Role of Bell violation and local filtering in quantum key distribution

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In this article, we analyse the relationship between the Bell violation and the secure key rate of entanglement assisted quantum key distribution (QKD) protocols. Specifically, we address the question whether Bell violation is necessary or sufficient for secure communication. We construct a class of states which do not show Bell violation, however, which can be used for secure communication after local filtering. Similarly, we identify another class of states which show Bell violation but cannot be used for generating secure key even after local filtering. The existence of these two classes of states demonstrates that Bell violation as an initial resource is neither necessary nor sufficient for QKD. Our work therefore forces a departure from traditional thinking that the degree of Bell violation is a key resource for quantum communication and brings out the role of local filtering.

I. INTRODUCTION

Quantum correlations have been instrumental in the development of quantum key distribution (QKD) protocols, where two parties Alice and Bob establish a secret key for secure communication [1–5]. QKD protocols can be classified into two different classes. The first class contains the prepare and measure schemes which involve Alice preparing the system in one of many possible states and transmitting it to Bob. Bob then performs a measurement on the same. Afterwards both the parties perform basis reconciliation and distill out a secret key. Examples of such schemes include BB84 [3], B92 [6], six state protocol [7] and SARG04 [8]. The second class of QKD schemes involves the use of entanglement [9] and are termed as entanglement assisted QKD protocols, e.g., the E91 protocol [10]. QKD Protocols in both classes are proven to be robust against eavesdropping [11–15] and are fundamentally secure as opposed to the classical key distribution protocols.

In the entanglement assisted QKD protocols Bell-violation is a necessary condition for the security of the protocol [16–20]. However, it has also been shown that there exist bipartite bound entangled states which can be used to distill a secure key but do not violate any known Bell’s inequalities [21]. Bell’s inequalities, like the CHSH inequality [22], I3322 inequality [23, 24] and the CGLMP inequality [25] characterize the non-classicality of the correlations. It is also known that entanglement is necessary but not sufficient to violate a Bell’s inequality [26]. This renders a huge class of entangled states unusable for QKD. On the other hand, the question whether Bell violation is sufficient for the security of QKD is also not settled and has been a matter of debate [14]. Since entanglement is an expensive resource, it is important to ask if one can carry out QKD with states which are entangled but do violate Bell’s inequalities thereby exploiting entanglement.

In this article, we provide conclusive evidence that Bell-violation is neither necessary nor sufficient for the security of QKD protocols, thereby proving a conjecture put forward by Acin et. al. [14]. We devise a geometrical representation of correlations and relate the CHSH violation with the secure key rate for the protocols where the secure key rate is a function of the error rate only. Such a representation allows direct inference of secure key rate along with Bell violation, and states offering optimal security can be directly identified. Using this representation we identify a class of states showing Bell violation but not offering a secure key. These states are therefore of no use for QKD as they lead to a high error rate and no security.

Local filtering operations allow states to reveal hidden Bell non-locality [27] and can therefore increase the secure key rate. Classical distillation of key has been previously considered in the literature [28]. We, however, use quantum filtering processes to relate secure key rate and Bell violation. It is found that filtering can transform useless states into a useful resource for QKD. Our work is also expected to have an impact in device independent QKD, where Bell violation plays a major role [29] and may lead to feasible experimental implementation. There are states that violate Bell-CHSH inequality and still cannot be transformed into states useful for QKD. Since these states are entangled, under general multicopy entanglement distillation these states can in principle be made useful. However, we restrict ourselves to local filtering which is a special class of entanglement distillation and can be applied to one copy at a time. Experimentally, local filtering is much more accessible than multicopy entanglement distillation [30].

The article is arranged as follows: In Section II we review the Bell-CHSH inequality, outline a general entanglement assisted QKD protocol and give a brief description of local filtering operations. In Section III we de-
velop a geometrical representation of correlations which provides a clear picture of how various states would fare for QKD and show that application of local filtering operations is indeed advantageous. In Section IV we offer concluding remarks and discussions.

II. BACKGROUND

In this section we provide the relevant background with an aim to calculate various quantities such as the Bell-CHSH violation and the secure key rate in entanglement assisted QKD protocols. In Section II C we briefly outline local filtering operations on two-qubit systems.

A. Bell-CHSH inequality

The Bell-CHSH inequality quantifies the correlations arising from measurements on two-qubit states. All correlations which violate the inequality are termed as non-local correlations as they defy explanation by any local realistic hidden variable model (LRHVM).

The CHSH inequality involves two parties Alice and Bob sharing an entangled state \( \rho \). Each party performs two measurements, having two outcomes \( \pm 1 \) on their respective subsystem. Let \( \{A_0, A_1\} \) be the measurement operators in Alice’s lab and \( \{B_0, B_1\} \) be the measurement operators in Bob’s lab. We can define a joint operator \( B = A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1 \) which is called the Bell operator. The Bell-CHSH inequality states that the expectation value of the Bell operator \( B \) for the classical situations describable by LRHVM is bounded between 2 and \(-2\), i.e.,

\[
S = |B| \leq 2. \tag{1}
\]

However, some quantum states violate this bound implying that there is no LRHVM for the corresponding physical situations.

An arbitrary two-qubit state \( \rho \) can be written in the Bloch form as [31]

\[
\rho = \frac{1}{4} \left[ I \otimes I + \mathbf{r} \cdot \mathbf{\sigma} \otimes I + I \otimes \mathbf{s} \cdot \mathbf{\sigma} + \sum_{i,j=1}^{3} T_{ij} \mathbf{\sigma}_i \otimes \mathbf{\sigma}_j \right], \tag{2}
\]

where \( \mathbf{r} \) and \( \mathbf{s} \) are three-dimensional real vectors characterizing the reduced density matrices of the first and the second qubit and \( T \) is a \( 3 \times 3 \) real matrix representing the correlations between the two qubits. The state \( \rho \) can also be parameterized linearly with real parameters as

\[
\rho = \frac{1}{4} \sum_{i,j=0}^{3} M_{ij} \mathbf{\sigma}_i \otimes \mathbf{\sigma}_j, \tag{3}
\]

where \( \mathbf{\sigma}_0 \) is the \( 2 \times 2 \) identity matrix and \( M_{ij} \) is the Mueller matrix [32], with \( M_{11} = \text{Tr}(\rho) \), \( M_{0j} = \mathbf{s}^j \), \( M_{i0} = \mathbf{r} \) and \( M_{ij} = T \forall i, j \in \{1, 2, 3\} \). This representation turns out to be quite useful as will become evident.

The measurement operators \( \{A_0, A_1\} \) and \( \{B_0, B_1\} \) are defined as

\[
A_i = a_i \cdot \mathbf{\sigma}, \quad B_i = b_i \cdot \mathbf{\sigma}, \tag{4}
\]

where \( a_i \) and \( b_i \) are normalized three-dimensional real vectors. In this new notation, we can calculate the expectation value of the Bell operator as

\[
S = a_0^i \mathbf{b}_0 + a_0^i \mathbf{b}_1 + a_1^i \mathbf{b}_0 - a_1^i \mathbf{b}_1. \tag{5}
\]

Simple algebra shows that for a given two-qubit state \( \rho \) the maximum value of \( S \) that can be achieved for optimal measurements is [33]

\[
\max \{S\} = 2\sqrt{\lambda_1^2 + \lambda_2^2}, \tag{6}
\]

where \( \lambda_1 \) and \( \lambda_2 \) are the two largest singular values of the correlation matrix \( T \) with their value bounded from above by 1. Therefore, the maximum Bell violation is achieved when \( S = 2\sqrt{2} \) [34].

An interesting point to note is that the violation of the Bell-CHSH inequality does not depend on the Bloch vectors \( \mathbf{r} \) and \( \mathbf{s} \), but only on the correlation matrix \( T \). Therefore, different state with the same correlation matrix result in the same value of \( S \) which itself is determined by the two parameters \( \lambda_1 \) and \( \lambda_2 \) only.

Therefore, if we fix the optimized Bell violation parameter \( S \) we obtain a relation between \( \lambda_1 \) and \( \lambda_2 \) giving us a way to describe the family of states with this particular value of Bell violation by only one effective parameter.

B. Entanglement assisted QKD protocols

In this subsection, we consider a simple but general entanglement assisted QKD protocol for which the minimum secure key rate is a function of the quantum bit error rate (QBER), which is the mismatch between the prepared and measured states. We establish that the minimum QBER \( Q \) is a function of the two largest singular values \( \lambda_1 \) and \( \lambda_2 \) of the correlation matrix \( T \) of the two-qubit state \( \rho \) given in Eq. (2).

The QBER dictates the minimum achievable secure key rate \( r_{\text{min}} \), thereby quantifying the minimum non-local correlations required for security. It is also essential for security purposes that the entangled state shared by Alice and Bob violate the CHSH inequality (6) [14, 15]. To ensure the same both the parties can perform measurements as detailed in Sec. II A to check for the violation of the CHSH inequality prior to proceeding for key distribution.

In the scheme that we consider, the two parties Alice and Bob who want to establish a secure key, share a two-qubit entangled state. Each of them have a choice of two measurement settings with binary outcomes. Both the
parties perform measurements on their respective subsystems and keep a record of the obtained outcomes. Afterwards, they publicly compare their measurement bases and keep only those outcomes for which their bases matched as the key.

In the ideal scenario, Alice and Bob are left with perfectly identical keys. However, imperfections in state preparation, transmission and measurement processes can yield differences in their key strings. Alice and Bob can estimate the average error by calculating the QBER $Q$ after comparing a small portion of their secret key.

Formally, the QBER $Q$ for a given state $\rho$ is defined as the average probability for Alice and Bob not to get correlated outcomes when the measurements are performed in the same basis and is given as

$$Q = \frac{1}{L} \sum_{\alpha=1}^{L} \sum_{i \neq j} \langle \psi_i^\alpha \phi_j^\alpha | \rho | \psi_i^\alpha \phi_j^\alpha \rangle$$

$$= \frac{1}{4} (2 - x_0^i T y_0 - x_i^i T y_1),$$

where $\{|\psi_i^\alpha\rangle\}_{i=1}^{L}$ and $\{|\phi_j^\alpha\rangle\}_{j=1}^{L}$ denote the $i$th and $j$th element of the measurement basis $\alpha$ with Alice and Bob respectively. The second equality can be achieved by assuming $x_i$ and $y_j$ to be the Bloch vectors of the elements of the measurement basis with Alice and Bob. The terms can then be seen as singular value decomposition of the correlation matrix $T$ with two largest singular values as $\lambda_1$ and $\lambda_2$.

Therefore, the minimum QBER is,

$$Q = \frac{1}{4} (2 - |\lambda_1| - |\lambda_2|),$$

for $\lambda_1$ and $\lambda_2$ the two largest singular values of the correlation matrix $T$.

The minimum secure key rate $r_{\min}$ is defined as the average number of secret bits that can be distilled from each run of the protocol when Alice and Bob measured in the same basis. The minimum secure key rate is a protocol dependent quantity and for entanglement-assisted protocols like the one presented above it is given by \[35, 36\]

$$r_{\min} = 1 + 2(1 - Q) \log_2 (1 - Q) + 2Q \log_2 Q.$$  

(9)

Only when $r_{\min} > 0$, can a secure key be distilled from a protocol. The secure key rate is also required to be as high as possible and as can be seen from Eq. (9) this amounts to minimizing the $Q$ over various values of $\lambda_1$ and $\lambda_2$.

We are now left with two separate conditions for the security of a QKD protocol. One being $r_{\min} > 0$ for a secure key to be distilled while the second is the requirement that the underlying entangled state violates the CHSH inequality.

\[\rho_{\min}\]

C. Local filtering

In this subsection, we present a special class of local quantum operations which is useful to concentrate entanglement and non-local correlations in two-qubit systems.

Local filtering are operations which transform a state $\rho$ to $\rho'$ which has a higher concentration of entanglement and Bell non-local correlations. Consider the local single-qubit measurements on a two-qubit system where the measurement operators $M_1, M_2$ for the first qubit and $N_1, N_2$ for the second qubit. For simplicity, we choose $M_2 = \sqrt{1 - M_1^\dagger M_1}$ and $N_2 = \sqrt{1 - N_1^\dagger N_1}$.

The state after measuring $M_1$ and $N_1$ is given by,

$$\rho' = \frac{(M_1 \otimes N_1) \rho (M_1 \otimes N_1)^\dagger}{\text{Tr}((M_1 \otimes N_1) \rho (M_1 \otimes N_1)^\dagger)},$$

(10)

and can be made to increase if we consider operations with $|\text{det}(M_1)| \neq 0$ and $|\text{det}(N_1)| \neq 0$ and $|\text{det}(M_1)||\text{det}(N_1)| > \text{Tr}(M_1^\dagger M_1 \otimes N_1^\dagger N_1 \rho)$. Consequently, it can be shown that for a certain class of states, Bell violation can also be made to increase \[38\]. Specifically, the state $\rho$ can be filtered to a state $\rho'$ which is Bell diagonal or a special form of the ‘$X$’ state \[27, 32\] and has higher entanglement and exhibits higher Bell non-local correlations.

Following Ref. \[27\], we briefly illustrate the method to obtain a two-qubit filtered state. Any valid operations on the state $\rho$ can be seen as proper orthochronous Lorentz transformations on the Mueller matrix $M$ [Eq. (3)] as

$$M' = L_{M_1} M L_{N_1}^T.$$  

(12)

The Lorentz transformations are given in terms of the measurement operators as:

$$L_{M_1} = T(M_1 \otimes M_1^\dagger)T^\dagger,$$

$$L_{N_1} = T(N_1 \otimes N_1^\dagger)T^\dagger,$$  

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$  

(13)

Further, the Mueller matrix can be brought to a diagonal or a special form by Lorentz transformations $L_1$ and $L_2$ as

$$M = L_1 \Sigma L_2^T,$$  

(14)
where $\Sigma$ is a diagonal Mueller matrix corresponding to a Bell diagonal state or of the form

$$
\Sigma = \begin{pmatrix}
a & 0 & 0 & b \\
0 & d & 0 & 0 \\
0 & 0 & -d & 0 \\
c & 0 & 0 & a + c - b
\end{pmatrix}.
$$

(15)

The latter of the forms can be brought close to a Bell diagonal state for $d \neq 0$ by repeated application of local filtering operations, while $d = 0$ corresponds to a separable initial state.

The matrix representation of these optimal local filtering operations applied on the corresponding Mueller matrix can be constructed by considering its columns as the eigenvectors of $MGM^T G$ and its transposition respectively, where $G = \text{diag}(1, -1, -1, -1)$ is the Minkowski metric. The Mueller matrix under these optimal Lorentz transformations then transforms as

$$
M' = L_1^T G M G L_2.
$$

(16)

Finally, the singular values of the correlation matrix $T$ are the singular values of the matrix $M_{ij}$, $i, j \in \{2, 3, 4\}$.

It is to be noted that the state corresponding to non-diagonal $\Sigma$ is a subset of measure zero and thus has zero probability of occurrence. Therefore, physically states can be brought to a Bell diagonal form which has higher entanglement content and Bell non-local correlations.

It should be noted that local filtering for two qubits is a special case of entanglement distillation [9, 37] when local operations are performed on the level of single copy of the quantum state. In the present article we restrict access of Alice and Bob to single copies and then calculate the secure key rate after local filtering.

### III. RESULTS

In this section, we first develop a geometrical representation of correlations to study the Bell-CHSH violation and QBER for arbitrary two-qubit states. We apply this representation to explicitly identify states which are optimally secure and insecure for a fixed Bell-CHSH violation for QKD. This geometrical representation offers a useful visualization of two-qubit states from a QKD perspective. Next we present a new QKD protocol which involves local filtering to improve the key rate. We conclude this section with explicit examples of states which shows improved key rate upon local filtering.

#### A. Geometrical representation of correlations

As detailed in Sec. II A, all two-qubit states can be parameterized by the two largest singular values of the real correlation matrix $T$ as far as the violation of the Bell-CHSH inequality is concerned. For a bonafide quantum state all the singular values of the $T$ matrix must satisfy $|\lambda_i| \leq 1$ and $\sum_i \lambda_i^2 \leq 3$. States lying outside this constrained region are unphysical and do not correspond to valid density matrices. For the sake of simplicity we only consider the region $0 \leq \lambda_1 \leq 1$ and $0 \leq \lambda_2 \leq 1$ as all the arguments presented below apply equally well to the entire region.

The geometrical representation of the two-qubit states parameterized by the two largest singular values of the correlation matrix is depicted in Fig. 1. Here all physical states are represented by shaded regions while the unshaded region corresponds to parameter range with no corresponding bonafide quantum state. In this representation all the states with fixed value $S$ of the expectation value of the CHSH operator lie on the circular arc $\lambda_1^2 + \lambda_2^2 = S^2/4$. Therefore, all the physical states that do not violate the Bell-CHSH inequality lie within the disc of unit radius $\lambda_1^2 + \lambda_2^2 \leq 1$, as can be seen from Eq. (6), while all physical states lying outside this region show a violation. Thus, for a given physical state its distance from the origin quantifies the Bell-CHSH correlation and if this distance is above 1 the state violates the CHSH inequality. In this geometric representation, the QBER $Q$ is represented by straight lines with slope $-1$, i.e., $\lambda_1 + \lambda_2 = m$ (Fig. 1), where $m$ is the $y$-intercept. These states offer the same $Q = \frac{1}{2} (2 - m)$. Increasing values of $m$ for the straight lines corresponds to a decreasing QBER.

![FIG. 1. A geometrical representation of the Bell-CHSH inequality and the QBER $Q$ parameterized by $\lambda_1$ and $\lambda_2$. The dark grey region corresponds to states which violate the Bell-CHSH inequality but offer $Q > Q_{crit}$. These states are therefore unusable for QKD. Only the states lying in the light grey region offer a secure key rate while also violating the Bell-CHSH inequality.](image-url)
B. Characterization of states based on the geometrical representation

We are now ready to use the geometrical representation described above to identify a set of states which exhibit a violation of the Bell-CHSH inequality, but cannot be used to distill a secure key rate. This way of identifying states which is useless for QKD, is stricter than the one identified earlier [35].

We also identify a set of states most suitable for experimentally implementing entanglement assisted QKD protocols with fixed violation of the Bell-CHSH inequality. For this characterization we consider only those protocols for which the secure key rate \( r_{\min} \) is a function of the error rate \( Q \) alone.

It is clear from Fig. 1 that the set of states having the same Bell-CHSH value \( S \) do not share the same error rate \( Q \). Hence the minimum secure key rate \( r_{\min} \) is different. Considering entanglement as an expensive resource, the variation in the error rate for the same value of \( S \) indicates that some states are more suitable for performing QKD than others despite having the same Bell non-locality. This also implies that the violation of Bell-CHSH inequality alone cannot provide a characterization of the security in an entanglement assisted QKD protocol.

Note that all the classical states saturating the Bell-CHSH bound lie on the circle \( \lambda^2_1 + \lambda^2_2 = 1 \). However, as noted in Sec. III A all these states do not share the same error rate \( Q \). The set of states offering the least error rate \( Q \) for a given value of \( S \) lie on the line which is tangent to the circle of radius \( S/2 \) and will satisfy \( \lambda_1 = \lambda_2 = S/2\sqrt{2} \). Therefore the set of local states saturating the Bell-CHSH inequality and offering the least error rate lie on the point \( \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \) (Fig. 1). Since Bell-CHSH violation is necessary for the security of the QKD protocol, the states corresponding to the point \( \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \) offer no security. We define the error rate at this point as the critical error rate and is given as \( Q_{\text{crit}} = \frac{1}{2} (2 - \sqrt{2}) \approx 0.14 \). All the states on the line \( \lambda_1 + \lambda_2 = \sqrt{2} \) have the same critical error rate (Fig. 1). All valid quantum states lying below this line posses higher error rate and therefore can not be used for secure QKD. To summarize, all the states above \( \lambda^2_1 + \lambda^2_2 = 1 \) violate Bell-CHSH inequality and all the states below \( \lambda_1 + \lambda_2 = \sqrt{2} \) have QBER more than \( Q_{\text{crit}} \); hence unusable for secure QKD. The region of intersection between these two regions contain states which are useless for QKD, even though they are Bell non-local.

The critical line, \( \lambda_1 + \lambda_2 = \sqrt{2} \) provides the theoretically maximum tolerable error rate for carrying out secure QKD using Bell-CHSH violation as a necessary requirement. It may happen that for a given QKD protocol \( Q_{\text{crit}} \) is smaller than the one we obtained. For example, \( Q_{\text{crit}} \) in the protocol presented in [35] is \( Q'_{\text{crit}} = 0.11 \) which is smaller than the theoretical critical value. Therefore, using this protocol even a larger set of states will be obtained which violate Bell-CHSH inequality but do not provide a secure key rate.

Since it is not always possible to achieve the quantum maximum of the Bell-CHSH inequality experimentally, it is desirable to identify states most optimum for QKD for a particular violation. These states should have the property of offering the least error rate for a fixed violation of the CHSH inequality. As detailed in Sec. III A these states are identified with the points \( |\lambda_1| = |\lambda_2| > 1/\sqrt{2} \).

In the non-ideal QKD scenario, Alice and Bob may share states violating the Bell-CHSH inequality but with an error rate higher than \( Q_{\text{crit}} \). It is then desirable to transform these states such that the error rate is reduced below the critical value and the states can be used to distill a secure key. Since we were carrying out QKD between remote locations, such transformations will have to be local operations performed by Alice and Bob. However, local operations cannot increase the violation of the Bell-CHSH inequality unless we sacrifice some of the copies from the ensemble. In the following subsection, we present a QKD protocol which incorporates local-filtering operations, to concentrate the Bell-CHSH correlations in order to enhance the secure key rate.

C. Secure key rate under local filtering operations

In the new QKD protocol, Alice and Bob share entangled pairs of qubits in the states \( \rho \) with Bell-CHSH value \( S \). Let \( M_1 \) and \( N_1 \) be the optimal filtering operators and \( M_2 = \sqrt{1 - M_1^2} M_1 \) and \( N_2 = \sqrt{1 - N_1^2} N_1 \) (as described in II C). The modified protocol then consists of the following steps:

1. First Alice and Bob perform local measurement using \( \{M_i\} \) and \( \{N_i\} \) measurement settings. Which is followed by followed by the measurement of the \( \{A_1, A_2\} \) and \( \{B_1, B_2\} \) which have binary outcomes \( \pm 1 \).

2. Alice and Bob announce the outcome of the measurement in \( \{M_i\} \) and \( \{N_i\} \) measurement settings and the choice of the measurement operators \( \{A_1, A_2\} \) and \( \{B_1, B_2\} \) for each of the qubit pair.

3. They consider only the qubit pairs for which \( M_1 \) and \( N_1 \) clicked, i.e., the pairs for which the local filtering was successful. Then they reconcile their measurement basis \( \{A_1, A_2\} \) and \( \{B_1, B_2\} \) and discard the qubits for which the measurement was performed in different bases.

The QKD protocol presented above relies on the fact that we can successfully filter an ensemble of two qubit partially entangled states into a smaller ensemble with higher entanglement. The states with enhanced entanglement are used for QKD while the other states are discarded. In this process one can transform states useless for QKD into states useful for QKD, specifically the state which violates Bell inequality but fall below the critical
error rate line. The probability of success in the filtering process is

\[ P_{\text{succ}} = \text{Tr}[(M_1 \otimes N_1)\rho(M_1' \otimes N_1')] \]

which implies that the state shows very little Bell-CHSH violation and has error rate higher than the critical value. Since these states are two-qubit entangled and therefore distillable states, in principle, one can perform the entanglement distillation (single-copy or multi-copy) and extract pure Bell states. In this case the entanglement of formation is the same as distillable entanglement which can be readily calculated as,

\[ E(\psi) = -\text{Tr}(\rho_A \log_2 \rho_A) = 0.005, \]

where \( \rho_A \) is the reduced state of Alice. This gives us a qualitative indication that the state can be used for quantum communication. However, how best to harness this resource remains to be figured out.

After performing local filtering operations, the state \( |\psi\rangle \) can be brought into a Bell-diagonal form with the following properties

\[ \lambda_1' + \lambda_2' = 1.7354 > \sqrt{2}, \]

\[ \lambda_1'^2 + \lambda_2'^2 = 1.2271, \]

where \( \lambda_i' \) are the singular values of the correlation matrix for the state transformed after local filtering.

The above properties state that the filtered state has higher Bell-CHSH violation and lower error rate than the critical value. Therefore, the states belonging to the region where Bell-CHSH violation is observed but having \( Q > Q_{\text{crit}} \) can be filtered to the region where a higher Bell-CHSH violation is observed along with \( Q < Q_{\text{crit}} \).

The secure keyrate \( r \) of the protocol with these states can be calculated and turns out to be

\[ r = P_{\text{succ}}r_{\text{min}} \]

\[ = 0.2565 \text{ bits}, \]

where \( P_{\text{succ}} = 0.8638 \) and \( r_{\text{min}} = 0.2972 \text{ bits} \).

Next, consider a class of mixed states with density operator given by \( \rho \)

\[ \rho = \frac{1}{4} [I \otimes I + \lambda(\sigma^2 - \beta^2)(\sigma_z \otimes I - I \otimes \sigma_z) + (1 - 2\lambda)\sigma_x \otimes \sigma_x - 2\lambda\alpha(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y)], \]

where \( \alpha, \beta \in \mathcal{R} \), \( \alpha^2 + \beta^2 = 1 \) and \( 0 < \lambda \leq 1 \).

These states have been studied extensively under local filtering operations [38]. As an example we consider the randomly generated state \( \rho \) with \( \alpha = -0.9789, \beta = -0.2043 \) and \( \lambda = 0.9 \), which has the following properties,

\[ \lambda_1^2 + \lambda_2^2 = 0.7696, \]

\[ \lambda_1 + \lambda_2 = 1.16. \]

This is an example of a state that does not violate Bell-CHSH inequality and is therefore useless for QKD. The distillable entanglement from this state turns out to be

\[ E(C(\rho)) = h \left( \frac{1 + \sqrt{1 - C^2}}{2} \right) \]

\[ = 0.0471, \]
where $C$ is the concurrence of the quantum state and
\[ h(x) = -x \log_2 x - (1-x) \log_2 (1-x) \] is the binary entropy.
This again states that the state $\rho$ considered above can provide some secure key.

After applying local filtering operations as detailed above, we get the state $\rho'$ with the following properties:
\[
\begin{align*}
\lambda_1^2 + \lambda_2^2 &= 1.2249, \\
\lambda_1' + \lambda_2' &= 1.5652.
\end{align*}
\] (27)

The resultant state $\rho'$ is seen to violate Bell-CHSH inequality with $Q < Q_{\text{crit}}$, indicating that it is now a useful state for QKD. Consequently, the keyrate $r$ for the transformed state can be calculated as
\[
\begin{align*}
\bar{r} &= \frac{P_{\text{succ}} r_{\text{min}}}{P_{\text{succ}}} \\
&= 0.0071 \text{ bits},
\end{align*}
\] (28)
where $P_{\text{succ}} = 0.8894$ and $r_{\text{min}} = 0.0080$ bits.

It should also be noted that at the level of single-copy distillation, the local filtering operations considered above have been shown to be optimal for concentrating entanglement and Bell non-locality [27]. Therefore the key rates obtained after applying local filtering, are the best that can be achieved, given access to individual copies only.

Further, according to [27], Bell-diagonal states cannot be filtered further. From Fig. 1 it can be easily seen that there exist such Bell diagonal states which exhibit Bell-CHSH violation, having $Q > Q_{\text{crit}}$ and which cannot be filtered. These states remain useless for QKD even after filtering, thereby indicating that Bell-CHSH violation is not a sufficient condition either.

IV. CONCLUSION

We develop a geometrical representation for two-qubit correlations to quantitatively analyse the relationship between the secure key rate of a QKD protocol and the violation of the Bell-CHSH inequality.

The usefulness of this geometrical representation is demonstrated by showing that states sharing the same non-local correlations do not necessarily share the same secure key rate. This leads to an important conclusion that some states are more apt for performing QKD efficiently than others, even when they share the same non-local correlations.

For fixed (non-maximal) Bell-CHSH violation, the states that are optimally suited for performing QKD are located, which can be useful when Alice and Bob share a non-maximally entangled state. We use the threshold error rate requirement for security to identify a class of states which cannot be used for QKD, even though they exhibit a violation of the Bell-CHSH inequality. This is an improvement over a previous result and has profound experimental implications to develop QKD protocols with non-maximal violation of Bell-CHSH inequality. Such states which are deemed useless for QKD can be seen as a result of the specificity of the protocol considered or because of errors arising due to preparation, transmission or measurements. To harness the entanglement present in states that do not violate Bell-CHSH inequality we employed local filtering operations and found that the performance of such states can be greatly improved in terms of providing key rate for QKD. The local filtering operations considered is a special subclass of entanglement distillation dealing with single copies. Under the paradigm of single copy distillation not all entangled states can provide a secure key as compared to multi-copy distillation in which all two qubit entangled states can be used to distill some secure key. However, multi-copy distillation is harder to achieve experimentally than local filtering. Further, the protocol for local filtering described has been shown to be optimal in the case of single copies [27] and the secure key rate obtained under these operations is the best that can be achieved. We explicitly provided examples when the original state exhibits Bell-CHSH violation but has $Q > Q_{\text{crit}}$ and states which do not violate the Bell-CHSH inequality. It is seen that in both cases local filtering offers improvement in terms of secure key rate.

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[1] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Rev. Mod. Phys. 74, 145 (2002).
[2] V. Scarani, H. Bechmann-Pasquinucci, N. J. Cerf, M. Dušek, N. Lütkenhaus, and M. Peev, Rev. Mod. Phys. 81, 1301 (2009).
[3] C. H. Bennett and G. Brassard, in Proceedings of the IEEE International Conference on Computers, Systems and Signal Processing (IEEE Press, New York, 1984) pp. 175–179.
[4] C. Kumar, J. Singh, S. Bose, and Arvind, Phys. Rev. A 100, 052329 (2019).
[5] J. Singh, K. Bharti, and Arvind, Phys. Rev. A 95, 062333 (2017).
[6] C. H. Bennett, Phys. Rev. Lett. 68, 3121 (1992).
[7] D. Bruß, Phys. Rev. Lett. 81, 3018 (1998).
[8] V. Scarani, A. Acín, G. Ribordy, and N. Gisin, Phys. Rev. Lett. 92, 057901 (2004).
[9] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. 81, 865 (2009).
[10] A. K. Ekert, Phys. Rev. Lett. 67, 661 (1991).
[11] P. W. Shor and J. Preskill, Phys. Rev. Lett. 85, 441 (2000).
[12] C.-H. F. Fung, K. Tamaki, and H.-K. Lo, Phys. Rev. A 73, 012337 (2006).
[13] C. Branciard, N. Gisin, B. Kraus, and V. Scarani, Phys. Rev. A 72, 032301 (2005).
[14] A. Acín, N. Gisin, and L. Masanes, Phys. Rev. Lett. 97, 120405 (2006).
[15] M. Pawłowski, Phys. Rev. A 82, 032313 (2010).
[16] R. Augusiak, D. Cavalcanti, G. Prettico, and A. Acín, Phys. Rev. Lett. 104, 230401 (2010).
[17] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935).
[18] J. S. Bell, Physics 1, 195 (1964).
[19] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Rev. Mod. Phys. 86, 419 (2014).
[20] Č. Brukner and M. Žukowski, “Bell’s inequalities — foundations and quantum communication,” in Handbook of Natural Computing, edited by G. Rozenberg, T. Bäck, and J. N. Kok (Springer Berlin Heidelberg, Berlin, Heidelberg, 2012) pp. 1413–1450.
[21] K. Horodecki, M. Horodecki, P. Horodecki, and J. Oppenheim, Phys. Rev. Lett. 94, 160502 (2005).
[22] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. 23, 880 (1969).
[23] M. Froissart, Il Nuovo Cimento B (1971-1996) 64, 241 (1981).
[24] D. Collins and N. Gisin, Journal of Physics A: Mathematical and General 37, 1775 (2004).
[25] D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, Phys. Rev. Lett. 88, 040404 (2002).
[26] R. F. Werner, Phys. Rev. A 40, 4277 (1989).
[27] F. Verstraete, J. Dehaene, and B. DeMoor, Phys. Rev. A 64, 010101 (2001).
[28] A. Acín, L. Masanes, and N. Gisin, Phys. Rev. Lett. 91, 167901 (2003).
[29] G. Murta, S. B. van Dam, J. Ribeiro, R. Hanson, and S. Wehner, Quantum Science and Technology 4, 035011 (2019).
[30] Z.-W. Wang, X.-F. Zhou, Y.-F. Huang, Y.-S. Zhang, X.-F. Ren, and G.-C. Guo, Phys. Rev. Lett. 96, 220505 (2006).
[31] M. M. Wilde, Quantum Information Theory (Cambridge University Press, 2013).
[32] R. Sridhar and R. Simon, Journal of Modern Optics 41, 1903 (1994).
[33] R. Horodecki, P. Horodecki, and M. Horodecki, Physics Letters A 200, 340 (1995).
[34] B. S. Cirel’son, Letters in Mathematical Physics 4, 93 (1980).
[35] A. Ferenczi and N. Lütkenhaus, Phys. Rev. A 85, 052310 (2012).
[36] L. Sheridan and V. Scarani, Phys. Rev. A 82, 030301 (2010).
[37] W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[38] N. Gisin, Physics Letters A 210, 151 (1996).