JACOB’S LADDER AS GENERATOR OF NEW CLASS OF ITERATED $L_2$-ORTHOGONAL SYSTEMS AND THEIR DEPENDENCE ON THE RIEMANN’S FUNCTION $\zeta\left(\frac{1}{2} + it\right)$

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Abstract. In this paper new classes of $L_2$-orthogonal functions are constructed as iterated $L_2$-orthogonal systems. In order to do this we use the theory of the Riemann’s zeta-function as well as our theory of Jacob’s ladders. The main result is new one in the theory of the Riemann’s zeta-function and simultaneously in the theory of $L_2$-orthogonal systems.

DEDICATED TO THE MEMORY OF FOURIER’S EGYPTIAN ANABASE

1. Main result

1.1. Let us remind the following notions:

(a) Jacob’s ladder $\varphi_1(t)$,

(b) the function

$$\tilde{Z}^2(t) = \frac{d\varphi_1(t)}{dt} = \frac{1}{\omega(t)} \left| \zeta\left(\frac{1}{2} + it\right) \right|^2,$$

$$\omega(t) = \left\{ 1 + O\left(\frac{\ln \ln t}{\ln t}\right) \right\} \ln t, \quad t \to \infty,$$

(1.1)

(c) the (direct) iterations of the Jacob’s ladder

$$\varphi_1^0(t) = t, \quad \varphi_1^1(t) = \varphi_1(t), \quad \varphi_1^2(t) = \varphi_1(\varphi_1(t)), \ldots, \varphi_1^k(t) = \varphi_1(\varphi_1^{k-1}(t))$$

for every fixed $k \in \mathbb{N}$,

(d) the reverse iterations (by means of $\varphi_1(t)$)

$$U = o\left(\frac{T}{\ln T}\right), \quad T \to \infty,$$

of the basic segment

$$[T, T + U] = [T, T + U], \quad [T, T + U], \ldots, [T, T + U],$$

where

(1.2)

$$[0, T + U] \prec \left[\frac{1}{T}, T + U\right] \prec \cdots \prec \left[\frac{k}{T}, T + U\right],$$

that we have introduced into the theory of the Riemann’s zeta-function in our papers [1] – [4].

Key words and phrases. Riemann zeta-function.
1.2. The following theorem is the main result of this paper.

**Theorem 1.** For every fixed $L_2$-orthogonal system
\[
\{f_n(t)\}_{n=0}^{\infty}, \ t \in [a, a + 2l], \ a \in \mathbb{R}, \ l \in \mathbb{R}^+
\]
and for every fixed $k \in \mathbb{N}$ there is the set of $k$ new iterated $L_2$-orthogonal systems
\[
\{f_n^p(t)\}_{n=0}^{\infty}, \ t \in [a, a + 2l], \ p = 1, 2, \ldots, k,
\]
where
\[
f_n^p(t) = f_n\left(\varphi_1^p\left(\frac{p}{T + 2l - T}(t - a) + \frac{p}{T}\right) - T + a\right) \times
\left|\prod_{r=0}^{p-1} \tilde{Z}\left(\varphi_1^p\left(\frac{p}{T + 2l - T}(t - a) + \frac{p}{T}\right)\right)\right|,
\]
for all sufficiently big $T > 0$.

1.3. For example, by Theorem 1, we can assign the following set of $k$ new iterated orthogonal systems
\[
P_n^p(t) = P_n\left(\varphi_1^p\left(\frac{1}{T + 2l - T}(t + 1) + \frac{1}{T}\right) - T - 1\right) \times
\left|\prod_{r=0}^{p-1} \tilde{Z}\left(\varphi_1^p\left(\frac{1}{T + 2l - T}(t + 1) + \frac{1}{T}\right)\right)\right|,
\]
for all sufficiently big $T$ to the classical Legendre’s orthogonal system
\[
\{P_n(t)\}_{n=0}^{\infty}, \ t \in [-1, 1]; \ a = -1, \ l = 1.
\]

For example
\[
P_n^1(t) = P_n\left(\varphi_1\left(\frac{1}{T + 2l - T}(t + 1) + \frac{1}{T}\right) - T - 1\right) \times
\left|\tilde{Z}\left(\varphi_1\left(\frac{1}{T + 2l - T}(t + 1) + \frac{1}{T}\right)\right)\right|, \ t \in [-1, 1].
\]

1.4. Restating Theorem 1 we have the following

**Corollary 1.** For $L_2$-orthogonal system \[1.3\] there is the set of $k$ new iterated $L_2$-orthogonal systems
\[
\left\{f_n^p\left(\varphi_1^p\left(\frac{p}{T + 2l - T}(t - a) + \frac{p}{T}\right) - T + a\right)\right\}_{n=0}^{\infty},
\]
$t \in [a, a + 2l], \ p = 1, \ldots, k,$
with weights
\[ \prod_{r=0}^{p-1} \tilde{Z} \left( \varphi_1 \left( \frac{T + 2l - T}{2l} (t + 1) + \frac{T}{T} \right) \right) \]
and this last is (see (1.1))
\[ \sim \frac{1}{\ln^p T} \prod_{r=0}^{p-1} \tilde{Z}^2 \left( \varphi_1 \left( \frac{T + 2l - T}{2l} (t - a) + \frac{T}{T} \right) \right), \quad T \to \infty. \]

1.5. Now we give some remarks.

Remark 1. Theorem 1 represents completely new result in the theory of the Riemann’s zeta-function and simultaneously in the theory of \( L^2 \)-orthogonal systems.

Remark 2. The last row for all sufficiently big \( T > 0 \) in the Theorem 1 gives the continuum set of possibilities for construction of new \( k \)-tuples of iterated \( L^2 \)-orthogonal systems for every fixed system (1.3).

Remark 3. Dependence of iterated \( L^2 \)-orthogonal systems (1.4) on the Riemann’s zeta-function \( \zeta \left( \frac{1}{2} + it \right) \) is evident one, see (1.1), (1.5).

Remark 4. This paper is the continuation of 54 papers concerning Jacob’s ladders. These can be found in arXiv [math.CA] starting with the paper [1].

2. Jacob’s ladders

2.1. Let us remind the following non-linear integral equation
\[ \int_0^{\mu[Z(T)]} Z^2(t) e^{-\tilde{Z} \varphi(t)} dt = \int_0^T Z^2(t) dt \]
we have introduced in the paper [1], where
\[ Z(t) = e^{i\tilde{Z}(t)} \zeta \left( \frac{1}{2} + it \right), \]
\[ \tilde{Z}(t) = -\frac{t}{2} \ln \pi + \text{Im} \left\{ \ln \Gamma \left( \frac{1}{4} + i \frac{t}{2} \right) \right\}, \]
and the class of functions \( \{\mu\} \) specified as
\[ \mu \in C^\infty ([y_0, +\infty)) \]
monotonically increasing and unbounded from above and obeying the inequality
\[ \mu(y) \geq 7y \ln y. \]

2.2. The following statement holds (see [1]).

**Lemma 1.** For any \( \mu \in \{\mu\} \) there is exactly one solution to the integral equation (2.1):
\[ \varphi(T) = \varphi(T, \mu), \quad T \in [T_0, +\infty), \quad T_0 = T_0[\varphi] > 0, \]
\[ \varphi(T) \xrightarrow{T \to \infty} \infty. \]
Let the symbol \( \{ \varphi \} \) denote the system of these solutions. The function \( \varphi(T) \) is related to the zeroes of the Riemann’s zeta-function on the critical line by the following way. Let \( t = \gamma \) be such a zero of
\[
\zeta \left( \frac{1}{2} + it \right)
\]
and of the order \( n(\gamma) \), where
\[
n(\gamma) = \mathcal{O}(\ln \gamma), \quad \gamma \to \infty,
\]
of course. Then the following holds true.

**Remark 5.** The points \([\gamma, \varphi(\gamma)], \gamma > T_0\) (and only these points) are inflection points with the horizontal tangent. In more detail, the following system of equations holds true:
\[
\varphi'(\gamma) = \varphi''(\gamma) = \cdots = \varphi^{(2n)}(\gamma) = 0, \quad \varphi^{(2n+1)}(\gamma) \neq 0, \quad n = n(\gamma).
\]

**Definition 1.** With respect to the above mentioned property, an element \( \varphi \in \{ \varphi \} \) is called Jacob’s ladder leading to \([+\infty, +\infty]\). The rungs of that ladder are segments of the curve \( \varphi \) lying in the neighbourhoods of the points \([\gamma, \varphi(\gamma)], \gamma > T_0\).

**Remark 6.** We call \( \varphi(T) \) as Jacob’s ladder in analogy with Jacob’s dream in Chumash, Bereishis, 28:12.

**Remark 7.** Finally, the composite function \( g[\varphi(T)] \) is also called Jacob’s ladder for any function \( g \) that is increasing, \( C^\infty \) on \([y_0, +\infty)\) and unbounded from above. For example, the function
\[
(2.4) \quad \varphi_1(T) = \frac{1}{2} \varphi(T)
\]
as composition of \( g(y) = \frac{1}{2} y, \ y \geq y_0, \ y = \varphi(T), \ T \geq T_0[\varphi] = T_0[\varphi_1], \ g'_0 = \frac{1}{2} > 0 \) is the Jacob’s ladder.

3. Basic property of Jacob’s ladders: existence of almost exact expressions for the classical Hardy-Littlewood integral (1918)

3.1. Let us remind the Hardy-Littlewood integral
\[
(3.1) \quad \int_0^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt
\]
can be expressed as follows:
\[
(3.2) \quad \int_0^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt = T \ln T + (2c - 1 - \ln 2\pi)T + R(T),
\]
with, for example, Ingham’s error term
\[
(3.3) \quad R(T) = \mathcal{O}(T^{\frac{1}{2}} \ln T) = \mathcal{O}(T^{\frac{1}{2}+\delta}), \ \delta > 0, \ T \to \infty
\]
for arbitrarily small \( \delta \).

Next, by Good’s \( \Omega \)-theorem (1977) we have that
\[
(3.4) \quad R(T) = \Omega(T^{\frac{1}{2}}), \ T \to \infty.
\]
Remark 8. Let
\[ R_a(T) = \mathcal{O}(T^{\frac{1}{4} + \alpha}), \quad a \in [\delta, \frac{1}{4} + \delta], \quad T \to \infty. \]
Then, by (3.4), it is true that for every valid estimate of type (3.5) one obtains:
\[ \limsup_{T \to \infty} |R_a(T)| = +\infty. \]
In other words, every expression of the type (3.2) possesses an unbounded error term at infinity.

3.2. Under the circumstances (3.2) and (3.6) we have proved in [1] that the Hardy-Littlewood integral (3.1) has an infinite set of other completely new and almost exact representations expressed by the following:

**Property 1.**
\[ \int_0^T \left| c \left( \frac{1}{2} + it \right) \right|^2 \, dt = \varphi_1(T) \ln \{ \varphi_1(T) \} + \]
\[ + (c - \ln 2\pi) \varphi_1(T) + c_0 + \mathcal{O} \left( \frac{\ln T}{T} \right), \quad T \to \infty, \]
(comp. (2.4)) with the following
\[ \lim_{T \to \infty} \hat{R}(T) = \lim_{T \to \infty} \mathcal{O} \left( \frac{\ln T}{T} \right) = 0, \]
where \( c \) is the Euler’s constant and \( c_0 \) is the constant from the Titchmarsh-Kober-Atkinson formula.

Remark 9. Comparison of (3.6) and (3.8) completely characterizes the level of exactness of our representation (3.7) of the Hardy-Littlewood integral (3.1).

4. Asymptotic relation between Jacob’s ladder and the prime-counting function

4.1. Further, in the paper [1], (6.2), we have obtained the following formula.

**Lemma 2.**
\[ T - \varphi_1(T) \sim (1 - c)\pi(T); \quad \pi(T) \sim \frac{T}{\ln T}, \quad T \to \infty. \]

Remark 10. As a consequence, the Jacob’s ladder can be viewed as complementary function to the function
\[ (1 - c)\pi(T) \]
in the sense that
\[ \varphi_1(T) + (1 - c)\pi(T) \sim T, \quad T \to \infty. \]

4.2. Let (see [3], (1.11))
\[ y = \varphi_1(t): \quad \varphi_0^k(t) = t, \quad \varphi_1^k(t) = \varphi_1(t), \quad \varphi_2^k(t) = \varphi_1(\varphi_1(t)), \ldots, \]
\[ \varphi_1^k(t) = \varphi_1(\varphi_1^{k-1}(t)), \ldots, \quad t \in [T, T + U], \quad T \geq T_0[\varphi_1], \]
of course (see [4.11])
\[ T_0 > \varphi_1(T_0), \]
and the symbol \( \varphi_1^k(t) \) represents the k-th iteration of the Jacob’s ladder.
Remark 11. Let us remind that the functions
\[ \varphi_k^1(t), \text{ } k = 2, 3, \ldots \]
are increasing since \( \varphi_1^1(t) \) is increasing.

4.3. In the case
\[ t \mapsto \varphi_k^1(t), \text{ } t \in [T, T + U] \]
it follows from Lemma 2 that:

\[ \varphi_k^1(t) - \varphi_{k+1}^1(t) \sim (1 - c) \frac{\varphi_k^1(t)}{\ln \varphi_k^1(t)}, \text{ } k = 0, 1, \ldots, n, \text{ } t \to \infty, \]

where \( n \in \mathbb{N} \) is arbitrary and fixed one. Now formulae (4.5) imply the following properties of the set \( \{\varphi_k^1(t)\}_{k=0}^{n+1} \):

Lemma 3. For
\[ t \in [T, T + U], \text{ } U = o \left( \frac{T}{\ln T} \right), \text{ } T \to \infty \]
the following statements hold true:

\[ t \sim \varphi_1^1(t) \sim \varphi_2^1(t) \sim \cdots \sim \varphi_{n+1}^1(t), \]
\[ t > \varphi_1^1(t) > \varphi_2^1(t) > \cdots > \varphi_{n+1}^1(t), \]
\[ \varphi_k^1(T) > (1 - \epsilon)T, \text{ } k = 0, 1, \ldots, n + 1, \text{ } \epsilon > 0, \text{ } \epsilon \text{ small and fixed}, \]
\[ \varphi_k^1(T + U) - \varphi_k^1(T) < \frac{1}{2n + 5 \ln T}, \text{ } k = 1, \ldots, n + 1, \]
\[ \varphi_k^1(T) - \varphi_{k+1}^1(T + U) > 0.18 \times \frac{T}{\ln T}, \text{ } k = 0, 1, \ldots, n. \]

4.4. Further, we have introduced (see [3], (2.2)) the following set
\[ D(T, U, n) = \bigcup_{k=0}^{n+1} [\varphi_k^1(T), \varphi_k^1(T + U)]. \]

Remark 12. We list here the properties of the set (4.12):

(a) It is disconnected set (see (4.11)) for every admissible \( U \), (see (4.6));
(b) Components of the set \( D \) are distributed as follows: (see (4.11))
\[ [\varphi_{n+1}^1(T), \varphi_{n+1}^1(T + U)] \prec [\varphi_n^1(T), \varphi_n^1(T + U)] \prec \cdots \prec [\varphi_1^1(T), \varphi_1^1(T + U)] \prec [\varphi_0^1(T), \varphi_0^1(T + U)] = [T, T + U]. \]

Remark 13. Asymptotic behaviour of the set \( D \) is as follows: at \( T \to \infty \) its components receding unboundedly each from other (see (4.11)) and all together recede to infinity. Hence at large \( T \) the set (4.12) behaves like one-dimensional Friedmann-Hubble expanding universe.
5. On the function $\hat{Z}^2(t)$

Let us recall the following formula we have proved in [1]:
\begin{equation}
Z^2(t) = \Phi'_{\varphi}(\varphi(t)) \frac{d\varphi(t)}{dt}, \quad t \in [T, T + U], \quad U = o\left(\frac{T}{\ln T}\right),
\end{equation}
where
\begin{equation}
\Phi'_{\varphi}[\varphi] = \frac{2}{\varphi^2} \int_0^{\mu[\varphi]} t e^{-\frac{2}{\varphi} t} Z^2(t) dt + \frac{Z^2(\mu[\varphi])}{\varphi} \frac{d\mu[\varphi]}{d\varphi},
\end{equation}
(see [1], (3.5), (3.9)). Now we put (see (2.4) and [2], (9.1))
\begin{equation}
\hat{Z}^2(t) = \frac{d\varphi(t)}{dt}, \quad t \geq T_0[\varphi_1].
\end{equation}
In the next step we present just the result (see [2], Lemma 1, (7.7) – (7.9), (9.2)):

**Lemma 4.** If
\begin{equation}
\mu_a[\varphi] = a \varphi \ln \varphi, \quad a \in [7, 8],
\end{equation}
\begin{equation}
t \in [T, T + U], \quad U = o\left(\frac{T}{\ln T}\right),
\end{equation}
then
\begin{equation}
\Phi'_{\varphi}[\varphi(t)] = \frac{1}{2} \left\{ 1 + \mathcal{O}\left(\frac{\ln \ln t}{\ln t}\right) \right\} \ln t,
\end{equation}
i.e. (see (5.1), (5.2))
\begin{equation}
\hat{Z}^2(t) = \frac{d\varphi(t)}{dt} = \frac{1}{\omega(t)} \left| \zeta\left(\frac{1}{2} + it\right) \right|^2,
\end{equation}
where
\begin{equation}
\omega(t) = 2\Phi'_{\varphi}[\varphi(t)] = \left\{ 1 + \mathcal{O}\left(\frac{\ln \ln t}{\ln t}\right) \right\} \ln t, \quad t \to \infty.
\end{equation}

**Remark 14.** The segment $[7, 8]$ is sufficient one for our purposes since the continuum set $\mu_a[\varphi]$ corresponds to this one (comp. [23], (5.1)).

6. Reverse iterations

6.1. Next, in our paper [4] we have introduced the reverse iterations by means of the Jacob’s ladder. First, we define the sequence
\begin{equation}
\{ T \}_k^{k_0}
\end{equation}
by the formula
\begin{equation}
\varphi^k_1(T) = T_k^{k-1}, \quad k = 1, 2, \ldots, k_0, \quad T = T_0[\varphi_1]
\end{equation}
(where $k_0 \in \mathbb{N}$ is an arbitrary and fixed numer) since the function $\varphi_1(T)$, $T \to \infty$ increases to $+\infty$. Further we have
\begin{equation}
\varphi_1(T) = T_k^{k-1} \Rightarrow \ldots \Rightarrow \varphi_1^k(T) = T, \quad k = 1, \ldots, k_0.
\end{equation}
Since
\[ \varphi_1(T) = T \Rightarrow \frac{T}{\varphi_1(T)} = T = \varphi_1^{-1}(T), \]
and then we may use the inverse function \( \varphi_1^{-1}(T) \) to generate a reverse iterations. Namely we have:
\[ \varphi_1(T) = T \Rightarrow \frac{T}{\varphi_1^{-1}(T)} = \varphi_1^{-1}(\varphi_1^{-1}(T)) = \varphi_1^{-2}(T), \ldots, \]
\[ \frac{T}{\varphi_1^{-k}(T)} = \varphi_1^{-k}(\varphi_1^{-k}(T)) = \varphi_1^{-k}(T), \]
where the last row gives the \( k \)-th reverse iteration of the point \( T = \frac{0}{0} \). Of course, we have
\[ \varphi_1^{k}(T) = \varphi_1^{k}(\varphi_1^{-k}(T)) = T. \]

6.2. Now, the basic formula (4.1) gives the following properties of the reverse iterations (see [4], (5.1) – (5.13)).

**Lemma 5.** If
\[ U = o \left( \frac{T}{\ln T} \right), \quad T \to \infty \]
then
\[ \varphi_1^{k}[\frac{0}{0}, T + U] = \varphi_1^{k} \left( \frac{0}{0}, \frac{T}{T + U} \right), \quad T = [T, T + U], \]
\[ \| \frac{T}{T + U} \| = \frac{0}{T} - \frac{k}{k} = \frac{0}{T} - \frac{0}{T}, \quad k = 1, \ldots, k_0, \]
\[ \frac{0}{T} - \frac{k}{k} - \frac{0}{T} = \frac{0}{T} - \frac{k}{k} = o \left( \frac{T}{\ln T} \right), \]
\[ \| \frac{k}{k} - \frac{k}{k} \| = \frac{0}{T} - \frac{k}{k} - \frac{0}{T} = \frac{0}{T} - \frac{k}{k} - \frac{0}{T} \sim (1 - c) \frac{T}{\ln T}, \]
\[ ||[T, T + U]|| \prec \frac{1}{T} - \frac{1}{T + U} \prec \cdots \prec \frac{k}{k} - \frac{k}{k}, \quad k = 1, \ldots, k_0, \]

(comp. Lemma 3 and (4.13)).

From (6.6) – (6.10) we obtain the following property of the Jacob’s ladders.

**Property 2.** For every segment
\[ [T, T + U], \quad U = o \left( \frac{T}{\ln T} \right), \quad T \to \infty \]
there is the following class of disconnected sets (comp. (4.12))
\[ \Delta(T, U) = \bigcup_{r=0}^{k} [T, T + U], \quad 1 \leq k \leq k_0, \]

generated by the Jacob’s ladder \( \varphi_1(T) \).

**Remark 15.** Asymptotic behaviour of the set \( \Delta \) is the same as behavior of the set (4.12), i.e. at \( T \to \infty \) its components receding unboundedly each from other and all together recede to infinity. Hence at large \( T \) the set (6.11) behaves like one-dimensional Friedmann-Hubble expanding universe.
6.3. Further, we have the following statement, see [4], (6.4):

**Lemma 6.** If

\[(6.12) \quad t \in [\varphi_1^{-k}(T), \varphi_1^{-k}(T + U)], \quad k = 1, \ldots, k_0,\]

then (see (6.5))

\[(6.13) \quad \varphi_1^r(t) \in [\varphi_1^{-r-k}(T), \varphi_1^{-r-k}(T + U)], \quad r = 0, 1, \ldots, k,\]

i.e.

\[\varphi_1^0(t) = t \in [\varphi_1^{-k}(T), \varphi_1^{-k}(T + U)] = [T, T + U],\]
\[\varphi_1^1(t) \in [\varphi_1^{-k+1}(T), \varphi_1^{-k+1}(T + U)] = [T, T + U],\]
\[\vdots\]
\[\varphi_1^{k-1}(t) \in [\varphi_1^{-1}(T), \varphi_1^{-1}(T + U)] = [T, T + U],\]
\[\varphi_1^k(t) \in [\varphi_1^0(T), \varphi_1^0(T + U)] = [T, T + U].\]

7. Main lemma and proof of Theorem 1

7.1. In connection with direct and reverse iterations we have proved (see [4], (7.1), (7.2)) the following

**Lemma 7.** If

\[(7.1) \quad U = o \left( \frac{T}{\ln T} \right), \quad T \to \infty,\]

then for every Lebesgue-integrable function

\[g(t), \quad t \in [T, T + U]\]

the following holds true:

\[(7.2) \quad \int_T^{T+U} g(t)dt = \int_T^{T+U} \sum_{r=0}^{k-1} Z^2([\varphi_1^r(t)]|dt, \quad k = 1, \ldots, k_0.\]

**Remark 16.** We have obtained the case \(k = 1:\)

\[(7.3) \quad \int_T^{T+U} g(t)dt = \int_T^{1+U} g(\varphi_1(t))Z^2(t)dt\]

in our paper [2], (9.5).

7.2. Now we proceed to the proof of our Theorem 1.
Proof of Theorem 1. Since the system (1.3) is fixed one then the corresponding \( l \) is also fixed and consequently the condition (7.1)

\[
l = o \left( \frac{T}{\ln T} \right), \quad T \to \infty
\]

is fulfilled for all sufficiently big positive \( T \). Now we have

\[
m \neq n : \quad 0 = \int_a^{a+2l} f_m(t)f_n(t)\,dt = \int_T^{T+2l} f_m(\tau - T + a)f_n(\tau - a)\,d\tau =
\]

and next, by our Lemma 7 for sufficiently big \( T > 0 \), one obtains

\[
= \int_T^{p+2l} f_m[\varphi_1^p(\rho) - T + a]f_n[\varphi_1^p(\rho) - T + a] \prod_{r=0}^{p-1} \tilde{Z}^2[\varphi_1^p(\rho)]d\rho =
\]

and next, by simple substitution

\[
\rho = \frac{p}{T + 2l - T}(t - a) + T, \quad t \in [a, a+2l], \quad \rho \in [T, T + 2l]
\]

we obtain

\[
\frac{p}{T + 2l - T} \int_a^{a+2l} f_m[\varphi_1^p(\frac{p}{T + 2l - T}(t - a) + T) - T + a] \times
\]

\[
\times f_n[\varphi_1^p(\frac{p}{T + 2l - T}(t - a) + T) - T + a] \times
\]

\[
\times \prod_{r=0}^{k-1} \tilde{Z}^2[\varphi_1^p(\frac{p}{T + 2l - T}(t - a) + T)]\,dt =
\]

and, in the next step, we finish with (see (1.5))

\[
(7.5) = \frac{p}{T + 2l - T} \int_a^{a+2l} f_m^p(t)f_n^p(t)\,dt \Rightarrow \int_a^{a+2l} f_m^p(t)f_n^p(t)\,dt = 0.
\]

7.3. We give at once the following.

Remark 17. If we use the last formula in (7.5) as the origin of a new process (an analogue of this in the subsection 7.2), then we obtain \( k^2 \) new iterated \( L_2 \)-orthogonal systems

\[
\{f_n^{p_1,p_2}(t)\}_{n=0}^{\infty}, \quad t \in [a, a + 2l], \quad p_1, p_2 = 1, \ldots, k
\]
JACOB'S LADDER AS GENERATOR OF NEW CLASS OF ITERATED L₂-ORTHOGONAL SYSTEMS AND THEIR DEPENDENCE ON THE RIEMANN'S FUNCTION ζ(\frac{1}{2} + it)

where

\[ f_n^{p_1,p_2}(t) = \]

\[ = f_n[\varphi_1^{p_1}(\frac{T + 2l - T}{2l}(\varphi_1^{p_2}(\frac{T + 2l - T}{2l}(t - a) + \frac{p_2}{T} - \frac{p_1}{T} - T + a) - T + a)\times\]

\[ \times \prod_{r=0}^{p_1-1} |\tilde{Z}[\varphi_1^{p_1}(\frac{T + 2l - T}{2l}(\varphi_1^{p_2}(\frac{T + 2l - T}{2l}(t - a) + \frac{p_2}{T} - \frac{p_1}{T} - T + a))|\times\]

\[ \times \prod_{r=0}^{p_2-1} |\tilde{Z}[\varphi_1^{p_2}(\frac{T + 2l - T}{2l}(t - a) + \frac{p_2}{T})|,\]

and so on up to \( k^l \) new iterated \( L₂ \)-orthogonal systems

\( \{f_n^{p_1,p_2,...,p_l}(t)\}, \ t \in [a, a + 2l], \ p_1, \ldots, p_l = 1, \ldots, k \)

for every fixed \( l \in \mathbb{N} \).

7.4. Let us notice that the transformation (7.6)

\[ w = w(t) = \varphi_1^{p}(\frac{T + 2l - T}{2l}(t - a) - T + a), \ t \in [a, a + 2l] \]

has the following properties:

(a) by the subsection 6.1:

\[ w(a) = \varphi_1^{p}(\frac{T}{2l} - \frac{p}{T}(t - a)) - T + a, \ t \in [a, a + 2l] \]

(b) since the function \( \varphi_1^{p}(u) \) is increasing one and

\[ u = \frac{T + 2l - T}{2l}(t - a) + \frac{p}{T}, \ u \in [a, a + 2l] \]

is evident then the composed function

\[ w(t), \ t \in [a, a + 2l] \]

is increasing that is

\[ w(t) \in [a, a + 2l]. \]

Remark 18. Consequently, it follows from (a) and (b) that by the one-to-one correspondence (7.6) we have defined new automorphism on \([a, a + 2l]\), i.e. the \( k, k^2, \ldots, k^l \) of new automorphisms for every fixed sufficiently big positive \( T \).

I would like to thank Michal Demetrian for his moral support of my study of Jacob's ladders.
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