Dilatation structures II.
Linearity, self-similarity and the Cantor set

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Introduction

In this paper we continue the study of dilatation structures, introduced in [5].

A dilatation structure on a metric space is a kind of enhanced self-similarity. By way of examples this is explained here with the help of the middle-thirds Cantor set.

Linear and self-similar dilatation structures are introduced and studied on ultra-metric spaces, especially on the boundary of the dyadic tree (same as the middle-thirds Cantor set).

Some other examples of dilatation structures, which share some common features, are given. Another class of examples, coming from sub-Riemannian geometry, will make the subject of an article in preparation.

In the particular case of ultrametric spaces the axioms of dilatation structures take a simplified form, leading to a description of all possible weak dilatation structures on the Cantor set.

As an application we prove that there is more than one linear and self-similar dilatation structure on the Cantor set, compatible with the iterated functions system which defines the Cantor set.

Applications to self-similar groups are reserved for a further paper.

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1 Basics

Here we collect well-known facts and notations which we shall need further.

1.1 Notations

Let $\Gamma$ be a topological separated commutative group endowed with a continuous group morphism

$$\nu : \Gamma \to (0, +\infty)$$

with $\inf \nu(\Gamma) = 0$. Here $(0, +\infty)$ is taken as a group with multiplication. The neutral element of $\Gamma$ is denoted by 1. We use the multiplicative notation for the operation in $\Gamma$.

The morphism $\nu$ defines an invariant topological filter on $\Gamma$ (equivalently, an end). Indeed, this is the filter generated by the open sets $\nu^{-1}(0,a)$, $a > 0$. From now on we shall name this topological filter (end) by "0" and we shall write $\varepsilon \in \Gamma \to 0$ for $\nu(\varepsilon) \in (0, +\infty) \to 0$.

The set $\Gamma_1 = \nu^{-1}(0,1]$ is a semigroup. We note $\bar{\Gamma}_1 = \Gamma_1 \cup \{0\}$ On the set $\bar{\Gamma} = \Gamma \cup \{0\}$ we extend the operation on $\Gamma$ by adding the rules $00 = 0$ and $\varepsilon 0 = 0$ for any $\varepsilon \in \Gamma$. This is in agreement with the invariance of the end 0 with respect to translations in $\Gamma$. 
1.2 Words and the Cantor middle-thirds set

Let $X$ be a finite, non empty set. The elements of $X$ are called letters. The collection of words of finite length in the alphabet $X$ is denoted by $X^*$. The empty word $\emptyset$ is an element of $X^*$.

In this paper we shall work mainly with the alphabet $X = \{0, 1\}$. In this particular case one can define a conjugate function (from $X$ to $x$), given by $\bar{0} = 1$, $\bar{1} = 0$.

The length of any word $w \in X^*$

$$w = a_1...a_m, \quad a_k \in X \quad \forall k = 1, ..., m$$

is denoted by $|w| = m$.

The set of words which are infinite at right is denoted by $X^\omega = \{f | f : \mathbb{N}^* \rightarrow X\} = X^{\mathbb{N}^*}$. Concatenation of words is naturally defined. If $q_1, q_2 \in X^*$ and $w \in X^\omega$ then $q_1 q_2 \in X^*$ and $q_1 w \in X^\omega$.

The shift map $s : X^\omega \rightarrow X^\omega$ is defined by

$$w = w_1 s(w)$$

for any word $w \in X^\omega$. For any $k \in \mathbb{N}^*$ we define $[w]_k \in X^k \subset X^*$, $\{w\}_k \in X^\omega$ by

$$w = [w]_k s^k(w), \quad \{w\}_k = s^k(w).$$

The topology on $X^\omega$ is generated by cylindrical sets $qX^\omega$, for all $q \in X^*$. The topological space $X^\omega$ is compact.

To any $q \in X^*$ is associated a continuous injective transformation $\hat{q} : X^\omega \rightarrow X^\omega$, $\hat{q}(w) = qw$. The semigroup $X^*$ (with respect to concatenation) can be identified with the semigroup (with respect to function composition) of these transformations. This semigroup is obviously generated by $X$. The empty word corresponds to the identity function.

The topological space $X^\omega$ is metrizable. Indeed, denote by $N$ the number of letters in the alphabet $X$. Let us fix a bijection between $X$ and $\{1, ..., N\}$, which allows us to identify $X$ with $\{1, ..., N\}$.

For any $w \in X^\omega$ and $i \in \mathbb{N}^*$, the symbol $w_i \in X = \{1, ..., N\}$ denotes the $i$-th letter in the word $w$. Define then the injective continuous function

$$\Phi : X^\omega \rightarrow \mathbb{R}, \quad \Phi(w) = \sum_{i=1}^{\infty} \frac{w_i}{2^i}.$$ 

If $N$ is a prime number then we may equally define a homeomorphism between $X^\omega$ and the group of $N$-adic integers. We shall first recall some basic facts about $p$-adic integers.

3
$Q_p$ is the closure of $Q$ with respect to $p$-adic norm. For $x \in Z$, $x \neq 0$ the $p$-norm is defined by

$$|x|_p = \inf \{p^{-r} \mid p^r \text{ divides } x\}.$$  

For $x/y \in Q$ ($x, y \in Z$, $y \neq 0$) the $p$-adic norm is defined by

$$|x/y|_p = \frac{|x|_p}{|y|_p}.$$  

The norm induces an ultrametric distance on $Q_p$. The closed unit disk of $Q_p$ is called the ring of $p$-adic integers, denoted $Z_p$. The set $Z_p$ is compact.

**Proposition 1.1** Any element of $x \in Z_p$ admits an unique $p$-adic expansion

$$x = \sum_{i=1}^{\infty} x_i p^i,$$

with all $x_i \in \{0, ..., p-1\}$. Any element of $x \in Q_p$ admits an unique $p$-adic expansion

$$x = \sum_{i=r}^{\infty} x_i p^i,$$

starting from some $r \in Z$, with all $x_i \in \{0, ..., p-1\}$.

The addition and multiplication of $p$-adic numbers is done using the $p$-adic expansion and the standard algorithms (with some simple modifications: with addition or multiplications of "digits" $x_i$, modulo $p$, with remainders, from left to right).

The function

$$\Psi : X^\omega \to Z_N, \quad \Phi(w) = \sum_{i=1}^{\infty} (w_i - 1) N^i$$

is then a homeomorphism.

The elements of $X^*$, seen as transformations of $X^\omega$, are contractions, with respect to the distances induced by $\Phi$ and (if $N$ is prime) $\Psi$. Indeed, it is enough to check this for the transformations associated with the letters in the alphabet $X$. Let $a \in X$ and $w \in X^\omega$. We have then

$$\Phi(aw) = a + \frac{1}{2} \Phi(w) \quad \text{and} \quad \Psi(aw) = (a - 1) + N \Psi(w),$$

which implies that for any $w_1, w_2 \in X^\omega$ we have

$$|\Phi(aw_1) - \Phi(aw_2)| = \frac{1}{2} |\Phi(w_1) - \Phi(w_2)|,$$

$$|\Psi(aw_1) - \Psi(aw_2)|_N = \frac{1}{N} |\Phi(w_1) - \Phi(w_2)|_N.$$  

Therefore $\Phi(X^\omega), \Psi(X^\omega)$ are invariant sets of iterated functions systems of contractions.

In the particular case $X = \{0,1\}$, up to a multiplicative factor $\Phi(X^\omega)$ is the middle-thirds Cantor set.
1.3 IFS of contractions

In order to put things into perspective, we shall recall simple facts about iterated functions systems of contractions and their invariant sets, following Hutchinson [6].

**Definition 1.2** A contraction is a Lipschitz map \( \phi : (X, d) \to (X, d) \) with Lipschitz constant smaller than 1.

An iterated system of contractions \( S \) is a finite collection of contractions on a complete metric space \( (X, d) \).

An invariant set of \( S \) is a set \( M \subset X \) such that

\[
M = \bigcup_{\phi \in S} \phi(M) .
\]

**Theorem 1.3** There exists and it is unique a non empty bounded invariant set of \( S \), denoted by \( K(S) \). Moreover \( K(S) \) is compact.

For any bounded, non empty set \( A \subset X \), let us define

\[
S(A) = \bigcup_{\phi \in S} \phi(A) .
\]

Then \( S^n(A) \) converges in the Hausdorff distance to \( K(S) \), as \( n \to \infty \).

If \( \phi : X \to X \) is a contraction and \( (X, d) \) is compact then it has an unique fixed point \( x_\phi \in X \), that is \( x_\phi \) exists and it is unique with the property \( \phi(x_\phi) = x_\phi \).

Let \( S^* \) be the semigroup (with function composition) generated by \( S \) and \( Fix(S) \) be the collection of fixed points of elements of \( S^* \) (recall that each element of \( S^* \) is a contraction which preserves the compact set \( M(S) \)).

**Theorem 1.4** The set \( Fix(S) \) is dense in \( M(S) \).

We can give codes to elements of \( M(S) \). Indeed, let us start by remarking that we already used a notation similar to one in the previous subsection, namely \( S^* \). There is a surjective morphism from \( S^* \), as the semigroup of finite words with concatenation, to \( S^* \), as the semigroup of contractions generated by \( S \), with function composition. This morphism induces a surjective function \( \Lambda : S^\omega \to K(S) \) with the property that for any \( q = \phi_1...\phi_m \in S^* \) (finite word) and any \( w \in S^\omega \) we have

\[
\Lambda(qw) = \phi_1...\phi_m(\Lambda(w)) .
\]

The function \( \Lambda \) is constructed like this: for any \( w \in S^\omega \) and \( n \in \mathbb{N}^* \), let \( [w]_n \in S^* \) be the \( n \)-letter word from the beginning of \( w \). There exists \( w' \in S^\omega \) such that \( w = [w]_n w' \). Define \( \Lambda(w) \) by

\[
\Lambda(w) \in \bigcap_{n \in \mathbb{N}^*} [w]_n(K(S)) ,
\]
(where $[w]_n$ from the right hand side of the previous relation is understood as the composition of the first $n$ letters of the word $w$).

The definition is good because $\text{diam} [w]_n(K(S)) \to 0$ as $n \to \infty$ and for any $n \in \mathbb{N}^*$ we have
\[ [w]_{n+1}(K(S)) \subset [w]_n(K(S)) \]
Therefore the intersection of all $[w]_n(K(S))$ is a singleton.

The function $\Lambda$ is not generally bijective. It is, though, if the following condition is satisfied.

**Definition 1.5** $S$ satisfies the open set condition if there exists a non empty open set $U$ such that

(a) $\bigcup_{\phi \in S} \phi(U) \subset U,$

(b) if $\phi \neq \psi$, $\phi, \psi \in S$, then $\phi(U) \cap \psi(U) = \emptyset$.

### 1.4 Isometries of the dyadic tree

The dyadic tree $T$ is the infinite rooted binary tree, with any node having two descendants. The nodes are coded by elements of $X^*$, $X = \{0, 1\}$. The root is coded by the empty word $\emptyset$ and if a node is coded by $x \in X^*$ then its left hand side descendant has the code $x0$ and its right hand side descendant has the code $x1$. We shall therefore identify the dyadic tree with $X^*$ and we put on the dyadic tree the natural (ultrametric) distance on $X^*$. The boundary (or the set of ends) of the dyadic tree is then the same as the compact ultrametric space $X^\omega$.

An isometry of $T$ is just an invertible transformation which preserves the structure of the tree. It is well known that isometries of $(X^\omega, d)$ are the same as isometries of $T$.

Let $A \in Isom(X^\omega, d)$ be such an isometry. For any finite word $q \in X^*$ we may define $A_q \in Isom(X^\omega, d)$ by
\[ A(qw) = A(q)A_q(w) \]
for any $w \in X^\omega$. Note that in the previous relation $A(q)$ makes sense because $A$ is also an isometry of $T$.

### 2 Dilatation structures

The first two sections contain notions and results introduced or proved in Buliga [5]. The space $(X, d)$ is a complete, locally compact metric space.
2.1 Axioms of dilatation structures

The axioms of a dilatation structure \((X,d,\delta)\) are listed further. The first axiom is merely a preparation for the next axioms. That is why we counted it as axiom 0.

**A0.** The dilatations

\[
\delta^\varepsilon_x : U(x) \rightarrow V_\varepsilon(x)
\]

are defined for any \(\varepsilon \in \Gamma, \nu(\varepsilon) \leq 1\). All dilatations are homeomorphisms (invertible, continuous, with continuous inverse).

We suppose that there is \(1 < A\) such that for any \(x \in X\) we have

\[
\bar{B}_d(x,A) \subset U(x)
\]

We suppose that for all \(\varepsilon \in \Gamma, \nu(\varepsilon) \in (0,1)\), we have

\[
B_d(x,\varepsilon) \subset \delta^\varepsilon_x B_d(x,A) \subset V_\varepsilon(x) \subset U(x)
\]

For \(\nu(\varepsilon) \in (1, +\infty)\) the associated dilatation

\[
\delta^\varepsilon_x : W_\varepsilon(x) \rightarrow B_d(x,B)
\]

is injective, invertible on the image. We shall suppose that \(W_\varepsilon(x)\) is open,

\[
V_{\varepsilon^{-1}}(x) \subset W_\varepsilon(x)
\]

and that for all \(\varepsilon \in \Gamma_1\) and \(u \in U(x)\) we have

\[
\delta^\varepsilon_{\varepsilon^{-1}} \delta^\varepsilon_x u = u
\]

We remark that we have the following string of inclusions, for any \(\varepsilon \in \Gamma, \nu(\varepsilon) \leq 1\),

and any \(x \in X:\)

\[
B_d(x,\varepsilon) \subset \delta^\varepsilon_x B_d(x,A) \subset V_\varepsilon(x) \subset W_{\varepsilon^{-1}}(x) \subset \delta^\varepsilon_x B_d(x,B)
\]

A further technical condition on the sets \(V_\varepsilon(x)\) and \(W_\varepsilon(x)\) will be given just before the axiom A4. (This condition will be counted as part of axiom A0.)

**A1.** We have \(\delta^\varepsilon_x x = x\) for any point \(x\). We also have \(\delta^1_x = \text{id}\) for any \(x \in X\).

Let us define the topological space

\[
dom \delta = \{(\varepsilon,x,y) \in \Gamma \times X \times X : \text{ if } \nu(\varepsilon) \leq 1 \text{ then } y \in U(x), \text{ else } y \in W_\varepsilon(x)\}
\]

with the topology inherited from the product topology on \(\Gamma \times X \times X\). Consider also \(\text{Cl}(dom \delta)\), the closure of \(dom \delta\) in \(\bar{\Gamma} \times X \times X\) with product topology. The function

\[
\delta : dom \delta \rightarrow X
\]

defined by \(\delta(\varepsilon,x,y) = \delta^\varepsilon_x y\) is continuous. Moreover, it can be continuously extended to \(\text{Cl}(dom \delta)\) and we have

\[
\lim_{\varepsilon \rightarrow 0} \delta^\varepsilon_x y = x
\]
A2. For any \( x, \varepsilon, \mu \in \Gamma_1 \) and \( u \in \bar{B}_d(x, A) \) we have:
\[
\delta^\varepsilon \delta^\mu u = \delta^\varepsilon_{\varepsilon \mu} u .
\]

A3. For any \( x \) there is a function \((u, v) \mapsto d^x(u, v)\), defined for any \( u, v \) in the closed ball (in distance \( d \)) \( \bar{B}(x, A) \), such that
\[
\lim_{\varepsilon \to 0} \sup \left\{ \left| \frac{1}{\varepsilon} d^\varepsilon u, \delta^\varepsilon v - d^x(u, v) \right| : u, v \in \bar{B}_d(x, A) \right\} = 0
\]
uniformly with respect to \( x \) in compact set.

Remark 2.1 The "distance" \( d^x \) can be degenerated. That means: there might be \( v, w \in \bar{B}_d(x, A) \) such that \( d^x(v, w) = 0 \) but \( v \neq w \). We shall use further the name "distance" for \( d^x \), essentially by commodity, but keep in mind the possible degeneracy of \( d^x \).

For the following axiom to make sense we impose a technical condition on the co-domains \( V_\varepsilon(x) \): for any compact set \( K \subset X \) there are \( R = R(K) > 0 \) and \( \varepsilon_0 = \varepsilon(K) \in (0, 1) \) such that for all \( u, v \in \bar{B}_d(x, R) \) and all \( \varepsilon \in \Gamma, \nu(\varepsilon) \in (0, \varepsilon_0) \), we have
\[
\delta^\varepsilon v \in W_{\varepsilon_0}(\delta^\varepsilon u) .
\]

With this assumption the following notation makes sense:
\[
\Delta^x_\varepsilon(u, v) = \delta^{-1}_{\varepsilon_0} \delta^x u v.
\]

The next axiom can now be stated:

A4. We have the limit
\[
\lim_{\varepsilon \to 0} \Delta^x_\varepsilon(u, v) = \Delta^x(u, v)
\]
uniformly with respect to \( x, u, v \) in compact set.

Definition 2.2 A triple \((X, d, \delta)\) which satisfies A0, A1, A2, A3, but \( d^x \) is degenerate for some \( x \in X \), is called degenerate dilatation structure.

If the triple \((X, d, \delta)\) satisfies A0, A1, A2, A3 and \( d^x \) is non-degenerate for any \( x \in X \), then we call it a weak dilatation structure.

If a weak dilatation structure satisfies A4 then we call it dilatation structure.

2.2 Groups with dilatations. Conical groups

Metric tangent spaces sometimes have a group structure which is compatible with dilatations. This structure, of a group with dilatations, is interesting by itself. The notion has been introduced in [4]; we describe it further.

Let \( G \) be a topological group endowed with an uniformity such that the operation is uniformly continuous. The following description is slightly non canonical,
but is nevertheless motivated by the case of a Lie group endowed with a Carnot-Caratheodory distance induced by a left invariant distribution (see for example [3], [4]).

We introduce first the double of $G$, as the group $G^{(2)} = G \times G$ with operation

$$(x,u)(y,v) = (xy, y^{-1}uyv)$$

The operation on the group $G$, seen as the function

$$\text{op}: G^{(2)} \to G, \text{ op}(x,y) = xy$$

is a group morphism. Also the inclusions:

$$i' : G \to G^{(2)}, \ i'(x) = (x,e)$$

$$i'' : G \to G^{(2)}, \ i''(x) = (x,x^{-1})$$

are group morphisms.

**Definition 2.3** 1. $G$ is an uniform group if we have two uniformity structures, on $G$ and $G \times G$, such that $\text{op}, i', i''$ are uniformly continuous.

2. A local action of a uniform group $G$ on a uniform pointed space $(X, x_0)$ is a function $\phi \in W \in \mathcal{V}(e) \mapsto \hat{\phi} : U_{\phi} \in \mathcal{V}(x_0) \to V_{\phi} \in \mathcal{V}(x_0)$ such that:

(a) the map $(\phi, x) \mapsto \hat{\phi}(x)$ is uniformly continuous from $G \times X$ (with product uniformity) to $X$,

(b) for any $\phi, \psi \in G$ there is $D \in \mathcal{V}(x_0)$ such that for any $x \in D \phi \hat{\psi}^{-1}(x)$ and $\hat{\phi}(\hat{\psi}^{-1}(x))$ make sense and $\hat{\phi} \hat{\psi}^{-1}(x) = \hat{\phi} \hat{\psi}^{-1}(x)$.

3. Finally, a local group is an uniform space $G$ with an operation defined in a neighbourhood of $(e,e) \subset G \times G$ which satisfies the uniform group axioms locally.

Remark that a local group acts locally at left (and also by conjugation) on itself.

An uniform group, according to the definition (2.3), is a group $G$ such that left translations are uniformly continuous functions and the left action of $G$ on itself is uniformly continuous too. In order to precisely formulate this we need two uniformities: one on $G$ and another on $G \times G$.

These uniformities should be compatible, which is achieved by saying that $i'$, $i''$ are uniformly continuous. The uniformity of the group operation is achieved by saying that the $\text{op}$ morphism is uniformly continuous.

**Definition 2.4** A group with dilatations $(G, \delta)$ is a local uniform group $G$ with a local action of $\Gamma$ (denoted by $\delta$), on $G$ such that
H0. the limit \( \lim_{\varepsilon \to 0} \delta_{\varepsilon}x = e \) exists and is uniform with respect to \( x \) in a compact neighbourhood of the identity \( e \).

H1. the limit

\[
\beta(x, y) = \lim_{\varepsilon \to 0} \delta_{\varepsilon}^{-1}((\delta_{\varepsilon}x)(\delta_{\varepsilon}y))
\]

is well defined in a compact neighbourhood of \( e \) and the limit is uniform.

H2. the following relation holds

\[
\lim_{\varepsilon \to 0} \delta_{\varepsilon}^{-1}((\delta_{\varepsilon}x)^{-1}) = x^{-1}
\]

where the limit from the left hand side exists in a neighbourhood of \( e \) and is uniform with respect to \( x \).

These axioms are in fact a particular version of the axioms for a dilatation structure. We shall explain this a bit later.

Further we define conical local uniform groups.

**Definition 2.5** A conical group \( N \) is a local group with a local action of \( \Gamma \) by morphisms \( \delta_{\varepsilon} \) such that \( \lim_{\varepsilon \to 0} \delta_{\varepsilon}x = e \) for any \( x \) in a neighbourhood of the neutral element \( e \).

The next proposition explains why a conical group is the infinitesimal version of a group with dilatations.

**Proposition 2.6** Under the hypotheses H0, H1, H2 \((G, \beta, \delta)\) is a conical group, with operation \( \beta \) and dilatations \( \delta \).

Any group with dilatations has an associated dilatation structure on it. In a group with dilatations \((G, \delta)\) we define dilatations based in any point \( x \in G \) by

\[
\delta_{\varepsilon}^x u = x\delta_{\varepsilon}(x^{-1}u).
\]  \(\text{(2.2.1)}\)

**Definition 2.7** A normed group with dilatations \((G, \delta, \| \cdot \|)\) is a group with dilatations \((G, \delta)\) endowed with a continuous norm function \( \| \cdot \| : G \to \mathbb{R} \) which satisfies (locally, in a neighbourhood of the neutral element \( e \)) the properties:

(a) for any \( x \) we have \( \| x \| \geq 0 \); if \( \| x \| = 0 \) then \( x = e \),

(b) for any \( x, y \) we have \( \| xy \| \leq \| x \| + \| y \| \),

(c) for any \( x \) we have \( \| x^{-1} \| = \| x \| \),

(d) the limit \( \lim_{\varepsilon \to 0} \frac{1}{\nu(\varepsilon)} \| \delta_{\varepsilon}x \| = \| x \|^N \) exists, is uniform with respect to \( x \) in compact set,
(e) if \(|x|^N = 0\) then \(x = e\).

It is easy to see that if \((G, \delta, \| \cdot \|)\) is a normed group with dilatations then \((G, \beta, \delta, \| \cdot \|)^N\) is a normed conical group. The norm \(\| \cdot \|^N\) satisfies the stronger form of property (d) definition 2.7, for any \(\varepsilon > 0\)
\[
\| \delta \varepsilon x \|^N = \varepsilon \|x\|^N.
\]

Normed conical groups generalize the notion of Carnot groups.

In a normed group with dilatations we have a natural left invariant distance given by
\[
d(x, y) = \| x^{-1} y \|.
\]

**Theorem 2.8** Let \((G, \delta, \| \cdot \|)\) be a locally compact normed group with dilatations. Then \((G, \delta, d)\) is a dilatation structure, where \(\delta\) are the dilatations defined by (2.2.1) and the distance \(d\) is induced by the norm as in (2.2.2).

### 2.3 Tangent bundle of a dilatation structure

**Theorem 2.9** Let \((X, d, \delta)\) be a weak dilatation structure. Then

(a) for all \(x \in X\), \(u, v \in X\) such that \(d(x, u) \leq 1\) and \(d(x, v) \leq 1\) and all \(\mu \in (0, A)\) we have:
\[
d^\mu(u, v) = \frac{1}{\mu} d^\mu(\delta^\mu u, \delta^\mu v).
\]

We shall say that \(d^\mu\) has the cone property with respect to dilatations.

(b) we have the following limit:
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sup \{ \left| d(u, v) - d^\mu(u, v) \right| : d(x, u) \leq \varepsilon, \ d(x, v) \leq \varepsilon \} = 0.
\]

Therefore \((X, d)\) admits a metric tangent space at \(x\), for any point \(x \in X\).

For the next theorem we need the previously introduced notion of a conical (local) group.

**Theorem 2.10** Let \((X, d, \delta)\) be a dilatation structure. Then for any \(x \in X\) the triple \((U(x), \Sigma^x, \delta^x)\) is a conical group. Moreover, left translations of this group are \(d^x\) isometries.

The conical group \((U(x), \Sigma^x, \delta^x)\) can be regarded as the tangent space of \((X, d, \delta)\) at \(x\). Further will be denoted by: \(T_x X = (U(x), \Sigma^x, \delta^x)\).

The following definition will be used in several further places.

**Definition 2.11** Let \((X, \delta, d)\) be a dilatation structure and \(x \in X\) a point. In a neighbourhood \(U(x)\) of \(x\), for any \(\mu \in (0, 1)\) we defined the distances:
\[
(\delta^x, \mu)(u, v) = \frac{1}{\mu} d(\delta^x \mu u, \delta^x \mu v).
\]
2.4 Topological considerations

In this subsection we compare various topologies and uniformities related to a dilatation structure.

The axiom A3 implies that for any \(x \in X\) the function \(d^x\) is continuous, therefore open sets with respect to \(d^x\) are open with respect to \(d\).

If \((X,d)\) is separable and \(d^x\) is non degenerate then \((U(x),d^x)\) is also separable and the topologies of \(d\) and \(d^x\) are the same. Therefore \((U(x),d^x)\) is also locally compact (and a set is compact with respect to \(d^x\) if and only if it is compact with respect to \(d\)).

If \((X,d)\) is separable and \(d^x\) is non degenerate then the uniformities induced by \(d\) and \(d^x\) are the same. Indeed, let

\[
\{u_n : n \in \mathbb{N}\}
\]

be a dense set in \(U(x)\), with \(x_0 = x\). We can embed \((U(x), (\delta^x, \varepsilon))\) isometrically in a separable Banach space, for any \(\varepsilon \in (0, 1)\), by the function

\[
\phi_{\varepsilon}(u) = \left(\frac{1}{\varepsilon} d(\delta^x u, \delta^x x_n) - \frac{1}{\varepsilon} d(\delta^x x, \delta^x x_n)\right)_n.
\]

A reformulation of point (a) in theorem 2.9 is that on compact sets \(\phi_{\varepsilon}\) uniformly converges to the isometric embedding of \((U(x), d^x)\)

\[
\phi(u) = (d^x(u, x_n) - d^x(x, x_n))_n.
\]

Remark that the uniformity induced by \((\delta, \varepsilon)\) is the same as the uniformity induced by \(d\), and that it is the same induced from the uniformity on the separable Banach space by the embedding \(\phi_{\varepsilon}\). We proved that the uniformities induced by \(d\) and \(d^x\) are the same.

2.5 Equivalent dilatation structures

Definition 2.12 Two dilatation structures \((X, \delta, d)\) and \((X, \delta', d')\) are equivalent if

(a) the identity map \(id : (X,d) \to (X,d)\) is bilipschitz and

(b) for any \(x \in X\) there are functions \(P^x, Q^x\) (defined for \(u \in X\) sufficiently close to \(x\)) such that

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} d(\delta^x u, \delta^x Q^x(u)) = 0, \quad (2.5.3)
\]

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} d(\delta^x u, \delta^x P^x(u)) = 0, \quad (2.5.4)
\]

uniformly with respect to \(x, u\) in compact sets.

Proposition 2.13 Two dilatation structures \((X, \delta, d)\) and \((X, \delta', d)\) are equivalent if and only if
(a) the identity map $id : (X, d) \to (X, \bar{d})$ is bilipschitz and

(b) for any $x \in X$ there are functions $P^x, Q^x$ (defined for $u \in X$ sufficiently close to $x$) such that

$$\lim_{\varepsilon \to 0} \left( \delta_\varepsilon^x \right)^{-1} \delta_\varepsilon^x (u) = Q^x(u), \quad (2.5.5)$$

$$\lim_{\varepsilon \to 0} \left( \delta_\varepsilon^x \right)^{-1} \bar{\delta}_\varepsilon^x (u) = P^x(u), \quad (2.5.6)$$

uniformly with respect to $x, u$ in compact sets.

The next theorem shows a link between the tangent bundles of equivalent dilatation structures.

**Theorem 2.14** Let $(X, \delta, d)$ and $(X, \bar{\delta}, \bar{d})$ be equivalent dilatation structures. Suppose that for any $x \in X$ the distance $d^x$ is non degenerate. Then for any $x \in X$ and any $u, v \in X$ sufficiently close to $x$ we have:

$$\Sigma^x (u, v) = Q^x \left( \Sigma^x (P^x(u), P^x(v)) \right). \quad (2.5.7)$$

The two tangent bundles are therefore isomorphic in a natural sense.

### 2.6 Differentiable bundles

Dilatation structures allow to define differentiable functions. The idea is to keep only one relation from definition 2.12, namely (2.5.3). We also renounce to uniform convergence with respect to $x$ and $u$, and we replace this with uniform convergence in the "$u$" variable, with a conical group morphism condition for the derivative.

**Definition 2.15** Let $(N, \delta)$ and $(M, \bar{\delta})$ be two conical groups. A function $f : N \to M$ is a conical group morphism if $f$ is a group morphism and for any $\varepsilon > 0$ and $u \in N$ we have $f(\delta_\varepsilon u) = \bar{\delta}_\varepsilon f(u)$.

**Definition 2.16** Let $(X, \delta, d)$ and $(Y, \bar{\delta}, \bar{d})$ be two dilatation structures and $f : X \to Y$ be a continuous function. The function $f$ is differentiable in $x$ if there exists a conical group morphism $Q^x : T_x X \to T_{f(x)} Y$, defined on a neighbourhood of $x$ with values in a neighbourhood of $f(x)$ such that

$$\lim_{\varepsilon \to 0} \sup \left\{ \frac{1}{\varepsilon} \mathcal{D} \left( f(\delta_\varepsilon^x u), \bar{\delta}_\varepsilon^{f(x)} Q^x(u) \right) : \bar{d}(x, u) \leq \varepsilon \right\} = 0, \quad (2.6.8)$$

The morphism $Q^x$ is called the derivative of $f$ at $x$ and will be sometimes denoted by $Df(x)$.

The function $f$ is uniformly differentiable if it is differentiable everywhere and the limit in (2.6.8) is uniform in $x$ in compact sets.
A trivial way to obtain a differentiable function (everywhere) is to modify the dilatation structure on the target space.

**Definition 2.17** Let \((X, \delta, d)\) be a dilatation structure and \(f : (X, d) \to (Y, \overline{d})\) be a bilipschitz and surjective function. We define then the transport of \((X, \delta, d)\) by \(f\), named \((Y, f \ast \delta, \overline{d})\), by:

\[
(f \ast \delta)^f(x) f(u) = f (\delta^x u).
\]

The relation of differentiability with equivalent dilatation structures is given by the following simple proposition.

**Proposition 2.18** Let \((X, \delta, d)\) and \((X, \overline{\delta}, \overline{d})\) be two dilatation structures and \(f : (X, d) \to (X, \overline{d})\) be a bilipschitz and surjective function. The dilatation structures \((X, \overline{\delta}, \overline{d})\) and \((X, f \ast \delta, \overline{d})\) are equivalent if and only if \(f\) and \(f^{-1}\) are uniformly differentiable.

We shall prove now the chain rule for derivatives, after we elaborate a bit over the definition \[2.16\].

Let \((X, \delta, d)\) and \((Y, \overline{\delta}, \overline{d})\) be two dilatation structures and \(f : X \to Y\) a function differentiable in \(x\). The derivative of \(f\) in \(x\) is a conical group morphism \(Df(x) : T_xX \to T_{f(x)}Y\), which means that \(Df(x)\) is defined on a open set around \(x\) with values in a open set around \(f(x)\), having the properties:

(a) for any \(u, v\) sufficiently close to \(x\)

\[
Df(x) (\Sigma^x (u, v)) = \Sigma^{f(x)} (Df(x)(u), Df(x)(v)),
\]

(b) for any \(u\) sufficiently close to \(x\) and any \(\varepsilon \in (0, 1]\)

\[
Df(x) (\delta^x u) = \delta^{f(x)} (Df(x)(u)),
\]

(c) the function \(Df(x)\) is continuous, as uniform limit of continuous functions. Indeed, the relation \[2.6.8\] is equivalent to the existence of the uniform limit (with respect to \(u\) in compact sets)

\[
Df(x)(u) = \lim_{\varepsilon \to 0} \delta^{f(x)} (f (\delta^x u)).
\]

From \[2.6.8\] alone and axioms of dilatation structures we can prove properties (b) and (c). We can reformulate therefore the definition of the derivative by asking that \(Df(x)\) exists as an uniform limit (as in point (c) above) and that \(Df(x)\) has the property (a) above.

From these considerations the chain rule for derivatives is straightforward.
Proposition 2.19 Let \((X, \delta, d)\), \((Y, \delta, d)\) and \((Z, \hat{\delta}, \hat{d})\) be three dilatation structures and \(f : X \rightarrow Y\) a continuous function differentiable in \(x\), \(g : Y \rightarrow Z\) a continuous function differentiable in \(f(x)\). Then \(gf : X \rightarrow Z\) is differentiable in \(x\) and

\[Dgf(x) = Dg(f(x))Df(x).\]

Proof. Use property (b) for proving that \(Dg(f(x))Df(x)\) satisfies (2.6.8) for the function \(gf\) and \(x\). Both \(Dg(f(x))\) and \(Df(x)\) are conical group morphisms, therefore \(Dg(f(x))Df(x)\) is a conical group morphism too. We deduce that \(Dg(f(x))Df(x)\) is the derivative of \(gf\) in \(x\).

2.7 Some induced dilatation structures

Proposition 2.20 For any \(u, v \in U(x)\) let us define

\[^\tilde{}\delta^u v = \Sigma^x_\mu(u, \delta^x_\mu \Delta^x_\mu(u, v)) = \delta^x_\mu - 1 \delta^x_\mu v.\]

Then \((U(x), \hat{\delta}, (\delta^x, \mu))\) is a dilatation structure.

Proof. We have to check the axioms. The first part of axiom A0 is an easy consequence of theorem 2.9 for \((X, \delta, d)\). The second part of A0, A1 and A2 are true based on simple computations.

The first interesting fact is related to axiom A3. Let us compute, for \(v, w \in U(x)\),

\[\frac{1}{\varepsilon} (\delta^x, \mu)(\hat{\delta^u v, \hat{\delta^u w}}) = \frac{1}{\varepsilon \mu} d(\delta^x_\mu \hat{\delta^u v, \delta^x_\mu \hat{\delta^u w}}) =
\]

\[= \frac{1}{\varepsilon \mu} d(\delta^x_\mu \Delta^x_\mu(u, v), \delta^x_\mu \Delta^x_\mu(u, w)) = \frac{1}{\varepsilon \mu} d(\delta^x_\mu \Delta^x_\mu(u, v), \delta^x_\mu \Delta^x_\mu(u, w)) =
\]

\[= (\delta^x_\mu, \mu)(\Delta^x_\mu(u, v), \Delta^x_\mu(u, w)).\]

The axiom A3 is then a consequence of axiom A3 for \((X, \delta, d)\) and we have

\[\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (\delta^x, \mu)(\hat{\delta^u v, \hat{\delta^u w}}) = d^{\delta^x_\mu, \mu}(\Delta^x_\mu(u, v), \Delta^x_\mu(u, w)).\]

The axiom A4 is also a straightforward consequence of A4 for \((X, \delta, d)\) and is left to the reader. □

The proof of the following proposition is an easy computation, of the same type as in the lines above, therefore we shall not write it here.

Proposition 2.21 With the same notations as in proposition 2.20, the transformation \(\Sigma^x_\mu(u, \cdot)\) is an isometry from \((\delta^x_\mu u, \mu)\) to \((\delta^x, \mu)\). Moreover, we have

\[\Sigma^x_\mu(u, \delta^x_\mu u) = u.\]
These two propositions show that on a dilatation structure we almost have translations (the infinitesimal sums), which are almost isometries (that is, not with respect to the distance $d$, but with respect to distances of type $(\delta^x, \mu)$). It is almost as if we were working with a conical group, only that we have to use families of distances and to make small shifts in the tangent space (as in the last formula in the proof of proposition 2.20). Moreover, in a very precise way everything converges as $\mu \to 0$ to the right thing.

3 The linear group of a dilatation structure

Definition 3.1 Let $(X, d, \delta)$ be a weak dilatation structure. A transformation $A : X \to X$ is linear if it is Lipschitz and it commutes with dilatations in the following sense: for any $x \in X$, $u \in U(x)$ and $\varepsilon \in \Gamma$, $\nu(\varepsilon) < 1$, if $A(u) \in U(A(x))$ then

$$A \delta^x = \delta^{A(x)} A(u).$$

The group of linear transformations, denoted by $GL(X, d, \delta)$ is formed by all invertible and bi-lipschitz linear transformations of $X$.

$GL(X, d, \delta)$ is a (local) group. Indeed, we start from the remark that if $A$ is Lipschitz then there exists $C > 0$ such that for all $x \in X$ and $u \in B(x, C)$ we have $A(u) \in U(A(x))$. The inverse of $A \in GL(X, d, \delta)$ is then linear. Same considerations apply for the composition of two linear, bi-lipschitz and invertible transformations.

In the particular case of example 4.1, namely $X$ finite dimensional real, normed vector space, $d$ the distance given by the norm, $\Gamma = (0, +\infty)$ and dilatations

$$\delta^x u = x + \varepsilon(u - x),$$

a linear transformations in the sense of definition 3.1 is an affine transformation of the vector space $X$.

Linear transformations have nice properties which justify the name ”linear”. We shall use further the (sum and difference) operations

$$\Sigma^x_{\varepsilon}(u, v) = (\delta^x)^{-1} \delta^x_{\varepsilon} u v, \quad \Delta^x_{\varepsilon}(u, v) = \left(\delta^x_{\varepsilon}\right)^{-1} \delta^x_{\varepsilon} v$$

and the inverse function $\text{inv}^x_{\varepsilon} u = \Delta^x_{\varepsilon}(u, x)$.

Proposition 3.2 Let $(X, d, \delta)$ be a weak dilatation structure and $A : X \to X$ a linear transformation. Then:

(a) for all $x \in X$, $u, v \in U(x)$ sufficiently close to $x$, we have:

$$A \Sigma^x_{\varepsilon}(u, v) = \Sigma^x_{\varepsilon}(A(u), A(v)).$$

(b) or all $x \in X$, $u \in U(x)$ sufficiently close to $x$, we have:

$$A \text{inv}^x_{\varepsilon}(u) = \text{inv}^{A(x)} A(u).$$

(c) for all $x \in X$ the transformation $A$ is derivable and the derivative equals $A$. 
Proof. Straightforward, just use the commutation with dilatation. □

This is important because the sum, difference, inverse operations induced by a dilatation structure give to the space $X$ almost the structure of an affine space. We collect some results from [5] section 4.2, regarding the properties of these operations. Only the last point is new, but with straightforward proof.

**Theorem 3.3** Let $(X, d, \delta)$ be a weak dilatation structure. Then, for any $x \in X$, $\varepsilon \in \Gamma$, $\nu(\varepsilon) < 1$, we have:

(a) for any $u \in U(x)$, $\Sigma^x_\varepsilon(x, u) = u$.

(b) for any $u \in U(x)$ the functions $\Sigma^x_\varepsilon(u, \cdot)$ and $\Delta^x_\varepsilon(u, \cdot)$ are inverse one to another.

(c) the inverse function is shifted involutive: for any $u \in U(x)$,

\[
\text{inv}^x_\varepsilon u \text{ inv}^x_\varepsilon (u) = u.
\]

(d) the sum operation is shifted associative: for any $u, v, w$ sufficiently close to $x$ we have

\[
\Sigma^x_\varepsilon \left( u, \Sigma^x_\varepsilon (v, w) \right) = \Sigma^x_\varepsilon (\Sigma^x_\varepsilon (u, v), w).
\]

(e) the difference, inverse and sum operations are related by

\[
\Delta^x_\varepsilon (u, v) = \Sigma^x_\varepsilon (\text{inv}^x_\varepsilon (u), v),
\]

for any $u, v$ sufficiently close to $x$.

(f) for any $u, v$ sufficiently close to $x$ and $\mu \in \Gamma$, $\nu(\mu) < 1$, we have:

\[
\Delta^x_\varepsilon (\delta^x_\mu u, \delta^x_\mu v) = \delta^x_\mu \Delta^x_\varepsilon (u, v).
\]

Remark that in principle the "translations" $\Sigma^x_\varepsilon(u, \cdot)$ are not linear. Nevertheless, they commute with dilatation in a known way, according to point (f) theorem 3.3. This is important, because the transformations $\Sigma^x_\varepsilon(u, \cdot)$ really behave as translations, as explained in subsection 2.7.

The reason for which translations are not linear is that dilatations are not linear. In the case of strong dilatation structures, this happens only when we are in a conical group.

**Theorem 3.4** Let $(X, d, \delta)$ be a weak dilatation structure.

(a) If dilatations are linear then all transformations $\Delta^x_\varepsilon(u, \cdot)$ are linear for any $u \in X$.

(b) If the dilatation structure is strong then dilatations are linear if and only if the dilatations come from the dilatation structure of a conical group.
Proof. (a) If dilatations are linear, then let \( \varepsilon, \mu \in \Gamma, \nu(\varepsilon), \nu(\mu) \leq 1 \), and \( x, y, u, v \in X \) such that the following computations make sense. We have:

\[
\Delta^x_{\varepsilon}(u, \delta^y_{\mu} v) = \delta^{\delta^x_{\varepsilon} u}_{\delta^y_{\mu} \delta^y_{\mu} v} 
\]

Let \( A_\varepsilon = \delta^{\delta^x_{\varepsilon} u}_{\delta^y_{\mu} \delta^y_{\mu} v} \). We compute:

\[
\delta^{\Delta^x_{\mu}(u,y)}_{\mu} \Delta^x_{\varepsilon}(u, v) = \delta^{A_\varepsilon \delta^x_{\mu} y}_{\delta^y_{\mu} \delta^x_{\mu} v} 
\]

We use twice the linearity of dilatations:

\[
\delta^{\Delta^x_{\mu}(u,y)}_{\mu} \Delta^x_{\varepsilon}(u, v) = \delta^{A_\varepsilon \delta^x_{\mu} y}_{\delta^y_{\mu} \delta^x_{\mu} v} = \delta^{\delta^x_{\varepsilon} u}_{\delta^y_{\mu} \delta^y_{\mu} v} 
\]

We proved that:

\[
\Delta^x_{\varepsilon}(u, \delta^y_{\mu} v) = \delta^{\Delta^x_{\mp}(u,y)}_{\mu} \Delta^x_{\varepsilon}(u, v) 
\]

which is the conclusion of the part (a).

(b) Suppose that the dilatation structure is strong. If dilatations are linear, then by point (a) the transformations \( \Delta^x_{\varepsilon}(u, \cdot) \delta \) are linear for any \( u \in X \). Then, with notations made before, for \( y = u \) we get

\[
\Delta^x_{\varepsilon}(u, \delta^y_{\mu} v) = \delta^{\delta^x_{\varepsilon} u}_{\delta^y_{\mu} \delta^y_{\mu} v} \Delta^x_{\varepsilon}(u, v) 
\]

which implies

\[
\delta^y_{\mu} v = \Sigma^y_{\varepsilon}(u, \delta^y_{\mu} \Delta^x_{\varepsilon}(u, v)) 
\]

We pass to the limit with \( \varepsilon \to 0 \) and we obtain:

\[
\delta^y_{\mu} v = \Sigma^y(\mu, \delta^y_{\mu} \Delta^x(u, v)) 
\]

We recognize at the right hand side the dilatations associated to the conical group \( T_x X \).

The opposite implication is straightforward, because the dilatation structure of any conical group is linear. \( \Box \)

4 First examples of dilatation structures

In this section we give several examples of dilatation structures, which share some common features.

Example 4.1 The first example is known to everybody: take \((X,d) = (\mathbb{R}^n, d_E)\), with usual (euclidean) dilatations \( \delta^x_{\varepsilon} \), that is:

\[
d_E(x,y) = ||x-y|| \quad , \quad \delta^x_{\varepsilon} y = x + \varepsilon (y-x) \)

Dilatations are defined everywhere. The group \( \Gamma \) is \((0, +\infty)\) and the function \( \nu \) is the identity.
There are few things to check: axioms 0,1,2 are obviously true. For axiom A3, remark that for any \( \varepsilon > 0 \), \( x, u, v \in X \) we have:

\[
\frac{1}{\varepsilon}d_E(\delta_\varepsilon^x u, \delta_\varepsilon^x v) = d_E(u, v),
\]

therefore for any \( x \in X \) we have \( d^x = d_E \).

Finally, let us check the axiom A4. For any \( \varepsilon > 0 \) and \( x, u, v \in X \) we have

\[
\delta_\varepsilon^{\delta_{\varepsilon^{-1}}^x y} = x + \varepsilon(u - x) + \frac{1}{\varepsilon}(x + \varepsilon(v - x) - x - \varepsilon(u - x)) = x + \varepsilon(u - x) + v - u
\]

therefore this quantity converges to

\[
x + v - u = x + (v - x) - (u - x)
\]
as \( \varepsilon \to 0 \). The axiom A4 is verified.

\[
\square
\]

4.1 Standard dilatation structures

**Example 4.2** Take now \( \phi : \mathbb{R}^n \to \mathbb{R}^n \) a bi-Lipschitz diffeomorphism. Then we can define the dilatation structure: \( X = \mathbb{R}^n \),

\[
d_\phi(x, y) = \|\phi(x) - \phi(y)\| , \quad \delta_\varepsilon^x y = x + \varepsilon(y - x)
\]
or the equivalent dilatation structure: \( X = \mathbb{R}^n \),

\[
d_\phi(x, y) = \|x - y\| , \quad \delta_\varepsilon^x y = \phi^{-1}(\phi(x) + \varepsilon(\phi(y) - \phi(x))).
\]

In this example (look at its first version) the distance \( d_\phi \) is not equal to \( d^x \). Indeed, a direct calculation shows that

\[
d^x(u, v) = \|D\phi(x)(v - u)\|
\]
The axiom A4 gives the same result as previously.

\[
\square
\]

**Example 4.3** Because dilatation structures are defined by local requirements, we can easily define dilatation structures on riemannian manifolds, using particular atlases of the manifold and the riemannian distance (infimum of length of curves joining two points). Note that any finite dimensional manifold can be endowed with a riemannian metric. This class of examples covers all dilatation structures used in differential geometry. The axiom A4 gives an operation of addition of vectors in the tangent space (compare with Bellaïche [2] last section).

\[
\square
\]

There is a version of the snowflake construction for dilatation structures. It is stated in the next proposition, which has a straightforward proof.
**Proposition 4.1** If \((X,d,\delta)\) is a dilatation structure then \((X,d_a,\delta(a))\) is also a dilatation structure, for any \(a \in (0,1]\), where
\[
d_a(x,y) = d(x,y)^a, \quad \delta(a)^\frac{\epsilon}{\epsilon+1} = \delta(x_a^\epsilon) .
\]

**Example 4.4** In particular we get a snowflake variation of the euclidean case: \(X = \mathbb{R}^n\) and for any \(a \in (0,1]\) take
\[
d_a(x,y) = \|x - y\|^a, \quad \delta^\frac{\epsilon}{\epsilon+1} = x + \epsilon\frac{1}{\epsilon+1}(y - x) .
\]
\[\square\]

### 4.2 Nonstandard dilatations in the euclidean space

**Example 4.5** Take \(X = \mathbb{R}^2\) with the euclidean distance. For any \(z \in \mathbb{C}\) of the form \(z = 1 + i\theta\) we define dilatations
\[
\delta^\epsilon x = \epsilon^z x .
\]
It is easy to check that \((X,\delta,+,d)\) is a conical group, equivalently that the dilatations
\[
\delta^\epsilon y = x + \epsilon\frac{1}{\epsilon+1}(y - x) .
\]
form a linear dilatation structure with the euclidean distance.

Two such dilatation structures (constructed with the help of complex numbers \(1 + i\theta\) and \(1 + i\theta'\)) are equivalent if and only if \(\theta = \theta'\).

There are two other interesting properties of these dilatation structures. The first is that if \(\theta \neq 0\) then there are no non trivial Lipschitz curves in \(X\) which are differentiable almost everywhere.

The second property is that any holomorphic and Lipschitz function from \(X\) to \(X\) (holomorphic in the usual sense on \(X = \mathbb{R}^2 = \mathbb{C}\)) is differentiable almost everywhere, but there are Lipschitz functions from \(X\) to \(X\) which are not differentiable almost everywhere (suffices to take a \(C^\infty\) function from \(\mathbb{R}^2\) to \(\mathbb{R}^2\) which is not holomorphic).
\[\square\]

Take now a one parameter group of linear transformations \(s \in \mathbb{R} \mapsto A_s\) in \(X = \mathbb{R}^2\) such that \(A_s \rightarrow 0\) as \(s \rightarrow -\infty\). Such a group generates functions
\[
\delta^\epsilon = A_{\log \epsilon} .
\]
which can be used to construct dilatation structures.

It is well known that, up to conjugation, there are only three such one-parameter groups in \(\mathbb{R}^2\).
Example 4.6 The first group generates diagonal functions $\delta$ with the form

$$\delta_{\varepsilon}(x_1, x_2) = (\varepsilon^{\alpha} x_1, \varepsilon^{\beta} x_2).$$

By the snowflake construction we can find a distance on $X$ such that we get a dilatation structure. \hfill \Box

Example 4.7 The second group generates functions $\delta$ with the form

$$\delta_{\varepsilon}x = \varepsilon^{z} x,$$

with $Re\, z > 0$. We can choose a distance on $X$ such that we have again a dilatation structure (just combine the first example in this subsection with the snowflake construction). \hfill \Box

Finally, the third group generates functions $\delta$ with a different form. Modulo the snowflake construction the functions $\delta$ have the form:

$$\delta_{\varepsilon}(x_1, x_2) = (\varepsilon x_1 + \varepsilon \log(\varepsilon) x_2, \varepsilon x_2).$$

If we choose the distance $d$ to be the euclidean distance then we verify all the axioms excepting the axiom A3.

Let now $\delta_{\varepsilon}$ be a one parameter group of linear invertible transformations (in multiplicative form) on $\mathbb{R}^n$, such that $\delta_{\varepsilon}$ converges to 0 as $\varepsilon$ goes to 0. To any $x \in \mathbb{R}^n$ we associate, in a continuous way, a linear invertible transformation of $\mathbb{R}^n$, denoted by $A(x)$. We define now

$$\delta_{\varepsilon}^x y = x + A(x)\delta_{\varepsilon}A(x)^{-1}(y - x).$$

We want to know if this is a dilatation structure on $(\mathbb{R}^n, d)$, where $d$ is the euclidean distance.

We have to check only axioms A3 and A4. For this notice that for any $u, v \in \mathbb{R}^n$ and $\varepsilon > 0$

$$\frac{1}{\varepsilon} d(\delta_{\varepsilon}^x u, \delta_{\varepsilon}^x v) = \|A(x)\frac{1}{\varepsilon}\delta_{\varepsilon}A(x)^{-1}(u - v)\|,$$

therefore the axiom A3 is satisfied if $\frac{1}{\varepsilon}\delta_{\varepsilon}$ converges to an invertible transformation. Assume that there is invertible (and linear, as limit of linear transformations) function $P$ such that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\delta_{\varepsilon} = P.$$

In order to check the axiom A4 we compute:

$$\Delta_{\varepsilon}^x (u, v) = x + A(\delta_{\varepsilon}^x u)\delta_{\varepsilon-1} A(\delta_{\varepsilon}^x u)^{-1} A(x) \delta_{\varepsilon}A(x)^{-1}(v - x) =$$

$$= x + \varepsilon A(\delta_{\varepsilon}^x u)\delta_{\varepsilon-1} A(\delta_{\varepsilon}^x u)^{-1} \left( A(x)\frac{1}{\varepsilon}\delta_{\varepsilon}A(x)^{-1}(v - x) \right).$$
This expression converges if and only if the following limit exists:

$$\lim_{\varepsilon \to 0} \varepsilon A(\delta_{\varepsilon} u)\delta_{\varepsilon-1} A(\delta_{\varepsilon} u)^{-1}.$$  

The map $x \mapsto A(x)$ continuous, therefore we have the equivalence: $(\mathbb{R}^n, d, \delta)$ is a dilatation structure if and only if we have the limit

$$\lim_{\varepsilon \to 0} \varepsilon\delta_{\varepsilon-1} A(\delta_{\varepsilon} u)^{-1}.$$  

This limit may not exist (or it may be infinite), thus providing examples of a structure which satisfies all axioms excepting A4.

We state as a conclusion of this subsection:

**Theorem 4.2** The axioms A3 and A4 are independent of the rest of axioms.

### 4.3 Ultrametric valued fields

**Definition 4.3** Let $R$ be a commutative ring with unity 1. A function $N : R \to [0, +\infty)$ is a norm on $R$ if the following are true.

(a) $N(x) = 0$ if and only if $x = 0$.

(b) For all $x, y \in R$ $N(xy) = N(x)N(y)$.

(c) For all $x, y \in R$ $N(x + y) \leq N(x) + N(y)$.

In case of a field, the set of norms of non zero elements of $R$ is called value group and denoted by $\Theta \subset (0, +\infty)$.

Particular examples of complete valued fields are fields of $p$-adic numbers $\mathbb{Q}_p$ ($p$ prime). Good references are Schikhof [7], or Bachman [1].

For such fields the value group $\Theta$ is discrete and condition (c) in definition above is strengthened to

$$\forall x, y \in R \quad N(x + y) \leq \max \{N(x), N(y)\}.$$  

**Example 4.8** Let $K$ be an ultrametric, complete, valued field,

$$\bar{B}(0, 1) = \{x \in K \mid |x| \leq 1\}$$  

$$A = \{x \in K \mid x \neq 0, |x| \leq 1, |1 - x| \leq 1\}.$$  

Then $\bar{B}(0, 1)$ is a compact set, a valued ring, $A$ is a semigroup with multiplication and

$$\delta_{\varepsilon} : \bar{B}(0, 1) \to \bar{B}(0, 1), \quad \delta_{\varepsilon} y = x + \varepsilon(y - x),$$  

is well defined for any $x \in \bar{B}(0, 1), \varepsilon \in A.$
Moreover, the functions $\delta$ and the norm on $K$ define a dilatation structure $(\overline{B}(0,1), d, \delta)$.

Indeed, the norm is ultrametric, therefore $\overline{B}(0,1)$ is a compact set, a valued ring, and $A$ is a semigroup with multiplication. We have to check that $\delta^\varepsilon_x$ is well defined for any $x \in \overline{B}(0,1), \varepsilon \in A$. But this is straightforward: take any $y \in \overline{B}(0,1)$. Then

$$|\delta^\varepsilon_x y| \leq \max \{|\varepsilon| |y|, |1 - \varepsilon| |x|\} \leq 1.$$  

The proof that we have here a dilatation structure is formally identical with example 4.1. \square

5 Dilatation structures on the boundary of the dyadic tree

Dilatation structures on the boundary of the dyadic tree will have a simpler form than general, mainly because the distance is ultrametric.

The boundary of the dyadic tree identifies with $X^\omega$, for $X = \{0, 1\}$, and also with the ring of dyadic integers $\mathbb{Z}_2$. Use shall use the usual, ultrametric distance, denoted by $d$, on this set.

We take the group $\Gamma$ to be the set of integer powers of 2, seen as a subset of dyadic numbers. Thus for any $p \in \mathbb{Z}$ the element $2^p \in \mathbb{Q}_2$ belongs to $\Gamma$. The operation is the multiplication of dyadic numbers and the morphism $\nu : \Gamma \to (0, +\infty)$ is defined by

$$\nu(2^p) = d(0, 2^p) = \frac{1}{2^p} \in (0, +\infty).$$

Axiom A0. This axiom states that for any $p \in \mathbb{N}$ and any $x \in X^\omega$ the dilatation

$$\delta^\varepsilon_{2^p} : U(x) \to V_{2^p}(x)$$

is a homeomorphism, the sets $U(x)$ and $V_{2^p}(x)$ are open and there is $A > 1$ such that the ball centered in $x$ and radius $A$ is contained in $U(x)$. But this means that $U(x) = X^\omega$, because $X^\omega = B(x, 1)$.

Further, for any $p \in \mathbb{N}$ we have the inclusions:

$$B(x, \frac{1}{2^p}) \subset \delta^\varepsilon_{2^p} X^\omega \subset V_{2^p}(x).$$  \hspace{1cm} (5.0.1)

For any $p \in \mathbb{N}^*$ the associated dilatation

$$\delta^\varepsilon_{2^{-p}} : W_{2^{-p}}(x) \to B(x, B) = X^\omega,$$

is injective, invertible on the image. We suppose that $W_{2^{-p}}(x)$ is open,

$$V_{2^p}(x) \subset W_{2^{-p}}(x).$$  \hspace{1cm} (5.0.2)
and that for all \( p \in \mathbb{N}^* \) and \( u \in X^\omega \) we have
\[
\delta_{2^p}^x \delta_{2^p}^x u = u .
\]

We leave aside for the moment the interpretation of the technical condition before axiom A4.

**Axioms A1 and A2.** Nothing simplifies.

**Axiom A3.** Because \( d \) is an ultrametric distance and \( X^\omega \) is compact, this axiom has very strong consequences, for a non degenerate dilatation structure.

In this case the axiom A3 states that there is a non degenerate distance function \( d^x \) on \( X^\omega \) such that we have the limit
\[
\lim_{p \to \infty} 2^p d(\delta_{2^p}^x u, \delta_{2^p}^x v) = d^x (u, v)
\]
uniformly with respect to \( x, u, v \in X^\omega \).

We continue further with properties of weak dilatation structures.

**Lemma 5.1** There exists \( p_0 \in \mathbb{N} \) such that for any \( x, u, v \in X^\omega \) and for any \( p \in \mathbb{N} \), \( p \geq p_0 \), we have
\[
2^p d(\delta_{2^p}^x u, \delta_{2^p}^x v) = d^x (u, v)
\]

**Proof.** From the limit (5.0.3) and the non degeneracy of the distances \( d^x \) we deduce that
\[
\lim_{p \to \infty} \log_2 \left( 2^p d(\delta_{2^p}^x u, \delta_{2^p}^x v) \right) = \log_2 d^x (u, v)
\]
uniformly with respect to \( x, u, v \in X^\omega \), \( u \neq v \). The right hand side term is finite and the sequence from the limit at the left hand side is included in \( \mathbb{Z} \). Use this and the uniformity of the convergence to get the desired result. \( \square \)

In the sequel \( p_0 \) is the smallest natural number satisfying lemma 5.1

**Lemma 5.2** For any \( x \in X^\omega \) and for any \( p \in \mathbb{N} \), \( p \geq p_0 \), we have
\[
\delta_{2^p}^x X^\omega = [x]_{p} X^\omega
\]
Otherwise stated, for any \( x, y \in X^\omega \), any \( q \in X^* \), \( |q| \geq p_0 \) there exists \( w \in X^\omega \) such that
\[
\delta_{2^{|q|}}^x w = qy
\]
and for any \( z \in X^\omega \) there is \( y \in X^\omega \) such that
\[
\delta_{2^{|q|}}^x z = qy
\]
Moreover, for any \( x \in X^\omega \) and for any \( p \in \mathbb{N} \), \( p \geq p_0 \) the inclusions from (5.0.1), (5.0.2) are equalities.
Lemma 5.3
For any \( p \)
Moreover, under the same hypothesis, for any \( p \)
The first part of the lemma is proven. For the proof of the second part write again
Therefore, by lemma 5.1, we have
This is just the cone property for \( d \). From here we deduce that for any \( p \)
Finally, the last part of the lemma has a similar proof, only that we have to use also the last part of axiom A0. □

Proof. By lemma 5.1, lemma 5.2 and axiom A2. Indeed, from lemma 5.1 and axiom A2, for any \( p \) and any \( x, u, v \) we have
This is just the cone property for \( d^x \). From here we deduce that for any \( p \) we have
If \( 2^p d(x, u) \leq 1 \), \( 2^p d(x, v) \leq 1 \) then write \( x = q x' \), \( |q| = p_0 \), and use lemma 5.2 to get the existence of \( u', v' \) such that
Therefore, by lemma 5.1 we have
The first part of the lemma is proven. For the proof of the second part write again
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5.1 Weak dilatation structures

**Remark 5.4** The space $X^\omega$ decomposes into a disjoint union of $2^{p_0}$ balls which are isometric. There is no connection between the weak dilatation structures on these balls, therefore we shall study further only the case $p_0 = 0$.

The purpose of this subsection is to find the general form of a weak dilatation structure on $X^\omega$, with $p_0 = 0$.

**Definition 5.5** A function $W : \mathbb{N}^* \times X^\omega \to Isom(X^\omega)$ is smooth if for any $\varepsilon > 0$ there exists $\mu(\varepsilon) > 0$ such that for any $x, x' \in X^\omega$ such that $d(x, x') < \mu(\varepsilon)$ and for any $y \in X^\omega$ we have

$$\frac{1}{2^k} d(W^x_k(y), W^{x'}_k(y)) \leq \varepsilon$$

for an $k$ such that $d(x, x') < 1/2^k$.

**Theorem 5.6** Let $(X^\omega, d, \delta)$ be a weak dilatation structure on $(X^\omega, d)$, where $d$ is the standard distance on $X^\omega$, such that $p_0 = 0$. Then there exists a smooth (according to definition 5.5) function $W : \mathbb{N}^* \times X^\omega \to Isom(X^\omega)$, $W(n, x) = W^x_n$ such that for any $q \in X^*$, $\alpha \in X$, $x, y \in X^\omega$ we have

$$\delta_x^{q_\alpha} \bar{q} \bar{y} = q \alpha x [W^q_1(y)]$$

(5.1.4)

Conversely, to any smooth function $W : \mathbb{N}^* \times X^\omega \to Isom(X^\omega)$ is associated a weak dilatation structure $(X^\omega, d, \delta)$, with $p_0 = 0$, induced by functions $\delta_x^q$, defined by $\delta_x^q x = x$ and otherwise by relation (5.1.3).

**Proof.** Let $(X^\omega, d, \delta)$ be a weak dilatation structure on $(X^\omega, d)$, such that $p_0 = 0$. Any two different elements of $X^\omega$ can be written in the form $q \alpha x$ and $q \bar{\alpha} y$, with $q \in X^*$, $\alpha \in X$, $x, y \in X^\omega$. We also have

$$d(q \alpha x, q \bar{\alpha} y) = 2^{-|q|}.$$ 

From the following computation (using $p_0 = 0$ and axiom A1):

$$2^{-|q|-1} = \frac{1}{2} d(q \alpha x, q \bar{\alpha} y) = d(q \alpha x, \delta_x^{q \alpha} \bar{q} \bar{y})$$

we find that there exists $\delta_x^{q \alpha} \bar{q} \bar{y} \in X^\omega$ such that

$$\delta_x^{q \alpha} \bar{q} \bar{y} = q \alpha [W^q_{|q|+1}(y)]$$

Further on, we compute:

$$\frac{1}{2} d(q \bar{\alpha} x, q \bar{\alpha} y) = d(\delta_x^{q \alpha} q \alpha x, \delta_x^{q \alpha} q \bar{\alpha} y) = d(q \alpha [W^q_{|q|+1}(x)], q \alpha [W^q_{|q|+1}(y)])$$

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From this equality we find that
\[ 1 > \frac{1}{2} d(x, y) = d(w_{|q|+1}^{\alpha x}(x), w_{|q|+1}^{\alpha x}(y)) , \]
which means that the first letter of the word \( w_{|q|+1}^{\alpha x}(y) \) does not depend on \( y \), and is equal to the first letter of the word \( w_{|q|+1}^{\alpha x}(x) \). Let us denote this letter by \( \beta \) (which depends only on \( q, \alpha, x \)). Therefore we may write:
\[ w_{|q|+1}^{\alpha x}(y) = \beta W_{|q|+1}^{\alpha x}(y) , \]
where the properties of the function \( y \mapsto W_{|q|+1}^{\alpha x}(y) \) remain to be determined later.

We go back to the first computation in this proof:
\[ 2^{-|q|-1} = d(q \alpha x, \bar{\delta}_2^{\alpha x} q \bar{\alpha} y) = d(q \alpha x, q \alpha \beta W_{|q|+1}^{\alpha x}(y)) . \]
This shows that \( \bar{\beta} \) is the first letter of the word \( x \). We proved the relation (5.1.4), expecting the fact that the function \( y \mapsto W_{|q|+1}^{\alpha x}(y) \) is an isometry. But this is true. Indeed, for any \( u, v \in X^\omega \) we have
\[ \frac{1}{2} d(q \bar{\alpha} u, q \bar{\alpha} v) = d(\delta_2^{\alpha x} q \bar{\alpha} u, \delta_2^{\alpha x} q \bar{\alpha} v) = d(q \alpha x \bar{x}_1 W_{|q|+1}^{\alpha x}(x), q \alpha x_1 W_{|q|+1}^{\alpha x}(y)) . \]
This proves the isometry property.

The dilatations of coefficient 2 induce all dilatations (by axiom A2). In order to satisfy the continuity assumptions from axiom A1, the function \( W : \mathbb{N}^* \times X^\omega \to Isom(X^\omega) \) has to be smooth in the sense of definition 5.5. Indeed, axiom A1 is equivalent to the fact that \( \delta_2^{\alpha x}(y') \) converges uniformly to \( \delta_2^{\alpha x}(y) \), as \( d(x, x'), d(y, y') \) go to zero. There are two cases to study.

Case 1: \( d(x, x') \leq d(x, y), d(y, y') \leq d(x, y) \). It means that \( x = q \alpha q' \beta X, y = q \bar{\alpha} q' \gamma Y, x' = q \alpha q' \beta X', y' = q \bar{\alpha} q' \gamma Y' \), with \( d(x, y) = 1/2^k, k = |q| \).

Suppose that \( q' \neq \emptyset \). We compute then:
\[ \delta_2^{\alpha x}(y) = q \alpha q_1 W_{k+1}^{x} (q'' \gamma Y) , \quad \delta_2^{\alpha x}(y') = q \alpha q_1 W_{k+1}^{x'} (q'' \gamma Y') . \]
All the functions denoted by a capitalized ”W” are isometries, therefore we get the estimation:
\[ d(\delta_2^{\alpha x}(y), \delta_2^{\alpha x}(y')) = \frac{1}{2^{k+2}} d(W_{k+1}^{x} (q'' \gamma Y), W_{k+1}^{x'} (q'' \gamma Y')) \leq \frac{1}{2^{k+2}} d(q'' \gamma Y, q'' \gamma Y') + \frac{1}{2^{k+2}} d(W_{k+1}^{x} (q'' \gamma Y), W_{k+1}^{x'} (q'' \gamma Y)) = \frac{1}{2} d(y, y') + \frac{1}{2^{k+2}} d(W_{k+1}^{x} (q'' \gamma Y), W_{k+1}^{x'} (q'' \gamma Y)) . \]
We see that if \( W \) is smooth in the sense of definition 5.5 then the structure \( \delta \) satisfies the uniform continuity assumptions for this case. Conversely, if \( \delta \) satisfies A1 then \( W \) has to be smooth.
If \( q' = \emptyset \) then a similar computation leads to the same conclusion.

Case 2: \( d(x, x') > d(x, y) > d(y, y') \). It means that \( x = qαq' βX, x' = qαX' \), \( y = qαq' βq' γY, y' = qαq' βq' γY' \), with \( d(x, x') = 1 \ 2^k \), \( k = |q| \).

We compute then:

\[
\delta_2^x(y) = qαq' βX_1W^x_{k+2+|q'|}(q'' γY), \quad \delta_2^x(y') = qαX_1'W^x_{k+1}(q' βq'' γY') \leq \frac{1}{2^k} = d(x, x').
\]

Therefore in his case the continuity is satisfied, without any supplementary constraints on the function \( W \).

The first part of the theorem is proven.

For the proof of the second part of the theorem we start from the function \( W : \mathbb{N}^* \times X^\omega \to Isom(X^\omega) \). It is sufficient to prove for any \( x, y, z \in X^\omega \) the equality

\[
\frac{1}{2}d(y, z) = d(\delta_2^x y, \delta_2^x z).
\]

Indeed, then we can construct the all dilatations from the dilatations of coefficient 2 (thus we satisfy A2). All axioms, excepting A1, are satisfied. But A1 is equivalent with the smoothness of the function \( W \), as we proved earlier.

Let us prove now the before mentioned equality. If \( y = z \) there is nothing to prove. Suppose that \( y \neq z \). The distance \( d \) is ultrametric, therefore the proof splits in two cases.

Case 1: \( d(x, y) = d(x, z) > d(y, z) \). This is equivalent to \( x = qαx', y = qαq' βy', z = qαq' βz' \), with \( q, q' \in X^*, \alpha, \beta \in X \), \( x', y', z' \in X^\omega \). We compute:

\[
d(\delta_2^x y, \delta_2^x z) = d(\delta_2^0 x', qαq' βy', \delta_2^0 x', qαq' βz') =
\]

\[
d(qαx_1^x W^x_{|q|+1}(q' βy'), qαx_1^x W^x_{|q|+1}(q' βz')) = 2^{-|q|-1}d(W^x_{|q|+1}(q' βy'), W^x_{|q|+1}(q' βz')) =
\]

\[
= 2^{-|q|-1}d(q' βy', q' βz') = \frac{1}{2}d(qαq' βy', qαq' βz') = \frac{1}{2}d(y, z).
\]

Case 2: \( d(x, y) = d(y, z) > d(x, z) \). If \( x = z \) then we write \( x = qαu, y = qαv \) and we have

\[
d(\delta_2^x y, \delta_2^x z) = d(qαu_1 W^x_{|q|+1}(v), qαu) = 2^{-|q|+1} =
\]

\[
= \frac{1}{2}d(y, z).
\]

If \( x \neq z \) then we can write \( z = qαz', y = qαq' βy', x = qαq' βz' \), with \( q, q' \in X^*, \alpha, \beta \in X \), \( x', y', z' \in X^\omega \). We compute:

\[
d(\delta_2^x y, \delta_2^x z) = d(qαq' βx', qαq' βy', \delta_2^0 qαq' βx') =
\]

\[
d(qαq' βx_1^x W^x_{|q'|+2}(y'), qαγW^x_{|q|+1}(z')),
\]

with \( γ \in X, \gamma = q_1' \) if \( q' \neq \emptyset \), otherwise \( γ = β \). In both situations we have

\[
d(\delta_2^x y, \delta_2^x z) = 2^{-|q|} \leq \frac{1}{2}d(y, z).
\]

The proof is done. □
5.2 Self-similar dilatation structures

Let \((X^\omega,d,\delta)\) be a weak dilatation structure. There are induced dilatations structures on \(0X^\omega\) and \(0X^\omega\).

**Definition 5.7** For any \(\alpha \in X\) and \(x,y \in X^\omega\) we define \(\delta_2^{\alpha,x}y\) by the relation

\[
\delta_2^{\alpha,x}y = \alpha \delta_2^{\alpha}y .
\]

The following proposition has a straightforward proof, therefore we skip it.

**Proposition 5.8** If \((X^\omega,d,\delta)\) is a weak dilatation structure and \(\alpha \in X\) then \((X^\omega,d,\delta_2^\alpha)\) is a weak dilatation structure.

If \((X^\omega,d,\delta')\) and \((X^\omega,d,\delta'')\) are weak dilatation structures then \((X^\omega,d,\delta)\) is a weak dilatation structure, where \(\delta\) is uniquely defined by \(\delta^0 = \delta', \delta^1 = \delta''\).

The previous proposition justifies the next definition.

**Definition 5.9** A weak dilatation structure \((X^\omega,d,\delta)\) is self-similar if for any \(\alpha \in X\) and \(x,y \in X^\omega\) we have

\[
\delta_2^{\alpha,x}y = \alpha \delta_2^{\alpha}y .
\]

**Proposition 5.10** Let \((X^\omega,d,\delta)\) be a self-similar weak dilatation structure and \(W : \mathbb{N}^* \times X^\omega \to Isom(X^\omega)\) the function associated to it, according to theorem 5.6. Then there exists a function \(W : X^\omega \to Isom(X^\omega)\) such that:

(a) for any \(q \in \mathbb{N}^*\) and any \(x \in X^\omega\) we have

\[
W^x_{|q|+1} = W^x .
\]

(b) there exists \(C > 0\) such that for any \(x,x',y \in X^\omega\) and for any \(\lambda > 0\), if \(d(x,x') \leq \lambda\) then

\[
d(W^x(y),W^{x'}(y)) \leq C\lambda .
\]

**Proof.** We define \(W^x = W^x_1\) for any \(x \in X^\omega\). We want to prove that this function satisfies (a), (b).

(a) Let \(\beta \in X\) and any \(x,y \in X^\omega\), \(x = q\alpha u, y = q\bar{\alpha}v\). By self-similarity we obtain:

\[
\beta q\alpha u W^\beta_{|q|+2}(v) = \delta_2^\beta x \beta y = \beta \delta_2^\beta y = \beta q\alpha \bar{u}_1 W^{x}_{|q|+1}(v) .
\]

We proved that

\[
W^\beta_{|q|+2}(v) = W^x_{|q|+1}(v)
\]

for any \(x,v \in X^\omega\) and \(\beta \in X\). This implies (a).

(b) This is a consequence of smoothness, in the sense of definition 5.5 of the function \(W : \mathbb{N}^* \times X^\omega \to Isom(X^\omega)\). Indeed, \((X^\omega,d,\delta)\) is a weak dilatation structure, therefore by theorem 5.6 the previous mentioned function is smooth.
By (a) the smoothness condition becomes: for any \( \varepsilon > 0 \) there is \( \mu(\varepsilon) > 0 \) such that for any \( y \in X^\omega \), any \( k \in \mathbb{N} \) and any \( x, x' \in X^\omega \), if \( d(x, x') \leq 2^k \mu(\varepsilon) \) then
\[
d(W^x(y), W^{x'}(y)) \leq 2^k \varepsilon.
\]
Define then the modulus of continuity: for any \( \varepsilon > 0 \) let \( \bar{\mu}(\varepsilon) \) be given by
\[
\bar{\mu}(\varepsilon) = \sup \{ \mu : \forall x, x', y \in X^\omega \ d(x, x') \leq \mu \implies d(W^x(y), W^{x'}(y)) \leq \varepsilon \}.
\]
We see that the modulus of continuity \( \bar{\mu} \) has the property
\[
\bar{\mu}(2^k \varepsilon) = 2^k \bar{\mu}(\varepsilon)
\]
for any \( k \in \mathbb{N} \). Therefore there exists \( C > 0 \) such that \( \bar{\mu}(\varepsilon) = C^{-1} \varepsilon \) for any \( \varepsilon = 1/2^p \), \( p \in \mathbb{N} \). The point (b) follows immediately. \( \square \)

6 The Cantor set again

**Theorem 6.1** Let \((X^\omega, d, \delta)\) and \((X^\omega, d, \bar{\delta})\) be two linear and self-similar weak dilatation structures, with \( p_0 = 0 \). Suppose that there are two different points \( x_0, x_1 \in X^\omega \) such that \( \delta_2^{x_0} = \bar{\delta}_2^{x_1}, \ i = 0, 1 \). Then a dense set \( B \subset X^\omega \) exists such that for any \( x \in B \) there is a positive radius \( r(x) > 0 \) such that for any \( y \in X^\omega \) with \( d(x, y) \leq r(x) \) we have:
\[
\delta_2^x y = \bar{\delta}_2^x y.
\]

**Proof.** We use the self-similarity hypothesis to restrict to the case of \( x_0, x_1 \in X^\omega \), such that \( d(x_0, x_1) = 1 \).

Let \( q \in X^* \), \( q \neq \emptyset \), a non empty word. To \( q = q_1...q_n \) is associated
\[
f(q) = \delta_2^{x_1} x_0 = \delta_2^{x_2} \delta_2^{x_2} ... \delta_2^{x_n} (x_0) \in X^\omega.
\]

We shall use the next lemma.

**Lemma 6.2** Let \( F = \{0\} \cup X^*1 \) be the set of all non empty finite words which either are equal to 0 or they end with 1. Then the restriction of the previously defined function \( f \) to
\[
f : F \to X^\omega
\]
is injective and the image of \( f \) is dense in \( X^\omega \). Moreover, \( f(X^* \setminus \{\emptyset\}) = f(F) \).

The dilatation structure \((X^\omega, d, \delta)\) is linear by hypothesis, which implies that for any \( x = f(q), q \in F, \ |q| = n, \) and
\[
u \in \delta_2^{x_0} X^\omega = [f(q)]_n X^\omega
\]

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the value of the dilatation \( \delta_2^{f(q)}(u) \) is uniquely determined by the relation:

\[
\delta_2^{f(q)}(u) = \delta_2^{x_0} \delta_2^{x_0} (\delta_2^{x_0})^{-1} (u) .
\]

Form the hypothesis we deduce that in particular for \( q, u \) as previously we have

\[
\delta_2^{f(q)}(u) = \delta_2^{f(q)}(u) .
\]

From the self-similarity of the dilatation structures we deduce that we have the same equality for all \( u \in X^\omega \) and for all \( q \). From the continuity of dilatation structures and the density result stated in lemma 6.2 we get the result. □

**Proof of Lemma 6.2.** We shall suppose that \( x_0 = (0) = 0000 \ldots \) and \( x_1 = (1) = 1111 \ldots \), only for expository reasons. The proof which follows may be easily (but with longer notations) adapted to the general case.

Let \( q_1, q_2 \in F \) such that \( q_1 = z_0 q_1', q_2 = z_0 q_2' \). Then

\[
d(f(q_1), f(q_2)) = d(\delta_2^{x_0} \delta_2^{x_0} \delta_2^{x_0} (x_0), \delta_2^{x_0} \delta_2^{x_0} \delta_2^{x_0} (x_0)) =
\]

\[
= \frac{1}{2^{|z|}} d(\delta_2^{x_q} \delta_2^{x_q} (x_0), \delta_2^{x_q} \delta_2^{x_q} (x_0)) = \frac{1}{2^{|z|}} d(\delta_2^{x_q} (u), \delta_2^{x_q} (v)) ,
\]

for certain \( u, v \in X^\omega \). It follows that

\[
d(f(q_1), f(q_2)) = \frac{1}{2^{|z|}} .
\]

Suppose now that \( q_1, q_2 \in F \) , \( q_1 = q_2 q \), with \( q \neq \emptyset \). Then \( q \in F \), \( q \neq 0 \) and

\[
d(f(q_1), f(q_2)) = d(\delta_2^{x_q} \delta_2^{x_q} (x_0), \delta_2^{x_q} \delta_2^{x_q} (x_0)) = \frac{1}{2^{|q|}} d(\delta_2^{x_q} (x_0), x_0) .
\]

We want to prove that for any \( q \in F \) \( d(\delta_2^{x_q} (x_0), x_0) \neq 0 \). We know that \( q = q' 1 \) and we use a ping-pong type reasoning. Notice that \( f(1) \in 10X^\omega \). We shall prove that for any \( q \in X^* \), \( q \neq 0 \), we cannot have

\[
(0) = \delta_2^{x_q} f(1) .
\]

Indeed, for any \( p, r \in \mathbb{N}^* \) we have

\[
\delta_2^{(0)} 0^r 1X^\omega \subset 0^{p+r} 1X^\omega .
\]

Remark that if we apply \( \delta_2^{(1)} \) to \( X^\omega \) then we get \( 1X^\omega \). Therefore after application of a finite string of \( \delta_2^{(0)}, \delta_2^{(1)} \) to a word starting with 10, we shall always get a word containing 1. This proves the first part of the lemma.

For the second part remark that

\[
X^\omega = \delta_2^{(0)} X^\omega \cup \delta_2^{(1)} X^\omega .
\]

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and use the Hutchinson theorem 1.4. The lemma is proven. □

The standard IFS of the Cantor set $X^\omega$ is given by the pair of transformations $\phi_\alpha(x) = \alpha x$, $\alpha = 0, 1$. These two transformations are contractions of coefficient $1/2$, therefore we may think about them as dilatations of coefficient $2$ based at $(0) = 0000...$ and $(1) = 1111...$ respectively.

Suppose that $(X^\omega, d, \delta)$ is a linear, self-similar dilatation structure with the property that for any $x \in X^\omega$:

$$
\delta_2^{(0)} x = \phi_0(x) = 0x, \quad \delta_2^{(1)} x = \phi_1(x) = 1x.
$$

Then, according to theorem 6.1 the dilatation structure is uniquely determined on a dense subset of $X^\omega$, in the sense explained in the conclusion of the theorem.

Is this implying that there is only one such dilatation structure? (According to examples, there is at least one such dilatation structure, coming from dyadic integers).

We shall even put more demand on the dilatation structure $(X^\omega, d, \delta)$ by asking further that for any $n \in \mathbb{N}^*$, any finite word of length $n$, $q \in X^n$, the composition of functions $\phi_{q_n}...\phi_{q_1}$ is a dilatation of coefficient $2^n$, based at $(q) = qqqq...$, that is:

$$
\phi_{q_n}...\phi_{q_1} = \delta_{2^n}^{(q)}.
$$

Is there more than one linear, self similar dilatation structure which satisfies relations (6.0.1), (6.0.2)?

**Proposition 6.3** The standard IFS of the Cantor set does not uniquely determine a linear and self-similar dilatation structure satisfying (6.0.1), (6.0.2). There exist two different and non equivalent strong linear dilatation structures which satisfy these two relations.

**Proof.** There is a strong, linear and self-similar dilatation structure which has these two contractions $\phi_\alpha$, $\alpha = 0, 1$, as dilatations. This is coming from the interpretation of $X^\omega$ as the set of dyadic integers. Under this identification we have

$$
\delta_2^x y = x + 2(y - x) = 2y - x.
$$

The dilatation structure defined like this is strong, linear and self-similar. Let us check this.

Start with linearity: for any $x, u, v \in X^\omega \equiv \mathbb{Z}_2$ the following is true

$$
\delta_2^y \delta_2^u (v) = \delta_2^{\delta_2^y u} \delta_2 v.
$$

Indeed, the left hand side equals:

$$
\delta_2^y \delta_2^u (v) = \delta_2^y (2v - u) = 2(2v - u) - x,
$$

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and the right hand side is equal to
\[
\delta_2^{2x}u \delta_2^x v = \delta_2^{2x} (2v - x) = 2(2v - x) - (2u - x) = 2(2v - u) - x.
\]

Concerning the self-similarity: let \( q \in X^* \), non empty finite word of length \( n \), and \( u, v \in X^\omega \). Then we may identify \( q \) with the element of \( \mathbb{Z}_2 \) described by the word \( q(0) \), and we have
\[
qu = q + 2^nu, \quad qv = q + 2^nv,
\]
therefore the self-similarity comes from the string of equalities:
\[
\delta_2^{nq} qv = 2(q + 2^n v) - (q + 2^nu) = q + 2^n (2v - u) = q \delta_2^n v.
\]
The dilatation structure is strong because we have here a conical group.

We check now the relations (6.0.1), (6.0.2). These are equivalent with: for any \( x \in X^\omega \) and for any \( n \in \mathbb{N}^* \), any finite word of length \( n \) \( q \in X^n \), we have:
\[
\delta_2^{q} x = qx.
\]
This is true for our dilatation structure. Indeed, when we identify \( X^\omega \) with the dyadic integers, the word \( (q) \) is identified with
\[
(q) = \sum_{i=1}^{n} q_i 2^{i-1} \left( \sum_{k=0}^{\infty} 2^k \right) = \frac{q}{1-2^n},
\]
(here we used the identification \( q = \sum_{i=1}^{n} q_i 2^{i-1} \)). Therefore
\[
\delta_2^{q} x = \frac{q}{1-2^n} + 2^n \left( x - \frac{q}{1-2^n} \right) = q + 2^n x = qx.
\]
The second strong linear, self-similar dilatation structure which satisfies the relations (6.0.1), (6.0.2) is the dilatation structure associated to the infinite dihedral group \( \mathbb{D}_\infty \). This is the group \( (X^\omega, *) \) with the following group operation: for any \( x, y \in X^\omega \) the sum \( x * y \in X^\omega \) is the word with the letters \( (x * y)_i = x_i + y_i \mod 2 \). Remark that for any \( x \in X^\omega \) we have \( x^{-1} = x \).

Consider the group morphism: \( \delta_2 x = 0x \). This induces the dilatation structure
\[
\delta_2^*(y) = x * \delta_2(x * y).
\]
This is a strong linear dilatation structure. It is also self-similar and it satisfies (6.0.1), (6.0.2) by straightforward computations.

These two dilatation structures are not equivalent. Indeed, both dilatation structures come from conical groups. If they are equivalent then the groups (dyadic integers with addition and the infinite dihedral group) have to be isomorphic. But these two groups are not isomorphic therefore the dilatation structures cannot be equivalent. \( \Box \)

In contrast, on \( \mathbb{R} \) with euclidean distance, up to equivalence, there is only one strong linear dilatation structure coming from a conical group without small subgroups.
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