A CONSEQUENCE OF LITTLEWOOD’S CONDITIONAL ESTIMATES FOR THE RIEMANN ZETA-FUNCTION
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Assuming the Riemann hypothesis (RH) and using Littlewood’s conditional estimates for the Riemann zeta-function, we provide an estimate related to an approach of Y. Motohashi to the zero-free region.

Key words: Riemann zeta-function, Riemann hypothesis.

1. Introduction. The approach of Y. Motohashi [1] to the zero-free region of the Riemann zeta-function extended by the author in [2] may be modified to give regions free of large values of some products, which contain finite products \( \prod_j \zeta(s_j) \). On the Riemann hypothesis, one can obtain upper bounds for such products for \( s_j = 1 + t_j \) using the method of Littlewood. To prove our result on regions free of large values we also use an \( \Omega \)-theorem for \( \prod_j \frac{1}{(\zeta(s_j))} \), where \( s_j = \sigma_j + it_j + h_j \) with \( h_j \) lying in short intervals around \( t_j \) and \( \sigma_j \geq 1 \). The \( \Omega \)-theorem depends on a version of Kronecker’s theorem with an explicit upper bound.

2. Lemmas.

Lemma 1. On the Riemann hypothesis, uniformly for \( \frac{1}{2} < \sigma_0 \leq \sigma \leq \frac{5}{8} \) and \( t \geq e^{2T} \) we have

\[
\log \zeta(s) \ll \begin{cases} 
\log \frac{1}{\sigma - 1} & \text{if } 1 + \frac{1}{\log \log t} \leq \sigma \leq \frac{5}{8}, \\
(\log 1) \frac{1}{(1-\sigma) \log \log t} + \log \log t & \text{if } \sigma_0 \leq \sigma \leq 1 + \frac{1}{\log \log t},
\end{cases}
\]

and for \( \sigma > 1 - \frac{E}{\log \log t}, E > 0 \) fixed,

\[
\zeta(s) \ll e^{L(2^{\sigma})E} (\log \log t),
\]

where \( L = L(t) = \log \log \log \log t \) and the implied constant in the \( \ll \) depends on \( E \).

For the first estimate, see [3], Chapter XIV, §14.33. The second estimate is similar to the first and is obtained along the lines of [3], Chapter XIV, §14.9. For a more precise estimate, see [4].

Lemma 2. For \( \alpha \leq \sigma \leq \beta \) and \( t > 1 \) we have

\[
\Gamma(\sigma + it) = t^{\sigma-it-1/2} \exp \left( -\frac{\pi}{2} t - it + \frac{\pi}{2} \left( \sigma - \frac{1}{2} \right) \right) \sqrt{2\pi} \left( 1 + O \left( \frac{1}{t} \right) \right),
\]

with the constant in the big-O depending only on \( \alpha \) and \( \beta \).

For the proof, see e.g. [5], Appendix, §3.

Lemma 3. Let \( \sigma_a(n), a \in \mathbb{C} \), be the sum of \( a \)th powers of the divisors of \( n \). Let \( \xi(d) \) be an arbitrary bounded arithmetical function with the support in the set of square-free integers. Then for \( \sigma > 1, T_1, T_2 \in \mathbb{R} \) we have the identity

\[
\sum_{n=1}^{\infty} \sigma_{it_1}(n) \sigma_{-it_2}(n) \left( \sum_{d|n} \xi(d) \right) n^{-s} = \frac{\zeta(s) \zeta(s- iT_1) \zeta(s+ iT_2) \zeta(s- i(T_1- T_2))}{\zeta(2s- i(T_1- T_2))} \left( \xi(1) + \sum_{d=2}^{\infty} \xi(d) \mathcal{P}_d(s, T_1, T_2) \right),
\]
Changing the order of summation, we have

\[
P_d(s, T_1, T_2) = \prod_{p \mid d} \left( 1 - \left(1 - \frac{1}{p^s} \right) \left( 1 - \frac{1}{p^{s-iT_1}} \right) \left( 1 - \frac{1}{p^{s+iT_2}} \right) \left( 1 - \frac{1}{p^{s-\mu(T_1-T_2)}} \right) \left( 1 - \frac{1}{p^{2s-\mu(T_1-T_2)}} \right)^{-1} \right).
\]

**Proof.** This is a version of Lemma 3 of Y. Motohashi [1]. Let

\[
Z = \frac{\zeta(s) \zeta(s-iT_1) \zeta(s+iT_2) \zeta(s-i(T_1-T_2))}{\zeta(2s-i(T_1-T_2))}.
\]

Changing the order of summation, we have

\[
\sum_{n=1}^{\infty} \sigma_{iT_1}(n) \sigma_{iT_2}(n) \left( \sum_{d \mid n} \xi(d) \right) n^{-s} = \xi(1) Z + \sum_{d \geq 2, d \text{ square-free}} \xi(d) \left( \sum_{k=1}^{\infty} \sigma_{iT_1}(kp_{d_1} \cdots p_{d_r}) \sigma_{iT_2}(kp_{d_1} \cdots p_{d_r}) k^s p_{d_1}^s \cdots p_{d_r}^s \right).
\]

By an identity of Ramanujan—Wilson [3], (1.3.3),

\[
\prod_{p \mid d} \frac{(1+p^{iT_1})(1+p^{-iT_2})}{p^s} + \frac{(1-p^{iT_1})(1-p^{-iT_2})}{p^s} + \ldots = \prod_{p \mid d} \frac{(1-p^{iT_1})(1-p^{-iT_2})}{p^s} + \frac{(1-p^{iT_2})(1-p^{-iT_1})}{p^s} + \ldots.
\]

This obviously ends the proof of the lemma.

**Lemma 4.** Assume the truth of the Riemann hypothesis. Fix \( E > 0 \). Let

\[
\exp(A \log \log T \log \log \log T) \leq N \leq \exp(DA \log \log T \log \log \log T), \quad T \geq e^{27},
\]

with \( A = \frac{18+\varepsilon}{E} \) and a sufficiently large positive constant \( D \), and let us put \( T_1 = T, T_2 = T + H \),
with \( H = c(\log \log T)^{-1} \). Then we have

\[
\sum_{n \in N} |\sigma_i T_1(n)|^2 |\sigma_i T_2(n)|^2 \\
\ll_{A,D} N \\
\times \left( (\log \log \log T)^3(\log \log T)^7|\zeta(1 + iT_1)|^4|\zeta(1 + iT_2)|^4 \\
+ (\log \log T)^7|\zeta(1 + iT_1 + H)|^2|\zeta(1 - iT_1 - H)|^2|\zeta(1 + iT_2 + H)|^2|\zeta(1 - iT_2 - H)|^2 \\
+ (\log \log T)^7|\zeta(1 + iT_1 - H)|^2|\zeta(1 - iT_1 + H)|^2|\zeta(1 + iT_2 - H)|^2|\zeta(1 - iT_2 + H)|^2 \right) \\
+ O \left( N(\log \log T)^{-1} \right).
\]

Proof. Let

\[
F_0(s, T_1, T_2) = \sum_{n=1}^{\infty} |\sigma_i T_1(n)|^2 |\sigma_i T_2(n)|^2 n^{-s} \quad (\sigma > 1).
\]

By the identity of U. Balakrishnan \[6\], we have

\[
F_0(s, T_1, T_2) = \zeta(s)\zeta(s+iT_1)\zeta(s+iT_2)^2 \zeta(s+iT_2) \zeta(s+iT_2) \\
\times \zeta(s+i(T_1-T_2))\zeta(s-i(T_1-T_2))\zeta(s+i(T_1+T_2))\zeta(s-i(T_1+T_2)) G(s, T_1, T_2),
\]

where \( G(s, T_1, T_2) \) is regular and bounded for \( \sigma \geq \sigma_0 > 1/2 \), uniformly in \( T_1, T_2 \). The limiting case \( T_1 = T_2 \) gives the identity of Y. Motohashi, which is connected with the famous nonnegative trigonometric polynomial \( 3 + 4 \cos \varphi + \cos 2\varphi \) and the inequality of Mertens. Littlewood’s bound \[1\] and Perron’s inversion formula for the height \( U = N^{1+\epsilon} \) give

\[
\sum_{n \in N} |\sigma_i T_1(n)|^2 |\sigma_i T_2(n)|^2 = \text{Res} \left( F_0(s, T_1, T_2)N^{s-1} \right)_{s=1,1+iH} \\
+ O \left( \left( e^L(T)e^{(2+\epsilon)N \log \log T} \right)^{10} \right) \left( \log \log T \right)^6 N^{\eta \log \log T} \\
= \text{Res} \left( F_0(s, T_1, T_2)N^{s-1} \right)_{s=1,1+iH} + O \left( N(\log \log T)^{1-\epsilon} \right),
\]

where we have put

\[
\eta = 1 - \frac{E}{\log \log T}.
\]

Also, \( \text{Res} \left( F_0(s, T_1, T_2)N^{s-1} \right)_{s=1} \ll N \sum_{k=0}^{3} |(\partial s)^k_{s=1} H(s, T_1, T_2)(\log N)^{3-k} |, \)

where

\[
H(s, T_1, T_2) = \zeta(s+iT_1)\zeta(s-iT_1)\zeta(s+iT_2)^2 \zeta(s-iT_2)^2 \\
\times \zeta(s+i(T_1-T_2))\zeta(s-i(T_1-T_2))\zeta(s+i(T_1+T_2))\zeta(s-i(T_1+T_2)).
\]

By taking the logarithmic derivative, we get

\[
(\partial s)^k_{s=1} H(s, T_1, T_2) \ll H(1, T_1, T_2)(\log \log T \log \log T)^k.
\]

From the theorem of Littlewood and the definition of \( H \) we see that

\[
\zeta(1+i(T_1-T_2))\zeta(1-i(T_1-T_2))\zeta(1+i(T_1+T_2))\zeta(1-i(T_1+T_2)) \ll (\log \log T)^4,
\]

which implies the assertion of the lemma.
Lemma 5. Let μ(d) be the Möbius function, and let

\[ \lambda_d(z) = \begin{cases} 
\mu(d) & \text{if } d < z, \\
\mu(d) \frac{\log(z^2/d)}{\log z} & \text{if } z \leq d < z^2, \\
0 & \text{otherwise,}
\end{cases} \]

where \( z > 1 \) is arbitrary. Then we have, uniformly in \( N > 1 \) and in \( z, \)

\[ \sum_{n \leq N} \left( \sum_{d | n} \lambda_d(z) \right)^2 \ll \frac{N}{\log z}. \]

This lemma is due to Barban—Vehov [7] and appears as Lemma 5 in Y. Motohashi [1]. For the proof, see [8] and [9].

Lemma 6. For any large \( y \), and fixed \( a, q > 1, (a, q) = 1, \)

\[ \sum_{\substack{p \leq y \equiv a \pmod{q}}} \text{sgn} \left( \cos(2h \log p) \right) \frac{\cos(h \log p)}{p} = \frac{1}{\varphi(q)} \log \left( \min \left( h^{-1}, \log y \right) \right) + O(1) \quad \text{for } 0 < h < c. \]

This and related estimates can be proved by using PNT in arithmetic progressions and Stieltjes integration. A similar lemma can be found in [10].

3. Proof of Theorem. We put

\[ X = \exp(0.5DA \log \log T \log \log \log T), \quad z = \exp(A \log \log T \log \log \log T) \]

with the same \( A \) and \( D \) as in Lemma 4 set \( \xi(d) = \lambda_d(z) \) in Lemma 3 and for \( T_1 = T, T_2 = T + H \) with \( H = c(\log \log T)^{-1}, \)

write

\[ J(s, T_1, T_2) = \frac{\zeta(s) \zeta(s - iT_1) \zeta(s + iT_2) \zeta(s - iT_1 - T_2)}{\zeta(2s - iT_1 - T_2)}, \]

\[ K(s, T_1, T_2) = \sum_{d \leq z^2} \lambda_d(z) P_d(s, T_1, T_2). \]

Theorem 1. Assume the Riemann hypothesis. Then there exists an infinite sequence of pairs of real numbers \( (T_1, T_2), T_1 = T, T_2 = T + H, \) with arbitrarily large values of \( T \) and \( H = c(\log \log T)^{-1}, \) such that

\[ |\zeta(1 + iT_1)||\zeta(1 + iT_2)| \ll (\log \log T)^{-2} \]

and

\[ (\log \log T)^7 |\zeta(1 + iT_1)|^4 |\zeta(1 + iT_2)|^4 \\
+ (\log \log T)^7 \zeta(1 + iT_1 + H)^2 \zeta(1 - iT_1 - H)^2 \zeta(1 + iT_2 + H)^2 \zeta(1 - iT_2 - H)^2 \\
+ (\log \log T)^7 \zeta(1 + iT_1 - H)^2 \zeta(1 - iT_1 + H)^2 \zeta(1 + iT_2 - H)^2 \zeta(1 - iT_2 + H)^2 \\
\ll (\log \log T)^{-1}. \]

Let \( s_0 = \sigma_0 + it_0 \) be a point such that

\[ |J(s_0, T_1, T_2) K(s_0, T_1, T_2)| \geq (\log \log T)^\epsilon \]

with arbitrarily small fixed \( \epsilon > 0, \) and

\[ \sigma_0 = 1 - \frac{E_0}{\log \log T} \geq 1 - \frac{E}{\log \log T}, \quad C \log \log T \leq |t_0| \leq T/2. \]

Then \( E_0 \geq c_2(\epsilon) > 0. \)
Proof. By Mellin’s inversion formula, when \( c - \sigma_0 > 0 \),
\[
e^{-n/X} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s-s_0) \frac{X^{s-s_0}}{n^{s-s_0}} \, ds.
\]
Hence for \( c > 1 \) and \( c > \sigma_0 \) by Lemma 3 we have that
\[
e^{-1/X} + \sum_{n \geq z} \sigma_{T_1}(n) \sigma_{-T_2}(n) n^{-s_0} a(n) e^{-n/X} = \frac{X^{-s_0}}{2\pi i} \int_{(a=c)} J(s,T_1,T_2) K(s,T_1,T_2) \Gamma(s-s_0) X^s \, ds,
\]
where
\[
a(n) = \sum_{d|n} \lambda_d(z).
\]
We now move the line of integration to the line
\[
\sigma = \eta = 1 - \frac{E}{\log \log T}.
\]
There are simple poles at \( s = 1, 1 + iT_1, 1 - iT_2, 1 + i(T_1 - T_2) \), but by (3) and Lemma 2 they
leave residues that are all bounded by \( O((\log \log T)^{-2}) \). Now we consider the estimation of
the integral along \( \sigma = \eta \). For the estimation of \( K(s,T_1,T_2) \) we define the generating Dirichlet series
\[
M_w(s,T_1,T_2) = 1 + \sum_{d=2}^{\infty} \mu(d) P_d(s,T_1,T_2) d^{-w}
\]
\[
= \prod_p \left( 1 - \frac{1}{p^w} \left( 1 - \left( 1 - \frac{1}{p^s} \right) \left( 1 - \frac{1}{p^{s-T_1}} \right) \left( 1 - \frac{1}{p^{s+iT_2}} \right) \right) \right) \times \left( 1 - \frac{1}{p^{2s-i(T_1-T_2)}} \right)^{-1}.
\]
Using a version of Perron’s inversion formula, we get
\[
\frac{1}{d!} \sum_{d \leq z^2} \mu(d) P_d(s,T_1,T_2) \log (z^2/d) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M_w(s,T_1,T_2) z^{2w} \frac{d^w}{w^2} \, dw,
\]
with \( c = 1 - \Re s + \frac{1}{\log z} \), which implies that on the line \( \Re s(= \sigma) = \eta \) we have
\[
K(s,T_1,T_2) \ll z^{2(1-\eta)} (\log z)^{10} \ll \exp(2AE \log \log \log T)(\log \log T \log \log \log T)^{10}.
\]
Thus recalling (1), (3) and (2) we get, as in the proof of Lemma 1 that
\[
\left| \text{Res} \left( X^{-s_0} J(s,T_1,T_2) K(s,T_1,T_2) \Gamma(s-s_0) X^s \right)_{s=s_0} \right|
\]
\[
+ \frac{X^{-s_0}}{2\pi i} \int_{(\sigma=\eta)} J(s,T_1,T_2) K(s,T_1,T_2) \Gamma(s-s_0) X^s \, ds - e^{-1/X}
\]
\[
\geq (\log \log T)^{\varepsilon} + O \left( \exp \left( 0.5DA \log \log \log T(E_0 - E) \right) \frac{\log \log T}{E - E_0} \right)
\]
\[
\times \left( e^{L(T)e^{(2+\varepsilon)}T}} \log \log T \right)^4 (\log \log T)^{2AE + 10 + \varepsilon}.
\]
Hence there is an \( N \) such that \( z \leq N \leq X^2 \), and

\[
\sum_{N \leq n \leq 2N} |\sigma_{\tau_{T_1}}(n)||\sigma_{-\tau_{T_2}}(n)||a(n)|n^{-\sigma_0} \gg (\log \log T)^{-1+\varepsilon},
\]

since the range of the summation \( z \leq n \leq X^2 \) may be divided into the intervals \( N \leq n \leq 2N \) so that the number of the intervals is \( \ll X^2/z \ll \log \log T \log \log \log T \) and the sum over the entire range must be \( \gg (\log \log T)^\varepsilon \). By the Cauchy inequality and by Lemma 5 we get

\[
(\log \log T)^{-2+\varepsilon} \log z \ll \sum_{N \leq n \leq 2N} |\sigma_{\tau_{T_1}}(n)|^2|\sigma_{\tau_{T_2}}(n)|^2 N^{1-2\sigma_0}.
\]

Finally, by Lemma 4 with \( T_1 = T \), \( T_2 = T + H \) we establish that

\[
N^{2(1-\sigma_0)} \gg (\log \log T)^3(\log \log T)^7|\zeta(1+iT_1)|^4|\zeta(1+iT_2)|^4 \\
+ (\log \log T)^7|\zeta(1+i(T_1+H))^2\zeta(1-i(T_1-H))^2\zeta(1+i(T_2+H))^2\zeta(1-i(T_2-H))^2 \\
+ (\log \log T)^7|\zeta(1+i(T_1-H))^2\zeta(1-i(T_1+H))^2\zeta(1+i(T_2-H))^2\zeta(1-i(T_2+H))^2 \\
+ O((\log \log T)^{-1})^{-1}(\log \log T)^{-1+\varepsilon}.
\]

Next we prove existence of the infinite sequence of pairs of real numbers \((T_1, T_2)\), claimed in the theorem. We may choose \( T_1 = T \) and \( T_2 = T + H \) in the following way: As in [3], Chapter VIII, §8.6, for \( \sigma > 1 \)

\[
\log \frac{1}{|\zeta(s)|} = -\sum \frac{\cos(t \log p_n)}{p_n^\sigma} + O(1).
\]

Also, we have the identity

\[
\cos((t + h) \log p_n) = \cos(t \log p_n) \cos(h \log p_n) - \sin(t \log p_n) \sin(h \log p_n).
\]

So, we want to choose \( t \) such that for, say, every \( p_n \equiv \pm 1 \pmod{7} \) and \( n \leq N_2 \)

\[
\cos(t \log p_n) < -1 + \frac{1}{N_2},
\]

for every \( p_n \equiv \pm 2 \pmod{7} \) and \( n \leq N_2 \)

\[
\cos(t \log p_n) \begin{cases} < -1 + \frac{1}{N_2} & \text{if} \cos(H \log p_n) \geq 0, \\ > 1 - \frac{1}{N_2} & \text{if} \cos(H \log p_n) < 0, \end{cases}
\]

and for every \( p_n \equiv \pm 3 \pmod{7} \) and \( n \leq N_2 \)

\[
\cos(t \log p_n) \begin{cases} < -1 + \frac{1}{N_2} & \text{if} \cos(2H \log p_n) \geq 0, \\ > 1 - \frac{1}{N_2} & \text{if} \cos(2H \log p_n) < 0. \end{cases}
\]

This may be done as in Lemma \( \delta \) of [3], Chapter VIII, §8.8. Now existence of the sequence \((T_1, T_2)\) follows from this and estimates as in Lemma 6 by the Phragmén–Lindelöf method. Thus,

\[
N^{2(1-\sigma_0)} \gg (\log \log T)^3(\log \log T)^{-1} + O((\log \log T)^{-1})^{-1} \\
\times (\log \log T)^{-1+\varepsilon}.
\]

This ends the proof of the theorem.
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