A study of the maximum group velocity in a one-dimensional model with a sinusoidally varying staggered potential

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We use Floquet theory to study the maximum value of the stroboscopic group velocity in a one-dimensional tight-binding model subjected to an on-site staggered potential varying sinusoidally in time. The results obtained by numerically diagonalizing the Floquet operator are analyzed using a variety of analytical schemes. In the low frequency limit we use adiabatic theory, while in the high frequency limit the Magnus expansion of the Floquet Hamiltonian turns out to be appropriate. When the magnitude of the staggered potential is much greater or much less than the hopping, we use degenerate Floquet perturbation theory; we find that dynamical localization occurs in the former case when the maximum group velocity vanishes. Finally, starting from an “engineered” initial state where the particles (taken to be hard core bosons) are localized in one part of the chain, we demonstrate that the existence of a maximum stroboscopic group velocity manifests in a light cone like spreading of the particles in real space.

I. INTRODUCTION

In the real world, the speed at which information can propagate is limited by the speed of light; this results in the light cone effect as postulated by the special theory of relativity. Is there a similar upper bound of the speed at which correlations (information) can propagate in interacting quantum many-body systems? Following the seminal work by Lieb and Robinson\textsuperscript{1,2}, which established the existence of a maximum group velocity in a one-dimensional spin chain with a finite range interaction, some recent studies have explored this conjecture in several interacting many-body systems; these studies do indeed exhibit an effective light cone that sets a bound on the speed of propagation of correlations. This is reflected for example, in the growth of block entanglement entropy following a quench\textsuperscript{3,4}, or the collapse and revival of the Loschmidt echo\textsuperscript{5,6}. The light cone like propagation of quantum correlations has also been observed experimentally by quenching a one-dimensional quantum gas in an optical lattice\textsuperscript{7,8}.

In parallel, there have been a plethora of studies of closed quantum systems driven periodically in time in an optical lattice\textsuperscript{9,10}-\textsuperscript{12} and localization\textsuperscript{13,14}, dynamical fidelity\textsuperscript{15}, as well as thermalization\textsuperscript{16}. The study of periodically perturbed many-body systems have also gained importance because of the proposal of Floquet (irradiated) graphene\textsuperscript{17,18}, Floquet topological insulators and the generation of topologically protected edge states\textsuperscript{19-21} some of which have been experimentally studied\textsuperscript{22,23}.

In this work, we use Floquet theory to explore the stroboscopic (i.e., measured at the end of each complete period) group velocity of a system of hard core bosons residing on a one-dimensional lattice in the presence of a staggered potential which is varying sinusoidally in time\textsuperscript{24,25}. In particular we study the maximum value of the group velocity to observe the consequent light cone effect. Although the time-independent version of the model is integrable, the periodic sinusoidal perturbation renders the situation rather complicated since the corresponding Floquet operator cannot be obtained in a closed analytical form unlike the case of periodic perturbations which are piecewise continuous in time\textsuperscript{26,27}. One therefore has to use various approximation schemes valid in the appropriate regions of the parameter space to analyze the behavior of the stroboscopic group velocity.

The paper is organized in the following way. In Sec. II we present the Hamiltonian of the model under consideration and discuss the generic behavior of the maximum value of the stroboscopic group velocity ($V_{\text{max}}$) as a function of the amplitude and frequency of the periodic perturbation; this is derived using the numerically obtained Floquet quasi-energies. In Sec. III we use adiabatic theory to find the behavior of $V_{\text{max}}$ in the low frequency limit while the high frequency limit is treated within a Magnus expansion in Sec. IV. In Sec. V (Sec. VI), we use a Floquet perturbation theory\textsuperscript{28} when the hopping term is much smaller (greater) than the amplitude of the staggered potential. In the case of a large magnitude of the staggered potential, we point to the situations when the maximum stroboscopic group velocity vanishes resulting in the so-called dynamical localization. We demonstrate a light cone like propagation of particles in real space in Sec. VII. Concluding remarks are presented in Sec. VIII.

II. MODEL AND THE STROBOSCOPIC GROUP VELOCITY

We consider a Hamiltonian of the tight-binding form

$$H = -\gamma \sum_{l=1}^{L} (b_l^\dagger b_{l+1} + b_{l+1}^\dagger b_l),$$

(1)
and Planck’s constant \( \hbar \) and spatially alternating potential \( V \). The Hamiltonian in Eq. (1) reduces in momentum space to a 2 \( \times \) 2 matrix form in terms of the momenta \( k \) and \( k + \pi \),

\[
H_k(t) = -2\gamma \cos k \sigma^z + V_0 \sin(\omega t) \sigma^x,
\]

where \( \sigma^x, \sigma^z \) denote pseudo-spin Pauli matrices. Clearly the spectrum is gapless for \( V_0 = 0 \). We will set \( \gamma = 1 \) and Planck’s constant \( \hbar = 1 \) for rest of the paper.

Defining the time period \( T = 2\pi/\omega \), the stroboscopic Floquet operator for each momenta mode is given by the unitary operator \( \mathcal{F}_k(V_0, T) = T \exp \left( -i \int_0^T dtH_k(t) \right) \), where \( T \) denotes the time-ordering. This operator cannot be computed analytically for a sinusoidal driving. One can however numerically calculate \( \mathcal{F}_k \) and find its eigenvalues which take the form \( \exp(-i\mu_k T) \) where \( \mu_k \) are the quasi-energies. The group velocity can be obtained from the quasi-energies as \( v_k = \partial \mu_k / \partial k \). The maximum of \( v_k \) as a function of \( k \) gives the quantity \( V_{\text{max}} \), which is the main object of interest in this paper. This object is presented in Fig. 1 as a function of \( V_0 \) and \( \omega \).

Upon inspecting the results presented in Fig. 1, one finds that \( V_{\text{max}} \) tends to saturate at some value for large values of \( \omega \) and small or intermediate values of \( V_0 \) (\( V_0 \leq 1 \)); see Fig. 1(a). The maximum group velocity shows an interesting behavior when both \( V_0 \) and \( \omega \) become large, as shown in Fig. 1(b). In this limit, one finds that for a given frequency \( \omega \), \( V_{\text{max}} \) vanishes in regular intervals of \( V_0 \). On the other hand, for a given \( V_0 \), the zeros of \( V_{\text{max}} \) lie in increasing intervals of \( \omega \). In this regime \( V_{\text{max}} \) is given by the zeros of a Bessel function as we will show below. But when \( \omega \) is small, \( V_{\text{max}} \) never becomes zero though, in the limit \( \omega \to 0 \), \( V_{\text{max}} \) becomes very small irrespective of \( V_0 \); see the bottom regions of Figs. 1(c) and (d). The maximum group velocity gradually decreases with \( V_0 \) if one keeps \( \omega \) fixed at a lower value, as shown in Fig. 1(d). In subsequent sections, we will use different analytical methods to analyze the
III. ADIABATIC LIMIT OF SMALL FREQUENCY

The behavior of the quasi-energy can be explained in the small frequency limit where the adiabatic theory holds. We will choose the basis states as the eigenstates of the pseudo-spin operator $\sigma^z$, i.e., $(1\ 0)^T$ and $(0\ 1)^T$. In this limit, the product of the time period $T$ and the Floquet quasi-energies $\mu_k^\pm$ is equal to the dynamical phase $\epsilon_k^\pm$ accumulated over a complete time period $T$; this is given by

$$
\epsilon_k^\pm = \pm \int_0^{2\pi/\omega} dt \sqrt{4\cos^2 k + V_0^2 \sin^2 \omega t} = \pm \left[ \frac{2\sqrt{4\cos^2 k + V_0^2}}{\omega} E\left(\frac{V_0^2}{V_0^2 + 4\cos^2 k}\right) + \frac{4\cos k}{\omega} E\left(-\frac{1}{4}V_0^2 \sec^2 k\right) \right],
$$

where $E(x)$ is the elliptic integral. (We note that the Berry phase term vanishes in this problem since the closed path traced out by the Hamiltonian in $\mathbf{2}$ as $t$ goes from 0 to $T$ is a line in the $x-z$ pseudo-spin space; such a line covers zero solid angle at the origin $(x, y, z) = (0, 0, 0)$). Interestingly, the behavior of the quasi-energy can be qualitatively explained up to a certain values of $V_0$. Let us elaborate on this below.

In the limit $V_0 \ll 1$ and very small $\omega$, Eq. (3) reduces to

$$
\epsilon_k^\pm \simeq \pm \left[ \frac{4\pi \cos k}{\omega} + \frac{\pi V_0^2 \sec k}{4\omega} + O(V_0^2) \right].
$$

A quasi-degeneracy occurs when this dynamical phase $\epsilon_k^\pm = m\pi$. This condition successfully gives the number of quasi-degenerate points along with values of the quasi-degenerate momenta, $k = \pm \cos[1/8 (m\omega + \sqrt{-4V_0^2 + m^2\omega^2})]$; see Fig. 2 (a).

In the intermediate potential range $V_0 \sim 1$, the behavior of the quasi-energy is again determined by the adiabatic evolution of two-level systems. The number of quasi-degenerate points is successfully given by $\epsilon_k^\pm = m\pi$, where $\epsilon_k^\pm$ is given by

$$
\epsilon_k^\pm \simeq \pm \left[ \frac{4\cos^2 k}{V_0^2\omega} + \frac{8\cos^2 k \log(4V_0)}{V_0^2\omega} + \frac{4V_0^2}{\omega} \right].
$$

From Eqs. (4) and (5), we conclude that $V_{\text{max}}$ is (nearly) independent of $\omega$ for small frequencies; see Fig. 2. We find that $V_{\text{max}} = 2$ for very small $V_0$ while for $V_0 \gg 1, V_{\text{max}} \sim 1/V_0$ as shown in Fig. 1 for small values of $\omega$.

IV. MAGNUS EXPANSION FOR LARGE FREQUENCY

The time evolution operator describing the Schrödinger evolution of a quantum system is given by $\mathcal{T} \exp[-i \int_0^t dt' H(t')] = \exp[\Lambda(t)]$. In the Magnus expansion, the operator $\Lambda(t)$ is decomposed in the form $\Lambda(t) = \sum_{k=1}^\infty \Lambda_k(t)$. The advantage of using a Hamiltonian periodic in time is that the stroboscopic unitary operator (i.e., the Floquet operator) can be expressed in the form $F(T) = \exp(-i H_F T)$, where $H_F$ is the corresponding Floquet Hamiltonian. Thus, for a periodically driven system, the Magnus expansion enables us to express the Floquet Hamiltonian in the form $H_F = \sum_{n=1}^\infty H_F^{(n)}$, where the $H_F^{(n)}$’s can be expressed in

FIG. 2: Plots showing the variation of the quasi-energy $\mu_k^\pm$ as a function of $k$ with the potential for $V_0 < 1$. In this parameter regime we choose two cases: (a) $V_0 > \omega$ and (b) $V_0 <\omega$. The behavior observed in Fig. (a) can be explained using adiabatic theory with small $V_0$. The locations of the quasi-degenerate points in Fig. (b) is discussed in Sec. VII.
Hamiltonian given by terms up to the order $1/\omega$ and $H_{\text{model}}$ given by Eq. (2), and is therefore vanishingly small for the following form $n\omega V$. We then arrive at an effective Hamiltonian given by

$$H_{\text{eff}}^{(1)} = \frac{1}{T} \int_0^T dt H(t),$$

$$H_{\text{eff}}^{(2)} = \frac{1}{2T} \int_0^T dt_1 \int_0^{t_2} dt_2 [H(t_1), H(t_2)],$$

$$H_{\text{eff}}^{(3)} = \frac{-1}{6T} \int_0^T dt_1 \int_0^{t_2} dt_2 \int_0^{t_2} dt_3 [(H(t_1), H(t_2), H(t_3)] + 1 \leftrightarrow 3).$$

(6)

As shown below, the $n$-th order term decreases as $1/\omega^n$, and is therefore vanishingly small for $\omega \to \infty$. For the model given by Eq. (2), $H_{\text{eff}}^{(1)} = \alpha \sigma^z$, $H_{\text{eff}}^{(2)} = 2V_0 \alpha/\omega \sigma^y$ and $H_{\text{eff}}^{(3)} = -(4\alpha^2V_0^2/3\omega^2) \sigma^x - (\alpha V_0^2/\omega^2) \sigma^z$.

We will work in the high frequency limit and retain terms up to the order $1/\omega$. We then arrive at an effective Hamiltonian given by

$$H_{\text{eff}}^{(1)} = \frac{\alpha}{\omega} V_0^2 \sigma^z + \frac{2\alpha V_0}{\omega} \sigma^y - \frac{4\alpha^2 V_0^2}{3\omega^2} \sigma^x,$$

$$\approx \alpha \sigma^z + \frac{2\alpha V_0}{\omega} \sigma^y,$$

(7)

where $\alpha = 2\cos k$. The effective quasi-energy, obtained by diagonalizing (7), is given by $\mu_{\text{eff}}^\pm = \pm \sqrt{\alpha^2 + 4\alpha^2 V_0^2/\omega^2} \approx \pm \alpha(1 + 2V_0^2/\omega^2)$; the quasi-degeneracy points occur at $k = \pm \pi/2$ where $\alpha$ vanishes. In the limit $\omega \to \infty$, $\mu_{\text{eff}}^\pm \approx \pm \alpha$. Therefore, the maximum group velocity $V_{\text{max}}$ becomes 2 irrespective of $V_0$.

This can also be explained simply by noting that the periodically varying perturbation in the Hamiltonian (2) vanishes on the average in the high frequency limit. Moreover, for smaller values of $V_0$, $V_{\text{max}}$ reaches its saturation value $V_{\text{max}} = 2$ at a smaller value of $\omega$ as compared to higher values of $V_0$.

V. LARGE POTENTIAL COMPARED TO THE HOPPING AMPLITUDE: $V_0 \gg \gamma$

Let us now examine the case where the hopping amplitude $\gamma$ can be treated as a perturbing parameter in the Hamiltonian given by Eq. (2). It is useful to consider a unitary transformation which shifts the time dependence of the Hamiltonian to the diagonal term, so that the transformed Hamiltonian takes the form $H_k(t) = V_0 \sin(\omega t) / \omega + 2\gamma \cos k \sigma^z$. (We now set $\gamma = 1$ as usual). The time dependent Schrödinger equation in the new basis can then be written as $i|\phi_k^\pm(t)⟩ = H_k(t)|\phi_k^\pm(t)⟩$. Dividing both sides of the equation by $V_0$ and rescaling $t$ to $tV_0$, the Schrödinger equation can be rewritten as

$$i|\phi_k^+(t)⟩ = \frac{\omega}{V_0} |\phi_k^+(t)⟩ + \frac{2\cos k}{V_0} |\phi_k^-(t)⟩,$$

$$i|\phi_k^-(t)⟩ = -\frac{\omega}{V_0} |\phi_k^-(t)⟩ + \frac{2\cos k}{V_0} |\phi_k^+(t)⟩.$$ (8)

We will set $\omega/V_0 = a$ and $2\cos k/V_0 = b$ in subsequent calculations. The solutions in the zeroth order of $w$ are given by

$$|\phi_k^+(t)⟩ = \begin{pmatrix} c_k^+(0) \cdot e^{i\cos(at)/a} \\ 0 \end{pmatrix},$$

$$|\phi_k^-(t)⟩ = \begin{pmatrix} c_k^-(0) \cdot e^{-i\cos(at)/a} \\ 0 \end{pmatrix},$$

(9)

where $c_k^+(0)$ denote the probability amplitudes of the states $|\phi_k^+(T)⟩$ at time $t = 0$. We find that $|\phi_k^+(T)⟩ = |\phi_k^+(0)⟩$ and $|\phi_k^-(T)⟩ = |\phi_k^-(0)⟩$, implying that these solutions are degenerate in Floquet theory. We therefore employ a degenerate perturbation theory to include the hopping term perturbatively and find the time dependent coefficients $c_k^\pm(t)$, which satisfy the evolution equations

$$i\dot{c}_k^+(t) = bc_k^-(t) \cdot e^{-i2\cos(at)/a},$$

$$i\dot{c}_k^-(t) = bc_k^+(t) \cdot e^{i2\cos(at)/a}.$$ (10)

To incorporate the correction up to first order in the hopping, we substitute $c_k^\pm(t)$ appearing on the right sides of the equations (10) by $c_k^\pm(0)$, respectively. The solution at time $t = T = 2\pi/a$ is then given by

$$c_k^-(T) = c_k^+(0) - (i2\pi bc_-)(0)/a) J_0(2V_0/\omega),$$

$$c_k^+(T) = c_k^-(0) - (i2\pi bc_+)(0)/a) J_0(2V_0/\omega).$$ (11)

Up to first order in the hopping $\gamma$, the Floquet operator is given by

$$F_k(T) = \begin{bmatrix} 1 & -i(2\pi b/a) J_0(2V_0/\omega) \\ -(i2\pi b/a) J_0(2V_0/\omega) & 1 \end{bmatrix},$$

(12)

so that $(c_k^+(T) c_k^-(T))^T = F_k(T)(c_k^+(0) c_k^-(0))^T = \exp(i\theta_k^\pm)(c_k^+(0) c_k^-(0))^T$, where $(\ldots )^T$ denotes transpose. Diagonalizing the matrix (12), one obtains
the eigenvalues $\varepsilon^{\pm}_{k}$, and hence $\theta_{k}^{\pm} = \mu_{k}^{\pm}T = \pm(2\pi b/\alpha)J_{0}(2V_{0}/\omega) = 2\cos kJ_{0}(2V_{0}/\omega)$.

For large $V_{0}$ and large $\omega$, the quasi-energy is given by $\mu_{k}^{\pm} \approx \pm 2\cos k\sqrt{\omega/\pi V_{0}}\cos(2V_{0}/\omega - \pi/4)$. The maximum group velocity is $V_{k}^{\text{max}} = 2\sqrt{\omega/\pi V_{0}}\cos(2V_{0}/\omega - \pi/4)$ which that vanishes at $2V_{0}/\omega = (n+3/4)\pi$. This matches with the observed numerical results presented in Fig. 4.

We note that the maximum group velocity vanishes when $J_{0}(2/\omega) = 0$; this situation corresponds to the coherent destruction of tunneling and dynamical freezing.

Furthermore, given $\mu_{k}^{\pm} = \pm 2\cos kJ_{0}(2V_{0}/\omega)$, one can find the values of the momenta for which quasi-degeneracies occur in the Floquet spectrum given by $\mu_{k}T = m\pi$, namely, $k = \text{acos}\{m\omega/(4J_{0}(2V_{0}/\omega))\}$. A solution for $k = \pm \pi/2$ can be found only for $m = 0$. This behavior is identical to that obtained from the Magnus expansion for $\omega \gg V_{0}$. From Fig. 4, we find that for high $V_{0}$, that Floquet perturbation theory works better at higher frequencies where $V_{\text{max}} = 2J_{0}(2V_{0}/\omega)$. We note that for very small values of $\omega$, $V_{\text{max}}$ falls off as $1/V_{0}$ as predicted by the adiabatic theory. Therefore, a crossover in the behavior of $V_{\text{max}}$ as a function of $V_{0}$ and $\omega$ is expected. The crossover happens between two types of behaviors of the maximum group velocity, i.e., $V_{\text{max}} \propto V_{0}^{-1}$ and $V_{\text{max}} \propto 2J_{0}(2V_{0}/\omega)$. Although $V_{\text{max}}$ never becomes zero (but shows a dip) at the zeros of a Bessel function for small $\omega$, we find that $V_{\text{max}}$ indeed vanishes at these points for higher values of $\omega$.

![Graph showing the variation of maximum group velocity $V_{\text{max}}$ as a function of $V_{0}$](image.png)

**FIG. 4:** Plot shows the variation of the maximum group velocity as a function of $V_{0}$. For higher values of $V_{0}$ and frequency $\omega$, the numerically obtained $V_{\text{max}}$ is found to match the Bessel function given by $2J_{0}(2V_{0}/\omega)$. On the other hand, for very small values of the frequency $V_{\text{max}}$ does not match the Bessel function.

**VI. SMALL POTENTIAL COMPARED TO THE HOPPING AMPLITUDE: $V_{0} \ll \gamma$**

We now consider the other limit, $V_{0} \ll \gamma = 1$, when it can be treated perturbatively. At the zeroth order in $V_{0}$, the Hamiltonian reduces to $H_{k} = -2\cos k\sigma^{z}$ with eigenfunctions

$$|\phi_{k}^{+}(t)\rangle = \begin{pmatrix} e^{-i2\cos k\sigma^{z} t} \\ 0 \end{pmatrix}, \quad |\phi_{k}^{-}(t)\rangle = \begin{pmatrix} 0 \\ e^{i2\cos k\sigma^{z} t} \end{pmatrix}. \quad (13)$$

In the non-degenerate case when $e^{-i2\cos k\sigma^{z} t} \neq e^{i2\cos k\sigma^{z} t}$, $T = 2\pi/\omega$, it can be easily shown that the first order correction in the quasi-energy vanishes since $(\sigma^{z}) = 0$ when the expectation values are calculated with the eigenfunctions in Eq. (13). This necessitates the application of a degenerate perturbation theory when the condition $e^{-i2\cos k\sigma^{z} T} = \pm 1$ is satisfied, implying that $4\cos k = m\omega$.

We will distinguish between two situations, $4\cos k/\omega \neq 1$ and $4\cos k/\omega = 1$; as we will show below, in the former case there is a correction of order $V_{0}$, while in the latter case a correction of order $V_{0}$ emerges.

Let us first discuss the situation when $4\cos k/\omega \neq 1$. In the same spirit as in Sec. VII the quasi-states are chosen to be

$$|\phi_{k}^{+}(t)\rangle = \begin{pmatrix} c_{k}^{+}(t) e^{-i2\cos k\sigma^{z} t} \\ 0 \end{pmatrix}, \quad |\phi_{k}^{-}(t)\rangle = \begin{pmatrix} 0 \\ c_{k}^{-}(t) e^{i2\cos k\sigma^{z} t} \end{pmatrix}. \quad (14)$$

We note that $|\phi_{k}^{+}(T)\rangle = e^{-i2T\cos k}|\phi_{k}^{+}(0)\rangle$ and $|\phi_{k}^{-}(T)\rangle = e^{i2T\cos k}|\phi_{k}^{-}(0)\rangle$. The time-dependent coefficients satisfy the Schrödinger equation

$$i\dot{c}_{k}^{+}(t) = -iV_{0}\sin(\omega t) c_{k}^{-}(t) e^{-i4t\cos k},$$

$$i\dot{c}_{k}^{-}(t) = -iV_{0}\sin(\omega t) c_{k}^{+}(t) e^{i4t\cos k}. \quad (15)$$

Within the first order perturbative approximation, we substitute $c_{k}^{+}(t) = c_{k}^{+}(0)$ on the right hand side of the above equations. At $t = T$, we find that

$$c_{k}^{+}(T) = c_{k}^{+}(0) - \frac{\omega V_{0}c_{k}^{-}(0)}{\omega^{2} - 16\cos^{2}k}\sin(4T\cos k),$$

$$c_{k}^{-}(T) = c_{k}^{-}(0) + \frac{\omega V_{0}c_{k}^{+}(0)}{\omega^{2} - 16\cos^{2}k}\sin(4T\cos k). \quad (16)$$

Considering first the situation $4\cos k \neq \omega$, the Floquet operator up to the first order in $V_{0}$ at time $t = T$ is given by

$$F_{k}(T) = \begin{bmatrix} e^{-i2T\cos k} \frac{\omega V_{0}c_{k}^{+}(0)}{\omega^{2} - 16\cos^{2}k}\sin(4T\cos k)e^{i2T\cos k} - \frac{\omega V_{0}}{\omega^{2} - 16\cos^{2}k}\sin(4T\cos k)e^{-i2T\cos k} \\ \omega V_{0}\sin(4T\cos k)\end{bmatrix}. \quad (17)$$
Diagonalizing the Floquet operator \( \mu_k \), we get the Floquet quasi-energies \( \exp(i\mu_k^+ T) = \cos(2T \cos k) \pm i \sqrt{\sin^2(2T \cos k) + [\omega V_0 \sin(4T \cos k)/(\omega^2 - 16 \cos^2 k)]^2} \), and hence \( \mu_k^\pm = \cos k \pm O(V_0^2) \). The maximum group velocity becomes equal to \( 2 \) for small \( V_0 \).

On the other hand, when \( 4 \cos k = \omega \), we have \( |\phi_k^+(T)| = |\phi_0^k(0)| \) and \( |\phi_k^-(T)| = |\phi_0^{-k}(0)| \) to zeroth order in \( V_0 \). Solving the Schrödinger equations within the first order approximation, one finds that the time dependent coefficients are given by

\[
c_k^+(T) = c_k^+(0) \frac{\pi V_0 c_k^{-}(0)}{\omega}, \quad c_k^-(T) = c_k^-(0) - \frac{\pi V_0 c_k^+(0)}{\omega}.
\]

The Floquet operator is given by

\[
F_k(T) = \begin{bmatrix} 1 & V_0\pi/\omega \\ -V_0\pi/\omega & 1 \end{bmatrix}. \tag{18}
\]

The eigenvalues of the Floquet operator are \( e^{i\mu_k^+ T} = 1 \pm iV_0\pi/\omega \). The quasi-energy \( \mu_k^\pm = \log(1 \pm iV_0\pi/\omega) / T \approx \pm V_0/2 \), leading to a first-order correction to the quasi-energy unlike the previous case \( 4 \cos k \neq \omega \). The group velocity vanishes for \( \omega = 4 \). We also find that the quasi-degenerate momentum modes are given by \( k = \pm \alpha \cos(m\omega/4) \). Referring to Fig. (b) for \( V_0 = 0.5 \) and \( \omega = 2\pi/10 \), we note that the number of quasi-degenerate points is successfully predicted by this theory. A correction to the quasi-energy of the order of \( V_0 \) appears at only \( m = 1 \).

**VII. LIGHT CONE LIKE PROPAGATION OF PARTICLES IN REAL SPACE WITH STROBOSCOPIC TIME \( T \)**

In the earlier sections, we discussed the maximum group velocity \( V_{\max} \) for a given set of parameter values \( V_0 \) and \( \omega \) as presented in Fig. (1). Here we illustrate how the light cone effect arising due to the existence of an upper bound to the group velocity manifests in the real space propagation of particles as shown in Fig. (2), we see that there is a dynamical localization when \( V_{\max} \to 0 \). This is illustrated by choosing an initial state at \( t = 0 \) of a 200-site system in which the sites labeled 51 to 150 are filled (shown by the light region) and the remaining sites are empty (shown by the dark region); this initial state evolves with the total Hamiltonian, i.e., the tight-binding part as well as the sinusoidal driving of the staggered potential. For every stroboscopic instant \( (t = nT) \), we can find the particle density at each site by numerically studying the time evolution of the initial density matrix \( \rho(0) \), namely, \( \rho(nT) = F(nT)\rho(0)F^\dagger(nT) \), where \( F(nT) \) is the real space Floquet operator at time \( t = nT \). The slope of the red dotted line separating the occupied and unoccupied regions in Fig. (3) is proportional to \( \pm V_{\max} \); this clearly demonstrates the light cone like propagation.

**VIII. CONCLUDING REMARKS**

We have analyzed the behavior of a one-dimensional system of hard core bosons which have a nearest neighbor hopping amplitude \( \gamma = 1 \) and are driven by a sinusoidally varying staggered potential with magnitude \( V_0 \) and frequency \( \omega \). We have derived the maximum group velocity \( V_{\max} \) from the quasi-energies computed numerically from the Floquet operator. A number of analytical approximation methods have been used to study \( V_{\max} \) in different regions in the parameter space. Within the adiabatic approximation (which is valid when \( \omega \to 0 \)), we find that \( V_{\max} \) is independent of \( \omega \) for small \( V_0 \) and scales as \( 1/V_0 \) for large \( V_0 \). For large frequencies we use the Magnus expansion of the Floquet Hamiltonian and find that \( V_{\max} = 2 \), independent of the magnitude of \( V_0 \). In this limit, the periodic perturbation vanishes on the average and only the tight-binding part of the Hamiltonian contributes to the group velocity. In the limit \( V_0 \gg \gamma \), we show that the Floquet perturbation theory correctly predicts the vanishing of \( V_{\max} \) for \( J_0(2V_0/\omega) = 0 \); this dynamical localization is particularly prominent in the limit of large \( V_0 \) and \( \omega \) with \( V_0/\omega \sim 1 \). In the other limit \( V_0 \ll \gamma \), there is a correction to the group velocity at first order in \( V_0 \) when the condition \( 4\gamma \cos k = m\omega \) is satisfied.

None of the analytical methods work in the intermediate region when \( V_0 \) and \( \omega \) are both of the order of \( \gamma \) (shown by the central region in the right panel of Fig. (4)). An analysis of the behavior of \( V_{\max} \) in this region may be an interesting subject for future research.

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FIG. 5: Density plots showing the density of particles as a function of the stroboscopic time $nT$ and lattice site $L$. The red dotted line signifies the localization-delocalization boundary. One can determine the maximum group velocity from the slope of this line ($V_{\text{max}} = dL/(ndT)$) which is found to agree well with the $V_{\text{max}}$ obtained analytically from a Bessel function. Fig. (a) depicts a situation where a dynamical localization (dynamical freezing) nearly happens, for $\omega = 4\pi$ and $V_0 = 15$; $V_{\text{max}}$ nearly vanishes when $J_0(2V_0/\omega) = 0$. Fig. (b) shows that particles move with a maximum group velocity of $V_{\text{max}} = 0.814$ for $\omega = 4\pi$ and $V_0 = 24$. Figs. (c) and (d) show that no dynamical localization is observed for $\omega = 2\pi/3$ with $V_0 = 12$ and $V_0 = 14$, respectively.

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