0$. This is called a real mass term and can be understood as a modification of the Kähler potential:

$$\int d^4 \theta e^{m_R \bar{Q}^i} \sim \left( \frac{m_R}{2} |\phi_Q|^2 + i m_R e^{a\beta} \bar{\psi}_a \psi_b \right).$$

(3.1.18)

Note that the complex mass term preserves parity, while the real real term breaks parity. The physical mass of the chiral superfield is given by the combination

$$m = \sqrt{m_R^2 + m_C^2}.$$  

(3.1.19)

A real mass term can be induced by turning on a vector superfield $\bar{V}$ as a susy-preserving background field,

$$\sigma = \frac{m_R}{e_3}, \ \bar{A}_i = \bar{\lambda} = \bar{\lambda} = \bar{D} = 0.$$  

(3.1.20)

Note, that since the real masses come from a background vector multiplet, they cannot appear in holomorphic objects such as the superpotential.

**Moduli space.** The moduli space of vacua of three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories contains in general a Coulomb branch and a Higgs branch. By $\mathcal{N} = 2$ supersymmetry, both moduli spaces are Kähler manifolds, but neither is protected against loop or perturbative corrections. The scalars of the quark multiplets $\phi_Q$ parameterize the Higgs branch, while vevs of $\sigma$ parameterize the Coulomb branch. Due to the term $\phi_Q^* \sigma^2 \phi_Q$ appearing in the scalar potential (see the matter action Eq. (3.1.16)), $\langle \sigma \rangle \neq 0$ in general requires $\langle \sigma \rangle = 0$ and vice versa. From this, we see that the moduli space splits up into distinct branches.

The low-energy effective action has the form

$$S_{\text{eff}} = \int d^3 x \int d^4 \theta K(\Phi_i, \bar{\Phi}_i) + \int d^2 \theta (W(\Phi_i) + \text{c.c.}) .$$

(3.1.21)

On the Higgs branch, the $\Phi_i$ are suitable gauge-invariant combinations of the matter fields $Q, \bar{Q}$. At a generic point of the Coulomb branch, the gauge group is broken to its maximal torus, as the scalar $\sigma$ is in the adjoint representation of $G$ and a generic vev breaks $G \to U(1)^{\text{rk}(G)}$.

The action contains the terms

$$\text{Tr} [A_i, A_j] \quad \text{and} \quad \text{Tr} [A_i, \sigma].$$

(3.1.22)

For a supersymmetric vacuum, they must be zero, which is the case if we choose both $A_i$ and $\sigma$ to lie in the Cartan subalgebra of $G$.

We cannot dualize $A_i$ to $\gamma$ for non-Abelian gauge theories, but at a generic point of the Coulomb branch, this is nonetheless possible. We can dualize the
rk(G) massless vector multiplets into chiral multiplets \( Y_n, n = 1, \ldots, \text{rk}(G) \). The low-energy theory at a generic point of the Coulomb branch thus includes these \( \text{rk}(G) \) chiral multiplets \( Y_n \). The classical Coulomb branch can be parameterized by the vevs of \( \sigma_n, \gamma_n, n = 1, \ldots, \text{rk}(G) \), or by the chiral operators

\[
Y_n \sim \exp \left( \frac{\sigma_n}{e^2} + i \gamma_n \right),
\]

where we choose \( \sigma_1 \geq \cdots \geq \sigma_{\text{rk}(G)} \) in order to fix the Weyl freedom.

The metrics of the classical moduli spaces receive both loop and non-perturbative corrections. A superpotential can only be generated non-perturbatively via instanton effects. These instantons, which correspond to monopoles in four dimensions, can for non-Abelian cases lift most or all of the Coulomb branch, and sometimes also the Higgs branch.

It has been shown that for a pure \( SU(2) \) sym, the vev of \( Y_1 \) classically breaks \( SU(2) \) to \( U(1) \). In four dimensions, this breaking is associated to a static monopole solution, while in three dimensions, it corresponds to an instanton. This instanton generates the Affleck–Harvey–Witten \([57]\) (AHW) superpotential

\[
W_{\text{AHW}} = \frac{1}{Y_1}.
\]

In pure \( U(N) \) or \( SU(N) \) sym, the same effect happens whenever two eigenvalues of \( \sigma \) approach each other, resulting in an unbroken \( SU(2) \). In these cases, the superpotential generated by the instantons takes the form

\[
W_{\text{AHW}} = \sum_{n=1}^{N-1} \frac{1}{Y_n}.
\]

This superpotential lifts the entire Coulomb branch. When matter multiplets are present, this behavior is modified as extra fermion zero modes can prevent instantons from generating a superpotential. The zero-mode counting analysis can be performed using the Callias index theorem \([58]\) (see also \([48]\) for a more physical approach). The global charges of the monopole operators can be computed following \([59]\).

**Parity anomaly.** Note that the Chern–Simons terms breaks parity. This is referred to as a *parity anomaly* and is related to an induced term at one loop with charged fermions running in the loop \([60,61]\).

Let us consider a \( U(1)^r \) gauge theory. The Chern–Simons term takes the form

\[
\sum_{m,n=1}^{r} k_{mn} \int d^4 \theta \Sigma_m V_n.
\]

If we integrate out the charged fermions, an additional contribution to the Chern–Simons term is induced:

\[
k_{mn} + \frac{1}{2} \sum_Q (q_Q)_m (q_Q)_n \text{sign}(m_Q) \in \mathbb{Z},
\]
where the sum runs over all fermions in the chiral matter multiplets \(Q_i\), \((q_Q)_n\) is the integer charge of the fermion \(\phi_Q\) under \(U(1)_n\), and \(m_Q\) is the real mass of \(\phi_Q\), i.e.

\[
m_Q = m_R + \sum_{n=1}^{r} (q_Q)_n \sigma_n.
\] (3.1.28)

If \(\sum_Q (q_Q)_n (\sigma_Q)_n\) is odd, \(k_{ij} \neq 0\) and the parity is broken. An induced Chern–Simons term can be avoided by choosing the matter content such that the parity anomaly does not arise.

A similar parity anomaly arises for non-Abelian theories.

### 3.1.2 SQCD with \(G = U(N_c)\)

Here we will study three-dimensional \(\text{sqcd}\) with gauge group \(U(N_c)\). The matter consists of \(N_f\) chiral multiplets \(Q_i\) in the fundamental representation \(N_c\) and \(N_f\) chiral multiplets \(\tilde{Q}_i\) in the anti-fundamental representation \(\overline{N_c}\). As the matter multiplets are paired, this choice of matter fields does not lead to a parity anomaly and thus no Chern–Simons term is present.

A topological global symmetry similar to the \(U(1)_J\) in Abelian gauge theories can be formed from the \(U(1)\) gauge group contained in \(U(N_c)\). It acts by shifting the dual photon, such that the operators \(Y_i\) are charged under it.

For \(G = U(N_c)\), the adjoint scalar \(\sigma\) can be diagonalized to

\[
\sigma = \text{diag}(\sigma_1, \ldots, \sigma_{N_c}).
\] (3.1.29)

On generic points of the Coulomb branch, we can as discussed above, dualize the massless gauge fields into \(N_c\) dual photons \(\gamma_i\).

After taking into account the instanton effects, only the part of the Coulomb branch remains on which

\[
\sigma_1 > 0 = \sigma_2 = \cdots = \sigma_{N_c-1} > \sigma_{N_c}.
\] (3.1.30)

The Coulomb branch that remains after instanton corrections is thus a two-dimensional manifold and is parameterized by

\[
T = \exp \left( \frac{\sigma_1}{\epsilon_f^2} + i\gamma_1 \right), \quad \tilde{T} = \exp \left( -\frac{\sigma_{N_c}}{\epsilon_f^2} + i\gamma_{N_c} \right).
\] (3.1.31)

The monopole operators \(T(\tilde{T})\) carry charge \(1(-1)\) under the \(U(1)_f\) symmetry.

The Higgs branch of \(\mathcal{M}_q\) is parameterized by the gauge-invariant superfields \(M_i^j = Q_i \tilde{Q}_j^j\).

Additional non-perturbative effects lifting the Coulomb branch come into play for \(N_f < N_c - 1\). At the end of the day, the quantum moduli space \(\mathcal{M}_q\) is very similar to its four-dimensional counterpart:

- \(N_f < N_c - 1\): the classical moduli space is lifted completely, there is no supersymmetric vacuum.
- \(N_f = N_c - 1\): a smooth quantum moduli space exists.
\( N_f = N_c \): a quantum moduli space exists with a dual description at the origin using only chiral multiplets.

\( N_f \geq N_c + 1 \): a quantum moduli space exists, but it has a singularity at the origin.

Under the (quantum corrected) global symmetries, the fields transform as given in Table 3.1.

### 3.2 Dualities in three dimensions

In this section we discuss some general aspects of dualities between three-dimensional \( \mathcal{N} = 2 \) theories. We distinguish three classes of dualities:

- Aharony-like dualities
- Giveon–Kutasov-like dualities
- Mirror symmetry.

The name of the first two classes originates from the prototypical examples of those dualities, \textit{i.e.} the duality found by Aharony in [6] and the one found by Giveon and Kutasov in [17]. The main difference between these dualities consists in the presence of a \( cs \) term in the action of the latter type. As we will see later, the two dualities can be connected via an \( rg \) flow triggered by a real mass term. The third class of dualities is a generalization of \( \mathcal{N} = 4 \) mirror symmetry to \( \mathcal{N} = 2 \). This last duality will play an interesting role in our discussion.

In the first part of this section, we will review the main aspects of the three dualities just introduced, while in the second part we will enumerate the generalizations of Aharony and Giveon–Kutasov duality that have been found in the literature. We will not review the dualities among theories with chiral matter content described in [62].

To conclude, observe that in three dimensions, there is no notion of chirality. When we call a three-dimensional theory chiral, we are referring to its four-dimensional parent theory. In the following, we will refer to a four-dimensional theory as a parent theory of a three-dimensional theory if they share the same field content, \textit{i.e.} the same amount of \( \mathcal{N} = 1 \) vector and matter multiplets, and the same interactions among the matter fields. In this identification, we do not

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**Table 3.1:** Transformation properties of the fields under the global symmetries.

| Field | \( SU(N_f)_L \) | \( SU(N_f)_R \) | \( U(1)_I \) | \( U(1)_A \) | \( U(1)_R \) |
|-------|-----------------|-----------------|--------------|--------------|--------------|
| \( Q^a_i \) | \( N_f \) | \( 1 \) | 0 | 1 | \( \Delta \) |
| \( \tilde{Q}^{\dagger a}_i \) | \( 1 \) | \( \bar{N}_f \) | 0 | 1 | \( \Delta \) |
| \( M^i_j \) | \( N_f \) | \( \bar{N}_f \) | 0 | 2 | 2\( \Delta \) |
| \( T \) | \( 1 \) | \( 1 \) | 1 | \( -N_f \) | \( N_f(1 - \Delta) - N_c + 1 \) |
| \( \tilde{T} \) | \( 1 \) | \( 1 \) | \( -1 \) | \( -N_f \) | \( N_f(1 - \Delta) - N_c + 1 \) |

Note, that often, the choice \( \Delta = 0 \) is used in the literature.
refer to the gauge interactions and to the Coulomb branch of the three-dimensional theory. We call for example four-dimensional $\mathcal{N} = 1$ $U(N_c)$ SQCD with $N_f$ pairs of fundamental and antifundamental quarks the parent theory of both $U(N_c)_k$ and $U(N_c)_0$ three-dimensional $\mathcal{N} = 2$ SQCD with $N_f$ pairs of fundamental and antifundamental quarks and $W = 0$.

### 3.2.1 Aharony duality

This duality was originally discussed in [6]. The electric theory consists of a $U(N_c)$ gauge theory with $N_f$ pairs of fundamental and antifundamental quarks.

Observe that, differently from the four-dimensional Seiberg duality, in this case the $U(1)$ factor of the gauge group does not decouple in the IR. In other words, it is possible to consider a gauging of the baryonic symmetry. The dual theory has gauge group $U(N_f - N_c)$, $N_f$ pairs of dual fundamentals $\widetilde{q}$ and antifundamentals $\overline{q}$ and a singlet $M$ corresponding to the meson of the electric theory. The superpotential term $W = Mq\overline{q}$ is present also in this case.

So far we have presented a version of the duality identical to four-dimensional Seiberg duality in presence of a gauging of the baryonic symmetry. Nevertheless, in the three-dimensional case this is not the end of the story. One can see that something is missing e.g. by looking at the global symmetries. In the three-dimensional case, there are two global currents that do not arise in four dimensions. The first one is the axial symmetry, $U(1)_A$: while this symmetry is anomalous in four dimensions, the absence of global anomalies in three dimensions forces us to consider it here. We will see that this symmetry plays a crucial role in the reduction of Seiberg duality to three dimensions and also its interpretation at the brane level will look quite interesting.

The second symmetry that one has to consider in three dimensions is a topological symmetry, i.e. a symmetry conserved because of the Bianchi identity. This topological $U(1)_J$ symmetry does not charge the elementary fields but it shifts the dual photon associated to the monopole operators. Let us denote the monopole operators with flux $(\pm 1, 0, \ldots, 0)$ of the electric theory as $T$ and $\overline{T}$ and the one of the magnetic theory as $t$ and $\overline{t}$. It has been argued in [6] that the monopoles $t$ and $\overline{t}$ in the dual theory have to be set to zero in the chiral ring for the duality to hold. This can be achieved by adding a superpotential

$$\Delta W = tT + \overline{t}\overline{T}. \quad (3.2.1)$$

where the electric monopoles $T$ and $\overline{T}$ are treated as singlets in the dual theory, imposing the constraints on $t$ and $\overline{t}$ from their F-terms. Observe that the monopole operators $t$ and $\overline{t}$ of the dual theory are not elementary degrees of freedom, i.e. the above term does not have to be interpreted as a mass term but has the role of enforcing the constraints on the chiral ring as discussed above.

We conclude this review of the duality with a table of the charges of the various fields in the magnetic phase (Tab. 3.2).

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2 There is also a version of this duality with an $SU(N_c)$ gauge group that we will not discuss here [25].
### 3.2.2 Giveon–Kutasov duality

This duality was originally worked out in [17] using a brane construction which we will discuss in Sec. 3.3.2. The electric and the magnetic sides of this duality are as follows. The electric theory has gauge group $U(N_c)_k$, where $k$ represents the integer $c_\text{s}$ level. There are again $N_f$ pairs of fundamentals $Q$ and antifundamentals $\tilde{Q}$ in the spectrum. The superpotential is

$$W = 0.$$  \hspace{1cm} (3.2.2)

The dual theory in this case has gauge group $U(N_f - N_c + |k|)_k$, $N_f$ pairs of fundamentals $q$ and antifundamentals $\tilde{q}$ in the action and mesons $M_i = Q_i \tilde{Q}_i$, interacting through a superpotential

$$W = Mq\tilde{q}.$$  \hspace{1cm} (3.2.3)

### 3.2.3 There and back again

Aharony duality and Giveon–Kutasov duality are related and it has been shown that they can be connected via a real mass flow in both directions.

The first flow (from Aharony to Giveon–Kutasov) is quite easy to understand, it has been first discussed in [63]. Here one turns on a background gauge field for the global symmetries and assigns a (large) vev to the background scalars $\sigma$. This turns into a large real mass for some of the fundamental fields that are integrated out. Integrating out these fields generates $c_\text{s}$ levels and contact terms (or BF couplings). More in detail, we start from $U(N_c)$ SQCD with $N_f + k$ pairs of fundamentals and antifundamentals and assign a large real mass to $k$ fundamentals and $k$ antifundamentals. These large real masses have to be chosen with the same sign. By integrating out the massive fermions the theory becomes three-dimensional $\mathcal{N} = 2 U(N_c)_{\pm k}$ SQCD with $N_f$ pairs of fundamentals and antifundamentals, with vanishing superpotential, i.e. the electric theory of the Giveon–Kutasov duality. The sign of the $c_\text{s}$ level $\pm k$ depends on the sign of the real masses. The same flow has to be realized on the dual side. In this case, the dual theory before turning on the background gauge fields is $U(N_f + k - N_c)$ with $N_f + k$ pairs of fundamentals and antifundamentals, $q$ and $\tilde{q}$ interacting with a meson with $(N_f + k)^2$ component. One has to turn on the background gauge fields for the global symmetries consistently with the electric theory. This implies that $k$ pairs of fundamentals and antifundamentals acquire a large real mass, as opposed to

| $SU(N_f)_L$ | $SU(N_f)_R$ | $U(1)_I$ | $U(1)_A$ | $U(1)_R$ |
|-------------|-------------|---------|--------|--------|
| $q^i_a$     | $\overline{N}_f$ | $1$     | $0$   | $-1$   | $1 - \Delta$ |
| $q_i^a$     | $1$         | $N_f$   | $0$   | $-1$   | $1 - \Delta$ |
| $M_i^j$     | $N_f$       | $\overline{N}_f$ | $0$ | $2$ | $2\Delta$ |
| $t$         | $1$         | $1$     | $-1$  | $N_f$  | $N_f(\Delta - 1) + N_c + 1$ |
| $\bar{t}$   | $1$         | $1$     | $1$   | $N_f$  | $N_f(\Delta - 1) + N_c + 1$ |

Table 3.2: Transformation properties of the magnetic fields under the global symmetries.
the single one of the electric fields. There are $N_f^2$ components left in the mesonic field and the electric monopoles are massive. By integrating the massive fermions out a cs level $\mp k$ is generated. The dual gauge group is $U(N_f + k - N_c)_{\mp k}$. At the end of the day, the electric gauge group is $U(N_c)_k$ and the magnetic one is $U(N_f + |k| - N_c)_{-k}$, with $k \neq 0$. This corresponds precisely to the duality of Giveon and Kutasov.

The flow from Giveon–Kutasov duality to Aharony’s is more involved and it requires the notion of topological vacua, discussed in [64]. Let us review briefly this construction. In this case one starts with $U(N_c)_k$ with $N_f + k$ pairs of fundamentals and antifundamentals. One can integrate the $k$ pairs of fundamentals and antifundamentals by turning on some background gauge fields as before. The theory becomes $U(N_c)_0$ if the sign of these masses is chosen such that the cs term is canceled. The novelty is in the dual flow: in this case the duality is obtained not only by turning on the masses in the dual phases correctly, but we also need a Higgsing of the gauge group. This is due to the presence of a non-trivial vev for the real scalars in the $U(N_f + |k| - N_c)_0$ dual theory. After the flow, the dual gauge group becomes $U(N_f - N_c) \times U(k)^2$. By integrating out the massive fermions and keeping the massless matter fields, the first sector has vanishing cs level with $N_f$ pairs of fundamentals and antifundamentals, while the two $U(k)$ sector have cs level $k/2$, plus $k$ massless fundamentals and no antifundamentals. In a slight abuse of language, borrowed from the four-dimensional case, we can can think of them as chiral theories. The $U(k)_{k/2}$ theories can be dualized to singlets, and these singlets interact with the monopoles of the $U(N_f - N_c)_0$ theory through an anhw superpotential. This completely reconstructs the dual phase of the Aharony duality.

### 3.2.4 Mirror symmetry

Three-dimensional mirror symmetry, in its original formulation, is a duality of theories with $\mathcal{N} = 4$ supersymmetry [67]. It essentially exchanges the Higgs and the Coulomb branches of moduli space. At the level of the algebra, it exchangers the two $SU(2)$ R–symmetry groups inside $SO(4)_R = SU(2)_L \times SU(2)_R$. The origin of this duality can be easily visualized in the brane description: it corresponds to exchanging the NS and the R sector, i.e. NS5 and D5 branes. In this sense, mirror symmetry is a consequence of S-duality.

Here, we are interested in a less supersymmetric version of mirror symmetry, i.e. in theories with $\mathcal{N} = 2$ supersymmetry [13, 68, 71]. In this case, the action of mirror symmetry is less clear than above, as the R-symmetry group is Abelian and there are mixed branches in the moduli space. However, we can still think of the action of this symmetry as exchanging the R and the NS sectors. The prototypical example of $\mathcal{N} = 2$ mirror symmetry corresponds to the duality between $\mathcal{N} = 2$ supersymmetric quantum electrodynamics (SQED) with one flavor and a model with three chiral fields, named X, Y and Z with the superpotential $W = XYZ$. This last model is named XYZ model in the literature.

Observe that this example can be thought of as a limiting case of Aharony duality if the electric gauge group is $U(1)$ and $N_f = 1$, the dual theory consists of one monopole, one anti-monopole and the meson $M$, interacting through the

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3 The case with $k = 1$ was studied in [64], and it was generalized to higher $k$ in [65, 66].
superpotential. In the following we will make a lot of use of this relation, by conjecturing that similar relations hold also for the limiting cases of generalized Aharony dualities with tensor matter content. In this case, the effective superpotential can be written as

$$W = \tilde{t} \det M,$$

(3.2.4)

corresponding to the one of the XYZ model, by identifying $X$ with $t$, $Y$ with $\tilde{t}$ and $Z$ with $\det M$. Note that the topological symmetry of the electric theory becomes non-topological in the mirror description, i.e. a symmetry of the Coulomb branch on one side is a symmetry of the Higgs branch on the other side. Interestingly, mirror symmetry maps the Fayet–Iliopoulos ($f_i$) term of sqed to a real mass in the dual model.

### 3.2.5 Generalizations of the dualities: real groups and tensor matter

As discussed in the case of Seiberg duality, also in the three-dimensional case the dualities have been extended to cases with tensor matter and real gauge group. In this section, we will review the extensions that have been worked out in the literature for the case of Aharony and Giveon–Kutasov duality. This is a necessary step in order to study the reduction of the four-dimensional case to the three-dimensional ones.

**Tensor matter**

**$U(N_c)_0$ with adjoint matter**

The simplest generalization of Aharony duality with tensor matter is the case of $U(N_c)_0$ sqcd with an adjoint field $X$ with superpotential (2.2.7). For $N_c < nN_f$, this theory has a dual description in terms of an $U(nN_f - N_c)_0$ gauge theory, with $N_f$ dual quarks, an adjoint $Y$ and a set of singlets. There are two types of singlets, the meson

$$M_j = QX^j\tilde{Q}, \quad j = 0, \ldots, n - 1,$$

(3.2.5)

and a set of fields with the same quantum charges of the electric monopoles with flux $(\pm 1, \ldots, 0)$. From the quantum charges on can identify these fields with the electric monopoles, denoted here as $T_0$ and $\tilde{T}_0$, to stress that they are the bare monopoles discussed above, dressed by the adjoint field $X$. These singlets are defined as

$$T_j = T_0X^j \quad \text{and} \quad \tilde{T}_j = \tilde{T}_0X^j, \quad j = 0, \ldots, n - 1,$$

(3.2.6)

consistently with the constraints on the chiral ring imposed by the superpotential of the electric theory. These singlets play the role of enforcing the correct quantum constraints on the Coulomb branch of the dual theory. These relations are enforced by a superpotential relation, as in the Aharony duality. The superpotential of the dual theory is given by

$$W = Y^{n+1} + \sum_{j=1}^{n-1} M_jqY^{n-1-j}q + \sum_{j=1}^{n-1} (t_jT_{n-1-j} + \tilde{t}_j\tilde{T}_{n-1-j}),$$

(3.2.7)
where \( t_j \) and \( \tilde{t}_j \) are the monopoles of the magnetic theory, with flux \((\pm 1, \ldots, 0)\).
The first two terms of the superpotential correspond to the generalization of KSS duality in three dimensions, and enforce the constraints on the chiral ring for the matter fields. The last two terms generalize the constraints on the chiral ring to impose on the monopole sector, taking into account the presence of the adjoint field in the counting of the zero modes. This duality was first derived by Kim and Park in [24] (KP) (see also [72, 73] for related discussions).

**U(\(N_c\))\(_0\) with symmetric flavor**

This duality has never been discussed in the literature so far, but it can be derived from the case of \( U(\(N_c\))\(_k\) with symmetric flavor discussed in [23]. Another derivation of this duality comes from the dimensional reduction of the four-dimensional case as will be discussed in Section 4. For completeness, we present here the final result.

This duality has gauge group \( U(\(N_c\))\(_0\), \(N_f\) pairs of fundamentals and antifundamentals and two tensor matter fields, the field \( S \) in the symmetric representation of the gauge group, and its conjugate \( \tilde{S} \), with an interaction

\[
W = \text{Tr}(S\tilde{S})^{n+1}.
\]  

(3.2.8)

This induces a truncation on the chiral ring, such that the mesons that we have to consider are

\[
M = Q(S\tilde{S})^{j+1}\tilde{Q}, \quad P = Q(S\tilde{S})^j\tilde{S}Q, \quad \tilde{P} = \tilde{Q}S(\tilde{S})^j\tilde{Q}, \quad j = 0, \ldots, n - 1,
\]  

(3.2.9)

where \( M \) is a bifundamental operator, while \( P \) and \( \tilde{P} \) are symmetric in the flavor indices.

The dual theory has \( U((2n + 1)N_f + 2n - N_c) \) gauge group, there are dual fundamentals and antifundamentals, \( q \) and \( \tilde{q} \), dual symmetric and conjugate fields \( s \) and \( \tilde{s} \), and the singlet fields are the mesons \( M, P \) and \( \tilde{P} \).

Also in this case one can work out the details on the monopole sector and study how the electric monopoles modify the structure of the dual superpotential. We leave this derivation to the interested reader.

**U(\(N_c\))\(_0\) with antisymmetric flavor**

Also this duality has not been discussed in the literature but it can be derived from the case of \( U(\(N_c\))\(_k\) with symmetric flavor discussed in [23]. Another derivation of this duality comes from the dimensional reduction of the four-dimensional case we will discuss in Section 4. For completeness, we present here the final result.

This duality has gauge group \( U(\(N_c\))\(_0\), \(N_f\) pairs of fundamentals and antifundamentals and two tensor matter fields, in the antisymmetric and in the conjugate antisymmetric representations. There is also a superpotential term

\[
\text{Tr}(A\tilde{A})^{n+1}.
\]  

(3.2.10)

This induces a truncation on the chiral ring, such that the mesons that we have to consider are

\[
M = Q(A\tilde{A})^{j+1}\tilde{Q}, \quad P = Q(A\tilde{A})^j\tilde{A}Q, \quad \tilde{P} = \tilde{Q}A(\tilde{A}A)^j\tilde{Q}, \quad j = 0, \ldots, n - 1,
\]  

(3.2.11)
where $M$ is a bifundamental operator, while $P$ and $\tilde{P}$ are antisymmetric in the flavor indices.

The dual theory has $U((2n + 1)\, N_f - 2n - N_c)$ gauge group, there are dual fundamentals and antifundamentals $q$ and $\tilde{q}$, dual antisymmetric and conjugate fields $a$ and $\tilde{a}$, and the singlet fields are the mesons $M$, $P$ and $\tilde{P}$. The details on the monopole sector are left to the reader also here.

**Orthogonal case**

In this section we discuss three-dimensional dualities for gauge group $SO(N)$. Before starting the analysis we will make a small digression about the global properties of theories with $so(n)$ gauge algebra. In presence of $N_f$ vectors, there are four possible dual pairs of Aharony type (at zero $cs$ level):

\begin{align}
O(N_c)_{+} & \leftrightarrow O(N_f - N_c + 2)_{+}, \\
O(N_c)_{-} & \leftrightarrow Spin(N_f - N_c + 2), \\
Spin(N_c) & \leftrightarrow O(N_f - N_c + 2)_{-}, \\
Pin(N_c) & \leftrightarrow Pin(N_f - N_c + 2),
\end{align}

(3.2.12)

where the $Pin(N_c)$ group is obtained by gauging the charge conjugation symmetries in the $Spin(N_c)$ group. In the discussion of the D brane constructions, we will ignore the global properties and refer to the duality between an $SO(N_c)$ theory and an $SO(N_f - N_c + 2)$ theory.

**$SO(N_c)_0$ with $N_f$ vectors**

This duality was originally proposed by [23] (see also [24] for further discussions). The electric theory has $SO(N_c)_0$ gauge group and $N_f$ vectors $Q$, with vanishing superpotential. It is dual to an $SO(N_f - N_c + 1)$ theory, with $N_f$ dual vectors $q$, the meson $M = q^2$ in the symmetric representation of the $SU(N_f)$ flavor symmetry group and the extra singlets $T_0$. The latter have the same quantum numbers as the monopole operators that parameterize the Coulomb branch of the electric theory and they appear as singlets of the dual theory enforcing the constraint of $t_0 = 0$ on the magnetic monopole. The superpotential of this dual theory is

\begin{equation}
W = Mqq + t_0 T_0.
\end{equation}

(3.2.13)

**$SO(N_c)_0$ with $N_f$ fundamentals and an adjoint**

This duality has been studied in [24]. In this case one has an $SO(N_c)_0$ theory with $N_f$ vectors and one adjoint $X$, with superpotential

\begin{equation}
W = \text{Tr} \, X^{2(n+1)}.
\end{equation}

(3.2.14)

The dual theory is an $SO((2n + 1)\, N_f - N_c + 2)$ gauge theory with $N_f$ vectors and singlets. The mesonic singlets are $M_j = q X^{j} \bar{q}$ with $j \leq 2n$, (anti-)symmetric in the flavor indices for (odd) even $n$. There are also singlets associated to the dressed monopole operators of the electric theory. They
are of the form $T_j = X_i T_0$, for $j \leq 2n$, where $T_0$ is the bare monopole, and impose the necessary constraint on the Coulomb branch of the dual theory.

This theory has superpotential

$$W = Y^{2(n+1)} + \sum_{j=0}^{2n} M_j q Y^{2n-j} q + \sum_{j=0}^{2n} T_j t_{2n-j}.$$  \hfill (3.2.15)

**$SO(N_c)_0$ with $N_f$ fundamentals and a symmetric tensor**

This duality has not been discussed in the literature. It may be derived by a real mass flow from the case discussed in [23], in which the electric theory has a $cs$ term. Another possibility consists in deriving it from dimensional reduction, see Section 4. In this case the electric theory has $SO(N_c)$ gauge symmetry, with $N_f$ vectors $Q$ and a traceless symmetric tensor $S$ with superpotential

$$W = \text{Tr} S^{n+1}.$$  \hfill (3.2.16)

The mesons in the chiral ring are identified as $M_j = QS^j \tilde{Q}$, with $j = 0, \ldots, n - 1$. The dual theory has $SO(n(N_f + 2) - N_c)$ gauge group, $N_f$ vectors $q$, a traceless symmetric tensor $s$ and the mesons $M_j$. There are also dressed monopole operators acting as singlets in the dual phase, of the form $T_j = T_0 S^j$, with $j = 0, \ldots, n - 1$. The superpotential of this dual theory is

$$W = s^{n+1} + \sum_{j=0}^{n-1} M_{n-j} q s^j q + \sum_{j=0}^{n-1} t_j t_{n-1-j},$$  \hfill (3.2.17)

where $t_j$ are the monopoles that parameterize the Coulomb branch of the dual theory.

**Symplectic case**

**$Sp(2N_c)_0$ with $2N_f$ fundamentals**

This case was first discussed in Aharony’s original paper [6]. The electric theory has $Sp(2N_c)_0$ gauge group with $2N_f$ fundamentals $Q$ and $W = 0$. The magnetic theory has $Sp(2(N_f - N_c - 1))_0$ gauge group, $2N_f$ dual fundamentals and a meson $M = Q^2$ in the antisymmetric representation of the $SU(2N_f)$ flavor symmetry group. There is also an extra singlet, corresponding to the electric monopole $T_0$, necessary to impose the constraint $t_0 = 0$ on the monopole that parameterizes the Coulomb branch of the dual theory. The superpotential of the dual theory is

$$W = Mqq + t_0 T_0.$$  \hfill (3.2.18)

**$Sp(2N_c)_0$ with $2N_f$ fundamentals and an adjoint**

This duality has been studied in [24]. In this case one has an $Sp(2N_c)_0$ theory with $2N_f$ fundamentals and one adjoint $X$, with superpotential

$$W = \text{Tr} X^{2(n+1)}.$$  \hfill (3.2.19)
The dual theory is an $Sp(2(2n+1)N_f - N_c - 1))$ gauge theory with $2N_f$ fundamentals and singlets. The mesonic singlets are $M_j = qXj\bar{q}$ with $j \leq 2n$, (anti)-symmetric in the flavor indices for (even) odd $n$. There are also singlets associated to the dressed monopole operators of the electric theory. They are of the form $T_j = X^j T_0$, for $j \leq 2n$, where $T_0$ is the bare monopole, and impose the necessary constraint on the Coulomb branch of the dual theory. This theory has superpotential

$$W = Y^{2(n+1)} + \sum_{j=0}^{2n} M_j q Y^{2n-j} q + \sum_{j=0}^{2n} T_j t_{2n-j}. \quad (3.2.20)$$

**$Sp(2N_c)$ with $2N_f$ fundamentals and an antisymmetric tensor**

Here we can consider the case of the traceless antisymmetric tensor. This duality has not been described in the literature, but it can be derived from the reduction of the four-dimensional case as we will discuss later. In this case the electric theory has gauge group $Sp(2N_c)$ and there are $2N_f$ fundamental quarks $Q$ and a traceless antisymmetric tensor $A$. The superpotential is

$$W = \text{Tr} A^{n+1}, \quad (3.2.21)$$

and there are mesons $M_j = QA^j Q$ with $j = 0, \ldots, n-1$. The dual theory has $Sp(2(n(N_f - 1) - N_c))$ gauge group with $2N_f$ fundamentals $q$, a dual antisymmetric field $a$, and the mesons $M_j$. There are also dressed monopole operators acting as singlets, $T_j = \text{Tr} T_0 A^j$ for $j = 0, \ldots, k-1$. The superpotential of this dual theory is

$$W = \text{Tr} a^{n+1} + \sum_{j=0}^{n-1} \text{Tr} M_{n-j} q a^j q + \sum_{j=0}^{n-1} t_j T_{n-1-j}. \quad (3.2.22)$$

where $t_j$ are the dressed monopole operators of the dual theory.

### 3.2.6 Dualities with Chern–Simons terms

Another generalization of the dualities discussed above consists in adding a level $k$ cs term for the electric gauge group, i.e. dealing with $U(N_c)_k$, $Sp(2N_c)_k$ and $O(N_c)_k$. Many of these dualities can been derived from a brane construction and we will review these constructions at the of this chapter. Another possible way to derive such dualities consists in assigning a large real mass (the same, or at least same sign) to $k$ pairs of quarks and antiquarks. This induces a real mass flow, reducing the number of flavors and generating at one loop a level $k$ cs level. In the dual theory one can assign the real masses consistently to the dual fields and trigger the dual of this flow. The prototypical example of such a flow is the derivation of Giveon–Kutasov duality from Aharony duality [63], as discussed in Sec. 3.2.3. Many dualities involving also chiral theories have been constructed in a similar fashion in [62]. Here we will not review these constructions but present a different example: we show how to obtain the duality of Niarchos from the duality

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4 At the end of this section, we also discuss the case with non-vanishing trace, which is just a singlet that modifies the duality slightly.
of unitary theories and adjoint matter with superpotential $X^ {n+1}$. We will leave the derivation of the other dualities to the reader, and limit ourselves to a listing of the various three-dimensional dualities found in the literature.

**$U(N_c)_k$ with the adjoint**

Let us consider an electric theory with $U(N_c)_0$ gauge group, $N_f + k$ fundamental flavors and an adjoint field $X$ with superpotential (2.2.7). We can assign a large mass $m$ to $k$ fundamental flavor pairs. This can be done by combining the effect of the $U(N_f + k)_r \times U(N_f + k)_l$ symmetries. We can parameterize them as follows:

$$m(Q_a) = \mu_a + m_A + m_B \quad m(\bar{Q}_a) = \nu_a + m_A - m_B,$$

(3.2.23)

where $\mu_a$ and $\nu_a$ refer to the $SU(N_f + k)_r$ and the $SU(N_f + k)_l$ part of the symmetry group, and are subjected to the constraint $\sum_a \mu_a = \sum_a \nu_a = 0$, while $m_A$ and $m_B$ refer to the real mass of the axial and of the baryonic symmetry respectively. We assign the masses as

$$\mu_{a_1} = \tilde{\mu}_{a_1} - \frac{km}{k-N_f} \quad \text{with} \quad \sum_{a_1=1}^{N_f} \tilde{\mu}_{a_1} = 0 \quad \text{and} \quad \mu_{a_2} = -\frac{km}{k-N_f}$$

(3.2.24)

$$\mu_{a_1} = \tilde{\nu}_{a_2} - \frac{N_f m}{k-N_f} \quad \text{with} \quad \sum_{a_2=1}^{N_f} \tilde{\nu}_{a_2} = 0 \quad \text{and} \quad \mu_{a_2} = -\frac{N_f m}{k-N_f}$$

(3.2.25)

and $m_A = \frac{km}{k-N_f} + m_A$ where $a_1 = 1, \ldots, N_f$ and $a_2 = 1, \ldots, k$. By integrating these fields at one loop we are left with a model with $U(N_c)_k$ gauge group, $N_f$ flavors and an adjoint with the superpotential (2.2.7). The dual theory, before considering the large-mass limit, has gauge group $U(n(N_f + k) - N_c)_0$. The charged matter fields where $N_f + k$ dual fundamental flavors and adjoint $Y$. There were also singlets, the mesons $M_j = QX|\bar{Q}$ and the dressed monopoles $T_j = T_0 X|$. The dual real mass flow can be understood from the spectrum of the global symmetries. By performing the analysis, one obtains $k$ fundamental flavors with a large mass, $-m$. By integrating them out one obtains a cs theory at level $-k$. Each meson $M_j$ has $2kN_f$ components with real mass $m$ and $k^2$ components with mass $2m$. The monopoles are charged under $U(1)_A$ and they are all massive. The final theory has $U(n(N_f + k) - N_c)_{-k}$ gauge group, with $N_f$ fundamental pairs and one adjoint $Y$. Observe that so far we have been agnostic on the sign of $m$. Both signs are possible, and by specifying the sign, the analysis one ends up with the magnetic gauge group in the form $U(n(N_f + |k|) - N_c)_{-k}$. There are also mesons $M_j = QX|\bar{Q}$, each with $N_f^2$ components, and the superpotential is (2.2.9). This corresponds to the dual phase originally found in [18].

**Other dualities**

We can perform an analogous real mass flow on the other dualities without the cs terms discussed above. The flows generate cs terms in the action and one can end up with various generalizations of $ck$ dualities.

We will not discuss these cases in detail but just review the final results in Table 3.3, by specifying the electric and the magnetic gauge groups and the charged
fields. We also specify the paper in which the duality of interest has been first introduced.

3.2.7 Checks

We conclude this overview of $\mathcal{N} = 2$ three-dimensional dualities with a discussion of possible checks. In three dimensions, many of the tools used for the four-dimensional case are not available due to the absence of anomalies for the continuous symmetries. Also at the level of the moduli space, the checks are more involved because of the presence of a Coulomb branch, parameterized by the monopole operators. We refer the reader to some relevant papers for the analysis of these checks [13, 64]. In the recent years, new checks have come from the computation of the Witten index [64, 75], matching the number of vacua of the dual theories. Another class of checks consists in matching the parity anomalies. Moreover, in presence of $cs$ terms, one can further check the contact terms for the global symmetries [51, 76]. Many new techniques have been possible thanks to the development of localization techniques. The partition function on the three-sphere [49, 72, 78] and the superconformal index [79] can be used to match the dualities discussed above (see for example [62, 63, 80]). Exact mathematical results play a crucial role in these matchings. Matching the partition function on the three-sphere for example corresponds to some identities among hyperbolic hypergeometric integrals [81]. More recently, the topologically twisted index of [82] has been used to test some of the dualities presented above.

3.3 The brane picture in three dimensions

In this section, we discuss the realization of the three-dimensional dualities in presence of $cs$ terms in terms of D and NS branes.

We begin our analysis in Sec. 3.3.1 by reviewing the generation of $cs$ terms in the brane picture in terms of bound states of 1 NS and $p$ D5 branes, i.e. $(1, p)$–fivebranes. In Section 3.3.2, we review the simplest example of a three-dimensional $\mathcal{N} = 2$ duality, namely $gk$ duality. The generalization to the case of an adjoint with a polynomial potential is discussed in Sec. 3.3.3. We discuss how these results are extended to configurations with orientifolds in Section 3.3.4. We conclude this section by discussing the problems in extending these results to the cases with vanishing $cs$ level $k_{CS} = 0$ in Section 3.3.5.

3.3.1 Chern–Simons action from fivebranes

Three-dimensional field theories are engineered in ten-dimensional type IIB string theory. The NS branes are the same as in the type IIA case (Section 2.3), while the $D_p$ branes have $p = -1, 1, 3, 5, 7, 9$. We will mostly consider NS branes extended along $x_{012345}$, NS’ along $x_{012389}$, D3 branes extended along $x_{0126}$, D5 branes along $x_{012789}$, and D5’ along $x_{012457}$ (see Table 3.1).

A new ingredient, not present in four dimensions, is the notion of the $cs$ term, which can also be realized in the brane engineering. Let us consider first a D5 brane and an NS brane. Let us suppose that the two branes intersect in the directions $x_{37}$. Then one can break the D5 brane on the NS brane into two semi-infinite D5 branes
Table 3.3: Summary of the three dimensional dualities with $cs$ action discussed here. $F$ is the fundamental representation, $\tilde{F}$ the antifundamental, $S$ the symmetric, and $\tilde{S}$ to its conjugate, $A$ to the antisymmetric, $\tilde{A}$ to its conjugate, $X$ to the adjoint and $V$ to the vector. There are also singlets in the spectrum of the magnetic phases, corresponding to the generalized mesons that can be read off from the case without the $cs$ term or from the four-dimensional parent theory. We do not report these singlets here. Electric and Magnetic superpotentials coincide with those of the four-dimensional parent theory.

| $G$                  | Charged Fields                       | $\tilde{G}$                  | $W_e$ | $W_m$ | Ref. |
|----------------------|-------------------------------------|-----------------------------|-------|-------|------|
| $U(N_c)_k$           | $N_f(F \oplus \tilde{F})$          | $U(N_f - N_c + |k|)_k$      | 2.2.1 | (2.2.2) | 17   |
| $U(N_c)_k$           | $N_f(F \oplus \tilde{F}) \oplus S$ | $U(n(N_f + |k|) - N_c)_k$   | 2.2.7 | (2.2.9) | 18   |
| $U(N_c)_k$           | $N_f(F \oplus \tilde{F}) \oplus (S \oplus \tilde{S})$ | $U((2n + 1)(N_f + |k|) + 2n - N_c)_k$ | 2.2.10 | (2.2.12) | 23   |
| $U(N_c)_k$           | $N_f(F \oplus \tilde{F}) \oplus (A \oplus \tilde{A})$ | $U((2n + 1)(N_f + |k|) - 2n - N_c)_k$ | 2.2.13 | (2.2.15) | 23   |
| $Sp(2N_c)_{2k}$      | $2N_fF$                             | $Sp(2(N_f + |k| - N_c - 1))_k$ | 2.2.25 | (2.2.27) | 63   |
| $Sp(2N_c)_{2k}$      | $2N_fF \oplus X$                   | $Sp(2((2n + 1)(N_f + |k|) - N_c - 1))_k$ | 2.2.28 | (2.2.30) | 23   |
| $Sp(2N_c)_{2k}$      | $2N_fF \oplus A$                   | $Sp(2(n(N_f + |k| - 1) - N_c))_k$ | 2.2.31 | (2.2.33) | 74   |
| $O(N_c)_k$           | $N_fV$                              | $O(N_f + |k| - N_c + 2)_k$  | 2.2.16 | (2.2.18) | 20   |
| $O(N_c)_k$           | $N_fV \oplus X$                    | $O((2n + 1)(N_f + |k|) - N_c + 2)_k$ | 2.2.19 | (2.2.21) | 23   |
| $O(N_c)_k$           | $N_fV \oplus S$                    | $O(n(N_f + |k| + 2) - N_c)_k$ | 2.2.22 | (2.2.24) | 23   |
Table 3.4: Brane configuration for three-dimensional brane engineering.

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| NS | × | × | × | × | × | × | × | × | × |
| NS' | × | × | × | × | × | × | × | × | × |
| D3 | × | × | × | × | × | × | × | × | × |
| D5 | × | × | × | × | × | × | × | × | × |
| D5' | × | × | × | × | × | × | × | × | × |

on opposite sides of the NS. The two halves can slide along the same or opposite directions along $x_3$. The first case corresponds to generating opposite real masses for the quarks and the antiquarks, such that there is no $cs$ generated. In the other case, a $cs$ term at level one is generated for each pair of D5 branes sliding along opposite directions. In this case, supersymmetry is preserved by replacing the NS segment between the two configurations with a bound state of one D5 and one NS’ brane, i.e. a $(1, 1)$-fivebrane. This fivebrane is rotated by an angle $\tan \phi = g_s$ with respect to the original configuration (see Figure 3.1). In general, starting with $k$ D5 branes we end up with a $(1, k)$ fivebrane and the angle is $\tan \phi = kg_s$.

Figure 3.1: Brane engineering of the $cs$ term. Start with a D5–NS system (a). The D5 brane is broken at the intersection with the NS into two semi-infinite branes (b). Supersymmetry is preserved if the NS segment is replaced by a $(1, 1)$ bound state (c).

As anticipated in footnote 6 on the bottom of page 25, this configuration shows that the global symmetry is $U(N_f)^2$, such that the two $U(N_f)$ act on each stack of semi-infinite D5 branes separately.
Let us now consider a configuration with an NS brane and a \((1, k)\) fivebrane and stretch along \(x_6\) a stack of \(N_c\) D3 branes. The resulting theory corresponds to a level \(k, \mathcal{N} = 2\) \(U(N_c)_k\) SYM-\(cs\) theory. To see this, start by looking at the original configuration with \(k\) D5 branes, before the brane recombination. In this case, ignoring the Abelian factors, there should be an \(SU(k)^2\) flavor symmetry on the field theory side\(^5\) corresponding to the independent rotations of the \(Q\) and the \(\tilde{Q}\) flavors. This symmetry is visible at the brane level by the fact that we can break the D5 on the NS' and treat the two semi-infinite branes separately. Moving the semi-infinite D5 branes corresponds to integrating out the corresponding superfields with a real mass. The \(cs\) is generated precisely by the real massive fermions in the one loop diagrams. In this case the fivebrane is constructed by moving the two semi-infinite stacks on opposite directions. They have opposite charge under the gauge symmetry implying that a level \(k\) is generated.

In general, by considering a configuration with two fivebranes of the form \((p_1, q_1)\) and \((p_2, q_2)\) we can realize a \(cs\) theory with level \(k = q_1 p_2 - p_1 q_2\). Observe that the sign of \(k\) in our case depends from having sent to infinity the fundamentals or the antifundamental flavors; the opposite sign would reflect into an different choice of the angle, \(\tan(\pi - \phi) = kg_s\).

### 3.3.2 Giveon–Kutasov duality

The duality of Giveon and Kutasov can be now formulated as in the original presentation, in terms of its brane engineering. The electric theory is obtained by considering the \(U(N_c)_k\) theory discussed above and by adding \(N_f\) D3 branes on the right side of the NS brane ending on a stack of \(N_f\) D5 branes. This configuration corresponds to considering \(N_f\) pairs of fundamentals and antifundamentals \(Q\) and \(\tilde{Q}\). The dual configuration is obtained by considering the \(hw\) transition. Note, however that the presence of \(k\) D5 branes in the \((1, k)\) bound state modifies the rules of the duality: when the \((1, k)\) fivebrane crosses the NS brane, one has to add \(|k|\) D3 branes, independently of the sign of \(k\), because there are always D5 branes in the bound state. The \(s\)-rule is not violated as long as \(N_f + |k| > N_c\) in this case. The dual configuration is shown in Figure 5.2. In this case the \(cs\) level is \(-k\) and the gauge group becomes \(U(N_f - N_c + |k|)_{-k}\). There are \(N_f\) dual pairs of fundamentals \(q\) and antifundamentals \(\tilde{q}\) and a meson \(M\) as in the four-dimensional case, with a superpotential \(W = Mq\tilde{q}\). Note the following subtlety: in this case the \(U(1)\) factor of the gauge group does not decouple in the \(ir\), and we are allowed to consider the baryonic symmetry as gauged. Since from type type II\(B\), there is no uplift to \(M\)-theory we do not expect any bending effect for the NS branes. This reflects the fact that the axial symmetry is not anomalous in three dimensions, and the two rotations \(x_{45}\) and \(x_{89}\) are both realized in the quantum theory.

### 3.3.3 Niarchos duality

A natural extension of \(gk\) duality has been studied in \([18]\), by adding a power law superpotential for the adjoint field \(X\). This generalizes the Giveon–Kutasov duality in the same way as \(kss\) duality generalizes Seiberg duality. In this case,

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\(^6\) Remember that there are no anomalies in three dimensions.
one considers a configuration with one NS’ and \( n \) NS branes and \( N_c \) D3s stretched along the segment connecting them. One can also add flavor by considering \( N_f \) D3 branes ending on \( N_f \) D5s on the right of the \( n \) NS branes. By replacing the NS’ brane with a \((1,k)\) fivebrane one realizes the electric theory, i.e. SU\((N_c)_k\) CS matter theory with \( N_f \) pairs of fundamentals and antifundamentals and one adjoint with superpotential \( W = \text{Tr} X^{n+1} \). The duality is obtained as in the four-dimensional case, by the HW transition. In this case we can still proceed as above, first we separate the NS branes along the \( x_{89} \) direction, breaking the gauge group into \( SU(r_1)_k \times SU(r_n)_k \) with \( r_1 + \cdots + r_n = N_c \). Then we dualize each single sector by considering the brane creation effect as above. We have a gauge group \( SU(N_f - r_1 + |k|)_-k \times SU(N_f - r_n + |k|)_-k \) as we can see from Figure 3.3. By reconnecting the NS branes we end up with an \( SU(n(N_f + |k|) - N_c)_-k \) gauge group as expected.
3.3.4 Adding orientifolds

One can also study three-dimensional dualities with cs terms in presence of orientifold planes. We can construct the electric theories by considering the four-dimensional parent theories and substituting $D_p$ branes and $O_p$ planes with $D(p-1)$ and $O(p-1)$ branes. The direction on which these branes are not extended anymore can be fixed to be $x_3$. On the other hand the NS branes are still as above, i.e. extended along the non-compact direction $x_3$. Observe that this is just a mnemonic at this point and it is not related to any compactification or dimensional reduction of a Seiberg duality.

The cs levels are obtained by substituting NS branes with $(1,k)$ fivebranes, consistently with the constraints from the parity anomalies and the s-rule. The dual theories are then constructed by the $hw$ transition. The authors are not aware of any of such construction in the literature so far. For this reason, we will present some simple example in the following to clarify the construction. The reader can
work out further possibilities.

3.3.5 Discussion of the case $k = 0$

We conclude this review section with presenting a problem in describing the Aharony duality and its generalization in terms of the brane engineering. One can imagine to engineer an $U(N_c)_0$ electric theory in terms of $N_c$ D3 branes stretched between an NS and an NS’ brane and adding the usual $N_f$ D3 flavor branes ending on $N_f$ heavy D5 branes. The naive guess would result in a dual theory with $U(N_f - N_c)_0$ gauge group and superpotential $W = Mq\tilde{q}$, but in this description there is no trace of any extra superpotential involving the monopole operators as singlets in the dual description. In order to understand the problem we remind the reader that the monopole operators can be associated to D1 branes extended along the directions $x_3$ and $x_6$ between pairs of D3 branes ending on the NS and the NS’ brane. It is unclear how to enforce such operators as singlets of the dual description. There is however a physical description of the role of the dual superpotential: it should constrain the monopole operators $t$ and $\tilde{t}$. This constrains the freedom of moving the first and the last gauge D4 brane along the infinite
Figure 3.6: Brane engineering for $U(N)$ theories with symmetric and antisymmetric flavors.

Figure 3.7: Brane engineering for $SO$ and $Sp$ theories with adjoint matter.
direction $x_3$. In principle, this is a quantum effect, but in this case we cannot use the help of M-theory to visualize it on the brane picture. We will see that there is another way to understand this effect via a T-duality to type IIA.
4 Reduction of four-dimensional dualities to three dimensions

In this section we review the reduction of four-dimensional Seiberg duality to three dimensions.

In Section 4.1, we review the field theory aspects of the reduction, following largely [7]. We then take the brane perspective in Section 4.2, where in the first step, we explain how to obtain the $\eta$ superpotential from T–duality (Sec. 4.2.1). In Section 4.2.2, we discuss in detail the reduction pf $\text{sqcd}$ from four to three dimensions. The reduction of four-dimensional Seiberg duality with adjoint matter to three dimensions is discussed in Section 4.2.3. Finally, we discuss the reduction of four-dimensional dualities in the presence orientifold planes, leading to real gauge groups and tensor matter.

4.1 Field theory aspects

Here we review the work of [7], where Seiberg duality was reduced to three dimensions. A four-dimensional theory can be naively reduced to three dimensions by compactifying one dimension on a circle and shrinking its radius to zero size. This procedure is well defined and leads to a consistent three-dimensional theory. It turns out however, that Seiberg duality is not preserved under this naive dimensional reduction. Roughly speaking, the duality does not commute with the zero-size limit of the circle. We have seen that Seiberg duality is an IR property, valid for energies much below the holomorphic scales of the electric and of the magnetic theory,

$$E \ll \Lambda, \tilde{\Lambda}. \quad (4.1.1)$$

These scales are related to the four-dimensional coupling through the relation (2.2.4). The gauge coupling of the dimensionally reduced theory is related to the one of its parent theory via

$$g_4^2 = r g_3^2. \quad (4.1.2)$$

We see that in the limit $r \to 0$, the strong coupling scales of the electric and of the magnetic theory are vanishing. For this reason the limit $E \ll \Lambda, \tilde{\Lambda}$ cannot be defined. Postulating a three-dimensional duality inherited from four-dimensional Seiberg duality becomes an empty statement.

That the duality cannot be preserved under dimensional reduction is also obvious when we consider the global symmetries. In the case of four-dimensional $\text{sqcd}$, there is an anomalous axial symmetry $U(1)_A$. When reducing the theory on the circle, this symmetry is however preserved also at the quantum level in three dimensions. In order to preserve the four-dimensional duality in three dimensions, we need a mechanism that prevents the generation of this symmetry.

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1 See also to [83] for an early attempt.
The above problems can be avoided by reducing the dual phases on the circle, but at finite size $r$, and considering the theories at an energy $E \ll 1/r$, where the physics is effectively three dimensional. We have seen that when reducing a vector multiplet to three dimensions, the fourth component $A_3$ becomes a real scalar $\sigma$. Gauge invariance of $\int_S A_3$ requires the identification of $\sigma$ with $\sigma + r^{-1}$, i.e. a compact scalar. The presence of this compact scalar is reflected in the presence of an additional interaction for the unlifted coordinates on the Coulomb branch. This interaction is called the AHW superpotential. If we consider for example a $U(N_c)$ theory, this term becomes

$$W_{\text{AHW}} = \eta T_0 \tilde{T}_0,$$  \hspace{1cm} (4.1.3)

where $\eta = e^{-(g_s)^{-2}}$. Observe that the superpotential (4.1.3) breaks the axial symmetry explicitly.

One can perform the same reduction on the Seiberg-dual theory, and generate the superpotential (4.1.3) for the Coulomb branch coordinates on the magnetic side. This leads to an effective three-dimensional duality between the two theories considered at energies much below the finite radius of $S^1$.

It is possible to obtain Aharony duality from this new effective duality in three dimensions by sending the superpotential (4.1.3) to zero consistently in the dual phases. This limit requires a real mass flow. This is however not enough because, for consistency, often a non-trivial map between the vacua of the electric and the magnetic theory has to be considered. If the vacuum of the electric theory is at the origin of the Coulomb branch, i.e. $\langle \sigma \rangle = 0$, one typically has $\langle \tilde{\sigma} \rangle \neq 0$ for the dual theory. This corresponds to the topological vacuum discussed in [64].

Consider for example $U(N_c)$ SQCD with $N_f + 2$ flavors. We assign real masses to two flavors on the electric side using opposite masses for the fundamental and the antifundamental, such that no $cs$ term is generated in the action. In the dual theory, one can assign the masses to the dual quarks and to the mesons consistently with the symmetries. One is also forced to consider a non-zero VEV for the last two diagonal entries of $\tilde{\sigma}$, breaking the gauge group to $SU(N_f - N_c) \times U(1)^2$. The dual superpotential becomes

$$W = Mq\bar{q} + M_1q_1\bar{q}_1 + M_2q_2\bar{q}_2,$$ \hspace{1cm} (4.1.4)

where the first term is associated to the $U(N_f - N_c)$ sector, while the extra two belong to the two $U(1)$ sectors. The two $U(1)$ sectors correspond to SQED with one flavor and can be dualized to a set of singlets. The superpotential after this duality is given by

$$W = Mq\bar{q} + M_1N_1 + M_2N_2 + N_1t_1\bar{t}_1 + N_2t_2\bar{t}_2,$$ \hspace{1cm} (4.1.5)

where we identified the fields $N_i$, $t_i$ and $\tilde{t}_i$ in each sector with $X$, $Y$ and $Z$. By carefully studying the interaction, we see that there are AHW interactions between the monopoles of the $U(N_f - N_c)$ sector and the monopoles of the $U(1)^2$ sector. This interaction is given by

$$W_{\text{AHW}} = t\bar{t}_1 + t_1\bar{t}_2 + t_2\bar{t}.$$ \hspace{1cm} (4.1.6)

By integrating out the massive fields and by identifying the fields $t_1 = T$ and $\bar{t}_2 = \bar{T}$, i.e. with the monopoles of the electric theory acting as singlets in the magnetic
phase, we get
\[ W = Mq\tilde{q} + tT + i\tilde{t}\tilde{T}. \] (4.1.7)

This shows that the three-dimensional Aharony duality can be obtained by a stepwise procedure from four-dimensional Seiberg duality. First one reduces the duality on the circle, at finite size. Then one has to perform a real mass flow on the electric side while carefully analyzing the vacuum structure of the dual theory in order to recover the right constraint on the Coulomb branch. To conclude, we can also recover the Giveon–Kutasov duality: this is done by assigning some real masses to the flavors of the electric theory, with the same sign, such as to generate a cs term. The real masses of the dual are dictated by the duality and in this case, no topological vacua arise. See [63] for details on this derivation.

### 4.1.1 $U(N_c)$ with adjoint matter

In this section, we generalize the reduction of Seiberg duality to the case of $kss$ duality discussed in Section 2.3.4. By reducing the theory on the circle in presence of an adjoint field with superpotential $\text{Tr} X^{n+1}$, we have $2n$ unlifted directions on the Coulomb branch of the $U(N_c)$ theory.

There is a superpotential contribution coming from the Kaluza–Klein (kk) monopoles in this case as well. As observed in [72], in this case the kk monopoles have too many zero modes and should not give rise to a superpotential. As one can see from the Callias index theorem, there are in total four fermionic zero modes, two coming from the gauginos and two from the adjoint matter fermions. Nevertheless the extra fermionic zero modes from the adjoint fermion can be contracted because the superpotential gives rise to a potential term $X^{n-1}\psi_X^2$. After taking this contraction into account, we are left with the two fermionic zero modes coming from the gauginos, and the kk superpotential for the (dressed) monopoles is generated. We can parameterize these directions in terms of a set of operators $T_i$ and $\tilde{T}_i$ and the $\eta$-superpotential is of the form
\[ W_\eta = \eta \sum_{j=0}^{n-1} T_j\tilde{T}_{n-j-1}. \] (4.1.8)

There is a similar contribution in the magnetic theory, necessary to preserve the duality at finite radius of the circle. We now can proceed as above with a real mass flow, and by identifying the topological vacuum of the dual theory one can reduce this effective duality to the duality of [24]. We can also reduce the duality in a slightly different way, along the lines of [72], by deforming the duality by considering a polynomial superpotential for the adjoint and considering the four-dimensional theory in a vacuum of the adjoint. In this way one has a set of $n$ decoupled $sqcd$-like theories. One can perform the reduction in each sector separately, in both the electric and the magnetic phase. The real mass flow to the duality of [24] can be done in each sector separately. This gives rise to the expected duality in a broken phase. The agreement between the two procedures has been shown in [73]. This second approach will be useful in deriving the reduction of the dualities from the brane dynamics.
4.1.2 Generalizations

The discussion above can be generalized to cases with real gauge groups and with tensor matter.

Let us first summarize the case with real gauge groups and fundamental matter, discussed in [7] for the symplectic case and in [32] for the orthogonal one.

The second case is more involved because it requires the analysis of the global properties of the gauge group and we will not elaborate on this subtlety here. We will restrict the analysis to the gauge algebras $sp(2N_c)$ and $so(N_c)$.

When reducing a theory on the circle, the contribution of the $\text{AHW}$ superpotential can be formulated algebraically as

$$W_\eta = \frac{2\eta}{\alpha_0^*} \Phi,$$  \hspace{1cm} (4.1.9)

where $\alpha_0$ corresponds to the affine root of the Lie algebra and $\alpha_0^*$ corresponds to the associated coroot. We also identified the Coulomb branch coordinates with $\phi = \sigma / e_3^2 + i \gamma$, \hspace{1cm} (4.1.10)

where $e_3$ is the gauge coupling and $\gamma$ is the dual photon (3.1.7).

Once the $\eta$-superpotential is added, we can claim an effective duality on the circle as in the case discussed above. The cases of interest are the $A_N$, $B_N$, $C_N$ and $D_N$ series corresponding to the $su(N)$, $so(2N+1)$, $sp(2N)$ and $so(2N)$ gauge algebras, respectively. Note that here, we refer to the $su(N)$ case, while above we discussed the $u(N)$ case. On the brane side, we will comment on the difference of the interpretation of these cases.

The conventional Aharony dualities are obtained by a real mass flow. In the case of $sp(2N)$, we do not have to consider a topological vacuum on the dual side. The meson $M$ of the dual description in this case is broken into two massless fields. One of them corresponds to the meson of the Aharony dual theory, while the other corresponds to the monopole $T$ that parameterizes the Coulomb branch of the electric theory. By a scale matching relation, one can show that the $\eta$ superpotential becomes the extra contribution of Aharony duality. The final form of the superpotential is

$$W = Mqq + tT.$$ \hspace{1cm} (4.1.11)

In the orthogonal case, the three-dimensional Aharony duality is obtained without considering any real mass flow. In this case, one can eliminate the $\eta$ superpotential in the electric theory safely and in the dual theory one has to consider a topological vacuum. We refer the reader to [32] for details.

Further generalizations are possible. The picture that has emerged so far serves as a guideline for studying the reduction of all the cases discussed in Section 2 in presence of tensor matter and unitary or real gauge group. In presence of tensor matter, one can break the gauge group into $\text{SQCD}$ sectors by deforming the theory with a polynomial superpotential for the tensors. These sectors can have a unitary or real gauge group depending on the details of the model. Then one has to reduce each sector separately, according to the rules explained above. This results in new effective three-dimensional dualities on the circle. One can also flow in each
broken sector to an Aharony-like duality by assigning real masses and choosing the topological vacua when necessary.

This procedure has been used in [84] to study the reduction of the four-dimensional duality for the $Sp(2N_c)$ gauge theory with $2N_f$ fundamentals and one adjoint discussed in Section 2.2.3. The polynomial superpotentials in the electric and in the magnetic theory are given by

$$W_e = \sum_{i=1}^{n+1} \lambda_i \text{Tr} S^{2i} \quad \rightarrow \quad W_m = \sum_{i=1}^{n+1} \tilde{\lambda}_i \text{Tr} S^{2i}.$$  \hspace{1cm} (4.1.12)

In the electric theory we can break $Sp(2N_c)$ to $Sp(2r_0) \times \prod U(r_i)$, with $\sum r_i = N_c$. In each sector, there is an $\eta$-superpotential generated on the circle: for $Sp(2r_0)$ $W = \eta_0 T$ is generated, while for each $U(r_i)$ sector, a superpotential $W = \eta_i T_i \tilde{T}_i$ is generated. The coupling constant $\eta_i$ corresponds to $\Lambda^b_i$. In the magnetic theory, the gauge group $Sp(2N_c)$ is broken to $Sp(2\tilde{r}_0) \times \prod U(\tilde{r}_i)$, where $\sum \tilde{r}_i = \tilde{N}_c$. Also in this case, a $\eta$-superpotential is generated in each broken sector. The reduction to three dimensions is performed via a real mass flow. In the $Sp(2\tilde{r}_0)$ sectors, there is no Higgsing, while in the $U(\tilde{r}_i)$ case, we have to consider the topological vacua. By considering the interactions of the monopoles in each Higgsed phase and by using local mirror symmetry when necessary, we obtain the dual Aharony-like theory. More concretely, the massive singlets are integrated out and the massless ones interact with the monopoles of the $U(\tilde{r}_i)$ theories. These singlets become the electric monopoles expected in the dual superpotential.

### 4.2 The brane picture

In this section, we study the reduction of four-dimensional dualities from the perspective of brane engineering. We will translate the field theory constructions into the brane language, which offers a unified version of the results that have emerged on the field theory side. The reduction is performed via a T-duality along a circle that we will keep finite also after the duality. In this way, we are able to capture the quantum effects leading to the $\eta$-superpotential in the brane picture. It also offers an explanation of the breaking of the axial symmetry in a geometric language. The real mass flow leading to Aharony-like dualities has a clear explanation in the brane picture and the relation to the dualities in the new sectors obtained from the topological vacua can be explained in terms of a local mirror symmetry. In this way, we are able to give a brane realization of Aharony duality explaining the role of the extra monopoles in the dual phase.

We start our analysis with a general review of the reduction by discussing the generation of the non-perturbative superpotential for $\text{sqCD}$ at finite radius from the brane perspective. We start by discussing the case of pure $\text{sym} (N_f = 0)$. This analysis introduces two of the main characters, namely T–duality and D1 monopoles, that will be very useful in the following. After this, we extend the discussion to the case with fundamental matter and describe the reduction of the four-dimensional duality for $U(N_c) \text{ sqCD}$, essentially following the discussion in [85]. Then we explain the generalization to the case with an adjoint field. In the
Our goal is to explain the different three-dimensional limits discussed on the field theory side. From the brane description, we will see that we naturally require a double-scaling limit of the radius of the circle and the real mass in order to recover the standard three-dimensional dualities.

### 4.2.1 The $\eta$ superpotential from T-duality

#### Gaugino condensate

To introduce the relation between D branes and monopole operators in three dimensions, we start with a discussion of gaugino condensation in four dimensional $\mathcal{N} = 1$ sym in terms of brane constructions. This derivation has been presented in [87] and we refer the reader there for details.

Consider four-dimensional $\mathcal{N} = 1$ $SU(N_c)$ sym on $\mathbb{R}^3 \times S^1$. The supersymmetric vacuum can be expressed in terms of a gaugino condensate. This gaugino bilinear has been obtained in [88, 89] from an instantonic contribution in four dimensions. Equivalently, on $\mathbb{R}^3 \times S^1$, one obtains the same result by splitting the instantons in terms of $N_c$ monopoles [90]. In the three-dimensional effective description, $N_c - 1$ of them are Bogomol’nyi–Prasad–Sommerfield (bps) monopoles while the $N_c$-th one is usually called the kk monopole. We will not discuss this construction and refer the reader to the original reference for details.

Interestingly, the same effect can be reproduced via the brane engineering of the theory in type iia. As discussed in Sec. 2.3, the $\mathcal{N} = 1$ sym theory is described by an NS5 brane, an NS5' brane and $N_c$ D4 branes extended as shown in Table 4.1. The D4s are suspended between the NS5 and the NS5'. The four-dimensional gauge coupling is $e^2_4 = g_4^2 / \ell_6 = (2\pi)^2 \sqrt{\alpha'} / \ell_6$, where $\ell_6$ is the distance between the NS5 branes and $g_4^2 = (2\pi)^2 \sqrt{\alpha'}$ is the D brane coupling constant.

#### $\eta$ superpotential

If $x_3$ is compact, $x_3 \sim x_3 + 2\pi R_3$, we can perform a T-duality along that direction. In the resulting type iib frame, the D4s have turned into D3 branes while the NS and NS' branes are unchanged. This configuration describes $U(N_c)$ sym on $\mathbb{R}^3 \times S^1$. As shown in [34], the entire Coulomb branch is lifted and there are $N_c$ isolated vacua. To see that in the string theory picture, observe that the vacua correspond to stable supersymmetric configurations of the brane system. Again, in absence of D5 branes there is a repulsive force between the D3s so that the unique stable configuration is obtained when the D3 branes are distributed along $x_3$ at equal
All moduli are lifted as the D3 branes cannot move freely due to the repulsive force.

The repulsive force is a non-perturbative quantum effect. From the three-dimensional field theory point of view, a non-perturbative superpotential induced by three-dimensional instantons is generated. These instantons are monopole configurations of the four-dimensional theory. In the brane picture, these monopoles are represented by Euclidean D1–strings stretched between each pair of D3 branes and the NS and NS‘ branes, as depicted in Figure 4 as shaded area along $x_6$ and $x_3$, respectively. The effect of the monopoles has been computed in [27, 70] and is given by the exponential $e^{-S}$, where $S$ is the D1 world-sheet action. This action has two pieces, the Nambu–Goto part and a contribution from the boundary of the D1s:

- The Nambu–Goto term is proportional to the area of the D1 and involves the scalar $\sigma$ that parameterizes the position of the D3;
- The boundary term is proportional to the dual photon $\gamma$.

Putting them together, we obtain

$$W = \sum_{i=1}^{N_c-1} e^{\Phi_i - \Phi_{i+1}},$$

where $\Phi_i = \sigma_i / e_3^2 + i \gamma_i$ and $e_3^2 = (\sqrt{\alpha'} / R_3)(g_5^2 / \ell_6) = 2\pi \sqrt{\alpha'} / (R_3 \ell_6)$ is the three-dimensional gauge coupling. The result can be naturally expressed in terms of Coulomb branch coordinates and we see explicitly the monopole operators $e^{\Phi_i}$.

Now, if $x^3$ is compact there is another contribution from D1 branes stretching from the $N_c$th to the 1st D3 branes, which are respectively at $\sigma_{N_c}$ and $\sigma_1 + 2\pi R_3$. By following the calculation of [87, 91] one finds

$$W_\eta = \eta e^{\Phi_{N_c} - \Phi_1},$$

where $\eta$ incorporates the radius dependence from the position of the first D3 brane. This is the $\eta$–superpotential (4.1.9) which plays an essential role in the reduction from four to three dimensions as shown in Section 4.1. From the point of view of field theory, the Euclidean D1 branes are the BPS monopoles and the one coming from the extra D1 is the KK monopole.

**Affine algebra and Toda chain**

There is an algebraic approach which allows an elegant description of this construction. The fundamental (in the sense of [92, 93]) BPS monopoles are labeled by the simple co-roots of the Lie co-algebra. For unitary gauge groups $G$ this corresponds to placing the $i$–th D1 brane between the $i$–th and the $(i + 1)$–th D3 brane (for $i = 1, \ldots, \text{rank}(G)$). It is useful to study the S–dual configurations where D1 branes become F1–strings. In this picture the D3 branes are still distributed on the circle and connected by F1–strings, as depicted in the upper left corner of Figure 4.2. The spectrum of the allowed F1–strings is given by the simple roots of the corresponding Lie algebra [94, 95]. For a unitary gauge group the simple roots of

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2 Here $g_5^2 = 2\pi$, and $\sqrt{\alpha'}/R_3$ is the contribution of the type IIB dilaton so that $e_4^2/e_3^2 = 2\pi R_3$. 



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Figure 4.1: D1 branes (in gray) stretched between D3 branes with compact $x_3$. The D5 branes are represented by $\otimes$ symbols, the NS5 by a continuous line and the NS5' by a dashed line.

the $A_N$ series correspond to $\sigma_i - \sigma_{i+1}$, i.e. to the difference between the positions of two consecutive D3 branes.

The picture incorporates very naturally the $\mathbb{K}\mathbb{K}$ monopoles due to the compact direction. It turns out that the extra F1 string, which winds around the circle connecting, for unitary $G$, the 1st and the $n$-th D3 brane, can be accounted for by extending the Dynkin diagram to its affine version.

Summing the contributions from the bps and the $\mathbb{K}\mathbb{K}$ monopoles we obtain the superpotential on the compact Coulomb branch, finding an affine Toda potential for the associated affine algebra $\tilde{A}_N$.

\[
W(\Sigma) = \sum_{i=1}^{\text{rank}(G)} \frac{2}{a_i^2} \exp[\alpha_i^* \cdot \Phi] + \frac{2\eta}{a_0^2} \exp[\alpha_0^* \cdot \Phi],
\]

(4.2.3)

where the scalar component of $\Phi$ is $\phi = \sigma/e_3^2 + i\gamma$, $\alpha_i$ are the simple roots and $\alpha_i^*$ are the associated co-roots. The extra simple root $\alpha_0$ corresponds to the $\mathbb{K}\mathbb{K}$ monopole and the corresponding contribution to (4.2.3) is commonly referred to as $\eta$--superpotential.

In the unitary case, we have the affine algebra $\tilde{A}_N$, where the extra simple root is associated to the combination $\sigma_N - \sigma_1$ (see Figure 4.2).

For $SU(N_c)$ theories, the superpotential associated to this diagram is

\[
W = \sum_{i=1}^{N_c-1} \frac{1}{Y_i} + \eta Y_{N_c},
\]

(4.2.4)

where $Y_i = e^{i(\sigma_i - \sigma_{i+1})/e_3^2 + i(\gamma_i - \gamma_{i+1})}$. The last term in (4.2.4) breaks explicitly the $U(1)_A$ symmetry in the three-dimensional field theory. This symmetry is associated to the rotation in the (4,5) plane in the brane picture. The geometric realization of the breaking of this symmetry for compact $x_3$ has been discussed in [85].

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In the $U(N_c)$ case the same result holds but the last (affine) root splits in two terms $T = e^{\Phi_1}$ and $\tilde{T} = e^{-\Phi_0}$.
Figure 4.2: Branes and Dynkin diagrams for $A_N$ and $\tilde{A}_N$. The left column shows the S-dual configuration of F1–strings stretched between D3 branes. In the right column we depict the corresponding $A_N$ and $\tilde{A}_N$ Dynkin diagrams. After compactification a new string appears between $\sigma_1$ and $\sigma_N$ and corresponds to the affine node (in blue in the brane cartoon and in the Dynkin diagram).

**Matter fields**

The picture is richer if we introduce matter fields. Now, two directions of the moduli space remain unlifted, which can be seen from the brane picture as follows. In the type IIA frame, fundamental matter is associated to $N_f$ D6 branes extended along 0123789 and sitting on the NS$'$ brane (see Sec. 2.3.2). In the T–dual frame they become D5 branes. Strings stretched between the stack of D3 branes and the D5s correspond to $N_f$ massless fundamentals $Q$ and anti-fundamentals $\tilde{Q}$ (see Sec. 3.3).

When D5 branes sitting at $x_3 = 0$ intersect the worldsheet of the D1–strings, they contribute two additional zero modes to the D1–instanton and the superpotential in Eq. (4.2.1) is not generated. In this sense, the D5 branes screen the repulsive force between the D3 branes [48, 97]. Without loss of generality, we can chose the screening to happen between the first and the $N_c$th D3 brane, which are now free to move without being subjected to any force. One modulus is nevertheless lifted by the interaction mediated by the superpotential Eq. (4.2.2).

This is different from the non-compact case where there is no superpotential (4.2.2) and the forceless motion of the 1st and the $N_c$th D3 brane leads to a two-dimensional moduli space.

### 4.2.2 Reduction of SQCD

Having understood the origin of the $\eta$ superpotential, we can now consider the brane realizations of the four-dimensional dualities and relate them to three dimensions via T–duality.

As a first example, consider the reduction of four-dimensional Seiberg duality
Table 4.2: Brane configuration for $\mathcal{N} = 1, d = 4$ sqcd with $N_f$ flavors. The $N_c$ D4 branes are extended between the NS5 and the $N_f$ D6 branes sit on the NS5'.

|       | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-------|---|---|---|---|---|---|---|---|---|---|
| NS    | × | × | × | × | × |   |   |   |   |   |
| NS'   | × | × | × | × |   |   |   | × |   |   |
| D4    | × | × | × | × |   |   |   |   | × |   |
| D6    | × | × | × | × |   |   |   |   |   | × |

Table 4.3: Brane configuration for $\mathcal{N} = 2, d = 3$ sqcd. The $N_c$ D3 branes are extended between the NS5 and the $N_f$ D5 branes sit on the NS5'.

|       | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-------|---|---|---|---|---|---|---|---|---|---|
| NS    | × | × | × | × | × |   |   |   |   |   |
| NS'   | × | × | × | × |   |   |   |   | × |   |
| D3    | × | × | × |   |   |   |   |   | × |   |
| D5    | × | × | × |   |   |   |   | × |   |   |

for sqcd to three dimensions. It is convenient to consider the four-dimensional gauge symmetry to be $U(N_c)$ rather than $SU(N_c)$. In field theory, this enhancement is obtained by gauging the baryonic symmetry (see Sec. 2.1.2).

As in Sec. 2.3.2, we start by considering a stack of $N_c$ D4 branes, one NS5 brane, one NS5' brane and $N_f$ D6 branes. In this type IIA description, the branes are extended as shown in Table 4.2. The two possible configurations with the NS5 either on the left or the right of the NS5' are related by a $hw$ transition (Sec. 2.3.3). What we want to do is to compactify the two phases along $x_3$ and study the resulting theories at finite radius.

**Duality at finite radius**

If we consider $x_3$ to be compact, we can perform a T–duality and obtain an effective three-dimensional $\mathcal{N} = 2$ theory. In this case, the D4 and the D6 branes become D3s and D5s respectively, while the NS and NS' branes are left unchanged. In Figure 2.3, the electric and magnetic Seiberg–dual theories are shown. In the type IIB description the branes are extended as shown in Table 4.3. This is a $U(N_c)$ gauge theory with $N_f$ fundamentals and anti-fundamentals as discussed in Section 4.2.1 with global symmetry group $SU(N_f)_L \times SU(N_f)_R \times U(1)_R \times U(1)_J$. The axial symmetry $U(1)_A$ under which $Q$ and $\tilde{Q}$ have the same charge is broken by the superpotential in Eq. (4.2.2).

Let us see how the global symmetries are realized in the brane picture and how having a compact $x_3$ direction is the geometric interpretation of the fact that the three-dimensional theory does not have an axial symmetry, as discussed in Sec. 4.1. For ease of exposition, we concentrate on the electric frame.

- The rotations in the $(4,5)$ and in the $(8,9)$ plane discussed in section 2.3.2 still correspond to the $R$–symmetry and to the axial symmetry. While admissible in
three dimensions, on $\mathbb{R}^3 \times S^1$ the latter is broken as will be discussed in the next paragraph.

- The non-Abelian flavor symmetry comes from the stack of $N_f$ D5 branes. The branes can be broken at the intersection with the NS5', leading to two semi-infinite stacks, $D5_L$ and $D5_R$, one extended in $x_7 > 0$ and the other in $x_7 < 0$: this corresponds to the fact that the symmetry has a notion of chirality. The flavor group is $U(N_f) \times U(N_f)$. One combination of the two $U(1)$ factors appears as baryonic and one as axial symmetry. The $U(1) \subset U(N_f)$ real mass, i.e. the vev of the scalar in the corresponding $U(1)$ vector multiplet, is a collective shift of the stack of semi-infinite D5s. The axial $U(1)$ subgroup in $U(N_f) \times U(N_f)$ comes from separating the two stacks and moving them in opposite directions; the diagonal subgroup corresponds to shifting both stacks of D5 together. This is the same as a shift of the stack of the D3 branes, which is a $U(1) \subset U(N_c)$ gauge transformation (the gauged baryonic symmetry). The separation of the stacks acts on the fundamentals and the anti-fundamentals (this is the axial symmetry) and corresponds to turning on a real mass for $U(1)_A$.

On a circle (when $x_3$ is compact) the $U(1)_A$ is broken. In the brane picture this breaking can be visualized as follows. When moving the stack of $D5_{LS}$ in $x_3 < 0$ and the stack of $D5_{RS}$ in $x_3 > 0$, charge conservation requires the generation of a $(1,N_f)$ fivebrane along the directions $x_3$ and $x_7$ (98,99) (see Figure 4.3). The NS brane now cannot close anymore on the circle without breaking supersymmetry. This obstruction precludes the realization of the $U(1)_A$ symmetry in the brane setup. As we have seen before, a compact $x_3$ leads to an $\eta$-superpotential, here we find a geometrical origin for the breaking the $U(1)_A$ symmetry.

The situation is different for $U(1)_B$. This symmetry is realized by the motion of the entire stack of D5 branes with respect to the stack of D3 branes along $x_3$. The D5 branes can slide together on the NS brane along $x_3$, without generating any $(1,N_f)$ fivebrane, and the symmetry is still realized in the brane picture. Observe that in the three-dimensional picture this symmetry is gauged since we can understand it as coming from the motion of the D3 branes as opposed to the motion of the D5 branes.
• The last symmetry is the topological $U(1)_J$ that shifts the dual photon. This is an effect that is not visible in the type IIB description and a geometric interpretation would require the lift of the configuration to M–theory where the dual photon is associated to the $x_{10}$ direction. This is not surprising, since already in gauge theory this symmetry is not manifest in the Lagrangian but comes from the Bianchi identity. In the type IIB description, however we still have control of the real mass parameter associated to this symmetry. This is the $\tilde{t}$ term coming from to the displacement of the NS and NS$'$ branes along $x_7$.

The same reduction can be performed in the magnetic picture where the discussion is essentially equivalent. The superpotential $M \tilde{q} \tilde{q}$ is combined with the $\eta'$ term. At finite radius, this theory can be treated as an effective three-dimensional theory if the radius of the T–dual circle is large enough, and in this sense we have a new three-dimensional duality.

Flowing to Aharony duality

In Section 4.4 we have seen how to flow to Aharony duality by turning on real mass terms for some of the quarks. Now we would like to reproduce this flow in terms of the brane engineering, where the real masses correspond to the motion of the D5 branes in the $x_3$ direction, as discussed in the previous section.

Start with $N_f + 2$ flavors. The flow is generated by breaking the $SU(N_f + 2)^2$ flavor symmetry down to $SU(N_f)^2 \times U(1)_A$. At energy scale $E < 1/\tilde{R}_3$ (where $\tilde{R}_3 = \alpha'/R_3$ is the T–dual radius), $x_3$ is effectively non-compact. This is the large-mass limit in which the $\eta'$–superpotential disappears and the axial $U(1)_A$ is restored. In the brane description we obtain this breaking by moving one D5 brane in the $x_3 > 0$ direction and one D5 in the $x_3 < 0$ direction (we put the stack of $N_c$ D3 branes at $x_3 = 0$). This is the configuration in Figure 4.4(a). The result is a $U(N_c)$ gauge theory with $N_f$ fundamentals and anti-fundamentals and vanishing superpotential, namely the electric theory studied by Aharony in [7].

The dual picture is shown in Figure 4.4(b). Now each D5 brane drags one D3 along the $x_3$ direction and we have three separate gauge sectors that form
$U(N_f - N_c) \times U(1)^2$, as expected from field theory. The chiral multiplets connecting the two sectors acquire a large mass whenever the two D5 branes are separated along $x_3$. Now we have one fundamental and one anti-fundamental massless flavor and a meson with a superpotential interaction in each $U(1)$ sector. The three sectors interact on the Coulomb branch. As we discussed above, the D3s separate along $x_3$ at equal distance. In the large-mass limit, when the two D5s are far away in the $x_3$ direction, two D3s of the stack of $N_f - N_c$ D3 branes are free to move, and this parameterizes the Coulomb branch. While in the electric phase, these branes can be pushed to infinity, in the magnetic phase there are the two extra D3-branes representing the $U(1)$ sectors. This explains the interaction between the $U(N_f - N_c)$ sector and the $U(1)$ sectors, mediated by an AHW superpotential, coming from the broken $U(N_f - N_c + 2)$ theory. In the brane picture this can be understood as coming from D1-strings extended between the NS branes, the two D3s that are pushed far away in the $x_3$ direction by the D5s, and the two D3s that parameterize the Coulomb branch of the $U(N_f - N_c)$ sector.

This can be made precise in analogy with the construction discussed in Section 4.2.4. The non-perturbative AHW superpotential due to the D1 branes is

$$W_{\text{AHW}} = e^\Phi - \Phi_N + e^{\Phi_1 - \Phi_N} + \eta e^{\Phi_2 - \Phi_N} = \tilde{t}_1 t + \tilde{t}_2 t + \eta \tilde{t}_2 t_1,$$  

(4.2.5)

where $\Phi^1$ ($\Phi^2$) is the Coulomb branch coordinate of the 1st (2nd) $U(1)$ sector. Similarly, $\Phi_1$ and $\Phi_N$ are the Coulomb branch coordinates that remain unlifted by the potential (4.2.1).

In the limit of large $R_3$, the two $U(1)$ sectors at $x_3 > 0$ and $x_3 < 0$ can be dualized separately. It is known that the hw exchange of the NS and the D5 branes in a $U(1)$ gauge theory with one pair of fundamental and anti fundamental leads to the xyz model. Our $U(1)$ sectors are similar to the gauge theory dual of the xyz models, but for the additional superpotential interaction $M^{(i)} q^{(i)} \bar{q}^{(i)}$ in each sector $i = 1, 2$. After the duality we obtain a deformation of the xyz model, where the singlets $X, Y$ and $Z$ are identified with the meson $N^{(i)} = q^{(i)} \bar{q}^{(i)}$ and the Coulomb branch coordinates $t, \tilde{t}_i$. At the end of the day, the superpotential in each sector is

$$W_m^{(i)} = M^{(i)} N^{(i)} + N^{(i)} t \tilde{t}_i.$$  

(4.2.6)

Putting it all together, after integrating out the massive fields, the superpotential $W^{(1)}_m + W^{(2)}_m + W_{\text{AHW}}$ becomes the interaction $t T + \tilde{t} \tilde{T}$ expected for the Aharony duality. The fields $I_1$ and $I_2$ have become the singlets $T$ and $\tilde{T}$ describing the monopole operators of the electric phase.

If we do not take $R_3$ to be infinite, the two branes reconnect on the circle, at the mirror point $x_3 = x_3^0$. The magnetic dual is then obtained by a hw transition, generating a confining gauge theory at $x_3^0$. This is a $U(2)$ gauge theory with two flavors. The three-dimensional limit is recovered with a double scaling on the relative positions of these D branes at $x_3^0$ and the radius of the circle. The new three-dimensional confining sector in the magnetic theories is locally dualized to its confined counterpart. This is a local mirror symmetry⁴. The singlets of this

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⁴ See [100] for a related discussion of local mirror symmetry.
confining sector are coupled to the monopole of the gauge symmetry living at \( x_3 = 0 \), reproducing the gauge theory duality in the pure three-dimensional limit.

In the reduction of \( \text{sqcd} \), the double-scaling limit is just an alternative description. It will become more important because it provides a unifying description when discussing more general dualities.

**Giveon–Kutasov duality**

We can now move on to discuss the \( \mathcal{R} \mathcal{C} \) flow of four-dimensional Seiberg duality to \( \mathcal{GK} \) duality for three-dimensional \( \text{U}(N_c)_k \text{ sqcd} \). Start with \( N_f + k \) D5 branes and separate \( k \) D5s on the NS' brane to obtain two semi-infinite stacks. If we move half of the stack to \( x_3 > 0 \) and the other half to \( x_3 < 0 \), a \((1,k)\) five-brane is created and in the limit when the \((1,k)\) fivebrane becomes infinite, a \( cs \) term is generated \[98, 99\] (we had already encountered this effect when discussing the breaking of axial symmetry in Sec.\[4.2.2\]). This reproduces the electric side of the \( \mathcal{GK} \) duality. In the dual phase, the construction is more involved. The number of D3s created by this process is still \( N_f + k + 2 \), the gauge theory is now \( U(1)^2 \times U(N_f - N_c + k) \). The motion of the D5s along \( x_3 \) generates a non-trivial \( \mathcal{F} \) term for the \( U(1) \) sectors. This term is proportional to the axial real mass (since in the flow \( U(1)_j \) and \( U(1)_A \) mix). This gives a mass to the monopoles and finally one is left with the dual \( \mathcal{GK} \) configuration.

**4.2.3 The case of Adjoint SQCD**

Consider now the reduction of four-dimensional Seiberg duality with adjoint matter to three dimensions.

In the brane description we introduce a set of \( n \) NS5 branes instead of only one. As discussed above, this induces the superpotential \( W = \text{Tr} X^{n+1} \). The electric theory is given in Figure \[4.5(a)\]. First we separate the \( n \) NS branes to induce a superpotential in the adjoint and break the gauge symmetry to \( \prod_{i=1}^{n} U(r_i) \). This corresponds to separating the \( N_c \) D3 branes along \((8,9)\). Finally, we move the NSs and the D3 branes in \((8,9)\) and come back to the original stack (Figure \[4.5(b)\]).
Repeating the procedure outlined above, we T–dualize along $x_3$ so that the D4 and the D6 branes turn into D3s and D5s, while the NS branes are unchanged (Figure 4.6). The electric configuration (Figure 4.6(a)) consists of a set of $n$ decoupled SQCD models, each with $N_f$ flavors. The corresponding gauge groups are $U(r_i)$, with $\sum_{i=1}^{n} r_i = N_c$. There is no adjoint matter anymore. Just like in the previous construction, the superpotential (1.1.9) is generated since $x_3$ is compact. The same procedure can be implemented in the magnetic case. This reproduces the duality with the $\eta$–superpotential that we have discussed above. The flow to the KP duality is obtained by separating the D5 branes along $x_3$, just like in the case of SQCD. The final duality is obtained by reconstructing the stack of $n$ NS branes, which sends the $\eta$-superpotential to zero.

**S-confining theories**

We wish to conclude this section with a comment on the case $N_f = N_c + 1$. In this case, there is no Seiberg duality in four dimensions, but at low energy there is an effective description of this confining theory in terms of mesonic and baryonic operators. This case can also be visualized as a limiting case of Seiberg duality. At the brane level, one can understand the effective theory by studying the role of stringy D0 instantons on the D4 gauge branes after performing an HW transition. The reduction of confining gauge theories in three dimensions has been discussed in [101,102], while the interpretation in terms of D–branes has been given in [103], by elaborating on the role of the stringy instanton after T-duality.
4.2.4 Orientifolds

In this final section we discuss the reduction of four-dimensional dualities in presence of real gauge groups and tensor matter. In the geometric picture this implies adding orientifold planes to the brane systems.

First we will discuss how the orientifold modifies the Coulomb branch and then we will see how this affects the reduction of the dualities, comparing to the unitary cases discussed above.

Monopole superpotentials from the branes

As we have seen in Section 4.2.1, the repulsive interaction between the D3 branes can be understood in terms of Euclidean D1 branes stretched between the NS and the D3 branes and the η-superpotential is reproduced by the extra D1 brane between the first and the last D3. Moreover, an algebraic analysis of the superpotential leads to the notion of the Toda chain for the affine ˜Aₙ algebra (Eq. (4.2.3)).

A similar analysis is possible for real gauge groups realized with O₃ or O₅ planes. Let us first discuss the configurations with O₃ planes on R³ × S¹. As reviewed in Sec. 2.3.5, the four possible orientifolds O₃⁺, O₃⁻, O₃⁺ and O₃⁻ project a unitary group to Sp(2Nₖ), SO(2Nₖ), Sp(2Nₖ) and SO(2Nₖ + 1). If one of the directions is compact, orientifolds must come in pairs and in our case there are six possible pairs [72]. Brane configurations with (O₃⁻, O₃⁻), (O₃⁻, O₃⁻) and (O₃⁺, O₃⁺) are associated to affine Dynkin diagrams. The other pairs (O₃⁺, O₃⁻), (O₃⁺, O₃⁻) and (O₃⁻, O₃⁻) correspond to twisted affine Dynkin diagrams. Here we are interested only in the “affine” pairs, since they can be obtained via T-duality from type IIA configurations with O₄ planes. The same holds for O₆ planes. Now a O₅⁺ is associated to an SO(Nₖ) and O₅⁻ to an Sp(2Nₖ) gauge group. The pairs (O₅⁺, O₅⁻) are obtained by T-duality from a type IIA configuration with O₆ planes.

In the presence of matter fields, we have seen how the two stacks of D5 branes reconnect at the mirror point x₃, and in the limit R₃ → 0 they correspond to a set of massive singlets. On the magnetic side, a D3 is created whenever a D5 crosses an NS in the hw transition, generating an extra gauge theory at x₃, which reproduces the topological vacua of the magnetic field theory.

In the following we consider the effects of orientifold planes in this picture. Their D-brane charge modifies the standard brane creation effect in the hw transition. The extra sector at x₃ in the magnetic theory remains confining, and we can use the same picture to describe the reduction of dualities with orientifolds from four to three dimensions.

Sp(2Nₖ) with fundamentals

Let us move now to the duality for Sp(2Nₖ) with 2N_f fundamentals.

There are two ways to engineer this theory, by either considering an O4⁺ plane, or an O6⁻ plane. The two resulting theories differ in the representation of the matter fields under the global symmetries. When adding a larger number of NS branes,

5 For simplicity we do not distinguish between O₃⁺ and O₃⁻ planes.
the two constructions give rise to different two-index matter fields, adjoint or antisymmetric, which we will study separately.

- In the $O4^+$ plane case we consider a stack of $2N_c$ D4 branes and an $O4^+$ plane stretched between an NS and an NS$'$ brane. We consider also $2N_f$ D6 branes on the NS$'$ brane. The naive construction has an $SO(2N_f)$ global symmetry that is enhanced to $SU(2N_f)$ by the same mechanism that doubles the global symmetry for unitary gauge groups (see Sec. 3.3.1). The dual theory is obtained by a $hw$ transition that exchanges the NS and the NS$'$ branes. Every time a D6 brane crosses a non-parallel fivebrane, a D4 brane is generated. We need to pay attention to the linking number: an $O4^+$ has to be treated as a stack of $-4$ D6 branes [97]. After the transition we obtain the dual picture, in which the net number of D4 branes is $2(N_f - N_c - 2)$.

- A similar theory can be constructed by using an $O6^-$ plane. Consider two NS5 branes, $2N_c$ D4s, $2N_f$ D6’s and an $O6^-$ plane as in Figure 4.8(a). If all the NS5 branes are parallel, the system has $\mathcal{N} = 2$ supersymmetry. The orientifold projects the $SU(2N_c)$ gauge group to $Sp(2N_c)$ (where, as usual, $Sp(2) \simeq SU(2)$). The theory has an $SU(N_f)$ global symmetry with $F$ flavors. We expect this symmetry to be enhanced to $SU(N_f)^2$. Here we rotate the NS5 branes and the D6$'$ branes by an angle $\theta$ as in Figure 4.8(a), and we have two stacks of NS$\pm\theta$ and D6$\pm\theta$. For generic angles, the $\mathcal{N} = 2$ adjoint is massive. If $\theta = \pi/2$ the orientifold is parallel to the NS$\pm\theta$ branes and this field is massless and has to be considered in the low-energy spectrum. This model (for $\theta \neq \pi/2$) has a dual description as discussed above. In this case the $O6^-$ behaves like a stack of $-4$ D6 branes in the $hw$ transition. The brane picture becomes the one shown in Figure 4.8(b) where the dual gauge group is again $Sp(2(N_f - N_c - 2))$.

**O3 planes.** Once more, the three-dimensional system is obtained by compactifying the $x_3$-direction and $T$-dualizing. The novelty is that the $O4^+$ plane becomes a pair of $(O3^+, O3^+)$ planes. In this case the superpotential can be read off from the top half of Figure 4.9:

$$W = \sum_{i=1}^{N_c-1} \frac{2}{Y_i} + \frac{1}{Y_N}, \quad (4.2.7)$$
Figure 4.8: Brane cartoon for the realization of an $Sp(2N_c)$ theory in the (a) electric and (b) magnetic phase with an O6 plane.

Figure 4.9: Dynkin and affine Dynkin diagrams and spectrum of BPS F1–strings associated to the fundamental monopoles for $Sp(2N)$ theories in the linear case and on the circle. The affine root is represented in blue on the affine Dynkin diagram and in the brane cartoon.
where \( Y_i = e^{(\sigma_i - \sigma_i+1)/\epsilon_3^2 + i(\gamma_i - \gamma_{i+1})} \) and \( Y_{Nc} = e^{2(\sigma_{Nc}/\epsilon_3^2 + i\gamma_{Nc})} \). The extra root in the affine case is proportional to the variable \( Y_0 = e^{2(\sigma_1/\epsilon_3^2 + i\gamma_1)} \) (shown in blue on the bottom half of Figure 4.9) and it gives rise to the \( \eta \)-superpotential

\[
W = \eta e^{2\gamma_1/\epsilon_3^2 + 2i\gamma_1}.
\]  
(4.2.8)

The same result is obtained after the hw transition.

Now we want to flow to Aharony duality. We start by considering 2\((N_f + 1)\) D5 branes in the electric theory. We separate two D5 branes from the stack and move them in opposite directions along the circle to reconnect them on the other side. Since the D5s intersect the NS brane in this configuration, there are no massless fields in this extra sector. If we take the \( r \to 0 \) limit for this configuration, we obtain an \( Sp(2N_c) \) theory with 2\( N_f \) fundamentals.

Next we turn to the dual theory. In this case, if we perform a hw transition, there are 2\((N_f - N_c - 1)\) D3s at the origin. On the other side of the circle, the two D3s created by the D5s crossing the NS brane are destroyed by the extra orientifold plane located there. This example shows one of the general aspects of our analysis. In principle, it is not necessary to reconnect the D5 branes at \( x_3 = x_3' \). Here, in order to avoid the orientifold creating a negative number of D3 branes in the hw transition, we have to reconnect the D5s at \( x_3' \).

Even if there is no gauge symmetry, an extra meson arising from the D5 brane remains massless. This suggests that we cannot simply decouple this sector before considering the effect of this massless field in the ordinary dual gauge theory. In fact in this case, the two D5 branes attract the branes labeled by \( \sigma_1 \) and \( -\sigma_1 \) in the dual gauge sector. This attractive force is reflected in the scale-matching relation between \( Y_1 \) and the meson \( M_{2N_f+1,2N_f+2} \). It corresponds to the superpotential interaction

\[
W = \tilde{\eta} y_{low} M_{2N_f+1,2N_f+2} ,
\]  
(4.2.9)
i.e. the low-energy description of the \( \eta \)-superpotential. In the large-mass limit the effect of this interaction has to be considered.

This reproduces the field theory expectation: the dual theory is an \( Sp(2(N_f - N_c - 1)) \) theory with 2\( N_f \) fundamentals, an antisymmetric meson \( M \) and superpotential

\[
W = Mqq + tT ,
\]  
(4.2.10)

where we identified the broken component of the electric singlet \( M \) that parameterizes a direction in the dual Higgs branch, with the electric monopole \( T \) that parameterizes the Coulomb branch of the electric phase. This is commonly the case when dealing with mirror symmetry. In fact, the electric singlet describes the Higgs branch of the dual phase, i.e. the Coulomb branch of the electric theory.

**O5 planes.** In the case of the O6 plane realization the reduction works essentially in the same way., We reduce the duality to three dimensions by compactifying the \( x_3 \)-direction. After T-duality, the type IIB system contains a pair of O5 planes and describes a theory with the same \( \eta \)-superpotential as above. By considering \( N_f + 2 \) flavors and integrating out two of them, we recover the usual Aharony duality. At
the brane level, this is obtained by introducing $N_f + 2 \, D5_{\pm \theta}$. The real masses are realized like the construction with the O3 plane. The orientifold identification is however different: in this case we have a unitary symmetry. Moving a pair of $D5_{\pm \theta}$ along $x_3$ gives a mass to one flavor.

One can flow to Aharony duality by taking the double scaling limit as above (see Figure 4.10). First we move the D5 branes in the $x_3$--direction, assigning the real masses. We reconnect them on the other side of the circle. They reconnect at $x_3 = x_3^0$, where the second orientifold plane is located. The extra sector does not have massless degrees of freedom, and we can take the $r \to 0$ limit to obtain a three-dimensional $Sp(2N_c)$ theory with $2N_f$ flavors. In the dual picture, after the $\text{hw}$ transition, D3s are created when the branes cross each other. The orientifold cancels two D3s every time an NS5 brane crosses it and the net effect at $x_3 = x_3^0$ is that no D3 branes remain. The final configuration is reproduced in Figure 4.10.

Like in the case with O3 planes, here we have an extra sector with massless singlets (coming from the original mesons). The $r \to 0$ limit has to be taken by considering the effect of this sector on the $Sp(2(N_f - N_c - 1))$ theory. This is the same mechanism introduced above: the $\eta$–superpotential is absorbed in a scale matching, the meson couples with the magnetic monopoles, and in the final three-dimensional dual theory the extra interaction between the electric and magnetic monopoles takes place.

### Other cases

The prescription explained so far can be generalized to other examples with $Sp(2N_c)$ gauge group and tensor matter (see [86] for details). Theories with $SO(N_c)$ gauge groups deserve some more comments. In this case, one has to consider $O^+$ and $\tilde{O}^+$ orientifolds. In the reduction on the circle, the $\eta$ superpotential is associated to an affine $B_N$ algebra for the $SO(2N + 1)$ case and a $D_N$ affine algebra for $SO(2N)$. The flow to Aharony duality does not require any real masses. In the electric theory, one can simply take the large $r'$ limit. In the dual picture, an extra D3 brane is generated by the $\text{hw}$ transition (the orientifold carries D brane charge and for consistency, an extra D3 has to be considered at the mirror point). This reproduces the Higgsing required in field theory. The final form of the Aharony duality is eventually recovered with a local mirror symmetry on this sector.

By now it is clear that we can extract a general recipe to construct three-
dimensional $\mathcal{N} = 2$ dualities from four-dimensional $\mathcal{N} = 1$ Seiberg-like dualities. Consider a system with D4, D6, NS branes (and possibly orientifolds). Separate the stacks of NS branes if present, in order to fix a set of $\text{sqCD}$ sectors. Then compactify the direction $x_3$ and T-dualize along it. By following the rules of the reduction in each single sector, one obtains an effective three-dimensional duality at finite size. The flow to Aharony duality is performed in the $U(N)$, $Sp(2N)$ and $SO(N)$ sectors as explained above and this reconstructs the dualities discussed in Section 3.2. One can also generate $c_3$ terms and flow to the generalizations of the Giveon–Kutasov duality that we presented in Section 3.2.2.
5 4d/3d reduction and localization

A further check for susy dualities independent of the brane picture is provided by the one-loop exact results obtained from localization (see for example \([104]\)). This procedure corresponds to the computation of the partition function of a supersymmetric gauge theory on a curved manifold. The supersymmetric action on the curved manifold can be obtained by a background (super)-gravitational setup and in general it is possible to turn on background vector multiplets for the global symmetries as well \([105]\). The partition function is exact at one loop and can be obtained by modifying the action by a \(Q\)-exact deformation (\(Q\) being the preserved supercharge). This leads to a dramatic simplification of the modes contributing to the partition function and the final result is expressed in terms of the classical action and the one-loop determinants for the matter and for the gauge fields. In some cases, depending on the choice of the background, non-perturbative contributions may arise as well.

5.1 Reducing the four-dimensional superconformal index to the three-dimensional partition function

The partition function appears in general as a matrix integral over the gauge group (in some cases also a sum over the monopole sector may be necessary). Its dependence on the global symmetries is encoded in the dependence on some fugacities or real masses, depending on the manifold and on the dimensionality. Here we focus on the squashed three-sphere \(S^3_b\) partition function for three-dimensional \(\mathcal{N} = 2\) theories \([106]\). The sphere is defined by its action on the Euclidean coordinates \(x^i\),

\[
\frac{x_1^2 + x_2^2}{b^2} + \frac{x_3^2 + x_4^2}{1/b^2} = 1,
\]

where \(b\) represents the real squashing parameter.

Before discussing the details of the three-dimensional partition function we discuss its derivation from four dimensions, along the lines of \([7]\). It has been observed that \(Z_{S^3_0}\) can be formally obtained by suitable limit on the fugacities of the superconformal index. The four-dimensional superconformal index is a Witten-like index, computed by placing a four-dimensional theory on \(S^3 \times \mathbb{R}\) \([107]\) \([108]\). Alternatively it can be defined as the supersymmetric partition function on \(S^3 \times S^1\). The two definitions differ by an overall contribution of the supersymmetric Casimir energy \([109]\), but this discrepancy is irrelevant when checking a duality (provided the anomalies match).

The definition of the Witten index on \(S^3 \times \mathbb{R}\) is given by

\[
I = Tr(-1)^Fe^{-\beta H}(pq)^{\frac{H}{2}}p^{\mu}q^{\nu} - R/2 q^{\mu}p^{\nu} - R/2 \prod_a d_\alpha^a.
\]

(5.1.2)
In the formula above, \( F \) stands for the fermion number, while \( H \) corresponds to the Hamiltonian on the curved manifold. The fugacities \( p \) and \( q \) refer to the \( SU(2)_t \times SU(2)_r \) isometry of the three-sphere and their third spin components are \( j_1 \) and \( j_2 \). The fugacities of \( SU(2)_t \times SU(2)_r \) satisfy the constraints
\[
\text{Re}(pq) = 0, \quad |p/q| = 1, \quad |pq| < 1. \quad (5.1.3)
\]
The \( U(1) \) R-charge is \( R \). The fugacities \( u_\alpha \) are in the Cartan of the global (and gauge) symmetry group. The index is non-vanishing only on states with \( H = 0 \). The index can be computed in two steps: first one calculates the single particle index and then takes the so-called plethystic exponential. In localization, this is equivalent to calculating the one-loop determinants for the matter and the vector multiplets. They can be formulated in terms of elliptic Gamma functions by
\[
\Gamma_e(y; p, q) = \frac{\prod_{j=0}^{\infty} (1 - p^{j+1}q^{j+1}/y)}{\prod_{j=0}^{\infty} (1 - p^jq/\pi^j)}. \quad (5.1.4)
\]
Here \( y \) is a collective fugacity for both global and local symmetries. The last step consists in integrating the index over the holonomy of the gauge group, obtaining
\[
I_G = \frac{\kappa G}{|W|} \int_{T^G} \frac{dz_i}{2\pi i z_i} \prod_{\alpha \in G_+} \Gamma_e^{-1}(z^{\pm\alpha}) \prod_{\rho \in R_l, \tilde{\rho} \in R_l} \Gamma_e(z^{\rho}(p)^{\tilde{\rho}}), \quad (5.1.5)
\]
with \( \kappa = (p; p)(q; q) \) and \( (x; p) = \prod_{k=0}^{\infty} (1 - xp^k) \). We used the notation \( \alpha \in G_+ \) for the positive roots of the gauge group \( G \), while \( |W| \) is the dimension of the Weyl group. The weight \( \rho_l \) refers to the representation of each multiplet under the gauge group, while \( \tilde{\rho}_l \) stands for the flavor symmetry. The fugacities \( z \) and \( u \) are in the Cartan of the gauge and of the flavor symmetry, respectively. The integral in (5.1.5) is over \( T^G \), corresponding to the maximal Abelian torus of the gauge group. The \( R \) charges are denoted by \( R_l \).

The next step consists in obtaining the partition function on \( Z_{S^3_b} \). One can use a direct approach and derive it from a purely three-dimensional perspective, as done in [49, 77, 78, 106]. Here, we prefer to review the derivation from the \( \kappa \kappa \) reduction of the superconformal index on \( S^3_b \times S^1 \) following the discussion of [2] (see also [24, 83, 110, 111]). Let \( \tilde{r}_1 \) be the radius of \( S^1 \), the fugacities that enter in the definition of the index become
\[
p = e^{2\pi i \tilde{r}_1 \omega_1}, \quad q = e^{2\pi i \tilde{r}_1 \omega_2}, \quad u_\alpha = e^{2\pi i \tilde{r}_1 \mu_\alpha}, \quad z_i = e^{2\pi i \tilde{r}_1 \omega_i}. \quad (5.1.6)
\]
The parameters in the RH sides of (5.1.6) represent the real scalars of the local and of the background symmetries of the three-dimensional vector multiplets. The parameters \( \omega_1, 2 \) satisfy the relations \( \omega_1 = ib \) and \( \omega_2 = ib^{-1} \) and \( \omega \equiv \omega_1^2/\omega_2 \).

The reduction of the four-dimensional superconformal index to the three-dimensional partition function on \( S^3_b \) corresponds to the \( \tilde{r}_1 \to 0 \) limit,
\[
\lim_{\tilde{r}_1 \to 0} \Gamma_e(e^{2\pi i \tilde{r}_1 x}, e^{2\pi i \tilde{r}_1 \omega_1}, e^{2\pi i \tilde{r}_1 \omega_2}) = e^{\sigma_1 \omega_1^2/\omega_2} (x - \omega)^\mu_1 \Gamma_h(x; \omega_1^2, \omega_2). \quad (5.1.7)
\]
This formula corresponds to the \( \kappa \kappa \) reduction of the states contributing to the index to the states contributing to \( X_{S^3_b} \). In the language of localization it gives the
one-loop determinants for the matter and for the vector multiplets contributing to the three-dimensional partition function. It corresponds to the reduction of the elliptic gamma functions $\Gamma_e$, representing the one-loop determinants for the four-dimensional fields to the hyperbolic gamma function $\Gamma_h$ (see for example [81]). This is defined as

$$\Gamma_h(x; \omega_1, \omega_2) \equiv \Gamma_h(x) \equiv e^{\frac{\pi i}{2} \omega_1 \omega_2} \prod_{j=0}^{\infty} \frac{1 - e^{\frac{2\pi i}{\omega_1} (\omega_2 - x)} e^{\frac{2\pi i \omega_1 j}{\omega_2}}}{1 - e^{\frac{2\pi i}{\omega_2} x} e^{-\frac{2\pi i \omega_1 j}{\omega_2}}}. \quad (5.1.8)$$

There is a divergent prefactor in (5.1.7) corresponding to the four-dimensional gravitational anomalies.

Summarizing, the partition function $Z_{S^3}$ is given by

$$Z_{S^3}(\lambda; \bar{\mu}) = \frac{1}{|W|} \int \prod_{i=1}^{G} \frac{d\sigma_i}{\sqrt{-\omega_1 \omega_2}} e^{\frac{\pi i}{2} \omega_1 \omega_2} \prod_I \frac{\Gamma_h \left( \omega \Delta_I + \rho_I(\sigma) + \tilde{\rho}_I(\mu) \right)}{\prod_{\alpha \in G_i} \Gamma_h \left( \pm \alpha(\sigma) \right)}, \quad (5.1.9)$$

where $\sigma$ and $\mu$ are real parameters in the Cartan of the gauge and of the flavor symmetries respectively. The $\Omega$ term is identified with $\lambda$ and the $R$ charge by $\Delta_I$. A last comment is needed regarding the Gaussian factor in the integrand. It refers to the contribution of a possible $cs$ term at level $k$. It cannot be obtained from the dimensional reduction, and it corresponds, in localization, to the contribution of the classical action. Similarly, one can turn on a $cs$ term for the flavor symmetry, associated to the contact terms of the flavor symmetry [51, 76].

Despite the fact that the three-dimensional $cs$ term cannot be obtained from the reduction of the index, one can mimic its generation by integrating out charged fermions with a large real mass. This boils down to the following limit on the hyperbolic Gamma functions:

$$\lim_{x \to \pm \infty} \Gamma_h(x) = e^{\pm \frac{\pi i}{2} (x - \omega)^2}. \quad (5.1.10)$$

There is another interesting point regarding the $R$-charges $\Delta_I$: it is not necessary to impose the exact $R$-charge, obtained by minimizing the partition function, in the analysis. This is because one can always turn on an imaginary part for the fugacities of the flavor symmetries, in order to parameterize the mixing of a trial $U(1)_R$ with the other global symmetries.

## 5.2 Relation to the reduction of dualities

The derivation of the partition function as a $kk$ reduction of the four-dimensional superconformal index can be used to test the reduction of four-dimensional Seiberg duality to three dimensions studied above. One of the main ingredients that has not been discussed yet is the generation of the $\eta$-superpotential leading to the new effective dualities on the circle. Such a superpotential prevents the generation of the axial symmetry and is necessary in order to preserve the four-dimensional duality in three dimensions. On the index/partition function side however, the presence of a superpotential is not captured by localization. Nevertheless, the constraints imposed on the symmetries can be reformulated as a balancing condition on the fugacities.
The condition in four dimensions corresponds to the existence of an anomaly-free R-symmetry. When reducing the index to the partition function, this condition does not signal an anomaly-free R-symmetry anymore, due to the absence of anomalies in three dimensions. It signals instead the presence of the $\eta$-superpotential discussed on the field theory side. Imposing this condition on the four-dimensional identity between two Seiberg-dual phases then reduces the identity to the one of an effective duality in three dimensions. Observe that the divergent contributions, signaling the presence of the four-dimensional gravitational anomalies, cancel among the dual phases, leading to an identity for effective three-dimensional theories. The more conventional relation for Aharony and Giveon–Kutasov like dualities are obtained by a real-mass flow, exploiting the formula $(5.1.10)$. Note that in this case, it is also necessary to match the divergent prefactors to ensure that the correct leading saddles are picked up. This strategy has been used in [7] to reduce the identity for Seiberg duality with fundamental matter for unitary and symplectic gauge group. In [32], it has been observed that an analogous discussion for the orthogonal case fails, due to the fact that the partition function for the effective duality is divergent. In this case, a double-scaling limit may be necessary to obtain the expected three-dimensional identity. These results can be extended to more complicated setups.

5.2.1 Application: reduction of the KSS duality

Here we discuss as an example the reduction for KSS duality to three dimensions, following the derivation of [73]. By dimensionally reducing the integral identity between the indices of the four-dimensional KSS duality, we obtain an identity for the partition functions of the effective three-dimensional duality with the $\eta$-superpotential. The identity for Kim–Park duality is then obtained by a real mass deformation. The duality of [18,112] with cs terms is obtained by a further real mass flow.

Let us consider four-dimensional KSS duality. The electric phase is a $U(N_c)$ gauge theory with one adjoint and $N_f + 2$ (anti-)fundamental flavors. Its superconformal index is

$$I_{el} = \frac{(p;p)^{N_c}(q;q)^{N_c}}{N_c!} \Gamma_e((pq)^{1+\tau}) \int \prod_{i=1}^{N_c} \frac{dz_i}{2\pi i z_i} \prod_{i<j} \frac{\Gamma_e((z_i/z_j)^{\pm 1})}{\Gamma_e((z_i/z_j)^{\pm 1})} \times \prod_{a,b=1}^{N_f+2} \prod_{i=1}^{N_c} \Gamma_e((pq)^{s_a z_i}) \Gamma_e((pq)^{t_b^{-1} z_i^{-1}}), \quad (5.2.1)$$

where $\Gamma_e((z)^{\pm 1}) \equiv \Gamma_e(z)\Gamma_e(1/z)$. The fugacities $z_i$ refer to the $U(N_c)$ gauge symmetry while $s_a$ and $t_b$ to $SU(N_f)_L$ and $SU(N_f)_R$, respectively.

The magnetic theory has gauge group $U(k(N_f + 2) - N_c) \equiv U(\tilde{N}_c + 2k)$, one adjoint, $N_f + 2$ (anti-)fundamental flavors and the $k$ electric mesons. Its supercon-
formal index is

\[ I_{mag} = \frac{(p; p)^{N_c+2k} (q; q)^{N_c+2k}}{(N_c + 2k)!} \prod_{j=0}^{k-1} \prod_{a,b=1}^{N_f} \Gamma_p((pq)^{k+1}) \sum_{\tau} e^{\frac{\pi i}{16} \sum_{\tau} R_\tau J_{\tau}} \]

\[ \times \int \prod_{i=1}^{N_f+2k} \frac{dz_i}{2\pi i z_i} \prod_{i<j}^{N_f+2k} \Gamma_p((pq)^{k+1}) \gamma (z_i/z_j)^{\pm 1} \]

\[ \times \prod_{a,b=1}^{N_f+2k} \prod_{i=1}^{N_f} \Gamma_p((pq)^{k+1}) \gamma (z_i)^{\pm 1} \].

(5.2.2)

\( kss \) duality predicts the integral identity \( I_{el} = I_{mag} \), provided that the chemical potentials satisfy the balancing condition

\[ \prod_{a=1}^{N_f+2k} s_a t_a^{-1} = (pq)^{N_f+2-2N_c/(k+1)}, \]

(5.2.3)

equivalent to the requirement of an anomaly free R-current.

The identity (5.2.2) can be reduced to an identity between a pair of three-dimensional partition functions with the procedure described above. First we redefine the chemical potentials as

\[ p = e^{2\pi i \omega_1}, \quad q = e^{2\pi i \omega_2}, \quad z = e^{2\pi i \sigma}, \quad s_a = e^{2\pi i m_a}, \quad t_a = e^{2\pi i \tilde{m}_a}, \]

(5.2.4)

where \( \sigma, m_a \) and \( \tilde{m}_a \) parameterize the Cartan of the \( U(N_c) \) gauge and the \( SU(N_f)_L \times SU(N_f)_R \) flavor group respectively.

By taking the limit \( r \to 0 \) and dropping the divergent prefactors, corresponding to the contributions of the gravitational anomaly, that match in the dual phases, we obtain the partition function for the three-dimensional electric theory with the \( \eta \) superpotential

\[ Z_{el} = W_{N_c,0}(\mu; \nu; \omega \Delta_X; \lambda), \]

(5.2.5)

where the function \( W_{N_c,0} \) is

\[ W_{N_c,0}(\mu; \nu; \tau; \lambda) = \frac{\Gamma_h(\tau)^{N_c}}{N_c!} \int \prod_{i=1}^{N_c} d\sigma_i e^{i \sum_{i=1}^{N_c} (2\lambda i \sigma_i - 2k i \nu^2)} \]

\[ \times \prod_{1 \leq i < j \leq N_c} \Gamma_h(\tau \pm (\sigma_i - \sigma_j)) \prod_{i=1}^{N_c} \prod_{a,b=1}^{N_f} \Gamma_h(\mu_a + \sigma_i) \Gamma_h(v_b - \sigma_i) \]

(5.2.6)

and \( \Delta_X = 2/(k+1) \) is the R charge of the adjoint field. From (5.2.4) we obtain

\[ \mu_a = \omega \Delta_Q + m_a \quad \nu_a = \omega \Delta_Q - \tilde{m}_a. \]

(5.2.7)

The balancing condition (5.2.3) reduces to

\[ \sum_{a=1}^{N_f+2} (\mu_a + \nu_a) = \omega(N_f + 2 - N_c \Delta_X), \]

(5.2.8)
reflecting the presence of the $\eta$-superpotential. In the magnetic phase the index reduces to the partition function

\[ Z_{\text{mag}} = \prod_{a,b=1}^{N_f+2} \prod_{j=0}^{k-1} \Gamma_h(\mu_a + v_b + j\omega \Delta X) W_{N_f+2}^j (\omega \Delta X - \nu; \omega \Delta X - \mu; \omega \Delta X; -\lambda). \] (5.2.9)

The four-dimensional integral identity $I_{el} = I_{mag}$ reduces to $Z_{el} = Z_{mag}$, with the constraint \[5.2.8\].

Next we flow to Kim–Park duality by turning on some large real masses. The parameters $\mu$ and $v$ become

\[
\begin{align*}
\mu_a &= \begin{cases} 
  m_a + m_A + \omega \Delta_Q \\
  M - \frac{m_A N_f}{2} + \omega \Delta_{Q_m} \\
  -M - \frac{m_A N_f}{2} + \omega \Delta_{Q_m}
\end{cases} \\
\nu_a &= \begin{cases} 
  -\tilde{m}_a + \omega \Delta_Q \\
  -M + \frac{m_A N_f}{2} + \omega \Delta_{Q_m} \\
  M - \frac{m_A N_f}{2} + \omega \Delta_{Q_m}
\end{cases}
\end{align*}
\]

(5.2.10)

where $a = 1, \ldots, N_f$. The global $SU(N_f + 2)^2$ symmetry gets broken to $SU(N_f)^2 \times U(1)_A$ in the large-$M$ limit, where \[5.2.5\] becomes

\[ Z_{el} = e^{-\frac{in}{\kappa^2} (4MN_c(m_A N_f - 2\omega(\Delta_{Q_m} - 1)))} W_{N_f,0}^i (\mu; \nu; \omega \Delta X; \lambda). \] (5.2.11)

In the magnetic case the situation is more complicated. While the real mass follows from the one discussed in the electric theory, through the duality map, there is a non-trivial Higgsing in the dual gauge symmetry,

\[
\begin{align*}
s_i &= \begin{cases} 
  0 & i = 0, \ldots, kN_f - N_c \\
  M & i = kN_f - N_c + 1, \ldots, k(N_f + 1) - N_c \\
  -M & i = k(N_f + 1) - N_c + 1, \ldots, k(N_f + 2) - N_c
\end{cases}
\end{align*}
\]

(5.2.12)

In the large-$M$ limit \[5.2.9\] becomes

\[ Z_{\text{mag}} = e^{-\frac{in}{\kappa^2} (4MN_c(m_A N_f - 2\omega(\Delta_{Q_m} - 1)))} \prod_{j=0}^{k-1} \prod_{a,b=1}^{N_f} \Gamma_h(\mu_a + v_b + j\omega \Delta X) \\
\times W_{kN_f - N_c,0}^i (\omega \Delta X - \nu_a; \omega \Delta X - \mu_a; \omega \Delta X; -\lambda) \times Z_+ Z_- \] (5.2.13)

The additional terms $Z_{\pm}$ are the partition functions of the two $U(k)$ sectors

\[ Z_{\pm} = \prod_{j=0}^{k-1} \Gamma_h(m_{j,\pm}) W_{k,0} (\mu_{\pm}, \nu_{\pm}, \omega \Delta X, \lambda_{\pm}) \] (5.2.14)

with

\[
m_{j,\pm} = -m_A N_f + 2\omega \Delta Q + j\omega \Delta X, \quad \mu_{\pm} = \nu_{\pm} = \frac{1}{2} m_A N_f + \omega (\Delta X - \Delta_{Q_m})
\]

(5.2.15)

and effective $\text{FI}$ terms

\[ \lambda_{\pm} = -\lambda \pm (N_f m_A + \omega (2N_f \Delta Q + 2\Delta_{Q_m} - (N_f + 1)(k - 1)\Delta X)). \] (5.2.16)
The $U(k)^2$ sector can be dualized to a set of $6k$ singlets. On the partition function it corresponds to the integral identity \cite{81}

$$W_{N_i,0}(\mu; \nu; \omega \Delta X; \lambda) = \prod_{j=0}^{N_i-1} \Gamma_h\left( \omega - \frac{\mu + \nu}{2} - j \omega \Delta X + \frac{\lambda}{2} \right) \Gamma_h(\mu + \nu + j \omega \Delta X). \quad (5.2.17)$$

In this case, there are $4k$ massive singlets that are integrated out. This is reflected in the relation $\Gamma_h(z) \Gamma_h(2 \omega - z) = 1$ on the partition function. We are left with the relation

$$Z_+ Z_- = \prod_{j=0}^{k-1} \Gamma_h\left( \pm \frac{\lambda}{2} - m_A N_f + \omega \left( (j - N_c + 1) \Delta X + N_f (1 - \Delta_Q) \right) \right). \quad (5.2.18)$$

This corresponds to the $2k$ singlets which remain light. They have exactly the correct global charges for the superpotential coupling with the magnetic monopoles, and we identify them with the electric monopoles, acting as singlet in the dual phase. The final identity for the Kim–Park duality is then

$$W_{N_i,0}(\mu; \nu; \omega \Delta X; \lambda) = \prod_{j=0}^{k-1} \prod_{a,b=1}^{N_f} \Gamma_h(\mu_a + \nu_b + j \omega \Delta X)$$

$$\times \prod_{j=0}^{k-1} \Gamma_h\left( \pm \frac{\lambda}{2} - m_A N_f + \omega \left( (j - N_c + 1) \Delta X + N_f (1 - \Delta_Q) \right) \right)$$

$$\times W_{kN_i - N_i,0}(\omega \Delta X - \nu; \omega \Delta X - \mu; \omega \Delta X; -\lambda). \quad (5.2.19)$$

Eventually, the duality of \cite{18,112} is obtained by a further real mass flow. By integrating out the massive fields with (5.1.10) we obtain

$$W_{N_i,K}(\mu; \nu; \omega \Delta X; \lambda) = e^{\frac{im_{\lambda}}{2} \phi^{(2+K^2)}} \prod_{j=0}^{k-1} \prod_{a,b=1}^{N_f} (\mu_a + \nu_b + j \omega \Delta X)$$

$$\times W_{k(N_i + K) - N_i, -K}(\omega \Delta X - \nu; \omega \Delta X - \mu; \omega \Delta X; -\lambda), \quad (5.2.20)$$

where $\zeta = \exp\left(\frac{im_{\lambda}^2 + \omega^2}{2m_{\lambda} \omega^2}\right)$ and the extra phase is

$$\phi = \omega m_A (2kN_f (2(K - N_f) \Delta Q - 2N_c \Delta X - 2N_f + K(k - 1) \Delta X))$$

$$+ 2km_{A}^2 N_f (K - N_f) - \frac{k \lambda^2}{2} + k \sum_{a=1}^{N_f} (\mu_a^2 + v_a^2)$$

$$- k \omega^2 \left(2N_c (N_c + (k + 1) N_f (\Delta Q - 1) - kK) + \frac{k^2 (11K^2 + 2) + K^2 - 2}{12} \right) \Delta X^2$$

$$- 2N_f \left(K (\Delta Q^2 + (k + 1) \Delta Q \Delta X + \frac{5}{3} (\Delta X - 1) - 1) - N_f (\Delta Q - 1)^2 \right). \quad (5.2.21)$$

Equation (5.2.20) is the identity between the electric and the magnetic partition functions of the duality in \cite{18,112}. 
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