Resumations with renormalon effects for the hadronic vacuum polarization contribution to the muon (g-2)

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Abstract

The hadronic vacuum polarization contribution to the muon (g-2) value is calculated by considering a known dispersion integral which involves the $R_{e\gamma}$ (s) ratio. The theoretical part stemming from the region below 1.8 GeV is the largest contribution in our approach, and is calculated by using a contour integral involving the associated Adler function $D(Q^2)$. In the resumations, we explicitly take into account the exactly known renormalon singularity of the leading infrared renormalon in the usual and in the modified Borel transform of $D(Q^2)$, and map further away from the origin the other renormalon singularities by employing judiciously chosen conformal transformations. The renormalon effect increases the predicted value of the hadronic vacuum polarization contribution to the muon (g-2), and therefore diminishes the difference between the recently measured and the SM/QCD-predicted value of (g-2). It is also shown that the total QED correction to the hadronic vacuum polarization is very small, about 0.06%.
I. INTRODUCTION

The new precise measurement of the muon anomalous magnetic moment \( g-2 = 2 \) allows for detailed testing of the standard model, and therefore for the possibility of looking into physics beyond the standard model as well. In fact, comparison of the experimental result with some theoretical calculations shows a 2.6 difference \([1]\). This has been suggested as the appearance of new physics\[2\]. Since the advertised discrepancy comes from the calculation of Ref. [8] of the hadronic vacuum polarization contributions \( a^{(v+p)} \) to \( g-2 \), we re-evaluate in the present paper the theoretical (pQCD + OPE) parts of this quantity. We use the same theoretical approach as in Ref. [8], which in turn is based on the approach of Ref. [9]. However, in addition, we take into account the known renormalon structure of the Adler function.

II. FORMALISM

According to Ref. [10], the hadronic vacuum polarization contribution to the muon anomalous magnetic moment \( g-2 = 2 \) appears in the following dispersion integral:

\[
a^{(v+p)} = \frac{2}{3} \int_0^{\infty} ds \frac{K(s) R_{e^+e^-}(s)}{4m^2 s} ;
\]

where \( K(s) \) is the QED kernel \([11]\),

\[
K(s) = x^2 \frac{x^2}{2} + 1 + (1 + x)^2 \frac{x^2}{2} \ln(1 + x) + \frac{x^2}{2} \left( \frac{1}{1 - x} \right) x^2 \ln x ;
\]

Here, \( x = (1 - y)/(1 + y) \) with \( y = (4m^2 - s)^{1/2} \). The largest part (about 92%) of \( a^{(v+p)} \) comes from the region with CMS energy \( \sqrt{s} < \sqrt{s_0} = 1.8 \text{ GeV} \). Following the approach of Ref. [9], applied in Ref. [9] to \( a^{(v+p)} \), we rewrite the dispersion integral \([4]\) (with \( s_{\text{max}} = s_0 \)) in the form

\[
a^{(v+p)}(s, s_0) = Z_{s_0} \int_{4m^2}^{s_0} ds R_{e^+e^-}(s) \frac{K(s)}{s} C_1 \frac{s}{s_0} \left( Z_{s_0} \int_{4m^2}^{s_0} ds R_{e^+e^-}(s) \right) ;
\]

Here, \( C_1 \) is in principle an arbitrary constant, which however, according to the philosophy of Ref. [9], may be chosen in such a way as to minimize the rest (\"data\") integral and maximize the second (\"theory\") integral. It is known that \( R_{e^+e^-}(s) \approx 12 \ln(s + 11) \), with \( s \) being the hadronic part of the (vector) photon vacuum polarization function which has no poles.

\[1\] For an (incomplete) list of works investigating the possibility to explain and/or to find implications of this difference within various frameworks of new physics, see: Refs. [2] (supersymmetric models), [3] (non-supersymmetric extended Higgs sectors), [4] (composites models), [5] (models with extra dimensions), [6] (leptoquarks), and [7] (other models).
in the interval $D; 4m^2$; further, $l = s = s_0$ also has no poles in that interval, in contrast to the function $K(s)$. Therefore, the Cauchy theorem can be applied to the second ("theory") integral, with the path of Fig. 1. Carrying subsequently integration by parts, and using the identity $D \left( Q^2 = 12 \frac{s}{d} (s) = ds \right)$, leads to

$$
\frac{3}{2}e^2 \int_0^s ds' R_{ee}(s) = \frac{Z_{s_0}}{4m^2} K(s) \int_0^{s_0} \frac{ds}{s} C_1 \frac{s}{s_0}^{\#},
$$

where the associated vector Adler function $D \left( Q^2 \right)$ can be written in the following way:

$$
D \left( Q^2 \right) = N_c \sum_{f = u \neq d} X_{f} Q^2 + D_{\text{can}} : Q^2 + 4^2 \sum_{d = 2; J = 0}^{3} \frac{Q^2}{Q_{f}^2} \left[ D_{f f f f} (Q^2) \right]^{5}.
$$

Here, $D_{\text{can}} : Q^2$ is the canonically normalized massless QCD part with dimension $d = 0$, whose power expansion in $a_{\text{MS}}(Q^2)$ is

$$
D_{\text{can}} : Q^2 = a_2^{\text{MS}} Q^2 + d_1^{(0)} a_2^{\text{MS}} (Q^2) + d_2^{(0)} a_2^{\text{MS}} (Q^2)^2 + d_3^{(0)} a_2^{\text{MS}} (Q^2)^3 + \cdots.
$$

with $d_1^{(0)} = 1.6398$,$^{13}$ $d_2^{(0)} = 63710$,$^{12}$ and $d_3^{(0)}$ is estimated to be $d_3^{(0)} = 25 - 10$ $^{13}$. The renormalization scheme is $\overline{\text{MS}}$. The number of active quark flavors is $n_f = 3$. The $d = 2$ contributions are

$$
4^2 D_{f f f f} (Q^2) = \frac{6m_f^2}{Q^2} \left( Q^2 \right) + \frac{14}{3} a_2^{\text{MS}} (Q^2) + 43 \times 581 \ a_2^{\text{MS}} (Q^2)^2 + O (a^3);
$$

where only the $s$ quark contributes appreciably. The $d = 4$ contributions are those of the gluon condensates

$$
4^2 D_{f f f f} (Q^2) = \frac{2^2}{3(Q^2)^2} h_{\text{GF}} \frac{11}{18} a_2^{\text{MS}} (Q^2) + O (a^2);
$$

those of the quark mass condensates

$$
4^2 D_{f f f f} (Q^2) = \frac{16}{Q^2} \left( Q^2 \right)^2 + \frac{1}{3} a_2^{\text{MS}} (Q^2) + \frac{11}{2} a_2^{\text{MS}} (Q^2)^2 + O (a^3) \text{Im} \ e_{f f f f} i
$$

$$
+ \frac{8}{(Q^2)^2} \left( \frac{4}{27} a_2^{\text{MS}} (Q^2) + 1.074 a_2^{\text{MS}} (Q^2)^2 + O (a^3) \text{Im} \ e_{f f f f} i \right);
$$

and those proportional to $m^4$ $^{14}$

\footnote{This Adler function does not contain QED radiative corrections. The effects of them will be added later on.}

3
Here, we used the complex momentum contour integrals

\[ A_{\text{can}} = \frac{1}{2} \int \frac{d^2 \mathbf{p}}{2\pi} \cdot (1 + e^{i\mathbf{k}})^2 D_{\text{can}}(Q^2 = s_0 e^{i\mathbf{k}}); \]

\[ A_{n,2k} = \int \frac{d^2 \mathbf{p}}{2\pi} (1 + e^{i\mathbf{k}})^n e^{i\mathbf{k}\cdot \mathbf{v}} \left[ a_{\overline{\text{MS}}}^g (Q^2 = s_0 e^{i\mathbf{k}}) \right]^n; \]

\[ M_{n,2k} = \int \frac{d^2 \mathbf{p}}{2\pi} (1 + e^{i\mathbf{k}})^n e^{i\mathbf{k}\cdot \mathbf{v}} \left[ \frac{m_a(s_0 e^{i\mathbf{k}})}{m_a(s_0)} \right]^{1+2k} a_{\overline{\text{MS}}}^g (Q^2 = s_0 e^{i\mathbf{k}})^n; \]

For RGE evolution of \( a_{\overline{\text{MS}}}^g (Q^2) \) we use the four-loop truncated perturbation series (TPS) [17] of the \( \overline{\text{MS}} \) beta function, with \( n_c = 3 \). In addition, for the RGE evolution of \( m_a (Q^2) \) we use the \( \overline{\text{MS}} \) four-loop TPS quark mass anomalous dimension [18]. For example, the RGE evolution along the complex momentum contour gives

\[ \frac{m_a(s_0 e^{i\mathbf{k}})}{m_a(s_0)} = \exp \int_0^z \frac{d^2 \mathbf{p}}{2\pi} a_{\overline{\text{MS}}}^g (s_0 e^{i\mathbf{k}}) + \sum_{n=1}^3 e_n a_{\overline{\text{MS}}}^g (s_0 e^{i\mathbf{k}})^n; \]
III. EVALUATION

We first use the input values as used in Ref. [3]

$$g_G \equiv (0.015, 0.020) \text{ GeV}^4;$$

$$\frac{m_s}{m} (\text{ in GeV}) = (0.333, 0.017); \quad \frac{m_s}{m} (\text{ in GeV}) = (0.1201, 0.0020);$$

$$m_s (\text{ in GeV}) = (0.20, 0.007); \quad m_s (\text{ in GeV}) = (0.151, 0.053) \text{ GeV}.$$  \hspace{1cm} (18)

Further, we use for $D_{\text{can}}: (Q^2)$ the NNLO TPS (i.e., with $d_s^{(0)} = 0$) and with the normalization scale $Q^2 = Q^2$, i.e., the approach apparently used by [3]. The result for their input values, and for $C_1 = 0.007 \text{ GeV}^2$, is then:

$$10^{11} a_{\text{exp}}^3 (s \rightarrow \text{part; NNLO } D_{\text{can}}) = 4752 \quad 108;$$

(21)

in contrast to their value (46862 1132). The central values of these theory parts are thus higher by 14% than those given in [3]. This percentage does not change at different values of the parameter $C_1$, since the results are linear with respect to $C_1$. The uncertainty in [21] is obtained by adding in quadrature the uncertainty from $m_s$ (61), from $m_s$ (74), and from $g_G$ (49). The separate contributions to the central value 4752 in (21) are:

- 4079 from the leading term; 749 from the (resummed) canonical part [3] [1] (14); 87 from the $d = 2$ strange mass term (3); 37 from the $d = 4$ gluon condensate term (3); 25 from the $d = 4$ quark condensate terms $s_1$ (3); 1 from the $d = 4$ quark mass terms $s_1$ (10).

We can, however, re(calculate the canonical part (14) by using methods which account for the renormalon structure of the Adler function. The latter structure has been shown [13] to have numerically significant effects in the hadronic decay width ratio. Ref. [13] uses ordinary Borel transform $s_1$, and Ref. [13] mod ed Borel transform $s_1$ of $D_{\text{can}}$. In these two references, the exactly known leading infrared renormalon singularity of the Borel transform $s_1$ (3), at the value of the Borel variable $b = 2$, has been fully taken into account as the prefactor function of the form $1 = (1 + b = 2)$, and $1 = (1 + b = 2)$, respectively. This procedure allows us to describe the Borel transform $s_1$ more accurately in the most important region of the Borel integration, namely, the interval between the origin and the leading IR renormalon at $b = 2$. Furthermore, judiciously chosen conformal transformations $b = b(w)$ have been employed there to map the singularities of other renormalons further away from the origin and thus to reduce their numerical significance. For details of the calculation procedures, we refer to these two references.

If we re(calculate the canonical part (14) by using the ordinary Borel transform method of Ref. [13], the result [21] increases further to 4817 116, as a consequence of the renormalon

\footnote{In Refs. [13,19], the massless QCD quantity $r$ was calculated (resummed). It is obtained from the hadronic decay width ratio $R$ by subtracting from it the strangeness-changing contributions, factoring out the CKM (element $y_{ud}$) and the electroweak correction factors, and subtracting the quark mass $m_{ud}$ (60) contributions. The quantity $r$ is the same kind of contour integral in the complex $m_{ud}$ mass plane as the quantity $A_{\text{can}}$ of Eq. (14), but with the contour factor $(1 + e^{iy})^2$ replaced by $(1 + e^{iy})^2 (1 - e^{iy})$. Therefore, the analyses of Refs. [13,19] can be repeated, without any major changes, for the resummation of the quantity $A_{\text{can}}$ of Eq. (14).}
electrons. In $D_{\text{can}}$, we took $d_3^{(0)} = 25$, and for the renormalization scale ($R_{\text{Sc}l}$) in $D_{\text{can}}$, we took the value which gives us the local insensitivity of the result with respect to $R_{\text{Sc}l} (2^2 - 1 \times Q^2)$, as argued in [13].

If we apply to expression (14) the method of resummation of Ref. [19] which employs modified Borel transforms with the same input and with $d_3^{(0)} = 25$ we obtain a value very similar to the aforementioned one

$$10^{11} \alpha_{\text{QCD}}(s_0; \text{th: part; meth: [19]}) = 4820 \pm 120$$

The uncertainty in (22) is obtained by adding in quadrature the uncertainty from $s$ (74), from $m_s$ (75), from $M_{\text{Gg}}$ (49), as well as from the resummation method uncertainty and the uncertainty of $d_3^{(0)}$ (28). This calculation and the uncertainty estimates procedure are carried out in complete analogy with Ref. [19], to which we refer for details. The separate contributions to the central value 4820 in (22) are the same as those to (21), except that the contribution from the (resummed) canonical part [4] is now 817 (before: 749).

In order to isolate the contribution of the renormalon structure included in the value (22), we should compare the latter central value with the one obtained by using for $D_{\text{can}}$ the N$^3$LO TPS with $d_3^{(0)} = 25$. The central value result in this case is 4772. The obtained renormalon structure effect is thus 48 units, or 1.0%.

The input values (19) (23), used by the authors of Ref. [8], and taken up until now in the present work, can be replaced by what we believe to be more updated values

$$m_s^2(m_Z) = 0.3254 \pm 0.0124 \quad m_s^2(M_Z) = 0.1192 \pm 0.0015$$

(23)

$$m_s^2(m_Z) = 0.119 \pm 0.024 \text{ GeV}$$

(24)

The values (23) were obtained in [19] by a detailed analysis of the $R$ ratio, involving modified Borel transforms $s$, and accounting for the renormalon structure of the associated Adler function via an explicit ansatz and with judiciously chosen conformal transformations. This result virtually agrees with the one obtained in the $R_{\text{Sc}l}$ analysis of Ref. [13] where ordinary Borel transforms were used instead. The values (23) are shifted downwards and the uncertainties are reduced, in comparison to the values (19). The latter values are based largely on the ALEPH analysis of the $R$ decay [20]. The latter analysis did not account for the renormalon structure of the associated Adler function. The trend towards smaller values of $s$ and towards smaller uncertainties appears also in the analysis of the $R$ ratio of the authors of Ref. [21], who accounted for the renormalon structure via a large ($d_3^{(0)}$ resummation of the ordinary Borel transform and employed a resummation related to the effective charge [ECH]) method (they obtained $s_{\text{resum}} = 0.330 \pm 0.014$. The values (24) for the strange quark mass, which are significantly lower than those in (23) were obtained in the recent analysis of Ref. [13].

Furthermore, ALEPH analysis [20] of the $J/\psi$ decays predicts the gluon condensate term to be consistent with zero.

\footnote{We refer to [13] for a detailed discussion of this estimate of $d_3^{(0)}$ values.}
in contrast with the input [19]. We will take, in addition to the $(s;m_s)$ (inputs [23] [24]), either the input [14] or [25] for the gluon condensate term. Small values of the gluon condensate close to the ALEPH values [25] are also suggested in the formalism of Ref. [23], where the power-suppressed terms are obtained from the knowledge of the perturbation series of $D_{\text{can}}$ [9] and of its infrared renormalon structure.

Applying then again the resummation method of Ref. [19] to expression (14), we obtain the prediction

$$10^{11} a^{(\nu \nu)}(s, s_0; \text{theory part}) = 4823 \quad 77 \quad \text{for } h_{\text{G G}} \text{ value Eq: (18)} \quad (26)$$

$$= 4789 \quad 71 \quad \text{for } h_{\text{G G}} \text{ value Eq: (25)} \quad (27)$$

The uncertainties 77; 71 in (26) (27) are obtained by adding in quadrature the uncertainty from $s$ (49), from $m_s$ (24), from $h_{\text{G G}}$ (49; 37), and from the resummation method uncertainty and $d_1^{(*)}$ (uncertainty (25). The separate contributions to the central value 4823 in (26) are: 4079 from the leading term 787 from the (resummed) canonical part 4; 54 from the $d = 2$ strange mass term 1; 37 if using (25) from the d = 4 gluon condensate term 3; 25 from the d = 4 quark condensate terms 3; 0.5 from the d = 4 quark mass term 3.

We note that the uncertainties 77; 71 as obtained by us in (26) (27) are lower than the uncertainty 1132 for that quantity obtained by the authors of Ref. [8] (for $C_1 = 0.007 \text{ GeV}^2$). The main reason for this is that we used different, in our opinion more updated, values of $\frac{h_{\text{G G}}}{m_s}$ and $m_s$ (23) (24) than the values [13] (20) used by Ref. [8]. The uncertainties 77; 71 are about the same as the uncertainties obtained in Ref. [8] for the "data" part, i.e., the first integral in (4), for $C_1 = 0.000(0.007 \text{ GeV}^2$ (see Table 1 of [8]). The authors of Ref. [8] chose $C_1 = 0.001(0.002 \text{ GeV}^2$, i.e., their "theory" contribution was very small in comparison to their "data" contribution. The argument for the virtual exclusion of the theory from their considerations was that the uncertainties from their "theory" part were too high when $C_1$ is appreciable. We believe that a different approach, which emphasizes the "theory" part more than the "data" part, is legitimate as well, especially because the uncertainties from our analysis of the "theory" part are reasonable and comparable to the uncertainties of the "data" part even for large values of $C_1$. The authors of Ref. [8] obtained the values of the "data" parts for $C_1 = 0.007 \text{ GeV}^2$, by using the available $e^+e^-$ and $\mu^-\mu^+$ decay data. Therefore, we choose the largest $C_1 = 0.007 \text{ GeV}^2$ listed in their Table 1.

The "data" part given in their Table 1, for $C_1 = 0.007 \text{ GeV}^2$, is

$$10^{11} a^{(\nu \nu)}(s, s_0; \text{data part}) = 1622 \quad 56 \quad (28)$$

It corresponds to the first integral in Eq. (4).

Our evaluation of the theoretical part [24] (27) of $s$, $s_0$, contributions excludes any QED corrections to the hadronic vacuum polarization. The leading term of such QED corrections corresponds to the exchange of a virtual photon within the hadronic blob in Fig. 1. This diagram induces in $R_{\mu^-\mu^+}$ in the master formula [3] the real photon emissions and the virtual (plus soft photon emissions) corrections to the pure hadronic final states.
Since the theory part in Eq. (4) (2nd line) is dominated by the perturbative QCD (pQCD) Adler function, this QED correction can be easily implemented by replacing \((1 + D_{\text{can}}(Q^2))\) in (4) with the QED corrected one:

\[
1 + D_{\text{can}}(Q^2) \rightarrow 1 + \left(1 + \frac{3 \alpha^2}{4} \right) \text{ em}(Q^2),
\]

(29)

The QED corrections to the power-suppressed terms in (4) are negligibly small. Then, since the running of \(\text{em}(Q^2)\) along the integration contour can be safely ignored, the QED correction can be seen to shift the theory part by the factor:

\[
1 + \frac{3}{4} \left[ Q^2 \right]_{\text{em}}(s_0) = 1 + \frac{3}{4} \text{ em}(s_0) = 1 + 6 \times 10^{-4}
\]

(30)

Thus, the QED correction to the theory part of \(a^{(\nu,\varphi)}\) is approximately \((6 \times 10^{-4})\) 4800 3. It is small.

On the other hand, in the data part [23], the aforementioned class of QED contributions is, in principle, already included. This data part is taken from Ref. [3], and corresponds to the 1st integral in Eq. (1), for \(C_1 = 0.007 \text{GeV}^2\), with \(R_{\text{e}^+\text{e}^-}\) (s) there based largely on the experimental \(\text{e}^+\text{e}^-\) and (decay data. In this connection, we mention that there is some controversy on whether the specific photon decay channels via meson \(e^+e^-\) are included in the data or should be added to them. The contribution of these channels to the data part (23), in the narrow width approximation, would be about \((18\ 4)\). However, the (decay data already include this decay channel, while the situation with the \(\text{e}^+\text{e}^-\) data in this respect is less clear. Since the authors of Ref. [3] included in their analysis both types of data, we presume that the additional possible contributions from the aforementioned photonic decay channels to the data part [23] are significantly lower than \((18\ 4)\), and we will therefore ignore such contributions.

We note that the inclusive, total QED correction to \(a^{(\nu,\varphi)}\) is very small when compared to the contribution from the real photon emissions. From the above QED correction to the theory part, we can estimate the total QED correction to \(a^{(\nu,\varphi)}\) as \(a^{(\nu,\varphi)}\) \(\text{em}(s_0) = 1 + 6 \times 10^{-4}\) 7000 \((6 \times 10^{-4})\) 4. This is much smaller than the contribution from the real photon emissions, which is according to [2] about 90 units from the photonic decay channels of mesons, e.g., \(\pi, \rho, \omega\), and \(\phi\). These two contributions may appear incompatible, but there is no real contradiction. It can be seen most easily in the narrow width approximation (n.w.a.) of the resonances. The n.w.a. formulas for the portions of \(R_{\text{e}^+\text{e}^-}\) in (1) from the various decay channels of a resonance contain as factors the corresponding branching ratios, whose sum is always one. Thus, an increase in the photonic decay channel contributions must be compensated by the decrease in the pure hadronic channel contributions, by exactly the same amount. The contributing QED correction factor in the Adler function in Eq. (29) thus accounts for the QED corrections in the continuum only. Beyond the n.w.a. approximation, however, the cancellation between the photonic channel contributions and the pure hadronic channel contributions cannot be exact, since the decay widths of the resonances are sizable, and so a small QED correction survives after the cancellation. This remaining QED correction, along with the QED corrections in the continuum, are those accounted for by the QED correction factor in the Adler function in Eq. (29).
We now include in our theory the aforementioned QED contributions \( \text{(26)} \) \( \text{(27)} \).

\[
10^{11} \ a^{(pQCD)}(s \ s_0; \text{part};\text{meth: [19]}) = 4826 \ 77 \ \text{for} \ \text{hagG i value Eq: [18]} \quad (31)
= 4792 \ 71 \ \text{for} \ \text{hagG i value Eq: [25]} \ : \quad (32)
\]

This implies for the sum of the \( \text{theory} \) \( \text{(31)} \) \( \text{(32)} \) and the \( \text{data} \) \( \text{(28)} \) parts

\[
10^{11} \ a^{(pQCD)}(s \ s_0) = 6448 \ 95 \ \text{for} \ \text{hagG i value Eq: [18]} \quad (33)
= 6414 \ 90 \ \text{for} \ \text{hagG i value Eq: [25]} \ : \quad (34)
\]

The authors of Ref. \[8\] obtained for this quantity the value 6343 \ 60, by almost excluding their \( \text{theory} \) part (complete exclusion of the \( \text{theory} \) part gave them 6350 \ 74).

If we decreased parameter \( C_1 \) from 0.007 GeV \(^2\) toward zero, our central values in \( \text{(33)} \) \( \text{(34)} \) would go down toward the pure \( \text{data} \) value 6351, i.e., by 97 and 63 units, respectively. We will comment on this point in the next Section.

If we add to the obtained quantities \( \text{(33)} \) \( \text{(34)} \) the hadronic vacuum polarization parts from the region \( s > s_0 \) as obtained in Ref. \[8\] we obtain

\[
10^{11} \ a^{(pQCD)} = 7029 \ 96 \ \text{for} \ \text{hagG i value Eq: [18]} \quad (35)
= 6995 \ 91 \ \text{for} \ \text{hagG i value Eq: [25]} \ : \quad (36)
\]

The result obtained in the analysis of Ref. \[8\] is

\[
10^{11} \ a^{(pQCD)} = 6924 \ 62 \ \text{[H: [8]]} \ : \quad (37)
\]

While the bulk of the result \( \text{(37)} \) of Ref. \[8\] was obtained by taking into account the data on \( e^+ e^- \) and the \( \text{decays} \), the bulk of the result \( \text{(33)} \) \( \text{(34)} \) is obtained here by careful resummation of the contour integral of \( \text{(4)} \) where we account for the renormalization structure of the associated Adler function and for the \( d > 0 \) terms.

IV. COMPARISONS

How do our results compare with recent results of others on \( a^{(pQCD)} \)?

\(^5\) They obtained 581 \ 15 for this contribution, using for the pQCD parts apparently the value \( \text{(19)} \) for \( s \). The authors of Ref. \[23\] obtained slightly higher values 584 \ 9, once we subtract from their values the (pQCD) contributions from \( (3 \ \text{GeV}^2 < s < s_0) \) \( (s_0 = 18^2 \ \text{GeV}^2) \). They used lower values for \( s : \ s(m^2) = 0.0308 \ 0.0245 \). This small discrepancy would apparently increase once we adjusted \( s \) to the same value, say \( [19] \), for the two approaches of Refs. \[8, 23\] \). Then the prediction of Ref. \[23\] would be about 592 \ 10. We decide to take the value of Ref. \[8\]: 581 \ 15. We can reproduce their pQCD (parts with the simple TPS approach for \( R_{ee}(s) \)). When we use, instead, the approaches that account for the renormal structure of \( R_{ee}(s) \), and/or we use the value \( [23] \) instead of \( [19] \) for \( s(m^2) \), the values of these contributions change insignificantly. QED corrections are negligible.
In Table I, we show the values of $a^{(\nu \phi)}$ as predicted by others, along with our values. We see that our values are consistent with the values obtained by most of the other authors. We recall the fact that the bulk of our result is obtained by the pQCD + renormalon calculation, while the bulk of the results of the others was obtained by data integration. We observe that the consistency with the results of the others is even slightly stronger when we take the value of the gluon condensate of Eq. (25), i.e., the small gluon condensate value obtained by the ALEPH analysis [20] and suggested also by the renormalon formalism of [22]. Several of the entries in the Table are based on inclusion of the decay data.

How do our results (35) (36) compare with the experimental predictions for $a^{(\nu \phi)}$? This question remains somewhat unclear due to theoretical uncertainties of several higher order hadronic contributions, as argued by the authors of Ref. [23]. The largest theoretical uncertainty is in the calculation of the hadronic light (by light (LL)) contributions. The chiral model (ch.m.) approaches would predict $10^{11} a^{(\text{LL})} = 86 \pm 25$ [22]; the quark constituent model (qcm.) would predict $92 \pm 20$ [23]. The quark constituent model is valid for large values of the virtual photon momentum only, so we will give results for the chiral model unless otherwise stated. The QED corrections of the type of Fig. 2 have already been included in [35] (36). In addition, there are other QED radiative corrections where the photon propagator does not have both ends attached to the hadronic blob (the "rest") $10^{11} a^{(\text{rad corr}; \text{rest})} = 101 \pm 6$ [33]. These two radiative corrections ($86 \pm 26$ and $101 \pm 6$) have to be added to the entries of Table I to obtain the total hadronic contribution $a^{(\text{had})}$. In our case, we add them to the hadronic vacuum polarization contribution [35] (36) (we add the uncertainties in quadrature), and obtain

$$10^{11} a^{(\text{had})} = 6842 \pm 100 \text{ for } h a G G \text{ value Eq. (18)} \quad (38)$$

$$= 6808 \pm 95 \text{ for } h a G G \text{ value Eq. (29)}.$$

If we took, instead, the quark constituent model result for the light (LL) contributions, we would obtain $7014 \pm 98$ and $6980 \pm 93$, respectively.

We can now add the results (35) (36) to the well known electroweak contribution (34] and references therein

$$10^{11} a^{(\text{EW})} = 116,584,858 \pm 5 \quad (40)$$

and obtain the predictions

$$10^{11} a^{(\text{predicted})} = 116,591,700 \pm 100 \text{ for } h a G G \text{ value Eq. (18)} \quad (41)$$

$$= 116,591,666 \pm 95 \text{ for } h a G G \text{ value Eq. (29)}.$$

The actual experimental number [1], when averaged with the older measurements [34], is

6 We did not include in the Table the results of some earlier analyses [23].

7 The entries of Table I are based on the use of the dispersion relation [1]. The latter has been questioned in Ref. [29], because there is no proof that the photon propagator is at least polynomially bounded. The inclusion of the decay data for calculation of $a^{(\nu \phi)}$ has been questioned in Ref. [30], and the difficulties connected with this inclusion have been investigated in Ref. [31].
\[10^{11} \ a \ (\text{experiment; averaged}) = 116 \ 592 \ 030 \ 152 : \quad (43)\]

The predictions \((41)\), \((42)\) thus differ from the experimental result by \(330 \ 182 \ (1.81)\) and \(364 \ 179 \ (2.03)\), respectively, where \(= 182; 179\) is obtained by adding in quadrature the experimental \((\text{exp.} = 152)\) and the theoretical uncertainty \((\text{th.} = 100; 95)\). If we took, instead of the chiral model result, the quark constituent model result for the light-by-light contributions, the corresponding deviations would be \(152 \ 181 \ (0.84)\) and \(186 \ 178 \ (1.04)\), respectively.

If we take only the newest experimental number \((41)\)

\[10^{11} \ a \ (\text{experiment; new}) = 116 \ 592 \ 020 \ 152 ; \quad (44)\]

then the predictions \((41)\), \((42)\) differ from it by \(320 \ 182 \ (1.76)\) and \(354 \ 179 \ (1.98)\), respectively. These deviations would be \(0.78\) and \(0.99\), respectively, if we used the quark constituent model results for the light-by-light contributions.

V. SUMMARY

We obtain clear deviations of the theoretical results from the experimental ones when we use the chiral approach results for the light-by-light contributions. However, these deviations nonetheless are significantly smaller than the \(430 \ 165 \ (2.6)\) difference \((40)\) that has been suggested as the appearance of new physics. The deviations of our predictions from the new experimental values \((43), \ (44)\), range from \(320 \ 182 \ (1.76)\) to \(364 \ 179 \ (2.03)\), depending mostly on the taken values of the gluon condensate. Our results are consistent with those of most of the other authors who used data integration for the bulk of their results. In contrast, the bulk of our results was obtained by pQCD + renormal calculation.

A major contribution to the aforementioned reduction of the deviations are our values for the hadronic vacuum polarization contributions (CLS1 and CLS2 in Table \(\text{I}\)), which are significantly higher than those of Davier and Hocker \(\text{DH}\) in Table \(\text{I}\). Our values originate from resummation of a major part of this contribution via a contour integral in the complex energy plane and accounting for the renormalon structure of the associated Adler function in the integrand. The accounting for the renormalon structure contributes about 50 units to \(10^{11} \ a^{\nu P 0}\), and this is larger than the expected future experimental uncertainties.

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8 The decrease of \(C_1\) to zero would contribute, in quadrature, additional 97 and 63 units to the uncertainties, respectively (as mentioned in the previous Section. In such a case, the predictions would deviate from the experimental result by 330 \ 206\) and \(364 \ 190\), respectively.

9 There, \(165\) was obtained by adding in quadrature \(\text{exp.} \quad 150\) and \(\text{th.} \quad 67\), where the value of \(\text{th.}\) was taken from Ref. \(\text{I}\).
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Fig. 1. Application of the Cauchy theorem to the depicted integration contour in the $Q^2$ plane leads to the contour integral in Eq. (4).
**FIG. 2.** QED correction to the hadronic $\mu\mu$.

| authors | $10^{11}$ | $a^{(\mu\mu)}$ | method                        |
|---------|-----------|-----------------|-------------------------------|
| ADH [24] | 7011 94  | $e^+e$ + data   |                               |
| BW [25]  | 7026 160 | $e^+e$ data     |                               |
| J [24]   | 6974 105 | mostly $e^+e$ data |                           |
| N1 [27]  | 7031 77  | $e^+e$ + data   |                               |
| N2 [27]  | 7011 117 | $e^+e$ data     |                               |
| TY1 [23] | 7002 65  | $e^+e$ + spacialF (t) data |                     |
| TY2 [23] | 6982 97  | $e^+e$ + spacialF (t) data |                     |
| DH [8]    | 6924 62  | mostly $e^+e$ + data |                           |
| CLS1      | 7029 96  | mostly theory, and [18] value |                  |
| CLS2      | 6995 91  | mostly theory, and [25] value |                  |

**TABLE I.** Comparison of predictions of the hadronic vacuum polarization contributions to $a$ by various authors.