String Theory on $AdS_3/Z_N$

John Son

Department of Physics
Harvard University
Cambridge, MA 02138

json@pauli.harvard.edu

Abstract

We study string theory on singular $Z_N$ quotients of $AdS_3$, corresponding to spaces with conical defects. The spectrum is computed using the orbifold procedure. It is shown that spectral flow may be used to generate the twisted sectors. We further compute the thermal partition function and show that it correctly reproduces the spectrum.
1. Introduction

General Relativity in three dimensions with a negative cosmological constant,

\[ S = \frac{1}{2\pi} \int d^3x \sqrt{-g} \left( R + \frac{2}{l^2} \right) + \text{surface terms}, \tag{1.1} \]

has a family of solutions labelled by two parameters \( M \) and \( J \).\[ \quad \]

\[ ds^2 = -N^2 dt^2 + N^{-2}dr^2 + r^2(N^\phi dt + d\phi)^2, \]
\[ N^2 = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \tag{1.2} \]
\[ N^\phi = -\frac{J}{2r^2}. \]

What the resulting spacetimes look like depend on the values of the two parameters. When \( M > 0 \) and \( Ml > |J| \), these spacetimes correspond to black holes. The second condition ensures that a horizon exists. The constants are then identified with the mass and angular momentum of the black hole, respectively. These spaces may be thought of as excitations of the \( M = 0 \) case.

However, \( M = 0 \) is not the lowest energy state possible. It is known that when \( M = -1 \) the space (1.2) corresponds to the three-dimensional anti-de Sitter space, \( AdS_3 \).

For the spacetimes with \( -1 < M < 0 \) (and \( J = 0 \)), a rescaling of the coordinates brings the metric into the form

\[ ds^2 = -\left(1 + \frac{r^2}{l^2}\right) dt^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\phi^2, \tag{1.3} \]

which is the same as \( AdS_3 \), but with a deficit angle \( \delta = 2\pi(1 - \sqrt{|M|}) \) for \( \phi \). Thus, these spaces correspond to \( AdS_3 \) with conical singularities.

In fact, even the black holes corresponding to (1.2) with \( M > 0 \) are locally \( AdS_3 \), and can be obtained from \( AdS_3 \) by a quotient. This is consistent with the equations of motion resulting from (1.1), which implies that the curvature is constant. The black hole solutions do not have a curvature singularity, and differ from \( AdS_3 \) only by some global identifications.

The solutions that are being discussed here are easily lifted to solutions of string theory. By including a three form \( H \), which must be proportional to the volume form in three dimensions, these spaces provide a background in which it is possible to describe string propagation via the \( SL(2,R) \) WZW model. At the level of low energy effective
action, (1.1) arises when one takes the action for the massless fields of string theory $g_{\mu\nu}, H, \phi$ and sets $H_{\mu\nu\sigma} = \frac{2}{l} \epsilon_{\mu\nu\sigma}$, and $\phi = 0$ \[3\].

The purpose of this paper is to study string propagation on the conical spaces. For the special values of the opening angle $2\pi/N$, where $N$ is an integer, the spaces may be obtained as a $Z_N$ orbifold of $AdS_3$ \[4\]. The singularity present is then just an orbifold singularity, and it is possible to formulate a consistent string theory on this background given the knowledge of string theory on $AdS_3$.

There are many reasons for studying this theory. Such spaces can be formed by adding mass to empty $AdS_3$ \[4\]. Relative to the $AdS_3$ vacuum, an object of mass less than 1 would create a conical singularity. One can imagine a process where a collision taking place inside of $AdS_3$ leaves a lump of stable matter, not enough to produce a black hole but distorting the geometry to what we are studying here. Indeed, this provides a controlled setting to study black hole formation, as in \[5\].

Another reason is that a consistent description of strings propagating on $AdS_3$ has emerged only recently in \[6\]. The crucial observation made in that work is the existence of spectral flow as a symmetry of the $SL(2,R)$ WZW model. One would like to gain further insight into this spectral flow symmetry. In this paper we will learn that on the conical spaces spectral flow acts as a twist, in the orbifold sense.

Finally, $AdS_3$/CFT$_2$ correspondence \[7,8,9\] stands apart from the correspondence in other dimensions \[10\] in that it may be studied in a fully string-theoretic context, and not just in the gravity approximation. Therefore, it will be useful to understand as much as possible string theory on $AdS_3$ and its extensions.

After this work was completed, we received \[11\] which has some overlap with this paper.

2. String theory on $AdS_3$

We begin by briefly summarizing the known results of string theory on $AdS_3$ \[8\], as this will serve as the theory on the covering space of the orbifold. From this point on, we set $l = 1$.

$1$ $\phi$ corresponds to rotation in $X-Y$ plane in the covering space $ds^2 = -dU^2-dV^2+dX^2+dY^2$, and is always a space-like killing vector, ensuring causality in the resulting quotient space.
The action is given by that of the $SL(2, R)$ WZW model

$$S = \frac{k}{8\pi} \int d^2\sigma \ Tr \left( g^{-1}\partial a g^{-1}\partial a g \right) + \frac{ik}{12\pi} \int \Tr \left( g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg \right)$$  \hspace{1cm} (2.1)$$

where the level $k$ need not be quantized as $H^3$ vanishes for $SL(2, R)$. We choose the parametrization for the group element

$$g = e^{iu\sigma_2} e^{p\sigma_3} e^{iu\sigma_2},$$  \hspace{1cm} (2.2)$$

where $\sigma$’s are the Pauli matrices, and upon setting $u = \frac{1}{2}(t + \phi), v = \frac{1}{2}(t - \phi)$ we obtain the coordinates of (1.3) by the transformation $r = \sinh \rho$. To avoid closed timelike curves we take the universal cover of this space to be $AdS_3$.

The key property of the WZW model is invariance under the action of left and right multiplication by group elements. This implies the existence of two sets of conserved currents, the right-moving

$$J^3_R = k(\partial_+ u + \cosh 2\rho \partial_+ v),$$  \hspace{1cm} (2.3)$$

$$J^\pm_R = k(\partial_+ \rho \pm i \sinh 2\rho \partial_+ v)e^{\mp i2u},$$

and the left-moving counter parts which are obtained by the replacement $\partial_+ \rightarrow \partial_- \text{ and } u \leftrightarrow v$. The zero modes of $J^3_R$ and $J^3_L$ will be denoted $J^3_0$ and $\bar{J}^3_0$ respectively, and they are related to the spacetime energy and angular momentum of a state in $AdS_3$ by

$$J^3_0 = \frac{1}{2}(E + \ell),$$

$$\bar{J}^3_0 = \frac{1}{2}(E - \ell).$$  \hspace{1cm} (2.4)$$

The modes of $J_{R,L}$ each generate the current algebra $\hat{SL}(2, R)$. Hence, the Hilbert space of $SL(2, R)$ WZW model is a sum of representations of $\hat{SL}(2, R)_L \times \hat{SL}(2, R)_R$. The Hilbert space of string theory on $AdS_3$ is the subspace obtained after imposing the Virasoro constraints $(L_n - \delta_{n,0})|\text{physical} = 0$. In general, we will consider a spacetime of the form $AdS_3 \times \mathcal{M}$ in which $\mathcal{M}$ is described by an appropriate CFT.

The representations that appear in the Hilbert space of $SL(2, R)$ WZW model are the following. Start with the discrete representations $\hat{D}_{j}^+, \hat{D}_{j}^+$, where $\hat{D}_{j}^+$ is the representation obtained by acting with the raising operators $J^{3,\pm}_{n<0}$ on the unitary $SL(2, R)$ representation
and also include the representations obtained by spectral flow (same amount on left and right) \[6\]

\[
\begin{align*}
J^3_n &\rightarrow \tilde{J}^3_n = J^3_n - \frac{k}{2} w \delta_{n,0} \\
J^+_n &\rightarrow \tilde{J}^+_n = J^+_n + w \\
J^-_n &\rightarrow \tilde{J}^-_n = J^-_n - w,
\end{align*}
\]

which changes the Virasoro generators to

\[
\tilde{L}_n = L_n + w J^3_n - \frac{k}{4} w^2 \delta_{n,0}.
\]

The resulting representations are denoted \(\hat{D}^{+,w}_j\) \(\times\) \(\hat{D}^{+,w}_j\). Spectral flow by \(-1\) gives the charge conjugated representations, \(\tilde{D}^{+,w=-1}_j = \tilde{D}^{-,w}_{k/2-j}\). In all these representations the \(SL(2,R)\) spin \(j\) must be in the range \(\frac{1}{2} < j < \frac{k-1}{2}\), which is more restrictive than what is allowed by the no-ghost theorem \([13,14,15,16,17,18,19,20,12]\). In the context of string theory on \(AdS_3\), these representations correspond to the short strings. A state is labelled as \(|j, m, \bar{m}, N, \bar{N}, h, \bar{h}\rangle\), where \(m\) is the \(J^3_0\) eigenvalue given by \(j + q\) where \(q\) is some integer. \(N\) denotes the level of the current algebra and \(h\) is the conformal weight coming from the CFT of \(\mathcal{M}\). Imposing the Virasoro constraints and using \((2.4)\) the energy of this state is found to be\[3\]

\[
E = 1 + q + \bar{q} + 2w + \sqrt{1 + 4(k-2) \left( N - wq + h - 1 - \frac{1}{2} w(w+1) \right)}.
\]

The other representations in the Hilbert space of \(SL(2,R)\) WZW model are the spectral flows \((2.5)\) of continuous representations \(\hat{C}^{\alpha,w}_{1/2+i s} \times \hat{C}^{\alpha,w}_{1/2+i s}\) where \(\hat{C}^{\alpha}_{1/2+i s}\) is built with \(J^3_{n<0}\) acting on the unitary \(SL(2,R)\) representation \(C^\alpha_{j=1/2+i s}\). These states correspond to the long strings in \(AdS_3\) \([21,22]\), and \(s\) is a real number representing the momentum in the radial direction. For the continuous representations, there is no relation between \(m\) and \(j\). Therefore the expression for energy \(J^3_0 + \tilde{J}^3_0\) will look different. It is given by

\[
E = \frac{k}{2} w + \frac{1}{w} \left( \frac{2s^2 + \frac{1}{k} + N + h + \bar{N} + \bar{h} - 2}{k-2} \right).
\]

See \([12]\) for details on representations of \(SL(2,R)\).

This equation does not look symmetric between the left and right quantum numbers because the level matching condition implies that the expression in the square root is the same for left and right.
It is possible to give an independent derivation of this spectrum, via a path integral on thermal $AdS_3$ [23]. Going to Euclidean coordinates in both the worldsheet and spacetime, one periodically identifies time with period $\beta = 1/T$. Then one evaluates the 1-loop path integral including the topologically non-trivial sectors corresponding to a string wrapping the thermal circle. After integrating over the torus moduli, the amplitude is identified with the spacetime free energy of string states in Lorentzian $AdS_3$. In this way one can simply read off the spectrum which agrees with (2.7) and (2.8).

3. Conical spaces as orbifolds of $AdS_3$

3.1. $Z_N$ quotient

Taking string theory on $AdS_3$ as the starting point, the conical spaces with opening angles $2\pi/N$ are obtained by taking a $Z_N$ orbifold. Let us first note how spectral flow acts on this quotient space. The effect of spectral flow is to take a solution of the WZW equation of motion

$$g = g_+(x^+)g_-(x^-)$$

and generate a new solution [3]

$$g_+(x^+) \to e^{\frac{i}{2}w x^+ \sigma_2} g_+(x^+), \quad g_-(x^-) \to g_-(x^-) e^{\frac{i}{2}w x^- \sigma_2}.$$  \hspace{1cm} (3.2)

Under this operation, $t \to t + w\tau$ and $\phi \to \phi + w\sigma$. In regular $AdS_3$ closure of the string worldsheet required that $w$ be an integer, but now we see that $w$ only needs to be a multiple of $1/N$.

When we spectral flow by a fractional amount the $SL(2,R)$ currents obey twisted boundary conditions. Consider the $n$th twisted sector:

$$J^+(x^+ + 2\pi) = J^+(x^+) \ e^{-2\pi in/N}, \quad J^-(x^+ + 2\pi) = J^-(x^+) \ e^{2\pi in/N}. \quad (3.3)$$

Then the mode expansion is

$$J^+(z) = \sum_{r\in\mathbb{Z}+n/N} J^+_r z^{-r-1}, \quad J^-(z) = \sum_{s\in\mathbb{Z}-n/N} J^-_s z^{-s-1}, \quad (3.4)$$

where $z = e^{ix^+}$. The commutation relations are

$$[J^+_r, J^-_s] = -2J^3_{r+s} + k r \delta_{r+s}$$

$$[J^3_m, J^\pm_r] = \pm J^\pm_{m+r}$$

$$[J^3_m, J^3_l] = -\frac{k}{2} m \delta_{m+l}. \quad (3.5)$$

There is a total of $N$ sectors to consider, and in each sector $J^\pm$ are quantized with different periodicity. We now turn to the first step in taking an orbifold, which is to construct the twisted states. Once again, spectral flow will be seen to play a crucial role.
3.2. Twisted states and spectral flow

Consider a state obtained by repeated applications of the raising operators on a lowest weight state,
\[ \prod_{m_i} J_{m_i}^3 \prod_{r_j} J_{r_j}^+ \prod_{s_k} J_{s_k}^- |j, j\rangle . \] (3.6)

If necessary, commutation relations may be used to change the order in which the generators appear. However, in what follows the ordering will be immaterial. Such a state has
\[ L_0 = -(\sum m_i + \sum r_j + \sum s_k) \] and \[ J_0^3 = j + N^+ - N^- \] where \( N^+ \) (\( N^- \)) is the number of times \( J^+ \) (\( J^- \)) appears in the above expression. Also note that the fractional part of the level is given by \( (N^+ - N^-)n/N \). If we take this state and spectral flow \((2.5)\) by \( w = -n/N \), we find that the new generators acting on it are integrally moded. Thus, one can think of this state as belonging to \( \hat{D}^+_{\tilde{j}}, \hat{C}_{\alpha}/2^+ \). To obtain a string state in spacetime (including \( M \)), we impose the Virasoro constraints
\[ (L_0 - 1)|\tilde{j}, \tilde{m}, \tilde{N}, h\rangle = 0 \]
and obtain the same expression for the energy that was found in \( AdS_3 \times M \), \((2.7)\). The discussion for the continuous states is similar and once again we conclude that the energy is given by \((2.8)\).

Normally, twisting the currents as in \((3.3)\) gets rid of the zero mode and the corresponding total charge \( Q^\pm = \frac{1}{2\pi} \int J^\pm d\sigma \) vanishes. This results in breaking of the gauge symmetry \([24,25]\). What we have found here, in the case of \( AdS_3 \), is that such twists are nothing but fractional spectral flows. One might worry that there is still a distinction between those states built with integrally moded \( J^\pm \) and those states built with fractionally moded \( J^\pm \), in that the latter are expected to have a different ground state energy. However, in the next section we will show from the partition function calculation that this does not happen. As such, by taking \( \hat{D}^+_{\tilde{j}}, \hat{C}_{\alpha}/2^+ \) and their images under fractional spectral flows, we automatically include the twisted states. Of course, the integer-valued spectral flows are still allowed and all the flowed sectors are treated in equal footing. In particular, the form of the Virasoro constraints remains the same and so does the expression for the energy and angular momentum. It is tempting to think that even in the case of \( AdS_3 \), spectral flow arises as a kind of twisting of some underlying theory, possibly with \( \phi \) noncompact. But one probably needs a better understanding of the \( SL(2, R)/U(1) \) parafermion theory \([20]\) in order to pursue this idea.
3.3. Invariant subspace

Having constructed the twisted sectors, only the states that are invariant under the identification $\phi \sim \phi + 2\pi/N$ are to be retained in the spectrum. There is a simple way to see what one should expect. If one considers the wave equation for a scalar field in the background (1.3), the solution may be expressed as $\Psi = \sum R(r, \omega, m) e^{-i\omega t + im\phi}$. Then single-valuedness of the wave function implies $m = N \times \text{integer}$. The effect of the projection, then, is to restrict the angular momentum to be a multiple of $N$.

It is straightforward to see how this condition comes about. For all the sectors that we have, we are to project on to the states invariant under the operator

$$e^{-2\pi i (J_0^3 - \bar{J}_0^3)/N}.$$  (3.8)

Therefore, the states that remain satisfy the condition $\ell = J_0^3 - \bar{J}_0^3 = N \times \text{integer}$, for both the discrete and continuous representations.

3.4. Thermal partition function

As in the case of $AdS_3$, we can check that the spectrum derived above agrees with what one gets by evaluating the finite temperature partition function. The calculation in the case of conical spaces is a minor extension of the calculation in [23] for $AdS_3$ and our focus will only be on the effects due to the conical singularity. The reader is referred to that paper for greater detail.

We first transform to the coordinates that are well suited for carrying out the path integral. They are given by [27]

$$v = \sinh \rho \ e^{i\phi}$$
$$\bar{v} = \sinh \rho \ e^{-i\phi}$$
$$\theta = t - \log \cosh \rho.$$  (3.9)

Under the identification $\phi \sim \phi + 2\pi/N$, the fields are identified as $v \sim v e^{2\pi i/N}$ and $\bar{v} \sim \bar{v} e^{-2\pi i/N}$. We take the worldsheet to be a torus with modular parameter $\tau$. Then the boundary conditions are

$$v(z + 2\pi) = v(z) e^{2\pi i a/N}, \quad v(z + 2\pi \tau) = v(z) e^{2\pi i b/N}.$$  (3.10)
We will denote by $Z_{ab}$ the path integral $\int e^{-S_{D\theta}DvD\bar{v}}$ with the above boundary conditions. We remind the reader that these boundary conditions are in addition to those introduced by identifying the Euclidean time $t \sim t + \beta$.

Let us first calculate $Z_{a0}$. We can implement the right boundary condition by setting $v(z) = \tilde{v} \exp\left(-\frac{a}{2N\tau_2}(z\bar{\tau} - \bar{z}\tau)\right)$, \hspace{1cm} (3.11)

with $\tilde{v}$ periodic. Then $\tilde{U}_{n,m}$, as defined in eqn. (23) of [23], picks up an additional term, $\tilde{U}_{n,m} \rightarrow \tilde{U}_{n,m} + a\tau/N$. With this change, we can repeat the calculation that was done in [23], and obtain the partition function as written in eqn. (27) of that paper:

$$Z = \frac{\beta}{8\pi} \left(\frac{k - 2}{\tau_2}\right)^{\frac{3}{2}} \times \sum_{n,m} \frac{e^{-k\beta^2|m-n\tau|^2/4\pi\tau_2 + 2\pi(\text{Im}U_{n,m})^2/\tau_2}}{|\sin(\pi U_{n,m})|^2 \left[\prod_{r=1}^{\infty} (1 - e^{2\pi i r\tau})(1 - e^{2\pi i r\tau+2\pi i U_{n,m}})(1 - e^{2\pi i r\tau-2\pi i U_{n,m}})\right]^2}.$$ \hspace{1cm} (3.12)

Similarly, for $Z_{0b}$ all we need to do is twist along the other direction of the torus, to obtain $\tilde{U}_{n,m} \rightarrow \tilde{U}_{n,m} + b/N$ and once again the partition function takes the same form. In this way we obtain for the partition function of thermal $AdS_3/Z_N$,

$$Z = 1/N \sum_{a,b} Z_{ab}.$$ \hspace{1cm} (3.13)

To obtain the free energy of strings on $AdS_3/Z_N \times \mathcal{M}$, we multiply (3.13) by the partition function of the CFT on $\mathcal{M}$ and the reparametrization ghosts, and integrate $\tau$ over the fundamental domain:

$$\int_{F_0} Z_{AdS_3/Z_N} Z_{\mathcal{M}} Z_{bc} = -\beta F = -\sum_{\text{physical}} \log(1 - e^{-\beta E}).$$ \hspace{1cm} (3.14)

However, as shown in [28,23], one can obtain an equivalent expression where the integration is over a larger domain while restricting the sum in (3.12) to $n = 0$. Then one can follow exactly the same steps as in [23] to reproduce the spectrum. We will explain some of the new features that arise in the course of this computation.

As usual the sum over $a$ represents the twisted sectors and the sum over $b$ serves as a projection down to the invariant states. Consider $Z_{a0}$ and its expansion along the lines of [23]. From $\exp\{2\pi(\text{Im}U_{0,1})^2/\tau_2\}$ and $|\sin(\pi U_{0,1})|^{-2}$ we obtain the additional factor

$$\exp\left[2\pi\tau_2\left(\frac{a^2}{N^2} - \frac{a}{N}\right)\right].$$ \hspace{1cm} (3.15)
A conformal field theory of 2 bosons with periodicity $\theta$ has ground state energy

$$ (q\bar{q})^{-\frac{1}{2}}(\theta^2-\theta-\frac{\theta}{4}), \quad (3.16) $$

so we have reproduced what might have been the expected shift in the ground state energy. However, this is not the end of story. The oscillator terms are changed to

$$ \prod_{n=1}^{\infty} (1 - e^{\beta+2\pi i \tau(n-a/N)}) (1 - e^{-\beta+2\pi i (n+a/N)}) \bigg|^{-2} , \quad (3.17) $$

which has poles when $\tau_2 = \frac{\beta}{2\pi(1-a/N)}$. In [23] it was shown that the location of the poles correspond to spectral flow parameters. So we see that $w$ is given by $w = n - a/N$ with $n$ being positive integers. It will be explained shortly that $w = -a/N$ arises from $\tau_2$ above the first pole at $\frac{\beta}{2\pi(1-a/N)}$. The shift in the location of the poles also causes the expansion of (3.17) to be slightly different from the $AdS_3$ case. One finds the terms (compare to eqn. (47) of [23])

$$ \ldots \exp \left[ 2\pi \tau_2 \left( w(w + 1) - \frac{a^2}{N^2} + \frac{a}{N} \right) \right] \ldots \quad (3.18) $$

The extra terms on the right serve to cancel the shift in ground state energy, (3.15), and we are left with the correct expression for the energy. Note that this cancellation is in agreement with what we found in the previous section. What appears to be twisting is actually a fractional spectral flow.

To see that summing over $b$ corresponds to a projection down to the invariant states, take $Z_{ab}$ and its expansion. The only additional change is the appearance of a new term

$$ \exp \left[ -\frac{2\pi ib(q - \bar{q})}{N} \right] \quad (3.19) $$

in every state. Hence, $\frac{1}{N} \sum_b Z_{ab}$ only includes the states with the correct condition on angular momentum. This shows that from (3.13) we obtain the spectrum that agrees with what was found in the algebraic analysis.

### 3.5. Bound on $\tilde{j}$

In expanding the partition function, the presence of poles in the oscillator terms meant that the range of $\tau_2$ was broken up into

$$ \frac{\beta}{2\pi(w + 1)} < \tau_2 < \frac{\beta}{2\pi w} , \quad (3.20) $$
and a different expansion was used in each interval. This gave rise to the states with spectral flow by amount \( w \). In the case of \( AdS_3 \), this included the sector with \( w = 0 \). But now that \( w \) is no longer limited to be an integer, we need to re-examine the special case

\[
\frac{\beta}{2\pi(1 - a/N)} < \tau_2 < \infty. \quad (3.21)
\]

In this range, the energy is found to be

\[
E = 1 + q + \bar{q} - \frac{2a}{N} + \sqrt{1 + 4(k - 2) \left( N_w + h - 1 - \frac{1}{2} \left( \frac{a^2}{N^2} - \frac{a}{N} \right) \right)}. \quad (3.22)
\]

So we see that these states are in the sector flowed by \( w = -a/N \). Thus, the allowed values of spectral flow are \( w = n - a/N \), including \( n = 0 \). We expect that these states will have a different range of \( \tilde{j} \), because the integral over \( \tau_2 \) is broken up in a different way from all the other states.\(^4\) Repeating the saddle point calculation as was done in \[23\], \( \tilde{j} \) is seen to satisfy

\[
\frac{(k - 2) \frac{a}{N} + 1}{2} < \tilde{j} < \frac{k - 1}{2}. \quad (3.23)
\]

On the algebraic side, this change in the lower bound can be seen from solving the physical state condition \((3.7)\)

\[
\tilde{j} = \frac{1}{2} - \frac{k - 2}{2} w + \sqrt{\cdots}, \quad (3.24)
\]

with \( w = -a/N \). The semi-classical limit (large \( k, h \)) of this bound translates into

\[
0 < \sqrt{\frac{4h}{k}} < 1 - \frac{a}{N}, \quad (3.25)
\]

which is consistent with the analysis of \[3\], extended to negative values of \( w \) (see section 3 of that paper). In \( AdS_3 \), states with negative \( w \) automatically had negative energy, but now we find that in the quotient space it is possible for states with negative fractional spectral flow to have positive energy.

\[4\] That is to say, in the variable \( 1/\tau_2 \) these states occupy a strip of length less than \( 2\pi/\beta \).
4. Summary and discussion

We have formulated a description of strings moving on $AdS_3$ but with an opening angle of $2\pi/N$ for $\phi$. The twisted states arising from the orbifold construction found a natural description as states with fractional spectral flow. Specifically, we have shown that the $n$th twisted sector is obtained by taking spectral flow with $w = n/N$. Rather than thinking of the original states with integral $w$ as being “untwisted” and fractional $w$ as being “twisted”, we have proposed that there is only one untwisted sector, namely those with $w = 0$, and all the spectral flowed sectors should be thought of as being twisted.

We have also computed the thermal partition function on this background and extracted the spectrum that agrees with the results of the algebraic description. Despite the fact that there are states constructed by acting with fractionally-moded generators, it was shown that this does not cause a change in the ground state energy.

The fact that twisted states may be obtained by spectral flow means that we are also able to write down the corresponding vertex operators, by bosonizing the $J^3$ current $\mathbf{30,3}$. Thus, unlike what usually happens in orbifolds we have explicit formulas for the twisted state vertex operators. Using these vertex operators we can compute the long string scattering amplitude on $AdS_3/Z_N$, following $\mathbf{3}$. One might wonder whether we can extend our analysis to the case with rational values of the opening angle. Indeed, it is fairly simple to generalize the algebraic construction given here, by first going to the covering space in which $\phi$ has period $2\pi P$ and taking a $Z_Q$ orbifold. The resulting space would have an opening angle $2\pi P/Q$. However, it is not clear whether one can calculate the partition function with this geometry, and that prevents us from concluding at present that such descriptions are possible.

Acknowledgements

I am grateful to J. Maldacena for helpful discussions and comments on the manuscript. This work was supported in part by DOE grant DE-FG02-91ER40654.

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5 See $\mathbf{1}$ for a detailed discussion.
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