One-loop divergences in the six-dimensional $\mathcal{N} = (1,0)$ hypermultiplet self-coupling model

A.S. Budekhina$^{a,b}$, B.S. Merzlikin$^{c,a}$

$^a$ Center for Theoretical Physics, Tomsk State Pedagogical University, 634061, Tomsk, Russia
$^b$ National Research Tomsk State University, 634050, Tomsk, Russia
$^c$ Tomsk State University of Control Systems and Radioelectronics, 634050, Tomsk, Russia

Abstract

We study the six-dimensional $\mathcal{N} = (1,0)$ supersymmetric hypermultiplet model with arbitrary self-coupling. The model is considered in the external classical gauge superfield background. Using the harmonic superspace formulation we study the one-loop effective action of the model. We calculate the one-loop divergences in the theory as a gauge-invariant function of the external gauge multiplet. We demonstrate that the one-loop divergences in the theory do not vanish even in the case of on-shell hypermultiplet background. We briefly discuss an application of the developed technique to the hypermultiplet model in four dimensions.

1 budekhina@tspu.edu.ru
2 merzlikin@tspu.edu.ru
1 Introduction

The study of the scalar field models in diverse dimensions attracted much interest in the last few years. The $O(N)$ vector model of the scalar fields with the relevant interactions in six dimensions has been extensively studied in the context of a critical phenomenon and large $N$ expansion [1, 2] (see, also, the reviews [3–5]). Supersymmetry restricts possible interaction of the scalar fields, which are described in the general case by the supersymmetric nonlinear sigma-models. These theories highlight the close connections between supersymmetry and complex geometry. For the case of the simplest supersymmetric sigma-model with four supercharges the target space is known to be a Kähler manifold [6]. The rigid supersymmetric sigma-model with eight supercharges in diverse dimensions possesses by huper-Kähler manifolds as target space. Such sigma-model has $\mathcal{N} = 2$ supersymmetry in four dimensions and $\mathcal{N} = (1, 0)$ supersymmetry in the six ones [7, 8]. In its turn, the target space of the locally supersymmetric sigma-model with eight supercharges is quaternionic Kähler manifold [9] (see, also, [10, 11] and references wherein).

The formulation of the supersymmetric sigma-models in terms of unconstrained superfields allows one to deeply understand the relationship between geometry and supersymmetry. The superfield technique is a convenient tool to study quantum properties of the model, which preserves explicit supersymmetry during quantization (see, e.g., [12]). In the present paper, we study the six-dimensional $\mathcal{N} = (1, 0)$ supersymmetric hypermultiplet model minimally interacting with external classical gauge multiplet. We consider the case of arbitrary gauge-invariant potential function for the hypermultiplet [13]. For the vanish gauge multiplet, this theory describes the supersymmetric sigma-model with general multicenter hyper-Kähler metric [15] (see, also, for a review [16]). We formulate the model in the six-dimensional $\mathcal{N} = (1, 0)$ harmonic superspace [16–19] and develop the covariant technique to study the effective action. We construct the one-loop effective action in the theory using the proper-time technique in the harmonic superspace [20–23].

The study of the ultraviolet behavior of supersymmetric field theories in six dimensions has received considerable attention in recent years. The analysis of the UV divergences is carried out using both symmetry considerations [24–28] and direct calculations of the effective action [20, 21, 29–33] (see also earlier analysis [34–37] and scattering amplitudes [38–42]. As an application of the developed technique, we calculate the one-loop divergent contribution to the effective action in the gauge-invariant $\mathcal{N} = (1, 0)$ supersymmetric sigma-model in six dimensions [14]. The obtaining result depends on both background hypermultiplet and external gauge multiplet. We also demonstrate that the leading finite contribution to the effective potential of the scalar fields in the model vanishes.

The work is organized as follows. In Section 2 we discuss the basic properties of six-dimensional $\mathcal{N} = (1, 0)$ harmonic superspace and formulate the model. Section 3 contains the quantization procedure in the $\mathcal{N} = (1, 0)$ harmonic superspace and the evaluation of the one-loop effective action for the model. Analysis of divergent contributions to the one-loop effective action is collected in Section 4. Finally, in Section 5, we briefly discuss the application of the developed technique to the $\mathcal{N} = 2$ supersymmetric hypermultiplet model in four dimensions.

1See also [13], where the quantum corrections for general four-dimensional supersymmetric Kähler sigma models were studied in the superspace.
2 Hypermultiplet theory in 6D

Throughout the paper, we follow the notations and conventions from the work [25]. We denote the 6D Minkowski space coordinates by \( x^M \) and the Grassmann ones by \( \theta^a \), where \( M = 0, \ldots, 5 \), \( a = 1, \ldots, 4 \). The additional \( SU(2) \) index \( i = 1, 2 \) corresponds to the R-symmetry group of the simplest \( \mathcal{N} = (1, 0) \) supersymmetry in six dimensions.

We use the harmonic superspace formulation [16] for the theory. The six-dimensional \( \mathcal{N} = (1, 0) \) harmonic superspace coordinates \( (x^M, \theta^a, u^{\pm i}) \) include additional harmonic variables \( u^{\pm i} \). They parameterize the coset \( SU(2)/U(1) \) and obey the constraints \( u^+ u^- = 1, u^{i+} \equiv (u^{i+})^* \). The harmonic superspace naturally admits the analytic basis, which is parameterized by the coordinates \( \zeta = (x^M_A, \theta^{\pm a}) \)

\[
x^M_A = x^M + \frac{i}{2} \theta^{+a} \gamma^M_{ab} \theta^{-b}, \quad \theta^{\pm a} = u_k^{\pm a} \theta^{ak}.
\]

In the analytic basis the spinor and harmonic derivatives are defined as follows [25]

\[
D_a^+ = \partial_{-a}, \quad D_a^- = -\partial_{+a} = 2i\theta^{-b}\partial_{ab},
\]

\[
D^0 = u^{i+} \partial_{-i} - u^{-i} \partial_{+i} + \theta^{+a} \partial_{+a} - \theta^{-a} \partial_{-a}, \quad D^{\pm \pm} = \delta^{\pm \pm} + i\theta^{\pm a} \theta^{b\pm}\partial_{ab} + \theta^{\pm a} \partial_{\pm a}.
\]

Here we denoted \( \partial_{\pm a} \theta^{\mp b} = \delta^b_a \), \( \partial^{\pm \pm} = u^{i+} \partial_{\mp i}, \partial_{\pm i} = \frac{\partial}{\partial u_{\pm i}} \) and \( \partial_{ab} = \frac{1}{2}(\gamma^M_{ab}) \partial_M \). The six-dimensional Weyl matrices \( (\gamma^M_{ab}) \) are chosen as

\[
(\gamma^M_{ab}) = -(\gamma^M_{ba}), \quad \tilde{\gamma}^M_{ab} = \frac{1}{2} \epsilon^{abcd} (\gamma^M_{cd})\delta_{ab}, \quad (\gamma^M_{ac})\tilde{\gamma}^M_{cb} + (\gamma^M_{ac})\tilde{\gamma}^N_{cb} = -2\delta^b_a \eta_{MN},
\]

where \( \epsilon^{abcd} \) is the totally antisymmetric symbol. The derivatives listed above satisfy the algebra

\[
\{D^+_a, D^-_b\} = 2i\partial_{ab}, \quad [D^{++}, D^{--}] = D^0, \quad [D^{\pm \pm}, D^+_a] = 0, \quad [D^{\pm \pm}, D^-_a] = D^+_a.
\]

We include the integration over harmonic variables \( u^{i+} \) into the full and analytic superspace integration measures,

\[
d^{14}z = d^6x_A du(D^+)^4(D^-)^4, \quad d\zeta^{(-4)} = d^6x_A du(D^-)^4,
\]

where \( (D^{\pm})^4 = -\frac{1}{24} \epsilon^{abcd} D^+_a D^+_b D^+_c D^+_d \).

We consider the \( \mathcal{N} = (1, 0) \) \( q \)-hypermultiplet model in six dimensions. We assume the standard kinetic term and general self-coupling potential for the superfield \( q^+ \) [16]. The action for the model is

\[
S = -\int d\zeta^{(-4)} \left[ \tilde{q}^+ D^{++} q^+ + L^{++}(q^+, \tilde{q}^+, u) \right],
\]

where \( L^{++} \) is an arbitrary analytic function of the hypermultiplet \( q^+ \), which is real one \( \tilde{L}^{++} = L^{++} \). The kinetic term is also real under tilde conjugation [16] because of the property \( \tilde{q}^+ = -q^+ \). In the simplest quartic interaction, then \( L^{++} = \frac{1}{4} (\tilde{q}^+ q^+)^2 \), the action (7) describes the Taub-NUT sigma-model for the physical bosons (see, e.g., [16] for details).

Let us introduce an abelian analytic gauge superfield \( V^{++} \), which is transformed as

\[
\delta V^{++} = -D^{++}\lambda,
\]
where $\lambda = \lambda(\zeta, u)$ is a real analytic gauge parameter. We assume the superfield $V^{++}$ is an external background superfield minimally interacting with hypermultiplet. The action describing this model is similar to (7) and can be written as

$$S[q^+; V^{++}] = - \int d\zeta^{(-4)} \left[ q^+(D^{++} + iV^{++})q^+ + L^+(\tilde{q}^+ q^+) \right].$$  \hspace{1cm} (9)$$

The transformations

$$\delta q^+ = i\lambda q^+, \quad \delta \tilde{q}^+ = -i\lambda \tilde{q}^+.$$  \hspace{1cm} (10)$$
together with (8) hold the action (7) invariant if the potential function for hypermultiplet depends on the invariant combination of superfields, $L^+ = L^+(\tilde{q}^+ q^+)$. The action (9) can be considered as a part of a more general theory describing the supersymmetric Yang-Mills theory in the broken phase interacting with the self-coupling hypermultiplet.

Finally, it is interesting to discuss a case of $4N_f$ hypermultiplets $q_{AI}^+$. Here we have denoted $q_{AI}^+ = (q_1^+ ... q_{N_f}^+; -\tilde{q}_1^+ ... -\tilde{q}_{N_f}^+)$ for each index $I = 1, 2$. Also, the conjugation of the hypermultiplet $q_{AI}^+$ reads $\tilde{q}_{AI}^+ = q_{I}^{+A} = \Omega^{AB} q_{BI}^+$, where $\Omega^{AB} = -\Omega^{BA}$ is the $Sp(N_f)$ invariant tensor. We consider the model with the classical action

$$S_{N_f}^+[q^+; V^{++}] = - \int d\zeta^{(-4)} \left( q_{I}^{+A} D^{++} q_{AI}^+ + V^{++}\epsilon_{IJ} q_{I}^{+A} q_{AJ}^+ + V^{++}\zeta^{++} \right),$$  \hspace{1cm} (11)$$

where the summations over indexes $A$ and $I$ are assumed and $\epsilon_{12} = -\epsilon_{21} = 1$. The last term in (11) contains the constant vector $\xi^{++} = \xi^i u_i^+ u_j^+$, which is under the constraint $D^{++}\xi^{++} = 0$. This contribution corresponds to the Fayet-Iliopoulos term in the harmonic superspace. The action (11) is invariant under the local $SO(2)$ symmetry

$$\delta V^{++} = -D^{++}\lambda, \quad \delta \tilde{q}_{AI}^+ = \lambda \epsilon_{IJ} q_{AJ}^+,$$  \hspace{1cm} (12)$$
where $\lambda = \lambda(\zeta, u)$ is a real analytic gauge parameter.

The bosonic part of the action (11) has the form

$$S_{bos}^{N_f} = \int d^6 x \left( \partial^ab \phi_I^{iA} \partial_{ab} \phi_{iAI} - iA^{ab} \epsilon_{IJ} (\phi_I^{iA} \partial_{ab} \phi_{iAJ} - \partial_{ab} \phi_I^{iA} \phi_{iAJ}) \right)$$

$$- A^{ab} A_{ab} \phi_I^{iA} \phi_{iAI} + D_{ij} (\epsilon_{IJ} \phi_I^{iA} \phi_{jAI} + \tilde{\xi}_{ij}),$$  \hspace{1cm} (13)$$

where $\phi_{iAI}^j$ is the complex scalar component of hypermultiplet $q_{AI}^+$ and $\phi_{iAI} = \epsilon_{ij} \phi_{jAI}$, $\epsilon_{ij} = -\epsilon_{ji}$. The bosonic components of the vector multiplet $V^{++}$ in the Wess-Zumino gauge include the gauge field $A_{ab}$ and the triplet of auxiliary scalar fields $D_{ij}$ [25]. The action (13) takes a simpler form if we introduce the covariant derivative $\nabla_{ab} = \partial_{ab} - iA_{ab}$

$$S_{bos}^{N_f} = \int d^6 x \left( \nabla^ab \phi_I^{iA} \nabla_{ab} \phi_{iAI} + D_{ij} (\epsilon_{IJ} \phi_I^{iA} \phi_{jAI} + \tilde{\xi}_{ij}) \right).$$  \hspace{1cm} (14)$$

The fields $A_{ab}$ and $D_{ij}$ in the action (14) one can exclude using the algebraic equations of motion. In this case, the equation for the field $D_{ij}$ can be considered as a constraint on the dynamical field $\phi_{AI}^i$. 

3
3 One-loop effective action

To study the effective action in model (9) we will use the loop expansion at leading order. Setting
\[ q^+ \to q^+ + Q^+, \quad q^- \to q^- + \tilde{Q}^+, \]
where \(Q^+\) and \(\tilde{Q}^+\) are classical fields, we decompose the classical action up to the second order over \(q^+\) and \(\tilde{q}^+\)
\[ S_2 = -\int d\zeta (-4) \left\{ q^+ \mathcal{D}^{++} q^+ + \frac{1}{2} \Psi \left( (\tilde{Q}^+ q^+)^2 + 2\tilde{Q}^+ Q^+ \tilde{q}^+ q^+ + (\tilde{q}^+ Q^+)^2 \right) \right\}. \]

Here we introduce the covariant harmonic derivative
\[ \mathcal{D}^{++} = D^{++} + iV^{++} + \Psi, \]
and use the notations
\[ \Psi^{++} = -i \frac{\partial L^{+4}(\tilde{Q}^+ Q^+)}{\partial (\tilde{Q}^+ Q^+)^2}, \quad \Psi = \frac{\partial^2 L^{+4}(\tilde{Q}^+ Q^+)}{\partial (\tilde{Q}^+ Q^+)^2}. \]

We rewrite the action (16) into the matrix form
\[ S_2 = -\frac{1}{2} \int d\zeta (-4) \left( q^+ \tilde{q}^+ \right) \left( \begin{array}{cc} \mathcal{D}^{++} + \Psi \tilde{Q}^+ Q^+ & \Psi (Q^+)^2 \\ -\Psi (\tilde{Q}^+)^2 & \mathcal{D}^{++} - \Psi \tilde{Q}^+ Q^+ \end{array} \right) \left( \begin{array}{c} q^+ \\ \tilde{q}^+ \end{array} \right). \]

Using the last expression it is easy to obtain the one-loop contribution \(\Gamma^{(1)}\) to the effective action of the theory (9) by integrating over quantum fields. We have
\[ \Gamma^{(1)}[Q^+; V^{++}] = \frac{i}{2} \text{Tr}_{(3,1)} \ln S''_2[Q^+; V^{++}]. \]

In the (20) the functional trace includes the matrix trace and integration over harmonic superspace
\[ \text{Tr}_{(q,4-q)} \mathcal{O} = \text{tr} \int d\zeta_{-4} d\zeta_2 (-4) \delta(q,4-q) \mathcal{O}(q,4-q) (1|2), \]
where \(\delta_{(q,4-q)}(1|2)\) is an analytic delta-function (16) and \(\mathcal{O}(q,4-q)\) is the kernel of an operator \(\mathcal{O}\) acting in the space of analytic superfields with the harmonic U(1) charge \(q\) (16). The expression (20) includes the second variation derivative of the action \(S_2\) (19) and can be represented in the form
\[ \Gamma^{(1)}[Q^+; V^{++}] = i \text{Tr}_{(3,1)} \ln \mathcal{D}^{++} + \frac{i}{2} \text{Tr}_{(3,1)} \ln \left( 1 + \mathcal{G}^{(1,1)} \Psi Q^{++} \right). \]

Here we introduce the matrix
\[ Q^{++} = \left( \begin{array}{cc} \tilde{Q}^+ Q^+ & (Q^+)^2 \\ -\tilde{Q}^+ Q^+ & -\tilde{Q}^+ Q^+ \end{array} \right), \]
and the Green function \(\mathcal{G}^{(1,1)}\), which satisfies the equation
\[ \mathcal{D}^{++}_1 \mathcal{G}^{(1,1)}(1|2) = \delta^{(3,1)}(1|2). \]
We consider the second term in the one-loop contribution to effective action (21) in more detail. This part of effective action is defined as a series
\[
\text{Tr}_{\mathfrak{g}} \ln \left( 1 + \mathcal{G}^{(1,1)} \Psi \Psi^{++} \right) = \text{Tr} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left( \mathcal{G}^{(1,1)} \Psi \Psi^{++} \right)^n.
\]  
(24)

The matrix power in the last expression also includes the integration over analytic subspace. Every term in the series (24) including \((\Psi^{++})^n = \Psi^{++1} \Psi^{++2} \cdots \Psi^{++n} \), \(n > 1\), contains the second power of the matrix \(\Psi^{++}\) depending on the same argument due to the trace \(\text{Tr}\). However the matrix \(\Psi^{++}\) is a traceless and a nilpotent one, \((\Psi^{++})^2 = 0\). Hence the contribution (24) equals zero.

Thus the one-loop effective action (21) is reduced to
\[
\Gamma^{(1)}[\Psi^{++}; V^{++}] = i \text{Tr}_{\mathfrak{g}} \ln \mathcal{D}^{++}.
\]  
(25)

We have to note that the effective action (25) depends on the two superfields, namely the external classical analytic gauge superfield \(V^{++}\) and the background hypermultiplet \(\Psi^{++}\). We do not assume any restriction on these fields. Our aim now is to study one-loop divergencies of the action (25), but first of all, we consider the properties of covariant harmonic derivative \(\mathcal{D}^{++}\).

The operator \(\mathcal{D}^{++}\) includes beside the analytic gauge connection \(V^{++}\) also the superfield \(\Psi^{++}\) [18]. The analytic superfield \(\Psi^{++}\) by construction is an ordinary real function of the hypermultiplet. One can consider the superfield \(\Psi^{++}\) as an additional analytic gauge connection. Hence we have the harmonic covariant derivative depending on the two abelian gauge connections.

As a next step, we have to derive the algebra of covariant derivatives including both gauge superfield connections. For convenience, we introduce a notation
\[
\mathcal{V}^{++} = V^{++} + \Psi^{++}.
\]  
(26)

Then following standard procedure [16] we consider the non-analytic gauge connection \(\mathcal{V}^{--}\) by the rule
\[
\mathcal{D}^{--} \mathcal{V}^{++} = \mathcal{D}^{++} \mathcal{V}^{--},
\]  
(27)

which in our abelian case can be solved exactly
\[
\mathcal{V}^{--} = \int du \frac{\mathcal{V}^{++}(u_1)}{(u^+u_1^+)^2}.
\]  
(28)

The commutation relations of the algebra [5] should be rewritten and up to the definitions coincide with the algebra of covariant derivatives of six-dimensional \(\mathcal{N} = (1, 0)\) supersymmetric gauge theory [25][28]
\[
[D^{--}, D_a^+] = \nabla_a^-, \quad [\nabla^{++}, \nabla_a^-] = D_a^+, \quad [\nabla^{++}, D_a^+] = [\nabla^{--}, \nabla_a^-] = 0,
\]
\[
[D_a^+, \nabla_b^-] = 2 \nabla_{ab}^-, \quad [D_a^+, \nabla_{bc}] = \frac{i}{2} \varepsilon_{abcd} \mathcal{W}^{+d}, \quad [\nabla_a^-, \nabla_{bc}] = \frac{i}{2} \varepsilon_{abcd} \mathcal{W}^{-d}.
\]  
(29)

Here we have introduced the gauge field strength \(\mathcal{W}^{+-}\)
\[
\mathcal{W}^{+a} = -i \varepsilon^{abcd} D_b^+ D_c^+ D_d^+ \mathcal{V}^{--}, \quad \mathcal{W}^{-a} := \nabla^{--} \mathcal{W}^{+a}.
\]  
(30)
Let us define the analytic superfield \( \mathcal{F}^{++} = D^+_a \mathcal{W}^a \). The superfields \( \mathcal{F}^{++} \) and \( \mathcal{W}^a \) possess some useful properties \[28\]

\[
D^{++} \mathcal{F} = D^{++} \mathcal{W}^a = D^{++} \mathcal{W}^- = 0, \quad \mathcal{W}^- = D^- \mathcal{W}^a.
\]

We have to note that superfields \( \mathcal{F}^{++} \) and \( \mathcal{W}^a \) depend on the superfields \( V^{++} \) and \( Q^+ \). One can separate both \( \mathcal{F}^{++} \) and \( \mathcal{W}^a \) in two independent parts

\[
\mathcal{W}^a = \mathcal{W}_{V}^a + \mathcal{W}_{Q}^a, \quad \mathcal{F}^{++} = F_{V}^{++} + F_{Q}^{++}.
\]

Each of these superfields corresponds to the connection \( V^{++} \) and \( \Psi^{++} \) respectively.

The covariant analytic d’Alembertian \[28\] in six dimensions reads

\[
\Box = \frac{1}{2} (D^+)^4 (D^-)^2.
\]

Acting in a space of analytic superfields the operator \[33\] transforms to \[25\]

\[
\Box = \eta^{MN} D_M \mathcal{D}_N + \mathcal{W}^a D_a^- + F^{++} D^- - \frac{1}{2} (D^- F^{++}),
\]

where \( \eta_{MN} \) is 6D Minkowski metric and \( D_M = \partial_M - i \mathcal{A}_M \) is a space-time covariant derivative, \( \mathcal{A}_M = \frac{i}{2} (\gamma_M)^{ab} D_a^\dagger D_b^\dagger \mathcal{V}^{--} \). Also we have introduced the covariant harmonic derivative \( D^- = D^- + i \mathcal{V}^{--} \), which is constructed using non-analytic connection \( \mathcal{V}^{--} \).

### 4 One-loop divergencies

The one-loop contribution to effective action \[25\] contains the first order differential operator \( D^{++} = D^{++} + i \mathcal{V}^{++} \). To calculate the divergent contributions we following \[23\] and vary the one-loop effective action \[25\] with respect to the superfield \( \mathcal{V}^{++} \)

\[
\delta \Gamma^{(1)} = - \text{Tr}_{(3,1)} \delta \mathcal{V}^{++} \mathcal{G}^{(1,1)} = - \int d\zeta_{1}^{(-4)} \delta \mathcal{V}^{++}_{1} \mathcal{G}^{(1,1)}(1|2) \bigg|_{2 \to 1}.
\]

The Green function \( \mathcal{G}^{(1,1)}(1|2) \) solves the equation \[23\] and has the following formal expression (see, e.g., \[20\], \[30\])

\[
\mathcal{G}^{(1,1)}(1|2) = \frac{(D_1^+)^4 (D_2^+)^4 \delta^{14}(z_1 - z_2)}{\Box_{1}} \left( u_1^+ u_2^+ \right)^{3},
\]

where \( \delta^{14}(z_1 - z_2) = \delta^6(x_1 - x_2) \delta^8(\theta_1 - \theta_2) \) is a full \( N = (1,0) \) superspace delta-function.

The analysis of divergent contributions of the expression \[35\] is similar to what was carried out for six-dimensional \( N = (1,0) \) supersymmetric abelian gauge theory in \[20\]. First of all we use the proper time technique for the inverse operator \( \Box^{-1} \) and rewrite the expression \[35\] as follows:

\[
\delta \Gamma^{(1)} = - \int d\zeta_{1}^{(-4)} \delta \mathcal{V}^{++} \int_{0}^{\infty} d(is)(i\mu)^{2} e^{is\zeta_{1}} (D_1^+)^4 (D_2^+)^4 \delta^{14}(z_1 - z_2) \left( u_1^+ u_2^+ \right)^{3} 2^{1-\varepsilon}.
\]

In the last expression, we have introduced the proper-time parameter \( s \) and the arbitrary parameter \( \mu \) of mass dimension. We use a dimensional regularization scheme, where divergencies appear as a pole \( \frac{1}{\varepsilon} \) under the condition \( \varepsilon \to 0 \).
The integrand in the expression (37) contains eight $D$-factors acting on the Grassmann delta-function $\delta^8(\theta_1 - \theta_2)$ in coinciding points limit. To calculate this limit one can use the identity

$$(D^+_1)^4(D^+_2)^4\delta^8(\theta_1 - \theta_2)|_{2 \to 1} = (u^+_1 u^+_2)^4.$$  

After that, it transforms to

$$\delta \Gamma^{(1)}_{\text{div}} = - \int d\zeta \int_0^\infty d(is)(is\mu^2)^2 e^{is\frac{\Box^1}{4}}(u^+_1 u^+_2)\delta^6(x_1 - x_2)|_{2 \to 1} = (u^+_1 u^+_2)^4.$$  

Operator $\Box$ contains the harmonic derivative $D^{--}$ which acting on the factor $(u^+_1 u^+_2)$ produces non-zero contribution due to the identity $D^{--}_1 (u^+_1 u^+_2)|_{2 \to 1} = (u^+_1 u^+_2)|_{2 \to 1} = -1$. Commuting the operator $e^{is\Box}$ with the factor $(u^+_1 u^+_2)$ we act on the space-time delta-function and count integral over proper time (see the details in [20]). We have

$$\delta \Gamma^{(1)}_{\text{div}} = \frac{1}{3(4\pi)^3\varepsilon} \int d\zeta \delta \mathcal{V}^{++} \partial^2 \mathcal{F}^{++},$$  

where $\partial^2 = \partial_M \partial^M = \frac{1}{2} (D^+)^4 (D^{--})^2$. One can show that the last expression can be evaluated from the functional

$$\Gamma^{(1)}_{\text{div}} = \frac{1}{6(4\pi)^3\varepsilon} \int d\zeta \mathcal{V}^{++} \mathcal{F}^{++},$$  

taking into account the connection of the variation $\delta \mathcal{V}^{++}$ with $\delta \mathcal{V}^{--}$ [25]

$$\delta \mathcal{V}^{--} = \frac{1}{2} (D^{--})^2 \delta \mathcal{V}^{++} - \frac{1}{2} D^{++}(D^{--} \delta \mathcal{V}^{--}),$$  

and the properties (31) for the superfield $\mathcal{F}^{++}$.

The one-loop divergent contribution (40) to the effective action of the theory (9) depends on the external gauge multiplet $V^{++}$ and background hypermultiplet $Q^+$. Using the definitions (32) we can rewrite (40) in the form

$$\Gamma^{(1)}_{\text{div}}[V^{++}, Q^+] = \frac{1}{6(4\pi)^3\varepsilon} \int d\zeta \mathcal{F}^{++}_{V^{++}} \mathcal{F}^{++}_{Q^+}.$$  

Let us discuss the final result (42). First, we consider possible constraints on the background superfields. The gauge superfield $V^{++}$ is an arbitrary external classical superfield. One can restrict the superfield $V^{++}$ requiring, e.g.,

$$\mathcal{F}^{++}_V = 0.$$  

This condition corresponds to the Maxwell equation for the physical boson component of superfield $V^{++}$. If we assume that external superfield $V^{++}$ satisfies the constraint (43) the divergent contribution will depend only on off-shell background superfield $Q^+$. Hence, the one-loop divergent contribution (42) under the condition (43) can be rewritten in the form

$$\Gamma^{(1)}_{\text{div}}[Q^+] = \frac{1}{6(4\pi)^3\varepsilon} \int d\zeta \mathcal{F}^{++}_{Q^+} = \frac{1}{6(4\pi)^3\varepsilon} \int d^4x \Psi^{--} \partial^2 \Psi^{++}.$$  

Here we use the properties $\mathcal{F}^{++} = \frac{1}{2} D^{++} D^{--} \mathcal{F}^{++}$ and $D^{++} \mathcal{F}^{++} = 0$. 

The superfield $F_{Q^+}$ depends on the dynamical classical background hypermultiplet $Q^+$. The classical equation of motion for the background hypermultiplet

$$D^{++}Q^+ = (D^{++} + iV^{++} + i\Psi^{++})Q^+ = 0,$$

(45)

includes self-interaction term $\Psi^{++}$ and external gauge multiplet $V^{++}$. This equation also should be completed by the equation for Grassmann conjugate hypermultiplet $\tilde{Q}^+$. Equation (45) gives the nontrivial connection between physical components of hypermultiplet $Q^+$ and external gauge multiplet $V^{++}$ [16]. If we take into account condition (45), divergent contribution (44) will contain the dependence on the external gauge multiplet in the component action.

As an example, we consider the case when external gauge multiplet $V^{++} = 0$ and the potential function $L^{++}$ in the initial model (9) has a simple form $L^{++} = \frac{1}{2}(\tilde{Q}^+Q^+)^2$. The analytic $\Psi^{++}$ and non-analytic $\Psi^{--}$ connections for this potential are

$$\Psi^{++} = -i\tilde{Q}^+Q^+, \quad \Psi^{--} = \int du_2 \frac{\Psi^{++}_2}{(u_1^+u_2^+)^2}.$$

(46)

The divergent contribution (44) in the theory with such self-interaction takes the form

$$\Gamma_{\text{div}}^{(1)}[Q^+] = -\frac{1}{6(4\pi)^3\varepsilon} \int d^{14}z \frac{du_1 du_2}{(u_1^+u_2^+)^2} (\tilde{Q}^+Q^+)_{1\sigma} \partial^2(\tilde{Q}^+Q^+)_2.$$

(47)

The last expression is non-local in the harmonic space. However for the on-shell background hypermultiplet the non-locality of the divergent contribution can be removed. In this case, $\Psi^{--} = -i\tilde{Q}^-Q^-$, where $Q^- = D^{--}Q^+$, and we finally have

$$\Gamma_{\text{div}}^{(1)}[Q^+] = -\frac{1}{6(4\pi)^3\varepsilon} \int d^{14}z \tilde{Q}^-Q^- \partial^2(\tilde{Q}^+Q^+).$$

(48)

As one can see the divergent contribution (48) vanishes if we suppose the slowly varying background hypermultiplet, $\partial Q^+ \approx 0$. To obtain the leading finite contribution to effective action (37) in this case one can use the method developed in [44]. The result is

$$\Gamma_{\text{lead}}^{(1)}[Q^+] \sim \int d\zeta (W_{Q^+})^4 = 0.$$

(49)

Indeed, the superfield $W_{Q^+}$ introduced in (30) includes the term $D_a^+ D_{\alpha}^+ Q^-$, which after some algebra transforms to $2i\partial_{\alpha\beta}Q^+$ and vanishes for the constant background. We can consider this as a consequence of the absence of a scalar potential in the gauge-invariant supersymmetric $\sigma$-model in six dimensions [14].

The developed technique in application to the model (11) takes the following result

$$\Gamma_{\text{div}}^{(1)}[V^{++}] = \frac{N_f}{3(4\pi)^3\varepsilon} \int d\zeta (F^{++})^2.$$

(50)

It is remarkable that the divergent contribution (50) to the one-loop effective action for the model (11) is fully determined by the external superfield $V^{++}$ in the case of the off-shell background hypermultiplet. The expression (50) corresponds to the action for gauge multiplet with high-derivatives, which was introduced and studied in detail in [25]. To obtain the divergent contribution to the effective action
action of the physical scalar fields $\phi^i_{A_M}$ in model (11), we take from the action (50) the part, which depends only on the gauge field $A_M$ [25]. After that one has to exclude the gauge field $A_M$ using the classical equation of motion. We have

$$\Gamma_{\text{div}}^{(1) N_f}[\phi] = \frac{2N_f}{3(4\pi)^2 \varepsilon} \int d^6x (\partial^M F_{MN})^2 \bigg|_{A_M = \frac{i\epsilon_{IJ} A_I \phi^i}{\sqrt{2} \phi^A}} ,$$

where we denote $\partial_M = \frac{1}{2}(\tilde{\gamma}_M)^{ab} \partial_{ab}$ and $F_{MN}$ is the abelian gauge field strength.

5 Concluding remarks

In the present paper, we studied the divergent contributions to the one-loop effective action of the hypermultiplet model with arbitrary self-interaction potential function in six dimensions. We considered in detail the model of the hypermultiplet $q^+$ interacting with the external gauge multiplet $V^{++}$ and assume the arbitrary-gauge invariant self-coupling for hypermultiplet (9). In the six-dimensional $\mathcal{N} = (1,0)$ harmonic superspace we derived the one-loop contribution to the effective action in the theory and calculate the divergent contribution to the effective action (42). Also, we briefly discussed the one-loop divergent contribution to effective action (50) of model (11) with $N_f$ number of hypermultiplets interacting with the external gauge multiplet.

The developed above technique is a universal and can be applied to the hypermultiplet models with rigid supersymmetry in lower dimensions. Technically, the main distinction in these cases will be in the calculation of the momentum and proper-time integrals in expression (38). The five-dimensional hypermultiplet theory does not have one-loop logarithmic divergencies. Hence it is interesting to study the finite contribution to the effective action. However, in four dimensions the one-loop effective action of the hypermultiplet with gauge-invariant self-interacting term contains the divergent contribution. The developed technique in application to this model takes the result

$$\Gamma_{4D, \text{div}}^{(1)}[Q^+] = \frac{1}{32\pi^2 \varepsilon} \int d^2z \tilde{\Psi}^- - \tilde{\Psi}^+ .$$

For the case of potential function $L^{+4} = \frac{1}{2}(\bar{Q}^+ Q^+)^2$ and the on-shell background the last expression becomes much simpler

$$\Gamma_{4D, \text{div}}^{(1)}[Q^+] = -\frac{1}{32\pi^2 \varepsilon} \int d^2z \tilde{Q}^- Q^- \tilde{Q}^+ Q^+ .$$

In the present paper, we considered the interaction of the hypermultiplet with an abelian gauge multiplet in six dimensions. It is natural to generalize the above consideration to the arbitrary non-abelian case and to study the divergent contribution to the effective action of such model. It would be interesting to calculate the finite contributions to the effective action of the non-abelian hypermultiplet theory in six dimensions and consider the dimensional reduction of such contributions to dimensions five and four. We are going to examine these problems in the forthcoming works.

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\footnote{Many aspects of the $\mathcal{N} = 2$ hypermultiplet theory in four dimensions were considered in the works [15 50].}
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