A Brief Proof of Bochner’s Tube Theorem and
a Generalized Tube

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Abstract

The aim of this note is firstly to give a new brief proof of classical Bochner’s Tube Theorem (1938) by making use of K. Oka’s Boundary Distance Theorem (1942), showing directly that two points of the envelope of holomorphy of a tube can be connected by a line segment. We then apply the same idea to show that if an unramified domain $D := A_1 + iA_2 \to \mathbb{R}^n + i\mathbb{R}^n = \mathbb{C}^n$ with unramified real domains $A_j \to \mathbb{R}^n$ is pseudoconvex, then both $A_j$ are univalent and convex (a generalization of Kajiwara’s theorem). From the viewpoint of this result we discuss a generalization by M. Abe with giving an example of a finite tube over $\mathbb{C}^n$ for which Abe’s theorem no longer holds. The present method may clarify the point where the (affine) convexity comes from.

Keywords: tube domain; Oka’s boundary distance theorem; Kajiwara’s Theorem; analytic continuation; envelope of holomorphy.

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1. Introduction

The following statement is classical and well-known as Bochner’s Tube Theorem:

**Theorem 1.1** (Bochner [2, 3], Stein [18] $(n = 2)$). Let $T_R = R + i\mathbb{R}^n$ be a tube (domain) of $\mathbb{C}^n$ with a domain (open, connected) $R \subset \mathbb{R}^n$ as real base. Then the envelope of holomorphy of $T_R$ is $T_{\text{co}(R)}$, where $\text{co}(R)$ denotes the (affine) convex hull of $R$.

Our first aim is to give a new brief simple proof of this theorem, based on Oka’s Boundary Distance Theorem (§2).

We then deal with a generalized tube $\pi : A_1 + iA_2 \to \mathbb{C}^n$ with real unramified domains $\pi_j : A_j \to \mathbb{R}^n (j = 1, 2)$ and $\pi = \pi_1 + i\pi_2$. In §3 we also give another proof to Porten [17], Theorem 1.1:

**Theorem 1.2** (Generalized tube). If $A_1 + iA_2$ is pseudoconvex, then the both $A_j$ are univalent and convex subdomains of $\mathbb{R}^n$.

The case where $A_j$ are univalent was obtained by J. Kajiwara [12], and the case where $A_2 = \mathbb{R}^n$ was dealt with by M. Abe [1]. We will give counter-examples such that M. Abe’s Theorem does not holds for a finite tube; i.e., the part $A_2$ is bounded in $\mathbb{R}^n$ (see §4).

We will see the point where the (affine) convexity comes from (see Remark [24]).

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2. Proof of Theorem 1.1

To be precise, a ‘domain’ of $\mathbb{R}^n$ (or $\mathbb{C}^n$) is an open and connected subset of $\mathbb{R}^n$ (or $\mathbb{C}^n$). If $X$ is a connected Hausdorff topological space with a local homeomorphism $p : X \to \mathbb{R}^n$ (or $\mathbb{C}^n$), we call $p : X \to \mathbb{R}^n$ (or $\mathbb{C}^n$) or simply $X$ ap domain over $\mathbb{R}^n$ (or $\mathbb{C}^n$). If $p$ is injective, it is said to be univalent (schlicht) or otherwise multivalent in general; a univalent domain over $\mathbb{R}^n$ (or $\mathbb{C}^n$) may be identified with a domain of $\mathbb{R}^n$ (or $\mathbb{C}^n$). In this paper, domains are always unramified.

For our proof we use the next two basic theorems: As for the envelope of holomorphy we add the constructive existence for a convenience as an appendix (cf. §5 Appendix (1) at the end).

Theorem 2.1. Every holomorphically separable domain $\mathcal{D}$ over $\mathbb{C}^n$ admits an envelope of holomorphy, containing $\mathcal{D}$ as a subdomain. In particular, a univalent domain $\Omega$ of $\mathbb{C}^n$ admits an envelope of holomorphy (multivalent in general), containing $\Omega$ as a subdomain.

Theorem 2.2 (Boundary distance: Oka [15, 16 VI (1942), IX (1953); 9, 13]). If $\mathcal{D}$ is a domain of holomorphy over $\mathbb{C}^n$, then $-\log \delta(\zeta, \partial \mathcal{D})$ ($\zeta \in \mathcal{D}$) is a continuous plurisubharmonic function, where $\delta(\zeta, \partial \mathcal{D})$ denotes the distance function to the boundary (cf. §5 Appendix (2)).

Let $\pi : \hat{T} \to \mathbb{C}^n$ be the envelope of holomorphy of $T_R$ by Theorem 2.1. With $\hat{R} := \hat{T} \cap \pi^{-1}\mathbb{R}^n$, $\varpi = \pi|_{\hat{R}} : \hat{R} \to \mathbb{R}^n$ is a real domain over $\mathbb{R}^n$ and $\varpi(R) \subset \text{co}(R)$. Then $T$ has a structure of a tube in the following sense:

\[(\mathbb{R}^n+i\mathbb{R}^n) \to \mathbb{R}^n + i\mathbb{R}^n = \mathbb{C}^n.\]

It follows from Oka’s Boundary Distance Theorem 2.2 that $-\log \delta(\zeta, \partial \hat{T})$ is plurisubharmonic and satisfies

\[\tag{2.4} -\log \delta(\zeta, \partial \hat{T}) = -\log(\zeta + iy, \partial \hat{T}), \quad \forall y \in \mathbb{R}^n.\]

With the local coordinates $\pi(p) = (x_j + iy_j)$, if $\delta(p, \partial \hat{T})$ is of $C^2$-class, it satisfies the semi-positive definiteness:

\[\tag{2.5} \left( \frac{\partial^2}{\partial z_j \partial z_k} - \log \delta(\zeta, \partial \hat{T}) \right)_{j,k} = \left( \frac{\partial^2}{\partial x_j \partial x_k} - \log \delta(\zeta, \partial \hat{T}) \right)_{j,k} \geq 0.\]

We define a line segment $L[p,q] \subset \hat{R}$ connecting two points $p, q \in \hat{R}$ as follows. Let $L[\varpi(p), \varpi(q)] \subset \mathbb{R}^n$ be a line segment connecting $\varpi(p)$ and $\varpi(q)$. Then there is a unique connected component $L_p$ of the inverse $\pi^{-1} L[\varpi(p), \varpi(q)]$, containing $p$. If $L_p \ni q$, we write $L_p = L[p,q] \subset \hat{R}$. For mutually close $p, q \in \hat{R}$, $L[p,q]$ exists, but in general the existence is unknown at this moment. If $p = q$, then $L[p,q] = \{p\}$ is considered as a special case of degenerate line segment. Assuming the existence of $L[p,q]$, we see by (2.5) that the restricted function $-\log \delta(\zeta, \partial \hat{T})|_{L[p,q]}$, even if it is not differentiable, is a convex function on the line segment $L[p,q]$. Therefore we have

\[\tag{2.6} \min_{L[p,q]} \delta(\zeta, \partial \hat{T}) = \min_{L[p,q]} \delta(\zeta, \partial \hat{T}).\]

Claim 2.7. If $S := \{(p, q) \in \hat{R}^2 : \exists L[p,q] \subset \hat{R} \} \subset \hat{R}^2$, then $S = \hat{R}^2$.

Firstly, $S$ is non-empty and open. It suffices to show that $S$ is closed in $\hat{R}^2$. Let $(p, q) \in \hat{R}^2$ be an accumulation point of $S$. Then there is a sequence of points $(p_\nu, q_\nu) \in S$ ($\nu = 1, 2, \ldots$) such that

\[\lim_{\nu \to \infty} p_\nu = p, \quad \lim_{\nu \to \infty} q_\nu = q, \quad L[p_\nu, q_\nu] \subset \hat{R}.\]

By (2.6) there is a constant $r_0 > 0$ independent of $\nu$ such that the tubular neighborhood $U_\nu$ (univalent) of every $L[p_\nu, q_\nu]$ with width $r_0$ is contained in $\hat{R}$. Then for every sufficiently large $\nu$, $U_\nu \ni p, q$. Therefore $L[p, q] \subset U_\nu \subset \hat{R}$; thus, $(p, q) \in S$ and hence $S = \hat{R}^2$.

It follows that $\varpi : \hat{R} \to \mathbb{R}^n$ is univalent. For, otherwise, there were two points, $p, q \in \hat{R}$ such that $p \neq q$ and $\varpi(p) = \varpi(q)$. But there would be no line segment $L[p,q]$; contradiction. Moreover, for arbitrary distinct $p, q \in \hat{R}$, $L[p,q] \subset \hat{R}$, and hence $\hat{R}$ is convex. Thus, $\hat{R} = \text{co}(R)$ and $\hat{T} = T_{\text{co}(R)}$. \hfill \Box

The above proof immediately implies the following generalization due to M. Abe [1].
Theorem 2.8. Let \( \varpi : R \to R^n \) be a real domain over \( R^n \) and let \( \pi : T_R = R + iR^n \to C^n \) be a domain as in (2.3). Then, \( T_R \) is a domain of holomorphy if and only if \( T_R \) is univalent and convex.

Remark 2.9. In the above proof, it was the point to deduce the (affine) convexity from (2.5), provided that the domain is pseudoconvex or a domain of holomorphy.

Notes. Theorem 1.1 was proved by S. Bochner [2, 3], and by K. Stein [13] (Hilfssatz 1) in \( n = 2 \). Since then there have been many papers dealing with the proof (cf. Jarnicki–Pflug [10], §3.2 for more informations). The proofs were rather technically involved (cf., e.g., [4] Chap. V, [9] Chap. II). The methods may be classified into four kinds:

(i) By Legendre polynomial expansions (Bochner [3], Bochner–Martin [4]).
(ii) By a family of ellipses (Stein [18] (n = 2), S. Hitotsumatsu [7], L. Hörmander [9] (Theorem 2.5.10), etc.)
(iii) By the boundary distance function (H.J. Bremermann [5] in the case of \( n = 2 \)).
(iv) An approximation theorem of Bauendi-Treves (J. Hounie [8]).

The present proof may belong to (iii) and was inspired by Fritzsche–Grauert [6] p. 87 Exercise 1, while in the textbook the notion of unramified domains is presented in the subsequent section after it; so the supposed situation might be different to the present one. It is also noticed that the presentation of (2.5) goes back to Bremermann [5] §3.5. In the present proof as above, the univalence of the envelope of holomorphy \( T \) and the convexity are proved at the same time.

3. Proof of Theorem 1.2

Let \( \pi_j : A_j \to R^n \) \( (j = 1, 2) \) be domains and put

\[ \pi : D := A_1 + iA_2 \ni x + iy \to \pi_1(x) + i\pi_2(y) \in C^n. \]

We call \( D \) a generalized tube (domain). Let \( y_0 \in A_2 \) be arbitrarily fixed point, and take a univalent ball neighborhood \( B(y_0; 2\rho_0) \subset A_2 \) with center \( y_0 \) and radius \( 2\rho_0 > 0 \). The assumption implies that the continuous function \( \varphi(z) := -\log \delta(z, \partial\Omega) \) is plurisubharmonic in \( D \). Set

\[ V = \{ x \in A_1 : \delta(x, \partial A_1) < \rho_0 \}. \]

Then the function \( \varphi(x + iy) \) in \( x + iy \in V + iB_0(y_0; \rho_0) \) is a function only in \( x \). Therefore, \( \varphi(x + iy) = \varphi(x + iy_0) \) is convex in \( x \in V \). We set

\[ \psi(x) = \max\{\varphi(x + iy_0), -\log \rho_0\}, \quad x \in A_1. \]

Then \( \psi(x) \) is a continuous convex function in \( A_1 \). The same arguments as in (2) with \( \psi(x) \) imply that \( A_1 \) is univalent and convex; the same is applied to \( A_2 \).

4. Counter-examples of Abe’s Theorem 2.8 for finite tubes

Here we give examples of ‘finite tubes’ by replacing the imaginary part \( R^n \) in Theorem 2.8 by a bounded domain, to say, an open ball, for which the theorem no longer holds.

Let \( 0 < R_1 < R_2 \leq \infty \) and set

\[ A = \{ x = (x_1, x_2) \in R^2 : R_1 < \| x \| := (x_1^2 + x_2^2)^{1/2} < R_2 \}, \]
\[ B = \{ y = (y_1, y_2) \in R^2 : \| y \| < R_1 \}. \]

With complex coordinates \( z_j = x_j + iy_j \) \( (j = 1, 2) \) we define a ‘finite tube’ or a ‘tube of finite length’ by

\[ \Omega = A + iB \subset C^2. \]

We consider a holomorphic function \( f(z) = z_1 + iz_2 \in O(\Omega) \) (it is the same with \( f(z) = z_1 - iz_2 \)). Since

\[ |f(z)| = \| x_1 + ix_2 + i(y_1 + iy_2) \| \geq \| x_1 + ix_2 \| - \| y_1 + iy_2 \| > 0, \]

\[ g(z) = 1/f(z) \in O(\Omega); \]

in particular, \( g(z) \) is not holomorphic at the origin 0. Therefore we first note:
Remark 4.1. The envelope of holomorphy $\Omega$ of $\Omega$ is not equal to $\text{co}(A) + iB$. This gives a counterexample for Kajiwara [12], from which $\hat{\Omega} = \text{co}(A) + iB$ should follow. Cf. Jarnicki–Pflug [10], §3.3 for more examples and discussions.

Let $2 \leq \nu \leq \infty$. For $2 \leq \nu < \infty$ we put

$$A_\nu = \left\{ u = (u_1, u_2) \in \mathbb{R}^2 : R_1^{1/\nu} < \|u\| < R_2^{1/\nu} \right\},$$

$$p_\nu : A_\nu \ni u = u_1 + iu_2 \mapsto u' = x_1 + ix_2 = (x_1, x_2) = x \in A,$$

where the complex structures of `$u_1 + iu_2$' and `$x_1 + ix_2$' are different and independent to that of $(z_1, z_2) \in \mathbb{C}^2$. It follows that $p_\nu$ is a local real analytic diffeomorphism between the annuli. We put

$$\pi_\nu : \Omega_\nu = A_\nu \times B \ni (u, y) \mapsto p_\nu(u) + iy \in \Omega \hookrightarrow \mathbb{C}^2.$$

Then $\pi_\nu : \Omega_\nu \rightarrow \mathbb{C}^2$ is a local real analytic diffeomorphism and hence an unramified domain over $\mathbb{C}^2$. We consider $f_\nu(z) = (f(z))^{1/\nu} = (x_1 + ix_2 + iy_1 + iy_2)^{1/\nu}$, which is $\nu$-valued holomorphic in $z \in \Omega$. Note that

$$f_\nu(z) = (x_1 + ix_2)^{1/\nu} \left( 1 + i \frac{y_1 + iy_2}{x_1 + ix_2} \right)^{1/\nu} :$$

Here the latter product factor $\left( 1 + i \frac{y_1 + iy_2}{x_1 + ix_2} \right)^{1/\nu}$ has a 1-valued branch in $\Omega$, because

$$\left| \frac{y_1 + iy_2}{x_1 + ix_2} \right| < 1.$$

Whereas the first factor $(x_1 + ix_2)^{1/\nu}$ is defined to be 1-valued in $A_\nu$, and hence $f_\nu(z)$ is 1-valued holomorphic in $\Omega_\nu$. It follows that the domain $\pi_\nu : \Omega_\nu \rightarrow \mathbb{C}^2$ is holomorphically separable and $g_\nu = 1/f_\nu \in \mathcal{O}(\Omega_\nu)$.

For $\nu = \infty$, we put

$$p_\infty : A_\infty = \left\{ (u_1, u_2) \in \mathbb{R}^2 : \log R_1 < u_1 < \log R_2, \ u_2 \in \mathbb{R} \right\} \ni (u_1, u_2) \mapsto e^{u_1} e^{iu_2} = (e^{u_1} \cos u_2, e^{u_1} \sin u_2).$$

Then $p_\infty : A_\infty \rightarrow A$ is a local real analytic diffeomorphism. Set

$$\pi_\infty : \Omega_\infty = A_\infty \times B \ni (u, y) \mapsto p_\infty(u) + iy \in \Omega \hookrightarrow \mathbb{C}^2.$$

Then, $\pi_\infty : \Omega_\infty \rightarrow \mathbb{C}^2$ is an infinitely-sheeted unramified domain over $\mathbb{C}^2$.

We take $f_\infty(z) = \log f(z)$. Then we have

$$f_\infty(z) = \log(x_1 + ix_2) + \log \left( 1 + i \frac{y_1 + iy_2}{x_1 + ix_2} \right), \quad z \in \Omega :$$

Here, because of (4.2) the second term $\log \left( 1 + i \frac{y_1 + iy_2}{x_1 + ix_2} \right)$ has a 1-valued branch in $\Omega$ and the first term $\log(x_1 + ix_2)$ is 1-valued in $\Omega_\infty$, so that $f_\infty \in \mathcal{O}(\Omega_\infty)$. Therefore, the unramified domain $\pi_\infty : \Omega_\infty \rightarrow \mathbb{C}^2$ is holomorphically separable. Since $f_\infty$ has no zero in $\Omega_\infty$, $1/f_\infty \in \mathcal{O}(\Omega_\infty)$.

Thus we have:

**Proposition 4.3.** Let the notation be as above. For every $\nu$ with $2 \leq \nu \leq \infty$, $\pi_\nu : \Omega_\nu \rightarrow \mathbb{C}^2$ is a $\nu$-sheeted holomorphically separable unramified domain over $\mathbb{C}^2$, and the envelope of holomorphy $\hat{\pi}_\nu : \hat{\Omega}_\nu \rightarrow \mathbb{C}^2$ of $\Omega_\nu$ is never univalent over $\mathbb{C}^2$ and $\hat{\pi}_\nu(\hat{\Omega}_\nu) \neq 0$.

We may propose at the end:

**Problem 4.4.** Let $\Omega = A_1 + iB$ be a univalent generalized tube with $A_1 \subset \mathbb{R}^n$ and an open ball $B \subset \mathbb{R}^n$.  

(i) What is the envelope of holomorphy $\hat{\Omega}$ of $\Omega$?
(ii) What is the condition of $A_1$ with which $\hat{\Omega}$ is univalent. For example, if $A_1$ is simply connected or contractible, is $\hat{\Omega}$ univalent?

Remark 4.5. Very lately, Jarnicki–Pflug [11] dealt with the above problem for $\Omega = A_1 + iB$ with

$$A_1 = \{ x \in \mathbb{R}^n : R_1 < \|x\| < 1 \}, \quad B = \{ y \in \mathbb{R}^n : \|y\| < R_2 \}.$$ 

This case is interesting in view of the above counter-example, and they proved that $\hat{\Omega}$ is univalent and given by

$$\hat{\Omega} = \{ x + iy \in \mathbb{R}^n + i\mathbb{R}^n : \|x\| < 1, \|y\| < R_2, \|y\|^2 < \|x\|^2 + R_2^2 - R_1^2 \}.$$ 

5. Appendix

(1) Envelope of holomorphy. In quite a few references, the notion of the envelope of holomorphy of domains over $C^n$ are presented in a rather sophisticated manner. For our aim the following simple-minded constructive existence is sufficient.

We first fix a notation. If $D$ is a connected Hausdorff space and $\pi : D \rightarrow C^n$ is a local homeomorphism, $\pi : D \rightarrow C^n$ or simply $D$ is called a (unramified Riemann) domain over $C^n$. If $\pi$ is injective, $D$ is said to be univalent. A domain $D$ over $C^n$ naturally admits a structure of complex manifold such that $\pi$ is a local biholomorphism; the set of all holomorphic functions on $D$ is denoted by $O(D)$.

For an element $f \in O(D)$ and a point $p \in D$ there is a small polydisk neighborhood of $a = \pi(p)$ which is identified with a neighborhood of $p$, and $f$ is written there as a convergent power series in the local coordinate $z$:

$$f_p := f(z) = \sum_\alpha c_\alpha(z - a)\alpha.$$ 

If for two points $p, q \in D$ with $p \neq q$ and $\pi(p) = \pi(q)$ there is an element $f \in O(D)$ such that $f_p \neq f_q$, then $\pi : D \rightarrow C^n$ is said to be holomorphically separable.

We fix a point $p_0 \in D$. We consider a curve $C^b$ in $C^n$ with the initial point $a = \pi(p_0)$ and the end point $b \in C^n$ such that every analytic function $f_{C^b}$ at $a$ defined by $f \in O(D)$ can be analytically continued along $C^b$, and defines an analytic function, denoted by $f_{C^b}(z)$, at the end point $b$. Let $\Gamma$ denote the set of all such curves $C^b$. If $C^b, C'^b \in \Gamma$ are homotope through a continuous family of curves belonging to $\Gamma$, then $f_{C^b} = f_{C'^b}$. We denote by $\{C^b\}$ the homotopy class in the above sense, and write $f_{\{C^b\}} := f_{C^b}$.

We fix a polydisk $P_{\Delta} \subset C^n$ with center at the origin. For $f \in O(D)$ and $C^b \in \Gamma$ there is a polydisk neighborhood $b + rP_{\Delta} (r > 0)$ of $b$ where $f_{\{C^b\}}(z)$ converges. Let $r(\{C^b\}, f)$ be the supremum of such $r$, and let $\Gamma^r$ denote all of $\{C^b\}$ such that $\inf_{f \in O(D)} r(\{C^b\}, f) > 0$.

For two element $\{C^b\}, \{C'^b\}$ of $\Gamma^r$ we define an equivalence relation $\{C^b\} \sim \{C'^b\}$ by

$$b = b', \quad f_{\{C^b\}} = f_{\{C'^b\}}, \quad \forall f \in O(D).$$

Let $\{\{C^b\}\}$ stand for the equivalence class, and let $D = \Gamma^r / \sim$, $\hat{\pi} : \{\{C^b\}\} \rightarrow b \in C^n$ be respectively the quotient set and the natural map. It follows from the construction that $\hat{\pi} : D \rightarrow C^n$ gives rise to a holomorphically separable (unramified) domain over $C^n$. Since $D$ is arc-wise connected, $D$ is independent of the choice of $p_0 \in D$. There is a natural holomorphic map $\eta : D \rightarrow \hat{D}$ with $\pi = \hat{\pi} \circ \eta$. If $D$ is holomorphically separable, then $\eta$ is an inclusion map and $D$ is a subdomain of $\hat{D}$.

We call $\hat{\pi} : D \rightarrow C^n$ the envelope of holomorphy of $D$. In the case of $n \geq 2$, even if $D$ is univalent, the envelope of holomorphy $\hat{D}$ of $D$ may be (infinitely) multi-sheeted over $C^n$ in general. If $\eta : D \rightarrow \hat{D}$ is biholomorphic ($D = \hat{D}$), $D$ is called a domain of holomorphy.

(2) Boundary distance. The boundary distance $d(\zeta, \partial D)$ is defined as follows. For a point $\zeta \in D$ there is an open ball $B(\pi(\zeta); r) \subset C^n$ with center $\pi(\zeta)$ and radius $r (> 0)$ such that the connected component $U(\zeta; r)$ of $\pi^{-1}B(\pi(\zeta); r)$ containing $\zeta$ is biholomorphically mapped onto $B(\pi(\zeta); r)$ by $\pi$. We
write \( \delta(\zeta, \partial D) \) for the supremum of such \( r \), which is called the \textit{boundary distance}. The proof of Theorem 2.2 is similar to the case of univalent domains.

In place of an open ball we may use a polydisk \( P\Delta \) with center at \( 0 \) in the above definition. Then the boundary distance is denoted by \( \delta_{P\Delta}(\zeta, \partial D) \); Theorem 2.2 holds with \( -\log_{P\Delta}(\zeta, \partial D) \).

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