Dirac operators on noncommutative hypersurfaces

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Abstract

This paper studies geometric structures on noncommutative hypersurfaces within a module-theoretic approach to noncommutative Riemannian (spin) geometry. A construction to induce differential, Riemannian and spinorial structures from a noncommutative embedding space to a noncommutative hypersurface is developed and applied to obtain noncommutative hypersurface Dirac operators. The general construction is illustrated by studying the sequence $T^2_\theta \hookrightarrow S^3_\theta \twoheadrightarrow \mathbb{R}^4_\theta$ of noncommutative hypersurface embeddings.

Keywords: noncommutative geometry, bimodule connections, Dirac operators, noncommutative hypersurfaces

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1 Introduction and summary

Dirac operators play a fundamental role in both quantum physics and noncommutative geometry. From the point of view of Connes’ axiomatization of noncommutative Riemannian spin manifolds in terms of spectral triples \cite{Con94}, a Dirac operator is the basic object that is supposed to encode all geometric information about the noncommutative space. However, the way in which a Dirac operator encodes this geometric data is rather implicit, hence it is in general difficult to extract information about the metric or curvature of a noncommutative space, see e.g. \cite{CM14}.
An alternative approach to noncommutative Riemannian (spin) geometry is to encode the relevant geometric data layer by layer in terms of noncommutative generalizations of differential calculi, metrics, connections and spinorial structures, see e.g. [Lan97], [D-V01] and [BM19]. This module-theoretic approach maintains closer ties to the structures familiar from classical differential geometry, which can be very beneficial for constructing, analyzing and also interpreting examples of noncommutative spaces. Moreover, due to results by Beggs and Majid [BM17], this approach leads under certain additional hypotheses to examples of spectral triples in the sense of [Con94].

The aim of the present paper is to develop techniques that allow us to induce differential, Riemannian and spinorial structures on a noncommutative embedding space to a noncommutative hypersurface. Our construction is a noncommutative generalization of well-known results in classical differential geometry, see e.g. [Bur93], [Tra95], [Bar96], [HMZ02], and it results in an explicit formula for the Dirac operator on the noncommutative hypersurface. In particular, our techniques and results can be applied to construct examples of curved noncommutative hypersurfaces and their Dirac operators from very simple flat noncommutative embedding spaces.

The outline of the remainder of this paper is as follows: In Section 2 we provide a brief review of the relevant algebraic and geometric preliminaries from the module-theoretic approach to noncommutative Riemannian (spin) geometry. Section 3 presents our main results on induced differential, Riemannian and spinorial structures on noncommutative hypersurfaces. Our construction requires certain additional hypotheses on the structure of the noncommutative hypersurface under consideration, which we will introduce consecutively as soon as they are needed. We refer the reader to Assumptions 3.2, 3.5, 3.9 and 3.11 for a complete list of these hypotheses. The main result of this section is Proposition 3.13 where we derive an explicit expression for the Dirac operator on the noncommutative hypersurface. In Section 4 we illustrate our constructions and results by applying them to the sequence of noncommutative hypersurface embeddings $T^2_0 \hookrightarrow S^3_0 \hookrightarrow \mathbb{R}^4$ studied by Arnlind and Norkvist [AN19]. Starting from a very simple flat noncommutative geometry on $\mathbb{R}^4$, we compute the induced geometric structures on both the Connes-Landi sphere $S^3_0$ and the noncommutative torus $T^2_0$. Our induced hypersurface Dirac operators on $S^3_0$ and $T^2_0$ are isospectral to the commutative ones and hence also to the Dirac operators obtained from toric noncommutative deformations in [CL01], [CD-V02], [BLvS13].

## 2 Algebraic and geometric preliminaries

In this paper all vector spaces, algebras, modules, etc., will be over the field $\mathbb{C}$ of complex numbers. Given an (associative and unital) algebra $A$, we denote by $A \text{Mod}$ the category of left $A$-modules and by $A \text{Mod}^\text{op}$ the category of right $A$-modules. Recall that the latter category is monoidal with respect to the relative tensor product $V \otimes_A W \in A \text{Mod}^\text{op}$ of $A$-bimodules $V, W \in A \text{Mod}^\text{op}$ and monoidal unit given by the 1-dimensional free $A$-bimodule $A \in A \text{Mod}$. Furthermore, $A \text{Mod}$ is a (left) module category over the monoidal category $(A \text{Mod}, \otimes_A, A)$, with left action given by the relative tensor product $V \otimes_A \mathcal{E} \in A \text{Mod}$, for all $V \in A \text{Mod}$ and $\mathcal{E} \in A \text{Mod}$.

Let us recall briefly some basic concepts from noncommutative geometry, see e.g. [Lan97], [D-V01] and [BM19] for a detailed introduction to the relevant frameworks.

**Definition 2.1.** A (first-order) differential calculus on an algebra $A$ is a pair $(\Omega^1_A, d)$ consisting of an $A$-bimodule $\Omega^1_A \in A \text{Mod}^\text{op}$ and a linear map $d : A \to \Omega^1_A$ (called differential), such that

1. $d(aa') = (da)a' + a(da')$, for all $a, a' \in A$,
2. $\Omega^1_A = A d(A) := \{ \sum_i a_i da'_i : a_i, a'_i \in A \}.$

We call $\Omega^1_A$ the $A$-bimodule of 1-forms on $A$.

**Definition 2.2.** Let $(\Omega^1_A, d)$ be a differential calculus on an algebra $A$. 


A connection on a left $A$-module $E \in \mathcal{A} \text{Mod}$ is a linear map \( \nabla : E \to \Omega^1_A \otimes_A E \) that satisfies the left Leibniz rule
\[
\nabla(a s) = a \nabla(s) + da \otimes_A s \quad , \tag{2.1}
\]
for all $a \in A$ and $s \in E$.

(ii) A bimodule connection on an $A$-bimodule $V \in \mathcal{A} \text{Mod}_A$ is a pair \( (\nabla, \sigma) \) consisting of a linear map \( \nabla : V \to \Omega^1_A \otimes_A V \) and an $A$-bimodule isomorphism \( \sigma : V \otimes_A \Omega^1_A \to \Omega^1_A \otimes_A V \), such that the following left and right Leibniz rules
\[
\nabla(a v) = a \nabla(v) + da \otimes_A v \quad , \tag{2.2a}
\]
\[
\nabla(v a) = \nabla(v) a + \sigma(v \otimes_A da) \quad , \tag{2.2b}
\]
are satisfied, for all $a \in A$ and $v \in V$.

The concept of bimodule connections is motivated by the following standard result, see e.g. [D-V01] Section 10.

**Proposition 2.3.** Let \( (\Omega^1_A, d) \) be a differential calculus on an algebra $A$.

(i) Let \( \nabla^E \) be a connection on a left $A$-module $E \in \mathcal{A} \text{Mod}$ and \( (\nabla^V, \sigma^V) \) a bimodule connection on an $A$-bimodule $V \in \mathcal{A} \text{Mod}_A$. Then
\[
\nabla^\otimes(v \otimes_A s) := \nabla^V(v) \otimes_A s + (\sigma^V \otimes_A \text{id})(v \otimes_A \nabla^E(s)) \quad , \tag{2.3}
\]
for all $v \in V$ and $s \in E$, defines a connection on the tensor product module $V \otimes_A E \in \mathcal{A} \text{Mod}$.

(ii) Let \( (\nabla^V, \sigma^V) \) and \( (\nabla^W, \sigma^W) \) be bimodule connections on two $A$-bimodules $V, W \in \mathcal{A} \text{Mod}_A$. Then
\[
\nabla^\otimes(v \otimes_A w) := \nabla^V(v) \otimes_A w + (\sigma^V \otimes_A \text{id})(v \otimes_A \nabla^W(w)) \quad , \tag{2.4a}
\]
for all $v \in V$ and $w \in W$, and the composite $A$-bimodule isomorphism
\[
\sigma^\otimes : V \otimes_A W \otimes_A \Omega^1_A \xrightarrow{id \otimes_A \sigma^W} V \otimes_A \Omega^1_A \otimes_A W \xrightarrow{\sigma^V \otimes_A \text{id}} \Omega^1_A \otimes_A V \otimes_A W \quad \tag{2.4b}
\]
defines a bimodule connection on the tensor product bimodule $V \otimes_A W \in \mathcal{A} \text{Mod}_A$.

**Definition 2.4.** A (generalized) metric on $\Omega^1_A$ is an $A$-bimodule map \( g : A \to \Omega^1_A \otimes_A \Omega^1_A \) for which there exists an $A$-bimodule map \( g^{-1} : \Omega^1_A \otimes_A \Omega^1_A \to A \), such that the two compositions
\[
\Omega^1_A \cong \Omega^1_A \otimes_A A \xrightarrow{id \otimes_A g} \Omega^1_A \otimes_A \Omega^1_A \otimes_A \Omega^1_A \xrightarrow{g^{-1} \otimes_A \text{id}} A \otimes_A \Omega^1_A \cong \Omega^1_A \quad \tag{2.5}
\]
\[
\Omega^1_A \cong A \otimes_A \Omega^1_A \xrightarrow{g \otimes_A \text{id}} \Omega^1_A \otimes_A \Omega^1_A \otimes_A \Omega^1_A \xrightarrow{\text{id} \otimes_A g^{-1}} \Omega^1_A \otimes_A A \cong \Omega^1_A \quad \tag{2.5}
\]
are the identity morphisms. We call $g^{-1}$ the inverse metric.

**Remark 2.5.** Since $A$ is a free module with a basis given by the unit element $1 \in A$, the datum of an $A$-bimodule map \( g : A \to \Omega^1_A \otimes_A \Omega^1_A \) is equivalent to that of a central element \( g(1) \in \Omega^1_A \otimes_A \Omega^1_A \), i.e. \( a g(1) = g(1) a \) for all $a \in A$. Writing this element as \( g(1) = \sum_{\alpha} g^\alpha \otimes_A g_\alpha \), the two conditions in (2.5) read as
\[
\sum_{\alpha} g^{-1}(\omega \otimes_A g^\alpha) g_\alpha = \omega = \sum_{\alpha} g^\alpha g^{-1}(g_\alpha \otimes_A \omega) \quad , \tag{2.6}
\]
for all $\omega \in \Omega^1_A$. Using these identities it is easy to prove that, provided it exists, the inverse metric $g^{-1}$ is unique.
Definition 2.6. Let $(Ω_1^A, d)$ be a differential calculus on an algebra $A$. A Riemannian structure on $(Ω_1^A, d)$ is a pair $(g, (\nabla, \sigma))$ consisting of a (generalized) metric $g$ on $Ω_1^A$ and a bimodule connection $(\nabla, \sigma)$ on $Ω_1^A$ that satisfies the following properties:

(i) **Symmetry:** The diagram

$$
\begin{array}{ccc}
Ω_1^A \otimes_A Ω_1^A & \xrightarrow{\sigma} & Ω_1^A \otimes_A Ω_1^A \\
\downarrow_{g^{-1}} & & \downarrow_{g^{-1}} \\
\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc & & \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc
\end{array}
$$

commutes.

(ii) **Metric compatibility:** The diagram

$$
\begin{array}{ccc}
Ω_1^A \otimes_A Ω_1^A & \xrightarrow{\nabla \otimes} & Ω_1^A \otimes_A Ω_1^A \otimes_A Ω_1^A \\
\downarrow_{g^{-1}} & & \downarrow_{\text{id} \otimes_A g^{-1}} \\
\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc & & Ω_1^A \cong Ω_1^A \otimes_A A
\end{array}
$$

commutes, where $\nabla \otimes$ is the tensor product connection from Proposition 2.3.

Remark 2.7. Note that our definition of Riemannian structures does not include a torsion-free condition for the connection $\nabla$. In those cases where one has a second-order differential calculus $Ω_2^A$, one can supplement Definition 2.6 with the torsion-free condition $\mathcal{T} = 0$, where $\mathcal{T} := \wedge \circ \nabla - d : Ω_1^A \to Ω_2^A$ is the torsion tensor, see [BM19]. The reason why we do not consider the torsion-free condition is that our constructions in this paper apply to connections with torsion too, hence this condition is not needed.

Next, we introduce a suitable concept of spinorial structure based on the module-theoretic approach by Beggs and Majid [BM17, BM19]. Let $(Ω_1^A, d)$ be a differential calculus on an algebra $A$ and $(g, (\nabla, \sigma))$ a Riemannian structure on $(Ω_1^A, d)$. Consider a left $A$-module $E ∈ A\text{Mod}$, which we interpret as the module of sections of a spinor bundle. This module should come endowed with a connection $\nabla^\text{sp} : E → Ω_1^A \otimes_A E$, which we interpret as spin connection, and an $A$-module map $γ : Ω_1^A \otimes_A E → E$, which we interpret as Clifford multiplication. These data will be required to be compatible (in the sense defined below) with the Riemannian structure $(g, (\nabla, \sigma))$. For later use, let us introduce the notation

$$
γ_{[2]} : Ω_1^A \otimes_A Ω_1^A \otimes_A E \xrightarrow{\text{id} \otimes_A γ} Ω_1^A \otimes_A E \xrightarrow{γ} E
$$

for the $A$-module map obtained by iterated application of $γ$. Analogously, one can define $γ_{[n]} : Ω_1^A \otimes_A^n \otimes_A E → E$, for all $n ∈ \mathbb{N}$.

Definition 2.8. Let $(Ω_1^A, d)$ be a differential calculus on an algebra $A$ and $(g, (\nabla, \sigma))$ a Riemannian structure on $(Ω_1^A, d)$. A **spinorial structure** on $(g, (\nabla, \sigma))$ is a triple $(E, \nabla^\text{sp}, γ)$ consisting of a left $A$-module $E ∈ A\text{Mod}$, a connection $\nabla^\text{sp}$ on $E$ and an $A$-module map $γ : Ω_1^A \otimes_A E → E$ that satisfies the following properties:

(i) **Clifford relations:** The diagram

$$
\begin{array}{ccc}
Ω_1^A \otimes_A Ω_1^A \otimes_A E & \xrightarrow{-2g^{-1} \otimes_A \text{id}} & A \otimes_A E \\
γ_{[2]} + γ_{[2]}(σ \otimes_A \text{id}) \downarrow & & \downarrow \cong \\
E & & E
\end{array}
$$

commutes.
(ii) **Clifford compatibility**: The diagram

\[
\begin{array}{c}
\Omega^1_A \otimes_A \mathcal{E} \xrightarrow{\nabla^\otimes} \Omega^1_A \otimes_A \Omega^1_A \otimes_A \mathcal{E} \\
\downarrow \gamma \quad \downarrow \text{id} \otimes_A \gamma \\
\mathcal{E} \xrightarrow{\nabla^{sp}} \Omega^1_A \otimes_A \mathcal{E}
\end{array}
\]

(2.11)

commutes, where $\nabla^\otimes$ is the tensor product connection from Proposition 2.3.

We shall call the composite

\[
D : \mathcal{E} \xrightarrow{\nabla^{sp}} \Omega^1_A \otimes_A \mathcal{E} \xrightarrow{\gamma} \mathcal{E}
\]

(2.12)

the *Dirac operator* associated with the given spinorial structure.

**Remark 2.9.** We would like to emphasize that our definition of spinorial structures is less general than the one by Beggs and Majid [BM17, BM19], which does not assume the Clifford compatibility property (2.11). We decided to include this additional axiom in our definition, because it is an important guiding principle for our construction of Dirac operators on noncommutative hypersurfaces in Section 3 and it is satisfied in our examples of interest in Section 4.

## 3 Induced structures on noncommutative hypersurfaces

Throughout the whole section, we fix an algebra $A$, a differential calculus $(\Omega^1_A, d)$ on $A$ (see Definition 2.1), a Riemannian structure $(g, (\nabla, \sigma))$ on $(\Omega^1_A, d)$ (see Definition 2.6) and a spinorial structure $(\mathcal{E}, \nabla^{sp}, \gamma)$ on $(g, (\nabla, \sigma))$ (see Definition 2.8). We interpret $A$ as (the algebra of functions on) a noncommutative embedding space, which is endowed with a differential, Riemannian and spinorial structure. Given any *central* element $f \in Z(A) \subseteq A$, i.e. $fa = af$ for all $a \in A$, we define the quotient algebra

\[
B := A/(f)
\]

(3.1)

where $(f) \subseteq A$ denotes the ideal generated by $f \in A$. One should interpret $B$ as (the algebra of functions on) the noncommutative hypersurface determined heuristically by $f = 0$. The quotient algebra map $q : A \to B$ should be thought of as (the dual of) the embedding of the noncommutative hypersurface into the embedding space. We assume that our noncommutative hypersurface admits a “normalized normal vector field”, which in the dual language of forms means that there exists a central element $\lambda \in Z(A) \subseteq A$, such that the 1-form

\[
\eta := \lambda df \in \Omega^1_A
\]

(3.2a)

is normalized according to

\[
[g^{-1}(\eta \otimes_A \eta)] = 1 \in B
\]

(3.2b)

Here and in the following, we use square brackets to denote equivalence classes in quotient spaces. The aim of this section is to induce along the quotient map $q : A \to B$ the differential, Riemannian and spinorial structures from the noncommutative embedding space $A$ to the noncommutative hypersurface $B$. Our construction requires certain additional assumptions that will be introduced below as soon as they are needed.
3.1 Induced differential structure

Our first aim is to induce a differential calculus $(\Omega^1_B, d)$ on the quotient algebra $B = A/(f)$. For this we recall that associated with the algebra map $q : A \to B$ is a change of base functor $q : \mathcal{A} \text{Mod}_A \to \mathcal{B} \text{Mod}_B$ for bimodules, which is given by $q(V) = B \otimes_A V \otimes_A B \in \mathcal{B} \text{Mod}_B$, for all $V \in \mathcal{A} \text{Mod}_A$. Because $B = A/(f)$ is a quotient algebra, with $f \in \mathcal{Z}(A) \subseteq A$ a central element, and $q : A \to B$ is the corresponding quotient map, there exists a natural $B$-bimodule isomorphism

$$q(V) \xrightarrow{\cong} \frac{V}{fV \cup Vf}, \ [a] \otimes_A v \otimes_A [a'] \mapsto [av a'], \quad (3.3)$$

where $fV := \{fv : v \in V\} \subseteq V$ and $Vf := \{vf : v \in V\} \subseteq V$. Applying the change of base functor on the $A$-bimodule of 1-forms $\Omega^1_A \in \mathcal{A} \text{Mod}_A$ is however not sufficient to define a differential calculus on $B$, because the differential $d : A \to q(\Omega^1_A)$ does not in general descend to the quotient $B = A/(f)$. This problem is solved by introducing the quotient $B$-bimodule

$$\Omega^1_B := \frac{q(\Omega^1_A)}{B[df]B} \in \mathcal{B} \text{Mod}_B, \quad (3.4)$$

where $B[df]B := \{\sum_i b_i [df] b'_i : b_i, b'_i \in B\}$ is the $B$-subbimodule generated by $[df] \in q(\Omega^1_A)$. The differential $d : A \to \Omega^1_A$ then descends to a linear map

$$d_B : B \to \Omega^1_B, \quad [a] \mapsto [da]. \quad (3.5)$$

**Proposition 3.1.** $(\Omega^1_B, d_B)$ is a differential calculus on the quotient algebra $B = A/(f)$. 

**Proof.** The necessary properties of Definition 2.1 are inherited from the differential calculus $(\Omega^1_A, d)$ on $A$. \hfill \square

As a preparation for the following subsections, we construct a $B$-bimodule isomorphism between $\Omega^1_B \in \mathcal{B} \text{Mod}_B$ and a certain $B$-subbimodule of $q(\Omega^1_A) \in \mathcal{B} \text{Mod}_B$. This construction requires the following

**Assumption 3.2.** The element $df \in \Omega^1_A$ is central, i.e. $(df)a = a(df)$ for all $a \in A$.

**Lemma 3.3.** Assumption 3.2 implies that $f \omega = \omega f$, for all $\omega \in \Omega^1_A$. As a consequence, the $B$-bimodule isomorphism in 3.3 specializes to

$$q!(\Omega^1_A) \cong \frac{\Omega^1_A}{f\Omega^1_A}. \quad (3.6)$$

**Proof.** By Definition 2.1 (ii), we can write $\omega = \sum_i a_i da'_i$. Using centrality of $f \in A$, the Leibniz rule for $d$ and Assumption 3.2, we compute

$$f \omega = \sum_i a_i f(da'_i) = \sum_i a_i (d(fa'_i) - (df)a'_i) = \sum_i a_i (d(a'_i f) - a'_i (df)) = \omega f, \quad (3.7)$$

which proves our claim. \hfill \square

Using the inverse metric $g^{-1} : \Omega^1_A \otimes_A \Omega^1_A \to A$ and the normalized 1-form $\eta = \lambda df \in \Omega^1_A$ from 3.2, we define the $A$-bimodule endomorphism

$$\Pi : \Omega^1_A \to \Omega^1_A, \quad \omega \mapsto \omega - g^{-1}(\omega \otimes_A \eta) \eta. \quad (3.8a)$$

Note that Assumption 3.2 implies that also $\eta$ is central, hence $\Pi$ is indeed right $A$-linear. Using the change of base functor, $\Pi$ defines a $B$-bimodule endomorphism

$$\Pi : q!(\Omega^1_A) \to q!(\Omega^1_A), \quad [\omega] \mapsto \Pi([\omega]) := [\Pi(\omega)]. \quad (3.8b)$$
Proposition 3.4. The $B$-bimodule endomorphism $\Pi$ from (3.8) satisfies the following properties:

(i) $\Pi([df]) = 0$.

(ii) $\Pi^2 = \Pi$, i.e. $\Pi$ is a projector.

(iii) The induced $B$-bimodule map $\Pi : \Omega^1_B \rightarrow q_!(\Omega^1_A)$ on $\Omega^1_B$ (cf. (3.1)) is a section of the quotient $B$-bimodule map $q_!(\Omega^1_A) \rightarrow \Omega^1_B$. In particular, it defines an isomorphism $\Omega^1_B \cong \Pi q_!(\Omega^1_A)$.

Proof. Item (i) follows directly from the normalization condition (3.2) and item (ii) is a direct consequence of (i). To prove item (iii), note that the induced map $\Pi : \Omega^1_B \rightarrow q_!(\Omega^1_A)$ is well-defined by (i). It is a section of the quotient map as the latter maps $\eta = [\lambda df]$ to 0 by the relations in (3.4). This in particular implies that the induced map $\Pi : \Omega^1_B \rightarrow q_!(\Omega^1_A)$ is injective, hence it defines an isomorphism onto its image $\Pi q_!(\Omega^1_A)$.

\[\square\]

3.2 Induced Riemannian structure

Our next aim is to induce a Riemannian structure $(g_B, (\nabla_B, \sigma_B))$ on the differential calculus $(\Omega^1_B, d)$ from Section 3.1. Using the change of base functor, the metric $g : A \rightarrow \Omega^1_A \otimes_A \Omega^1_A$ on $\Omega^1_A$ defines a $B$-bimodule map

\[g : B \rightarrow q_!(\Omega^1_A) \otimes_B q_!(\Omega^1_A), \quad [a] \mapsto [g(a)] \quad (3.9a)\]

and the inverse metric $g^{-1} : \Omega^1_A \otimes_A \Omega^1_A \rightarrow A$ defines a $B$-bimodule map

\[g^{-1} : q_!(\Omega^1_A) \otimes_B q_!(\Omega^1_A) \rightarrow B, \quad [\omega] \otimes_B [\zeta] \mapsto [g^{-1}(\omega \otimes_A \zeta)] \quad . \quad (3.9b)\]

Using also the quotient map $q_!(\Omega^1_A) \rightarrow \Omega^1_B$ and its section $\Pi : \Omega^1_B \rightarrow q_!(\Omega^1_A)$ from Proposition 3.4 (see also (3.5)), we define the composite $B$-bimodule maps

\[g_B : B \xrightarrow{g} q_!(\Omega^1_A) \otimes_B q_!(\Omega^1_A) \xrightarrow{\Pi \otimes_B \Pi} \Omega^1_B \otimes_B \Omega^1_B \quad (3.10a)\]

and

\[g_B^{-1} : \Omega^1_B \otimes_B \Omega^1_B \xrightarrow{g^{-1}} q_!(\Omega^1_A) \otimes_B q_!(\Omega^1_A) \xrightarrow{g^{-1}} B \quad . \quad (3.10b)\]

For our studies below, we shall need the following

Assumption 3.5. The $A$-bimodule isomorphism $\sigma : \Omega^1_A \otimes_A \Omega^1_A \rightarrow \Omega^1_A \otimes_A \Omega^1_A$ associated with the bimodule connection $(\nabla, \sigma)$ on $\Omega^1_A$ satisfies

\[\sigma(\omega \otimes_A df) = df \otimes_A \omega, \quad \sigma(df \otimes_A \omega) = \omega \otimes_A df \quad , \quad (3.11)\]

for all $\omega \in \Omega^1_A$.

Lemma 3.6. Assumption 3.5 implies the following properties:

(i) $g^{-1}(\omega \otimes_A df) = g^{-1}(df \otimes_A \omega)$, for all $\omega \in \Omega^1_A$.

(ii) $[g^{-1}(\Pi(\omega) \otimes_A df)] = 0$ and $[g^{-1}(df \otimes_A \Pi(\omega))] = 0$ in $B = A/(f)$, for all $\omega \in \Omega^1_A$. This implies that

\[g^{-1}(\Pi(\omega) \otimes_A \Pi(\zeta)) = [g^{-1}(\omega \otimes_A \Pi(\zeta))] = [g^{-1}(\Pi(\omega) \otimes_A \zeta)] \quad , \quad (3.12)\]

for all $\omega, \zeta \in \Omega^1_A$.

(iii) $[[\text{id} \otimes_A g^{-1}](\nabla(\eta) \otimes_A \eta)] = 0$. 

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(iv) The two $B$-bimodule maps in (3.10) define a metric $g$ and its inverse $g_B^{-1}$ on $\Omega_B^1$.

Proof. Item (i) is a direct consequence of the symmetry property of $g^{-1}$ (cf. Definition 2.6 and Assumption 3.5). The first equality of item (ii) follows from a short calculation

$$
[g^{-1}(\Pi(\omega) \otimes_A d\jmath)] = [g^{-1}((\omega - g^{-1}(\omega \otimes_A \eta) \otimes_A d\jmath))]
= [g^{-1}(\omega \otimes_A d\jmath) - g^{-1}(\omega \otimes_A \eta) g^{-1}(\eta \otimes_A d\jmath)] = 0,
$$

where we used the normalization condition (3.22) for $\eta = \lambda d\jmath$. The second equality in item (ii) follows from this and (i).

Item (iii) follows from the calculation

$$
[(\text{id} \otimes_A g^{-1})(\nabla(\eta) \otimes_A \eta)] = [d(g^{-1}(\eta \otimes_A \eta)) - (\text{id} \otimes_A g^{-1})(\sigma \otimes_A \text{id})(\eta \otimes_A \nabla(\eta))]
= -[(\text{id} \otimes_A g^{-1})(\nabla(\eta) \otimes_A \eta)],
$$

where in the first step we used metric compatibility (2.8) and in the second step we used the normalization condition (3.12) and item (i).

To prove item (iv), we use the same notations as in Remark 2.5 to write $g_B(1) = [g(1)] = [\sum \alpha g^\alpha \otimes_A g_\alpha] = [\sum \alpha (g^\alpha) \otimes_B [g_\alpha]]$ for $g_B^{-1}([\omega] \otimes_B [\zeta]) = [g^{-1}(\Pi(\omega) \otimes_A \Pi(\zeta))]$. We then compute

$$
\sum \alpha g_B^{-1}([\omega] \otimes_B [g^\alpha]) [g_\alpha] = \sum \alpha g^{-1}(\Pi(\omega) \otimes_A g^\alpha) [g_\alpha] = [\Pi(\omega)] = [\omega],
$$

where in the first step we used (ii). The second step follows from $g^{-1}$ being the inverse metric of $g$ and the last step uses that $\Pi$ is a section of the quotient map (cf. Proposition 3.4). The second property $\sum \alpha [g^\alpha] g_B^{-1}([g_\alpha] \otimes_B [\omega]) = [\omega]$ follows from a similar calculation.

Let us now focus on the bimodule connection $(\nabla, \sigma)$ on $\Omega_A^1$. From (3.6) and (3.4), we observe that the connection $\nabla : \Omega_A^1 \rightarrow B \otimes_B \Omega_A^1 \otimes_B \Omega_A^1 \rightarrow \Omega_B^1 \otimes_B \Omega_B^1 \otimes_B \Omega_B^1$ descends to a connection $\nabla : q_!(\Omega_A^1) \rightarrow B \otimes_B \Omega_B^1 \otimes_B \Omega_B^1$ on $q_!(\Omega_A^1) \in \mathcal{B}\text{Mod}_B$. Indeed, from the left Leibniz rule we conclude that $[\nabla(f \omega)] = [f \nabla(\omega) + d\jmath \otimes_A \omega] = 0$, hence this map is well-defined on the quotient. Using the quotient map $q_!(\Omega_A^1) \rightarrow \Omega_B^1$ and its section $\Pi : \Omega_B^1 \rightarrow q_!(\Omega_A^1)$ from Proposition 3.4 (see also 3.8), we define the composite linear map

$$
\nabla_B : \Omega_B^1 \xrightarrow{\Pi} q_!(\Omega_A^1) \xrightarrow{\nabla} \Omega_B^1 \otimes_B q_!(\Omega_A^1) \xrightarrow{\sigma} \Omega_B^1 \otimes_B \Omega_B^1.
$$

The $A$-bimodule isomorphism $\sigma : \Omega_A^1 \otimes_A \Omega_A^1 \rightarrow \Omega_B^1 \otimes_B \Omega_B^1$ associated with the bimodule connection $(\nabla, \sigma)$ on $\Omega_A^1$ descends, due to Assumption 3.5, to the $B$-bimodule isomorphism

$$
\sigma_B : \Omega_B^1 \otimes_B \Omega_B^1 \rightarrow \Omega_B^1 \otimes_B \Omega_B^1, \ [\omega] \otimes_B [\zeta] \rightarrow \left[\sigma(\omega \otimes_A \zeta]\right].
$$

Lemma 3.7. The pair $(\nabla_B, \sigma_B)$ defined by (3.16) and (3.17) is a bimodule connection on $\Omega_B^1$. It reads explicitly as

$$
\nabla_B([\omega]) = \left[\nabla(\omega) - g^{-1}(\omega \otimes_A \eta) \nabla(\eta)\right],
$$

for all $[\omega] \in \Omega_B^1$.

Proof. The explicit expression (3.18) is obtained by a short calculation

$$
\nabla_B([\omega]) = \left[\nabla(\omega - g^{-1}(\omega \otimes_A \eta))\right] = \left[\nabla(\omega) - g^{-1}(\omega \otimes_A \eta) \nabla(\eta) - d\jmath(g^{-1}(\omega \otimes_A \eta) \otimes_A \eta)\right]
= \left[\nabla(\omega) - g^{-1}(\omega \otimes_A \eta) \nabla(\eta)\right],
$$

where in the second step we used the left Leibniz rule for $\nabla$ and in the third step that $\eta = \lambda d\jmath$ is identified with 0 in $\Omega_B$ (cf. (3.4)). The left Leibniz rule is a direct consequence of this expression...
and the right Leibniz rule follows from the fact that $\nabla(\eta) \in \Omega^1_A \otimes_A \Omega^1_A$ is a central element. The latter statement is proven as follows
\begin{equation}
a \nabla(\eta) = \nabla(a \eta) - da \otimes_A \eta = \nabla(\eta) a - \sigma(\eta \otimes_A da) = \nabla(\eta) a, \tag{3.20}
\end{equation}
where we used the left and right Leibniz rules for $(\nabla, \sigma)$, centrality of $\eta = \lambda df$ (cf. Assumption 3.2) and also Assumption 3.3.

**Remark 3.8.** Note that (3.18) is a noncommutative analog of the usual Gauss formula for connections on Riemannian submanifolds, see e.g. [KN96, Chapter VII.3].

In order to prove the main result of this subsection, we require an additional

**Assumption 3.9.** The diagrams
\begin{align}
q_i(\Omega^1_A) \otimes_B q_i(\Omega^1_A) \xrightarrow{\Pi \otimes_B \text{id}} q_i(\Omega^1_A) \otimes_B q_i(\Omega^1_A) \tag{3.21a}
\end{align}
\begin{align}
q_i(\Omega^1_A) \otimes_B q_i(\Omega^1_A) \xrightarrow{\text{id} \otimes_B \Pi} q_i(\Omega^1_A) \otimes_B q_i(\Omega^1_A)
\end{align}
and
\begin{align}
q_i(\Omega^1_A) \otimes_B q_i(\Omega^1_A) \xrightarrow{\Pi \otimes_B \text{id}} q_i(\Omega^1_A) \otimes_B q_i(\Omega^1_A) \tag{3.21b}
\end{align}

commute.

**Proposition 3.10.** The pair $(g_B, (\nabla_B, \sigma_B))$ defined in (3.10), (3.16) and (3.17) is a Riemannian structure on $(\Omega^1_B, d)$.

**Proof.** It remains to prove the symmetry and metric compatibility properties from Definition 2.6. The symmetry property (2.7) for $(g_B, (\nabla_B, \sigma_B))$ follows immediately from Assumption 3.9 and symmetry of $g^{-1}$. To verify metric compatibility (2.8) for $(g_B, (\nabla_B, \sigma_B))$, we compute by using metric compatibility of the original Riemannian structure $(g, (\nabla, \sigma))$.
\begin{align}
d_B(g_B^{-1}(\omega) \otimes_B \zeta)) = \left[(\text{id} \otimes_A g^{-1})(\nabla\Pi(\omega) \otimes_A \Pi(\zeta) + \sigma_{12}(\Pi(\omega) \otimes_A \nabla\Pi(\zeta))\right], \tag{3.22}
\end{align}
where $\sigma_{12} := \sigma \otimes_A \text{id}$. Using Lemma 3.6 (ii), we can in the first term replace $\nabla\Pi(\omega)$ with $(\text{id} \otimes_A \Pi)\nabla\Pi(\omega)$. Using also Assumption 3.9, we can replace in the second term $\sigma_{12}(\Pi(\omega) \otimes_A \nabla\Pi(\zeta))$ with $(\text{id} \otimes_A \Pi \otimes_A \text{id})\sigma_{12}(\omega \otimes_A \nabla\Pi(\zeta))$ and hence via Lemma 3.6 (ii) with $(\text{id} \otimes_A \Pi \otimes_A \Pi)\sigma_{12}(\omega \otimes_A \nabla\Pi(\zeta))$. The resulting expression proves the metric compatibility property for $(g_B, (\nabla_B, \sigma_B))$.

### 3.3 Induced spinorial structure

We now induce a spinorial structure $(\mathcal{E}_B, \nabla_B^H, \gamma_B)$ on the Riemannian structure $(g_B, (\nabla_B, \sigma_B))$ from Section 3.3. Our definitions and constructions below are inspired by well-known results on spinorial structures on hypersurfaces in classical differential geometry, see e.g. [Bur93, Tra95, Bar96] and also [HMZ02] for a good review. As the first step, we use the change of base functor (for left modules) to define the $B$-module
\begin{equation}
\mathcal{E}_B := q_i(\mathcal{E}) \cong \frac{\mathcal{E}}{j \mathcal{E}} \in \text{BMod} \tag{3.23}
\end{equation}
To introduce a suitable Clifford multiplication \( \gamma_B : \Omega^1_B \otimes B \mathcal{E}_B \to \mathcal{E}_B \), we recall the classical case from [HMZ02, Eqn. (3.4)] and define the \( B \)-module map

\[
\gamma_B : \Omega^1_B \otimes B \mathcal{E}_B \to \mathcal{E}_B, \quad [\omega] \otimes [s] \mapsto [\gamma_2](\Pi(\omega) \circ \lambda_A \eta \circ \lambda_A s),
\]

where \( \gamma_2 \) was defined in (3.5). Note that this map is well-defined since the \( \lambda \)-form \( \eta = \lambda df \) is central by Assumption 3.2.

The given connection \( \nabla^{sp} : \mathcal{E} \to \Omega^1_A \otimes A \mathcal{E} \) on \( \mathcal{E} \in A \text{Mod} \) descends to a connection \( \nabla^{sp} : \mathcal{E}_B \to \Omega^1_B \otimes B \mathcal{E}_B \) on \( \mathcal{E}_B \in B \text{Mod} \) because \( \nabla^{sp}(f s) = f \nabla^{sp}(s) + df \otimes_A s \) = 0 as a consequence of the relations in (3.4) and (3.23). This is however not yet the correct induced spin connection on \( \mathcal{E}_B \). Motivated by the classical spinorial Gauss formula from [HMZ02, Eqn. (3.5)], we define

\[
\nabla^{sp}_B : \mathcal{E}_B \to \Omega^1_B \otimes B \mathcal{E}_B, \quad [s] \mapsto \left[ \nabla^{sp}(s) + \frac{1}{2}(\text{id} \otimes_A \gamma_2)(\nabla(\eta) \otimes_A \eta \circ \lambda_A s) \right].
\]

This defines a connection on the left \( B \)-module \( \mathcal{E}_B \in B \text{Mod} \) since both \( \eta \in \Omega^1_A \) and \( \nabla(\eta) \in \Omega^1_A \otimes \Omega^1_A \) are central. (The latter statement was proven in (3.20).)

In order to prove that these data define a spinorial structure in the sense of Definition 2.8, we require an additional

**Assumption 3.11.** The element \( \nabla(\eta) \in \Omega^1_A \otimes \Omega^1_A \) satisfies

\[
[\sigma_{23} \sigma_{12}(\Pi(\omega) \circ \lambda_A \nabla(\eta))] = [\nabla(\eta) \otimes_A \Pi(\omega)] \in \Omega^1_B \otimes B q(\Omega^1_A) \otimes B q(\Omega^1_A),
\]

for all \( \omega \in \Omega^1_A \), where \( \sigma_{12} := \sigma \circ \lambda_A \text{id} \) and \( \sigma_{23} := \text{id} \circ \lambda_A \sigma \).

**Proposition 3.12.** The triple \( (\mathcal{E}_B, \nabla^{sp}_B, \gamma_B) \) defined in (3.23), (3.25) and (3.24) is a spinorial structure on the Riemannian structure \( (g_B, (\nabla_B, \sigma_B)) \).

**Proof.** It remains to prove the Clifford relations and Clifford compatibility properties from Definition 2.8. In these calculations we frequently use the identities

\[
[\gamma_2](\Pi(\omega) \circ \lambda_A \eta \circ \lambda_A s) = -[\gamma_2](\eta \circ \lambda_A \Pi(\omega) \circ \lambda_A s)]
\]

and

\[
[\gamma_2](\eta \circ \lambda_A \eta \circ \lambda_A s) = -[s],
\]

which follow from the Clifford relations (2.10) for \( \gamma \), Assumption 3.5, Lemma 3.6 (ii) and the normalization condition (3.2).

The Clifford relations (2.10) for \( \gamma_B \) follow from a direct calculation, for which we introduce the convenient short notation \( \sigma(\omega \otimes \zeta) = \sum_\alpha \zeta^\alpha \otimes \omega_\alpha \). We compute

\[
\gamma_B[2]([\omega] \otimes [\zeta] \otimes [s] + \sigma_{B12}([\omega] \otimes [\zeta] \otimes [s]))
\]

\[
= [\gamma_4]\Pi(\omega) \circ \lambda_A \eta \circ \lambda_A \Pi(\zeta) \circ \lambda_A \eta \circ \lambda_A s + \sum_\alpha \Pi(\zeta^\alpha) \circ \lambda_A \eta \circ \lambda_A \Pi(\omega_\alpha) \circ \lambda_A \eta \circ \lambda_A s
\]

\[
= [\gamma_2]\Pi(\omega) \circ \lambda_A \Pi(\zeta) \circ \lambda_A s + \sum_\alpha \Pi(\zeta^\alpha) \circ \lambda_A \Pi(\omega_\alpha) \circ \lambda_A s
\]

\[
= -2 g_B^{-1}([\omega] \otimes [\zeta])[s],
\]

where in the second step we used (3.27). The last step follows from Assumption 3.9, the Clifford relations for \( \gamma \) and the definition of \( g_B^{-1} \) in (3.10).

Proving the Clifford compatibility property (2.11) for \( \nabla_B, \nabla^{sp}_B \) and \( \gamma_B \) is a lengthier computation. Using as above (3.27), Assumption 3.9 and also Clifford compatibility for \( \nabla, \nabla^{sp} \) and \( \gamma \),
one finds that the desired equality \( \nabla_{sB}^\text{sp} \gamma_B([\omega] \otimes_B [s]) = (\text{id} \otimes_B \gamma_B)(\nabla_{sB}^\text{sp}([\omega] \otimes_B [s])) \) is equivalent to the statement that the two expressions

\[
(id \otimes_A \gamma_{[2]})(\sigma_{12}(\Pi(\omega) \otimes_A \nabla(\eta) \otimes_A s) + \frac{1}{2} \nabla(\eta) \otimes_A \Pi(\omega) \otimes_A s)
\]

(3.29a)

and

\[
(id \otimes_A g^{-1})(\nabla \Pi(\omega) \otimes_A \eta) \otimes_A s + \frac{1}{2}(id \otimes_A \gamma_{[4]})(\sigma_{12}\sigma_{23}(\Pi(\omega) \otimes_A \eta \otimes_A \nabla(\eta) \otimes_A \eta \otimes_A s))
\]

(3.29b)

are equal. (The term with \( g^{-1} \) in (3.29b) arises from computing \((id \otimes_A \Pi)\nabla \Pi(\omega) = \nabla \Pi(\omega) - (id \otimes_A g^{-1})(\nabla \Pi(\omega) \otimes_A \eta) \otimes_A \eta \) via (3.8).) Using metric compatibility (2.8) for \((g, (\nabla, \sigma))\), Lemma 3.6 (ii) and the Clifford relations for \(\gamma\), we can rewrite the first term of (3.29b) as

\[
(3.29b)_{1\text{st}} = \left[ -(id \otimes_A g^{-1})\sigma_{12}(\Pi(\omega) \otimes_A \nabla(\eta)) \otimes_A s \right]
\]

\[
= \left[ \frac{1}{2}(id \otimes_A \gamma_{[2]})(\sigma_{12}(\Pi(\omega) \otimes_A \nabla(\eta) \otimes_A s) + \sigma_{23}\sigma_{12}(\Pi(\omega) \otimes_A \nabla(\eta) \otimes_A s)) \right].
\]

(3.30)

Concerning the second term of (3.29b), we use the Clifford relations for \(\gamma\) to bring the left factor of \(\eta\) to the right and observe that there is no \(g^{-1}\) contribution as a result of Lemma 3.6 (iii). Hence, we can rewrite the second term of (3.29b) as

\[
(3.29b)_{2\text{nd}} = \left[ \frac{1}{2}(id \otimes_A \gamma_{[2]})(\sigma_{12}(\Pi(\omega) \otimes_A \nabla(\eta) \otimes_A s)) \right].
\]

(3.31)

From these simplifications and Assumption 3.11 it follows that the expressions in (3.29b) and (3.29a) are equal. This completes our proof of the Clifford compatibility property.

We conclude this section by presenting an explicit expression for the induced Dirac operator

\[
D_B : \mathcal{E}_B \xrightarrow{\nabla_{sB}^\text{sp}} \Omega_B^1 \otimes_B \mathcal{E}_B \xrightarrow{\gamma_B} \mathcal{E}_B.
\]

(3.32)

**Proposition 3.13.** The induced Dirac operator (3.32) reads explicitly as

\[
D_B([s]) = \left[ -\frac{1}{2}\gamma_{[2]} - \gamma_{[2]}(\sigma \otimes_A \text{id})(\eta \otimes_A \nabla_{sB}(s)) + \frac{1}{2} \gamma_{[2]}((\Pi \otimes_A \text{id})\nabla(\eta) \otimes_A s) \right],
\]

(3.33)

for all \([s] \in \mathcal{E}_B\).

**Proof.** This is a straightforward calculation using (3.25), (3.24), the projector (3.8) and the Clifford relations (2.10) for \(\gamma\), in particular (3.27). Since the relevant steps are similar to those in the proof of Proposition 3.12 we do not have to write out the details of this calculation.

4 Examples

In this section we will illustrate our framework by applying it to the sequence of noncommutative hypersurface embeddings \(T^2_\emptyset \hookrightarrow S^3_\emptyset \rightarrow \mathbb{R}^4_\emptyset\) studied by Arnlind and Norkvist [AN19]. We describe first the relevant differential, Riemannian and spinorial structures on the noncommutative embedding space \(\mathbb{R}^4_\emptyset\) following our definitions in Section 2. We then use our constructions from Section 3 to induce these structures to the noncommutative hypersurface \(S^3_\emptyset \hookrightarrow \mathbb{R}^4_\emptyset\) and in a second step to the noncommutative hypersurface \(T^2_\emptyset \hookrightarrow S^3_\emptyset\). These studies result in explicit expressions for the Dirac operators (in the sense of Definition 2.8) on these noncommutative hypersurfaces.
4.1 Noncommutative embedding space $\mathbb{R}^4_\theta$

We begin by introducing the noncommutative embedding space $\mathbb{R}^4_\theta$. Instead of working with real coordinates $x^\mu$, for $\mu = 1, \ldots, 4$, it will be more convenient to introduce the complex coordinates $z^1 := x^1 + ix^2$ and $z^2 := x^3 + ix^4$, together with their complex conjugates $\overline{z}^1 = x^1 - ix^2$ and $\overline{z}^2 = x^3 - ix^4$. The noncommutative embedding space $\mathbb{R}^4_\theta \cong \mathbb{C}^2_\theta$ is then described by the noncommutative algebra

$$A := \mathbb{C}[z^1, z^2, z^3, z^4] / (z^i z^j - R^{ji} z^j z^i)$$

(4.1)

that is freely generated by the complex coordinates, modulo the ideal generated by the commutation relations determined by the entries $R^{ji}$ of the matrix

$$R := \begin{pmatrix}
1 & e^{-i\theta} & 1 & e^{i\theta} \\
e^{i\theta} & 1 & e^{i\theta} & 1 \\
e^{-i\theta} & 1 & e^{-i\theta} & 1 \\
1 & e^{-i\theta} & 1 & e^{i\theta}
\end{pmatrix}, \quad \theta \in \mathbb{R} \quad (4.2)$$

For later use, we note that the entries of the matrix $R$ satisfy

$$R^{ij} = R^{ji}$$

(4.3a)

and

$$R^{ij} R^{ji} = 1$$

(4.3b)

where in the latter equation there is no summation over $i$ and $j$.

To define a differential calculus on $A$, let us introduce the free left $A$-module

$$\Omega^1_A := \bigoplus_{i=1}^4 A \, dz^i$$

(4.4a)

which we endow with the right $A$-action determined by

$$dz^i z^j := R^{ji} z^j dz^i$$

(4.4b)

(Note that this is analogous to the commutation relations in (4.1).) This defines an $A$-bimodule $\Omega^1_A \in A\text{Mod}_A$, which we endow with a differential $d : A \to \Omega^1_A$ by setting $d : z^i \mapsto dz^i$ for the generators and extending to all of $A$ via the Leibniz rule.

**Proposition 4.1.** The pair $(\Omega^1_A, d)$ is a differential calculus on $A$.

**Proof.** The statement holds by construction. 

The next step is to introduce a Riemannian structure. For this we consider the standard (flat) Euclidean metric on $\mathbb{R}^4_\theta$, which in complex coordinates reads as

$$g := \sum_{i,j=1}^4 g_{ij} \, dz^i \otimes_A dz^j \in \Omega^1_A \otimes_A \Omega^1_A \quad (4.5a)$$

where $g_{ij}$ are the entries of the matrix

$$(g_{ij}) := \frac{1}{2} \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix} \quad (4.5b)$$

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Using (4.3), (4.3b) and (12), one easily checks that \( g \in \Omega^1_A \otimes_A \Omega^1_A \) is a central element, hence it defines an \( A \)-bimodule map \( g : A \to \Omega^1_A \otimes_A \Omega^1_A \), see Remark 2.5. The inverse metric \( g^{-1} : \Omega^1_A \otimes_A \Omega^1_A \to A \) is defined on the basis \( \{ dz^i \otimes_A dz^j : i, j = 1, \ldots, 4 \} \) of \( \Omega^1_A \otimes_A \Omega^1_A \) by

\[
g^{-1}(dz^i \otimes_A dz^j) = g^{ij} \tag{4.6a}
\]

where \( g^{ij} \) are the entries of the matrix

\[
(g^{ij}) = 2 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} . \tag{4.6b}
\]

Observe that

\[
\sum_{j=1}^4 g_{ij} g^{jk} = \delta_i^k . \tag{4.7}
\]

**Lemma 4.2.** The element \( g \in \Omega^1_A \otimes_A \Omega^1_A \) in (4.3) defines a (generalized) metric with inverse metric \( g^{-1} : \Omega^1_A \otimes_A \Omega^1_A \to A \) defined by (4.6).

**Proof.** It is sufficient to verify the conditions in Remark 2.5 on the basis \( \{ dz^k \in \Omega^1_A \} \). Using (4.7), we compute

\[
\sum_{i,j=1}^4 g_{ij} dz^j g^{-1}(dz^j \otimes_A dz^k) = \sum_{i,j=1}^4 g_{ij} dz^i g^{jk} = dz^k . \tag{4.8}
\]

The second condition in Remark 2.5 is confirmed through a similar calculation.

We define a connection \( \nabla : \Omega^1_A \to \Omega^1_A \otimes_A \Omega^1_A \) on \( \Omega^1_A \) by

\[
\nabla(dz^i) := 0 \tag{4.9}
\]

and the left Leibniz rule. Furthermore, we define an \( A \)-bimodule isomorphism \( \sigma : \Omega^1_A \otimes_A \Omega^1_A \to \Omega^1_A \otimes_A \Omega^1_A \) by

\[
\sigma(dz^i \otimes_A dz^j) := R^{ij} dz^j \otimes_A dz^i \tag{4.10}
\]

and left \( A \)-linear extension to all of \( \Omega^1_A \otimes_A \Omega^1_A \). (Note that this is analogous to the commutation relations in (4.1).)

**Lemma 4.3.** The pair \((\nabla, \sigma)\) introduced in (4.9) and (4.10) defines a bimodule connection.

**Proof.** It remains to confirm the right Leibniz rule from Definition 2.2 (ii). For this it is sufficient to consider homogeneous elements \( a = z^{j_1} \cdots z^{j_n} \in A \), for some \( n \in \mathbb{Z}_{\geq 0} \). We compute

\[
\nabla(dz^{j_1} \cdots dz^{j_n}) = \nabla(R^{j_1i} \cdots R^{j_ni} z^{j_1} \cdots z^{j_n} dz^i)
\]

\[
= R^{j_1i} \cdots R^{j_ni} d(z^{j_1} \cdots z^{j_n}) \otimes_A dz^i
\]

\[
= \sigma(dz^{j_1} \otimes_A d(z^{j_1} \cdots z^{j_n})) \tag{4.11}
\]

where in the first equality we used (4.4) and in the second equality we used the left Leibniz rule for the connection (4.9). The last equality follows by using the Leibniz rule to write \( d(z^{j_1} \cdots z^{j_n}) = \sum_{k=1}^n z^{j_1} \cdots z^{j_k-1} dz^k z^{j_k+1} \cdots z^{j_n} \) and then using (4.4), (4.3b) and the definition of \( \sigma \) in (4.10) in order to rearrange these terms.

**Proposition 4.4.** The pair \((g, (\nabla, \sigma))\) defined above is a Riemannian structure on \((\Omega^1_A, d)\).
the deformed gamma matrices

By construction, these gamma matrices satisfy the Clifford relations

where the first equality follows from (4.9) and the second equality from \( g^{-1}(dz^i \otimes_A dz^j) \in \mathbb{C} \).

The final step is to introduce a spinorial structure. For the spinor module, we take the 4-dimensional free left \( A \)-module

We denote by \( \{ e_\alpha \in \mathcal{E} : \alpha = 1, \ldots, 4 \} \) the standard basis for \( A^4 \), i.e. \( e_\alpha \) is the vector with 1 in the entry \( \alpha \) and 0 elsewhere. We define a spin connection \( \nabla^{sp} : \mathcal{E} \to \Omega^1_A \otimes_A \mathcal{E} \) by

and the left Leibniz rule. Introducing a Clifford multiplication requires some preparations. First, let us recall that the standard Euclidean gamma matrices in Cartesian coordinates on \( \mathbb{R}^4 \) can be expressed in terms of the 2 \( \times \) 2 identity matrix \( I_2 \) and the three Pauli matrices

Transforming the standard gamma matrices from Cartesian coordinates to our complex coordinates \( z^i \), we obtain

By construction, these gamma matrices satisfy the Clifford relations \( \{ \gamma^i, \gamma^j \} := \gamma^i \gamma^j + \gamma^j \gamma^i = -2 g^{ij} I_4 \), with \( g^{ij} \) given in (4.6). These gamma matrices are however not directly applicable to our noncommutative space \( \mathbb{R}^4 \), because the noncommutative Clifford relations (2.10) are given by an anticommutator involving the isomorphism \( \sigma \) in (4.10). To address this issue, we introduce the deformed gamma matrices

which can be obtained from the cocycle deformation techniques developed in [BLS13] [Ast14]. We define the associated Clifford multiplication \( \gamma : \Omega^1_A \otimes_A \mathcal{E} \to \mathcal{E} \) by

and left \( A \)-linear extension to all of \( \Omega^1_A \otimes_A \mathcal{E} \), where \( \gamma^i_0 e_\alpha \) denotes the action of the matrix \( \gamma^i_0 \) on the basis vector \( e_\alpha \in \mathcal{E} = A^4 \). Let us record some useful identities that we will use later.

**Lemma 4.5.** Define the \( \theta \)-anticommutator \( \{ \gamma^i_0, \gamma^j_0 \}_\theta := \gamma^i_0 \gamma^j_0 + R^{ij} \gamma^j_0 \gamma^i_0 \) and the \( \theta \)-commutator \( [\gamma^i_0, \gamma^j_0]_\theta := \gamma^i_0 \gamma^j_0 - R^{ij} \gamma^j_0 \gamma^i_0 \). Then the following properties hold true:
(i) \( \{ \gamma_0^i, \gamma_0^j \}_\partial = R^{ij} \{ \gamma_0^i, \gamma_0^j \}_\partial \)
(ii) \( [\gamma_0^i, \gamma_0^j]_\partial = -R^{ij} [\gamma_0^i, \gamma_0^j]_\partial \)
(iii) \( \{ \gamma_0^i, \gamma_0^j \}_\partial = -2 g^{ij} I_4 \)

Proof. Items (i) and (ii) follow directly from the definitions and (1.31). Item (iii) is a straight-forward calculation.

**Proposition 4.6.** The triple \((E, \nabla^{sp}, \gamma)\) defined in (1.13), (1.14) and (1.18) is a spinorial structure on the Riemannian structure \((g, (\nabla, \sigma))\).

**Proof.** We have to verify the two properties of Definition 2.8. The Clifford relations (2.10) follow directly from Lemma 4.5 (iii), because

\[(\gamma_2^i + \gamma_2^j)(\sigma \otimes_A d\alpha) (dz^i \otimes_A dz^j) e_\alpha = (\gamma_0^i \gamma_0^j + R^{ij} \gamma_0^i \gamma_0^j) e_\alpha = \{\gamma_0^i, \gamma_0^j\}_\partial e_\alpha = -2 g^{ij} e_\alpha = -2 g(dz^i \otimes_A d\alpha) e_\alpha .\]

Clifford compatibility (2.11) follows from

\[(id \otimes_A \gamma) \nabla^{\otimes} (dz^i \otimes_A e_\alpha) = 0 = \nabla^{sp} \gamma (dz^i \otimes_A e_\alpha) ,\]

where we used (1.9), (1.14) and (1.18).

We can now compute the Dirac operator (2.12) associated with our spinorial structure on \(R^4_\theta\). Expressing elements \(s = \sum_{\alpha=1}^4 s^\alpha e_\alpha \in E\) in terms of our basis and introducing the notation \(ds = \sum_{i=1}^4 \partial_\alpha dz^i\), for all \(a \in A\), we obtain

\[D(s) = \gamma (\nabla^{sp}(s)) = \sum_{\alpha=1}^4 \gamma (ds^\alpha \otimes_A e_\alpha) = \sum_{\alpha=1}^4 \sum_{i=1}^4 \partial_i s^\alpha \gamma_i^\alpha e_\alpha = \sum_{i=1}^4 \gamma_i^\alpha \partial_i s ,\]

where in the last equality we used the shorthand notation \(\partial_i s := \sum_{\alpha=1}^4 \partial_i s^\alpha e_\alpha\). Observe that this Dirac operator differs from the one of the commutative Euclidean space \(R^4\), because (4.21) involves the deformed gamma matrices (4.17).

**4.2 Noncommutative hypersurface \(S^3_\theta \rightarrow R^4_\theta\)**

In this section we apply our construction from Section 3 to induce the differential, Riemannian and spinorial structure on \(R^4_\theta\) to the noncommutative 3-sphere \(S^3_\theta \rightarrow R^4_\theta\). This amounts to verifying for this example that the Assumptions 3.2, 3.5, 3.9 and 3.11 for our general construction hold true. We shall also provide explicit expressions for these induced structures and in particular for the induced Dirac operator. To simplify our notation, we will suppress in what follows the square brackets denoting equivalence classes. It will be clear from the context and our general construction in Section 3 which of the following expressions are considered in quotient spaces.

The algebra \(B = B_{S^3_\theta}\) of the noncommutative Connes-Landi 3-sphere [CL01, CD-V02] is defined as the quotient

\[B := A/(f)\]

of the algebra \(A = A_{R^4_\theta}\) of \(R^4_\theta\) (cf. (4.1)) by the ideal generated by the sphere relation

\[f := \sum_{i,j=1}^4 g_{ij} z^i z^j - 1 = z^1 \bar{z}^1 + z^2 \bar{z}^2 - 1 .\]
From the commutation relations given by (4.1) and (4.2), one checks that \( f \in Z(A) \subseteq A \) is central. The normalized 1-form (3.2) for this example reads as
\[
\eta := \frac{1}{2} df = \sum_{i,j=1}^{4} g_{ij} z^{i} dz^{j} . \tag{4.24}
\]

**Proposition 4.7.** The normalized 1-form \( \eta \in \Omega^{1}_{A} \), and hence also \( df = 2 \eta \in \Omega^{1}_{A} \), are central, i.e. Assumption 3.2 holds true. The projector \( \Pi : q_{1}(\Omega^{1}_{A}) \rightarrow q_{1}(\Omega^{1}_{A}) \) from Proposition 3.4 reads explicitly as
\[
\Pi(d z^{i}) = d z^{i} - z^{i} \eta . \tag{4.25}
\]

**Proof.** Centrality of \( \eta \), or equivalently of \( df = 2 \eta \), is a simple check using (4.4) and (4.2). The explicit expression for the projector is obtained from a short calculation
\[
\Pi(d z^{i}) = d z^{i} - \sum_{j,k=1}^{4} g_{jk} z^{j} dz^{k} . \tag{4.26}
\]
where in the second step we used (4.4), (4.2) and (4.6) in order to write \( \sum_{j,k=1}^{4} g_{jk} z^{j} dz^{k} = \sum_{j,k=1}^{4} dz^{k} g_{jk} z^{j} \).

**Proposition 4.8.** Assumptions 3.5 and 3.9 hold true. The induced Riemannian structure from Proposition 3.10 reads explicitly as
\[
\sigma_{B}(d z^{i} \otimes_{B} d z^{j}) = R^{ji} dz^{j} \otimes_{B} dz^{i} . \tag{4.27d}
\]

**Proof.** Verifying Assumption 3.5 is a simple check using (4.4), (4.10) and (4.2). To prove commutativity of the top diagram in Assumption 3.9 we use (4.26) and compute
\[
\sigma(\Pi(d z^{i}) \otimes_{A} dz^{j}) = \sigma(d z^{i} \otimes_{A} dz^{j}) - \sigma(z^{i} \eta \otimes_{A} dz^{j})
\]
\[
= R^{ji} dz^{j} \otimes_{A} dz^{i} - R^{ji} dz^{j} \otimes_{A} z^{i} \eta
\]
\[
= R^{ji} dz^{j} \otimes_{A} \Pi(d z^{i})
\]
\[
= (id \otimes_{A} \Pi) \sigma(d z^{i} \otimes_{A} dz^{j}) , \tag{4.28}
\]
where in the second step we used also (3.11) and (4.4). Commutativity of the bottom diagram in Assumption 3.9 is proven by a similar calculation.

Concerning the explicit expressions for the induced Riemannian structure, we observe that (4.27a) follows trivially from (3.10a). Equation (4.27b) follows from (3.10b), (4.25) and a straightforward calculation. Equation (4.27c) follows from (3.18) and (4.9) by a short calculation
\[
\nabla_{B}(dz^{i}) = \nabla(d z^{i}) - g^{-1}(d z_{i} A \eta) \nabla(\eta) = -z^{i} \nabla(\eta) = -z^{i} \sum_{k,l=1}^{4} g_{kl} dz^{k} \otimes_{B} dz^{l} , \tag{4.29}
\]
where in the last step we used that
\[
\nabla \left( \eta \right) = \sum_{k,l=1}^{4} g_{kl} \nabla \left( z^k z^l \right) = \sum_{k,l=1}^{4} g_{kl} \partial K d z^k \otimes_A d z^l
\] (4.30)
via the left Leibniz rule. Finally, (4.27d) follows trivially from (4.17) and (4.10).

**Proposition 4.9.** Assumption 3.11 holds true. The induced spinorial structure from Proposition 3.12 reads explicitly as
\[
\mathcal{E}_B = \mathcal{E} \frac{\theta}{f \tilde{\mathcal{E}}} ,
\] (4.31a)
\[
\gamma_B \left( dz^i \otimes_B e_\alpha \right) = -\left( \sum_{k,l=1}^{4} g_{kl} z^k \gamma^l_\theta \gamma^i_\theta + z^i \right) e_\alpha ,
\] (4.31b)
\[
\nabla^B_{sp} \left( e_\alpha \right) = \frac{1}{2} \sum_{i,j,k,l=1}^{4} g_{ij} g_{kl} z^k d z^i \otimes_B \gamma^j_\theta \gamma^l_\theta e_\alpha .
\] (4.31c)

**Proof.** Recalling (4.30), Assumption 3.11 is verified by a similar calculation as the one that proves centrality of the metric \(g\).

Concerning the explicit expressions for the induced spinorial structure, we observe that (4.31a) is just the definition in (3.23). Equation (4.31b) follows from (3.24) by a short calculation
\[
\gamma_B \left( dz^i \otimes_B e_\alpha \right) = -\gamma_2 \left( \eta \otimes_A \Pi \left( dz^i \right) \otimes e_\alpha \right) + g^{-1} \left( dz^i \otimes_A \eta \right) \gamma_2 \left( \eta \otimes_A \eta \otimes_A e_\alpha \right)
\]
\[
= -\left( \sum_{k,l=1}^{4} g_{kl} z^k \gamma^l_\theta \gamma^i_\theta + z^i \right) e_\alpha ,
\] (4.32)
where in the first step we used (3.27a) and in the third step we used (3.27b). Finally, equation (4.31c) follows from writing out (3.25) and using (4.14) and (4.30).

We now have all the building blocks for computing the induced Dirac operator on \(S^3_\theta\).

**Proposition 4.10.** The induced Dirac operator \((3.32)\) on \(S^3_\theta\) is given by
\[
D_B \left( s \right) = -\frac{3}{2} \sum_{i,j=1}^{4} \left[ \gamma^j_\theta, \gamma^i_\theta \right] \theta \partial_i s z_j - \frac{3}{2} s ,
\] (4.33)
where \(z_i := \sum_{k=1}^{4} \eta_{ik} z^k, \partial_i s := \sum_{\alpha=1}^{4} \partial_i s^\alpha e_\alpha,\) and \(\left[ \gamma^j_\theta, \gamma^i_\theta \right] \theta\) is the \(\theta\)-commutator from Lemma 4.7.

**Proof.** We have to compute the induced Dirac operator from Proposition 3.13 for our example. Using (4.9) and (4.24), the first term of (3.33) is given by
\[
(\Pi \otimes_A \text{id}) \nabla \left( \eta \right) = \sum_{i,j=1}^{4} g_{ij} \Pi \left( dz^i \right) \otimes_A d z^j = \sum_{i,j=1}^{4} \left( g_{ij} - z_j z_i \right) d z^i \otimes_A d z^j .
\] (4.35)
where in the first step we used (4.30) and in the second step (4.25). This element is invariant under applying $\sigma$, i.e. $\sigma(\Pi \otimes A id)\nabla(\eta) = (\Pi \otimes A id)\nabla(\eta)$, hence we can write

$$(\Pi \otimes A id)\nabla(\eta) = \frac{1}{2} \left( (\Pi \otimes A id)\nabla(\eta) + \sigma(\Pi \otimes A id)\nabla(\eta) \right)$$

(4.36)

in the second term of (4.33). Using the Clifford relations (2.10), we obtain

$$
\sum_{2nd} = -\frac{1}{2} \sum_{i,j=1}^{4} (g_{ij} - z_{j} z_{i}) g^{ij} s = -\frac{1}{2} (4 - 1) s = -\frac{3}{2} s ,
$$

(4.37)

where in the second step we used $\sum_{i,j=1}^{4} g_{ij} g^{ij} = \sum_{i=1}^{4} \delta_{i}^{i} = 4$ (cf. (4.7)) and the sphere relation $\sum_{i,j=1}^{4} z_{j} z_{i} g^{ij} = \sum_{i,j=1}^{4} g_{ij} z^{i} z^{j} = 1$ (cf. (4.25)).

Remark 4.11. For vanishing deformation parameter $\theta = 0$, our Dirac operator (4.33) on $S_{3}^{3}$ reduces to the usual Dirac operator on the commutative 3-sphere $S_{3}^{3} \subseteq \mathbb{R}^{4}$, see e.g. [Tra93] Section 7.1.

In order to better understand our Dirac operator (4.33) on the noncommutative sphere $S_{3}^{3}$, we compute its spectrum by adopting the techniques from [Tra93]. For this we first rewrite (4.38) by using the Clifford relations (in the form of Lemma 4.5 (iii)) as

$$
D_{B}(s) = -\gamma(\eta \otimes A D(s)) - \left( \partial_{r} + \frac{3}{2} \right) s ,
$$

(4.38)

where we introduced the radial derivation $\partial_{r} := \sum_{i=1}^{4} \partial_{i} s z^{i}$ and $D(s) = \sum_{i=1}^{4} \gamma_{i}^{i} \partial_{i} s$ is the Dirac operator on the embedding space $\mathbb{R}_{g}^{4}$. To find the eigenvalues of $D_{B}$ and their corresponding eigenvectors, we consider as in [Tra93] spinor-valued harmonic polynomials of degree $l + 1$ on the embedding space $\mathbb{R}_{g}^{4}$. These are elements $\Phi = \sum_{\alpha} \Phi^{\alpha} e_{\alpha} \in \mathcal{E} = A^{4}$ of the spinor module, such that each $\Phi^{\alpha} \in A$ is a homogeneous polynomial of degree $l + 1$ and $D^{2}(\Phi) = 0$. The square of the Dirac operator $D$ on $\mathbb{R}_{g}^{4}$ (cf. (4.21)) is given by

$$
D^{2}(\Phi) = \sum_{i,j=1}^{4} \gamma_{i}^{i} \partial_{i} \partial_{j} \Phi = -\sum_{i,j=1}^{4} g^{ij} \partial_{i} \partial_{j} \Phi .
$$

(4.39)

We define $\Psi := D(\Phi) \in \mathcal{E}$ and note that $\Psi$ is a homogeneous polynomial of degree $l$. Observe that $D(\Psi) = D^{2}(\Phi) = 0$ by construction and that $\partial_{r} \Psi = l \Psi$, because $\partial_{r}$ is a derivation on $A$ that satisfies $\partial_{r} z^{j} = z^{j}$, for all $j$. Regarding $\Psi$ as an element of the quotient module $\mathcal{E}_{B}$ (cf. (4.22)) and applying the Dirac operator (4.38), we compute

$$
D_{B}(\Psi) = - \left( l + \frac{3}{2} \right) \Psi .
$$

(4.40)

Hence, the spectrum of $D_{B}$ contains the eigenvalues $\lambda = -(l + \frac{3}{2})$, for all $l \in \mathbb{Z}_{\geq 0}$. Using also the identity

$$
D_{B}(\gamma(\eta \otimes B s)) = -\gamma(\eta \otimes B D_{B}(s)) ,
$$

(4.41)

which one can prove easily from the definitions in (4.33) and (4.25), we compute

$$
D_{B}(\gamma(\eta \otimes \Psi)) = -\gamma(\eta \otimes B D_{B}(\Psi)) = (l + \frac{3}{2}) \gamma(\eta \otimes \Psi) .
$$

(4.42)

Hence, the spectrum of $D_{B}$ contains also the eigenvalues $\lambda = +(l + \frac{3}{2})$, for all $l \in \mathbb{Z}_{\geq 0}$.

Summing up, we obtain that the spectrum of our Dirac operator (4.33) on the noncommutative sphere $S_{3}^{3}$ coincides with the spectrum of the Dirac operator on the commutative 3-sphere [Tra93]. As a consequence, it also coincides with the spectrum of the Connes-Landi Dirac operator on $S_{3}^{3}$ [CL01][CD-V02], which is obtained from an isospectral deformation quantization [BLvS13].
4.3 Noncommutative hypersurface $\mathbb{T}_\theta^2 \hookrightarrow S_\theta^3$

In this section we apply our construction from Section 3 to induce the differential, Riemannian and spinorial structure on $S_\theta^3$ (cf. Section 4.2) to the noncommutative 2-torus $\mathbb{T}_\theta^2 \hookrightarrow S_\theta^3$. In analogy to Section 4.2 this amounts to verifying for this example that the Assumptions 3.2, 3.5, 3.9 and 3.11 for our general construction hold true. We shall also provide explicit expressions for these induced structures and in particular for the induced Dirac operator. We will again suppress in what follows the square brackets denoting equivalence classes in order to simplify our notations.

Consider the quotient

$$C := B / (\tilde{f})$$

of the algebra $B = B_{S_\theta^3}$ of $S_\theta^3$ (cf. (4.22)) by the ideal generated by

$$\tilde{f} := \sum_{i,j=1}^4 h_{ij} z^i z^j = z^1 z^1 - z^2 z^2$$

where $h_{ij}$ are the entries of the matrix

$$\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}.$$ (4.45)

From the commutation relations given by (4.1) and (4.2), one checks that $\tilde{f} \in Z(B) \subseteq B$ is central. We denote the quotient map

$$\tilde{q} : B \longrightarrow C$$

by a tilde in order to distinguish it from the quotient map $q : A \rightarrow B$ in Section 4.2. To recognize $C = C_{T_\theta^2}$ as the algebra of the noncommutative 2-torus $\mathbb{T}_\theta^2$, let us recall from (4.22) that $B = A/(f)$, hence $C = A/(f, \tilde{f})$ is the quotient of the algebra $A = A_{R^4}$ of $R^4$ by the ideal generated by the two relations $f$ and $\tilde{f}$ in (4.23) and (4.44). The usual torus relations for the rescaled coordinates $u := \sqrt{2} z^1$ and $v := \sqrt{2} z^2$ are then obtained from the linear combinations

$$f + \tilde{f} = 2 z^1 z^1 - 1 = u \overline{u} - 1, \quad f - \tilde{f} = 2 z^2 z^2 - 1 = v \overline{v} - 1.$$ (4.47)

The normalized 1-form (3.2) for this example reads as

$$\tilde{\eta} := \frac{1}{2} \text{d}\tilde{f} = \sum_{i,j=1}^4 h_{ij} z^i \text{d}z^j.$$ (4.48)

**Proposition 4.12.** The normalized 1-form $\tilde{\eta} \in \Omega^1_B$, and hence also $\text{d}\tilde{f} = 2 \tilde{\eta} \in \Omega^1_B$, are central, i.e. Assumption 3.2 holds true. The projector $\tilde{\Pi} : \tilde{q}_!(\Omega^1_B) \rightarrow \tilde{q}_!(\Omega^1_B)$ from Proposition 3.4 reads explicitly as

$$\tilde{\Pi}(\text{d}z^i) = \text{d}z^i + (-1)^i z^i \tilde{\eta}.$$ (4.49)

**Proof.** Let us confirm first that (4.48) is indeed normalized. Using (4.27b), we compute

$$g_B^{-1}(\tilde{\eta} \otimes_B \tilde{\eta}) = \sum_{i,j,k,l=1}^4 h_{ij} z^i (g^{jk} - z^j z^l) h_{kl} z^k = \sum_{i,k=1}^4 g_{ik} z^i z^k = 1.$$ (4.50)
where in the second step we used (4.44) and the identity
\[
\sum_{j,l=1}^{4} h_{ij} g^{jl} h_{kl} = g_{ik} ,
\] (4.51)
and in the last step we used (4.23). Centrality of \( \tilde{\eta} \), or equivalently of \( d\tilde{f} = 2 \tilde{\eta} \), is a simple check using (4.4) and (4.2). The explicit expression for the projector is obtained from a short calculation
\[
\tilde{\Pi}(dz^i) = dz^i - g_B^{-1}(dz^i \otimes_B \tilde{\eta}) \tilde{\eta} = dz^i - \sum_{k,l=1}^{4} (g^{il} - z^i z^j) h_{kl} z^k \tilde{\eta}
\]
\[
= dz^i - \sum_{k,l=1}^{4} g^{il} h_{kl} z^k \tilde{\eta} = dz^i + (-1)^i z^i \tilde{\eta} ,
\] (4.52)
where in the second step we used (4.27b), in the third step we used (4.44) and the last step follows from \( \sum_{l=1}^{4} g^{il} h_{kl} = -(-1)^i \delta^i_k \).

Proposition 4.13. Assumptions 3.5 and 3.9 hold true. The induced Riemannian structure from Proposition 3.10 reads explicitly as
\[
g_C = \sum_{i,j=1}^{4} g_{ij} dz^i \otimes_C dz^j \in \Omega^1_C \otimes_C \Omega^1_C ,
\] (4.53a)
\[
g_C^{-1}(dz^i \otimes_C dz^j) = g^{ij} - (1 + (-1)^i (-1)^j) z^i z^j ,
\] (4.53b)
\[
\nabla_C(dz^i) = -z^i \sum_{k,l=1}^{4} (g_{kl} - (-1)^i h_{kl}) dz^k \otimes_C dz^l ,
\] (4.53c)
\[
\sigma_C(dz^i \otimes_C dz^j) = R^{ij} dz^j \otimes_C dz^i .
\] (4.53d)

Proof. Verifying Assumption 3.5 is a simple check using (4.4), (4.10) and (4.2). To prove commutativity of the top diagram in Assumption 3.9 we use (4.49) and compute
\[
\sigma_B(\tilde{\Pi}(dz^i) \otimes_B dz^j) = R^{ij} dz^j \otimes_B dz^i + (-1)^i z^i dz^j \otimes_B \tilde{\eta}
\]
\[
= R^{ij} dz^j \otimes_B dz^i + R^{ij} dz^j \otimes_B (-1)^i z^i \tilde{\eta}
\]
\[
= (id \otimes_B \tilde{\Pi}) \sigma_B(dz^i \otimes_B dz^j) ,
\] (4.54)
where in the second step we used (4.4). Commutativity of the bottom diagram in Assumption 3.9 is proven by a similar calculation.

We observe that (4.53a) follows trivially from (3.10a) and (4.53b) follows from (3.10b), (4.49) and a straightforward calculation. Equation (4.53c) follows from (3.18), (4.27c) and (4.55) by a short calculation. Finally, (4.53d) follows trivially from (3.17) and (4.27d). \( \square \)
Proposition 4.14. Assumption 3.11 holds true. The induced spinorial structure from Proposition 3.12 reads explicitly as

$$\mathcal{E}_C = \frac{\mathcal{E}_B}{f \mathcal{E}_B} = \frac{\mathcal{E}}{f \mathcal{E} \cup f \mathcal{E}} ,$$

(4.56a)

$$\gamma_C(dz^i \otimes C e_\alpha) = \left( z^i \sum_{k,l,m,n=1}^4 g_{mn} z^m h_{kl} z^k \gamma_\theta^{ij} \gamma_\theta^{kl} - \sum_{k,l=1}^4 h_{kl} z^k \gamma_\theta^{ij} + (-1)^i z^i \right) e_\alpha , \quad (4.56b)$$

$$\nabla^{sp}_C(e_\alpha) = \frac{1}{2} \sum_{i,j,k,l=1}^4 \left( g_{kl} z^k g_{ij} dz^i + h_{kl} z^k h_{ij} dz^i \right) \otimes C \gamma_\theta^{ij} \theta^{kl} e_\alpha . \quad (4.56c)$$

Proof. Recalling (4.55), Assumption 3.11 is verified by a similar calculation as the one that proves centrality of $f$ given in (4.41). The explicit expressions in (4.56a), (4.56b) and (4.56c) follow easily from the definitions (cf. (3.23), (3.24) and (3.25)) by straightforward calculations. (To obtain (4.56c), one has to recall that $\tilde{\eta} = 2 df = 0$ in $\Omega_C$.)

Proposition 4.15. The induced Dirac operator (3.32) on $\mathbb{T}_\theta^2$ is given by

$$D_C(s) = -\frac{1}{2} \sum_{i,j=1}^4 [\gamma_\theta^{ij}, \gamma_\theta^{ij}] \left( \partial_i s \, z_j - \frac{4}{d} \partial_k s \, z_k \right) \, (4.57)$$

where $z_i := \sum_{k=1}^4 g_{ik} z^k$, $\tilde{z}_i := \sum_{k=1}^4 h_{ik} z^k$, $\partial_i s := \sum_{\alpha=1}^4 \partial_i s^\alpha e_\alpha$ and $[\gamma_\theta^{ij}, \gamma_\theta^{ij}]$ is the $\theta$-commutator from Lemma 4.5.

Proof. The proof is a straightforward but slightly lengthy calculation and hence will not be written out in detail.

From our presentation given in (4.57), it is not easy to interpret and understand $D_C$ as a Dirac operator on the flat noncommutative torus $\mathbb{T}_\theta^2$. We will now simplify (4.57) to a form that admits an obvious interpretation. For this it will be useful to introduce the standard generators

$$u := \sqrt{2} z^i , \quad v := \sqrt{2} \bar{z}^i , \quad \Pi := \sqrt{2} \bar{z}^1 , \quad \bar{\Pi} := \sqrt{2} z^2 \quad (4.58a)$$

of the algebra $C$ of $\mathbb{T}_\theta^2$, which satisfy the relations

$$\Pi u = 1 , \quad \bar{\Pi} v = 1 , \quad u v = e^{i \phi} \, v u . \quad (4.58b)$$

The module $\Omega_C$ of 1-forms on $C$ is a 2-dimensional free module with central basis

$$d\phi^1 := \frac{1}{i} \bar{\Pi} du , \quad d\phi^2 := \frac{1}{i} \Pi dv , \quad (4.59)$$

where $i \in \mathbb{C}$ denotes the imaginary unit. (Our notation is inspired by thinking of $u = e^{i \phi^1}$ and $v = e^{i \phi^2}$ as exponential functions.) The inverse metric (4.53b) in this basis reads as

$$g_C^{-1}(d\phi^i \otimes d\phi^j) = 2 \delta^{ij} \quad (4.60)$$

where the factor 2 is due to the fact that our embedded noncommutative torus $\mathbb{T}_\theta^2 \to S^3$ has radius $\frac{1}{\sqrt{2}}$, see (4.47). The differential $da = \partial_\phi^i a \, d\phi^i + \partial_\phi^2 a \, d\phi^2$ of any $a \in C$ can be expressed in the basis (4.59). Comparing this to $da = \sum_{i=1}^4 \partial_i a \, dz^i \in \Omega_C$, we find

$$\partial_1 a = \frac{2}{i} \partial_\phi^1 a \, \bar{\Pi} , \quad \partial_2 a = \frac{2}{i} \partial_\phi^2 a \, \bar{\Pi} , \quad \partial_3 a = 0 , \quad \partial_4 a = 0 \quad (4.61)$$
for the noncommutative partial derivatives along $z^i$.

To simplify the induced Dirac operator \((4.61)\) on $\mathbb{T}^2_\theta$, we use the Clifford relations in the form of Lemma 4.5 (iii) and obtain after a short calculation

$$D_C(s) = -\gamma \left( \tilde{\eta} \otimes C \sum_{i=1}^4 \left( \gamma_\theta^i \partial_i s - \gamma_\theta^i s z_i \right) \right) + \gamma_2 \left( \tilde{\eta} \otimes C \tilde{\eta} \otimes C \sum_{i=1}^4 \partial_i s z^i \right) + \sum_{i=1}^4 (-1)^i \partial_i s z^i \ .$$

(4.62)

Applying the map $\gamma(\tilde{\eta} \otimes C (-)) : \mathcal{E}_C \to \mathcal{E}_C$ to this expression, which squares to $-\text{id}$ because $\tilde{\eta}$ is normalized, we define

$$\tilde{D}_C(s) := \gamma(\tilde{\eta} \otimes C D_C(s))$$

$$= \sum_{i=1}^4 \left( \gamma_\theta^i \partial_i s - \gamma_\theta^i s z_i \right) - \gamma \left( \eta \otimes C \sum_{i=1}^4 \partial_i s z^i \right) + \gamma(\tilde{\eta} \otimes C \sum_{i=1}^4 (-1)^i \partial_i s z^i) \ .$$

(4.63)

Inserting (4.61), (4.24) and (4.48) into this expression and carrying out all summations, one obtains

$$\tilde{D}_C(s) = \frac{1}{i} \left( \gamma_\theta^1 \partial_1 s \sqrt{\tau} - \gamma_\theta^3 \partial_3 s z^1 \right) - \frac{1}{2} \left( \gamma_\theta^2 s z^2 + \gamma_\theta^3 s z^1 \right)$$

$$+ \frac{1}{i} \left( \gamma_\theta^3 \partial_2 s \sqrt{\tau} - \gamma_\theta^4 \partial_4 s z^2 \right) - \frac{1}{2} \left( \gamma_\theta^2 s z^2 + \gamma_\theta^4 s z^3 \right) \ .$$

(4.64)

Let us introduce the $C$-module map $\tilde{\gamma} : \Omega^1_{\mathcal{C}} \otimes C \mathcal{E}_C \to \mathcal{E}_C$ by defining

$$\tilde{\gamma}(d\phi^i \otimes C s) := \frac{1}{i} \left( \gamma_\theta^i \partial_1 s \sqrt{\tau} - \gamma_\theta^3 \partial_3 s z^1 \right) \ , \ \tilde{\gamma}(d\phi^2 \otimes C s) := \frac{1}{i} \left( \gamma_\theta^2 s z^2 + \gamma_\theta^3 s z^1 \right) \ ,$$

(4.65)

for all $s \in \mathcal{E}_C$. One easily shows that $\tilde{\gamma}$ satisfies the Clifford relations

$$\tilde{\gamma}_2(d\phi^i \otimes C d\phi^j \otimes C s) + \tilde{\gamma}_2(d\phi^i \otimes C d\phi^j \otimes C s) = -2 g_{C}^{-1}(d\phi^i \otimes d\phi^j) s = -4 \delta^{ij} s$$

(4.66)

for the inverse metric (4.60). (Note that there is no $\sigma$ in this expression because $\sigma(d\phi^i \otimes C d\phi^j) = d\phi^j \otimes C d\phi^i$.) This allows us to write (4.64) as

$$\tilde{D}_C(s) = \tilde{\gamma}\left( d\phi^1 \otimes C \left( \partial_1 s + \frac{1}{4} \tilde{\gamma}(d\phi^1 \otimes C (\gamma_\theta^1 \partial_1 s \sqrt{\tau} + \gamma_\theta^3 s z^1)) \right) \right)$$

$$+ \tilde{\gamma}\left( d\phi^2 \otimes C \left( \partial_2 s + \frac{1}{4} \tilde{\gamma}(d\phi^2 \otimes C (\gamma_\theta^2 s z^2 + \gamma_\theta^4 s z^3)) \right) \right)$$

$$= \tilde{\gamma}\left( d\phi^1 \otimes C \left( \partial_1 s + \frac{1}{8i} [\gamma_\theta^1 \partial_1 s, \gamma_\theta^3 s z^1] \right) \right) + \tilde{\gamma}\left( d\phi^2 \otimes C \left( \partial_2 s + \frac{1}{8i} [\gamma_\theta^2 s z^2 + \gamma_\theta^4 s z^3]) \right) \right) \ ,$$

(4.67)

which we recognize as the Dirac operator on $\mathbb{T}^2_\theta$ corresponding to a rotating frame spin structure, see [BG19]. By a direct calculation, one shows that the spectrum of this operator, and hence the spectrum of the Dirac operator $D_C$ in (4.57) on the noncommutative torus $\mathbb{T}^2_\theta$, is given by

$$\left\{ \pm \sqrt{2} \left( (m + \frac{1}{2})^2 + (n + \frac{1}{2})^2 \right) : m, n \in \mathbb{Z} \right\} \ .$$

(4.68)

We note that this coincides with the spectrum of the Dirac operator corresponding to the $(1, 1)$ spin structure on the commutative 2-torus $\mathbb{T}^2$, see e.g. [PT18]. (The factor $\sqrt{2}$ in (4.68) is because our noncommutative torus $\mathbb{T}^2_\theta \to S^3_\theta$ has radius $\frac{1}{\sqrt{2}}$.)

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