Automorphisms of relative Quot schemes

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Abstract. Let \( X \rightarrow S \) be a smooth family of projective curves over an algebraically closed field \( k \) of characteristic zero. Assume that both \( X \) and \( S \) are smooth projective varieties and let \( E \) be a vector bundle of rank \( r \) over \( X \) and \( \mathbb{P}(E) \) be its projectivization. Fix \( d \geq 1 \). Let \( Q(E, d) \) be the relative quot scheme of torsion quotients of \( E \) of degree \( d \). Then we show that if \( r \geq 3 \), then the identity component of the group of automorphisms of \( Q(E, d) \) over \( S \) is isomorphic to the identity component of the group of automorphisms of \( \mathbb{P}(E) \) over \( S \). We also show that under additional hypotheses, the same statement is true in the case when \( r = 2 \). As a corollary, the identity component of the automorphism group of flag scheme of filtrations of torsion quotients of \( \mathcal{O}_C^r \), where \( r \geq 3 \) and \( C \) a smooth projective curve of genus \( \geq 2 \) is computed.

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1. Introduction

Let \( p_S : X \rightarrow S \) be a smooth family of projective curves over an algebraically closed field \( k \) of characteristic zero. Assume \( X \) and \( S \) are smooth projective varieties. Let \( E \) be a vector bundle over \( X \) of rank \( r \).

Fix \( d \geq 1 \). Let \( \pi_S : Q(E, d) \rightarrow S \) be the relative Quot scheme [6, Theorem 2.2.4] whose closed points correspond to quotients \( E|_{X_s} \rightarrow B_d, \forall s \in S \), where \( X_s \) is the fibre of \( p_S \) over \( s \), and \( B_d \) is a torsion sheaf over the smooth projective curve \( X_s \) of degree \( d \).

Notation. If \( f : Y \rightarrow Z \) is a smooth morphism between two projective varieties, let us denote the automorphism group of \( Y \) over \( Z \) by \( \text{Aut}(Y/Z) \). It is known that \( \text{Aut}(Y/Z) \) is a group scheme and let its identity component be denoted by \( \text{Aut}^0(Y/Z) \). Let \( T_{Y/Z} \) be the relative tangent bundle of \( f \). Then \( \text{Lie}(\text{Aut}^0(Y/Z)) = H^0(Y, T_{Y/Z}) \) [3, Theorem 3.7].

Let us denote the projective bundle associated to \( E \) by \( \mathbb{P}(E) \). Note that \( Q(E, 1) \cong \mathbb{P}(E) \). Then, we will prove the following theorem.

Theorem 2.5(a). If rank \( E \geq 3 \), then

(i) \( \text{Aut}^0(Q(E, d)) \cong \text{Aut}^0(\mathbb{P}(E)/S) \),
(ii) \( H^0(\mathbb{P}(E), T_{\mathbb{P}(E)/S}) \cong H^0(Q(E, d), T_{Q(E,d)/S}) \).
Let \( \mathcal{O}_X(1) \) be a \( p_S \)-ample line bundle and \( \mathcal{O}_S(1) \) be an ample line bundle on \( S \). Then \( \mathcal{O}_X(1) \otimes \mathcal{O}_S(a) \) is ample over \( X \) for \( a \gg 0 \). Fix such an ample line bundle \( \mathcal{M} \) over \( X \).

Then we have as follows.

**Theorem 2.5(b).** Let \( E \) be a vector bundle of rank 2 over \( X \) such that \( E \) is semistable with respect to \( \mathcal{M} \). Further, assume that genus of each fibre of \( X \to S \) is \( \geq 2 \). Then both (i) and (ii) are also true in this case.

Note that a special case of the above theorem was proved in [2].

In the third section, we will use Theorem 2.5 to compute the automorphism group in certain specific cases.

2. Main Theorem

**Lemma 2.1.**

(i) \( \text{Aut}^o(Q(E, d)) \leftrightarrow \text{Aut}^o(\mathbb{P}(E)/S) \),

(ii) \( H^0(\mathbb{P}(E), T_{\mathbb{P}(E)/S}) \leftrightarrow H^0(Q(E, d), T_{Q(E, d)/S}) \).

**Proof.** Note that by [3, Corollary 2.2], any automorphism \( g \in \text{Aut}^o(\mathbb{P}(E)/S) \) descends to an automorphism \( g' \in \text{Aut}^o(Q(E, d)/S) \). Therefore, we have the following diagram:

\[
\begin{array}{ccc}
\mathbb{P}(E) & \xrightarrow{g} & \mathbb{P}(E) \\
\downarrow^p & & \downarrow^p \\
X & \xrightarrow{g'} & X
\end{array}
\]

Then \( E \cong (g')^*E \otimes p^*L \) for some line bundle \( L \) on \( X \). Let us denote this isomorphism of bundles by \( \Psi_g \).

It is clear that \( \Psi_g \) induces an automorphism of \( Q(E, d) \) by sending \( [E|_{p_S^{-1}(s)} \to B_d \to 0] \) to \( [E|_{p_S^{-1}(s)} \xrightarrow{(g')^*} (E|_{p_S^{-1}(s)} \to B_d \to 0) \otimes L|_{p_S^{-1}(s)}] \). Hence, we have a homomorphism \( \text{Aut}^o(\mathbb{P}(E)/S) \to \text{Aut}^o(Q(E, d)/S) \) and clearly it is injective. Hence, we have a morphism of Lie algebras \( H^0(X, T_{\mathbb{P}(E)/S}) \leftrightarrow H^0(Q(E, d), T_{Q(E, d)/S}) \). \( \square \)

Let \( \mathcal{Z} \) be the fibered product of \( d \) copies of \( \mathbb{P}(E) \) over \( S \). Then

**Theorem 2.2.**

(a) There exists an open subset \( \mathcal{U} \subseteq \mathcal{Z} \) and a dominant map \( \Phi : \mathcal{U} \to Q(E, d) \) over \( S \) such that \( \text{codim}_S(\mathcal{Z} \setminus \mathcal{U}) \geq 2 \).

(b) If either \( r \geq 3 \) or the hypothesis of Theorem 2.5(b) holds, then

(i) \( H^0(\mathcal{U}, \Phi^*T_{Q(E, d)/S}) = \bigoplus_{i=1}^d H^0(\mathbb{P}(E), T_{\mathbb{P}(E)/S}) \),

(ii) The natural map \( H^0(Q(E, d), T_{Q(E, d)/S}) \to H^0(\mathcal{U}, \Phi^*T_{Q(E, d)/S}) = \bigoplus_{i=1}^d H^0(\mathbb{P}(E), T_{\mathbb{P}(E)/S}) \) is an injection and is invariant under permutation of the components of \( \bigoplus_{i=1}^d H^0(\mathbb{P}(E), T_{\mathbb{P}(E)/S}) \), i.e. we have an injection \( H^0(Q(E, d), T_{Q(E, d)/S}) \hookrightarrow H^0(\mathbb{P}(E), T_{\mathbb{P}(E)/S}) \).
By Grauert’s theorem [5, Corollary 12.9], we have
\[ \phi^* T_{Q(E,d)/S} = (\phi^* (p_Q)_* \text{Hom}(A(E,d), B(E,d))) \]
\[ \cong (\pi_1)_* (id_X \times \phi)^* \text{Hom}(A(E,d), B(E,d)). \]

Now, since \( A(E,d) \) is a vector bundle, we have
\[ (id_X \times \phi)^* \text{Hom}(A(E,d), B(E,d)) = \text{Hom}(\phi^* A(E,d), \phi^* B(E,d)). \]
Applying \( \Phi_1 \), we have \( \Phi^*B(E, d) \cong (\bigoplus_{i=1}^d \pi_{2,i}^*\mathcal{O}(1)|_{\Delta_i})|_{X \times \mathcal{U}} \). Also, \( \Phi^*A(E, d) \cong \mathcal{F}(E, d)|_{X \times \mathcal{U}} \), since by [4, Lemma 2.2], \( \mathcal{F}(E, d)|_{X \times \mathcal{U}} \) is again a vector bundle of rank \( r \), and there exists a surjection \( \Phi^*A(E, d) \twoheadrightarrow \mathcal{F}(E, d)|_{X \times \mathcal{U}} \). Therefore, \( \Phi^*\mathcal{T}_q(E, d)/S = \mathcal{H}\text{om}(\mathcal{F}(E, d), \bigoplus_{i=1}^d \pi_{2,i}^*\mathcal{O}(1)|_{\Delta_i})|_{X \times \mathcal{U}} \). Now \( b(i) \) follows from Lemma 2.3.

(b(ii)) Note that since \( \Delta_1 \) is dominant, then the induced map \( H^0(\mathcal{Q}(E, d), \mathcal{T}_q(E, d)/S) \to H^0(\mathcal{U}, \Phi^*\mathcal{T}_q(E, d)/S) = \bigoplus_{i=1}^d H^0(\mathbb{P}(E), \mathcal{T}_{\mathbb{P}(E)}/S) \) is an injection.

Now, since \( \Phi : \mathcal{U} \to \mathcal{Q}(E, d) \) is invariant under any permutation of the various \( \mathbb{P}(E) \) factors of \( \mathcal{U} \), hence, the map \( H^0(\mathcal{Q}(E, d), \mathcal{T}_q(E, d)/S) \hookrightarrow \bigoplus_{i=1}^d H^0(\mathbb{P}(E), \mathcal{T}_{\mathbb{P}(E)}/S) \) factors through the \( H^0(\mathbb{P}(E), \mathcal{T}_{\mathbb{P}(E)}/S) \) factors in a way that is again a vector bundle of rank \( r \).

Lemma 2.3. If either \( r \geq 3 \) or the hypothesis of Theorem 2.5(b) holds, then \( H^0(X \times_S \mathcal{U}, \mathcal{H}\text{om}(\mathcal{F}(E, d), \pi_{2,j}^*\mathcal{O}(1)|_{\Delta_j})|_{X \times \mathcal{U}}) = H^0(\mathbb{P}(E), \mathcal{T}_{\mathbb{P}(E)}/S) \) \( \forall \; j \).

Proof. Over \( X \times_S \mathcal{U} \), we have the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \downarrow & & & \downarrow & \\
0 & \downarrow & & & \downarrow & \\
0 & \rightarrow & \mathcal{F}(E, d) & \rightarrow & \pi_1^*E & \rightarrow \bigoplus_{i=1}^d \pi_{2,i}^*\mathcal{O}(1)|_{\Delta_i} & \rightarrow 0 \\
& \downarrow & & & \downarrow & \\
0 & \rightarrow & (\pi_1 \times \pi_{2,j})^*\mathcal{F}(E, 1) & \rightarrow & \pi_1^*E & \rightarrow \pi_{2,j}^*\mathcal{O}(1)|_{\Delta_j} & \rightarrow 0 \\
& & & \downarrow & & \downarrow & \\
& & & 0 & & 0 & \\
\end{array}
\]

Now, using snake lemma for the above diagram, we get the following exact sequence over \( X \times_S \mathcal{U} \):

\[
0 \rightarrow \mathcal{F}(E, d) \rightarrow (\pi_1 \times \pi_{2,j})^*\mathcal{F}(E, 1) \rightarrow \bigoplus_{i=1}^d \pi_{2,i}^*\mathcal{O}(1)|_{\Delta_i} \rightarrow 0. \tag{2.1}
\]

Applying \( \mathcal{H}\text{om}(\cdot, \pi_{2,j}^*\mathcal{O}(1)|_{\Delta_j}) \) and Lemma 2.4(i) to the exact sequence (2.3), we get that over \( X \times_S \mathcal{U} \), we have the following exact sequence:

\[
0 \rightarrow \mathcal{H}\text{om}((\pi_1 \times \pi_{2,j})^*\mathcal{F}(E, 1), \pi_{2,j}^*\mathcal{O}(1)|_{\Delta_j}) \rightarrow \mathcal{H}\text{om}(\mathcal{F}(E, d), \pi_{2,j}^*\mathcal{O}(1)|_{\Delta_j}) \rightarrow \mathcal{E}\text{xt}^1\left( \bigoplus_{i=1, i \neq j}^d \pi_{2,i}^*\mathcal{O}(1)|_{\Delta_i}, \pi_{2,j}^*\mathcal{O}(1)|_{\Delta_j} \right).
\]

Applying \( H^0 \) to the above exact sequence and using Lemma 2.4(i) and (ii), we get that \( H^0(X \times_S \mathcal{U}, \mathcal{H}\text{om}(\mathcal{F}(E, d), \pi_{2,j}^*\mathcal{O}(1)|_{\Delta_j}) = H^0(\mathbb{P}(E), \mathcal{T}_{\mathbb{P}(E)}/S). \)
Lemma 2.4.

(i) $\mathcal{H}om(\bigoplus_{i=1, i \neq j}^d \pi_{2,i}^*, O(1)|_{\Delta_i}, \pi_{2,j}^*, O(1)|_{\Delta_j}) = 0$.

(ii) If either $r \geq 3$ or the hypothesis of Theorem 2.5(b) holds, then $H^0(X \times_S U, E_{\text{ext}}(\bigoplus_{i=1, i \neq j}^d \pi_{2,i}^*, O(1)|_{\Delta_i}, \pi_{2,j}^*, O(1)|_{\Delta_j})) = 0$.

(iii) $H^0(X \times_S U, \mathcal{H}om((\pi_1 \times \pi_{2,j})*F(E, 1), \pi_{2,j}^*, O(1)|_{\Delta_j})) = H^0(\mathbb{P}(E), T_{\mathbb{P}(E)/S})$.

Proof.

(i) $\mathcal{H}om(\bigoplus_{i=1, i \neq j}^d \pi_{2,i}^*, O(1)|_{\Delta_i}, \pi_{2,j}^*, O(1)|_{\Delta_j}) = \bigoplus_{i=1, i \neq j}^d \mathcal{H}om(\pi_{2,i}^*, O(1)|_{\Delta_i}, \pi_{2,j}^*, O(1)|_{\Delta_j}) = \bigoplus_{i=1, i \neq j}^d \mathcal{H}om(\pi_{2,i}^*, O(1)|_{\Delta_i}, \pi_{2,j}^*, O(1)|_{\Delta_j})$. Now, by adjunction, $\mathcal{H}om_{O_X \times_S \mathbb{P}}(O_{\Delta_i}, O_{\Delta_j}) = \mathcal{H}om_{O_{\Delta_i}}(O_{\Delta_i \cap \Delta_j}, O_{\Delta_j})$. Now, since $\Delta_j$ is an integral scheme, and $\Delta_i \cap \Delta_j$ is a proper subset of $\Delta_j$, so $\mathcal{H}om_{O_{\Delta_i}}(O_{\Delta_i \cap \Delta_j}, O_{\Delta_j})=0$.

(ii) $E_{\text{ext}}(\bigoplus_{i=1, i \neq j}^d \pi_{2,i}^*, O(1)|_{\Delta_i}, \pi_{2,j}^*, O(1)|_{\Delta_j}) = \bigoplus_{i=1, i \neq j}^d E_{\text{ext}}(\pi_{2,i}^*, O(1)|_{\Delta_i}, O(1)|_{\Delta_j}) = \bigoplus_{i=1, i \neq j}^d (\pi_{2,i}^*, O(1)|_{\Delta_i}, O(1)|_{\Delta_j}) E_{\text{ext}}(O_{\Delta_i}, O_{\Delta_j})$. 

Now, consider the exact sequence

$$0 \to O(-\Delta_i) \to O_{X \times_S U} \to O_{\Delta_i} \to 0.$$ 

Applying $\mathcal{H}om(\cdot, O_{\Delta_j})$ to the above exact sequence, we get

$$0 \to O_{\Delta_j} \to O(\Delta_i)|_{\Delta_j} \to E_{\text{ext}}(O_{\Delta_i}, O_{\Delta_j}) \to 0,$$

i.e., $E_{\text{ext}}(O_{\Delta_i}, O_{\Delta_j}) \cong \pi_{1,j}^* T_{X/S}|_{\Delta_i \cap \Delta_j}$. Therefore,

$$H^0(X \times_S U, (\pi_{2,j}^*, O(-1) \otimes \pi_{2,i}^*, O(1))) E_{\text{ext}}(O_{\Delta_i}, O_{\Delta_j})$$

$$= H^0(X \times_S U, (\pi_{2,j}^*, O(-1) \otimes \pi_{2,i}^*, O(1)) \otimes \pi_{1,i}^* T_{X/S}|_{\Delta_i \cap \Delta_j})$$

$$= H^0((X \times_S U) \cap \Delta_i \cap \Delta_j, (\pi_{2,j}^*, O(-1) \otimes \pi_{2,i}^*, O(1)) \otimes \pi_{1,i}^* T_{X/S}|_{\Delta_i \cap \Delta_j}).$$

Without loss of generality, assume that $i = 1, j = 2$. Then $\Delta_1 \cong \mathbb{Z}$, where this isomorphism is given by $\Delta_1 \hookrightarrow X \times_S \mathbb{Z} \to \mathbb{Z}$. Under this isomorphism $(X \times_S U) \cap \Delta_1 \cong \mathbb{Z} \times \mathbb{Z} \cong \mathbb{Z}$ and $\Delta_1 \cap \Delta_2 \cong (\mathbb{P}(E) \times_X \mathbb{P}(E)) \times_S (\mathbb{P}(E))^{d-2}$, and the line bundle $(\pi_{2,j}^*, O(-1) \otimes \pi_{2,i}^*, O(1)) \otimes \pi_{1,i}^* T_{X/S}|_{\Delta_i \cap \Delta_j} \cong p_{1,j}^* O(-1) \otimes p_{2,j}^* O(1) \times (p \circ p_1)^* T_{X/S}$.

Let us denote the subscheme $(\mathbb{P}(E) \times_X \mathbb{P}(E)) \times_S (\mathbb{P}(E))^{d-2} \hookrightarrow \mathbb{Z}$ by $Y$, and the line bundle $p_{1,j}^* O(-1) \otimes p_{2,j}^* O(1) \times (p \circ p_1)^* T_{X/S}|_Y$ by $L$.

We need to show that $H^0(Y \cup U, L) = 0$.

Now $Y \setminus U = \bigcup_{i,j} (\Delta_i \cap Y) \cup \bigcup_{i,j,k} (\Delta_i \cap \Delta_j \cap Y)$.

Claim.

(a) $\text{codim}_Y (\Delta_i \cap Y) \geq 2$ if $i, j \neq \{1, 2\}$,
(b) $\text{codim}_Y (\Delta_{1,2} \cap Y) = r$,
(c) If $\{1, 2\} \notin \{i, j, k\}$, then $\text{codim}_Y (\Delta_i \cap \Delta_j \cap Y) \geq 2$,
(d) $\Delta_{1,2,k} \cap Y$ has codimension 1 in $Y$. 

Proof of Claim.

(a) If \( \{i, j\} \neq \{1, 2\} \), then \( \Delta_{i,j} \cap Y \cong (\mathbb{P}(E) \times_X \mathbb{P}(E)) \times_S (\mathbb{P}(E))^{d-3}_S \), and hence has codimension \( \geq 2 \) subset in \( Y \).

(b) If \( \{i, j\} = \{1, 2\} \), then \( \Delta_{1,2} \cong \mathbb{P}(E) \times_S (\mathbb{P}(E))^{d-2}_S \), and since rank \( E \geq 3 \), it has codimension \( r \) in \( Y \).

(c) If \( \{1, 2\} \cap \{i, j, k\} = \emptyset \), then

\[ \Delta_{i,j,k,X} \cap Y \cong (\mathbb{P}(E) \times_X \mathbb{P}(E)) \times_S (\mathbb{P}(E) \times_X \mathbb{P}(E)) \times_S (\mathbb{P}(E))^{d-5}_S, \]

and if \( i = 1 \) and \( \{j, k\} \cap \{1, 2\} = \emptyset \), then

\[ \Delta_{i,j,k,X} \cap Y \cong (\mathbb{P}(E) \times_X \mathbb{P}(E)) \times_S (\mathbb{P}(E) \times_X \mathbb{P}(E)) \times_S (\mathbb{P}(E))^{d-4}_S. \]

Hence, in both these cases, we have codimension of \( \Delta_{i,j,k,X} \cap Y \geq 2 \) in \( Y \).

(d) \( \Delta_{1,2,k,X} \cong \mathbb{P}(E) \times_X \mathbb{P}(E) \times_S (\mathbb{P}(E))^{d-3}_S \). Hence, it has codimension 1 in \( Y \).

For notational convenience, we will denote \( \Delta_{i,j} \cap Y \) and \( \Delta_{i,j,k,X} \cap Y \) by \( \Delta_{i,j} \) and \( \Delta_{i,j,k,X} \) respectively.

Case \( r \geq 3 \). Let us denote the open set \( Y \setminus (\bigcup_{k \geq 3} \Delta_{1,2,k,X}) \) by \( V \). Then, by the above claim \( H^0(Y \cap U, \mathcal{L}) = H^0(V, \mathcal{L}) \). Now, if \( s \in H^0(V, \mathcal{L}|_V) \), then for some \( t \in H^0(Y, \mathcal{O}(\sum_{k=3}^d n \Delta_{1,2,k,X})) \), \( st^n \) extends to a global section of \( \mathcal{L} \otimes \mathcal{O}(\sum_{k=3}^d n \Delta_{1,2,k,X}) \), for some \( n \geq 0 \). So, it is enough to show that \( H^0(Y, \mathcal{L} \otimes \mathcal{O}(\sum_{k=3}^d n \Delta_{1,2,k,X})) = 0 \), \( \forall n \geq 0 \). Now, this follows from the next claim.

Claim. \( (p_2 \times p_3 \times \cdots \times p_d)_* (\mathcal{L} \otimes \mathcal{O}(\sum_{k=3}^d n \Delta_{1,2,k,X})) = 0. \)

Proof of Claim. Let us denote \( \mathbb{P}(E) \times_S (\mathbb{P}(E))^{d-2}_S \to \mathbb{P}(E) \), the \( k \)-th projection to \( \mathbb{P}(E) \) by \( p'_k \), where \( 3 \leq k \leq d \). Now, consider the following diagram:

\[
\begin{array}{ccc}
(\mathbb{P}(E) \times_X \mathbb{P}(E)) \times_S (\mathbb{P}(E))^{d-2}_S & \xrightarrow{p_2 \times p_3 \times \cdots \times p_d =: f} & (\mathbb{P}(E)) \times_S (\mathbb{P}(E))^{d-2}_S \\
& \downarrow & \downarrow \quad (p_2p'_2) \times (p_3p'_3) \times \cdots \times (p_dp'_d) =: g \\
X \times_S (X)^{d-2}_S.
\end{array}
\]

Let us denote the map \( p_2 \times p_3 \times \cdots \times p_d \) by \( f \), and \( (p_2p'_2) \times (p_3p'_3) \times \cdots \times (p_dp'_d) \) by \( g \). Then, if we denote by \( X_k \subseteq X \times_S (X)^{d-2}_S \) the closed subscheme whose points are of the form \( (x_1, x_2, \ldots, x_{d-1}) \) with \( x_1 = x_k \), then \( \mathcal{O}(\sum_{k=3}^d n \Delta_{1,2,k,X}) = (g \circ f)^* \mathcal{O}(\sum_{k=2}^d nX_k) \). Hence by projection formula, we have

\[
f_* \left( \mathcal{L} \otimes \mathcal{O} \left( \sum_{k=3}^d n \Delta_{1,2,k,X} \right) \right) = f_* ((p_1)_* \mathcal{O}(-1) \otimes (p_2)^* \mathcal{O}(1)) \otimes (p \circ p_2)^* \mathcal{T}_{X/S} \otimes \mathcal{O} \left( \sum_{k=3}^d n \Delta_{1,2,k,X} \right)
\]

\[
\otimes (p \circ p_2)^* \mathcal{T}_{X/S} \otimes \mathcal{O} \left( \sum_{k=3}^d n \Delta_{1,2,k,X} \right)
\]
Claim. $H_0$

Proof. Let us denote the diagram:

$$\sum_{k=1}^d nX_k$$

Hence, it is enough to show that $f_*(p_1)^*\mathcal{O}(-1) = 0$. Now consider the following fibered sequence:

$$\begin{array}{ccc}
\mathbb{P}(E) \times_{\mathbb{P}(E)} \mathbb{P}(E) & \xrightarrow{p_1} & \mathbb{P}(E) \\
\downarrow f & & \downarrow p \\
\mathbb{P}(E) \times_{\mathbb{P}(E)} \mathbb{P}(E) & \xrightarrow{p \circ p_2'} & X.
\end{array}$$

Since $p \circ p_2'$ is flat, we have by [5, Proposition 9.3], $f_* p_1^* \mathcal{O}(1) \cong (p \circ p_2')_* p_* \mathcal{O}(-1) = 0$.

Case $r = 2$. As in the previous case, let us denote by $V$ the open set $Y \setminus (\Delta_{1,2} \cup \bigcup_{k=2}^d \Delta_{1,2,k,x})$ in $Y$. Then $H^0(Y \cap U, \mathcal{L}) = H^0(V, \mathcal{L})$. Therefore, to show that $H^0(Y \cap U, \mathcal{L}) = H^0(V, \mathcal{L}) = 0$, we need to show that $H^0(Y, \mathcal{L}(n(\Delta_{1,2} + \sum_{i=1}^d \Delta_{1,2,k,x}))) = 0 \forall n \geq 0$.

Now consider the following exact sequence:

$$0 \longrightarrow \mathcal{O}(n\Delta_{1,2}) \longrightarrow \mathcal{O}((n+1)\Delta_{1,2}) \longrightarrow \mathcal{O}(n\Delta_{1,2})|_{\Delta_{1,2}} \longrightarrow 0.$$  

Tensoring the above exact sequence by $\mathcal{L}(n \sum_{i=1}^d \Delta_{1,2,k,x})$ and applying $H^0$, it is enough to show that $H^0(\Delta_{1,2}, \mathcal{L}(n(\Delta_{1,2} + \sum_{i=1}^d \Delta_{1,2,k,x})))|_{\Delta_{1,2}} = 0$. Note that $\mathcal{O}(\Delta_{1,2})|_{\Delta_{1,2}} = p_1^* \mathcal{O}(\mathbb{P}(E)/S)|_{\Delta_{1,2}}$. Then

$$\mathcal{L}
\left(n(\Delta_{1,2} + \sum_{i=1}^d \Delta_{1,2,k,x})\right)|_{\Delta_{1,2}} = p_1^* \mathcal{O}(-1) \otimes p_2^* \mathcal{O}(1)$$

$$\otimes (p \circ p_1)^* \mathcal{T}_{X/S} \otimes p_1^* \mathcal{T}_{\mathbb{P}(E)/X} \otimes \mathcal{O}(\sum_{i=1}^d n\Delta_{1,2,k,x})|_{\Delta_{1,2}}$$

$$= p_1^* \mathcal{T}_{\mathbb{P}(E)/X} \otimes (p \circ p_1)^* \mathcal{T}_{X/S} \mathcal{O}(\sum_{i=1}^d n\Delta_{1,2,k,x})|_{\Delta_{1,2}}.$$  

Therefore, identifying $\Delta_{1,2}$ with $(\mathbb{P}(E))^d_{S}$, the result follows from the following claim.

Claim. $H^0(\mathcal{Z}, p_1^* \mathcal{T}_{\mathbb{P}(E)/X} \otimes (p \circ p_1)^* \mathcal{T}_{X/S} \otimes \mathcal{O}(\sum_{k=2}^d n\Delta_{1,k,x}) = 0 \forall n \geq 0$.

Proof. Let us denote the $i$-th projection from $(X)^d_{S}$ to $X$ by $p_{i,X}$. Recall that $X_k \subseteq (X)^d_{S}$ is the closed set defined by the equation $p_{1,X} = p_{k,X}$.

Note that by the projection formula,

$$((p \circ p_1) \times (p \circ p_2) \times \cdots \times (p \circ p_d))_* (p_1^* \mathcal{T}_{\mathbb{P}(E)/X} \otimes (p \circ p_1)^* \mathcal{T}_{X/S} \otimes \mathcal{O}(\sum_{k=2}^d n\Delta_{1,k,x}))$$
Under the hypothesis of Theorem 2.5, we have the following left exact sequence of algebraic groups:

\[ p_1^* (S^{2n}(E) \otimes (\det(E^\vee))^n) \otimes p_{1,X}^* T_X/S \]
\[ \otimes \mathcal{O} \left( \sum_{k=2}^{d} nX_k \right). \]

Now we have the following exact sequence:

\[ 0 \longrightarrow \mathcal{O}(nX_k) \longrightarrow \mathcal{O}((n + 1)X_k) \longrightarrow \mathcal{O}((n + 1)X_k)|_{X_k} \longrightarrow 0. \]

Now \( \mathcal{O}((n + 1)X_k)|_{X_k} = p_{1,X}^* T_X^{n+1} \). Using this and tensoring the above exact sequence with \( p_{1,X}^* (S^{2n}(E) \otimes (\det(E^\vee))^n) \otimes p_{1,X}^* T_X/S \), we have \( H^0(\mathcal{Z}, p_{1,X}^* (S^{2n}(E) \otimes (\det(E^\vee))^n) \otimes p_{1,X}^* T_X/S) = 0 \) \( \forall n \geq 0, m \geq 1 \).

Again, applying projection formula for the morphism \( p_{1,X}^* (S^{2n}(E) \otimes (\det(E^\vee))^n) \otimes p_{1,X}^* T_X/S = S^{2n}(E) \otimes (\det(E^\vee))^n \otimes T_X^m/S \). Hence, it is enough to show that \( H^0(X, S^{2n}(E) \otimes (\det(E^\vee))^n \otimes T_X^m/S) = 0 \) \( \forall n \geq 0, m \geq 1 \).

Now, \( \deg S^{2n}(E) = \frac{2 + 2n - 1}{2} \deg E = n(2n + 1) \deg E \) and rank \( S^{2n}(E) = 2n + 1 \).
Therefore, \( \deg S^{2n}(E) \otimes (\det(E^\vee))^n \otimes T_X^m/S = n(2n + 1) \deg E + (2n + 1)(-n(\deg E) + m(\deg T_X/S)) = m(2n + 1)(\deg T_X/S). \)

Now, since genus of each fibre of \( X \rightarrow S \) is \( \geq 2 \), \( \deg T_X/S < 0 \). Hence, \( \deg S^{2n}(E) \otimes (\det(E^\vee))^n \otimes T_X^m/S < 0 \). Since, by assumption, \( E \) is semistable, hence \( S^{2n}(E) \otimes (\det(E^\vee))^n \otimes T_X^m/S \) is also semistable with negative degree. Therefore, it does not have any global sections.

Proof of Lemma 2.4(iii). As before, we identify \( \Delta_j \) with \( \mathcal{Z} \). Then, we have

\[ H^0(X \times_S \mathcal{U}, \mathcal{H}\text{om}(p_1 \times \pi_{2,j})^* \mathcal{F}(E, 1, \pi_{2,j}^* \mathcal{O}(1)|_{\Delta_j})) \]
\[ = H^0(\mathcal{U}, \mathcal{H}\text{om}(p_j^* \mathcal{F}(E, 1)|_{\Delta_1}, p_j^* \mathcal{O}(1))) \]
\[ = H^0(\mathcal{U}, p_j^* (\mathcal{F}(E, 1)^\vee|_{\Delta_1} \otimes \mathcal{O}(1))). \]

Now, since \( p_j^* (\mathcal{F}(E, 1)^\vee|_{\Delta_1} \otimes \mathcal{O}(1)) \) is vector bundle over \( \mathcal{Z} \) and codimension of \( \mathcal{Z} \setminus \mathcal{U} \geq 2 \), hence, \( H^0(\mathcal{U}, p_j^* (\mathcal{F}(E, 1)^\vee|_{\Delta_1} \otimes \mathcal{O}(1))) = H^0(\mathcal{Z}, p_j^* (\mathcal{F}(E, 1)^\vee|_{\Delta_1} \otimes \mathcal{O}(1))). \)
Using the projection formula for the morphism \( p_j \), we get that \( H^0(\mathcal{U}, p_j^* (\mathcal{F}(E, 1)^\vee|_{\Delta_1} \otimes \mathcal{O}(1))) = H^0(\mathbb{P}(E), (\mathcal{F}(E, 1)^\vee|_{\Delta_1} \otimes \mathcal{O}(1))) \).
Now, over \( \mathbb{P}(E), \mathcal{F}(E, d)^\vee|_{\Delta_1} \otimes \mathcal{O}(1) \cong \mathcal{T}_{\mathbb{P}(E)/S}. \) Hence, we have the result.

Proof of Theorem 2.5. By Lemma 2.1, we have an inclusion \( \text{Aut}^e(\mathbb{P}(E)/S) \hookrightarrow \text{Aut}^e(Q(E, d)/S) \) and the corresponding map of Lie algebras \( H^0(\mathbb{P}(E), \mathcal{T}_{\mathbb{P}(E)/S}) \hookrightarrow H^0(Q(E, d), \mathcal{T}_{Q(E, d)/S}). \) Now, by Theorem 2.2, we get that this morphism of Lie algebras is an isomorphism. Since, the characteristic of \( k = 0 \), both \( \text{Aut}^e(\mathbb{P}(E)/S) \) and \( \text{Aut}^e(Q(E, d)/S) \) are reduced. Therefore the inclusion \( \text{Aut}^e(\mathbb{P}(E)/S) \hookrightarrow \text{Aut}^e(Q(E, d)/S) \) is an isomorphism.

3. Applications

COROLLARY 3.1

Under the hypothesis of Theorem 2.5, we have the following left exact sequence of algebraic groups.
0 \rightarrow GL(E)/k^* \rightarrow \text{Aut}^0(Q(E, d)/S) \rightarrow \text{Aut}^0(X/S). \quad (3.1)

The corresponding sequence of Lie algebras is given by

0 \rightarrow H^0(X, ad E) \rightarrow H^0(Q(E, d), T_{Q(E,d)/S}) \rightarrow H^0(X, T_{X/S}). \quad (3.2)

Proof. The left exactness of the above sequences follow from Theorem 2.5 and from the fact that Aut^0(\mathbb{P}(E)/S) and its Lie algebra fits into the above exact sequences.

COROLLARY 3.2

If genus of the fibres of X \rightarrow S is \geq 2 and the hypothesis of Theorem 2.5 holds, then

(i) \text{Aut}^0(Q(E, d)/S) = GL(E)/k^*,
(ii) H^0(Q(E, d), T_{Q(E,d)/S}) = H^0(X, ad E).

Proof. If genus of each fibre is \geq 2, then (p_S)_*T_{X/S} = 0. In particular, H^0(X, T_{X/S}) = 0. Hence, Aut^0(X/S) = 0. Now the corollary follows from Theorem 2.5.

COROLLARY 3.3

Let C be a smooth projective curve of genus \geq 2 over an algebraically closed field k of characteristic zero. Fix \vec{d} = (d_1, d_2, \ldots, d_k) \in \mathbb{N}^k with d_1 > d_2 > \cdots > d_k and r \geq 1. Let D(r, \vec{d}) be the flag scheme of filtration of quotients of \mathcal{O}_C^r \rightarrow B_1 \rightarrow B_2 \rightarrow \cdots \rightarrow B_d, where \mathcal{O}_C^r \rightarrow B_1 is a torsion quotient of degree d_1 [6, 2.A.1]. Then

(i) H^0(D(r, \vec{d}), T_{D(r,\vec{d})}) = sl(r),
(ii) \text{Aut}^0(D(r, \vec{d})) = PGL(r).

Proof. Over C \times D(r, (d_2, d_3, \ldots, d_k)), we have the universal chain of filtrations:

\mathcal{A}(r, d_2) \subseteq \mathcal{A}(r, d_3) \subseteq \cdots \subseteq \mathcal{A}(r, d_k) \subseteq \mathcal{O}_{C \times D(r, (d_1, d_2, \ldots, d_k))}^r.

Then D(r, \vec{d}) is the relative quot scheme of torsion quotients of degree d_1 - d_2 of the vector bundle \mathcal{A}(r, d_1) for the map C \times D(r, (d_2, d_3, \ldots, d_k)) \rightarrow D(r, (d_2, d_3, \ldots, d_k)). Then, by Corollary 3.1, we get that H^0(D(r, \vec{d}), T_{D(r,\vec{d})}) = H^0(C \times D(r, (d_2, d_3, \ldots, d_k)), ad \mathcal{A}(r, d_2)). Now, we know that for k \geq 1, \mathcal{A}(r, d_2) is stable with respect to certain polarizations on C \times D(r, (d_2, d_3, \ldots, d_k)) [4, Theorem 3.2.4, Theorem 5.1], so H^0(C \times D(r, (d_2, d_3, \ldots, d_k)), ad \mathcal{A}(r, d_2)) = 0. Now, by induction on k, we get that H^0(D(r, \vec{d}), T_{D(r, \vec{d})}) = H^0(C, ad \mathcal{O}_C^r) = sl(r). Also, (ii) follows immediately from this fact.

Note that we can apply Theorem 2.5(b) since \mathcal{A}(r, d_2) is stable.

□
COROLLARY 3.4

Let \( C \) be a smooth projective curve over an algebraically closed field \( k \). Let \( E \) be a vector bundle of rank \( \geq 3 \) over \( C \). Fix \( d \geq 1 \). Let \( \mathcal{Q}(E, d) \) be the quot scheme of torsion quotients of \( E \) of degree \( d \). Then

(i) We have the following:

(a) If genus of \( C = 0 \), i.e. \( C \cong \mathbb{P}^1 \), then \( \text{Aut}^{0}(\mathcal{Q}(E, d)) = PGL(2, k) \times GL(E)/k^{*} \).

(b) If genus of \( C = 1 \) and if \( E \) is semistable, then we have the following sequence of algebraic groups:

\[
0 \longrightarrow GL(E)/k^{*} \longrightarrow \text{Aut}^{0}(\mathcal{Q}(E, d)) \longrightarrow \text{Aut}^{0}(C) \longrightarrow 0.
\]

In both of these cases, we have the exact sequence of Lie algebras:

\[
0 \longrightarrow H^{0}(C, \text{ad } E) \longrightarrow H^{0}(\mathcal{Q}(E, d), \mathcal{T}_{\mathcal{Q}(E, d)}) \longrightarrow H^{0}(C, \mathcal{T}_{C}) \longrightarrow 0.
\]

(ii) If \( E \) is not semistable, then \( \text{Aut}^{0}(\mathcal{Q}(E, d)) = GL(E)/k^{*} \) and \( H^{0}(\mathcal{Q}(E, d), \mathcal{T}_{\mathcal{Q}(E, d)}) = H^{0}(C, \text{ad } E) \).

Proof.

(i) We have the following:

(a) If \( C \cong \mathbb{P}^1 \), then any vector bundle \( E \) admits a \( GL(2) \) linearization. In particular, we have a homomorphism \( GL(2) \rightarrow \text{Aut}^{0}(\mathcal{Q}(E, d)) \). This homomorphism factors through \( PGL(2, k) \) and gives a section to the map \( \text{Aut}^{0}(\mathcal{Q}(E, d)) \rightarrow PGL(2, k) \). Therefore, the left exact sequence (3.1) is exact in this case and it splits.

(b) We show that for any \( g \in \text{Aut}^{0}(C) \), \( g^{*}E \cong E \). This will show that the sequence (3.1) is also right exact in this case.

We know that \( E \cong \bigoplus E_{i} \), where \( E_{i} \)'s are indecomposable vector bundles. Since \( E \) is semistable, \( \mu(E_{i}) = \mu(E_{j}) \forall i, j \). Since \( \mathcal{Q}(E, d) \cong \mathcal{Q}(E \otimes \mathcal{L}, d) \) canonically for any line bundle \( \mathcal{L} \), after twisting \( E \) by a line bundle of appropriate degree, we can assume \( \mu(E) = \mu(E_{i}) = 0 \forall i \). By [1, Theorem 5(i)], \( E_{i} \cong F_{r_{i}} \otimes \mathcal{L}_{i} \), where \( F_{r_{i}} \) is the unique indecomposable vector bundle of rank \( r_{i} \) with \( H^{0}(C, E_{r_{i}}) \neq 0 \) and \( \mathcal{L} \) is a line bundle of degree 0.

It follows that for any \( g \in \text{Aut}^{0}(C) \), \( g^{*}F_{r} \cong F_{r} \), since \( g^{*}F_{r} \) is also an indecomposable bundle of degree 0 and rank \( r \) with \( H^{0}(C, g^{*}F_{r}) \neq 0 \). So, we need to show that \( g^{*}\mathcal{L} \cong \mathcal{L} \) for any \( \mathcal{L} \in \text{Pic}^{0}(C) \).

Fix a base point \( x_{0} \in C \). Then, under the group structure of \( C \) with \( x_{0} \) as the identity, \( \text{Aut}^{0}(C) \cong C \) i.e. any element of \( \text{Aut}^{0}(C) \) is given by \( y \mapsto y + c x \) for a fixed \( x \in C \). (Here we denote the group addition of \( C \) by \( +_{C} \).) Fix such an automorphism \( x \in C \cong \text{Aut}^{0}(C) \).

Now \( (C, x_{0}, +_{C}) \cong \text{Pic}^{0}(C) \) with the morphism given by \( z \mapsto \mathcal{O}(z - x_{0}) \).

Let us assume \( \mathcal{L} = \mathcal{O}(z - x_{0}) \) for \( z \in C \). Then \( x^{*}\mathcal{L} = \mathcal{O}((z + c x) - (x_{0} - c x)) \).

Since \( (C, x_{0}, +_{C}) \cong \text{Pic}^{0}(C) \) is a homomorphism, it follows that

\[
x^{*}\mathcal{L} \cong \mathcal{O}([((z + c x) - x_{0}) - ((x_{0} - c x) - x_{0})])
\]

\[
= \mathcal{O}(((z - c x) - x_{0}) - ((x_{0} - c x) - x_{0}))
\]

\[
= \mathcal{O}((z - x_{0}) - (x - x_{0}) - (x_{0} - x_{0}) + (x - x_{0}))
\]
= \mathcal{O}(z - x_0) = \mathcal{L}.

Hence, it follows that for any \( g \in \text{Aut}^o(C) \), \( g^* E \cong E \) for any \( E \) semistable.

(ii) By [7, Proposition 6.13], every semi-homogeneous vector bundle [7, Definition 5.2] is semistable. In particular, if \( E \) is not semistable, then the map \( H^0(\mathbb{P}(E), T_{\mathbb{P}(E)}) \rightarrow H^0(C, T_C) \) is zero. Hence, using the sequence (3.2), we get \( H^0(C, \text{ad} E) \rightarrow H^0(\mathbb{Q}(E, d), T_{\mathbb{Q}(E, d)}) \) is an isomorphism. From this, it follows that \( \text{Aut}^o(\mathbb{Q}(E, d)) = GL(E)/k^* \). \( \square \)

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