Gravitational Faraday rotation in a weak gravitational field

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Abstract

We examine the rotation of the plane of polarization for linearly polarized light rays by the weak gravitational field of an isolated physical system. Based on the rotation of inertial frames, we review the general integral expression for the net rotation. We apply this formula, analogue to the usual electromagnetic Faraday effect, to some interesting astrophysical systems: uniformly shifting mass monopoles and a spinning external shell.

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I. INTRODUCTION

Electromagnetic theory in a curved space-time, in the approximation of geometric optics, provides some of the most well-known and stringent tests of Einstein’s general theory of gravitation. Under geometric optics, a ray follows a null geodesic regardless of its polarization state and the polarization vector is parallel transported along the ray [1]. In the last decades, observations of both bending of light and gravitational time delay have revealed themselves as a powerful tool in observational astrophysics and cosmology. These phenomena are fully accounted for in the gravitational lensing theory [2, 3]. On the other hand, effects of polarization along the light path have not yet been measured.

The polarization vector of a linearly polarized electromagnetic wave rotates due to the properties of the space-time. The gravitational rotation of the plane of polarization in stationary space-times is a gravitational analogue of the electro-magnetic Faraday effect, i.e., the rotation that a light ray undergoes when passing through plasma in the presence of a magnetic field. The analogy was first noted in [4], where the problem of nonlinear interaction on gravitational radiation was considered.

The first discussion of this relativistic effect goes back to 1957, when Skrotskii [5] applied a method previously developed by Rytov [6] to consider geometric optics in a curved space-time. For this historical reason, the gravitational effect on the polarization of light rays is also known as Skrotskii or Rytov effect. In 1958, Balazs [7] further stressed how the gravitational field of a rotating body may behave as an optically active medium. In 1960, Plebanski [8] solved the Maxwell’s equations in the gravitational field of an isolated physical system. He showed how the polarization vector changes its direction due to the deflection of the light ray, and, in addition to this change, how a rotation of the plane of polarization around the propagation vector may occur. Ten years later, Godfrey [9] took a very different approach. Following Mach’s principle, he considered dragging of inertial frames along with a rotating body and obtained an approximate expression for the rotation of the polarization vector of a light ray propagating along the rotation axis of a Kerr black hole. Trajectories initially propagating parallel to the symmetry axis of a central spinning body were studied in [10], where the problem was formulated in a cylindrical-like Kerr solution. A different situation was considered in [11], where it was discussed how the polarization features of X-ray radiation emitted from an accretion disk surrounding a rotating black hole are also
strongly affected by general-relativistic effects. The relativistic rotation of the plane of polarization was further studied in [12]. Solving the equations of motion of a light ray in the first post-Minkowskian approximation, a formula describing the Skrotskii effect for arbitrary translational and rotational motion of gravitating bodies was derived.

Finally, the Skrotskii effect on light rays propagating in the vacuum region outside the event horizon of a Kerr black hole has been discussed in [13, 14]. In particular, the formulation in [14] stressed in an illuminating way the analogy with the usual Faraday effect.

In this paper, we explore the gravitational Faraday rotation by the gravitational field of an isolated system (lens) when the source of radiation and the observer are remote from the gravitational lens. We restrict to the weak gravitational field far from the lens, and analyze it using linearized theory. This approximation holds for almost all gravitational lensing phenomena. We consider gravitational Faraday rotation by usual astrophysical systems, such as a system of shifting stars acting as lenses or a galaxy deflecting light rays emitted from background sources.

The paper is as follows. In Section II we extend the argument of Godfrey [9] on dragging of inertial frames to reobtain the well known general formula for the angle of rotation of the plane of polarization of a linearly polarized electromagnetic wave in a stationary space-time. This heuristic approach allows us to face the problem without integrating the equation of motion. In Sec. III the weak-field, slow motion approximation is introduced and the weak field limit of the gravitational Faraday rotation is performed. In Section IV we evaluate the Faraday rotation for some systems of astrophysical interest. We examine a system of uniformly moving lenses and a rotating external shell. Section V is devoted to some final considerations.

II. DERIVATION OF THE GRAVITATIONAL FARADAY ROTATION

Let us consider a stationary space-time embedded with a metric $g_{\alpha\beta}$ [21]. Such a metric can be written as

$$ds^2 = h \left( dx^0 - A_i dx^i \right)^2 - dl_p^2$$

(1)

where we have introduced the notation

$$h \equiv g_{00}, \quad A_i \equiv -\frac{g_{0i}}{g_{00}},$$

(2)
and

\[ dl_p^2 \equiv \left( -g_{ij} + \frac{g_{0i}g_{0j}}{g_{00}} \right) dx^i dx^j \equiv \gamma_{ij} dx^i dx^j \]

is the spatial distance in terms of the spatial metric \( \gamma_{ij} \) [15]. As suggested in [9], the rotation of the photon’s plane of polarization may be computed using an argument on the dragging of inertial frames. Due to the gravito-magnetic field, inertial frames near moving bodies are dragged in the direction of motion of the sources of the gravitational field. A differential rotation between adjacent frames results from the variation of the rate of dragging with position. The angular velocity with which a local inertial frame, instantaneously at rest in a stationary frame, rotates relative to the stationary frame can be expressed as [15],

\[ \Omega_{Sk} = -\frac{1}{2} \sqrt{h} \nabla \times \vec{A}, \]  

Following [9], we assume the polarization vector to be dragged along by the rotation of the inertial frames. This view agrees with the result in [16]. The net angle of rotation around the tangent three-vector \( \hat{k} \) [22] along the path between the source and the observer turns out to be

\[ \Omega_{Sk} = -\frac{1}{2} \int_{\text{source}}^{\text{observer}} \sqrt{h} \nabla \times \vec{A} \cdot d\vec{x}, \]

where \( d\vec{x} = \hat{k} dl_P \), in agreement with equation (20) in [14]. Identifying \( \nabla \times \vec{A} \) with the gravito-magnetic field \( \vec{B}_g \) [14], we see how Eq. (4) takes the same form of the usual Faraday effect, i.e., it is proportional to the integral of the component of the gravito-magnetic field along the propagation path. The analogy is only formal since the origins of the two effects are completely different. The gravitational rotation of the plane of polarization is a purely geometrical effect due to the structure of space-time, and no frequency dependence occurs [13, 14].

### III. THE WEAK FIELD LIMIT

Standard hypotheses of gravitational lensing [2, 3] assume that the gravitational lens is localized in a very small region of the sky and its lensing effect is weak. The deflector changes its position slowly with respect to the coordinate system, i.e., the matter velocity is much less than \( c \), the speed of light; matter stresses are also small (the pressure is much smaller than the energy density). In this weak field regime and slow motion approximation, space-time is nearly flat near the lens. As can be seen from Eq. (4), the order of approximation is
determined by off-diagonal components of the metric. To our aim, it is enough to write
\[
    ds^2 \simeq \left(1 + 2\frac{\phi}{c^2} + O(\varepsilon^4)\right) c^2 dt^2 - 8cdt \frac{\vec{V} \cdot d\vec{x}}{c^3} - \left(1 - 2\frac{\phi}{c^2} + O(\varepsilon^4)\right) d\vec{x}^2, \tag{5}
\]
where \( \varepsilon \ll 1 \) denotes the order of approximation. \( \phi \) is the Newtonian potential,
\[
    \phi(t, \vec{x}) \simeq -G \int_{\mathbb{R}^3} \frac{\rho(t, \vec{x}')}{|\vec{x} - \vec{x}'|} d^3 x'; \tag{6}
\]
\( \phi/c^2 \) is of order \( \sim O(\varepsilon^2) \).

\( \vec{V} \) is a vector potential taking into account the gravito-magnetic field produced by mass currents. To \( O(\varepsilon^3) \),
\[
    \vec{V}(t, \vec{x}) \simeq -G \int_{\mathbb{R}^3} \frac{\rho \vec{v}(t, \vec{x}')}{|\vec{x} - \vec{x}'|} d^3 x', \tag{7}
\]
where \( \vec{v} \) is the velocity field of the mass elements of the deflector.

We assume that, during the time light rays take to traverse the lens, the potentials in Eqs. (6, 7) vary negligibly little. Then, the lens can be treated as stationary. In Eqs. (6, 7), we have neglected the retardation \( [3] \).

In the weak field limit, \( h \) and \( \vec{A} \) are simply related to the gravitational potentials. It is
\[
    h \simeq 1 + 2\frac{\phi}{c^2} + O(\varepsilon^4), \tag{8}
\]
\[
    A_i \simeq \frac{4}{c^3} V_i + O(\varepsilon^5); \tag{9}
\]
the proper arc length reads
\[
    dl_p \simeq \left\{1 - \frac{\phi}{c^2} + O(\varepsilon^3)\right\} dl_E, \tag{10}
\]
where \( dl_E \equiv \sqrt{\delta_{ij} dx^i dx^j} \) is the Euclidean arc length.

Inserting Eqs. (8) \( \text{[9]} \) \( \text{[10]} \) in Eq. (4), we get
\[
    \Omega^{Sk} \simeq -\frac{2}{c^3} \int_p \nabla \times \vec{V} \cdot \hat{k} \ dl_E + O(\varepsilon^5), \tag{11}
\]
where \( p \) is the spatial projection of the null geodesics and \( \hat{k} \) is the unit tangent vector.

It is useful to employ the spatial orthogonal coordinates \((l, \xi_1, \xi_2) \equiv (l, \vec{\xi})\), centred on the lens and such that the \( l \)-axis is along the incoming, unperturbed light ray direction \( \vec{e}_m \). The lens plane, \((\xi_1, \xi_2)\), corresponds to \( l = 0 \). The three-dimensional position vector to the light ray \( \vec{x} \) can be written as \( \vec{x} = \vec{\xi} + le_m \).
To calculate the Skrotskii effect to order $\mathcal{O}(\varepsilon^3)$ we can adopt the Born approximation, which assumes that rays of electromagnetic radiation propagate along straight lines, i.e., the bending of the ray may be neglected. The integration along the line of sight (l.o.s.) is accurate enough to evaluate the main contribution to the net rotation. To this order, we can employ the unperturbed Minkowski metric $\eta_{\alpha\beta} = (1, -1, -1, -1)$ and a constant unit propagation vector of the signal, $\hat{k}(0) = (1, 0, 0)$.

The Faraday rotation to order $\mathcal{O}(\varepsilon^3)$ reads

$$\Omega_{sk} \simeq -\frac{2}{c^3} \int_{l.o.s.} \nabla \times \mathbf{V} \big|_{l.o.s.} \, dl_E + \mathcal{O}(\varepsilon^5). \tag{12}$$

Eq. (12) gives the main contribution to the gravitational Faraday rotation.

IV. APPLICATIONS

In this section, we provide explicit formulas for the gravitational Faraday rotation for some relevant astrophysical systems.

A. System of shifting lenses

A system of point-like lenses moving with constant velocities induces a gravitational Faraday rotation. Stars in motion, where the velocities can be treated as slow, provide a suitable representation of such a system.

The gravito-magnetic potential generated by a point-like lens of mass $M_i$ with vector position $\mathbf{x}_i$, shifting with a constant velocity $\mathbf{v}_i$, reduces to

$$\mathbf{V}_i(\mathbf{x}) \simeq -GM_i \frac{\mathbf{v}_i}{|\mathbf{x} - \mathbf{x}_i|}. \tag{13}$$

The rotation angle from a system of $N$ shifting lenses is

$$\Omega_{sk} \simeq -\frac{2}{c^3} \sum_i^N \int_{l.o.s.} \nabla \times \mathbf{V}_i \big|_{l.o.s.} \, dl_E + \mathcal{O}(\varepsilon^5)$$

$$= -\frac{4G}{c^3} \sum_i^N M_i \frac{\Delta \xi_{(i)1} v_{(i)2} - \Delta \xi_{(i)2} v_{(i)1}}{|\Delta \xi_{(i)}|^2} + \mathcal{O}(\varepsilon^5) \tag{14}$$

where $\Delta \xi_{(i)} \equiv \xi - \xi_{(i)}$. To the lowest order, only the projection along the line-of-sight of the total angular momentum enters the effect.
B. Rotating shell

The gravito-magnetic potential takes a very simple form in the case of a spherically symmetric distribution of matter in rigid rotation. We limit to a slow rotation so that the deformation caused by rotation is negligible and the body has a nearly spherical symmetry. Taking the centre of the source as the spatial origin of a background inertial frame, we get

\[ \vec{V} \simeq -\frac{4\pi}{3} G \left\{ \frac{1}{x^3} \int_0^x \rho(r) r^4 dr + \int_x^{+\infty} \rho(r) r dr \right\} \vec{\omega} \times \vec{x}, \]

where \( \vec{\omega} = \text{const.} \) is the angular velocity and

\[ \vec{J}(x) = \frac{8\pi}{3} \left( \int_0^x \rho(r) r^4 dr \right) \vec{\omega} \]

is the angular momentum contributed from the matter within a radius \( x \equiv |\vec{x}| \).

Einstein’s gravitational theory predicts peculiar phenomena inside a rotating shell. It is interesting to calculate the gravito-magnetic potential for such a system. The gravito-magnetic potential in Equation (15), inside a uniform spherical shell of mass \( M \), radius \( R \) and rotating with constant frequency, reduces to (see also [18]):

\[ \vec{V}_{\text{in}}(\vec{x}) \simeq -\frac{GM}{3R} \vec{\omega} \times \vec{x}. \]  

Outside the rotating shell \( (x > R) \),

\[ \vec{V}_{\text{out}}(\vec{x}) \simeq -\frac{GM R^2}{3} \vec{\omega} \times \vec{x} = -\frac{G \vec{J} \times \vec{x}}{2 x^3}, \]

where \( \vec{J} = \frac{2}{3} M R^2 \vec{\omega} \).

It is

\[ \nabla \times \vec{V}_{\text{in}}(\vec{x}) \simeq -\frac{2GM}{3R} \vec{\omega} \]

\[ \nabla \times \vec{V}_{\text{out}}(\vec{x}) \simeq \frac{G}{2} \left[ \frac{\vec{J} - 3(\vec{J} \cdot \hat{x}) \hat{x}}{x^3} \right] \]

Let us consider a light ray which enters the shell, i.e. with impact parameter \( \xi \leq R \); the light ray enters and leaves the sphere at \( l_{\text{in}} = -\sqrt{R^2 - \xi^2} \) and \( l_{\text{out}} = +\sqrt{R^2 - \xi^2} \), respectively. The net rotation of the polarization vector is

\[ \Omega_{\text{Sk}} \simeq -\frac{2}{c^3} \left\{ \int_{-\infty}^{l_{\text{in}}} \nabla \times \vec{V}_{\text{out}} \bigg|_{\text{l.o.s.}} dl_{E} + \int_{l_{\text{in}}}^{l_{\text{out}}} \nabla \times \vec{V}_{\text{in}} \bigg|_{\text{l.o.s.}} dl_{E} + \int_{l_{\text{out}}}^{+\infty} \nabla \times \vec{V}_{\text{out}} \bigg|_{\text{l.o.s.}} dl_{E} \right\} + O(\varepsilon^5) \]

\[ = \frac{4GM}{c^3} \omega_{\text{l.o.s.}} \sqrt{1 - \left( \frac{\xi}{R} \right)^2} + O(\varepsilon^5). \]
The result vanishes if the angular velocity lies in the lens plane. Since the gravitational Faraday rotation outside a rotating body, when the light path does not enter the lens, is \( \sim \frac{G^2 M \text{rot}}{c^2} \), i.e., of order \( \mathcal{O}(\varepsilon^3) \), the effect on the light ray can be neglected at this order of approximation.

The case of a rotating external sphere of finite thickness can be easily solved just integrating the result in Eq. (20). Every infinitesimal shell of radius \( r' \) with mass \( dM = 4\pi \rho(r') r'^2 dr' \) and angular velocity \( \omega(r') \) contributes an angle

\[
    d\Omega_{\text{Sk}} \simeq \frac{16\pi G}{c^3} \rho(r') \omega_{\text{l.o.s.}}(r') \sqrt{r'^2 - \xi^2} r' dr' + \mathcal{O}(\varepsilon^5).
\]  

Integrating from the impact parameter, \( \xi \), to the external shell radius \( R \), we get the total gravitational Faraday rotation which a light ray undergoes because of the spin of the external shell. We get

\[
    \Omega_{\text{Sk}} = \int d\Omega_{\text{Sk}} \\
    \simeq \frac{16\pi G}{c^3} \int_{\xi}^{R} \rho(r') \omega_{\text{l.o.s.}}(r') \sqrt{r'^2 - \xi^2} r' dr' + \mathcal{O}(\varepsilon^5). 
\]  

Let us consider a homogeneous sphere of constant density in rigid rotation. The plane of polarization of a light ray, that penetrates through this rotating body, is rotated of

\[
    \Omega_{\text{Sk}} \simeq \frac{16\pi G}{3c^3} \rho \omega_{\text{l.o.s.}} \left( R^2 - \xi^2 \right)^{3/2} + \mathcal{O}(\varepsilon^5) \\
    = \frac{10G}{c^3} J_{\text{l.o.s.}} \frac{(R^2 - \xi^2)^{3/2}}{R^5} + \mathcal{O}(\varepsilon^5),
\]  

where \( J_{\text{l.o.s.}} \) is the component along the line of sight of the total angular momentum of the sphere (see also [8]).

V. CONCLUSIONS

We have discussed the theory of the gravitational Faraday rotation in the weak field limit. To the lowest order of approximation, only the projection along the line-of-sight of the total angular momentum contributes to the Skrotskii effect.

The rotation angle of the plane of polarization of a linearly polarized electromagnetic wave is of order \( \mathcal{O}(\varepsilon^3) \) in both cases of a system of shifting sources and inside a rotating shell. These models suitably apply to well known gravitational lensing systems. During
microlensing events on the Galactic scale, a star, acting as deflector, moves relatively to a background source. This is the case of a shifting lens.

A distant quasar lensed by a foreground galaxy may form images inside the galaxy radius. In such an astrophysical configuration, photons propagate inside a rotating shell. Since the Faraday rotation due to external rotating shell is of order $O(\varepsilon^3)$, it could induce a detectable effect. High quality data in total flux density, percentage polarization and polarization position angle at radio frequencies already exist for multiple images of some gravitational lensing systems, like B0218+357 [20].

The prospects to detect the gravitational Faraday rotation will be the argument of a forthcoming paper.

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Latin indices run from 1 to 3, whereas Greek indices run from 0 to 3.

The contravariant components of spatial three-vectors are equal to the spatial components of the corresponding four-vectors. Operations on such three-vectors are defined in the three-dimensional space with metric $\gamma_{\alpha\beta}$. 