ON THE CAUCHY PROBLEM FOR A HIGHER-ORDER $\mu$-CAMASSA-HOLM EQUATION

FENG WANG
School of Mathematics and Statistics, Xidian University
Xi’an 710071, China

FENGQUAN LI
School of Mathematical Sciences, Dalian University of Technology
Dalian 116024, China

ZHIJUN QIAO
School of Mathematical & Statistical Sciences, University of Texas-Rio Grande Valley
Texas 78539, USA

(Communicated by Adrian Constantin)

Abstract. In this paper, we study the Cauchy problem of a higher-order $\mu$-Camassa-Holm equation. We first establish the Green’s function of $(\mu - \partial_x^2 + \partial_x^4)^{-1}$ and local well-posedness for the equation in Sobolev spaces $H^s(S)$, $s > \frac{7}{2}$. Then we provide the global existence results for strong solutions and weak solutions. Moreover, we show that the solution map is non-uniformly continuous in $H^s(S)$, $s \geq 4$. Finally, we prove that the equation admits single peakon solutions which have continuous second derivatives and jump discontinuities in the third derivatives.

1. Introduction. The Camassa-Holm equation

$$m_t + 2mu_x + m_xu = 0, \quad m = (1 - \partial_x^2)u$$

was introduced in [2] to model the unidirectional propagation of shallow water waves over a flat bottom. $u(t,x)$ represents the fluid velocity at time $t$ and in the spatial direction $x$. It is a re-expression of the geodesic flow both on the diffeomorphism group of the circle [14] and on the Bott-Virasoro group [31]. Eq. (1.1) has a bi-Hamiltonian structure [25] and is completely integrable [2, 9]. Moreover, it has been extended to an entire integrable hierarchy including both negative and positive flows and shown to admit algebro-geometric solutions on a symplectic submanifold [36]. The Cauchy problem of (1.1), in particular its well-posedness, blow-up behavior and global existence, have been well-studied both on the real line and on the circle, e.g., [1, 8, 10, 11, 12, 13, 16, 17, 18, 19, 26, 27, 33, 43]. Eq. (1.1) with weakly dissipative term was studied in [42].

2010 Mathematics Subject Classification. 35G25, 35L05, 35B30.

Key words and phrases. Higher-order $\mu$-Camassa-Holm equation, global existence, weak solutions, non-uniformly continuous, peakon solutions.

Authors to whom correspondence should be addressed. E-mails: wangfeng@xidian.edu.cn, fqli@dlut.edu.cn, zhijun.qiao@utrgv.edu.
Equation (1.1) has been recently generalized into some $\mu$-versions and higher order forms. Khesin et al. in [30] introduced a $\mu$-version of Camassa-Holm equation as follows

$$m_t + 2\mu u_x + m_x u = 0, \quad m = (\mu - \partial_x^2)u,$$

(1.2)

where $u(t, x)$ is a time-dependent function on the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ and $\mu(u) = \int_\mathbb{S} u dx$ denotes its mean. This equation describes the propagation of weakly non-linear orientation waves in a massive nematic liquid crystal with external magnetic filed and self-interaction. Moreover, Eq.(1.2) is also an Euler equation on $\mathcal{D}^s(\mathbb{S})$ and it describes the geodesic flow on $\mathcal{D}^s(\mathbb{S})$ with the right-invariant metric given at the identity by the inner product [30]

$$\langle f, g \rangle_\mu = \mu(f)\mu(g) + \int_\mathbb{S} f'(x)g'(x)dx.$$ 

In [30, 32], the authors showed that Eq.(1.2) is bi-Hamiltonian and admits both cusped and smooth travelling wave solutions which are natural candidates for solitons. The orbit stability of periodic peakons was studied in [3]. A weakly dissipative $\mu$-Camassa-Holm equation was studied in [34].

For the higher order Camassa-Holm equation, [5, 15] considered the following equation

$$m_t + 2\mu u_x + m_x u = 0, \quad m = k \sum_{j=0}^\infty (-1)^j \partial_x^{2j}u,$$

(1.3)

which describes exponential curves of the manifold of smooth orientation-preserving diffeomorphisms of the unit circle in the plane. In [5], Coclite et al. established the existence of global weak solutions and presented some invariant spaces under the action of the equation. Tian et al. [40] investigated the global existence of strong solutions to Equation (1.3) with $k = 2$. Ding and Lv [21] studied the existence of global conservative solutions to (1.3). Recently, Coclite and Ruvo [7] showed the convergence of the solution to (1.3). Ding et al. [20, 22] discussed traveling solutions of (1.3) and their evolution properties.

In this paper, we will consider a $\mu$-version of (1.3) with $k = 2$ as follows

$$m_t + 2\mu u_x + m_x u = 0, \quad m = (\mu - \partial_x^2 + \partial_x^4)u,$$

(1.4)

where $u(t, x)$ is a time-dependent spatially periodic function on the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ and $\mu(u) = \int_\mathbb{S} u dx$ denotes its mean.

We first give the Green’s function of the operator $(\mu - \partial_x^2 + \partial_x^4)^{-1}$ and local well-posedness of (1.4). Then we show the global existence of strong solutions to (1.4). Next, for any $T_0 > 0$ and $s \geq 4$, we prove that the data-to-solution map is Hölder continuous from any bounded subset of $H^s(\mathbb{S})$ into $C([0, T_0]; H^r(\mathbb{S}))$ with $0 \leq r < s$, but is not uniformly continuous from any bounded subset of $H^s(\mathbb{S})$ into $C([0, T_0]; H^s(\mathbb{S}))$. Motivated by the recent work [6], we establish the existence of global weak solutions in $H^2(\mathbb{S})$ without using an Oleinik-type estimate (see [4, 43]), which is not easy to be verified in numerical experiment. Lastly, we show the existence of single peakon solutions which have continuous second derivatives and jump discontinuities in the third derivatives.

We noticed that McIachlan and Zhang [35] have studied another higher-order Camassa-Holm equation as follows

$$m_t + 2\mu u_x + m_x u = 0, \quad m = (1 - \partial_x^2)^k u,$$

(1.5)
which is derived as the Euler-Poincaré differential equation on the Bott-Virasoro group with respect to the $H^k$ metric. A $\mu$-version of (1.5) with $k = 2$, first proposed in [24], was very recently studied in our recent paper [41], in which we also established the Green’s function of the operator $(\mu - \partial_x^2 + \partial_x^4)^{-1}$ and showed it admits single peakon solutions, but they are completely different from the results in the present paper.

The rest of the paper is organized as follows. In Section 2, the Green’s function of the operator $(\mu - \partial_x^2 + \partial_x^4)^{-1}$ and local well-posedness for (1.4) with initial data in $H^s(\mathbb{S}), s > \frac{7}{2}$, are established. In Section 3, we show the global existence of strong solutions. The Hölder continuity and non-uniform continuity of solution map for the equation are established in Section 4. In Section 5, we show the global existence of weak solutions. The existence of single peakon solutions is proved in Section 6.

2. Preliminaries. In this section, we will give the Green’s function of the operator $A^{-1}_\mu := (\mu - \partial_x^2 + \partial_x^4)^{-1}$ and establish the local well-posedness for Eq. (1.4).

2.1. Green’s function. To construct the peaked solutions in the last section, we need to investigate the Green’s function of the operator $A^{-1}_\mu$. We denote the Fourier transform of $f$ by $\hat{f}$.

For a periodic function $g$ on the circle $S = \mathbb{R}/\mathbb{Z}$, we have

$$\widehat{\mu(g)}(k) = \int_S \mu(g)(x) e^{-2\pi i k x} dx = \mu(g) \int_S e^{-2\pi i k x} dx = \mu(g) \delta_0(k),$$

where

$$\delta_0(k) = \begin{cases} 1, & k = 0, \\ 0, & k \neq 0. \end{cases}$$

Since $\mu(g) = \hat{g}(0)$, we have $\mu(g)(k) = \delta_0(k) \hat{g}(k)$. Thus,

$$A^{-1}_\mu g(k) = (\mu - \partial_x^2 + \partial_x^4) g(k) = [\delta_0(k) + (2\pi k)^2 + (2\pi k)^4] \hat{g}(k).$$

If $g$ is the Green’s function of the operator $A^{-1}_\mu$, that is, $g$ satisfies $A^{-1}_\mu g = \delta(x)$, then $[\delta_0(k) + (2\pi k)^2 + (2\pi k)^4] \hat{g}(k) = 1$. Thus,

$$g(x) = \sum_{k \in \mathbb{Z}} \hat{g}(k) e^{2\pi i k x} = \sum_{k \in \mathbb{Z}} \frac{1}{\delta_0(k) + (2\pi k)^2 + (2\pi k)^4} e^{2\pi i k x} = 1 + 2 \sum_{k=1}^{\infty} \frac{\cos 2\pi k x}{(2\pi k)^2 + (2\pi k)^4},$$

By Weierstrass’s criterion, we know the series

$$- \sum_{k=1}^{\infty} \frac{\sin 2\pi k x}{2\pi k + (2\pi k)^2}, \quad - \sum_{k=1}^{\infty} \frac{\cos 2\pi k x}{1 + (2\pi k)^2}$$

uniformly converge in $[0, 1) \simeq \mathbb{S}$. From Dirichlet’s criterion, we know that the series

$$\sum_{k=1}^{\infty} \frac{2\pi k x}{1 + (2\pi k)^2}$$

converge for any $x \in [0, 1) \simeq \mathbb{S}$, and uniformly converge in any closed interval $[\alpha, \beta] \subset (0, 1)$. Thus, $g(x)$ is two-times continuously differentiable on $[0, 1) \simeq \mathbb{S}$ and three-times continuously differentiable on $[\alpha, \beta] \subset (0, 1)$. It follows that $\|\partial_x^2 g\|_{L^\infty(\mathbb{S})} < \infty$ ($i = 0, 1, 2, 3$).

Note that

$$g(x) = 1 + 2 \sum_{k=1}^{\infty} \frac{\cos 2\pi k x}{(2\pi k)^2 + (2\pi k)^4},$$

$$= 1 + 2 \sum_{k=1}^{\infty} \frac{1}{(2\pi k)^2 + (2\pi k)^4} - \frac{1}{1 + (2\pi k)^2} \cos 2\pi k x,$$

$$= 1 + 2 \sum_{k=1}^{\infty} \frac{\cos 2\pi k x}{(2\pi k)^2} - 2 \sum_{k=1}^{\infty} \frac{\cos 2\pi k x}{1 + (2\pi k)^2}.$$
Since the Green’s functions of \((\mu - \partial_x^2)^{-1}\) and \((1 - \partial_x^2)^{-1}\) are 
\[g_\mu(x) = \frac{\cosh(x - \frac{x}{2})}{2 \sinh(\frac{x}{2})}\] and \(g_1(x) = \frac{\cosh(x - \frac{x}{2})}{2 \sinh(\frac{x}{2})}\) respectively, that is,
\[g_\mu(x) = 1 + 2 \sum_{k=1}^{\infty} \frac{\cos 2\pi k x}{(2\pi k)^2}, \quad g_1(x) = 1 + 2 \sum_{k=1}^{\infty} \frac{\cos 2\pi k x}{1 + (2\pi k)^2},\]
the Green’s function \(g(x)\) is given by
\[g(x) = g_\mu(x) - g_1(x) + 1\]
\[= \frac{1}{2} (x - \frac{1}{2})^2 - \frac{\cosh(x - \frac{x}{2})}{2 \sinh(\frac{x}{2})} + \frac{47}{24}, \quad x \in [0, 1) \simeq S,\]
and is extended periodically to the real line, that is
\[g(x) = \frac{1}{2} (x - [x] - \frac{1}{2})^2 - \frac{\cosh(x - [x] - \frac{x}{2})}{2 \sinh(\frac{x}{2})} + \frac{47}{24}, \quad x \in \mathbb{R},\]
where \([x]\) denotes the largest integer part of \(x\). The graph of \(g_1(x) - g_\mu(x)\) can be seen in Fig.3 in [32]. Note that \(\mu(g) = 1\).

The inverse \(v = A_\mu^{-1}w\) is given by
\[v(x) = (g * w)(x) = (g_\mu * w)(x) - (g_1 * w)(x) + (1 * w)(x)\]
\[= (\mu - \partial_x^2)^{-1}w - (1 - \partial_x^2)^{-1}w + \int_0^1 w(x)dx\]
\[= \left(\frac{x^2}{2} - \frac{z}{2} + \frac{25}{24}\right)\mu(w) + (x - \frac{1}{2}) \int_0^1 \int_0^y w(s)dsdy + \int_0^1 \int_0^y \int_0^y w(r)dxdy\]
\[-\int_0^1 \int_0^y \int_0^y w(s)dsdy - (1 - \partial_x^2)^{-1}w.\]
Since \((\mu - \partial_x^2)^{-1}\) and \((1 - \partial_x^2)^{-1}\) commute with \(\partial_x\), the following identity holds
\[A_\mu^{-1}\partial_x w = \partial_x A_\mu^{-1} w,\]
that is, \(A_\mu^{-1}\) commutes with \(\partial_x\).

For any \(s \in \mathbb{R}\), \(H^s(\mathbb{S})\) is defined by the Sobolev space of periodic functions
\[H^s(\mathbb{S}) = \left\{ v = \sum_k \tilde{v}(k)e^{2\pi i k x} : \|v\|_{H^s(\mathbb{S})} = \sum_k |\tilde{\Lambda}^s v(k)|^2 < \infty \right\},\]
where the pseudodifferential operator \(\Lambda^s = (1 - \partial_x^2)^{\frac{s}{2}}\) is defined by
\[\tilde{\Lambda}^s v(k) = (1 + 4\pi^2 k^2)^{\frac{s}{2}} \tilde{v}(k).\]
We can check that \(A_\mu = \mu - \partial_x^2 + \partial_x^4\) is an isomorphism between \(H^s(\mathbb{S})\) and \(H^{s-4}(\mathbb{S})\). Moreover, when \(w \in H^{r+j-4}(\mathbb{S})\) for \(j = 0, 1, 2, 3\), we have \(A_\mu^{-1}\partial_x^j w \in H^r(\mathbb{S})\) with
\[
\|A_\mu^{-1}\partial_x^j w\|_{H^r(\mathbb{S})}^2 = \sum_k (1 + 4\pi^2 k^2)^r |\tilde{\Lambda}^s_{\mu} \partial_x^j w(k)|^2
\]
\[
= \sum_k (1 + 4\pi^2 k^2)^r \left| \frac{(2\pi k)^j}{\delta_0(k) + (2\pi k)^j + (2\pi k)^j - (2\pi k)^j} \tilde{\Lambda}^s v(k) \right|^2
\]
\[
= \sum_k (1 + 4\pi^2 k^2)^{r+j-4} (1 + 4\pi^2 k^2)^{4-j} |\tilde{\Lambda}^s_{\mu} \partial_x^j w(k)|^2
\]
\[
\leq 2^{4-j} \sum_k (1 + 4\pi^2 k^2)^{r+j-4} \|\tilde{\Lambda}^s v(k)\|^2
\]
\[
= 2^{4-j} \|w\|_{H^{r+j-4}(\mathbb{S})}^2. \quad (2.1)
\]
2.2. Local well-posedness. The initial-value problem associated to Eq.(1.4) can be rewritten in the following form:

\[
\begin{align*}
\mu(u_t) - uu_{xx} + u_{xxxx} + 2\mu(u)u_x - 2uu_{xx} - uu_{xxx} + 2u_xu_{xxxx} + uu_{xxxx} &= 0, \quad t > 0, \quad x \in \mathbb{R}, \\
u(t, x + 1) &= u(t, x), \quad t \geq 0, \quad x \in \mathbb{R}, \\
u(0, x) &= \nu_0(x), \quad x \in \mathbb{R}.
\end{align*}
\]

or, equivalently,

\[
\begin{align*}
u_t + uu_x + \partial_x A_\mu^{-1} (2\mu(u)u + \frac{1}{2}u_x^2 - 3uu_{xx} - \frac{7}{2}u_{xx}^2) &= 0, \quad t > 0, \quad x \in \mathbb{S}, \\
u(t, x + 1) &= \nu(t, x), \quad t \geq 0, \quad x \in \mathbb{S}, \\
u(0, x) &= \nu_0(x), \quad x \in \mathbb{S}.
\end{align*}
\]

On the other hand, integrating both sides of Eq.(2.3) over S with respect to x, we obtain

\[
\frac{d}{dt} \nu(u) = 0.
\]

Then it follows that \(\nu(u) = \nu(u_0) := \mu_0\).

Thus, Eq.(2.3) can be rewritten as

\[
\begin{align*}
u_t + uu_x + \partial_x A_\mu^{-1} (2\mu_0u + \frac{1}{2}u_x^2 - 3uu_{xx} - \frac{7}{2}u_{xx}^2) &= 0, \quad t > 0, \quad x \in \mathbb{S}, \\
u(0, x) &= \nu_0(x), \quad x \in \mathbb{S}.
\end{align*}
\]

Applying the Kato’s theorem [28], one may follow the similar argument as in [34] to obtain the following local well-posedness result for Eq.(2.4).

**Theorem 2.1.** Given \(\nu_0 \in H^s(\mathbb{S}), \ s > \frac{3}{2}\), there exist a maximal \(T = T(\nu_0) > 0\), and a unique solution \(\nu\) to Eq.(2.4) such that

\[
\nu = \nu(\cdot, \nu_0) \in C([0, T); H^s(\mathbb{S})) \cap C^1([0, T); H^{s-1}(\mathbb{S})).
\]

Moreover, the solution depends continuously on the initial data, and T is independent of \(\nu\).

**Lemma 2.2.** (See [10]) Assume \(f(x) \in H^1(\mathbb{S})\) satisfies that \(\int_\mathbb{S} f(x)dx = a_0^2\). Then, for any \(\varepsilon > 0\), we have

\[
\max_{x \in \mathbb{S}} f^2(x) \leq \frac{\varepsilon + 2}{24} \int_\mathbb{S} f^2(x)dx + \frac{\varepsilon + 2}{4\varepsilon} a_0^2.
\]

**Corollary 1.** For \(f(x) \in H^1(\mathbb{S})\), if \(\int_\mathbb{S} f(x)dx = 0\), then we have

\[
\max_{x \in \mathbb{S}} f^2(x) \leq \frac{1}{12} \int_\mathbb{S} f^2(x)dx.
\]

**Lemma 2.3.** Let \(\nu_0 \in H^s(\mathbb{S}), \ s > \frac{3}{2}\), and let \(T\) be the maximal existence time of the solution \(\nu\) to Eq.(2.4) with the initial data \(\nu_0\). Then we have

\[
\int_\mathbb{S} (u_x^2 + uu_{xx})dx = \int_\mathbb{S} (u_{\nu_0,x}^2 + uu_{\nu_0,xx})dx := \mu_1^2, \quad \forall \ t \in [0, T).
\]

Moreover, we have

\[
\|u_x(t, \cdot)\|_{L^\infty(\mathbb{S})} \leq \frac{\sqrt{3}}{6} \mu_1
\]
and
\[ \|u(t, \cdot)\|_{L^\infty(S)} \leq |\mu_0| + \frac{1}{12} \mu_1. \]

Proof. A direct computation gives
\[ \frac{d}{dt} \int_S (u_x^2 + u_{xx}^2) dx = 0, \]
which implies (2.5).

Since \( u(t, \cdot) \in H^s(S) \subset C^2(S) \) for \( s > \frac{7}{2} \), and \( \int_S u_x dx = 0 \), Corollary 1 and (2.5) imply that
\[ \max_{x \in S} u_x^2(t, x) \leq \frac{1}{12} \int_S u_x^2(t, x) dx \leq \frac{1}{12} \|u_x(t, \cdot)\|_{L^\infty(S)}^2. \]
It then follows that
\[ \|u_x(t, \cdot)\|_{L^\infty(S)} \leq \frac{\sqrt{3}}{6} \mu_1. \]

Note that
\[ \int_S (u(t, x) - \mu_0) = 0. \]
By Corollary 1, we have
\[ \max_{x \in S} (u(t, x) - \mu_0)^2 \leq \frac{1}{12} \int_S u_x^2(t, x) dx \leq \frac{1}{12} \|u_x(t, \cdot)\|_{L^\infty(S)}^2, \]
which implies that
\[ \|u(t, \cdot)\|_{L^\infty(S)} - |\mu_0| \leq \|u(t, \cdot) - \mu_0\|_{L^\infty(S)} \leq \frac{1}{12} \mu_1. \]
Hence, we get
\[ \|u(t, \cdot)\|_{L^\infty(S)} \leq |\mu_0| + \frac{1}{12} \mu_1. \]
This completes the proof of the lemma. \( \square \)

3. Global existence of strong solution. In this section, we present the global existence of strong solution to Eq.(2.4). Firstly, we will give some useful lemmas.

Lemma 3.1. (see [29]) If \( r > 0 \), then \( H^r(S) \cap L^\infty(S) \) is an algebra. Moreover,
\[ \|fg\|_{H^r(S)} \leq c_r(\|f\|_{L^\infty(S)} \|g\|_{H^r(S)} + \|f\|_{H^r(S)} \|g\|_{L^\infty(S)}), \]
where \( c_r \) is a positive constant depending only on \( r \).

Lemma 3.2. (see [29]) If \( r > 0 \), then
\[ \|\Lambda^r f\|_{L^2(S)} \leq c_r(\|\partial_x f\|_{L^\infty(S)} \|\Lambda^{-1} g\|_{L^2(S)} + \|\Lambda f\|_{L^2(S)} \|g\|_{L^\infty(S)}), \]
where \( \Lambda^r = (1 - \partial_x^2)^{r/2} \) and \( c_r \) is a positive constant depending only on \( r \).

Lemma 3.3. (see [23, 38]) If \( f \in H^s(S) \) with \( s > \frac{3}{2} \), then there exists a constant \( c > 0 \) such that for any \( g \in L^2(S) \) we have
\[ \|J_\varepsilon f\|_{L^2(S)} \leq c\|f\|_{C^1(S)} \|g\|_{L^2(S)}, \]
in which for each \( \varepsilon \in (0, 1] \), the operator \( J_\varepsilon \) is the Friedrichs mollifier defined by
\[ J_\varepsilon f(x) = j_\varepsilon * f(x), \]
where \( j_\varepsilon(x) = \frac{1}{\varepsilon} j(\frac{x}{\varepsilon}) \) and \( j(x) \) is a nonnegative, even, smooth bump function supported in the interval \((-\frac{1}{2}, \frac{1}{2})\) such that \( \int_{\mathbb{R}} j(x) dx = 1 \). For any \( f \in H^s(S) \) with \( s \geq 0 \), we have \( J_\varepsilon f \to f \) in \( H^s(S) \) as \( \varepsilon \to 0 \). Moreover, for any \( p \geq 1 \), the Young’s
inequality \( \|j \ast f\|_{L^p(S)} \leq \|j\|_{L^1(\mathbb{R})} \|f\|_{L^p(S)} = \|f\|_{L^p(S)} \) holds since \( j(x) \) is supported in the interval \((-\frac{1}{2}, \frac{1}{2})\).

Now we establish the global existence of strong solution to Eq. (2.4).

**Theorem 3.4.** Let \( u_0 \in H^s(S), s > \frac{7}{4}, \) and let \( T \) be the maximal existence time of the solution \( u \) to Eq. (2.4) with the initial data \( u_0 \). Then there exists a constant \( c > 0 \) depending on \( s \) and \( u_0 \) such that

\[
\|u\|_{H^s(S)} \leq e^{c(1+\|u_{xxx}\|_{L^2(S)}}\|u_0\|_{H^s(S)}, \quad t \in [0, T).
\]

**Proof.** Note that the product \( uu_x \) only has the regularity of \( H^{s-1}(S) \) when \( u \in H^s(S) \). To deal with this problem, we will consider the following modified equation

\[
(J_u)\partial_x J_{u} + J_{\varepsilon}(uu_x) + \partial_x A^{-4}_\mu(2\mu_0 J_{\varepsilon} u + \frac{1}{2} J_{\varepsilon}(u_x^2) - 3 J_{\varepsilon}(u_{xx} u_{xxx}) - \frac{7}{2} J_{\varepsilon}(u_{xx}^2)) = 0,
\]

(3.2)

where \( J_{\varepsilon} \) is defined in (3.1).

Applying the operator \( A^s = (1-\partial^2_x)^{s/2} \) to Eq. (3.2), then multiplying the resulting equation by \( A^s J_{\varepsilon} u \) and integrating with respect to \( t \in [0, T) \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|A^s J_{\varepsilon} u\|^2_{H^s(S)} = -(A^s J_{\varepsilon}(uu_x), A^s J_{\varepsilon} u) \tag{3.3}
\]

\[
= \langle (A^s, u) u_x, J_{\varepsilon} A^s J_{\varepsilon} u \rangle + \langle (u A^s u_x, J_{\varepsilon} A^s J_{\varepsilon} u) \rangle 
\]

\[
= \langle (u A^s J_{\varepsilon} u_x, J_{\varepsilon} A^s J_{\varepsilon} u) \rangle + \langle (u J_{\varepsilon} u_x, A^s J_{\varepsilon} u) \rangle
\]

\[
\leq \|u_{x} A^s J_{\varepsilon} u\| \|u_{xx} A^s J_{\varepsilon} u\|_{L^2(S)} + \left\| J_{\varepsilon}, \partial_x A^s u, u \right\|_{L^2(S)} \|A^s J_{\varepsilon} u\|_{L^2(S)}
\]

(3.4)

where we have used Lemma 3.2 with \( r = s \) and Lemma 3.3. Here and in what follows, we use "\( \lesssim \)" to denote inequality up to a positive constant. Furthermore, we estimate the second term of the right hand side of (3.3) in the following way

\[
\|\partial_x A^{-4}_\mu(2\mu_0 J_{\varepsilon} u + \frac{1}{2} J_{\varepsilon}(u_x^2) - 3 J_{\varepsilon}(u_{xx} u_{xxx}) - \frac{7}{2} J_{\varepsilon}(u_{xx}^2))\|_{H^s(S)} \|u\|_{H^s(S)}
\]

\[
\lesssim (\|u\|_{H^s(S)} \|u\|_{H^s(S)} \|u\|_{H^s(S)} \|u\|_{H^s(S)} + \|u_{xxx}\|_{L^2(S)} \|u\|_{H^s(S)} \|u_{xxx}\|_{L^2(S)} \|u\|_{H^s(S)} + \|u_{xx}\|_{L^2(S)} \|u\|_{H^s(S)} \|u_{xx}\|_{L^2(S)} \|u\|_{H^s(S)}
\]

(3.5)
Since $u(t, \cdot) \in H^s(\mathbb{S}) \subset C^2(\mathbb{S})$ for $s > \frac{7}{2}$, and \( \int_\mathbb{S} u_{xx} dx = 0 \), Corollary 1 implies that
\[
\|u_{xx}(t, \cdot)\|_{L^\infty(\mathbb{S})} \leq \frac{\sqrt{3}}{6} \|u_{xxxx}(t, \cdot)\|_{L^2(\mathbb{S})} \leq \frac{\sqrt{3}}{6} \|u_{xxxx}(t, \cdot)\|_{L^\infty(\mathbb{S})}, \quad t \in [0, T).
\]
Thus,
\[
\left| \left( \Lambda^* \partial_x \Lambda^* A_n^{-1} \left( 2 \mu_0 \partial_x u + \frac{1}{2} J_e(u_x^2) - 3 J_e(u_{xxx}) - 7 J_e(u_{xx}^2) \right) \right) \right| \leq (\|u_0\| + \|u_x\|_{L^\infty(\mathbb{S})} + \|u_{xxx}\|_{L^\infty(\mathbb{S})}) \|u\|_{H^s(\mathbb{S})},
\]
Combining (3.4) and (3.5) and using Lemma \( 2.3 \), we have
\[
\frac{1}{2} \frac{d}{dt} \|u\|_{H^s(\mathbb{S})}^2 \leq \left( (\|u_0\| + \|u\|_{L^\infty(\mathbb{S})} + \|u_{xxx}\|_{L^\infty(\mathbb{S})}) \|u\|_{H^s(\mathbb{S})} \right)^2
\]
Letting \( \varepsilon \to 0 \), we get
\[
\frac{1}{2} \frac{d}{dt} \|u\|_{H^s(\mathbb{S})}^2 \leq c(1 + \|u_{xxx}\|_{L^\infty(\mathbb{S})}) \|u\|_{H^s(\mathbb{S})}^2,
\]
where \( c \) is a constant depending on \( s \) and \( u_0 \). An application of Gronwall’s inequality yields
\[
\|u\|_{H^s(\mathbb{S})}^2 \leq e^{2c(1+\|u_{xxx}\|_{L^\infty(\mathbb{S})})t} \|u_0\|_{H^s(\mathbb{S})}^2,
\]
which completes the proof of the theorem. \( \square \)

**Theorem 3.5.** Let \( u_0 \in H^s(\mathbb{S}), s > \frac{7}{2} \). Then the corresponding strong solution \( u \) of the initial value \( u_0 \) exists globally in time.

**Proof.** By using the local well-posedness theorem and a density argument, it suffices to show the theorem for \( s \geq 5 \). Assume that \( u_0 \in H^s(\mathbb{S}), s \geq 5 \). Let \( u \) be the corresponding solution of Eq.(2.4) on \([0, T) \times \mathbb{S}\), which is guaranteed by Theorem 2.1. Multiplying Eq.(1.4) by \( m \) and integrating over \( \mathbb{S} \) with respect to \( x \) yield
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{S}} m^2 dx = \int_{\mathbb{S}} m(-2mu_x - um_x) dx = -3 \int_{\mathbb{S}} u_{xx} m^2 dx.
\]
Note that in Lemma 2.3 we have
\[
\|u_x(t, \cdot)\|_{L^\infty(\mathbb{S})} \leq \frac{\sqrt{3}}{6} \mu_1.
\]
Then
\[
\frac{d}{dt} \int_{\mathbb{S}} m^2 dx \leq \sqrt{3} \mu_1 \int_{\mathbb{S}} m^2 dx.
\]
By Gronwall’s inequality, we have
\[
\int_{\mathbb{S}} m^2 dx \leq e^{\sqrt{3} \mu_1 t} \int_{\mathbb{S}} m_0^2 dx.
\]
Note that
\[
\int_{\mathbb{S}} m^2 = \mu(u)^2 + \int_{\mathbb{S}} u_{xx}^2 + 2 \int_{\mathbb{S}} u_{xxx}^2 + \int_{\mathbb{S}} u_{xxxx}^2 \geq \|u_{xxxx}\|_{L^2(\mathbb{S})}^2.
\]
Since \( u \in H^5(\mathbb{S}) \subset C^4(\mathbb{S}) \) and \( \int_{\mathbb{S}} u_{xxx} dx = 0 \), Corollary 1 implies that
\[
\|u_{xxx}\|_{L^\infty(\mathbb{S})} \leq \frac{\sqrt{3}}{6} \|u_{xxxx}\|_{L^2(\mathbb{S})} \leq \frac{\sqrt{3}}{6} \|u\|_{L^2(\mathbb{S})} \leq \frac{\sqrt{3}}{6} e^{\sqrt{3} \mu_1 t} \|m_0\|_{L^2(\mathbb{S})}.
\]
By Theorem 3.4, we have
\[
\|u\|_{H^s(\mathbb{S})} \leq e^{c(1+\frac{2}{3} e^{\sqrt{3} \mu_1 T})} \|m_0\|_{L^2(\mathbb{S})} T \|u_0\|_{H^s(\mathbb{S})}.
\]
On the other hand, we can deduce the following estimate from Eq. (2.4)
\[ \|u_t\|_{H^{r-1}(S)} \leq \|u_w\|_{H^{r-1}(S)} + \|\partial_x A_u^{-1}(2\mu_0 u + \frac{1}{2}u_x^2 - 3u_x u_{xxx} - \frac{7}{2}u_{xxx}^2)\|_{H^{r-1}(S)} \]
\[ \lesssim \|u\|_{H^r(S)}^2 + 2\mu_0 u + \frac{1}{2}u_x^2 - 3u_x u_{xxx} - \frac{7}{2}u_{xxx}^2 \|u\|_{H^{r-1}(S)} \]
\[ \lesssim \|u\|_{H^r(S)}. \]

Thus, for any \( t \in [0, T) \),
\[ \|u\|_{H^r(S)} + \|u_t\|_{H^{r-1}(S)} < \infty, \]
which ensures that the solution \( u \) does not blow up in finite time, that is, \( T = \infty \).

This completes the proof of Theorem 3.5. \( \square \)

4. **Non-uniform dependence on initial data.** In this section, we will first give an estimate of the solution size in time interval \([0, T_0]\) for any fixed \( T_0 > 0 \), and then we show that, for any \( s \geq 4 \), the data-to-solution map is H"older continuous from any bounded subset of \( H^s(S) \) into \( C([0, T_0]; H^r(S)) \) with \( 0 \leq r < s \), but is not uniformly continuous from any bounded subset of \( H^s(S) \) into \( C([0, T_0]; H^r(S)) \).

Similar as the proofs of in [41], we can obtain the following Lemma 4.1 and Theorem 4.2.

**Lemma 4.1.** Let \( u \) be the solution of Eq. (2.4) with initial data \( u_0 \in H^s(S) \), \( s \geq 4 \). Then, for any fixed \( T_0 > 0 \), we have
\[ \|u(t)\|_{H^r(S)} \leq e^{cT_0}\|u_0\|_{H^r(S)}, \quad t \in [0, T_0], \] (4.1)
where \( c = c(s, T_0, u_0) \) is a constant depending on \( s, T_0 \) and \( \|u_0\|_{H^r(S)} \).

**Theorem 4.2.** Assume \( s \geq 4 \) and \( 0 \leq r < s \). Then the solution map for Eq. (2.4) is H"older continuous with exponent
\[ \alpha = \begin{cases} 1, & \text{if } 0 \leq r \leq s - 1, \\ s - r, & \text{if } s - 1 < r < s \end{cases} \]
as a map from \( B(0, h) := \{ u \in H^s(S) : \|u\|_{H^r(S)} \leq h \} \) with \( H^r(S) \)-norm to \( C([0, T_0]; H^r(S)) \) for any fixed \( T_0 > 0 \). More precisely, we have
\[ \|u(t) - w(t)\|_{C([0, T_0]; H^r(S))} \leq c\|u(0) - w(0)\|_{H^r(S)}, \]
for all \( u(0), w(0) \in B(0, h) \) and \( u(t), w(t) \) the solutions corresponding to the initial data \( u(0), w(0) \), respectively. The constant \( c \) depends on \( s, r, T_0 \) and \( h \).

Next, we prove that the data-to-solution map is not uniformly continuous. Firstly, we will recall some useful lemmas.

**Lemma 4.3.** (see [27]) Let \( \sigma, \alpha \in \mathbb{R} \). If \( n \in \mathbb{Z}^+ \) and \( n \gg 1 \), then
\[ \|\cos(nx - \alpha)\|_{H^r(S)} \approx n^\sigma. \]
Relation is also true if \( \cos(nx - \alpha) \) is replaced by \( \sin(nx - \alpha) \).

**Lemma 4.4.** (see [39]) If \( s > \frac{3}{2} \) and \( 0 \leq \sigma + 1 \leq s \), then there exists a constant \( c > 0 \) such that
\[ \|\Lambda^\sigma \partial_x f\|_{L^2(S)} \leq c\|f\|_{H^r(S)}\|v\|_{H^r(S)}. \]

**Lemma 4.5.** (see [27]) If \( r > \frac{1}{2} \), then there exists a constant \( c_r > 0 \) depending only on \( r \) such that
\[ \|fg\|_{H^{r-1}(S)} \leq c_r\|f\|_{H^r(S)}\|g\|_{H^{r-1}(S)}. \]
Lemma 4.6. If \( \omega \) is bounded, then for \( n \gg 1 \), we have
\[
\| F(t, \cdot) \|_{H^s(\mathbb{S})} \lesssim n^{-2s+1+\sigma} + n^{-s-4+\sigma} + n^{-2s-4+\sigma} + n^{-2s-1+\sigma}.
\]
In particular, if \( s > \frac{1+\sigma}{2} \), then
\[
\| F(t, \cdot) \|_{H^s(\mathbb{S})} \lesssim n^{-r_s},
\]
where \( r_s > 0 \) and
\[
r_s = \begin{cases} 
2s - 1 - \sigma, & \text{if } \frac{1+\sigma}{2} < s \leq 5, \\
 s + 4 - \sigma, & \text{if } s > 5.
\end{cases}
\]
4.2. Difference between approximate and actual solutions. Let \( u_{\omega,n} \) be the solution of Eq. (2.4) with initial data given by the approximate solution \( u^{\omega,n} \) evaluated at time zero. That is, \( u_{\omega,n} \) solves the following Cauchy problem

\[
\begin{cases}
\partial_t u_{\omega,n} + u_{\omega,n}\partial_x u_{\omega,n} + \partial_x A_{\mu}^{-1}(2\mu(u_{\omega,n})u_{\omega,n} + \frac{1}{2}(\partial_x u_{\omega,n})^2)

-3\partial_x u_{\omega,n}\partial_x^2 u_{\omega,n} - \frac{7}{2}(\partial_x^2 u_{\omega,n})^2) = 0, & t > 0, \ x \in \mathbb{S},

u_{\omega,n}(0,x) = u^{\omega,n}(0,x) = \omega n^{-1} + n^{-s} \cos(nx), & x \in \mathbb{S}.
\end{cases}
\]

Using Lemma 4.3, we obtain

\[
\|u_{\omega,n}(0, \cdot)\|_{H^r(\mathbb{S})} = \|u^{\omega,n}(0, \cdot)\|_{H^r(\mathbb{S})} = \|\omega n^{-1} + n^{-s} \cos(nx)\|_{H^r(\mathbb{S})} \lesssim 1.
\]

By Theorem 2.1, we know that \( u_{\omega,n} \) is the unique solution of (4.2) and exists globally in time. To estimate the difference between the approximate and actual solutions, we let \( v = u^{\omega,n} - u_{\omega,n} \), then for \( t > 0 \) and \( x \in \mathbb{S} \), \( v \) satisfies the following Cauchy problem

\[
\begin{cases}
\partial_x v = F - \frac{1}{2}\partial_x [(u^{\omega,n} + u_{\omega,n})v] - 2\mu(u_{\omega,n})\partial_x A_{\mu}^{-1}v

-2\mu(v)\partial_x A_{\mu}^{-1}u^{\omega,n} - \frac{1}{2}\partial_x A_{\mu}^{-1}[\partial_x (u^{\omega,n} + u_{\omega,n})\partial_x v]

+3\partial_x A_{\mu}^{-1}(\partial_x u^{\omega,n}\partial_x^2 v) + 3\partial_x A_{\mu}^{-1}(\partial_x u_{\omega,n}\partial_x v)

+\frac{7}{2}\partial_x A_{\mu}^{-1}[\partial_x^2 (u^{\omega,n} + u_{\omega,n})\partial_x v], & t > 0, \ x \in \mathbb{S},

v(0,x) = 0, & x \in \mathbb{S}.
\end{cases}
\]

Lemma 4.7. If \( n \gg 1, s \geq 4 \) and \( \frac{5}{2} < \sigma \leq s \), then for any fixed \( T_0 > 0 \), we have

\[
\|v(t, \cdot)\|_{H^r(\mathbb{S})} \lesssim n^{-r_s}, \ t \in [0, T_0].
\]

Proof. Applying \( \Lambda^\sigma \) to both sides of (4.3), multiplying the resulting equation by \( \Lambda^\sigma v \) and integrating it with respect to \( x \), we obtain

\[
\begin{align*}
\frac{1}{2} \int_{\mathbb{S}} \Lambda^\sigma F \cdot \Lambda^\sigma v dx - \frac{1}{2} \int_{\mathbb{S}} \Lambda^\sigma \partial_x [(u^{\omega,n} + u_{\omega,n})v] \cdot \Lambda^\sigma v dx

-2\mu(u_{\omega,n}) \int_{\mathbb{S}} \Lambda^\sigma \partial_x A_{\mu}^{-1}v \cdot \Lambda^\sigma v dx - 2\mu(v) \int_{\mathbb{S}} \Lambda^\sigma \partial_x A_{\mu}^{-1}u^{\omega,n} \cdot \Lambda^\sigma v dx

-\frac{1}{2} \int_{\mathbb{S}} \Lambda^\sigma \partial_x A_{\mu}^{-1}[\partial_x (u^{\omega,n} + u_{\omega,n})\partial_x v] \cdot \Lambda^\sigma v dx

+3 \int_{\mathbb{S}} \Lambda^\sigma \partial_x A_{\mu}^{-1}(\partial_x u^{\omega,n}\partial_x^2 v) \cdot \Lambda^\sigma v dx

+3 \int_{\mathbb{S}} \Lambda^\sigma \partial_x A_{\mu}^{-1}(\partial_x u_{\omega,n}\partial_x v) \cdot \Lambda^\sigma v dx

+\frac{7}{2} \int_{\mathbb{S}} \Lambda^\sigma \partial_x A_{\mu}^{-1}[\partial_x^2 (u^{\omega,n} + u_{\omega,n})\partial_x v] \cdot \Lambda^\sigma v dx

:= \sum_{i=1}^{8} G_i.
\end{align*}
\]

By Hölder inequality, we know

\[
|G_1| = | \int_{\mathbb{S}} \Lambda^\sigma F \cdot \Lambda^\sigma v dx | \leq \| \Lambda^\sigma F \|_{L^2(\mathbb{S})} \| \Lambda^\sigma v \|_{L^2(\mathbb{S})}.
\]

For \( G_2 \), we have

\[
|G_2| = \left| -\frac{1}{2} \int_{\mathbb{S}} \Lambda^\sigma \partial_x [(u^{\omega,n} + u_{\omega,n})v] \cdot \Lambda^\sigma v dx \right|
\]

\[
= \left| -\frac{1}{2} \int_{\mathbb{S}} [\Lambda^\sigma \partial_x (u^{\omega,n} + u_{\omega,n})]v \cdot \Lambda^\sigma v dx - \frac{1}{2} \int_{\mathbb{S}} (u^{\omega,n} + u_{\omega,n}) \Lambda^\sigma \partial_x v \cdot \Lambda^\sigma v dx \right|
\]

\[
= \left| -\frac{1}{2} \int_{\mathbb{S}} \Lambda^\sigma \partial_x (u^{\omega,n} + u_{\omega,n}) v \cdot \Lambda^\sigma v dx + \frac{1}{4} \int_{\mathbb{S}} \partial_x (u^{\omega,n} + u_{\omega,n}) \cdot (\Lambda^\sigma v)^2 dx \right|
\]
\[ \begin{align*}
\frac{1}{2} \| \Lambda^s \partial_x (u^{\omega,n} + u_{\omega,n}) \|_{L^2(\mathbb{S})} & \leq \frac{1}{2} \| \Lambda^s \partial_x (u^{\omega,n} + u_{\omega,n}) \|_{L^2(\mathbb{S})} \\
+ \frac{1}{2} \| \partial_x (u^{\omega,n} + u_{\omega,n}) \|_{L^2(\mathbb{S})} & \leq \frac{1}{2} \| \Lambda^s \partial_x \|_{L^2(\mathbb{S})} \| \Lambda^s v \|_{L^2(\mathbb{S})} \\
\| \Lambda^s \partial_x (u^{\omega,n} + u_{\omega,n}) \|_{L^2(\mathbb{S})} & \leq \| \Lambda^s \partial_x \|_{L^2(\mathbb{S})} \| \Lambda^s v \|_{L^2(\mathbb{S})} \\
\| u^{\omega,n} + u_{\omega,n} \|_{H^s(\mathbb{S})} & \leq \left( \| u^{\omega,n} \|_{H^s(\mathbb{S})} + \| u_{\omega,n} \|_{H^s(\mathbb{S})} \right) \| v \|_{H^s(\mathbb{S})}.
\end{align*} \]

where we have used integrating by parts, the Sobolev imbedding theorem and Lemma 4.4.

According to (2.1), we have

\[ |G_3| + |G_4| \]

\[ = \left| 2\mu (u_{\omega,n}) \int_{\mathbb{S}} \Lambda^s \partial_x A_{\mu}^{-1} v \cdot \Lambda^s v dx \right| \\
+ \left| 2\mu (v) \int_{\mathbb{S}} \Lambda^s \partial_x A_{\mu}^{-1} u^{\omega,n} \cdot \Lambda^s v dx \right| \\
\leq 2 \| u_{\omega,n} \|_{L^2(\mathbb{S})} \| \Lambda^s \partial_x A_{\mu}^{-1} v \|_{L^2(\mathbb{S})} \| \Lambda^s v \|_{L^2(\mathbb{S})} \\
+ 2 \| v \|_{L^2(\mathbb{S})} \| \Lambda^s \partial_x A_{\mu}^{-1} u^{\omega,n} \|_{L^2(\mathbb{S})} \| \Lambda^s v \|_{L^2(\mathbb{S})} \\
= 2 \| u_{\omega,n} \|_{L^2(\mathbb{S})} \| \partial_x A_{\mu}^{-1} v \|_{H^s(\mathbb{S})} \| v \|_{H^s(\mathbb{S})} \\
+ 2 \| v \|_{L^2(\mathbb{S})} \| \partial_x A_{\mu}^{-1} u^{\omega,n} \|_{H^s(\mathbb{S})} \| v \|_{H^s(\mathbb{S})} \\
\lesssim \| u_{\omega,n} \|_{L^2(\mathbb{S})} \| v \|_{H^{s-3}(\mathbb{S})} \| v \|_{H^s(\mathbb{S})} + \| v \|_{L^2(\mathbb{S})} \| u^{\omega,n} \|_{H^{s-3}(\mathbb{S})} \| v \|_{H^s(\mathbb{S})} \\
\lesssim \left( \| u_{\omega,n} \|_{H^s(\mathbb{S})} + \| u^{\omega,n} \|_{H^s(\mathbb{S})} \right) \| v \|_{H^s(\mathbb{S})}^2.
\]

Since \( \frac{3}{2} < \sigma \leq s \), by (2.1) and Lemma 4.5, we get

\[ |G_5| + |G_8| \]

\[ = \left| \frac{1}{2} \int_{\mathbb{S}} \Lambda^s \partial_x A_{\mu}^{-1} \left[ \partial_x (u^{\omega,n} + u_{\omega,n}) \partial_x v \right] \cdot \Lambda^s v dx \right| \\
+ \left| \frac{1}{2} \int_{\mathbb{S}} \Lambda^s \partial_x A_{\mu}^{-1} \left[ \partial_x (u^{\omega,n} + u_{\omega,n}) \partial_x v \right] \cdot \Lambda^s v dx \right| \\
\leq \frac{1}{2} \| \Lambda^s \partial_x A_{\mu}^{-1} \|_{L^2(\mathbb{S})} \| \partial_x (u^{\omega,n} + u_{\omega,n}) \partial_x v \|_{L^2(\mathbb{S})} \| \Lambda^s v \|_{L^2(\mathbb{S})} \\
+ \frac{1}{2} \| \Lambda^s \partial_x A_{\mu}^{-1} \|_{L^2(\mathbb{S})} \| \partial_x (u^{\omega,n} + u_{\omega,n}) \partial_x v \|_{L^2(\mathbb{S})} \| \Lambda^s v \|_{L^2(\mathbb{S})} \\
= \frac{1}{2} \| \partial_x A_{\mu}^{-1} \|_{L^2(\mathbb{S})} \| \partial_x (u^{\omega,n} + u_{\omega,n}) \partial_x v \|_{H^s(\mathbb{S})} \| v \|_{H^s(\mathbb{S})} \\
+ \frac{1}{2} \| \partial_x A_{\mu}^{-1} \|_{L^2(\mathbb{S})} \| \partial_x (u^{\omega,n} + u_{\omega,n}) \partial_x v \|_{H^s(\mathbb{S})} \| v \|_{H^s(\mathbb{S})} \\
\lesssim \| \partial_x (u^{\omega,n} + u_{\omega,n}) \|_{H^{s-3}(\mathbb{S})} \| \partial_x v \|_{H^s(\mathbb{S})} \| v \|_{H^s(\mathbb{S})} \\
+ \| \partial_x (u^{\omega,n} + u_{\omega,n}) \|_{H^{s-3}(\mathbb{S})} \| \partial_x v \|_{H^s(\mathbb{S})} \| v \|_{H^s(\mathbb{S})} \\
\lesssim \left( \| u_{\omega,n} \|_{H^s(\mathbb{S})} + \| u^{\omega,n} \|_{H^s(\mathbb{S})} \right) \| v \|_{H^s(\mathbb{S})}^2.
\]

and

\[ |G_6| + |G_7| \]

\[ = \left| 3 \int_{\mathbb{S}} \Lambda^s \partial_x A_{\mu}^{-1} \left[ \partial_x (\partial_x u^{\omega,n}) \partial_x v \right] \cdot \Lambda^s v dx \right| \\
+ \left| 3 \int_{\mathbb{S}} \Lambda^s \partial_x A_{\mu}^{-1} \left[ \partial_x (\partial_x u^{\omega,n}) \partial_x v \right] \cdot \Lambda^s v dx \right| \\
\leq 3 \| \Lambda^s \partial_x A_{\mu}^{-1} \|_{L^2(\mathbb{S})} \| \partial_x (\partial_x u^{\omega,n}) \partial_x v \|_{L^2(\mathbb{S})} \| \Lambda^s v \|_{L^2(\mathbb{S})} \\
+ 3 \| \Lambda^s \partial_x A_{\mu}^{-1} \|_{L^2(\mathbb{S})} \| \partial_x (\partial_x u^{\omega,n}) \partial_x v \|_{L^2(\mathbb{S})} \| \Lambda^s v \|_{L^2(\mathbb{S})} \\
= 3 \| \partial_x A_{\mu}^{-1} \|_{L^2(\mathbb{S})} \| \partial_x (\partial_x u^{\omega,n}) \partial_x v \|_{H^s(\mathbb{S})} \| v \|_{H^s(\mathbb{S})} + 3 \| \partial_x A_{\mu}^{-1} \|_{L^2(\mathbb{S})} \| \partial_x (\partial_x u^{\omega,n}) \partial_x v \|_{H^s(\mathbb{S})} \| v \|_{H^s(\mathbb{S})} \\
\lesssim \| \partial_x u^{\omega,n} \|_{H^{s-3}(\mathbb{S})} \| \partial_x v \|_{H^s(\mathbb{S})} + \| \partial_x u^{\omega,n} \|_{H^{s-3}(\mathbb{S})} \| \partial_x v \|_{H^s(\mathbb{S})} \\
\lesssim \| \partial_x u^{\omega,n} \|_{H^{s-3}(\mathbb{S})} \| \partial_x v \|_{H^s(\mathbb{S})} + \| \partial_x u^{\omega,n} \|_{H^{s-3}(\mathbb{S})} \| \partial_x v \|_{H^s(\mathbb{S})} \\
\lesssim \left( \| u_{\omega,n} \|_{H^s(\mathbb{S})} + \| u^{\omega,n} \|_{H^s(\mathbb{S})} \right) \| v \|_{H^s(\mathbb{S})}^2.
\]
Thus,
\[ \frac{1}{2} \frac{d}{dt} \|v(t, \cdot)\|_{H^s(S)}^2 \lesssim \left( \|u_{\omega,n}\|_{H^s(S)} + \|u^{\omega,n}\|_{H^s(S)} \right) \|v\|_{H^s(S)}^2 + \|F\|_{H^s(S)} \|v\|_{H^s(S)}. \]
By (4.1), we have
\[ \|u_{\omega,n}(t, \cdot)\|_{H^s(S)} + \|u^{\omega,n}(t, \cdot)\|_{H^s(S)} \leq e^{cT_0} \|u_{\omega,n}(0, \cdot)\|_{H^s(S)} + \|u^{\omega,n}(0, \cdot)\|_{H^s(S)} \lesssim 1, \quad t \in [0, T_0]. \]
According to Lemma 4.6, we obtain
\[ \frac{1}{2} \frac{d}{dt} \|v(t, \cdot)\|_{H^s(S)}^2 \lesssim \|v\|_{H^s(S)}^2 + n^{-r_s} \|v\|_{H^s(S)}. \]
That is,
\[ \frac{d}{dt} \|v(t, \cdot)\|_{H^s(S)} \lesssim \|v\|_{H^s(S)} + n^{-r_s}. \]
Since \( v(0, x) = 0 \), the Gronwall’s inequality implies the desired result. \( \square \)

4.3. Non-uniform dependence. The following theorem is our main result in this section.

**Theorem 4.8.** If \( s \geq 4 \), then for any fixed \( T_0 > 0 \), the solution map \( u_0 \to u(t) \) of Eq. (2.4) is not uniformly continuous from any bounded subset of \( H^s(S) \) into \( C([0, T_0]; H^s(S)) \). More precisely, there exist two sequences of \( u_n(t) \) and \( v_n(t) \) in \( C([0, T_0]; H^s(S)) \) such that
\[ \|u_n(t)\|_{H^s(S)} + \|v_n(t)\|_{H^s(S)} \lesssim 1, \]
\[ \lim_{n \to \infty} \|u_n(0) - v_n(0)\|_{H^s(S)} = 0, \]
and
\[ \liminf_{n \to \infty} \|u_n(t) - v_n(t)\|_{H^s(S)} \gtrsim |\sin t|, \quad t \in [0, T_0]. \]

**Proof.** Let \( u_{1,n}(t, x) \) and \( u_{-1,n}(t, x) \) be the unique solutions to Eq. (2.4) with the initial data \( u_{1,n}(0, x) \) and \( u_{-1,n}(0, x) \), respectively. Using Lemma 4.1, we have
\[ \|u_{1,n}(t)\|_{H^s(S)} + \|u_{-1,n}(t)\|_{H^s(S)} \lesssim \|u_{1,n}(0)\|_{H^s(S)} + \|u_{-1,n}(0)\|_{H^s(S)} \lesssim 1. \]
Moreover,
\[ \lim_{n \to \infty} \|u_{1,n}(0, x) - u_{-1,n}(0, x)\|_{H^s(S)} = \lim_{n \to \infty} \|2n^{-1}\|_{H^s(S)} = 0. \]
Since \( 2s - \sigma \geq s \geq 4 \), applying Lemma 4.7 and the interpolation inequality
\[ \|f\|_{H^s(S)} \leq \|f\|_{H^{s_1}(S)}^{\frac{1}{2}} \|f\|_{H^{s_2}(S)}^{\frac{1}{2}}, \]
we have
\[ \|u_{\pm 1,n}(t) - u_{\pm 1,n}(t)\|_{H^s(S)} \]
\[ \lesssim \|u_{\pm 1,n}(t) - u_{\pm 1,n}(t)\|_{H^{s_1}(S)}^{\frac{1}{2}} \|u_{\pm 1,n}(t) - u_{\pm 1,n}(t)\|_{H^{s_2}(S)}^{\frac{1}{2}} \]
\[ \lesssim n^{-\frac{1}{2}r_s} \|u_{\pm 1,n}(t)\|_{H^{2s-s}(S)} + \|u_{\pm 1,n}(t)\|_{H^{2s-s}(S)} \]
\[ \lesssim n^{-\frac{1}{2}r_s} \|u_{\pm 1,n}(t)\|_{H^{2s-s}(S)} + \|u_{\pm 1,n}(t)\|_{H^{2s-s}(S)} \]
\[ = n^{-\frac{1}{2}r_s} \|u_{\pm 1,n}(t)\|_{H^{2s-s}(S)} + \|u_{\pm 1,n}(t)\|_{H^{2s-s}(S)} \]
\[ \lesssim n^{-\frac{1}{2}r_s} \|u_{\pm 1,n}(t)\|_{H^{2s-s}(S)} \]
\[ \lesssim n^{-\frac{1}{2}r_s} (n^{-1} + n^{s-s}) \frac{1}{2}. \]
(4.4)
By Lemma 4.3,
\[
\liminf_{n \to \infty} \|u^{1,n}(t) - u^{-1,n}(t)\|_{H^s(S)} = \liminf_{n \to \infty} \|2n^{-1} + n^{-s}[\cos(nx + t) - \cos(nx - t)]\|_{H^s(S)} = \liminf_{n \to \infty} \|2n^{-1} + 2n^{-s} \sin(nx) \sin t\|_{H^s(S)}
\geq \liminf_{n \to \infty} \|2n^{-s} \sin(nx) \sin t\|_{H^s(S)} \geq 2s - m > 0.
\]
(4.5)

Therefore, by (4.4) and (4.5), we know
\[
\liminf_{n \to \infty} \|u_{1,n}(t) - u_{-1,n}(t)\|_{H^s(S)} = \liminf_{n \to \infty} \|u^{1,n}(t) - u^{-1,n}(t)\|_{H^s(S)} - \|u^{1,n}(t) - u_{1,n}(t)\|_{H^s(S)} - \|u^{-1,n}(t) - u_{-1,n}(t)\|_{H^s(S)}
\geq \liminf_{n \to \infty} \|u^{1,n}(t) - u^{-1,n}(t)\|_{H^s(S)}
\geq \liminf_{n \to \infty} \|u^{1,n}(t) - u_{1,n}(t)\|_{H^s(S)} + \|u^{-1,n}(t) - u_{-1,n}(t)\|_{H^s(S)}
\geq |\sin t|, \]
which completes the proof. \(\square\)

**Remark 1.** If we consider the solution in a small time interval, we can extend the condition \(s \geq 4\) in Theorem 4.2 to \(s > \frac{7}{2}\). In fact, the restriction condition \(s \geq 4\) is assumed in Lemma 4.1 to get the estimate (4.1). For \(s > \frac{7}{2}\), by (3.6), we know
\[
\frac{1}{2} \frac{d}{dt} \|u\|_{H^s(S)}^2 \leq c_s(\|u\|_{H^s(S)}^2 + \|u\|_{H^s(S)}^3),
\]
which implies
\[
\|u(t)\|_{H^s(S)}^2 \leq 2e^{c_s T_0} \|u_0\|_{H^s(S)}^2, \quad \forall t \in [0, T_0] \quad \text{with} \quad T_0 := \frac{1}{2c_s} \ln(1 + \frac{1}{\|u_0\|_{H^s(S)}^3}).
\]

5. **Global existence of weak solution.** In this section, we establish the existence of global weak solution in \(H^2(S)\). Firstly, the Cauchy problem (2.4) can be rewritten as follows
\[
\begin{align*}
\partial_t u + u \partial_x u + \partial_x P &= 0, \quad t > 0, \quad x \in \mathbb{R}, \\
A_\mu P &= 2\mu(u)u + \frac{1}{2}(\partial_x u)^2 - \frac{1}{2}(\partial_x^2 u)^2 - 3\partial_x(\partial_x u \partial_x^2 u), \\
u(t, x + 1) &= u(t, x), \quad t \geq 0, \quad x \in \mathbb{R}, \\
u(0, x) &= u_0(x), \quad x \in \mathbb{R}.
\end{align*}
\]
(5.1)

Now we introduce the definition of a weak solution to the Cauchy problem (5.1).

**Definition 5.1.** We call \(u : \mathbb{R}_+ \times S \to \mathbb{R}\) an admissible global weak solution of the Cauchy problem (5.1) if
(i) \(u(t, x) \in C(\mathbb{R}_+; C^1(S)) \cap L^\infty(\mathbb{R}_+; H^2(S))\) and
\[
\|\partial_x u(t, \cdot)\|_{H^1(S)} \leq \|\partial_x u_0\|_{H^1(S)} \quad \text{for each} \ t > 0.
\]
(5.2)
(ii) \(u(t, x)\) satisfies Eq.(5.1) in the sense of distributions and takes on the initial data pointwise.

The main result of this section is as follows.

**Theorem 5.2.** Let \(p > 2\). For any \(u_0 \in H^2(S)\) satisfying \(\partial_x^2 u_0 \in L^p(S)\), the Cauchy problem (5.1) has an admissible global weak solution in the sense of Definition 5.1.
5.1. Viscous approximate solutions. In this subsection, we construct the approximation solution sequence \( u_\varepsilon = u_\varepsilon(t,x) \). Hence, we consider the viscous problem of Eq. (5.1) as follows

\[
\begin{aligned}
\partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon + \partial_x P_\varepsilon &= \varepsilon \partial_x^2 u_\varepsilon, \quad t > 0, \ x \in \mathbb{R}, \\
A_\mu P_\varepsilon &= 2\mu(u_\varepsilon) + \frac{1}{2}(\partial_x u_\varepsilon)^2 - \frac{1}{2}(\partial_x^2 u_\varepsilon)^2 - 3\partial_x(\partial_x u_\varepsilon \partial_x^2 u_\varepsilon), \quad t \geq 0, \ x \in \mathbb{R}, \\
u_\varepsilon(t,x+1) &= u_\varepsilon(t,x), \quad t \geq 0, \ x \in \mathbb{R}, \\
u_\varepsilon(0,x) &= \nu_{\varepsilon,0}(x), \quad x \in \mathbb{R},
\end{aligned}
\]  

(5.3)

where \( u_{\varepsilon,0}(x) = (j_\varepsilon * u_0)(x) \) and \( j_\varepsilon, j_\varepsilon \) are defined in (3.1). By Lemma 3.3, we have

\[
\begin{aligned}
\|u_{\varepsilon,0}\|_{L^2(S)} &\leq \|u_0\|_{L^2(S)}, \quad \|\partial_x u_{\varepsilon,0}\|_{L^2(S)} \leq \|\partial_x u_0\|_{L^2(S)}, \\
\|\partial_x^2 u_{\varepsilon,0}\|_{L^2(S)} &\leq \|\partial_x^2 u_0\|_{L^2(S)}, \quad \|\partial_x^2 u_{\varepsilon,0}\|_{L^p(S)} \leq \|\partial_x^2 u_0\|_{L^p(S)}
\end{aligned}
\]

(5.4)

and \( u_{\varepsilon,0} \to u_0 \) in \( H^2(S) \) as \( \varepsilon \to 0 \).

**Lemma 5.3.** Let \( \varepsilon > 0 \) and \( u_{\varepsilon,0} \in H^s(S), \ s \geq 5 \). Then there exists a unique \( u_\varepsilon \in C([0,T_0];H^s(S)) \) to Eq. (5.3). Moreover, for each \( t \geq 0 \) and \( \varepsilon > 0 \), it holds that

\[
\begin{aligned}
\int_{S}[(\partial_x u_\varepsilon)^2 + (\partial_x^2 u_\varepsilon)^2](t,x)dx + 2\varepsilon \int_{0}^{t}\int_{S}[(\partial_x u_\varepsilon)^2 + (\partial_x^2 u_\varepsilon)^2](s,x)dxds
= \int_{S}[(\partial_x u_{\varepsilon,0})^2 + (\partial_x^2 u_{\varepsilon,0})^2]dx,
\end{aligned}
\]

(5.5)

and for each \( \varepsilon > 0 \),

\[
\|u_\varepsilon\|_{L^\infty(\mathbb{R} \times S)} \leq \|u_0\|_{H^2(S)}, \quad \|\partial_x u_\varepsilon\|_{L^\infty(\mathbb{R} \times S)} \leq 2\|u_0\|_{H^2(S)}.
\]

**Proof.** First, following the standard argument for a nonlinear parabolic equation, one can obtain the local well-posedness result that, for \( u_{\varepsilon,0} \in H^s(S), \ s \geq 4 \), there exists a positive constant \( T_0 \) such that Eq. (5.3) has a unique solution \( u_\varepsilon = u_\varepsilon(t,x) \in C([0,T_0];H^s(S)) \cap L^2([0,T_0];H^{s+1}(S)) \). We denote the life span of the solution \( u_\varepsilon(t,x) \) by \( T \). Note that \( (A_\mu f,g)_{L^2(S)} = (f,A_\mu g)_{L^2(S)} \). Multiplying Eq. (5.3) by \( A_\mu u_\varepsilon \) and integrating over \( S \), we obtain

\[
\begin{aligned}
\frac{1}{2} \int_{S}[(\partial_x u_\varepsilon)^2 + (\partial_x^2 u_\varepsilon)^2](t,x)dx = \varepsilon \int_{S}[(\partial_x u_\varepsilon)^2 + (\partial_x^2 u_\varepsilon)^2](t,x)dx.
\end{aligned}
\]

Then (5.5) holds for all \( 0 \leq t < T \).

By Corollary 1, (5.4) and (5.5), we obtain that

\[
\begin{aligned}
\max_{x \in S}(\partial_x u_\varepsilon)^2(t,x) &\leq \frac{1}{12} \int_{S}[(\partial_x^2 u_\varepsilon)^2](t,x)dx \leq \frac{1}{12} \int_{S}[(\partial_x u_{\varepsilon,0})^2 + (\partial_x^2 u_{\varepsilon,0})^2]dx \\
&\leq \frac{1}{12} \left( \|\partial_x u_0\|_{L^2(S)}^2 + \|\partial_x^2 u_0\|_{L^2(S)}^2 \right).
\end{aligned}
\]

This in turn implies that

\[
\|\partial_x u_\varepsilon\|_{L^\infty(S)} \leq \frac{\sqrt{3}}{\mu_1} \mu_1 \leq \|u_0\|_{H^2(S)},
\]

where \( \mu_1 \) is defined in (2.5). Note that \( \int_{S}(u_\varepsilon - u_{\varepsilon,0}) = 0 \). By Corollary 1,

\[
\max_{x \in S}(u_\varepsilon(t,x) - u_{\varepsilon,0})^2 \leq \frac{1}{12} \int_{S}(\partial_x u_\varepsilon)^2(t,x)dx \leq \frac{1}{12} \|\partial_x u_{\varepsilon,0}(\cdot,\cdot)\|_{L^\infty(S)}^2.
\]

Hence, we get

\[
\|u_\varepsilon\|_{L^\infty(S)} \leq \frac{1}{12} \mu_1 + |u_{\varepsilon,0}| \leq \frac{1}{12} \mu_1 + \|u_0\|_{L^2(S)} + 2\|u_0\|_{H^2(S)}.
\]

Next, we prove \( T = \infty \), that is, the solution \( u_\varepsilon \) exists globally for each \( \varepsilon > 0 \). Similar to the proof of Theorems 3.4-3.5, we can show that the \( H^s(S) \)-norm of \( u_\varepsilon(t,\cdot) \) does not blow up on \([0,T]\) if \( \|\partial_x^2 u_\varepsilon(t,\cdot)\|_{L^\infty(S)} < \infty \) holds. By Corollary 1,
we get \(\|\partial^3_x u_\epsilon(t, \cdot)\|_{L^\infty(S)} \leq \frac{\sqrt{3}}{6} \|\partial^1_x u_\epsilon(t, \cdot)\|_{L^2(S)}\), so we only need to derive an a priori estimate on \(\|\partial^3_x u_\epsilon\|_{L^2(S)}\). Applying the operator \(A_\mu\) to Eq. (3.3), then multiplying both sides by \(A_\mu u_\epsilon\) and integrating over \(S\) with respect to \(x\), we get

\[
\frac{1}{2} \frac{d}{dt} \int_S \left(\partial^2_x u_\epsilon \right)^2 + 2\left(\partial^3_x u_\epsilon \right)^2 + \left(\partial^1_x u_\epsilon \right)^2 \left(\partial^1_x u_\epsilon \right) \left(\partial^3_x u_\epsilon \right) \right) \, dx
\]

\[
+ \varepsilon \int_S \left(\partial^2_x u_\epsilon \right)^2 + 2\left(\partial^3_x u_\epsilon \right)^2 + \left(\partial^1_x u_\epsilon \right)^2 \left(\partial^3_x u_\epsilon \right) \right) \, dx
\]

\[
- \int_S u_\epsilon \partial_x u_\epsilon \partial^2_x u_\epsilon \, dx + \int_S \partial_x u_\epsilon \partial^2_x u_\epsilon \, dx - \int_S \partial_x u_\epsilon \left(\partial^1_x u_\epsilon \right)^2 \, dx
\]

\[
+ 2\int_S \partial_x u_\epsilon \partial^2_x u_\epsilon \partial^2_x u_\epsilon \, dx + \varepsilon \int_S u_\epsilon \partial^3_x u_\epsilon \partial^3_x u_\epsilon \, dx
\]

\[
\leq \|u_\epsilon\|_{L^\infty(S)} \|\partial_x u_\epsilon\|_{L^\infty(S)} \|\partial^2_x u_\epsilon\|_{L^2(S)} + \|\partial_x u_\epsilon\|_{L^\infty(S)} \|\partial^2_x u_\epsilon\|_{L^2(S)}
\]

\[
+ 2\|u_\epsilon\|_{L^\infty(S)} \|\partial^2_x u_\epsilon\|_{L^2(S)} + \varepsilon \|\partial^3_x u_\epsilon\|_{L^2(S)}
\]

\[
+ \frac{1}{4\pi} \|\partial^3_x u_\epsilon\|_{L^2(S)} \|\partial^3_x u_\epsilon\|_{L^2(S)}
\]

By Gronwall’s inequality and the above estimates of \(\|u_\epsilon\|_{L^\infty(S)}\) and \(\|\partial_x u_\epsilon\|_{L^\infty(S)}\), we know, for each \(\varepsilon > 0\) and \(t \in [0, T]\), there exists a constant \(C(\varepsilon, u_0) > 0\) depending on \(\varepsilon\) and \(u_0\) such that \(\|\partial^1_x u_\epsilon(t, \cdot)\|_{L^2(S)} \leq C(\varepsilon, u_0)\). Thus, we have \(T = \infty\), which completes the proof of the lemma.

5.2. Precompactness. In this subsection, we are ready to obtain the necessary compactness of the viscous approximation solutions \(u_\epsilon(t, x)\).

For convenience, we denote \(P_\varepsilon = P_1, \varepsilon + P_2, \varepsilon\), where \(P_1, \varepsilon, P_2, \varepsilon\) are defined by

\[
P_1, \varepsilon = A_\mu^{-1}[2\mu(u_\epsilon) u_\epsilon + \frac{1}{2}(\partial_x u_\epsilon)^2 - \frac{1}{2}(\partial^2_x u_\epsilon)^2]
\]

\[
P_2, \varepsilon = -3\partial_x A_\mu^{-1}(\partial_x u_\epsilon \partial_x^2 u_\epsilon).
\]

Lemma 5.4. Assume \(u_0 \in H^2(S)\). For each \(t \geq 0\) and \(\varepsilon > 0\), the following inequalities hold

\[
\|\partial^1_x P_1, \varepsilon (t, \cdot)\|_{L^\infty(S)} \leq C_0 \|u_0\|_{H^2(S)}^2,
\]

\[
\|\partial^2_x P_1, \varepsilon (t, \cdot)\|_{L^\infty(S)} \leq C_0 \|u_0\|_{H^2(S)}^2,
\]

\[
\|\partial^2_x P_2, \varepsilon (t, \cdot)\|_{L^1(S)} \leq C_0 \|u_0\|_{H^2(S)}^2.
\]

Here and in what follows, we use \(C_0\) to denote a generic positive constant, independent of \(\varepsilon\), which may change from line to line.

Proof. For \(\sigma = 1\) or \(\infty\), by Young’s inequality, we have

\[
\|\partial^1_x P_1, \varepsilon (t, \cdot)\|_{L^\infty(S)} = \|\partial^1_x A_\mu^{-1}[2\mu(u_\epsilon) u_\epsilon + \frac{1}{2}(\partial_x u_\epsilon)^2 - \frac{1}{2}(\partial^2_x u_\epsilon)^2]\|_{L^\infty(S)}
\]

\[
= \|\partial^1_x g \cdot [2\mu(u_\epsilon) u_\epsilon + \frac{1}{2}(\partial_x u_\epsilon)^2 - \frac{1}{2}(\partial^2_x u_\epsilon)^2]\|_{L^\infty(S)}
\]

\[
\leq \|\partial^1_x g\|_{L^\infty(S)} \|2\mu(u_\epsilon) u_\epsilon + \frac{1}{2}(\partial_x u_\epsilon)^2 - \frac{1}{2}(\partial^2_x u_\epsilon)^2\|_{L^1(S)}
\]

\[
\leq C_0 \|u_0\|_{H^2(S)}^2.
\]

and

\[
\|\partial^2_x P_2, \varepsilon (t, \cdot)\|_{L^\infty(S)} = \|-3\partial^2_x A_\mu^{-1}(\partial_x u_\epsilon \partial_x^2 u_\epsilon)\|_{L^\infty(S)}
\]

\[
= \|-3\partial^2_x g \cdot (\partial_x u_\epsilon \partial_x^2 u_\epsilon)\|_{L^\infty(S)}
\]

\[
\leq 3\|\partial^2_x g\|_{L^\infty(S)} \|\partial_x u_\epsilon \partial_x^2 u_\epsilon\|_{L^1(S)}
\]

\[
\leq C_0 \|u_0\|_{H^2(S)}^2.
\]
where $i = 0, 1, 2, 3$ and $j = 0, 1, 2$. The estimate of $\|\partial_x^i P_{1, \varepsilon}\|_{L^1(S)}$ follows from the above estimates and the fact
\[
\partial_x^i P_{1, \varepsilon} = -\mu(P_{1, \varepsilon}) + \partial_x^i P_{1, \varepsilon} + 2\mu(u_{\varepsilon})u_{\varepsilon} + \frac{1}{2}(\partial_x u_{\varepsilon})^2 - \frac{1}{2}(\partial_x^2 u_{\varepsilon})^2.
\]
Since $P_{\varepsilon} = P_{1, \varepsilon} + P_{2, \varepsilon}$, we can directly deduce the estimate of $\|P_{\varepsilon}(t, \cdot)\|_{W^{2, \varepsilon}(S)}$.
Moreover,
\[
\begin{align*}
\partial_x^3 P_{\varepsilon} &= \partial_x^3 P_{1, \varepsilon} + \partial_x^3 P_{2, \varepsilon} = \partial_x^3 P_{1, \varepsilon} - 3\partial_x^4 A_1^{-1}(\partial_x u_{\varepsilon}\partial_x^3 u_{\varepsilon}) \\
&= \partial_x^3 P_{1, \varepsilon} - 3\partial_x u_{\varepsilon}\partial_x^2 u_{\varepsilon} + 3\mu(A_1^{-1}(\partial_x u_{\varepsilon}\partial_x^2 u_{\varepsilon})) - 3\partial_x^4 A_1^{-1}(\partial_x u_{\varepsilon}\partial_x^3 u_{\varepsilon}),
\end{align*}
\]
then by Young’s inequality, we have
\[
\begin{align*}
\|\partial_x^3 P_{\varepsilon}\|_{L^1(S)} &\leq \|\partial_x^3 P_{1, \varepsilon}\|_{L^1(S)} + 3\|\partial_x u_{\varepsilon}\partial_x^2 u_{\varepsilon}\|_{L^1(S)} + 3\|A_1^{-1}(\partial_x u_{\varepsilon}\partial_x^2 u_{\varepsilon})\|_{L^1(S)} \\
&\leq \|\partial_x^3 P_{1, \varepsilon}\|_{L^1(S)} + 3\|\partial_x u_{\varepsilon}\|_{L^1(S)}\|\partial_x^2 u_{\varepsilon}\|_{L^1(S)} \\
&\leq C_0\|u_0\|^2_{H^2(S)},
\end{align*}
\]
which completes the proof. □

Next we turn to estimates of time derivatives.

**Lemma 5.5.** Assume $u_0 \in H^2(S)$. For each $T, t > 0$ and $0 < \varepsilon < 1$, the following inequalities hold
\[
\begin{align*}
\|\partial_t u_{\varepsilon}(t, \cdot)\|_{L^2(S)} &\leq C_0\|u_0\|^2_{H^2(S)} + \|u_0\|^2_{H^2(S)}, \\
\|\partial_t \partial_x u_{\varepsilon}\|_{L^2([0, T] \times S)} &\leq C_0\sqrt{T}\|u_0\|^2_{H^2(S)} + \frac{\varepsilon^2}{2}\|u_0\|^2_{H^2(S)}, \\
\|\partial_t \partial_x^2 P_{1, \varepsilon}\|_{L^1([0, T] \times S)} &\leq C_0(T + 1)(\|u_0\|^2_{H^2(S)} + \|u_0\|^2_{H^2(S)}).
\end{align*}
\]

**Proof.** By the first equation of (5.3) and Lemmas 5.3-5.4, we have
\[
\begin{align*}
\|\partial_t u_{\varepsilon}(t, \cdot)\|_{L^2(S)} &\leq \|u_{\varepsilon}\|_{L^2([0, T] \times S)} + \|\partial_x P_{\varepsilon}\|_{L^2(S)} + \varepsilon\|\partial_x^2 u_{\varepsilon}\|_{L^2(S)} \\
&\leq C_0\|u_0\|^2_{H^2(S)} + \varepsilon\|u_0\|^2_{H^2(S)} \\
&\leq C_0\|u_0\|^2_{H^2(S)} + \|u_0\|^2_{H^2(S)}.
\end{align*}
\]
Differentiating the first equation of (5.3) with respect to $x$, one obtains
\[
\partial_t \partial_x u_{\varepsilon} + u_{\varepsilon}\partial_x^2 u_{\varepsilon} + (\partial_x u_{\varepsilon})^2 + \partial_x^2 P_{\varepsilon} = \varepsilon\partial_x^3 u_{\varepsilon}.
\]
Thus,
\[
\begin{align*}
\|\partial_t \partial_x u_{\varepsilon}\|_{L^2([0, T] \times S)} &\leq \|u_{\varepsilon}\|_{L^2([0, T] \times S)} + \|\partial_x u_{\varepsilon}\|_{L^2([0, T] \times S)} + \varepsilon\|\partial_x^2 u_{\varepsilon}\|_{L^2([0, T] \times S)} \\
&\leq C_0\|u_0\|^2_{H^2(S)} + \sqrt{T}\|\partial_x u_{\varepsilon}\|^2_{L^2([0, T] \times S)} + \varepsilon\|\partial_x^2 u_{\varepsilon}\|_{L^2([0, T] \times S)} \\
&\leq C_0\sqrt{T}\|u_0\|^2_{H^2(S)} + \frac{\varepsilon}{2}\|u_0\|^2_{H^2(S)}.
\end{align*}
\]
Moreover, differentiating the first equation of (5.3) with respect to $x$ two times, we have
\[
\partial_t \partial_x^2 u_{\varepsilon} + u_{\varepsilon}\partial_x^3 u_{\varepsilon} + 3\partial_x u_{\varepsilon}\partial_x^2 u_{\varepsilon} + \partial_x^3 P_{\varepsilon} = \varepsilon\partial_x^4 u_{\varepsilon},
\]
and then
\[
-\partial_x^2 u \partial_t \partial_x^2 u = \partial_x^2 u (u \partial_x^2 u + 3 \partial_x u \partial_x^2 u + \partial_t^2 P_e - \varepsilon \partial_x^4 u)
\]
\[
= \frac{1}{2} \partial_x u (\partial_x^2 u)^2 - \frac{1}{2} \partial_x u (\partial_x^2 u)^2 + \partial_x^2 u \partial_x^4 P_e
+ 3 \partial_x^2 u \mu (A^{-1}_\mu (\partial_x u \partial_x^2 u)) - 3 \partial_x^2 u \partial_x^4 A^{-1}_\mu (\partial_x u \partial_x^2 u)
- \varepsilon \partial_x (\partial_x^2 u \partial_x^2 u) + \varepsilon (\partial_x^4 u)^2.
\]

By the definition of \( P_{1,e} \), we know
\[
\partial_t \partial_x^4 P_{1,e} = \partial_x^3 A^{-1}_\mu \left[ 2 \mu (u_x) \partial_t u_x + \partial_x u_x \partial_x \partial_t u_x - \partial_x^2 u_x \partial_t \partial_x^2 u_x \right]
\]
\[
= \partial_x^3 A^{-1}_\mu \left[ 2 \mu (u_x) \partial_t u_x + \partial_x u_x \partial_t u_x - \frac{1}{2} \partial_x u_x (\partial_x^2 u_x)^2 + \partial_x^2 u_x \partial_x^4 P_{1,e}
+ 3 \partial_x^2 u_x \mu (A^{-1}_\mu (\partial_x u \partial_x^2 u)) - 3 \partial_x^2 u_x \partial_x^4 A^{-1}_\mu (\partial_x u \partial_x^2 u)
+ \varepsilon (\partial_x^4 u_x)^2 \right]
\]
\[
+ \partial_x^4 A^{-1}_\mu \left[ \frac{1}{2} u_x (\partial_x^2 u_x)^2 - \varepsilon \partial_x^2 u_x \partial_x^4 u_x \right]
\]
\[
:= \partial_x^4 A^{-1}_\mu (E_1) + \partial_x^4 A^{-1}_\mu (E_2).
\]

Note that
\[
\partial_x^4 A^{-1}_\mu (E_2) = - \mu (A^{-1}_\mu (E_2)) + \partial_x^2 A^{-1}_\mu (E_2) + E_2.
\]

By Lemmas 5.3-5.4, (5.6), (5.7) and Young’s inequality, we get
\[
\int_0^T \int_\Sigma |\partial_t \partial_x^4 P_{1,e}| dx dt = \int_0^T \int_\Sigma |\partial_x^3 g (E_1 - \mu (g \ast E_2) + \partial_x^2 g \ast E_2 + \partial_x g) dx dt
\]
\[
\leq C_0 (\|E_1\|_{L^1([0,T] \times \Sigma)} + \|E_2\|_{L^1([0,T] \times \Sigma)})
\]
\[
\leq C_0 (T + 1)(\|u_0\|^3_{H^2(S)} + \|u_0\|^2_{H^3(S)}),
\]
which completes the proof of Lemma 5.5.

\[\square\]

**Lemma 5.6.** Let \( u_0 \in H^2(S) \) and \( \partial_x^2 u_0 \in L^p(S) \) for some \( p > 2 \). Then the following inequality holds
\[
\|\partial_x^2 u_x (t, \cdot)\|_{L^p(S)} \leq C_0 \|u_0\|_{H^2(S)} (e^{C_0 \|u_0\|_{H^2(S)} t} - 1).
\]

**Proof.** Denote \( q_x := \partial_x^2 u_x \), then \( q_x \) satisfies
\[
\partial_t q_x + u_x \partial_x q_x + \partial_x^4 P_{1,e} + 3 \mu (A^{-1}_\mu (q_x \partial_x u_x)) - 3 \partial_x^4 A^{-1}_\mu (q_x \partial_x u_x) = \varepsilon \partial_x^4 q_x.
\]

Multiplying the above equation by \( pq_x |q_x|^{p-2} \), we have
\[
\partial_t (pq_x |q_x|^{p-2}) + u_x \partial_x (pq_x |q_x|^{p-2}) + pq_x |q_x|^{p-2} \partial_x^4 P_{1,e}
+ 3pq_x |q_x|^{p-2} \mu (A^{-1}_\mu (q_x \partial_x u_x)) - 3pq_x |q_x|^{p-2} \partial_x^4 A^{-1}_\mu (q_x \partial_x u_x)
= \varepsilon pq_x |q_x|^{p-2} \partial_x^4 q_x + \varepsilon \partial_x^4 (pq_x |q_x|^{p-2} \partial_x q_x)^2.
\]

By Lemmas 5.3-5.4 and Young’s inequality, we know
\[
\int_\Sigma |q_x|^{p-2} \partial_x u_x dx + p \int_\Sigma |q_x|^{p-2} \partial_x^4 P_{1,e} dx
+ 3p \int_\Sigma |q_x|^{p-2} \mu (A^{-1}_\mu (q_x \partial_x u_x)) dx + 3p \int_\Sigma |q_x|^{p-2} \partial_x^4 A^{-1}_\mu (q_x \partial_x u_x) dx
\leq \|\partial_x u_x\|_{L^p(S)} \|q_x\|^{p-1}_{L^p(S)} \|\partial_x^4 P_{1,e}\|_{L^p(S)}
+ 3p \|q_x\|^{p-1}_{L^p(S)} \|\partial_x^4 A^{-1}_\mu (q_x \partial_x u_x)\|_{L^p(S)} + 3p \|q_x\|^{p-1}_{L^p(S)} \|\partial_x^4 A^{-1}_\mu (q_x \partial_x u_x)\|_{L^p(S)}
\leq \|\partial_x u_x\|_{L^\infty(S)} \|q_x\|^{p-1}_{L^p(S)} \|\partial_x^4 P_{1,e}\|_{L^p(S)}
+ 3p \|q_x\|^{p-1}_{L^p(S)} \|\partial_x^4 A^{-1}_\mu (q_x \partial_x u_x)\|_{L^p(S)} + 3p \|q_x\|^{p-1}_{L^p(S)} \|\partial_x^4 A^{-1}_\mu (q_x \partial_x u_x)\|_{L^p(S)}
\leq \|u_0\|_{H^2(S)} \|q_x\|^{p-1}_{L^p(S)} \|\partial_x^4 P_{1,e}\|_{L^p(S)}
+ 3p \|q_x\|^{p-1}_{L^p(S)} \|\partial_x^4 A^{-1}_\mu (q_x \partial_x u_x)\|_{L^p(S)} + 3p \|q_x\|^{p-1}_{L^p(S)} \|\partial_x^4 A^{-1}_\mu (q_x \partial_x u_x)\|_{L^p(S)}
\leq \|u_0\|_{H^2(S)} \|q_x\|^{p-1}_{L^p(S)} \|\partial_x^4 P_{1,e}\|_{L^p(S)}
+ 3p \|q_x\|^{p-1}_{L^p(S)} \|\partial_x^4 A^{-1}_\mu (q_x \partial_x u_x)\|_{L^p(S)} + 3p \|q_x\|^{p-1}_{L^p(S)} \|\partial_x^4 A^{-1}_\mu (q_x \partial_x u_x)\|_{L^p(S)}.
\]
Note that
\[
\frac{d}{dt} \int_{\mathcal{S}} |q_{\varepsilon}|^p dx = p \|q_{\varepsilon}\|_{L^p(\mathcal{S})}^{p-1} \frac{d}{dt} \|q_{\varepsilon}\|_{L^p(\mathcal{S})}.
\]
Thus,
\[
\frac{d}{dt} \|q_{\varepsilon}\|_{L^p(\mathcal{S})} \leq \|u_0\|_{H^2(\mathcal{S})} \|q_{\varepsilon}\|_{L^p(\mathcal{S})} + pC_0 \|u_0\|^2_{L^2(\mathcal{S})}.
\]
The Gronwall inequality implies the desired result.\(\square\)

To convenient, we define
\[
Q_{\varepsilon} := 3 \partial_x u_{\varepsilon}\partial_x^2 u_{\varepsilon} + \partial_x^3 P_{\varepsilon}
= \partial_x^3 P_{1,\varepsilon} + 3 \mu (A_{\mu}^{-1}(\partial_x u_{\varepsilon} \partial_x^2 u_{\varepsilon})) - 3 \partial_x^2 A_{\mu}^{-1}(\partial_x u_{\varepsilon} \partial_x^2 u_{\varepsilon}).
\]

**Lemma 5.7.** Let \(u_0 \in H^2(\mathcal{S})\) and \(\partial_x^2 u_0 \in L^p(\mathcal{S})\) for some \(p > 2\). There exist a positive sequence \(\{\varepsilon_k\}_{k \in \mathbb{N}}\) decreasing to zero and three functions \(u \in L^\infty(\mathbb{R}_+; H^2(\mathcal{S})) \cap H^1(\mathbb{R}_+ \times \mathcal{S}) \subseteq C(\mathbb{R}_+; C^1(\mathcal{S}))\) for each \(T > 0\), \(P \in L^\infty(\mathbb{R}_+; W^{2,\infty}(\mathcal{S}))\) and \(Q \in L^\infty(\mathbb{R}_+; W^{1,1}(\mathcal{S}) \cap L^\infty(\mathcal{S}))\) such that
\[
\begin{align*}
\{u_{\varepsilon_k}\} & \quad \text{weakly in } H^1([0, T] \times \mathcal{S}) \text{ for each } T \geq 0; \\
u_{\varepsilon_k} & \quad \text{strongly in } L^\infty_{\text{loc}}(\mathbb{R}_+; H^1(\mathcal{S})); \\
P_{\varepsilon_k} & \quad \text{weakly in } L^\sigma_{\text{loc}}(\mathbb{R}_+ \times \mathcal{S}) \text{ for each } 1 < \sigma < \infty; \\
Q_{\varepsilon_k} & \quad \text{strongly in } L^\sigma_{\text{loc}}(\mathbb{R}_+ \times \mathcal{S}) \text{ for each } 1 \leq \sigma < \infty.
\end{align*}
\]

**Proof.** Due to Lemmas 5.3 and 5.5, we have that
\[
\{u_{\varepsilon}\}, \{\partial_t u_{\varepsilon}\}, \text{ is uniformly bounded in } L^\infty(\mathbb{R}_+; H^2(\mathcal{S})),
\]
\[
\{\partial_t u_{\varepsilon}\} \text{ is uniformly bounded in } L^2([0, T]; H^1(\mathcal{S})) \text{ for each } T > 0.
\]
In particular, \(\{u_{\varepsilon}\}\) is uniformly bounded in \(H^1([0, T] \times \mathcal{S})\) and then we have \(u_{\varepsilon_k} \rightarrow u\) weakly in \(H^1([0, T] \times \mathcal{S})\). Moreover, Using the fact \(H^2(\mathcal{S}) \subseteq H^1(\mathcal{S}) \subseteq L^2(\mathcal{S})\) and Corollary 4 in [37], we know that \(u_{\varepsilon_k} \rightarrow u\) strongly in \(L^\infty_{\text{loc}}(\mathbb{R}_+; H^1(\mathcal{S}))\).

Due to Lemma 5.4, we obtain that \(\{P_{\varepsilon}\}\) is uniformly bounded in \(L^\infty(\mathbb{R}_+; W^{2,\infty}(\mathcal{S}))\). In particular, \(\{P_{\varepsilon}\}\) is uniformly bounded in \(L^\sigma([0, T] \times \mathcal{S})\) with \(1 < \sigma < \infty\) and then we have \(P_{\varepsilon_k} \rightarrow P\) weakly in \(L^\sigma_{\text{loc}}(\mathbb{R}_+ \times \mathcal{S})\) for each \(1 < \sigma < \infty\).

Moreover, due to Lemmas 5.4-5.5, we know
\[
\|Q_{\varepsilon}\|_{L^\infty(\mathcal{S})} = \|\partial_x^3 P_{1,\varepsilon} + 3 \mu (A_{\mu}^{-1}(\partial_x u_{\varepsilon} \partial_x^2 u_{\varepsilon})) + \partial_x P_{2,\varepsilon}\|_{L^\infty(\mathcal{S})}
\leq \|\partial_x^3 P_{1,\varepsilon}\|_{L^\infty(\mathcal{S})} + 3 \|u_{\varepsilon}\|_{L^1(\mathcal{S})} \|\partial_x^2 u_{\varepsilon}\|_{L^\infty(\mathcal{S})} + \|\partial_x P_{2,\varepsilon}\|_{L^\infty(\mathcal{S})}
\leq C_0 \|u_0\|^2_{H^2(\mathcal{S})},
\]
and
\[
\|\partial_x Q_{\varepsilon}\|_{L^1(\mathcal{S})} = \|\partial_x^3 P_{1,\varepsilon} + \partial_x^2 P_{2,\varepsilon}\|_{L^1(\mathcal{S})}
\leq \|\partial_x^3 P_{1,\varepsilon}\|_{L^1(\mathcal{S})} + \|\partial_x^2 P_{2,\varepsilon}\|_{L^1(\mathcal{S})}
\leq C_0 \|u_0\|^2_{H^2(\mathcal{S})},
\]
then \(\{Q_{\varepsilon}\}\) is uniformly bounded in \(L^\infty(\mathbb{R}_+; W^{1,1}(\mathcal{S}) \cap L^\infty(\mathcal{S}))\). Note that
\[
\begin{align*}
\partial_x u_{\varepsilon}\partial_x^2 u_{\varepsilon} &= \partial_x u_{\varepsilon}(u_{\varepsilon}\partial_x^2 u_{\varepsilon} + 3 \partial_x u_{\varepsilon}\partial_x^2 u_{\varepsilon} + \partial_x^3 P_{\varepsilon} - \varepsilon \partial_x^4 u_{\varepsilon}) \\
&= \partial_x u_{\varepsilon}(\partial_x u_{\varepsilon}\partial_x^2 u_{\varepsilon}) - u_{\varepsilon}(\partial_x^2 u_{\varepsilon})^2 - (\partial_x u_{\varepsilon})^2 u_{\varepsilon} + \partial_x u_{\varepsilon}\partial_x^2 P_{1,\varepsilon} \\
&\quad + 3 \partial_x u_{\varepsilon} \mu(A_{\mu}^{-1}(\partial_x u_{\varepsilon} \partial_x^2 u_{\varepsilon})) - 3 \partial_x u_{\varepsilon} \partial_x^2 A_{\mu}^{-1}(\partial_x u_{\varepsilon} \partial_x^2 u_{\varepsilon}) \\
&\quad - \varepsilon \partial_x(u_{\varepsilon}\partial_x^2 u_{\varepsilon}) + \varepsilon \partial_x^2 u_{\varepsilon}\partial_x^2 u_{\varepsilon}.
\end{align*}
\]
By Young’s inequality and Lemma 5.3,\[
\|\partial_t Q_z\|_{L^1([0,T] \times S)}
\leq \|\partial_t \partial_s^2 P_{1,r}\|_{L^1([0,T] \times S)} + 3\mu(A_\mu^{-1}\partial_t (\partial_x u_r \partial_s^2 u_r)) - 3\partial_s^2 A_\mu^{-1}\partial_t (\partial_x u_r \partial_s^2 u_r)\|_{L^1([0,T] \times S)}
\leq \|\partial_t \partial_s^2 P_{1,r}\|_{L^1([0,T] \times S)} + \|\partial_t \partial_s^2 u_r\|_{L^1([0,T] \times S)} + \|\partial_t \partial_s^2 A_\mu^{-1}(\partial_x u_r \partial_s^2 u_r)\|_{L^1([0,T] \times S)}
\leq \|\partial_t \partial_s^2 P_{1,r}\|_{L^1([0,T] \times S)} + \|\partial_t \partial_s^2 u_r\|_{L^1([0,T] \times S)} + \|\partial_t \partial_s^2 A_\mu^{-1}(\partial_x u_r \partial_s^2 u_r)\|_{L^1([0,T] \times S)}
\leq C_0(T + 1)(\|u_0\|_{H^2(S)} + \|u_0\|_{H^2(S)}),
\]
which implies that \(\{\partial_t Q_z\}_r\) is uniformly bounded in \(L^1([0, T] \times S)\) for each \(T > 0\).

Using the fact \(W^{1,1}(S) \subset L^1(S) \subset L^1(S), 1 \leq \sigma < \infty\) and Corollary 4 in [37], we know that \(Q_{z_k} \rightarrow Q\) strongly in \(L_{loc}(\mathbb{R}^+ \times S)\) for each \(1 \leq \sigma < \infty\).

5.3. Existence of solutions. From Lemmas 5.3 and 5.6, we can deduce that there exist two functions \(q \in L^p_{loc}(\mathbb{R}^+ \times S)\) and \(\tilde{q}^2 \in L^r_{loc}(\mathbb{R}^+ \times S)\) such that
\[
q_{z_k} \rightarrow q \quad \text{in } L^p_{loc}(\mathbb{R}^+ \times S), \quad q_{z_k}^2 \rightarrow \tilde{q}^2 \quad \text{in } L^r_{loc}(\mathbb{R}^+ \times S),
\]
for every \(1 < \rho < p\) and \(1 < r < \frac{4}{\rho}\). Moreover,
\[
\tilde{q}^2(t, x) \leq \tilde{q}^2(t, x), \quad \text{a.e. } (t, x) \in \mathbb{R}^+ \times S.
\]

In view of (5.9), we conclude that for any \(\eta \in C^1(\mathbb{R})\) with \(\eta'\) bounded, Lipschitz continuous on \(\mathbb{R}\), \(\eta(0) = 0\) and any \(1 < \rho < p\), we have
\[
\eta(q_{z_k}) \rightarrow \eta(q) \quad \text{in } L^p_{loc}(\mathbb{R}^+ \times S).
\]

Here and in what follows, we use overbars to denote weak limits in spaces to be understood from the context.

Lemma 5.8. The following inequality holds in the sense of distributions
\[
\int_S \left( (q^+_\epsilon)^2 - (q^+)^2 \right) dx
\leq \int_0^t \int_S u \left( (q^+_\epsilon)^2 - (q^+)^2 \right) dtdx - 2 \int_0^t \int_S Q(\bar{q}_\epsilon - q^+)dtdx.
\]

Proof. Taking \(\xi \in C^2(\mathbb{R})\) convex and multiplying (5.8) by \(\xi'(q_{z_k})\), we have
\[
\partial_t \xi(q_{z_k}) + \partial_x (u_{z_k} \xi(q_{z_k})) - \xi(q_{z_k}) \partial_t u_{z_k} + \xi'(q_{z_k})Q_{z_k}
= \varepsilon_k \partial_x^2 \xi(q_{z_k}) - \varepsilon_k \xi''(q_{z_k}) (\partial_x q_{z_k})^2 \leq \varepsilon_k \partial_x^2 \xi(q_{z_k}).
\]

In particular, we can use the entropy \(q \mapsto (q^+)^2/2\) and get
\[
\partial_t (q_{z_k}^+) + \partial_x \left( u_{z_k} (q_{z_k}^+) \right) - \frac{(q_{z_k}^+)^2}{2} \partial_x u_{z_k} + q_{z_k} + Q_{z_k} \leq \varepsilon_k \partial_x^2 (q_{z_k}^+)^2.
\]

Letting \(k \rightarrow \infty\), we have
\[
\partial_t (q^+) + \partial_x \left( u (q^+) \right) - \frac{(q^+)^2}{2} \partial_x u + Q^+ \leq 0.
\]
The rest of the proof is the same as in [41].
Similar as the proof of Lemmas 5.7-5.8 in [41], we can obtain the following two lemmas by using the entropy $\eta_R(\xi) := \eta_R(\xi)|_{-\infty,0}(\xi)$, where $R > 0$, $\chi_E$ is the characteristic function in set $E$ and

$$\eta_R(\xi) := \begin{cases} \frac{1}{2}\xi^2, & \text{if } |\xi| \leq R, \\ R|\xi| - \frac{1}{2}R^2, & \text{if } |\xi| > R. \end{cases}$$

**Lemma 5.9.** For any $t > 0$ and any $R > 0$,

$$\int_0^t \int_0^1 \left( \eta_R(q) - \eta_R(\xi) \right) \, dx \leq \int_0^t \int_0^1 \partial_x u \left( \eta_R(q) - \eta_R(\xi) \right) \, dx \, dt + \int_0^t \int_0^1 Q \left( (\eta_R^+)'(q) - (\eta_R^-)'(\xi) \right) \, dx \, dt.$$

**Lemma 5.10.** There holds $q^2 = q^2$, a.e. on $\mathbb{R}^+ \times S$.

**Proof of Theorem 5.2.** Let $u(t, x)$ be the limit of the viscous approximation solutions $u_\varepsilon(t, x)$ as $\varepsilon \to 0$. It then follows from Lemmas 5.3 and 5.7 that $u(t, x) \in C([\mathbb{R}^+; C^1(S)) \cap L^\infty(\mathbb{R}^+; H^2(S))$ and (5.2) holds.

Let $\mu(x, \lambda)$ be the Young measure associated with $\{q_\varepsilon\} = \{\partial_x^2 u_\varepsilon\}$, see more details in [43]. By Lemma 5.10, we have $\mu(x, \lambda) = \delta_{\lambda}(\partial_x^2 u)$ a.e. $(t, x) \in \mathbb{R}^+ \times S$, then

$$q_\varepsilon = \partial_x^2 u_\varepsilon \to q = \partial_x^2 u \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^+ \times S).$$

Taking $\varepsilon \to 0$ in Eq.(5.3), one finds from (5.10) and Lemma 5.7 that $u(t, x)$ is an admissible weak solution to Eq.(5.1). This completes the proof of Theorem 5.2. \Box

6. **Peaked solutions.** In this section, we show the existence of single peakon solutions to Eq.(1.4).

**Theorem 6.1.** For any $c > 0$, Eq.(1.4) admits the peaked periodic-one traveling-wave solutions $u_c(t, x) = \phi(\xi)$, $\xi = x - ct$, where

$$\phi(\xi) = \frac{12 \sinh(2)}{25 \sinh(2) - 6 \cosh(\frac{1}{2})} c \left[ \frac{1}{2} \left( \xi - \frac{1}{2} \right)^2 - \frac{\cosh(\xi - \frac{1}{2})}{2 \sinh(\frac{1}{2})} + \frac{47}{24} \right].$$

**Proof.** Motivated by the forms of periodic peakons for the $\mu$-Camassa-Holm equation [32], we assume that the periodic peakon of (1.4) is given by

$$u_c(t, x) = a \left[ \frac{1}{2} \left( \xi - \frac{1}{2} \right)^2 - \frac{\cosh(\xi - \frac{1}{2})}{2 \sinh(\frac{1}{2})} + \frac{47}{24} \right].$$

According to the definition of weak solutions, $u_c(t, x)$ satisfies the following equation

$$\sum_{j=1}^{4} I_j := \int_0^T \int_0^1 u_{c,x} \varphi \, dx \, dt + \int_0^T \int_0^1 u_{c} u_{c,x} \varphi \, dx \, dt + \int_0^T \int_0^1 g_x \ast [2\mu(u_c)u_c + \frac{1}{2}u_{c,x}^2 - \frac{1}{2}u_{c,xx}^2] \varphi \, dx \, dt - \int_0^T \int_0^1 3g_{xx} \ast (u_{c,x} u_{c,xx}) \varphi \, dx \, dt = 0,$$

for some $T > 0$ and any test function $\varphi(t, x) \in C^\infty_c([0, T) \times S)$, where $g(x) = \frac{1}{2}(x - |x| - \frac{1}{2})^2 - \frac{\cosh(x - |x| - \frac{1}{2})}{2 \sinh(\frac{1}{2})} + \frac{47}{24}$. One can obtain that

$$\mu(u_c) = a \int_{-\infty}^\infty \left[ \frac{1}{2} \left( \xi + \frac{1}{2} \right)^2 - \frac{\cosh(\xi + \frac{1}{2})}{2 \sinh(\frac{1}{2})} + \frac{47}{24} \right] \, dx + a \int_{-\infty}^\infty \left[ \frac{1}{2} \left( \xi - \frac{1}{2} \right)^2 - \frac{\cosh(\xi - \frac{1}{2})}{2 \sinh(\frac{1}{2})} + \frac{47}{24} \right] \, dx = a.$$
For $x > ct$, we get

\[
g_x = \int_0^\infty \left( x - y - \frac{1}{2} \right)^2 (y - ct - \frac{1}{2})^2 \frac{\sinh(y - ct - \frac{1}{2})}{2 \sinh(\frac{y}{2})} dy
\]

and

\[
g_{x,x} = \int_0^\infty \left( x - y - \frac{1}{2} \right)^2 (y - ct - \frac{1}{2})^2 \frac{\sinh(y - ct - \frac{1}{2})}{2 \sinh(\frac{y}{2})} dy
\]

By calculations, we have

\[
g_x = a^2 \int_0^\infty \left( x - y - \frac{1}{2} \right)^3 (y - ct - \frac{1}{2})^2 \frac{\sinh(y - ct - \frac{1}{2})}{2 \sinh(\frac{y}{2})} dy
\]

and

\[
u_{x,x} = a^2 \int_0^\infty \left( x - y - \frac{1}{2} \right)^3 (y - ct - \frac{1}{2})^2 \frac{\sinh(y - ct - \frac{1}{2})}{2 \sinh(\frac{y}{2})} dy
\]

we have

\[
\sum_{j=1}^4 I_j = \int_0^T \int_0^\infty \left[ c - \frac{25}{48} \frac{\cos(\frac{y}{2})}{2 \sinh(\frac{y}{2})} \right] \left( (x - ct - \frac{1}{2}) + \frac{\sinh(x - ct - \frac{1}{2})}{2 \sinh(\frac{y}{2})} \right) \varphi dx dt.
\]
Similarly, for $x \leq ct$, we have

\[
u_{c,t} = ac \left[ -(x - ct + \frac{1}{2}) + \frac{\sinh(x - ct + \frac{1}{2})}{2 \sinh(\frac{1}{2})} \right],
\]

\[
u_{u,c} = a^2 \left[ \frac{1}{2} (x - ct + \frac{1}{2})^3 - (x - ct + \frac{1}{2})^2 \frac{\sinh(x - ct + \frac{1}{2})}{4 \sinh(\frac{1}{2})} - (x - ct + \frac{1}{2}) \frac{\cosh(x - ct + \frac{1}{2})}{2 \sinh(\frac{1}{2})} \right]
+ \frac{\sinh(x - ct + \frac{1}{2}) \cosh(x - ct + \frac{1}{2})}{4(\sinh(\frac{1}{2}))^2} + \frac{47}{24} (x - ct + \frac{1}{2}) - \frac{47}{48} \frac{\sinh(x - ct + \frac{1}{2})}{\sinh(\frac{1}{2})}
\]

and

\[
\frac{\partial}{\partial x} \left[ 2\mu(u_c)u_c - \frac{1}{2} u_{c,xx} \right] - 3g_{xx} \left( u_{c,x}u_{c,xx} \right)
= a^2 \left[ \frac{-1}{2} (x - ct + \frac{1}{2})^3 + (x - ct + \frac{1}{2})(\frac{5}{8} - \frac{\cosh(\frac{1}{2})}{2 \sinh(\frac{1}{2})}) + \frac{\cosh(x - ct + \frac{1}{2})}{2 \sinh(\frac{1}{2})} \right]
+ (x - ct)^2 \frac{\sinh(x - ct + \frac{1}{2})}{4 \sinh(\frac{1}{2})} + (x - ct) \frac{\sinh(x - ct + \frac{1}{2})}{4 \sinh(\frac{1}{2})}
- \frac{\sinh(x - ct + \frac{1}{2}) \cosh(x - ct + \frac{1}{2})}{4(\sinh(\frac{1}{2}))^2} + \frac{\sinh(x - ct + \frac{1}{2}) \cosh(x - ct + \frac{1}{2})}{4(\sinh(\frac{1}{2}))^2}.
\]

Thus,

\[
\sum_{j=1}^4 \int_0^T \int_0^a \left[ c - \left( \frac{25}{12} - \frac{\cosh(\frac{1}{2})}{2 \sinh(\frac{1}{2})} \right) a \right] \left[ -(x - ct + \frac{1}{2}) + \frac{\sinh(x - ct + \frac{1}{2})}{2 \sinh(\frac{1}{2})} \right] \varphi dx dt.
\]

Since $\varphi$ is arbitrary, both cases imply $a$ satisfies $c - \left( \frac{25}{12} - \frac{\cosh(\frac{1}{2})}{2 \sinh(\frac{1}{2})} \right) a = 0$, which completes the proof.

**Acknowledgments.** The authors are very grateful to the referees for their helpful comments that improve this paper. Wang’s work was supported by the Fundamental Research Funds for the Central Universities. Li’s work was supported by NSFC (No:11571057). Qiao’s work was partially supported by the President’s Endowed Professorship program of the University of Texas system.

**REFERENCES**

[1] A. Bressan and A. Constantin, Global dissipative solutions of the Camassa-Holm equation, *Anal. Appl.*, 5 (2007), 1–27.

[2] R. Camassa and D. D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.*, 71 (1993), 1661–1664.

[3] R. Chen, J. Lenells and Y. Liu, Stability of the $\mu$-Camassa-Holm peakons, *J. Nonlinear Sci.*, 23 (2013), 97–112.

[4] G. M. Coclite, H. Holden and K. H. Karlsen, Global weak solutions to a generalized hyperelastic-rod wave equation, *SIAM J. Math. Anal.*, 37 (2005), 1044–1069.

[5] G. M. Coclite, H. Holden and K. H. Karlsen, Well-posedness of higher-order Camassa-Holm equations, *J. Diff. Equ.*, 246 (2009), 929–963.

[6] G. M. Coclite and K. H. Karlsen, A note on the Camassa-Holm equation, *J. Diff. Equ.*, 259 (2015), 2158–2166.

[7] G. M. Coclite and L. Ruvo, A note on the convergence of the solution of the high order Camassa-Holm equation to the entropy ones of a scalar conservation law, *Discrete Contin. Dyn. Syst.*, 37 (2017), 1247–1282.

[8] A. Constantin, On the Cauchy problem for the periodic Camassa-Holm equation, *J. Diff. Equ.*, 141 (1997), 218–235.

[9] A. Constantin, On the inverse spectral problem for the Camassa-Holm equation, *J. Funct. Anal.*, 155 (1998), 352–363.

[10] A. Constantin, On the blow-up of solutions of a periodic shallow water equation, *J. Nonlinear Sci.*, 10 (2000), 391–399.

[11] A. Constantin and J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Math.*, 181 (1998), 229–243.

[12] A. Constantin and J. Escher, Global existence and blow-up for a shallow water equation, *Ann. Scuola Norm. Sup. Pisa*, 26 (1998), 303–328.
[13] A. Constantin and J. Escher, On the blow-up rate and the blow-up set of breaking waves for a shallow water equation, *Math. Z.*, 233 (2000), 75–91.

[14] A. Constantin and B. Kolev, On the geometric approach to the motion of inertial mechanical systems, *J. Phys. A.*, 35 (2002), R51–R79.

[15] A. Constantin and B. Kolev, Geodesic flow on the diffeomorphism group of the circle, *Comment. Math. Helv.*, 78 (2003), 787–804.

[16] A. Constantin and H. P. McKean, A shallow water equation on the circle, *Comm. Pure Appl. Math.*, 52 (1999), 949–982.

[17] A. Constantin and L. Molinet, Global weak solution solutions for a shallow water equation, *Comm. Math. Phys.*, 211 (2000), 45–61.

[18] A. Constantin and W. Strauss, Stability of peakons, *Comm. Pure Appl. Math.*, 53 (2000), 603–610.

[19] R. Danchin, A few remarks on the Camassa-Holm equation, *Diff. Int. Equ.*, 14 (2001), 953–988.

[20] D. Ding, Traveling solutions and evolution properties of the higher order Camassa-Holm equation, *Nonlinear Anal.*, 152 (2017), 1–11.

[21] D. Ding and P. Lv, Conservative solutions for higher-order Camassa-Holm equations, *J. Math. Phys.*, 51 (2010), 072701, 15pp.

[22] D. Ding and S. Zhang, Lipschitz metric for the periodic second-order Camassa-Holm equation, *J. Math. Anal. Appl.*, 451 (2017), 990–1025.

[23] J. Escher and B. Kolev, Geodesic completeness for Sobolev $H^s$-metrics on the diffeomorphism group of the circle, *J. Evol. Equ.*, 14 (2014), 949–968.

[24] J. Escher and B. Kolev, Right-invariant Sobolev metrics of fractional order on the diffeomorphism group of the circle, *J. Geom. Mech.*, 6 (2014), 335–372.

[25] A. Fokas and B. Fuchssteiner, Symplectic structures, their Bäcklund transformations and hereditary symmetries, *Phys. D*, 4 (1981/82), 47–66.

[26] A. Himonas and C. Kenig, Non-uniform dependence on initial data for the CH equation on the line, *Diff. Int. Eqs.*, 22 (2009), 201–224.

[27] A. Himonas, C. Kenig and G. Misiołek, Non-uniform dependence for the periodic CH equation, *Comm. Partial Differential Equations*, 35 (2010), 1145–1162.

[28] T. Kato, Quasi-linear equations of evolution, with applications to partial differential equations, In: *Spectral Theory and Differential Equations*, Lecture Notes in Mathematics, Springer, Berlin, 448 (1975), 25–70.

[29] T. Kato and G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, *Comm. Pure Appl. Math.*, 41 (1988), 203–208.

[30] B. Khesin, J. Lenells and G. Misiołek, Generalized Hunter-Saxton equation and the geometry of the group of circle diffeomorphisms, *Math. Ann.*, 342 (2008), 617–656.

[31] B. Kolev, Poisson brackets in hydrodynamics, *Discrete Contin. Dyn. Syst.*, 19 (2007), 555–574.

[32] J. Lenells, G. Misiołek and F. Tiğlay, Integrable evolution equations on spaces of tensor densities and their peakon solutions, *Commun. Math. Phys.*, 299 (2010), 129–161.

[33] Y. Li and P. Olver, Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation, *J. Diff. Equ.*, 162 (2000), 27–63.

[34] J. Liu and Z. Yin, On the Cauchy problem of a weakly dissipative $\mu$-Hunter-Saxton equation, *Ann. I. H. Poincaré-AN.*, 31 (2014), 267–279.

[35] R. McLachlan and X. Zhang, Well-posedness of a modified Camassa-Holm equations, *J. Diff. Equ.*, 246 (2009), 3241–3259.

[36] Z. Qiao, The Camassa-Holm hierarchy, N-dimensional integrable systems, and algebro-geometric solution on a symplectic submanifold, *Commun. Math. Phys.*, 239 (2003), 309–341.

[37] J. Simon, Compact sets in the space $L^p(0,T;B)$, *Ann. Mat. Pura Appl.*, 146 (1987), 65–96.

[38] M. Taylor, *Pseudodifferential Operators and Nonlinear PDE*, Birkhäuser, Boston, 1991.

[39] M. Taylor, Commutator estimates, *Proc. Amer. Math. Soc.*, 131 (2003), 1501–1507.

[40] L. Tian, P. Zhang and L. Xia, Global existence for the higher-order Camassa-Holm shallow water equation, *Nonlinear Anal.*, 74 (2011), 2468–2474.

[41] F. Wang, F. Li and Z. Qiao, Well-posedness and peakons for a higher-order $\mu$-Camassa-Holm equation, *arXiv:1712.07996*.

[42] S. Wu and Z. Yin, Global existence and blow-up phenomena for the weakly dissipative Camassa-Holm equation, *J. Diff. Equ.*, 246 (2009), 4309–4321.
[43] Z. Xin and P. Zhang, On the weak solutions to a shallow water equation, Comm. Pure Appl. Math., 53 (2000), 1411–1433.

Received December 2017; revised February 2018.

E-mail address: wangfeng@xidian.edu.cn
E-mail address: fqli@dlut.edu.cn
E-mail address: zhijun.qiao@utrgv.edu