Lagrangian submanifolds of Adjoint semisimple orbits given by real forms

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Abstract

We found some Lagrangian submanifolds of the adjoint semisimple orbit in two cases. For the first, the compact case, also known as the Generalized flag manifolds, we prove that the real flags can be seen as infinitesimally tight Lagrangian submanifolds with respect to the KKS symplectic form and we give a complete classification. And for the second, the complex case, we prove that the orbits of real forms are Lagrangian submanifolds with respect to the Hermitian symplectic form.

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1 Introduction

A Lagrangian submanifold of a $2n$-dimensional symplectic manifold $(M, \omega)$ is an $n$-dimensional submanifold on which the symplectic form $\omega$ vanishes. Lagrangian submanifolds play an important role in symplectic geometry and topology. We study some applications of the semisimple Lie theory to Symplectic geometry, in particular to find Lagrangian submanifolds on Adjoint orbits. Our motivation to study Lagrangian submanifolds and their classification comes from questions related to the Homological Mirror Symmetry conjecture and in particular from concepts of objects and morphisms in the so called Fukaya–Seidel categories, which are generated by Lagrangian vanishing cycles (and their thimbles) with prescribed behavior inside of symplectic fibrations. In Section 2 we fix the notation that will be used throughout this paper and we give some references.

The purpose of this paper is to study Lagrangian submanifolds of adjoint orbits of semisimple Lie groups. We consider compact as well as non compact groups. In the compact case the orbits are the so called generalized complex flag manifolds and are homogeneous spaces of the complex group obtained by complexifying the compact one. We endow the complex flag manifolds with the Kostant-Kirillov-Souriau (KKS) symplectic form and look at compact orbits of the real forms of the complex group. We provide a classification of the complex flag manifolds and real forms having Lagrangian compact orbits. This is done in a case by case analysis via the Satake diagrams of the real forms. The result is presented at Table 3.1 at the end of Subsection 3.1 and the case by case proof is done in Subsection 3.2. Subsection 3.3 is dedicated to a special kind of Lagrangian submanifolds, these are tight submanifolds, these submanifolds are widely studied in [8] and [14] for compact Hermitian symmetric spaces. Recently, it was studied in [7] for the case of homogeneous spaces, and it was presented an infinitesimal version known as infinitesimally tight.

Furthermore, in Section 4 we apply the symplectic deformation given in [3] to built Lagrangian submanifolds given for real forms of $\mathfrak{g}^C$, which admits the Cartan decomposition $\mathfrak{g}^C = \mathfrak{u} + i\mathfrak{u}$. Given $H$ on the closure of the Weyl chamber, the semi-direct product orbit $U_{\text{ad}} \cdot H$ has the cylindrical shape

$$U_{\text{ad}} \cdot H = \bigcup_{X \in \text{Ad}(U) \cdot H} (X + \text{ad}(X) i\mathfrak{u})$$

where $\text{ad}(X) i\mathfrak{u}$ is a subspace of $\mathfrak{u}$. We use a particular class of automorphisms on $\mathfrak{u}$ and we build some families of Lagrangian submanifolds on $\text{Ad}(G^C) \cdot H$ with respect to the Hermitian symplectic form $\Omega_\tau$.

2 Generalities

Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$. The adjoint orbit of $G$ passing through $H \in \mathfrak{g}$ is the set

$$\text{Ad}(G) \cdot H = \{ \text{Ad}(g) \cdot H : g \in G \} \subset \mathfrak{g},$$

which can be determined by the adjoint action of $G$ on $\mathfrak{g}$. The adjoint action of $G$ is a Hamiltonian action, where the induced vector fields are $\tilde{X} = \text{ad}(X)$ with Hamiltonian function $H_X(\cdot) = \langle X, \cdot \rangle$. Hence, the adjoint orbit $\text{Ad}(G) \cdot H$ admits the Kirillov-Konstant-Souriau symplectic form (briefly KKS form), given by

$$\omega_x \left( \tilde{X}(x), \tilde{Y}(x) \right) = \langle x, [X, Y] \rangle \quad X, Y \in \mathfrak{g}, \quad x \in \text{Ad}(G) \cdot H,$$

where $\langle \cdot, \cdot \rangle$ is the Cartan-Killing form on $\mathfrak{g}$. 

Other properties of this type of manifolds will depend on the properties of the Lie algebra $\mathfrak{g}$ or the Lie group $G$, in particular we will see what happens in the compact and the non-compact cases.

- Let $K$ be a compact semisimple Lie group with Lie algebra $\mathfrak{k}$. Our purpose is to study the homogeneous spaces $K/Z_H$ with $Z_H$ the centralizer in $K$ of an element $H$, where $H$ can be chosen as follows (see [10] and [11]):

  - Let $\mathfrak{g}$ be a non-compact real semisimple Lie algebra and $G$ a connected Lie group with Lie algebra $\mathfrak{g}$. Given an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and $G = KAN$ a global decomposition, with $K = \langle \exp \mathfrak{k} \rangle$, $A = \langle \exp \mathfrak{a} \rangle$, $N = \langle \exp \mathfrak{n} \rangle$ and $\mathfrak{a}$ a maximal abelian subalgebra, there exists $h$ a Cartan subalgebra of $\mathfrak{g}$, such that $\mathfrak{a} \subset h$ (where $h$ is the realification of $h_C$). For $H \in \mathfrak{a}$:

    $$K/K_H = \text{Ad}(K) \cdot H,$$

    where $K_H$ is the centralizer in $K$ of $H$. The adjoint orbit $\text{Ad}(K) \cdot H$ is called real flag manifold. If $b_H = 1 \cdot K_H$ denotes the origin of $\text{Ad}(K) \cdot H$, then

    $$T_{b_H} \text{Ad}(K) \cdot H = \sum_{\alpha(H) < 0} g_\alpha,$$

    where $g_\alpha$ is the root space of $\mathfrak{g}$, for $\alpha \in \Pi$ and $\Pi$ is the root system of $\mathfrak{a}$.

- Let $\mathfrak{g}^C$ be a complex semisimple Lie algebra and $G^C$ a connected Lie group with Lie algebra $\mathfrak{g}^C$ (the complexifications of $\mathfrak{g}$ real semisimple Lie algebra and $G$ connected Lie group with Lie algebra $\mathfrak{g}$). If $u$ is a compact form of $\mathfrak{g}^C$ and $U = \langle \exp u \rangle$, take $h_C$ the Cartan subalgebra of $\mathfrak{g}^C$, then $t = i h_R$ is the Lie algebra of a maximal torus $T = \exp t$ in $U$. For $H \in h_R$:

    $$U/U_H = \text{Ad}(U) \cdot iH = F_H,$$

    where $U_H$ is the centralizer in $U$ of $H$. The adjoint orbit $\text{Ad}(U) \cdot iH$ is called complex flag manifold. If $b_H = 1 \cdot U_H$ denotes the origin of the flag $F_H$, then

    $$T_{b_H} F_H = \sum_{\alpha(H) < 0} g^C_\alpha,$$

    where $g^C_\alpha$ is the root space of $\mathfrak{g}^C$, for $\alpha \in \Pi$ and $\Pi_C$ is the root system of $h_C$.

Furthermore, the element $H$ may be chosen in the closure of the respective Weyl Chamber, denoted by $a^+$ and determined by $a$ (or $h_C$, depending on the case). In the same way, the element $H \in \text{cl}(a^+)$ can determine a subset $\Theta_H \subset \Sigma$, where $\Sigma$ is the simple roots system of $\Pi$ (or $\Pi_C$, depending on the case) such that:

$$\Theta_H = \{ \alpha \in \Sigma : \alpha(H) = 0 \}.$$

Conversely, given $\Theta \subset \Sigma$, there is $H_\Theta \in \text{cl}(a^+)$ such that:

$$H_\Theta = \sum_{\beta \in \Sigma \setminus \Theta} H_\beta.$$
where $H_\beta \in \mathfrak{a}$ (or $h_C$) is determined by $\beta \in \Sigma$ (identified with a dual basis). Therefore, the complex flag $F_H$ can be denoted by $F_{\Theta_H}$ and is called flag manifold of type $\Theta_H$. When $\Theta = \emptyset$ or $H$ is a regular element, then $F_H = F_{\emptyset}$ will be denoted by $F$, and is called maximal flag manifold.

- For the non-compact case, we have that the adjoint orbit $\text{Ad}(G) \cdot H$ is a union of affine subspaces (see [5]), that is the consequence of the global Iwasawa decomposition $G = KAN$, where $AN \subset P_H$ and the adjoint action of $P_H$ on $H$ is given by $\text{Ad}(P_H) \cdot H = H + n^+_H$. Thus

$$\text{Ad}(G) \cdot H = \bigcup_{k \in K} \text{Ad}(k)(H + n^+_H),$$

where

$$n^+_H = \sum_{\alpha(H) > 0} \mathfrak{g}_\alpha.$$

When $\mathfrak{g}^C$ is a complex semisimple Lie algebra and $\mathfrak{u}$ its compact real form with Cartan involution $\tau$, then

$$H_\tau(X, Y) = -\langle X, \tau Y \rangle \quad X, Y \in \mathfrak{g}^C$$

is a Hermitian form of $\mathfrak{g}^C$, where $\langle \cdot, \cdot \rangle$ is the complex Cartan-Killing form of $\mathfrak{g}^C$. The imaginary part of $H_\tau$ will be denoted by $\Omega_\tau$, that is

$$\Omega_\tau(\cdot, \cdot) = \text{im}(H_\tau(\cdot, \cdot))$$

and will be called symplectic Hermitian form determined by $\tau$, that was studied on [3] and [6].

3 Real flags as infinitesimally tight Lagrangian submanifold of complex flag

In this section, we prove that the real flag manifolds can be seen as Lagrangian submanifolds of their respective complex flag and we give the complete classification of complex flag manifolds that admit as Lagrangian submanifold each real flag manifold determined by the different symmetric pairs. Furthermore, we show that these Lagrangian submanifolds are infinitesimally Tight.

3.1 Lagrangian immersion of Real Flags

Let $U$ be a compact semisimple Lie group with Lie algebra $\mathfrak{u}$ and $\mathfrak{k} \subset \mathfrak{u}$ a Lie subalgebra. We say that $(\mathfrak{u}, \mathfrak{k})$ is a symmetric pair if $[\mathfrak{k}, \mathfrak{t}^\perp] \subset \mathfrak{t}^\perp$ and $[\mathfrak{t}^\perp, \mathfrak{t}^\perp] \subset \mathfrak{k}$. In particular, given the symmetric pair $(\mathfrak{u}, \mathfrak{k})$ and $K = \langle \exp \mathfrak{k} \rangle$, then $U/K$ is a symmetric space. The dual symmetric pair is $(\mathfrak{g}, \mathfrak{t})$, where $\mathfrak{g}$ is a non-compact semisimple Lie algebra (real form of $\mathfrak{u}^C$) with Cartan decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{s}$, such that $\mathfrak{s} = \mathfrak{i} \mathfrak{t}^\perp \subset \mathfrak{u}^C = \mathfrak{g}^C$.

By the section 2, the $K$-isotropy representation orbits on $\mathfrak{s}$ (or $\mathfrak{t}^\perp$) are the flag manifolds of $\mathfrak{g}$. For $H \in \mathfrak{t}^\perp$ we have the usual construction of Lagrangian immersion of real flags on the (corresponding) complex flag in the following sense: Let $\mathfrak{a} \subset \mathfrak{s}$ be a maximal abelian subalgebra, there is a Cartan
subalgebra \( h \) of \( g \), such that \( a \in h \) and \( h_C \) is a Cartan subalgebra of \( g^\mathbb{C} \) (specifically, \( a \in h_R \)). For \( H \in a \):

\[
K/K_H = \text{Ad}(K) \cdot H \subset \text{Ad}(U) \cdot iH = U/U_H = \mathbb{F}_H.
\]

Therefore the flags of \( g \) are determined by the adjoint action of \( K \) through \( H \) and are immersed on the flags of \( g_C \) given by the adjoint action of \( U \) through \( iH \). We will see that the immersion is a Lagrangian submanifold given by the adjoint action \( \text{Ad} : U \to (\mathbb{F}_H, \Omega) \).

In fact, the action is Hamiltonian with equivariant moment map \( \mu = \text{id} \), and:

- The symplectic form \( \Omega \) at \( x \in \mathbb{F}_H \) is the Kirillov-Konstant-Souriau (KKS) form:

\[
\Omega_x \left( \tilde{X}(x), \tilde{Y}(x) \right) = \langle x, [X, Y] \rangle \quad X, Y \in u.
\]

- \( \tilde{X} = \text{ad}(x) \) is a Hamiltonian field, with Hamiltonian function \( H_X(x) = \langle x, X \rangle \), for \( X \in u \).

Moreover, as \( u \) is compact we have that \( (t')^\circ \) corresponds to the orthogonal complement of \( t' \) with respect to the invariant scalar product of \( u \). Then we can conclude:

**Proposition 1.** Given a symmetric pair \((u, t)\) and \( H \in a \subset i\mathfrak{t} \), a real flag manifold \( \text{Ad}(K) \cdot H \) is a Lagrangian submanifold of \( \mathbb{F}_H \) with respect to the KKS form.

**Proof.** Since \( t' \subset t \), then \( t' \subset (t')^\circ \) and \( \text{Ad}(K) \cdot H \subset t' = i\mathfrak{s} \), then \( \text{Ad}(K) \cdot H \cap (t')^\circ \neq \emptyset \) and by Proposition 4 of [7] the adjoint \( K \)-orbit (real flag) is an isotropic submanifold.

Furthermore, if \( b_H = 1 \cdot K \), we have that

\[
\dim (T_{b_H} \text{Ad}(K) \cdot H) = \dim \left( \sum_{\alpha(H) < 0} \mathfrak{g}_\alpha \right) = \# \{ \alpha \in \Pi_C : \alpha(H) < 0 \},
\]

as the roots spaces of \( g_C \) are 1-dimensional complex spaces (i.e., 2-dimensional real spaces), then

\[
2 \dim_R (\text{Ad}(K) \cdot H) = \dim_R (\mathbb{F}_H).
\]

Hence \( \text{Ad}(K) \cdot H \) is a Lagrangian submanifold of \( \mathbb{F}_H \). \( \square \)

Now, our interest is to determine the complex flags of \( g_C \) that admit, as Lagrangian submanifold, a real flag given by the action of \( K = \langle \exp t \rangle \) for the symmetric pair \((u, t)\). For \( a \subset \mathfrak{s} \) a maximal abelian subalgebra there is a Cartan subalgebra \( h \) of \( g \) such that \( a \subset h \). Take \( \Pi_C \) the set of roots of \( h_C \) such that the roots of \( a \) are the restrictions of the roots on \( h_C \). If \( \theta \) is a Cartan involution associated with the Cartan decomposition \( g = \mathfrak{t} \oplus \mathfrak{s} \), then exists an involutive extension of \( \theta \) in \( g_C \), we will also denote this extension by \( \theta \). Therefore the restriction of \( \Pi_C \) in \( a \) is given by

\[
P = \frac{1}{2} (1 - \theta^*), \quad \text{where} \quad \theta^* \alpha = \alpha \circ \theta.
\]

Let \( \Pi_{im} \subset \Pi_C \) be the set of imaginary roots, such that \( \alpha \in \Pi_{im} \) if and only if \( P(\alpha) = 0 \). Set \( \Pi_{co} = \Pi_C \setminus \Pi_{im} \), then the set of restricted roots is given by \( P(\Pi_{co}) \). Soon given a proper order (with respect to the lexicographic order in \( a^* \)), take \( \Sigma_{im} \) the system of imaginary simple roots and \( \Sigma_{co} \) its complement such that the projection of \( \Sigma_{co} \) on \( a^* \) is a system of restricted roots \( \Sigma \) and \( a^* \) the positive Weyl chamber of \( g \) determined by \( \Sigma \). For \( H \in \mathfrak{c} (a^*) \)

\[
\Theta_H = \{ \beta \in \Sigma : \beta(H) = 0 \} \subset \Sigma.
\]
Define $\tilde{\Theta}_H \subset \Sigma_C$, given by
\[
\tilde{\Theta}_H = P^{-1}(\Theta_H) \cup \Sigma_{\text{im}},
\]
i.e., $\tilde{\Theta}_H$ is determined by the Satake diagram of $\mathfrak{g}$ (see \cite{10}). Therefore

**Proposition 2.** $\tilde{\Theta}_H = \{\alpha \in \Sigma_C : \alpha(H) = 0\}$.

**Proof.** If $H \in \mathfrak{a}$, then for all $\alpha \in \Sigma_C$
\[
\theta^* \alpha(H) = \alpha \circ \theta(H) = -\alpha(H),
\]
because $\theta|_\mathfrak{a} = -\text{id}$. Also, if $\alpha \in \Sigma_{\text{im}}$, then $\theta^* \alpha = \alpha$, and by \cite{5} we have that $\alpha(H) = 0$. Therefore it is enough to see the roots in $\Sigma_{\text{co}}$. If $\alpha \in P^{-1}(\Theta_H)$, then $(\alpha - \theta^* \alpha)(H) = 0$ implies that $\alpha(H) = \theta^* \alpha(H)$, and by \cite{5} we have that $\alpha(H) = 0$. Thus $\tilde{\Theta}_H \subseteq \{\alpha \in \Sigma_C : \alpha(H) = 0\}$.

Conversely, if $\alpha \in \Sigma_{\text{co}}$ such that $\alpha(H) = 0$, then $\theta^* \alpha(H) = -\alpha(H) = 0$, thus $P(\alpha)(H) = 0$ and implies that $P(\alpha) \in \Theta_H$, i.e. $\alpha \in P^{-1}(\Theta_H)$. \hfill $\square$

Therefore

**Theorem 3.** Given a symmetric pair $(\mathfrak{u}, \mathfrak{t})$, the complex flags of $\mathfrak{u}_C$ of type $\tilde{\Theta} \subset \Sigma_C$ admits as Lagrangian submanifold the real flag of type $\Theta \subset \Sigma$ if and only if
\[
\tilde{\Theta} = P^{-1}(\Theta) \cup \Sigma_{\text{im}}.
\]
That is, $\tilde{\Theta}$ is determined by the Satake diagram of $\mathfrak{g}$.

In particular

**Corollary 4.** A maximal flag $F$ of $\mathfrak{g}_C$ admits a real flag $\text{Ad}(K) \cdot H$ as Lagrangian submanifold if and only if $\Sigma_{\text{im}} = \varnothing$ and $\varnothing = \Theta_H \subset \Sigma$.

**Example 5.** Let $\mathfrak{u} = \mathfrak{su}(7)$, $\mathfrak{t} = S(\mathfrak{u}(2) \times \mathfrak{u}(5))$ and $\mathfrak{g} = \mathfrak{su}(2,5)$ that determine the symmetric pair $(\mathfrak{u}, \mathfrak{t})$ and its respective dual symmetric pair $(\mathfrak{g}, \mathfrak{t})$. The Satake diagram of $\mathfrak{su}(2,5)$ is

\[
\begin{array}{ccc}
\alpha_1 & \alpha_2 & \alpha_3 \\
\alpha_6 & \alpha_5 & \alpha_4 \\
\end{array}
\]

By Theorem \cite{5} the flags of type $\tilde{\Theta} \subset \Sigma_C$ that admit as Lagrangian submanifold a real flag of type $\Theta \subset \Sigma$ are

- If $\Theta_0 = \varnothing$, then $\tilde{\Theta}_0 = \Sigma_{\text{im}} = \{\alpha_3, \alpha_4\}$.
- If $\Theta_1 = \{\beta_1\}$, then $\tilde{\Theta}_1 = \{\alpha_1, \alpha_3, \alpha_4, \alpha_6\}$.
- If $\Theta_2 = \{\beta_2\}$, then $\tilde{\Theta}_2 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$.

Analogously, this is equivalent to that given in the table \cite{3,1} for $n = 7$:
\begin{itemize}
  \item \( \tilde{\Theta}_0 = \Sigma_C \setminus \{\alpha_1, \alpha_2, \alpha_{n-2}, \alpha_{n-1}\} \),
  \item \( \tilde{\Theta}_1 = \Sigma_C \setminus \{\alpha_2, \alpha_{n-2}\} \),
  \item \( \tilde{\Theta}_2 = \Sigma_C \setminus \{\alpha_1, \alpha_{n-1}\} \).
\end{itemize}

To facilitate notation, we use a convenient notation for partitioning an integer, that is, let’s define \( \flat(n) \) for \( n \in \mathbb{N} \), as the set of ordered \( l \)-tuples of integers \( (n_1, \ldots, n_l) \) such that \( 0 < n_1 < \cdots < n_l \leq n \), for example:

\[
\flat(3) = \{(1), (2), (3), (1, 2), (1, 3), (2, 3), (1, 2, 3)\}.
\]

Hence, using the Satake diagrams we can determine which are the complex flags of type \( \tilde{\Theta} \subset \Sigma_C \) that satisfy the theorem 3.

**Corollary 6.** The complex flags of type \( \tilde{\Theta} \subset \Sigma_C \) admits as Lagrangian submanifold a real flag given by the K-adjoint orbit if and only if \( \tilde{\Theta} \) appears in Table 3.1.

The proof of this result is given in the next Subsection.

### 3.2 The case-by-case proof

We will determine on a case-by-case Satake diagram all the complex flags that admit the Lagrangian immersion of the corresponding real flag, determined by the possible symmetric pairs. We will see the construction of the table 3.1 where for normal cases: \( AI, CI, G2, F4, E6, E7 \) and \( E8 \) all possible \( \tilde{\Theta} \subset \Sigma_C \) are admissible.

#### Type \( AII \)

In this case \( g = \mathfrak{sl}(n, \mathbb{H}) \), such that \( g_C = \mathfrak{sl}(2n, \mathbb{C}) \). Then the Satake diagram is

\[
\begin{array}{cccccccc}
\alpha_1 & \alpha_2 & \cdots & \alpha_{n-3} & \alpha_{n-2} & \alpha_{n-1} \\
\end{array}
\]

As \( \Sigma_{im} = \{\alpha_{2j-1} : 1 \leq j \leq n\} \) and \( \Sigma = \{\beta_j = P(\alpha_{2j}) : 1 \leq j \leq n - 1\} \). Therefore the possible \( \tilde{\Theta} \) that satisfy the Theorem 3 are:

\[
\tilde{\Theta} = \Sigma_C \setminus \{\alpha_{2s_1}, \ldots, \alpha_{2s_l} : (s_1, \ldots, s_l) \in \flat(n - 1)\}. \tag{6}
\]

#### Type \( AIII \)

In this case \( g = \mathfrak{su}(k, n-k) \)

- If \( k < n-k \), the Satake diagram is
| Group | Flags type $\Theta \in \Sigma_C$ |
|-------|---------------------------------|
| $\mathfrak{g}_C$ | $\mathfrak{g}$ | $\mathfrak{k}$ |
| $\mathfrak{sl}(n, \mathbb{C})$ | $\mathfrak{su}(n, \mathbb{R})$ | $\mathfrak{so}(n)$ | All possibilities |
| $\mathfrak{su}(2n, \mathbb{C})$ | $\mathfrak{su}^*(2n)$ | $\mathfrak{sp}(n)$ | $\Sigma_C \smallsetminus \{\alpha_{2n_1}, \ldots, \alpha_{2n_j} : (n_1, \ldots, n_j) \in \mathfrak{k}(n-1)\}$ |
| $\mathfrak{sl}(2n, \mathbb{C})$ | $\mathfrak{su}(n, n)$ | $\mathfrak{so}(n) \oplus \mathfrak{so}(n)$ | $\Sigma_C \smallsetminus \{\alpha_{n_1}, \ldots, \alpha_{n_j}, \alpha_{n-n_1}, \ldots, \alpha_{n-n_1} : (n_1, \ldots, n_j) \in \mathfrak{k}(n-1)\}$ |
| $\mathfrak{su}(2n, \mathbb{C})$ | $\mathfrak{su}(n, n)$ | $\mathfrak{so}(n) \oplus \mathfrak{so}(n)$ | $\Sigma_C \smallsetminus \{\alpha_{2n_1}, \ldots, \alpha_{2n_j} : (n_1, \ldots, n_j) \in \mathfrak{k}(n-1)\}$ |

Table 1: Complex flags that admit a Lagrangian immersion of the real flag determined by the action of $K = \exp F$.
As $\Sigma_{im} = \{\alpha_j : k < j < n - k\}$ and $\Sigma = \{\beta_j = P(\alpha_j) = P(\alpha_{n-j}) : 1 \leq j \leq k\}$. Therefore the possible $\tilde{\Theta}$ that satisfy the Theorem 3 are:

$$\tilde{\Theta} = \Sigma_C \setminus \{\alpha_{s_1}, \ldots, \alpha_{s_l}, \alpha_{n-s_l}, \ldots, \alpha_{n-s_1} : (s_1, \ldots, s_l) \in \bar{\nu}(k)\}. \quad (7)$$

- If $k = n - k$, the Satake diagram is

As $\Sigma_{im} = \emptyset$ and $\Sigma = \{\beta_j = P(\alpha_j) = P(\alpha_{n-j}), \beta_k = P(\alpha_k) : 1 \leq j \leq k - 1\}$. Therefore the possible $\tilde{\Theta}$ that satisfy the Theorem 3 are:

$$\tilde{\Theta} = \Sigma_C \setminus \{\alpha_{s_1}, \ldots, \alpha_{s_l}, \alpha_{n-s_l}, \ldots, \alpha_{n-s_1} : (s_1, \ldots, s_l) \in \bar{\nu}(k-1)\}, \quad (8)$$

or

$$\tilde{\Theta} = \Sigma_C \setminus \{\alpha_{s_1}, \ldots, \alpha_{s_l}, \alpha_k, \alpha_{n-s_l}, \ldots, \alpha_{n-s_1} : (s_1, \ldots, s_l) \in \bar{\nu}(k-1)\}. \quad (9)$$

**Type B**

In this case $g = \mathfrak{so}(k, 2n + 1 - k)$, then the Satake diagram is

As $\Sigma_{im} = \emptyset$ and $\Sigma = \{\beta_j = P(\alpha_j) : 1 \leq j \leq k\}$. If $k = n$ then $g$ is normal, but in general the possible $\tilde{\Theta}$ that satisfy the Theorem 3 are:

$$\tilde{\Theta} = \Sigma_C \setminus \{\alpha_{s_1}, \ldots, \alpha_{s_l} : (s_1, \ldots, s_l) \in \bar{\nu}(k)\}. \quad (10)$$

**Type CII**

In this case $g = \mathfrak{sp}(k, n - k)$.

- If $k < n - k$, the Satake diagram is

As $\Sigma_{im} = \{\alpha_j : k < j \leq n\}$ and $\Sigma = \{\beta_j = P(\alpha_j) : 1 \leq j \leq k\}$. Therefore the possible $\tilde{\Theta}$ that satisfy the Theorem 3 are:

$$\tilde{\Theta} = \Sigma_C \setminus \{\alpha_{2s_1}, \ldots, \alpha_{2s_l} : (s_1, \ldots, s_l) \in \bar{\nu}(k)\}. \quad (11)$$
• If $n = 2m$ and $k = m$, the Satake diagram is

\[
\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \cdots \quad \alpha_{n-2} \quad \alpha_{n-1} \quad \alpha_n
\]

As $\Sigma_{im} = \{\alpha_{2j-1} : 1 \leq j \leq m\}$ and $\Sigma = \{\beta_j = P(\alpha_{2j}) : 1 \leq j \leq m\}$. Therefore the possible $\widetilde{\Theta}$ that satisfy the Theorem are:

\[
\widetilde{\Theta} = \Sigma_{C} \setminus \{\alpha_{2s_1}, \ldots, \alpha_{2s_l} : (s_1, \ldots, s_l) \in \mathbb{1}(k)\}.
\]  

(12)

**Type DI**

In this case $\mathfrak{g} = \mathfrak{so}(k, 2n - k)$.

• If $k = n$ then $\mathfrak{g}$ is a normal form.

• If $k < n - 1$ then the Satake diagram is

\[
\alpha_1 \quad \cdots \quad \alpha_k \quad \alpha_{k+1} \quad \alpha_2 \quad \cdots \quad \alpha_{n-2} \quad \alpha_{n-1} \quad \alpha_n
\]

As $\Sigma_{im} = \{\alpha_j : j > k\}$ and $\Sigma = \{\beta_j = P(\alpha_j) : 1 \leq j \leq k\}$. Therefore the possible $\widetilde{\Theta}$ that satisfy the Theorem are:

\[
\widetilde{\Theta} = \Sigma_{C} \setminus \{\alpha_{s_1}, \ldots, \alpha_{s_l} : (s_1, \ldots, s_l) \in \mathbb{1}(k)\}.
\]

(13)

• If $k = n - 1$ then the Satake diagram is

\[
\alpha_1 \quad \cdots \quad \alpha_2 \quad \alpha_{k-2} \quad \alpha_{k-1} \quad \alpha_k
\]

As $\Sigma_{im} = \emptyset$ and $\Sigma = \{\beta_j = P(\alpha_j), \beta_k = P(\alpha_k) = P(\alpha_n) : 1 \leq j < k\}$. Therefore the possible $\widetilde{\Theta}$ that satisfy the Theorem are:

\[
\widetilde{\Theta} = \Sigma_{C} \setminus \{\alpha_{s_1}, \ldots, \alpha_{s_l} : (s_1, \ldots, s_l) \in \mathbb{1}(k - 1)\},
\]  

(14)

or

\[
\widetilde{\Theta} = \Sigma_{C} \setminus \{\alpha_{s_1}, \ldots, \alpha_{s_l}, \alpha_k, \alpha_n : (s_1, \ldots, s_l) \in \mathbb{1}(k - 1)\}.
\]

(15)
**Type DII**

In this case $\mathfrak{g} = \mathfrak{so}^*(2n)$.

- If $n$ is even, the Satake diagram is

\[
\begin{array}{ccccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n-3} & \alpha_{n-2} & \alpha_{n-1} \\
\end{array}
\]

As $\Sigma_{im} = \{\alpha_j : j \text{ is odd}\}$ and $\Sigma = \{\beta_j = P(\alpha_{2j}) : 1 \leq j \leq n\}$. Therefore the possible $\tilde{\Theta}$ that satisfy the Theorem \[\[\] are:

$$\tilde{\Theta} = \Sigma_{\mathbb{C}} \setminus \{\alpha_{2s_1}, \ldots, \alpha_{2s_l} : (s_1, \ldots, s_l) \in \frac{k}{n}\}.$$  \quad (16)

- If $n$ is odd, the Satake diagram is

\[
\begin{array}{ccccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n-3} & \alpha_{n-2} & \alpha_{n-1} \\
\end{array}
\]

As $\Sigma_{im} = \{\alpha_j : j \text{ is odd and } j < n\}$ and $\Sigma = \{\beta_j = P(\alpha_{2j}), \beta_k = P(\alpha_{n-1}) = P(\alpha_n) : 1 \leq j \leq k, \ k = (n - 1)/2\}$. Therefore the possible $\tilde{\Theta}$ that satisfy the Theorem \[\[\] are:

$$\tilde{\Theta} = \Sigma_{\mathbb{C}} \setminus \left\{\alpha_{2s_1}, \ldots, \alpha_{2s_l}, \alpha_{n-1}, \alpha_n : (s_1, \ldots, s_l) \in \frac{k}{n}\right\}.$$  \quad (17)

or

$$\tilde{\Theta} = \Sigma_{\mathbb{C}} \setminus \left\{\alpha_{2s_1}, \ldots, \alpha_{2s_l}, \alpha_{n-1}, \alpha_n : (s_1, \ldots, s_l) \in \frac{k}{n}\right\}.$$  \quad (18)

### 3.2.1 Exceptional cases

**Type F4II**

In this case $\mathfrak{g} = F_{4}^{-20}$, then the Satake diagram is

\[
\begin{array}{cccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
\end{array}
\]

Therefore the only non-trivial possibility of $\tilde{\Theta}$ that satisfy the Theorem \[\[\] is

$$\tilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3\} = \Sigma_{im}. \quad (19)$$
**Type $E_6II$**

In this case $g = E_6^2$, then the Satake diagram is

![Satake diagram for $E_6II$](image)

Therefore the non-trivial possibilities for $\tilde{\Theta}$ that satisfy Theorem 3 are:

- $\tilde{\Theta} = \emptyset$,
- $\tilde{\Theta} = \{\alpha_6\}$,
- $\tilde{\Theta} = \{\alpha_3\}$,
- $\tilde{\Theta} = \{\alpha_2, \alpha_4\}$,
- $\tilde{\Theta} = \{\alpha_1, \alpha_5\}$,
- $\tilde{\Theta} = \{\alpha_3, \alpha_6\}$,
- $\tilde{\Theta} = \{\alpha_2, \alpha_4, \alpha_6\}$,
- $\tilde{\Theta} = \{\alpha_1, \alpha_5, \alpha_6\}$,
- $\tilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_4\}$,
- $\tilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_5\}$.

**Type $E_6III$**

In this case $g = E_6^{-14}$, then the Satake diagram is

![Satake diagram for $E_6III$](image)

Therefore the non-trivial possibilities for $\tilde{\Theta}$ that satisfy the Theorem 3 are:
• $\tilde{\Theta} = \{\alpha_2, \alpha_3 \alpha_4\}$, \quad $\tilde{\Theta} = \{\alpha_3, \alpha_4, \alpha_6\}$, \quad $\tilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$.

**Type $E6\text{IV}$**

In this case $g = E_6^{-26}$, then the Satake diagram is

\[
\begin{array}{cccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\
\end{array}
\]

Therefore the non-trivial possibilities for $\tilde{\Theta}$ that satisfy the Theorem 3 are:

• $\tilde{\Theta} = \{\alpha_2, \alpha_3 \alpha_4, \alpha_6\}$, \quad $\tilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6\}$, \quad $\tilde{\Theta} = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$.

**Type $E7\text{II}$**

In this case $g = E_7^{-5}$, then the Satake diagram is

\[
\begin{array}{cccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\
\end{array}
\]

Therefore the non-trivial possibilities for $\tilde{\Theta}$ that satisfy the Theorem 3 are:

• $\tilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_7\}$, \quad $\tilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_7\}$, \quad $\tilde{\Theta} = \{\alpha_1, \alpha_3, \alpha_4, \alpha_7\}$, \quad $\tilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_7\}$, \quad $\tilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_7\}$, \quad $\tilde{\Theta} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_7\}$.

**Type $E7\text{III}$**

In this case $g = E_7^{-25}$, then the Satake diagram is

\[
\begin{array}{cccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\
\end{array}
\]

Therefore the non-trivial possibilities for $\tilde{\Theta}$ that satisfy the Theorem 3 are:
Therefore the non-trivial possibilities for $\tilde{\Theta}$ that satisfy the Theorem 3 are:

- $\tilde{\Theta} = \{a_4, a_5, a_6, a_8\}$,
- $\tilde{\Theta} = \{a_1, a_4, a_5, a_6, a_8\}$,
- $\tilde{\Theta} = \{a_2, a_4, a_5, a_6, a_8\}$,
- $\tilde{\Theta} = \{a_3, a_4, a_5, a_6, a_8\}$,
- $\tilde{\Theta} = \{a_4, a_5, a_6, a_7, a_8\}$,
- $\tilde{\Theta} = \{a_1, a_2, a_4, a_5, a_6, a_8\}$,
- $\tilde{\Theta} = \{a_1, a_3, a_4, a_5, a_6, a_8\}$,
- $\tilde{\Theta} = \{a_2, a_3, a_4, a_5, a_6, a_8\}$,
- $\tilde{\Theta} = \{a_3, a_4, a_5, a_6, a_7, a_8\}$,
- $\tilde{\Theta} = \{a_1, a_2, a_3, a_4, a_5, a_6, a_8\}$,
- $\tilde{\Theta} = \{a_1, a_3, a_4, a_5, a_6, a_7, a_8\}$,
- $\tilde{\Theta} = \{a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$.

### 3.3 Infinitesimally Tight

In 1991, Y.-G. Oh [14] introduced the notion of tightness of closed Lagrangian submanifolds in compact Hermitian symmetric spaces. Let $(M, \omega, J)$ be a Hermitian symmetric space of a compact type and $\mathcal{L}$ be a closed embedded Lagrangian submanifold of $M$. Then $\mathcal{L}$ is said to be **globally tight** (resp. **tight**) if it satisfies

$$\# (\mathcal{L} \cap g \cdot \mathcal{L}) = \text{SB}(\mathcal{L}, \mathbb{Z}_2)$$

for any isometry $g \in G$ (resp. close to identity) such that $\mathcal{L}$ transversely intersects with $g \cdot \mathcal{L}$. Here $\text{SB}(\mathcal{L}, \mathbb{Z}_2)$ denotes the sum of $\mathbb{Z}_2$-Betti numbers of $\mathcal{L}$. The concept of tightness has applications to the problem of Hamiltonian volume minimization. In particular, Oh showed that the standard $\mathbb{R}P^n$ inside $\mathbb{C}P^n$ is tight and has the least volume among all its Hamiltonian deformations. Tight submanifolds are highly researched in articles [8] and [14], but the infinitesimally Tight case is developed in the article [7] for product of flags manifolds, where they prove in Theorem 35 that it is equivalent to be infinitesimally tight and local tight.

Also, in Proposition 38 of [7] they prove that a Lagrangian orbit $\mathcal{L} = S^3$ of $U(2)$ in the flag $\mathbb{F}(1, 2)$ is infinitesimally tight, but $S^3$ is a real flag of $AIII$ type, then we can propose a generalization...
using the results seen above. In this section we will look at the case of Lagrangian immersions of real flags in the corresponding complex flag. In general, let $G$ be a Lie group and $M = G/H$ a homogeneous space together with a $G$-invariant symplectic form $\omega$, that is, the action of $G$ on $(M, \omega)$ is symplectic.

**Definition 7.** Let $\mathcal{L}$ in $M = G/H$ be a submanifold. An element $X \in \mathfrak{g} = \text{Lie}(G)$ is called transversal to $\mathcal{L}$ if it satisfies the following two conditions

1. for any $x \in \mathcal{L}$, if $\tilde{X}(x) \in T_xN$ then $\tilde{X}(x) = 0$, and

2. the set

$$f_N(X) = \{ x \in N : 0 = \tilde{X}(x) \in T_xN \}$$

is finite.

That is, $\tilde{X}$ is only tangent to $\mathcal{L}$ at most at finitely many points where it vanishes.

A Lagrangian submanifold $L$ in $M = G/H$ is called infinitesimally tight if the equality

$$\#(f_\mathcal{L}(X)) = \text{SB}(\mathcal{L}, \mathbb{Z}_2)$$

holds for any $X \in \mathfrak{g}$ such that $\tilde{X}$ is transversal to $\mathcal{L}$.

By [9]: Let $H \in \text{cl}(\mathfrak{a}^*)$, the $\mathbb{Z}_2$-homology of $\text{Ad}(K) \cdot H$ is freely generated by the Schubert cells $S_{[w]}^{\Theta_H}$, for $[w] \in \mathcal{W}/\text{slash.l} \mathcal{W}$. Therefore

$$\text{SB}(\text{Ad}(K) \cdot H, \mathbb{Z}_2) = \#(\mathcal{W}/\text{W}_{\Theta_H}) \cdot (20)$$

As $\text{Ad}(K) \cdot H \subseteq \mathfrak{s} = \mathfrak{k} \perp$, for $x \in \text{Ad}(K) \cdot H$ we have that:

$$T_x(\text{Ad}(K) \cdot H) = \{ \tilde{A}(x) : A \in \mathfrak{k} \} ,$$

where $\tilde{A} = \text{ad}(A)$. Then

- If $X \in \mathfrak{t}^\perp$, then $\tilde{X} = \text{ad}(X)$ is a Hamiltonian field of the function $H_X = \langle X, x \rangle$. Thus the singularities of $X$ are the singularities of $H_X$, and their number is finite, if and only if $X$ is regular.

Therefore, the transversal elements are the regular elements $X$, and they satisfies

$$\#(f_{\text{Ad}(K) \cdot H}(X)) = \#(\mathcal{W}/\text{W}_{\Theta_H}) .$$

- If $Y \in \mathfrak{t}$, then $\tilde{Y}$ is tangent, thus it cannot be transversal.

- If $Z = X + Y$ for $X \in \mathfrak{t}^\perp$ and $Y \in \mathfrak{t}$, then $\tilde{Z}(x) \notin T_x\text{Ad}(K) \cdot H$ if $\tilde{X}(x) \neq 0$, so for $Z$ to have singularity in $x$ we need that $\tilde{X}(x) = \tilde{Y}(x) = 0$ in a finite quantity. But this only happens for $X$ regular, such that $[X, Y] = 0$. Thus:

$$\#(f_{\text{Ad}(K) \cdot H}(Z)) = \#(\mathcal{W}/\text{W}_{\Theta_H}) .$$

Hence:

**Theorem 8.** Real flags are infinitesimally Tight submanifolds of their corresponding complex flag, given in the Table 3.1.

As a consequence of Theorem 35 of [7]:

**Corollary 9.** Real flags are local Tight submanifolds of their corresponding complex flag.
4 Adjoint semisimple orbit and Hermitian symplectic form

Given a non-compact semisimple Lie algebra \( \mathfrak{g} \) with Cartan decomposition \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s} \). Then \([\mathfrak{k}, \mathfrak{s}] \subset \mathfrak{s}\), implies that the subalgebra \( \mathfrak{k} \) can be represented on \( \mathfrak{s} \) by the adjoint representation. As in [3] we denote the semi-direct product by

\[
\mathfrak{k}_{\text{ad}} := \mathfrak{k} \rtimes_{\rho} \mathfrak{s} \quad \text{where} \quad \rho = \text{ad}|_{\mathfrak{k}},
\]

where \( \mathfrak{s} \) can be seen as an abelian subalgebra. This is a new Lie algebra structure on the same vector space \( \mathfrak{g} \) where the brackets \([X, Y]\) are the same when \( X \) and \( Y \) are in \( \mathfrak{k} \), but the bracket changes when \( X, Y \in \mathfrak{s} \). Let \( G \) be a connected semisimple Lie group with lie algebra \( \mathfrak{g} \) and take \( K \subset G \) the subgroup given by \( K = \langle \exp \mathfrak{t} \rangle \). The semi-direct product of \( K \) in \( \mathfrak{s} \) will be denoted by

\[
K_{\text{ad}} := K \times_{\text{Ad}} \mathfrak{s}.
\]

As the dual space \( \mathfrak{k}^*_{\text{ad}} = \mathfrak{k}^* \times \mathfrak{s}^* \) can be identified with \( \mathfrak{k}_{\text{ad}} = \mathfrak{k} \times \mathfrak{s} \) by the inner product

\[
B_\theta (X, Y) = -(X, \theta Y) \quad X, Y \in \mathfrak{g},
\]

where \( \langle \cdot, \cdot \rangle \) is the Cartan-Killing form of \( \mathfrak{g} \). Then the coadjoint orbit of \( K_{\text{ad}} \) passing through \( H \in \text{cl}(\mathfrak{a}^*) \) is diffeomorphic to the cotangent bundle of the flag manifold \( \text{Ad}(K)H \), thus the \( K_{\text{ad}} \)-orbit itself is the union of the fibers \( \text{ad}(Y)(\mathfrak{s}) \), with \( Y \in \text{F}_H \). That is

\[
K_{\text{ad}} \cdot H = \bigcup_{Y \in \text{Ad}(K)H} Y + \text{ad}(Y)(\mathfrak{s}).
\]  

In [3] was showed that the adjoint orbit \( \text{Ad}(G) \cdot H \) deforms in \( K_{\text{ad}} \cdot H \), by diffeomorphism. When \( \mathfrak{g} \) is a complex semisimple Lie algebra, we have that \( \mathfrak{g} = \mathfrak{u} \oplus i\mathfrak{u} \) is a Cartan decomposition with Cartan involution \( \tau \), for \( \mathfrak{u} \) the compact real form of \( \mathfrak{g} \). Take \( U \subset G \) the compact subgroup with Lie algebra \( \mathfrak{u} \). If \( H \in \mathfrak{s} = i\mathfrak{u} \), its semi-direct orbit is denoted by \( U_{\text{ad}} \cdot H \). Then, the form \( \Omega_\tau \) of \( \mathfrak{g} \) restricted to \( U_{\text{ad}} \cdot H \) is a symplectic form, for \( H \in \text{cl}(\mathfrak{a}^*) \) and the deformation of \( \text{Ad}(G) \cdot H \) on \( U_{\text{ad}} \cdot H \) is a symplectomorphism with respect to \( \Omega_\tau \).

Now, we are going to use those constructions to find Lagrangian submanifolds of complex adjoint orbits with respect to the Hermitian symplectic form. For that, let \( \mathfrak{g} \) be a real semisimple non-compact Lie algebra such that is a real form of \( \mathfrak{g}_C \), and \( \mathfrak{u} \) a compact real form of \( \mathfrak{g}_C \) with Cartan involution \( \tau \), such that

\[
\mathfrak{g} = \left( \mathfrak{g} \cap \mathfrak{u} \right) \oplus \left( \mathfrak{g} \cap i\mathfrak{u} \right)
\]

is a Cartan decomposition of \( \mathfrak{g} \). As the restriction of \( \mathcal{H}_\tau \) to \( \mathfrak{g} \) is real we claim that \( \Omega_\tau|_{\mathfrak{g}} = 0 \).

Moreover, let \( G^C \) be a Lie group with Lie algebra \( \mathfrak{g}_C \). Then

Proposition 10. Let \( M_H \) be a submanifold of \( \text{Ad}_r(G^C) \cdot H \) or \( U_{\text{ad}} \cdot H \) contained on \( \mathfrak{g} \), then \( M_H \) is an isotropic submanifold of \( \text{Ad}_r(G^C) \cdot H \) or \( U_{\text{ad}} \cdot H \), respectively.

With the Cartan decomposition of \( \mathfrak{g} \) given above, we have that

\[
\mathfrak{k}_{\text{ad}} = \mathfrak{k} \times_{\text{ad}} \mathfrak{s} \leq \mathfrak{u} \times_{\text{ad}} i\mathfrak{u} = \mathfrak{u}_{\text{ad}},
\]

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take $K = \langle \exp t \rangle$, then $K_{ad} \cdot H$ is an immersed submanifold on $U_{ad} \cdot H$, for $H \in \mathfrak{a}$. Then
\[ T_xK_{ad} \cdot H \subseteq \mathfrak{t}_{ad} \quad \forall x \in K_{ad} \cdot H, \]
where $\mathfrak{t}_{ad}$ can be identified with $\mathfrak{g}$ as a vector space, as the restriction of $\mathcal{H}_x$ to $\mathfrak{g}$ is real, thus
\[ \Omega_x|_{\mathfrak{t}_{ad}} \equiv 0. \]

Therefore $K_{ad} \cdot H$ is an isotropic submanifold of $U_{ad} \cdot H$, we want to see that $K_{ad} \cdot H$ is a Lagrangian submanifold of $U_{ad} \cdot H$, as we can see in the following example.

**Example 11.** For $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{k} = \mathfrak{so}(2)$ and $u = \mathfrak{su}(2)$. Given
\[ H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{a}, \]
we have that $K_{ad} \cdot H$ (cylinder) is a 2 dimensional isotropic submanifold of $U_{ad} \cdot H$, a 4-dimensional manifold. Hence, the cylinder $K_{ad} \cdot H$ is a Lagrangian submanifold of $U_{ad} \cdot H$.

Let $\sigma$ be an **anti-linear involutive conjugation** on $\mathfrak{g}_{\mathbb{C}}$, such that $\mathfrak{g}$ is the subspace of fixed points of $\sigma$, that is
\[ \mathfrak{g} = \{ X \in \mathfrak{g}_{\mathbb{C}} : \sigma(X) = X \}. \]

Then, if we have that $\mathcal{A} := \{ X \in U_{ad} \cdot H : \sigma(X) = X \}$ coincides with $K_{ad} \cdot H$ then we can conclude that $K_{ad} \cdot H$ is a Lagrangian submanifold of $U_{ad} \cdot H$ with respect to the Hermitian symplectic form, for $H \in \mathfrak{a}$. As $K_{ad} \cdot H$ is contained on $\mathfrak{g}$ and it is a submanifold of $U_{ad} \cdot H$, we have that $K_{ad} \cdot H \subseteq \mathcal{A}$. For the opposite inclusion, by equation [21] we have that
\[ U_{ad} \cdot H = \bigcup_{Y \in \text{Ad}(U) \cdot H} Y + \text{ad}(Y)(iu), \]
then given an element $x \in U_{ad} \cdot H$ implies that
\[ x = Y + \big[ Y, iZ \big] \quad \text{where} \quad Y = \text{Ad}(u) \cdot H, \ u \in U, \ Z \in \mathfrak{u}. \]

As $u = \mathfrak{k} \oplus i\mathfrak{s}$, we have the following possibilities:

- **Take** $X \in \mathfrak{k}$ then $e^{tX} \in U$
\[
\sigma(x) = \sigma \left( \text{Ad}(e^{tX}) \cdot H \right) + \sigma \left( i[\text{Ad}(e^{tX}) \cdot H, Z] \right) = \text{Ad}(e^{tX}) \cdot H - i \left( [\text{Ad}(e^{tX}) \cdot H, \sigma Z] \right),
\]
if $Z \in \mathfrak{k}$, we have that $\sigma(Z) = Z$ and if $Z \in i\mathfrak{s}$, we have that $\sigma(Z) = -Z$, then $\sigma(x) = x$ if and only if $Z \in i\mathfrak{s}$. Thus $x$ is a fixed point if and only if $x \in K_{ad} \cdot H$.

- **Take** $X \in i\mathfrak{s}$ then $e^{tX} \in U$
\[
\sigma(x) = \sigma \left( \text{Ad}(e^{tX}) \cdot H \right) + \sigma \left( i[\text{Ad}(e^{tX}) \cdot H, Z] \right) = -\text{Ad}(e^{tX}) \cdot H + i \left( [\text{Ad}(e^{tX}) \cdot H, \sigma Z] \right),
\]
for $Z \in \mathfrak{u}$, we have that $\sigma(x) \neq x$, then in this case is impossible to have fixed points.
Any other possible choice of $X \in \mathfrak{u}$, we do not have fixed points because it would be a combination of the cases above.

Therefore $\mathcal{A} = K_{\text{ad}} \cdot H$, and $K_{\text{ad}} \cdot H$ is the set of fixed points of $\sigma$, its dimension is half the dimension of $U_{\text{ad}} \cdot H$. Hence

**Proposition 12.** For $H \in \mathfrak{a}$, the coadjoint orbit $K_{\text{ad}} \cdot H$ is a Lagrangian submanifold of $U_{\text{ad}} \cdot H$ with respect to the Hermitian symplectic form.

By [3], the coadjoint orbit $K_{\text{ad}}$ deforms into $\text{Ad}(G) \cdot H$ and $U_{\text{ad}}$ deforms into $\text{Ad}(G^C) \cdot H$. Then we can conclude that

**Corollary 13.** For $H \in \mathfrak{a}$, the orbit $\text{Ad}(G) \cdot H$ is a Lagrangian submanifold of $\text{Ad}(G^C) \cdot H$ with respect to the Hermitian symplectic form.

Furthermore, as the coadjoint orbit $U_{\text{ad}} \cdot H$ is invariant by automorphism of $\mathfrak{u}$, because any automorphism of $\mathfrak{u}$ leaves invariant its Cartan subalgebra (see [10]). Given $k \in \text{Aut}(\mathfrak{t})$ we know that the $k$-action on $\mathfrak{g}$ leaves invariant the Cartan decomposition of $\mathfrak{g}$, its maximal abelian subalgebra and $\mathfrak{u}$ (because $\mathfrak{t}$ is contained in $\mathfrak{u}$). If $\exp$ is the exponential between the Lie algebra $\mathfrak{u}$ and the Lie group $\text{Aut}(\mathfrak{u})$, then for any $X \in i\mathfrak{s}$ we have that $\mathfrak{g}^{tX} = \exp(tX) \cdot \mathfrak{g}$ is a real form of $G^C$ with Cartan decomposition $\mathfrak{g}^{tX} = \mathfrak{k}^{tX} \oplus \mathfrak{s}^{tX}$. Take $G^{tX}$ a Lie group with Lie algebra $\mathfrak{g}^{tX} \subset \mathfrak{u}$, then we can conclude that

**Corollary 14.** For $X \in i\mathfrak{s} \subset \mathfrak{u}$, the adjoint orbit $\text{Ad}(G^{tX}) \cdot \tilde{H}$, where $\tilde{H} = \exp(tX) \cdot H$ is a Lagrangian submanifold of $\text{Ad}(G^C) \cdot H$ with respect to the Hermitian symplectic form.

In fact, we have associated a family of Lagrangian submanifolds of $\text{Ad}(G^C) \cdot H$ determined by $\mathfrak{g}$, and given by the $i\mathfrak{s}$-conjugated real forms of $\mathfrak{g}$.

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