BASES FOR LOCAL WEYL MODULES FOR THE HYPER AND TRUNCATED CURRENT $\mathfrak{sl}_2$–ALGEBRAS

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Abstract. We use the theory of Gröbner-Shirshov bases for ideals to construct linear bases for graded local Weyl modules for the (hyper) current and the truncated current algebras associated to the finite-dimensional complex simple Lie algebra $\mathfrak{sl}_2$. The main result is a characteristic-free construction of bases for this important family of modules for the hyper current $\mathfrak{sl}_2$-algebra. In the positive characteristic setting this work represents the first construction in the literature. In the characteristic zero setting, the method provides a different construction of the Chari-Pressley (also Kus-Littelmann) bases and the Chari-Venkatesh bases for local Weyl modules for the current $\mathfrak{sl}_2$-algebra. Our construction allows us to obtain new bases for the local Weyl modules for truncated current $\mathfrak{sl}_2$-algebras with very particular properties.

Introduction

The category of level zero representations of affine (and quantum affine) algebras and its full subcategory of finite-dimensional representations have been extensively studied in the last three decades. The representation theory of these algebras gives a significant contribution to identifying interesting families of finite-dimensional representations of loop and current algebras, such as the universal finite-dimensional highest weight modules called local Weyl modules, which became objects of independent and deep interest in the last ten years (cf. [3, 6, 7, 14, 17, 26, 27, 28]). Among the generalizations of current algebras (as in [3]) special attention is given to the truncated current algebras (cf. [11, 21, 33]), which are finite-dimensional quotients of the current algebra.

During the last decade, the study of the positive characteristic analogues of these local Weyl modules for current and loop algebras was also developed in [2, 3, 10], where we can see that the characteristic zero and positive characteristic cases share several properties. To differentiate the positive characteristic case for the current algebra, we will refer to it as hyper current algebra.

Even with a large number of papers dedicated to the study of structure, character, decomposition, tensor product, fusion product, and reducibility of Weyl modules, only few are devoted to constructing bases for these modules (cf. [8, 9, 10, 21]). Additionally, some recent papers (cf. [29, 30, 31]) focused in studying properties of the bases constructed in [8, 9].

A basis for local Weyl modules for the current $\mathfrak{sl}_2$-algebra was first constructed by Chari-Pressley [9] and it was used by Chari-Loktev [8] in the construction of a basis for local Weyl modules for current algebras associated to $\mathfrak{sl}_{n+1}$, $n > 1$. Recently, two constructions came up: Kus-Littelmann [21] constructed a basis for truncated local Weyl modules for the current $\mathfrak{sl}_2$-algebra whose construction contemplates the Chari-Pressley basis for graded local Weyl modules by using a very different approach; and Chari-Venkatesh [10] provided the construction of a different basis for graded local Weyl modules for the current $\mathfrak{sl}_2$-algebra. Unfortunately, we still do not see how to adapt any of those constructions to the positive characteristic setting. This fact already appeared with many other results first proved in characteristic zero and then generalized to positive characteristic setting by using very different tools as in the pairs of papers [9, 16] and [21, 27].
The present paper was originally intended to present a different approach to obtain some new bases for graded local Weyl modules for the current \( \mathfrak{sl}_2 \)-algebra. While this work was developed, we noticed that we could provide a characteristic-free construction of a basis for graded local Weyl modules for the current and hyper current \( \mathfrak{sl}_2 \)-algebra. This work provides the first explicit construction of bases for hyper current algebras (see Corollary 5.3) and it is also an alternative construction to get the Chari-Venkatesh basis (see Theorem 6.1). Further, our method is also an alternative construction (see Theorem 6.1) to get Chari-Pressley (cf. [9]) and Kus-Littelmann (cf. [21]) bases in the characteristic zero setting. Further, for hyper current algebras (see Corollary 5.3) and it is also an alternative construction to get the Chari-Pressley bases.

In this section we fix some notation and review basic facts about the base algebras. We always denote by \( \mathbb{C}, \mathbb{Z}, \mathbb{Z}_+ \) and \( \mathbb{N} \) the sets of complex, integer, nonnegative integer and natural numbers, respectively.

### 1. Preliminaries

In this section we fix some notation and review basic facts about the base algebras. We always denote by \( \mathbb{C}, \mathbb{Z}, \mathbb{Z}_+ \) and \( \mathbb{N} \) the sets of complex, integer, nonnegative integer and natural numbers, respectively.

#### 1.1. Simple Lie algebras and their current algebras

Let \( \mathfrak{g} \) be a finite-dimensional complex simple Lie algebra with a fixed Cartan subalgebra \( \mathfrak{h} \). Denote by \( R \) the associated root system, fix a choice of positive roots \( R^+ \subseteq R \) and let \( \mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \) be the corresponding triangular decomposition. The associated simple roots and fundamental weights will be denoted respectively by \( \alpha_i \) and \( \omega_i, i \in I \), where \( I \) is an indexing set for the nodes of the Dynkin diagram of \( \mathfrak{g} \). Denote by \( P \) and \( Q \) the weight and root lattices of \( \mathfrak{g} \). Let \( P^+ \) and \( Q^+ \) be the \( \mathbb{Z}_+ \)-span of the fundamental weights and simple roots of \( \mathfrak{g} \), respectively. For notational convenience, fix a Chevalley basis \( \{ x^\pm_i, h_i : \alpha \in R^+, i \in I \} \) of \( \mathfrak{g} \).

Let \( \mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t] \) denote the corresponding current algebra (namely, the extension by scalars of \( \mathfrak{g} \) to the polynomial ring \( \mathbb{C}[t] \)), where the bracket is given by \( [x \otimes f, y \otimes g] = [x, y] \otimes fg \) for \( x, y \in \mathfrak{g} \) and \( f, g \in \mathbb{C}[t] \). Notice that \( \mathfrak{g} \otimes 1 \) is a subalgebra of \( \mathfrak{g}[t] \) isomorphic to \( \mathfrak{g} \) and, if \( \mathfrak{b} \) is a subalgebra of \( \mathfrak{g} \), then \( \mathfrak{b}[t] = \mathfrak{b} \otimes \mathbb{C}[t] \) is naturally a subalgebra of \( \mathfrak{g}[t] \). In particular, we have \( \mathfrak{g}[t] = \mathfrak{n}^-[t] \oplus \mathfrak{h}[t] \oplus \mathfrak{n}^+[t] \) and \( \mathfrak{h}[t] \) is an abelian subalgebra of \( \mathfrak{g}[t] \).

For a Lie algebra \( \mathfrak{a} \), we denote by \( \mathcal{U}(\mathfrak{a}) \) the corresponding universal enveloping algebra of the Lie algebra \( \mathfrak{a} \). The PBW theorem implies that the multiplication establishes isomorphisms \( \mathcal{U}(\mathfrak{g}[t]) \cong \mathcal{U}(\mathfrak{n}^-[t]) \otimes \mathcal{U}(\mathfrak{h}[t]) \otimes \mathcal{U}(\mathfrak{n}^+[t]) \).

#### 1.2. Integral forms and hyperalgebras

For a fixed order on the Chevalley basis of \( \mathfrak{g}[t] \) and a PBW monomial with respect to this order, we construct an ordered monomial in the elements of the set

\[
\mathcal{M}[t] = \{(x^\pm_i \otimes t^r)^{(k)} : \Lambda_{\alpha_i k}^{(h)} \} : \alpha \in R^+, i \in I, k, r \in \mathbb{Z}_+\}
\]

where

\[
(x^\pm_i \otimes t^r)^{(k)} := \frac{(x^\pm_i \otimes t^r)^k}{k!}, \quad \left( \frac{h_i}{k} \right) := \frac{h_i(h_i - 1) \ldots (h_i - k + 1)}{k!}
\]
and
\[ \Lambda_\alpha(u) := \sum_{r=0}^\infty \Lambda_{\alpha,r} u^r := \exp \left( -\sum_{s=1}^\infty \frac{h_\alpha \otimes t^s}{s} u^s \right), \]
by using the correspondence \((x_\alpha^+ \otimes t^r)^k \leftrightarrow (x_\alpha^- \otimes t^r)^{(k)\ell} \), \((h_i \otimes t^r)^k \leftrightarrow (k_i)^{(h_i)\lambda} \) and \((h_i \otimes t^r)^k \leftrightarrow (\Lambda_{\alpha,i})^k\). Using a similar correspondence we also consider monomials in \(U(\mathfrak{g})\) formed by elements of
\[ \mathcal{M} = \left\{ (x_\alpha^+)^{(k)}, (h_i)^{(k)} : \alpha \in \mathcal{R}^+, i \in I, k \in \mathbb{Z}_+ \right\}. \]
Notice that \(\mathcal{M}\) can be naturally regarded as a subset of \(\mathcal{M}[t]\). The set of ordered monomials thus obtained are bases of \(U(\mathfrak{g}[t])\) and \(U(\mathfrak{g})\), respectively.

Let \(U_Z(\mathfrak{g}[t]) \subset U(\mathfrak{g}[t])\) and \(U_Z(\mathfrak{g}) \subset U(\mathfrak{g})\) be the \(\mathbb{Z}\)-subalgebras generated respectively by \(\{(x_\alpha^+ \otimes t^r)^k : \alpha \in \mathcal{R}^+, r, k \in \mathbb{Z}_+\}\) and \(\{(x_\alpha^+)^{(k)} : \alpha \in \mathcal{R}^+, k \in \mathbb{Z}_+\}\). The following crucial theorem was proved in [20], in the \(U(\mathfrak{g})\) case, and in [14, 23] (see also [4]) for the \(U(\mathfrak{g}[t])\) case.

**Theorem 1.1.** The subalgebras \(U_Z(\mathfrak{g})\) and \(U_Z(\mathfrak{g}[t])\) are free \(\mathbb{Z}\)-modules and the sets of ordered monomials constructed from \(\mathcal{M}\) and \(\mathcal{M}[t]\) are \(\mathbb{Z}\)-bases of \(U_Z(\mathfrak{g})\) and \(U_Z(\mathfrak{g}[t])\), respectively.

In other words, \(U_Z(\mathfrak{g}[t])\) is an integral form of \(U(\mathfrak{g}[t])\) (similarly for \(U_Z(\mathfrak{g})\)).

If \(\mathfrak{a}\) is any subalgebra of \(\mathfrak{g}\), set \(U_Z(\mathfrak{a}[t]) := U(\mathfrak{a}[t]) \cap U_Z(\mathfrak{g}[t])\) and \(U_Z(\mathfrak{a}) := U(\mathfrak{a}) \cap U_Z(\mathfrak{g})\). Then,
\[ \mathfrak{a} \in \{ \mathfrak{g}, \mathfrak{n}^\pm[t], \mathfrak{h}[t], \mathfrak{n}^\pm, \mathfrak{h} \} \quad \Rightarrow \quad \mathbb{C} \otimes_\mathbb{Z} U_Z(\mathfrak{a}) \xrightarrow{\cong} U(\mathfrak{a}). \]

Given a field \(\mathbb{F}\), the \(\mathbb{F}\)-hyperalgebra of \(\mathfrak{a}\) is defined by
\[ U_{\mathbb{F}}(\mathfrak{a}) := \mathbb{F} \otimes_\mathbb{Z} U_Z(\mathfrak{a}). \]
We will refer to \(U_{\mathbb{F}}(\mathfrak{g}[t])\) as the hyper current algebra of \(\mathfrak{g}\) over \(\mathbb{F}\).

Notice that we have
\[ (1.1) \quad U_{\mathbb{F}}(\mathfrak{g}) = U_{\mathbb{F}}(\mathfrak{n}^-)U_{\mathbb{F}}(\mathfrak{h})U_{\mathbb{F}}(\mathfrak{n}^+) \quad \text{and} \quad U_{\mathbb{F}}(\mathfrak{g}[t]) = U_{\mathbb{F}}(\mathfrak{n}^-[t])U_{\mathbb{F}}(\mathfrak{h}[t])U_{\mathbb{F}}(\mathfrak{n}^+[t]). \]

**Remark 1.** More discussion about hyperalgebras can be found in [2, 3, 14]. We recall some important facts:

1. If the characteristic of \(\mathbb{F}\) is zero, the algebra \(U_{\mathbb{F}}(\mathfrak{g}[t])\) is naturally isomorphic to \(U(\mathfrak{g}[t]_{\mathbb{F}})\) where \(\mathfrak{g}[t]_{\mathbb{F}} = \mathbb{F} \otimes_\mathbb{Z} \mathfrak{g}[t]_\mathbb{Z}\) and \(\mathfrak{g}[t]_\mathbb{Z}\) is the \(\mathbb{Z}\)-span of the Chevalley basis of \(\mathfrak{g}[t]\). If the characteristic of \(\mathbb{F}\) is positive, we only have an algebra homomorphism \(U(\mathfrak{a}_\mathbb{F}) \to U_{\mathbb{F}}(\mathfrak{a})\) which is neither injective nor surjective.

2. Notice that the Hopf algebra structure of the universal enveloping algebras induce such structure on the hyperalgebras. For any Hopf algebra \(H\), we denote by \(H^0\) its augmentation ideal.

3. The integral form of \(\mathfrak{g}\) (cf. [20]) coincides with its intersection with the integral form of \(U(\mathfrak{g}[t])\) (cf. [15, 23]), i.e \(U_Z(\mathfrak{g}) = U(\mathfrak{g}) \cap U_Z(\mathfrak{g}[t])\), which allows us to regard \(U_Z(\mathfrak{g})\) as a \(\mathbb{Z}\)-subalgebra of \(U_Z(\mathfrak{g}[t])\).

1.3. **Partitions.** The language of partitions will be freely used in the text in a natural way associated with exponents of polynomials. A partition of a positive integer number is a sequence \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)\) of non increasing positive integers (\(\lambda_i \geq \lambda_{i+1}\) and \(\lambda_i > 0\) for \(1 \leq i \leq k\)). We say \(\lambda\) is a partition of \(r\), symbolically \(\lambda \vdash r\), if \(\sum_{i=1}^k \lambda_i = r\). We also say \(|\lambda| = r\) is the sum of the partition and \(\ell(\lambda) = k\) is its length. A useful way of graphically representing partitions is the Ferrers diagram which associates a left justified row of \(\lambda_i\) boxes for each \(i \in \{1, 2, \ldots, k\}\), decreasing from top to bottom. The transpose of \(\lambda\), denoted \(\lambda^\top\) is the partition given by changing rows and columns in \(\lambda\). The partial order \(\preceq\) known as dominance order is defined by \(\mu \preceq \lambda\) if, and only if, \(\sum_{i=1}^j \mu_i \leq \sum_{i=1}^j \lambda_i\), \(1 \leq j \leq \ell(\lambda)\).
2. SOME COMMUTATION FORMULAS: A GENERALIZATION OF GARLAND'S FORMULA.

We state some useful commutation identities in hyperalgebras. First we recall a famous formula due to Garland and then we present a generalization of it.

Given $\alpha \in R^+$, consider the following power series with coefficients in $U_Z(n^{-}[t])$:

$$X_{\alpha; (m,n)}(u) := \sum_{j=0}^{\infty} (x_\alpha^- \otimes t^{(m+n)j+m}) u^{j+1}.$$ 

In what follows we denote $h \otimes t \mathbb{C}[t]$ by $t h[t]$ for shortness.

**Lemma 2.1.** [23] Let $\alpha \in R^+$ and $\ell, k, m, n \in \mathbb{Z}_+$. We have

$$(x_\alpha^+ \otimes t^n)^{(\ell)} (x_\alpha^- \otimes t^m)^{(k)} = (X_{\alpha; (m,n)}(u))^{(k-\ell)} \mod U_Z(g[t]) (U_Z(n^+[t])^0 + U_Z(t h[t])^0)$$

where the index $k$ means the coefficient of $u^k$ of the above power series.

We now extend this result. Let $w, u_1, u_2, \dotsc$ be algebraically independent variables which will be used as formal variables. For any ring $R$, the formal power series ring in the variables $u_1, u_2, \dotsc$ will be denoted $R[[u_i]]$. Similarly, the formal power series ring in the variables $w, u_1, u_2, \dotsc$ will be denoted $R[[u; w]]$. Set $S := \mathbb{C}[[u; w]]$

Let $\alpha \in R^+$. We define

$$X_{\alpha}(u) := \sum_{n=0}^{\infty} (x_\alpha^- \otimes t^n) u^{n+1} = X_{\alpha; (0,1)}(u) \quad \text{and} \quad H_{\alpha}(u) := \sum_{n=0}^{\infty} (h_\alpha \otimes t^n) u^{n+1}.$$ 

In $U_Z(g[t])[u]$ we define the formal series $Y_{\alpha}[s]$, for $s \geq 0$, by

$$Y_{\alpha}[s] = \text{Res}_w(G[s](w)X_{\alpha}(w^{-1}))$$

where

$$G[s](w) = \begin{cases} \frac{1}{1+u_1 w + u_2 w^2 + \dotsc + u_s w^s}, & \text{if } s \geq 1 \\ 1, & \text{if } s = 0 \end{cases}$$

and $\text{Res}_w(f)$ denotes the coefficient of $w^{-1}$ in $f \in S$ (that is the residue at 0 with respect to the formal variable $w$) understood as a formal series in $w$ with coefficients in the polynomial ring $\mathbb{C}[u_1, \ldots, u_s]$. In particular, if $s = 0$ we have

$$Y_{\alpha}[s] = \text{Res}_w(X_{\alpha}(w^{-1})) = \text{Res}_w \left( \sum_{i=0}^{\infty} (x_\alpha^- \otimes t^i) w^{-i-1} \right) = x_\alpha^-.$$ 

Let us record some useful results in formal series. The first is an analogue of a well-known result from vertex operators for loop algebras. Let $\pi_w$ denote projection onto the negative powers in $w$. We
have:

$$[H_\alpha(w^{-1}), X_\alpha(z^{-1})] = \left[ \sum_{i=0}^{\infty} (h_\alpha \otimes t^i) w^{-i-1}, \sum_{j=0}^{\infty} (x^-_\alpha \otimes t^j) z^{-j-1} \right]$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} [h_\alpha \otimes t^i, x^-_\alpha \otimes t^j] w^{-i-1} z^{-j-1}$$

$$= -2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (x^-_\alpha \otimes t^{i+j}) w^{-i-1} z^{-j-1}$$

$$= -2 \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (x^-_\alpha \otimes t^k) w^{-(k-j)-1} z^{-j-1}$$

$$= -2 \pi^- w \left( \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (x^-_\alpha \otimes t^k) w^{-(k-j)-1} z^{-j-1} \right)$$

$$= -2 \pi^- w \left( \sum_{k=0}^{\infty} \left( (x^-_\alpha \otimes t^k) w^{-k-1} \sum_{j=0}^{\infty} w^j z^{-j-1} \right) \right)$$

$$= \pi^- w \left( X_\alpha(w^{-1}) \frac{-2}{z-w} \right)$$

with $\frac{1}{z-w}$ being considered a formal series in $w$.

Observe that, for $s \geq 1$ we have:

$$(2.1) \quad G[s](w) = \frac{G[s-1](w)}{1 + u_s w^s G[s-1](w)}.$$ 

from which we see that

$$(2.2) \quad \partial_{u_s} G[s](w) = -w^s G[s](w)^2.$$ 

Let $f(u_1, \ldots, u_s; w), g(u_1, \ldots, u_s; w) \in S$. In the following, we suppress the notational dependence on $u_1, \ldots, u_s$. We have:

$$(2.3) \quad [\Res_w(f(w)H_\alpha(w^{-1})), \Res_w(g(w)X_\alpha(w^{-1}))] = [\Res_w(f(w)H_\alpha(w^{-1})), \Res_z(g(z)X_\alpha(z^{-1}))]$$

$$= \Res_w(f(w)\Res_z(g(z)[H_\alpha(w^{-1}), X_\alpha(z^{-1})]))$$

$$= \Res_w \left( f(w)\Res_z \left( g(z)\pi^- w \left( \frac{-2}{z-w} X_\alpha(w^{-1}) \right) \right) \right)$$

$$= -2 \Res_w \left( f(w)\pi^- w \left( X_\alpha(w^{-1})\Res_z \left( \frac{g(z)}{z-w} \right) \right) \right)$$

$$= -2 \Res_w \left( f(w)\pi^- w (g(w)X_\alpha(w^{-1})) \right)$$

$$= -2 \Res_w \left( f(w)g(w)X_\alpha(w^{-1}) \right)$$

where the first equation is a change of variables, in the second we used the linearity properties of the bracket and residue, in the third we used the previous result, and in the fifth we used a standard property of the residue. In the sixth equation, $\pi^- w$ disappears in the residue since $f(w)$ contains only positive powers of $w$.

The main result we are going to prove in this subsection is:
Proposition 2.2. Let \( \alpha \in R^+, k \geq 0, \) and \( a_1, \ldots, a_s \in \mathbb{Z}_{\geq 0}, s \geq 1 \) such that \( \sum_{i=1}^{s} a_i \leq k. \) We have that
\[
(x_\alpha^+ \otimes t^s)^{(a_1)} \cdots (x_\alpha^+ \otimes t^{a_2})(x_\alpha^-)^{(a_1)}(x_\alpha^-)^{(k)} = \text{coeff}_{a_1, a_2, \ldots, a_s} Y[s]^{(k-a_1-a_2-\cdots-a_s)} \mod U_z(g[t]) \left( U_z(t h[t])^0 + U_z(n^+ t)^0 \right)
\]
where \( \text{coeff}_{a_1, a_2, \ldots, a_s} Y[a]^s \) denotes the coefficient of \( u_1^{a_1} u_2^{a_2} \cdots u_s^{a_s} \) in the respective series.

It will be a direct consequence of the following lemma:

Lemma 2.3. Let \( \alpha \in R^+. \)
(1) In \( U_z(g[t])[u] \) we have, for \( r \geq 1, s \geq 1: \)
\[
[x_\alpha^+ \otimes t^s, Y[a][s]]^{(r)} = \partial_u Y[a][s]^{(r-1)} \mod U_z(g[t])[u] \left( U_z(t h[t])^0[u] + U_z(n^+ t)^0[u] \right).
\]
(2) In \( U_z(g[t])[u] \) we have, for \( r \geq 1, 0 \leq n \leq r, s \geq 1: \)
\[
(x_\alpha^+ \otimes t^s)^{(n)} Y[a][s-1]^{(r)} = (Y[a][s]^{(r-n)})^{(n)} \mod U_z(g[t])[u] \left( U_z(t h[t])^0[u] + U_z(n^+ t)^0[u] \right)
\]
where the subscript \( n \) means the coefficient of \( u^n. \)
(3) For all \( s \geq 1 \) we have
\[
Y[a][s] = \sum_{\eta : \ell(\eta) \leq s} (-1)^{t(\eta)} \frac{\ell(\eta)!}{\prod_{i=1}^{\ell(\eta)} m_i(\eta)} (x_\alpha^- \otimes t^{|\eta|}) \prod_{i=1}^{s} u_i^{m_i(\eta)}.
\]

Proof. Throughout this proof, let \( \mathcal{I} = U_z(g[t])[u] \left( U_z(t h[t])^0[u] + U_z(n^+ t)^0[u] \right). \)

(1) We use induction on \( r. \) When \( r = 1 \) we have
\[
[x_\alpha^+ \otimes t^s, Y[a][s]] = [x_\alpha^+ \otimes t^s, \text{Res}_w(G[s](w) X[a](w^{-1}))] = 0 \mod \mathcal{I}.
\]
We now assume that statement (1) of the lemma is true for \( r \) and compute:
\[
(x_\alpha^+ \otimes t^s, (r+1) Y[a][s]^{(r+1)}) = [x_\alpha^+ \otimes t^s, Y[a][s]^{(r+1)}] = [x_\alpha^+ \otimes t^s, Y[a][s]^{(r)}] + Y[a][s][x_\alpha^+ \otimes t^s, Y[a][s]^{(r)}]
\]
In the first term above we have
\[
[x_\alpha^+ \otimes t^s, Y[a][s]^{(r)}] = [x_\alpha^+ \otimes t^s, \text{Res}_w(G[s](w) X[a](w^{-1}))] Y[a][s]^{(r)}
\]
\[
= \text{Res}_w(w^s G[s](w) H[a](w^{-1})) Y[a][s]^{(r)}
\]
\[
= Y[a][s]^{(r-1)} \left[ \text{Res}_w(w^s G[s](w) H[a](w^{-1})), Y[a][s] \right] \mod \mathcal{I}
\]
\[
= Y[a][s]^{(r-1)} \text{Res}_w(-2 w^s G[s](w)^2 X[a](w^{-1})) \mod \mathcal{I}
\]
\[
= 2 Y[a][s]^{(r-1)} \partial_u Y[a][s] \mod \mathcal{I},
\]
where in the third equation we used induction on \( r, \) in the fourth we used (2.3) and in the fifth we used (2.2).

In the second term of (2.4) we have:
\[
Y[a][s][x_\alpha^+ \otimes t^s, Y[a][s]^{(r)}] = Y[a][s] \partial_u (Y[a][s]^{(r-1)}) \mod \mathcal{I}
\]
\[
= Y[a][s] Y[a][s]^{(r-2)} \partial_u Y[a][s] \mod \mathcal{I}
\]
\[
= (r-1) Y[a][s]^{(r-1)} \partial_u Y[a][s] \mod \mathcal{I}.
\]
Combining terms, we get
\[
(r+1)[x_\alpha^+ \otimes t^s, Y[a][s]^{(r+1)}] = (r+1) \partial_u Y[a][s]^{(r)} \mod \mathcal{I}.
\]
Since \( r+1 \neq 0 \) we see that \([x_\alpha^+ \otimes t^s, Y[a][s]^{(r+1)}] - \partial_u Y[a][s]^{(r)} \in \mathcal{I}.\)
(2) We use induction on \( n \). For the \( n = 0 \) case, we want to show

\[
Y_\alpha[s - 1]^{(r)} = (Y_\alpha[s]^{(r)})_0.
\]

We compute:

\[
(Y_\alpha[s]^{(r)})_0 = Y_\alpha[s]^{(r)}|_{u_s=0}
\]

\[
= \text{Res}_w(G[s](w))X_\alpha(w^{-1})^{(r)}|_{u_s=0}
\]

\[
= \text{Res}_w \left( \frac{G[s - 1](w)}{1 + u_sw^sG[s - 1](w)} X_\alpha(w^{-1}) \right)^{(r)}|_{u_s=0}
\]

\[
= \text{Res}_w(G[s - 1](w))X_\alpha(w^{-1})^{(r)}
\]

\[
= Y_\alpha[s - 1]^{(r)}
\]

using (2.1) in the third equation.

Now assume the statement is true for all \( n < r \). We compute:

\[
(n + 1)(x_\alpha^+ \otimes t^s)^{(n+1)}Y_\alpha[s - 1]^{(r)} = (x_\alpha^+ \otimes t^s)(x_\alpha^+ \otimes t^s)^{(n)}Y_\alpha[s - 1]^{(r)}
\]

\[
= (x_\alpha^+ \otimes t^s)(Y_\alpha[s]^{(r-n)})_n \mod \mathcal{I}
\]

\[
= ((x_\alpha^+ \otimes t^s)Y_\alpha[s]^{(r-n)})_n \mod \mathcal{I}
\]

\[
= (\partial_{u_s} Y_\alpha[s]^{(r-n-1)})_n \mod \mathcal{I}
\]

\[
= (n + 1)(Y_\alpha[s]^{(r-n-1)})_{n+1} \mod \mathcal{I},
\]

where in the fourth equation, we use the result of (1) and in the fifth we use the “power rule” property of the formal derivative. Therefore, \((x_\alpha^+ \otimes t^s)^{(n+1)}Y_\alpha[s - 1]^{(r)} - (Y_\alpha[s]^{(r-n-1)})_{n+1} \in \mathcal{I}\).

(3) We compute:

\[
G[s](w) = \frac{1}{1 + u_1w + u_2w^2 + \cdots + u_sw^s}
\]

\[
= \sum_{n=0}^{\infty} (-1)^n (u_1w + u_2w^2 + \cdots + u_sw^s)^n
\]

\[
= \sum_{n=0}^{\infty} (-1)^n \sum_{\nu_1 + \cdots + \nu_s = n} \frac{n!}{\prod_{i=1}^{s} \nu_i!} \prod_{j=1}^{s} (u_jw^j)^{\nu_j}
\]

\[
= \sum_{\eta} (-1)^{\ell(\eta)} \frac{\ell(\eta)!}{\prod_{i=1}^{\ell(\eta)} m_i(\eta)!} \prod_{i=1}^{\ell(\eta)} u_i w_i^{m_i(\eta)}
\]

using multinomial expansion in the third equality, and where the sum is over all partitions \( \eta \) with \( \ell(\eta) \leq s \). The result then follows. \( \Box \)

3. POLYNOMIALS, DIVIDED POWER ALGEBRA, AND GRÖBNER-SHIRSHOV BASES

In the Subsection 1.2 we introduced the hyperalgebras. In order to get the main result of this paper, we will need to pass to the setting of polynomial rings and the divided power polynomial algebra, which we introduce now. Further, we recall the definition and main properties of Gröbner bases for polynomial rings and Gröbner-Shirshov bases for divided power polynomial algebra. For additional details we refer to [1] [18] [19].
3.1. Polynomial rings and Gröbner bases. Given a field \( \mathbb{K} \) and \( n \in \mathbb{N} \) we denote by \( \mathbb{K}_n = \mathbb{K}[x_0, \ldots, x_{n-1}] \) the polynomial ring in \( n \) variables \( x_0, \ldots, x_{n-1} \). For any set of elements \( S \subseteq \mathbb{K}_n \) we shall denote by \( \langle S \rangle \) the ideal generated by \( S \) in \( \mathbb{K}_n \). Any element of the form \( x_0^{a_0} \cdots x_{n-1}^{a_{n-1}} \), with \((a_0, \ldots, a_{n-1}) \in \mathbb{Z}_n^*\), is called a monomial in \( \mathbb{K}_n \). Let \( \prec \) be a monomial order in \( \mathbb{K}_n \). Given \( f \in \mathbb{K}_n \), we write \( f = \sum_{i=0}^k c_i m_i \), where \( m_i \) are monomials in \( \mathbb{K}_n \) and \( c_i \in \mathbb{K} \) for each \( i \in \{1, \ldots, n\} \). We denote by \( LM(f) \) the leading monomial of \( f \) with respect to \( \prec \), and, for any \( S \subseteq \mathbb{K}_n \), we denote the ideal of leading terms of \( S \) by \( LM(S) = \langle LM(s) \mid s \in S \rangle \). Given an ideal \( I \subseteq \mathbb{K}_n \), a set \( G = \{g_1, \ldots, g_s\} \subseteq I \) is called a Gröbner basis of \( I \) if \( LM(G) = LM(I) \). A Gröbner basis \( G = \{g_1, \ldots, g_s\} \) is called a reduced Gröbner basis if, for all \( i \in \{1, \ldots, s\} \), \( g_i \) is monic and no nonzero term in \( g_i \) is divisible by any \( LM(g_j) \) for any \( j \in \{1, \ldots, s\} \), \( j \neq i \).

Similarly, we will denote by \( \mathbb{K}_\infty \) the polynomial ring in infinitely many variables \( x_i, i \in \mathbb{Z}_+ \), and we naturally consider the same definitions of the previous paragraph.

We state two classical results which we shall use in this work.

Theorem 3.1 (\([1]\)). Fix an monomial order in \( \mathbb{K}_n \). Let \( I \) be a nonzero ideal in \( \mathbb{K}_n \).

(a) The ideal \( I \) has a unique (finite) reduced Gröbner basis.

(b) The set of monomials in \( \mathbb{K}_n \) which are not divisible by any of the leading terms of a Gröbner basis for \( I \) forms a \( \mathbb{K} \)-basis of \( \mathbb{K}_n/I \).

Given a polynomial ring \( \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_m] =: \mathbb{K}[[X], \{Y\}] \) in which the variables are split into two subsets \( X = \{x_1, \ldots, x_n\} \) and \( Y = \{y_1, \ldots, y_m\} \), a monomial ordering “eliminating the variables” \( \{x_1, \ldots, x_n\} \) is a monomial ordering for which two monomials are compared by first comparing the variables \( \{x_1, \ldots, x_n\} \), and, in case of equality only, considering the variables \( \{y_1, \ldots, y_m\} \). This implies that a monomial containing a variable from \( X \) is greater than every monomial independent of the variables from \( X \). The next theorem simplifies some computations in \( G \cap \mathbb{K}[y_1, \ldots, y_m] \):

Theorem 3.2. \([1]\), Elimination Theorem] If \( G \) is a Gröbner basis of an ideal \( I \subseteq \mathbb{K}[[X], \{Y\}] \) for an elimination monomial ordering, then \( G \cap \mathbb{K}[y_1, \ldots, y_m] \) is a Gröbner basis of the elimination ideal \( I \cap \mathbb{K}[y_1, \ldots, y_m] \). Moreover, a polynomial belongs to \( G \cap \mathbb{K}[y_1, \ldots, y_m] \) if, and only if, its leading term belongs to \( G \cap \mathbb{K}[y_1, \ldots, y_m] \).

Remark 2. The lexicographical orderings such that \( x_1 > \cdots > x_n \) and \( x_n > \cdots > x_1 \) are elimination orderings for every partition \( \{x_1, \ldots, x_k\} \cup \{x_{k+1}, \ldots, x_n\} \) but the crucial difference for each partition of this type is that the first order is eliminating \( \{x_1, \ldots, x_k\} \) and the second eliminates \( \{x_{k+1}, \ldots, x_n\} \).

3.2. The divided power polynomial algebra and reduction modulo a prime \( p \). Suppose \( \text{char}(\mathbb{K}) = 0 \), let \( \mathbb{Z}_n^D \subseteq \mathbb{K}_n \) be the divided power polynomial algebra in \( n \) variables \( x_0, \ldots, x_{n-1} \) over \( \mathbb{Z} \), i.e the \( \mathbb{Z} \)-algebra generated by

\[
\left\{(x_i)^{(a_i)} := x_i^{a_i}/a_i! \mid i = 0, \ldots, n-1, \ a_i \in \mathbb{Z}_+\right\}.
\]

For any set of elements \( S \subseteq \mathbb{Z}_n^D \) we also denote by \( \langle S \rangle \) the ideal generated by \( S \) in \( \mathbb{Z}_n^D \). Any element of the form \( x_0^{(a_0)} \cdots x_{n-1}^{(a_{n-1})} \), with \((a_0, \ldots, a_{n-1}) \in \mathbb{Z}_n^+ \), will also be called a monomial in \( \mathbb{Z}_n^D \). Further, any monomial order \( \prec \) in \( \mathbb{K}_n \) induces a monomial order in \( \mathbb{Z}_n^D \) in a obvious way and we keep denoting this induced order by \( \prec \). Similarly, we also define \( \mathbb{Z}_n^D \subseteq \mathbb{K}_\infty \).

The reduction of \( \mathbb{Z}_n^D \) modulo a prime \( p \) is just the change of scalars of \( \mathbb{Z}_n^D \) by an algebraically closed field \( \mathbb{F} \) of characteristic \( p \), that is the tensor product \( \mathbb{Z}_n^D \otimes_{\mathbb{Z}} \mathbb{F} =: \mathbb{D}_p \mathbb{F}_n \) as \( \mathbb{Z} \)-modules. Equivalently, we can define \( \mathbb{D}_p \mathbb{F}_n \) as the commutative algebra quotient \( \mathbb{F}[x_i^{(k)}]_{0 \leq i \leq n-1, k \geq 0}/I \), where

\[
I = \left\{(x_i^{(j)}x_j^{(k)} - \binom{j+k}{j}x_i^{(j+k)}) \mid 0 \leq i \leq n-1, j, k \geq 0\right\},
\]

which allows us to consider a similar Gröbner bases theory for \( \mathbb{D}_p \mathbb{F}_n \). In fact it is known as Gröbner-Shirshov theory (for historical reasons) and we present it the next subsection.
Remark 3. One of the most relevant theoretical differences between $\mathbb{F}_n$ and $\mathbb{D}_n$ comes from the fact that $\mathbb{D}_n$ is not Noetherian.

3.3. Monomials associated to partitions. Following Subsection [19] we let $m_i(\lambda)$ denote the number of parts in $\lambda$ which are exactly equal to $i$. In certain cases, abusing notation, we may allow $\lambda$ to have a specified number of parts equal to 0. In this case, it is clear that for any partition $\lambda$ with $\ell(\lambda) = k \leq m - 1$, we may associate the monomials

$$x^{(\lambda)} = x_0^{(m_0(\lambda))}x_1^{(m_1(\lambda))}\cdots x_k^{(m_k(\lambda))} \in \mathbb{D}_m$$

and

$$x^\lambda = x_0^{m_0(\lambda)}x_1^{m_1(\lambda)}\cdots x_k^{m_k(\lambda)} \in \mathbb{F}_m.$$

3.4. Gröbner-Shirshov bases. Let $X = \{x_1, x_2, \ldots\}$ be an enumerable set and let $X^*$ be the free monoid of associative monomials on $X$. Fix a monomial order $\prec$ on $X^*$ and let $\mathbb{F}_X$ be the free associative algebra generated by $X$ over a field $\mathbb{F}$. Given a nonzero element $p \in \mathbb{F}_X$, we denote by $LM(p)$ the maximal monomial appearing in $p$ under the ordering $\prec$. In this case, $p = \alpha LM(p) + \sum_i \beta_i w_i$ with $\alpha, \beta_i \in \mathbb{F}$, $w_i \in X^*$, $\alpha \neq 0$ and $w_i \prec LM(p)$ for all $i$. If $\alpha = 1$, $p$ is said to be monic.

Let $(S, T)$ be a pair of subsets of $\mathbb{F}_X$. Denote by $\langle S \rangle$ the ideal generated by $S$ in $\mathbb{F}_X$, and by $I_T$ the ideal of $\mathbb{F}_X$ generated by the image of $T$ in $\mathbb{F}_X/\{S\}$. We say that the algebra $A = \mathbb{F}_X/\langle S \rangle$ is defined by $S$ and the $A$-module $M = \frac{\mathbb{F}_X}{I}$ is defined by the pair $(S, T)$.

A monomial $u \in X^*$ is said to be $(S, T)$-reduced if $u \neq fLM(s)$ and $u \neq gLM(t)$ for any $s \in S$, $t \in T$ and $f, g \in X^*$. Otherwise, the monomial $u$ is said to be $(S, T)$-reducible.

Let $p, q \in \mathbb{F}_X$, we define the composition of $p$ and $q$ as the polynomial

$$S(p, q) = \frac{w_{p,q}}{LT(p)}p - \frac{w_{p,q}}{LT(q)}q$$

where $w_{p,q} = \text{lcm}(LM(p), LM(q))$. We say that $p, q \in \mathbb{F}_X$ are congruent with respect to the pair $(S, T)$, and denote $p \equiv q \mod (S, T)$ if $p - q = \sum_i \alpha_i a_i s_i + \sum_j \beta_j b_j t_j$, where $\alpha_i, \beta_j \in \mathbb{F}$, $a_i, b_j \in X^*$, $s_j \in S$, $t_j \in T$, $a_i LT(s_i) \prec w_{p,q}$, and $b_j LT(t_j) \prec w_{p,q}$. When $T = \emptyset$, we simply write $p \equiv q \mod (S)$.

A pair $(S, T)$ of subsets of monic elements of $A_X$ is called a Gröbner-Shirshov pair if $S(p, q) \equiv 0 \mod (S)$ for any $p, q \in S$, $S(p, q) \equiv 0 \mod (S, T)$ for any $p, q \in T$, and $S(p, q) \equiv 0 \mod (S, T)$ for any $p \in S$ and $q \in T$.

Theorem 3.3 ([18]). Let $(S, T)$ be a pair of subsets of monic elements in $A_X$. Let $A = A_X/\langle S \rangle$ be the associative algebra defined by $S$ and let $M = A/I_T$ be the $A$-module defined by $(S, T)$.

(a) The pair $(S, T)$ can be completed to a Gröbner-Shirshov pair $(S, T)$ for the $A$-module $M$.

(b) If $(S, T)$ is a Gröbner-Shirshov pair for the $A$-module $M$, then the set of $(S, T)$-reduced monomials forms a linear basis of $M$. □

4. Graded Local Weyl modules

We now recall the definition of graded local Weyl modules for hyper current algebras. We refer to [3, 5, 7] for details on finite-dimensional representations, Weyl modules and related topics recently developed.

During this section, $\mathbb{F}$ will always denote an algebraically closed field of any characteristic.

Definition 4.1. Given $\lambda = \sum_{i=1}^r m_i \omega_i \in P^+$, the graded local Weyl module $W_\mathbb{F}(\lambda)$ is the $U_\mathbb{F}(\mathfrak{g}[t])$-module generated by the element $v_\lambda$ with defining relations

$$x_\alpha^+ \otimes t^r v_\lambda = A_{\lambda,i} v_\lambda = h - \lambda(h) v_\lambda = (x^-_\alpha)^{(k)} v_\lambda = 0,$$

for all $h \in U_\mathbb{F}(\mathfrak{h})$, $\alpha \in R^+$, $i \in I$, $r \geq 0$, $s > 0$, $k > \lambda(h_\alpha)$.

In other words, defining $R$ as the left ideal of $U_\mathbb{F}(\mathfrak{g}[t])$ generated by

$$x_\alpha^+ \otimes t^r, \ A_{\lambda,i}, \ h - \lambda(h), \ (x^-_\alpha)^{(k)},$$
for all \( h \in U_\mathbb{F}(h), \alpha \in R^+, i \in I, r \geq 0, s > 0, k > \lambda(h_\alpha) \), then

\[(4.3) \quad W_\mathbb{F}(\lambda) = \frac{U_\mathbb{F}(g[t])}{\mathcal{R}}.\]

Remark 4. In the case \( \mathbb{F} = \mathbb{C} \), we simply denote \( W_\mathbb{C}(\lambda) = W(\lambda) \) following the traditional notation for the graded local Weyl module for the current algebra \( g \otimes \mathbb{C}[t] \).

We recall the main result on local Weyl modules, that they are universal modules in the category of finite-dimensional \( U_\mathbb{F}(g[t]) \)-modules:

**Theorem 4.2.** [3 Theorem 3.3.4] For all \( \lambda \in P^+ \), the modules \( W_\mathbb{F}(\lambda) \) are finite-dimensional and indecomposable. Moreover, any graded finite-dimensional \( U_\mathbb{F}(g[t]) \)-module generated by a vector \( v_\lambda \) satisfying the relations \( (x_\alpha^+ \otimes t^r)(s)v_\lambda = (h - \lambda(h))v_\lambda = \Lambda_{\alpha,s}v_\lambda = 0, s \geq 1 \), is a quotient of \( W_\mathbb{F}(\lambda) \). □

5. A characteristic-free basis for Weyl modules for \( U_\mathbb{F}(\mathfrak{sl}_2[t]) \)

From now we will consider only the case \( \mathfrak{g} = \mathfrak{sl}_2 \). Since \( I \) is singleton for \( \mathfrak{sl}_2 \), we shall denote \( \omega = \omega_1, x^+ = x^+_{\alpha_1}, h = h_1, \Lambda_{\alpha_1} = \Lambda, \) and \( \Lambda_{\alpha_1, k} = \Lambda_k \). Similarly, for \( \alpha = \alpha_1 \) we simply denote by \( X(u) \) the series \( X_\alpha(u) \) and \( Y(u_1, \ldots, u_s) \) will denote \( Y_{\alpha}(u_1, \ldots, u_s) \), both defined in Section 2. In this case \( P^+ \) is in a bijective correspondence with \( \mathbb{Z}_+ \) and we denote by \( m \) the weight \( m\omega \) and \( v_m = v_{m\omega} \).

We now state some useful identities for local Weyl modules which will be relevant for our purposes. From the defining relation of \( W_\mathbb{F}(m) \) and Proposition 2.2 we get

\[(5.1) \quad \text{coeff}_{a_1, a_2, \ldots, a_s} Y[s]^{(k-\ell(a))}v_m = 0 \quad \text{for} \quad 0 \leq \ell(a) \leq k, \quad k > m \]

where \( \ell(a) = \sum_{i=1}^s a_i \).

Let \( \mathcal{J} \) be the ideal of \( U_\mathbb{F}(n^{-}[t]) \) generated by the set

\[ \left\{ \text{coeff}_{a_1, a_2, \ldots, a_s} Y[s]^{(k-\ell(a))} | 0 \leq \ell(a) \leq k, \quad m + 1 \leq k \right\}. \]

Since \( W_\mathbb{F}(m) = U_\mathbb{F}(n^{-}[t])v_m \), and \( U_\mathbb{F}(n^{-}[t]) \) is commutative, it follows from (4.1) and (5.1) that \( \mathcal{J} \cdot W_\mathbb{F}(m) = 0 \). Further, by (1.1), (4.2) and (4.3)

\[(5.2) \quad W_\mathbb{F}(m) = \frac{U_\mathbb{F}(g[t])}{\mathcal{R}} \cong \frac{U_\mathbb{F}(n^{-}[t])}{\mathcal{J}}. \]

In order to calculate the dimension and to construct a \( \mathbb{F} \)-basis of \( W_\mathbb{F}(m) \) we will pass to a purely polynomial setting.

First, note that we have an isomorphism \( \phi : U_\mathbb{F}(n^{-}[t]) \to \mathbb{D}_\mathbb{F}_\infty \) defined by \( (x^- \otimes t^r)(k) \mapsto x_r^k \) for all \( r, k \in \mathbb{Z}_+ \). Now, consider the series \( Y_\infty[s] \) with coefficients in \( \mathbb{D}_\mathbb{F}_\infty \) given by

\[(5.3) \quad \sum_{\eta} (-1)^{\ell(\eta)} \prod_{i=1}^{s} \frac{\ell(\eta)!}{m_i(\eta)!} x_{\eta|n|} \prod_{i=1}^{s} u_i^{m_i(\eta)} \]

where the sum is over all partitions \( \eta \) of length \( s \). Respectively, we define \( Y[s] \) with coefficients in \( \mathbb{D}_\mathbb{F}_m \) by (5.3) with the restriction that \( |\eta| \leq m - 1 \).

Consider the ideals

\[ J_\infty := \langle \text{coeff}_{a_1, a_2, \ldots, a_s} Y_\infty[s]^{(k-\ell(a))} | 0 \leq \ell(a) \leq k, \quad m + 1 \leq k \rangle \]
\[ J_m := \langle \text{coeff}_{a_1, a_2, \ldots, a_s} Y[s]^{(k-\ell(a))} | 0 \leq \ell(a) \leq k, \quad m + 1 \leq k \rangle. \]

From the isomorphism \( \phi \) and the inclusion \( \mathbb{D}_\mathbb{F}_m \hookrightarrow \mathbb{D}_\mathbb{F}_\infty \), we get isomorphisms

\[(5.4) \quad \frac{\mathbb{D}_\mathbb{F}_m}{J_m} \cong \frac{\mathbb{D}_\mathbb{F}_\infty}{J_\infty} \cong \frac{U_\mathbb{F}(n^{-}[t])}{\mathcal{J}}. \]
In particular, (5.2) and (5.4) implies

\[ \dim W_F(m) = \dim \frac{D\mathcal{F}_m}{J_m} \]

as \( \mathbb{F} \)-vector spaces.

The rest of this section is devoted to showing that in fact we have \( \dim W_F(m) = 2^m \) (in particular, the dimension is independent of the ground field), and to construct an explicit basis for \( W_F(m) \).

We now discuss a Gröbner-Shirshov pair of \( J_m \). First, we need to define lexicographic ordering for the divided power algebra \( D\mathcal{F}_m \). This is not the same as the usual lexicographic order, but it is adapted for reducing our infinite alphabet \( \{ x_i^{(j)} \mid 0 \leq i \leq m - 1, j \geq 1 \} \) modulo the relations \( x_i^{(r)} x_i^{(s)} - x_i^{(r+s)} x_i^{(r-s)} \). First, we write down a monomial in increasing order of its variables and for each variable in increasing order of its upper index. For monomials of the form \( x_i^{(\mu_1)} x_i^{(\mu_2)} \cdots x_i^{(\mu_k)} \) with \( \mu \) a partition, we compare them by degree-graded lexicographic ordering, then we extend this ordering to a product ordering on arbitrary monomials by using lexicographic order on the index of variables appearing in the monomials. This is a well-defined monomial ordering, and it can be seen as a natural extension of the usual lexicographic ordering.

**Example 1.** Let \( \lambda \) and \( \mu \) two partitions as in Subsection 1.3 and 3.3.

1. If \( |\lambda| > |\mu| \), then \( x_i^{(\lambda)} \) is greater than \( x_i^{(\mu)} \) for any \( 0 \leq i \leq m - 1 \). Further, \( x_i^{(k)} \), \( 0 \leq i \leq m - 1 \) and \( k > 0 \), is greater than any polynomial involving only \( x_j \) with \( j > i \).
2. \( (x_i^{(1)})^{m_1(\lambda)} (x_i^{(2)})^{m_2(\lambda)} \cdots (x_i^{(\lambda)}))^{m_{\lambda_1}(\lambda)} \geq_{\text{lex}} x_i^{(|\lambda|)} \). In other words, \( x_i^{(|\lambda|)} \) is the minimum monomial over all partitions \( \mu \vdash |\lambda| \).
3. \( x_i^{(\lambda)} \leq_{\text{lex}} x_i^{(\mu)} \) if, and only if, \( \lambda \leq_{\text{lex}} \mu \) using the usual lexicographic order for partitions, i.e. \( m_i(\lambda) < m_i(\mu) \) where \( i \) is the least index such that \( m_i(\lambda) \neq m_i(\mu) \).

Before we continue, notice that to find an \( \mathbb{F} \)-basis for \( D\mathcal{F}_m / J_m \) by using Gröbner-Shirshov theory, it is not necessary to compute a Gröbner-Shirshov basis for \( D\mathcal{F}_m \) completely. Recall that, if \( I \) is an ideal of \( D\mathcal{F}_m \) with Gröbner-Shirshov pair \( (S, G) \) with respect to a monomial order, then a linear basis of \( D\mathcal{F}_m / I \) is the set of unreduced monomials with respect to \( (S, G) \).

The main theorem of this paper is to find a Gröbner-Shirshov pair \( (S, G_m) \) for \( J_m \) with respect to the ordering defined above, and thereby to find a basis of \( D\mathcal{F}_m / J_m \), where

\[
S = \left\{ x_i^{(j)} x_i^{(k)} - \binom{j + k}{j} x_i^{(j+k)} \mid 0 \leq i \leq n - 1, j, k \geq 0 \right\}
\]

and the set \( G_m \) will be given later, but for now we give only the set of leading monomials (which also determine the basis elements). We are ready to state our main result:

**Theorem 5.1.** The set of leading monomials of \( G_m \) is

\[
\{ x_0^{(a_0)} \cdots x_s^{(a_s)} \mid 0 \leq s \leq m - 1, a_0 + a_1 + \cdots + a_s > m - s, a_s \neq 0 \}.
\]

**Corollary 5.2.** The set \( \{ x_0^{(a_0)} \cdots x_s^{(a_s)} \mid 0 \leq s \leq m - 1, a_0 + a_1 + \cdots + a_s \leq m - s \} \) is a monomial \( \mathbb{F} \)-linear basis of \( D\mathcal{F}_m / J_m \). In particular, \( \dim (D\mathcal{F}_m / J_m) = \dim W_F(m) = 2^m \).

**Proof.** First part follows from Theorems 5.1 and 3.3(b). Last part is immediate by counting the elements given in the first part. \( \square \)

Putting in the original notation we get:

**Corollary 5.3.** The set \( \{ (x^{-\otimes 1})^{(a_0)} \cdots (x^{-\otimes t^s})^{(a_s)} v_m \mid 0 \leq s \leq m - 1, a_0 + a_1 + \cdots + a_s \leq m - s \} \) is a monomial \( \mathbb{F} \)-linear basis of \( W_F(m) \). \( \square \)
Remark 5. The basis described here is the same basis obtained by Chari-Pressley \[9\] and by Kus-Littelmann \[21\] in the case of $W_C(m)$. The present construction can be thought as a third construction of the same basis for $W_C(m)$. However, it never appeared in the literature in the hyper current context (positive characteristic case), which makes relevant the achievement.

6. Some Applications

In the next two subsections we suppose $\mathbb{F}$ is an algebraically closed field of characteristic zero.

6.1. The basis for $W_\mathbb{F}(m)$ coming from the reverse lexicographic order. The simple combinatorial description of the ideal $J_m$ and its leading terms with respect to a fixed order is also useful to construct bases from other monomial orders in $\mathbb{F}_m$. We can describe a basis for $W_\mathbb{F}(m)$ whenever we can describe the leading term of the elements in a Gröbner basis for $J_m$ with respect to this order similarly to the method of the famous Gröbner Walk algorithm.

The calculation and procedure to construct the basis with respect to the reverse lexicographic order through the basis in the lexicographic case is very technical and we present it separately in Subsection \[7.2\]. We give here only the statement of our second main result. The explanation for disregarding positive characteristic will be given then.

Theorem 6.1. Let

$$ \mathcal{R}_1 = \left\{ x_0^{f_0} x_1^{f_1} \cdots x_{m-1}^{f_{m-1}} v_m \mid f_i \geq 0, \sum_{i=0}^{m-1} f_i \leq \frac{m}{2}, (i-1)f_i + i f_{i+1} \leq m - 2 \sum_{j=i}^{m-1} f_j, \text{ for } 1 \leq i \leq m-2 \right\}, $$

and

$$ \mathcal{R}_2 = \left\{ x_0^{f_0} x_1^{f_1} \cdots x_{m-1}^{f_{m-1}} v_m \mid f_i \geq 0, \sum_{i=0}^{m-1} f_i = k > \frac{m}{2} \text{ and } x_0^{m-2k+f_0} x_1^{f_1} \cdots x_{m-1}^{f_{m-1}} \in \mathcal{R}_1 \right\}. $$

The set $\mathcal{R}_m = \mathcal{R}_1 \cup \mathcal{R}_2$ is a basis for $W_\mathbb{F}(m)$.

6.2. The Chari-Venkatesh construction in \[10\] provides a basis for $W_C(m)$ as follows (we are suiting their original notation to correspond to ours). Let $\mathcal{S}_m$ be the set of $m$-tuples $(i_0, \ldots, i_{m-1}) \in (\mathbb{Z}^+)^m$ such that for all $0 \leq k \leq m-1$ and $1 \leq j \leq k + 1$ we have

$$ ji_k + (j + 1)i_{k+1} + 2 \sum_{p=k+2}^{m-1} i_p \leq m - k + j + 1. $$

Then the elements

$$ \{(x^- \otimes 1)^{i_0} \cdots (x^- \otimes t^{m-1})^{i_{m-1}} \mid (i_0, \ldots, i_{m-1}) \in \mathcal{S}_m \} $$

form a basis for $W_\mathbb{F}(m)$ (cf. \[10\] Theorem 5]).

Surprisingly this basis and the basis coming from the reverse lexicographic order with our Gröbner basis approach are the same.

Proposition 6.2. The Chari-Venkatesh basis and the basis of Theorem 6.1 are the same.

Proof. Let $i = (i_0, i_2, \ldots, i_{m-1})$ and $|i| = \sum_{p=0}^{m-1} i_p \leq \frac{m}{2}$. Then, for $0 \leq k \leq m-1$, and $1 \leq j \leq k + 1$:

$$ ji_k + (j + 1)i_{k+1} + 2 \sum_{p=k+2}^{m-1} i_p \leq m - k + j + 1 $$

In particular, for $j = k + 1$ we have:

$$ (k + 1)i_k + (k + 2)i_{k+1} + 2 \sum_{p=k+2}^{m-1} i_p \leq m $$

which implies
\[(k - 1)i_k + ki_{k+1} + 2 \sum_{p=k}^{m-1} i_p \leq m.\]

Therefore, \(x^i \in \mathcal{R}_1\). Now, let \(i = (i_0, i_2, \ldots, i_{m-1}) \in \mathcal{S}_m\) satisfy \(r = |i| > \frac{m}{2}\). Then, by (6.2) with \(k = 0, j = 1\) we have:

\[(6.3) \quad i_0 + 2i_1 + 2 \sum_{p=2}^{m-1} i_p \leq m.\]

Therefore:
\[
m - 2r + i_0 + 2 \sum_{p=1}^{m-1} i_p \leq 2(m - r) < 2\left(\frac{m}{2}\right) = m
\]

and
\[
m - 2r + i_0 = m + i_0 - 2 \sum_{p=0}^{m-1} i_p = m - i_0 - 2 \sum_{p=1}^{m-1} i_p \geq 0,
\]

where the last inequality follows from condition (6.3). Hence, \((m - 2r + i_0, i_1, \ldots, i_{m-1}) \in \mathcal{S}_m\), since \(i_0\) only enters (6.2) when \(k = 0\).

Finally,
\[
m - 2r + i_0 + \sum_{p=1}^{m-1} i_p = m - 2r + r = m - r \leq \frac{m}{2}
\]

Therefore, we have \(x^{m-2r+i_0}x_1^{i_1} \cdots x_{m-1}^{i_{m-1}} \in \mathcal{R}_1\). Hence, \(x^i \in \mathcal{R}_2\).

We conclude that the Chari-Venkatesh basis is a subset of our basis. In particular, since they are both bases, we see that both sets are equal. \(\square\)

**Remark 6.** In view of this proposition, it is simpler to consider the presentation obtained by Chari-Venkatesh (cf. (6.1)) than the presentation in Theorem 6.1 remembering its linkage with the reverse lexicographical order. The advantage of our interpretation becomes clear from its use in Theorem 6.4. The basis has some interesting properties:

(1) It differs from the basis for \(W_C(m)\) presented in Corollary 5.3 (also Chari-Pressley and Kus-Littelmann bases).

(2) By letting \(\Theta_m\) be the index set of elements of \(\mathcal{R}_m\), the combinatorial description allow us to conclude that these sets are well-behaved with respect to inclusions:

\[\Theta_0 \subseteq \Theta_1 \subseteq \cdots \subseteq \Theta_m.\]

(3) It shows the combinatorial skeleton of the \(\mathfrak{sl}_2\)–module structure. Monomials not divisible by \(x_0\) correspond bijectively with \(\mathfrak{sl}_2\)–highest weight vectors. Such a monomial not involving \(x_0\) has weight \(m - k + 1\) where \(k\) is its degree. The sets \(\mathcal{R}_1\) and \(\mathcal{R}_2\) realize the symmetry between positive and negative \(\mathfrak{sl}_2\)–weight spaces.

(4) The next application is a good reason to consider this basis as special and also well-behaved with respect to truncations.

### 6.3. A basis for truncated local Weyl modules for \(\mathfrak{sl}_2\).

Let \(N \in \mathbb{N}\). For any Lie algebra \(\mathfrak{g}\) as in Subsection \(\Pi\) set \(\mathfrak{g}[t]_N = \mathfrak{g} \otimes \mathbb{F}[t]/t^N\mathbb{F}[t]\). We recall the definition of the \(N\)-truncated local Weyl module for \(U_{\mathbb{F}}(\mathfrak{g}[t]_N)\).
**Definition 6.3.** Given \( \lambda = \sum_{i=1}^r m_i \omega_i \in P^+ \), the \( N \)-truncated local Weyl module \( W_{\lambda}(\lambda, N) \) is the \( U_{\lambda}(g[t]_N) \)-module generated by the element \( v_{\lambda,N} \) with defining relations

\[
(n^+ \otimes F[t]/tN F[t]) \cdot v_{\lambda,N} = (h \otimes tF[t]/tN F[t]) \cdot v_{\lambda,N} = 0 \]

\[
(h - \lambda(h)) \cdot v_{\lambda,N} = (x_\alpha^-)^k \cdot v_{\lambda,N} = 0,
\]

for all \( h \in U(\mathfrak{h}) \), \( \alpha \in R^+ \), \( k > \lambda(h_\alpha) \).

Notice that \( g[t]_N \cong \frac{g[t]}{g[t]_N} \) and \( W_{\lambda}(\lambda, N) \) naturally becomes a \( g[t] \)-module. The universal properties of Weyl modules gives us epimorphisms of \( U \)-basis for \( W_{\lambda}(\lambda, N) \). The universal properties of Weyl modules gives us epimorphisms of \( g[t] \)-modules

\[
W_{\lambda}(\lambda, N) \to W_{\lambda}(\lambda, N) \quad \text{and} \quad W_{\lambda}(\lambda, N) \to W_{\lambda}(\lambda, N') \quad \text{for} \quad N \geq N'.
\]

It is now easy to see that relevant truncations are given by \( N < \lambda(h_\alpha) \), for \( \alpha \in R^+ \), since

\[
W_{\lambda}(\lambda, N) \cong W_{\lambda}(\lambda, N) \quad \text{for} \quad N \geq \max \{ \lambda(h_\alpha) \mid \alpha \in R^+ \}.
\]

In particular, if \( g = sl_2 \) and \( \lambda = m \omega_1 \), denoting \( v_{\lambda,N} \) by \( v_{m,N} \), we get

\[
W_{\lambda}(m, N) \cong \frac{W_{\lambda}(m)}{(x^- \otimes tN v_{m,N})}.
\]

We can use Theorem 6.1 to get a basis for \( W_{\lambda}(m, N) \):

**Theorem 6.4.** If \( N < m \), the set of elements in \( R_m \) which do not involves \( x_N, \ldots, x_{m-1} \) forms a basis for \( W_{\lambda}(m, N) \) for \( U_{\lambda}(sl_2[t]_N) \).

**Proof.** It directly follows from [43] and the Elimination Theorem 3.2 by setting \( Y = \{ x_0, \ldots, x_{N-1} \} \) and \( X = \{ x_N, \ldots, x_{m-1} \} \) and considering the reverse lexicographic order \( x_0 < \cdots < x_{N-1} < x_N < \cdots < x_{m-1} \).

**Remark 7.** The Kus-Littelmann basis for the truncated local Weyl module \( W_{\lambda}(m, N) \) described in [21] is different from the basis constructed here. The construction in that paper uses an advanced realization of these Weyl modules as fusion products and it is an intrinsic construction. The crucial difference comes from the fact that our procedure is based on the construction of a specific basis for graded local Weyl modules (with no truncation) and then we cut off some elements in a very simple way. In particular, by letting \( \Theta_m^N \) be the index set of elements of a basis for \( W_{\lambda}(m, N) \), we conclude that we have the following natural chain of inclusions:

\[
\Theta_m^1 \subseteq \Theta_m^2 \subseteq \cdots \subseteq \Theta_m^{m-1} \subseteq \Theta_m.
\]

**7. Proofs of the main results**

In this section, we will prove the main results.

**7.1. Proof of Theorem 6.1** In what follows all unadorned tensor product symbol \( \otimes \) are considered over the field \( F \) (i.e. \( \otimes_F \)). One of the main problems in module theory is showing when a given ring element acts as zero on a given module element. Sending the module in question to another, more well-understood module can give insight into this problem. Therefore, we use the following helpful construction from [9], now adapted to the positive characteristic setting, as a guide.

Define \( V = \text{span}_F \{ v_+, v_- \} \). We give \( V \otimes F[t] \) a \( U_{\lambda}(sl_2[t]) \)-module structure by the following:

\[
(x^- \otimes t^k) \cdot (v_+ \otimes t^r) = v_- \otimes t^{k+r}, \quad (x^+ \otimes t^k) \cdot (v_- \otimes t^r) = v_+ \otimes t^{k+r},
\]

\[
(x^- \otimes t^k) \cdot (v_- \otimes t^r) = 0, \quad (x^+ \otimes t^k) \cdot (v_+ \otimes t^r) = 0,
\]

\[
(x^- \otimes t^k) \cdot (V \otimes F[t]) = (x^+ \otimes t^k) \cdot (V \otimes F[t]) = 0,
\]

for all \( s > 1, r \geq 0, \) and \( k \geq 0 \).
Thus we also consider the $U_{\mathbb{F}}(\mathfrak{sl}_2[t])$-module $(V \otimes \mathbb{F}[t])^\otimes m$ with the usual tensor product action:

$$(x^\pm \otimes t^k)^{(r)} \cdot (v_1 \otimes v_2) = \sum_{i=0}^{r} (x^\pm \otimes t^k)^{(i)} \cdot v_1 \otimes (x^\pm \otimes t^k)^{(r-i)} \cdot v_2,$$

$$\Lambda_r \cdot (v_1 \otimes v_2) = \sum_{i=0}^{r} \Lambda_i \cdot v_1 \otimes \Lambda_{r-i} \cdot v_2,$$

$$\left( \frac{h}{r} \right) \cdot (v_1 \otimes v_2) = \sum_{i=0}^{r} \left( \frac{h}{i} \right) \cdot v_1 \otimes \left( \frac{h}{r-i} \right) \cdot v_2.$$

If $M$ is a $U_{\mathbb{F}}(\mathfrak{sl}_2[t])$-module then by $(M^\otimes k)^\Sigma_k$ we mean the $U_{\mathbb{F}}(\mathfrak{sl}_2[t])$-submodule of $M^\otimes k$ of invariants under permutation of tensor factors. Also, note that

$$(V \otimes \mathbb{F}[t])^\otimes m \rightarrow ((V \otimes \mathbb{F}[t])^\otimes m)^{\Sigma_m}$$

under the homomorphism $\text{Sym}$, which maps a tensor to the sum of all distinct permutations of the positions of the vectors. The $U_{\mathbb{F}}(\mathfrak{sl}_2[t])$-module action on $((V \otimes \mathbb{F}[t])^\otimes m)^{\Sigma_m}$ induces an isomorphism

$$(V \otimes \mathbb{F}[t])^\otimes m \cong (V^\otimes m \otimes \mathbb{F}[t_1, \ldots, t_m])^{\Sigma_m}$$

which maps $\text{Sym}((v_1 \otimes t_1^1) \otimes (v_2 \otimes t_2^2) \otimes \cdots \otimes (v_k \otimes t_k^k))$ to $\text{Sym}(v_1 \otimes v_2 \otimes \cdots \otimes v_k) \otimes M_{(r_1, r_2, \ldots, r_k)}(t_1, t_2, \ldots, t_m)$. Here and further $M_\lambda(t_1, t_2, \ldots, t_m)$ is the monomial symmetric polynomial in $m$ variables associated with the partition $\lambda$. Now, consider the submodule

$$W_m := (V^\otimes m \otimes \mathbb{F}[t_1, \ldots, t_m])^{\Sigma_m}$$

where

$$\mathbb{F}[t_1, \ldots, t_m]^{\Sigma_m} = \frac{\mathbb{F}[t_1, \ldots, t_m]}{(\mathbb{F}[t_1, \ldots, t_m])^+}$$

is the coinvariant algebra of $\Sigma_m$.

There is a $\mathcal{DF}_m$-module homomorphism $\mathcal{DF}_m \xrightarrow{\psi} W_m$ defined by $1 \mapsto v_+^\otimes m \otimes 1$, where $W_m$ is considered a $\mathcal{DF}_m$-module by $\mathcal{DF}_m \hookrightarrow \mathcal{DF}_\infty \cong U_{\mathbb{F}}(n-1)[t]$ as discussed in Section 5. We here and further assume the polynomial in the second tensor factor to be reduced in $\mathbb{F}[t_1, \ldots, t_m]^{\Sigma_m}$. We can prove that $\psi$ is surjective using an argument similar to [9 Lemma 6.3].

Let $\lambda$ be a partition satisfying $\lambda_1 \leq m - 1$ and $\ell(\lambda) = r \leq m$, where $0$ may be included as a part. Let $x^{(\lambda)} = x_1^{(j_0)} x_1^{(j_1)} \cdots x_1^{(j_{m-1})}$ with $j_k = m_k(\lambda)$ be the monomial corresponding to $\lambda$ in $\mathcal{DF}_m$. Let $\Sigma_{j_0, j_1, \ldots, j_{m-1}}$ be the subgroup of all $\rho \in \Sigma_r$ satisfying $\rho(s_k + 1) < \rho(s_k + 2) < \cdots < \rho(s_{k+1})$, $k = 0, 1, m - 2$ where $s_k = \sum_{i=0}^{k-1} j_i$ and $s_0 = 0$. This is called the group of $(j_0, j_1, \ldots, j_{m-1})$-shuffles of $r$. We compute

$$\psi(x^{(\lambda)}) = x_1^{(j_0)} x_1^{(j_1)} \cdots x_1^{(j_{m-1})} \cdot (v_+ \otimes 1)^\otimes m = \sum_{\rho \in \Sigma^m_{j_{m-1} \ldots j_0}} \rho((v_- \otimes t^{s_{m-1} - 1}) \otimes \cdots \otimes (v_- \otimes 1)^\otimes j_0 \otimes (v_+ \otimes 1)^\otimes r) = \sum_{\rho \in \Sigma^m_{j_{m-1} \ldots j_0}} \rho(v_-^\otimes \otimes v_+^\otimes \otimes t_1^{\lambda_1} \otimes \cdots \otimes t_r^{\lambda_r})$$

$$= \sum_{\sigma \in \Sigma_{\lambda, m-r}} \sigma(v_-^\otimes \otimes v_+^\otimes) \otimes \sum_{\pi \in \Sigma_{j_{m-1} \ldots j_0}} t_1^{\lambda_1} t_2^{\lambda_2} \cdots t_r^{\lambda_r}$$

$$= \sum_{\sigma \in \Sigma_{\lambda, m-r}} \sigma(v_-^\otimes \otimes v_+^\otimes) \otimes M_\lambda(t_\sigma(1), t_\sigma(2), \ldots, t_\sigma(r)).$$
The notation $\eta$ is the symmetric polynomial defined by

$$
(7.1) \quad s_\lambda(t_1, t_2, \ldots, t_r) = \sum_{\mu \leq \lambda} K_\lambda \mu M_\mu(t_1, t_2, \ldots, t_r),
$$

where the positive integers $K_\lambda \mu$ are the Kostka numbers (see [22]), satisfying $K_{\lambda \lambda} = 1$. We also define, letting $\lambda$ be a partition with at most $r$ parts with highest part $\lambda_1 < m$, which we extend to have $r$ parts by adding 0 where necessary:

$$
s_{\lambda, r}(x_0, x_1, \ldots, x_{m-1}) = \sum_{\mu \leq \lambda} K_{\lambda \mu} x^{(\mu)}
$$

and compute

$$
\psi(s_{\lambda, r}(x_0, x_1, \ldots, x_{m-1})) = \text{Sym} \left( v^{\otimes r}_- \otimes v^{\otimes m-r}_+ \otimes \sum_{\mu \leq \lambda} K_{\lambda \mu} M_\mu(t_1, t_2, \ldots, t_r) \right)
$$

$$
= \text{Sym} \left( v^{\otimes r}_- \otimes v^{\otimes m-r}_+ \otimes s_\lambda(t_1, t_2, \ldots, t_r) \right).
$$

The complete homogeneous symmetric polynomial in $r$ variables is defined by

$$
h_k(t_1, t_2, \ldots, t_r) = \sum_{j_1 + j_2 + \cdots + j_r = k; \ j_i \geq 0} t_1^{j_1} t_2^{j_2} \cdots t_r^{j_r}.
$$

They are important to us because of the following result, which gives us our first application of Gröbner basis theory:

**Theorem 7.1** ([24]). For any field, the set of polynomials $\{h_{m-r+1}(t_1, t_2, \ldots, t_r) | 1 \leq r \leq m\}$ is a Gröbner basis of $\mathbb{F}[t_1, \ldots, t_m]_{\Sigma m}$ with respect to lexicographic order such that $t_1 < t_2 < \cdots < t_m$. □

**Proposition 7.2.** Fix $0 \leq k \leq m$. Let $\lambda$ be a partition with highest part $\lambda_1 \leq m - k$ and $\ell(\lambda) \leq k$. Then

$$
\psi(s_{\lambda, k}(x_0, x_1, \ldots, x_{m-1})) \neq 0.
$$

**Proof.** We have $\psi(s_{\lambda, r}(x_0, x_1, \ldots, x_{m-1})) = 0$ if, and only if, $s_\lambda(t_1, t_2, \ldots, t_k) = 0 \in \mathbb{F}[t_1, t_2, \ldots, t_m]_{\Sigma m}$. The leading term of $s_\lambda(t_1, t_2, \ldots, t_k)$ is $t_{\lambda_1}^{\lambda_1} \cdots t_{\lambda_k}^{\lambda_k}$. For $r \leq k$, consider $h_{m-r+1}(t_1, t_2, \ldots, t_r)$ whose leading monomial is $t_{r+1}^{m-r+1}$. Since $m - r + 1 \geq m - k + 1 > \lambda_1$, $t_{r+1}^{m-r+1}$ does not divide the leading monomial of $s_\lambda$. But if $r > k$ then $t_r$ does not appear in $s_\lambda(t_1, t_2, \ldots, t_k)$. Therefore, $\psi(s_{\lambda, k}(x_0, x_1, \ldots, x_{m-1})) \neq 0$ with the given conditions on $\lambda$. □

Let $k \in \{2, \ldots, m + 1\}$ and $\lambda$ be a partition such that all the parts of $\lambda$ are $\leq m - 1$. Here, $\ell(\lambda) \leq m + 1$ and no other restriction is made. Note that $\lambda$ is independent of $k$. Define $f_{\lambda, k}$ to be the following homogeneous function of degree $k$ (the sum is over partitions $\mu$ of length $k$ each of whose parts is $\leq m - 1$ with 0 possibly included):

$$
f_{\lambda, k}(x_0, x_1, \ldots, x_{m-1}) = \sum_{\mu \geq \lambda} D_{\lambda \mu} x^{(\mu)},
$$

where

$$
D_{\lambda \mu} = (-1)^{\vert \mu \vert - \ell(\lambda)} \sum_{\eta^1, \eta^2, \ldots, \eta^{\ell(\mu)}} \frac{\ell(\eta^i)!}{\prod_{j=1}^{\ell(\mu)} m_{\mu_j}(\eta^j)!}
$$

and the sum is over all sequences of partitions such that $\eta^1 \uplus \eta^2 \uplus \cdots \uplus \eta^{\ell(\mu)} = \lambda$ and $\eta^i \vdash \mu_i$. The notation $\eta^1 \uplus \eta^2$ denotes the partition $(\eta_{11}^1, \eta_{21}^1, \ldots, \eta_{11}^i), (\eta_{11}^2, \eta_{21}^2, \ldots, \eta_{11}^i)$ with terms arranged in decreasing order. Observe that $\mu \geq \lambda$ implies $\mu \leq_{\text{revlex}} \lambda$. The $D_{\lambda \mu}$ appear in the following expression for the “forgotten symmetric polynomials” (see [22] page 22, [32] Equation (2.41)):
\[ f_\lambda(t_1, t_2, \ldots) = \sum_{\mu \geq \lambda} D_{\lambda\mu} M_\mu(t_1, t_2, \ldots). \]

We now use Proposition 2.2 to prove:

**Proposition 7.3.** \( f_{\lambda, k}(x_0, x_1, \ldots, x_{m-1}) \) is in \( J_m \) if \( \ell(\lambda) > m - k \).

**Proof.** We want to show that \( f_{\lambda, k}(x_0, x_1, \ldots, x_{m-1}) \) is, up to a sign, given by the coefficient of \( u_1^{m_1(\lambda)} u_2^{m_2(\lambda)} \cdots u_{\lambda_1}^{m_{\lambda_1}(\lambda)} \) in \( Y[\lambda_1]^{(k)} \). By expanding, we have

\[
Y[\lambda_1]^{(k)} = \left( \sum_{\eta} (-1)^{\ell(\eta)} \frac{\ell(\eta)!}{\prod_{i=1}^{\lambda_1} m_i(\eta)!} \prod_{i=1}^{\lambda_1} u_i^{m_i(\eta)} \right)^{(k)}
= \sum_{\{\nu \mid k\}} \sum_{\nu, \nu^1, \nu^2, \ldots, \nu^{\ell(\mu)}} \left( (-1)^{\ell(\nu^j)} \frac{\ell(\nu^j)!}{\prod_{i=1}^{\lambda_1} m_i(\nu^j)!} x_{\nu^j} \prod_{i=1}^{\lambda_1} u_i^{m_i(\nu^j)} \right)^{(v_j)}
= \sum_{\{\nu \mid k\}} \sum_{\nu, \nu^1, \nu^2, \ldots, \nu^{\ell(\mu)}} \left( (-1)^{\ell(\nu^j)} \frac{\ell(\nu^j)!}{\prod_{i=1}^{\lambda_1} m_i(\nu^j)!} x_{\nu^j} \prod_{i=1}^{\lambda_1} u_i^{m_i(\nu^j)} \right)^{(v_j)}
\]

In the above, the inner sum is over distinct partitions. Now we change variables by letting \( \mu \) be such that \( m_{\nu^j}(\mu) = \nu_j \) and \( (\nu^j)_{j=1}^{\ell(\mu)} \) be the sequence of partitions resulting taking \( \nu^j \) repeated \( m_{\nu^j}(\mu) \) times. We get

\[
\sum_{\{\mu \mid \ell(\mu) = k\}} \sum_{\nu^1, \nu^2, \ldots, \nu^{\ell(\mu)}} \prod_{j=1}^{\ell(\mu)} \left( (-1)^{\ell(\nu^j)} \frac{\ell(\nu^j)!}{\prod_{i=1}^{\lambda_1} m_i(\nu^j)!} x_{\nu^j} \prod_{i=1}^{\lambda_1} u_i^{m_i(\nu^j)} \right) x^{(\mu)} \prod_{i=1}^{\lambda_1} u_i^{m_i(\nu^j)}. \]

It is important to note that \( \mu \) may contain 0 in the sum above. We take the coefficient of the term \( u_1^{m_1(\lambda)} u_2^{m_2(\lambda)} \cdots u_{\lambda_1}^{m_{\lambda_1}(\lambda)} \). This implies that, for \( 1 \leq i \leq \lambda_1 \):

\[
m_i(\lambda) = \sum_{j=1}^{\ell(\mu)} m_i(\nu^j). \]

Therefore, \( \lambda = \nu^1 \uplus \nu^2 \uplus \cdots \uplus \nu^{\ell(\mu)} \), and \( |\mu| = |\lambda| \). Hence, the coefficient of \( u_1^{m_1(\lambda)} u_2^{m_2(\lambda)} \cdots u_{\lambda_1}^{m_{\lambda_1}(\lambda)} \) is

\[
\sum_{\{\mu \mid \ell(\mu) = k\}} (-1)^{\ell(\lambda)} \sum_{\nu^1, \nu^2, \ldots, \nu^{\ell(\mu)}} \prod_{j=1}^{\ell(\mu)} \left( \frac{\ell(\nu^j)!}{\prod_{i=1}^{\lambda_1} m_i(\nu^j)!} \right) x^{(\mu)},
\]

where the inner sum is over all \( \nu^1, \nu^2, \ldots, \nu^{\ell(\mu)} \) such that \( \nu^j \vdash \mu_j \) and \( \lambda = \nu^1 \uplus \nu^2 \uplus \cdots \uplus \nu^{\ell(\mu)} \). But this is just:

\[
\sum_{\{\mu \mid \ell(\mu) = k\}} (-1)^{|\lambda|} D_{\lambda\mu} x^{(\mu)} = (-1)^{|\lambda|} f_{\lambda, k}(x_0, x_1, \ldots, x_{m-1}).
\]

By Proposition 2.2, elements in \( J_m \) correspond to coefficients of \( u^\lambda \) in \( Y[\lambda_1]^{(k)} \) where \( k + \ell(\lambda) > m \). □

The transition matrix from \( f_\mu \) to \( s_\lambda \) (22 Table 1) gives:

\[
(7.2) \quad s_\lambda(t_1, t_2, \ldots, t_k) = \sum_{\mu \geq \lambda'} K_{\lambda'\mu} f_\mu(t_1, t_2, \ldots, t_k).
\]

The identity in (22 page 22) holds in infinitely many variables, therefore we get (7.2) by restricting to finitely many variables. We see that

\[
(7.3) \quad s_{\lambda, k}(x_0, x_1, \ldots, x_{m-1}) = \sum_{\mu \geq \lambda'} K_{\lambda'\mu, k} f_{\mu, k}(x_0, x_1, \ldots, x_{m-1}).
\]
Now let $\lambda$ satisfy $\lambda_1 + k > m$. We want to show that for all $\mu \leq \lambda'$ we have $\ell(\mu) + k > m$. We have $\mu' \geq \lambda$, hence $\ell(\mu) + k = \mu' + k \geq \lambda_1 + k > m$. Therefore, all $f_{\mu,k}$ appearing in the right sum are in $J_m$, and all the $K_{\lambda',\mu}$ are integers. Therefore, $s_{\lambda,k}(x_0, x_1, \ldots, x_{m-1}) \in J_m$. We now finish the proof of Theorem 5.1.

**Proof.** Let $G_m = \{s_{\lambda,k}(x_0, x_1, \ldots, x_{m-1})|\lambda_1 + k > m\}$. We verify that $(S, G_m)$ is a Gröbner-Shirshov basis. First, $G_m$ is a generating set of $J_m$. Next, by Proposition 7.3 Equation 7.3 and the fact that every monomial $x(\mu)$ in $D_{F,m}$ can be written as an integral combination of $s_{\mu,k}(x_0, x_1, \ldots, x_{m-1})$ with $\lambda > \mu$, we see that every polynomial $f \in J_m$ has a unique integral expansion in terms of $s_{\lambda,k}(x_0, x_1, \ldots, x_{m-1}) \in G_m$. Let $p, q \in G_m$ have a composition. Then $S(p, q)$ can be written as a linear combination of some terms in $S$ to make the monomials reduced and $s_{\mu,k}(x_0, x_1, \ldots, x_{m-1}) \in G_m$ where we have $w \geq \lambda \equiv LT(s_{\mu,k}(x_0, x_1, \ldots, x_{m-1}))$. Therefore, $S(p, q) \equiv 0 \mod (S, G_m)$. The remaining compositions are easily seen to be $\equiv 0 \mod S$ or $(S, G_m)$, respectively. Therefore, $(S, G_m)$ is closed under compositions, hence a Gröbner-Shirshov pair for $W_m$. □

7.2. **Proof of Theorem 6.1.** Now, fix an integer $k$ and a partition $\lambda$ such that $\ell(\lambda) + k > m$. We can see that $D_{\lambda} \mu$ are integers satisfying $D_{\lambda} \lambda \equiv \pm 1$ and $D_{\lambda} \mu = 0$ unless $\mu > \lambda$. Unfortunately, if $\ell(\lambda) > m/2$ then some terms will drop out of the expression for $f_{\lambda,k}$, and the remaining leading coefficient can be something other than 1. Therefore, we only consider the characteristic 0 case in this section. In this case, notice that the notion of Gröbner-Shirshov bases is purely replaced by Gröbner bases as in Subsection 3.1 due to Remark 11.

Let $S_{\text{revlex}} = \{f_{\lambda,k}(x_0, x_1, \ldots, x_{m-1}) | 2 \leq k \leq m + 1, \ell(\lambda) \geq m - k + 1\}$. We show that $S_{\text{revlex}}$ is a Gröbner basis of $W_\ell'(m)$. We remark that $S_{\text{revlex}}$ is larger than we need to be a Gröbner basis of $W_\ell'(m)$ for all $m \geq 1$. It needs to be proven that $S_{\text{revlex}}$ gives the correct dimension for the basis of $W_\ell'(m)$.

We first give an alternative characterization of the monomials appearing in $LM(S_{\text{revlex}})$. Let $\mu$ be a partition such that $2 \leq \ell(\mu) \leq m/2$. Let $\eta \geq \text{revlex} \mu$ be the least partition of $|\mu|$ with respect to reverse lexicographic order such that $\ell(\eta) = m - \ell(\mu) + 1$. We have following:

**Lemma 7.4.** Let $i^*$ be the least index such that $\eta_{i^*} < \mu_{i^*}$. Then $\eta_i = 1$ for all $i \in \{i^*, i^* + 1, \ldots, \ell(\eta)\}$.

**Proof.** Suppose $\eta_i > 1$ for some $i > i^*$ and that $i$ is the maximum with this property. Then we could move a box in $\eta$ from column $i$ to column $i^*$ to get a partition $\rho \vdash |\mu|$ such that $\mu \leq \text{revlex} \rho < \text{revlex} \eta$ and $\ell(\rho) = \ell(\eta) = m - \ell(\mu) + 1$ contradicting our choice of $\eta$. □

The above lemma can be used to prove that $\eta$ is given by the following algorithm:

**Algorithm 1**

Let $r := m - 2\ell(\mu) + 1$
Let $\eta := \mu$
Let $i := \ell(\mu)$
while $i \geq 1$ do
  if $\mu_i \leq r$ then
    $r := r - \mu_i + 1$
    Append 1 to the end of $\eta$ ($\mu_i - 1$ times)
    $\eta_i := 1$
    $i := i - 1$
  else
    $\eta_i := \mu_i - r$
    Append 1 to the end of $\eta$ ($r - 1$ times)
    break
  end if
end do
return $\eta$
Lemma 7.5. Let $i^*$ be the least index such that $\eta_{i^*} < \mu_{i^*}$. Then $\mu = \nu$ if, and only if, $\mu_{i^*} - \mu_{\ell(\mu)} \leq 1$.

Proof. If $1 \leq i' < i^*$ then $\eta_{i'} = \mu_{i'}$, hence $\mu_{i'} = \nu_{i'}$. Now suppose that $\mu_{i^*} - \mu_{\ell(\mu)} > 1$. Then we can move one box from the $i^*$ column of $\mu$ to another non-empty column to obtain a new partition $\rho$. In the worst case $\rho_{i^*} = \eta_{i^*}$ and $\rho_{i^*+1} > \eta_{i^*+1} = 1$. Therefore, $\eta > \rho_{\text{revlex}} \rho > \rho_{\text{revlex}} \mu$ but $\nu$ is the greatest partition of $|\mu|$ that is $\leq_{\text{revlex}} \eta$, hence $\mu \neq \nu$. Conversely, if $\mu_{i^*} - \mu_{\ell(\mu)} \leq 1$ then no such move can be made. It follows that $\mu = \nu$. \hfill $\Box$

Next, we give a characterization of the monomials that are unreduced with respect to the leading monomials of polynomials in $S_{\text{revlex}}$.

Lemma 7.6. (1) Let $x^\mu = x_0^m x_1^{f_1} \cdots x_{m-1}^{f_{m-1}}$ with $f_i = m_i(\mu)$ be a monomial of degree $k \leq m/2$ that is not in $LM(S_{\text{revlex}})$. Then for $1 \leq i \leq m-1$ we have $(i-1)f_i + if_{i+1} \leq m - 2 \sum_{j=i}^{m-1} f_j$. The converse is also true.

(2) If $x^\lambda$ is unreduced and $\deg(x^\lambda) > m/2$ and $\lambda^+$ is the subsequence of $\lambda$ with 0 removed, then $x^{\lambda^+}$ is in the set $R_1$.

Proof. (1) Suppose it were the case that $(i-1)f_i + if_{i+1} > m - 2 \sum_{j=i}^{m-1} f_j$ for some $i \in \{1, 2, \ldots, m-2\}$. Consider the subsequence $x_i^{f_i} x_{i+1}^{f_{i+1}} \cdots x_{m-1}^{f_{m-1}}$ of $x^\mu$. On $\mu$, Algorithm 1 would terminate when $\mu_{i^*} = i$ or $i+1$. In either case $\mu_{i^*} - \mu_{\ell(\mu)} \leq 1$, which would imply (by Lemma [7.5]) that $x^\mu$ is the leading monomial of $f_{\eta,\ell(\mu)} \in S_{\text{revlex}}$. We can see that the coefficient of $x^\mu$ is non-zero using the definition of $D_{\eta\mu}$. In that expression, we have $\eta = (\eta_1) \cup (\eta_2) \cup \cdots \cup (\eta_{i^*}) \cup (\eta_{i^*+1}, 1^{\mu_{i^*+1} - \eta_{i^*+1}}) \cup (1^{\mu_{i^*+2}}) \cup \cdots \cup (1^{\mu_{\ell(\mu)}})$. Therefore, for $x^\mu$ to be reduced we must have $(i-1)f_i + if_{i+1} \leq m - 2 \sum_{j=i}^{m-1} f_{j}$ for $1 \leq i \leq m - 2$.

Now, suppose that $(i-1)f_i + if_{i+1} \leq m - 2 \sum_{j=i}^{m-1} f_j$ for all $i \in \{1, 2, \ldots, m-2\}$. Let $g_i \leq f_i$, $0 \leq i \leq m - 2$. Then

$$(i-1)g_i + ig_{i+1} \leq (i-1)f_i + if_{i+1} \leq m - 2 \sum_{j=i}^{m-1} f_j \leq m - 2 \sum_{j=i}^{m-1} g_j.$$ 

Therefore, no subsequence of $x^\mu$ is in $LM(S_{\text{revlex}})$, which implies that $x^\mu$ is unreduced in $S_{\text{revlex}}$.

(2) Now suppose $\deg(x^\lambda) > m/2$. Let $\lambda^+$ be the subsequence of $\lambda$ without 0 as a part. If $\ell(\lambda^+) > m/2$ then $f_{\lambda^+,k} \in S_{\text{revlex}}$ and $LM(f_{\lambda^+,k}) = x^\lambda$. Therefore, for $x^\lambda$ to be reduced we must have $\ell(\lambda^+) \leq m/2$. But any subsequence of a reduced monomial must be reduced, hence $x^{\lambda^+}$ is reduced and is in $R_1$. \hfill $\Box$

Theorem 7.7. There are $2^m$ monomials in $R_1 \cup R_2$. In particular, $R_1 \cup R_2$ is a basis of $W(m)$.

Proof. We start by enumerating the monomials whose degree in $x_0$ is 0. For $t \geq 1$, $0 \leq \ell \leq m/2$, $s \geq 0$ define $g_{t, \ell, s, m}$ to be the number of monomials of degree $\ell$ in $R_1$ for which each $x_i$ satisfies $t \leq i \leq m-1$, and such that the degree of $x_t$ is $s$. For reasons that will become clear, we also define $g_{t, \ell, -1, m} := g_{t, \ell+1, 0, m}$.

We now prove that the function $g_{t, \ell, s, m}$ satisfies the following recursion for $0 \leq \ell \leq m/2$, $1 \leq t \leq m - 1$, and $-1 \leq s \leq \ell$:

$$g_{t, \ell, s, m} = \begin{cases} \sum_{j=0}^{\ell-s} H(m - 2\ell - tj - (t-1)s)g_{t+1, \ell-s, j, m} & \text{if } m - 2\ell \geq (t-1)s, \\ 0 & \text{otherwise}, \end{cases}$$

where

$$H(n) = \begin{cases} 1 & \text{if } n \geq 0, \\ 0 & \text{otherwise}. \end{cases}$$
The boundary conditions are:

\[ g_{m,\ell,s,m} = \begin{cases} 
1 & \text{if } \ell = s = 0, \\
0 & \text{otherwise.}
\end{cases} \]

These are easily verified. If \( s \geq 0 \) then Equation (7.4) follows if we let \( X \) be a monomial counted by \( g_{t,\ell,s,m} \) under the given conditions on \( t, \ell, s, \) and \( m \). Then \( m - 2\ell \leq (t-1)s \), and if we delete all the occurrences of \( x_t \) from \( X \) the resulting monomial satisfies \( m - 2\ell \geq tj + (t-1)s \), is of degree \( \ell - s \), and has only terms \( x_i \) where \( i \geq t+1 \). Conversely, if \( X' \) is a monomial counted by \( g_{t+1,\ell-s,j,m} \) for a \( 0 \leq j \leq \ell - s \), and \( m - 2\ell \leq (t-1)s \) and \( m - 2\ell \geq tj + (t-1)s \), then we can and do multiply \( X' \) by \( x_t^s \) to get a monomial counted by \( g_{t,\ell,s,m} \).

If \( s = -1 \) then Equation (7.4) formally gives:

\[ g_{t,\ell,-1,m} = \sum_{j=0}^{\ell+1} H(m-2\ell-tj + (t-1))g_{t+1,\ell+1,j,m} \]

\[ = \sum_{j=0}^{\ell+1} H(m-2(\ell+1)-tj)g_{t+1,\ell+1,j,m} \]

\[ = g_{t,\ell+1,0,m}. \]

In the second line, we have used the fact that if \( m - 2\ell < tj - (t - 1) \) then \( m - 2(\ell+1) < tj \), but if \( m - 2\ell \geq tj - (t - 1) \) and \( m - 2(\ell+1) < tj \) then \( g_{t+1,\ell+1,j,m} = 0 \). This explains why it makes sense to define \( g_{t,\ell,-1,m} \) as we have done, and proves Equation (7.4) in the case \( s = -1 \).

The recursion in (7.4) is inductive since each term involving \( g \) the right increases in \( t \). Therefore, it determines the values of \( g \).

Next, we prove the following relation, where \( 1 \leq t \leq m-1, \ell \geq 1 \) and \( 0 \leq s \leq \ell \) and \( m \geq 1 \):

\[ g_{t,\ell,s,m} = \begin{cases} 
g_{t,\ell,s+1,m-1} + g_{t,\ell-1,s-1,m-1}, & \text{if } m - 2\ell \geq (t-1)s \\
0, & \text{otherwise.}
\end{cases} \tag{7.5} \]

If \( m - 2\ell < (t-1)s \) then \( g(t,\ell,s,m) = 0 \) so there is nothing to prove. Suppose \( m - 2\ell \geq (t-1)s \).

We use reverse induction on \( t \). We verify

\[ g_{m-1,\ell,s,m} = g_{m-1,\ell,s+1,m-1} + g_{m-1,\ell-1,s-1,m-1} = 0, \text{ if } (\ell,s) \neq (1,1) \]

\[ g_{m-1,1,1,m} = g_{m-1,1,2,m-1} + g_{m-1,0,0,m-1} = 1. \]

Suppose, now, that Equation (7.5) holds for \( t + 1 \). By (7.4) we have:

\[ g_{t,\ell,s,m} = g_{t+1,\ell-s,0,m} + \sum_{j=1}^{\ell-s} H(m-2\ell-tj-(t-1)s)g_{t+1,\ell-s,j,m} \]

\[ = g_{t+1,\ell-s,0,m} + \sum_{j=1}^{\ell-s} H(m-2\ell-tj-(t-1)s)g_{t+1,\ell-s,j+1,m-1} \]

\[ + \sum_{j=1}^{\ell-s} H(m-2\ell-tj-(t-1)s)g_{t+1,\ell-s-1,j-1,m-1} \]

\[ = g_{t+1,\ell-s,1,m-1} + g_{t+1,\ell-s-1,-1,m-1} \]

\[ + \sum_{j=1}^{\ell-s} H(m-1-2(\ell-1)-t(j+1)-(t-1)(s-1))g_{t+1,\ell-s,j+1,m-1} \]

\[ + \sum_{j=1}^{\ell-s} H(m-1-2(\ell-1)-t(j+1)-(t-1)(s-1))g_{t+1,\ell-s,j+1,m-1} \]
We rearrange the sums and use the fact that
\[
B \ell = 0 \quad \text{in the second equality, we have used the inductive hypothesis,} \quad \text{in the third we use a simple algebraic equivalence of the terms inside} \quad H, \quad \text{in the fourth we shift the index of the sums, in the fifth we rearrange the sums and use the fact that} \quad g(t + 1, \ell - s, \ell - s + 1, m - 1) = 0, \quad \text{and in the last we used} \quad (7.31) \quad \text{again.}
\]

We now use recursion (7.35) to give a familiar recursion for the number of monomials in \( R_1 \) with degree \( \ell \) and \( x_0 \) degree 0. Let \( B_{m, \ell} \) denote the cardinality of this set. Then, for \( 1 \leq \ell \leq \lfloor m/2 \rfloor \),

\[
B_{m, \ell} = g_{1, \ell, 0, m} + \sum_{s=1}^{\ell} g_{1, \ell, s, m}
\]

\[
= g_{1, \ell, 1, m-1} + g_{1, \ell-1, s, m} + \sum_{s=1}^{\ell} g_{1, \ell, s+1, m} + \sum_{s=1}^{\ell} g_{1, \ell-1, 1, m}
\]

\[
= B_{m-1, \ell} + B_{m-1, \ell-1}.
\]

When \( \ell = 0 \), there is just the monomial 1 so \( B_{0, m} = 1 \). There are \( m - 1 \) monomials of degree 1, hence \( B_{1, m} = m - 1 \) when \( m > 0 \). These boundary conditions and recursions have the solution: \( B_{m, \ell} = \binom{m}{\ell} - \binom{m}{\ell-1} \). The total number of monomials of degree \( k \) satisfying (1) is hence \( \sum_{\ell=0}^{k} B_{m, \ell} = \binom{m}{k} \). By definition, the total number of monomials of degree \( k \) in \( R_2 \) is \( \binom{m}{m-k} = \binom{m}{k} \). The total number of monomials in \( R_1 \cup R_2 \) is \( \sum_{k=0}^{m} \binom{m}{k} = 2^m \). \( \square \)

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