SURFACES IN $D^4$ WITH THE SAME BOUNDARY AND FUNDAMENTAL GROUP

TAKAHIRO OBA

Abstract. We construct a family of pairs of non-isotopic symplectic surfaces in the standard symplectic 4-disk $(D^4, \omega_{st})$ such that they are bounded by the same transverse knot in the standard contact 3-sphere and fundamental groups of their complements are isomorphic. In the appendix, we prove explicitly that one can obtain a symplectic surface in $(D^4, \omega_{st})$ from a braided surface in a bidisk.

1. INTRODUCTION

This paper is concerned with symplectic surfaces in the standard symplectic 4-disk $(D^4, \omega_{st})$ bounded by the same transverse link in the standard contact 3-sphere $(S^3, \xi_{st})$. Such surfaces have been studied in some papers [7, 6, 2, 4]. Up to the present, the provided families of distinct symplectic surfaces bounded by the same transverse knot (or link) can be distinguished by the fundamental groups of their complements. Hence it is natural to ask whether there is a pair of non-isotopic symplectic surfaces in $D^4$ bounded by the same transverse knot such that complements of two surfaces have isomorphic fundamental groups.

The main result of this paper is the following:

Theorem 1.1. There is a family $\{(S_1(n), S_2(n))\}_{n \in \mathbb{Z}_{\geq 0}}$ of pairs of symplectic surfaces in the standard symplectic 4-disk $(D^4, \omega_{st})$ with contact boundary such that:

1. For a fixed $n \in \mathbb{Z}_{\geq 0}$,
   a. their boundaries $\partial S_j(n)$ ($j = 1, 2$) are the same transverse knot up to isotopy in the boundary $(S^3, \xi_{st})$,
   b. two fundamental groups $\pi_1(D^4 \setminus S_j(n))$ are isomorphic, and
   c. double branched covers $X_j(n)$ of $D^4$ branched along $S_j(n)$ are not homeomorphic, and, particularly, two surfaces $S_j(n)$ are not isotopic;

2. The boundaries $\partial S_j(n)$ and $\partial S_j(n')$ are not smoothly isotopic in $\partial D^4$ if $n \neq n'$.

Date: August 9, 2017.
2010 Mathematics Subject Classification. Primary 57R17; Secondary 57R65.
This work was partially supported by JSPS KAKENHI Grant Number 15J05214.
Rudolph exhibited two braid factorizations of a fixed 3-braid in [14]. Based on this example, we construct inequivalent braid factorizations, which provide symplectic surfaces in the above family. Note that it does not directly follow from the argument in [13] that from a braided surface in a bidisk one can obtain a symplectic surface in \((D^4, \omega_{st})\) isotopic to the given braided surface. To complete this, we prove it in Appendix A explicitly.

A Stein filling of a contact manifold is a sublevel set of a proper, bounded below strictly plurisubharmonic function on a complex manifold whose convex boundary is contactomorphic to the given one (see [12] for more details). We obtain the following corollary from the above theorem combined with an argument about contact and Stein structures.

**Corollary 1.2.** There is a family of contact 3-manifolds \(\{(M(n), \xi(n))\}_{n \in \mathbb{Z}_{\geq 0}}\) such that each contact manifold admits two non-homeomorphic Stein fillings \(X_1(n)\) and \(X_2(n)\) which have the same fundamental group and homology group but non-isomorphic intersection forms.

**Acknowledgements** The author would like to express his gratitude to Professor Hisaaki Endo for his continuous support and encouragement. He also thanks Dennis Auroux and Kyle Hayden for many fruitful suggestions and comments on an early draft of the paper, and Josh Sabloff for his kind e-mail correspondence.

## 2. Braided Surfaces

### 2.1. Braid groups

We here briefly review braid groups (see [15, Section 2.2] for example). Let \(D^2\) be a closed disk in \(\mathbb{R}^2\) equipped with the standard orientation and \(K \subset \text{Int}D^2\) a finite set. Suppose that \#\(K = m\).

**Definition 2.1.** The braid group with respect to \(D^2\) and \(K\), denoted by \(B_m[D^2, K]\), is the group of isotopy classes of orientation-preserving diffeomorphisms \(\beta\) of \(D^2\) such that \(\beta|_{\partial D^2} = \text{id}_{\partial D^2}\) and \(\beta(K) = K\). The elements of this group are called braids.

Let \(\sigma\) be a smooth simple path in \(\text{Int}D^2\) with distinct end points \(a, b \in K\) and \(\sigma \cap K = \{a, b\}\). Choose a small tubular neighborhood \(U \subset \text{Int}D^2\) of \(\sigma\) such that \(U \cap K = \{a, b\}\).

**Definition 2.2.** The half-twist \(H(\sigma)\) along \(\sigma\) is an element of the braid group \(B_m[D^2, K]\) which switches the end points \(a\) and \(b\) of \(\sigma\) by a counterclockwise 180\(^o\) rotation and whose support is contained in \(U\).

### 2.2. Braided surfaces and their descriptions

Let \(D^2_1\) and \(D^2_2\) be two oriented closed disks.

**Definition 2.3.** A **braided surface** in the bidisk \(D^2_1 \times D^2_2\) is a properly embedded surface \(S\) in \(D^2_1 \times D^2_2\) such that:

1. The restriction of the first projection \(pr_1|_S : S \to D^2_1\) is a simple branched covering.
(2) For each branch point \( x \in S \) of \( pr_1|_S \), there are complex coordinates \((z, w)\) and \( \zeta \) around \( x \) and \( pr_1(x) \), respectively, compatible with orientations of \( D^2_1 \times D^2_2 \) and \( D^2_1 \) such that \( pr_1 \) can be written as \( \zeta = pr_1(z, w) = z \) and locally the set \( \{(z, w) | z = w^2\} \) coincides with \( S \).

Suppose that \( S \) is a braided surface in \( D^2_1 \times D^2_2 \). Let \( \Delta(S) \subset \text{Int}D^2_1 \) denote the set of branch points of the covering \( pr_1|_S \). For a point \( y \) of \( D^2_1 \setminus \Delta(S) \), the number \( m = \#(S \cap pr_1^{-1}(y)) \) is called the degree of the braided surface \( S \).

One can read off the fundamental group of the complement of a braided surface \( S \subset D^2_1 \times D^2_2 \) from its braid monodromy. Fix a base point \( y_0 \in \partial D^2_1 \) and set \( D_{y_0} = pr_1^{-1}(y_0) \) and \( K(y_0) = D_{y_0} \cap S = \{x_1, \ldots, x_m\} \). For a point \( y \) of \( \Delta(S) \), consider a smooth simple loop \( \gamma : [0, 1] \to D^2_1 \setminus \Delta(S) \) around \( y \) based at \( y_0 \) whose bounding region does not contain any other branch points. This loop lifts to \( (\gamma([0, 1]) \times D^2_2) \cap S \) as a motion \( pr_2(\{x_1(t), \ldots, x_m(t) | t \in [0, 1]\}) \) of \( m \) distinct points of \( D^2_2 \), where \( pr_2 : D^2_1 \times D^2_2 \to D^2_2 \) is the second projection. When \( t = 0, 1 \), it is nothing but \( K(y_0) \). Hence this motion defines a braid \( \beta(\gamma) \in B_m[D_{y_0}, K(y_0)] \), called a braid monodromy (with respect to \( y_0 \)) around the branch point \( y \). It is known that this braid is the half-twist \( H(\sigma) \) along a smooth simple path \( \sigma \) connecting two distinct points of \( K(y_0) \). One can associate an element of \( B_m[D_{y_0}, K(y_0)] \) to any loop in \( D^2_1 \setminus \Delta(S) \) based at \( y_0 \), and define the homomorphism

\[ \varphi : \pi_1(D^2_1 \setminus \Delta(S), y_0) \to B_m[D(y_0), K(y_0)] \]

Set \( \Delta(S) = \{y_1, \ldots, y_k\} \). Take smooth simple loops \( \gamma_i \in \pi_1(D^2_1 \setminus \Delta(S), y_0) \) around \( y_i \), as we did before, so that the composition \( \gamma_1 \cdots \gamma_k \) is homotopic to \( \partial D^2_1 \). Obviously, \( \{\gamma_1, \ldots, \gamma_k\} \) serves as a free basis for \( \pi_1(D^2_1 \setminus \Delta(S), y_0) \), and it is called a geometric basis for the group. Then, the braid \( \varphi(\partial D^2_1) = \varphi(\gamma_1 \cdots \gamma_k) \) can be factorized into \( k \) half-twists as

\[ \beta(\gamma_1) \cdots \beta(\gamma_k) \]

which is a braid monodromy factorization of \( \varphi(\partial D^2_1) \). As a similar notion, a braid factorization of a braid \( \beta \) is a factorization \( \beta = \beta_1 \cdots \beta_k \) into half-twists \( \beta_j \). Remark that given a braid factorization \( \beta_1 \cdots \beta_k \) of a braid, one can construct a braided surface \( S \) with \( k \) branch points whose braid monodromy around each branch point \( y_i \) is \( \beta_i \) for some geometric basis for \( \pi_1(D^2_1 \setminus \Delta(S), y_0) \).

Now we explain how to compute the fundamental group of the complement of a braided surface as the special case of [15, Theorem 2.5]. Let \( S \) be a braided surface in \( D^2_1 \times D^2_2 \). Suppose that \( \{\gamma_1, \ldots, \gamma_k\} \) is a geometric basis for the fundamental group \( \pi_1(D^2_1 \setminus \Delta(S), y_0) \) and the ordered \( k \)-tuple \((H(\sigma_1), \ldots, H(\sigma_k))\) consists of braid monodromies \( \varphi(\gamma_1), \ldots, \varphi(\gamma_k) \) of \( S \), where each \( \varphi(\gamma_j) \) is a smooth simple path connecting two distinct points of \( K(y_0) \). Fix a point \( x_0 \) of \( \partial D_{y_0} \). Label the points of \( K(y_0) \) as \( x_1, \ldots, x_m \) and let \( \{\gamma'_1, \ldots, \gamma'_m\} \) be a geometric basis for \( \pi_1(D_{y_0} \setminus K(y_0), x_0) \) constructed in the same way we did for \( \pi_1(D^2_1 \setminus \Delta(S), y_0) \). For each \( j = 1, \ldots, k \), set \( A_j = \gamma'_i \), where \( x_i \) is either of end points of \( \sigma_j \) and \( B_j = H(\sigma_j)(A_j) \) (see Figure 1). It is clear that \( B_j \) can be expressed in terms of \( \gamma_1', \ldots, \gamma'_m \) because they form a geometric basis for \( \pi_1(D_{y_0} \setminus K(y_0), x_0) \). By using Zariski-Van Kampen’s theorem, we have the following
When \((3.2)\)
\[ \tau \]
\[ \text{Here is a Lefschetz fibration diffeomorphic to the surface } F. \]

Proof of Theorem 1.1.

3.1. Double branched covers and Lefschetz fibrations. Let \(S\) be a braided surface of degree \(m\) in a bidisk \(D_1^2 \times D_2^2\) whose braid monodromy factorization with respect to some base point \(y_0\) and geometric basis for \(\pi_1(D_1^2 \setminus \Delta(S), y_0)\) is
\[
H(\sigma_1) \cdots H(\sigma_k).
\]

Consider the double branched covering \(p : X \to D_1^2 \times D_2^2\) whose branch set is \(S\). The covering \(p\) restricts to the double branched covering \(p|_{F_{y_0}} : F_{y_0} = p^{-1}(D_{y_0}) \to D_{y_0}\). Each path \(\sigma_j\) lifts, with respect to \(p|_{F_{y_0}}\), to a unique simple closed curve \(c_j\) on the surface \(F_{y_0}\) up to isotopy. Then, according to [10, Proposition 1], the composition \(pr_1 \circ p : X \to D_1^2\) is a Lefschetz fibration (see [8, Chapter 8] for the precise definition) whose fibers are diffeomorphic to the surface \(F_{y_0}\) and monodromy factorization is
\[
\tau(c_k) \circ \cdots \circ \tau(c_1).
\]

Here \(\tau(c)\) denotes the isotopy class of a right-handed Dehn twist along \(c\). Throughout this paper, we use the functional notation for the products in the mapping class group of \(F_{y_0}\), i.e. \(f \circ g\) means that we apply \(g\) first and then \(f\).

3. Proof of Results

3.1. Proof of Theorem 1.1. Fix an integer \(n \in \mathbb{Z}_{\geq 0}\). Let \(D^2\) be the closed unit disk in \(\mathbb{C}\) and \(K_{n+3}\) the set of \(n + 3\) points of \(\text{Int } \mathbb{D}^2\) on the real axis. Let \(a, b, c_n, d_1, \ldots, d_{n+2}\) be smooth simple paths in \(\mathbb{D}^2\) as shown in Figure 2. Define two braids \(\beta_1(n), \beta_2(n) \in B_{n+3}[\mathbb{D}^2, K_{n+3}]\) with factorizations given by
\[
\beta_1(n) = H(a) \cdot H(b) \cdot H(d_1) \cdot H(c_n) \cdot H(d_{n+2}) \cdots \cdot H(d_3),
\]
\[
\beta_2(n) = H(H(d_2)(a)) \cdot H(H(d_2)(b)) \cdot H(d_1) \cdot H(c_n) \cdot H(d_{n+2}) \cdots \cdot H(d_3).
\]

When \(n = 0\), we set \(\beta_1(0) = H(a) \cdot H(b) \cdot H(d_1) \cdot H(c_0)\) and apply the same manner to \(\beta_2(0)\). We obtain two braided surfaces \(S_1(n)\) and \(S_2(n)\) whose braid monodromy

![Figure 1. (a) Path \(\sigma_j\). (b) Loops \(A_j\) and \(B_j\) associated to \(\sigma_j\).](image)
factorizations are the given braid factorizations (3.1) and (3.2), respectively. From the argument in Appendix A, these braided surfaces can be considered as symplectic surfaces in the 4-disk $(D^4, \omega_{st})$ with transverse knot boundaries.

The transverse knots $\partial S_1(n)$ and $\partial S_2(n)$ in $(S^3, \xi_{st})$ are represented by the closure of braids $\beta_1(n)$ and $\beta_2(n)$, respectively. It can be easily checked that $H(d_2)$ commutes with the product $H(a) \cdot H(b)$, which proves that $\beta_1(n) = \beta_2(n)$. Hence two boundaries are transversely isotopic. Hereafter, for the sake of simplicity, set $\beta(n) = \beta_1(n) = \beta_2(n)$.

Next, we show that the fundamental groups of complements $D^4 \setminus S_1(n)$ and $D^4 \setminus S_2(n)$ are isomorphic. Fixing a base point $x_0$ in a fiber of the projection $pr_1$ of the bidisk, by the formula (2.1), $\pi_1(D^4 \setminus S_1(n), x_0)$ is isomorphic to the group generated by $\gamma_1, \ldots, \gamma_{n+3}$ with relations

$$\gamma_1 = \gamma_2 \gamma_3 \gamma_2^{-1}, \quad \gamma_1 = \gamma_3, \quad \gamma_1 = \gamma_2,$$

$$(\gamma_1 \gamma_2 \cdots \gamma_{n+2}) \gamma_{n+3} (\gamma_1 \gamma_2 \cdots \gamma_{n+2})^{-1} = \gamma_2, \quad \gamma_j = \gamma_{j+1} (j = 3, \ldots, n + 2).$$

Hence $\pi_1(D^4 \setminus S_1(n), x_0) \cong \langle \gamma_1, \ldots, \gamma_{n+3} \rangle \cong \mathbb{Z}$. On the other hand, $\pi_1(D^4 \setminus S_2(n), x_0)$ is isomorphic to the group generated by $\gamma_1, \ldots, \gamma_{n+3}$ with relations

$$\gamma_2 = \gamma_3 \gamma_2^{-1} \gamma_1 \gamma_2 \gamma_3, \quad \gamma_1 = \gamma_2, \quad \gamma_1 = \gamma_2,$$

$$(\gamma_1 \gamma_2 \cdots \gamma_{n+2}) \gamma_{n+3} (\gamma_1 \gamma_2 \cdots \gamma_{n+2})^{-1} = \gamma_2, \quad \gamma_j = \gamma_{j+1} (j = 3, \ldots, n + 2).$$

Thus, $\pi_1(D^4 \setminus S_2(n), x_0) \cong \langle \gamma_1, \ldots, \gamma_{n+3} \rangle \cong \mathbb{Z}$ that is isomorphic to $\pi_1(D^4 \setminus S_1(n), x_0)$.

For each $j = 1, 2$ let $p_j(n) : X_j(n) \to D^2_x \times D^2_y$ be the double branched covering whose branch set is $S_j(n)$. As we discussed in Section 2.3, $X_j(n)$ is considered as the total space of the Lefschetz fibration $f_j(n) = pr_1 \circ p_j(n)$. Let $A, B, C_n, D_i$ be lifts of arcs $a, b, c_n, d_i$, respectively, with respect to the covering $p_j(n)|_{F_{yo}}$, where $F_{yo}$ is the preimage of $y_0 \in D^2_y$ under $f_j(n)$ (see Figure 3). Fibers of the Lefschetz fibration $f_j(n)$ are diffeomorphic to $F_{yo}$ and its monodromy factorization is

$$\tau(D_3) \circ \cdots \circ \tau(D_{n+2}) \circ \tau(C_n) \circ \tau(D_1) \circ \tau(B) \circ \tau(A) \quad \text{if } j = 1,$$

$$\tau(D_3) \circ \cdots \circ \tau(D_{n+2}) \circ \tau(C_n) \circ \tau(D_1) \circ \tau(\tau(D_2)(B)) \circ \tau(\tau(D_2)(A)) \quad \text{if } j = 2.$$
Figure 3. Surface $F_{y_0}$ as the double branched cover and lift $C_n$: Each rounded arrow indicates the orientation of $F_{y_0}$.

Figure 4. Handle diagrams of $X_1(n)$ and $X_2(n)$: All 2-handle framings which are not written here are $-2$.

From these data, one can draw handle diagrams (or Kirby diagrams) of $X_1(n)$ and $X_2(n)$ as in Figure 4. Here we use the standard Seifert surface for the $(2, n+3)$-torus link as the fiber surface to see the monodromy curves more easily. We should note that the surface framing of each curve does not always coincide with its blackboard framing (see [8, Section 6.3], which explains the way to draw handle diagrams of Milnor fibers in the same manner as ours). After sliding 2-handles and cancelling 1-/2-handle pairs as indicated in Figure 5 and 6, we obtain handle diagrams of $X_1(n)$ and $X_2(n)$ each of which consists of only one 0-handle and two 2-handles.

From the bottom left diagram of Figure 6 one can see that $X_2(n)$ contains a smooth surface with self-intersection number $-2$. In contrast, we will show below that the double cover $X_1(n)$ contains no such surfaces: Let $\{e_1, e_2\}$ be the basis for the homology group $H_2(X_1(n); \mathbb{Z})$, where each $e_j$ is the homology class represented by the 2-handle depicted in the bottom left diagram of Figure 5.
Figure 5. Handle calculus for $X_1(n)$: Each dashed arrow in the diagram indicates how we slide a 2-handle over another one.

Figure 6. Handle calculus for $X_2(n)$: Each dashed arrow in the diagram indicates how we slide a 2-handle over another one.
Suppose for the sake of contradiction that there are integers $\alpha_1, \alpha_2 \in \mathbb{Z}$ such that $(\alpha_1 e_1 + \alpha_2 e_2)^2 = -2$. The matrix $Q(n)$ of the intersection form $Q_{X_1(n)}$ with respect to the basis $\{e_1, e_2\}$ can be read off from the handle diagram of $X_1(n)$, and

$$Q(n) = \begin{bmatrix} -2n - 4 & -1 \\ -1 & -8 \end{bmatrix}. $$

Then, by using this matrix, we have $-(2n - 4)\alpha_1^2 - 2\alpha_1\alpha_2 - 8\alpha_2^2 = -2$. The left-hand side also can be written as the form

$$-(2n + 4)(\alpha_1 + \frac{1}{2n + 4}\alpha_2)^2 - (8 - \frac{1}{2n + 4})\alpha_2^2. $$

Since the above two terms are non-positive, we have $-(8 - 1/(2n + 4))\alpha_2^2 \geq -2$, or $(8 - 1/(2n + 4))\alpha_2^2 \leq 2$. The coefficient $8 - 1/(2n + 4)$ is greater than 2, and hence $\alpha_2^2 < 1$, that is, $\alpha_2 = 0$. Thus

$$-(2n + 4)\alpha_1^2 = -2 \quad \text{and} \quad \alpha_1 \in \mathbb{Z} \setminus \{0\}. $$

However, $-(2n+4)\alpha_1^2 \leq -(2n+4) \leq -4$ for $\alpha_1 \in \mathbb{Z} \setminus \{0\}$, which contradicts the equation (3.5). Thus, we conclude that $X_1(n)$ and $X_2(n)$ are not homeomorphic.

To distinguish the closure $\hat{\beta}(n)$ from $\hat{\beta}(n')$ for $n \neq n'$, we use the determinant of a knot, defined by $|\det(V + V^T)|$, where $V$ is a Seifert matrix for the knot. It is known that it equals the order of the first homology group of the double branched cover of $S^3$ branched along the knot. Moreover, let $X$ be a compact 4-manifold admitting a handle decomposition with only one 0-handle and 2-handles. Then, the determinant of a matrix for the intersection form $Q_X$ coincides with $|H_1(\partial X; \mathbb{Z})|$ up to sign (see [8, Corollary 5.3.12]). Thus the determinant of the closure $\hat{\beta}(n)$ is

$$\det Q(n) = \det \begin{bmatrix} -2n - 4 & -1 \\ -1 & -8 \end{bmatrix} = 16n + 31, $$

which proves that all $\hat{\beta}(n)$ ($n \in \mathbb{Z}_{\geq 0}$) are mutually non-isotopic. This finishes the proof.

**Remark 3.1**. Braid factorizations $H(a) \cdot H(b)$ and $H(H(d_2)(a)) \cdot H(H(d_2)(b))$ we used above are essentially found by Rudolph in [14, Example 1.13], where he showed that two factorizations are ones of the same braid. This pair was also used in [2, Proposition 3.2] (see also Example 4.2 in the same paper).

**Remark 3.2**. Two mapping class factorizations (3.3) and (3.4) are related by a partial conjugation, twisting the last two factors by $\tau(D_2)$. This implies that two corresponding double covers are related by a Luttinger surgery along a torus built by parallel transport of the curve $D_2$ along a loop in $D_4^2$ (see [3]).

3.2. **Proof of Corollary 1.2.** Let each $S_j(n)$ ($j = 1, 2$) be the braided surface constructed above and $p_j(n) : X_j(n) \to D^4$ ($j = 1, 2$) the double branched covering whose branch set is $S_j(n)$. As we mentioned before, the covering $p_j(n)$ induces the Lefschetz fibration $f_j(n)$ on $X_j(n)$. According to [1, 10], $X_j(n)$ admits a Stein structure, and the contact structure $\xi_j(n)$ on the boundary $M_j(n) = \partial X_j(n)$ induced from the Stein
structure is compatible with the open book determined by the Lefschetz fibration. We see that the monodromy \( \phi_j(n) \) of this open book is isotopic to the composition
\[
\phi_j(n) = \begin{cases} 
\tau(D_3) \circ \cdots \circ \tau(D_{n+2}) \circ \tau(C_n) \circ \tau(D_1) \circ \tau(B) \circ \tau(A) & \text{if } j = 1, \\
\tau(D_3) \circ \cdots \circ \tau(D_{n+2}) \circ \tau(C_n) \circ \tau(D_1) \circ \tau(\tau(D_2)(B)) \circ \tau(\tau(D_2)(A)) & \text{if } j = 2.
\end{cases}
\]
Since \( \tau(D_2) \) commutes with \( \tau(B) \circ \tau(A) \), we have
\[
\tau(B) \circ \tau(A) = \tau(\tau(D_2)(B)) \circ \tau(\tau(D_2)(A)).
\]
Hence \( \phi_1(n) = \phi_2(n) \), which proves that the contact manifolds \( (M_1(n), \xi_1(n)) \) and \( (M_2(n), \xi_2(n)) \) are mutually contactomorphic. Therefore, \( X_1(n) \) and \( X_2(n) \) serve as Stein fillings of the contact manifold \( (M(n), \xi(n)) := (M_1(n), \xi_1(n)) \) whose intersection forms are non-isomorphic by Theorem 1.1. Moreover, from handle diagrams depicted in Figure 5 and 6, it is obvious that \( X_1(n) \) and \( X_2(n) \) are simply connected and have the same homology group. This completes the proof.

**Appendix A. From braided surfaces to symplectic surfaces**

In this appendix, we explain how to obtain a symplectic surface in the standard symplectic 4-disk \( (D^4, \omega_{sl}) \) from a braided surface in a bidisk. Here \( D^4 = \{(z, w) \in \mathbb{C}^2 | |z|^2 + |w|^2 \leq 1\} \subset \mathbb{C}^2 \) and \( \omega_{sl} = \sqrt{-1} (dz \wedge d\bar{z} + dw \wedge d\bar{w})/2 \). Although in fact it has been used as a well-known fact implicitly, here for future use we prove it explicitly.

We assume that the reader is familiar with basics of contact and symplectic geometry. If necessary, we refer the reader to [11].

Let \( D^2(r) \) be the closed disk in \( \mathbb{C} \) of radius \( r \) centered at the origin with the complex orientation. In particular, \( D^2(1) \) is the closed unit disk \( \mathbb{D}^2 \). One can construct a braided surface \( S \) in a bidisk whose braid monodromy factorization coincides with a given braid factorization. Furthermore, by a result of Rudolph [13, Section 4], for the braided surface \( S \), there is a polynomial \( f(z, w) \in \mathbb{C}[z, w] \) such that \( S = \{ f(z, w) = 0 \} \cap (\mathbb{D}^2 \times D^2(r)) \) is a braided surface (with respect to the first projection of \( \mathbb{D}^2 \times D^2(r) \)) isotopic to \( S \) for some \( r > 0 \). Without loss of generality, we may assume that \( S \) is smooth by slightly perturbing \( f \). Moreover, possibly after making \( D^2(r) \) larger by \( D^2(r) : w \mapsto Kw \in D^2(Kr) \) for some large \( K > 0 \) and considering \( \{ f(z, Kw) = 0 \} \cap (\mathbb{D}^2 \times D^2(Kr)) \), the boundary \( \partial S \) can be assumed to be a transverse link in \( (\partial D^2 \times D^2(r), \alpha|_{\partial D^2 \times D^2(r)}, \lambda (\partial D^2 \times D^2(r))) \), where \( \alpha = \sqrt{-1} (zd\bar{z} - \bar{z}d z + w\partial w - \bar{w}\partial \bar{w})/4 \).

In addition, we may assume that the Liouville vector field \( V = z\partial/\partial z - \bar{z}\partial/\partial \bar{z} + w\partial/\partial w - \bar{w}\partial/\partial \bar{w} \) for the symplectic form \( \omega_{sl} \) is tangent to the symplectic surface \( S \). This can be achieved as follows: Since \( \partial S \) is a transverse link, there is a neighborhood \( N(\partial S) \) of \( \partial S \) in \( \mathbb{D}^2 \times D^2(r) \) endowed with a symplectomorphism
\[
\varphi : (N(\partial S), \omega) \to ((-\epsilon, 0] \times S^1 \times \mathbb{R}^2, d(\epsilon^2(d\theta + xdy - ydx)))
\]
for some \( \epsilon \), where \( (t, \theta, x, y) \in (-\epsilon, 0] \times S^1 \times \mathbb{R}^2 \) and \( \varphi(\partial S) = \{0\} \times S^1 \times \{0\} \). The push-forward map \( \varphi_* \) maps the vector field \( V \) to \( \partial/\partial t \) on \( (-\epsilon, 0] \times S^1 \times \mathbb{R}^2 \). Then, one can find an embedding given by
\[
\psi : (-\epsilon, 0] \times S^1 \to (-\epsilon, 0] \times S^1 \times \mathbb{R}^2, \psi(t, \theta) = (t, \theta, x(t, \theta), y(t, \theta))
\]
whose image coincides with \( \varphi(S \cap N(\partial S)) \). Note that \( \psi(0, \theta) = (0, \theta, 0, 0) \) for any \( \theta \in S^1 \). Since \( \psi((0, \theta) \times S^1) = \varphi(S \cap N(\partial S)) \) is symplectic,
\[
\psi^* d(e^t(d\theta + xdy - ydx)) = (1 + x\dot{y} - x\dot{\theta}y + 2(x_t y_\theta - x_\theta y_t))dt \wedge d\theta > 0,
\]
where \( x_t = \partial x/\partial t, x_\theta = \partial x/\partial \theta, y_t = \partial y/\partial t, y_\theta = \partial y/\partial \theta \). Now, we perturb \( \psi((0, \theta) \times S^1) \) so that \( \partial/\partial t \) is tangent to the surface near the boundary. Since \( x(t, \theta), y(t, \theta) \) and their partial derivatives are continuous on \((-\epsilon, 0) \times S^1 \), there exists \( \epsilon' \in (0, \epsilon) \) such that
\[
|x(t, \theta)y_\theta(t, \theta) - x_\theta(t, \theta)y(t, \theta)| < 1/4, \quad |x_t(t, \theta)y_\theta(t, \theta) - x_\theta(t, \theta)y_t(t, \theta)| < 1/4
\]
for any \((t, \theta) \in (-\epsilon', 0) \times S^1 \). Let \( \tau : (-\epsilon', 0) \to \mathbb{R} \) be a smooth function such that (see Figure 7):

- \( \tau(t) = 0 \) near \( t = 0 \);
- \( \tau(t) = t \) near \( t = -\epsilon' \);
- \( \tau'(t) = 3/2 \) near \( t = -\epsilon'/2 \);
- \( 0 \leq \tau'(t) \leq 3/2 \) for any \( t \in (-\epsilon', 0] \).

![Figure 7. Graph of the function \( \tau : (-\epsilon', 0) \to \mathbb{R} \)](image)

Using this function, set
\[
\psi_s(t, \theta) = (t, \theta, x(s\tau(t) + (1 - s)t, \theta), y(s\tau(t) + (1 - s)t, \theta))
\]
for \( s \in [0, 1] \) as a perturbation of \( \psi((-\epsilon', 0] \times S^1) \). Then,
\[
\psi_s^* d(e^t(d\theta + xdy - ydx)) = (1 + xy_\theta - x_\theta y + 2(s\tau' + (1 - s))(x_t y_\theta - x_\theta y_t))dt \wedge d\theta,
\]
and hence all we have to do is to see that the coefficient is positive. Indeed,
\[
1 + (xy_\theta - x_\theta y) + 2(s\tau' + (1 - s))(x_t y_\theta - x_\theta y_t) > 1 - 1/4 + 2 \cdot 3/2 \cdot (-1/4) = 0,
\]
which shows that \( S_s = ((\varphi^{-1} \circ \psi_s)((-\epsilon', 0] \times S^1)) \cup (S \setminus ((\varphi^{-1} \circ \psi_s)((-\epsilon', 0] \times S^1))) \) is symplectic. In particular, \( S_1 \) satisfies the desired condition because \( \psi_1(t, \theta) = (t, \theta, 0, 0) \) near \( t = 0 \). From now on, we think of the surface \( S_1 \) as \( S \).

Symplectically embed the bidisk \( \mathbb{D}^2 \times D(r) \) into the symplectic manifold \((\mathbb{C}^2, \omega)\), where \( \omega = \sqrt{-1}(dz \wedge d\bar{z} + dw \wedge d\bar{w})/2 \). Using the flow of the Liouville vector field \( V \), extend
the surface $S$ and obtain the completion $\hat{S} \subset \mathbb{C}^2$ of $S$, that is, $S \cup (0, \infty) \times \partial S$. For a sufficient large $R$, let $S'$ be the intersection of the surface $\hat{S}$ and the round 4-ball $D^4(R)$ of radius $R$. Then, $S' = \hat{S} \cap D^4(R)$ is a symplectic surface in $(D^4(R),\omega|_{D^4(R)})$. From this it follows that
\[(\iota_{\nu}\omega)_p(v) = \omega_p(V_p, v) > 0\]
at any point $p \in \partial S'$ for a tangent vector $v \in T_p\partial S'$ such that the vector space $\langle v \rangle$ spanned by $v$ is isomorphic to $T_p\partial S'$ as oriented vector spaces. This implies that the boundary $\partial S'$ is a transverse link in the boundary $(\partial D^4(R),\ker(\alpha|_{\partial D^4(R)}))$, where $\alpha = \iota_{\nu}\omega$.

Define a diffeomorphism $\Phi_R : D^4 = D^4(1) \rightarrow D^4(R)$ by $\Phi_R(z,w) = (Rz,Rw)$. Since $S'$ is symplectic in $(D^4(R),\omega|_{D^4(R)})$, so is $S'' = \Phi^{-1}_R(S')$ in $(D^4, (\Phi_R)^{*}\omega)$. The restriction of $\Phi_R$ on $\partial D^4$ gives a contactomorphism between $(\partial D^4,\ker((\Phi_R)^{*}\alpha|_{\partial D^4}))$ and $(\partial D^4(R),\ker(\alpha|_{\partial D^4(R)}))$. Thus, the boundary $\partial S'' = \Phi^{-1}_R(\partial S')$ is a transverse link in $(\partial D^4,\ker((\Phi_R)^{*}\alpha|_{\partial D^4}))$. Now, we have $(\Phi_R)^{*}\omega = R^2\omega$ and $(\Phi_R)^{*}\alpha = R^{2}\alpha$. The coefficients $R^2 > 0$ do not affect whether the surface $S''$ (resp. its boundary $\partial S''$) is symplectic (resp. transverse), and so $S''$ is a symplectic surface in the standard symplectic 4-disk $(D^4,\omega|_{D^4} = \omega_{st})$ and $\partial S''$ is a transverse link in the standard contact 3-sphere $(\partial D^4,\ker(\alpha|_{\partial D^4}) = \xi_{st})$. Note that our procedure here preserves the link type of the transverse link in the boundary. Therefore, $S'' \subset (D^4,\omega_{st})$ is a desired symplectic surface.

**Remark A.1.** We would like to point out that in his recent paper [9], Hayden obtains a similar result of *Stein quasipositive* links in general Stein fillable contact 3-manifolds. The proof is based on *ascending surfaces*, originated from the work of Boileau and Orevkov [5].

**References**

[1] S. Akbulut and B. Ozbagci. Lefschetz fibrations on compact Stein surfaces. *Geom. Topol.*, 5:319–334, 2001.
[2] D. Auroux. Factorizations in $SL(2,\mathbb{Z})$ and simple examples of inequivalent Stein fillings. *J. Symplectic Geom.*, 13(2):261–277, 2015.
[3] D. Auroux, S. K. Donaldson, and L. Katzarkov. Luttinger surgery along Lagrangian tori and non-isotopy for singular symplectic plane curves. *Math. Ann.*, 326(1):185–203, 2003.
[4] R. I. Baykur and J. Van Horn-Morris. Fillings of genus–1 open books and 4–braids. *Int. Math. Res. Not. IMRN*, rnw281, 2016.
[5] M. Boileau and S. Orevkov. Quasi-positivité d’une courbe analytique dans une boule pseudo-convexe. *C. R. Acad. Sci. Paris Sér. I Math.*, 332(9):825–830, 2001.
[6] C. Cao, N. Gallup, K. Hayden, and J. M. Sabloff. Topologically distinct Lagrangian and symplectic fillings. *Math. Res. Lett.*, 21(1):85–99, 2014.
[7] A. Geng. Two surfaces in $D^4$ bounded by the same knot. *J. Symplectic Geom.*, 9(2):119–122, 2011.
[8] R. E. Gompf and A. I. Stipsicz. 4-manifolds and Kirby calculus, volume 20 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1999.
[9] K. Hayden. Quasipositive links and stein surfaces. *preprint*, arXiv:1703.10150.
[10] A. Loi and R. Piergallini. Compact Stein surfaces with boundary as branched covers of $B^4$. *Invent. Math.*, 143(2):325–348, 2001.
[11] D. McDuff and D. Salamon. *Introduction to symplectic topology*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, second edition, 1998.
[12] B. Ozbagci. On the topology of fillings of contact 3-manifolds. *Geometry & Topology Monographs*, 19(1):73–123, 2015.
[13] L. Rudolph. Algebraic functions and closed braids. *Topology*, 22(2):191–202, 1983.
[14] L. Rudolph. Braided surfaces and Seifert ribbons for closed braids. *Comment. Math. Helv.*, 58(1):1–37, 1983.
[15] M. Teicher and M. Friedman. On non fundamental group equivalent surfaces. *Algebr. Geom. Topol.*, 8(1):397–433, 2008.

Department of Mathematics, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguroku, Tokyo 152-8551, Japan

E-mail address: oba.t.ac@m.titech.ac.jp