Pseudo asymptotically periodic solutions for fractional integro-differential neutral equations

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Abstract In this paper, we study the existence and uniqueness of pseudo $S$-asymptotically $\omega$-periodic mild solutions of class $r$ for fractional integro-differential neutral equations. An example is presented to illustrate the application of the abstract results.

Keywords fractional integro-differential neutral equations, asymptotic periodicity, mild solutions, $S$-asymptotically $\omega$-periodic solutions

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1 Introduction

In the literature, several concepts were presented to study the approximate periodic function. The existence of periodic solutions to different kinds of differential equations and its various generalizations like the existence of almost periodic solutions, almost automorphic solutions, asymptotically almost periodic solutions, asymptotically almost automorphic solutions, pseudo-almost periodic solutions and pseudo-almost automorphic solutions have been extensively studied, see the monographs of Corduneanu [11], N’Guérékata [29], the works [1-5, 8-10, 19, 20, 25, 27, 28, 30, 33, 34, 38, 39] and references therein.

Asymptotic periodicity problem is an active topic of the current research in the behavior theory of solutions of differential equations. We note that many concrete systems usually are submitted to non-periodic external perturbations. In many practical situations, we can assume that these perturbations are approximately periodic in a broad sense. Recently, the $S$-asymptotically $\omega$-periodic functions were formally introduced by Henriquez et al. [24] and after that, its wild applications in functional differential equations, integro-differential equations, fractional differential equations and the existence and uniqueness of $S$-asymptotically $\omega$-periodic solutions to these equations have been well studied. See, for instance, [6,15-18,21,23,24,31]. In 2013, Pierri and Rolnik [32] introduced the concept of pseudo $S$-asymptotically $\omega$-periodic ($PSAP_{\omega}$, in short) functions, which is a natural generalization of $S$-asymptotically $\omega$-periodic functions. Also, they studied the existence and uniqueness of pseudo $S$-asymptotically $\omega$-periodic mild solutions.

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solutions for abstract neutral functional equations. The theory of pseudo $S$-asymptotically $\omega$-periodic functions has been shown to have interesting applications in several branches of differential equations, and because of this fact, this theory has been attracting the attention of several mathematicians and the interest in the topic remains growing. We refer the reader to [14,22,36] and the references therein. In [6] the authors established the sufficient conditions to ensure the existence and uniqueness of pseudo $S$-asymptotically $\omega$-periodic mild solutions for hyperbolic evolution equations respectively by hyperbolic semigroup and uniformly stable semigroup and considered the $PSA\bar{P}_\omega$-mild solutions in intermediate spaces. Xia [37] discussed the pseudo asymptotically periodic solutions of two-term time fractional differential equations with delay. In [22] the authors established the criteria to guarantee the existence and uniqueness of pseudo $S$-asymptotically periodic solutions of second-order abstract Cauchy problems. In [14] the authors have studied qualitative properties of pseudo $S$-asymptotically $\omega$-periodic functions. Also they discussed the existence of pseudo $S$-asymptotically $\omega$-periodic mild solutions for fractional differential equations in the following form

$$\begin{aligned}
\frac{d}{dt}v(t) &= \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} A v(s) ds + f(t, v(t)), \quad t \geq 0, \\
v(0) &= v_0 \in X,
\end{aligned}$$

(1.1)

where $1 < \alpha < 2$, $A : D(A) \subseteq X \to X$ is a linear densely defined operator of sectorial type on a complex Banach space $(X, \| \cdot \|)$ and $f : [0, \infty) \times X \to X$ is an appropriate function.

Fractional calculus and its applications have gained a lot of attention due to their applications in the fields such as physics, fluid mechanics viscoelasticity, heat conduction in materials with memory, chemistry and engineering. In recent years, notable contributions have been made in theory and applications of fractional differential equations, one can refer to [26,40,41].

However, pseudo $S$-asymptotically $\omega$-periodic mild solutions for fractional neutral differential equations have still rarely been treated in the literature. Motivated by these facts, in this paper, we mainly investigate the existence and uniqueness of pseudo $S$-asymptotically $\omega$-periodic mild solutions of class $p$ to the system

$$\begin{aligned}
\frac{d}{dt}D(t, u_t) &= \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} AD(s, u_s) ds + f(t, u_t), \quad t \geq 0, \\
u_0 &= \varphi \in C,
\end{aligned}$$

(1.2)

where $1 < \alpha < 2$, $D(t, \varphi) = \varphi(0) + g(t, \varphi)$, $A : D(A) \subseteq X \to X$ is a linear densely defined operator of sectorial type on a complex Banach space $X$, the history $u_t : [-r, 0] \to C = C([-r, 0], X)$ is defined by $u_t(\theta) = u(t + \theta)$ for $\theta \in [-r, 0]$ and $f, g$ are functions satisfying some additional conditions to be specified later. The convolution integral in (1.2) is understood in the Riemann–Liouville sense.

We organize this paper as follows. In the next section, we will introduce the notions of $S$-asymptotically $\omega$-periodic and pseudo $S$-asymptotically $\omega$-periodic functions and give some basic properties. In Section 3, we obtain very general results on the existence and uniqueness of pseudo $S$-asymptotically $\omega$-periodic mild solutions for the semilinear problem (1.2) under Lipschitz type hypothesis on the nonlinearity. Finally, in Section 4, we apply our theory to concrete applications. We present an example, leading to a better understanding of the work and hence attract the attention to researchers who are entering the subject.

2 Preliminaries

Assume $a$ is an arbitrary real number, $r > 0$. Let $C_b([a, \infty), X)$, $C_b([0, \infty), \mathbb{R}^+)$ and $C = C([-r, 0], X)$ stand for the space formed by all bounded continuous functions from $[a, \infty)$ into $X$ endowed with the sup norm $\| \cdot \|_{C_b([a, \infty), X)}$, the space formed by all bounded continuous functions from $[0, \infty)$ into the positive real number set $\mathbb{R}^+$ endowed with the sup norm $\| \cdot \|_{C_b([0, \infty), \mathbb{R}^+)}$ and the space of continuous functions from $[-r, 0]$ into $X$ with sup norm $\| \cdot \|_C$, respectively. The notation $B(X)$ stands for the space of bounded
linear operators from $X$ into $X$ endowed with the uniform operator norm denoted $\| \cdot \|_{\mathcal{B}(X)}$.

A closed linear operator $A$ is said to be sectorial of type $\mu$ if there exist $0 < \theta < \frac{\mu}{2}, M > 0$ and $\mu \in \mathbb{R}$ such that the spectrum of $A$ is contained in the sector $\mu + S_\theta = \{ \mu + \lambda : \lambda \in \mathbb{C} \setminus \{0\}, |\arg(-\lambda)| < \theta \}$, and $\| \lambda I - A \|^{-1} \leq \frac{M}{|\lambda|^\mu}$ for all $\lambda \notin \mu + S_\theta$. (see [14])

In order to give an operator analytic approach to system (1.2) we recall the following definition.

**Definition 2.1** (See [14]). Let $A$ be a closed and linear operator with domain $D(A)$ defined on a Banach space $X$. We call $A$ the generator of a solution operator if there exist $\mu \in \mathbb{R}$ and a strongly continuous function $S_\alpha : \mathbb{R}^+ \to \mathcal{B}(X)$ such that $\{ \lambda^\alpha : Re\lambda > \mu \} \subseteq \rho(A)$ and $\lambda^\alpha - 1 (\lambda^\alpha I - A)^{-1} x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x dt$ for all $Re\lambda > \mu, x \in X$. In this case, $S_\alpha(t)$ is called the solution operator generated by $A$ which satisfies $S_\alpha(0) = I$. We observe that the power function $\lambda^\alpha$ is uniquely defined as $\lambda^\alpha = |\lambda|^{\alpha} e^{i \alpha \arg(\lambda)}$ with $-\pi < \arg(\lambda) < \pi$.

We note that if $A$ is a sectorial operator of type $\mu$ with $0 < \theta < \pi(1 - \frac{\mu}{2})$, then $A$ is the generator of a solution operator given by $S_\alpha(t) := \frac{1}{2\pi i} \int_{\gamma} e^{\lambda t} \lambda^{\alpha - 1} (\lambda^\alpha I - A)^{-1} d\lambda$, $t > 0$, where $\gamma$ is a suitable path lying outside the sector $\mu + S_\theta$ (cf. [12]). In 2007, Cuesta [12, Theorem 1] proved that if $A$ is a sectorial operator of type $\mu < 0$, for $M > 0$ and $0 < \theta < \pi(1 - \frac{\mu}{2})$, then there exists $C > 0$ such that

$$
\| S_\alpha(t) \| \leq \frac{CM}{1 + |t|^\alpha}, \quad t \geq 0.
$$

(2.1)

**Remark 2.1.** In the rest of this paper, we always suppose that $A$ is a sectorial of type $\mu < 0$ with angle $\theta$ satisfying $0 < \theta \leq \pi(1 - \frac{\mu}{2})$, $M$ and $C$ are the constants introduced above.

We now recall some notations and properties related to $S$-asymptotically $\omega$-periodic functions.

**Definition 2.2** (See [25]). A set $\Omega \subseteq \mathbb{R}$ is said to be an ergodic zero set if $\lim_{r \to \infty} \mes(Q_r \cap \Omega) = 0$, where $\mes$ denotes the Lebesgue measure and $Q_r = [-r, r]$.

**Definition 2.3** (See [19]). The Bochner transform $f^B(t, s), t, s \in [0, 1]$, of a functions $f : \mathbb{R} \to X$ is defined by $f^B(t, s) := f(t + s)$.

**Definition 2.4** (See [19]). Let $p \in [1, +\infty)$, the space $BS^p(X)$ of all Stepanov bounded functions, with the exponent $p$, consists of all measurable functions $f : \mathbb{R} \to X$ such that $f^B \in L^\infty(\mathbb{R}, L^p(0, 1; X))$. This is a Banach space with the norm

$$
\| f \|_{S^p} = \| f^B \|_{L^\infty(\mathbb{R}, L^p)} = \sup_{t \in \mathbb{R}} \left( \int_{t}^{t+1} \| f(\tau) \|^p d\tau \right)^{\frac{1}{p}}.
$$

**Definition 2.5** (See [24]). A function $f \in C_b([0, \infty), X)$ is said to be $S$-asymptotically $\omega$-periodic if $\lim_{t \to \infty} \| f(t + \omega) - f(t) \| = 0$. In this case, we say that $\omega$ is an asymptotic period of $f$.

We use the notation $SAP_{\omega}(X)$ (respectively, $AP_{\omega}(X)$) to represent the subspace of $C_b([0, +\infty), X)$ formed by all $S$-asymptotically $\omega$-periodic functions (respectively, asymptotically $\omega$-periodic). We note that $SAP_{\omega}(X)$ and $AP_{\omega}(X)$ endowed with the norm of uniform convergence are Banach spaces.

**Definition 2.6** (See [14]). A function $f \in C_b([0, \infty), X)$ (respectively, $f \in C_b(\mathbb{R}, X)$) is said to be pseudo-$S$-asymptotically $\omega$-periodic if $\lim_{t \to \infty} \frac{1}{t} \int_0^t \| f(s + \omega) - f(s) \| ds = 0$ (respectively, $\lim_{t \to \infty} \frac{1}{2t} \int_{-t}^t \| f(s + \omega) - f(s) \| ds = 0$). In this case, we say that $\omega$ is an asymptotic period of $f$.

We use the notation $PSAP_{\omega}(X)$ to represent the subspace of $C_b([0, \infty), X)$ formed by all pseudo-$S$-asymptotically $\omega$-periodic functions. We observe that $PSAP_{\omega}(X)$ endowed with the norm of uniform
Proof. Let \(\omega\) be a point in \(\mathbb{R}\). We observe that \(u \in \text{PSAP}_\omega(X)\) if and only if for each \(\varepsilon > 0\), the set \(C_\varepsilon = \{t \in [0, \infty) : \|u(t + \omega) - u(t)\| \geq \varepsilon\}\) is an ergodic zero set.

**Definition 2.7** (See [32]). Let \(\omega\). We say that a function \(u \in \text{PSAP}_\omega(X)\). We say that \(u\) is pseudo \(S\)-asymptotically \(\omega\)-periodic of class \(r\) if

\[
\lim_{T \to \infty} \frac{1}{T} \int_r^T \sup_{\tau \in [s-r,s]} \|u(\tau + \omega) - u(\tau)\| ds = 0. \tag{2.2}
\]

We denote by \(\text{PSAP}_{\omega,r}(X)\) the subspace of \(C_b([0, \infty), X)\) formed by all the pseudo \(S\)-asymptotically \(\omega\)-periodic functions of class \(r\). We note that \(\text{PSAP}_{\omega,r}(X)\) endowed with the norm of uniform convergence is a Banach space.

**Remark 2.3** (See [37]). Assume \(u \in C_b([0, +\infty), X)\), then \(u \in \text{PSAP}_{\omega,r}(X)\) if and only for all \(\varepsilon > 0\),

\[
\lim_{T \to \infty} \frac{1}{T} \text{mes}(\text{sup}_{t \in [0,T]} \|u(t + \omega) - u(t)\| \geq \varepsilon) = 0,
\]

where mes denotes the Lebesgue measure and

\[
M_{T,\varepsilon}(u) = \left\{ t \in [0,T] : \sup_{t \in [0,T]} \|u(t + \omega) - u(t)\| > \varepsilon \right\}.
\]

**Lemma 2.1** (See [37]). Assume \(r \geq 0, r_1 > 0, r_2 > 0\), then

(i) \(\text{PSAP}_{\omega,r}(X) \subseteq \text{PSAP}_{\omega,r_1}(X)\).

(ii) \(\text{PSAP}_{\omega,r}(X)\) is a closed subspace of \(C_b([0, +\infty), X)\).

(iii) \(\text{PSAP}_{\omega,r_1}(X) = \text{PSAP}_{\omega,r_2}(X)\).

**Lemma 2.2.** The space \(\text{PSAP}_{\omega,r}(X)\) is translation invariant on \(\mathbb{R}^+\).

**Proof.** Let \(u \in \text{PSAP}_{\omega,r}(X)\). For \(s \in \mathbb{R}^+\), we have the estimate

\[
\frac{1}{t} \int_r^s \sup_{\rho \in [\xi-r,\xi]} \|u(\rho + \omega + s) - u(\rho + s)\| d\xi
\]

\[
= (1 + \frac{s}{t}) \left( \frac{1}{t+s} \int_r^{r+s} \sup_{\rho \in [\xi-r,\xi]} \|u(\rho + \omega) - u(\rho)\| d\xi \right)
\]

\[
\leq (1 + \frac{s}{t}) \left( \frac{1}{t+s} \int_r^{r+s} \sup_{\rho \in [\xi-r,\xi]} \|u(\rho + \omega) - u(\rho)\| d\xi \right).
\]

Thus we show that the function \(\rho \to u(\rho + s)\) belongs to \(\text{PSAP}_{\omega,r}(X)\). \(\square\)

**Definition 2.8** (See [32]). We say that a function \(F \in C([0, \infty) \times C, X)\) is uniformly \((C, X)\) pseudo \(S\)-asymptotically \(\omega\)-periodic of class \(r\) if

\[
\lim_{T \to \infty} \frac{1}{T} \int_r^T \sup_{\tau \in [s-r,s]} \sup_{\|x\| \leq L} \|F(\tau + \omega, x) - F(\tau, x)\| ds = 0. \tag{2.3}
\]

for all \(L > 0\). Denote by \(\text{PSAP}_{\omega,r}(C, X)\) the set formed by functions of this type.

**Lemma 2.3** (See [32]). Let \(u \in C_b([-r, \infty), X)\) and assume that \(u \big|_{[0, \infty)} \in \text{PSAP}_{\omega,r}(X)\). Then the function \(s \to u_s\) belongs to \(\text{PSAP}_{\omega,r}(C)\).

**Lemma 2.4.** Assume that \(F \in \text{PSAP}_{\omega,r}(C, X)\) and there exists \(L_F \in C_b([0, \infty), \mathbb{R}^+)\) such that \(\|F(t, \psi_1) - F(t, \psi_2)\| \leq L_F(t)\|\psi_1 - \psi_2\|\) for all \(t, \psi_i \in [0, \infty) \times C\). If \(u \in C_b([-r, \infty), X)\) and \(u \big|_{[0, \infty)} \in \text{PSAP}_{\omega,r}(X)\), then the function \(s \to F(s, u_s)\) belongs to \(\text{PSAP}_{\omega,r}(C)\).

**Proof.** Let \(Q = \|u\|_{C_b([-r, \infty), X)}\). Since the function \(s \to u_s\) belongs to \(\text{PSAP}_{\omega,r}(C)\) (see Lemma 2.3),
then for all $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that for each $T > T_\varepsilon$, we have

$$\|LF\|_{C_h([0,\infty) , \mathbb{R}^+)} \frac{1}{T} \int_r^T \sup_{t \in [s-r,s]} \|u_{t+\omega} - u_t\| \, ds \leq \varepsilon.$$ 

Moreover, by $F \in PSAP_{\omega,r}(C, X)$, we have

$$\frac{1}{T} \int_r^T \sup_{t \in [s-r,s]} \sup_{\|x\| \leq Q} \|F(t+\omega, x) - F(t, x)\| \, ds \leq \varepsilon.$$ 

Thus for all $T > T_\varepsilon$, we can obtain

$$\frac{1}{T} \int_r^T \sup_{t \in [s-r,s]} \|F(t+\omega, u_{t+\omega}) - F(t, u_t)\| \, ds$$

$$\leq \frac{1}{T} \int_r^T \sup_{t \in [s-r,s]} \|F(t+\omega, u_{t+\omega}) - F(t, u_{t+\omega})\| \, ds$$

$$+ \frac{1}{T} \int_r^T \sup_{t \in [s-r,s]} \|F(t, u_{t+\omega}) - F(t, u_t)\| \, ds$$

$$\leq \frac{1}{T} \int_r^T \sup_{t \in [s-r,s]} \|F(t+\omega, x) - F(t, x)\| \, ds$$

$$+ \|LF\|_{C_h([0,\infty) , \mathbb{R}^+)} \frac{1}{T} \int_r^T \sup_{t \in [s-r,s]} \|u_{t+\omega} - u_t\| \, ds$$

$$\leq 2\varepsilon,$$

which claims the assertion. \( \square \)

**Lemma 2.5.** Let $u \in C_h([−r, \infty), X)$ and $\kappa : [−r, \infty) \rightarrow X$ be the function defined by $\kappa(t) = 0$ for $t \in [−r, 0]$ and $\kappa(t) = \int_0^t S_\alpha(t-s)u(s)\, ds$ for $t \geq 0$. If $u \in [0,\infty) \in PSAP_{\omega,r}(X)$, then $\kappa \in PSAP_{\omega,r}(X)$.

**Proof.** We have the estimate

$$\int_0^t \|S_\alpha(t-s)u(s)\| \, ds \leq CM\|\mu_{\frac{1}{\alpha}}\pi \alpha \sin(\frac{\pi}{\alpha})\| u\|_{C_h([0,\infty), X)},$$

which shows that $\kappa \in C_h([−r, \infty), X)$. Now we shall show that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_r^T \sup_{t \in [\xi-r, \xi]} \|\kappa(t+\omega) - \kappa(t)\| \, d\xi = 0.$$ 

We first choose positive constant $q$ such that $T > q > r > 0$, then we can obtain

$$\frac{1}{T} \int_r^T \sup_{t \in [\xi-r, \xi]} \|\kappa(t+\omega) - \kappa(t)\| \, d\xi$$

$$\leq \frac{1}{T} \int_r^T \sup_{t \in [\xi-r, \xi]} \int_0^t \|S_\alpha(t-s)[u(s+\omega) - u(s)]\| \, ds \, d\xi$$

$$+ \frac{1}{T} \int_r^T \sup_{t \in [\xi-r, \xi]} \int_0^\omega \|S_\alpha(t+\omega-s)u(s)\| \, ds \, d\xi$$

$$\leq \frac{1}{T} \int_q^q \sup_{t \in [\xi-r, \xi]} \int_0^t \|S_\alpha(t-s)[u(s+\omega) - u(s)]\| \, ds \, d\xi$$

$$+ \frac{1}{T} \int_q^T \sup_{t \in [\xi-r, \xi]} \int_0^t \|S_\alpha(t-s)[u(s+\omega) - u(s)]\| \, ds \, d\xi$$
where and

\[ K_1(T) = \frac{1}{T} \int_{r}^{T} \sup_{t \in (\xi - r, \xi]} \int_{0}^{\omega} \|S_\alpha(t + \omega - s)u(s)\|dsd\xi := \sum_{i=1}^{3} K_i(T). \]

Next, we will estimate the terms \( K_i(T), 1 \leq i \leq 3 \), separately.

For the term \( K_1(T) \), we aim to prove that \( \lim_{T \to \infty} K_1(T) = 0 \). Since \( u \in C_0([-r, \infty), X) \), then we can get

\[
K_1(T) = \frac{1}{T} \int_{r}^{T} \sup_{t \in (\xi - r, \xi]} \int_{0}^{t} \|S_\alpha(t - s)[u(s + \omega) - u(s)]\|dsd\xi
\]

\[
\leq \frac{1}{T} \int_{r}^{T} \sup_{t \in (\xi - r, \xi]} \int_{0}^{t-2} \|S_\alpha(t - s)[u(s + \omega) - u(s)]\|dsd\xi
\]

\[
+ \frac{1}{T} \int_{r}^{T} \sup_{t \in (\xi - r, \xi]} \int_{t-2}^{t} \|S_\alpha(t - s)[u(s + \omega) - u(s)]\|dsd\xi
\]

\[
= \sum_{i=1}^{2} K_i^2(T),
\]

where

\[
K_1^2(T) = \frac{1}{T} \int_{r}^{T} \sup_{t \in (\xi - r, \xi]} \int_{0}^{t-2} \|S_\alpha(t - s)[u(s + \omega) - u(s)]\|dsd\xi,
\]

and

\[
K_2^2(T) = \frac{1}{T} \int_{r}^{T} \sup_{t \in (\xi - r, \xi]} \int_{t-2}^{t} \|S_\alpha(t - s)[u(s + \omega) - u(s)]\|dsd\xi.
\]

For the term \( K_1^1(T) \), we have

\[
K_1^1(T) = \frac{1}{T} \int_{r}^{T} \sup_{t \in (\xi - r, \xi]} \int_{0}^{t-2} \left| \sum_{\alpha = 2}^{3} K_\alpha(t - s)[u(s + \omega) - u(s)] \right|dsd\xi
\]

\[
\leq \frac{1}{T} \int_{r}^{T} \sup_{t \in (\xi - r, \xi]} \int_{0}^{t-2} \left( \sum_{\alpha = 2}^{3} \frac{CM}{1 + |\mu[(t - s)^\alpha + 1 - \alpha] u(s + \omega) - u(s)|s d\xi d s \right)
\]

\[
\leq \frac{1}{T} \int_{r}^{T} \sup_{t \in (\xi - r, \xi]} \int_{0}^{t-2} \frac{CM}{1 + |\mu[(t - s)^\alpha + 1 - \alpha] u(s + \omega) - u(s)|}dsd\xi d s \right)
\]

\[
\leq \frac{1}{T} \int_{r}^{T} \sup_{t \in (\xi - r, \xi]} \int_{0}^{t-2} \frac{CM}{1 + |\mu[(t - s)^\alpha + 1 - \alpha] dsd\xi d s \right)
\]

\[
\leq \frac{1}{T} \int_{r}^{T} \sup_{t \in (\xi - r, \xi]} \int_{0}^{t-2} \frac{CM}{1 + |\mu[(t - s)^\alpha + 1 - \alpha] u(s + \omega) - u(s)|}dsd\xi d s \right)
\]

\[
\leq \frac{2CM}{T} \left( \frac{2(T - 2r)^{2-\alpha}}{\alpha(\alpha - 1)(2 - \alpha)} + \frac{(T - r)^{2-\alpha}}{\alpha(\alpha - 1)(2 - \alpha)} + \frac{r^{2-\alpha}}{1 - \alpha} + \frac{r^{2-\alpha}}{\alpha(\alpha - 1)(2 - \alpha)} \right)
\]

\[
\times \|u\|_{C_0([0, \infty), X)} + \frac{CM}{|\mu|} \frac{r^{1-\alpha}}{1 - \alpha} \int_{r}^{T} \|u(s + \omega) - u(s)\|ds.
\]
From the estimate \( \frac{1}{T} \int_T^T \| u(s+\omega) - u(s) \| ds \leq \frac{1}{T} \int_T^T \sup_{s \in [r-r,T]} \| u(s+r+\omega) - u(s+r) \| d\tau \) and \( PSAP_{\omega,r}(X) \) is translation invariant on \( \mathbb{R}^+ \), we can infer that

\[
\lim_{T \to \infty} \frac{1}{T} \int_T^T \| u(s+\omega) - u(s) \| ds = 0. \tag{2.5}
\]

Also, since \( 1 < \alpha < 2 \), then we can easily get \( \lim_{T \to \infty} K_2(T) = 0 \).

Now the term \( K_2(T) \) can be estimated as,

\[
K_2(T) = \frac{1}{|\mu|} \int_0^T \int_{t \in [\xi-r,\xi]} \int_t^T \| S_\alpha(t-s)[u(s+\omega) - u(s)] \| dsd\xi
\]

\[
\leq CM \frac{1}{|\mu|} \int_0^T \int_{\xi-r}^{\xi} \int_0^{\xi-r-2} \left( \frac{1}{(\xi-r-s)^\alpha} \right) \| u(s+\omega) - u(s) \| dsd\xi
\]

\[
= CM \frac{1}{|\mu|} \left[ \int_0^{T-r-2} \int_0^{s+r+2} \frac{1}{(\xi-r-s)^\alpha} \| u(s+\omega) - u(s) \| d\xi ds
\]

\[
+ \int_0^{T-r-2} \int_s^{s+r+2} \frac{1}{(\xi-r-s)^\alpha} \| u(s+\omega) - u(s) \| d\xi ds
\]

\[
+ \int_0^{T-r-2} \int_s^{T-r-s} \frac{1}{(\xi-r-s)^\alpha} \| u(s+\omega) - u(s) \| d\xi ds
\]

\[
= CM \frac{1}{|\mu|} \left[ \int_0^{T-r-2} \left( \frac{2^{1-\alpha} - \left( \frac{-r-s}{1-\alpha} \right)^{1-\alpha} }{1-\alpha} \right) \| u(s+\omega) - u(s) \| ds
\]

\[
+ \int_0^{T-r-2} \left( \frac{2^{1-\alpha} - \left( \frac{-r}{1-\alpha} \right)^{1-\alpha} }{1-\alpha} \right) \| u(s+\omega) - u(s) \| ds
\]

\[
+ \int_0^{T-r-2} \left( \frac{2^{1-\alpha} - \left( \frac{r}{1-\alpha} \right)^{1-\alpha} }{1-\alpha} \right) \| u(s+\omega) - u(s) \| ds
\]

\[
= \sum_{i=1}^5 I_i.
\]

Now we estimate \( I_i \) separately.

\[
I_1 = CM \frac{1}{|\mu|} \int_0^{T-r-2} \left( \frac{2^{1-\alpha} - \left( \frac{-r-s}{1-\alpha} \right)^{1-\alpha} }{1-\alpha} \right) \| u(s+\omega) - u(s) \| ds
\]

\[
\leq CM \frac{2(r+2)2^{1-\alpha}}{|\mu|} \| u \|_{\mathcal{C}_\lambda(0,\infty),X} \to 0, \quad \text{as } T \to 0.
\]

For \( I_2 \), by a standard calculation and \( u \mid_{\mathbb{R}^+} \in PSAP_{\omega,r}(X) \), we have

\[
I_2 = CM \frac{1}{|\mu|} \int_0^{T-r-2} \left( \frac{2^{1-\alpha} - \left( \frac{-r}{1-\alpha} \right)^{1-\alpha} }{1-\alpha} \right) \| u(s+\omega) - u(s) \| ds
\]

Existence and uniqueness of pseudo

This section is mainly concerned with the existence and uniqueness results of pseudo

ω

Definition 3.1.

For

I

3

= \frac{CM}{|\mu|} \frac{2^{1-\alpha}}{1-\alpha} \frac{(-r)^{1-\alpha}}{1-\alpha} \|u + u - u\| \|ds\|

\leq \frac{CM}{|\mu|} \frac{2^{1-\alpha}}{1-\alpha} \frac{(-r)^{1-\alpha}}{1-\alpha} \|T\| \|u + u - u\| \|ds\|

\leq \frac{CM}{|\mu|} \frac{2^{1-\alpha}}{1-\alpha} \frac{(-r)^{1-\alpha}}{1-\alpha} \|I_2\| \|u + u - u\| \|ds\| \rightarrow 0, \quad \text{as } T \rightarrow 0.

For

I_4

and

I_5,

we can derive

\begin{align*}
I_4 &= \frac{CM}{|\mu|} \frac{1}{T} \int_{-r-2}^{0} \frac{(-r-s)^{1-\alpha}}{\alpha - 1} \|u + u - u\| \|ds\| \\
&= \frac{CM}{|\mu|} \frac{1}{T} \int_{-r-2}^{0} \frac{(-r)^{(2-\alpha)(1-\alpha)}}{(2-\alpha)(\alpha - 1)} \|u + u - u\| \|ds\|,
\end{align*}

\begin{align*}
I_5 &= \frac{CM}{|\mu|} \frac{1}{T} \int_{T-r-2}^{T} \frac{(T-r-s)^{1-\alpha}}{1-\alpha} \|u + u - u\| \|ds\| \\
&= \frac{CM}{|\mu|} \frac{1}{T} \int_{T-r-2}^{T} \frac{(-r)^{(2-\alpha)(1-\alpha)}}{(2-\alpha)(1-\alpha)} \|u + u - u\| \|ds\|.
\end{align*}

We can get

\[ I_4 + I_5 = 0. \]

Combing with the estimates above, expression (2.5) and \( 1 < \alpha < 2 \), we can infer that \[ \lim_{T \rightarrow \infty} K^2(T) = 0. \]

Finally,

\[ K_3(T) = \frac{1}{T} \int_{r}^{T} \sup_{t \in [\xi - r, \xi]} \int_{0}^{\omega} \|S(t + \omega - s)u(s)\|dsd\xi \]

\[ \leq \frac{1}{T} \int_{r}^{T} \frac{CM}{1 + |\mu|(|\xi - r|)^{\alpha}} \int_{0}^{\omega} \|u(s)\|dsd\xi \]

\[ \leq \frac{1}{T} \omega \|u\|_{C_h([0, \infty), X)} \int_{r}^{T} \frac{CM}{1 + |\mu|(|\xi - r|)^{\alpha}} d\xi \]

\[ \leq \frac{1}{T} \omega \|u\|_{C_h([0, \infty), X)} \frac{CM |\mu| \pi}{\alpha \sin\left(\frac{\pi}{\alpha}\right)} \rightarrow 0, \quad \text{as } T \rightarrow \infty. \]

Consequently, we prove that \( \kappa \in PSAP_{\omega,(r)(X)}. \)

3 Existence and uniqueness of pseudo S-asymptotically \( \omega \)-periodic solutions

This section is mainly concerned with the existence and uniqueness results of pseudo S-asymptotically \( \omega \)-periodic mild solutions of class \( r \).

Definition 3.1. A function \( u \in C_h([-r, \infty), X) \) is said to be a mild solution of system (1.2) if \( u_0 = \varphi \)

and

\[ u(t) = S_\alpha(t)[\varphi(0) - g(0, \varphi)] + g(t, u_t) + \int_{0}^{t} S_\alpha(t - s)f(s, u_s)ds, \quad \forall \ t \geq 0. \]

To establish our results, we require the following assumptions:
(H1) The functions $g, f \in PSAP_{\omega,r}(C,X)$ and $f(\cdot,0),g(\cdot,0)$ belong to $C_b([0,\infty),X)$;
(H2) There exist positive constants $L_g, L_f$ such that for any $\psi, \varphi \in C$ and all $t \in [0,\infty)$,
$$
\|g(t,\psi) - g(t,\varphi)\| \leq L_g \|\psi - \varphi\|_C, \quad \|f(t,\psi) - f(t,\varphi)\| \leq L_f \|\psi - \varphi\|_C;
$$
(H3) There exist $L_g(t), L_f(t) \in C_b([0,\infty),\mathbb{R}^+)$ such that for any $\psi, \varphi \in C$ and all $t \in [0,\infty)$,
$$
\|g(t,\psi) - g(t,\varphi)\| \leq L_g(t) \|\psi - \varphi\|_C, \quad \|f(t,\psi) - f(t,\varphi)\| \leq L_f(t) \|\psi - \varphi\|_C;
$$
(H4) There exist $L_g(t) \in C_b([0,\infty),\mathbb{R}^+)$ and $L_f(t) \in C_b([0,\infty),\mathbb{R}^+) \cap L^1_{loc}([0,\infty),\mathbb{R}^+)$ such that for any $\psi, \varphi \in C$ and all $t \in [0,\infty)$,
$$
\|g(t,\psi) - g(t,\varphi)\| \leq L_g(t) \|\psi - \varphi\|_C, \quad \|f(t,\psi) - f(t,\varphi)\| \leq L_f(t) \|\psi - \varphi\|_C;
$$
(H5) There exist $L_g(t) \in C_b([0,\infty),\mathbb{R}^+)$ and $L_f(t) \in BS^p(\mathbb{R}^+,\mathbb{R}^+) \cap L^1_{loc}([0,\infty),\mathbb{R}^+)$ such that for any $\psi, \varphi \in C$ and all $t \in [0,\infty)$,
$$
\|g(t,\psi) - g(t,\varphi)\| \leq L_g(t) \|\psi - \varphi\|_C, \quad \|f(t,\psi) - f(t,\varphi)\| \leq L_f(t) \|\psi - \varphi\|_C.
$$

We are now in a position to establish our first existence theorem.

**Theorem 3.1.** Assume that hypotheses (H1) and (H3) hold, and
$$
\|L_g\|_{C_b([0,\infty),\mathbb{R}^+)} + \|L_f\|_{C_b([0,\infty),\mathbb{R}^+)} \frac{CM\|\frac{-1}{\alpha}\|_1}{\alpha \sin(\frac{\pi}{\alpha})} < 1,
$$
then system (1.2) has a unique mild solution $u \in PSAP_{\omega,r}(X)$.

**Proof.** Let $\mathcal{B} = \{ u : [-r,\infty) \rightarrow X \mid u_0 = \varphi, u|_{[0,\infty)} \in PSAP_{\omega,r}(X) \}$ endowed with the metric $d(u,z) = \|u - z\|_{C_b([-r,\infty),X)}$ and let $\Phi_\alpha$ be the map defined by
$$
\Phi_\alpha u(t) = S_\alpha(t)[\varphi(0) - g(0,\varphi)] + g(t,u_t) + \int_0^t S_\alpha(t-s)f(s,u_s)ds. \quad \forall t \geq 0. \tag{3.2}
$$
We next prove that $\Phi_\alpha$ is a contraction on $\mathcal{B}$.

Assume $u \in \mathcal{B}$, we now show that $\Phi_\alpha u \in \mathcal{B}$.

From (H3) we know that $f,g$ satisfy the Lipschitz condition. Moreover, by a straightforward computation, we obtain the following estimate
$$
\|\Phi_\alpha u(t)\| \leq \|S_\alpha(t)\|([\|\varphi(0)\| + \|g(0,\varphi)\|]) + \left[\|g(t,u_t) - g(t,0)\| + \|g(\cdot,0)\|_{C_b([0,\infty),X)}\right]
$$
$$
+ \int_0^t S_\alpha(t-s)[\|f(s,u_s) - f(s,0)\| + \|f(\cdot,0)\|_{C_b([0,\infty),X)}]ds
$$
$$
\leq CM([\|\varphi(0)\| + \|g(0,\varphi)\|]) + \left[\|L_g\|_{C_b([0,\infty),\mathbb{R}^+)}\|u\|_{C_b([-r,\infty),X)} + \|g(\cdot,0)\|_{C_b([0,\infty),X)}\right]
$$
$$
+ \frac{CM\|\frac{-1}{\alpha}\|_1}{\alpha \sin(\frac{\pi}{\alpha})} \left[\|L_f\|_{C_b([0,\infty),\mathbb{R}^+)}\|u\|_{C_b([-r,\infty),X)} + \|f(\cdot,0)\|_{C_b([0,\infty),X)}\right]. \tag{3.3}
$$

On the other hand, it is obvious that
$$
\frac{1}{T} \int_0^T \sup_{\tau \in [s-r,s]} \left[\|S_\alpha(\tau + \omega)[\varphi(0) - g(0,\varphi)] - S_\alpha(\tau)[\varphi(0) - g(0,\varphi)]\right]ds
$$
$$
\leq \frac{2CM}{T} \int_0^T \frac{1}{1 + |\mu|^\alpha} d\tau \|\varphi(0)\| + \|g(0,\varphi)\| \tag{3.4}
$$
$$
\leq \frac{2CM\|\frac{-1}{\alpha}\|_1}{\alpha \sin(\frac{\pi}{\alpha})} \times \frac{1}{T} \|\varphi(0)\| + \|g(0,\varphi)\| \rightarrow 0, \quad \text{as } T \rightarrow \infty.
$$
Hence, we infer that $S_{α}(\cdot)\varphi(0) - g(0, \varphi) \in PSAP_{ω, r}(X)$. Using now the estimates (3.3), (3.4) and Lemmas 2.4-2.5, we can claim that $Φ_α u \in B$ whenever $u \in B$.

Furthermore, for all $u, v \in B$ and $t \geq 0$, we can deduce that

$$
\|Φ_α u(t) - Φ_α v(t)\| \\
\leq L_g(t)\|u_t - v_t\| + CM \int_0^t \frac{L_f(s)}{1 + |μ|(t - s)^α} ds\|u_s - v_s\|c
$$

which follows from (3.1) that $Φ_α$ is a contraction on $B$. Therefore we can affirm that $Φ_α$ has a unique fixed point $u \in PSAP_{ω, r}(X)$ in $B$, which is the mild solution of (1.2). The proof is complete.

The next result is an immediate consequence of Theorem 3.1.

**Corollary 3.2.** Assume that hypotheses (H1) and (H2) hold, and

$$
L_g + \frac{CM L_f |μ|^\frac{π}{α}}{α sin(\frac{π}{α})} < 1,
$$

then system (1.2) has a unique mild solution $u \in PSAP_{ω, r}(X)$.

A similar result can be established when $f$ satisfies a local Lipschitz condition.

**Theorem 3.3.** Assume that hypotheses (H1) and (H4) hold, and

$$
\sup_{t \geq 0} W_f(t) + L_g\|c_{\alpha}(0, \infty, R^+)< 1,
$$

where $W_f(t) = \int_0^t \frac{L_f(s)}{1 + |μ|(t - s)^α} ds$. Then system (1.2) has a unique mild solution $u \in PSAP_{ω, r}(X)$.

**Proof:** We still define the map $Φ_α$ and the set $B$ as in Theorem 3.1. Next, we prove that $Φ_α$ is a contraction. We initially prove that $Φ_α$ is well defined. By the expression (3.4), we can claim that $S_{α}(\cdot)\varphi(0) - g(0, \varphi)$ belongs to $PSAP_{ω, r}(X)$. Moreover, for $∀ u \in B$ and $∀ t \geq 0$, we have

$$
\|Φ_α u(t)\| \leq \|S_{α}(t)(\|\varphi(0)\| + \|g(0, \varphi)\|) + \left[\|g(t, u_t) - g(t, 0)\| + \|g(\cdot, 0)\|c_{\alpha}(0, \infty, X)\right]
$$

$$
+ \int_0^t \frac{CM}{1 + |μ|(t - s)^α} \left[\|f(s, u_s) - f(s, 0)\| + \|f(\cdot, 0)\|c_{\alpha}(0, \infty, X)\right]
$$

$$
\leq CM(\|\varphi(0)\| + \|g(0, \varphi)\|) + \left[\|L_g\|c_{\alpha}(0, \infty, R^+)\|u\|c_{\alpha}(-r, \infty, X) + \|g(\cdot, 0)\|c_{\alpha}(0, \infty, X)\right]
$$

$$
+ \sup_{t \geq 0} W_f(t)\|u\|c_{\alpha}(-r, \infty, X) + \frac{CM |μ|^\frac{π}{α}}{α sin(\frac{π}{α})} \|f(\cdot, 0)\|c_{\alpha}(0, \infty, X).
$$

Combing with Lemmas 2.4-2.5 and the arguments above, it follows that $Φ_α u$ is well defined on $B$.

Furthermore, for all $u, v \in B$ and $t \geq 0$, we have

$$
\|(Φ_α u)(t) - (Φ_α v)(t)\| \\
\leq L_g(t)\|u_t - v_t\| + CM \int_0^t \frac{L_f(s)}{1 + |μ|(t - s)^α} ds\|u_s - v_s\|c ds
$$

$$
\leq \left[\|L_g\|c_{\alpha}(0, \infty, R^+) + \sup_{t \geq 0} W_f(t)\right]\|u - v\|c_{\alpha}(-r, \infty, X).
$$

The assumption (3.5) enables us to claim our assertion. This completes the proof of Theorem 3.3. □
Theorem 3.4. Assume that hypotheses (H₁) and (H₅) hold, and

\[ \|L_{g}\|_{C_{\alpha}([0,\infty),\mathbb{R}^{+})} + CM \left(1 + \frac{|\mu|^{-\frac{1}{\alpha}}}{\alpha \sin\left(\frac{\alpha}{2}\right)}\right) \|L_{f}\|_{S^{p}} < 1, \]  

then system (1.2) has a unique mild solution \( u \in PSAP_{\omega,r}(X) \).

Proof. The operator \( \Phi_{\alpha} \) and the set \( \mathcal{B} \) are defined as in Theorem 3.1. From the assumption (H₅), according to Theorem 3.2 in [37], we know that \( f(s, u_{s}) \in PSAP_{\omega,r}(X) \). Proceeding as in the proof of Theorem 3.3, it is easy to claim that \( \Phi_{\alpha}u \in \mathcal{B} \) whenever \( u \in \mathcal{B} \).

Furthermore, for all \( u, v \in \mathcal{B} \) and each \( t \geq 0 \), one has

\[
\| \Phi_{\alpha}u(t) - \Phi_{\alpha}v(t) \| \leq L_{g}(t)\|u_{t} - v_{t}\|_{c} + C_{1} \int_{0}^{t} S_{\alpha}(t-s)\|f(s, u_{s}) - f(s, v_{s})\|ds
\]

If \( t = m \in \mathbb{N} \), in this case

\[
\| \Phi_{\alpha}u(t) - \Phi_{\alpha}v(t) \|
\leq L_{g}(t)\|u_{t} - v_{t}\|_{c} + CM \sum_{k=0}^{m-1} \int_{k}^{k+1} \frac{L_{f}(s)}{1 + |\mu|(m-s)^{\alpha}} \|u_{s} - v_{s}\|_{c} ds.
\]

We now deal with more general case. If \( t = m - h \), where \( 0 < h < 1 \), \( m \in \mathbb{N} \), we have

\[
\int_{0}^{t} S_{\alpha}(t-s)L_{f}(s)ds \leq \int_{0}^{m-h} \frac{CM}{1 + |\mu|(m-h-s)^{\alpha}}L_{f}(s)ds
\]

where \( \tilde{L}_{f} \) is defined by

\[
\tilde{L}_{f}(s) := \begin{cases} 
0, & 0 \leq s < h, \\
L_{f}(s-h), & s \geq h.
\end{cases}
\]
Thus we have $\|\tilde{L}_f\|_{S^p} = \|L_f\|_{S^p}$. Moreover, we can easily get
\[
\int_0^t \frac{1}{1 + |\mu|(t - s)^\alpha} L_f(s)ds \leq \left(1 + \frac{|\mu|^{-1}}{\alpha \sin\left(\frac{\pi}{\alpha}\right)}\right) \|\tilde{L}_f\|_{S^p}.
\]
From the preceding estimates, we have
\[
\|\Phi_\alpha u - \Phi_\alpha v\|_{C_k((-r, \infty), X)} \leq \left[\|L_0\|_{C_k((0, \infty), \mathbb{R}^+) + CM\left(1 + \frac{|\mu|^{-1}}{\alpha \sin\left(\frac{\pi}{\alpha}\right)}\right) \|L_f\|_{S^p}\right] \|u - v\|_{C_k((-r, \infty), X)}.
\]
By (3.6) and the Banach contraction mapping principle, we know that $\Phi_\alpha$ has a unique fixed point $u \in PSAP_{\omega,r}(X)$ in $B$, which is the mild solution of system (1.2). The proof is complete. \(\square\)

### 4 Applications

In what follows, we use the previous theory to verify the existence and uniqueness of pseudo $S$-asymptotically $\omega$-periodic mild solutions to a fractional partial integro-differential neutral equation. We are concerned with the following system
\[
\begin{align*}
\frac{\partial}{\partial t} \left[v(t, \vartheta) - h(t) \int_{-r}^{t} k(s)v(t + s, \vartheta)ds\right] &= J_{\alpha}^{1-1}(\frac{\partial^2}{\partial \vartheta^2} - c) \left[v(t, \vartheta) - h(t) \int_{-r}^{t} k(s)v(t + s, \vartheta)ds\right] \\
&\quad + j(t) \int_{-r}^{t} m(s)v(t + s, \vartheta)ds, \quad t \geq 0, \ c > 0, \ \vartheta \in [0, \pi],
\end{align*}
\]
(4.1)

where $1 < \alpha < 2$, $h(t), j(t) \in PSAP_{\omega,r}([0, +\infty), \mathbb{R}), k, m \in C_b([-r, 0], \mathbb{R})$ and satisfy some particular conditions specified later. Let $X = L^2([0, \pi], \mathbb{R}, \|\cdot\|_{L^2})$ and define the operator $A$ on $X$ by $Av = (\frac{\partial^2}{\partial \vartheta^2} - c)v$. $D(A) = \{v \in X = L^2([0, \pi], \mathbb{R}) : v'' \in L^2[0, \pi], v(0) = v(\pi) = 0\}$. It is well known that $A$ is sectorial of type $\omega = -c < 0$. Let $u(t)(\vartheta) = v(t, \vartheta)$ for $t \in [-r, \infty)$, $g(t, u_t)(\vartheta) = h(t) \int_{-r}^{t} k(s)v(t + s, \vartheta)ds$, $f(t, u_t)(\vartheta) = j(t) \int_{-r}^{t} m(s)v(t + s, \vartheta)ds$, $t \in [0, \infty), \ \vartheta \in [0, \pi]$, and $J_{\alpha}^{\alpha-1}u(t) = \int_{0}^{t} (t - s)^{\alpha-2} \gamma(s)ds$. Then (4.1) can be rewritten as an abstract system of the form (1.2).

For reader's convenience, we only give a brief outline of the proof. We have the following estimates
\[
\|g(t, \varphi)\| \leq r_1 \|h\|_1 \left(\int_{-r}^{0} |k(s)|^2 ds\right)^{\frac{1}{2}} \|\varphi\|_C,
\]
(4.2)
\[
\|g(t, \varphi) - g(t, \psi)\| \leq r_2 \|h\|_1 \left(\int_{-r}^{0} |k(s)|^2 ds\right)^{\frac{1}{2}} \|\varphi - \psi\|_C,
\]
(4.3)
\[
\frac{1}{T} \int_{T}^{T} \sup_{t \in \left[0, t\right]} \|g(t + \omega, u_t) - g(t, u_t)\| dt \\
\leq \frac{1}{T} \int_{T}^{T} \sup_{t \in \left[0, t\right]} \|h(t + \omega) - h(t)\| dt \times \|r_1\|_1 \left(\int_{-r}^{0} |k(s)|^2 ds\right)^{\frac{1}{2}} |u_t|_C.
\]
(4.4)

We note that estimate (4.2) implies that $g$ is bounded. Since $h$ belongs to $PSAP_{\omega,r}(X)$, then estimates (4.3) and (4.4) imply that $g \in PSAP_{\omega,r}(C, X)$ with $L_g = r_1 \|h\|_1 \left(\int_{-r}^{0} |k(s)|^2 ds\right)^{\frac{1}{2}}$. Similarly, we can verify that $f$ is bounded and $f \in PSAP_{\omega,r}(C, X)$ with $L_f = r_2 \|j\| \left(\int_{-r}^{0} |m(s)|^2 ds\right)^{\frac{1}{2}}$. Thus we can deduce the following result as an immediate consequence of Corollary 3.2.

**Proposition 4.1.** Under previous assumptions, if we assume $L_f$ and $L_g$ are constants small enough
such that \( L_g + L_f \frac{CM|\mu|^{-\pi}}{\alpha \sin(\frac{\pi}{\alpha})} < 1 \), where \( L_f \) and \( L_g \) are introduced above. Then there exists a unique mild solution \( v \in \text{PSAP}_{\omega,r}(X) \) of system (4.1).

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