An Efficient Cyclic Entailment Procedure in a Fragment of Separation Logic

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Abstract. An efficient entailment proof system is essential to compositional verification using separation logic. Unfortunately, existing decision procedures are either inexpressive or inefficient. For example, Smallfoot is an efficient procedure but only works with hardwired lists and trees. Other procedures that can support general inductive predicates run exponentially in time as their proof search requires back-tracking to deal with disjunction in the consequent.

In this paper, we present a decision procedure that can derive cyclic entailment proofs for general inductive predicates in polynomial time. Our procedure is efficient and does not require back-tracking; it uses normalisation rules that help avoid the introduction of disjunction in the consequent. Moreover, our decidable fragment is sufficiently expressive: It is based on compositive predicates and can capture a wide range of data structures, including sorted and nested list segments, skip lists with fast forward pointers, and binary search trees. We have implemented the proposal in a prototype tool and evaluated it over challenging problems taken from a recent separation logic competition. The experimental results confirm the efficiency of the proposed system.

Keywords: Cyclic Proofs, Entailment Procedure, Separation Logic.

1 Introduction

Separation logic [20,35] has been very successful in automatically reasoning about programs that manipulate pointer structures. Separation logic empowers reusability and scalability through compositional reasoning [6,7]. Moreover, a compositional verification system relies on a bi-abduction procedure which is essentially based on the entailment proof system. Entailment is defined as: Given an antecedent A and a consequent C where A and C are formulas in separation logic, entailment problem is the act of checking whether A \models C is valid. Thus, an efficient decision procedure for entailments is the vital ingredient of an automatic verification system in separation logic.

To enhance the expressiveness of the assertion language, for example, to specify unbounded heaps and interesting pure properties (e.g., sortedness, parent pointers), separation logic is typically combined with user-defined inductive predicates [9,29,33]. In this setting, one key challenge of an entailment procedure is the ability to support induction reasoning over the combination of heaps and data content. The problem of induction is very difficult, especially for an automated inductive theorem prover, where the induction rules are not explicitly stated. In fact, this problem is undecidable [1].
Developing a sound and complete entailment procedure that could be used for compositional reasoning is not trivial. While it is unknown how model-based systems e.g., [14,15,17,18,22,23], could support compositional reasoning, there was evidence that proof-based decision procedures, e.g., Smallfoot [2] and its variant [12], and Cycomp [40], can be extended to solve the bi-abduction problem, which enables compositional reasoning and scalability [7,25]. In fact, Smallfoot was the center of the biabductive procedure deployed in Infer [7], which achieved great impact in both academia and industry [13]. Furthermore, Smallfoot is very efficient due to its use of “exclude-the-middle” rule in which it can avoid the proof search over the disjunction in the consequent. However, Smallfoot works for hardwired lists and binary trees only. In contrast, Cycomp, a recent complete entailment procedure, is a cyclic proof system without “exclude-the-middle”, can support general inductive predicates, but has double exponential time complexity due to the proof search (and back-tracking) in the consequent.

In this paper, we introduce a cyclic proof system with an “exclude-the-middle”-styled decision procedure for decidable yet expressive inductive predicates. Especially, we show that our procedure runs in polynomial time when the maximum number of fields of data structures is bounded by a constant. The decidable fragment, called SHLIDe, contains inductive definitions of compositional predicates and pure properties. These predicates can capture nested list segments, skip lists and trees. The pure properties of small models can model a wide range of common data structures e.g., a list with fast forward pointers, a nested list being sorted, a tree being a binary search tree [22,30]. This fragment is much more expressive than Smallfoot’s fragment and is non-overlapping with Cycomp’s one [40]: there exist some entailments which our system can handle but Cycomp could not, and vice versa.

Our procedure is a variant of the cyclic proof system, which was first introduced by Brotherston [3] and has become one of the main solutions to induction reasoning. Intuitively, a cyclic proof is naturally represented as a tree of statements (entailments in this paper): the leaves are either axioms or nodes which are linked back to inner nodes, the root of the tree is the theorem to be proven, and nodes are connected to one or more children by locally sound proof rules. Alternatively, a cyclic proof can be viewed as a tree possibly containing some backlinks (a.k.a. cycles, e.g., “C, if B, if C”) such that the proof satisfies some global soundness condition. This condition ensures that the proof can be viewed as a proof of infinite descent. Particularly, for a cyclic entailment proof with inductive definitions, if every cycle contains an unfolding of some inductive predicate, then that predicate is infinitely often reduced into a strictly “smaller” predicate; this is impossible as the semantics of inductive definitions only allows finite steps of unfolding. Hence, that proof path with the cycle can be disregarded.

The proposed system advances Brotherston’s system in three ways. First, the proposed proof search algorithm is specialized to SHLIDe in which it includes “exclude-the-middle” rules and excludes any back-tracking. The existing proof procedures typically search for a proof (and back-track) over disjunctive cases generated from unfolding inductive predicates in the RHS of an entailment. To avoid such costly searches, we propose a “exclude-the-middle”-styled normalized rules in which unfolding of inductive predicates in the RHS always produces one disjunct. Therefore, our system is much more efficient than existing systems. Second, while a standard Brotherston sys-
tem is incomplete, our proof search is complete in SHLDe: If it is stuck (i.e., it can not apply any inference rules) then the root entailment is invalid.

Lastly, while the global soundness in [5] must be checked globally and explicitly, every backlink generated in SHLDe is sound by design. We note that Cycomp, introduced in [40], was the first work to show completeness of a cyclic proof system. However, in contrast to ours, it did not discuss the global soundness condition, which is the key idea attributing to the soundness of cyclic proofs.

Contributions Our primary contributions are summarized as follows.
– We present a novel decision procedure, called S2S\textsubscript{Lin}, for the entailment problem in separation logic with inductive definitions of compositional predicates.
– We provide a complexity analysis of the procedure.
– We have implemented the proposal in a prototype tool, called S2S\textsubscript{Lin}, and tested it with the SL-COMP 2022 benchmarks [36,37]. The experimental results show that S2S\textsubscript{Lin} is effective and efficient when compared with the state-of-the-art solvers.

Organization The remainder of the paper is organised as follows. Sect. 2 describes syntax of formulas in fragment SHLDe. Sect. 3 presents the basics of an “exclude-the-middle” proof system and cyclic proofs. Sect. 4 elaborates the result, the novel cyclic proof system including an illustrative example. Sect. 5 discusses the soundness and completeness. Sect. 6 presents the implementation and evaluation. Sect. 7 discusses related work. Finally, Sect. 8 concludes the work. All proofs are available in Appendix.

2 Decidable Fragment SHLDe

Subsection 2.1 presents syntax of separation logic formulae and recursive definitions of linear predicates and local properties. Subsection 2.2 shows semantics.

2.1 Separation Logic Formulas

Concrete heap models assume a fixed finite collection Node, a fixed finite collection Fields, a set Loc of locations (heap addresses), a set of non-addressable values Val, with the requirement that Val \cap Loc = \emptyset (i.e., no pointer arithmetic). null is a special element of Val. \mathbb{Z} denotes the set of integers (\mathbb{Z} \subseteq Val) and k denotes integer numbers. Var an infinite set of variables, \bar{v} a sequence of variables.

Syntax Disjunctive formula \Phi, symbolic heaps \Delta, spatial formula \kappa, pure formula \pi, pointer (dis)equality \phi, and (in)equality formula \alpha are as follows.
\begin{align*}
\Phi &::= \Delta \mid \Phi \lor \Phi \\
\Delta &::= \kappa \land \pi \mid \exists v. \kappa \land \pi \\
\phi &::= v=v \mid v=null \\
\kappa &::= \text{emp} \mid x\rightarrow c(f:v, .., f:v) \mid P(\bar{v}) \mid \kappa \ast \kappa \\
\pi &::= \text{true} \mid \alpha \mid \phi \mid \neg \pi \mid \exists v. \pi \mid \pi \land \pi \\
\alpha &::= a=a \mid a \leq a
\end{align*}

where \(v \in \text{Var}, c \in \text{Node} \) and \(f \in \text{Fields} \). Note that we often discard field names \(f \) of points-to predicates \(x \rightarrow c(f:v, .., f:v) \) and use the short form as \(x \rightarrow c(\bar{v}) \). \(v_1 \neq v_2 \) is the short form of \(\neg(v_1 = v_2) \). \(E \) denotes for either a variable or null. \(\Delta[E/v] \) denotes the formula obtained from \(\Delta \) by substituting \(v \) by \(E \). A symbolic heap is referred as a base, denoted as \(\Delta^b \), if it does not contain any occurrence of inductive predicates.
Inductive Definitions  We write \( \mathcal{P} \) to denote a set of \( n \) defined predicates \( \mathcal{P} = \{ P_1, \ldots, P_n \} \) in our system. Each inductive predicate has following types of parameters: a pair of root and segment defining segment-based linked points to heaps, reference parameters (e.g., parent pointers, fast-forwarding pointers), transitivity parameters (e.g., singly-linked lists where every heap cell contains the same value \( a \)) and pairs of ordering parameters (e.g., trees being binary search trees). An inductive predicate is defined as

\[
\text{pred } P(r,F,\bar{B},u,sc,tg) \equiv \text{emp} \land r=F \land sc= tg \\
\lor \exists X_{tl}, Z, sc', r \mapsto c(X_{tl}, \bar{p}, u, sc') \ast \kappa' \ast P(X_{tl}, F, \bar{B}, u, sc', tg) \land r \neq F \land sc \circ sc'
\]

where \( r \) is the root, \( F \) the segment, \( \bar{B} \) the borders, \( u \) the parameter for a transitivity property, \( sc \) and \( tg \) source and target, respectively, parameters of an order property, \( r \mapsto c(X_{tl}, \bar{p}, u, sc') \ast \kappa' \) the matrix of the heaps, and \( \circ \in \{=, \geq, \leq \}. \) (The extension for multiple local properties is straightforward.) Moreover, this definition is constrained by the following three conditions on heap connectivity, establishment, and termination.

**Condition C1.** In the recursive rule, \( \bar{p} = \{ \text{null} \} \cup \bar{Z} \). This condition implies that if two variables points to the same heap, their content must be the same. For instance, the following definition of singly-linked lists of even length does not satisfy this condition.

\[
\text{pred } \ell\ell(r,F) \equiv \text{emp} \land r=F \lor \exists x_1,X.r \mapsto c_1(x_1) \ast x_1 \mapsto c_1(X) \ast \ell\ell(X,F) \land r \neq F
\]
as \( n_3 \) and \( X \) are not field variables of the node pointed-to by \( r \).

**Condition C2.** The matrix heap defines nested and connected list segments as:

\[
\kappa':= Q(Z,U) \mid \kappa' \ast \kappa' \mid \text{emp}
\]

where \( Z \in \bar{p} \) and \((\bar{U} \setminus \bar{p}) \cap Z = \emptyset \). This condition ensures connectivity (i.e. all allocated heaps are connected to the root) and establishment (i.e. every existential quantifier either is allocated or equals to a parameter).

**Condition C3.** There is no mutual recursion. We define an order \( \prec_{\mathcal{P}} \) on inductive predicates as: \( P \prec_{\mathcal{P}} Q \) if at least one occurrence of predicate \( Q \) appears in the definition of \( P \) and \( Q \) is called a direct sub-term of \( P \). We use \( \prec_{\mathcal{P}}^* \) to denote the transitive closure of \( \prec_{\mathcal{P}} \).

Several definition examples are shown as follows.

\[
\text{pred } \ell\ell(r,F) \equiv \text{emp} \land r=F \lor \exists X_{tl}.r \mapsto c_1(X_{tl}) \ast \ell\ell(X_{tl}, F) \land r \neq F
\]

\[
\text{pred } \text{nll}(r,F,B) \equiv \text{emp} \land r=F \\
\lor \exists X_{tl}, Z, r \mapsto c_3(X_{tl}, Z) \ast \text{nll}(X_{tl}, F,B) \land r \neq F
\]

\[
\text{pred } \text{skl1}(r,F) \equiv \text{emp} \land r=F \lor \exists X_{tl}, r \mapsto c_4(X_{tl}, \text{null}, \text{null}) \ast \text{skl1}(X_{tl}, F) \land r \neq F
\]

\[
\text{pred } \text{skl2}(r,F) \equiv \text{emp} \land r=F \\
\lor \exists X_{tl}, Z_1, r \mapsto c_4(Z_1, X_{tl}, \text{null}) \ast \text{skl1}(Z_1, X_{tl}) \ast \text{skl2}(X_{tl}, F) \land r \neq F
\]

\[
\text{pred } \text{skl3}(r,F) \equiv \text{emp} \land r=F \\
\lor \exists X_{tl}, Z_1, Z_2, r \mapsto c_4(Z_1, Z_2, X_{tl}) \ast \text{skl1}(Z_1, Z_2) \ast \text{skl2}(Z_2, X_{tl}) \ast \text{skl3}(X_{tl}, F) \land r \neq F
\]

\[
\text{pred } \text{tree}(r,B) \equiv \text{emp} \land r=B \\
\lor \exists r_1, r_2, r_3 \mapsto c_4(r_1, r_2) \ast \text{tree}(r_1, B) \ast \text{tree}(r_2, B) \land r \neq B
\]

\( \ell\ell \) defines singly-linked lists, \( \text{nll} \) defines lists of acyclic lists, \( \text{skl1} \), \( \text{skl2} \) and \( \text{skl3} \) define skip-lists. Finally, \( \text{tree} \) defines binary trees. We extend predicate \( \ell\ell \) with transi-
tivity and order parameters to obtain predicate \(11a\) and \(11s\), respectively, as follows.

\[
\text{pred } 11a(r,F,a) \equiv \text{emp} \land r = F \lor \exists X_{tl}. r \rightarrow c_2(X_{tl}, a) \land 11a(X_{tl}, F,a) \land r \neq F
\]

\[
\text{pred } 11s(r,F,mi,ma) \equiv \text{emp} \land r = F \land ma = mi \\
\lor \exists X_{tl},mi_1.r \rightarrow c_4(X_{tl},mi_1) \land 11s(X_{tl}, F,mi_1,ma) \land r \neq F \land mi \leq mi_1
\]

**Unfolding** Given \(\text{pred } P(t) \equiv \Phi\) and a formula \(P(\bar{v}) \Delta\), then unfolding \(P(\bar{v})\) means replacing \(P(\bar{v})\) by \(\Phi[\bar{v}/t]\). We annotate a number, called unfolding number, for each occurrence of inductive predicates. Suppose \(\exists \bar{w}. r \rightarrow c(\bar{p}) \ast Q_1(\bar{v}_1) \ast \ldots \ast Q_m(\bar{v}_m) \ast P(\bar{v}_0) \land \pi\) be the recursive rule, then in the unfolded formula, if \(P(\bar{v}_0[\bar{v}/t])^{k_1}\) and \(Q_1(\ldots)^{k_2}\) are direct sub-terms of \(P(\bar{v})^{k}\) like above, then \(k_1 = k + 1\) and \(k_2 = 0\). When it is unambiguous, we discard the annotation of the unfolding number for simplicity.

### 2.2 Semantics

The program state is interpreted by a pair \((s,h)\) where \(s \in \text{Stacks}, h \in \text{Heaps}\) and stack \(\text{Stacks}\) and heap \(\text{Heaps}\) are defined as:

\[
\text{Heaps} \equiv \text{Loc} \rightarrow f_{\text{fin}}(\text{Node} \rightarrow (\text{Fields} \rightarrow \text{Val} \cup \text{Loc})^m)
\]

\[
\text{Stacks} \equiv \text{Var} \rightarrow \text{Val} \cup \text{Loc}
\]

Note that we assume that every data structure contains at most \(m\) fields. Given a formula \(\Phi\), its semantics is given by a relation: \(s,h \models \Phi\) in which the stack \(s\) and the heap \(h\) satisfy the constraint \(\Phi\). The semantics is shown below:

\[
s,h \models \text{emp} \iff \text{dom}(h) = \emptyset
\]

\[
s,h \models v \rightarrow c(f_i : v_i) \iff \text{dom}(h) = \{s(v)v\}, h(s(v)) = g, g(c,f_i) = s(v_i)
\]

\[
s,h \models P(\bar{v}) \iff (h, s(\bar{v}_1), \ldots, s(\bar{v}_m)) \in [P]
\]

\[
s,h \models \kappa_1 \ast \kappa_2 \iff \exists h_1, h_2. s.t. h_1 \# h_2, h = h_1 \cdot h_2, \cdot s, h_1 \models \kappa_1\text{ and } h_2 \models \kappa_2
\]

\[
s,h \models \text{true} \iff \text{always}
\]

\[
s,h \models \kappa \land \pi \iff s, h \models \kappa\text{ and } h \models \pi
\]

\[
s,h \models \exists \bar{v}.\Delta \iff \exists \alpha. s[\bar{v} \rightarrow \alpha], h \models \Delta
\]

\[
s,h \models \Phi_1 \lor \Phi_2 \iff s, h \models \Phi_1\text{ or } s, h \models \Phi_2
\]

\(\text{dom}(g)\) is the domain of \(g\), \(h_1 \# h_2\) denotes disjoint heaps \(h_1\) and \(h_2\) i.e., \(\text{dom}(h_1) \cap \text{dom}(h_2) = \emptyset\), and \(h_1 \cdot h_2\) denotes the union of two disjoint heaps. If \(s\) is a stack, \(v \in \text{Var}\), and \(\alpha \in \text{Val} \cup \text{Loc}\), we write \(s[\bar{v} \rightarrow \alpha] = s\) if \(v \in \text{dom}(s)\), otherwise \(s[\bar{v} \rightarrow \alpha] = s \cup \{(v, \alpha)\}\). Semantics of non-heap (pure) formulas is omitted for simplicity. The interpretation of an inductive predicate \(P(t)\) is based on the least fixed point semantics \([P]\).

Entailment \(\Delta \models \Delta'\) holds iff for all \(s\) and \(h\), if \(s, h \models \Delta\) then \(s, h \models \Delta'\).

### 3 Entailment Problem & Overview

Throughout this work, we consider the following problem.

**PROBLEM:** \(\text{QF\_ENT\_SLIB}\).

**INPUT:** \(\Delta_\alpha \equiv \kappa_c \land \pi_a\) and \(\Delta_c \equiv \kappa_c \land \pi_c\) where \(\text{FV}(\Delta_c) \subseteq \text{FV}(\Delta_\alpha) \cup \{\text{null}\}\).

**QUESTION:** Does \(\Delta_\alpha \models \Delta_c\) hold?
An entailment, denoted as $\Delta_1 \vdash \Delta_2$, is syntactically formalized as: $\Delta_a \vdash \Delta_c$ where $\Delta_a$ and $\Delta_c$ are quantifier-free formulas whose syntax are defined in the preceding section.

In Sect. 3.1, we present the basis of an exclude-the-middle proof system and our approach to $\text{QF-ENT-SL-LIN}$. In Sect. 3.2, we describe the foundation of cyclic proofs.

### 3.1 Exclude-the-middle proof system

Given a goal $\Delta_a \vdash \Delta_c$, an entailment proof system might derive entailments with disjunction in the right-hand side (RHS). Such an entailment can be obtained by a proof rule that replaces an inductive predicate by its definition rules. Authors of Smallfoot [2] introduced a normal form and proof rules to prevent such entailments when the predicate are lists or trees. Basically, Smallfoot considers the following two scenarios.

- **Case 1 (Exclude-the-middle and Frame):** The inductive predicate matches with a points-to predicate in the left hand side (LHS). For instance, the entailment is of the form $e_1 : x \rightarrow c(z) \ast \Delta \vdash 11(x, y) \ast \Delta'$, where 11 is singly-linked lists and 11(x, y) matches with $x \rightarrow c(z)$ as they have the same root $x$. To discharge $e_1$, a typical proof system might search for a proof through two definition rules of predicate 11 (i.e., by unfolding 11(x, y) into two disjuncts): One includes the base case with $x = y$ and another contains the recursive case with $x \neq y$. Smallfoot prevents such unfolding by excluding the middle in the LHS: It reduces the entailment into two premises: $x \rightarrow c(z) \ast \Delta \land x = y \vdash 11(x, y) \ast \Delta'$ and $x \rightarrow c(z) \ast \Delta \land x \neq y \vdash 11(x, y) \ast \Delta'$. The first one considers the base case of the list (that is, 11(x, x)) and is equivalent to $x \rightarrow c(z) \ast \Delta \land x = y \vdash \Delta'$. And the second premise checks the inductive case of the list and is equivalent to $\Delta \land x = y \vdash \Delta' \ast \Delta'$.

- **Case 2 (Induction proving via hard-wired Lemma).** The inductive predicate matches other inductive predicates in the LHS. For example, the entailment is of the form $e_2 : 11(x, z) \ast \Delta \vdash 11(x, \text{null}) \ast \Delta'$. Smallfoot handle $e_2$ by using a proof rule as the consequence of applying the following hard-wired lemma $11(x, z) \ast 11(z, \text{null}) \vdash 11(x, \text{null})$ and reduces the entailment to $\Delta \vdash 11(z, \text{null}) \ast \Delta'$.

In doing so, Smallfoot does not introduce a disjunction in the RHS. However, as it uses specific lemmas in the induction reasoning, it only works for the hardwired lists.

In this paper, we propose $\text{S2S}_{\text{US}}$ as an exclude-the-middle’s system for user-defined predicates, those in SHILDe. In stead of using hardwired lemmas, we apply cyclic proofs for induction reasoning. For instance, to discharge the entailment $e_2$ above, $\text{S2S}_{\text{US}}$ first unfolds $11(x, z)$ in the LHS and obtains two premises:

- $e_{21} : (\text{emp} \land x = z) \ast \Delta \vdash 11(x, \text{null}) \ast \Delta'$; and
- $e_{22} : (x \rightarrow c(y) \ast 11(y, z) \land x \neq z) \ast \Delta \vdash 11(x, \text{null}) \ast \Delta'$

While it reduces $e_{21}$ to $\Delta[z/x] \vdash 11(z, \text{null}) \ast \Delta'[z/x]$, for $e_{22}$, it further applies the frame rule as in Case 1 above and obtains $11(y, z) \ast \Delta \land x \neq z \vdash 11(y, \text{null}) \ast \Delta'$. Then, it makes a backlink between the latter and $e_2$ and closes this path. By doing so, it does not introduce disjunctions in the RHS and can handle user-defined predicates.
3.2 Cyclic proofs

Central to our work is a procedure that constructs a cyclic proof for an entailment. Given an entailment $\Delta \vdash \Delta'$, if our system can derive a cyclic proof, then $\Delta \models \Delta'$ holds. If, instead, it is stuck without a proof, then $\Delta \models \Delta'$ is not valid.

The procedure includes proof rules, each of which is of the form:

$$
\frac{\text{PR}_0 \ e_1 \ldots \ e_n}{\text{cond}}
$$

where entailment $e$ (called the conclusion) is reduced to entailments $e_1, \ldots, e_n$ (called the premises) through inference rule $\text{PR}_0$ given that the side condition $\text{cond}$ holds.

A cyclic proof is a proof tree $T_i$ which is a tuple $(V, E, C)$ where

- $V$ is a finite set of nodes representing entailments derived during the proof search;
- A directed edges $(e, \text{PR}, e') \in E$ (where $e'$ is a child of $e$) means that the premise $e'$ is derived from the conclusion $e$ via inference rule $\text{PR}$. For instance, suppose that the rule $\text{PR}_0$ above has been applied, then the following $n$ edges are generated:
  
  - $(e, \text{PR}_0, e_1)$, ..., $(e, \text{PR}_0, e_n)$;
- $C$ is a partial relation which captures back-links in the proof tree. If $C(e_c \rightarrow e_b, \sigma)$ holds, then $e_b$ is linked back to its ancestor $e_c$ through the substitution $\sigma$ (where $e_b$ is referred as a bud and $e_c$ is referred as a companion). In particular, $e_c$ is of the form: $\Delta \vdash \Delta'$ and $e_b$ is of the form: $\Delta_1 \land \pi \vdash \Delta'_1$ where $\Delta \equiv \Delta_1 \sigma$ and $\Delta' \equiv \Delta'_1 \sigma$.

A leaf node is marked as closed if it is evaluated as valid (i.e. the node is applied with an axiom) or invalid (i.e. no rule can apply), or it is linked back. Otherwise, it is marked as open. A proof tree is invalid if it contains at least one invalid leaf node. It is a pre-proof if all its leaf nodes are either valid or linked back. A pre-proof is a cyclic proof if a global soundness condition is imposed in the tree. Intuitively, this soundness condition requires that for every $C(e_c \rightarrow e_b, \sigma)$, there exist inductive predicates $P(t_i)$ in $e_c$ and $Q(t_2)$ in $e_b$ such that $Q(t_2)$ is a subterm of $P(t_1)$.

Definition 1 (Trace) Let $T_i$ be a pre-proof of $\Delta_a \vdash \Delta_c$ and $(\Delta_{a_i} \vdash \Delta_{c_i})_{i \geq 0}$ be a path of $T_i$. A trace following $(\Delta_{a_0} \vdash \Delta_{c_0})_{i \geq 0}$ is a sequence $(\alpha_i)_{i \geq 0}$ such that each $\alpha_i$ (for all $i \geq 0$) is a subformula of $\Delta_{a_i}$, containing predicate $P(t_i)^n$ and either:

- $\alpha_{i+1}$ is the subformula occurrence in $\Delta_{a_{i+1}}$ corresponding to $\alpha_i$ in $\Delta_{a_i}$.
- or $\Delta_{a_i} \vdash \Delta_{c_i}$ is the conclusion of a left-unfolding rule, $\alpha_i \equiv P(t_i)^n$ is unfolded, and $\alpha_{i+1}$ is a subformula in $\Delta_{a_{i+1}}$ and is the definition rule of $P(\bar{x})^n[t/\bar{x}]$. In this case, $i$ is said to be a progressing point of the trace.

Definition 2 (Cyclic proof) A pre-proof $T_i$ of $\Delta_a \vdash \Delta_c$ is a cyclic proof if, for every infinite path $(\Delta_{a_i} \vdash \Delta_{c_i})_{i \geq 0}$ of $T_i$, there is a tail of the path $p=(\Delta_{a_i} \vdash \Delta_{c_i})_{i \geq n}$ such that there is a trace following $p$ which has infinitely progressing points.

Suppose that all proof rules are (locally) sound (i.e., if the premises are valid then the conclusion is valid), the following Theorem shows the global soundness.

Theorem 3.1 (Soundness [5]). If there is a cyclic proof of $\Delta_a \vdash \Delta_c$, then $\Delta_a \models \Delta_c$.

The proof is by contraction and can be found in [5]. Intuitively, if we can derive a cyclic proof for $\Delta_a \vdash \Delta_c$ and $\Delta_a \not\models \Delta_c$, then the inductive predicate at the progress points can be unfolded infinite often. This contradicts with the least semantics of the predicate.

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4 Cyclic Entailment Procedure

In this section, we present our main proposal, the entailment procedure $\omega$-ENT with the proposed inference rules (subsection 4.1), and an illustrative example in subsection 4.2.

4.1 Proof Search

The proof search algorithm $\omega$-ENT is presented in Fig. 1. $\omega$-ENT takes $e_0$ as input, produces cyclic proofs and based on that decides whether the input is valid or invalid. Initially, for every $P(i)^k \in e_0$, $k$ is reset to 0 and $T_0$ only has $e_0$ as an open leaf, the root. The overall idea of $\omega$-ENT is to iteratively reduce $T_0$ into a sequence of cyclic proof trees $T_i$, $i \geq 0$. On line 3, through procedure $\text{is\_closed}(T_i)$, $\omega$-ENT chooses an open leaf node $e_i$ and a proof rule $PR_i$ to apply. If $\text{is\_closed}(T_i)$ returns valid (that is, every leaf is applied to an axiom rule or involved in a backlink), $\omega$-ENT returns valid on line 4. If it returns invalid, then $\omega$-ENT returns invalid (one line 5). Otherwise, it tries to link $e_i$ back to an internal node (on line 6). If this attempt fails, it applies the rule (line 7).

Note that for each leaf, $\text{is\_closed}$ attempts rules in the following order: normalization rules, axiom rules, and reduction rules. A rule $PR_i$ is chosen if its conclusion can be unified with the leaf, through some substitution $\sigma$. Then, on line 7, for each premise of $PR_i$, procedure $\text{apply}$ creates a new open node and connects the node to $e_i$ via a new edge. If $PR_i$ is an axiom, procedure $\text{apply}$ marks $e_i$ as closed and returns.

Procedure $\text{is\_closed}(T_i)$ This procedure examines the following three cases.

1. First, if all leaf nodes are marked closed and none of them is invalid then $\text{is\_closed}$ returns valid.
2. Secondly, $\text{is\_closed}$ returns invalid if there exists an open leaf node $e_i : \Delta \vdash \Delta'$ in NF such that one of the four following conditions holds:
   (a) $e_i$ could not be applied by any inference rule.
   (b) there exists a predicate $op_1(E) \in \Delta$ such that $op_2(E) \notin \Delta'$ and one of the following conditions holds:
      - either $P(E', E, ..., E) \in \Delta$ and $E' \in E \in c(E, ...)$ are in both sides
      - both $P(E', E, ..., E) \notin \Delta$ and $E' \in c(E, ...) \notin \Delta$
   (c) there exists a predicate $op_1(E) \in \Delta'$ such that $G(op_1(E)) \in \Delta$ and $op_2(E) \notin \Delta$.
   (d) there exist $x \rightarrow c_1(v_1) \in \Delta$, $x \rightarrow c_2(v_2) \in \Delta'$ such that $c_1 \neq c_2$ or $v_1 \neq v_2$.
3. Lastly, there exists an open leaf node $e_i$ that could be applied by an inference rule (e.g. $PR_i$), $\text{is\_closed}$ returns the triple (unknown, $e_i$, $PR_i$).

In the rest, we discuss the proof rules and the auxiliary procedures in detail.
and marked as closed. The remaining ones in Fig. 3 are reduction rules. Axiom rules include

\[ \text{If each of these rules is applied into a leaf node, the node is evaluated as} \]

where \( c \) is in normal form if:

1. \( \text{op}(E) \in \kappa \) implies \( G(\text{op}(E)) \in \phi \)
2. \( \text{op}(E) \in \kappa \) implies \( E \neq \text{null} \) \( \in \phi \)
3. \( \text{op}_1(E_1) \ast \text{op}_2(E_2) \in \kappa \) implies \( E_1 \neq E_2 \) \( \in \phi \)
4. \( E_1 = E_2 \notin \phi \)
5. \( E \neq E \notin \phi \)
6. \( a \) is satisfiable

Fig. 2: Normalization rules

Normalization An entailment is in the normal form (NF) if its LHS is in NF. We write \( \text{op}(E) \) to denote for either \( E \rightarrow c(\bar{v}) \) or \( P(E,F,\bar{v}) \). Furthermore, the guard \( G(\text{op}(E)) \) is defined by: \( G(E \rightarrow c(\bar{v})) \) defined \( G(P(E,F,\bar{v})) \) \( \defeq E \neq F \).

Definition 3 (Normal Form) A formula \( \kappa \wedge \phi \wedge a \) is in normal form if:

Inconsistency

Normalization An entailment is in the normal form (NF) if its LHS is in NF. We write \( \text{op}(E) \) to denote for either \( E \rightarrow c(\bar{v}) \) or \( P(E,F,\bar{v}) \). Furthermore, the guard \( G(\text{op}(E)) \) is defined by: \( G(E \rightarrow c(\bar{v})) \) defined \( G(P(E,F,\bar{v})) \) \( \defeq E \neq F \).

Definition 3 (Normal Form) A formula \( \kappa \wedge \phi \wedge a \) is in normal form if:

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2. \( \text{op}(E) \in \kappa \) implies \( E \neq \text{null} \) \( \in \phi \)
3. \( \text{op}_1(E_1) \ast \text{op}_2(E_2) \in \kappa \) implies \( E_1 \neq E_2 \) \( \in \phi \)
4. \( E_1 = E_2 \notin \phi \)
5. \( E \neq E \notin \phi \)
6. \( a \) is satisfiable

If \( \Delta \) is in NF and for any \( s, h \models \Delta \), then \( \text{dom}(h) \) is uniquely defined by \( s \).

The normalisation rules are presented in Fig. 2. Basically, \( \omega \)-ENT applies these rules to a leaf exhaustively and transforms it into NF before other. Given an inductive predicate \( P(E,F,...) \), rule \( \text{ExM} \) excludes the middle by doing case analysis for the predicate between base-case (i.e., \( E = F \)) and recursive-case (i.e., \( E \neq F \)). The normalisation rule \#null follows the following facts: \( E \rightarrow c(\bar{y}) \Rightarrow E \neq \text{null} \) and \( P(E,F,\bar{y}) \Rightarrow E \neq \text{null} \). Similarly, rule \# follows the following facts: \( x \rightarrow y \Rightarrow y \neq y \Rightarrow x \neq y \), and \( P_1(x,F_1,y) \Rightarrow P_2(y,F_2,y) \Rightarrow x \neq y \).

Axiom and Reduction Axiom rules include \( \text{Emp}, \text{Inconsistency} \) and \( \text{Id} \) presented in Fig. 3. If each of these rules is applied into a leaf node, the node is evaluated as valid and marked as closed. The remaining ones in Fig. 3 are reduction rules.

To simplify the presentation, the unfoldings in rules \( \text{Frame}, \text{RInd} \), and \( \text{LInd} \) are applied with the following definition of inductive predicates:

\[ P(x,F,\bar{B},u,s,c,tg) \equiv \text{emp} \wedge F \wedge sc = tg \]

\[ \lor \exists X,s',d_1,d_2,x \rightarrow c(X,d_1,d_2,u,s) \ast \text{Q}_1(d_1,B) \ast \text{Q}_2(d_2,X) \ast P(X,F,\bar{B},u,s',tg) \wedge \pi_0 \]

where \( B \in \bar{B} \), the matrix \( \kappa' \) contains two nested predicates \( Q_1 \) and \( Q_2 \), and the heap cell \( c \in \text{Node} \) is defined as data \( c \{ c_{\text{next}}; c_1 \text{down}_1; c_2 \text{down}_2; \tau_s \text{sedata}; \tau_u \text{udata} \} \)

where \( c_1, c_2 \in \text{Node}, \text{down}_1, \text{down}_2 \) fields are for the nested predicates in the matrix
heaps, *adata* field is for the transitivity data, and *scdata* field are for ordering data. The formalism of these rules for general form of the matrix heaps $\kappa'$ is presented in App. A.

$=\mathbb{R}$ and Hypothesis eliminate pure constraints in the RHS. In rule $\star$, roots($\kappa$) is defined inductively as: roots(emp) := $\{\}$, roots(r→) := $\{r\}$, roots(P(r, F, ..., ) ) := $\{r\}$ and roots($\kappa_1 \cup \kappa_2$) := roots($\kappa_1$) $\cup$ roots($\kappa_2$). This rule is applied in three ways. First, it is applied into an entailment which is of the form $\kappa \land \pi \vdash \kappa \land \pi'$. It matches and discards the identified heap predicates between the two sides so as to generate a premise with empty heaps. As a result, this premise may be applied with the axiom rule Emp. Secondly, it is applied into an entailment whose LHS is a base formula e.g., $x_1 \rightarrow c_1(\bar{v}_i) \land \ldots \land x_n \rightarrow c_n(\bar{v}_n) \land \pi \vdash \kappa' \land \pi'$. For each points-to predicate $x_i \rightarrow c_i(\bar{v}_i) \in \kappa'$, $\omega$-ENT searches for one points-to predicate $x_j \rightarrow c_j(\bar{v}_j)$ in the LHS such that $x_j \rightarrow c_j(\bar{v}_j) \equiv x_i \rightarrow c_i(\bar{v}_i)$. Likewise, for each occurrence of inductive predicates P(r, F, B, u, sc, tg) in the RHS, $\omega$-ENT searches for a points-to predicate r→ in the LHS such that rule RInd could be applied. If any of these searches fails, $\omega$-ENT decides the conclusion as invalid. Lastly, it is applied into an entailment that is of the form $\Delta_1 \ast \Delta \vdash \Delta_2 \ast \Delta'$ where either $\Delta_1 \vdash \Delta_2$ or $\Delta \vdash \Delta'$ could be linked back into an internal node.

Rule LInd unfolds the inductive predicates in the LHS. We notice that every LHS of entailments in this rule also captures the unfolding numbers for subterm relationship and generates the progressing point in the cyclic proofs afterward. These numbers are essential for our system to construct cyclic proofs. This rule is applied in a depth-first manner i.e., if there are more than one occurrences of inductive predicates in the LHS that could be applied by this rule, the one with the greatest unfolding number is chosen.
We emphasize that the last five rules still work well when the predicate in the RHS contains only a subset of the local properties wrt. the predicate in the LHS.

**Back-Link Generation** Procedure link\textsubscript{back} generates a back-link as follows. In a pre-proof, given a path containing a back-link, say \(e_1, e_2, \ldots, e_m\) where \(e_1\) is a companion and \(e_m\) a bud, then \(e_1\) is in NF and of the following form:

- \(e_1 \equiv \text{P}(x, F, \bar{B}, u, sc, tg)\) \(^* k \wedge \pi \wedge x \neq F \wedge x \neq \text{null} \vdash \text{Q}(x, F_2, \bar{B}, u, sc, tg_2)\) \(^* k' \wedge \pi'\).

- \(e_2\) is obtained from applying \text{LInd} into \(e_1\). \(e_2\) is of the form:

\[
x \mapsto \text{c}(x, \bar{p}, u, sc) \equiv \kappa' \equiv \text{P}(x, F, \bar{B}, u, sc', tg) \equiv \kappa'' \equiv \text{P}(x, F, \bar{B}, u, sc', tg_2) \equiv \kappa''' \equiv \text{Q}(x, F_2, \bar{B}, u, sc, tg_2) \equiv \kappa' \wedge \pi' \\
\vdash \text{Q}(x, F_2, \bar{B}, u, sc, tg_2) \equiv \kappa' \wedge \pi'
\]

We remark that \(sc \circ sc' \in \pi_1\) and if \(k \geq 1\) then \(sc \circ sc \in \pi\)

- \(e_3, \ldots, e_{m-4}\) are obtained from applications of normalization rules in order to normalize the LHS of \(e_2\) due to the presence of \(\kappa'\). We note that as the roots of inductive predicates in \(\kappa'\) are fresh variables, the applications of the normalization rules above do not affect the RHS of \(e_2\). That means RHS of \(e_3, \ldots, e_{m-4}\) are the same with the RHS of \(e_2\). As a result, \(e_{m-4}\) is of the form:

\[
x \mapsto \text{c}(x, \bar{p}, u, sc) \equiv \kappa'' \equiv \text{P}(x, F, \bar{B}, u, sc', tg) \equiv \kappa''' \equiv \text{P}(x, F, \bar{B}, u, sc', tg_2) \equiv \kappa'''' \equiv \text{Q}(x, F_2, \bar{B}, u, sc, tg_2) \equiv \kappa' \wedge \pi' \\
\vdash \text{Q}(x, F_2, \bar{B}, u, sc, tg_2) \equiv \kappa' \wedge \pi'
\]

where \(\kappa'''\) may be \text{emp} and \(\pi_2\) is a conjunction of disequalities coming from \text{Ext}\text{M}.

- \(e_{m-3}\) is obtained from application of \text{Ext}\text{M} over \(x\) and \(F_2\) and of the form:

\[
x \mapsto \text{c}(x, \bar{p}, u, sc) \equiv \kappa'' \equiv \text{P}(x, F, \bar{B}, u, sc', tg) \equiv \kappa''' \equiv \text{P}(x, F, \bar{B}, u, sc', tg_2) \equiv \kappa'''' \equiv \text{Q}(x, F_2, \bar{B}, u, sc, tg_2) \equiv \kappa' \wedge \pi' \\
\vdash \text{Q}(x, F_2, \bar{B}, u, sc, tg_2) \equiv \kappa' \wedge \pi'
\]

(For the case \(x=F_2\), the rule \text{Ext}\text{M} is kept applying until either \(F \equiv F_2\), that is two sides are reaching the end of the same heap segment, or it is stuck.)

- \(e_{m-2}\) is obtained from application of \text{RInd} and is of the form:

\[
x \mapsto \text{c}(x, \bar{p}, u, sc) \equiv \kappa'' \equiv \text{P}(x, F, \bar{B}, u, sc', tg) \equiv \kappa''' \equiv \text{P}(x, F, \bar{B}, u, sc', tg_2) \equiv \kappa'''' \equiv \text{Q}(x, F_2, \bar{B}, u, sc, tg_2) \equiv \kappa' \wedge \pi' \\
\vdash \text{Q}(x, F_2, \bar{B}, u, sc, tg_2) \equiv \kappa' \wedge \pi'
\]

- \(e_{m-1}\) is obtained from application of \text{Hypothesis} to eliminate \(\pi_2'\) (otherwise, it is stuck) and is of the form:

\[
x \mapsto \text{c}(x, \bar{p}, u, sc) \equiv \kappa'' \equiv \text{P}(x, F, \bar{B}, u, sc', tg) \equiv \kappa''' \equiv \text{P}(x, F, \bar{B}, u, sc', tg_2) \equiv \kappa'''' \equiv \text{Q}(x, F_2, \bar{B}, u, sc, tg_2) \equiv \kappa' \wedge \pi' \\
\vdash \text{Q}(x, F_2, \bar{B}, u, sc, tg_2) \equiv \kappa' \wedge \pi'
\]

- \(e_m\) is obtained from application of \(\ast\) and is of the form:

\[
\text{P}(x, F, \bar{B}, u, sc', tg) \equiv \kappa' \wedge \pi' \wedge x \neq F \wedge x \neq \text{null} \wedge \pi_1 \wedge \pi_2 \wedge x \neq F_2 \\
\vdash \text{Q}(x, F_2, \bar{B}, u, sc', tg_2) \equiv \kappa' \wedge \pi'
\]

When \(k \geq 1\) it is always possible to link \(e_m\) back to \(e_1\) through the substitution is \(\sigma \equiv [x/X, sc/sc']\) after weakening some pure constraints in its LHS.
4.2 Illustrative Example

We illustrate our system through the following example:

\[ e_0 \vdash \text{lls}(x, \text{null}, mi, ma)^0 \land x \neq \text{null} \vdash \text{llb}(x, \text{null}, mi) \]

where the sorted linked-list \text{lls} (\text{mi} is the minimum value and \text{ma} is the maximum value) is defined in Sect. 2.1 and \text{llb} defines singly-linked lists whose values are greater than or equal to a constant number. Particularly, predicate \text{llb} is defined as follows.

\[
\text{pred llb}(r, F, b) \equiv \text{emp} \lor \exists X, d \rightarrow c_4(X, d) \land (r \neq F \lor b \leq d)
\]

Since the LHS is stronger than the RHS, this entailment is valid. Our system could generate the cyclic proof (shown in Fig. 4) to prove the validity of \( e_0 \). In the following, we present step-by-step to show how the proof was created. Firstly, \( e_0 \), which is in NF, is applied with rule \( \text{LInd} \) to unfold predicate \text{lls}(x, \text{null}, mi, ma)^0 and obtain \( e_1 \) as:

\[
e_1: x \rightarrow c_4(X, m') \land x \neq \text{null} \land mi \leq m' \vdash \text{llb}(x, \text{null}, mi)
\]

We remark that the unfolding number of the recursive predicate \text{lls} in the LHS is increased by 1. Next, our system normalizes \( e_1 \) by applying rule \( \text{ExM} \) into \( X \) and \( \text{null} \) to generate two children \( e_2 \) and \( e_3 \) as follows.

\[
e_2: x \rightarrow c_4(X, m') \land x \neq \text{null} \land mi \leq m' \land X = \text{null} \vdash \text{llb}(x, \text{null}, mi)
\]

\[
e_3: x \rightarrow c_4(X, m') \land x \neq \text{null} \land mi \leq m' \land X \neq \text{null} \vdash \text{llb}(x, \text{null}, mi)
\]

For the left child, it applies normalization rules to obtain

\[
e_4: x \rightarrow c_4(\text{null}, m') \land x \neq \text{null} \land mi \leq m' \vdash \text{llb}(x, \text{null}, mi)
\]

\[
e_5: x \rightarrow c_4(\text{null}, ma) \land x \neq \text{null} \land mi \leq ma \vdash \text{llb}(x, \text{null}, mi)
\]
Now, $e_5$ is in NF. $S_{2\omega_{1\text{u}}}$ applies $\text{RInd}$ and then $\text{RBase}$ to $llb$ in the RHS as:

$$e_6: x\rightarrow c_4(\text{null, } ma) \land x\neq \text{null} \land mi \leq ma$$

$$\vdash x\rightarrow c_4(\text{null, } ma) \land x\neq \text{null} \land mi \leq ma \vdash x\rightarrow c_4(\text{null, } ma) \land mi \leq ma$$

After that, as $mi \leq ma \Rightarrow mi \leq ma$, $e_6$ is applied with Hypothesis to obtain $e_7$.

$$e_7: x\rightarrow c_4(\text{null, } ma) \land x\neq \text{null} \land mi \leq ma \vdash x\rightarrow c_4(\text{null, } ma)$$

As the LHS of $e_7$ is in NF and a base formula, it is sound and complete to apply rule $*$ to have $e_8$ as: $\text{emp} \land x\neq \text{null} \land mi \leq ma \vdash \text{emp}$. By $\text{Emp}$, $e_8$ is decided as valid. For the right branch of the proof, $e_3$ is applied with rule $\neq^*$ and then $\text{RInd}$ to obtain $e_9$:

$$e_9: x\rightarrow c_4(X, m') \land llb(X, \text{null, } m', ma)^1 \land x\neq \text{null} \land mi \leq m' \land X \neq \text{null} \land x \neq X$$

$$\vdash x\rightarrow c_4(X, m') \land llb(X, \text{null, } mi)^1 \land mi \leq m'$$

After that, $e_9$ is applied with Hypothesis to eliminate the pure constraint in the RHS:

$$e_{10}: x\rightarrow c_4(X, m') \land llb(X, \text{null, } m', ma)^1 \land x\neq \text{null} \land mi \leq m' \land X \neq \text{null} \land x \neq X$$

$$\vdash x\rightarrow c_4(X, m') \land llb(X, \text{null, } mi)$$

$e_{10}$ is then applied with $*$ to obtain $e_{11}$ and $e_{12}$ as follows.

$$e_{11}: x\rightarrow c_4(X, m')$$

$$e_{12}: llb(X, \text{null, } m', ma)^1 \land x\neq \text{null} \land mi \leq m' \land X \neq \text{null} \land x \neq X \vdash llb(X, \text{null, } mi)$$

$e_{11}$ is valid by I.d. $e_{12}$ is successfully linked back to $e_{10}$ to form a pre-proof as

$$(llb(X, \text{null, } m', ma)^1 \land X \neq \text{null})[x/X, mi/m'] \vdash llb(X, \text{null, } mi)[x/X, mi/m']$$

is identical to $e_6$. Since $llb(X, \text{null, } m', ma)^1$ in $e_{12}$ is the subterm of $llb(X, \text{null, } mi, ma)^0$ in $e_6$, our system decided that $e_6$ is valid with the cyclic proof presented in Fig. 4.

\section{Soundness, Completeness, and Complexity}

We describe the soundness, termination, and completeness of $\omega$-ENT. First, we need to show the invariant about the quantifier-free entailments of our system.

\textbf{Corollary 5.1.} Every entailment derived from $\omega$-ENT is quantifier-free.

The following lemma shows the soundness of the proof rules.

\textbf{Lemma 5.2 (Soundness).} For each proof rule, if all premises are valid, then the conclusion is valid.

As every backlink generated contains at least one pair of inductive predicate occurrences in a subterm relationship, the global soundness condition holds in our system.

\textbf{Lemma 5.3 (Global Soundness).} A pre-proof derived is indeed a cyclic proof.
The termination relies on the number of premises/entailments generated by rule ∗. As the number of inductive symbols and their arities are finite, there is a finite number of equivalent classes of these entailments in which any two entailments in the same class are equivalent under some substitution and linked back together. Therefore, the number of premises generated by rule ∗ is finite considering the generation of backlinks.

Lemma 5.4. \( \omega \text{-ENT} \) terminates.

In the following, we show the complexity analysis. First, we show that every occurrence of inductive predicates in the LHS is unfolded at most two times.

Lemma 5.5. Given any entailment \( P(\bar{v})^k \triangleleft \Delta_c \), then \( 0 \leq k \leq 2 \).

Let \( n \) be the maximum number of predicates (both inductive predicates and points-to predicates) among the LHS of the input and the definitions in \( P \), and \( m \) be the maximum number of fields of data structures. Then, the complexity is defined as follows.

Proposition 5.6 (Complexity). \( \mathbb{QF} \text{-ENT} \rightarrow \mathbb{SL}_{\text{LIN}} \) is \( \mathcal{O}(n \times 2^m + n^3) \).

As such, if \( m \) is bounded by a constant, the complexity becomes polynomial in time.

Our completeness proofs are shown in two steps. First, we show the proofs for an entailment whose LHS is a base formula. Second, we show the correctness when the LHS contains inductive predicates. In the following, we first define the base formulas of the LHS of an entailment are obtained once every inductive predicate has been unfolded once.

Lemma 5.7. If \( \kappa \land \pi \) is in NF then \( \kappa \land \pi \) is in NF, and \( \kappa \land \pi \vdash \Delta \) is valid.

In other words, \( \kappa \land \pi \) is an under-approximation of \( \kappa \land \pi \); invalidity of \( \kappa \land \pi \vdash \Delta' \) implies invalidity of \( \kappa \land \pi \vdash \Delta' \).

Definition 5 (Bad Model) The bad model for \( \kappa \land \phi \land a \) in NF is obtained by assigning

- a distinct non-null value to each variable in \( \text{FV}(\kappa \land \phi) \); and
- a value to each variable in \( \text{FV}(a) \) such that \( a \) is satisfiable.

Lemma 5.8. 1. For every proof rule except rule ∗, all premises are valid only if the conclusion is valid.

2. For rule ∗ where the conclusion is of the form \( \Delta^b \vdash \kappa' \), all premises are valid only if the conclusion is valid and \( \Delta^b \) is in NF.
The following lemma states the correctness of the procedure \texttt{is\_closed} for cases 2(b-d).

**Lemma 5.9 (Stuck Invalidity).** Given $\kappa \land \pi \vdash \Delta'$ in NF, it is invalid if procedure \texttt{is\_closed} returns invalid for cases 2(b-d).

A bad model of the $\pi \land \pi$ is a counter-model. Cases 2b) and 2c) show that the heaps of bad models are not connected and thus accordingly to conditions C1 and C2, any model of the LHS could not be a model of the RHS. Case 2d) shows that heaps of the two sides could not be matched. Now, we show the correctness of Case 2(a) of procedure \texttt{is\_closed} and invalidity is preserved during the proof search in $\omega$-ENT.

**Proposition 5.10 (Invalidity Preservation).** If $\omega$-ENT is stuck, the input is invalid.

**Theorem 5.11.** QF\_ENT$\dashv$SL\_LIN is decidable.

## 6 Implementation and Evaluation

We implement S2S\_Lin using OCaml. This implementation is an instantiation of a general framework for cyclic proofs. To discharge satisfiability for a separation logic formula, we utilize the cyclic proof systems to derive bases for inductive predicates in the decidable fragment shown in \cite{24}. For those formulas beyond this fragment, we use the solver presented in \cite{27,29}. We also develop a built-in solver for discharging equalities.

We evaluated S2S\_Lin to show that i) it can discharge problems in SHLI\_de effectively; and ii) its performance is compatible to the state-of-the-art solvers.

### Experiment settings

We have evaluated S2S\_Lin on entailment problems taken from SL-COMP 2022 \cite{36}, a competition of separation logic solvers. We take the problems in two divisions of the SL-COMP 2022, \texttt{qf\_shls\_entl} and \texttt{qf\_shlid\_entl}, and one new division \texttt{qf\_shlid2\_entl}. All these problems semantically belongs to our decidable fragment and their syntax are written in SMT 2.6 format \cite{37}.

- Division \texttt{qf\_shls\_entl} includes 296 entailment problems, 122 invalid problems and 174 valid problems, with only singly linked lists. They were randomly generated by the authors in \cite{31}.
- Division \texttt{qf\_shlid\_entl} contains 60 entailment problems which were mostly handcrafted by the authors in \cite{15}. They include singly-linked lists, doubly-linked lists, lists of singly-linked lists or skip lists. Furthermore, the system of inductive predicates must satisfy the following condition: For two different predicates $P, Q$ in the system of definitions, either $P \prec^* Q$ or $Q \prec^* P$.
- In the third division, we introduce new benchmarks, with 27 problems, that are beyond the problems in the previous two divisions. In particular, in every system of predicate definitions, there exist two predicates $P, Q$ such that they are semantically equivalent. We have submitted this division to the Github repository of SL-COMP.

To evaluate S2S\_Lin’s performance, we compared it with the state-of-the-art tools such as Cyclist\_SL \cite{5}, Spen \cite{15}, Songbird \cite{38}, SLS \cite{39} and Harrsh \cite{23}. We did not include Cycomp \cite{40}, as these benchmarks are beyond its decidable fragment. Note
Table 1: Experimental results

| Tool         | $qf_{shls\	ext{entl}}$ | $qf_{shlid\	ext{entl}}$ | $qf_{shlid2\	ext{entl}}$ |
|--------------|--------------------------|---------------------------|---------------------------|
|              | invalid                  | valid                     | invalid                   |
|              | (122)                    | (174)                     | (296)                     |
| SLS          | 12                       | 174                       | 507m42s                   |
| Spen         | 122                      | 174                       | 10.78s                    |
| CyclistSL    | 0                        | 58                        | 520m5s                    |
| Harrsh       | 39                       | 116                       | 425m19s                   |
| Songbird     | 12                       | 174                       | 237m25s                   |
| S2S Lin      | 122                      | 174                       | 6.22s                     |

|              | Time                      | Time                      | Time                      |
|--------------|---------------------------|---------------------------|---------------------------|
|              | (m)                      | (s)                       | (m)                      |
| SLS          | 507m42s                   | 10.78s                    | 533m28s                   |
| Spen         | 10.78s                    | 3.44s                     | 82                        |
| CyclistSL    | 520m5s                    | 425m19s                   | 53m56s                    |
| Harrsh       | 237m25s                   | 240m38s                   | 47m11s                    |
| Songbird     | 6.22s                     | 1.20s                     | 1.69s                     |

that CyclistSL, Songbird and SLS are not complete; for non-valid problems, while CyclistSL returns unknown, Songbird, and SLS use some heuristic to guess the outcome. For each division, we report the number of correct outputs (invalid, valid) and the time (in minutes and seconds) taken by each tool. Note that we use the status (invalid, valid) annotated with each problem in the SL-COMP benchmark as the ground truth. If an output is the same with the status, we classify it as correct; otherwise, it is marked as incorrect. We also note that in these experiments we used the competition pre-processing tool [37] to transform the SMT 2.6 format into the corresponding formats of the tools before running them. All experiments were performed on a machine with Intel Core i7-6700 CPU 3.4Gh and 8GB RAM. The CPU timeout is 600 seconds.

*Experiment results* The experimental results are reported in Table 1. In this table, the first column presents the names of the tools. The next three columns show the results of the first division including the number of correct *invalid* outputs, the number of correct *valid* outputs and the time taken (where *m* for minutes and *s* for seconds), respectively. In the third row, the number between each pair of brackets (...) shows the number of problems in the corresponding column. Similarly, the next two groups of six columns describe the results of the second and third divisions, respectively.

In general, the experimental results show that S2S Lin is the one (and only one) that could produce all the correct results. Other solvers either produced wrong results or could discharge a fraction of the experiments. Moreover, S2S Lin took a short time for the experiments (8.38 seconds compared to 15.91 seconds for Spen, 324 minutes for Songbird, 635 minutes for Harrsh, 739 minutes for SLS and 2120 minutes for CyclistSL). While SLS returned 14 false negatives, Spen reported 20 false positives. CyclistSL, Songbird and Harrsh did not produce any wrong result. Of 569 tests, while CyclistSL could handle 85 tests (15%), Harrsh could handle 215 tests (38%) and Songbird could decide 235 tests (41.3%). In total of 223 valid tests, while CyclistSL could handle 85 problems (38%), Songbird could decide 222 problems (99.5%).

Now we examine the results for each division in details. For $qf_{shls\text{entl}}$, Spen returned all correct, Songbird 186, Harrsh 155, and CyclistSL 58. If we set the timeout to 2400 seconds, both Songbird and Harrsh produced all the correct results. For division $qf_{shlid\text{entl}}$ includes 24 invalid problems and 36 valid problems. While Songbird produced 37 problems correctly, CyclistSL produced 24 correct results. Spen reported 27 correct results and 13 false positives ($skl2$−$vc\{01−04\}$ $skl3$−$vc01$, $skl3$−$vc\{03−...
For the last division \texttt{qf\_shlid2\_entl} includes 14 invalid test problems and 13 valid test problems. While Songbird decided only 12 problems correctly, Cyclist \texttt{sl} produced 3 correct outcomes. Spen reported 10 correct results. However, it produced 7 false positives (\texttt{ls\_mul\_vc\{01 - 03\}}, \texttt{ls\_mul\_vc05}, \texttt{nll\_mul\_vc\{01 - 03\}}).

Since our experiments provide break-down results of the two divisions of SL-COMP competition, we hope that they provide an initial understanding of the SL-COMP benchmarks and tools. Consequently, this might reduce the effort to prepare experiments over these benchmarks to evaluate new SL solvers. Finally, one might point out that S2S\texttt{lin} performed well because the entailments in the experiments are within its scope. We do not totally disagree with this argument, but would like to emphasize that tools do not always work well on favorable benchmarks. For example, Spen introduced wrong results on \texttt{qf\_shlid\_entl}, and Harrsh did not handle \texttt{qf\_shlid\_entl} and \texttt{qf\_shlid2\_entl} well although these problems are in their decidable fragments. We believe that engineering design and effort play an important role along side with theory development.

7 Related Work

S2S\texttt{lin} is a variant of the cyclic proof systems \cite{3,4,26} and \cite{40}. Unlike existing cyclic proof systems, the soundness of S2S\texttt{lin} is local, and the proof search is not backtracking. The work presented in \cite{40} shows the completeness of the cyclic proof system. Its main contribution is the introduction of rule \texttt{*} for those entailments with disjunction in the RHS obtained from predicate unfolding. In contrast to \cite{40}, our work includes normalization to soundly and completely avoid disjunction in the RHS during unfolding. Our work also presents how to obtain the global soundness condition for cyclic proofs. Moreover, our decidable fragment SHLIDe is non-overlapping to the cone predicates introduced in \cite{40}. Furthermore, due to the empty heap in the base cases, the matching rule in \cite{40} cannot be applied to the predicates in SHLIDe.

Our work relates to the inductive theorem provers introduced in \cite{10}, \cite{38} and Smallfoot \cite{2}. While \cite{10} is based on structural induction, \cite{38} is based on mathematical induction. Smallfoot \cite{2} proposed a decision procedure for a fragment with linked lists and trees (and without arithmetic). To handle inductive entailments, this system made use of a fixed compositional rule as consequences of induction reasoning. This technique was further explored by the authors in \cite{31}. Compared with Smallfoot, our proof system replaces the compositional rule by the combination of rule \texttt{LInd} and the back-link construction. In doing so, our system could support induction reasoning on a much more expressive fragment of inductive predicates.

Our proposal also relates to works that use lemmas as consequences of induction reasoning \cite{2,16,28,39}. These works in \cite{16,25,28,39} automatically generate lemmas for some classes of inductive predicates. S2 \cite{25} generated lemmas to normalize (such as split, equivalence) the shapes of the synthesized data structures. \cite{16} proposed to generate several sets of lemmas not only for compositional predicates, but also for different predicates (e.g., completion lemmas, stronger lemmas and static parameter contraction lemmas). To prove an entailment, SLS \cite{39} aims to infer general lemmas. Similarly, S2ENT \cite{28} solves a more generic problem, frame inference, using cyclic proofs and lemma synthesis. It first infers shape-based residual frame in the LHS and then synthe-
sizes the pure constraints over the two sides. It would be a future work to integrate the pure constraint synthesis into $S2S_{\text{Lin}}$ to support non-local pure properties.

$S2S_{\text{Lin}}$ relates to model-based decision procedures that reduce the entailment problem in separation logic to a well-studied problem in other domains. For instance, in [8,11,17] the entailment problem including singly-linked lists and their invariants is reduced to the problem of inclusion checking in a graph theory. The authors in [18] reduced the entailment problem to the satisfiability problem in second-order monadic logic. This reduction could handle an expressive fragment of spatial-based predicates, called bounded-tree width. Recently, the work presented in [23] show a model-based decision procedure for a subfragment of the bounded-tree width. Furthermore, while the work in [15,19] reduced the entailment problem to the tree automata inclusion checking problem, [21] presented an idea to reduce the problem to the heap automata inclusion checking problem. Moreover, while the procedure in [15] supported well compositional predicates (single and double links), the procedure in [19] could handle predicates satisfying local properties (e.g., trees with parent pointers). Our decidable fragment subsumes the one described in [2,11,15] but is incompatible to the ones presented in [8,17,18,19]. Works in [32] and [33,34] reduced the entailment problem in separation logic into the satisfiability problem in SMT. While GRASShoper [33,34] could handle transitive closure pure properties, $S2S_{\text{Lin}}$ is capable of supporting local ones. Unlike GRASShoper, which reduces entailment into SMT problems, $S2S_{\text{Lin}}$ reduces an entailment to admissible entailments and detects repetitions via cyclic proofs.

Our work relates to decidable fragments and complexity results of the entailment problem in separation logic with inductive predicates. The entailment is 2-EXPTIME in cone predicates [40], the bounded tree width predicates and beyond [18,14], and EXPTIME in a sub-fragment of cone predicates [19]. In the other class, entailment is in polynomial time for singly-linked lists [11], semantically linear inductive predicates [15], and its extensions with arithmetic [17] (but becomes EXPTIME when the lists are extended with double links [8]). Our fragment (with nested lists, trees and arithmetic properties) is roughly in the “middle” of the two classes above where the entailment is EXPTIME and becomes polynomial under the upper bound restriction.

8 Conclusion

We have presented a novel decision procedure for the quantifier-free entailment problem in separation logic combining with inductive definitions of compositional predicates and pure properties. Our proposal is the first complete cyclic proof system for the problem in separation logic without back-tracking. We have implemented the proposal in $S2S_{\text{Lin}}$ and evaluated it over the set of nontrivial entailments taken from the SL-COMP competition. The experimental results show that our proposal is both effective and efficient when compared against the state-of-the-art solvers.

For future work, we plan to combine this proposal with the cyclic frame inference procedure presented in [28] for a bi-abductive procedure. This is a basic step to obtain a compositional shape analysis beyond the lists and trees. Another work is to formally prove that our system is as strong as Smallfoot in the decidable fragment with lists and trees [2]: Given an entailment, if Smallfoot can produce a proof, so is $S2S_{\text{Lin}}$. 

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References

1. Timos Antonopoulos, Nikos Gorogiannis, Christoph Haase, Max Kanovich, and Joël Ouaknine. Foundations for decision problems in separation logic with general inductive predicates. In Anca Muscholl, editor, Foundations of Software Science and Computation Structures, pages 411–425, Berlin, Heidelberg, 2014. Springer Berlin Heidelberg.

2. J. Berdine, C. Calcagno, and P. W. O’Hearn. Symbolic Execution with Separation Logic. In APLAS, volume 3780, pages 52–68, November 2005.

3. J. Brotherston. Cyclic proofs for first-order logic with inductive definitions. In Proceedings of TABLEAUX-14, volume 3702 of LNAI, pages 78–92. Springer-Verlag, 2005.

4. J. Brotherston, N. Gorogiannis, and R. L. Petersen. A generic cyclic theorem prover. In Proceedings of APLAS-10, LNCS, pages 350–367. Springer, 2012.

5. James Brotherston, Dino Distefano, and Rasmus Lerchedahl Petersen. Automated cyclic entailment proofs in separation logic. In Proceedings of the 23rd International Conference on Automated Deduction, CADE’11, page 131–146, Berlin, Heidelberg, 2011. Springer-Verlag.

6. Cristiano Calcagno, Dino Distefano, Jeremy Dubreil, Dominik Gabi, Pieter Hooimeijer, Martino Luca, Peter O’Hearn, Irene Papakonstantinou, Jim Purbrick, and Dulma Rodriguez. Moving fast with software verification. In Klaus Havelund, Gerard Holzmann, and Rajeev Joshi, editors, NASA Formal Methods, pages 3–11, Cham, 2015. Springer International Publishing.

7. Cristiano Calcagno, Dino Distefano, Peter W. O’Hearn, and Hongseok Yang. Compositional shape analysis by means of bi-abduction. In POPL, pages 289–300, 2009.

8. Taolue Chen, Fu Song, and Zhilin Wu. Tractability of Separation Logic with Inductive Definitions: Beyond Lists. In Roland Meyer and Uwe Nestmann, editors, 28th International Conference on Concurrency Theory (CONCUR 2017), volume 85 of Leibniz International Proceedings in Informatics (LIPIcs), pages 37:1–37:17, Dagstuhl, Germany, 2017. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

9. W.-N. Chin, C. Gherghina, R. Voicu, Q.-L. Le, F. Craciun, and S. Qin. A specialization calculus for pruning disjunctive predicates to support verification. In CA V. 2011.

10. Duc-Hiep Chu, Joxan Jaffar, and Minh-Thai Trinh. Automatic induction proofs of data-structures in imperative programs. In Proceedings of PLDI, PLDI ’15, pages 457–466, New York, NY, USA, 2015. ACM.

11. B. Cook, C. Haase, J. Ouaknine, M. Parkinson, and J. Worrell. Tractable reasoning in a fragment of separation logic. In CONCUR, volume 6901, pages 235–249, 2011.

12. Christopher Curry, Quang Loc Le, and Shengchao Qin. Bi-abductive inference for shape and ordering properties. In 2019 24th International Conference on Engineering of Complex Computer Systems (ICECCS), pages 220–225, 2019.

13. Dino Distefano, Manuel Fähndrich, Francesco Logozzo, and Peter W. O’Hearn. Scaling static analyses at facebook. Commun. ACM, 62(8):62–70, Jul 2019.

14. Mnacho Echenim, Radu Iosif, and Nicolas Peltier. Unifying decidable entailments in separation logic with inductive definitions. In Automated Deduction-CADE 28-28th International Conference on Automated Deduction, Virtual Event, July 12-15, 2021, Proceedings, pages 183–199, 2021.

15. Constantin Enea, Ondrej Lengál, Mihaela Sighireanu, and Tomáš Vojnar. Compositional entailment checking for a fragment of separation logic. Formal Methods in System Design, 51(3):575–607, 2017.

16. Constantin Enea, Mihaela Sighireanu, and Zhilin Wu. On automated lemma generation for separation logic with inductive definitions. ATVA, 2015.

17. Xincan Gu, Taolue Chen, and Zhilin Wu. A Complete Decision Procedure for Linearly Compositional Separation Logic with Data Constraints, pages 532–549. Springer International Publishing, Cham, 2016.
18. R. Iosif, A. Rogalewicz, and J. Simácek. The tree width of separation logic with recursive definitions. In CADE, pages 21–38, 2013.
19. Radu Iosif, Adam Rogalewicz, and Tomáš Vojnar. Deciding entailments in inductive separation logic with tree automata. ATVA, 2014.
20. S. Işıtaq and P.W. O’Hearn. BI as an assertion language for mutable data structures. In ACM POPL, pages 14–26, London, January 2001.
21. Christina Jansen, Jens Katelaan, Christoph Matheja, Thomas Noll, and Florian Zuleger. Unified Reasoning About Robustness Properties of Symbolic-Heap Separation Logic, pages 611–638. Springer Berlin Heidelberg, Berlin, Heidelberg, 2017.
22. Katelaan Jens, Jovanovic Dejan, and Weissenebacher Georg. A separation logic with data: Small models and automation. In IJCAI, 2018.
23. Jens Katelaan, Christoph Matheja, and Florian Zuleger. Effective entailment checking for separation logic with inductive definitions. In Tomáš Vojnar and Lijun Zhang, editors, Tools and Algorithms for the Construction and Analysis of Systems, pages 319–336, Cham, 2019. Springer International Publishing.
24. Quang Loc Le. Compositional satisfiability solving in separation logic. In Fritz Henglein, Sharon Shoham, and Yakir Vizel, editors, Verification, Model Checking, and Abstract Interpretation, pages 578–602, Cham, 2021. Springer International Publishing.
25. Quang Loc Le, Cristian Gherghina, Shengchao Qin, and Wei-Ngan Chin. Shape analysis via second-order bi-abduction. In CAV, volume 8559, pages 52–68, 2014.
26. Quang Loc Le and Mengda He. A decision procedure for string logic with quadratic equations, regular expressions and length constraints. In Sukyoung Ryu, editor, Programming Languages and Systems, pages 350–372, Cham, 2018. Springer International Publishing.
27. Quang Loc Le, Jun Sun, and Wei-Ngan Chin. Satisfiability modulo heap-based programs. In CAV, 2016.
28. Quang Loc Le, Jun Sun, and Shengchao Qin. Frame inference for inductive entailment proofs in separation logic. In Dirk Beyer and Marieke Huisman, editors, Tools and Algorithms for the Construction and Analysis of Systems, pages 41–60, 2018.
29. Quang Loc Le, Makoto Tatsuta, Jun Sun, and Wei-Ngan Chin. A decidable fragment in separation logic with inductive predicates and arithmetic. In CAV, pages 495–517, 2017.
30. Scott McPeak and George C. Necula. Data structure specifications via local equality axioms. In Kousha Etessami and Sriram K. Rajamani, editors, Computer Aided Verification, pages 476–490, Berlin, Heidelberg, 2005. Springer Berlin Heidelberg.
31. Juan Antonio Navarro Pérez and Andrey Rybalchenko. Separation logic + superposition calculus = heap theorem prover. In Proceedings of the 32nd ACM SIGPLAN Conference on Programming Language Design and Implementation, PLDI ’11, page 556–566, New York, NY, USA, 2011. Association for Computing Machinery.
32. Juan Antonio Navarro Pérez and Andrey Rybalchenko. Separation logic modulo theories. In APLAS, volume 8301, pages 90–106, 2013.
33. R. Piskac, T. Wies, and D. Zufferey. Automating separation logic using smt. In Natasha Sharygina and Helmut Veith, editors, CAV, volume 8044, pages 773–789, 2013.
34. Ruzica Piskac, Thomas Wies, and Damien Zufferey. Automating separation logic with trees and data. In CAV, volume 8559, pages 711–728, 2014.
35. J. Reynolds. Separation Logic: A Logic for Shared Mutable Data Structures. In IEEE IICS, pages 55–74, 2002.
36. Mihaela Sighireanu and Quang Loc Le. SL-COMP 2022. https://sl-comp.github.io/, 2022. [Online; accessed Jun-2022].
37. Mihaela Sighireanu, Juan Antonio Navarro Pérez, Andrey Rybalchenko, Nikos Gorogiannis, Radu Iosif, Andrew Reynolds, Cristina Serban, Jens Katelaan, Christoph Matheja, Thomas Noll, Florian Zuleger, Wei-Ngan Chin, Quang Loc Le, Quang-Trung Ta, Ton-Chanh Le,
Thanh-Toan Nguyen, Siu-Cheng Khoo, Michal Cyprian, Adam Rogalewicz, Tomáš Vojnar, Constantin Enea, Ondrej Lengál, Chong Gao, and Zhilin Wu. SL-COMP: competition of solvers for separation logic. In *Tools and Algorithms for the Construction and Analysis of Systems - 25 Years of TACAS: TOOLympics*, pages 116–132, 2019.

38. Quang-Trung Ta, Ton Chanh Le, Siu-Cheng Khoo, and Wei-Ngan Chin. Automated mutual explicit induction proof in separation logic. In John Fitzgerald, Constance Heitmeyer, Stefania Gnesi, and Anna Philippou, editors, *FM 2016: Proceedings*, pages 659–676, 2016.

39. Quang-Trung Ta, Ton Chanh Le, Siu-Cheng Khoo, and Wei-Ngan Chin. Automated lemma synthesis in symbolic-heap separation logic. *POPL*, 2018.

40. Makoto Tatsuta, Koji Nakazawa, and Daisuke Kimura. Completeness of cyclic proofs for symbolic heaps with inductive definitions. In Anthony Widjaja Lin, editor, *Programming Languages and Systems*, pages 367–387, Cham, 2019. Springer International Publishing.
\[
\sigma = \{ \bar{v}_i/\bar{p}_i \mid \bar{p}_i \in \bar{w} \land \bar{p}_i \neq \text{null} \}
\]

\[
\begin{array}{c}
x \mapsto c(\bar{v}) \ast \kappa_1 \land \pi_1 \land x \neq F \vdash (\exists (\bar{w} \setminus \bar{p}) \cdot x \mapsto c(\bar{p}) \ast \kappa' \ast P(w,F,\bar{B},u,sc',tg) \land \pi_0) \sigma \ast \kappa_2 \land \pi_2 \\
\end{array}
\]

\[
\begin{array}{c}
x \mapsto c(\bar{v}) \ast \kappa_1 \land \pi_1 \land x \neq F \vdash P(x,F,\bar{B},u,sc,tg) \ast \kappa_2 \land \pi_2 \\
\end{array}
\]

\[
\begin{array}{c}
\vdash (\forall (\bar{w} \setminus \bar{p}) \cdot x \mapsto c(\bar{p}) \ast \kappa' \ast P(w,F,\bar{B},u,sc',tg) \land \pi_0) \ast \kappa_2 \land \pi_2 \\
\end{array}
\]

\[
\begin{array}{c}
\vdash Q(x,F_3,\bar{B},u,sc_2,tg) \ast \kappa_2 \land \pi_2 \\
\end{array}
\]

\[
\begin{array}{c}
\vdash RInd \\
\end{array}
\]

\[
\begin{array}{c}
LInd \\
\end{array}
\]

**Fig. 5: Reduction Rules**

Where \( \vdash x \mapsto c(\bar{v}) \notin \kappa_2 \)

**A Reduction Rules for Compositional Predicates in General Form**

In Figure 5, we present rules \( \text{RInd} \) and \( \text{LInd} \) for the following definitions of compositional predicates:

\[
P(x,F,\bar{B},u,sc,tg) \equiv \text{emp} \land x = F \land sc = tg \lor \exists \bar{w}. x \mapsto c(\bar{p}) \ast \kappa' \ast P(w,F,\bar{B},u,sc',tg) \land \pi_0;
\]

where \( \bar{w} \) are fresh variables.

To define \( \text{SHLDe} \) base in a general form, we further assume every heap cells \( c_i \in \text{Node} \) used in definitions of compositional predicates \( P(r,F,\bar{B},u,sc,tg) \) are defined in the form of data \( c_i \{ c_i \text{ next}; c_i \text{ down}_1; \ldots; c_i \text{ down}_j; \tau_u \text{ udata}; \tau_s \text{ scdata} \} \) where \( c_{i1}, \ldots, c_{ij} \in \text{Node}, \text{ down}_1, \ldots, \text{ down}_j \) fields are for the nested structures in the matrix heaps, \( \text{udata} \) field is for the transitivity data, and \( \text{scdata} \) field are for ordering data.

Then, \( \text{SHLDe} \) base of an occurrence of the compositional predicates is defined as:

\[
P(E,F,\bar{B},u,sc,tg) \overset{\text{def}}{=} E \mapsto (\bar{d},tg,u)[\bar{v}/\bar{d}] \land \pi_0[\text{tg/scd}] \ast \kappa'([\bar{v}/\bar{d}] \circ [\text{tg/scd}])
\]

**B Proof of Corollary 5.1**

*Proof.* We need to show that the premises in rule \( \text{RInd} \) and rule \( \text{LInd} \) are quantifier-free. The condition \( C1 \) in section 2.1 ensures that \( \bar{w} \subseteq \bar{p} \). Hence, \( \bar{w} \setminus \bar{p} \equiv \emptyset \). Thus, the RHS of the premise in \( \text{RInd} \) and the LHS of the premise in \( \text{LInd} \) are quantifier-free.

**C Proof of Soundness**

We show the correctness of the soundness of the proof system.

**C.1 Soundness of proof rules: Lemma 5.2**

For each rule, we show that if all the premises hold, so is the conclusion
Rule $\text{Subst}$ . First, we consider the case $E$ is a variable. Suppose $\Delta[v/x] \vdash \Delta'[v/x]$. That is for any $s, h$, if $s, h \models \Delta[v/x]$ then $s, h \models \Delta'[v/x]$. As $x \notin \text{FV}(\Delta[v/x])$, we extend the domain of stack with $x$ as: $s' = s[x \rightarrow s(v)]$. As so, $s', h \models \Delta \land v = x$ and $s', h \models \Delta'$. Therefore $\Delta \land v = x \models \Delta'$ holds.

The case $E$ is $\text{null}$ is similar.

Rule $\text{ExM}$ For simplicity, we assume that $E_1$ and $E_2$ are both variables. Suppose $\Delta \land v_1 = v_2 \vdash \Delta'$ and $\Delta \land v_1 \neq v_2 \vdash \Delta'$.

Suppose $s, h \models \Delta$.

- Case 1: if $s(v_1) = s(v_2)$ then $s, h \models \Delta \land v_1 = v_2$. As $\Delta \land v_1 = v_2 \vdash \Delta'$, $s, h \models \Delta'$.
- Case 1: if $s(v_1) \neq s(v_2)$ then $s, h \models \Delta \land v_1 \neq v_2$. As $\Delta \land v_1 \neq v_2 \vdash \Delta'$, $s, h \models \Delta'$.

Rule $\rightarrow_L$, $\rightarrow_R$, and rule Hypothesis Trivial.

Rule $\text{LBase}$ and rule $\text{RBase}$ based on the fact that given a compositional predicate $P(E,F,B,u,sc,tg)$ where $F$ is a dangling pointer, then $P(E,E,B,u,sc,tg)$ implies the base rule with $\text{emp}$ heap predicate.

Rule $\neq \text{null}$ Follows semantics of points-to predicate where $\text{null} \notin \text{Loc}$.

Rule $\neq$ Follows semantics of the spatial conjunction $\ast$.

Rule $\ast$ Suppose $\kappa_1 \land \pi \models \kappa_2 \land \pi'$ and $\kappa \land \pi \models \kappa' \land \pi'$.

For any $s, h_1 \models \kappa_1 \land \pi$, $s, h_1 \models \kappa_2 \land \pi'$. And any $s, h_2 \models \kappa \land \pi$, $s, h_2 \models \kappa' \land \pi$. 

As $\text{roots}(\kappa_1) \cap \text{roots}(\kappa) = \emptyset$ $\text{dom}(h_1) \cap \text{dom}(h_2) = \emptyset$. Hence $s, h_1 h_2 \models \kappa_1 \ast \kappa_2 \land \pi$ (a). Similarly, $s, h_1 h_2 \models \kappa_1 \ast \kappa_2 \land \pi'$ (b).

From (a), (b), $\kappa_1 \ast \kappa_2 \land \pi \models \kappa_2 \ast \kappa' \land \pi'$.

Rule $\text{LInd}$ and $\text{RInd}$ . Based on the least semantics of the inductive predicates and the base case could not happen due to constraint $x \neq F$ in $\text{RInd}$ (respectively $x \neq F_3$ in $\text{LInd}$).

C.2 Global Soundness: Lemma 5.3

As our system always generates back-links with progressing points (via rule $\text{LInd}$), there are infinitely progressing points in any infinite trace.

We now show that all cycles are pairwise disjoint (such that in the path between a companion and a bud of every back-link, no rule can ever “delete” an inductive predicate formula on which the soundness relies). We prove by contradiction.

In intuition, the soundness replies on a pair of inductive predicates in a sub-term relationship. Given an inductive predicate $P(E,F,B,v)$ only rule $\text{ExM}$ is able to generate the constraint $E=F$ such that $P(E,F,B,v)$ can be transformed into $P(F,F,B,v')$ via rule $\text{Subst}$ and finally eliminated by rule $\text{LBase}$. We now show that every companion node of a back-link involving a bud that is in the branch $E \neq F$ of rule $\text{ExM}$ is below the node including the applications of rule $\text{ExM}$.
Assume that our system generates back-links with non-disjoint cycles. Two cycles are non-disjoint only when their both companion nodes are above at least one branch (applications of rule ExM). The non-disjoint cycles is similar to the one as shown in the proof tree in Fig. 6 where there is no branch in the path between $\Delta_2$ and $\Delta_3$. (The proof for the case where $\Delta_5$ is linked with $\Delta_2$ and $\Delta_6$ is linked with $\Delta_3$ is similar. We discuss this proof below.)

We prove the contradiction by case analysis on the pair of variables $E_1$ and $E_2$ applied with rule ExM at node $\Delta_4$. As ExM is applied to introduce $G(op(E))$ for every $op(E)$ in the LHS of entailments. We proceed case analysis on $op(E)$.

1. Case 1. $op(E) \equiv E \mapsto \null$ and ExM at node $\Delta_4$ does case split $E = \null$ and $E \neq \null$ to obtain two children. Assume that the left child (on the path from $\Delta_4$ to $\Delta_5$) is $\Delta_4 \land E = \null$. After substitution, LHS of this node is reduced to $\Delta_4' \land E = \null$ which is equivalent to $\false$. Thus, the back-link from $\Delta_5$ to $\Delta_3$ could not established.

2. Case 2. $op(E) \equiv P(E,F,\vec{B},\vec{v})$ and ExM at node $\Delta_4$ does case split $E = F$ and $E \neq F$ to obtain two children. Assume that the left child (on the path from $\Delta_4$ to $\Delta_5$) is $\Delta_4 \land E = F$. After substitution, LHS of this node is reduced to $\Delta_4' \land E = F$. In turn, this entailment is applied with normalization rule LBase to eliminate $P(F,F,\vec{B},\vec{v})$. Next, we consider two sub-cases of the inductive predicate in any application of rule LInd applied into a node between $\Delta_3$ and $\Delta_5$.

(a) the predicate applied is $P(E',F,\vec{B},\vec{v})$. We note that in the recursive rule of definitions of compositional predicates

$$\exists X,sc',d_1,d_2.x\rightarrow c(X,d_1,d_2,u,sc)\ast Q_1(d_1,B)\ast Q_2(d_2,X)\ast P(X,F,\vec{B},u,sc',tg)\land \pi_0$$

all nested predicates $Q_1, Q_2$ are syntactically different to $P$. $\Delta_5$ is missing one occurrence of predicate $P$. Hence, it could not be linked back to $\Delta_4$.

(b) the predicate applied is $Q(E',F',F,\vec{v'})$ such that $P(U,F,\vec{B},u',sc',tg')$ is a nested predicate in the definition of $Q$. However, $u',tg'$ are fresh variables and in any
back-links they are never substituted to become $u$ and $tg$, respectively. Hence, $\Delta_5$ could not be linked back to $\Delta_3$.

3. Case 3. op($E$) $\equiv$ tree($E, \bar{B}, \bar{v}$) and Ext at node $\Delta_4$ does case split $E = \text{null}$ and $E \neq \text{null}$ to obtain two children. As LInd only applies for compositional predicates, tree($E, \bar{B}, \bar{v}$) could not be a fresh formula. It has been normalised in $\Delta_3$ already. This case could not be occurred.

The proof for the case where $\Delta_5$ is linked with $\Delta_2$ and $\Delta_6$ is linked with $\Delta_3$ is similar. The main difference is that we need to show that predicate $P(E, F, \bar{B}, \bar{v})$ is a sub-formula of $\Delta_2$ in the proof of Case 2 like above. That means $P(E, F, \bar{B}, \bar{v})$ has not been eliminated by rule Ext in the path between $\Delta_2$ and $\Delta_3$. This is straightforward as no branch exists in the path between $\Delta_2$ and $\Delta_3$.

D Proofs of Termination

D.1 Proof of Lemma 5.4

Proof. Termination of our system is based on the size of an entailment which is defined as:

**Definition 6 (Size)** The size of an entailment $\phi: \kappa_\alpha \wedge \phi_\alpha \wedge a_\alpha \vdash \kappa_c \wedge \phi_c \wedge a_c$ is a triple of:

1. $N_p$ $-$ $n_p$ where $N_p$ is the maximal number of both points-to predicates and occurrences of inductive predicates that the RHS of any entailments derived (by $\omega$-ENT) from $\phi$ may contain, and $n_p$ is the total number of both points-to predicates and occurrences of inductive predicates in $\kappa_c$.
2. $N_e$ $-$ $n_e$ where $N_e$ is the maximal number of both disequalities and non-trivial equalities that the LHS of any entailments derived (by $\omega$-ENT) from $\phi$ may contain, and $n_e$ is the number of both disequalities and non-trivial equalities in $\phi_\alpha$.
3. the sum of the length of $\kappa_\alpha \wedge \phi_\alpha \wedge a_\alpha \vdash \kappa_c \wedge \phi_c \wedge a_c$, where length is defined in the obvious way taking all simple formulas to have length 1.
4. $N_a$: the number of constraints on arithmetic properties generated by the recursive rules of inductive definition.

If $N_p$ and $N_e$ are bounded, applying any rules except LInd makes progress since the size of each premise of any rule application is lexicographically less than the size of the conclusion. $N_p$ and $N_a$ rely on the number of applications of rule LInd. $N_e$ depends on the number applications of rule Ext. In turn, the application of Ext relies on the number of spatial variables. Thus, $N_e$ also relies on the number applications of rule LInd. To show the termination, we show that the number applications of rule LInd is bounded. In consequence, this bound is achieved if the number of applications of $*$ is finite. As the number of inductive symbols as their arities are finite, rule $*$ indeed generates a finite number of equivalent classes of entailments in which two entailments in the same class are equivalent after some substitution. Thus, all entailments in the same class are linked back together through a finite number of steps.
D.2 Proof of Lemma 5.5

Suppose we have an entailment $P(E,F,\overline{B},\overline{v})^k \ast \kappa \land \pi \vdash \kappa' \land \pi'$. If $\pi \not\vdash \pi'$ then exhaustively applying rule $\mathtt{false}$ our system decides it as invalid through the base cases like $E=F$.

If $\pi \vdash \pi'$, then our system applies rule $\mathtt{Hypothesis}$ to obtain $P(E,F,\overline{B},\overline{v})^k \ast \kappa \land \pi \vdash \kappa'$. Hence, in the following proof, we only consider the later form of the entailment in conclusion of rule $\text{LInd}$.

Without loss of generality, we assume $\mathcal{P}$ includes 4 predicates definitions: $P_1$, $P_2$, $Q_1$, and $Q_2$ where $P_1 \not\triangleright P_2$ (that is the recursive branch of predicate definition $P_2$ contains one and only one occurrence of predicate $P_1$ and $P_1$ is self-recursive), $Q_1 \not\triangleright Q_2$ (that is the recursive branch of predicate definition $Q_2$ contains one and only one occurrence of predicate $Q_1$ and $Q_1$ is self-recursive), $P_1 \not\triangleright P Q_1$, $P_1 \not\triangleright P Q_2$, $P_2 \not\triangleright P Q_1$, and $P_2 \not\triangleright P Q_2$. For instance, the definitions of these predicates could be as follows.

\[
pred P_1(r,F,u) \equiv \text{emp} \land r=F
\]
\[
\lor \exists X,sc',r \rightarrow c_1(X,F,u_d,u) \ast P_1(X,F,u) \land r \neq F \land \alpha_1
\]
\[
pred P_2(r,F,B_1,u,sc,tg) \equiv \text{emp} \land r=F \land sc=\text{tg}
\]
\[
\lor \exists X,d,u',sc',r \rightarrow c_1(X,d,u,u',sc') \ast P_1(d,B,u') \ast P_2(X,F,B_1,u,sc',tg) \land r \neq F \land \alpha_2
\]
\[
pred Q_1(r,F,u) \equiv \text{emp} \land r=F
\]
\[
\lor \exists X,sc',r \rightarrow c_2(X,F,u_d,u) \ast Q_1(X,F,u) \land r \neq F \land \alpha_3
\]
\[
pred Q_2(r,F,B_1,u,sc,tg) \equiv \text{emp} \land r=F \land sc=\text{tg}
\]
\[
\lor \exists X,d,u',sc',r \rightarrow c_2(X,d,u,u',sc') \ast Q_1(d,B,u') \ast Q_2(X,F,B_1,u,sc',tg) \land r \neq F \land \alpha_4
\]

We notice that in the definitions of $P_2$ and $Q_2$, we assume that in the recursive rule $sc'$ is a variable of a field of the root points-to predicate. In general, it may be a parameter of $P_2$ and $Q_2$ as well.

If the input entailment is in NF and of the form: $P(x,F,B,u,sc,tg)^0 \ast \Delta \vdash \Delta'$ and there does not exist an occurrence of inductive predicate $Q(x,F_2,B,...) \in \Delta'$ then this entailment satisfy the case 2c in Sect. 4.1 and is classified as invalid immediately. Thus, the Lemma holds. In the rest, to prove this Lemma, we only need to consider the application of rule $\text{LInd}$ where the entailment is in NF and of the form in the conclusion of $\text{LInd}$ as:

\[
\mathcal{e}_0: P(x,F,B,u,sc,tg)^0 \ast \kappa \land \pi \land x \neq F \land x \neq F_2 \vdash Q(x,F_2,B,u,sc,tg_2) \land \kappa'
\]

Furthermore, it is safe to assume that $P(x,F,B,u,sc,tg)^0$ is the only one with the smallest unfolding number (i.e., 1) in the LHS of $\mathcal{e}_0$ could be applied with rule $\text{LInd}$. We prove it by the structural induction on the number of occurrences of inductive predicates in the LHS of the input entailment. We do case splits.

D.3 Case 1: $P$ and $Q$ have the same definition.

We consider two cases where the definition contains nested structures or not.
Case 1.1 For the simpliest scenario, we assume both definitions of $P$ and $Q$ are self-recursive and do not contain nested structures i.e., $P \equiv Q \equiv P_1$. Then, $e_0$ becomes:

$$e_0 : P_1(x,F,u)^0 \ast \kappa \land \pi \land x \neq F \land x \neq F_2 \vdash P_1(x,F_2,u) \ast \kappa'$$

After applied with rule $\text{LInd}$, our system generates a premise as follows.

$$e_{1.1} : x \rightarrow c_1(X,d,u,sc',uprm) \ast P_1(X,F,u)^1 \ast \kappa \land \pi \land x \neq F \land x \neq F_2 \land a_1 \land x \neq \text{null}$$

$$\vdash P_1(x,F_2,u) \ast \kappa'$$

where $X$, $d$, $sc'$ and $uprm$ are two fresh variables and $a_1$ is the arithmetical constraint obtained by substituting actual/formal parameters into the constraint $a_1$ of the recursive rule of the definition of $P_1$. Next, entailment $e_{1.1}$ is normalized by applying rule $\neq \text{null}$ to obtain:

$$e_{1.2} : x \rightarrow c_1(X,d,u,sc',uprm) \ast P_1(X,F,u)^1 \ast \kappa \land \pi \land x \neq F \land x \neq F_2 \land a_1 \land x \neq \text{null}$$

$$\vdash P_1(x,F_2,u) \ast \kappa'$$

Now, $e_{1.2}$ is applied with rule $\text{RInd}$ to obtain:

$$e_{1.3} : x \rightarrow c_1(X,d,u,sc',uprm) \ast P_1(X,F,u)^1 \ast \kappa \land \pi \land x \neq F \land x \neq F_2 \land a_1 \land x \neq \text{null}$$

$$\vdash x \rightarrow c_1(X,d,u,sc',uprm) \ast P_1(x,F_2,u) \ast \kappa'$$

We note that for completeness applications of rule $\ast$ are always performed after all other rules. As so, next, hypothesis is applied to eliminate the arithmetical constraint in the RHS to obtain:

$$e_{1.4} : x \rightarrow c_1(X,d,u,sc',uprm) \ast P_1(X,F,u)^1 \ast \kappa \land \pi \land x \neq F \land x \neq F_2 \land a_1$$

$$\vdash x \rightarrow c_1(X,d,u,sc',uprm) \ast P_1(x,F_2,u) \ast \kappa'$$

Our system now applies rules $\text{Ext}$ and $\neq \ast$ to normalize the LHS where application of the latter rule generates two premises.

$$e_{1.5} : x \rightarrow c_1(X,d,u,sc',uprm) \ast P_1(X,F,u)^1 \ast \kappa \land \pi \land x \neq F \land x \neq F_2 \land a_1 \land x \neq \text{null} \land X=F$$

$$\vdash x \rightarrow c_1(X,d,u,sc',uprm) \ast P_1(x,F_2,u) \ast \kappa'$$

$$e_{1.6} : x \rightarrow c_1(X,d,u,sc',uprm) \ast P_1(X,F,u)^1 \ast \kappa \land \pi \land x \neq F \land x \neq F_2 \land a_1 \land x \neq \text{null} \land X \neq F$$

$$\land \land X \vdash x \rightarrow c_1(X,d,u,sc',uprm) \ast P_1(x,F_2,u) \ast \kappa'$$

1. For the premise $e_{1.5}$, our system applies rules $\text{Subst}$ and $L = \ast$ to eliminate $X=F$ and obtain:

$$e_{1.5.1} : x \rightarrow c_1(F,d,u,sc',uprm) \ast P_1(F,F,u)^1 \ast \kappa \land \pi \land x \neq F \land x \neq F_2 \land a_1 \land x \neq \text{null}$$

$$\vdash x \rightarrow c_1(F,d,u,sc',uprm) \ast P_1(x,F_2,u) \ast \kappa'$$

Next, rule $\text{LBase}$ is applied to discard the inductive predicate in the LHS and obtain the following premise:

$$e_{1.5.2} : x \rightarrow c_1(F,d,u,sc',uprm) \ast \kappa \land \pi \land x \neq F \land x \neq F_2 \land a_1 \land x \neq \text{null}$$

$$\vdash x \rightarrow c_1(X,d,u,sc',uprm) \ast P_1(x,F_2,u) \ast \kappa'$$

As the number of inductive predicates in the LHS of $e_{1.5.2}$ is reduced, by induction, this Lemma holds.
2. For the premise $e_{16}$, we have two cases.
   (a) If $FV(\pi) \cap FV(a_{11}) = \emptyset$. Our system links $e_{16}$ back to $e_1$, as follows. First, it weakens (a.k.a. discards) two matched points-to predicates in the two sides and the following pure constraints in LHS: $x \not= F$, $a_1$, $x \not= \text{null}$, and $x \not= X$. After that, it substitutes the remaining entailment with $\sigma = \{x / X\}$ to obtain the identical entailment with $e_1$. We notice that as $X$ is a fresh variable, it does not appear in $\kappa \land \pi$ and $\kappa'$. Then, the Lemma holds for this case.
   (b) $FV(\pi) \cap FV(a_{11}) \not= \emptyset$. As the substitution $[sc'/sc'']$ could not be applied, our system could not link $e_{16}$ back to $e_1$. It applies the same proof search as applied for $e_1$ to unfold $e_{16}$. As this time, $e_{16}$ contains respective $a_1'$ and $sc''$ and where $a_1' = a_1'[sc'/sc'']$. Now, our system could link $e_{16}$ back to $e_{16}$.

Case 1.2 For a more general case, we assume $P \equiv Q \equiv P_2$. Then, $e_0$ becomes:

$$e_{12} : P_2(x,F,B,u,sc,tg)^0 \ast \kappa \land \pi \land x \not= F \land x \not= F_2 \vdash P_2(x,F_2,B,u,sc,tg_2)^* \kappa'$$

The first four steps are similar to Case 1.1. After applied with rule $\text{LInd}$, our system generates a premise as follows.

$$e_{121} : x \mapsto c_1(X,d,u',sc') + P_1(d,B,u')^1 + P_2(X,F,B,u,sc',tg)^1 \ast \kappa \land \pi \land x \not= F \land x \not= F_2 \not= \alpha_2 \vdash P_2(x,F_2,B,u,sc,tg_2)^* \kappa'$$

where $\alpha_2$ is obtained by substituting actual/formal parameters into the arithmetical constraint $\alpha_2$ of the recursive rule of the definition of $P_2$. Next, this entailment is normalized by applying rule $\neq \text{null}$ to obtain:

$$e_{122} : x \mapsto c_1(X,d,u',sc') + P_1(d,B,u')^1 + P_2(X,F,B,u,sc',tg)^1 \ast \kappa \land \pi \land x \not= F \land x \not= F_2 \not= \alpha_2 \vdash x \not= \text{null} \vdash P_2(x,F_2,B,u,sc,tg_2)^* \kappa'$$

Next, $e_{122}$ is applied with rule $\text{RInd}$ to obtain:

$$e_{123} : x \mapsto c_1(X,d,u',sc') + P_1(d,B,u')^1 + P_2(X,F,B,u,sc',tg)^1 \ast \kappa \land \pi \land x \not= F \land x \not= F_2 \not= \alpha_2 \vdash x \not= \text{null} \vdash x \mapsto c_1(X,d,u',sc') + P_1(d,B,u')^0 + P_2(X,F_2,B,u,sc',tg_2)^* \kappa'$$

We note that rule $\ast$ is always applied after all other rules. As so, next, Hypothesis is applied to eliminate the arithmetical constraint in the RHS to obtain:

$$e_{124} : x \mapsto c_1(X,d,u',sc') + P_1(d,B,u')^1 + P_2(X,F,B,u,sc',tg)^1 \ast \kappa \land \pi \land x \not= F \land x \not= F_2 \not= \alpha_2 \vdash x \not= \text{null} \vdash x \mapsto c_1(X,d,u',sc') + P_1(d,B,u')^0 + P_2(X,F_2,B,u,sc',tg_2)^* \kappa'$$

Our system now applies rules $\text{Ext}$ and $\neq \ast$ to normalize the LHS. Particularly, applying rule $\text{Ext}$ for $d$ and $B$ generates two premises:

$$e_{125} : x \mapsto c_1(X,d,u',sc') + P_1(d,B,u')^1 + P_2(X,F,B,u,sc',tg)^1 \ast \kappa \land \pi \land x \not= F \land x \not= F_2 \not= \alpha_2 \vdash x \not= \text{null} \land d = B \vdash x \mapsto c_1(X,d,u',sc') + P_1(d,B,u')^0 + P_2(X,F_2,B,u,sc',tg_2)^* \kappa'$$

$$e_{126} : x \mapsto c_1(X,d,u',sc') + P_1(d,B,u')^1 + P_2(X,F,B,u,sc',tg)^1 \ast \kappa \land \pi \land x \not= F \land x \not= F_2 \not= \alpha_2 \vdash x \not= \text{null} \land d \not= B \vdash x \mapsto c_1(X,d,u',sc') + P_1(d,B,u')^0 + P_2(X,F_2,B,u,sc',tg_2)^* \kappa'$$
1. For the first premise \( e_{125} \), our system first applies rules Sub\( \tau \) to obtain and \( L =: \)

\[
\begin{align*}
e_{125_1} : & \rightarrow c_1(X,B,u,u',sc') \cdot P_1(B,B,u') \cdot P_2(X,F,B,u,sc',tg) \cdot k \wedge \pi \wedge x \neq F \wedge x \neq F_2 \wedge a_2 \wedge x \neq null \wedge x \rightarrow c_1(X,B,u,u',sc') \cdot P_1(B,B,u') \cdot P_2(X,F_2,B,u,sc',tg_2) \cdot k' \\
\end{align*}
\]

After that, it applies rules LBase and RBase to eliminate inductive predicates \( P_1 \) in the LHS and RHS, respectively. Afterward, the premise is obtained as:

\[
\begin{align*}
e_{125_2} : & \rightarrow c_1(X,B,u,u',sc') \cdot P_2(X,F,B,u,sc',tg) \cdot k \wedge \pi \wedge x \neq F \wedge x \neq F_2 \wedge a_2 \wedge x \neq null \wedge x \rightarrow c_1(X,B,u,u',sc') \cdot P_2(X,F_2,B,u,sc',tg_2) \cdot k' \\
\end{align*}
\]

Now, it generates a back-link between \( e_{125_2} \) and \( e_{12} \). Hence, the Lemma holds.

2. For the second premise \( e_{126} \), the system applies rule \( \neq * \) and then rule Ext to obtain two following premises:

\[
\begin{align*}
e_{127} : & \rightarrow c_1(X,d,u,u',sc') \cdot P_1(d,B,u') \cdot P_2(X,F,B,u,sc',tg) \cdot k \wedge \pi \wedge x \neq F \wedge x \neq d \wedge d = F \\
& \rightarrow \rightarrow c_1(X,d,u,u',sc') \cdot P_1(d,B,u') \cdot P_2(X,F_2,B,u,sc',tg_2) \cdot k' \\
e_{128} : & \rightarrow c_1(X,d,u,u',sc') \cdot P_1(d,B,u') \cdot P_2(X,F,B,u,sc',tg) \cdot k \wedge \pi \wedge x \neq F \wedge x \neq d \wedge d = F \\
& \rightarrow \rightarrow c_1(X,d,u,u',sc') \cdot P_1(d,B,u') \cdot P_2(X,F_2,B,u,sc',tg_2) \cdot k' \\
\end{align*}
\]

(a) For the premise \( e_{127} \), our system applies rules Sub\( \tau \) and \( L =: \) first and then rule LBase to eliminate inductive predicates \( P_2 \) in the LHS. Afterward, the premise is obtained as:

\[
\begin{align*}
e_{127_3} : & \rightarrow c_1(F,d,u,u',sc') \cdot P_1(d,B,u') \cdot P_2(X,F,B,u,sc',tg) \cdot k \wedge \pi \wedge x \neq F \wedge x \neq d \\
& \rightarrow \rightarrow c_1(F,d,u,u',sc') \cdot P_1(d,B,u') \cdot P_2(F,F_2,B,u,sc',tg_2) \cdot k' \\
\end{align*}
\]

Similarly to Case 1.1, the Lemma holds for \( e_{127_3} \).

(b) For the premise \( e_{128} \), our system processes similarly to \( \square \) in Case 1.1: the predicate in the LHS is unfolded at most two times. Hence, the Lemma holds.

D.4 Case 2: \( P \) and \( Q \) have different definitions and they are syntactically dependent.

We consider two sub-cases. In the first case, we assume \( P \prec_P Q \). In the second case, we assume \( Q \prec_P P \).

Case 2.1: \( P \prec_P Q \) For a general case, we assume \( P \equiv P_1' \) and \( Q \equiv P_2' \) where \( P_1' \) is defined similarly to \( P_1 \) except it contains an additional local property (Otherwise, the proof for \( P \equiv P_1 \) and \( Q \equiv P_2 \) is quite trivial.)

\[
\begin{align*}
\text{pred } P_1'(r,F,u,sc,tg) & \equiv \text{emp} \lor r = F \wedge sc = tg \\
& \lor \exists X,sc',r \rightarrow c_1(X,u,u',sc') \cdot P_1'(X,F,u,sc',tg) \wedge r \neq F \wedge a_1 \\
\text{pred } P_2'(r,F,B_1,u,sc,tg) & \equiv \text{emp} \lor r = F \wedge sc = tg \\
& \lor \exists X,d,u',sc',r \rightarrow c_1(X,d,u,u',sc') \cdot P_1(d,B,u',sc,tg) \wedge r \neq F \wedge a_2 \\
\end{align*}
\]

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Then, e₀ becomes:

\[ e₂₁: P'_1(x,F,B,u,sc,tg) \vdash \kappa' \]

After applying three rules \( L\text{Ind, } \neq \text{null } \) and \( R\text{Ind} \) in sequence (and similarly to \textbf{Case 1.1} and \textbf{Case 1.2} above), our system generates the following premise.

\[ e₂₁: x \rightarrow c₁(X,d,u,u',sc') \vdash P'_1(X,F,B,u,sc',tg) \vdash \kappa' \]

where \( X, d, sc' \) and \( u' \) are two fresh variables. Our system applies rule \( \text{Exm} \) for \( d \) and \( B \) to generate the following two premises.

\[ e₂₁: x \rightarrow c₁(X,d,u,u',sc') \vdash P'_1(X,F,B,u,sc',tg) \vdash \kappa' \]

As \( d \) is a fresh variable, the predicate \( P'_1(d,B,u',sc,tg) \) does not appear in the LHS of \( e₂₁ \). Hence, \( e₂₁ \) is classified as invalid and \( \omega\text{-ENT} \) returns invalid. Hence, the Lemma holds.

\textbf{Case 2.2:} \( Q \prec P \) For a general case, we assume \( P \equiv P₂ \) and \( Q \equiv P₁ \). Then, \( e₀ \) becomes:

\[ e₂₂: P₂(x,F,B,u,sc,tg) \vdash \kappa' \]

The proof for this case is similar to \textbf{Case 1.1}. After applying three rules \( L\text{Ind, } \neq \text{null } \) and \( R\text{Ind} \) in sequence, our system generates the following premise.

\[ e₂₂: x \rightarrow c₁(X,d,u,u',sc') \vdash P₁(d,B,u,sc,tg) \vdash \kappa' \]

where \( X, d, sc' \) and \( u' \) are two fresh variables. Our system applies rule \( \text{Exm} \) for \( d \) and \( B \) to generate the following two premises.

\[ e₂₂: x \rightarrow c₁(X,d,u,u',sc') \vdash P₁(d,B,u,sc,tg) \vdash \kappa' \]

As \( d \) is a fresh variable, the predicate \( P₁(d,B,u') \) does not appear in the RHS of \( e₂₂ \). Hence, \( e₂₂ \) is classified as invalid and \( \omega\text{-ENT} \) returns invalid. Hence, the Lemma holds.

\textbf{D.5 Case 3:} \( P \) and \( Q \) have different definitions and they are syntactically independent.

We consider three sub-cases based on the positions of inductive predicates in the dependency hierarchies. In the first case, we assume \( P \) is “smaller” than \( Q \). In the second case, we assume \( P \) is “bigger” than \( Q \). And in the last case, we assume \( P \) is “equal” to \( Q \).
Case 3.1: We assume \( P \equiv P_2 \) and \( Q \equiv Q_1 \). Then, \( e_0 \) becomes:

\[
e_3: P_2(x,F,B,u,sc,tg) 0 \ast \kappa \land \pi \land x \neq F \land x \neq F_2 \vdash Q_1(x,F_2,B,u) \ast \kappa'
\]

If \( c_1 \neq c_2 \), \( \text{is\_closed} \) returns invalid (Case 2d in Sect. 4.1). Otherwise, the proof for this case is similar to Case 2.2.

Case 3.2: We assume \( P \equiv P'_1 \) and \( Q \equiv Q_2 \). Then, \( e_0 \) becomes:

\[
e_3': P'_1(x,F,B,u,sc,tg) 0 \ast \kappa \land \pi \land x \neq F \land x \neq F_2 \vdash Q_2(x,F_2,B,u,sc,tg_2) \ast \kappa'
\]

If \( c_1 \neq c_2 \), the proof is straightforward. Otherwise, the proof for this case is similar to Case 2.1.

Case 3.3: We assume \( P \equiv P_1 \) and \( Q \equiv Q_1 \). Then, \( e_0 \) becomes:

\[
e_3'': P_2(x,F,B,u) 0 \ast \kappa \land \pi \land x \neq F \land x \neq F_2 \vdash Q_2(x,F_2,B,u) \ast \kappa'
\]

If \( c_1 \neq c_2 \), the proof is straightforward. Otherwise, the proof for this case is similar to Case 1.1.

Case 3.4: We assume \( P \equiv P_2 \) and \( Q \equiv Q_2 \). Then, \( e_0 \) becomes:

\[
e_3'': P_2(x,F,B,u,sc,tg) 0 \ast \kappa \land \pi \land x \neq F \land x \neq F_2 \vdash Q_2(x,F_2,B,u,sc,tg_2) \ast \kappa'
\]

If \( c_1 \neq c_2 \), the proof is straightforward. Otherwise, the proof for this case is similar to Case 1.2.

\[\square.\]

E Complexity Analysis - Proposition 5.6

Suppose that \( n \) is the maximum number of predicates (both inductive predicates and points-to predicates) among the LHS of the input entailment and those definitions in \( \mathcal{P} \), and \( m \) is the maximum number of fields of data structures. Then, the complexity is defined as follows.

First, we analyze the number of computation when all inductive predicates in the LHS are unfolded at most once. Let \( P(n,m) \) be the time complexity function under this assumption. Each pair of the root and segment parameters, say \( r \) and \( F \), of an inductive predicate is applied with rule \( \text{Ext} \) at most one. For the first premise where \( r = F \), after applied with \( \text{LBase} \) the number of inductive predicates is \( n - 1 \) and its running time is \( P(n - 1, m) \).

For the second premise, say \( e_c \), where \( r \neq F \), after applied with \( \text{LInd, normalization rules and RInd} \), it is applied with * to create two premises. While one of them is linked back to \( e_c \), the second is of the form: \( \kappa' \vdash \kappa'' \) where \( \kappa' \) (respective \( \kappa'' \)) is the matrix heap of the unfolded predicate in the LHS (respective RHS). \( \text{Ext} \) is applied at most
\[(n - 1) + (n - 2) + \ldots + 1 = \mathcal{O}(n^2)\] times. Moreover, as (i) all roots of inductive predicates in a matrix heap must not be aliasing (ensured by the normalization rule \(\exists \kappa \Phi\)) and (ii) they are must be in the fields of the root points-to predicate of the recursive definition rule, the number of inductive predicates in both \(\kappa'\) and \(\kappa''\) must be less than \(m\). Suppose that the running time of such an entailment of \(m\) inductive predicates of matrix heap is \(q(m)\), then \(P(n, m) = P(n - 1, m) + q(m) + \mathcal{O}(n^2)\).

\[
P(n, m) = P(n - 1, m) + q(m) + \mathcal{O}(n^2) \\
= P(n - 2, m) + 2q(m) + 2\mathcal{O}(n^2) \\
= \ldots \\
= P(1, m) + (n - 1) \times q(m) + (n - 1)\mathcal{O}(n^2) \\
= 1 + n \times q(m) + \mathcal{O}(n^3)
\]

(We presume that the running time of entailment without any inductive predicates is 1.)

We remark that if a formula contains two inductive predicates which has the same root parameters i.e., \(P(r, F_1 \ldots) \ast Q(r, F_2 \ldots) \ast \Delta\), then at least one of them must be reduced into base case with the empty heap. As \(m\) is the maximum number of fields of data structures and the roots parameters of \(\kappa'\) must be one of these variables of the fields, the number of inductive predicates of the LHS of any entailment that is derived from \(\kappa' \vdash \kappa''\), is less than or equal to \(m\). Thus, under modular substitution the number of combination of such \(m\) inductive predicates is \(\mathcal{O}(2^m)\).

Therefore, \(P(n, m) = \mathcal{O}(n \times 2^m + n^3)\).

The unfolding is depth-first and the steps for the second unfolding are similar. As the proof is linear, then the number of computation when all inductive predicates are unfolded at most two times is at most as \(2 \times P(n, m) = \mathcal{O}(n \times 2^m + n^3)\).

\section{Completeness of proof rules - Lemma \ref{lem:completeness}}

The completeness of all rules except rule \(\ast\) is straightforward. In the following, we prove the completeness of rule \(\ast\). The proof is based on the following auxiliary Lemma.

\begin{lemma}
If \(\kappa \land \phi \land a\) is in \(\mathcal{N}F\) and \(x \not\in \mathcal{E} \not\in \phi\), then \(\langle \kappa \land \phi \land [E/x] \land a\rangle\) is in \(\mathcal{N}F\).
\end{lemma}

\begin{proof}
All but the fifth clause in the definition are invariant under substitution. Moreover, \(x \not\in \mathcal{E} \not\in \phi\) exclude the violation of the fifth clause under substitution as well.
\end{proof}

First, we provide proofs for pure part when pure contraints in LHS does not imply pure contraints in RHS.

\begin{proposition}
If \(\kappa \land \phi \land a \vdash \kappa_m \land \phi' \land a'\) is in \(\mathcal{N}F\) and \(\phi': \mathcal{E} \mathcal{P} \land \phi \land a \vdash \mathcal{E} \mathcal{P} \land \phi' \land a'\) is not derivable, then \(\kappa \land \phi \land a \vdash \kappa' \land \phi' \land a'\) is invalid.
\end{proposition}

\begin{proof}
We show that there is a model of the LHS that satisfies either \(\neg \phi'\) or \(\neg a'\) holds. We proceed cases for each predicate in the RHS.

1. Case \(\phi' \equiv E_1 = E_2\). As the LHS is in \(\mathcal{N}F\), any bad model of \(\kappa_m \land \phi \land a\) implies that \(E_1 \neq E_2\). In other words, \(\kappa_m \land \phi \land a\) implies that \(\neg E_1 = E_2\).
2. Case \( \phi' \equiv E_1 \not\equiv E_2 \). As \( \phi' \) is not derivable, then the side condition of rule \emph{Hypothesis} does not hold. This means \( E_1 \not\equiv E_2 \not\models \phi \).

We note that if \( \kappa_m \land \phi \land a \) is in NF and \( E_1 \not\equiv E_2 \not\models \phi \), then \( (\kappa_m \land \phi \land a)[E_1/E_2] \) is also in NF (assuming that \( E_1 \) is a variable - Lemma [F.1]). Then suppose \( s, h \) be a bad model of \( (\kappa_m \land \phi \land a)[E_1/E_2] \), then \( s[E_1 \mapsto s(E_2)] \), \( h \) is a model of \( (\kappa_m \land \phi \land a) \). \( s[E_1 \mapsto s(E_2)] \), \( h \) implies that \( \neg E_1 \not\equiv E_2 \). Therefore, \( (\kappa_m \land \phi \land a) \) does not imply \( E_1 \not\equiv E_2 \). Neither is \( (\kappa_m \land \phi \land a) \).

3. Case \( \phi' \equiv \text{true} \). As \( \phi' \) is not derivable, then the side condition of rule \emph{Hypothesis} does not hold. We consider two cases.

(a) \( a \land a' \) is unsatisfiable. Hence, any model of \( a \) implies \( \neg a' \).
(b) \( a \land a' \) is satisfiable. Hence, \( a \land \neg a' \) is also satisfiable and is in NF. Moreover, any model of \( a \land \neg a' \) implies \( \neg a' \). As \( a \land \neg a' \) is an under-approximation of \( a \), from any model \( a \land \neg a' \) we can construct a model satisfying \( a \) implies \( \neg a' \).

Therefore, any bad model of \( \kappa_m \land \phi \land a \) is a counter-model. \( \square \).

Secondly, we prove the completeness of rule \( * \) when the LHS of the conclusion in NF is a base formula.

\[
\frac{\kappa_1 \land \pi \vdash \kappa_2 \quad \kappa \land \pi \vdash \kappa'}{\kappa_1 \land \pi \vdash \kappa_2 \land \kappa'}
\]

\textbf{Proof.} We prove that if the rule’s conclusion is derivable then the rule’s premises are derivable.

We prove by induction on the number \( n \) of points-to predicates in the LHS of the conclusion.

1. Base case: \( n = 0 \) and \( n = 1 \), the proof is trivial.
2. Inductive case: Assume that it is true for \( n = k \).

Suppose \( \kappa_1 \ast \kappa = x_1 \mapsto c_1(\bar{v}_1) \ast \ldots \ast x_{k+1} \mapsto c_{k+1}(\bar{v}_{k+1}) \) and \( \pi \) contains enough disequalities for NF.

We proceed by cases on \( \kappa_2 \).

(a) Case 1: \( \kappa_2 \) is a points-to predicate. If \( \kappa_2 \equiv x_j \mapsto \omega(\cdot) \) where \( x_j \notin \{x_1, \ldots, x_{k+1}\} \).

Then procedure is \textit{closed} has also returned \textit{invalid} already and the conclusion is not derivable. Contradiction. Therefore, \( \kappa_2 \) must be one of the points-to predicates in the LHS. Assume that \( \kappa_2 \equiv x_1 \mapsto c_1(\bar{v}_1) \). Then, \( \kappa_1 \models \kappa_2 \) and by induction \( \kappa \land \pi \vdash \kappa' \) is also derivable.

(b) Case 2: \( \kappa_2 \) is an inductive predicate; assume \( \kappa_2 \equiv P(r,F) \). Similarly to the above case, \( r \in \{x_1; \ldots; x_{k+1}\} \). Otherwise, procedure is \textit{closed} has returned \textit{invalid} already. Assume \( r \equiv x_1 \). Secondly, \( x_1 \not\in \pi \). Otherwise, \( \omega\text{-}\text{ENT} \) is stuck (it could not apply rule \textit{RInd} and procedure is \textit{closed} has returned \textit{invalid} already. Third, \( x_1 \mapsto c_1(\cdot) \not\in \kappa' \). Otherwise, the RHS of the conclusion is \textit{false} the conclusion is not derivable. Now, the conclusion could be applied with rule \textit{RInd} to generate \( x_1 \mapsto c_1(\cdot) \ast \kappa'' \ast P(F_2,F) \). Now, it comes back to Case 1 above.

\( \square \).
We prove the correctness of Proposition 5.10 through two steps:

1. proofs for the case where LHS is a base formula. Those entailments are reduced without \( \text{LInd} \).
2. proofs for the case where LHS is a general formula. Those entailments are reduced with \( \text{LInd} \) prior to applying other rules.

In the proofs, we make use of the following auxiliary Lemmas.

**Lemma G.1.** If \( \pi \land \pi \vdash \kappa_1 * \kappa_2 \) in NF is derivable, then there exist \( \kappa_1, \kappa_2 \) such that \( \kappa \equiv \kappa_1 * \kappa_2 \) and both \( \pi_1 \land \pi \vdash \kappa_1 \) and \( \pi_2 \land \pi \vdash \kappa_2 \) are derivable.

**Lemma G.2.** If \( \kappa_1 * \kappa_2 \land \pi \) is in NF and \( \kappa_1 \land \pi \vdash \kappa_1' \) is valid, then \( \kappa_1 * \kappa_2 \land \pi \vdash \kappa_2' \) is valid iff \( \kappa_2 \land \pi \vdash \kappa_2' \) is valid.

Based on the fact that heaps of a normalized base formula is precise. The proof is straightforward based on the semantics of the separating conjunction \(*\).

**G.1 Base-Formula LHS**

First, we show the correctness of case 2a) of procedure \texttt{is\_closed} i.e., an entailment is stuck then it is invalid. After that, we show the invalidity is preserved through proof search.

As the LHS is a base formula, rule \( \text{LInd} \) (and rule \( \text{LBase} \)) is never be applied. We prove case 2a) by induction on the number of disequalities missing from the LHS and generated by rule \( \text{ExM} \). First, we prove the case where the RHS is an occurrence of compositional predicate \( P \) assuming that the points-to predicate in the definition of \( P \) is \( c \).

**Lemma G.3.** If \( e_0 \colon x \mapsto c(F_2,\bar{p},u) * \pi \land \phi \land a \vdash P(x,F,\bar{B},u,sc,tg) \) is in NF and is stuck, then it is invalid.

**Proof.** Due to the stuckness, \( \omega\text{-ENT} \) could not applies rule \( \text{RInd} \). Hence, \( x \notin F \not\in \phi \). As the the entailment is in NF, \( (x \mapsto c(F_2,\bar{p},u) * \pi \land \phi)(F/x) \land a \vdash P(x,F,\bar{B},u,sc,tg)(F/x) \) is in NF (by Lemma F.1). As all models satisfying the LHS \( (x \mapsto c(F_2,\bar{p},u) * \pi \land \phi)(F/x) \land a \) are non-empty heap and in NF, all models satisfying the RHS \( P(x,F,\bar{B},u,sc,tg)(F/x) \equiv P(F,F,\bar{B},u,sc,tg) \) are empty heap, this entailment is invalid. As the substitution law is sound and complete, \( e_0 \) is invalid. \( \square \).

**Lemma G.4.** If \( e_0 \colon \pi \land \phi \land a \vdash \kappa' \) is in NF and is stuck, then it is invalid.

**Proof.** By induction on the number of disequalities missing from \( \phi \). We proceed by cases.

1. \( \kappa' \equiv P(x,F,\bar{B},u,sc,tg) * \kappa'' \) assuming that the points-to predicate in the definition of \( P \) is \( c \).
(a) $op(x) \not\in \pi$. This case is the case 2c) of procedure is_closed. The bad model of the LHS is the counter-model.
(b) $op(x) \in \pi$. $\omega$-ENT reduces the entailments by first applying rule ExM prior to applying rule LInd. We are considering the case $\omega$-ENT could not apply rule LInd. We proceed cases for the LHS.

i. $\kappa \equiv x \rightarrow \bar{c}(F_2, \bar{v}) \ast \pi_0$ and $x \not\in F \land \phi$.

If $\epsilon_1 \equiv x \rightarrow \bar{c}(F_2, \bar{v}) \ast \pi_0 \land \phi \land a \land x \not\in F \vdash p(x, F, \bar{B}, u, sc, tg) \ast \kappa''$ is stuck, rule LInd could not be applied. Hence, $x \rightarrow \bar{c}(F_2, \bar{v}) \in \kappa''$. Therefore, there is no model that satisfies the RHS. This entailment $\epsilon_1$ is thus invalid. As rule ExM is complete (Lemma G.8), $\epsilon_0$ is invalid.

If $\epsilon_1$ is derivable, following Lemma G.1, there exist $\kappa_1, \kappa_2$ such that $\kappa_0 \equiv \kappa_1 \ast \kappa_2$ and both $\epsilon_2 \equiv x \rightarrow \bar{c}(F_1, \bar{v}) \ast \pi_1 \land \phi \land a \land x \not\in F \vdash p(x, F, \bar{B}, u, sc, tg)$ and $\epsilon_3 \equiv \pi_2 \land \phi \land a \land x \not\in F \vdash \kappa''$ are derivable. We proceed two sub-cases:

A. $\epsilon_4 \equiv \pi_2 \land \phi \land a \land x \not\in F \vdash \kappa''$ is stuck. Hence either $op_1 x \in \pi_2$ or $op_2 F \in \pi_2$.

This implies either $op x \ast op_1 x$ in the LHS of $\epsilon_0$ or $op x \ast op_2 F$ in the LHS of $\epsilon_0$. As $\epsilon_0$ is in NF, either $x \not\in F \land \phi$ or $x \not\in F$ and $\phi$. Both can’t not happen as the first scenario contradicts with assumption that LHS is in LHS and the second one contradicts with assumption $x \not\in F \not\in \phi$.

B. $\epsilon_4 \equiv \pi_2 \land \phi \land a \land x \not\in F \vdash \kappa''$ is derivable. Hence, by soundness (Lemma 5.3), it is valid. (2a)

As $\epsilon_0$ is stuck and $\epsilon_4$ is derivable, we deduce that $\epsilon'_4 \equiv x \rightarrow \bar{c}(F_1, \bar{v}) \ast \pi_1 \land \phi \land a \vdash p(x, F, \bar{B}, u, sc, tg)$ is stuck (Otherwise, $\epsilon_4$ is derivable as well, contradiction). By Lemma G.3, $\epsilon'_4$ is invalid. (2b)

By (2a), (2b) and Lemma G.2, $\epsilon_0$ is invalid.

ii. $\kappa \equiv x \rightarrow \bar{c}'(F_2, \bar{v}) \ast \pi_0$ and $x \not\in \phi$ and $c' \not\in c$. The proof is similar to Case 2d of procedure is_closed. The bad model of the LHS is the counter-model.

2. $\kappa' \equiv x \rightarrow \bar{c}(\bar{v}) \ast \kappa''$. Straightforward.

3. $\kappa' \equiv \mathsf{emp}$. Straightforward.

\[ \square \]

**Proposition G.5.** If $\pi \land \phi \land a \vdash \kappa'$ is in NF and is not derivable, then it is invalid.

**Proof.** In an incomplete proof, if a leaf node in NF is stuck then it is invalid (Lemma G.4 and Lemma G.2). The invalidity is preserved up to the root based on Lemma G.8. \[ \square \]

**G.2 General LHS**

By induction on the RHS. We proceed cases on the RHS. In the proofs, for convenient, we write $\kappa_a \land \pi_a \vdash \kappa_c \land \pi_c$ as a shorthand of $\kappa_a \ast \kappa \land \pi_a \vdash \kappa_c \ast \kappa \land \pi_c$ and and no any matching heaps between $\kappa_a$ and $\kappa_c$ could be found through the application of rule $\ast$.

**Lemma G.6.** If $\epsilon \equiv \kappa \land \pi \vdash \kappa'$ in NF where $\epsilon_0 \equiv \kappa \land \pi \vdash \kappa$ is stuck, then $\epsilon_0$ is invalid.

**Proof.** We first show $\epsilon_0$ is invalid. After that by using Lemma G.2, we could deduce the invalidity of $\epsilon$. To show invalidity of $\epsilon_0$, we proceed cases on the possible base formula of the LHS.
1. \( e_1: \pi \land \pi \vdash \kappa' \) is stuck. By Proposition G.5, \( e_1 \) is invalid. As \( \pi \land \pi \) is an approximation of \( \kappa \land \pi \), \( e_0 \) is invalid.

2. \( e_2: \pi \land \pi \vdash \kappa' \) is derivable. And \( \kappa' \equiv \mathcal{O}(E) \ast \kappa'' \). We proceed cases on \( \mathcal{O}(E) \).
   
   - \( \mathcal{O}(E) \equiv Q(x,F_1,B,u,sc,\text{tg}_1) \). As the LHS is in NF, \( e_2 \) could be reduced by \( \text{RInd} \). This implies that \( x\mapsto c(F,\bar{d},tg,u)[\bar{v}/\bar{d}] \land \pi_0[\text{tg}/\text{scd}] \in \pi \) and \( x \neq F_3 \in \pi \).
   
   This implies that there are two possible sub-cases.
   
   - Sub-case 1: \( x\mapsto c(F,\bar{d},tg,u)[\bar{v}/\bar{d}] \in \kappa \). As \( x \neq F_3 \in \pi \), \( e_0 \) could be applied with \( \text{RInd} \). It is impossible as it contradicts with the assumption that \( e_0 \) is stuck.
   
   - Sub-case 2: \( P(x,F,B,..) \in \kappa \). As \( x \neq F_3 \in \pi \), \( e_0 \) could be applied with \( \text{LInd} \). As \( x \neq F_3 \in \pi \), \( e_0 \) could be applied with \( \text{LInd} \).

   - \( \mathcal{O}(E) \equiv x\mapsto c(\text{next}:F,\bar{v}) \). Based on \( x\mapsto c(\text{next}:F,\bar{v}) \in \pi \), there are two cases.
   
   - Sub-case 1: \( x\mapsto c(\text{next}:F,\bar{v}) \in \kappa \). This contradicts with the assumption that \( x\mapsto c(\text{next}:F,\bar{v}) \) could not be matched with any predicate in \( \kappa \). This case is impossible.

   - Sub-case 2: \( P(x,E,..) \in \kappa \). Any model satisfying the LHS when replacing \( P(x,E,..) \) by three-time unfolding (with two points-to predicates e.g., \( x\mapsto c(\text{next}:F_1,..) \ast F_1\mapsto c(\text{next}:F,..) \)) is a counter-model.

\( \Box \).

**Proposition G.7 (Incompleteness Preservation).** Given an input entailment \( e_0: \Delta \vdash \Delta' \), and there is an leaf node \( e_i: \Delta_l \vdash_{\kappa_m} \Delta'_l \) in its incomplete proof tree where

- the leaf node \( e_i \) is in NF; and
- none of application of rule \( \text{FR} \) from the root \( e_0 \) to the leaf node \( e_i \); and
- \( \Delta_l \vdash \Delta'_l \) is not derivable.

then \( e_0 \) is invalid.

**Proof.** By Lemma G.6, \( e_i \) is invalid. By Lemma 5.8, \( e_0 \) is invalid. \( \Box \).