Interpolations between Jordanian twists, the Poincaré-Weyl algebra and dispersion relations

D. Meljanac\textsuperscript{1}, S. Meljanac\textsuperscript{2}, Z. Škoda\textsuperscript{3}, and R. Štrajn\textsuperscript{4}

\textsuperscript{1}Division of Materials Physics, Ruđer Bošković Institute, Bijenička cesta 54, 10002 Zagreb, Croatia
\textsuperscript{2}Division of Theoretical Physics, Ruđer Bošković Institute, Bijenička cesta 54, 10002 Zagreb, Croatia
\textsuperscript{3}Faculty of Science, University of Hradec Králové, Rokitanského 62, Hradec Králové, Czech Republic
\textsuperscript{4}Department of Electrical Engineering and Computing, University of Dubrovnik, Ćira Carića 4, 20000 Dubrovnik, Croatia

\textbf{Abstract}

We consider a two parameter family of Drinfeld twists generated from a simple Jordanian twist further twisted by 1-cochains. Twists from this family interpolate between two simple Jordanian twists. Relations between them are constructed and discussed. It is proved that there exists a one parameter family of twists identical to a simple Jordanian twist. The twisted coalgebra, star product and coordinate realizations of the $\kappa$-Minkowski noncommutative space time are presented. Real forms of Jordanian deformations are also discussed. The method of similarity transformations is applied to the Poincaré-Weyl Hopf algebra and two types of one parameter families of dispersion relations are constructed. Mathematically equivalent deformations may lead to differences in physical phenomena.

\begin{flushleft}
\textsuperscript{1}Daniel.Meljanac@irb.hr
\textsuperscript{2}meljanac@irb.hr
\textsuperscript{3}zoran.skoda@uhk.cz
\textsuperscript{4}rina.strajn@unidu.hr
\end{flushleft}
1 Introduction

Many proposals to resolve fundamental issues at the Planck scale involve the development of models of field theories on noncommutative (NC) spaces, most notably the $\kappa$-Minkowski spacetime. The parameter $\kappa$ is here usually interpreted as the Planck mass or the quantum gravity scale. One of the possible quantum symmetries of the $\kappa$-Minkowski NC space time is the $\kappa$-Poincaré quantum group [1, 2] and it constitutes one of the examples of deformed relativistic spacetime symmetries and the corresponding dispersion relations. Similar models exhibit Hopf algebra symmetries.

For Hopf algebras there is a remarkable systematic twisting procedure, invented by Drinfeld [3, 4]. Namely, from a given Hopf algebra $H$ and a twist element $F \in H \otimes H$ satisfying the 2-cocycle condition, one produces a new Hopf algebra $H_F$ with the same algebra sector, but a different coalgebra sector, with coproduct $\Delta_F(h) = F \Delta(h) F^{-1}$. If a spacetime has a Hopf algebra symmetry, and the Hopf algebra is twisted, then the spacetime can also be twisted using the same twist and preserving the covariance properties. Moreover, many other constructions like differential calculi and, to some extent, some basic constructions of field theories can be systematically deformed by procedures involving only a twist. Thus, the Drinfeld twist provides a laboratory for systematic deformation of space time and for investigating its deformed relativistic symmetry, geometric and physical structures.

Drinfeld 2-cocycles can be further modified by 1-cochains [3, 4]. Given a Drinfeld twist $F$ and an invertible 1-cochain $\omega \in H$, the expression $F_\omega = (\omega^{-1} \otimes \omega^{-1}) F \Delta(\omega)$ defines a new Drinfeld twist cohomologous to $F$. If we start from a trivial 2-cocycle $1 \otimes 1$, we obtain a twist, $(\omega^{-1} \otimes \omega^{-1}) \Delta(\omega)$, which is a 2-coboundary in the sense of nonabelian cohomology [4]. Cohomologous 2-cocycles induce isomorphic deformed Hopf algebras and equivalent related mathematical constructions. Hence, we may say that the transformation of changing a 2-cocycle by a 1-cochain is a gauge transformation of the 2-cocycle.

In the late 1980-s remarkable deformations of $R$-matrices and related quantum groups have been found under the name of Jordan(ian) $R$-matrices and Jordanian deformations [5, 6]. The corresponding Drinfeld twist has been written out independently in [7] and [8]. These examples involve the universal enveloping algebra of the two dimensional solvable Lie algebra (with generators $H, E$, $[H, E] = E$), some Hopf algebra which contains it ($U(\mathfrak{sl}(2))$, Yangian $Y(\mathfrak{gl}(2))$ etc.) or their duals. A special example of a Jordanian twist is $F_0 = \exp(\ln(1 + aE) \otimes H) = 1 \otimes 1 + aE \otimes H + O(a^2)$ with $a$ a deformation parameter [8]. Also, $r$-symmetric versions of the Jordanian twist, where $r = E \otimes H - H \otimes E$ is the classical $r$-matrix, were introduced in [9] and [10].

The Jordanian twist reappeared in the context of the $\kappa$-Minkowski NC space time [11, 12, 13, 14]. A relation with the symmetry of the $\kappa$-Minkowski space time is established with the introduction of the generators of relativistic symmetries, dilatation $D$ and momenta $p_\alpha$ (instead of generators $H$ and $E$ of the $\mathfrak{sl}(2)$ algebra) satisfying the same commutation relation, $[p_\alpha, D] = p_\alpha$. Dilatation $D$ is included in a minimal extension of the relativistic space time symmetry, the so-called Poincaré-Weyl algebra generated by $(M_{\mu\nu}, p_\mu, D)$, where $M_{\mu\nu}$ denote the Lorentz generators. One parameter interpolations between Jordanian twists, which are generated from a simple Jordanian twist $F_0$ by twisting with 1-cochains, were studied in [15, 16, 17].
Applications of Jordanian twists have been of interest in recent literature. For example, Jordanian deformations of the conformal algebra were considered in [18] and [19]. In [19], deformations of the anti de Sitter and the de Sitter algebra were investigated. Jordanian deformations have also been considered within applications in the AdS/CFT correspondence [20]. Integrable deformations of sigma models in relation to deformations of AdS\(_5\) and supergravity were investigated [21]. Jordanian twists have been applied in the deformation of spacetime metrics [22], dispersion relations [13, 14] and gauge theories [23].

In this paper, we consider a two parameter family of Drinfeld twists generated from a simple Jordanian twist by further twisting by 1-cochains. It leads to the \(\kappa\)-Minkowski spacetime and produces the same deformation of the Poincaré-Weyl symmetry algebra. We show that twists from this family interpolate between two simple Jordanian twists. Using the method of similarity transformations, we construct one parameter families of dispersion relations related to each other by inverse transformation. We point out that mathematically equivalent deformations may lead to differences in physical phenomena.

In Section 2, two special families of twists induced by 1-cochains are presented. These twists interpolate between two simple Jordanian twists. In Section 3, we define a two parameter family of Jordanian twists. Relations between them are presented and discussed. In 4.1, the twisted coalgebra sector and, in 4.2, coordinate realizations and the star product are presented. Real forms of new Jordanian deformations are also discussed in Subsection 4.3. In Section 5, a method of similarity transformations is applied to the Poincaré-Weyl Hopf algebra and one parameter families of dispersion relations are constructed. At the end of Section 5, concluding remarks are given.

\section{Families \(\mathcal{F}_{L,u}\) and \(\mathcal{F}_{R,u}\) of Jordanian twists and the relation between them}

In [15], the following family of Drinfeld twists was considered

\[ \mathcal{F}_{L,u} = \exp(-u(DA \otimes 1 + 1 \otimes DA)) \exp(-\ln(1 + A) \otimes D) \exp(\Delta(uDA)), \]

\[ = \exp\left(-\frac{u}{k}(DP \otimes 1 + 1 \otimes DP)\right) \exp\left(-\ln\left(1 - \frac{1}{k}P\right) \otimes D\right) \exp\left(-\Delta\left(u\frac{k}{D_P}\right)\right), \]

where generators of dilatation \(D\) and momenta \(P\) satisfy \([P, D] = P\) and \(A = -\frac{1}{k}P\). The deformation parameter \(\kappa\) is of the order of the Planck mass, \(u\) is a real dimensionless parameter and \(\Delta\) is the undeformed coproduct. These twists are constructed using a 1-cochain \(\omega_L = \exp\left(-\frac{2}{k}DP\right)\) and they satisfy the normalization and cocycle condition.
The corresponding deformed Hopf algebra is given by

\[ \Delta F_{L,u} = \mathcal{F}_{L,u} \Delta p_\mu F_{L,u}^{-1} = p_\mu \otimes \left( 1 + \frac{u}{k} P \right) + \left( 1 - \frac{1}{k} \right) \frac{1}{1 + u(1 - u) \frac{1}{k^2} P \otimes P} \]

(3)

\[ \Delta F_{L,u}(D) = \mathcal{F}_{L,u} \Delta D F_{L,u}^{-1} = \left( D \otimes \frac{1}{1 + \frac{u}{k} P} + \frac{1}{1 - \frac{1}{k} P \otimes D} \right) \left( 1 \otimes 1 + u(1 - u) \frac{1}{k^2} P \otimes P \right) \]

(4)

\[ S F_{L,u} = p_\mu \frac{1}{1 - (1 - 2u) \frac{1}{k} P} \]

(5)

\[ S F_{L,u}(D) = \left( \frac{1 - (1 - 2u) \frac{1}{k} P}{1 + \frac{u}{k} P} \right) D \left( 1 + \frac{u}{k} P \right), \]

(6)

where \( p_\mu, \mu \in \{0, 1, ..., n-1\} \) are momenta in the Minkowski spacetime and \( P = v^\alpha p_\alpha \), where \( v^\alpha v_{\alpha} \in \{1, 0, -1\} \).

For \( u = 0 \), \( F_{L,u=0} \) reduces to the Jordanian twist

\[ F_0 = \exp(-\ln(1 - \frac{1}{k} P) \otimes D) \]

\[ = \sum_{k=0}^{\infty} \left( -\frac{P}{k} \right)^k \otimes \left( -\frac{D}{k} \right) = \exp\left(-\ln\left(1 - \frac{1}{k} P\right) \otimes D\right). \]

(7)

For \( u = 1/2 \), \( F_{L,u=1/2} \) corresponds to the twist proposed in [9]. For \( u = 1 \), it follows from in [16] and [17] that \( F_{L,u=1} \) is identical to the Jordanian twist

\[ F_1 = \exp(-D \otimes \ln(1 + \frac{1}{k} P)) \]

\[ = \sum_{l=0}^{\infty} \left( -\frac{D}{l} \right) \otimes \left( \frac{P}{k} \right)^l = \exp\left(-D \otimes \ln\left(1 + \frac{1}{k} P\right) \right). \]

(8)

Hence, the family of twists \( F_{L,u} \) interpolates between twists \( F_0 \) and \( F_1 \).

Note that from \([P, D] = P\) it follows that \( DP = P(D - 1) \), and generally

\[ f(D)P = Pf(D - 1) \]

(9)

\[ f(D)P^m = P^m f(D - m). \]

(10)

Another family of twists induced with a 1-cochain \( \omega_R = \exp\left(-\frac{2}{P} PD\right) \) is [16]

\[ F_{R,u} = e^{\frac{2}{P} (PD\otimes 1 + 1 \otimes PD)} e^{-(1 - \frac{1}{k} P) \otimes D} e^{-\lambda(PD)}. \]

(11)

These twists satisfy the normalization and cocycle condition.
The corresponding deformed Hopf algebra is given by

\[
\Delta \mathcal{F}_{R,u}(p_{\mu}) = \mathcal{F}_{R,u} \Delta p_{\mu} \mathcal{F}_{R,u}^{-1} = \frac{p_{\mu} \otimes \left( 1 + \frac{u}{\kappa} P \right) + \left( 1 - \frac{1-u}{\kappa} P \right) \otimes p_{\mu}}{1 \otimes 1 + \frac{u(1-u)}{\kappa^2} P \otimes P}
\]

(12)

\[
\Delta \mathcal{F}_{R,u}(D) = \mathcal{F}_{R,u} \Delta D \mathcal{F}_{R,u}^{-1} = \left( 1 \otimes 1 + \frac{u(1-u)}{\kappa^2} P \otimes P \right) \left( D \otimes \frac{1}{1 + \frac{2}{\kappa} P} + \frac{1}{1 - \frac{1-u}{\kappa} P} \right)
\]

(13)

\[
S \mathcal{F}_{R,u}(p_{\mu}) = -\frac{p_{\mu}}{1 - \frac{2u}{\kappa} P}
\]

(14)

\[
S \mathcal{F}_{R,u}(D) = -\left( 1 - (1-u) \frac{P}{\kappa} \right) D \left( \frac{1 - (1-2u) \frac{P}{\kappa}}{1 - (1-u) \frac{P}{\kappa}} \right)
\]

(15)

For \( u = 0 \), \( \mathcal{F}_{R,u=0} \) reduces to the Jordanian twist \( \mathcal{F}_0 \) and for \( u = 1 \) it was shown in \([17]\) that \( \mathcal{F}_{R,u=1} = \mathcal{F}_1 \). Hence, the family of twists \( \mathcal{F}_{R,u} \) interpolates between two Jordanian twists \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \).

We point out that for \( u = 1/2 \), \( \mathcal{F}_{R,u=1/2}^{-1} = \mathcal{F}_{GZ}^{-1} \), where \( \mathcal{F}_{GZ}^{-1} \) is the twist proposed in \([10]\), theorem 2.20. Hence, the twist \( \mathcal{F}_{R,u=1/2} \) is given by \([11]\) and satisfies the normalization and cocycle condition automatically, as it is obtained as the cochain twist of a normalized 2-cocycle. It was shown in \([17]\) that the twist \( \mathcal{F}_{R,u}^{-1} \) can be written as

\[
\mathcal{F}_{R,u}^{-1} = \sum_{k,l=0}^{\infty} (u-1)^{k} \frac{P^k}{\kappa^k} \begin{pmatrix} D \end{pmatrix} \otimes \left( \frac{P}{\kappa} \right)^l \begin{pmatrix} D \end{pmatrix}.
\]

(16)

Starting with

\[
\mathcal{F}_{R,u}^{-1} = e^{\mathcal{M}(\frac{u}{\kappa} PD)} e^{-\mathcal{M}(\frac{1}{\kappa} DP)} \mathcal{F}_{L,u}^{-1} e^{\mathcal{M}(\frac{1}{\kappa} DP \otimes 1 + 1 \otimes DP)} e^{-\mathcal{M}(\frac{1}{\kappa} PD \otimes 1 + 1 \otimes PD)},
\]

(17)

and using (see also Eq. (39) in \([16]\))

\[
\exp \left( \frac{u}{\kappa} PD \right) \exp \left( -\frac{1}{\kappa} DP \right) = 1 + \frac{u}{\kappa} P,
\]

(18)

we find

\[
\mathcal{F}_{R,u}^{-1} = \Delta \left( 1 + \frac{u}{\kappa} P \right) \mathcal{F}_{L,u}^{-1} \frac{1}{1 \otimes 1 + \frac{u}{\kappa} P \otimes 1} \frac{1}{1 \otimes 1 - 1 \otimes \left( -\frac{u}{\kappa} P \right)}
\]

(19)

and

\[
\mathcal{F}_{L,u}^{-1} = \Delta \left( 1 + \frac{u}{\kappa} P \right) \frac{1}{1 \otimes 1 + \frac{u}{\kappa} P \otimes 1} \frac{1}{1 \otimes 1 - 1 \otimes \left( -\frac{u}{\kappa} P \right)} = \frac{1}{1 \otimes 1 + \frac{u(1-u)}{\kappa^2} P \otimes P}.
\]

(20)

Hence

\[
\mathcal{F}_{R,u}^{-1} = \mathcal{F}_{L,u}^{-1} \frac{1}{1 \otimes 1 + \frac{u(1-u)}{\kappa^2} P \otimes P}.
\]

(21)

Note that for \( u = 1 \), \( \mathcal{F}_{L,u=1} = \mathcal{F}_{R,u=1} \).
3 Two-parameter family of Jordanian twists and relations between them

More generally we can define a two parameter family of twists \( \mathcal{F}_{w,u} \), with coboundary twist \( \omega_{w,u} = \exp \left( -\frac{\kappa}{2} (D + w) P \right) \)

\[
\mathcal{F}_{w,u} = \exp \left( \frac{\kappa}{2} ((D + w) P \otimes 1 + 1 \otimes (D + w) P) \right) \exp \left( -\ln \left( 1 - \frac{P}{\kappa} \right) \otimes D \right) \exp \left( -\Delta \left( \frac{\kappa}{2} (D + w) P \right) \right). \tag{22}
\]

For \( w = 1 \) it coincides with \( \mathcal{F}_{R,u} \) and for \( w = 0 \) with \( \mathcal{F}_{L,u} \). Let us use relation (18). For \( u = 1 \), it yields

\[
e^{-\frac{\kappa}{2}D} e^{-D\kappa} = e^{-\frac{\kappa}{2}(D + P)} e^{-D\kappa} = e^{\left( 1 + \frac{\kappa}{2} P \right)}. \tag{23}
\]

Generalizing this relation for arbitrary \( w \) and using the BCH formula, it follows

\[
\exp \left( D\frac{P}{\kappa} + w\frac{P}{\kappa} \right) \exp \left( -D\frac{P}{\kappa} \right) = \exp \left( w\ln \left( 1 + \frac{P}{\kappa} \right) \right) = \left( 1 + \frac{w}{\kappa} P \right)^w. \tag{24}
\]

After rescaling \( P \to uP, D \to D, w \to w \), it follows

\[
\exp \left( \frac{\kappa}{2} (D + w) P \right) \exp \left( \frac{\kappa}{2} D P \right) = \left( 1 + \frac{u}{\kappa} P \right)^w. \tag{25}
\]

Using the above identity, the relation between \( \mathcal{F}_{w,u} \) and \( \mathcal{F}_{w=0,u} = \mathcal{F}_{L,u} \), one gets

\[
\mathcal{F}_{w,u}^{-1} = \Delta \left( 1 + \frac{u}{\kappa} P \right)^w \mathcal{F}_{L,u}^{-1} \left( 1 \otimes 1 + \frac{u}{\kappa} P \otimes 1 \right) \left( 1 \otimes 1 + \frac{w}{\kappa} P \otimes 1 \right)^w \tag{26}
\]

\[
\mathcal{F}_{L,u} \mathcal{F}_{w,u}^{-1} = \Delta^{\mathcal{F}_{u}} \left( 1 + \frac{u}{\kappa} P \right)^w \left( 1 \otimes 1 + \frac{w}{\kappa} P \otimes 1 \right)^w \left( 1 \otimes 1 + \frac{u}{\kappa} P \otimes 1 \right)^w.
\]

\[
= \left( 1 \otimes 1 + \frac{u(1-u)}{\kappa^2} P \otimes P \right)^w. \tag{27}
\]

Hence,

\[
\mathcal{F}_{w,u}^{-1} = \left( 1 \otimes 1 + \frac{u(1-u)}{\kappa^2} P \otimes P \right)^w \tag{28}
\]

\[
\mathcal{F}_{w,u}^{-1} = \mathcal{F}_{L,u}^{-1} \left( 1 \otimes 1 + \frac{u(1-u)}{\kappa^2} P \otimes P \right)^w. \tag{29}
\]

Hence, the above one parameter family of twists \( \mathcal{F}_{w,u} \) is identically equal to the simple Jordanian twist \( \mathcal{F}_1 \), generalizing the results from Sections 2 and 3. Note that for \( u = 1/2 \) twists \( \mathcal{F}_{w,u=1/2} \) are \( r \)-symmetric in the first order in \( 1/\kappa \) for all values of \( w \).

The quantum \( R \)-matrices are

\[
R_{w,u} = \left( 1 \otimes 1 + \frac{u(1-u)}{\kappa^2} P \otimes P \right)^w R_{L,u} \left( 1 \otimes 1 + \frac{u(1-u)}{\kappa^2} P \otimes P \right)^{-w}. \tag{30}
\]
Note that the classical $r$-matrix does not depend on the parameters $w$ and $u$.

\[ r = \frac{1}{\kappa} (D \otimes P - P \otimes D). \]  

\[ (31) \]

4 Twisted coalgebra, star product and realizations

4.1 Twisted coalgebra sector

The corresponding Hopf algebra is defined with

\[ \Delta^{F_w}(p_{\mu}) = \Delta^{F_w}(p_{\mu}) \]  

\[ (32) \]

\[ S^{F_w}(p_{\mu}) = S^{F_w}(p_{\mu}), \]  

for arbitrary $w$

\[ \Delta^{F_w}(D) = \left(1 \otimes 1 + \frac{u(1-u)}{\kappa^2} P \otimes P\right)^w \Delta^{F_w}(D) \left(1 \otimes 1 + \frac{u(1-u)}{\kappa^2} P \otimes P\right)^{-w} \]  

\[ (34) \]

\[ S^{F_w}(D) = -\left(1 - \frac{1-2uP}{1 + \frac{u}{\kappa} P}\right)^{-w} \]  

\[ \left(1 - \frac{1-u}{\kappa} P\right)^w \Delta^{F_w}(D) \left(1 - \frac{1-2uP}{1 + \frac{u}{\kappa} P}\right)^{-w} \]  

\[ (35) \]

Note that

\[ S^{F_w}(D) = -\left(1 - \frac{P}{\kappa}\right) D, \quad \forall w \]  

\[ (36) \]

\[ S^{F_w}(D) = -D \left(1 + \frac{P}{\kappa}\right), \quad \forall w. \]  

\[ (37) \]

4.2 Coordinate realizations of the $\kappa$-Minkowski spacetime and star product

Let us define the Heisenberg(-Weyl) algebra generated by commutative coordinates $x_\mu$ and the corresponding momenta $p_\mu$, satisfying

\[ [x_\mu, x_\nu] = 0, \quad [p_\mu, p_\nu] = 0, \quad [p_\mu, x_\nu] = -i\eta_{\mu\nu}. \]  

\[ (38) \]

The Heisenberg algebra acts on the space of functions $f(x) = f(x_\mu)$ of the commutative coordinates, where $x_\mu$ act by multiplication and the action of generators $p_\mu$ and dilatation operator $D = ix_\mu p_\mu$ is given by

\[ (p_\mu \triangleright f)(x) = -i \frac{\partial f(x)}{\partial x_\mu} \]  

\[ (D \triangleright f)(x) = x_\mu \frac{\partial f(x)}{\partial x_\mu} \]  

\[ (39) \]

\[ (40) \]

The subalgebra of coordinates becomes noncommutative due to a twist deformation, replacing the usual multiplication with star product multiplication. This star product is associative, because the twist $F_{w,u}$ satisfies the 2-cocycle condition. When the functions are exponentials $e^{ikx}$ and $e^{iqx}$, we find their star product

\[ e^{ikx} \ast e^{iqx} = e^{iD_{x}(k,q)w} G(k, q), \]  

\[ (41) \]
where

$$D_\mu(k, q) = \frac{k_\mu(1 + \frac{\mu}{\kappa}(v \cdot q)) + (1 - \frac{\mu}{\kappa}(v \cdot k))q_\mu}{1 + \frac{\mu(1 - u)}{\kappa^2}(v \cdot k)(v \cdot q)},$$  \hspace{1cm} (42)$$

$$G(k, q) = \left(1 + \frac{u(1 - u)}{\kappa^2}(v \cdot k)(v \cdot q)\right)^{-w},$$  \hspace{1cm} (43)$$

and \(kx = k_\mu x_\mu, qx = q_\mu x_\mu, v \cdot k = v_\mu k_\mu, v \cdot q = v_\mu q_\mu\). Results (41), (42), (43) follow also from the methods in [24][25].

Directly from the twist \(\mathcal{F}_{w,a}\), we also calculate the realizations of the noncommutative coordinates \(\hat{x}_\mu\) in terms of the Heisenberg algebra generators,

$$\hat{x}_\mu = m \mathcal{F}_w^{-1}(\hbar \otimes 1)(x_\mu \otimes 1) = \left(x_\mu + iv_\mu \frac{1 - u}{\kappa}D\right)\left(1 + \frac{u}{\kappa}P\right) + iw\frac{u(1 - u)}{\kappa^2}v_\mu P,$$  \hspace{1cm} (44)$$

where \(m\) is the multiplication map \(m : a \otimes b \mapsto ab\) of the Heisenberg algebra. These realizations are generalizations of those discussed in [26][27][28][29].

In the case \(u = 0\), \(\hat{x}_\mu = x_\mu + \frac{i}{\kappa}v_\mu D\) and in the case \(u = 1\), \(\hat{x}_\mu = x_\mu + \frac{1}{\kappa}P\). Note that for \(u = 0\) and \(u = 1\) realizations do not depend on \(w\). Noncommutative coordinates \(\hat{x}_\mu\), (44), generate the \(\kappa\)-Minkowski spacetime and satisfy

$$[\hat{x}_\mu, \hat{x}_\nu] = \frac{i}{\kappa} \left(v_\mu \hat{x}_\nu - v_\nu \hat{x}_\mu\right)$$  \hspace{1cm} (45)$$

$$[p_\mu, \hat{x}_\nu] = \left(-i\eta_\nu + \frac{i}{\kappa}v_\nu \frac{1 - u}{\kappa}p_\mu\right)\left(1 + \frac{u}{\kappa}P\right)$$  \hspace{1cm} (46)$$

Note that the term in \(\hat{x}_\mu\) proportional to \(w\) does not influence the commutation relations for \([\hat{x}_\mu, \hat{x}_\nu]\) and \([p_\mu, \hat{x}_\nu]\).

### 4.3 Real forms of Jordanian deformations

For physical applications it is important to address the question if the symmetry Hopf algebras can be endowed with (compatible) \(*\)-structures. Hopf \(*\)-algebras are Hopf algebras with an involution \(a \mapsto a^*\) satisfying identities, needed to treat unitarity and hermiticity in physical applications. In our case, to construct the \(*\)-involution, we start with \(P^* = P, D^* = -D\) and find

$$\mathcal{F}_{w,a}^{*\otimes w} = \mathcal{F}_w^{-1}(1 \otimes 1 + \frac{u(1 - u)}{\kappa^2}P \otimes P)^{2n-1}$$  \hspace{1cm} (47)$$

$$\left(S^{\mathcal{F}_{w,a}}(g)\right)^* = S^{\mathcal{F}_{w,a}}(g^*)_{-\kappa}, \forall g$$  \hspace{1cm} (48)$$

For \(w = 1/2\), it follows

$$\mathcal{F}_{1/2,a}^{*\otimes 1/2} = \mathcal{F}_{1/2,a}^{-1}$$  \hspace{1cm} (49)$$

$$\left(S^{\mathcal{F}_{1/2,a}}(D)\right)^* = S^{\mathcal{F}_{1/2,a}}(D^*)_{-\kappa}.$$  \hspace{1cm} (50)
Hence the corresponding twist is unitary for \( w = 1/2, u \in \mathbb{R} \) and also for \( u = 0, u = 1 \) and for arbitrary \( w \).

Generally, the twist \( \mathcal{T}_{w,u} \) is not unitary. If one wants to obtain a Hopf *-algebra structure on the deformation, there is a construction [4] provided the condition

\[
(S \otimes S)(\mathcal{T}^{\otimes w}) = \mathcal{T}^\tau
\]

is satisfied, where \( \mathcal{T}^\tau \) denotes the flipped (transposed) twist \( \mathcal{T} \). In our case, for \( \mathcal{T}_{w,u} \), we check

\[
(S \otimes S)(\mathcal{T}_{w,u})^{\otimes w} = \mathcal{T}_{w,u|\ldots} = \mathcal{T}^\tau_{w,(1-u)},
\]

and the condition (51) is satisfied for \( u = 1/2 \).

5 Similarity transformations, the Poincaré-Weyl algebra and dispersion relations

Let us consider the twist \( \mathcal{T}_{w,u} \), for \( u = 0 \) and denote the corresponding generators with \( P^0, D^0 \), i.e.

\[
\mathcal{T}_0 = \exp \left( -\ln \left( 1 - \frac{P_0}{\kappa} \right) \otimes D_0 \right)
\]

leading to the following Hopf algebra

\[
\Delta^{\mathcal{T}_0}(P^0_\mu) = P^0_\mu \otimes 1 + \left( 1 - \frac{P_0}{\kappa} \right) \otimes P^0_\mu
\]

\[
\Delta^{\mathcal{T}_0}(D_0) = D_0 \otimes 1 + \frac{1}{1 - \frac{1}{\kappa} P_0} \otimes D_0
\]

\[
S^{\mathcal{T}_0}(P^0_\mu) = -\frac{P^0_\mu}{1 - \frac{1}{\kappa} P_0}
\]

\[
S^{\mathcal{T}_0}(D_0) = -\left( 1 - \frac{1}{\kappa} P_0 \right) D_0.
\]

We define Lorentz generators \( M^{0}_{\mu\nu} \) generating the Poincaré-Weyl algebra together with \( p^0_\mu \) and \( D^0 \)

\[
[M^0_{\mu\nu}, M^0_{\rho\sigma}] = -(\eta_{\mu\rho}M^0_{\nu\sigma} - \eta_{\mu\sigma}M^0_{\nu\rho} + \eta_{\nu\rho}M^0_{\mu\sigma} - \eta_{\nu\sigma}M^0_{\mu\rho})
\]

\[
[M^0_{\mu\nu}, p^0_\rho] = -(\eta_{\rho\lambda}p^0_\nu - \eta_{\nu\lambda}p^0_\rho)
\]

\[
[D^0, M^0_{\mu\nu}] = [p^0_\mu, p^0_\nu] = 0
\]

\[
[D^0, p^0_\mu] = -p^0_\mu
\]

Then the coproduct and antipodes are

\[
\Delta^{\mathcal{T}_0}(M^0_{\mu\nu}) = M^0_{\mu\nu} \otimes 1 + 1 \otimes M^0_{\mu\nu} + \frac{1}{\kappa} \left( v_\mu p^0_\nu - v_\nu p^0_\mu \right) \otimes D^0
\]

\[
S^{\mathcal{T}_0}(M^0_{\mu\nu}) = -M^0_{\mu\nu} + \frac{1}{\kappa} (v_\mu p^0_\nu - v_\nu p^0_\mu) D^0
\]
Generalizing [14], and using similarity transformations induced by $\omega_w$, we obtain new generators

$$p_\mu = \exp\left(\frac{u}{k}(D_0 + w)P_0\right)p_\mu^0 \exp\left(\frac{u}{k}(D_0 + w)P_0\right) = \frac{p_\mu^0}{1 - \frac{u}{k}P_0} \quad (60)$$

$$D = \exp\left(-\frac{u}{k}(D_0 + w)P_0\right)D_0 \exp\left(-\frac{u}{k}(D_0 + w)P_0\right) = D_0 \left(1 - \frac{u}{k}P_0\right) - \frac{u}{k}wP_0 \quad (61)$$

$$M_{\mu\nu} = \exp\left(-\frac{u}{k}(D_0 + w)P_0\right)M_{\mu\nu}^0 \exp\left(-\frac{u}{k}(D_0 + w)P_0\right) = M_{\mu\nu}^0 - \frac{u}{k}(D^0 + w)(v_\mu p_\nu^0 - v_\nu p_\mu^0) \quad (62)$$

and the inverse relations

$$p_\mu^0 = \exp\left(\frac{u}{k}(D_0 + w)P_0\right)p_\mu \exp\left(-\frac{u}{k}(D_0 + w)P_0\right) = \frac{p_\mu}{1 + \frac{u}{k}P} \quad (63)$$

$$D^0 = \exp\left(\frac{u}{k}(D_0 + w)P_0\right)D \exp\left(\frac{u}{k}(D_0 + w)P_0\right) = D \left(1 + \frac{u}{k}P\right) + \frac{u}{k}wP \quad (64)$$

$$M_{\mu\nu}^0 = \exp\left(\frac{u}{k}(D_0 + w)P_0\right)M_{\mu\nu} \exp\left(-\frac{u}{k}(D_0 + w)P_0\right) = M_{\mu\nu} + \frac{u}{k}(D + w)(v_\mu p_\nu - v_\nu p_\mu). \quad (65)$$

Note that

$$(D + w)P = (D_0 + w)P_0. \quad (66)$$

Using (60), (61), (62), we express $M_{\mu\nu}, D, p_\mu$, in terms of $M_{\mu\nu}^0, D_0, p_\mu^0$, and calculate the deformed coproducts of $M_{\mu\nu}, D, p_\mu$ in terms of $M_{\mu\nu}^0, D_0$ and $p_\mu^0$, and then we use (63), (64), (65) to reexpress the resulting coproducts in terms of $M_{\mu\nu}, D$ and $p_\mu$, obtaining

$$\DeltaT_{\infty}(p_\mu) = \DeltaT_0\left(\frac{p_\mu^0}{1 - \frac{u}{k}P_0}\right) = \frac{p_\mu \otimes \left(1 + \frac{u}{k}P\right) + \left(1 - \frac{u}{k}P\right) \otimes p_\mu}{1 \otimes 1 + \frac{u(1 - u)}{k}P \otimes P} \quad (67)$$

$$\DeltaT_{\infty}(D) = \DeltaT_0\left(D_0 \left(1 - \frac{u}{k}P_0\right) - \frac{u}{k}wP_0\right)$$

$$= \left(1 \otimes 1 + \frac{u(1 - u)}{k^2}P \otimes P\right)^w D \otimes \left(1 + \frac{u}{k}P\right) + \frac{1 - \frac{u}{k}P}{1 + \frac{u}{k}P} \otimes D \left(1 \otimes 1 + \frac{u(1 - u)}{k^2}P \otimes P\right)^{1-w} \quad (68)$$

$$\DeltaT_{\infty}(M_{\mu\nu}) = \DeltaT_0\left(M_{\mu\nu}^0 - \frac{u}{k}(D^0 + w)(v_\mu p_\nu^0 - v_\nu p_\mu^0)\right) = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} - \frac{1}{k} \left[u - 1 \right] \frac{v_\mu p_\nu - v_\nu p_\mu}{1 + \frac{u}{k}P} \otimes \left(D(1 - \frac{u}{k}P) + \frac{u}{k}wP\right) + u \left(D + (D + w)\frac{u}{k}P - 1\right) \otimes \frac{v_\mu p_\nu - v_\nu p_\mu}{1 + \frac{u}{k}P}. \quad (69)$$
and similarly for the antipode,

\[
S^{T^+\alpha}(p_\mu) = S^{T_0} \left( \frac{p_\mu^0}{1 - \frac{w}{k} P_0} \right) = -\frac{p_\mu}{1 + \frac{2w-1}{k} P} \tag{70}
\]

\[
S^{T_0}(D) = S^{T_0} \left( D_0 \left( 1 - \frac{u}{k} P_0 \right) - \frac{w}{k} P_0 \right) = \frac{1 + \frac{2w-1}{k} P}{1 + \frac{w}{k} P} D \left( 1 + \frac{u}{k} P \right) + w \frac{u(1-u)}{k^2} P^2 \left( 1 + \frac{w}{k} P \right) \left( 1 + \frac{2w-1}{k} P \right) \tag{71}
\]

\[
S^{T^+\alpha}(M_{\mu\nu}) = S^{T_0} \left( M_{\mu\nu}^0 - \frac{u}{k} (D^0 + w) (v_\mu p_\nu - v_\nu p_\mu) \right) = -M_{\mu\nu} - \frac{u}{k} (D + w) (v_\mu p_\nu - v_\nu p_\mu) - \frac{v_\mu p_\nu - v_\nu p_\mu}{k} \left[ (u-1) D + u \frac{(u-1)(w-1) \frac{P}{k} - w}{1 + \frac{w}{k} P} \right] \tag{72}
\]

in agreement with the results \(62\) - \(65\) in Section 4. We point out that the above equations \(67\) - \(72\) for \(u = 1\) do not depend on the parameter \(w\), in accordance with \(29\).

For the Poincaré algebra generated by Lorentz generators \(M_{\mu\nu}^0\) and \(p_\mu^0\), the corresponding quadratic Casimir is

\[
(p^{0})^2 = \frac{p^2}{(1 + \frac{w}{k} P)} \tag{73}
\]

with respect to the realization of \(\hat{x}_\mu\) in terms of \(x_\mu, p_\mu\) \(44\). On the other hand, the quadratic Casimir for the Poincaré algebra generated by \(M_{\mu\nu}\) and \(p_\mu\) is

\[
(P)^2 = \frac{(p^{0})^2}{(1 - \frac{w}{k} P_0)^2} \tag{74}
\]

with respect to the realization of \(\hat{x}_\mu\) expressed in terms of \(x_\mu^0, p_\mu^0\). These two families of dispersion relations are related to each other by inverse transformation, with a parameter change \(u \mapsto -u\).

The addition of momenta \(k_\mu \oplus q_\mu\) also depends on the parameter \(u\)

\[
k_\mu \oplus q_\mu = \frac{k_\mu \left( 1 + \frac{w}{k} v \cdot q \right) + \left( 1 + \frac{2w-1}{k} v \cdot k \right) q_\mu}{1 + \frac{u(1-u)(v \cdot k)}{k^2} (v \cdot q)} \tag{75}
\]

Note that dispersion relations \(73\), \(74\) and addition of momenta \(75\) do not depend on parameter \(w\), whereas the realization of \(\hat{x}_\mu\) \(44\), the star product \(41\), \(\Delta^{T^+\alpha}(D)\) \(34\), \(\Delta^{T^+\alpha}(M_{\mu\nu})\) \(69\), \(S^{T^+\alpha}(D)\) \(35\) and \(S^{T_0}(M_{\mu\nu})\) \(72\) do depend on \(w\).

**Concluding remarks.** The physical interpretation depends on the realization of the generators of the Poincaré-Weyl algebra \(14, 24, 30\). Particularly, the spectrum of the relativistic hydrogen atom depending on the parameter \(u\) was investigated in \(15\). Differences in realizations of NC coordinates could also lead to different physical predictions, see e.g. \(31\), where dispersion relations and the time delay parameter were investigated. Mathematically equivalent deformations may lead to differences in physical phenomena.

**Acknowledgements.** Z. Š. has been partly supported by grant no. 18-00496S of the Czech Science Foundation.
References

[1] J. Lukierski, A. Nowicki, H. Ruegg, V. N. Tolstoy, “q-deformation of Poincaré algebra”, Phys. Lett. B264, 331 (1991)

[2] J. Lukierski, A. Nowicki, H. Ruegg, “New quantum Poincaré algebra and κ-deformed field theory”, Phys. Lett. B293, 344 (1992)

[3] V. G. Drinfel’d, “Hopf algebras and the quantum Yang-Baxter equation”, Soviet Math. Dokl. 32, 254 (1985)

[4] S. Majid, “Foundations of Quantum Group Theory”, Cambridge University Press, 1995

[5] V. Lyubashenko, “Hopf algebras and vector symmetries”, Russian Mathematical Surveys 41:5 (1986).

[6] E.E. Demidov, Yu.I. Manin, E.E. Mukhin, D.V. Zhdanovich, “Nonstandard quantum deformations of GL(n) and constant solutions of the Yang-Baxter equation”, Prog. Theor. Phys. Suppl. 102 (1990) 203–218.

[7] V. Coll, M. Gerstenhaber, A. Giaquinto, “An explicit deformation formula with non-commuting derivations”, Ring Theory, Vol. 1, Israel Mathematical Conference Proceedings (1989) 396–403

[8] O. Ogievetsky, “Hopf structures on the Borel subalgebra of sl(2)”, in: Proceedings of Winter School in Geometry and Physics, Zdiok, January 1993, Supplemento ai Rendiconti del Circolo Matematico di Palermo, Serie II 37 (1994) 185.

[9] V. N. Tolstoy, “Twisted Quantum Deformations of Lorentz and Poincaré algebras”, Bulg. J. Phys. 35 441-459 (2008), arXiv:0712.3962.

V. N. Tolstoy, “Quantum Deformations of Relativistic Symmetries”, Invited talk at the XXII Max Born Symposium "Quantum, Super and Twistor", September 27-29, 2006, Wroclaw (Poland), in honour of Jerzy Lukierski, arXiv:0704.0081

[10] A. Giaquinto, J. J. Zhang, “Bialgebra actions, twists, and universal deformation formulas”, J. Pure Appl. Algebra 128, 133-151 (1998), arXiv:hep-th/9411140

[11] A. Borowiec, A. Pachol, “kappa-Minkowski spacetime as the result of Jordanian twist deformation”, Phys. Rev. D79 045012 (2009), arXiv:0812.0576

[12] J.-G. Bu, J. H. Yee, H.-C. Kim, “Differential Structure on kappa-Minkowski Spacetime Realized as Module of Twisted Weyl Algebra”, Phys. Lett. B679, 486-490 (2009), arXiv:0903.0040

[13] P. Aschieri, A. Borowiec, A. Pachol, “Observables and dispersion relations in κ-Minkowski spacetime”, JHEP 1710 (2017) 152, arXiv:1703.08726

[14] D. Meljanac, S. Meljanac, S. Mignemi, R. Štrajn, “Kappa-deformed phase space, Jordanian twists, Lorentz-Weyl algebra and dispersion relations”, Phys. Rev. D99 126012 (2019), arXiv:1903.08679
[15] S. Meljanac, D. Meljanac, A. Pachoł, D. Pikutić, “Remarks on simple interpolation between Jordanian twists”, J. Phys. A50, no.26, 265201 (2017), arXiv:1612.07984

[16] A. Borowiec, S. Meljanac, D. Meljanac, A. Pachoł, "Interpolations between Jordanian twists induced by coboundary twists", SIGMA 15 (2019) 054, arXiv:1812.05535

[17] D. Meljanac, S. Meljanac, Z. Škoda, R. Štrajn, “One parameter family of Jordanian twists”, SIGMA 15 (2019) 082, arXiv:1904.03993

[18] S. Meljanac, A. Pachoł, D. Pikutić, “Twisted conformal algebra related to κ-Minkowski space”, Phys. Rev. D92, 105015 (2015), arXiv:1509.02115

[19] F. J. Herranz, “New quantum (anti)de Sitter algebras and discrete symmetries”, Phys. Lett. B543 (2002) 89-97, arXiv:hep-ph/0205190
A. Ballesteros, N. R. Bruno, F. J. Herranz, “A non-commutative Minkowskian spacetime from a quantum AdS algebra”, Phys. Lett. B574 (2003) 276-282, arXiv:hep-th/0306089.
A. Ballesteros, N. R. Bruno, F. J. Herranz, “Quantum (anti)de Sitter algebras and generalizations of the kappa-Minkowski space”, The XI Int. Conf. on Symmetry Methods in Physics, June 21-24, 2004, Prague, Czech Republic; Published in “Symmetry methods in Physics”, edited by C. Burdik, O. Navratil and S. Posta, Joint Institute for Nuclear Research, Dubna (Russia), pp. 1-20, (2004), arXiv:hep-th/0409295

[20] I. Kawaguchi, T. Matsumoto, K. Yoshida, “Jordanian deformations of the AdS $S^5$ superstring”, JHEP 1404 (2014) 153, arXiv:1401.4855
T. Matsumoto, K. Yoshida, “Lunin-Maldacena backgrounds from the classical Yang-Baxter equation – Towards the gravity/CYBE correspondence”, JHEP 1406 (2014) 135, arXiv:1404.1838
I. Kawaguchi, T. Matsumoto, K. Yoshida, “A Jordanian deformation of AdS space in type IIB supergravity”, JHEP 1406 (2014) 146, arXiv:1402.6147
T. Matsumoto, K. Yoshida, “Yang-Baxter deformations and string dualities”, JHEP 1503 (2015) 137, arXiv:1412.3658

[21] S. J. van Tongeren, “Yang-Baxter deformations, AdS/CFT, and twist-noncommutative gauge theory”, Nucl. Phys. B904, 148 (2016), arXiv:1506.01023
B. Hoare and S. J. van Tongeren, “On jordanian deformations of AdS5 and supergravity”, J. Phys. A49, 434006 (2016), arXiv:1605.03554

[22] A. Borowiec, T. Juric, S. Meljanac, A. Pachoł, “Central tetrads and quantum spacetimes”, Int. J. Geom. Methods Mod. Phys. 13, 1640005 (2016), arXiv:1602.01292

[23] M. Dimitrijevic, L. Jonke and A. Pachoł, “Gauge Theory on Twisted κ-Minkowski: Old Problems and Possible Solutions”, SIGMA 10, 063 (2014), arXiv:1403.1857
[24] S. Meljanac, D. Meljanac, F. Mercati, D. Pikutić, “Noncommutative Spaces and Poincaré Symmetry”, Phys. Lett. B766 181-185 (2017), arXiv:1610.06716

[25] D. Meljanac, S. Meljanac, S. Mignemi, R. Štrajn, “Snyder-type spaces, twisted Poincaré algebra and addition of momenta”, Int. J. Mod. Phys. A32 1750172 (2017), arXiv:1608.06207

[26] T. R. Govindarajan, K. S. Gupta, E. Harikumar, S. Meljanac, D. Meljanac, “Twisted Statistics in kappa-Minkowski Spacetime”, Phys. Rev. D77 105010, (2008), arXiv:0802.1576

[27] S. Meljanac, M. Stojić, “New realizations of Lie algebra kappa-deformed Euclidean space”, Eur. Phys. J. C47, 531 (2006), arXiv:hep-th/0605133

[28] S. Meljanac, S. Kresic-Juric, “Differential structure on kappa-Minkowski space, and kappa-Poincaré algebra”, Int. J. Mod. Phys. A26 (20) (2011), 3385-3402, arXiv:1004.4647

D. Meljanac, S. Meljanac, D. Pikutić, “Families of vector-like deformed relativistic quantum phase spaces, twists and symmetries”, Eur. Phys. J. C77 (2017) 830, arXiv:1709.04745

[29] S. Meljanac, D. Meljanac, A. Samsarov, M. Stojić, “Kappa-deformed Snyder spacetime”, Mod. Phys. Lett. A25 579-590, (2010), arXiv:0912.5087

[30] D. Kovačević, S. Meljanac, A. Pachol, R. Štrajn, “Generalized Poincaré algebras, Hopf algebras and kappa-Minkowski spacetime”, Phys. Lett. B711, 122-127 (2012), arXiv:1202.3305

[31] A. Borowiec, K. S. Gupta, S. Meljanac and A. Pachol, “Constraints on the quantum gravity scale from kappa-Minkowski spacetime”, EPL 92, 20006 (2010), arXiv:0912.3299