VEECH GROUPS WITHOUT PARABOLIC ELEMENTS

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Abstract. We prove that a translation surface which has two transverse parabolic elements has totally real trace field. As a corollary, non trivial Veech groups which have no parabolic elements do exist.

The proof follows Veech’s viewpoint on Thurston’s construction of pseudo-Anosov diffeomorphisms.

1. Introduction

For a long time, it has been known that the ergodic properties of linear flows on a translation surface are strongly related to the behavior of its $SL_2(\mathbb{R})$-orbit in the moduli space of holomorphic one forms (see [MaTa, Zo] for surveys of the literature on this subject). The $SL_2(\mathbb{R})$-orbit of a translation surface is called its Teichmüller disc. Its stabilizer under the action of $SL_2(\mathbb{R})$ is a Fuchsian group called the Veech group.

In 1989, Veech proved that a translation surface whose stabilizer is a lattice has optimal dynamical properties: the directional flows are periodic or uniquely ergodic (see [Ve2]). Since then, much effort has gone into the study of the geometry of Teichmüller discs ([Ve3, Yo, Wa]). Hubert and Schmidt [HuSc1, HuSc2] found the first examples of infinitely generated Veech groups. Just after that, McMullen [Mc1, Mc2] proved that, in genus 2, the existence of a pseudo-Anosov diffeomorphism in the affine group implies that the Veech group is a Fuchsian group of the first kind (which means that either it is a lattice or it is infinitely generated; moreover McMullen proved that both cases occur). See also [Ca] for related results. McMullen’s proof uses the existence of infinitely many parabolic elements. By contrast, we will give examples with a very different behavior in genus $g \geq 3$.

The trace field is a natural invariant of the Veech group. Thurston proved that the trace of the derivative of any pseudo-Anosov diffeomorphisms is an algebraic integer over $\mathbb{Q}$ with degree less that the dimension of the Teichmüller space divided by 2 (see [Th, p. 427]). In [GalJi] it was shown that a translation surface is a covering of the torus ramified over one point if and only if its trace field equals to $\mathbb{Q}$. Kenyon and Smillie [KeSm] gave a simple criterion ensuring this property: if the Veech group of a translation surface contains a hyperbolic element whose trace belongs to $\mathbb{Q}$, then this group is commensurable to $SL_2(\mathbb{Z})$. In fact they showed that the trace field is generated by the trace of the derivative of any pseudo-Anosov diffeomorphisms. Moreover, if $K$ is the trace field of $(X, \omega)$ then the Veech group is commensurable to a subgroup of $SL_2(\mathcal{O}_K)$ where $\mathcal{O}_K$ is the ring of integers of $K$.

An interesting problem is to determine which Fuchsian group can occur as affine group of some surface. Up to now, there are no general methods to compute a Veech group. To date there are two methods to produce pseudo-Anosov diffeomorphisms in the coordinates of the flat surface. In the first one, due to Thurston, a pseudo-Anosov diffeomorphism is obtained as a product of two parabolic elements (see [FLP, Th, Ve2]). Veech computed the first non trivial examples of affine groups by making calculations with a pair of parabolic elements (see [Ve2]). Independently, a very general construction of pseudo-Anosov diffeomorphisms was discovered by Veech [Ve1]. It is based on the Rauzy induction of interval exchange transformations (see also [ArYo] for specific examples of such pseudo-Anosov diffeomorphisms, for any genus $g \geq 3$). A simple consequence of our result is that some
pseudo-Anosov diffeomorphisms are not given by Thurston’s construction (see [Le] for another proof).

In fact, we prove a stronger result:

**Theorem 1.1.** Let \((X, \omega)\) be a translation surface. Let us assume that the Veech group \(SL(X, \omega)\) contains two transverse parabolic elements \(^1\). Then the trace field

\[ \mathbb{Q}[\text{Trace}(A) \mid A \in SL(X, \omega)] \]

is totally real.

When a pseudo-Anosov diffeomorphism \(\phi\) acts linearly on a translation surface by the diagonal matrix \(D\phi = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \) (with \(\lambda^{-1} < 1 < \lambda\)), its expansion factor is \(\lambda(\phi) = \lambda\) (and \(1/\lambda\) is the contraction factor).

From Theorem 1.1 we draw the following results.

**Theorem 1.2.** Let \((X, \omega)\) be a translation surface endowed with a pseudo-Anosov diffeomorphism \(\phi\) with expansion factor \(\lambda\). Let us assume that the field \(\mathbb{Q}[\lambda + \lambda^{-1}]\) is not totally real. Then \(SL(X, \omega)\) does not contain any parabolic elements.

Arnoux and Yoccoz [ArYo] discovered a family \(\phi_n, n \geq 3\) of pseudo-Anosov diffeomorphisms with expansion factor \(\lambda_n = \lambda(\phi_n)\) the Pisot root of the irreducible polynomial \(P_n\) with

\[ P_n(X) = X^n - X^{n-1} - \cdots - X - 1. \]

The pseudo-Anosov \(\phi_n\) acts linearly on a genus \(n\) surface (the corresponding Abelian differential having two zeroes of order \(n - 1\)).

**Corollary 1.3.** The Teichmüller disc stabilized by the Arnoux–Yoccoz pseudo-Anosov \(\phi_n, n \geq 3\) does not contain any parabolic direction. Therefore, for any genus \(g \geq 3\), there exists a genus \(g\) translation surface such that its Veech group has (at least) one hyperbolic element and no parabolic elements.

**Corollary 1.4.** The trace field of any Veech surface is totally real.

**Remark 1.1.** Möller proved Corollary 1.4 by very different methods (see [Mö]).

**Corollary 1.5.** There exists a Veech group which is commensurable to a Fuchsian group which only contains hyperbolic elements.

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2. Background

In order to establish notations and preparatory material, we review basic notions concerning translation surfaces, affine automorphisms groups and trace fields. We will end this section by recalling Veech’s viewpoint on Thurston’s construction. See say [KeSm], [MaTa], [Mc1, Mc2], [Th], [Ve2] for more details; See also [Mc3, Mc4, Mc5], for recent related developments. For a general reference on Fuchsian groups, see [Ka].

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\(^1\) The surface \((X, \omega)\) is then called a *prelattice surface* [HuSc1] or a “bouillabaisse surface” in honor of John Hubbard’s Lecture at the CIRM in July 2003.
2.1. Translation surfaces and affine diffeomorphisms groups. A translation surface is a (real) genus $g$ surface with an atlas such that all transition functions are translations. As usual, we consider maximal atlases. These surfaces are precisely those given by a Riemann surface $X$ and a holomorphic (non-null) one form $\omega \in \Omega(X)$; see [MaTa] for a general reference on translation surfaces and holomorphic one forms.

We denote by $X'$ the surface that arises from $X$ by deleting the zeroes of the form $\omega$ on $X$. The translation structure defines on $X'$ a Riemannian structure; we therefore have notions of geodesic, length, angle, flow, measure... Orbits of the directional flows meeting singularities are called separatrices. Orbits of the flow going from a singularity to another one (possibly the same) are called saddle connections.

Given any matrix $A \in \text{SL}_2(\mathbb{R})$, we can post-compose the coordinate functions of the charts of $(X, \omega)$ by $A$. One easily checks that this gives a new translation surface, denoted by $A \cdot (X, \omega)$. We therefore get an $\text{SL}_2(\mathbb{R})$-action on these translation surfaces.

An affine diffeomorphism $f : X \to X$ is a homeomorphism of $X$ such that $f$ restricts to a diffeomorphism on $X'$ of constant derivative. It is equivalent to say that $f$ restricts to an isomorphism of $X'$ which preserves the induced affine structure given by $\omega$. Usually, one denotes by $\text{Aff}(X, \omega)$ the group of orientation preserving affine diffeomorphisms. The function which takes an affine diffeomorphism $f$ to its derivative $Df$ gives a homomorphism from $\text{Aff}(X, \omega)$ into $\text{SL}_2(\mathbb{R})$. The image of $\text{Aff}(X, \omega)$ is the Veech group $\text{SL}(X, \omega)$ of the surface $(X, \omega)$ – this is a discrete subgroup and, when $X$ has genus greater than one, the kernel of the homomorphism is finite.

One easily checks that the Veech group $\text{SL}(X, \omega)$ is the $\text{SL}_2(\mathbb{R})$-stabilizer of $(X, \omega)$. Thus, for any matrix $A \in \text{SL}_2(\mathbb{R})$, the Veech group of $(X, \omega)$ and $A \cdot (X, \omega) = (Y, \alpha)$ are conjugate in $\text{SL}_2(\mathbb{R})$:

$$\text{SL}(Y, \alpha) = A \cdot \text{SL}(X, \omega) \cdot A^{-1}$$

2.2. Classification of affine diffeomorphisms. There is a standard classification of the elements of $\text{SL}_2(\mathbb{R})$ into three types: elliptic, parabolic and hyperbolic. This induces a classification of affine diffeomorphisms.

An affine diffeomorphism is respectively parabolic, elliptic or pseudo-Anosov if respectively $|\text{trace}(Df)| = 2$, $|\text{trace}(Df)| < 2$ or $|\text{trace}(Df)| > 2$.

Remark 2.1. If an elliptic element belongs to a Fuchsian group, its order is finite.

Remark 2.2. In a Fuchsian group, a parabolic direction (invariant direction of a parabolic element) is never fixed by a hyperbolic element. More precisely, if a hyperbolic element $H$ fixes a parabolic direction of a parabolic element $P$ then one can easily check that $H^n P H^{-n}$ converges to $\text{Id}$ as $n$ tends to $+\infty$ (or $-\infty$), which is impossible in a discrete group.

2.3. Cylinders decomposition and parabolic element. A cylinder on $(X, \omega)$ is a maximal connected set of homotopic simple closed geodesics. If the genus of $X$ is greater than one then every cylinder is bounded by saddle connections. A cylinder has a width (or circumference) $x$ and a height $y$. The modulus of a cylinder is $\mu = y/x$. Veech [Ve2] proved the following:

Proposition (Veech). If a translation surface $(X, \omega)$ has a parabolic affine diffeomorphism $f$, then there is a decomposition of $X$ into metric cylinders parallel to the fixed direction of $Df$. Furthermore, the moduli of the cylinders are commensurable (have rational ratios).

Remark 2.3. In the above proposition, up to take a power of the affine diffeomorphism, we can assume that $f$ acts as a power of the affine Dehn twist on each cylinder. Therefore the boundary of each cylinder is fixed by $f$.

Conversely, a cylinder decomposition into cylinders of commensurable moduli produces parabolic elements. Namely, the following holds:
Proposition (Veech). If \((X, \omega)\) has a decomposition into metric cylinders for the horizontal direction, with commensurable moduli, then the Veech group \(SL(X, \omega)\) contains
\[
Df = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}
\]
where \(c\) is the least common multiple of the inverse of the moduli.

2.4. Trace fields. In this section we recall some general properties of the trace field of a group; see [GuJu], [KeSm], [Mc1, Mc2].

The trace field of a group \(\Gamma \subset SL_2(\mathbb{R})\) is the subfield of \(\mathbb{R}\) generated by \(tr(A), A \in \Gamma\). One defines the trace field of a flat surface \((X, \omega)\) to be the trace field of its Veech group \(SL(X, \omega) \subset SL_2(\mathbb{R})\).

Let \((X, \omega)\) be a genus \(g\) translation surface. Then the following holds:

Theorem A. (Kenyon, Smillie) The trace field of \((X, \omega)\) has degree at most \(g\) over \(\mathbb{Q}\). Assume that the affine diffeomorphisms group of \((X, \omega)\) contains a pseudo-Anosov element \(f\) with expansion factor \(\lambda\). Then the trace field of \((X, \omega)\) is \(\mathbb{Q}[\lambda + \lambda^{-1}]\).

One defines the holonomy vectors to be the integrals of \(\omega\) along the saddle connections. Let us denote \(\Lambda = \Lambda(\omega)\) the subgroup of \(\mathbb{R}^2\) generated by holonomy vectors
\[
\Lambda = \int_{H \subset (X, \omega)} \omega
\]
Let \(e_1, e_2 \in \Lambda\) be non-parallel vectors in \(\mathbb{R}^2\). One defines the holonomy field \(k\) to be the smallest subfield of \(\mathbb{R}\) such that every element of \(\Lambda\) can be written \(ae_1 + be_2\) with \(a, b \in k\).

Theorem B. (Kenyon, Smillie) The trace field of \((X, \omega)\) coincides with \(k\). The space \(\Lambda \otimes \mathbb{Q} \subset \mathbb{C}\) is a 2-dimensional vector space over \(k\).

See also [GuJu] for a different approach of these notions. Note that these results have been reproved in [Mc1, Mc2].

Pisot numbers. An algebraic integer \(\beta\) is a Pisot number if \(\beta \in \mathbb{R}, \beta > 1\) and all of its conjugates belong to the unit disc \(\mathbb{D} = \{ z \in \mathbb{C}, |z| < 1 \}\).

2.5. Veech’s viewpoint on Thurston’s construction. Let us recall the Thurston construction [Th]. We will follow the notations of the paper of Veech [Ve2], section §9.

Let \((Y, \alpha)\) be a translation surface with vertical and horizontal parabolic directions. Up to taking a power of the parabolic elements, one can assume that the corresponding parabolic \(P_v\) (resp \(P_h\)) is a multiple of the Dehn twist of each vertical (resp horizontal) cylinder (see Remark 2.3).

In these coordinates our two parabolic elements are
\[
P_h = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad P_v = \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix}
\]

Without loss of the generality, we may assume that \(c\) and \(d\) are positive real numbers.

Claim 2.1. Let \(t = cd > 0\); then the trace field of \(SL(Y, \alpha)\) is \(\mathbb{Q}[t]\).

Proof of Claim 2.1 The matrix \(P_hP_v\) has trace \(2 + t > 2\), thus this is a hyperbolic element and, following [KeSm] (see section 2.3 Theorem A), the trace field of \(SL(Y, \alpha)\) is \(\mathbb{Q}[t]\). So the claim is proven.

Let us denote by \(H_i, 1 \leq i \leq r\) and \(V_j, 1 \leq j \leq s\) the horizontal and vertical cylinders. Let us denote the width and heights of \(H_i\) and \(V_j\) respectively by \((x_i, y_i)\) and \((\eta_j, \xi_j)\). We insist that the first coordinate is the width and the second one is the height even for vertical cylinders.

With these notations, let \(E\) be the \(r \times s\) integer matrix whose entry \(E_{i,j}\) is the number of rectangles \((\xi_j \times y_i)\) in the intersection \(H_i \cap V_j\). All of these rectangles have width \(y_j\) and heights \(\xi_i\). In other words, \(E_{i,j}\) is the intersection number of the core curves of the cylinders \(H_i\) and \(V_j\).
Let us introduce the following notations of linear algebra: $\vec{x} = (x_1, \ldots, x_r)$, $\vec{y} = (y_1, \ldots, y_r)$, $\vec{\xi} = (\xi_1, \ldots, \xi_s)$ and $\vec{\eta} = (\eta_1, \ldots, \eta_r)$. Then one can summarize the above discussion by the matrix relations:

\[
\begin{cases}
E \vec{\xi} = \vec{x} \\
E \vec{\eta} = \vec{y}
\end{cases}
\]

The moduli of the vertical cylinder $V_j$ (resp horizontal cylinder $H_i$) is commensurable with $d$ (resp with $c$). More precisely, there exist integers $m_i$, $1 \leq i \leq r$, and $n_j$, $1 \leq j \leq s$, such that

\[
\begin{cases}
m_i x_i = c y_i \\
n_j \eta_j = d \xi_j
\end{cases}
\]

Let us denote by $D_m = \Diag(m_1, \ldots, m_r)$ and $D_n = \Diag(n_1, \ldots, n_s)$ the diagonal matrices. Then the above equation (2) becomes:

\[
\begin{cases}
D_m \vec{x} = c \vec{y} \\
D_n \vec{\eta} = d \vec{\xi}
\end{cases}
\]

From equations (1) and (3) one gets the following new one:

\[
\begin{cases}
E D_n \vec{\eta} = d \vec{x} \\
t E D_m \vec{x} = c \vec{y}
\end{cases}
\]

and therefore we deduce:

\[
\begin{cases}
E D_n t E D_m \vec{x} = c d \vec{x} \\
t E D_m E D_n \vec{\eta} = c d \vec{\eta}
\end{cases}
\]

Now, in order to follow Veech’s notations, let us introduce the two matrices $F_n = E D_n$ and $F_m = t E D_m$. As remarked in \[Ve2\], the matrices $F_n F_m$ and $F_m F_n$ have a power with positive entries (see \[HuLe\] Appendix C for a proof). The vector $\vec{x}$ is a non negative eigenvector of the Perron–Frobenius matrix $F_n F_m$, therefore $t = c d > 0$ is the unique Perron–Frobenius eigenvalue of $F_n F_m$ (the same is true for the Perron–Frobenius matrix $F_m F_n$). Thus, up to renormalization of the area of the surface, the coordinates of $\vec{x}$ and $\vec{\eta}$ belong to $\mathbb{Q}[t]$ (see section 2.4 Theorem B).

Now we have all necessary tools to prove the announced results.

3. Proofs

We first prove Theorem 1.2 assuming Theorem 1.1.

Proof of Theorem 1.2. Let us assume that there is a parabolic element $P$ in $\SL(X, \omega)$. Let us denote by $H$ the derivative of the pseudo-Anosov $\phi$. Then the conjugate $HPH^{-1}$ is another parabolic element in $\SL(X, \omega)$. Let $x \in \partial \mathbb{H}$ be the fixed point of $P$. Thus, $H(x)$ is a fixed point of $HPH^{-1}$. But by Remark 2.2 $H(x) \neq x$, then $HPH^{-1} \in \SL(X, \omega)$ is certainly a parabolic element transverse to the parabolic $P$. Therefore Theorem 1.1 applies. \[\square\]

Proof of Theorem 1.1. Let us assume that the surface $(X, \omega)$ has two transverse parabolic elements. By a standard argument, one can find a matrix $A \in \SL_2(\mathbb{R})$ which sends the two invariant directions of our parabolic elements into horizontal and vertical direction. The Veech group $\SL(Y, \alpha) = A \cdot \SL(X, \omega) \cdot A^{-1}$ possesses the same trace field as $\SL(X, \omega)$.

Now up to taking a power of the parabolic element, one can assume that they are a multiple of the Dehn twist on each vertical (resp horizontal) cylinder (see Remark 2.3). Thus one can apply Veech’s viewpoint on Thurston’s construction, section 2.5. In particular we follow the notations introduced in that section.

Recall that the trace field of $\SL(Y, \alpha)$ is $\mathbb{Q}[t]$ (see Claim 2.1). Now let us prove that $\mathbb{Q}[t]$ is totally real.
Let $\sigma$ be an embedding of $\mathbb{Q}[t]$ into $\mathbb{C}$ and $t' = \sigma(t) \in \mathbb{C}$ be a conjugate of $t$. Applying $\sigma$ to the first part of equation (4), $F_n F_m \vec{x} = t' \vec{x}$ and recalling that $F_n F_m$ is an integer matrix, one gets
\begin{equation}
F_n F_m \sigma(\vec{x}) = t' \sigma(\vec{x})
\end{equation}
Now, let us denote by $D_\sqrt{m}$ = $\text{Diag}(\sqrt{m}, \ldots, \sqrt{m})$ and $D_\sqrt{n}$ = $\text{Diag}(\sqrt{n}, \ldots, \sqrt{n})$ the diagonal matrices. Then

$$F_n F_m = ED_\sqrt{m} t' ED_\sqrt{n} = ED_\sqrt{m} D_\sqrt{m} E D_\sqrt{m} D_\sqrt{n} = ED_\sqrt{m} t'(ED_\sqrt{n}) D_\sqrt{n} D_\sqrt{m}$$

Let us set $A = ED_\sqrt{m}$. Substituting this into the last equation, yields:

$$F_n F_m = A^t A D_\sqrt{m} D_\sqrt{n}$$

Letting $M = D_\sqrt{m} A$, it becomes:
\begin{equation}
F_n F_m = D_\sqrt{m}^{-1} D_\sqrt{m} A^t A D_\sqrt{m} D_\sqrt{n} = \frac{1}{n} D_\sqrt{m} A^t (D_\sqrt{m} A) D_\sqrt{m} = \frac{1}{n} D_\sqrt{m}^{-1} M A^t M D_\sqrt{m}
\end{equation}

Now equation (6) and the fact that $\sigma(\vec{x}) \neq 0$ imply that $t'$ is an eigenvalue of $F_n F_m$. But by equation (5), the two matrices $F_n F_m$ and $M A^t M$ are similar, thus they have the same eigenvalues. But $M A^t M$ is symmetric, thus all of its eigenvalues are real, and so $t' \in \mathbb{R}$.

Finally the trace field $\mathbb{Q}[t]$ of $(Y, \alpha)$, and that of $(X, \omega)$, is totally real. Theorem 1.4 is proved.

**Proof of Corollary 1.3**. Let $n \geq 3$ be any odd integer. We denote by $(X_n, \omega_n)$ a flat surface in the Teichmüller disc stabilized by the Arnoux–Yoccoz pseudo-Anosov $\phi_n$. By Theorem A (see section 2.4), the trace field of $(X_n, \omega_n)$ is $\mathbb{Q}[\lambda_n + \lambda_n^{-1}]$.

**Claim 3.1.** The polynomial $X^n - X^{n-1} - \cdots - 1$ has 2 real roots if $n$ is even and 1 if $n$ is odd.

**Proof of the Claim.** Following [AYYo], we introduce the polynomial

$$Q_n(X) = (X^n - X^{n-1} - \cdots - 1)(X - 1) = X^{n+1} - 2X^n + 1$$

One can directly check, by calculating $Q'_n$, that $Q_n(X)$ has 2 real roots if $n$ is odd and 3 if $n$ is even. This proves the claim.

Above Claim 3.1 asserts that $\mathbb{Q}[\lambda_n]$ is not totally real. Recall that $\lambda_n$ is a Pisot number (see [AYYo]). Applying next Lemma 3.2 we get that $\mathbb{Q}[\lambda_n + \lambda_n^{-1}]$ is not totally real.

Thus Corollary 1.3 follows from Theorem 1.2.

**Lemma 3.2.** Let $\beta$ be any Pisot number. Let us assume that $\mathbb{Q}[\beta]$ is not totally real. Then the field $\mathbb{Q}[$ $\beta + \beta^{-1}]$ is not totally real.

**Proof of Lemma 3.2**. Let $\delta$ be a conjugate of $\beta$ which is not real. Galois theory ensures that there is a field homomorphism $\chi : \mathbb{Q}[:\beta] \rightarrow \mathbb{Q}[:\delta]$. The complex number $\chi(\beta + \beta^{-1}) = \delta + \delta^{-1}$ is a conjugate of $\beta + \beta^{-1}$. It is enough to show that $\delta + \delta^{-1}$ is not real to prove that $\mathbb{Q}[\beta + \beta^{-1}]$ is not totally real. Writing $\delta = \rho e^{i\theta}$ (with $\sin(\theta) \neq 0$), we have $\Im(\delta + \delta^{-1}) = (\rho - \rho^{-1}) \sin(\theta)$. As $\beta$ is a Pisot number, $\rho = |\delta| < 1$. Therefore $\delta + \delta^{-1}$ is not real. So Lemma 3.2 is proven.

**Proof of Corollary 1.4**. On a Veech surface, the direction of every saddle connection is a parabolic direction. There are thus at least two transverse parabolic elements in the Veech group and Theorem 1.4 applies.

**Proof of Corollary 1.5**. Let $(X, \omega)$ be any genus $g \geq 3$ translation surface whose Veech group only contains hyperbolic and elliptic elements. Any elliptic element in $\text{SL}(X, \omega)$ is conjugate in $\text{SL}_2(\mathbb{R})$ to a rotation. As a rotation preserves the underlying complex structure of the Riemann surface $X$, it is an automorphism of a genus $g$ Riemann surface. Therefore, by a Theorem of Hurwitz, the order of any elliptic element belonging to $\text{SL}(X, \omega)$ is bounded by $84(g - 1)$, see [FaKr] §5 p.242.

Now we recall a Theorem of Purzitsky on Fuchsian groups (see [Pu] Theorem 7 p.241):
Theorem (Purzitsky). Let $\Gamma$ be a Fuchsian group. Then $\Gamma$ contains a finite index subgroup without elliptic elements if and only if there exists a constant $N$ such that the order of any elliptic element of $\Gamma$ is less than $N$.

Now recalling that any elliptic element belonging to a Fuchsian group has finite order (see Remark 2.1), Corollary 1.5 follows from Purzitsky’s Theorem taking $\Gamma = \text{SL}(X, \omega)$.

\[ \square \]

REFERENCES

[ArYo] P. Arnoux, J.C. Yoccoz, Construction de difféomorphismes pseudo-Anosov. (French) C. R. Acad. Sci. Paris Sr. I Math. 292 no. 1, (1981) 75–78.

[Ca] K. Calta, Veech surfaces and complete periodicity in genus two, J. Amer. Math. Soc. 17 no. 4, (2004) 871–908.

[FaKr] H.M. Farkas, I. Kra, Riemann surfaces, Graduate Texts in Mathematics 71, Springer-Verlag, New York-Berlin, (1980).

[FLP] A. Fathi, F. Laudenbach, V. Poénaru, Travaux de Thurston sur les surfaces, Astérisque 66–67, (1979).

[GuJu] E. Gutkin, C. Judge, Affine mappings of translation surfaces: Geometry and arithmetic, Duke Math. J. 103, (2000) 191–213.

[HuLe] P. Hubert, S. Leliève, Prime arithmetic Teichmüller discs in $H(2)$, Israel Journal of Math., (2005) to appear.

[HuSc1] P. Hubert, T.A. Schmidt, Infinitely generated Veech groups, Duke Math. J. 123, (2004) 49–69.

[HuSc2] P. Hubert, T.A. Schmidt, Geometry of infinitely generated Veech groups, ArXiv [math.GT/0410132] preprint (2004).

[Ka] S. Katok, Fuchsian Groups, U. Chicago Press, Chicago, (1992).

[KeSm] R. Kenyon, J. Smillie, Billiards in rational-angled triangles, Comment. Math. Helv. 75, (2000) 65–108.

[Le] C. Leininger, On groups generated by two positive multi-twists: Teichmüller curves and Lehner’s number, Geom. Topol. 8, (2004) 1301–1359.

[MaTa] H. Masur, S. Tabachnikov, Rational billiards and flat structures, in Handbook of dynamical systems, Vol. 1A, North-Holland, Amsterdam, (2002) 1015–1089.

[Mc1] C. McMullen, Billiards and Teichmüller curves on Hilbert modular surfaces, J. Amer. Math. Soc. 16 no. 4, (2003) 857–885.

[Mc2] C. McMullen, Teichmüller geodesics of infinite complexity, Acta Math. 191 no. 2, (2003) 191–223.

[Mc3] C. McMullen, Teichmüller curves in genus two: Discriminant and spin, preprint (2004).

[Mc4] C. McMullen, Teichmüller curves in genus two: The deagon and beyond, preprint (2004).

[Mc5] C. McMullen, Teichmüller curves in genus two: Torsion divisors and ratios of sines, preprint (2004).

[Mö] M. Möller, Variations of Hodge structure of Teichmüller curves, ArXiv [math.AG/0410250] preprint (2004).

[Pu] N. Purzitsky, A cutting and pasting of noncompact polygons with applications to Fuchsian groups, Acta Math. 143, (1979) 233–250.

[Th] W. Thurston, On the geometry and dynamics of diffeomorphisms of surfaces, Bull. A.M.S. 19, (1988) 417–431.

[Ve1] W. Veech, Gauss measures for transformations on the space of interval exchange maps, Ann. Math. 115, (1982) 201–242.

[Ve2] W. Veech, Teichmüller curves in modular space, Eisenstein series, and an application to triangular billiards, Inv. Math. 97, (1989) 553–583.

[Ve3] W. Veech, The billiard in regular polygon, Geom. and Func. Analysis 2, (1992) 341–379.

[Vo] Y. Vorobets, Planar structure and billiards in rational polygons: the Veech alternative, Russ. Math. Surv. 51, (1996) 779–817.

[Wa] C. Ward, Calculation of Fuchsian groups associated to billiards in a rational triangle, Ergod. Th. Dynam. Sys. 18, (1998) 1019–1042.

[Zo] A. Zorich, Flat Surfaces, Frontiers in Number Theory, Physics and Geometry. Volume 1: On random matrices, zeta functions and dynamical systems, École de physique des Houches, France, March 9-21 2003, Springer-Verlag, Berlin, (2006) 1–149.

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