TEMPERED FRACTIONAL ORDER COMPARTMENT MODELS
AND APPLICATIONS IN BIOLOGY

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Abstract. Compartment models with classical derivatives have diverse applications and attracted a lot of interest among scientists. To model the dynamical behavior of the particles that existed in the system for a long period of time with little chance to be removed, a power-law waiting time technique was introduced in the most recent work of Angstmann et al. [2]. The divergent first moment makes the power-law waiting time distribution less physical because of the finite lifespan of the particles. In this work, we take the tempered power-law function as the waiting time distribution, which has finite first moment while keeping the power-law properties. From the underlying physical stochastic process with the exponentially truncated power-law waiting time distribution, we build the tempered fractional compartment model. As an application, the tempered fractional SEIR epidemic model is proposed to simulate the real data of confirmed cases of pandemic AH1N1/09 influenza from Bogotá D.C. (Colombia). Some analysis and numerical simulations are carried out around the equilibrium behavior.

1. Introduction. Compartment models play a central role in the modeling of a wide variety of phenomena such as epidemics [28, 29], in-host pathogen dynamics [38] and the transport of chemicals (nutrients, drugs, hormones, etc.) between different parts in the human body [9, 16]. Compared with the integer order models, fractional order compartment models are a powerful instrument for incorporating memory and hereditary properties into evolution systems. By replacing integer order derivatives with Caputo fractional derivatives, fractional epidemiological model have been used to study epidemiological dynamics [1, 5, 6, 7, 14, 19, 21, 22, 23, 24, 25, 39, 43, 44, 46].
The power law distributions of the waiting time or jump length of the particles lead to the fractional operators in the macroscopic governing equation \[18\]. In recent years, fractional order compartment models were introduced by Angstmann et al. from an underlying physical stochastic process \[2\]. Fractional order recovery SIR models from a stochastic process were also derived by Angstmann et al. in \[3, 4\]. It is well known that in the stochastic process models, particles enter a compartment, waiting for random time, and then leave the compartment. Fractional order compartment models were formulated in \[3, 4, 2\] when the waiting time distribution \(\phi(t)\) has a power-law asymptotic decay \[8\] as \(t \to \infty\), i.e., \(\phi(t) \sim t^{-1-\alpha}(0 < \alpha < 1)\). Notice that in this case the waiting time distribution \(\phi(t)\) has divergent first moment. However because of the finite lifespan of the particles or the finite observation time of the experimentalist, it is more reasonable to make \(\phi(t)\) finite first moment, which can be done by using technical advantages of exponential truncating \[31, 36, 45\].

The technique of exponential truncating can not only realize the finite first moment of the wait time distribution, but also offers both mathematical and practical advantages, i.e., the tempered process is still an infinitely divisible Lévy process \[34, 42\], and tempering the positive stable law is equivalent to tempering its Lévy measure \[34\]. It is worth noting that the exponential truncation stability processes were first introduced in statistical physics to model turbulence and are known in physics literature as the truncated Lévy flight model \[31, 33, 37\], and then further developed by Cartea and del Castillo-negrete \[13\]; Carr, Genan, Madan and Yor introduced those processes to model stochastic volatility in mathematical finance \[11, 12\]; Boyarchenko, Levendorskii \[10\] and Hagen \[30\] considered such processes into option pricing; Meerschaert, Zhang and Baeumer applied the law of tempered stability to various problems in geophysics \[35\]. These processes also play an important role in the construction of certain Poisson-Dirichlet laws studied in \[40\]. A general theory of tempered stable laws in \(R^d\) has been developed by Rosiński \[41\].

Taking the tempered power-law function \(\phi(t, \lambda)\) as the waiting time distribution for the non-Markovian removal process, i.e., \(\phi(t, \lambda) \sim e^{-\lambda t}t^{-1-\alpha}\) as \(t \to \infty\) where \(0 < \alpha < 1\), \(\lambda > 0\) and the asymptotic behavior is in the sense of equivalent infinitesimal, in this paper we derive a tempered fractional order compartment model from an underlying physical stochastic process, and discuss the intercorrelation in the compartment models with tempered fractional order, or fractional order or classical ones. Furthermore, the equilibrium behavior of the tempered fractional order master equation is analysed by using some techniques of Laplace transform and binomial expansion proposed in \[3, 2\], and the relations among the different master equations are addressed.

By using the general framework of tempered fractional order compartment models, we construct a tempered fractional SEIR epidemic model and compute the equilibrium solutions to the system. Some numerical simulations are carried out for the tempered fractional SEIR epidemic model to show that its solutions converge to equilibrium states. In particular, to explain and understand the outbreaks of influenza A(H1N1), we fit the AH1N1/09 virus data by using the classical, fractional and tempered fractional SEIR epidemic models. It is interesting to note that we do not need to adjust the parameter values with regard to the incubation and recovery period of influenza A(H1N1), just taking the appropriate fractional parameters and tempering parameters, so that a good simulation can be obtained by using the tempered fractional SEIR epidemic model.
This paper is organized as follows. In Section 2, we derive the tempered fractional order master equation for an ensemble of particles in a single compartment from a stochastic process, where a tempered power-law distribution is used for the time that a particle remains in the compartment. Furthermore, we analyze the equilibrium behavior of the tempered fractional order single compartment model. In addition, the tempered fractional order multiple compartment model is given by linking multiple tempered fractional order single compartment models. In Section 3, we present the construction of the tempered fractional SEIR epidemic model and the analysis of the equilibrium behavior. Some numerical simulations are also carried out for the tempered fractional SEIR epidemic model. In Section 4, we fit the AH1N1/09 virus data by using the classical, fractional and tempered fractional SEIR epidemic models. A summary of this work is provided in Section 5. Finally, the finite difference methods are presented in the appendix for the tempered fractional SEIR epidemic model.

2. Tempered fractional order single compartment model. In this section, we investigate the dynamics of a single compartment model. In particular, we derive an master equation associated with a tempered power-law tailed waiting time distribution for the non-Markovian removal process. Furthermore, the relations among tempered fractional order, fractional order and classical master equations are presented.

Following [2] we assume that the creation of the particles in a single compartment is governed by $N_C$ distinct creation processes, i.e., the total arrival flux $q(t) = \sum_{i=1}^{N_C} \beta_i(t)$, where $\beta_i(t)$ is the arrival flux of particles due to the $i$th creation process. Similarly we suppose that the particles in the single compartment can be removed by $N_R$ Markovian removal processes and a non-Markovian removal process with a tempered power-law tailed waiting time distribution.

It is well known that for the Markovian removal process, the probability of a particle being removed from the compartment at time $t$ only depends on the state of the system at time $t$. Without loss of generality we can express the probability, that a particle will be removed by the $i$th Markovian removal process in the time interval $t$ to $t + \delta t$, as $\lambda_i(t) \delta t + o(\delta t)$. Then the survival function for $N_R$ Markovian removal processes can be written as

$$\theta(t, t_0) = \exp \left( - \int_{t_0}^{t} \omega(s) ds \right),$$

(2.1)

where

$$\omega(t) = \sum_{i=1}^{N_R} \lambda_i(t).$$

(2.2)

Furthermore, we notice that

$$\theta(t, t_0) = \theta(t, u) \theta(u, t_0),$$

(2.3)

for any $t_0 \leq u \leq t$, and

$$\frac{d\theta(t, t_0)}{dt} = -\omega(t) \theta(t, t_0).$$

(2.4)

It is clear that $\theta(t_0, t_0) = 1$ since a particle cannot be created and removed in the same instance.

In a non-Markovian removal process, the probability of a particle being removed from the compartment at time $t$ depends on the interval $t - t_0$, where $t_0$ is the
initial time when the particle enters the compartment. Denote by $\Phi(t)$ the survival probability for the non-Markovian removal process. Noticing that $\Phi(0) = 1$, the survival function can be expressed as

$$
\Phi(t) = 1 - \int_0^t \phi(u)du,
$$

where $\phi(t)$ is a waiting time probability density function. Then it follows from (2.5) that

$$
\frac{d\Phi(t)}{dt} = -\phi(t).
$$

In order to derive the tempered fractional order master equation that governs the dynamics of the number of particles in the compartment, we need to consider a tempered power-law waiting time distribution for the non-Markovian removal process. The power-law waiting time distribution for the non-Markovian removal process used in [2] implies that the longer particles have been in the compartment the slower their rate of removal by the non-Markovian removal process. However, the lifespan of particles or the observation time of the experimentalist is finite. It is more reasonable to use the tempered power-law waiting time distribution with finite first moment [31, 36, 45].

We represent the flux of particles entering the compartment at time $t$, by $q(t)$, which can be constructed by an injection of flux into the compartment at time $t = 0$ and the flux of a continuous function for $t > 0$. More precisely we have

$$
q(t) = i_0 \delta(t) + q^+(t),
$$

where $i_0$ is the initial injection, $\delta(t)$ is the Dirac delta function, $q^+(t)$ is right continuous at $t = 0$ and continuous for all $t > 0$. Noticing that for a particle in the compartment at time $t$, it must have entered the compartment at some prior time $t_0$ and survived until time $t$. If we assume that there are no particles in the compartment before time zero, then the number of particles in the compartment at time $t$, $\rho(t)$, can be given by

$$
\rho(t) = \int_0^t \Phi(t - t_0)\theta(t, t_0)q(t_0)dt_0 = i_0 \Phi(t)\theta(t, 0) + \int_0^t \Phi(t - t_0)\theta(t, t_0)q^+(t_0)dt_0,
$$

where we have assumed that the various removal processes are independent and thus the probability of surviving all of the removal processes from time $t_0$ to $t$ can be written as $\Phi(t - t_0)\theta(t, t_0)$.

When particles are in the compartment for a long period of time, there is little chance to be removed. This kind of situation can not be described by exponentially distributed waiting times. Since the power-law tailed waiting time distribution ($\phi(t) \sim t^{-1-\alpha}$ as $t \to \infty$ with $0 < \alpha < 1$) has divergent first moment, the expected waiting time diverges. Using such a distribution in our system will lead to particles being “trapped” in the compartment for a long period of time. In view of the finite lifespan of the particles on the finite observation time of the experimentalist, it is more reasonable to use the exponentially truncated power-law waiting time distribution. Hence here we choose the non-Markovian waiting time density to be the tempered Mittag-Leffler function

$$
\phi(t, \lambda) = ((\tau\lambda)^\alpha + 1) \exp(-\lambda t) \frac{t^\alpha - 1}{\tau^\alpha} \mathcal{E}_{\alpha, \alpha} \left(-\left(\frac{t}{\tau}\right)^\alpha\right),
$$

where

$$
\mathcal{E}_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}
$$
with an exponent $0 < \alpha \leq 1$, time scale parameter $\tau > 0$ and tempering parameter $\lambda > 0$. For the Mittag-Leffler probability density case, see [26] for more details. Here $E_{\alpha, \beta}(z)$ is the two parameter Mittag-Leffler function defined by

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \ z \in \mathbb{C},$$

which generalizes the exponential function in the sense that $E_{1,1}(z) = e^z$. For $0 < \alpha < 1$ the density has the exponentially truncated power-law tail, i.e., $\phi(t, \lambda) \sim e^{-\lambda t^{-1-\alpha}}$ for $t \to \infty$.

Laplace transforming (2.9) from $t$ to $s$, we find that

$$\mathcal{L}_t\{\phi(t, \lambda)\} = \frac{(\tau \lambda)^\alpha + 1}{\tau^\alpha (\lambda + s)^\alpha + 1}. \quad (2.10)$$

This implies that the Laplace transform of the corresponding survival function is given by

$$\mathcal{L}_t\{\Phi(t, \lambda)\} = \frac{\tau^\alpha (\lambda + s)^\alpha - \tau^\alpha \lambda^\alpha}{s (\tau^\alpha (\lambda + s)^\alpha + 1)}, \quad (2.11)$$

where $\Phi$ is given in (2.5), and

$$\mathcal{L}_t\{\Phi(t, \lambda)\} = \frac{1 - \mathcal{L}_t\{\phi(t, \lambda)\}}{s}.$$

Now we are ready to derive the tempered fractional order master equation. By (2.3) we obtain from (2.8) that

$$\frac{\rho(t)}{\theta(t, 0)} = i_0 \Phi(t, \lambda) + \int_0^t \Phi(t - t_0, \lambda) \theta(t, t_0) \frac{q^+(t_0)}{\theta(t, 0)} dt_0$$

$$= i_0 \Phi(t, \lambda) + \int_0^t \Phi(t - t_0, \lambda) \frac{q^+(t_0)}{\theta(t_0, 0)} dt_0. \quad (2.12)$$

Laplace transforming (2.12) from $t$ to $s$, by the convolution theorem and (2.11) we have

$$\mathcal{L}_t \left\{ \frac{\rho(t)}{\theta(t, 0)} \right\} = i_0 \mathcal{L}_t\{\Phi(t, \lambda)\} + \mathcal{L}_t\{\Phi(t, \lambda)\} \mathcal{L}_t\left\{ \frac{q^+(t)}{\theta(t, 0)} \right\}$$

$$= \frac{\tau^\alpha (\lambda + s)^\alpha - \tau^\alpha \lambda^\alpha}{s ((\tau (\lambda + s))^\alpha + 1)} \left( i_0 + \mathcal{L}_t\left\{ \frac{q^+(t)}{\theta(t, 0)} \right\} \right)$$

$$= \frac{1 - \lambda^\alpha (\lambda + s)^{-\alpha}}{s + \tau^{-\alpha} (\lambda + s)^{1-\alpha} - \lambda \tau^{-\alpha} (\lambda + s)^{-\alpha}} \left( i_0 + \mathcal{L}_t\left\{ \frac{q^+(t)}{\theta(t, 0)} \right\} \right). \quad (2.13)$$

Rearranging (2.13), we obtain

$$s \mathcal{L}_t \left\{ \frac{\rho(t)}{\theta(t, 0)} \right\} = i_0 + \mathcal{L}_t\left\{ \frac{q^+(t)}{\theta(t, 0)} \right\} - \lambda^\alpha (\lambda + s)^{-\alpha} \left( i_0 + \mathcal{L}_t\left\{ \frac{q^+(t)}{\theta(t, 0)} \right\} \right)$$

$$- \tau^{-\alpha} (\lambda + s)^{1-\alpha} \mathcal{L}_t\left\{ \frac{\rho(t)}{\theta(t, 0)} \right\}$$

$$+ \lambda \tau^{-\alpha} (\lambda + s)^{-\alpha} \mathcal{L}_t\left\{ \frac{\rho(t)}{\theta(t, 0)} \right\}. \quad (2.14)$$
Taking the inverse Laplace transform of (2.14), in view of $\theta(0,0) = \Phi(0) = 1$ and $\rho(0) = i_0$, we have

$$\frac{d\rho(t)}{dt} = q^+(t) - \omega(t)\rho(t) - \lambda^\alpha \theta(t,0) \, \mathcal{D}_t^{-\alpha,\lambda} \left( \delta(t)i_0 + \frac{q^+(t)}{\theta(t,0)} \right)$$

$$- \tau^{-\alpha} \theta(t,0) \, \mathcal{D}_t^{-\alpha,\lambda} \left( \frac{\rho(t)}{\theta(t,0)} \right)$$

$$+ \lambda \tau^{-\alpha} \theta(t,0) \, \mathcal{D}_t^{-\alpha,\lambda} \left( \frac{\rho(t)}{\theta(t,0)} \right).$$

(2.15)

The symbols $\mathcal{D}_t^{-\alpha,\lambda}$ and $\mathcal{D}_t^{1-\alpha,\lambda}$ represent the tempered fractional integral and the Riemann-Liouville tempered fractional derivative defined by [15, 17, 42],

$$\mathcal{D}_t^{-\alpha,\lambda} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{e^{-\lambda(t-\tau)}}{(t-\tau)^{1-\alpha}} f(\tau) d\tau,$$

with $\alpha > 0$, and

$$\mathcal{D}_t^{1-\alpha,\lambda} f(t) = \frac{1}{\Gamma(\alpha)} \left( \lambda + \frac{d}{dt} \right) \int_0^t \frac{e^{-\lambda(t-\tau)}}{(t-\tau)^{1-\alpha}} f(\tau) d\tau,$$

with $0 < \alpha \leq 1$ and $\lambda > 0$. Taking the Laplace transform of the tempered fractional integral and the Riemann-Liouville fractional derivative from $t$ to $s$ results in

$$\mathcal{L}_t \left\{ \mathcal{D}_t^{-\alpha,\lambda} f(t) \right\} = [\lambda + s]^{-\alpha} \mathcal{L}_t \{ f(t) \},$$

(2.18)

$$\mathcal{L}_t \left\{ \mathcal{D}_t^{1-\alpha,\lambda} f(t) \right\} = [\lambda + s]^{1-\alpha} \mathcal{L}_t \{ f(t) \}.$$  

(2.19)

Consider the case that particles are in the compartment for a long period of time with little chance to be removed. In such case, we take the non-Markovian waiting time to be Mittag-Leffler distributed, i.e.,

$$\phi(t) = \frac{t^{\alpha-1}}{\tau^\alpha} E_{\alpha,\alpha} \left( - \frac{t}{\tau} \right)^\alpha,$$

(2.20)

with $0 < \alpha \leq 1$ and $\tau > 0$. Replacing the exponentially truncated power-law waiting time distribution $\phi(t,\lambda)$ with the power-law distribution $\phi(t)$, by similar arguments as above, we obtain

$$\frac{d\rho(t)}{dt} = q^+(t) - \omega(t)\rho(t) - \tau^{-\alpha} \theta(t,0) \, \mathcal{D}_t^{1-\alpha} \left( \frac{\rho(t)}{\theta(t,0)} \right),$$

(2.21)

where $\mathcal{D}_t^{1-\alpha}$ is the Riemann-Liouville fractional derivative given by

$$\mathcal{D}_t^{1-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau,$$

(2.22)

which is equal to $s^{1-\alpha}$ in Laplace $s$ space. It is worth noticing that the fractional order master equation (2.21) coincides with the result established in [2].

The fractional order master equations reduce to the classical case, if we take the waiting time to be exponentially distributed, i.e., $\phi(t) = \frac{1}{\lambda} e^{-\frac{t}{\lambda}}$. In this case the probability of a particle being removed from the compartment can not be dependent on the amount of time that the the particle has already been in the compartment. Note that $E_{1,1} \left( -\frac{t}{\lambda} \right) = e^{-\frac{t}{\lambda}}$ and $\phi(t,\lambda) = \frac{1}{\lambda} e^{-\frac{t}{\lambda}}$ for the special case $\lambda = 0$ and $\alpha = 1$. Replacing the exponentially truncated power-law waiting time distribution
\(\rho(t, \lambda)\) by the exponential distribution \(\frac{1}{\lambda} e^{-\lambda t}\), by a similar argument as above we deduce that

\[
\frac{d\rho(t)}{dt} = q^+(t) - \omega(t)\rho(t) - \frac{1}{\tau}\rho(t). \tag{2.23}
\]

Now we consider the equilibrium behavior of the tempered fractional order master equation. It is worth noticing that the Riemann-Liouville derivative of a constant is nonzero. We shall use some techniques of Laplace transform and binomial expansion proposed in [3, 2] to overcome the difficulties caused by the fractional derivative.

We first assume that there exists the equilibrium solution \(\rho^*\) for the number of particles in the compartment, i.e.,

\[
\lim_{t \to +\infty} \rho(t) = \rho^*. \tag{2.24}
\]

Since this limit may depend on the initial condition of the system, it is possible that the equilibrium solution is not unique. In order to calculate the equilibrium solution \(\rho^*\), we further assume that the rate \(\omega(t)\) associated with the Markovian removal processes and the incoming flux \(q^+(t)\) approach to constants as \(t \to \infty\), i.e.,

\[
\lim_{t \to +\infty} \omega(t) = \omega^*, \quad \lim_{t \to +\infty} q^+(t) = q^*. \tag{2.25}
\]

For simplicity we set \(\omega(t) = \omega^*\), and thus it follows from (2.1) that

\[
\theta(t, 0) = \exp(-\omega^* t). \tag{2.26}
\]

Taking the limit of (2.15), by (2.24)-(2.26) we have

\[
0 = q^* - \omega^*\rho^* - \tau^{-\alpha} \lim_{t \to +\infty} \exp(-\omega^* t) D_t^{1-\alpha, \lambda} (\exp(\omega^* t)\rho(t)) + \lambda \tau^{-\alpha} \lim_{t \to +\infty} \exp(-\omega^* t) D_t^{1-\alpha, \lambda} (\exp(\omega^* t)\rho(t)) \tag{2.27}
\]

\[
- \lambda^\alpha \lim_{t \to +\infty} \exp(-\omega^* t) D_t^{1-\alpha, \lambda} (\delta(t)i_0 + \exp(\omega^* t)q^+(t)).
\]

We proceed to calculate the last three terms on the right-hand side of (2.27). First, by taking the Laplace transform and using the well-known shift identity, as well as the binomial expansion, we deduce that

\[
\mathcal{L}_t \left\{ \exp(-\omega^* t) D_t^{1-\alpha, \lambda} (\exp(\omega^* t)\rho(t)) \right\}
\]

\[
= \mathcal{L}_t \left\{ D_t^{1-\alpha, \lambda} (\exp(\omega^* t)\rho(t)) ; s + \omega^* \right\}
\]

\[
= (s + \omega^* + \lambda)^{1-\alpha} \mathcal{L}_t \left\{ \exp(\omega^* t)\rho(t) ; s + \omega^* \right\}
\]

\[
= (s + \omega^* + \lambda)^{1-\alpha} \mathcal{L}_t \left\{ \rho(t) ; s \right\}
\]

\[
= \mathcal{L}_t \{ \rho(t) \} \left( (\omega^* + \lambda)^{1-\alpha} + (1 - \alpha)(\omega^* + \lambda)^{-\alpha} s + \mathcal{O}(s^2) \right). \tag{2.28}
\]

Using the final value theorem of Laplace transform and (2.28) yields

\[
\lim_{t \to +\infty} \exp(-\omega^* t) D_t^{1-\alpha, \lambda} (\exp(\omega^* t)\rho(t))
\]

\[
= \lim_{s \to 0} s \mathcal{L}_t \left\{ \exp(-\omega^* t) D_t^{1-\alpha, \lambda} (\exp(\omega^* t)\rho(t)) \right\}
\]

\[
= \lim_{s \to 0} s \mathcal{L}_t \{ \rho(t) \} \left( (\omega^* + \lambda)^{1-\alpha} + (1 - \alpha)(\omega^* + \lambda)^{-\alpha} s + \mathcal{O}(s^2) \right)
\]

\[
= \rho^* (\omega^* + \lambda)^{1-\alpha}. \tag{2.29}
\]
In the similar way as above, we obtain that
\[
\lim_{t \to +\infty} \exp(-\omega^* t) \, D_t^{-\alpha, \lambda} \left( \exp(\omega^* t) \rho(t) \right) = (\omega^* + \lambda)^{-\alpha} \rho^*,
\]  
(2.30)
and
\[
\lim_{t \to +\infty} \exp(-\omega^* t) \, D_t^{-\alpha, \lambda} \left( \delta(t) \rho(t) + \exp(\omega^* t) q^+(t) \right) = (\omega^* + \lambda)^{-\alpha} q^*.
\]  
(2.31)
Substituting (2.29)-(2.31) into (2.27) results in
\[
\rho^* = \frac{q^* - \lambda^\alpha (\omega^* + \lambda)^{-\alpha} q^*}{\omega^* + \tau^{-\alpha} (\omega^* + \lambda)^{1-\alpha} - \lambda \tau^{-\alpha} (\omega^* + \lambda)^{-\alpha}}.
\]  
(2.32)
Note that when \( \lambda = 0 \), the equilibrium point \( \rho^* \) for the tempered fractional system (2.15) reduces to \( \rho^* = \frac{q^*}{\omega^* + \tau^{-\alpha} (\omega^* + \lambda)^{1-\alpha}} \), which is the equilibrium solution of the fractional system (2.21); see also [2]. In particular, when \( \lambda = 0 \) and \( \alpha = 1 \), the equilibrium point \( \rho^* = \frac{q^*}{\omega^* + \tau^{-1}} \) is the fixed point of the classical system (2.23).

The stability of the equilibrium point will be presented in Section 3 for the specific example of a tempered fractional order SEIR model.

2.1. Tempered fractional order multiple compartment model. Based on the tempered fractional order single compartment model, we can construct the master equations for any given multiple compartment model with tempered fractional dynamics. More precisely, for a \( N \) compartments model, the dynamics of each compartment can be governed by the following tempered fractional order master equation
\[
\frac{d\rho_k(t)}{dt} = q^+_k(t) - \omega_k(t) \rho_k(t) - \tau_k^{-\alpha_k} \theta_k(t, 0) \, D_t^{1-\alpha_k, \lambda_k} \left( \frac{\rho_k(t)}{\theta_k(t, 0)} \right) + \tau_k^{-\alpha_k} \lambda_k \theta_k(t, 0) \, D_t^{-\alpha_k, \lambda_k} \left( \frac{\rho_k(t)}{\theta_k(t, 0)} \right)
- \lambda_k^{\alpha_k} \theta_k(t, 0) \, D_t^{\alpha_k, \lambda_k} \left( \delta(t) \rho_k(t) + \frac{q^+_k(t)}{\theta_k(t, 0)} \right),
\]  
(2.33)
where \( k = 1, \ldots, N \) denotes the compartment.

Similar to the previous analysis, in view of the fractional order master equation (2.21) and the classical master equation (2.23), a fractional order \( N \) compartments model and a classical \( N \) compartments model, respectively, can be expressed as
\[
\frac{d\rho_k(t)}{dt} = q^+_k(t) - \omega_k(t) \rho_k(t) - \tau_k^{-\alpha_k} \theta_k(t, 0) \, D_t^{1-\alpha_k} \left( \frac{\rho_k(t)}{\theta_k(t, 0)} \right),
\]  
(2.34)
and
\[
\frac{d\rho_k(t)}{dt} = q^+_k(t) - \omega_k(t) \rho_k(t) - \frac{1}{\tau_k} \rho_k(t),
\]  
(2.35)
where \( k = 1, \ldots, N \) denotes the compartment.

In general, the Markovian rates, \( \omega_k(t) \), are functions of time and may be dependent on the population in the compartment. A example will be given to explain in detail how to build a tempered fractional order multiple compartment model by using the above mentioned method.
3. **Tempered fractional SEIR epidemic model.** Using the general framework we have established in previous sections, we construct a tempered fractional SEIR epidemic model to explain and understand the outbreaks of influenza A(H1N1) worldwide. We extend the standard SEIR model to a tempered fractional SEIR model, where we assume that there is a tempered fractional order recovery of individuals from the disease and tempered fractional order latent individuals become infected. When one consider an infectious disease transmission with long latency, it is more appropriate to use the tempered fractional SEIR epidemic model to capture its dynamical behavior. By using the general tempered fractional order multiple compartment model given in (2.33), we can construct a tempered fractional order four compartment SEIR epidemic model: the susceptible individual $S$ (those able to contract the disease), the exposed individual $E$ (those who have been infected but are not yet infectious and may not have symptoms), the infective individuals $I$ (those capable of transmitting the disease), and the recovered individuals $R$ (those who have recovered and become immune). The disease transmission flow is shown in Figure 1, where $q_3^+(t)$ and $q_4^+(t)$ are given in (3.3) and (3.5), respectively.

Let $\rho_1 = S$, $\rho_2 = E$, $\rho_3 = I$ and $\rho_4 = R$. Taking $q_1(t) = S_0 \delta(t) + \mu$ with $q_1^+(t) = \mu$, and setting $\omega_1(t) = \mu_1 + \beta I(t)$. We assume that there is no non-Markovian removal process, i.e., $\Phi_1(t) = 1$, and then establish the equation for the susceptible compartment,

$$\frac{dS(t)}{dt} = \mu - \mu_1 S(t) - \beta S(t) I(t). \quad (3.1)$$

Since the flux into the $E$ compartment comes from the $S$ compartment, we have $q_2(t) = E_0\delta(t) + q_2^+(t)$ with $q_2^+(t) = \beta S(t) I(t)$. We separate the removals from $E$ into two parts: Markovian/non-Markovian which corresponds to the death of individuals and the exposed ones who become infectious, respectively. More specifically, we choose $\omega_2(t) = \mu_2$ and $\tau_2^{-1} = \Omega_2$. By (2.1) we have $\theta_2(t,0) = \exp(-\mu_1 t)$. Then the tempered fractional order master equation for the exposed compartment can be written as

$$\frac{dE(t)}{dt} = \beta S(t) I(t) - \mu_1 E(t) - \Omega_2^\alpha \exp(-\mu_1 t) \, \delta_0 D_1^{1-\alpha_2,\lambda_2} (\exp(\mu_1 t) E(t))$$

$$+ \lambda_2 \Omega_2^\alpha \exp(-\mu_1 t) \, \delta_0 D_1^{1-\alpha_2,\lambda_2} (\exp(\mu_1 t) E(t))$$

$$- \lambda_2^\alpha \exp(-\mu_1 t) \, \delta_0 D_1^{1-\alpha_2,\lambda_2} (E_0 \delta(t) + \beta \exp(\mu_1 t) S(t) I(t)). \quad (3.2)$$

Similarly, the flux in compartment $I$ is $q_3(t) = I_0 \delta(t) + q_3^+(t)$ with $q_3^+(t) = \Omega_2^\alpha \exp(-\mu_1 t) \, \delta_0 D_1^{1-\alpha_2,\lambda_2} (\exp(\mu_1 t) E(t))$.

Figure 1. Flux flow of tempered fractional SEIR model.
\[ \omega \text{ compartment is } R \]

Therefore the equations (3.1)-(3.2), (3.4) and (3.6), subject to the initial conditions respectively, can be given by the fractional order multiple compartment model (2.34) and the classical multi-

\[ \theta \text{ compartment model (2.35)}, \text{ the frSEIR model and the classical SEIR model}, \]

where

\[ S(t), E(t), I(t), R(t) \]

\[ q^+_4(t) = \Omega_3^{\alpha_3} \exp(-\mu_1 t) D_t^{1-\alpha_3,\lambda_3} (\exp(\mu t) I(t)) \]

\[ - \lambda_3^{\alpha_3} \exp(-\mu_1 t) D_t^{-\alpha_3,\lambda_3} (I_0 \delta(t) + \exp(\mu_1 t) q^+_3(t)). \]

Finally, by the single flux balance consideration we obtain that the flux into the R compartment is

\[ \frac{dR(t)}{dt} = q^+_4(t) - \mu_1 R(t). \]

Therefore the equations (3.1)-(3.2), (3.4) and (3.6), subject to the initial conditions

\[ S(0) = S_0, \quad E(0) = E_0, \quad I(0) = I_0, \quad R(0) = 0, \quad S_0, \quad E_0, \quad I_0 \in \mathbb{R}_0^+, \]

where \( \mathbb{R}_0^+ = \{ x \in \mathbb{R} : x \geq 0 \} \), give the ftrSEIR model. In particular, by using the fractional order multiple compartment model (2.34) and the classical multi-

\[ \frac{dS(t)}{dt} = \mu - \mu_1 S(t) - \beta S(t) I(t), \]

\[ \frac{dE(t)}{dt} = \beta S(t) I(t) - \mu_1 E(t) - \Omega_2^{\alpha_2} \exp(-\mu_1 t) D_t^{1-\alpha_2} (\exp(\mu t) E(t)), \]

\[ \frac{dI(t)}{dt} = \Omega_2^{\alpha_2} \exp(-\mu_1 t) D_t^{-\alpha_2} (\exp(\mu t) E(t)) - \mu_1 I(t) \]

\[ \frac{dR(t)}{dt} = \Omega_3^{\alpha_3} \exp(-\mu_1 t) D_t^{1-\alpha_3} (\exp(\mu t) I(t)) - \mu_1 R(t), \]

and

\[ \frac{dS(t)}{dt} = \mu - \mu_1 S(t) - \beta S(t) I(t), \]

\[ \frac{dE(t)}{dt} = \beta S(t) I(t) - \mu_1 E(t) - \Omega_2 E(t), \]

\[ \frac{dI(t)}{dt} = \Omega_2 E(t) - \mu_1 I(t) - \Omega_3 I(t), \]

\[ \frac{dR(t)}{dt} = \Omega_3 I(t) - \mu_1 R(t). \]

In the following, we analyze the equilibrium behavior of the ftrSEIR model. Let \( (S^*, E^*, I^*, R^*) \) be the equilibrium state of the ftrSEIR system, such that

\[ \lim_{t \to +\infty} S(t) = S^*, \quad \lim_{t \to +\infty} E(t) = E^*, \quad \lim_{t \to +\infty} I(t) = I^*, \quad \lim_{t \to +\infty} R(t) = R^*. \]
Taking the limits of Eqs. (3.1)-(3.2), (3.4) and (3.6), in view of (2.29)-(2.31) and (3.10), we deduce that

\[
0 = \mu - \mu_1 S^* - \beta S^* I^*, \\
0 = \beta S^* I^* - \mu_1 E^* - q_3^*, \\
0 = q_3^* - \mu_1 I^* - q_4^*, \\
0 = q_4^* - \mu_1 R^*,
\]

where

\[
q_3^* = \Omega_2^{\alpha_2}(\mu_1 + \lambda_2)^{1-\alpha_2} E^* - \lambda_2 \Omega_2^{\alpha_2}(\mu_1 + \lambda_2)^{-\alpha_2} E^* \\
+ \lambda_2^{\alpha_2}(\mu_1 + \lambda_2)^{-\alpha_2} \beta S^* I^*, \\
q_4^* = \Omega_3^{\alpha_3}(\mu_1 + \lambda_3)^{1-\alpha_3} I^* - \lambda_3 \Omega_3^{\alpha_3}(\mu_1 + \lambda_3)^{-\alpha_3} I^* \\
+ \lambda_3^{\alpha_3}(\mu_1 + \lambda_3)^{-\alpha_3} q_3^*.
\]

Then the equilibrium solutions to the tfrSEIR system (3.1)-(3.2), (3.4) and (3.6) can be found by solving (3.11)-(3.14), which yields two equilibrium solutions, the disease free solution,

\[
S^* = \frac{\mu}{\mu_1}, \quad E^* = 0, \quad I^* = 0, \quad R^* = 0,
\]

and the equilibrium solution,

\[
S^* = \frac{\mu_1 + \Omega_2^{\alpha_2}(\mu_1 + \lambda_2)^{1-\alpha_2} - \lambda_2 \Omega_2^{\alpha_2}(\mu_1 + \lambda_2)^{-\alpha_2}}{\beta(1 - \lambda_3^{\alpha_3}(\mu_1 + \lambda_3)^{-\alpha_3}) \left[\lambda_2^{\alpha_2}(\mu_1 + \lambda_2)^{-\alpha_2} + S_{part}^*\right]}, \quad (3.18)
\]

where

\[
S_{part}^* = \frac{(\Omega_2^{\alpha_2}(\mu_1 + \lambda_2)^{1-\alpha_2} - \lambda_2 \Omega_2^{\alpha_2}(\mu_1 + \lambda_2)^{-\alpha_2})(1 - \lambda_3^{\alpha_3}(\mu_1 + \lambda_3)^{-\alpha_3})}{\mu_1 + \Omega_2^{\alpha_2}(\mu_1 + \lambda_2)^{1-\alpha_2} - \lambda_2 \Omega_2^{\alpha_2}(\mu_1 + \lambda_2)^{-\alpha_2}}, \quad (3.19)
\]

\[
E^* = \frac{1 - \lambda_3^{\alpha_3}(\mu_1 + \lambda_2)^{1-\alpha_2} - \lambda_2 \Omega_2^{\alpha_2}(\mu_1 + \lambda_2)^{-\alpha_2}}{\mu_1 + \Omega_2^{\alpha_2}(\mu_1 + \lambda_2)^{1-\alpha_2} - \lambda_2 \Omega_2^{\alpha_2}(\mu_1 + \lambda_2)^{-\alpha_2}}, \quad (3.19)
\]

\[
I^* = \frac{\mu - \mu_1 S^*}{\beta S^*}, \quad (3.20)
\]

\[
R^* = \frac{\lambda_3^{\alpha_3}(\mu_1 + \lambda_3)^{1-\alpha_3}(\mu_1 S^* - \mu_1 E^*)}{\mu_1} \lambda_3^{\alpha_3}(\mu_1 + \lambda_3)^{1-\alpha_3} I^* - \lambda_3 \Omega_3^{\alpha_3}(\mu_1 + \lambda_3)^{-\alpha_3} I^*. \quad (3.21)
\]

Since the population in each compartment can not be negative, the endemic equilibrium exist if

\[
S^* < \frac{\mu}{\mu_1}. \quad (3.22)
\]

Notice that when \(\alpha_2 = \alpha_3 = 1, \lambda_2 = \lambda_3 = 0\), the endemic equilibrium \((S^*, E^*, I^*, R^*)\) of frSEIR system reduces to

\[
S^* = \frac{(\mu_1 + \Omega_2)(\mu_1 + \Omega_3)}{\beta \Omega_2}, \quad E^* = \frac{\mu - \mu_1 S^*}{\mu_1 + \Omega_2}, \quad I^* = \frac{\Omega_2(\mu - \mu_1 S^*)}{(\mu_1 + \Omega_3)(\mu_1 + \Omega_2)}, \quad R^* = \frac{\Omega_3 I^*}{\mu_1}, \quad (3.23)
\]

which is the fixed point of the classical SEIR system (3.9). For the case \(0 < \alpha_2, \alpha_3 < 1\) and \(\lambda_2 = \lambda_3 = 0\), the endemic equilibrium \((S^*, E^*, I^*, R^*)\) of tfrSEIR system degenerates into the equilibrium solution of the frSEIR system (3.8).

Since the tfrSEIR model (3.1)-(3.2), (3.4) and (3.6) involves tempered fractional derivatives, we can give a good simulation by choosing the appropriate fractional
parameters and tempering parameters, which are different with the classical SEIR model. Fix the following parameters, $\mu = 5$, $\mu_1 = 0.01$, $\beta = 0.1$, $\Omega_2 = 0.6$, $\Omega_3 = 0.1$. We consider $S(0) = 100$, $E(0) = 0$, $I(0) = 1$ and $R(0) = 0$ as initial conditions. It is worth to notice that the tfrSEIR system has two possible steady states, the disease free steady state given in (3.17) and the endemic steady state given in (3.18)-(3.21). Figure 2 shows the different trends of the infected individuals with the vary of $\lambda_2$ or $\lambda_3$, when all the other parameters are fixed at: $\alpha_3 = 0.7$, $\lambda_3 = 0.0001$, $\alpha_2 = 0.7$, $\lambda_2 = 0.0001$. While from Figure 3, we can observe that for fixed $\lambda_3 = 0.0001$, $\alpha_3 = 0.7$, and each $\lambda_2 = 0.0001$, 0.01, the number of the endemic steady state for the infected compartment falls sharply to a minimum before rising slowly with increasing $\alpha_2$, however for fixed $\lambda_2 = 0.0001$, $\alpha_2 = 0.7$, and each $\lambda_3 = 0.0001$, 0.01, the number of the endemic steady state for the infected compartment rises slowly to a maximum before falling slowly with increasing $\alpha_3$.

To obtain the numerical solution of the tfrSEIR system, we use the discrete schemes (A.5)-(A.8) given in the appendix. Set $h = 0.02$, $S(0) = 100$, $E(0) = 0$, $I(0) = 1$ and $R(0) = 0$. From Figures 4 and 5, we find that the number in the infected compartment rise rapidly to a maximum before decreasing slowly towards a long time steady state. It can be observed that the peak level and the long time level of infection increase with the increase of $\alpha_2$ or $\alpha_3$, however this is opposite to the tempering parameters $\lambda_2$ and $\lambda_3$. Figure 4 shows that the parameter $\alpha_3$ plays a vital role in the model dynamics, since a small change in $\alpha_3$ can produce big variation in the long time levels of infection, which is different from $\alpha_2$. While from Figure 5, we see that the trends of the infected individuals are similar with the vary of $\lambda_2$ or $\lambda_3$.

**Figure 2.** The endemic steady state plotted as the functions of $\lambda_2$ (Left) and $\lambda_3$ (Right) for the tfrSEIR model.

**Figure 3.** The endemic steady state plotted as the functions of $\alpha_2$ (Left) and $\alpha_3$ (Right) for the tfrSEIR model.
4. AH1N1/09 virus data and model fitting. In this section, we fit the real data of confirmed cases of pandemic from Bogotá D.C. (Colombia) by using the classical, fractional and tempered fractional SEIR epidemic models.

Following [24], we use the data originally collected from state health institutions. Table 1 shows the number of registered people who tested positive in the metropolitan area of Bogotá D.C. (Colombia). Thanks to [24] and the papers cited therein, we set the parameters and initial conditions of the tfrSEIR model (3.1)-(3.2), (3.4) and (3.6), see Table 2.

In Figure 6, Figure 7 and Figure 8, respectively, we consider the effects of the parameters $\alpha_2$, $\alpha_3$, $\beta$ and $\lambda_2$, $\lambda_3$ on the dynamics of the AH1N1/09 influenza epidemic by varying the values of these parameters. The other parameters are the same as in Table 2. From Figure 6(a), we find that the impact of the parameter $\alpha_2$ on the epidemic peak is not obvious. The numerical results on Figure 6(b) show that, with the increase of the parameter $\alpha_3$, the epidemic peak is wider and lower. In addition, we notice that the epidemic develops slowly with decreasing $\alpha_3$. For the parameter $\beta$, it can be observed from Figure 7 that the situation is the same as in Figure 6. However from Figure 8 we see that for higher values of $\lambda_2$ or $\lambda_3$, the epidemic peak is higher, and the epidemic develops more slowly.

Comparing with the real data of confirmed cases of pandemic AH1N1/09 influenza from Bogotá D.C. (Colombia), Figure 9 shows the numerical solutions of the classical SEIR model (a) and the frSEIR model (b) with $\alpha_2 = 0.6$, $\alpha_3 = 0.88$. By choosing $\alpha_2 = 0.6$, $\alpha_3 = 0.88$, $\lambda_2 = 0.015$ and $\lambda_3 = 0.09$, comparing with Figure 9(b), Figure 10 indicates that the tfrSEIR model (3.1)-(3.2), (3.4) and (3.6) produces a good simulation by adjusting the tempering parameters $\lambda_2$, $\lambda_3$. It is
worth to notice that in the tempered fractional model case, we do not need to select the unreal parameter values with regard to the incubation and recovery period of influenza A(H1N1).

Table 1. Data provided by the Secretaria de Salud Distrital de Bogotá D.C. First row corresponds to the week number of year 2009. Second row presents the number of infectious individuals by AH1N1/09 detected in each week.

| Week | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
|------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| Cases| 4  | 1  | 0  | 1  | 2  | 5  | 12 | 17 | 22 | 16 | 15 | 53 | 55 | 45 |
| Week | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 |
| Cases| 84 | 102| 127| 261| 210| 155| 109| 116| 125| 76 | 52 | 19 | 15 | 1  |

Table 2. Values of various parameters and initial conditions.

| Model  | $\lambda_2$ | $\lambda_3$ | $\beta$ | $\Omega_2$ | $\Omega_3$ | $\alpha_2$ | $\alpha_3$ |
|--------|-------------|-------------|----------|------------|------------|------------|------------|
| cSEIR  | 5.038e+01  | 1.0e-08     | 2.89e-03 | 2.25e-01   | 1.373e+00  | 1          | 1          |
| frSEIR | 5.038e+01  | 1.0e-08     | 2.89e-03 | 2.25e-01   | 1.373e+00  | 0.6        | 0.88       |
| tfrSEIR| 5.038e+01  | 1.0e-08     | 2.89e-03 | 2.25e-01   | 1.373e+00  | 0.6        | 0.88       |

| Model  | $\lambda_2$ | $\lambda_3$ | $S_0$ | $E_0$ | $I_0$ | $R_0$ |
|--------|-------------|-------------|-------|-------|-------|-------|
| cSEIR  | 0           | 0           | 100   | 0     | 1     | 0     |
| frSEIR | 0           | 0           | 100   | 0     | 1     | 0     |
| tfrSEIR| 0.015       | 0.09        | 100   | 0     | 1     | 0     |

Figure 6. The variation of Infected $I(t)$ in the tfrSEIR model for $\alpha_2 = 0.6, 0.8, 1$ (Left) and $\alpha_3 = 0.8, 0.9, 1$ (Right).

5. Summary. In this work we derive tempered fractional order compartment models from the stochastic process of particles undergoing a continuous time random walk with tempered power-law waiting time distribution. Because of the finite lifespan of biological particles or the finite observation time of the experimentalist, the more natural choice for the distribution of the waiting time is tempered power-law instead of power-law. Tempered fractional order evolution equations are obtained by considering tempered power-law distribution for the time that a particle remains in the compartment. The presence of tempered fractional derivatives makes the analysis of the equilibrium behavior more complicated. Another highlight of the
Figure 7. The variation of Infected $I(t)$ in the tfrSEIR model for $\beta = 2.69e-3, 2.79e-3, 2.89e-3$.

Figure 8. The variation of Infected $I(t)$ in the tfrSEIR model for $\lambda_2 = 0.2, 0.05, 0.001$ (Left) and $\lambda_3 = 0.09, 0.07, 0.05$ (Right).

Figure 9. Comparing with the real data of confirmed cases of pandemic AH1N1/09 influenza from Bogotá D.C. (Colombia), the numerical solutions of the classical SEIR model (Left) and the fr-SEIR model (Right) with $\alpha_2 = 0.6, \alpha_3 = 0.88$.

work is the construction of the tempered fractional SEIR epidemic model, which can be used to simulate the real data of confirmed cases of pandemic AH1N1/09 influenza from Bogotá D.C. (Colombia).

Appendix A. Finite difference methods for the tfrSEIR model. First, we recall the definitions of the Riemann-Liouville tempered fractional derivative and
Let \( T > 0 \) be given arbitrarily, and set \( h = \frac{T}{N}, \) \( t_n = nh, n = 0, 1, 2 \ldots, N \in \mathbb{Z}^+ \).

The tempered fractional integral \( _0D_t^{\alpha,\lambda} f(t) \) can be approximated by

\[
_0D_t^{\alpha,\lambda} f(t_n) = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (t_n - \tau)^{\alpha-1} e^{\lambda\tau} f(t_j) d\tau
\]

(3.5)

where \( b_{j,n,\alpha} = (n - j)^\alpha - (n - 1 - j)^\alpha, j = 0, \ldots, n - 1 \).

If the integral \( \int_{t_j}^{t_{j+1}} (t_n - \tau)^{\alpha-1} \frac{d}{d\tau} (e^{\lambda\tau} f(t)) d\tau \) is approximated by \( \int_{t_j}^{t_{j+1}} (t_n - \tau)^{\alpha-1} e^{\lambda\tau} f(t_{j+1}) - e^{\lambda\tau} f(t_j) d\tau \),
then by (A.2) $0D_t^{1-\alpha,\lambda} f(t) - \lambda_0 D_t^{-\alpha,\lambda} f(t)$ can be discretized as

$$
\begin{align*}
0D_t^{1-\alpha,\lambda} f(t_n) - \lambda_0 D_t^{-\alpha,\lambda} f(t_n) &= e^{-\lambda_n} \frac{n-1}{\Gamma(\alpha)} \sum_{j=0}^{t_{j+1}} (t_n - \tau)^{\alpha-1} e^{\lambda_j \tau} f(t_j) - e^{\lambda_1 f(t_j)} d\tau \\
&+ f(0) e^{-\lambda_n t_n^{-1+\alpha}} \frac{n-1}{\Gamma(\alpha)} \sum_{j=0}^{t_{j+1}} (t_n - \tau)^{\alpha-1} e^{\lambda_j f(t_j)} d\tau \\
&= e^{-\lambda_n} \frac{n-1}{\Gamma(\alpha+1)} \sum_{j=0}^{t_{j+1}} b_{j,n,\alpha} (e^{\lambda_j f(t_j+1) - e^{\lambda_j f(t_j)})} \\
&+ f(0) e^{-\lambda_n t_n^{-1+\alpha}} \frac{n-1}{\Gamma(\alpha+1)} \sum_{j=0}^{t_{j+1}} b_{j,n,\alpha} e^{\lambda_j f(t_j)}.
\end{align*}
\tag{A.4}
$$

Let $S(0), I(0) \in \mathbb{R}_0^+, E(0) = R(0) = 0$. Then by using (A.3)-(A.4), the scheme of the tfrSEIR system (3.1)-(3.2), (3.4) and (3.6) is given by

$$
\begin{align*}
S(t_{n+1}) &= h\mu - (h\mu_1 - 1)S(t_n) - h\beta S(t_n)I(t_n), \quad \tag{A.5}
E(t_{n+1}) &= h\beta S(t_n)I(t_n) - (h\mu_1 - 1)E(t_n) - hq_3^+ (t_n), \quad \tag{A.6}
I(t_{n+1}) &= hq_3^+ (t_n) - (h\mu_1 - 1)I(t_n) - hq_3^+ (t_n), \quad \tag{A.7}
R(t_{n+1}) &= hq_3^+ (t_n) - (h\mu_1 - 1)R(t_n), \quad \tag{A.8}
\end{align*}
$$

where

$$
\begin{align*}
q_3^+(t_n) &= \Omega_2^{\alpha_2} \exp(-\mu_1 t_n) \frac{1}{\Gamma(\alpha_2 + 1)} \exp((\mu_1 + \lambda_2) t_n)E(t_n)) \\
&- \lambda_2 \Omega_2^{\alpha_2} \exp(-\mu_1 t_n) \frac{1}{\Gamma(\alpha_2 + 1)} \exp((\mu_1 + \lambda_2) t_n)E(t_n)) \\
&+ \lambda_2 \Omega_2^{\alpha_2} \exp(-\mu_1 t_n) \frac{1}{\Gamma(\alpha_2 + 1)} \exp((\mu_1 + \lambda_2) t_n)S(t_n)I(t_n)) \\
&= \frac{\Omega_2^{\alpha_2} h^{\alpha_2 - 1}}{\Gamma(\alpha_2 + 1)} \exp(-(\lambda_2 + \mu_1) t_n) \\
&\times \sum_{j=0}^{n-1} b_{j,n,\alpha_2} (\exp((\mu_1 + \lambda_2) t_{j+1})E(t_{j+1}) - \exp((\mu_1 + \lambda_2) t_j)E(t_j)) \\
&- \lambda_2 \Omega_2^{\alpha_2} \exp(-\lambda_2 + \mu_1) t_n) \frac{1}{\Gamma(\alpha_2 + 1)} \sum_{j=0}^{n-1} b_{j,n,\alpha_2} \exp((\lambda_2 + \mu_1) t_j)E(t_j) \\
&+ \lambda_2 \Omega_2^{\alpha_2} \frac{1}{\Gamma(\alpha_2 + 1)} \exp(-\lambda_2 + \mu_1) t_n) \sum_{j=0}^{n-1} b_{j,n,\alpha_2} \beta \exp((\lambda_2 + \mu_1) t_j)S(t_j)I(t_j),
\end{align*}
\tag{A.9}
$$

$$
\begin{align*}
q_3^+(t_n) &= \Omega_3^{\alpha_3} \exp(-\mu_1 t_n) \frac{1}{\Gamma(\alpha_3 + 1)} \exp((\mu_1 + \lambda_3) t_n)I(t_n)) \\
&- \lambda_3 \Omega_3^{\alpha_3} \exp(-\mu_1 t_n) \frac{1}{\Gamma(\alpha_3 + 1)} \exp((\mu_1 + \lambda_3) t_n)I(t_n)) \\
&+ \lambda_3 \exp(-\mu_1 t_n) \frac{1}{\Gamma(\alpha_3 + 1)} (I_0 \delta(t_n) + \exp(\mu_1 t_n)q_3^+(t_n))
\end{align*}
$$
\[ \frac{\Omega_3^{\alpha_3} h^{\alpha_3-1}}{\Gamma(\alpha_3 + 1)} \exp(- (\lambda_3 + \mu_1) t_n) \]
\[ \times \sum_{j=0}^{n-1} b_{j,n,\alpha_3} (\exp((\mu_1 + \lambda_3) t_{j+1}) I(t_{j+1}) - \exp((\mu_1 + \lambda_3) t_j) I(t_j)) \]
\[ + \frac{I(0) \Omega_3^{\alpha_3} \exp(- (\lambda_3 + \mu_1) t_n)}{\Gamma(\alpha_3)} t_n^{-1+\alpha_3} - \frac{\lambda_3 \Omega_3^{\alpha_3} \exp(- (\lambda_3 + \mu_1) t_n)}{\Gamma(\alpha_3 + 1)} h^{\alpha_3} \]
\[ \times \sum_{j=0}^{n-1} b_{j,n,\alpha_3} \exp((\lambda_3 + \mu_1) t_j) I(t_j) \]
\[ + \frac{\lambda_3^{\alpha_3} I^{\alpha_3}}{\Gamma(\alpha_3 + 1)} \exp(- (\lambda_3 + \mu_1) t_n) \]
\[ \times \sum_{j=0}^{n-1} b_{j,n,\alpha_3} \left( \exp((\lambda_3 + \mu_1) t_j) q_3^+(t_j) + \exp(\lambda_3 t_j) \exp\left( -\frac{t_j^2}{\alpha^2} \right) \frac{I_0}{\alpha \sqrt{\pi}} \right), \]  
(A.12)

where \( a > 0 \) is sufficiently small.

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