Extremal $\mathcal{N} = (2, 2)$ 2D Conformal Field Theories and Constraints of Modularity

Matthias R. Gaberdiel, Sergei Gukov, Christoph A. Keller, Gregory W. Moore, and Hirosi Ooguri

1 Institut für Theoretische Physik, ETH Zurich, CH-8093 Zürich, Switzerland
2 School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA
3 Department of Physics and Department of Mathematics, University of California, Santa Barbara, CA 93106, USA
4 NHETC and Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08855-0849, USA
5 California Institute of Technology, Pasadena, CA 91125, USA
6 Institute for the Physics and Mathematics of the Universe, University of Tokyo, Kashiwa, Chiba 277-8586, Japan

ABSTRACT: We explore the constraints on the spectrum of primary fields implied by modularity of the elliptic genus of $\mathcal{N} = (2, 2)$ 2D CFT’s. We show that such constraints have nontrivial implications for the existence of “extremal” $\mathcal{N} = (2, 2)$ conformal field theories. Applications to $\text{AdS}_3$ supergravity and flux compactifications are addressed.

CALT-68-2685, IPMU 08-0031, ITEP-TH-20/08

*On leave from California Institute of Technology.
Contents

1. Introduction and summary 2
2. Polar states and the elliptic genus 3
   2.1 Counting weight zero weak Jacobi forms 4
   2.2 Counting polar monomials 7
3. Extremal $\mathcal{N} = (2, 2)$ conformal field theories 10
   3.1 Definition 10
   3.2 The extremal polar polynomial 13
4. Experimental search for the extremal elliptic genus 15
5. The extremal elliptic genus does not exist for $m$ sufficiently large 17
   5.1 NS-sector elliptic genus 18
   5.2 A nontrivial constraint 20
   5.3 A constraint for $m = 0 \mod 4$ 22
   5.4 What are the exceptional values of $m$? 24
6. Near-extremal $\mathcal{N} = 2$ conformal field theories 25
   6.1 A constraint on the spectrum of $\mathcal{N} = 2$ theories with integral $U(1)$ charges 27
7. Construction of nearly extremal elliptic genera 28
8. Discussion: quantum corrections to the cosmic censorship bound 30
9. Extremal $\mathcal{N} = 4$ theories 32
10. Applications to flux compactifications 34
A. Growth properties 35
   A.1 Analysis of the constraint for $m$ odd 35
   A.2 Analysis of the constraint for $m = 2 \mod 4$ 38
   A.3 Analysis of the constraint for $m = 0 \mod 4$ 38
B. Rademacher expansions 40
1. Introduction and summary

In recent work [49] Witten has revived the study of 2 + 1 dimensional quantum gravity. In particular, he has defined a notion of pure AdS$_3$ quantum gravity and investigated its properties in light of the AdS/CFT correspondence. These considerations naturally lead to a notion of an extremal conformal field theory. Extremality means that the partition function of the boundary CFT is as close as possible to the Virasoro character of the vacuum. The reason for this is that there are two kinds of excitations in pure gravity: the perturbative excitations and the black holes. The perturbative excitations are identified with Virasoro descendants of the vacuum following [8] while the Virasoro primaries correspond to the BTZ black holes. Since black holes are parametrically heavy there is a large gap from the vacuum to the first nontrivial Virasoro primary. The present paper addresses similar questions for pure quantum gravity with extended $\mathcal{N} = 2$ supersymmetry. Our main tool will be the elliptic genus of an $\mathcal{N} = 2$ superconformal field theory. As we recall below, this is a weak Jacobi form, and its modular properties impose tight constraints on the partition function. The advantage of this approach is that, unlike the case of [49], we do not have to assume the complete holomorphic factorization of the partition function. The holomorphy and modularity of the elliptic genus holds for any conformal field theory with $\mathcal{N} = 2$ supersymmetry. Thus, we can test the hypothetical existence of a theory of pure AdS$_3$ supergravity without relying on the additional assumption of holomorphic factorization. We will show that there is some tension between these modular properties and the notion of extremality.

A brief summary of our results is the following:

1. In section 3.1 we give a definition of an extremal (2, 2) superconformal field theory which, one might expect would constitute a holographic dual to “pure (2, 2) AdS$_3$ supergravity.” In any case, it is a natural generalization of the notion of extremality to (2, 2) supersymmetry. In this paper we restrict attention to theories with $c \equiv 0 \pmod{6}$ and integral $U(1)_R$ charges for the left- and right-moving $\mathcal{N} = 2$ algebras. Relaxing this assumption is an interesting open problem.

2. In section 4 we give numerical evidence that only a finite number of “exceptional” examples of extremal (2, 2) theories can exist. Then in section 5 we give an analytic proof that this is indeed the case. We also present very strong evidence that the extremal elliptic genus only exists for nine values of $c$, namely

$$6, 12, 18, 24, 30, 42, 48, 66, 78.$$ (1.1)

3. In section 6 we then introduce the notion of a “nearly extremal (2, 2) superconformal theory,” whose spectrum only approximates that of pure (2, 2) supergravity. We show that if the degree of approximation is relaxed then candidate elliptic genera do indeed exist.

4. By quantifying the degree of approximation required to produce candidate elliptic genera we are able to constrain the spectrum as follows. Consider states (in the
NSNS sector) which are right-chiral-primary and left $\mathcal{N} = 2$ primary with $(L_0, J_0)$ eigenvalue $(h, \ell)$. In section 6.1, equation (6.11) we show that for $c$ large any theory with modular elliptic genus must have some such state with

$$h < \frac{c}{24} + \frac{3\ell^2}{2c} - \frac{1}{8} + \mathcal{O}(c^{-1/2}) .$$

(1.2)

This result is conjectural. It is supported by numerical evidence described in section 4. Finding a rigorous justification of (6.11) (or a counterexample) is an interesting open problem raised by the present paper.

5. On the other hand, in section 7 we show that it is possible to construct an elliptic genus which is compatible with the spectrum of an extremal $(2, 2)$ superconformal theory for conformal weights $h \leq \frac{c}{24}$.

6. In section 8 we comment on a partial generalization of our results to $\mathcal{N} = 4$ theories.

In the remainder of the paper we discuss some implications of the above results. First, in section 8 we discuss the implications for the existence of pure $(2, 2)$ $AdS_3$ supergravity. While our results cast some doubt on the existence of such theories, they are not conclusive. It is conceivable that quantum corrections to the cosmic censorship bound for the existence of black holes imply that one should identify a near-extremal rather than an extremal $(2, 2)$ CFT as a holographic dual of pure supergravity. We leave this question for future work. Of course, even when a candidate Jacobi form exists that does not mean a corresponding $(2, 2)$ supergravity necessarily exists. In the analogous $\mathcal{N} = 0$ case the relevant partition functions can readily be constructed, but it is not known whether the corresponding extremal CFT’s exist for general Chern-Simons levels $k$. Indeed, there is an argument based on the modular differential equation of these partition functions [24, 25] that suggests that the theories are in fact inconsistent for sufficiently large $k$.

A second motivation for the present work is that constraints on conformal field theory spectra implied by modular invariance might have interesting applications to flux compactifications of string theory and M-theory. This is briefly explained in section 11. Again, the development of this idea is left to future work.

2. Polar states and the elliptic genus

We will focus on theories with $\mathcal{N} = (2, 2)$ two-dimensional superconformal symmetry. It will be convenient to parametrize the (left = right) central charge as $c = 6m$. A simple example of such a theory that the reader might wish to keep in mind is an $\mathcal{N} = (2, 2)$ sigma-model based on a Calabi-Yau target space of complex dimension $2m$. In the present paper we only consider integer values of $m$, and thus the relevant Calabi-Yau manifolds have even complex dimension.1 In particular, the smallest non-trivial value of $m$ corresponds to a Calabi-Yau 2-fold, that is a torus $T^4$ or a $K3$ surface.

1A generalization to half-integer values of $m$ should be possible, but we will not attempt it in the present paper. For $m = 1$ the resulting theory actually has $(4, 4)$ supersymmetry, but we will not use this fact.
We assume that the Hilbert space of our theory is a direct sum of unitary highest weight representations of the $\mathcal{N} = 2$ algebra. This allows us to define the RR-sector partition function
\[ Z_{RR}(\tau, z; \bar{\tau}, \bar{z}) := \text{Tr}_{H_{RR}} q^{L_0 - c/24} e^{2\pi i \phi J_0} \bar{q}^{\bar{L}_0 - c/24} e^{2\pi i \bar{\phi} \bar{J}_0} e^{i\pi (J_0 - \bar{J}_0)} \] (2.1)
which has good modular properties under the $SL(2, \mathbb{Z})$ action $(\tau, z) \to (a\tau + b, c\tau + d)$, Here, as usual, $q = e^{2\pi i \tau}$ and $y = e^{2\pi i z}$, and similarly for $\bar{q}$ and $\bar{y}$.

In these conventions, the elliptic genus of an $\mathcal{N} = (2, 2)$ superconformal field theory $\mathcal{C}$ is defined to be
\[ \chi(\tau, z; \mathcal{C}) := Z_{RR}(\tau, z; \bar{\tau}, 0). \] (2.2)
It is holomorphic in $(\tau, z)$ by the standard properties of the Witten index. For references on the elliptic genus see [4, 5, 19, 29, 30, 31, 40, 41, 42, 43, 44, 46, 47].

$\mathcal{N} = 2$ algebras have the crucial spectral flow isomorphism [45], which allows us to relate the NS and R-sector partition functions. Recall that spectral flow $SF_\theta$ for $\theta \in \frac{1}{2} \mathbb{Z}$ is an isomorphism of $\mathcal{N} = 2$ superconformal algebras which maps eigenvalues
\[ L_0 \to L_0 + \theta J_0 + \theta^2 m \] (2.3)
\[ J_0 \to J_0 + 2\theta m. \] (2.4)
The spectral flow operators act on $Z = Z_{RR}$ as:
\[ (SF_\theta SF_{\bar{\theta}}) Z = e \left( m\theta^2 \tau + 2m\theta(z + \frac{1}{2}) \right) \cdot e \left( m\bar{\theta}^2 \bar{\tau} + 2m\bar{\theta}(\bar{z} - \frac{1}{2}) \right) Z(\tau, z + \tau \theta; \bar{\tau}, \bar{z} + \bar{\theta} \bar{\tau}), \] (2.5)
where $e(x) := e^{2\pi i x}$. For simplicity we restrict our attention to theories with integral spectrum of left- and right-moving $U(1)$ charges $J_0, \bar{J}_0$. Again, it should be possible, and would be interesting, to relax this assumption. Spectral-flow invariant theories with integral $U(1)$ charges satisfy
\[ Z_{RR} = (SF_\theta SF_{\bar{\theta}}) Z_{RR} \quad \theta, \bar{\theta} \in \mathbb{Z} \] (2.6)
\[ Z_{NSNS} = (SF_\theta SF_{\bar{\theta}}) Z_{RR} \quad \theta, \bar{\theta} \in \mathbb{Z} + \frac{1}{2}. \] (2.7)
As is well-known [29], the modularity properties of $Z_{RR}$ together with spectral flow invariance and unitarity imply that the elliptic genus is a weak Jacobi form of index $m$ and weight zero [22]. A weak Jacobi form $\phi(\tau, z)$ of weight $w$ and index $m \in \mathbb{Z}$, with $(\tau, z) \in \mathbb{H} \times \mathbb{C}$, satisfies the transformation laws
\[ \phi\left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = (c\tau + d)^w e^{2\pi im \frac{cz^2}{c}\tau + \frac{ad}{c} \tau + \frac{bd}{c} \tau + \frac{ad}{c} z} \phi(\tau, z) \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{Z}), \] (2.8)
\[ \phi(\tau, z + \ell \tau + \ell') = e^{-2\pi i m (\ell^2 \tau + 2\ell z)} \phi(\tau, z) \quad \ell, \ell' \in \mathbb{Z}, \] (2.9)
and has a Fourier expansion
\[ \phi(\tau, z) = \sum_{n \geq 0, \ell \in \mathbb{Z}} c(n, \ell) q^n y^\ell \] (2.10)
with \(c(n, \ell) = (-1)^w c(n, -\ell)\). It follows from the spectral flow identity that \(c(n, \ell) = 0\) for \(4mn - \ell^2 < -m^2\). Following \cite{22}, we denote by \(\tilde{J}_{w,m}\) the vector space of weak Jacobi forms of weight \(w\) and index \(m\). A Jacobi form is then a weak Jacobi form whose polar part vanishes (see below).

Suppose we are given an integer \(m \in \mathbb{Z}_+\). If \((\ell, n) \in \mathbb{Z}^2\) is a lattice point we refer to its polarity as \(p = 4mn - \ell^2\). If \(\phi \in \tilde{J}_{0,m}\) let us define the polar part of \(\phi\), denoted \(\phi^-\), to be the sum of the terms in the Fourier expansion corresponding to lattice points of negative polarity. By spectral flow one can always relate the degeneracies to those in the fundamental domain with \(|\ell| \leq m\). If we impose the modular transformation (2.8) with \(-1 \in SL(2, \mathbb{Z})\), which implements charge conjugation, then \(c(n, \ell) = c(n, -\ell)\) and therefore the polar coefficients which cannot be related to each other by spectral flow and charge conjugation are \(c(n, \ell)\) where \((\ell, n)\) is valued in the polar region \(\mathcal{P}\) (of index \(m\)), defined to be

\[
\mathcal{P}^{(m)} := \{(\ell, n) : 1 \leq \ell \leq m, \ 0 \leq n, \ p = 4mn - \ell^2 < 0\}.
\]

(2.11)

For an example, see figure 1.

![Figure 1: A cartoon showing polar states (represented by “•”) in the region \(\mathcal{P}^{(m)}\). Spectral flow by \(\theta = \frac{\pi}{2}\) relates these states to particle states in the NS sector of an \(\mathcal{N} = 2\) superconformal field theory which are holographically dual to particle states in \(AdS_3\).](image)

Given any Fourier expansion

\[
\psi(\tau, z) = \sum_{\ell, n \in \mathbb{Z}} \hat{\psi}(n, \ell) q^n y^\ell
\]

(2.12)

we define its polar polynomial (of index \(m\)) to be the sum restricted to the polar region \(\mathcal{P}^{(m)}\):

\[
\text{Pol}(\psi) := \sum_{(\ell, n) \in \mathcal{P}^{(m)}} \hat{\psi}(n, \ell) q^n y^\ell.
\]

(2.13)

Let us moreover denote by \(V_m\) the space of polar polynomials, i.e. the vector space generated by the monomials \(q^n y^\ell\) with \((\ell, n) \in \mathcal{P}^{(m)}\).
The key mathematical fact we need follows from the theory of “periods of modular forms.” The upshot is that one can reconstruct a weak Jacobi form of weight zero from its polar polynomial. Moreover there is a sequence
\[ 0 \to \tilde{J}_{0,m} \overset{\text{Pol}}{\to} V_m \overset{\text{Per}}{\to} S_{5/2} \tag{2.14} \]
exact at \( V_m \), where Per is a “period map” to a certain space of vector-valued cusp forms of weight \( 5/2 \). A nonzero image in the space of cusp forms means that the polar polynomial cannot be realized by a true weak Jacobi form. For an explanation of these facts in the physics literature, together with references to the mathematical literature, see [16, 38, 35, 36]. The reader interested in these matters should also consult [7].

In the next two sections we will show that there can indeed be nontrivial obstructions simply by computing the dimensions of \( \tilde{J}_{0,m} \) and \( V_m \).

Returning to the conformal field theory \( C \), an eigenstate of \( L_0, J_0 \) is called a polar state if it has negative polarity:
\[ p = 4mL_0 - J_0^2 - m^2 = 4m(L_0 - \frac{c}{24}) - J_0^2 < 0 \ . \tag{2.15} \]
One checks that \( 4mL_0 - J_0^2 \) is spectral flow invariant, so we can speak of polar states in both the R and NS sector. Using the mathematical results explained above we see that the significance of polar states is that the polar degeneracies of the elliptic genus determine all the other Fourier coefficients of the elliptic genus.

### 2.1 Counting weight zero weak Jacobi forms

Let \( \tilde{J}_{ev,*} = \oplus_{w, m \in \mathbb{Z}} \tilde{J}_{w,m} \) denote the bigraded ring of weak Jacobi forms of even weight. According to [22], Theorem 9.3, \( \tilde{J}_{ev,*} \) is a polynomial algebra on four generators of degree
\[ (w, m) = (4, 0), \ (6, 0), \ (-2, 1), \ (0, 1) \ . \tag{2.16} \]

The first two generators correspond to the Eisenstein series
\[ E_4 = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n = 1 + 240q + 2160q^2 + 6720q^3 + \ldots \tag{2.17} \]
and
\[ E_6 = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n = 1 - 504q - 16632q^2 - 122976q^3 - \ldots \ , \tag{2.18} \]
where the Fourier coefficients \( \sigma_k(n) := \sum_{d|n} d^k \) are defined to be the \( k \)-th powers of the positive divisors of \( n \). A generalization of Eisenstein series to Jacobi forms is described in [22]:
\[ E_{k,m}(\tau, z) = \frac{1}{2} \sum_{c,d \in \mathbb{Z}, (c,d)=1} \sum_{\ell \in \mathbb{Z}} (c\tau + d)^{-k} e^{2\pi i m \left( \frac{\ell^2 a + \ell d}{c^2 + d^2} + \frac{2 \ell a c}{c^2 + d^2} - \frac{c^2}{c^2 + d^2} \right)} \ . \tag{2.19} \]
In terms of these generalized Eisenstein series one can write the remaining two generators in (2.16) as

\[
\tilde{\phi}_{-2,1} = \frac{\phi_{10,1}}{\Delta} \in \tilde{J}_{-2,1} \quad \tilde{\phi}_{0,1} = \frac{\phi_{12,1}}{\Delta} \in \tilde{J}_{0,1},
\]

(2.20)

where the first subscript on \( \tilde{\phi} \) denotes the weight and the second denotes the index. Here, \( \Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24} \) and

\[
\phi_{10,1} = \frac{1}{144} (E_6 E_{4,1} - E_{4,0}) = (y - 2 + y^{-1})q + (-2y^2 - 16y + 36 - 16y^{-1} - 2y^{-2})q^2 + \ldots ,
\]

\[
\phi_{12,1} = \frac{1}{144} (E_4^2 E_{4,1} - E_6 E_{6,1}) = (y + 10 + y^{-1})q + (10y^2 - 88y - 132 - 88y^{-1} + 10y^{-2})q^2 + \ldots .
\]

Thus the two weak Jacobi forms \( \tilde{\phi}_{-2,1} \) and \( \tilde{\phi}_{0,1} \) have the series expansion

\[
\tilde{\phi}_{-2,1} = (y - 2 + y^{-1}) + (-2y^2 + 8y - 12 + 8y^{-1} - 2y^{-2})q + \ldots ,
\]

\[
\tilde{\phi}_{0,1} = (y + 10 + y^{-1}) + (10y^2 - 64y + 108 - 64y^{-1} + 10y^{-2})q + \ldots .
\]

(2.22)

Much useful information about Jacobi forms can be found in [22].

To summarize, a natural vector space basis of \( \tilde{J}_{0,m} \) is given by

\[
(\tilde{\phi}_{-2,1})^a(\tilde{\phi}_{0,1})^b E_4^c E_6^d,
\]

(2.23)

where \( a, b, c, d \) are nonnegative integers such that \( a + b = m \), and \( a = 2c + 3d \). It is straightforward to compute the number of solutions to these constraints and thereby show that

\[
j(m) := \dim \tilde{J}_{0,m} = \frac{m^2}{12} + \frac{m}{2} + \left( \delta_{s,0} + \frac{s^2}{2} - \frac{s^2}{12} \right),
\]

(2.24)

where \( m = 6\rho + s \) with \( \rho \geq 0 \) and \( 0 \leq s \leq 5 \). Specifically,

\[
j(m) = \begin{cases} 
  m^2/12 + m/2 + 1 & m = 0 \mod 6 \\
  m^2/12 + m/2 + 5/12 & m = 1, 5 \mod 6 \\
  m^2/12 + m/2 + 2/3 & m = 2, 4 \mod 6 \\
  m^2/12 + m/2 + 3/4 & m = 3 \mod 6 .
\end{cases}
\]

(2.25)

2.2 Counting polar monomials

Let us now compute the dimension of the space \( V_m \), and compare it to \( j(m) \). In other words, we wish to count the number of integer points in the \((\ell, n)\) plane bounded (on one side) by the parabola \( 4mn - \ell^2 = 0 \), as shown in figure 1. We have

\[
P(m) := \dim V_m = \sum_{\ell=1}^{m} \left\lceil \frac{\ell^2}{4m} \right\rceil.
\]

(2.26)
Note that we want the ceiling function and not the floor function, as we include \( n = 0 \) up to the largest \( n \) with \( n < \ell^2/(4m) \) for each \( \ell = 1, \ldots, m \).

To compute this we follow [22] and write our sum as a sum of three terms.

\[
\sum_{\ell=1}^{m} \left\lceil \frac{\ell^2}{4m} \right\rceil = \sum_{\ell=1}^{m} \frac{\ell^2}{4m} - \sum_{\ell=1}^{m} \left( \left\lceil \frac{\ell^2}{4m} \right\rceil - \left\lfloor \frac{\ell^2}{4m} \right\rfloor \right), \tag{2.27}
\]

where

\[
((x)) := x - \frac{1}{2} [x] + [x] = \begin{cases} 0 & x \in \mathbb{Z} \\ \alpha - \frac{1}{2} & x = n + \alpha, 0 < \alpha < 1 . \end{cases} \tag{2.28}
\]

Note that \( ((x)) \) is the sawtooth function. It is periodic of period 1.

Now we evaluate the three terms. The main term comes from the elementary formula

\[
\sum_{\ell=1}^{m} \frac{\ell^2}{4m} = \frac{m^2}{12} + \frac{m}{8} + \frac{1}{24}. \tag{2.29}
\]

Next, note that the number of integers \( \ell \) with \( 1 \leq \ell \leq m \) with \( \ell^2 \equiv 0 \mod 4m \) is \( \left\lfloor \frac{b}{2} \right\rfloor \) where \( b \) is the largest integer with \( b^2 | 4m \). This follows from the prime factorization of \( m \). Thus, we obtain:

\[
\sum_{\ell=1}^{m} \left\lceil \frac{\ell^2}{4m} \right\rceil - \sum_{\ell=1}^{m} \left\lfloor \frac{\ell^2}{4m} \right\rfloor = m - \left\lfloor \frac{b}{2} \right\rfloor. \tag{2.30}
\]

Finally we come to the most subtle term \( \sum_{\ell=1}^{m} ((\frac{\ell^2}{4m})) \). The numbers \( ((\frac{\ell^2}{4m})) \) are, very roughly speaking, randomly distributed between \(-1/2\) and \(+1/2\), Therefore, the average will go to zero. In fact, they roughly make a random walk so we expect a quantity on the order of \( m^{1/2} \). To be more precise the discussion of [22], pp. 122-124 shows that

\[
\sum_{\ell=1}^{m} ((\frac{\ell^2}{4m})) = -\frac{1}{4} \sum_{d|4m} h'(-d) + \frac{1}{2} \left( \frac{m}{4} \right),
\]

where \( h'(-d) \) is the class number of a quadratic imaginary field of discriminant \(-d\) (with the exception of \( d = 3, 4 \)).

Putting the three terms together we obtain:

\[
P(m) = \frac{m^2}{12} + \frac{5m}{8} + A(m) \tag{2.31}
\]

where \( A(m) \) is the arithmetic function

\[
A(m) = \frac{1}{4} \sum_{d|4m} h'(-d) - \frac{1}{2} \left\lfloor \frac{b}{2} \right\rfloor - \frac{1}{2} \left( \frac{m}{4} \right) + \frac{1}{24}. \tag{2.32}
\]

Very roughly speaking, \( A(m) \) grows like \( O(m^{1/2}) \) so for large \( m \) we have

\[
P(m) - j(m) = \frac{m}{8} + O(m^{1/2}). \tag{2.33}
\]
The reader should be warned that we are not using the $O$ symbol in its precise mathematical sense here, but rather as a heuristic order-of-magnitude for the “average” value of the subleading term to the linear behavior in $(2.33)$.  

| $m$ | $\dim J_{0,m}$ | $\dim V_m$ |
|-----|----------------|------------|
| $m=0$ | 1 | 0 |
| $m=1$ | 1 | 1 |
| $m=2$ | 2 | 2 |
| $m=3$ | 3 | 3 |
| $m=4$ | 4 | 4 |
| $m=5$ | 5 | 6 |
| $m=6$ | 7 | 8 |
| $m=7$ | 8 | 9 |
| $m=8$ | 10 | 11 |
| $m=9$ | 12 | 13 |
| $m=10$ | 14 | 16 |
| $m=11$ | 16 | 18 |
| $m=12$ | 19 | 21 |

(2.34)

The first few values of $P(m)$ and $j(m)$ are shown in the above table. Note that $P(m) > j(m)$ for $m \geq 5$, and it is straightforward to check with a computer that $(P(m) - j(m) - \frac{m}{8})m^{-1/2}$ is positive and roughly order one for values of $m$ out to order several thousand. See figure 2 below.

The important conclusion that we draw is that for large $m$ there are on the order of $\frac{m}{8}$ linear constraints on the polar coefficients of the elliptic genus expressing modularity.

Remarks

1. The action of charge conjugation together with spectral flow defines an action of $D_\infty$ on the $(\ell, n)$ plane which preserves the space $Q$ of polar values $-m^2 \leq 4mn - \ell^2 < 0$. A fundamental domain is given by the polar region $P^{(m)}$, but the quotient $Q/D_\infty$ has fixed points: for $\ell = -m$ the spectral flow to $\ell = +m$ can be undone by charge conjugation. Therefore, if we compute the orbifold Euler character of $Q/D_\infty$ the line of states $(\ell, h)$ with $\ell = m$ should be counted with weight $\frac{1}{2}$. There are precisely $m/4$

\footnote{This is a subtle issue which, while fascinating, we believe is a distraction from our main exposition. A theorem of Siegel states that $\lim_{d \to \infty} \frac{\log h'(d)}{\log d} = \frac{1}{2}$ as $d$ runs through discriminants of quadratic imaginary fields, but $h'(-d)$ itself does not have a simple asymptotic expansion. This follows from its relation to the Dirichlet Series $L_d(s)$ at $s = 1$. For a discussion of these and related matters, together with their possible applications to black holes and with references to the math literature see \cite{37}. For a rigorous discussion of the probability distribution of $h'(-d)$ see \cite{4,23}.}
Figure 2: A plot of the first few hundred values of \((P(m) - j(m) - m/8)m^{-1/2}\) shows the quantity remains on the order of 1. The points do not tend to a limiting value - as would be the case for an asymptotic expansion, but are distributed about a mean value and exhibit a considerable amount of scatter. Detailed results on the distribution are available in the math literature, but we will not need these.

states on this line and hence \(\chi_{orb}(Q/D_{\infty}) = P(m) - m/8\), which is a much closer approximation to \(j(m)\).

2. Recently, J. Manschot [36] has reproduced the formula for \(P(m) - j(m)\) by directly computing the dimension of the image of the period map \(\text{Per}\) in (2.14).

3. Extremal \(\mathcal{N} = (2, 2)\) conformal field theories

3.1 Definition

In [49] Witten suggested that the holographic dual of pure 2+1 dimensional quantum gravity should be an “extremal conformal field theory.” The latter is defined to be a conformal field theory whose modular invariant partition function is “as close as possible” to the Virasoro character of the vacuum. When \(c = 24k\) the vacuum character is

\[
\chi^{(k)}_{\text{Vac}}(\tau) = q^{-k} \prod_{n=2}^{\infty} \frac{1}{1 - q^n} \cdot \frac{1}{1 - q^n}.
\]  

(3.1)

The partition function \(Z_k(\tau)\) has weight zero. Unlike the elliptic genus case, there is no obstruction to completing an arbitrary polynomial in \(q^{-1}\) to a modular function by adding nonpolar terms. Therefore, Witten defines \(Z_k(\tau)\) to be the unique modular function with no singularities for \(\tau \in \mathbb{H}\) such that the expansion around the cusp at infinity satisfies

\[
Z_k(\tau) := [q^{-k} \prod_{n=2}^{\infty} \frac{1}{1 - q^n}]_{q \leq 0} + \mathcal{O}(q) \cdot \frac{1}{1 - q^n}.
\]  

(3.2)
Following [16], Witten interprets the first Virasoro primary above the vacuum representation to be a state corresponding to the lightest possible BTZ black hole in AdS$_3$.

Following Witten [49] let us consider “pure $\mathcal{N} = (2, 2)$ supergravity” with negative cosmological constant. This is the hypothetical quantum theory whose classical action is a supersymmetric completion of the Einstein-Hilbert action,

$$I_{\text{sugra}} = \frac{1}{16\pi G} \int d^3x \sqrt{g} \left( R(g) + \frac{2}{R^2} + \ldots \right).$$

(3.3)

Here, $R$ is the AdS length scale and the ellipses denote contributions of other fields in the $\mathcal{N} = 2$ supergravity multiplet. Specifically, apart from the metric, these fields include real spin-$\frac{3}{2}$ gravitino fields, $\psi^i_L$ and $\psi^i_R$, $i = 1, 2$ as well as two abelian gauge fields, $a_L$ and $a_R$. In general, if we were interested in $\mathcal{N} = (p, q)$ supergravity theory, the corresponding gauge group would be $SO(p) \times SO(q)$. Thus, in the present context of $\mathcal{N} = (2, 2)$ theory we have $SO(2) \times SO(2)$ gauge fields.

In fact, by enlarging the gauge group one can write the entire supergravity action (3.3) as the Chern-Simons action [1, 2]:

$$I_{\text{CS}} = \frac{k_L}{4\pi} \int \text{tr} \left( A_L \wedge dA_L + \frac{2}{3} A_L \wedge A_L \wedge A_L \right)$$

$$- \frac{k_R}{4\pi} \int \text{tr} \left( A_R \wedge dA_R + \frac{2}{3} A_R \wedge A_R \wedge A_R \right)$$

(3.4)

where the gauge fields $A_L$ and $A_R$ take values in the Lie algebra of the supergroup

$$G = G_L \times G_R = OSp(2|2)_L \times OSp(2|2)_R.$$  

(3.5)

Since the bosonic part of the supergroup $OSp(2|2)$ is $SO(2) \times SL(2, \mathbb{R})$, the gauge group (3.3) contains the classical symmetry$^3$ group, $SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$, of the three-dimensional AdS space. In the simple case $k_L = k_R$, which will be of interest to us in the present paper, one finds the following relation between the parameters:

$$k_L = k_R = \frac{R}{16G}.$$  

(3.6)

Combining this with the Brown-Henneaux formula $c_L = c_R = \frac{2R}{2\pi}$ and using our expression for the central charge $c_L = c_R = 6m$, we can conveniently write (3.6) as

$$k_L = k_R = \frac{m}{4}.$$  

(3.7)

Since we take $m$ to be integer, it follows that $k_L$ and $k_R$ take values in $\frac{1}{4}\mathbb{Z}$. This is consistent with the fact that the bosonic part of our supergroup $OSp(2|2)$ contains $SL(2, \mathbb{R})$, which is a double cover of the identity component of $SO(2, 1)$; see section 2.1 of [19] for further details on the allowed values of $k_L$ and $k_R$.

The equivalence of $\mathcal{N} = (2, 2)$ supergravity and Chern-Simons theory based on the supergroup (3.3) is valid not only classically, but to all orders in perturbation theory, as

---

$^3$This symmetry group is the gauge group of the analogous formulation of $\mathcal{N} = 0$ gravity theory.
long as the perturbative expansion starts with a non-degenerate classical solution. This way of formulating perturbative $\mathcal{N} = (2, 2)$ supergravity will be useful to us in what follows, in particular, in section 8 where we discuss quantum corrections.

The $\mathcal{N} = (2, 2)$ case is similar to the $\mathcal{N} = 0$ case of Chern-Simons gravity: There are no local degrees of freedom, but the Chern-Simons theory does give rise to “edge states.” These are $\mathcal{N} = 2$ descendants of the vacuum representation, that is, the irreducible highest weight representation defined by $(h = 0, q = 0)$.

The natural generalization of Witten’s proposal to $(2, 2)$ supergravity in $2 + 1$ dimensions is that the holographic dual should be an “extremal $(2, 2)$ superconformal field theory,” where we define the latter to be a theory whose partition function is “as close as possible” to the vacuum character of the $\mathcal{N} = 2$ algebra. The vacuum character of the $\mathcal{N} = 2$ algebra is

$$\chi^{(m)}_{\text{vac}}(\tau, z) := \text{Tr}_{V_{0,0}} q^{L_0-c/24} e^{2\pi i(z+\frac{c}{24})J_0} = q^{-m/4} (1 - q^2)^{-1} \prod_{n=1}^{\infty} \frac{(1 - yq^{n+1/2})(1 - y^{-1}q^{n+1/2})}{(1 - q^n)^2}.$$ \hspace{1cm} (3.8)

We have shifted $z$ by 1/2 relative to the standard definition for later convenience. The expression in (3.8) is neither spectral flow invariant, nor modular invariant, and hence more terms must certainly be added to get a physical partition function.

In [16] the near horizon geometry of the D1D5 system was investigated and it was observed that the cosmic censorship bound for the BTZ black hole, which requires $r_\pm \geq 0$ for the two roots of the lapse function, can be translated into the holographic conformal field theory as the bounds

$$4m(L_0 - \frac{c}{24}) - J_0^2 \geq 0 \quad \& \quad 4m(\bar{L}_0 - \frac{c}{24}) - \bar{J}_0^2 \geq 0.$$ \hspace{1cm} (3.9)

In [16] the connection of these inequalities to the conditions on polarity of terms in the partition function was pointed out. We will assume here that for general $\mathcal{N} = (2, 2)$ supergravity the cosmic censorship bound continues to be (3.9). That is, black hole states must have $p, \bar{p} \geq 0$, where $p$ and $\bar{p}$ refer to the polarity of the left- and right-moving states (i.e., $p = 4mn - \ell^2$). In a theory of “pure supergravity” we would certainly want to require that all states with $p < 0$ and $\bar{p} < 0$ are $\mathcal{N} = 2$ descendents of the vacuum (or their spectral flow images). These considerations, then, motivate our definition of an $\mathcal{N} = (2, 2)$ extremal conformal field theory to be:

**Definition:** An $\mathcal{N} = (2, 2)$ extremal conformal field theory of level $m$ (“$\mathcal{N} = 2$ ECFT” for short) is a hypothetical theory whose partition function is of the form:

$$Z_{\text{NSNS}}(\tau, z; \bar{\tau}, \bar{z}) := \text{Tr}_{H_{\text{NSNS}}} q^{L_0-c/24} e^{2\pi i z J_0} e^{2\pi i \bar{z} \bar{J}_0} e^{i\pi (J_0 - \bar{J}_0)}$$

$$= \sum_{s, \tilde{s} \in Z} SF_s \chi^{(m)}_{\text{vac}}(\tau, z) SF_{\tilde{s}} \chi^{(m)}_{\text{vac}}(\bar{\tau}, \bar{z})$$

$$+ \sum_{s \in Z} SF_s \chi^{(m)}_{\text{vac}}(\tau, z) \bar{f}(\bar{\tau}, \bar{z}) + \sum_{\tilde{s} \in Z} f(\tau, z) SF_{\tilde{s}} \chi^{(m)}_{\text{vac}}(\bar{\tau}, \bar{z})$$

$$+ \sum_{p, \tilde{p} \geq 0} a(n, \ell; \tilde{n}, \tilde{\ell}) q^n y^\ell \bar{q}^{\tilde{n}} \bar{y}^{\tilde{\ell}}.$$ \hspace{1cm} (3.10)
Here the coefficients $a(n, \ell; \tilde{n}, \tilde{\ell})$ are integers, and the sum over nonpolar states in the last line means that both the left and right polarity of the state is non-negative. The functions $f(\tau, z)$ and $\bar{f}(\bar{\tau}, \bar{z})$ describe the contribution of terms with non-negative polarity with respect to the left and right polarity, respectively. We need to include such terms since states with either $p \geq 0$ or $\tilde{p} \geq 0$ are not polar and are allowed by the extremality condition.

Using spectral flow (2.2) we can compute $Z_{RR}(\tau, z; \bar{\tau}, \bar{z})$ for an $\mathcal{N} = 2$ ECFT from (3.10). The elliptic genus is then obtained upon setting $\bar{z} = 0$. In this limit only those terms that have $\bar{q}^0$ contribute. All of these terms have negative polarity, with the exception of the $\bar{q}^0\bar{y}^0$ term that has polarity zero. Thus the elliptic genus of an $\mathcal{N} = 2$ ECFT of level $m$ is of the form

$$
(2(-1)^m + u) \sum_{\theta \in \mathbb{Z} + \frac{1}{2}} SF_{\theta} \chi_{\text{vac}}^{(m)} + \text{Nonpolar},
$$

where $u$ is the coefficient of the $\bar{q}^0\bar{y}^0$ term coming from $\bar{f}(\bar{\tau}, \bar{z})$. The factor $2(-1)^m$ is the limit $\bar{z} \to 0$ of the first term in (3.10), as we will see momentarily. Using (5.21) below one can determine the constant to be $u = 12m - 2$. For convenience we drop the overall constant factor from the right-movers and define:

$$
\chi_{\text{ext}}^{(m)}(\tau, z) := \sum_{\theta \in \mathbb{Z} + \frac{1}{2}} SF_{\theta} \chi_{\text{vac}}^{(m)} + \text{Nonpolar}.
$$

We will call a weak Jacobi form that satisfies (3.12) an extremal elliptic genus. Because the only unknown terms in (3.12) are nonpolar terms we can compute the polar polynomial of such an extremal elliptic genus. We will give an explicit formula for it in section 3.2. Then, in section 4 we investigate whether such a polar polynomial is consistent with modularity.

### 3.2 The extremal polar polynomial

Let us compute the polar polynomial of a would-be extremal elliptic genus. We begin by demonstrating the following useful fact:

$$
\text{Pol}\left( \sum_{\theta \in \mathbb{Z} + \frac{1}{2}} SF_{\theta} \chi_{\text{vac}}^{(m)} \right) = \text{Pol}(SF_{1/2}\chi_{\text{vac}}^{(m)}).
$$

Indeed, if we apply the spectral flow by $\theta = l + \frac{1}{2}$ to the vacuum character (3.8) we obtain an expression of the form

$$
(-1)^m q^{l(l+1)m}y^{2(l+1)m}(1 - q) \prod_{n=1}^{\infty} \frac{(1 - yq^{n+l+1})(1 - y^{-1}q^{n-l})}{(1 - q^n)^2}.
$$

We wish to show that this expression contains no polar terms in the fundamental domain (2.11) for $l \neq 0$. Without loss of generality, we can assume $l > 0$. Note that it is not true that (3.14) has no polar terms. In fact, already the first term $q^{l(l+1)m}y^{2(l+1)m}$ is polar for every $l$; it has polarity $p = -m^2$. However, it does not belong the polar region $\mathcal{P}^{(m)}$ since the power of $y$ is not in the allowed range $1 \leq \ell \leq m$. 

---
On the other hand, there are terms in (3.14) with $1 \leq \ell \leq m$ but, as we show momentarily, these terms are not polar. We can simplify the problem a little bit and omit the denominator in (3.14) and the factor $(1 - q)$ which can only increase the polarity. Then, our goal is to show that

$$q^{l(l+1)m} y^{(2l+1)m} \prod_{n=1}^{\infty} \frac{1 - q^n}{1 - y q^{n+1}} \frac{1 - y^{-1} q^{n-1}}{1 - y^{-1} q^{n-l}}$$

has no polar terms in the range $1 \leq \ell \leq m$. From the above discussion, we already know that the term $q^{l(l+1)m} y^{(2l+1)m}$ is polar. We can combine it with the terms from factors $(1 - y q^{n+1})$ and $(1 - y^{-1} q^{n-l})$ for various $n$ to bring the power of $y$ to the desired range. Since $l$ is assumed to be positive, it is easy to see that the terms coming from factors $(1 - y q^{n+1})$ can be ignored, while from $\prod_{n=1}^{\infty} (1 - y^{-1} q^{n-l})$ we need to collect at least $2lm$ factors of $y^{-1}$ to bring the overall power of $y$ to the desired range. The most economical way to do this (which yields the minimal increase in polarity) is to collect the factors in the infinite product with the smallest powers of $q$. These are the terms with $n = 1, \ldots, 2lm$:

$$q^{l(l+1)m} y^{(2l+1)m} \prod_{n=1}^{2lm} y^{-1} q^{n-l} = q^{(2lm - l + 2)lm} y^m.$$  

(3.16)

The resulting term has polarity $p = 4(2lm - l + 2)lm^2 - m^2$ which satisfies $p > 0$ for any $l, m \geq 1$. It is easy to see that including other factors from the infinite product in (3.17) only increases the polarity further.

Having proven (3.13) we now define

$$p_{ext}^{(m)} := (-1)^m \text{PolSF}_{1/2 \chi_{vac}}^{(m)}.$$  

(3.17)

On the other hand, setting $l = 0$ in (3.14) one finds

$$(-1)^m \text{SF}_{1/2 \chi_{vac}}^{(m)} = (1 - q) y^m \prod_{n=1}^{\infty} \frac{(1 - y q^{n+1})(1 - y^{-1} q^n)}{(1 - q^n)^2}.$$  

(3.18)

The Fourier expansion of (3.18) begins:

$$y^m + q(y^m - y^{m-1}) + q^2(-2y^{-1+m} + 3y^m - y^{1+m}) + \cdots.$$  

(3.19)

The first few polar polynomials follow easily from (3.19) since the polar terms for index
\( m \) have \( n \leq \left\lfloor \frac{m}{4} \right\rfloor \). In this way we find that the first few polar polynomials are:

\[
\begin{align*}
  p_1^{\text{ext}} &= y \\
  p_2^{\text{ext}} &= y^2 \\
  p_3^{\text{ext}} &= y^3 \\
  p_4^{\text{ext}} &= y^4 \\
  p_5^{\text{ext}} &= (1 + q)y^5 \\
  p_6^{\text{ext}} &= (1 + q)y^6 - qy^5 \\
  p_7^{\text{ext}} &= (1 + q)y^7 - qy^6 \\
  p_8^{\text{ext}} &= (1 + q)y^8 - qy^7 \\
  p_9^{\text{ext}} &= (1 + q + 3q^2)y^9 - qy^8 \\
  p_{10}^{\text{ext}} &= (1 + q + 3q^2)y^{10} - (q + 2q^2)y^9 \\
  p_{11}^{\text{ext}} &= (1 + q + 3q^2)y^{11} - (q + 2q^2)y^{10} \\
  p_{12}^{\text{ext}} &= (1 + q + 3q^2)y^{12} - (q + 2q^2)y^{11}.
\end{align*}
\]

4. Experimental search for the extremal elliptic genus

Since \( P(m) > j(m) \) for \( m \geq 5 \), and since eq. (3.18) does not have any obvious modular properties, it is far from obvious that (3.13) is the polar polynomial of a true weak Jacobi form. In this section we describe numerical results suggesting that in fact, for all but finitely many \( m \) it is not in the image of Pol applied to \( \tilde{J}_{0,m} \). We will find that there are actually some “exceptional” cases where it is in the image for \( m \geq 5 \). In section 5 we will show analytically that there can only be a finite number of such exceptional cases. That might seem to obviate the need for the present section, but the methods we employ here will prove very useful when we come to describe nearly extremal theories in section 6.

Choose a basis \( \phi_i, i = 1, \ldots, j(m) \) for \( \tilde{J}_{0,m} \). We are searching for real numbers \( x_i \) such that

\[
\sum_{i=1}^{j(m)} x_i \text{Pol}(\phi_i) = p_{\text{ext}}^{(m)}.
\]

A useful way of trying to solve this equation is the following. We choose a polarity-ordered basis of monomials \( q^n y^\ell \) for \( V_m \), that is the basis monomials \( q^{n(a)} y^{l(a)} \) where \( a = 1, \ldots, \dim V_m = P(m) \) so that polarity increases as \( a \) increases, and terms with the same polarity are ordered in increasing powers of \( y \). For example for \( a = 1 \) the most polar term is \( y^m \). A polarity-ordered basis for \( V_5 \) would be

\[
y^5, y^4, y^3, qy^5, y^2, y^1
\]

with \( a = 1, \ldots, 6 \). The polarity-ordered basis will be very useful for our discussion of \( \beta \)-extremal \( \mathcal{N} = 2 \) conformal field theories in section 6.
Having chosen these two bases we can define a matrix \( N_{ia} \) of dimensions \( j(m) \times P(m) \) from the expansion
\[
\text{Pol}(\phi_i) = \sum_{a=1}^{P(m)} N_{ia} q^{n(a)} y^{\ell(a)}.
\] (4.3)

Similarly, we can define coefficients \( d_a \) by
\[
P^{(m)}_{\text{ext}} = \sum_{a=1}^{P(m)} d_a q^{n(a)} y^{\ell(a)}.
\] (4.4)

Thus, we are trying to solve the linear equations
\[
\sum_{i=1}^{j(m)} x_i N_{ia} = d_a, \quad a = 1, \ldots, P(m).
\] (4.5)

It should be stressed that even if we can find a solution \( x_i \) to (4.5) we are far from establishing the existence of an \( \mathcal{N} = 2 \) extremal theory. If a solution exists then the next test we should apply is to see whether the resulting form \( \sum x_i \phi_i \) has integral Fourier coefficients. Integrality is clearly a necessary condition for any candidate elliptic genus since it arises in conformal field theory from the trace on a Hilbert space.

Using a computer (and the explicit basis (2.23) above) we have examined equation (4.3) for \( 1 \leq m \leq 36 \). We have found that there is a solution \( x_i \) in rational numbers for \( 1 \leq m \leq 5 \) and for \( m = 7, 8, 11, 13 \), but there is no solution for \( m = 6, 9, 10 \) and \( 14 \leq m \leq 36 \).\(^4\) Moreover, remarkably, for those values of \( m \) which give a solution, the Fourier coefficients we have explicitly evaluated turn out to be integral.

The simplest example is the case \( m = 1 \), in which case \( \chi^{(1)}_{\text{ext}} = \bar{\phi}_{0,1} \). The next simplest case, \( m = 2 \) yields
\[
\chi^{(2)}_{\text{ext}} = \frac{1}{6}(\bar{\phi}_{0,1})^2 + \frac{5}{6}(\bar{\phi}_{-2,1})^2 E_4.
\] (4.6)

Although it is not obvious, one can prove that the Fourier coefficients are all integral. Indeed, the claim that this expression has integer Fourier coefficients is equivalent to the statement
\[
(\bar{\phi}_{0,1})^2 + 5(\bar{\phi}_{-2,1})^2 E_4 = 0 \mod 6.
\] (4.7)

In order to prove this, it is convenient to note (see (2.17) and (2.18)) that:
\[
E_4 = 1 \mod 6, \quad E_6 = 1 \mod 6.
\]

Moreover, from (2.21) it also follows that \( \bar{\phi}_{10,1} = \bar{\phi}_{12,1} \mod 6 \), which in turn implies \( \bar{\phi}_{-2,1} = \bar{\phi}_{0,1} \mod 6 \), cf. (2.22). Substituting this into \( (\bar{\phi}_{0,1})^2 + 5(\bar{\phi}_{-2,1})^2 \) and using the fact that \( \bar{\phi}_{0,1} \) and \( \bar{\phi}_{-2,1} \) have integer Fourier coefficients we therefore demonstrate (4.7).

\(^4\)The arguments of section 5 demonstrate that there can only be finitely many solutions. Using the constraints of that section it is easy to check that there are no further solutions up to \( m \leq 400 \). This suggests that the above list is in fact complete.
When we use the basis (2.23) the solutions $x_i$ are rational numbers with increasingly large denominators as $m$ increases. For example, already the next case, $m = 3$, looks like

$$\chi_{\text{ext}}^{(3)} = \frac{1}{48}(\tilde{\phi}_{0,1})^3 + \frac{7}{16}\tilde{\phi}_{0,1}(\tilde{\phi}_{-2,1})^2E_4 + \frac{13}{24}(\tilde{\phi}_{-2,1})^3E_6 .$$

(4.8)

Even though the coefficients $x_i$ of every monomial $(\tilde{\phi}_{-2,1})^a(\tilde{\phi}_{0,1})^bE_4^cE_6^d$ are rational numbers, the Fourier coefficients $c(n,\ell)$ are integers. In order to show this, as in the previous example, we express this as the following statement

$$(\tilde{\phi}_{0,1})^3 + 21\tilde{\phi}_{0,1}(\tilde{\phi}_{-2,1})^2E_4 + 26(\tilde{\phi}_{-2,1})^3E_6 = 0 \mod 48 .$$

(4.9)

Then, using (2.17) we note that $E_4 = 1 \mod 48$, so we can ignore $E_4$ in this computation. It is not true, however, that $E_6 = 1 \mod 48$. Instead, from (2.18) we find that $E_6^3 = 1 \mod 48$. According to (2.21) and (2.21), this implies the following identity,

$$\tilde{\phi}_{-2,1} = \tilde{\phi}_{0,1}E_6 \mod 48 ,$$

which, after substituting in the LHS of (4.9), proves the desired result.

Using the basis of weak Jacobi forms described in section 7 below one can check that for the “miraculous” values $m = 5, 7, 8, 11, 13$ the solution does indeed have the property that all the Fourier coefficients $c(n,\ell)$ are integers.

5. The extremal elliptic genus does not exist for $m$ sufficiently large

In this section we give an analytic proof that there is no weak Jacobi form in $\tilde{J}_{0,m}$ satisfying (3.12) for $m$ sufficiently large. Since this section is rather long and technical let us summarize the main idea here. Using the spectral flow symmetry one can determine the NS sector character (without an insertion of $y_J^0$ or $(-1)^F$) from the elliptic genus. This character is a modular form for a congruence subgroup $\Gamma_\theta$ of the modular group. It is therefore highly constrained, and as in the case discussed in [49], determined by the coefficients of the negative powers of $q$, which in turn are fixed by the polar terms of the original elliptic genus. On the other hand, given the full NS sector character, we can also determine from it, by a suitable modular transformation, the R-sector character (without an insertion of $(-1)^F$), and thus, in particular, its leading term in the $q$-expansion. This coefficient is however also directly determined by the extremal hypothesis and a sum rule (5.20) for Fourier coefficients. The two ways of evaluating the same coefficient lead to a non-trivial constraint on $m$, equation (5.19). Using properties of modular forms one can show that this constraint is violated for sufficiently large $m$. The argument must be broken up into cases: $m$ odd, $m = 2 \mod 4$ and $m = 0 \mod 4$, of which the last case is technically the most difficult. In this section we will give the main line of argument, whereas the technical details can be found in appendix A.
5.1 NS-sector elliptic genus

Suppose $\chi(\tau, z)$ is the elliptic genus of a CFT with $\chi \in \tilde{J}_{0,m}$. By spectral flow we define the “NS sector elliptic genus” to be

$$\chi_{NS}(\tau, z) := e \left[ m \left( \frac{\tau}{4} + z + \frac{1}{2} \right) \right] \chi \left( \tau, z + \frac{\tau}{2} + \frac{1}{2} \right). \quad (5.1)$$

Using the transformation properties of a Jacobi form it follows easily that

$$\begin{align*}
\chi_{NS}(-1/\tau, z/\tau) &= (-1)^m e \left( \frac{mz^2}{\tau} \right) \chi_{NS}(\tau, z) \\
\chi_{NS}(\tau + 2, z) &= (-1)^m \chi_{NS}(\tau, z). 
\end{align*} \quad (5.2)$$

If we put $z = 0$ we thus obtain simple transformation laws for $\chi_{NS}(\tau) := \chi_{NS}(\tau, 0)$ under the congruence subgroup $\Gamma_\theta = \langle T^2, S \rangle$. (In this section we consider the modular group to be $PSL(2, \mathbb{Z})$.) For $m$ even we have a strict modular function and for $m$ odd we have a function with multiplier system given by $-1$ on the two generators.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fundamental_domain.png}
\caption{The fundamental domain $\mathcal{F}_\theta$ of the genus zero subgroup $\Gamma_\theta$ of $\Gamma$.}
\end{figure}

To begin, let us sketch a few mathematical facts. The group $\Gamma_\theta$ is a genus zero subgroup of $\Gamma$. It has modular domain $\mathcal{F}_\theta = \mathcal{F} \cup T \cdot \mathcal{F} \cup TS \cdot \mathcal{F}$ shown in figure 3. Note there are two cusps, equivalent to $\tau = i\infty$ and $\tau = 1$.

Since $\mathbb{H}/\Gamma_\theta$ is genus zero the function field has a generator $\hat{K}(\tau)$ which can be uniquely specified (up to an additive and multiplicative constant) by demanding that $\hat{K}$ takes $i\infty$ to $\infty$. An explicit choice is:

$$\hat{K}(\tau) := \frac{\vartheta_{3}^{12}(\tau)}{\eta^{12}(\tau)} = \frac{\Delta^2(\tau)}{\Delta(2\tau)\Delta(\tau/2)} = q^{-1/2} + 24 + 276q^{1/2} + \cdots . \quad (5.3)$$

5Note that unlike the NS vacuum character $\chi_{NS}(\tau, z)$ does not involve the shift of $z$ by $1/2$.

6Such a function for a genus zero congruence subgroup is often referred to as a “Hauptmodul.”
The expansion of $\hat{K}$ around the cusp at $\tau = 1$ is obtained by writing $\tau = 1 - \frac{1}{r}$ and observing that
\[ \hat{K}(\tau) := \hat{K}(\tau_r) = -\frac{\vartheta_2^4}{\eta_1^2} (\tau_r) = -2^{12} q_r + \cdots, \] (5.4)
where $q_r = e(\tau_r)$.

In order to work with the case of $m$ odd it will be useful to consider the index two subgroup\(^7\) $\tilde{\Gamma}_\theta := \langle T^4, ST^2 \rangle$ such that $\Gamma_\theta = \tilde{\Gamma}_\theta \cup S \cdot \tilde{\Gamma}_\theta$. This is again a genus zero subgroup, and its Hauptmodul is the NS-sector character of $\tilde{\phi}_{0,1}$ (i.e. the elliptic genus for $K3$ divided by two). Using the definition (5.1) with $m = 1$ and $\chi = \tilde{\phi}_{0,1}$ and putting $z = 0$ one finds
\[ \kappa(\tau) := \left(\frac{2 \vartheta_4}{\vartheta_2}\right)^2 - \left(\frac{2 \vartheta_2}{\vartheta_4}\right)^2 = q^{-1/4} (1 - 20q^{1/2} + \cdots) . \] (5.5)

This function satisfies $\kappa|_S = -\kappa$ and $\kappa|_{T_2} = -\kappa$, and is thus odd under the Deck transformation $\mathbb{H}/\tilde{\Gamma}_\theta \rightarrow \mathbb{H}/\Gamma_\theta$. Indeed,
\[ \kappa^2(\tau) = \hat{K}(\tau) - 64 , \] (5.6)
giving the explicit double cover. Near the Ramond cusp $\kappa$ has the expansion
\[ \kappa(1 - 1/\tau_r) := \tilde{\kappa}(\tau_r) = -4i \left[ \left(\frac{\vartheta_3}{\vartheta_4}\right)^2 + \left(\frac{\vartheta_4}{\vartheta_3}\right)^2 \right] (\tau_r) = -8i \left[ 1 + 32q_r + O(q_r^2) \right] . \] (5.7)

Now, $\chi_{NS}$ has no singularities for $\tau \in \mathbb{H}$, and, moreover, using again the transformation laws of a Jacobi form
\[ \chi_{NS}(1 - 1/\tau_r) = e \left( -\frac{m}{4} \right) \chi(\tau_r, \frac{1}{2}) = e \left( -\frac{m}{4} \right) \sum_{n,\ell} c(n, \ell) (-1)^\ell q_r^n . \] (5.8)

By unitarity the sum is over $n \geq 0$ and hence $\chi_{NS}(\tau)$ must be a polynomial in $\kappa(\tau)$. This polynomial will be even for $m$ even and odd for $m$ odd. Moreover, the polynomial is fixed by the coefficients of the nonpositive powers of $q$. Those coefficients in turn are related to the polar contributions to $\chi(\tau, z)$. To demonstrate the relationship note that
\[ \chi_{NS}(\tau) = \sum_{n,\ell} c(n, \ell) (-1)^{m+\ell} q_r^{m+n+\frac{\ell}{2}} . \] (5.9)

Now write:
\[ 4mn - \ell^2 = 4m \left(\frac{m}{4} + n + \frac{\ell}{2}\right) - (m + \ell)^2 . \] (5.10)

The nonpolar terms in $\chi(\tau, z)$ have $4mn - \ell^2 \geq 0$ and therefore from (5.9) contribute only nonnegative powers of $q$ in (5.10). In fact, they always contribute positive powers with precisely one exception: when $4mn - \ell^2 = 0$ and $\ell = -m$. In that case $n = m/4$. Note that this cannot happen if $m \neq 0 \mod 4$ because $n$ is integral.

---

\(^7\)To prove the subgroup is index two note that for all $n \in \mathbb{Z}$, $T^{4n}$, $ST^{4n+2}$, $T^{4n+2}S$ and $ST^{4n}S$ are in $\Gamma_\theta$. Then use induction on the length of the word in $S, T^2$. Recall that in this section modular transformations are regarded as elements of $PSL(2, \mathbb{Z})$. 
5.2 A nontrivial constraint

In this subsection we assume \( m \neq 0 \mod 4 \). We return to a discussion of the case \( m = 0 \mod 4 \) in subsection 5.3 below.

Our conclusion thus far is that for \( m \neq 0 \mod 4 \), \( \chi_{NS}(\tau) \) is a modular function for \( \tilde{\Gamma}_\theta \) such that

\[
\chi_{NS}(\tau) = \sum_{\theta \in \mathbb{Z}} SF_\theta \chi_{vac}^{(m)}(\tau, z) \big|_{z = \frac{1}{2}} + O(q^{1/4}).
\] (5.11)

One easily finds that only \( \theta = 0 \) can contribute to negative powers of \( q \) and hence we can simplify this equation to

\[
\chi_{NS}(\tau) = q^{-m/4} \left( 1 - q \right) \prod_{n=1}^{\infty} \frac{(1 + q^{n+1/2})^2}{(1 - q^n)^2} + O(q^{1/4}).
\] (5.12)

This has expansion

\[
q^{-m/4} \left( 1 + q + 2q^{3/2} + 3q^2 + 4q^{5/2} + 6q^3 + \cdots \right).
\] (5.13)

While the expression on the RHS of (5.12) is not modular, it can be written as:

\[
\chi_{NS}(\tau) = q^{-m/4} \frac{1 - q^{1/2}}{1 + q^{1/2}} \varphi_3 + O(q^{1/4}).
\] (5.14)

Now we can write an explicit formula for \( \chi_{NS}(\tau) \). Define expansion coefficients:

\[
q^{-m/4} 1 - q^{1/2} \varphi_3 = \sum_{\alpha = -m/4}^{\infty} \tilde{h}(\alpha)q^\alpha.
\] (5.15)

Note that \( \tilde{h}(\alpha) \) is only nonzero for \( \alpha \in \frac{1}{2}\mathbb{Z} \), for \( m \) even and \( \frac{1}{4} + \frac{1}{2}\mathbb{Z} \) for \( m \) odd. For \( \alpha \in \frac{1}{4}\mathbb{Z}_+ \) let \( \varphi_\alpha \) be the unique polynomial of degree 4\( \alpha \) such that

\[
\varphi_\alpha(\kappa) = q^{-\alpha} + O(q^{1/4}) \, , \quad \alpha \in \frac{1}{4}\mathbb{Z}_+.
\] (5.16)

Then for \( m \neq 0 \mod 4 \)

\[
\chi_{NS} = \sum_{\alpha = -m/4}^{0} \tilde{h}(\alpha) \varphi_{-\alpha}(\kappa).
\] (5.17)

On the other hand, if we expand around the cusp \( \tau = 1 \) then, by (5.8)

\[
\sum_{\alpha = -m/4}^{0} \tilde{h}(\alpha) \varphi_{-\alpha}(\kappa(\tau_r)) = e^{-i\pi m/2} \sum_{n, \ell} c(n, \ell) (-1)^\ell q_r^n.
\] (5.18)

In particular, if we take \( \tau_r \to i\infty \), then we arrive at the key constraint:

\[
L := \sum_{\alpha = -m/4}^{0} \tilde{h}(\alpha) \varphi_{-\alpha}(-8i) = e^{-i\pi m/2} \sum_{\ell} c(0, \ell) (-1)^\ell.
\] (5.19)
The argument for the non-existence of the extremal elliptic genus is based on showing that, for large $m$, the left-hand side and right-hand side of (5.19) have different growth rates. As we shall see momentarily, the right-hand side is always an affine linear function of $m$, while the left-hand side grows exponentially for $m = 2 \mod 4$; for $m$ odd, the left-hand side grows also linearly in $m$, but the coefficient is different.

Let us first establish the growth property of the right-hand side. By the ansatz for pure supergravity we know that the only nonzero polar coefficients $c(0, \ell)$ occur for $\ell = \pm m$ and are given by 1. The coefficient $c(0, 0)$ is not polar. Fortunately, Gritsenko has proven a useful identity for the Fourier coefficients of weak Jacobi forms of index $m$ [26]:

$$m \sum_\ell c(0, \ell) = 6 \sum_\ell \ell^2 c(0, \ell).$$  (5.20)

Using (5.20) and (3.18) we can solve for $c(0, 0)$ to get

$$c(0, 0) = 12m - 2,$$

and therefore

$$\sum_\ell c(0, \ell)(-1)^\ell = 12m - 2 + 2(-1)^m = \begin{cases} 12m & m \text{ even} \\ 12m - 4 & m \text{ odd.} \end{cases}$$  (5.21)

In particular, the right-hand side of (5.19) grows linearly with $m$.

Now let us turn to the left-hand side of (5.19). Observe that this is the $q^0$ term in the $q$-expansion of

$$\left( \sum_{\alpha \geq -m/4} \tilde{h}(\alpha) q^\alpha \right) \left( \sum_{n \geq 0} q^{n/4} \varphi_{n/4}(-8i) \right).$$  (5.22)

On the other hand, using the fact that $\kappa$ is a Hauptmodul one can show, that $^9$

$$\sum_{n=0}^{\infty} q^{n/4} \varphi_{n/4}(z) = \frac{4 q^{d/2} \kappa}{z - \kappa}.$$  (5.23)

---

^8The proof is very simple: $\exp[-8\pi^2 m \tau_2^2(\tau)z^2] \chi(\tau, z)$ transforms as a weight zero modular form. Therefore the coefficients of $z^{2n}$ in the Taylor series around $z = 0$ transform like forms of weight $2n$. In particular the coefficient of $z^2$ must vanish, since there are no modular forms of weight two.

^9Write $\varphi_\alpha(z) = \frac{\ell_C}{\ell(\tau)} \frac{e^{i \pi \alpha}}{2 \pi i}$ where the contour is on a large circle $C$ in the $\ell$ plane. Now make the change of variables $\ell \rightarrow \ell(r) := r^{-1} - 20r + \cdots$ so that $\ell(q^{1/4}) = \kappa$. This gives a one-one map of $C$ to a small circle $C'$ around the origin. Using $\varphi_\alpha(\ell(r)) = r^{-4\alpha} + O(r)$, and taking the circle to be small we see $\varphi_\alpha(z) = - \frac{\ell'(r)}{\ell(r)} r^{-4\alpha} \frac{dr}{2\pi i}$. It is now straightforward to sum the series and apply Cauchy’s theorem to arrive at (5.23). We thank Terry Gannon for pointing out this crucial identity to us.
In order to apply this to our problem we use the identities

\[ 24q \frac{d}{dq} \log \vartheta_4 = E_2 - (\vartheta_2^4 + \vartheta_3^4) \]
\[ 24q \frac{d}{dq} \log \vartheta_3 = E_2 + (\vartheta_2^4 - \vartheta_4^4) \]
\[ 24q \frac{d}{dq} \log \vartheta_2 = E_2 + (\vartheta_3^4 + \vartheta_4^4) \]

(5.24)
to compute \( 4q \frac{d}{dq} \kappa = -4\vartheta_2^3/(\vartheta_2^2\vartheta_4^2) \). Using the “abstruse identity” \( \vartheta_2^3 = \vartheta_2^4 + \vartheta_4^4 \) it follows that

\[ \sum_{n=0}^{\infty} q^{n/4} \varphi_{n/4}(-8i) = (\vartheta_2^2 - i\vartheta_4^2)^2 . \]

(5.25)

Thus, we need to estimate the large \( m \) behavior of

\[ L := \left[ q^{-m/4} \frac{1 - q^{1/2} \vartheta_3}{1 + q^{1/2} \vartheta_3} (\vartheta_4^2 - i\vartheta_2^2)^2 \right] . \]

(5.26)

We estimate the growth behavior of \( L \) in appendix \( A \), and it turns out to be quite different for even and odd.

For \( m \) odd, \( e^{i\pi m/2} L \) is positive, and is bounded below by

\[ e^{i\pi m/2} L \geq 4\pi m - 8\pi \sqrt{m - \frac{5}{2}} - 6\pi . \]

(5.27)

Since \( 4\pi > 12 \), this will asymptotically (i.e. for \( m \geq 2000 \)) grow more quickly than (5.21). We have checked that among the first 2000 terms, the two numbers only agree for \( m = 1, 3, 5, 7, 11, 13, 19, 31, 41 \). For \( m = 1, 3, 5, 7, 11, 13 \) there exists indeed a sugra elliptic genus, while for \( m = 19, 31, 41 \) there does not, as we have verified explicitly. (Note that the fact that the two numbers agree does not imply that there must exist a sugra elliptic genus!)

For \( m = 2 \mod 4 \), \( L \) turns out to grow exponentially, so that (5.19) cannot be satisfied for \( m \) large enough. For details of the calculation, see again appendix \( A \).

5.3 A constraint for \( m = 0 \mod 4 \)

We now turn to the case \( m = 0 \mod 4 \). As we have pointed out above, in this case non-polar terms contribute to the constant term of \( \chi_{NS} \). We thus need to make the more general

---

10To prove these identities note that \( (24q \frac{d}{dq} - E_2) \vartheta_2 \) must be a weight 5/2 modular form for \( \Gamma(2) \) and hence is a polynomial of degree 5 in \( \vartheta_2, \vartheta_3, \vartheta_4 \). Moreover, the \( q \) expansion has only coefficients \( q^{k+n} \) with \( n \) integer. Together with the transformation property under \( \tau \to \tau + 1 \) this fixes it to be of the form \( \vartheta_2(a\vartheta_2^3 + \vartheta_4^3) + b\vartheta_3\vartheta_4(\vartheta_2^4 + \vartheta_3^4) + c\vartheta_2^4\vartheta_4^2 \) for some constants \( a, b, c \). Now, matching the first 3 coefficients of the \( q \) expansion on the left and right hand sides we find \( a = 1, b = c = 0 \). The other two equations now follow by modular transformations. These identities also have nice interpretations in terms of massless free fermions on a two-dimensional torus. One can compute the expectation value of their energy either by differentiating their partition function or by evaluating the energy-momentum tensor using the fermion two-point function. Requiring that these two methods produce the same answer implies these identities [13].

---
ansatz
\[ \chi_{NS}(\tau, z) = q^{-\frac{1}{2} + \frac{1}{4}} \frac{1 - q}{(1 + y q^{1/2})(1 + y^{-1} q^{1/2})} \frac{\vartheta_3(\tau, z)}{\eta^3} + d + \mathcal{O}(q^{1/2}) . \] (5.28)

Instead of (5.17) we obtain
\[ \chi_{NS} = \sum_{\alpha = -m/4}^{0} \tilde{h}(\alpha) \varphi_{-\alpha}(\kappa) + d . \] (5.29)

The argument of section 5.2 can then be used to fix the value of \( d \):
\[ d = 12m - \left[ q^{-\frac{1}{2} + \frac{1}{4}} \frac{1 - q^{1/2}}{1 + q^{1/2}} \frac{\vartheta_3(\tau)}{\eta^3} (\vartheta_4^4 - \vartheta_2^4) \right] q^0 . \] (5.30)

We obtain an additional constraint on the theory in the following way: Let
\[ \hat{D} := \left( y d \frac{d}{dy} \right)^2 \frac{m}{6} E_2 . \] (5.31)

Then \( \hat{\chi}_{NS}(\tau) := \hat{D}(\chi_{NS}(\tau, z)) |_{z=0} \) is a weight two weakly holomorphic modular form for \( \Gamma_\theta \) which moreover satisfies
\[ \hat{\chi}_{NS}(1 - 1/\tau_r) = \tau_r^2 \hat{D}(\chi(\tau, z)) |_{z=1/2} . \] (5.32)

The \( q_r \to 0 \) limit of the coefficient of \( \tau_r^2 \) of the right-hand-side of (5.32) is
\[ \sum c(0, \ell)(-1)^\ell \ell^2 - \frac{m}{6} \sum c(0, \ell)(-1)^\ell = 2m^2 - \frac{m}{6} 12m = 0 . \] (5.33)

On the other hand, weakly holomorphic modular forms of weight two for \( \Gamma_\theta \) are of the form
\[ (\vartheta_4^4 - \vartheta_2^4) \times L(\hat{K}) , \] (5.34)

where \( L(\hat{K}) \) is a Laurent series in \( \hat{K} \). By examining the Ramond cusp we see that \( L(\hat{K}) \) must be a polynomial in \( \hat{K} \). Define polynomials \( \hat{P}_a(\hat{K}) = q^{-a/2} + \mathcal{O}(q^{1/2}) \) for \( a \geq 0 \) and
\[ \hat{P}_a(\hat{K})(\vartheta_4^4 - \vartheta_2^4) = \begin{cases} 1 + \mathcal{O}(q^{1/2}) & a = 0 \\ a q^{-a/2} + \mathcal{O}(q^{1/2}) & a > 0 . \end{cases} \] (5.35)

Using (5.24) we find
\[ 2q d \frac{d}{dq} \hat{K} = \hat{K}(\vartheta_2^4 - \vartheta_4^4) , \] (5.36)

from which we deduce
\[ \hat{P}_a(z) = \begin{cases} -1 & a = 0 \\ -z P'_a(z) & a > 0 . \end{cases} \] (5.37)

Define expansion coefficients
\[ \hat{\chi}_{NS}(\tau) = \sum_{\alpha = -m/4}^{\alpha}(2\alpha)x(\alpha)q^\alpha + X(0) . \] (5.38)
If the extremal elliptic genus exists then

\[ \hat{\chi}_{NS}(\tau) = \sum_{\alpha < 0} x(\alpha) \tilde{P}_{-\alpha}(\tilde{K})(\vartheta_2^4 - \vartheta_4^4) - X(0)(\vartheta_2^4 - \vartheta_4^4) . \]  

(5.39)

Evaluating at the Ramond cusp we have

\[ \tau_r^2 \left( X(0)(\vartheta_2^4 + \vartheta_3^4) - \sum_{\alpha < 0} x(\alpha) \tilde{P}_{-\alpha}(\tilde{K})(\vartheta_2^4 + \vartheta_3^4) \right) , \]  

(5.40)

and evaluating at \( q_r \to 0 \) the coefficient of \( \tau_r^2 \) becomes simply \( 2X(0) \) since \( \tilde{P}_\alpha(0) = 0 \) for \( \alpha > 0 \). Therefore, \( X(0) = 0 \).

On the other hand, we can deduce the coefficient \( X(0) \) directly from the \( q^0 \) term of \( \hat{D}\chi_{NS} \). Expressing \( \chi_{NS} \) by (5.28) and (5.30) and then using

\[ \frac{1}{(1 + yq^{1/2})(1 + y^{-1}q^{1/2})} \bigg|_{y=1} = -\frac{2q^{1/2}}{(1 + q^{1/2})^4} , \]  

(5.41)

\[ y \partial_y \vartheta_3 \bigg|_{y=1} = 0 , \]  

(5.42)

\[ (y \partial_y)^2 \vartheta_3 \bigg|_{y=1} = 2q \partial_q \vartheta_3 , \]  

(5.43)

and (5.24), we obtain the constraint

\[ 0 = \left[ \hat{D}\chi_{NS} \right]_{q^0} = \left[ (y \partial_y)^2 \chi_{NS} - \frac{m}{6} E_2 \chi_{NS} \right]_{q^0} = -2m^2 + \left[ q^{-m/4+1/8} \frac{1 - q^{1/2}}{1 + q^{1/2} (1 + q^{1/2})^2 \eta^3} \right]_{q^0} \]

\[ - (4m - 2) \left[ q^{-m/4+1/8} \frac{1 - q^{1/2} q \partial_q \vartheta_3}{1 + q^{1/2} \eta^3} \right]_{q^0} \]

\[ = -2m^2 - R_1 - (4m - 2) R_2 , \]  

(5.44)

where \( R_1 \) and \( R_2 \) are defined as

\[ R_1 = \left[ 2q^{1/2} \frac{(1 - q^{1/2})^2 \vartheta_3}{(1 - q)^3 \eta^3} \right]_{\eta^{m/4 - \frac{1}{8}}} \]  

(5.45)

\[ R_2 = \left[ \frac{(1 - q^{1/2})^2 q \partial_q \vartheta_3}{1 - q \eta^3} \right]_{\eta^{m/4 - \frac{1}{8}}} . \]  

(5.46)

In appendix A we show that for large enough \( m \) both \( R_1 \) and \( R_2 \) are positive. It is then clear that (5.44) cannot be satisfied.

### 5.4 What are the exceptional values of \( m \)?

The results of the previous subsections establish rigorously that there are at most a finite number of values of \( m \) for which a candidate extremal elliptic genus can exist. The results
of section 4 suggest that there are in fact precisely 9 such values namely $1 \leq m \leq 5$, and $m = 7, 8, 11, 13$. Although we do not have a rigorous proof, we strongly believe this list to be complete.

As we have mentioned, for $m$ odd we have studied the first 2000 terms and the only possibilities are the values mentioned above. For $m \sim 2000$ we are well within the regime for which our asymptotic bounds are valid. For $m$ even we have also examined the constraints numerically and it appears that $m \geq 36$ is well within the range of validity of our bounds. See figures 4 and 5 above.

6. Near-extremal $\mathcal{N} = 2$ conformal field theories

In section 3 we showed that $\mathcal{N} = 2$ ECFT’s, as we have defined them, at best exist only
for a finite number of exceptional values of \( m \). One might object that our definition is too narrow, and that we should simply modify the definition of an extremal theory.

In this section we consider one way of modifying the notion of an extremal theory, by demanding only that some “significant” fraction of the polar degeneracies \( c(n, \ell) \) coincide with those predicted from the vacuum character.

Returning to the system of equations (4.5), for fixed \( m \) define \( k(m) \) to be the largest integer such that

\[
\sum_{i=1}^{j(m)} x_i N_{ia} = d_a , \quad a = 1, \ldots, k(m) \tag{6.1}
\]

admits a solution \( x_i \) for which the elliptic genus \( \sum x_i \phi_i \) has an integral Fourier expansion.

We would like to show that we can choose \( k(m) \) to be “close” to \( P(m) \).

Turning again to a numerical analysis, we studied the truncation of (6.1) to the first \( j(m) \) equations: \( 1 \leq a \leq j(m) \) where we ordered the polar terms via their polarity. We found that in all cases \( 1 \leq m \leq 36 \) there is indeed a solution \( x_i \) in rational numbers. Moreover, for all \( m \) except \( m = 17 \) the Fourier expansion coefficients are integral — in so far as we have tested them. This indicates that \( k(m) = j(m) + O(1) \).

We conjecture that this is the case in general, and in section 6.1, assuming this conjecture to be true, we derive an interesting constraint on the spectrum of \( \mathcal{N} = 2 \) CFTs.

For the analysis in section 6.1 it turns out to be more convenient to define a “\( \beta \)-extremal \( \mathcal{N} = 2 \) CFT” by imposing the less restrictive condition of only requiring that polar degeneracies are predicted from the vacuum character in the \( \beta \)-truncated polar region:

\[
P_{\beta} := \{ (\ell, n) : 1 \leq \ell \leq m, n \geq 0, 4mn - \ell^2 \leq -\beta \} . \tag{6.2}
\]

We know that for suitable \( \beta \) candidate elliptic genera exist. For example, if we take \( \beta = m^2 \) then we can always construct a candidate elliptic genus. We get a better approximation to an extremal theory by lowering the value of \( \beta \). Therefore, let \( P_\beta(m) \) be the number of independent polar monomials of polarity \( \leq -\beta \), and let \( \beta_* \) be the smallest integer \( \beta \) such that

\[
\sum_{i=1}^{j(m)} x_i N_{ia} = d_a , \quad a = 1, \ldots, P_\beta(m) \tag{6.3}
\]

admits a solution \( x_i \) for which \( \sum x_i \phi_i \) has integral coefficients in its Fourier expansion. According to our conjecture \( P_{\beta_*}(m) \cong j(m) \). We would therefore like to estimate the value of \( \beta \) for which \( P_\beta(m) = j(m) + O(m^{1/2}) \) for large \( m \). The computation follows closely the analysis of section 2.2.

We now have

\[
P_\beta(m) = \sum_{r=r_0}^{m} \left[ \frac{r^2 - \beta}{4m} \right] , \tag{6.4}
\]

where \( r_0 := \lceil \sqrt{\beta} \rceil \). As before, we write this as a sum of three terms,

\[
P_\beta(m) = \sum_{r=r_0}^{m} \frac{r^2 - \beta}{4m} - \sum_{r=r_0}^{m} \left( \frac{r^2 - \beta}{4m} \right) + \frac{1}{2} \sum_{r=r_0}^{m} \left( \left\lceil \frac{r^2 - \beta}{4m} \right\rceil - \left\lfloor \frac{r^2 - \beta}{4m} \right\rfloor \right) . \tag{6.5}
\]

\footnote{Note that at least for the exceptional solutions \( m = 7, 8, 11, 13 \) we have \( k(m) > j(m) \).}
The first term is
\[
\sum_{r=r_0}^{m} \frac{r^2 - \beta}{4m} = \frac{m^2}{12} + \frac{m}{8} + \frac{1}{24} - \frac{r_0(2r_0 - 1)(r_0 - 1)}{24m} - \beta \frac{(m - r_0 + 1)}{4m}. \tag{6.6}
\]

Denote the number of integers \(r\) such that \(r_0 \leq r \leq m\) with \(r^2 = \beta \mod 4m\) by \(\nu(m, \beta)\). Unlike the case \(\beta = 0\) we cannot write down an exact formula, but it is clear that asymptotically \(\nu(m, \beta) \sim m^{1/2}\). The second term is
\[
\sum_{r=r_0}^{m} \left\lfloor \frac{r^2 - \beta}{4m} \right\rfloor - \sum_{r=r_0}^{m} \left\lceil \frac{r^2 - \beta}{4m} \right\rceil = m + 1 - r_0 - \nu(m, \beta). \tag{6.7}
\]

For the third term we again use the argument that the numbers \((\frac{r^2 - \beta}{4m})\) are randomly distributed. We thus have a random walk between \(-1/2\) and \(+1/2\) and the sum is expected to be of order \(m^{1/2}\).

To conclude, note that for \(\beta = \alpha m\) with \(\alpha\) a constant \(0 < \alpha < 1\) we have \(r_0 \sim m^{1/2}\), so the large \(m\) asymptotics are
\[
P_\beta(m) = \frac{m^2}{12} + \left(\frac{5}{8} - \frac{\alpha}{4}\right) m + O(m^{1/2}). \tag{6.8}
\]

Comparing to (2.25) we see that for large \(m\) the reduction of polarity to obtain the truncated supergravity elliptic genus is given by \(\beta = \frac{1}{2}m + O(m^{1/2})\).

As in equation (2.33) above the symbol \(O(m^{1/2})\) is to be understood heuristically. It would be worthwhile being more rigorous about this point.

6.1 A constraint on the spectrum of \(\mathcal{N} = 2\) theories with integral \(U(1)\) charges

In the previous sections we have found strong evidence that we must have \(P_\beta_*(m) \equiv j(m)\), and hence by (6.8)
\[
\beta_* \geq \frac{m}{2} + O(m^{1/2}) \tag{6.9}
\]

for large \(m\).

Now a monomial \(q^n y^\ell\) of polarity \(\beta\) corresponds by spectral flow to a state in the NS sector that contributes as \(q^{h - \frac{\beta m}{4}} y^\ell\) with
\[
h = \frac{m}{4} + \frac{\ell^2}{4m} - \frac{\beta}{4m}. \tag{6.10}
\]

Therefore, if we accept (6.9) then we can obtain an interesting constraint on the spectrum of a \((2, 2)\) \(AdS_3\) supergravity with a holographically dual CFT: It must contain at least one state which is a left-moving \(\mathcal{N} = 2\) primary (not necessarily chiral primary) tensored with a right-moving chiral primary such that
\[
h < \frac{m}{4} + \frac{\ell^2}{4m} - \frac{1}{8} + O(m^{-1/2}). \tag{6.11}
\]

It would be interesting and useful to sharpen this bound. However, we will show in section 7 below that it is possible to construct elliptic genera, which, after spectral flow, do match
the spectrum of the vacuum character for all conformal weights with $h \leq \frac{m}{4}$. There is no contradiction between this result and (6.11) because under $1/2$ unit of spectral flow $0 \leq |\ell| \leq 2m$ and hence $\frac{\ell^2}{4m}$ could be as large as $m$, and thus the bound can be as large as $\frac{5m}{4} - \frac{1}{8} + O(m^{-1/2})$.

7. Construction of nearly extremal elliptic genera

In this section we consider an alternative basis for the weak Jacobi forms which has a “triangular” nature, allowing us to replace the polar region $P(m)$ by an alternative region $S$. We will see that for large $m$, $S$ “approximates” $P(m)$. In the next section we discuss the possible physical significance of this fact.

It is shown in [26] that there is an integral basis of the ring of weak Jacobi forms of weight zero with integral coefficients

$$\tilde{J}_0^* = \mathbb{Z}[\phi_{0,1}, \phi_{0,2}, \phi_{0,3}, \phi_{0,4}]/I ,$$

where $I$ is the ideal generated by the relation

$$\phi_{0,1}\phi_{0,3} = 4\phi_{0,4} + \phi_{0,2}^2 .$$

The generators are elliptic genera of Calabi-Yau manifolds, and explicit formulae are given in [26]. In the basis (2.23) they can be expressed as\(^{12}\)

$$\phi_{0,1} = \tilde{\phi}_{0,1}$$

$$\phi_{0,2} = \frac{1}{24} \tilde{\phi}_{0,1}^2 - \frac{1}{24} \tilde{\phi}_{-2,1}^2 E_4$$

$$\phi_{0,3} = \frac{1}{432} \tilde{\phi}_{0,1}^3 - \frac{1}{144} \tilde{\phi}_{0,1} \tilde{\phi}_{-2,1}^2 E_4 + \frac{1}{216} \tilde{\phi}_{-2,1}^3 E_6$$

$$\phi_{0,4} = \frac{1}{6912} \tilde{\phi}_{0,1}^4 - \frac{1}{1152} \tilde{\phi}_{0,1}^2 \tilde{\phi}_{-2,1}^2 E_4 + \frac{1}{864} \tilde{\phi}_{0,1} \tilde{\phi}_{-2,1}^3 E_6 - \frac{1}{2304} \tilde{\phi}_{-2,1}^4 E_4^2 .$$

To make the triangular nature of this basis manifest it is useful to consider the NS sector images of the generators,

$$\hat{\phi}_{0,m} = (-1)^m q^{m/4} y^m \phi_{0,m}(\tau, z + \frac{\tau}{2} + \frac{1}{2}) .$$

We now consider ordering the $q, y$ expansion by the leading power of $q$ and, for each power of $q$ by the largest positive power of $y$. (Recall that $\chi_{NS}(\tau, z)$ is an even function of $z$, so the positive powers of $y$ determine the negative powers of $y$.) With this ordering of terms we have

$$\hat{\phi}_{0,1} = q^{-1/4} + O(q^{1/4})$$

$$\hat{\phi}_{0,2} = (y + y^{-1}) + O(q^{1/2})$$

$$\hat{\phi}_{0,3} = q^{1/4}(y - y^{-1})^2 + O(q^{3/4})$$

$$\hat{\phi}_{0,4} = 1 + O(q^{1/2}) .$$

\(^{12}\)We have redefined $\phi_{0,4}$ in [26] by a factor of $-1$. 

\[ -28 - \]
By (7.1) an overcomplete linear basis of $\tilde{J}_{0,m}$ is given by

$$
(\hat{\phi}_{0,1})^i(\hat{\phi}_{0,2})^j(\hat{\phi}_{0,3})^k(\hat{\phi}_{0,4})^l
$$

(7.9)

with $i + 2j + 3k + 4l = m$, $i, j, k, l \geq 0$. In order to obtain a set of linearly independent basis vectors we distinguish the monomials in (7.9) according to whether $i > k$ or $i \leq k$ and then use identity (7.2) to eliminate $\hat{\phi}_{0,3}$ or $\hat{\phi}_{0,1}$, respectively. The result is that there exists a vector space basis for $\tilde{J}_{0,m}$ which is a disjoint union of two sets $A \cup B$ with

$$
A := \{(\hat{\phi}_{0,1})^i(\hat{\phi}_{0,2})^j(\hat{\phi}_{0,4})^k | i > 0, j \geq 0, k \geq 0, l \geq 0, i + 2j + 4l = m\},
$$

(7.10)

$$
B := \{(\hat{\phi}_{0,2})^j(\hat{\phi}_{0,3})^k(\hat{\phi}_{0,4})^l | j \geq 0, k \geq 0, l \geq 0, 2j + 3k + 4l = m\}.
$$

(7.11)

A tedious but elementary counting argument shows that

$$
|A| = \begin{cases} 
\frac{m^2}{16} + \frac{3m}{8} - \frac{s^2}{16} + \frac{s}{4} + \frac{1}{2} & m = s \text{ mod } 4, s = 1, 3 \\
\frac{m^2}{16} + \frac{m}{4} - \frac{s^2}{16} + \frac{s}{4} & m = s \text{ mod } 4, s = 0, 2
\end{cases}
$$

(7.12)

and $|A| + |B| = j(m)$.

**Figure 6:** A comparison of the polar region $P^{(m)}$ and the region $S$. The NS sector polar region is bounded by $\ell \geq 0, h \geq \ell/2, h \leq \frac{\ell^2}{4} + \frac{\ell}{4m}$. The region $S$ is the triangular region, $\ell \geq 0, h \geq \frac{\ell}{2}, h - \frac{\ell}{2} \leq \frac{\ell}{8}$, which itself is a union of two triangular regions $S_A$ and $S_B$, where $S_A$ is the subregion of $S$ with $h < \frac{m}{4}$. The polar region contains $S_A$, while $S_B$ is an “approximation” to the remainder.

Now note that the leading expression in the $q,y$ expansion of an element in the set $A$ is $q^{-i/4}y^j$, while that in the set $B$ is $q^{k/4}y^{j+2k}$. It thus follows that an (NS-sector) Jacobi form of weight zero and index $m$ with integral Fourier coefficients is uniquely determined by the coefficients of $q^n y^\ell$ where $(\ell, n)$ run over the set:

$$
S = S_A \cup S_B
$$

(7.13)
where
\[ S_A = \{ (\ell, n) | n < 0, \ 0 \leq \ell, \ n + \frac{m}{4} \geq \frac{\ell}{2} \} \] (7.14)
and
\[ S_B = \{ (\ell, n) | 0 \leq n, \ 8n \leq \ell, \ n + \frac{m}{4} \geq \frac{\ell}{2} \} \] (7.15)

In both \( S_A \) and \( S_B \) the \((\ell, n)\) are in the lattice \((\ell, n) \in \mathbb{Z} \times \frac{1}{4}\mathbb{Z}\), subject to the quantization condition
\[ \left( n + \frac{m}{4} \right) - \frac{\ell}{2} = 0 \mod 1 \] (7.16)

(This quantization is equivalent to the statement that in the Ramond sector the elliptic genus has a Fourier expansion in \( q, y \) with integral powers of \( q, y \).) The regions \( S_A \) and \( S_B \) in the \((\ell, h)\) plane are triangles and their union is a triangle. The full region \( S \) can serve as a surrogate for the polar region \( \mathcal{P}^{(m)} \), as explained in figure 6.

Recall that \( n \), the power of \( q \) in the NS sector character, is related to \( h \) as \( n = h - \frac{m}{4} \).

It then follows from (7.14) that \( S_A \) contains all possible points with \( h < m/4 \) that occur in the NS vacuum character (3.8). Thus it is possible to construct a weak Jacobi form with integral coefficients whose \( q \)-expansion agrees with that of an extremal theory for all NS-sector Virasoro weights up to \( h = m/4 \) (for \( m \) even) and \( h = (m - 1)/4 \) (for \( m \) odd). This fits in very nicely with the bound (6.11), which puts an upper bound on the range of \( h \) where all states can be descendants of the vacuum.

8. Discussion: quantum corrections to the cosmic censorship bound

If the pure \( \mathcal{N} = (2, 2) \) supergravity is a consistent quantum theory, its Hilbert space should be spanned by states which can be identified as excitations of the supergravity fields. One class of such states are perturbative and normalizable excitations of the supergravity fields in \( AdS_3 \), which generate the vacuum representation in the boundary CFT [8]. It is expected that these are the only states up to the cosmic censorship bound. We define this bound to be the boundary of the region in the space of energy and charges in which states corresponding semiclassically to black hole solutions can exist. In the classical limit the cosmic censorship bound is the condition on mass and charges of a black hole such that there is a regular horizon.

It turns out that the classical cosmic censorship bound is exactly equal to the upper bound of the polar part of the CFT spectrum [10]. This was the motivation for the definition of \( \mathcal{N} = (2, 2) \) extremal conformal field theory in section 3.1. On the other hand, in section 5, we proved that such a conformal field theory does not exist for sufficiently large \( m \). This result, however, does not immediately rule out the conjectured existence of pure \( \mathcal{N} = (2, 2) \) supergravity since the cosmic censorship bound might receive quantum corrections. That is, there might be quantum corrections to the relation between the values of the mass and charges of those quantum states whose semiclassical manifestation are black holes. There are two potential sources for such corrections, and we will discuss each of them below.

As far as perturbative effects are concerned, the pure supergravity theory can be treated as the Chern-Simons gauge theory with the gauge group \( \text{U}(1) \). Since the classical
equations of motion of the Chern-Simons theory imply vanishing of the gauge field strength and since any perturbative corrections to the equations of motion can be expressed as a polynomial of the field strength and its covariant derivatives, black hole solutions are not corrected to all orders in the perturbative (i.e. $1/m$) expansion. However, values of the mass and charge of a given black hole solution can receive corrections since computing them requires knowing the action as well as the equations of motion. In particular, the “level” $m$, whose inverse appears in front of the action, can be corrected. The leading discrepancy between the dimension of the space of polar polynomials, $P(m)$, and the dimension of the space of weak Jacobi forms, $j(m)$,

$$P(m) - j(m) = \frac{m}{8} + O(m^{1/2}), \quad (8.1)$$

can be explained if $m$ is shifted by an appropriate constant by quantum effects. Such a shift is known to occur in perturbative Chern-Simons gauge theory \[48\], where the level $k$ is shifted at one loop by the dual Coxeter number of the gauge group, $C_2(G)$. For the supergroup $OSp(2|2)$, we have $C_2 = -2$, so that in the present case both $k_L$ and $k_R$ are shifted as\[13\]

$$k_L \rightarrow k_L - 2. \quad (8.2)$$

Combining this with equation (3.7), we can express this as the shift of $m$,

$$m \rightarrow m - 8, \quad (8.3)$$

which, unfortunately, does not account for the difference in (8.1). Furthermore, it seems difficult to attribute sub-leading terms in $(P(m) - j(m))$ to higher order perturbative effects since sub-leading terms in $P(m)$ contains the arithmetic function $A(m)$, which does not have a nice $1/m$ expansion (see footnote 2).

There is another source of corrections which are non-perturbative in nature. To see this, we note that conformal weights $h$ for states counted by the elliptic genus are integers, as required by modular invariance. This granularity, which is smeared out in any perturbative analysis, gives rise to an intrinsic ambiguity in the cosmic censorship bound of $O(1)$ in $h$. Since the boundary of the polar region in the $(L_0, J_0)$ plane has a length of order $m$, it is possible that the discrepancy of $P(m)$ and $j(m)$ mentioned in the above is entirely attributed to this granularity. For example, the bound on $h$ for a new primary state found in (6.11) is within $O(1)$ of the cosmic censorship bound.

It is possible that a combination of these two effects resolves the apparent contradiction between the conjectured existence of pure $\mathcal{N} = (2, 2)$ supergravity and the properties of the elliptic genus we found in this paper.

---

\[13\]One way to think about this shift is as follows. The supergroup $OSp(2|2)$ is the superconformal group of $AdS_2$, and its dual Coxeter number, $C_2$, can be thought of as the beta-function of the world-sheet sigma-model defining $AdS_2$ space-time. If instead of $AdS_2$ we consider a positive curvature space, that is a 2-sphere $S^2$, the contribution to the beta-function of the world-sheet theory should have opposite sign and, hence, the opposite shift of $k$. In particular, for $S^2$, which has the isometry group $SU(2)$, the shift $k \rightarrow k + 2$ is familiar in the study of $SU(2)$ Chern-Simons theory \[48\]. In the case of $OSp(2|2)$ Chern-Simons theory the shift should have opposite sign, therefore justifying (8.2).
Given the close resemblance of the region \( S_A \cup S_B \) identified in section [7] with the polar region it is natural to ask whether the boundary of that region might in fact constitute the quantum-corrected cosmic censorship bound. This seems unlikely to us. Along the line \( h = \ell + \frac{m}{4} \), \( 0 \leq \ell \leq \frac{2m}{3} \) the polarity becomes as great as \( p = \frac{m^2}{16} \). It seems unlikely that quantum corrections will modify the mass and charge in such a way as to change a semi-classical black hole state with such a polarity to a descendent of the vacuum.

9. Extremal \( \mathcal{N} = 4 \) theories

The analysis for the case of the pure \( \mathcal{N} = (2,2) \) supergravity theories is somewhat inconclusive since we cannot rule out that there are quantum corrections to the classical supergravity ansatz. The situation is sharper for the case with \( \mathcal{N} = (4,4) \) superconformal symmetry since the possible quantum corrections of these theories are well constrained [12]. Therefore, in this section we shall begin to address whether modular invariance allows for a pure \( \mathcal{N} = (4,4) \) supergravity theory. Unfortunately, our results are somewhat incomplete.

Following the earlier definition we define an extremal \( \mathcal{N} = (4,4) \) theory to be a theory whose partition function is of the form (3.10) where \( \chi^{(m)}_{\text{vac}} \) is now the vacuum character of the \( \mathcal{N} = 4 \) algebra [20, 21]:

\[
\chi^{(m)}_{\text{vac}} = q^{-m/4} \prod_{n=1}^{\infty} \frac{(1-yq^{n-1/2})^2(1-y^{-1}q^{n-1/2})^2}{(1-q^n)} \chi(q,y), \tag{9.1}
\]

with

\[
\chi(q,y) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)(1-y^2q^n)(1-y^{-2}q^{n-1})} \times \sum_{j \in \mathbb{Z}} q^{(m+1)j^2+j} \left( \frac{y^{2(m+1)j}}{(1-yq^{j+1/2})^2} - \frac{y^{-2(m+1)j-2}}{(1-y^{-1}q^{j+1/2})^2} \right). \tag{9.2}
\]

As in the case of the \( \mathcal{N} = 2 \) vacuum character, we have evaluated this expression at \( z + \frac{1}{2} \). To get rid of the negative powers of \( q \) in the denominator, we can rewrite it as two separate sums over positive \( j \),

\[
\chi(q,y) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)(1-y^2q^n)(1-y^{-2}q^{n-1})} \times \\
\left[ \sum_{j \geq 0} q^{(m+1)j^2+j} \left( \frac{y^{2(m+1)j}}{(1-yq^{j+1/2})^2} - \frac{y^{-2(m+1)j-2}}{(1-y^{-1}q^{j+1/2})^2} \right) \right. \\
\left. + \sum_{j \geq 1} q^{(m+1)j^2+j-1} \left( \frac{y^{-2(m+1)j-2}}{(1-y^{-1}q^{j-1/2})^2} - \frac{y^{2(m+1)j}}{(1-yq^{j-1/2})^2} \right) \right]. \tag{9.3}
\]

It is straightforward to read off the polar polynomial from this expression.

Using the same methods as in section [4], we have analyzed whether this polar polynomial can be completed to a weak Jacobi form. We have performed the analysis for \( 1 \leq m \leq 20 \), and we have found that the only cases where this is possible are \( m = 1, 2, 3, 4, 5 \). (Note
that for $1 \leq m \leq 4$ this is automatic since $P(m) = j(m)$. Thus, apart from a few low level exceptions, we expect that the pure $\mathcal{N} = (4,4)$ sugra ansatz is incompatible with modular invariance. It might be possible to prove this assertion by suitably modifying the methods of section [3], but the expressions appear to be challenging and we have not attempted to do so.

An important loophole in our argument is the possibility that there are zero-modes making the elliptic genus vanish. This might happen when there is an extension of the chiral algebra and $m$ is odd. In order to demonstrate this write the character expansion of the RR sector partition function as

$$Z_{RR} = \sum_{1 \leq \ell, \tilde{\ell} \leq m} c_{\ell\tilde{\ell}}\chi_\ell \overline{\chi_{\tilde{\ell}}} + \sum_{1 \leq \ell \leq m} c_{0\ell}\chi_\ell \overline{\chi_0} + \sum_{1 \leq \ell \leq m} c_{0\ell}\chi_0 \overline{\chi_{\ell}} + \cdots$$ (9.4)

Here $\chi_\ell$ denote the characters of the unitary massless representations, with $0 \leq \ell \leq m$ denoting twice the spin of the highest weight vector, and $+ \cdots$ refers to terms with a massive representation on the left or the right. The reason for separating out the $\ell = 0$ spin as special is that its highest weight vector is not a polar state, whereas the highest weight vectors of all the other massless representations are polar states. An extremal theory must have an expansion of the form

$$Z_{RR} = \chi_m \overline{\chi_m} + \sum_{1 \leq \ell \leq m} c_{0\ell}\chi_\ell \overline{\chi_0} + \sum_{1 \leq \ell \leq m} c_{0\ell}\chi_0 \overline{\chi_{\ell}} + \cdots$$ (9.5)

since $\chi_m$ is the spectral flow image of the NS vacuum. Now, the elliptic genus of $\chi_\ell$ is $(-1)^\ell(\ell + 1)$, while that of the massive representations is zero. Thus, if the elliptic genus vanishes then, comparing the coefficient of the left-moving vacuum character $\chi_m$ we see that

$$c_{m0} = (-1)^{m+1}(m + 1).$$ (9.6)

Note that a non-vanishing coefficient $c_{m0}$ implies that the right-moving chiral algebra is enhanced, as claimed. Also, since $c_{m0}$ is a positive integer this can only happen when $m$ is odd. Moreover, by comparing the coefficients of the other left-moving characters we find the constraints $c_{0\ell} = 0$ for $1 \leq \ell \leq m - 1$ and $\sum_{\ell=0}^{m} c_{0\ell}(-1)^\ell(\ell + 1) = 0$. Since our no-go theorem would apply if either the holomorphic or anti-holomorphic elliptic genus is non-vanishing we might as well assume the anti-holomorphic elliptic genus also vanishes. In this case we find that $c_{0\ell} = 0$ for $1 \leq \ell \leq m - 1$ and hence $c_{00} = (m + 1)^2$, so that $Z_{RR} = |\chi_m + (m + 1)\chi_0|^2 + \cdots$. Thus, for extremal theories of this type our arguments fail, and further investigation is necessary.

It should be noted that a vanishing elliptic genus does indeed occur in some important examples. One example arises in $AdS_3 \times S^3 \times T^4$ compactifications [33]. A second example is in the MSW conformal field theory with $(0,4)$ supersymmetry, which is dual to an $AdS_3 \times S^2 \times X$ compactification, where $X$ is Calabi-Yau [34, 38]. In all these cases there is an extended chiral algebra due to singleton modes. In such a case one must take derivatives with respect to $\bar{z}$ and set $\bar{z} = 0$ [14, 33]. The resulting modular object is a non-holomorphic generalization of a Jacobi theta function [13, 15]. A similar phenomenon happens in the
analog of the elliptic genus for the large $\mathcal{N} = 4$ superconformal algebra \[27\]. Of course, the examples we have just cited are not extremal theories. However, these examples do suggest that it would be useful to extend the investigation of extremal theories to the cases of vanishing elliptic genera, or $(0, 4)$ supersymmetry, or large $\mathcal{N} = 4$ supersymmetry.

10. Applications to flux compactifications

Flux compactifications of M-theory and string theory have been a very popular subject of investigation in recent years \[17, 4\]. Unfortunately, these compactifications are in general very complicated and it is difficult to be sure that they are valid solutions of string theory within a controlled approximation scheme. The demonstration of holographically dual conformal field theories would definitively settle such difficulties, at least for flux compactifications to anti-de Sitter spacetimes. The considerations and techniques of this paper might put interesting constraints on the allowed spectra of some classes of flux compactifications, namely compactifications to $AdS_3$ with a holographically dual $(2, 2)$ conformal field theory. One could imagine, for example, flux compactifications of M-theory on a suitable Calabi-Yau 4-fold, where one includes $M5$ instanton effects, in order to exclude no-scale compactifications.

The compactifications of greatest interest are those with a small cosmological constant and a large gap from the ground state to the Kaluza-Klein scale. These simple aspects of the spectrum already have implications for the conformal field theory. If the cosmological constant is small then the Brown-Henneaux central charge $c = \frac{3}{2}RM_{pl}^{(3)}$ is large. This implies that the level

$$m = \frac{RM_{pl}^{(3)}}{4} \quad (10.1)$$

is large.

Now let us consider the spectrum of the theory. The supergravity multiplet corresponds to the super-Virasoro descendants. Next, if $V_{8}$ is the volume of the Calabi-Yau 4-fold in 11-dimensional Planck units then

$$[V_{8}(M_{pl}^{(11)})^{8}]M_{pl}^{(11)} = M_{pl}^{(3)} \quad (10.2)$$

and therefore, $M_{pl}^{(11)} \sim M_{pl}^{(3)}$ unless $V_{8}$ is unnaturally large, and hence in AdS units, the KK scale is of order $m$. Thus, we naturally expect a large gap to the primary fields corresponding to the KK modes.

In addition to the supergravity multiplet and the KK modes there will typically be other primary fields, for example the moduli fields, many of which might have acquired masses in the compactification scheme. Our conjectured bound (6.11) might possibly put constraints on the masses which the moduli acquire.

It would clearly be of interest to make these considerations more precise, and moreover to extend them to theories with holographic duals with only $(1, 1)$ supersymmetry. Indeed, one does not expect generic flux compactifications to lead to $\mathcal{N} = 2$ supersymmetry since there is no candidate isometry for the $U(1)$ current algebra. Given the $\mathcal{N} = 1$ supersymmetry one can still form a holomorphic elliptic genus, but the existence of the Hauptmodul
\(\hat{K}\) for \(\Gamma_{\theta}\) (see eq. (5.3) above) shows that the techniques of this paper cannot be used to exclude compactifications just based on the polar polynomial of the elliptic genus. Further work is needed to see whether modularity, combined with other ideas, puts any interesting constraints on the landscape of three-dimensional AdS compactifications.\(^{14}\)

Acknowledgments

We would like to thank C. Vafa for collaboration at an earlier stage of this project. GM would also like to thank M. Douglas for a past collaboration on closely related issues. We would like to thank F. Denef, T. Gannon, S. Kachru, J. Maldacena, J. Manschot, P. Sarnak, and D. Zagier for useful discussions.

GM and HO thank the organizers of the 37th Paris Summer Institute on Black Holes, Black Rings and Modular Forms, which stimulated progress in this work. SG acknowledges the hospitality of Institut für Theoretische Physik, ETH Zurich, Institute for Advanced Study, Harvard University, Banff Center, the Aspen Center for Physics, the Simons Workshops in 2005, 2006, and 2007, where part of this work was carried out. HO also thanks the Aspen Center for Physics, the Kavli Institute for Theoretical Physics in Santa Barbara, Harvard University, the Simons Workshops in 2005 and 2006 in Stony Brook, the Banff International Research Station, the University of Tokyo, the Galileo Galilei Institute in Florence, the CERN theory institute, and the Ettore Majorana Centre for Scientific Culture in Erice, where part of this work was carried out.

The work of MRG and CAK is supported by the Swiss National Science Foundation. GM is supported by DOE grant DE-FG02-96ER40949. The work of SG and HO is supported in part by DOE grant DE-FG03-92-ER40701. The work of SG is also supported in part by NSF grant DMS-0635607 and by the Alfred P. Sloan Foundation. The work of HO is also supported in part by NSF grant OISE-0403366, by a Grant-in-Aid for Scientific Research (C) 20540256 from the Japan Society for the Promotion of Science, by the 21st Century COE Visiting Professorship at the University of Tokyo, by the World Premier International Research Center Initiative of MEXT of Japan, and by the Kavli Foundation. Opinions and conclusions expressed here are those of the authors and do not necessarily reflect the views of funding agencies.

A. Growth properties

A.1 Analysis of the constraint for \(m\) odd

For \(m\) odd we have

\[
L = \left[ -2iq^{-\frac{m}{2}+\frac{1}{8}} \frac{1 - q^{1/2} \vartheta_3(\tau) \vartheta_2^2 \vartheta_4^2}{q^{1/2}} \right] \ . \quad (A.1)
\]

where \(L\) was defined in (5.26). We can simplify this significantly using the triple product identity \(\vartheta_2 \vartheta_3 \vartheta_4 = 2\eta^3\). Next, shifting \(\tau \to \tau + 1\) (which cannot change the \(q^0\) coefficient)
we obtain:

\[ L = 4e^{-i\pi m/2} \left[ q^{-\frac{m+1}{4}} \left( \frac{1+q^{1/2}}{1-q^{1/2}} \vartheta_2 \vartheta_3 \right) \right] . \] (A.2)

Now use the usual sum formula for \( \vartheta_2 \) and \( \vartheta_3 \) to obtain

\[ \vartheta_2 \vartheta_3 = \sum_{r,s \in \mathbb{Z}} q^{(r-1/2)^2/2 + s^2/2} = \sum_{r,s \in \mathbb{Z}} q^{(2r-1)^2/8 + (2s)^2/8} = \sum_{n \in \mathbb{N}_0} B(n) q^{n/8} , \] (A.3)

where \( B(n) \) is the number of ways of writing \( n \) as a sum of an even and an odd integer squared, i.e. \( n = (2r - 1)^2 + (2s)^2 \) with both \( r \) and \( s \) integer. We also observe that the series expansion of the other factor is

\[ \frac{1 + q^{1/2}}{1-q^{1/2}} = 1 + 2 \sum_{\ell=1}^{\infty} q^{\ell/2} . \] (A.4)

Thus the exact result for (5.26) is

\[ L = 4e^{-i\pi m/2} \left[ B(2m - 1) + 2 \sum_{\ell=1}^{2m-1} B(2m - 1 - 4\ell) \right] . \] (A.5)

The dominant contribution comes from the second term. This sum is precisely equal to all combinations of an odd and an even integer whose square sum up to a number less or equal to \( 2m - 5 \). Now draw a rectangular lattice whose unit cell is a square with length 2, where we shift the lattice by one unit in the \( x_1 \)-direction say, so that the centers of the cells are at \( (x_1, x_2) = (2r - 1, 2s) \). Consider the area of all those unit cells for which the corresponding center point \( (2r - 1, 2s) \) has the property that \( (2r - 1)^2 + (2s)^2 \leq 2m - 5 \). It follows from elementary geometry that this area is bigger than the area of the disk with radius \( \sqrt{2m - 5} - \sqrt{2} \) (see figure 5). Since each unit cell has area 4, it follows that

\[ \sum_{\ell=1}^{2m-1} B(2m - 1 - 4\ell) \geq 1 + \frac{1}{4} \pi \left( \sqrt{2m - 5} - \sqrt{2} \right)^2 = \frac{\pi}{2} m - \pi \sqrt{m - \frac{5}{2}} - \frac{3}{4} \pi . \] (A.6)

Thus it follows that \( e^{i\pi m/2} L \), which is positive, is bounded below by

\[ e^{i\pi m/2} L \geq 4\pi m - 8\pi \sqrt{m - \frac{5}{2}} - 6\pi . \] (A.7)

**A.2 Analysis of the constraint for \( m = 2 \mod 4 \)**

In the case of \( m \) odd we saw that \( L \) only grew linearly. Since the original expression contained exponentially growing function such as \( \eta^{-3} \), this means that there had to occur cancellations. We will now show that for \( m = 2 \mod 4 \) such cancellation do not occur, i.e. that

\[ L = \left[ q^{-\frac{m+1}{4}} + \frac{1-q^{1/2}}{1+q^{1/2} \eta^{\frac{1}{2}} (\vartheta_4 - \vartheta_2^4)} \right] q^0 . \] (A.8)
grows exponentially with \( m \). To this end, use (5.24) to write

\[
q^{-\frac{1}{2}+\frac{1}{2}} \left( 1 - q^{1/2} \right)^2 \left( -\frac{q\vartheta_3\vartheta_3}{\eta^2} + \frac{E_2\vartheta_3}{\eta^2} \right) q^N , \tag{A.9}
\]

where \( N = m/4 - 1/2 \). The following form of \( E_2 \) will be useful:

\[
E_2(\tau) = 1 - 24 \sum_{k=1}^{\infty} \sigma_1(k)q^k , \tag{A.10}
\]

where \( \sigma_1(k) \) is the divisor function.

Let us first consider the second term of (A.9). We will show that this is negative and grows exponentially fast with \( N \). We introduce the expansion coefficients of \( \vartheta_3/\eta^3 \),

\[
\frac{\vartheta_3}{\eta^3} = q^{-1/8} \sum_{n \geq 0} \left( F_1(n)q^n + F_2(n)q^{n+1/2} \right) . \tag{A.11}
\]

From these we obtain the discrete derivative \((1 - q^{1/2})^2 \vartheta_3/\eta^3\),

\[
q^{-\frac{1}{2}+\frac{1}{2}}(1-q^{1/2})^2 \frac{\vartheta_3}{\eta^3} = \sum_{n \geq 0} \left( \hat{K}(n)q^n + \hat{K}'(n)q^{n+1/2} \right) \tag{A.12}
\]

with \( \hat{K}(n) = F_2(n) - 2F_1(n) + F_2(n-1) \), and, including \( E_2 \),

\[
q^{-\frac{1}{2}+\frac{1}{2}}E_2(1-q^{1/2})^2 \frac{\vartheta_3}{\eta^3} = \sum_{n \geq 0} \left( \hat{K}(n)q^n + \hat{K}'(n)q^{n+1/2} \right) \tag{A.13}
\]

with

\[
\hat{K}(n) = K(n) - 24 \sum_{s=1}^{n} \sigma_1(s)K(n-s) . \tag{A.14}
\]

Finally, the desired second term of (A.9) is \( \sum K(n) \). It will therefore suffice to show that \( \hat{K}(n) \) grows exponentially and is negative for large \( n \).

To examine the large \( n \) behavior we begin with the Rademacher expansions for \( F_1(n) \) and \( F_2(n) \). These are summarized in appendix B with the result that

\[
F_1(n) = (8n)^{-5/4} e^{\pi \sqrt{2n}} \left( 1 - \frac{15 + \pi^2}{8\sqrt{2\pi}} n^{-1/2} + \frac{105 + 10\pi^2 + \pi^4}{256\pi^2} n^{-1} + O(n^{-3/2}) \right) ,
\]

\[
F_2(n) = (8n)^{-5/4} e^{\pi \sqrt{2n}} \left( 1 + \frac{3(\pi^2 - 5)}{8\sqrt{2\pi}} n^{-1/2} + \frac{3(35 - 10\pi^2 + 3\pi^4)}{256\pi^2} n^{-1} + O(n^{-3/2}) \right) .
\]

From this we compute the discrete derivative:

\[
K(n) = \pi^2 (8n)^{-9/4} e^{\pi \sqrt{2n}} (1 + O(n^{-1/2})) , \tag{A.15}
\]

\[
\text{Figure 7: The grey area is given by those boxes whose centers lie within the outer circle of radius } \sqrt{2m-5} \text{. The inner circle has radius } \sqrt{2m-5} - \sqrt{2} \text{ and is completely contained in the grey area.}
\]
Note the exponential growth with \( n \). Now write

\[
\dot{K}(n) = K(n) - 24K(n-1) - 24S
\]  

(A.16)

with \( S := \sum_{s=2}^{n} \sigma(s)K(n-s) \). It is straightforward to see that the sum \( S \) is positive definite for large \( n \): first note that because of (A.15) \( K(n) \) is negative for at most finitely many \( n \). Since \( K(n) \) grows exponentially and \( \sigma(s) \) only grows like \( \sigma(s) \sim e^{\gamma s \ln \ln s} \), where \( \gamma \) is the Euler-Mascheroni constant [28], it follows that the first terms of the sum dominate the (potentially negative) terms at its tail. The first two terms on the RHS of (A.16) clearly grow like \(-23\pi^2(8n)^{-9/4}e^{\pi \sqrt{2n}}\), hence \( \dot{K}(n) \) is negative and exponentially growing for large \( n \). Therefore the same is true for \( \sum_{N} \dot{K}(n) \).

In the analysis of the case \( m = 0 \mod 4 \) below we will show that the first term of (A.9) is negative, so that there can be no cancellations between the two. We thus conclude that (A.9) grows exponentially.

**A.3 Analysis of the constraint for \( m = 0 \mod 4 \)**

Define

\[
R_1 = \left[ 2 q^{1/2} \frac{(1-q^{1/2})^4 \vartheta_3}{(1-q)^3 \eta^3} \right]_{q^{\frac{\vartheta_3}{\eta^3}}} \quad (A.17)
\]

\[
R_2 = \left[ \frac{(1-q^{1/2})^2 q^2 \vartheta_3 \vartheta_3}{1-q} \eta^3 \right]_{q^{\frac{\vartheta_3^2}{\eta}}} \quad (A.18)
\]

We shall show that for large enough \( m \) both \( R_1 \) and \( R_2 \) are positive. Consider first \( R_2 \). Note that the only negative coefficients that can appear are due to the factor \((1-q^{1/2})^2\). It will suffice to show that the coefficients

\[
\left[ \frac{(1-q^{1/2})^2}{(1-q)^3(1-q^2)^3} q^2 \vartheta_3 \vartheta_3 \right]_{q^{N}}
\]

are positive for \( N \) large enough. We have dropped the factor of \((1-q)^{-1}\) and included only the first two factors of \( \eta^3 \), which will turn out to be sufficient to ensure positivity. Defining

\[
\frac{1}{(1-q)^3(1-q^2)^3} = \sum_{n=0}^{\infty} b(n)q^n \quad (A.20)
\]

it is straightforward to calculate

\[
b(n) = \begin{cases} \frac{1}{1920}(2+n)(4+n)(6+n)(8+n)(5+2n) & n \text{ even} \\ \frac{1}{1920}(1+n)(3+n)(5+n)(7+n)(13+2n) & n \text{ odd}. \end{cases}
\]

(A.21)

Note in particular that

\[
b(n) = \frac{n^5}{960} + \frac{3n^4}{128} + \frac{19n^3}{96} + \mathcal{O}(n^2)
\]

(A.22)
We now want to calculate the coefficients $p(N)$ of
\[
\frac{1}{(1-q)^3(1-q^2)^3} q \partial q^3 = \sum_{N \in \mathbb{Z}} p(N) q^N .
\] (A.23)

We need to distinguish the cases $N \in \mathbb{N}$ and $N \in \mathbb{N} + \frac{1}{2}$:
\[
N \in \mathbb{N} : p(N, K) = \sum_{s=0}^{K} b(N - 2s^2) 4s^2
\] (A.24)
\[
N \in \mathbb{N} + \frac{1}{2} : p(N, K) = \sum_{s=0}^{K} b(N - (2s + 1)^2/2) (2s + 1)^2
\] (A.25)

In principle, the upper bound $K$ is given by the requirement that the argument of $b$ be non-negative, and its explicit expression will involve some floor function of a square root of $N$. For the moment, we will leave $K$ as an auxiliary integer parameter. One can then evaluate the sums explicitly to obtain polynomials in both $N$ and $K$, again distinguishing the cases $N$ odd and $N$ even. As the resulting expressions are rather lengthy, we refrain from writing them down explicitly. To determine the $N$th coefficient of (A.19), we then need to evaluate
\[
p(N, K_1) - 2p(N - 1/2, K_2) + p(N - 1, K_3) .
\] (A.26)

In principle, we would now have to determine the exact values of $K_i$, which are complicated step functions of $N^{1/2}$. For our purposes however it is enough to know their leading behavior. In particular, we know that $K_i = \sqrt{\frac{N}{2}} - \epsilon_i$, where $0 \leq \epsilon_i < 2$, so that $\epsilon_i$ is of order one. We then obtain for (A.26) the expression
\[
\frac{N^{9/2}}{1890\sqrt{2}} + \mathcal{O}(N^{7/2}) .
\] (A.27)

Note that this holds for all $N \in \frac{1}{2}\mathbb{N}$. (Hence, our estimates can also be applied to the analysis of section A.2.) This shows that the leading term has a positive coefficient and that it is independent of the $\epsilon_i$, which only appear in the subleading terms. This then shows that (A.19) has positive coefficients for $N$ large enough.

Note that for low values of $N$ the coefficients of (A.19) can still be negative. To complete the argument, we thus have to show that after convolution with the remaining factors in (A.18) the potentially negative coefficients for $N < N_0$ cannot render negative the coefficients at arbitrarily large $N$. To see this, note that it follows from the Rademacher expansion that, for any set of positive integers $a_1, \ldots, a_k$, the Fourier coefficients of
\[
(1-q)^{a_1}(1-q^2)^{a_2} \cdots (1-q^k)^{a_k} \eta^{-3}
\] (A.28)
will have the asymptotic behavior $\sim n^{a_1} e^{\pi \sqrt{2m}}$. For example in Appendix B we show that for the case of interest, $(1-q)^3(1-q^2)^3 \eta^{-3}$ the leading asymptotics is given by
\[
\frac{\pi^6}{8\sqrt{2}} n^{-9/2} e^{\pi \sqrt{2m}} .
\] (A.29)
We approximate the convolution sum as the integral
\[ \int_0^N ds \left. \frac{9/2}{\sqrt{(N-s)}} \right. e^{\pi \sqrt{2(N-s)}}. \] (A.30)
The position of the saddle point of this integral grows as
\[ s_0 \sim N^{1/2}. \] (A.31)
This means that for \( N \) large enough the contribution of the negative coefficients around \( s \sim 1 \) will be negligible, so that the total coefficient is positive.

Turning to \( R_1 \), we need to consider
\[ (1-q)^{-3}(1-q^2)^{-3}(1-q^3)^{-3}(1-q^4)^{-3} = \sum_{n=0}^{\infty} \tilde{b}(n)q^n. \] (A.32)
A straightforward, but somewhat tedious calculation then gives expressions similar to (A.21) whose explicit forms depend on \( n \) mod 12. Again, the leading terms are independent of this, so that we can write
\[ \tilde{b}(n) = \frac{n^{11}}{551809843200} + \frac{n^{10}}{3344302080} + \frac{29 n^9}{1337720832} + \frac{5 n^8}{55050524} + \frac{16949 n^7}{696729600} + O(n^6). \] (A.33)
We can now define \( \tilde{p}(N,K) \) analogously to (A.24), (A.25) and evaluate
\[ \tilde{p}(N,K_1) - 4\tilde{p}(N-1/2,K_2) + 6\tilde{p}(N-1,K_3) - 4\tilde{p}(N-3/2,K_4) + \tilde{p}(N-2,K_5), \] (A.34)
which leads to
\[ \frac{N^{15/2}}{1751349600\sqrt{2}} + O(N^{13/2}). \] (A.35)
Since sums of terms of order \( n^6 \) give contributions of at most \( N^7 \), this also shows that it was sufficient to consider (A.33) only up to \( n^6 \). The coefficients of the truncated \( \eta^{-3} \) expansion grow as in (B.21), and the rest of the argument is then completely analogous to the case of \( R_2 \).

**B. Rademacher expansions**

The proofs in appendix A require some asymptotic expansions for coefficients of some modular forms. We collect these here.

First, we apply the expansion to the modular vector
\[ f_1 = \frac{1}{2} \frac{\vartheta_3 + \vartheta_4}{\eta^3} = q^{-1/8} \sum_{n=0}^{\infty} F_1(n)q^n \] (B.1)
\[ f_2 = \frac{1}{2} \frac{\vartheta_3 - \vartheta_4}{\eta^3} = q^{3/8} \sum_{n=0}^{\infty} F_2(n)q^n \] (B.2)
\[ f_3 = \frac{\vartheta_2}{\eta^3} = \sum_{j=0}^{\infty} F_3(n)q^n. \] (B.3)
We have weight \( w = -1 \), the representation is manifest for \( T \), and for \( S \) it is computed from

\[
\begin{align*}
  f_1(-1/\tau) &= (-i\tau)^{-1/2} (f_1 + f_2 + f_3)(\tau) \quad (B.4) \\
  f_2(-1/\tau) &= (-i\tau)^{-1/2} (f_1 + f_2 - f_3)(\tau) \quad (B.5) \\
  f_3(-1/\tau) &= (-i\tau)^{-1} (f_1 - f_2)(\tau) \quad . (B.6)
\end{align*}
\]

We now have convergent expansions

\[
\begin{align*}
  F_1(n) &= \frac{\pi}{8} (n - 1/8)^{-1} I_2(4\pi \sqrt{1/8(n - 1/8)}) + O(e^{2\pi \sqrt{n/8}}) \quad (B.8) \\
  F_2(n) &= \frac{\pi}{8} (n + 3/8)^{-1} I_2(4\pi \sqrt{1/8(n + 3/8)}) + O(e^{2\pi \sqrt{n/8}}) \quad (B.9) \\
  F_3(n) &= \frac{\pi}{8} (n)^{-1} I_2(4\pi \sqrt{1/8n}) + O(e^{2\pi \sqrt{n/8}}) . \quad (B.10)
\end{align*}
\]

Now use

\[
I_\nu(x) \sim \frac{1}{\sqrt{2\pi x}} e^x \left( 1 - \frac{4\nu^2 - 1}{8x} + \frac{(4\nu^2 - 1)(4\nu^2 - 9)}{128x^2} + \cdots \right) \quad (B.11)
\]

for \( x \to +\infty \) to get

\[
\begin{align*}
  F_1(n) &= (8n)^{-5/4} e^{4\pi \sqrt{\frac{1}{8}}} \left( 1 - \frac{\pi^2 + 15}{8\sqrt{2\pi}} \frac{1}{n^{1/2}} + \frac{\pi^4 + 70\pi^2 + 105}{256\pi^2} \frac{1}{n} + \cdots \right) \quad (B.12) \\
  F_2(n) &= (8n)^{-5/4} e^{4\pi \sqrt{\frac{1}{8}}} \left( 1 + \frac{3(\pi^2 - 5)}{8\sqrt{2\pi}} \frac{1}{n^{1/2}} + \frac{3(3\pi^4 - 70\pi^2 + 35)}{256\pi^2} \frac{1}{n} + \cdots \right) . \quad (B.13)
\end{align*}
\]

We also need the asymptotic expansion of functions that are obtained from \( \eta^{-3} \) by dropping the first few factors in the product formula. Defining

\[
\eta^{-3} = q^{-1/8} \sum_n p_3(n)q^n \quad (B.14)
\]

(with \( p_3(n) = 0 \) for \( n < 0 \)), we have the Rademacher formula

\[
p_3(n) = 2\pi(8n - 1)^{-5/4} I_{3/2}(\pi \sqrt{2(n - 1/8)}) + O(e^{\pi \sqrt{n/2}}) . \quad (B.15)
\]

Note that the Bessel function is elementary

\[
I_{3/2}(x) = \frac{2}{\sqrt{2\pi x}} (\cosh x - \frac{\sinh x}{x}) . \quad (B.16)
\]

Define

\[
(1 - q)^3(1 - q^2)^3 \eta^{-3} = q^{-1/8} \sum_n \hat{p}_3(n)q^n , \quad (B.17)
\]
which is a kind of sixth-order discrete derivative:

\[ \hat{p}_3(n) = p_3(n) - 3p_3(n - 1) + 8p_3(n - 3) \]
\[ - 6p_3(n - 4) - 6p_3(n - 5) + 8p_3(n - 6) - 3p_3(n - 8) + p_3(n - 9). \] (B.18)

Substituting the asymptotic expansion (B.15) one finds after some algebraic manipulations

\[ \hat{p}_3(n) = \left( \frac{\pi^6}{8\sqrt{2}} n^{-9/2} + O(n^{-5}) \right) e^{\pi \sqrt{2n}}. \] (B.19)

Similarly, the coefficients

\[ (1 - q)^3(1 - q^2)^3(1 - q^3)^3(1 - q^4)^3\eta^{-3} = q^{-1/8} \sum_n \hat{p}_3(n)q^n \] (B.20)

have leading asymptotics

\[ \hat{p}_3(n) \sim \left( \frac{27\pi^{12}}{\sqrt{2}} n^{-15/2} + O(n^{-8}) \right) e^{\pi \sqrt{2n}}. \] (B.21)

References

[1] A. Achucarro and P.K. Townsend, *A Chern-Simons action for three-dimensional anti-De Sitter supergravity theories*, Phys. Lett. B 180, 89 (1986).

[2] A. Achucarro and P.K. Townsend, *Extended supergravities in d=(2+1) as Chern-Simons theories*, Phys. Lett. B 229, 383 (1989).

[3] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large N field theories, string theory and gravity*, Phys. Rept. 323, 183 (2000) [arXiv:hep-th/9905111].

[4] O. Alvarez, T.P. Killingback, M.L. Mangano and P. Windey, *String theory and loop space index theorems*, Commun. Math. Phys. 111, 1 (1987).

[5] O. Alvarez, T.P. Killingback, M.L. Mangano and P. Windey, *The Dirac-Ramond operator in string theory and loop space index theorems*, UCB-PTH-87/1, invited talk presented at the Irvine Conference on ‘Non-Perturbative Methods in Physics’, Irvine, California, Jan 5-9, 1987.

[6] M.B. Barban, *The ‘large sieve’ method and its applications in the theory of numbers*, Russ. Math. Surv. 21 (1), 49 (1966).

[7] R.E. Borcherds, *The Gross-Kohnen-Zagier theorem in higher dimensions*, Duke mathematical journal 97 no. 2, 219 (1999) [arXiv:alg-geom/971002v3].

[8] J.D. Brown and M. Henneaux, *Central charges in the canonical realization of asymptotic symmetries: an example from three-dimensional gravity*, Commun. Math. Phys. 104, 207 (1986).

[9] W. Boucher, D. Friedan and A. Kent, *Determinant formulae and unitarity for the $N = 2$ superconformal algebras in two-dimensions or exact results on string compactification*, Phys. Lett. B 172, 316 (1986).

[10] S. Cecotti, P. Fendley, K.A. Intriligator and C. Vafa, *A new supersymmetric index*, Nucl. Phys. B 386, 405 (1992) [arXiv:hep-th/9204102].
[11] M. Cvetic and F. Larsen, Near horizon geometry of rotating black holes in five dimensions, Nucl. Phys. B 531, 239 (1998) [arXiv:hep-th/9805097].

[12] J.R. David, B. Sahoo and A. Sen, AdS$_3$, black holes and higher derivative corrections, JHEP 0707, 058 (2007) [arXiv:0705.0735].

[13] J. de Boer, M.C.N. Cheng, R. Dijkgraaf, J. Manschot and E. Verlinde, A farey tail for attractor black holes, JHEP 0611, 024 (2006) [arXiv:hep-th/0608059]. E. Verlinde, A Farey tail for N=2 black holes, http://strings06.itp.ac.cn/.

[14] F. Denef, M.R. Douglas and S. Kachru, Physics of string flux compactifications, Ann. Rev. Nucl. Part. Sci. 57, 119 (2007) [arXiv:hep-th/0701050].

[15] F. Denef and G.W. Moore, Split states, entropy enigmas, holes and halos, hep-th/0702146.

[16] R. Dijkgraaf, J.M. Maldacena, G.W. Moore, and E.P. Verlinde, A black hole farey tail, hep-th/0005003.

[17] M.R. Douglas and S. Kachru, Flux compactification, Rev. Mod. Phys. 79, 733 (2007) [arXiv:hep-th/0610102].

[18] T. Eguchi and H. Ooguri, Conformal and current algebras on general Riemann surface, Nucl. Phys. B 282, 308 (1987).

[19] T. Eguchi, H. Ooguri, A. Taormina and S. K. Yang, Superconformal algebras and string compactification on manifolds with SU(N) holonomy, Nucl. Phys. B 315, 193 (1989).

[20] T. Eguchi and A. Taormina, Character formulas for the N = 4 superconformal algebra, Phys. Lett. B 200, 315 (1988).

[21] T. Eguchi and A. Taormina, On the unitary representations of N = 2 and N = 4 superconformal algebras, Phys. Lett. B 210, 125 (1988).

[22] M. Eichler and D. Zagier, The Theory of Jacobi Forms, Birkhäuser, 1985.

[23] P.D.T.A. Elliot, Probabilistic Number Theory II: Central Limit Theorems, Grundlehren der Mathematische Wissenschaften 240, Springer Verlag, New York, 1980.

[24] M.R. Gaberdiel, Constraints on extremal self-dual CFTs, JHEP 0711, 087 (2007) [arXiv:0707.4073].

[25] M.R. Gaberdiel and C.A. Keller, Modular differential equations and null vectors, JHEP 0809, 079 (2008) [arXiv:0804.0489].

[26] V. Gritsenko, Elliptic genus of Calabi-Yau manifolds and Jacobi and Siegel modular forms, St. Petersburg Math. J. 11, 781 (2000) [math/9906190].

[27] S. Gukov, E. Martinec, G.W. Moore and A. Strominger, An index for 2D field theories with large $\mathcal{N} = 4$ superconformal symmetry, arXiv:hep-th/0404023.

[28] G. Hardy and E. Wright, An Introduction to the Theory of Numbers, Clarendon Press, 1968.

[29] T. Kawai, Y. Yamada, and S.-K. Yang, Elliptic genera and $\mathcal{N} = 2$ superconformal field theory, Nucl. Phys. B 414, 191 (1994) hep-th/9306098.

[30] W. Lerche, B.E.W. Nilsson, A.N. Schellekens and N.P. Warner, Anomaly cancelling terms from the elliptic genus, Nucl. Phys. B 299, 91 (1988).

[31] W. Lerche and N.P. Warner, Index theorems in $\mathcal{N} = 2$ superconformal theories, Phys. Lett. B 205, 471 (1988).
[32] J.M. Maldacena and A. Strominger, \textit{AdS(3) black holes and a stringy exclusion principle}, JHEP \textbf{9812}, 005 (1998) [arXiv:hep-th/9804085].

[33] J.M. Maldacena, G.W. Moore and A. Strominger, \textit{Counting BPS black holes in toroidal type II string theory}, arXiv:hep-th/9903163.

[34] J.M. Maldacena, A. Strominger and E. Witten, \textit{Black hole entropy in M-theory}, JHEP \textbf{9712}, 002 (1997) [arXiv:hep-th/9711053].

[35] J. Manschot and G.W. Moore, \textit{A modern Farey tail}, arXiv:0712.0573.

[36] J. Manschot, \textit{On the space of elliptic genera}, arXiv:0805.4333 [hep-th].

[37] S.D. Miller and G.W. Moore, \textit{Landau-Siegel zeroes and black hole entropy}, arXiv:hep-th/9903267.

[38] R. Minasian, G.W. Moore and D. Tsimpis, \textit{Calabi-Yau black holes and (0,4) sigma models}, Commun. Math. Phys. \textbf{209}, 325 (2000) [arXiv:hep-th/9904217].

[39] G.W. Moore, \textit{Les Houches lectures on strings and arithmetic}, arXiv:hep-th/0401049.

[40] H. Ooguri, \textit{Superconformal symmetry and geometry of Ricci flat Kähler manifolds}, Int. J. Mod. Phys. A \textbf{4}, 4303 (1989).

[41] K. Pilch, A.N. Schellekens and N.P. Warner, \textit{Path integral calculation of string anomalies}, Nucl. Phys. B \textbf{287}, 362 (1987).

[42] K. Pilch and N.P. Warner, \textit{String structures and the index of the Dirac-Ramond operator on orbifolds}, Commun. Math. Phys. \textbf{115}, 191 (1988).

[43] A.N. Schellekens and N.P. Warner, \textit{Anomaly cancellation and selfdual lattices}, Phys. Lett. B \textbf{181}, 339 (1986).

[44] A.N. Schellekens and N.P. Warner, \textit{Anomalies and modular invariance in string theory}, Phys. Lett. B \textbf{177}, 317 (1986).

[45] A. Schwimmer and N. Seiberg, \textit{Comments on the $N = 2, N = 3, N = 4$ superconformal algebras in two-dimensions}, Phys. Lett. B \textbf{184}, 191 (1987).

[46] P. Windey, \textit{The new loop space index theorems and string theory}, Lectures given at 25th Ettore Majorana Summer School for Subnuclear Physics, Erice, Italy, Aug 6-14, 1987.

[47] E. Witten, \textit{Elliptic genera and quantum field theory}, Commun. Math. Phys. \textbf{109}, 525 (1987); \textit{The index of the Dirac operator in loop space}, Proceedings of the conference on ‘Elliptic curves and modular forms in algebraic topology’, Princeton NJ, 1986.

[48] E. Witten, \textit{Quantum field theory and the Jones polynomial}, Commun. Math. Phys. \textbf{121}, 351 (1989).

[49] E. Witten, \textit{Three-dimensional gravity revisited}, arXiv:0706.3355.