A DECOMPOSITION FORMULA FOR FRACTIONAL HESTON JUMP DIFFUSION MODELS

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Abstract. We present an option pricing formula for European options in a stochastic volatility model. In particular, the volatility process is defined using a fractional integral of a diffusion process and both the stock price and the volatility processes have jumps in order to capture the market effect known as leverage effect. We show how to compute a martingale representation for the volatility process. Finally, using Itô calculus for processes with discontinuous trajectories, we develop a first order approximation formula for option prices. There are two main advantages in the usage of such approximating formulas to traditional pricing methods. First, to improve computational efficiency, and second, to have a deeper understanding of the option price changes in terms of changes in the model parameters.

1. INTRODUCTION

Classical stochastic volatility models, where the volatility also follows a diffusion process, have been proven to be capable of reproducing some important features of the implied volatility together with its variation with respect to the strike price. These features are usually described through the smile or skew, see [13]. One of the main downsides from these class of models is their inability to explain what is known as the term structure of the skew, i.e. the dependence of the skew to the time to maturity.

For instance, it can be observed in [14], how a decrease of the smile amplitude when time to maturity increases, turns out to be much slower than it should be according to standard stochastic volatility models. On the other hand, the observed short-time implied volatility skew slope tends to infinity as time to maturity tends to zero, whereas this limit is a constant under classical stochastic volatility models.

On one hand, the long-memory features for the volatility process can be achieved by the introduction of fractional noises with a Hurst parameter $H > 1/2$ in the volatility process, as introduced by Comte and Renault in [7] and deeply studied in [4]. This allows to endow the volatility with high persistence in the long-run, showing the steepness of long-term volatility smiles without over increasing the short-run persistence. On the other hand, it was proved in [3] that these models fail to describe the short-time behavior of implied volatility. In order to overcome this limitation, we present one of the possible approaches, consisting in adding a jump term to both the stock price and the volatility processes with a correlation factor between both jumps in order to reproduce the so-called leverage effect. This results into a combination between the fractional setup of Alòs and Yang [4] and the jump diffusion framework, first studied by Merton in [21]. In the present work, we build an option price approximation methodology for a combination of the fractional

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\footnote{This is a well known effect observed in most markets that shows how most measures of volatility of an asset are negatively correlated with the returns of that asset.}
and the jump diffusion models previously mentioned, which capture the short-time and long-time behavior of implied volatility. Hence, we will study what we will call fractional stochastic volatility jump diffusion (FSVJJ) models with both jumps in the price and volatility processes.

The first stochastic volatility models with jumps were introduced by Bates in [6] and were achieved by incorporating jumps to stochastic variance processes, previously introduced by Heston (1993) in [16]. In our case of study, the variance of stock prices follows a combination of a CIR process [9], a fractional integral of the stochastic term in the CIR process and a jump term driven by a Lévy-type process. Adding the jump framework to the model should improve the market fit for short-term maturity options, overcoming the original problem of the Heston stochastic volatility model. The last would require unrealistically high values for the vol-of-vol parameter in order to obtain a reasonable fit of short-term smiles. An alternative approach to model short-term smiles is the use of rough fractional volatility models, see for instance [14, 11].

Pricing derivatives under stochastic volatility jump diffusion models involves, naturally, an extra degree of complexity compared to the standard Black-Scholes pricing framework. This has motivated the development of approximating formulas in the literature such as [17, 1, 2, 15, 20]. These formulas provide good intuition on the behavior of the smiles and a better understanding of the effects of changes in the model parameters onto the price of a derivative. Despite not being closed pricing formulae, they bring clarity to the practitioner to understand the effects of model parameters in the option price. As well as, speed up the calibration process as proved in [15]. We will use this idea to find a general decomposition formula for a fractional Heston model with jumps in the price and the volatility processes under basic integrability conditions. In a recent paper [19], the authors find decomposition formulas in the setup of rough volatility models.

This paper is organized as follows. First, we introduce in Section 2 a detailed description of the model that will be used throughout the article. Section 3 is devoted to introduce some preliminary concepts and notation needed in later parts of our study. Then, in Section 4 we present an exact expansion formula for option pricing in terms of the Black-Scholes formula adjusted by extra terms that depend on the future expected volatility, the jumps and correlation parameters. We provide a martingale representation of future expected volatility in Section 5 by means of the Clark-Ocone-Haussmann formula and end the section providing its dynamics. We conclude by developing a first order approximating formula of a call option under our fractional Heston jump diffusion model. Finally, we provide an appendix in Section 7 with some additional technical results.

2. A Fractional Heston Model with Jumps in the Stock Log-Prices and Volatility (FSVJJ) Model

Let $T > 0$ be fixed time horizon and let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a complete probability space. Assume that in $(\Omega, \mathcal{F}, \mathbb{Q})$ there are defined $W = \{W_t\}_{t \in [0,T]}$ and $\tilde{W} = \{\tilde{W}_t\}_{t \in [0,T]}$, two independent standard Brownian motions. Moreover, assume that in $(\Omega, \mathcal{F}, \mathbb{Q})$ there is defined a Lévy subordinator $J = \{J_t\}_{t \in [0,T]}$ which is independent of the Brownian motions $W$ and $\tilde{W}$. We define the filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t \in [0,T]}$ to be the minimal augmented filtration generated

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2Subordinators are Lévy processes with increasing paths. Alternatively, they are Lévy processes with finite variation paths and positive jumps. These type of processes are often used in Lévy-based financial models. See [8].
by $W, \tilde{W}$ and $J$. We assume that $J$ is a Lévy subordinator with generating triplet $(0, \ell, \gamma)$, that is,

$$
\mathbb{E} [e^{iyJ}] = \exp \left( t \left( iy + \int_0^\infty (e^{iyz} - 1 - z1_{(0<z<\infty)}) \ell(dz) \right) \right),
$$

with $y \in \mathbb{R}, b = \gamma - \int_0^1 z \ell(dz) \geq 0$, and $\ell$ a Lévy measure with support on $(0, \infty)$ and satisfying $\int_0^1 z \ell(dz) < \infty$. Note that we are not assuming $\ell((0, \infty)) < \infty$ and, therefore, the process $J$ may have infinite activity, that is, the $J$ may have an infinite number of small jumps on any finite time interval. We will assume that for some $C > 0$ we have $\int_1^\infty e^{Cz} \ell(dz) < \infty$. Then $J$ has moments of all orders. In particular, $J$ has finite expectation which is equivalent to $\int_1^\infty z \ell(dz) < \infty$, see Proposition 3.13 in [8]. By the Lévy-Itô decomposition we have that $J$ can be written as

$$
J_t = \gamma t + \int_0^t \int_0^1 z \tilde{N}(ds, dz) + \int_0^t \int_1^\infty z N(ds, dz)
$$

$$
= \gamma t - \int_0^t \int_0^1 z \ell(dz) ds + \int_0^t \int_0^\infty z N(ds, dz)
$$

$$
= bt + \int_0^t \int_0^\infty z N(ds, dz)
$$

$$
= \left( b + \int_0^\infty z \ell(dz) \right) t + \int_0^t \int_0^\infty z \tilde{N}(ds, dz)
$$

$$
= \left( b + \int_0^\infty z \ell(dz) \right) t + \tilde{J}_t
$$

$$
= \left( \gamma + \int_1^\infty z \ell(dz) \right) t + \tilde{J}_t,
$$

where $N(ds, dz)$ denotes the Poisson random measure with Lévy measure $\ell$ and $\tilde{N}(ds, dz) \triangleq N(ds, dz) - \ell(dz) ds$ denotes its associated compensated Poisson random measure.

Next we introduce the volatility process that we will use in our model. Following [4], we start by considering a CIR process $\tilde{\sigma}^2 = \{\tilde{\sigma}_t^2\}_{t \in [0,T]}$ of the following form

$$
d\tilde{\sigma}_t^2 = \kappa (\theta - \tilde{\sigma}_t^2) dt + \nu \sqrt{\tilde{\sigma}_t^2} dW_t,
$$

with $\tilde{\sigma}_0^2, \theta, \kappa, \nu > 0$ and satisfying the so called Feller condition $2\kappa \theta \geq \nu^2$, in order to ensure positivity for the variance process, see [11]. Applying Itô formula to the process $e^{\kappa t} \tilde{\sigma}_t^2$ one can easily see that the following holds

$$
\tilde{\sigma}_t^2 = \theta + e^{-\kappa t} (\tilde{\sigma}_0^2 - \theta) + \nu \int_0^t e^{-\kappa(t-s)} \sqrt{\tilde{\sigma}_s^2} dW_s.
$$

Note that we can write $\tilde{\sigma}^2$ as the sum of two processes $Y = \{Y_t\}_{t \in [0,T]}$ and $Z = \{Z_t\}_{t \in [0,T]}$, defined by

$$
U_t \triangleq \theta + e^{-\kappa t} (\tilde{\sigma}_0^2 - \theta), \quad t \in [0, T],
$$

and

$$
Z_t \triangleq \int_0^t e^{-\kappa(t-s)} \sqrt{\tilde{\sigma}_s^2} dW_s, \quad t \in [0, T].
$$
Finally, we will consider a fractional volatility model with only positive jumps built on a combination of the processes $Y, Z$ and a fractional integral of $Z$. Let us denote this fractional volatility process with jumps by $\sigma^2 = \{\sigma^2_t\}_{t \in [0,T]}$, where

$$\sigma^2_t \triangleq U_t + c_1 \nu Z_t + c_2 \nu I^{H-\frac{1}{2}}_{0+} Z_t + c_3 \eta J_t, \quad t \in [0,T],$$

where $H \in (\frac{1}{2}, 1), c_1, c_2, c_3, \eta \geq 0$ and $I^{H-\frac{1}{2}}_{0+}$ is the left-sided fractional Riemann-Liouville integral of the path $Z(\omega)$ of order $H - \frac{1}{2}$ on $[0,T]$. Recall that, given $f \in L^1([0,T])$, the left-sided fractional Riemann-Liouville integral of $f$ of order $\alpha \in \mathbb{R}_+$ on $[0,T]$ is defined as

$$\left(I^{\alpha}_{0+} f\right)(t) \triangleq \frac{1}{\Gamma(\alpha)} \int_0^t f(u)(t-u)^{\alpha-1} du.$$

On the other hand let $\rho_1 \in [-1,1], \rho_2 \leq 0$ and consider a Lévy model for the dynamics of the stock log-price in the time interval $[0,T]$ given by the following equation

$$X_t = x + rt - \frac{1}{2} \int_0^t \sigma^2_s ds + \int_0^t \sigma_s \left( \rho_1 dW_s + \sqrt{1-\rho_2^2} dB_s \right) + \rho_2 \eta J_t,$$

where $r$ is the risk-free rate. For the process $e^{-rt} e^{X_t}$ to be a local martingale, we must assume, see Corollary 5.2.2 in [5], that

$$\rho_2 \eta + \int_0^1 \left( e^{\rho_2 \eta z} - 1 - \rho_2 \eta z 1_{(0<z<1)} \right) \ell(z) + \int_1^{\infty} \left( e^{\rho_2 \eta z} - 1 \right) \ell(z) = 0.$$

Note that $\gamma = -\frac{1}{\rho_2 \eta} \int_0^\infty \left( e^{\rho_2 \eta z} - 1 - \rho_2 \eta z 1_{(0<z<1)} \right) \ell(z)$. Since we have already proved that $J_t = (\gamma + \int_1^{\infty} z \ell(z)) t + J_t$, it is trivial to see that

$$\rho_2 \eta J_t = -\int_0^\infty \left( e^{\rho_2 \eta z} - 1 - \rho_2 \eta z 1_{(0<z<1)} \right) \ell(z) + \rho_2 \eta \int_1^{\infty} z \ell(z) t + \rho_2 \eta J_t$$

$$= -\int_0^\infty \left( e^{\rho_2 \eta z} - 1 - \rho_2 \eta z \right) \ell(z) t + \rho_2 \eta J_t$$

$$\triangleq -\zeta(\rho_2, \eta) t + \rho_2 \eta J_t$$

Now we have that, whenever a jump occurs in the volatility process, since $\rho_2 \leq 0$ we get a negative jump in log-prices, so we can model the leverage effect. We can also specify the model through the following equations

$$X_t = x + (r - \zeta(\rho_2, \eta)) t - \frac{1}{2} \int_0^t \sigma^2_s ds + \int_0^t \sigma_s \left( \rho_1 dW_s + \sqrt{1-\rho_2^2} dB_s \right) + \rho_2 \eta J_t,$$

$$\sigma^2_t = Y_t + c_1 \nu Z_t + c_2 \nu I^{H-\frac{1}{2}}_{0+} Z_t + c_3 \eta J_t.$$

3. Preliminaries and notation

Following similar ideas to the ones found in [3], we will extend the decomposition formula to a fractional Heston model with infinite activity jumps in both prices and volatility. It is well known that $V_t$, the value at time $t$ of a derivative whose payoff is $h(X_T)$, is given by the risk neutral pricing formula

$$V_t(h) = e^{-r(T-t)} \mathbb{E} [h(X_T) \mid \mathcal{F}_t].$$

We now proceed to introduce some definitions and notations which will be used throughout the paper:

- We will denote $\mathbb{E}_\mathcal{F} [\cdot] \triangleq \mathbb{E} [\cdot \mid \mathcal{F}_t]$. 

Let $BS(t, x, \sigma)$ denote the price of a plain vanilla European call option under the classical Black-Scholes pricing formula with constant volatility $\sigma$, stock log-price $x$, strike price $K$, time to maturity $T - t$, and constant interest rate $r$. In this case,

$$BS(t, x, \sigma) = e^{\gamma} (d_+ - Ke^{-r(T-t)}),$$

where $\Phi$ denotes the standard normal cumulative probability function and $d_+$ is defined as

$$d_+ \triangleq \frac{x - \ln K - r(T - t)}{\sigma \sqrt{T - t}} + \frac{\sigma}{2} \sqrt{T - t}.$$

In our setting, the price of a call option at time $t$ is given by

$$V_t = e^{-r(T-t)} E_t [(e^{X_T} - K)^+].$$

We recall from the Feynman-Kac formula for the continuous version of the model (2.3), the operator

$$\mathcal{L}_\sigma \triangleq \partial_t + \frac{1}{2} \sigma^2 \partial^2_{xx} + \left( r - \frac{1}{2} \sigma^2 \right) \partial_x - r.$$

Note that $\mathcal{L}_\sigma BS(\cdot, \cdot, \sigma) = 0$ by construction.

We will use an adapted projection of the future average variance defined by

$$v_t^2 \triangleq \frac{1}{T - t} \int_t^T E_t [\sigma_s^2] \, ds,$$

to obtain a decomposition of $V_t$ in terms of $v_t$. This idea, used in [2], switches an anticipative problem into a non-anticipative one, related to the adapted process $v_t$.

We define $M_t \triangleq \int_t^T E_t [\sigma_s^2] \, ds$. Notice then that the projected future average variance can be written as $v_t^2 = \frac{1}{T - t} (M_t - \int_0^t \sigma_s^2 \, ds)$. Recall that, by definition, $M$ is a martingale with respect to the filtration generated by $W$ and $J$, is also $\mathcal{F}_W$-independent and its dynamics is given by

$$dM_t = \nu A(T, t) \sqrt{\sigma_t^2} \, dW_t + c_3 \eta dJ_t,$$

as it is later proved in Proposition [3]. It will also be useful to introduce $M^c$, the continuous part of the process $M$, with dynamics given by

$$dM^c_t = \nu A(T, t) \sqrt{\sigma_t^2} \, dW_t.$$

Let $\{X_t\}_{t \in [0, T]}$ and $\{Y_t\}_{t \in [0, T]}$ be two Itô-Lévy processes given by the following dynamics

$$dX_t = \alpha_x(t) \, dt + \beta_x(t) \, dW_t + \int_0^\infty \gamma_x(t, z) \, \tilde{N}(dt, dz),$$

$$dY_t = \alpha_y(t) \, dt + \beta_y(t) \, dW_t + \int_0^\infty \gamma_y(t, z) \, \tilde{N}(dt, dz).$$

Given a function $F \in C^{0,1,1}([0, T] \times \mathbb{R} \times \mathbb{R})$, we define

$$\Delta_x F(t, X_{t-}, Y_{t-}) \triangleq F(t, X_{t-} + \gamma_x(t, z), Y_{t-}) - F(t, X_{t-}, Y_{t-}),$$

$$\Delta_y F(t, X_{t-}, Y_{t-}) \triangleq F(t, X_{t-} + \gamma_y(t, z), Y_{t-}) - F(t, X_{t-}, Y_{t-}),$$

$$\Delta^2_x F(t, X_{t-}, Y_{t-}) \triangleq \Delta_x F(t, X_{t-}, Y_{t-}) - \gamma_x(t, z) (\partial_x F)(t, X_{t-}, Y_{t-}),$$

$$\Delta^2_y F(t, X_{t-}, Y_{t-}) \triangleq \Delta_y F(t, X_{t-}, Y_{t-}) - \gamma_y(t, z) (\partial_y F)(t, X_{t-}, Y_{t-}).$$
The introduction of the following differential operators is very convenient for notational purposes, and both expressions will be used indistinctly throughout the article.

\[ \Lambda \triangleq \partial_x, \]
\[ \Gamma \triangleq (\partial_x^2 - \partial_x); \quad \Gamma^2 = \Gamma \circ \Gamma = (\partial_x^4 - 2\partial_x^2 + \partial_x^2). \]

- Given two continuous semimartingales \( X \) and \( Y \), we define the following processes

\[
L[X,Y]_t \triangleq \mathbb{E}_t \left[ \int_t^T \sigma_u d[X,Y]_u \right],
\]
\[
D[X,Y]_t \triangleq \mathbb{E}_t \left[ \int_t^T d[X,Y]_u \right],
\]

for \( t \in [0,T] \).

Since the derivatives of \( BS(t,X_t,v_t) \) are not bounded, we will make use of an approximating argument. Consider the approximation \( v_t^\delta \) of \( v_t \) for a fixed \( \delta > 0 \), given by

\[
v_t^\delta \triangleq \sqrt{\frac{1}{T-t} \left( \delta + \int_t^T \mathbb{E}_t [\sigma_u^2] du \right)} = \sqrt{\frac{1}{T-t} \left( \delta + M_t - \int_0^t \sigma_u^2 du \right)}.
\]

The following proposition shows how the dynamics of this process is obtained.

**Proposition 1.** For fixed \( \delta > 0 \), let \( v_t^\delta \) be the approximation of the adapted projection of the average future volatility process, defined in (3.2). Then, the dynamics of \( v_t^\delta \) is given by

\[
dv_t^\delta = \frac{(v_t^\delta)^2 - \sigma_t^2}{2v_t^\delta (T-t)} dt + \frac{dM_t^\delta}{2v_t^\delta (T-t)} - \frac{[M,c,M]_t}{8 (v_t^\delta)^3 (T-t)^2} \]
\[
\quad + \int_0^\infty \Delta_m^2 g^\delta (t, M_{t-}, Y_{t-}) \ell (dz) dt + \int_0^\infty \Delta_m g^\delta (t, M_{t-}, Y_{t-}) \tilde{N} (dt, dz),
\]

where \( g^\delta (t,m,y) = \sqrt{\frac{1}{T-t} (\delta + m - y)} \).

**Proof.** Define the realized variance as \( Y_t \triangleq \int_0^t \sigma_u^2 du \) for every \( t \in [0,T] \). Note that \( v_t^\delta = g^\delta (t, M_t, Y_t) \), where the dynamics of \( Y \) and \( M \) are given by

\[
dY_t = \sigma_t^2 dt,
\]
\[
dM_t = \nu A (T,t) \sqrt{\sigma_t^2} dW_t + \int_0^\infty c_3 \eta z \tilde{N} (dt, dz).
\]

Now following [22], we can apply the multidimensional Itô formula for Itô-Lévy processes to obtain

\[
dv_t^\delta = dg^\delta (t, M_t, Y_t)
\]
\[
\quad = \frac{\partial g^\delta}{\partial t} (t, M_t, Y_t) dt + \frac{\partial g^\delta}{\partial m} (t, M_t, Y_t) dM_t^\delta + \frac{\partial g^\delta}{\partial y} (t, M_t, Y_t) dY_t
\]
\[
\quad + \frac{1}{2} \frac{\partial^2 g^\delta}{\partial m^2} (t, M_t, Y_t) d[M^c,M^c]_t + \int_0^\infty \Delta_m g^\delta (t, M_{t-}, Y_{t-}) \ell (dz) dt
\]
\[
\quad + \int_0^\infty \Delta_m g^\delta (t, M_{t-}, Y_{t-}) \tilde{N} (dt, dz)
\]
\[
\quad = \frac{(v_t^\delta)^2}{2v_t^\delta (T-t)} dt + \frac{dM_t^\delta}{2v_t^\delta (T-t)} - \frac{\sigma_t^2}{2v_t^\delta (T-t)} dt - \frac{[M^c,M^c]_t}{8 (v_t^\delta)^3 (T-t)^2} \]
filtration

expectation of depending on the future expected variance, the Lévy measure of the jumps and correlation formula for option pricing in terms of the Black-Scholes formula adjusted by extra terms.

In this section we present the main result of the paper. We provide an exact expansion parts of (3.4)

\[ X_t = \int_0^t \left( r - \zeta (\rho_2, \eta) - \frac{1}{2} \sigma_t^2 \right) dt + \sigma_t \left( \rho_1 dW_t + \sqrt{1 - \rho_1^2} d\tilde{W}_t \right), \]

where \( \gamma_m (t, z) \triangleq c_3 \eta z \) in the expression for \( \Delta_m^2 g^\delta (t, M_{t-}, Y_{t-}) \).

**Remark 2.** It will be useful in further results to write down the dynamics for the continuous parts of \( X_t \) and \( v_t^\delta \), respectively as

\[ dX_t^e = \left( r - \zeta (\rho_2, \eta) - \frac{1}{2} \sigma_t^2 \right) dt + \sigma_t \left( \rho_1 dW_t + \sqrt{1 - \rho_1^2} d\tilde{W}_t \right); \]

\[ d\left( v_t^\delta \right)^e = \left[ \frac{(v_t^\delta)^2 - \sigma_t^2}{2v_t^\delta (T-t)} + \int_0^t \Delta_m^2 g^\delta (t, M_{t-}, Y_{t-}) \ell (dz) \right] dt \]

\[ + \frac{dM_t^e}{2v_t^\delta (T-t)} - \frac{d [M^c, M^c]_t}{8 (v_t^\delta)^3 (T-t)^2}. \]

4. General expansion formulas

In this section we present the main result of the paper. We provide an exact expansion formula for option pricing in terms of the Black-Scholes formula adjusted by extra terms depending on the future expected variance, the Lévy measure of the jumps and correlation parameters.

**Theorem 3.** Let \( B = \{B_t, t \in [0, T]\} \) be a continuous semimartingale with respect to the filtration \( \mathcal{F}^W \vee \mathcal{F}^I \), let \( A (t, x, y) \) be a \( C^{1,2,2} ([0, T] \times \mathbb{R} \times \mathbb{R}) \) function such that

\[ \mathcal{L}_y A = \left( \partial_t + \frac{1}{2} y^2 \partial_x^2 + \left( r - \frac{1}{2} y^2 \right) \partial_x - r \right) A = 0, \]

and let \( v_t \) and \( M_t \) be defined as in the previous section. Then, for every \( t \in [0, T] \), the expectation of \( e^{-r(T-t)} A (T, X_T, v_T) B_T \) can be written as follows:

\[ e^{-r(T-t)} \mathbb{E}_t [ A (T, X_T, v_T) B_T ] \]

\[ = A (t, X_t, v_t) B_t - \zeta (\rho_2, \eta) \mathbb{E}_t \left[ \int_t^T e^{-rs} \partial_s A (s, X_s, v_s) B_s ds \right] \]

\[ + \frac{1}{2} \mathbb{E}_t \left[ \int_t^T e^{-r(s-t)} \left( \partial_z^2 - \partial_x \right) A (s, X_s, v_s) B_s \left( \sigma_s^2 - v_s^2 \right) ds \right] \]

\[ + \frac{1}{2} \mathbb{E}_t \left[ \int_t^T e^{-r(s-t)} \partial_y A (s, X_s, v_s) B_s \left( v_s^2 - \sigma_s^2 \right) ds \right] \]

\[ + \frac{1}{8} \mathbb{E}_t \left[ \int_t^T e^{-r(s-t)} \partial_y A (s, X_s, v_s) B_s \left( \frac{d [M^c, M^c]_s}{v_s^3 (T-s)^2} \right) \right] \]

\[ + \mathbb{E}_t \left[ \int_t^T e^{-r(s-t)} A (s, X_s, v_s) dB_s \right] \]

\[ + \frac{\rho_1}{2} \mathbb{E}_t \left[ \int_t^T e^{-r(s-t)} \partial_{xy} A (s, X_s, v_s) B_s \frac{\sigma_s}{v_s (T-s)} d [W, M^c]_s \right] \]
\[ + \frac{1}{8} \mathbb{E}_t \left[ \int_t^T e^{-r(s-t)} \partial_y^2 A(s, X_s, v_s) B_s \frac{d[M^c, M^c]_s}{v_s^2 (T-s)^2} \right] + \rho_1 \mathbb{E}_t \left[ \int_t^T e^{-r(s-t)} \partial_x A(s, X_s, v_s) \sigma_s d[W, B]_s \right] \\
+ \sqrt{1 - \rho_1^2} \mathbb{E}_t \left[ \int_t^T e^{-r(s-t)} \partial_x A(s, X_s, v_s) \sigma_s d[\tilde{W}, B]_s \right] + \mathbb{E}_t \left[ \int_t^T \int_0^\infty e^{-r(s-t)} B_s \left[ \Delta_x^2 A(s, X_{s-}, v_{s-}) + \Delta_y^2 A(s, X_{s-}, v_{s-}) \right] \ell(dz)ds \right]. \]

**Proof.** Let \( F(t, X_t, v_t^\delta) \equiv e^{-rt} A(t, X_t, v_t^\delta) B_t \), and apply again, the multidimensional Itô formula for Lévy processes to \( F(t, X_t, v_t^\delta) \). To do so, we will consider the continuous parts of \( X_t \) and \( v_t^\delta \), respectively given by equations (3.4) and (3.5). Therefore we can write the Itô formula in its integral version over the time interval \([t, T]\) as follows:

\[ e^{-rT} A(T, X_T, v_T^\delta) B_T = e^{-rt} A(t, X_t, v_t^\delta) B_t - r \int_t^T e^{-rs} A(s, X_s, v_s^\delta) B_s ds + \int_t^T e^{-r\rho s} \partial_x A(s, X_s, v_s^\delta) B_s dX^c_s + \int_t^T e^{-r\rho s} \partial_y A(s, X_s, v_s^\delta) B_s dX^c_s \]

\[ + \frac{1}{2} \int_t^T e^{-r\rho s} \partial_x^2 A(s, X_s, v_s^\delta) B_s d[X^c, X^c]_s + \int_t^T e^{-r\rho s} \partial_y^2 A(s, X_s, v_s^\delta) B_s d[X^c, X^c]_s + \frac{1}{2} \int_t^T \int_0^\infty e^{-r\rho s} B_s \left[ \Delta_x^2 A(s, X_{s-}, v_{s-}^\delta) + \Delta_y^2 A(s, X_{s-}, v_{s-}^\delta) \right] \ell(dz)ds \]

Now, recalling the definitions of \( d[X^c, X^c]_s \) and \( d(v^\delta)^c \), the fact that

\[ d[X^c, X^c]_s = \sigma_s^2 ds, \]

\[ d[(v^\delta)^c, (v^\delta)^c]_s = \frac{\rho_1 d[M^c, M^c]_s}{2v_s'(T-s)} + \sqrt{1 - \rho_1^2} d[\tilde{W}, M^c]_s, \]

\[ d[(v^\delta)^c, (v^\delta)^c]_s = \frac{d[M^c, M^c]_s}{4(v^\delta)^2 (T-s)^2}, \]
\[ d\left([X^c,B]_s\right) = \sigma_s \left( \rho_1 d\left([W,B]_s\right) + \sqrt{1 - \rho_1^2} d\left[\tilde{W},B\right]_s \right), \]
\[ d\left([v^\delta]^c,B\right)_s = \frac{d[M^c,B]_s}{2v^\delta \left(T-s\right)}, \]

and the independence between \( M \) and \( \tilde{W} \), we can rewrite \( e^{-rT} A \left(T, X_T, v^\delta_T\right) B_T \) as
\[ e^{-rT} A \left(T, X_T, v^\delta_T\right) B_T \]
\[ = e^{-rt} A \left(t, X_t, v^\delta_t\right) B_t \]
\[ + \int_t^T e^{-rs} \left( \partial_s + \frac{1}{2} \sigma_s^2 \partial^2_s + \left(r - \frac{1}{2} \sigma_s^2 \right) \partial_x - r \right) A \left(s, X_s, v^\delta_s\right) B_s ds \]
\[ - \zeta \left(\rho_2, \eta\right) \int_t^T e^{-rs} \partial_x A \left(s, X_s, v^\delta_s\right) B_s ds \]
\[ + \int_t^T e^{-rs} \partial_x A \left(s, X_s, v^\delta_s\right) B_s \left[ \sigma_s \left(\rho_1 dW_s + \sqrt{1 - \rho_1^2} d\tilde{W}_s\right) \right] \]
\[ + \frac{1}{2} \int_t^T e^{-rs} \partial_y A \left(s, X_s, v^\delta_s\right) B_s \left[ \frac{(v^\delta_s)^2 - \sigma_s^2}{v^\delta_s \left(T-s\right)} + 2 \int_0^\infty \Delta^2_s g^\delta \left(s, M_{s-}, Y_{s-}\right) \ell \left( dz\right) \right] ds \]
\[ + \frac{1}{2} \int_t^T e^{-rs} \partial_y A \left(s, X_s, v^\delta_s\right) B_s \left[ \frac{d[M^c,M^c]_s}{(v^\delta_s)^3 \left(T-s\right)^2} \right] \]
\[ - \frac{1}{8} \int_t^T e^{-rs} \partial_y A \left(s, X_s, v^\delta_s\right) B_s \left[ \frac{d[M^c,B]_s}{2v^\delta \left(T-s\right)} \right] \]
\[ + \int_t^T e^{-rs} \partial_x A \left(s, X_s, v^\delta_s\right) dB_s \]
\[ + \frac{\rho_1}{2} \int_t^T e^{-rs} \partial^2_x A \left(s, X_s, v^\delta_s\right) B_s \left[ \sigma_s \left(\rho_1 dW_s + \sqrt{1 - \rho_1^2} d\tilde{W}_s\right) \right] \]
\[ + \frac{1}{2} \int_t^T e^{-rs} \partial^2_x A \left(s, X_s, v^\delta_s\right) B_s \left[ \frac{d[M^c,M^c]_s}{4(v^\delta_s)^2 \left(T-s\right)^2} \right] \]
\[ + \rho_1 \int_t^T e^{-rs} \partial_x A \left(s, X_s, v^\delta_s\right) \sigma_s d[W,B]_s \]
\[ + \sqrt{1 - \rho_1^2} \int_t^T e^{-rs} \partial_x A \left(s, X_s, v^\delta_s\right) \sigma_s d[\tilde{W},B]_s \]
\[ + \int_t^T e^{-rs} \partial_y A \left(s, X_s, v^\delta_s\right) d[M^c,B]_s \]
\[ + \int_t^T \int_0^\infty e^{-rs} B_s \left[ \Delta^2_x A \left(s, X_{s-}, v^\delta_{s-}\right) + \Delta^2_y A \left(s, X_{s-}, v^\delta_{s-}\right) \right] \ell \left( dz\right) ds \]
\[ + \int_t^T \int_0^\infty e^{-rs} B_s \left[ \Delta_x A \left(s, X_{s-}, v^\delta_{s-}\right) + \Delta_y A \left(s, X_{s-}, v^\delta_{s-}\right) \right] \tilde{N} \left( ds, dz\right). \]

We can identify in the previous expression, the operator \( \mathcal{L}_{\sigma_s} \). We know from [18], that the following is true,
\[ \mathcal{L}_{\sigma_s} = \mathcal{L}_{v^\delta} + \frac{1}{2} \left( \sigma_s^2 \left(\rho_1 dW_s + \sqrt{1 - \rho_1^2} d\tilde{W}_s\right) \right) \left(\partial_x^2 - \partial_x \right), \]
where \( \mathcal{L}_{v^\delta} A = 0 \). Therefore, multiplying by \( e^{rt} \), taking conditional expectation and letting \( \delta \to 0 \), combined with the use of the dominated convergence theorem, we obtain result follows, ending the proof. \( \Box \)
If we assume additional properties on the function $A$ one can simplify the formula given in the previous theorem. In the following corollary, we assume certain relationship of the partial derivative with respect to $y$ and the first and second order partial derivatives with respect to $x$. This relation is often referred in the literature as the Delta-Gamma-Vega relationship and it is satisfied by the Black-Scholes function.

**Corollary 4.** Let the function $A$ and the process $B$, be defined as in Theorem 3. Suppose that the function $A$ satisfies the Delta-Gamma-Vega relationship given by

$$
\frac{\partial_y A(t, x, y)}{y(T-t)} = \frac{1}{y(T-t)} \left( \partial_{xx}^2 - \partial_x\right) A(t, x, y).
$$

Then, for every $t \in [0, T]$, the following formula holds:

$$
e^{-r(T-t)}\mathbb{E}_t[A(T, X_T, v_T) B_T] = A(t, X_t, v_t) B_t - \zeta (\rho_2, \eta) \mathbb{E}_t \left[ \int_t^T e^{-r \Delta s} \Lambda A(s, X_s, v_s) B_s ds \right] + \mathbb{E}_t \left[ \int_t^T e^{-r \Delta s} \Gamma A(s, X_s, v_s) B_s ds \right]
+ \mathbb{E}_t \left[ \int_t^T e^{-r \Delta s} \Lambda A(s, X_s, v_s) \sigma_s dB_s \right]
+ \frac{\rho_1}{2} \mathbb{E}_t \left[ \int_t^T e^{-r \Delta s} \Delta A(s, X_s, v_s) \sigma_s d[W, B]_s \right]
+ \frac{1}{8} \mathbb{E}_t \left[ \int_t^T e^{-r \Delta s} \Gamma^2 A(s, X_s, v_s) dB_s d[M^c, B]_s \right]
+ \frac{1}{2} \mathbb{E}_t \left[ \int_t^T e^{-r \Delta s} \Gamma A(s, X_s, v_s) \sigma_s d[W, B]_s \right]
+ \mathbb{E}_t \left[ \int_t^T \int_0^\infty e^{-r \Delta s} B_s \left[ \Delta_x^2 A(s, X_s, v_s) + \Delta_y^2 A(s, X_s, v_s) \right] \ell(dz) ds \right].
$$

**Proof.** If (4.1) holds, then it is trivial to see that

$$
\partial_{xy}^2 A(t, x, y) = y (T-t) \left( \partial_{xx}^3 - \partial_{xx}^2 \right) A(t, x, y),
$$

$$
\partial_{yy}^2 A(t, x, y) = \frac{y^2 (T-t)}{y^2 (T-t)} \left( \partial_{xx}^2 - \partial_x \right) A(t, x, y)
+ y^2 (T-t) \left( \partial_{xx}^2 - \partial_x \right) A(t, x, y).
$$

Therefore, by replacing (4.1) together with the previous equalities in Theorem 3, the result is straightforward. \[\square\]

The next result yields an analogous formula to Proposition 9 in [4], that contains the continuous part of the formula and the discontinuous terms coming from the jumps assumed in the model.
Theorem 5. Assume the model given by equations (2.3) such that \(2k\theta \geq \nu^2\),
\[
\left(1 - c_1 - c_2 \frac{T^\alpha}{\alpha \Gamma(\alpha)}\right) \geq 0,
\]
and let \(A(t, X_t, v_t) = BS(t, X_t, v_t)\) and \(B_t \equiv 1\). Then, for every \(t \in [0, T]\),
\[
V_t = BS(t, X_t, v_t) - \zeta(\rho_2, \eta) \mathbb{E}_t \left[ \int_t^T e^{-r\tau} \Delta BS(s, X_s, v_s) \, ds \right]
+ \frac{\rho_1}{2} \mathbb{E}_t \left[ \int_t^T e^{-r(s-t)} \Delta BS(s, X_s, v_s) \sigma_s \, d\mathbb{M} \right]
+ \frac{1}{8} \mathbb{E}_t \left[ \int_t^T e^{-r(s-t)} \Gamma^2 BS(s, X_s, v_s) \, d\mathbb{M} \right]
+ \mathbb{E}_t \left[ \int_t^T \int_0^\infty e^{-r(s-t)} \left[ \Delta^2 BS(s, X_{s-}, v_{s-}) + \Delta^2 BS(s, X_{s-}, v_{s-}) \right] \ell(dz) \, ds \right]
\]
(4.2)
\[
\Delta \triangleq BS(t, X_t, v_t) - \zeta(\rho_2, \eta) \int_t^T e^{-r\tau} \Delta BS(s, X_s, v_s) \, ds + (I) + (II) + (III) + (IV).
\]

Proof. The result is trivially achieved by replacing \(A(t, X_t, v_t) = BS(t, X_t, v_t)\) and \(B_t \equiv 1\) in Corollary 4 and noticing that \(V_t = e^{-r(T-t)} \mathbb{E}_t \left[ BS(T, X_T, v_T^T) \right]\).

5. Martingale Representation of the Future Expected Volatility

This section is devoted to derive an expression for the dynamics of the integrated projected future variance \(M\). In the next proposition we show that, under certain conditions on the parameters, the variance process is bounded away from zero (lower bounded by a strictly positive function).

Proposition 6. Consider \(\alpha \in \left(0, \frac{1}{2}\right)\) and \(T \geq 0\). Assuming that \(2k\theta \geq \nu^2\) and
\[
\left(1 - c_1 - c_2 \frac{T^\alpha}{\alpha \Gamma(\alpha)}\right) \geq 0.
\]
Then for all \(0 < t < T\)
\[
\sigma_t^2 \geq \sigma_0^2 e^{-\kappa t} + \theta (1 - e^{-\kappa t}) \left(1 - c_1 - c_2 \frac{T^\alpha}{\alpha \Gamma(\alpha)}\right) \text{ a.s.}
\]
(5.1)

Proof. We know by definition of the volatility equation given by (2.3), that
\[
\sigma_t^2 = U_t + c_1 \nu Z_t + c_2 \nu \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} Z_r \, dr + c_3 \eta J_t.
\]

Knowing that jumps are only positive and that \(c_3 \geq 0, \eta > 0\), we can lower bound the volatility process by
\[
\sigma_t^2 \geq U_t + c_1 \nu Z_t + c_2 \nu \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} Z_r \, dr.
\]
Back to the Heston’s volatility process, we know that it has to be positive, therefore $\nu Z_t > -U_t = -\theta + e^{-\kappa t}(\tilde{\sigma}_0^2 - \theta)$ for all initial condition $\tilde{\sigma}_0^2$. And letting $\tilde{\sigma}_0^2 \to 0$ we have that $\nu Z_t \geq -\theta(1 - e^{-\kappa t})$ a.s. Now we can write

$$
\sigma_t^2 \geq U_t - c_1(1 - e^{-\kappa t}) - c_2 \frac{\theta}{\Gamma(\alpha)} \int_0^t (t - r)^{\alpha-1}(1 - e^{-\kappa r})dr
$$

$$
\geq U_t - c_1(1 - e^{-\kappa t}) - c_2 \frac{\theta(1 - e^{-\kappa t})}{\Gamma(\alpha)} \int_0^t (t - r)^{\alpha-1}dr
$$

$$
= U_t - c_1(1 - e^{-\kappa t}) - c_2 \frac{\theta(1 - e^{-\kappa t})\kappa t}{\alpha\Gamma(\alpha)}
$$

$$
= \tilde{\sigma}_0^2 e^{-\kappa t} + \theta(1 - e^{-\kappa t}) \left(1 - c_1 - c_2 \frac{t^\alpha}{\alpha\Gamma(\alpha)}\right)
$$

$$
\geq \tilde{\sigma}_0^2 e^{-\kappa t} + \theta(1 - e^{-\kappa t}) \left(1 - c_1 - c_2 \frac{T^\alpha}{\alpha\Gamma(\alpha)}\right).
$$

And it is a positive quantity since we have by hypothesis that

$$
\left(1 - c_1 - c_2 \frac{T^\alpha}{\alpha\Gamma(\alpha)}\right) \geq 0.
$$

Now as we have defined $v_t^2 = \frac{1}{\kappa t} \left(M_t - \int_0^t \sigma_t^2 ds\right)$, when applying the Itô Lemma we need to compute $dv_t^2$ which implies computing $dM_t$, for $M_t = \int_0^T E_1[\sigma_s^2] ds$. The computation of this last derivative, needs to be done by means of Clark-Ocone-Haussmann formula and Malliavin calculus techniques. A good reference for this topic is, for instance, [22].

**Proposition 7.** We have that $\sigma_t^2 \in \mathbb{D}^{1,2}_N$ and $D_{s,z}^N \sigma_t^2 = c_3 \eta z$.

**Proof.** Note that $\sigma_t^2 = Y_t + c_1\nu Z_t + c_2\nu \frac{1}{\Gamma(\alpha)} \int_0^t (t - r)^{\alpha-1}Z_r dr + c_3\eta J_t$. Now we have that

$$
D_{s,z}^N \sigma_t^2 = D_{s,z}^N \left(Y_t + c_1\nu Z_t + c_2\nu \frac{1}{\Gamma(\alpha)} \int_0^t (t - r)^{\alpha-1}Z_r dr + c_3\eta J_t\right)
$$

$$
= D_{s,z}^N c_3\eta J_t,
$$

and since $J_t$ is a pure jump Lévy process which can be represented as $J_t = -\zeta(\rho, \eta) t + \int_0^t \int_0^\infty z\tilde{N}(ds,dz)$, following [22], we have that

$$
D_{s,z}^N \sigma_t^2 = D_{s,z}^N c_3\eta J_t^0 = D_{s,z}^N c_3\eta \left(-\zeta(\rho, \eta) t + \int_0^t \int_0^\infty z\tilde{N}(ds,dz)\right)
$$

$$
= D_{s,z}^N \left(\int_0^t \int_0^\infty c_3\eta z\tilde{N}(ds,dz)\right) = c_3\eta z.
$$

**Proposition 8.** Assume that $2k\theta \geq \nu^2$. Then, we have that $\sigma_t^2 \in \mathbb{D}^{1,2}_W$ and

$$
D_{s}^W \sigma_t^2 = c_1 D_{s}^W \sigma_t^2 + \frac{c_2}{\Gamma(\alpha)} \int_s^t (t - r)^{\alpha-1}D_{s}^W \sigma_r^2 dr.
$$

**Proof.** Starting from the definition of $\sigma_t^2$, we have that

$$
D_{s}^W \sigma_t^2 = D_{s}^W (Y_t + c_1\nu Z_t + c_2\nu I_{0+}^\alpha Z_t + c_3\eta J_t)
$$

$$
= c_1\nu D_{s}^W Z_t + c_2 I_{0+]^\alpha \nu D_{s}^W Z_t.
$$
We have to note that $D^W_t J_t = 0$ and that $D^W_t \sigma^2_t = \nu D^W_t Z_t$, hence we can rewrite the previous expression as

$$D^W_t \sigma^2_t = c_1 D^W_t \bar{\sigma}^2_t + \frac{c_2}{\Gamma(\alpha)} \int_s^t (t - r)^{\alpha - 1} D^W_s \bar{\sigma}^2_r dr.$$ 

Now, for instance, we have to solve $D^W_s \sigma^2_t$. To achieve this we will make use of Theorem 2.1 in [10] where in our particular case $\mu(s, \sigma^2_t) = \kappa (\bar{\sigma}^2 - \sigma^2_t)$ and $\sigma(s, \sigma^2_t) = \nu \sqrt{\sigma^2_t}$. Therefore, we have that by replacing and doing some basic algebraic manipulations we obtain the following expression.

$$D^W_s \sigma^2_t = \sigma \exp \left\{ \int_s^t \left[ \partial_2 \mu - \frac{\mu \partial_3 \sigma}{\sigma} - \frac{1}{2} (\partial_2 \sigma) \partial_1 \sigma - \partial_1 \sigma \right] (u, \bar{\sigma}^2_t) du \right\}$$

$$= \nu \sqrt{\bar{\sigma}^2_t} \exp \left\{ \int_s^t \left[ -\frac{\kappa}{2} - \left( \frac{\kappa}{2} - \frac{\nu^2}{8} \right) \frac{1}{\bar{\sigma}^2_t} \right] du \right\}$$

$$= \nu \sqrt{\bar{\sigma}^2_t} f(t, s),$$

where $f(t, s) \triangleq \exp \left\{ \int_t^s \left[ -\frac{\kappa}{2} - \left( \frac{\kappa}{2} - \frac{\nu^2}{8} \right) \frac{1}{\sigma^2_U} \right] du \right\}$. 

**Proposition 9.** Assume that $2 \kappa \theta \geq \nu^2$ and $\left( 1 - c_1 - c_2 \frac{T^n}{\alpha \Gamma(\alpha)} \right) \geq 0$. Then,

$$(5.3) \quad dM_t = \nu A(T, t) \sqrt{\bar{\sigma}^2_t} dW_t + \nu dJ_t.$$ 

**Proof.** Using the Clark-Ocone-Haussmann formula we have

$$(5.4) \quad \sigma^2_t = E[\sigma^2_t] + \int_0^t E_s [D^W_s \sigma^2_t] dW_s + \int_0^t \int_0^\infty E_s [D^{\nu}_{s, z} \sigma^2_t] \tilde{N}(ds, dz).$$

Now we will treat the second addend using (5.2) and doing some basic algebraic manipulations, to obtain the following relationship

$$\int_0^t E_s [D^W_s \sigma^2_t] dW_s = \int_0^t E_s \left[ c_1 D^W_s \sigma^2_t + \frac{c_2}{\Gamma(\alpha)} \int_s^t (t - r)^{\alpha - 1} D^W_s \sigma^2_r dr \right] dW_s$$

$$= \int_0^t \left[ c_1 E_s [D^W_s \sigma^2_t] + \frac{c_2}{\Gamma(\alpha)} \int_s^t (t - r)^{\alpha - 1} E_s [D^W_s \sigma^2_r] dr \right] dW_s.$$ 

Remembering from proposition (5) that $D^W_t \sigma^2_t = \nu D^W_t Z_t$ we can rewrite the previous equation as

$$\int_0^t E_s [D^W_s \sigma^2_t] dW_s$$

$$= \int_0^t \left[ c_1 \nu E_s [D^W_t Z_t] + \frac{c_2 \nu}{\Gamma(\alpha)} \int_s^t (t - r)^{\alpha - 1} E_s [D^W_s Z_r] dr \right] dW_s$$

$$= \int_0^t \left[ c_1 \nu \exp \{-\kappa (t - s)\} + \left( \frac{c_2 \nu}{\Gamma(\alpha)} \int_s^t (t - r)^{\alpha - 1} \exp \{-\kappa (t - s)\} dr \right) \right] \sqrt{\bar{\sigma}^2_t} dW_s,$$

as $Z_t = \int_0^t e^{-\kappa (t - s)} \sqrt{\bar{\sigma}^2_s} dW_s$. Finally if we set

$$a(t, s) := \left[ c_1 \nu \exp \{-\kappa (t - s)\} + \left( \frac{c_2 \nu}{\Gamma(\alpha)} \int_s^t (t - r)^{\alpha - 1} \exp \{-\kappa (t - s)\} dr \right) \right] \sqrt{\bar{\sigma}^2_s},$$

we can write $\int_0^t E_s [D^W_s \sigma^2_t] dW_s$ in a more compact way as

$$\int_0^t E_s [D^W_s \sigma^2_t] dW_s = \int_0^t a(t, s) dW_s.$$
We will continue by treating the third addend in equation (5.4) using proposition (7).

\[
\int_0^t \int_0^\infty \mathbb{E}_s [D_{\kappa,s}^{N,\sigma^2} \tilde{N}(ds, dz)] = \int_0^t \int_0^\infty c_3 \eta z \tilde{N}(ds, dz),
\]

which is a martingale process. Therefore, recalling that we defined \( M_t = \int_0^T \mathbb{E}_t [\sigma_s^2] ds \), we can write \( dM_t \) as

\[
dM_t = \left( \int_t^T a(r) dr \right) dW_t + c_3 \eta d\tilde{J}_t
\]

\[
= \nu \int_t^T c_1 \exp \{ -\kappa (r - t) \} \sqrt{\bar{\sigma}_r^2} dr dW_t
\]

\[
+ \frac{c_2 \nu}{\Gamma(\alpha)} \int_t^T \int_t^r (r - u)^{\alpha - 1} \exp \{ -\kappa (u - t) \} \sqrt{\bar{\sigma}_r^2} dr du dW_t + c_3 \eta d\tilde{J}_t.
\]

Using Fubini’s Theorem and integrating the expression we obtain

\[
dM_t = \nu \int_t^T c_1 \exp \{ -\kappa (u - t) \} \sqrt{\bar{\sigma}_r^2} dr
\]

\[
+ \frac{c_2 \nu}{\Gamma(\alpha)} \int_t^T \int_t^r (r - u)^{\alpha - 1} \exp \{ -\kappa (u - t) \} \sqrt{\bar{\sigma}_r^2} dr du dW_t + c_3 \eta d\tilde{J}_t
\]

\[
= \nu \int_t^T c_1 \exp \{ -\kappa (u - t) \} \sqrt{\bar{\sigma}_r^2} dr dW_t
\]

\[
+ \frac{c_2 \nu}{\Gamma(\alpha)} \int_t^T \left( T - u \right)^\alpha \exp \{ -\kappa (u - t) \} \sqrt{\bar{\sigma}_r^2} du dW_t + c_3 \eta d\tilde{J}_t
\]

\[
= \nu \left( \int_t^T \left( \frac{c_2}{\alpha \Gamma(\alpha)} (T - u)^\alpha + c_1 \right) \exp \{ -\kappa (u - t) \} du \right) \sqrt{\bar{\sigma}_r^2} dW_t + c_3 \eta d\tilde{J}_t.
\]

Defining \( A(T, t) = \int_t^T \left( \frac{c_2}{\alpha \Gamma(\alpha)} (T - u)^\alpha + c_1 \right) \exp \{ -\kappa (u - t) \} du \), we can rewrite the previous expression as

\[
dM_t = \nu A(T, t) \sqrt{\bar{\sigma}_r^2} dW_t + c_3 \eta d\tilde{J}_t.
\]

\[\square\]

6. Call Price Approximations in the Fractional Heston Model with Jumps

This section is aimed at providing an approximation formula for the pricing of vanilla options presented in Theorem 5, where we assumed the FSVJJ model presented in previous sections. In particular, we will deduce a first order approximation formula with respect to the vol-of-vol parameter \( \nu \), and the jump parameter \( \eta \). In order to do so, we will need to introduce a series of technical lemmas. These will help us bound the conditional expectation of the integrated future variance, needed to bound the error terms of the approximating formula.

**Lemma 10.** The following results regarding some of the expressions considered in previous sections hold.

1. \( A(T, t) = \int_t^T \left( \frac{c_2}{\alpha \Gamma(\alpha)} (T - u)^\alpha + c_1 \right) e^{-\kappa (u - t)} du \).
2. \( \mathbb{E}_t [\bar{\sigma}_s^2] = \theta + (\bar{\sigma}_r^2 - \theta) e^{-\kappa (s - t)} = \bar{\sigma}_r^2 e^{-\kappa (s - t)} + \theta \left( 1 - e^{-\kappa (s - t)} \right) \).
3. \( \mathbb{E}_t [\sigma_s \sqrt{\bar{\sigma}_s^2}] = \bar{\sigma}_r^2 e^{-\kappa (s - t)} + \theta \left( 1 - e^{-\kappa (s - t)} \right) + \mathcal{O} (\nu^2 + \eta^2) \).
4. \( dM_t = \nu A(T, t) \sqrt{\bar{\sigma}_r^2} dW_t \).
exists $C$
The proof of this result can be found in the subsection 7.1 of the Appendix.

Lemma 11. Let $0 \leq t < s \leq T$ and $\mathcal{G}_t \triangleq \mathcal{F}_t \vee \mathcal{F}_T^W \vee \mathcal{F}_T^P$. Then for every $n \geq 0$, there exists $C = C(n, \rho_1)$ such that

$$|\mathbb{E} [\partial^n G (s, X_s + a, v_s) | \mathcal{G}_t]| \leq C \left( \int_s^T \mathbb{E}_s [\sigma_0] d\theta \right)^{-\frac{1}{2} (n+1)} ,$$

where $a \geq 0$, and $G (t, x, y) \triangleq (\partial_x^2 - \partial_y) BS (t, x, y)$.

Proof. By means of simple computations we know that

$$G (s, X_s + a, v_s) = K e^{-r(T-s)} \phi \left( X_s + a - \mu_-, v_s \sqrt{T-s} \right) ,$$

where $\mu_- \triangleq \ln (K) - \left( r - \frac{v_\theta^2}{2} \right) (T-s)$ and $\phi$ is the normal density distribution function. Notice that $v_s \sqrt{T-s} = \sqrt{\int_s^T \mathbb{E}_s [\sigma_0^2] d\theta}$. The properties of $\phi$ under the derivation sign, allow us to write the following equality

$$\partial^n G (s, X_s + a, v_s) = (-1)^n K e^{-r(T-s)} \partial^n_{\mu^} \phi \left( X_s + a - \mu_-, v_s \sqrt{T-s} \right) .$$

Now consider the conditional expectation with respect to the filtration $\mathcal{G}_t$ that allow us to know the trajectories of the instantaneous variance and the jump process up to time $T$. Therefore, we can write the following,

$$\mathbb{E} [\partial^n G (s, X_s + a, v_s) | \mathcal{G}_t] = (-1)^n K e^{-r(T-s)} \partial^n_{\mu^} \mathbb{E} \left[ \phi \left( X_s + a - \mu_-, v_s \sqrt{T-s} \right) | \mathcal{G}_t \right] .$$

The law of $X_s$ given $\mathcal{G}_t$ is a normal random variable with mean

$$\tilde{\mu} \triangleq X_t + \int_t^s \left( r - \frac{1}{2} \sigma_0^2 \right) d\theta + \rho_2 \eta \left( \tilde{J}_s - \tilde{J}_t \right) - \zeta (\rho_2, \eta) (s-t) + \rho_1 \int_t^s \sigma_0 dW_\theta ,$$

and variance $\tilde{\Sigma} \triangleq (1 - \rho_1^2) \int_t^s \sigma_0^2 d\theta = (1 - \rho_1^2) \int_t^s \mathbb{E}_s [\sigma_0^2] d\theta$. Now, we can compute the conditional expectation as follows,

$$\mathbb{E} \left[ \phi \left( X_s + a - \mu_-, v_s \sqrt{T-s} \right) | \mathcal{G}_t \right] = \int_\mathbb{R} \phi \left( x + a - \mu_-, v_s \sqrt{T-s} \right) f_X (x) dx .$$
\[ = \int_{\mathbb{R}} \phi \left( x + a - \mu_- , \sqrt{T - s} \right) \phi \left( x - \bar{\mu} , \bar{\Sigma} \right) dx \]

\[ = \int_{\mathbb{R}} \left[ \frac{1}{\sqrt{2\pi (T - s)}} \exp \left\{ -\frac{1}{2} \left( \frac{x + a - \mu_-}{\sqrt{T - s}} \right)^2 \right\} \right] \left[ \frac{1}{\sqrt{2\pi \Sigma}} \exp \left\{ -\frac{1}{2} \left( \frac{x - \bar{\mu}}{\sqrt{\Sigma}} \right)^2 \right\} \right] dx. \]

We know that the product of two Gaussian probability density functions, results in the expression given by equation (7.3) from proposition 17 in the Appendix 7. Therefore, we can rewrite the previous conditional expectation as

\[
\mathbb{E} \left[ \phi \left( X_s + a - \mu_- , v_s \sqrt{T - s} \right) \mid \mathcal{G}_t \right]
\]

\[ = \frac{1}{\sqrt{2\pi (\Sigma + v_s^2 (T - s))}} \exp \left\{ -\frac{1}{2} \left( \frac{\bar{\mu} + a - \mu_-}{\Sigma + v_s^2 (T - s)} \right)^2 \right\} \]

\[ \times \int_{\mathbb{R}} \left[ \frac{1}{\sqrt{2\pi \Sigma + v_s^2 (T - s)}} \exp \left\{ -\frac{1}{2} \left( \frac{x - \bar{\mu}v_s^2 (T - s) + (\mu_- - a)\Sigma}{\Sigma + v_s^2 (T - s)} \right)^2 \right\} \right] dx \]

\[ = \frac{1}{\sqrt{2\pi (\Sigma + v_s^2 (T - s))}} \exp \left\{ -\frac{1}{2} \left( \frac{\bar{\mu} + a - \mu_-}{\Sigma + v_s^2 (T - s)} \right)^2 \right\} \]

\[ = \phi \left( \bar{\mu} + a - \mu_- , \sqrt{\Sigma + v_s^2 (T - s)} \right). \]

The last equality results from the fact that the integral of the Gaussian density equals to one.

Putting this result in (6.1), we have

\[ \| \mathbb{E} [\partial^m_x G(s, X_s, v_s) \mid \mathcal{G}_t] \| = \left| (-1)^m K e^{-r(T-s)} \partial^m_{\mu_-} \phi \left( \bar{\mu} + a - \mu_- , \sqrt{\Sigma + v_s^2 (T - s)} \right) \right|. \]

Now, notice that \( |\partial^m_x \phi (x, \sigma) | \leq \frac{x^m}{\sigma^{2m+1}} e^{-\frac{x^2}{2\sigma^2}} \), for all \( x \in \mathbb{R} \), \( \sigma \in \mathbb{R}^+ \) and \( m \geq 1 \). Let \( C(n) = n + 1 \) be a positive constant, and \( d = \frac{1}{\sigma^2} \). Then, trivially the following holds

\[ |\partial^m_x \phi (x, \sigma) | \leq x^C e^{-dx^2}. \]

Note that the function \( \psi (x) = x^C e^{-dx^2} \) has a global maximum at \( x = \pm \sqrt{\frac{C}{2d}} \) and therefore, \( |\psi (x) | \leq \psi \left( \sqrt{\frac{C}{2d}} \right) \leq C\sigma^{(n+1)}. \) Therefore, since \( \sigma = \sqrt{\Sigma + v_s^2 (T - s)} \), we can write

\[ \mathbb{E} \left[ \partial^m_x G(s, X_s, v_s) \mid \mathcal{G}_t \right] \]

\[ \leq C \left( \Sigma + v_s^2 (T - s) \right)^{-\frac{1}{2}(n+1)} \]

\[ \leq C \left( (1 - \rho_1^2) \int_t^s \mathbb{E}_\theta \left[ \sigma_\theta^2 \right] d\theta + (1 - \rho_2^2) \int_s^T \mathbb{E}_\theta \left[ \sigma_\theta^2 \right] d\theta \right)^{-\frac{1}{2}(n+1)} \]

\[ \leq C \left( (1 - \rho_1^2) \int_t^T \mathbb{E}_s \left[ \sigma_\theta^2 \right] d\theta - (1 - \rho_2^2) \int_s^T \mathbb{E}_s \left[ \sigma_\theta^2 \right] d\theta \right)^{-\frac{1}{2}(n+1)} \]
\[
\leq C \left( (1 - \rho^2_1) \int_t^T E_s [\sigma^2_\theta] d\theta + \rho^2_1 \int_s^T E_s [\sigma^2_\theta] d\theta \right)^{-\frac{1}{2}(n+1)} \\
\leq C \left( \int_s^T E_s [\sigma^2_\theta] d\theta \right)^{-\frac{1}{2}(n+1)}.
\]

**Lemma 12.** Assume that \(2k\theta > \nu^2\) and let \(\varphi(t) \triangleq \int_t^T e^{-\kappa(z-t)}dz = \frac{1}{\kappa} \left(1 - e^{-\kappa(T-t)}\right)\). Then, for all \(0 \leq s < t \leq T\)

1. \( \int_s^T E_s [\sigma^2_u] du \geq \sigma^2 \varphi(s) \).
2. \( \int_s^T E_s [\sigma^2_u] du \geq \frac{\theta\kappa}{2} \varphi(s)^2 \).

**Proof.** From statement 2 in Lemma 10 we know that \(E_t [\sigma^2_s] = \bar{\sigma}^2_t e^{-\kappa(s-t)} + \theta \left(1 - e^{-\kappa(s-t)}\right)\). Now (i) results from lower bounding the conditional expectation by considering only the first term of the equality and integrating in the interval \([s, T]\) as follows,

\[
\int_s^T E_s [\sigma^2_u] du \geq \int_s^T \sigma^2_s e^{-\kappa(u-s)} du \geq \frac{\bar{\sigma}^2_t}{\kappa} \left(1 - e^{-\kappa(T-s)}\right).
\]

In order to prove (ii), we lower bound \(E_t [\sigma^2_s] \geq \theta\kappa\varphi(s)\) and integrate in the interval \([s, T]\). Therefore, we can write

\[
\int_s^T E_s [\sigma^2_u] du \geq \frac{\theta\kappa}{2} \varphi(s)^2.
\]

**Lemma 13.** Let \(g(t, m, y) \triangleq \sqrt{\frac{m-y}{T-t}}\). Then, the following inequality holds,

\[
|\partial^2_m g(t, M_t, Y_t)| \leq \frac{(\bar{\sigma}^2_t \varphi(t))^{-\frac{3}{2}}}{4\sqrt{T-t}}.
\]

**Proof.** By a simple calculation, we know that

\[
\partial^2_m g(t, m, y) = \frac{-1}{4(T-t)^2} \left(\frac{m-y}{T-t}\right)^{-\frac{3}{2}}.
\]

Recalling the definition of processes \(M_t\) and \(Y_t\), the following holds

\[
|\partial^2_m g(t, M_t, Y_t)| = \left| \frac{-1}{4(T-t)^2} \left(\frac{M_t - Y_t}{T-t}\right)^{-\frac{3}{2}} \right| = \frac{1}{4\sqrt{T-t}} (M_t - Y_t)^{-\frac{3}{2}} = \frac{1}{4\sqrt{T-t}} \left(\int_t^T E_t [\sigma^2_s] ds\right)^{-\frac{3}{2}}.
\]

From Lemma 12 (i), we know that \(\int_t^T E_t [\sigma^2_s] ds \geq \bar{\sigma}^2_t \varphi(t)\), finishing the proof.
Theorem 14. (1st order approximation formula). Fix $T > 0$. Assume the model \[ \text{Corollary 4} \] where the volatility process $\sigma = \{\sigma_s, s \in [0, T]\}$ satisfies the conditions $2k\theta > \nu^2$ and $(1 - c_1 - c_2 \frac{T}{\theta^2} > C$ for some positive constant $C$. Then
\[
V_t = e^{-r(T-t)}E_t \left[ (e^{X_T} - K)^+ \right] 
\]
\[
= BS(t, X_t, v_t) + \frac{\rho_1}{2} \Lambda \Gamma BS(t, X_t, v_t) L[W, M^c] + E_t \left[ \int_t^T \int_0^\infty e^{-r(s-t)} \left( 1 + \frac{\rho_1}{2} L[W, M^c]_s \right) \Delta^2 \Lambda \Gamma BS(s, X_{s-}, v_{s-}) \ell (dz) ds \right] + \epsilon_t,
\]
where $\epsilon_t$ is the error term and satisfies $|\epsilon_t| \in O(\nu^2 + \eta^2)$.

Proof. This proof relies on applying Corollary \[ \text{Corollary 4} \] iteratively to the different terms appearing in the call price formula given by equation \[ (4) \] in Theorem \[ \text{Theorem 5} \]. This way, the resulting formula will only contain terms of order $O(\nu^2 + \eta^2)$ which will be incorporated into the error term.

Note that we will omit the term $-\zeta (\rho_2, \eta) E_t \left[ \int_t^T e^{-rs} \Lambda A(s, X_s, v_s) B_s ds \right]$ in the application of Corollary \[ Corollary 4 \] and treat it as part of the error term in the approximating formula. Since $E_t \left[ \int_t^T e^{-rs} \Lambda A(s, X_s, v_s) B_s ds \right] < +\infty$ and from Lemma \[ Lemma 13 \] we know that
\[
\zeta (\rho_2, \eta) = \int_0^\infty (e^{\rho_2 \eta z} - 1 - \rho_2 \eta z) \ell (dz) 
\]
\[
= \frac{\rho_2}{2} \int_0^\infty z^2 e^{\rho_2 \eta z} (1 - \lambda) d\lambda \ell (dz) \leq \rho^2 \int_0^\infty z^2 \ell (dz),
\]
where we have used that $\rho_2 \leq 0$ so $e^{\rho_2 \eta z} (1 - \lambda) \leq 1$, and we also have that $\int_0^\infty z^2 \ell (dz) < \infty$. \[ \square \]

• Step 1: Applying Corollary \[ Corollary 4 \] to term (I) in equation \[ (4.2) \] with $A(t, X_t, v_t) = \Lambda \Gamma BS(t, X_t, v_t)$ and $B_t = \frac{\rho_1}{2} L[W, M^c]_t$ and recalling that $B_T = 0$ by definition, this gives
\[
(6.2) \quad (I) = \frac{\rho_1}{2} \Lambda \Gamma BS(t, X_t, v_t) L[W, M^c]_t + \frac{\rho_1}{4} E_t \left[ \int_t^T e^{-r(s-t)} \Lambda \Gamma^2 BS(s, X_s, v_s) L[W, M^c]_s \right] 
\]
\[
+ \frac{\rho_1}{4} E_t \left[ \int_t^T e^{-r(s-t)} \Lambda \Gamma BS(s, X_s, v_s) L[W, M^c]_s \right] 
\]
\[
+ \frac{\rho_1}{2} E_t \left[ \int_t^T e^{-r(s-t)} \Lambda \Gamma BS(s, X_s, v_s) \sigma_s d[W, M^c]_s \right] 
\]
\[
+ \frac{\rho_1}{4} E_t \left[ \int_t^T e^{-r(s-t)} \Lambda \Gamma^2 BS(s, X_s, v_s) \ell [M^c, L[W, M^c]]_s \right] 
\]
\[
+ \frac{\rho_1}{2} E_t \left[ \int_t^T e^{-r(s-t)} \Lambda \Gamma^2 BS(s, X_s, v_s) L[W, M^c]_s \right] 
\]
\[
\times v_s (T - s) \int_0^\infty \Delta^2 \Lambda \Gamma \theta (s, M_{s-}, Y_{s-}) \ell (dz) ds
\]
\[ + \frac{\rho_3}{2} \mathbb{E}_t \left[ \int_t^T \int_0^\infty e^{-r(s-t)} L [W, M^c]_s \right] \times \left[ \Delta_x^2 \Lambda_x BS \left( s, X_{s-}, v_{s-} \right) + \Delta_y^2 \Lambda_y BS \left( s, X_{s-}, v_{s-} \right) \right] \ell(dz) ds \]

(6.3) \quad \Delta \equiv (I.I) + (I.II) + \ldots + (I.VII) .

Notice also, that we can apply Lemma [III] since we are working under the fractional Heston model with jumps. Therefore the previous equation can be rewritten as

\[
\begin{align*}
(I) &= \frac{\rho_1 \nu^2}{2} \Lambda BS (t, X_t, v_t) \left( \int_t^T A(T, s) \mathbb{E}_s \left[ \sqrt{\sigma_x^2 \sigma_x^2} \right] ds \right) \\
&+ \frac{\rho_2 \nu^2}{4} \mathbb{E}_t \left[ \int_t^T e^{-r(s-t)} \Lambda^2 \Gamma^2 BS (s, X_s, v_s) \left( \int_s^T A(T, z) \mathbb{E}_s \left[ \sqrt{\sigma_z^2 \sigma_z^2} \right] dz \right) A(T, s) \sigma_s \sqrt{\sigma_z^2 dz} \right] \\
&+ \frac{\rho_3}{16} \mathbb{E}_t \left[ \int_t^T e^{-r(s-t)} \Lambda^3 BS (s, X_s, v_s) \left( \int_s^T A(T, z) \mathbb{E}_s \left[ \sqrt{\sigma_z^2 \sigma_z^2} \right] dz \right) A^2(T, s) \sigma_s^2 ds \right] \\
&+ \frac{\rho_4 \nu^2}{2} \mathbb{E}_t \left[ \int_t^T e^{-r(s-t)} \Lambda^2 \Gamma BS (s, X_s, v_s) \left( \int_s^T A(T, z) e^{-\kappa(z-t)} dz \right) \sigma_s \sigma_s ds \right] \\
&+ \frac{\rho_5 \nu^2}{4} \mathbb{E}_t \left[ \int_t^T e^{-r(s-t)} \Lambda^2 BS (s, X_s, v_s) \left( \int_s^T A(T, u) \mathbb{E}_s \left[ \sqrt{\sigma_u^2 \sigma_u^2} \right] du \right) \right. \\
&\quad \times \left. v_s (T - s) \int_0^\infty \Delta_y^2 g \left( s, M_{s-}, Y_{s-} \right) \ell(dz) ds \right] \\
&+ \frac{\rho_6 \nu^2}{2} \mathbb{E}_t \left[ \int_t^T \int_0^\infty e^{-r(s-t)} \left( \int_t^T A(T, u) \mathbb{E}_s \left[ \sqrt{\sigma_u^2 \sigma_u^2} \right] du \right) \right. \\
&\quad \times \left. \left[ \Delta_x^2 \Lambda_x BS (s, X_{s-}, v_{s-}) + \Delta_y^2 \Lambda_y BS (s, X_{s-}, v_{s-}) \right] \ell(dz) ds \right].
\end{align*}
\]

The terms (I.II) \ldots (I.VII), belong to the error term \( \mathcal{O} \left( \nu^2 + \nu^2 \right) \). We will prove the previous statement for the terms (I.III) and (I.V), the proof for the rest of the terms is analogous.

\[
\begin{align*}
(I.III) + (I.V) \\
&= \frac{\rho_1 \nu^3}{16} \mathbb{E}_t \left[ \int_t^T e^{-r(s-t)} \Lambda^3 BS (s, X_s, v_s) \left( \int_s^T A(T, z) \mathbb{E}_s \left[ \sqrt{\sigma_z^2 \sigma_z^2} \right] dz \right) A^2(T, s) \sigma_s^2 ds \right] \\
&+ \frac{\rho_3}{4} \mathbb{E}_t \left[ \int_t^T e^{-r(s-t)} \Lambda^2 BS (s, X_s, v_s) \left( \int_s^T A(T, z) e^{-\kappa(z-t)} dz \right) A(T, s) \sigma_s^2 ds \right].
\end{align*}
\]

Using the fact that \( A(T, z) \) is a decreasing function, defining \( a_s \equiv v_s \sqrt{T - s} \), and using the inequality from Lemma (III), we can upper bound the previous expression by

\[
\begin{align*}
|\{I.III) + (I.V)\} | \\
&\leq C \frac{\rho_1 \nu^3}{16} \mathbb{E}_t \left[ \int_t^T e^{-r(s-t)} \left( \frac{1}{a_s^2} + \frac{2}{a_s^2} + \frac{1}{a_s^2} \right) \left( \int_s^T \mathbb{E}_s \left[ \sqrt{\sigma_z^2 \sigma_z^2} \right] dz \right) A^3(T, s) \sigma_s^2 ds \right] \\
&\quad + C \frac{\rho_3}{4} \mathbb{E}_t \left[ \int_t^T e^{-r(s-t)} \left( \frac{1}{a_s^2} + \frac{1}{a_s^2} \right) A^3(T, s) \sigma_s^2 ds \right].
\end{align*}
\]
Remember that it follows from Lemma (10) that \( \int_s^T E_s \left[ \sqrt{\sigma^2_s \sigma^2_r} \right] dz \leq \int_s^T E_s \left[ \sigma^2_r \right] dz + C(T-s)\nu^2 = a^2_s + C(T-s)\nu^2 \), and from Lemma (12) (i), that \( \frac{a^2_s}{\nu} \geq \bar{\sigma}^2_s \). Hence

\[
| (I.III) + (I.IV) | 
\leq C \frac{\nu^3}{16} \int_t^T e^{-r(s-t)} \left( \frac{1}{a^2_s} + \frac{2}{a^2_s} + \frac{1}{a^2_s} \right) \frac{a^4_s}{\varphi(s)} A^3(T,s) ds 
\]

\[
+ C \frac{\nu^5}{16} \int_t^T e^{-r(s-t)} \left( \frac{1}{a^2_s} + \frac{2}{a^2_s} + \frac{1}{a^2_s} \right) \frac{a^2_s}{\varphi(s)} C(T-s) A^3(T,s) ds 
\]

\[
+ C \frac{\nu^3}{16} \int_t^T e^{-r(s-t)} \left( \frac{4}{a^2_s} + \frac{4}{a^2_s} \right) A^3(T,s) \frac{a^2_s}{\varphi(s)} ds 
\]

\[
\leq C \frac{\nu^3}{16} \int_t^T e^{-r(s-t)} \left( \frac{5}{a^2_s} + \frac{6}{a^2_s} + 1 \right) \frac{a^4_s}{\varphi(s)} (T-s) A^3(T,s) ds 
\]

\[
+ C \frac{\nu^5}{16} \int_t^T e^{-r(s-t)} \left( \frac{1}{a^2_s} + \frac{2}{a^2_s} + \frac{1}{a^2_s} \right) \frac{a^2_s}{\varphi(s)} C(T-s) A^3(T,s) ds .
\]

From (ii) in Lemma (12), we know that \( a_s \geq \sqrt{\frac{\nu}{T}} \varphi(s) \). Note also that \( \varphi(s) \leq \frac{1}{\nu} \) for all \( s \in [0,T] \). Taking into account that \( A(T,s) \leq \varphi(s) \left( \frac{\nu^2}{\nu^2} (T-t) + c_1 \right) \), we can finally upper bound the previous sum as follows

\[
| (I.III) + (I.IV) | \leq C \frac{\nu^3}{16} \int_t^T e^{-r(s-t)} ds.
\]

The same reasoning applies to obtain an upper bound of terms (I.I), (I.IV). Notice the term (I.I), depends linearly on \( \nu \) and therefore, it is part of the first order approximation formula. We will now provide upper bounds for the discontinuous term (I.IV). We start applying Lemma (13) to the functions

\[
G_1(x) = g(s,x,Y_{s-}),
\]

\[
\Delta^2 G_1(M_{s-},c_3 \eta z) = c^2_3 \eta^2 z^2 \int_0^1 \partial^2_{m} g(s,M_{s-} + \lambda c_3 \eta z, Y_{s-}) (1 - \lambda) d\lambda ,
\]

\[
G_2(x) = \Lambda^G B S (s,X_{s-},x)
\]

\[
\Delta^2 G_2(v_{s-},c_3 \eta z) = c^2_3 \eta^2 z^2 \int_0^1 \partial^2_{c} \Lambda^G B S (s,X_{s-},v_{s-} + \lambda c_3 \eta z) (1 - \lambda) d\lambda .
\]

Given that the terms \( \Delta^2 G_i \) are proportional to \( \eta^2 \), all we need to prove is that the integrals in the term (I.IV) have an upper bound, in order to properly justify that they belong to the error term.

- Using the inequality from Lemma (11), we can upper bound the term (I.IV) as follows

\[
| (I.IV) | \leq C \frac{\nu^3}{2} c^2_3 \eta^2 z^2 \int_t^T e^{-r(s-t)} \left( \frac{1}{a^2_s} + \frac{1}{a^2_s} \right) L [W, M^c]_s 
\]

\[
\times v_s(T-s) \int_0^\infty \int_0^1 \partial^2_{m} g(s,M_{s-} + \lambda c_3 \eta z, Y_{s-}) (1 - \lambda) d\lambda d(\nu dz) .
\]

From Lemma (13) we can upper bound \( \left| \partial^2_{m} g(s,M_{s-} + \lambda c_3 \eta z, Y_{s-}) \right| \). We also know that \( L [W, M^c]_s = \nu \int_s^T A(T,r) E_s \left[ \sqrt{\sigma^2_r \sigma^2_r} \right] dr \leq \nu A(T,s) \int_s^T E_s \left[ \sqrt{\sigma^2_r \sigma^2_r} \right] dr \), as \( A(T,t) \) is a decreasing function of \( t \in [0,T] \). From Lemma (10) follows that \( \int_s^T E_s \left[ \sqrt{\sigma^2_r \sigma^2_r} \right] dr \geq \int_s^T E_s \left[ \sigma^2_r \right] dr \triangleq a^2_s \), and from Lemma (12) (i), that
\[
\frac{a^2}{\varphi(s)} \geq \sigma^2. \text{ Therefore we can upper bound the term } (I.VI) \text{ as follows }
\]

\[
| (I.VI) | \leq C \frac{\rho_1}{2} \frac{3 \eta^2 z^2 \mathbb{E}_t}{2} \left[ \int_t^T e^{-r(s-t)} \left( \frac{1}{a_s^4} + \frac{1}{a_s^2} \right) L [W, M^c]_s \right. \\
& \left. \times v_s (T - s) \int_0^\infty \left[ \int_0^1 \frac{1}{4\sqrt{T-s}} (\sigma^2 \varphi(s))^{-\frac{3}{2}} (1 - \lambda) \, d\lambda \right] \ell (dz) \, ds \right]
\]

\[
\leq C \frac{\rho_1}{8} \frac{3 \eta^2 z^3 \mathbb{E}_t}{2} \left[ \int_t^T e^{-r(s-t)} \left( \frac{1}{a_s^4} + \frac{1}{a_s^2} \right) L [W, M^c]_s v_s \sqrt{T-s} \left( \sigma^2 \varphi(s) \right)^{-\frac{3}{2}} \, ds \right]
\]

\[
\leq C \frac{\rho_1}{8} \frac{3 \eta^2 z^3 \mathbb{E}_t}{2} \left[ \int_t^T e^{-r(s-t)} \left( \frac{1}{a_s^4} + \frac{1}{a_s^2} \right) \left( \frac{a_s^2}{\varphi(s)} \right)^{-\frac{3}{2}} A(T, s) \, ds \right]
\]

\[
\leq C \frac{\rho_1}{8} \frac{3 \eta^2 z^3 \mathbb{E}_t}{2} \left[ \left( \frac{C_2}{\Gamma(\alpha)} (T-t)^\alpha + c_1 \right) \int_t^T e^{-r(s-t)} \left( \frac{1}{a_s^4} + \frac{1}{a_s^2} \right) \, ds \right].
\]

Again, from (ii) in Lemma 12, we know that \( a_s \geq \sqrt{\frac{\sigma^2}{3}} \varphi(s) \) Replacing it in the previous inequality and taking into account that \( \varphi(s) \leq \frac{1}{\kappa} \) for all \( s \in [0, T] \), proves that the term \( (I.VI) \in O(\eta^2) \).

We can rewrite the term \( (I.VII) \) recalling \( \Delta^2 G_2 \) and the Delta-Gamma-Vega relationship given by equation (7.2). If we also make use of Lemma (11) and use the same bounding techniques we have already been using along the proof, we can rewrite the term as follows,

\[
(I.VII) = \frac{\rho_1}{2} \mathbb{E}_t \left[ \int_t^T \int_0^\infty e^{-r(s-t)} L [W, M^c]_s \Delta^2 \Delta \Gamma BS (s, X_{s-}, v_{s-}) \ell (dz) \, ds \right]
\]

\[
+ \frac{\rho_1}{2} \frac{3 \eta^2 z^2 \mathbb{E}_t}{2} \left[ \int_t^T e^{-r(s-t)} L [W, M^c]_s \right. \\
& \left. \times \left[ z^2 \int_0^1 \Lambda^3 BS (s, X_{s-}, v_{s-} + \lambda c_3 \eta z) (v_{s-} + \lambda c_3 \eta z)^2 (T - s)^2 (1 - \lambda) \, d\lambda \right] \ell (dz) \, ds \right]
\]

\[
\leq \frac{\rho_1}{2} \mathbb{E}_t \left[ \int_t^T \int_0^\infty e^{-r(s-t)} L [W, M^c]_s \Delta^2 \Delta \Gamma BS (s, X_{s-}, v_{s-}) \ell (dz) \, ds \right]
\]

\[
+ C \frac{\rho_1 \lambda c_3}{2} \mathbb{E}_t \left[ \int_t^T e^{-r(s-t)} \left( \frac{A(T, s)}{\varphi(s)} \right) \left( \frac{1}{a_s^4} + \frac{2}{a_s^3} + \frac{1}{a_s^2} \right) \right.
\]

\[
\left. \times \int_0^\infty \left( (v_{s-} + \lambda c_3 \eta z)^2 (T - s)^2 \right) \ell (dz) \, ds \right].
\]

The bounds used in the proof of the previous term also apply here, ending the proof that shows the term \( (I.VII) \) can be written as

\[
(I.VII) = \frac{\rho_1}{2} \mathbb{E}_t \left[ \int_t^T \int_0^\infty e^{-r(s-t)} L [W, M^c]_s \Delta^2 \Delta \Gamma BS (s, X_{s-}, v_{s-}) \ell (dz) \, ds \right] + O (\eta^2)
\]

This concludes the proof to show that

\[
(I) = \frac{\rho_1}{2} \Delta \Gamma BS (t, X_t, v_t) \left( \int_t^T A(T, s) \mathbb{E}_t \left[ \sqrt{\sigma_s^2 \sigma_z^2} \right] \, ds \right)
\]

\[
+ \frac{\rho_1}{2} \mathbb{E}_t \left[ \int_t^T \int_0^\infty e^{-r(s-t)} \left( \int_s^T A(u, s) \mathbb{E}_s \left[ \sqrt{\sigma_u^2 \sigma_z^2} \right] \, du \right) \Delta^2 \Delta \Gamma BS (s, X_{s-}, v_{s-}) \ell (dz) \, ds \right] + O (\eta^2 + \eta^2).
\]
Step 2: Applying Corollary 4 to term (II) in equation (4.22) with $A(t, X_t, v_t) = \Gamma^2 BS(t, X_t, v_t)$ and $B_t = \frac{1}{8} D [M^c, M^c]_t$ and recalling that $B_T = 0$ by definition, we have

\[(II) = \frac{1}{8} \Gamma^2 BS(t, X_t, v_t) D [M^c, M^c]_t \]

\[+ \frac{\nu^3}{16} I_t^T e^{-r(s-t)} \Gamma^3 BS \left( s, X_s, v_s \right) \left( \int_s^T A^2(T, z) \mathbb{E}_s [\sigma^2] dz \right) A(T, s) \sigma_s \sqrt{\sigma^2} ds \]

\[+ \frac{\nu^4}{64} I_t^T e^{-r(s-t)} \Gamma^4 BS \left( s, X_s, v_s \right) \left( \int_s^T A^2(T, z) \mathbb{E}_s [\sigma^2] dz \right) A^2(T, s) \sigma_s^2 ds \]

\[+ \frac{\nu^3}{8} I_t^T e^{-r(s-t)} \Gamma^2 BS \left( s, X_s, v_s \right) \left( \int_s^T A^2(T, z) e^{-\kappa(z-t)} dz \right) \sigma_s \sigma_s ds \]

\[+ \frac{\nu^4}{16} I_t^T e^{-r(s-t)} \Gamma^3 BS \left( s, X_s, v_s \right) \left( \int_s^T A^2(T, z) e^{-\kappa(z-t)} dz \right) A(T, s) \sigma_s^2 ds \]

\[+ \frac{\nu^2}{8} I_t^T e^{-r(s-t)} \Gamma^2 BS \left( s, X_s, v_s \right) \left( \int_s^T A^2(T, u) \mathbb{E}_s [\sigma^2] du \right) \sigma_s \sigma_s \]

\[\times v_s (T - s) \int_0^\infty \Delta^2_m g \left( s, M_{s-}, Y_{s-} \right) \ell (dz) ds \]

\[+ \frac{\nu^2}{8} I_t^T e^{-r(s-t)} \left( \int_s^T A^2(T, u) \mathbb{E}_s [\sigma^2] du \right) \sigma_s \sigma_s \]

\[\times \left( \Delta^2_{m} \Gamma^2 BS \left( s, X_{s-}, v_{s-} \right) + \Delta^2_{m} \Gamma^2 BS \left( s, X_{s-}, v_{s-} \right) \right) \ell (dz) ds \].

Applying Lemma 10, the previous equation is rewritten as

\[(II) = \frac{\nu^2}{8} \Gamma^2 BS(t, X_t, v_t) \left( \int_t^T A^2(T, s) \mathbb{E}_t [\sigma^2] ds \right) \]

\[+ \frac{\nu^3}{16} I_t^T e^{-r(s-t)} \Gamma^3 BS \left( s, X_s, v_s \right) \left( \int_s^T A^2(T, z) \mathbb{E}_s [\sigma^2] dz \right) A(T, s) \sigma_s \sqrt{\sigma^2} ds \]

\[+ \frac{\nu^4}{64} I_t^T e^{-r(s-t)} \Gamma^4 BS \left( s, X_s, v_s \right) \left( \int_s^T A^2(T, z) \mathbb{E}_s [\sigma^2] dz \right) A^2(T, s) \sigma_s^2 ds \]

\[+ \frac{\nu^3}{8} I_t^T e^{-r(s-t)} \Gamma^2 BS \left( s, X_s, v_s \right) \left( \int_s^T A^2(T, z) e^{-\kappa(z-t)} dz \right) \sigma_s \sigma_s ds \]

\[+ \frac{\nu^4}{16} I_t^T e^{-r(s-t)} \Gamma^3 BS \left( s, X_s, v_s \right) \left( \int_s^T A^2(T, z) e^{-\kappa(z-t)} dz \right) A(T, s) \sigma_s^2 ds \]

\[+ \frac{\nu^2}{8} I_t^T e^{-r(s-t)} \Gamma^2 BS \left( s, X_s, v_s \right) \left( \int_s^T A^2(T, u) \mathbb{E}_s [\sigma^2] du \right) \sigma_s \sigma_s \]

\[\times v_s (T - s) \int_0^\infty \Delta^2_m g \left( s, M_{s-}, Y_{s-} \right) \ell (dz) ds \]

\[+ \frac{\nu^2}{8} I_t^T e^{-r(s-t)} \left( \int_s^T A^2(T, u) \mathbb{E}_s [\sigma^2] du \right) \sigma_s \sigma_s \]

\[\times \left( \Delta^2_{m} \Gamma^2 BS \left( s, X_{s-}, v_{s-} \right) + \Delta^2_{m} \Gamma^2 BS \left( s, X_{s-}, v_{s-} \right) \right) \ell (dz) ds \].

Observe that all the terms in (II) can be incorporated into the error term, i.e. (II) = $O (\nu^2)$. This is due to the dependency of each term on higher order of $\nu$ and the fact that all terms can be upper bounded following the reasoning we provided for terms (I.III) and (I.V).

Step 3: Proving the term (III) in equation (4.22) belongs to the error term $O (\eta^2)$, is analogous to the discussion we did for the term (I.VI).
Step 4: Again, an analogous discussion as the one performed in the study of $(I.VII)$ applies here, showing that
\[
(IV) = \mathbb{E}_t \left[ \int_t^T \int_0^\infty e^{-r(s-t)} \Delta^2 \Gamma BS(s, X_{s-}, v_{s-}) \ell(dz) \, ds \right] + \mathcal{O}(\eta^2).
\]

7. Appendix

In this appendix we gather additional technical lemmas.

Lemma 15. Let $G \in C^2(\mathbb{R})$ and consider the expression $\Delta^2 G(x, h)$ defined as
\[
\Delta^2 G(x, h) \triangleq G(x + h) - G(x) - hG'(x).
\]
Then, the following equality holds
\[
\Delta^2 G(x, h) = h^2 \int_0^1 G''(x + \lambda h) (1 - \lambda) \, d\lambda.
\]
(7.1)

Proof. It is Taylor’s Theorem with integral remainder. \qed

Proposition 16. (Delta-Gamma-Vega Relationship) Let
\[
BS(t, x, y) = e^{x \Phi(d_+)} - e^{-r(T-t)} K \Phi(d_-),
\]
where $\Phi$ denotes the cumulative distribution function of a standard normal distribution and
\[
d_\pm = \frac{x - \ln K + (r \pm \frac{y^2}{2})(T-t)}{y \sqrt{T-t}}; \quad d_+ = d_- + y \sqrt{T-t}.
\]
Then for every $t \in [0, T]$, the following formula, known as the Delta-Gamma-Vega relationship, holds.
\[
(7.2)
\]
\[
\partial_y BS(t, x, y) \frac{1}{y(T-t)} = (\partial^2_{xx} - \partial_x) BS(t, x, y).
\]

Proof. We start by computing the log-Delta
\[
\partial_x BS(t, x, y) = e^{x \Phi(d_+)}.
\]
Now the log-Gamma is computed as
\[
\partial^2_{xx} BS(t, x, y) = e^{x \Phi(d_+)} \frac{e^{x \phi(d_+)} + y \sqrt{T-t}}{y \sqrt{T-t}}.
\]
Therefore, we have that
\[
(\partial^2_{xx} - \partial_x) BS(t, x, y) = e^{x \phi(d_+)} \frac{y \sqrt{T-t}}{y \sqrt{T-t}}.
\]

On the other hand, the Vega is derived as follows
\[
\partial_y BS(t, x, y) = e^{x \phi(d_+)} \partial_y d_+ - e^{-r(T-t)} K \phi(d_-) \partial_y d_- - e^{-r(T-t)} K \phi(d_-) \partial_y d_- - e^{x \phi(d_+)} \partial_y \left( y \sqrt{T-t} \right) - e^{-r(T-t)} K \phi(d_-) \partial_y d_-.
\]

We will name log-Delta the change in the option price with respect to the change in the underlying asset log-price. This is a different sensitivity of what is commonly referred to as the Delta, i.e. the change in the option price with respect to the change in the underlying asset price.
The relationship follows trivially from these computations. □

**Proposition 17.** Let \( \phi(x, \sigma) \) denote the density function of the normal law with mean zero and standard deviation \( \sigma \). Then, for \( \mu_1, \mu_2 \in \mathbb{R} \) and \( \sigma_1, \sigma_2 \) strictly positive real numbers we have

\[
(7.3) \quad \phi(x - \mu_1, \sigma_1) \phi(x - \mu_2, \sigma_2) = \phi \left( x - \frac{\mu_1 \sigma_2^2 + \mu_2 \sigma_1^2}{\sigma_1^2 + \sigma_2^2}, \sqrt{\frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}} \right) \phi \left( \mu_1 - \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2} \right).
\]

**Proof.** This proof results from basic algebraic manipulations. Note that we can trivially write the product of densities as follows

\[
\phi(x - \mu_1, \sigma_1) \phi(x - \mu_2, \sigma_2) = \frac{1}{2\pi \sigma_1^2} \exp \left\{ -\frac{(x - \mu_1)^2}{2\sigma_1^2} \right\} \frac{1}{2\pi \sigma_2^2} \exp \left\{ -\frac{(x - \mu_2)^2}{2\sigma_2^2} \right\}.
\]

\[= \frac{1}{2\pi \sigma_1^2 \sigma_2^2} \exp \left\{ -\frac{(x - \mu_1)^2}{2\sigma_1^2} - \frac{(x - \mu_2)^2}{2\sigma_2^2} \right\}.
\]

Since \( \sigma_1^2 + \sigma_2^2 > 0 \) we can rewrite the previous expression as

\[
\phi(x - \mu_1, \sigma_1) \phi(x - \mu_2, \sigma_2) = \frac{1}{\sqrt{2\pi \sigma_1^2 \sigma_2^2} (\sigma_1^2 + \sigma_2^2) 2\pi} \exp \left\{ -\frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} (x - \mu_1)^2 - \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} (x - \mu_2)^2 \right\}.
\]

Notice from the previous expression that

\[
\sqrt{2\pi \sigma_1^2 \sigma_2^2} (\sigma_1^2 + \sigma_2^2) 2\pi = \sqrt{2\pi (\sigma_1^2 + \sigma_2^2) 2\pi (\sigma_1^2 + \sigma_2^2)},
\]

therefore, replacing this in the product of the two pdf’s and considering that \( x = e^{\ln x} \), one obtains that

\[
\phi(x - \mu_1, \sigma_1) \phi(x - \mu_2, \sigma_2) = \exp \left\{ \ln \frac{1}{\sqrt{2\pi (\sigma_1^2 + \sigma_2^2)}} \exp \left\{ -\frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} (x - \mu_1)^2 - \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} (x - \mu_2)^2 \right\} \right\}.
\]

Focusing on the exponential term, we will perform some further algebraic manipulations on it as follows.

\[-\frac{1}{2} \ln (2\pi (\sigma_1^2 + \sigma_2^2)) + \frac{-\sigma_1^2 (x - \mu_1)^2 \sigma_2^2 (x - \mu_2)^2}{2 \sigma_1^2 \sigma_2^2} (-\sigma_1^2 + \sigma_2^2) (x - \mu_2)^2.
\]
Now, one can write the previous equality as a second order polynomial of the form $-x^2 + 2Ax - A^2 - C = -(x - A)^2 - C$, where $A$ and $C$ are given by

$$A = \frac{\mu_1 \sigma_1^2 + \mu_2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2},$$

$$C = \frac{\sigma_1^2 \sigma_2^2}{(\sigma_1^2 + \sigma_2^2)^2} (\mu_1 - \mu_2)^2 + \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \ln \left(2\pi \left(\frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}\right)\right).$$

Equipped with this, we can now rewrite the product of the two Gaussian densities as

$$\phi(x - \mu_1, \sigma_1) \phi(x - \mu_2, \sigma_2) = \frac{1}{\sqrt{2\pi \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}}} \exp \left\{ -\frac{(x - A)^2 - C}{2 \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}} \right\},$$

$$= \phi \left( x - A, \sqrt{\frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}} \right) \exp \left\{ -\frac{-C}{2 \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}} \right\}.$$

We will focus on the second exponential term to further expand it.

$$\exp \left\{ \frac{-C}{2 \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}} \right\} = \exp \left\{ -\frac{\sigma_1^2 \sigma_2^2}{(\sigma_1^2 + \sigma_2^2)^2} (\mu_1 - \mu_2)^2 + \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \ln \left(2\pi \left(\frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}\right)\right) \right\}$$

$$= \frac{1}{\sqrt{2\pi (\sigma_1^2 + \sigma_2^2)}} \exp \left\{ -\frac{(\mu_1 - \mu_2)^2}{2 (\sigma_1^2 + \sigma_2^2)} \right\}$$

$$= \phi \left( \mu_1 - \mu_2, \sqrt{\sigma_1^2 + \sigma_2^2} \right).$$

This ends the proof. \qed

7.1. **Proof of Lemma** [10]. We will only prove by order statements 3, 9 and 10, as the rest are trivially deduced from the definitions and results provided in previous sections.

- In order to see that $\mathbb{E}_t \left[ \sigma_s \sqrt{\sigma_s^2} \right] = \sigma_t^2 e^{-\kappa(s-t)} + \theta \left(1 - e^{-\kappa(s-t)}\right) + O(\nu^2 + \eta^2)$ we consider the process

  $$u_r \triangleq \sqrt{(U_s + \nu Z_r^s) \left( U_s + c_1 \nu Z_r^s + c_2 \nu Z_r^{H-\frac{1}{2}} + Z_r^c + c_3 \eta J_r^c \right)},$$

  where

  $$U_t \triangleq \theta + e^{-\kappa t} (\sigma_0^2 - \theta),$$
Consider the following process
\[ Z^s_r = \int_0^r e^{-\kappa(s-u)} \sqrt{\sigma_u^2} dW_u, \]
\[ I_{0+}^{H-\frac{1}{2}} Z^s_r = \frac{1}{\Gamma(H-\frac{1}{2})} \int_0^r \int_0^{r(u)} \left( \int_u^{r(u)} e^{-\kappa(u-v)} \sqrt{\sigma_v^2} dW_v \right) (s-u)^{H-\frac{3}{2}} du, \quad r \in [0, s]. \]

Note that \( u_s = \sigma_s^2 Z^s_r, Z^s_r = Z_s, s \in [0, T] \) and \( I_{0+}^{H-\frac{1}{2}} Z^s_r = I_{0+}^{H-\frac{1}{2}} Z_s \). Applying Fubini, we can write
\[ I_{0+}^{H-\frac{1}{2}} Z^s_r = \frac{1}{\Gamma(H-\frac{1}{2})} \int_0^r \left( \int_u^{r(s-u)^{H-\frac{3}{2}} du} \right) \sqrt{\sigma_u^2} dW_u, \]
where
\[ \psi(r, s, u) = \frac{1}{\Gamma(H-\frac{1}{2})} \int_u^{r(s-u)^{H-\frac{3}{2}} du} \sqrt{\sigma_u^2} dW_u, \]
\[ \zeta(r, s, u) = c_1 \nu e^{-\kappa(s-u)} + c_2 \psi(r, s, u). \]

Next, consider the process
\[ \Pi_r^s = c_1 \nu Z^s_r + c_2 \nu I_{0+}^{H-\frac{1}{2}} Z^s_r \]
\[ = c_1 \nu \int_0^r e^{-\kappa(s-u)} \sqrt{\sigma_u^2} dW_u + c_2 \nu \int_0^r \psi(r, s, u) \sqrt{\sigma_u^2} dW_u, \]
where
\[ \zeta(r, s, u) = c_1 \nu e^{-\kappa(s-u)} + c_2 \psi(r, s, u). \]

Note that
\[ \zeta(r, s, r) = c_1 \nu e^{-\kappa(s-r)} + c_2 \psi(r, s, r) = c_1 \nu e^{-\kappa(s-r)}, \]
and
\[ \partial_t \zeta(r, s, u) = c_2 \nu \partial_t \psi(r, s, u) = \frac{c_2 \nu}{\Gamma(H-\frac{1}{2})} e^{-\kappa(r-u)} (s-r)^{H-\frac{3}{2}}. \]
Therefore,
\[ d\Pi_r^s = \left( c_1 \nu e^{-\kappa(s-u)} + c_2 \nu \psi(r, s, u) \right) \sqrt{\sigma_u^2} dW_r + \left( \frac{c_2 \nu}{\Gamma(H-\frac{1}{2})} \right) \int_0^r e^{-\kappa(r-u)} (s-r)^{H-\frac{3}{2}} \sqrt{\sigma_u^2} dW_u dr. \]

Consider the following process \( \Theta^s_r = \Pi^s_r + c_3 \eta J_r \). We have that
\[ dZ^s_r = e^{-\kappa(s-r)} \sqrt{\sigma_r^2} dW_r, \]
\[ d\Theta^s_r = d\Pi^s_r + c_3 \eta dJ_r, \]
\[ d\Theta^{s,c}_r = d\Pi^s_r, \]
\[ d(Z^s, Z^s)_r = e^{-2\kappa(s-r)} \sigma_r^2 dr, \]
\[ d(Z^s, \Theta^{s,c})_r = \left\{ c_1 \nu e^{-\kappa(s-u)} + c_2 \nu \psi(r, s, u) \right\} e^{-\kappa(s-r)} \sigma_r^2 dr. \]
\[ d(\Theta^s, \Theta^c) = \left( c_1 \nu e^{-\kappa(s-u)} + c_2 \nu \psi(r, s, u) \right)^2 \sigma_r^2 dr. \]

Next, we can apply Itô formula to \( f(Z^s, \Theta^s) \), where \( f \) is the function defined by

\[ f(x, y) = \sqrt{U_x + \nu x} (U_x + y). \]

We have that

\[
\begin{align*}
\partial_1 f(x, y) &= \frac{\nu}{2} \left( \frac{U_x + y}{U_x + \nu x} \right)^{1/2}, \\
\partial_2 f(x, y) &= \frac{1}{2} \left( \frac{U_x + \nu x}{U_x + y} \right)^{1/2}, \\
\partial_{11} f(x, y) &= -\frac{\nu^2}{4} \left( \frac{U_x + y}{U_x + \nu x} \right)^{1/2} (U_x + \nu x)^{-3/2}, \\
\partial_{12} f(x, y) &= \frac{\nu}{4} \left( U_x + y \right)^{-1/2} (U_x + \nu x)^{-1/2}, \\
\partial_{22} f(x, y) &= -\frac{1}{4} (U_x + y)^{-3/2} (U_x + \nu x)^{1/2}.
\end{align*}
\]

Note that

\[
f(Z^s, \Theta^s) = \sqrt{\sigma_s^2 \sigma_c^2} = \sqrt{(U_x + \nu Z^c) (U_x + c_1 \nu Z^s + c_2 \nu I_{0+}^{H-\frac{1}{2}} Z^s + c_3 \eta J_s)}.
\]

Now, an application of the Itô formula yields the following expression

\[
f(Z^s, \Theta^s)
= f(Z^s, \Theta^s) + \int_0^s \partial_1 f(Z^s, \Theta^s) dZ^s + \int_0^s \partial_2 f(Z^s, \Theta^s) d\Theta^s
+ \int_0^s \partial_{11} f(Z^s, \Theta^s) d(Z^s, \Theta^s)_r + \int_0^s \partial_{12} f(Z^s, \Theta^s) d(Z^s, \Theta^c),
+ \int_0^s \frac{1}{2} \partial_{22} f(Z^s, \Theta^s) d(\Theta^s, \Theta^c)_r
+ \int_0^s \int_0^\infty \Delta_y f(Z^s, \Theta^s_{r-}) \tilde{N}(dr, dz) + \int_0^s \int_0^\infty \Delta_y^2 f(Z^s, \Theta^s_{r-}) \ell (dz) dr.
\]

Now, taking expectations and writing the terms with \( \nu^2 \) as an error term of order \( O(\nu^2) \), we can rewrite the previous equation as

\[
\mathbb{E}[f(Z^s, \Theta^s)] = \mathbb{E}[f(Z^s, \Theta^s_0)].
\]
In order to see that 
•
we proceed in an analogous way as in the previous set of computations. 

\[ \mathbb{E} \left[ f \left( Z_{t}^{s}, \Theta_{r}^{s} \right) \right] = \mathbb{E} \left[ U_{s} \right] + \mathbb{E} \left[ \int_{0}^{s} \int_{0}^{\infty} \Delta_{g}^{2} f \left( Z_{r}^{s}, \Theta_{r}^{s} \right) \ell \left( dz \right) \right] + \mathcal{O} \left( \nu^{2} \right), \]

where \( f \left( Z_{0}^{s}, \Theta_{0}^{s} \right) = U_{s} \). Finally, it only remains to notice that the term \( \Delta_{g}^{2} f \left( Z_{r}^{s}, \Theta_{r}^{s} \right) = \partial_{22} f \left( Z_{r}^{s}, \Theta_{r}^{s} \right) c_{3}^{2} \eta^{2} z^{2} \in \mathcal{O} \left( \eta^{2} \right) \), allowing us to write

\[ \mathbb{E} \left[ f \left( Z_{t}^{s}, \Theta_{t}^{s} \right) \right] = \mathbb{E} \left[ \sqrt{\sigma_{t}^{2} \sigma_{s}^{2}} \right] = \mathbb{E} \left[ U_{s} \right] + \mathcal{O} \left( \nu^{2} + \eta^{2} \right). \]

• In order to see that \( dL \left[ W, M^{c} \right]_{t} = \nu^{2} \left( \int_{t}^{T} A(T, s) e^{-\kappa(s-t)} ds \right) \bar{\sigma}_{t} dW_{t} - \nu \bar{\sigma}_{t}^{2} A(T, t) dt + \mathcal{O} \left( \nu^{3} + \nu \eta^{2} \right) dt \), we can compute the following:

\[ dL \left[ W, M^{c} \right]_{t} = d \left( \nu \int_{t}^{T} A(T, s) \mathbb{E}_{t} \left[ \sqrt{\sigma_{t}^{2} \sigma_{s}^{2}} \right] ds \right) \]

\[ = d \left( \nu \int_{t}^{T} A(T, s) \left[ \bar{\sigma}_{t}^{2} e^{-\kappa(s-t)} + \theta \left( 1 - e^{-\kappa(s-t)} \right) \right] ds \right) \]

\[ = d \left( \nu \bar{\sigma}_{t}^{2} \int_{t}^{T} A(T, s) e^{-\kappa(s-t)} ds \right) + d \left( \nu \theta \int_{t}^{T} A(T, s) \left( 1 - e^{-\kappa(s-t)} \right) ds \right) \]

\[ + d \left( \nu \int_{t}^{T} A(T, s) \mathcal{O} \left( \nu^{2} + \eta^{2} \right) ds \right) \]

\[ = \nu d \bar{\sigma}_{t}^{2} \int_{t}^{T} A(T, s) e^{-\kappa(s-t)} ds + \nu \bar{\sigma}_{t}^{2} d \left( \int_{t}^{T} A(T, s) e^{-\kappa(s-t)} ds \right) \]

\[ + \nu \theta d \left( \int_{t}^{T} A(T, s) \left( 1 - e^{-\kappa(s-t)} \right) ds \right) + \nu \mathcal{O} \left( \nu^{2} + \eta^{2} \right) d \left( \int_{t}^{T} A(T, s) ds \right). \]

Applying Leibniz rule to derive under the integral sign we can rewrite the previous expression as

\[ dL \left[ W, M^{c} \right]_{t} = \nu d \bar{\sigma}_{t}^{2} \int_{t}^{T} A(T, s) e^{-\kappa(s-t)} ds \]

\[ + \nu \bar{\sigma}_{t}^{2} \left[ -A(T, t) dt + \kappa \left( \int_{t}^{T} A(T, s) e^{-\kappa(s-t)} ds \right) dt \right] \]

\[ + \nu \theta \left[ -\kappa \left( \int_{t}^{T} A(T, s) e^{-\kappa(s-t)} ds \right) dt \right] \]

\[ - \nu \mathcal{O} \left( \nu^{3} + \eta^{3} \right) A(T, t) dt \]

\[ = \nu^{2} \left( \int_{t}^{T} A(T, s) e^{-\kappa(s-t)} ds \right) \bar{\sigma}_{t} dW_{t} - \nu \bar{\sigma}_{t}^{2} A(T, t) dt + \mathcal{O} \left( \nu^{3} + \nu \eta^{2} \right). \]

• In order to see that \( dD \left[ M^{c}, M^{c} \right]_{t} = \nu^{3} \left( \int_{t}^{T} A^{2}(T, s) e^{-\kappa(s-t)} ds \right) \bar{\sigma}_{t} dW_{t} - \nu^{2} \bar{\sigma}_{t}^{2} A^{2}(T, t) dt \), we proceed in an analogous way as in the previous set of computations.

\[ dD \left[ M^{c}, M^{c} \right]_{t} = d \left( \nu^{2} \int_{t}^{T} A^{2}(T, s) \mathbb{E}_{t} \left[ \sigma_{s}^{2} \right] ds \right) \]
\[
\nu^2 \left[ d \left( \sigma^2 \int_t^T A^2(T,s) e^{-\kappa(s-t)} ds \right) + \theta d \left( \int_t^T A^2(T,s) \left( 1 - e^{-\kappa(s-t)} \right) ds \right) \right] \\
= \nu^2 \left[ d \left( \sigma^2 \int_t^T A^2(T,s) e^{-\kappa(s-t)} ds \right) + \theta d \left( \int_t^T A^2(T,s) \left( 1 - e^{-\kappa(s-t)} \right) ds \right) \right] \\
+ \nu^2 \theta d \left( \int_t^T A^2(T,s) \left( 1 - e^{-\kappa(s-t)} \right) ds \right) \\
= \nu^2 \left( \int_t^T A^2(T,s) e^{-\kappa(s-t)} ds \right) \hat{\sigma}_t dW_t - \nu^2 \sigma^2 A^2(T,t) dt.
\]

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