Orders occurring as endomorphism ring of a Drinfeld module in some isogeny classes of Drinfeld modules of higher ranks

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Abstract

The question we propose to answer throughout this paper is the following: Given an isogeny class of Drinfeld modules over a finite field, what are the orders of the corresponding endomorphism algebra (which is an isogeny invariant) that occur as endomorphism ring of a Drinfeld module in that isogeny class?

It is worth mentioning that this question is different from the ones investigated by the authors Kuhn, Pink in [6] and Garai, Papikian in [3]. The former authors rather provided an answer to the question, given a Drinfeld module $\phi$, how does one efficiently compute the endomorphism ring of $\phi$?

The importance of our question resides in the fact that it might be very helpful to better understand isogeny graphs of Drinfeld modules of higher rank ($r \geq 3$) and may be reopen the debate concerning the application to isogeny-based cryptography.

We answer that question for the case whereby the endomorphism algebra is a field by providing a necessary and sufficient condition for a given order to be the endomorphism ring of a Drinfeld module. We apply our result to rank $r = 3$ Drinfeld modules and explicitly compute those orders occurring as endomorphism rings of rank 3 Drinfeld modules over a finite field.

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1 Endomorphism rings of Drinfeld modules of higher ranks

Notations:

- $A = \mathbb{F}_q[T]$ : Ring of univariate polynomials in $T$ over a finite field $\mathbb{F}_q = \mathbb{F}_{p^r}$, $p$ prime.
- $k = \mathbb{F}_q(T)$ : Rational function field over $\mathbb{F}_q$.
- $L$ : Finite $A$-field.
- $p_v$ : (Generator of the) Kernel of the $\mathbb{F}_q$-algebra homomorphism $\gamma$ defining the $A$-field $L$.
- $v$ : The place of $k$ defined by $p_v$.
- $\infty$ : The place at infinity of $k$.
- $m$ : The degree of $L$ over $A/p_v$, i.e. $m = [L : A/p_v]$.

Nkotto proved in [1], that a general rank $r$ Weil polynomial defining an isogeny class of rank $r$ Drinfeld modules has the form

$$M(x) = x^{r_1} + a_1 x^{r_1-1} + \cdots + a_{r_1-1} x + \mu p_v^{r_2}$$

where $r_1 = [k(\pi) : k]$, $r_2 = \sqrt{\dim k(\pi)} \cdot \text{End} \otimes_A k$ and $r = r_1 r_2$.

Therefore our restriction on the endomorphism algebra (of the corresponding isogeny class) that must be a field, leads to the restriction to isogeny classes defined by Weil polynomials of the form

$$M(x) = x^r + a_1 x^{r-1} + \cdots + a_{r-1} x + \mu p_v^m$$

We aim in this part to prove the following theorem.

**Theorem 1.1.** $A = \mathbb{F}_q[T]$, $k = \mathbb{F}_q(T)$ and $p_v$ is the (generator of the) kernel of the characteristic morphism $\gamma : A \rightarrow L$ defining the finite $A$-field $L$. $M(x) = x^r + a_1 x^{r-1} + \cdots + a_{r-1} x + \mu p_v^m \in A[x]$ is a Weil polynomial, where $m = [L : A/p_v, A]$. Let $O$ be an $A$-order of the function field $k(\pi) = k[x]/M(x) \cdot k[x]$. Let $v_0$ be the unique zero of the Frobenius endomorphism $\pi$ in $k(\pi)$ lying over the place $v$ of $k$.

$O$ is the endomorphism ring of a Drinfeld module in the isogeny class defined by the Weil polynomial $M(x)$ if and only if $O$ contains $\pi$ and $O$ is maximal at the place $v_0$.

Before proving it, let us recall the notions of Tate modules and Dieudonné modules which are very important for the proof.

1.1 Tate module of a Drinfeld module

Let $\psi$ be a Drinfeld module over the $A$-field $L$ with $A$-characteristic $p_v$. $v$ denotes the place of $k$ associated to the prime $p_v$. Let $\omega$ be a place of $k$ different from $v$ and $p_\omega$ denotes the corresponding prime. $\psi[p_\omega^n]$ denotes the group of $p_\omega^n$-torsion points of $\psi$.

**Definition 1.1.** The Tate module of $\psi$ at $\omega$ is defined by the inverse limit

$$T_\omega \psi := \varprojlim_{n \geq 1} \psi[p_\omega^n] = \text{Hom}_{A_{\omega}}(k_{\omega}/A_{\omega}, \psi[p_\omega^n])$$

where $\psi[p_\omega^n] = \bigcup_{n \geq 1} \psi[p_\omega^n]$. 


1.2 Dieudonné module of a Drinfeld module

Remark 1.1. (Recall)
Let \( \phi \) and \( \psi \) be two isogenous Drinfeld modules defined over the \( A \)-field \( L \). \( \text{Hom}_L(\phi, \psi) \) denotes the group of isogenies from \( \phi \) to \( \psi \). Let \( u : \phi \rightarrow \psi \) be an isogeny. If \( y \in \phi[p^n] \) then \( u(y) \in \psi[p^n] \).
To \( u \in \text{Hom}_L(\phi, \psi) \otimes A_\omega \), corresponds therefore a canonical morphism of \( A_\omega \)-modules \( u^* \in \text{Hom}_{A_\omega}(T_\omega \phi, T_\omega \psi) \).

Theorem 1.2. [Tate, see theorem 4.12.12]
Let \( \phi \) and \( \psi \) be two isogenous Drinfeld modules over the finite \( A \)-field \( L \) as mentioned in the previous remark. Let \( G = \text{Gal}(\overline{L}/L) \). The canonical map
\[
\text{Hom}_L(\phi, \psi) \otimes A_\omega \rightarrow \text{Hom}_{A_\omega[G]}(T_\omega \phi, T_\omega \psi)
\]
is a bijection (as morphism of \( A_\omega \)-modules).

Corollary 1.1.
- If \( \phi = \psi \) then we have the bijection
  \[
  \text{End}_L \phi \otimes A_\omega \rightarrow \text{End}_{A_\omega[G]} T_\omega \phi
  \]
- We denote \( V_\omega \phi := T_\omega \phi \otimes k_\omega \).
  \[
  \text{End}_L \phi \otimes k_\omega \cong \text{End}_{k_\omega[G]} V_\omega \phi
  \]
as \( k_\omega \)-algebras.

Remark 1.2. Let \( \pi \) be the Frobenius endomorphism of the Drinfeld module \( \phi \). We denote \( M(x) \) the minimal polynomial of \( \pi \) over \( k \).
The characteristic polynomial of the action of \( \pi \) on the Tate module \( T_\omega \phi \) is \( M(x)^t \) where \( t = \dim_k(\pi) \text{End}_A \phi \otimes k \). If \( t = 1 \) as it will be the case in the sequel, then \( M(x) \) is the characteristic polynomial of the action of \( \pi \) on \( T_\omega \phi \).

1.2 Dieudonné module of a Drinfeld module

We want now to discuss what the so-called Tate’s theory says when one works at the place \( v \) defined by the \( A \)-characteristic of the Drinfeld module \( \phi \) defined over the finite \( A \)-field \( L \).
Let us recall that the Tate’s theory at the other places \( \omega \), strongly relies on the fact that the polynomial \( \phi[p^n](x) \) is separable. That means \( \phi[p^n] \) (as group scheme) is étale. This is not true anymore at the place \( v \). That difficulty is overcome by considering the notion of Dieudonné modules. Before moving forward, let us recall the following theorem known as Dieudonné-Cartier-Oda theorem.

Theorem 1.3. Let \( m \in \mathbb{N} \) and \( L \) be a degree \( m \) field extension of \( A/p_v \). Let \( K_v \) be the unique degree \( m \) unramified extension of the completion field \( k_v \) of \( k \) at the place \( v \). Let \( W \) be the ring of integers of \( K_v \). Let \( F \) and \( V \) be indeterminates such that
- \( FV = VF = p_v \)
- \( FX = \sigma(\lambda)F \) and \( \lambda V = V \sigma(\lambda) \forall \lambda \in W \)
where \( \sigma : W \rightarrow W \) is the unique automorphism induced by the Frobenius \( \tau^{\deg p_v} \) of \( L \).
There is an anti-equivalence of categories between the category of finite commutative group scheme over \( L \) of finite \( A/p_v \)-rank and the category of left \( W[F, V] \)-modules of finite \( W \)-length.

**Remark 1.3.**

- Given a finite commutative \( L \)-group scheme \( S \) of finite \( A/p_v \)-rank, we denote \( D(S) \) the corresponding left \( W[F, V] \)-module of finite \( W \)-length.
- \( D(S) \) is \( W \)-free and \( \text{rank}_{A/p_v} S = \text{rank}_W D(S) \).
- \( W \) is also known as the ring of Witt vectors over the field \( L \) and since \( L \) is finite (and therefore perfect), \( W \) is a discrete valuation ring and \( L \) is its residue field.

**Definition 1.2** (Dieudonné module at the place \( v \)).

Let \( \psi \) be a Drinfeld module over the finite \( A \)-field \( L \) with \( m = [L : A/p_v] \).

The Dieudonné module of \( \psi \) is defined by the direct limit

\[
T_v \psi := \varprojlim_D(\psi[p^n_v])
\]

where \( D(\psi[p^n_v]) \) is the left \( W[F, V] \)-module associated to the \( L \)-group scheme \( \psi[p^n_v] \) as mentioned in the previous remark.

The corresponding Tate theorem is given below.

**Theorem 1.4.** [Serre-Tate, [3, proposition 8.2, corollary 8.3, theorem 8.4]]

The canonical map

\[
\text{Hom}_L (\phi, \psi) \otimes A_v \sim \text{Hom}_{W[F, V]} (T_v \phi, T_v \psi)
\]

is a bijection (as morphism of \( A_v \)-modules).

**Remark 1.4.** [see [3]]

- If \( \phi = \psi \) then we have \( \text{End}_L \psi \otimes A_v \sim \text{End}_{W[F, V]} T_v \psi \).
- We denote \( V_v \psi = T_v \psi \otimes k_v \). We have \( \text{End}_L \psi \otimes k_v \sim \text{End}_{K_v[F, F^{-1}]} V_v \psi \).
- \( T_v \psi/p_v^n T_v \psi \) can be identified to \( D(\psi[p^n_v]) \).
- The \( W[F, V] \)-module \( D(\psi[p^n_v]) \) can be decomposed into its étale and local parts. \( D(\psi[p^n_v]) = D(\psi[p^n_v])_{\text{loc}} \oplus D(\psi[p^n_v])_{\text{ét}} \).

Actually the polynomial \( \psi[p^n_v](x) = x^{\text{deg } \psi[p^n_v]} g_n(x) \) where \( g_n(x) \) is a separable polynomial.

\[
D(\psi[p^n_v])_{\text{loc}} = D(\psi[p^n_v]_{\text{loc}}) \quad \text{and} \quad D(\psi[p^n_v])_{\text{ét}} = D(\psi[p^n_v]_{\text{ét}})
\]

where \( \psi[p^n_v]_{\text{loc}} = \text{Spec } (L[x]/(x^{\text{deg } \psi[p^n_v]}) \) and \( \psi[p^n_v]_{\text{ét}} = \text{Spec } (L[x]/(g_n(x))) \).

That means the Dieudonné module can also be decomposed as \( T_v \psi = (T_v \psi)_{\text{loc}} \oplus (T_v \psi)_{\text{ét}} \).

- The Frobenius \( \pi \) of \( \psi \) acts on \( T_v \psi \) via \( \pi = F^m \).
- \( F \) (and therefore \( \pi = F^m \)) acts on the local part \( D(\psi[p^n_v])_{\text{loc}} \) as a nilpotent element and acts on the étale part \( D(\psi[p^n_v])_{\text{ét}} \) as an isomorphism.
1.3 Main theorem

For more details on this part, one can follow [5, §6, 7 and 8].

The following dictionary can be helpful:

- \( Q = \mathbb{F}_q(C) \Rightarrow k = \mathbb{F}_q(T) \)
- \( \Gamma (C', \mathcal{O}_{C'}) \Rightarrow A = \mathbb{F}_q[T] \)
- \( z \Rightarrow p_v \)
- \( \mathbb{F}_q[[z]] \Rightarrow A_v \)
- \( \mathcal{O}_S[[z]] \Rightarrow W \)
- \( \mathcal{O}_S[[z]][\mathbb{A}] \Rightarrow W[V] \)
- Abelian sheaf \( \mathcal{F} \Rightarrow \) Drinfeld module \( \phi \)
- Dieudonné module \( (\hat{\mathcal{F}}, F) \Rightarrow \) Dieudonné \( \mathbb{F}[F, V] \)-module \( T_v \phi \)

1.3 Main theorem

Before giving the main theorem, let us lay the groundwork with the following lemmas and remarks.

**Lemma 1.1.** Let \( M(x) = x^r + a_1 x^{r-1} + \cdots + a_{r-1} x + \mu p_v^m \) be a Weil polynomial as described in the previous chapter.

The height \( h \) (in the sense of [4, Definition 4.5.8]) of the isogeny class defined by \( M(x) \) is the sub-degree of the polynomial \( M(x) \mod p_v \). That is \( M(x) \equiv x^r + a_1 x^{r-1} + \cdots + a_{r-1} x + \mu p_v^m \mod p_v \).

**Proof:** Let us first of all recall that the height is an isogeny invariant. That means two isogenous Drinfeld modules share the same height. Let \( \psi \) be a Drinfeld module in our isogeny class. We recall that the Dieudonné module \( T_v \psi \) of \( \psi \) is a \( \mathbb{F}_q[V] \)-module and the Frobenius endomorphism \( \pi \) acts on it via \( \pi = F_m \) as we mentioned before.

\( \pi = F_m \) acts \( \mathbb{F}_q[V] \)-linearly on the Dieudonné module \( T_v \psi \) with the same characteristic polynomial (in \( \mathbb{F}_q[V] \)) as it does as \( A_\psi \)-linear endomorphism of the Tate module \( T_\omega \psi \) for any \( \omega \neq v \) (see [2, proof of theorem A1.1.1] or replacing Tate modules by Dieudonné modules in the proof of theorem 4 in [9, page 167]).

But the characteristic polynomial of the action of \( \pi \) on the Tate module \( T_\omega \psi \) is the minimal polynomial \( M(x) \) of \( \pi \) over \( k \) (since \( \text{End}_\phi \otimes k = k(\pi) \) see remark 1.2).

Therefore \( M(x) \) is also the characteristic polynomial of the action of the Frobenius endomorphism \( \pi = F_m \) on the Dieudonné module \( T_v \psi \).

One gets from there that \( M(x) \mod p_v \) is the characteristic polynomial of the action of \( \pi \) on \( T_v \psi / p_v T_v \psi = D^0(\psi[p_v]) \) (see remark 1.3).

As mentioned in remark 1.3, we also know that \( D(\psi[p_v]) \) decomposes (via the corresponding group scheme) into its étale and local parts i.e. \( D(\psi[p_v]) = D(\psi[p_v])_{\text{loc}} \oplus D(\psi[p_v])_{\text{ét}} \).

Therefore the characteristic polynomial also splits into

\[
M(x) \equiv M_{\text{loc}}(x) \cdot M_{\text{ét}}(x) \mod p_v
\]
1.3 Main theorem

where $M_{\text{loc}}(x) \mod p_v$ (resp. $M_{\text{et}}(x) \mod p_v$) is the characteristic polynomial of the action of $\pi$ on the local part $D(\psi[p_v])_{\text{loc}}$ (resp. on the étale part $D(\psi[p_v])_{\text{et}}$). That means,

$$\deg(M_{\text{loc}}(x) \mod p_v) = \text{rank}_W D(\psi[p_v])_{\text{loc}}$$

and

$$\deg(M_{\text{et}}(x) \mod p_v) = \text{rank}_W D(\psi[p_v])_{\text{et}}$$

But we have by the definition of the height of $\psi$ (see definition ??)

$$\psi_{p_v} = \tau^{\deg p_v} + \alpha_1 \tau^{\deg p_v - 1} + \cdots + \alpha_{r-h} \deg p_v \tau^{h \deg p_v}$$

with $\alpha_{r-h} \deg p_v \neq 0$. That is,

$$\psi_{p_v}(x) = \left(x^{g(r-h) \deg p_v} + \alpha_1 x^{g(r-h) \deg p_v - 1} + \cdots + \alpha_{r-h} \deg p_v \right) x^{h \deg p_v} = g(x) \cdot x^{h \deg p_v}$$

where $g(x)$ is a separable polynomial (since $\alpha_{r-h} \deg p_v \neq 0$) and

$$\psi[p_v]_{\text{et}} = \text{Spec} \left( \mathcal{T}[x]/(g(x)) \right)$$

and

$$\psi[p_v]_{\text{loc}} = \text{Spec} \left( \mathcal{T}[x]/(x^{h \deg p_v}) \right)$$

where $\mathcal{T}$ is an algebraic closure of $L$.

As we have mentioned in remark 1.4, $\pi$ acts on $D(\psi[p_v])_{\text{loc}}$ (resp. $D(\psi[p_v])_{\text{et}}$) as a nilpotent element (resp. as an isomorphism). That means the characteristic polynomial $M_{\text{loc}}(x) \mod p_v$ is a power of $x$ and the characteristic polynomial $M_{\text{et}}(x) \mod p_v$ has only non-zero roots (non-zero eigenvalues). In addition,

$$\deg(M_{\text{et}}(x) \mod p_v) = \text{rank}_W D(\psi[p_v])_{\text{et}} = r - h$$

Therefore

$$M(x) \equiv M_{\text{loc}}(x) \cdot M_{\text{et}}(x) \equiv x^h \left(x^{r-h} + a_1 x^{r-h-1} + \cdots + a_{r-h} \right) \mod p_v$$

and the result follows.

\[\diamondsuit\]

Corollary 1.2. Let $M(x)$ be as in the previous lemma.

$M_{\text{loc}}(x)$ is the irreducible factor of $M(x)$ in $k_v[x]$ that describes the unique zero of $\pi$ in $k(\pi)$

Proof:

- First of all, $M_{\text{loc}}(x)$ is an irreducible factor of $M(x)$ in $k_v[x]$. Indeed, if $M_{\text{loc}}(x) = f_1(x) \cdot f_2(x) \in k_v[x]$ is a product of two irreducible factors of $M(x)$ in $k_v[x]$, then since $M_{\text{loc}}(x) \equiv x^h \mod p_v$, $f_1(x)$ and $f_2(x)$ would have a common zero modulo $p_v$. That is not possible since $M(x)$ is a Weil polynomial.

- If $f_{i_0}(x)$ is the factor of $M(x)$ in $k_v[x]$ describing the zero $p_{i_0}$ of $\pi$ in $k(\pi)$, then the constant coefficient $a_{0,i_0}$ of $f_{i_0}(x)$ must be divisible by $p_v$. Indeed,

$$v_{i_0}(\pi) > 0 \text{ i.e. } \mathcal{T} \circ \tau_{i_0}(\pi) > 0.$$  

In other words, $\mathcal{T}(\tau_{i_0}(\pi)) > 0$,

where $\tau_{i_0}$ denotes a root of $f_{i_0}(x)$.

That means, $\mathcal{T} \circ \tau_{i_0}(\pi_{i_0}) > 0 \text{ i.e. } v_{i_0}(\pi_{i_0}) > 0$.

As a result $v_{i_0}(N_{k_v(\pi_{i_0})/k_v}(\pi_{i_0})) > 0$ and thus

$$v \left(N_{k_v(\pi_{i_0})/k_v}(\pi_{i_0}) \right) > 0 \text{ since } N_{k_v(\pi_{i_0})/k_v}(\pi_{i_0}) \in k_v.$$  

But the constant coefficient of $f_{i_0}(x)$, $a_{0,i_0} = (-1)^{\deg f_{i_0}(x)} N_{k_v(\pi_{i_0})/k_v}(\pi_{i_0})$.

That means we also have $v(a_{0,i_0}) > 0$ and the claim follows.

- Since $M(x)$ is a Weil polynomial, there must be only one such factor $f_{i_0}(x)$ of $M(x)$ in $k_v[x]$. Since $M_{\text{loc}}(x) \equiv x^h \mod p_v$, the constant coefficient of
Before moving forward, let us formulate the problem.

Formulation of the problem:

Yu in [1] basically showed that for an isogeny class of rank 2 Drinfeld modules, the orders occurring as endomorphism ring of a Drinfeld module are either (in case the endomorphism algebra is not a field) the maximal orders in the quaternion algebra over \( k \) ramified at exactly the places \( v \) and \( \infty \), or those orders \( \mathcal{O} \) of \( k(\pi) \) containing \( \pi \) that are maximal at all the places lying over \( v \) i.e. such that \( \mathcal{O} \otimes A_v \) is a maximal \( A_v \)-order of the \( k_v \)-algebra \( k_v(\pi) \).

Now the question is: What about Drinfeld modules of higher rank \( (r \geq 3) \)? Of course for an order \( \mathcal{O} \) of (the endomorphism algebra) \( k(\pi) \) to be the endomorphism ring of a Drinfeld module, it is necessary that the Frobenius \( \pi \in \mathcal{O} \).

But must we have \( \mathcal{O} \) maximal at all the places of \( k(\pi) \) lying over the place \( v \)? In other words, must we have \( \mathcal{O} \otimes A_v \) maximal \( A_v \)-order of the \( k_v \)-algebra \( k_v(\pi) \)?

The answer is No! and we provide below an example of a rank 3 Drinfeld module whose endomorphism ring is not at all places of \( k(\pi) \) lying over the place \( v \) maximal.

Before the example, let us recall the definition and a fact concerning the notion of conductor of an order.

Definition 1.3 (Recall). \( A = \mathbb{F}_q[T] \), \( k = \mathbb{F}_q(T) \)

Let \( F/k \) be a function field and \( \mathcal{O}_{\text{max}} \) be the ring of integers of \( F \). Let \( \mathcal{O} \) be an \( A \)-order of \( F \). The conductor \( \mathfrak{c} \) of \( \mathcal{O} \) is the maximal ideal of \( \mathcal{O} \) which is also an ideal of \( \mathcal{O}_{\text{max}} \). It is defined by \( \mathfrak{c} = \{ x \in F \mid x\mathcal{O}_{\text{max}} \subseteq \mathcal{O} \} \).

Remark 1.5. As a very well known fact, \( \text{disc} \mathcal{O} = N_{F/k}(\mathfrak{c}) \text{disc} \mathcal{O}_{\text{max}} \).

Where \( \text{disc}(?) \) denotes the discriminant of a basis of the corresponding free \( A \)-lattice and \( N_{F/k}(?) \) denotes the norm of the ideal in argument. We recall that if \( \mathfrak{p} \) is a prime of \( F \) above the prime \( \mathfrak{p} \) of \( k \) then \( N_{F/k}(\mathfrak{p}) = \mathfrak{p}^f \) where \( f \) denotes the residual degree of \( \mathfrak{p} \mid \mathfrak{p} \). In addition \( N_{F/k}(?) \) is multiplicative i.e. \( N_{F/k}(\mathfrak{p}_1, \mathfrak{p}_2) = N_{F/k}(\mathfrak{p}_1)N_{F/k}(\mathfrak{p}_2) \).

Example 1.1.

\( A = \mathbb{F}_5[T] \), \( k = \mathbb{F}_5(T) \), \( L = \mathbb{F}_{125} = \mathbb{F}_5(\alpha) \) with \( \alpha^3 + 3\alpha + 3 = 0 \).

\( p = \ker \gamma = (T) \). \( M(x) = x^3 + (T+1)x^2 + (T^2+3T+4)x + 4T^3 \).

One easily shows that \( M(x) \) is a Weil polynomial (see [1])

\( \text{disc}(M(x)) = T^2(T+4)^2(T^2+4T+2) \). Following the paper [1] one computes the following:

The discriminant of the cubic function field \( k(\pi) \) is

\( \Delta = \text{disc}(k(\pi)) = (T+4)^2(T^2+4T+2) \). We set \( I = \sqrt{\frac{\text{disc}(M(x))}{\Delta}} = T \).

The maximal order of the function field \( k(\pi)/k \) is the order generated by \( \langle \omega_0, \omega_1, \omega_2 \rangle \),

where \( \omega_0 = 1 \), \( \omega_1 = \tilde{\pi} = \pi + 2T + 2 \), \( \omega_2 = \frac{\alpha_1 + \beta_2 \tilde{\pi} + \tilde{\pi}^2}{\tilde{\pi}} = \frac{\alpha_2 + \beta_2 \tilde{\pi} + \tilde{\pi}^2}{\tilde{\pi}} \)

With
The norm of the conductor is
$$p \equiv \pi \pmod{T} \cdot \mathcal{O}$$
respective.
That is, $$\omega_2 = \frac{3 + 4(\pi + 2T + 2) + (\pi + 2T + 2)^2}{2p} \pmod{T}$$

We now claim that the conductor $$c$$ of $$\mathcal{O} = A[\pi]$$ is $$c = T \cdot \mathcal{O} + (\pi - 3T + 3) \cdot \mathcal{O}$$. Indeed,
$$M(x) \equiv x(x - 3T + 3)^2 \pmod{T}$$
We also have $$(\pi - 3T + 3)\lambda_0\omega_0 + (\pi - 3T + 3)\lambda_1\omega_1 + (\pi - 3T + 3)\lambda_2\omega_2 \in A[\pi]$$ for $$\lambda_i \in A$$. Because $$(\pi - 3T + 3)\omega_2 = (T + 1)\pi + 4T^2 + 4T + 3 \in A[\pi]$$. That means $$\pi - 3T + 3 \in c$$.

Therefore $$T \cdot \mathcal{O} + (\pi - 3T + 3) \cdot \mathcal{O} \subseteq c \subseteq \mathcal{O}$$. Let us consider the canonical morphisms

$$A[\pi] \simeq \frac{A[x]/M(x) \cdot A[x]}{\varphi(x)} \xrightarrow{\varphi_1} \frac{(A/T \cdot A)[x]}{M(x) \cdot (A/T \cdot A)[x]} \xrightarrow{\varphi_2} \frac{(A/T \cdot A)[x]}{(x - 3T + 3) \cdot (A/T \cdot A)[x]} \simeq A/T \cdot A$$

$$M(x) \equiv x(x + 3)^2 \pmod{T}$$. Since $$M(x)$$ is a Weil polynomial, the irreducible decomposition of $$M(x)$$ over the completion field $$k_v$$ is of the form
$$M(x) = M_1(x) \cdot M_2(x) \in k_v[x]$$. That means $$p_v = T$$ splits into two primes $$p_1$$ and $$p_2$$ in $$k(\pi)$$. As a matter of fact, any prime ideal $$p$$ of $$\mathcal{O}$$ containing $$T$$ is either $$T \cdot \mathcal{O}$$ or $$T \cdot \mathcal{O} + \pi \cdot \mathcal{O}$$. Indeed,
First of all $$T \cdot \mathcal{O} + (\pi - 3T + 3) \cdot \mathcal{O}$$ and $$T \cdot \mathcal{O} + \pi \cdot \mathcal{O}$$ are maximal ideals of $$\mathcal{O} = A[\pi]$$ as kernel of the canonical morphisms

$$A[\pi] \simeq \frac{A[x]/M(x) \cdot A[x]}{\varphi(x)} \xrightarrow{\varphi_1} \frac{(A/T \cdot A)[x]}{M(x) \cdot (A/T \cdot A)[x]} \xrightarrow{\varphi_2} \frac{(A/T \cdot A)[x]}{(x - 3T + 3) \cdot (A/T \cdot A)[x]} \simeq A/T \cdot A$$

and

$$A[\pi] \simeq \frac{A[x]/M(x) \cdot A[x]}{\varphi(x)} \xrightarrow{\varphi_1} \frac{(A/T \cdot A)[x]}{M(x) \cdot (A/T \cdot A)[x]} \xrightarrow{\varphi_2} \frac{(A/T \cdot A)[x]}{(x - 3T + 3) \cdot (A/T \cdot A)[x]} \simeq A/T \cdot A$$

respectively.

Since $$M(x) \equiv x(x - 3T + 3)^2 \pmod{T}$$ and $$M(\pi) = 0$$, we have
$$\pi(\pi - 3T + 3)^2 \in T \cdot A[\pi] \subseteq p$$. But $$p$$ is a prime ideal of $$\mathcal{O}$$. That means $$\pi \in p$$ or $$\pi - 3T + 3 \in p$$. In other words
$$T \cdot \mathcal{O} + (\pi - 3T + 3) \cdot \mathcal{O} \subseteq p$$ or $$T \cdot \mathcal{O} + \pi \cdot \mathcal{O} \subseteq p$$

From the maximality of these ideals we conclude that $$p = T \cdot \mathcal{O} + (\pi - 3T + 3) \cdot \mathcal{O}$$ or $$p = T \cdot \mathcal{O} + \pi \cdot \mathcal{O}$$. We assume then WLOG that $$p_2 \cap \mathcal{O} = T \cdot \mathcal{O} + (\pi - 3T + 3) \cdot \mathcal{O} = c$$.
That is, $$p_2 \mid c$$ and $$p_1 \nmid c$$.
The norm of the conductor is
1.3 Main theorem

$N_{k(n)/k}(c) = T^2$ since $\text{disc}(M(x)) = T^2 \cdot \text{disc}(k(n))$.

Therefore we have only two possibilities for orders occurring as endomorphism of a Drinfeld module: $A[\pi]$ and the maximal order $\mathcal{O}_{\text{max}}$. This is due to the fact that the norm of the conductor of any order $\mathcal{O}$ containing properly $A[\pi]$ (i.e. $A[\pi] \subset \mathcal{O} \subset \mathcal{O}_{\text{max}}$) is a square of a proper divisor of $T^2$ and thus must be a unit. In other words $\text{disc}(\mathcal{O}) = \text{disc}(\mathcal{O}_{\text{max}})$. i.e. $\mathcal{O} = \mathcal{O}_{\text{max}}$.

After some computations (using a code we implemented in the computer algebra system SAGE) we found the following:

- For $\phi_T = -\alpha^2 \tau^3 + 2\alpha^2 \tau^2 + \alpha \tau$ we have:
  $$\omega_2 = \frac{3 + 4(\tau^3 + 2\phi_T + 2) + (\tau^3 + 2\phi_T + 2)^2}{\phi_T} \in L(\tau)$$

  In other words $\omega_2 \in \text{End}\phi$. Therefore $\text{End}\phi = \mathcal{O}_{\text{max}}$.

- For $\psi_T = r^3 + r^2 + r$ we have:
  $$\omega_2 = \frac{3 + 4(\tau^3 + 2\psi_T + 2) + (\tau^3 + 2\psi_T + 2)^2}{\psi_T} \not\in L(\tau)$$

Since we have only two possibilities for $\text{End}\psi$, we can conclude that $\text{End}\psi = A[\pi]$. $A[\pi]$ is therefore the endomorphism ring of a Drinfeld module but $A[\pi]$ is not maximal at at least one of the places of $\mathcal{k}$ lying over the place $v$ because its conductor $c$ is not relatively prime to $p_v = T$.

One can notice in the example above that $M_{\text{loc}}(x) = M_1(x) \equiv x \mod p_v$. That means $\text{deg} M_{\text{loc}}(x) = 1$. Thus any order containing $\pi$ is maximal at the corresponding place $v_1$ (which represent the zero of $\pi$ in $k(\pi)$).

Concerning the étale part, $M_{\text{et}}(x) = M_2(x) \equiv (x + 3)^2 \mod p_v$, i.e. $\text{deg} M_{\text{et}}(x) = 2$.

We have then here “enough” $p_v$-torsion points.

This example already encodes some tips for the generalization.

**Definition 1.4.** [4, remark 4.7.12.1]/recall/

Let $\phi$ and $\psi$ be two isogenous Drinfeld modules over $L$. Let $u : \phi \rightarrow \psi$, $u \in L(\tau)$ be an isogeny from $\phi$ to $\psi$. $\psi$ is called the quotient of the Drinfeld module $\phi$ by the kernel $G$ of $u$ and denoted $\psi := \phi/G$.

**Lemma 1.2.** Let $\phi$ be a Drinfeld module over the finite $A$-field $L$ whose endomorphism algebra is a field i.e. $\text{End}\phi \otimes k = k(\pi)$, where $\pi$ is the Frobenius endomorphism of $\phi$. Let $\mathcal{O}$ be an $A$-order of $k(\pi)$ containing $\pi$. We choose a place $\omega$ of $k$ different from $v$.

If $\text{End}\phi \otimes A_\omega \not\cong \mathcal{O} \otimes A_\omega$ as $A_\omega$-module then there exists a Drinfeld module quotient $\psi = \phi/G_\omega$ such that $\text{End}\psi \otimes A_\omega \cong \mathcal{O} \otimes A_\omega$ and $\text{End}\psi \otimes A_v \cong \text{End}\phi \otimes A_v$ for all places $v \neq \omega$.

Proof: With the hypotheses of the lemma, let us assume that $\text{End}\phi \otimes A_\omega \not\cong \mathcal{O} \otimes A_\omega$. We are looking for an isogeny $u$ that changes (via its kernel) the Drinfeld module $\phi$ into a Drinfeld module $\psi$ so that the endomorphism ring of the resulting Drinfeld module coincides at $\omega$ with $\mathcal{O}$. 
Let \( \mathcal{O} \) be an \( A \)-order of \( k(\pi) \) containing \( \pi \). That means \( \mathcal{O} \otimes A_\omega \) is an \( A_\omega \)-order of the \( k_\omega \)-algebra \( k_\omega(\pi) = \text{End}_\phi \otimes k_\omega \). We also know from the corollary of the Tate theorem that there is a canonical isomorphism of \( k_\omega \)-algebras \( \text{End}_\phi \otimes k_\omega \cong \text{End}_{k_\omega[\pi]} V_\phi \), where \( V_\phi = T_\phi \otimes k_\omega \).

Since in addition \( \pi \in \mathcal{O}, V_\phi \) therefore contains an \( A_\omega \)-lattice \( \mathcal{L} \) containing \( T_\phi \phi \) and stable under the action of \( \pi \) such that the corresponding order \( \text{End}_{A_\omega[\pi]} \mathcal{L} \cong \mathcal{O} \otimes A_\omega \) as \( A_\omega \)-modules. We consider then such an \( A_\omega \)-lattice \( \mathcal{L} \). We have then \( T_\phi \phi \subseteq \mathcal{L} \subseteq V_\phi \).

Let \( (t_1, \ldots, t_r) \) be an \( A_\omega \)-basis of \( T_\phi \phi \) and \( (z_1, \ldots, z_r) \) be an \( A_\omega \)-basis of \( \mathcal{L} \), where \( r = \text{rank} \phi \). \( M_0 \) denotes the matrix in \( M_{r \times r}(A_\omega) \) such that

\[
\begin{pmatrix}
t_1 \\
\vdots \\
t_r \\
\end{pmatrix} = M_0 
\begin{pmatrix}
z_1 \\
\vdots \\
z_r \\
\end{pmatrix}
\]

Let \( s = \omega(\text{det}M_0) \) be the valuation (wrt \( \omega \)) of the determinant \( \text{det}M_0 \).

\[
\text{det}M_0 = \alpha_0 p_\omega^s, \quad \text{where } p_\omega \text{ is the uniformizing element of the place } \omega \text{ and } \alpha_0 \text{ is a unit in } A_\omega.
\]

We can consider the following map

\[
\mathcal{O}(M_0)^t : T_\phi \phi \longrightarrow \mathcal{L}
\]

\[
\begin{pmatrix}
t_1 \\
\vdots \\
t_r \\
\end{pmatrix} \longrightarrow \alpha_0 p_\omega^s 
\begin{pmatrix}
z_1 \\
\vdots \\
z_r \\
\end{pmatrix}
\]

The kernel of this map is \( \ker \mathcal{O}(M_0)^t = M_0 \cdot \phi[p_\omega^s] \).

We recall that \( \mathcal{O}(M_0)^t \) (as one can guess) denotes the transpose of the co-matrix of the matrix \( M_0 \).

Indeed, if \( \lambda_1 t_1 + \cdots + \lambda_r t_r \in M_0 \cdot \phi[p_\omega^s] \) then

\[
\mathcal{O}(M_0)^t \cdot (\lambda_1 t_1 + \cdots + \lambda_r t_r) \in \mathcal{O}(M_0)^t \cdot M_0 \cdot \phi[p_\omega^s] = p_\omega^s \cdot \phi[p_\omega^s] = \{0\}.
\]

That is, \( \mathcal{O}(M_0)^t \cdot (\lambda_1 t_1 + \cdots + \lambda_r t_r) = 0 \) and thus

\[
\lambda_1 t_1 + \cdots + \lambda_r t_r \in \ker \mathcal{O}(M_0)^t.
\]

Conversely if \( \lambda_1 t_1 + \cdots + \lambda_r t_r \in \ker \mathcal{O}(M_0)^t \) then \( \mathcal{O}(M_0)^t \cdot (\lambda_1 t_1 + \cdots + \lambda_r t_r) = 0 \) i.e. \( \alpha_0 p_\omega^s (\lambda_1 z_1 + \cdots + \lambda_r z_r) = 0 \) and therefore \( \lambda_1 z_1 + \cdots + \lambda_r z_r \in \phi[p_\omega^s] \).

That means \( \lambda_1 t_1 + \cdots + \lambda_r t_r = M_0 \cdot (\lambda_1 z_1 + \cdots + \lambda_r z_r) \in M_0 \cdot \phi[p_\omega^s] \).

Hence \( \ker \mathcal{O}(M_0)^t = M_0 \cdot \phi[p_\omega^s] \).

Applying the first isomorphism theorem to the morphism of \( A_\omega \)-modules, one gets \( T_\phi \phi/M_0 \cdot \phi[p_\omega^s] \cong \text{Im} \left( \mathcal{O}(M_0)^t \right) = \langle p_\omega^s z_1, \cdots, p_\omega^s z_r \rangle \).

Let \( \mathcal{L}_s = \langle p_\omega^s z_1, \cdots, p_\omega^s z_r \rangle \) be the \( A_\omega \)-lattice generated by \( (p_\omega^s z_1, \cdots, p_\omega^s z_r) \).

\( T_\phi \phi/M_0 \cdot \phi[p_\omega^s] \cong \mathcal{L}_s = p_\omega^s \cdot \mathcal{L} \).

We set \( G_L = M_0 \cdot \phi[p_\omega^s] \) and we consider the Drinfeld module quotient \( \psi = \phi/G_L \) defined over \( L \).

The existence of the Drinfeld module \( \psi \) is guaranteed by the fact that the separable additive polynomial

\[
u = x \prod_{\alpha \in G_L} \left(1 - \frac{x}{\alpha}\right)
\]

whose kernel \( G_L \) (which is stable under the action of the Frobenius endomorphism \( \pi \) mainly because \( \pi \in \mathcal{O} \)), lie in \( L \{ \tau \} \) (see \cite{4}, proposition 1.1.5 and
corollary 1.2.2), in addition to the fact that the local part of the group scheme $H = \text{Spec}(\mathbb{T}[x]/(u(x)))$ is trivial because $u \in L\{\tau\}$ is separable (see [4], proposition 4.7.11, for $t=0$).

We have then $T_{\nu}\psi \cong T_{\nu}\phi/M_0 \cdot \phi[p^*_w] \cong L_s = p^*_w \cdot \mathcal{L}$ as $\mathcal{A}_e$-modules.

Since $G_\mathcal{L} = M_0 \cdot \phi[p^*_w]$ and $\mathcal{L}$ are stable under the action of $\nu$, so are $T_{\nu}\psi$ and $L_s$. In other words $T_{\nu}\psi \cong L_s$ as $A_\nu[\pi]-$modules.

That means $\text{End}_{A_\nu}[\pi]T_{\nu}\psi \cong \text{End}_{A_\nu}[\pi]L_s.$

One also easily checks that (since $L_s = p^*_w \cdot \mathcal{L}$ and $L_s$ generate the same order i.e. $\text{End}_{A_\nu}L_s = \text{End}_{A_\nu}\mathcal{L}$).

Therefore $\text{End}_{A_\nu}[\pi]T_{\nu}\psi \cong \text{End}_{A_\nu}[\pi]L_s$. Applying the Tate theorem [1,2] one gets then $\text{End}_\psi \otimes A_\nu \cong \text{End}_{A_\nu}[\pi]T_{\nu}\psi \cong \text{End}_{A_\nu}[\pi]L \cong \mathcal{O} \otimes A_\nu.$

At all the other places $\nu \neq \omega, v$ of $k$, we have the following:

0 $\rightarrow$ $G_\mathcal{L} = M_0 \cdot \phi[p^*_w]$ $\begin{array}{c} \longrightarrow \phi \end{array}$ $\begin{array}{c} \longrightarrow \psi \end{array}$ $\rightarrow$ 0 is an exact sequence.

$G_\mathcal{L}$ has no non-trivial $p_\nu$-torsion points. Applying the Tate theorem at the place $\nu$ to this short exact sequence, one gets the exact sequence

0 $\rightarrow$ $T_{\nu}\phi$ $\begin{array}{c} \longrightarrow \end{array}$ $T_{\nu}\psi$ $\rightarrow$ 0. That means $T_{\nu}\phi \cong T_{\nu}\psi$ as $A_\nu$-modules.

In other words $\text{End}_\psi \otimes A_\nu \cong \text{End}_{A_\nu}[\pi]T_{\nu}\psi \cong \text{End}_{A_\nu}[\pi]T_{\nu}\phi \cong \text{End}_\phi \otimes A_\nu.$

\[\diamond\]

**Lemma 1.3.** Let $\phi$ be a Drinfeld module over the finite $A$-field $L$ whose endomorphism algebra $\text{End}_\phi \otimes k = k(\pi)$ is a field, where $\pi$ denotes the Frobenius endomorphism of $\phi$. Let $\mathcal{O}$ be an $A$-order of $k(\pi)$ containing $\pi$ and such that $\mathcal{O}$ is maximal at the unique zero $v_0$ of $\pi$ in $k(\pi)$ lying over the place $v$ of $k$.

If $\mathcal{O} \otimes A_\nu \neq \text{End}_\phi \otimes A_\nu$ then there exists a quotient Drinfeld module $\psi = \phi/G_\mathcal{L}$ such that $\text{End}_\psi \otimes A_\nu \cong \mathcal{O} \otimes A_\nu$ and $\text{End}_\psi \otimes A_\nu \cong \text{End}_\phi \otimes A_\nu$ at all the other places $\nu \neq v$ of $k$.

Proof: With the hypothesis of the lemma, we assume that $\text{End}_\phi \otimes A_\nu \neq \mathcal{O} \otimes A_\nu$ as $A_\nu$-modules. That means there must exist at least one other place $v_1 \neq v_0$ of $k(\pi)$ lying over the place $v$ of $k$ (i.e. $\phi$ is not supersingular) such that the completion $\mathcal{O}_{v_1}$ of $\mathcal{O}$ at the place $v_1$ is different from the completion $(\text{End}_\phi)_{v_1}$ of $\text{End}_\phi$ at that same place $v_1$.

Let $v_0, v_1, \ldots, v_s$ be the places of $k(\pi)$ lying over the place $v$ of $k$. We choose $v_0$ here to be the unique zero of $\pi$ in $k(\pi)$ lying over the place $v$.

We are looking for a quotient Drinfeld module $\psi = \phi/G_\mathcal{L}$ such that $\text{End}_\psi \otimes A_\nu \cong \mathcal{O} \otimes A_\nu$ and $\text{End}_\psi \otimes A_\nu \cong \text{End}_\phi \otimes A_\nu$ at all the other places $\nu \neq v$.

The idea here is to act on the étale part of the Dieudonné module $T_{\nu}\phi$ of $\phi$ so that the resulting endomorphism ring meets our needs.

Let then $M(x)$ be the minimal polynomial (Weil polynomial) of $\pi$ over $k$.

We know that the places $v_0, v_1, \ldots, v_s$ are described by the irreducible factors of $M(x)$ in $k_v[x]$. Let then $M(x) = M_0(x) \cdot M_1(x) \cdots M_s(x) \in k_v[x]$ be the irreducible decomposition of $M(x)$ over the completion field $k_v$.

We also know that the irreducible factor $M_0(x) = M_{\text{loc}}(x)$ describing the zero $v_0$ of $\pi$ in $k(\pi)$ is the characteristic polynomial of the action of $\pi$ on the local part of the Dieudonné module $(T_{\nu}\phi)_{\text{loc}}$ (see corollary [1,2]).

In addition, $M_0(x) \equiv x^h \mod p_{v_0}$, where $h$ is the height of $\phi$ (see lemma [1,4]).

$M_{\text{et}}(x) = M_1(x) \cdots M_s(x)$ is the characteristic polynomial of the action of $\pi$ on the étale part of the Dieudonné module $(T_{\nu}\phi)_{\text{et}}$. In this case, we therefore clearly see that $\text{rank}_W(T_{\nu}\phi)_{\text{ét}} = \deg M_{\text{et}}(x) \geq 2$. Because if we had $\deg M_{\text{et}}(x) = 0$, $\phi$
would be supersingular and if we had \( \deg M_\alpha(x) = 1 \), \( \text{End} \phi \otimes A_v \) and \( \mathcal{O} \otimes A_v \) would be both maximal orders of the \( k_v \)-algebra \( k_v(\pi) \) and thus we would have \( \text{End} \phi \otimes A_v \cong \mathcal{O} \otimes A_v \), which in either case contradicts our assumption.

We recall the notation \( K_v \) which is the unique degree \( m \) unramified extension of \( k_v \) and \( W \) its ring of integers.

We know that \( \mathcal{O} \otimes A_v = \prod_{v_i \mid v} \mathcal{O}_{v_i} \) is an \( A_v \)-order of the \( k_v \)-algebra \( k_v(\pi) = \text{End} \phi \otimes k_v \cong \text{End}_{K_v,[F,V]} V_\phi \) (see remark 1.3).

i.e. \( \mathcal{O} \otimes A_v \subseteq k_v(\pi) \cong \text{End}_{K_v,[F,V]} V_\phi \).

Also, \( \mathcal{O} \) is maximal at \( v_0 \) i.e. the completion \( \mathcal{O}_{v_0} \) is the maximal order of the field \( k_v(\pi_0) = k_v[x]/M_0(x) \cdot k_v[x] \).

Thus there exists a \( W \)-lattice \( L_0 \) of \( (V_\phi)_\text{ét}(T_\phi)_\text{ét} \otimes K_v \) containing \( (T_\phi)_\text{ét} \) stable under the actions of \( F \) and \( V \),

(i.e. \( T_\phi = (T_\phi)_\text{loc} \otimes (T_\phi)_\text{ét} \subseteq (T_\phi)_\text{loc} \oplus L_0 \leq V_\phi = (V_\phi)_\text{loc} \oplus (V_\phi)_\text{ét} \)) such that the corresponding order \( \text{End}_{W}((T_\phi)_\text{loc} \oplus L_0) \cong \mathcal{O} \otimes A_v \).

We set \( l = r - h = \deg M_\alpha(x) \geq 2 \). Let \( (t_1, \ldots, t_l) \) be a \( W \)-basis of \( (T_\phi)_\text{loc} \) and \( (z_1, \ldots, z_l) \) be a \( W \)-basis of \( L_0 \). \( N_0 \) denotes the matrix in \( M_{l \times l}(W) \) such that

\[
\begin{pmatrix}
(t_1) \\
\vdots \\
(t_l)
\end{pmatrix}
\begin{pmatrix}
(z_1) \\
\vdots \\
(z_l)
\end{pmatrix}
= N_0
\]

Let \( s_0 = v(\det N_0) \). Since \( \text{End} \phi \otimes A_v \cong \text{End}_{W,[F,V]} T_\phi \otimes \mathcal{O} \otimes A_v \), \( s_0 \geq 1 \).

Since \( K_v \) is an unramified extension of \( k_v \) and the corresponding ring of integers \( W \) is a discrete valuation ring, we keep (by abuse of language) the same notation \( v \) for the place of \( K_v \) extending the place \( v \) of \( k_v \). \( p_v \) denotes the corresponding prime.

\[
\det N_0 = \beta_0 p_v^{s_0}
\]

where \( \beta_0 \) is a unit in \( W \). The same way we did before, let us consider the morphism

\[
\begin{array}{ccc}
\text{Co}(N_0)^t & : & \text{(T_\phi)}_\text{ét} \\
\begin{pmatrix}
(t_1) \\
\vdots \\
(t_l)
\end{pmatrix} & \rightarrow & L_0 \\
\rightarrow & \text{Co}(N_0)^t & \begin{pmatrix}
(z_1) \\
\vdots \\
(z_l)
\end{pmatrix}
\end{array}
\]

where \( \text{Co}(N_0)^t \) denotes the transpose of the co-matrix of \( N_0 \). We recall that \( \text{Co}(N_0)^t \cdot N_0 = \det N_0 \cdot \text{IdentityMatrix} \).

The kernel of \( \text{Co}(N_0)^t \) is given by \( \ker(\text{Co}(N_0)^t) = N_0 \cdot D(\phi[p_v^{s_0}])_\text{ét} \).

\( D(\phi[p_v^{s_0}])_\text{ét} \) is the \( W[F,V] \)-module associated to the group-scheme \( \phi[p_v^{s_0}]_\text{ét} \) (see remark 1.3). Indeed,

Let \( \lambda_1 t_1 + \cdots + \lambda_l t_l \in N_0 \cdot D(\phi[p_v^{s_0}])_\text{ét} \). We have then, \( \text{Co}(N_0)^t \cdot (\lambda_1 t_1 + \cdots + \lambda_l t_l) \in \text{Co}(N_0)^t \cdot N_0 \cdot D(\phi[p_v^{s_0}])_\text{ét} = p_v^{s_0} \cdot D(\phi[p_v^{s_0}])_\text{ét} = \{0\} \).

We recall that \( D(\phi[p_v^{s_0}])_\text{ét} \) can be identified to \( T_\phi/p_v^{s_0} : T_\phi \) for any \( n \in N \).

Conversely, let \( \lambda_1 t_1 + \cdots + \lambda_l t_l \in \ker(\text{Co}(N_0)^t) \) i.e. \( \text{Co}(N_0)^t \cdot (\lambda_1 t_1 + \cdots + \lambda_l t_l) = 0 \) That means \( \beta_0 p_v^{s_0} (\lambda_1 z_1 + \cdots + \lambda_l z_l) = 0 \) and then \( \lambda_1 z_1 + \cdots + \lambda_l z_l \in D(\phi[p_v^{s_0}])_\text{ét} \).

But \( \lambda_1 t_1 + \cdots + \lambda_l t_l = N_0 \cdot (\lambda_1 z_1 + \cdots + \lambda_l z_l) \in N_0 \cdot D(\phi[p_v^{s_0}])_\text{ét} \).

Therefore \( \ker(\text{Co}(N_0)^t) = N_0 \cdot D(\phi[p_v^{s_0}])_\text{ét} \).

Applying the first isomorphism theorem to our morphism, one gets that \( (T_\phi)_\text{ét}/N_0 \cdot D(\phi[p_v^{s_0}])_\text{ét} \cong \text{Im} \text{(Co}(N_0)^t) = (p_v^{s_0} z_1, \cdots, p_v^{s_0} z_l) \).
Let $\mathcal{L}_{s_0}$ be the $W$-lattice generated by $(p_v^{m_0} z_1, \ldots, p_v^{m_0} z_l)$.

i.e. $(T_\nu \phi)_{\text{et}} / N_0 \cdot D (\phi[p_v^{m_0}])_{\text{et}} \cong \mathcal{L}_{s_0}$.

$N_0 \cdot D (\phi[p_v^{m_0}])_{\text{et}}$ is stable under the actions of $F$ and $V$ because $N_0$ commutes with the actions of $F$ and $V$ (via the stability of $(T_\nu \phi)_{\text{et}}$ and $\mathcal{L}_0$ under those actions) and $D (\phi[p_v^{m_0}])$ is by definition stable under those actions (see theorem 1.6).

Let $G_{s_0}$ be the finite commutative $L$-group scheme associated to the $W[F, V]$-module $N_0 \cdot D (\phi[p_v^{m_0}])_{\text{et}}$ (theorem 1.3). We consider the additive separable polynomial

$$u = x \prod_{\alpha \in G_{s_0} \setminus \{0\}} \left(1 - \frac{x}{\alpha}\right)$$

whose kernel is $G_{s_0}$. By definition, $G_{s_0}$ is stable under the action of $\pi = F^m$.

For the same reason as the case $\omega \neq v$ in lemma 1.2, $\alpha \in L(\tau)$ and $\nu$ is an isogeny from the Drinfeld module $\phi$ to a Drinfeld module $\psi$. That is, $\phi \cdot u = u \cdot \psi$. In fact $\psi := \phi / G_{s_0}$.

The Dieudonné module of $\psi$ is given as follows:

$$T_\nu \psi = T_\nu (\phi / G_{s_0}) \cong T_\nu / D(G_{s_0}) = ((T_\nu \phi)_{\text{loc}} \oplus (T_\nu \phi)_{\text{et}}) / N_0 \cdot D (\phi[p_v^{m_0}])_{\text{et}}.$$ 

That is,

$$T_\nu \psi \equiv (T_\nu \phi)_{\text{loc}} \oplus (T_\nu \phi)_{\text{et}} / N_0 \cdot D (\phi[p_v^{m_0}])_{\text{et}} \cong (T_\nu \phi)_{\text{loc}} \oplus \mathcal{L}_{s_0}.$$

One easily checks that since $\mathcal{L}_{s_0} = p_v^{m_0} \cdot \mathcal{L}_0$, $\text{End}_W \mathcal{L}_{s_0} = \text{End}_W \mathcal{L}_0$.

Therefore $\text{End}_W ((T_\nu \phi)_{\text{loc}} \oplus \mathcal{L}_{s_0}) \cong \text{End}_W ((T_\nu \phi)_{\text{loc}} \oplus \mathcal{L}_0) \cong \mathcal{O} \otimes A_v$ and from the stability under the actions of $F$ and $V$, one gets

$$\mathcal{O} \otimes A_v \cong \text{End}_{W[F, V]} ((T_\nu \phi)_{\text{loc}} \oplus \mathcal{L}_0) \cong \text{End}_{W[F, V]} ((T_\nu \phi)_{\text{loc}} \oplus \mathcal{L}_{s_0}) \cong \text{End}_{W[F, V]} T_\nu \phi.$$

Hence $\mathcal{O} \otimes A_v \cong \text{End}_{\psi} \oplus A_v$. (Thanks to the Tate’s theorem 1.3).

At all the other places $\omega \neq v$ we have the exact sequence

$$0 \rightarrow \mathcal{O}_{s_0} \rightarrow \phi \rightarrow \psi \rightarrow \phi / G_{s_0} \rightarrow 0.$$

Applying the Tate’s theorem at the place $\omega$, we get

$$0 \rightarrow T_\nu \phi \rightarrow T_\nu \psi \rightarrow 0.$$

In fact by definition of the Dieudonné functor in theorem 1.3 and from the Lagrange theorem for finite group scheme, we have the following:

If $\pi = \text{rank} (N_0 \cdot D (\phi[p_v^{m_0}]))$ then $p_v^{m_0} \cdot G_{s_0} = \{0\}$ i.e. $G_{s_0} \subseteq \phi[p_v^{m_0}]$. That means the Tate module $T_\nu G_{s_0} = \{0\}$ for any place $\omega \neq v$.

Hence we get from the above exact sequence that $T_\nu \phi \cong T_\nu \psi$.

In other words

$$\text{End}_{\psi} \oplus A_v \cong \text{End}_{A_v[\pi]} T_\nu \phi \cong \text{End}_{A_v[\pi]} T_\nu \psi \cong \text{End}_{\psi} \oplus A_v.$$

Let us recall the theorem we want to prove.

**Theorem 1.5.** $A = \mathbb{F}_q[T], \ k = \mathbb{F}_q(T)$ and $p_v$ is the (generator of the) kernel of the characteristic morphism $\gamma : A \rightarrow L$ defining the finite $A$-field $L$.

$M(x) = x^m + a_1 x^{m-1} + \cdots + a_m x + \mu p_v^n \in A[x]$ is a Weil polynomial, where $m = [L : A / p_v, A]$. Let $O$ be an $A$-order of the function field $k(\pi) = k[\pi] / M(x) \cdot k[\pi]$. Let $v_0$ be the unique zero of $\pi$ in $k(\pi)$ lying over the place $v$ of $k$.

$O$ is the endomorphism ring of a Drinfeld module in the isogeny class defined by the Weil polynomial $M(x)$ if and only if $O$ contains $\pi$ and $O$ is maximal at the place $v_0$.  

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Proof: With the hypotheses of the theorem, we have the following:

If $O = \text{End}\phi$ then it is clear that $O$ contains the Frobenius endomorphism $\pi$. Yu proved in [11] that $\text{End}\phi$ is maximal at the zero $v_0$ of $\pi$ in $k(\pi)$.

Conversely, let us assume that $O$ contains $\pi$ and $O$ is maximal at the place $v_0$.

Let $\phi$ be any Drinfeld module over $L$ in the isogeny class defined by $M(x)$. We know that $O$ and $\text{End}\phi$ differ at only finitely many places, since both are orders of the same function field $k(\pi)$. That means there exist finitely many places $\omega_1, \cdots, \omega_s$ such that $O \otimes A_\omega \cong \text{End}\phi \otimes A_\omega$ for all places $\omega$ except (may be) at $\omega \in \{v, \omega_1, \omega_2, \cdots, \omega_s\}$.

For $\omega = \omega_1$, one can get from lemma 1.2 a Drinfeld module $\phi_1$ defined over $L$ such that $\text{End}\phi_1 \otimes A_{\omega_1} \cong O \otimes A_{\omega_1}$ and $\text{End}\phi_1 \otimes A_\nu \cong \text{End}\phi \otimes A_\nu$ at all other places $\nu \neq \omega_1, v$.

Repeating the process at all the places $\omega_i$, one gets from lemma 1.2 a Drinfeld module $\phi$ defined over $L$ such that $\text{End}\phi \otimes A_\omega \cong O \otimes A_\omega$ for all places $\omega$ with $\omega \neq v$.

Concerning the place $v$, we know in addition that $O$ is maximal at the unique zero $v_0$ of $\pi$ in $k(\pi)$ lying over the place $v$.

We can therefore apply lemma 1.3 and get the following:

- If $\phi$ (equivalently our isogeny class) is supersingular, then we already have $\text{End}\phi \otimes A_v \cong O \otimes A_v$ as maximal order of the $k_v$-algebra (which is actually in this case a field) $k_v(\pi)$.

- If $\phi$ (equivalently our isogeny class) is not supersingular and $\text{End}\phi \otimes A_v \not\cong O \otimes A_v$, then there exists (see lemma 1.3) a Drinfeld module $\psi = \phi/G_L$ such that $\text{End}\psi \otimes A_v \cong O \otimes A_v$ and $\text{End}\psi \otimes A_\omega \cong \text{End}\phi \otimes A_\omega$ at all the other places $\omega \neq v$.

In any case, we get a Drinfeld module $\psi$ such that $\text{End}\psi \otimes A_\omega \cong O \otimes A_\omega$ at all the places $\omega$ of $k$.

Hence $O = \text{End}\psi$.

\[\Diamond\]

2 Application: Endomorphism rings in some isogeny classes of rank 3 Drinfeld modules

We give in this part a more specific description of the orders occurring as endomorphism of a Drinfeld module in the special case of an isogeny class of rank 3 Drinfeld modules. As a direct consequence of theorem 1.4 we have the following:

Proposition 2.1. We keep the same notation we have in the above mentioned theorem.

Let $M(x) = x^3 + a_1x^2 + a_2x + \mu \beta^m$ be a rank 3 Weil polynomial.
1) If \( p_v \nmid a_2 \) then an \( A \)-order \( O \) of \( k(\pi) \) is the endomorphism ring of a Drinfeld module in the isogeny class defined by \( M(x) \) if and only if it contains the Frobenius \( \pi \in \mathcal{O} \).

2) Otherwise (i.e. if \( p_v \mid a_2 \)), an order \( O \) of \( k(\pi) \) occurs as endomorphism ring of a Drinfeld module in the isogeny class defined by \( M(x) \) if and only if the Frobenius endomorphism \( \pi \in \mathcal{O} \) and \( O \) is maximal at all the places of \( k(\pi) \) lying over \( v \) (i.e. \( O \otimes A_v \) is a maximal order of the \( k_v \)-algebra \( k_v(\pi) \)).

Proof:

1) If \( p_v \nmid a_2 \) then \( M(x) \equiv x(x^2 + a_1 x + a_2) \mod p_v \). That means (see corollary \[1.2\]) the irreducible factor \( M_{loc}(x) \) of \( M(x) \) in \( k_v[x] \) describing the unique zero \( v_0 \) of \( \pi \) in \( k(\pi) \) is a degree 1 polynomial. Therefore any \( A \)-order of \( k(\pi) \) containing \( \pi \) is already maximal at \( v_0 \). The statement follows then from theorem \[1.5\].

2) If \( p_v \mid a_2 \) then we have two sub-cases.

- If \( p_v \nmid a_1 \) then \( M(x) \equiv x^2(x + a_1) \mod p_v \).
  That means there are two places of \( k(\pi) \) lying over the place \( v \). The zero \( v_0 \) of \( \pi \) which is described by the irreducible factor \( M_{loc}(x) \) of \( M(x) \) in \( k_v[x] \) fulfilling \( M_{loc}(x) \equiv x^2 \mod p_v \), and another place \( v_1 \) described by the irreducible factor \( M_1(x) \) of \( M(x) \) in \( k_v[x] \) fulfilling \( M_1(x) \equiv x + a_1 \mod p_v \).
  As a consequence, \( \deg M_1(x) = 1 \). That means the completion of any \( A \)-order \( \mathcal{O} \) of \( k(\pi) \) containing \( \pi \) at the place \( v_1 \) must be maximal. It follows that, \( \mathcal{O} \) is maximal at the zero \( v_0 \) of \( \pi \) if and only if \( \mathcal{O} \) is maximal at all the places \( (v_0 \text{ and } v_1) \) of \( k(\pi) \) lying over \( v \) and the statement follows.

- If \( p_v \mid a_1 \) then \( M(x) \equiv x^3 \mod p_v \). That means the isogeny class defined by \( M(x) \) is supersingular. In other words there is a unique place (the zero \( v_0 \) of \( \pi \)) of \( k(\pi) \) lying over \( v \) and the statement follows from theorem \[1.6\].

\[\Diamond\]

Remark 2.1 (Recall).

To check that \( \mathcal{O} \otimes A_v \) is a maximal \( A_v \)-order in the \( k_v \)-algebra \( k_v(\pi) \) one can just check that the norm of the conductor \( c \) of \( \mathcal{O} \) is not divisible by \( p_v \). We recall that the norm of the conductor can be gotten from the relationship between the discriminant of the order \( \mathcal{O} \) and the discriminant of the field \( k(\pi) \).

\[
disc(\mathcal{O}) = N_{k(\pi)/k}(c) \cdot disc(k(\pi))
\]

In the upcoming part, we want to explicitly compute the maximal order of the cubic function field \( k(\pi) \) and all the sub-orders occurring as endomorphism ring of a rank-3 Drinfeld module.

Proposition 2.2. \[2, \text{Corollary 5.2}\]
Let \( M_0(x) = x^3 + c_1 x + c_2 \) be the standard form of the polynomial \( M(x) = x^3 + a_1 x^2 + a_2 x + \mu Q \). Where \( c_1 \) and \( c_2 \) are like computed in the algorithm ??.
Let \( \text{disc}(M_0(x)) = \lambda \prod_{i=1}^t D_i^2 \) be the square-free factorization of \( \text{disc}(M_0(x)) \).

The discriminant of the function field \( k(\pi) \) is given by

\[
\text{disc}(k(\pi)) = \lambda D \gcd(D_2D_4,c_2)^2 \quad \text{where} \quad D = \prod_{i \text{ odd}} D_i, \quad \lambda \in \mathbb{F}_q^*,
\]

We will not give the proof in details since it has already been done in \([7]\).

We just remind that the proof strongly relies on the fact that \( M_0(x) = x^3 + c_1 x + c_2 \) is given in the standard form. That is, for any prime element \( p \in A \), \( v_p(c_1) < 2 \) or \( v_p(c_2) < 3 \). This condition forces the valuation of the discriminant \( v_p(\text{disc}(M_0(x))) = v_p(-4c_1^3 - 27c_2^2) \) to be bounded and leads to the following lemma.

**Lemma 2.1.** \([8\text{ theorem 2}]

Let \( k(\pi)/k \) be a cubic function field defined by the irreducible polynomial \( M_0(x) = x^3 + c_1 x + c_2 \) given in the standard form. Let \( D_0 = \text{disc}(M_0(x)) \) and \( \Delta_0 = \text{disc}(k(\pi)) \). For any prime \( p \) of \( k \) we have the following:

1. \( v_p(\Delta_0) = 2 \) if and only if \( v_p(c_1) \geq v_p(c_2) \geq 1 \).
2. \( v_p(\Delta_0) = 1 \) if and only if \( v_p(D_0) \) is odd.
3. \( v_p(\Delta_0) = 0 \) otherwise.

**Remark 2.2.** The index of \( \tilde{\pi} \) can therefore be computed using the fact that \( \text{disc}(M_0(x)) = \text{ind}(\tilde{\pi})^2 \text{disc}(k(\pi)) \) i.e.

\[
I := \text{ind}(\tilde{\pi}) = \sqrt{\frac{\text{disc}(M_0(x))}{\text{disc}(k(\pi))}}
\]

We recall that \( \tilde{\pi} \) and \( \pi \) define the same function field \( k(\pi) = k(\tilde{\pi}) \).

**Proposition 2.3.** \([2\text{ theorem 6.4}] \) and \([10\text{ lemma 3.1}] \)

Let \( M_0(x) = x^3 + c_1 x + c_2 \) be the standard form of the Weil polynomial \( M(x) = x^3 + a_1 x^2 + a_2 x + \mu Q \). \( \pi \) denotes a root of \( M(x) \) and \( \tilde{\pi} = \frac{\pi + \frac{\Delta}{\gcd(g_1,g_2)}}{2} \) is a root of \( M_0(x) \). Let \( \omega_1 = \alpha_1 + \tilde{\pi} \) and \( \omega_2 = \frac{\alpha_2 + \beta_2 \tilde{\pi} + \tilde{\pi}^2}{I} \), where \( \alpha_1, \alpha_2 \) and \( \beta_2 \) are elements of \( A \).

(1, \omega_1, \omega_2) is an integral basis of the cubic function field \( k(\pi) = k(\tilde{\pi}) \) if and only if

\[
\begin{cases}
3\beta_2^2 + c_1 \equiv 0 \mod I \\
\beta_2^2 + c_1 \beta_2 + c_2 \equiv 0 \mod I^2 \\
\alpha_2 \equiv -2\beta_2^2 \equiv 2c_1/3 \mod I
\end{cases}
\]

Proof: The proof mainly relies on the following two facts:

- \( \text{disc}(1, \tilde{\pi}, \tilde{\pi}^2) = I^2 \text{disc}(k(\pi)/k) \)

- For \( \omega_2 = \frac{2a_2 + \beta_2 \tilde{\pi} + \tilde{\pi}^2}{I} \) to be integral it is necessary that

\[
\omega_2^2 = \left( \frac{\alpha_2 + \beta_2 \tilde{\pi} + \tilde{\pi}^2}{I^2} \right)^2 \quad \text{and} \quad (\alpha_1 + \tilde{\pi})\omega_2 \quad \text{both lie in} \quad A[1, \tilde{\pi}, \omega_2] \]
In other words there exist \( \lambda_0, \mu_0, \lambda_1, \mu_1 \) and \( \lambda_2, \mu_2 \in A \) such that \( \omega_2^2 = \lambda_0 + \lambda_1 \pi + \lambda_2 \omega_2 \) and \( \pi \omega_2 = \mu_0 + \mu_1 \pi + \mu_2 \omega_2 \).

**Corollary 2.1.** \( \alpha_1 \) in the previous proposition can be assumed to be \( 0 \) because if
\[
\begin{pmatrix}
1, \alpha_1 + \pi, \frac{\alpha_2 + \beta_2 \pi + \pi^2}{I}
\end{pmatrix}
\]
is an integral basis, then so is
\[
\begin{pmatrix}
1, \tilde{\pi}, \frac{\alpha_2 + \beta_2 \tilde{\pi} + \tilde{\pi}^2}{I}
\end{pmatrix}.
\]

This is simply due to the fact that both triples have the same discriminant.

**Remark 2.3.** One can therefore, given an isogeny class of Drinfeld modules described by the Weil polynomial \( M(x) = x^3 + a_1 x^2 + a_2 x + \mu Q \), compute the corresponding maximal order \( \mathcal{O}_{\text{max}} \) which is the \( A \)-module generated by \( (1, \omega_1, \omega_2) \) as mentioned before.

Let \( \mathcal{O}_{\text{max}} = \langle 1, \omega_1, \omega_2 \rangle = \left\{ (X, Y, Z) \begin{pmatrix} 1 \\ \omega_1 \\ \omega_2 \end{pmatrix} \left| X, Y, Z \in A \right. \right\} \).

We want now to give a complete list of sub-orders of \( \mathcal{O}_{\text{max}} \) occurring as endomorphism rings of Drinfeld modules. We know from proposition 2.1 that this is equivalent to looking for sub-orders containing \( \pi \) and whose conductor’s norm (in case \( p_n \mid a_2 \)) is relatively prime to \( p_n \).

Let then \( \mathcal{O} = \langle \omega_0, \omega_1, \omega_2 \rangle \) be a sub-order of \( \mathcal{O}_{\text{max}} \).

1 \( \in \mathcal{O} \). That means one can write without loss of generality
\[
\mathcal{O} = \langle 1, \omega_1, \omega_2 \rangle = \left\{ (X, Y, Z) \begin{pmatrix} 1 \\ \omega_1 \\ \omega_2 \end{pmatrix} \left| X, Y, Z \in A \right. \right\}.
\]

But \( \omega_1 \) and \( \omega_2 \in \mathcal{O}_{\text{max}} \). That means
\[
\omega_1 = \alpha_1 + \beta_1 \omega_1 + \gamma_1 \omega_2 \quad \text{and} \quad \omega_2 = \alpha_2 + \beta_2 \omega_1 + \gamma_2 \omega_2
\]
for some \( \alpha_i, \beta_i, \gamma_i \in A \) \( i = 1, 2 \). In other words,
\[
\begin{pmatrix}
1 \\
\omega_1 \\
\omega_2
\end{pmatrix} = \begin{pmatrix}
\alpha_1 & \beta_1 & \gamma_1 \\
\alpha_2 & \beta_2 & \gamma_2 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 \\
\omega_1 \\
\omega_2
\end{pmatrix}.
\]

Let \( M = \begin{pmatrix}
1 & 0 & 0 \\
\alpha_1 & \beta_1 & \gamma_1 \\
\alpha_2 & \beta_2 & \gamma_2
\end{pmatrix} \in \mathcal{M}_3(A) \)
Where \( \mathcal{M}_3(A) \) denotes the ring of \( 3 \times 3 \) -matrices with entries in \( A \).

\( M \) can be transformed into the so-called Hermite normal form. That means there the exists a matrix \( U \in GL_3(A) \) and an upper triangular matrix \( H \) such that \( U \cdot M = H \).

Some simple row operations show that the Hermite normal form of \( M \) looks like
\[
H = \begin{pmatrix}
1 & 0 & 0 \\
0 & c & b \\
0 & 0 & a
\end{pmatrix} \quad \text{with deg}_T(b) < \text{deg}_T(a) \quad (1)
\]

We therefore redefine \( \tilde{\omega}_1 \) and \( \tilde{\omega}_2 \) as \( \tilde{\omega}_1 = c \omega_1 + b \omega_2 \) and \( \tilde{\omega}_2 = a \omega_2 \).

The sub-lattice \( \mathcal{O} \) can then be written as
\[
\mathcal{O} = \langle 1, \tilde{\omega}_1, \tilde{\omega}_2 \rangle = \left\{ (X, Y, Z) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ c \\ b \\ 0 \\ 0 \\ a \end{pmatrix} \begin{pmatrix} 1 \\ \omega_1 \\ \omega_2 \end{pmatrix} \left| X, Y, Z \in A \right. \right\}
\]

**Remark 2.4.** One clearly notices that the sub-lattice \( \mathcal{O} \) above is an order if and only if \( \omega_1^2, \omega_2^2 \) and \( \omega_1 \omega_2 \) belong to \( \mathcal{O} \).
One can therefore compute \( \omega_1^2 = (c\omega_1 + b\omega_2)^2 = c^2\omega_1^2 + 2bc\omega_1\omega_2 + b^2\omega_2^2 \)

But
\[
\begin{align*}
\omega_2^2 &= (a\omega_2)^2 = a^2\omega_2^2 \\
\omega_1\omega_1 &= (c\omega_1 + b\omega_2)(a\omega_2) = ac\omega_1\omega_2 + ab\omega_2^2
\end{align*}
\]

Thus
\[
\begin{pmatrix}
\omega_1^2 \\
\omega_2^2 \\
\omega_1\omega_2
\end{pmatrix}
= 
\begin{pmatrix}
c^2 & b^2 & 2bc \\
0 & a^2 & 0 \\
0 & ab & ac
\end{pmatrix}
\begin{pmatrix}
\omega_1^2 \\
\omega_2^2 \\
\omega_1\omega_2
\end{pmatrix}
\]

As we have seen in proposition 2.3 and its corollary, \( \omega_1 = \bar{\pi} \) and \( \omega_2 = \frac{\alpha_2 + \beta_2\bar{\pi} + \bar{\pi}^2}{I} \) where \( \bar{\pi}^3 + c_1\bar{\pi} + c_2 = 0 \)

One can therefore compute \( \omega_1^2, \omega_2^2 \) and \( \omega_1\omega_2 \) in terms of \( \omega_1 \) and \( \omega_2 \). One gets
\[
\begin{pmatrix}
\omega_1^2 \\
\omega_2^2 \\
\omega_1\omega_2
\end{pmatrix}
= 
\begin{pmatrix}
X_{11} & X_{12} & X_{13} \\
X_{21} & X_{22} & X_{23} \\
X_{31} & X_{32} & X_{33}
\end{pmatrix}
\begin{pmatrix}
1 \\
\omega_1 \\
\omega_2
\end{pmatrix}
\]

Therefore
\[
\begin{align*}
X_{11} &= -\alpha_2, X_{12} = -\beta_2, X_{13} = I \\
X_{21} &= \frac{\alpha_2\beta_2 - c_1\alpha_2 + 3\alpha_2^2 - 2\alpha_2\beta_2}{I^2}, X_{22} = \frac{-\beta_2^3 - c_1\beta_2 - c_2}{I^2}, X_{23} = \frac{\beta_2^2 - c_1 + 2\alpha_2}{I} \\
X_{31} &= \frac{\alpha_2\beta_2 - c_2}{I}, X_{32} = \frac{-\beta_2^2 - c_1 + \alpha_2}{I} \quad \text{and} \quad X_{33} = \beta_2.
\end{align*}
\]

Remark 2.5. \( \mathcal{O} \) is an order if and only if \( M_1M_2H^{-1} \in \mathcal{O}(A) \)

Let us now investigate the orders occurring as endomorphism ring of a rank 3 Drinfeld module.

We know that in addition to the above mentioned condition, \( \mathcal{O} = (1, \omega_1, \omega_2) \) must contain the Frobenius \( \pi \). In other words, there should exist \( a_0, b_0, c_0 \in A \) such that
\[
\pi = a_0 + b_0\omega_1 + c_0\omega_2.
\]

But
\[
\omega_1 = \bar{\pi} \quad \text{and} \quad \bar{\pi} = \frac{\pi + \frac{a_0}{\gcd(g_1, g_2)}}{\gcd(g_1, g_2)}
\]

Also \( \omega_1 = c\omega_1 + b\omega_2 \) and \( \omega_2 = a\omega_2 \). Therefore
\[
-\frac{a_1}{3} + \gcd(g_1, g_2) \cdot \omega_1 = a_0 + b_0c \cdot \omega_1 + (b_0b + c_0a) \cdot \omega_2
\]

Thus
\[
b_0c = \gcd(g_1, g_2) \quad \text{and} \quad b_0b = -c_0a
\]

That is,
\[
\begin{align*}
c \quad &\text{divides} \quad \gcd(g_1, g_2) \\
a \quad &\text{divides} \quad b\gcd(g_1, g_2)
\end{align*}
\]

We summarize our discussion in the following theorem:
Let \( M(x) = x^3 + a_1 x^2 + a_2 x + \mu Q \in \mathbb{F}_q[x] \) be a Weil polynomial. In order to put \( M(x) \) in a simple form \( x^3 + b_1 x + b_2 \), let \( b_1 = \frac{-a_1^2}{3} + a_2 \) and \( b_2 = \frac{2a_1^3}{27} - \frac{a_1 a_2}{3} + \mu Q \) whose square-free factorizations are given by

\[
b_1 = \mu_1 \prod_{i=1}^{n_1} b_{i_1} \quad \text{and} \quad b_2 = \mu_2 \prod_{j=1}^{n_2} b_{j_2}, \quad \mu_1, \mu_2 \in \mathbb{F}_q^*
\]

In order to get the standard form \( M_0(x) = x^3 + c_1 x + c_2 \) of \( M(x) \) (as defined in ??), we consider \( g_1 \) and \( g_2 \) the elements of \( A \) defined by

\[
g_1 = \prod_{i=1}^{n_1} b_{i_1}^{\frac{2}{c_1}} \quad \text{and} \quad g_2 = \prod_{j=1}^{n_2} b_{j_2}^{\frac{2}{c_2}}
\]

We remove out from \( b_1 \) and \( b_2 \) resp. the highest square common divisor and the highest cubic common divisor by setting

\[
c_1 = \frac{b_1}{\gcd(g_1, g_2)^2} \quad \text{and} \quad c_2 = \frac{b_2}{\gcd(g_1, g_2)^2}
\]

Let \( \bar{\pi} = \frac{\pi + 2^{\frac{1}{c_1}} \alpha}{\gcd(g_1, g_2)} \) be a root of the standard polynomial \( x^3 + c_1 x + c_2 \).

Let \( I = \text{ind}(\bar{\pi}) = \frac{\text{ind}(\pi)}{\gcd(g_1, g_2)^2}, \alpha_2 \) and \( \beta_2 \in A \) such that

\[
\begin{align*}
3\beta_2^2 + c_1 &\equiv 0 \mod I \\
\beta_2^2 + c_1 \beta_2 + c_2 &\equiv 0 \mod I^2 \\
\alpha_2 &\equiv -2\beta_2^3 \equiv 2c_1/3 \mod I
\end{align*}
\]

We consider the matrix \( M_2 \in \mathbb{M}_3(k) \) defined by

\[
M_2 = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix} \text{ where}
\]

\[
X_{11} = -\alpha_2, \quad X_{12} = -\beta_2, \quad X_{13} = I \\
X_{21} = \alpha_2 \beta_2^2 - c_1 \beta_2 + 3 \alpha_2^2 - 2c_2 \beta_2, \quad X_{22} = -\frac{\beta_2^2}{I^2} - c_1 \beta_2 - c_2, \quad X_{23} = \frac{\beta_2^2 - c_1 + 2\alpha_2}{I} \\
X_{31} = \frac{\alpha_2 \beta_2^2 - c_2}{I}, \quad X_{32} = -\frac{\beta_2^2 - c_1 + \alpha_2}{I} \text{ and } X_{33} = \beta_2.
\]

The Endomorphism rings of Drinfeld modules in the isogeny class defined by the Weil polynomial \( M(x) \) are:

\[
O = A + A \cdot \left( c\bar{\pi} + b \left( \frac{\alpha_2 + \beta_2 \bar{\pi} + \bar{\pi}^2}{I} \right) \right) + A \cdot \left( \frac{\alpha_2 + \beta_2 \bar{\pi} + \bar{\pi}^2}{I} \right)
\]

such that \( M_1 M_2 H^{-1} \in \mathbb{M}_3(A) \) and in addition \( \gcd(p_v, ac) = 1 \) if \( p_v \mid a_2 \). Where

\[
M_1 = \begin{pmatrix} c^2 & b^2 & 2bc \\ 0 & a^2 & 2bc \\ 0 & ab & ac \end{pmatrix} \text{ and } H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & b \\ 0 & 0 & a \end{pmatrix}
\]
\[
\begin{cases}
c \text{ runs through the divisors of } \gcd(g_1, g_2) \\
a \text{ runs through the divisors of } I \\
b \in A \text{ such that } \deg_T b < \deg_T a \text{ and } a \mid b^{\gcd(g_1, g_2)}/c
\end{cases}
\]

Proof: The proof follows straightforwardly from our discussion before. The condition \(\gcd(p_v, ac) = 1\) comes from the fact that in case \(p_v \mid a_2\), the norm of the conductor of \(O\) must be prime to \(p_v\) (see Proposition 2.1).

\[\Box\]

Corollary 2.2. Let \(M(x) = x^3 + a_1 x^2 + a_2 x + \mu Q \in A[x]\) be a Weil polynomial. \(\pi\) is a root of \(M(x)\) and \(\tilde{\pi} = \pi + \alpha_2^{1/3} + \beta_2^{1/3} \mu Q\). If there is no prime \(p \in A\) such that \(p^2 \mid b_1\) and \(p^3 \mid b_2\) (in particular if \(b_1\) and \(b_2\) are coprime or \(b_1\) is square-free or \(b_2\) is cubic-free) then the endomorphism rings of Drinfeld modules in the isogeny class defined by the Weil polynomial \(M(x)\) are

\[
\mathcal{O}_a = A + A \cdot \tilde{\pi} + A \cdot a \left(\frac{\alpha_2 + \beta_2 \tilde{\pi} + \tilde{\pi}^2}{I}\right)
\]

such that \(M_a M_2 H_a^{-1} \in \mathcal{M}_3(A)\) and in addition \(\gcd(p_v, a) = 1\) if \(p_v \mid a_2\). Where

\[
M_a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{pmatrix} \quad \text{and} \quad H_a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}
\]

Here \(a \) runs through the divisors of the index \(I = \text{ind}(\tilde{\pi})\).

Proof: One can just reconsider the equation (2) right after remark 2.3. Here \(\gcd(g_1, g_2) = 1\). Thus \(b_0 c = 1\) i.e. \(b_0\) and \(c\) are units. In addition \(b_0 b = -c_0 a\) and \(b_0\) is a unit. That means \(a \mid b\). But \(\deg_T b < \deg_T a\) (see equation (1)). Therefore \(b = 0\). Hence the matrix \(H\) in equation (1) and the matrix \(M_1\) become

\[
H_a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix} \quad \text{and} \quad M_a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{pmatrix}
\]

and the result follows.

\[\Box\]
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