Global well-posedness for the 3D incompressible inhomogeneous Navier-Stokes equations and MHD equations

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Abstract

The present paper is dedicated to the global well-posedness for the 3D inhomogeneous incompressible Navier-Stokes equations, in critical Besov spaces without smallness assumption on the variation of the density. We aim at extending the work by Abidi, Gui and Zhang (Arch. Ration. Mech. Anal. 204 (1):189–230, 2012, and J. Math. Pures Appl. 100 (1):166–203, 2013) to a more lower regularity index about the initial velocity. The key to that improvement is a new a priori estimate for an elliptic equation with nonconstant coefficients in Besov spaces which have the same degree as $L^2$ in $\mathbb{R}^3$. Finally, we also generalize our well-posedness result to the inhomogeneous incompressible MHD equations.

Key Words: Well-posedness · Navier-Stokes equations · MHD equations · Besov spaces

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1. Introduction and the main results

The first part of this paper is devoted to studying the Cauchy problem of the 3D incompressible inhomogeneous Navier-Stokes equations in critical Besov spaces

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \rho(\partial_t u + u \cdot \nabla u) - \text{div} \left( 2\mu(\rho)M(u) \right) + \nabla \Pi = 0, \\ \text{div } u = 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0). \end{cases}$$

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where $\rho, u = (u_1, u_2, u_3)$, stand for the density, velocity, $\mathcal{M}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$, $\Pi$ is a scalar pressure function, the viscosity coefficient $\mu(\rho)$ is smooth, positive on $[0, \infty)$. This system describes a fluid which is obtained by mixing two miscible fluids that are incompressible and that have different densities. It may also describe a fluid containing a melted substance. One may check [33] for the detailed derivation.

A lot of recent works have been dedicated to the mathematical study of the above system. In the case of smooth data with no vacuum, Ladyženskaja and Solonnikov first addressed in [29] the question of unique solvability of (1.1), similar results were obtained by Danchin [15] in $\mathbb{R}^n$ with initial data in the almost critical Sobolev spaces. Global weak solutions with finite energy were constructed by Simon in [40] (see also the book by Lions [33] for the variable viscosity case). Yet the regularity and uniqueness of such weak solutions are big open problems. Recently, Danchin and Mucha [19] proved by using a Lagrangian approach that the system (1.1) has a unique local solution with initial data $(\rho_0, u_0) \in L^\infty(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$ if initial vacuum does not occur. When the density $\rho$ is away from zero, we denote by $a = \frac{1}{\rho} - 1$ and $\mu(\rho) = 1$ which allow us to work with the following system:

$$\begin{align}
\partial_t a + u \cdot \nabla a &= 0, \\
\partial_t u + u \cdot \nabla u - (1 + a) \Delta u + (1 + a) \nabla \Pi &= 0, \\
\text{div } u &= 0, \\
(a, u)|_{t=0} &= (a_0, u_0).
\end{align}$$

(1.2)

Similar to the classical Navier-Stokes equations, the above system also has a scaling. It is easy to see that the transformations:

$$(a_\lambda, u_\lambda)(t, x) = (a(\lambda^2 \cdot, \lambda \cdot), \lambda u(\lambda^2 \cdot, \lambda \cdot))$$

have that property, provided that the pressure term $\Pi$ and the initial data have been changed accordingly. We can also verify that the space $\dot{B}^{\frac{p}{2p}}_{q,1}(\mathbb{R}^n) \times \dot{B}^{-1+\frac{p}{2}}_{p,1}(\mathbb{R}^n)$ is the critical space for the system.

When the initial density is close enough to a positive constant, Danchin in [14] proved that if initial data $a_0 \in \dot{B}^{\frac{p}{2}}_{2,\infty}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $u_0 \in \dot{B}^{-1+\frac{p}{2}}_{2,1}(\mathbb{R}^n)$, then the system (1.1) has a unique local-in-time solution. This result was improved by Abidi in [1], where he proved that if initial data $a_0 \in \dot{B}^{\frac{p}{2}}_{p,1}(\mathbb{R}^n)$, $u_0 \in \dot{B}^{-1+\frac{p}{2}}_{p,1}(\mathbb{R}^n)$ is small enough $1 < p < 2n$, then (1.2) has a global solution, moreover,
this solution is unique if \(1 < p \leq n\). The results in [1], [14], [18] also were improved by Abidi and Paicu in [4] to a more general case, i.e., if \(a_0 \in \dot{B}^{\frac{3}{4}}_{q,1}(\mathbb{R}^n)\), \(u_0 \in \dot{B}^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^n)\) for \(p, q\) satisfying some technical assumptions. Very recently, Danchin and Mucha [18] improved the uniqueness result in [1] for \(p \in (n, 2n)\) through Lagrange approach. Huang, Paicu and Zhang in [28] first proved the global existence of weak solutions to the system (1.2). The regularity of the initial velocity in [28] is critical to the scaling of this system and is general enough to generate non-Lipschitz velocity fields. Furthermore, with additional regularity assumptions on the initial velocity or on the initial density, they also proved the uniqueness of such a solution. Paicu and Zhang in [35] could also get the global well-posedness only under the assumption that horizontal components of the initial velocity are small exponentially small compared with the third component of the initial velocity. Chemin, Paicu and Zhang in [10] generalized the result in [35] to the critical anisotropic Besov spaces.

When the initial density is not close enough to a positive constant, Abidi, Gui and Zhang in [2] firstly proved the global well-posedness of (1.2) in the energy space if the initial \(\|u_0\|_{\dot{B}^{1+\frac{2}{p}}_{p,1}}\) is small enough. Lately, this results was improved in [3] to the critical \(L^p\) framework. More precisely, Abidi, Gui and Zhang in [3] proved the following theorem:

**Theorem 1.1.** Let \(q \in [1, 2]\), \(p \in [3, 4]\) and \(\frac{1}{p} + \frac{1}{q} > \frac{5}{6} - \frac{1}{3}\), \((a_0, u_0) \in \dot{B}^{\frac{3}{4}}_{q,1}(\mathbb{R}^3) \times \dot{B}^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^3)\) with \(\text{div } u_0 = 0 \) and \(1 + \inf_{x \in \mathbb{R}^3} a_0(x) \geq \kappa > 0\). Then (1.2) has a unique local solution \((a, u, \nabla \Pi)\) on \([0, T]\) such that

\[
a \in C([0, T]; \dot{B}^{\frac{3}{q}}_{q,1}(\mathbb{R}^3)) \cap \dot{L}^\infty_T (\dot{B}^{\frac{3}{q}}_{q,1}(\mathbb{R}^3)), \quad \nabla \Pi \in L^1_T (\dot{B}^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^3)),
\]

\[
u \in C([0, T]; \dot{B}^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^3)) \cap \dot{L}^\infty_T (\dot{B}^{-1+\frac{2}{p}}_{p,1}(\mathbb{R}^3)) \cap L^1_T (\dot{B}^{1+\frac{2}{p}}_{p,1}(\mathbb{R}^3)).
\] (1.3)

Moreover, if there exists a small constant \(c\) depending \(\|a_0\|_{\dot{B}^{\frac{3}{q}}_{q,1}}\) such that

\[
\|u_0\|_{\dot{B}^{-1+\frac{2}{p}}_{p,1}} \leq c,
\]

then (1.2) has a global solution \((a, u, \nabla \Pi)\) such that for any \(t > 0:\)

\[
\|a\|_{\dot{L}^\infty_T (\dot{B}^{\frac{3}{q}}_{q,1})} + \|u\|_{\dot{L}^\infty_T (\dot{B}^{-1+\frac{2}{p}}_{p,1})} + \|u\|_{L^1_T (\dot{B}^{1+\frac{2}{p}}_{p,1})} + \|\nabla \Pi\|_{L^1_T (\dot{B}^{1+\frac{2}{p}}_{p,1})} \\
\leq C (\|a_0\|_{\dot{B}^{\frac{3}{q}}_{q,1}} + \|u_0\|_{\dot{B}^{-1+\frac{2}{p}}_{p,1}}) \exp \left\{ C \exp (C t^\frac{1}{2}) \right\},
\] (1.4)

for some time independent of constant \(C\).
The main purpose of this paper is to improve the well-posedness results from \[2, 3\]. In particular, we want to improve the index \( p \) in \[3\] to an ideal range, i.e. \( 1 < p < 6 \). The main difficulty in \[3\] is to deal with the pressure function. Their main observations are that with \( p, q \) satisfying the restrictions \( p \in [3, 4], q \in [1, 2], \frac{1}{q} + \frac{1}{p} > \frac{5}{6} \) and \( \frac{1}{q} - \frac{1}{p} \leq \frac{1}{3} \), they can use the \( L^2 \) estimate of \( \nabla \Pi \). One can sketch more details about the estimate \( \| \nabla \Pi \|_{L^2} \) in \[2, 3\]. In the present paper, we will instead using \( \| \nabla \Pi \|_{L^2} \) by \( \| \nabla \Pi \|_{B^{\frac{3}{p} - \frac{3}{q} + \frac{1}{2}}_{p,2}} \). This is reasonable for the two spaces have the same degree in \( \mathbb{R}^3 \). By using a new commutator of integral type, we can get the solution mapping \( \mathcal{H}_a : F \mapsto \nabla \Pi \) to the 3D elliptic equation \( \text{div}((1 + a)\nabla \Pi) = \text{div}F \) is bounded on \( B^{\frac{3}{p} - \frac{3}{q}}_{q,1}(\mathbb{R}^3) \). More precisely, we prove, if \( 1 < q \leq p \) with \( p \in \left(1 + \frac{\sqrt{17}}{4}, 1 + \frac{\sqrt{17}}{2}\right) \) and \( \frac{1}{p} + \frac{1}{q} \geq \frac{1}{2}, a \in B^{\frac{3}{q}, 1}_{q,1}(\mathbb{R}^3) \) with \( 1 + a \geq \kappa > 0 \),

\[
\| \nabla \Pi \|_{B^{\frac{3}{p} - \frac{3}{q} + \frac{1}{2}}_{p,2}} \leq C \left(1 + \|a\|_{B^{\frac{3}{q}, 1}_{q,1}}\right) \|F\|_{B^{\frac{3}{p} - \frac{3}{q} + \frac{1}{2}}_{p,2}}.
\]

However, it’s a pity that we still cannot fill the gap \( p \) in \( \left[\frac{5 + \sqrt{17}}{2}, 6\right) \).

The first main result of the present paper is stated in the following theorem:

**Theorem 1.2.** Let \( 1 < q \leq p \) with \( p \in \left(1 + \frac{\sqrt{17}}{4}, 1 + \frac{\sqrt{17}}{2}\right) \) and \( \frac{1}{2} - \frac{1}{p} \leq \frac{1}{q} \leq \frac{1}{p} + \frac{1}{3}, (a_0, u_0) \in B^{\frac{3}{q}, 1}_{q,1}(\mathbb{R}^3) \times B^{-1 + \frac{3}{2}}_{p,1}(\mathbb{R}^3) \) with \( \text{div} u_0 = 0 \) and \( 1 + \inf_{x \in \mathbb{R}^3} a_0(x) \geq \kappa > 0 \). Then \( (1.2) \) has a local solution \((a, u, \nabla \Pi)\) on \([0, T] \) such that

\[
\begin{align*}
a &\in C([0, T]; B^{\frac{3}{q}, 1}_{q,1}(\mathbb{R}^3)) \cap \tilde{L}^\infty_T(B^{\frac{3}{q}, 1}_{q,1}(\mathbb{R}^3)), \ 
\nabla \Pi &\in L^1_T(B^{-1 + \frac{3}{2}}_{p,1}(\mathbb{R}^3)), \\
 u &\in C([0, T]; B^{-1 + \frac{3}{2}}_{p,1}(\mathbb{R}^3)) \cap \tilde{L}^\infty_T(B^{-1 + \frac{3}{2}}_{p,1}(\mathbb{R}^3)) \cap L^1_T(B^{1 + \frac{3}{2}}_{p,1}(\mathbb{R}^3)).
\end{align*}
\]

Especially, if \( a_0 \in B^{\frac{3}{q}, 1}_{q,1}(\mathbb{R}^3), p \in [3, 4], q \in [1, 2], \) this solution is unique. Moreover, if there exists a small constant \( c \) depending \( \|a_0\|_{B^{\frac{3}{q}, 1}_{q,1}} \) such that \( \|u_0\|_{B^{-1 + \frac{3}{2}}_{p,1}} \leq c \),

then \( (1.2) \) has a global solution \((a, u, \nabla \Pi)\) such that for any \( t > 0 \):

\[
\begin{align*}
\|a\|_{\tilde{L}^\infty_T(B^{\frac{3}{q}, 1}_{q,1})} + \|u\|_{L^\infty_T(B^{-1 + \frac{3}{2}}_{p,1})} + \|u\|_{L^1_T(B^{1 + \frac{3}{2}}_{p,1})} + \|\nabla \Pi\|_{L^1_T(B^{1 + \frac{3}{2}}_{p,1})} \\
\lesssim C(\|a_0\|_{B^{\frac{3}{q}, 1}_{q,1}} + \|u_0\|_{B^{-1 + \frac{3}{2}}_{p,1}}) \exp \left\{ C \exp \left( Ct^2 \right) \right\},
\end{align*}
\]

for some time independent of constant \( C \).
Remark 1.3. The existence of our theorem only requires that the initial density belongs to the homogeneous Besov space $\dot{B}_{q,1}^{3,q}(\mathbb{R}^3)$ and also extends the $p,q$ in [3] to be a more larger range. The uniqueness of our theorem has removed the technical assumption $\frac{1}{p} + \frac{1}{q} > \frac{5}{6}$ when $p \in [3,4], q \in [1,2]$ in [3]. The key to this improvement is that we do not need to make the $L^2$ estimate for the pressure. Thus, we will not use the $L^2$ estimate for the nonlinear terms in the momentum equation.

Remark 1.4. Our main result also allows the initial velocity field and initial density function to be highly oscillatory just as in [2,3]. We also generalize the above result to the 3D incompressible inhomogeneous MHD system [21] with variable electrical conductivity which has the following form:

\[
\begin{align*}
\partial_t \rho + u \cdot \nabla \rho &= 0, \\
\rho (\partial_t u + u \cdot \nabla u) - \Delta u + \nabla \Pi &= B \cdot \nabla B, \\
\partial_t B + u \cdot \nabla B - \text{div} \left( \frac{\nabla B}{\sigma(\rho)} \right) &= B \cdot \nabla u, \\
\text{div} u &= \text{div} B = 0, \\
\rho(u,B)|_{t=0} &= (\rho_0,u_0,B_0),
\end{align*}
\]  
(1.7)

where $\rho$ is the density and $u$ is the velocity field, $B$ is the magnetic field, $\mathcal{M}(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ is the symmetrical part of the gradient, $\Pi(x,t)$ is the scalar pressure, $\sigma(\rho) > 0$ is the electrical conductivity of the field. Moreover, we suppose that $\sigma(\rho)$ is a $C^\infty$ function and that $0 < \tilde{\sigma} \leq \frac{1}{\sigma(\rho)} \leq \hat{\sigma} < \infty$.

Similar to (1.2), when the density $\rho$ is away from zero, we can also use the transform $a = \frac{1}{\rho} - 1$ to change (1.7) into the following system:

\[
\begin{align*}
\partial_t a + u \cdot \nabla a &= 0, \\
\partial_t u + u \cdot \nabla u - (1 + a)\Delta u + (1 + a)\nabla \Pi &= (1 + a)B \cdot \nabla B, \\
\partial_t B + u \cdot \nabla B - \text{div} (\tilde{\sigma}(a) \nabla B) &= B \cdot \nabla u, \\
\text{div} u &= \text{div} B = 0, \\
(a,u,B)|_{t=0} &= (a_0,u_0,B_0),
\end{align*}
\]  
(1.8)

where $\tilde{\sigma}(a) = \frac{1}{\sigma(1+a)}$ is a smooth function.

Compared with the Navier-Stokes equations, the dynamic motion of the fluid and the magnetic field interact on each other and both the hydrodynamic and electrodynamic effects in the motion are strongly coupled, the problems of MHD system are considerably more complicated. Even through,
in the past several years, there are also many mathematical results related to the incompressible MHD system (see [8], [22], [23], [26], [32], [34], [39]). For the homogeneous viscous incompressible MHD system (i.e. $\rho(t, x) = \text{constant}$), Duvaut and Lions [23] established the local existence and uniqueness of solution in the classical Sobolev space $H^s(\mathbb{R}^n)$, $s \geq n$, they also proved the global existence of solutions to this system with small initial data. Sermange and Temam [39] proved the global unique solution in $\mathbb{R}^2$. With mixed partial dissipation and additional magnetic diffusion in the two-dimensional MHD system, Cao and Wu [9] proved that such a system is globally well-posed for any data in $H^2(\mathbb{R}^2)$. In a recent remarkable paper Lin, Xu and Zhang [31] proved the global existence of smooth solution of the 2-D MHD system around the trivial solution $(x_2, 0)$ (see [32] for 3-D case). In [38], Ren, Wu, Xiang and Zhang got the global existence and the decay estimates of small smooth solution for the 2-D MHD equations without magnetic diffusion. When the fluid is nonhomogeneous, Gerbeau and Le Bris [24] (see also Desjardins and Le Bris [22]) studied the global existence of weak solutions of finite energy in the whole space or in the torus. Abidi and Paicu [5] established the global existence of strong solutions with small initial data in the critical Besov spaces. Moreover, they allowed variable viscosity and conductivity coefficients but required an essential assumption that there is no vacuum (more precisely, the initial data are closed to a constant state). Zhai, Li and Yan in [42] also considered the global well-posedness for (1.2) in critical Besov spaces. By using the Gagliardo-Nirenberg inequality, they obtained the global existence for this system without any small conditions imposed on the third components of the initial velocity field and magnetic field, which can be regarded as an improvement of [5]. Chen, Tan, and Wang [12] extended the local existence in presence of vacuum by using the Galerkin method, energy method and the domain expansion technique. Lately, with initial data satisfying some compatibility conditions, by using a critical Sobolev inequality of logarithmic type, Huang and Wang [27] got the global strong solution to the 2-D nonhomogeneous incompressible MHD system. Recently, Gui [25] studied the Cauchy problem of the 2-D magnetohydrodynamic system with inhomogeneous density and electrical conductivity. He showed that this system with a constant viscosity is globally well-posed for a generic family of the variations of the initial data and an inhomogeneous electrical conductivity. Moreover, He established that the system is globally well-posed if the electrical conductivity is homogeneous.
Theorem 1.5. Let $1 < q \leq p$ with $p \in (1, \frac{5 + \sqrt{17}}{2})$ and $\frac{1}{q} - \frac{1}{p} \leq \frac{1}{4} - \frac{1}{p} + \frac{1}{q}$, $(a_0, u_0, B_0) \in \dot{B}_{p,1}^{\frac{3}{p}}(\mathbb{R}^3) \times \dot{B}_{p,1}^{-1 + \frac{3}{p}}(\mathbb{R}^3) \times \dot{B}_{p,1}^{-1 + \frac{3}{p}}(\mathbb{R}^3)$ with $\| u_0 - B_0 = 0$ and $1 + \inf_{x \in \mathbb{R}^3} a_0(x) \geq \kappa > 0$. Assume that $\sigma(a)$ is a smooth, positive function on $[0, \infty)$. Then (1.8) has a local solution $(a, u, B, \nabla \Pi)$ on $[0, T]$ such that

$$a \in \mathcal{C}([0, T]; \dot{B}_{q,1}^{\frac{3}{q}}(\mathbb{R}^3)) \cap \dot{L}_{T}^{\infty}(\dot{B}_{q,1}^{\frac{3}{q}}(\mathbb{R}^3)), \quad \nabla \Pi \in L_{T}^{1}(\dot{B}_{p,1}^{-1 + \frac{3}{p}}(\mathbb{R}^3)),
$$

$$u \in \mathcal{C}([0, T]; \dot{B}_{p,1}^{-1 + \frac{3}{p}}(\mathbb{R}^3)) \cap \dot{L}_{T}^{\infty}(\dot{B}_{p,1}^{-1 + \frac{3}{p}}(\mathbb{R}^3)), \quad L_{T}^{1}(\dot{B}_{p,1}^{-1 + \frac{3}{p}}(\mathbb{R}^3)),
$$

$$B \in \mathcal{C}([0, T]; \dot{B}_{p,1}^{-1 + \frac{3}{p}}(\mathbb{R}^3)) \cap \dot{L}_{T}^{\infty}(\dot{B}_{p,1}^{-1 + \frac{3}{p}}(\mathbb{R}^3)), \quad L_{T}^{1}(\dot{B}_{p,1}^{-1 + \frac{3}{p}}(\mathbb{R}^3)).$$

(1.9)

Especially, if $a_0 \in \dot{B}_{q,1}^{\frac{3}{q}}(\mathbb{R}^3)$, $p \in [3, 4], q \in [1, 2]$, this solution is unique. Moreover, when $\sigma(a)$ is a positive constant, and there exists a small constant $c$ depending $\| a_0 \|_{\dot{B}_{q,1}^{\frac{3}{q}}}$ such that

$$\| u_0 \|_{\dot{B}_{p,1}^{-1 + \frac{3}{p}}} + \| B_0 \|_{\dot{B}_{p,1}^{-1 + \frac{3}{p}}} \leq c,$$

then (1.8) has a global solution $(a, u, B, \nabla \Pi)$ such that for any $t > 0$:

$$\| a \|_{L_{T}^{\infty}(\dot{B}_{p,1}^{\frac{3}{p}})} + \| (u, B) \|_{L_{T}^{\infty}(\dot{B}_{p,1}^{-1 + \frac{3}{p}})} + \| (u, B) \|_{L_{T}^{1}(\dot{B}_{p,1}^{-1 + \frac{3}{p}})} + \| \nabla \Pi \|_{L_{T}^{1}(\dot{B}_{p,1}^{-1 + \frac{3}{p}})} \lesssim C(\| a_0 \|_{\dot{B}_{p,1}^{\frac{3}{p}}} + \| u_0 \|_{\dot{B}_{p,1}^{-1 + \frac{3}{p}}} + \| B_0 \|_{\dot{B}_{p,1}^{-1 + \frac{3}{p}}}) \exp \left\{ C \exp \left( Ct^\frac{1}{2} \right) \right\},$$

(1.10)

for some time independent of constant $C$.

Remark 1.6. When the magnetic $B = 0$, Theorem 1.5 coincides with Theorem 1.2. Thus, in the following, we only take the a priori estimate for the system (1.8), the local existence, uniqueness, and the global solution under the small initial data of Theorem 1.2 will be proved together with Theorem 1.5.

Remark 1.7. By the embedding relation, in what follows we only concern the case $p \in [3, \frac{5 + \sqrt{17}}{2})$, the case where $1 < p < 3$ being easier.

The paper is organized as follows. In the second section, we shall first collect some basic facts on Littlewood-Paley theory and various product laws and commutator’s estimates in Besov spaces; then present the estimates to the free transport equation and heat equation. In Section 3, we use a new commutator of integral type to give a new elliptic estimates with the nonconstant coefficients.
In Section 4, we apply the elliptic estimates with the nonconstant coefficients to get the linear estimates of the inhomogeneous Navier-Stokes-type equations and MHD equations respectively. With these estimates in hand, we shall proved the local well-posedness part of Theorem 1.5 in Section 5. In Section 6, we present the proof of the global existence part of Theorem 1.5.

Let us complete this section by describing the notations which will be used in the sequel. Notations: For two operators $A$ and $B$, we denote $[A, B] = AB - BA$, the commutator between $A$ and $B$. The letter $C$ stands for a generic constant whose meaning is clear from the context. We write $a \lesssim b$ instead of $a \leq Cb$. Given a Banach space $X$, we shall denote by $\langle a, b \rangle$ the $L^2(\mathbb{R}^2)$ inner product of $a$ and $b$, and $\| (a, b) \|_X = \|a\|_X + \|b\|_X$.

For $X$ a Banach space and $I$ an interval of $\mathbb{R}$, we denote by $C(I; X)$ the set of continuous functions on $I$ with values in $X$, and by $C^b(I; X)$ the subset of bounded functions of $C(I; X)$. For $q \in [1, +\infty]$, $L^q(I; X)$ stands for the set of measurable functions on $I$ with values in $X$, such that $t \mapsto \|f(t)\|_X$ belongs to $L^q(I)$. For short, we write $L^q_T(X)$ instead of $L^q((0, T); X)$. We always let $(d_j)_{j \in \mathbb{Z}}$ be a generic element of $l^1(\mathbb{Z})$ so that $\sum_{j \in \mathbb{Z}} d_j = 1$, and $(c_{j,r})_{j \in \mathbb{Z}}$ to be a generic element of $l^r(\mathbb{Z})$ so that $c_{j,r} \geq 0$ and $\sum_{j \in \mathbb{Z}} c_{j,r} = 1$.

2. Preliminaries

Let $(\chi, \phi)$ be two smooth radial functions, $0 \leq (\chi, \phi) \leq 1$, such that $\chi$ is supported in the ball $B = \{\xi \in \mathbb{R}^3, |\xi| \leq \frac{4}{3}\}$ and $\phi$ is supported in the ring $C = \{\xi \in \mathbb{R}^3, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$. Moreover, there hold

$$\forall \xi \in \mathbb{R}^3 \setminus \{0\}, \quad \sum_{j \in \mathbb{Z}} \phi(2^{-j} \xi) = 1,$$

$$\forall \xi \in \mathbb{R}^3, \quad \chi(\xi) + \sum_{j \geq 0} \phi(2^{-j} \xi) = 1.$$

Let $h = \mathcal{F}^{-1} \phi$ and $\tilde{h} = \mathcal{F}^{-1} \chi$, the inhomogeneous dyadic blocks $\Delta_j$ are defined as follows:

$$\text{if } j \leq -2, \quad \Delta_j f = 0,$$

$$\text{if } j = -1, \quad \Delta_j f = \Delta_{-1} f = \int_{\mathbb{R}^3} \tilde{h}(y) f(x - y) dy,$$

$$\text{if } j \geq 0, \quad \Delta_j f = 2^j \int_{\mathbb{R}^3} h(2^j y) f(x - y) dy.$$
The inhomogeneous low-frequency cut-off operator \( S_j \) is defined by

\[
S_j f = \sum_{j' \leq j-1} \Delta_j f .
\]

For \( j \in \mathbb{Z} \), the homogeneous dyadic blocks \( \hat{\Delta}_j \) and the homogeneous low-frequency cut-off operator \( \hat{S}_j \) are defined as follows:

\[
\hat{\Delta}_j f = \varphi(2^{-j}D)f = 2^{3j} \int_{\mathbb{R}^3} h(2^jy)f(x-y)dy ,
\]

\[
\hat{S}_j f = \chi(2^{-j}D)f = 2^{3j} \int_{\mathbb{R}^3} \tilde{h}(2^jy)f(x-y)dy .
\]

Denote by \( \mathcal{S}'_h(\mathbb{R}^3) \) the space of tempered distributions \( f \) such that

\[
\lim_{j \to -\infty} \hat{S}_j f = 0 \text{ in } \mathcal{S}'(\mathbb{R}^3).
\]

Then we have the formal decomposition

\[
f = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j f , \quad \forall f \in \mathcal{S}'_h(\mathbb{R}^3).
\]

Now we recall the definition of homogeneous Besov spaces.

**Definition 2.1.** Let \((p,r) \in [1, +\infty]^2\), \( s \in \mathbb{R} \) and \( u \in \mathcal{S}'_h(\mathbb{R}^3) \), which means that \( u \in \mathcal{S}'(\mathbb{R}^3) \) and \( \lim_{j \to -\infty} \| \hat{S}_j u \|_{L^\infty} = 0 \) (see Definition 1.26 of [6]); we set

\[
\| u \|_{\dot{B}^s_{p,r}(\mathbb{R}^3)} \triangleq (2^{js} \| \Delta_j u \|_{L^p})_r .
\]

- For \( s < \frac{3}{p} \) (or \( s = \frac{3}{p} \) if \( r = 1 \)), we define \( \dot{B}^s_{p,r}(\mathbb{R}^3) \triangleq \{ u \in \mathcal{S}'_h(\mathbb{R}^3) \mid \| u \|_{\dot{B}^s_{p,r}} < \infty \} \).
- If \( k \in \mathbb{N} \) and \( \frac{3}{p} + k \leq s < \frac{3}{p} + k + 1 \) (or \( s = \frac{3}{p} + k + 1 \) if \( r = 1 \)), then \( \dot{B}^s_{p,r}(\mathbb{R}^3) \) is defined as the subset of distributions \( u \in \mathcal{S}'_h(\mathbb{R}^3) \) such that \( \partial^\beta u \in \dot{B}^{s-k}_{p,r} (\mathbb{R}^3) \).

We also define the inhomogeneous Besov space \( \dot{B}^s_{p,r}(\mathbb{R}^3) \) as the space of those distributions \( u \in \mathcal{S}'_h(\mathbb{R}^3) \) such that

\[
\| u \|_{\dot{B}^s_{p,r}} \triangleq (2^{js} \| \Delta_j u \|_{L^p})_r < \infty .
\]

We are going to define the space of Chemin-Lerner (see [6]) in which we will work, which is a refinement of the space \( L^1_T(\dot{B}^s_{p,r}(\mathbb{R}^3)) \).
Definition 2.2. (see [6]) Let \( s \leq \frac{3}{p} \) (respectively \( s \in \mathbb{R} \)), \((r, \lambda, p) \in [1, +\infty]^3 \) and \( T \in (0, +\infty] \). We define \( \widetilde{L}^s_T(\dot{B}^p_{r,p}(\mathbb{R}^3)) \) as the completion of \( C([0,T]; S(\mathbb{R}^3)) \) by the norm

\[
\| f \|_{\widetilde{L}^s_T(\dot{B}^p_{r,p})} = \left\{ \sum_{q \in \mathbb{Z}} 2^{qs} \left( \int_0^T \| \hat{\Delta}_q f(t) \|_{\dot{L}^p_r}^2 \, dt \right)^{\frac{1}{2}} \right\}^{\frac{1}{s}} < \infty,
\]

with the usual change if \( r = \infty \). For short, we just denote this space by \( \widetilde{L}^s_T(\dot{B}^p_{r,p}) \).

Remark 2.3. It is easy to observe that for \( 0 < s_1 < s_2, \theta \in [0,1] \), \( p, r, \lambda, \lambda_1, \lambda_2 \in [1, +\infty] \), we have the following interpolation inequality in the Chemin-Lerner space (see [6]):

\[
\| u \|_{\dot{L}^s_T(\dot{B}^p_{r,p})} \leq \| u \|_{\dot{L}^3_T(\dot{B}^p_{3,p})} \| u \|_{(1-\theta) L^{s_2}_T(\dot{B}^{s_2}_{p_2,p_2})}^{\theta} + \| u \|_{\dot{L}^3_T(\dot{B}^p_{3,p})}^{1-\theta},
\]

with \( \frac{1}{s} = \frac{\theta}{s_2} + \frac{1-\theta}{s_1} \) and \( s = \theta s_1 + (1-\theta)s_2 \).

Let us emphasize that, according to the Minkowski inequality, we have

\[
\| u \|_{\dot{L}^3_T(\dot{B}^p_{3,p})} \leq \| u \|_{\dot{L}^1_T(\dot{B}^p_{1,p})}, \quad \text{if } \lambda \leq r,
\]

\[
\| u \|_{\dot{L}^3_T(\dot{B}^p_{3,p})} \geq \| u \|_{\dot{L}^1_T(\dot{B}^p_{1,p})}, \quad \text{if } \lambda \geq r.
\]

The following Bernstein’s lemma will be repeatedly used throughout this paper.

Lemma 2.4. (see [6]) Let \( B \) be a ball and \( C \) a ring of \( \mathbb{R}^3 \). A constant \( C \) exists so that for any positive real number \( \lambda \), any non-negative integer \( k \), any smooth homogeneous function \( \sigma \) of degree \( m \), and any couple of real numbers \((a,b)\) with \( 1 \leq a \leq b \), there hold

\[
\text{Supp } \hat{u} \subset \lambda B \Rightarrow \sup_{|a|=k} \| \partial^a u \|_{L^b} \leq C^{k+1} \lambda^{k+3(1/a-1/b)} \| u \|_{L^b},
\]

\[
\text{Supp } \hat{u} \subset \lambda C \Rightarrow C^{-k-1} \lambda^k \| u \|_{L^b} \leq \sup_{|a|=k} \| \partial^a u \|_{L^b} \leq C^{k+1} \lambda^k \| u \|_{L^b},
\]

\[
\text{Supp } \hat{u} \subset \lambda C \Rightarrow \| \sigma(D) u \|_{L^b} \leq C_{\sigma,m} \lambda^{m+3(1/a-1/b)} \| u \|_{L^b}.
\]

In the sequel, we shall frequently use Bony’s decomposition from [7] in the homogeneous context:

\[
uv = T_u v + T_v u + R(u,v) = T_u v + \mathcal{R}(u,v),
\]

where

\[
T_u v \triangleq \sum_{j \in \mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v, \quad R(u,v) \triangleq \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \Delta_j v,
\]

(2.1)
and
\[ \tilde{\Delta}_j v \triangleq \sum_{|j-j'| \leq 1} \tilde{\Delta}_{j'} v, \quad \tilde{R}(u,v) \triangleq \sum_{j \in \mathbb{Z}} \tilde{\Delta}_{j+2} v \tilde{\Delta}_j u. \]

As an application of the above basic facts on the Littlewood-Paley theory, we present the following different product laws in Besov spaces.

**Lemma 2.5.** Let \( 1 \leq p, q \leq \infty, s_1 \leq \frac{3}{q}, s_2 \leq 3 \min\{\frac{1}{p}, \frac{1}{q}\} \) and \( s_1 + s_2 > 3 \max\{0, \frac{1}{p} + \frac{1}{q} - 1\} \). For \( \forall (a, b) \in \dot{B}^s_{q,1}(\mathbb{R}^3) \times \dot{B}^p_{p,1}(\mathbb{R}^3) \), we have
\[
\|ab\|_{\dot{B}^{s_1+s_2-\frac{3}{q}}_{p,1}} \lesssim \|a\|_{\dot{B}^{s_1}_{q,1}} \|b\|_{\dot{B}^{s_2}_{p,1}}. \tag{2.2}
\]

**Proof.** This lemma is proved in \[35\] in the case when \( q \leq p \). We shall only prove (2.2) for the case \( q > p \). Applying Bony’s decomposition, we have
\[ ab = \tilde{T}_a b + T_b a + \tilde{R}(a,b). \]

Then applying Lemma 2.4, we get for \( s_1 \leq \frac{3}{q} \)
\[
\|\tilde{\Delta}_j (\tilde{T}_a b)\|_{L^p} \lesssim \sum_{|j-j'| \leq 1} \|\tilde{\Delta}_{j'} a\|_{L^\infty} \|	ilde{\Delta}_j b\|_{L^p} \lesssim d_j 2^{-j(s_1+s_2-\frac{2}{q})}\|a\|_{\dot{B}^{s_1}_{q,1}} \|b\|_{\dot{B}^{s_2}_{p,1}},
\]
and for \( s_2 \leq \frac{3}{q} \)
\[
\|\tilde{\Delta}_j (T_b a)\|_{L^p} \lesssim \sum_{|j-j'| \leq 1} \|\tilde{\Delta}_{j'} a\|_{L^\infty} \|	ilde{\Delta}_j b\|_{L^p} \lesssim d_j 2^{-j(s_1+s_2-\frac{2}{q})}\|a\|_{\dot{B}^{s_1}_{q,1}} \|b\|_{\dot{B}^{s_2}_{p,1}}.
\]

If \( \frac{1}{p} + \frac{1}{q} \geq 1 = \frac{1}{p} + \frac{1}{q} \), we infer
\[
\|\tilde{\Delta}_j (\tilde{R}(a,b))\|_{L^p} \lesssim d_j 2^{3(1-\frac{1}{p})} \sum_{j' \geq j-3} \|\tilde{\Delta}_{j'} a\|_{L^p} \|	ilde{\Delta}_j b\|_{L^p} \lesssim d_j 2^{3(1-\frac{1}{p})} \|a\|_{\dot{B}^{s_1}_{q,1}} \|b\|_{\dot{B}^{s_2}_{p,1}} \sum_{j' \geq j-3} d_j 2^{-j'(s_1+s_2-3(\frac{1}{p}+\frac{1}{q}-1))} \lesssim d_j 2^{-j(s_1+s_2-\frac{3}{q})} \|a\|_{\dot{B}^{s_1}_{q,1}} \|b\|_{\dot{B}^{s_2}_{p,1}},
\]
for \( s_1 + s_2 > 3(\frac{1}{p} + \frac{1}{q} - 1) \). Finally, in the case when \( \frac{1}{p} + \frac{1}{q} \) def \( \frac{1}{r} \) \( < 1 \), noticing that \( s_1 + s_2 > 0 \), one has
\[
\|\tilde{\Delta}_j (\tilde{R}(a,b))\|_{L^p} \lesssim d_j 2^{3(\frac{1}{r}-\frac{1}{p})} \sum_{j' \geq j-3} \|\tilde{\Delta}_{j'} a\|_{L^r} \|	ilde{\Delta}_j b\|_{L^p} \lesssim d_j 2^{-j(s_1+s_2-\frac{3}{q})} \|a\|_{\dot{B}^{s_1}_{q,1}} \|b\|_{\dot{B}^{s_2}_{p,1}}.
\]
This completes the proof of the lemma. \( \square \)
Lemma 2.8. (see [6]) Let
\[ \| ab \|_{B^{s}_{p,1}^{1+\frac{2}{p}}} \lesssim \| a \|_{B^{s}_{q,1}^{1+\frac{2}{p}}} \| b \|_{B^{s}_{q,1}^{1+\frac{2}{p}}}, \quad \| ab \|_{B^{s-1}_{p,1}^{1+\frac{2}{p}}} \lesssim \| a \|_{B^{s-1}_{q,1}^{1+\frac{2}{p}}} \| b \|_{B^{s-1}_{q,1}^{1+\frac{2}{p}}}. \] (2.3)

(2) Let $1 < q \leq p$ and $\frac{1}{p} + \frac{1}{q} \geq \frac{1}{2}$, then there hold
\[ \| ab \|_{B^{s}_{p,1}^{1+\frac{2}{p}}} \lesssim \| a \|_{B^{s}_{q,1}^{1+\frac{2}{p}}} \| b \|_{B^{s}_{q,1}^{1+\frac{2}{p}}}, \quad \| ab \|_{B^{s-1}_{p,1}^{1+\frac{2}{p}}} \lesssim \| a \|_{B^{s-1}_{q,1}^{1+\frac{2}{p}}} \| b \|_{B^{s-1}_{q,1}^{1+\frac{2}{p}}}. \] (2.4)

Lemma 2.7. Let $1 < q \leq p$ and $\frac{1}{p} + \frac{1}{q} \geq \frac{1}{2}$, then there hold
\[ \| \mathcal{P}(a \nabla \Pi) \|_{B^{s}_{p,1}^{1+\frac{2}{p}}} \lesssim \| \nabla a \|_{B^{s}_{q,1}^{1+\frac{2}{p}}} \| \nabla \Pi \|_{B^{s-1}_{p,1}^{1+\frac{2}{p}}}, \] (2.5)
\[ \| \mathcal{P}(a \nabla \Pi) \|_{L^{q}_{t}(B^{s-1}_{p,1}^{1+\frac{2}{p}})} \lesssim \| a \|_{B^{q}_{r,1}^{1+\frac{2}{r}}} \| \nabla \Pi \|_{L^{q}_{t}(B^{s-1}_{p,2}^{1+\frac{2}{p}})} . \]

Proof. We only treat the second inequality, since the proof of the first inequality is similar. For
\[ \mathcal{P}(a \nabla \Pi) = \mathcal{P}(\nabla(a \Pi) - \nabla \Pi a) = \mathcal{P}(\nabla a \Pi), \] thus by Lemma 2.5 we have
\[ \| \mathcal{P}(a \nabla \Pi) \|_{L^{q}_{t}(B^{s-1}_{p,1}^{1+\frac{2}{p}})} \lesssim \| \mathcal{P}(\nabla a \Pi) \|_{L^{q}_{t}(B^{s-1}_{p,1}^{1+\frac{2}{p}})} \lesssim \| \nabla a \|_{B^{q}_{r,1}^{1+\frac{2}{r}}} \| \nabla \Pi \|_{L^{q}_{t}(B^{s-1}_{p,2}^{1+\frac{2}{p}})} . \] (2.6)

from which the desired inequality follows.

Let us also recall the following commutator estimate from [6].

Lemma 2.8. (see [6]) Let $1 \leq p, q \leq \infty$, $-3 \min\{\frac{1}{p}, 1 - \frac{1}{q}\} < s \leq 1 + 3 \min\{\frac{1}{p}, \frac{1}{q}\}$, \( \nabla a \in \dot{B}^{\frac{3}{p}+\frac{s}{2}}_{p,1}(\mathbb{R}^3) \) and \( b \in \dot{B}^{s-1}_{q,1}(\mathbb{R}^3) \). Then there holds
\[ \| [\dot{\Delta}_j, a] b \|_{L^q} \lesssim d_j 2^{-js} \| \nabla a \|_{B^{\frac{3}{p}+\frac{s}{2}}_{p,1}} \| b \|_{B^{s-1}_{q,1}} . \]

Lemma 2.9. Let $1 < q \leq p < 6$, \( a \in \dot{B}^{\frac{3}{q}+\frac{s}{2}}_{q,1}(\mathbb{R}^3) \) and \( \nabla \Pi \in \dot{B}^{\frac{3}{p}+\frac{s}{2}}_{p,2}(\mathbb{R}^3) \). Then there holds
\[ \| [\dot{\Delta}_j, a] \nabla \Pi \|_{L^p} \lesssim 2^{-\left(\frac{3}{p}-1\right)j} d_j \| a \|_{\dot{B}^{\frac{3}{q}+\frac{s}{2}}_{q,1}} \| \nabla \Pi \|_{\dot{B}^{\frac{3}{p}+\frac{s}{2}}_{p,2}} . \] (2.7)

Proof. Taking advantage of the Bony’s decomposition (2.1), we rewrite the commutator as:
\[ [\dot{\Delta}_j, a] \nabla \Pi = \dot{\Delta}_j (a \nabla \Pi) - a \dot{\Delta}_j \nabla \Pi = [\dot{\Delta}_j, T_a] \nabla \Pi + \dot{\Delta}_j (T_{\nabla \Pi} a) - T_{\nabla \Pi} \dot{\Delta}_j a. \] (2.8)
By the definition of Bony’s decomposition again, we have

\[
[\hat{\Delta}_j, \tilde{T}_a] \nabla \Pi = -2^{3j} \sum_{|j' - j| \leq 4} \int_{\mathbb{R}^3} h(2^jy) \hat{\Delta}_{j'} \nabla \Pi(x - y)dy \int_0^1 y \cdot \nabla \hat{S}_{j'-1}a(x - \tau y)d\tau,
\]

from which we can get by Lemma 2.4 and the Hölder inequality that

\[
\| [\hat{\Delta}_j, \tilde{T}_a] \nabla \Pi \|_{L^p} \lesssim \sum_{|j' - j| \leq 4} \| \nabla \hat{S}_{j'-1}a \|_{L^\infty} \| \hat{\Delta}_{j'} \nabla \Pi \|_{L^p}
\]

\[
\lesssim \sum_{|j' - j| \leq 4} ( \sum_{k \leq j' - 2} \| \hat{\Delta}_k a \|_{L^4} 2^\frac{3k}{4} ) \| \hat{\Delta}_{j'} \nabla \Pi \|_{L^p}
\]

\[
\lesssim c_j 2^{-\frac{(j' - j)}{2}} \| a \|_{B^{\frac{3}{2} + \frac{3}{4}, \frac{3}{4}}_{q, 2}} \| \nabla \Pi \|_{B^{\frac{3}{2} + \frac{3}{4}, \frac{3}{4}}_{p, 2}}.
\]

(2.9)

When \(1 < q \leq p \leq 2\), we get from

\[
\hat{\Delta}_j(\tilde{T}_a' \nabla \Pi) = \sum_{j' \geq j - 3} \hat{\Delta}_j(\hat{\Delta}_{j'} a \hat{S}_{j'+2} \nabla \Pi)
\]

that

\[
\| \hat{\Delta}_j(\tilde{T}_a' \nabla \Pi) \|_{L^p} \lesssim 2^{3j(1 - \frac{1}{p})} \sum_{j' \geq j - 3} \| \hat{\Delta}_{j'} a \|_{L^p} \| \hat{S}_{j'+2} \nabla \Pi \|_{L^\infty}
\]

\[
\lesssim 2^{3j(1 - \frac{1}{p})} \sum_{j' \geq j - 3} c_{j'} 2^j 2^{\frac{3(j' - j)}{4} + \frac{3}{4}} \| \hat{\Delta}_{j'} a \|_{L^p} ( \sum_{k \leq j' + 1} \| \hat{\Delta}_k \nabla \Pi \|_{L^p} 2^\frac{6k}{p - 3})
\]

\[
\lesssim 2^{3j(1 - \frac{1}{p})} \sum_{j' \geq j - 3} c_{j'} 2^j 2^{-\frac{(j' - j)}{2}} \| a \|_{B^{\frac{3}{2} + \frac{3}{4}, \frac{3}{4}}_{q, 2}} ( \sum_{k \leq j' + 1} c_{k, 2} \| \nabla \Pi \|_{B^{\frac{3}{2} + \frac{3}{4}, \frac{3}{4}}_{p, 2}} 2^\frac{(j' - j)}{2} 2^\frac{k}{3})
\]

\[
\lesssim d_j 2^{-\frac{(j' - j)}{2}} \| a \|_{B^{\frac{3}{2} + \frac{3}{4}, \frac{3}{4}}_{q, 2}} \| \nabla \Pi \|_{B^{\frac{3}{2} + \frac{3}{4}, \frac{3}{4}}_{p, 2}}.
\]

(2.10)

Similarly, when \(1 < q \leq p \in [2, 6)\), we have

\[
\| \hat{\Delta}_j(\tilde{T}_a' \nabla \Pi) \|_{L^p} \lesssim 2^{3j} \sum_{j' \geq j - 3} \| \hat{\Delta}_{j'} a \|_{L^p} \| \hat{S}_{j'+2} \nabla \Pi \|_{L^p}
\]

\[
\lesssim 2^{3j} \sum_{j' \geq j - 3} c_{j'} 2^{3j(1 - \frac{1}{p})} \| \hat{\Delta}_{j'} a \|_{L^p} ( \sum_{k \leq j' + 1} \| \hat{\Delta}_k \nabla \Pi \|_{L^p})
\]

\[
\lesssim 2^{3j} \sum_{j' \geq j - 3} c_{j'} 2^j 2^{-\frac{(j' - j)}{2}} \| a \|_{B^{\frac{3}{2} + \frac{3}{4}, \frac{3}{4}}_{q, 2}} ( \sum_{k \leq j' + 1} c_{k, 2} \| \nabla \Pi \|_{B^{\frac{3}{2} + \frac{3}{4}, \frac{3}{4}}_{p, 2}} 2^\frac{(j' - j)}{2} 2^\frac{k}{3})
\]

\[
\lesssim 2^{3j} \sum_{j' \geq j - 3} d_j 2^{(1 - \frac{1}{p})} \| a \|_{B^{\frac{3}{2} + \frac{3}{4}, \frac{3}{4}}_{q, 2}} \| \nabla \Pi \|_{B^{\frac{3}{2} + \frac{3}{4}, \frac{3}{4}}_{p, 2}}
\]

\[
\lesssim d_j 2^{-\frac{(j' - j)}{2}} \| a \|_{B^{\frac{3}{2} + \frac{3}{4}, \frac{3}{4}}_{q, 2}} \| \nabla \Pi \|_{B^{\frac{3}{2} + \frac{3}{4}, \frac{3}{4}}_{p, 2}}.
\]

(2.11)

For the last term on the right hand side of (2.9), we write

\[
\tilde{T}_a' \nabla \Pi = \sum_{j' \geq j - 3} \hat{\Delta}_j \hat{\Delta}_{j'} a \hat{S}_{j'+2} \nabla \Pi.
\]
Thus

\[ \| T'_{\Delta_j} \nabla^2 a \|_{L^p} \lesssim \sum_{j' \geq j - 3} \| \Delta_j a \|_{L^2} \| S_{j' + 2} \Delta_j \nabla^2 \|_{L^p} \]
\[ \lesssim \sum_{j' \geq j - 3} 2^{-\frac{j'}{2}} \| \Delta_j a \|_{L^2} \| \Delta_j \nabla^2 \|_{L^p} \]
\[ \lesssim \sum_{j' \geq j - 3} c_{j,2} 2^{-\frac{j'}{2}} \| a \|_{B_{p/2,1}^{3/2}} \| \nabla \|_{B_{p/2,1}^{3/2}} \]
\[ \lesssim d_1 2^{-\frac{(p-1)}{2}} \| a \|_{B_{p/2,1}^{3/2}} \| \nabla \|_{B_{p/2,1}^{3/2}} \cdot \] \tag{2.12}

Combining with the above estimates (2.8)–(2.12), we can complete the proof of (2.7). \qed

**Proposition 2.10. (see [2])** Let \( 1 \leq q \leq p \leq 6 \) with \( \frac{1}{q} - \frac{1}{p} \leq \frac{1}{3} \), and \( m \in \mathbb{Z} \), \( a_0 \in \dot{B}_{q,1}^{3/2} \), \( \nabla u \in L_t^1(B_{p,1}^{3/2}) \) with \( \text{div} u = 0 \), and \( a \in C([0, T]; \dot{B}_{q,1}^{3/2}) \) such that \((a, u)\) solves

\[
\begin{align*}
\partial_t a + u \cdot \nabla a &= 0, \\
a(x, 0) &= a_0.
\end{align*}
\]

Then there hold for \( \forall t \leq T \)

\[ \| a \|_{L_t^\infty(B_{q,1}^{3/2})} \leq \| a_0 \|_{B_{q,1}^{3/2}} e^{CU(t)}, \] \tag{2.13}

\[ \| a - S_u a \|_{L_t^\infty(B_{q,1}^{3/2})} \leq \sum_{q \geq m} 2^{3q/2} \| \Delta_q a_0 \|_{L^2} + \| a_0 \|_{B_{q,1}^{3/2}} (e^{CU(t)} - 1), \] \tag{2.14}

with \( U(t) = \| \nabla u \|_{L_t^1(B_{p,1}^{3/2})} \).

Similar inequality holds for the inhomogeneous Besov norm \( B_{q,1}^{3/2}(\mathbb{R}^3) \).

In order to prove the uniqueness of the main theorems, we need the following proposition in [3], we omit the details here for its proof.

**Proposition 2.11. (see [3])** Let \( \alpha \in (0, 1/4), p \in [3, 4], u_0 \in B_{2,1}^{1/2}(\mathbb{R}^3) \) and \( v \) be a divergence free vector field satisfying \( \nabla v \in L_t^1(B_{p,1}^{3/2}) \). And let \( f \in L_t^1(B_{2,1}^{1/2}) \) and \( a \in L_t^\infty(H^2) \) with \( 1 + a \geq \frac{p}{2} > 0 \). We assume that \( u \in C([0, T]; B_{2,1}^{1/2}) \cap L_t^1(B_{2,1}^{3/2}) \) and \( \nabla \Pi \in L_t^1(H^{-1/2 - a}) \) solve

\[
\begin{align*}
\partial_t u + v \cdot \nabla u - (1 + a)(\Delta u - \nabla \Pi) &= f, \\
\text{div} u &= 0, \\
u \bigg|_{t=0} &= u_0.
\end{align*}
\]
Then for all $t \leq T$, there holds:

$$
\|u\|_{L^p_t(B^{\frac{3}{2}}_{2,1})} + \|u\|_{L^1_t(B^{\frac{3}{2}}_{2,1})} \leq \left\{ \|u_0\|_{B^{\frac{1}{2}}_{2,1}} + \int_0^t \|u\|_{B^{\frac{1}{2}}_{2,1}} \|\nabla v\|_{B^{\frac{3}{2}}_{p,1}} d\tau + \|a\|_{L^p_t(H^{\frac{3}{2}+\alpha})} \|\nabla \Pi\|_{L^1_t(H^{\frac{3}{2}-\alpha})} + \|a\|_{L^\infty_t(H^{3_2})} \|\nabla \Pi\|_{L^1_t(H^{-1_2})} \right\}.
$$

3. Elliptic estimates with variable coefficients

This section is devoted to the proof of new estimates for the elliptic equation with variable coefficients.

**Proposition 3.1.** Assume $1 < q \leq p$ with $p \in \left(\frac{3+\sqrt{17}}{4}, \frac{5+\sqrt{17}}{2}\right)$ and $\frac{1}{p} + \frac{1}{q} \geq \frac{1}{2}$, $a \in \dot{B}^{\frac{3}{q_1}}_{q_1,1}(\mathbb{R}^3)$ with $1 + a \geq \kappa > 0$.

Let $F = (F_1, F_2, F_3) \in \dot{B}^{-\frac{3}{2}+\frac{3}{p}}_{p,2}(\mathbb{R}^3)$ and $\nabla \Pi \in \dot{B}^{-\frac{3}{2}+\frac{3}{p}}_{p,2}(\mathbb{R}^3)$ solve

$$
\text{div} \left( (1+a) \nabla \Pi \right) = \text{div} F. \quad (3.1)
$$

Then, we have

$$
\|\nabla \Pi\|_{\dot{B}^{\frac{3}{2}+\frac{3}{p}}_{p,2}} \leq (1 + \|a\|_{\dot{B}^{\frac{3}{q_1}}_{q_1,1}}) \|QF\|_{\dot{B}^{\frac{3}{2}+\frac{3}{p}}_{p,2}}. \quad (3.2)
$$

**Proof.** Thanks to $1 + a \geq \kappa > 0$ and $\text{div} F = \text{div} QF$, similar to the proof of Lemma 2 in [17], we readily deduce from (3.1) that

$$
\kappa \|\nabla \Pi\|_{L^2} \leq \|QF\|_{L^2}. \quad (3.3)
$$

Applying $\hat{\Delta}_j$ to (3.1) gives

$$
\text{div} \left( (1+a)\hat{\Delta}_j \nabla \Pi \right) = \text{div} \hat{\Delta}_j F - \text{div} \left( [\hat{\Delta}_j, a] \nabla \Pi \right). \quad (3.4)
$$

We next multiply the above equation by $-|\hat{\Delta}_j \Pi|^{p-2} \hat{\Delta}_j \Pi$ and integrate over $\mathbb{R}^3$. Then applying Lemma 8 in Appendix B of [17] implies for some constants $c$ and $C$

$$
ck2^j \|\hat{\Delta}_j \Pi\|_{L^p}^p \leq C2^j \|\hat{\Delta}_j QF\|_{L^p} \|\hat{\Delta}_j \Pi\|_{L^p}^{p-1} + \int_{\mathbb{R}^3} \text{div} \left( [\hat{\Delta}_j, a] \nabla \Pi \right) \cdot |\hat{\Delta}_j \Pi|^{p-2} \hat{\Delta}_j \Pi dx. \quad (3.5)
$$

In order to estimate the last term on the right hand side of (3.5), we need the following commutator estimates of integral type. The estimates of the following lemma have no restrict on the size of the relationship of $p, q$ which are somewhat more general than the necessary one in the present paper.
Lemma 3.2. Let \((p, q) \in \left(\frac{1 + \sqrt{7}}{4}, 2\right) \times [1, \infty)\) with \(\frac{1}{p} - \frac{1}{q} \leq \frac{1}{2}\). Then we have

\[
I_j \overset{\text{def}}{=} \int_{\mathbb{R}^3} \text{div} \left( [\Delta_j, a] \nabla \Pi \right) \cdot |\Delta_j \Pi|^{p-2} \Delta_j \Pi dx \lesssim d_j 2^{j\left(\frac{2}{p} - \frac{2}{q}\right)} \|a\|_{B^\frac{3}{q+1}} \|\nabla \Pi\|_{L^2} \|\Delta_j \Pi\|_{L^p}^{p-1}. \tag{3.6}
\]

Proof. Noticing that we can not directly use integration by parts. For this, we first get by using Bony’s decomposition

\[
I_j = \int_{\mathbb{R}^3} \text{div} \left( [\Delta_j, T_a] \nabla \Pi \right) \cdot |\Delta_j \Pi|^{p-2} \Delta_j \Pi dx + \int_{\mathbb{R}^3} \text{div} \left( \Delta_j (T_{\nu} \tau a) \right) \cdot |\Delta_j \Pi|^{p-2} \Delta_j \Pi dx \nonumber
\]

\[- \int_{\mathbb{R}^3} \text{div} \left( T_{\Delta_j \nu} \tau a \right) \cdot |\Delta_j \Pi|^{p-2} \Delta_j \Pi dx \nonumber
\]

\[= I_j^1 + I_j^2 + I_j^3. \tag{3.7}
\]

By the definition of Bony’s decomposition, we have

\[
[\Delta_j, T_a] \nabla \Pi = -2^{2j} \sum_{|j' - j| \leq 4} \int_{\mathbb{R}^3} h(2y) \Delta_{j'} \nabla \Pi(x - y) dy \int_0^1 \frac{1}{2} y \cdot \nabla S_{j'} a(x - \tau y) d\tau,
\]

from which, we get by using the Hölder inequality and Lemma 2.4 that

\[
\| [\Delta_j, T_a] \nabla \Pi \|_{L^p} \lesssim 2^{-j} \sum_{|j' - j| \leq 4} \| \nabla S_{j'} a\|_{L^{2^p}} \| \Delta_{j'} \nabla \Pi \|_{L^2} \lesssim c_j 2^{-j\left(\frac{2}{p} - \frac{2}{q}\right)} \|a\|_{B^\frac{3}{q+1}} \|\nabla \Pi\|_{L^2}, \tag{3.9}
\]

where we have used

\[
\| \nabla S_{j'} a\|_{L^{2^p}} \lesssim d_j 2^{j\left(\frac{2}{p} - \frac{2}{q}\right)} \|a\|_{B^\frac{3}{q+1}} \text{ for } p > \frac{6}{5} \text{ and } \frac{1}{p} - \frac{1}{q} \leq \frac{1}{2}.
\]

Note that \([\Delta_j, T_a] \nabla \Pi\) is spectrally supported in an annulus of size \(2^j\). Whence we infer

\[
I_j^1 \lesssim d_j 2^{j\left(\frac{2}{p} - \frac{2}{q}\right)} \|a\|_{B^\frac{3}{q+1}} \|\nabla \Pi\|_{L^2} \|\Delta_j \Pi\|_{L^p}^{p-1}. \tag{3.10}
\]

Owing to the localization properties of the Littlewood-Paley decomposition, we have

\[
\Delta_j (T_{\nu} \tau a) = \sum_{j' \geq j-3} \Delta_j (\Delta_{j'} a \nabla \Pi).
\]

If \(q \geq 2\), we denote \(\frac{1}{q} \overset{\text{def}}{=} \frac{1}{2} + \frac{1}{q} \geq \frac{1}{p}\) and apply Lemma 2.4 to obtain

\[
\|\Delta_j (T_{\nu} \tau a)\|_{L^p} \lesssim 2^{3j\left(\frac{2}{q} - \frac{1}{q'}\right)} \sum_{j' \geq j-3} \|\Delta_{j'} a\|_{L^q} \|\nabla \Pi\|_{L^2} \lesssim d_j 2^{j\left(\frac{2}{p} - \frac{2}{q}\right)} \|a\|_{B^\frac{3}{q+1}} \|\nabla \Pi\|_{L^2}. \tag{3.11}
\]

While if \(q < 2\), the embedding \(B^\frac{3}{q+1}_q(\mathbb{R}^3) \hookrightarrow B^\frac{2}{q+1}_q(\mathbb{R}^3)\) ensures that the above inequality still holds. Thus we obtain

\[
I_j^2 \lesssim d_j 2^{j\left(\frac{2}{p} - \frac{2}{q}\right)} \|a\|_{B^\frac{3}{q+1}} \|\nabla \Pi\|_{L^2} \|\Delta_j \Pi\|_{L^p}^{p-1}. \tag{3.12}
\]
For $I_j^3$, due to the fact that

$$
\sum_{j' \geq j} \hat{\Delta}_j \nabla \Pi \hat{\Delta}_{j'} a = \sum_{j' \geq j} \left( \sum_{k \leq j' + 1} \hat{\Delta}_k \hat{\Delta}_{j'} \nabla \Pi \right) \hat{\Delta}_{j'} a \\
= \sum_{j' \geq j} \left( (I - \sum_{k \geq j' + 2} \hat{\Delta}_k) \hat{\Delta}_{j'} \nabla \Pi \right) \hat{\Delta}_{j'} a = \sum_{j' \geq j} \hat{\Delta}_{j'} \nabla \Pi \hat{\Delta}_{j'} a,
$$

due to the fact that $j \cdot q \leq 2^{j+1} j' \cdot q'$, we can write

$$
I_j^3 = - \sum_{j' - j = -1, -2} \int_{R^3} \text{div} \left( \hat{\Delta}_{j'} a \hat{\Delta}_{j'} \nabla \Pi \right) \cdot |\hat{\Delta}_j \Pi|^{p-2} \hat{\Delta}_j \Pi dx + (p - 1) \sum_{j' \geq j} \int_{R^3} \hat{\Delta}_j a |\hat{\Delta}_j \nabla \Pi|^2 \cdot |\hat{\Delta}_j \Pi|^{p-2} dx
$$

$\overset{\text{def}}{=} I_j^{3,1} + I_j^{3,2}.$

Then it is easy to observe for $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{2}$ that

$$
I_j^{3,1} \lesssim 2^j \sum_{j' - j = -1, -2} \|\hat{\Delta}_{j'} a\|_{L^\frac{2p}{2-p}} \|\hat{\Delta}_j \nabla \Pi\|_{L^2} \|\hat{\Delta}_j \Pi\|_{L^p}^{p-1}
$$

$$
\lesssim d_j 2^{j\left(\frac{5}{2} - \frac{3}{p}\right)} \|a\|_{B^{\frac{1}{2},1}_q \nabla \Pi} \|\nabla \Pi\|_{L^2} \|\hat{\Delta}_j \Pi\|_{L^p}^{p-1}. \quad (3.14)
$$

While the assumption $p \in (\frac{1+\sqrt{p}}{2}, 2]$ ensures that $\frac{1}{p - 1} < \frac{2p}{2-p}$. In the case when $\max\{p, \frac{1}{p-1}\} < q \leq \frac{2p}{2-p}$, we have $(p - 2)q' + 1 > 0$ so that we can use a similar approximate argument as in the proof of Lemma A.5 in the appendix of [13] to obtain

$$
\| |\hat{\Delta}_j \nabla \Pi|^2 \cdot |\hat{\Delta}_j \Pi|^{p-2}\|_{L^{q'}}
$$

$$
= \int_{R^3} |\hat{\Delta}_j \Pi|^{(p-2)q'} \hat{\Delta}_j \nabla \Pi \cdot \hat{\Delta}_j \nabla \Pi |\hat{\Delta}_j \nabla \Pi|^{2q' - 2} dx
$$

$$
= - \frac{1}{(p - 2)q' + 1} \int_{R^3} |\hat{\Delta}_j \Pi|^{(p-2)q'} \hat{\Delta}_j \Pi \cdot \text{div} \left( \hat{\Delta}_j \nabla \Pi |\hat{\Delta}_j \nabla \Pi|^{2q' - 2} \right) dx.
$$

Denoting $\frac{1}{p} \overset{\text{def}}{=} \frac{1}{p} - \frac{1}{q} \leq \frac{1}{2}$ and using the Hölder inequality and Lemma 2.4 gives

$$
\| |\hat{\Delta}_j \nabla \Pi|^2 \cdot |\hat{\Delta}_j \Pi|^{p-2}\|_{L^{q'}}
$$

$$
\lesssim \| |\hat{\Delta}_j \Pi|^{(p-2)q' + 1}\|_{L^{\frac{p}{p-2q' + 1}}} \| |\hat{\Delta}_j \nabla \Pi|^{q' - 1}\|_{L^{\frac{p}{q' - 1}}} \| |\hat{\Delta}_j \nabla \Pi|^{q' - 1}\|_{L^{\frac{p}{q' - 1}}} \| \nabla^2 \hat{\Delta}_j \Pi\|_{L^{q'}}
$$

$$
\lesssim 2^{jq' \left(\frac{5}{2} - \frac{3}{p}\right)} \| \hat{\Delta}_j \Pi\|_{L^p}^{(p-1)q'} \| \nabla \hat{\Delta}_j \Pi\|_{L^2}^{q'}. \quad (3.15)
$$
which implies
\begin{align*}
I_j^{3,2} & \lesssim \sum_{j' \geq j} \| \hat{A}_j a \|_{L^p} \| \hat{A}_j \nabla \Pi \|^2 \cdot \| \hat{A}_j \Pi \|_{L^q}^{p-2} \| \nabla \Pi \|_{L^q} \| \hat{A}_j \Pi \|_{L^q}^{p-1}. 
\end{align*}

Similarly, (3.15) is valid for \( q \leq \max \{ p, \frac{1}{p'+1} \} \) according to embedding. Summing up the inequalities (3.10)–(3.15) results in (3.6). \( \square \)

Now, let us go back to the estimate of (3.5).

(i) When \( p \in (\frac{1-\sqrt{17}}{4}, 2] \), substituting (3.6) into (3.5) leads to
\begin{align*}
\| \hat{A}_j \nabla \Pi \|_{L^p} \lesssim \| \hat{A}_j Q F \|_{L^p} + d_j 2^{j \left( \frac{3}{2} - \frac{3}{q} \right)} \| a \|_{\dot{B}^{\frac{3}{2}}_{p,2}} \| \nabla \Pi \|_{L^q},
\end{align*}
which along with (3.3) and the embedding \( \dot{B}^\frac{3}{2} \cap \dot{B}^{\frac{1}{2} + \frac{1}{q}}_{p',2} (\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3), I^1 \hookrightarrow I^2 \) gives
\begin{align*}
\| \nabla \Pi \|_{\dot{B}^{\frac{3}{2} + \frac{3}{p'}}_{p',2}} \lesssim (1 + \| a \|_{\dot{B}^{\frac{q}{2}}_{p,1}}) \| Q F \|_{\dot{B}^{\frac{3}{2} - \frac{3}{q'}}_{p',2}}. \tag{3.16}
\end{align*}

Next, we consider the case when \( p \in [2, \frac{5+\sqrt{17}}{2}] \) with \( \frac{1}{p} + \frac{1}{q} \geq \frac{1}{2} \). In this case, motivated by [2, 3], we shall use a duality argument:
\begin{align*}
\| \nabla \Pi \|_{\dot{B}^{\frac{3}{2} + \frac{3}{p'}}_{p',2}} = \sup_{\| g \|_{\dot{B}^{\frac{q}{2}}_{p,1}} = 1} \langle \nabla \Pi, g \rangle = \sup_{\| g \|_{\dot{B}^{\frac{3}{2} - \frac{3}{q'}}_{p',2}} = 1} -\langle \Pi, \text{div } g \rangle, \tag{3.17}
\end{align*}

where \( \langle \cdot, \cdot \rangle \) denotes the duality bracket between \( \mathcal{S}'(\mathbb{R}^3) \) and \( \mathcal{S}(\mathbb{R}^3) \). Noticing that \( p' \in (\frac{1+\sqrt{17}}{4}, 2] \)
and \( \frac{1}{p} - \frac{1}{q} \leq \frac{1}{2} \), then applying (3.16) ensures that for any \( g \in \dot{B}^{\frac{3}{2} + \frac{3}{p'}}_{p',2} (\mathbb{R}^3) \), there exists a unique solution \( \nabla P_g \in \dot{B}^{\frac{3}{2} + \frac{3}{p'}}_{p',2} (\mathbb{R}^3) \) to the elliptic equation
\begin{align*}
\text{div} \left( (1 + a) \nabla P_g \right) = \text{div} g,
\end{align*}
such that
\begin{align*}
\| \nabla P_g \|_{\dot{B}^{\frac{3}{2} + \frac{3}{p'}}_{p',2}} \lesssim (1 + \| a \|_{\dot{B}^{\frac{q}{2}}_{p,1}}) \| g \|_{\dot{B}^{\frac{3}{2} - \frac{3}{q'}}_{p',2}}. \tag{3.18}
\end{align*}

We proceed
\begin{align*}
-\langle \Pi, \text{div } g \rangle &= -\langle \Pi, \text{div} \left( (1 + a) \nabla P_g \right) \rangle = -\langle \text{div} \left( (1 + a) \nabla \Pi \right), P_g \rangle \\
&= -\langle \text{div } F, P_g \rangle = \langle Q F, \nabla P_g \rangle \leq \| Q F \|_{\dot{B}^{\frac{3}{2} + \frac{3}{p'}}_{p',2}} \| \nabla P_g \|_{\dot{B}^{\frac{3}{2} - \frac{3}{q'}}_{p',2}}. \tag{3.19}
\end{align*}
which along with (3.17) and (3.18) implies (3.2).

This completes the proof of the proposition. \(\square\)

4. Linear estimates

With the pressure estimates in hand, now, we are going to give the linear estimates for the inhomogeneous incompressible Navier-Stokes equations, more precisely, we can get the following proposition:

**Proposition 4.1.** Assume \(1 < q \leq p \) with \( p \in [3, \frac{5+\sqrt{17}}{2}] \) and \( \frac{1}{p} + \frac{1}{q} \geq \frac{1}{2} \), \( u_0 \in \dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3) \) and \( a \in L_T^\infty(\dot{B}_{q,1}^{-\frac{3}{2}}(\mathbb{R}^3)) \), with \(1 + a \geq \kappa > 0\). Let \( f \in L_T^1(\dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)) \cap L_T^1(\dot{B}_{p,1}^{-\frac{3}{2}+\frac{3}{p}}(\mathbb{R}^3)) \), \((u, \nabla \Pi) \in C([0, T]; \dot{B}_{p,1}^{-1+\frac{3}{p}}(\mathbb{R}^3)) \cap L_T^1(\dot{B}_{p,1}^{-\frac{3}{2}+\frac{3}{p}}(\mathbb{R}^3)) \times L_T^1(\dot{B}_{p,1}^{-\frac{3}{2}+\frac{3}{p}}(\mathbb{R}^3)) \) solve

\[
\begin{aligned}
\partial_t u - \text{div}((1 + a) \nabla u) + (1 + a) \nabla \Pi &= f, \\
\text{div} u &= 0, \\
u|_{t=0} &= u_0.
\end{aligned}
\]

Then there holds for \( t \in [0, T] \)

\[
\|u\|_{L_T^\infty(\dot{B}_{p,1}^{-1+\frac{3}{p}})} + \|u\|_{L_T^1(\dot{B}_{p,1}^{-\frac{3}{2}+\frac{3}{p}})} + \|\nabla \Pi\|_{L_T^1(\dot{B}_{p,1}^{-1+\frac{3}{p}})} \\
\lesssim & \|u_0\|_{\dot{B}_{p,1}^{-1+\frac{3}{p}}} + \|f\|_{L_T^1(\dot{B}_{p,1}^{-1+\frac{3}{p}})} + 2^n \|a\|_{L_T^\infty(\dot{B}_{q,1}^{-\frac{3}{2}})} \|u\|_{L_T^1(\dot{B}_{p,1}^{-\frac{3}{2}+\frac{3}{p}})} \\
&+ 2^n (1 + \|a\|_{L_T^\infty(\dot{B}_{q,1}^{-\frac{3}{2}})})^2 \|f\|_{L_T^1(\dot{B}_{p,1}^{-\frac{3}{2}+\frac{3}{p}})} + \|a\|_{L_T^\infty(\dot{B}_{q,1}^{-\frac{3}{2}})} \|u\|_{L_T^1(\dot{B}_{p,1}^{-\frac{3}{2}+\frac{3}{p}})},
\]

provided that

\[
(1 + \|a\|_{L_T^\infty(\dot{B}_{q,1}^{-\frac{3}{2}})})^2 \|a - \dot{S}_m a\|_{L_T^\infty(\dot{B}_{q,1}^{-\frac{3}{2}})} \leq c_0
\]

for some sufficiently small positive constant \(c_0\) and some integer \(m \in \mathbb{Z}\).

**Proof.** We first use the decomposition \(\text{Id} = \dot{S}_m + (\text{Id} - \dot{S}_m)\) to turn the \(u\) equation of (4.1) into

\[
\partial_t u - \text{div}((1 + \dot{S}_m a) \nabla u) + (1 + \dot{S}_m a) \nabla \Pi = f + \dot{E}_m, \tag{4.4}
\]

with \(\dot{E}_m \overset{\text{def}}{=} \text{div}((a - \dot{S}_m a) \nabla u) - (a - \dot{S}_m a) \nabla \Pi\). Then we infer from \(1 + a \geq \kappa > 0\) and (4.3) that

\[
1 + \dot{S}_m a \geq \frac{1}{2} \kappa. \tag{4.5}
\]
**Step 1.** The estimate of $$\|u\|_{\ell^{p_4}(B_{p_1}^{1/\alpha,\frac{3}{P}})} + \|u\|_{L^3_t(B_{p_1}^{1/\alpha,\frac{3}{P}})}$$

Applying $$\check{\Delta}_j \mathbb{P}$$ to (4.4), we arrive at

$$\partial_t \Delta_j u - \text{div} ((1 + \check{S}_m a) \Delta_j \nabla u) = \check{\Delta}_j \mathbb{P} (f + \check{E}_m - \check{S}_m a \nabla \Pi)$$

$$\quad + \check{\Delta}_j Q (\nabla \check{S}_m a \cdot \nabla u - \check{S}_m a \Delta u) + \text{div} ([\check{\Delta}_j, \check{S}_m a] \nabla u), \quad (4.6)$$

where we have used the fact that:

$$\mathbb{P} \nabla \Pi = 0, \quad \text{div} ((1 + \check{S}_m a) \nabla u) = \partial_t ((1 + \check{S}_m a) \partial_t u_j).$$

Applying Lemma 8 in the appendix of [17] and using the Hölder inequality, we get for some positive constant $$c$$ that

$$\frac{d}{dt} \| \check{\Delta}_j u \|_{L^p} + c 2^{2j} \| \check{\Delta}_j u \|_{L^p}$$

$$\lesssim \| \check{\Delta}_j f \|_{L^p} + \| \check{\Delta}_j \check{E}_m \|_{L^p} + \| \check{\Delta}_j \mathbb{P} (\check{S}_m a \nabla \Pi) \|_{L^p}$$

$$\quad + \| \check{\Delta}_j (\nabla \check{S}_m a \cdot \nabla u) \|_{L^p} + \| \check{\Delta}_j Q (\check{S}_m a \Delta u) \|_{L^p} + 2^j [\check{\Delta}_j, \check{S}_m a] \nabla u \|_{L^p}. \quad (4.7)$$

After time integration, multiplying $$2^{\frac{3}{2} - 1/j}$$ and summing up over $$j$$, we infer

$$\| u \|_{\ell^{p_4}(B_{p_1}^{1/\alpha,\frac{3}{P}})} + \| u \|_{L^3_t(B_{p_1}^{1/\alpha,\frac{3}{P}})}$$

$$\lesssim \| u_0 \|_{B_{p_1}^{1/\alpha,\frac{3}{P}}} + \| f \|_{L^1_t(B_{p_1}^{1/\alpha,\frac{3}{P}})} + \| \check{E}_m \|_{L^1_t(B_{p_1}^{1/\alpha,\frac{3}{P}})} + \| \mathbb{P} (\check{S}_m a \nabla \Pi) \|_{L^1_t(B_{p_1}^{1/\alpha,\frac{3}{P}})}$$

$$\quad + \| \nabla \check{S}_m a \cdot \nabla u \|_{L^1_t(B_{p_1}^{1/\alpha,\frac{3}{P}})} + \| Q (\check{S}_m a \Delta u) \|_{L^1_t(B_{p_1}^{1/\alpha,\frac{3}{P}})} + \sum_{j \in \mathbb{Z}} 2^{3j} \| [\check{\Delta}_j, \check{S}_m a] \nabla u \|_{L^1_t(L^p)}. \quad (4.8)$$

Applying Lemmas 2.5 and 2.6 one has

$$\| \nabla \check{S}_m a \cdot \nabla u \|_{L^1_t(B_{p_1}^{1/\alpha,\frac{3}{P}})} \lesssim 2^m \| a \|_{L^{p_4}(B_{p_1}^{1/\alpha,\frac{3}{P}})} \| u \|_{L^3_t(B_{p_1}^{1/\alpha,\frac{3}{P}})}. \quad (4.9)$$

$$\| \mathbb{P} (\check{S}_m a \nabla \Pi) \|_{L^1_t(B_{p_1}^{1/\alpha,\frac{3}{P}})} \lesssim 2^m \| a \|_{L^{p_4}(B_{p_1}^{1/\alpha,\frac{3}{P}})} \| \nabla \Pi \|_{L^1_t(B_{p_2}^{1/\alpha,\frac{3}{P}})}. \quad (4.10)$$

$$\| \check{E}_m \|_{L^1_t(B_{p_1}^{1/\alpha,\frac{3}{P}})} \lesssim \| a - \check{S}_m a \|_{L^{p_4}(B_{p_1}^{1/\alpha,\frac{3}{P}})} \| u \|_{L^3_t(B_{p_1}^{1/\alpha,\frac{3}{P}})} + \| u - \check{S}_m a \|_{L^{p_4}(B_{p_1}^{1/\alpha,\frac{3}{P}})} \| \nabla \Pi \|_{L^1_t(B_{p_2}^{1/\alpha,\frac{3}{P}})}. \quad (4.11)$$

Yet noticing that $$Q = -\nabla (-\Delta)^{-1} \text{div}$$ and $$\text{div} u = 0$$, we get by applying Bony’s decomposition that

$$Q (\check{S}_m a \Delta u) = -\nabla (-\Delta)^{-1} (\check{T}_\nabla \check{S}_m a \Delta u) + Q (\check{T}_\Delta \check{S}_m a) + Q (\check{R} (\check{S}_m a, \Delta u)).$$

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Then it is easy to get

$$
\| \hat{\Delta}_j (\hat{T} \cdot \nabla S_m a \Delta u) \|_{L^p} \lesssim \sum_{|j'|-j| \leq 4} \| \hat{S}_{p-1} \nabla \hat{S}_m a \|_{L^\infty} \| \hat{\Delta}_j \Delta u \|_{L^p} \lesssim d_j 2^{j(\frac{2}{\tilde{p}}) + m} \| a \|_{\mathcal{B}^{\frac{1}{2}}_{p,1}} \| u \|_{\mathcal{B}^{\frac{1}{2}}_{p,1}}, \quad (4.12)
$$

$$
\| \hat{\Delta}_j (\hat{T} \cdot \Delta u \hat{S}_m a) \|_{L^p} \lesssim \sum_{|j'-j| \leq 4} \| \hat{\Delta}_j \hat{S}_m a \|_{L^p} \| \hat{S}_{p-1} \Delta u \|_{L^\infty} \\
\lesssim \sum_{|j'-j| \leq 4} 2^{j(\frac{1}{\tilde{p}}) - \frac{1}{\tilde{q}'}} \| \hat{\Delta}_j \hat{S}_m a \|_{L^p} \| \hat{S}_{p-1} \Delta u \|_{L^\infty} \\
\lesssim d_j 2^{j(1 - \frac{1}{\tilde{p}}) + m} \| a \|_{\mathcal{B}^{\frac{1}{2}}_{p,1}} \| u \|_{\mathcal{B}^{\frac{1}{2}}_{p,1}}.
$$

Let $\frac{1}{\tilde{q}} = \frac{1}{p} + \frac{1}{q'}$ for the fact that $\frac{1}{p} + \frac{1}{q} \geq \frac{1}{\tilde{q}}$ then

$$
\| \hat{\Delta}_j \hat{R} (\hat{S}_m a, \Delta u) \|_{L^p} \lesssim 2^{j(\frac{1}{\tilde{p}}) - \frac{1}{\tilde{q}'}} \sum_{j' \geq j - 3} \| \hat{\Delta}_j \hat{S}_m a \|_{L^p} \| \hat{\Delta}_j \Delta u \|_{L^p} \\
\lesssim 2^{\frac{1}{\tilde{q}}} \sum_{j' \geq j - 3} \| \hat{\Delta}_j \hat{S}_m a \|_{L^p} \| \hat{\Delta}_j \Delta u \|_{L^p} \\
\lesssim d_j 2^{j(1 - \frac{1}{\tilde{p}}) + m} \| a \|_{\mathcal{B}^{\frac{1}{2}}_{p,1}} \| u \|_{\mathcal{B}^{\frac{1}{2}}_{p,1}}.
$$

Whence we conclude that

$$
\| Q (\hat{S}_m a \Delta u) \|_{L^p_1 (\mathcal{B}^{\frac{1}{2}}_{p,1})} \lesssim 2^m \| a \|_{\mathcal{B}^{\frac{1}{2}}_{p,1}} \| u \|_{\mathcal{B}^{\frac{1}{2}}_{p,1}}.
$$

While applying Lemma 2.8 leads to

$$
\sum_{j \in \mathbb{Z}} 2^{\frac{\tilde{q}}{2}} \| [\hat{\Delta}_j \hat{S}_m a] \nabla u \|_{L^p_1} \lesssim 2^m \| a \|_{\mathcal{B}^{\frac{1}{2}}_{p,1}} \| u \|_{\mathcal{B}^{\frac{1}{2}}_{p,1}}.
$$

Plugging the above estimates (4.9)–(4.11), (4.15), (4.16) into (4.8) and using (4.3) yield

$$
\| u \|_{\mathcal{L}^{\infty}_T (\mathcal{B}^{\frac{1}{2}}_{p,1})} + \| u \|_{L^p_1 (\mathcal{B}^{\frac{1}{2}}_{p,1})} \\
\lesssim \| u_0 \|_{\mathcal{B}^{\frac{1}{2}}_{p,1}} + \| f \|_{L^p_1 (\mathcal{B}^{\frac{1}{2}}_{p,1})} + \| a - \hat{S}_m a \|_{\mathcal{L}^{\infty}_T (\mathcal{B}^{\frac{1}{2}}_{p,1})} \| \nabla \Pi \|_{L^p_1 (\mathcal{B}^{\frac{1}{2}}_{p,1})} \\
+ 2^{\frac{\tilde{q}}{2}} \| a \|_{\mathcal{L}^{\infty}_T (\mathcal{B}^{\frac{1}{2}}_{p,1})} \| \nabla /I \|_{L^p_1 (\mathcal{B}^{\frac{1}{2}}_{p,1})} + 2^m \| a \|_{\mathcal{L}^{\infty}_T (\mathcal{B}^{\frac{1}{2}}_{p,1})} \| u \|_{L^p_1 (\mathcal{B}^{\frac{1}{2}}_{p,1})}.
$$

Step 2. The estimate of $\| \nabla \Pi \|_{L^p_1 (\mathcal{B}^{\frac{1}{2}}_{p,1})}$.

We first get by taking div to (4.4) that

$$
\text{div}(1 + \hat{S}_m a) \nabla \Pi = \text{div}(f + \hat{E}_m + \nabla \hat{S}_m a \cdot \nabla u + \hat{S}_m a \Delta u),
$$

which implies

$$
\text{div}((1 + \hat{S}_m a) \hat{\Delta}_j \nabla \Pi) = \text{div} \hat{\Delta}_j (f + \hat{E}_m + \nabla \hat{S}_m a \cdot \nabla u + \hat{S}_m a \Delta u) - \text{div}([\hat{\Delta}_j \hat{S}_m a] \nabla \Pi). \quad (4.18)
$$
A similar argument as in (4.8) results in
\[
\|\nabla \Pi\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})} \lesssim \|f\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})} + \|\dot{E}_m\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})} + \|\nabla S_m a \cdot \nabla u\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})}
\]
\[+ \|Q(\dot{S}_m a \Delta u)\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})} + \sum_{j \in \mathbb{Z}} 2^{(\frac{3}{2} - 1)} j \|[\dot{\Delta}_j, \dot{S}_m a] \nabla \Pi\|_{L^1_t(L^p)}.
\]
(4.19)

Again thanks to Lemma 2.9, one has
\[
\sum_{j \in \mathbb{Z}} 2^{(\frac{3}{2} - 1)} j \|[\dot{\Delta}_j, \dot{S}_m a] \nabla \Pi\|_{L^1_t(L^p)} \lesssim 2^{\frac{5}{2}} \|a\|_{L^\infty_t(B^{\frac{3}{p},\frac{3}{2}})} \|[\dot{\Delta}_j, \dot{S}_m a] \nabla \Pi\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})}.
\]
Whence we obtain
\[
\|\nabla \Pi\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})} \lesssim \|f\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})} + \|a - \dot{S}_m a\|_{L^\infty_t(B^{\frac{3}{p},\frac{3}{2}})} \|[\dot{\Delta}_j, \dot{S}_m a] \nabla \Pi\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})}
\]
\[+ 2^m \|a\|_{L^\infty_t(B^{\frac{3}{p},\frac{3}{2}})} \|[\dot{\Delta}_j, \dot{S}_m a] \nabla \Pi\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})} + 2^{\frac{5}{2}} \|a\|_{L^\infty_t(B^{\frac{3}{p},\frac{3}{2}})} \|[\dot{\Delta}_j, \dot{S}_m a] \nabla \Pi\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})}.
\]
(4.20)
which along with (4.3), (4.17) gives rise to
\[
\|u\|_{L^\infty_t(B^{\frac{3}{p},\frac{3}{2}})} + \|u\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})} + \|[\dot{\Delta}_j, \dot{S}_m a] \nabla \Pi\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})}
\lesssim \|u_0\|_{B^{\frac{3}{p},\frac{3}{2}}} + \|f\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})} + 2^m \|a\|_{L^\infty_t(B^{\frac{3}{p},\frac{3}{2}})} \|[\dot{\Delta}_j, \dot{S}_m a] \nabla \Pi\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})} + 2^{\frac{5}{2}} \|a\|_{L^\infty_t(B^{\frac{3}{p},\frac{3}{2}})} \|[\dot{\Delta}_j, \dot{S}_m a] \nabla \Pi\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})}.
\]
(4.21)

Step 3. The estimate of \(|\nabla \Pi\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})}\).

Applying \(\text{div}\) to the first equation of (4.1) implies that
\[
\text{div}((1 + a) \nabla \Pi) = \text{div}(f + \nabla a \cdot \nabla u + a \Delta u).
\]
Whence we get by applying Proposition 3.1 to the above equation that
\[
\|\nabla \Pi\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})} \lesssim (1 + \|a\|_{L^\infty_t(B^{\frac{3}{p},\frac{3}{2}})}) (\|f\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})} + \|\nabla a \cdot \nabla u\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})} + \|Q(a \Delta u)\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})}).
\]
(4.22)
Yet applying Remark 2.6 yields
\[
\|\nabla a \cdot \nabla u\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})} + \|Q(a \Delta u)\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})} \lesssim \|a\|_{L^\infty_t(B^{\frac{3}{p},\frac{3}{2}})} \|\nabla u\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})} + \|\Delta u\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})}
\]
\[\lesssim \|a\|_{L^\infty_t(B^{\frac{3}{p},\frac{3}{2}})} \|[\dot{\Delta}_j, \dot{S}_m a] \nabla \Pi\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})}.
\]
This gives rise to
\[
\|\nabla \Pi\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})} \lesssim (1 + \|a\|_{L^\infty_t(B^{\frac{3}{p},\frac{3}{2}})}) (\|f\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})} + \|a\|_{L^\infty_t(B^{\frac{3}{p},\frac{3}{2}})} \|u\|_{L^1_t(B^{\frac{3}{p},\frac{3}{2}})}).
\]
(4.23)
Substituting the above inequality into (4.21) we have

\[
\|u\|_{L^p_B(B_{p,1}^{\frac{3}{2}})} + \|f\|_{L^q_B(B_{p,1}^{\frac{3}{2}})} + 2^m \|a\|_{L^\infty_B(B_{q,1}^{\frac{3}{2}})} \|u\|_{L^1_B(B_{p,1}^{\frac{3}{2}})} \\
+ 2^\frac{1}{2} \left( 1 + \|a\|_{L^q_B(B_{q,1}^{\frac{3}{2}})} \right)^2 \left( \|f\|_{L^q_B(B_{q,1}^{\frac{3}{2}})} + \|a\|_{L^\infty_B(B_{q,1}^{\frac{3}{2}})} \|u\|_{L^1_B(B_{p,1}^{\frac{3}{2}})} \right).
\]

This completes the proof of the proposition. □

Next, we give the linear estimates for the following magnetic field equation:

\[
\begin{cases}
\partial_t B - \text{div}(\bar{\sigma}(a) \nabla B) + v \cdot \nabla B = g, \\
\text{div} B = 0, \\
B|_{t=0} = B_0.
\end{cases}
\]

(4.25)

**Proposition 4.2.** Assume \(1 < p < 6, 1 \le q \le \infty, 1 - \frac{1}{q} \le \frac{1}{2}, B_0 \in B_{q,1}^{\frac{3}{2}}(\mathbb{R}^3), \bar{\sigma}(a) \) be a smooth, positive function on \([0, \infty), \) and \(a \in L^\infty_T(B_{q,1}^{\frac{3}{2}}(\mathbb{R}^3)), \) with \(1 + a \ge \kappa > 0. \) Let \(g \in L^1_T(B_{q,1}^{\frac{3}{2}}(\mathbb{R}^3)), \) and \(B \in C([0, T]; B_{p,1}^{\frac{3}{2}}(\mathbb{R}^3)) \cap L^1_T(B_{p,1}^{\frac{3}{2}}(\mathbb{R}^3))\) solve (4.25).

Then there holds for \(t \in [0, T]\)

\[
\|B\|_{L^p_T(B_{p,1}^{\frac{3}{2}})} + \|B\|_{L^1_T(B_{p,1}^{\frac{3}{2}})} \\
\lesssim \|B_0\|_{B_{p,1}^{\frac{3}{2}}} + \|g\|_{L^1_T(B_{p,1}^{\frac{3}{2}})} + 2^m \|a\|_{L^\infty_T(B_{q,1}^{\frac{3}{2}})} \|B\|_{L^1_T(B_{p,1}^{\frac{3}{2}})} + \int_0^T \|v\|_{B_{p,1}^{\frac{3}{2}}} \|B\|_{B_{p,1}^{\frac{3}{2}}} \, d\tau,
\]

(4.26)

provided that

\[
\|a - \hat{\sigma}_m a\|_{L^p_T(B_{q,1}^{\frac{3}{2}})} + \|\bar{\sigma}(a) - \bar{\sigma}(\hat{\sigma}_m a)\|_{L^p_T(B_{q,1}^{\frac{3}{2}})} \le c_0
\]

for some sufficiently small positive constant \(c_0\) and some integer \(m \in \mathbb{Z}.\)

**Proof.** We may rewrite the system (4.25), after decomposing \(\bar{\sigma}(a) = \bar{\sigma}(\hat{\sigma}_m a) + \bar{\sigma}(a) - \bar{\sigma}(\hat{\sigma}_m a), \) that

\[
\partial_t B - \text{div}(\bar{\sigma}(a) \nabla B) + v \cdot \nabla B + \text{div}((\bar{\sigma}(a) - \bar{\sigma}(\hat{\sigma}_m a)) \nabla B) = g.
\]

(4.28)

Applying the operator \(\Delta_j\) to (4.28), using a standard commutator’s process, we get

\[
\partial_t \Delta_j B - \text{div}(\bar{\sigma}(\hat{\sigma}_m a) \Delta_j \nabla B) \\
= \Delta_j g + \Delta_j (v \cdot \nabla B) + \Delta_j \text{div}((\bar{\sigma}(a) - \bar{\sigma}(\hat{\sigma}_m a)) \nabla B) + \text{div}(|\Delta_j, \bar{\sigma}(\hat{\sigma}_m a)| \nabla B).
\]
Noticing that \( \overline{\sigma}(a) - \overline{\sigma}(\bar{S}_m a) \) is small enough in norm \( L^\infty_t(B_{p,1}^{\frac{3}{q}+\frac{3}{p}}) \), it follows that \( \overline{\sigma}(\bar{S}_m a) \geq \frac{\nu}{2} \). Taking \( L^2 \) inner product with \( |\Delta_j B|^p - 2|\bar{\Delta}_j B| \) and applying Lemma 8 in the appendix of [17], we get

\[
\|B\|_{L^\infty_t(B_{p,1}^{1+\frac{3}{p}})} + \|B\|_{L^1_t(B_{p,1}^{1+\frac{3}{p}})} \\
\leq \|B_0\|_{B_{p,1}^{1+\frac{3}{p}}} + \|g\|_{L^1_t(B_{p,1}^{1+\frac{3}{p}})} + \|\nabla B\|_{L^1_t(B_{p,1}^{1+\frac{3}{p}})} + \|
abla((\overline{\sigma}(a) - \overline{\sigma}(\bar{S}_m a)) \nabla B)\|_{L^1_t(B_{p,1}^{1+\frac{3}{p}})} + \sum_{j \in \mathbb{Z}} 2^{\frac{3}{p}} \| [\Delta_j, \overline{\sigma}(\bar{S}_m a)] \nabla |u| \|_{L^1_t(L^p)}.
\]

(4.29)

By Lemma 2.8, one has

\[
\|\nabla \cdot \nabla B\|_{L^1_t(B_{p,1}^{1+\frac{3}{p}})} \lesssim \int_0^t \|v\|_{B_{p,1}^{\frac{3}{p}}} \|B\|_{B_{p,1}^{\frac{3}{p}}} \ d\tau,
\]

(4.30)

\[
\sum_{j \in \mathbb{Z}} 2^{\frac{3}{p}} \| [\Delta_j, \overline{\sigma}(\bar{S}_m a)] \nabla |B| \|_{L^1_t(L^p)} \lesssim 2^m \|a\|_{L^\infty_t(B_{p,1}^{\frac{3}{q}+\frac{3}{p}})} \|B\|_{L^1_t(B_{p,1}^{1+\frac{3}{p}})}.
\]

(4.31)

By Lemma 2.4, we have

\[
\|\nabla((\overline{\sigma}(a) - \overline{\sigma}(\bar{S}_m a)) \nabla B)\|_{L^1_t(B_{p,1}^{1+\frac{3}{p}})} \lesssim \|
abla((\overline{\sigma}(a) - \overline{\sigma}(\bar{S}_m a))\|_{L^\infty_t(B_{p,1}^{\frac{3}{q}+\frac{3}{p}})} \|B\|_{L^1_t(B_{p,1}^{1+\frac{3}{p}})}.
\]

(4.32)

Inserting the above estimates (4.30) – (4.32) into (4.29), one can finally get

\[
\|B\|_{L^\infty_t(B_{p,1}^{1+\frac{3}{p}})} + \|B\|_{L^1_t(B_{p,1}^{1+\frac{3}{p}})} \\
\leq \|B_0\|_{B_{p,1}^{1+\frac{3}{p}}} + \|g\|_{L^1_t(B_{p,1}^{1+\frac{3}{p}})} + 2^m \|a\|_{L^\infty_t(B_{p,1}^{\frac{3}{q}+\frac{3}{p}})} \|B\|_{L^1_t(B_{p,1}^{1+\frac{3}{p}})} + \int_0^t \|v\|_{B_{p,1}^{\frac{3}{p}}} \|B\|_{B_{p,1}^{\frac{3}{p}}} \ d\tau.
\]

(4.33)

This completes the proof of the proposition. \( \square \)

5. Local well-posedness of Theorem 1.5

5.1. Local existence

Step 1. Construction of smooth approximate solutions.

Firstly, there exists a sequence \( \{(a_0^n, u_0^n, B_0^n)\}_{n \in \mathbb{N}} \subset \mathcal{S}(\mathbb{R}^3) \) such that \((a_0^n, u_0^n, B_0^n)\) converges to \((a_0, u_0, B_0)\) in \( B_{p,1}^{\frac{3}{q}+\frac{3}{p}}(\mathbb{R}^3) \times (B_{p,1}^{1+\frac{3}{p}}(\mathbb{R}^3))^2 \). Define \( u_0^n \overset{\text{def}}{=} P u_0^n, B_0^n \overset{\text{def}}{=} P B_0^n \), so that \( \text{div} u_0^n = \text{div} B_0^n = 0 \). Then \((u_0^n, B_0^n)\) belongs to \( H^\infty(\mathbb{R}^3) \times H^\infty(\mathbb{R}^3) \) and converges to \((u_0, B_0)\) in \( B_{p,1}^{1+\frac{3}{p}}(\mathbb{R}^3) \times B_{p,1}^{1+\frac{3}{p}}(\mathbb{R}^3) \).

Furthermore, we could assume that

\[
\|a_0^n\|_{L^\infty} \leq 2\|a_0\|_{L^\infty}, \quad \|a_0^n\|_{B_{p,1}^{1+\frac{3}{p}}} \leq 2\|a_0\|_{B_{p,1}^{1+\frac{3}{p}}}, \quad \|u_0^n\|_{B_{p,1}^{1+\frac{3}{p}}+\frac{3}{p}} \leq 2\|u_0\|_{B_{p,1}^{1+\frac{3}{p}}+\frac{3}{p}}, \quad \|B_0^n\|_{B_{p,1}^{1+\frac{3}{p}}} \leq 2\|B_0\|_{B_{p,1}^{1+\frac{3}{p}}}.
\]

(5.1)
and

\[ 1 + a_0^n = 1 + a_0 + (a_0^n - a_0) \geq \frac{1}{2} \kappa. \]  \hspace{1cm} (5.2)

Therefor, applying Theorem 1.1 of [4] ensures that the MHD system (1.8) with the initial data \((a_0^n, u_0^n, B_0^n)\) admits a unique local in time solution \((a^n, u^n, B^n, \nabla \Pi^n)\) belonging to

\[
\begin{align*}
C([0, T^n]; H^{\alpha + 1} (\mathbb{R}^3)) \times C([0, T^n]; H^\alpha (\mathbb{R}^3)) \cap \bar{L}_{loc}^1 (0, T^n; H^{\alpha + 2} (\mathbb{R}^3)) \\
\times C([0, T^n); H^\alpha (\mathbb{R}^3)) \cap \bar{L}_{loc}^1 (0, T^n; H^{\alpha + 2} (\mathbb{R}^3)) \times \bar{L}_{loc}^1 (0, T^n; H^\alpha (\mathbb{R}^3))
\end{align*}
\]  \hspace{1cm} (5.3)

with \(\alpha > \frac{1}{2}\). Moreover, from the transport equation of (1.8), we deduce that

\[
\|a^n(t)\|_{L^\infty} = \|a^n_0\|_{L^\infty} \leq 2 \|a_0\|_{L^\infty}, \quad \forall t \in [0, T^n),
\]  \hspace{1cm} (5.4)

\[
1 + \inf_{(t,x) \in [0, T^n) \times \mathbb{R}^3} a^n(t, x) = 1 + \inf_{y \in \mathbb{R}^3} a_0^n(y) \geq \frac{1}{2} \kappa.
\]  \hspace{1cm} (5.5)

**Step 2. Uniform estimates to the approximate solutions.**

Next let us turn to the uniform estimates for the approximate solutions consequences, that is, we shall prove that there exists a positive time \(T < \inf_{n \in \mathbb{N}} T^n\) such that \((a^n, u^n, B^n, \nabla \Pi^n)\) is uniformly bounded in the space

\[
E_T \defeq \bar{L}_{loc}^\infty (B_{3,1}, 3) \times \bar{L}_{loc}^\infty (B_{1,1}^{-1+\frac{3}{p}}) \cap L_{loc}^1 (B_{1,1}^{1+\frac{3}{p}}) \times \bar{L}_{loc}^\infty (B_{1,1}^{-1+\frac{3}{p}}) \cap L_{loc}^1 (B_{1,1}^{1+\frac{3}{p}}) \times \bar{L}_{loc}^\infty (B_{1,1}^{-1+\frac{3}{p}}).
\]

For this, let \((u_F(t), u^n_F(t)) \defeq (e^{\sigma t} u_0, e^{\sigma t} u^n_0), (B_F(t), B^n_F(t)) \defeq (e^{\sigma t} B_0, e^{\sigma t} B^n_0)\) with \(\sigma \defeq \sigma(0)\).

Then it is easy to observe that

\[
\|(u^n_F, B^n_F)\|_{L^\infty (\mathbb{R}^+; B_{p,1}^{-1+\frac{3}{p}})} + \|(u_F, \sigma B^n_F)\|_{L^1 (\mathbb{R}^+; B_{p,1}^{1+\frac{3}{p}})} \lesssim \|(u^n_0, B^n_0)\|_{B_{p,1}^{-1+\frac{3}{p}}} \lesssim \|(u_0, B_0)\|_{B_{p,1}^{-1+\frac{3}{p}}} \]  \hspace{1cm} (5.6)

and

\[
\begin{align*}
\|u^n_F\|_{L_1^1 (B_{p,1}^{1+\frac{3}{p}})} &\lesssim \|u_F\|_{L_1^1 (B_{p,1}^{1+\frac{3}{p}})} + \|u^n_F - u_F\|_{L_1^1 (B_{p,1}^{1+\frac{3}{p}})} \\
&\lesssim \|u_F\|_{L_1^1 (B_{p,1}^{1+\frac{3}{p}})} + \|u^n_0 - u_0\|_{B_{p,1}^{-1+\frac{3}{p}}},
\end{align*}
\]  \hspace{1cm} (5.7)

\[
\begin{align*}
\|B^n_F\|_{L_1^1 (B_{p,1}^{1+\frac{3}{p}})} &\lesssim \|B_F\|_{L_1^1 (B_{p,1}^{1+\frac{3}{p}})} + \|B^n_F - B_F\|_{L_1^1 (B_{p,1}^{1+\frac{3}{p}})} \\
&\lesssim \|B_F\|_{L_1^1 (B_{p,1}^{1+\frac{3}{p}})} + \|B^n_0 - B_0\|_{B_{p,1}^{-1+\frac{3}{p}}},
\end{align*}
\]  \hspace{1cm} (5.8)
Thus, for any $\epsilon > 0$, there exist a number $k = k(\epsilon) \in \mathbb{N}$ and a positive time $T = T(\epsilon, u_0)$ such that

$$\sup_{n \geq k} \| (u^n, B^n) \|_{L^1_t(B^{1/\gamma}_{p,1})} \leq \epsilon. \tag{5.9}$$

Denote by $\bar{u}^n \equiv u^n - u^n_F$, $\bar{B}^n \equiv B^n - B^n_F$. Then the system for $(a^n, \bar{u}^n, \bar{B}^n, \nabla \Pi^n)$ reads

$$\begin{aligned}
\begin{cases}
\partial_t a^n + (u^n_F + \bar{u}^n) \cdot \nabla a^n = 0, \\
\partial_t \bar{u}^n - \nabla ((1 + a^n) \nabla \bar{u}^n) + (1 + a^n) \nabla \Pi^n = F_n, \\
\partial_t \bar{B}^n - \nabla (\bar{\sigma}(a^n) \nabla \bar{B}^n) + u^n_F \cdot \nabla \bar{B}^n = G_n, \\
\text{div} \bar{u}^n = \bar{B}^n = 0, \\
(a^n, \bar{u}^n, \bar{B}^n)|_{t=0} = (a^n_0, 0, 0),
\end{cases}
\end{aligned} \tag{5.10}$$

where

$$F_n = - \nabla a^n \cdot \nabla \bar{u}^n - u^n_F \cdot \nabla \bar{u}^n - u^n_F \cdot \nabla u^n_F - \bar{u}^n \cdot \nabla u^n_F - \bar{u}^n \cdot \nabla \bar{u}^n + a^n \Delta u^n_F + (1 + a^n)(B^n_F \cdot \nabla B^n + B^n_F \cdot \nabla B^n + B^n \cdot \nabla B^n_F + B^n \cdot \nabla B^n),$$

$$G_n = \bar{B} \cdot \nabla u^n_F - \bar{u}^n \cdot \nabla \bar{B}^n + \bar{B} \cdot \nabla \bar{B}^n = a^n \cdot \nabla B^n_F + B^n_F \cdot \nabla u^n_F + B^n_0 \cdot \nabla \bar{u}^n + \text{div} ((\bar{\sigma}(a^n) - \bar{\sigma}(0)) \nabla \bar{B}^n).$$

For notational simplicity, we denote by $A^n(t) \equiv \| a^n \|_{L^\infty_t(B^{1/\gamma}_{p,1})}$ and

$$Z^n(t) \equiv \| (a^n, B^n) \|_{L^\infty_t(B^{1-1/\gamma}_{p,1})} + \| (a^n, B^n, \nabla \Pi^n) \|_{L^1_t(B^{1/\gamma}_{p,1})} + \| \nabla \Pi^n \|_{L^1_t(B^{1-1/\gamma}_{p,1})}.$$

By Propositions 4.1, 4.2 we have

$$Z^n(t) \lesssim \| (F_n, G_n) \|_{L^1_t(B^{1-1/\gamma}_{p,1})} + 2^\frac{3}{\gamma} (1 + \| a^n \|_{L^\infty_t(B^{1/\gamma}_{p,1})})^2 \| F_n \|_{L^1_t(B^{1/\gamma}_{p,1})}$$

$$+ 2^m \| a^n \|_{L^\infty_t(B^{1/\gamma}_{p,1})} \| (a^n, B^n) \|_{L^1_t(B^{1/\gamma}_{p,1})} + \int_0^1 \| u^n_F \|_{L^1(B^{1/\gamma}_{p,1})} \| B^n \|_{L^1(B^{1/\gamma}_{p,1})} d\tau$$

$$+ 2^\frac{3}{\gamma} (1 + \| a^n \|_{L^\infty_t(B^{1/\gamma}_{p,1})})^3 \| a^n \|_{L^1_t(B^{1+1/\gamma}_{p,1})}.$$

According to $\text{Id} = \hat{S}_m + (\text{Id} - \hat{S}_m)$ and Lemma 2.5 one has

$$\| \nabla a^n \cdot \nabla \bar{u}^n \|_{L^1_t(B^{1-1/\gamma}_{p,1})} \lesssim \| \nabla \hat{S}_m a^n \cdot \nabla \bar{u}^n \|_{L^1_t(B^{1+1/\gamma}_{p,1})} + \| \nabla (a^n - \hat{S}_m a^n) \cdot \nabla \bar{u}^n \|_{L^1_t(B^{1+1/\gamma}_{p,1})}$$

$$\lesssim 2^m \| a^n \|_{L^\infty_t(B^{1/\gamma}_{p,1})} \| \bar{u}^n \|_{L^1_t(B^{1/\gamma}_{p,1})} + \| a^n - \hat{S}_m a^n \|_{L^\infty_t(B^{1/\gamma}_{p,1})} \| \bar{u}^n \|_{L^1_t(B^{1+1/\gamma}_{p,1})}.$$

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By Remark 2.6 we have
\[
\|\nabla a^n \cdot \nabla \bar{u}^n\|_{L^1_t(B_{p,1}^{-\frac{1}{5}+\frac{2}{p}})} 
\lesssim \|\nabla \bar{S}_m a^n \cdot \nabla u^n\|_{L^1_t(B_{p,1}^{-\frac{1}{5}+\frac{2}{p}})} + \|\nabla (a^n - \bar{S}_m a^n) \cdot \nabla \bar{u}^n\|_{L^1_t(B_{p,1}^{-\frac{1}{5}+\frac{2}{p}})}
\]
\[
\lesssim \|\nabla \bar{S}_m a^n\|_{L^\infty_t(B_{3,1}^{-\frac{1}{5}+\frac{2}{p}})} \|\nabla \bar{a}^n\|_{L^1_t(B_{p,1}^{-\frac{1}{5}+\frac{2}{p}})} + \|\nabla (a^n - \bar{S}_m a^n)\|_{L^\infty_t(B_{3,1}^{-\frac{1}{5}+\frac{2}{p}})} \|\nabla \bar{u}^n\|_{L^1_t(B_{p,1}^{-\frac{1}{5}+\frac{2}{p}})}
\]
\[
\lesssim t^\frac{1}{2} \|\bar{a}^n\|_{L^1_t(B_{p,1}^{-\frac{1}{5}+\frac{2}{p}})} \|\bar{u}^n\|_{L^1_t(B_{p,1}^{-\frac{1}{5}+\frac{2}{p}})} + \|\bar{a}^n\|_{L^1_t(B_{p,1}^{-\frac{1}{5}+\frac{2}{p}})} \|\bar{u}^n\|_{L^1_t(B_{p,1}^{-\frac{1}{5}+\frac{2}{p}})}.
\]
\[
\text{(5.13)}
\]
in which we have used the following interpolation inequality:
\[
\|\bar{a}^n\|_{L^1_t(B_{p,1}^{-\frac{1}{5}+\frac{2}{p}})} \leq t^\frac{1}{2} \|\bar{a}^n\|_{L^1_t(B_{p,1}^{-\frac{1}{5}+\frac{2}{p}})} \|\bar{a}^n\|_{L^1_t(B_{p,1}^{-\frac{1}{5}+\frac{2}{p}})}.
\]

Similarly, we can get
\[
\|a^n \Delta u^n_F\|_{L^1_t(B_{p,1}^{-\frac{1}{5}+\frac{2}{p}})} + \|a^n \Delta u^n_F\|_{L^1_t(B_{p,1}^{-\frac{1}{5}+\frac{2}{p}})} \lesssim \|a^n\|_{L^\infty_t(B_{3,1}^{-\frac{1}{5}+\frac{2}{p}})} \left(\|u^n_F\|_{L^1_t(B_{p,1}^{-\frac{1}{5}+\frac{2}{p}})} + \|u^n_F\|_{L^1_t(B_{p,1}^{-\frac{1}{5}+\frac{2}{p}})}\right).
\]
\[
\text{(5.14)}
\]
Yet thanks to \(\text{div} u^n_F = \text{div} \bar{u}^n = 0\), we get by using product laws and interpolation inequality in Besov spaces
\[
\|\bar{a}^n \cdot \nabla \bar{u}^n + u^n_F \cdot \nabla u^n_F + \bar{u}^n \cdot \nabla u^n_F + u^n_F \cdot \nabla \bar{u}^n\|_{L^1_t(B_{p,1}^{-\frac{1}{5}+\frac{2}{p}})}
\]
\[
\lesssim \|\bar{a}^n \otimes \bar{u}^n + u^n_F \otimes u^n_F + \bar{u}^n \otimes u^n_F + u^n_F \otimes \bar{u}^n\|_{L^1_t(B_{p,1}^{-\frac{1}{5}+\frac{2}{p}})}
\]
\[
\lesssim \int_0^t \|\bar{a}^n\|_{B_{p,1}^{-\frac{1}{5}+\frac{2}{p}}} \|\bar{u}^n\|_{B_{p,1}^{-\frac{1}{5}+\frac{2}{p}}} + \|u^n_F\|_{B_{p,1}^{-\frac{1}{5}+\frac{2}{p}}} \|u^n_F\|_{B_{p,1}^{-\frac{1}{5}+\frac{2}{p}}} + \|\bar{a}^n\|_{B_{p,1}^{-\frac{1}{5}+\frac{2}{p}}} \|u^n_F\|_{B_{p,1}^{-\frac{1}{5}+\frac{2}{p}}} \|u^n_F\|_{B_{p,1}^{-\frac{1}{5}+\frac{2}{p}}} \, d\tau
\]
\[
\lesssim \int_0^t \|\bar{a}^n\|_{B_{p,1}^{-\frac{1}{5}+\frac{2}{p}}} \|\bar{a}^n\|_{B_{p,1}^{-\frac{1}{5}+\frac{2}{p}}} + \|u^n_F\|_{B_{p,1}^{-\frac{1}{5}+\frac{2}{p}}} \|u^n_F\|_{B_{p,1}^{-\frac{1}{5}+\frac{2}{p}}} \|u^n_F\|_{B_{p,1}^{-\frac{1}{5}+\frac{2}{p}}} \, d\tau
\]
\[
\lesssim (Z^n(t))^2 + \|u_0\|_{B_{p,1}^{-\frac{1}{5}+\frac{2}{p}}} \|u^n_F\|_{L^1_t(B_{p,1}^{-\frac{1}{5}+\frac{2}{p}})}.
\]
\[
\text{(5.15)}
\]
Similarly,
\[
\|\text{div} \left( (\tilde{a}(a^n) - \tilde{a}(0)) \nabla B^n_F \right)\|_{L^1_t(B_{p,1}^{-\frac{1}{5}+\frac{2}{p}})} \lesssim \|a^n\|_{L^\infty_t(B_{3,1}^{-\frac{1}{5}+\frac{2}{p}})} \|B^n_F\|_{L^1_t(B_{p,1}^{-\frac{1}{5}+\frac{2}{p}})}
\]
\[
\text{(5.16)}
\]
\[
\|(1 + a^n)(B^n \cdot \nabla B^n + B^n \cdot \nabla B^n + B^n \cdot \nabla B^n + B^n \cdot \nabla B^n)\|_{L^1_t(B_{p,1}^{-\frac{1}{5}+\frac{2}{p}})}
\]
\[
\lesssim \left(1 + \|a^n\|_{L^\infty_t(B_{3,1}^{-\frac{1}{5}+\frac{2}{p}})}\right) (Z^n(t))^2 + \|B_0\|_{B_{p,1}^{-\frac{1}{5}+\frac{2}{p}}} \|u^n_F\|_{L^1_t(B_{p,1}^{-\frac{1}{5}+\frac{2}{p}})},
\]
\[
\text{(5.17)}
\]
\[
\|B^n \cdot \nabla u^n_F - \bar{u}^n \cdot \nabla B^n + B^n \cdot \nabla \bar{u}^n - u^n_F \cdot \nabla B^n - \bar{u}^n \cdot \nabla B^n + B^n_F \cdot \nabla u^n_F + B^n_F \cdot \nabla \bar{u}^n\|_{L^1_t(B_{p,1}^{-\frac{1}{5}+\frac{2}{p}})}
\]
\[
\lesssim (Z^n(t))^2 + \||u_0, B_0\|_{B_{p,1}^{-\frac{1}{5}+\frac{2}{p}}} \|u^n_F, B^n_F\|_{L^1_t(B_{p,1}^{-\frac{1}{5}+\frac{2}{p}})}.
\]
\[
\text{(5.18)}
\]
Thus
\[
\|G_0\|_{L^1_t(B^{-\frac{3}{2}}_{p,1})} \lesssim (Z^n(t))^2 + \|(u_0, B_0)\|_{B^{-\frac{3}{2}}_{p,1}} \big(\|u^n_F, B^n_F\|_{L^2_t(B^{-\frac{3}{2}}_{p,1})} + \|a^n\|_{L^\infty_t(B^{-\frac{3}{2}}_{p,1})} \big) \lesssim \|B^n_F\|_{L^1_t(B^{-\frac{3}{2}}_{p,1})}.
\] (5.19)

By Lemma 2.5 and interpolation inequality in Chemin-Lerner spaces, we have
\[
\|a^n \cdot \nabla a^n + u^n_F \cdot \nabla u^n_F + a^n \cdot \nabla u^n_F + u^n_F \cdot \nabla a^n\|_{L^1_t(B^{-\frac{3}{2}}_{p,1})} 
\lesssim \int_0^t \|a^n \cdot \nabla a^n\|_{B^{-\frac{3}{2}}_{p,1}} + \|u^n_F \cdot \nabla u^n_F\|_{B^{-\frac{3}{2}}_{p,1}} + \|a^n \cdot u^n_F\|_{B^{-\frac{3}{2}}_{p,1}} + \|u^n_F \cdot a^n\|_{B^{-\frac{3}{2}}_{p,1}} \, d\tau 
\lesssim \int_0^t \|a^n\|_{B^{-\frac{3}{2}}_{p,1}}^\frac{1}{2} + \|u^n_F\|_{B^{-\frac{3}{2}}_{p,1}}^\frac{1}{2} + \|u^n_F\|_{B^{-\frac{3}{2}}_{p,1}} \, d\tau + \int_0^t \|u^n_F\|_{B^{-\frac{3}{2}}_{p,1}} + \|a^n\|_{B^{-\frac{3}{2}}_{p,1}} \, d\tau 
\lesssim t^{\frac{1}{2}}((Z^n(t))^2 + \|u_0\|_{B^{-\frac{3}{2}}_{p,1}}) + \|u_0\|_{B^{-\frac{3}{2}}_{p,1}} \|u^n_F\|_{L^1_t(B^{-\frac{3}{2}}_{p,1})} + (Z^n(t))^2.
\] (5.20)

Similarly,
\[
\|(1 + a^n)(B^n \cdot \nabla B^n + B^n_F \cdot \nabla B^n_F + B^n \cdot \nabla B^n + B^n_F \cdot \nabla B^n)\|_{L^1_t(B^{-\frac{3}{2}}_{p,1})} 
\lesssim (1 + \|a^n\|_{L^\infty_t(B^{-\frac{3}{2}}_{p,1})}) \left(t^{\frac{1}{2}}((Z^n(t))^2 + \|B_0\|_{B^{-\frac{3}{2}}_{p,1}}) + \|B_0\|_{B^{-\frac{3}{2}}_{p,1}} \|B^n_F\|_{L^1_t(B^{-\frac{3}{2}}_{p,1})} + (Z^n(t))^2 \right). \] (5.21)

As a consequence, we obtain
\[
\|(F_n, G_n)\|_{L^1_t(B^{-\frac{3}{2}}_{p,1})} \lesssim \|a^n\|_{L^\infty_t(B^{-\frac{3}{2}}_{p,1})}^\frac{1}{2} \|a^n\|_{L^2_t(B^{-\frac{3}{2}}_{p,1})} + \|a^n \cdot \hat{S}_m a^n\|_{L^\infty_t(B^{-\frac{3}{2}}_{p,1})} + \|a^n\|_{L^1_t(B^{-\frac{3}{2}}_{p,1})} + (1 + \|a^n\|_{L^\infty_t(B^{-\frac{3}{2}}_{p,1})}) ((Z^n(t))^2 + \|(u_0, B_0)\|_{B^{-\frac{3}{2}}_{p,1}}^\frac{1}{2} \|u^n_F\|_{L^1_t(B^{-\frac{3}{2}}_{p,1})}) 
\lesssim \varepsilon Z^n(t) + \|a^n \cdot \hat{S}_m a^n\|_{L^\infty_t(B^{-\frac{3}{2}}_{p,1})} Z^n(t)
\]
\[
\quad + (1 + 2^n(A^n(t))^2)(Z^n(t))^2 + (1 + A^n(t)) \|(u_0, B_0)\|_{B^{-\frac{3}{2}}_{p,1}}^\frac{1}{2} \|u^n_F, B^n_F\|_{L^1_t(B^{-\frac{3}{2}}_{p,1})} \lesssim \varepsilon Z^n(t) + A^n(t)Z^n(t) 
\quad + (1 + A^n(t)) t^{\frac{1}{2}} A^n(t)Z^n(t) 
\quad + (1 + A^n(t)) t^{\frac{1}{2}} ((u_0, B_0) \|B^n_F\|_{L^1_t(B^{-\frac{3}{2}}_{p,1})} + (Z^n(t))^2). \] (5.22)

Choosing \(\varepsilon\) small enough, using the Young inequality and
\[
(1 + \|a\|_{L^\infty_t(B^{-\frac{3}{2}}_{p,1})})^2 \|a - \hat{S}_m a\|_{L^\infty_t(B^{-\frac{3}{2}}_{p,1})} \leq c_0,
\] (5.24)
we have

\[ Z^n(t) \lesssim (1 + 2^{2m}(A^n(t))^2)(Z^n(t))^2 \]

\[ + 2^{\frac{3}{16}} (1 + A^n(t))^2 \left[ t^{\frac{3}{16}} A^n(t) Z^n(t) + A^n(t) \| u^n_F \|_{L^1_t(B^{1+\frac{1}{16}}_{p,1})} \right] \]

\[ + (1 + A^n(t)) \left( t^{\frac{1}{16}} (Z^n(t))^2 + \| (u_0, B_0) \|_{B^{-1+\frac{1}{16}}_{p,1}} + \| (u_0, B_0) \|_{B^{-1+\frac{1}{16}}_{p,1}} (u^n_F, B^n_F) \|_{L^1_t(B^{1+\frac{1}{16}}_{p,1})} \right). \]

(5.25)

On the other hand, applying Proposition 2.10 to the transport equation of (5.10), we have for \( t \in [0, T^n) \)

\[ A^n(t) \leq \| a^n_0 \|_{L^\infty_t(B^{\frac{3}{16}}_{q,1})} (\| u^n_F \|_{L^1_t(B^{1+\frac{1}{16}}_{p,1})}) \]

\[ \leq C \| a_0 \|_{L^\infty_t(B^{\frac{3}{16}}_{q,1})} \exp (C Z^n(t) + \| u_0 \|_{B^{-1+\frac{1}{16}}_{p,1}})), \]

(5.26)

and

\[ \| A^n - \dot{S}_m A^n \|_{L^\infty_t(B^{\frac{3}{16}}_{q,1})} \]

\[ \leq \sum_{j \geq m} 2^{\frac{3j}{16}} \| \Delta_j a^n_0 \|_{L^0_t} + \| a^n_0 \|_{L^\infty_t(B^{\frac{3}{16}}_{q,1})} \left( \exp (C \| u^n_F \|_{L^1_t(B^{1+\frac{1}{16}}_{p,1})}) - 1 \right) \]

\[ \leq \sum_{j \geq m} 2^{\frac{3j}{16}} \| \Delta_j a^n_0 \|_{L^0_t} + \| a^n_0 - a^n_0 \|_{L^\infty_t(B^{\frac{3}{16}}_{q,1})} \]

\[ + C \| a_0 \|_{L^\infty_t(B^{\frac{3}{16}}_{q,1})} \left( \exp (C Z^n(t) + C \| u^n_F \|_{L^1_t(B^{1+\frac{1}{16}}_{p,1})}) - 1 \right). \]

(5.27)

However, for any function \( \chi \in \mathcal{D}(\mathbb{R}) \) with \( \chi(0) = 0 \), the composite function \( \chi(a^n) \) with initial data \( \chi(a^n_0) \) also solves the renormalized transport equation

\[ \partial_t \chi(a^n) + (u^n_F + \dot{a}^n) \cdot \nabla \chi(a^n) = 0. \]

Whence a similar argument as in (5.27) leads to

\[ \| \chi(a^n) - \dot{S}_m \chi(a^n) \|_{L^\infty_t(B^{\frac{3}{16}}_{q,1})} \]

\[ \leq \sum_{j \geq m} 2^{\frac{3j}{16}} \| \Delta_j \chi(a^n_0) \|_{L^0_t} + \| \chi(a^n_0) \|_{L^\infty_t(B^{\frac{3}{16}}_{q,1})} \left( \exp (C Z^n(t) + C \| u^n_F \|_{L^1_t(B^{1+\frac{1}{16}}_{p,1})}) - 1 \right) \]

\[ \leq \sum_{j \geq m} 2^{\frac{3j}{16}} \| \Delta_j \chi(a_0) \|_{L^5_t} + C \| a_0 \|_{L^\infty_t(B^{\frac{3}{16}}_{q,1})} \| a^n_0 - a^n_0 \|_{L^\infty_t(B^{\frac{3}{16}}_{q,1})} \]

\[ + C \| a_0 \|_{L^\infty_t(B^{\frac{3}{16}}_{q,1})} \left( \exp (C Z^n(t) + C \| u^n_F \|_{L^1_t(B^{1+\frac{1}{16}}_{p,1})}) - 1 \right), \]

(5.28)
where we used
\[
\|\chi(a_0^n) - \chi(a_0)\|_{\dot{B}_{q,1}^{3}} = \left\| (a_0^n - a_0) \int_0^1 \chi'(\tau a_0^n + (1 - \tau)a_0) d\tau \right\|_{\dot{B}_{q,1}^{3}} \\
\leq C (1 + \|a_0\|_{\dot{B}_{q,1}^{3}}) \|a_0^n - a_0\|_{\dot{B}_{q,1}^{3}}.
\] (5.29)

As a consequence, we obtain for \(t \in [0, T^n]\)
\[
\|(\Id - \dot{S}_m) a^n\|_{L_t^q(\dot{B}_{q,1}^{3})} \leq \sum_{j \geq m} 2^{3j} \left( \|\Delta_j a_0\|_{L^q} + C (1 + \|a_0\|_{\dot{B}_{q,1}^{3}}) \|a_0^n - a_0\|_{\dot{B}_{q,1}^{3}} \right. \\
+ C \|a_0\|_{\dot{B}_{q,1}^{3}} \left. \left( \exp (CZ^n(t) + C\|u^n_p\|_{L_t^1(\dot{B}_{p,1}^{1 + \frac{3}{p}})}) - 1 \right) \right). \] (5.30)

Next, for any \(n \in \mathbb{N}\), we define
\[
T^n_* \defeq \sup\{t \in (0, T^n) : Z^n(t) \leq 2\epsilon_0\},
\] (5.31)
with \(\epsilon_0 \in (0, \frac{1}{2})\) to be determined. We shall prove \(\inf_{n \in \mathbb{N}} T^n_* > 0\).

Firstly, we deduce from (5.26) for \(t \leq T^n_*\) that
\[
A^n(t) \leq C \|a_0\|_{\dot{B}_{q,1}^{3}} \exp \left( C (1 + \|u_0\|_{L_t^1(\dot{B}_{p,1}^{1 + \frac{3}{p}})}) \right) \defeq A_0.
\] (5.32)

Noticing that \(a_0 \in \dot{B}_{q,1}^{3}(\mathbb{R}^3)\), there exist \(m = m(c_0) \in \mathbb{Z}\) and \(n_0 = n_0(c_0) \in \mathbb{N}\) such that
\[
(1 + A_0)^2 \left( \sum_{j \geq m} 2^{3j} \left( \|\Delta_j a_0\|_{L^q} + \sup_{n \geq n_0} C (1 + \|a_0\|_{\dot{B}_{q,1}^{3}}) \|a_0^n - a_0\|_{\dot{B}_{q,1}^{3}} \right) \right) \leq \frac{1}{2} c_0.
\] (5.33)

Yet thanks to (5.9), taking \(\epsilon_0\) and \(T_0\) small enough and \(n_1 \geq n_0\) large enough ensures
\[
C (1 + A_0)^2 \|a_0\|_{\dot{B}_{q,1}^{3}} \left( \exp (2C\epsilon_0 + C\|u^n_p\|_{L_t^1(\dot{B}_{p,1}^{1 + \frac{3}{p}})}) - 1 \right) \leq \frac{1}{2} c_0
\] (5.34)
for any \(n \geq n_1\). Combining (5.32)–(5.34) implies that (5.31) with \(T = \min(T^n_*, T_0)\) is fulfilled for any \(n \geq n_1\). Without loss of generality, we may assume \(T^n_* \leq T_0\). Then for any \(t \leq T^n_*\), we deduce from (5.25) that
\[
Z^n(t) \lesssim (1 + 2^{2m} A_0^2) \epsilon_0 Z^n(t) \\
+ 2^{3j} (1 + A_0)^4 \left[ t^\frac{1}{2} A_0 \epsilon_0 + A_0 \|u^n_p\|_{L_t^1(\dot{B}_{p,1}^{1 + \frac{3}{p}})} \right. \\
+ (1 + A_0) \left( t^\frac{1}{2} (\epsilon_0^2 + \|u_0, B_0\|_{L_t^1(\dot{B}_{p,1}^{1 + \frac{3}{p}})}) + \|u_0, B_0\|_{L_t^1(\dot{B}_{p,1}^{1 + \frac{3}{p}})} \right). \] (5.35)
Finally, taking \( \varepsilon_0 \) and \( T_1 \) small enough and \( n_2 \geq n_1 \) large enough ensures for any \( n \geq n_2 \)
\[
(1 + 2^{2n} A_0^2) \varepsilon_0 \leq \frac{1}{2},
\]
\[
2^n (1 + A_0) \quad T_1 \quad A_0 + (1 + A_0) T_1 \quad (\varepsilon_0^2 + \| (u_0, B_0) \|_{B_{p,1}^{\frac{1}{2}},1}) \leq \frac{\varepsilon_0}{4},
\]
and
\[
2^n (1 + A_0)^2 \| (u_0, B_0) \|_{B_{p,1}^{1+\frac{1}{p}},1} \| (u^n, B^n) \|_{L_1^1(B_{p,1}^{1+\frac{1}{p}})} \leq \frac{\varepsilon_0}{4},
\]
which together with (5.35) implies
\[
Z^n(t) \leq \varepsilon_0, \quad \forall t \leq \min(T^n_*, T_1), \quad n \geq n_2.
\]

However, by the definition of \( T^n_* \), we eventually infer \( T^n_* \geq T_1 \) and \( \sup_{n \geq n_2} Z^n(T_1) \leq \varepsilon_0 \), which along with (5.6) and (5.32) ensures that \( (a^n, u^n, B^n, \nabla \Pi^n) \) is uniformly bounded in \( E_{T_1} \).

**Step 3. Convergence.**

The convergence of \( (u^n, B^n) \) to \( (u, B) \) readily stems from the definition of Besov spaces. As for the convergence of \( (a^n, \dot{u}^n, \dot{B}^n) \), it relies upon Ascoli’s theorem compactness properties of the consequence, which are obtained by considering the time derivative of the solution, we omit the details here.

**Step 4. Uniqueness.**

The goal of this section is to prove the uniqueness of our main theorem. We will follow the method used in [3] to complete our proof. Before giving the details, we also need the following lemma which can be proved similarly as Lemma 4.1 in [3].

**Lemma 5.1.** (see [3]) Let \( p \in [3, 4], q \in [1, 2], \quad \frac{1}{q} - \frac{1}{p} \leq \frac{1}{3} \), and
\[
(a^i, u^i, B^i, \nabla \Pi^i) \in C_b(0, T; B_{q,1}^{\frac{3}{2}}) \times (C_b(0, T; B_{p,1}^{-1+\frac{1}{p}}) \cap L_1^1(B_{p,1}^{1+\frac{1}{p}}))^2 \times L_1^1(B_{p,1}^{-1+\frac{1}{p}}),
\]
for \( i = 1, 2 \), be two solutions of the system (1.8) and satisfy (1.3) for some \( m \). Denote
\[
(\delta a, \delta u, \delta B, \nabla \delta \Pi) \triangleq (a^2 - a^1, u^2 - u^1, B^2 - B^1, \nabla \Pi^2 - \nabla \Pi^1).
\]
Then there holds
\[
(\delta a, \delta u, \delta B, \nabla \delta \Pi) \in C_b([0, T]; B_{2,1}^3) \times C_b([0, T]; B_{2,1}^{-1/2} \cap L^1([0, T]; B_{2,1}^3)) \times C_b([0, T]; B_{2,1}^{-1/2})
\]
\[
\cap L^1([0, T]; B_{2,1}^3) \times L^1([0, T]; B_{2,1}^{-1/2}).
\]
(5.39)
Now, let us begin to prove our uniqueness, it’s easy to get \((\delta a, \delta u, \delta B, \nabla \delta \Pi)\) solves

\[
\begin{aligned}
\partial_t \delta a + u^2 \cdot \nabla \delta a &= -\delta u \cdot \nabla a^1, \\
\partial_t \delta u + u^2 \cdot \nabla \delta u - (1 + S_m a^2)(\Delta \delta u - \nabla \delta \Pi) &= F_1(a^1, u^1, B^1, \nabla \Pi^1), \\
\partial_t \delta B + u^2 \cdot \nabla \delta B - \text{div}(\tilde{\sigma}(a^2) \nabla \delta B) &= F_2(a^1, u^1, B^1), \\
\text{div} u &= \text{div} B = 0,
\end{aligned}
\]

\((\delta a, \delta u, \delta B)|_{t=0} = (0, 0, 0),\)

where

\[
F_1(a^1, u^1, B^1, \nabla \Pi^1) = (a^2 - S_m a^2)(\Delta \delta u - \nabla \delta \Pi) - \delta u \cdot \nabla u^1 + \delta a(\Delta u^1 - \nabla \Pi^1)
+ (1 + a^2)(\delta B \cdot \nabla B^2 + B^1 \cdot \nabla \delta B) + \delta a(B^1 \cdot \nabla B^1),
\]

\[
F_2(a^1, u^1, B^1) = -\delta u \cdot \nabla B^1 + \delta B \cdot \nabla u^2 + B^1 \cdot \nabla \delta u + \text{div}((\tilde{\sigma}(a^2) - \tilde{\sigma}(a^1)) \nabla B^1).
\]

Using a similar method as in Proposition 4.2, we can get from the third equation of (5.40) that

\[
\|\delta B\|_{L^\infty_t(B^\frac{1}{2}_{2,1})} + \|\delta B\|_{L^1_t(B^\frac{3}{2}_{2,1})} 
\lesssim \|\Delta_j u^2 \cdot \nabla \delta B\|_{L^1_t(B^\frac{1}{2}_{2,1})} + \|F_2\|_{L^1_t(B^\frac{1}{2}_{2,1})} + \sum_{j \in \mathcal{Z}} 2^j \|\Delta_j \tilde{\sigma}(S_m a^2) \nabla \delta B\|_{L^1_t(L^2)} + \|\text{div}((\tilde{\sigma}(a^2) - \tilde{\sigma}(S_m a^2)) \nabla B)\|_{L^1_t(B^\frac{1}{2}_{2,1})}.
\]

By Lemmas 2.4, 2.5, 2.8 one has

\[
\|\Delta_j u^2 \cdot \nabla \delta B\|_{L^1_t(B^\frac{1}{2}_{2,1})} \lesssim \int_0^t \|\nabla u^2\|_{B^{\frac{1}{2}}_{p,1}} \|\delta B\|_{B_{2,1}^{\frac{1}{2}}} d\tau,
\]

\[
\|\text{div}((\tilde{\sigma}(a^2) - \tilde{\sigma}(S_m a^2)) \nabla B)\|_{L^1_t(B^\frac{1}{2}_{2,1})} \lesssim \|\tilde{\sigma}(a^2) - \tilde{\sigma}(S_m a^2)\|_{L^1_t(B^\frac{1}{2}_{2,1})} \lesssim C \|a^2 - S_m a^2\|_{L^\infty_t(B^\frac{1}{2}_{2,1})} \|\delta B\|_{L^1_t(B^\frac{1}{2}_{2,1})},
\]

\[
n \sum_{j \in \mathcal{Z}} 2^j \|\Delta_j \tilde{\sigma}(S_m a^2) \nabla \delta B\|_{L^1_t(L^2)} \lesssim \int_0^t \|\nabla \tilde{\sigma}(S_m a^2)\|_{B^{\frac{3}{2}}_{2,1}} \|\delta B\|_{B_{2,1}^{\frac{1}{2}}} d\tau
\lesssim \epsilon \|\delta B\|_{L^1_t(B^\frac{1}{2}_{2,1})} + 2^n \int_0^t \|a^2\|_{B^{\frac{1}{2}}_{2,1}} \|\delta B\|_{B_{2,1}^{\frac{1}{2}}} d\tau,
\]
Thus, inserting the above estimates (5.42)–(5.46) into (5.41) and taking \( \varepsilon \) small enough, we have

\[
\| \delta B \|^2_{L^\infty_t(B^{\frac{1}{2},1})} + \| \delta B \|^2_{L^1_t(B^{\frac{1}{2},1})} \lesssim \int_0^t \| \delta u \|^2_{B^{\frac{1}{2},1}} + \| \delta B \|^2_{B^{\frac{1}{2},1}} + \| \delta B \|^2_{B^{\frac{1}{2},1}} (\| \delta B \|^2_{B^{\frac{1}{2},1}} + \| u^2 \|^2_{B^{\frac{1}{2},1}}) \, d\tau
\]

(5.47)

Applying Proposition 2.11 to the second equation of (5.40), we can get

\[
\| \delta u \|^2_{L^\infty_t(B^{\frac{1}{2},1})} + \| \delta u \|^2_{L^1_t(B^{\frac{1}{2},1})} \lesssim \left\{ \int_0^t \| \delta u \|^2_{B^{\frac{1}{2},1}} \| \nabla \delta u \|^2_{H^{-1,\frac{1}{2}}} + \| S_m a^2 \|^2_{L^\infty_t(H^{\frac{3}{2}+\varepsilon})} \| \nabla \delta \Pi \|^2_{L^1_t(H^{-\frac{1}{2}-\varepsilon})} \right. \\
+ \| S_m a^2 \|^2_{L^\infty_t(H^2)} \| \nabla \delta u \|^2_{L^1_t(L^2)} + \| \delta u \|^2_{L^1_t(L^2)} + \left. \| F_1(a^i, u^i, B^i, \nabla \Pi^i) \|^2_{L^1_t(B^{\frac{1}{2},1})} \right\},
\]

(5.48)

where

\[
\begin{align*}
&\| F_1 \|^2_{L^1_t(B^{\frac{1}{2},1})} \lesssim \| a^2 - S_m a^2 \|^2_{L^\infty_t(B^{\frac{1}{2},1})} (\| \delta u \|^2_{L^1_t(B^{\frac{1}{2},1})} + \| \nabla \delta \Pi \|^2_{L^1_t(B^{\frac{1}{2},1})}) \\
&\quad + \| \delta u \cdot \nabla u \|^2_{L^1_t(B^{\frac{1}{2},1})} + \| \delta a \Delta u - \nabla \Pi^i \|^2_{L^1_t(B^{\frac{1}{2},1})} + \| (1 + a^2) \delta B \cdot \nabla B^2 \|^2_{L^1_t(B^{\frac{1}{2},1})} \\
&\quad + \| (1 + a^2) (B^1 \cdot \nabla \delta B) \|^2_{L^1_t(B^{\frac{1}{2},1})} + \| \delta a (B^1 \cdot \nabla B^1) \|^2_{L^1_t(B^{\frac{1}{2},1})}.
\end{align*}
\]

(5.49)
Taking divergence to the second equation of (5.41), we have

\[ \text{div}((1 + S_m a^2) \nabla \delta \Pi) = \text{div} F_3, \quad (5.50) \]

with

\[ F_3 = -u^2 \cdot \nabla \delta u + S_m a^2 \Delta \delta u + F_1. \]

By Proposition 3.4 in [3], we have

\[ \| \nabla \delta \Pi \|_{L^1_t(B_{2,1}^+)} \lesssim \| F_3 \|_{L^1_t(B_{2,1}^+)} + \| S_m a^2 \|_{L^\infty_t(H^{3/2 + \alpha})} \| \nabla \delta \Pi \|_{L^1_t(H^{3/2 + \alpha})} \]
\[ \lesssim \| u^2 \cdot \nabla \delta u \|_{L^1_t(B_{2,1}^+)} + \| S_m a^2 \|_{L^\infty_t(B_{2,1}^+)} \| \Delta \delta u \|_{L^1_t(B_{2,1}^+)} \]
\[ + \| F_1 \|_{L^1_t(B_{2,1}^+)} + \| S_m a^2 \|_{L^\infty_t(H^{3/2 + \alpha})} \| \nabla \delta \Pi \|_{L^1_t(\dot{H}^{3/2 + \alpha})}. \quad (5.51) \]

Taking the above estimate into (5.49), we get from (5.48) that

\[ \| \delta u \|_{L^\infty_t(B_{2,1}^+)} + \| \delta u \|_{L^1_t(B_{2,1}^+)} \]
\[ \lesssim \int_0^t \| \delta u \|_{B_{2,1}^+} \| \nabla u^2 \|_{B_{2,1}^+} + \| S_m a^2 \|_{L^\infty_t(H^{3/2 + \alpha})} \| \nabla \delta \Pi \|_{L^1_t(H^{3/2 + \alpha})} \]
\[ + \| S_m a^2 \|_{L^\infty_t(H^2)} \| \nabla \delta u \|_{L^1_t(L^2)} + \| \delta u \|_{L^1_t(L^2)} + \| u^2 \cdot \nabla \delta u \|_{L^1_t(B_{2,1}^+)} \]
\[ + \| \delta u \cdot \nabla u^1 \|_{L^1_t(B_{2,1}^+)} + \| \delta a(\Delta u^1 - \nabla \Pi^1) \|_{L^1_t(B_{2,1}^+)} + \| \delta a(B^1 \cdot \nabla B^1) \|_{L^1_t(B_{2,1}^+)} \]
\[ + \| (1 + a^2) \delta B \cdot \nabla B^2 \|_{L^1_t(B_{2,1}^+)} + \| (1 + a^2)(B^1 \cdot \nabla B) \|_{L^1_t(B_{2,1}^+)} \quad (5.52) \]

Firstly, it follows from Proposition 2.10 that

\[ \| \delta a \|_{L^\infty_t(B_{2,1}^+)} \lesssim \exp \left\{ C \| \nabla u^2 \|_{L^1_t(B_{2,1}^+)} \| a^1 \|_{L^\infty_t(B_{2,1}^+)} \| \delta u \|_{L^1_t(B_{2,1}^+)} \right\} \quad (5.53) \]

In what follows, we will estimate the terms on the right hand side of (5.52). By Lemma 2.5, Young’s inequality and (5.53), we have

\[ \| \delta a(B^1 \cdot \nabla B^1) \|_{L^1_t(B_{2,1}^+)} + \| \delta a(\Delta u^1 - \nabla \Pi^1) \|_{L^1_t(B_{2,1}^+)} \]
\[ \lesssim \| \delta a(\Delta u^1 - \nabla \Pi^1) \|_{L^1_t(B_{2,1}^+)} + \| \delta a(B^1 \cdot \nabla B^1) \|_{L^1_t(B_{2,1}^+)} \]
\[ \lesssim \int_0^t \| \delta a \|_{B_{2,1}^+} \| \Delta u^1 \|_{B_{3/2,1}^+} + \| \nabla \Pi^1 \|_{B_{3/2,1}^+} + \| B^1 \cdot \nabla B^1 \|_{B_{3/2,1}^+} d\tau \]
\[ \lesssim \int_0^t \exp \left\{ C \| \nabla u^2 \|_{L^1_t(B_{3/2,1}^+)} \| a^1 \|_{L^\infty_t(B_{2,1}^+)} \| \delta u \|_{L^1_t(B_{2,1}^+)} \right\} d\tau \]

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\[
\times (\| \Delta u^1 \|_{B_{p,1}^{\frac{2}{p} + \frac{1}{2}}} + \| \nabla \Pi^1 \|_{B_{p,1}^{\frac{2}{p} + \frac{1}{2}}} + \| B^1 \|_{B_{p,1}^{\frac{2}{p} + \frac{1}{2}}} ) d\tau \\
\lesssim \int_0^t \| \delta u \|_{L^1_t(B_{2,1}^1)} (\| \Delta u^1 \|_{B_{p,1}^{\frac{2}{p} + \frac{1}{2}}} + \| \nabla \Pi^1 \|_{B_{p,1}^{\frac{2}{p} + \frac{1}{2}}} + \| B^1 \|_{B_{p,1}^{\frac{2}{p} + \frac{1}{2}}} ) d\tau,
\]

(5.54)

and

\[
\| u^2 \cdot \nabla u \|_{L^1_t(B_{2,1}^1)} + \| \delta u \cdot \nabla u \|_{L^1_t(B_{2,1}^1)} + \| (1 + a^2) \delta B \cdot \nabla B \|_{L^1_t(B_{2,1}^1)} \\
+ \| (1 + a^2) (B \cdot \nabla \delta B) \|_{L^1_t(B_{2,1}^1)} \\
\lesssim \int_0^t (\| u^2 \|_{B_{p,1}^{\frac{2}{p} + \frac{1}{2}}} + \| u^2 \|_{B_{p,1}^{\frac{2}{p} + \frac{1}{2}}} ) d\tau \\
+ \| (1 + a^2) (B \cdot \nabla \delta B) \|_{L^1_t(B_{2,1}^1)} \\
\leq \epsilon (\| \delta u \|_{L^1_t(B_{2,1}^1)} + \| \delta B \|_{L^1_t(B_{2,1}^1)} ) + C_\epsilon \int_0^t \| \delta u \|_{B_{2,1}^1} (\| u^2 \|_{B_{p,1}^{\frac{2}{p} + \frac{1}{2}}} + \| u^2 \|_{B_{p,1}^{\frac{2}{p} + \frac{1}{2}}} ) d\tau \\
+ C_\epsilon \int_0^t \| \delta B \|_{B_{2,1}^1} (1 + \| a^2 \|_{L^\infty_t(B_{2,1}^1)} )^2 (\| B^1 \|_{B_{p,1}^{\frac{2}{p} + \frac{1}{2}}} + \| B^2 \|_{B_{p,1}^{\frac{2}{p} + \frac{1}{2}}} ) d\tau.
\]

(5.55)

Plugging the estimates (5.54), (5.55) into (5.52), we can get

\[
\| \delta u \|_{L^\infty_t(B_{2,1}^1)} + \| \delta u \|_{L^1_t(B_{2,1}^1)} \\
\lesssim \| \mathcal{S}_m a^2 \|_{L^\infty_t(H^{\frac{1}{2} - a})} \| \nabla \delta \Pi \|_{L^1_t(H^\frac{1}{2} - a)} + (1 + \| \mathcal{S}_m a^2 \|_{L^\infty_t(H^\frac{1}{2})} ) \| \delta u \|_{L^1_t(H^\frac{1}{2})} \\
+ \int_0^t \| \delta u \|_{B_{2,1}^1} (\| u^2 \|_{B_{p,1}^{\frac{2}{p} + \frac{1}{2}}} + \| u^2 \|_{B_{p,1}^{\frac{2}{p} + \frac{1}{2}}} ) d\tau \\
+ \int_0^t \| \delta B \|_{B_{2,1}^1} (1 + \| a^2 \|_{L^\infty_t(B_{2,1}^1)} )^2 (\| B^1 \|_{B_{p,1}^{\frac{2}{p} + \frac{1}{2}}} + \| B^2 \|_{B_{p,1}^{\frac{2}{p} + \frac{1}{2}}} ) d\tau \\
+ \int_0^t \| \delta u \|_{L^1_t(B_{2,1}^1)} (\| \Delta u^1 \|_{B_{p,1}^{\frac{2}{p} + \frac{1}{2}}} + \| \nabla \Pi^1 \|_{B_{p,1}^{\frac{2}{p} + \frac{1}{2}}} + \| B^1 \|_{B_{p,1}^{\frac{2}{p} + \frac{1}{2}}} ) d\tau.
\]

(5.56)

In the following, we have to estimate \( \| \nabla \delta \Pi \|_{H^{-\frac{1}{2} - a}} \). We get from (5.40) that

\[
\text{div}( (1 + a^2) \nabla \delta \Pi ) = \text{div} F_4,
\]

with

\[
F_4 = a^2 \Delta \delta u - u^2 \cdot \nabla \delta u - \delta u \cdot \nabla u^1 + \delta a (\Delta u^1 - \nabla \Pi^1) \\
+ (1 + a^2) (\delta B \cdot \nabla B^2 + B^1 \cdot \nabla \delta B) + \delta a (B^1 \cdot \nabla B^1).
\]

Hence, by Proposition 3.5 in [3], we have

\[
\| \nabla \delta \Pi \|_{H^{-\frac{1}{2} - a}} \lesssim (1 + 2^{m(\frac{1}{2} + \frac{1}{p})} ) \| F_4 \|_{H^{-\frac{1}{2} - a}}.
\]

(5.57)
By Lemma 2.5 and Young’s inequality, we have

\[
\|F_4\|_{L^2_t(H^{-\frac{1}{2}})} \lesssim \|a^2\|_{\mathcal{A}^{2}} \|\Delta \delta u\|_{H^{-\frac{1}{2}}} + \|\delta u\|_{H^{-\frac{1}{2}}} (\|u^1\|_{B_{p,1}^{1+\frac{1}{p}}} + \|u^2\|_{B_{p,1}^{1+\frac{1}{p}}})
\]

\[
+ \|\delta B\|_{H^{-\frac{1}{2}}} (1 + \|a^2\|_{L^p_t(B_{2,1}^{2})}) (\|B_1\|_{B_{p,1}^{\frac{1}{p}}} + \|B_2\|_{B_{p,1}^{\frac{1}{p}}})
\]

\[
+ \|\delta a\|_{B_{1,2}^{\frac{1}{2}}} (\|u^1\|_{B_{p,1}^{\frac{1}{p}}} + \|\nabla \delta u\|_{B_{p,1}^{1+\frac{1}{p}}} + \|B_1\|_{B_{p,1}^{1+\frac{1}{p}}})
\]

\[
+ \|\delta B\|_{B_{1,2}^{\frac{1}{2}}} (1 + \|a^2\|_{L^p_t(B_{2,1}^{2})}) (\|B_1\|_{B_{p,1}^{\frac{1}{p}}} + \|B_2\|_{B_{p,1}^{\frac{1}{p}}})
\]

\[
\lesssim \|\delta u\|_{H^{-\frac{1}{2}}} (\|a^2\|_{B_{2,1}^{2}} + \|u^1\|_{B_{p,1}^{\frac{1}{p}}} + \|u^2\|_{B_{p,1}^{1+\frac{1}{p}}})
\]

\[
+ \|\nabla \delta u\|_{B_{p,1}^{1+\frac{1}{p}}} + \|\nabla \delta u\|_{B_{p,1}^{1+\frac{1}{p}}} + \|B_1\|_{B_{p,1}^{1+\frac{1}{p}}})
\]

\[
\lesssim \|\delta u\|_{H^{-\frac{1}{2}}} (\|a^2\|_{B_{2,1}^{2}} + \|u^1\|_{B_{p,1}^{\frac{1}{p}}} + \|u^2\|_{B_{p,1}^{1+\frac{1}{p}}})
\]

\[
+ \|\nabla \delta u\|_{B_{p,1}^{1+\frac{1}{p}}} + \|\nabla \delta u\|_{B_{p,1}^{1+\frac{1}{p}}} + \|B_1\|_{B_{p,1}^{1+\frac{1}{p}}})
\]

\[
\lesssim \|\delta u\|_{H^{-\frac{1}{2}}} (\|a^2\|_{B_{2,1}^{2}} + \|u^1\|_{B_{p,1}^{\frac{1}{p}}} + \|u^2\|_{B_{p,1}^{1+\frac{1}{p}}})
\]

\[
+ \|\nabla \delta u\|_{B_{p,1}^{1+\frac{1}{p}}} + \|\nabla \delta u\|_{B_{p,1}^{1+\frac{1}{p}}} + \|B_1\|_{B_{p,1}^{1+\frac{1}{p}}})
\]

\[
\lesssim \|\delta u\|_{L^2_t(H^{-\frac{1}{2}})} + \|\delta u\|_{L^2_t(B_{2,1}^{2})} + \|\delta B\|_{L^2_t(B_{2,1}^{2})} + \|\delta B\|_{L^2_t(B_{2,1}^{2})}
\]

\[
\lesssim \int_0^t (\|\delta u\|_{B_{2,1}^{2}} + \|\delta u\|_{B_{2,1}^{2}} + \|\delta B\|_{L^2_t(B_{2,1}^{2})}) w(\tau) d\tau,
\]

\[
\text{Thus, taking the above estimate into (5.57), we have}
\]

\[
\|\nabla \delta u\|_{L^1_t(H^{-\frac{1}{2}})} \lesssim (1 + 2^{m(\frac{1}{2} + \frac{1}{q})}) \|a^2\|_{L^\infty_t(B_{2,1}^{2})} \left\{ \int_0^t \|\delta u\|_{H^{-\frac{1}{2}}} (\|a^2\|_{B_{2,1}^{2}} + \|u^1\|_{B_{p,1}^{\frac{1}{p}}} + \|u^2\|_{B_{p,1}^{1+\frac{1}{p}}}) d\tau \right\}
\]

\[
+ \int_0^t \|\delta B\|_{H^{-\frac{1}{2}}} (1 + \|a^2\|_{L^\infty_t(B_{2,1}^{2})}) (\|B_1\|_{B_{p,1}^{\frac{1}{p}}} + \|B_2\|_{B_{p,1}^{\frac{1}{p}}}) d\tau
\]

\[
+ \int_0^t \|\delta a\|_{B_{p,1}^{1+\frac{1}{p}}} (\|u^1\|_{B_{p,1}^{\frac{1}{p}}} + \|\nabla \delta u\|_{B_{p,1}^{1+\frac{1}{p}}} + \|B_1\|_{B_{p,1}^{1+\frac{1}{p}}}) d\tau.
\]

By interpolation inequality, one has

\[
\|\delta u\|_{L^1_t(H^{-\frac{1}{2}})} \leq \|\delta u\|_{L^1_t(B_{2,1}^{2})} + C \|\delta u\|_{L^1_t(B_{2,1}^{2})},
\]

\[
\|\delta B\|_{L^1_t(H^{-\frac{1}{2}})} \leq \|\delta B\|_{L^1_t(B_{2,1}^{2})} + C \|\delta B\|_{L^1_t(B_{2,1}^{2})}.
\]

Taking the above estimates into (5.59) and choosing \(\varepsilon\) small enough, we can get from (5.56) that

\[
\|\delta u\|_{L^\infty_t(B_{2,1}^{2})} + \|\delta u\|_{L^1_t(B_{2,1}^{2})} \lesssim \|S m a^2\|_{L^\infty_t(H^{-\frac{1}{2}})} \|\nabla \delta u\|_{L^1_t(H^{-\frac{1}{2}})} + (1 + \|S m a^2\|_{L^\infty_t(H^1)}) \|\delta u\|_{L^1_t(H^1)}
\]

\[
+ \int_0^t \|\delta u\|_{B_{2,1}^{1+\frac{1}{p}}} (\|u^2\|_{B_{p,1}^{\frac{1}{p}}} + \|u^1\|_{B_{p,1}^{\frac{1}{p}}} + \|u^2\|_{B_{p,1}^{\frac{1}{p}}}) d\tau
\]

\[
+ \int_0^t \|\delta B\|_{B_{2,1}^{1+\frac{1}{p}}} (1 + \|a^2\|_{L^\infty_t(B_{2,1}^{2})})^2 (\|B_1\|_{B_{p,1}^{\frac{1}{p}}} + \|B_2\|_{B_{p,1}^{\frac{1}{p}}}) d\tau
\]

\[
+ \int_0^t \|\delta u\|_{B_{p,1}^{1+\frac{1}{p}}} (\|\Delta u\|_{B_{p,1}^{1+\frac{1}{p}}} + \|\nabla \delta u\|_{B_{p,1}^{1+\frac{1}{p}}} + \|B_1\|_{B_{p,1}^{1+\frac{1}{p}}} + \|B_1\|_{B_{p,1}^{1+\frac{1}{p}}}) d\tau.
\]

Thanks to estimates (5.47), (5.61), one can finally get by choosing \(\varepsilon\) small enough that

\[
\|\delta u\|_{L^\infty_t(B_{2,1}^{2})} + \|\delta u\|_{L^1_t(B_{2,1}^{2})} + \|\delta B\|_{L^\infty_t(B_{2,1}^{2})} + \|\delta B\|_{L^1_t(B_{2,1}^{2})}
\]

\[
\lesssim \int_0^t (\|\delta u\|_{B_{2,1}^{2}} + \|\delta u\|_{B_{2,1}^{2}} + \|\delta B\|_{B_{2,1}^{2}} + \|\delta u\|_{L^1_t(B_{2,1}^{2})}) w(\tau) d\tau,
\]

\[
\text{36}
\]
where
\[
\omega(\tau) = \|u^n\|_{B^{1+\frac{3}{p}}_{p,1}} + 2^{2m}\|a^n\|_{B^{1+\frac{3}{p}}_{p,1}} + (1 + \|a^n\|_{L^p(B^{1+\frac{3}{p}}_{p,1})})^2(\|B^n\|_{B^{1+\frac{3}{p}}_{p,1}} + \|B^n\|_{B^{1+\frac{3}{p}}_{p,1}} + \|u^n\|_{B^{1+\frac{3}{p}}_{p,1}})
\]
+ \|u^n\|_{B^{1+\frac{3}{p}}_{p,1}} + \|u^n\|_{B^{1+\frac{3}{p}}_{p,1}} + \|u^n\|_{B^{1+\frac{3}{p}}_{p,1}}
\]
+ (\|\Delta u^n\|_{B^{1+\frac{3}{p}}_{p,1}} + \|\nabla \Pi^n\|_{B^{1+\frac{3}{p}}_{p,1}} + \|B^n\|_{B^{1+\frac{3}{p}}_{p,1}}\|B^n\|_{B^{1+\frac{3}{p}}_{p,1}}).
\]

(5.63)

Applying Gronwall’s inequality and using (5.53) implies \(\delta a = \delta u = \delta B = 0\) for all \(t \in [0, T]\). This concludes the proof to the uniqueness part of Theorem 1.5.

5.2. Higher regularity part of the solution

Let \((a^n, u^n, B^n, \Pi^n)\) be the approximate solutions of (1.8) constructed in Step 2 of Subsection 5.1. Then for \(0 < \tau < t_0 < t \leq T^*\), with \(T^*\) being determined by (5.31), we deduce by a similar proof of (4.26), (4.2) that

\[
\|\langle u^n, B^n \rangle \|_{L^p([\tau, t]; B^{\frac{3}{p}, 1}_{p,1})} + \|\langle u^n, B^n \rangle \|_{L^1([\tau, t]; B^{\frac{3}{p}, 1}_{p,1})} + \|\Pi^n\|_{L^p([\tau, t]; B^{\frac{3}{p}, 1}_{p,1})}
\]
\[
\lesssim \|a_0\|_{B^{\frac{3}{p}, 1}_{p,1}} \left\{ \|u^n(\tau), B^n(\tau)\|_{B^{\frac{3}{p}, 1}_{p,1}} + \|\langle u^n(\tau), B^n(\tau) \rangle \|_{B^{\frac{3}{p}, 1}_{p,1}} \right\}
\]
\[
\times \exp\{\|\langle u^n, B^n \rangle \|_{L^1([\tau, t]; B^{\frac{3}{p}, 1}_{p,1})}\}.
\]

(5.64)

Integrating the above inequality for \(\tau\) over \([0, t_0]\), and then dividing the resulting inequality by \(t_0\) lead to

\[
\|\langle u^n, B^n \rangle \|_{L^p([0, t_0]; B^{\frac{3}{p}, 1}_{p,1})} + \|\langle u^n, B^n \rangle \|_{L^1([0, t_0]; B^{\frac{3}{p}, 1}_{p,1})} + \|\Pi^n\|_{L^p([0, t_0]; B^{\frac{3}{p}, 1}_{p,1})}
\]
\[
\lesssim \|a_0\|_{B^{\frac{3}{p}, 1}_{p,1}} \left\{ \|u(0, B_0)\|_{B^{-1+\frac{3}{p}}_{p,1}} \|u(0, B_0)\|_{B^{-1+\frac{3}{p}}_{p,1}} (1 + 1/\sqrt{t_0}) \exp\{\|\langle u(0, B_0) \rangle \|_{B^{-1+\frac{3}{p}}_{p,1}}\} \right\}.
\]

(5.65)

6. Global Well-posedness of Theorem 1.5

In this section, we will give the proof of the global well-posedness of Theorem 1.5 under the assumption that \(\|u_0\|_{B^{-1+\frac{3}{p}_{p,1}}} + \|B_0\|_{B^{-1+\frac{3}{p}_{p,1}}}\) is sufficiently small. By a similar proof of Theorem 4.1 in [3], we can get, for \(a_0 \in B^{\frac{3}{q}, 1}_{q,1}\) if \(u_0 \in B^{-1+\frac{3}{p}_{p,1}}, B_0 \in B^{-1+\frac{3}{p}_{p,1}}\), sufficiently small, (1.8) has a unique local solution \((a^n, u^n, B^n)\) satisfying

\[
a \in C_b([0, T^*]; B^{\frac{3}{q}, 1}_{q,1}(\mathbb{R}^3)), \quad u \in C_b([0, T^*]; B^{-1+\frac{3}{p}_{p,1}}(\mathbb{R}^3) \cap L^1([0, T^*]; B^{\frac{3}{p}, 1}_{p,1}(\mathbb{R}^3)) ,
\]
\[
B \in C_b([0, T^*]; B^{-1+\frac{3}{p}_{p,1}}(\mathbb{R}^3) \cap L^1([0, T^*]; B^{\frac{3}{p}, 1}_{p,1}(\mathbb{R}^3)) ,
\]

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for some $T^* > 1$. In what follows, we will prove $T^* = \infty$. Let $u = v + w, B = h + b$ where $(v, h)$ satisfies the following equations

\[
\begin{aligned}
    v_t + v \cdot \nabla v - \Delta v - h \cdot \nabla h + \nabla P_v &= 0, \\
    h_t - \Delta h + v \cdot \nabla h - h \cdot \nabla v &= 0, \\
    \text{div} v &= 0, \text{div} h = 0, \\
    v|_{t=t_1} &= u(t_1), h|_{t=t_1} = B(t_1).
\end{aligned}
\]

(6.1)

Then $(\rho, w, b)$ solves the equations

\[
\begin{aligned}
    \rho w_t + \rho(v + w) \cdot \nabla w - \Delta w + \nabla P_w &= h \cdot \nabla b + b \cdot \nabla h + b \cdot \nabla b \\
    &\quad + (1 - \rho)(v_t + v \cdot \nabla v) - \rho w \cdot \nabla v, \\
    b_t - \Delta b + w \cdot \nabla h + w \cdot \nabla b + v \cdot \nabla b - h \cdot \nabla w - b \cdot \nabla w - b \cdot \nabla v &= 0, \\
    \text{div} w &= 0, \text{div} b = 0, \\
    \rho|_{t=t_1} &= \rho(t_1), w|_{t=t_1} = 0, b|_{t=t_1} = 0.
\end{aligned}
\]

(6.2)

Note that $\|u(t_1)\|_{B_{p,1}^{1+\frac{3}{p}} \cap B_{p,1}^{\frac{1}{p}}} + \|B(t_1)\|_{B_{p,1}^{1+\frac{3}{p}} \cap B_{p,1}^{\frac{1}{p}}}$ is very small, provided that $\|u_0\|_{B_{p,1}^{1+\frac{3}{p}}} + \|B_0\|_{B_{p,1}^{1+\frac{3}{p}}}$ is sufficiently small. It follows from the classical theory of MHD equations (see [34]) that (6.1) has a unique global solution

$$(v, h) \in C([t_1, +\infty); B_{p,1}^{1+\frac{3}{p}} \cap L^1([t_1, +\infty); B_{p,1}^{1+\frac{3}{p}}) \times ([t_1, +\infty); B_{p,1}^{1+\frac{3}{p}}))$$

satisfying

\[
\begin{aligned}
    \|v\|_{L^\infty([t_1, +\infty); B_{p,1}^{1+\frac{3}{p}})} + \|h\|_{L^\infty([t_1, +\infty); B_{p,1}^{1+\frac{3}{p}})} + \|v_t\|_{L^1([t_1, +\infty); B_{p,1}^{1+\frac{3}{p}})} \\
    + \|h_t\|_{L^1([t_1, +\infty); B_{p,1}^{1+\frac{3}{p}})} + \|\nabla P_v\|_{L^1([t_1, +\infty); B_{p,1}^{1+\frac{3}{p}})} \\
    \leq (\|u(t_1)\|_{B_{p,1}^{1+\frac{3}{p}}} + \|B(t_1)\|_{B_{p,1}^{1+\frac{3}{p}}})
\end{aligned}
\]

(6.3)

and

\[
\begin{aligned}
    \|v_t\|_{L^1([t_1, +\infty); B_{p,1}^{1+\frac{3}{p}})} + \|h_t\|_{L^1([t_1, +\infty); B_{p,1}^{1+\frac{3}{p}})} \\
    \leq (\|u(t_1)\|_{B_{p,1}^{1+\frac{3}{p}}} + \|B(t_1)\|_{B_{p,1}^{1+\frac{3}{p}}}) + \|\text{div} (v \otimes v)\|_{L^1([t_1, +\infty); B_{p,1}^{1+\frac{3}{p}})} \\
    + \|\text{div}(v \otimes h)\|_{L^1([t_1, +\infty); B_{p,1}^{1+\frac{3}{p}})} + \|\text{div}(h \otimes v)\|_{L^1([t_1, +\infty); B_{p,1}^{1+\frac{3}{p}})} \\
    \leq (\|u(t_1)\|_{B_{p,1}^{1+\frac{3}{p}}} + \|B(t_1)\|_{B_{p,1}^{1+\frac{3}{p}}}.
\end{aligned}
\]

(6.4)
With \((v, h)\) thus obtained, we denote \(w = u - v, b = B - h\). Then thanks to (6.1) and (6.2). The proof of Theorem 1.5 reduces to proving the global well-posedness of (6.2). For simplicity, in what follows, we just present the a priori estimates for smooth enough solutions of (6.2) on \([0, T^*]\).

6.1. The higher regularities of \((v, h)\).

**Proposition 6.1.** Let \((v, h, \nabla P_v)\) be the unique global solution of (6.7) which satisfies (6.3) and (6.4). Then for \(s_1 \in [\frac{3}{p}, 2 + \frac{3}{p}]\) and \(s_2 \in [-1 + \frac{3}{p}, \frac{3}{p}]\), there hold

\[
\|v\|_{L^\infty([t_1, +\infty); \dot{B}^{s_1}_{p, 1})} + \|h\|_{L^\infty([t_1, +\infty); \dot{B}^{s_2}_{p, 1})} + \|\nabla P_v\|_{L^1([t_1, +\infty]; \dot{B}^{s_1}_{p, 1})} \\
+ \|\nabla h\|_{L^1([t_1, +\infty]; \dot{B}^{s_2}_{p, 1})} + \|\nabla^2 P_v\|_{L^1([t_1, +\infty]; \dot{B}^{s_2}_{p, 1})} \\
\leq C(\|u_0\|_{B^{s_1}_{p, \frac{3}{p}}} + \|v_0\|_{B^{s_1}_{p, \frac{3}{p}}})
\]

and

\[
\|v_t\|_{L^\infty([t_1, +\infty); \dot{B}^{s_1}_{p, 2})} + \|h_t\|_{L^\infty([t_1, +\infty); \dot{B}^{s_2}_{p, 2})} + \|v_t\|_{L^1([t_1, +\infty]; \dot{B}^{s_1}_{p, 2})} \\
+ \|h_t\|_{L^1([t_1, +\infty]; \dot{B}^{s_2}_{p, 2})} + \|\partial_t \nabla P_v\|_{L^1([t_1, +\infty]; \dot{B}^{s_2}_{p, 1})} \\
\leq C(\|u_0\|_{B^{s_1}_{p, \frac{3}{p}}} + \|v_0\|_{B^{s_1}_{p, \frac{3}{p}}}).
\]

(6.5) and

(6.6)

The proof of this proposition is rather standard, we omit the details here. By Proposition 6.1 we can easily get the following corollary:

**Corollary 6.2.** Under the assumptions of Proposition 6.1 one has

\[
\|\nabla v\|_{L^2([t_1, +\infty]; L^\infty)} + \|\nabla h\|_{L^2([t_1, +\infty]; L^\infty)} + \|v_t + v \cdot \nabla v\|_{L^2([t_1, +\infty]; L^\infty)} \\
\leq C(\|u_0\|_{B^{s_1}_{p, \frac{3}{p}}} + \|v_0\|_{B^{s_1}_{p, \frac{3}{p}}}).
\]

(6.7)

6.2. The \(L^2\) estimate of \((w, b)\).

**Proposition 6.3.** If the conditions of Theorem 1.5 are satisfied, there holds for \(t_1 < t < T^*\)

\[
\|w\|_{L^\infty([t_1, t]; L^2)} + \|b\|_{L^\infty([t_1, t]; L^2)} + \|\nabla w\|_{L^2([t_1, t]; L^2)} + \|\nabla b\|_{L^2([t_1, t]; L^2)} \\
\leq C(\|u_0\|_{B^{s_1}_{p, \frac{3}{p}}} + \|v_0\|_{B^{s_1}_{p, \frac{3}{p}}}),
\]

(6.8)

with \(C\) being independent of \(t\).
Proof. Firstly, thanks to $1 + \inf_{x \in \mathbb{R}^3} a_0(x) \geq \kappa > 0$, we can get from the transport equation of (1.8) that

$$(1 + \|a_0\|_{\mathbb{B}^1_{\kappa,1}}^{-1})^{-1} \leq \rho(t, x) \leq d^{-1},$$

(6.9)

from which and $1 - \rho = \rho \alpha$, we get by taking the $L^2$ inner product of the $w$ equation of (6.2) with $w$ and of the $b$ equation of (6.2) with $b$

$$\frac{1}{2} \frac{d}{dt} (\|\sqrt{\rho} w\|_{L^2}^2 + \|b\|_{L^2}^2)^2 + \|\nabla w\|_{L^2}^2 + \|\nabla b\|_{L^2}^2$$

$$= \int_{\mathbb{R}^3} ((1 - \rho)(v_t + v \cdot \nabla v) - \rho w \cdot \nabla v) \cdot wdx + \int_{\mathbb{R}^3} h \cdot \nabla b \cdot wdx + \int_{\mathbb{R}^3} b \cdot \nabla h \cdot wdx$$

$$+ \int_{\mathbb{R}^3} b \cdot \nabla b \cdot wdx - \int_{\mathbb{R}^3} w \cdot \nabla h \cdot bdx - \int_{\mathbb{R}^3} h \cdot \nabla v \cdot bdx$$

$$+ \int_{\mathbb{R}^3} h \cdot \nabla w \cdot bdx + \int_{\mathbb{R}^3} b \cdot \nabla w \cdot bdx + \int_{\mathbb{R}^3} b \cdot \nabla v \cdot bdx$$

$$\triangleq \sum_{j=1}^{10} I_j.$$  

(6.10)

Integrating by parts, we can get $I_2 + I_8 = I_4 + I_9 = I_6 = I_7 = 0$. Using the Hölder inequality, we have

$$|I_1| = \left| \int_{\mathbb{R}^3} ((1 - \rho)(v_t + v \cdot \nabla v) - \rho w \cdot \nabla v) \cdot wdx \right| \leq C \|\sqrt{\rho} w\|_{L^2} \|a\|_{L^2} \|v_t + v \cdot \nabla v\|_{L^\infty},$$

(6.11)

$$|I_3 + I_5 + I_{10}| = \left| \int_{\mathbb{R}^3} b \cdot \nabla h \cdot wdx + \int_{\mathbb{R}^3} w \cdot \nabla h \cdot bdx + \int_{\mathbb{R}^3} b \cdot \nabla v \cdot bdx \right|$$

$$\leq C (\|b\|_{L^2} \|\sqrt{\rho} w\|_{L^2} \|\nabla h\|_{L^\infty} + \|b\|_{L^2}^2 \|\nabla v\|_{L^\infty}).$$

(6.12)

Substituting the above estimates into (6.10), we have

$$\frac{1}{2} \frac{d}{dt} (\|\sqrt{\rho} w\|_{L^2}^2 + \|b\|_{L^2}^2)^2 + \|\nabla w\|_{L^2}^2 + \|\nabla b\|_{L^2}^2$$

$$\leq C (\|\sqrt{\rho} w\|_{L^2} \|a\|_{L^2} \|v_t + v \cdot \nabla v\|_{L^\infty}) + C (\|\sqrt{\rho} w\|_{L^2} + \|b\|_{L^2}^2) (\|\nabla v\|_{L^\infty} + \|\nabla h\|_{L^\infty}),$$

(6.13)

from which, we infer for $t \in (t_1, T^*)$ that

$$\frac{d}{dt} (e^{-2 \int_{t_1}^t (\|\nabla v\|_{L^\infty} + \|\nabla h\|_{L^\infty})d\tau} (\|\sqrt{\rho} w\|_{L^2}^2 + \|b\|_{L^2}^2))$$

$$\leq C \|a_0\|_{L^2} e^{-2 \int_{t_1}^t (\|\nabla v\|_{L^\infty} + \|\nabla h\|_{L^\infty})d\tau} \|\sqrt{\rho} w\|_{L^2} \|v_t + v \cdot \nabla v\|_{\mathbb{B}^1_{\kappa,1}}^2.$$
This, along with (6.5), implies

\[ \| \sqrt{p}w \|^2_{L^\infty([t_1,t];L^2)} + \| b \|^2_{L^\infty([t_1,t];L^2)} \leq C e^{H_{p'}(\| \nabla v \|_{L^\infty} + \| \nabla b \|_{L^\infty})} \| v_t + v \cdot \nabla v \|_{L^1([t_1,t];B_{p,1}^\frac{3}{p})} \]
\[ \leq C(\| u(t_1) \|_{B_{p,1}^{\frac{3}{p}}} + \| B(t_1) \|_{B_{p,1}^{\frac{3}{p}}}) \exp\{ C(\| u(t_1) \|_{B_{p,1}^{\frac{1}{p}}} + \| B(t_1) \|_{B_{p,1}^{\frac{1}{p}}}) \} \]
\[ \leq C(\| u_0 \|_{B_{p,1}^{\frac{1}{p}}} + \| B_0 \|_{B_{p,1}^{\frac{1}{p}}}). \] (6.15)

Taking the above estimate into (6.13) gives rise to

\[ \| \nabla w \|^2_{L^2([t_1,t];L^2)} + \| \nabla b \|^2_{L^2([t_1,t];L^2)} \leq C(\| u_0 \|_{B_{p,1}^{\frac{1}{p}}} + \| B_0 \|_{B_{p,1}^{\frac{1}{p}}}). \] (6.16)

6.3. The $H^1$ estimate of $(w,b)$.

**Proposition 6.4.** Under the assumptions of Theorem 6.3 there exist two positive constants $e_1, e_2$ so that for $t_1 < t < T^*$

\[ \| \nabla w \|^2_{L^\infty([t_1,t];L^2)} + \| \nabla b \|^2_{L^\infty([t_1,t];L^2)} \]
\[ + \int_{t_1}^t (e_1(\| \partial_t w \|^2_{L^2} + \| \partial_t b \|^2_{L^2}) + e_2(\| \nabla^2 w \|^2_{L^2} + \| \nabla^2 b \|^2_{L^2}) + \| \nabla P_w \|^2_{L^2}) dt' \]
\[ \leq C(\| u_0 \|^2_{B_{p,1}^{\frac{1}{p}}} + \| B_0 \|^2_{B_{p,1}^{\frac{1}{p}}}) \] (6.17)

with $C$ being independent of $t$.

**Proof.** We first get, by taking the $L^2$ inner product of (6.2)$_1$, (6.2)$_2$ with $\frac{1}{p} \Delta w, \Delta b$ respectively and using the Hölder inequality, Young inequality, (6.2) that,

\[ \frac{1}{2} \frac{d}{dt} (\| \nabla w \|^2_{L^2} + \| b \|^2_{L^2}) + \| \frac{1}{\sqrt{p}} \Delta w \|^2_{L^2} + \| \Delta b \|^2_{L^2} \]
\[ \leq C \| \frac{1}{\sqrt{p}} \Delta w \|^2_{L^2} (\| \nabla P_w \|^2_{L^2} + \| v \|^2_{L^\infty} \| \Delta w \|^2_{L^2} + \| w \|^2_{L^3} \| \nabla w \|^2_{L^6} + \| a \|_{L^2} \| v_t + v \cdot \nabla v \|_{L^\infty} + \| w \|^2_{L^2} \| \Delta w \|^2_{L^2} + \| \Delta b \|^2_{L^2} + \| \nabla \|_{L^\infty} + \| h \|^2_{L^2} \| \Delta w \|^2_{L^2} + \| \Delta b \|^2_{L^2}) \]
\[ + C \| \Delta b \|^2_{L^2} (\| b \|^2_{L^\infty} \| \Delta b \|^2_{L^2} + \| |w||_{L^6} \| \Delta h \|^2_{L^\infty} + \| h \|^2_{L^2} \| \Delta w \|^2_{L^2} + \| |w||_{L^6} \| \Delta b \|^2_{L^2}) \]
\[ \leq \frac{1}{16} \| \frac{1}{\sqrt{p}} \Delta w \|^2_{L^2} + \frac{1}{16} \| \Delta b \|^2_{L^2} + C \| \nabla P_w \|^2_{L^2} + C \| v_t + v \cdot \nabla v \|^2_{L^\infty} \]
\[ + C(\| \nabla w \|^2_{L^\infty} + \| h \|^2_{L^2}) (\| \nabla w \|^2_{L^2} + \| \nabla b \|^2_{L^2}) + C(\| \nabla v \|^2_{L^\infty} + \| h \|^2_{L^2} (\| \nabla w \|^2_{L^2} + \| b \|^2_{L^2}) + C(\| \nabla w \|^2_{L^\infty} + \| b \|^2_{L^2}) \] (6.18)
Now, we give the estimate of pressure function $\nabla P_w$. Thanks to $\text{div} w = 0$, we obtain from the momentum equation in (6.2) that

$$
\|\Delta w\|_{L^2}^2 + \|\nabla P_w\|_{L^2}^2 \leq 2\|\Delta w - \nabla P_w\|_{L^2}^2
$$

$$
\leq C_1 \|\sqrt{\rho} w_t\|_{L^2}^2 + C_1 (\|\tau v\|_{L^2}^2 + \|v\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \|\nabla b\|_{L^2}^2)
$$

$$
+ C_1 (\|\nabla v\|_{L^\infty}^2 + \|\nabla h\|_{L^\infty}^2)(\|w\|_{L^2}^2 + \|b\|_{L^2}^2)
$$

$$
+ C_1 (\|w\|_{L^2}^2 + \|b\|_{L^2}^2) (\|\nabla^2 w\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^2),
$$

(6.19)

which along with (6.18) leads to

$$
\frac{d}{dt} (\|\nabla w\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + c_1 (\|\nabla^2 w\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^2)
$$

$$
\leq C_1 \|\sqrt{\rho} w_t\|_{L^2}^2 + C_1 (\|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla h\|_{L^2}^2 + \|w\|_{L^2}^2 + \|b\|_{L^2}^2)
$$

$$
+ C_1 (\|\nabla v\|_{L^\infty}^2 + \|\nabla h\|_{L^\infty}^2)(\|w\|_{L^2}^2 + \|b\|_{L^2}^2) + C_1 (\|w\|_{L^2}^2 + \|b\|_{L^2}^2) (\|\nabla^2 w\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^2).
$$

(6.20)

Along the same line, we get by taking the $L^2$ inner-product of the equation of (6.2) with $w_t, b_t$ respectively that

$$
\frac{d}{dt} (\|\nabla w\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + \|\sqrt{\rho} w_t\|_{L^2}^2 + \|b_t\|_{L^2}^2
$$

$$
\leq C_1 (\|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\nabla h\|_{L^2}^2 + \|w\|_{L^2}^2 + \|b\|_{L^2}^2)
$$

$$
+ C_1 (\|\nabla v\|_{L^\infty}^2 + \|\nabla h\|_{L^\infty}^2)(\|w\|_{L^2}^2 + \|b\|_{L^2}^2) + C_1 (\|w\|_{L^2}^2 + \|b\|_{L^2}^2) (\|\nabla^2 w\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^2).
$$

(6.21)

Combining (6.20) with (6.21), we deduce that there is a positive constant $c_2$ such that

$$
\frac{d}{dt} (\|\nabla w\|_{L^2}^2 + \|\nabla b\|_{L^2}^2) + c_2 (\|w_t\|_{L^2}^2 + \|b_t\|_{L^2}^2)
$$

$$
+ (\frac{c_1}{2C_1} - C_11 (\|w\|_{L^2}^2 + \|b\|_{L^2}^2))(\|\nabla^2 w\|_{L^2}^2 + \|\nabla^2 b\|_{L^2}^2)
$$

$$
\leq C_11(\|v_t + v \cdot \nabla v\|_{L^\infty}^2 + \|v\|_{L^\infty}^2 + \|\nabla h\|_{L^\infty}^2)(\|w\|_{L^2}^2 + \|b\|_{L^2}^2)
$$

$$
+ (\|\nabla v\|_{L^\infty}^2 + \|\nabla h\|_{L^\infty}^2)(\|w\|_{L^2}^2 + \|b\|_{L^2}^2).
$$

(6.22)

Denote

$$
\tau^* \Delta \sup \left\{ t \geq t_1, \quad \|w\|_{L^2}^2 + \|b\|_{L^2}^2 \leq \frac{c_1}{2C_1} \right\}.
$$

(6.23)

We claim that $\tau^* = T^*$ provided that $\|u_0\|_{B_{p,1}^{-1 + \frac{3}{p}}} + \|B_0\|_{B_{p,1}^{-1 + \frac{3}{p}}}$ is sufficiently small.
Otherwise for \( t \in [t_1, \tau^*) \), we can get from (6.22) that

\[
\frac{d}{dt} \left( ||\nabla w||_{L^2}^2 + ||b||_{L^2}^2 \right) + c_2( ||w||_{L^2}^2 + ||b_t||_{L^2}^2 ) + \frac{c_1}{4C_{11}} ( ||\nabla^2 w||_{L^2}^2 + ||\nabla^2 b||_{L^2}^2 ) \\
\leq C_{12} \left( ||v_t + v \cdot \nabla v||_{L^{\infty}}^2 + ||v||_{L^{\infty}}^2 + ||h||_{L^{\infty}}^2 \right) ( ||\nabla w||_{L^2}^2 + ||\nabla b||_{L^2}^2 ) \\
+ \left( ||\nabla v||_{L^{\infty}}^2 + ||\nabla h||_{L^{\infty}}^2 \right) ( ||w||_{L^2}^2 + ||b||_{L^2}^2 ) .
\]

(6.24)

Applying Gronwall’s inequality to (6.24) and using (6.27) give rise to

\[
||\nabla w||_{L^2}^2 + ||b||_{L^2}^2 \leq C_{12} \exp \left\{ C_{12} \int_{t_1}^t ( ||v||_{L^{\infty}}^2 + ||h||_{L^{\infty}}^2 ) dt' \right\} \\
\times \int_{t_1}^t ( ||v_t + v \cdot \nabla v||_{L^{\infty}}^2 + ||\nabla v||_{L^2}^2 + ||\nabla h||_{L^2}^2 ) dt'
\leq C_{13} \left( ||u_0||_{\dot{B}^{-1+\frac{3}{p}}_{p,1}} + ||B_0||_{\dot{B}^{-1+\frac{3}{p}}_{p,1}} \right) .
\]

(6.25)

However, (6.3) and (6.25) tell us that

\[
||w||_{L^2}^2 + ||b||_{L^2}^2 \leq C ( ||w||_{L^2} ||\nabla w||_{L^2} + ||b||_{L^2} ||\nabla b||_{L^2} ) \\
\leq C_{14} \left( ||u_0||_{\dot{B}^{-1+\frac{3}{p}}_{p,1}} + ||B_0||_{\dot{B}^{-1+\frac{3}{p}}_{p,1}} \right) \leq \frac{c_1}{4C_{11}}
\]

(6.26)

for \( t \in [t_1, \tau^*) \), provided that \( ||u_0||_{\dot{B}^{-1+\frac{3}{p}}_{p,1}} + ||B_0||_{\dot{B}^{-1+\frac{3}{p}}_{p,1}} \leq \frac{c_1}{4C_{11}C_{14}} \), which contradicts (6.23). This, in turn, shows that \( \tau^* = T^* \). Then integrating (6.24) and using (6.27) lead to (6.17). This completes the proof of the proposition. \( \square \)

6.4. The \( H^2 \) estimate of \((w, b)\).

**Proposition 6.5.** Under the assumptions of Theorem 7.3 there exists a time independent constant \( C \) such that for \( t_1 < t < T^* \)

\[
||\nabla^2 w||_{L^\infty([t_1, t]; L^2)} + ||\nabla^2 b||_{L^\infty([t_1, t]; L^2)} + ||\nabla w_t||_{L^2([t_1, t]; L^2)} + ||\nabla b_t||_{L^2([t_1, t]; L^2)} \\
+ ||\nabla^2 w||_{L^2([t_1, t]; L^6)} + ||\nabla^2 b||_{L^2([t_1, t]; L^6)} \leq C.
\]

(6.27)

**Proof.** We get by first applying \( \partial_t \) to the first equation and second equation of (6.2) respectively and then taking the \( L^2 \) inner product of the resulting equation with \((w_t, b_t)\), that

\[
\frac{1}{2} \frac{d}{dt} \left( ||\sqrt{\rho} w_t||_{L^2}^2 + ||b_t||_{L^2}^2 \right) + ||\nabla w_t||_{L^2}^2 + ||\nabla b_t||_{L^2}^2 \\
= \int_{\mathbb{R}^3} (1 - \rho) w_t \cdot \partial_t (\Delta v - \nabla P_v - h \cdot \nabla h) dx \\
- \int_{\mathbb{R}^3} \rho_t w_t \cdot (w_t + (v + w) \cdot w + w \cdot \nabla v + (\Delta v - \nabla P_v - h \cdot \nabla h)) dx
\]

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Taking above estimates (6.29)-(6.35) into (6.28), we have

\[ - \int_{\mathbb{R}^3} \rho w_i \cdot ((v + w)_t \cdot \nabla w + w_i \cdot \nabla v + w \cdot \nabla v_t) dx \]
\[ + \int_{\mathbb{R}^3} h_i \cdot \nabla b \cdot w_i dx + \int_{\mathbb{R}^3} b_i \cdot \nabla h \cdot w_i dx + \int_{\mathbb{R}^3} b \cdot \nabla h_i \cdot w_i dx \]
\[ + \int_{\mathbb{R}^3} h_i \cdot \nabla w \cdot b_i dx + \int_{\mathbb{R}^3} b_i \cdot \nabla v \cdot b_i dx + \int_{\mathbb{R}^3} b \cdot \nabla v_t \cdot b_i dx + \int_{\mathbb{R}^3} b_i \cdot \nabla w \cdot b_i dx \]
\[ = \Delta \sum_{i=1}^{13} J_i. \quad (6.28) \]

\( J_1, J_2, J_3 \) can be estimated the same as in [2], that is,

\[ |J_1| \leq C \|a_0\|_{L^\infty} \|\sqrt{\rho} w_t\|_{L^2} \|\partial_t (\Delta v - \nabla P_v - h \cdot \nabla h)\|_{L^2}, \quad (6.29) \]

\[ |J_2| \leq \frac{1}{4} \|\nabla w_t\|_{L^2}^2 + C \left( \|v\|_{L^\infty}^4 + \|\nabla v\|_{L^6}^2 + \|\Delta w\|_{L^2}^2 + \|v_t + v \cdot \nabla v\|_{L^2}^2 \right) \]
\[ + \|\nabla^2 v\|_{L^6}^2 + \|\sqrt{\rho} w_t\|_{L^2}^2 \left( \|v\|_{L^\infty}^4 + \|\nabla w\|_{L^2} \|\nabla^2 w\|_{L^2} \right) \]
\[ + \|\sqrt{\rho} w_t\|_{L^2} + \|\nabla (\Delta v - \nabla P_v - h \cdot \nabla h)\|_{L^4} \right), \quad (6.30) \]

\[ |J_3| \leq \frac{1}{4} \|\nabla w_t\|_{L^2}^2 + C \left( \|\sqrt{\rho} w_t\|_{L^2}^2 \left( \|\nabla v\|_{L^\infty} + \|\nabla w\|_{L^2} \|\nabla^2 w\|_{L^2} \right) \right. \]
\[ + \|\sqrt{\rho} w_t\|_{L^2} \left( \|v_t\|_{L^\infty} + \|\nabla v_t\|_{L^4} \right) \right), \quad (6.31) \]

Using the Hölder inequality and Young’s inequality implies

\[ |J_4 + J_{10}| \leq C \|h_t\|_{L^\infty} \left( \|\nabla b\|_{L^2} \|\sqrt{\rho} w_t\|_{L^2} + \|\nabla w\|_{L^2} \|b_t\|_{L^2} \right) \leq C \|h_t\|_{L^\infty} \left( \|\sqrt{\rho} w_t\|_{L^2} + \|b_t\|_{L^2} \right), \quad (6.32) \]

\[ |J_6 + J_9 + J_{12}| \leq C \left( \|b_t\|_{L^4} \|\nabla h_t\|_{L^4} \|\sqrt{\rho} w_t\|_{L^2} + \|w\|_{L^4} \|\nabla h_t\|_{L^4} \|b_t\|_{L^2} + \|b\|_{L^4} \|\nabla v_t\|_{L^4} \|b_t\|_{L^2} \right), \quad (6.33) \]

\[ |J_5 + J_8 + J_{11}| \leq C \left( \|b_t\|_{L^2} \|\sqrt{\rho} w_t\|_{L^2} \|\nabla h\|_{L^\infty} + \|b_t\|_{L^2} \|\nabla v\|_{L^\infty} \right), \quad (6.34) \]

\[ |J_7 + J_{13}| \leq C \left( \|b_t\|_{L^4} \|\sqrt{\rho} w_t\|_{L^2} \|\nabla b\|_{L^3} + \|b_t\|_{L^4} \|b_t\|_{L^2} \|\nabla w\|_{L^3} \right) \]
\[ \leq \frac{1}{16} \|\nabla b_t\|_{L^2}^2 + C \|\sqrt{\rho} w_t\|_{L^2}^2 \|\nabla b\|_{L^2} \|\nabla^2 b\|_{L^2} + C \|b_t\|_{L^2}^2 \|\nabla w\|_{L^2} \|\nabla^2 w\|_{L^2}. \quad (6.35) \]

Taking above estimates (6.29)-(6.35) into (6.28), we have

\[ \frac{d}{dt} \left( \|\sqrt{\rho} w_t\|_{L^2}^2 + \|b_t\|_{L^2}^2 \right) + \|\nabla w_t\|_{L^2}^2 + \|\nabla b_t\|_{L^2}^2 \]
\[ \leq C \left( \frac{f_1(t) + f_3(t)}{2} \right) \left( \|\sqrt{\rho} w_t\|_{L^2}^2 + \|b_t\|_{L^2}^2 \right) + f_2(t) + f_3(t) \right), \quad (6.36) \]
with
\[
\begin{align*}
    f_1(t) &= \|v\|_{L^2}^2 + \|\nabla v\|_{L^\infty} + \|\nabla h\|_{L^\infty} + \|\nabla w\|_{L^2} + \|\nabla^2 w\|_{L^2} + \|\nabla b\|_{L^2} + \|\nabla^2 b\|_{L^2}, \\
    f_2(t) &= \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + \|\Delta w\|_{L^2}^2 + \|\nabla v + v \cdot \nabla v\|_{L^2}^2 + \|\nabla^2 v\|_{H^s}^2, \\
    f_3(t) &= \|\nabla(\Delta v - \nabla P_v - h \cdot \nabla h)\|_{L^4} + \|\partial_t (\Delta v - \nabla P_v - h \cdot \nabla h)\|_{L^2} + \|v_t\|_{L^\infty} + \|\nabla v_t\|_{L^4} \\
    &+ \|h_t\|_{L^\infty} + \|\nabla h_t\|_{L^4}.
\end{align*}
\]

Gronwall’s inequality helps us to get from (6.36) for \( t \in (t_1, T^*) \) that
\[
\begin{align*}
    &\|\sqrt{\rho} w_t\|_{L^2}^2 + \|b_t\|_{L^2}^2 + \int_{t_1}^t (\|\nabla w_t\|_{L^2}^2 + \|\nabla b_t\|_{L^2}^2) dt' \\
    \leq &\ C \exp \left\{ \int_{t_1}^t (f_1(t') + f_3(t')) dt' \right\} (\|\sqrt{\rho} w_t(t_1)\|_{L^2}^2 + \|b_t(t_1)\|_{L^2}^2 + \int_{t_1}^t (f_2(t') + f_3(t')) dt'). \quad (6.37)
\end{align*}
\]
However, by embedding relations
\[
\begin{align*}
    &\hat{B}_{p,1}^{3/4}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3), \hat{B}_{p,1}^{3/2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3), \hat{B}_{p,1}^{3/4}(\mathbb{R}^3) \hookrightarrow L^4(\mathbb{R}^3),
\end{align*}
\]
we deduce from (6.5), (6.6), (6.8) and (6.17) that
\[
\int_{t_1}^t (f_1(t') + f_2(t') + f_3(t')) dt' \leq C,
\]
with C being independent of t. Whereas taking the \( L^2 \) inner product of the first and second equation of (6.2) with \((w_t, b_t)\) at \( t = t_1 \) respectively and using the higher regularity of \((a, u, B, \nabla \Pi)\) give rise to
\[
\begin{align*}
    &\|\sqrt{\rho} w_t(t_1)\|_{L^2}^2 + \|b_t(t_1)\|_{L^2}^2 \leq C \|a_t(t_1)\|_{L^2}^2 \|(v_t + v \cdot \nabla v)(t_1)\|_{L^\infty} \\
    &\leq C \|(\Delta v - \nabla P_v - h \cdot \nabla h)(t_1)\|_{\hat{B}_{p,1}^{3/4}}^2 \\
    &\leq C \|v(t_1)\|_{\hat{B}_{p,1}^{2+\theta}}^2 + \|v(t_1)\|_{\hat{B}_{p,1}^{2+\theta}}^2 \|v(t_1)\|_{\hat{B}_{p,1}^{1+\theta}}^2 + \|h(t_1)\|_{\hat{B}_{p,1}^{1+\theta}}^2 \\
    &\leq C.
\end{align*}
\]
As a consequence, we deduce from (6.37) that
\[
\begin{align*}
    &\sup_{t \in [t_1, T^*]} (\|\sqrt{\rho} w_t\|_{L^2}^2 + \|b_t\|_{L^2}^2) + \int_{t_1}^{T^*} \left( \|\nabla w_t(t')\|_{L^2}^2 + \|\nabla b_t(t')\|_{L^2}^2 \right) dt'. \quad (6.38)
\end{align*}
\]
In the following, we give the estimates of the second space derivative estimate of \((w, b)\).
We first observe from the equation of (6.2) that

\[
\begin{align*}
\|\nabla^2 w\|_{L^2} + \|\nabla P_w\|_{L^2} + \|\nabla^2 b\|_{L^2} \\
\lesssim \|\rho w_x\|_{L^2} + \|\rho w \cdot \nabla w\|_{L^2} + \|\rho v \cdot \nabla w\|_{L^2} + \|\rho w \cdot \nabla v\|_{L^2} + \|h \cdot \nabla b\|_{L^2} \\
+ \|b \cdot \nabla h\|_{L^2} + \|b \cdot \nabla b\|_{L^2} + \|(1 - \rho)(v_t + v \cdot \nabla v)\|_{L^2} + \|w \cdot \nabla h\|_{L^2} + \|w \cdot \nabla b\|_{L^2} \\
+ \|v \cdot \nabla b\|_{L^2} + \|h \cdot \nabla w\|_{L^2} + \|b \cdot \nabla w\|_{L^2} + \|b \cdot \nabla v\|_{L^2} + \|b_t\|_{L^2} \\
\lesssim \|\rho w_t\|_{L^2} + \|b_t\|_{L^2} + \|w\|_{L^6} \|\nabla w\| + \|\nabla^2 w\|_{L^2} + \|\nabla^2 b\|_{L^2} \\
+ \|\nabla v\|_{L^\infty} \|w\|_{L^2} + \|h\|_{L^\infty} \|\nabla b\|_{L^2} + \|\nabla h\|_{L^\infty} \|b\|_{L^2} + \|b\|_{L^6} \|\nabla b\|_{L^2} + \|w\|_{L^6} \|\nabla b\|_{L^2} + \|v\|_{L^\infty} \|\nabla b\|_{L^2} \\
+ \|b\|_{L^6} \|\nabla w\|_{L^2} \|\nabla^2 w\|_{L^2} + \|\nabla v\|_{L^\infty} \|b\|_{L^2} \\
\lesssim \|\rho w_t\|_{L^2} + \|b_t\|_{L^2} + \|\nabla w\|_{L^2} + \|\nabla b\|_{L^2} + \|\nabla w\|_{L^2} \|\nabla b\|_{L^2} + \|w\|_{L^6} \|\nabla b\|_{L^2} + \|h\|_{L^\infty} \|\nabla b\|_{L^2} \\
+ \|b\|_{L^6} \|\nabla w\|_{L^2} \|\nabla^2 w\|_{L^2} + \|\nabla v\|_{L^\infty} \|b\|_{L^2} \\
\lesssim \|\rho w_t\|_{L^2} + \|b_t\|_{L^2} + \|\nabla w\|_{L^2} + \|\nabla b\|_{L^2} + \|\nabla w\|_{L^2} \|\nabla b\|_{L^2} + \|w\|_{L^6} \|\nabla b\|_{L^2} + \|h\|_{L^\infty} \|\nabla b\|_{L^2} \\
\lesssim \|\rho w_t\|_{L^2} + \|b_t\|_{L^2} + \|\nabla w\|_{L^2} + \|\nabla b\|_{L^2} + \|\nabla w\|_{L^2} \|\nabla b\|_{L^2} + \|w\|_{L^6} \|\nabla b\|_{L^2} + \|h\|_{L^\infty} \|\nabla b\|_{L^2} \\
\lesssim \|\rho w_t\|_{L^2} + \|b_t\|_{L^2} + \|\nabla w\|_{L^2} + \|\nabla b\|_{L^2} + \|\nabla w\|_{L^2} \|\nabla b\|_{L^2} + \|w\|_{L^6} \|\nabla b\|_{L^2} + \|h\|_{L^\infty} \|\nabla b\|_{L^2} \\
+ \frac{1}{8}(\|\nabla^2 w\|_{L^2} + \|\nabla^2 b\|_{L^2}) + (\|v\|_{L^\infty} + \|h\|_{L^\infty})(\|\nabla w\|_{L^2} + \|\nabla b\|_{L^2}) \\
+ (\|\nabla v\|_{L^\infty} + \|\nabla h\|_{L^\infty})(\|w\|_{L^2} + \|b\|_{L^2}) + \|v\|_{B^{2,\frac{3}{p}}_{p,1}} + \|v\|_{B^{1,\frac{3}{p}}_{p,1}} + \|h\|_{B^{\frac{3}{p}}_{p,1}} \|h\|_{B^{1,\frac{3}{p}}_{p,1}},
\end{align*}
\]

where we have used the following fact:

\[
\|\partial_t v\|_{B^{\frac{3}{p}}_{p,1}} = \|\Delta v - P(v \cdot \nabla v) + P(h \cdot \nabla h)\|_{B^{\frac{3}{p}}_{p,1}} \leq \|v\|_{B^{2,\frac{3}{p}}_{p,1}} + \|v\|_{B^{1,\frac{3}{p}}_{p,1}} + \|h\|_{B^{\frac{3}{p}}_{p,1}} \|h\|_{B^{1,\frac{3}{p}}_{p,1}} \]  

which along with (6.5), (6.6), (6.8), (6.17) and (6.38) ensures that

\[
\sup_{t \in [t_1, T^*]} (\|\nabla^2 w\|_{L^2} + \|\nabla P_w\|_{L^2} + \|\nabla^2 b\|_{L^2}) \leq C.
\]  

On the other hand, let \((v, q)\) solve

\[
-\Delta v + \nabla q = f, \quad \text{div} v = 0.
\]

Then one has \(\nabla q = -\nabla (\Delta)^{-1} \text{div} f\), and for any \(r \in (1, \infty)\),

\[
\|\nabla q\|_{L^r} \leq C\|f\|_{L^r}, \quad \|\Delta v\|_{L^r} \leq C\|f\|_{L^r}.
\]

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From this and the equation of (6.2) we infer

\[
\|\nabla^2 w\|_{L^6} + \|\nabla P_w\|_{L^6} + \|\nabla^2 b\|_{L^6}
\]

\[
\lesssim \|\nabla w_t\|_{L^2} + \|\nabla b_t\|_{L^2} + \|\nabla w\|_{L^2}^2 \|\nabla^2 w\|_{L^2} + \|\nabla^2 w\|_{L^2} \|\nabla b\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \|\nabla^2 b\|_{L^2}
\]

\[+ \frac{1}{8} (\|\nabla^2 w\|_{L^6} + \|\nabla^2 b\|_{L^6}) + (\|v\|_{L^\infty} + \|h\|_{L^\infty})(\|\nabla^2 w\|_{L^2} + \|\nabla b^2\|_{L^2})
\]

\[+ (\|\nabla v\|_{L^\infty} + \|\nabla h\|_{L^\infty})(\|\nabla w\|_{L^2} + \|\nabla b\|_{L^2}) + \|v\|_{p,1}^\frac{2}{p} \|v\|_{p,1}^\frac{2}{p} + \|h\|_{p,1}^\frac{3}{p} \|h\|_{p,1}^\frac{3}{p},
\]

which implies

\[
\|\nabla^2 w\|_{L^6} + \|\nabla P_w\|_{L^6} + \|\nabla^2 b\|_{L^6}
\]

\[
\lesssim \|\nabla w_t\|_{L^2} + \|\nabla b_t\|_{L^2} + \|\nabla w\|_{L^2}^2 \|\nabla^2 w\|_{L^2} + \|\nabla^2 w\|_{L^2} \|\nabla b\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 \|\nabla^2 b\|_{L^2}
\]

\[+ (\|v\|_{L^\infty} + \|h\|_{L^\infty})(\|\nabla^2 w\|_{L^2} + \|\nabla b^2\|_{L^2}) + (\|\nabla v\|_{L^\infty} + \|\nabla h\|_{L^\infty})(\|\nabla w\|_{L^2} + \|\nabla b\|_{L^2})
\]

\[+ \|v\|_{p,1}^\frac{2}{p} \|v\|_{p,1}^\frac{2}{p} + \|h\|_{p,1}^\frac{3}{p} \|h\|_{p,1}^\frac{3}{p}.
\]

Taking the \(L^2\) norm for the time variables on \([t_1, t]\), we get by using (6.38), (6.38) and (6.41) that

\[
\|\nabla^2 w\|^2_{L^2([t_1,t];L^6)} + \|\nabla P_w\|^2_{L^2([t_1,t];L^6)} + \|\nabla^2 b\|^2_{L^2([t_1,t];L^6)}
\]

\[\leq C(\|\nabla w_t\|^2_{L^2([t_1,t];L^2)} + \|\nabla b_t\|^2_{L^2([t_1,t];L^2)}) + C(\|\nabla^2 w\|^2_{L^2([t_1,t];L^2)} + \|\nabla^2 b\|^2_{L^2([t_1,t];L^2)})
\]

\[+ C\|v\|^2_{L^2([t_1,t];B_{p,1}^{2+\frac{3}{p}})} + C\|v\|^2_{L^2([t_1,t];B_{p,1}^{2+\frac{3}{p}})} + C\|h\|^2_{L^2([t_1,t];B_{p,1}^{1+\frac{3}{p}})}
\]

\[\leq C.
\]

This completes the proof of the proposition.

\[
\square
\]

6.5. Proof of Theorem 1.5

We then rewrite the equations for \(u\) and \(B\) in (1.8) as

\[
\begin{cases}
\partial_t u - \Delta u + \nabla \Pi = B \cdot \nabla B - u \cdot \nabla u + \frac{a}{1+a} (\partial_t u + u \cdot \nabla u), \\
\partial_t B - \Delta B = B \cdot \nabla u - u \cdot \nabla B.
\end{cases}
\]

Then it is easy to observe that for \(t \in [t_1, T^*]\)

\[
\|(u, B)\|_{L^\infty([t_1,T];B_{p,1}^{1+\frac{3}{p}})} + \|(\Delta u, \Delta B, \nabla \Pi)\|_{L^1([t_1,T];B_{p,1}^{1+\frac{3}{p}})}
\]

\[
\lesssim \|(u(t_1), B(t_1))\|_{B_{p,1}^{1+\frac{3}{p}}} + \left\| \frac{a}{1+a} (\partial_t u + u \cdot \nabla u) \right\|_{L^1([t_1,T];B_{p,1}^{1+\frac{3}{p}})}
\]

\[+ \|(u \cdot \nabla u, B \cdot \nabla B, B \cdot \nabla u, u \cdot \nabla B)\|_{L^1([t_1,T];B_{p,1}^{1+\frac{3}{p}})}.
\]

(6.44)
By the product law in Besov spaces gives
\[
\left\| \frac{a}{1+a} \left( \partial_t u + u \cdot \nabla u \right) \right\|_{L^1([t_1,t];B_{p,1}^{\frac{3}{p}+\frac{3}{2}})} \lesssim \left( 1 + \|a\|_{L^\infty([t_1,t];B_{p,1}^{\frac{3}{p}+\frac{3}{2}})} \right) \left\| \partial_t u + u \cdot \nabla u \right\|_{L^1([t_1,t];B_{p,1}^{\frac{3}{p}+\frac{3}{2}})}.
\]

Yet thanks to Lemma 2.4 and (6.27), one has
\[
\| \partial_t u \|_{L^1([t_1,t];B_{p,1}^{\frac{3}{p}+\frac{3}{2}})} \leq C(\|t^{\frac{1}{2}}\| \| \partial_t w \|_{L^2([t_1,t];H^1)} + \| u(t_1) \|_{B_{p,1}^{\frac{3}{p}+\frac{3}{2}}}) \leq C(1 + t^\frac{1}{2})
\]
and
\[
\| u \cdot \nabla u \|_{L^1([t_1,t];B_{p,1}^{\frac{3}{p}+\frac{3}{2}})} + \| B \cdot \nabla B \|_{L^1([t_1,t];B_{p,1}^{\frac{3}{p}+\frac{3}{2}})} + \| u \cdot \nabla B \|_{L^1([t_1,t];B_{p,1}^{\frac{3}{p}+\frac{3}{2}})} + \| B \cdot \nabla u \|_{L^1([t_1,t];B_{p,1}^{\frac{3}{p}+\frac{3}{2}})} 
\leq C \int_{t_1}^t (\| \nabla w \|_{L^2} \| \Delta w \|_{L^2} + \| v \|_{B_{p,1}^{1+\frac{3}{2}}}^2) \, dt' + C \int_{t_1}^t (\| \nabla b \|_{L^2} \| \Delta b \|_{L^2} + \| h \|_{B_{p,1}^{1+\frac{3}{2}}}^2) \, dt' \leq C. \tag{6.45}
\]

Thanks to Theorem 2.87 in [6] and Proposition 2.10 we have
\[
\left\| \frac{a}{1+a} \right\|_{L^\infty([t_1,t];B_{p,1}^{\frac{3}{p}+\frac{3}{2}})} \leq C \| a \|_{L^\infty([t_1,t];B_{p,1}^{\frac{3}{p}+\frac{3}{2}})} \leq C \| a(t_1) \|_{B_{p,1}^{\frac{3}{p}+\frac{3}{2}}} \exp \{ C \int_{t_1}^t \| u(t') \|_{B_{6,1}^{\frac{3}{p}}} \, dt' \}. \tag{6.46}
\]

By Lemma 2.4 we have
\[
\| u \|_{B_{6,1}^{\frac{3}{2}}} \leq C \| \nabla w \|_{L^2}^{\frac{1}{2}} \| \nabla^2 w \|_{L^6}^{\frac{1}{2}} + C \| v \|_{B_{p,1}^{1+\frac{3}{2}}},
\]
which along with (6.17) and (6.27) implies
\[
\| \nabla u \|_{L^1([t_1,t];L^\infty)} + \| u \|_{L^1([t_1,t];B_{6,1}^{\frac{3}{2}})} \leq C \| u \|_{L^1([t_1,t];B_{6,1}^{\frac{3}{2}})} 
\leq C \| v \|_{L^1([t_1,t];B_{6,1}^{1+\frac{3}{2}})} + t^\frac{3}{2} \| \Delta w \|_{L^2([t_1,t];L^\infty)} \| \Delta w \|_{L^2([t_1,t];L^6)} 
\leq C(1 + t^{\frac{1}{2}}).
\]

Therefore, we obtain
\[
\left\| \frac{a}{1+a} \left( \partial_t u + u \cdot \nabla u \right) \right\|_{L^1([t_1,t];B_{p,1}^{\frac{3}{p}+\frac{3}{2}})} \leq C \| a(t_1) \|_{B_{p,1}^{\frac{3}{p}}} \exp \{ Ct^\frac{1}{2} \}. \tag{6.47}
\]

Taking estimates (6.45), (6.47) into (6.44) and applying Gronwall’s inequality, one can finally get
\[
\| (u, B) \|_{L^\infty([t_1,t];B_{p,1}^{\frac{3}{p}+\frac{3}{2}})} + \| (u, B) \|_{L^1([t_1,t];B_{p,1}^{\frac{3}{p}+\frac{3}{2}})} + \| \nabla \Pi \|_{L^1([t_1,t];B_{p,1}^{\frac{3}{p}+\frac{3}{2}})} 
\leq C(\| a_0 \|_{B_{p,1}^{\frac{3}{p}}} + \| (u_0, B_0) \|_{B_{p,1}^{\frac{3}{p}+\frac{3}{2}}}) \exp \{ Ct^\frac{1}{2} \}, \tag{6.48}
\]

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from which and \([6.46]\), we can complete the proof of Theorem[1.5] by a standard argument.

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