Large-scale Regularity of Nearly Incompressible Elasticity in Stochastic Homogenization

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Communicated by N. MASMOUDI

Abstract

In this paper, we systematically study the regularity theory of the linear system of nearly incompressible elasticity. In the setting of stochastic homogenization, we develop new techniques to establish the large-scale estimates of displacement and pressure, which are uniform in both the scale parameter and the incompressibility parameter. In particular, we obtain the boundary estimates in a new class of Lipschitz domains whose boundaries are smooth at large scales and bumpy at small scales.

1. Introduction

1.1. Motivations

The system of linear elasticity for a homogeneous isotropic material is called Lamé system given by

$$\mu \Delta u + (\mu + \lambda) \nabla (\nabla \cdot u) = F,$$

where the vector-valued function $u = (u^1, u^2, \ldots, u^d)$ represents the displacement field in equilibrium state, and the scalar constants $\mu$ and $\lambda$ are usually referred as Lamé’s first and second parameters. In terms of the modulus of elasticity $E$ and Poisson ratio $\nu \left(-1 < \nu < \frac{1}{2}\right)$, one has

$$\mu = \frac{E}{2(1 + \nu)}, \quad \lambda = \frac{E \nu}{(1 + \nu)(1 - 2\nu)}.$$ 

The quantity $\frac{2}{3} \mu + \lambda = \frac{1}{3} E/(1 - 2\nu)$ is called the bulk modulus, which measures the compressibility. A material tends to be incompressible, if the bulk modulus is large, or equivalently, if the Poisson ratio $\nu$ is close to $\frac{1}{2}$. The endpoint $\nu = \frac{1}{2}$ corresponds to the incompressible materials and reduces the system into a Stokes system. In the real world, all materials are more or less compressible, while the
Poisson ratio for some materials could be very close to \( \frac{1}{2} \). For example, the typical nearly incompressible material, natural rubber, has a Poisson ratio of 0.4999 [33]. Due to many applications of nearly incompressible materials in engineering, the Lamé system (1.1) with large \( \lambda \) has been studied extensively in physics and numerical analysis (see, for example, [13,23,24,31,32,39,40] and references therein). However, to the best of the authors’ knowledge, only sparse results are known on the theoretical analysis of the nearly incompressible elasticity.

In this paper, we study the system of linear elasticity for inhomogeneous, anisotropic, nearly incompressible materials [26,35]. Precisely, let \( D \subset \mathbb{R}^d \) be a bounded domain occupied by a material body and consider the system with Dirichlet boundary condition

\[
\begin{aligned}
\begin{cases}
\nabla \cdot (A(x) \nabla u) + \nabla (\lambda(x) \nabla \cdot u) = F & \text{in } D, \\
\quad u = f & \text{on } \partial D,
\end{cases}
\end{aligned}
\tag{1.2}
\]

where \( A(\cdot) = (a_{ij}^{\alpha \beta}(\cdot)) : \mathbb{R}^d \mapsto \mathbb{R}^{d^2 \times d^2} \) is a tensor-valued function and \( \lambda(\cdot) : \mathbb{R}^d \mapsto [0, \infty) \) is a scalar function. We point out that \( \lambda(\cdot) \) plays a role, similar as Lamé’s second parameter or bulk modulus, in measuring the incompressibility of the material.

Our primary hypothesis for the coefficients \( A \) and \( \lambda \) (measurable and deterministic) are as follows:

- **Ellipticity condition:** there exists a fixed constant \( \Lambda > 0 \) so that

\[
\Lambda^{-1} |\xi|^2 \leq a_{ij}^{\alpha \beta}(x) \xi_\alpha \xi_\beta \leq \Lambda |\xi|^2 \quad \text{for any } x \in \mathbb{R}^d, \xi \in \mathbb{R}^{d \times d}.
\tag{1.3}
\]

(The Einstein summation convention will be used throughout the paper.)

- **Symmetry condition:**

\[
a_{ij}^{\alpha \beta} = a_{ji}^{\beta \alpha} \quad \text{for any } 1 \leq i, j, \alpha, \beta \leq d.
\tag{1.4}
\]

- **Compressibility condition:** there is a constant \( \lambda_0 \geq 0 \) so that

\[
\lambda_0 \leq \lambda(x) \leq \lambda_0 + \Lambda.
\tag{1.5}
\]

Note that (1.3) and (1.4) are the usual ellipticity and symmetry conditions for the system of elasticity. However, the “incompressibility parameter” \( \lambda_0 \) in the compressibility condition (1.5) is allowed to be arbitrarily large which makes the system (1.2) very ill-conditioned. In this paper, we are interested in the regularity estimates that are uniform in \( \lambda_0 \).

Observe that the upper bound of (1.5) also implies that the oscillation of \( \lambda \) is bounded by a fixed constant \( \Lambda \), though its magnitude could be arbitrarily large. This assumption is purely technical but crucial for our analysis, because it allows us to reduce (1.2) to a simpler situation. Indeed, we may write \( \lambda(x) = \lambda_0 + b(x) \) and

\[
\nabla \cdot (A(x) \nabla u) + \nabla (\lambda(x) \nabla \cdot u) = \left[ \nabla \cdot (A(x) \nabla u) + \nabla (b(x) \nabla \cdot u) \right] + \nabla (\lambda_0 \nabla \cdot u)
\]

\[
= \nabla \cdot (\tilde{A}(x) \nabla u) + \nabla (\lambda_0 \nabla \cdot u),
\tag{1.6}
\]
where \( \tilde{A} = (\tilde{a}^{ij}_{\alpha\beta}) \) is defined by
\[
\tilde{a}^{ij}_{\alpha\beta}(x) = a^{ij}_{\alpha\beta}(x) + b(x)\delta^\alpha_i \delta^\beta_j.
\] (1.7)

Clearly, \( \tilde{A} \) still satisfies the ellipticity and symmetry conditions. Hence, without loss of generality, we may simply assume \( \lambda \geq 0 \) is constant and concentrate on the following Dirichlet boundary value problem
\[
\begin{cases}
\nabla \cdot (A(x)\nabla u_\lambda) + \lambda \nabla \cdot (\nabla u_\lambda) = F \quad \text{in} \; D, \\
u_\lambda = f \quad \text{on} \; \partial D,
\end{cases}
\] (1.8)

where \( A \) satisfies (1.3) and (1.4) and \( \lambda \geq 0 \) is an arbitrary constant.

### 1.2. General Regularity Theory

Nowadays, the regularity theory for elliptic equation/system and Stokes system has been well-understood. The system (1.8) can be viewed as an intermediate state between elliptic system and Stokes system. Intuitively, the uniform regularity should be expected thanks to the fine regularity of the endpoint systems. However, the regularity for (1.8) uniform in \( \lambda \) seems to be a mathematically different and harder problem, compared to the usual elliptic or the Stokes systems. For example, the fundamental Caccioppoli inequality for (1.8) does not hold uniform in \( \lambda \).

Actually, if \( \nabla \cdot (A(x)\nabla u_\lambda) + \lambda \nabla \cdot (\nabla u_\lambda) = 0 \) in \( B_{2r} \), we can only show that
\[
\int_{B_r} |\nabla u_\lambda|^2 \leq \frac{C \lambda}{r^2} \int_{B_{2r}} |u_\lambda|^2.
\] (1.9)

Note that \( \lambda \), appearing in front of the \( L^2 \) norm of \( u_\lambda \), makes the inequality useless in the study of the uniform regularity. Fortunately, we invent a novel variation of (1.9), which will be called the generalized Caccioppoli inequality:
\[
\int_{B_r} |\nabla u_\lambda|^2 \leq \frac{C}{r^2} \int_{B_{2r}} |u_\lambda|^2 + \frac{C}{r^2} \|\lambda \nabla \cdot u_\lambda - \int_{B_{2r}} \lambda \nabla \cdot u_\lambda\|_{H^{-1}(B_{2r})}^2 + C \sup_{k, \ell \in [1/4, 1]} \left| \int_{B_{2r}} \lambda \nabla \cdot u_\lambda - \int_{B_{2r}} \lambda \nabla \cdot u_\lambda \right|^2.
\] (1.10)

We emphasize that, in (1.10) and all the estimates involved in this paper, \( \lambda \) is harmless when it comes together with \( \nabla \cdot u_\lambda \). As a whole, \( \lambda \nabla \cdot u_\lambda \) has an obvious physical and mathematical meaning, namely, the “pressure”. The additional structures with the pressure in (1.10) gives a taste why the uniform regularity of (1.8) should be expected, but meanwhile more complicated.

The first part of this paper is devoted to the uniform regularity of (1.2) or (1.8) in the non-homogenization setting. Besides the energy estimates and the generalized Caccioppoli inequality mentioned above, our main tool to study the uniform regularity for large \( \lambda \) is the asymptotic expansion (also see [39])
\[
u_\lambda = \sum_{k=0}^{\infty} \lambda^{-k} v_k \quad \text{in} \; H^1 \quad \text{and} \quad \lambda \nabla \cdot u_\lambda - \lambda \langle f \rangle_D = \sum_{k=0}^{\infty} \lambda^{-k} p_k \quad \text{in} \; L^2, \] (1.11)
where \((v_k, p_k), k \geq 0,\) are the solutions of certain \(\lambda\)-independent iterative Stokes systems whose regularity are known (see Theorem 3.2), and \((f)_D\) is a constant defined in (3.2). In particular, the pair \((u_\lambda, \lambda \nabla \cdot u_\lambda - \lambda (f)_D)\), as \(\lambda \to \infty,\) converges quantitatively to \((v_0, p_0)\), which is the solution of a Stokes system

\[
\begin{aligned}
\nabla \cdot (A(x)\nabla v_0) + \nabla p_0 &= F \quad \text{in} \ D, \\
\nabla \cdot v_0 &= (f)_D \quad \text{in} \ D, \\
v_0 &= f \quad \text{on} \ \partial D.
\end{aligned}
\]

The above asymptotic behavior of \(u_\lambda\) provides us with at least two approaches to study the uniform regularity: (1) exploring the regularity for all \((v_k, p_k)\) in (1.11); (2) a real variable perturbation argument. It turns out that these two approaches are effective in different situations. For instance, the first approach is useful for the Schauder estimate (Theorem 3.8), while the second one is more efficient for the Calderón–Zygmund estimate (Theorem 3.10). These basic regularity estimates uniform in \(\lambda\) are of independent interest and crucial for studying the uniform regularity in homogenization.

1.3. Regularity in Homogenization

The larger part of this paper will be devoted to the uniform regularity in stochastic homogenization. Recently, there has been lots of interest in the uniform regularity in homogenization theory in either random (for example, [2–5,7,17,18,34]) or deterministic settings (for example, [6,8,9,20,22,28]). In particular, the uniform regularity for Stokes system in periodic homogenization has been studied in [20–22,41]. In this paper, we consider the system of nearly incompressible elasticity in a bounded Lipschitz domain with an additional tiny scale parameter \(\varepsilon > 0:\)

\[
\nabla \cdot (A^\varepsilon \nabla u^\varepsilon_\lambda) + \nabla (\lambda^\varepsilon \nabla \cdot u^\varepsilon_\lambda) = 0 \quad \text{in} \ D, \tag{1.12}
\]

where \(A^\varepsilon(x) = A(x/\varepsilon), \lambda^\varepsilon(x) = \lambda(x/\varepsilon)\) and the solution \(u^\varepsilon_\lambda\) depends both on \(\varepsilon\) and \(\lambda_0.\)\(^1\) We are interested in the interior and boundary uniform estimates that are independent of both \(\varepsilon \in (0, 1)\) and \(\lambda_0 \in (0, \infty).\) Notice that the expansion (1.11) also applies to the system (1.12). Therefore, we expect to obtain the uniform estimate for the system (1.12) by Theorem 3.2, as long as the same uniform estimate holds for the Stokes system. This straightforward strategy actually works for \(W^{1,p}\) estimate with \(p \in (1, \infty);\) see Theorem 3.4 for example (in the periodic setting). However, it fails for the Lipschitz estimate of \(u^\varepsilon_\lambda,\) due to the following two essential reasons: (1) the Lipschitz estimate is optimal in homogenization. This means that it is impossible to prove a higher regularity, say \(C^{1,\alpha}\) estimate, that implies the Lipschitz estimates. (2) The \(L^\infty\) estimate of the pressure (corresponding to the Lipschitz estimate of \(u^\varepsilon_\lambda\)) is not preserved through the iterative Stokes system (3.8), since the map \(p_{k-1} \mapsto p_k\) is a singular integral which definitely is not bounded in \(L^\infty.\)

\(^1\) Without ambiguity, we write \(u^\varepsilon_\lambda,\) instead of \(u^\varepsilon_{\lambda_0},\) for short.
In this paper, we will make substantial modifications to the excess decay method developed recently to establish the uniform Lipschitz and pressure estimates for (1.12) in stochastic homogenization. Moreover, we generalize the large-scale boundary estimates to a class of Lipschitz domains whose boundaries are "smooth" only above $\varepsilon$-scale and could be very rough at or below $\varepsilon$-scale.

Before stating the main result, we first set up the random environment in quantitative homogenization theory. We will follow the approach developed recently in, for example, [2,3,5,7], which is based on the natural stationarity and ergodicity assumptions on the coefficient fields. Precisely, denote the set of all the possible coefficient fields by

$$\Omega := \{(A, \lambda) : \mathbb{R}^d \mapsto \mathbb{R}^{d^2 \times 2d^2} \times \mathbb{R} \text{ satisfying (1.3)-(1.5)}\},$$

which is endowed with a family of $\sigma$-algebras $\{\mathcal{F}_D\}$, where $\mathcal{F}_D$ is the $\sigma$-algebra representing the information of the coefficient fields restricted in $D$, formally generated by

$$\mathcal{F}_D := \text{the } \sigma\text{-algebra generated by the random elements}(A, \lambda) \mapsto \left(\int_{\mathbb{R}^d} a^{\alpha\beta}_{ij}(x)\phi(x), \int_{\mathbb{R}^d} \lambda(x)\psi(x)\right), \phi, \psi \in C_0^\infty(D), 1 \leq i, j, \alpha, \beta \leq d.$$

Let $\mathcal{F} = \mathcal{F}_{\mathbb{R}^d}$ be the largest $\sigma$-algebra in the family $\{\mathcal{F}_D\}$. We further assume that there is a probability measure $\mathbb{P}$ satisfying the following assumptions:

- Stationarity with respect to $\mathbb{Z}^d$-translations:
  $$\mathbb{P} \circ T_z = \mathbb{P}, \text{ where } (T_z(A, \lambda))(x) = (A(x+z), \lambda(x+z)).$$
  (1.14)

- Unit range of dependence:
  $$\mathcal{F}_D \text{ and } \mathcal{F}_E \text{ are } \mathbb{P}\text{-independent for every Borel subset pair } D, E \subset \mathbb{R}^d \text{ satisfying } \text{dist}(D, E) \geq 1.$$
  (1.15)

Throughout this paper, we will use the following notation, which has been commonly used in many recent references, to control the size of a random variable. For a random variable $X : \Omega \to [1, \infty)$, we write $X \leq O_s(\theta)$ for some $s > 0, \theta > 0$, if

$$\mathbb{E}\left[\exp\left((X/\theta)^s\right)\right] \leq 2,$$

where $\mathbb{E}[Y]$ denotes the expectation of the random variable $Y$. Note that (1.16) implies that, for any $t \geq 1$,

$$\mathbb{P}[X \geq t] \leq 2 \exp((-t/\theta)^s).$$

Conversely, (1.17) implies $X \leq O_s(2^{1/s}\theta)$. As a convention, we write $X \leq Y + O_s(\theta)$ if $X - Y \leq O_s(\theta)$.

Now we state the main theorem for the interior estimate.
**Theorem 1.1.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be as above. For any \(s \in (0, d)\) and \(\lambda_0 \in [0, \infty)\), there exist a constant \(C_0 = C_0(s, d, \Lambda)\) and a random variable \(\mathcal{X} = \mathcal{X}_{s, \lambda} : \Omega \mapsto [1, \infty)\) satisfying

\[
\mathcal{X} \leq \mathcal{O}_s(C_0), \tag{1.18}
\]

such that if \(u^\varepsilon_\lambda \in H^1(B_2; \mathbb{R}^d)\) is a weak solution of

\[
\nabla \cdot (A^\varepsilon \nabla u^\varepsilon_\lambda) + \nabla (\lambda^\varepsilon \nabla \cdot u^\varepsilon_\lambda) = 0 \quad \text{in} \quad B_2 = B_2(0), \tag{1.19}
\]

then for any \(r \in [\varepsilon \mathcal{X}, 1]\), we have

\[
\left( \frac{1}{2} \int_{B_r} |\nabla u^\varepsilon_\lambda|^2 \right)^{1/2} + \left( \frac{1}{2} \int_{B_r} |\lambda^\varepsilon \nabla \cdot u^\varepsilon_\lambda - \int_{B_2} \lambda^\varepsilon \nabla \cdot u^\varepsilon_\lambda|^2 \right)^{1/2} \leq C \left( \frac{1}{2} \int_{B_2} |\nabla u^\varepsilon_\lambda|^2 \right)^{1/2}, \tag{1.20}
\]

where \(B_r = B_r(0)\) and \(C\) depends only on \(s, d\) and \(\Lambda\).

Before anything else, we should emphasize that the nontrivial point of Theorem 1.1 is that the constants involved are independent of both \(\varepsilon\) and \(\lambda_0\). The first part of (1.20) is called the large-scale Lipschitz estimate of the displacement. In other words, the “average deformation” of the material at relatively large scales \((r \geq \varepsilon \mathcal{X})\) is controlled by the average deformation at macroscopic scale \((r = 2)\). Particularly, this guarantees the continuity of the material and no “cracks” will be seen at scales greater than \(\varepsilon \mathcal{X}\). Since there is no regularity assumption on the coefficients \((A, \lambda)\), we cannot expect any continuity for \(u^\varepsilon_\lambda\) at scales less than \(\varepsilon\). The second part of (1.20) is called the large-scale oscillation estimate of the pressure, which is an exclusive feature of the system of elasticity or Stokes system. This particularly implies that the “average pressure” at a relatively large scale \((r \geq \varepsilon \mathcal{X})\) has uniform bounded oscillation, that is,

\[
\left| \int_{B_r(x)} \lambda^\varepsilon \nabla \cdot u^\varepsilon_\lambda - \int_{B_r(y)} \lambda^\varepsilon \nabla \cdot u^\varepsilon_\lambda \right| \leq C \left( \frac{1}{2} \int_{B_2} |\nabla u^\varepsilon_\lambda|^2 \right)^{1/2},
\]

for any \(x, y \in B_1(0)\) and \(r \in [\varepsilon \mathcal{X}, 1]\). If \(\lambda_0\) is large, this estimate actually implies that the spatial change of \(\nabla \cdot u^\varepsilon_\lambda\) is small as we expected for the nearly incompressible materials. We also mention that the estimate (1.20) together with the ranges of \(s \in (0, d)\) and \(r \in [\varepsilon \mathcal{X}, 1]\) is optimal in terms of stochastic integrability [4]. Of course, because of (1.18) and (1.17), the random variable \(\mathcal{X}\) is large only with a small probability (decaying exponentially). Precisely, Theorem 1.1 implies that

\[
\mathbb{P}[\text{(1.20) holds for } r] \geq 1 - 2 \exp \left( - \left( \frac{r}{C_0 \varepsilon} \right)^s \right). \tag{1.21}
\]

Note that this probability (depending on \(r/\varepsilon\)) is independent of \(\lambda_0\).

Now, let us consider the boundary estimates. As we mentioned earlier, we will establish the boundary estimates in a class of Lipschitz domains, defined as follows:
Definition 1.2. Let $\alpha \in (0, 1]$ and $D$ be a bounded Lipschitz domain with $0 \in \partial D$. We say that $D$ satisfies the $\varepsilon$-scale $C^{1, \alpha}$ condition at $0$, if there exist $C_0 > 0$ and $r_0 > 0$ such that for any $r \in (\varepsilon, r_0)$, there exists a unit vector $n_r$ such that
\[
\{ y \in \mathbb{R}^d : y \cdot n_r < -C_0 r [r^\alpha + (\varepsilon/r)\alpha] \} \cap B_r(0) 
\subset D \cap B_r(0) \subset \{ y \in \mathbb{R}^d : y \cdot n_r < C_0 r [r^\alpha + (\varepsilon/r)\alpha] \} \cap B_r(0).
\]
(1.22)

From the above definition, we see that the local boundary $\partial D \cap B_r(0)$ is contained between two parallel hyperplanes with distance comparable to $r \zeta_\alpha(r, \varepsilon)$, where $\zeta_\alpha(r, \varepsilon) := r^\alpha + (\varepsilon/r)\alpha$. In particular, this class of domains covers the classical $C^{1, \alpha}$ domains and the so called bumpy Lipschitz domains. Obviously, the $C^{1, \alpha}$ domains correspond to the case $r \zeta_\alpha(r, \varepsilon) = r^{1+\alpha}$. On the other hand, in [29] and [30], Kenig and Prange studied the Lipschitz estimate by the compactness argument in the bumpy Lipschitz domain whose boundary is the graph of the function
\[
x_d = \varepsilon \psi(x'/\varepsilon),
\]
where $\psi \in W^{1, \infty}(\mathbb{R}^{d-1})$. This actually corresponds to the special case $r \zeta_1(r, \varepsilon) = \varepsilon = r(\varepsilon/r)$ in Definition 1.2. From these two typical examples, we notice that the two parts of the function $\zeta_\alpha$ come from two different sources, namely, smoothness and small bumps, which dominates at large-scales and small scales, respectively. In particular, Definition 1.2 includes the domain whose boundary is the local graph of
\[
x_d = \psi_0(x') + \varepsilon \psi_1(x'/\varepsilon),
\]
where $\psi_0 \in C^{1, \alpha}(\mathbb{R}^{d-1})$ and $\psi_1 \in W^{1, \infty}(\mathbb{R}^{d-1})$.

Fig. 1. A Lipschitz domain bumpy at small scales
Remark 1.3. In Definition 1.2, we assume in priori that $D$ is a Lipschitz domain. This assumption actually is not essential in the proof of the large-scale regularity. It is only required for Lemma 2.1 which affects the basic energy estimate for the Stokes system and the system of elasticity with large $\lambda_0$. It is possible to relax this assumption to even more general domains with fractals (such as John domains [1,27,36]). For simplicity, however, we will not explore this direction in the present paper.

Remark 1.4. Oscillating boundaries given by (1.23), with additional structure such as periodicity, have been widely studied in the analysis of the wall laws for the Navier–Stokes equations with rough boundaries; see [10–12,14,15] for some recent references. Most recently, Higaki and Prange [25] obtained the large-scale Lipschitz estimate for the stationary Navier–Stokes equations over bumpy Lipschitz boundaries without any structure by a compactness method.

Our main result for the boundary estimate is stated as follows:

**Theorem 1.5.** Let $D$ be a bounded Lipschitz domain satisfying the $\varepsilon$-scale $C^{1,\alpha}$ condition at $0 \in \partial D$. Define $D_r = D \cap B_r(0)$ and $\Delta_r = \partial D \cap B_r(0)$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be as before. For any $s \in (0, d)$ and $\lambda_0 \in [0, \infty)$, there exist a constant $C_0 = C_0(s, d, \Lambda)$ and a random variable $\mathcal{X} = \mathcal{X}_{s, \lambda} : \Omega \mapsto [1, \infty)$ satisfying

$$\mathcal{X} \leq O_{\varepsilon}(C_0),$$

such that if $u^\varepsilon_\lambda \in H^1(D_2; \mathbb{R}^d)$ is a weak solution of

$$\begin{cases}
\nabla \cdot (A^\varepsilon \nabla u^\varepsilon_\lambda) + \nabla (\lambda^\varepsilon \nabla \cdot u^\varepsilon_\lambda) = 0 & \text{in } D_2, \\
u^\varepsilon_\lambda = 0 & \text{on } \Delta_2,
\end{cases}$$

(1.24)

then for any $r \in [\varepsilon \mathcal{X}, 1]$,

$$\left( \int_{D_r} |\nabla u^\varepsilon_\lambda|^2 \right)^{1/2} + \left( \int_{D_r} |\lambda^\varepsilon \nabla \cdot u^\varepsilon_\lambda - \int_{D_2} \lambda^\varepsilon \nabla \cdot u^\varepsilon_\lambda|^2 \right)^{1/2} \leq C \left( \int_{D_2} |\nabla u^\varepsilon_\lambda|^2 \right)^{1/2}.$$  

(1.25)

The above theorem gives the expected boundary estimates parallel to Theorem 1.1. The main novelty of Theorem 1.5 is that the boundary is not necessarily smooth below $\varepsilon$-scale. This phenomenon seems physically and experimentally natural, as the microscopic structure of the bumpy boundary (which is always the case in reality) should not have a visible influence if only the large-scale or macroscopic regularity is concerned. In other words, the following philosophy should be valid:

The large-scale smoothness of the boundary

$\implies$ the large-scale smoothness of the solutions.

Furthermore, it seems very promising that the quantitative method in this paper may also apply to other types of equations, such as Navier–Stokes equations.
Remark 1.6. By forcing $\lambda \equiv 0$, our main theorems recover the results for the usual elliptic system. Particularly, Theorem 1.5 recovers Kenig and Prange’s work [30] for the large-scale boundary estimate. On the other hand, by taking $\lambda \to \infty$ (literally, replacing $\lambda \epsilon \nabla \cdot u_\lambda^\epsilon$ by $p_\epsilon$ in (1.20) and (1.25)), the results in this paper also implies the large-scale regularity for Stokes system (namely, the system of completely incompressible elasticity).

1.4. New Ingredients of the Proofs

We will prove our main theorems by a method of excess decay iteration. The key step in this method is to show an algebraic rate of convergence for (1.12) (which will first be reduced to the case with constant $\lambda$). Precisely, if $u_\lambda^\epsilon \in H^1(D; \mathbb{R}^d)$ is a weak solution of

$$
\begin{align*}
\nabla \cdot (A^\epsilon \nabla u_\lambda^\epsilon) + \lambda \nabla (\nabla \cdot u_\lambda^\epsilon) &= 0 & \text{in } D, \\
\nabla \cdot f &= 0 & \text{on } \partial D,
\end{align*}
$$

(1.26)

for some $f \in W^{1,2+\delta}(D; \mathbb{R}^d)$ with $\delta > 0$, then we show that this system homogenizes to

$$
\begin{align*}
\nabla \cdot (A^\lambda \nabla u_\lambda^0) + \lambda \nabla (\nabla \cdot u_\lambda^0) &= 0 & \text{in } D, \\
u_\lambda^0 &= f & \text{on } \partial D,
\end{align*}
$$

(1.27)

with a rate of convergence

$$
\|u_\lambda^\epsilon - u_\lambda^0\|_{L^2(D)} + \|\lambda \nabla \cdot u_\lambda^\epsilon - \lambda \nabla \cdot u_\lambda^0\|_{H^{-1}(D)} \leq C(\epsilon^{\beta(d-\sigma)} + (X_{\epsilon})^{\alpha s}) \|\nabla f\|_{L^{2+s}(D)},
$$

(1.28)

for some $\alpha, \beta \in (0, 1)$, where the random variable satisfies $X = X_{\epsilon, \lambda} \leq O_s(C_0)$. Note that the pressures only have a weak convergence in $H^{-1}(D)$. Surprisingly, this is sufficient for us to establish the optimal pressure estimates. Also, it should be pointed out that the homogenized tensor $A^\lambda$, depending on $\lambda$ implicitly, satisfies the ellipticity condition (1.3) uniformly in $\lambda$. Moreover, $A^\lambda = \hat{A} + O(\lambda^{-1})$ as $\lambda \to \infty$, where $\hat{A}$ is the homogenized tensor of a Stokes system; see Section 4.

A crucial principle in our mind to prove (1.28) is that (1.12) could be viewed as an elliptic system for relatively small $\lambda$ and could be approximated by a Stokes system for relatively large $\lambda$, due to (1.11). The precise threshold we will use is $\lambda = \epsilon^{-\sigma}$ for some small $\sigma \in (0, 1)$ independent of $\epsilon$ and $\lambda$. If $\lambda < \epsilon^{-\sigma}$, the convergence rate follows from the result for elliptic system. In this case, we need to track how the constant $C$ depends on $\lambda$. If $\lambda > \epsilon^{-\sigma}$, with an explicit error, (1.11) may be first reduced to a Stokes system for which a convergence rate may be obtained similarly as elliptic system (Theorem 4.8). This process may be described by the diagram in Fig. 2.

Fig. 2. A sketch of the proof of convergence rate
With the explicit convergence rate, we are able to control the excess decays in an iterative argument. Our method follows from Shen’s elegant framework in [37], which originates from [6, 7]. Because of the generalized Caccioppoli inequality (1.10), however, our argument is much more complicated than the usual elliptic/Stokes system. In the following context, we would like to describe our main idea to tackle the boundary estimate (the interior estimate is similar). In [29] and [30], Kenig and Prange introduced the boundary layer correctors to prove the large-scale Lipschitz estimate by a compactness argument in bumpy Lipschitz domains. In this paper, we adopt a more effective quantitative perturbation argument which can be beautifully unified into the aforementioned method of excess decay iteration. The quantified excess we are going to use for the boundary estimate is defined by

$$H(t) = \frac{1}{t} \inf_{q \in \mathbb{R}^d} \left( \int_{D_t} |u^\varepsilon_{\lambda} - (n_t \cdot x)q|^2 \right)^{1/2} + \frac{1}{t} \| \lambda \nabla \cdot u^\varepsilon_{\lambda} - \int_{D_t} \lambda \nabla \cdot u^\varepsilon_{\lambda} \|_{H^{-1}(D_t)}$$

\[ (1.29) \]

where $n_t$ is the unit vector given in Definition 1.2 which may be understood as an approximate outer normal to the large-scale smooth boundary. We do need this specified directions because of the lack of regularity of the real normal or tangential directions along the Lipschitz boundary at small scales. Also, the particular structure is designed corresponding to the generalized Caccioppoli inequality (1.10). The advantage of this new structure in (1.29) involving the pressure is that we can obtain the Lipschitz estimate for the displacement and the pressure estimate simultaneously, which were proved separately in the previous work [20, 22] for the Stokes system.

With the excess quantity given as above, we show that there exists some constant $\theta \in (0, 1/4)$ such that, for $r \in (\varepsilon X, 1)$,

$$H(\theta r) \leq \frac{1}{2} H(r) + \text{small error.} \quad (1.30)$$

This eventually leads to the desired estimates by Lemma 6.7, which is an iteration argument generalizing [37, Lemma 8.5]. Finally, let us explain the key idea for proving (1.30). First of all, Definition 1.2 implies that for any mesoscopic scale $r \in (\varepsilon, 1)$, the localized boundary is close to be flat with a controllable error. This fact allows us to construct an approximate solution $v^\varepsilon_{\lambda}$ in a nicer domain with flat boundary, in which the excess decay estimate for the approximate solution could be established in a familiar way. Meanwhile, the errors between the approximate and real solutions could be estimated quantitatively via the Meyers’ estimate which holds in any Lipschitz domains. Collecting all these errors, we obtain (1.30).

1.5. Organization of the Paper

The organization of the paper is as follows: in Section 2, we give the definitions of variational solutions, energy estimates and the generalized Caccioppoli inequality. In Section 3, we prove the asymptotic expansion and apply it to the regularity
theory in non-homogenization setting. In Sections 4 and 5, we establish the algebraic convergence rates for Stokes system and system of elasticity, respectively. Finally, in Sections 6 and 7, we prove Theorems 1.1 and 1.5, respectively.

2. Variational Solutions and Energy Estimates

In this section, we will define the variational solution for the system of elasticity and establish the energy estimates. The classical theory for the Stokes system may be found in [16,39]. We begin with an important lemma for the Stokes system.

Denote by $H^{-1}(D)$ and $W^{-1,p'}(D)$ the dual spaces of $H_0^1(D)$ and $W_0^{1,p}(D)$, respectively. Define $L^2_0(D) = \{ f \in L^2(D) : \int_D f = 0 \}$.

Lemma 2.1. Let $D$ be a Lipschitz domain and $f \in L^p(D)$. Then

$$C^{-1}\|\nabla f\|_{W^{-1,p'}(D)} \leq \| f - \int_D f \|_{L^p(D)} \leq C\|\nabla f\|_{W^{-1,p'}(D)},$$

where $C$ depends only on $d$ and $D$. In particular, if $p = 2$,

$$C^{-1}\|\nabla f\|_{H^{-1}(D)} \leq \| f - \int_D f \|_{L^2(D)} \leq C\|\nabla f\|_{H^{-1}(D)}.$$

Proof. This is the duality of the solvability of the divergence equation $\nabla \cdot u = g \in L^p(D)$, $u \in W_0^{1,p}(D)$, which has been proved in [1] in any bounded John domains (including Lipschitz domains).

2.1. Stokes System

Let $A = A(x) : D \mapsto \mathbb{R}^{d^2 \times d^2}$ satisfy (1.3). Consider the general (compressible) Stokes system

$$\begin{cases}
\nabla \cdot (A(x)\nabla v) + \nabla p = F & \text{in } D, \\
\nabla \cdot v = g & \text{in } D, \\
v = f & \text{on } \partial D.
\end{cases}$$

We say that a pair $(v, p) \in H^1(D; \mathbb{R}^d) \times L_0^2(D)$ is a weak solution of (2.1) if it holds that

$$\int_D A(x)\nabla v \cdot \nabla w + \int_D p \nabla \cdot w = -\langle F, w \rangle$$

for any $w \in H_0^1(D; \mathbb{R}^d)$, and $\nabla \cdot v = g$ in $L^2(D)$, $v - f \in H_0^1(D; \mathbb{R}^d)$.

The following theorem includes the wellposedness of (2.1) and the energy estimate:
Theorem 2.2. Let $D$ be a bounded Lipschitz domain and the compatibility condition
\[
\int_D g \, dx = \int_{\partial D} f \cdot n \, d\sigma \tag{2.2}
\]
be satisfied. Then the Stokes system (2.1) has a unique weak solution $(v, p) \in H^1(D; \mathbb{R}^d) \times L^2_0(D)$. Moreover,
\[
\|\nabla v\|_{L^2(D)} + \|p\|_{L^2(D)} \leq C \left( \|F\|_{H^{-1}(D)} + \|g\|_{L^2(D)} + \|f\|_{H^{1/2}(\partial D)} \right),
\]
where $C$ depends only on $d$, $\Lambda_1$ and $D$.

2.2. System of Elasticity

Let $A$ satisfy (1.3) and consider (1.8) with constant $\lambda \geq 0$. We say $u_\lambda \in H^1(D; \mathbb{R}^d)$ is the weak solution of (1.8), if
\[
\int_D A(x) \nabla u_\lambda \cdot \nabla w + \int_D \lambda \nabla \cdot u_\lambda - \int_D \lambda \nabla \cdot u_\lambda \nabla \cdot (u_\lambda - f) = \langle F, w \rangle
\]
for any $w \in H^1_0(D; \mathbb{R}^d)$ and $u_\lambda - f \in H^1(D; \mathbb{R}^d)$. The following theorem gives the energy estimate of the elasticity system with arbitrary $\lambda \geq 0$:

Theorem 2.3. Let $D$ be a bounded Lipschitz domain. Then the elasticity system (1.8) has a unique weak solution $u_\lambda \in H^1(D; \mathbb{R}^d)$ satisfying
\[
\|u_\lambda\|_{H^1(D)} + \|\lambda \nabla \cdot u_\lambda\|_{L^2(D)} \leq C \left( \|F\|_{H^{-1}(D)} + \|\nabla u_\lambda\|_{L^2(D)} \right), \tag{2.3}
\]
where $C$ depends only on $d$, $\Lambda$ and $D$.

Proof. If we view (1.8) as an elliptic system with a large ellipticity constant, then the classical Lax–Milgram theorem implies that $u_\lambda \in H^1(D; \mathbb{R}^d)$. It suffices to show (2.3) with constant $C$ independent of $\lambda$. By Lemma 2.1:
\[
\|\lambda \nabla \cdot u_\lambda - \int_D \lambda \nabla \cdot u_\lambda\|_{L^2(D)} \leq C \left( \|F\|_{H^{-1}(D)} + \|\nabla u_\lambda\|_{L^2(D)} \right). \tag{2.4}
\]

Now, by adding a constant, we write the system in (1.8) as
\[
\nabla \cdot (A(x) \nabla u_\lambda) + \nabla \left( \lambda \nabla \cdot u_\lambda - \int_D \lambda \nabla \cdot u_\lambda \right) = F \quad \text{in} \quad D.
\]

Integrating this system against $u_\lambda - f$ and using the integration by parts, we arrive at
\[
\int_D A(x) \nabla u_\lambda \cdot \nabla u_\lambda + \int_D \left( \lambda \nabla \cdot u_\lambda - \int_D \lambda \nabla \cdot u_\lambda \right) \nabla \cdot (u_\lambda - f) = \langle F, u_\lambda - f \rangle + \int_D A(x) \nabla u_\lambda \cdot \nabla f. \tag{2.5}
\]
Substituting
\[ \nabla \cdot (u_\lambda - f) = \left( \nabla \cdot u_\lambda - \int_D \nabla \cdot u_\lambda \right) + \int_D \nabla \cdot u_\lambda - \nabla \cdot f \]
into the second term of (2.5), we obtain

\[
\int_D A(\lambda) \nabla u_\lambda \cdot \nabla u_\lambda + \int_D \lambda \left( \nabla \cdot u_\lambda - \int_D \nabla \cdot u_\lambda \right)^2 \\
\leq C \| F \|_{H^{-1}(D)} (\| \nabla u_\lambda \|_{L^2(D)} + \| \nabla f \|_{L^2(D)}) \\
+ \Lambda \| \nabla u_\lambda \|_{L^2(D)} \| \nabla f \|_{L^2(D)} \\
\leq \frac{1}{2\Lambda} \| \nabla u_\lambda \|^2_{L^2(D)} + C \left( \| F \|^2_{H^{-1}(D)} + \| \nabla f \|^2_{L^2(D)} \right),
\]
where in the second inequality, we have used (2.4), the Cauchy–Schwarz inequality and the fact

\[
\int_D \nabla \cdot u_\lambda = \frac{1}{|D|} \int_{\partial D} f \cdot n d\sigma = \int_D \nabla \cdot f.
\]

It follows from the ellipticity condition that

\[
\| \nabla u_\lambda \|_{L^2(D)} \leq C \left( \| F \|_{H^{-1}(D)} + \| \nabla f \|_{L^2(D)} \right).
\]

Finally, the estimate (2.3) follows from the Poincaré inequality and (2.4). \[\square\]

### 2.3. A Generalized Caccioppoli Inequality

We introduce the scale-invariant space \( H^{-1} \). Let \( D \) be a Lipschitz domain. Define the scale-invariant \( H^1 \) norm by

\[
\| v \|_{H^1(D)} := |D|^{-1/d} \left( \int_D |v|^2 \right)^{1/2} + \left( \int_D |\nabla v|^2 \right)^{1/2}.
\]

Then, we define

\[
\| u \|_{H^{-1}(D)} := \sup \left\{ \int_D u v : v \in H^1_0(D) \text{ and } \| v \|_{H^1} \leq 1 \right\}.
\]

Observe that if \( u \in L^2(D) \), then

\[
\| u \|_{H^{-1}(D)} \leq |D|^{1/d} \left( \int_D |u|^2 \right)^{1/2}.
\]
Theorem 2.4. Let \( u_\lambda \in H^1(B_2; \mathbb{R}^d) \) be a weak solution of
\[
\nabla \cdot A(x) \nabla u_\lambda + \nabla (\lambda \nabla \cdot u_\lambda) = 0 \quad \text{in} \quad B_2.
\] (2.9)

Then there exists a constant \( C \) depending only on \( \Lambda \) and \( d \) such that
\[
\int_{B_2} |\nabla u_\lambda|^2 + \int_{B_2} |\lambda \nabla \cdot u_\lambda| \leq C \left( \int_{B_2} \lambda \nabla \cdot u_\lambda - \lambda T(t) \right)^2 + CT(t)^2.
\]

Proof. Since \( \frac{\lambda}{|x|} \cdot u_\lambda \in L^2(B_2) \), the co-area formula implies
\[
\int_{1/2}^2 \int_{\partial B_t} \left( \frac{x}{|x|} \cdot u_\lambda \right)^2 \, dt = \int_{1/2}^2 \int_{\partial B_t} (n \cdot u_\lambda)^2 \, dt \leq C \int_{B_2} |u_\lambda|^2,
\]
where \( n \) is the unit outer normal of \( \partial B_t \). Let
\[
T(t) = \int_{B_t} \nabla \cdot u_\lambda.
\]

Then the divergence theorem yields
\[
\int_{1/2}^2 T(t)^2 \, dt \leq C \int_{1/2}^2 \int_{\partial B_t} (n \cdot u_\lambda)^2 \, dt \leq C \int_{B_2} |u_\lambda|^2.
\] (2.10)

Let \( \phi \in C_0^\infty(B_t) \) be a nonnegative cut-off function so that \( \phi(x) = 1 \) for \( x \in B_1 \) and \( \sum_{k=0}^2 \int_{B_t} \lambda |\nabla^k \phi|^2 \leq C \). By inserting the constant \( \lambda T(t) \) to the divergence part of the equation, we obtain
\[
\nabla \cdot A(x) \nabla u_\lambda + \nabla (\lambda \nabla \cdot u_\lambda - \lambda T(t)) = 0 \quad \text{in} \quad B_2.
\]

Now, integrating the above system against \( u_\lambda \phi^2 \) and using the integration by parts, we obtain
\[
\int_{B_2} A \nabla u_\lambda \cdot \nabla u_\lambda \phi^2 + \int_{B_2} \lambda (\nabla \cdot u_\lambda - T(t))^2 \phi^2
\]
\[
= -2 \int_{B_2} A \nabla u_\lambda \phi \cdot (u_\lambda \otimes \nabla \phi) - \int_{B_2} \lambda (\nabla \cdot u_\lambda - T(t))(u_\lambda \cdot 2\phi \nabla \phi)
\]
\[
- \int_{B_2} \lambda (\nabla \cdot u_\lambda - T(t)) T(t) \phi^2.
\]

This implies that
\[
\int_{B_2} |\nabla u_\lambda|^2 \phi^2
\]
\[
\leq C \int_{B_2} |u_\lambda|^2 + \|\lambda \nabla \cdot u_\lambda - \lambda T(t)\|^2_{H^{-1}(B_2)} + CT(t)^2
\]
\[
\leq C \int_{B_2} |u_\lambda|^2 |\nabla \phi|^2 + \|\lambda \nabla \cdot u_\lambda - \lambda T(2)\|^2_{H^{-1}(B_2)} + C |\lambda T(t) - \lambda T(2)|^2 + CT(t)^2.
\]
Finally, integrating in $t$ over $[1/2, 2]$ and using (2.10), we obtain the desired estimate for $\nabla u_\lambda$. The estimate for the pressure follows from (2.1).

**Remark 2.5.** By considering rescaling and the fact that $u_\lambda - q$ with any constant $q \in \mathbb{R}^d$ is also a solution, we actually prove that

$$\int_{B_r} |\nabla u_\lambda|^2 + \int_{B_r} |\lambda \nabla \cdot u_\lambda| + \int_{B_r} |\nabla \cdot u_\lambda|^2 \leq C \inf_{q \in \mathbb{R}^d} \int_{B_{2r}} |u_\lambda - q|^2 + \frac{C}{r^2} \|\lambda \nabla \cdot u_\lambda\|_{H^{-1}(B_{2r})}^2 \quad (2.11)$$

$$+ C \sup_{k, \ell \in [1/4, 1]} \left| \int_{B_{2kr}} \lambda \nabla \cdot u_\lambda - \int_{B_{2r}} \lambda \nabla \cdot u_\lambda \right|^2.$$

Similar to the proof of Theorem 2.4, one may also show the generalized boundary Caccioppoli inequality. Let $0 \in \partial D, D_t = D \cap B_t(0)$ and $\Delta_t = \partial D \cap B_t(0).$ Let $u_\lambda \in H^1(D_2; \mathbb{R}^d)$ be a weak solution of

$$\left\{ \begin{array}{l}
\nabla \cdot A(x) \nabla u_\lambda + \lambda \nabla (\nabla \cdot u_\lambda) = 0 \quad \text{in} \quad D_2, \\
u_\lambda = 0 \quad \text{on} \quad \Delta_2.
\end{array} \right. \quad (2.12)$$

Then, we have

$$\int_{D_2} |\nabla u_\lambda|^2 + \int_{D_2} |\lambda \nabla \cdot u_\lambda| + \int_{D_2} |\nabla \cdot u_\lambda|^2 \leq C \int_{D_{2r}} |u_\lambda|^2 + \frac{C}{r^2} \|\lambda \nabla \cdot u_\lambda\|_{H^{-1}(D_{2r})}^2 \quad (2.13)$$

$$+ C \sup_{t \in [1/4, 1]} \left| \int_{D_{2t}} \lambda \nabla \cdot u_\lambda - \int_{D_{2}} \lambda \nabla \cdot u_\lambda \right|^2.$$

The generalized Caccioppoli inequalities (2.11) and (2.13) will be useful for us.

### 3. Asymptotic Behaviors and General Regularity

It is well-known in physics and numerical analysis that the solution $u_\lambda$ of

$$\left\{ \begin{array}{l}
\nabla \cdot (A(x) \nabla u_\lambda) + \lambda \nabla (\nabla \cdot u_\lambda) = F \quad \text{in} \quad D, \\
u_\lambda = f \quad \text{on} \quad \partial D,
\end{array} \right. \quad (3.1)$$

converges to the solution of a Stokes system as constant $\lambda$ approaches infinity. This property allows people to design efficient numerical algorithm to solve the system of nearly incompressible elasticity [39]. In this section, we will prove a complete asymptotic expansion in terms of the solutions of certain iterative Stokes systems and use it to study the uniform regularity of the system of nearly incompressible elasticity.
3.1. A Proof of the Asymptotic Expansion

To describe the limiting system of (3.1), we define

$$\langle f \rangle_D = \frac{1}{|D|} \int_{\partial D} f \cdot n \, d\sigma. \quad (3.2)$$

Roughly speaking, the quantity $\langle f \rangle_D$ represents the averaged volume change of the material body. For nearly incompressible materials, $\langle f \rangle_D$ is small under a mild physical condition, and could be large under high pressure.

**Lemma 3.1.** Let $D$ be a Lipschitz domain and $u_\lambda$ the weak solution of (3.1). Let $v_0$ be the weak solution of

$$\begin{cases}
\nabla \cdot (A(x)\nabla v_0) + \nabla p_0 = F \quad &\text{in } D, \\
\nabla \cdot v_0 = \langle f \rangle_D \quad &\text{in } D, \\
v_0 = f \quad &\text{on } \partial D.
\end{cases} \quad (3.3)$$

Then

$$\|u_\lambda - v_0\|_{H^1(D)} + \|\lambda \nabla \cdot u_\lambda - \int_D \lambda \nabla \cdot u_\lambda - p_0\|_{L^2(D)} \leq C\lambda^{-1} \left(\|F\|_{H^{-1}(D)} + \|f\|_{H^{1/2}(\partial D)}\right), \quad (3.4)$$

where $C$ depends only on $d$, $\Lambda$ and $D$.

**Proof.** The proof is well-known. We provide a proof for completeness. By the divergence theorem, $\langle f \rangle_D = \int_D \nabla \cdot u_\lambda$. Let $w_0 = u_\lambda - v_0$ and consider the Stokes system for $w_0$

$$\begin{cases}
\nabla \cdot (A(x)\nabla w_0) + \nabla (\lambda \nabla \cdot w_0 - p_0) = 0 \quad &\text{in } D, \\
\nabla \cdot w_0 = \nabla \cdot u_\lambda - \int_D \lambda \nabla \cdot u_\lambda \quad &\text{in } D, \\
w_0 = 0 \quad &\text{on } \partial D. \quad (3.5)
\end{cases}$$

As a consequence, it follows from Theorems 2.2 and 2.3 that

$$\begin{align*}
\|u_\lambda - v_0\|_{H^1(D)} + \|\lambda \nabla \cdot u_\lambda - \int_D \lambda \nabla \cdot u_\lambda - p_0\|_{L^2(D)} \\
\leq C\|\nabla \cdot u_\lambda - \int_D \lambda \nabla \cdot u_\lambda\|_{L^2(D)} \\
\leq C\lambda^{-1} \left(\|F\|_{H^{-1}(D)} + \|f\|_{H^{1/2}(\partial D)}\right).
\end{align*}$$

This completes the proof. \qed

Divided by $\lambda$, (3.4) yields

$$\nabla \cdot u_\lambda - \int_D \nabla \cdot u_\lambda - \lambda^{-1} p_0 \leq O(\lambda^{-2}). \quad (3.6)$$

Observe that $\lambda^{-1} p_0$ may be used to correct system (3.5) with a higher-order error $O(\lambda^{-2})$ and thus the first-order term may be determined. It turns out that iterating this argument leads to a complete asymptotic expansion in $\lambda$ for the solution $u_\lambda$. 
Theorem 3.2. Let $D$ be a bounded Lipschitz domain and $A$ satisfy (1.3) and (1.4). Suppose $u_\lambda$ is the weak solution of (3.1) with $F \in H^{-1}(D; \mathbb{R}^d)$ and $f \in H^{1/2}(D; \mathbb{R}^d)$. Then there exists $C_0 > 0$ depending only on $d$, $\Lambda$ and $D$, such that if the constant $\lambda > C_0$,

$$u_\lambda = \sum_{k=0}^{\infty} \lambda^{-k} v_k \quad \text{in } H^1 \quad \text{and} \quad \lambda \nabla \cdot u_\lambda - \lambda \langle f \rangle_D = \sum_{k=0}^{\infty} \lambda^{-k} p_k \quad \text{in } L^2_0. \quad (3.7)$$

where $(v_0, p_0)$ is the weak solution of (3.3) and $(v_k, p_k)$ with $k \geq 1$ are the solutions of a sequence of iterative Stokes systems

$$\begin{cases}
\nabla \cdot (A(x) \nabla v_k) + \nabla p_k = 0 & \text{in } D, \\
\nabla \cdot v_k = p_{k-1} & \text{in } D, \\
v_k = 0 & \text{on } \partial D.
\end{cases} \quad (3.8)$$

Proof. The theorem is proved by induction with Lemma 3.1 being the base case. Let $w_\ell = u_\lambda - \sum_{k=0}^{\ell} \lambda^{-k} v_k$ and $\pi_\ell = \lambda \nabla \cdot u_\lambda - \lambda \langle f \rangle_D - \sum_{k=0}^{\ell} \lambda^{-k} p_k$. We prove that

$$\|w_\ell\|_{H^1(D)} + \|\pi_\ell\|_{L^2(\Omega)} \leq C_0 C_1^{\ell} \lambda^{-\ell-1}, \quad (3.9)$$

where $C_0$ depends only on $d$, $\Lambda$, $D$ and the data $(F, f)$, and $C_1$ depends only on $d$, $\Lambda$ and $D$. Definitely, if we let $\lambda > \lambda_0 := C_0$, then the right-hand side of (3.9) converges to zero as $\ell \to \infty$, which leads to (3.7).

To show (3.9), we first consider the base case $\ell = 0$. Note that $(w_0, \pi_0) = (u_\lambda - v_0, \lambda \nabla \cdot u_\lambda - \lambda \langle f \rangle_D - p_0)$ satisfies

$$\begin{cases}
\nabla \cdot (A(x) \nabla w_0) + \nabla \pi_0 = 0 & \text{in } D, \\
\nabla \cdot w_0 = \nabla \cdot u_\lambda - \langle f \rangle_D & \text{in } D, \\
w_0 = 0 & \text{on } \partial D.
\end{cases} \quad (3.8)$$

Thus, Lemma 3.1 implies

$$\|w_0\|_{H^1(D)} + \|\pi_0\|_{L^2(\Omega)} \leq C_0 \lambda^{-1},$$

where $C_0$ depends only on $d$, $\Lambda$, $D$ and the data $(F, f)$.

To clearly see our idea, let us work out the first iteration step for $(w_1, \pi_1) = (w_0, \pi_0) - \lambda^{-1}(v_1, p_1)$. By the definition of $(v_1, p_1)$ in (3.8), one has

$$\begin{cases}
\nabla \cdot (A(x) \nabla w_1) + \nabla \pi_1 = 0 & \text{in } D, \\
\nabla \cdot w_1 = \lambda^{-1} \pi_0 & \text{in } D, \\
w_1 = 0 & \text{on } \partial D.
\end{cases}$$

Then the energy estimate for the Stokes system yields

$$\|w_1\|_{H^1(D)} + \|\pi_1\|_{L^2(\Omega)} \leq C_1 \lambda^{-1} \|\pi_0\|_{L^2(D)} \leq C_0 C_1 \lambda^{-2},$$

where $C_1$ depends only on $d$, $\Lambda$ and $D$. 

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In general, assume that 

\[(w_\ell, \pi_\ell) = (w_{\ell-1}, \pi_{\ell-1}) - \lambda^{-\ell}(v_\ell, p_\ell)\]

with \(\ell \geq 1\) satisfies

\[
\begin{cases}
\nabla \cdot (A(x) \nabla w_\ell) + \nabla \pi_\ell = 0 & \text{in } D, \\
\nabla \cdot w_\ell = \lambda^{-1} \pi_{\ell-1} & \text{in } D, \\
w_\ell = 0 & \text{on } \partial D,
\end{cases}
\]

and

\[
\|w_\ell\|_{H^1(D)} + \|\pi_\ell\|_{L^2(\Omega)} \leq C_0 \lambda^{-\ell-1}.
\]

Now, let 

\[(w_{\ell+1}, \pi_{\ell+1}) = (w_\ell, \pi_\ell) - \lambda^{-\ell-1}(v_{\ell+1}, p_{\ell+1}).\]

In view of (3.10) and (3.8), we see that 

\[(w_{\ell+1}, \pi_{\ell+1})\]

is the solution of

\[
\begin{cases}
\nabla \cdot (A(x) \nabla w_{\ell+1}) + \nabla \pi_{\ell+1} = 0 & \text{in } D, \\
\nabla \cdot w_{\ell+1} = \lambda^{-1} \pi_\ell & \text{in } D, \\
w_{\ell+1} = 0 & \text{on } \partial D,
\end{cases}
\]

where we have used the construction \(\pi_\ell = \pi_{\ell-1} - \lambda^{-\ell} p_\ell\). By (3.11), the last system implies

\[
\|w_{\ell+1}\|_{H^1(D)} + \|\pi_{\ell+1}\|_{L^2(\Omega)} \leq C_1 \lambda^{-1} \|\pi_\ell\|_{L^2(D)} \leq C_0 \lambda^{-1} \|\pi_\ell\|_{L^2(D)}.
\]

This proves (3.9).

3.2. Global Estimates

The asymptotic expansion in Theorem 3.2 is a powerful tool to study the global regularity of the system of nearly incompressible elasticity, provided the same regularity holds for the Stokes systems. The result may be described in an abstract setting. Let \(X_0(D; \mathbb{R}^d)\) be a subspace of \(L^1(D; \mathbb{R}^d)\) endowed with the norm \(\|\cdot\|_{X_0}\), namely,

\[
X_0(D; \mathbb{R}^d) = \{ f \in L^1(D; \mathbb{R}^d) : \|f\|_{X_0} < \infty \}.
\]

Define \(X_1(D; \mathbb{R}^d) = \{ f \in X_0(D; \mathbb{R}^d) : \nabla f \in X_0(D; \mathbb{R}^d \times \mathbb{R}^d) \}\) and \(\|f\|_{X_1} = \|f\|_{X_0} + \|\nabla f\|_{X_0}\).

**Theorem 3.3.** Let \(D\) be a Lipschitz domain. Suppose there exists a constant \(M > 0\) such that for any \(h \in X_0(D; \mathbb{R}^d), g \in X_0(D; \mathbb{R})\) and \(f \in X_1(D; \mathbb{R}^d)\), the solution \((v, p)\) of the Stokes system

\[
\begin{cases}
\nabla \cdot (A(x) \nabla v) + \nabla p = \nabla \cdot h & \text{in } D, \\
\nabla \cdot v = g & \text{in } D, \\
v = f & \text{on } \partial D,
\end{cases}
\]

satisfies

\[
\|v\|_{X_1} + \|p\|_{X_0} \leq M(\|f\|_{X_1} + \|h\|_{X_0} + \|g\|_{X_0}).
\]
Then if \( \lambda > 2M \), and \( u_\lambda \) is the weak solution of
\[
\begin{align*}
  \nabla \cdot (A(\xi) \nabla u_\lambda) + \nabla (\lambda \nabla \cdot u_\lambda) &= \nabla \cdot h \quad \text{in } D, \\
  u_\lambda &= f \quad \text{on } \partial D,
\end{align*}
\]
then
\[
\|u_\lambda\|_{X_1} + \lambda \|\nabla \cdot u_\lambda - \langle f \rangle_D\|_{X_0} \leq 2M (\|f\|_{X_1} + \|h\|_{X_0}).
\]  

(3.16)

Theorem 3.3 may be proved directly by using Theorem 3.2. Note that Theorem 3.3 applies only for \( \lambda \geq 2M \). However, for \( 0 \leq \lambda \leq 2M \), the system (3.15) is the classical elliptic system whose regularity theory is well-understood.

Particularly, Theorem 3.3 applies to \( X_0 = W^{k,p} \), \( X_1 = W^{k+1,p} \) or \( X_0 = C^{k,\alpha} \), \( X_1 = C^{k+1,\alpha} \) for \( p \in (1, \infty) \) and \( k \in \mathbb{N}, \alpha \in (0, 1) \). To give a concrete example, in the following, we apply Theorem 3.3 to show the
\[
\text{Theorem 3.4. Let } D \text{ and } (A, \lambda) \text{ satisfy the above assumptions. Suppose } p \in (1, \infty), h \in L^p(D; \mathbb{R}^{d \times d}) \text{ and } f \in W^{1,p}(D; \mathbb{R}^d). \text{ Then the weak solution of (3.18) satisfies}
\]
\[
\|u_\lambda^\varepsilon\|_{W^{1,p}(D)} + \|\lambda^\varepsilon \nabla \cdot u_\lambda^\varepsilon - \lambda_0 \langle f \rangle_D\|_{L^p(D)} \leq C (\|f\|_{W^{1,p}(D)} + \|h\|_{L^p(D)}),
\]  

(3.19)

where \( C \) depends only on \( d, D, \Lambda \) and the VMO modulus of \( (A, \lambda) \). In particular, \( C \) is independent of \( \lambda_0 \) and \( \varepsilon \).

Proof. By (1.6) and (1.7), \( \lambda^\varepsilon \) may be reduced to \( \lambda_0 \). The result then follows from Theorem 3.3 and [20, Theorem 1.4].

Remark 3.5. We should emphasize that the asymptotic expansion does not apply to the Lipschitz estimate of \( u_\lambda^\varepsilon \) for (3.18), because the uniform boundedness of \( |\nabla u_\lambda^\varepsilon| \) is not preserved under the iterative Stokes system (3.8). In other words, (3.14) is generally wrong for \( X_0 = L^\infty \). This failure is due to a well-known fact that the singular integral (or Riesz transform) is not bounded in \( L^\infty \). In the rest of the paper, we will develop a new approach, using the regularity theory for Stokes system only in the case of constant coefficients, to resolve this problem.
3.3. Local Estimates

In this subsection, we will prove two local estimates, that is, the Meyers’ estimate and $C^{1,\alpha}$ estimate, which are crucial in the study of quantitative homogenization. For the system of elasticity with large $\lambda$, the local estimates are technically more involving.

To show the local $C^{1,\alpha}$ regularity of the system of elasticity, we need the same regularity for the Stokes system. For convenience, we define

$$[f]_{C^\alpha(D)} := \sup_{x,y \in D} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

**Theorem 3.6.** [16] Let $D$ be a bounded $C^{1,\alpha}$ domain and $A$ be constant. There exits $C > 0$ depending only on $d$, $\Lambda$ and $D$ such that if $(v, p)$ is a weak solution of

$$\begin{aligned}
\nabla \cdot (A \nabla v) + \nabla p &= \nabla \cdot h \quad \text{in} \quad D_{2r}, \\
\nabla \cdot v &= g \quad \text{in} \quad D_{2r}, \\
v &= f \quad \text{on} \quad \Delta_{2r},
\end{aligned}$$

(3.20)

with $h \in C^\alpha(D_{2r}; \mathbb{R}^{d \times d})$, $g \in C^\alpha(D_{2r})$ and $f \in C^{1,\alpha}(\Delta_{2r}; \mathbb{R}^d)$, then $(\nabla v, p) \in C^\alpha(D_r; \mathbb{R}^d \times \mathbb{R})$ and

$$[\nabla v]_{C^\alpha(D_r)} + [p]_{C^\alpha(D_r)} \leq C \left( \frac{1}{r^\alpha} \left( \int_{D_{2r}} |\nabla v|^2 \right)^{1/2} + [h]_{C^\alpha(D_{2r})} + [g]_{C^\alpha(D_{2r})} + [\nabla f]_{C^\alpha(\Delta_{2r})} \right).$$

As a corollary, we may show

**Lemma 3.7.** Let the same conditions as in Theorem 3.6 hold. Let $(v, p)$ be the solution of (3.20) with $r = 1$. Then for any $s \in (0, 2)$

$$[\nabla v]_{C^\alpha(D_s)} + [p]_{C^\alpha(D_s)} \leq \frac{C}{(2-s)^{d/2+\alpha}} \left( \int_{D_2} |\nabla v|^2 \right)^{1/2}$$

$$+ C \left( [h]_{C^\alpha(D_{(2+s)/2})} + [g]_{C^\alpha(D_{(2+s)/2})} + [\nabla f]_{C^\alpha(\Delta_{(2+s)/2})} \right).$$

**Proof.** The case $s \in (0, 1)$ is obvious. Fix $s \in (1, 2)$. Let $x, y \in D_s$. Then $\text{dist}(x, \partial D_{(2+s)/2} \setminus \Delta_2) \geq (2-s)/2$ and $\text{dist}(y, \partial D_{(2+s)/2} \setminus \Delta_2) \geq (2-s)/2$. Now, if $|x - y| < (2-s)/4$, we can find a ball $B$ with radius $(2-s)/8$ containing both $x$ and $y$ so that $2B \subset B_{(2+s)/2}$. It follows from Theorem 3.6 that

$$[\nabla v]_{C^\alpha(\tilde{B} \cap D_2)} + [p]_{C^\alpha(\tilde{B} \cap D_2)} \leq C \left( \frac{1}{(2-s)^a} \left( \int_{2\tilde{B} \cap D_2} |\nabla v|^2 \right)^{1/2} + [h]_{C^\alpha(2\tilde{B} \cap D_2)} + [g]_{C^\alpha(2\tilde{B} \cap D_2)} + [\nabla f]_{C^\alpha(\tilde{B} \cap \Delta_2)} \right)$$

$$\leq C \left( \frac{1}{(2-s)^{d/2+\alpha}} \left( \int_{D_2} |\nabla v|^2 \right)^{1/2} + [h]_{C^\alpha(D_{(2+s)/2})} + [g]_{C^\alpha(D_{(2+s)/2})} + [\nabla f]_{C^\alpha(\Delta_{(2+s)/2})} \right).$$
Now, assume $|x - y| > (2 - s)/4$. Because $D_{(2-s)/2}$ is a Lipschitz domain, we can find a sequence of balls $\{B_{r_k}(x_k)\}_{k=M}^{N}$ (a Harnack chain) connecting the points $x$ and $y$. Moreover, the radius $r_k$ are comparable to $\theta^k(2-s)$ for some $\theta > 1$ and

$$2B_{r_k}(x_k) \subset B_{(2-s)/2} \text{ for all } k = M, M + 1, \ldots, N.$$  

The largest radius is comparable to $|x - y|$, that is, $\theta^N(2-s) \simeq |x - y|$. The idea is that we apply Theorem 3.6 on each ball $B_{r_k}(x_k)$ and then estimate $|\nabla v(x) - \nabla v(y)|$ and $|p(x) - p(y)|$ by connecting a path through the chain of balls. Precisely, we have

$$|\nabla v(x) - \nabla v(y)| + |p(x) - p(y)|$$

$$\leq C \sum_k \left( \int_{2B_{r_k}(x_k) \cap D_2} |\nabla v|^2 \right)^{1/2}
+ C \sum_k r_k^\alpha \left( [h] C^\alpha(2B_{r_k}(x_k) \cap D_2) + [g] C^\alpha(2B_{r_k}(x_k) \cap \Delta_2) + [\nabla f] C^\alpha(2B_{r_k}(x_k) \cap \Delta_2) \right)$$

$$\leq \frac{C}{(2-s)^{d/2}} \left( \int_{D_2} |\nabla v|^2 \right)^{1/2}
+ C |x - y|^\alpha \left( [h] C^\alpha(D_{(2-s)/2}) + [g] C^\alpha(D_{(2-s)/2}) + [\nabla f] C^\alpha(\Delta_{(2-s)/2}) \right).$$

This implies the desired estimate since $|x - y| > (2-s)/4$. \hfill $\square$

**Theorem 3.8.** Let $D$ be a bounded $C^{1,\alpha}$ domain and $A$ be constant. There exists $C > 0$ depending only on $d$, $\Lambda$ and $D$ such that if $u_\lambda$ is a weak solution of

$$\begin{cases}
\nabla \cdot (A \nabla u_\lambda) + \nabla (\lambda \nabla \cdot u_\lambda) = \nabla \cdot h & \text{in } D_{2r}, \\
u_\lambda = f & \text{on } \partial D_{2r},
\end{cases}$$

with $h \in C^\alpha(D_{2r}; \mathbb{R}^{d \times d})$ and $f \in C^{1,\alpha}(\Delta_{2r}; \mathbb{R}^d)$, then $u_\lambda \in C^{1,\alpha}(D_r; \mathbb{R}^d)$ and

$$[\nabla u_\lambda] C^\alpha(D_r) + [\lambda \nabla \cdot u_\lambda] C^\alpha(D_r)$$

$$\leq C \left\{ \frac{1}{r} \left( \int_{D_{2r}} |\nabla u_\lambda|^2 \right)^{1/2} + [h] C^\alpha(D_{2r}) + [\nabla f] C^\alpha(\Delta_{2r}) \right\}.$$

**Proof.** It suffices to show the estimate for $\lambda > 2C$ for some constant $C > 0$, while the case with small $\lambda$ follows from the classical Schauder estimate for the elliptic system.

By rescaling and normalization, we assume $r = 1$ and

$$\left( \int_{D_2} |\nabla u_\lambda|^2 \right)^{1/2} + [h] C^\alpha(D_2) + [\nabla f] C^\alpha(\Delta_2) = 1.$$  

Applying Theorem 3.2 in $D_2$, we may write

$$u_\lambda = \sum_{k=0}^\infty \lambda^{-k} v_k \in H^1 \text{ and } \lambda \nabla \cdot u_\lambda - \lambda (f)_D = \sum_{k=0}^\infty \lambda^{-k} p_k \in L^2.$$
where \((v_0, p_0)\) solves (3.3) with \(F = \nabla \cdot h\) and \((v_k, p_k)\) with \(k \geq 1\) solves (3.8). By the energy estimate, it is not hard to see
\[
\|\nabla v_k\|_{L^2(D_2)} + \|p_k\|_{L^2(D_2)} \leq C^{k+1},
\]
where \(C\) is the constant in Theorem 2.2.

Now, applying Lemma 3.7 to (3.3), we have
\[
[\nabla v_0]C^\alpha(D_3) + [p_0]C^\alpha(D_3) \leq \frac{C^2}{(2 - \sigma)^{d/2 + \alpha}}.
\]

On the other hand, applying Lemma 3.7 to (3.8) and using (3.22), we obtain, for any \(s \in (0, 2)\),
\[
[\nabla v_k]C^\alpha(D_3) + [p_k]C^\alpha(D_3) \leq \frac{C^{k+2}}{(2 - s)^{d/2 + \alpha}} + C[p_{k-1}]C^\alpha(D_{(2+s)/2}).
\]

Next, without much difficulty, we may prove by induction that
\[
[\nabla v_k]C^\alpha(D_3) + [p_k]C^\alpha(D_3) \leq \frac{(k + 1)C^{k+2}}{(2 - s)^{d/2 + \alpha}} \quad \text{for all} \quad k \geq 0, s \in (0, 2).
\]

This implies the desired estimate if \(\lambda\) is large. Indeed, if \(\lambda > 2C\),
\[
[\nabla u_\lambda]C^\alpha(D_3) + [\lambda \nabla \cdot u_\lambda]C^\alpha(D_3) \leq \sum_{k=0}^{\infty} \lambda^{-k} \left([\nabla v_k]C^\alpha(D_3) + [p_k]C^\alpha(D_3)\right)
\leq \sum_{k=0}^{\infty} (2C)^{-k} \frac{(k + 1)C^{k+2}}{(2 - s)^{d/2 + \alpha}}
\leq \frac{C}{(2 - s)^{d/2 + \alpha}}.
\]

This implies the desired estimate by setting \(s = 1\). \(\square\)

Next, we are going to show the Meyers’ estimate for the system of elasticity which is independent of \(\lambda\). To this end, let us recall the local Meyers’ estimate of Stokes system with bounded measurable coefficients.

**Theorem 3.9.** (Meyers’ estimate for Stokes system) Let \(D\) be a Lipschitz domain. There exists \(p_0 > 2\), depending only on \(d\), \(\Lambda\) and Lip\((D)\) so that if \((v, p)\) is a weak solution of
\[
\begin{cases}
\nabla \cdot (A(x)\nabla v) + \nabla p = \nabla \cdot h & \text{in } D_{2r}, \\
\n \cdot v = g & \text{in } D_{2r}, \\
v = f & \text{on } \Delta_{2r},
\end{cases}
\]
with \(h \in W^{1,p_0}(D_{2r}; \mathbb{R}^d\times \mathbb{R}^d)\), \(g \in L^{p_0}(D_{2r})\) and \(f \in W^{1,p_0}(D_{2r}; \mathbb{R}^d)\), then \((v, p)\) is weak in \(W^{1,p_0}(D_r; \mathbb{R}^d) \times L^{p_0}(D_r)\) and
\[
\left(\int_{D_r} |\nabla v|^{p_0}\right)^{1/p_0} + \left(\int_{D_r} |p - f_{D_r}|^{p_0}\right)^{1/p_0}
\leq C \left\{ \left(\int_{D_{2r}} |\nabla u|^2\right)^{1/2} + \left(\int_{D_{2r}} |h|^{p_0}\right)^{1/p_0} + \left(\int_{D_{2r}} |g|^{p_0}\right)^{1/p_0} + \left(\int_{D_{2r}} |\nabla f|^{p_0}\right)^{1/p_0}\right\},
\]
where \(C\) depends only on \(d\), \(\Lambda\) and Lip\((D)\).
Of course, the Meyers’ estimate for the system of elasticity could be proved by the similar strategy as Theorem 3.8. Here we will use an alternative approach which takes advantage of a real variable perturbation argument by Shen [38, Chapter 3].

**Theorem 3.10.** Let $D$ be a Lipschitz domain. There exists $q_0 > 2$, depending only on $d$, $\Lambda$ and $D$ so that if $u_{\lambda}$ is the weak solution of

$$
\begin{cases}
\nabla \cdot (A(x)\nabla u_{\lambda}) + \nabla (\lambda \nabla \cdot u_{\lambda}) = \nabla \cdot h & \text{in } D_{2r}, \\
u_{\lambda} = f & \text{on } \partial D_{2r},
\end{cases}
$$

with $h \in W^{1,q_0}(D_{2r}; \mathbb{R}^{d \times d})$ and $f \in W^{1,q_0}(D_{r}; \mathbb{R}^{d})$, then $u_{\lambda} \in W^{1,q_0}(D_{r}; \mathbb{R}^{d})$ and

$$
\left( \int_{D_{2r}} |\nabla u_{\lambda}|^{q_0} \right)^{1/q_0} + \left( \int_{D_{2r}} |\lambda \nabla \cdot u_{\lambda} - \int_{D_{2r}} \lambda \nabla \cdot u_{\lambda}|^{q_0} \right)^{1/q_0}
\leq C \left\{ \left( \int_{D_{2r}} |\nabla u_{\lambda}|^{2} \right)^{1/2} + \left( \int_{D_{2r}} |h|^{q_0} \right)^{1/q_0} + \left( \int_{D_{2r}} |\nabla f|^{q_0} \right)^{1/q_0} \right\},
$$

where $C$ depends only on $d$, $\Lambda$ and $D$.

**Proof.** Again, it suffices to consider the case when $\lambda$ is large. By rescaling, assume $r = 1$. Now, let $x \in \Delta_2$ and $D_{2r}(x) \subset D_2$. We construct an approximation of $u_{\lambda}$ in $D_{2r}(x)$. Actually, let $(v^r_{\lambda}, p^r_{\lambda})$ be the weak solution of

$$
\begin{cases}
\nabla \cdot (A(x)\nabla v^r_{\lambda}) + \nabla p^r_{\lambda} = \nabla \cdot h & \text{in } D_{2r}(x), \\
abla \cdot v^r_{\lambda} = \langle u_{\lambda} \rangle_{D_{2r}(x)} & \text{in } D_{2r}(x), \\
v^r_{\lambda} = u_{\lambda} & \text{on } \partial D_{2r}(x).
\end{cases}
$$

Then Lemma 3.1 implies

$$
\left( \int_{D_{2r}(x)} |\nabla u_{\lambda} - \nabla v^r_{\lambda}|^{2} \right)^{1/2} \leq \frac{C}{\lambda} \left\{ \left( \int_{D_{2r}(x)} |\nabla u_{\lambda}|^{2} \right)^{1/2} + \left( \int_{D_{2r}(x)} |h|^{2} \right)^{1/2} \right\}. \tag{3.25}
$$

Next, we will reduce (3.24) to a homogeneous system. To this end, let $w^r_{\lambda}$ be the solution of

$$
\begin{cases}
\nabla \cdot (A(x)\nabla w^r_{\lambda}) + \nabla \pi^r_{\lambda} = \nabla \cdot h & \text{in } D_{2r}(x), \\
abla \cdot w^r_{\lambda} = \langle f \rangle_{D_{2r}(x)} & \text{in } D_{2r}(x), \\
w^r_{\lambda} = f & \text{on } \partial D_{2r}(x).
\end{cases}
$$

Clearly, the energy estimate implies

$$
\left( \int_{D_{2r}(x)} |\nabla w^r_{\lambda}|^{2} \right)^{1/2} \leq C \left\{ \left( \int_{D_{2r}(x)} |h|^{2} \right)^{1/2} + \left( \int_{D_{2r}(x)} |\nabla f|^{2} \right)^{1/2} \right\}
$$

Combined with (3.25), this leads to

$$
\left( \int_{D_{2r}(x)} |\nabla u_{\lambda} - \nabla (v^r_{\lambda} - w^r_{\lambda})|^{2} \right)^{1/2} \leq \frac{C}{\lambda} \left( \int_{D_{2r}(x)} |\nabla u_{\lambda}|^{2} \right)^{1/2}
+ C \left\{ \left( \int_{D_{2r}(x)} |h|^{2} \right)^{1/2} + \left( \int_{D_{2r}(x)} |\nabla f|^{2} \right)^{1/2} \right\}. \tag{3.27}
$$
On the other hand, observe that the difference \( v_x^r - w_x^r \) satisfies
\[
\begin{align*}
\nabla \cdot (A(x) \nabla (v_x^r - w_x^r)) + \nabla (p_x^r - \pi_x^r) &= 0 & \text{in } D_{2s}(x), \\
\nabla \cdot (v_x^r - w_x^r) &= (u_\lambda - f)_{D_{2s}(x)} & \text{in } D_{2s}(x), \\
v_x^r - w_x^r &= 0 & \text{on } \partial D_{2s}(x) \cap D_2.
\end{align*}
\]

Now, Theorem 3.9 implies that there exists some \( p_0 > 2 \) depending only on \( d, \Lambda \) and \( D \) so that
\[
\left( \int_{D_{2s}(x)} |\nabla (v_x^r - w_x^r)|^{p_0} \right)^{1/p_0} \leq C \left\{ \left( \int_{D_{2s}(x)} |\nabla (v_x^r - w_x^r)|^2 \right)^{1/2} + |(u_\lambda - f)_{D_{2s}(x)}| \right\}
\]
\[
\leq C \left\{ \left( \int_{D_{2s}(x)} |\nabla u_\lambda|^2 \right)^{1/2} + \left( \int_{D_{2s}(x)} |h|^2 \right)^{1/2} + \left( \int_{D_{2s}(x)} |\nabla f|^2 \right)^{1/2} \right\}.
\]

In view of (3.27) and (3.29), which actually holds in any \( D_{2s}(x) \subset D_2 \), we may apply [38, Theorem 4.2.6] to conclude that for any \( q_0 \in (2, p_0) \), there exists \( \eta > 0 \) such that if \( C/\lambda < \eta \), then
\[
\left( \int_{D_{2s}(x)} |\nabla u_\lambda|^{q_0} \right)^{1/q_0} \leq C \left\{ \left( \int_{D_{2s}(x)} |\nabla u_\lambda|^2 \right)^{1/2} + \left( \int_{D_{2s}(x)} |h|^{q_0} \right)^{1/q_0} + \left( \int_{D_{2s}(x)} |\nabla f|^{q_0} \right)^{1/q_0} \right\},
\]
for any \( D_s(x) \) so that \( D_{4s}(x) \subset D_2 \). This particularly gives the desired estimate for \( \nabla u_\lambda \) with \( r = 1 \). Finally, the estimate of the pressure follows from Lemma 2.1. \( \square \)

4. Homogenization of the Stokes System

In this section, we study the quantitative homogenization of the Stokes system with coefficient tensor in a probability measure space \((\Omega, \mathcal{F}, \mathbb{P})\), where
\[
\Omega := \{ A : \mathbb{R}^d \mapsto \mathbb{R}^{d \times d} \text{ satisfying (1.3)–(1.4)} \}.
\]

Notice that we redefined \((\Omega, \mathcal{F}, \mathbb{P})\) with a slight abuse of notation since it has been defined as the probability measure space for the system of elasticity in Section 1.3. Fortunately, this will cause no ambiguity in this section.

Given an open set \( D \subset \mathbb{R}^d \). Let \( \mathcal{F}_D \) be the \( \sigma \)-algebra generated by the random elements
\[
A \mapsto \int_{\mathbb{R}^d} a_{ij}^{\alpha \beta}(x) \phi(x), \quad \phi \in C_0^\infty(D), \quad 1 \leq i, j, \alpha, \beta \leq d.
\]

Let \( \mathcal{F} \) be the largest \( \sigma \)-algebra containing all \( \mathcal{F}_D \) with \( D \subset \mathbb{R}^d \). We assume the probability measure \( \mathbb{P} \) satisfies the following assumptions:
• Stationarity with respect to \( \mathbb{Z}^d \)-translations:
\[
P \circ T_z = P, \quad \text{where } T_z(A)(x) = A(x + z).
\]

• Unit range of dependence:
\[\mathcal{F}_D \text{ and } \mathcal{F}_E \text{ are } \mathbb{P}\text{-independent for every Borel subset pair } D, \ E \subset \mathbb{R}^d \text{ satisfying } \text{dist}(D, E) \geq 1.\]

4.1. Finite Volume Correctors

In this subsection, we will introduce several correctors and obtain some quantitative estimates in the case of Stokes system, following the standard work in [4] for the elliptic equation. Since the key results and their proofs are actually very similar to the elliptic equation, we will briefly describe this process with a few key steps and skip the most details.

First of all, we define the solenoidal space \( H^0_0 \) by
\[
H^0_0(D) := H^1_0(D) \cap L^2_{\text{sol}}(D) = \{ f \in H^1_0(D, \mathbb{R}^d) : \nabla \cdot f = 0 \}
\]
and \( H^{\text{sol}} \) by
\[
H^{\text{sol}}(D) := H^1(D) \cap L^2_{\text{sol}}(D) = \{ f \in H^1(D, \mathbb{R}^d) : \nabla \cdot f = 0 \}.
\]

For each \( P \in \mathbb{R}^{d \times d} \), we introduce the subadditive quantity \( \mu(D, P) \) defined by
\[
\mu(D, P) := \inf_{v \in \ell_P + H^0_0(D)} \int_D \frac{1}{2} \nabla v \cdot A \nabla v = \inf_{w \in H^0_0(D)} \int_D \frac{1}{2} (P + \nabla w) \cdot A(P + \nabla w),
\]
where \( \ell_P := Px \) is the affine function with slope \( P \in \mathbb{R}^{d \times d} \). The quantity \( \mu(D, P) \) is the energy of its unique minimizer
\[
v(\cdot, D, P) := \arg\min_{v \in \ell_P + H^0_0(D)} \mu(D, P),
\]
which turns out to be the Dirichlet corrector, that is, the weak solution of
\[
\begin{cases}
\nabla \cdot (A \nabla v) + \nabla \zeta = 0 & \text{in } D, \\
\nabla \cdot v = \text{Tr}(P) & \text{in } D, \\
v = \ell_P & \text{on } \partial D.
\end{cases}
\]
Note that the pressure \( \zeta \) in the above system is not defined directly in (4.1).

**Proposition 4.1.** Let \( D \) be a bounded Lipschitz domain in \( \mathbb{R}^d \). Then \( \mu(D, P) \) and its minimizer satisfy the following properties:
• Representation as a quadratic form: there exists a symmetric $A_D$ such that
\[ \Lambda^{-1} I \leq A_D \leq \Lambda I, \]
and, for $P \in \mathbb{R}^{d \times d}$,
\[ \mu(D, P) = \frac{1}{2} P \cdot A_D P. \tag{4.3} \]

• Subadditivity. Let $\{D_i\}_{i=1}^N \subset D$ be bounded Lipschitz domains that form a partition of $D$, in the sense that $D_i \cap D_j = \emptyset$ if $i \neq j$ and
\[ |D \setminus \bigcup_{i=1}^N D_i| = 0. \]
Then, for every $P \in \mathbb{R}^{d \times d}$,
\[ \mu(D, P) \leq \sum_{i=1}^N \frac{|D_i|}{|D|} \mu(D_i, P). \]

• Quadratic response. For every $w \in \ell_P + H^{sol}_0(D)$,
\[ \frac{1}{2\Lambda} \int_D |\nabla w - \nabla v(\cdot, D, P)|^2 \leq \int_D \frac{1}{2} \nabla w \cdot A \nabla w - \mu(D, P) \]
\[ \leq \frac{\Lambda}{2} \int_D |\nabla w - \nabla v(\cdot, D, P)|^2. \]

For each integer $m \geq 1$, define the triadic cube
\[ \square_m := \left( -\frac{1}{2} 3^m, \frac{1}{2} 3^m \right)^d \subset \mathbb{R}^d. \]

If $1 \leq n < m$, then $\square_m$ can be partitioned (up to a set of zero Lebesgue measure) into exactly $3^{d(m-n)}$ subcubes which are $\mathbb{Z}^d$-translations of $\square_n$.

Now, for each $P \in \mathbb{R}^{d \times d}$, the subadditivity of $\mu$ and stationarity of the probability measure space $(\Omega, \mathcal{F}, P)$ imply the monotonicity of $\mathbb{E}[\mu(\square_m, P)]$ in $m$, namely, $\mathbb{E}[\mu(\square_{m+1}, P)] \leq \mathbb{E}[\mu(\square_m, P)]$ for all $m \geq 1$. By the monotone convergence theorem, we may define
\[ \hat{\mu}(P) := \lim_{m \to \infty} \mathbb{E}[\mu(\square_m, P)]. \]

By taking expectation and limit to (4.3), one sees that there is a unique symmetric tensor $\hat{A}$ such that for any $P \in \mathbb{R}^{d \times d}$,
\[ \hat{\mu}(P) = \frac{1}{2} P \cdot \hat{A} P. \]
Moreover, $\Lambda^{-1} I \leq \hat{A} \leq \Lambda I$. We will call $\hat{A}$ the homogenized tensor of the Stokes system with stochastic coefficient tensor $A$. 
The homogenized tensor $\hat{A}$ defined above is sufficient for us to establish the qualitative homogenization for Stokes system. However, it is not enough for quantitative analysis. To this end, for each $Q \in \mathbb{R}^{d \times d}$, we define the dual subadditive quantity $\mu^*(D, Q)$ by

$$\mu^*(D, Q) := \sup_{w \in H^{sol}(D)} \int_D \left( -\frac{1}{2} \nabla w \cdot A \nabla w + Q \nabla w \right),$$ (4.4)

The quantity $\mu^*(D, Q)$ is the energy of its unique maximizer

$$v^*(\cdot, D, Q) := \arg\max_{w \in H^{sol}(D)} \mu^*(D, Q),$$

where $(v^*, \zeta^*)$ turns out to be the Neumann corrector, that is, the weak solution of

$$\begin{cases} \nabla \cdot (A \nabla v^*) + \nabla \zeta^* = 0 & \text{in } D, \\ \nabla \cdot v^* = 0 & \text{in } D, \\ n \cdot A \nabla v^* + n \zeta^* = nQ & \text{on } \partial D. \end{cases}$$ (4.5)

Proceeding as [4], we can verify $\mu(D, P) + \mu^*(D, Q) \geq P \cdot Q$ and then define a crucial quantity

$$J(D, P, Q) = \mu(D, P) + \mu^*(D, Q) - P \cdot Q,$$ (4.6)

which allows to carry out quantitative analysis of correctors. To study the basic properties of $J$, we introduce the set of “$A$-Stokes functions” in $D$

$$\mathcal{A}(D) = \{ u \in H^1(D) : \nabla \cdot A \nabla u + \nabla \pi = 0 \text{ in } D \text{ for some } \pi \in L^2(D) \}.$$ Then, we have the following properties for $J(D, P, Q)$.

**Proposition 4.2.** Let $D$ be a bounded Lipschitz domain in $\mathbb{R}^d$ and $P, Q \in \mathbb{R}^{d \times d}$. Then

$$J(D, P, Q) = \sup_{w \in \mathcal{A}(D)} \int_D \left( -\frac{1}{2} \nabla w \cdot A \nabla w - P \cdot A \nabla w + Q \cdot \nabla w \right).$$

Moreover, the maximizer $v(\cdot, D, P, Q)$ is equal to $v^*(D, Q) - v(D, P)$.

**Proposition 4.3.** Let $D$ be a bounded Lipschitz domain in $\mathbb{R}^d$. Then the quantity $J(D, P, Q)$ and its maximizer $v(\cdot, D, P, Q)$ satisfy the following properties:

- Representation as quadratic form. The mapping $(P, Q) \mapsto J(D, P, Q)$ is a quadratic form and there exist $A_D, A_{*D} \in \mathbb{R}^{d^2 \times d^2}$ such that

$$\Lambda^{-1} I \leq A_{*D} \leq A_D \leq \Lambda I,$$

and

$$J(U, P, Q) = \frac{1}{2} P \cdot A_D P + \frac{1}{2} Q \cdot A_{*D}^{-1} Q - P \cdot Q.$$
Moreover, the matrices $A_D$ and $A^*_{D}$ are characterized by the following relationships:

$$A_D P = - \int_D A \nabla v(\cdot, D, P, 0),$$

$$A^{-1}_{*D} Q = \int_D \nabla v(\cdot, D, 0, Q).$$

- **Subadditivity.** Let $\{D_i\}_{i=1}^{N} \subset D$ be bounded Lipschitz domains that forms a partition of $D$, in the sense that $D_i \cap D_j = \emptyset$ if $i \neq j$ and

$$\left| D \setminus \bigcup_{i=1}^{N} D_i \right| = 0.$$

Then, for every $P, Q \in \mathbb{R}^{d \times d}$,

$$J(D, P, Q) \leq \sum_{i=1}^{N} \frac{|D_i|}{|D|} J(D_i, P, Q).$$

- **Quadratic response.** For every $w \in \mathcal{A}(D)$ and $P, Q \in \mathbb{R}^{d \times d}$,

$$\int_D \frac{1}{2} (\nabla w - \nabla v(\cdot, D, P, Q)) \cdot A (\nabla w - \nabla v(\cdot, D, P, Q)) = J(D, P, Q) - \int_D \left( - \frac{1}{2} \nabla w \cdot A \nabla w - P \cdot A \nabla w + Q \cdot \nabla w \right).$$

Propositions 4.2 and 4.3 may be proved by the similar method as [4, Lemmas 2.1 and 2.2]. Then, following the argument there, we are able to establish all the quantitative estimates of the correctors. Most importantly, we can show the following rate of convergence for the Dirichlet correctors.

**Theorem 4.4.** Let $s \in (0, d)$. There exists $C(d, \Lambda) < \infty$ such that, for every $m \in \mathbb{N}$ and $\|P\|_{2} \leq 1$,

$$3^{-m} \|\nabla v(\cdot, \Box_m, P) - P\|_{\bar{H}^{-1}(\Box_m)} + 3^{-m} \|A \nabla v(\cdot, \Box_m, P) - \hat{A} P\|_{\bar{H}^{-1}(\Box_m)} \leq C 3^{-m \sigma(d-s)} + O_1(C 3^{-ms}).$$

Note that the pressure term does not appear explicitly in the variational method of Stokes system and therefore no estimate of the pressure can be derived directly. In Lemma 4.6, we will see that a weaker estimate for the pressure may be reduced to the estimate of the displacement.

Next, we construct the “finite volume corrector” (an approximation of the true corrector) which is crucial in the study of the convergence rates in homogenization. For each $n \in \mathbb{N}$ and $P \in \mathbb{R}^{d \times d}$, define $(\Phi_{n,j}^\beta, \Pi_{n,j}^\beta)$ by

$$\Phi_{n,j}^\beta(x) = v(x, \Box_n, P_j^\beta) - P_j^\beta x, \quad \Pi_{n,j}^\beta(x) = \pi(x, \Box_n, P_j^\beta).$$
where \( P^\beta_j = e^\beta \otimes e_j \). Observe that \( (\Phi^\beta_{n,j}, \Pi^\beta_{n,j}) \) solves
\[
\begin{align*}
\nabla \cdot (A\nabla \Phi^\beta_{n,j}) + \nabla \Pi^\beta_{n,j} &= -\nabla \cdot (AP^\beta_j) \quad \text{in } \square_n, \\
\nabla \cdot \Phi^\beta_{n,j} &= 0 \quad \text{in } \square_n, \\
\Phi^\beta_{n,j} &= 0 \quad \text{on } \partial \square_n.
\end{align*}
\]
(4.7)

The following theorem is a rescaling of Theorem 4.4:

**Theorem 4.5.** Fix \( s \in (0, d) \). There exists \( \sigma = \sigma(d, \Lambda) > 0 \) so that
\[
\varepsilon \| \Phi^\beta_{n,j} \|_{L^2(\varepsilon \square_n)} + \| A(\cdot/\varepsilon)(\nabla \Phi^\beta_{n,j}(\cdot/\varepsilon) + P^\beta_j) - \hat{A}P^\beta_j \|_{H^{-1}(\varepsilon \square_n)} \leq C \varepsilon^{\sigma(d-s)} + C \varepsilon^s \mathcal{O}_1(1),
\]
where \( \varepsilon = 3^{-n} \).

Again, the above theorem does not include the estimate of the pressure \( \Pi_n \). The following lemma reduces the estimate of \( \Pi_n \) to the estimate of \( \Phi_n \).

**Lemma 4.6.** Let \( D \) be a bounded Lipschitz domain. Let \( W \in L^2(D; \mathbb{R}^{d \times d}) \) and \( \pi \in L^2_0(D) \) satisfy \( \nabla \cdot W + \nabla \pi = 0 \) in \( D \). Then, there exists \( \delta = \delta(d, \Lambda, \text{Lip}(D)) > 0 \) so that
\[
\| \pi \|_{H^{-1}(D)} \leq C \| W \|_{H^{-1}(D)} \| W \|_{L^2(D)},
\]
where \( C \) depends only on \( d, \Lambda \) and \( \text{Lip}(D) \).

**Proof.** This can be proved by duality and the Meyers’ estimate. By rescaling, we may assume \( |D| = 1 \). Let \( \phi \in H^1(D) \) with \( \int_D \phi = 0 \) and let \( (u, p) \) solve
\[
\begin{align*}
\Delta u + \nabla p &= 0 \quad \text{in } D, \\
\nabla \cdot u &= \phi \quad \text{in } D, \\
u &= 0 \quad \text{on } \partial D.
\end{align*}
\]
(4.8)

Note that \( \phi \in H^1(D) \) implies \( \phi \in L^{p_1} \) where \( p_1 = 2d/(d-2) \). Since \( D \) is Lipschitz, by the global version of Theorem 3.9, there exists \( p_0 \in (2, p_1] \) so that
\[
\| \nabla u \|_{L^{p_0}(D)} + \| p \|_{L^{p_0}(D)} \leq C \| \phi \|_{L^{p_0}(D)} \leq C \| \phi \|_{H^1(D)}.
\]
(4.9)

On the other hand, the interior \( H^2 \) regularity implies \( u \in H^2_{\text{loc}}(D; \mathbb{R}^d) \). Moreover, if \( D' \subset D \)
\[
\| \nabla^2 u \|_{L^2(D')} \leq \frac{C}{\text{dist}(D', \partial D)} \| \phi \|_{H^1(D)}.
\]
(4.10)

Now, given any \( t \in (0, 1) \), define \( D_t = \{ x \in D : \text{dist}(x, \partial D) \geq t \} \). Let \( \eta_t \in C^\infty_0(D) \) be a cut-off function so that \( \eta_t(x) = 0 \) if \( x \in D \setminus D_t \) and \( \eta_t(x) = 1 \)
if \( x \in D_2 \). Moreover, \( |\nabla \eta_t(x)| \leq Ct^{-1} \). Recall that \( \nabla \cdot W + \nabla \pi = 0 \) in \( D \). Then the integration by parts yields

\[
\int_D \pi \phi = \int_D \pi \nabla \cdot u = - \int_D \nabla \pi \cdot u = \int_D (\nabla \cdot W) \cdot u = - \int_{D_t} W \cdot \eta_t \nabla u - \int_{D_2 \setminus D_t} W \cdot (1 - \eta_t) \nabla u.
\]

The first integral is bounded by

\[
\|W\|_{H^{-1}(D)} \|\eta_t \nabla u\|_{H^1(D_t)} \leq Ct^{-1} \|W\|_{H^{-1}(D)} \|\phi\|_{H^1(D)},
\]

where we have used Theorem 2.2 and (4.10). The second integral is bounded by

\[
\|W\|_{L^2(D)} \|\nabla u\|_{L^2(D_2 \setminus D_t)} \leq C|D \setminus D_2|^{\frac{1}{2} - \frac{1}{p_0}} \|W\|_{L^2(D)} \|\nabla u\|_{L^{p_0}(D)} \leq Ct^{\frac{1}{2} - \frac{1}{p_0}} \|W\|_{L^2(D)} \|\phi\|_{H^1(D)},
\]

where we have used (4.9) in the last inequality. It follows that

\[
\left| \int_D \pi \phi \right| \leq \left( Ct^{-1} \|W\|_{H^{-1}(D)} + Ct^{\frac{1}{2} - \frac{1}{p_0}} \|W\|_{L^2(D)} \right) \|\phi\|_{H^1(D)}. \tag{4.11}
\]

Now choose

\[
t = \left( \frac{\|W\|_{H^{-1}(D)}}{\|W\|_{L^2(D)}} \right)^{\frac{1}{1 + \frac{1}{2} - \frac{1}{p_0}}}, \quad \beta = \frac{\frac{1}{2} - \frac{1}{p_0}}{1 + \frac{1}{2} - \frac{1}{p_0}}.
\]

Then we obtain the desired estimate from (4.11) by duality. \( \square \)

**Theorem 4.7.** Fix \( s \in (0, d) \). There exists \( \sigma > 0, \delta > 0 \) so that

\[
\|\Pi_{n,j}^\beta (\cdot / \varepsilon)\|_{H^{-1}(\varepsilon \square_n)} \leq C \varepsilon^{\sigma(d-s)} + C(\varepsilon^{s} O_1(1))^{\delta},
\]

where \( \varepsilon = 3^{-n} \).

**Proof.** This follows from Theorem 4.5 and Lemma 4.6. \( \square \)
4.2. Convergence Rates

With Theorems 4.5 and 4.7, we are able to prove the an algebraic rate of convergence for the Stokes system in Lipschitz domains.

**Theorem 4.8.** Let \( \delta > 0, \ s \in (0, d), \) and \( D \) be a bounded Lipschitz domain. There exist exponents \( \alpha, \beta > 0 \) (depending only on \( \delta, s, d, \Lambda \) and \( \text{Lip}(D) \)) and a random variable \( X = X_\epsilon : \Omega \to [0, \infty) \) satisfying

\[
X \leq O_1(C),
\]

such that for every \( \epsilon \in (0, 1] \) and the solution pairs \((u^\epsilon, p^\epsilon), (u^0, p^0)\) of the Stokes systems

\[
\begin{align*}
\nabla \cdot (A^\epsilon \nabla u^\epsilon) + \nabla p^\epsilon &= 0 \quad \text{in } D, \\
\nabla \cdot u^\epsilon &= (f)_D \quad \text{in } D, \\
u^\epsilon &= f \quad \text{on } \partial D,
\end{align*}
\]

and

\[
\begin{align*}
\nabla \cdot (\tilde{A} \nabla u^0) + \nabla p^0 &= 0 \quad \text{in } D, \\
\nabla \cdot u^0 &= (f)_D \quad \text{in } D, \\
u^0 &= f \quad \text{on } \partial D,
\end{align*}
\]

with \( u^0 \in W^{1, 2 + \delta}(D; \mathbb{R}^d) \), we have

\[
\|u^\epsilon - u^0\|_{L^2(D)} + \|\nabla u^\epsilon - \nabla u^0\|_{H^{-1}(D)} + \|A^\epsilon \nabla u^\epsilon - \tilde{A} \nabla u^0\|_{H^{-1}(D)} + \|p^\epsilon - p^0\|_{H^{-1}(D)} \leq C \left( \epsilon^{\delta(d-s)} + (\epsilon^s X)^{\alpha} \right) \|\nabla u^0\|_{L^{2+\delta}(D)}.
\]

The proof of Theorem 4.8 uses the same idea from [4, Theorem 1.17], as well as some useful computation for Stokes system from [19]. We omit the detailed proof.

5. Homogenization of Elasticity System

In this section, we will establish the convergence rate for the linear system of nearly incompressible elasticity.

5.1. Reduction to Constant \( \lambda \)

Let the probability measure space \((\Omega, \mathcal{F}, \mathbb{P})\) satisfy (1.13), (1.14) and (1.15). Let \( u^\epsilon_\lambda \) be the weak solution of

\[
\begin{align*}
\nabla \cdot (A^\epsilon \nabla u^\epsilon_\lambda) + \nabla (\lambda^\epsilon \nabla \cdot u^\epsilon_\lambda) &= 0 \quad \text{in } D, \\
u^\epsilon_\lambda &= f \quad \text{on } \partial D.
\end{align*}
\]

In order to reduce the system to the case with constant \( \lambda \), use the splitting technique in (1.6) and (1.7), and write (1.12) as

\[
\begin{align*}
\nabla \cdot (\tilde{A}^\epsilon \nabla u^\epsilon_\lambda) + \lambda_0 \nabla (\nabla \cdot u^\epsilon_\lambda) &= 0 \quad \text{in } D, \\
u^\epsilon_\lambda &= f \quad \text{on } \partial D,
\end{align*}
\] (5.1)
where $\lambda(x) = \lambda_0 + b(x)$ and $\tilde{A} = (\tilde{a}_{ij}^{\alpha \beta})$ is defined by

$$\tilde{a}_{ij}^{\alpha \beta}(x) = a_{ij}^{\alpha \beta}(x) + b(x)\delta_i^\alpha \delta_j^\beta.$$  \hspace{1cm} (5.2)

Denote the map $((A, \lambda) \mapsto (\tilde{A}, \lambda_0))$ by $S$, that is, $(\tilde{A}, \lambda_0) = S(A, \lambda)$. Now, note that in (5.1), $\lambda_0$ is a fixed constant and $\tilde{A}$ is a new (tensor-valued) random variable on $\Omega$ satisfying the similar hypotheses as $A$. Define a new probability measure space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ endowed by $(\Omega, \mathcal{F}, \mathbb{P})$ via the map $S$, where

$$(\tilde{\Omega} = \{(\tilde{A}, \lambda_0) = S(A, \lambda) : (A, \lambda) \in \Omega\},$$

$$(\tilde{\mathcal{F}} = S(\mathcal{F})$$

and $\tilde{\mathbb{P}}[\omega] = \mathbb{P}[S^{-1}(\omega)]$ for every $\omega \in \tilde{\mathcal{F}}$.

The following proposition shows that the probability measure space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ satisfies the same conditions as in Section 4.

**Proposition 5.1.** The probability measure space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ satisfies the following conditions:

- **Stationarity with respect to $\mathbb{Z}^d$-translations:**
  $$\tilde{\mathbb{P}} \circ T_z = \tilde{\mathbb{P}},$$
  where $(T_z(\tilde{A}, \lambda_0))(x) = (\tilde{A}(x + z), \lambda_0)$.

- **Unit range of dependence:**
  $\tilde{\mathcal{F}}_D$ and $\tilde{\mathcal{F}}_E$ are $\mathbb{P}$-independent for every Borel subset pair $D, E \subset \mathbb{R}^d$ satisfying $\text{dist}(D, E) \geq 1$.

Because of the above reduction, without loss of generality, it is sufficient to consider the case with constant $\lambda$. In other words, we consider

$$\left\{ \begin{array}{ll}
\nabla \cdot (C(x) \nabla u^x_\lambda) + \lambda \nabla \cdot (\nabla \cdot u^x_\lambda) = 0 & \text{in } D, \\
u^x_\lambda = f & \text{on } \partial D,
\end{array} \right.$$  \hspace{1cm} (5.3)

where $\lambda \geq 0$ is constant and (1.13)–(1.15) are satisfied.

### 5.2. Small $\lambda$ Case

For $\lambda$ relatively small (compared to $\varepsilon^{-\sigma}$ for some $\sigma > 0$ determined in Remark 5.4), the system of elasticity may be viewed as the elliptic system. In this case, the homogenization theory may be established by the standard method developed in [2,4,5,7]. One of the goals in this subsection is to identify the structure of the homogenized operator depending on $\lambda$, which is not obvious.

Let $C_\lambda = (c_{\lambda,i,j}^{\alpha \beta})$ be defined by

$$c_{\lambda,i,j}^{\alpha \beta}(x) = a_{ij}^{\alpha \beta}(x) + \lambda \delta_i^\alpha \delta_j^\beta.$$  \hspace{1cm} (5.4)

Thus, the system (5.3) may be written as

$$\left\{ \begin{array}{ll}
\nabla \cdot (C(x) \nabla u^x_\lambda) = 0 & \text{in } D, \\
u^x_\lambda = f & \text{on } \partial D.
\end{array} \right.$$
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Clearly, the random variable $C_\lambda$ also satisfies the stationarity and the unit-range dependence assumptions. However, in this case, the ellipticity constant of $C_\lambda$ depends on $\lambda$, namely,

$$\Lambda^{-1} |\xi|^2 \leq c_{\lambda,ij}^\alpha (x) \xi^\alpha \xi^\beta_j \leq (2\Lambda + \lambda) |\xi|^2 \text{ for any } x \in \mathbb{R}^d, \quad \xi \in \mathbb{R}^{d \times d}. \quad (5.5)$$

In this case, we need to figure out how the homogenized operator and the convergence rate depend on the parameter $\lambda$. A sin[4], we define

$$\mu_\lambda (D, P) := \inf_{v \in \ell + H^1_0(D; \mathbb{R}^d)} \int_D \frac{1}{2} \nabla v \cdot C_\lambda \nabla v,$$

$$\mu_\lambda (D, P) = \inf_{v \in \ell + H^1_0(D; \mathbb{R}^d)} \int_D \left( \frac{1}{2} \nabla v \cdot A \nabla v + \frac{1}{2} \lambda (\nabla \cdot v)^2 \right), \quad (5.6)$$

where $\ell (P) := Px$ is an affine function. Recall that the unique minimizer $v_\lambda = v_\lambda (\cdot, D, P)$ (also called the Dirichlet corrector) is the solution of

$$\begin{cases}
\nabla \cdot (C_\lambda \nabla v_\lambda) = \nabla \cdot (A \nabla v_\lambda) + \lambda \nabla (\nabla \cdot v_\lambda) = 0 \text{ in } D, \\
v_\lambda = \ell (P) \text{ on } \partial D.
\end{cases} \quad (5.7)$$

**Proposition 5.2.** The minimum energy $\mu_\lambda (D, P)$ satisfies the following properties:

(i) **Representation as quadratic form:** there exists a symmetric $A_\lambda (D)$ such that

$$\Lambda^{-1} I \leq A_\lambda (D) \leq \Lambda I,$$

and for each $P \in \mathbb{R}^{d \times d}$

$$\mu_\lambda (D, P) - \frac{1}{2} \lambda \text{Tr}(P)^2 = \frac{1}{2} P \cdot A_\lambda (D) P.$$

(ii) **Subadditivity:** Let $\{D_i\}_{i=1}^N \subset D$ be bounded Lipschitz domains that forms a partition of $D$, in the sense that $D_i \cap D_j = \emptyset$ if $i \neq j$ and

$$\left| D \setminus \bigcup_{i=1}^N D_i \right| = 0.$$

Then, for every $P \in \mathbb{R}^{d \times d}$,

$$\mu_\lambda (D, P) \leq \sum_{i=1}^N \frac{|D_i|}{|D|} \mu_\lambda (D_i, P).$$

**Proof.** Part (ii) is the same as in [4] for elliptic equations. It suffices to prove Part (i). As in [4], we know in priori that $\mu_\lambda (D, P)$ is a symmetric quadratic form of $P$. For $v \in \ell + H^1_0(D; \mathbb{R}^d)$, using the divergence theorem, we have

$$\int_D \frac{1}{2} \lambda (\nabla \cdot v)^2 = \int_D \frac{1}{2} \lambda (\nabla \cdot v - \text{Tr}(P))^2 + \frac{1}{2} \lambda \text{Tr}(P)^2.$$
It follows that
\[
\mu_{\lambda}(D, P) - \frac{1}{2} \lambda \text{Tr}(P)^2 = \inf_{v \in \ell_p + H^1_0(D; \mathbb{R}^d)} \int_D \left( \frac{1}{2} \nabla v \cdot A \nabla v + \frac{1}{2} \lambda (\nabla \cdot v - \text{Tr}(P))^2 \right).
\]  
(5.8)

Now, by choosing \(v = \ell_P\), we have \(\nabla \cdot v = \text{Tr}(P)\) and therefore
\[
\mu_{\lambda}(D, P) - \frac{1}{2} \lambda \text{Tr}(P)^2 \leq \frac{1}{2} \int_D P \cdot A P \leq \frac{1}{2} \Lambda |P|^2.
\]  
(5.9)

On the other hand, let \(v = \ell_P + w\) with \(w \in H^1_0(D; \mathbb{R}^d)\). Then, the Hölder’s inequality and the divergence theorem imply
\[
\mu_{\lambda}(D, P) - \frac{1}{2} \lambda \text{Tr}(P)^2 \geq \inf_{w \in H^1_0(D; \mathbb{R}^d)} \int_D \frac{1}{2} \nabla (P + \nabla w) \cdot A (P + \nabla w)
\]
\[
\geq \frac{1}{2\Lambda} \inf_{w \in H^1_0(D; \mathbb{R}^d)} \int_D |P + \nabla w|^2
\]
\[
\geq \frac{1}{2\Lambda} \inf_{w \in H^1_0(D; \mathbb{R}^d)} \left[ \int_D (P + \nabla w)^2 \right]
\]
\[
= \frac{1}{2\Lambda} |P|^2.
\]  
(5.10)

Since we already know \(\mu_{\lambda}(D, P) - \frac{1}{2} \lambda \text{Tr}(P)^2\) is a symmetric quadratic form, (5.9) and (5.10) leads to the desired representation and estimate.

Let \(\box_m\) be the triadic cube defined as in Section 4. Then the subadditivity property implies that the limit of \(\mathbb{E}[\mu_{\lambda}(\box_m, P)]\) exists as \(m \to \infty\). Denote this limit by \(\overline{\mu}_{\lambda}(P)\). It turns out that the limit
\[
\overline{\mu}_{\lambda}(P) - \frac{1}{2} \lambda \text{Tr}(P)^2 = \lim_{m \to \infty} \mathbb{E} \left[ \frac{1}{2} P \cdot A_{\lambda}(\box_m) P \right]
\]
exists and is a quadratic form. Hence, there exists a constant \(\overline{A}_{\lambda}\) satisfying
\[
\Lambda^{-1} I \leq \overline{A}_{\lambda} \leq \Lambda I,
\]  
(5.11)

so that
\[
\overline{\mu}_{\lambda}(P) = \frac{1}{2} P \cdot \overline{A}_{\lambda} P + \frac{1}{2} \lambda \text{Tr}(P)^2.
\]

This particularly indicates that the homogenized operator of \(\nabla \cdot A^\epsilon \nabla + \lambda \nabla (\nabla \cdot)\) takes a form of \(\nabla \cdot \overline{A}_{\lambda} \nabla + \lambda \nabla (\nabla \cdot)\). In fact, we have the following rate of convergence.
Theorem 5.3. Let \( \delta > 0 \) and \( D \) be a bounded Lipschitz domain. Let \( s \in (0, d) \) and \( \lambda \in [0, \infty) \). There exists a random variable \( \mathcal{X} = \mathcal{X}_{s, \lambda} \) and constants \( \alpha, \beta, C_0 > 0 \) (independent of \( \varepsilon \) and \( \lambda \)) satisfying
\[
\mathcal{X} \leq \mathcal{O}_1(C_0),
\]
(5.12)
such that if \( u_\lambda^{\varepsilon} \) and \( u_\lambda^{0} \) are the weak solutions of
\[
\begin{align*}
\nabla \cdot (A^{\varepsilon} \nabla u_\lambda^{\varepsilon}) + \lambda \nabla \cdot u_\lambda^{\varepsilon} &= 0 & \text{in } D, \\
u_\lambda^{\varepsilon} &= f & \text{on } \partial D,
\end{align*}
\]
(5.13)
and
\[
\begin{align*}
\nabla \cdot (\overline{A}_\lambda \nabla u_\lambda^{0}) + \lambda \nabla \cdot u_\lambda^{0} &= 0 & \text{in } D, \\
u_\lambda^{0} &= f & \text{on } \partial D,
\end{align*}
\]
(5.14)
respectively, with \( f \in W^{1,2+\delta}(D; \mathbb{R}^d) \), then
\[
\|u_\lambda^{\varepsilon} - u_\lambda^{0}\|_{L^2(D)} + \|
abla u_\lambda^{\varepsilon} - \nabla u_\lambda^{0}\|_{H^{-1}(D)} \\
\leq C(\lambda + 1)(\varepsilon^{\beta(d-s)} + \varepsilon^{s} \mathcal{X}^{\varepsilon})(\|
abla f\|_{L^{2+s}(D)}),
\]
(5.15)

Proof. By viewing (5.13) as an elliptic system with ellipticity constant \( \Lambda + \lambda \), the result may be seen by examining the proof of [4, Theorem 2.18]. \( \square \)

Remark 5.4. Given \( s \in (0, d) \), let \( \sigma = \beta(d-s)/2 \) and \( s' = s + \sigma \in (0, d) \). If \( \lambda \leq \varepsilon^{-\sigma} \), the above theorem particularly implies
\[
\|u_\lambda^{\varepsilon} - u_\lambda^{0}\|_{L^2(D)} + \|
abla u_\lambda^{\varepsilon} - \nabla u_\lambda^{0}\|_{H^{-1}(D)} \\
\leq (\varepsilon^{\beta(d-s')-\sigma} + (\varepsilon \mathcal{X}')^{s'-\sigma})(\|
abla f\|_{L^{2+s}(D)}),
\]
(5.16)
where \( \mathcal{X}' = (\mathcal{X}')^{1/(s'-\sigma)} = \mathcal{X}'^{1/s} \leq \mathcal{O}_s(C'_0) \) (since \( \mathcal{X} \leq \mathcal{O}_1(C_0) \)). Now, without loss of generality, by making \( \beta \) smaller in (5.16), we have
\[
\|u_\lambda^{\varepsilon} - u_\lambda^{0}\|_{L^2(D)} + \|
abla u_\lambda^{\varepsilon} - \nabla u_\lambda^{0}\|_{H^{-1}(D)} \\
\leq (\varepsilon^{\beta(d-s)} + (\varepsilon \mathcal{X}')^{s})\|
abla f\|_{L^{2+s}(D)},
\]
(5.17)
with \( \mathcal{X}' = \mathcal{X}'_{s, \lambda} \leq \mathcal{O}_s(C) \).

For the same reason, if \( \lambda \leq \varepsilon^{-\sigma} \), (5.15) also implies the rate of convergence for the pressure
\[
\|\lambda \nabla \cdot u_\lambda^{\varepsilon} - \lambda \nabla \cdot u_\lambda^{0}\|_{H^{-1}(D)} \leq C(\varepsilon^{\beta(d-s)} + (\varepsilon \mathcal{X}')^{s})\|
abla f\|_{L^{2+s}(D)},
\]
(5.18)
with the same \( \mathcal{X}' \).

We should point out that even though the homogenized tensor \( \overline{A}_\lambda \) and the random variable \( \mathcal{X}_{s, \lambda} \) may vary as \( \lambda \) varies, both of them are fortunately in the classes that are in dependent of \( \lambda \); see (5.11) and (5.12).
5.3. Large $\lambda$ Case

To obtain a rate of convergence for the system of elasticity when $\lambda$ is relatively large, we have to employ the result of Stokes system. Let us state the main result of this subsection.

**Theorem 5.5.** Let the same assumptions of Theorem 5.3 hold. There exist a random variable $X'' = X''_s$ and constants $\alpha, \beta, C_0 > 0$ (independent of $\varepsilon$ and $\lambda$) satisfying

$$X'' \leq O_s(C_0),$$

so that

$$\|u_{\lambda}^\varepsilon - u_{\lambda}^0\|_{L^2(D)} + \|\lambda \nabla \cdot u_{\lambda}^\varepsilon - \lambda \nabla \cdot u_{\lambda}^0\|_{H^{-1}(D)} \leq C(\lambda^{-1} + \varepsilon^{\beta(d-s)} + (X''\varepsilon)^{\alpha s})\|
$$

where $u_{\lambda}^\varepsilon$ and $u_{\lambda}^0$ are the weak solutions of (5.13) and (5.14), respectively.

To prove this theorem, we need the following lemma.

**Lemma 5.6.** For any $\lambda \in (0, \infty)$, we have $|\overline{A}_{\lambda} - \widehat{A}| \leq C_\lambda^{-1}$.

**Proof.** Let $\mu(D, P)$ and $\mu_\lambda(D, P)$ be defined in (4.1) and (5.6), respectively. Let $v_\infty \in \ell_P + H_0^{s\text{sol}}(D; \mathbb{R}^d)$ and $v_{\lambda} \in \ell_P + H_0^1(D; \mathbb{R}^d)$ be the minimizers of (4.1) and (5.6), respectively. Because $H_0^{s\text{sol}}(D; \mathbb{R}^d) \subset H_0^1(D; \mathbb{R}^d)$, we have

$$\mu_\lambda(D, P) \leq \int_D \left( \frac{1}{2} \nabla v_\infty \cdot A \nabla v_\infty + \frac{1}{2} \lambda (\nabla \cdot v_\infty)^2 \right) = \int_D \left( \frac{1}{2} \nabla v_{\lambda} \cdot A \nabla v_{\lambda} \right) + \frac{1}{2} \lambda \text{Tr}(P)^2$$

$$\leq \mu_\lambda(D, P) - \frac{1}{2} \lambda \text{Tr}(P)^2$$

where we have used the fact $\nabla \cdot v_\infty = \text{Tr}(P)$.

On the other hand, by Lemma 3.1 and rescaling,

$$\left( \int_D |\nabla v_\lambda - \nabla v_\infty|^2 \right)^{1/2} \leq C\lambda^{-1}|P|.$$

Consequently,

$$\mu(D, P) = \int_D \frac{1}{2} \nabla v_\infty \cdot A \nabla v_\infty$$

$$= \int_D \frac{1}{2} \nabla v_\lambda \cdot A \nabla v_\lambda + \int_D \frac{1}{2} (\nabla v_\infty - \nabla v_\lambda) \cdot A (\nabla v_\infty + \nabla v_\lambda)$$

$$\leq \mu_\lambda(D, P) - \frac{1}{2} \lambda \text{Tr}(P)^2 + C\lambda^{-1}|P|^2,$$

where we have used (5.8) in the last inequality.
Combining (5.19) and (5.20),

\[ |\mu_\lambda(D, P) - \frac{1}{2} \lambda \text{Tr}(P)^2 - \mu(D, P)| \leq C\lambda^{-1}|P|^2. \]

Now, let \( D = \square_m \). Taking expectations and sending \( m \to \infty \), in view of the definition of \( \overline{\lambda} \) and \( \tilde{\lambda} \), we have

\[ \left| \frac{1}{2} P : \overline{\lambda} P - \frac{1}{2} P \cdot \tilde{\lambda} P \right| \leq C\lambda^{-1}|P|^2. \]

This implies the desired estimate.

\[ \square \]

**Proof of Theorem 5.5.** Let \( v_\varepsilon^\lambda \) be the weak solution of

\[
\begin{align*}
\nabla \cdot (A_\varepsilon \nabla v_\varepsilon^\lambda) + \nabla p_\varepsilon &= 0 \quad \text{in} \ D, \\
\nabla \cdot v_\varepsilon^\lambda &= \langle f \rangle_D \quad \text{in} \ D, \\
v_\varepsilon^\lambda &= f \quad \text{on} \ \partial D, 
\end{align*}
\]

and let \( v^0 \) be the weak solution of

\[
\begin{align*}
\nabla \cdot (\tilde{A} \nabla v^0) + \nabla p^0 &= 0 \quad \text{in} \ D, \\
\nabla \cdot v^0 &= \langle f \rangle_D \quad \text{in} \ D, \\
v^0 &= f \quad \text{on} \ \partial D. 
\end{align*}
\]

It follows from Lemma 3.1 that

\[ \| u_\varepsilon^\lambda - v_\varepsilon^\lambda \|_{H^1(D)} + \| \lambda \nabla \cdot u_\varepsilon^\lambda - \int_D \lambda \nabla \cdot v_\varepsilon^\lambda - p_\varepsilon \|_{L^2(D)} \leq C\lambda^{-1}\| \nabla f \|_{L^2(D)}. \]

On the other hand, Theorem 4.8 implies

\[ \| v_\varepsilon - v^0 \|_{L^2(D)} + \| p_\varepsilon - p^0 \|_{H^{-1}(D)} \leq C(e^{\beta(d-s)} + (e\lambda''')^{\alpha_1})\| \nabla f \|_{L^{2+s}(D)}, \]

for some random variable \( \lambda''' = \lambda''' \leq O_s(C_0) \) with some absolute constant \( C_0 \).

Next, using Lemma 3.1 again, we obtain

\[ \| v_\lambda^0 - v^0 \|_{H^1(D)} + \| \lambda \nabla \cdot v_\lambda^0 - \int_D \lambda \nabla \cdot v_\lambda^0 - p^0 \|_{L^2(D)} \leq C\lambda^{-1}\| \nabla f \|_{L^2(D)}, \]

where \( v_\lambda^0 \) is the weak solution of

\[
\begin{align*}
\nabla \cdot (\tilde{A} \nabla v_\lambda^0) + \lambda \nabla (\nabla \cdot v_\lambda^0) &= 0 \quad \text{in} \ D, \\
v_\lambda^0 &= f \quad \text{on} \ \partial D. 
\end{align*}
\]

Finally, comparing (5.14) and (5.26) and using Lemma 5.6 and the energy estimate, we obtain

\[ \| v_\lambda^0 - u_\lambda^0 \|_{H^1(D)} + \| \lambda \nabla \cdot v_\lambda^0 - \lambda \nabla \cdot u_\lambda^0 \|_{L^2(D)} \leq C\lambda^{-1}\| \nabla f \|_{L^2(D)}, \]

where we also used the fact

\[ \int_D (\lambda \nabla \cdot v_\lambda^0 - \lambda \nabla \cdot u_\lambda^0) = 0. \]

Combining (5.23), (5.24), (5.25) and (5.27), we obtain the announced estimate. \( \square \)
5.4. Global Convergence Rate

Combing Theorems 5.3 and 5.5, we obtain a global convergence rate uniform for any \( \lambda \geq 0 \).

**Theorem 5.7.** Let the same assumptions of Theorem 5.3 hold. There exist a random variable \( \mathcal{X} = \mathcal{X}_{s, \lambda} \) and constants \( \alpha, \beta, C > 0 \) (independent of \( \epsilon \) and \( \lambda \)) satisfying

\[
\mathcal{X} \leq O_s(C),
\]

so that

\[
\|u^\epsilon_{\lambda} - u^0_{\lambda}\|_{L^2(D)} + \|\lambda \nabla \cdot u^\epsilon_{\lambda} - \lambda \nabla \cdot u^0_{\lambda}\|_{H^{-1}(D)} \leq C(\epsilon^{\beta(d-s)} + (\mathcal{X} \epsilon)^{\alpha s}) \|\nabla f\|_{L^{2+s}(D)},
\]

where \( u^\epsilon_{\lambda} \) and \( u^0_{\lambda} \) are the weak solutions of (5.13) and (5.14), respectively.

**Proof.** Let \( \mathcal{X}' = \mathcal{X}'_{s, \lambda} \leq O_s(C_0) \) be the random variable in (5.17) and (5.18). Let \( \mathcal{X}'' = \mathcal{X}''_{s, \lambda} \leq O_s(C_0) \) be the random variable in Theorem 5.5. Choose \( \mathcal{X} = \mathcal{X}_{s, \lambda} = \max\{\mathcal{X}', \mathcal{X}'\} \). Observe that \( \mathcal{X} \leq O_s(C) \) for some absolute constant \( C \). In fact, since \( \mathcal{X}' \leq O_s(C_0) \) and \( \mathcal{X}'' \leq O_s(C_0) \),

\[
\mathbb{E}[\exp\left((\mathcal{X}' / C)^s\right)] \leq \mathbb{E}[\exp\left((\mathcal{X}' / C)^s\right) + \exp\left((\mathcal{X}'' / C)^s\right)] \leq 4,
\]

as desired.

Then, given any fixed \( \lambda \geq 0 \), by considering \( \lambda < \epsilon^{-\sigma} \) and \( \lambda > \epsilon^{-\sigma} \) separately with appropriate \( \sigma \) depending on \( s \in (0, d) \), Theorems 5.3 and 5.5 together implies that there exist \( \alpha, \beta > 0 \) so that

\[
\|u^\epsilon_{\lambda} - u^0_{\lambda}\|_{L^2(D)} + \|\lambda \nabla \cdot u^\epsilon_{\lambda} - \lambda \nabla \cdot u^0_{\lambda}\|_{H^{-1}(D)} \leq C(\epsilon^{\beta(d-s)} + (\mathcal{X} \epsilon)^{\alpha s}) \|\nabla f\|_{L^{2+s}(D)},
\]

where \( C \) is independent of \( \epsilon \) and \( \lambda \). \( \square \)

6. Large-Scale Regularity: Interior Estimates

In this section, we investigate the large-scale interior estimate for the system of elasticity

\[
\nabla \cdot (A^\epsilon \nabla u^\epsilon_{\lambda}) + \lambda \nabla (\nabla \cdot u^\epsilon_{\lambda}) = 0 \quad \text{in} \quad B_2,
\]

where \( B_2 = B_2(0) \) and \( \lambda \geq 0 \) is a constant.
6.1. Excess Quantities and Properties

Let $u^\varepsilon_\lambda$ be a weak solution of (6.1). We define two critical excess quantities. Define

$$
\Phi(t) = \frac{1}{t} \inf_{q \in \mathbb{R}^d} \left( \int_{B_t} |u^\varepsilon_\lambda - q|^2 \right)^{1/2}
$$

$$
+ \frac{1}{t} \|\lambda \nabla \cdot u^\varepsilon_\lambda - \frac{1}{t} \int_{B_t} \lambda \nabla \cdot u^\varepsilon_\lambda \|_{H^{-1}(B_t)} + \sup_{k, \ell \in [1/4, 1]} \left( \int_{B_{kt}} |\lambda \nabla \cdot u^\varepsilon_\lambda - \int_{B_{kt}} \lambda \nabla \cdot u^\varepsilon_\lambda | \right).
$$

For any $v \in H^1(B_t; \mathbb{R}^d)$, define

$$
H(t; v) = \frac{1}{t} \inf_{M \in \mathbb{R}^{d \times d}} \left( \int_{B_t} |v - Mx - q|^2 \right)^{1/2}
$$

$$
+ \frac{1}{t} \|\lambda \nabla \cdot v - \frac{1}{t} \int_{B_t} \lambda \nabla \cdot v \|_{H^{-1}(B_t)} + \sup_{k, \ell \in [1/4, 1]} \left( \int_{B_{kt}} |\lambda \nabla \cdot v - \int_{B_{kt}} \lambda \nabla \cdot v | \right).
$$

For simplicity, if $v = u^\varepsilon_\lambda$, we also write $H(r) = H(r; u^\varepsilon_\lambda)$. Then it is not hard to see

$$
H(r) \leq \Phi(r) \leq C \left( \int_{B_r} |\nabla u^\varepsilon_\lambda|^2 \right)^{1/2} \leq C \Phi(2r),
$$

(6.2)

where the second inequality follows from the Poincaré inequality and Lemma 2.1, and the third inequality follows from (2.11).

Two useful properties of $H$ and $\Phi$ are given below.

**Lemma 6.1.** There exists a function $h : (0, 2) \mapsto [0, \infty)$ so that for any $r \in (0, 1)$

$$
\begin{cases}
    h(r) \leq C(H(r) + \Phi(r)) \\
    \Phi(r) \leq H(r) + h(r) \\
    \sup_{r \leq s, t \leq 2r} |h(s) - h(t)| \leq CH(2r).
\end{cases}
$$

**Proof.** Let $M_r$ be the matrix in $\mathbb{R}^{d \times d}$ that minimizes $H(r)$, namely,

$$
H(r) = \frac{1}{r} \inf_{q \in \mathbb{R}^d} \left( \int_{B_r} |u^\varepsilon_\lambda - M_r x - q|^2 \right)^{1/2}
$$

$$
+ \frac{1}{r} \|\lambda \nabla \cdot u^\varepsilon_\lambda - \frac{1}{r} \int_{B_r} \lambda \nabla \cdot u^\varepsilon_\lambda \|_{H^{-1}(B_r)} + \sup_{k, \ell \in [1/4, 1]} \left( \int_{B_{kr}} |\lambda \nabla \cdot u^\varepsilon_\lambda - \int_{B_{kr}} \lambda \nabla \cdot u^\varepsilon_\lambda | \right).
$$

Define $h(r) = |M_r|$. In view of the definition, it is obvious that $\Phi(r) \leq H(r) + h(r)$. Also,

$$
h(r) \leq \frac{C}{r} \inf_{q \in \mathbb{R}^d} \left( \int_{B_r} |M_r x + q|^2 \right)^{1/2}
$$

$$
\leq \frac{C}{r} \inf_{q \in \mathbb{R}^d} \left( \int_{B_r} |u^\varepsilon_\lambda - M_r x + q|^2 \right)^{1/2} + \frac{1}{r} \inf_{q \in \mathbb{R}^d} \left( \int_{B_r} |u^\varepsilon_\lambda - q|^2 \right)^{1/2}
$$

$$
\leq C(H(r) + \Phi(r)).
$$
Finally, if \( r \leq s, t \leq 2r \),
\[
|h(s) - h(t)| \leq |M_s - M_t|
\]
\[
\leq \frac{C}{r} \inf_{q \in \mathbb{R}^d} \left( \int_{B_r} |(M_s - M_t)x + q|^2 \right)^{1/2}
\]
\[
\leq \frac{1}{r} \inf_{q \in \mathbb{R}^d} \left( \int_{B_r} |u^\varepsilon_{\lambda} - M_s x - q|^2 \right)^{1/2}
\]
\[
+ \frac{1}{r} \inf_{q \in \mathbb{R}^d} \left( \int_{B_r} |u^\varepsilon_{\lambda} - M_t x - q|^2 \right)^{1/2}
\]
\[
\leq \frac{C}{s} \inf_{M \in \mathbb{R}^d \times \mathbb{R}^d} \left( \int_{B_s} |u^\varepsilon_{\lambda} - M x - q|^2 \right)^{1/2}
\]
\[
+ \frac{C}{r} \inf_{M \in \mathbb{R}^d \times \mathbb{R}^d} \left( \int_{B_r} |u^\varepsilon_{\lambda} - M x - q|^2 \right)^{1/2}
\]
\[
\leq \frac{C}{2r} \inf_{M \in \mathbb{R}^d \times \mathbb{R}^d} \left( \int_{B_{2r}} |u^\varepsilon_{\lambda} - M x - q|^2 \right)^{1/2} \leq CH(2r).
\]

The proof is complete. \( \square \)

**Lemma 6.2.** Suppose \( u^\varepsilon_{\lambda} \) is a weak solution of (6.1). There exists a constant \( C \) so that for any \( r \in (0, 1) \), we have
\[
\sup_{s \in [r, 2r]} \Phi(s) \leq C \Phi(2r).
\]

**Proof.** Assume \( s \in [r, 2r] \). We estimate the three parts of \( H(s) \) separately. First, it is obvious that
\[
\frac{1}{s} \inf_{q \in \mathbb{R}^d} \left( \int_{B_s} |u^\varepsilon_{\lambda} - q|^2 \right)^{1/2} \leq \frac{C}{2r} \inf_{q \in \mathbb{R}^d} \left( \int_{B_{2r}} |u^\varepsilon_{\lambda} - q|^2 \right)^{1/2} \leq C \Phi(2r).
\]

Second, using the fact that \( H^1_0(B_s) \subset H^1_0(B_{2r}) \) and (2.8), we have
\[
\frac{1}{s} \| \nabla \cdot u^\varepsilon_{\lambda} \|_{H^{-1}(B_s)} = \frac{1}{s} \left( \int_{B_s} \| \nabla \cdot u^\varepsilon_{\lambda} \|_{H^{-1}(B_s)}^2 \right)^{1/2}
\]
\[
\leq \frac{1}{s} \left( \int_{B_{2r}} \| \nabla \cdot u^\varepsilon_{\lambda} \|_{H^{-1}(B_{2r})}^2 \right)^{1/2} \leq \frac{C}{s} \sum_{k, \ell \in [1/4, 1]} \sup_{k, \ell \in [1/4, 1]} \left| \int_{B_{k\ell}} \nabla \cdot u^\varepsilon_{\lambda} - \int_{B_{k\ell}} \nabla \cdot u^\varepsilon_{\lambda} \right|
\]
\[
\leq C \Phi(2r).
\]

Finally, it suffices to estimate
\[
\sup_{k, \ell \in [1/4, 1]} \left| \int_{B_{k\ell}} \nabla \cdot u^\varepsilon_{\lambda} - \int_{B_{k\ell}} \nabla \cdot u^\varepsilon_{\lambda} \right| \leq 2 \sup_{k \in [1/4, 1]} \left| \int_{B_{k\ell}} \nabla \cdot u^\varepsilon_{\lambda} - \int_{B_{k\ell}} \nabla \cdot u^\varepsilon_{\lambda} \right|.
\]
If \( r \leq ks \leq 2r \), then by definition of \( \Phi(2r) \), the above inequality is bounded by \( 2\Phi(2r) \). If \( r/4 < ks < r \), we have

\[
\left| \int_{B_{ks}} \lambda \nabla \cdot u^g_\lambda - \int_{B_r} \lambda \nabla \cdot u^g_\lambda \right| \leq C \left( \int_{B_r} |\lambda \nabla \cdot u^g_\lambda - \int_{B_r} \lambda \nabla \cdot u^g_\lambda|^2 \right)^{1/2}
\]
\[
\leq C \left( \int_{B_r} |\nabla u^g_\lambda|^2 \right)^{1/2}
\]
\[
\leq C \Phi(2r),
\]

where we have used Lemma 2.1 and (2.11). The desired estimate then follows readily.

\[ \square \]

6.2. Excess Decay Estimates

This is done via a sequence of lemmas.

**Lemma 6.3.** For each \( s \in (0, d) \), there exists a random variable \( \mathcal{X} = \mathcal{X}_{s, \lambda} : \Omega \to [1, \infty) \) and a constant \( C > 0 \) satisfying

\[ \mathcal{X} \leq \mathcal{O}_s(C), \]

such that for each \( r \in (\varepsilon, 1) \), we have

\[
\left( \int_{B_r} |u^g_\lambda - u^{0,r}_\lambda|^2 \right)^{1/2} + \|\lambda \nabla \cdot u^g_\lambda - \lambda \nabla \cdot u^{0,r}_\lambda\|_{H^{-1}(B_r)} \leq C r \eta(\varepsilon X/r) \Phi(4r), \quad (6.3)
\]

where

\[ \eta(\rho) = \rho^{\frac{1}{3}} \min\{\alpha s, \beta(d-s)\}, \quad (6.4) \]

and \( u^{0,r}_\lambda \) is the weak solution of

\[
\begin{aligned}
\nabla \cdot (\Omega_\lambda \nabla u^{0,r}_\lambda) + \lambda \nabla \cdot u^{0,r}_\lambda &= 0 & \quad & \text{in } B_r, \\
u^{0,r}_\lambda &= u^g_\lambda & \quad & \text{on } \partial B_r.
\end{aligned} \quad (6.5)
\]

**Proof.** By rescaling, it suffices to prove the result for \( r = 1 \). By the Meyers’ estimate (Theorem 3.10), we know that \( u^g_\lambda \in W^{1,2+\delta}_0(B_1; \mathbb{R}^d) \) for some \( \delta > 0 \) and

\[ \|\nabla u^g_\lambda\|_{L^{2+\delta}(B_1)} \leq C \|\nabla u^g_\lambda\|_{L^2(B_2)}. \]

Then the desired estimate follows immediately from Theorem 5.7 and (2.11). \[ \square \]

**Lemma 6.4.** Let \( u^{0,r}_\lambda \) be a solution of (6.5). There exists some \( \theta \in (0, \frac{1}{4}) \), depending only on \( \Lambda \) and \( d \), so that

\[ H(\theta r; u^{0,r}_\lambda) \leq \frac{1}{2} H(r; u^{0,r}_\lambda). \quad (6.6) \]
First of all, for any $M \in \mathbb{R}^{d \times d}$ and $q \in \mathbb{R}^d$, $u^{0,r}_\lambda - Mx - q$ is a weak solution of (6.5) with constant coefficients. By the interior $C^{1,\alpha}$ estimate uniform in $\lambda$ (analogous to Theorem 3.8), for any $\theta \in (0, 1/4)$, we have

$$H(\theta r; u^{0,r}_\lambda) \leq C\theta^\alpha \inf_{M \in \mathbb{R}^{d \times d}} \inf_{q \in \mathbb{R}^d} \left( \frac{1}{2} \int_{B_r/2} |\nabla (u^{0,r}_\lambda - Mx - q)|^2 \right),$$

where we also used the fact (see (2.8))

$$\|\lambda \nabla \cdot v - \int_{B_r} \lambda \nabla \cdot v\|_{H^{-1}(B_r)} \leq Ct \left( \int_{B_r} |\lambda \nabla \cdot v - \int_{B_r} \lambda \nabla \cdot v|^2 \right)^{1/2}.$$

Combined with the generalized Caccioppoli inequality (2.11), we obtain

$$H(\theta r; u^{0,r}_\lambda) \leq C\theta^\alpha H(r; u^{0,r}_\lambda).$$

Finally, the desired estimate follows by choosing a proper $\theta$ so that $C\theta^\alpha \leq 1/2$. □

**Lemma 6.5.** There exists $\theta \in (0, 1/4)$, for each $r \in (\varepsilon \mathcal{X}, 1)$,

$$H(\theta r) \leq \frac{1}{2} H(r) + C\sqrt[3]{\eta(\varepsilon \mathcal{X}^\alpha/r)} \Phi(4r). \quad (6.7)$$

**Proof.** This is a corollary of Lemmas 6.3 and 6.4. Let $u^{0,r}_\lambda$ and $\theta \in (0, 1/4)$ be as in Lemma 6.4. Observe that the triangle inequality for $H$ implies

$$|H(t; f) - H(t; g)| \leq H(t; f - g).$$

Applying this to (6.6), we get

$$H(\theta r; u^\varepsilon_\lambda) \leq H(\theta r; u^0_\lambda) + H(\theta r; u^\varepsilon_\lambda - u^0_\lambda)$$

$$\leq \frac{1}{2} H(r; u^0_\lambda) + H(\theta r; u^\varepsilon_\lambda - u^0_\lambda)$$

$$\leq \frac{1}{2} H(r; u^\varepsilon_\lambda) + H(\theta r; u^\varepsilon_\lambda - u^0_\lambda) + \frac{1}{2} H(r; u^\varepsilon_\lambda - u^0_\lambda).$$

Then it suffices to estimate $H(\theta r; u^\varepsilon_\lambda - u^0_\lambda)$ with $\theta \in (0, 1]$ by Lemma 6.3. Let us consider the three terms of the function $H(\theta r; u^\varepsilon_\lambda - u^0_\lambda)$ separately.

First, it is clear from (6.3) that

$$\frac{1}{\theta r} \inf_{M \in \mathbb{R}^{d \times d}} \left( \frac{1}{2} \int_{B_{\theta r}} |u^\varepsilon_\lambda - u^0_\lambda| - Mx - q|^2 \right)^{1/2} \leq C\theta \left( \frac{1}{\theta r} \int_{B_r} |u^\varepsilon_\lambda - u^0_\lambda|^2 \right)^{1/2} \leq C\eta(\varepsilon \mathcal{X}^\alpha/r) \Phi(4r). \quad (6.8)$$

Next, since $H^1_0(B_{\theta r})$ is continuously embedded into $H^1_0(B_r)$, we trivially have

$$\frac{1}{\theta r} \|\lambda \nabla \cdot u^\varepsilon_\lambda - \lambda \nabla \cdot u^0_\lambda\|_{H^{-1}(B_{\theta r})} \leq C\theta \frac{1}{\theta r} \|\lambda \nabla \cdot u^\varepsilon_\lambda - \lambda \nabla \cdot u^0_\lambda\|_{H^{-1}(B_r)} \leq C\eta(\varepsilon \mathcal{X}^\alpha/r) \Phi(4r). \quad (6.9)$$
Finally, it suffices to estimate
\[
\int_{B_{k\theta r}} (\lambda \nabla \cdot u^\varepsilon - \lambda \nabla \cdot u^{0,r}_\lambda)
\] (6.10)
for any \( k \in (1/4, 1) \). This follows from the following lemma:

**Lemma 6.6.** Let \( F \in L^2(B_r) \). Then for any \( \theta \in (0, 1) \),
\[
\left\| \int_{B_{\theta r}} F \right\|^3 \leq \frac{C_{\theta, k}}{r} \| F \|_{H^{-1}(B_r)} \int_{B_r} |F|^2.
\]

We postpone the proof of Lemma 6.6 and apply it to estimate (6.10) to arrive at
\[
\left\| \int_{B_{k\theta r}} (\lambda \nabla \cdot u^\varepsilon - \lambda \nabla \cdot u^{0,r}_\lambda) \right\| \leq C_{\theta, k} \frac{3}{4} \eta(\varepsilon \chi/r) \Phi(4r)
\] (6.11)

Now, let \( t \in [3/2, 2] \) be given as in Lemma 6.3. From the boundary condition of (6.5)
\[
\int_{B_{tr}} (\lambda \nabla \cdot u^\varepsilon - \lambda \nabla \cdot u^{0,r}_\lambda) = 0.
\]

Consequently, by the triangle inequality and Lemma 2.1, one has
\[
\left( \int_{B_{tr}} |\lambda \nabla \cdot u^\varepsilon - \lambda \nabla \cdot u^{0,r}_\lambda|^2 \right)^{1/2}
\leq C \left( \int_{B_{tr}} |\lambda \nabla \cdot u^\varepsilon - \lambda \nabla \cdot u^{0,r}_\lambda|^2 \right)^{1/2}
\leq C \left( \int_{B_{tr}} |\lambda \nabla \cdot u^\varepsilon - \int_{B_{tr}} \lambda \nabla \cdot u^{0,r}_\lambda|^2 \right)^{1/2}
\leq C \left( \int_{B_{tr}} |\nabla u^\varepsilon|^2 \right)^{1/2} \leq C \Phi(4r).
\]

Inserting this into (6.11), we obtain
\[
\left\| \int_{B_{k\theta r}} (\lambda \nabla \cdot u^\varepsilon - \lambda \nabla \cdot u^{0,r}_\lambda) \right\| \leq C \frac{3}{4} \eta(\varepsilon \chi/r) \Phi(4r).
\] (6.12)

Combining (6.8), (6.9) and (6.12), we obtain the estimate for \( H(\theta r; u^\varepsilon - u^0_\lambda) \) and hence complete the proof.
Proof of Lemma 6.6. By rescaling, it suffices to prove the case \( r = 1 \). Let \( \theta \in (0, 1) \) be fixed and \( t \in (0, \theta) \) to be determined. Suppose \( \phi \in C_0^\infty(B_\theta) \) is a test function so that \( \phi(x) = 1 \) for \( x \in B_{\theta-t} \) and \( |\nabla \phi| \leq C/t \). Then
\[
\begin{align*}
\left| \int_{B_\theta} F \right| &= \left| \int_{B_\theta} F \phi \right| + \left| \int_{B_\theta \setminus B_{\theta-t}} F(1 - \phi) \right| \\
&\leq C \| F \|_{H^{-1}(B_1)} \| \phi \|_{H^1(B_1)} + |B_\theta \setminus B_{\theta-t}|^{1/2} \| F \|_{L^2(B_1)} \\
&\leq Ct^{-1} \| F \|_{H^{-1}(B_1)} + Ct^{1/2} \| F \|_{L^2(B_1)}.
\end{align*}
\]
Choosing
\[
t = \frac{\| F \|_{H^{-1}(B_1)}^{2/3}}{\| F \|_{L^2(B_1)}^{2/3}},
\]
we obtain
\[
\left| \int_{B_\theta} F \right|^3 \leq C \| F \|_{H^{-1}(B_1)} \| F \|_{L^2(B_1)}^2,
\]
as desired.

6.3. Iteration

The following lemma is a generalization of [37, Lemma 8.5]:

**Lemma 6.7.** Suppose \( \eta_i : (0, 1] \rightarrow [0, 1] \), with \( i = 1, 2 \), are increasing continuous functions so that \( \eta_i(0) = 0 \) and
\[
\int_0^1 \frac{\eta_i(r)}{r} \, dr < \infty \quad \text{for} \quad i = 1, 2.
\]  

Let \( H, \Phi, h : (0, 2] \rightarrow [0, \infty) \) be nonnegative functions. Suppose that there exist \( \theta \in (0, 1/4) \) and \( C_0 > 0 \) so that
\[
\begin{align*}
H(\theta r) &\leq \frac{1}{2} H(r) + C_0 \{ \eta_1(\epsilon/r) + \eta_2(r) \} \Phi(4r) \tag{6.14a} \\
H(r) &\leq C_0 \Phi(r) \tag{6.14b} \\
h(r) &\leq C_0 (H(r) + \Phi(r)) \tag{6.14c} \\
\Phi(r) &\leq C_0 (H(r) + h(r)) \tag{6.14d} \\
\sup_{r \leq t \leq 2r} \Phi(t) &\leq C_0 \Phi(2r) \tag{6.14e} \\
\sup_{r \leq s, t \leq 2r} |h(s) - h(t)| &\leq C_0 H(2r) \tag{6.14f}
\end{align*}
\]
for all \( r \in [\epsilon, 1] \). Then there is a constant \( C > 0 \) depending only on \( C_0 \) and \( \eta_i (i = 1, 2) \) such that
\[
\int_\epsilon^2 \frac{H(r)}{r} \, dr \leq C \Phi(2). \tag{6.15}
\]
Proof. We start from an estimate of $h$. The assumption (6.14f) on $h$ implies $h(r) \leq h(2r) + CH(2r)$. Hence, given any $t \in (\varepsilon, 1)$

$$\int_t^1 \frac{h(r)}{r} dr \leq \int_t^1 \frac{h(2r)}{r} dr + C_0 \int_t^1 \frac{H(2r)}{r} dr$$

$$= \int_{2t}^2 \frac{h(r)}{r} dr + C_0 \int_{2t}^2 \frac{H(r)}{r} dr$$

It follows in sequence from (6.14c), (6.14b) and (6.14e) that

$$\int_t^{2t} \frac{h(r)}{r} dr \leq C \Phi(2) + C \int_{2t}^2 \frac{H(r)}{r} dr.$$

Hence, by using (6.14f) again, for every $t \in (\varepsilon, 1)$,

$$h(t) \leq C \Phi(2) + C \int_t^2 \frac{H(r)}{r} dr. \quad (6.16)$$

Let $\alpha > 1$ be a large number and $\beta < 1/2$ be a small number to be determined. Without loss of generality, assume $\varepsilon < \alpha^{-1}\beta$. Integrating (6.14a) over the interval $[\alpha \varepsilon, \beta]$, we have

$$\int_{\alpha \varepsilon}^{\beta} \frac{H(\theta r)}{r} dr \leq \frac{1}{2} \int_{\alpha \varepsilon}^{\beta} \frac{H(r)}{r} dr + C_0 \int_{\alpha \varepsilon}^{\beta} \left\{ \eta_1(\varepsilon/r) + \eta_2(r) \right\} \Phi(4r) \frac{dr}{r}. \quad (6.17)$$

Using the condition (6.14d), we have

$$\int_{\alpha \varepsilon}^{\beta} \left\{ \eta_1(\varepsilon/r) + \eta_2(r) \right\} \Phi(4r) \frac{dr}{r} \leq C_0 \int_{\alpha \varepsilon}^{\beta} \left\{ \eta_1(\varepsilon/r) + \eta_2(r) \right\} \left( H(4r) + h(4r) \right) \frac{dr}{r}.$$

Now, we observe that the monotonicity of $\eta_i$ and (6.16) imply

$$\int_{\alpha \varepsilon}^{\beta} \left\{ \eta_1(\varepsilon/r) + \eta_2(r) \right\} H(4r) \frac{dr}{r} = \int_{4\alpha \varepsilon}^{4\beta} \left\{ \eta_1(4\varepsilon/r) + \eta_2(r/4) \right\} H(r) \frac{dr}{r} \leq \left\{ \eta(\alpha^{-1}) + \eta(\beta) \right\} \int_{4\alpha \varepsilon}^{4\beta} H(r) \frac{dr}{r},$$

and

$$\int_{\alpha \varepsilon}^{\beta} \left\{ \eta_1(\varepsilon/r) + \eta_2(r) \right\} h(4r) \frac{dr}{r} \leq C \Phi(2) \int_{\alpha \varepsilon}^{\beta} \left\{ \eta_1(\varepsilon/r) + \eta_2(r) \right\} \frac{dr}{r} + C \int_{\alpha \varepsilon}^{\beta} \left\{ \eta_1(\varepsilon/r) + \eta_2(r) \right\} \int_r^2 \frac{H(s)}{s} ds dr$$

$$\leq C \Phi(2) + C \left\{ \int_0^{\alpha^{-1}} \frac{\eta_1(r)}{r} dr + \int_0^{\beta} \frac{\eta_2(r)}{r} dr \right\} \int_{\alpha \varepsilon}^{\beta} \frac{H(s)}{s} ds.$$

Combining the last four inequalities, we obtain

$$\int_{\theta \alpha \varepsilon}^{\theta \beta} \frac{H(r)}{r} dr \leq C \Phi(2) + C \left\{ \int_0^{\alpha^{-1}} \frac{\eta_1(r)}{r} dr + \int_0^{\beta} \frac{\eta_2(r)}{r} dr \right\} \int_{\alpha \varepsilon}^{\beta} \frac{H(s)}{s} ds.$$
Next, by choosing $\alpha$ sufficiently large and $\beta$ sufficiently small so that
\[
\frac{1}{2} + C \left\{ \eta_1(\alpha^{-1}) + \eta_2(\beta) + \int_0^{\alpha^{-1}} \frac{\eta_1(r)}{r} \, dr + \int_0^\beta \frac{\eta_2(r)}{r} \, dr \right\} \leq \frac{3}{4},
\]
we have
\[
\int_{\theta \alpha \varepsilon}^{2 \theta \alpha \varepsilon} \frac{H(r)}{r} \, dr \leq C \Phi(2) + 3 \int_{\theta \beta}^{2 \theta \beta} \frac{H(r)}{r} \, dr \leq C \Phi(2), \tag{6.18}
\]
where we also used (6.14b) and (6.14e) in the last inequality. In view of (6.16), this gives
\[
h(r) \leq C \Phi(2), \quad \text{for any} \quad r \in (\theta \alpha \varepsilon, 2). \tag{6.19}
\]
Therefore, for any $t \in (\theta \alpha \varepsilon, 2)$, by (6.14d), (6.18) and (6.19),
\[
\int_t^{2t} \frac{\Phi(r)}{r} \, dr \leq C_0 \int_t^{2t} \frac{H(r)}{r} \, dr + C_0 \int_t^{2t} \frac{h(r)}{r} \, dr \leq C \Phi(2).
\]
In view of (6.14e) again, this implies that
\[
\Phi(r) \leq C \Phi(2), \quad \text{for any} \quad r \in (\theta \alpha \varepsilon, 2). \tag{6.20}
\]
Note that (6.18) and (6.20) almost give the desired estimate (6.15), except for the interval $(\varepsilon, \theta \alpha \varepsilon)$. However, since $\theta \alpha$ is a fixed number depending only on $C_0, \eta_1$ and $\eta_2$, by repeatedly using (6.14e) finitely many times, we recover the estimate (6.20) for $r \in (\varepsilon, \theta \alpha \varepsilon)$. Also, using (6.14b), we recover
\[
\int_{\varepsilon}^{\theta \alpha \varepsilon} \frac{H(r)}{r} \, dr \leq C_0 \int_{\varepsilon}^{\theta \alpha \varepsilon} \frac{\Phi(r)}{r} \, dr \leq C_0(\theta \alpha - 1) \sup_{r \in (\varepsilon, \theta \alpha)} \Phi(r) \leq C \Phi(2).
\]
This completes the proof. \hfill \Box

**Theorem 6.8.** For any $s \in (0, d), \lambda \in [0, \infty)$, there exists a random variable $X = X_{s, \lambda} : \Omega \mapsto [1, \infty)$ satisfying
\[
X \leq O_s(C),
\]
such that if $r \in [\varepsilon X, 1]$, then
\[
\left( \frac{1}{B_r} \int_{B_r} |\nabla u^\xi_r|^2 \right)^{1/2} + \left( \frac{1}{B_r} \int_{B_r} |\lambda \nabla \cdot u^\xi_r - \int_{B_2} \lambda \nabla \cdot u^\xi_\lambda|^2 \right)^{1/2} \leq C \left( \frac{1}{B_2} \int_{B_2} |\nabla u^\xi_\lambda|^2 \right)^{1/2}. \tag{6.21}
\]
Proof. Let $H$ and $\Phi$ be defined as in Section 6.1. Note that the conditions of $H$ and $\Phi$ are verified by Lemmas 6.1, 6.4 and 6.5. Let $\eta_1(r) = \frac{3}{\sqrt{\eta(r)}} = r^\sigma$ with $\sigma > 0$ and $\eta_2 = 0$. Applying Lemma 6.7 with $\varepsilon$ replaced by $\mathcal{X}\varepsilon$, we obtain

$$
\int_0^2 \frac{H(r)}{r} \, dr + \sup_{(\varepsilon \mathcal{X})} \Phi(r) \leq C \Phi(2). \quad (6.22)
$$

This, together with (6.2), implies

$$
\left( \int_{B_r} |\nabla u^\varepsilon_\lambda|^2 \right)^{1/2} \leq C \left( \int_{B_2} |\nabla u^\varepsilon_\lambda|^2 \right)^{1/2}. \quad (6.23)
$$

To estimate the pressure, note that the first part of (6.22) gives

$$
\int_0^2 \sup_{(\varepsilon \mathcal{X})} \left| \int_{B_{2r}} \lambda \nabla \cdot u^\varepsilon_\lambda - \int_{B_{2r}} \lambda \nabla \cdot u^\varepsilon \right| \, dr \leq C \left( \int_{B_2} |\nabla u^\varepsilon_\lambda|^2 \right)^{1/2}.
$$

For a given $r \in (\varepsilon \mathcal{X}, 1)$, there is an integer $N \geq 0$ so that $2^N r \in (1/2, 1]$. Now, for any $0 \leq j \leq N - 1$, note that

$$
\left| \int_{B_{2^{j+1}r}} \lambda \nabla \cdot u^\varepsilon_\lambda - \int_{B_{2^j r}} \lambda \nabla \cdot u^\varepsilon_\lambda \right| 
\leq 2 \int_{2^{j+2r}}^{2^{j+1}r} \sup_{k, \ell \in [1/4, 1]} \left| \int_{B_{kt}} \lambda \nabla \cdot u^\varepsilon_\lambda - \int_{B_{kt}} \lambda \nabla \cdot u^\varepsilon_\lambda \right| \, dr.
$$

It follows that

$$
\left| \int_{B_r} \lambda \nabla \cdot u^\varepsilon_\lambda - \int_{B_{2^N r}} \lambda \nabla \cdot u^\varepsilon_\lambda \right| 
\leq \sum_{j=0}^{N-1} \left| \int_{B_{2^{j+1}r}} \lambda \nabla \cdot u^\varepsilon_\lambda - \int_{B_{2^j r}} \lambda \nabla \cdot u^\varepsilon_\lambda \right|
\leq 2 \int_{2r}^{2^{N+1}r} \sup_{k, \ell \in [1/4, 1]} \left| \int_{B_{kt}} \lambda \nabla \cdot u^\varepsilon_\lambda - \int_{B_{kt}} \lambda \nabla \cdot u^\varepsilon_\lambda \right| \, dr.
$$

$$
\leq C \left( \int_{B_2} |\nabla u^\varepsilon_\lambda|^2 \right)^{1/2}.
$$

On the other hand, since $2^N r \in (1, 2]$, Lemma 2.1 implies

$$
\left| \int_{B_{2^N r}} \lambda \nabla \cdot u^\varepsilon_\lambda - \int_{B_r} \lambda \nabla \cdot u^\varepsilon_\lambda \right| \leq C \left( \int_{B_2} |\nabla u^\varepsilon_\lambda|^2 \right)^{1/2}.
$$

Consequently, for any $r \in (\varepsilon \mathcal{X}, 1)$,

$$
\left| \int_{B_r} \lambda \nabla \cdot u^\varepsilon_\lambda - \int_{B_2} \lambda \nabla \cdot u^\varepsilon_\lambda \right| \leq C \left( \int_{B_2} |\nabla u^\varepsilon_\lambda|^2 \right)^{1/2}. \quad (6.24)
$$

Finally, using (6.24), Lemma 2.1 and (6.23), we obtain

$$
\left( \int_{B_r} |\lambda \nabla \cdot u^\varepsilon_\lambda - \int_{B_2} \lambda \nabla \cdot u^\varepsilon_\lambda|^2 \right)^{1/2} \leq \left( \int_{B_r} |\lambda \nabla \cdot u^\varepsilon_\lambda - \int_{B_2} \lambda \nabla \cdot u^\varepsilon_\lambda|^2 \right)^{1/2} + \int_{B_r} \lambda \nabla \cdot u^\varepsilon_\lambda - \int_{B_r} \lambda \nabla \cdot u^\varepsilon_\lambda
$$
\[
\leq C \left( \int_{B_r} |\nabla u^\varepsilon|^2 \right)^{1/2} + C \left( \int_{B_2} |\nabla u^\lambda|^2 \right)^{1/2}
\]
\[
\leq C \left( \int_{B_2} |\nabla u^\lambda|^2 \right)^{1/2},
\]
as desired. \(\square\)

**Proof of Theorem 1.1.** By (1.6) and (1.7), as well as the reduction in Section 5.1, the system may be reduced to the case with constant \(\lambda = \lambda_0\). Then, by Theorem 6.8, we obtain
\[
\left( \int_{B_r} |\nabla u^\lambda|^2 \right)^{1/2} + \left( \int_{B_2} |\lambda_0 \nabla \cdot u^\varepsilon - \lambda_0 \nabla \cdot u^\lambda|^2 \right)^{1/2} \leq C \left( \int_{B_2} |\nabla u^\lambda|^2 \right)^{1/2}.
\]
Finally, \(\lambda_0\) may be replaced by \(\lambda^\varepsilon\) due to the assumption (1.5) and the estimate of \(|\nabla u^\lambda|\) in the last inequality. This finishes the proof. \(\square\)

# 7. Large-Scale Regularity: Boundary Estimates

In this section, we study the uniform large-scale estimate near the boundary which is of \(\varepsilon\)-scale \(C^{1,\alpha}\) at 0 \(\in \partial D\). Unfortunately, the previous argument for the interior estimate does not apply identically since the boundary below \(\varepsilon\)-scale is only Lipschitz, which means the local \(C^{1,\alpha}\) regularity cannot be expected even for the homogenized system. To overcome this difficulty, we introduce a new idea of boundary perturbation to modify the excess decay iteration method which allows the boundary to be bumpy at microscopic scales.

## 7.1. Boundary Geometry

Let \(\alpha \in (0, 1)\) be fixed and \(D\) a bounded \(\varepsilon\)-scale \(C^{1,\alpha}\) domain and 0 \(\in \partial D\). As usual, define \(D_t = D \cap B_t(0)\) and \(\Delta_t = \partial D \cap B_t(0)\). By Definition 1.2, there exists a constant \(C_0\) so that for any \(t \in (\varepsilon, 1)\), there exists a unit “outer normal” vector \(n_t \in \mathbb{S}^{d-1}\) such that
\[
T_t^- := \{x \in \mathbb{R}^d : x \cdot n_t < -C_0 t \xi(t, \varepsilon)\} \cap B_t(0)
\]
\[
\subset D_t \subset T_t^+ := \{x \in \mathbb{R}^d : x \cdot n_t < C_0 t \xi(t, \varepsilon)\} \cap B_t(0),
\]
where \(\xi(t, \varepsilon) = t^\alpha + (\varepsilon/t)^\alpha\). This particularly implies that both \(T_t^-\) and \(T_t^+\) approximate \(D_t\) well at almost all scales with \(\varepsilon \ll t \ll 1\). Moreover,
\[
|T_t^+ \setminus T_t^-| \lesssim t^d \xi(t, \varepsilon) \simeq \xi(t, \varepsilon)|D_t|.
\]

The outer normal \(n_t\) of the flat boundary of \(T_t^\pm\) will play an important role in our proof. Intuitively, it represents a macroscopically approximate direction perpendicular to the boundary near 0 at \(t\)-scale and coincides with the usual outer normal if the boundary is smooth (that is, \(C^{1,\alpha}\)). The next lemma shows that \(n_t\) changes gently with \(t \in (\varepsilon, 1)\).
Lemma 7.1. Let $\varepsilon \leq s \leq r \leq 1$, then

$$|n_r - n_s| \leq C \frac{r \xi(r, \varepsilon)}{s}.$$ 

This is a simple geometric observation and the proof will be omitted.

7.2. Excess Quantities and Properties

Let $\lambda \geq 0$ and $u^\varepsilon_\lambda \in H^1(D_2; \mathbb{R}^d)$ be the weak solution of

$$\begin{cases}
\nabla \cdot (A^\varepsilon \nabla u^\varepsilon_\lambda) + \nabla (\lambda \nabla \cdot u^\varepsilon_\lambda) = 0 & \text{in } D_2, \\
u^\varepsilon_\lambda = 0 & \text{on } \Delta_2.
\end{cases} \quad (7.1)$$

We redefine $\Phi$ and $H$ in the boundary case as

$$\begin{align*}
\Phi(t) &= \frac{1}{t} \left( \int_{D_t} |u^\varepsilon_\lambda|^2 \right)^{1/2} + \frac{1}{t} \| \lambda \nabla \cdot u^\varepsilon_\lambda - \int_{D_t} \lambda \nabla \cdot u^\varepsilon_\lambda \|_{H^{-1}(D_t)} \\
&\quad + \sup_{k, \ell \in [1/4, 1]} \left| \int_{D_{kt}} \lambda \nabla \cdot u^\varepsilon_\lambda - \int_{D_{\ell t}} \lambda \nabla \cdot u^\varepsilon_\lambda \right|,
\end{align*} \quad (7.2)$$

and for $v \in H^1(D_t; \mathbb{R}^d)$,

$$\begin{align*}
H(t; v) &= \frac{1}{t} \inf_q \left( \int_{D_t} |v - (n_t \cdot x)q|^2 \right)^{1/2} + \frac{1}{t} \| \lambda \nabla \cdot v - \int_{D_t} \lambda \nabla \cdot v \|_{H^{-1}(D_t)} \\
&\quad + \sup_{k, \ell \in [1/4, 1]} \left| \int_{D_{kt}} \lambda \nabla \cdot v - \int_{D_{\ell t}} \lambda \nabla \cdot v \right|.
\end{align*} \quad (7.3)$$

We put $H(t) = H(t; u^\varepsilon_\lambda)$ for short.

To apply Lemma 6.7, as before, we need some basic properties of $\Phi$ and $H$. As in the interior case, we still have

$$H(r) \leq \Phi(r) \quad \text{and} \quad \sup_{r \leq s \leq 2r} \Phi(s) \leq C \Phi(2r). \quad (7.4)$$

Also, we have the following key property which is slightly different from Lemma 6.1:

Lemma 7.2. There exists a function $h : (0, 2) \mapsto [0, \infty)$ so that, for any $r \in (0, 1)$,

$$\begin{cases}
h(r) \leq C (H(r) + \Phi(r)) \\
\Phi(r) \leq H(r) + h(r) \\
\sup_{r \leq s, t \leq 2r} |h(s) - h(t)| \leq C H(2r) + C \xi(r, \varepsilon) \Phi(2r).
\end{cases}$$

Proof. The proofs for the first two inequality are similar to Lemma 6.1. We only prove the third inequality here. Let $q_r$ be the vector that minimizes $H(r)$, namely,

$$\begin{align*}
H(r) &= \frac{1}{r} \left( \int_{D_r} |v - (n_r \cdot x)q_r|^2 \right)^{1/2} + \frac{1}{r} \| \lambda \nabla \cdot v - \int_{D_r} \lambda \nabla \cdot v \|_{H^{-1}(D_r)} \\
&\quad + \sup_{k, \ell \in [1/4, 1]} \left| \int_{D_{kr}} \lambda \nabla \cdot v - \int_{D_{\ell r}} \lambda \nabla \cdot v \right|.
\end{align*} \quad (7.5)$$
Define \( h(r) = |q_r| \). Then \( h(r) \leq C (H(r) + \Phi(r)) \leq C \Phi(r) \). Let \( s, t \in [r, 2r] \), one has

\[
|q_s - q_t| \leq \frac{C}{r} \left( \int_{D_r} |(n_{2r} \cdot x)(q_s - q_t)|^2 \right)^{1/2} \\
\leq \frac{C}{r} \left( \int_{D_r} |u_{\lambda}^e - (n_{2r} \cdot x)q_s|^2 \right)^{1/2} \\
+ \frac{C}{r} \left( \int_{D_r} |u_{\lambda}^e - (n_{2r} \cdot x)q_t|^2 \right)^{1/2}.
\]

(7.6)

We estimate the first term. Using Lemma 7.1 and (7.4), we have

\[
\frac{1}{r} \left( \int_{D_r} |u_{\lambda}^e - (n_{2r} \cdot x)q_s|^2 \right)^{1/2} \\
\leq \frac{C}{r} \left( \int_{D_s} |u_{\lambda}^e - (n_s \cdot x)q_s|^2 \right)^{1/2} + |n_{2r} - n_s||q_s| \\
\leq \frac{C}{s} \inf_{q \in \mathbb{R}^d} \left( \int_{D_s} |u_{\lambda}^e - (n_s \cdot x)q|^2 \right)^{1/2} + C \zeta(2r, \varepsilon) \Phi(2r) \\
\leq \frac{C}{2r} \left( \int_{D_{2r}} |u_{\lambda}^e - (n_{2r} \cdot x)q_{2r}|^2 \right)^{1/2} + |n_{2r} - n_s||q_{2r}| + C \zeta(2r, \varepsilon) \Phi(2r) \\
\leq CH(2r) + C \zeta(r, \varepsilon) \Phi(2r),
\]

where we have used the fact \( \zeta(2r, \varepsilon) \leq C \zeta(r, \varepsilon) \) since \( \zeta(r, \varepsilon) = r^\alpha + (\varepsilon/r)^\alpha \). The estimate for the second term of (7.6) is similar. Hence, for any \( s, t \in [r, 2r] \),

\[
|h(s) - h(t)| \leq |q_s - q_t| \leq CH(2r) + C \zeta(r, \varepsilon) \Phi(2r),
\]

as desired. \( \square \)

### 7.3. Boundary Perturbation

We construct an approximate solution in \( T_i^+ \) at each scale \( t > \varepsilon \) which admits a better regularity due to the smoothness of the boundary. Let \( u_{\lambda}^e \in H^1(D_2; \mathbb{R}^d) \) solve (7.1). We extend the function \( u_{\lambda}^e \) to the entire \( B_2 \) by

\[
\tilde{u}_{\lambda}^e(x) = \begin{cases} 
  u_{\lambda}^e(x) & \text{for } x \in D_2 \\
  0 & \text{for } x \in B_2 \setminus D_2.
\end{cases}
\]

(7.7)

Then \( \tilde{u}_{\lambda}^e \in H^1(B_2; \mathbb{R}^d) \) and \( \|\tilde{u}_{\lambda}^e\|_{H^1(B_2)} = \|u_{\lambda}^e\|_{H^1(D_r)} \) for any \( r \in (0, 2) \). Now, fix \( r \in (\varepsilon, \lambda_s, 1) \), let \( v_{\lambda}^e \) be the weak solution of

\[
\begin{cases} 
  \nabla \cdot (A^e \nabla v_{\lambda}^e) + \lambda \nabla \cdot v_{\lambda}^e = 0 & \text{in } T_{2r}^+, \\
  v_{\lambda}^e = \tilde{u}_{\lambda}^e & \text{on } \partial T_{2r}^+.
\end{cases}
\]

(7.8)

The following is the key lemma:
Lemma 7.3. There exists \( \gamma > 0 \), depending only on \( d \), \( \Lambda \) and the Lipschitz constant of \( \Delta_2 \), so that for any \( \lambda > 0 \),

\[
\left( \int_{T_{2r}^+} |\nabla \tilde{u}_\lambda^e - \nabla v_\lambda^e|^2 \right)^{1/2} + \left( \int_{T_{2r}^+} |\lambda \nabla \cdot \tilde{u}_\lambda^e - \lambda \nabla \cdot v_\lambda^e|^2 \right)^{1/2} \\
\leq C \zeta(r, \varepsilon)^\gamma \left( \int_{D_{2r}} |\nabla u_\lambda^e|^2 \right)^{1/2}.
\]

(7.9)

Proof. First of all, by the system (7.8) and Theorem 2.3, we have

\[
\|\nabla v_\lambda^e\|_{L^2(T_{2r}^+)} + \lambda \|\nabla \cdot v_\lambda^e - \int_{T_{2r}^+} \nabla \cdot \tilde{u}_\lambda^e\|_{L^2(T_{2r}^+)} \\
\leq C \|\nabla \tilde{u}_\lambda^e\|_{L^2(T_{2r}^+)} = C \|\nabla u_\lambda^e\|_{L^2(D_{2r})}.
\]

(7.10)

Let \( \phi_r \) be a cut-off function so that \( \phi_r(x) = 1 \) on \( \{x \cdot n_{2r} \leq -4C_0r \zeta(2r, \varepsilon)\} \) and \( \phi_r(x) = 0 \) on \( \{x \cdot n_{2r} > -2C_0r \zeta(2r, \varepsilon)\} \) and \( |\nabla \phi_r(x)| \leq C[r \zeta(r, \varepsilon)]^{-1} \). Then it is not hard to show

\[
\int_{B_{2r}} A^e \nabla (\tilde{u}_\lambda^e - v_\lambda^e) \cdot \nabla (\tilde{u}_\lambda^e - v_\lambda^e) + \lambda \int_{B_{2r}} |\nabla \cdot (\tilde{u}_\lambda^e - v_\lambda^e)|^2 \\
= \int_{B_{2r}} A^e \nabla (\tilde{u}_\lambda^e - v_\lambda^e) \cdot \nabla \left( (\tilde{u}_\lambda^e - v_\lambda^e) \phi_r \right) + \lambda \int_{B_{2r}} |\nabla \cdot (\tilde{u}_\lambda^e - v_\lambda^e)|^2 + \int_{B_{2r}} A^e \nabla (\tilde{u}_\lambda^e - v_\lambda^e) \cdot \nabla \left( (\tilde{u}_\lambda^e - v_\lambda^e)(1 - \phi_r) \right) \\
+ \lambda \int_{B_{2r}} \nabla \cdot (\tilde{u}_\lambda^e - v_\lambda^e) \cdot \nabla \left( (\tilde{u}_\lambda^e - v_\lambda^e)(1 - \phi_r) \right).
\]

(7.11)

Now observe that \((\tilde{u}_\lambda^e - v_\lambda^e)\phi_r \in H_0^1(T_{2r}^-; \mathbb{R}^d)\). Hence, by the integration by parts, the first two terms in the right-hand side of the last identity vanishes since \(\tilde{u}_\lambda^e - v_\lambda^e\) is a weak solution in \(T_{2r}^-\) which is a subset of \(D_{2r} \subset T_{2r}^+\). So it suffices to estimate the third and fourth terms. We will estimate the fourth term only and the estimate of the third term is similar and easier. Note that \(\tilde{u}_\lambda^e\) and \(v_\lambda^e\) are supported in \(D_{2r}\) and \(T_{2r}^+\), respectively. Thus \(\tilde{u}_\lambda^e - v_\lambda^e \in H_0^1(B_{2r}; \mathbb{R}^d)\) yields

\[
\int_{B_{2r}} \nabla \cdot (\tilde{u}_\lambda^e - v_\lambda^e)(1 - \phi_r) = 0.
\]

Hence, by inserting constants in the fourth term of (7.11) and using the triangle inequality, the Cauchy–Schwarz inequality and (7.10), we obtain

\[
\lambda \int_{B_{2r}} \nabla \cdot (\tilde{u}_\lambda^e - v_\lambda^e) \nabla \cdot (\tilde{u}_\lambda^e - v_\lambda^e)(1 - \phi_r) \\
\leq \lambda \left( \int_{D_{2r}} |\nabla \cdot \tilde{u}_\lambda^e| - \int_{D_{2r}} |\nabla \cdot \tilde{u}_\lambda^e|^2 \right)^{1/2} + \lambda \left( \int_{T_{2r}^+} |\nabla \cdot v_\lambda^e - \int_{T_{2r}^+} |\nabla \cdot v_\lambda^e|^2 \right)^{1/2} \\
\times \left( \int_{T_{2r}^+} |\nabla \left( (\tilde{u}_\lambda^e - v_\lambda^e)(1 - \phi_r) \right)|^2 \right)^{1/2} \\
\leq C \|\nabla u_\lambda^e\|_{L^2(D_{2r})} \times \left( \int_{T_{2r}^+} |\nabla \left( (\tilde{u}_\lambda^e - v_\lambda^e)(1 - \phi_r) \right)|^2 \right)^{1/2}.
\]

(7.12)
Next, by Poincaré’s inequality
\[
\left( \int_{T_{2r}^+} \left| \nabla \left[ \tilde{u}_\lambda^e (1 - \phi_r) \right] \right|^2 \right)^{1/2} \leq \left( \int_{D_{2r} \cap \{ x \cdot n_{2r} \leq -4C_0 r \xi (2r, \varepsilon) \}} \left| \nabla \tilde{u}_\lambda^e \right|^2 \right)^{1/2} \\
+ C \left[ \xi (2r, \varepsilon) \right]^{-1} \left( \int_{D_{2r} \cap \{ x \cdot n_{2r} \leq -4C_0 r \xi (2r, \varepsilon) \}} \left| \tilde{u}_\lambda^e \right|^2 \right)^{1/2} \\
\leq C \left( \int_{D_{2r} \cap \{ x \cdot n_{2r} \leq -4C_0 r \xi (2r, \varepsilon) \}} \left| \nabla \tilde{u}_\lambda^e \right|^2 \right)^{1/2}.
\]

Note that \( D_{2r} \cap \{ x \cdot n_{2r} \geq -4C_0 r \xi (2r, \varepsilon) \} \subset B_{2r} \cap \{ -4C_0 r \xi (2r, \varepsilon) \leq x \cdot n_{2r} < 2C_0 r \xi (2r, \varepsilon) \} \), which implies \( |D_{2r} \cap \{ x \cdot n_{2r} \geq -2C_0 r \xi (2r, \varepsilon) \}| \leq C r^d \xi (r, \varepsilon) \).

It follows from the Hölder’s inequality and the Meyers’ estimate (Theorem 3.10) that
\[
\left( \int_{D_{2r} \cap \{ x \cdot n_{2r} \leq -4C_0 r \xi (2r, \varepsilon) \}} \left| \nabla \tilde{u}_\lambda^e \right|^2 \right)^{1/2} \\
\leq |D_{2r} \cap \{ x \cdot n_{2r} \geq -4C_0 r \xi (2r, \varepsilon) \}| \frac{1}{2} - \frac{1}{p_0} \left( \int_{D_{2r}} \left| \nabla \tilde{u}_\lambda^e \right|^{p_0} \right)^{1/p_0} \\
\leq C \left[ \xi (r, \varepsilon) \right]^{\frac{1}{2} - \frac{1}{p_0}} \left( \int_{D_{2r}} \left| \nabla \tilde{u}_\lambda^e \right|^2 \right)^{1/2}.
\]

Consequently, we arrive at
\[
\left( \int_{T_{2r}^+} \left| \nabla \left[ (\tilde{u}_\lambda^e - v_\lambda^e)(1 - \phi_r) \right] \right|^2 \right)^{1/2} \leq C \left[ \xi (r, \varepsilon) \right]^{\frac{1}{2} - \frac{1}{p_0}} \left( \int_{D_{2r}} \left| \nabla \tilde{u}_\lambda^e \right|^2 \right)^{1/2}.
\]

Substituting this into (7.12), we obtain the estimate of (7.11). In particular,
\[
\left( \int_{T_{2r}^+} \left| \nabla \tilde{u}_\lambda^e - \nabla v_\lambda^e \right|^2 \right)^{1/2} \leq C \left[ \xi (r, \varepsilon) \right]^{\frac{1}{2} - \frac{1}{p_0}} \left( \int_{D_{2r}} \left| \nabla u_\lambda^e \right|^2 \right)^{1/2}.
\]

This implies the first part of (7.9).

To obtain the estimate for pressure, note that it follows easily from (7.13) and Lemma 2.1 that
\[
\left( \int_{D_{2r}} \left| \lambda \nabla \cdot \tilde{u}_\lambda^e - \lambda \nabla \cdot v_\lambda^e \right|^2 \right)^{1/2} \leq C \left[ \xi (r, \varepsilon) \right]^{\frac{1}{2} - \frac{1}{p_0}} \left( \int_{D_{2r}} \left| \nabla u_\lambda^e \right|^2 \right)^{1/2} \\
+ \left( \int_{D_{2r}} \left| \lambda \nabla \cdot \tilde{u}_\lambda^e - \lambda \nabla \cdot v_\lambda^e \right| \right).
\]

By the boundary condition of (7.8) and the divergence theorem,
\[
\int_{T_{2r}^+} (\lambda \nabla \cdot \tilde{u}_\lambda^e - \lambda \nabla \cdot v_\lambda^e) = 0.
\]

Because \( \tilde{u}_\lambda^e = 0 \) in \( T_{2r}^+ \setminus D_{2r} \), the above fact implies that
\[
\left| \int_{D_{2r}} (\lambda \nabla \cdot \tilde{u}_\lambda^e - \lambda \nabla \cdot v_\lambda^e) \right| = \frac{1}{|D_{2r}|} \int_{T_{2r}^+ \setminus D_{2r}} \lambda \nabla \cdot v_\lambda^e.
\]

(7.15)
Next, in $T_{2r}^+ \setminus D_{2r}$, using the Meyers’ estimate for $\lambda \nabla \cdot v^\varepsilon_\lambda$ again and the fact $|T_{2r}^+ \setminus D_{2r}| \leq Cr^d \zeta(r, \varepsilon)$, we have

$$\left( \frac{1}{T_{2r}^+ \setminus D_{2r}} |\lambda \nabla \cdot v^\varepsilon_\lambda|^2 \right)^{1/2} \leq C \zeta(r, \varepsilon)^r \left( \frac{1}{D_{2r}} |\nabla u^\varepsilon_\lambda|^2 \right)^{1/2}. \quad (7.16)$$

Combining estimates (7.14), (7.15), (7.16) and using the fact $\tilde{u}^\varepsilon_\lambda = 0$ in $T_{2r}^+ \setminus D_{2r}$, we obtain

$$\left( \frac{1}{T_{2r}^+} |\lambda \nabla \cdot \tilde{u}^\varepsilon_\lambda - \lambda \nabla \cdot v^\varepsilon_\lambda|^2 \right)^{1/2} \leq C \zeta(r, \varepsilon)^r \left( \frac{1}{D_{2r}} |\nabla u^\varepsilon_\lambda|^2 \right)^{1/2}. \quad (7.17)$$

This proves the second part of (7.9). \qed

**Remark 7.4.** Since $\tilde{u}^\varepsilon_\lambda - v^\varepsilon_\lambda = 0$ on $B_{2r} \setminus T_{2r}^+$, whose volume is comparable to $r^d$, the Poincaré inequality implies

$$\left( \frac{1}{T_{2r}^+} |u^\varepsilon_\lambda - v^\varepsilon_\lambda|^2 \right)^{1/2} \leq Cr\zeta(r, \varepsilon)^r \left( \frac{1}{D_{2r}} |\nabla u^\varepsilon_\lambda|^2 \right)^{1/2}. \quad (7.18)$$

Now we consider the homogenization of the approximate solution $v^\varepsilon_\lambda$. Let $v^0_\lambda$ be the weak solution of the corresponding homogenized system

$$\begin{cases}
\nabla \cdot (A_\lambda \nabla v^0_\lambda) + \lambda \nabla (\nabla \cdot v^0_\lambda) = 0 & \text{in } T_{2r}^+,

v^0_\lambda = v^\varepsilon_\lambda & \text{on } \partial(T_{2r}^+). 
\end{cases} \quad (7.19)$$

Note that Theorem 3.10 implies $\tilde{u}^\varepsilon_\lambda \in W^{1,2+\delta}(T_{2r}^+; \mathbb{R}^d)$ and $v^\varepsilon_\lambda \in W^{1,2+\delta}(T_{2r}^+; \mathbb{R}^d)$. Therefore, Theorem 5.7 implies that there exists a random variable $\lambda' = \lambda'_{\varepsilon, \lambda} \leq O_{\delta}(C)$ with $s \in (0, d)$ such that

$$\left( \frac{1}{T_{2r}^+} |v^\varepsilon_\lambda - v^0_\lambda|^2 \right)^{1/2} + \|\lambda \nabla \cdot v^\varepsilon_\lambda - \lambda \nabla \cdot v^0_\lambda\|_{H^{-1}(T_{2r}^+)} \leq Cr\zeta(r, \lambda') \left( \frac{1}{D_{2r}} |\nabla u^\varepsilon_\lambda|^2 \right)^{1/2}, \quad (7.20)$$

where $\zeta(\rho)$ is the same as (6.4).

Next, we consider the smoothness of the homogenized solution. To this end, we will take a special form adapted to the boundary situation:

$$\tilde{H}(r, \theta; v^0_\lambda) = \frac{1}{\theta r} \inf_{q \in \mathbb{R}^d} \left( \frac{1}{T_{2r}^+ \cap B_{\theta r}} |v^0_\lambda - (n_{\theta r} \cdot x)q|^2 \right)^{1/2}$$

$$+ \frac{1}{\theta r} \|\lambda \nabla \cdot v^0_\lambda - \int_{T_{2r}^+ \cap B_{\theta r}} \lambda \nabla \cdot v^0_\lambda\|_{H^{-1}(T_{2r}^+ \cap B_{\theta r})}$$

$$+ \sup_{k, \epsilon \in [1/4, 1]} \left| \int_{T_{2r}^+ \cap B_{k\theta r}} \lambda \nabla \cdot v^0_\lambda - \int_{T_{2r}^+ \cap B_{\theta r}} \lambda \nabla \cdot v^0_\lambda \right|.$$  

Recall that $n_{2r}$ is roughly the outward normal of the flat boundary of $T_{2r}^+$, which can be viewed as a large-scale normal vector of $\partial D$ at 0. The linear function $(n_{\theta r} \cdot x)q$ is particularly effective in approximating $v^0_\lambda$, since $v^0_\lambda$ vanishes on the flat boundary and hence the tangent derivatives on the flat boundary vanishes.
Lemma 7.5. There exists a constant $\theta \in (0, 1/4)$, depending only on $d$ and $\Lambda$, so that for any $r \in (\varepsilon, 1)$

$$
\tilde{H}(r, \theta; v_\lambda^0) \leq \frac{1}{2} \tilde{H}(r, 2; v_\lambda^0) + C \zeta(r, \varepsilon) \left( \int_{T_3^r} |\nabla v_\lambda^0|^2 \right)^{1/2}.
$$

Proof. First of all, by the $C^{1,\alpha}$ regularity of $v_\lambda^0$ (Theorem 3.8), we know

$$
[\nabla v_\lambda^0] c^\alpha(T_{2r} \cap B_r) + [\lambda \nabla \cdot v_\lambda^0] c^\alpha(T_{2r} \cap B_r) \leq C r^{-\alpha} \left( \int_{T_{2r}^+ \cap B_r} |\nabla v_\lambda^0|^2 \right)^{1/2} \tag{7.20}
$$

Let $x_{2r}$ be the point on $\partial T_{2r}^+ \cap B_r$ (flat boundary) that is the closest point to the origin. By our assumption, $|x_{2r}| \leq C r \zeta(r, \varepsilon)$. Clearly, since $v_\lambda^0$ is identically zero on the flat boundary, $v_\lambda^0(x_{2r}) = 0$ and the tangential derivative vanishes, that is,

$$
(I_{d \times d} - n_{2r} \otimes n_{2r}) \nabla v_\lambda^0(x_{2r}) = 0,
$$

where $I_{d \times d}$ is the $d \times d$ identity matrix. It follows that

$$
\nabla v_\lambda^0(x_{2r}) = (n_{2r} \otimes n_{2r}) \nabla v_\lambda^0(x_{2r}) = n_{2r} (n_{2r} \cdot \nabla v_\lambda^0(x_{2r})). \tag{7.21}
$$

Now, if $\theta \in (0, 1/4)$ (to be determined) and $x \in T_{2r}^+ \cap B_{\theta r}$, (7.20) implies

$$
|v_\lambda^0(x) - v_\lambda^0(x_{2r}) - (x - x_{2r}) \cdot \nabla v_\lambda^0(x_{2r})| \leq C(\theta r)^{1+\alpha} r^{-\alpha} \left( \int_{T_{2r}^+ \cap B_r} |\nabla v_\lambda^0|^2 \right)^{1/2}.
$$

Combined with (7.21), this implies

$$
|v_\lambda^0(x) - (x \cdot n_{2r}) (n_{2r} \cdot \nabla v_\lambda^0(x_{2r}))| \leq C \theta^\alpha(\theta r) \left( \int_{T_{2r}^+ \cap B_r} |\nabla v_\lambda^0|^2 \right)^{1/2} + |x_{2r}| \left( \int_{T_{2r}^+ \cap B_r} |\nabla v_\lambda^0|^2 \right)^{1/2} \leq C \theta r \left( \theta^\alpha + \theta^{-1} \zeta(r, \varepsilon) \right) \left( \int_{T_{2r}^+ \cap B_r} |\nabla v_\lambda^0|^2 \right)^{1/2}.
$$

Moreover, in view of Lemma 7.1, we may replace $(x \cdot n_{2r})$ by $(x \cdot n_{\theta r})$ with the same error as above. Consequently, one arrives at

$$
\frac{1}{\theta r} \inf_{q \in \mathbb{H}^d} \left( \int_{T_{2r}^+ \cap B_{\theta r}} |v_\lambda^0 - (n_{\theta r} \cdot x) q|^2 \right)^{1/2} \leq C(\theta^\alpha + \theta^{-1} \zeta(r, \varepsilon)) \left( \int_{T_{2r}^+ \cap B_r} |\nabla v_\lambda^0|^2 \right)^{1/2}.
$$

Together with the estimates of the pressure (which is obvious due to (7.20)), we have

$$
\tilde{H}(r, \theta; v_\lambda^0) \leq C(\theta^\alpha + \theta^{-1} \zeta(r, \varepsilon)) \left( \int_{T_{2r}^+ \cap B_r} |\nabla v_\lambda^0|^2 \right)^{1/2}.
$$
Next, observe that \( v_\lambda^0 - n_{2r} \cdot (x - x_{2r})q \) is a weak solution vanishing on the flat boundary for any \( q \in \mathbb{R}^d \). Then (2.13) gives

\[
\left( \int_{T^+_{2r} \cap B_r} |\nabla v_\lambda^0|^2 \right)^{1/2} \leq \frac{C}{r} \inf_{q \in \mathbb{R}^d} \left( \int_{T^+_{2r} \cap B_{r/2}} |v_\lambda^0 - n_{2r} \cdot (x - x_{2r})q|^2 \right)^{1/2} + \frac{C}{r} \| \lambda \nabla \cdot v_\lambda^0 - \int_{T^+_{2r}} \lambda \nabla \cdot v_\lambda^0 \|_{H^{-1}(T^+_{2r})} + C \sup_{t \in [1/2, 2]} \left( \int_{T^+_{2r} \cap B_{2r}} |\lambda \nabla \cdot v_\lambda^0 - \int_{T^+_{2r} \cap B_{2r}} \lambda \nabla \cdot v_\lambda^0| \right) \leq C \tilde{H}(r, 2; v_\lambda^0) + C |\nabla v_\lambda^0|^2 \right)^{1/2}
\]

Combining the previous two estimates, we obtain

\[
\tilde{H}(r, \theta; v_\lambda^0) \leq C \theta^\alpha \tilde{H}(r, \theta; v_\lambda^0) + C_\delta \xi(r, \epsilon) \left( \int_{T^+_{2r}} |\nabla v_\lambda^0|^2 \right)^{1/2}
\]

Choosing \( \theta \in (0, 1/4) \) small enough, we obtain the desired estimate. \( \square \)

**Lemma 7.6.** There exists \( \theta \in (0, 1/4) \) so that for any \( r \in (\epsilon \mathcal{X}, 1) \)

\[
\tilde{H}(r, \theta; v_\xi^0) \leq \frac{1}{2} \tilde{H}(r, 2; v_\xi^0) + C \left( \xi(r, \epsilon) + \sqrt[3]{\eta(\epsilon \mathcal{X} / r)} \right) \left( \int_{D_{2r}} |\nabla u_\xi^0|^2 \right)^{1/2}
\]

**Proof.** This follows from a similar argument as Lemma 6.5 by using (7.19) and Lemma 7.5. \( \square \)

Combining Lemmas 7.3 and 7.6, we obtain

**Lemma 7.7.** There exists \( \theta \in (0, 1/4) \) so that for each \( r \in (\epsilon \mathcal{X}, 1) \), we have

\[
H(\theta r) \leq \frac{1}{2} H(r) + C \left( \xi(r, \epsilon)^{\nu} + \sqrt[3]{\eta(\epsilon \mathcal{X} / r)} \right) \Phi(8r).
\]

**Proof.** By the triangle inequality, Lemma 7.6 and (2.13),

\[
H(\theta r) = H(\theta r; u_\lambda^0) \leq H(\theta r; v_\lambda^0) + H(\theta r; u_\lambda^0 - v_\lambda^0) \\
\leq \tilde{H}(r, \theta; v_\lambda^0) + |\tilde{H}(r, \theta; v_\lambda^0)| - H(\theta; v_\lambda^0) + H(\theta; u_\lambda^0 - v_\lambda^0) \\
\leq \frac{1}{2} \tilde{H}(r, 2; v_\lambda^0) + C \left( \xi(r, \epsilon)^{\nu} + \sqrt[3]{\eta(\epsilon \mathcal{X} / r)} \right) \Phi(8r) + |\tilde{H}(r, \theta; v_\lambda^0)| - H(\theta; v_\lambda^0) + H(\theta; u_\lambda^0 - v_\lambda^0) \\
\leq \frac{1}{2} H(2r; u_\lambda^0) + C \left( \xi(r, \epsilon)^{\nu} + \sqrt[3]{\eta(\epsilon \mathcal{X} / r)} \right) \Phi(8r) + |\tilde{H}(r, \theta; v_\lambda^0)| - H(\theta; u_\lambda^0 - v_\lambda^0) + H(\theta; u_\lambda^0 - v_\lambda^0) = \frac{1}{2} H(2r; u_\lambda^0) + C \left( \xi(r, \epsilon)^{\nu} + \sqrt[3]{\eta(\epsilon \mathcal{X} / r)} \right) \Phi(8r) + \text{I} + \text{II} + \text{III} + \text{IV}.
\]
It suffices to estimate I and II, while the estimates of III and IV are the same.

Part (i): Estimate of I. We will compare the three terms of \( \tilde{H}(r, \theta; v_\lambda^e) \) and \( H(\theta r; v_\lambda^e) \) separately. Note that the quantities \( \tilde{H} \) and \( H \) are defined for the same function \( v_\lambda^e \) over different domains \( D_{\theta r} \) and \( T_{2r} \cap B_{\theta r} \). Recall that \( D_{\theta r} \subset T_{2r}^+ \cap B_{\theta r} \) and \( |T_{2r}^+ \cap B_{\theta r} \setminus D_{\theta r}| \leq C r^d + \varepsilon \approx \varepsilon |D_{\theta r}| \approx \varepsilon |T_{2r} \cap B_{\theta r}| \). This fact and the Meyers' estimate will be our main ingredients for the estimate of I. We begin with the first term of \( H(\theta r; v_\lambda^e) \):

\[
\frac{1}{\theta r} \inf_{q \in \mathbb{R}^d} \left( \int_{D_{\theta r}} |v_\lambda^e - (n_{\theta r} \cdot x) q|^2 \right)^{1/2} \leq \frac{1}{\theta r} \inf_{q \in \mathbb{R}^d} \left( \int_{T_{2r}^+ \cap B_{\theta r}} |v_\lambda^e - (n_{\theta r} \cdot x) q|^2 \right)^{1/2} \left( \frac{|T_{2r}^+ \cap B_{\theta r}|}{|D_{\theta r}|} \right)^{1/2} \leq \frac{1}{\theta r} \inf_{q \in \mathbb{R}^d} \left( \int_{T_{2r}^+ \cap B_{\theta r}} |v_\lambda^e - (n_{\theta r} \cdot x) q|^2 \right)^{1/2} \left( 1 + C\zeta(r, \varepsilon) \right)^{1/2} \leq \frac{1}{\theta r} \inf_{q \in \mathbb{R}^d} \left( \int_{T_{2r}^+ \cap B_{\theta r}} |v_\lambda^e - (n_{\theta r} \cdot x) q|^2 \right)^{1/2} + C\zeta(r, \varepsilon)^{1/2} \left( \int_{T_{2r}^+ \cap B_{\theta r}} |\nabla v_\lambda^e|^2 \right)^{1/2} \leq \frac{1}{\theta r} \inf_{q \in \mathbb{R}^d} \left( \int_{T_{2r}^+ \cap B_{\theta r}} |v_\lambda^e - (n_{\theta r} \cdot x) q|^2 \right)^{1/2} + C\zeta(r, \varepsilon)^{1/2} \Phi(4r),
\]

where we also used the energy estimate of \( v_\lambda^e \) and the generalized Caccioppoli inequality (2.13) in the last inequality. To prove the other direction, let \( q_{\theta r} \) be the vector that minimizes

\[
\frac{1}{\theta r} \inf_{q \in \mathbb{R}^d} \left( \int_{D_{\theta r}} |v_\lambda^e - (n_{\theta r} \cdot x) q|^2 \right)^{1/2}.
\]

Then

\[
|q_{\theta r}| \leq \frac{C}{\theta r} \left( \int_{D_{\theta r}} |(n_{\theta r} \cdot x) q_{\theta r}|^2 \right)^{1/2} \leq \frac{C}{\theta r} \left( \int_{D_{\theta r}} |v_\lambda^e|^2 \right)^{1/2} \leq C \Phi(4r).
\]

Hence,

\[
\frac{1}{\theta r} \inf_{q \in \mathbb{R}^d} \left( \int_{T_{2r}^+ \cap B_{\theta r}} |v_\lambda^e - (n_{\theta r} \cdot x) q|^2 \right)^{1/2} \leq \frac{1}{\theta r} \left( \int_{T_{2r}^+ \cap B_{\theta r}} |v_\lambda^e - (n_{\theta r} \cdot x) q_{\theta r}|^2 \right)^{1/2} \leq \frac{1}{\theta r} \left( \int_{D_{\theta r}} \frac{1}{|T_{2r}^+ \cap B_{\theta r}|} \int_{T_{2r}^+ \cap B_{\theta r} \setminus D_{\theta r}} |v_\lambda^e - (n_{\theta r} \cdot x) q_{\theta r}|^2 \right)^{1/2} + \frac{1}{\theta r} \left( \int_{T_{2r}^+ \cap B_{\theta r}} \int_{D_{\theta r}} |v_\lambda^e - (n_{\theta r} \cdot x) q_{\theta r}|^2 \right)^{1/2} \leq \frac{1}{\theta r} \inf_{q \in \mathbb{R}^d} \left( \int_{D_{\theta r}} |v_\lambda^e - (n_{\theta r} \cdot x) q|^2 \right)^{1/2} + C\zeta(r, \varepsilon) \Phi(8r).
\]
Consequently,
\[
\left| \inf_{q \in \mathbb{R}^d} \left( \int_{D_{B_r}} |v^e_\lambda - (n_{\partial r} \cdot x)q|^2 \right)^{1/2} - \inf_{q \in \mathbb{R}^d} \left( \int_{T^+_2 \cap B_{2r}} |v^e_\lambda - (n_{\partial r} \cdot x)q|^2 \right)^{1/2} \right| \\
\leq C \zeta(r, \varepsilon)^{1/2} \Phi(8r).
\] (7.23)

Next, we consider the second part of \( \tilde{H} \) and \( H \). We would like to show
\[
\left| \| \lambda \nabla \cdot v^e_\lambda - \int_{D_{B_r}} \lambda \nabla \cdot v^e_\lambda \|_{H^{-1}(D_{B_r})} - \| \lambda \nabla \cdot v^e_\lambda - \int_{T^+_2 \cap B_{2r}} \lambda \nabla \cdot v^e_\lambda \|_{H^{-1}(T^+_2 \cap B_{2r})} \right|
\leq C \zeta(r, \varepsilon)^{1/2} \Phi(4r).
\] (7.24)

By the definition of \( H^{-1} \),
\[
\| \lambda \nabla \cdot v^e_\lambda - \int_{D_{B_r}} \lambda \nabla \cdot v^e_\lambda \|_{H^{-1}(D_{B_r})}
\leq \| \lambda \nabla \cdot v^e_\lambda - \int_{T^+_2 \cap B_{2r}} \lambda \nabla \cdot v^e_\lambda \|_{H^{-1}(D_{B_r})} + \left| \int_{T^+_2 \cap B_{2r}} \lambda \nabla \cdot v^e_\lambda - \int_{D_{B_r}} \lambda \nabla \cdot v^e_\lambda \right|
\leq \frac{|T^+_2 \cap B_{2r}|}{|D_{B_r}|} \| \lambda \nabla \cdot v^e_\lambda - \int_{T^+_2 \cap B_{2r}} \lambda \nabla \cdot v^e_\lambda \|_{H^{-1}(T^+_2 \cap B_{2r})}
+ \left| \int_{T^+_2 \cap B_{2r}} \lambda \nabla \cdot v^e_\lambda - \int_{D_{B_r}} \lambda \nabla \cdot v^e_\lambda \right|.
\]

The second term on the right-hand side of the last inequality is bounded by \( \zeta(r, \varepsilon)^{1/2} \Phi(4r) \) due to the fact \( |T^+_2 \cap B_{2r} \setminus D_{B_r}| \leq C r^d \zeta(r, \varepsilon) \). Also observe that \( |T^+_2 \cap B_{2r}|/|D_{B_r}| = 1 + C \zeta(r, \varepsilon) \). It follows that
\[
\| \lambda \nabla \cdot v^e_\lambda - \int_{D_{B_r}} \lambda \nabla \cdot v^e_\lambda \|_{H^{-1}(D_{B_r})}
\leq \| \lambda \nabla \cdot v^e_\lambda - \int_{T^+_2 \cap B_{2r}} \lambda \nabla \cdot v^e_\lambda \|_{H^{-1}(T^+_2 \cap B_{2r})} + C \zeta(r, \varepsilon)^{1/2} \Phi(4r).
\] (7.25)

Conversely, to prove the other direction, we claim the following fact
\[
\| F \|_{H^{-1}(T^+_2 \cap B_{2r})} \leq \| F \|_{H^{-1}(D_{B_r})} + C \zeta(r, \varepsilon)\gamma(\int_{T^+_2 \cap B_{2r}} |F|^p)^{1/p},
\]
with \( \gamma = 1/2 - 1/p \) and \( p > 2 \). Applying the above fact to \( F = \lambda \nabla \cdot v^e_\lambda - \int_{T^+_2 \cap B_{2r}} \lambda \nabla \cdot v^e_\lambda \) and using the Meyers estimate of \( v^e_\lambda \), we have
\[
\| \lambda \nabla \cdot v^e_\lambda - \int_{T^+_2 \cap B_{2r}} \lambda \nabla \cdot v^e_\lambda \|_{H^{-1}(T^+_2 \cap B_{2r})}
\leq \| \lambda \nabla \cdot v^e_\lambda - \int_{T^+_2 \cap B_{2r}} \lambda \nabla \cdot v^e_\lambda \|_{H^{-1}(D_{B_r})} + C \zeta(r, \varepsilon)\gamma(\int_{T^+_2 \cap B_{2r}} |F|^p)^{1/p}
\leq \| \lambda \nabla \cdot v^e_\lambda - \int_{D_{B_r}} \lambda \nabla \cdot v^e_\lambda \|_{H^{-1}(D_{B_r})} + C \zeta(r, \varepsilon)^{1/2} \Phi(4r).
\]
This, combined with (7.25), gives (7.24).

Finally, for any $k, \ell \in [1/4, 1]$, it is not hard to show

$$
\left| \int_{D_k} \lambda \nabla \cdot v^{\varepsilon}_\lambda - \int_{D_\ell} \lambda \nabla \cdot v^{\varepsilon}_\lambda \right| - \left| \int_{T_{2}^- \cap B_{k_0}} \lambda \nabla \cdot v^{\varepsilon}_\lambda - \int_{T_{2}^+ \cap B_{\ell_0}} \lambda \nabla \cdot v^{\varepsilon}_\lambda \right| \\
\leq 2 \sup_{k \in [1/4, 1]} \left| \int_{D_k} \lambda \nabla \cdot v^{\varepsilon}_\lambda - \int_{T_{2}^- \cap B_{k_0}} \lambda \nabla \cdot v^{\varepsilon}_\lambda \right| \\
\leq C \zeta (r, \varepsilon)^{1/2} \Phi (4r).
$$

This, together with (7.23) and (7.24), implies

$$
|\tilde{H}(r, \theta; v^{\varepsilon}_\lambda) - H(\theta r; v^{\varepsilon}_\lambda)| \leq C \zeta (r, \varepsilon)^{\gamma/\Phi_1 (8r)},
$$

which is the desired estimate of $I$.

Part (ii): Estimate of $II$. Recall that

$$
H(\theta r; u^{\varepsilon}_\lambda - v^{\varepsilon}_\lambda) = \frac{1}{\theta r} \inf_{q \in \mathbb{R}^d} \left( \int_{D_{br}} |u^{\varepsilon}_\lambda - v^{\varepsilon}_\lambda - (n_{\theta r} \cdot x) q|^2 \right)^{1/2} \\
+ \frac{1}{\theta r} \| \lambda \nabla \cdot (u^{\varepsilon}_\lambda - v^{\varepsilon}_\lambda) - \int_{D_{br}} \lambda \nabla \cdot (u^{\varepsilon}_\lambda - v^{\varepsilon}_\lambda) \|_{L^1 (D_{br})} \\
+ \sup_{k, \ell \in [1/4, 1]} \left| \int_{D_{k_0}} \lambda \nabla \cdot (u^{\varepsilon}_\lambda - v^{\varepsilon}_\lambda) - \int_{D_{\ell_0}} \lambda \nabla \cdot (u^{\varepsilon}_\lambda - v^{\varepsilon}_\lambda) \right|.
$$

Then it follows from Lemma 7.3 and Remark 7.4 that

$$
H(\theta r; u^{\varepsilon}_\lambda - v^{\varepsilon}_\lambda) \leq C \zeta (r, \varepsilon)^{\gamma/\Phi_1 (8r)}.
$$

Hence, we have proved the estimates for $I$ and $II$. Similar argument gives the same estimates for $III$ and $IV$. Therefore, the desired estimate follows from (7.22).

Finally, following the similar argument as the interior estimate and using the full version of Lemma 6.7, we obtain

**Theorem 7.8.** For any $s \in (0, d)$, $\lambda \in [0, \infty)$, there exists a random variable $X = X_{s, \lambda} : \Omega \mapsto [1, \infty)$ satisfying

$$
X \leq O_s (C),
$$

such that if $r \in [\varepsilon X, 1]$, then

$$
\left( \int_{D_r} |\nabla u^{\varepsilon}_\lambda|^2 \right)^{1/2} + \left( \int_{D_2} |\lambda \nabla \cdot u^{\varepsilon}_\lambda - \int_{D_2} \lambda \nabla \cdot u^{\varepsilon}_\lambda|^2 \right)^{1/2} \leq C \left( \int_{D_r} |\nabla u^{\varepsilon}_\lambda|^2 \right)^{1/2}. \quad (7.26)
$$
 Proof. Let $H$, $\Phi$ and $h$ be defined as before. In view of Lemma 7.2, the condition (6.14f) in Lemma 6.7 is not satisfied exactly. However, if we set $H^*(r) = H(r) + \zeta(r, \varepsilon) \Phi(r)$. Thus, $H^*$, $\Phi$ and $h$ satisfy

\[
H^*(\theta r) \leq \frac{1}{2} H(r) + C_0 \left\{ \zeta(r, \varepsilon) \gamma + \zeta(\theta^{-1} r, \varepsilon) + \sqrt[3]{\eta(\varepsilon \mathcal{X}/r)} \right\} \Phi(8r) \tag{7.27a}
\]

\[
H^*(r) \leq C_0 \Phi(r) \tag{7.27b}
\]

\[
h(r) \leq C_0 (H^*(r) + \Phi(r)) \tag{7.27c}
\]

\[
\Phi(r) \leq C_0 (H^*(r) + h(r)) \tag{7.27d}
\]

\[
\sup_{r \leq t \leq 2r} \Phi(t) \leq C_0 \Phi(2r) \tag{7.27e}
\]

\[
\sup_{r \leq s, t \leq 2r} |h(s) - h(t)| \leq C_0 H^*(2r). \tag{7.27f}
\]

Recall that for any $r \in (\varepsilon \mathcal{X}, 1)$

\[
\zeta(r, \varepsilon) \gamma + \zeta(\theta^{-1} r, \varepsilon) + \sqrt[3]{\eta(\varepsilon \mathcal{X}/r)} \leq Cr^\alpha \gamma + C(\varepsilon/r)^\alpha \gamma + (\varepsilon \mathcal{X}/r)^{\rho/3}.
\]

Thus, Lemma 6.7 applies (with $\varepsilon$ replaced by $\varepsilon \mathcal{X}$) and gives

\[
\int_{\varepsilon \mathcal{X}}^2 \frac{H^*(r)}{r} dr + \sup_{\varepsilon \mathcal{X} \leq r \leq 2} \Phi(r) \leq C \Phi(2).
\]

Now, we may proceed as the proof of Theorem 6.8 and obtain the desired estimate. \qed

Proof of Theorem 1.5. The proof is the same as Theorem 1.1 by using Theorem 7.8. \qed

Acknowledgements. Both of the authors would like to thank Professor Fanghua Lin for the helpful comments after the second author reporting the results of this paper in the SUSTech PDE Workshop in Shenzhen. Both of the authors would like to thank Professor Hongjie Dong for pointing out a mistake in an early version of this paper.

Data Availability Statement Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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References

1. Acosta, G., Durán, R.G., Muschietti, M.A.: Solutions of the divergence operator on John domains. *Adv. Math.* **206**(2), 373–401, 2006
2. Armstrong, S.N., Kuusi, T., Mourrat, J.-C.: Mesoscopic higher regularity and subadditivity in elliptic homogenization. *Commun. Math. Phys.* **347**(2), 315–361, 2016
3. Armstrong, S.N., Kuusi, T., Mourrat, J.-C.: The additive structure of elliptic homogenization. *Invent. Math.* **208**(3), 999–1154, 2017
4. Armstrong, S.N., Kuusi, T., Mourrat, J.-C.: Quantitative stochastic homogenization and large-scale regularity. In: Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 352. Springer, Cham (2019)
5. Armstrong, S.N., Mourrat, J.-C.: Lipschitz regularity for elliptic equations with random coefficients. *Arch. Ration. Mech. Anal.* **219**(1), 255–348, 2016
6. Armstrong, S.N., Shen, Z.: Lipschitz estimates in almost-periodic homogenization. *Commun. Pure Appl. Math.* **69**(10), 1882–1923, 2016
7. Armstrong, S.N., Smart, C.K.: Quantitative stochastic homogenization of convex integral functionals. *Ann. Sci. Éc. Norm. Supér.* (4) **49**(2), 423–481, 2016
8. Avellaneda, M., Lin, F.: Compactness methods in the theory of homogenization. *Commun. Pure Appl. Math.* **40**(6), 803–847, 1987
9. Avellaneda, M., Lin, F.: Compactness methods in the theory of homogenization. II. Equations in nondivergence form. *Commun. Pure Appl. Math.* **42**(2), 139–172, 1989
10. Basson, A., Gérard-Varet, D.: Wall laws for fluid flows at a boundary with random roughness. *Commun. Pure Appl. Math.* **61**(7), 941–987, 2008
11. Dalibard, A., Gérard-Varet, D.: Effective boundary condition at a rough surface starting from a slip condition. *J. Differ. Equ.* **251**(12), 3450–3487, 2011
12. Dalibard, A., Prange, C.: Well-posedness of the Stokes–Coriolis system in the half-space over a rough surface. *Anal. PDE* 7(6), 1253–1315, 2014
13. Dohrmann, C.R., Widlund, O.B.: An overlapping Schwarz algorithm for almost incompressible elasticity. *SIAM J. Numer. Anal.* **47**(4), 2897–2923, 2009
14. Gérard-Varet, D.: The Navier wall law at a boundary with random roughness. *Commun. Math. Phys.* **286**(1), 81–110, 2009
15. Gérard-Varet, D., Masmoudi, N.: Relevance of the slip condition for fluid flows near an irregular boundary. *Commun. Math. Phys.* **295**(1), 99–137, 2010
16. Giaquinta, M., Modica, G.: Nonlinear systems of the type of the stationary Navier–Stokes system. *J. Reine Angew. Math.* **330**, 173–214, 1982
17. Gloria, A., Neukamm, S., Otto, F.: Quantification of ergodicity in stochastic homogenization: optimal bounds via spectral gap on Glauber dynamics. *Invent. Math.* **199**(2), 455–515, 2015
18. Gloria, A., Otto, F.: Quantitative results on the corrector equation in stochastic homogenization. *J. Eur. Math. Soc. (JEMS)* **19**(11), 3489–3548, 2017
19. Gu, S.: Convergence rates in homogenization of Stokes systems. *J. Differ. Equ.* **260**(7), 5796–5815, 2016
20. Gu, S., Shen, Z.: Homogenization of Stokes systems and uniform regularity estimates. *SIAM J. Math. Anal.* **47**(5), 4025–4057, 2015
21. Gu, S., Xu, Q.: Optimal boundary estimates for Stokes systems in homogenization theory. *SIAM J. Math. Anal.* **49**(5), 3831–3853, 2017
22. Gu, S., Zhuge, J.: Periodic homogenization of Green’s functions for Stokes systems. *Calc. Var. Partial Differ. Equ.* **58**(3), 114, 46, 2019
23. Hansbo, P., Larson, M.G.: Discontinuous Galerkin methods for incompressible and nearly incompressible elasticity by Nitsche’s method. *Comput. Methods Appl. Mech. Eng.* **191**(17–18), 1895–1908, 2002
24. Herrmann, L.R.: Elasticity equations of incompressible and nearly incompressible materials by a variational theorem. *AIAA J.* **3**, 1896–1900, 1965
25. Higaki, M., Prange, C.: Regularity for the stationary Navier–Stokes equations over bumpy boundaries and a local wall law. *Calc. Var. Partial Differ. Equ.* **59**(4), 131, 46, 2020
26. Jikov, V.V., Kozlov, S.M., Olenik, O.A.: Homogenization of Differential Operators and Integral Functionals. Springer, Berlin (1994)
27. Jones, P.W.: Quasiconformal mappings and extendability of functions in Sobolev spaces. *Acta Math.* **147**(1–2), 71–88, 1981
28. Kenig, C.E., Lin, F., Shen, Z.: Homogenization of elliptic systems with Neumann boundary conditions. *J. Am. Math. Soc.* **26**(4), 901–937, 2013
29. Kenig, C.E., Prange, C.: Uniform Lipschitz estimates in bumpy half-spaces. *Arch. Ration. Mech. Anal.* **216**(3), 703–765, 2015
30. Kenig, C.E., Prange, C.: Improved regularity in bumpy Lipschitz domains. *J. Math. Pures Appl. (9)* **113**, 1–36, 2018
31. Kouhia, R., Stenberg, R.: A linear nonconforming finite element method for nearly incompressible elasticity and Stokes flow. *Comput. Methods Appl. Mech. Eng.* **124**(3), 195–212, 1995
32. Mott, P.H., Dorgan, J.R., Roland, C.M.: The bulk modulus and Poisson’s ratio of “incompressible” materials. *J. Sound Vib.* **312**(4–5), 572–575, 2008
33. Mott, P.H., Roland, C.M.: Limits to Poisson’s ratio in isotropic materials. *Phys. Rev. B* **80**(132104), 1–4, 2009
34. Mourrat, J.-C., Nolen, J.: Scaling limit of the corrector in stochastic homogenization. *Ann. Appl. Probab.* **27**(2), 944–959, 2017
35. Olenik, O.A., Shamaev, A.S., Yosifian, G.A.: Mathematical problems in elasticity and homogenization. In: Studies in Mathematics and its Applications, vol. 26. North-Holland Publishing Co., Amsterdam (1992)
36. Rogers, L.G.: Degree-independent Sobolev extension on locally uniform domains. *J. Funct. Anal.* **235**(2), 619–665, 2006
37. Shen, Z.: Boundary estimates in elliptic homogenization. *Anal. PDE* **10**(3), 653–694, 2017
38. Shen, Z.: Periodic homogenization of elliptic systems. In: Operator Theory: Advances and Applications, vol. 269. Birkhäuser/Springer, Cham, Advances in Partial Differential Equations (Basel) (2018)
39. Temam, R.: Navier–Stokes equations. In: Theory and Numerical Analysis. AMS Chelsea Publishing, Providence, RI, (2001), Reprint of the 1984 edition
40. Vogelius, M.: An analysis of the $p$-version of the finite element method for nearly incompressible materials. Uniformly valid, optimal error estimates. *Numer. Math.* **41**(1), 39–53, 1983
41. Xu, Q.: Convergence rates and $W^{1,p}$ estimates in homogenization theory of Stokes systems in Lipschitz domains. *J. Differ. Equ.* **263**(1), 398–450, 2017
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(Received May 24, 2020 / Accepted March 18, 2022)
Published online April 25, 2022
© The Author(s), under exclusive licence to Springer-Verlag GmbH, DE, part of Springer
Nature (2022)