Generalized Teichmüller space of non-compact 3–manifolds and Mostow rigidity

Charalampos Charitos and Ioannis Papadoperakis

July 15, 2010

Abstract

Consider a 3–dimensional manifold \( N \) obtained by gluing a finite number of ideal hyperbolic tetrahedra via isometries along their faces. By varying the isometry type of each tetrahedron but keeping fixed the gluing pattern we define a space \( T \) of complete hyperbolic metrics on \( N \) with cone singularities along the edges of the tetrahedra. We prove that \( T \) is homeomorphic to a Euclidean space and we compute its dimension. By means of examples, we examine if the elements of \( T \) are uniquely determined by the angles around the edges of \( N \).

2000 Mathematics Subject Classification: 57M50

1 Introduction

In [1], [2] spaces \( X \) which are called ideal simplicial complexes, are considered. These spaces \( X \) are obtained by gluing along their edges finitely many ideal hyperbolic triangles. The Teichmüller space \( T(X) \) of \( X \) is defined and parametrized via the shifts parameters.

In the present work we consider orientable, compact 3–manifolds with non-empty boundary \( \partial M \). The interior \( \text{Int}(M) \) of \( M \) always has a triangulation \( D \) by ideal tetrahedra. Fixing \( D \) we define ideal hyperbolic structures with axial singularities on \( M \), as well as, the generalized Teichmüller space \( T_D(M) \) of \( M \). The 2–skeleton of \( D \), equipped with a hyperbolic structure induced from an element of \( T_D(M) \), is a 2–dimensional ideal simplicial complex \( X \) and we show that the shift parameters which parametrize \( T(X) \) also parametrize \( T_D(M) \). Thus \( T_D(M) \) is homeomorphic to a Euclidean space \( \mathbb{R}^d \) and we prove that \( d \) is equal to the number of edges minus the number of vertices of \( D \).

By Mostow rigidity theorem, if \( h, h' \) are ideal hyperbolic structures on \( M \) and the angle around each edge of \( D \) is equal to 2\( \pi \), then \( h, h' \) represent the same element in \( T(M) \). In this work we give examples of 3–manifolds equipped with ideal hyperbolic structures and we show that all these structures are parametrized by the angles around the edges of \( D \). An interesting problem for further investigation, is to consider ideal hyperbolic structures on \( M \), i.e. complete metrics \( h \) in the interior of \( M \) of curvature \( \leq -1 \), and examine if the angles around the edges of \( D \) uniquely determine \( h \) as an element of \( T_D(M) \).

2 Definitions and Preliminaries

In his pioneering work [11], Thurston constructed a hyperbolic structure on the complement of certain knots by realizing them as a union of finitely many ideal hyperbolic tetrahedra. In the present paper, inspired from Thurston’s work and from the work of other mathematicians, see for example [3], [4], we glue a finite number of ideal hyperbolic tetrahedra and we consider, in the resulting manifold, hyperbolic structures in a broader sense.

Definition 1 Assume that \( M \) is a compact, orientable 3–manifold with \( \partial M \neq \emptyset \). A topological ideal triangulation of \( M \) consists of two finite sets \( D \) and \( F \) which satisfy the following two conditions:

(1) Each element \( \Delta \in D \) is a standard tetrahedron and each element \( f \in F \) is a simplicial homeomorphism \( f : A \to B \), where \( A \) and \( B \) are triangular faces of two tetrahedra \( \Delta \) and \( \Delta' \) of \( D \). The elements of \( F \) are called gluing maps and they are orientation reversing simplicial maps. Furthermore, for each face \( A \) of a tetrahedron \( \Delta \in D \), there exists precisely one \( f \in F \) and a face \( B \) of some tetrahedron \( \Delta' \in D \); such that \( f \) maps \( A \) onto \( B \) or \( B \) onto \( A \).
(2) If \( Y \) is the quotient space of the disjoint union of all tetrahedra in \( \mathcal{D} \) by the relation which identifies any two points \( x \in A \) and \( y \in B \) by a map \( f \in \mathcal{F} \) satisfying \( f(x) = y \) and if we remove from \( Y \) all vertices of tetrahedra then we obtain a space homeomorphic to the interior \( \text{Int}(M) \) of \( M \).

The subdivision of \( \text{Int}(M) \) into tetrahedra of \( \mathcal{D} \) with its vertices deleted, will be called topological ideal triangulation of \( M \) and will be also denoted by \( \mathcal{D} \). Each tetrahedron \( \Delta \in \mathcal{D} \) will be called an ideal tetrahedron. A face (resp. an edge) of some \( \Delta \) in \( \mathcal{D} \) will be called a face (resp. an edge) of \( \mathcal{D} \). The deleted vertices of \( Y \) will be called ideal vertices of \( \text{Int}(M) \) or vertices of \( \mathcal{D} \).

Remark A more accurate picture is obtained, rather than by removing the vertices, by truncating the tetrahedra; that is, by removing an open neighborhood of each vertex of tetrahedra. Then we recover, not only \( \text{Int}(M) \), but the whole \( M \) by these truncated tetrahedra.

Henceforth, for each manifold \( M \) we will denote by \( M^{o} \) its interior. We shall be interested in metrics \( h \) on \( M^{o} \) which are obtained in the following manner: each tetrahedron \( \Delta \in \mathcal{D} \) is equipped with a metric which makes it isometric to an ideal hyperbolic tetrahedron. These tetrahedra are glued among them along isometries and thus \( M^{o} \) is equipped naturally with the length metric. The subdivision of \( M^{o} \) into ideal hyperbolic tetrahedra will be called hyperbolic ideal triangulation of \( M \) and will also be denoted by \( \mathcal{D} \). The length metric \( h \) on \( M^{o} \) will be called an ideal metric.

If \( e \) is an edge of \( \mathcal{D} \), we denote by \( \theta_{h}(e) \) the sum of all dihedral angles formed by the faces of ideal hyperbolic tetrahedra which have \( e \) as a common edge. Then we distinguish two cases:

1. If \( \theta_{h}(e) \neq 2\pi \), the edge \( e \) is called singular or axis.
2. If \( \theta_{h}(e) = 2\pi \) the edge \( e \) is called regular. \( \theta_{h}(e) \) will be called the angle around the edge \( e \).

We shall henceforth assume that the metric \( h \) on \( M^{o} \) is complete. This metric \( h \) has singularities along the axes of \( M^{o} \) which will be called axial singularities. Besides the axes of \( M^{o} \), the curvature of \( h \) is constant, equal to \(-1\). Such a complete metric \( h \) will be called an ideal structure on \( M \), with respect to \( \mathcal{D} \). In what follows the topological ideal triangulation \( \mathcal{D} \) will be fixed so the specification “with respect to \( \mathcal{D} \)” is omitted.

The completeness of \( h \) imposes some restrictions on the gluing maps, which can easily be described in terms of a geometrical property at the ideal vertices or cusps of \( M^{o} \). (When an ideal metric \( h \) is considered on \( M^{o} \), an ideal vertex of \( M^{o} \) will be also referred as cusp of \( M^{o} \).) For each cusp \( v \) of \( M^{o} \), we can associate a natural foliation of a subset of \( M^{o} \) (a “neighborhood” of \( v \)). The definition is as follows. Consider an ideal hyperbolic tetrahedron \( \Delta \) in \( \mathcal{D} \), having \( v \) as one of its ideal vertices. Consider a foliation of a horoball neighborhood of \( v \) in \( \Delta \), whose leaves are pieces of horodiscs which are centered at \( v \). Then, \( (M^{o}, h) \) is complete as a metric space if and only if, the horodiscs on each ideal hyperbolic tetrahedron abutting at \( v \), fit together properly so that they form a product foliation \( o \) which will be called an axial singularities.

Besides the axes of \( M^{o} \) we will denote by \( \mathcal{D}^{o} \) those \( \mathcal{D} \) with deleted vertices. An \( \mathcal{D}^{o} \) is complete as a metric space if and only if, the horodiscs on each ideal hyperbolic tetrahedron abutting at \( v \), fit together properly so that they form a product foliation \( o \) which will be called an ideal structure on \( M \), with respect to \( \mathcal{D} \). In what follows the topological ideal triangulation \( \mathcal{D} \) will be fixed so the specification “with respect to \( \mathcal{D} \)” is omitted.

The completeness of \( h \) imposes some restrictions on the gluing maps, which can easily be described in terms of a geometrical property at the ideal vertices or cusps of \( M^{o} \). (When an ideal metric \( h \) is considered on \( M^{o} \), an ideal vertex of \( M^{o} \) will be also referred as cusp of \( M^{o} \).) For each cusp \( v \) of \( M^{o} \), we can associate a natural foliation of a subset of \( M^{o} \) (a “neighborhood” of \( v \)). The definition is as follows. Consider an ideal hyperbolic tetrahedron \( \Delta \) in \( \mathcal{D} \), having \( v \) as one of its ideal vertices. Consider a foliation of a horoball neighborhood of \( v \) in \( \Delta \), whose leaves are pieces of horodiscs which are centered at \( v \). Then, \( (M^{o}, h) \) is complete as a metric space if and only if, the horodiscs on each ideal hyperbolic tetrahedron abutting at \( v \), fit together properly so that they form a product foliation \( K \times \{ t \} \), \( t \in [0, \infty) \) defined in a “horoball neighborhood” \( V \) of \( v \) in \( M^{o} \). The complete metric \( h \) defined on \( M^{o} \), induces on every fiber \( K_{t} = K \times \{ t \} \), which is a closed surface, a Euclidean structure \( h_{t} \) with conical singularities, see [12] for the precise definition and for a thorough discussion of structures \( h_{t} \). In fact, each \( K_{t} \) is naturally triangulated by the horospherical section of the ideal tetrahedra. The conical singularities arise exactly at the points where the singular edges (axes) intersect \( K_{t} \). These sections are Euclidean triangles and, since \( h \) is complete, they are glued by isometries. Obviously, for each \( t \in [0, \infty) \), \( h_{t} \) is a rescaling of the metric \( h_{0} \). A surface \( S \) which coincides with some \( K_{t} \), \( t \in [0, \infty) \) will be referred to as the geometrical link of \( v \), with respect to \( h \).

3 The Teichmüller space of the 2–skeleton of a hyperbolic ideal triangulation

Consider the topological ideal triangulation \( \mathcal{D} \) of \( M \) and let \( \mathcal{D}^{(2)} \) be the 2–skeleton of \( \mathcal{D} \). Let \( h \) be an ideal structure on \( M^{o} \). With respect to \( h \), every tetrahedron of \( \mathcal{D} \) with its vertices deleted becomes an ideal hyperbolic tetrahedron and so every face of \( \mathcal{D}^{(2)} \) is isometric to an ideal hyperbolic triangle. Denote by \( X = |\mathcal{D}^{(2)}| \) the support of \( \mathcal{D}^{(2)} \) and let \( \overline{h} \) be the metric induced by \( h \) on \( X \). Then \( X \) equipped with \( \overline{h} \), is an ideal 2–dimensional simplicial complex in the sense of [2 Def. 3.1]. An edge of \( X \) is a 1–simplex of \( \mathcal{D}^{(2)} \) and it is isometric to a line. A face of \( X \) is a 2–simplex of \( \mathcal{D}^{(2)} \) and it is isometric to a hyperbolic ideal triangle. The deleted vertices of \( \mathcal{D}^{(2)} \) are called cusps of \( X \).

Let \( v \) be a cusp of \( X \). The link \( \Gamma = \Gamma(v) \) of \( v \) in \( X \) is a simplicial graph embedded in \( X \), which is defined by taking one vertex on each half-edge of \( X \) abutting on \( v \) and then joining two such vertices by an edge contained in a face of \( X \). Notice also that there exists a closed neighborhood \( V \) in \( X \) “centered”
at $v$ which has a natural structure of a geometric cone $v \cdot \Gamma - \{v\}$ (see Def. 2.2 in [1]). We will be calling $V$ a neighborhood of $v$. We have the following lemma.

**Lemma 2** The ideal metric $h$ on $M^\circ$ is complete if and only if the induced metric $\overline{h}$ is complete on $X$.

**Proof.** First assume that $h$ is complete. Since $X$ is a closed subset of $M^\circ$ we have that the induced metric $\overline{h}$ on $X$ is complete. Assume now that $\overline{h}$ is complete. Then, for each cusp $v$ of $X$ the horocycles on each ideal triangle which have $v$ as an ideal vertex, fit together properly so that they form, in a neighborhood of $v$, a connected graph whose edges are horocycle segments, see Proposition 3.4.18 in [10]. Actually, this proposition is proven for cusped surfaces but the same method of proof applies for ideal 2-dimensional simplicial complexes. This implies that the horospherical sections of every ideal hyperbolic tetrahedron on a neighborhood of $v$, fit together forming a closed surface which is the geometrical link of $v$. From the discussion in the section above, we deduce that $h$ is complete.

By considering various ideal structures $h$ on $M$ we obtain various ideal hyperbolic structures $\overline{h}$ on $X$ i.e. complete metrics such that $(X, \overline{h})$ is a local CAT($-1$) space, see Prop. 1.4 of [1]. This leads us to consider the Teichmüller space $\mathcal{T}(X)$ of $X$ and relate it to the generalized Teichmüller space $\mathcal{T}_D(M)$ of $M$ which will be defined in the next section.

We recall the definition of $\mathcal{T}(X)$ (see Def. 2.1 in [1]).

**Definition 3** The Teichmüller space $\mathcal{T}(X)$ of $X$ is the set of equivalence classes of ideal hyperbolic structures on $X$; such two structures $h$ and $h'$ are considered equivalent if there is a homeomorphism $F : X \rightarrow X$ which preserves each edge and each face of $X$ and which satisfies $F^*(h) = h'$, where $F^*(h)$ denotes the pull-back of the metric $h$ via $F$. Remark that such an $F$ is isotopic to the identity map.

Let $T$ be an ideal hyperbolic triangle. Then $T$ has a distinguished point which is the barycentre of $T$. Each edge of $T$ is also equipped with a distinguished point, namely, the foot of the perpendicular drawn from the barycentre of $T$ to that edge. We shall call this point the centre of the edge.

There are several ways of describing the topology of $\mathcal{T}(X)$, and we shall use here the shift parameters. Let $X$ be an ideal 2-dimensional simplicial complex equipped with an ideal hyperbolic structure $h$ and let $V$ be the set of cusps of $X$. In order to describe the shift parameters, we start by choosing once and for all an orientation on each edge of $X$. If $T, T'$ are two faces of $X$ with $e \subset T \cap T'$, we define the quantity $x_h(T, T', e)$ as the algebraic distance on $e$ from the centre $p$ of $e$ associated to $T$ to the centre $p'$ of $e$ associated to $T'$, and we call it the shift parameter on the ordered triad $(T, T', e)$. More precisely, if the direction from $p$ to $p'$ coincides with the orientation of $e$ then $x_h(T, T', e)$ is positive, otherwise it is negative (see Def. 3.2 in [2]).

Let $\mathcal{B}$ be the set of ordered triads $(T, T', e)$ where $T, T'$ are triangles of $X$. The shift parameter defines a map $\mathcal{I} : \mathcal{T}(X) \rightarrow \mathbb{R}^\mathcal{B}$, by the formula

$$\mathcal{I}(h)(T, T', e) = x_h(T, T', e).$$

The map $\mathcal{I}$ is clearly injective but not necessarily onto. We equip $\mathcal{T}(X)$ with the topology induced from the embedding $\mathcal{I} : \mathcal{T}(X) \rightarrow \mathbb{R}^\mathcal{B}$. Thus, $\mathcal{T}(X)$ is parametrized by the shift parameters of the elements of $\mathcal{B}$. These shift parameters satisfy certain linear equations and therefore $\mathcal{T}(X)$ is homeomorphic to a Euclidean space. By induction on the number of triangles of $X$ we obtain the following Proposition, which is stated without proof in [1].

**Proposition 4** Let $d_0$ be the number of gluing maps $\phi$ appearing in the construction of $X$ and let $r_i$ be the rank of $\pi_1(\Gamma_i)$. Then $\mathcal{T}(X)$ is homeomorphic to a Euclidean space and its dimension is equal to $d_0 - \sum_{i=1}^k r_i$.

## 4 The generalized Teichmüller space of $M$

In this section we will prove that if $h'$ is an ideal structure on $M$ such that $\theta_{h'}(e) \geq 2\pi$ for each edge $e$ of $\mathcal{D}$ then any other ideal structure $h$ on $M$ is hyperbolic in the sense of Gromov [6]. After that, we will define the generalized Teichmüller space of $M$ and we will compute its dimension.

A map $f : X \rightarrow Y$ between metric spaces is Lipschitz if and only if there is a constant $K > 0$ such that

$$d_Y(f(x), f(y)) \leq Kd_X(x, y) \text{ for all } x, y \text{ in } X.$$  

If $f$ is a homeomorphism then $f$ is called bi-Lipschitz if $f$ and $f^{-1}$ are Lipschitz.
Let $h$ be an ideal structure on $M$. Denote by $(M^o, h)$ the interior of $M$ equipped with the complete metric $h$ and by $(\hat{M}^o, \hat{h})$ the universal covering of $M^o$ equipped with a metric $\hat{h}$ such that the covering projection $\pi : (\hat{M}^o, \hat{h}) \to (M^o, h)$ is a local isometry. Let $h, h'$ be two ideal structures on $M$. We have the following proposition.

**Proposition 5**  
(1) The identity map $Id : (M^o, h) \to (M^o, h')$ is bi-Lipschitz.  
(2) There is a lifting $\hat{Id} : (\hat{M}^o, \hat{h}) \to (\hat{M}^o, \hat{h}')$ which is bi-Lipschitz.

**Proof.**  
(1) Let $\Delta$ be a tetrahedron of $D$. Obviously $Id|_\Delta$ sends $(\Delta, h)$ onto $(\Delta, h')$. Furthermore, using appropriate coordinates, $Id|_\Delta$ preserves the hyperbolic height on the link of each vertex of $\Delta$. This immediately implies that $Id|_\Delta$ is bi-Lipschitz.

The spaces $(M^o, h)$ and $(\hat{M}^o, \hat{h}')$ are geodesic. Let $x, y$ be two arbitrary points of $M^o$ and let $\gamma_h[x, y]$ be a geodesic segment joining $x$ and $y$, which realizes the distance of $x, y$ with respect to $h'$. Let $x_1 = x, x_2, ..., x_n = y$ be a subdivision of $\gamma_h[x, y]$ such that $x_i, x_{i+1}$ belong to the same ideal tetrahedron of $D$ for each $i$. Then $h(x, y) \leq \sum_i h(x_i, x_{i+1})$ and since $Id|_\Delta$ is bi-Lipschitz we have that there exists $K > 0$ such that

$$h(x, y) \leq \sum_i h(x_i, x_{i+1}) \leq \sum_i K h'(x_i, x_{i+1}) = K h'(x, y)$$

This last inequality proves that $Id$ is Lipschitz. Similarly we prove that $\hat{Id}$ is bi-Lipschitz.  
(2) Obviously $Id$ lifts to Lipschitz homeomorphism $\hat{Id} : (M^o, h) \to (\hat{M}^o, \hat{h}')$ which is also bi-Lipschitz. \(\square\)

Let now $H(M)$ be the set of all ideal structures on $M$.

**Corollary 6** Assume that there exists $h' \in H(M)$ such that that $\theta_{h'}(e) \geq 2\pi$ for each edge $e$ of $D$. Then for any metric $h \in H(M)$ the metric space $(\hat{M}^o, \hat{h})$ is hyperbolic in the sense of Gromov.

**Proof.** The metric space $(M^o, h')$ is of curvature less than or equal to $-1$ i.e. $(M^o, h')$ satisfies locally the $\text{CAT}(-1)$ inequality (see Thm. 3.13 in [8]). Therefore $(\hat{M}^o, \hat{h})'$ is a $\text{CAT}(-1)$ space (see [8], page 119) which implies that $(\hat{M}^o, \hat{h})$ is hyperbolic in the sense of Gromov. Now, from Proposition 5 there is a bi-Lipschitz mapping between $(\hat{M}^o, \hat{h})$ and $(\hat{M}^o, \hat{h}')$. Therefore $(\hat{M}^o, \hat{h})$ is hyperbolic in the sense of Gromov, see Thm. 2.2 in [7]. \(\square\)

Such an ideal structure $h$ will be referred to as an ideal hyperbolic structure. So, the previous lemma asserts that if $M$ admits an ideal hyperbolic structure then all ideal structures on $M$ are hyperbolic. For these manifolds $M$ we will define the generalized Teichmüller space $T_D(M)$ of $M$.

**Definition 7** The generalized Teichmüller space $T(M)$ of $M$ is the set of equivalence classes of ideal hyperbolic structures on $M$; such two structures $h$ and $h'$ are considered equivalent if there is a homeomorphism $F : M^o \to M^o$ which preserves each ideal tetrahedron, each face and each edge of $D$ and which satisfies $F^\ast(h) = h'$.

Let $h \in T_D(M)$. If $\pi$ is the induced metric on $X = |D(2)|$ then, from the definition of spaces $T_D(M)$ and $T(X)$ and from Lemma 2 we may immediately deduce that the mapping $\Psi : T_D(M) \to T(X)$ which sends the equivalence class of $h$ in $T_D(M)$ to the equivalence class of $\pi$ in $T(X)$ is a bijection. Therefore, we may equip $T_D(M)$ with the topology induced from $\Psi$, so that $\Psi$ becomes a homeomorphism and $T_D(M)$ homeomorphic to $\mathbb{R}^d$ for some $d$.

We have the following theorem.

**Theorem 8** The generalized Teichmüller space $T_D(M)$ of $M$ is homeomorphic to $\mathbb{R}^d$, where $d$ is equal to the number of edges minus the number of vertices of $D$.

**Proof.** Let $E$, $V$ and $F$ be the set of edges, vertices and faces, respectively, of the topological ideal triangulation $D$ of $M$.

Consider an element $h \in T_D(M)$. For every cusp $v_i$ of $M^o$, let $S^i_h$ be the geometrical link of $v_i$ with respect to $h$. $D$ induces a loose Euclidean triangulation $D^i_c$ by Euclidean pseudo-triangles on every $S^i_h$, i.e. each pseudo-triangle of $D^i_c$ is isometric to a Euclidean triangle which may have two or three vertices indentified to one point. Remark also that the triangles of $D^i_c$ can be multiply incident to each other. Denote by $T_i, A_i, K_i$ the sets of triangles, edges, vertices of $S^i_h$ respectively. Denote also by $T, A, K$ the sets of all triangles, edges and vertices of $S = \cup_i S^i_h$. 
Now, for the number $d_0$ of Proposition 4 we have that,

$$d_0 = 3\text{card}(F) - \text{card}(E)$$  \hspace{1cm} (1)

This follows from the fact that, if an edge $e$ of $X$ belongs to $n$ different faces of $D$, then the number of gluing isometries that identify these faces along $e$ is $n - 1$.

Now, every edge of $D$ intersects $S$ in two vertices and every face of $D$ intersects $S$ in three edges. Therefore, we have that, \text{card}(F) = \text{card}(A)$ and \text{card}(E) = \text{card}(K). By replacing these relations to (1) we have that,

$$d_0 = \text{card}(A) - \frac{\text{card}(K)}{2}$$ \hspace{1cm} (2)

Now, from Proposition 4 we have that $\text{dim} T(X) = d_0 - \sum_{i=1}^{k} r_i$. On the other hand, a well known fact from graph theory asserts that $r_i = \text{card}(A_i) - \text{card}(K_i) + 1$, $\forall i$. Therefore, from relation (2) we have that $\text{dim} T(X) = \frac{\text{card}(K)}{2} - \text{card}(V) = \text{card}(E) - \text{card}(V)$ which proves the theorem.  

![Figure 1](image.png)

**5 The angles of axes**

Let $\Delta$ be an ideal hyperbolic tetrahedron in $\mathbb{H}^3$ which has an ideal vertex at $\infty$. We equip the edges of $\Delta$ with an orientation such that the edges $e_1$, $e_2$, $e_3$ abutting on $\infty$ are oriented towards $\infty$. Let also $\alpha$, $\beta$, $\gamma$ be the dihedral angles of $\Delta$ corresponding to $e_1$, $e_2$, $e_3$ and let $T_1 = (C, B, \infty)$, $T_2 = (A, C, \infty)$, $T_3 = (A, B, \infty)$, see Figure 1.

The tetrahedron $\Delta$ is parametrized by the angles $\alpha$, $\beta$, $\gamma$ which satisfy the relations $0 < \alpha, \beta, \gamma < \pi$ and $\alpha + \beta + \gamma = \pi$. Therefore, each $(\alpha, \beta, \gamma)$ determines a unique point in the interior of a triangle $T \subset (0, \pi)^3$ whose vertices are the points $(\pi, 0, 0)$, $(0, \pi, 0)$, $(0, 0, \pi)$. On the other hand, the boundary $\partial \Delta$ of $\Delta$ equipped with the hyperbolic metric from $\Delta$, say $h$, is an ideal 2-dimensional simplicial complex which is homeomorphic to the sphere $S^2 - \{3 \text{ points}\}$. If we consider the shift parameters $\xi_1 = x_h(T_3, T_1, e_1)$, $\xi_2 = x_h(T_1, T_3, e_2)$, $\xi_3 = x_h(T_2, T_1, e_3)$, then, from Proposition 4 the hyperbolic metric $h$ on $\partial \Delta$ is parametrized by two of them, say $\xi_1$, $\xi_2$. This implies that $\Delta$ is also parametrized by $\xi_1$, $\xi_2$. It is not difficult to express analytically $\alpha$, $\beta$, $\gamma$ as a function of $\xi_1$, $\xi_2$, $\xi_3$ and inversely. Therefore, we may derive the existence of a diffeomorphism $\phi : \mathbb{R}^2 \to \text{Int}(T)$ which can be chosen to send $\xi_1$, $\xi_2$ to $\alpha$, $\beta$. The expression of the angles $\alpha$, $\beta$, $\gamma$ as a function of $\xi_1$, $\xi_2$, $\xi_3$ is indicated below.
Assume that $\Delta$ is projected to a Euclidean triangle $ABC$ in the $(x, y)$-plane and the angle at the vertex $A$ (resp. $B, C$) of $ABC$ is equal to $\alpha$ (resp. $\beta, \gamma$), see Figure 1. Assuming, without loss of generality, that the Euclidean length of $BC$ is equal to 1, we have that

$$|AB| = \frac{\sin \gamma}{\sin \alpha}, \quad |AC| = \frac{\sin \beta}{\sin \alpha}$$

The shift parameters $\kappa_1, \kappa_2, \kappa_3$ are given by the formulas:

$$\kappa_1 = \log \frac{\sin \beta}{\sin \alpha} - \log \frac{\sin \gamma}{\sin \alpha}, \quad \kappa_2 = \log 1 - \log \frac{\sin \beta}{\sin \alpha}, \quad \kappa_3 = \log \frac{\sin \alpha}{\sin \beta}$$

Therefore,

$$\kappa_1 = \log \frac{\sin \beta}{\sin \gamma}, \quad \kappa_2 = \log \frac{\sin \gamma}{\sin \alpha}, \quad \kappa_3 = \log \frac{\sin \alpha}{\sin \beta} \quad (3)$$

Now we have

$$e^{\kappa_2} = \frac{\sin \gamma}{\sin \alpha} = \frac{\sin(\pi - \alpha - \beta)}{\sin \alpha} = \cos \beta + \cos \alpha \frac{\sin \beta}{\sin \alpha} \implies e^{\kappa_2} = \cos \beta + \cos \alpha \cdot e^{-\kappa_3} \quad (4)$$

From relation (3) and replacing $\cos \beta$ from (4) we have that

$$\cos \alpha = \frac{e^{2\kappa_2} + e^{-2\kappa_3} - 1}{2e^{\kappa_2 - \kappa_3}} = \frac{e^{\kappa_2 + \kappa_3} + e^{-\kappa_2 - \kappa_3} - e^{\kappa_3 - \kappa_2}}{2}$$

Therefore

$$\alpha = \text{Arc} \cos \left( \frac{e^{\kappa_2 + \kappa_3} + e^{-\kappa_2 - \kappa_3} - e^{\kappa_3 - \kappa_2}}{2} \right)$$

In a similar way we may express $\beta$ and $\gamma$ as a function of $\kappa_2$ and $\kappa_3$.

Now, fix an ideal hyperbolic structure $h$ on $M$ and let $X = [\mathcal{D}^{(2)}]$. Let $e_i, \; i = 1, \ldots, n$ be the edges of $M^o$ and let $\theta_h(e_i)$ be the angle around the edge $e_i$. From the discussion above the angles $\theta_h(e_i)$ can be expressed as a function of shift parameters of $X$, but it is difficult to express the shift parameters as a function of the angles $\theta_h(e_i)$. Therefore it is an interesting problem to see, at least in some cases, whether or not $\theta_h(e_i)$ determine the hyperbolic structure $h$. In the next section we give examples which explore this problem.

In the following proposition we investigate the linear relation among the angles $\theta_h(e_i)$.

**Proposition 9** Let $h$ be a hyperbolic metric on $M^o$. Assuming that dim $\mathcal{D}(M) = d$, we may choose $d$ edges $e_1, \ldots, e_d$ such that $\theta_h(e_1), \ldots, \theta_h(e_d)$ determine all $\theta_h(e)$ for each edge $e$.

**Proof.** The space $M^o$ is obtained by gluing ideal hyperbolic tetrahedra by isometries along their faces. Recall that $M^o$ does not have boundary, i.e. each face of a tetrahedron is glued necessarily with another face.

Our lemma will be proved by induction on the number of pairs of faces which are glued together in order to construct $M^o$. For this reason, we are obliged to prove the lemma in a more general context. We consider spaces $N$ which are constructed as follows:

(1) $N$ is always obtained by gluing ideal tetrahedra but we permit $N$ to have free faces i.e. faces which is not glued to another face.

(2) $N$ is complete but not necessarily connected.

Let $v_1, \ldots, v_l$ be the cusps of $N$ and let $S^i_1, \ldots, S^i_l$ be the geometrical links of these cusps. Let $S_h = \cup_i S^i_h$. Each edge $e$ of $\mathcal{D}$ intersects $S_h$ into two points which appear as vertices of $S_h$. Let $s$ be such an intersection point of $e$ with $S_h$. If $\theta(s)$ denotes the angle around $s$ in $S_h$, then $\theta(s) = \theta_h(e)$. For each $i$, $S^i_h$ is a compact Euclidean surface with conical singularities, probably with boundary. Let $s_1^i, \ldots, s_n^i, \; i = 1, 2, \ldots, l$ be the vertices of each $S^i_h$. Then we may prove easily that

$$\sum_{j=1, \ldots, n_i} \phi_i^j = \pi \chi(DS^i_h), \forall S^i_h \quad (*)$$

where,
\[ \phi_i^j = \begin{cases} \pi - \theta(s_i^j), & \text{if } s_i^j \in \partial S_h^i \\ 2\pi - \theta(s_i^j), & \text{otherwise} \end{cases} \]

and \(DS_h^i\) is the double of the surface \(S_h^i\) obtained by gluing two copies of \(S_h^i\) along their boundaries provided that \(\partial S_h^i \neq \emptyset\).

Remark that for each variable \(\phi_i^j\) in the set of variables \(\{\phi_i^j\}_{i,j}\) there exists exactly one variable \(\phi_i^{j2}\) equal to \(\phi_i^{j1}\). These two variables correspond to vertices of \(S_h = \bigcup_i S_h^i\) which are induced by the same edge of \(N\).

In order to prove our lemma it suffices to prove that the \(l\) equations \((*)\) are linearly independent. Actually, the constants in the right hand side of equations \((*)\) do not affect the linear independence of them. So, it is sufficient to prove that the sums \(\sum_{j=1,\ldots,n} \phi_i^j, \ i = 1,\ldots,l\) are linearly independent in the sense that we cannot obtain one of them as a linear combination of the others. We will prove the lemma by induction on the number of gluings between faces of ideal tetrahedra.

Let \(n = 0\). This means that we do not have any gluing between tetrahedra. So, it suffices to examine the case where our space consists only of one tetrahedron, because the variables that correspond to distinct tetrahedra are distinct.

For an ideal hyperbolic tetrahedron \(\Delta = v_1v_2v_3v_4\) the left hand side of our equations are

\[
\begin{align*}
\varphi_1 + \varphi_2 + \varphi_3 \\
\varphi_1 + \varphi_2 + \varphi_3 \\
\varphi_4 + \varphi_2 + \varphi_3 \\
\varphi_4 + \varphi_2 + \varphi_3
\end{align*}
\]

Therefore, the matrix that correspond to these variables is

\[
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 0
\end{bmatrix}
\]

which has rank 4.

Now, let \(\{\sum_j \phi_i^j\}_i\) be the set of sums which correspond to a space \(N\). For each \(i\) we correspond a row and we assume that all rows are linearly independent. We will show that if we glue two free faces of tetrahedra so that the resulting space is complete the corresponding set of sums, say \(\{\sum_m \tilde{\phi}_n^m\}_n\) is linearly independent.

We remark that the set \(\{\sum_m \tilde{\phi}_n^m\}_n\) results from the relations \(\{\sum_j \phi_i^j\}_i\) by applying successively the following two transformations:

(A) Replace two rows by their sums. That is,

\[
\{\sum_j \phi_i^j\}_i \rightarrow \{\sum_j \phi_i^j\}_{i \neq i_1, i_2} \cup \{\sum_j \phi_{i_1}^j + \sum_j \phi_{i_2}^j\}.
\]

(B) Replace two variables, by a new one, say \(\phi_0\). That is,

\[
\phi_{i_1}^{j1} + \phi_{i_2}^{j2} \rightarrow \phi_0
\]

Remark that \(\phi_{i_1}^{j1} + \phi_{i_2}^{j2}\) can appear either once in two different rows or twice in the same row. Indeed, assuming that two free faces of \(N\) are glued, then two different edges, say \(e', e''\), match together and give a common edge \(e\). In the level of geometrical links of cusps we have the following two possibilities:

(i) two surfaces \(S_{h1}, S_{h2}\) which are the geometrical links of two distinct cusps of \(N\), are glued along an edge in their boundaries or,

(ii) two edges in the boundary of a surface \(S_{h1}\) which is the geometrical link of a cusp of \(N\), are glued together.

In both cases we obtain the relations \(\{\sum_m \tilde{\phi}_n^m\}_n\) from the relations \(\{\sum_j \phi_i^j\}_i\) by applying the rules (A) and (B).

Obviously, every transformation of type (A) or (B) gives a system of linearly independent sums and therefore our proof is complete. \(\square\)
6 Examples

Let $h \in \mathcal{T}_D(M)$ and let $S_h$ be the geometrical link (not necessarily connected) of $M^o$ with respect to $h$. The fixed triangulation $\mathcal{D}$ of $M$ induces on $S_h$ a (fixed) loose triangulation $\mathcal{D}_e$. Every ideal hyperbolic tetrahedron of $\mathcal{D}$ induces four similar pseudo-triangles on $S_h$. Therefore $\mathcal{D}_e$ is equipped with a pattern which indicates which Euclidean pseudo-triangles are pairwise similar. The similarity of these pseudo-triangles imposes a system of equations between the lengths of edges of $\mathcal{D}_e$. The solutions of this system is in 1−1 correspondence with the elements $h \in \mathcal{T}_D(M)$. This gives rise to a parametrization of $\mathcal{T}_D(M)$.

The above idea is used to study $\mathcal{T}_D(M)$ in the examples 2, 3 and 4 below.

Example 1

In the example 3.3.12 of [10], Thurston glues two ideal hyperbolic tetrahedra so that the resulting space is a manifold $N$ with one axis and one cusp. By truncating the tetrahedra we obtain a compact manifold $M$ with boundary and we have that $N$ is homeomorphic to $M^o$. From Theorem 5 the dimension of $\mathcal{T}_D(M)$ is equal to 0. Therefore $M$ admits a unique ideal hyperbolic structure modulo the equivalence relation of Definition 7. It is interesting to remark here that if $\dim(\mathcal{T}_D(M)) = 0$, then the unique hyperbolic structure in $\mathcal{T}_D(M)$ is always obtained by gluing regular ideal hyperbolic tetrahedra.

In [5], all manifolds with one cusp and one axis are constructed by gluing precisely two ideal hyperbolic tetrahedra. Using more than two ideal tetrahedra it is easy to construct manifolds whose the number of edges minus the number of cusps is zero. For all these manifolds, a similar analysis and a similar result as above is valid.
Example 2

Consider two ideal hyperbolic tetrahedra $\Delta_i = A_i B_i C_i D_i$, $i = 1, 2$, whose edges are labeled by the letters $b$ and $c$ and directed as it is shown in Figure 2(a). There is a unique way to glue, via isometries, the faces of $\Delta_1, \Delta_2$ so that the directed edges labeled with $b$ (resp. $c$) are identified to an edge, say $b$ (resp. $c$). The obtained manifold $N$ has one cusp and is homeomorphic to the interior $M^o$ of the compact manifold $M = S^3 - V$, where $V$ is an open tubular neighborhood of the figure eight knot in the sphere $S^3$, see Example 1.4.8 in [10]. From Theorem 8 the dimension of $T(M)$ is equal to 1.

Fix an arbitrary ideal hyperbolic structure $h$ on $M$ and let $S_h$ be the geometrical link of $v$. The surface $S_h$ is obtained from the polygon of Figure 2(b), by identifying the following directed segments:

$$AB \equiv KI, \ BC \equiv IO, \ CD \equiv OH, \ DE \equiv HZ, \ EZ \equiv AK.$$
We use the following notation: if $A, B, \Gamma$ are the vertices of a Euclidean triangle $AB\Gamma$ and $|AB|, |A\Gamma|, |B\Gamma|$ the lengths of its sides, then we set $|AB\Gamma| = (|AB|, |B\Gamma|, |A\Gamma|)$. Remark now that all four triangles in $S_h$ induced by the tetrahedron $\Delta_1$ (resp. $\Delta_2$) are similar. Therefore there are positive numbers $\lambda, \mu, \nu, l, m, n$ such that we have the following system of equations:

$$
|KBA| = \lambda|BIC| = \mu|OIC| = \nu|DHE|
$$

$$
|IBK| = l|CIO| = m|IDO| = n|EZD|
\tag{5}
$$

We may assume that $|AB| = 1$ and $|KB| = r$. Then, using the equalities\footnote{5} we may express successively all quantities in the system\footnote{5} as a function of the parameter $r$, as follows:

$|AB| = |KI| = |BC| = |IO| = |CD| = |OH| = |DE| = |HZ| = m = \mu = 1$

$|KB| = |IC| = |OD| = |HE| = l = n = \lambda = \nu = r$

$|BI| = |DH| = r^2$, $|AK| = |EZ| = |CO| = \frac{1}{r}$

Now, it is immediate to verify that the previous expressions of $r$ verify all the equations of system\footnote{5} Also, from Theorem\footnote{5} we know that the dimension $d$ of $T_D(M)$ is equal to one. This implies that $T_D(M)$ can be parametrized by the parameter $r$.

We also deduce that all triangles of $S_h$ are similar. We set $\overrightarrow{AKB} = x$, $\overrightarrow{ABK} = y$, $\overrightarrow{BAK} = z$ see Figure 2(b). The parameters $1, r, \frac{1}{r}$ are lengths of sides of a Euclidean triangle. Therefore these quantities must satisfy the triangle inequalities which imply that $\sqrt{\frac{5}{2}} - 1 < r < \sqrt{\frac{5}{2}}$. If $r = 1/r$, i.e. $r = 1$ we deduce that all triangles of $S_h$ are equilateral and therefore the tetrahedra $\Delta_1$ and $\Delta_2$ are regular and the angles around the edges $b$ and $c$ are equal to $2\pi$.

Generally, the angle $\theta$ around the axis $c$ is equal to

$$
\theta = 2x + 2y + 2y = 2(\pi - z) + 2y = 2\pi + 2(y - z) \tag{**}
$$

We consider the function $\varphi(r) = z - y$. Then, from the cosine law, we have

$$
\varphi(r) = \arccos \frac{1 + \frac{1}{r} - r^2}{2\frac{1}{r}} - \arccos \frac{1 + r^2 - \frac{1}{r}}{2r} = \\
\arccos \frac{1}{2}(r + \frac{1}{r} - r^3) - \arccos \frac{1}{2}(r + \frac{1}{r} - r^3)
$$

We have that

$$
\varphi'(r) = -\frac{1}{\sqrt{1 - \frac{1}{4}(r + \frac{1}{r} - r^3)^2}} \frac{1}{2}(1 - \frac{1}{r^2} - 3r^2) + \\
+ \frac{1}{\sqrt{1 - \frac{1}{4}(r + \frac{1}{r} - r^3)^2}} \frac{1}{2}(1 - \frac{1}{r^2} + 1 + 3r^{-4})
$$

But

$$
-1 + \frac{1}{r^2} + 3r^2 = \frac{1}{r^2}(3r^4 - r^2 + 1) > 0
$$

$$
-\frac{1}{r^2} + 1 + 3r^4 = \frac{1}{r^2}(r^4 - r^2 + 3) > 0
$$

Therefore, $\varphi'(r) > 0$ which implies that $\varphi$ is $1 - 1$. Therefore, $r$ determines uniquely the angle $y - z$ and from equation $(**)$, $r$ determines uniquely the angle $\theta$ around the edge $c$. This proves that $T_D(M)$ is parametrized by the angle around the edge $c$ of $N$. 
Example 3

Consider four ideal hyperbolic tetrahedra \( \Delta_i = A_i B_i C_i D_i, \ i = 1, 2, 3, 4 \) whose edges are labeled by the letters \( b \) and \( c \) and directed as it is shown in Figure 3. The faces of \( \Delta_i \) are glued, via isometries, so that the directed edges labeled with the same letter are identified. More precisely the faces are glued as follows:

- \( B_1 C_1 D_1 = C_2 D_2 B_2, A_1 C_1 D_1 = C_2 D_2 A_2, D_2 B_2 A_2 = B_3 A_3 D_3, \)
- \( C_2 B_2 A_2 = B_3 A_3 C_3, B_3 C_3 D_3 = C_4 D_4 B_4, A_3 C_3 D_3 = C_4 D_4 A_4, \)
- \( D_4 B_4 A_4 = B_1 A_1 D_1, A_4 B_4 A_4 = B_1 A_1 C_1. \)

Denote by \( N \) the obtained manifold and we remark that \( N \) has two edges \( b, c \) and one cusp, say \( v \). By truncating the tetrahedra we obtain a compact manifold \( M \) with boundary. We have that \( N \) is homeomorphic to \( M^o \). It is not difficult to show that all the ideal structures on \( M \) are hyperbolic and we will prove that \( T_D(M) \) is parametrized by the angle around an edge of \( N \).

Fix an arbitrary ideal hyperbolic structure \( h \) on \( M \) and let \( S_h \) be the geometrical link of \( v \). We may verify that \( S_h \) is a closed surface obtained from the polygon of Figure 4, by identifying its sides which are labeled by the same number. By computing the Euler characteristic of \( S_h \) we can see that \( S_h \) is a surface of genus three.

\( S_h \) contains four groups of four triangles which are similar because they are induced by the same tetrahedron. Therefore, there are positive numbers \( \lambda, \mu, \nu, l, m, n, r, s, t, \rho, \sigma, \tau \) such that:

\[
\begin{align*}
|IJK| &= \lambda|KML| = \mu|ZVE| = \nu|BAQ| \\
|ABN| &= l|BQP| = m|UKL| = n|VOE| \\
|BTN| &= r|RBP| = s|YPQ| = t|KUS| \\
|BRF| &= \rho|RPW| = \sigma|UGS| = \tau|XYQ|
\end{align*}
\]

On the other hand, due to the identifications of the faces of tetrahedra we have also the following equalities:

\[
\begin{align*}
|MK| &= |LU|, |HJ| = |OE|, |OV| = |SK|, |NA| = |TB|, |TN| = |SG|, |PY| = |QX|, |VZ| = |XY|, \\
|IH| &= |RF|, |AQ| = |RW|, |LM| = |PW|, |EZ| = |GU|, |JI| = |FB|
\end{align*}
\]

Without loss of generality, we may assume that \( |IH| = 1 \). Then, using the above equalities, we may express successively all quantities in the system \( \mathbb{R} \) as a function of the parameter \( r \), as follows:

![Figure 4]
Example 4

Consider four ideal hyperbolic tetrahedra $\Delta_i = A_iB_iC_iD_i$, $i = 1, 2, 3, 4$ whose edges are labeled by the letters $e, b, c, d$ and directed as it is shown in Figure 5. There is a unique way to glue, via isometries, the faces of $\Delta_i$ so that the directed edges labeled with the same letter are identified. The obtained manifold $N$ is homeomorphic to the interior $M^\circ$ of the compact manifold $M = S^3 - V$, where $V$ is an open tubular neighborhood of the Whitehead link in $S^3$ (see p. 452 in [1]). Fix an arbitrary ideal hyperbolic structure

$$|IH| = |KL| = |VE| = |AQ| = |BN| = |BP| = |YQ| = |US| = |RF| = |RW| = \tau = \lambda = m = s = 1$$

$$|IJ| = |KM| = |ZE| = |BQ| = |AN| = |UL| = |VO| = |BT| = |RP| = |YP| = |KS| = |BF| = |UG| = |XQ| = n = \sigma = l = \rho = 1/r$$

$$|ZV| = |BA| = |UK| = |RB| = |XY| = 1/r^2$$

$$|JH| = |ML| = |QP| = |OE| = |TN| = |PW| = |GS| = t = \nu = \mu = r$$

Now, it is immediate to verify that the previous expressions of $r$ verify all the equations of system (6). Also, from Theorem 8 we know that the dimension $d$ of $TD(M)$ is equal to one. This implies that $r$ parametrizes $TD(M)$.

The Euclidean triangles induced on $S_h$ are all similar triangles and the lengths of their sides are either $(1, r, 1/r)$ or $(1, 1/r, 1/r^2)$. Consider the triangle $IJK$, see Figure 3, and set $x = \overline{IJK}$, $y = \overline{JKI}$, $z = \overline{HIJ}$. Therefore, all the triangles in $S_h$ have angles equal to $x, y, z$. Furthermore, we have that $|IJK| = |BQP| = |BTN| = |RPW| = (1/r, r, 1)$. So, the angle around the axis $b$ is easily computed and is equal to $4(x + 2y) = 4\pi + 4(y - z)$.

As is Example 2, we consider the function $\varphi(r) = z - y$ and we prove in the same way that $\varphi'(r) > 0$ and so $\varphi$ is $1 - 1$. Therefore $TD(M)$ is parametrized by the angle around the edge $b$.

![Figure 5](image-url)

Figure 5
h on M. The manifold N has two cusps, say v, w, and four edges so \( \dim(T_D(M)) = 2 \). The geometrical link \( S^h_w \) of w (resp. \( S^v_w \) of v) is a Euclidean torus with conical singularities, see Figure 6(a) (resp. Figure 6(b)). There are four groups of four triangles which are similar because they are induced by the same tetrahedron. Therefore, there are positive numbers \( k_i, l_i, m_i, n_i, i = 1, 2, 3 \) such that:

\[
\begin{align*}
|T_T T_3 T_4| &= k_1 |T_T T_10 Q_2| = k_2 |T_6 T_3 Q_1| = k_3 |S_3 S_5 S_4| \\
|T_T T_3| &= l_1 |T_4 Q_1 T_5| = l_2 |S_3 T_1 S_2| = l_3 |T_7 Q_2 T_8| \\
|Q_2 T_5 T_10| &= m_1 |T_8 T_7 T_5| = m_2 |Q_2 T_6 T_3| = m_3 |S_6 S_3 S_5| \\
|Q_2 T_6 T_9| &= n_1 |T_1 T_3 T_4| = n_2 |S_3 S_2 S_4| = n_3 |T_6 T_5 T_7|
\end{align*}
\]

(7)

![Figure 6](https://via.placeholder.com/150)

Without loss of generality we assume that two edges in \( S^w_h \) and \( S^v_h \) respectively have length equal to 1. For example \( |S_1 S_3| = 1, |T_5 T_6| = 1 \). Using these two relations and the system of equations (7) the lengths of edges of \( S_h = S^w_h \cup S^v_h \) are computed as follows:

- \( |S_4 S_5| = |T_3 T_6| = |T_1 T_4| = |T_1 T_3| = |T_6 T_7| = l_2 = k_3 = 1 \)
- \( |T_1 T_2| = |T_5 T_10| = |T_5 T_6| = |T_5 T_7| = |S_2 S_3| = |S_1 S_4| = |T_3 T_4| = |S_3 S_6| = |S_1 S_2| = t^2 \)
- \( |T_2 S_4| = |T_1 T_3| = |T_5 T_7| = |S_6 S_3| = l_1 = s \)
- \( |T_4 Q_1| = |Q_1 T_6| = m_3 = 1/l_3 = 1/k_2 = m_1 = t/s \)
- \( |T_5 Q_2| = |Q_2 T_3| = |Q_2 T_7| = |T_5 Q_1| = n_3 = 1/k_1 = n_2 = t^2/s \)
- \( |T_6 Q_2| = |Q_2 T_10| = t^4/s \)

Since the above expressions satisfy system\[7\] and \( \dim(T(M)) = 2 \) we deduce that the parameters \( t, s \) parametrize \( T_D(M) \).

The angles \( \theta_h(c), \theta_h(d) \) around the axis \( c \) and \( d \) are respectively equal to

\[
\begin{align*}
\theta_h(c) &= 2(T_1 T_4 T_3 + T_1 T_4 T_3 + T_2 T_4 T_3 + T_2 T_4 T_3 + T_7 T_4 T_3 + T_7 T_4 T_3 + T_6 T_4 T_3) = 2(T_1 T_4 T_3 + T_2 T_4 T_3) = 2(\pi - T_1 T_4 T_3 + T_2 T_4 T_3) \\
\theta_h(d) &= T_1 T_4 T_3 + T_1 T_4 T_3 + T_2 T_4 T_3 + T_2 T_4 T_3 + T_7 T_4 T_3 + T_7 T_4 T_3 + T_6 T_4 T_3 = 2(T_1 T_4 T_3 + T_2 T_4 T_3)
\end{align*}
\]

If we set \( x = T_1 T_4 T_3 + T_1 T_4 T_3 \) and \( y = T_2 T_4 T_3 - T_1 T_4 T_3 \) we have that \( \theta_h(c) = 2(\pi + y) \) and \( \theta_h(d) = 2x \). Obviously \( 0 < x < 2\pi, -\pi < y < \pi \) and the angle around each edge of \( N \) can be expressed as a function of \( x, y \). Now, in order to prove that \( T_D(M) \) is parametrized by the angles \( x \) and \( y \) it is sufficient to prove the following claim.
Claim: The mapping \( \Theta : (t, s) \to (x, y) \) is \( 1 - 1 \).

Proof of Claim. From the expressions of edges of \( S_h \) as a function of \( t, s \) we have that \( |T_1T_3T_4| = (s, t, 1) \) and \( |T_1T_2T_3| = (t, t^2, s) \). We glue the triangles \( T_1T_3T_4 \) and \( T_1T_2T_3 \) by identifying \( T_1T_4 \) with \( T_1T_2 \). Thus, we take a quadrilateral \( ABCD \) such that \( AC \) is a diagonal with \( |AC| = t \) and \( |DCA| = (s, t, 1) \), \( |CAB| = (t, t^2, s) \), \( x = \overline{DAB} \), \( y = \overline{ACB} - \overline{ACD} \), see Figure 7. Consider the perpendicular bisector \( \zeta \) of \( BD \). Then \( C \in \zeta \) and \( AC \) and \( \zeta \) form an angle equal to \( \frac{\pi}{2} \). Let \( E = \zeta \cap AB \).

If \( t \geq 1 \), we have that \( y \geq 0 \). Let \( z = \overline{ABD} \), see Figure 7. From the law of sines in the triangle \( ACE \) we have that

\[
\frac{\sin \frac{y}{2}}{t^2 - \sqrt{t^4 + 1 - 2t^2 \cos x}} = \frac{\cos z}{t}
\]

So,

\[
\sin \frac{y}{2} = \Phi(t, x) = t \sqrt{1 - \frac{\sin^2 x}{t^4 + 1 - 2t^2 \cos x}} - \frac{\sqrt{t^4 + 1 - 2t^2 \cos x}}{2t} = \frac{t(t^2 - \cos x)}{\sqrt{t^4 + 1 - 2t^2 \cos x}} - \frac{\sqrt{t^4 + 1 - 2t^2 \cos x}}{2t} = \frac{1}{2} \left( t^3 - 1 \right) \left( t^4 + 1 - 2t^2 \cos x \right)^{-\frac{1}{2}}
\]

Therefore,

\[
\frac{\partial}{\partial t} \left( \sin \frac{y}{2} \right) = \frac{1}{2} \left( 3t^2 + \frac{1}{t^2} \right) \left( t^4 + 1 - 2t^2 \cos x \right)^{-\frac{3}{2}} + \left( \frac{1}{t} - t^3 \right) \left( t^4 + 1 - 2t^2 \cos x \right)^{-\frac{3}{2}} \left( t^3 - t \cos x \right) = \frac{1}{2} \left( t^4 + 1 - 2t^2 \cos x \right)^{-\frac{3}{2}} \left( 3t^4 + 1 \right) \left( t^4 + 1 - 2t^2 \cos x \right) + 2t^2 \left( 1 - t^4 \right) \left( t^2 - \cos x \right) = \frac{1}{2} \left( t^4 + 1 - 2t^2 \cos x \right)^{-\frac{3}{2}} \left( t^8 + 4t^6 \cos x + 6t^4 - 4t^2 \cos x + 1 \right) \geq \frac{1}{2} \left( t^4 + 1 - 2t^2 \cos x \right)^{-\frac{3}{2}} \left( t^8 - 4t^6 \cos x + 6t^4 - 4t^2 \right) = \frac{1}{2} \left( t^4 + 1 - 2t^2 \cos x \right)^{-\frac{3}{2}} \left( t^2 - 1 \right)^2 \geq 0.
\]

Therefore, we conclude that \( \frac{\partial}{\partial t} \left( \sin \frac{y}{2} \right) > 0 \), \( \forall t \geq 1 \). This implies that the mapping which express \( \sin \frac{y}{2} \) as function of \( t \) is \( 1 - 1 \). On the other hand, we have that \( 0 \leq \frac{y}{2} \leq \frac{\pi}{2} \), therefore the mapping \( f \) which express \( y \) as a function of \( t \) is \( 1 - 1 \).
Now, we assume first that the angles $x$, $y$ satisfy $0 < x < 2\pi$ and $0 \leq y < \pi$. Then, using the fact that $f$ is $1-1$, we may construct a unique quadrilateral $ABCD$, as well as, a point $E \in AB$ such that: $|AD| = 1$, $|CD| = |CB|$, $\angle BAD = x$ and $\angle ACE = \frac{y}{2}$, see Figure 7. Therefore the parameters $t = |AC|$ and $s = |CB|$ are uniquely determined from $x$ and $y$. Therefore $\Theta$ is $1-1$ in this case.

If $t \leq 1$ then $-\pi < y \leq 0$ and we have that $\sin \frac{y}{2} = -\Phi(\frac{1}{2}, x)$. An easy computation shows again that $\frac{\partial}{\partial t}(\sin \frac{y}{2}) > 0$. So in this case, we also have that $t$, $s$ are uniquely determined from $x$ and $y$. Therefore we conclude that $\Theta$ is $1-1$ and the claim is proved.

Finally, the following question arises naturally:

**Question.** Let $M$ be an orientable compact manifold with boundary equipped with an ideal triangulation $\mathcal{D}$ and assume that $\dim(\mathcal{T}_\mathcal{D}(M)) \geq 1$. Do there exist distinct elements $h, h' \in \mathcal{T}_\mathcal{D}(M)$ which have the same angles around the edges of $\mathcal{D}$?

Note that that even if the ideal structures on $M$ are not hyperbolic, the authors do not know examples where these structures are not uniquely determined by the angles around the edges.

**Acknowledgement** The authors would like to thank the referee for his remarks which improved significantly the paper in many ways.

**References**

[1] Ch. Charitos, A. Papadopoulos, *The geometry of ideal 2-dimensional simplicial complexes*, Glasgow Math. J. 43, 39-66, 2001.

[2] Ch. Charitos, A. Papadopoulos, *Hyperbolic structures and measured foliations on 2-dimensional complexes*, Monatsh. Math. 139, 1-17, 2003.

[3] R. Frigerio, B. Martelli, C. Petronio, *Small hyperbolic 3-manifolds with geodesic boundary*, Experiment Math. 13, 171-184, 2004.

[4] R. Frigerio, C. Petronio, *Constructions and recognition of hyperbolic 3-manifolds with geodesic boundary*, Trans. Amer. Math. Soc. 356, 3243-3282, 2004.

[5] M. Fujii, Hyperbolic 3-Manifolds with Totally Geodesic Boundary which are decomposed into Hyperbolic Truncated Tetrahedra, Tokyo J. Math. 13(2), 355-373, 1990.

[6] M. Gromov, *Hyperbolic groups*, MSRI Publications, 1988.

[7] M. Coornaert, T. Delzant, A. Papadopoulos, *Géométrie et théorie des groupes*, Lecture Notes in Mathematics 1441, Springer-Verlag, 1990.

[8] F. Paulin, *Constructions of hyperbolic groups via hyperbolization of polyhedra*, In group theory from a geometrical viewpoint. ICTP, Trieste, Italy, 1990, E. Ghys and A. Haefliger eds, 1991.

[9] J. Ratcliffe, *Foundations of Hyperbolic Manifolds*, Springer, 1994.

[10] W. Thurston, *Three-Dimensional Geometry and Topology*, Edited by Silvio Levy, Princeton University Press, NJ 1997.

[11] W. Thurston, *The Geometry and Topology of 3-Manifolds*, Lecture Notes, Princeton Univ. Princeton, NJ, 1979.

[12] M. Troyanov, *Les surfaces Euclidiennes à singularités coniques*, L’ Enseignement Math. 32, 79-84, 1986.