Convergence and path divergence sets for bounded analytic functions in the disk.

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September 29, 2016

Abstract
Let $f : \mathbb{D} \to \mathbb{C}$ be a bounded analytic function, such that $f$ has a non-tangential limit at 1. A set $K \subset \mathbb{D}$ which contains 1 in its closure is called a convergence set for $f$ at 1 if $f(z)$ converges to some value $\zeta$ as $z \to 1$ with $z$ in $K$. $K$ is called a path divergence set for $f$ at 1 if $f$ diverges along every path $\gamma$ which lies in $K$ and approaches 1. In this paper, we show that if $\gamma_1$ and $\gamma_2$ are paths in the disk approaching 1, and $f$ converges along $\gamma_1$ and $\gamma_2$, then the region between $\gamma_1$ and $\gamma_2$ is a convergence set for $f$ at 1. On the other hand, if $\gamma_3$ is any path in the disk approaching 1 along which $f$ diverges, then either the region above $\gamma_3$ or the region below $\gamma_3$ is a path divergence set for $f$ at 1. We conclude the paper with an examination of the convergence and path divergence sets for the function $e^{\frac{z+1}{z-1}}$ at 1.

1 Introduction
Let $f : \mathbb{D} \to \mathbb{C}$ be analytic and bounded. It is well known (see for example Theorem 5.2 in [2]) that the non-tangential limit of $f$ exists at almost every point in $\partial \mathbb{D}$. The sectorial limit theorem (see for example Theorem 5.4 in [2]) states that if $f$ has a limit $\zeta$ along any path $\gamma$ in $\mathbb{D}$ which approaches some point $w \in \partial \mathbb{D}$, then the non-tangential limit of $f$ at $w$ exists and equals $\zeta$. This paper is concerned with local convergence properties of $f$, so throughout we will assume that the point $w$ being approached is just the point 1. We will also assume for the sake of convenience that unless otherwise specified every path $\gamma$ mentioned in this paper is a path in $\mathbb{D}$ from $-1$ to 1. That is, $\gamma : [0, 1] \to \mathbb{C}$ with $\gamma(0) = -1$, $\gamma(1) = 1$, and for all $s \in (0, 1)$, $\gamma(s) \in \mathbb{D}$. We will also use the symbol “$\gamma$” at times to denote the trace of $\gamma$ as a subset of the plane.

To say that $f$ has the non-tangential limit $\zeta$ at 1 is to say that for any Stolz region $S$ with vertex at 1,

$$\lim_{z \to 1, z \in S} f(z) = \zeta.$$
In this paper we generalize the notion of convergence or divergence in a Stolz region as follows.

**Definition** Let $K \subset \mathbb{D}$ be a set, and assume that $1 \in \partial K$.

- If \( \lim_{z \to 1, z \in K} f(z) \) exists, then we call $K$ a **convergence set** for $f$ at 1.
- If $\lim_{n \to \infty} f(z_n)$ does not converge for every sequence $\{z_n\} \subset K$, then we call $K$ a **divergence set** for $f$ at 1.
- If $\lim_{s \to 1^-} f(\gamma(s))$ exists for every path $\gamma \subset K$, then we call $K$ a **path convergence set** for $f$ at 1.
- If $\lim_{s \to 1^-} f(\gamma(s))$ does not exist for every path $\gamma \subset K$, then we call $K$ a **path divergence set** for $f$ at 1.

Although we made the definition above for a divergence set for $f$ at 1 for the sake of symmetry in the definitions, we observe that since $f$ is bounded, compactness considerations immediately imply that $f$ does not have any divergence sets at 1 (or anywhere on the boundary of $\mathbb{D}$).

We can now restate the sectorial limit theorem by saying that if there is a path $\gamma$ along which $f$ has a limit, then every Stolz region with vertex at 1 is a convergence set for $f$ at 1. In Section 2, we will prove the following theorem which guarantees the existence of convergence and path divergence sets for $f$ at 1 in relation to paths along which $f$ converges or diverges respectively. We must first make a definition.

**Definition** Let $\gamma_1$, $\gamma_2$, and $\gamma_3$ be paths.

- By the **region between $\gamma_1$ and $\gamma_2$** we mean the set of all points $z \in \mathbb{D}$ such that $z$ is in $\gamma_1$, $z$ is in $\gamma_2$, or $z$ is in a bounded component of $(\gamma_1 \cup \gamma_2)^c$.
- The component of $\mathbb{D} \setminus \gamma_3$ which contains $i$ in its boundary is called the **region above $\gamma_3$**, and the component of $\mathbb{D} \setminus \gamma_3$ which contains $-i$ in its boundary is called the **region below $\gamma_3$**.

**Theorem 1.1.** Let $\gamma_1$, $\gamma_2$, and $\gamma_3$ be paths such that the limit of $f$ exists along $\gamma_1$ and along $\gamma_2$, and does not exist along $\gamma_3$. Then the following hold.

- The region between $\gamma_1$ and $\gamma_2$ is a convergence set for $f$ at 1.
- Either the region above $\gamma_3$ or the region below $\gamma_3$ is a path divergence set for $f$ at 1.

Note that the first item in the above theorem immediately implies the second item.

In Section 3, we will examine the convergence and path convergence/divergence sets for the function $g(z) = e^{\frac{z}{1+z}}$ at 1. Although no open disk $U \subset \mathbb{D}$ which is tangent to the unit circle at 1 can be a convergence set for $g$ at 1, we will find a convergence set for $g$ at 1 with smooth boundary close to 1 that is tangent to the unit circle at 1.
2 Main Results

In our proof of Theorem 1.1, we will use a result having to do with the cluster sets of a disk function, so we make the following definition.

Definition Let $h : \mathbb{D} \to \mathbb{C}$ be a meromorphic function such that $h$ extends continuously to each point in $\partial \mathbb{D}$ in a neighborhood of 1 except possibly at 1.

- The cluster set of $h$ at 1, denoted $\mathcal{C}(h,1)$, is the set of all values $\zeta \in \hat{\mathbb{C}}$ such that there is some sequence of points $\{z_n\}$ contained in the disk such that $z_n \to 1$ and $h(z_n) \to \zeta$.

- The boundary cluster set of $h$ at 1, denoted $\mathcal{C}_B(h,1)$, is the set of all values $\zeta \in \hat{\mathbb{C}}$ such that there is some sequence of points $\{z_n\}$ contained in the unit circle such that $z_n \to 1$ and $h(z_n) \to \zeta$.

We will use the following fact regarding these cluster sets (which appears as Theorem 5.2 in [1]), which says that the boundary of the cluster set for a meromorphic disk function $h$ at 1 is contained in the boundary cluster set of $h$ at 1.

Fact 2.1. Let $h : \mathbb{D} \to \hat{\mathbb{C}}$ be a meromorphic function such that $h$ extends continuously to each point in $\partial \mathbb{D}$ in a neighborhood of 1 except possibly at 1. Then

$$\partial \mathcal{C}(h,1) \subset \mathcal{C}_B(h,1).$$

We continue with several lemmas.

Lemma 2.2. Let $\gamma$ be a path. Then $\lim_{z \in \gamma, z \to 1} \gamma^{-1}(z) = 1$, in the sense that as $z \in \gamma$ approaches 1, $\max(1 - s : s \in \gamma^{-1}(z)) \to 0$.

Proof. This follows from basic compactness and continuity properties. \hfill \square

Lemma 2.3. Let $E_1$ and $E_2$ be disjoint circles in $\mathbb{C}$, and let $\gamma : [0,1] \to \mathbb{C}$ be any path. Then only finitely many of the components of $\gamma^c$ can intersect both $E_1$ and $E_2$.

Proof. For $r > 0$, define $C_r$ to be the circle centered at the origin, with radius $r$. By first applying a homeomorphism to the sphere, we may assume without loss of generality that $E_1 = C_1$ and $E_2 = C_2$.

Suppose by way of contradiction that there are infinitely many components of $\gamma^c$ which intersect both $C_1$ and $C_2$. Since each component of $\gamma^c$ is open, there are only countably many of them. Let $\{F_j\}_{j=0}^{\infty}$ be an enumeration of the components of $\gamma^c$ which intersect both $C_1$ and $C_2$. Since each $F_j$ is open and connected in $\mathbb{C}$, it is also path connected. Therefore for each $j$, we may define a path $\varphi_j : [0,1] \to F_j$ such that $\varphi_j(0) \in C_1$ and $\varphi_j(1) \in C_2$. By restricting $\varphi_j$ if necessary, we may also assume that for each $t \in (0,1)$, $|\varphi_j(t)| \in (1,2)$. Define $a_j = \varphi_j(0)$ and $c_j = \varphi_j(1)$. Define $r_j \in (0,1)$ to be the smallest number such that $|\varphi_j(r_j)| = 1.5$, and define $b_j = \varphi_j(r_j)$.

It is immediate from their definition that, for any $j,k \in \mathbb{N}$, if $j \neq k$, then $\varphi_j$ and $\varphi_k$ are disjoint from each other. Since they do not cross in the region $\{z : 1 < |z| < 2\}$, it follows that the orientation of the $a_j$ around $C_1$ is the same as the orientation of the
$c_j$ around $C_2$ and the $b_j$ around $C_{1.5}$. That is, for any distinct $j, k, l \in \mathbb{N}$, if $a_j, a_k, a_l$ is the order in which those points appear when $C_1$ is traversed with positive orientation, then $c_j, c_k, c_l$ is the order in which those points appear when $C_2$ is traversed with positive orientation, and $b_j, b_k, b_l$ is the order in which those points appear when $C_{1.5}$ is traversed with positive orientation.

Since $\gamma$ is continuous on the compact set $[0, 1]$, it is uniformly continuous, therefore we may choose a $\delta > 0$ such that for any $s, t \in [0, 1]$, if $|\gamma(s) - \gamma(t)| > 1/4$, then $|s - t| > \delta$. Choose an $N > 0$ such that $N\delta > 1$. By reordering the elements described above, we may assume that the points $a_0, a_1, \ldots, a_N$ appear in precisely this order as $C_1$ is traversed with positive orientation. For the remainder of the proof, addition is done modulo $N + 1$.

For each $j \in \{0, 1, \ldots, N\}$, let $G_j$ denote the domain which is bounded by the paths $\varphi_j, \varphi_{j+1}$, the arc of the circle $C_1$ with end points $a_j$ and $a_{j+1}$ which does not contain $a_{j+2}$, and the arc of the circle $C_2$ with end points $c_j$ and $c_{j+1}$ which does not contain $c_{j+2}$. If $\gamma(1) \in G_j$ for some $0 \leq j \leq N - 1$, we just remove this region from our list, and shift all indices greater than $j$ down by 1. Thereby we may assume that $\gamma(1) \notin G_j$ for all $0 \leq j \leq N - 1$.

Since $b_j$ and $b_{j+1}$ are in different faces of $\gamma$, the arc of $C_{1.5}$ with end points $b_j$ and $b_{j+1}$ which does not contain $b_{j+2}$, must intersect $\gamma$ in some point $\alpha_j \in G_j$. Define $s_j \in (0, 1)$ to be the smallest number such that $\gamma(s_j) = \alpha_j$. Since $\gamma(1) \notin G_j$, we may define $t_j \in (s_j, 1)$ to be the smallest number such that $\gamma(t_j) \in \partial G_j$. Define $\beta_j = \gamma(t_j)$.

The point $\beta_j \in \partial G_j$ must be in either $C_1$ or $C_2$, since the portion of $\partial G_j$ which is not in either $C_1$ or $C_2$ (namely $\varphi_j$ and $\varphi_{j+1}$) is contained in $\gamma^c$. Since $|\alpha_j| = 1.5$ and $|\beta_j| = 1$ or 2, it follows that $|\gamma(s_j) - \gamma(t_j)| = |\alpha_j - \beta_j| > 1/4$, and thus $t_j - s_j > \delta$. Note that for any distinct $j, k \in \{0, 2, \ldots, N - 1\}$, since $G_j$ and $G_k$ are disjoint, $\gamma([s_j, t_j])$ and $\gamma([s_k, t_k])$ are disjoint, and thus $[s_j, t_j]$ and $[s_k, t_k]$ are disjoint. We thus have that $\{[s_j, t_j]\}_{i=0}^{N-1}$ is a sequence of disjoint intervals in $[0, 1]$, giving us the contradiction

$$1 = \lambda([0, 1]) \geq \sum_{j=0}^{N-1} t_j - s_j \geq \sum_{i=0}^{N-1} \delta = N\delta > 1.$$

This concludes the proof.

\textbf{Lemma 2.4.} Let $\gamma : [0, 1] \to \mathbb{C}$ be a path, and let $F$ be a bounded component of $\gamma^c$. Then for any Riemann map $\tau : \mathbb{T} \to F$, $\tau$ extends to a homeomorphism $\tau : \text{cl}(\mathbb{D}) \to \text{cl}(F)$.

\textbf{Proof.} Let $E$ denote the union of every component of $\gamma^c$ other than $F$. Since $E$ is open, $E$ is locally connected. By the Hahn–Mazurkiewicz theorem (see for example Theorem 3-30 in [3]) since $\gamma$ is the continuous image of the unit interval, $\gamma$ is locally connected. Thus $\mathbb{C} \setminus F$ is the disjoint union of two locally connected sets, and is thus locally connected.

Now the Caratheodory–Torhorst theorem gives that any Riemann map $\tau : \mathbb{D} \to F$ extends to a homeomorphism from the closure of the unit disk to the closure of $F$. \hfill $\Box$

\textbf{Lemma 2.5.} Let $\gamma$ be a path. If $\lim_{s \to 1^-} f(\gamma(s)) = 0$, then $\lim_{z \to 1, z \in \gamma} f(z) = 0$.

\textbf{Proof.} Fix an $\epsilon > 0$, and choose a $\delta > 0$ small enough that for every $s \in (1 - \delta, 1)$, $|f(\gamma(s))| < \epsilon$. Define $\epsilon = \min_{s \in [0, 1 - \delta]} |f(\gamma(s) - 1)|$. Then for any $z \in \gamma$ with $|z - 1| < \epsilon$, we must have $\gamma^{-1}(z) \subset (1 - \delta, 1)$, and thus $|f(z)| < \epsilon$. We conclude that $\lim_{z \to 1, z \in \gamma} f(z) = 0$. \hfill $\Box$
Proof of Theorem 1.1. Let $\gamma_1$, $\gamma_2$, and $\gamma_3$ be paths (subject to the assumptions mentioned in the introduction, that each path lies in the unit disk and interpolates from $-1$ to 1.

Let $\{z_n\}_{n=1}^\infty$ be a sequence of points contained in the region between $\gamma_1$ and $\gamma_2$. The sectorial limit theorem immediately implies that the limit $\zeta$ of $f$ along $\gamma_1$ must equal the limit of $f$ along $\gamma_2$. Replacing $f$ with $f - \zeta$, we assume that this common limit is 0. We wish to show that $f(z_n) \to 0$ as $n \to \infty$. Partition the natural numbers two sets $A$ and $B$ by the rule that if $n \in A$ if $z_n$ is in either $\gamma_1$ or $\gamma_2$, and $n \in B$ if $z_n$ is contained in a bounded component of $(\gamma_1 \cup \gamma_2)^c$. It is immediately clear from Lemma 2.2 that $\lim_{n \to \infty, n \in A} f(z_n) = 0$.

It remains to show that $\lim_{n \to \infty, n \in B} f(z_n) = 0$.

Fix an $\epsilon > 0$. Choose an $\iota_1 > 0$ such that for all $s \in (1 - \iota_1, 1)$, $|f(\gamma_1(s))| < \epsilon$ and $|f(\gamma_2(s))| < \epsilon$. Define $\delta_1 = \min_{s \in [0,1-\iota_1]}(|\gamma_1(s) - 1|, |\gamma_2(s) - 1|)$. By Lemma 2.3 there are only finitely many faces of $(\gamma_1 \cup \gamma_2)^c$ that intersect both the domain $\{z : |z - 1| > \delta_1\}$ and the disk $\{z : |z - 1| < \delta_1/2\}$. Choose a $\delta_2 \in (0, \delta_1/2)$ small enough so that every bounded component of $(\gamma_1 \cup \gamma_2)^c$ which intersects the disk $\{z : |z - 1| < \delta_2\}$ is either contained in the disk $\{z : |z - 1| < \delta_1\}$, or contains 1 in its boundary.

Let $F$ be some component of $(\gamma_1 \cup \gamma_2)^c$ which intersects $\{z : |z - 1| < \delta_2\}$.

**Case 2.5.1. $F$ is not contained in the disk $\{z : |z - 1| < \delta_1\}$.**

Let $\tau : \mathbb{D} \to F$ be some Riemann map for $F$. Lemma 2.4 implies that $\tau$ extends to a homeomorphism $\tau : \partial \mathbb{D} \to \partial F$. We adopt the normalization $\tau(1) = 1$. Since $f$ is analytic on $\partial F \setminus \{1\}$, it thus follows that $f \circ \tau$ extends continuously to every point on $\partial \mathbb{D}$ except possibly to 1. Moreover, as $\theta \to 0$, $\tau(e^{i\theta})$ approaches 1 in $\partial F$ (which is in turn contained in $\gamma_1 \cup \gamma_2$), so that $f \circ \tau(e^{i\theta})$ approaches 0 (by Lemma 2.5).

Therefore $C_B(f \circ \tau, 1)$ consists of the single point 0 only. Since $f \circ \tau$ is bounded in the disk, compactness considerations imply that $C(f \circ \tau, 1)$ is non-empty. Fact 2.1 now immediately implies that $C(f \circ \tau, 1) = \{0\}$, and thus that $f \circ \tau(z) \to 0$ as $z \to 0$ in the disk. Finally we conclude that $f(z) \to 0$ as $z \to 1$ in $F$.

**Case 2.5.2. $F$ is contained in the disk $\{z : |z - 1| < \delta_1\}$.**

By choice of $\delta_1$, for all $z \in \partial F \cap \mathbb{D}$, $|f(z)| < \epsilon$. Compactness and continuity considerations show that if $\partial F \cap \partial \mathbb{D} \neq \emptyset$, then $\partial F \cap \partial \mathbb{D} = \{1\}$, and work in the previous case shows that $\lim_{z \to 1, z \in F} f(z) = 0$. Thus we have that for all $w \in \partial F$, either $|f(w)| < \epsilon$ or $\lim_{z \to w, z \in F} f(z) = 0$. The maximum modulus principle now implies that for all $z \in F$, $|f(z)| < \epsilon$.

Since $f(z)$ converges to 0 as $z$ approaches 1 in each of the finitely many components of $(\gamma_1 \cup \gamma_2)^c$ which intersect both $B(1; \delta_1)^c$ and $B(1; \delta_2)$, and $|f(z)| < \epsilon$ for all $z$ in the components of $(\gamma_1 \cup \gamma_2)^c$ which are contained entirely in $B(1; \delta_1)$, we can choose an $\delta_3 \in (0, \delta_2)$ small enough so that for all $z$ contained in a bounded face of $(\gamma_1 \cup \gamma_2)^c$, if $|z - 1| < \delta_3$, then $|f(z)| < \epsilon$.

Now choose a $M \in \mathbb{N}$ large enough so that for all $n > M$, $|z_n - 1| < \delta_3$. If $n > M$ is in $B$, then $|f(z_n)| < \epsilon$. We conclude that $\lim_{n \to \infty, n \in B} f(z_n) = 0$, concluding the proof of the first item of the theorem.

In order to prove the second item, suppose by way of contradiction that neither the region above $\gamma_3$ nor the region below $\gamma_3$ is a domain of path divergence. Then there are
two paths $\psi_1$ and $\psi_2$ along which $f$ converges, such that $\psi_1$ lies above $\gamma_3$ and $\psi_2$ lies below $\gamma_3$. By the first item of the theorem, the region between $\psi_1$ and $\psi_2$ is a convergence set for $f$ at 1, but $\gamma_3$ lies in the region between $\psi_1$ and $\psi_2$, providing us with the desired contradiction.

\section{Convergence and Divergence Sets for $g(z) = e^{\frac{z+1}{z-1}}$}

In this section we will explore the convergence and path convergence/divergence sets of the function $g(z) = e^{\frac{z+1}{z-1}}$. We start with a definition.

\textbf{Definition}

- For any $-\infty \leq p < q \leq \infty$, define $H_{Re}(p, q) = \{ w \in \mathbb{C} : p < Re(w) < q \}$.
- For any $-\infty \leq p < q \leq \infty$, define $H_{Im}(p, q) = \{ w \in \mathbb{C} : p < Im(w) < q \}$.

Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ denote the M"{o}bius transformation $R(z) = \frac{z+1}{z-1}$. $R$ is a conformal map from the unit disk $\mathbb{D}$ to the half plane $H_{Re}(-\infty, 0)$. Moreover if $\{z_n\}$ is any sequence of points in $\mathbb{D}$, $z_n \to 1$ if and only if $R(z_n) \to \infty$ in $H_{Re}(-\infty, 0)$. Thus in order to study the convergence and path convergence/divergence sets of $g$ at 1, it suffices to study the convergence and path convergence/divergence sets of the function $h : H_{Re}(-\infty, 0) \to \mathbb{D}$ defined by $h(w) = e^w$, as $w \to \infty$ in $H_{Re}(-\infty, 0)$, and we will treat the two settings as interchangable in what follows.

It is easy to see that as $w \to \infty$ in $H_{Re}(-\infty, 0)$ in the real line, $h(w) \to 0$. Therefore by the sectorial limit theorem, if $\psi$ is any path in $H_{Re}(-\infty, 0)$ approaching $\infty$, and $h$ converges along $\psi$, then $h$ must converge to 0 along $\psi$.

Note that for $w = x + iy \in H_{Re}(-\infty, 0)$, $|h(w)| = e^x$. Therefore in the following subsections we will use the fact that $h(w) \to 0$ in $H_{Re}(-\infty, 0)$ if any only if $x \to -\infty$.

\subsection{Convergence Sets for $g$ at 1}

While it is possible to construct a convergence set for $h$ at $\infty$ in which $h$ approaches any fixed value in the disk, we are interested in large sets in $H_{Re}(-\infty, 0)$ in which $h$ approaches 0. A set $G \subset H_{Re}(-\infty, 0)$ is a convergence set for $h$ at $\infty$ in which $h$ approaches 0 (equivalently $K = R^{-1}(G)$ is a convergence set for $g$ at 1 in which $g$ approaches 0) if and only if for every sequence of points $\{w_n = x_n + iy_n\}$ contained in $G$, if $w_n \to \infty$, then $x_n \to -\infty$.

Therefore $G$ is a convergence set for $h$ at $\infty$ in which $h$ approaches 0 if and only if for every number $p \in (-\infty, 0)$, the set $G \cap H_{Re}(p, 0)$ is bounded. Since the region to the left of a verticle line in $H_{Re}(-\infty, 0)$ is mapped by $\tau^{-1}$ to the interior of a circle contained in the unit disk which is tangent to the unit circle at 1, we may translate the above observation to the disk as follows.

\textbf{Proposition 3.1.} Let $K \subset \mathbb{D}$ be a set with $1 \in \partial K$. $K$ is a convergence set for $g$ at 1 in which $g$ approaches 0 if and only for every disk $D \subset \mathbb{D}$ which is tangent to the unit circle at 1, there is an $\epsilon > 0$ such that $K \cap B(1; \epsilon) \subset D$. 


3.2 Path Convergence Sets for \( g \) at 1

It is of course easier for a set to be a path convergence set for \( h \) than it is to be a convergence set. Consider the following example. For any \( n \in \mathbb{N} \), let \( E_n \) be defined by

\[
E_n = \left\{ -\frac{1}{n} + iy : 0 \leq y \leq n \right\} \cup \left\{ x + in : -\infty < x \leq -\frac{1}{n} \right\}.
\]

Define \( E \) to be the union of the set \( \bigcup_{n=1}^{\infty} E_n \) with the interval in the real line \([-1, 0)\), and let \( G \) be a small neighborhood of \( E \). \( G \) is an open, unbounded, simply connected set, and any path \( \psi \) in \( G \) which approaches \( \infty \) must have bounded imaginary part, and thus \( g \) must converge to 0 along \( \psi \).

Thus \( G \) is a path convergence set for \( h \) at \( \infty \). However, for any \( p \in (-\infty, 0) \), \( H_{Re}(p, 0) \cap G \) is unbounded, so \( G \) is not a convergence set for \( h \) at \( \infty \). There does not appear to be a concise analytic characterization of the path convergence sets for \( g \) at 1.

3.3 Path Divergence Sets for \( g \) at 1

As mentioned above, for a path \( \psi \) approaching \( \infty \) in \( H_{Re}(-\infty, 0) \), by the sectorial limit theorem if \( h \) converges along \( \psi \), then \( h \) converges to 0 along \( \psi \). Therefore for a given path \( \psi \) approaching \( \infty \) in \( H_{Re}(-\infty, 0) \), \( h \) will not converge along \( \psi \) if and only if for some fixed \( p \in (-\infty, 0) \), \( \psi(s) \) revisits \( H_{Re}(p, 0) \) infinitely often as \( s \to 1^- \). Certainly for any \( p \in (-\infty, 0) \), \( H_{Re}(p, 0) \) itself will be a path divergence set for \( h \) at \( \infty \) (and thus given any disk \( D \) contained in the unit disk which is tangent to the unit circle at 1, \( D \setminus D \) is a path divergence set for \( g \) at 1). However as in the previous subsection, we will give a more interesting example. Let \( E \) denote the piecewise linear path in \( H_{Re}(-\infty, 0) \) obtained by concatenating the line segments with the following vertices listed in order:

\[-1 + i, -2 + 2i, -1 + 2i, -3 + 3i, -1 + 3i, -4 + 4i, -1 + 4i, \ldots \]

Let \( G \) be a small neighborhood of \( E \) in \( H_{Re}(-\infty, 0) \). \( G \) is an open, unbounded, simply connected set, and any path \( \psi \) in \( G \) which approaches \( \infty \) must revisit the vertical line \( \{ x + iy \in \mathbb{C} : x = 2 \} \) infinitely often, and thus \( g \) does not converge along \( \psi \).

3.4 Convergence Set for \( g \) at 1 with Vertical Tangent Line at 1

We will finish by considering a specific convergence set for \( g \) at 1. First let us translate the problem to the upper half plane \( H_{Im}(0, \infty) \) via the Möbius transformation \( S : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) defined by \( S(z) = -iz + i \).

\( S \) will transform any disk contained in \( \mathbb{D} \) which is tangent to the unit circle at 1 to a disk contained in \( H_{Im}(0, \infty) \) which is tangent to the real line at 0. Let \( G \subset H_{Im}(0, \infty) \) be defined by

\[
G = \left\{ x + iy : y > |x|^\frac{3}{2} \right\}.
\]

\( G \) is the region above the graph \( y = T(x) = |x|^\frac{3}{2} \). \( T(0) = 0 \) and \( T'(0) = 0 \), so \( y = T(x) \) has the real line as its tangent line at \( x = 0 \). An easy calculation shows that the curvature
of $y = T(x)$ at $x = 0$ is $\infty$, and that for any $s > 0$, the half circle $y = s - \sqrt{s^2 - x^2}$ lies below $y = T(x)$ in a neighborhood of $x = 0$.

Figure 1: The convergence set $K$.

Therefore defining $K = S^{-1}(G)$ (depicted in Figure 1), we now have that $K$ is tangent to the unit circle at 1, and for any circle $C$ in the disk which is tangent to the unit circle at 1, the restriction of $K$ to some small neighborhood of 1 is contained in the bounded face of $C$. By Proposition 3.1, $K$ is a domain of convergence for $g$ at 1 in which $g$ approaches 0.

References

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