MATRIX $p$-NORMS ARE NP-HARD TO APPROXIMATE IF $p \neq 1, 2, \infty$

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Abstract. We show that, for any rational $p \in [1, \infty)$ except $p = 1, 2$, unless $P = NP$, there is no polynomial time algorithm which approximates the matrix $p$-norm to arbitrary relative precision. We also show that, for any rational $p \in [1, \infty)$ including $p = 1, 2$, unless $P = NP$, there is no polynomial-time algorithm which approximates the $\infty, p$ mixed norm to some fixed relative precision.

Key words. matrix norms, complexity, NP-hardness

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1. Introduction. The $p$-norm of a matrix $A$ is defined as

$$||A||_p = \max_{||x||_p = 1} ||Ax||_p.$$ 

We consider the problem of computing the matrix $p$-norm to relative error $\epsilon$, defined as follows: given the inputs (i) a matrix $A \in \mathbb{R}^{n \times n}$ with rational entries and (ii) an error tolerance $\epsilon$ which is a positive rational number, output a rational number $r$ satisfying

$$|r - ||A||_p| \leq \epsilon||A||_p.$$ 

We will use the standard bit model of computation. When $p = \infty$ or $p = 1$, the $p$-matrix norm is the largest of the row/column sums and thus may be easily computed exactly. When $p = 2$, this problem reduces to computing an eigenvalue of $A^T A$ and thus can be solved in polynomial time in $n, \log \frac{1}{\epsilon}$ and the bit size of the entries of $A$. Our main result suggests that the case of $p \notin \{1, 2, \infty\}$ may be different.

Theorem 1.1. For any rational $p \in [1, \infty)$ except $p = 1, 2$, unless $P = NP$, there is no algorithm which computes the $p$-norm of a matrix with entries in $\{-1, 0, 1\}$ to relative error $\epsilon$ with running time polynomial in $n, \frac{1}{\epsilon}$.

On the way to our result, we also slightly improve the NP-hardness result for the mixed norm $||A||_{\infty, p} = \max_{||x||_{\infty} \leq 1} ||Ax||_p$ from [5]. Specifically, we show that, for every rational $p \geq 1$, there exists an error tolerance $\epsilon(p)$ such that, unless $P = NP$, there is no polynomial time algorithm approximating $||A||_{\infty, p}$ with a relative error smaller than $\epsilon(p)$.

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1.1. Previous work. When $p$ is an integer, computing the matrix norm can be recast as solving a polynomial optimization problem. These are known to be hard to solve in general [3]; however, because the matrix norm problem has a special structure, one cannot immediately rule out the possibility of a polynomial time solution. A few hardness results are available in the literature for mixed matrix norms $\|A\|_{p,q} = \max_{\|x\|_p \leq 1} \|Ax\|_q$. Rohn has shown in [4] that computing the $\|A\|_{\infty,1}$ norm is NP-hard. In her thesis, Steinberg [5] proved more generally that computing $\|A\|_{p,q}$ is NP-hard when $1 \leq q < p \leq \infty$. We refer the reader to [5] for a discussion of applications of the mixed matrix norm problems to robust optimization.

It is conjectured in [5] that there are only three cases in which mixed norms are computable in polynomial time: First, $p = 1$, and $q$ is any rational number larger than or equal to 1. Second, $q = \infty$, and $p$ is any rational number larger than or equal to 1. Third, $p = q = 2$. Our work makes progress on this question by settling the “diagonal” case of $p = q$; however, the case of $p < q$, as far as the authors are aware, is open.

1.2. Outline. We begin in section 2 by providing a proof of the NP-hardness of approximating the mixed norm $\|\cdot\|_{\infty,p}$ within some fixed relative error for any rational $p \geq 1$. The proof may be summarized as follows: observe that, for any matrix $M$, $\max_{\|x\|_\infty=1} \|Mx\|_p$ is always attained at one of the $2^n$ points of $\{-1,1\}^n$. So by appropriately choosing $M$, one can encode an NP-hard problem of maximization over the latter set. This argument will prove that computing the $\|\cdot\|_{\infty,p}$ norm is NP-hard.

Next, in section 3, we exhibit a class of matrices $A$ such that $\max_{\|x\|_p=1} \|Ax\|_p$ is attained at each of the $2^n$ points of $\{-1,1\}^n$ (up to scaling) and nowhere else. These two elements are combined in section 4 to prove Theorem 1.1. More precisely, we define the matrix $Z = (M^T \alpha A^T)^T$, where we will pick $\alpha$ to be a large number depending on $n,p$ ensuring that the maximum of $\|Zx\|_p/\|x\|_p$ occurs very close to vectors $x \in \{-1,1\}^n$. As mentioned several sentences ago, the value of $\|Ax\|_p$ is the same for every vector $x \in \{-1,1\}^n$; as a result, the maximum of $\|Zx\|_p/\|x\|_p$ is determined by the maximum of $\|Mx\|_p$ on $\{-1,1\}^n$, which is proved in section 2 to be hard to compute. We conclude with some remarks on the proof in section 5.

2. The $\|\cdot\|_{\infty,p}$ norm. We now describe a simple construction which relates the $\infty,p$ norm to the maximum cut in a graph.

Suppose $G = (\{1,\ldots,n\},E)$ is an undirected, connected graph. We will use $M(G)$ to denote the edge-vertex incidence matrix of $G$; that is, $M(G) \in \mathbb{R}^{E \times n}$. We will think of columns of $M(G)$ as corresponding to nodes of $G$ and rows of $M(G)$ as corresponding to the edges of $G$. The entries of $M(G)$ are as follows: orient the edges of $G$ arbitrarily, and let the $i$th row of $M(G)$ have $+1$ in the column corresponding to the origin of the $i$th edge, $-1$ in the column corresponding to the endpoint of the $i$th edge, and 0 at all other columns.

Given any partition of $\{1,\ldots,n\} = S \cup S^c$, we define $cut(G,S)$ to be the number of edges with exactly one endpoint in $S$. Furthermore, we define $\text{maxcut}(G) = \max_{S \subseteq \{1,\ldots,n\}} \text{cut}(G,S)$. The indicator vector of a cut $(S,S^c)$ is the vector $x$ with $x_i = 1$ when $i \in S$ and $x_i = -1$ when $i \in S^c$. We will use $cut(x)$ for vectors $x \in \{-1,1\}^n$ to denote the value of the cut whose indicator vector is $x$.

**Proposition 2.1.** For any $p \geq 1$,

$$\max_{\|x\|_\infty \leq 1} \|M(G)x\|_p = 2\text{maxcut}(G)^{1/p}. \quad \Box$$
Observing that the former inequality implies is close to maxcut(\(G\)).

Next, we introduce an error term into this proposition. Define \(f^*\) to be the optimal value \(f^* = \max_{\|x\|_\infty \leq 1} \|M(G)x\|_p\); the above proposition implies that \((f^*/2)^p = \maxcut(G)\). We want to argue that if \(f_{\text{approx}}\) is close enough to \(f^*\), then \((f_{\text{approx}}/2)^p\) is close to \(\text{maxcut}(G)\).

**Proposition 2.2.** If \(p \geq 1\), \(|f^* - f_{\text{approx}}| < \epsilon f^*\) with \(\epsilon < 1\), then

\[
\left| \left( \frac{f_{\text{approx}}}{2} \right)^p - \text{maxcut}(G) \right| \leq \frac{1}{2} |f^* - f_{\text{approx}}| p \max \left( \frac{f^*}{2}, \frac{f_{\text{approx}}}{2} \right)^{p-1}.
\]

It follows from \(\epsilon < 1\) that \(f_{\text{approx}} \leq 2f^*\). We have therefore

\[
\left| \left( \frac{f_{\text{approx}}}{2} \right)^p - \text{maxcut}(G) \right| \leq \frac{1}{2} |f^* - f_{\text{approx}}| \cdot p \cdot (f^*)^{p-1} \leq \frac{\epsilon}{2} (f^*)^p,
\]

where we have used the assumption that \(|f^* - f_{\text{approx}}| \leq \epsilon f^*\). The result follows then from \(\text{maxcut}(G) = (f^*/2)^p\).

We now put together the previous two propositions to prove that approximating the \(\| \cdot \|_{\infty,p}\) norm within some fixed relative error is NP-hard.

**Theorem 2.3.** For any rational \(p \geq 1\) and \(\delta > 0\), unless \(P = NP\), there is no algorithm which, given a matrix with entries in \([-1,0,1]\), computes its \(p\)-norm to relative error \(\epsilon = ((33 + \delta)p2^{p-1})^{-1}\) with running time polynomial in the dimensions of the matrix.

**Proof.** Suppose there was such an algorithm. Call \(f^*\) its output on the \(|E| \times n\) matrix \(M(G)\) for a given connected graph \(G\) on \(n\) vertices. It follows from Proposition 2.2 that

\[
\left| \left( \frac{f_{\text{approx}}}{2} \right)^p - \text{maxcut}(G) \right| \leq \frac{2^{p-1}p}{(33 + \delta)p2^{p-1}} \text{maxcut}(G) = \frac{1}{33 + \delta} \text{maxcut}(G).
\]

Observing that

\[
\frac{32 + \delta}{34 + \delta} \text{maxcut}(G) = \frac{33 + \delta}{34 + \delta} \left( \text{maxcut}(G) - \frac{1}{33 + \delta} \text{maxcut}(G) \right),
\]

the former inequality implies

\[
\frac{32 + \delta}{34 + \delta} \text{maxcut}(G) \leq \frac{33 + \delta}{34 + \delta} \left( \frac{f_{\text{approx}}}{2} \right)^p \leq \text{maxcut}(G).
\]

Since \(p\) is rational, one can compute in polynomial time a lower bound \(V\) for \(\frac{33 + \delta}{34 + \delta} (f_{\text{approx}}/2)^p\) sufficiently accurate so that \(V > \frac{32 + \delta/2}{34 + \delta/2} \text{maxcut}(G) > \frac{16}{17} \text{maxcut}(G)\).
However, it has been established in [2] that, unless \( P = NP \), for any \( \delta' > 0 \), there is no algorithm producing a quantity \( V \) in polynomial time in \( n \) such that

\[
\left( \frac{16}{17} + \delta' \right) \maxcut(G) \leq V \leq \maxcut(G). \tag*{□}
\]

**Remark.** Observe that the matrix \( M(G) \) is not square. If one desires to prove hardiness of computing the \( \infty \)-norm for square matrices, one can simply add \( |E| - n \) zeros to every row of \( M(G) \). The resulting matrix has the same \( \infty \)-norm as \( M(G) \) and is square, and its dimensions are at most \( n^2 \times n^2 \).

3. **A discrete set of exponential size.** Let us now fix \( n \) and a rational \( p > 2 \). We denote by \( X \) the set \(-1, 1 \)^n and use \( S(a, r) = \{ x \in \mathbb{R}^n \mid ||x - a||_p = r \} \) to stand for the sphere of radius \( r \) around \( a \) in the \( p \)-norm. We consider the following matrix in \( \mathbb{R}^{2n \times n}:

\[
A = \begin{pmatrix}
1 & -1 \\
1 & 1 \\
-1 & 1 \\
1 & 1 \\
. & . \\
. & . \\
1 & -1 \\
1 & 1
\end{pmatrix}
\]

and show that the maximum of \( ||Ax||_p \) for \( x \in S(0, n^{1/p}) \) is attained at the \( 2^n \) vectors in \( X \) and at no other points. For this, we will need the following lemma.

**Lemma 3.1.** For any real numbers \( x, y \) and \( p \geq 2 \)

\[
|x + y|^p + |x - y|^p \leq 2^{p-1} (|x|^p + |y|^p).
\]

In fact, \( |x + y|^p + |x - y|^p \) is upper bounded by

\[
2^{p-1} (|x|^p + |y|^p) - \frac{(|x| - |y|)^2}{4} \left( p(p-1) |x|^p + |y|^{p-2} - 2 |x| - |y|^{p-2} \right),
\]

where the last term on the right is always nonnegative.

**Proof.** By symmetry we can assume that \( x \geq y \geq 0 \). In that case, we need to prove

\[
(x + y)^p + (x - y)^p \leq 2^{p-1} (x^p + y^p) - \frac{(x - y)^2}{4} \left( p(p-1)(x + y)^{p-2} - 2(x - y)^{p-2} \right).
\]

Divide both sides by \( (x + y)^p \), and change the variables to \( z = (x - y)/(x + y) \):

\[
1 + z^p \leq \frac{(1 + z)^p + (1 - z)^p}{2} - \left( \frac{p(p-1)}{4} z^2 - \frac{1}{2} z^p \right).
\]

The original inequality holds if this inequality holds for \( z \in [0, 1] \). Let’s simplify:

\[
2 + z^p \leq (1 + z)^p + (1 - z)^p - \frac{p(p-1)}{2} z^2.
\]
Observe that we have equality when \( z = 0 \), so it suffices to show that the right-hand side grows faster than the left-hand side, namely,

\[
z^{p-1} \leq (1 + z)^{p-1} - (1 - z)^{p-1} - (p - 1)z,
\]

and this follows from

\[
(1 + z)^{p-1} \geq 1 + (p - 1)z \geq (1 - z)^{p-1} + z^{p-1} + (p - 1)z,
\]

where we have used the convexity of \( f(a) = a^{p-1} \).

Now we prove that every vector of \( X \) optimizes \( \|Ax\|_p/\|x\|_p \) or, equivalently, optimizes \( \|Ax\|_p^p \) over the sphere \( S(0, n^{1/p}) \).

**Lemma 3.2.** For any \( p \geq 2 \), the supremum of \( \|Ax\|_p^p \) over \( S(0, n^{1/p}) \) is achieved by any vector in \( X \).

**Proof.** Observe that \( \|Ax\|_p^p = n2^p \) for any \( x \in X \). To prove that this is the largest possible value, we write

\[
\|Ax\|_p^p = \sum_{i=1}^{n} |x_i - x_{i+1}|^p + |x_i + x_{i+1}|^p,
\]

using the convention \( n + 1 = 1 \) for the indices. Lemma 3.1 implies that

\[
|x_i - x_{i+1}|^p + |x_i + x_{i+1}|^p \leq 2^{p-1} (|x_i|^p + |x_{i+1}|^p).
\]

By applying this inequality to each term of (3.1) and by using \( \|x\|_p^n = n \), we obtain

\[
\|Ax\|_p^p \leq \sum_{i=1}^{n} 2^{p-1} (|x_i|^p + |x_{i+1}|^p) = 2^p \sum_{i=1}^{n} |x_i|^p = 2^pn. \]

Next we refine the previous lemma by including a bound on how fast \( \|Ax\|_p^p \) decreases as we move a little bit away from the set \( X \) while staying on \( S(0, n^{1/p}) \).

**Lemma 3.3.** Let \( p \geq 2, c \in (0, 1/2] \), and suppose \( y \in S(0, n^{1/p}) \) has the property that

\[
\min_{x \in X} \|y - x\|_\infty \geq c.
\]

Then

\[
\|Ay\|_p^p \leq n2^p - \frac{3(p - 2)}{2pn^2} c^2.
\]

**Proof.** We proceed as before in the proof of Lemma 3.2 until the time comes to apply Lemma 3.1, when we include the error term which we had previously ignored:

\[
\|Ay\|_p^p \leq n2^p - \frac{1}{4} \sum_{i} (|y_i| - |y_{i+1}|)^2 \left( p(p - 1) \|y_i| + |y_{i+1}| \right)^{p-2} - 2 \|y_i| - |y_{i+1}| \right)^{p-2},
\]

Note that on the right-hand side, we are subtracting a sum of nonnegative terms. The upper bound will still hold if we subtract only one of these terms, so we conclude that, for each \( k \),

\[
\|Ay\|_p^p \leq n2^p - \frac{1}{4} (|y_k| - |y_{k+1}|)^2 \left( p(p - 1) \|y_k| + |y_{k+1}| \right)^{p-2} - 2 \|y_i| - |y_{i+1}| \right)^{p-2}.
\]
By assumption, there is at least one \( y_k \) with \( |y_k| - 1 |\geq c \). Suppose first that \( |y_k| > 1 \). Then we have \( |y_k| > 1 + c \), and there must be a \( y_j \) with \( |y_j| < 1 \), for otherwise \( y \) would not be in \( S(0,n^{1/p}) \). Similarly, if \( |y_k| < 1 \), then \( |y_k| < 1 - c \) and there is a \( j \) for which \( |y_j| > 1 \). In both cases, this implies the existence of an index \( m \) with \( |y_m| \) and \( |y_{m+1}| \) differing by at least \( c/n \) and such that at least one of \( |y_m| \) and \( |y_{m+1}| \) is larger than or equal to \( 1 - c \). Therefore,

\[
||Ay||^p \leq n^{2p} - \frac{1 + c^2}{4 n^2} [p(p-1) \langle |y_m| + |y_{m+1}| \rangle^{p-2} - 2 \langle |y_m| - |y_{m+1}| \rangle^{p-2}].
\]

Now observe that \( |y_m| - |y_{m+1}| \leq |y_m| + |y_{m+1}| \) and that \( |y_m| + |y_{m+1}| \geq (1-c) \geq 1/2 \) because \( c \in (0,1/2) \). These two inequalities suffice to establish that the term in square brackets is at least \( (1/2)^{p-2}(p(p-1) - 2) \geq (3/2p)(p-2) \) so that

\[
||Ay||^p \leq n^{2p} - \frac{3(p-2)}{2p^2} e^2. \]

4. Proof of Theorem 1.1. We now relate the results of the last two sections to the problem of the \( p \)-norm. For a suitably defined matrix \( Z \) combining \( A \) and \( M(G) \), we want to argue that the optimizer of \( ||Zx||_p/||x||_p \) is very close to satisfying \( x_i = x_j \) for every \( i, j \).

**Proposition 4.1.** Let \( p > 2 \) and \( G \) be a graph on \( n \) vertices. Consider the matrix

\[
\hat{Z} = \left( \frac{p-2}{64pn^8} M(G) \right)
\]

with \( M(G) \) and \( A \) as in sections 2 and 3, respectively. If \( x^* \) is the vector at which the optimization problem \( \max_{x \in S(0,n^{1/p})} ||\hat{Z}x||_p \) achieves its supremum, then

\[
\min_{x \in X} ||x^* - x||_\infty \leq \frac{1}{4pn^6}.
\]

**Proof.** Suppose the conclusion is false. Then using Lemma 3.3 with \( c = 1/4pn^6 \), we obtain

\[
||Ax^*||_p^p \leq n^{2p} - \frac{3(p-2)}{2p4p^2n^{14}} = n^{2p} - \frac{3(p-2)}{32p^2n^{14}}.
\]

It follows from Proposition 2.1 that

\[
||Mx^*||_p^p \leq 2^p \maxcut(G) \leq 2^p n^2
\]

so that

\[
||\hat{Z}x^*||_p^p = ||Ax^*||_p^p + \left( \frac{p-2}{64pn^8} \right)^p ||Mx^*||_p^p \leq 2^p n^2 - \frac{3(p-2)}{32p^2n^{14}} + \frac{2^p(p-2)n^2}{64p^{p^2n^{8p}}.}
\]

Observe that the last term in this inequality is smaller than the previous one (in absolute value). Indeed, for \( p > 2 \), we have that \( 3/32p > (2/64)^p \), \( p-2 > [(p-2)/p]^p \), and \( 1/n^{14} > n^2/n^{8p} \). We therefore have \( ||\hat{Z}x^*||_p^p < 2^p n^2 \). By contrast, let \( x \) be any vector in \( \{-1,1\}^n \). Then \( x \in S(0,n^{1/p}) \) and

\[
||\hat{Z}x||_p^p \geq ||Ax||_p^p \geq 2^p n,
\]
which contradicts the optimality of $x^*$.

Next we seek to translate the fact that the optimizer $x^*$ is close to $X$ to the fact that the objective value $\|Zx\|_p/\|x\|_p$ is close to the largest objective value at $X$.

**Proposition 4.2.** Let $p > 2$, $G$ be a graph on $n$ vertices, and

$$Z = \begin{pmatrix} \frac{64pn^8}{p-2}A \\ M(G) \end{pmatrix}.$$ 

If $x^*$ is the vector at which the optimization problem

$$\max_{x \in S(0,n^{1/p})} \|Zx\|_p$$

achieves its supremum and $x_r$ is the rounded version of $x^*$ in which every component is rounded to the closest of $-1$ and $1$, then

$$\||Zx^*||_p^p - \|Zx_r\|_p^p \leq \frac{1}{n^2}.$$

**Proof.** Observe that $x^*$ is the same as the extremizer of the corresponding problem with $\tilde{Z}$ instead of $Z$ so that $x$ satisfies the conclusion of Proposition 4.1. Consequently every component of $x^*$ is closer to one of $\pm 1$ than to the other, and so $x_r$ is well defined. We have,

$$\|Zx^*\|_p^p - \|Zx_r\|_p^p = \left(1 + \frac{64}{p-2}n^8\right)^p (\|Ax^*\|_p^p - \|Ax_r\|_p^p) + (\|Mx^*\|_p - \|Mx_r\|_p)^p.$$

This entire quantity is nonnegative since $x^*$ is the maximum of $\|Zx\|$ on $S(0,n^{1/p})$. Moreover, $\|Ax^*\|_p^p - \|Ax_r\|_p^p$ is nonpositive since, by Proposition 3.2, $\|Ax\|_p$ achieves its maximum over $S(0,n^{1/p})$ on all the elements of $X$. Consequently,

$$\|Zx^*\|_p^p - \|Zx_r\|_p^p \leq (\|Mx^*\|_p - \|Mx_r\|_p)(p\max(\|Mx^*\|_p,\|Mx_r\|_p)^p - 1).$$

We now bound all the terms in the last equation. First

$$\|Mx^*\|_p - \|Mx_r\|_p \leq \|M\|_2 \|x^* - x_r\|_2 \leq \|M\|_F \sqrt{n} \|x^* - x_r\|_\infty = \frac{n\sqrt{n}}{4pn^6},$$

where we have used $\|M(G)\|_F = \sqrt{2|E|} < n$ and Proposition 4.1 for the last inequality. Now that we have a bound on the first term in (4.1), we proceed to the last term.

It follows from the definition of $M$ that

$$\|Mx_r\|_p^p \leq 2^p \cdot \frac{n}{2} \leq 2^p n^2.$$

Next we bound $\|Mx^*\|_p^p$. Observe that a particular case of (4.2) is

$$\|Mx^*\|_p \leq \|Mx_r\|_p + 1.$$ 

Moreover, observe that $\|Mx_r\|_p \geq 1$. (The only way this does not hold is if every entry of $x_r$ is the same, i.e., $\|Mx_r\|_p = 0$. But then (4.3) implies that $\|Mx^*\|_p < 1$, which is
impossible since $G$ has at least one edge.), So (4.3) implies that $\|Mx^*\|_p \leq 2\|Mx\|_p$, and so

$$\|Mx^*\|_p \leq 4^p n^2.$$ 

Thus

$$\max(\{\|Mx^*\|_p, \|Mx\|_p\}) \leq 4^p n^2,$$

and therefore $\max(\|Mx^*\|_p, \|Mx\|_p) \leq 4^p n^2$. Indeed, this bound is trivially valid if $\max(\|Mx^*\|_p, \|Mx\|_p) = 1$ and follows from $a^{p-1} < a^p$ for $a \geq 1$ otherwise. Using this bound and the inequality (4.2), we finally obtain

$$\|Zx^*\|_p - \|Zx\|_p \leq \frac{n^{1.5}}{4^p n^2} \cdot 4^p n^2 \leq \frac{1}{n^2}.$$ 

Finally let us bring it all together by arguing that if we can approximately compute the $p$-norm of $Z$, we can approximately compute the maximum cut.

**Proposition 4.3.** Let $p > 2$. Consider a graph $G$ on $n > 2$ vertices and the matrix

$$Z = \left( \begin{array}{c} 64 \frac{n^8}{n^2} A \\ M(G) \end{array} \right),$$

and let $f^* = \|Z\|_p$. If

$$|f_{\text{approx}} - f^*| \leq \frac{(p - 2)p}{132^p p^p n^p + 3^p},$$

then

$$\left| \left( \frac{n^{1.5}}{2^p} f_{\text{approx}}^p - \frac{n^{1.5}}{2^p} \frac{64 pn^8}{n - 2} \right) - \text{maxcut}(G) \right| \leq \frac{1}{n^2}.$$ 

**Proof.** Observe that $n^{\frac{1}{p}} f^* = \max_{x \in S(0, n^{1/p})} \|Zx\|_p$. It follows thus from Proposition 4.2 that

$$\left| n^{f^p} - \max_{x \in X} \|Zx\|_p \right| < \frac{1}{n^2}.$$ 

Recall that $\|Zx\|_p = \|Mx\|_p + \left( \frac{64pn^8}{n - 2} \right)^p \|Ax\|_p$ and that $\|Ax\|_p = n^{2^p}$ for every $x \in X$. Therefore,

$$\max_{x \in X} \|Zx\|_p = \left( \frac{64pn^8}{n - 2} \right)^p n^{2^p} + \max_{x \in X} \|Mx\|_p = \left( \frac{64pn^8}{n - 2} \right)^p n^{2^p} + 2^p \text{maxcut}(G),$$

and by combining the last two equations, we have

\begin{equation}
(4.4) \quad \left| \left( \frac{n^{1.5}}{2^p} f_{\text{approx}}^p - \frac{n^{1.5}}{2^p} \frac{64 pn^8}{n - 2} \right) - \text{maxcut}(G) \right| \leq \frac{1}{2^{p^2} n^2}.
\end{equation}

Let us now evaluate the error introduced by the approximation $f_{\text{approx}}$:

$$\left| \left( \frac{n^{1.5}}{2^p} f_{\text{approx}}^p - \frac{n^{1.5}}{2^p} \frac{64 pn^8}{n - 2} \right) - \text{maxcut}(G) \right| \leq \frac{1}{2^{p^2} n^2} + \frac{n}{2^p} |f_{\text{approx}}^p - f^p|$$

$$\leq \frac{1}{2^{p^2} n^2} + \frac{n}{2^p} |f_{\text{approx}} - f^*| p \max(f^*, f_{\text{approx}})^{p-1}.$$
It remains to bound the last term of this inequality. First we use the fact that \( f^* \geq 1 \) and (4.4) to argue

\[
(5.1) \quad f^{(p-1)} \leq f_*^p \leq 2^p \left( \frac{64pn^8}{p-2} \right)^p + \frac{2^p}{n} \maxcut(G) + \frac{1}{n^2} \leq 2^p \left( \frac{66pn^8}{p-2} \right)^p,
\]

where we have used \( \maxcut(G) < n^2 \) and \( 1 \leq p/(p-2) \) for the last inequality. By assumption, \( |f_{\text{approx}} - f^*| \leq 1 \), and since \( f^* \geq 1 \),

\[
f_{\text{approx}}^{(p-1)} \leq (2f_*^*)^{p-1} \leq (2f^*)^p \leq 4^p \left( \frac{66pn^8}{p-2} \right)^p.
\]

Putting it all together and using the bound on \( |f_{\text{approx}} - f^*| \), we obtain (assuming \( n > 1 \))

\[
\left| \left( \frac{n}{2^p f_{\text{approx}}} - n \left( \frac{64pn^8}{p-2} \right)^p \right) - \maxcut(G) \right| \leq \frac{1}{2^p n^2} + \frac{(p-2)^p}{132^p p^p n^{p+3}} 2^{p} np \left( \frac{66pn^8}{p-2} \right)^p
\]

\[
\leq \frac{1}{2^p n^2} + \frac{1}{n^2}
\]

\[
\leq \frac{1}{n}. \quad \square
\]

**Proposition 4.4.** Fix a rational \( p \in [1, \infty) \) with \( p \neq 1, 2 \). Unless \( P = NP \), there is no algorithm which, given input \( \epsilon > 0 \) and a matrix \( Z \), computes \( ||Z||_p \) to a relative accuracy \( \epsilon \), in time which is polynomial in \( 1/\epsilon \), the dimensions of \( Z \), and the bit size of the entries of \( Z \).

**Proof.** Suppose first that \( p > 2 \). We show that such an algorithm could be used to build a polynomial time algorithm solving the maximum cut problem. For a graph \( G \) on \( n \) vertices, fix

\[
\epsilon = \left( 132^p \left( \frac{p}{p-2} \right)^p n^{p+3} \right)^{-1} \left( 132 \left( \frac{p}{p-2} \right) n^8 \right)^{-1},
\]

build the matrix \( Z \) as in Proposition 4.3, and compute the norm of \( Z \); let \( f_{\text{approx}} \) be the output of the algorithm. Observe that, by (4.5),

\[
||Z||_p \leq \frac{132pn^8}{p-2},
\]

so

\[
|f_{\text{approx}} - ||Z||_p| \leq \epsilon ||Z||_p \leq \epsilon \left( 132 \frac{p}{p-2} n^8 \right) \leq \left( 132^p \left( \frac{p}{p-2} \right)^p n^{p+3} \right)^{-1}.
\]

It follows then from Proposition 4.3 that

\[
n \left( \frac{f_{\text{approx}}}{2} \right)^p - n \left( 64 \cdot \left( \frac{p}{p-2} \right)^p n^8 \right)^p
\]

is an approximation of the maximum cut with an additive error at most \( 1/n \). Once we have \( f_{\text{approx}} \), we can approximate this number in polynomial time to an additive accuracy of \( 1/4 \). This gives an additive error \( 1/4 + 1/n \) approximation algorithm for
maximum cut, and since the maximum cut is always an integer, this means we can compute it exactly when \( n > 4 \). However, maximum cut is an NP-hard problem [1].

For the case of \( p \in (1, 2) \), NP-hardness follows from the analysis of the case of \( p > 2 \) since, for any matrix \( Z, ||Z||_p = ||Z^T||_{p'} \), where \( 1/p + 1/p' = 1 \). \( \square \)

**Remark.** In contrast to Theorem 2.3 which proves the NP-hardness of computing the matrix \( \infty, k \)-norm to relative accuracy \( \epsilon = 1/C(p) \), for some function \( C(p) \), Proposition 4.4 proves the NP-hardness of computing the \( p \)-norm to accuracy \( 1/C'(p)n^{8p+11} \) for some function \( C'(p) \). In the latter case, \( \epsilon \) depends on \( n \).

Our final theorem demonstrates that the \( p \)-norm is still hard to compute when restricted to matrices with entries in \([-1, 0, 1]\).

**Theorem 4.5.** Fix a rational \( p \in [1, \infty) \) with \( p \neq 1, 2 \). Unless \( P = NP \), there is no algorithm which, given input \( \epsilon \) and a matrix \( M \) with entries in \([-1, 0, 1]\), computes \( ||M||_p \) to relative accuracy \( \epsilon \), in time which is polynomial in \( \epsilon^{-1} \) and the dimensions of the matrix.

**Proof.** As before, it suffices to prove the theorem for the case of \( p > 2 \); the case of \( p \in (1, 2) \) follows because \( ||Z||_p = ||Z^T||_{p'} \), where \( 1/p + 1/p' = 1 \).

Define

\[
Z^* = \left( \left[ \left( \frac{64}{p-r^2} n^8 \right)^A \right] M(G) \right),
\]

where \([\cdot]\) refers to rounding up to the closest integer. Observe that, by an argument similar to the proof of the previous proposition, computing \( ||Z^*||_p \) to an accuracy \( \epsilon = (C(p)n^{8p+11})^{-1} \) is NP-hard for some function \( C(p) \). But if we define

\[
Z^{**} = \begin{pmatrix}
A \\
A \\
\vdots \\
A \\
M
\end{pmatrix},
\]

where \( A \) is repeated \( \left\lfloor \left( \frac{64}{p-r^2} n^8 \right)^P \right\rfloor \) times, then

\[
||Z^{**}||_p = ||Z^*||_p.
\]

The matrix \( Z^{**} \) has entries in \([-1, 0, 1]\), and its size is polynomial in \( n \), so it follows that it is NP-hard to compute \( ||Z^{**}||_p \) within the same \( \epsilon \). \( \square \)

**Remark.** Observe that the argument also suffices to show that computing the \( p \)-norm of square matrices with entries in \([-1, 0, 1]\) is NP-hard: simply pad each row of \( Z^{**} \) with enough zeros to make it square. Note that this trick was also used in section 2.

5. **Concluding remarks.** We have proved the NP-hardness of computing the matrix \( p \)-norm approximately with relative error \( \epsilon = 1/C(p)n^{8p+11} \), where \( C(p) \) is some function of \( p \), and the NP-hardness of computing the matrix \( \infty, p \)-norm to some fixed relative accuracy depending on \( p \). We finish with some technical remarks about various possible extensions of the theorem:

- Due to the linear property of the norm \( ||\alpha A|| = ||A|| \alpha|| \), our results also imply the NP-hardness of approximating the matrix \( p \)-norm with any fixed or polynomially growing additive error.
Our construction also implies the hardness of computing the matrix $p$-norm for any irrational number $p > 1$ for which a polynomial time algorithm to approximate $x^p$ is available.

Our construction may also be used to provide a new proof of the NP-hardness of the $|| \cdot ||_{p,q}$ norm when $p > q$, which has been established in [5]. Indeed, it rests on the matrix $A$ with the property that $\max ||Ax||_p/||x||_p$ occurs at the vectors $x \in \{ -1, 1 \}^n$. We use this matrix $A$ to construct the matrix $Z = (\alpha A \ M)^T$ for large $\alpha$ and argue that $\max ||Zx||_p/||x||_p$ occurs close to the vectors $x \in \{ -1, 1 \}^n$. At these vectors, it happens $Ax$ is a constant, so we are effectively maximizing $||Mx||_p$, which is hard as shown in section 2.

If one could come up with such a matrix for the case of the mixed $|| \cdot ||_{p,q}$ norm, one could prove NP-hardness by following the same argument. However, when $p > q$, actually the same matrix $A$ works. Indeed, one could simply argue that

$$||A||_{p,q} = \max_{x \neq 0} \frac{||Ax||_q}{||x||_p} = \max_{x \neq 0} \frac{||Ax||_q}{||x||_q} \frac{||x||_q}{||x||_p},$$

and since the maximum of $||x||_q/||x||_p$ when $1 \leq q < p \leq \infty$ occurs at the vectors $x \in \{ -1, 1 \}^n$, we have that both terms on the right are maximized at $x = \in \{ -1, 1 \}^n$, and that is where $||Ax||_q/||x||_p$ is maximized.

Finally we note that our goal was only to show the existence of a polynomial time reduction from the maximum cut problem to the problem of matrix $p$-norm computation. It is possible that more economical reductions which scale more gracefully with $n$ and $p$ exist.

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