A geometric approach to equivariant factorization homology and nonabelian Poincaré duality

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Received: 26 July 2022 / Accepted: 16 February 2023
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Abstract
Fix a finite group $G$ and an $n$-dimensional orthogonal $G$-representation $V$. We define the equivariant factorization homology of a $V$-framed smooth $G$-manifold with coefficients in an $E_V$-algebra using a two-sided bar construction, generalizing (Andrade, From manifolds to invariants of $E_n$-algebras. PhD thesis, Massachusetts Institute of Technology, 2010; Kupers and Miller, Math Ann 370(1–2):209–269, 2018). This construction uses minimal categorical background and aims for maximal concreteness, allowing convenient proofs of key properties, including invariance of equivariant factorization homology under change of tangential structures. Using a geometrically-seen scanning map, we prove an equivariant version (eNPD) of the nonabelian Poincaré duality theorem due to several authors. The eNPD states that the scanning map gives a $G$-equivalence from the equivariant factorization homology to mapping spaces out the one-point compactification of the $G$-manifolds, when the coefficients are $G$-connected. For non-$G$-connected coefficients, when the $G$-manifolds have suitable copies of $\mathbb{R}$ in them, the scanning map gives group completions. This generalizes the recognition principle for $V$-fold loop spaces in Guillou and May (Algebr Geom Topol 17(6):3259–3339, 2017).

Keywords  Factorization homology · Bar construction · Scanning map · Nonabelian Poincaré duality

Mathematics Subject Classification  55P91 · 55P48 · 57R91

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Published online: 20 March 2023
1 Introduction

1.1 Factorization homology: history

Factorization homology is a theory of invariants on manifolds with coefficients in suitable \( E_n \)-algebras. The language has been used to formulate and solve questions in many areas of mathematics. For example, there are homological stability results in [19, 21], a reconstruction of the cyclotomic trace in [2] and the study of quantum field theory in [7, 8].

Non-equivariantly, factorization homology has multiple origins. The most well-known approach started in Beilinson–Drinfeld’s study of an algebraic geometric approach to conformal field theory [4] under the name of chiral homology. Lurie [25, 5.5] and Ayala–Francis [1] introduced and extensively studied the algebraic topology analogue, named as factorization homology. This route relies heavily on \( \infty \)-categorical foundations. An alternative geometric model is Salvatore’s configuration spaces with summable labels [36]. This construction is close to the geometric intuition, but is not homotopical. Yet another model, using the bar construction and developed by Andrade [3], Miller [31] and Kupers–Miller [19], is homotopically well-behaved while staying close to the geometric intuition of configuration spaces. It is this last approach that we will generalize equivariantly.

To give context, we first give an introduction to this approach to non-equivariant factorization homology. It is a classical theorem by Dold–Thom [11] that the ordinary integral homology groups of a connected space \( M \) are exactly the homotopy groups the configuration space on \( M \) with summable labels in \( \mathbb{N} \), the commutative monoid of natural numbers. Salvatore [36] observed that one can form the configuration space on \( M \) with summable labels in an \( E_n \)-algebra \( A \), which has less structure than a commutative monoid, if the space \( M \) has the structure of a framed smooth manifold of dimension \( n \). This is because the local Euclidean chart of \( M \) offers the way to sum the labels in the \( E_n \)-algebra \( A \). In [3, 19], the authors used this idea and defined the factorization homology of a framed smooth manifold.
$M$ with coefficients in an $E_n$-algebra $A$ to be the two sided bar construction

$$\int_M A = B(D_M, D_n, A),$$

where $D_n$ is the monad associated to the little $n$-disks operad and $D_M$ is a certain functor associated to embeddings of disks in $M$.

This bar construction definition (1.1) is a concrete point-set level model of the $\infty$-categorical definition of [1, 25]. One can construct a topological category $\text{Mfld}_{fr}^n$ of framed smooth $n$-dimensional manifolds and framed embeddings, which is a common ground for both definitions. It is a symmetric monoidal category under disjoint unions. Let $\text{Disk}_{fr}^n$ be the full subcategory spanned by objects equivalent to $\bigcup_k \mathbb{R}^n_k$ for some $k \geq 0$. An $E_n$-algebra $A$ can be viewed as a symmetric monoidal topological functor out of $\text{Disk}_{fr}^n$. The $\infty$-categorical factorization homology [1, definition 3.2] is the derived symmetric monoidal topological left Kan extension of $A$ along the inclusion:

$$\text{Disk}_{fr}^n \xrightarrow{A} (\text{Top}, \times) \xrightarrow{\int_{fr}} \text{Mfld}_{fr}^n \xrightarrow{\int_{fr} A}$$

(1.2)

Horel [15, 7.7] showed the equivalence of (1.1) and (1.2).

### 1.2 The definition of equivariant factorization homology

We fix an integer $n$ and a finite group $G$ throughout. An equivariant version of an $E_n$-algebra is an $E_V$-algebra, where $E_V$ is a monad associated to a $G$-operad that is equivalent to the little $V$-disks operad $\mathcal{D}_V$ (see Sect. 3.4). The $E_V$-algebras give the correct concrete coefficient input of equivariant factorization homology on $V$-framed smooth $G$-manifolds. Here, a smooth $G$-manifold $M$ is $V$-framed if it comes with an equivariant trivialization

$$TM \cong M \times V$$

(1.3)

of its tangent bundle.

In line with (1.1), we define the equivariant factorization homology of a $V$-framed smooth $G$-manifold $M$ with coefficients in an $E_V$-algebra $A$ to be (Definition 3.14):

$$\int^\text{fr}_V M A = B(D^\text{fr}_M, D^\text{fr}_V, A).$$

(1.4)

**Remark 1.5** As will be made clear in [20],

$$B(D^\text{fr}_M, D^\text{fr}_V, A) \simeq D^\text{fr}_M \otimes D^\text{fr}_V \int^\text{fr}_V A,$$

where $\int^\text{fr}_V A = B(D^\text{fr}_V, D^\text{fr}_V, A)$. This bar construction is a cofibrant replacing of $A$ in $D^\text{fr}_V$-algebra, and thus the equivariant factorization homology could be understood as first taking a cofibrant replacement, and then extending from local to global by tensoring with $D^\text{fr}_M$ over $D^\text{fr}_V$.
We explain the definition (1.4) in a conveniently generalized context. A tangential structure is a \( G \)-map \( \theta : B \to B_G O(n) \) for some well-chosen \( G \)-space \( B \).\(^1\) A morphism of two tangential structures is a \( G \)-map over \( B_G O(n) \). All tangential structures form a category \( TS \), which is simply the over category \( G\text{Top}/B_G O(n) \).

Denote by \( \zeta_n \) the universal \( G \)-\( n \)-vector bundle over \( B_G O(n) \). Pulling back along \( \theta \) gives a bundle \( \theta^* \zeta_n \) over \( B \). A \( \theta \)-framing on a smooth \( G \)-manifold \( M \) is an equivariant bundle map \( \phi_M : TM \to \theta^* \zeta_n \). The \( G \)-manifold \( M \) has a \( \theta \)-framing if and only if the classifying map of its tangent bundle \( \tau : M \to B_G O(n) \) factors up to \( G \)-homotopy through \( \theta : B \to B_G O(n) \). Indeed, a \( \theta \)-framing on \( M \) is the same data as a map \( \tau_B : M \to B \) plus a homotopy between the two classifying maps \( \tau \) and \( \theta \circ \tau_B \) from \( M \) to \( B_G O(n) \) (see Corollary B.10 with Definition B.4). The \( V \)-framing (1.3) is a special case: it is \( \text{fr}_V \)-framing for a particular tangential structure \( \text{fr}_V : * \to B_G O(n) \).

In Sect. 3.1, we construct a \( G\text{Top} \)-enriched category \( \text{Mfld}^{\theta}_{G,n} \), the category of smooth \( n \)-dimensional \( \theta \)-framed \( G \)-manifolds and \( \theta \)-framed embeddings. In particular, there is the category of \( V \)-framed smooth \( G \)-manifold \( \text{Mfld}^{\text{fr}_V}_{G,n} \). It takes some effort to define the morphisms in the category. For example, \( V \)-framed embeddings between little \( V \)-disks should be just the linear embeddings in the definition of the little \( V \)-disks operad. However, we do not have the notion of linear embeddings between general \( V \)-framed manifolds. The solution is to allow all embeddings and to add in path data to correct the homotopy type, so that we do not see the unwanted rotations. This idea goes back to Steiner [37] and was used non-equivariantly by Andrade [3] and Kupers–Miller [19]. Using paths in the framing space, we define the \( \theta \)-framed embedding space of \( \theta \)-framed manifolds (Definition 3.6). This construction is covariant as a functor of \( \theta \).

In Sect. 3.2, we use the \( G\text{Top} \)-enriched category \( \text{Mfld}^{\text{fr}_V}_{G,n} \) to build the \( V \)-framed factorization homology by the bar construction (1.4). The representation \( V \) can be viewed as a \( G \)-manifold with a canonical \( V \)-framing, so each \( U_k V \) also has a canonical \( V \)-framing. Let \( \Lambda \) be the category of based finite sets \( k = \{0, 1, 2, \ldots, k \} \) with base point 0 and based injections\(^2\). For any \( M \in \text{Mfld}^{\text{fr}_V}_{G,n} \), \( \mathcal{D}^{\text{fr}_V}_{M} (k) = \text{Emb}^{\text{fr}_V} (U_k V, M) \) gives a functor \( \Lambda^{\text{op}} \to G\text{Top} \). Such functors \( \mathcal{E} : \Lambda^{\text{op}} \to G\text{Top} \) and their associated functors \( \text{E} : G\text{Top}_u \to G\text{Top}_u \) (Construction 2.4) give a convenient context for reduced operads and monads, which we explain in Sect. 2.1.

Taking \( M = V \), compositions in \( \text{Mfld}^{\text{fr}_V}_{G,n} \), equip the sequence \( \mathcal{D}^{\text{fr}_V}_{M} \) with the structure of a reduced \( G \)-operad. It is the endomorphism operad of the object \( V \). Moreover, it is equivalent to the little \( V \)-disks operad \( \mathcal{D}_V \) (Proposition 3.33), so it is an \( E_V \)-operad. The functors associated to \( \mathcal{D}^{\text{fr}_V}_{M} \) and \( \mathcal{D}^{\text{fr}_V}_{M} \) give a monad \( \mathcal{D}^{\text{fr}_V}_{M} \) and a right \( \mathcal{D}^{\text{fr}_V}_{M} \)-module functor \( \mathcal{D}^{\text{fr}_V}_{M} \), and thus (1.4) makes sense for a \( \mathcal{D}^{\text{fr}_V}_{M} \)-algebra \( A \).

For a tangential structure \( \theta \) so that \( V \) is \( \theta \)-framed (possible under the conditions on \( \theta \) prescribed in Proposition 3.10), one can define the \( \theta \)-framed equivariant factorization homology with coefficient in a \( \mathcal{D}^{\text{fr}_V}_{M} \)-algebra \( A \) as

\[
\int_{M}^{\theta} A = B(D^{\text{fr}_V}_{M}, \mathcal{D}^{\text{fr}_V}_{M}, A). \tag{1.6}
\]

Specializing to \( \theta = \text{fr}_V \), (1.6) gives (1.4). This construction is homotopically well-behaved.

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\(^1\) Non-equivariantly, \( \theta \) is usually taken to be \( B\Pi \to BO(n) \) for a subgroup \( \Pi \subset O(n) \).

\(^2\) This \( \Lambda \) is not the cyclic category defined by Connes; it is isomorphic to the category FI used in representation stability.
Proposition 1.7 (Proposition 3.15). The functor $\int_{M}^{\theta} - : D_{V}^{\theta}[G\text{Top}_{\ast}] \to G\text{Top}_{\ast}$ preserves weak equivalences.

1.3 Main results

In Sect. 3.3, we prove that the embedding space in $\text{Mfld}^{\text{fr}}_{G,n}$ has a close connection to the configuration space.

Proposition 1.8 (Proposition 3.30) Evaluating at 0 of the embedding gives a $(G \times \Sigma_{k})$-homotopy equivalence:

$$ev_{0} : \mathcal{F}_{M}^{\text{fr}}(k) = \text{Emb}^{\text{fr}}(\cup_{k} V, M) \sim \mathcal{F}_{M}(k).$$

Here, $\mathcal{F}_{M}(k)$ is the ordered configuration space of $k$ points in $M$. This is used to justify that $\mathcal{F}_{V}^{\text{fr}}$ is an $E_{V}$-operad.

We also prove an invariance result in the equivariant setting. Such a result is known non-equivariantly [1, Proposition 3.9] and expected equivariantly.

Theorem 1.9 (Theorem 3.20) Let $q : \theta_{1} \to \theta_{2}$ be a morphism of tangential structures and $V$ be $\theta_{1}$-framed. We also write $V$ for the $\theta_{2}$-framed $G$-manifold $q_{*}V$. Then for a $\theta_{1}$-framed $G$-manifold $M$ and a $D_{V}^{\theta_{2}}$-algebra $A$, there is a $G$-equivalence

$$\int_{M}^{\theta_{1}} q^{*} A \simeq \int_{q_{*}M}^{\theta_{2}} A.$$ 

Due to the invariance, we may drop the $\theta$ from the notation $\int^{\theta}$ when the context is clear.

The bar construction definition (1.6) stays close to the geometric origin, which readily leads to proofs of the following results using classical techniques.

Proposition 1.10 Equivariant factorization homology satisfies the following properties:

(1) (Proposition 3.16)

$$\int_{V}^{\theta} A \simeq A.$$

$$\int_{M}^{\theta} D_{V}^{\theta} A \simeq D_{M}^{\theta} A.$$

(2) (Proposition 3.17)

$$\int_{M \sqcup N}^{\theta} A \simeq \int_{M}^{\theta} A \times \int_{N}^{\theta} A.$$

In Sect. 4, we prove that our definition satisfies the following theorem.

Theorem 1.11 (Theorem 4.7 and Theorem 4.41) Let $M$ be a $V$-framed manifold and $A$ be a $D_{V}^{\text{fr}}$-algebra in $G\text{Top}$. There is a $G$-map:

$$p_{M} : \int_{M}^{\theta} A \to \text{Map}_{\ast}(M^{+}, B^{V} A).$$

(1) (eNPD) If $A$ is $G$-connected, $p_{M}$ is a weak $G$-equivalence.
(2) If \( V = W \oplus \mathbb{R} \) and \( M \cong N \times \mathbb{R} \) for a \( W \)-framed manifold \( N \), then \( p_M \) is a weak group completion (in the sense of Definition 4.37).

(3) If \( V = U \oplus \mathbb{R}^2 \) and \( M \cong N \times \mathbb{R}^2 \) for a \( U \)-framed manifold \( N \), then \( p_M \) is a group completion (in the sense of Definition 4.38).

Here, \( M^+ \) is the one-point compactification of \( M \); \( B^V A \) is a model for the \( V \)-fold deloop of \( A \) defined in Sect. 4.2.

In Theorem 1.11, part (1) is an equivariant version of the nonabelian Poincaré duality theorem due to several authors, including [36, Theorem 6.6] and [25, 5.5.6.6]; specializing to \( M = V \) in Theorem 1.11, it recovers the equivariant recognition principle of [13, Theorem 1.14]. In particular, if the \( E_V \)-algebra \( A \) is grouplike, then \( A \simeq \Omega^V B^V A \). This justifies the definition of \( B^V A \).

**Corollary 1.12** Let \( M \) and \( A \) be as in Theorem 1.11 and \( A \) be \( G \)-connected.

Then we have \( \int_{G/H \times V} A \simeq \text{Map}_*(G/H_+, A) \). Therefore, \( (\int_{G/H \times V} A)^G \simeq A^H \).

The map \( p_M \) in the eNPD theorem is induced by a scanning map, a natural transformation of right \( D^r_V \)-functors:

\[
D^r_V (\cdot) \to \text{Map}_*(M^+, \Sigma^V -).
\] (1.13)

The scanning map has been studied in various forms in [5, 30, 33]. In particular, Rourke–Sanderson [35] proved that McDuff’s scanning map is a weak \( G \)-equivalence on \( G \)-connected objects. Classically, given a configuration of \( k \) points in \( M \), regarded as an embedding of \( k \) to \( M \), the Pontryagin-Thom collapse gives an element of \( \text{Map}_*(M^+, \vee_k S^n) \). Note that the \( i \)-th wedge component \( S^n \) is in fact the fiber at the image of \( i \in k \) of the sphere bundle \( \text{Sph}(TM) \).

The scanning map pushes the target further to the codomain \( \text{Section}_c(M, \text{Sph}(TM)) \) independent of \( k \), so that the individual Pontryagin-Thom maps vary continuously for the configurations. To do this, one needs an identification of the normal bundle of the embedded points with the tangent bundle of the manifold. There are conceptually two ways to do this: to use geodesics to generate a canonical local vector field ([30]), or to fatten the configuration space to include the data of a tubular neighborhood [33].

In the \( V \)-framed case, we can give an easy definition of the scanning map (4.2). In Appendix A, we compare our scanning map to the scanning maps in the literature. In particular, we prove in Proposition A.10 that equivariant versions of the scanning maps in [30] and [33] are homotopic, which is claimed without proof in [33, Remark 3.2].

Our proof of eNPD has two steps. We sketch it out when \( A \) is \( G \)-connected. The first step is to use the scanning map (1.13). It assembles to a simplicial map

\[
B_* (D^r_M, D^r_V, A) \to \text{Map}_*(M^+, K_*)
\]

for a simplicial \( G \)-space \( K_* \) that realizes to \( B^V A \). Using the Rourke–Sanderson result, the induced map on the geometric realization is a weak \( G \)-equivalence

\[
\int_M A = |B_* (D^r_M, D^r_V, A)| \to |\text{Map}_*(M^+, K_*)|.
\]

The second step is to pull the \( M^+ \) out of the geometric realization. The map

\[
|\text{Map}_*(M^+, K_*)| \to \text{Map}_*(M^+, |K_*|)
\] (1.14)

is a \( G \)-equivalence only when \( K_* \) satisfies some connectivity conditions. Non-equivariantly, for \( M = \mathbb{R} \) so that \( M^+ = S^1 \), a sufficient connectivity condition is given in [26, Theorem
12.3]. Let \( v \) be a function from the conjugacy classes of subgroups of \( G \) to \( \mathbb{Z}_{\geq -1} \). We say a finite-dimensional based \( G \)-CW complex \( X \) has cell dimension \( v \) if its cells of the form of \( G/H \times D^n \) have highest dimension \( v(H) \). We define the function \( \dim(X) \) to be
\[
\dim(X)(H) = \max_{H \subseteq L} v(L).
\]

Combining the non-equivariant result with induction shows:

**Theorem 1.15** (Theorem 4.30) If \( X \) is a finite-dimensional based \( G \)-CW complex and \( K_\bullet \) is a simplicial \( G \)-space such that for all \( n \) and \( H \subseteq G \), \( K_n^H \) is \( \dim(X)(H) \)-connected, then 
\[
|\text{Map}_*(X, K_\bullet)| \rightarrow |\text{Map}_*(X, |K_\bullet|)| \text{ is a weak } G\text{-equivalence.}
\]

When \( A \) is \( G \)-connected, the \( K_\bullet \) constructed out of it satisfies this connectivity condition, so the eNPD theorem follows.

### 1.4 Comparison to other work

In this paper we give a homotopical point set definition of equivariant factorization homology generalizing [3]. There are axiomatic approaches to \( \infty \)-categorical equivariant factorization homology [16, 39] using \( G\)-\( \infty \)-categories and \( \infty \)-\( G \)-categories respectively. Our definition and [39], being generalizations of (1.1) and (1.2) respectively, are equivalent. The definition of equivariant factorization homology in [16] is called “genuine”, meaning that it considers \( H \)-manifolds for all subgroups \( H \subseteq G \). Restricted to \( G \)-manifolds, a theory of [16] gives a theory of [39].

In joint work with Horev and Klang [14], the author studies equivariant factorization homology of Thom \( G \)-spectra in the context of [16]. There, a very different proof of the eNPD theorem adapted to the \( \infty \)-categorical context is given, generalizing Corollary 4.6 of [1]. The alternative proof is an axiomatic one, based on equivariant handle-body decompositions of the \( G \)-manifold \( M \). In contrast, we provide a geometrically-seen scanning map that gives the equivalence in this paper. The scanning map was used to prove homological stability properties of non-equivariant configuration spaces and factorization homology in [19, 30, 31]. The approach in our paper should lead to equivariant stability results.

Another advantage of our approach to the equivariant factorization homology and the eNPD theorem is that it gives a simplicial filtration on the mapping space \( |\text{Map}_*(X, K_\bullet)| \) (taking \( A = \Omega^V Y \)), thus offering a spectral sequence. It could be useful for obtaining equivariant generalizations of [9]. However, as computations of equivariant homology of the free \( E_V \)-algebra on \( A \), \( H_*^G(D_M^{\text{frv}} A) \), and in general, \( H_*^G(D_M^{\text{frv}} A) \), remains open for any coefficients, this computational tool has not yet been explored.

Our definition of \( \text{Mfld}_{G,n}^\theta \) in Sect. 3.1 is closely related to Ayala–Francis [1], which we compare in Appendix B. For the trivial tangential structure id : \( B_G \text{O}(n) \rightarrow B_G \text{O}(n) \), we have \( \text{Mfld}_{G,n}^{\text{id}} \simeq \text{Mfld}_{G,n}^\theta \). The category \( \text{Mfld}_{G,n}^\theta \) is a pullback of \( \text{Mfld}_{G,n}^{\text{id}} \) induced by the map tangential structure \( \theta \rightarrow \text{id} \). We also identify the automorphism \( G \)-space \( \text{Emb}^\theta(V, V) \) in Theorem B.15.

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3 Note that [3] is non-equivariant: their \( G \) in \( \text{Emb}^G \) is a subgroup of \( GL_n(\mathbb{R}) \) and therefore refers to a tangential structure \( \theta : B_G \rightarrow BGL_n(\mathbb{R}) \).

4 In [39], their \( G \) is our \( \theta : B_G \rightarrow BO(n) \); their \( \Gamma \) is our \( G \); their \( \rho \) is our \( V \); their \( \Gamma^\rho \text{Orb}_{G,n}^G \) is our \( \text{Mfld}_{G,n}^{\text{frv}} \) with the adjustment that the morphisms are replaced by the \( G \)-fixed points of the morphisms; their \( \Gamma^\rho \text{Disk}_{G,n}^G \) algebra is defined in a symmetric monoidal category \( \mathcal{C} \) whose objects do not necessarily have \( G \)-actions, and a \( D_{V}^{\text{frv}} \) algebra \( A \) in \( G\text{Top} \) in our sense gives a \( \Gamma^\rho \text{Disk}_{n}^G \)-algebra in \( \mathcal{C} = G\text{Top} \) in their sense by sending \( G \times_H V \in \Gamma^\rho \text{Disk}_{n}^G \) to \( \text{Map}(G/H, A) \).
1.5 Notations

- $G\text{Top}$ is the Top-enriched category of $G$-spaces and $G$-equivariant maps.
- $\text{Top}_G$ is the $G\text{Top}$-enriched category of $G$-spaces and non-equivariant maps where $G$ acts by conjugation on the mapping space.

For a space $M$ and a fiber bundle $E \to M$,

- $\mathcal{F}_M(k)$ is the ordered configuration space of $k$ points in $M$.
- $\mathcal{F}_{E \downarrow M}(k)$ is the ordered configuration space of $k$ points in $E$ whose images are $k$ distinct points in $M$.

2 Preliminaries on operads and equivariant bundles

2.1 $\Lambda$-sequences and operads

To streamline the monadic bar construction in the main body, we use an elementary categorical framework of $\Lambda$-objects. This framework is studied in more detail in a paper with May and Zhang [34]. This subsection is a summary of the relevant content towards Example 2.10 and Proposition 2.11, which are used in later sections.

Let $\Lambda$ be the category of based finite sets $k = \{0, 1, 2, \cdots, k\}$ with base point 0 and based injections. The morphisms of $\Lambda$ are generated by permutations and the ordered injections $s^i_k : k - 1 \to k$ that skip $i$ for $1 \leq i \leq k$. It is a symmetric monoidal category with wedge sum as the symmetric monoidal product.

For a symmetric monoidal category $(\mathcal{V}, \otimes, I)$, let $\mathcal{V}_I$ be the category under the unit. In [34], $\mathcal{V}$ is more general, but here we will work only with the Cartesian monoidal category $(G\text{Top}, \times, \ast)$. The empty $G$-space $\varnothing$ is an initial object.

**Definition 2.1** A $\Lambda$-sequence in $G\text{Top}$ is a functor $\mathcal{E} : \Lambda^{op} \to G\text{Top}$. We write $\mathcal{E}(k)$ for $\mathcal{E}(k)$. It is called unital if $\mathcal{E}(0) = \ast$. The category of all $\Lambda$-sequences in $G\text{Top}$ is denoted $\Lambda^{op}[G\text{Top}]$, where morphisms are natural transformations of functors. The category of all unital $\Lambda$-sequences in $G\text{Top}$ is denoted $\Lambda_0^{op}[G\text{Top}]$, where morphisms are natural transformations of functors that are identity at level zero.

The category $\Lambda^{op}[G\text{Top}]$ admits a symmetric monoidal structure $(\mathcal{V}, \otimes, I)$, let $\mathcal{V}_I$ be the category under the unit. In [34], $\mathcal{V}$ is more general, but here we will work only with the Cartesian monoidal category $(G\text{Top}, \times, \ast)$. The empty $G$-space $\varnothing$ is an initial object.

The category $\Lambda^{op}[G\text{Top}]$ admits a symmetric monoidal structure $(\Lambda^{op}[G\text{Top}], \boxtimes, \mathcal{I}_0)$. Here, $\boxtimes$ is the Day convolution of functors on the closed symmetric monoidal category $\Lambda^{op}$. The unit is given by

$$\mathcal{I}_0(n) = \begin{cases} \ast, & n = 0; \\ \varnothing, & n > 0; \end{cases}$$

The symmetric monoidal product $\boxtimes$ on $\Lambda^{op}[G\text{Top}]$ induces a symmetric monoidal product on $\Lambda^{op}[G\text{Top}], \mathcal{I}_0$ and $\Lambda_0^{op}[G\text{Top}]$, which we still denote by $\boxtimes$.

The categories $\Lambda^{op}[G\text{Top}], \mathcal{I}_0$ and $\Lambda_0^{op}[G\text{Top}]$ admit a second (nonsymmetric) monoidal product $\odot$ in addition to $\boxtimes$, called the circle product. It is analogous to Kelly’s circle product on symmetric sequences [18]. The unit for $\odot$ is given by

$$\mathcal{I}_1(n) = \begin{cases} \ast, & n = 0, 1; \\ \varnothing, & n > 1; \end{cases}$$

where the only non-trivial morphism $\mathcal{I}_1(1) \to \mathcal{I}_1(0)$ is the identity. For a brief definition of $\odot$, see Construction 2.6 (2).
An operad in $G\text{Top}$, as defined in [29], gives an example of a symmetric sequence in $G\text{Top}$. If the operad is unital, meaning the 0-space of the operad is the unit, it has the structure of a $\Lambda$-sequence in $G\text{Top}$. A unital operad in $\text{Top}$ or $G\text{Top}$, is also called a reduced operad in [29]. In fact, we have the unital variant of Kelly’s observation [18]:

**Theorem 2.2** ([34, Theorem 0.10]) A unital operad in $G\text{Top}$ is a monoid in the monoidal category $(\Lambda_*^{op}[G\text{Top}], \circ, \mathcal{F}_1)$.

We give a construction which will be used in the definition of equivariant factorization homology: the associated functor of a unital $\Lambda_1^{op}$-sequence. This construction specializes to the monad associated to a reduced operad of [29]; it also appears in the definition of the circle product $\circ$.

Assume that $(\mathcal{W}, \otimes, \mathcal{J})$ is a cocomplete symmetric monoidal category tensored over $G\text{Top}$.

**Construction 2.3** Let $X \in \mathcal{W}\mathcal{J}$ be an object under the unit. Define $X^* : \Lambda \to \mathcal{W}$ to be the covariant functor that sends $n$ to $X^\otimes n$. On morphisms, it sends the permutations to permutations of the $X$’s and sends the injection $s^k_i : k-1 \to k$ for $1 \leq i \leq k$ to the map

$$(s^k_i)_* : X^\otimes k-1 \cong X^\otimes i-1 \otimes \mathcal{J} \otimes X^\otimes k-\eta\otimes s^k_i \to X^\otimes k,$$

where $\eta : \mathcal{J} \to X$ is the unit map of $X$. By convention, $X^\otimes 0 = \mathcal{J}$.

This defines a functor $(-)^* : \mathcal{W}\mathcal{J} \to \text{Fun}(\Lambda, \mathcal{W})$. Then one can form the categorical tensor product over $\Lambda$ of the contravariant functor $E$ and the covariant functor $X^*$.

**Construction 2.4** Let $E \in \Lambda_*^{op}[G\text{Top}]$ be a unital $\Lambda_1^{op}$-sequence. The functor

$$E : \mathcal{W}\mathcal{J} \to \mathcal{W}\mathcal{J}$$

associated to $E$ is defined to be

$$E(X) = E \otimes_\Lambda X^* = \bigsqcup_{k \geq 0} E(k) \otimes X^\otimes k / \sim,$$

where $(\alpha^* f, x) \approx (f, \alpha_* x)$ for all $f \in E(m)$, $x \in X^\otimes n$ and $\alpha \in \Lambda(n, m)$. The unit map of $E(X)$ is given by $\mathcal{J} \equiv \ast \otimes \mathcal{J} \equiv E(0) \otimes X^\otimes 0 \to E(X)$.

**Remark 2.5** It is sometimes useful to take the quotient in two steps and use the following alternative formula for $E$:

$$E(X) = \bigsqcup_{k \geq 0} E(k) \otimes \Sigma_k X^\otimes k / \sim,$$

where $[(s^k)^* f, x] \sim [f, (s^k)_* x]$ for all $f \in E(k)$, $x \in X^\otimes k-1$. We will use $\approx$ or $\sim$ for the equivalence relation to be clear which formula we are using and refer to $\sim$ as the base point identification.

**Construction 2.6** We focus on the following context of Construction 2.4.

(1) Letting $\mathcal{W} \equiv G\text{Top}$, one gets from $E \in \Lambda_*^{op}[G\text{Top}]$ an associated functor:

$$C : G\text{Top}_* \to G\text{Top}_*.$$
Let \( \mathcal{W} = (\Lambda^{op}[\text{GTop}], \boxtimes, \mathcal{J}) \) with the Day monoidal structure. Then \( \mathcal{W} \) is tensored over \( \text{GTop} \) in the obvious way by levelwise tensoring. One gets the circle product for \( \mathcal{E} \in \Lambda^{op}_{+}[\text{GTop}] \) and \( \mathcal{F} \in \Lambda^{op}[\text{GTop}], \mathcal{J} \):

\[
\mathcal{E} \circ \mathcal{F} := \mathcal{E} \otimes_{\Lambda} \mathcal{F}^* \in \Lambda^{op}[\text{GTop}], \mathcal{J}.
\]

These two cases are further related: the 0-th level functor

\[
i_0: \text{GTop}_* \to \Lambda^{op}[\text{GTop}], \mathcal{J}, (i_0X)(n) = \begin{cases} X, & n = 0; \\ \varnothing, & n > 0; \end{cases}
\]

gives an inclusion of a full symmetric monoidal subcategory, so we have

\[
i_0(\mathcal{E} \otimes \mathcal{X}^*) \cong \mathcal{E} \otimes \mathcal{X} (i_0(\mathcal{X}^*)) = \mathcal{E} \circ i_0X.
\]

(2.7)

In words, the reduced monad construction is what happens at the 0-space of the circle product. Using this, one can show:

**Proposition 2.8** ([34, Proposition 6.2]) Let \( E, F : \text{GTop}_* \to \text{GTop} \) be the functors associated to \( \mathcal{E} \) and \( \mathcal{F} \). Then the functor associated to \( \mathcal{E} \circ \mathcal{F} \) is \( E \circ F \).

A monad is a monoid in the functor category. Using the associativity of the circle product and (2.7), it is easy to prove that when \( \mathcal{E} \) is an operad, the associated functor \( C \) is a monad; and that when \( \mathcal{F} \) is a left/right module over the monoid \( \mathcal{E} \) in \( (\Lambda_{+}^{op}[\text{GTop}], \circ) \), the associated functor \( F \) is a left/right module over \( C \). The following construction gives examples.

**Construction 2.9** ([34, Section 8]) Suppose that we have a \( \text{GTop} \)-enriched symmetric monoidal category \( (\mathcal{W}, \otimes, \mathcal{J}) \) such that \( \mathcal{W}(\mathcal{J}, Y) \cong \ast \) for all objects \( Y \) of \( \mathcal{W} \). Then we can construct a \( \Lambda_{+}^{op}[\text{GTop}] \)-enriched category \( \mathcal{H}_{\mathcal{W}} \). The objects are the same as those of \( \mathcal{W} \), while the enrichment is given by

\[
\mathcal{H}_{\mathcal{W}}(X, Y) = \mathcal{W}(X^0, Y).
\]

The definition of the composition in \( \mathcal{H}_{\mathcal{W}} \) is similar to the structure maps of an endomorphism operad. So, for any objects \( X, Y, Z \) of \( \mathcal{W} \), \( \mathcal{H}_{\mathcal{W}}(Y, Z) \) is a monoid in \( (\Lambda_{+}^{op}[\text{GTop}], \otimes) \), \( \mathcal{H}_{\mathcal{W}}(X, Y) \) is a left module over it, and \( \mathcal{H}_{\mathcal{W}}(Y, Z) \) is a right module. By Theorem 2.2, \( \mathcal{H}_{\mathcal{W}}(Y, Z) \) is a unital operad, and it is called the endomorphism operad of \( Y \). The assumption \( \mathcal{W}(\mathcal{J}, Y) \cong \ast \) is automatically satisfied if \( \mathcal{W} \) is coCartesian monoidal.

**Example 2.10** In Sect. 3.1, we construct a \( \text{GTop} \)-enriched category \( (\text{Mfld}^0_{G,n}, \Pi, \otimes) \) with a designated element \( V \in \text{Mfld}^0_{G,n} \). Applying Construction 2.9 to \( \mathcal{W} = \text{Mfld}^0_{G,n} \), we obtain for any \( M \in \text{Mfld}^0_{G,n} \) a \( \Lambda \)-sequence

\[
\mathcal{D}_M^0 = \mathcal{H}_{\mathcal{W}}(V, M).
\]

Then, \( \mathcal{D}_V^0 = \mathcal{H}_{\mathcal{W}}(V, V) \) is a monoid in \( (\Lambda_{+}^{op}[\text{GTop}], \otimes) \) and \( \mathcal{D}_M^0 \) is a right module over it. Translating by Theorem 2.2, \( \mathcal{D}_V^0 \) is a reduced operad in \( (\text{GTop}, \times) \). By Proposition 2.8, \( D_V^0 \) is a monad and \( D_M^0 \) is a right module over \( D_V^0 \).

We will use that the circle product is strong symmetric monoidal in the first variable:

**Proposition 2.11** ([34, Proposition 4.7]) For any \( \mathcal{E} \in \Lambda^{op}[\text{GTop}], \mathcal{J} \), the functor \( - \otimes \mathcal{E} \) on \( (\Lambda^{op}(\text{GTop}), \boxtimes, \mathcal{J}) \) is strong symmetric monoidal. That is, the circle product distributes over the Day convolution: for any \( \mathcal{D}, \mathcal{D}' \in \Lambda^{op}(\text{GTop)}, \mathcal{J} \), we have

\[
(\mathcal{D} \boxtimes \mathcal{D}') \circ \mathcal{E} \cong (\mathcal{D} \circ \mathcal{E}) \boxtimes (\mathcal{D}' \circ \mathcal{E}).
\]
2.2 Equivariant bundles

As pointed out in the introduction, we define $\theta$-framed embeddings using maps between equivariant bundles. In this subsection, we list some preliminary results on equivariant vector bundles for the reader’s reference. The proofs of the results as well as a clarification of different notions of equivariant fiber bundles can be found in [41].

Let $G$ and $\Pi$ be compact Lie groups, where $G$ is the ambient action group and $\Pi$ is the structure group.

**Definition 2.12** A $G$-$n$-vector bundle is a map $p : E \rightarrow B$ such that the following statements hold:

1. The map $p$ is a non-equivariant $n$-dimensional vector bundle;
2. Both $E$ and $B$ are $G$-spaces and $p$ is $G$-equivariant;
3. The $G$-action is linear on fibers.

**Definition 2.13** A principal $G$-$\Pi$-bundle is a map $p : P \rightarrow B$ such that the following statements hold:

1. The map $p$ is a non-equivariant principal $\Pi$-bundle;
2. Both $P$ and $B$ are $G$-spaces and $p$ is $G$-equivariant;
3. The actions of $G$ and $\Pi$ commute on $P$.

**Remark 2.14** This is called a principal $(G, \Pi)$-bundle in [24, IV1].

**Theorem 2.15** There is an equivalence of categories between \{G-$n$-vector bundles over $B$\} and \{principal $G$-$O(n)$-bundles over $B$\}.

The classical procedure of passing from $n$-vector bundles to principal $O(n)$-bundles is called taking the space of admissible maps. The equivariant bundles mentioned are both just non-equivariant bundles with $G$-actions, and the classical procedure is compatible with the $G$-actions.

A $G$-vector bundle $E \rightarrow B$ is $V$-trivial for some $n$-dimensional $G$-representation $V$ if there is a $G$-vector bundle isomorphism $E \cong B \times V$. Such an isomorphism is called a $V$-trivialization or $V$-framing of the bundle. This is analogous to the case of non-equivariant vector bundles, except that equivariance adds in the representation $V$ that’s part of the data. However, the representation $V$ in the equivariant trivialization of a fixed vector bundle may not be unique.

**Example 2.16** ([41, Examples 3.4 and 3.5])

1. Let $G = C_2$, $\sigma$ be the sign representation. The unit sphere, $S(2\sigma)$, is $S^1$ with the 180 degree rotation action. As $C_2$-vector bundles,
   \[ S(2\sigma) \times \mathbb{R}^2 \cong S(2\sigma) \times 2\sigma. \]
   This is related to the fact that the tangent bundle of the $C_2$-manifold $S(2\sigma) \times \mathbb{R}$ is both $\mathbb{R}^2$-framed and 2$\sigma$-framed.
2. Take $V$ and $W$ to be any two representation of $G$ that are of the same dimension and take $B$ to be the space $G$ with the left multiplication action. Then
   \[ G \times V \cong G \times W. \]
   We do have the uniqueness of $V$ in the following case ([41, Corollary 3.2]).
Proposition 2.17 If $B$ has a $G$-fixed point, then $B \times V \cong B \times W$ if and only if $V \cong W$.

Equivariantly, $G$-representations serve the role of $\mathbb{R}^n$. So it is natural to consider the $V$-framing bundle $\text{Fr}_V(E)$ for an orthogonal $n$-dimensonal representation $V$. Write $\rho : G \to O(n)$ for a group homomorphism defining the representation $V$.

Definition 2.18 Let $p : E \to B$ be a $G$-n-vector bundle. Let $\text{Fr}_V(E)$ be the space of admissible maps with the $G$-action $g(\psi) = g\psi g(g)^{-1}$.

In other words, $\text{Fr}_V(E)$ has the same underlying space as $\text{Fr}_\mathbb{R}^n(E)$, but we think of admissible maps as mapping out of $V$ instead of $\mathbb{R}^n$. As $\text{Fr}_\mathbb{R}^n(E) \cong \text{Hom}(\mathbb{R}^n, E)$, taking into account the $G$-action, there is an equivariant homeomorphism

$$\text{Fr}_V(E) \cong \text{Hom}(V, E).$$

Here, $\text{Hom}(V, E)$ is the space of bundle maps from $V \to *$ to $E \to B$, with $G$ acting by conjugation; evaluating at $*$ gives the map $\text{Hom}(V, E) \to B$.

Let $H \subset G$ be a subgroup and $\text{Rep}(H, \Pi)$ be the set:

$$\text{Rep}(H, \Pi) = \{ \text{group homomorphism } \rho : H \to \Pi \}/\Pi\text{-conjugation}.$$

A group homomorphism $\rho : H \to \Pi$ gives a subgroup $\Lambda_\rho \subset (\Pi \times G)$ via its graph:

$$\Lambda_\rho = \{ (\rho(h), h) | h \in H \}.$$

Denote the centralizer of the image of $\rho$ in $\Pi$ by $Z_\Pi(\rho)$. It is a closed subgroup of $\Pi$, defined as $Z_\Pi(\rho) = \Pi \cap Z_{\Pi \times G}(\Lambda_\rho) = \{ v \in \Pi | v\rho(h) = \rho(h)v \text{ for all } h \in H \}$.

Take $p : P \to B$ to be a principal $G$-$\Pi$-bundle. Then each component $B_0 \subset B^H$ is associated to a homomorphism $[\rho] \in \text{Rep}(H, \Pi)$:

Theorem 2.19 There is a well-defined map $\pi^H_\rho (B) \to \text{Rep}(H, \Pi)$ by

$$B_0 \mapsto \{ \rho : H \to \Pi | (p^{-1}(B_0))^{\Lambda_\rho} \neq \emptyset \}.$$ 

Furthermore, for any fixed representative $\rho$, $(p^{-1}(B_0))^{\Lambda_\rho} \to B_0$ is a principal $Z_\Pi(\rho)$-bundle and $p^{-1}(B_0) \cong \Pi \times_{Z_\Pi(\rho)} (p^{-1}(B_0))^{\Lambda_\rho}$.

This is essentially [23, Theorem 12] and is explained in [41, Section 2.6]. Note that a principal $G$-$\Pi$-bundle morphism preserves the associated homomorphism $[\rho]$.

There is a notion of the universal $G$-$\Pi$-bundle $E_G \Pi \to B_G \Pi$, so that principal $G$-$\Pi$-bundles over a base $G$-space $B$ are classified by $G$-homotopy classes of maps from $B$ to $B_G \Pi$. We denote the universal $G$-$n$-vector bundle by $\zeta_n \to B_G O(n)$, where

$$\zeta_n = E_G O(n) \times_{O(n)} \mathbb{R}^n.$$

The $G$-homotopy type of the universal base can be obtained from information about the fixed-point spaces of total space. We have

Theorem 2.20 ([22, Theorem 2.17])

$$(B_G O(n))^G \simeq \bigsqcup_{[\rho] \in \text{Rep}(G, O(n))} BZ_{O(n)}(\rho);$$

$$\simeq \bigsqcup_{[V] \in \text{Rep}(G, O(n))} B(O(V)^G).$$
Here, $O(V)$ is the space of isometric self maps of $V$ with $G$ acting by conjugation.

**Example 2.21** Take $H = G = C_2$ and $\Pi = O(2)$. Then

$$\text{Rep}(C_2, O(2)) = \{\id, \text{rotation}, \text{reflection}\}.$$  

For $\rho = \id$ or $\rho$ = rotation, $Z_\Pi(\rho) = O(2)$. For $\rho$ = reflection, $Z_\Pi(\rho) \cong \Z/2 \times \Z/2$. So

$$(B_{C_2}O(2))^C_2 \cong BO(2) \amalg BO(2) \amalg B(\Z/2 \times \Z/2).$$

One can make explicit the classifying maps of $V$-trivial bundles as follows. A $G$-map $\theta : \ast \to BG O(n)$ gives the following data: it lands in one of the $G$-fixed components of $BG O(n)$, indexed by a representation class $[V]$; its image is a $G$-fixed point $b \in B_G O(n)$.

**Proposition 2.22** The pullback of the universal bundle along this map is exactly $\theta^* \zeta_n \cong V$ as a $G$-vector bundle over $\ast$.

The loop space of $B_G O(n)$ at the base point $b$, $\Omega_b B_G O(n)$, is a $G$-space with the pointwise $G$-action on the loops. Via concatenation of loops, it is an $A_\infty$-algebra in $G$-spaces. Using the Moore loop space

$$\Lambda_b B_G O(n) = \{(l, \alpha) \in \R_{\geq 0} \times \text{Map}(\R_{\geq 0}, B_G O(n)) | \alpha(0) = b, \ \alpha(t) = b \text{ for } t \geq l\},$$

we may strictify $\Omega_b B_G O(n)$ to a $G$-monoid.

**Definition 2.23** A $G$-monoid is a monoid in $G$-spaces, that is, an underlying monoid such that the multiplication is $G$-equivariant. A morphism of $G$-monoids is an equivalence if it is a weak $G$-equivalence.

**Theorem 2.24** ([41, Theorem 3.12]) Let $b$ be a fixed point in the $V$-indexed component of $(B_G O(n))^G$.

1. There is a $G$-homotopy equivalence $\Omega_b B_G O(n) \cong O(V)$;
2. There is an equivalence of $G$-monoids $\Lambda_b B_G O(n) \cong O(V)$.

The equivalence of $G$-monoids is explicitly given by a zigzag (see Remark B.17). Theorem 2.24 is used in Theorem B.15 to understand the automorphism space of a framed disk $V$.

### 3 Tangential structures and factorization homology

#### 3.1 Equivariant tangential structures

In this subsection we fix a tangential structure $\theta$ and construct two categories. The first one is $\text{Vec}^\theta_{G,n}$, the category of $n$-dimensional $\theta$-framed equivariant bundles and $\theta$-framed bundle maps. The second one is $\text{Mfld}^\theta_{G,n}$, the category of smooth $n$-dimensional $\theta$-framed $G$-manifolds and $\theta$-framed embeddings. The category $\text{Mfld}^\theta_{G,n}$ is a subcategory of $\text{Vec}^\theta_{G,n}$; both $\text{Mfld}^\theta_{G,n}$ and $\text{Vec}^\theta_{G,n}$ are enriched over $G\text{Top}$. Recall from the introduction that all tangential structures $\theta$ form a category $\mathcal{T}S = G\text{Top}/B_G O(n)$; both constructions define covariant functors from $\mathcal{T}S$ to categories.

Recall that $\zeta_n$ is the universal $G\cdot n$-vector bundle over $B_G O(n)$. Pulling back along the tangential structure $\theta : B \to B_G O(n)$ gives a bundle $\theta^* \zeta_n$ over $B$. This is meant to be the
universal $\theta$-framed vector bundle. For an $n$-dimensional smooth $G$-manifold $M$, the tangent bundle of $M$ is a $G$-$n$-vector bundle. It is classified by a $G$-map up to $G$-homotopy:

$$\tau : M \to B_{GO(n)}.$$  

**Definition 3.1** A $\theta$-framing on a $G$-$n$-vector bundle $E \to M$ is a $G$-$n$-vector bundle map $\phi_E : E \to \theta^*\zeta_n$. A $\theta$-framing on a smooth $G$-manifold $M$ is a $\theta$-framing $\phi_M$ on its tangent bundle. We abuse notations and refer to the map on the base spaces as $\phi_M$ as well.

Note that for a manifold $M$ to be $\theta$-framed, it must be of dimension $n$. We consider the empty set to be uniquely $\theta$-framed for any $n$ and any $\theta : B \to B_{GO(n)}$.

A bundle has a $\theta$-framing if and only if its classifying map $\tau : M \to B_{GO(n)}$ has a factorization up to $G$-homotopy through $B$; see diagram (3.2). However, a factorization $\tau_B : M \to B$ does not uniquely determine a $\theta$-framing $\phi_E : E \to \theta^*(\zeta_n)$. Indeed, a bundle map $\phi_E : E \to \theta^*(\zeta_n)$ is the same data as a map $\tau_B : M \to B$ on the base plus a homotopy between the two classifying maps from $M$ to $B_{GO(n)}$. For a detailed proof, see Corollary B.10 with Definition B.4.

\[
\begin{array}{ccc}
M & \xrightarrow{\tau} & B_{GO(n)} \\
\downarrow{h} & \searrow{\theta} & \downarrow{\tau_B} \\
B & \xrightarrow{B} & B_{GO(n)}
\end{array}
\]  

(3.2)

**Example 3.3** As seen in Proposition 2.22, the tangential structure $fr_V : * \to B_{GO(n)}$ characterizes $V$-trivializations. We call it the $V$-framing tangential structure, and emphasize that is not only a space $B = *$ but also a map $fr_V$.

**Definition 3.4** Given two $\theta$-framed bundles $E_1, E_2$ with framings $\phi_1, \phi_2$, the space of $\theta$-framed bundle maps between them is defined as:

$$\text{Hom}^\theta(E_1, E_2) := \text{hofib}(\text{Hom}(E_1, E_2) \xrightarrow{\phi_2 \circ -} \text{Hom}(E_1, \theta^*\zeta_n)), \quad (3.5)$$

where $\text{Hom}(E_1, \theta^*\zeta_n)$ is based at $\phi_1$.

We use the following model for the homotopy fiber in (3.5):

$$\text{Hom}^\theta(E_1, E_2) = \{(f, \alpha, l) \mid f \in \text{Hom}(E_1, E_2), \alpha \in \text{Map}(\mathbb{R}_{\geq 0}, \text{Hom}(E_1, \theta^*\zeta_n)), l \in \text{Map}(\text{Hom}(E_1, E_2), \mathbb{R}_{\geq 0}) \text{ such that }$$

\begin{align*}
&l \text{ is locally constant,} \\
&\alpha(0) = \phi_1, \alpha(t) = \phi_2 \circ f \text{ for } t \geq l(f)\}.
\end{align*}

Here, the function $l$ is the length of the Moore paths and locally constant means being constant on path components. The $\theta$-framed bundle maps have unital and associative composition, with the unit in $\text{Hom}^\theta(E, E)$ given by $(\text{id}_E, \phi_{\text{const}}, 0_{\text{const}})$. Ignoring the path data $l$, the composition is defined up to homotopy as:

$$\text{Hom}^\theta(E_2, E_3) \times \text{Hom}^\theta(E_1, E_2) \to \text{Hom}^\theta(E_1, E_3);$$

\[
(g, \beta), (f, \alpha) \mapsto (g \circ f, \lambda),
\]

where $\lambda(t) = \begin{cases} 
\alpha(2t), & \text{when } 0 \leq t \leq 1/2; \\
\beta(2t - 1) \circ f, & \text{when } 1/2 < t \leq 1.
\end{cases}$
Note that in the definition of \( \text{Hom}^\theta(E_1, E_2) \), everything is taken non-equivariantly. The spaces \( \text{Hom}(E_1, E_2) \) and \( \text{Hom}(E_1, \theta^* \zeta_n) \) have \( G \)-actions by conjugation. Since \( \phi_1 \) and \( \phi_2 \) are \( G \)-maps, the homotopy fiber \( \text{Hom}^\theta(E_1, E_2) \) inherits the conjugation \( G \)-action. So we have built a \( G \text{-Top-enriched category} \) \( \text{Vec}^{\theta}_{G,n} \) of \( \theta \)-framed bundles and \( \theta \)-framed bundle maps.

**Definition 3.6** The space of \( \theta \)-framed embeddings between two \( \theta \)-framed manifolds is defined as the pullback displayed in the following diagram of \( G \)-spaces:

\[
\begin{array}{ccc}
\text{Emb}^\theta(M, N) & \longrightarrow & \text{Hom}^\theta(TM, TN) \\
\downarrow & & \downarrow \\
\text{Emb}(M, N) & \xrightarrow{d} & \text{Hom}(TM, TN)
\end{array}
\]  
(3.7)

Here, \( \text{Emb}(M, N) \) is the space of smooth embeddings and the map \( d \) takes an embedding to its derivative. For the empty manifold, we define \( \text{Emb}^\theta(\emptyset, N) = \ast \) and \( \text{Emb}^\theta(M, \emptyset) = \emptyset \) unless \( M = \emptyset \). The category \( \text{Mfld}^\theta_{G,n} \) has objects \( \theta \)-framed manifolds (including the empty set) and morphism spaces \( \text{Emb}^\theta \).

**Remark 3.8** Most of the time, we drop the Moore-path-length data and write an element of \( \text{Emb}^\theta(M, N) \) as a package of a map \( f \) and a homotopy \( \tilde{f} = (f, \alpha) \), with \( f \in \text{Emb}(M, N) \) and \( \alpha : [0, 1] \to \text{Hom}(TM, TN) \) satisfying \( \alpha(0) = \phi_M \) and \( \alpha(1) = \phi_N \circ df \). There is a functor \( \text{Mfld}^\theta_{G,n} \to \text{Mfld}_{G,n} \) by forgetting the tangential structure. It sends \( \tilde{f} \in \text{Emb}^\theta(M, N) \) to \( f \in \text{Emb}(M, N) \).

Let \( \mathbb{L} \) be the disjoint union of \( \theta \)-framed vector bundles or manifolds and \( \emptyset \) be the empty bundle or manifold. Both \( (\text{Vec}^\theta_{G,n}, \mathbb{L}, \emptyset) \) and \( (\text{Mfld}^\theta_{G,n}, \mathbb{L}, \emptyset) \) are \( G \text{-Top-enriched symmetric monoidal categories} \). In both categories, \( \emptyset \) is the initial object. In \( \text{Vec}^\theta_{G,n} \), \( \mathbb{L} \) is the coproduct, but it is not so in \( \text{Mfld}^\theta_{G,n} \).

**Remark 3.9** We need the length of the Moore path to be locally constant as opposed to constant for the enrichment to work. Namely, the map

\[
\text{Hom}^\theta(E_1, E'_1) \times \text{Hom}^\theta(E_2, E'_2) \to \text{Hom}^\theta(E_1 \sqcup E_2, E'_1 \sqcup E'_2)
\]

is given by first post-composing with the obvious \( \theta \)-framed map \( E'_i \to E'_i \sqcup E'_2 \) for \( i = 1, 2 \), then using a homeomorphism, as follows:

\[
\text{Hom}^\theta(E_1, E'_1) \times \text{Hom}^\theta(E_2, E'_2) \to \text{Hom}^\theta(E_1, E'_1 \sqcup E'_2) \times \text{Hom}^\theta(E_2, E'_1 \sqcup E'_2) 
\cong \text{Hom}^\theta(E_1 \sqcup E_2, E'_1 \sqcup E'_2)
\]

If the length of the Moore path were constant, the displayed homeomorphism would only be a homotopy equivalence, as the length of a Moore path can be different on the two parts.

To set up factorization homology in Sect. 3.2, we fix an \( n \)-dimensional orthogonal \( G \)-representation \( V \); in addition, we suppose that \( V \) is \( \theta \)-framed and fix a \( \theta \)-framing on \( V \)

\[
\phi : TV \to \theta^* \zeta_n.
\]

From the proof of the next proposition, we may assume without loss of generality that the base of \( \phi : V \to B \) is the constant map to \( \phi(0) \in B^G \) (which is a \( V \)-indexed component in the sense of Theorem 2.19).
Proposition 3.10 Write $\rho : G \to O(n)$ for a matrix representation of $V$ and write $\Lambda_\rho = \{(\rho(g), g) \in O(n) \times G | g \in G \}$. For a tangential structure $\theta : B \to B_G O(n)$, the space of $\theta$-framings on the $G$-manifold $V$ is equivalent to $(\theta^* E_G O(n))^{\Lambda_\rho} \cong (\theta^* (E_G O(n)))^{\Lambda_\rho}$. So a $\theta$-framing on $V$ exists, if and only if the intersection of $\theta(B)$ and the $V$-indexed component of $(B_G O(n))^{\Lambda_\rho}$ as introduced in Theorem 2.20 is non-empty.

Proof Since $TV \cong V \times V$, the space of $\theta$-framings on $V$ is

$$\text{Hom}(TV, \theta^* \zeta_n)^G \cong \text{Map}_G(V, \text{Hom}(V, \theta^* \zeta_n)) \cong \text{Hom}(V, \theta^* \zeta_n)^G.$$ 

The conjugation $G$-action on $\text{Hom}(V, \theta^* \zeta_n)$ is the same thing as the $\Lambda_\rho$-action on $\text{Hom}(\mathbb{R}^n, \theta^* \zeta_n)$, where $\Lambda_\rho \subset O(n) \times G$, $O(n)$ acts on $\mathbb{R}^n$ and $G$ acts on $\theta^* \zeta_n$. So we can identify

$$\text{Hom}(V, \theta^* \zeta_n)^G = \text{Hom}(\mathbb{R}^n, \theta^* \zeta_n)^{\Lambda_\rho} \cong (\theta^* E_G O(n))^{\Lambda_\rho},$$

and then

$$(\theta^* E_G O(n))^{\Lambda_\rho} \cong (\theta^* (E_G O(n)))^{\Lambda_\rho}$$

by applying Theorem 2.19 to the principal $G$-$O(n)$-bundles

$$\theta^* E_G O(n) \to B$$

and similarly for $B_G O(n)$.

\[\square\]

Corollary 3.11 Let $V, W$ be $n$-dimensional $G$-representations.

1. The $G$-manifold $W$ can be $\text{fr}_V$-framed if and only if $W \cong V$ as $G$-representations.
2. For a tangential structure $\theta$ so that $V$ and $W$ are both $\theta$-framed and $H \subset G$,

$$(\text{Emb}^\theta (V, W))^H \neq \varnothing$$

if and only if $\text{Res}^G_H V \cong \text{Res}^G_H W$ as $H$-representations.

Proof (1) follows from Proposition 3.10 and Theorem 2.20. (2) can be proved directly; here we give a more structured proof. By Theorem 2.15, we can apply Theorem 2.19 to $G$-vector bundles as well: We have $\pi_0^H(V) = *$, and the map on $\pi_0^H$ defined in Theorem 2.19 sends $*$ to $\pi_0^H V$, and similarly for $W$. Suppose $\text{Hom}(TV, TW)^H \neq \varnothing$, then by examining $\pi_0^H$ we must have $\text{Res}^G_H V \cong \text{Res}^G_H W$. So $(\text{Emb}^\theta (V, W))^H \neq \varnothing$ implies $\text{Res}^G_H V \cong \text{Res}^G_H W$. The other direction is easy.

\[\square\]

We also describe the change of tangential structures. Let $q$ be a morphism from $\theta_1 : B_1 \to B_G O(n)$ to $\theta_2 : B_2 \to B_G O(n)$, equivalently, a $G$-map $q : B_1 \to B_2$ satisfying $\theta_2 q = \theta_1$. Then a $\theta_1$-framed vector bundle $E \to B$ with $\phi_E : E \to \theta_1^* \zeta_n$ is $\theta_2$-framed by

$$E \to \theta_1^* \zeta_n = q^* \theta_2^* \zeta_n \to \theta_2^* \zeta_n.$$ 

The morphism $q$ also induces a map on framed-morphisms. So we have a functor

$$q_* : \text{Vec}^\theta_{G,n} \to \text{Vec}^\theta_{G,n},$$

and similarly $q_* : \text{Mfld}^\theta_{G,n} \to \text{Mfld}^\theta_{G,n}$. 

3.2 Equivariant factorization homology

In this subsection, we use the $G\text{Top}$-enriched category $\text{Mfld}^\theta_{G,n}$ developed in Sect. 3.1 to define the equivariant factorization homology as a monadic bar construction. We have fixed an $n$-dimensional orthogonal $G$-representation $V$ and a $\theta$-framing $\phi : TV \to \theta^* \zeta_n$ on the $G$-manifold $V$.
Recall that $\Lambda$ is the category of finite based sets $k$ and based injections. From Example 2.10, we have a $\Lambda$-sequence $\mathcal{D}^{\theta}_M$ for a $\theta$-framed manifold $M$. Explicitly, on objects, we have

$$\mathcal{D}^{\theta}_M(k) = \text{Emb}^\theta(\amalg V_k, M);$$

(3.12)

On morphisms, $\Sigma_k$ acts by permuting the copies of $V$, and $s_i^k : k - 1 \rightarrow k$ induces $(s_i^k)^* : \mathcal{D}^{\theta}_M(k) \rightarrow \mathcal{D}^{\theta}_M(k - 1)$ by forgetting the $i$-th $V$ in the embeddings for $1 \leq i \leq k$.

Taking $M = V$, $\mathcal{D}^{\theta}_V$ is a reduced $G$-operad. Using Construction 2.6, we get associated functors of $\mathcal{D}^{\theta}_M$ and $\mathcal{D}^{\theta}_V$, which we denote by $D^{\theta}_M, D^{\theta}_V : G\text{Top}_* \rightarrow G\text{Top}_*$.

The associated functor $D^{\theta}_V$ is a monad (with natural transformations $\eta : \text{id} \rightarrow D^{\theta}_V$ and $m : D^{\theta}_V \circ D^{\theta}_V \rightarrow D^{\theta}_V$) and $D^{\theta}_M$ is a right $D^{\theta}_V$-module (with a natural transformation $m_L : D^{\theta}_M \circ D^{\theta}_V \rightarrow D^{\theta}_M$). The following is a standard definition:

**Definition 3.13** Let $C$ be a reduced operad in $(G\text{Top}, \times)$ and $C$ be the associated monad. An object $A \in G\text{Top}_*$ is a $C$-algebra, or equivalently a $C$-algebra, if there is a map $\gamma : CA \rightarrow A$ such that the following diagrams commute, where the unlabeled maps are the unit and multiplication map of the monad $C$:

$$\begin{array}{ccc}
CCA & \xrightarrow{C\gamma} & CA \\
\downarrow & & \downarrow \gamma \\
CA & \xrightarrow{\gamma} & A
\end{array}$$

In what follows, let $A$ be a $\mathcal{D}^{\theta}_V$-algebra in $G\text{Top}_*$. It is conceptually a left $D^{\theta}_V$-module. We have a simplicial $G$-space, whose $q$-th level is

$$B_q(D^{\theta}_M, D^{\theta}_V, A) := D^{\theta}_M(D^{\theta}_V)^q A.$$

The face maps are induced by the above-given structure maps

$$m_L : D^{\theta}_M D^{\theta}_V \rightarrow D^{\theta}_M, \quad m : D^{\theta}_V D^{\theta}_V \rightarrow D^{\theta}_V \quad \text{and} \quad \gamma : D^{\theta}_V A \rightarrow A.$$

The degeneracy maps are induced by $\eta : \text{id} \rightarrow D^{\theta}_V$.

We have the following definition after the non-equivariant work of [3, IX.1.5]:

**Definition 3.14** The factorization homology of $M$ with coefficients $A$ is

$$\int^\theta_M A := B(D^{\theta}_M, D^{\theta}_V, A).$$

The category of algebras $\mathcal{D}^{\theta}_V[G\text{Top}_*]$ has a transferred model structure via the forgetful functor $\mathcal{D}^{\theta}_V[G\text{Top}_*] \rightarrow G\text{Top}_*$ ([6, 3.2, 4.1]), so that weak equivalences of maps between algebras are just underlying weak equivalences.

**Proposition 3.15** The functor $\int^\theta_M : \mathcal{D}^{\theta}_V[G\text{Top}_*] \rightarrow G\text{Top}_*$ is homotopical.
The proof is a formal argument assembling the literature. We show that the bar construction is Reedy cofibrant in the deferred Lemma 4.18 and Corollary 4.19. The claim then follows since geometric realization preserves levelwise weak equivalences between Reedy cofibrant simplicial \( G \)-spaces, as quoted in the deferred Theorem 4.14.

We have the following properties of the factorization homology.

**Proposition 3.16**

\[
\int^\theta_V A \simeq A.
\]

\[
\int^\theta_M D_V^\theta A \simeq D_M^\theta A.
\]

**Proof** Both follow from the extra degeneracy argument of [26, Propositions 9.8 and 9.9]. For the first equivalence, the extra degeneracy coming from the unit map of the first \( D_V^\theta \) establishes \( A \) as a retract of \( B(D_V^\theta, D_V^\theta, A) \), which is just \( \int^\theta_V A \). For the second equivalence, the unit map \( A \to D_V^\theta A \) establishes \( D_M^\theta A \) as a retract of \( B(D_M^\theta, D_V^\theta, D_V^\theta A) \). \( \square \)

**Proposition 3.17** For \( \theta \)-framed manifolds \( M \) and \( N \),

\[
\int_{M \sqcup N}^\theta A \cong \int_M^\theta A \times \int_N^\theta A.
\]

**Proof** Without loss of generality, we may assume that both \( M \) and \( N \) are connected. Then

\[
\mathcal{D}_{M \sqcup N}^\theta(k) \cong \text{Emb}^\theta(\sqcup_k V, M \sqcup N)
\]

\[
\cong \prod_{i=0}^k \left( \text{Emb}^\theta(\sqcup_i V, M) \times \text{Emb}^\theta(\sqcup_{k-i} V, N) \right) \times \Sigma_i \times \Sigma_{k-i} \Sigma_k
\]

\[
\cong \prod_{i=0}^k \left( \mathcal{D}_{M}^\theta(i) \times \mathcal{D}_{N}^\theta(k-i) \right) \times \Sigma_i \times \Sigma_{k-i} \Sigma_k
\]

This is the formula of the Day convolution of \( \mathcal{D}_{M}^\theta \) and \( \mathcal{D}_{N}^\theta \). So we have

\[
\mathcal{D}_{M \sqcup N}^\theta \cong \mathcal{D}_{M}^\theta \boxtimes \mathcal{D}_{N}^\theta. \tag{3.18}
\]

We drop the \( \theta \) in the rest of the proof. By (3.18) and iterated use of Proposition 2.11, there is an isomorphism in \( \Lambda_{op}^{op}(G\text{Top}) \) for each \( q \):

\[
B_q(\mathcal{D}_{M \sqcup N}, \mathcal{D}_V, t_0(A)) \cong B_q(\mathcal{D}_M, \mathcal{D}_V, t_0(A)) \boxtimes B_q(\mathcal{D}_N, \mathcal{D}_V, t_0(A)). \tag{3.19}
\]

Iterated use of (2.7) identifies

\[
t_0(B_q(D_M, D_V, A)) \cong B_q(D_M, D_V, t_0(A)),
\]

so evaluating on the 0-th level of (3.19) gives an equivalence of simplicial \( G \)-spaces:

\[
B_s(D_{M \sqcup N}, D_V, A) \cong B_s(D_M, D_V, A) \times B_s(D_N, D_V, A).
\]

The claim follows from passing to geometric realization and commuting the geometric realization with the product. \( \square \)
Theorem 3.20 Let \( q : \theta_1 \to \theta_2 \) be a morphism of tangential structures and \( V = (V, \phi_1) \) be \( \theta_1 \)-framed. We also write \( V \) for the \( \theta_2 \)-framed \( G \)-manifold \( q_* V = (V, q \phi_1) \). For a \( \theta_1 \)-framed \( G \)-manifold \( M \) and a \( D^\theta_V \)-algebra \( A \), there is a \( G \)-equivalence

\[
\int_M^{\theta_1} q^* A \simeq \int_{q_* M}^{\theta_2} A.
\]

Here, \( q^* A \) is \( A \) with the \( D^\theta_V \)-algebra structure via the map of monads \( q_* : D^\theta_V \to D^\theta_V \) induced by \( q \).

The proof is deferred to the end of Sect. 3.4.

Notation 3.21 From now on, we consider \( \theta \) implicit and write \( \int_M^\theta A \) as \( \int_M A \).

3.3 Relation to configuration spaces

Now we restrict our attention to the \( V \)-framed case for an orthogonal \( n \)-dimensional \( G \)-representation \( V \). We give \( V \) the canonical \( V \)-framing \( TV \cong V \times V \) and let \( M \) be a \( G \)-manifold of dimension \( n \). When \( M \) is \( V \)-framed, we denote the \( V \)-framing by \( \phi_M : TM \to (fr V)^*(\zeta_n) = V \).

In this subsection, we first prove that a smooth embedding of \( \sqcup_k V \) into \( M \) is determined by its images and derivatives at the origin up to a contractible choice of homotopy (Proposition 3.26). Then we proceed to prove that a \( V \)-framed embedding space of \( \sqcup_k V \) into \( M \) as defined in (3.7) is homotopically the same as choosing the center points (Proposition 3.30).

To formulate the result, we first define the suitable equivariant configuration space related to a manifold, which will be “the space of points and derivatives”.

We use \( \mathcal{F}_E (k) \) to denote the ordered configuration space of \( k \) distinct points in \( E \), topologized as a subspace of \( E^k \). When \( E \) is a \( G \)-space, \( \mathcal{F}_E (k) \) has a \( G \)-action by pointwise acting. It commutes with the \( \Sigma_k \)-action that permutes the points.

Definition 3.22 For a fiber bundle \( p : E \to M \), define \( \mathcal{F}_{E|M}(k) \) to be configurations of \( k \)-ordered distinct points in \( E \) with distinct images in \( M \). \( \mathcal{F}_{E|M}(k) \) is a subspace of \( \mathcal{F}_E (k) \) and inherits a free \( \Sigma_k \)-action. When \( p \) is a \( G \)-fiber bundle, \( \mathcal{F}_{E|M}(k) \) is a \( G \)-space.

Example 3.23 When \( k = 1 \), \( \mathcal{F}_{E|M}(1) \cong \mathcal{F}_E (1) \).

Example 3.24 When \( E = M \times F \) is a trivial bundle over \( M \) with fiber \( F \),

\[
\mathcal{F}_{E|M}(k) \cong \mathcal{F}_M (k) \times F^k.
\]

In general, we have the following pullback diagram:

\[
\begin{array}{ccc}
\mathcal{F}_{E|M}(k) & \hookrightarrow & E^k \\
\downarrow & & \downarrow p^k \\
\mathcal{F}_M (k) & \hookrightarrow & M^k.
\end{array}
\]

Now, we take \( E = Fr V (TM) \). Recall that \( Fr V (TM) = \text{Hom}(V, TM) \) is a \( G \)-bundle over \( M \). For an embedding \( \sqcup_k V \to M \), we take its derivative and evaluate at \( 0 \in V \). We will get \( k \)-points in \( Fr V (TM) \) with different images projecting to \( M \). In other words, the composition

\[
\text{Emb}(\sqcup_k V, M) \xrightarrow{d} \text{Hom}(\sqcup_k TV, TM) \xrightarrow{ev_0} \text{Hom}(\sqcup_k V, TM) = Fr V (TM)^k
\]
factors as
\[
\text{Emb}(\bigcup_k V, M) \xrightarrow{d_0} \mathcal{F}_{\text{Fr}_V(TM)}(k) \hookrightarrow \text{Fr}_V(TM)^k.
\] (3.25)

**Proposition 3.26** The map \(d_0\) in (3.25) is a \(G\)-Hurewicz fibration and \((G \times \Sigma_k)\)-homotopy equivalence.

One can find an equivariant local trivialization of \(d_0\). The proof is tedious and can be found in [40, Prop 5.5.5].

A section and homotopy inverse exists uniquely up to homotopy:
\[
\sigma : \mathcal{F}_{\text{Fr}_V(TM)}(k) \rightarrow \text{Emb}(\bigcup_k V, M).
\] (3.27)

For \(k = 1\), it is given by the exponential map:
\[
\sigma : \text{Fr}_V(TM) \rightarrow \text{Emb}(V, M).
\]

Since there is a (contractible) choice of the radius at each point for the exponential map to be homeomorphism, \(\sigma\) is unique only up to homotopy.

**Lemma 3.28** For a \(V\)-framed manifold \(M\), the projection
\[
\mathcal{F}_{\text{Fr}_V(TM)}(k) \rightarrow \mathcal{F}_M(k)
\]
is a trivial bundle with fiber \((O(V))^k\). We call the section that selects \((\text{id}_V)^k\) in each fiber the zero section \(z\).

**Proof** Regarding \(V\) as a bundle over \(*\), we may identify \(\text{Fr}_V(V) \cong \text{Hom}(V, V) \cong O(V)\). Since \(M\) is \(V\)-framed, \(\text{Fr}_V(TM) \cong \text{Fr}_V(M \times V) \cong M \times \text{Fr}_V(V)\) as equivariant bundles. The claim follows from Example 3.24. \(\square\)

We can restrict the exponential map (3.27) to the zero section in Lemma 3.28 to get
\[
\sigma_0 : \mathcal{F}_M(k) \rightarrow \text{Emb}(\bigcup_k V, M).
\] (3.29)

Now we are ready to justify the equivalence of \(\text{Emb}^{\text{fr}_V}(\bigcup_k V, M)\) and the configuration spaces of \(M\). Moreover, we show that this equivalence is compatible over \(\text{Emb}(\bigcup_k V, M)\). This will be used in later sections to compare different scanning maps.

**Proposition 3.30** For a \(V\)-framed manifold \(M\), we have:

(1) Evaluating at 0 of the embedding gives a \((G \times \Sigma_k)\)-homotopy equivalence:
\[
ev_0 : \mathcal{F}_M(k) \equiv \text{Emb}^{\text{fr}_V}(\bigcup_k V, M) \rightarrow \mathcal{F}_M(k).
\]

(2) The forgetful map \(\text{Emb}^{\text{fr}_V}(\bigcup_k V, M) \rightarrow \text{Emb}(\bigcup_k V, M)\) is homotopic to (3.29) in the sense that the following diagram is \((G \times \Sigma_k)\)-homotopy commutative:
\[
\begin{array}{ccc}
\text{Emb}^{\text{fr}_V}(\bigcup_k V, M) & \rightarrow & \text{Emb}(\bigcup_k V, M) \\
\downarrow\ev_0 & & \downarrow\ev_0 \\
\mathcal{F}_M(k) & \xrightarrow{\sigma_0} & \mathcal{F}_M(k)
\end{array}
\]
**Proof** (1) By Definition 3.6 and (3.12), $\text{Emb}_{\text{fr}}^{*}(\mathcal{U}_{k}V, M)$ is the homotopy fiber of the composite:

$$D : \text{Emb}(\mathcal{U}_{k}V, M) \xrightarrow{d} \text{Hom}(\mathcal{U}_{k}TV, TM) \xrightarrow{(\phi_{M})^{*}} \text{Hom}(\mathcal{U}_{k}TV, V).$$

We would like to restrict the composite at $[0] \sqcup \cdots \sqcup [0] \subset V \sqcup \cdots \sqcup V$. Since $\text{Hom}(\mathcal{U}_{k}TV, TM) \cong \prod_{k} \text{Hom}(TV, TM)$ and $i_{0} : V \to TV$ is a $G$-homotopy equivalence of $G$-vector bundles,

$$ev_{0} : \text{Hom}(\mathcal{U}_{k}TV, TM) \xrightarrow{(i_{0})^{*}} \prod_{k} \text{Hom}(V, TM) \cong (\text{Fr}_{V}(TM))^{k}$$

is a $(G \times \Sigma_{k})$-homotopy equivalence. So in the following commutative diagram, the vertical maps are all $(G \times \Sigma_{k})$-homotopy equivalences:

$$
\begin{array}{ccc}
\text{Emb}(\mathcal{U}_{k}V, M) & \xrightarrow{d} & \text{Hom}(\mathcal{U}_{k}TV, TM) \xrightarrow{(\phi_{M})^{*}} \text{Hom}(\mathcal{U}_{k}TV, V) \\
\downarrow\varepsilon_{d} & & \downarrow\varepsilon_{(\phi_{M})^{*}} \\
\mathcal{F}_{\text{Fr}_{V}(TM)\downarrow M}(k) & \xleftarrow{\varepsilon_{v}} & \text{Fr}_{V}(TM)^{k} \xrightarrow{(\phi_{M})^{*}} \text{Fr}_{V}(V)^{k} \\
\downarrow\cong & & \downarrow\cong \\
\mathcal{F}_{M}(k) \times \text{Fr}_{V}(V)^{k} & \xrightarrow{\text{proj}_{2}} & \text{Fr}_{V}(V)^{k}.
\end{array}
$$

We focus on the top composition $D$ and the bottom map $\text{proj}_{2}$. The map $ev_{0}$ between their codomains is a based map. Indeed, the base point of $\text{Hom}(\mathcal{U}_{k}TV, V)$ is from the $V$-framing of $\mathcal{U}_{k}V$ and is $(G \times \Sigma_{k})$-fixed. It is mapped to $id^{k}$, the base point of $\text{Fr}_{V}(V)^{k}$. Consequently, there is a $(G \times \Sigma_{k})$-homotopy equivalence between the homotopy fibers of these two maps.

$$\text{Emb}_{\text{fr}}^{*}(\mathcal{U}_{k}V, M) = \text{hofib}(D) \cong \text{hofib}(\text{proj}_{2}). \tag{3.32}$$

Our desired $ev_{0}$ in question is the composite of (3.32) and the following map:

$$X : \text{hofib}(\text{proj}_{2}) \to \mathcal{F}_{M}(k) \times \text{Fr}_{V}(V)^{k} \xrightarrow{\text{proj}_{1}} \mathcal{F}_{M}(k).$$

It suffices to show that $X$ is a $(G \times \Sigma_{k})$-equivalence. Indeed, $X$ is the comparison of the homotopy fiber and the actual fiber of $\text{proj}_{2}$. Write temporarily $F = \mathcal{F}_{M}(k)$ and $B = \text{Fr}_{V}(V)^{k}$ with the $(G \times \Sigma_{k})$-fixed base point $b$. Then the map $X$ is projection to $F$:

$$\text{hofib}(\text{proj}_{2}) \cong P_{b}B \times F \to F.$$  

The claim follows from the fact that $P_{b}B$ is $(G \times \Sigma_{k})$-contractible.

(2) We examine the following diagram, where $\gamma$ is the zero section in Lemma 3.28:

$$
\begin{array}{ccc}
\text{Emb}_{\text{fr}}^{*}(\mathcal{U}_{k}V, M) & \xrightarrow{ev_{0}} & \text{Emb}(\mathcal{U}_{k}V, M) \\
\downarrow & & \downarrow \\
\mathcal{F}_{M}(k) & \xrightarrow{\gamma} & \mathcal{F}_{\text{Fr}_{V}(TM)\downarrow M}(k).
\end{array}
$$

The left column is given by the (homotopy) fibers of the first and second rows of (3.31), so the solid diagram is $(G \times \Sigma_{k})$-homotopy commutative. As $\sigma_{0} = \sigma \circ \gamma$ and $\sigma$ is a $(G \times \Sigma_{k})$-homotopy inverse of $d_{0}$ by Proposition 3.26, the upper triangle with the dotted arrow is homotopy commutative.  

\(\square\)
3.4 Comparison of operads and the invariance theorem

In this subsection, we study the $\theta$-framed little $V$-disk operad $\mathcal{D}^\theta_V$.

For $\theta = \text{fr}_V$, $\mathcal{D}^\text{fr}_V$ is equivalent to the little $V$-disks operad $\mathcal{D}_V$. For background, $\mathcal{D}_V$ is a well-studied operad introduced for recognizing $V$-fold loop spaces; see [13, 1.1]. Roughly speaking, $\mathcal{D}_V(k)$ is the space of non-equivariant embeddings of $k$ copies of the open unit disks $D(V)$ to $D(V)$, each of which takes only the form $v \mapsto av + b$ for some $0 < a \leq 1$ and $b \in D(V)$, called linear. In particular, the spaces are the same as those of the non-equivariant little $n$-disks operad, and so are the structure maps. The $G$-action on $\mathcal{D}_V(k)$ is by conjugation. It is well-defined, commutes with the $\Sigma_k$-action and the structure maps are $G$-equivariant.

**Proposition 3.33** There is an equivalence of $G$-operads $\beta : \mathcal{D}_V \to \mathcal{D}^\text{fr}_V$.

**Proof** We prepare an equivariant diffeomorphism

$$\eta : D(V) \to V; \ v \mapsto \tan\left(\frac{\pi |v|}{2}\right) \frac{v}{|v|},$$

$$\eta^{-1} : V \to D(V); \ v \mapsto \frac{2}{\pi} \arctan(|v|) \frac{v}{|v|}.$$  

The two maps are differentiable for functions of the form $k(r) \frac{v}{r}$ where $r = |v|$, its derivative

$$\nabla \left( k(r) \frac{v}{r} \right) = \left( k'(r) - \frac{k(r)}{r} \right) \frac{v \cdot v^T}{r^2} + \frac{k(r)}{r} \mathbf{I}$$

is continuous as long as $k(0) = 0$ and $k'(r)$ is continuous. This diffeomorphism does not preserve the canonical $\text{fr}_V$-framing on $V$ and $D(V) \subset V$. For technical reasons, we change the $\text{fr}_V$-framing on $V$ to be the one induced by $\eta^{-1}$ and the canonical framing on $D(V)$. Since all $\text{fr}_V$-framings on $V$ are equivalent by an automorphism of $TV$, this does not affect the homeomorphism type of the $G$-operad $\mathcal{D}^\text{fr}_V$.

Then, we define $\beta(1)$. Take $e \in \mathcal{D}_V(1)$; we must give $\beta(1)(e) = (f, l, \alpha) \in \mathcal{D}^\text{fr}_V(1)$. For $e : D(V) \to D(V)$ is $e(v) = av + b$ for some $0 < a \leq 1$ and $b \in D(V)$, we define a homotopy

$$l \in \mathbb{R}_{\geq 0} \text{ to be } l = -\ln(a);$$

$$H : \mathbb{R}_{\geq 0} \to \text{Emb}(D(V), D(V)) \text{ to be } H(t) = \begin{cases} \exp(-t)v + \frac{1 - \exp(-t)}{1-a}b & \text{for } t < l; \\ av + b & \text{for } t \geq l. \end{cases}$$

This homotopy satisfies that $H(0) = id$ and $H(l) = e$. Note that $l$ depends multiplicatively on $e$. Due to the shift by $b$, the data $H$ is not multiplicative, but $\mathbb{R}_{\geq 0} \times TD(V) \to TD(V) \to V$, the derivative of $H$ composed with the framing of $D(V)$, is multiplicative in terms of $e$. Finally, we define $\beta(1)(e) = (f, l, \alpha)$ as

$$f : V \to V \text{ to be } f = \eta \circ e \circ \eta^{-1};$$

$$l \in \mathbb{R}_{\geq 0} \text{ to be } l = -\ln(a);$$

$$\alpha : \mathbb{R}_{\geq 0} \to \text{Hom}(TV, V) \text{ to be } \alpha(t) = \text{proj} \circ D(\eta \circ H(t) \circ \eta^{-1}).$$

Here, $\text{proj} : TV \to V$ is the framing on $V$ induced by $\eta^{-1}$ and the canonical framing on $D(V)$. This makes $\beta(1)$ as defined a map of $G$-monoids.

For a $G$-monoid $X$, there is an obvious $G$-operad $X^*$ with $X^*(n) = X^n$. The map $\beta(1)$ induces a map of $G$-operads $\beta(1)^* : \mathcal{D}_V(1)^* \to \mathcal{D}^\text{fr}_V(1)^*$. Since $\mathcal{D}_V$ and $\mathcal{D}^\text{fr}_V$ are suboperads.
of $D_V(1)^*$ and $(D_V^{frv}(1))^*$ containing only embeddings with disjoint images, restricting $\beta(1)^*$ to the subspaces $D_V(k) \subset D_V(1)^k$, we get $\beta : D_V \to D_V^{frv}$.

The composite $ev_0 \circ \beta : D_V \to D_V^{frv} \to F_V$ is a levelwise homotopy equivalence by [13, Lemma 1.2]. We have shown that $ev_0$ is a levelwise equivalence (Proposition 3.30 (1)). So $\beta$ is also a levelwise homotopy equivalence.

For general $\theta$, $D_V^\theta$ also allows $\theta$-framed automorphisms of the embedded $V$-disks. By Theorem B.15, the $\theta$-framed automorphism space of $V$ is equivalent to $A_\phi B$, the Moore loop space of $B$ based at $\phi(0)$.

**Proposition 3.34** ([40, B.2.8]) There is a $G$-monoid $\widetilde{A}B$ equivalent to $A_\phi B$ which acts on $D_V$. Furthermore, there is an equivalence of $G$-operads $D_V \times \widetilde{A}B \to D_V^\theta$.

**Explanation.** Here, $D_V \times \widetilde{A}B$ is a $G$-operad whose $k$-th space is $D_V \times (\widetilde{A}B)^k$, and the semi-direct product is a notation introduced in [38] to indicate a twisting in the structure maps.

Without loss of generality we assume $V$ is $\theta$-framed by a constant map. Recall $D_M^\theta(k) = \text{Emb}^\theta(\Sigma_k V, M)$. Note that $fr_V$ is initial for such tangential structures, so we have

$$D_M^{frv}(k) \to D_M^\theta(k).$$

We claim that the composition map

$$D_M^{frv}(k) \times (\text{Emb}^\theta(V, V))^k \to D_M^\theta(k) \times (\text{Emb}^\theta(V, V))^k \to D_M^\theta(k) \quad (3.35)$$

is a $G \times \Sigma_k$-equivalence. In fact, the composite

$$\text{Emb}^{frv}(\Sigma_k V, M) \longrightarrow \text{Emb}^\theta(\Sigma_k V, M) \xrightarrow{ev_0} F_M(k)$$

is an equivalence by Proposition 3.30. Here, the map $ev_0$ is evaluation at 0 and is a $G \times \Sigma_k$ fibration and its fiber is equivalent to $(\text{Emb}^\theta(V, V))^k$. So it follows that (3.35) is an equivalence.

Combining Proposition 3.33 with (3.35), there is a $G \times \Sigma_k$-equivalence

$$D_V(k) \times (A_\phi B)^k \simeq D_V^{frv}(k) \times (\text{Emb}^\theta(V, V))^k \to D_V^\theta(k)$$

for each $k$. In [40, Appendix B], this equivalence is upgraded to an equivalence of $G$-operads $D_V \times \widetilde{A}B \to D_V^\theta$, where $\widetilde{A}B$ is a replacement of $A_\phi B$ that acts on $D_V(k)$.

**Proof of Theorem 3.20** Without loss of generality we assume $V$ is $\theta_1$-framed by a constant map. We omit the $q_*$ and $q^*$ in the proof. As $B(D_V^{\theta_2}, D_V^{\theta_2}, A) \simeq A$ as $D_V^{\theta_2}$-algebra, we have

$$\int_{q_*, M} \vartheta_2 A = B(D_M^{\theta_1}, D_V^{\theta_1}, A) \simeq B(D_M^{\theta_1}, D_V^{\theta_1}, B(D_V^{\theta_2}, D_V^{\theta_2}, A)) \simeq B(B(D_M^{\theta_1}, D_V^{\theta_1}, D_V^{\theta_2}), D_V^{\theta_2}, A).$$

It suffices to show that natural map of right $D_V^{\theta_2}$-functors

$$\epsilon : B(D_M^{\theta_1}, D_V^{\theta_1}, D_V^{\theta_2}) \to D_M^{\theta_2} \quad (3.36)$$

is an equivalence.
Using (3.35), one can already construct a retract of (3.36). To construct a deformation retract, we need the full strength of Proposition 3.34. There are equivalences of $G$-operads fitting in a commutative diagram

\[
\begin{array}{ccc}
\mathcal{D}_V \times \tilde{\Lambda}B_1 & \sim & \mathcal{D}_V^\theta_1 \\
\downarrow & & \downarrow \\
\mathcal{D}_V \times \tilde{\Lambda}B_2 & \sim & \mathcal{D}_V^\theta_2
\end{array}
\]

(3.37)

The monad associated to $\mathcal{D}_V \times \tilde{\Lambda}B_i$ for $i = 1, 2$ is

\[\overline{D}_V^\theta_i(A) = \mathcal{D}_V(\tilde{\Lambda}B_i \times A).\]

And similarly the associated functors for $k \mapsto \mathcal{D}_M(k) \times (\tilde{\Lambda}B_i)^k$ are given by

\[\overline{D}_M^\theta_i(A) = \mathcal{D}_M(\tilde{\Lambda}B_i \times A).\]

Note that $\tilde{\Lambda}B_i$ is a $G$-monoid, so the functor $A \mapsto \tilde{\Lambda}B_i \times A$ is a monad, which we still write as $\tilde{\Lambda}B_i$. We have

\[
\overline{\epsilon} : B(\overline{D}_M^\theta_1, \overline{D}_V^\theta_1, \overline{D}_V^\theta_2) \simeq B(\mathcal{D}_M \circ \tilde{\Lambda}B_1, \mathcal{D}_V \circ \tilde{\Lambda}B_1, \mathcal{D}_V \circ \tilde{\Lambda}B_2) \\
\simeq B(\mathcal{D}_M \circ \tilde{\Lambda}B_1 \circ \mathcal{D}_V \circ \tilde{\Lambda}B_1, \mathcal{D}_V \circ \tilde{\Lambda}B_2) \\
\simeq \mathcal{D}_M \circ \tilde{\Lambda}B_2 = \overline{D}_M^\theta_2
\]

is an equivalence. Here, the last equivalence is given by a deformation retract using an extra degeneracy argument [26, Proposition 9.9]. Now, in the following commutative diagram whose vertical maps are equivalences induced by the approximation (3.37),

\[
\begin{array}{ccc}
B(\overline{D}_M^\theta_1, \overline{D}_V^\theta_1, \overline{D}_V^\theta_2) & \xrightarrow{\overline{\epsilon}} & \overline{D}_M^\theta_2 \\
\downarrow & & \downarrow \\
B(\overline{D}_M^\theta_1, \overline{D}_V^\theta_1, \overline{D}_V^\theta_2) & \xrightarrow{\epsilon} & \overline{D}_M^\theta_2
\end{array}
\]

we see that $\epsilon$ is an equivalence.

\[\square\]

### 4 Nonabelian Poincaré Duality for $V$-framed manifolds

Configuration spaces have scanning maps out of them. It turns out that equivariantly the scanning map is an equivalence in the case of $G$-connected labels $X$. Since the factorization homology is built up simplicially by the configuration spaces, we can upgrade the scanning equivalence to what is known as the nonabelian Poincaré duality theorem.

#### 4.1 Scanning map for $V$-framed manifolds

In this subsection we construct the scanning map for $V$-framed manifolds, a natural transformation of right $D_M^{fr_V}$-functors:

\[s : D_M^{fr_V}(-) \to \text{Map}_c(M, \Sigma^V -).\]
Here, $\text{Map}_c(X, Y)$ for a based space $Y$ denotes the space of maps $f$ so that the support $f^{-1}(Y \setminus \ast)$ is compact. The map, especially (4.2), is simple and does not use the $V$-framing on $M$. However, the $V$-framing is needed to compare our scanning map to the existing different constructions in the literature, done in Appendix A. This allows us to utilize known results about equivariant scanning maps to give Theorem 4.5, a key input to the nonabelian Poincaré duality theorem in Sect. 4.2.

Assume that the scanning map (4.1) has been constructed for a moment. When we take $M = V$, (4.1) gives a map of monads $s : D^\text{fr}_V(M) \to \Omega^V \Sigma^V$. The adjoint natural transformation $\Sigma^V D^\text{fr}_V \xi \to \Sigma^V \Omega^V \Sigma^V \text{counit} \to \Sigma^V$ induces the right $D^\text{fr}_V$-module structure for the functor $\text{Map}_c(M, \Sigma^V X)$.

Now we construct the scanning map. For any $G$-space $X$, recall that

$$D^\text{fr}_M(X) = \coprod_{k \geq 0} D^\text{fr}_M(k) \times_{\Sigma_k} X^k / \sim,$$

where $\sim$ is the base point identification. Take an element

$$P = [f_1, \ldots, f_k, x_1, \ldots, x_k] \in D^\text{fr}_M(k) \times_{\Sigma_k} X^k.$$

Here, each $f_i = (f_i, \alpha_i)$ consists of an embedding $f_i : V \to M$ and a homotopy $\alpha_i$ of two bundle maps $TV \to V$, see Definition 3.6. We use only the embeddings $f_i$ to define an element $s_X(P) \in \text{Map}_c(M, \Sigma^V X)$:

$$s_X(P)(m) = \begin{cases} f_{i}^{-1}(m) \wedge x_i & \text{when } m \in M \text{ is in the image of some } f_i; \\ \ast & \text{otherwise}. \end{cases} \quad (4.2)$$

Notice that if $x_i$ is the base point, $f_{i}^{-1}(m) \wedge x_i$ is the base point regardless of what $f_i$ is. So passing to the quotient, (4.2) yields a well-defined map

$$s_X : D^\text{fr}_M(X) \to \text{Map}_c(M, \Sigma^V X). \quad (4.3)$$

In particular, taking $X = S^0$, we get

$$s_{S^0} : \coprod_{k \geq 0} D^\text{fr}_M(k) / \Sigma_k \to \text{Map}_c(M, S^V), \quad (4.4)$$

and $s_X$ is simply a labeled version of it. A more categorical construction of the scanning map $s_X$, as the composition of the Pontryagin–Thom collapse map and a “folding” map $\vee_k S^V \times X^k \to \Sigma^V X$ is given in [34, Section 9].

We use the following results of Rourke–Sanderson [35], which are proved using equivariant transversality. To translate from their context to ours, see Theorems A.2 and A.11.

**Theorem 4.5** The scanning map $s_X : D^\text{fr}_M X \to \text{Map}_c(M, \Sigma^V X)$ is:

1. a weak $G$-equivalence if $X$ is $G$-connected,
2. or a weak group completion if $V \cong W \oplus \mathbb{R}$ and $M \cong N \times \mathbb{R}$. Here, $W$ is a $(n-1)$-dimensional $G$-representation and $N$ is a $W$-framed compact manifold, so that $N \times \mathbb{R}$ is $V$-framed.
4.2 Equivariant nonabelian Poincaré duality (eNPD) theorem

We have seen that the scanning map is an equivalence for $G$-connected labels $X$. Since the factorization homology is built up simplicially by the configuration spaces, we can upgrade the scanning equivalence to the eNPD theorem. The proof in this subsection is motivated by the non-equivariant treatment [31].

Let $A$ be a $D_{V}^{fr}$-algebra in $G$Top throughout this subsection. Assume that $A$ is non-degenerately based, meaning that the structure map $\mathcal{D}_{V}^{fr}(0) = \ast \rightarrow A$ gives a non-degenerate base point of $A$. This is a mild assumption for homotopical purposes. We use the following $V$-fold delooping model of $A$.

**Definition 4.6** The $V$-fold delooping of $A$, denoted as $B^{V}A$, is the monadic two sided bar construction $B(\Sigma^{V}, D_{V}^{fr}, A)$. 5

Here, $B_{q}(\Sigma^{V}, D_{V}^{fr}, A) = \Sigma^{V}(D_{V}^{fr})^{q}A$. The first face map $\Sigma^{V}D_{V}^{fr} \rightarrow \Sigma^{V}$ is induced by the scanning map of monads $D_{V}^{fr} \rightarrow \Omega^{V}\Sigma^{V}$. The last face map $D_{V}^{fr}A \rightarrow A$ is the structure map of the algebra. The middle face maps and degeneracy maps are induced by the structure map of the monad $D_{V}^{fr}D_{V}^{fr} \rightarrow D_{V}^{fr}$ and by its unit map Id $\rightarrow D_{V}^{fr}$.

**Theorem 4.7** (eNPD) Let $M$ be a $V$-framed manifold and $A$ be a $D_{V}^{fr}$-algebra in $G$Top. Then there is a $G$-map, which is a weak $G$-equivalence if $A$ is $G$-connected:

$$PM : \int_{M} A \rightarrow |B_{\bullet}(D_{M}^{fr}, D_{V}^{fr}, A)| \rightarrow \text{Map}_{\ast}(M^{+}, B^{V}A).$$

Here, $M^{+}$ is the one-point-compactification of $M$.

**Proof** We give the proof assuming some lemmas that are proven in the remainder of this section. First, from (4.1), we have a scanning map for each $q \geq 0$:

$$D_{M}^{fr}(D_{V}^{fr})^{q}A \rightarrow \text{Map}_{\ast}(M, \Sigma^{V}(D_{V}^{fr})^{q}A).$$

They assemble to a simplicial scanning map, which is a levelwise weak $G$-equivalence as shown in Corollary 4.13:

$$B(s, \text{id}, \text{id}) : B_{\bullet}(D_{M}^{fr}, D_{V}^{fr}, A) \rightarrow \text{Map}_{\ast}(M, \Sigma^{V}(D_{V}^{fr})\ast A). \quad (4.8)$$

One can identify the space of compactly supported maps with the space of based maps out of the one point compactification:

$$\text{Map}_{\ast}(M, \Sigma^{V}(D_{V}^{fr})\ast A) \approx \text{Map}_{\ast}(M^{+}, \Sigma^{V}(D_{V}^{fr})\ast A).$$

With some cofibrancy argument in Theorem 4.14 and Corollary 4.19, this map induces a weak $G$-equivalence on the geometric realization:

$$B(D_{M}^{fr}, D_{V}^{fr}, A) \rightarrow |\text{Map}_{\ast}(M^{+}, \Sigma^{V}(D_{V}^{fr})\ast A)|.$$

Next, we change the order of the mapping space and the geometric realization. There is a natural map (4.27):

$$\zeta : |\text{Map}_{\ast}(M^{+}, \Sigma^{V}(D_{V}^{fr})\ast A)| \rightarrow \text{Map}_{\ast}(M^{+}, |\Sigma^{V}(D_{V}^{fr})\ast A|).$$

---

5 A $D_{V}^{fr}$-algebra $A$ has a $D_{V}$-algebra structure by pulling back along the equivalence of $G$-operads $\mathcal{D}_{V} \rightarrow \mathcal{D}_{V}^{fr}$ (Proposition 3.33), and there is an equivalence from the delooping $B(\Sigma^{V}, D_{V}, A)$ in [13] to our delooping $B(\Sigma^{V}, D_{V}^{fr}, A)$. 

[Springer]
Taking $X = M^{+}$ and $K_{*} = \Sigma V (D_{V}^{frv})^{*} A$, Theorem 4.30 gives a sufficient connectivity condition for it to be a weak $G$-equivalence. This connectivity condition is then checked in Lemma 4.26.

Finally, $|\Sigma V (D_{V}^{frv})^{*} A| = B^{V} A$ by Definition 4.6. This finishes the proof of the theorem. \qed

When $A$ is not $G$-connected but $M \cong N \times \mathbb{R}$ or $M \cong N \times \mathbb{R}^{2}$, there is also a group completion version of Theorem 4.7 in Theorem 4.41.

Remark 4.9 If we take $M = V$ in the theorem and use Proposition 3.16, we get that $A \cong \Omega V B^{V} A$ for a $G$-connected $E_{V}$-algebra $A$. This recovers [13, Theorem 1.14] and justifies the definition of $B^{V} A$.

4.3 $G$-connectedness

Definition 4.10 A $G$-space $X$ is $G$-connected if $X^{H}$ is connected for all subgroups $H \subset G$.

To show that the scanning map is an equivalence in each simplicial level, we need:

Lemma 3.29 If $X$ is $G$-connected, then $D_{V}^{frv} X$ is also $G$-connected.

Proof By Proposition 3.30, $D_{V}^{frv} X$ is $G$-homotopy equivalent to $F_{V} X$. It suffices to show that $F_{V} X$ is $G$-connected. Fix any subgroup $H \subset G$; we must show that $(F_{V} X)^{H}$ is connected. This is the space of $H$-equivariant unordered configuration on $V$ with based labels in $X$.

Intuitively, this is true because the space of labels $X$ is $G$-connected, so that one can always move the labels of a configuration to the base point. Nevertheless, we give a proof here by carefully writing down the fixed points of $F_{V} X$ in terms of the fixed points of $F_{V} (k)$ and $X$.

We have:

$$(F_{V} X)^{H} = \left( \prod_{k \geq 0} \frac{F_{V} (k) \times \Sigma k X^{k}}{\sim} \right)^{H} = \prod_{k \geq 0} \frac{(F_{V} (k) \times \Sigma k X^{k})^{H}}{\sim_{H}}$$

Here, $\sim$ is the equivalence relation in Remark 2.5 and $\sim_{H}$ is $\sim$ restricted on $H$-fixed points. They are explicitly forgetting a point in the configuration if the corresponding label is the base point in $X$. Notice that taking $H$-fixed points will not commute with $\cong$ in Construction 2.4, but commutes with $\sim$. This is because the $H$-action preserves the filtration and $\sim$ only identifies elements of different filtrations. The single point at filtration $k = 0$, or equivalently the point at any $k$ with all labels being the base point of $X$, is the base point of $(F_{V} X)^{H}$.

Since the $\Sigma k$-action is free on $F_{V} (k) \times X^{k}$ and commutes with the $G$-action, we have a principal $G$-$\Sigma k$-bundle

$$F_{V} (k) \times X^{k} \rightarrow F_{V} (k) \times \Sigma k X^{k}.$$ To get $H$-fixed points on the base space, we need to consider the $\Lambda_{k}$-fixed points on the total space for all the subgroups $\Lambda_{k} \subset G \times \Sigma k$ that are the graphs of some group homomorphisms $\alpha : H \rightarrow \Sigma k$. More precisely, by Theorem 2.19, we have

$$(F_{V} (k) \times \Sigma k X^{k})^{H} = \coprod_{[\alpha : H \rightarrow \Sigma k]} ((F_{V} (k) \times X^{k})^{\Lambda_{\alpha}} / Z_{\Sigma k} (\alpha)).$$

Here, the coproduct is taken over $\Sigma k$-conjugacy classes of group homomorphisms and $Z_{\Sigma k} (\alpha)$ is the centralizer of the image of $\alpha$ in $\Sigma k$.  

\(@ Springer\)
We would like to make the expression coordinate-free for $k$. A homomorphism $\alpha$ can be identified with an $H$-action on the set $\{1, \cdots, k\}$. For an $H$-set $S$, write $X^S = \text{Map}(S, X)$ and $\mathcal{F}_V(S) = \text{Emb}(S, V)$. Then

$$(\mathcal{F}_V(k) \times X^k)^{\Delta_0} = (\mathcal{F}_V(S) \times X^S)^H$$ and $Z_{\Sigma_k}(\alpha) = \text{Aut}_H(S)$.

So we have:

$$(\mathcal{F}_V(k) \times \Sigma_k X^k)^H = \bigsqcup_{[S]: \text{iso classes of } H\text{-set}, |S| = k} \left((\mathcal{F}_V(S) \times X^S)^H / \text{Aut}_H(S)\right).$$

If we take care of the base point identification, we end up with:

$$(F_V X)^H = \left(\bigsqcup_{[S]: \text{iso classes of finite } H\text{-set}} (\mathcal{F}_V(S) \times X^S)^H / \text{Aut}_H(S)\right) / \sim_H. \quad (4.12)$$

Suppose that the $H$-set $S$ breaks into orbits as $S = \bigsqcup_{i=1}^s r_i(H/K_i)$ for $i = 1, \cdots, s$, where $K_i$’s are in distinct conjugacy classes of subgroups of $H$ and $r_i > 0$, then we know explicitly each coproduct component is:

$$(\mathcal{F}_V(S) \times X^S)^H / \text{Aut}_H S = \left((\text{Emb}_H(S, V) \times \text{Map}_H(S, X)) / \text{Aut}_H S\right)$$

$$= (\text{Emb}_H(\bigsqcup_{i} r_i(H/K_i), \text{Emb}_{\Sigma_i}) \times \prod_i (X^{K_i})^{r_i}) / \prod_i (W_H(K_i) : \Sigma_{r_i}).$$

Since $X^{K_i}$ are all connected, so are the spaces $\prod_i (X^{K_i})^{r_i}$. They contain the base point of the labels $* = \prod_{i} \prod_{r_i} * \rightarrow \prod_{i} (X^{K_i})^{r_i}$. So after the gluing $\sim_H$, each component in (4.12) is in the same component as the base point of $F_V X$. Thus $(F_V X)^H$ is connected. \hfill $\square$

**Corollary 4.13** The map $B_\ast(D_M^{\text{fr}}, D_V^{\text{fr}}, A) \rightarrow \text{Map}_c(M, \Sigma^V(D_V^{\text{fr}}, A))$ in (4.8) is a levelwise weak $G$-equivalence of simplicial $G$-spaces if $A$ is $G$-connected.

**Proof** This is a consequence of Theorem 4.5 and Lemma 3.29. \hfill $\square$

For geometric realization, we have:

**Theorem 4.14** (Theorem 1.10 of [32]) A levelwise weak $G$-equivalence between Reedy cofibrant simplicial objects realizes to a weak $G$-equivalence.

### 4.4 Cofibrancy

We take care of the cofibrancy issues in this part, following details in [26]. We first show that some functors preserve $G$-cofibrations. One who is willing to take it as a blackbox may skip directly to Definition 4.17. We uses NDR data, which give a hands-on way to handle cofibrations.

**Definition 4.15** (Definition A.1 of [26]) A pair $(X, A)$ of $G$-spaces with $A \subset X$ is an NDR pair if there exists a $G$-invariant map $u : X \rightarrow I = [0, 1]$ such that $A = u^{-1}(0)$ and a homotopy given by a map $h : I \rightarrow \text{Map}_G(X, X)$ satisfying

- $h_0(x) = x$ for all $x \in X$;
- $h_t(a) = a$ for all $t \in I$ and $a \in A$;
- $h_1(x) \in A$ for all $x \in u^{-1}[0, 1]$.
The pair \((h, u)\) is said to a representation of \((X, A)\) as an NDR pair. A pair \((X, A)\) of based 
\(G\)-spaces is an NDR pair if it is an NDR pair of \(G\)-spaces with the \(h_t\) being based maps for 
all \(t \in I\).

An NDR pair gives a \(G\)-cofibration \(A \to X\). The function \(u\) gives an open neighborhood 
\(U\) of \(A\) by taking \(U = u^{-1}[0, 1)\). The function \(h\) restricts on \(I \times U\) to a neighborhood 
deformation retract of \(A\) in \(X\).

We have the following lemma by elaborating the NDR data. Its proof is tedious and omitted 
here (See [40, Section 6.4]).

**Lemma 4.16** Any functor \(F\) associated to \(\mathcal{F} \in \Lambda^\text{op}_*[G\text{Top}]\), in particular both \(D^{\text{fr}_V}_V\) and 
\(D^{\text{fr}_M}_M\), sends NDR pairs to NDR pairs. The functors \(\text{Map}_c(M, -), \text{Map}_s(M^+, -, \Sigma V)\) and \(\Sigma V\) all 
send NDR pairs to NDR pairs.

**Definition 4.17** (Lemma 1.9 of [32]) A simplicial \(G\)-space \(X_*\) is Reedy cofibrant if all degeneracy operators \(s_i\) are \(G\)-cofibrations.

The following lemma shows that monadic bar constructions are Reedy cofibrant.

**Lemma 4.18** (adaptation of Proposition A.10 of [26]) Let \(\mathcal{C}\) be a reduced operad in \(G\)-spaces 
such that the unit map \(\eta: * \to \mathcal{C}(1)\) gives a non-degenerate base point. Let \(C\) be the reduced 
monad associated to \(\mathcal{C}\). Let \(A\) be a \(C\)-algebra in \(G\text{Top}_s\) and \(F: G\text{Top}_s \to G\text{Top}_s\) be a 
right-\(C\)-module functor. Suppose that \(F\) sends NDR pairs to NDR pairs. Then \(B_*(F, C, A)\) is 
Reedy cofibrant.

**Proof** We need to show that for any \(n \geq 0\) and \(0 \leq i \leq n\), the degeneracy map 
\[s^i_n = FC^i\eta_{C^n-i A}: FC^n A \to FC^{n+1} A\]
is a \(G\)-cofibration. Write \(X = C^{n-i} A\). By Lemma 4.16, \(C\) sends NDR pairs to NDR pairs. 
Starting from the NDR pair \((A, *)\) and applying this functor \((n-i)\) times, we get an NDR pair 
\((C^{n-i} A, *) = (X, \ast)\). Together with the assumption that \(\mathcal{C}(1)\) is non-degenerately based, 
we can show \((C X, X)\) is an NDR pair with \(X\) is identified with the image \(\eta_X: X \to CX\) 
(see the proof of [26, A.10]). Applying \(C\) another \(i\) times and then \(F\), we get the NDR pair 
\((FC^{i+1} X, FC^i X) = (FC^{n+1} A, FC^n A)\). Thus \(s^i_n = FC^i \eta_X\) is a \(G\)-cofibration. \(\square\)

**Corollary 4.19** Let \(M, V, A\) be as in Theorem 4.7. Then the following are Reedy cofibrant 
simplicial \(G\)-spaces:

\[B_*(D^\text{fr}_V, D^\text{fr}_V, A), \text{Map}_c(M, \Sigma V (D^\text{fr}_V)^\bullet A) \text{ and } \text{Map}_s(M^+, \Sigma V (D^\text{fr}_V)^\bullet A).\]

**Proof** In Lemma 4.18, we take \(C = D^{\text{fr}_V}_V\) and respectively \(F = D^{\text{fr}_V}_M\), \(F = \text{Map}_c(M, \Sigma V)\) 
or \(F = \text{Map}_s(M^+, \Sigma V)\). By Lemma 4.16, each \(F\) does send NDR pairs to NDR pairs. \(\square\)

### 4.5 Dimension

We start by recalling some facts about \(G\)-CW complexes and equivariant dimensions following 
[28, I.3]. A \(G\)-CW complex \(X\) is a union of \(G\)-spaces \(X^n\), where \(X^0\) is a disjoint union 
of orbits, and \(X^n\) is obtained by inductively gluing cells \(G/K \times D^n\) for subgroups \(K \subset G\) 
via \(G\)-maps along their boundaries \(G/K \times S^{n-1}\) to the previous skeleton \(X^{n-1}\).

We shall look at functions from the conjugacy classes of subgroups of \(G\) to \(\mathbb{Z}_{\geq -1}\) and 
typically denote such a function by \(v\). We say that a \(G\)-CW complex \(X\) has dimension \(\leq v\) if
its cells of orbit type $G/H$ all have dimensions $\leq v(H)$, and that a $G$-space $X$ is $v$-connected if $X^H$ is $v(H)$-connected for all subgroups $H \subset G$, that is, $\pi_k(X^H) = 0$ for $k \leq v(H)$. We allow $v(H) = -1$ for the case $X^H = \emptyset$.

It is worth pointing out that this notion of dimension should be more appropriately called the cell dimension. (It is not the dimension of $X^H$, as we explain shortly.) It gives information on which cells to consider in an induction. For the purpose of induction, we use the following ad hoc definition in this paper:

**Definition 4.20** A based $G$-CW complex is a union of $G$-spaces $X^n$ obtained by inductively gluing cells to $X^0$, a disjoint union of orbits plus a disjoint base point $\ast$. (The gluing maps are non-based maps.) In a based map out of $X$, the base point $\ast$ has no freedom but to be sent to the base point. So we do NOT count it as a cell for a based $G$-CW complex, excluding it from counting the dimension as well. It then makes sense to write $X^{-1} = \ast$. This is not the same as a based $G$-CW complex in [28, Page 18], where the base point is put in the 0-skeleton $X^0$.

Fix a subgroup $H \subset G$. A function $v$ from the conjugacy classes of subgroups of $G$ to $\mathbb{Z}_{\geq -1}$ induces a function from the conjugacy classes of subgroups of $H$ to $\mathbb{Z}_{\geq -1}$, which we still call $v$. We have the double coset formula

$$G/K \cong \bigsqcup_{1 \leq i \leq |H/K|} H/K_i \text{ as } H\text{-sets,}$$

where each $K_i = H \cap g_i K g_i^{-1}$ for some element $g_i \in G$. So a (based) $G$-CW structure on $X$ restricts to a (based) $H$-CW structure on the $H$-space $\text{Res}_H^G X$. However, for $X$ of cell dimension $\leq v$, $\text{Res}_H^G X$ may not be of cell dimension $\leq v$, as we see in (4.21) that an $H/K_i$-cell can come from a $G/K$-cell for a larger group $K$.\footnote{For a trivial example, let $G$ be a non-trivial finite group. The orbit $G/e$ is of cell dimension $v$ for $v(G) = 0$ and $v(H) = -1$ for $H \neq G$; but $\text{Res}_e^G(G/e)$ is of dimension 0 whereas $\text{Res}(v) = -1$.}

For a function $v$, we define the function $d_v$ to be

$$d_v(K) = \max_{K \subset L} v(L).$$

(4.22)

Then $\text{Res}_H^G X$ is of cell dimension $\leq d_v$.

**Remark 4.23** More specifically, we define the cell dimension of a (based) $G$-CW complex $X$ to be the minimum $v$ such that $X$ is of cell dimension $\leq v$. Suppose that $X$ has cell dimension $v$. From (4.21), we get:

(i) The (based) $H$-CW complex $\text{Res}_H^G X$ has cell dimension $v_H$, where

$$v_H(K) = \max_{K \subset L} v(L).$$

We have $v_H(K) \leq d_v(K)$, and it can be strictly less. (For a trivial example, take $H = G$.)

(ii) The (based) CW-complex $X^H$ has dimension $v_H(H) = d_v(H) \geq v(H)$. (In the based case, we also exclude the base point from counting the dimension of $X^H$, so that if $X^H = \ast$, the dimension of $X^H$ is $-1$.)

**Definition 4.24** (1) For a (based) $G$-CW complex $X$ of cell dimension $v$, $\dim(X)$ is the function $d_v$.

(2) For a $G$-representation $V$, $\dim(V)$ is the function $\dim(V)(H) = \dim(V^H)$.
From Remark 4.23, we have two observations: First, \( \dim(X)(H) \) is equal to the dimension of the CW-complex \( X^H \). So \( \dim(X) \) is independent of the \( G \)-CW decomposition of the underlying \( G \)-space of \( X \). Second, for an unbased \( G \)-CW complex \( X \), the based \( G \)-CW complex \( X_+ = X \sqcup \ast \) satisfies \( \dim(X_+) = \dim(X) \) because \( \ast \) is excluded from cells in the based case.

We prepare the following results regarding dimension for the next subsection.

**Theorem 4.25** (Theorem 3.6 of [17]) For a smooth \( G \)-manifold \( M \) and a closed smooth \( G \)-submanifold \( N \), there exists a smooth \( G \)-equivariant triangulation of \( (M, N) \).

**Lemma 4.26** Let \( M \) be a \( V \)-framed manifold and \( A \) be a \( G \)-space, then

1. \( M^+ \) has the homotopy type of a \( G \)-CW complex of cell dimension \( \leq \dim(V) \).
2. \( K_n = \Sigma^V (D^V(V))^n A \) is \((\dim(V) - 1)\)-connected. If furthermore \( A \) is \( G \)-connected, then \( K_n \) is \((\dim(V))\)-connected.

**Proof** (1) Since \( M \) is a \( V \)-framed, the exponential maps give local coordinate charts of \( M^H \) as a (possibly empty) manifold of dimension \( \dim(V^H) \). If \( M \) is compact we take \( W = M \), otherwise we take a manifold \( W \) with boundary such that \( M \) is diffeomorphic to the interior of \( W \). By Theorem 4.25, \((W, \partial W)\) has a \( G \)-equivariant triangulation. It gives a relative \( G \)-CW structure on \((W, \partial W)\) with relative cells of type \( G/H \) of dimension \( \leq \dim(V^H) \). The quotient \( W/\partial W \) gives the desired \( G \)-CW model for \( M^+ \).

(2) For any subgroup \( H \subset G \), we have \( K^H_n = (\Sigma^V (D^V(V))^n A)^H = \Sigma^V ((D^V(V))^n A)^H \). Then \( (K_n)^H \) is obviously \((\dim(V^H) - 1)\)-connected. When \( A \) is \( G \)-connected, by Lemma 3.29, \((D^V(V))^n A)^H \) is connected, so that \( K_n \) is \((\dim(V^H))\)-connected.

\[ \square \]

### 4.6 Commuting mapping space and geometric realization

Let \( X \) be a based \( G \)-CW complex and \( K_\bullet \) be a based simplicial \( G \)-space. Then the levelwise evaluation is a \( G \)-map

\[ |\text{Map}_n(X, K_\bullet)| \wedge X \cong |\text{Map}_n(X, K_\bullet) \wedge X| \to |K_\bullet|, \]

whose adjoint gives a \( G \)-map

\[ \zeta : |\text{Map}_n(X, K_\bullet)| \to |\text{Map}_n(X, K_\bullet)|. \tag{4.27} \]

Non-equivariantly, it is one of the key steps in May’s recognition principal that (4.27) is a weak equivalence when each \( K_n \) is \( \dim(X) \)-connected [26, Theorem 12.3]. The goal of this subsection is to give a sufficient condition for \( \zeta \) to be a weak \( G \)-equivalence.

The strategy is to induce on cells. However, the geometric realization of a levelwise fibration is not necessarily a fibration. Dold–Thom came up with the notion of quasi-fibrations, which is good enough for handling the homotopy groups.

**Definition 4.28** A map \( p : Y \to W \) of spaces is a quasi-fibration if \( p \) is onto and it induces an isomorphism on homotopy groups \( \pi_n(Y, p^{-1}(w), y) \to \pi_n(W, w) \) for all \( w \in W \) and \( y \in p^{-1}(w) \). In other words, there is a long exact sequence on homotopy groups of the sequence \( p^{-1}(w) \to Y \to W \) for any \( w \in W \).

**Theorem 4.29** ([26, Theorem 12.7]) Let \( p : E_\bullet \to B_\bullet \) be a levelwise Hurewicz fibration of pointed simplicial spaces such that \( B_\bullet \) is Reedy cofibrant and \( B_n \) is connected for all \( n \). Set \( F_\bullet = p^{-1}(\ast) \). Then the realization \( |E_\bullet| \to |B_\bullet| \) is a quasi-fibration with fiber \( |F_\bullet| \).
Theorem 4.30 Let $G$ be a finite group. If $X$ is a finite-dimensional based $G$-CW complex and $K_\bullet$ is a based simplicial $G$-space such that for any $n$, $K_n$ is $\dim(X)$-connected, then the natural map (4.27)
\[ \zeta : |\text{Map}_*(X, K_\bullet)| \to \text{Map}_*(X, |K_\bullet|) \]
is a weak $G$-equivalence.

Proof Suppose that $X$ is of cell dimension $\nu$, so $\dim(X) = d_\nu$. (See (4.22) for $d_\nu$.) Let $\ast = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X^{d_\nu(e)} = X$ be the $G$-CW skeleton of $X$. We use induction on $k$ to show that

(i) $|\text{Map}_*(X^k, K_n)|^H$ is connected for all $n$ and $H \subset G$.
(ii) $|\text{Map}_*(X^k, K_\bullet)|^H \to |\text{Map}_*(X^k, |K_\bullet|)|^H$ is a weak equivalence for all $H \subset G$;

The base case $k = -1$ is obvious. Suppose that (i) and (ii) hold for $k$. Take the cofiber sequence
\[ X^k \to X^{k+1} \to X^{k+1}/X^k \]
and map it into $K_\bullet$. We then apply (4.27) and get the following commutative diagram:
\[ \begin{array}{ccc}
|\text{Map}_*(X^{k+1}/X^k, K_n)|^H & \to & |\text{Map}_*(X^{k+1}, K_\bullet)|^H \\
\downarrow & & \downarrow \\
\text{Map}_*(X^{k+1}/X^k, |K_\bullet|)^H & \to & \text{Map}_*(X^{k+1}, |K_\bullet|)^H
\end{array} \] (4.31)

Since maps out of a cofiber sequence form a fiber sequence, we have a fiber sequence in the second row and a realization of the following levelwise fiber sequence in the first row:
\[ \begin{array}{ccc}
\text{Map}_*(X^{k+1}/X^k, K_n)^H & \to & \text{Map}_*(X^{k+1}, K_\bullet)^H \\
& & \to \text{Map}_*(X^k, K_\bullet)^H
\end{array} \] (4.32)

By the inductive hypothesis (i) and Theorem 4.29, it realizes to a quasi-fibration.

We first show the inductive case of (i). We can write
\[ X^{k+1}/X^k = \vee_i (G/L_i)_+ \wedge S^{k+1}, \]
where each $L_i$ is a subgroup of $G$. When $L_i$ is presented, $\nu(L_i) \geq k + 1$. From (4.21), we can further write $X^{k+1}/X^k \cong \vee_i \vee_j (H/L_{i,j})_+ \wedge S^{k+1}$ as spaces with $H$-action, where each $L_{i,j}$ is $G$-conjugate to a subgroup of $L_i$. Then $d_\nu(L_{i,j}) \geq \nu(L_i) \geq k + 1$, and the following space is connected by assumption:
\[ \text{Map}_*(X^{k+1}/X^k, K_n)^H = \prod_i \text{Map}_*(S^{k+1}, K_n^L_{i,j}). \]

This space is the fiber in (4.32). The connectedness of the base space given by (i) then implies the connectedness of the total space.

We next show the inductive case of (ii). Commuting geometric realization with finite products and with fixed points, the left vertical map of (4.31) is a product of maps
\[ |\text{Map}_*(S^{k+1}, K_n^L_{i,j})| \to |\text{Map}_*(S^{k+1}, |K_n^L_{i,j}|)|. \]

Since we have $d_\nu(L_{i,j}) \geq k + 1$, these maps are weak equivalences by [26, Theorem 12.3]. By (ii), the right vertical map is a weak equivalence. Comparing the long exact sequences of homotopy groups, this implies that the middle vertical map is also a weak equivalence. \qed
Remark 4.33 Non-equivariantly, Miller [31, Cor 2.22] observed that the theorem is also true if $K_n$ is only $(\dim(X) - 1)$-connected for all $n$, since the only thing that fails in the proof is the claim (i) for $k = \dim(X^e)$. Equivariantly, one needs (i) to hold for all inductive steps of $k < d_\nu(e)$. So we can only relax the assumption to the following extent: If $K_n^H$ is $\min\{d_\nu(H), d_\nu(e) - 1\}$-connected for all $n$ and $H$, then the natural map (4.27) is a weak $G$-equivalence. This is an improvement only when $d_\nu(H) = d_\nu(e)$, that is $d_\nu(H) \geq \nu(K)$ for all $K \subset H$.

Nevertheless, when $X = \Sigma Z$ and $Z$ is of cell dimension $\nu$, so that $X$ is of cell dimension $\nu + 1$, we can relax the assumption further.

Corollary 4.34 If $Z$ is a finite-dimensional based $G$-$CW$ complex and $K_\bullet$ is a based simplicial $G$-space such that for any $n$, $K_n$ is $\dim(Z)$-connected, then the natural map (4.27)
\[ \xi : |\text{Map}_*(\Sigma Z, K_\bullet)| \to \text{Map}_*(\Sigma Z, |K_\bullet|) \]
is a weak $G$-equivalence.

Proof The cofiber sequence $S^0 \vee S^0 \to S^0 \to S^1$ gives a levelwise fiber sequence
\[ \text{Map}_*(\Sigma Z, K_\bullet) \to \text{Map}_*(Z, K_\bullet) \to \text{Map}_*(Z, K_\bullet) \times \text{Map}_*(Z, K_\bullet). \quad (4.35) \]
By Theorem 4.30 and its proof, (4.35) has a $G$-connected base and realizes to a quasi-fibration; the same method will show the claim. □

The unbased version of Theorem 4.30 is due to Hauschild and written down by Costenoble–Waner [10, Lemma 5.4], stated as:

Theorem 4.36 Let $G$ be a finite group. If $Y$ is a finite unbased $G$-$CW$ complex and $K_\bullet$ is a simplicial $G$-space such that for any $n$, $K_n$ is $\dim(Y)$-connected, then the natural map
\[ |\text{Map}(Y, K_\bullet)| \to \text{Map}(Y, |K_\bullet|) \]
is a weak $G$-equivalence.

Theorem 4.30 improves Theorem 4.36 slightly in the case when $X^G = \ast$. On one hand, taking $X$ in Theorem 4.36 to be $Y \amalg \{\ast\}$ recovers Theorem 4.30. On the other hand, for a based $G$-$CW$ complex $X$ we have the levelwise fibration sequence
\[ \text{Map}_*(X, K_\bullet) \to \text{Map}(X, K_\bullet) \to K_\bullet. \]
If the cell dimension of $X$ satisfies $\nu(H) \geq 0$ for all $H$, then $\dim(X)(H) = d_\nu(H) \geq 0$. The assumptions imply that $K_n$ is $G$-connected, we can use the quasi-fibration technique to deduce Theorem 4.30 from Theorem 4.36 (with $Y = X$). But there are also cases when the assumption in Theorem 4.30 is weaker, for example, when $X = (G/H)_+ \wedge S^n$ for some $H \neq G$. In this case, $d_\nu(G) = \dim(X^G) = -1$, so the $K_n^G$ are required to be connected in Theorem 4.36 but not in Theorem 4.30.

4.7 Group completion

Recall that an $E_n$-structure on a $G$-space is an algebra structure over the little disk operad $\mathcal{D}_n$ for the trivial representation $\mathbb{R}^n$. As pointed out in [13, Section 1.2], there are two notions of group completion, one topological, one computational, which we recall now.
Definition 4.37 Let \( C \) and \( D \) be \( E_1 \)-\( G \)-spaces. An \( E_1 \)-\( G \)-map \( f : C \to D \) is called a weak group completion if for any subgroup \( H \subset G \), there is a weak equivalence \( \Omega B(C^H) \simeq D^H \) and \( f^H : C^H \to D^H \) is equivalent to the natural group completion map \( C^H \to \Omega B(C^H) \).

When \( C \) is an \( E_1 \)-\( G \)-space and \( H \subset G \), the fixed point space \( C^H \) is an \( E_1 \)-space; so \( f^H \) is up to homotopy a weak group completion of \( C^H \).

Definition 4.38 Let \( C \) and \( D \) be \( E_2 \)-\( G \)-spaces.\(^7\)

1. \( D \) is called grouplike if for any subgroup \( H \subset G \), \( \pi^H_0(D) \) is a group.
2. A \( E_2 \)-\( G \)-map \( f : C \to D \) is called a group completion if \( D \) is grouplike and for any subgroup \( H \subset G \), \( f^H \) induces an isomorphism \( H_*(C^H)[\pi^H_0(C)^{-1}] \cong H_*(D^H) \) for any field coefficients.

Theorem 4.39 Let \( C \) and \( D \) be \( E_2 \)-\( G \)-spaces. Then an \( E_2 \)-\( G \)-map \( f : C \to D \) is a group completion if and only if it is a weak group completion.

Proof The “if” direction is [27, 15.1]. For the “only if” direction, the assumption that \( D \) is grouplike implies that \( f \) factors as \( C \to \Omega B C \overset{\bar{f}}{\to} D \). Now for each \( H \subset G \), \( \bar{f}^H \) is a homology equivalence between simple spaces and thus a weak equivalence.

Lemma 4.40 Let \( C_* \) and \( D_* \) be Reedy cofibrant simplicial \( E_1 \)-\( G \)-spaces. Suppose that \( f : C_* \to D_* \) is a levelwise weak group completion. Then \( f \) induces a weak group completion \( \left| C_* \right| \to \left| D_* \right| \). If \( C_* \) and \( D_* \) are levelwise \( E_2 \), then \( f \) induces a group completion.

Proof The \( E_n \)-\( G \)-space structures are algebra structures over certain monads and thus preserved by geometric realization ([12, Lemma 8.17]). The functor \( B \) is the geometric realization of a simplicial construction \( B_m(\cdot) \). So \( B(C^H) \) and \( |BC^H| \), being two ways of realizing the bisimplicial space \( X_{m,n} = B_mC_n^H \), are homeomorphic. We have the following commutative diagram:

\[
\begin{array}{ccc}
| C_*^H | & \longrightarrow & \left| \Omega B C_*^H \right| \\
\downarrow & & \Rightarrow \downarrow \zeta \\
\left| \Omega B C_*^H \right| & \overset{\cong}{\longrightarrow} & \Omega |BC_*^H| \\
\end{array}
\]

The top right map is induced by \( \Omega B C_*^H \to D_*^H \). From the assumptions, it is a levelwise equivalence between Reedy cofibrant simplicial spaces, so the top right map is a weak equivalence. Each \( BC^H_n \) is connected, so the vertical map \( \zeta \) is a weak equivalence by Corollary 4.34. This proves that the top composite is homotopic to the left arrow up to equivalence.

Theorem 4.41 Let \( M \) be a \( V \)-framed manifold and \( A \) be a \( D^{fr}_V \)-algebra in \( G \)\( \text{Top} \). There is a \( G \)-map from Theorem 4.7

\[
p_M : \int_M A \to \text{Map}_s(M^+, B^V_V A).
\]

1. If \( V = W \oplus \mathbb{R} \) and \( M \cong N \times \mathbb{R} \) for a \( W \)-framed manifold \( N \), then \( p_M \) is a weak group completion.
2. If \( V = U \oplus \mathbb{R}^2 \) and \( M \cong N \times \mathbb{R}^2 \) for a \( U \)-framed manifold \( N \), then \( p_M \) is a group completion.

\(^7\) This definition makes sense for homotopy associative and commutative \( G \)-monoids, for which \( E_2 \)-\( G \)-spaces are examples.

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Proof From the proof of Theorem 4.7, the map $p_M$ is a composite

$$\left| B_\ast(D^{fr}_{M}, D^{fr}_{V}, A) \right| \xrightarrow{\alpha_M} \left| \text{Map}_\ast(M^+, \Sigma^V(D^{fr}_V)^\ast A) \right| \xrightarrow{\zeta} \text{Map}_\ast(M^+, B^V A)$$

We first examine $\alpha_M$. By Theorem 4.5, $\alpha_M$ is the realization of a levelwise weak group completion between simplicial $E_1$-$G$-spaces in case of (1) and $E_2$-$G$-spaces in case of (2). Then by Lemma 4.40, $\alpha_M$ is a weak group completion in case of (1) and a group completion in case of (2).

Next we prove that $\zeta$ is a weak $G$-equivalence in case (1), and case (2) will follow. By Lemma 4.26 (2), $\Sigma^V(D^{fr}_V)^\ast A$ is $(\dim(V) - 1) = \dim(W)$-connected. Applying Lemma 4.26 (1) to $N$, it has a $G$-CW structure of cell dimension $\leq \dim(W)$. By Corollary 4.34 and the fact that $M^+ \simeq \Sigma(N^+)$, $\zeta$ is a weak $G$-equivalence. This finishes the proof.

Acknowledgements This paper is mostly based on my thesis. I would like to express my deepest gratitude to my advisor Peter May, who raises me up from the kindergarten of mathematics. I am indebted to Inbar Klang, Alexander Kupers and Jeremy Miller, whose work motivates my research. I would like to thank Asaf Horev and Haynes Miller for helpful conversations, Shmuel Weinberger for being my committee member, and the referees for making useful comments and catching small mistakes.

Appendix A. A comparison of scanning maps

The scanning map studied in Sect. 4.1 is a key input to the eNPD theorem. In this section we compare our scanning map (4.3) to other constructions.

Notation A.1 For a $G$-manifold $M$, $\text{Sph}(TM)$ is the $G$-space obtained by fiberwise one-point compatification of the tangent bundle of $M$. It is a fiber bundle over $M$ with based fiber $S^n$, where the base point in each fiber is the point at infinity.

Non-equivariantly, people have used the name scanning map to refer to different but related constructions. In slogan, it is a map from the (fattened) configuration spaces of a manifold $M$ to compactly defined sections of $TM$, or compactly supported sections of $\text{Sph}(TM)$. McDuff [30] was probably the first to study the scanning map for general manifolds. She thought of it as the field of the point charges and proved homological stability properties of this map.

When $TM \cong M \times V$, the situation is simpler and we have defined a scanning map in (4.4):

$$s_{\Sigma_0} : \coprod_{k \geq 0} \mathcal{D}^{fr}_M(k)/\Sigma_k \to \text{Map}_c(M, S^V).$$

The left hand side is a model of the configuration space as justified in Proposition 3.30 (1); the right hand side is equivalent to the compactly supported sections of $\text{Sph}(TM) \cong M \times S^V$.

We are interested in the scanning maps of Manthorpe–Tillman and McDuff, both of which can be made equivariant without pain. The following table is a summary of the natural domains and codomains of each construction:
In this section, we focus on the case of $V$-framed manifolds $M$. Then these maps have equivalent domains and identical codomains. We will show in Proposition A.7 and Proposition A.10 that:

**Theorem A.2** The scanning maps $s_X, s_X^{MD}$ and $s_X^{MT}$ are $G$-homotopic after the change of domain.

**Notation A.3** In the above and subsequent paragraphs,

- We use the letter $s$ for scanning maps without labels and $s_X$ for labels in $X$.
- A tilde is put on $s$ to denote when the codomain is the sections of $\text{Sph}(TM)$, that is, before composition with the framing.
- A superscript is put on $s$ to distinguish between the different authors in the literature.

### A.1. Scanning map from tubular neighborhood

Manthorpe–Tillman [33, Section 3.1] gave a map

$$\gamma^+ : \left( \bigsqcup_{k \geq 0} \text{Emb}(\mathbb{L}_k \mathbb{R}^n, M) \times \Sigma_k X^k \right) / \sim \to \text{Sect}_c(M, \text{Sph}(TM) \wedge_M \tau_X).$$

Here, $\text{Sect}_c$ is the space of compactly supported sections; $\tau_X$ is the constant parametrized base space $X \times M$ over $M$ and $\text{Sph}(TM) \wedge_M \tau_X$ is the fiberwise smashing of $\text{Sph}(TM)$ with $X$. (To translate, take their $M_0 = \emptyset$, $Y = W \times X$. Their $E_k(M, \pi)$ is the space $\text{Emb}(\mathbb{L}_k \mathbb{R}^n, M) \times \Sigma_k X^k$, and their $\Gamma(W \setminus M_0, W \setminus M, \pi)$ is $\text{Sect}_c(M, \text{Sph}(TM) \wedge_M \tau_X)$.)

The key feature of their construction is to exploit the data of the tubular neighborhood, so a framing on $M$ is not needed. For example, when $k = 1$, we start with an embedding $f \in \text{Emb}(\mathbb{R}^n, M)$ and want to define $\gamma^+(f)$, a compactly supported section of $\text{Sph}(TM)$. The image of $f$ is a tubular neighborhood of the image of $0 \in V$ in $M$, and $f$ induces an inclusion of bundles $df : T\mathbb{R}^n \to TM$. There is a canonical diagonal section $\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n \cong T\mathbb{R}^n$. Pushing this section by $df$ gives $\gamma^+(f)$.

We can modify their $\gamma^+$ by replacing $\mathbb{R}^n$ by the representation $V$ to get

$$\gamma_V^+ : \text{Emb}_M(X) \equiv \left( \bigsqcup_{k \geq 0} \text{Emb}(\mathbb{L}_k V, M) \times \Sigma_k X^k \right) / \sim \to \text{Sect}_c(M, \text{Sph}(TM) \wedge_M \tau_X).$$

We then precompose with the forgetting map $D_{fr}^{fr} M \to \text{Emb}_M(X)$ in Remark 3.8 to get

$$\tilde{s}_X^{MT} : D_{fr}^{fr} M(X) \to \text{Sect}_c(M, \text{Sph}(TM) \wedge_M \tau_X). \quad \text{(A.4)}$$

We describe how $\tilde{s}_X^{MT}$ works on the subspace $k = 1$ and it is similar on the whole space. For the element $\tilde{f} = (f, \alpha) \in \text{Emb}_{fr} V(M)$, we take the embedding $f : V \to M$. The derivative map of $f$ is $df : TV \cong V \times V \to TM$. For each $m \in \text{image}(f)$, we need a vector $s^{MT}(f) \in T_m M$ that is determined by $f$. Denote $v = f^{-1}(m) \in V$. We have

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$df(v, -) : V \cong T_v V \to T_m M$. Then the explicit formulas without or with labels are given by

$$s^\text{MT} (\bar{f})(m) = df(v, v) \quad \text{and} \quad \bar{s}^\text{MT} (\bar{f}, x)(m) = df(v, v) \wedge x.$$  \hfill (A.5)

Both of them are $G$-maps.

The $V$-framing $\phi_M : TM \to V$ induces $\text{Sph}(TM) \wedge_M \tau_X \cong M \times \Sigma^V X$. So we obtain a map which we still call the scanning map:

$$s^\text{MT}_X : \text{DFr}_M^V (X) \to \text{Map}_c(M, \Sigma^V X).$$  \hfill (A.6)

A prior, this scanning map is different from the scanning map (4.2) in Sect. 4.1. For an element $\bar{f} = (f, \alpha)$ where $f : V \to M$ with $f(v) = m$, we have $s(\bar{f})(m) = v \in V$ in (4.2), while $s^\text{MT}(\bar{f})(m) = df(v, v) \in T_m M$ in (A.5). However, the data of a homotopy in defining the $V$-framed embedding ensures that the two approaches give homotopic scanning maps:

**Proposition A.7** The map $s_X$ defined by the formula (4.2) is $G$-homotopic to the map $s^\text{MT}_X$ defined in (A.6).

**Proof** We show that $s \simeq s^\text{MT} : \text{DFr}_M^V (k) \to \text{Map}_c(M, S^V)$. We write the homotopy explicitly for $k = 1$ and the case for general $k$ is similar. To unravel the data, an element $\bar{f} = (f, \alpha) \in \text{DFr}_M^V (1)$ consists of an embedding $f : V \to M$ and a homotopy $\alpha$ of two maps $TV \to V$, where $\alpha(0)$ is the standard framing on $V$ and $\alpha(1)$ is $\phi_M \circ df$.

The two scanning maps use the two endpoints of this homotopy. Namely, for $m$ in $\text{Image}(f)$, write $v = f^{-1}(m) \in V \cong T_v V$. Then the first approach can be written as

$$s(\bar{f})(m) = v = \alpha(0)(v, v)$$

and the $df$-shifted-approach can be written as

$$s^\text{MT}_X (\bar{f})(m) = \phi_M df(v, v) = \alpha(1)(v, v).$$

Now it is clear that we can define a homotopy

$$H : \text{DFr}_M^V (1) \times I \to \text{Map}_c(M, S^V);$$

$$H(\bar{f}, t)(m) = \alpha(t)f^{-1}(m)(f^{-1}(m)).$$

It is $G$-equivariant and gives a homotopy between $H(-, 0) = s$ and $H(-, 1) = s^\text{MT}$. The claim follows from observing that this homotopy is compatible with forgetting from $k$ to $k - 1$. \hfill $\square$

### A.2. Scanning map using geodesic

McDuff gave a geometric construction for

$$\tilde{s}^\text{MD} : F_M(S^0) = \bigsqcup_{k \geq 0} \mathcal{F}_M(k) \to \text{Sect}_c(M, \text{Sph}(TM)),$$

Recall that $\mathcal{F}_M(k)$ is the configuration space of $k$ points in $M$. Note that the base point in each fiber of $\text{Sph}(TM)$ is the point at infinity. A compactly supported section of $\text{Sph}(TM)$ is just a vector field defined in the interior of a compact set on $M$ that blows up to infinity towards the boundary.
We first review McDuff’s construction and fit it into a neat comparison with the previously defined scanning maps. We focus on the case when \( M \) is without boundary. Then we can translate her \( M_\varepsilon \) to our \( M \); her \( E_M \) can be identified with our \( \text{Sph}(TM) \); her \( \tilde{C}_M \) to our \( F_M(S^0) \); her \( \tilde{C}_\varepsilon(M) \) to a subspace of our \( \text{Emb}_M(S^0) \).

In summary, \( s^{\text{MD}} \) goes in two steps: fatten up the configurations ([30, Lemma 2.3]) and use geodesics to give compactly supported vector fields ([30, p95]).

\[
s^{\text{MD}} : F_M(S^0) \xrightarrow{\text{fatten}} \tilde{C}_\varepsilon(M) \xrightarrow{\phi_\varepsilon} \text{Sect}_\varepsilon(M, E_M) \xrightarrow{\eta_1} \text{Emb}_M(S^0) \xrightarrow{\gamma^+} \text{Sect}_\varepsilon(M, \text{Sph}(TM))
\]

The commutative diagram (A.8) is central in this section. In the first row, fatten and \( \phi_\varepsilon \) are the two steps in McDuff’s scanning map. The map \( \gamma^+ \) is from Section A.1. We will define the undefined spaces and maps as we go along.

Put a Riemannian metric on \( M \). Define

\[
\tilde{C}_\varepsilon(M)_1 = \{ \exp_{m_0} : T_{m_0}M \to M \text{ such that it is a diffeomorphism on the } \varepsilon \text{-ball} \};
\]

\[
\tilde{C}_\varepsilon(M) = \{ (\delta, e_1, \cdots, e_k) | 0 < \delta \leq \varepsilon, k \in \mathbb{N}, e_i \in \tilde{C}_\varepsilon(M)_1 \text{ for } 1 \leq i \leq k, \text{ images of } e_i \text{ on the } \delta \text{-balls are disjoint in } M \}.
\]

For preparation, we write down an explicit homeomorphism

\[
\eta_\varepsilon : D_{\varepsilon}(\mathbb{R}^n) \to \mathbb{R}^n; \quad v \mapsto \tan \left( \frac{\pi |v|}{2\varepsilon} \right) \frac{v}{|v|}.
\]

Here, \( D_{\varepsilon}(\mathbb{R}^n) \) is the disk of radius \( \varepsilon \) in \( \mathbb{R}^n \). Then, applying the same formula to \( T_m M \) we also have

\[
\eta_1 : D_1(T_m M)/\partial D_1(T_m M) \cong T_m M \cup \{ \infty \} \equiv \text{Sph}(T_m M).
\]

Define \( E_M \) to be the bundle over \( M \) whose fiber over \( m \) is \( D_1(T_m M)/\partial D_1(T_m M) \), which is identified with \( \text{Sph}(T_m M) \) through \( \eta_1 \). This is the right vertical map in (A.8).

We give the vertical map in the middle of (A.8). For an element \( \exp_{m_0} \in \tilde{C}_\varepsilon(M)_1 \), the composite \( \exp_{m_0} \circ \eta_{\varepsilon}^{-1} \) is an embedding \( \mathbb{R}^n \to M \), so we can identify \( \tilde{C}_\varepsilon(M)_1 \) with a subspace of \( \text{Emb}(\mathbb{R}^n, M) \). Similarly, we can include as subspace:

\[
\tilde{C}_\varepsilon(M) \to \text{Emb}_M(S^0);
\]

\[
(\delta, e_1, \cdots, e_k) \mapsto (e_1 \circ \eta_{\delta}^{-1}, \cdots, e_k \circ \eta_{\delta}^{-1})
\]

In McDuff’s first step, let us define \( \phi_\varepsilon \) and compare it to the map \( \gamma^+ \) locally. The input for \( \phi_\varepsilon \) are the exponential maps in \( \tilde{C}_\varepsilon(M)_1 \). Define

\[
\phi_\varepsilon(\exp_{m_0})(m) = \begin{cases} * & \text{if } \dist(m, m_0) > \varepsilon; \\ \frac{\dist(m, m_0)}{\varepsilon} \cdot t(m, m_0) & \text{if } \dist(m, m_0) \leq \varepsilon. \end{cases}
\]

Here, the values are vectors in \( D_1(T_m M) \); \( t(m, m_0) \) is the unit tangent at \( m \) of the minimal geodesic from \( m_0 \) to \( m \); \( \dist(m, m_0) \) is the distance between \( m \) and \( m_0 \). Now, it can be easily verified that

\[
\gamma^+(\exp_{m_0} \circ \eta_{\varepsilon}^{-1}) = \eta_1 \circ \phi_\varepsilon(\exp_{m_0}).
\]
We can work the same way to extend $\phi_\epsilon$ to $\tilde{C}_\epsilon(M)$ and we have the commutativity part of (A.8):

$$\gamma^+ (e_1 \circ \eta^{-1}_\delta, \cdots, e_k \circ \eta^{-1}_\delta) = \eta_1 \circ \phi_\epsilon (\delta, e_1, \cdots, e_k).$$

In McDuff’s second step, we describe the fattening map in (A.8). We can take a continuous positive function $\epsilon$ on $M$ such that for any $m_0 \in M$, the exponential map $\exp_{m_0} : T_{m_0} M \to M$ is always a diffeomorphism on the $\epsilon(m_0)$-ball. (We note the change here: $\epsilon(m_0)$ is going to serve as the $\epsilon$ in the first step. It does not harm to think as if $\epsilon(m_0) = \epsilon$ for all $m_0$.) Then, as is easily checked, we can choose a continuous positive function $\bar{\epsilon}$ on $FM(S^0)$ such that at any $p = (m_1, \cdots, m_k) \in \mathcal{F}_M(k)$,

(i) for all $i = 1, \cdots, k, \bar{\epsilon}(p) \leq \epsilon(m_i);
(ii) the $m_i$’s are at least $2\bar{\epsilon}(p)$ apart from each other.

The fattening map in (A.8) sends $p = (m_1, \cdots, m_k) \in \mathcal{F}_M(k)$ to $(\bar{\epsilon}(p), \exp_{m_1}, \cdots, \exp_{m_k}) \in \tilde{C}_\epsilon(M)$. The continuity of $s^{\text{MD}}$ follows from the continuity of $\bar{\epsilon}$.

**Remark A.9** The composite

$$F_M(S^0) \xrightarrow{\text{fatten}} \tilde{C}_\epsilon(M) \xrightarrow{\text{include}} \text{Emb}_M(S^0)$$

in (A.8) is up to homotopy the $\sigma_0$ in (3.29).

Equivariantly, we can take all of the Riemannian metric, $\epsilon$ and $\bar{\epsilon}$ to be $G$-invariant because $G$ is finite: for example, replacing $\epsilon$ by $\Sigma_{g \in G} \epsilon(g-)/|G|$ will do. Then $s^{\text{MD}}$ defined by (A.8) is $G$-equivariant. We can fiberwise smash with labels to get

$$s^{\text{MD}}_X : F_M(X) \to \text{Sect}_c(M, \text{Sph}(TM) \wedge_M \tau_X).$$

We note that there is no $V$ involved in $s^{\text{MD}}_X$. When $M$ is $V$-framed, we can compose it with the $V$-framing on $M$ to get

$$s^{\text{MD}}_X : F_M(X) \to \text{Map}_c(M, \Sigma^V X).$$

This scanning map $s^{\text{MD}}_X$ is good only for studying the configuration spaces, possibly with labels. It depends on the fattening-up radius $\bar{\epsilon}$, which is not recorded explicitly in the data. The choice does not matter because a different choice of the fattening-up will give a homotopic scanning map. But for the purpose of a scanning map out of “configuration spaces with summable labels” or the factorization homology, remembering the radius is important to sum the labels.

We have seen three scanning maps so far: $s_X$ in (4.2), $s^{\text{MT}}_X$ in (A.5) and $s^{\text{MD}}_X$ in (A.8). We have shown that $s_X$ and $s^{\text{MT}}_X$ are $G$-homotopic in Proposition A.7. We compare $s^{\text{MD}}_X$ and $s^{\text{MT}}_X$ in the following proposition

**Proposition A.10** The following diagram is $G$-homotopy commutative:

$$
\begin{array}{ccc}
F_M^V X & \xrightarrow{s_X^{\text{MT}}} & \text{Map}_c(M, \Sigma^V X) \\
\downarrow_{\epsilon^0} & & \\
F_M X & \xrightarrow{s_X^{\text{MD}}} & \\
\end{array}
$$
Proof Recall that \( s_X^{MT} \) is the composite of the forgetting map and \( \gamma^+_V \):

\[
\gamma^+_V: D_{MV} X \to \text{Emb}_M(X) \xrightarrow{\gamma^+_V} \text{Map}_c(M, \Sigma V X).
\]

By (A.8) and Remark A.9, we have a homotopy commutative diagram:

\[
\begin{array}{ccc}
\text{Emb}_M(X) & \xrightarrow{\gamma^+_V} & \text{Map}_c(M, \Sigma V X) \\
\sigma_0 \uparrow & & \downarrow s_X^{MD} \\
F_M(X) & & \end{array}
\]

By Proposition 3.30(2), \( \sigma_0 \circ ev_0 \) is \( G \)-homotopic to the forgetting map \( D_{MV} X \to \text{Emb}_M(X) \).

So the claim follows. \( \square \)

A.3. Scanning equivalence

We are interested in when the scanning map is an equivalence. In this subsection, we list Rourke–Sanderson’s results from [35]. Their work is based on McDuff’s scanning map. The \( C_M X \) in their paper is our \( (F_M X)^G \).

Theorem A.11 The scanning map \( s_X^{MD}: F_M X \to \text{Map}_c(M, \Sigma V X) \) is:

1. a weak \( G \)-equivalence if \( X \) is \( G \)-connected,
2. or a weak group completion if \( V \cong W \oplus \mathbb{R} \) and \( M \cong N \times \mathbb{R} \). Here, \( W \) is a \((n - 1)\)-dimensional \( G \)-representation and \( N \) is a \( W \)-framed \( G \)-manifold, so that \( N \times \mathbb{R} \) is \( V \)-framed.

Proof (1) is [35, Theorem 5]. For (2), we first note that when \( M \cong N \times \mathbb{R} \), the map \( s_X^{MD} \) factors in steps as:

\[
\begin{align*}
F_M X &= F_{\mathbb{R}}(F_N X) \to \text{Map}_c(\mathbb{R}, \Sigma F_N(X)) \quad (A.12) \\
&\to \text{Map}_c(\mathbb{R}, F_N(\Sigma X))) \quad (A.13) \\
&\to \text{Map}_c(\mathbb{R}, \text{Map}_c(N, \Sigma^{1+W} X)). \quad (A.14)
\end{align*}
\]

Here, (A.12) and (A.14) are scanning maps for manifolds \( \mathbb{R} \) and \( N \); (A.13) sends an element \( p \wedge t \) for a configuration \( p \) on \( N \) with labels in \( X \) and \( t \in S^1 \) to the same configuration on \( N \) with labels suspended all by \( t \) in \( \Sigma X \). All spaces presented have \( A_\infty \)-structures from the factor \( \mathbb{R} \) in \( M \): for any space \( Y \), both the labeled configuration space \( F_{\mathbb{R}} Y \) and the mapping space \( \text{Map}_c(\mathbb{R}, Y) \simeq \Omega Y \) have obvious \( A_\infty \)-structures.

The map (A.14) is a weak \( G \)-equivalence by applying part (1) with \( M \) replaced by \( N \) and \( X \) replaced by \( \Sigma X \), which is \( G \)-connected. It suffices to show the composite of (A.12) and (A.13), denoted as \( j \), is a weak group completion.

[35, Theorem 3] constructed a homotopy equivalence

\[
q: B((F_M X)^G) \simeq (F_N(\Sigma X))^G.
\]

Moreover, in Page 548, they established a homotopy commutative diagram:

\[
\begin{array}{ccc}
(F_M X)^G & \xrightarrow{j^G} & \text{Map}_c(\mathbb{R}, (F_N(\Sigma X))^G) \\
\downarrow & & \parallel \\
\text{Map}_c(\mathbb{R}, B((F_M X)^G)) & \xrightarrow{\Omega q} & \text{Map}_c(\mathbb{R}, (F_N(\Sigma X))^G)
\end{array}
\]
The left column is the group completion map for the \( A_\infty \)-space \((FM X)^G \). Since \( q \) is a homotopy equivalence, \( j^G \) is a weak group completion. This remains true for any subgroup \( H \subset G \) replacing \( G \). Therefore, \( j \) is a weak group completion.

**Remark A.15** [35] does not assume the manifold \( M \) to be framed. Without the framing on \( M \), Theorem A.11 is true in the following form:

The scanning map \( s^{MD}_X : FM X \to \text{Sect}_c(\mathcal{M}, \text{Sph}(TM) \wedge_M \tau X) \) is

1. a weak \( G \)-equivalence if \( X \) is \( G \)-connected,
2. or a weak group completion if \( M \cong N \times \mathbb{R} \).

**Appendix B. A comparison of \( \theta \)-framed morphisms**

In Sect. 3.1, we defined the \( \theta \)-framed embedding space of \( \theta \)-framed bundles using paths in the \( \theta \)-framing. In this appendix, we compare this approach to an alternative definition following Ayala–Francis [1, Definition 2.7] in Proposition B.11. With this alternative definition, we identify the automorphism \( G \)-space \( \text{Emb}^\theta(V, V) \) of \( V \) in \( \text{Mfld}^\theta_{G,n} \) in Theorem B.15; the special case \( \theta = \text{fr}_V \) has been treated directly in Sect. 3.3.

**B.1. The \( \theta \)-framed maps.**

The classification theorem says that isomorphism classes of vector bundles are in bijection with homotopy classes of maps to a classifying space. Passing to the classification maps seems to lose the information about morphisms between bundles, but it turns out not to. We show that the space of morphisms between bundles is equivalent to the space of homotopies between their classifying maps in Corollary B.10. To this end, we first define a suitable “over category up to homotopy”.

Let \( B \) be a \( G \)-space. A typical example is to take \( B = BG O(n) \). Then we have a Top-enriched over category \( G\text{Top}_/B \): the objects are \( G \)-spaces over \( B \), and the morphisms are \( G \)-maps over \( B \). Explicitly, for \( G \)-spaces over \( B \) given by \( G \)-maps \( \phi_M : M \to B \) and \( \phi_N : N \to B \), the space \( \text{Hom}_{G\text{Top}/B}(M, N) \) is the pullback displayed in the following diagram: (note that we have \( \text{Hom}_{G\text{Top}} = \text{Map}_G \))

\[
\begin{array}{ccc}
\text{Hom}_{G\text{Top}/B}(M, N) & \longrightarrow & \text{Map}_G(M, N) \\
\downarrow & & \downarrow_{\phi_N \circ -} \\
* & \underset{[\phi_M]}{\longrightarrow} & \text{Map}_G(M, B)
\end{array}
\]  

(B.1)

Now we want to work with \( G \)-spaces over \( B \) up to homotopy. We modify the morphism space by taking the homotopy pullback in (B.1). Just like the difference between \( G\text{Top} \) and \( \text{Top}_G \), we have two versions: the Top-enriched \( G\text{Top}^h/B \) and the \( G\text{Top}\)-enriched \( \text{Top}^G_{\text{Top}}/B \). That is, we have homotopy pullback diagrams of spaces in (B.2) and of \( G \)-spaces in (B.3):

\[
\begin{array}{ccc}
\text{Hom}_{G\text{Top}^h/B}(M, N) & \longrightarrow & \text{Map}_G(M, N) \\
\downarrow & & \downarrow_{\phi_N \circ -} \\
* & \underset{[\phi_M]}{\longrightarrow} & \text{Map}_G(M, B)
\end{array}
\]  

(B.2)

\[
\begin{array}{ccc}
\text{Hom}_{G\text{Top}^G_{\text{Top}}/B}(M, N) & \longrightarrow & \text{Map}_G(M, N) \\
\downarrow & & \downarrow_{\phi_N \circ -} \\
* & \underset{[\phi_M]}{\longrightarrow} & \text{Map}_G(M, B)
\end{array}
\]  

(B.3)
Using the Moore path space model for the homotopy fiber as given in the following definition, one can define unital and associative compositions to make $G\text{Toph}_G/B$ and $\text{Toph}_G/B$ categories.

**Definition B.4** For $\phi_M : M \to B$ and $\phi_N : N \to B$, the space $\text{Hom}_{\text{Toph}_G/B} (M, N)$ and the $G$-space $\text{Hom}_{\text{Toph}_G/B} ^* (M, N)$ are given by:

$$\text{Hom}_{\text{Toph}_G/B} (M, N) = \{ (f, \alpha, l) | f \in \text{Map}_G(M, N), \alpha \in \text{Map}(\mathbb{R}_{\geq 0}, \text{Map}_G(M, B)), l \in \text{Map}(\text{Map}_G(M, N), \mathbb{R}_{\geq 0}) \text{ such that}$$

$$\alpha(0) = \phi_M, \alpha(t) = \phi_N \circ f \text{ for } t \geq l(f) \}.$$ 

$$\text{Hom}_{\text{Toph}_G/B} ^* (M, N) = \{ (f, \alpha, l) | f \in \text{Map}(M, N), \alpha \in \text{Map}(\mathbb{R}_{\geq 0}, \text{Map}(M, B)), l \in \text{Map}(\text{Map}(M, N), \mathbb{R}_{\geq 0}) \text{ such that}$$

$$\alpha(0) = \phi_M, \alpha(t) = \phi_N \circ f \text{ for } t \geq l(f) \}.$$ 

**Remark B.5** Roughly speaking, a point in the morphism space $\text{Toph}_G/B$ is a $G$-map $f \in \text{Map}_G(M, N)$ and a $G$-homotopy from $\phi_M$ to $\phi_N \circ f$ in the following diagram:

$$\begin{array}{ccc}
N & \xrightarrow{\phi_N} & B \\
\downarrow \phi_M & & \\
M & \xrightarrow{f} & B
\end{array}$$

A point in the morphism space $\text{Top}_G/B$ is a map $f \in \text{Map}(M, N)$ and a homotopy from $\phi_M$ to $\phi_N \circ f$; the map $f$ is not necessarily a $G$-map, but we do require $\phi_M$ and $\phi_N$ to be $G$-maps. And we have

$$\text{Hom}_{\text{Top}_G/B} (M, N) \cong (\text{Hom}_{\text{Top}_G/B} (M, N))^G.$$ 

We show that the category $\text{Top}_G/B$ models $\theta$-framed bundles. One can restrict bundle maps to get maps on the base spaces. We denote this restriction map by $\pi$.

**Proposition B.6** For $i = 1, 2$, let $E_i \to B_i$ be $G$-n-vector bundles with $\theta$-framings $\phi_i : E_i \to \theta^* \zeta_n$. We have the following equivalences of $G$-spaces that are natural with respect to the two variables as well as the tangential structure:

$$\beta : \text{Hom}^* (E_1, E_2) \xrightarrow{\sim} \text{Hom}_{\text{Top}_G/B} (B_1, B_2).$$
Proof From our definition of \( \text{Hom}^\theta \) in Definition 3.4 and \( \text{Hom}_{\text{Top}}^g G/B \) in Definition B.4, \( \pi \) induces the map \( \beta \) and they fit in the following commutative diagram of \( G \)-spaces:

\[
\begin{array}{ccc}
\text{Hom}^\theta (E_1, E_2) & \xrightarrow{\beta} & \text{Hom}_{\text{Top}}^g G/B (B_1, B_2) \\
\downarrow & & \downarrow \\
\text{Hom}(E_1, E_2) & \xrightarrow{\pi} & \text{Map}(B_1, B_2) \\
\phi_2 \circ \downarrow & & \phi_2 \circ \downarrow \\
\text{Hom}(E_1, \theta^* \zeta_n) & \xrightarrow{\pi} & \text{Map}(B_1, B) \\
\end{array}
\]

(B.7)

We claim that the bottom square is a pullback. The isomorphism \( \phi_2 : E_2 \cong \phi_2 \theta^* \zeta_n \) establishes \( E_2 \) as a pullback of \( \theta^* \zeta_n \) over \( \phi_2 \). So a bundle map \( E_1 \to E_2 \) is determined by \( \phi \in \text{Map}(B_1, B_2) \).

\[
\begin{array}{c}
E_1 \xrightarrow{\phi} E_2 \\
\downarrow & & \downarrow \\
B_1 \xrightarrow{f} B_2 \xrightarrow{\phi_2} B
\end{array}
\]

It remains to show that the map \( \pi \) is a \( G \)-fibration, as then each column of (B.7) being a homotopy fiber sequence will imply that \( \beta \) is a \( G \)-equivalence. We denote the image of \( \pi \) by \( \text{Map}_p(B_1, B_2) \) etc. Then

\[
\text{Map}_p(B_1, B_2) = \{ f : B_1 \to B_2 | \text{non-equivariantly, } (\theta \phi_2) \circ f \simeq \theta \phi_1 : B_1 \to B G O(n) \}
\]

We can replace \( \text{Map}(B_1, B_2) \) etc. in (B.7) by \( \text{Map}_p(B_1, B_2) \) etc. without changing the fiber and the homotopy fiber. It then suffices to show that the map

\[
\pi : \text{Hom}(E_1, \zeta_n) \to \text{Map}_p(B_1, B G O(n))
\]

is a \( G \)-fibration. This is true because \( \pi \) is an equivariant principal bundle as quoted in Lemma B.8 by taking \( \Pi = O(n) \) and using Theorem 2.15. \( \square \)

Lemma B.8 ([41, Theorem 3.17 and Lemma 3.18]) Let \( p : P \to B \) be any principal \( G \)-bundle and \( \text{Hom}(P, E G \Pi) \) be the space of (non-equivariant) principal \( \Pi \)-bundle morphisms with \( G \) acting by conjugation. Then

(1) The space \( \text{Hom}(P, E G \Pi) \) is \( G \)-contractible.
(2) The map \( \pi : \text{Hom}(P, E G \Pi) \to \text{Map}_p(B, B G \Pi) \) is an equivariant principal bundle.

We remark that in Proposition B.6, \( \pi \) is not a homotopy equivalence to its image. In other words, a vector bundle map is not just a map on the bases. In contrast, a \( \theta \)-framed vector bundle map can be seen as a map on the bases as \( \beta \) is an equivalence.

The “classical” bundle maps are the \( \theta \)-framed bundle maps for the tangential structure \( \theta = \text{id} : B G O(n) \to B G O(n) \):

Lemma B.9 For \( G \)-vector bundles \( E_i \to B_i, \ i = 1, 2 \), we have an equivalence of \( G \)-spaces:

\[
\varphi : \text{Hom}^\text{id}(E_1, E_2) \xrightarrow{\sim} \text{Hom}(E_1, E_2).
\]
By definition, $\text{Hom}^{\text{id}}(E_1, E_2)$ is the homotopy fiber of $\phi_2 \circ -$ , so we have a homotopy fiber sequence of $G$-spaces:

$$\text{Hom}^{\text{id}}(E_1, E_2) \xrightarrow{\theta} \text{Hom}(E_1, E_2) \xrightarrow{\phi_2 \circ -} \text{Hom}(E_1, \zeta_n).$$

By Lemma B.8, we know $\text{Hom}(E_1, \zeta_n)$ is $G$-contractible. So $\varrho$ is a $G$-equivalence. □

**Corollary B.10** For $G$-vector bundles $E_i \to B_i$, $i = 1, 2$, we have an equivalence of $G$-spaces:

$$\text{Hom}(E_1, E_2) \simeq \text{Hom}_{\text{Top}_{G/O}^b}(B_1, B_2).$$

**Proof** This follows from Proposition B.6 and Lemma B.9. □

**Proposition B.11** The $G$-space $\text{Emb}^\theta(M, N)$ as defined in Definition 3.6 is the homotopy pullback displayed in the following diagram of $G$-spaces:

$$\begin{array}{ccc}
\text{Emb}^\theta(M, N) & \longrightarrow & \text{Hom}_{\text{Top}_{G/O}^b}(M, N) \\
\downarrow & & \downarrow \\
\text{Emb}(M, N) & \longrightarrow & \text{Hom}_{\text{Top}_{G/O}^b}(B_1, B_2).
\end{array}$$

**Proof** The lower horizontal map in (B.12) is neither obvious nor canonical. We take it as the composite in the following commutative diagram with a chosen $G$-homotopy inverse to $\varrho$.

The maps $\varrho$ and $\beta$ are $G$-equivalences by Proposition B.6 and Lemma B.9.

$$\begin{array}{ccc}
\text{Emb}^\theta(M, N) & \longrightarrow & \text{Hom}^{\text{id}}(TM, TN) \xrightarrow{\beta} \text{Hom}_{\text{Top}_{G/O}^b}(M, N) \\
\downarrow & & \downarrow \\
\text{Emb}(M, N) & \longrightarrow & \text{Hom}_{\text{Top}_{G/O}^b}(B_1, B_2).
\end{array}$$

As defined in Definition 3.6, $\text{Emb}^\theta(M, N)$ is the pullback in the left square. It is clear that it is also equivalent to the homotopy pullback of the whole square. □

We can take (B.12) as an alternative definition to (3.7). In practice, (3.7) is easier to deal with. First, the right vertical map in the square is a fibration so the diagram is an actual pullback. Second, the map $d$ is easy to describe. On the other hand, (B.12) has a conceptual advantage. It can be viewed as a comparison of the $\theta$-framing to the trivial framing $\text{id} : B_{G/O(n)} \to B_{G/O(n)}$.

**B.2. Automorphism space of $(V, \phi)$**

With this alternative description of $\theta$-framed mapping spaces in Section B.1, we can identify the automorphism $G$-space $\text{Emb}^\theta(V, V)$ of $V$ in $\text{Mfld}_{G, n}^\theta$ by first identifying of the automorphism $G$-space $\text{Hom}^\theta(TV, TV)$ of $TV$ in $\text{Vec}_{G, n}^\theta$.

**Notation B.14** As $\phi$ is an equivariant map, $\phi(0)$ for the origin $0 \in V$ is a $G$-fixed point in $B$. We denote by $\Lambda_\phi B$ the Moore loop space of $B$ at the base point $\phi(0)$. 

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Theorem B.15  We have the following:

(1) There is an equivalence of monoids in G-spaces

\[ \text{Hom}^\theta(\mathcal{T}V, \mathcal{T}V) \sim \Lambda_\phi B, \]

which is natural with respect to tangential structures \( \theta : B \to B_G \text{O}(n) \). Here, the group \( G \) acts on both sides by conjugation.

(2) The automorphism G-space \( \text{Emb}^\theta(V, V) \) of \((V, \phi)\) in \( \text{Mfld}^\theta_{G,n} \) fits in the following homotopy pullback diagram of G-spaces:

\[
\begin{array}{ccc}
\text{Emb}^\theta(V, V) & \longrightarrow & \Lambda_\phi B \\
\downarrow & & \downarrow \\
\text{Emb}(V, V) & \longrightarrow_{d_0} & \text{O}(V)
\end{array}
\]

Consequently, \( \text{Emb}^\theta(V, V) \simeq \Lambda_\phi B \).

Proof (1) We have \( \text{Hom}_{\text{Top}^h_G/B}(V, V) \) from Definition B.4 and showed in Proposition B.6 that restriction-to-the-base gives a natural \( G \)-equivalence:

\[ \beta : \text{Hom}^\theta(\mathcal{T}V, \mathcal{T}V) \sim \text{Hom}_{\text{Top}^h_G/B}(V, V). \]

Let \( \ast \) be the \( G \)-space over \( B \) given by \( \phi(0) : \ast \to B \). We claim that the two maps \( \text{inc} : 0 \to V \) and \( \text{proj} : V \to \ast \) can be lifted to give equivalences of \( V \simeq \ast \) in \( \text{Top}^h_G/B \). If so, pre-composing with \( \text{inc} \) and post-composing with \( \text{proj} \) give

\[ \text{Hom}_{\text{Top}^h_G/B}(V, V) \sim \text{Hom}_{\text{Top}^h_G/B}(\ast, \ast) \simeq \Lambda_\phi B. \]

It remains to verify the claim, which is a routine job. We choose the lifts of \( \text{inc} \) and \( \text{proj} \) given by

\[ I = (\text{inc}, \alpha_1, 0) \in \text{Hom}_{\text{Top}^h_G/B}(\ast, V), \quad \text{where} \quad \alpha_1(t) = \phi(0) \quad \text{for all} \quad t \geq 0. \]

\[ P = (\text{proj}, \alpha_2, 1) \in \text{Hom}_{\text{Top}^h_G/B}(V, \ast), \quad \text{where} \quad \alpha_2(t) = \begin{cases} 
\phi \circ h_t, & 0 \leq t < 1; \\
\phi(0), & t \geq 1;
\end{cases} \]

where \( h_t : V \to V \) is any chosen homotopy from \( h_0 = \text{id} \) to \( h_1 = \text{proj} \). Then we have an obvious homotopy:

\[ P \circ I = (\text{id}, \text{const}_{\phi(0)}), 1) \simeq (\text{id}, \text{const}_{\phi(0)}), 0) = \text{id}_\ast. \]

and using the contraction \( h_t \), we can also construct a homotopy:

\[ I \circ P = (\text{proj}, \alpha_2, 1) \simeq (\text{id}, \text{const}_{\phi}), 0) = \text{id}_V. \]

(2) This is an assembly of part (1), Proposition B.11 and Theorem 2.24. However, we note that the map \( \Lambda_\phi B \to \text{O}(V) \) is only a non-canonical \( G \)-equivalence. The author does not know how to upgrade it to a map of \( G \)-monoids. So although all spaces displayed in the pullback diagram are \( G \)-monoids, it is not obvious whether one can write \( \text{Emb}^\theta(V, V) \) as a pullback of \( G \)-monoids.

To be more precise, we show how the quoted results assemble. We have the following large commutative diagram (B.16) expanding (B.13). Note that this is a commutative diagram of
$G$-monoids.

\[
\begin{array}{cccc}
\text{Emb}^g(V, V) & \longrightarrow & \text{Hom}^g(TV, TV) & \begin{array}{c}
\beta \\
\sim
\end{array} & \text{Hom}_{\text{Top}^h_B}(V, V) \\
\downarrow & & \downarrow & \sim & \downarrow \\
\text{Hom}^{id}(TV, TV) & \begin{array}{c}
\beta \\
\sim
\end{array} & \text{Hom}_{\text{Top}^h_B O(n)}(V, V) & \begin{array}{c}
\sim \\
\sim
\end{array} & \Lambda_{\phi} B \\
\downarrow & & \downarrow & \sim & \downarrow \\
\text{Emb}(V, V) & \longrightarrow & \text{Hom}(TV, TV) & \begin{array}{c}
\beta \\
\sim
\end{array} & \Lambda_{\phi} B G O(n) \\
\downarrow & & \downarrow & \sim & \downarrow \\
\text{Hom}(V, V) = O(V)
\end{array}
\] (B.16)

The map $\phi$ is studied in Lemma B.9. The map $\beta$ and the square $\Box$ are in Proposition B.6. The diagonal unlabeled maps are all induced by the inclusion $V \rightarrow TV$ and the projection $TV \rightarrow V$. In particularly, the parallelogram $\diamond$ is in part (1). Naturality of $\phi$ and $\beta$ gives the commutativity of $\Box$ and $\diamond$. Now, $d_0$ in the theorem is the composite

\[\text{Emb}(V, V) \xrightarrow{d} \text{Hom}(TV, TV) \xrightarrow{\sim} \text{Hom}(V, V),\]

and $d_0$ is an equivalence by taking $k = 1$ and $M = V$ in Proposition 3.26. It can be seen that the vertical map in the theorem involves choosing an inverse of the $\beta$ displayed in the third line.

\[\Lambda_{\phi} B G O(n) \xleftarrow{\beta} \text{Hom}^{id}(V, V) \xrightarrow{\phi} \text{Hom}(V, V) = O(V).\]

(See also [40, 4.4.12, 5.3.4]).

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