ON CATEGORIFICATION

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Abstract. We review several known categorification procedures, and introduce a functorial categorification of group extensions with applications to non-abelian group cohomology. Categorification of acyclic models and of topological spaces are briefly mentioned.

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1. Introduction

The term categorification was invented by L. Crane [1, 2] to denote a process of associating category-theoretic concepts to set-theoretic notions and relations (see also [3]).

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An often used correspondence is the following:

| set elements | objects |
|-------------|---------|
| equality between | isomorphisms |
| set elements | between objects |
| | |
| objects | categories |
| | |
| morphisms | functors |
| equality between | natural isomorphisms |
| morphisms | of functors |

The role of such a procedure is to introduce new structures, for example Hopf categories for constructing 4-D TQFTs [1]. It provides examples, for instance monoidal categories with prescribed fusion rules [2], and and toy models, for example invariants of 3-D manifolds associated to finite gauge groups as a special case of more general constructions ([12], pp. 12).

We consider categorification as a "bridge" allowing to transport constructions from set-algebra to higher-dimensional algebra [3], and allowing to exploit the category theory results for a better understanding of the former, e.g. defining the cohomology of monoidal categories [4, 5] and then using it to understand non-abelian group cohomology [6].

It is also a method to "force" the right approach in solving a problem:

```
problem  generalize  solve
          ↓              ↓
solution  specialize
```

since categorification may lead to surprising clarifications.

We will consider correspondences as defined below.

**Definition 1.1.** A categorification is a functor \( C \) defined on a concrete category \( C_1 \) to a category \( C_2 \) with small categories as objects (0-arrows), functors as morphisms (1-arrows), associating categories to objects and functors to morphisms.

Often the target is a 2-category with natural transformations as 2-arrows.

The categorification procedures considered may be extended to group rings, which provide much more structure as Hopf rings.

2. Groups as symmetries

From the beginning of category theory, groups were interpreted as categories by Eilenberg-Mac Lane [3] (1942).

2.1. Tautological categorification. Let \( G \) be a group. Define the category \( C_G \) with one object \( *_G \) having as morphisms the elements of the group. The group multiplication is used to define the composition of morphisms in the category. Note that \( C_G \) is a groupoid. A
group homomorphism \( f : G \to H \) defines in a tautological way a functor \( C_f \) between the one object groupoids associated to the source and target of \( f \).

\[
C_f(\ast_G) = \ast_H, \quad C_f(x) = f(x), \quad x \in G
\]

The above correspondence is functorial, with domain the category of groups \( \mathcal{G}rp \) and valued in the category \( \mathcal{G}rpoid \) of groupoids.

**Definition 2.1.** The categorification functor \( C \) is called the *tautological categorification*. It is an isomorphism between the categories of groups and one object groupoids.

Natural transformations are one benefit of categorification. Let \( f, g : G \to H \) be group homomorphisms, or in categorical language, functors between the corresponding one object groupoids. A natural transformation \( \eta : C_f \to C_g \) (functorial morphism) is determined by one element \( h = \eta_{\ast_G} \) of \( H \), satisfying the relation \( g(x) = hf(x)h^{-1} \):

\[
\begin{array}{ccc}
of(\ast_G) & \overset{h}{\longrightarrow} & g(\ast_G) \\
f(x) \downarrow & & \downarrow g(x) \\
f(\ast_G) & \overset{h}{\longrightarrow} & g(\ast_G)
\end{array}
\]

Then the set of conjugacy classes of group homomorphisms is the set of equivalence class of functors modulo natural isomorphisms and it will be denoted by \([C_G, C_H] = \text{Hom}(C_G, C_H)/\sim\).

**Notation 2.1.** We will call natural isomorphisms of functors *homotopies* and isomorphism classes of functors, homotopy classes.

The map sending an object to its isomorphism class is denoted by \( \pi_0 \). The functor associating the homotopy class of a functor will be denoted by \( \pi_1 \).

The relevance of the homotopy class needs not be explained (fundamental groupoid of a topological space, holonomy groups, monodromy, etc.).

**2.2. Group homomorphisms as representations.** Another benefit of tautological categorification is the representation of a group as the group of symmetries of an abstract object. Identifying \( G \) with \( \text{Aut}(\ast_G) \), the elements of \( G \) are “abstract” symmetries of \( \ast_G \). Here “identifying” means the “identity” morphism:

\[
G \xrightarrow{id} \text{Aut}_{C_G}(\ast_G)
\]

i.e. the fundamental representation of \( \text{Aut}(\ast_G) \) on \( \ast_G \).

More general, the set of group homomorphisms \( \text{Hom}(G, H) \) can be interpreted as the category \( G - C_H \) of representations of the group \( G \) on \( \ast_H \), i.e. on the one-object of the category \( C_H \). Similar to the case of representations of a group in the category of vector spaces, \( \mathcal{V}ect \), the morphisms are \( G \)-equivariant \( C_H \)-morphisms, i.e. elements of the group \( H \) which intertwine the two morphisms. In this way conjugacy classes of group homomorphisms are interpreted as homotopy classes of functors \([C_G, C_H]\) or isomorphism classes of representations.
As an example, we mention the set of outer morphisms of the group $H$:

$$\text{Out}(H) = \text{Aut}(H)/\sim = [H, H]$$

and outer actions $\text{Hom}(G, \text{Aut}(H))/\sim$ of $G$ on $H$, as a homotopy class of functors $\text{Hom}(C_G, G - C_H)$.

To accommodate quasi-actions and to obtain examples of monoidal categories, one may interpret the group composition as a monoidal product, rather than a composition of morphisms.

### 3. Groups as monoidal categories

The notion of monoidal category $(\mathcal{M}, \otimes, \alpha)$ is itself a categorification. It is the category analog of a monoid.

Recall that an equivalence relation on a set $X$ defines a groupoid with objects the elements of the set and with a morphism $x \rightarrow y$ for each ordered pair of equivalent elements $x \sim y$. The composition of morphisms is actually the transitivity property.

There are two important, “antipodal” cases [14].

**Definition 3.1.** The discrete groupoid on the set $X$ is the groupoid associated to the minimal equivalence relation defined by the diagonal of $X$. It is denoted by $\mathcal{D}_X$.

The simplicial groupoid on the set $X$ is the groupoid associated to the maximal equivalence relation defined by the product $X \times X$. It is denoted by $\mathcal{S}_X$.

**3.1. Discrete categorification.** Let $G$ be a group. Define $\mathcal{D}_G$ as the discrete groupoid on the set $G$. It is a skeletal category, i.e. with only one object in each isomorphism class.

There is a natural monoidal product on the category $\mathcal{D}_G$ defined by the multiplication law in the group $G$. Then $(\mathcal{D}_G, \otimes)$ is a strict monoidal category.

Group homomorphisms define in an obvious way strict monoidal functors. The correspondence $G \mapsto \mathcal{D}_G$ is functorial, faithful and full, since in this version of categorification there are no natural transformations, except identities.

**Definition 3.2.** The functor $\mathcal{D} : \text{Grp} \rightarrow \otimes\text{-Grpoid}$ is called discrete categorification, where $\otimes\text{-Grpoid}$ is the category of monoidal groupoids.

It is a construction used in relation to fusion rings, as it will be briefly mentioned below.

### 3.2. K-categorification.** A fusion rule [11, 4] is a set $S$ with a family of non-negative coefficients defining a semi-ring structure on the free abelian group generated by the set $S$: the enveloping semi-ring.

The “linear version” of discrete categorification was introduced by L. Crane and D. Yetter [12] to provide examples of semi-simple monoidal categories with a prescribed set of fusion rules. A $K$-categorification of a fusion rule [12], with $K$ a field, is a $K$-linear tensor category with Grothendieck ring the enveloping ring corresponding to the fusion rule. It is a section for the isomorphism class functor $\pi_0 : \otimes\text{-Grpoid} \rightarrow \text{Mon}$, which associates to a monoidal category the monoid of isomorphism classes.
Changing coefficients in the discrete monoidal category associated to a group by “tensoring” with $K$ is a categorification of the group ring $KG$. It is a $K$-categorification of the fusion rule defined by the group ring. The set of such categorifications is in bijection with the set of 3-cocycle of the group $G$ with coefficients in the multiplicative group of the field, as associators [12].

3.3. **Simplicial categorification.** Let $G$ be a group. Consider the simplicial groupoid $S_G$. There is only one isomorphism class, and in fact it is a *contractible groupoid*:

```
0 \rightarrow \ast \rightarrow x
```

i.e. the functor collapsing the category to a chosen base point is homotopic to the identity functor.

Define the monoidal product on objects as group multiplication. Since between each pair of objects there is precisely one morphism only, the extension of the product on morphisms is unique. We obtain a strict monoidal category $(S_G, \otimes)$ where any diagram commutes!

Group homomorphisms define in an obvious way strict monoidal functors between the corresponding simplicial groupoids, and the correspondence $G \mapsto S_G$ is functorial.

**Definition 3.3.** The functor $S_G : \text{Grp} \rightarrow \otimes\text{-Grpoid}$ is called *simplicial categorification*.

3.4. **The covering transformation.** Define the functor $R_G : S_G \rightarrow C_G$ between the simplicial and tautological categorification of the group $G$:

$$R_G(x) = \ast_G, \quad R_G(x \rightarrow y) = yx^{-1}, \quad x, y \in \text{Ob}(C_G)$$

Disregarding the monoidal structures on $S_G$, the correspondence defines a natural transformation $R : S \rightarrow C$ between simplicial and tautological categorification functors.

The group $G$ acts freely on $S_G$ through left and right multiplication on objects and morphisms. It also acts on $C_G$ as inner conjugation from the left and right trivial.

The functor $R_G$ is $G$-biequivariant:

$$R_G(I_a \otimes x \downarrow y) = aR_G(x \downarrow y)a^{-1}$$

$$R_G(x \downarrow y \otimes I_a) = R_G(x \downarrow y)$$

while the corresponding relation on objects is trivial.
4. Applications to group extensions

Recall that a group extension $E$ of $G$ by $N$ is a short exact sequence in the category of groups:

$$1 \to N \overset{j}{\to} E \overset{p}{\to} G \to 1 \quad (\mathcal{E})$$

It is a pointed fibered object with distinguished fiber $N \cong p^{-1}(1)$. It is natural to categorify the discrete group $G$ as a base $\mathcal{D}_G$, using discrete categorification and the group $N$ as a fiber $\mathcal{S}_G$, using simplicial categorification.

4.1. Bundle categorification. There is a natural lift of the group extension $\mathcal{E}$ to groupoids, defining the categorification $\mathcal{B}_E$ of $E$ as a disjoint fibration over $\mathcal{D}_G$ of the simplicial groupoids associated to the fibers of $E$.

The multiplication in $E$ extends in a unique way to a monoidal product. Similarly, the maps $j$ and $p$ extend in a unique way to strict monoidal functors.

Then $\mathcal{B}_E \to \mathcal{D}_G$ is a fibered monoidal category, part of a (short exact) sequence of strict monoidal groupoids with unit. Morphisms of group extensions define in an obvious way monoidal functors between the corresponding fibered categories, and the correspondence $(E \to G) \mapsto (\mathcal{B}_E \to \mathcal{D}_G)$ is functorial.

**Definition 4.1.** The functor associating the monoidal groupoid $\mathcal{B}_E$ to a group extension $\mathcal{E}$ is called bundle categorification and it is denoted by $\mathcal{B}$.

The categorification functors are clearly compatible in the following sense.

**Theorem 4.1.** Discrete, Simplicial and Bundle Categorification maps $\mathcal{D}, \mathcal{S}$ and $\mathcal{B}$ are functors from the category of groups and group extensions $\text{Ext}$, to the category of monoidal groupoids.

$\mathcal{D}$ and $\mathcal{S}$ are the restrictions of $\mathcal{B}$ corresponding to the two natural embeddings:

$$G \mapsto (1 \to G \to G \to 1 \to 1) \quad (\text{trivial base})$$

$$G \mapsto (1 \to 1 \to G \to G \to 1) \quad (\text{trivial fiber})$$

embedding the category of groups into the category of group extensions:

```
\begin{align*}
\text{Grp} & \xrightarrow{\mathcal{S}} \text{Grpoids} \\
\text{Grp} & \xrightarrow{\mathcal{T}_b} \mathcal{B} \\
\text{Grp} & \xrightarrow{\mathcal{T}_j} \mathcal{D}
\end{align*}
```
4.2. Relation to group cohomology. Consider the group extension $\mathcal{C}$. Categorifying
the extension as defined in the previous section, a set-theoretic section is equivalent to a
splitting in the category $\otimes$-Grpoid, as explained below.

Choose such a section $s : G \to E$ of $p$. There is a unique functor over the section $s$,
denoted by $S$. Since any fiber of $\mathcal{B}_E$ is simplicial, there is a unique natural isomorphism $\eta$
between the functors $s \circ \otimes$ and $s^* (\otimes)$:

$$\eta(x, y) : S(x \otimes y) \to S(x) \otimes S(y), \quad x, y \in G = \text{Ob}(\mathcal{D}_G)$$

Define the functor $\mathcal{R}_E : \mathcal{B}_E \to \mathcal{C}_N$, the fiberwise analog of the covering transformation
defined in section 3.4:

$$\mathcal{R}_E(x \downarrow y) = n \text{ iff } j(n) = yx^{-1}$$

The natural isomorphism defines what in group language is called the factor set of the
function $s$:

$$f(x, y) = \mathcal{R}_E(\eta(x, y)) = s(x)s(y)s(xy)^{-1}$$

Since any diagram in $\mathcal{B}_E$ commutes, the functorial morphism $\eta$ is a monoidal structure:

$$S((ab)c) \xrightarrow{\eta(ab,c)} S(ab) \otimes S(c) \xrightarrow{\eta(a,b) \otimes I_{S(c)}} (S(a) \otimes S(b)) \otimes S(c)$$

$$S(ab) \otimes S(a) \otimes S(b) \xrightarrow{I_{S(a)} \otimes \eta(b,c)} S(a) \otimes (S(b) \otimes S(c))$$

and the pair $(S, \eta)$ is a monoidal functor. In the above diagram the monoidal product in $G$
was denoted as concatenation.

Inner conjugation $C : E \to \text{Aut}(E)$ in $E$ defines a left quasi-action $L : G \to \text{Aut}(N)$
of $G$ on $N$: $L(x)(n) = C_{s(x)}(n)$. Recall that the covering transformation $\mathcal{R}_E$
interwines left multiplication by identity morphisms with inner conjugation and right multiplication
by identity morphisms with the trivial right action of $G$ (see 3.4). Applying the functor $\mathcal{R}_E$
to the above monoidality diagram, one obtains that the factor set $f$ is a group 2-cocycle
relative to the left quasi-action $L$:

$$f(ab,c)f(a, b) = L_a(f(b, c))f(a, bc)$$

Isomorphic group extensions inducing the same quasi-action of $G$ on $N$ define cohomologous
2-cocycles. To see this, consider an isomorphism of extensions $\gamma' : E \to E'$, i.e. acting
trivial on $N$ and inducing identity on $G$. It defines a section $s' = \gamma' \circ s$ of $E'$
(functor $S'$), and a corresponding factor set $f'$ (monoidal structure $\eta'$). The induced quasi-action
$L'(g)(n) = s'(g)ns'(g)^{-1}$ has the same conjugacy class $[L'] = [L]$ as the one induced by the
section $s$. Since $\gamma$ is a group homomorphism, the unique functor $\Gamma$ extending $\gamma$
on objects is a strict monoidal functor, and $(S', \eta') = (\Gamma, id) \circ (S, \eta)$ as monoidal functors. Then the
two 2-cocycles $f$ and $f'$ (relative to the different quasi-actions $L$ and $L'$), coincide.
To prove that the cohomology class \([f]\) corresponding to the extension \(E\) is well defined, we may thus assume \(E' = E\). If \(s'\) is another section inducing the same outer action \([L]\), define the 1-cochain \(\gamma(x) : G \to N\) by:

\[
\gamma(x) = n \quad \text{iff} \quad j(n) = s(x)s'(x)^{-1}
\]

Then \(\gamma\) corresponds through the covering transformation to the unique natural isomorphism \(\Gamma(x) : S(x) \to S'(x)\). Since any diagram in \(B_E\) commutes:

\[
\begin{array}{ccc}
S(ab) & \xrightarrow{\Gamma(ab)} & S'(ab) \\
\eta(ab) \downarrow & & \downarrow \eta'(ab) \\
S(a) \otimes S(b) & \xrightarrow{\Gamma(a) \otimes \Gamma(b)} & S'(a) \otimes S'(b)
\end{array}
\]

and \(\Gamma\) is an isomorphism of monoidal functors. The relation obtained by applying the covering transformation:

\[
f'(ab) \gamma(ab) = R_E(\Gamma(a) \otimes \Gamma(b)) f(ab)
\]

shows that the two 2-cocycles \((f, L)\) and \((f', L')\) are weak cohomologous as defined in [8]:

\[
f'(x, y) \partial_L \gamma = \partial_L' \gamma f(x, y)
\]

If \(s\) and \(s'\) induce the same quasi-action \(L = L'\), then \(\gamma\) is central and one obtains the usual cohomology relation [13, 8]:

\[
f'(ab) \gamma(ab) = \gamma(a) \gamma(b) f(ab)
\]

**Remark 1.** The above categorical interpretation of groups and functions suggests to consider the full subcategory of group objects in the category \(Sets\) rather than just groups and group homomorphisms. Then the natural multiplication of group valued functions provides an internal \(\text{Hom}\).

### 5. Categorification and Topology

#### 5.1. Nerve of a category

The nerve of a category is another example of categorification, as it will be explain below.

Let \(\mathcal{C}\) be a small category. The categorical analog of the model spaces of singular homology are the semi-simplicial category denoted \(\Delta_n\) associated to the total ordered sets \(\{0, 1, \ldots, n\}\) [13]. The analog of singular n-cochains are the \(\mathcal{C}\)-valued functors \(C^n(\mathcal{C})\) defined on \(\Delta_n\).

The semi-simplicial structure coboundary and degeneracy functors are defined by duality:

\[
\partial^c c = c \circ \partial_i, \quad \epsilon^c c = c \circ \epsilon_i
\]

with the natural definition of boundary and degeneracy functors defined on \(\Delta_n\).

Let \(\Delta\) the quiver of categories and functors defined by the above categories \(\Delta_n\) and simplicial functors. The nerve \(\mathcal{N}C\) of the category \(\mathcal{C}\) is the value on \(\mathcal{C}\) of the representation functor \(\text{Hom}(\Delta, \cdot)\) of \(\Delta\).
Thus the nerve of a category is a categorification of acyclic model from topology, and it may be characterized as the “singular homology” of the category $\mathcal{C}$’. The correspondence is given by the geometric realization $BC$ of the semi-simplicial set $NC$ \cite{10}, which is called the classifying space of $\mathcal{C}$.

As an example, the nerve of $S_G$ is the simplicial structure underlying the bar construction for the group $G$. Its geometric realization $EG$ is a contractible, free $G$-space \cite{15}.

The classifying space $BC_G$ of the tautological groupoid associated to the group $G$ is a classifying space $BG$ for $G$ as a discrete group. All its homotopy groups are trivial except the fundamental group which is $G$. At the level of groupoids $S_G/G \cong C_G$ through the covering transformation, as well as for the corresponding topological spaces $EG/G \cong BG$ \cite{15}, by taking their geometric realization.

5.2. **Topological spaces as categories.** The above procedure associates a topological space to a small category. We will briefly recall the categorification which, in the other direction, associates a category to a topological space.

Let $(X, \tau)$ be a topological space. Define the category $\mathcal{C}_X$ of open sets with inclusions as morphisms. Intersection of open sets is a monoidal product. It is a category with final object $X$ and with direct limits. Continuous functions $f$ define functors $f^{-1}$ in the obvious way, and the construction is functorial.

Families of open sets are the (full) subcategories and covers are cofinal subcategories. As usual “subobjects” should be viewed as monomorphisms, so covers will be viewed as full and faithful functors $U : I \to \mathcal{C}_X$, defined on posets ($i \leq j$ iff $U_i \to U_j$).

Then morphisms of covers:

$$
\begin{array}{ccc}
J \xrightarrow{\phi} I & & J \xrightarrow{f^*U} \mathcal{C}_X \\
V \xrightarrow{\phi} U & & V \xrightarrow{r} C_X \\
\mathcal{C}_X \xrightarrow{id} \mathcal{C}_X & & \mathcal{C}_X \xrightarrow{\psi} \mathcal{C}_X
\end{array}
$$

are precisely refinements of open covers. Alternatively, the map $r$ is a natural transformation between the pull-back cover $\phi^*U$ and the cover $U$.

The above categorification shows that, in many cases, 2-arrows more general then natural transformations are needed. The usual approach to $n$-categories, the “globular approach”:

$$
\begin{array}{ccc}
A \xrightarrow{f} B \xrightarrow{f'} C & & \\
g \xrightarrow{\eta} \xrightarrow{\phi} \xrightarrow{\psi g'} C
\end{array}
$$
corresponds to interpreting 2-arrows as fix end homotopies, while more general transformations corresponding to arbitrary homotopies are often present:

\[
\begin{array}{c}
\partial^+ \downarrow \downarrow \partial^- \\
\text{source} \quad \downarrow H \quad \downarrow H \quad \downarrow \text{target}
\end{array}
\]

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