ON THE EXISTENCE OF MAXIMAL ORDERS

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Abstract. We generalize the existence of maximal orders in a semi-simple algebra for general ground rings. We also improve several statements in Chapter 5 and 6 of Reiner’s book [10] concerning separable algebras by removing the separability condition, provided the ground ring is only assumed to be Japanese, a very mild condition. Finally, we show the existence of maximal orders as endomorphism rings of abelian varieties in each isogeny class.

1. Introduction

Maximal orders are basic objects in the integral theory of semi-simple algebras. As a generalization of the rings of integers in number fields, they are also main interests in number theory. A classical result states that the existence of maximal orders, not just for the ring of integers in a number field, may hold in a quite general setting, which we describe now (Theorem 1.1).

Let $R$ be a Noetherian integral domain with quotient field $K$. We only consider $K$-algebras which are finite-dimensional. A (finite-dimensional) $K$-algebra $A$ is said to be separable over $K$ if it is semi-simple and the center $Z(A)$ of $A$ is a separable (commutative) semi-simple $K$-algebra, that is, $Z(A)$ is a finite product of finite separable field extensions of $K$. Clearly any central simple $K$-algebra is separable. For a $K$-algebra $A$, an $R$-order $\Lambda$ in $A$ is a finite $R$-subring of $A$ which spans $A$ over $K$. An $R$-order $\Lambda$ of $A$ is said to be maximal if there is no $R$-order $\Lambda'$ of $A$ which strictly contains $\Lambda$. The following is a classical result about the existence of maximal orders; see [10, Corollary 10.4 and Theorem 10.5 (iv), p. 127–8].

Theorem 1.1. Let $R$ and $K$ be as above, and $A$ a semi-simple algebra over $K$. Then there exists a maximal $R$-order of $A$ provided one of the following conditions hold:

1. $R$ is normal and $A$ is separable over $K$.
2. $R$ is a complete discrete valuation ring.

When char $K = 0$ or even $K$ is a number field, Theorem 1.1 provides most of the situations we may encounter. However, when $K$ is a global function field, the assumption of the separability of $A$ seems to be superfluous. We would like to find a necessary and sufficient condition for the ground ring $R$ so that maximal orders in any semi-simple $K$ algebra exists. In this Note we prove the following result, which removes the separability assumption in Theorem 1.1 (1) for rather general ground rings in positive characteristics.

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Theorem 1.2. Let $R$ be a Noetherian integral domain and $K$ be its quotient field.

(1) Assume that $R$ is a Japanese ring. Then any $R$-order of a semi-simple $K$-algebra $A$ is contained in a maximal $R$-order. In particular, every semi-simple $K$-algebra contains a maximal $R$-order.

(2) Conversely, if every semi-simple $K$-algebra $A$ contains a maximal $R$-order, then $R$ is a Japanese ring.

We shall recall the definition of Japanese and Nagata rings as well as some of their properties and the relationship to (quasi-)excellent rings. Nagata domains are special cases of Japanese rings. Examples of Nagata rings include commutative rings of finite type over $\mathbb{Z}$ and their localizations, commutative rings of finite type over any field $k$ and their localizations, and Noetherian complete semi-local rings. Noetherian normal domains with quotient field of characteristic zero are Japanese rings.

In the second part of this Note we give a description of maximal orders in a semi-simple $K$-algebra, where the ground ring $R$ is either a Noetherian Japanese ring or an excellent ring. We reduce the description to the case when $R$ is a Noetherian normal domain, whose description becomes well-known. For the convenience of the reader, we also include the expository account of this important theory. Our reference is the well-written book by I. Reiner [10].

Note that the results of this Note generalize all statements concerning separable $K$-algebras $A$ in Chapters 5 and 6 of Reiner’s book [10]. We remove the separability condition for $A$ provided the Dedekind domain $R$ is assumed to be either excellent or Japanese; see the reduction step in Section 3 or Subsection 4.1. For number theorists, this assumption is harmless.

Our motivation of entering the integral theory of semi-simple algebras is due to the basic fact that the endomorphism ring of an abelian variety is an order of a semi-simple $\mathbb{Q}$-algebra. An abelian variety whose endomorphism ring is maximal should be distinguished from others in its isogeny class. The last part of this Note shows that in any isogeny class of abelian varieties there is an abelian variety whose endomorphism ring is maximal, a result about the existence of maximal orders. More precisely, we show the following result.

Theorem 1.3. Let $A_0$ be an abelian variety over an arbitrary field $k$. Let $O' \subset \text{End}_k^0(A_0) = \text{End}_k(A_0) \otimes_{\mathbb{Z}} \mathbb{Q}$ be a maximal order containing $\text{End}_k(A_0)$. Then there exist an abelian variety $A'$ over $k$ and an isogeny $\varphi : A_0 \to A'$ over $k$ such that with the identification $\text{End}_k^0(A_0) = \text{End}_k^0(A')$ by $\varphi$ one has $\text{End}_k(A') = O'$. Moreover, the isogeny $\varphi$ can be chosen to be minimal with respect to $O'$ in the following sense: if $(A_1, \varphi_1)$ is another pair such that $\text{End}_k(A_1) = O'$, then there is an (necessarily unique) isogeny $\alpha : A' \to A_1$ such that $\varphi_1 = \alpha \circ \varphi$.

A local version of Theorem 1.3 (where $O'$ is a maximal order of $\text{End}(A_0) \otimes \mathbb{Q}_p$ containing $\text{End}(A_0) \otimes \mathbb{Z}_p$ when $k$ is an algebraically closed field of characteristic $p > 0$ is used in the proof of the reduction step [12, Lemma 2.4].

This Note is organized as follows. In Section 2 we review the properties of Japanese, Nagata, and excellent rings, and their relationship as well. We also discuss the relationship between the properties (regularity, normality and some others) of a local ring and its completion. In Section 3 we show the existence of
maximal orders in a semi-simple algebra with Japanese ground rings. In Section 3 we attempted to describe maximal orders in these semi-simple algebras $A$ and show that the description can be reduced to the case where the ground ring is a complete discrete valuation ring and $A$ may be assumed to be central simple. We collect the description of maximal orders in a central simple algebra over a discrete valuation ring and a Dedekind domain, following Reiner [10]. In the last section we give the proof of Theorem 1.3.

2. Nagata and excellent rings

In this section we recall the definition of Nagata rings and excellent rings, as well as their properties. Our references are Matsumura [6], and EGA IV [4, 5].

Notations here are independent of Section 1, as we prefer to follow closely Matsumura [6] and EGA IV. All rings and algebras in this section are commutative with identity.

2.1. Nagata rings.

Definition 2.1. Let $A$ be an integral domain with quotient field $K$.

(1) We say that $A$ is N-1 if the integral closure $A'$ of $A$ in its quotient field $K$ is a finite $A$-module.

(2) We say that $A$ is N-2 if for any finite extension field $L$ over $K$, the integral closure $A_L$ of $A$ in $L$ is a finite $A$-module.

If $A$ is N-1 (resp. N-2), then so is any localization of $A$. If $A$ is a Noetherian domain of characteristic zero, then $A$ is N-2 if and only if $A$ is N-1. This follows immediately from the basic theorem that if $L$ is a finite separable field extension of $K$ and $A$ is a Noetherian normal domain, then the integral closure $A_L$ of $A$ in $L$ is finite over $A$ (cf. [6, Proposition 31.B, p. 232]).

Definition 2.2. A ring $A$ is said to be Nagata if

(1) $A$ is Noetherian, and

(2) $A/p$ is N-2 for any prime ideal $p$ of $A$.

If $A$ is Nagata, then any localization of $A$ and any finite $A$-algebra are all Nagata. Nagata uses the term “pseudo-geometric rings” for such rings as coordinate rings of varieties over any field all share this property. Nagata rings are the same as what are called Noetherian universal Japanese rings in EGA IV [4, 23.1.1, p. 213], which we recall now.

Definition 2.3.

(1) An integral domain $A$ is said to be Japanese if it is N-2.

(2) An ring $A$ is said to be universal Japanese if any finitely generated integral domain over $A$ is Japanese.

From the definition, a universal Japanese ring is not required to be an integral domain nor to be Noetherian. A universal Japanese domain is Japanese. It follows from the definition that any Noetherian universal Japanese ring is Nagata. Conversely, the following theorem [6, Theorem 72, p. 240], due to Nagata, shows that any Nagata ring is also a Noetherian universal Japanese ring.

Theorem 2.4. If $A$ is a Nagata ring, then so is any finitely generated $A$-algebra.
The proof of this theorem is quite involved and the reader is referred to Matsumura [6 § 31]. The following provides some more examples (see [6] Corollaries 1 and 2, p. 234).

**Proposition 2.5.**

1. If \( A \) is a Noetherian normal domain which is N-2, then the formal power series ring \( A[[X_1, \ldots, X_n]] \) is N-2 also.
2. Any Noetherian complete local ring \( A \) is a Nagata ring.

For any scheme \( X \), let Nor(\( X \)) denote the subset of \( X \) that consists of normal points.

**Lemma 2.6.** Let \( A \) be a Noetherian domain and \( X := \text{Spec} \ A \).

1. If there is a non-zero element \( f \in A \) such that \( A_f := A[1/f] \) is normal, then Nor(\( X \)) is open in \( X \).
2. If \( A \) is N-1, then Nor(\( X \)) is open in \( X \).

**Proof.** (1) This is Lemma 3 of Matsumura [6 § 31.G, p. 238] and its proof is sketched there. We provide more details for the convenience of the reader. Using a criterion for normality [6, Theorem 39, p. 125], for \( q \in \text{Spec} \ A \), the integral domain \( A_q \) is normal if and only if it satisfies the conditions (\( R_1 \)) and (\( S_2 \)), that is, \( A_p \) is regular for all prime ideals \( p \subseteq q \) with \( \text{ht}(p) = 1 \), and \( \text{Ass}(A_q/f') \), the set of associated prime ideals of \( A_q/f' \) for all \( 0 \neq f' \in q \), has no embedded prime ideals (cf. [6] p. 125). Let

\[ E := \{ p \in \text{Ass}(A/f) \mid \text{ht}(p) = 1 \text{ and } A_p \text{ is not regular, or } \text{ht}(p) > 1 \} \]

Clearly \( E \) is a finite subset. We claim that

\[ \text{Nor}(X) = X - \bigcup_{p \in E} V(p). \]

Let \( q \) be a prime ideal not contained in \( \bigcup_{p \in E} V(p) \). We shall show that \( q \) is a normal point. If \( q \in \text{Spec} \ A[1/f] \), then \( q \) is a normal point by our assumption. Suppose that \( f \in q \). If \( p \in \text{Ass}(A_q/f) \), then \( \text{ht}(p) = 1 \). This means \( \text{Ass}(A_q/f) \) has no embedded prime ideals and hence \( A_q \) satisfies (\( S_2 \)). On the other hand, let \( p \subseteq q \) be prime ideals with \( \text{ht}(p) = 1 \). If \( f \notin p \), then \( p \) is a normal point and \( (A_q)_p = A_p \) is regular. If \( f \in p \), then \( (A_q)_p = A_p \) is regular, by the definition of \( E \). This shows that \( q \in \text{Nor}(X) \) and the proof of (1) is completed.

(2) Let \( A' \) be the normalization of \( A \), and let \( X' := \text{Spec} \ A' \). Since \( A \) is N-1, the natural morphism \( X' \to X \) is a finite dominant birational morphism. Then there is a non-zero element \( f \in A \) such that the restriction to the open subset

\[ X'_f := \text{Spec} \ A'[1/f] \to X_f := \text{Spec} \ A_f \]

is an isomorphism. In particular, \( A_f \) is normal. It follows from (1) that Nor(\( X \)) is open in \( X \).

We provide another simpler proof of (2), which is not based on (1). Put \( M := A'/A; \) this is a finite \( A \)-module as \( A \) is N-1. For each \( p \in X \), we have

\[ M_p = (A')_p/A_p = (A_p)'/A_p, \]

as the operations localization and normalization commute. It follows that

\[ \text{Nor}(X) = \{ p \in X \mid M_p = 0 \}. \]
Since \( A \) is Noetherian and \( M \) is finite over \( A \), \( \text{Nor}(X) \) is open in \( X \).

Let \( A \) be a Noetherian semi-local ring and \( A^\ast \) its completion. If \( A^\ast \) is reduced, then \( A \) is said to be \emph{analytically reduced}.

**Theorem 2.7.** \([6, \text{Theorem 70, p. 236}]\) Any Nagata semi-local domain is analytically reduced.

This is useful for checking non-Nagata rings; see Example 2.22.

2.2. G-rings and closedness of singular loci. Recall that a Noetherian local ring \((A, \mathfrak{m}, k)\) is said to be a \emph{regular local ring} if \( \dim A = \dim_k \mathfrak{m}/\mathfrak{m}^2 \) \([6, \text{p. 78}]\). A Noetherian ring \( A \) is said to be \emph{regular} if all local rings \( A_p \) are regular local for \( p \in \text{Spec} \ A \). One can show that if the local ring \( A_m \) is regular for all maximal ideals \( m \) of \( A \), then \( A \) is regular \([6, \text{§18.G Corollary, p. 139}]\). Any regular local ring is an integral domain.

**Definition 2.8.** \([6, \text{§33, p. 249}]\).

1. Let \( A \) be a Noetherian ring containing a field \( k \). We say that \( A \) is \emph{geometrically regular over} \( k \) if for any finite field extension \( k' \) over \( k \), the ring \( A \otimes_k k' \) is regular. This is equivalent to saying that the local ring \( A_m \) has the same property for all maximal ideals \( m \) of \( A \).

2. Let \( \phi : A \to B \) be a homomorphism (not necessarily of finite type) of Noetherian rings. We say that \( \phi \) is \emph{regular} if it is flat and for each \( p \in \text{Spec} \ A \), the fiber ring \( B \otimes_A k(p) \) is geometrically regular over the residue field \( k(p) \).

3. A Noetherian ring \( A \) is said to be a \emph{G-ring} if for each \( p \in \text{Spec} \ A \), the natural map \( \phi_p : A_p \to (A_p)^\ast \) is regular, where \( (A_p)^\ast \) denotes the completion of the local ring \( A_p \).

Note that the natural map \( \phi_p : A_p \to (A_p)^\ast \) is faithfully flat. The fibers of the natural morphism \( \text{Spec} \ (A_p)^\ast \to \text{Spec} \ A_p \) are called formal fibers. To say a Noetherian ring \( A \) is a G-ring then is equivalent to saying that all formal fibers of the canonical map \( \phi_p \) for each prime ideal \( p \) of \( A \) are geometrically regular. It is clear that, if \( A \) is a G-ring, then any localization \( S^{-1}A \) of \( A \) and any homomorphism image \( A/I \) of \( A \) are G-rings.

**Theorem 2.9.** \([6, \text{Theorem 93, p. 279}]\). Let \((A, \mathfrak{m})\) be a Noetherian local ring containing a field \( k \). Then \( A \) is geometrically regular over \( k \) if and only if \( A \) is formally smooth over \( k \) in the \( \mathfrak{m} \)-adic topology.

**Lemma 2.10.** \([6, \text{Lemma 2, p. 251}]\). Let \( \phi : A \to B \) be a faithfully flat, regular homomorphism. Then

1. \( A \) is regular (resp. normal, resp. Cohen-Macaulay, resp. reduced) if and only if \( B \) has the same property;
2. If \( B \) is a G-ring, then so is \( A \).

For a Noetherian scheme \( X \), let \( \text{Reg}(X) \) denote the subset of \( X \) that consists of regular points.

**Definition 2.11.** Let \( A \) be a Noetherian ring.
(1) We say that \( A \) is \( J-0 \) if \( \text{Reg} (\text{Spec} A) \) contains a non-empty open set of \( \text{Spec} A \).

(2) We say that \( A \) is \( J-1 \) if \( \text{Reg} (\text{Spec} A) \) is open in \( \text{Spec} A \).

If \( A \) is a integral domain, then \( \text{Reg} (\text{Spec} A) \) is non-empty and hence the condition \( J-1 \) implies \( J-0 \). Indeed, the the generic point of \( \text{Spec} A \) is a regular point as the localization of \( A \) is its quotient field, which is a regular local ring.

**Theorem 2.12.** \([6, \text{Theorem 73, p. 246}]\). For a Noetherian ring \( A \), the following conditions are equivalent

1. any finitely generated \( A \)-algebra \( B \) is \( J-1 \);
2. any finite \( A \)-algebra \( B \) is \( J-1 \);
3. for any \( p \in \text{Spec} A \), and for any finite radical extension \( K' \) of \( k(p) \), there exists a finite \( A \)-algebra \( A' \) satisfying \( A/p \subseteq A' \subseteq K' \) which is \( J-0 \) and whose quotient field is \( K' \).

**Definition 2.13.** A Noetherian ring \( A \) is \( J-2 \) if it satisfies one of the equivalent conditions in Theorem 2.12.

**Remark 2.14.** The condition (3) of Theorem 2.12 is satisfied if \( A \) is a Nagata ring of dimension one. Indeed, \( A/p \) is either a field or a Nagata domain of dimension one. In the first case, (3) is trivial. In the second case, the integral closure \( A' \) of \( A \) in \( K' \) is finite over \( A \) and is a regular ring. Therefore, any Nagata ring of dimension one is a \( J-2 \). On the other hand, we have Theorem 2.16 (2).

We gather some properties of G-rings.

**Theorem 2.15.**

1. Any complete Noetherian local ring is a G-ring.
2. If for any maximal ideal \( \mathfrak{m} \) of a Noetherian ring \( A \), the natural map \( A_\mathfrak{m} \to (A_\mathfrak{m})^* \) is regular, then \( A \) is a G-ring
3. Let \( A \) and \( B \) be Noetherian rings, and let \( \phi : A \to B \) be a faithfully flat and regular homomorphism. If \( B \) is \( J-1 \), then so is \( A \).
4. A semi-local G-ring is \( J-1 \).

**Proof.** (1) See \([6, \text{Theorem 68, p. 225 and p. 250}]\). (2) See \([6, \text{Theorem 75, p. 251}]\). (3) and (4) See \([6, \text{Theorem 76, p. 252}]\).

**Theorem 2.16.**

1. Let \( A \) be a G-ring and \( B \) a finitely generated \( A \)-algebra. Then \( B \) is a G-ring.
2. Let \( A \) be a G-ring which is \( J-2 \). Then \( A \) is a Nagata ring.

**Proof.** (1) See \([6, \text{Theorem 77, p. 254}]\). (2) See \([6, \text{Theorem 78, p. 257}]\).

**Theorem 2.17.** (Analytic normality of normal G-rings). Let \( A \) be a G-ring and \( I \) an ideal of \( A \). Let \( B \) be the \( I \)-adic completion of \( A \). Then the canonical map \( A \to B \) is regular. Consequently, if \( B \) is normal (resp. regular, resp. Cohen-Macaulay, resp. reduced) so is \( A \).

**Proof.** See \([6, \text{Theorem 79, p. 258}]\).
2.3. Excellent rings.

**Definition 2.18.** [6, § 34, p. 259]. Let $A$ be a Noetherian ring.

1. We say that $A$ is *quasi-excellent* if the following conditions are satisfied:
   
   (i) $A$ is a G-ring;
   
   (ii) $A$ is J-2.

2. We say that $A$ is *excellent* if it satisfies (i), (ii) and the following condition
   
   (iii) $A$ is universally catenary.

We recall the following [6, p. 84]:

**Definition 2.19.**

1. A ring $A$ is said to be *catenary* if for any two prime ideals $p \subseteq q$, the relative height $ht(q/p)$ is finite and is equal to the length of any maximal chain of prime ideals between them.

2. A Noetherian ring $A$ is said to be *universally catenary* if any finitely generated $A$-algebra is catenary.

**Remark 2.20.**

1. Each of the conditions (i), (ii), and (iii) is stable under the localization and passage to a finitely generated algebra (Theorems 2.12 (1) and 2.16 (1)).

2. Note that (i), (ii), (iii) are conditions on $A/p$, $p \in \text{Spec } A$. Thus a Noetherian ring $A$ is (quasi-)excellent if and only if $A_{\text{red}}$ is so.

3. The conditions (i) and (iii) are of local nature (in the sense that if they hold for $A_p$ for all $p \in \text{Spec } A$, then they hold for $A$), while (ii) is not.

4. Theorem 2.16 (2) states that any quasi-excellent ring is a Nagata ring.

5. It follows from Theorems 2.12 and 2.15 (4) that any Noetherian local G-ring is quasi-excellent.

**Example 2.21.** [6, § 34.B, p. 260].

1. Any complete Noetherian semi-local ring is excellent.

2. Convergent power series rings over $\mathbb{R}$ or $\mathbb{C}$ are excellent.

3. Any Dedekind domain $A$ of characteristic zero is excellent.

**Example 2.22.** (cf. [6] § 34.B, p. 260).

There exists a regular local domain of dimension one, that is, a discrete valuation ring, in characteristic $p > 0$ which is not excellent. Take a field $k$ of characteristic $p > 0$ with $[k : kp] = \infty$. Put $R := k[[t]]$ and let $A$ be the subring of $R$ consisting of power series

$$
\sum_{n=0}^{\infty} a_n t^n, \quad \text{with} \quad [kp(a_0, a_1, \ldots) : kp] < \infty.
$$

Then $A$ is a regular local ring of dimension one with uniformizer $t$ and the completion $A^*$ is equal to $R$. Let $K$ be the quotient field of $A$. The formal fiber of the natural map $A \to A^* = R$ at the generic point is given by $K \to R[1/t] = k((t))$. Since $R^p \subset A$, the quotient field $k((t))$ of $R$ is purely inseparable over $K$. Note that a field extension $K'$ over a field $k'$ is geometrically regular if and only if $K'/k'$ is separable. Therefore, $k((t))$ is not geometrically regular over $K$. This shows that $A$ is not a G-ring.

We show that $A$ is not a Nagata ring either. Suppose on the contrary that $A$ is a Nagata ring. Choose an element $c \in R - A$. Then $b := c^p$ lies in $A$ and the ring
\( A[c] \) is finite over \( A \) and hence a Noetherian semi-local ring. By Theorem 2.4, \( A[c] \) is again a Nagata ring. By Theorem 2.4, the completion \( A[c]^* \) should be reduced. However, the completion
\[
A[c]^* = A^* \otimes_A A[c] = A^*[t]/(t^p - b) = A^*[t]/(t - c)^p
\]
is not reduced, contradiction.

Note that the ring \( A[c] \) is not a discrete valuation ring as it is not integrally closed. If let \( L \) be the quotient field of \( A[c] \) and \( B \) the integral closure of \( A \) in \( L \), then \( B \) is not a finite \( A \)-module. Indeed, write
\[
c = a_0 + a_1 t + a_2 t^2 + \cdots
\]
and define, for \( i \geq 0 \),
\[
c_i := a_i + a_{i+1} t + a_{i+2} t^2 + \cdots.
\]
Then
\[
c_i = a_i + t c_{i+1}, \quad \forall i \geq 0,
\]
and each element \( c_i \) is contained in \( L \) and integral over \( A \). We now have an increasing sequence of subrings in \( L \)
\[
A[c] = A[c_0] \subseteq A[c_0, c_1] = A[c_1] \subseteq A[c_0, c_1, c_2] = A[c_2] \subseteq \cdots
\]
which is not stationary as \( c \notin A \). This shows that \( B \) is not a finite \( A \)-module, and hence \( A \) is not Japanese.

Remark 2.23. We explain that if there exists a non-Japanese discrete valuation ring \( A \), then \( A \) is essentially of the form as in Example 2.22. Let \( k \) be the residue field of \( A \), then the completion \( A^* \) of \( A \), by Cohen’s structure theorem for complete regular local rings [3, Corollary 2, p. 205], is isomorphic to \( k[[t]] =: R \) and one has \( \text{char} \, k = p > 0 \) (otherwise \( A \) is an excellent ring; see Example 2.21 (3)). Since the natural map \( A \to A^* \) is faithfully flat, we may regard \( A \) as a dense subring of \( R \), and may also choose \( t \) as a uniformizing element of \( A \). Since the quotient field \( k((t)) \) should be inseparable over \( K \), it is natural to expect that \( A \) contains the subring \( R^p = k^p[[t^p]] \). Since \( A \) contains \( k \) and \( t \), \( A \) should contain the image \( C \) of \( k \otimes_{k^p} k^p[[t]] \) in \( R \). If \( [k : k^p] < \infty \), then \( R = C \subseteq A \) and there is no such an example. So we have to assume \( [k : k^p] = \infty \). Under this assumption, the image \( C \) of \( k \otimes_{k^p} k^p[[t]] \) in \( R \) is exactly the example constructed in Example 2.22.

2.4. Relation with the completion. Let \( A \) be a local ring. We have
\[
\text{regular} \implies \text{normal} \implies \text{integral} \implies \text{reduced}.
\]
We discuss the relationship of \( A \) with its completion \( A^* \) through these properties.

Proposition 2.24. Let \( f : A \to B \) be a flat local homomorphism of local rings.

(1) If \( B \) is reduced (resp. integral and integrally closed), then so is \( A \).

(2) If \( B \) is a regular local ring, then so is \( A \).

Proof. (1) This is elementary; we keep a proof simply for the convenience of the reader. Since \( f \) is faithfully flat, we may regard \( A \) as subring of \( B \). Thus, \( A \) is reduced or integral if \( B \) is so. Assume \( B \) is a normal domain. Let \( L \) (resp. \( K \)) be the quotient field of \( B \) (resp. \( A \)), and we have \( K \subset L \). Since \( B \) is integrally closed, the integral closure \( A' \) of \( K \) is equal to \( B \cap K \). Since \( B \) has no non-zero
A-torsion element, the natural map $K \otimes_A B \to KB \subset L$ is injective. This shows that the map $A \otimes_A B \to A' \otimes_A B \cong B$ is an isomorphism. By faithful flatness, we get $A = A'$.

(2) We learned the proof from C.-L. Chai. First, we show $A$ is Noetherian. Let $I_1 \subset I_2 \subset \cdots$ be an increasing sequence of ideals of $A$. Then $I_i \otimes_A B \cong I_i B$, $i = 1, \ldots$, form an increasing sequence of ideals of $B$. As $B$ is Noetherian and $B$ is faithfully flat over $A$, the (ACC) holds for $A$.

To show the regularity of $A$, we recall the following definitions and results. The projective dimension of a module $M$ over a ring $A$, denoted by $\text{proj}. \dim M$, is defined to be the length of a shortest projective resolution of $M$. The global dimension of $A$, denoted by $\text{gl}. \dim A$, is defined to be $\text{supp} M \{\text{proj}. \dim M\}$, where $M$ runs through all $A$-modules, or equivalently all finite $A$-modules [6, Lemma 2, p. 128]. When $A$ is Noetherian, we have

$$\text{gl}. \dim A \leq n \iff \text{Tor}^{A}_{n+1}(M, N) = 0$$

for all finite $A$-modules $M$ and $N$ [6, Lemma 5, p. 130]. A theorem of Serre states that a Noetherian local ring $A$ is regular if and only if the global dimension of $A$ is finite (cf. [6, Theorem 45, p. 139]). Now we are ready to prove the regularity. Since $B$ is flat over $A$, the functor $B \otimes_A$ commutes with the Tor functors, and hence we have the canonical isomorphism

$$B \otimes_A \text{Tor}^{A}_{i}(M, N) \cong \text{Tor}^{B}_{i}(B \otimes_A M, B \otimes_A N)$$

for any $A$-modules $M$ and $N$. Since $B$ has finite global dimension, the latter vanishes for all $M$ and $N$ when $i > \text{gl}. \dim B$. By faithful flatness, we have $\text{Tor}^{A}_{i}(M, N) = 0$ for $i > \text{gl}. \dim B$ and hence $A$ has finite global dimension. By Serre’s theorem, $A$ is regular.

**Corollary 2.25.** Let $f : Y \to X$ be a flat morphism of schemes. If $y$ is a point of $Y$ and $O_{y}$ is reduced (resp. normal, resp. regular), then so is $O_{f(y)}$.

**Lemma 2.26.** Let $A$ be a local ring. Then $A$ is regular if and only if its completion $A^*$ is so.

**Proof.** The implication $\implies$ follows from

$$\dim A^* = \dim A = \dim m_A/m^2_A = \dim m_{A^*}/m^2_{A^*},$$

where $m_A$ (resp. $m_{A^*}$) is the maximal ideal of $A$ (resp. $A^*$). The other implication follows from Proposition 2.24 (2).

We now have the implications

$$A^* \text{ is } P \implies A \text{ is } P,$$

where $P$ is normal, integral, or reduced, and

$$A^* \text{ is regular } \iff A \text{ is regular},$$

without any condition for the local ring $A$.

It is also well-known that the implication

$$A \text{ is an integral domain } \implies A^* \text{ is an integral domain}$$
is wrong even when $A$ is excellent. For example, take $A = k[x, y]/(y^2 - x^2 - x^3)$, where $k$ is any field of characteristic $p \neq 2$.

When $P$ is reduced or normal, the implications

$$A^* \text{ is } P \implies A \text{ is } P,$$

need some conditions on $A$, for example, the morphism $\varphi : A \to A^*$ is reduced (resp. normal), i.e. it is flat and the fibers are geometrically reduced (resp. geometrically normal). As we have the implications

$$\text{regular } \implies \text{ normal } \implies \text{ reduced}$$

for morphisms (this follows from the definition), we have the implications

$$A \text{ is reduced (resp. normal) } \implies A^* \text{ is reduced (resp. normal)}$$

when $A$ is a G-ring.

Recall that a local ring $A$ is said to be unibranched if its reduced ring $A_{\text{red}}$ is an integral domain, and the normalization $A'_{\text{red}}$ of $A_{\text{red}}$ is again local (EGA IV [4, 23.2.1, p. 217]). It is clear that a normal local domain is unibranched. We know that the completion of a Noetherian local normal domain may not be reduced. Is the reduced ring of its completion still an integral domain, or even unibranched? We find in [S E7.1, p. 210] that Nagata constructed a Noetherian local normal domain $A$ such that (1) its completion $A^*$ is reduced, and (2) $A^*$ is not an integral domain. Therefore, the completion $A^*$ is not unibranched.

3. Existence of maximal orders

In this section we give the proof of Theorem 1.2.

Let $R$ be a Noetherian integral domain and $K$ its quotient field. Assume that $R$ is a Japanese ring. Let $A$ be a semi-simple algebra over $K$ and $\Lambda$ be an $R$-order of $A$. Let $Z$ be the center of $A$ and write

$$Z = \prod_{i=1}^{r} Z_i$$

as a product of finite field extensions of $K$. This gives rise to a decomposition of the semi-simple algebra

$$A = \prod_{i=1}^{r} A_i$$

into simple factors, and each simple factor $A_i$ is central simple over $Z_i$. Let $R'$ be the integral closure of $R$ in $Z$. Then

$$R' = \prod_{i=1}^{r} R'_i,$$

where $R'_i$ is the integral closure of $R$ in $Z_i$ for each $i$. Choose a system of generators $x_1, \ldots, x_m$ of $\Lambda$ over $R$ (as $R$-modules). Let $\Lambda'$ be the $R'$-submodule of $\Lambda$ generated by these $x_j$’s; clearly $\Lambda'$ is an $R'$-subring. Since $R$ is Japanese, the ring $R'$ is a finite $R$-module. Then any $R'$-subring of $A$ is an $R'$-order if and only if it is an $R$-order,
in particular, $\Lambda'$ is an $R$-order containing $\Lambda$. The decomposition (3.3) gives rise to the decomposition

$$\Lambda' = \prod_{i=1}^{r} \Lambda_i'$$

and each factor $\Lambda_i'$ is an $R'_i$-order in $A_i$. Since $A_i$ is central simple over $Z_i$ and $R'_i$ is a Noetherian normal domain, by Theorem 1.1 (cf. [10, Corollary 10.4]) there exists a maximal $R'_i$-order $\Lambda''_i$ of $A_i$ containing $\Lambda_i'$ for each $i$. Then the product

$$\Lambda'' := \prod_{i=1}^{r} \Lambda''_i$$

is a maximal $R'$-order and hence $R$-order of $A$ containing $\Lambda$. This completes the proof of Theorem 1.2 (1).

We show the second statement. Let $A$ be a finite field extension of $K$. Any $R$-order in $A$ is contained in the integral closure $R_A$ of $R$ in $A$. Let $\Lambda$ be a maximal $R$-order in $A$. Then one has $\Lambda = R_A$, otherwise one can make a bigger $R$-order $\Lambda[c]$ by adding an element $c \in R_A \setminus \Lambda$. This shows that the integral closure $R_A$ is the unique maximal $R$-order in $A$. Thus, $R_A$ is a finite $R$-module. Therefore, the ring $R$ is Japanese. This completes the proof of Theorem 1.2.

4. Description of maximal orders

Keep the notation of Section 1. In this section we attempted to describe maximal $R$-orders in $A$ when $R$ is either a Noetherian Japanese ring or an excellent ring.

4.1. Reduction to normal domains. Let $\Lambda$ be an $R$-order of a semi-simple $K$-algebra $A$, where $R$ is a Noetherian Japanese ring or an excellent domain. Let

$$Z = \prod_{i=1}^{r} Z_i, \quad A = \prod_{i=1}^{r} A_i, \quad \text{and} \quad R' = \prod_{i=1}^{r} R'_i$$

be as in Section 3. If $\Lambda$ is a maximal $R$-order, then $\Lambda$ contains the subring $R'$, that gives rise to the decomposition

$$\Lambda = \prod_{i=1}^{r} \Lambda_i,$$

and each factor $\Lambda_i$ is a maximal $R'_i$-order of $A_i$. Conversely, if we are given a maximal $R'_i$-order $\Lambda_i$ of $A_i$ for each $i$, then the product $\prod_{i=1}^{r} \Lambda_i$ is a maximal $R'$-order of $A$, and is also a maximal $R$-order of $A$ as $R'$ is finite over $R$.

Therefore, the description of maximal $R$-orders in $A$ can be reduced to the case where $A$ is central simple over $K$ and $R$ is a Noetherian normal domain, or an excellent normal domain if the initial ground ring is excellent.

4.2. Noetherian normal domain cases. Let $R$ be a Noetherian normal domain with quotient field $K$ and $A$ a central simple algebra over $K$. We recall the following results.

**Proposition 4.1.** An $R$-order $\Lambda$ in $A$ is maximal if and only if for each maximal ideal $m$ of $R$, the localization $\Lambda_m$ is a maximal $R_m$-order in $A$.

**Proof.** See [10] Corollary 11.2, p. 132.,
We say that an $R$-order $\Lambda$ in $A$ is \textit{reflexive} if the inclusion $\Lambda \subseteq \Lambda^{**}$ is an equality, where
\[
\Lambda^* = \text{Hom}_R(\Lambda, R), \quad \Lambda^{**} = \text{Hom}_R(\Lambda^*, R).
\]

**Theorem 4.2 (Auslander-Goldman).** An $R$-order $\Lambda$ is maximal if and only if $\Lambda$ is reflexive and for each minimal non-zero prime $p$ of $R$, the localization $\Lambda_p$ is a maximal $R_p$-order.

**Proof.** See [10, Theorem 11.4, p. 133].

Using Theorem 4.2, we may even reduce the description of maximal orders to the case where $R$ is a discrete valuation ring. By the following theorem, we may even pass to their completions.

**Theorem 4.3.** Assume that $R$ is an excellent local normal domain. Let $\hat{R}$ be the completion of $R$ and $\hat{K}$ the quotient field of $\hat{R}$. Let $\Lambda$ be $R$-order in $A$, and set
\[
\hat{\Lambda} := \hat{R} \otimes_R \Lambda, \quad \hat{A} := \hat{K} \otimes_K A,
\]
so $\hat{\Lambda}$ is an $\hat{R}$-order in $\hat{A}$. Then $\Lambda$ is a maximal $R$-order in $A$ if and only if $\hat{\Lambda}$ is a maximal $\hat{R}$-order in $\hat{A}$.

**Proof.** This is [10, Theorem 11.5, p. 133].

Note that the assumption of excellence for $R$ is not stated in [10, Theorem 11.5]. That would cause a problem as the completion of $R$ may not be a domain (see Nagata [8, Appendix] for the examples). For the special case where $R$ is a discrete valuation ring, the original statement holds, as the completion $\hat{R}$ is again a discrete valuation ring. See also Subsection 2.4 for more details about the relationship of a Noetherian local ring with its completion.

**4.3. Complete discrete valuation ring cases.** Let $R$ be a complete discrete valuation ring with the unique maximal ideal $P = \pi R \neq 0$, $K$ its quotient field, and $\overline{R} = R/P$. Let $A$ be a central simple $K$-algebra, and $V$ be a minimal left ideal of $A$. Set $D := \text{Hom}_A(V, V)$. Then $D$ is a division algebra, by Schur’s Lemma, whose center is equal to $K$. The minimal left ideal $V$ naturally forms a right $D$-vector space, and one has $A = \text{Mat}_r(D)$, where $r := \text{dim}_DV$. Let $v$ be the normalized $P$-adic valuation on $K$, that is, $v(\pi) = 1$. Let $N_{D/K}$ be the reduced norm on $D$ and define
\[
w(a) := [D : K]^{-1/2}v(N_{D/K}(a)), \quad a \in D.
\]

It is easy to see the following (see [10, Theorem 12.8, p. 137], [10, Theorem 12.10, p. 138] and [10, Theorem 13.2, p. 139]).

**Lemma 4.4.**

1. The valuation $w$ is the unique extension of $v$ to $D$ and the ring of integers
\[
\Delta = \{a \in D : w(a) \geq 0\}
\]
is the unique maximal $R$-order in $D$. 

(2) Let $\pi_D$ be a prime element of $\Delta$, and set $\phi = \pi_D \Delta$. Then every non-zero one-sided ideal of $\Delta$ is a two-sided ideal, and is a power of $\phi$. The residue class ring $\overline{\Delta} := \Delta/\phi$ is again a division algebra over the field $\overline{R}$, and $\phi \cap R = P$.

Lemma 4.4 describes the maximal orders $\Lambda$ in $D$ (in fact $\Lambda$ is unique) and all ideals of $\Lambda$. Now we look at the central simple case.

**Theorem 4.5.**

1. Let $\Lambda = \text{Mat}_r(\Delta)$. Then $\Lambda$ is a maximal $R$-order in $A$, and has a unique maximal two-sided ideal $\pi_D \Lambda$. The powers
   \[(\pi_D \Lambda)^m = \pi_D^m \Lambda, \quad m = 0, 1, 2, \ldots,\]
   give all of the nonzero two-sided ideals of $\Lambda$.
2. Every maximal $R$-order in $A$ is of the form $u \Lambda u^{-1}$ for some unit $u \in A$, and each such ring is a maximal $R$-order.
3. Every maximal order $\Lambda'$ is left and right hereditary, and each of its one-sided ideals is principal. The unique maximal two-sided ideal of $u \Lambda u^{-1}$ is $u \cdot \pi_D \Lambda \cdot u^{-1}$.
4. Let $\Lambda$ be any maximal $R$-order in $A$. Then there exists a full free $\Delta$-lattice $M$ in $V$ such that $\Lambda = \text{Hom}_\Delta(M, M)$. Conversely, each such $\Lambda$ is maximal.

**Proof.** See [10, Theorem 17.3, p. 171] and [10, Corollary 17.4, p. 172].

See Subsection 5.1 for the definition of hereditary rings. Note that in Theorem 4.5 \(4\) any $\Delta$-lattice $M$ in $V$ is free automatically, and hence any two full $\Delta$-lattices in $V$ are isomorphic. Therefore, the statements \(2\) and \(4\) in Theorem 4.5 are equivalent. For more general ground rings, the analogue of \(4\) is weaker than that of \(2\) in general; see also Theorem 5.6.

### 4.4. Discrete valuation rings cases.

Keep the notations as in Subsection 4.3 but the ground ring $R$ is now only assumed to be a discrete valuation ring. Let $\hat{R}$, $\hat{\Delta}$, $\hat{\Lambda}$ be the same as in Theorem 4.3. If $\Lambda$ is an $R$-order in $A$, then $\hat{\Lambda} := \hat{R} \otimes_R \Lambda$ is again $\hat{R}$-order $\hat{\Lambda}$ in $\hat{A}$, and it is maximal if and only if so is $\Lambda$.

**Theorem 4.6.** Let $\Lambda$ be a maximal $R$-order in a central simple $K$-algebra $A$. Then $\Lambda$ has a unique maximal two-sided ideal $\mathfrak{P}$, given by $\mathfrak{P} = \Lambda \cap \text{rad} \Lambda$. Then $\text{rad} \Lambda = \mathfrak{P}$, and every nonzero two-sided ideal of $\Lambda$ is a power of $\mathfrak{P}$. Further, $\text{rad} \hat{\Lambda}$ is the $P$-adic completion of $\text{rad} \Lambda$.

**Proof.** See [10, Theorem 18.3, p. 176].

The following provides a criterion for an $R$-order to be maximal.

**Theorem 4.7 (Auslander-Goldman).** Let $\Lambda$ be an $R$-order in the central simple $K$-algebra $A$. Then $\Lambda$ is maximal if and only if $\Lambda$ is hereditary, and $\text{rad} \Lambda$ is its unique maximal two-sided ideal.

**Proof.** See [11, Theorem 18.4, p. 176].

**Theorem 4.8.** Let $\Lambda$ be a maximal $R$-order in a central simple $K$-algebra $A$.

1. The ring $\Lambda$ is left and right hereditary.
2. Every maximal $R$-order in $A$ is of the form $u \Lambda u^{-1}$ for some unit $u \in A$, and each such ring is a maximal $R$-order.
3. Every maximal order $\Lambda'$ is left and right hereditary, and each of its one-sided ideals is principal. The unique maximal two-sided ideal of $u \Lambda u^{-1}$ is $u \cdot \pi_D \Lambda \cdot u^{-1}$.
4. Let $\Lambda$ be any maximal $R$-order in $A$. Then there exists a full free $\Delta$-lattice $M$ in $V$ such that $\Lambda = \text{Hom}_\Delta(M, M)$. Conversely, each such $\Lambda$ is maximal.

**Proof.** See [10, Theorem 17.3, p. 171] and [10, Corollary 17.4, p. 172].
(2) Let $M$ and $N$ be left $\Lambda$-lattices. Then $M \cong N$ if and only if $M$ and $N$ have the same rank.

(3) Every one-sided ideal of $\Lambda$ is principal.

(4) Every maximal $R$-order in $A$ is a conjugate $u\Lambda u^{-1}$ of $\Lambda$, where $u$ is a unit of $A$.

(5) Let $\hat{A} = \hat{K} \otimes A \cong \text{Mat}_t(E)$, where $E$ is a division algebra with center $\hat{K}$, and let $\Omega$ be the unique maximal $\hat{R}$-order in $E$. Then
$$\Lambda/\text{rad } \Lambda \cong \text{Mat}_t(\Omega/\text{rad } \Omega)$$
and $\Omega/\text{rad } \Omega$ is a division algebra.

**Proof.** See [10, Theorem 18.1, p. 175] and [10, Theorem 18.7, p. 179].

We see that the description of maximal orders $\Lambda$ and that of all ideals of $\Lambda$ are similar to the case where $R$ is complete. However, the maximal orders $\Delta$ in $D$, the division part of $A$, may not be unique, as the valuation $v$ may not be extended to $D$ uniquely. It is the case exactly when the completion $\hat{D} := D \otimes \hat{K}$ remains a division algebra.

5. **Maximal orders over Dedekind domains**

In this section, we give the expository description of maximal $R$-orders in a central simple algebra, where $R$ is a Dedekind domain. Our reference is again I. Reiner [10].

5.1. **Hereditary rings.** We recall

**Definition 5.1.** [10 § 2f, p. 27] A (not necessarily commutative) ring $\Lambda$ with identity is said to be left hereditary (resp. right hereditary) if every left (resp. right) ideal of $\Lambda$ is a projective $\Lambda$-module.

**Lemma 5.2.** Let $\Lambda$ be a left hereditary ring, and $N$ a $\Lambda$-submodule of a finite free left $\Lambda$-module. Then $N$ is isomorphic to an external finite direct sum of left ideals of $\Lambda$, and is therefore projective.

**Proof.** One can show this easily by induction; see [10 Theorem 2.44, p. 28].

**Remark 5.3.**

(1) If the ring $\Lambda$ has the property that every submodule of finite free left modules are projective, then in particular all left ideals of $\Lambda$ are projective. Therefore, by Lemma 5.2 $\Lambda$ is left hereditary if and only if submodules of finite free left modules are projective.

(2) There are examples of rings which are left hereditary but not right hereditary. However, if $\Lambda$ is left and right Noetherian, then $\Lambda$ is left hereditary if and only if $\Lambda$ is right hereditary (this fact is due to Auslander, cf. [10 p. 29]). Therefore, we may simply say $\Lambda$ hereditary in this case.

(3) Lemma 5.2 also holds without the finiteness for the free module. One can use transfinite induction to prove this.

**Theorem 5.4** (Steinitz). (cf. [10] (4.1), p. 45 and Theorem 4.13, p. 49)
(1) Every Dedekind domain $R$ is hereditary. Therefore, every finitely generated $R$-module $M$ without torsion elements is isomorphic to an external finite direct sum

$$M \cong J_1 \oplus \ldots \oplus J_n,$$

where $\{J_i\}$ are ideals of $R$, and $n = \text{rank}_RM := \dim_K M \otimes_R K$. In particular, $M$ is a projective $R$-module.

(2) Two such sums $\bigoplus_{i=1}^n J_i$ and $\bigoplus_{i=1}^m J'_i$ are $R$-isomorphic if and only if $m = n$, and the products $J_1 \cdots J_n$ and $J'_1 \cdots J'_m$ are in the same ideal class.

See [9] Lemma 3 for a simple proof of Theorem 5.4 (2).

Theorem 5.5. Let $R$ be a Dedekind domain with quotient field $K$, and $A$ a central simple $K$-algebra. Let $\Lambda$ be an $R$-order in $A$. Then $\Lambda$ is hereditary if and only if the localization $\Lambda_p$ is hereditary for every prime ideal $p$ of $R$.

Proof. See [10] Theorem 40.5, p. 368.

5.2. Maximal orders over Dedekind domains. Let $R$ be a Dedekind domain with quotient field $K$, and assume $K \neq R$. Let $A$ be a central simple algebra over $K$. We may and do identify $A$ as $\text{Hom}_D(V, V)$, where $D$ is a central simple division algebra over $K$, and $V$ is a finite-dimensional right vector space over $D$. Choose a maximal $R$-order $\Delta$ of $D$; it exists by Theorem 1.1 or 1.2 though not necessarily unique.

Theorem 5.6. Notation as above. If $M$ is a full right $\Delta$-lattice in $V$, then $\Lambda := \text{Hom}_\Delta(M, M)$ is a maximal $R$-order in $A$. Conversely, for any maximal $R$-order $\Lambda'$ in $A$, there exists a full right $\Delta$-lattice $N$ in $V$ such that $\Lambda' = \text{Hom}_\Delta(N, N)$.

Proof. See [10] Theorem 21.6, p. 189.

If $M$ and $N$ are $\Delta$-isomorphic, then there is an element $g \in A$ such that $N = gM$. In this case, $\Lambda' = g\Lambda g^{-1}$. Conversely, if $\Lambda'$ is conjugate to $\Lambda$ by an element in $A$, then any $\Delta$-module $N$ with $\Lambda' = \text{Hom}_\Delta(N, N)$ is isomorphic to $M$ as $\Delta$-modules.

In general, the set of conjugacy classes of maximal $R$-orders may not be singleton; its cardinality, if is finite, is called the type number of $A$.

The description of maximal orders (Theorem 5.6) is generalized by Auslander and Goldman [11] to the case where $R$ is a regular domain but $A$ is a matrix algebra over $K$. They show that any maximal order $\Lambda$ in $A = \text{End}_K(V, V)$, where $V$ is a finite-dimensional $K$-vector space, is of the form $\text{Hom}_R(M, M)$, where $M$ is a full projective $R$-lattice in $V$.

An important property of maximal $R$-orders in $A$ is the following. It plays an important role in the integral theory which generalizes the ideal theory for Dedekind domains.

Theorem 5.7. Every maximal $R$-order $\Lambda$ in $A$ is hereditary.

Proof. This is a consequence of Theorems 5.5 and 4.8 (1).

The reader is referred to the last chapter of [11] for the explicit description of global hereditary $R$-orders in $A$, which is beyond the scope of this Note.
6. Maximal orders and abelian varieties

In this section we give a proof of Theorem 1.3. Theorem 1.3 follows from a more general statement (Theorem 6.5) where the ring $\text{End}_k(A_0)$ is replaced by any subring $\mathcal{O}$ in it and $\mathcal{O}'$ by any order of $\mathcal{O} \otimes \mathbb{Q}$ containing $\mathcal{O}$.

We are grateful to the referee for his/her kind suggestion of using Serre’s tensor product construction, which improves our earlier result (Proposition 6.6). The construction is explained in [2, 1.6 and 4.2].

6.1. A construction of Serre and properties. Let $A$ be an abelian variety over a field $k$. Let $\mathcal{O} \subset \text{End}_k(A)$ be any subring, not necessarily be commutative. Note that $\mathcal{O}$ is finite and free as a $\mathbb{Z}$-module, so it is both left and right Noetherian as a ring.

Let $M$ be a finite right $\mathcal{O}$-module. Consider the functor $T$ from the category of $k$-schemes to the category of abelian groups defined by $T(S) := M \otimes_{\mathcal{O}} A(S)$ for any $k$-scheme $S$.

**Lemma 6.1.** Notations as above. The fppf sheaf associated to the group functor $T$ is representable by an abelian variety $M \otimes_{\mathcal{O}} A$ over $k$.

**Proof.** This is [2, Proposition 1.6.4.3] (the assumption that $\mathcal{O}$ is commutative there is superfluous). We provide the proof for the reader’s convenience. Choose a finite presentation of $M$ as $\mathcal{O}$-modules:

$$\mathcal{O}^r \to \mathcal{O}^s \to M \to 0.$$  

Note that $M$ is the quotient of the abelian group $\mathcal{O}^r$ by its subgroup $\alpha(\mathcal{O}^r)$. Tensoring $\otimes_{\mathcal{O}} A$, we obtain a morphism $\alpha_A : A^r \to A^s$ of abelian varieties over $k$ and a short exact sequence of abelian groups

$$A^r(S) \to A^s(S) \to T(S) \to 0,$$

for any $k$-scheme $S$. The abelian group $T(S)$ is equal to the cokernel of the map $\alpha_S$. On the other hand, the quotient abelian variety $C := A^r/\alpha(\mathcal{O}^r)$ represents the cokernel of $\alpha_A$ as a fppf abelian sheaf over $k$. This shows the representability.

We examine some basic properties of this construction.

**Lemma 6.2.**

1. Let $M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of finite right $\mathcal{O}$-modules. Then the associated morphisms of abelian varieties over $k$

$$M_1 \otimes_{\mathcal{O}} A \to M_2 \otimes_{\mathcal{O}} A \to M_3 \otimes_{\mathcal{O}} A \to 0$$

form an exact sequence.

2. If $M$ is a $\mathbb{Z}$-torsion $\mathcal{O}$-module, then the abelian variety $M \otimes_{\mathcal{O}} A$ is zero. Therefore, the map $M \to M/M_{\text{tors}}$ of $\mathcal{O}$-modules induces an isomorphism

$$M \otimes_{\mathcal{O}} A \simeq (M/M_{\text{tors}}) \otimes_{\mathcal{O}} A$$

of abelian varieties over $k$, where

$$M_{\text{tors}} := \{x \in M \mid nx = 0 \text{ for some } n \neq 0 \in \mathbb{Z}\}$$

is the $\mathbb{Z}$-torsion $\mathcal{O}$-submodule of $M$. 

Let \( \alpha : \text{M}_1 \to \text{M}_2 \) be a map of finite right \( \mathcal{O} \)-modules. Then the induced morphism \( \alpha_A : \text{M}_1 \otimes \mathcal{O} \to \text{M}_2 \otimes \mathcal{O} \) is an isogeny if the map \( \alpha_Q : \text{M}_1 \otimes \mathbb{Q} \to \text{M}_2 \otimes \mathbb{Q} \) is an isomorphism.

**Proof.**

1. First, the sequence of abelian groups
   \[
   \text{M}_1 \otimes \mathcal{O} \to (A(S) \to \text{M}_2 \otimes \mathcal{O}) \to \text{M}_3 \otimes \mathcal{O} \to 0
   \]
   is exact for any \( k \)-scheme \( S \). Since the fppf sheafification is the inductive limit of the equalizers of all fppf covers and the inductive limit is an exact functor, the sequence of the sheafifications of \( T_i(S) = \text{M}_i \otimes \mathcal{O}(S) \) is exact. That is, the sequence \((6.1)\) is exact as fppf abelian sheaves over \( k \).

2. The natural morphism \( \text{M} \otimes \mathbb{Z} \to \text{M} \otimes \mathcal{O} \) is faithfully flat. Therefore, it suffices to show the case where \( \mathcal{O} = \mathbb{Z} \) and we can even assume that \( M = \mathbb{Z}/n\mathbb{Z} \) because any finite abelian group is a finite product of finite cyclic groups. Then we get an exact sequence of abelian varieties (from \( n : \mathbb{Z} \to \mathbb{Z} \) with cokernel \( \mathbb{Z}/n\mathbb{Z} \))
   \[
   A^n \to A \to \mathbb{Z}/n \otimes \mathbb{Z} A \to 0.
   \]
   It follows that the abelian variety \( \mathbb{Z}/n \otimes \mathbb{Z} A \) is zero.

3. Using (2) we may assume that \( \text{M}_1 \) and \( \text{M}_2 \) are free \( \mathbb{Z} \)-modules. If \( \alpha_Q \) is an isomorphism, then \( \alpha \) is injective with finite cokernel. Then \( \alpha_A \) is an isogeny by Proposition 1.6.4.3 of \([2]\). ■

**Example 6.3.** The converse of Lemma \(6.2\) (2) does not hold. That is, there may be a map \( \alpha \) such that the map \( \alpha_Q \) is not isomorphic but the morphism \( \alpha_A \) can be an isogeny. We give an example. Let \( E \) be an elliptic curve with \( \text{End}(E) = \mathbb{Z} \). Put \( A := E^{\otimes 2} \) and \( R := \text{End}(A) = \text{Mat}_2(\mathbb{Z}) \). Let
   \[
   \mathcal{O} := \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{Z} \right\}.
   \]
   Put \( \text{M}_2 = \mathbb{Z}^{\otimes 2} = \mathbb{Z}e_1 + \mathbb{Z}e_2 \), the free module of row \( \mathbb{Z} \)-vectors endowed with the natural right \( \mathcal{O} \)-module. Let \( \text{M}_1 := \mathbb{Z}e_2 \) be the invariant \( \mathcal{O} \)-submodule of \( \text{M}_2 \) and \( \alpha : \text{M}_1 \to \text{M}_2 \) be the inclusion map. Let \( \text{M}_3 = \mathbb{Z} \) be the cokernel of \( \alpha \) and \( \beta : \text{M}_2 \to \text{M}_3 \), \( (a, b) \mapsto a \), be the natural projection. The induced action of \( \mathcal{O} \) on \( \text{M}_3 \) is given by \( 1 \cdot \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = a \). We have
   \[
   \text{M}_1 = e_2 \mathcal{O} = \mathcal{O}/I_1, \quad \text{M}_2 = e_1 \mathcal{O} = \mathcal{O}/I_2, \quad \text{M}_3 = 1\mathcal{O} = \mathcal{O}/I_3,
   \]
   where
   \[
   I_1 = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \right\}, \quad I_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \quad I_3 = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \right\}.
   \]
   Now \( \text{M}_1 \otimes \mathcal{O} A \simeq (\text{M}_1 \otimes \mathcal{O} R) \otimes_R A \) and \( \text{M}_3 \otimes \mathcal{O} R \simeq R/I_3 R \). We easily see that
   \[
   I_1 R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \right\}, \quad I_2 R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \quad I_3 R = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \right\}.
   \]
   Therefore, \( \text{M}_1 \otimes \mathcal{O} A \simeq E, \text{M}_2 \otimes \mathcal{O} A \simeq E \) and \( \text{M}_3 \otimes \mathcal{O} A = 0 \) and the morphism \( \alpha_A \) is an isogeny.

In this example we see how to compute \( \mathcal{O} \otimes A \) when \( M \) is monogenetic. It seems that if \( \text{M}_2 \) is indecomposable and \( \text{M}_1 \subset \text{M}_2 \) is a nonzero \( \mathcal{O} \)-submodule, then the morphism \( \alpha_A \) induced from the inclusion \( \alpha \) is an isogeny.
6.2. Relations with Tate and Dieudonné modules. Let $A$, $O$ and $M$ be as above. For any prime $\ell \not= \text{char}(k)$, denote by $T_\ell(A)$ the $\ell$-adic Tate module of $A$ viewed as a $\text{Gal}(k^s/k)$-module, where $k^s$ is a separable closure of $k$. If $\text{char}(k) = p > 0$ and $k$ is perfect, then denote by $M(A)$ the covariant Dieudonné module of $A$.

**Proposition 6.4.** Let $A$, $O$ and $M$ be as in §6.1. Then there exist surjective maps

$$\xi_{t,M} : M \otimes_O T_\ell(A) \rightarrow T_\ell(M \otimes_O A)$$

with finite kernel for any prime $\ell \not= \text{char}(k)$, and a surjective map

$$\xi_{p,M} : M \otimes_O M(A) \rightarrow M(M \otimes_O A)$$

with finite length kernel if $\text{char}(k) = p > 0$ and $k$ is perfect. Moreover, if $\alpha : M_1 \rightarrow M_2$ is a map of finite $O$-modules, then the following diagram for the associated Tate modules for any prime $\ell \not= \text{char}(k)$

$$
\begin{array}{c}
M_1 \otimes T_\ell(A) \xrightarrow{\alpha \otimes 1} M_2 \otimes T_\ell(A) \\
\downarrow \xi_{t,M_1} \quad \quad \quad \downarrow \xi_{t,M_2} \\
T_\ell(M_1 \otimes A) \xrightarrow{T_\ell(\alpha_A)} T_\ell(M_2 \otimes A)
\end{array}
$$

(resp. for the associated Dieudonné modules) commutes.

**Proof.** Choose a finite presentation of $M$ as $O$-modules:

$$
\begin{array}{c}
\mathcal{O}^r \xrightarrow{\alpha} \mathcal{O}^s \xrightarrow{\beta} M \xrightarrow{\gamma} 0
\end{array}
$$

and get a morphism $\alpha_A : A^r \rightarrow A^s$ of abelian varieties over $k$. Let $B$ be the image of $\alpha_A$. We have a short exact sequence of abelian varieties over $k$:

$$
0 \rightarrow B \rightarrow A^s \rightarrow M \otimes_O A \rightarrow 0.
$$

This gives rise to a short exact sequence of Tate modules

$$
0 \rightarrow T_\ell(B) \rightarrow T_\ell(A^s) \rightarrow T_\ell(M \otimes_O A) \rightarrow 0
$$

and Dieudonné modules

$$
0 \rightarrow M(B) \rightarrow M(A^s) \rightarrow M(M \otimes_O A) \rightarrow 0.
$$

On the other hand, tensoring the exact sequence (6.3) over the Tate module $T_\ell(A)$ and the Dieudonné module $M(A)$, respectively, we get exact sequences

$$
\begin{array}{c}
T_\ell(A) \xrightarrow{T_\ell(\alpha)} T_\ell(A)^{\otimes s} \xrightarrow{\beta} M \otimes_O T_\ell(A) \xrightarrow{\gamma} 0,
\end{array}
$$

and

$$
\begin{array}{c}
M(A)^{\otimes s} \xrightarrow{M(\alpha)} M(A)^{\otimes s} \xrightarrow{\beta} M \otimes_O M(A) \xrightarrow{\gamma} 0.
\end{array}
$$

This gives a surjective map $\xi_{t,M} : M \otimes T_\ell(A) \rightarrow T_\ell(M \otimes O A)$ and a surjective map $\xi_{p,M} : M \otimes M(A) \rightarrow M(M \otimes O A)$. The kernel of the map $\xi_{t,M}$ (resp. $\xi_{p,M}$) is the cokernel of the map $T_\ell(\alpha) : T_\ell(A)^{\otimes r} \rightarrow T_\ell(B)$ (resp. $M(\alpha) : M(A)^{\otimes r} \rightarrow M(B)$).

This proves the first part of the proposition.

The natural map $\xi_{t,M}$ induces an natural isomorphism

$$
\xi_{t,M} : M \otimes T_\ell(A)/(\text{torsion}) \simeq T_\ell(M \otimes O A).
$$

From this it follows that the diagram (6.2) commutes. The proof of the assertion for Dieudonné modules is the same.
6.3. Computation of $M \otimes_\mathcal{O} A$ up to isogeny. Let $[A]$ denote the isogeny class of an abelian variety $A$ over a field $k$. Let $C$ be a $\mathbb{Q}$-subalgebra of the semi-simple algebra $\text{End}_k^0(A)$ and let $V$ be a finite right $C$-module. It is not hard to see from Lemma 6.2 that the isogeny class $[M \otimes_\mathcal{O} A]$, for a $\mathbb{Z}$-order $\mathcal{O}$ in $C$ contained in $\text{End}_k(A)$ and an $\mathcal{O}$-lattice $M$, does not depend on the choice of $\mathcal{O}$ and $M$ (and also the choice of $A$ in $[A]$). We denote this isogeny class by $V \otimes_\mathcal{O} [A]$.

Write $[A] = [\prod_{i=1}^r B_i^{m_i}]$ into a finite product of isotypic components, where each $B_i$ is a $k$-simple abelian variety and $B_i$ is not isogenous to $B_j$ for $i \neq j$. The endomorphism algebra $E := \text{End}_k^0(A) \cong \prod_{i=1}^r \text{Mat}_{n_i}(D_i)$ decomposes into the product of its simple factors, where $D_i$ is the endomorphism algebra of $B_i$. Observe that

$$V \otimes_C [A] = V \otimes_C E \otimes E [A].$$

If we write

$$V \otimes_C E \cong \bigoplus_{i=1}^r I_i^{m_i},$$

as $E$-modules, where $I_i$ is a minimal non-zero ideal of $\text{Mat}_{n_i}(D_i)$, then

$$V \otimes_C [A] \cong \left( \bigoplus_{i=1}^r I_i^{m_i} \right) \otimes_E [A] = \prod_{i=1}^r I_i^{m_i} \otimes_{\text{Mat}_{n_i}(D_i)} [B_i^{m_i}] \cong \prod_{i=1}^r [B_i^{m_i}].$$

Therefore, the computation of the abelian variety $M \otimes_\mathcal{O} A$ up to isogeny is reduced to the simple algebra problem (6.5) of decomposing the module $V \otimes C E$ into simple $E$-modules.

The dimension of $M \otimes_\mathcal{O} A$ is given by the formula:

$$\dim M \otimes_\mathcal{O} A = \sum_{i=1}^r m_i \dim B_i,$$

where $m_i$ are the integers in (6.5).

6.4. On minimal isogenies for abelian varieties. The main result of this section is the following theorem.

**Theorem 6.5.** Let $A_0$ be an abelian variety over $k$, $\mathcal{O}$ a subring of $\text{End}_k(A_0)$, and $\mathcal{O}'$ a $\mathbb{Z}$-order of $\mathcal{O} \otimes \mathbb{Q}$ containing $\mathcal{O}$. Then the isogeny $\iota : A_0 \rightarrow \mathcal{O}' \otimes_\mathcal{O} A_0$ satisfies the following property: for any pair $(\varphi_1, A_1)$ where $\varphi_1 : A_0 \rightarrow A_1$ is an isogeny of abelian varieties over $k$ such that with the identification $\text{End}_k^0(A_0) = \text{End}_k^0(A_1)$ by $\varphi_1$ one has $\mathcal{O}' \subset \text{End}(A_1)$, then there is a unique isogeny $\alpha : \mathcal{O}' \otimes \mathcal{O} A_0 \rightarrow A_1$ over $k$ such that $\varphi_1 = \alpha \circ \iota$.

Theorem 1.3 is the special case of Theorem 6.5 where $\mathcal{O} = \text{End}_k(A_0)$ and $\mathcal{O}'$ is a maximal order containing $\mathcal{O}$. We first prove a weaker statement.

**Proposition 6.6.** Let $A_0$, $\mathcal{O}$, $\mathcal{O}'$ be as in Theorem 6.5. Then there exist a finite purely inseparable extension $k'$ of $k$, an abelian variety $A'$ over $k'$, and an isogeny $\varphi : A_0 \otimes k' \rightarrow A'$ over $k'$ such that with the identification $\text{End}_k^0(A \otimes k') = \text{End}_{k'}^0(A')$ one has $\mathcal{O}' \subset \text{End}_{k'}(A')$. Moreover, the isogeny $\varphi$ can be chosen to be minimal with respect to $\mathcal{O}'$ in the following sense: if $(k_1, A_1, \varphi_1)$ is another triple with the
property $O' \subset \End_{k_1}(A_1)$, then there are a finite purely inseparable extension $k''$ of $k$ containing both $k'$ and $k_1$ and a unique isogeny

$$\alpha : A' \otimes_{k'} k'' \to A_1 \otimes_{k_1} k''$$

such that $\varphi_{1,k''} = \alpha \circ \varphi_{k''}$.

**Proof.** Replacing $k$ by a finitely generated subfield over its prime field and replacing $A_0$ by a model of $A_0$ defined over this subfield whose endomorphism ring is equal to $\End_B(A_0)$, we may assume that the ground field $k$ is finitely generated over its prime field. Put $G := \Gal(k^s/k)$, where $k^s$ is a separable closure of $k$. For any prime $\ell \neq \text{char } k$, let $T_\ell := T_\ell(A_0)$ be the associated Tate module of $A_0$, and

$$\rho_\ell : G \to \Aut(T_\ell)$$

be the associated Galois representation. Let $G_\ell \subset \Aut(T_\ell)$ be the image of the map $\rho_\ell$, and write $O_\ell := O \otimes_k \mathbb{Z}_\ell$ and $O'_\ell := O' \otimes_k \mathbb{Z}_\ell$, respectively. By the theorem of Tate, Zarhin and Faltings [11, 13, 3] on homomorphisms of abelian varieties, we have

$$O_\ell \subset \End_B(A_0) \otimes_k \mathbb{Z}_\ell = C_{\End(T_\ell)} G_\ell,$$

the centralizer of $G_\ell$ in $\End(T_\ell)$. Let $T'_\ell$ be the $O'_\ell$-submodule in $V_\ell := T_\ell \otimes_k \mathbb{Q}_\ell$ generated by $T_\ell$. Since the action of $O_\ell$ on $T_\ell$ commutes with that of $G_\ell$, the lattice $T'_\ell$ is stable under the $G_\ell$-action. As $O'_\ell = O_\ell$ for almost all primes $\ell$, we have $T'_\ell = T_\ell$ for such primes $\ell$. If $\text{char } k = 0$, then by a theorem of Tate, there are an abelian variety $A'$ over $k$ and an isogeny $\varphi : A_0 \to A'$ over $k$ such that the image of $T_\ell(A')$ in $V_\ell$ by $\varphi$ is equal to $T'_\ell$ for all primes $\ell$. We have

$$O'_\ell \subset \End_{G_\ell}(T'_\ell) = \End_B(A') \otimes_k \mathbb{Z}_\ell,$$

and hence $O' \subset \End_B(A')$.

Suppose $\text{char } k = p > 0$. Let $k^{pf}$ be the perfect closure of $k$. Let $M_0 := \mathcal{M}(A_0 \otimes_k k^{pf})$ be the covariant Dieudonné module of $A_0 \otimes_k k^{pf}$, on which the ring $O_p := O \otimes_k \mathbb{Z}_p$ acts. Let $W$ be the ring of Witt vectors over $k^{pf}$ and let $B(k^{pf})$ be its fraction field. Let $M'$ be the $O_p \otimes_k W$-submodule of $M_0 \otimes W B(k^{pf})$ generated by $M_0$. Since the action of $O_p$ on $M_0$ commutes with the Frobenius map $F$, the submodule $M'$ is a Dieudonné module containing $M_0$. By a theorem of Tate, there are an abelian variety $A$ over $k^{pf}$ and an isogeny $\varphi_{k^{pf}} : A_0 \otimes_k k^{pf} \to A$ over $k^{pf}$ such that the image of the Tate module $T(A)$ in $V_\ell$ and the Dieudonné module $\mathcal{M}(A)$ in $M_0 \otimes W B(k^{pf})$ by the isogeny $\varphi$ is equal to $T'_\ell$ for all primes $\ell \neq p$ and equal to $M'$ at the prime $p$, respectively. Similarly, we show $O' \subset \End_{k^{pf}}(A)$. Since $\ker \varphi_{k^{pf}}$ is of finite type, there is a model $A'$ of $A$ over a finite extension $k'$ of $k$ in $k^{pf}$ so that the isogeny $\varphi_{k^{pf}}$ is defined over $k'$.

If we have another triple $(k_1, A_1, \varphi_1)$ with $O' \subset \End_{k_1}(A_1)$, then the Tate module $T_\ell(A_1)$ viewed as a lattice in $V_\ell$ by the isogeny $\varphi_1$ is an $O'_\ell[G_\ell]$-stable lattice, and hence it contains $T'_\ell$. Similarly, one shows that the Dieudonné module $M_1$ of $A_1 \otimes k^{pf}$ as a Dieudonné sublattice in $M_0 \otimes B(k^{pf})$ by the isogeny $\varphi_1$ contains $M_0$. Therefore, there is an isogeny $\alpha : A' \otimes_{k'} k^{pf} \to A_1 \otimes_{k_1} k^{pf}$ such that $\varphi_{1,k^{pf}} = \alpha \circ \varphi_{k^{pf}}$. Clearly, the morphism $\alpha$ is defined over some finite extension of $k$ in $k^{pf}$ containing $k'$ and $k_1$. This proves the proposition. 

**Lemma 6.7 (Chow).** Let $A$ and $B$ be two abelian varieties over a field $k$, and let $K/k$ be a primary field extension (i.e. $k$ is separably algebraically closed in $K$).
Then the natural map

\[ \text{Hom}_K(A, B) \rightarrow \text{Hom}_K(A \otimes K, B \otimes K) \]

is bijective.

**Proof.** See [2, Lemma 1.2.1.2].

We are ready to prove Theorem 6.5. We show that the isogeny \( \iota : A_0 \rightarrow O' \otimes O A_0 \) satisfies the property of Proposition 6.6. Put \( A'_0 := O' \otimes O A_0 \). By Proposition 6.4 the Tate module \( T_\ell(A'_0) \) of \( A'_0 \) in \( V_\ell = T_\ell(A_0) \otimes \mathbb{Q}_\ell \) through the isogeny \( \iota \) is equal to \( T'_\ell \) (in the proof of Proposition 6.6), and its Dieudonné module \( \mathbf{M}(A'_0 \otimes k_{pf}) \) of the abelian variety \( A'_0 \otimes k_{pf} \) in \( M_0 \otimes W B(k_{pf}) \) through \( \iota \) is equal to \( M' \) (in the proof of Proposition 6.6), where \( M_0 = \mathbf{M}(A_0) \). Therefore, there is an isomorphism \( A'_0 \otimes k_{pf} \simeq A' \otimes k'_{pf} \) which transforms \( \iota \) to \( \varphi \). So the isogeny \( \iota \) satisfies the property of Proposition 6.6.

Now by Lemma 6.7 the morphism \( \alpha \) in Proposition 6.6, which is a priori defined over some finite purely inseparable extension of \( k \), is defined over \( k \). The proof of Theorem 6.5 is complete.

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