The Analytic Quantum Information Manifold

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Abstract

Let \( H \geq I \) be a self-adjoint operator on a Hilbert space, such that \( e^{-\beta H} \) is of trace-class for some \( \beta < 1 \). Let \( V \) be a symmetric operator such that \( \| V \|_\omega := \| R V \| < \infty \), where \( R = H^{-1} \). We show that the partition function \( \text{Tr} e^{-\lambda (H + \lambda V)} \) is analytic in \( \lambda \) in a hood of the origin in the sense of Fréchet, in the Banach space with norm \( \| \cdot \|_\omega \). This is applied to the quantum information manifold defined by \( H \).

Keywords: Fisher metric, Bogoliubov Kubo Mori Green function, Orlicz space.

1 Introduction

The information manifold in non-parametric estimation theory has been introduced and studied by Pistone and Sempi [20, 5]. For a finite number of parameters, the theory is described in the books [1, 9]. The quantum version has been introduced by Hasagawa [7, 8] and by Nagaoka [14], and by Petz [16, 17]. This study is strictly valid only if the Hilbert space \( \mathcal{H} \) of states is finite dimensional. It provides a geometrical picture, as well as rigour, to the theory of linear response [10, 13, 4, 12, 3, 22]. For \( \dim \mathcal{H} = \infty \), but limited to bounded potentials, the theory has been developed by Araki [4] in the context of Tomita-Takesaki theory. Our more concrete Hilbert-space version of this goes as follows.

Denote by \( \mathcal{C}_1 \) the set of trace-class operators on \( \mathcal{H} \) and let \( \Sigma \subseteq \mathcal{C}_1 \) denote the set of density operators. Given \( \rho_0 \in \Sigma \) let \( H_0 = -\log \rho_0 + cI \geq I \) be a self-adjoint operator with domain \( \mathcal{D}(H_0) \) such that

\[
\rho_0 = Z_0^{-1} e^{-H_0}.
\]
We recognise $H_0$, which is unique up to the constant $c$, as the Hamiltonian, and $Z_0$ as the partition function, whose finiteness expresses thermodynamic stability. Let $X$ be a symmetric bounded operator, so $X \in B(\mathcal{H})$, and perturb $H_0$ to $H_X = H_0 + X$, with corresponding state $\rho_X = Z_X^{-1} \exp(-H_X)$ and free energy $\psi_X = \log Z_X$. Araki proves that $\psi_X$ is an analytic functional with a convergent Taylor series, whose first few terms are

$$
\nabla_Y^+ \psi_X = \rho_X Y := \text{Tr}(\rho_X Y) \tag{2}
$$

$$
\nabla_Y^+ \nabla_Z^+ \psi_X = \int_0^1 d\lambda \text{Tr} \left\{ \rho_X^\lambda \hat{Y} \rho_X^{\lambda'} \hat{Z} \right\} := g_X(Y, Z) \tag{3}
$$

$$
\nabla_Y^+ \nabla_Z^+ \nabla_W^+ \psi_X = \int_0^1 \prod d\alpha_i \delta(\sum \alpha_i - 1) \text{Tr} (\rho_X^{\alpha_1} Y \rho_X^{\alpha_2} Z \rho_X^{\alpha_3} W) \nonumber \\
\phantom{=} := t_X(Y, Z, W). \tag{4}
$$

Here, $\nabla_Y^+$ denotes the Gateaux derivative of $\psi_X$ in the direction $Y \in B(\mathcal{H})$, $\hat{Y} = Y - \rho_X Y$ is the centred observable (the so-called “score”), $g_X(Y, Z)$ is the Bogoliubov-Kubo-Mori (BKM) metric, and $\text{Tr}(\ldots)_c$ denotes the cumulant of third order. We adopt the convention throughout this paper that if $\lambda \in [0, 1]$ then $\lambda' = 1 - \lambda$. The torsion, $t$, measures the failure of $g$ to be invariant under $(+1)$-parallel transport.

In [23, 24, 15] we began a study of the unbounded case. Let $C_p, 0 < p < 1$ denote the set of compact maps $A : \mathcal{H} \to \mathcal{H}$ such that $|A|^p \in \mathcal{C}_1$, and put

$$
\mathcal{C}_{<1} := \bigcup_{0 < p < 1} C_p. \tag{5}
$$

It is known [14] that if $p < q$ then $C_p \subset C_q$ and that $C_p$ is a complete linear space with quasinorm

$$
\|A\|_p := [\text{Tr}(A^* A)^{p/2}]^{1/p}. \tag{6}
$$

In [24] we took the underlying set of the quantum information manifold to be

$$
\mathcal{M} = \mathcal{C}_{<1} \cap \Sigma. \tag{7}
$$

$\mathcal{M}$ has a natural affine structure, coming from the linear structure of each $C_p$; it is called the $(-1)$-affine structure. Thus if $\rho_i \in C_{p_i} \cap \Sigma, i = 1, 2$ let $p = \max\{p_1, p_2\}$; then $\rho_i \in C_p \cap \Sigma, i = 1, 2$. We define $\lambda \rho_1 + \lambda' \rho_2, 0 \leq \lambda \leq 1$ as the usual sum of operators in $C_p$. We see that $\mathcal{M}$ is $(-1)$-convex.
There is a quantum analogue of the classical Orlicz space $L \log L$: let

$$C_1 \log C_1 := \{ \rho \in C_1 : S(\rho) := -\sum_{i=1}^{\infty} \lambda_i \log \lambda_i < \infty \}$$

(8)

where $\{ \lambda_i \}$ are the singular numbers of $\rho$. Thus the set of normal states of finite entropy is $C_1 \log C_1 \cap \Sigma$. It is easy to show that $C_{<1} \subset C_1 \log C_1$.

The topology given to $\mathcal{M}$ in [24] is not that induced on it as a subset of $C_1$. Following an idea in [20] we constructed a hood of $\rho_0 \in \mathcal{M}$ consisting of all $\rho \in \mathcal{M}$ which are ‘connected to $\rho_0$ by a one-parameter exponential family of states’. In [24] this is the family

$$\rho_{\lambda X} = Z_{\lambda}^{-1} e^{-(H_0 + \lambda X + c(\lambda))} \in \mathcal{M}, \quad 0 \leq \lambda \leq 1,$$

(9)

where $X$ is a symmetric quadratic form defined on $\mathcal{D}(H_1^{1/2})$ and form-bounded relative to the energy form $q_0(\phi, \psi) := \langle H_0^{1/2} \phi, H_0^{1/2} \psi \rangle$.

The state $\rho_{\lambda X}$ does not depend on the choice of $c(\lambda)$ and we do this so that $H := H_0 + \lambda X + c(\lambda) \geq I$. We write $R = R_X := H^{-1}$ for the inverse of $H$ at $\lambda = 1$. If $\rho_0 \in C_{\beta_0}$, to make sense we need that the $q_0$-bound of $X$ be less than $1 - \beta_0$. We then showed that $\rho_X \in C_{\beta_X}$ for some $\beta_X < 1$. The set of $q_0$-bounded quadratic forms $X$ is a Banach space $\mathcal{T}_0$ with norm

$$\|X\|_0 := \|R_0^{1/2}X R_0^{1/2}\|,$$

(11)

where $\| \cdot \|$ is the operator norm for $\mathcal{H}$, and $R_0 = H_0^{-1}$. We showed that sufficient for $\rho_X \in \mathcal{M}$ is that $\|X\|_0 < 1 - \beta_0$, and that $Z_X$ is a Lipschitz function of $X$; however, we could not show that the first moment $\text{Tr}(\rho_0 X)$ is finite. We introduced the regularised mean of $X \in \mathcal{T}_0$,

$$\rho_0.X := \text{Tr}[\rho_0^\lambda X \rho_0^{\lambda'}]$$

(12)

and showed that it is finite and independent of $\lambda \in (0, 1)$.

The hood of $\rho_0$ defined by $\{ \rho_X : \|X\|_0 < 1 - \beta_0 \}$ can be mapped bijectively onto the open ball of radius $1 - \beta_0$ in the Banach space

$$\hat{T}_0 := \{ X \in \mathcal{T}_0 : \rho_0.X = 0 \};$$

(13)

thus, $\hat{T}_0$ consists of score variables. This map makes the hood into a manifold modelled on $\hat{T}_0$. The linear structure of $\hat{T}_0$ induces a local affine structure.
on $\mathcal{M}$, called the (+1)-affine structure. To any point $\rho_X$, we may similarly define a hood of states of $\rho_X$ to be

$$\left\{ \rho_Y : \|Y\|_X := \|R_X^{1/2}YR_X^{1/2}\| < a \leq 1 - \beta_X \right\},$$

which is in bijection with an open ball in the Banach space

$$\tilde{T}_X := \{Y: \|Y\|_X < \infty \text{ and } \rho_X.Y = 0\}.$$  

We showed that on the overlap region, the norms $\|\cdot\|_0$ and $\|\cdot\|_X$ are equivalent norms; the (+1)-affine structures on the region of overlap are also the same. Thus we have covered $\mathcal{M}$ by a Banach manifold with a local affine structure. We showed that the connected component containing $\rho_0$ is (+1)-convex.

In the present paper we consider the problem of furnishing $\mathcal{M}$ with a Riemannian metric. As shown by Petz [17], in the finite-dimensional case there are several candidates, but for geometric reasons we choose the BKM metric. Indeed, the studies of Nagaoka [14] show that +1 and −1-affine structures are not dual relative to the ‘Uhlmann’ metric, (the real part of $\text{Tr} \rho X^*Y$). They are dual, however, relative to the BKM-metric. Since this is the second derivative of $\psi$, in infinite dimensions we must impose more regularity than we have so far. Apart from having this ‘Amari’ duality, similar to the classical Fisher-Rao metric, the BKM-metric is the one entering Kubo’s theory of linear response, as well as the one arising in quantum statistical dynamics.

In this paper we assume that $X$ is an operator which is $H_0$-bounded. We then say that the tangent direction $X$ is analytic. We show in §(2) that this leads to the Fréchet differentiability of $\psi_0$ in the analytic direction $X$, with Lipschitz continuous derivative

$$\nabla_X \psi_0 = \rho_0.X,$$

and that $\rho_0.X$ is of trace class. This means that the regularisation of the mean is not necessary. We show that a metric $g$ can be defined by eq. (3).

In §(3) we show that the $n$-point Kubo cumulant is finite, and that for small $|\lambda|$ the Taylor series for $Z_{\lambda V}$ converges if $V$ is an analytic direction. We show that when we furnish $\mathcal{M}$ with the analytic topology, it becomes a real analytic Banach manifold.
2 \( H_0 \)-bounded perturbations

Let \( H_0 \geq I \) be a selfadjoint operator with domain \( \mathcal{D}(H_0) \) and quadratic form \( q_0 \). A necessary and sufficient condition for a symmetric operator \( X : \mathcal{D}(H_0) \to \mathcal{H} \) to be \( H_0 \)-bounded is that \( R_0 X \) be bounded (21). One can show that

\[
\|X\|_0 := \|R_0^{1/2} X R_0^{1/2}\| \leq \|R_0 X\| \tag{17}
\]

It follows that if \( X \) is \( H_0 \)-bounded, then its form is \( q_0 \)-bounded. Suppose now that \( \rho_0 := Z_0^{-1} e^{-H_0} \in C_{\beta_0}, \beta_0 < 1 \), and that \( \|R_0 X\| < 1 - \beta_0 \). Then by eq. (17), \( \|X\|_0 < 1 - \beta_0 \) and so by (24) the perturbed state

\[
\rho_X = Z_X^{-1} e^{-H_X} \in C_{\beta_X} \subset C_{<1} \tag{18}
\]

lies in a hood of \( \rho_0 \) in the topology induced by \( \mathcal{T}_0 \). Let us define a stronger topology than this, by taking hoods of \( \rho_0 \) to be

\[
\{\rho_X : \|X\|_\omega := \|R_0 X\| < a \leq 1 - \beta_0\}. \tag{19}
\]

We call this the analytic or \( \omega \)-topology, and \( \| \cdot \|_\omega \) the \( \omega \)-norm. Similarly, if \( \rho_X \) is in the analytic hood of \( \rho_0 \), and \( Y \) is \( H_X \)-bounded with \( \|R_X Y\| < 1 - \beta_X \), then \( \rho_{X+Y} \in \mathcal{M} \) and the analytic hoods of \( \rho_X \) can be defined by

\[
\{\rho_{X+Y} : \|R_X Y\| < a \leq 1 - \beta_X\}. \tag{20}
\]

Let \( \hat{\mathcal{T}}_\omega(\rho_X) := \{Y : \|Y R_X\| < \infty, \rho_X Y = 0\} \). Then the construction of the \( \omega \)-hoods covers \( \mathcal{M} \) by sets homeomorphic to balls in the Banach spaces \( \hat{\mathcal{T}}_\omega(\rho) \); these have mutually equivalent norms on the overlaps of the hoods. To see this, suppose that \( Y \) is in the hood of \( \rho_0 \) and also of \( \rho_X \). The charts around \( \rho_0 \) and \( \rho_X \) take \( \rho_Y \) to balls in the Banach spaces \( \hat{\mathcal{T}}_\omega(\rho_0) \) and \( \hat{\mathcal{T}}_\omega(\rho_X) \), with norms \( \|R_0 Y\| \) and \( \|R_X Y\| \). These are equivalent, since

\[
m \|Y R_0\| \leq \|Y R_X\| \leq M \|R_0 Y\| \quad \text{for all } Y, \tag{21}
\]

where \( m = \|(H_X + I) R_0\|^{-1} \) and \( M = \|(H_0 + I) R_0\| \). Thus \( \mathcal{M} \) is made into a Banach manifold, called the \( \omega \) quantum information manifold. This name will be justified in the next section. It is easy to see that the local \((+1)\)-affine structure coming from the linear structure of \( \hat{\mathcal{T}}_\omega(\rho_X) \) is compatible with that given in (24).

The following lemma will be needed later. It is clear that \( R_{X+Y} \) maps \( \mathcal{H} \) into \( \mathcal{D}(H_0) = \mathcal{D}(H_X) = \mathcal{D}(H_{X+Y}) \). It follows from lemma 6, (24) that \( H_X R_{X+Y} \) is bounded. We find

\[
(I + Y R_X) (H_X R_{X+Y}) = H_X R_{X+Y} + Y R_{X+Y} = I. \tag{22}
\]
Hence \( H_X R_{X+Y} = (I + Y R_X)^{-1} \), and we get if \( \| Y R_X \| < 1 \),
\[
\| H_X R_{X+Y} \| \leq (1 - \| Y R_X \|)^{-1}. \tag{23}
\]
From this we see

**Lemma 2.1** If \( a < 1 \) and \( X \) is \( H_0 \)-small, then \( \| H_X R_{X+Y} \| \) is bounded in the set \( \{ Y : \| Y R_X \| \leq a \} \). \( \blacksquare \)

Now suppose that \( X \) perturbs \( H_0 \) and \( Y \) perturbs \( H_X \), as above. Then

**Theorem 2.2** If \( 1 - 2\delta > \beta_X \), then \( \rho_X^\delta Y \) has a trace-class extension.

Proof. Choose \( \delta > 0 \) so that \( \rho_X^{1-2\delta} \in \mathcal{M} \). Then
\[
\| \rho_X^{1-\delta} Y \|_1 = \| Z_X^{\delta} e^{-(1-\delta) H_X Y} \|_1 \leq C \| \rho_X^{1-2\delta} \|_1 \| e^{-\delta H_X} H_X \|_\infty \| R_X Y \|_\infty \leq \infty.
\]

Corollary. The regularised mean coincides with the true mean:
\[
\text{Tr}(\rho_X^\delta Y \rho_X^{1-\lambda}) = \text{Tr}(\rho_X Y), \quad \text{for all} \ \lambda \in (0, 1). \tag{24}
\]
For, we can write \( \rho_X Y = \rho_X^\delta \rho_X^{\delta Y} \) and use the cyclicity of the trace.

**Theorem 2.3** In the analytic manifold, \( \psi_X := \log Z_X \) is \( \nabla_Y^+ \) Fréchet differentiable, and
\[
\nabla_Y^+ \psi_X = \rho_X Y \tag{25}
\]

Proof. By two applications of Duhamel’s formula [24], Th. 9, we get for the difference quotient,
\[
\lambda^{-1} \text{Tr} \left[ e^{-(H_X + \lambda Y)} - e^{-H_X} \right] - \text{Tr} \left[ e^{-H_X Y} \right] = \\
= \text{Tr} \int_0^1 \alpha d\alpha \int_0^1 d\beta \left\{ e^{-\alpha \beta (H_X + \lambda Y)} \lambda Y e^{-\alpha' H_X Y} e^{-\alpha' H_X} \right\}.
\]
We can put the trace inside the integral, by Fubini’s theorem, if we can show that by doing so we get an absolutely convergent integral of the trace norm. The right hand side is then obviously \( O(\lambda) \), with constant equal to the trace of an operator of the form
\[
C = \int \int \int_0^1 \int_0^1 \rho_1^{\alpha_1} Y \rho_2^{\alpha_2} Z \rho_3^{\alpha_3} \prod_{i=1}^3 d\alpha_i \delta \left( \sum_{i=1}^3 \alpha_i - 1 \right), \tag{26}
\]
which is the quantum version of the integral remainder in Taylor’s theorem.

We divide the region of integration into three (overlapping) parts; in region $i$, $\alpha_i \geq 1/3, (i = 1, 2, 3)$. The estimate is similar in each case. The hardest case is when the variable larger than 1/3, say $\alpha_1$, involves the state $\rho_1 = Z\lambda^{-1} e^{-\lambda X}$, as this requires an estimate independent of $\lambda$. We therefore do this case in detail. So take $\alpha_1 \geq 1/3$, and $\alpha_2, \alpha_3 > 0$, and $H_i \geq I$. Put $R_i = H_i^{-1}$. We now show that for small $\delta > 0$, $\rho_1^{1-\delta} Y \rho_2^{\alpha_2} Z \rho_3^{\alpha_3}$ is of trace class. Indeed, by Hölder,

$$\|\rho_1^{1-\delta} Y \rho_2^{\alpha_2} Z \rho_3^{\alpha_3}\|_1 \leq \|\rho_1^{1-2\delta}\|_{1/\alpha_1}\|Y \rho_1\|_1\|\rho_2^{\alpha_2}\|_{1/\alpha_2}\|Z \rho_3\|_1\|\rho_3^{\alpha_3}\|_{1/\alpha_3} \leq \|\rho_1^{1-2\delta/\alpha_1}\|_{1/\alpha_1}\|Y R_1\|_1\|H_1 \rho_1\|_1\|Z R_3\|_1\|H_3 \rho_3\|_1\|\rho_3^{1-\gamma/\alpha_3}\|_{1/\alpha_3}.$$

If $2\delta < \beta_1 \alpha_1$ and $\gamma < \alpha_3 \beta_3$, then all the norms are finite; here, the $\beta_i$ are such that $\rho_i \in C_{\beta_i}$. The size of $\gamma$ depends on $\alpha_3$, but in the region $\alpha_3 \geq 1/3$, $\delta$ can be chosen independent of $\alpha_3$. Now take $2\delta < \beta_1 \alpha_1 / 3$. Then for each $\alpha_1 \geq 1/3, \alpha_2 > 0, \alpha_3 > 0$ we can take a bit, $\rho_1^{\delta}$, of the dominant factor to the right end. We get, using $\sum \alpha_i = 1$ and Hölder,

$$\|C\|_1 = \|\rho_1^{1-\delta/\alpha_1} (\rho_1^{1/\alpha_1}(Z \rho_3^{\alpha_3}(H_3 \rho_1^{\delta}))\|_1 \leq \|\rho_1^{1-\delta}\|_{1/\alpha_1}\|Y \rho_1\|_{1/\alpha_1}\|\rho_2^{\alpha_2}\|_{1/\alpha_2}\|Z R_3\|_1\|\rho_3^{\alpha_3}\|_{1/\alpha_3}\|H_3 R_1\|_1\|H_1 \rho_1^{\delta}\|_1\|.$$

The first norm is bounded by

$$\|\rho_1^{1-2\delta/\alpha_1}\|_{1/\alpha_1} \leq \|\rho_1^{1-6\delta}\|_1 < \infty$$

as $1 - 6\delta > \beta_1$. The second factor,

$$\|\rho_1^{\delta} Y\|_1 \leq \|\rho_1^{\delta} H_1\|_1\|R_1 Y\|_1$$

is finite, independent of $\lambda$, by the spectral theorem, and by lemma (2.1). The same can be said of the factors $\|H_1 \rho_1^{\delta}\|_1$ and $\|H_3 R_1\|_1$. The factors involving $\rho_2$ and $\rho_3$ are unity, as they are states. Hence the trace norm of $C$ is bounded (in the region 1) independent of $\alpha$ and $\lambda$, and its integral is bounded. The other regions, 2 and 3, are treated similarly. 

Corollary. The BKM metric

$$g_X(Y, Z) = \int_0^1 d\alpha \text{Tr} \left( \rho_X^\alpha Y \rho_X^{1-\alpha} Z \right)$$

(27)
is finite if $Y, Z$ are $\omega$-directions. Moreover, $\psi_X$ is Fréchet differentiable in the $Y$ direction in the analytic manifold, with derivative $\rho_X.Y$; for

$$|\rho_X.Y| = |\text{Tr}[\rho_X^{1-\delta}(\rho_X H_X)(R_X Y)]| \leq C\|R_X Y\|$$

(28)
is continuous in the $\omega$-norm.

## 3 The analyticity of the free energy

In this section, we show that the free energy $\psi_{\lambda X}$ is an analytic function in a hood of $\lambda = 0$ if $X$ is an $\omega$-direction. Suppose $\rho = Z^{-1}\exp(-H) \in C_\beta \in M$, and $V_j$ are $H$-bounded, $j = 1, \ldots, n$. As before, we assume that $H \geq 1$ and write $R = H^{-1}$.

We start with the $n$-point function

$$M_n := Z_0 \text{Tr} \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \ldots \int_0^1 d\alpha_{n-1} [\rho^{\alpha_1} V_1 \rho^{\alpha_2} V_2 \ldots \rho^{\alpha_n} V_n].$$

Here, $\alpha_n = 1 - \alpha_1 - \ldots - \alpha_{n-1}$. Write this trace as

$$\left[\rho^{\alpha_1} \beta\right] \left[H^{1-\delta_1 + \delta_1 \rho(1-\beta)\alpha_1}\right] \left[R^{\delta_1} V_1 R^{1-\delta_1}\right] \left[\rho^{\alpha_2} \beta\right] \left[H^{1-\delta_2 + \delta_2 \rho(1-\beta)\alpha_2}\right] \left[R^{\delta_2} V_2 R^{1-\delta_2}\right] \ldots \left[\rho^{\alpha_n} \beta\right] \left[H^{1-\delta_n + \delta_n \rho(1-\beta)\alpha_n}\right] \left[R^{\delta_n} V_n R^{1-\delta_n}\right],$$

where $\delta_j \in (0,1)$ are yet to be chosen. This is the product of $n$ factors of the form $\left[\rho^{\alpha_j} \beta\right]$, $n$ factors of the form $\left[H^{1-\delta_j + \delta_j \rho(1-\beta)\alpha_j}\right]$, where $\delta_0$ means $\delta_n$, and $n$ factors of the form $\left[R^{\delta_j} V_j R^{1-\delta_j}\right]$. By interpolation, the last type is bounded in operator norm by [21]

$$\|R^{\delta_j} V_j R^{1-\delta_j}\| \leq \|RV_j\| = \|V_j\|_\omega.$$  

This bound is independent of $\alpha$. The factors $\rho^{\alpha_j} \beta$ are bounded in trace-norm by the Hölder inequality

$$\|A_1 \ldots A_n\|_1 \leq \|A_1\|_{p_1}^{1/p_1} \ldots \|A_n\|_{p_n}^{1/p_n}.$$  

Since $\sum \alpha_j = 1$ we may put $p_j = 1/\alpha_j$ to get

$$\|\left[\rho^{\alpha_1} \beta\right] \ldots \left[\rho^{\alpha_n} \beta\right]\|_1 \leq \|\rho^{\beta}\|_{1}^{1} \ldots \|\rho^{\beta}\|_{1}^{n} = \|\rho^{\beta}\|_{1} < \infty.$$
This bound is also independent of the $\alpha'$s. We bound the remaining factors in operator norm. By the spectral theorem we have the bound

$$
\|\rho^{(1-\beta)\alpha_j} H^{1-\delta_j-1+\delta_j}\|_\infty = Z^{-\alpha_j(1-\beta)} \sup_{x \geq 1} \{e^{-(1-\beta)\alpha_j x}\}^{-\delta_j-1+\delta_j} (1 - \frac{1}{(1-\beta)\alpha_j})^{1-\delta_j-1+\delta_j} e^{-(1-\delta_j-1+\delta_j)} (29)
$$

$$
\leq Z^{-\alpha_j(1-\beta)} (1 - \frac{1}{\delta_j-1+\delta_j})^{1-\delta_j-1+\delta_j} e^{-(1-\delta_j-1+\delta_j)} (30)
$$

We have to integrate $\alpha_j^{-\delta_j-1+\delta_j} d\alpha_j$; the region of integration is the union of $n$ (overlapping) regions $S_j := \{\alpha : \alpha_j \geq 1/n\}$. We treat each in a similar manner. For $S_n$ we can ensure integrability at $\alpha_j = 0$, $j = 1, \ldots, n-1$ if we choose $\delta_j$ so that $1 - \delta_j-1 + \delta_j < 1$, that is, $\delta_j < \delta_j-1$. So we choose $\delta_n = \delta_0 > \delta_1 > \delta_2 > \ldots > \delta_{n-1}$. To ensure that $\delta_j \in [0, 1]$ it is enough to choose $\delta_n = 1$, $\delta_1 = 1 - 1/n$, $\ldots$, $\delta_{n-1} = 1/n$. Then each of the $n-1$ integrals for $j = 1, \ldots n-1$,

$$
\int_0^1 d\alpha_j \alpha_j^{-\delta_j-1+\delta_j} = (\delta_j-1 - \delta_j)^{-1}
$$

is equal to $n$, giving a factor $n^{n-1}$. The remaining $\alpha$ factor in the region $S_n$ is $\alpha_j^{-\delta_j-1+\delta_j} \leq n^2$. We get the same bound in the other regions, $S_j$ $j = 1, \ldots n-1$, so we get the bound $n^3n^{n-1} = n^2 n^n$. The other factors in the estimate (30) can be bounded independent of $\alpha$:

$$
\prod_{j=1}^n Z^{-\alpha_j(1-\beta)} (1 - \delta_j-1 + \delta_j)^{1-\delta_j-1+\delta_j}(1 - \beta)^{-\delta_j-1+\delta_j} e^{-(1-\delta_j-1+\delta_j)}
$$

$$
\leq 4Z^{-\delta_j-1+\delta_j} (1 - \beta)^{-n} e^{-n}
$$

since $(1 - \delta_j-1 + \delta_j) < 1$ except for one term, when it is less than 2. Thus we get for the $n$-point function the bound

$$
4\|\rho^{\beta}\|_1 Z^{-\beta} n^2 n^n e^{-n} \prod_j \|V_j/(1 - \beta)\|_\infty
$$

It follows that the partition function, $Z_{\lambda V}$, has a convergent Taylor series

$$
\sum \lambda^n M_n/n! x \text{ if } |\lambda| \|V\|_\infty < 1 - \beta.
$$

Recall that this is also the condition that $\rho_{\lambda V}$ lie in $\mathcal{M}$ in an $\omega$-hood of $\rho$. Since $Z_{\lambda V} > 0$, it follows that the free energy, $\psi := \log Z_{\lambda V}$, is real analytic in $\lambda$ in the same interval.
4 Conclusion

Suppose that $\beta < 1$. We have seen that the condition that the perturbation of the Hamiltonian $H \geq I$ of a state $\rho \in C_\beta$ by a small enough symmetric operator $V$ leads to another state $\rho_V \in M$. We defined a topology on $M$ in a hood of $\rho$ using the norm $\|V\|_\omega(\rho) = \|RV\|$, where $R \equiv I$; this hood $U$ is set of states $\rho_V$ coming from the ball of operators $V$ with $\|V\|_\omega < 1 - \beta$. This is finer than the topology defined by the norm $\|V\| = \|R^{1/2}VR^{1/2}\|$ which is finite for all form-bounded perturbations. We saw that the Taylor series for the partition function converges in the same hood of $\rho$. The $\omega$-norms associated with different points of $M$ are equivalent on the overlap of two hoods, so together we have a Banach manifold modelled on the Banach spaces of $H$-bounded symmetric operators. The question arises, is there an analytic structure of this manifold? In fact, an analytic structure is deemed to be provided if we specify the ring of germs of analytic functions at each point. Let us say that a map $\psi : U \to \mathbb{C}$ is $+1$-analytic if in $U$ it is infinitely often Fréchet differentiable and $\psi(\rho_\lambda V)$ has a convergent Taylor expansion in $\lambda$, for all $\omega$-directions $V$ in the tangent space. Because of the equivalence of norms, this concept is coordinate-free. Thus an analytic structure for $M$ has been specified. In particular, the mixture coordinates $\eta_X = \rho.X$ are analytic, as they are derivatives of the free energy. In his theory of expansionals, Araki \cite{2} showed that the free energy $\psi_X$ is an entire function in the Banach space $B(\mathcal{H})$ if the perturbation $X$ is bounded, in the more general context of Tomita-Takesaki theory. With unbounded perturbations we cannot hope for entire functions; for we need only take $V = -H_0$ to hit a singularity in $\psi_X$. However, it is likely that some analyticity remains, even in the thermodynamic limit, for relatively bounded perturbations.

Compared with \cite{23}, we have improved the result in several ways. We show analyticity instead of twice-differentiability; we have dropped the commutator condition altogether; the manifold is infinite dimensional instead of finite-dimensional; and we have enlarged the class of states to $C_{<1}$. Further results are obtained in \cite{6}.

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