Teleparallel Killing Vectors of Spherically Symmetric Spacetimes

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Abstract

In this paper, Killing vectors of spherically spacetimes have been evaluated in the context of teleparallel theory of gravitation. Further, we investigate the Killing vectors of the Friedmann metrics. It is found that for static spherically spacetimes the number of Killing vectors turn out to be seven while for the Friedmann models, we obtain six teleparallel Killing vectors. The results are then compared with those of General Relativity. We conclude that both of these descriptions of gravity do not provide the consistent results in general. However, these results may coincide under certain conditions for a particular spacetime.

Keywords: Teleparallel Theory, Killing Vectors.

1 Introduction

The concept of symmetry is much helpful to find the solution of many problems in physics. The connection between symmetries and conservation laws in physics has both fundamental significance and great practical utility. The symmetries of curved spacetimes are generated by Killing vectors (KVs) also called isometries. Some other spacetime symmetries are Ricci collineations

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(RCs), curvature collineations (CCs) and matter collineations (MCs). These symmetries help to solve the Einstein field equations (EFEs) and to classify their solutions. Thus symmetries play a significant role in describing the geometry of a spacetime. Much work has been done to explore these symmetries during the last two decades. Petrov [1] was the first who considered the four-dimensional spacetime to solve the Killing equations. Bokhari and Qadir [2,3] gave a complete classification of static spherically symmetric spacetimes. Ziad and Qadir [4,5] extended this work and obtained a complete classification of non-static spherically symmetric spacetimes.

Teleparallel theory of gravity (TPG) was introduced by Einstein in his unsuccessful attempt to unify electromagnetism and gravitation [6]. This is considered as an alternative description of gravitation which corresponds to a gauge theory for the translation group [7] based on Weitzenböck geometry [8]. Here the non-zero torsion plays the role of force [9] and curvature tensor vanishes identically. In this approach, there are no geodesics and the gravitational interaction is described by force equations similar to the Lorentz force equations of electrodynamics [10]. Thus we can say that the gravitational interaction can be described in terms of curvature as is usually done in GR or in terms of torsion as in TPG.

In this theory, tetrad plays the role of basic entity instead of the metric tensor as in the case of GR. The tetrad formulation of gravitation was considered by Møller with attempts to define the energy of the gravitational field. In 1961, Møller revived Einstein’s idea of unification of gravitation and electromagnetism and presented a theory which based on tetrad field instead of metric tensor [11]. Later, Pellegrini and Plebanski [12] found a Lagrangian formulation for absolute parallelism. Hayashi and Nakano [13] independently worked to formulate the gauge theory of the spacetime translation group. Hayashi [14] also pointed out the connection between the gauge theory of spacetime translation group and absolute parallelism. During the last two decades, much attention has been given to analyze the basic solutions of GR with TPG and comparing the results. There is a great literature available [15-24] on the study of TP versions of the exact solutions of GR.

Recently, Sharif and Jamil [25,26] have found the TP versions of the Friedmann models, Lewis-Papapetrou spacetimes and stationary axisymmetric solutions of the Einstein-Maxwell field equations. They have explored the energy-momentum distribution of these solutions [27] by using certain energy-momentum density developed from Møller’s tetrad theory. Further, they have evaluated the energy contents of static axially symmet-
ric spacetimes [28] by using TP version of four prescriptions, namely, Einstein, Landau-Lifshitz, Bergmann-Thomson and prescription developed from Møller’s tetrad theory. In a recent paper, the same authors [29] introduced the idea of symmetry of a spacetime with torsion only. For this purpose, they defined the TP version of Lie derivative and found the corresponding Killing equations. They evaluated the TP KVs of the Einstein universe.

This paper extends the above work by evaluating the TP KVs of static spherically symmetric and the Friedmann spacetimes. The results are compared with those available in GR. The layout of the paper is as follows. In section 2, we shall give brief review of fundamental concepts of TPG. Section 3 is devoted to explore the KVs of static spherically symmetric spacetimes. In section 4, the TP KVs of the Friedmann metrics have been investigated. The last section contains discussion and conclusion of the results obtained.

2 A Brief Review of Teleparallel Theory of Gravity

In TPG, the gravitational field is described by the tetrad field \( h^a_\mu \) which has been used to define the Weitzenböck connection [10] as

\[
\Gamma^\theta_{\mu\nu} = h^a_\theta \partial_{\nu} h^a_\mu ,
\]

with respect to which tetrad is parallel:

\[
\nabla_\nu h^a_\mu \equiv h^a_{\mu,\nu} - \Gamma^a_{\mu\nu} h^a_\theta = 0.
\]

This is called the absolute parallelism condition, i.e., tetrad are parallelly transported in the Weitzenböck spacetime. Greek alphabets \((\mu, \nu, \rho, \ldots = 0, 1, 2, 3)\) denote spacetime indices and Latin alphabets \((a, b, c, \ldots = 0, 1, 2, 3)\) represent tangent space indices. Spacetime indices and tangent space indices can be replaced by tetrad field satisfying

\[
h^a_\mu h^\nu_a = \delta^\nu_\mu , \quad h^a_\mu h^\mu_b = \delta^a_b.
\]

The presence of non-trivial tetrad field induces both Riemannian and TP structures on a spacetime and gives rise to the Riemannian metric as a by product [10], i.e.,

\[
g_{\mu\nu} = \eta_{ab} h^a_\mu h^b_\nu.
\]
The torsion of the Weitzenböck connection is defined as
\[ T^\theta_{\mu\nu} = \Gamma^\theta_{\nu\mu} - \Gamma^\theta_{\mu\nu}. \] (5)

The relationship between the Weitzenböck and Levi-Civita connections is
\[ \Gamma^\theta_{\mu\nu} = \Gamma^{0\theta}_{\mu\nu} + K^\theta_{\mu\nu}, \] (6)
where
\[ K^\theta_{\mu\nu} = \frac{1}{2} [T^\theta_{\mu\nu} + T^\theta_{\nu\mu} - T^\theta_{\theta\mu\nu}] \] (7)
is a tensor quantity known as contortion tensor and \( \Gamma^{0\theta}_{\mu\nu} \) is the Levi-Civita connection of GR
\[ \Gamma^{0\theta}_{\mu\nu} = \frac{1}{2} g^{\theta\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} + \partial_\sigma g_{\mu\nu}). \] (8)

The TP Lie derivative of a covariant tensor of rank 2 along a vector field \( \xi \) is defined [29] as
\[ (\mathcal{L}_\xi A)_{\mu\nu} = \xi^\rho \nabla_\rho A_{\mu\nu} + (\nabla_\rho \xi^\rho) A_{\mu\nu} + (\nabla_\nu \xi^\rho) A_{\mu\rho}, \] (9)
where \( \nabla_\rho \) represents TP covariant derivative given as
\[ \nabla_\rho A_{\mu\nu} = A_{\mu\nu,\rho} - \Gamma^\theta_{\rho\mu} A_{\theta\nu} - \Gamma^\theta_{\rho\nu} A_{\theta\mu}. \] (10)

In terms of torsion tensor, this can be written as
\[ (\mathcal{L}_\xi A)_{\mu\nu} = \xi^\rho A_{\mu\nu,\rho} - A_{\rho\nu} \xi^\rho_{\mu} + A_{\rho\mu} \xi^\rho_{\nu} + \xi^\rho (A^\theta_{\theta\mu} T^\theta_{\rho\nu} + A^\theta_{\rho\theta} T^\theta_{\nu\rho}). \] (11)
Similarly, for a contravariant tensor of rank 2, the TP Lie derivative turns out to be
\[ (\mathcal{L}_\xi A)^{\mu\nu} = \xi^\rho A^{\mu\nu,\rho} - A^{\mu\nu} \xi^\rho_{\mu} - A^{\mu\rho} \xi^\rho_{\nu} - \xi^\rho (A^\theta_{\theta\mu} T^\theta_{\rho\nu} + A^\theta_{\rho\theta} T^\theta_{\nu\rho}). \] (12)

The extension of this definition to a mixed tensor of rank \( r + s \) is given as
\[ (\mathcal{L}_\xi A)^{\rho...\sigma}_{\mu...\nu} = \xi^\alpha A^{\rho...\sigma}_{\mu...\nu,\alpha} + A^{\rho...\sigma}_{\mu...\nu\alpha} \xi^\alpha_{\rho} + A^{\rho...\sigma}_{\mu...\nu\alpha} \xi^\alpha_{\nu} - A^{\alpha...\sigma}_{\mu...\nu} \xi^\alpha_{\rho} \xi^\alpha_{\nu} - A^{\rho...\sigma}_{\mu...\nu} \xi^\alpha_{\rho} - A^{\rho...\sigma}_{\mu...\nu} \xi^\alpha_{\nu} + \xi^\alpha (A^{\rho...\sigma}_{\beta...\nu} T^\beta_{\mu\alpha} + ... + A^{\rho...\sigma}_{\mu...\beta} T^\beta_{\nu\alpha}) - A^{\beta...\sigma}_{\mu...\nu} T^\beta_{\rho\alpha} - ... - A^{\rho...\beta}_{\mu...\nu} T^\sigma_{\beta\alpha}), \] (13)
where \( |\rho...\sigma| = r \) and \( |\mu...\nu| = s \).

The Killing equations in TPG are defined as
\[ \mathcal{L}_\xi g_{\mu\nu} = 0. \] (14)
In component form, it can be expressed as
\[ g_{\mu\nu,\rho} \xi^\rho + g_{\rho\nu} \xi^\rho_{\mu} + g_{\rho\mu} \xi^\rho_{\nu} + \xi^\rho (g_{\theta\mu} T^\theta_{\rho\nu} + g_{\mu\theta} T^\theta_{\rho\nu}) = 0. \] (15)
3 TP Killing Vectors of Static Spherically Symmetric Spacetimes

The most general static spherically symmetric spacetime is given [30] as
\[
    ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2.
\]  
(16)

Using the procedure [29], the non-vanishing components of the Weitzenböck connection are of the form
\[
\begin{align*}
    \Gamma_{01}^0 &= [\ln \sqrt{|g_{00}|}, r], \quad \Gamma_{12}^2 = \frac{\sqrt{-g_{11}}}{r} = \Gamma_{13}^3, \quad \Gamma_{11}^1 = [\ln \sqrt{-g_{11}}, r], \\
    \Gamma_{21}^2 &= \frac{1}{r} = \Gamma_{31}^3, \quad \Gamma_{22}^1 = -\frac{r}{\sqrt{-g_{11}}} = \Gamma_{32}^3 = \cot \theta = \Gamma_{33}^3, \\
    \Gamma_{33}^2 &= -\sin \theta \cos \theta, \quad \Gamma_{33}^1 = \Gamma_{22}^3 \sin^2 \theta.
\end{align*}
\]  
(17)

The only non-zero components of the torsion tensor are
\[
\begin{align*}
    T_{01}^0 &= -[\ln \sqrt{g_{00}}, r], \quad T_{21}^2 = -\frac{1}{r} (1 - \sqrt{-g_{11}}) = T_{31}^3.
\end{align*}
\]  
(18)

Using these values of the components of torsion tensor in Eq. (15), we get the following system of ten partial differential equations
\[
\begin{align*}
    \xi^0 &= A(r, \theta, \phi), \quad (19) \\
    e^{\lambda} \xi^1_{,0} - e^{\nu} \xi^0_{,1} - \frac{\nu}{2} e^\nu \xi^0 &= 0, \quad (20) \\
    r^2 \xi^2_{,0} - e^{\nu} \xi^0_{,2} &= 0, \quad (21) \\
    r^2 \sin^2 \theta \xi^3_{,0} - e^{\nu} \xi^0_{,3} &= 0, \quad (22) \\
    \lambda \xi^1 + 2 \xi^1_{,1} &= 0, \quad (23) \\
    \xi^2_{,1} + \frac{e^{\lambda}}{r^2} \xi^1_{,2} + \frac{1 - \sqrt{e^\lambda}}{r} \xi^2 &= 0, \quad (24) \\
    \xi^3_{,1} + \frac{e^{\lambda}}{r^2 \sin^2 \theta} \xi^1_{,3} + \frac{1 - \sqrt{e^\lambda}}{r} \xi^3 &= 0, \quad (25) \\
    \xi^1 + \frac{r}{\sqrt{e^\lambda}} \xi^2_{,2} &= 0, \quad (26) \\
    \xi^2_{,3} + \sin^2 \theta \xi^3_{,2} &= 0, \quad (27) \\
    \sqrt{e^\lambda} \xi^1 + r \cot \theta \xi^2 + r \xi^3_{,3} &= 0. \quad (28)
\end{align*}
\]
where prime represents derivative with respect to $r$.

Solving Eqs. (23), (24) and (26) simultaneously, we get

$$\xi^1 = e^{-\frac{r}{2}} \{B_1(t, \phi) \cos \theta + B_2(t, \phi) \sin \theta\},$$  \hspace{1cm} (29)

$$\xi^2 = -\frac{1}{r} \{B_1(t, \phi) \sin \theta - B_2(t, \phi) \cos \theta\} + C(t, \phi) e^{-F(r)},$$  \hspace{1cm} (30)

where

$$F(r) = \int \frac{1 - e^{\frac{r}{2}}}{r} dr.$$  \hspace{1cm} (31)

Substituting this value of $\xi^2$ in Eq. (27), we have

$$\xi^3 = \frac{1}{r} \{B_{1,3}(t, \phi) \ln |\csc \theta - \cot \theta| + B_{2,3}(t, \phi) \csc \theta\}
- C_{,3}(t, \phi) e^{-F(r)} \cot \theta + D(t, r, \phi).$$  \hspace{1cm} (32)

Using these values in the remaining equations, it follows that

$$\xi^0 = e^{-\frac{r}{2}} c_0,$$

$$\xi^1 = e^{-\frac{r}{2}} \{c_1 \cos \theta + (c_2 \cos \phi + c_3 \sin \phi) \sin \theta\},$$

$$\xi^2 = -\frac{1}{r} \{c_1 \sin \theta - (c_2 \cos \phi + c_3 \sin \phi) \cos \theta\} + e^{-F(r)}(c_4 \cos \phi + c_5 \sin \phi),$$

$$\xi^3 = \frac{1}{r} \{(c_3 \cos \phi - c_2 \sin \phi) \csc \theta\}
- e^{-F(r)} \cot \theta(c_4 \sin \phi - c_5 \cos \phi) + c_6 e^{-F(r)}.$$  \hspace{1cm} (33)

Thus we obtain seven TP KVs of static spherically symmetric spacetimes given as

$$\xi_0 = e^{-\frac{r}{2}} \frac{\partial}{\partial t},$$

$$\xi_1 = e^{-F(r)} \frac{\partial}{\partial \phi},$$

$$\xi_2 = e^{-F(r)}(\cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi}),$$

$$\xi_3 = e^{-F(r)}(\sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi}),$$

$$\xi_4 = e^{-\frac{r}{2}} \cos \theta \frac{\partial}{\partial r} - \frac{1}{r} \sin \theta \frac{\partial}{\partial \theta},$$
\[ \xi_{(5)} = e^{-\frac{\dot{a}}{a}} \sin \theta \cos \phi \frac{\partial}{\partial r} + \frac{1}{r} \cos \phi \cos \theta \frac{\partial}{\partial \theta} - \frac{1}{r} \sin \phi \csc \theta \frac{\partial}{\partial \phi}, \]
\[ \xi_{(6)} = e^{-\frac{\dot{a}}{a}} \sin \theta \sin \phi \frac{\partial}{\partial r} + \frac{1}{r} \sin \phi \cos \theta \frac{\partial}{\partial \theta} + \frac{1}{r} \cos \phi \csc \theta \frac{\partial}{\partial \phi}. \]

4 TP Killing Vectors of the Friedmann Models

The metric representing the FRW models is given [31] as
\[ ds^2 = dt^2 - a^2(t)[d\chi^2 + f_k^2(\chi)(d\theta^2 + \sin^2 \theta d\phi^2)], \]
where
\[ f_k(\chi) = \sinh \chi, \quad k = -1, \]
\[ = \chi, \quad k = 0, \]
\[ = \sin \chi, \quad k = +1, \]

The non-zero components of the Weitzenböck connection are
\[ \Gamma^1_{10} = \Gamma^2_{20} = \Gamma^3_{30} = \frac{\dot{a}}{a}, \]
\[ \Gamma^1_{22} = -f_k, \quad \Gamma^1_{33} = \Gamma^1_{22} \sin^2 \theta, \]
\[ \Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{f_k}, \quad \Gamma^2_{21} = \Gamma^3_{31} = \Gamma^2_{12} f'_k, \]
\[ \Gamma^2_{33} = -\sin \theta \cos \theta, \quad \Gamma^3_{23} = \cot \theta = \Gamma^3_{32}, \]

where dot and prime denote the derivatives with respect to \( t \) and \( \chi \) respectively. Consequently, the non-vanishing components of the torsion tensor are
\[ T^1_{10} = T^2_{20} = T^3_{30} = -T^1_{01} = -T^2_{02} = -T^3_{03} = -\frac{\dot{a}}{a}, \]
\[ T^2_{21} = T^3_{31} = -T^2_{12} = -T^3_{13} = \frac{1}{f_k}(1 - f'_k). \]

In component form the TP Killing equations for this metric turn out to be
\[ \xi^0 = A(\chi, \theta, \phi), \]
\[ \xi^1 = B(t, \theta, \phi), \]  
\[ \xi^2,3 + \sin^2 \theta \xi^3,2 = 0, \]  
\[ a^2 \xi^1,0 - \xi^0,1 + a \dot{\xi} = 0, \]  
\[ \dot{\xi}_k + a \xi^3,2 + a^2 \xi^1 = 0, \]  
\[ f_k^2 \xi^1,1 + \xi^1,2 - f_k \{1 - f_k^2\} \xi^2 = 0, \]  
\[ f_k^2 \sin^2 \theta \xi^3,1 + \xi^1,3 - f_k \{1 - f_k^2\} \sin^2 \theta \xi^3 = 0, \]  
\[ \dot{a} f_k \xi^0 + a \xi^1 + a f_k \cot \theta \xi^2 + a f_k \xi^3,3 = 0, \]  
\[ a^2 f_k^2 \xi^2,0 - \xi^0,2 + a \dot{f}_k \xi^2 = 0, \]  
\[ a^2 f_k^2 \sin^2 \theta \xi^3,0 - \xi^3,3 + a \dot{f}_k \sin^2 \theta \xi^3 = 0. \]  

Solving Eqs. (43), (44) and (47) along with Eqs. (39) and (40), we get

\[ a = c_1 t + c_2 \]  
and

\[ \xi^0 = e^{F(\chi)} \{A_1(\phi) \theta + A_2(\phi)\}, \]
\[ \xi^1 = \frac{1}{a} \{B_1(\phi) \cos \theta + B_2(\phi) \sin \theta\}, \]
\[ \xi^2 = -\frac{\dot{a}}{a} e^{F(\chi)} \{A_1(\phi) \theta^2 + A_2(\phi) \theta\} \]
\[ - \frac{1}{a f_k} \{B_1(\phi) \sin \theta - B_2(\phi) \cos \theta\} + C(t, \chi, \phi), \]  
where

\[ F(\chi) = \int \frac{1 - f_k}{f_k} d\chi. \]  

Now solving Eqs. (41) and (46) along with Eq. (49), we have

\[ \xi^0 = e^{F(\chi)} (c_3 \phi + c_4) \theta + c_5 \phi + c_6, \]
\[ \xi^1 = \frac{1}{a} \{(c_7 \phi + c_8) \cos \theta + (c_9 \cos \phi + c_{10} \sin \phi) \sin \theta\}, \]
\[ \xi^2 = -\frac{\dot{a}}{a} e^{F(\chi)} \{(c_3 \phi + c_4) \theta^2 + (c_5 \phi + c_6) \theta\} \]
\[ - \frac{1}{a f_k} \{(c_7 \phi + c_8) \sin \theta - (c_9 \cos \phi + c_{10} \sin \phi) \cos \theta\} \]
\[ + C_1(t, \chi) \cos \phi + C_2(t, \chi) \sin \phi, \]
\[ \xi^3 = -\frac{\dot{a}}{a} e^{F(\chi)} \{(c_3 \phi^2 + c_4) \theta + (c_5 \phi^2 + c_6) \phi\} \]
\[- \frac{1}{af_k} \{(c_9 \sin \phi - c_{10} \cos \phi) \sin \theta \}
\]
\[+ \cot \theta \frac{\dot{\alpha}}{a} e^{F(\chi)} \{(c_3 \frac{\phi^2}{\phi} + c_4 \phi) \frac{\theta^2}{2} + (c_5 \frac{\phi^2}{2} + c_6 \phi) \theta \}
\]
\[- \frac{\cot \theta}{af_k} \{(c_9 \sin \phi - c_{10} \cos \phi) \cos \theta \}
\]
\[- \cot \theta \{C_1(t, \chi) \sin \phi - C_2(t, \chi) \cos \phi \} + D(t, \chi, \theta), \quad (51)\]

Using these values of \(\xi^0, \xi^1, \xi^2\) and \(\xi^3\) in the remaining equations, we obtain

\[\xi^0 = 0,\]
\[\xi^1 = \frac{1}{a} \{c_1 \cos \theta + (c_2 \cos \phi + c_3 \sin \phi) \sin \theta \},\]
\[\xi^2 = - \frac{1}{af_k} \{c_1 \sin \theta - (c_2 \cos \phi + c_3 \sin \phi) \cos \theta \} + \frac{e^{F(\chi)}}{a} \{c_4 \cos \phi + c_5 \sin \phi \},\]
\[\xi^3 = - \frac{1}{af_k} \{(c_2 \sin \phi - c_3 \cos \phi) \sin \theta \} - \frac{\cot \theta}{af_k} \{(c_2 \sin \phi - c_3 \cos \phi) \cos \theta \}
\[- \frac{e^{F(\chi)}}{a} \cot \theta \{c_4 \sin \phi - c_5 \cos \phi \} + c_6 \frac{e^{F(\chi)}}{a}. \quad (52)\]

This leads to six TP KVs of FRW spacetimes

\[\xi(0) = \frac{e^{F(\chi)}}{a} \frac{\partial}{\partial \phi},\]
\[\xi(1) = \frac{e^{F(\chi)}}{a} \{\sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi} \},\]
\[\xi(2) = \frac{e^{F(\chi)}}{a} \{\cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi} \},\]
\[\xi(3) = \frac{1}{a} \{\cos \theta \frac{\partial}{\partial \chi} - \frac{1}{f_k} \sin \theta \frac{\partial}{\partial \theta} \},\]
\[\xi(4) = \frac{1}{a} \{(\sin \theta \frac{\partial}{\partial \chi} + \frac{1}{f_k} \cos \theta \frac{\partial}{\partial \theta}) \cos \phi - \frac{1}{f_k} (\sin \theta + \cot \theta \cos \phi) \sin \phi \frac{\partial}{\partial \phi} \},\]
\[\xi(5) = \frac{1}{a} \{(\sin \theta \frac{\partial}{\partial \chi} + \frac{1}{f_k} \cos \theta \frac{\partial}{\partial \theta}) \sin \phi + \frac{1}{f_k} (\sin \theta + \cot \theta \cos \phi) \cos \phi \frac{\partial}{\partial \phi} \}. \quad (53)\]
5 Summary and Discussion

This paper is devoted to investigate the symmetries in the teleparallel theory of gravitation. We have evaluated TP KVs for static spherically symmetric spacetimes and the Friedmann metrics. We find that there exist seven TP KVs of static spherically symmetric spacetimes while there are four KVs in GR. If we compare KVs of TPG and GR, it follows that $\xi^{(0)}$ and $\xi^{(\rho)} (\rho = 1, 2, 3)$ in TPG are multiple of the corresponding KVs in GR by $e^{-\frac{\nu}{2}}$ and $e^{-F(r)}$ respectively. The first four TP KVs coincide to the four KVs of GR (as given in appendix A) for $\nu = 0$ and $F(r) = 0$, i.e., for $\lambda = 0$. These conditions reduce the metric (16) to Minkowski spacetime in spherical polar coordinates. On the other hand, under these assumptions, Eq. (18) implies that all components of the torsion tensor will become zero and hence the TP Killing equations will reduce to the Killing equations of GR. This yields the trivial four KVs of the spherical symmetry. The remaining three TP KVs are due to the non-vanishing components of the torsion tensor involving in the TP Killing equations. In view of these results, it can be conjectured that:

In TPG, the number of independent KVs are restricted to seven for all static spherically symmetric spacetimes.

We have also explored TP KVs of the Friedman models. It is found that these are six TP KVs for each model similar to GR [32]. However, the comparison of these results with those in GR shows that, for $k = 0$, the first four TP KVs $\xi^{(\rho)} (\rho = 0, 1, 2, 3)$ are multiple of the corresponding KVs of GR by $\frac{1}{a}$ while the remaining KVs are different. This difference occurs because of the presence of the torsion components in TP Killing equations. For $a(t) = 1$, the first four KVs are the same in both GR (as given in appendix A) and TPG which leads to Minkowski spacetime. In TPG, the value of scale factor is found to be $a(t) = c_1 t + c_2$ which exactly matches with the value found [30] in GR for $c_1 = \frac{1}{b}$ and $c_2 = 0$. For both open and closed FRW models, the KVs do not match in both the theories. This difference is due to the same reason and hence the Lie algebra of these TP KVs is not closed.

We conclude that the results of KVs do not coincide for the two theories. It seems that the reason is the non-vanishing components of torsion. It would be interesting to explore the compatibility of the two theories TPG and GR and interpret exactly the TP KVs corresponding to GR.
Appendix A

Killing Vectors of Static Spherically Symmetric Spacetimes in GR

Killing vectors for static spherically symmetric spacetimes in GR are

\[ \xi^{(0)} = \frac{\partial}{\partial t}, \]
\[ \xi^{(1)} = \frac{\partial}{\partial \phi}, \]
\[ \xi^{(2)} = \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi}, \]
\[ \xi^{(3)} = \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi}. \]  
(A1)

Killing Vectors of Friedmann Spacetimes in GR

Linearly independent KVs associated with the FRW models are [43]:

For \( k = 0 \)

\[ \xi^{(0)} = \frac{\partial}{\partial \phi}, \]
\[ \xi^{(1)} = \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi}, \]
\[ \xi^{(2)} = \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi}, \]
\[ \xi^{(3)} = \left( \cos \theta \frac{\partial}{\partial \chi} - \frac{1}{\chi} \sin \theta \frac{\partial}{\partial \phi} \right), \]
\[ \xi^{(4)} = \left( \sin \theta \frac{\partial}{\partial \chi} + \frac{1}{\chi} \cos \theta \frac{\partial}{\partial \theta} \right) \cos \phi - \frac{1}{\chi} \csc \theta \sin \phi \frac{\partial}{\partial \phi}, \]
\[ \xi^{(5)} = \left( \sin \theta \frac{\partial}{\partial \chi} + \frac{1}{\chi} \cos \theta \frac{\partial}{\partial \theta} \right) \sin \phi + \frac{1}{\chi} \csc \theta \cos \phi \frac{\partial}{\partial \phi}. \]  
(A2)

For \( k = 1 \)

\[ \xi^{(0)} = \frac{\partial}{\partial \phi}, \]
\[ \xi_{(1)} = \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi}, \]
\[ \xi_{(2)} = \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi}, \]
\[ \xi_{(3)} = (\cos \theta \frac{\partial}{\partial \chi} - \cot \chi \sin \theta \frac{\partial}{\partial \theta}), \]
\[ \xi_{(4)} = (\sin \theta \frac{\partial}{\partial \chi} + \cot \chi \cos \theta \frac{\partial}{\partial \theta}) \cos \phi - \cot \chi \csc \theta \sin \phi \frac{\partial}{\partial \phi}, \]
\[ \xi_{(5)} = (\sin \theta \frac{\partial}{\partial \chi} + \cot \chi \cos \theta \frac{\partial}{\partial \theta}) \sin \phi + \cot \chi \csc \theta \cos \phi \frac{\partial}{\partial \phi}. \]  

For \( k = -1 \)
\[ \xi_{(0)} = \frac{\partial}{\partial \phi}, \]
\[ \xi_{(1)} = \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi}, \]
\[ \xi_{(2)} = \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi}, \]
\[ \xi_{(3)} = (\cos \theta \frac{\partial}{\partial \chi} - \coth \chi \sin \theta \frac{\partial}{\partial \theta}), \]
\[ \xi_{(4)} = (\sin \theta \frac{\partial}{\partial \chi} + \coth \chi \cos \theta \frac{\partial}{\partial \theta}) \cos \phi - \coth \chi \csc \theta \sin \phi \frac{\partial}{\partial \phi}, \]
\[ \xi_{(5)} = (\sin \theta \frac{\partial}{\partial \chi} + \coth \chi \cos \theta \frac{\partial}{\partial \theta}) \sin \phi + \coth \chi \csc \theta \cos \phi \frac{\partial}{\partial \phi}. \]

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**References**

[1] Petrov, A.Z.: *Einstein Spaces* (Pergamon, Oxford University Press, 1989).

[2] Bokhari, A.H. and Qadir, A.: J. Math. Phys. 31(1990)1463.

[3] Bokhari, A.H. and Qadir, A.: J. Math. Phys. 34(1993)3543.

[4] Ziad, M., Ph.D. Thesis (Quaid-i-Azam University, 1990).
[5] Qadir, A. and Ziad, M.: Nuovo Cimento B110(1995)317.
[6] Howard, D. and Stachel, J. (editors): Einstein and the History of General Relativity (Birkhauser, Boston, 1989).
[7] Hayashi, K. and Shirafuji, T.: Phys. Rev. D19(1979)3524.
[8] Weitzenböck, R.: Invarianten Theorie (Gronningen: Noordhoft, 1923).
[9] De Andrade, V.C. and Pereira, J.G.: Phys. Rev. D56(1997)4689.
[10] Aldrovandi, R. and Pereira, J.G.: An Introduction to Gravitation Theory (preprint).
[11] Møller, C.: K. Dan. Vidensk. Selsk. Mat. Fys. Skr. 1(1961)10.
[12] Pellegrini, C. and Plebanski, J.: K. Dan. Vidensk. Selsk. Mat. Fys. Skr. 2(1962)4.
[13] Hayashi, K. and Nakano, T.: Prog. Theor. Phys. 38(1967)491.
[14] Hayashi, K.: Phys. Lett. B69(1977)441.
[15] Hehl, F.W. and Macias, A.: Int. J. Mod. Phys. D8(1999)399.
[16] Obukhov, Yu N., Vlachynsky, E.J., Esser, W., Tresguerres, R. and Hehl, F.W.: Phys. Lett. A220(1996)1.
[17] Baekler, P., Gurses, M., Hehl, F.W. and McCrea, J.D.: Phys. Lett. A128(1988)245.
[18] Vlachynski, E.J., Esser, W., Tresguerres, R. and Hehl, F.W.: Class. Quantum Grav. 13(1996)3253.
[19] Ho, J.K., Chern, D.C. and Nester, J.M.: Chin. J. Phys. 35(1997)640.
[20] Hehl, F.W., Lord, E.A. and Smalley, L.L.: Gen. Relativ. Gravit. 13(1981)1037.
[21] Kawai, T. and Toma, N.: Prog. Theor. Phys. 87(1992)583.
[22] Pereira, J.G., Vargas, T. and Zhang, C.M.: Class. Quantum Grav. 18(2001)833.

13
[23] Nashed, G.G.L.: Phys. Rev. D66(2002)064015.

[24] Kawai, T. and Toma, N.: Prog. Theor. Phys. 38(1992)583.

[25] Sharif, M. and Amir, M. Jamil.: Gen. Relativ. Gravit. 38(2006)1735.

[26] Sharif, M. and Amir, M. Jamil.: Gen. Relativ. Gravit. 39(2007)989.

[27] Sharif, M. and Amir, M. Jamil.: Mod. Phys. Lett. A22(2007)425.

[28] Sharif, M. and Amir, M. Jamil.: Mod. Phys. Lett. A(2008, to appear).

[29] Sharif, M. and Amir, M. Jamil.: Mod. Phys. Lett. A23(2008)963.

[30] Stephani, H., Kramer, D., MacCallum, M.A.H., Hoenselaers, C. and Hearlt, E.: Exact Solutions of Einstein Field Equations (Cambridge University Press, 2003).

[31] Hartle, J.B.: Gravity: An Introduction to Einstein’s General Relativity (Baba Barkha Nath Printers, India, 2006).

[32] Maartens, R. and Maharaj, S.D.: Class. Quantum Grav. 3(1986)1005.