GEOMETRIC REALIZATIONS OF QUIVER ALGEBRAS

DMITRI ORLOV

Dedicated to the blessed memory of my late adviser Andrei Nikolaevich Tyurin on the occasion of his 75th birthday

Abstract. In this paper we construct strong exceptional collections of vector bundles on smooth projective varieties that have a prescribed endomorphism algebra. We prove the construction problem always have a solution. We consider some applications to noncommutative projective planes and to the quiver connected with the 3-point Ising function.

Introduction

The main purpose of this paper is to provide constructions of strong exceptional collections on smooth projective varieties which consist of vector bundles and have a prescribed endomorphism algebra. We are interested in triangulated categories \( T \) with a full exceptional collections \( \sigma = (E_1, \ldots, E_n) \). Recently, we showed that whenever such a \( T \) has an enhancement, i.e. is equivalent to the homotopy category \( \mathcal{H}^0(\mathcal{A}) \) of some differential graded category \( \mathcal{A} \), then it can be realized as an admissible subcategory of the bounded derived category of coherent sheaves on a smooth projective variety (see \[O3\] Th.5.8)). Recall that a triangulated subcategory \( \mathcal{N} \subset \mathcal{D}^b(\text{coh} X) \), where \( X \) is smooth and projective, is called admissible if it is full and the inclusion functor has right and left adjoint projections. Admissible subcategories have many good properties and provide a large selection of smooth and proper noncommutative schemes that are called geometric noncommutative schemes [O3].

To provide some context we recall a result of [O3] in more detail. Suppose the homotopy category \( \mathcal{T} = \mathcal{H}^0(\mathcal{A}) \) of a small differential graded category \( \mathcal{A} \) has a full exceptional collection \( \mathcal{T} = \langle E_1, \ldots, E_n \rangle \). Then there exist a smooth projective scheme \( X \) and an exceptional collection of line bundles \( \sigma = (L_1, \ldots, L_n) \) on \( X \) such that the full subcategory of \( \mathcal{D}^b(\text{coh} X) \), generated by \( \sigma \), is equivalent to \( \mathcal{T} \). In [O3] we give an explicit construction of the variety \( X \) as a tower of projective bundles. It follows that \( X \) itself has a full exceptional collection. Furthermore, we show that a full exceptional collection on \( X \) can be chosen so that it contains the collection \( \sigma = (L_1, \ldots, L_n) \) as a subcollection. In this case we obtain a functor from the triangulated category \( \mathcal{T} \) to the derived category \( \mathcal{D}^b(\text{coh} X) \) that sends the exceptional objects \( E_i \) to shifts of the line bundles \( L_i[r_i] \) for some integers \( r_i \). Of course, we can not expect in general that \( E_i \) go to unshifted line bundles.

2010 Mathematics Subject Classification. 14F05, 18E30.

Key words and phrases. Coherent sheaves, triangulated categories, quiver algebras, noncommutative schemes.

This work is supported by the Russian Science Foundation (RSF) under grant 14-50-00005.
On the other hand, in the most important case when the exceptional collection \((E_1, \ldots, E_n)\) is strong it is desirable to find a realizations of this collection in form of vector bundles (without shifts). In this paper we will deal with strong exceptional collections and will discuss different constructions of geometric realizations of these collections in term of vector bundles on smooth projective varieties. We show that for a triangulated category \(T\) with a strong exceptional collection \(\sigma = (E_1, \ldots, E_n)\) it is always possible to find a smooth projective variety \(X\) and a fully faithful functor from \(T\) to the bounded derived category of coherent sheaves \(D^b(\text{coh}\ X)\) such that the exceptional objects \(E_i\) go to vector bundles \(E_i\) on \(X\) (see Theorem 2.6 and Corollary 2.7). In this way, we obtain a strong exceptional collection \((E_1, \ldots, E_n)\) of vector bundles on \(X\) with the same endomorphism algebra that the initial collection \(\sigma\) has.

In the last section we consider some applications of our constructions to specific interesting exceptional collections of three objects. The first example is a quiver related to the 3-point Ising function, while the second example is the family of quivers describing the noncommutative projective planes.

The author is very grateful to Anton Fonarev, Alexander Kuznetsov, and Valery Lunts for very useful discussions and to Tony Pantev for a large number of valuable comments.

1. Exceptional collections, triangulated and differential graded categories

1.1. Exceptional collections. We begin be recalling some definitions and facts concerning admissible subcategories, semi-orthogonal decompositions, and exceptional collections (see [BO]). Let \(T\) be a \(k\)-linear triangulated category, where \(k\) is a base field. Let \(\mathcal{N} \subset T\) be a full triangulated subcategory. Recall that the right orthogonal (resp. left orthogonal) to \(\mathcal{N}\) is the full subcategory \(\mathcal{N}^\perp \subset T\) (resp. \(\perp \mathcal{N}\)) consisting of all objects \(X\) such that \(\text{Hom}(Y, X) = 0\) (resp. \(\text{Hom}(X, Y) = 0\)) for any \(Y \in \mathcal{N}\). It is clear that the orthogonals are triangulated subcategories.

**Definition 1.1.** Let \(j: \mathcal{N} \hookrightarrow T\) be a full embedding of triangulated categories. We say that \(\mathcal{N}\) is right admissible (resp. left admissible) if there is a right (resp. left) adjoint functor \(q: T \to \mathcal{N}\). The subcategory \(\mathcal{N}\) will be called admissible if it is both right and left admissible.

It is well-know that a subcategory \(\mathcal{N}\) is right admissible if and only if for each object \(Z \in T\) there is an exact triangle \(Y \to Z \to X\), with \(Y \in \mathcal{N}\), \(X \in \mathcal{N}^\perp\).

Let \(\mathcal{N} \subset T\) be a full triangulated subcategory. If \(\mathcal{N}\) is right (resp. left) admissible, then the quotient category \(T/\mathcal{N}\) is equivalent to \(\mathcal{N}^\perp\) (resp. \(\perp \mathcal{N}\)). Conversely, if the quotient functor \(T \to T/\mathcal{N}\) has a left (resp. right) adjoint, then \(T/\mathcal{N}\) is equivalent to \(\mathcal{N}^\perp\) (resp. \(\perp \mathcal{N}\)).

**Definition 1.2.** A semi-orthogonal decomposition of a triangulated category \(T\) is a sequence of full triangulated subcategories \(\mathcal{N}_1, \ldots, \mathcal{N}_n\) in \(T\) such that there is an increasing filtration \(0 = \mathcal{T}_0 \subset \mathcal{T}_1 \subset \cdots \subset \mathcal{T}_n = T\) by left admissible subcategories for which the left orthogonals \(\perp \mathcal{T}_{p-1}\) in \(\mathcal{T}_p\) coincide with \(\mathcal{N}_p\). In particular, \(\mathcal{N}_p \cong \mathcal{T}_p/\mathcal{T}_{p-1}\). We write \(T = \langle \mathcal{N}_1, \ldots, \mathcal{N}_n \rangle\).
In some cases one can hope that $\mathcal{T}$ has a semi-orthogonal decomposition $\mathcal{T} = \langle N_1, \ldots, N_n \rangle$ in which each $N_p$ is as simple as possible, i.e. each of them is equivalent to the bounded derived category of finite-dimensional vector spaces.

**Definition 1.3.** An object $E$ of a $k$–linear triangulated category $\mathcal{T}$ is called exceptional if $\text{Hom}(E,E[l]) = 0$ whenever $l \neq 0$, and $\text{Hom}(E,E) = k$. An exceptional collection in $\mathcal{T}$ is a sequence of exceptional objects $\sigma = (E_1, \ldots, E_n)$ satisfying the semi-orthogonality condition $\text{Hom}(E_i, E_j[l]) = 0$ for all $l$ whenever $i > j$.

If a triangulated category $\mathcal{T}$ has an exceptional collection $\sigma = (E_1, \ldots, E_n)$ that generates the whole of $\mathcal{T}$, then this collection is called full. In this case $\mathcal{T}$ has a semi-orthogonal decomposition with $N_p = \langle E_p \rangle$. Since $E_p$ is exceptional, each of these categories is equivalent to the bounded derived category of finite dimensional $k$-vector spaces. In this case we write $\mathcal{T} = \langle E_1, \ldots, E_n \rangle$.

**Definition 1.4.** An exceptional collection $\sigma = (E_1, \ldots, E_n)$ is called strong if, in addition, $\text{Hom}(E_i, E_j[l]) = 0$ for all $i$ and $j$ when $l \neq 0$.

Let $\mathcal{T}$ be a triangulated category with a full strong exceptional collection $\sigma = (E_1, \ldots, E_n)$. The algebra $A = \text{End}(\bigoplus_{i=1}^n E_i)$ is called the algebra of endomorphisms of the exceptional collection $\sigma$.

### 1.2. Differential graded categories and enhancements.

Here we only introduce notations and recall some facts on differential graded (DG) categories. Our main references for DG categories are [Keller] and [Dwyer]. A differential graded or DG category is a $k$–linear category $\mathcal{A}$ whose morphism spaces $\text{Hom}(X,Y)$ are complexes of $k$–vector spaces, so that for any $X,Y,Z \in \text{Ob}(\mathcal{C})$ the composition $\text{Hom}(Y,Z) \otimes \text{Hom}(X,Y) \to \text{Hom}(X,Z)$ is a morphism of DG $k$–modules. The identity morphism $1_X \in \text{Hom}(X,X)$ is required to be closed of degree zero.

For a DG category $\mathcal{A}$ we denote by $\mathcal{H}^0(\mathcal{A})$ its homotopy category. The homotopy category $\mathcal{H}^0(\mathcal{A})$ has the same objects as the DG category $\mathcal{A}$ and its morphisms are defined by taking the 0-th cohomology $H^0(\text{Hom}_\mathcal{A}(X,Y))$ of the complex $\text{Hom}_\mathcal{A}(X,Y)$.

As usual, a DG functor $F : \mathcal{A} \to \mathcal{B}$ is given by a map $F : \text{Ob}(\mathcal{A}) \to \text{Ob}(\mathcal{B})$ and by morphisms of DG $k$–modules

$$F_{X,Y} : \text{Hom}_\mathcal{A}(X,Y) \to \text{Hom}_\mathcal{B}(FX,FY), \quad X,Y \in \text{Ob}(\mathcal{A})$$

compatible with the composition and units. A DG functor $F : \mathcal{A} \to \mathcal{B}$ is called a quasi-equivalence if $F_{X,Y}$ is a quasi-isomorphism for all pairs of objects $X,Y$ of $\mathcal{A}$ and the induced functor $H^0(F) : \mathcal{H}^0(\mathcal{A}) \to \mathcal{H}^0(\mathcal{B})$ is an equivalence. Two DG categories $\mathcal{A}$ and $\mathcal{B}$ are quasi-equivalent if there is a DG category $\mathcal{C}$ and quasi-equivalences $\mathcal{A} \xrightarrow{\sim} \mathcal{C} \xrightarrow{\sim} \mathcal{B}$.

Given a small DG category $\mathcal{A}$ we define a right DG $\mathcal{A}$–module as a DG functor $M : \mathcal{A}^{op} \to \mathcal{Mod}–k$, where $\mathcal{Mod}–k$ is the DG category of DG $k$–modules. We denote by $\mathcal{Mod}–\mathcal{A}$ the DG
category of right DG $\mathcal{A}$–modules. Each object $Y$ of $\mathcal{A}$ produces a right module $\text{Hom}_\mathcal{A}(-, Y)$ which is called a free DG module represented by $Y$. We obtain the Yoneda DG functor $\mathbf{h}^\bullet : \mathcal{A} \to \text{Mod–}\mathcal{A}$ that is fully faithful. The derived category $\mathcal{D}(\mathcal{A})$ is defined as the Verdier quotient

$$\mathcal{D}(\mathcal{A}) := \mathcal{H}^0(\text{Mod–}\mathcal{A}) / \mathcal{H}^0(\mathcal{A}c–\mathcal{A}),$$

where $\mathcal{A}c–\mathcal{A}$ is the full DG subcategory of $\text{Mod–}\mathcal{A}$ consisting of all acyclic DG modules, i.e. DG modules $M$ for which the complexes of $k$-modules $M(X)$ are acyclic for all $X \in \mathcal{A}$.

**Definition 1.5.** The triangulated category of perfect DG modules $\text{Perf–}\mathcal{A}$ is the smallest triangulated subcategory of $\mathcal{D}(\mathcal{A})$ that contains all free DG modules and is closed under direct summands.

The triangulated categories $\mathcal{D}(\mathcal{A})$ and $\text{Perf–}\mathcal{A}$ are invariant under quasi-equivalences of $\mathcal{A}$.

For any DG category $\mathcal{A}$ there exist a DG category $\mathcal{A}^{\text{pre-tr}}$ that is called the pretriangulated hull of $\mathcal{A}$ and a canonical fully faithful DG functor $\mathcal{A} \hookrightarrow \mathcal{A}^{\text{pre-tr}}$. The idea of the definition of $\mathcal{A}^{\text{pre-tr}}$ is to add to $\mathcal{A}$ all shifts, all cones, cones of morphisms between cones and etc. A DG category $\mathcal{A}$ is called pretriangulated if the canonical DG functor $\mathcal{A} \to \mathcal{A}^{\text{pre-tr}}$ is a quasi-equivalence. It is equivalent to require that the homotopy category $\mathcal{H}^0(\mathcal{A})$ is triangulated as a subcategory of $\mathcal{H}^0(\text{Mod–}\mathcal{A})$. The DG category $\mathcal{A}^{\text{pre-tr}}$ is always pretriangulated, so $\mathcal{H}^0(\mathcal{A}^{\text{pre-tr}})$ is a triangulated category.

**Definition 1.6.** Let $\mathcal{T}$ be a triangulated category. An enhancement of $\mathcal{T}$ is a pair $(\mathcal{A}, \varepsilon)$, where $\mathcal{A}$ is a pretriangulated DG category and $\varepsilon : \mathcal{H}^0(\mathcal{A}) \xrightarrow{\sim} \mathcal{T}$ is an exact equivalence.

For any quasi-compact and separated scheme $X$ over an arbitrary field $k$ the derived category $\mathcal{D}(\text{Qcoh} X)$ has an enhancement that is coming from $h$-injective complexes (see, e.g. [KS]), i.e. $\mathcal{H}^0(\mathcal{I}(X)) \cong \mathcal{D}(\text{Qcoh} X)$, where $\mathcal{I}(X)$ the DG category of $h$-injective complexes. As a consequence, we obtain an enhancements for any full triangulated subcategory of $\mathcal{D}(\text{Qcoh} X)$, for example for the triangulated category of perfect complexes $\text{Perf–}X$ and for the bounded derived category of coherent sheaves $\mathcal{D}^b(\text{coh} X)$ in noetherian case.

There are different notions of generators in triangulated categories. We recall the most useful notion of generating a triangulated category that is the notion of a classical generator.

**Definition 1.7.** An object $E \in \mathcal{T}$ is called a classical generator if the category $\mathcal{T}$ coincides with the smallest triangulated subcategory that contains $E$ and is closed under taking direct summands.

Note that the category of perfect complexes $\text{Perf–}X$ admits a classical generator for any quasi-compact and quasi-separated scheme $X$ (see [Ne] [BV]). If $X$ is quasi-projective of dimension $d$ then the object $\bigoplus_{p=0}^{d} \mathcal{L}^{\otimes p}$, where $\mathcal{L}$ is very ample line bundle, is a classical generator (see [O2]).

The following theorem shows that the notion of classical generator is very useful for triangulated categories admitting enhancements.
Theorem 1.8 ([K1], [K2]). Let $\mathcal{T}$ be an idempotent complete triangulated category that admits an enhancement $\mathcal{A}$ and let $E \in \mathcal{T}$ be a classical generator. Then the category $\mathcal{T}$ is equivalent to the triangulated category $\text{Perf} - A$, where $A = \text{Hom}_\mathcal{A}(E, E)$ is the DG algebra of endomorphisms of the object $E$ in the DG category $\mathcal{A}$.

This theorem implies the following corollary that we will use in the sequel.

Corollary 1.9. Let $\mathcal{T}$ be a triangulated category that admits an enhancement. Assume that $\mathcal{T}$ has a full strong exceptional collection $\sigma = (E_1, \ldots, E_n)$. Then the category $\mathcal{T}$ is equivalent to the derived category $D^b(\text{mod} - A)$, where $A = \text{End}(\bigoplus_{i=1}^n E_i)$ is the algebra of endomorphisms of the collection $\sigma$.

Proof. Since $\sigma$ is full, the object $E = \bigoplus_{i=1}^n E_i$ is a classical generator. As $\sigma$ is strong, the DG algebra of endomorphisms of the object $E$ has only 0-th cohomology. Hence, this DG algebra is quasi-isomorphic to the usual endomorphism algebra of the collection $\sigma$. As a triangulated category with a full exceptional collection the category $\mathcal{T}$ is idempotent complete. Now corollary follows from the Theorem 1.8 and the fact that for any algebra of finite global dimension the derived category $D^b(\text{mod} - A)$ is equivalent to the category of perfect complexes over $A$. □

In the paper [O3] we showed that any triangulated category with a full exceptional collection has a geometric realization as long as it has an enhancement. More precisely, we proved the following.

Theorem 1.10. [O3, Th.5.8] Let $\mathcal{A}$ be a small DG category over $k$ such that the homotopy category $\mathcal{T} = \mathcal{H}^0(\mathcal{A})$ has a full exceptional collection $\mathcal{T} = \langle E_1, \ldots, E_n \rangle$. Then there are a smooth projective scheme $X$ and an exceptional collection of line bundles $\sigma = (L_1, \ldots, L_n)$ on $X$ such that the subcategory of $D^b(\text{coh} X)$, generated by $\sigma$, is equivalent to $\mathcal{T}$. Moreover, $X$ is a sequence of projective bundles and has a full exceptional collection.

The scheme $X$ has a full exceptional collection as a tower of projective bundles (see [O1]). Furthermore, it follows from the construction that a full exceptional collection on $X$ can be chosen in a way that it contains the collection $\sigma = (L_1, \ldots, L_n)$ as a subcollection.

In the proof of this theorem we constructed a functor from the triangulated category $\mathcal{T}$ to the derived category $D^b(\text{coh} X)$ that sends the exceptional objects $E_i$ to shifts of the line bundles $L_i[r_i]$ for some integers $r_i$. Of course, we can not expect in general that $E_i$ go to line bundles without shifts. On the other hand, in the case of strong exceptional collections it is natural to seek realizations as collections of vector bundles (without shifts) on smooth projective varieties. It can be shown that in general we can not realize a strong exceptional collection as a collection of unshifted line bundles (see Remark 3.2), but trying to present it in terms of vector bundles seem quite reasonable.

In this paper we deal with strong exceptional collections and discuss different constructions of geometric realizations as strong exceptional collections of vector bundles on smooth projective varieties. We prove that such realizations always exist.
2. Geometric realizations

2.1. Quiver algebras. A quiver is a finite directed graph, possibly with multiple arrows and loops. More precisely, a quiver $Q$ consists of a data $(Q_0, Q_1, s, t)$, where $Q_0, Q_1$ are finite sets of vertices and arrows respectively, while $s, t : Q_1 \to Q_0$ are maps attaching to each arrow its source and target.

The path $k$-algebra of the quiver $Q$ is the algebra $kQ$ determined by the generators $e_q$ for $q \in Q_0$ and $a$ for $a \in Q_1$ with the following relations

$$e_q^2 = e_q, \quad e_re_q = 0, \quad \text{when} \quad r \neq q, \quad \text{and} \quad e_{t(a)}a = ae_{s(a)} = a.$$ 

In particular, the elements $e_q$ are orthogonal idempotents of the path algebra $kQ$. It follows from the relations above that $e_qa = 0$ unless $q = t(a)$ and $ae_q = 0$ unless $q = s(a)$.

As a $k$-vector space the path algebra $kQ$ has a basis consisting of the set of all paths in $Q$, where a path $\mathbf{p}$ is a possibly empty sequence $a_m a_{m-1} \cdots a_1$ of compatible arrows, i.e. $s(a_{i+1}) = t(a_i)$ for all $i = 1, \ldots, m - 1$. For an empty path we have to choose a vertex from $Q_0$. The composition of two paths $\mathbf{p}_1$ and $\mathbf{p}_2$ in $Q$ is defined naturally as $\mathbf{p}_2 \mathbf{p}_1$ if they are compatible and as $0$ if they are not compatible. This is a more natural definition of the product in paths algebra $kQ$.

To obtain a more general class of algebras, it is useful to introduce in consideration quivers with relations. A relation on a quiver $Q$ is a subspace of $kQ$ spanned by linear combinations of paths having a common source and a common target, and of length at least 2. A quiver with relations is a pair $(Q, I)$, where $Q$ is a quiver and $I$ is a two-sided ideal of the path algebra $kQ$ generated by relations. The quotient algebra $kQ/I$ will be called a quiver algebra of the quiver with relations $(Q, I)$. It can be shown that every module category $\text{mod} - A$, where $A$ is a finite dimensional algebra over $k$, is equivalent to $\text{mod} - kQ/I$ for some quiver with relations $(Q, I)$ (see e.g. [Ga]).

A quiver algebra $A = kQ/I$ viewed as right module over itself can be decomposed as a direct sum of projective modules $P_q = e_q A$ for $q \in Q_0$, i.e. $A = \bigoplus_{q \in Q_0} P_q$. The projective modules $P_q \subset A$ consist of all paths $\mathbf{p}$ with fixed target (or tail) $t(\mathbf{p}) = q$.

In this paper we consider quiver algebras for special type quivers with relations that are directly related to exceptional collections.

**Definition 2.1.** We say that $A$ is a quiver algebra on $n$ ordered vertices if it is a quiver algebra of a quiver with relations $(Q, I)$ for which $Q_0 = \{1, \ldots, n\}$ is the ordered set of $n$ elements and $s(a) < t(a)$ for any arrow $a \in Q_1$.

It is evident that the algebra of endomorphisms of any (strong) exceptional collection $\sigma = (E_1, \ldots, E_n)$ is a quiver algebra on $n$ ordered vertices. On the other hand, a quiver algebra $A$ on $n$ ordered vertices has finite global dimension and, moreover, its derived category $D^b(\text{mod} - A)$ has a strong full exceptional collection consisting of the projective modules $P_i$ for $i = 1, \ldots, n$. The algebra $A$ is exactly the algebra of endomorphisms of this full strong exceptional collection.
2.2. Strong exceptional collections and geometric realizations. Let $A$ be a quiver algebra on $n$ ordered vertices. Denote by $P_1, \ldots, P_n$ the respective right projective modules. As it was mentioned above, the collection of projective modules $(P_1, \ldots, P_n)$ is a full strong exceptional collection in the category $D^b(\text{mod-}A)$.

**Main Goal 2.2.** Let $A$ be a quiver algebra on $n$ ordered vertices. Our main goal is to construct a smooth projective variety $X$ and a strong (not full) exceptional collection $\sigma = (E_1, \ldots, E_n)$ of vector bundles on $X$ such that the algebra of endomorphisms $\text{End}(\bigoplus_{i=1}^{n}E_i)$ coincides with the algebra $A$. Assume that on a smooth projective variety $X$ there is a strong exceptional collection $\sigma = (E_1, \ldots, E_n)$, for which the algebra of endomorphisms $\text{End}(\bigoplus_{i=1}^{n}E_i)$ coincides with the algebra $A$. In this case by Corollary 1.9 the admissible subcategory of $D^b(\text{coh }X)$ generated by the collection $\sigma$ is equivalent to the derived category $D^b(\text{mod-}A)$ of the endomorphism algebra $A$. The inclusion functor sends the projective modules $P_i$ to the objects $E_i$. The right adjoint projection functor is $R\text{Hom}^\cdot(E, -) : D^b(\text{coh }X) \to D^b(\text{mod-}A)$, where $E = \bigoplus_{i=1}^{n}E_i$.

All of these statements are true for any objects $E_i$ not only for vector bundles. And it is proved in [O3] that such realizations exist for any exceptional collection. Our main aim in this paper is to give constructions of strong exceptional collections in terms of vector bundles. The main technique that we will use is an induction on the number of vertices and a passage from a quiver algebra $A$ to its ordinary extension by an $A$-module.

Let $A$ be a quiver algebra on $n$ ordered vertices. Denote by $S_i, i = 1, \ldots, n$, its respective simple modules. The set of simple modules $(S_n, \ldots, S_1)$ also forms a full (not strong) exceptional collection in $D^b(\text{mod-}A)$ called the dual to the collection of projective modules. Thus we have the following semi-orthogonal decompositions

$$D^b(\text{mod-}A) = \langle P_1, \ldots, P_n \rangle = \langle S_n, \ldots, S_1 \rangle.$$  

Note that for any finite right $A$-module $M$ there is a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that each successive quotient $M_p/M_{p-1}$ is isomorphic to a (finite) direct sum of copies of the corresponding simple module $S_p$.

Let us consider a quiver algebra $\tilde{A}$ on $n+1$ ordered vertices. Denote by $\tilde{P}_1, \ldots, \tilde{P}_{n+1}$ its right projective modules and denote by $\tilde{S}_i$ the respective simple modules.

Consider the first $n$ vertices and denote by $\tilde{A}$ the quiver algebra of the subquiver corresponding to these $n$ vertices. We have a full embedding $\text{mod-}A \to \text{mod-}\tilde{A}$ of abelian categories. It is an exact functor that sends the simple $A$-modules $S_i$ to the simple $\tilde{A}$-modules $\tilde{S}_i$ and the projective $A$-modules $P_i$ to the projective $\tilde{A}$-modules $\tilde{P}_i$ for all $i = 1, \ldots, n$. Furthermore, the induced derived functor $i : D^b(\text{mod-}A) \to D^b(\text{mod-}\tilde{A})$ is fully faithful.
Thus, there are semi-orthogonal decompositions
\[
\mathcal{D}^b(\text{mod-} \tilde{A}) = \left\langle \tilde{S}_{n+1}, \mathcal{D}^b(\text{mod-}A) \right\rangle = \left\langle \mathcal{D}^b(\text{mod-}A), \tilde{P}_{n+1} \right\rangle.
\]
Let us consider the natural surjection \( \tilde{P}_{n+1} \rightarrow \tilde{S}_{n+1} \) and denote by \( M \) the kernel of this map. In the exact sequence of \( \tilde{A} \)-modules
\[
0 \rightarrow M \rightarrow \tilde{P}_{n+1} \rightarrow \tilde{S}_{n+1} \rightarrow 0
\]
the module \( M \) belongs to the subcategory \( \text{i}(\mathcal{D}^b(\text{mod-}A)) \) and, hence, it can be considered as \( \tilde{A} \)-module. In particular, for all \( i = 1, \ldots, n \) we have isomorphisms
\[
\text{Hom}_{\tilde{A}}(\tilde{P}_i, \tilde{P}_{n+1}) \cong \text{Hom}_{\tilde{A}}(\tilde{P}_i, M) \cong \text{Hom}_A(P_i, M).
\]
This means that the algebra \( \tilde{A} \) can be constructed as low-triangular algebra of the following form
\[(1)
\tilde{A} = \begin{pmatrix} A & 0 \\ M & k \end{pmatrix},
\]
where \( A \) is a quiver algebra on \( n \) vertices and \( M \) is a right \( A \)-module. The algebra \( \tilde{A} \) is uniquely determined by the algebra \( A \) and the right \( A \)-module \( M \).

**Definition 2.3.** The algebra \( \tilde{A} \) defined by rule (1) will be called an ordinary extension of the algebra \( A \) via the \( A \)-module \( M \).

We start with a realization of the algebra \( A \) in terms of vector bundles and will try to extend this to a realization on the algebra \( \tilde{A} \). First, we record a simple fact that will be useful for applications.

**Proposition 2.4.** Let \( A \) be a quiver algebra on \( n \) ordered vertices and let \( \tilde{A} \) be its ordinary extension via an \( A \)-module \( M \). Suppose there are a smooth projective variety \( X \) and a fully faithful functor \( \mathcal{F} : \mathcal{D}^b(\text{mod-}A) \rightarrow \mathcal{D}^b(\text{coh}\ X) \) that sends the projective modules \( P_i \) to vector bundles \( \mathcal{E}_i \). Assume that there is a vector bundle \( \mathcal{M} \) on \( X \) such that \( R\text{Hom}(\bigoplus_{i=1}^n \mathcal{E}_i, \mathcal{M}) \cong M \) in \( \mathcal{D}^b(\text{mod-}A) \).

Then there are a smooth projective variety \( \tilde{X} \) and a fully faithful functor \( \mathcal{F} : \mathcal{D}^b(\text{mod-}A) \rightarrow \mathcal{D}^b(\text{coh}\ \tilde{X}) \) that sends the projective modules \( \tilde{P}_i \) to some vector bundles \( \tilde{\mathcal{E}}_i \) on \( \tilde{X} \).

**Proof.** The proof is direct. Let us put \( \tilde{X} = \mathbb{P}(\mathcal{M}^\vee) \) with the natural projection \( \pi : \tilde{X} \rightarrow X \). There is a canonical exact sequence on \( \tilde{X} \)
\[(2)
0 \rightarrow \Omega_{\tilde{X}/X}(1) \rightarrow \pi^*\mathcal{M} \rightarrow \mathcal{O}_{\tilde{X}}(1) \rightarrow 0,
\]
where \( \Omega_{\tilde{X}/X} \) is the relative cotangent bundle and \( \mathcal{O}_{\tilde{X}}(-1) \) is the tautological line bundle on \( \tilde{X} \).

Let us take \( \tilde{\mathcal{E}}_i = \pi^*(\mathcal{E}_i) \) for all \( i = 1, \ldots, n \) and put \( \tilde{\mathcal{E}}_{n+1} = \mathcal{O}_{\tilde{X}}(1) \). It is easy to see that the collection \( \tilde{\sigma} = (\tilde{\mathcal{E}}_1, \ldots, \tilde{\mathcal{E}}_n, \tilde{\mathcal{E}}_{n+1}) \) is exceptional. There is a sequence of isomorphisms
\[
\text{Ext}^i_X(\tilde{\mathcal{E}}_i, \tilde{\mathcal{E}}_{n+1}) = \text{Ext}^i_X(\pi^*\mathcal{E}_i, \mathcal{O}_{\tilde{X}}(1)) \cong \text{Ext}^i_X(\mathcal{E}_i, R\pi_*\mathcal{O}_{\tilde{X}}(1)) \cong \text{Ext}^i_X(\mathcal{E}_i, \mathcal{M}) \cong \text{Ext}^i_A(P_i, M)
\]
for all $i = 1, \ldots, n$. It implies that the exceptional collection $\sigma$ is strong and the algebra of endomorphisms of this collection is isomorphic to the extended algebra $\tilde{A}$.

There is another construction when one takes as $\tilde{X}$ the projective bundle $\mathbb{P}(\mathcal{M})$. In this case the dual exact sequence should be considered

$$0 \rightarrow \mathcal{O}_{\tilde{X}}(-1) \rightarrow \pi^*\mathcal{M} \rightarrow \mathcal{T}_{\tilde{X}/X}(-1) \rightarrow 0.$$  

As above we take $\tilde{E}_i = \pi^*(E_i)$ for all $i = 1, \ldots, n$, but put $\tilde{E}_{n+1} = \mathcal{T}_{\tilde{X}/X}(-1)$. The similar calculations give us that the collection $\tilde{\sigma} = (\tilde{E}_1, \ldots, \tilde{E}_n, \tilde{E}_{n+1})$ is strong exceptional and its algebra of endomorphisms is also isomorphic to the ordinary extended algebra $\tilde{A}$.

In both cases we obtain a fully faithful functor $\tilde{r} : D^b(\text{mod} - \tilde{A}) \rightarrow D^b(\text{coh} \tilde{X})$ that sends the projective modules $\tilde{P}_i$ to the vector bundles $\tilde{E}_i$ on $\tilde{X}$ for all $i = 1, \ldots, n + 1$.

These constructions can be useful in particular cases when one would like to represent a strong exceptional collection as a collection of vector bundles on a smooth projective variety. However, they can not help with the general proof, because we can not guarantee that a given module will be represented by a vector bundle in the next step of induction.

2.3. The main theorem. Let $A$ be a quiver algebra on $n$ ordered vertices. Suppose there is an exact functor $u : \text{mod} - A \rightarrow \text{coh}(X)$ between abelian categories. Denote by $u$ the derived functor from $D^b(\text{mod} - A)$ to $D^b(\text{coh} X)$. Since $u$ is exact the derived functor is defined and coincides with $u$ on the abelian category $\text{mod} - A$, i.e. $u(M) \cong \tilde{u}(M)$ for any $A$-module $M$.

We say that the exact functor $u : \text{mod} - A \rightarrow \text{coh}(X)$ satisfies property (V) if the following conditions hold

1) the induced derived functor $u : D^b(\text{mod} - A) \rightarrow D^b(\text{coh} X)$ is fully faithful;
2) simple modules $S_i$ go to line bundles $L_i$ on $X$ under $u$;
3) there is a line bundle $\mathcal{L}$ on $X$ such that $\mathcal{L} \in \mathcal{D}^1(u(D^b(\text{mod} - A)))$, the line bundles $\mathcal{L} \otimes L_i^{-1}$ are generated by global sections and $H^j(X, \mathcal{L} \otimes L_i^{-1}) = 0$ when $j \geq 1$ for all $i = 1, \ldots, n$.

Since any module $M$ has a composition filtration with successive quotient being simple modules, the condition (2) implies that any $A$-module $M$ goes to a vector bundle under the functor $u$. Moreover, by the same reasoning the vector bundle $u(M)$ has a filtration with successive quotients isomorphic to the line bundles $L_i$. Now it is not difficult to check that condition (3) of (V) implies the following condition

3') there is a line bundle $\mathcal{L}$ on $X$ such that $\mathcal{L} \in \mathcal{D}^1(u(D^b(\text{mod} - A)))$ and the vector bundles $\mathcal{L} \otimes u(M)^\vee$ are generated by global sections for all $M \in \text{mod} - A$.

Consider the exact functor $u : \text{mod} - A \rightarrow \text{coh}(X)$ and denote by $E_i$ the vector bundles $u(P_i)$ for all $i = 1, \ldots, n$. Since the derived functor $u$ is fully faithful, the sequence $\sigma = (E_1, \ldots, E_n)$ of vector bundles on $X$ forms a strong exceptional collection.
Proposition 2.5. Let $A$ be a quiver algebra on $n$ ordered vertices and let $\tilde{A}$ be its ordinary extension via an $A$-module $M$. Suppose there exist a smooth projective scheme $X$ and an exact functor $u : \text{mod} - A \to \text{coh}(X)$ that satisfies property (V). Then there are a vector bundle $F$ on $X$ and an exact functor $\tilde{u} : \text{mod} - \tilde{A} \to \text{coh}(\mathbb{P}(F))$ that also satisfies property (V). Moreover, the restriction of the functor $\tilde{u}$ on $\text{mod} - A$ is isomorphic to $\pi^* \circ u$, where $\pi : \mathbb{P}(F) \to X$ is the natural projection.

Proof. Let $u : \text{mod} - A \to \text{coh}(X)$ be an exact fully faithful functor that satisfies property (V). Denote by $\mathcal{M}$ the vector bundle $u(M)$. By assumption (3) and its consequence (3') there is a surjection $(\mathcal{L}^{-1})^{\oplus m} \to \mathcal{M}'$ for some $m \in \mathbb{N}$. Denote by $\mathcal{F}$ the vector bundle on $X$ that is dual to the kernel of this surjection. Thus, we have an exact sequence

$$(4) \quad 0 \to \mathcal{F}' \to (\mathcal{L}^{-1})^{\oplus m} \to \mathcal{M}' \to 0.$$ 

Taking sufficiently large $m$ we can assume that the rank of $\mathcal{F}$ is greater than 2. Let us consider the projective bundle $\pi : \mathbb{P}(\mathcal{F}) \to X$ and denote it as $\tilde{X}$. There are natural exact sequences on $\tilde{X}$ of the following form

$$0 \to \Omega_{\tilde{X}/X}(1) \to \pi^* \mathcal{F}' \to \mathcal{O}_{\tilde{X}}(1) \to 0, \quad \text{and} \quad 0 \to \mathcal{O}_{\tilde{X}}(-1) \to \pi^* \mathcal{F} \to \mathcal{T}_{\tilde{X}/X}(-1) \to 0,$$

where $\mathcal{O}_{\tilde{X}}(-1)$ is the tautological line bundle and $\mathcal{T}_{\tilde{X}/X}, \Omega_{\tilde{X}/X}$ are relative tangent and relative cotangent bundles, respectively.

Denote by $\tilde{\mathcal{L}}_i$ the pull back line bundles $\pi^* \mathcal{L}_i$ for $i = 1, \ldots, n$ and put $\tilde{\mathcal{L}}_{n+1} = \mathcal{O}_{\tilde{X}}(-1)$. Under the sequence of isomorphisms

$$\text{Ext}^1_X(\mathcal{M}', \mathcal{F}') \cong \text{Ext}^1_X(\pi^* \mathcal{M}', \mathcal{O}_{\tilde{X}}(1)) \cong \text{Ext}^1_X(\mathcal{O}_{\tilde{X}}(-1), \pi^* \mathcal{M})$$

the element $e \in \text{Ext}^1_X(\mathcal{M}', \mathcal{F}')$, which defines the short exact sequence (4), gives some element $e' \in \text{Ext}^1_X(\mathcal{O}(-1), \pi^* \mathcal{M})$. The element $e'$ defines the following extension

$$(5) \quad 0 \to \pi^* \mathcal{M} \to \tilde{\mathcal{L}}_{n+1} \to \mathcal{O}(-1) \to 0.$$ 

that can be considered as a definition of the vector bundle $\tilde{\mathcal{L}}_{n+1}$.

It follows from the definition of the exact sequence (5) that the dual sequence goes to the the exact sequence (4) under the direct image functor $R\pi_*$. In particular, there is an isomorphism $R\pi_*\tilde{\mathcal{L}}_{n+1} \cong (\mathcal{L}^{-1})^{\oplus m}$. This fact implies the following vanishing of Hom’s spaces

$$(6) \quad \text{Hom}^i_X(\tilde{\mathcal{L}}_{n+1}, \pi^* \mathcal{N}) \cong H^i(\tilde{X}, \pi^* \mathcal{N} \otimes \tilde{\mathcal{L}}_{n+1}) \cong H^i(X, \mathcal{N} \otimes R\pi_*\tilde{\mathcal{L}}_{n+1}) \cong$$

$$\cong H^i(X, \mathcal{N} \otimes (\mathcal{L}^{-1})^{\oplus m}) \cong \text{Hom}^i_X(\mathcal{L}^{\oplus m}, \mathcal{N}) = 0$$

for any object $\mathcal{N}$ from the image of the functor $u$ by (3) of (V). Hence, the vector bundle $\tilde{\mathcal{L}}_{n+1}$ belongs to the left orthogonal $\perp u^*(D^b(\text{mod} - A))$. 
Furthermore, since \( \text{Hom}^j_X(\mathcal{E}_{n+1}, \pi^*\mathcal{M}) = 0 \) and \( \text{Hom}^j_X(\pi^*\mathcal{M}, \mathcal{O}_X(-1)) = 0 \), the short exact sequence (5) induces the following isomorphisms

\[
(7) \quad \text{Hom}^j_X(\mathcal{E}_{n+1}, \mathcal{E}_{n+1}) \cong \text{Hom}^j_X(\mathcal{E}_{n+1}, \mathcal{O}_X(-1)) \cong \text{Hom}^j_X(\mathcal{O}_X(1), \mathcal{O}_X(-1)).
\]

This implies that \( \mathcal{E}_{n+1} \) is exceptional. Denote by \( \mathcal{E}_i \) the vector bundles \( \pi^*\mathcal{E}_i \) for all \( i = 1, \ldots, n \).

The vanishing properties (6) and the isomorphisms (7) imply that the ordered set \( (\mathcal{E}_1, \ldots, \mathcal{E}_{n+1}) \) forms an exceptional collection. Denote it by \( \mathcal{E} \).

Let us check that the collection \( \mathcal{E} \) is strong and calculate the endomorphism algebra of this collection. Taking into account the exact sequence (5) we obtain the following isomorphisms

\[
\text{Hom}^j_X(\pi^*\mathcal{N}, \mathcal{E}_{n+1}) \cong \text{Hom}^j_X(\pi^*\mathcal{N}, \pi^*\mathcal{M}) \cong \text{Hom}^j_X(\mathcal{N}, \mathcal{M}).
\]

If now \( \mathcal{N} = u(\mathcal{N}) \), where \( \mathcal{N} \) is an \( \mathcal{A} \)-module, then we obtain isomorphisms

\[
(8) \quad \text{Hom}^j_X(\pi^*\mathcal{N}, \mathcal{E}_{n+1}) \cong \text{Hom}^j_X(\mathcal{N}, \mathcal{M}).
\]

This implies that \( \text{Ext}^j_X(\mathcal{E}_i, \mathcal{E}_{n+1}) = 0 \) for all \( i = 1, \ldots, n \) and \( j \geq 1 \). Hence, the collection \( \mathcal{E} \) is strong. Moreover, the isomorphisms (8) for \( j = 0 \) allow us to calculate the endomorphism algebra of the collection \( \mathcal{E} \). Thus, the endomorphism algebra \( \text{End}(\bigoplus_{i=1}^{n+1} \mathcal{E}_i) \) of the collection \( \mathcal{E} \) is isomorphic to the ordinary extended algebra \( \mathcal{A} \).

Summarizing, we have an exact functor \( \tilde{u} : \text{mod} - \tilde{A} \to \text{coh}(\tilde{X}) \) that sends the projective modules \( P_i \) to the vector bundles \( \tilde{E}_i \) while simple modules \( S_i \) go to \( \tilde{L}_i \) for all \( i = 1, \ldots, n + 1 \). Since \( \text{End}(\bigoplus_{i=1}^{n+1} \tilde{E}_i) \cong \tilde{A} \) and the collection \( \mathcal{E} \) is strong, the derived functor \( \tilde{u} : \text{D}^b(\text{mod} - \tilde{A}) \to \text{D}^b(\text{coh} \tilde{X}) \) is fully faithful by Theorem 1.8. It is evident from the definition of \( \tilde{L}_i \) and \( \tilde{E}_i \) for \( i = 1, \ldots, n \) as pull backs of vector bundles from \( \tilde{X} \) that the restriction of the functor \( \tilde{u} \) on \( \text{mod} - \mathcal{A} \) is isomorphic to \( \pi^* \circ u \). This implies that the conditions (1) and (2) of (V) hold for the functor \( \tilde{u} \).

Finally, we have to show that the condition (3) also holds for an appropriate line bundle \( \tilde{L}' \) on \( \tilde{X} \). Choosing \( \tilde{L}' \) as a line bundle \( \mathcal{O}_X(1) \otimes \pi^*\mathcal{R}^\otimes s \), where \( \mathcal{R} \) is an ample line bundle on \( X \) and \( s \) is sufficiently large, we can guarantee that the condition (3) will hold. Indeed, since the rank of \( \mathcal{F} \) is greater than 2 the line bundle \( \tilde{L}' \) belongs to \( \mathcal{D}^b(\text{mod} - \tilde{A}) \). Besides, we have isomorphisms

\[
H^j(\tilde{X}, \tilde{L}_i^{-1} \otimes \tilde{L}') \cong H^j(X, \mathcal{L}_i^{-1} \otimes \mathcal{F}^\vee \otimes \mathcal{R}^\otimes s) \quad \text{and} \quad H^j(\tilde{X}, \tilde{L}_{n+1}^{-1} \otimes \tilde{L}') \cong H^j(X, S^2(\mathcal{F}^\vee) \otimes \mathcal{R}^\otimes s)
\]

for the cohomology of \( \tilde{L}_{n+1}^{-1} \otimes \tilde{L}' \) and of \( \tilde{L}_i^{-1} \otimes \tilde{L}' \), when \( i = 1, \ldots, n \).

Taking a sufficiently large \( s \) we obtain vanishing of cohomology for \( j > 0 \) and can guarantee that all these bundles are generated by global sections on \( X \) and on \( \tilde{X} \). \( \square \)

Proposition 2.5 as an induction step implies our Main Theorem.

**Theorem 2.6.** Let \( \mathcal{A} \) be a quiver algebra on \( n \) ordered vertices. Then there exist a smooth projective variety \( X \) and an exact functor \( u : \text{mod} - \mathcal{A} \to \text{coh}(X) \) such that the following conditions hold.
1) the induced derived functor \( u : D^b(\text{mod} - A) \to D^b(\text{coh} X) \) is fully faithful;
2) simple modules \( S_i \) go to line bundles \( L_i \) on \( X \) under \( u \);
3) any \( A \)-module \( M \) goes to a vector bundle on \( X \);
4) the variety \( X \) is a tower of projective bundles and has a full exceptional collection.

**Proof.** The proof proceeds by induction on \( n \). The base of induction is \( n = 1 \) and \( A = k \). In this case \( X = \mathbb{P}^1 \), the functor \( u \) sends \( A \) to \( \mathcal{O}_{\mathbb{P}^1} \), and \( \mathcal{L} = \mathcal{O}(1) \). The inductive step is Proposition 2.5. By construction, the variety \( X \) is a tower of projective bundles and, hence, has a full exceptional collection. \( \square \)

**Corollary 2.7.** Let \( \mathcal{A} \) be a small DG category over \( k \) such that the homotopy category \( \mathcal{T} = \mathcal{H}^0(\mathcal{A}) \) has a full exceptional collection \( \mathcal{T} = \langle E_1, \ldots, E_n \rangle \). Then there exist a smooth projective scheme \( X \) and fully faithful functor \( r : \mathcal{T} \to D^b(\text{coh} X) \) such that the functor \( r \) sends the exceptional objects \( E_i \) to vector bundles \( E_i \) on \( X \).

It follows directly from Theorem 2.6 taking in account Corollary 1.9.

**Corollary 2.8.** Let \( A \) be a quiver algebra on \( n \) ordered vertices. Then there exist a smooth projective variety \( X \) and a vector bundle \( E \) on \( X \) such that \( \text{End}_X(E) = A \) and \( \text{Ext}_X^j(E,E) = 0 \) for all \( j \neq 0 \). Moreover, they can be chosen so that the rank of \( E \) is equal to the dimension of \( A \).

**Proof.** It follows from Theorem 2.6. The vector bundles \( E \) is isomorphic to the direct sum \( \bigoplus_{i=1}^n E_i \). Since simple modules go to line bundles the rank of \( E \) is equal to the dimension of \( A \) over \( k \). \( \square \)

The quiver algebras on \( n \) vertices is a particular case of a finite dimensional algebra.

**Conjecture 2.9.** For any finite dimensional algebra \( \Lambda \) there exist a smooth projective variety \( X \) and a vector bundle \( E \) on \( X \) such that \( \text{End}_X(E) = \Lambda \) and \( \text{Ext}_X^j(E,E) = 0 \) for all \( j \geq 1 \).

This conjecture looks reasonable because of the following statement proven in [O3].

**Theorem 2.10.** [O3, Th.5.3] Let \( \Lambda \) be a finite dimensional algebra over \( k \). Assume that \( S = \Lambda/\mathfrak{R} \) is a separable \( k \)-algebra. Then there are a smooth projective scheme \( X \) and a perfect complex \( E \) such that \( \text{End}(E) \cong \Lambda \) and \( \text{Hom}(E,E[l]) = 0 \) for all \( l \neq 0 \).

3. Examples and applications

As applications we consider two examples of realizations of quiver algebras as endomorphism algebras of vector bundles on smooth projective varieties.

3.1. The quiver associated with the Ising 3-point function. We will combine the methods of proof of Proposition 2.5 and Proposition 2.4 to obtain varieties of small dimensions. Let us consider a quiver of type \( (2,2;2) \) that is defined by the following rule

\[
Q_I = \left( \begin{array}{c} \bullet \ar@<0.5ex>[r]^{a_1} \ar@<0.5ex>[l]^{b_1} & \bullet \ar@<0.5ex>[r]^{a_2} \ar@<0.5ex>[l]^{b_2} & \bullet \end{array} \right) \quad \bigg| \quad a_2 b_1 = 0, \ b_2 a_1 = 0 \right).
\]
The compositions of arrows \( a_2a_1 \) and \( b_2b_1 \) give two arrows from the first vertex to the third that will be denoted as \( a \) and \( b \), respectively. In this case, the three projective modules \( P_1, P_2, P_3 \) form an exceptional collection and all spaces of morphisms are 2-dimensional vector spaces. We denote by \( \widetilde{A} \) the algebra of this quiver \( Q_I \). It is an ordinary extension of the algebra \( A \) of the quiver \((\bullet \to \bullet)\) generated by projective modules \( P_1, P_2 \) via the \( A \)-module \( M = \text{Hom}_{\widetilde{A}}(P_1 \oplus P_2, P_3) \).

The quiver \( Q_I \) corresponds to the Ising 3-point function and is related to a Landau-Ginzburg model with superpotential the Weierstrass function \( W(z) = \varphi(z; \omega_1, \omega_2) \), and the identifications \( z \sim z + n_1 \omega_1 + n_2 \omega_2, \ n_i \in \mathbb{Z} \), where \( \omega_i \) are the two periods of \( \varphi(z; \omega_1, \omega_2) \) (see e.g. [CV]). In other words, the quiver \( Q_I \) in \((\bullet)\) is a quiver of the category of D-branes of type A in an LG model with the total space being an elliptic curve with a deleted flex point. Such a curve is given by an equation \( y^2 = 4x^2 - g_2x - g_3 \) in the affine plane and is equipped with the superpotential \( W(x, y) = x \).

Let us consider the two dimensional quadric \( Y = \mathbb{P}^1 \times \mathbb{P}^1 \) and take the two line bundles \( \mathcal{O}_Y \) and \( \mathcal{O}_Y(2, -1) \) on it. The pair \((\mathcal{O}(2, -1), \mathcal{O})\) is exceptional and there is only nontrivial two-dimensional \( \text{Ext}^1 \) from \( \mathcal{O}(2, -1) \) to \( \mathcal{O} \). Let us consider the universal extension

\[
0 \longrightarrow \mathcal{O}^{\oplus 2} \longrightarrow \mathcal{U} \longrightarrow \mathcal{O}(2, -1) \longrightarrow 0.
\]

As mutation in an exceptional pair the vector bundle \( \mathcal{U} \) is exceptional too and the pair \((\mathcal{O}, \mathcal{U})\) is an exceptional pair. Moreover, it is a strong exceptional pair, i.e. \( H^j(Y, \mathcal{U}) = 0 \) for \( j > 0 \) and \( H^0(Y, \mathcal{U}) = \mathcal{U} \) is the two-dimensional vector space.

Let us fix two projective lines \( L_1 \) and \( L_2 \) on \( Y \) that are fibers of the projection \( Y = \mathbb{P}^1 \times \mathbb{P}^1 \) on the first component and consider the following short exact sequence

\[
0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}(2, 0)^{\oplus 2} \overset{\phi}{\longrightarrow} \mathcal{O}_{L_1}(1) \oplus \mathcal{O}_{L_2}(1) \longrightarrow 0,
\]

where \( \phi \) is a general morphism and \( \mathcal{F} \) is the kernel of \( \phi \). The restriction of the line bundle \( \mathcal{O}(2, 0) \) on \( L_i \) is the trivial line bundle. Since each sheaf \( \mathcal{O}_{L_i}(1) \) is generated by global sections any general morphism \( \phi \) is surjective and consists of two components \( \phi_i : \mathcal{O}(2, 0)^{\oplus 2} \to \mathcal{O}_{L_i}(1) \) each of which is surjective. It is easy to see that the map on global sections

\[
H^0(Y, \mathcal{O}(2, 0)^{\oplus 2}) \longrightarrow H^0(Y, \mathcal{O}_{L_1}(1) \oplus \mathcal{O}_{L_2}(1))
\]

is also surjective. Hence, \( H^j(Y, \mathcal{F}) = 0 \) when \( j > 0 \) and \( H^0(Y, \mathcal{F}) \) is two-dimensional. It can also be checked that \( \text{Ext}^j(\mathcal{U}, \mathcal{F}) = 0 \) for \( j > 0 \) and \( \text{Hom}(\mathcal{U}, \mathcal{F}) \) is two-dimensional for general morphism \( \phi \). Moreover, if we consider the functor \( \mathbb{R} \text{Hom}(\mathcal{O} \oplus \mathcal{U}, -) \) from \( \mathbb{D}^b(\text{coh} Y) \) to \( \mathbb{D}^b(\text{mod} - A) \), where \( A = \text{End}_Y(\mathcal{O} \oplus \mathcal{U}) = \text{End}_{\widetilde{A}}(P_1 \oplus P_2) \), then this functor sends the bundle \( \mathcal{F} \) to the \( A \)-module \( M = \text{Hom}_{\widetilde{A}}(P_1 \oplus P_2, P_3) \).

Now we can apply Proposition \([2.3]\) and consider the projective bundle \( X = \mathbb{P}(\mathcal{F}^\vee) \) with the projection \( \pi \) on \( Y \). By Proposition \([2.3]\) the exceptional collection of vector bundles \((\mathcal{O}_X, \pi^* \mathcal{U}, \mathcal{O}_X(1))\)
is strong exceptional and the algebra of endomorphisms of this collection is the algebra ̃A that is
the quiver algebra of the quiver Q_I determined by the rule (9).

The variety X = \mathbb{P}(F^v) is 3-dimensional smooth projective variety, that is rational and possesses
a full exceptional collection. The quiver Q_I has an interesting property. It was checked many years
ago by A. Bondal that there is a module over the algebra of this quiver that is exceptional but the
semi-orthogonal complement to this module does not have any exceptional objects at all. Hence,
this exceptional module can not be include in a full exceptional collection in the derived category of
modules over this algebra (see [Kn]). This module as a representation of the quiver has 1-dimensional
vector spaces over each vertex and is such that the a-arrows act as isomorphisms while the b-arrows
act as 0. Thus, we obtain the following statement.

**Proposition 3.1.** There exist a smooth projective scheme X that is a projectivization of two
dimensional vector bundle over \( \mathbb{P}^1 \times \mathbb{P}^1 \) and a strong exceptional collection of vector bundles
(\( O_X, \pi^*U, O(1) \)) on it such that the algebra of endomorphisms of this collection is exactly the al-
gebra of the quiver Q_I (9). Moreover, the variety X possesses a full exceptional collection (of
length 8), on the one hand, and, on the other hand, there is an exceptional collection of length 6 that
can not be extended to a full exceptional collection on X.

**Remark 3.2.** It is also useful to note that the algebra of the quiver Q_I can not be realized as the
endomorphism algebra of a strong exceptional collection of line bundles on a variety. Indeed, any
morphism of line bundles on a smooth irreducible projective scheme is an isomorphism at the generic
point which contradicts the fact that \( b_2a_1 = 0 \).

3.2. Quivers of noncommutative projective planes. Noncommutative deformations of the projective
plane can be described in terms of exceptional collection [ATV, BP]. We know that the
derived category \( D^b(\text{coh} \mathbb{P}^2) \) has a full strong exceptional collection of line bundles (\( O, O(1), O(2) \)).
This means that the category \( D^b(\text{coh} \mathbb{P}^2) \) is equivalent to the derived category of finite modules of
the quiver algebra for the following quiver with relations

\[
Q_{\mathbb{P}^2} = \left( \begin{array}{ccc}
& a_1 & \\
1 & 2 & 3 \\
& b_1 & b_2 \\
c_1 & c_2 & \\
\end{array} \right) \quad \begin{array}{c}
\text{a}_2b_1 = b_2a_1, \quad a_2c_1 = c_2a_1, \quad b_2c_1 = c_2b_1 \end{array}
\]

A deformation of the category \( D^b(\text{coh} \mathbb{P}^2) \) is directly related to deformations of the relations of
the quiver \( Q_{\mathbb{P}^2} \). Namely, the derived category of coherent sheaves on a noncommutative projective
plane should be a triangulated category with a full strong exceptional collection (\( \mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2 \)), for
which the spaces of homomorphisms from \( \mathcal{F}_i \) to \( \mathcal{F}_j \) when \( j - i = 1 \) are 3-dimensional and the space
\( \text{Hom}(\mathcal{F}_0, \mathcal{F}_2) \) is a 6-dimensional vector space. Any such category is determined by a composition
tensor \( \mu : V \otimes U \rightarrow W \), where \( \dim V = \dim U = 3 \) and \( \dim W = 6 \). This map should be surjective.
Thus, the derive category of coherent sheaves $D^b(\text{coh } \mathbb{P}^2_\mu)$ on a noncommutative projective plane $\mathbb{P}^2_\mu$ is a category having a full strong exceptional collection with composition tensor $\mu$.

Denote by $I$ the relations, i.e. the kernel of $\mu$, and denote by $\nu$ the inclusion $I \to V \otimes U$. We will consider only the nondegenerate (geometric) case, where the restrictions $\nu_\bar{u} : I \to V$ and $\nu_\bar{v} : I \to U$ have rank at least two for all nonzero elements $\bar{u} \in U^\vee$ and $\bar{v} \in V^\vee$. The equations $\det \nu_\bar{u} = 0$ and $\det \nu_\bar{v} = 0$ define closed subschemes $\Gamma_U \subset \mathbb{P}(U^\vee)$ and $\Gamma_V \subset \mathbb{P}(V^\vee)$. It is easy to see that the correspondence which attaches the kernel of the map $\nu_\bar{v} : U^\vee \to I^\vee$ to a vector $\bar{v} \in V^\vee$ defines an isomorphism between $\Gamma_U$ and $\Gamma_V$. Moreover, under these circumstances $\Gamma_U$ is either a cubic in $\mathbb{P}(U^\vee)$ or the entire projective plane $\mathbb{P}(U^\vee)$. If $\Gamma_U = \mathbb{P}(U^\vee)$, then $\mu$ is the standard tensor $U \otimes U \to S^2U$. In this case we obtain the usual projective plane $\mathbb{P}^2$.

Thus, the non-trivial case is the situation where $\Gamma_V$ is a cubic. Let us denote it by $E$. The curve $E$ comes equipped with two embeddings into the projective planes $\mathbb{P}(U^\vee)$ and $\mathbb{P}(V^\vee)$, respectively. The restriction of the line bundles $O(1)$ these embeddings determine two line bundles $L_1$ and $L_2$ of degree 3 on $E$. This construction has an inverse.

**Construction 3.3.** The tensor $\mu$ can be reconstructed from the triple $(E, L_1, L_2)$. Namely, the spaces $U, V$ are isomorphic to $H^0(E, L_1)$ and $H^0(E, L_2)$, respectively, and the tensor $\mu : V \otimes U \to W$ is nothing but the canonical map $H^0(E, L_2) \otimes H^0(E, L_1) \to H^0(E, L_2 \otimes L_1)$.

Note also that the mirror symmetry relation for noncommutative planes is described in [AKO] as a special elliptic fibration over $\mathbb{A}^1$ with three ordinary critical points and with symplectic forms, variations of which are related to noncommutative deformations of $\mathbb{P}^2$.

Let us fix a noncommutative projective plane $\mathbb{P}_\mu$ that is defined by a tensor $\mu : V \otimes U \to W$, where $\dim V = \dim U = 3$ and $\dim W = 6$. Consider the usual projective plane $\mathbb{P}(U)$ and the vector bundle $T(-1)$ on it. The space of global section of $T(-1)$ is canonically isomorphic to $U$. The tensor $\mu$ defines the tensor $\nu : I \to V \otimes U$ as above, where $I$ is the kernel of $\mu$. They induce a morphism of vector bundles on $\mathbb{P}(U)$

$$\tilde{\nu} : I \otimes O_{\mathbb{P}^2} \to V \otimes T(-1),$$

the cokernel of which is a 3-dimensional vector bundle on the projective plane $\mathbb{P}(U)$ that we denote as $\mathcal{F}$. Let us take the projective bundle $X = \mathbb{P}(\mathcal{F}^\vee) \xrightarrow{\pi} \mathbb{P}(U)$ and consider the tautological line bundle $O_X(-1)$. Denote by $\mathcal{L}$ its dual $O_X(1)$. There is an isomorphism $R\pi_*\mathcal{L} = \mathcal{F}$. It was proved in Proposition 2.4 that the sequence $\sigma = (O_X, \pi^*T(-1), \mathcal{L})$ is strong exceptional. Moreover, it follows from the construction that

$$\text{Hom}_X(O_X, \pi^*T(-1)) = \text{Hom}_{\mathbb{P}(U)}(O_{\mathbb{P}^2}, T(-1)) = U,$$

$$\text{Hom}_X(\pi^*T(-1), \mathcal{L}) = \text{Hom}_{\mathbb{P}(U)}(T(-1), \mathcal{F}) = V,$$

$$\text{Hom}_X(O_X, \mathcal{L}) = \text{Hom}_{\mathbb{P}(U)}(O_{\mathbb{P}^2}, \mathcal{F}) = W,$$
and the tensor of this collection is exactly $\mu : V \otimes U \to W$. Hence, the subcategory of $D^b(\text{coh } X)$ generated by the collection $\sigma$ is equivalent to the derived category $D^b(\text{coh } \mathbb{P}^2_\mu)$ of the noncommutative projective plane $\mathbb{P}^2$.

Note that the derived category $D^b(\text{coh } X)$ has the following semi-orthogonal decomposition

$$D^b(\text{coh } X) = \langle \pi^*D^b(\text{coh } \mathbb{P}(U)) \otimes \mathcal{L}^{-1}, \pi^*D^b(\text{coh } \mathbb{P}(U)), \pi^*D^b(\text{coh } \mathbb{P}(U) \otimes \mathcal{L}), \rangle,$$

where all three pieces are the derived categories of coherent sheaves of the usual projective plane.

On the other hand, since $\text{Hom}_X(\pi^*\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{L}) = \text{Hom}_{\mathbb{P}(U)}(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{F}) = 0$, the line bundles $\pi^*\mathcal{O}_{\mathbb{P}^2}(1)$ and $\mathcal{L}$ are mutually orthogonal and we get the following decomposition

$$D^b(\text{coh } X) = \left\langle \pi^*D^b(\text{coh } \mathbb{P}(U)) \otimes \mathcal{L}^{-1}, D^b(\text{coh } \mathbb{P}^2_\mu), \langle \pi^*\mathcal{O}_{\mathbb{P}^2}(1), \pi^*\mathcal{T}(-1) \otimes \mathcal{L}, \pi^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{L} \rangle \right\rangle.$$  

The subcollection $\sigma' = (\pi^*\mathcal{O}_{\mathbb{P}^2}(1), \pi^*\mathcal{T}(-1) \otimes \mathcal{L}, \pi^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{L})$ is also strong exceptional and there are the following isomorphisms

$$\text{Hom}_Y(\pi^*\mathcal{O}_{\mathbb{P}^2}(1), \pi^*\mathcal{T}(-1) \otimes \mathcal{L}) = \text{Hom}_{\mathbb{P}(U)}(\mathcal{T}(-1), \mathcal{F}) = V,$$

$$\text{Hom}_Y(\pi^*\mathcal{T}(-1) \otimes \mathcal{L}, \pi^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{L}) = \text{Hom}_{\mathbb{P}(U)}(\mathcal{T}(-1), \mathcal{O}_{\mathbb{P}^2}(1)) = U,$$

$$\text{Hom}_Y(\pi^*\mathcal{O}_{\mathbb{P}^2}(1), \pi^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{L}) = \text{Hom}_{\mathbb{P}(U)}(\mathcal{O}_{\mathbb{P}^2}, \mathcal{F}) = W.$$

These isomorphisms show that the composition low in the collection $\sigma'$ is given by the tensor $\mu^\circ : U \otimes V \to W$ that is opposite to the tensor $\mu : V \otimes U \to W$, i.e. $\mu^\circ = \mu \cdot \iota$, where $\iota : V \otimes U \to U \otimes V$ is the commutativity isomorphism. Hence, the endomorphism algebra of the collection $\sigma'$ is opposite of the endomorphism algebra of the collection $\sigma$. We obtain the following semi-orthogonal decomposition for the derived category of coherent sheaves on $X$

$$D^b(\text{coh } X) = \left\langle D^b(\text{coh } \mathbb{P}(U)), D^b(\text{coh } \mathbb{P}^2_\mu), D^b(\text{coh } \mathbb{P}^2_{\mu^\circ}) \right\rangle.$$  

It should be noted that the noncommutative plane defined by the tensor $\mu^\circ$ is related to the triple $(E, \mathcal{L}_2, \mathcal{L}_1)$. It can be easily checked that the triple $(E, \mathcal{L}_2, \mathcal{L}_1)$ is isomorphic to the original triple $(E, \mathcal{L}_1, \mathcal{L}_2)$. Indeed, there is an automorphism $\tau$ of $E$ which is a multiplication by $-1$ with respect to a suitable pair of $E$ such that $\tau^*\mathcal{L}_1 \cong \mathcal{L}_2$ and $\tau^*\mathcal{L}_2 \cong \mathcal{L}_1$. Thus, as an abstract noncommutative schemes $\mathbb{P}^2_{\mu^\circ}$ is isomorphic to $\mathbb{P}^2_\mu$.

Finally, let us mention one interesting fact about the variety $X$. If $\mu$ is isomorphic to the usual tensor $U \otimes U \to S^2 U$ and we have the usual commutative projective plane $\mathbb{P}(U^\vee)$, then the vector bundle $\mathcal{F}$ constructed above is isomorphic to the symmetric square $S^2(\mathcal{T}(-1))$. In this case the variety $X = \mathbb{P}(\mathcal{F}^\vee)$ is isomorphic to the Hilbert scheme $\text{Hilb}^2 \mathbb{P}(U^\vee)$ of two points on the projective plane $\mathbb{P}(U^\vee)$. For a general tensor $\mu$ the variety $\mathbb{P}(\mathcal{F}^\vee)$ is a deformation of the Hilbert scheme $\text{Hilb}^2 \mathbb{P}(U^\vee)$. Thus, any noncommutative plane can be obtained as an admissible subcategory of a deformation of the Hilbert scheme of two points on the usual (dual) projective plane.
Proposition 3.4. For any noncommutative plane $\mathbb{P}^2_\mu$ there is a 4-dimensional smooth projective variety $X$ of the form $\mathbb{P}(\mathcal{F}^\vee)$, whose the derived category $D^b(\text{coh } X)$ has a semi-orthogonal decomposition of the following form

$$D^b(\text{coh } X) = \left\langle D^b(\text{coh } \mathbb{P}^2), D^b(\text{coh } \mathbb{P}^2_\mu), D^b(\text{coh } \mathbb{P}^2_{\mu^\circ}) \right\rangle,$$

where $\mu^\circ$ is the opposite tensor to the tensor $\mu$. Moreover, the variety $X$ is a deformation of the Hilbert scheme $\text{Hilb}^2 \mathbb{P}^2$ of two points on the projective plane.

References

[ATV] M. Artin, J. Tate, M. Van den Bergh, Some algebras associated to automorphisms of elliptic curves, The Grothendieck Festschrift, vol. I, Progr. Math. 86, Birkhäuser, Boston, (1990), 33–85.

[AKO] D. Auroux, L. Katzarkov, D. Orlov, Mirror symmetry for del Pezzo surfaces: vanishing cycles and coherent sheaves, Invent. Math., 166 (2006), 3, 537-582.

[BO] A. Bondal, D. Orlov, Semiorthogonal decomposition for algebraic varieties, Preprint MPIM 95/15 (1995), arXiv:math.AG/9506012.

[BP] A. Bondal, A. Polishchuk, Homological properties of associative algebras: the method of helices, Izv. Ross. Akad. Nauk, Ser. Mat. 57 (1993), 3–50; transl. in Russian Acad. Sci. Izv. Math. 42 (1994), 216–260.

[BV] A. Bondal, M. Van den Bergh, Generators and representability of functors in commutative and noncommutative geometry, Mosc. Math. J., 3 (2003), 1–36.

[CV] S. Cecotti, C. Vafa On classification of $N = 2$ Supersymmetric theories, Commun. Math. Phys., 158 (1993), 3, 569–644.

[Dr] V. Drinfeld, DG quotients of DG categories, J. of Algebra, 272 (2004), 5, 643-691.

[Ga] P. Gabriel, Auslander-Reiten sequences and representation-finite algebras, Representation theory I, Ottawa 1979. Lecture Notes in Mathematics 831, Springer-Verlag, Berlin/New York 1980.

[K1] B. Keller, Derived DG categories, Ann. Sci. École Norm. Sup. (4), 27 (1994), 63–102.

[K2] B. Keller, On differential graded categories, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, (2006), 151–190.

[Ku] A. Kuznetsov, A simple counterexample to the Jordan-Hölder property for derived categories, preprint arXiv:1304.0903

[LO] V. Lunts, D. Orlov, Uniqueness of enhancement for triangulated categories, J. Amer. Math. Soc., 23 (2010), 3, 853–908.

[Ne] A. Neeman, The Grothendieck duality theorem via Bousfield’s techniques and Brown representability, J. Amer. Math. Soc., 9 (1996), 205–236.

[O1] D. Orlov, Projective bundles, monoidal transforms and derived categories of coherent sheaves, Izv. Ross. Akad. Nauk Ser. Mat., 56 (1992), 4, 852–862; transl. in Russian Acad. Sci. Izv. Math., 41 (1993), 1, 133–141.

[O2] D. Orlov, Remarks on generators and dimensions of triangulated categories, Mosc. Math. J. 9 (2009), 153–159.

[O3] D. Orlov, Smooth and proper noncommutative schemes and gluing of DG categories, arXiv:1402.7364.