CANONICAL BASES ARISING FROM i-QUANTUM COVERING GROUPS OF KAC-MOODY TYPE

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Abstract. For quantum covering groups \((U, U^i)\) of super Kac-Moody type, we construct \(i\)-canonical bases for the highest weight integrable \(U\)-modules and their tensor products regarded as \(U^i\)-modules, as well as a canonical basis for the modified form \(\hat{U}^i\) of the \(i\)-quantum group \(U^i\), using the \(i^\pi\)-divided powers, rank one canonical basis for \(U^i\).

1. Introduction

1.1. Background. A quantum symmetric pair \((U, U^i)\) is a quantization of the symmetric pair of enveloping algebras \((U(g), U(g^\theta))\) where \(\theta : g \rightarrow g\) is an involution of the Lie algebra \(g\). Originally developed for applications in harmonic analysis for quantum group analogs of symmetric spaces, G. Letzter developed a comprehensive theory of quantum symmetric pairs for all semisimple \(g\) in [Le99]. The algebraic theory of quantum symmetric pairs was subsequently extended to the setting of Kac-Moody algebras in [Ko14]. The \(i\)-quantum group \(U^i\) is a subalgebra of the quantum group \(U\) satisfying a coideal property; coideal subalgebras provide important substructure for \(U\), since Hopf subalgebras are rare ‘in nature’.

More recent developments have made it apparent that quantum symmetric pairs play an important role in representation theory at large. In a series of papers, H. Bao and W. Wang proposed a program of canonical bases for quantum symmetric pairs [BW18a, BW18b, BW18c]. They performed their program for the Type AIII/IV symmetric pairs \((\mathfrak{sl}_{2N}, s(\mathfrak{gl}_N \times \mathfrak{gl}_N))\) and \((\mathfrak{sl}_{2N+1}, s(\mathfrak{gl}_N \times \mathfrak{gl}_{N+1}))\)
and applied it to tensor products of their $U$-modules, establishing a Kazhdan-Lusztig theory and irreducible character formula for the category $\mathcal{O}$ of the orthosymplectic Lie superalgebra $\mathfrak{osp}(2n+1|2m)$. Together with previously known results, these recent developments suggest that quantum symmetric pairs allow as deep a theory as quantized enveloping algebras themselves. In fact, $U$ can be viewed as a special type of quantum symmetric pair, the diagonal quantum symmetric pair $(U \otimes U, \iota(U))$ where $\iota = (\omega \otimes 1)\Delta : U \rightarrow U \otimes U$. It is thus reasonable to expect that many results about quantized groups have their counterparts in the realm of quantum symmetric pairs.

A quantum covering group $U_{\pi}$, introduced in [CHW13] is an algebra defined via a super Cartan datum $I$ (a finite indexing set associated to Kac-Moody superalgebras with no isotropic odd roots). $U_{\pi}$ depends on two parameters $q$ and $\pi$, where $\pi^2 = 1$. A quantum covering group specializes at $\pi = 1$ to the quantum group above, and at $\pi = -1$ to a quantum supergroup of anisotropic type (see [BKM98]). In addition to the usual Chevalley generators, we have generators $J_i$ for each $i \in I$. If one writes $K_i$ as $q^{h_i}$, then analogously we will have $J_i = \pi^{h_i}$. The parameter $\pi$ can be seen as a shadow of a parity shift functor in e.g. D. Hill and W. Wang’s ([HW15]) categorification of quantum groups by the spin quiver Hecke superalgebras introduced in [KKT16]. Since then, further progress has been made on the odd/spin/super categorification of quantum covering groups; see [KKO14, EL16, BE17].

Much of the theory for quantum groups, have parallel constructions in the realm of quantum covering groups. In particular, a theory of canonical bases for integrable modules of $U_{\pi}$ and its modified (idempotented) form $\hat{U}_{\pi}$ has been developed, in [CHW14, Cl14].

1.2. $\iota^\pi$-divided powers. For the negative half $U^-$ of the quantum group in rank one $U = U_q(\mathfrak{sl}_2)$, the Lusztig divided powers are monomials in a single variable $F$, and they form the canonical basis for $U^-$. The canonical basis for $U^\pi$ in rank one is formed by the $\iota$-divided powers, introduced in [BW18b, BW18c] and further explored in [BeW18]. Instead of being monomials, they are polynomials in a single variable $B$. They give bases for finite-dimensional simple $\mathfrak{sl}_2$-modules, and have two different formulas, $B^{(n)}_0$ and $B^{(n)}_1$, depending on the parity of the corresponding highest weight, which is a non-negative integer. The $\iota$-divided powers and their expansion formulas in [BeW18] formed a cornerstone of the construction of the Serre presentation for quasi-split $\iota$-quantum groups established in H. Chen, M. Lu and W. Wang in [CLW18]. In [BW18b, BW18c], $\iota$-divided powers for $i \in I$ with $\tau i = i$ were defined using the same formulas, and then shown to generate as an algebra the integral form $\hat{A}U^\pi$ of the modified quantum group. In [C19], the $\iota$-divided powers above are shown to have a generalization to $U_{\pi}^\iota$, the $\iota^\pi$-divided powers $B^{(m)}_{i,1}$ and $B^{(m)}_{i,0}$ which are given in the formulas (3.7) and (3.8) below for $i \in I$ with $\tau i = i$. The new facets $\pi$ and $J$ of quantum covering
groups are incorporated into these formulas, and when we specialize at \( \pi = 1 \) and \( J_i = 1 \), we obtain the \( i \)-divided powers above. The \( \pi \)-divided powers also satisfy a collection of expansion formulas which are used to give a Serre presentation for \( U^i_\pi \) and define a bar-involution on \( U^i \).

1.3. **Quasi-K-matrix and canonical basis for** \( U^i_\pi \). For regular quantum groups, the bar involutions \( \psi_i \) on \( U^i \) and \( \psi \) on \( U \) are not compatible; \( \psi_i \) is not simply the restriction of \( \psi \) to the subalgebra \( U^i \). However, one can define a quasi-\( K \)-matrix \( \Upsilon \) that ‘intertwines’ these two bar involutions. In the case of the diagonal quantum symmetric pair, the quasi-\( K \)-matrix arises naturally from Lusztig’s quasi-\( R \)-matrix. The quasi-\( K \)-matrix is applied in [BW18b, BW18c] to transform involutive based \( U \)-modules (\( U \)-modules with distinguished bases compatible with the bar-involution \( \psi \) on \( U \)), into involutive based \( U^i \)-modules, compatible with the bar-involution \( \psi_i \) on \( U^i \).

The quasi-\( K \)-matrix \( \Upsilon \) is invertible, and its inverse is obtained by applying the bar involution. Crucially, \( \Upsilon \) has the property that it preserves the integrality of the \( A \)-forms of integrable highest weight \( U^i_\pi \)-modules and their tensor products. Using this property of integrality of the action of their quasi-\( K \)-matrix, Bao and Wang defined in loc. cit. a new bar involution on based \( U \)-modules (modules \( M \) with a distinguished basis \( B \), and compatible involution \( \psi \)) thus enabling the construction of \( i \)-canonical bases of these modules (which are now based \( U^i \)-modules) from their canonical bases. With the \( \pi \)-divided powers above, these constructions also lead to a theory of canonical basis for integrable based \( U^i_\pi \)-modules, which we develop here in this article.

1.4. **Organization.** The rest of this article is organized as follows. In the next section, we introduce basic notation and notions for quantum covering groups. Then, in section 3 we describe \( U^i_\pi \) and the \( \pi \)-divided powers. In section 4, the quasi-\( K \)-matrix \( \Upsilon \) for \( U^i_\pi \) is constructed and in section 5 the integrality of its action is established, by which we mean that \( \Upsilon \) preserves the integral \( A \)-forms on integrable highest weight \( U^i_\pi \)-modules and their tensor products. We conclude by constructing the \( i \)-canonical basis for based \( U^i_\pi \)-modules in a section 6 followed by canonical basis for the modified form \( \hat{U}^i_\pi \) in section 7 generalizing [BW18b, BW18c].

**Remark on notation.** For the remaining sections we will drop the subscript \( \pi \) from \( U_\pi \) and related notation in the following chapters, so \( U \) will be understood to refer to the quantum covering group going forward. We will explicitly mention when we are referring to the usual quantum group e.g. when we specialize \( \pi = 1 \).

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2. Quantum covering groups and canonical bases

In this section, we will recall the definition of a quantum covering group from [CHW13] starting with a super Cartan datum and a root datum. A Cartan datum is a pair \((I, \cdot)\) consisting of a finite set \(I\) and a symmetric bilinear form \(\nu, \nu' \mapsto \nu \cdot \nu'\) on the free abelian group \(\mathbb{Z}[I]\) with values in \(\mathbb{Z}\) satisfying

(a) \(d_i = \frac{i \cdot i}{2} \in \mathbb{Z}_{>0}\);

(b) \(2\frac{i \cdot j}{\nu(i, j)} \in -\mathbb{N}\) for \(i \neq j\) in \(I\), where \(\mathbb{N} = \{0, 1, 2, \ldots\}\).

If the datum can be decomposed as \(I = I_0 \coprod I_1\) such that

(c) \(I_1 \neq \emptyset\),

(d) \(2\frac{i \cdot j}{\nu(i, j)} \in 2\mathbb{Z}\) if \(i \in I_1\),

(e) \(d_i \equiv p(i) \mod 2, \forall i \in I\).

then we will called it a (bar-consistent) super Cartan datum. Condition [(e)] is known as the ‘bar-consistency’ condition and is almost always satisfied for super Cartan data of finite or affine type (with one exception).

Note that (d) and (e) imply that

(f) \(i \cdot j \in 2\mathbb{Z}\) for all \(i, j \in I\).

The \(i \in I_0\) are called even, \(i \in I_1\) are called odd. We define a parity function \(p : I \to \{0, 1\}\) so that \(i \in I_{p(i)}\). We extend this function to the homomorphism \(p : \mathbb{Z}[I] \to \mathbb{Z}\). Then \(p\) induces a \(\mathbb{Z}_2\)-grading on \(\mathbb{Z}[I]\) which we shall call the parity grading.

A super Cartan datum \((I, \cdot)\) is said to be of finite (resp. affine) type exactly when \((I, \cdot)\) is of finite (resp. affine) type as a Cartan datum (cf. [Lu94, § 2.1.3]). In particular, the only super Cartan datum of finite type is the one corresponding to the Lie superalgebras of type \(B(0, n)\) for \(n \geq 1\) i.e. the orthosymplectic Lie superalgebras \(\mathfrak{osp}(1|2n)\).

A root datum associated to a super Cartan datum \((I, \cdot)\) consists of

(a) two finitely generated free abelian groups \(Y, X\) and a perfect bilinear pairing \(\langle \cdot, \cdot \rangle : Y \times X \to \mathbb{Z}\);

(b) an embedding \(I \subset X\) \((i \mapsto i')\) and an embedding \(I \subset Y\) \((i \mapsto i)\) satisfying

(c) \(\langle i, j' \rangle = 2\frac{i \cdot j}{\nu(i, j)}\) for all \(i, j \in I\).

We will always assume that the root datum is \(X\)-regular (respectively \(Y\)-regular) image of the embedding \(I \subset X\) (respectively, the image of the embedding \(I \subset Y\)) is linearly independent in \(X\) (respectively, in \(Y\)).

We also define a partial order \(\leq\) on the weight lattice \(X\) as follows: for \(\lambda, \lambda' \in X\),

\[
\lambda \leq \lambda' \text{ if and only if } \lambda' - \lambda \in \mathbb{N}[I].
\]

The matrix \(A := (a_{ij}) := \langle i, j' \rangle\) is a symmetrizable generalized super Cartan matrix: if \(D = \text{diag}(d_i \mid i \in I)\), then \(DA\) is symmetric.
Let $\pi$ be a parameter such that

$$\pi^2 = 1.$$ 

For any $i \in I$, we set

$$q_i = q^{i/2}, \quad \pi_i = \pi^{p(i)}.$$ 

Note that when the datum is consistent, $\pi_i = \pi^{i^2}$; by induction, we therefore have $\pi^{p(\nu)} = \pi^{\nu \nu/2}$ for $\nu \in \mathbb{Z}[I]$. We extend this notation so that if $\nu = \sum \nu_i i \in \mathbb{Z}[I]$, then

$$q_\nu = \prod_i q_i^{\nu_i}, \quad \pi_\nu = \prod_i \pi_i^{\nu_i}.$$ 

For any ring $R$ we define a new ring $R^\pi = R[\pi]/(\pi^2 - 1)$ (with $\pi$ commuting with $R$). Below, we will work over $\mathbb{Q}(q)^\pi$ where $\mathbb{Q}$ is a field of characteristic 0 and occasionally $A^\pi$ where $A := \mathbb{Z}[q, q^{-1}]$.

Recall also the $(q, \pi)$-integers and $(q, \pi)$-binomial coefficients in [CHW13]: we shall denote

$$[n] = \begin{bmatrix} n \\ 1 \end{bmatrix} = \frac{(\pi q)^n - q^{-n}}{\pi q - q^{-1}} \quad \text{for } n \in \mathbb{Z},$$

$$[n]! = \prod_{s=1}^{n} [s] \quad \text{for } n \in \mathbb{N},$$

and with this notation we have

$$\begin{bmatrix} m \\ n \end{bmatrix} = \frac{[m]!}{[n]! \cdot [m-n]!} \quad \text{for } 0 \leq n \leq m.$$ 

We denote by $[n]_i$, $[m]_i!$, and $\begin{bmatrix} n \\ m \end{bmatrix}_i$ the variants of $[n]$, $[m]!$, and $\begin{bmatrix} n \\ m \end{bmatrix}$ with $q$ replaced by $q_i$ and $\pi$ replaced by $\pi_i$, and $\begin{bmatrix} m \\ n \end{bmatrix}_{q^2}$ the variant with $q$ replacing $q^2$.

For any $i \neq j$ in $I$, we define the following polynomial in two (noncommutative) variables $x$ and $y$:

$$F_{ij}(x, y) = \sum_{n=0}^{1-a_{ij}} (-1)^n \pi_i^{np(j)} + \binom{n}{2} \left[ 1 - a_{ij} \right] \frac{x^n y x^{1-a_{ij}-n}}{n_i}.$$ 

Also, we have

Assume that a root datum $(Y, X, \langle , \rangle)$ of type $(I, \cdot)$ is given. The quantum covering group $U$ of type $(I, \cdot)$ is the associative $\mathbb{Q}(q)^\pi$-superalgebra with generators

$$E_i \quad (i \in I), \quad F_i \quad (i \in I), \quad J_\mu \quad (\mu \in Y), \quad K_\mu \quad (\mu \in Y),$$

with parity $p(E_i) = p(F_i) = p(i)$ and $p(K_\mu) = p(J_\mu) = 0$, subject to the relations (a)-(f) below for all $i, j \in I, \mu, \mu' \in Y$:

$$(R1) \quad K_0 = 1, \quad K_\mu K_{\mu'} = K_{\mu + \mu'},$$
\[ J_{2\mu} = 1, \quad J_{\mu} J_{\mu'} = J_{\mu + \mu'}, \]
\[ J_{\mu} K_{\mu'} = K_{\mu'} J_{\mu}, \]
\[ K_{\mu} E_i = q^{(\mu, i')} E_i K_{\mu}, \quad J_{\mu} E_i = \pi^{(\mu, i')} E_i J_{\mu}, \]
\[ K_{\mu} F_i = q^{-(\mu, i')} F_i K_{\mu}, \quad J_{\mu} F_i = \pi^{-(\mu, i')} F_i J_{\mu}, \]
\[ E_i F_j - \pi^{p(i)p(j)} F_j E_i = \delta_{i,j} \frac{\tilde{J}_i K_i - K_{\bar{i}}}{\pi q_i - q^{-1}}, \]
\[ (q, \pi)-\text{Serre relations} \quad F_{ij}(E_i, E_j) = 0 = F_{ij}(F_i, F_j), \quad \text{for all } i \neq j. \]

where for any element \( \nu = \sum_i \nu_i i \in \mathbb{Z}[I] \) we have set \( \tilde{K}_\nu = \prod_i K_{d_{\nu,i}}, \tilde{J}_\nu = \prod_i J_{d_{\nu,i}} \). In particular, \( \tilde{K}_i = K_{d_i i}, \tilde{J}_i = J_{d_i i} \). Under the bar-consistency condition, \( J_{\bar{i}} = 1 \) for \( i \in I_0 \) while \( \tilde{J}_i = J_i \) for \( i \in I_1 \). Note that by the same condition \( a_{ij} \) is always even for \( i \in I_1 \) and so \( J_i \) is central for all \( i \in I \). As usual, denote by \( U^-, U^+ \) and \( U^0 \) the subalgebras of \( U \) generated by \( \{ E_i \mid i \in I \}, \{ F_i \mid i \in I \} \) and \( \{ J_{\mu}, K_{\mu} \mid \mu \in Y \} \) respectively. Also denote \( U^{0'} = \{ J_i, K_i \mid i \in I \} \).

Note that the \((q, \pi)\)-Serre relations (R7) can be rewritten as
\[ \sum_{n=0}^{1-a_{ij}} (-1)^n \pi^{|p(j)+\frac{1}{2}|} F_i^{(n)} F_j F_i^{(1-a_{ij})-n} = 0 \]
and
\[ \sum_{n=0}^{1-a_{ij}} (-1)^n \pi^{|p(j)+\frac{1}{2}|} E_i^{(n)} E_j E_i^{(1-a_{ij})-n} = 0, \]
where we write \( F_i^{(n)} = F_i^n / [n]_i! \) and \( E_i^{(n)} = E_i^n / [n]_i! \) for \( n \geq 1 \) and \( i \geq 1 \).

Define \( f \) to be the free associative \( \mathbb{Q}(q)^{\pi} \)-superalgebra with 1 and with even generators \( \theta_i \) for \( i \in I_0 \) and odd generators \( \theta_i \) for \( i \in I_1 \). We abuse notation and define the parity grading on \( f \) by \( p(\theta_i) = p(i) \). We also have a weight grading \( | \cdot | \) on \( f \) defined by setting \( |\theta_i| = i \).

By [CHW13, Prop 1.4.1], there exists a unique symmetric bilinear form \( (\cdot, \cdot) \) on \( f \) with values in \( \mathbb{Q} \) such that \( (1, 1) = 1 \) and
\[ (\theta_i, \theta_j) = \delta_{ij}(1 - \pi_i q_i^{-2})^{-1} \] for all \( i, j \in I \).

Let \( J \) denote the radical of \((\cdot, \cdot)\) which is a 2-sided ideal of \( f \), and let \( f = f/\mathcal{J} \) be the quotient algebra of \( f \) by its radical. There exists well-defined algebra homomorphisms \( f \to U : x \mapsto x^+ \) with \( \theta_i^+ = E_i \) and image \( U^+ \), and \( x \mapsto x^- \) with \( \theta_i^- = F_i \) and image \( U^- \). The algebra \( f \) has weight space decomposition \( f = \bigoplus_{\nu} f_{\nu} \) where \( f_{\nu} \) is the image of \( f_{\nu} \), the weight space of \( f \) with weight \( \nu = \sum \nu_i i \in \mathbb{Z}[I] \).

We will denote the height of \( \nu \) by \( ht(\mu) = \sum_{i \in I} \nu_i \) and for any \( x \in f_{\nu} \), we set...
$|x| = \nu$. Each weight space is finite dimensional. The symmetric bilinear form on $\mathfrak{f}$ descends to a symmetric bilinear form on $\mathfrak{f}$ which is non-degenerate on each weight space.

2.1. The twisted derivations $r_i$ and $r_j$. Let $i \in I$. There exist unique $\mathbb{Q}(q)^{\pi}$-linear maps $r_{i,j}: \mathfrak{f} \to \mathfrak{f}$ such that $r_i(1) = i r(1) = 0$ and $r_i(\theta_j) = i r(\theta_j) = \delta_{ij}$ satisfying

$$i r(xy) = i r(x) y + \pi^{p(x)p(i)} q^{|x| \cdot i} x_i r(y)$$
$$r_i(xy) = \pi^{p(y)p(i)} q^{|y| \cdot i} r_i(x) y + x r_i(y)$$

for homogeneous $x, y \in \mathfrak{f}$. We see that if $x \in \mathfrak{f}_\nu$, then $i r(x), r_i(x) \in \mathfrak{f}_{\nu - 1}$ and moreover,

$$(\theta_i y, x) = (\theta_i, \theta_i)(y, i r(x)), \quad (y \theta_i, x) = (\theta_i, \theta_i)(y, r_i(x))$$

for all $x, y \in \mathfrak{f}$, and both maps descend to maps on $\mathfrak{f}$ cf. [CHW13, §1.5].

The following lemmas on the twisted derivation will be important tools for the construction of the quasi-K-matrix in part III. The first is [CHW13, Lemma 1.5.2], a direct generalization of [Lu94, Lemma 1.2.15] for quantum groups:

**Lemma 2.1.** Let $x \in \mathfrak{f}_\nu$ where $\nu \in \mathbb{N}[I]$ is nonzero.

(a) If $r_i(x) = 0$ for all $i \in I$, then $x = 0$.

(b) If $i r(x) = 0$ for all $i \in I$, then $x = 0$.

The following lemma is a generalization of [BW18a, Lemma 1.1] and will play a similar role in our setting:

**Lemma 2.2.** $r_i \circ r_j = r_i \circ r_j$ for all $i, j \in I$

**Proof.** It suffices to show this for homogeneous $x \in \mathfrak{f}_\mu$, using induction on the height of $\mu$; for $x = 1$ both sides are identically 0, and from their definition, we have

$$\begin{align*}
r_j \circ i r(xy) &= i r(x) r_j(y) + \pi^{p(y)p(j)} q^{|y| \cdot j} r_j(i r(x)) y + \pi^{p(x)p(i)} q^{|x| \cdot i} x_i r_j(i r(y)) \\
&\quad + \pi^{p(x)p(j)} q^{|x| \cdot i} x_i r_j(r_j(x)) y + \pi^{p(x)p(i)} q^{|x| \cdot i} x_i r_j(r_j(y)) + \pi^{p(y)p(j)} q^{|y| \cdot j} r_j(x) i r_j(x) r(y) \\

\end{align*}$$

and

$$\begin{align*}
ri \circ r_j(xy) &= i r(x) r_j(y) + \pi^{p(y)p(j)} q^{|y| \cdot j} r_j(i r(x)) y + \pi^{p(x)p(i)} q^{|x| \cdot i} x_i r_j(i r(y)) \\
&\quad + \pi^{p(y)p(j)} q^{|y| \cdot j} r_j(x) i r_j(x) r(y),
\end{align*}$$

and since $p(r_k(z)) = p(z) - p(k)$, the $\pi$ powers in the last term of each of the two expressions on the right are both equal to $p(x)p(i) + p(y)p(j) - p(i)p(j)$; similarly $|r_k(z)| = |z| - k$ so the $q$ powers are both $|x| \cdot i + |y| \cdot j - i \cdot j$, and so the two expressions agree by application of the inductive hypothesis. \qed

The following proposition from [CHW13] is a key ingredient in the construction of the quasi-$K$-matrix:
Proposition 2.3 (Prop 2.2.2 of [CHW13]). For \( x \in \mathcal{f} \) and \( i \in I \), we have in \( U \)

\[
\begin{align*}
(a) \quad x^+ F_i - \pi_i p(x) F_i x^+ &= \frac{r_i(x)^+ \bar{J}_i \bar{K}_i - \bar{K}_{-i} \pi_i p(-p(i)) r_i(x)^+}{\pi_i q_i - q_i^{-1}}, \\
(b) \quad E_i x^- - \pi_i p(x) x^- E_i &= \frac{\bar{J}_i \bar{K}_i r(x)^- - \pi_i p(-p(i)) r_i(x)^- \bar{K}_{-i}}{\pi_i q_i - q_i^{-1}}.
\end{align*}
\]

2.2. Bar-involution and Quasi-\( R \)-matrix for \( U \). There exists a unique \( \mathbb{Q} \)

algebra involution \( \overline{\cdot} \) (the bar-involution) on \( \mathbb{Q}(q)^{\pi} \) satisfying \( \overline{q} = \pi q^{-1} \) and \( \overline{\pi} = \pi \). For a bar-consistent super Cartan datum,

\[
\overline{q_i} = \pi_i q_i^{-1}.
\]

Furthermore, there exists a bar-involution \( \overline{\cdot} : \mathcal{f} \to \mathcal{f} \) such that \( \overline{\theta_i} = \theta_i \) for all \( i \in I \) and \( \overline{f x} = f \overline{x} \) for \( f \in \mathbb{Q}(q)^{\pi} \) and \( x \in \mathcal{f} \). This extends to a unique homomorphism of \( \mathbb{Q} \)-algebras \( \bar{\overline{\cdot}} : U \to U \) such that

\[
\begin{align*}
\overline{E_i} = E_i, \quad \overline{F_i} = F_i, \quad \overline{J_\mu} = J_\mu, \quad \overline{K_\mu} = J_\mu K_{-\mu},
\end{align*}
\]

and \( \overline{f x} = f \overline{x} \) for all \( f \in \mathbb{Q}(q)^{\pi} \) and \( x \in U \).

We remark here that our conventions for the comultiplication here are the same as in [CHW13]:

\[
\begin{align*}
(2.8) \quad \Delta(E_i) &= E_i \otimes 1 + \bar{J}_i \bar{K}_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes \bar{K}_{-i} + 1 \otimes F_i \quad (\text{for } i \in I), \\
(2.9) \quad \Delta(K_\mu) &= K_\mu \otimes K_\mu, \quad \Delta(J_\mu) = J_\mu \otimes J_\mu \quad (\text{for } \mu \in Y).
\end{align*}
\]

Let \( \widehat{U \otimes U} \) be the completion of the \( \mathbb{Q}(q)^{\pi} \)-modules \( U \otimes U \) with respect to the descending sequence of subspaces

\[
\begin{align*}
U^+ U^0 \left( \sum_{ht(\mu) \geq N} U_\mu^- \right) \otimes U &+ U \otimes U^- U^0 \left( \sum_{ht(\mu) \geq N} U_\mu^+ \right), \quad \text{for } N \geq 1, \mu \in \mathbb{Z}[I].
\end{align*}
\]

We have the obvious embedding of \( U \otimes U \) into \( \widehat{U \otimes U} \). By continuity the \( \mathbb{Q}(q)^{\pi} \)-
algebra structure on \( U \otimes U \) extends to a \( \mathbb{Q}(q)^{\pi} \)-algebra structure on \( \widehat{U \otimes U} \) cf. [CHW13, §3.1]. Let \( \overline{\cdot} : U \otimes U \to U \otimes U \) be the \( \mathbb{Q} \)-algebra homomorphism given by \( \overline{\overline{\cdot}} \). This extends to a \( \mathbb{Q} \)-algebra homomorphism on the completion. Let \( \overline{\Delta} : U \to U \otimes U \) be the \( \mathbb{Q}(q)^{\pi} \)-algebra homomorphism given by \( \overline{\Delta}(x) = \overline{\Delta(x)} \).

In [CHW13, §3.1], the quasi-\( R \)-matrix \( \Theta \) for \( U \) that intertwines \( \Delta \) and \( \overline{\Delta} \) is defined: For \( \nu = \sum_i \nu_i i \in \mathbb{N}[I] \), write \( \nu = \sum_{a=1}^{ht(\nu)} a_i \) for \( i_a \in I \). Then, set \( e(\nu) = \sum_{a<b} p(i_a)p(i_b) \in \mathbb{Z} \).

Proposition 2.4. There is a unique family of elements \( \Theta_\nu \in U_\nu^- \otimes U_\nu^+ \) (with \( \nu \in \mathbb{N}[I] \)) such that

\[
\begin{align*}
(a) \quad \Theta_0 &= 1 \otimes 1 \quad \text{and} \quad \Theta = \sum_\nu \Theta_\nu \in \widehat{U \otimes U} \quad \text{satisfies in } \widehat{U \otimes U} \quad \text{the identity} \\
\Delta(u) \Theta &= \Theta \overline{\Delta(u)} \quad \text{for all } u \in U.
\end{align*}
\]
Let $B$ be a $\mathbb{Q}(q)^n$-basis of $\mathfrak{g}$ such that $B_\nu = B \cap \mathfrak{g}_\nu$ is a basis of $\mathfrak{g}_\nu$ for any $\nu$. Let $\{b^* | b \in B_\nu\}$ be the basis of $\mathfrak{g}_\nu$ dual to $B_\nu$ under the bilinear form $(\cdot, \cdot)$. Then,

$$\Theta_\nu = (-1)^{ht(\nu)} q^{e(\nu)} \pi_\nu \sum_{b \in B_\nu} b^{-} \otimes b^{*+} \in \mathfrak{U}_\nu \otimes \mathfrak{U}_\nu^+.$$  

We will use $\Theta$ in the construction of the quasi-$\mathcal{R}$-matrix for $U^+$ in §6.1.

2.3. $\mathcal{A}$-form and modified form of $U$. For $i \in I$, let $\theta_i^{(m)}$ denote the divided power $\theta_i^m/[m]^2_i$ for $m \geq 0$. Let $\mathcal{A}_i\mathfrak{f}$ be the $\mathcal{A}^\pi$-subalgebra of $\mathfrak{f}$ generated by all divided powers $\theta_i^{(m)}$ for $m \geq 0$ and $i \in I$. Under the identification of $\mathfrak{f}$ with $U^-$ sending $\theta_i \mapsto F_i$, $U^-\mu$ can be identified with the image of $\mathfrak{f}_\mu$. Similarly, we can identify $\mathfrak{f} \cong U^+$ via $\theta_i$ with $E_i$. We let $\mathcal{A}_i U^-$ (respectively, $\mathcal{A}_i U^+$) denote the image of $\mathcal{A}_i\mathfrak{f}$ under this isomorphism, which is generated by all divided powers $F_i^{(m)}$ (respectively, $E_i^{(m)}$).

Recall from [CFLW, Definition 4.2] that the modified quantum covering group $\hat{U}$ is a non-unital $\mathbb{Q}(q)^{\pi}$-algebra generated by the symbols $1_\lambda$ (idempotents), $E_i 1_\lambda$ and $F_i 1_\lambda$, for $\lambda \in X$ and $i \in I$ and with relations:

$$1_\lambda 1_{\lambda'} = \delta_{\lambda, \lambda'} 1_\lambda, 
(E_1 1_\lambda) 1_{\lambda'} = \delta_{\lambda, \lambda'} E_1 1_{\lambda'}, 
1_{\lambda'} (E_i 1_\lambda) = \delta_{\lambda', \lambda + \nu} E_i 1_\lambda,$$

$$F_i (1_\lambda) 1_{\lambda'} = \delta_{\lambda, \lambda'} F_i 1_{\lambda'} ,
1_{\lambda'} (F_i 1_\lambda) = \delta_{\lambda', \lambda - \nu} F_i 1_\lambda,$$

$$\sum_{n+n' = 1} (-1)^{n'} \pi_i^{n'p(j)+\binom{n}{2}} E_i^n E_i^{n'} 1_\lambda = 0 \quad (i \neq j),$$

$$\sum_{n+n' = 1} (-1)^{n'} \pi_i^{n'p(j)+\binom{n}{2}} F_i^n F_i^{n'} 1_\lambda = 0 \quad (i \neq j),$$

where $i,j \in I$, $\lambda, \lambda' \in X$, and we use the notation $x y 1_\lambda = (x 1_{\lambda + |y|})(y 1_\lambda)$ for $x,y \in U$. The modified quantum covering group $\hat{U}$ admits an $\mathcal{A}^\pi$-form, $\mathcal{A}\hat{U}$ and so we can define $\mathcal{R}\hat{U} = R^\pi \otimes_{\mathcal{A}\hat{U}} \mathcal{A}\hat{U}$.

[Cl14, Lemma 3.5] goes here/after the following section.

2.4. Canonical basis and based $U$-modules. Here we recount some terminology and background on canonical basis and based $U$-modules.

Let $M(\lambda)$ be the Verma module of $U$ with highest weight $\lambda \in X$ and with a highest weight vector denoted by $\eta_\lambda$. Define a lowest weight $U$-module $\omega M(\lambda)$ with the same underlying vector space as $M(\lambda)$ but with the action twisted by the involution $\omega$ given in [CHW13, §2.2]. We will denote the lowest weight vector $\eta_\lambda$ in $\omega M(\lambda)$ by $\xi_{\lambda}$. Let $X^+ = \{ \lambda \in X | \langle i, \lambda \rangle \in \mathbb{N}, \forall i \in I \}$ be the set of dominant integral weights. By $\lambda \gg 0$ we shall mean that the integers $\langle i, \lambda \rangle$ for all $i$ are sufficiently large. The Verma module $M(\lambda)$ associated to $\lambda \in X$ has a unique simple quotient $U$-module, denoted by $\bar{L}(\lambda)$. We shall abuse the notation and
denote by $\eta_{\lambda} \in L(\lambda)$ the image of the highest weight vector $\eta_{\lambda} \in M(\lambda)$. Similarly we define the $U$-module $\omega L(\lambda)$ of lowest weight $-\lambda$ with lowest weight vector $\xi_{-\lambda}$. For $\lambda \in X^+$, we let $A L(\lambda) = A U^{-} \eta_{\lambda}$ and $\omega A L(\lambda) = A U^{+} \xi_{-\lambda}$ be the $A$-submodules of $L(\lambda)$ and $\omega L(\lambda)$, respectively.

We recall now the canonical basis for the half-quantum group developed in [CHW14]: Let $R$ be a ring. A $\pi$-basis for a free $R^{\pi}$-module $M$ is a set $S \subset M$ such that there exists an $R^{\pi}$-basis $B$ for $M$ with $S = B \cup \pi B$. Note that in [CHW14], this is called a maximal $\pi$-basis. We note that a $\pi$-basis of an $R^{\pi}$-module $M$ is an $R$-basis of $M$. The fundamental result on $\pi$-bases in loc. cit. is the following.

**Proposition 2.5 ([CHW14]).** There is a $\pi$-basis $B$ of $f$ with the following properties:

1. $B$ is a $\pi$-basis of $f$ over $A$.
2. Each $b \in B$ is homogeneous.
3. $b = b$ for all $b \in B$.
4. For $\lambda \in X^+$, there is a subset $B(\lambda)$ such that $B(L(\lambda)) = \{b\eta_{\lambda} : b \in B(\lambda)\}$ is a $\pi$-basis of $L(\lambda)$, and if $b \in B \setminus B(\lambda)$, $b^{-}\eta_{\lambda} = 0$.

We note that $B|_{\pi = 1} \subset f|_{\pi = 1}$ is precisely the Lusztig-Kashiwara canonical basis.

Thus, there is a canonical basis $\{b^{+}b \in B\}$ on $U^{+}$, and a canonical basis $\{b^{-}\eta_{\lambda}b \in B(\lambda)\}$ on $U^{-}$. For each $\lambda \in X^+$, there is a subset $B(\lambda)$ of $B$ so that $\{b^{-}\eta_{\lambda}b \in B(\lambda)\}$ (respectively, $\{b^{+}\xi_{-\lambda}b \in B(\lambda)\}$) forms a canonical basis of $L(\lambda)$ (respectively, $\omega L(\lambda)$). For any Weyl group element $w \in W$, let $\eta_{w\lambda}$ denote the unique canonical basis element of weight $w\lambda$.

Let $\hat{U}$ be the idempotented modified quantum group and $A \hat{U}$ its $A$-form. Then the sets $\{b^{+}1\lambda b^{-} : (b, b') \in B \times_{\pi} B\}$ and $\{b^{-}1\lambda b^{+} : (b, b') \in B \times_{\pi} B\}$ both form a $\pi$-basis of $A \hat{U}$ (cf. [CI14, Lemma 3.5]) and $A \hat{U}$ admits a canonical basis $\hat{B} = \{b \otimes_{\pi} \xi | (b, \xi) \in B \times_{\pi} X\}$ (cf. [CI14, Corollary 4.15]).

Recall the notion of based modules for finite type quantum groups [Lu94, chapter 27], and generalized to quantum groups of Kac-Moody type in [BW16], which is a module with a distinguished basis and compatible bar-involution. Like many results for quantum groups, these generalize to the quantum covering setting, see [CI14, §4] and also section 6 where we define based modules for $U^{e}$. Examples of based $U$-modules include $L(\lambda)$ and $\omega L(\lambda)$ with their $\pi$-basis $B(\lambda)$. We go through a few relevant results here:

**Proposition 2.6.** Let $(M, B), (M', B')$ be based modules, with either $M = \omega L(\lambda)$ or $M' = L(\lambda)$ for $\lambda \in X^+$. Let $L$ be the $\mathbb{Z}[q^{-1}]$-submodule of $M \otimes M'$ generated by $B \otimes B'$.

1. For any $(b, b') \in B \times B'$, there is a unique element $b \otimes b' \in L$ such that $\Psi(b \otimes b') = b \otimes b'$ and $b \otimes b' - b \otimes b' \in q^{-1} L$. 

(2) The element $b \otimes b'$ is equal to $b \otimes b'$ plus a $q^{-1}\mathbb{Z}[q^{-1}]$-linear combination of elements $b_2 \otimes b'_2$ with $(b_2, b'_2) \in B \times B'$ with $(b_2, b'_2) < (b, b')$.

(3) The elements $b \otimes b'$ with $(b, b') \in B \times B'$ form a $\mathbb{Q}(q)^\pi$-basis of $M \otimes M'$, an $A^\pi$-basis of $A^\pi \otimes \mathbb{Z}[q^{-1}]L$, and a $\mathbb{Z}[q^{-1}]$-basis of $L$.

Proof. The argument here is a direct generalization of [BW16, Theorem 2.7], using the quasi-$\mathcal{R}$-matrix $\Theta$ from §2.2 above and a similar construction to [Cl14, Corollary 4.2].

By applying this iteratively we have a direct generalization of [BW16, Proposition 2.9] and direct generalizations of constructions in [Lu94, §27] to the quantum covering setting leads to the quantum covering analogue of [BW16, Prop 2.11]:

Proposition 2.7. Let $\lambda_1, \ldots, \lambda_\ell \in X^+$. Let $\eta_i$ denote the highest weight vector of $L(\lambda_i)$ for each $i$ and let $\eta$ denote the highest weight vector of $L(\sum_{i=1}^\ell \lambda_i)$. Then the (unique) homomorphism of $U$-modules

$$\chi : L\left(\sum_{i=1}^\ell \lambda_i\right) \longrightarrow L(\lambda_1) \otimes \cdots \otimes L(\lambda_\ell), \quad \chi(\eta) = \eta_1 \otimes \cdots \otimes \eta_\ell$$

sends each canonical basis element to a canonical basis element.

For $\lambda, \mu \in X^+$ we define the $U$-submodule $L(\lambda, \mu) := U(\eta_\lambda \otimes \eta_\mu) \subset L(\lambda) \otimes L(\mu)$.

Proposition 2.8. Let $\lambda, \mu \in X^+$ and $w \in W$. Then, the $U$-submodule $L(\lambda, \mu)$ is a based $U$-submodule of $L(\lambda) \otimes L(\mu)$.

Proof. Write $\lambda = \lambda_1 - \nu$. From the results above, $L(\lambda_1) \otimes L(\mu)$ is a based $U$-module, and the map $\chi : L(\lambda_1 + \mu) \rightarrow L(\lambda_1) \otimes L(\mu)$ is a based $U$-module homomorphism, and so $\chi' := \text{id}_{L(\lambda)} \otimes \chi$ is a based module homomorphism. Similarly, the map $\phi : \omega L(\nu) \otimes L(\lambda) \rightarrow L(\lambda)$ is a based module homomorphism, and hence so is $\phi' := \phi \otimes \text{id}_{L(\mu)}$. Thus, the composition homomorphism $\phi' \chi' : \omega L(\nu) \otimes L(\lambda_1 + \mu) \rightarrow L(\lambda) \otimes L(\mu)$ sending $\xi \otimes \eta_{\lambda_1 + \mu} \mapsto \eta_\lambda \otimes \eta_\mu$ is a based $U$-module homomorphism. Since $\omega L(\nu) \otimes L(\lambda_1 + \mu)$ is cyclically generated by $\eta_\lambda \otimes \eta_\mu$, the $U$-module $L(\lambda, \mu)$ is the image of the based module homomorphism $\phi' \chi'$, and so $L(\lambda, \mu)$ is a based $U$-submodule of $L(\lambda) \otimes L(\mu)$. \qed

3. The quantum covering groups $U^\pi$

We begin with a definition (cf. [Cl19, Definition 2.2]):

Definition 3.1. The quasi-split quantum covering group, denoted by $U^\xi$, or just $U^\xi$, is the $\mathbb{Q}(q)^\pi$-subalgebra of $U$ generated by

\begin{equation}
B_i := F_i + \gamma_i E_i \tilde{K}_i^{-1}, \quad \tilde{J}_i \ (i \in I), \quad K_\mu \ (\mu \in Y^\pi).
\end{equation}

Here the parameters

\begin{equation}
\zeta = (\gamma_i)_{i \in I} \in (\mathbb{Q}(q)^\pi)^I,
\end{equation}

and
are assumed to satisfy Conditions (3.3)–(3.5) below:

\[(3.3)\] \[\overline{s_i q_i} = s_i q_i \text{ if } \tau i = i \text{ and } a_{ij} \neq 0 \text{ for some } j \in I \setminus \{i\};\]

\[(3.4)\] \[\overline{s_i} = s_i = \overline{s_{\tau i}}, \text{ if } \tau i \neq i \text{ and } a_{i, \tau i} = 0.\]

\[(3.5)\] \[s_{\tau i} = \pi_i q_i^{-a_{i, \tau i}} \overline{s_i} \text{ if } \tau i \neq i \text{ and } a_{i, \tau i} \neq 0.\]

The quantum covering group is a (right) coideal subalgebra of \(U\), since under the comultiplication \(\Delta : U^i \to U^i \otimes U\). We will occasionally denote the embedding by \(\iota : U^i \hookrightarrow U\); the \(\iota\) in the name and superscript originates from this convention. The conditions on the parameters ensure that \(U^i\) admits a suitable bar-involution (see \S 3.2).

### 3.1. The \(i^x\) divided powers.

Let \(U^t = U^t_\iota\) be an quantum group with parameter \(\varsigma\), for a given root datum \((Y, X, \langle \cdot, \cdot \rangle, \ldots)\).

**Definition 3.2.** For \(i \in I\) with \(\tau i \neq i\), imitating Lusztig’s divided powers, we define the \textit{divided power} of \(B_i\) to be

\[(3.6)\] \[B_i^{(m)} := B_i^m / [m]_i^1, \quad \forall m \geq 0, \quad \text{when } i \neq \tau i.\]

For \(i \in I\) with \(\tau i = i\), the \(i^x\)-\textit{divided powers} are defined to be

\[(3.7)\] \[B_{i, 1}^{(m)} = \frac{1}{[m]_i^1} \left\{ B_i \prod_{j=1}^{k} (B_i^2 - \varsigma_i q_i [2j - 1]_i^2 \tilde{J}_i) \right\} \text{ if } m = 2k + 1,
\]

\[(3.8)\] \[B_{i, 0}^{(m)} = \frac{1}{[m]_i^1} \left\{ B_i \prod_{j=1}^{k} (B_i^2 - \varsigma_i q_i [2j - 2]_i^2 \tilde{J}_i) \right\} \text{ if } m = 2k,\]

When we specialize \(\pi_i = 1\) and \(\tilde{J}_i = 1\), we obtain the \(i\)-divided powers in [CLW18] from the formulas above. These \(i^x\)-divided powers satisfy closed form expansion formulas when written in terms of the PBW basis for \(U\) (see [C19, \S 3.3–3.7]), which enables the formulation of a Serre presentation for \(U^t\) in [C19, Theorem 4.2], generalizing [CLW18, Theorem 3.1].

### 3.2. Bar involution on \(U^i\).

One application of the Serre presentation for \(U^t\) is that it enables us to establish the existence of the bar involution for the quasi-split quantum group \(U^t\) in [C19, Prop 4.10]:

**Proposition 3.3.** Assume the parameters \(\varsigma_i\), for \(i \in I\), satisfy the conditions (3.3)–(3.5) above. Then there exists a \(\mathbb{Q}\)-algebra automorphism \(i^{\prime} : U^t \to U^t\) (called a bar involution on \(U^t\)) such that

\[
\tau = \pi q^{-1}, \quad B_i = B_i, \quad \overline{J}_i = \tilde{J}_i, \quad K = J K^{-1}, \quad \forall \mu \in Y^t, i \in I.
\]

Note that bar-involution for \(U^t\) (which we will henceforth denote with \(\psi_i\)) differs from the bar-involution for \(U\) (which we will now call \(\psi\)) defined in \S 2.2.
previously when restricted to $U^\iota$: $\psi_\iota$ fixes $B_\iota$ but $\psi(F_\iota + c_\iota E_\iota K_\iota^{-1}) = F_\iota + c_\iota E_\iota J_\iota K_\iota$

In the next section, we will construct a quasi-$K$-matrix $\Upsilon$ intertwining the two involutions, which will lead to a theory of canonical bases in the following sections.

4. Quasi-$K$-matrix

The goal of this section will be the development of a quasi-$K$-matrix for $U^\iota$. Let $\hat{U}$ be the completion of $U$ with respect to the descending sequence of $Q(q)^\pi$-submodules $U^-U^0 \left( \sum_{ht(\mu) \geq N} U^\mu_+ \right)$. We have an embedding of $U$ into $\hat{U}$, and by continuity the $Q(q)^\pi$-algebra structure on $U$ extends to $\hat{U}$, and the bar-involution $\psi$ on $U$ extends to an involution on $\hat{U}$, which we also denote $\psi$. Let $\hat{U}^+$ denote the closure of $U^+$ in $\hat{U}$.

We will show that there exists a unique family of elements $\Upsilon_\mu \in U^\mu_+$ such that $\Upsilon_0 = 1$ and $\Upsilon = \sum_\mu \Upsilon_\mu$ satisfies the following identity in $\hat{U}$:

$$\psi_\iota(u) \Upsilon = \Upsilon \psi(u), \quad \text{for all } u \in U^\iota.$$

$\Upsilon$ is called the quasi-$K$-matrix cf. [BK15]; the terminology intertwiner also appears in the literature e.g. [BW18a, Chapter 2], since $\Upsilon$ "intertwines" the bar-involutions $\psi^\iota$ for $U^\iota$ and $\psi$ for $U$, which are not compatible under the embedding $\iota$.

4.1. A parity operator. A crucial ingredient of the quasi-$K$-matrix construction in [BW18a] is [Lu94, Prop 3.1.6]; its quantum covering analogue is Proposition 2.3 above. However, when attempting a similar computation in the quantum covering case, we run into the following issue: since $B_\iota = F_\iota + c_\iota E_\iota J_\iota K_\iota$ in $U$, we would like to have $\Upsilon = \sum_\mu \Upsilon_\mu \in \hat{U}^+$ satisfying

$$(F_\iota + c_\iota E_\iota K_\iota^{-1}) \Upsilon = \Upsilon (F_\iota + c_\iota E_\iota J_\iota K_\iota)$$

and so we have equivalently that

$$F_\iota \Upsilon_\mu - \Upsilon_\mu F_\iota = \Upsilon_{\mu-2i} c_\iota E_\iota J_\iota K_\iota - c_\iota E_\iota K_\iota^{-1} \Upsilon_{\mu-2i}.$$

Unfortunately here we cannot apply Prop 2.3 when $p(\mu) = \bar{1}$ due to an extraneous factor of $\pi_i$.

Borrowing inspiration from [BKK], we can get around this issue by enlarging our algebra slightly by introducing a parity operator $\sigma$ such that

$$\sigma E_\iota = \pi^{p(\iota)} E_\iota \sigma, \quad \sigma F_\iota = \pi^{p(\iota)} F_\iota \sigma, \quad \sigma K_\mu = K_\mu \sigma \text{ and } \sigma J_\mu = J_\mu \sigma$$

and separating odd and even parts $\Upsilon = \Upsilon_0 + \sigma \Upsilon_1$. 
4.2. Quasi-$K$-matrix for $\mathfrak{osp}(1|2n)$. In finite type rank $n$, we want to define $\Upsilon = \sum_\mu \sigma^\mu(\mu) \Upsilon_\mu \in \hat{U}^+$ satisfying

$$(F_i + c_i E_i K_i^{-1}) \Upsilon = \Upsilon(F_i + \overline{c_i} E_i J_i K_i)$$

which together with Proposition 2.3 yields equivalent conditions (which are the same for $p(\mu)$ even or odd) in terms of the twisted derivations $r_i$ and $i r$ defined as in § 2.1:

$$r_i(\Upsilon_\mu) = -(\pi_i q_i - q_i^{-1})(c_i \pi_i q_i^2) \Upsilon_{\mu - 2i} E_i$$

$$i r(\Upsilon_\mu) = -(\pi_i q_i - q_i^{-1})(c_i \pi_i q_i^2) E_i \Upsilon_{\mu - 2i}$$

where we have used the fact that $\pi_i^{p(\mu)} = \pi^{p(\mu)^2} = \pi_i$ since by the bar-consistency condition $p(i) \equiv d_i \pmod{2}$.

With this, we can use the methods in [BW18a, Section 2.4] (cf. also [BK18, Section 6.2]) to construct $\Upsilon$. Recall the non-degenerate symmetric bilinear form $\langle , \rangle$ on $\mathfrak{f}$ defined preceding § 2.1 above. We have

$$\langle r_i(\Upsilon_\mu), z \rangle = -(\pi_i q_i - q_i^{-1})(c_i \pi_i q_i^2) \langle \Upsilon_{\mu - 2i} E_i, z \rangle$$

$$\Leftrightarrow (\Upsilon_\mu, z E_i) = -(\pi_i q_i - q_i^{-1})(c_i \pi_i q_i^2) (E_i, E_i)^2 \langle \Upsilon_{\mu - 2i}, r_i(z) \rangle$$

Applying a similar argument to (4.3), we have:

**Lemma 4.1.** The conditions (4.2)–(4.3) yield the equivalent conditions

$$\langle \Upsilon_\mu, E_i z \rangle = -c_i q_i^2 (1 - \pi_i q_i^{-2})^{-1} \langle \Upsilon_{\mu - 2i}, i r(z) \rangle$$

$$\langle \Upsilon_\mu, z E_i \rangle = -c_i q_i^3 (1 - \pi_i q_i^{-2})^{-1} \langle \Upsilon_{\mu - 2i}, r_i(z) \rangle$$

Thus we may inductively define $\Upsilon^*_L$ and $\Upsilon^*_R$ in $\mathfrak{f}^*$ the non-restricted dual of $\mathfrak{f}$ such that $\Upsilon^*_L(1) = \Upsilon^*_R(1) = 1$ and

$$\Upsilon^*_L(E_i z) = -c_i q_i^3 (1 - \pi_i q_i^{-2})^{-1} \Upsilon^*_L(i r(z))$$

$$\Upsilon^*_R(z E_i) = -c_i q_i^3 (1 - \pi_i q_i^{-2})^{-1} \Upsilon^*_R(r_i(z))$$

Note that for all $i, j \in I$, we have from $i r(1) = 0$ and $i r(E_j) = \delta_{ij}$ that

$$\Upsilon^*_L(E_i) = 0 \quad \text{and} \quad \Upsilon^*_L(E_i E_j) = -c_i q_i^3 (1 - \pi_i q_i^{-2})^{-1} \delta_{ij},$$

and similarly for $\Upsilon^*_R$.

**Lemma 4.2.** For $x \in \mathfrak{f}_\mu$, if either $p(\mu)$ or $ht(\mu)$ is odd, then $\Upsilon^*_L(x) = \Upsilon^*_R(x) = 0$

**Proof.** We show this for odd $p(\mu)$ by induction on $ht(\mu)$ (the statement for odd $ht(\mu)$ is similar). The base cases $ht(\mu) = 1, 3$ are given above. For homogeneous such $x \in \mathfrak{f}_\mu$, $x = E_i z$ for some $z \in \mathfrak{f}_\nu$ so $i r(z) \in \mathfrak{f}_{\nu - i}$ where $p(\nu - i)$ is odd ($p(\nu)$ and $p(i)$ have opposite parity since $p(\mu) = p(\nu) + p(i)$ is odd), and so by induction
hypothesis, $\Upsilon_L^*(r(z)) = 0$, and hence by (4.6), $\Upsilon_L^*(x) = 0$ as well (similarly for $\Upsilon_R^*$).

Note that as a result, there will be no odd terms in $\Upsilon$ i.e. for $p(\mu) = 1$, $\Upsilon_\mu = 0$.

**Lemma 4.3.** We have $\Upsilon_L^* = \Upsilon_R^*$.

**Proof.** We will show that $\Upsilon_L^* = \Upsilon_R^*$ for all homogeneous $x \in \mathfrak{f}_\mu$ by induction on $\text{ht}(\mu)$, using Lemma 2.1 above.

The base cases $\text{ht}(|x|) = 0$ or 1 are trivial from the definition. Suppose that the identity holds for all homogeneous elements with height no greater than $k$ for $k \geq 1$, and let $x = E_i x' E_j$ with $\text{ht}(|x|) = k + 1 \geq 2$ for some $i,j \in I$. Let $\xi_k = -c_k q_k^2 (1 - \pi_k g_k^{-2})^{-1}$. Then,

$$
\Upsilon_L^*(E_i x' E_j) = \xi_i \Upsilon_L^* (r(x') E_j) \\
= \xi_i \left( \Upsilon_L^* (r(x') E_j) + \pi^{p(x')p(i)} q^{|x'|+\cdot} \Upsilon_L^* (x', r(E_j)) \right)
$$

and

$$
\Upsilon_R^*(E_i x' E_j) = \xi_j \Upsilon_R^* (r_j(E_i x')) \\
= \xi_j \left( \Upsilon_R^* (r_j(E_i x')) + \pi^{p(x')p(j)} q^{|x'|+\cdot} \Upsilon_R^* (x', r_j(E_i)) \right).
$$

The second terms of both of the final expressions above vanish unless $i = j$, in which case they are both equal (by application of the induction hypothesis to $x'$ of height $k - 1$), so it remains to show that

$$
\xi_i \Upsilon_L^* (r(x') E_j) = \xi_j \Upsilon_R^* (E_i r_j(x'))
$$

This can be done by applying the induction hypothesis to $r(x') E_j$ and $E_i r_j(x')$ to obtain

$$
\xi_i \Upsilon_L^* (r(x') E_j) = \xi_i \Upsilon_R^* (r(x') E_j) \quad (4.7)
$$

and

$$
\xi_j \Upsilon_R^* (E_i r_j(x')) = \xi_j \Upsilon_R^* (E_i r_j(x')) \quad (4.6)
$$

and from the fact that $r_j \circ r = i r \circ r_j$ by Lemma 2.2, and the induction hypothesis, since $r_j \circ r(x') = i r \circ r_j(x') \in \mathfrak{f}_{|x'|+1-j}$, the desired result follows.

Thus, we can denote $\Upsilon_L^* = \Upsilon_R^*$ by $\Upsilon^*$. For the Serre relators $S_{ij}$ for $\mathfrak{osp}(1|2n)$, $|S_{ij}|$ has height 3 when $(i,j) \neq (n,n-1)$, and $p(S_{n,n-1})$ is odd, so by 4.2:

$$
\Upsilon^*(S_{ij}) = 0
$$

and by the same induction argument in [BW18a, Lemma 2.17],

$$
\Upsilon^*(I) = 0
$$

where $I = \langle S_{ij} \rangle$ (cf. [BW18a, Lemma 2.17]) so $\Upsilon^*$ is an element in $(\mathbf{U}^+)^*$ (the unrestricted dual of $\mathbf{U}^+$).
Then, we can construct Υ in the same way [following the proof of Theorem 2.10]:

Let \( B = \{ b \} \) be a basis of \( U^- \) such that \( B_\mu = B \cap U^{+}_\mu \) is a basis for \( U^{+}_\mu \), and let \( B^* = \{ b^* \} \) be the dual basis of \( B \) with respect to \((\cdot, \cdot)\) and let

\[
\Upsilon := \sum_{b \in B} \Upsilon^*(b^*)b = \sum_{\mu} \Upsilon_{\mu} \in \hat{U}^+.
\]

As functions on \( U^+ \), we have \((\Upsilon, \cdot) = \Upsilon^*\), and \( \Upsilon_0 = 1 \). Also \( \Upsilon \) satisfies the identities in (4.2) and (4.3) by construction, because \( \Upsilon^* \) satisfies the equivalent identities in (4.6) and (4.7).

From this we see that \( r_1(\Upsilon_{\mu}) \) is determined by \( \Upsilon_{\nu} \) with weight \( \nu < \mu \). Together with Lemma 2.1, this implies the uniqueness of \( \Upsilon \).

**Remark 4.4.** For rank 2, we can generalize the above slightly by having \( \kappa_1 \neq 0 \) (i.e. \( B_1 = F_1 + c_1 E_1 K^{-1} + \kappa_1 K^{-1} \)). In this scenario, for \( \kappa = 1 \) and \( \pi_1 = \pi^{(1)} = 1 \), we have the following replacements for (4.2) and (4.3) (using \( \alpha_1 \) for \( i = 1 \in I \) to avoid confusion):

\[
\begin{align*}
(4.9) & \quad r_1(\Upsilon_{\mu}) = -(q_1 - q_1^{-1}) \left( (c_1 q_1^2 \Upsilon_{\mu - 2 \alpha_1} E_1 + \kappa_1 \Upsilon_{\mu - \alpha_1}) \right) \\
(4.10) & \quad 1r(\Upsilon_{\mu}) = -(q_1 - q_1^{-1}) \left( (c_1 q_1^2 \Upsilon_{\mu - 2 \alpha_1} + \kappa_1 \Upsilon_{\mu - \alpha_1}) \right)
\end{align*}
\]

This leads to the following replacements for \( i = 1 \) in the inductive definition for \( \Upsilon_L^* \) and \( \Upsilon_R^* \):

\[
\begin{align*}
(4.11) & \quad \Upsilon_L^*(E_1z) = -c_1 q_1^3 (1 - q_1^{-2})^{-1} \Upsilon_L^*(1r(z)) - \kappa_1 q_1 \Upsilon_L^*(z) \\
(4.12) & \quad \Upsilon_R^*(zE_1) = -c_1 q_1^3 (1 - q_1^{-2})^{-1} \Upsilon_R^*(r_1(z)) - \kappa_1 q_1 \Upsilon_R^*(z)
\end{align*}
\]

It can then be checked that \( \Upsilon_L^* = \Upsilon_R^* =: \Upsilon^* \) and \( \Upsilon^*(I) = 0 \) for \( I = \langle S_{12}, S_{21} \rangle \), and so the above construction for \( \Upsilon \) also holds. Note that \( \Upsilon \) is still even in this case since \( \kappa_1 \) is a coefficient for the long, even root.

**4.3. Example: rank 1 (single odd root).** Let

\[
\Upsilon = \sum_{k \geq 0} a_{2k} E^{(2k)} + a_{2k+1} \sigma E^{(2k+1)}
\]

Then, Proposition 2.3 in rank one gives

\[
E^{(N)} F - \pi^N FE^{(N)} = \pi^{N-1} \left[ K; 1 - N \right] E^{(N-1)} = \pi E^{(N-1)} (\pi q)^{1-N} JK - q^{N-1} K^{-1}
\]

We need to separate the computation for the condition \( B \Upsilon = \Upsilon B \) when \( N \) is even from when \( N \) is odd. When \( N = 2k \) is even, we have

\[
a_{2k}(E^{(2k)} F - \pi^N FE^{(2k)}) = a_{2k} \left( \sigma^2 q^2 K^{-1} EE^{(2k-2)} - \sigma EE^{(2k-2)} JK \right)
\]
and so using (4.3) and comparing coefficients of $E^{(2k-1)}JK$ and $E^{(2k-1)}K^{-1}$ respectively yield the (over-determined) system of solutions

$$a_{2k} = -c\pi q^2 (\pi q - q^{-1}) q^{1-2k}[2k - 1]a_{2k-2}$$

and

$$a_{2k} = -c\pi q^2 (\pi q - q^{-1}) q^{2k-1} q^{2(1-2k)}[2k - 1]a_{2k-2}.$$ 

Hence for $k$ even,

$$a_{2k} = (-c\pi q^2)^k (\pi q - q^{-1}) q^{-k^2}[2k - 1]!$$

where $[2k - 1]! = [2k - 1] \cdot [2k - 3] \cdot \ldots \cdot 1$ (normalization: $a_0 = 1$).

For $N$ odd, we also obtain an over-determined system of two solutions:

$$a_{2k+1} = (-c\pi q^2)(\pi q - q^{-1}) q^{-2k}[2k]a_{2k-1}$$

$$= (-c\pi q^2)^{k+1} (\pi q - q^{-1})^{k+1} q^{-2(k+1)}[2k]! a_{-1}$$

where $[2k]! = [2k] \cdot [2k - 2] \cdot \ldots \cdot 2$. Since $a_{-1} = 0$, we see that $\Upsilon$ has no odd part.

So we have

$$\Upsilon = \sum_{k \geq 0} (-c\pi q^2)^k (\pi q - q^{-1})^k q^{-k^2}[2k - 1]! E^{(2k)}$$

Note that $\Upsilon$ is a solution to the system of equations

(4.13) \hspace{1cm} \text{1}r(\Upsilon) = -c\pi q^2 (\pi q - q^{-1}) E \Upsilon,$

and

(4.14) \hspace{1cm} \text{r}_1(\Upsilon) = -c\pi q^2 (\pi q - q^{-1}) \Upsilon E,$

and indeed may be defined as the unique such solution (cf. [BK18, Proposition 6.3])

Existence: this can be verified for $\Upsilon$ defined above using $1_r(E^{(2k)}) = q^{2k-1} E^{(2k-1)}$ for the first equation:

$$1_r(\Upsilon_{2k}) = 1_r(a_{2k} E^{(2k)})$$

$$= a_{2k} q^{2k-1} E^{(2k-1)}$$

$$= -c\pi q^2 (\pi q - q^{-1}) a_{2k-2}[2k - 1] E E^{(2k-2)}$$

$$= (-c\pi q^2)(\pi q - q^{-1}) E \Upsilon_{2k-2},$$
and using \( r_1(E^{(2k)}) = q^{2k-1}E^{(2k-1)}(= 1r(E^{(2k)})) \) for the second.

Note that this definition implies no odd part for \( \Upsilon \), because
\[
1r(\Upsilon_{2k+1}) = 1r(a_{2k+1}\sigma E^{(2k+1)})
\]
\[
= a_{2k+1}q^{2k}\sigma E^{(2k)}
\]
\[
= -c\pi q^2(\pi q - q^{-1})a_{2k-1}\pi[2k]E\sigma \frac{E^{(2k-1)}}{[2k]} = \pi(-c\pi q^2)(\pi q - q^{-1})E\Upsilon_{2k-1}
\]

Remark 4.5 (rank 1 nonstandard). When we repeat the above computations with an additional term \( sK^{-1} \) in \( B \), we get the condition that
\[
a_N\left(1r(E^{(N)})\right) = a_N\left(r_1(E^{(N)})\right) = -(\pi q - q^{-1})(c\pi q^2[N-1]a_{N-2} + s\pi a_{N-1}\sigma E^{(N-1)}),
\]
and since \( 1r(E^{(N)}) = r_1(E^{(N)}) = q^{N-1}E^{(N-1)} \), there are no terms with \( \sigma E^{(N-1)} \) on the left hand side, and no solutions for \( s \neq 0 \).

4.4. Quasi-\( K \)-matrix for quasi-split QSP of general super Kac-Moody type. Now let \( \Upsilon \) be a general quantum covering group of super Kac-Moody type as defined in \( \S 2 \), and \((\Upsilon, \Upsilon')\) a quasi-split quantum symmetric pair for \( \Upsilon \), with bar-involutions \( \psi \) on \( \Upsilon \) and \( \psi \) on \( \Upsilon' \) respectively.

Theorem 4.6. There exists a unique family of elements \( \Upsilon_\mu \in \Upsilon_\mu^+ \) such that \( \Upsilon_0 = 1 \) and \( \Upsilon = \sum_\mu \Upsilon_\mu \) satisfies the following identity in \( \Upsilon' \):
\[
(4.15) \quad \psi_1(u)\Upsilon = \Upsilon_1\psi(u), \quad \text{for all } u \in \Upsilon'.
\]
Moreover, \( \Upsilon_\mu = 0 \) for all \( p(\mu) = 1 \).

Proof. The constructions in 4.2 are not particular to \( \text{osp}(1|2n) \) and so hold for quasi-split \( \Upsilon' \) of general super Kac-Moody type with \( E_{r_i} \) replacing \( E_i \), with the exception of checking that \( \Upsilon^*(S_{ij}) = 0 \) for general Serre relators. Using Remark 4.2 we have shown that this is the case for \( \text{ht}(S_{ij}) \) odd, and so it remains to show this for \( \text{ht}(S_{ij}) \) even. This can be done term-wise i.e. by showing that terms of the form
\[
(4.16) \quad \Upsilon^*(E_i^aE_jE_i^b) \quad \text{for } j \neq i \text{ and } a + b + 1 \text{ even}
\]
vanish. This can be done by induction using (4.6) or (4.7). For instance if \( a > 1 \), we may use (4.6) to show that (using \( \xi_k = -c_kq_k^3(1 - \pi_kq_k^{-2})^{-1} \) as above)
\[
\Upsilon^*(E_i^aE_jE_i^b) = \xi_i\Upsilon^*(1r(E_i^{a-1}E_jE_i^b))
\]
\[
= \Upsilon^*(1r(E_i^{a-1}E_j)E_i^b + \pi_i^{p(ai+j)}q^{(ai+j)q}E_i^{a-1}E_jr(E_i^b))
\]
\[
= \Upsilon^*(1r(E_i^{a-1})E_jE_i^b + \pi_i^{p(ai+j)}q^{(ai+j)q}E_i^{a-1}E_jr(E_i^b))
\]
and each of the two terms are of the form (4.16), and so the induction hypothesis applies; for \( a = 1 \) and not the base case we must have \( b > 1 \) so we can use 4.7.
on the other side. The base case here is $\Upsilon^*(E_i E_j) = 0$ for $i \neq j$ which has been computed above.

Note that $\Upsilon$ is invertible in $\hat{U}$ and in fact $\Upsilon^{-1} = \psi(\Upsilon) =: \Upsilon$.

**Corollary 4.7.** $\Upsilon \cdot \Upsilon = 1$

**Proof.** Multiplying by $\Upsilon^{-1}$ on the left and right on both sides of (4.1) gives us
$$\Upsilon^{-1} \psi_i(u) = \psi(u) \Upsilon^{-1},$$
for all $u \in U^i$.
Applying $\psi$ to both sides and replacing $u$ with $\psi_i(u)$, we have
$$\psi \Upsilon^{-1} \psi(u) = \psi_i(u) \psi \Upsilon^{-1},$$
for all $u \in U^i$ and so $\psi \Upsilon^{-1}$ also satisfies (4.1) hence by uniqueness $\psi \Upsilon^{-1} = \Upsilon$ and so $\Upsilon^{-1} = \Upsilon$. \hfill $\square$

5. **Integrality of actions of $\Upsilon$**

As observed in the non-quantum covering case, it is neither expected nor required that the quasi-$K$-matrix for $U^i$ beyond finite type is integral on its own cf. [BW16]. For quantum symmetric pairs of super Kac-Moody type, the correct formulation is the integrality of the action of the quasi-$K$-matrix $\Upsilon$ i.e. we will see in this section that $\Upsilon$ preserves the integral $A$-forms on integrable highest weight $U$-modules and their tensor products.

5.1. **Definitions and background.** We will use the following analogue of [BW16, Lemma 2.2].

**Lemma 5.1.** Let $(M, B(M))$ be a based $U$-module and let $\lambda \in X$. Then,

1. for $b \in B(M)$, the $\mathbb{Q}(q)$-linear map $\pi_b : U^{-1}_{b|b+\lambda} \rightarrow M \otimes M(\lambda)$, $u \mapsto u(b \otimes \eta_\lambda)$, restricts to an $A$-linear map $\pi_b : A U^{-1}_{b|b+\lambda} \rightarrow A M \otimes A A M(\lambda)$;
2. we have $\sum_{b \in B(M)} \pi_b(A U^{-1}_{b|b+\lambda}) = A M \otimes A A M(\lambda)$.

**Proof.** The proof is the almost identical to the one for [BW16, Lemma 2.2]: the comultiplication has the same general formula as [BW16, (2.1)], and the quantum covering analogue to (2.2) of [BW16] can be found in [CI14, (3.2)-(3.3)]. \hfill $\square$

The quantum covering group $U^i$ also has a modified form $\hat{U}^i$ with idempotents via a familiar construction cf. [BW18c, §3.5]. The bar-involution $\psi_i$ of $U^i$ then induces a bar-involution of the $\mathbb{Q}^\ast$-algebra $\hat{U}^i$, also denoted $\psi_i$, such that $\psi_i(q) = \pi q^{-1}$ and $\psi_i(B_1 1_\lambda) = B_1 1_\lambda$.

**Definition 5.2.** Just as in Definition 3.10 of *loc. cit.*, we define $A \hat{U}^i$ to be the set of elements $u \in \hat{U}^i$, such that $u \cdot m \in A \hat{U}^i$ for all $m \in A \hat{U}$. Then $A \hat{U}^i$ is clearly an $A$-subalgebra of $\hat{U}^i$ which contains all the idempotents $1_\zeta$ ($\zeta \in X_i$), and $A \hat{U}^i = \bigoplus_{\zeta \in X_i} A \hat{U}^i 1_\zeta$. 
Moreover, for \( u \in \hat{U}^i \), we have \( u \in A\hat{U} \) if and only if \( u \cdot 1_{\lambda} \in A\hat{U} \) for each \( \lambda \in X \) (cf. [BW18b, Lemma 3.20]).

As a consequence of the existence of the \( \kappa^\sigma \)-divided powers, we have the following proposition.

**Proposition 5.3.** For any \( i \in I \) and \( \mu \in X_i \), there exists an element \( B_{i,\zeta}^{(n)} \in A\hat{U}^i 1_{\zeta} \) satisfying the following 2 properties:

1. \( \psi_i(B_{i,\zeta}^{(n)}) = B_{i,\zeta}^{(n)} \);
2. \( B_{i,\zeta}^{(n)} 1_{\lambda} = F_i^{(n)} 1_{\lambda} + \sum_{a<n} F_i^{(a)} A\hat{U}^i 1_{\lambda}, \) for \( 1_{\lambda} \in A\hat{U}^i \) with \( \overline{\lambda} = \zeta \).

The elements \( B_{i,\zeta}^{(n)} \) can be thought of as the ‘leading term’ of the \( \kappa \)-canonical basis elements **Proposition 7.2** later.

**Definition 5.4.** Let \( A\hat{U}^i \) be the \( A \)-subalgebra of \( A\hat{U} \) generated by the \( \kappa^\sigma \)-divided powers \( B_{i,\zeta}^{(n)} \) \( (i \in I) \) for all \( n \geq 1 \) and \( \zeta \in X_i \).

Recall for \( \lambda \in X \), we denote by \( M(\lambda) \) the Verma module of highest weight \( \lambda \) (see [CHW13, Section 2.6]). We denote the highest weight vector by \( \eta_\lambda \). The following is an analogue of [BW18c, Lemma 6.3].

**Lemma 5.5.** Let \((M,B(M))\) be a based \( U \)-module. Let \( \lambda \in X \). Then,

1. for \( b \in B(M) \), the \( \mathbb{Q}(q) \)-linear map \( \pi_b : \hat{U}^i 1_{|\lambda|+\lambda} \to M \otimes M(\lambda), \ u \mapsto u(b \otimes \eta_\lambda) \), restricts to an \( A \)-linear map \( \pi_b : A\hat{U}^i 1_{|\lambda|+\lambda} \to A M \otimes_A A M(\lambda) \);

2. we have \( \sum_{b \in B(M)} \pi_b(A\hat{U}^i 1_{|\lambda|+\lambda}) = A M \otimes_A A M(\lambda) \).

**Proof.** Recall \( A\hat{U} \subset A\hat{U}^i \). Part (1) follows from **Definition 5.2**. Part (2) is proven in the same way as loc. cit. By part (1) we have \( \sum_{b \in B(M)} \pi_b(A\hat{U}^i 1_{|\lambda|+\lambda}) \subset A M \otimes_A A M(\lambda) \), and \( A\hat{U}^- \) has the increasing filtration

\[
A = A U_{\leq 0} \subseteq A U_{\leq 1} \subseteq \cdots \subseteq A U_{\leq N} \subseteq \cdots
\]

where \( A U_{\leq N} \) is the \( A \)-span of \( \{F^{(a_1)} \cdots F^{(a_n)} | a_1 + \cdots + a_n \leq N, i_1, \ldots, i_n \in I \} \), which induces an increasing filtration \( \{A M(\lambda)_{\leq N} \} \) on \( A M(\lambda) \).

We can prove by induction on \( N \) that \( A M \otimes_A A M(\lambda)_{\leq N} \subset \sum_{b \in B(M)} \pi_b(A\hat{U}^i 1_{|\lambda|+\lambda}) \):

Let \( b \otimes (F_{i_1}^{(a_1)} \cdots F_{i_n}^{(a_n)} \eta_\lambda) \in A M \otimes_A A M(\lambda)_{\leq N} \).

Now \( \Delta(B_{i,\zeta}^{(a_1)}) \) has the form \( 1 \otimes B_{i,\zeta}^{(a_1)} \) + terms lower in filtration degree and so by **Theorem 5.3** and appropriate \( \zeta \in X^i \) cf. [BW16, Lemma 2.2], we have

\[
B_{i,\zeta}^{(a_1)}(b \otimes (F_{i_2}^{(a_2)} \cdots F_{i_n}^{(a_n)} \eta_\lambda)) \in b \otimes (F_{i_1}^{(a_1)} \cdots F_{i_n}^{(a_n)} \eta_\lambda) + A M \otimes_A A M(\lambda)_{\leq N-1}.
\]

The lemma follows. \( \square \)
For \( \lambda \in X^+ \), we abuse the notation and denote also by \( \eta_\lambda \) the image of \( \eta_\lambda \) under the projection \( p_\lambda : M(\lambda) \rightarrow L(\lambda) \). Note that \( p_\lambda \) restricts to \( p_\lambda : A M(\lambda) \rightarrow A L(\lambda) \). The next corollary follows from Lemma 5.5.

**Corollary 5.6.** Let \( \lambda \in X^+ \), and let \((M, B(M))\) be a based \( U \)-module. Then,

1. for \( b \in B(M) \), the \( \mathbb{Q}(q) \)-linear map \( \pi_b : U^1_{|\lambda|+\lambda} \rightarrow M \otimes L(\lambda) \), \( u \mapsto u(b \otimes \eta_\lambda) \), restricts to an \( A \)-linear map \( \pi_b : \overset{'}A U^1_{|\lambda|+\lambda} \rightarrow A M \otimes_A A L(\lambda) \);
2. we have \( \sum_{b \in B(M)} \pi_b(\overset{'}A U^1_{|\lambda|+\lambda}) = A M \otimes_A A L(\lambda) \).

### 5.2. Integrality of actions of \( \Upsilon \)

#### 5.2.1. Just as in \textit{loc. cit.}, the quasi-\( K \)-matrix \( \Upsilon \in \hat{U}^+ \) induces a well-defined \( \mathbb{Q}(q) \)-linear map on \( M \otimes L(\lambda) \):

\[
(5.1) \quad \Upsilon : M \otimes L(\lambda) \rightarrow M \otimes L(\lambda),
\]

for any \( \lambda \in X^+ \) and any weight \( U \)-module \( M \) whose weights are bounded above.

Recall [BW18b, §5.1] that a \( U \)-module \( M \) equipped with an anti-linear involution \( \psi_i \) is called \textit{involutive} (or \( i \)-\textit{involutive}) if

\[
\psi_i(um) = \psi_i(u)\psi_i(m), \quad \forall u \in U^i, m \in M.
\]

**Proposition 5.7.** Let \((M, B)\) be a based \( U \)-module whose weights are bounded above. We denote the bar involution on \( M \) by \( \bar{\psi} \). Then \( M \) is an \( i \)-involutive \( U^i \)-module with involution

\[
(5.2) \quad \psi_i := \Upsilon \circ \psi.
\]

**Proof.** Just as in [BW18c], since the weights of \( M \) are bounded above, the action of \( \Upsilon : M \rightarrow M \) is well defined. The rest of the argument is analogous to the one found in the proof of [BW18b, Proposition 5.1] (also [BW18a, Proposition 3.10]): using Theorem 4.6, we have

\[
\psi_i(um) = \Upsilon \psi(um) = \Upsilon \psi(u)\psi(m) = \psi_i(u)\Upsilon \psi(m) = \psi_i(u)\psi_i(m)
\]

as required. \( \square \)

#### 5.2.2. Let \((M, B)\) be a based \( U \)-module whose weights are bounded above. Assume \( \Upsilon : M \rightarrow M \) preserves the \( A \)-submodule \( A M \).

**Proposition 5.8.** The \( \mathbb{Q}(q) \)-linear map \( \psi_i := \Upsilon \circ \psi \) preserves the \( A \)-submodule \( A M \otimes_A A L(\lambda) \), for any \( \lambda \in X^+ \).

**Proof.** The proof is again very similar: the \( U \)-module \( M \otimes L(\lambda) \) is involutive with the involution \( \psi := \Theta \circ (\sigma \otimes \tau) \) where \( \Theta \) is the quasi-\( R \)-matrix from Proposition 2.4. It follows by an argument similar to [BW16, Proposition 2.4] that \( \psi \) preserves the \( A \)-submodule \( A M \otimes_A A L(\lambda) \): The statement is that for \( \lambda \in X^+ \) and \((M, B(M))\) be a based \( U \)-module, the \( \mathbb{Q}(q) \)-linear map

\[
\Theta : M \otimes L(\lambda) \rightarrow M \otimes L(\lambda)
\]
preserves the $\mathcal{A}$-submodule $\mathcal{A}M \otimes \mathcal{A}L(\lambda)$.

We will write $\overline{\otimes}$ for $\overline{\otimes}^-$, which preserves the $\mathcal{A}$-lattice $\mathcal{A}M \otimes \mathcal{A}L(\lambda)$. Thus, any $x \in \mathcal{A}M \otimes \mathcal{A}L(\lambda)$ can be recognized as $x = \overline{x'}$ for some $x' \in \mathcal{A}M \otimes \mathcal{A}L(\lambda)$. By Lemma 5.1, $x' = \sum_i \pi_i(u'_i)$ (a finite sum), for some $b_i \in B(M)$ and $u'_i \in \mathcal{A}U^{-1}[b_i]$. Since $\mathcal{A}U^{-1}[b_i] \lambda$ is preserved by the bar involution on $\mathcal{U}$, we have $u'_i = \overline{u_i}$ for some $u_i \in \mathcal{A}U^{-1}[b_i] \lambda$. Hence,

$$x = \overline{x'} = \sum_i \overline{u_i}(b_i \otimes \eta_{\lambda}).$$

Using the property of the quasi-$\mathcal{R}$-matrix in Proposition 2.4, we have

$$u \Theta(m \otimes m') = \Theta(\overline{u(m \otimes m')}),$$

for $u \in \mathcal{U}$, $m \in M$ and $m' \in L(\lambda)$. Taking $m = b_i = \overline{b}_i$ and $m' = \eta_{\lambda} = \overline{\eta}_{\lambda}$, this gives

$$u(b_i \otimes \eta_{\lambda}) = \Theta(\overline{u(b_i \otimes \eta_{\lambda})})$$

since $\Theta(\overline{b}_i \otimes \overline{\eta}_{\lambda}) = \overline{b}_i \otimes \overline{\eta}_{\lambda}$ (by construction, $\Theta$ lies in a completion of $U^{-} \otimes U^{+}$), we have that

$$\Theta(x) = \sum_i \Theta(\overline{u(b_i \otimes \eta_{\lambda})}) = u_i(b_i \otimes \eta_{\lambda}) = \sum_i \pi_i(u_i),$$

where the latter lies in $\mathcal{A}M \otimes \mathcal{A}L(\lambda)$ by Lemma 5.1, which completes the proof.

Regarded as $\mathcal{U}^*$-module $M \otimes L(\lambda)$ is $\mathcal{I}$-involutive with the involution $\psi_i := \Upsilon \circ \psi_i$. We can now prove that $\psi_i$ preserves the $\mathcal{A}$-submodule $\mathcal{A}M \otimes \mathcal{A}L(\lambda)$.

By Corollary 5.6(2), for any $x \in \mathcal{A}M \otimes \mathcal{A}L(\lambda)$, we can write $x = \sum_k u_k (b_k \otimes \eta_{\lambda})$, for $u_k \in \mathcal{A}U$ and $b_k \in B$. Since $M \otimes L(\lambda)$ is $\mathcal{I}$-involutive, we have

$$\psi_i(x) = \sum_k \psi_i(u_k) \psi_i(b_k \otimes \eta_{\lambda}) = \sum_k \psi_i(u_k) \Upsilon \psi(b_k \otimes \eta_{\lambda}) = \sum_k \psi_i(u_k)(\Upsilon b_k \otimes \eta_{\lambda}),$$

where we have used the fact that $\Delta(\Upsilon) \in \Upsilon \otimes 1 + U \otimes U^r_{\lambda}$ and $\psi(b_k \otimes \eta_{\lambda}) = \Theta(b_k \otimes \eta_{\lambda}) = b_k \otimes \eta_{\lambda}$ since $\Theta$ is the sum of terms $\Theta_{\nu} \in U^r_{\nu} \otimes U^r_{\nu}$ and $\Theta_{\nu} = 1 \otimes 1$. By assumption we have $\Upsilon b_k \in \mathcal{A}M$ and it follows by definition of $\mathcal{A}U$ that $\psi_i(u_k) \in \mathcal{A}U$. Applying Corollary 5.6(2) again to (5.3), we obtain that $\psi_i(x) \in \mathcal{A}M \otimes \mathcal{A}L(\lambda)$. The proposition follows.

**Corollary 5.9.** The intertwiner $\Upsilon$ preserves the $\mathcal{A}$-submodule $\mathcal{A}M \otimes \mathcal{A}L(\lambda)$. In particular, $\Upsilon$ preserves the $\mathcal{A}$-submodule $\mathcal{A}L(\lambda)$ of $L(\lambda)$.

**Proof.** Recall $\Upsilon = \psi_i \circ \psi$. The corollary follows from Proposition 5.8 and the fact that $\psi$ preserves the $\mathcal{A}$-submodule $\mathcal{A}M \otimes \mathcal{A}L(\lambda)$. □

**Corollary 5.10.** Let $\lambda_i \in X^+$ for $1 \leq i \leq \ell$. The involution $\psi_i$ on the $\mathcal{I}$-involutive $\mathcal{U}^*$-module $L(\lambda_1) \otimes \ldots \otimes L(\lambda_\ell)$ preserves the $\mathcal{A}$-submodule $\mathcal{A}L(\lambda_1) \otimes \ldots \otimes \mathcal{A}L(\lambda_\ell)$. □
Proof. The module $L(\lambda_1) \otimes \ldots \otimes L(\lambda_r)$ is a based $U$-module whose weights are bounded above, and so the corollary follows by consecutive application of Proposition 5.8.

For finite type, we in fact have integrality of $\Upsilon$ and not just its action.

**Theorem 5.11.** Assume $(U, U^t)$ is of finite type. Write $\Upsilon = \sum_{\mu} \Upsilon_{\mu}$. Then we have $\Upsilon_{\mu} \in \mathcal{A}U^+$ for each $\mu$.

**Proof.** This follows by Corollary 5.9 and applying $\Upsilon$ to the lowest weight vector $\xi_{-\omega_0} \lambda \in \mathcal{A}L(\lambda)$, for $\lambda \gg 0$ (i.e., $\lambda \in X^+$ such that $\langle i, \lambda \rangle \gg 0$ for each $i$).

---

6. Canonical Basis on modules

We call a $U^t$-module $M$ a weight $U^t$-module if $M$ admits a direct sum decomposition $M = \oplus_{\lambda \in X^t} M_\lambda$ such that, for any $\mu \in Y^t$, $\lambda \in X^t$, $m \in M_\lambda$, we have $K_\mu m = q^{\langle \mu, \lambda \rangle} m$.

We will make the following definition of based $U^t$-modules (based on [BWW18, Definition 1]):

**Definition 6.1.** Let $M$ be a weight $U^t$-module over $\mathbb{Q}(q)^{\pi}$ with a given $\mathbb{Q}(q)^{\pi}$-basis $B^t$. The pair $(M, B^t)$ is called a based $U^t$-module if the following conditions are satisfied:

1. $B^t \cap M_\nu$ is a basis of $M_\nu$, for any $\nu \in X^t$;
2. The $\mathcal{A}$-submodule $\mathcal{A}M$ generated by $B^t$ is stable under $\mathcal{A}\hat{U}^t$;
3. $M$ is $t$-involutive; that is, the $\mathbb{Q}^\pi$-linear involution $\psi_i : M \to M$ defined by $\psi_i(q) = q^{-1}, \psi_i(b) = b$ for all $b \in B^t$ is compatible with the $\hat{U}^t$-action, i.e., $\psi_i(um) = \psi_i(u)\psi_i(m)$, for all $u \in \hat{U}^t, m \in M$;
4. Let $\mathcal{A} = \mathbb{Q}[[q^{-1}]]^\pi \cap \mathbb{Q}(q)^\pi$. Let $L(M)$ be the $\mathcal{A}$-submodule of $M$ generated by $B^t$. Then the image of $B^t$ in $L(M)/q^{-1}L(M)$ forms a $\mathbb{Q}^\pi$-basis in $L(M)/q^{-1}L(M)$.

We shall denote by $\mathcal{L}(M)$ the $\mathbb{Z}[q^{-1}]^\pi$-span of $B^t$; then $B^t$ forms a $\mathbb{Z}[q^{-1}]^\pi$-basis for $\mathcal{L}(M)$. We also define based $U^t$-submodules and based quotient $U^t$-modules in the obvious way.

By a standard argument using [Cl14, Lemma 9] (cf. [Lu94, Lemma 24.2.1]), we have the following generalization of [BW18, Theorem 6.12] (cf. [BW18b, Theorem 5.7]): Recall that the partial order here is the one given by (2.1), $\lambda \leq \lambda'$ iff $\lambda' - \lambda \in N[J]$.

**Theorem 6.2.** Let $(M, B)$ be a based $U$-module whose weights are bounded above. Assume the involution $\psi_i$ of $M$ from Proposition 5.7 preserves the $\mathcal{A}^\pi$-submodule $\mathcal{A}M$. 

(1) The $U^i$-module $M$ admits a unique $\pi$-basis (called the $\pi$-canonical basis) $B^i := \{ b | b \in B \}$, which is $\psi_i$-invariant and of the form

$$b^i = b + \sum_{b < b'} t_{b,b'} b', \quad \text{for} \quad t_{b,b'} \in \mathbb{Z}[q^{-1}].$$

(2) $B^i$ forms an $A^\pi$-basis for the $A^\pi$-lattice $A^i M$ (generated by $B$), and forms a $\mathbb{Z}[q^{-1}]$-basis for the $\mathbb{Z}[q^{-1}]$-lattice $M$ (generated by $B$).

(3) $(M, B^i)$ is a based $U^i$-module, where we call $B^i$ the $\pi$-canonical basis of $M$.

Recall the based $U$-submodule $L(\lambda, \mu)$, for $\lambda, \mu \in X^+$, which in light of Theorem 6.2 can be viewed as a based $U^i$-module. We denote this $U^i$-module $L^i(\lambda, \mu)$.

A corollary of the theorem is the following cf. [BW18c, §6]:

**Corollary 6.3.** Let $\lambda, \mu, \lambda_i \in X^+$ for $1 \leq i \leq \ell$, and $w \in W$.

(1) $L(\lambda_1) \otimes \ldots \otimes L(\lambda_\ell)$ is a based $U^i$-module, with the $\pi$-canonical basis defined as Theorem 6.2.

(2) $L(w\lambda, \mu)$ is a based $U^i$-submodule of $L(\lambda) \otimes L(\mu)$.

### 6.1. The element $\Theta^i$.

Recall the quasi-$R$ matrix $\Theta \in \widehat{U} \otimes \widehat{U}$ from §2.2 above. It follows from Theorem 4.6 that $\Upsilon^{-1} \otimes \text{id}$ and $\Delta(\Upsilon)$ are both in $\widehat{U} \otimes \widehat{U}$.

We define

$$\Theta^i = \Delta(\Upsilon) \cdot \Theta \cdot (\Upsilon^{-1} \otimes \text{id}) \in \widehat{U} \otimes \widehat{U}.$$

**Proposition 6.4** (cf. [BW18a, Proposition 3.2]). For any $b \in U^i$ one has

$$\Delta(\psi_i(b)) \cdot \Theta^i = \Theta^i \cdot (\psi_i \otimes \psi) \circ \Delta(b)$$

in $\widehat{U} \otimes \widehat{U}$.

**Proof.** Let $b \in U^i$. Using the intertwiner relations one calculates

$$\Theta^i \cdot (\psi_i \otimes \psi) \circ \Delta(b) = \Delta(\Upsilon) \cdot \Theta \cdot (\Upsilon^{-1} \otimes 1) \cdot (\psi_i \otimes \psi) \circ \Delta(b)$$

$$= \Delta(\Upsilon) \cdot \Theta \cdot (\psi_i \otimes \psi) \circ \Delta(b) \cdot (\Upsilon^{-1} \otimes 1) \quad \text{(using Theorem 4.6)}$$

$$= \Delta(\psi_i(b)) \cdot \Delta(\Upsilon) \cdot \Theta \cdot (\Upsilon^{-1} \otimes 1) \quad \text{(using Prop 2.4)}$$

$$= \Delta(\psi_i(b)) \cdot \Delta(\Upsilon) \cdot \Theta \cdot (\Upsilon^{-1} \otimes 1) \quad \text{(using Theorem 4.6 again)}$$

which proves the proposition. \qed

We can write

$$\Theta^i = \sum_{\mu \in NI} \Theta^i_{\mu}, \quad \text{where} \quad \Theta^i_{\mu} \in U \otimes U^+_{\mu}.$$ 

**Lemma 6.5.** The first and second tensor factors of each term in $\Theta^i_{\mu} \in U \otimes U^+_{\mu}$ share the same parity.
The following result is an analogue of [Ko17, Proposition 3.6], which first appeared in [BW18a, Proposition 3.5] for the quantum symmetric pairs of (quasisplit) type AIII/AIV.

**Lemma 6.6.** We have \( \Theta^i_\mu \in U^+_\mu U^+ \), for all \( \mu \). In particular, we have \( \Theta^i_0 = 1 \otimes 1 \).

**Proof.** For any \( i \in I \) one has

\[
\Delta(B_i) = B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i J_i \otimes E_i K_i^{-1}.
\]

Hence Proposition 6.4 implies that

\[
(B_i \otimes K_i^{-1} + 1 \otimes F_i + c_i J_i \otimes E_i K_i^{-1}) \cdot \Theta^i = \Theta^i \cdot (B_i \otimes J_i K_i + 1 \otimes F_i + \overline{\pi}_i J_i \otimes J_i K_i E_i).
\]

Rearranging this we obtain

\[
(\Theta^i (1 \otimes F_i) - (1 \otimes F_i) \Theta^i) = (B_i \otimes K_i^{-1} + c_i J_i \otimes E_i K_i^{-1}) \Theta^i - \Theta^i (B_i \otimes J_i K_i + \overline{\pi}_i J_i \otimes J_i K_i E_i)
\]

In each level \( \mu \), the left hand side is the sum of terms of the form

\[
((\Theta^i_\mu)^1 \otimes (\Theta^i_\mu)^2)(1 \otimes F_i) - (1 \otimes F_i)((\Theta^i_\mu)^1 \otimes (\Theta^i_\mu)^2)
\]

\[
= (\Theta^i_\mu)^1 \otimes (\Theta^i_\mu)^2 F_i - \pi_i^{p_i}(\Theta^i_\mu)^1 \otimes F_i(\Theta^i_\mu)^2 \quad \text{where} \quad p_k := p((\Theta^i_\mu)^k), \ k = 1, 2
\]

\[
= (\Theta^i_\mu)^1 \otimes [(\Theta^i_\mu)^2, F_i], \quad \text{since} \quad \pi_i^{p_1} = \pi_i^{p_2} \text{ by Lemma 6.5}
\]

\[
= (\Theta^i_\mu)^1 \otimes \left( \frac{r_i((\Theta^i_\mu)^2)J_i K_i - K_i^{-1} \pi_i^{r_i}(\Theta^i_\mu)^2)}{\pi_i q_i - q_i^{-1}} \right) \quad \text{by Proposition 2.3}
\]

Comparing this to terms on the right hand side of (6.4) with a factor of \( 1 \otimes J_i K_i \), we see that

\[
(1 \otimes r_i)(\Theta^i_\mu) = - (\pi_i q_i - q_i^{-1}) \Theta^i (B_i \otimes 1 + \overline{\pi}_i q_i^2 J_i \otimes E_i)
\]

Then, the same induction as in [Ko17, Proposition 3.6] completes the proof, this time using Lemma 2.2 as the appropriate analogue in the quantum covering group setting. \(\square\)

The following is an analogue of [BWW18, Lemma 3], used in the proof of a subsequent Theorem:

**Lemma 6.7.** We have \( \Theta^i_\mu \in \mathcal{A} U \otimes \mathcal{A} U^+ \) for all \( \mu \).

**Proof.** The argument is analogous, using integrality of \( \Theta \) by Proposition 2.4, together with Theorem 5.11 in the definition of \( \Theta^i \). \(\square\)
Theorem 6.8. Let $M$ be a based $U'$-module, and $\lambda \in X^+$. Then $\psi_1 \overset{\text{def}}{=} \Theta^i \circ (\psi_i \otimes \psi)$ is an anti-linear involution on $M \otimes L(\lambda)$, and $M \otimes L(\lambda)$ is a based $U'$-module with a bar involution $\psi_i$.

Proof. The anti-linear operator $\psi_1 = \Theta^i \circ (\psi_i \otimes \psi) : M \otimes L(\lambda) \to M \otimes L(\lambda)$ is well defined thanks to Lemma 6.6 and the fact that the weights of $L(\lambda)$ are bounded above. Then entirely similar to [BW18a, Proposition 3.13], we see that $\psi_i^2 = 1$ and $M \otimes L(\lambda)$ is $\iota$-involutive in the sense of Definition 6.1(3).

The proof that $\psi_i$ preserves the $\mathcal{A}$-submodule $\mathcal{A} M \otimes_{\mathcal{A}} \mathcal{A} L(\lambda)$ is the same as the proof of Proposition 5.8. By assumption, $(M, \mathcal{B}(M))$ is a based $U'$-module. For any $b \in \mathcal{B}(M)$, define

$$\pi_b : \hat{\mathcal{A}} \hat{U} \to \mathcal{A} \hat{U} \otimes_{\mathcal{A}} \mathcal{A} L(\lambda), \ u \mapsto u(b \otimes \eta_\lambda).$$

Then, $\pi_b$ is well defined, since by Definition 5.2 and the following remark the coproduct preserves the integral forms, that is, $\Delta(u)(1_\mu \otimes 1_\nu)$ preserves $\mathcal{A} M \otimes_{\mathcal{A}} \mathcal{A} L(\lambda)$, for any $\mu \in X'$ and $\nu \in X$.

We write $\mathcal{B} = \{b \otimes \eta_\lambda | b \in \mathcal{B}(\lambda)\}$ for the canonical basis of $L(\lambda)$. Following the same argument as for [BWW18, Theorem 4] i.e. using Lemma 6.6 and Lemma 6.7 and [Cl14, Lemma 9], we conclude that:

1. for $b_1 \in \mathcal{B}', b_2 \in \mathcal{B}$, there exists a unique element $b_1 \hat{\otimes} b_2$ which is $\psi_i$-invariant such that $b_1 \hat{\otimes} b_2 = b_1 \otimes b_2 + q^{-1} Z^g_b [q^{-1}] \mathcal{B}' \otimes \mathcal{B}$;
2. we have $b_1 \hat{\otimes} b_2 = b_1 \otimes b_2 + \sum_{(b_1', b_2') \in \mathcal{B}' \otimes \mathcal{B}, |b_1'| < |b_2'|} q^{-1} Z^g_b [q^{-1}] b_1' \otimes b_2'$;
3. $\mathcal{B}' \hat{\otimes} \mathcal{B} := \{b_1 \hat{\otimes} b_2 | b_1 \in \mathcal{B}', b_2 \in \mathcal{B}\}$ forms a $\mathbb{Q}(q)^g$-basis for $M \otimes L(\lambda)$, an $\mathcal{A}'$-basis for $\mathcal{A} \hat{U} \otimes \mathcal{A} L(\lambda)$, and a $\mathbb{Z}^g$-basis for $\mathcal{L}(\mathcal{M}) \otimes_{\mathbb{Z}^g} \mathcal{L}(\lambda)$;
4. $(M \otimes L(\lambda), \mathcal{B}' \hat{\otimes} \mathcal{B})$ is a based $U'$-module.

\[\square\]

7. Canonical basis on $\hat{U}'$

In this section, we formulate the main definition and theorems on canonical bases on the modified quantum groups. The formulations are based on [BW18c, Section 7], which in turn are generalizations of finite type counterparts in [BW18b, Section 6].

7.1. The modified quantum groups. Recall the partial order $\leq$ on the weight lattice $X$ in (2.1). The following proposition is a version of [BW18c, Proposition 7.1] in the quantum covering setting.
Proposition 7.1. Let $\lambda, \mu \in X^+$. 

(1) The $\iota$-canonical basis of the $U^\iota$-module $L^\iota(\lambda, \mu)$ is the basis 

$$B'(\lambda, \mu) = \{(b_1 \triangleleft_\zeta b_2)_{\lambda, \mu}^\iota|(b_1, b_2) \in B' \times B\}\setminus\{0\},$$

where $(b_1 \triangleleft_\zeta b_2)_{\lambda, \mu}^\iota$ is $\psi_\iota$-invariant and lies in 

$$(b_1 \triangleleft_\zeta b_2)(\eta_\lambda \otimes \eta_\mu) + \sum_{|b_1'| + |b_2'| \leq |b_1| + |b_2|} q^{-1}Z\{q^{-1}(b_1' \triangleleft_\zeta b_2')_{\lambda, \mu}^\iota\}(\eta_\lambda \otimes \eta_\mu).$$

(2) We have the projective system $\{L^\iota(\lambda + \nu^\iota, \mu + \nu)\}_{\nu \in X^+}$ of $U^\iota$-modules, where 

$$\pi_{\nu+\nu_1, \nu_1} : L^\iota(\lambda + \nu^\iota + \nu_1^\iota, \mu + \nu + \nu_1) \longrightarrow L^\iota(\lambda + \nu^{\iota}, \mu + \nu), \quad \nu, \nu_1 \in X^+,$$

is the unique homomorphism of $U^\iota$-modules such that 

$$\pi(\eta_{\lambda+\nu^\iota+\nu_1^\iota} \otimes \eta_{\mu+\nu+\nu_1}) = \eta_{\lambda+\nu^\iota} \otimes \eta_{\mu+\nu}.$$ 

(3) The projective system in (2) is asymptotically based in the following sense: for fixed $(b_1, b_2) \in B' \times B$ and any $\nu_1 \in X^+$, as long as $\nu \gg 0$, we have 

$$\pi_{\nu+\nu_1, \nu_1}(b_1 \triangleleft_\zeta b_2)^{\iota}_{\lambda+\nu^\iota+\nu_1^\iota, \mu+\nu+\nu_1} = ((b_1 \triangleleft_\zeta b_2)^{\iota}_{\lambda, \mu})^{\nu^\iota, \mu+\nu_1^\iota}.$$ 

Proof. Claim (1) is just a reformulation of 6.3. Claim (2) follows by the same proof as [BW18b, Proposition 6.12], using the quasi-$R$-matrix in Proposition 2.4. Claim (3) is the same as [BW18b, Proposition 6.16], and we can do without the mild modification needed in [BW18c] since the module $L(\nu^\iota + \nu)$ is finite dimensional. 

Proposition 7.1 is the main mechanism of proof in the following version of [BW18c, Theorem 7.2] (see also [BW18b, Theorem 6.17]), granting the $\iota$-canonical basis for $\hat{U}^\iota$: 

Proposition 7.2. Let $\zeta_i \in X_i$ and $(b_1, b_2) \in B \times B$. 

(1) There is a unique element $u = b_1 \triangleleft_\zeta b_2 \in \hat{U}^\iota$ such that 

$$u(\eta_\lambda \otimes \eta_\mu) = (b_1 \triangleleft_\zeta b_2)^{\iota}_{\lambda, \mu} \in L^\iota(\lambda, \mu),$$

for all $\lambda, \mu \gg 0$ with $\lambda + \mu = \zeta_i$. 

(2) The element $b_1 \triangleleft_\zeta b_2$ is $\psi_\iota$-invariant. 

(3) The set $\hat{B}^\iota = \{b_1 \triangleleft_\zeta b_2|\zeta_i \in X_i, (b_1, b_2) \in B_i \times B\}$ forms a $\mathbb{Q}(q)^\iota$-basis of $\hat{U}^\iota$ and an $A^\iota$-basis of $\hat{A} \hat{U}^\iota$. 


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