ULRICH BUNDLES ON DEL PEZZO THREEFOLDS

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ABSTRACT. We prove that for any $r \geq 2$ the moduli space of stable Ulrich bundles of rank $r$ and determinant $O_X(r)$ on any smooth Fano threefold $X$ of index two is smooth of dimension $r^2 + 1$ and that the same holds true for even $r$ when the index is four, in which case no odd-rank Ulrich bundles exist. In particular this shows that any such threefold is Ulrich wild. As a preliminary result, we give necessary and sufficient conditions for the existence of Ulrich bundles on any smooth projective threefold in terms of the existence of a curve in the threefold enjoying special properties.

1. INTRODUCTION

A vector bundle $E$ on a smooth (complex) projective variety $X \subset \mathbb{P}^m$ is said to be an Ulrich bundle if $h^i(E(-p)) = 0$ for all $i \geq 0$ and all $1 \leq p \leq \dim(X)$. The study of such bundles originates in the paper [Ul] from 1984, where they were considered in the framework of commutative algebra because they enjoy suitable extremal cohomological properties. The attention of algebraic geometers was drawn by the paper [ESW], where, among other things, the Chow form of a projective variety $X$ is computed using Ulrich bundles on $X$, if they exist.

In recent years there has been a lot of work on Ulrich bundles, mainly investigating the following problems: given a variety $X \subset \mathbb{P}^m$, does there always exist an Ulrich bundle on $X$? What are the possible ranks for Ulrich bundles? If Ulrich bundles exist, are they stable, and what are their moduli? (Recall that Ulrich bundles are always semistable and the notions of stability and slope-stability coincide, cf. [CH, Thm. 2.9]).

Although a lot is known about these problems for specific classes of varieties (e.g., curves, Segre and Grassmann varieties, rational scrolls, complete intersections, some classes of surfaces and threefolds, etc.) the above questions are still open in their full generality (for nice surveys on the subject, see for instance [Be, CH, Co, CMP]).

In this paper we consider smooth Fano threefolds of even index, also called Del Pezzo threefolds (cf. the beginning of $\S$4 for the definition and classification of such varieties).

In the index–two case our main result is:

**Theorem 1.1.** Let $X \subset \mathbb{P}^m$ be a smooth Fano threefold of index two with $\omega_X \cong O_X(-2)$. For every integer $r \geq 2$, the moduli space of stable Ulrich bundles of rank $r$ and determinant $O_X(r)$ on $X$ is smooth of dimension $r^2 + 1$.

In the remaining case of even index we prove:

**Theorem 1.2.** Let $X \subset \mathbb{P}^9$ be the 2–Veronese embedding of $\mathbb{P}^3$. Then:

(i) $X$ carries no Ulrich bundles of odd rank;

(ii) for every even integer $r \geq 2$, the moduli space of stable Ulrich bundles of rank $r$ and determinant $O_X(r)$ on $X$ is smooth of dimension $r^2 + 1$. 

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Our results in particular show that any such threefold $X$ is *Ulrich wild* (recall that, as suggested by an analogous definition in [DG], a variety $X$ is said to be Ulrich wild if it possesses families of dimension $p$ of pairwise non–isomorphic, indecomposable, Ulrich bundles for arbitrarily large $p$). Note that there are only very few cases of varieties known to carry stable Ulrich bundles of infinitely many ranks, and even fewer of *any* rank, namely curves, Del Pezzo surfaces and more recently general blow-ups of the plane (cf. [CFK2]).

Some special cases of the theorems were known already: the rank–two cases were proved by Beauville in [Be1 Prop. 6.1 and 6.4] and, in the case of the 2–Veronese, already in [ESW Prop. 5.10–5.11], without the statement about stability. The case of arbitrary rank on the cubic threefold in $\mathbb{P}^4$ was proved in [CH] Prop. 5.4 and Thm. 5.7 under the additional hypothesis that the threefold is *general*, and for *any* cubic threefold in [LMS] Thm. B; the case of intersections of two quadrics ($d = 4$) was proved in [CKL Thm. 1.1], and the case $d = 5$ was proved in [LP] Thm. 1.1. Our proof of Theorem 1.1 is different from the ones in these papers, and provides a uniform treatment of all Del Pezzo threefolds, whereas our proof of Theorem 1.2(ii) is similar to the one in [CH]. Theorem 1.1 proves the conjecture stated in [LP, p. 276].

Also note that the cases of rank–one Ulrich bundles on Del Pezzo threefolds are well–known (cf., e.g., [Be1, CFaM1, CFaM2, CFiM] and see the proof of Proposition 4.4). For other works regarding Ulrich bundles, or more generally *ACM* bundles, of rank two on Fano threefolds, we refer to [ACo, Be2, Be4, Be3, BF, CFaM1, CFaM2, CFiM, MT].

We do not claim the irreducibility of the moduli spaces in Theorems 1.1 and 1.2. Something is known in the rank–two case, cf. Remark 4.5; and irreducibility for all ranks in the cases $d = 4$ and 5 follows from the explicit description of the moduli spaces given in [CKL Thm. 1.1] and [LP Thm. 1.1], whereas it was very recently proved in the case of the cubic threefold in [FP Thm. 1.4]. Irreducibility in the remaining cases is an interesting open question.

Theorems 1.1 and 1.2(ii) will be proved in §4. Before that, in Theorem 3.1 we give necessary and sufficient conditions for the existence of a rank $r \geq 2$ Ulrich bundle on any smooth projective threefold $X \subset \mathbb{P}^m$ in terms of the existence of a quite peculiar curve $C$ on $X$, and the peculiarity of this curve may explain why Ulrich bundles often seem quite hard to find. Despite the complexity of the conditions that the curve has to verify, in certain cases they simplify a bit, as in the case of Fano threefolds: see Theorem 3.3 and Corollaries 3.7 and 3.9 (the latter proving Theorem 1.2(i)). We finish the paper with some remarks and speculations concerning interesting maps to moduli spaces of curves arising from this correspondence between Ulrich bundles and curves (cf. §5).

Throughout the paper we work over the field of complex numbers.

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2. A preliminary lemma

Lemma 2.1. Let $X \subset \mathbb{P}^n$ be a smooth, irreducible, projective variety with $\deg(X) > 1$ and let $\mathcal{E}$ be an Ulrich bundle on $X$. Then $h^0(\mathcal{E}^*) = 0$.

Proof. Assume by contradiction that $h^0(\mathcal{E}^*) > 0$ and pick a non-zero section $s^* \in H^0(\mathcal{E}^*)$. Since $h^0(\mathcal{E}) = \deg(X) \cdot \rk(\mathcal{E}) \geq \deg(X) > 1$ by [Be1, (3.1)], we have $\mathcal{E} \not\cong \mathcal{O}_X$. It follows that $\coker(s^*) \neq 0$, and by dualizing we obtain

$$0 \longrightarrow \mathcal{F} := \ker(s) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X,$$

where $\mathcal{F} \neq 0$. In particular, $\rk(\mathcal{F}) = \rk(\mathcal{E}) - 1 > 0$ and $\im(s)$ is an ideal sheaf, whence

$$c_1(\mathcal{F}) = c_1(\mathcal{E}) - c_1(\im(s)) = c_1(\mathcal{E}) + D,$$

for an effective divisor $D$ on $X$. Thus, denoting as usual by $\mu$ the slope of a sheaf, we have

$$\mu(\mathcal{F}) = \frac{c_1(\mathcal{F}) \cdot \mathcal{O}_X(1)^{\dim(X)-1}}{\rk(\mathcal{F})} > \frac{c_1(\mathcal{E}) \cdot \mathcal{O}_X(1)^{\dim(X)-1}}{\rk(\mathcal{E})} = \mu(\mathcal{E}),$$

contradicting that $\mathcal{E}$, being Ulrich, is slope-semistable ([CH Thm. 2.9(a)]).

Remark 2.2. The assumption $\deg(X) > 1$ in Lemma 2.1 is essential, as on $\mathbb{P}^n$ trivial bundles are Ulrich, and in fact the only Ulrich bundles (cf., e.g., [Be1 Thm. 2.3]).

3. Curves associated to Ulrich bundles on threefolds

In this section we will prove that the existence of Ulrich bundles on a smooth projective threefold is connected to the existence of smooth curves on the threefold with particular properties. Before giving the result, we fix some notation. Let $X$ be a smooth projective threefold, $C \subset X$ a curve and $D$ a divisor on $X$. The short exact sequence

$$0 \longrightarrow \mathcal{J}_{C/X} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0$$

tenced by $\mathcal{O}_X(D + K_X)$ determines a coboundary map

$$d : H^1(C, \mathcal{O}_C(D + K_X)) \longrightarrow H^2(X, \mathcal{J}_{C/X}(D + K_X)),$$

whose dual, by Serre duality, is

$$d^* : \Ext^1_X(\mathcal{J}_{C/X}(D), \mathcal{O}_X) \longrightarrow H^0(C, \omega_C(-D - K_X)).$$

Moreover, for any subspace

$$W \subseteq \Ext^1_X(\mathcal{J}_{C/X}(D), \mathcal{O}_X) \simeq (H^2(X, \mathcal{J}_{C/X}(D + K_X))^* \cdot$$

we obtain a surjection

$$H^2(X, \mathcal{J}_{C/X}(D + K_X)) \longrightarrow W^*.$$

Since, by Serre duality,

$$\Ext^1_X(\mathcal{J}_{C/X}(D), W^* \otimes \mathcal{O}_X) \simeq \Ext^1_X(\mathcal{J}_{C/X}(D + K_X), \omega_X) \otimes W^* \simeq$$

$$\simeq H^2(\mathcal{J}_{C/X}(D + K_X))^* \otimes W^* \simeq \Hom_X(H^2(\mathcal{J}_{C/X}(D + K_X), W^*),$$

we obtain an extension

$$0 \longrightarrow W^* \otimes \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{J}_{C/X}(D) \longrightarrow 0.$$
corresponding to the cocycle (5), that is, so that the coboundary map

$$H^2(J_{C/X}(D + K_X)) \longrightarrow H^3(W^* \otimes \omega_X) \simeq W^* \otimes H^3(\omega_X) \simeq W^*$$

of (4) tensored by $\omega_X$ is the map (3). For any $L \in \text{Pic}(X)$, we may twist (4) by $L$ and obtain a coboundary map

$$(5) \quad H^2(X, L(D) \otimes J_{C/X}) \longrightarrow W^* \otimes H^3(X, L),$$

which we will call the coboundary map induced by $(W, L)$.

**Theorem 3.1.** Let $X \subset \mathbb{P}^m$ be a smooth projective threefold with $\text{deg}(X) > 1$ and $D$ be a divisor on $X$. Let $r \geq 2$ be an integer. Set $H := O_X(1)$.

Then there is a bijection between the set of Ulrich bundles $E$ on $X$ satisfying

$$(6) \quad \text{rk}(E) = r \text{ and } \text{det}(E) = O_X(D)$$

and the set of pairs $(C, W)$, where $C \subset X$ is a smooth curve and $W \subseteq \text{Ext}^1_X(J_{C/X}(D), O_X)$ is an $(r - 1)$–dimensional subspace, such that

$$(7) \quad d^*(W) \text{ generates } \omega_C(-K_X - D) \quad \text{(cf. (2))},$$

$$H^2 \cdot D = \frac{1}{2}(H^2 \cdot K_X + 4H^3),$$

$$H \cdot C = \frac{1}{12}(K_X^2 \cdot H + c_2(X) \cdot H - 22H^3) - \frac{1}{2}(H \cdot D \cdot K_X - H \cdot D^2),$$

$$g(C) = rH^3 - r\chi(X, O_X) - \frac{1}{6}D^3 + \frac{1}{3}K_X \cdot D^2 - \frac{1}{12}(K_X^2 \cdot D + c_2(X) \cdot D) + C \cdot D + 1,$$

$$h^0(K_X + 3H - D) = 0,$$

$$h^0(J_{C/X}(D - H)) = 0,$$

$$h^1(J_{C/X}(D - pH)) = 0 \text{ for } p \in \{1, 2, 3\},$$

$$\text{the coboundary map } \delta : H^2(J_{C/X}(D - 3H)) \longrightarrow W^* \otimes H^3(O_X(-3H))$$

induced by $(W, O_X(-3H))$ is either injective or surjective.

Via the above bijection, $E$ sits in an exact sequence

$$(15) \quad 0 \longrightarrow W^* \otimes O_X \longrightarrow E \longrightarrow J_{C/X}(D) \longrightarrow 0.$$

**Remark 3.2.** As $C$ is the $(r - 2)$–degeneracy locus of $u_{r-1}$, it follows that its class of cohomology is $c_2(E)$, see for instance [Ot] (2.11)].

We point out that the curve $C$ could be the 0–curve, i.e., $C$ could be empty. In this case $H \cdot C = 0$, $g(C) = 1$, $J_{C/X} = O_X$ and (7) is vacuous. This case never happens if $D$ is big and nef, in particular if $\text{Pic}(X) \simeq \mathbb{Z}$, since in this case $\text{Ext}^1_X(O_X(D), O_X) = 0$.

For examples where $C$ is the 0–curve, see [LM].

**Proof.** Let $E$ be an Ulrich bundle of rank $r$ on $X$ with $\text{det}(E) = O_X(D)$. Then $E$ is globally generated by, e.g., [Bet] Thm. 2.3(i)]. We can pick a general $(r - 1)$–dimensional subspace $V_{r-1} \subset H^0(E)$ and let $V_{r-1} \otimes O_X \xrightarrow{u_{r-1}} E$ be the evaluation morphism. It is well–known, see, e.g., [Ot] Thm. 2.8 or [Ban] Thm. 1], that $\text{coker}(u_{r-1}) \simeq J_{C/X}(D)$ where $C$, the degeneracy locus of $u_{r-1}$, is a smooth curve, since $\text{dim}(X) = 3$. Setting $W := V_{r-1}^*$, we thus have an exact sequence like (15). We have $h^0(E(K_X)) = h^0(E^*) = 0$ by
Serre duality and Lemma\textsuperscript{2,11} Therefore ([15], tensored by $\mathcal{O}_X(K_X)$, yields a surjection $H^2(J_{C/X}(D + K_X)) \twoheadrightarrow W^*$, whence, by duality

$$W \subset (H^2(J_{C/X}(D + K_X)))^* \simeq \text{Ext}_X^1(J_{C/X}(D), \mathcal{O}_X).$$

The required correspondence associates to $\mathcal{E}$ the pair $(C, W)$.

Note that, dualizing ([15]), we find

\begin{equation}
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_X(-D) & \longrightarrow & \mathcal{E}^* & \longrightarrow & W \otimes \mathcal{O}_X & \mathcal{O}_X(-K_X - D) & \longrightarrow & 0,
\end{array}
\end{equation}

hence we have the surjection $W \otimes \mathcal{O}_X \twoheadrightarrow \omega_C(-K_X - D)$ that proves ([17]).

Conversely, assume we have a smooth curve $C \subset X$ and an $(r - 1)$–dimensional subspace $W \subseteq \text{Ext}_X^1(J_{C/X}(D), \mathcal{O}_X)$ such that ([17]) is satisfied. As explained in the beginning of the section, this defines a sheaf $\mathcal{E}$ on $X$ fitting in an exact sequence like ([15]). Tensoring ([1]) by $\mathcal{O}_X(D)$ gives

$$\text{Ext}_X^i(J_{C/X}(D), \mathcal{O}_X) = 0 \quad \text{for} \; i \geq 2$$

and

$$\text{Ext}_X^2(J_{C/X}(D), \mathcal{O}_X) \simeq \text{Ext}_X^2(\mathcal{O}_C(D), \mathcal{O}_X) \simeq \omega_C(-K_X - D).$$

From ([15]) we therefore find

$$\text{Ext}_X^i(\mathcal{E}, \mathcal{O}_X) = 0, \quad \text{for} \; i \geq 2,$$

and

$$\text{Ext}_X^2(\mathcal{E}, \mathcal{O}_X) = \text{coker} \left[ W \otimes \mathcal{O}_X \twoheadrightarrow \omega_C(-K_X - D) \right] = 0,$$

by ([17]). Therefore, $\mathcal{E}$ is locally free on $X$.

Thus, to finish the proof of the theorem, we have left to show that if $\mathcal{E}$ is a vector bundle verifying ([1]) and fitting into an exact sequence like ([15]), with $C \subset X$ a smooth curve, $D$ a divisor and $W$ an $(r - 1)$–dimensional subspace of $\text{Ext}_X^1(J_{C/X}(D), \mathcal{O}_X)$ satisfying ([17]), then $\mathcal{E}$ is Ulrich if and only if ([8])–([14]) hold.

Tensoring ([15]) by $\mathcal{O}_X(-pH)$, one sees that $\mathcal{E}$ is Ulrich if and only if

\begin{equation}
h^i(J_{C/X}(D - pH)) = 0, \quad \text{for} \; i \in \{0, 1, 3\}, \; p \in \{1, 2, 3\}
\end{equation}

and

\begin{equation}
\text{the coboundary maps} \; \delta_p : H^2(J_{C/X}(D - pH)) \longrightarrow W^* \otimes H^3(\mathcal{O}_X(-pH)) \quad \text{are isomorphisms for} \; p \in \{1, 2, 3\}.
\end{equation}

Condition ([17]) with $i = 3$ is equivalent to $h^3(\mathcal{O}_X(D - pH)) = 0$ for $p \in \{1, 2, 3\}$, which is equivalent to condition ([11]) by Serre duality. Condition ([17]) with $i = 0$ is equivalent to condition ([12]), whereas ([17]) with $i = 1$ is condition ([13]).

Consider now ([18]). Conditions ([11])–([13]), which as we just saw are equivalent to ([17]), yield that the domain of $\delta_p$ has dimension $\chi(J_{C/X}(D - pH))$. On the other hand, the target of $\delta_p$ has dimension $(r - 1)h^3(\mathcal{O}_X(-pH)) = (1 - r)\chi(\mathcal{O}_X(-pH))$ by Kodaira vanishing. Thus, ([18]) is equivalent to the conditions

\begin{equation}
\chi(J_{C/X}(D - pH)) = (1 - r)\chi(\mathcal{O}_X(-pH)) \quad \text{for} \; p \in \{1, 2, 3\}, \quad \text{and}
\end{equation}

\begin{equation}
\delta_p \text{ is either injective or surjective for} \; p \in \{1, 2, 3\}.
\end{equation}
Suppose for a moment that condition \((19)\) is verified. For each \(1 \leq p \leq 3\) we have a commutative diagram

\[
\begin{array}{ccc}
H^2(J_{C/X}(D - 3H)) \xrightarrow{\delta_3} V_r \otimes H^3(O_X(-3H)) & \xrightarrow{\delta_3} & V_r \otimes H^3(O_X(-pH))
\end{array}
\]

which shows that \(\delta_1, \delta_2, \delta_3\) are surjective (or, equivalently, injective) if and only if \(\delta_3\) is surjective (or, equivalently, injective). Thus, condition \((20)\) is equivalent to \((14)\).

Finally, consider condition \((19)\). It can be rewritten as

\[
(21) \quad 0 = -\chi(O_X(D - pH)) + \chi(O_C(D - pH)) + (1 - r)\chi(O_X(-pH)), \quad \text{for } p \in \{1, 2, 3\}.
\]

Using Riemann–Roch on \(X\) and \(C\), the right hand side can be rewritten as

\[
\frac{rH^3}{6} p^3 + \left(\frac{rK_X}{4} H^2 - \frac{D H^2}{2}\right) p^2 + \left(\frac{1}{12}(K_X^2 + c_2(X)) \cdot H - \frac{H \cdot K_X}{2} + \frac{H D^2}{2} - C \cdot H\right) p + \frac{1}{2} K_X \cdot D^2 + C \cdot D + 1 - g(C) - r\chi(O_X) - \frac{1}{6} D^3 - \frac{K_X \cdot D^2}{4} - \frac{1}{12}(K_X^2 + c_2(X)) \cdot D.
\]

Dividing by the leading coefficient \(\frac{rH^3}{6}\) gives a monic, cubic polynomial in \(p\), which, by condition \((19)\), must coincide with \((p - 1)(p - 2)(p - 3) = p^3 - 6p^2 + 11p - 6\).

Equating the coefficients of the terms of degrees 2, 1 and 0 gives, respectively, \((8)\), \((9)\), \((10)\) and \((11)\).

Despite the complexity of conditions \((7)\)–\((14)\) in Theorem 3.1, in certain cases some of them may be replaced by slightly easier ones, as shown in the following:

**Lemma 3.3.** In the same setting as in Theorem 3.1 we have:

(i) If \(h^i(O_X(-D)) = 0\) for \(i \in \{1, 2\}\), then \(W\) can be identified with a subspace of \(H^0(\omega_C(-K_X - D))\), and condition \((7)\) says that \(W\) generates \(\omega_C(-K_X - D)\).

(ii) If \(h^i(O_X(D - 3H)) = 0\) for \(i \in \{1, 2\}\), then condition \((14)\) is equivalent to

\[
\text{(22) the multiplication map } \nu : W \otimes H^0(O_X(K_X + 3H)) \longrightarrow H^0(\omega_C(3H - D))
\]

is either injective or surjective.

**Proof.** (i) The assumptions yield that \(d^r\) is an isomorphism, proving the assertion.

(ii) The assumptions yield that the coboundary map of \((11)\) tensored by \(O_X(D - 3H)\),

\[
\gamma : H^1(O_C(D - 3H)) \longrightarrow H^2(J_{C/X}(D - 3H)),
\]

is an isomorphism. The composed map \(\delta \circ \gamma : H^1(O_C(D - 3H)) \to V_{r-1} \otimes H^3(O_X(-3H))\) is the dual of the multiplication map \(\nu\), by Serre duality. Hence we see that condition \((14)\) is equivalent to \((22)\). \(\square\)

**Remark 3.4.** Let \(E\) be an Ulrich bundle of rank \(r\) sitting in an exact sequence like \((15)\), and set \(V_{r-1} := W^*\). Since \(E\) is generated by global sections ([Be1 Thm. 2.3(i)]) and is nontrivial (Lemma 2.1), the divisor \(D\) in Theorem 3.1 must be effective and nonzero (by Porteous). Pick a general \(r\)-dimensional subspace \(V_r \subseteq H^0(E)\) containing \(V_{r-1}\) and consider the evaluation morphism \(V_r \otimes O_X \xrightarrow{\nu_r} E\). It is well–known (see again,
e.g., [Or Thm. 2.8] or [Ban Thm. 1]) that $\mathcal{L} := \coker(u_r)$ is supported on a member $S$ of $|\det(\mathcal{E})| = |D|$ (the degeneracy locus of $u_r$), which is smooth as $\dim(X) = 3$, and that $\mathcal{L}$ is locally free of rank one on $S$. More precisely, by the Snake Lemma, we have a commutative diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
0 \\
\downarrow \\
0 \\
\downarrow \\
0 \\
\downarrow \\
\downarrow \\
0 \\
\downarrow \\
0 \\
\downarrow \\
0 \\
\downarrow \\
0 \\
\downarrow \\
0
\end{array}
\]

from which we see that $C \subset S$ (corresponding to the non-zero section of $\mathcal{J}_{C/X}(D)$ appearing in the rightmost column) and $\mathcal{L} \cong \mathcal{O}_S(D - C)$. Dualizing the lower horizontal sequence in (23) we find

\[
0 \to \mathcal{E}^* \to V_r^* \otimes \mathcal{O}_X \to \Ext^1_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \cong \mathcal{O}_S(C) \to 0,
\]

which shows that $\mathcal{O}_S(C)$ is globally generated by $r$ sections.

In the case of smooth embedded Fano threefolds, we get:

**Theorem 3.5.** Let $X \subset \mathbb{P}^m$ be a smooth (Fano) threefold of degree $d > 1$ with $\omega_X \cong \mathcal{O}_X(-\alpha)$, for $\alpha \in \{1, 2, 3\}$. Then $X$ carries an Ulrich bundle $\mathcal{E}$ satisfying

\[
\text{rk}(\mathcal{E}) = r \geq 2 \quad \text{and} \quad \det(\mathcal{E}) = \mathcal{O}_X(m), \quad \text{with} \ m \in \mathbb{Z}^+,
\]

if and only if

\[
m = \frac{r}{2}(4 - \alpha)
\]

and there exists a smooth curve $C \subset X$ such that

\[
\deg(C) = \frac{rd}{12}(\alpha^2 - 22) + \frac{2r}{\alpha} + \frac{md}{2}(\alpha + m)
\]

\[
g(C) = r(d - 1) + \frac{1}{3}m^3d + \frac{1}{4}am^2d + \frac{1}{12} \left[ r(\alpha^2 - 22) - \alpha^2 \right] md + \frac{2(r - 1)}{\alpha}m + 1,
\]

\[
\omega_C(\alpha - m) \text{ is generated by an } (r - 1)\text{-dimensional space } W,
\]

the multiplication map

\[
\nu : W \otimes H^0(\mathcal{O}_X(3 - \alpha)) \to H^0(\omega_C(3 - m))
\]

is either injective or surjective,

\[
\text{the restriction map } H^0(X, \mathcal{O}_X(m - p)) \to H^0(C, \mathcal{O}_C(m - p))
\]

is an isomorphism for $p \in \{1, 2, 3\}$.
Proof. We apply Theorem 3.1. Using Lemma 3.3(i), we see that condition (27) is (17). Conditions (9) and (10) in Theorem 3.1 are, respectively, (21), (24) and (26). (To compute $c_2(X) \cdot H$, we use $1 = \chi(X, \mathcal{O}_X) = \frac{1}{27}c_2 \cdot H$ by, e.g., [Fl, Exc. 15.2.5].)

Now consider condition (11) in Theorem 3.1. It reads $h^0(\mathcal{O}_X(3 - \alpha - m)) = 0$, which by (24) is equivalent to $3 - \alpha - \frac{r}{2} (4 - \alpha) < 0$, which is clearly satisfied.

Next consider conditions (12)–(13) in Theorem 3.1. Together they are equivalent to

$$h^i(X, \mathcal{J}_{C/X}(m - p)) = 0 \text{ for } i \in \{0, 1\}, \ p \in \{1, 2, 3\}.$$ (30)

From the short exact sequence

$$0 \longrightarrow \mathcal{J}_{C/X}(m - p) \longrightarrow \mathcal{O}_X(m - p) \longrightarrow \mathcal{O}_C(m - p) \longrightarrow 0,$$

and the fact that $h^1(\mathcal{O}_X(m - p)) = 0$, we see that (30) is equivalent to (29).

Finally, by Lemma 3.3(ii), condition (11) in Theorem 3.1 can be restated as (28). □

Remark 3.6. Suppose $\mathcal{E}$ is an Ulrich bundle on a smooth Fano threefold $X$ as in Theorem 3.5. Then $h^i(\mathcal{E}) = h^{3-i}(\mathcal{E}(K_X)) = h^{3-i}(\mathcal{E}(-\alpha)) = 0$ for all $0 \leq i \leq 3$, so that the cohomology of (16) yields $W = H^0(\omega_C(\alpha - m))$.

We will make use of Theorem 3.5 in the proof of our main results in the next section. In the rest of this section we give some consequences and examples.

Corollary 3.7. Let $X \subset \mathbb{P}^{g+1}$ be a smooth Fano threefold of degree $2g - 2$ such that $\omega_X \simeq \mathcal{O}_X(-1)$ and $\text{Pic}(X) \simeq \mathbb{Z}[\mathcal{O}_X(1)]$. Then $X$ carries no Ulrich bundles of odd rank.

Proof. It is well–known that varieties of degree $d > 1$ with Picard group generated by the hyperplane class cannot carry Ulrich line bundles (see, e.g., [Bo1 § 4]). As for rank $r \geq 2$, Theorem 3.5 (with $\alpha = 1$) yields that $m = \frac{2r}{r}$ must be an integer, cf. (24). □

Recall that Fano threefolds satisfying the hypotheses of the corollary (called prime Fano threefolds of index one) are classified and that $g \in \{3, \ldots, 10, 12\}$ (see [IP]). In [CFK1] we prove that all such Fano threefolds carry Ulrich bundles of all even ranks.

Example 3.8. Let $X \subset \mathbb{P}^{g+1}$ be a smooth Fano threefold of degree $2g - 2$ such that $\omega_X \simeq \mathcal{O}_X(-1)$. Theorem 3.5 yields that $X$ carries a rank–two Ulrich bundle $\mathcal{E}$ such that $\det(\mathcal{E}) = \mathcal{O}_X(m)$ if and only if $m = 3$ and there exists a smooth curve $C \subset X$ of degree $5g - 1$ such that $\omega_C \simeq \mathcal{O}_C(2)$ (in particular $g(C) = 5g$) and $C \subset \mathbb{P}^{g+1}$ is non–degenerate and linearly and quadratically normal.

Since in [CFK1] we prove the existence of such rank–two Ulrich bundles in the cases where $\text{Pic}(X) \simeq \mathbb{Z}[\mathcal{O}_X(1)]$, the existence of such a curve will follow as a consequence.

The following proves Theorem 1.2(i):

Corollary 3.9. Let $X \subset \mathbb{P}^9$ be the 2–Veronese embedding of $\mathbb{P}^3$. Then $X$ carries no Ulrich bundles of odd rank.

Proof. We have $\omega_X \simeq \mathcal{O}_X(-2)$. If there exists an Ulrich bundle of rank $r$ on $X$, then Theorem 3.5 (with $\alpha = 2$) yields the existence of a curve of degree $r(4r - 3)$ (since $m = r$ by (24)). As $\mathcal{O}_X(1)$ is $2$–divisible, the degree must be even, whence $r$ must be even. □

Example 3.10. Let $X \subset \mathbb{P}^{d+1}$ be a smooth Fano threefold of degree $d$ such that $\omega_X \simeq \mathcal{O}_X(-2)$.

(i) Theorem 3.5 yields that $X$ carries a rank–two Ulrich bundle $\mathcal{E}$ such that $\det(\mathcal{E}) \simeq \mathcal{O}_X(m)$ if and only if $m = 2$ and there exists a smooth elliptic normal curve
C ⊂ X of degree d + 2. The curve C is contained in a smooth surface S ∈ |O_X(2)| (cf. Remark 3.4), which is a K3 surface.

The existence of rank–two Ulrich bundles on X was proved in [Be1, Prop. 6.1] by proving the existence of such an elliptic curve (cf. Proposition 3.3 below). (The existence in the case of cubic threefolds in \( \mathbb{P}^4 \) had previously been proved in [CH, Prop. 5.1].)

(ii) Theorem 3.3 yields that X carries a rank–three Ulrich bundle \( E \) such that \( \det(E) \simeq \mathcal{O}_X(m) \) if and only if \( m = 3 \) and there exists a smooth curve \( C \subset X \) of degree \( 3d + 3 \) and genus \( 2d + 4 \) such that \( \omega_C(-1) \) is globally generated (by two sections), \( C \subset \mathbb{P}^{d+1} \) is linearly and quadratically normal and the Petri map

\[
\mu_{0,\mathcal{O}_X(1)} : H^0(\mathcal{O}_C(1)) \otimes H^0(\omega_C(-1)) \rightarrow H^0(\omega_C)
\]

is injective. (Indeed, by [29] and Remark 3.6 the Petri map equals the map \( \nu \) in (28).)

The curve C is contained in a smooth surface \( S \in |\mathcal{O}_X(3)| \) (cf. Remark 3.4), which is a canonical surface, i.e., \( \omega_S \simeq \mathcal{O}_S(1) \).

In the case where \( X \) is a general smooth cubic threefold in \( \mathbb{P}^4 \), i.e., \( d = 3 \), the existence of rank–three Ulrich bundles on X was proved in [CH, Prop. 5.4] by constructing a curve \( C \) satisfying the above conditions using Macaulay2 (cf. CH App. A.2). By contrast, we can deduce the existence of such a curve \( C \) as a consequence of Theorem 3.3.

4. Ulrich bundles on Del Pezzo threefolds

This section is devoted to the proofs of Theorems 1.1 and 1.2(ii), recalling that part (i) of the latter has been proved in Corollary 3.5.

We recall that a smooth Fano threefold \( X \) has even index if there exists an ample divisor \( H \) on \( X \) such that \( K_X = -2H \). If \( H \) is very ample, it is well–known that it embeds \( X \) as a subvariety of degree \( d = H^3 \) in \( \mathbb{P}^{d+1} \). The general curve section of \( X \) is an elliptic curve and the general hyperplane section is a smooth Del Pezzo surface. For this reason, such threefolds are also called Del Pezzo threefolds. The index of \( X \) is the order of divisibility of \( K_X \) in \( \text{Pic}(X) \). By the classification of such varieties (see, e.g., [IP]), there is one case of index four, namely the 2–Veronese embedding of \( \mathbb{P}^3 \) in \( \mathbb{P}^9 \) (where \( d = 8 \)), and the following cases of index two:

- \( d = 3 \): \( X \) is a smooth cubic hypersurface in \( \mathbb{P}^4 \), with \( \text{Pic}(X) = \mathbb{Z}[H] \);
- \( d = 4 \): \( X \) is a smooth complete intersection of type \( (2, 2) \) in \( \mathbb{P}^5 \), with \( \text{Pic}(X) = \mathbb{Z}[H] \);
- \( d = 5 \): \( X \) is a smooth section with a linear space \( \Lambda \simeq \mathbb{P}^6 \) of the Grassmannian \( G(1, 4) \) sitting in \( \mathbb{P}^9 \) via its Plücker embedding; one has \( \text{Pic}(X) = \mathbb{Z}[H] \).
- \( d = 6 \): two cases occur: either \( X \) is the Segre embedding of \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) in \( \mathbb{P}^7 \), in which case \( \text{Pic}(X) \simeq \mathbb{Z}^3 \), or \( X \) is a smooth hyperplane section of the Segre embedding of \( \mathbb{P}^2 \times \mathbb{P}^2 \) in \( \mathbb{P}^8 \), and \( \text{Pic}(X) \simeq \mathbb{Z}^2 \);
- \( d = 7 \): \( X \) is isomorphic to the blow–up of \( \mathbb{P}^3 \) at a point \( p \), embedded in \( \mathbb{P}^8 \) via the proper transform on \( X \) of the linear system of quadrics in \( \mathbb{P}^3 \) passing through the point \( p \). In this case one has \( \text{Pic}(X) \simeq \mathbb{Z}^2 \).

We will need the following lemma.

**Lemma 4.1.** Let \( X \subset \mathbb{P}^{d+1} \) be a smooth Del Pezzo threefold of degree \( d \).

(i) If \( U \) is an Ulrich bundle on \( X \) and \( E \) is any vector bundle on \( X \) satisfying

\[
h^0(E(-1)) = h^1(E(-2)) = 0, \text{ then } h^2(U \otimes E^*) = h^3(U \otimes E^*) = 0.
\]

In particular, for any Ulrich bundle \( U \) on \( X \) one has

\[
h^2(U \otimes U^*) = h^3(U \otimes U^*) = 0.
\]
(ii) If \( \mathcal{E}_i, \ i \in \{1, 2\} \), are vector bundles on \( X \) satisfying

\[
\text{rk}(\mathcal{E}_i) = r_i, \quad c_1(\mathcal{E}_i) = [\mathcal{O}_X(r_i)] \quad \text{and} \quad c_2(\mathcal{E}_i) \cdot \mathcal{O}_X(1) = r_i \left\lfloor r_i d - d + 2 \right\rfloor,
\]

then \( \chi(\mathcal{E}_1 \otimes \mathcal{E}_2^*) = -r_1 r_2 \).

\textbf{Proof.} (i) Since \( \mathcal{U} \) is Ulrich, there exists a resolution of the form

\[
\cdots \rightarrow \mathcal{O}_{\mathbb{P}^{d+1}}(-2) \oplus a_2 \rightarrow \mathcal{O}_{\mathbb{P}^{d+1}}(-1) \oplus a_1 \rightarrow \mathcal{O}_{\mathbb{P}^{d+1}} \rightarrow \mathcal{U} \rightarrow 0,
\]

where \( a_i \in \mathbb{Z}^+ \) (cf. e.g. [Be1, Thm. 1]). Tensoring by \( \mathcal{E}^* \), we obtain

\[
\cdots \rightarrow \mathcal{E}^*(-2) \oplus a_2 \rightarrow \mathcal{E}^*(-1) \oplus a_1 \rightarrow \mathcal{U} \otimes \mathcal{E}^* \rightarrow 0
\]

This implies, by Serre duality and the given assumptions, that

\[
\begin{align*}
\check{h}^3(\mathcal{U} \otimes \mathcal{E}^*) & \leq \check{h}^3(\mathcal{E}^*) \oplus a_0 = a_0 \check{h}^3(\mathcal{E}^*) = a_0 \check{h}^0(\mathcal{E}(-2)) = a_0 \check{h}^0(\mathcal{E}(-1)) = 0, \\
\check{h}^2(\mathcal{U} \otimes \mathcal{E}^*) & \leq a_0 \check{h}^2(\mathcal{E}^*) + a_1 \check{h}^3(\mathcal{E}^*(-1)) = a_0 \check{h}^1(\mathcal{E}(-2)) + a_1 \check{h}^0(\mathcal{E}(-1)) = 0.
\end{align*}
\]

(ii) We borrow an argument from [CH] Proposition 5.6. Set \( \mathcal{E}_i' := \mathcal{E}_i^*(2) \). Then one may check that properties (32) hold with \( \mathcal{E}_i \) replaced by \( \mathcal{E}_i' \), whence by Riemann–Roch

\[
\chi(\mathcal{E}_1 \otimes \mathcal{E}_2^*(-1)) = \chi(\mathcal{E}_1^* \otimes \mathcal{E}_2^*(-1))
\]

At the same time, Serre duality gives

\[
\chi(\mathcal{E}_1 \otimes \mathcal{E}_2^*(-1)) = -\chi(\mathcal{E}_1^* \otimes \mathcal{E}_2(-1)) = -\chi(\mathcal{E}_1^* \otimes \mathcal{E}_2^*(-1)),
\]

so that we must have \( \chi(\mathcal{E}_1 \otimes \mathcal{E}_2^*(-1)) = 0 \). Taking a general \( S \in |\mathcal{O}_X(1)| \), which is a Del Pezzo surface of degree \( d \), the restriction sequence thus yields \( \chi(\mathcal{E}_1 \otimes \mathcal{E}_2^*) = \chi(\mathcal{E}_1|_S \otimes \mathcal{E}_2^*|_S) \).

For vector bundles \( \mathcal{U} \) and \( \mathcal{V} \) on a surface \( S \), the Riemann–Roch theorem gives

\[
\chi(\mathcal{U} \otimes \mathcal{V}^*) = \frac{1}{2} \left( \text{rk}(\mathcal{V}) c_1(\mathcal{U})^2 + \text{rk}(\mathcal{U}) c_1(\mathcal{V})^2 - \text{rk}(\mathcal{V}) c_2(\mathcal{U}) - \text{rk}(\mathcal{U}) c_2(\mathcal{V}) \right)
\]

\[
- c_1(\mathcal{U}) c_1(\mathcal{V}) - \frac{1}{2} \left( \text{rk}(\mathcal{V}) c_1(\mathcal{U}) - \text{rk}(\mathcal{U}) c_1(\mathcal{V}) \right) \cdot K_S + \text{rk}(\mathcal{U}) \text{rk}(\mathcal{V}) \chi(\mathcal{O}_S).
\]

Inserting \( \mathcal{U} = \mathcal{E}_1|_S \) and \( \mathcal{V} = \mathcal{E}_2|_S \), and using that \( \text{rk}(\mathcal{E}_i|_S) = r_i \), \( \det(\mathcal{E}_i|_S) = \det(\mathcal{E}_i)|_S = \mathcal{O}_S(r_i) \) and \( c_2(\mathcal{E}_i|_S) = c_2(\mathcal{E}_i) \cdot S = \frac{r_i}{2} (r_i d - d + 2) \) by (32), the result follows. \( \square \)

The following settles all statements but the nonemptiness claims in Theorems 1.1 and 1.2(ii), recalling that stable bundles are simple:

\textbf{Proposition 4.2.} Let \( X \subset \mathbb{P}^{d+1} \) be a smooth Del Pezzo threefold of degree \( d \). If nonempty, the moduli space of simple rank-\( r \) Ulrich bundles with determinant \( \mathcal{O}_X(r) \) on \( X \) is smooth of dimension \( r^2 + 1 \).

\textbf{Proof.} Let \( \mathcal{E} \) be a member of the moduli space in question. By simplicity \( h^0(\mathcal{E} \otimes \mathcal{E}^*) = 1 \); moreover \( h^2(\mathcal{E} \otimes \mathcal{E}^*) = h^3(\mathcal{E} \otimes \mathcal{E}^*) = 0 \) by Lemma 4.1(i). It is then well-known that the moduli space is smooth at \( [\mathcal{E}] \) (see, e.g., [CH, Prop. 2.10]), of dimension

\[
\check{h}^1(\mathcal{E} \otimes \mathcal{E}^*) = -\chi(\mathcal{E} \otimes \mathcal{E}^*) + h^0(\mathcal{E} \otimes \mathcal{E}^*) = r^2 + h^0(\mathcal{E} \otimes \mathcal{E}^*) = r^2 + 1,
\]

using Lemma 4.1(ii). \( \square \)

We have left to prove the existence parts of Theorems 1.1 and 1.2(ii). Since both Ulrichness and stability are open conditions, it suffices to prove the existence of one stable Ulrich bundle on \( X \) (this observation had been done already in [CFAI] Lemma 2.3]). This will be proved by induction on the rank \( r \), starting with the case \( r = 2 \).
4.1. The cases of rank \( r = 2 \). The next result is due to Beauville \cite{Be1}; we will briefly recall part of the proof, because we will need it later.

**Proposition 4.3** (Beauville). Let \( X \subset \mathbb{P}^{d+1} \) be a smooth Del Pezzo threefold of degree \( d \). There exists a rank-two Ulrich bundle on \( X \) with determinant \( \mathcal{O}_X(2) \). Moreover, the moduli space parametrizing such bundles is 5-dimensional.

*Proof.* By Example 3.10(i) it suffices to prove that \( X \) contains a smooth elliptic normal curve of degree \( d+2 \). This is done in \cite{Be1}, Lemma 6.2. We recall the proof when \( d \leq 7 \): take a general \( S \in |\mathcal{O}_X(1)| \). This is a smooth Del Pezzo surface, obtained by blowing up 9 \(- d \) points on \( \mathbb{P}^2 \) in general position. Consider the linear system on \( S \) corresponding to quartic curves in \( \mathbb{P}^2 \) passing doubly through two of the blown-up points and simply through the rest. This is a \((d+1)\)-dimensional base point free linear system, whose general member is a smooth elliptic curve \( E \) of degree \( d+1 \). For any point \( p \in E_0 \), consider any line \( \ell_0 \) in \( X \) through \( p \) not lying on \( S \). As \( \ell_0 \cdot S = 1 \), the curves \( E_0 \) and \( \ell_0 \) intersect transversely at one point. In \cite{Be1} Pf. of Lemma 6.2 it is proved that \( E_0 \cup \ell_0 \) deforms to a smooth elliptic curve \( E \) in \( X \), which is the desired curve.

The statement about dimension of the moduli space is proved in \cite{Be1} Prop. 6.4. \( \square \)

The existence of a stable Ulrich bundle is proved in the following result, recalling that slope–stability implies stability.

**Proposition 4.4.** Let \( X \subset \mathbb{P}^{d+1} \) be a smooth Del Pezzo threefold of degree \( d \). There exists a rank-two slope-stable Ulrich bundle on \( X \) with determinant \( \mathcal{O}_X(2) \).

*Proof.* Let \( \mathcal{E} \) be a rank–two Ulrich bundle on \( X \) with determinant \( \mathcal{O}_X(2) \). If \( \mathcal{E} \) is not slope–stable, we have a destabilizing sequence

\[
0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0,
\]

with \( \mathcal{F} \) and \( \mathcal{G} \) of rank one. By saturating the sequence, we may assume that \( \mathcal{G} \) is torsion-free. It follows from \cite{CI} Thm. 2.9 that both \( \mathcal{F} \) and \( \mathcal{G} \) are Ulrich line bundles of slope \( \mu(\mathcal{E}) = d \) (see also \cite{Be1} (3.2)).

In the cases where \( \text{Pic}(X) \cong \mathbb{Z}[\mathcal{O}_X(1)] \), that is, when \( d \in \{3, 4, 5\} \), there exist no Ulrich line bundles on \( X \) (see again \cite{Be1} §4), so we have a contradiction. There are again no Ulrich line bundles on \( X \) when \( d = 7 \) (cf. \cite{CFM} Cor. 2.5 and Prop. 2.7) and \( d = 8 \) (cf. Corollary 3.9), and we have the same contradiction. We have left to treat the two cases with \( d = 6 \).

If \( X \) is a smooth hyperplane section of \( \mathbb{P}^2 \times \mathbb{P}^2 \) in its Segre embedding in \( \mathbb{P}^8 \), the only Ulrich line bundles on \( X \) are \( \pi_i^* \mathcal{O}_{\mathbb{P}^2}(2) \), \( i = 1, 2 \), where \( \pi_i : X \to \mathbb{P}^2 \) are the projections (cf. \cite{CFaM2} Cor. 2.7). Thus, in \( (33) \) we have \( \mathcal{F} \cong \pi_1^* \mathcal{O}_{\mathbb{P}^2}(2) \) and \( \mathcal{G} \cong \pi_2^* \mathcal{O}_{\mathbb{P}^2}(2) \), or vice–versa. Since

\[
\dim(\text{Ext}_X^1(\pi_1^* \mathcal{O}_{\mathbb{P}^2}(2), \pi_2^* \mathcal{O}_{\mathbb{P}^2}(2))) = 3,
\]

and similarly with \( \pi_1 \) and \( \pi_2 \) interchanged (cf. \cite{CFaM2} Example 5.4), bundles which are not slope–stable fill up a locus of dimension at most 2 in the moduli space, which is 5–dimensional by the last statement in Proposition 4.3. Hence there exist slope–stable rank–two Ulrich bundles on \( X \) with determinant \( \mathcal{O}_X(2) \).

If \( X \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \), one may similarly check that the only possibilities for \( \mathcal{F} \) and \( \mathcal{G} \) in \( (33) \) are \( \pi_1^* \mathcal{O}_{\mathbb{P}^1}(2) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(1) \) and \( \pi_3^* \mathcal{O}_{\mathbb{P}^1}(2) \otimes \pi_1^* \mathcal{O}_{\mathbb{P}^1}(1) \), where \( \pi_i : X \to \mathbb{P}^1 \) are the three projections, up to permutations (cf. \cite{CFaM1} Prop. 3.2). Since

\[
\dim(\text{Ext}_X^1(\pi_2^* \mathcal{O}_{\mathbb{P}^1}(2) \otimes \pi_3^* \mathcal{O}_{\mathbb{P}^1}(1), \pi_1^* \mathcal{O}_{\mathbb{P}^1}(2) \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(1))) = 3
\]
Remark 4.5. In the cases \(d \in \{6, 7\}\) the moduli space of rank–two Ulrich bundles with determinant \(O_X(2)\) is reducible. For \(X \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\), by \cite[Thm.B(3)]{CFaM1} and taking into account possible permutations of the generators \(h_1, h_2, h_3\) in the expressions of \(c_2(E)\), the moduli space consists of six irreducible components of dimension 5, three of which are generically smooth containing just one point representing the \(S\)–equivalence class of strictly slope–semistable Ulrich bundles whereas the general point corresponds to a slope–stable Ulrich bundle, and three components are smooth containing only slope–stable Ulrich bundles. Reducibility also occurs for a smooth hyperplane section of the Segre variety \(\mathbb{P}^2 \times \mathbb{P}^2\), cf. \cite[Thm. 5.6]{CFaM2}, and for the case \(d = 7\), cf. \cite[Thm.B(1)]{CFaM1}, where in both cases the components are characterized by different \(c_2(E)\).

By contrast, when \(d = 3\), the case when \(X \subset \mathbb{P}^4\) is a cubic threefold, the moduli space is birational to the intermediate Jacobian of \(X\), and is thus irreducible (cf. \cite{Be4}).

4.2. Conclusion of the proof of Theorem 1.1. Let \(X \subset \mathbb{P}^{d+1}\) be a smooth Fano threefold of degree \(d\) and index two with \(\omega_X \cong O_X(-2)\). Denote by \(H_1(X)\) the Hilbert scheme of lines contained in \(X\), which by \cite[Prop. 3.3.5(i)]{IP} is smooth, projective, of pure dimension two. (It is irreducible for \(3 \leq d \leq 5\) (cf. \cite[Thm. 1.1.1]{KPS}) and reducible for \(d = 6, 7\) (cf. \cite[Prop. 3.5.6]{IP}), but we will not need this.)

Lemma 4.6. Let \(X \subset \mathbb{P}^{d+1}\) be a smooth Fano threefold of degree \(d\) and index two with \(\omega_X \cong O_X(-2)\). Let \(E \subset S_2 \subset |O_X(2)|\) be a general curve and a general surface associated to a rank-two Ulrich bundle on \(X\) as in Theorem 3.5 and Remark 3.4, and let \(\ell\) be a general line in (a component of) \(H_1(X)\). Then the restriction map induces an isomorphism

\[ H^0(O_{S_2}(E)) \cong H^0(O_{\ell \cap S_2}(E)). \]

Proof. Consider the curve \(E_0\) contained in the smooth hyperplane section \(S\) of \(X\) and the line \(\ell_0\) as in the proof of Proposition 4.3. Then \(\ell_0\) is contained in a hyperplane section \(S' \neq S\) of \(X\). For \(\ell \subset X\) general line in (a component of) \(H_1(X)\), we have \(\ell \cap \ell_0 = \emptyset\) and \(\ell \cap S\) is one point. Hence \(\ell\) intersects \(E_0 \cup \ell_0\) in at most one point, and the same is true replacing \(E_0\) with any curve linearly equivalent to it on \(S\). Therefore, as one deforms the pair \((E_0 \cup \ell_0, S \cup S')\) to a pair \((E, S_2)\), where \(S_2 \in |O_X(2)|\) is a smooth, irreducible \(K\)–surface, \(E \subset S_2\) is a smooth elliptic normal curve of degree \(d + 2\), the line \(\ell\) intersects any member of the pencil \(|E|\) on \(S_2\) in at most one point. Since \(\ell \cap S_2\) consists of two points, we have \(H^0(J_{(\ell \cap S_2)/S_2}(E)) = 0\). Thus, the restriction map \(H^0(O_{S_2}(E)) \rightarrow H^0(O_{\ell \cap S_2}(E))\) is injective. As both spaces are two-dimensional, the map is an isomorphism.

Using induction on \(r\) we will prove the following statement, which concludes the proof of (the nonemptiness part of) Theorem 1.1.

Proposition 4.7. Let \(X \subset \mathbb{P}^{d+1}\) be a smooth Fano threefold of degree \(d\) and index two such that \(\omega_X \cong O_X(-2)\). For every integer \(r \geq 2\) there exists an irreducible component \(\Delta_r\) of the moduli space of slope–semistable vector bundles of rank \(r\) and determinant \(O_X(r)\) on \(X\) such that its general member is a slope–stable Ulrich bundle. Moreover, for
$r \geq 3$, $\mathcal{U}_r$ contains a closed subscheme $\mathcal{U}^\text{ext}_r$ of dimension at most $r^2 - r + 2$ consisting of extensions $\mathcal{E}$ of the form

$$0 \longrightarrow \mathcal{E}_{r-1} \longrightarrow \mathcal{E} \longrightarrow \mathcal{J}_{\ell/X}(1) \longrightarrow 0,$$

where $\mathcal{E}_{r-1}$ is a general member of $\mathcal{U}_{r-1}$ and $\ell$ is a general line in (a component of) $\mathcal{H}_1(X)$.

The case $r = 2$ has been proved in Propositions 4.3 and 4.4. Assume now we have proved Proposition 4.7 for some $r \geq 2$. We will prove that it also holds for $r + 1$.

We denote by $\mathcal{E}_r$ a general member of $\mathcal{U}_r$. Since $\mathcal{E}_r$ is Ulrich, by Theorem 3.5 and (15) it sits in a short exact sequence

$$0 \longrightarrow \mathcal{C}_r^{-1} \otimes \mathcal{O}_X \longrightarrow \mathcal{E}_r \longrightarrow \mathcal{J}_{\mathcal{C}_r/X}(r) \longrightarrow 0,$$

where $\mathcal{C}_r$ is a smooth irreducible curve in $X$ numerically equivalent to $c_2(\mathcal{E}_r)$, cf. Remark 3.2, with

$$\deg(\mathcal{C}_r) = \frac{r}{2} (rd - d + 2).$$

Pick a general line in (a component of) $\mathcal{H}_1(X)$.

**Lemma 4.8.** We have

(i) $\mathcal{E}_r|_{\ell} \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}$,

(ii) $h^2(\mathcal{E}_r^*(\mathcal{O}_{\ell}^{-1} \otimes \mathcal{J}_{\ell/X})) = r$.

**Proof.** (i) Recall from Theorem 3.1 and Remark 3.4 that there is a smooth surface $S_r \in |\mathcal{O}_X(r)|$ containing $\mathcal{C}_r$ and an exact sequence

$$0 \longrightarrow \mathcal{E}_r^* \longrightarrow \mathcal{O}_X^{\oplus r} \longrightarrow \mathcal{O}_{S_r}(C_r) \longrightarrow 0.$$

Restricting to $\ell$, and setting $Z_r := \ell \cap S_r$ (a scheme of length $r$), we obtain

$$0 \longrightarrow \mathcal{E}_r^* \longrightarrow \mathcal{O}_X^{\oplus r} \longrightarrow \mathcal{O}_{S_r}(C_r) \longrightarrow 0.$$

(The map $\mathcal{E}_r^*|_{\ell} \longrightarrow \mathcal{O}_X^{\oplus r}$ is injective because $\mathcal{E}_r^*|_{\ell}$ is locally free on $\ell$ and therefore it does not contain torsion subsheaves.) Since $\mathcal{E}_r$ is Ulrich, we have $H^i(\mathcal{E}_r^*) = H^{3-i}(\mathcal{E}_r(-2)) = 0$ for all $i$, whence we have a commutative diagram

$$
\begin{array}{ccc}
H^0(\mathcal{O}_X^{\oplus r}) & \xrightarrow{\pi} & H^0(\mathcal{O}_{S_r}(C_r)) \\
| & f & | \\
H^0(\mathcal{O}_\ell^{\oplus r}) & \xrightarrow{g} & H^0(\mathcal{O}_{Z_r}(C_r))
\end{array}
$$

where all four spaces are isomorphic to $\mathbb{C}^r$. Grant for the moment the following:

$$f \text{ is injective.}$$
Then $f$ is an isomorphism, and it follows that the map $g$ must be surjective, whence an isomorphism as well. Going back to (35), we see that $h^i(E_r|\ell) = 0$ for $i \in \{0, 1\}$. Since $\deg(E_r|\ell) = -c_1(E_r) \cdot \ell = -O_X(r \cdot \ell) = -r$, then $E_r|\ell \simeq O_{\mathbb{P}^1}(-1)^{3r}$, proving (i).

We now prove (37), which is equivalent to

$$H^0(J_{Z_r/S_r}(C_r)) = 0.$$  

The case $r = 2$ follows from Lemma 4.6 with $C_2 = E$.

If $r \geq 3$, we may use Proposition 4.7 (recalling that by the induction hypothesis we are assuming that it holds for $r$) and specialize $E_r$ in a one-parameter flat family over the disc $\mathbb{D}$ to a vector bundle $E_r'$ sitting in an extension

$$0 \longrightarrow E_{r-1} \longrightarrow E_r' \longrightarrow J_{\ell'/X}(1) \longrightarrow 0$$

with $E_{r-1}$ general in $U_{r-1}$ and $\ell'$ a general line in (a component of) $H_1(X)$. Then the surface $S_r$ in (35) specializes to $S_{r-1} \cup S_1$, where $S_{r-1} \in |O_X(r-1)|$ and $S_1 \in |O_X(1)|$ are smooth surfaces. In this process $Z_r$, the intersection scheme of $\ell$ with $S_r$, specializes to the disjoint union $Z_{r-1} \cup x$, where $Z_{r-1}$ is the intersection scheme of $\ell$ with $S_{r-1}$ and $x := \ell \cap S_1$. By generality of $\ell$, we have $x \not\in S_{r-1}$, so that $x \not\in Z_{r-1}$. By (39), we have

$$|C_r| = c_2(E_r) = c_2(E_r') = |C_{r-1} \cup \ell' \cup I|,$$

with $I := S_1 \cap S_{r-1}$, where $C_{r-1}$ sits in

$$0 \longrightarrow \mathbb{C}^{r-2} \otimes O_X \longrightarrow E_{r-1} \longrightarrow J_{C_{r-1}/X}(r-1) \longrightarrow 0.$$  

We want to prove that none of the curves in the limit linear systems of $|O_{S_r}(C_r)|$ contains the scheme $Z_{r-1} \cup x$. To do so, consider the family $\pi : S \to \mathbb{D}$, whose general fiber is a smooth surface $S_r \in |O_X(r)|$ and whose central fiber is $S_{r-1} \cup S_1$. Then, by [Se2, Chp. 2] or [FI, §2], the singular locus $3$ of $S$ is supported at a divisor in the linear system $|N_{I/S_1} \otimes N_{I/S_{r-1}}|$ on $I$, where $N_{I/S_j}$ denotes the normal bundle of $I$ in $S_j$ for $j = 1, r-1$, which has degree $(r-1)^2d + (r-1)d = r(r-1)d$. One can make $S$ smooth by making a small resolution at each of the singular points, cf., e.g., [CLM, p. 647], in such a way that the general fiber is unaltered and the central fibre is replaced by $S_{r-1} \cup \tilde{S}_1$, where $\tilde{S}_1 \to S_1$ is the blow–up along the points of $\tilde{3}$ on $I$. By abuse of notation, we still denote by $\ell'$ the strict transform of $\ell'$ on $\tilde{S}_1$. We also still denote $S_{r-1} \cap \tilde{S}_1$ by $I$.

A linear system of $|O_{S_r}(C_r)|$ consists of the union of suitable linear systems on $S_{r-1}$ and $\tilde{S}_1$ matching along $I$. Precisely, by (40), a linear system is of the form

$$\Lambda_k := \left\{(D_1, D_2) \in |O_{S_{r-1}}(C_{r-1} + I - kI)| \times |O_{\tilde{S}_1}(\ell' + kI)| : D_1 \cap I = D_2 \cap I\right\},$$

and we have $k \geq 0$ since $|O_{\tilde{S}_1}(\ell' - I)|$ is empty. By induction, no member of $|O_{S_{r-1}}(C_{r-1})|$ contains $Z_{r-1}$, whence also no member of $|O_{S_{r-1}}(C_{r-1} + I - kI)|$, for any $k > 0$, contains $Z_{r-1}$. Hence, for $k > 0$ no curve in $\Lambda_k$ contains $Z_{r-1} \cup x$. Let us see that also no curve in $\Lambda_0$ contains $Z_{r-1} \cup x$. In fact, as $\ell$ and $\ell'$ are general lines on $X$, they do not intersect, whence $x \not\in \ell'$. Since for $k = 0$ we have $|O_{\tilde{S}_1}(\ell' + kI)| = |O_{\tilde{S}_1}(\ell')| = \{\ell'\}$, we see that no curve in $\Lambda_0$ contains $Z_{r-1} \cup x$. Thus, (38) follows, proving (37).

(ii) Consider the short exact sequence

$$0 \longrightarrow E_r^*(-1) \otimes J_{\ell/X} \longrightarrow E_r^*(1) \longrightarrow E_r^*(-1)|_{\ell} \longrightarrow 0.$$
Since $\mathcal{E}_r$ is Ulrich, we have $h^i(\mathcal{E}_r^*(-1)) = h^{3-i}(\mathcal{E}_r(-1)) = 0$ for $i \in \{0, 1, 2, 3\}$. Moreover, by (i) we have $h^1(\mathcal{E}_r^*(-1)|\ell) = h^1(\mathcal{O}_\ell(-2)^{\oplus r}) = r$, and the result follows.

Since

$$
\text{(41)} \quad \text{Ext}^1_X(\mathcal{J}_{\ell/X}(1), \mathcal{E}_r) \simeq \text{Ext}^1_X(\mathcal{J}_{\ell/X}(1) \otimes \mathcal{E}_r^*, \omega_X) \simeq H^2(\mathcal{E}_r^*(-1) \otimes \mathcal{J}_{\ell/X})^* \simeq \mathbb{C}^r
$$

by Lemma 4.8(ii), there is an $(r - 1)$-dimensional family of nonsplit extensions

$$
\text{(42)} \quad 0 \rightarrow \mathcal{E}_r \rightarrow \mathcal{F} \rightarrow \mathcal{J}_{\ell/X}(1) \rightarrow 0.
$$

In the next three lemmas (4.9–4.11) we list a number of properties of any sheaf $\mathcal{F}$ obtained in this way.

**Lemma 4.9.** Any sheaf $\mathcal{F}$ as in (42) is locally free.

**Proof.** From the sequence

$$
\text{(43)} \quad 0 \rightarrow \mathcal{J}_{\ell/X} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_\ell \rightarrow 0
$$

twisted by $\mathcal{O}_X(1)$ we find that

$$
\text{ext}_i^{\mathcal{O}_X}(\mathcal{J}_{\ell/X}(1), \mathcal{O}_X) \simeq \text{ext}_i^{\mathcal{O}_X}(\mathcal{O}_\ell(1), \mathcal{O}_X) \cong 0 \quad \text{for} \quad i \geq 2,
$$

(as codim($\ell, X$) = 2) and

$$
\text{ext}_1^{\mathcal{O}_X}(\mathcal{J}_{\ell/X}(1), \mathcal{O}_X) \simeq \text{ext}_1^{\mathcal{O}_X}(\mathcal{O}_\ell(1), \mathcal{O}_X) \simeq \text{ext}_2^{\mathcal{O}_X}(\mathcal{O}_\ell, \omega_X) \otimes \mathcal{O}_X(1) \simeq \omega_\ell(1) \simeq \mathcal{O}_{\mathbb{P}^1}(-1).
$$

By (42) and the fact that $\mathcal{E}_r$ is locally free, we therefore get

$$
\text{ext}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \simeq \text{ext}_i^{\mathcal{O}_X}(\mathcal{J}_{\ell/X}(1), \mathcal{O}_X) = 0, \quad \text{for} \quad i \geq 2
$$

and

$$
\text{(44)} \quad 0 \rightarrow \mathcal{O}_X(-1) \rightarrow \mathcal{F}^* \rightarrow \mathcal{E}_r^* \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow \text{ext}_1^{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \rightarrow 0.
$$

To prove that $\mathcal{F}$ is locally free, we need to show that $\alpha$ is surjective. We argue by contradiction.

If $\alpha = 0$, then dualizing the left part of (44) we obtain

$$
\text{(45)} \quad 0 \rightarrow \mathcal{E}_r \rightarrow \mathcal{F}^* \rightarrow \mathcal{O}_X(1) \rightarrow 0.
$$

Ulrichness of $\mathcal{E}_r$ yields

$$
\text{Ext}^1_X(\mathcal{O}_X(1), \mathcal{E}_r) \simeq \text{Ext}^1_X(\mathcal{O}_X, \mathcal{E}_r(-1)) \simeq H^1(\mathcal{E}_r(-1)) = 0,
$$

whence (45) splits. Combining (42) and (45) with the natural map $h : \mathcal{F} \rightarrow \mathcal{F}^*$, we get

$$
\begin{array}{ccc}
0 & \rightarrow & \mathcal{E}_r \\
\downarrow & & \downarrow \alpha \\
\mathcal{F} & \rightarrow & \mathcal{J}_{\ell/X}(1) \\
\downarrow h & & \downarrow h \\
\mathcal{O}_X(1) & \rightarrow & 0,
\end{array}
$$

where $p : \mathcal{F}^* \rightarrow \mathcal{E}_r$ is the projection map induced by the splitting of (45). Since $p \circ j = \text{id}_{\mathcal{E}_r}$, we see that $p \circ h : \mathcal{F} \rightarrow \mathcal{E}_r$ induces a splitting of (42) as well, a contradiction.

If $\alpha \neq 0$ and $\alpha$ is not surjective, then $\text{im}(\alpha) \simeq \mathcal{O}_{\mathbb{P}^1}(-1 - a)$ for some $a > 0$. By Lemma 4.8(i) we would obtain a nonzero morphism $\mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-1 - a)$ for $a > 0$, which is impossible.

□
Lemma 4.10. Let \( \mathcal{F} \) be as in (42). Then:

(i) \( h^i(\mathcal{F}(-j)) = 0 \) for all \( i \geq 0 \) and \( j \in \{1, 2\} \),

(ii) \( h^i(\mathcal{F}(-3)) = \begin{cases} 0, & i \in \{0, 1\}, \\ 1, & i \in \{2, 3\} \end{cases} \),

(iii) \( h^3(\mathcal{F} \otimes \mathcal{F}^*) = 0 \),

(iv) \( \chi(\mathcal{F} \otimes \mathcal{F}^*) = -(r + 1)^2 \).

Proof. Items (i)-(ii) follow by straightforward computations using the sequences (42) and (43), suitably twisted.

By tensoring (42) by \( \mathcal{F}^* \) and taking cohomology we find

\[
h^3(\mathcal{F} \otimes \mathcal{F}^*) \leq h^3(\mathcal{E}_r \otimes \mathcal{F}^*) + h^3(\mathcal{F}^* \otimes \mathcal{J}_{\ell/X}(1)).
\]

By (i) and Lemma 4.1(i) we get \( h^3(\mathcal{E}_r \otimes \mathcal{F}^*) = 0 \). Moreover,

\[
h^3(\mathcal{F}^* \otimes \mathcal{J}_{\ell/X}(1)) \leq h^2(\mathcal{F}^*(1)_{|J}) + h^3(\mathcal{F}^*(1)) = h^0(\mathcal{F}(-3)) = 0,
\]

again by (i). This proves (iii).

Finally, by (42) we have:

\[
c_1(\mathcal{F}) = c_1(\mathcal{E}_r) + c_1(\mathcal{O}_X(1)) = c_1(\mathcal{O}_X(r + 1)),
\]

\[
c_2(\mathcal{F}) \cdot \mathcal{O}_X(1) = \left(c_2(\mathcal{E}_r) + c_2(\mathcal{J}_{\ell/X}(1)) + c_1(\mathcal{E}_r) \cdot c_1(\mathcal{J}_{\ell/X}(1))\right) \cdot \mathcal{O}_X(1)
\]

\[
= \left([C_r] + [\ell] + [\mathcal{O}_X(r) \cdot \mathcal{O}_X(1)]\right) \cdot \mathcal{O}_X(1) = \deg(C_r) + \deg(\ell) + rd
\]

\[
= \frac{r}{2} (rd - d + 2) + 1 + rd = \frac{r+1}{2} ((r + 1)d - d + 2)
\]

(where we have used (34)); thus (iv) follows from Lemma 4.1(ii). \( \square \)

Lemma 4.11. Let \( \mathcal{F} \) be as in (12) and let \( \mathcal{G} \) be a destabilizing subsheaf of \( \mathcal{F} \). Then \( \mathcal{G}^* \simeq \mathcal{E}_r^* \). In particular, \( \mu(\mathcal{G}) = \mu(\mathcal{F}) \) whence \( \mathcal{F} \) is slope-semistable.

Proof. We note that (42) yields \( \mu(\mathcal{F}) = \frac{\mathcal{O}_X(r+1) \cdot \mathcal{O}_X(1)^2}{r+1} = d \). Assume that \( \mathcal{G} \) is a destabilizing subsheaf of \( \mathcal{F} \), that is \( 0 < \text{rk}(\mathcal{G}) \leq \text{rk}(\mathcal{F}) - 1 = r \) and \( \mu(\mathcal{G}) \geq d \). Define

\[
Q := \text{im}\{\mathcal{G} \subset \mathcal{F} \rightarrow \mathcal{J}_{\ell/X}(1)\} \quad \text{and} \quad \mathcal{K} := \ker\{\mathcal{G} \rightarrow Q\}.
\]

Then we may put (12) into a commutative diagram with exact rows and columns:
Lemma 4.13. The general member in $\mathcal{U}_r$ is Ulrich and slope–stable. This will be done in Lemma 4.13 below, after an intermediate result:

Lemma 4.12. One has $\dim(\mathcal{U}_{r+1}^{\text{ext}}) < \dim(\mathcal{U}_{r+1})$.

Proof. By Lemma 4.10(iii)-(iv) and standard deformation theory we have

$$\dim(\mathcal{U}_{r+1}) \geq \dim(\mathcal{U}_r) + \dim \mathbb{P}(\text{Ext}^1_X(\mathcal{J}_{\ell/X}(1), \mathcal{E}_r)) + \dim(\mathcal{H}_1(X)) = (r^2 + 1) + (r - 1) + 2 = r^2 + r + 2.$$ 

On the other hand, using (41), we have

$$\dim(\mathcal{U}_{r+1}^{\text{ext}}) \leq \dim(\mathcal{U}_r) + \dim \mathbb{P}(\text{Ext}^1_X(\mathcal{J}_{\ell/X}(1), \mathcal{E}_r)) = \dim(\mathcal{U}_{r+1}) = r^2 + r + 2.$$ 

The result follows.

Lemma 4.13. The general member in $\mathcal{U}_{r+1}$ is Ulrich and slope–stable.
Proof. Let $\mathcal{E}$ be a general member of $\mathcal{U}_{r+1}$. By Lemma 4.10(i)-(ii) and semicontinuity we have $h^i(\mathcal{E}(j)) = 0$ for all $i \geq 0$ and $j \in \{1, 2\}$, $h^i(\mathcal{E}(-3)) = 0$ for $i \in \{0, 1\}$ and $\chi(\mathcal{E}(-3)) = 0$. To prove that $\mathcal{E}$ is Ulrich, we therefore have left to prove that $h^3(\mathcal{E}(-3)) = 0$, equivalently $h^0(\mathcal{E}^*(1)) = 0$.

Assume therefore, to get a contradiction, that $h^0(\mathcal{E}^*(1)) > 0$ for a general $\mathcal{E}$ in $\mathcal{U}_{r+1}$, and consider a nonzero section $s \in H^0(\mathcal{E}^*(1))$. Setting $Q := \text{coker}(s)$, we have

$$0 \longrightarrow \mathcal{O}_X \overset{s}{\longrightarrow} \mathcal{E}^*(1) \longrightarrow Q \longrightarrow 0. \tag{46}$$

We may assume that this specializes to a nonzero section $s_0 \in H^0(\mathcal{F}^*(1))$ for all $\mathcal{F}$ in $\mathcal{U}_{r+1}^{ext}$ and we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X \overset{s_0}{\longrightarrow} \mathcal{F}^*(1) \longrightarrow Q_0 \longrightarrow 0,$$

where $Q_0 := \text{coker}(s_0)$. Dualizing and twisting by $\mathcal{O}_X(1)$, we obtain

$$0 \longrightarrow Q_0^*(1) \longrightarrow \mathcal{F} \overset{s_0^*}{\longrightarrow} \mathcal{O}_X(1),$$

and one has $\text{im}(s_0^*) = J_{Z/X}(1)$ for some (possibly empty) subscheme $Z \subset X$. Since

$$c_1(Q_0^*(1)) = c_1(\mathcal{F}) - c_1(\mathcal{J}_{Z/X}(1)) = c_1(\mathcal{O}_X(r + 1)) - c_1(\mathcal{O}_X(1)) + c_1(\mathcal{O}_Z(1)) = c_1(\mathcal{O}_X(r)) + Z',$$

where $Z'$ is an effective divisor supported on the codimension-one locus of $Z$, we have

$$\mu(Q_0^*(1)) = \frac{(c_1(\mathcal{O}_X(r)) + Z') \cdot \mathcal{O}_X(1)^2}{\text{rk}(Q_0^*(1))} \geq \frac{c_1(\mathcal{O}_X(r)) \cdot \mathcal{O}_X(1)^2}{r} = \frac{rd}{r} = d.$$

Lemma 4.11 therefore yields $(Q_0^*(1))^* \simeq \mathcal{E}^*$, whence $Q_0^*(1) \simeq \mathcal{E}$, as $Q_0^*(1)$ is reflexive (by [Ha, Prop. 1.1]). It follows that $\mathcal{E}^*(1)$ is a deformation of $\mathcal{E}$, that is, $[\mathcal{E}^*(1)] \in \mathcal{U}_r$.

The dual of (46) twisted by $\mathcal{O}_X(1)$ gives

$$0 \longrightarrow \mathcal{E}^*(1) \longrightarrow \mathcal{E} \overset{s^*}{\longrightarrow} \mathcal{O}_X(1),$$

and for reasons of Chern classes we must have $\text{im}(s^*) \simeq J_{\ell/X}(1)$ for a line $\ell \subset X$. Therefore $[\mathcal{E}] \in \mathcal{U}_{r+1}^{ext}$, contradicting Lemma 4.12. We have thus proved that $\mathcal{E}$ is Ulrich.

If the general member of $\mathcal{U}_{r+1}$ were not slope–stable, we could find a one-parameter flat family of bundles $\{\mathcal{E}(t)\}$ over the disc $\mathbb{D}$ such that $\mathcal{E}(t)$ is a general member of $\mathcal{U}_{r+1}$ for $t \neq 0$ and $\mathcal{E}(0)$ lies in $\mathcal{U}_{r+1}^{ext}$, and such that we have, for $t \neq 0$, a destabilizing sequence

$$0 \longrightarrow \mathcal{G}(t) \longrightarrow \mathcal{E}(t) \longrightarrow \mathcal{F}(t) \longrightarrow 0, \tag{47}$$

which we can take to be saturated, that is, such that $\mathcal{F}(t)$ is torsion free, whence so that $\mathcal{F}(t)$ and $\mathcal{G}(t)$ are (Ulrich) vector bundles (see [CH Thm. 2.9] or [BeIl (3.2)]).

The limit of $\mathbb{P}(\mathcal{F}(t)) \subset \mathbb{P}(\mathcal{E}(t))$ defines a subvariety of $\mathbb{P}(\mathcal{E}(0))$ of the same dimension as $\mathbb{P}(\mathcal{F}(0))$, whence a coherent sheaf $\mathcal{F}(0)$ of rank $\text{rk}(\mathcal{F}(t))$ with a surjection $\mathcal{E}(0) \twoheadrightarrow \mathcal{F}(0)$. Denoting by $\mathcal{G}(0)$ its kernel, we have $\text{rk}(\mathcal{G}(0)) = \text{rk}(\mathcal{G}(t))$ and $c_1(\mathcal{G}(0)) = c_1(\mathcal{G}(t))$. Hence, (47) specializes to a destabilizing sequence for $t = 0$. Lemma 4.11 yields that $\mathcal{G}(0)^*$ is the dual of a member of $\mathcal{U}_r$. It follows that $\mathcal{G}^{(r)}$ is a deformation of the dual of a member of $\mathcal{U}_r$, whence $\mathcal{G}(t)$ is a deformation of a member of $\mathcal{U}_r$, as both are locally free. Therefore, $[\mathcal{G}(t)] \in \mathcal{U}_r$ and we obtain the same contradiction to Lemma 4.12 as above. \qed
4.3. Conclusion of the proof of Theorem 1.2(ii). Let $X \subset \mathbb{P}^9$ be the 2–Veronese embedding of $\mathbb{P}^3$. We imitate the proof of the existence of Ulrich bundles of rank $r \geq 2$ on a general cubic threefold in [CH, Thm. 5.7]. The strategy is as follows. We define $\mathcal{U}_2$ to be any component of the moduli space of slope–stable rank–2 Ulrich bundles of determinant $\mathcal{O}_X(2)$, which is nonempty by Propositions 1.3 and 1.4. Then we proceed to construct inductively (irreducible components of) moduli spaces $\mathcal{U}_{2h}$ of rank–2h Ulrich bundles, for all integers $h \geq 2$, in the following way: assume that we have constructed, for an integer $h \geq 1$, an irreducible moduli space $\mathcal{U}_{2h}$ parametrizing rank–2h Ulrich bundles of determinant $\mathcal{O}_X(2h)$ whose general member is slope–stable. In particular, for any $[\mathcal{E}_{2h}] \in \mathcal{U}_{2h}$, we have that

\begin{equation}
(49) \quad c_2(\mathcal{E}_{2h}) \cdot \mathcal{O}_X(1) = 2h(8h - 3)
\end{equation}

by (25) in Theorem 3.5 with $\alpha = 2$ (cf. Remark 3.2).

**Lemma 4.14.** For general $[\mathcal{E}_2] \in \mathcal{U}_2$ and $[\mathcal{E}'_{2h}] \in \mathcal{U}_{2h}$, such that $\mathcal{E}_2 \not\cong \mathcal{E}'_2$ when $h = 1$, one has $\dim(\text{Ext}^1(\mathcal{E}'_{2h}, \mathcal{E}_2)) = 4h$.

**Proof.** By Lemma 4.1(i) we have $h^2(\mathcal{E}_2 \otimes (\mathcal{E}'_{2h})^*) = h^3(\mathcal{E}_2 \otimes (\mathcal{E}'_{2h})^*) = 0$. Moreover $h^0(\mathcal{E}_2 \otimes (\mathcal{E}'_{2h})^*) = 0$, because $\mathcal{E}_2$ and $\mathcal{E}'_{2h}$ are slope–stable bundles of the same slope, namely 8, which are not isomorphic. Hence

$$\dim(\text{Ext}^1(\mathcal{E}'_{2h}, \mathcal{E}_2)) = h^1(\mathcal{E}_2 \otimes (\mathcal{E}'_{2h})^*) = -\chi(\mathcal{E}_2 \otimes (\mathcal{E}'_{2h})^*) = 4h,$$

by Lemma 4.1(ii), since conditions (92) hold true by (48). \hfill \Box

By the previous lemma, there exist nonsplit extensions of the form

\begin{equation}
(49) \quad 0 \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{F}_{2h+2} \longrightarrow \mathcal{E}'_{2h} \longrightarrow 0,
\end{equation}

for general $[\mathcal{E}_2] \in \mathcal{U}_2$ and $[\mathcal{E}'_{2h}] \in \mathcal{U}_{2h}$. We have that $\text{rk}(\mathcal{F}_{2h+2}) = 2h + 2$, $c_1(\mathcal{F}_{2h+2}) = [\mathcal{O}_X(2h + 2)]$ and $\mathcal{F}_{2h+2}$ is easily seen to be Ulrich. We denote by $\mathcal{U}_{2h+2}^{\text{ext}}$ the family of such nonsplit extensions and we define $\mathcal{U}_{2h+2}$ to be the component of the moduli space of Ulrich bundles containing $\mathcal{U}_{2h+2}^{\text{ext}}$.

To finish the proof of Theorem 1.2(ii), we have left to prove that the general element of $\mathcal{U}_{2h+2}$ is slope–stable. To this end, we need the following two lemmas.

**Lemma 4.15.** One has $\dim(\mathcal{U}_{2h+2}^{\text{ext}}) < \dim(\mathcal{U}_{2h+2})$.

**Proof.** One checks that in (49) one has $\mu(\mathcal{E}_2) = \mu(\mathcal{E}'_{2h}) = 8$. Thus, by [CH, Lemma 4.2], the bundle $\mathcal{F}_{2h+2}$ is simple, whence $\dim(\mathcal{U}_{2h+2}) = (2h + 2)^2 + 1 = 4h^2 + 8h + 5$ by Proposition 4.2. On the other hand, by Lemma 4.14 we have

$$\dim(\mathcal{U}_{2h+2}^{\text{ext}}) \leq \dim(\mathcal{U}_2) + \dim(\mathcal{U}_{2h}) + \dim(\mathbb{P}(\text{Ext}^1(\mathcal{E}'_{2h}, \mathcal{E}_2)))$$

$$= 5 + (4h^2 + 1) + (4h - 1) = 4h^2 + 4h + 5.$$

\hfill \Box

**Lemma 4.16.** Let $\mathcal{F}_{2h+2}$ be as in (49) and let $\mathcal{G}$ be a destabilizing subsheaf of $\mathcal{F}_{2h+2}$. Then $\mathcal{G}^* \cong \mathcal{E}_2^{\varphi}$.

**Proof.** This is identical to the proof of Lemma 4.11 and left to the reader. \hfill \Box
To prove that the general element of $\mathcal{U}_{2h+2}$ is slope–stable one applies almost verbatim the argument of the last part of the proof of Lemma 4.13 indeed, arguing by contradiction, if it were not slope–stable then one applies Lemma 4.16 to prove that it lies in $\mathcal{E}_{2h+2}^{\text{ext}}$, which contradicts Lemma 4.15. This completes the proof of Theorem 1.2.

5. Final remarks and speculations

Let $X \subset \mathbb{P}^{d+1}$ be a smooth Fano threefold of degree $d$ and index two with $\omega_X \simeq \mathcal{O}_X(-2)$. Theorem 4.1 proves that for every $r \geq 2$, there exists a smooth, $(r^2 + 1)$-dimensional moduli space $\mathcal{U}_{X,r}$, parametrizing a family of Ulrich bundles of rank $r$ and determinant $\mathcal{O}_X(r)$ on $X$. For any $\mathcal{E} \in \mathcal{U}_{X,r}$, one has $h^0(\mathcal{E}) = rd$ (cf., e.g., [Be1, (3.1)]).

By Theorems 3.1 and 3.5, given a general subspace $V$ of dimension $r - 1$ in $H^0(\mathcal{E})$, the inclusion $V \otimes \mathcal{O}_X \rightarrow \mathcal{E}$ drops rank along a smooth curve $C$ of genus

$$g_{r,d} = \frac{1}{3}rd(r^2 - 3r + 2) + (r - 1)^2$$

(cf. (24) and (26)).

Consider the moduli space $\mathfrak{S}_{2,d}$ of isomorphisms classes of smooth Fano threefolds of index two and degree $d$ and the universal moduli space $\mathcal{U}_{r,d}$ of Ulrich bundles of rank $r$ over $\mathfrak{S}_{2,d}$: a point in $\mathcal{U}_{r,d}$ is a pair $([X],[\mathcal{E}])$, where $[X] \in \mathfrak{S}_{2,d}$ and $[\mathcal{E}] \in \mathcal{U}_{X,r}$. Let $\mathbb{G}$ be the Grassmannian bundle over $\mathcal{U}_{r,d}$, whose fibre over $([X],[\mathcal{E}])$ is $\text{Grass}(r-1,H^0(X,\mathcal{E}))$. By the above considerations, we have a rational map

$$\varphi_{r,d} : \mathbb{G} \rightarrow \mathcal{M}_{g_{r,d}},$$

where $\mathcal{M}_g$ is the moduli space of smooth projective curves of genus $g$. Its image is clearly uniruled, and, as such, it is an interesting sublocus of $\mathcal{M}_{g_{r,d}}$. In particular, it is a natural question to understand for which $d$ and $r$ the map $\varphi_{r,d}$ is dominant.

Let us focus on low values of $r$. For $r = 2$ one has $g_{2,d} = 1$ and it is easy to deduce from the proof of Proposition 4.3 that $\varphi_{2,d}$ is dominant for all $d$.

The next step is $r = 3$. In this case $g_{3,d} = 2d + 4$, and, as we have seen in Example 3.10 through the embedding in $X$ the curve $C$ admits a linearly and quadratically normal embedding as a curve of degree $3d + 3$ in $\mathbb{P}^{d+1}$ with injective Petri map $\rho_{0,\mathcal{O}_C(1)}$ (cf. (31)). (We also note that the linear system $|\mathcal{O}_C(1)|$ has Brill–Noether number $\rho(2d + 4, d + 1, 3d + 3) = 0$.) This suggests that the map $\varphi_{3,d}$ might also be dominant.

A result in this direction is the following:

**Proposition 5.1.** The map $\varphi_{3,d}$ is dominant for $3 \leq d \leq 4$.

**Idea of the proof.** We give an idea in the case $d = 3$, where the target space is $\mathcal{M}_{10}$. The proof in the case $d = 4$, where the target space is $\mathcal{M}_{12}$, is similar.

Since $\rho(10,4,12) = 0$, there are linearly normal curves with general moduli in $\mathbb{P}^4$ of degree 12 and genus 10. It suffices to prove that such a general curve lies on a smooth cubic hypersurface; indeed the remaining properties as in Example 3.10(ii) are automatically satisfied since the curve has general moduli. Actually one proves that the general such curve lies on a general cubic hypersurface.

To do so, let $\mathcal{H}$ be the component of the Hilbert scheme containing the above curves with general moduli. An easy count of parameters shows that $\dim(\mathcal{H}) = 51$. Consider the incidence variety

$$I = \{(C,F) \in \mathcal{H} \times |\mathcal{O}_{\mathbb{P}^4}(3)| : C \subset F\}$$
with the two projections $p_1 : I \to \mathcal{H}$ and $p_2 : I \to |O_{\mathbb{P}^4}(3)|$. The map $p_1$ is dominant and the general fibre has dimension 7 (because the general curve $C \in \mathcal{H}$ is of maximal rank). So $\dim(I) = 58$. One proves that $p_2$ is dominant by proving that the general fibre of $p_2$ has dimension 24 (recall that $\dim(|O_{\mathbb{P}^4}(3)|) = 34$). It suffices to show that there is one $F \in |O_{\mathbb{P}^4}(3)|$ such that $p_2^{-1}(F)$ has dimension 24. One proves this by degeneration, showing that this is the case if $F$ is the general union of a quadric and a hyperplane, which is the technical and lengthy part of the proof that we skip. □

As a consequence, one has that $M_{10}$ and $M_{12}$ are uniruled, which is no big news, because we know since a long time that $M_{10}$ and $M_{12}$ are unirational (cf. [AS, Sc1]).

The proof of Proposition 5.1, based on delicate counts of parameters, is rather lengthy and we do not dwell on it here.

The question remains, how does $\varphi_{3,d}$ behave for $5 \leq d \leq 7$? We suspect that in these cases $\varphi_{3,d}$ is dominant onto a codimension 3 closed subset of $M_{2d+4}$. Let us briefly explain the reason for this expectation. First of all note that the moduli spaces $\mathfrak{F}_{2,d}$ consist of one isolated point for $5 \leq d \leq 7$ (see [IP]). Hence

$$\dim(\mathbb{G}) = 10 + 2(3d - 2) = 6d + 6.$$ 

Since $\dim(M_{2d+4}) = 6d + 9$, the map $\varphi_{3,d}$ cannot be dominant. We conjecture that it is generically finite onto its image, which would prove that $\varphi_{3,d}$ dominates a codimension 3 closed subset of $M_{2d+4}$.

References

[AS] E. Arbarello, E. Sernesi, The equation of a plane curve, Duke Math. J. 46 (2) (1979), 469–485.
[ACo] E. Arrondo, L. Costa, Vector bundles on Fano 3-folds without intermediate cohomology, Comm. Alg. 28 (2000), 3899–3911.
[Ban] C. Bănică, Smooth reflexive sheaves, Proceedings of the Colloquium on Complex Analysis and the Sixth Romanian-Finnish Seminar, Rev. Roumaine Math. Pures Appl. 36 (1991), 571-593.
[Be1] A. Beauville, An introduction to Ulrich bundles, Eur. J. Math. 4 (2018), 26–36.
[Be2] A. Beauville, Determinantal hypersurfaces, Michigan Math. J. 48 (2000), 39–64.
[Be3] A. Beauville, Vector bundles on Fano threefolds and K3 surfaces, Boll. Unione Mat. Ital. 15 (2022), 43–55.
[Be4] A. Beauville, Vector bundles on the cubic threefold, Contemp. Math. 312, Amer. Math. Soc., Providence, RI, 2002, 71–86.
[BF] M. C. Brambilla, D. Faenzi, Moduli spaces of rank-2 ACM bundles on prime Fano threefolds, Michigan Mathematical Journal 60 (2011), 113–148.
[CH] M. Casanellas, R. Hartshorne, with an appendix by F. Geiss and F.-O. Schreyer, Stable Ulrich bundles, Internat. J. Math. 23 (2012), 1250083, 50 pp.
[CFaM1] G. Casnati, D. Faenzi, F. Malaspina, Moduli spaces of rank two aCM bundles on the Segre product of three projective lines, J. Pure Appl. Algebra 220 (2016), 1554–1575.
[CFaM2] G. Casnati, D. Faenzi, F. Malaspina, Rank two aCM bundles on the del Pezzo fourfold of degree 6 and its general hyperplane section, J. Pure Appl Algebra 222 (2018), 585–609.
[CFiM] G. Casnati, M. Filip, F. Malaspina, Rank two aCM bundles on the del Pezzo threefold of degree 7, Rev. Math Complutense 37 (2017), 129–165.
[CKL] Y. Cho, Y. Kim and K.-S. Lee, Ulrich bundles on intersection of two 4-dimensional quadrics International Mathematics Research Notices 2021 (22) (2021), 17277–17303
[CFK1] C. Ciliberto, F. Flamini, A. L. Knutsen, Elliptic curves, ACM bundles and Ulrich bundles on prime Fano threefolds of index one, arxiv: 2206.09986, 2022.
[CFK2] C. Ciliberto, F. Flamini, A. L. Knutsen, Ulrich bundles on a general blow-up of the plane, Annali di Matematica Pura ed Applicata (2023), https://doi.org/10.1007/s10231-023-01303-4.
C. Ciliberto, A. F. Lopez, R. Miranda, Projective degenerations of $K3$ surfaces, Gaussian maps and Fano threefolds, Invent. Math. 114 (1993), 641–667.

E. Coskun, A Survey of Ulrich bundles, in “Analytic and Algebraic Geometry”, Springer, 2017, pp. 86–106.

L. Costa, R. M. Miró-Roig, J. Pons-Llopis, Ulrich bundles from commutative algebra to algebraic geometry, De Gruyter, Studies in Math. 77 (2021).

Y. Drozd, G. M. Greuel, Tame and wild projective curves and classification of vector bundles, J. Algebra 246 (2001), 1–54.

E. Coskun, A Survey of Ulrich bundles, in “Analytic and Algebraic Geometry”, Springer, 2017, pp. 86–106.

L. Costa, R. M. Miró-Roig, J. Pons-Llopis, Ulrich bundles from commutative algebra to algebraic geometry, De Gruyter, Studies in Math. 77 (2021).

D. Eisenbud, F. O. Schreyer, J. Weyman, Resultants and Chow forms via exterior syzygies, J. Amer. Math. Soc. 16 (2003), 537–579.

K. Friedman, Global smoothings of varieties with normal crossings, Ann. of Math. 118 (1983), 75–114.

W. Fulton, Intersection Theory, Springer-Verlag Berlin Heidelberg, Springer, New York, NY, 1988.

R. Hartshorne, Stable reflexive sheaves, Math. Ann. 254 (1980), 121–176.

V.A. Iskovskikh, Yu. G. Prokhorov, Fano varieties. Algebraic Geometry V, A.N. Parshin and I.R. Shafarevich eds., Encycl. Math. Sci. 47, Springer-Verlag, Berlin 1999.

A. G. Kuznetsov, Y. G. Prokhorov, C. A. Shramov, Hilbert schemes of lines and conics and automorphism groups of Fano threefolds, Jpn. J. Math. 13 (2018), 109–185.

M. Lahoz, E. Macrì, P. Stellari, Arithmetically Cohen-Macaulay bundles on cubic threefolds, arXiv:2109.13549 (2021).

K.-S. Lee, K.-D. Park, Moduli spaces of Ulrich bundles on the Fano 3-fold $V_5$, J. Algebra 574 (2021), 262–277.

A. F. Lopez, R. Muñoz, On the classification of non-big Ulrich vector bundles on surfaces and threefolds, Intern. J. of Math. 32 (2021), Paper no. 2150111.

D. Markushevich, A. S. Tikhomirov, The Abel-Jacobi map of a moduli component of vector bundles on the cubic threefold, J. Algebraic Geom. 10 (2001), 37–62.

G. Ottaviani, Varietà proiettive di codimensione piccola, Quaderni dell’Istituto Nazionale di Alta Matematica F. Severi, Aracne, 1995.

E. Sernesi, L’unirazionalità della varietà dei moduli delle curve di genere 12, Ann. Sc. norm. super. Pisa - Cl. sci. 8 (1981), 405–439.

E. Sernesi, Deformations of Algebraic Schemes, Grundlehren der mathematischen Wissenschaften, Volume 334 (Springer, Berlin Heidelberg, 2006).

B. Ulrich, Gorenstein rings and modules with high numbers of generators, Math. Z. 188 (1984), 23–32.

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