A NOTE ON SPHERES AND MINIMA

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Abstract. We write down a one-dimensional integral formula and compute large-\(n\) asymptotics for
\[
\int \frac{n}{\sum_{i=1}^{n} x_i^2} \min_{i=1}^{n} |x_i| d\sigma,
\]
where \(d\sigma\) is the usual rotationally invariant measure on \(S^{n-1}\). The method is general, and allows to write the mean over the sphere of an homogeneous function (Theorem 1) in terms of an expectation of a function of independent, identically distributed Gaussians. We also write down an asymptotic formula for the minimum of a large number of identical independent positive random variables (Theorem 3).

Introduction

Let \(S^{n-1}\) be the unit sphere in \(\mathbb{E}^n\). We would like to compute the following quantity:
\[
\text{Emin}(n) = \frac{1}{\text{vol} S^{n-1}} \int_{S^{n-1}} \min_{i=1}^{n} |x_i| d\sigma,
\]
where \(d\sigma\) is the standard measure on \(S^{n-1}\). In other words, we want to find the expected (absolute) value of the smallest coordinate of a unit vector in \(\mathbb{E}^n\). Direct integration seems to run into major computational difficulties, so instead we will compute \(\text{Nmin}(n)\), which we define as the expected value of the minimum of the absolute value of \(n\) independent, identically distributed random variables with mean 0 and variance \(1/2\). Before we compute \(\text{Nmin}\), we point the connection between \(\text{Emin}\) and \(\text{Nmin}\). First, observe that pretty much by definition,
\[
\text{Nmin}(n) = c_n \int_{\mathbb{E}^n} \exp \left( - \sum_{i=1}^{n} x_i^2 \right) \min_{i=1}^{n} |x_i| dx_1 \ldots dx_n,
\]

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where $c_n$ is such that

$$c_n \int_{\mathbb{R}^n} \exp \left( -\sum_{i=1}^{n} x_i^2 \right) \, dx_1 \ldots dx_n = 1. \tag{1}$$

Now we remark that $\min_{i=1}^{n} |x_i|$ is a 1-homogeneous function of the coordinates, hence we can rewrite the integral for $N_{\min}$ in polar coordinates as follows:

$$N_{\min}(n) = c_n \text{vol} S^{n-1} \int_{\mathbb{R}^n} e^{-r^2} r^n \, dr = c_n \text{Emin}(n) \int_{0}^{\infty} e^{-r^2} r^n \, dr. \tag{2}$$

Since, by the obvious substitution $u = r^2$,

$$\int_{0}^{\infty} e^{-r^2} r^n \, dr = \frac{1}{2} \int_{0}^{\infty} e^{-u^{(n-1)/2}} \, du = \frac{1}{2} \Gamma \left( \frac{n+1}{2} \right),$$

and Eq. (2) can be rewritten in polar coordinates as

$$1 = c_n \text{vol} S^{n-1} \int_{0}^{\infty} r^{n-1} \, dr = \frac{c_n \text{vol} S^{n-1}}{2} \Gamma \left( \frac{n}{2} \right),$$

we see that

$$\Gamma \left( \frac{n+1}{2} \right) \text{Emin}(n) = \Gamma \left( \frac{n}{2} \right) N_{\min}(n).$$

This implies, in particular (by Stirling’s formula) that

$$\frac{N_{\min}(n)}{\text{Emin}(n)} \sim \sqrt{\frac{n+1}{2}}.$$

It is clear that the above argument only depends on the homogeneity of the function, so it immediately generalizes to the following:

**Theorem 1.** Let $f(x_1, \ldots, x_n)$ be a homogeneous function on $\mathbb{R}^n$ of degree $d$ (in other words, $f(\lambda x_1, \ldots, \lambda x_n) = \lambda^d f(x_1, \ldots, x_n)$.) Then

$$\frac{\Gamma \left( \frac{n+d}{2} \right)}{\text{vol} S^{n-1}} \int_{S^{n-1}} f \, d\sigma = \Gamma \left( \frac{n}{2} \right) \mathbb{E} \left( f(X_1, \ldots, X_n) \right),$$

where $X_1, \ldots, X_n$ are independent random variables with probability density $e^{-x^2}$.

Now, to complete our computation, we must compute $N_{\min}(n)$. Let $X_1, \ldots, X_n$ be independent, identically distributed variables, whose common distribution $F$ is supported on $[0, \infty)$. What is the distribution of $X_{(1)} = \min(X_1, \ldots, X_n)$? The probability that $X_{(1)}$ is greater than $y$ is obviously $(1 - F(y))^n$, so the distribution function of $X_{(1)}$ is
obviously $1 - (1 - F(y))^n$. It follows (by integration by parts) that the expectation of $X_{(1)}$ is

$$E_{(1)} = \int_0^\infty (1 - F(y))^n dy.$$ 

In our particular case of $X_i = |Z_i|$, where $Z_i$ is normal with variance $1/2$,

$$F(y) = \frac{1}{\sqrt{\pi}} \int_{-y}^y \exp(-x^2) dx. \quad (3)$$

We thus have an integral formula for $E_{\text{min}}(n)$ promised in the abstract:

$$E_{\text{min}}(n) = \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \int_0^\infty \left(1 - \frac{1}{\sqrt{\pi}} \int_{-y}^y \exp(-x^2) dx\right)^n. \quad (4)$$

Since this formula is somewhat unwieldy, it is worthwhile to state an asymptotic result. Then,

**Theorem 2.** Let $F$ be as in Eq. (3). Then

$$\int_0^\infty (1 - F(y))^n dy = \frac{\sqrt{\pi}}{2(n+1)} + o(n^{-2}).$$

**Proof.** We write

$$\int_0^\infty (1 - F(y))^n dy = I_1 + I_2 + I_3, \quad (5)$$

where

$$I_1 = \int_0^{n^{-3/4}} (1 - F(y))^n dy, \quad (6)$$

$$I_2 = \int_{n^{-3/4}}^C (1 - F(y))^n dy, \quad (7)$$

$$I_3 = \int_C^\infty (1 - F(y))^n dy. \quad (8)$$

First, we show that $I_1$ and $I_2$ are asymptotically negligibly small. Indeed, $1 - F(y) = \frac{2}{\sqrt{\pi}} \int_y^\infty \exp(-x^2) dx$. For $x > 1$, $\exp(-x^2) \ll \exp(-x)$, and so $1 - F(y) \ll \exp(-x)$, so for a suitable choice of $C$, $I_3$ decreases exponentially in $n$. Furthermore $1 - F(y)$ is monotonically decreasing, so to estimate $I_2$, we write

$$\int_{n^{-3/4}}^C (1 - F(y))^n \leq C(1 - F(n^{-3/4}))^n \ll \exp(-C_1 n^{1/2}),$$

since for small $y$, $F(y) \approx 2y/\sqrt{\pi}$. 

Finally, to estimate the first integral, we expand $F(y)$ in a Taylor series, to obtain

$$1 - F(y) = 1 - \frac{2y}{\sqrt{\pi}} + O(y^3) = \left(1 - \frac{2y}{\sqrt{\pi}}\right)(1 + O(y^3)),$$

so that

$$(1 - F(y))^n = \left(1 - \frac{2y}{\sqrt{\pi}}\right)^n (1 + O(y^3))^n.$$

for $y < n^{-3/4}$, we know that $(1 + O(y^3))^n - 1 = O(n^{-5/4})$, so that

$$\int_0^{n^{-3/4}} (1 - F(y))^n dy = (1 + O(n^{-5/4})) \int_0^{n^{-3/4}} \left(1 - \frac{2y}{\sqrt{\pi}}\right)^n dy.$$

Now, since

$$\int_{n^{-3/4}}^{\sqrt{\pi}/2} \left(1 - \frac{2y}{\sqrt{\pi}}\right)^n dy \ll \exp\left(-\frac{\sqrt{\pi}}{2} n^{1/4}\right),$$

it follows that

$$\int_0^{n^{-3/4}} \left(1 - \frac{2y}{\sqrt{\pi}}\right)^n dy \sim \int_{n^{-3/4}}^{\sqrt{\pi}/2} \left(1 - \frac{2y}{\sqrt{\pi}}\right)^n dy = \frac{\sqrt{\pi}}{2(n + 1)}.$$

It is clear that in the above argument we don’t actually need $F$ to be the normal distribution, and it holds in much greater generality:

**Theorem 3.** Let $X_1, \ldots, X_n$ be identically independently distributed variables on $[0, \infty]$ with distribution function $F$. Suppose that the distribution $F$ satisfies the following conditions:

1. $F$ has a continuous nonvanishing density $f$ in a neighborhood of 0.
2. $1 - F \in L^p([0, \infty))$, for some $p > 0$.

Then, as $n \to \infty$,

$$\mathbb{E}(\min(X_1, \ldots, X_n)) \sim \frac{1}{f(0)(n + 1)}.$$

**Proof.** The argument goes through pretty much as above, except for the proof that the integral $I_3$ decreases exponentially with $n$. This, however, is easily fixed: The function $1 - F(y)$ is monotonically decreasing, so, for $y > C$, $(1 - F(y))^n \leq C^{n-p}(1 - F(y))^p$, where $p$ is as in the statement of the theorem. The result follows immediately. \qed
Remark 4. The second condition in the statement of Theorem 3 can be interpreted as saying that if the expectation of the minimum of some number of variables is finite, then we have the claimed asymptotics, and otherwise the expectation is always infinite, so, in a sense, we have complete asymptotic information in that case also.

Remark 5. Theorem 3 has no information about the error term, but this is due to the very weak regularity assumption on the density $f$ at the origin.

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