On the Equivalence of Two Non-Riemannian Curvatures in Warped Product Finsler Metrics

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Abstract In this paper we discuss the Busemann-Hausdorff volume form and Holmes-Thompson volume form for the warped product Finsler metrics. With the help of these volume forms we obtain the $E$-curvature and the $S$-curvature for this class of metrics. Further, we show that the notion of isotropic $E$-curvature and isotropic $S$-curvature are equivalent for this class of metrics.

Keywords Finsler warped product metrics · Finsler volume form · Isotropic $E$-curvature · Isotropic $S$-curvature

Significance of the paper The geometric quantities follow certain rule under change of transformation from one coordinate system to other. However, sometimes the components of a geometric quantity in a special coordinate system are more suitable for computation purpose. For instance, in the study of planetary motion the polar coordinate system is more suited than the Euclidean one. The warped product metric is a generalization of metric in polar coordinates and it was introduced by Bishop and O’Neill to construct a class of complete Riemannian manifolds of negative curvature. The notion of warped products plays important roles not only in geometry but also in mathematical physics, especially in general relativity. In present paper, the warped product metric has been studied in a more general setting than the Riemannian one and proved equivalency of two important non-Riemannian curvatures.

1 Introduction

The warped product Riemannian manifolds, introduced by Bishop and O’Neill [1], are the natural and fruitful generalization of Riemannian products of two manifolds. The notion of warped products plays important roles not only in geometry but also in mathematical physics, especially in general relativity. In fact, many basic solutions of the Einstein field equations, including the Schwarzschild and the Robertson-Walker models, are warped product metrics [2]. The famous Nash’s embedding theorem implies that every warped product Riemannian manifold can be realized as a warped product submanifold in a suitable Euclidean space [3, 4].

Later on, the warped product metric was extended to the case of Finsler manifolds by the work of Chen et al. and Kozma et al. [5, 6]. Recently, some significant progress has been made in the study of Finsler warped product metrics [5, 7]. It has been observed in [5] that spherically symmetric Finsler metrics have warped product structure. However, there are lot of Finsler warped product metrics that are not spherically symmetric [8–11]. Mo et al. [12] obtain the differential equation that characterizes the spherically symmetric Finsler metrics with vanishing Douglas curvature and obtain all the spherically symmetric Douglas metrics by solving this equation.

A Finsler metric on a smooth manifold is a smoothly varying family of Minkowski norms, one on each tangent space, rather than a family of inner products one on each tangent space, as in the case of Riemannian metrics. It turns out that every Finsler metric induces an inner product in each direction of a tangent space at each point of the...
manifold. However, in sharp contrast to the Riemannian case, these Finsler-inner products do not only depend on where we are, but also in which direction we are looking.

In this paper we consider the warped product Finsler metrics on the manifold $M = I \times \hat{M}$, which is a simple generalization of the Riemannian version, where $I$ is an open interval of $\mathbb{R}$ and $(\hat{M}, \tilde{a})$ is an $(n - 1)$ dimensional Riemannian manifold.

There are several non-Riemannian quantities in the Finsler literature, for instance, Cartan torsion, $S$-curvature, $\Xi$-curvature, $H$-curvature etc. These quantities become zero for a Riemannian metric. The concept of $S$-curvature was introduced by Shen [13] and it becomes a Riemannian metric for the other classes of Finsler metrics such as spherically symmetric Finsler metrics. The equivalence of isotropic $S$-curvature and isotropic $E$-curvature has been proved under certain condition for a general $(\alpha, \beta)$ metrics [8]. In the present paper we prove the following result:

**Theorem 1** A warped product Finsler metric $F = \tilde{a}\phi(r, s)$, defined in Eq. (2), has isotropic $S$-curvature if and only if it has isotropic $E$-curvature.

**Corollary 1** Any warped product Finsler metric $F = \tilde{a}\phi(r, s)$ has constant $S$-curvature if and only if it has constant $E$-curvature.

## 2 Preliminaries

Let $M$ be an $n$-dimensional smooth manifold. $T\mu M$ denotes the tangent space of $M$ at $u$. The tangent bundle of $M$ is the disjoint union of tangent spaces $TM := \bigcup_{u \in M} T_u M$. We denote the elements of $TM$ by $(u, v)$, where $v \in T_u M$ and $TM_0 := TM \setminus \{0\}$.

**Definition 1** [17] A Finsler metric on $M$ is a function $F : TM \to [0, \infty)$ satisfying the following conditions:

1. $F$ is smooth on $TM_0$.
2. $F$ is a positively 1-homogeneous on the fibers of tangent bundle $TM$.
3. The Hessian of $\frac{F^2}{2}$ with element $g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial u^i \partial u^j}$ is positive definite on $TM_0$.

The pair $(M, F)$ is called a Finsler space and $g_{ij}$ is called the fundamental tensor.

Let us consider the product manifold $M = I \times \hat{M}$, where $I$ is an interval of $\mathbb{R}$ and $(\hat{M}, \tilde{a})$ is an $(n - 1)$ dimensional Riemannian manifold with $\tilde{a}^2 = a_{ab}(u) \nu^a \nu^b$. Let $\{\theta^a\}_{a=2}^n$ be a local coordinate system on $M$. Then $\{u^a\}_{a=1}^n$ gives a local coordinate on $M$ by setting $u^1 = r$ and $u^a = \theta^a$. The indices $i, j, k, ...$ are ranging from 1 to $n$ while $a, b, c, ...$ are ranging from 2 to $n$. A vector $v$ on $M$ can be written as $v = v^a \partial / \partial u^a$ and its projection on $M$ is denoted by $\tilde{v} = v^a \partial / \partial \theta^a$. A warped product Finsler metric can be written in the form

$$F = \tilde{a} \sqrt{w(s, r)},$$

where $w$ is a suitable function defined on an open subset of $\mathbb{R}^2$ and $s = v^1 / \tilde{a}$. It can be rewritten as

$$F = \tilde{a} \phi(s, r),$$

where $\phi(s, r) = \sqrt{w(s, r)}$.

Shen et al. [5] also showed that the warped product Finsler metrics contain the class of spherically symmetric Finsler metrics.

The coefficients of fundamental metric tensor of the warped product Finsler metrics are given by

$$\begin{pmatrix}
  g_{11} & g_{1b} \\
  g_{a1} & g_{ab}
\end{pmatrix} = \begin{pmatrix}
  \frac{1}{2} w_{ss} & \frac{1}{2} \chi S \tilde{a} \tilde{v} \\
  \frac{1}{2} \chi S \tilde{a} \tilde{v} & \frac{1}{2} \chi S \tilde{a} \tilde{v} - \frac{1}{2} \chi S \tilde{a} \tilde{v} - \frac{1}{2} \chi S \tilde{a} \tilde{v}
\end{pmatrix},$$

where $\chi := 2w - sw_s$ and $\chi_S := w_s - sw_{ss}$. By some simple calculation we have

$$\det(g_{ij}) = \frac{1}{2} \chi\phi^{n-2} A,$$

where $A = 2w w_s - w_{ss}$. It can also be rewritten as

$$\det(g_{ij}) = \phi^{n+1} \phi_s (\phi - s \phi) \chi^{n-2}.$$  \hspace{1cm} (3)

The spray coefficients of the Finsler metric $F$ are defined by

$$G^i = \frac{1}{4} g^{ij} \left\{ [F^2]_{a^j} v^m - [F^2]_{a^j} \right\}$$

$$\begin{pmatrix}
  G^1 & G^2 \\
  G^3 & \Psi \tilde{a} \tilde{v}
\end{pmatrix}$$

where
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\( \Phi = \frac{1}{4} \left\{ (W, -\chi) U + s\chi V, \right\} \)
\( \Psi = \frac{1}{4} \left\{ (W, -\chi) V + s\chi (W + X) \right\} \)

and
\( \chi = 2w - sw, \quad A = 2ww_{ss} - w^2, \quad U = \frac{2\chi - 2sw}{A}, \)
\( V = \frac{-2\chi}{A}, \quad W = \frac{2}{A}, \quad X = \frac{2w\chi}{\chi A}. \) (7)

In Finsler geometry two volume forms are well known, namely Busemann-Hausdorff volume form \([13, 17]\) and Holmes-Thompson volume form. In a local coordinate system \((u', v')\), the Busemann-Hausdorff volume form is defined as
\( dV_{BH} = \sigma_{BH}(u) du, \)
where
\( \sigma_{BH}(u) = \frac{Vol(B^n(1))}{Vol\left\{ (v') \in \mathbb{R}^n : F(u, v', \frac{\partial}{\partial u'}) < 1 \right\}}. \) (8)

In a local coordinate system \((u', v')\), the Holmes-Thompson volume form is defined as
\( dV_{HT} = \sigma_{HT}(u) du, \)
where
\( \sigma_{HT}(u) = \frac{1}{Vol(B^n(1))} \int_{F(u, v', \frac{\partial}{\partial u'}) < 1} \det(g_{ij}(u, v)) dv). \) (9)

**Definition 2** For a vector \( v \in T_u M \setminus \{0\} \), let \( \gamma = \gamma(t) \) be the geodesic with \( \gamma(0) = u \) and \( \gamma(0) = v \). The \( S \)-curvature of the Finsler metric \( F \) with volume form \( dV = \sigma_{F}(u) du \) is defined by
\( S(u, v) = \frac{d}{dt} |_{t=0} [\gamma_{F}(\gamma(t), \gamma(t))]. \)
where \( \gamma_{F} \) is called distortion of the Finsler metric \( F \) and defined by
\( \gamma_{F} = \log \frac{\sqrt{\det(g_{ij})}}{\sigma_{F}}. \)

**Proposition 1** \([17]\) In a standard local coordinate system in \( TM \), the \( S \)-curvature of a Finsler metric \( F \) can also be written as
\( S = \frac{\partial G}{\partial v^m}(u, v) - v^m \frac{\partial}{\partial u^m} \left( \log \sigma_{F}(u) \right). \) (10)

**Definition 3** The \( E \)-curvature of a Finsler metric \( F \) is defined as
\( E_{ij} := \frac{1}{2} S'_{ij}(u, v) = \frac{\partial^2}{\partial v^i \partial v^j} \left( \frac{\partial G}{\partial v^m} \right), \) (11)

where \( G' \) are the spray coefficients of the Finsler metric \( F \).

### 3 Isotropic \( E \)-curvature and Isotropic \( S \)-curvature of the Warped Product Finsler Metrics

**Definition 4** A Finsler metric \( F \) is said to be of isotropic \( S \)-curvature if
\( S = (n + 1)c(u)F, \) (12)
where \( c(u) \) is scalar function on \( M \).

\( F \) is said to be of constant \( S \)-curvature if \( c(u) \) is constant.

**Definition 5** The Finsler metric \( F \) is said to be of isotropic \( E \)-curvature if there exists a scalar function \( c = c(u) \) on \( M \) such that
\( E_{ij} = \frac{n + 1}{2} c(u)F_{ij}. \) (13)

If \( c(u) \) is a constant, then \( F \) is said to be of constant \( E \)-curvature.

Before proving the results we need the followings:
\( \frac{\partial \tilde{a}^2}{\partial v^1} = 0, \quad \frac{\partial \tilde{a}^2}{\partial v^2} = 2\tilde{a}^2, \) (14)
\( s_{ij} = \frac{1}{\tilde{a}}, \quad s_{i1}v^1 = s, \quad s_{iv} = -\frac{s_{iv}}{\tilde{a}^2}, \quad s_{iv}v^0 = -s, \) (15)
\( \bar{a}_{i1} = 0, \quad \bar{a}_{i1} = \frac{\bar{a}^2a_{ab} - v_a v_b}{\bar{a}^2}, \quad \tilde{a}_{iv}v^0 = -\frac{v_{iv}}{\bar{a}^2}, \) (16)
\( s_{i1}v^1 = \frac{3sv_a v_b - s\bar{a}^2a_{ab}}{\bar{a}^4}. \)

**Proposition 2** Let \( F = \tilde{a}\phi(r, s) \) be a warped product Finsler metric on an \( n \)-dimensional manifold \( M = I \times \bar{M} \). Then the \( E \)-curvature of \( F \) is given by
\( E_{11} = \frac{1}{\tilde{a}} \left( \epsilon_{sss} + (n - 2)\psi_{ss} - s^2\psi_{ss} \right), \) (17)
\( E_{1i} = E_{i1} = \frac{1}{\tilde{a}^3} \left( \epsilon_{sss} + (n - 2)\psi_{ss} - s^2\psi_{ss} \right) v^i v^a, \) (18)
\( E_{ab} = \frac{1}{\tilde{a}} \left[ (\psi_{ss} - s\psi_{ss}) + (n - 1)s^2\psi_{ss} + s^2\Phi_{ss} \right] \)
\( + \frac{1}{\tilde{a}} \left[ s^3\psi_{sss} + (n - 1)^2s^2\psi_{ss} + s^2\Phi_{ss} \right] \)
\( + s\Phi_{ss} + (n - 2)\psi_{s} - (\psi_{s} + n\psi) \) (19)

**Proof** From Eq. (5) we have
\[ \frac{\partial G^i}{\partial \nu^i} = \tilde{\alpha} \Phi_i \]  
(20)

and

\[ \frac{\partial}{\partial \nu^i}(\Psi \tilde{\alpha} \nu^j) = \Psi_{x^\nu \nu} \tilde{\alpha} \nu^j + \Psi \tilde{\alpha} \nu^j \tilde{\alpha} \nu^j. \]  
(21)

Now putting \( a = b \) in Eq. (21) and taking summation over \( a \) we get

\[ \frac{\partial}{\partial \nu^i}(\Psi \tilde{\alpha} \nu^j) = \Psi_{x^\nu \nu} \tilde{\alpha} \nu^j + \Psi \tilde{\alpha} \nu^j \tilde{\alpha} \nu^j + (n-1)\Psi \tilde{\alpha} = -\tilde{\alpha} s \Psi_s + n \Psi \tilde{\alpha} \]

and therefore, we have

\[ \sum \frac{\partial G^m}{\partial \nu^m} = \tilde{\alpha} \Phi + n \Psi - s \Psi_s. \]  
(22)

Differentiating Eq. (22) with respect to \( \nu^i \) gives

\[ \frac{\partial}{\partial \nu^i} \left( \frac{\partial G^m}{\partial \nu^m} \right) = \tilde{\alpha}_i [\Phi_s + n \Psi - s \Psi_s] + \tilde{\alpha}_i [\Phi_s + (n-1)\Psi_s - s \Psi_s]. \]  
(23)

Again differentiating Eq. (23) with respect to \( \nu^i \) gives

\[ \frac{\partial}{\partial \nu^i} \frac{\partial}{\partial \nu^j} \left( \frac{\partial G^m}{\partial \nu^m} \right) = \left[ \tilde{\alpha}_{ij} [\Phi_s + n \Psi - s \Psi_s] + \tilde{\alpha}_{ij} [\Phi_s + (n-1)\Psi_s - s \Psi_s] \right] \]  
(24)

\[ + \tilde{\alpha}_s \tilde{\alpha}_{ij} \left[ \Phi_s + (n-2)\Psi_s - s \Psi_s \right] \]  
(25)

Putting \( i = j = 1 \) and using Eqs. (14), (15), (16)

\[ E_{11} = \frac{1}{\tilde{\alpha}} [\Phi_{xs} + (n-2)\Psi_{xs} - s \Psi_{xs}]. \]  
(26)

We also have,

\[ E_{1a} = E_{a1} = \frac{1}{\alpha^2} \left[ \Phi_{xs} + (n-2)\Psi_{xs} - s \Psi_{xs} \right] v_1 v_a \]  
(27)

and

\[ E_{ab} = \frac{\alpha_{ab}}{\tilde{\alpha}} \left[ (\Phi_s - s \Phi_{xs}) + (n-2)\Psi_s + n \Psi + s \Psi_{xs} \right] \]  
(28)

\[ + \frac{v_a v_b}{\tilde{\alpha}} \left[ s^3 \Psi_{xs} + (n-1)s^2 \Phi_{xs} + s^2 \Phi_{xs} + s \Phi_{xs} \right] \]  
(29)

\[ +(n-2)\Psi_s - (\Phi_s + n \Psi) \].

where \( \Phi, \Psi \) are given in Eq. (6) and \( \kappa \neq 0 \) is a scalar function on \( M \).

**Proof** Differentiating Eq. (2) with respect to \( \nu^i \), we have

\[ F_{\nu^i} = \tilde{\alpha}_i \phi + \tilde{\alpha}_i \phi_s. \]  
(30)

Differentiating again Eq. (29) with respect to \( \nu^i \) yields

\[ F_{\nu^i \nu^i} = \tilde{\alpha}_{i} \phi + \tilde{\alpha}_{i} \phi_s + \tilde{\alpha}_{i} \phi_s \phi_0 + \tilde{\alpha}_{i} \phi_s \phi_0 + \tilde{\alpha}_{i} \phi_s \phi_0. \]  
(31)

More, explicitly we have,

\[ F_{\nu^i \nu^i} = \frac{1}{\tilde{\alpha}^2} \left[ (\phi - s \phi_s) \tilde{\alpha}^2 a_{ab} - (\phi - s \phi_s - s^2 \phi_s) v_a v_b \right]. \]  
(32)

In the view of Eq. (13), (17) and (19) the warped product Finsler metric is of isotropic \( E \)-curvature if and only if we have,

\[ \Phi_{xs} + (n-2)\Psi_{xs} - s \Psi_{xs} = \frac{n+1}{2} c(x) \phi_s, \]  
(33)

\[ (\Phi_s - s \phi_s) + (n-2)\Psi_s + s^2 \Psi_{ss} = \frac{n+1}{2} c(x) (\phi - s \phi_s), \]  
(34)

\[ s \Psi_{ss} + (n-1)s^2 \Phi_{ss} + s^2 \Phi_{ss} \]  
(35)

\[ + s \Phi_{ss} + (n-2)\Psi_s - (\Phi_s + n \Psi) \]  
(36)

Now we will show that Eqs. (34) and (36) can be obtained from Eq. (35). Differentiating Eq. (35) with respect to \( s \) we get Eq. (34). Again multiplying Eq. (34) by \( s^2 \) and substituting it from Eq. (35) we obtain Eq. (36). Therefore, \( F \) has isotropic \( E \)-curvature if and only if Eq. (28) holds.

**Corollary 2** The warped product Finsler metric \( F = \tilde{\alpha} \phi(r, s) \) has isotropic \( E \)-curvature if and only if

\[ \Phi_{s} - s \Phi_{ss} \]  
(37)

where \( \Phi, \Psi \) are given in Eq. (6) and \( \kappa \) is a nonzero constant.

**Proof** Taking \( \kappa(s) \) as a constant in the previous theorem the proof is immediate.
Lemma 2  Let $F = \tilde{a}\phi(s, r)$, be a warped product Finsler metric on an $n$-dimensional manifold $M = I \times M$. Then the volume form $dV$ is given by $dV = k(r)dV_\tilde{a}$, where

$$k(r) = \begin{cases} \frac{1}{\int (1-s^2)^{n/2} ds} & \text{if, } dV = dV_{BH} \\ \frac{1}{\int \sin^{n-2}(t)dt} & \text{if, } dV = dV_{HT} \end{cases}$$

and $dV_\tilde{a} = \sqrt{\det(a_{ij})}du$ denotes the Riemannian volume form of $\tilde{a}$.

Proof  Fix an arbitrary point $u_0 \in U \subset \mathbb{R}^n$, in a suitable coordinate chart of $M$, consider an orthogonal basis at $u_0$ with respect to the Riemannian metric $\alpha$ so that

$$\alpha = \sqrt{\sum_{i=1}^{n} (\nu^i)^2}.$$  

Then the volume form $dV_\tilde{a}$ at the point $u_0$ is given by $dV_\tilde{a} = \sigma_a du$, where

$$\sigma_a = \sqrt{\det(a_{ij})} = 1.$$  

Let us consider the coordinate transformation:

$$\mu : (s, z^a) \rightarrow (\nu^i)$$

such that

$$\nu^1 = \frac{s}{\sqrt{1-s^2}} \tilde{a} \quad \text{and} \quad \nu^a = z^a,$$

where $\tilde{a} = \sqrt{\sum_{a=2}^{n} (z^a)^2}$.  

Then $\alpha = \frac{1}{\sqrt{1-s^2}} \tilde{a}$.  

Therefore,

$$F = a\phi(r, s) = \frac{\tilde{a}\phi}{\sqrt{1-s^2}}$$

and the Jacobian of the transformation $\mu : (s, z^a) \rightarrow (\nu^i)$ is given by $\frac{1}{(1-s^2)^{3/2}}$.  

Then

$$\text{Vol}\left\{ (\nu^i) \in \mathbb{R}^n : F(u, \nu^i \frac{\partial}{\partial \nu^i}) < 1 \right\}$$

$$= \int_{F(u, \nu^i) < 1} dv$$

$$= \int_{\tilde{a}\phi(r, s) < 1} dv$$

$$= \int_{\phi(r, s) < 1} (1-s^2)^{3/2} \tilde{a} ds$$

$$= \frac{1}{n} \text{Vol}(S^{n-2}) \int_{-1}^{1} (1-s^2)^{-1/2} ds$$

$$= \frac{1}{n} \text{Vol}(S^{n-2}) \int_{-1}^{1} (1-s^2)^{-1/2} \phi(r, s) ds$$

$$= \frac{1}{n} \text{Vol}(S^{n-2}) k(r),$$

where

$$k(r) = \int_{-1}^{1} (1-s^2)^{-1/2} \phi(r, s) ds.$$

Since, $\text{Vol}(B^n(1)) = \frac{1}{n} \text{Vol}(S^{n-2}) \int_{0}^{\pi} \sin^{n-2}(t) dt$, using Eq. (8) we have

$$\sigma_{BH} = \frac{1}{\int_{-1}^{1} (1-s^2)^{-1/2} ds}.$$

Let us consider $v = \phi \phi_\nu (\phi - s \phi_\nu)^{n-2}$. Hence, from Eqs. (9) and (3) we have,

$$\int_{F(u, v) < 1} \det(g_{ij}(u, v)) dv$$

$$= \int_{F(u, v) < 1} \phi^2(r, s) v(r, s) ds$$

$$= \frac{1}{n} \text{Vol}(S^{n-2}) \int_{-1}^{1} (1-s^2)^{(n-3)/2} v(r, s) ds$$

$$= \frac{1}{n} \text{Vol}(S^{n-2}) \int_{0}^{\pi} (\sin^{n-2}(t)) v(r^2, r \cos t) dt.$$  

Therefore,
\( \sigma_{HT} = \int_{0}^{\pi} (\sin^{n-2} t) v, r \cos t \, dt = \int_{0}^{\pi} \frac{\pi}{\sin^{n-2} t} \, dt. \) 

(41)

Hence we have the result.

**Theorem 3** A warped product Finsler metric \( F = \tilde{\alpha} \Phi(r, s) \) on a manifold \( M = I \times M \) has isotropic \( S \)-curvature with respect to volume form \( dV_{BH} \) or \( dV_{HT} \) if and only if

\[ \Phi_s + n \Psi - s \Psi_s + g(r)s = (n + 1)c(u), \] 

(42)

where \( c(u) \) is a scalar function on \( M \) and \( g(r) = -r k'(r) \). 

**Proof** Since, \( \tilde{G}^k = \frac{1}{2} \tilde{\Gamma}^k_{ij} \psi^i \psi^j \), we have

\[ [\tilde{G}^k]_{;i} = \tilde{\Gamma}^k_{ij} \tilde{\psi}^j = v^m \frac{\partial}{\partial u^m} (\log \sigma_a). \] 

(43)

By Lemma 2, we have \( dV = \sigma du = K(r) \sigma_a du \). Hence,

\[ \frac{v^m}{u^m} \frac{\partial}{\partial u^m} (\log \sigma_a) = \frac{k'(r)}{k(r)} v^m \frac{\partial r}{\partial u^m} + v^m \frac{\partial}{\partial u^m} (\log \sigma_a). \] 

(44)

The \( S \)-curvature of a warped product Finsler metric \( F = \tilde{\alpha} \Phi(r, s) \) is given by

\[ S = \frac{v^m}{u^m} \frac{\partial}{\partial u^m} (\log \sigma_a) + \tilde{\alpha} \left[ \Phi_s + n \Psi - s \Psi_s + g(r)s \right] - \frac{v^m}{u^m} \frac{\partial}{\partial u^m} (\log \sigma_a). \] 

(45)

Again

\[ \frac{v^m}{u^m} \frac{\partial}{\partial u^m} (\log \sigma_a) = \tilde{\alpha} sr. \] 

(46)

Now, from Eqs. (10), (45) and (46) we obtain

\[ S = \tilde{\alpha} \left[ \Phi_s + n \Psi - s \Psi_s + g(r)s \right], \] 

(47)

where

\[ g(r) = -r k'(r) \] 

(48)

Therefore, \( F \) has isotropic \( S \)-curvature if and only if Eq. (42) holds.

**Corollary 3** A warped product Finsler metric \( F = \tilde{\alpha} \Phi(r, s) \) on a manifold \( M = I \times M \) has constant \( S \)-curvature with respect to volume form \( dV_{BH} \) or \( dV_{HT} \) if and only if

\[ \Phi_s + n \Psi - s \Psi_s + g(r)s = c(n + 1)c(u), \] 

(49)

where \( c \) is a nonzero constant and \( g(r) = -\frac{k'(r)}{k(r)} \).

**Proof** Taking \( c(u) \) as a constant in the previous theorem the proof is immediate.

\[ \square \]

\[ \square \]

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