RIEMANN REARRANGEMENT THEOREM FOR SOME TYPES OF CONVERGENCE

YURIY DYBSKIY AND KONSTANTIN SLUTSKY

Abstract. We reexamine the Riemann Rearrangement Theorem for different types of convergence. We consider series convergence with respect to a filter. We describe the Sum Range (SR) of a series along the 2n-filter and for statistically convergent series.

1. Introduction

The classical Riemann Rearrangement Theorem says that the commutative law is no longer true for infinite sums. To be more precise it says the following:

Theorem 1.0.1 (Riemann Rearrangement Theorem). Let \( \sum_{k=1}^{\infty} x_k \) be a conditionally convergent series of real numbers. Then:

1. for any \( s \in \mathbb{R} \) one can find a permutation \( \pi \) such that \( \sum_{k=1}^{\infty} x_{\pi(k)} = s \);
2. one can find permutation \( \sigma \) such that \( \sum_{k=1}^{\infty} x_{\sigma(k)} = \infty \);
3. one can find permutation \( \sigma \) such that \( \sum_{k=1}^{\infty} x_{\sigma(k)} = -\infty \).

In the Riemann Rearrangement Theorem one considers the ordinary convergence of series. It looks natural to consider in this setting some weaker types of convergence. Interesting results in this direction are proved in [1] and [5] where generalizations of the Riemann theorem for Cesaro summation and other matrix summation methods were obtained. These generalizations are much more complicated than the original Riemann theorem, and even the statements strongly differ from the classical one: say for Cesaro summation it is possible that the set of sums under all permutations of summands forms an arithmetic sequence. V. Kadets posed to us the problem what effects appear if the ordinary convergence in the Riemann theorem statement is substituted by convergence with respect to a filter. In this paper we are doing the...
first two steps in this direction, considering statistical convergence and convergence of subsequence $\sum_{k=1}^{2n} x_k$ of partial sums.

2. Statistical convergence

2.1. Introduction. Statistical convergence is a generalization of the usual notion of convergence that parallels the usual theory of convergence. While statistical convergence has become an active area of research under the name of statistical convergence only recently, it has appeared in the literature in a variety of guises since the beginning of twentieth century. Statistical convergence has been discussed in number theory, trigonometric series and summability theory. A relation between statistical convergence and Banach space theory as well as a list of references one can find in [2]. The aim of this chapter is to generalize the Riemann’s Rearrangement Theorem to the case of the statistical convergence.

The object that is going to be investigated is $SR_{st.}(\sum x_k)$ and the sequence of definitions below leads to it.

**Definition 2.1.1.** $A \subset \mathbb{N}$ is said to be negligible if

$$\lim_{n \to \infty} \frac{|A \cap \{1, \ldots, n\}|}{n} = 0$$

**Definition 2.1.2.** The sequence $\{s_n\}_{n=1}^{\infty}$ statistically converges to $s$ ($s_n \to s_{\text{stat}}$) if for every $\varepsilon > 0$ the set $\{n : |s_n - s| > \varepsilon\}$ is negligible.

**Definition 2.1.3.** Series $\sum x_k$ is said to be convergent statistically to $s$ if the sequence $s_n = \sum_{k=1}^{n} x_k$ of partial sums converges statistically to $s$ (short notation is $\sum x_k \to s_{\text{stat}}$).

**Definition 2.1.4.** Point $s$ belongs to the statistical Sum Range of the series $\sum x_k$ if there exists a permutation $\pi$ such that $\sum_{k=1}^{n} x_{\pi(k)} \to s_{\text{stat}}$. The set of all such points is called the statistical Sum Range of the series and is denoted by $SR_{st.}(\sum x_k)$.

We will use also the following definition from [4].

**Definition 2.1.5.** A point is said to be a limit point for the series $\sum x_k$ if it is the limit point of some subsequence of the sequence of partial sums of some rearrangement of the series. The set of all such points, called the limit-point range of the series, will be denoted by $LPR(\sum x_k)$.

It is easy to see that $LPR(\sum x_k)$ is a closed set and $SR_{st.}(\sum x_k) \subset LPR(\sum x_k)$. H. Hadwiger [3] proved that $LPR(\sum x_k)$ is a shifted closed additive subgroup of the space in which the series lives. In particular
this is true for numerical series (see also [4], exercises 3.2.2, 2.1.2 and comments to these exercises).

By \( \mathbb{R} \) we denote the two point compactification of the real line:

\[
\mathbb{R} = \mathbb{R} \cup \{-\infty, \infty\}.
\]

2.2. Main theorem for \( SR_{st} \). The aim of this chapter is to prove the following result:

**Theorem 2.2.1.** Let \( \sum x_k \) be a statistical permutation. Then \( SR_{st}(\sum x_k) = LPR(\sum x_k) \). So \( SR_{st}(\sum x_k) \) is one of the following:

1. The only number \( a \);
2. \( \{a + \lambda \mathbb{Z}\} \) for some \( \lambda \in \mathbb{R} \);
3. The whole \( \mathbb{R} \).

**Proof.** Since series \( \sum x_k \) converges statistically there exists a subsequence \( x_{n_k} \) such that \( x_{n_k} \to 0 \). From the elements of \( x_{n_k} \) we can select a subsequence \( x_{n_{k_i}} \) such that \( \sum_{i=1}^{\infty} x_{n_{k_i}} < \infty \).

Now we can substitute all elements \( x_{n_{k_i}} \) in the original series for 0 and this will not affect the convergence since we are subtracting an absolutely convergent series. So without loss of generality we may assume that there are infinitely many zeros among the original series terms.

Let us write the definition of LPR in detail:

\[
LPR(\sum x_k) = \{x \mid \exists \pi \exists \{m_k\} : x = \lim_{k \to \infty} \sum_{j=1}^{m_k} x_{\pi(j)}\}
\]

where \( \pi \) is a permutation of \( \mathbb{N} \) and \( \{m_k\} \) is an increasing sequence of indices. Let \( b \) be an arbitrary element of LPR. Let \( \{m_k\} \) be a sequence from the definition corresponding to element \( b \), and such that \( m_{k+1}/m_k \to \infty \). We will arrange elements of our series in the following way:

\[
\begin{align*}
&x_{\pi(1)} + \cdots + x_{\pi(m_1)} + 0 \underbrace{+ \cdots + 0}_{(m_2)^2 \text{ times}} \\
&+ x_{\pi(m_1+1)} + \cdots + x_{\pi(m_2)} + 0 \underbrace{+ \cdots + 0 + \cdots}_{(m_3)^2 \text{ times}}
\end{align*}
\]

We get the permutation of the series that obviously statistically converges to \( b \).
2.3. Examples. We finish the proof by giving the examples which satisfy each case of the theorem 2.2.1.

Example 2.3.1. Any unconditionally convergent series in usual meaning gives us a series with \( SR_{st.} = \{a\} \), which corresponds case (1).

Example 2.3.2. Let the elements of series be the following:

\[
x_n = \begin{cases} 
0, & n \neq 10^k \land n \neq 10^k + 1 \\
\lambda, & n = 10^k \\
-\lambda, & n = 10^k + 1
\end{cases}, \quad k \in \mathbb{N}, \ n \in \mathbb{N}.
\]

Then \( SR_{st.} = \lambda \mathbb{Z} \) for some \( \lambda \in \mathbb{R} \), which corresponds case (2).

Example 2.3.3. Any conditionally convergent series in usual meaning gives us a series with \( SR_{st.} = \mathbb{R} \), which corresponds case (4).

Remark 2.3.4. In fact the statement \( SR_{st.}(\sum x_k) = LPR(\sum x_k) \) holds true for series in any Banach space. Thus one can prove that in any separable Banach space \( SR_{st.} \) can be any shifted closed subgroup.

Remark 2.3.5. If one wants to consider \( SR_{st.} \subset \mathbb{R} \) then modifying above argument one can prove

Theorem 2.3.6. Let \( \sum x_k \overset{st.}{=} a \) for the original permutation. Then \( SR_{st.}(\sum x_k) \) is one of the following:

1. The only number \( a \);
2. \( \{a + \lambda \mathbb{Z}\} \cup \{-\infty, \infty\} \) for some \( \lambda \in \mathbb{R} \);
3. The whole \( \mathbb{R} \);
4. The set \( \{-\infty, a, \infty\} \).

3. 2n-convergence

3.1. Introduction. Let us say that a series \( \sum_{k=1}^{\infty} x_k \) 2n-converges to \( c \) if \( \lim_{n \to \infty} \sum_{k=1}^{2n} x_k = c \)

Definition 3.1.1. Point \( s \in \mathbb{R} \) belongs to the 2n sum range of the series \( \sum x_k \) if there exists a permutation \( \pi : \mathbb{N} \to \mathbb{N} \) such that \( \lim_{n \to \infty} \sum_{k=1}^{2n} x_{\pi(k)} = s \). The set of all such points is called the 2n sum range of series \( \sum x_k \) and is denoted by \( SR_2(\sum x_k) \). When it is clear what series is considered we will denote this set by \( SR_2 \)

Consider first the following example:

Example 3.1.2. Series \( 1 + (-1) + 1 + (-1) + \cdots \)

It’s easy to see that this series diverges (reminder doesn’t tend to 0). But if we consider the subsequence \( S_n = \sum_{k=1}^{2n} x_k \) of it’s partial sums we see that \( \forall n \in \mathbb{N} : S_n = 0 \) and so this subsequence converges. Notice that in order to converge elements must go in strict pairs \( 1 + (-1) \) after some number of elements. Limit of \( S_n \) can be
It’s easy to prove that
\[ \{ S \in \mathbb{R} | \exists \pi - \text{permutation of } \mathbb{N} : \lim_{n \to \infty} \sum_{k=1}^{2n} x_{\pi(k)} = S \} = 2\mathbb{Z} \]

So the statement of Riemann Rearrangement Theorem in this case of convergence has to be modified. Surprisingly this modification and its proof appear to be rather non-trivial and much more complicated than in the case of statistical convergence.

Recall that \( X \) is said to be \( \varepsilon \)-separated if all pairwise distances between the elements of \( X \) are greater than \( \varepsilon \). \( X \) is said to be separated if it is \( \varepsilon \)-separated for some \( \varepsilon > 0 \).

The aim of this chapter is to prove the following result:

**Theorem 3.1.3 (Main Theorem).** Let \( \lim_{n \to \infty} \sum_{k=1}^{2n} x_k = a \in \mathbb{R} \). Then \( SR_2(\sum x_k) \) is one of the following:

1. Shifted additive subgroup of the form
   \[ a + \{ c_1 z_1 + \cdots + c_l z_l | z_k \in E, \ c_i \in \mathbb{Z}, \ \sum_{k=1}^{l} c_k \text{ is even} \} , \]
   where \( E \) of is an \( \varepsilon \)-separated set;
2. The whole \( \mathbb{R} \);
3. The only number \( a \).

**3.2. Reduction to a special form of the series.** We can represent the series in the following way:

(3.2.1) \[ x_1 + (-x_1 + \alpha_1) + x_3 + (-x_3 + \alpha_2) + x_5 + (-x_5 + \alpha_3) + \cdots \]

This can be done by denoting
\[ \alpha_k \overset{\text{def}}{=} x_{2k-1} + x_{2k} \ (\forall \ k \in \mathbb{N}) \]

Recall that the series \( \sum x_k \) 2n-converges in original order, i.e. \( \lim_{n \to \infty} \sum_{k=1}^{2n} x_k = a \), so \( \sum_{i=1}^{\infty} \alpha_i = a \).

**Theorem 3.2.1.** If \( \sum_{k=1}^{\infty} \alpha_k \) converges conditionally then

(3.2.2) \[ SR_2(\sum x_k) = \mathbb{R} \]

**Proof.** As Riemann Rearrangement Theorem says for a conditionally convergent series \( \sum_{k=1}^{\infty} \alpha_k \), for all \( c \in \mathbb{R} \) there exist a permutation of
indices $\pi$ such that $\sum_{k=1}^{\infty} \alpha_{\pi(k)} = c$. Consider the following arrangement of $\{x_k\}$:

$$x_{2\pi(1)-1} + (-x_{2\pi(1)-1} + \alpha_{\pi(1)}) + x_{2\pi(2)-1} + (-x_{2\pi(2)-1} + \alpha_{\pi(2)}) + \cdots$$

It's clear that this series converges to $c$. As $c$ was arbitrary we get (3.2.2). \hfill \Box

**Definition 3.2.2.** A series $\sum x_k$ is said to be equivalent to $\sum y_k$ if $\sum |x_k - y_k| < \infty$.

Remark, that if one of two equivalent series converges (converges) in some permutation then the same does the second series and that $SR_2(\sum x_k) = SR_2(\sum y_k) + \sum (x_k - y_k)$.

Theorem 3.2.1. corresponds to the case (2) of the main theorem.

Now consider what happens if $\sum_{k=1}^{\infty} \alpha_k$ converges unconditionally to $a$. In this case $\sum x_k$ is equivalent to the following simplified series:

$$x_1 + (-x_1) + x_3 + (-x_3) + x_5 + (-x_5) + \cdots$$

So we reduce the series (3.2.1) to (3.2.3). Changing notation we consider a series of the form:

$$x_1 + x_{-1} + x_2 + x_{-2} + x_3 + x_{-3} + \cdots$$

where $x_{-n} = -x_n$ and $x_n > 0$ for $n > 0$, $x_n < 0$ for $n < 0$. Denote by $X$ the set of all elements of the series (without repetitions) and enumerate the elements of $X$ as

$$X = \{e_i \mid i \in \mathbb{Z}\setminus\{0\}\},$$

$e_{-n} := -e_n$ and $e_i > 0$, $i \in \mathbb{N}$. By the order of an element $e \in X$ we mean

$$\chi(e) = | \{i \in \mathbb{Z}\setminus\{0\} \mid x_i = x \} | .$$

### 3.3. The (basic) case of separated $X$.

**Lemma 3.3.1.** Let $X$ be ε-separated and there are nonzero elements of infinite order. Then

$$SR_2 = \{c_1e_{j_1} + \cdots + c_re_{j_r} \mid \forall k : \chi(e_{j_k}) = \infty, \quad c_k \in \mathbb{Z}, \sum_{j=1}^{r} c_j \text{ is even} \}. \quad (3.3.5)$$

If there are no nonzero elements of infinite order then

$$SR_2 = \{0\}.$$
Proof. Denote right-hand side of (3.3.5) by $\mathcal{L}$. Let us prove that $SR_2 \subset \mathcal{L}$.

Let $\pi : \mathbb{N} \to \mathbb{Z}\setminus\{0\}$ be an arbitrary bijection such that

$$A = \lim_{n \to \infty} \sum_{j=1}^{2n} x_{\pi(j)} \in \mathbb{R}.$$  

Applying Cauchy Convergence Criterion to this series we get that there exist an even number $n_0$ such that for every even $n$ and $m$ greater or equal then $n_0$ the following inequality stands:

$$|S_n - S_m| < \varepsilon.$$  

Consider elements of $2n$ partial sum sequence. Presume $n > n_0$ and $n$ is even. We have $|S_{n+2} - S_n| = |x_{\pi(n+1)} + x_{\pi(n+2)}| < \varepsilon$ But $X$ is $\varepsilon$-separated. Since $x_j \in X$ we get that this inequality is true if and only if $|x_{\pi(n+1)} + x_{\pi(n+2)}| = 0$. In any other case this modulus is greater than $\varepsilon$.

$$((|x_{\pi(n+1)} + x_{\pi(n+2)}| = 0) \iff (x_{\pi(n+1)} = -x_{\pi(n+2)})$$

So the series has the following structure

$$A = x_{\pi(1)} + \cdots + x_{\pi(n_0)} + x_{k_1} + (-x_{k_1}) + x_{k_2} + (-x_{k_2}) + \cdots$$  

The first $n_0$ elements will be considered later. The other ones come in strict pairs and if $2n$ partial sums are considered it can be remarked that after element $S_{n_0}$ all of them are equal to $S_{n_0}$

Now in the sum $x_{\pi(1)} + \cdots + x_{\pi(n_0)}$ consider the elements $y_1, y_2, \ldots, y_j$ of finite order. Then for all $i \in \{1, \ldots, j\}$:

$$|\{k \in \{1, \ldots, n_0\} : x_{\pi(k)} = y_i\}| = |\{k \in \{1, \ldots, n_0\} : x_{\pi(k)} = -y_i\}|$$

That is because in the second part of the series all elements come in strict pairs. Same number of opposite elements in finite sum gives us zero. The only elements left are the elements with infinite order, so $A$ has requested form

$$A = c_1e_{j_1} + \cdots + c_re_{j_r},$$

where $c_i \in \mathbb{Z}$. Moreover considering that $n_0$ was even and although some number of pairs of elements was taken from it there still remains an even number, we deduce that $\sum_{k=1}^{t} c_k$ is even.

Why $\mathcal{L} \subset SR_2$?

Select a finite element $z \in \mathcal{L}$: $z = c_1e_{j_1} + \cdots + c_re_{j_r}$ and create a series starting with

$$\text{sign}(c_1)(e_{j_1} + \cdots + e_{j_1}) + \cdots + \text{sign}(c_r)(e_{j_r} + \cdots + e_{j_r})$$

$|c_1|$ times $|c_r|$ times
and after these elements the rest of $x_k$ is settled in pairs

$$x_{k_1} + (-x_{k_1}) + x_{k_2} + (-x_{k_2}) + \cdots$$

It is obvious that $2n$-sum of this series is $z$, and this series is a rearrangement of (3.2.3). With infinite elements the situation is obvious. So we have managed to prove that $\mathcal{L} \subset SR_2$. The second part was proved also because the only possible real value in $SR_2$ is 0.

End of Lemma’s proof.

3.4. Some combinatorial lemmas. Let $M$ be a set of indices. For a bounded sequence $(x_n)_{n \in M}$ introduce the following quantity:

$$(3.4.6) \quad \Delta(M) = \Delta(M, (x_n)_{n \in M}) = \inf_{a \in \mathbb{R}} \sum_{n \in M} |x_n - a|.$$

If occasionally a sum consists of empty set of summands, we mean here and below that the sum equals 0.

Lemma 3.4.1. If $\Delta(M) = \infty$ then one can select a finite collection of disjoint pairs $n_k, m_k \in M$, $k = 1, 2, \ldots, s$ with $\sum_{k=1}^s |x_{n_k} - x_{m_k}|$ being arbitrarily large.

Proof. We consider that $\{x_n\}_{n \in M}$ has only one limiting point (otherwise the statement is obvious). Denote it by $a$. The fact that $\Delta(M) = \infty$ implies that

$$\sum_{n \in M} |x_n - a| = \infty.$$ 

For every $K > 0$ and every $\delta > 0$ there exist such $s$ that

$$\sum_{k=1}^s |x_{n_k} - a| > K + \delta,$$

we just need to take a big finite part of the initial sum. Then we can select a subsequence $\{x_{m_k}\}$ disjoint with $\{x_{n_k}\}$ such that

$$\sum_{k=1}^s |x_{m_k} - a| < \delta,$$

this can be done since $a$ is a limit point of the sequence. This subsequences satisfy the following inequality: $\sum_{k=1}^s |x_{n_k} - x_{m_k}| > K$.

□

Lemma 3.4.2. If $\Delta(M) < \infty$ then

(1) Either $M$ is finite, or $(x_n)_{n \in M}$ has only one limiting point.
(2) The quantity $\sum_{n \in M} |x_n - a|$ in (3.4.6) attains its minimum in a point $a(M)$. If $M$ is infinite, then there is the only possibility for the $a(M)$ selection: $a(M)$ must be the only limiting point of $(x_n)_{n \in M}$. If $M$ is finite then the role of $a(M)$ can be played by any median of $(x_n)_{n \in M}$, i.e. by arbitrary point $a$ with the following property:

$$|\{n \in M : x_n < a\}| = |\{n \in M : x_n > a\}|.$$  

(3) For every $\varepsilon > 0$ one can select a finite collection of disjoint pairs $n_k, m_k \in M, k = 1, 2, \ldots, s$ for which

$$\sum_{k=1}^{s} |x_{n_k} - x_{m_k}| > \Delta(M) - \varepsilon.$$  

Proof. (1): Statement is obvious.

(2): Assume first that $M$ is infinite. Since $\Delta(M) < \infty$ there is at least one point $a$ such that $\sum_{n \in M} |x_n - a| < \infty$. Then $x_n - a$ tends to 0 along $M$, so $a$ is the only limiting point of $(x_n)_{n \in M}$. The case of finite $M$ is obvious.

(3): First we deal with a finite $M$. In this case we can chose $x_{n_1}$ to be the leftmost element with respect to $a(M)$ and $x_{m_1}$ to be the rightmost, then we define $x_{n_2}$ as the leftmost element from the remaining elements $x_{m_2}$ to be the rightmost and so on. We obtain that

$$\sum_{k=1}^{s} |x_{n_k} - x_{m_k}| = \Delta(M).$$

Suppose now $M$ to be infinite. In this case is we deal just like in Lemma (3.4.1).

Let $G = G_k, k \in \mathbb{N}$ be a disjoint collection of subsets in $\mathbb{R}$. We say that $G$ is an $\varepsilon$-collection, if diameters of all the $G_k$ do not exceed $\varepsilon$. Denote $M_k = \{n \in \mathbb{N} : x_n \in G_k\}; \Delta_G = \sum_{k \in \mathbb{N}} \Delta(M_k)$. Now the proof of the theorem splits into two cases.

3.5. Case 1 (reduction to the case of separated $X$). Below by distance between two sets $A, B$ of real numbers we call $d(A, B) = \inf\{|a - b| : a \in A, b \in B\}$. Infimum of the empty set we define to be $+\infty$, so if at least one of $A, B$ is empty, then $d(A, B) = +\infty$.

Lemma 3.5.1. Let $\{x_n\}$ have the following property: there is an $\varepsilon > 0$ such that $\Delta_G < \infty$ for every $\varepsilon$-collection $G$. Then the series $\sum x_n$ is equivalent to a series $\sum y_n$ with a separated set of elements (like in lemma 3.3.7), so it satisfies the statement of the Main theorem.
Proof. Let $\varepsilon$ satisfy the condition of the Lemma. We are going to cover the set of values $X^+ = X \cap \mathbb{R}^+$ by an $\varepsilon$-collection $G$ of intervals in such a way, that there is an $n_0$ such that for all $n, m > n_0$ all the distances between $G_n$ and $G_m$ are bigger than $\frac{\varepsilon}{4}$. If such $G$ is selected, put $M_k = \{n \in \mathbb{N} : x_n \in G_k\}$ and denote $a_k = a(M_k) \in G_k$, $k \in \mathbb{N}$ the number from Lemma 3.4.2. In such a case the sequence $a_k$ is separated, and we can define the required symmetric sequence $y_n, n \in \mathbb{Z} \setminus \{0\}$ as follows: $y_n = a_k$ for $n \in M_k$. The set of elements of $\sum y_n$ equals $\{a_1, -a_1, a_2, -a_2, \ldots\}$, so it is separated, and the mutual equivalence of $\sum x_n$ and $\sum a_n$ follows from the inequality $\sum_k \varepsilon k - y_k \leq 2\Delta_G < \infty$. So all what we need is to construct a $G$ with the property described above.

Consider covering of $X^+$ by $T_k = X^+ \cap [(k - 1)\varepsilon, k\varepsilon)$ and denote $t_k = \inf T_k$, $t^k = \sup T_k$ if $T_k \neq \emptyset$ and $t_k = t^k = (k - 1/2)\varepsilon$ if $T_k = \emptyset$. Since $T = (T_k)_{k \in \mathbb{N}}$ forms an $\varepsilon$-collection, we have
\[
\sum_k |t^k - t_k| \leq 2\Delta_T < \infty,
\]
so in particular $|t^k - t_k| \to 0$. Select the required $n_0$ in such a way that $|t^k - t_k| < \frac{\varepsilon}{4}$ for all $k > n_0$. For $k \leq n_0$ put $G_k = [(k - 1)\varepsilon, k\varepsilon)$. Before defining $G_k$ for $k > n_0$ let us explain the picture. We would like to take $G_k = [t_k, t^k]$, but this can be a wrong selection, because for some $k$ both $t^k$ and $t_{k+1}$ can be very close to $k\varepsilon$ and $t_{k+1} - t^k$ can be smaller than $\frac{\varepsilon}{4}$. But for such “bad” values of $k$ the segment $[t_k, t^{k+1}]$ is of the length at most $\frac{\varepsilon}{4}$, covers both the segments $[t_k, t^k]$ and $[t_{k+1}, t^{k+1}]$, and has at least distance $\frac{\varepsilon}{4}$ from the rest of $[t_j, t^j]$. So the required selection of $G_k$ for $k > n_0$ can be done as follows: take all those segments $[t_j, t^j]$, $j > n_0$, which are far from the others (i.e. the distances to the others are bigger than $\frac{\varepsilon}{4}$), and add all those segments $[t_j, t^{j+1}], j > n_0$, where $t_{j+1} - t^j < \frac{\varepsilon}{4}$.

3.6. The remaining case.

Lemma 3.6.1. Let $\{x_n\}$ have the the opposite to the case 1 property: for every $\varepsilon > 0$ there is an $\varepsilon$-collection $G$ such that $\Delta_G = \infty$. Then one can select a collection of disjoint pairs $n_k, m_k \in \mathbb{N}, k = 1, 2, \ldots$ such that $|x_{n_k} - x_{m_k}| \to 0$ as $k \to \infty$ and
\[
(3.6.8) \quad \sum_{k=1}^{\infty} |x_{n_k} - x_{m_k}| = \infty.
\]
In this case $SR_2(\sum x_k) = \mathbb{R}$, which satisfies the statement of the Main theorem.
Proof. For \( \varepsilon = 1 \) we can find an \( \varepsilon \)-collection \( G \) such that \( \Delta_G = \infty \).

Then applying (3) of Lemma 3.4.2 one can select a collection of disjoint pairs \( n_k, m_k \in \mathbb{N}, k = 1, 2, \ldots, n_1 \) such that \( |x_{n_k} - x_{m_k}| < 1 \) and \( \sum_{k=1}^{n_1} |x_{n_k} - x_{m_k}| > 1 \). Then for \( \varepsilon = 1/2 \) we select disjoint pairs \( n_k, m_k \in \mathbb{N}, k = n_1 + 1, \ldots, n_2 \) such that \( |x_{n_k} - x_{m_k}| < 1/2 \) and \( \sum_{k=1}^{n_2} |x_{n_k} - x_{m_k}| > 2 \). We can proceed to select the desirable sequence of pairs. To see that \( SR_2(\sum x_k) = \mathbb{R} \) we consider the pairs

\[
(x_{n_1} - x_{m_1}), (x_{m_1} - x_{n_1}), (x_{n_2} - x_{m_2}), (x_{m_2} - x_{n_2}), \ldots
\]

We add missing pairs of the form \( x_i - x_i \) to include all the elements into the series. Permuting pairs (like in the Riemann rearrangement theorem) we obtain \( SR_2(\sum x_k) = \mathbb{R} \). \( \square \)

3.7. Examples. To complete the paper we are going to demonstrate that for each of the cases (1) – (3) of the main theorem 3.1.3 there exists a series satisfying it. To write down such examples let us introduce more compact way of writing the series

\[
(Y, N) \overset{\text{def}}{=} \{(y_i, n_i) \mid y_i \in \mathbb{R}, n_i \in \mathbb{N} \cup \{\infty\}\}
\]

where \( n_i \) corresponds to the number of copies of \( y_i \) we have in the simplified series \( (n_i \) is the order of element \( y_i \)) and the following condition is satisfied

\[
y_i \neq y_j \ (\forall i \neq j).
\]

Say,

\[
SR_2((1, \infty), (-1, \infty)) = SR_2(1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots)
\]

General example: for any \( \varepsilon \)-separated set \( E = \{e_i\}_{i=1}^{\infty} \) and \( G = \{(e_i, \infty)\} \) we have

\[
SR_2(G) = \{c_1 e_1 + \cdots + c_l e_l \mid c_i \in E, c_i \in \mathbb{Z}, \sum_{k=1}^{l} c_k \text{ is even}\}.
\]

In particular consider the following two examples.

Example 3.7.1. \( G_1 = \{(1, \infty), (-1, \infty)\} \)

Here we get that \( SR_2(G_1) = \{2\mathbb{Z}\} \). Notice that \( 2\mathbb{Z} \) is 2-separated.

Example 3.7.2. \( G_2 = \{(1, \infty), (-1, \infty), (\sqrt{2}, \infty), (-\sqrt{2}, \infty)\} \)

Applying Lemma (3.3.1) here we get that \( SR_2(G_2) = \{a \cdot 1 + b \cdot \sqrt{2}\} \), where \( (a + b) \) is even. It’s obvious that \( SR_2(G_2) \) is dense in \( \mathbb{R} \).

Example 3.7.3. \( G_3 = \) any conditionally convergent series (in the usual sense). Obvious that it gives us \( SR_2(G_3) = \mathbb{R} \).
Example 3.7.4. $\mathcal{S}_4 = \{ (S_n, 1), (-S_n, 1) \mid n \in \mathbb{N} \}$, where $S_n = \sum_{i=1}^{n} \frac{1}{i}$.

This series too gives us $SR_2(\mathcal{S}_4) = \mathbb{R}$.

Example 3.7.5. Let $\mathcal{S}_5$ be an arbitrary unconditionally convergent series to $a \in \mathbb{R}$ (in usual sense). This series gives us $SR_2(\mathcal{S}_5) = \{a\}$.

Remark 3.7.6. If one considers convergence in $\overline{\mathbb{R}}$ the main theorem must be modified as follows:

Theorem 3.7.7. Let $\lim_{n \to \infty} \sum_{k=1}^{2n} x_k = a \in \mathbb{R}$. Then $SR_2(\sum x_k)$ is one of the following:

1. Shifted additive subgroup of the form
   
   $a + \{c_1 z_1 + \cdots + c_l z_l \mid z_k \in E, \ c_i \in \mathbb{Z}, \ \sum_{k=1}^{l} c_k \text{ is even} \} \cup \{-\infty, \infty\}$,

   where $E$ of is an $\varepsilon$-separated set;

2. The whole $\overline{\mathbb{R}}$;

3. The only number $a$;

4. The set $\{-\infty, a, \infty\}$.

Acknowledgment. The authors are grateful to V. Kadets for his support, fruitful communications, and help in preparing this text. We would also like to thank Professor Eve Oja and Professor Toivo Leiger from Tartu University for providing us references [1] and [5].

References

[1] Bagemihl, F. ; Erdős, P. Rearrangements of $C_1$-summable series. Acta Math. 92, (1954). 35-53

[2] Connor J., Ganichev M., Kadets V. A Characterization of Banach Spaces with Separable Duals via Weak Statistical Convergence. / Journal of Mathematical Analysis and Applications 244 (2000), 251 - 261.

[3] Hadwiger, H. Über das Umordnungsproblem im Hilbertschen Raum / Math. Zeitschrift 46 (1940), 70 - 79.

[4] Kadets, M. ; Kadets, V. Series in Banach spaces : conditional and unconditional convergence, Bazel: Birkhauser, (1997) (Operator theory advances and applications; Vol.94). 36

[5] Lorentz, G. G. ; Zeller, K. Series rearrangements and analytic sets. Acta Math. 100 (1958) 149–169.
(Y. Dybskiy) Department of Mathematics
Kharkov National University
4, Svobody sq. Kharkov, UKRAINE
E-mail address: dybskiy@gmail.com

(K. Slutsky) Department of Mathematics
Kharkov National University
4, Svobody sq. Kharkov, UKRAINE
E-mail address: kslutsky@gmail.com