ON UNIFORMIZATION OF COMPACT KÄHLER SPACES

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1. Introduction

The aim of the present note is to extend author’s uniformization theorem in [29] to compact Kähler spaces with mild singularities and establish a kind of rigidity of their universal coverings. We study compact Kähler spaces with large, residually finite and nonamenable fundamental groups and mild singularities by passing to their universal coverings where there are plenty of bounded harmonic functions (see Lyons and Sullivan[19] and Toledo [25]).

(1.1) Throughout the note $X$ will be a connected, compact, normal, Cohen-Macaulay Kähler space of dimension $n$. If $X$ is, in addition, projective we then assume the canonical class $K_X$ is a $\mathbb{Q}$-divisor. Also, we assume the fundamental group $\Gamma := \pi_1(X)$ is large, residually finite and nonamenable.

(1.2) Let $U_X$ denote the universal covering of $X$. Recall that $\pi_1(X)$ is large if and only if $U_X$ contains no proper holomorphic subsets of positive dimension [17].

Recall that a countable group $G$ is called amenable if there is on $G$ a finite additive, translation invariant nonnegative probability measure (defined for all subsets of $G$). Otherwise, $G$ is called nonamenable.

In 1970s, it became clear that in the classification of projective manifolds of dimension at least three, one has to consider varieties with mild singularities. Let $Z$ denote a normal holomorphic space. We say that $Z$ has at most $1$-rational singularities if $Z$ has a resolution of singularities $\psi : Y \to Z$ such that $R^1\psi_*\mathcal{O}_Y = 0$. It is well known that $Z$ has rational singularities if and only if $R^i\psi_*\mathcal{O}_Y = 0$ ($\forall i \geq 1$); equivalently, $Z$ is Cohen-Macaulay and $\psi_*\omega_Y = \omega_Z$, where $\omega$ denotes the dualizing sheaf. Set $Z^{\text{reg}} := Z \setminus Z_{\text{sing}}$.

Bounded domains in $\mathbb{C}^n$ have the (classical) Bergman metric. As was already observed by Bochner [2] (also see Kobayashi [15], [16]), one can view the Bergman metric in a bounded domain $D$ via a natural embedding of $D$ into a Fubini space $\mathbf{F}_C(\infty, 1)$ (infinite-dimensional projective space). Furthermore, the image of $D$ does not intersect a hyperplane at infinity. Taking, then, the Calabi flattening out of $\mathbf{F}_C(\infty, 1)$ into $\mathbf{F}_C(\infty, 0)$ (infinite-dimensional Hilbert space) [3, Chap. 4], our $D$ will be holomorphically embedded into $\mathbf{F}_C(\infty, 0)$.

Now, we say that $U_X$ has a generalized Bergman metric (Bergman metric in a singular space) if we get a natural embedding of $U_X$ into a Fubini space $\mathbf{F}_C(\infty, 1)$ given by functions-sections of a pluricanonical bundle. Furthermore, its image does not belong to a proper subspace of $\mathbf{F}_C(\infty, 1)$ and does not intersect a hyperplane at infinity.
The infinite-dimensional complex projective spaces $F_C(\infty, 1)$ as well as their real counterparts $F_R(2\infty, 1)$ play a prominent role in the present note. Often, $U_X$ can be flatten out into $F_C(r, 0)$ with $r < \infty$.

Let $X$ and $Z$ satisfy the assumptions (1.1) and are homeomorphic: $X \approx Z$. We assume they have generalized Bergman metrics and can be flatten out into $F_C(r_X, 0)$ and $F_C(r_Z, 0)$, respectively, with $r_X, r_Z < \infty$ (e.g., $X$ and $Z$ are nonsingular [27]). Then we say $U_X$ is rigid if the induced mapping $U_X \to U_Z$ is homotopic to a biholomorphism.

The aim of the present note is to prove the following theorem.

**Theorem (Uniformization).** We assume, in addition to the assumptions (1.1), that $X$ has at most rational singularities. Then

1. $U_X$ is Stein,
2. $K_X^m$ is very ample for an integer $m \gg 0$, and $U_X$ has a generalized Bergman metric,
3. $U_X$ is rigid.

First, we establish that $U_X$ is Stein provided $X$ is a projective variety. Second, we will show that our Kahler space $X$ is a Moishezon space. Therefore $X$ is a projective variety by a theorem of Namikawa [21] because $X$ has at most 1-rational singularities (a generalization of Moishezon’s theorem). We, then, establish (2) and (3).

### 2. Preliminaries

(2.1) **Kahler spaces.** Recall that Kahler spaces were defined by Grauert [10, Sect. 3.3]: Let $Y$ be an arbitrary reduced holomorphic space with a Hermitian metric $ds^2 = \sum g_{ik}dz_id\bar{z}_k$ in $Y \setminus Y_{\text{sing}}$ (with continuous coefficients). Then it is called a Kahler metric in $Y$ if and only if for every point $a \in Y$, there is an open neighborhood $Y$ of $a$ and a strictly plurisubharmonic function $p$ on $Y$ such that $g_{ik} = \frac{\partial^2 p}{\partial z_i \partial \bar{z}_k}$ in $Y \setminus Y_{\text{sing}}$.

A function $u : Y \to \mathbb{R} \cup \{-\infty\}$ is said to be weakly plurisubharmonic (equivalently, plurisubharmonic by Fornaess and Narasimhan [7]) if for any holomorphic map $\varepsilon : \Delta \to Y$ of the unit disk into $Y$, its pullback $\varepsilon^*u$ is either subharmonic or equal to $-\infty$.

Later, Moishezon [20] defined a Kahler space as follows: An arbitrary reduced holomorphic space $Y$ is said to have a Kahler metric if the exists and open covering $\{Y_\mu\}$ of $Y$ and a system of strictly plurisubharmonic functions $p_\mu$ of class $C^2$, with each $p_\mu$ defined on $Y_\mu$, such that $p_\mu - p_\nu$ is pluriharmonic function on $Y_\mu \cap Y_\nu$. Both definition coincide when $Y$ is normal.

Thus, a normal holomorphic space $Z$ has a Kahler metric (in the sense of Grauert or Moishezon) if there is a Kahler metric $h$ on $Z^{\text{reg}}$, and every singular point has a neighborhood $Z \subset Z$ and a closed embedding $Z \subset U$, where $U$ is an open subset of an affine space, such that there is a Kahler metric $h'$ on $U$ with $h'|_{Z^{\text{reg}}} = h$. Note that Grauert and Moishezon considered not necessary real-analytic Kahler metrics.

If we assume, in addition, that $X$ is a projective variety then $U_X$ will be equipped with $\pi_1(X)$-equivariant Kahler metric as in [27, Sect. 3]; see Lemma 3.1 below.

Recall the following well-known fact (see [9, Lemma 1]). Let $(Y, a)$ be a normal isolated singularity with $\dim(Y, a) \geq 2$. Let $f$ be a strictly plurisubharmonic func-
tion on \(Y \setminus a\). Then, for any sufficiently small neighborhood \(V\) of \(a\), there exists a strictly plurisubharmonic function \(f'\) on \(Y\) such that \(f = f'\) on \(Y \setminus V\).

(2.2) Diástasis. For complex manifolds \(M\) with a real-analytic Kahler metric, the diastasis was introduced by Calabi [3, Chap. 2] (see also [30, Appendix] as well as a brief review in [27]). This notion can be generalized to normal Cohen - Macaulay holomorphic spaces \(M\) with real-analytic Kahler metric. If \(\dim M = 1\) then we will consider (not necessary normal) reduced holomorphic spaces with real-analytic Kahler metric (in the sense of Moishezon).

We will mention only the fundamental property of the diastasis, namely, it is inductive on normal Cohen - Macaulay holomorphic subspaces with real-analytic Kahler metric (see a proof for manifolds in [3, Chap. 2, Prop. 6] as well as [30, Lemma 1.1]). Clearly, the other properties of the diastasis mentioned in [27, (2.2), (2.2.1), (2.3.2), (2.3.2.1), (2.3.2.2)] can be generalized to real-analytic Kahler spaces.

(2.2.1) Example. Let \(H\) denote a complex Hilbert space, i.e. \(F_C(0, N)\), where \(1 \leq N \leq \infty\). Then \(F_C(N, 0) \cong F_R(2N, 0)\). The diastasis of \(F_C(N, 0)\) with respect to the reference point \(Q\) is \(D(Q, p) = \sum_{i=1}^{N} |z_i(p)|^2\), It is equal to the diastasis of \(F_R(2N, 0)\), i.e., \(\sum_{i=1}^{N} (|x_i(p)|^2 + |y_i(p)|^2)\), where \(z_i = x_i + \sqrt{-1}y_i\).

Now, let \(P_C(H)\) be a complex projective space, i.e. \(F_C(N, 1)\). In the canonical coordinates around \(Q\), the diastasis of \(F_C(N, 1)\) is

\[
D(Q, p) = \log(1 + \sum_{i=1}^{N} |z_i(p)|^2).
\]

The diastasis of the real projective space \(F_R(2N, 1)\) equals the diastasis of \(F_C(N, 1)\). We can consider Calabi's flattening out of \(F_C(N, 1)\) as well as \(F_R(2N, 1)\).

In the present note, we often employ induction on dimension. Recall the following well-known local Bertini theorem: Let \(a \in M\) be a point of an arbitrary normal Cohen - Macaulay holomorphic space \(M\). If \(\dim M \geq 3\) then the general local (or global in the quasi-projective case) hyperplane section through \(a\) is also normal and Cohen - Macaulay.

(2.3) Harmonic mappings. For the early applications of harmonic mappings to the study of mapping between Kahler manifolds, see the surveys by Siu [23] and Yau [31] and references therein.

Also, Anderson [1] and Sullivan [24] constructed non-constant bounded harmonic functions on any simply-connected complete Riemannian manifold whose sectional curvature is bounded from above by a negative number.

Harmonic mappings between admissible Riemannian polyhedra were considered by Eells - Fuglede [6] and in papers by Serbinowski, Fuglede and others (see [8] and references therein) where the concept of energy is defined in the spirit of Korevaar - Schoen [18]. Also see the paper by Gromov - Schoen [14], and the survey articles by Toledo [26] and Delzant - Gromov [5].

Recall that an arbitrary normal holomorphic space is an admissible Riemannian polyhedron. It has been established the existence and uniqueness of a solution to the variational Dirichlet problem for harmonic mappings into the target space with nonpositive curvature [8]. The latter can be applied to rigidity questions.

(2.4) Reduction theory (also see [10, Sect. 2]). In 19th Century, Riemann and Poincaré studied compact complex manifolds of dimension one with non-Abelian
fundamental group by passing to the universal covering of the manifold. In dimension one, simply-connected manifolds are very simple. However, only in 1907, Koebe and Poincaré established, independently, that the universal coverings are discs $\Delta \subset \mathbb{C}$ provided the genus of the manifold is at least two.

If the dimension of the manifold is at least two then its universal covering can be very complicated. Furthermore, many compact Kahler manifolds can be simply connected.

So, one approaches the classification of higher-dimensional algebraic manifolds along the lines developed by the Italian school of algebraic geometry in the late 19th Century and the early 20th Century. In 1930s, Hodge employed transcendental methods in his study of algebraic varieties as well as more general Kahler manifolds bypassing the uniformization problem as well. His approach was proceeded by the Lefschetz topological methods in algebraic geometry.

In the 19th Century, the blowing downs were studied by Cremona, Del Pezzo and others, and were later applied to the classification of algebraic surfaces by Castelnuovo and Enriques. It is still one of the main tools in the classification of compact complex algebraic manifolds.

In 1956, Remmert proved that a holomorphically convex holomorphic space can be transformed into a holomorphically complete holomorphic space. Precisely, given a holomorphically convex holomorphic space $Y$, under some additional assumptions, Remmert proved that there exists a holomorphically complete $Z$ and proper holomorphic mapping $\Phi : Y \rightarrow Z$ such that if $U \subset Z$ is any open set, $V = \Phi^{-1}(U)$, and $f$ is a holomorphic function on $V$ then there exist a holomorphic function $g$ on $U$ such that $f = g \circ \Phi$.

The pair $(\Phi, Z)$ is called the holomorphic reduction of $Y$. If $Y$ is normal then $Z$ is also normal. In 1960, Cartan proved that Remmert’s additional assumptions can be removed. Moreover, for any $a \in Z$, $\Phi^{-1}(a)$ is a connected holomorphic space.

(2.5) Exceptional sets [10, Sect. 2.4, Def. 3]. Nowhere discrete compact holomorphic subset $X$ of a holomorphic space $Y$ is called exceptional if there is a holomorphic space $Z$ and a proper holomorphic mapping $\Phi : Y \rightarrow Z$ transforming $X$ onto a discrete set $D \subset Y$, mapping biholomorphically the set $Y \setminus X$ onto $Z \setminus D$, and, in addition, for every neighborhood $U = U(D) \subset Z$ and every holomorphic function $g$ on $V := \Phi^{-1}(U)$ there exists a holomorphic function $f$ on $U$ such that $g = f \circ \Phi$.

Grauert proved that $X \subset Y$ is exceptional if and only if there exists a strongly pseudoconvex neighborhood $U = U(X) \subset Y$ such that its maximal compact holomorphic subset equals $X$ [10, Sect. 2.4, Theorem 5].

According to Grauert, a vector bundle $V$ over $X$ is said to be weakly negative if its zero section $X$ has a strongly pseudoconvex neighborhood $U = U(X) \subset V$ [10, Sect. 3.1, Def. 1]. He also established that $V$ is weakly negative if and only if its zero section is exceptional (see [10, Sect. 3.1, Theorem 1]). Also, Grauert proved the following extension to holomorphic spaces of the celebrated Kodaira embedding theorem: Given a compact holomorphic space $Y$ with a weakly negative line bundle $L$ over $Y$, then $Y$ is a projective variety and $L^{-1}$ is an ample line bundle over $Y$ [10, Sect. 3.2, Theorem 2 and its proof].

(2.6) Shafarevich’s conjecture. In 1960s, H. Wu conjectured that the universal covering of a compact Kahler manifold with negative sectional curvature is a
bounded domain in an affine space. He had also established that the covering is Stein. The Wu conjecture was recently established by the author in [28] as well as in [29], provided the fundamental group is residually finite.

In 1971, Griffiths considered the following problem [11, Question 8.8]: Let $D$ be a complex manifold and $\Gamma \subset \text{Aut}(D)$ a properly discontinuous group of automorphisms such that $D/\Gamma$ is a quasi-projective algebraic variety. Then do the meromorphic functions separate points of $D$? Is $D$ meromorphically convex?

In the early 1970s, based on Remmert’s reduction, Shafarevich conjectured that the universal covering of any projective manifold is holomorphically convex [22]. According to Shafarevich, if the conjecture were established then, roughly speaking, one could reduce the study of the universal covering of a projective manifold to a compact case and a holomorphically complete case.

In the early 1990s, Campana (Kahler case) and Kollár (projective case) constructed independently a meromorphic map

$$Y \rightarrow \text{Sh}(Y) = \Gamma(Y)$$

such that for a general point $a \in Y$, the fiber $Y_a$ passing through $a$ is the largest among the connected holomorphic subsets $\mathcal{A}$ of $Y$ containing the point $a$ such that the natural map $i_* : \pi_1(\mathcal{A}^{\text{nor}}) \rightarrow \pi_1(Y)$ has a finite image, where $i_*$ is induced by the inclusion of $\mathcal{A}$ in $Y$ composed with the normalization map $\mathcal{A}^{\text{nor}} \rightarrow \mathcal{A}$.

This map is called the Shafarevich map by Kollár and the $\Gamma$-reduction of $Y$ by Campana. They also established the relationship of this map and the corresponding map for the universal covering $U_Y$ with (unsolved at the time) Shafarevich’s conjecture. The Shafarevich conjecture, established by the author a few years ago provided the fundamental group is residually finite, implies the meromorphic map is regular and $a \in Y$ can be any point.

3. Projective case

We will establish the following version of the Shafarevich conjecture for singular projective varieties. We need the following lemma.

**Lemma 3.1 (Extension of metric).** Let $Z$ be an arbitrary normal Cohen-Macaulay holomorphic space of dimension $n \geq 1$. Given an arbitrary real-analytic Kahler metric on $Z^{\text{reg}}$, we get a unique real-analytic Kahler metric on $Z$ whose restriction on $Z^{\text{reg}}$ is the given one.

**Proof of Lemma 3.1.** We will prove the lemma by induction on dimension of $Z$. If $Z = Z^{\text{reg}}$ there is nothing to prove. First, we consider the case $\dim Z = 2$. Then $Z$ has only isolated singularities. As in Section 2.1, we get a unique Kahler metric on $Z$.

This Kahler metric will be real analytic. Indeed, in a small neighborhood $\mathcal{V}$ of a point $q \in Z$, the potentials of this metric produce a unique holomorphic function in a neighborhood of the diagonal of $\mathcal{V}^{\text{reg}} \times \bar{\mathcal{V}}^{\text{reg}}$ as follows. This holomorphic function arises from the diastastic potential of $\mathcal{V}^{\text{reg}}$ which is a real-analytic function over the whole $\mathcal{V}^{\text{reg}}$. Next, the holomorphic function can be extended to a holomorphic function on the diagonal of $\mathcal{V} \times \bar{\mathcal{V}}$. The latter holomorphic function produces a real-analytic function on $\mathcal{V}$ by realification (i.e., the inverse of complexification) which is a potential of our Kahler metric at the point $q$. 

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If dim $Z \geq 3$ we apply the local Bertini theorem and induction on dimension. Let $p \in Z$ be a singular point. By induction hypothesis, we get a real-analytic potential (diastasic potential) in a small neighborhood inside a general local hyperplane section $v_p$ through $p \in Z$. We obtain a holomorphic function in a neighborhood of $p \times \bar{p}$ (in $v_p \times \bar{v}_p$) employing complexifications.

By Hartogs’ theorem (separate analyticity $\Rightarrow$ joint analyticity), we obtain holomorphic functions in the points of diagonal of $Z \times \bar{Z}$. Finally, we produce real-analytic potentials in every point of $Z$ by realifications because the obtained real-analytic functions will be strictly plurisubharmonic by Fornaess - Narasimhan [7].

This proves the lemma.

**Proposition 1.** Let $\phi : Y \hookrightarrow \mathbb{P}^r$ be a connected, normal, Cohen - Macaulay projective variety of dimension $n \geq 1$ with large and residually finite $\pi_1(Y)$. If the genus of a general curvilinear section of $Y$ is at least two then $U_Y$ is a Stein space.

**Proof of Proposition 1.** We assume the map $\phi$ is given by a very ample bundle $\mathcal{L}$. As in [27, Sect. 3], we obtain the real-analytic $\pi_1(Y)$-invariant Kahler metric $\Lambda_{\mathcal{L}}$ on $U_{Y_{\text{reg}}}$. By Lemma 3.1, we extend this metric to $U_Y$.

As in [27, Appendix, Lemma A], we establish the prolongation of the diastasic potential of our metric at a point of $U_Y$ over the whole $U_Y$.

Finally, as in [27, Appendix, (A.2)], we conclude the proof of holomorphic completeness by reduction to the one-dimensional case because one-dimensional non-compact holomorphic spaces are Stein.

4. Kahler case

We will establish the following Kahler version of Proposition 1.

**Proposition 1’.** Let $Y$ be a connected, compact, normal, Cohen - Macaulay, Kahler space of dimension $n \geq 1$ with large and residually finite $\pi_1(Y)$. If $\pi_1(Y)$ is nonamenable and $Y$ has at most 1-rational singularities then $U_Y$ is a Stein space and $Y$ is projective.

**Proof of Proposition 1’.** As in [29], we will establish that that $Y$ is projective by showing that $Y$ is a Moishezon space. Let $\text{Har}(U_Y) (\text{Har}^b(U_Y))$ be the space of harmonic functions (bounded harmonic functions) on $U_Y$. The space $\text{Har}^b(U_Y)$ contains non-constant harmonic functions by Lyons and Sullivan [19]. In fact, the space $\text{Har}^b(U_Y)$ is infinite dimensional by Toledo [25]. Their proof is stated for Riemannian manifolds but it works in our case. One can consider the transition density of Brownian motion on $U_{Y_{\text{reg}}}$ or $U_Y$.

We will integrate the bounded harmonic functions with respect to the measure

$$dv := p_{U_Y}(s, x, Q) d\mu = p_{U_Y}(x) d\mu,$$

where $Q \in U_Y$ is a fixed point (a so-called center of $U_Y$), $d\mu$ is the Riemannian measure and $p_{U_Y}(x) := p_{U_Y}(s, x, Q)$ is the heat kernel. We obtain a pre-Hilbert space of bounded harmonic functions (compare [27, Sect. 2.4 and Sect. 4]); note that all bounded harmonic functions are square integrable, i.e. in $L_2(dv)$.

We observe that the latter pre-Hilbert space has a completion in the Hilbert space $H$ of all harmonic $L_2(dv)$ functions:

$$H := \left\{ h \in \text{Har}(U_Y) \left| \| h \|^2_H := \int_{U_Y} |h(x)|^2 dv = \int_{U_Y} |h(x)|^2 p_{U_Y}(x) d\mu < \infty \right. \right\}.$$
Let $H^b \subseteq H$ be the Hilbert subspace generated by $Har^b(U_Y)$. These Hilbert spaces are separable infinite dimensional and have reproducing kernels.

Let $\{\phi_j\} \subset Har^b(U_Y)$ be an orthonormal basis of $H^b$. We obtain a continuous, even smooth, finite $\Gamma$-energy $\Gamma$-equivariant mapping into $(H^b)^*$:

$$ (1) \quad g : U_Y \longrightarrow (H^b)^* \quad (\subset P_R((H^b)^*)), \quad u \mapsto (\phi_0(u), \phi_1(u), \ldots). $$

The group $\Gamma$ acts isometrically on $(H^b)^*$ via $\psi \mapsto (\psi \circ \gamma)$.

We assume $g$ is harmonic; otherwise, we replace $g$ by a harmonic mapping homotopic to $g$. Clearly, the proposition is valid when dim $Y = 1$ even if $Y$ is not necessary normal. So we assume dim $Y \geq 2$ in the following lemmas.

**Lemma 4.1.** With assumptions of Proposition 1', $g$ will produce a pluriharmonic mapping $g^{fl}$. There exists a natural holomorphic mapping $g^h : U_Y \longrightarrow F_C(\infty, 0)$.

**Proof of Lemma 4.1.** We define a harmonic $\Gamma$-equivariant mapping as follows

$$ g^{fl} := S_{g(Q)} \circ g : U_Y \longrightarrow (H^b)^*. $$

We have applied the mapping $g$ followed by the Calabi flattening out $S_{g(Q)}$ of the real projective space $F_R(2\infty, 1)$ from $g(Q)$ into the Hilbert space [3, Chap. 4, p. 17].

By [3, Chap. 4, Cor. 1, p. 20], the whole $F_R(2\infty, 1)$, except the antipolar hyperplane $A$ of $g(Q)$, can be flatten out into $F_R(2\infty, 0)$. The image of $g$ does not intersect the antipolar hyperplane $A$ of $g(Q)$. Thus we have introduced a flat metric in a large (i.e. outside $A$) neighborhood of $g(Q)$ in $P_R((H^b)^*)$.

Since the mapping $g^{fl}$ has finite $\Gamma$-energy, it is pluriharmonic; this is a special case of the well-known theorem of Siu. Since $U_Y$ is simply connected, we obtain the natural holomorphic mapping $g^h : U_Y \longrightarrow F_C(\infty, 0)$.

**Lemma 4.2.** Construction of a complex line bundle $\mathcal{L}_Y$ on $Y$ and its pullback on $U_Y$, denoted by $\mathcal{L}$.

**Proof of Lemma 4.2.** We take a point $u \in U_Y$. Let $v := g^h(u) \in F_C(\infty, 1)$, where $F_C(\infty, 1)$ is the complex projective space (see (1)). We consider the linear system of hyperplanes in $F_C(\infty, 1)$ through $v$ and its proper transform on $U_Y$. We consider only the moving part. The projection on $Y$ of the latter linear system on $U_Y$ will produce a linear system on $Y$.

A connected component of a general member of the latter linear system on $Y$ will be an irreducible divisor $D$ on $Y$ by Bertini’s theorem. The corresponding line bundle will be $\mathcal{L}_Y := \mathcal{O}_Y(D)$ on $Y$.

**Lemma 4.3.** Conclusion of the proof of Proposition 1' by induction on dim $Y$.

**Proof of Lemma 4.3.** By the Campana-Deligne theorem [17, Theorem 2.14], $\pi_1(D)$ will be nonamenable. We proceed by induction on dim $Y$. As above, let dim $Y \geq 2$. Let $q = q(n)$ be an appropriate integer.

We get a global holomorphic function-section $f$ of $\mathcal{L}_q$ corresponding to a bounded pluriharmonic function (see Lemma 4.1 and [27, Sect. 4]).

We will define a $\Gamma$-invariant Hermitian quasi-metric on sections of $\mathcal{L}_q$ below. Furthermore, $f$ is $\ell^2$ on orbits of $\Gamma$, and it is not identically zero on any orbit because, otherwise, we could have replaced $U_Y$ by $U_Y \setminus B$, where the closed holomorphic subset $B \subset U_Y$ is the union of those orbits on which $f$ had vanished [17, Theorem 13.2, Proof of Theorem 13.9].
One can show that $f$ satisfies the above conditions by taking linear systems of curvilinear sections of $U_Y$ through $u \in U_Y$ and their projections on $Y$ (see the proof of Lemma 4.2 above), since the statements are trivial in dimension one. The required Hermitian quasi-metric on $L^q_{\text{reg}} := L^q|_{U_Y\text{reg}}$ is also defined by induction on dimension with the help of the Poincaré residue map [13, pp. 147-148].

The condition $\ell^2$ on orbits of $\Gamma$ is a local property on $Y_{\text{reg}}$. We get only a Hermitian quasi-metric on $L^q_{\text{reg}}$ (instead of a Hermitian metric). Precisely, we get Hermitian metrics over small neighborhoods of points of $Y_{\text{reg}}$, and on the intersections of neighborhoods, they will differ by constant multiples (see [17, Chap. 5.13]).

For $\forall k > N \gg 0$, the Poincaré series are holomorphic sections over $U_{Y\text{reg}}$:

$$P(f^k)(u) := \sum_{\gamma \in \Gamma} \gamma^* f^k(\gamma u) \quad (u \in U_{Y\text{reg}})$$

and they do not vanish for infinitely many $k$ (see [17, Sect. 13.1, Theorem 13.2]).

Finally, we can apply Gromov’s theorem, precisely, its generalization by Kollár (see [12, Corollary 3.2.B, Remark 3.2.B′] and [17, Theorem 13.8, Corollary 13.8.2, Theorem 13.9, Theorem 13.10]). So, $Y$ is a Moishezon space hence a projective variety.

The Lemma 4.3 and Proposition 1′ are established.

5. Conclusion of the proof of theorem

**Proposition 2.** With assumptions of the theorem, $\mathcal{K}_X^m$ is ample for an integer $m > 0$, and $U_X$ has a generalized Bergman metric.

**Proof of Proposition 2.** For simplicity, we assume that $\mathcal{K}_X$ is a line bundle.

It follows from the Kodaira embedding theorem that the sheaf $\mathcal{K}_X$ is ample, provided $X$ is nonsingular (see [27, Sect. 4]).

In the general case, the ampleness of $\mathcal{K}_X$ will follow from Grauert’s extension of the Kodaira theorem (see Section 2.5 and [27, Sections 4, 5] with $\mathcal{L} := \mathcal{K}_X^{-1}$).

Thus, we have to show $\mathcal{K}_X^{-1}$ is weakly negative, i.e., its zero section $\mathfrak{X}$ is exceptional. Namely, we have to prove the existence of a strongly pseudoconvex neighborhood $\mathcal{U} = \mathcal{U}(\mathfrak{X}) \in \mathcal{K}_X^{-1}$ such that its maximal compact holomorphic subset equals $\mathfrak{X}$.

We apply Grauert’s generalization of Kodaira’s result (see [10, Sect. 3, Proof of Theorem 1 (negativity $\implies$ exceptional)]). We will be able to choose a covering $\{\mathcal{V}_\tau\}$ of $X$ and strictly plurisubharmonic functions $-\log p_\tau$ on each $\mathcal{V}_\tau$ so that the following Hermitian form is strictly negative on $X$:

$$\sum \frac{\partial^2 \log p_\tau(z, \bar{z})}{\partial z_\alpha \partial \bar{z}_\beta} dz_\alpha dz_\beta.$$  

The latter form is negative of the Ricci form of the volume form on $X$ obtained as follows. First, we consider the resolution of singularities of $X$: $X' \to X$ (and the corresponding pullback $U'_X \to U_X$). As in Section 4, employing bounded harmonic functions, we consider the $\Gamma$-invariant volume form on $U'_X$ (compare [27, Sect. 4]).

We get a $\Gamma$-invariant Hermitian metric on $\mathcal{K}_X^{-1}|_{U'_X}$. We, then, get Hermitian metrics on $\mathcal{K}_X^{-1}$ and $\mathcal{K}_X^{-1}$ and obtain the desired $p_\tau$’s. Hence $\mathfrak{X}$ is an exceptional section.
Now, the existence of a generalized Bergman metric on $U_X$ is established as in [27, Sections 5.1-5.3] because, for a fixed large integer $q$, $K^q$ is very ample on every finite Galois covering of $X$. If $X$ is nonsingular the very ampleness follows from [4], and, in our case, we resolve the singularities of $X$ and apply the latter assertion.

**Proposition 3.** $U_X$ is rigid.

*Proof of Proposition 3.* Let $X$ and $Z$ satisfy the assumptions of the theorem and $X \approx Z$. We get a $\Gamma$-invariant topological equivalence $U_X \approx U_Z$. Both $U_X$ and $U_Z$ are equipped with the $\Gamma$-invariant generalized Bergman metrics and can be flattened out into the compact spaces $F_X^C(r_X, 0)$ and $F_Z^C(r_Z, 0)$, respectively. We get the embeddings into the corresponding Fubini spaces

$$g_X : U_X \hookrightarrow F_X^C(2r_X, 0) \quad \text{and} \quad g_Z : U_Z \hookrightarrow F_Z^C(2r_Z, 0).$$

We consider the towers of finite Galois coverings:

$$X \subset \cdots \subset X_i \subset \cdots \subset U_X \quad \text{and} \quad Z \subset \cdots \subset Z_i \subset \cdots \subset U_Z;$$

$$\bigcap_i \text{Gal}(U_X/X_i) = \bigcap_i \text{Gal}(U_Z/Z_i) = \{1\}.$$ 

We have assumed the topological equivalence $X_i \approx Z_i$. For the corresponding fundamental domains $F_{X_i} \subset U_X$ and $F_{Z_i} \subset U_Z$, we have the topological equivalences $F_{X_i} \approx F_{Z_i}$ for all $i$.

Now, we can and will assume the embeddings

$$F_{Z_i} \hookrightarrow U_Z \quad (\forall i),$$

where $U_Z$ and the interior of each $F_{Z_i}$ are considered with the above generalized Bergman metric, and $F_{Z_i}$ is the Dirichlet fundamental domain [27, (2.3.3.2)].

From existence and uniqueness of a solution to the variational Dirichlet problem for harmonic mappings, we get the mappings

$$F_{X_i} \rightarrow F_{Z_i} \quad (\forall i)$$

that are harmonic in the interiors of $F_{X_i}$ and coincide with the given continuous mappings on the boundaries. These mappings will be pluriharmonic in the interiors.

Thus we get a pluriharmonic mapping

$$U_X \rightarrow U_Z \subset F_Z^C(2r_Z, 0).$$

Similarly, we get a pluriharmonic mapping

$$U_Z \rightarrow U_X \subset F_X^C(2r_X, 0).$$

Since $U_X$ and $U_Z$ are simply connected, the above pluriharmonic mappings produce holomorphic mappings, and $U_X$ is biholomorphic to $U_Z$ by uniqueness of a solution to the variational Dirichlet problem:

$$F_X^C(r_X, 0) \supset U_X = U_Z \subset F_Z^C(r_Z, 0).$$

This proves Proposition 3 and the Theorem.
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