Initial states and decoherence of histories

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Abstract

We study decoherence properties of arbitrarily long histories constructed from a fixed projective partition of a finite dimensional Hilbert space. We show that decoherence of such histories for all initial states that are naturally induced by the projective partition implies decoherence for arbitrary initial states. In addition we generalize the simple necessary decoherence condition [Scherer et al., Phys. Lett. A (2004)] for such histories to the case of arbitrary coarse-graining.

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In the Copenhagen interpretation of quantum mechanics all properties of a quantum system are defined with respect to measurements performed by an external observer using classical measuring devices. This interpretation, however, cannot be used in the case of closed quantum systems, such as the Universe as a whole. In this case any observer must be a part of the system itself. A self-contained description of closed quantum systems that does not rely on either the external observer nor on the existence of classical devices is provided by the decoherent histories approach [2, 3, 4, 5, 6]. This approach predicts probabilities for quantum histories, i.e. ordered sequences of quantum-mechanical “propositions”. Mathematically, these propositions are represented by projectors: the same projectors that would define a quantum measurement in the Copenhagen approach. In particular, an exhaustive set of mutually exclusive propositions corresponds to a complete set of mutually orthogonal projectors.

Due to quantum interference, one cannot always assign probabilities to a set of histories in a consistent way. For this to be possible, the set of histories must be decoherent. Whether the corresponding decoherence condition is fulfilled or not depends on the initial state, the unitary dynamics of the system and the propositions from which the histories are constructed. In this paper we consider histories that are constructed from a fixed exhaustive set of mutually exclusive propositions, \( \{ P_\mu \} \), and investigate the question of how the choice of the initial state affects decoherence of such histories. We show that decoherence of arbitrarily long histories for all initial states that are induced by the projectors \( \{ P_\mu \} \) via normalization implies the decoherence for arbitrary initial states. It is relevant to note that, unlike the set of all possible states, the set \( \{ P_\mu \} \) is discrete and may contain as few as just two elements (for “yes-no” propositions). As an additional result, we obtain a generalization of the simple necessary decoherence condition that was derived for fine-grained histories in [1]. The new condition is applicable to arbitrary coarse-grainings.

The paper is organized as follows. After introducing our framework we present the mathematical content of our results in the form of a theorem. We prove the theorem, infer the results and conclude with a short summary.

**Definition 1:** A set of projectors \( \{ P_\mu \} \) on a Hilbert space \( \mathcal{H} \) is called a projective partition of \( \mathcal{H} \), if \( \forall \mu, \mu' : P_\mu P_{\mu'} = \delta_{\mu \mu'} P_\mu \) and \( \sum_\mu P_\mu = 1_{\mathcal{H}} \). Here, \( 1_{\mathcal{H}} \) denotes the unit operator. We call a projective partition fine-grained if all projectors are one-dimensional, i.e., \( \forall \mu \dim(\text{supp}(P_\mu)) = 1 \), and coarse-grained otherwise.
Definition 2: Given a projective partition \( \{ P_\mu \} \) of a Hilbert space \( \mathcal{H} \), we denote by \( \mathcal{K}[\{ P_\mu \}; k] := \{ h_\alpha \mid h_\alpha = (P_{\alpha_1}, P_{\alpha_2}, \ldots, P_{\alpha_k}) \in \{ P_\mu \}^k \} \) the corresponding exhaustive set of mutually exclusive histories of length \( k \). Histories are thus defined to be ordered sequences of projection operators, corresponding to quantum-mechanical propositions. Note that we restrict ourselves to histories constructed from a fixed projective partition: the projectors \( P_{\alpha_j} \) within the sequences are all chosen from the same partition for all “times” \( j = 1, \ldots, k \).

Definition 3: Given a Hilbert space \( \mathcal{H} \) and a projective partition \( \{ P_\mu \} \) of \( \mathcal{H} \), we denote by \( \mathcal{S} \) the set of all density operators on \( \mathcal{H} \) and by \( \mathcal{S}_{\{ P_\mu \}} \) the discrete set of “partition states” induced by the partition \( \{ P_\mu \} \) via normalization:

\[
\mathcal{S}_{\{ P_\mu \}} := \left\{ \frac{P_\nu}{\text{Tr}[P_\nu]} : P_\nu \in \{ P_\mu \} \right\}.
\]  

An initial state \( \rho \in \mathcal{S} \) and a unitary dynamics generated by a unitary map \( U : \mathcal{H} \to \mathcal{H} \) induce a probabilistic structure on the event algebra associated with \( \mathcal{K}[\{ P_\mu \}; k] \), if certain consistency conditions are fulfilled. These are given in terms of properties of the decoherence functional \( \mathcal{D}_{U, \rho} \) on \( \mathcal{K}[\{ P_\mu \}; k] \times \mathcal{K}[\{ P_\mu \}; k] \), defined by

\[
\mathcal{D}_{U, \rho} [h_\alpha, h_\beta] := \text{Tr} \left[ C_\alpha \rho C_\beta^\dagger \right],
\]

where

\[
C_\alpha := \left( U^\dagger P_{\alpha_k} U \right) \left( U^\dagger P_{\alpha_{k-1}} U \right) \ldots \left( U^\dagger P_{\alpha_1} U \right) = U^\dagger P_{\alpha_k} U P_{\alpha_{k-1}} U \ldots P_{\alpha_2} U P_{\alpha_1} U.
\]

The set \( \mathcal{K}[\{ P_\mu \}; k] \) is said to be decoherent or consistent with respect to a given unitary map \( U : \mathcal{H} \to \mathcal{H} \) and a given initial state \( \rho \in \mathcal{S} \), if

\[
\mathcal{D}_{U, \rho} [h_\alpha, h_\beta] \propto \delta_{\alpha\beta} \equiv \prod_{j=1}^{k} \delta_{\alpha_j \beta_j}
\]

for all \( h_\alpha, h_\beta \in \mathcal{K}[\{ P_\mu \}; k] \). These are the consistency conditions. If they are fulfilled, probabilities may be assigned to the histories and are given by the diagonal elements of the decoherence functional,

\[ p[h_\alpha] = \mathcal{D}_{U, \rho} [h_\alpha, h_\alpha]. \]

What we have just described is a slightly simplified version of the general decoherent histories formalism. In general, both the partition and the unitary may depend on the
parameter \( j = 1, \ldots, k \). Our setting based on a fixed partition and a fixed unitary is motivated by the analogy with the classical symbolic dynamics [1, 8]. In the literature several consistency conditions of different strength can be found [7]. The conditions given above are known as medium decoherence [6, 9].

**Theorem:** Let a projective partition \( \{ P_\mu \} \) of a finite dimensional Hilbert space \( \mathcal{H} \) and a unitary map \( U \) on \( \mathcal{H} \) be given. Then the following three statements are equivalent:

(a) \( \forall \rho \in \mathcal{S}(P_\mu) \ \forall k \in \mathbb{N} \ \forall h_\alpha, h_\beta \in \mathcal{K}[\{ P_\mu \} ; k] : \mathcal{D}_{U,\rho}[h_\alpha, h_\beta] \alpha \delta_{\alpha \beta} \)

(b) \( \forall P_{\mu'}', P_{\mu''} \in \{ P_\mu \} \ \forall n \in \mathbb{N} : [U^n P_{\mu'}'(U^\dagger)^n, P_{\mu''}] = 0 \)

(c) \( \forall \rho \in \mathcal{S} \ \forall k \in \mathbb{N} \ \forall h_\alpha, h_\beta \in \mathcal{K}[\{ P_\mu \} ; k] : \mathcal{D}_{U,\rho}[h_\alpha, h_\beta] \alpha \delta_{\alpha \beta} . \)

**Proof:** We will prove the theorem by showing that (a) implies (b), (b) implies (c), and (c) implies (a). The last implication, (c) \( \Rightarrow \) (a), is trivial, and the second implication, (b) \( \Rightarrow \) (c), can be easily shown using the notation of Eq. (3). It remains to prove the implication (a) \( \Rightarrow \) (b).

The proof is constructed as follows. We first show that the proposition

\[
\forall \rho \in \mathcal{S}(P_\mu) \ \forall n \in \mathbb{N} \ \forall \mu_0, \mu', \mu'' \ \text{with} \ \mu' \neq \mu'' : \\
\text{Tr} \left[ P_{\mu''}(U^n P_{\mu_0} U^\dagger^n) P_{\mu'}(U^n \rho U^\dagger^n) P_{\mu''} \right] = 0
\]

(5)

is a necessary consequence of the decoherence condition (a) and then conclude that this proposition implies the commutativity condition (b) of the theorem.

The first part of the proof will be accomplished by contradiction, i.e. we will assume that (5) is not satisfied, and then show that this assumption contradicts the decoherence condition (a) of the theorem.

Assume condition (5) is not satisfied. This means there exist a partition state \( \tilde{\rho} \in \mathcal{S}(P_\mu) \), an integer \( \tilde{n} \in \mathbb{N} \), and partition-element labels \( \mu_0, \mu', \mu'' \), with \( \mu' \neq \mu'' \), such that

\[
\text{Tr} \left[ P_{\mu''}(U^{\tilde{n}} P_{\mu_0} U^\dagger^{\tilde{n}}) P_{\mu'}(U^{\tilde{n}} \tilde{\rho} U^\dagger^{\tilde{n}}) P_{\mu''} \right] = c \neq 0 .
\]

(6)

This, as we will see, is in contradiction to decoherence condition (a). Written out, the decoherence condition (a) is

\[
\text{Tr} \left[ P_{\alpha_k} U P_{\alpha_{k-1}} U \ldots P_{\alpha_1} U \rho_0 U^\dagger P_{\beta_1} \ldots P_{\beta_{k-1}} U^\dagger P_{\beta_k} \right] \alpha \prod_{j=1}^{k} \delta_{\alpha_j \beta_j}
\]

(7)
for all $k \in \mathbb{N}$, all initial states $\rho_0 \in S_{(P_n)}$, and arbitrary histories $h_\alpha, h_\beta$. Since the length $k$ of the histories is arbitrary, we may choose $k = q\tilde{n}$ with arbitrary $q \in \mathbb{N}$. By summing over $\alpha_1, \ldots, \alpha_{\tilde{n}-1}, \alpha_{\tilde{n}+1}, \ldots, \alpha q\tilde{n}-1$ and $\beta_1, \ldots, \beta_{\tilde{n}-1}, \beta_{\tilde{n}+1}, \ldots, \beta q\tilde{n}-1$, and using $\sum_\mu P_\mu = \mathbb{1}_H$, we obtain

$$\text{Tr} \left[ P_{\alpha q} (U^{q-1})^{n} P_{\alpha n} U^{\tilde{n}} \rho_0 U^{\dagger \tilde{n}} P_{\beta n} (U^{\dagger q-1})^{n} P_{\beta q} \right] \propto \delta_{\alpha q, \beta q} \delta_{\alpha n, \beta n}$$

(8)

for all $q \in \mathbb{N}$, any $\rho_0 \in S_{(P_n)}$, and arbitrary $\alpha, \beta, \alpha q, \beta q$. In order to proceed we will need the following Lemma.

**Lemma:** Let $\mathcal{H}$ be a finite dimensional Hilbert space and $U : \mathcal{H} \to \mathcal{H}$ any unitary map on $\mathcal{H}$. Then $\forall \epsilon > 0 \exists q \in \mathbb{N}$ such that $\| U^q - \mathbb{1}_H \| < \epsilon$, $\| \cdot \|$ meaning the conventional operator norm, which is $\| A \| := \sup \{ \| Av \| : v \in \mathcal{H}, \| v \| = 1 \}$ for an operator $A$ on $\mathcal{H}$.

According to this Lemma, for any given arbitrarily small $\epsilon > 0$ we can always find a $q \in \mathbb{N}$ such that $U^q = \mathbb{1}_H + \hat{O}(\epsilon)$, where $\hat{O}(\epsilon)$ is some operator whose norm is of order $\epsilon$: $\| \hat{O}(\epsilon) \| < \epsilon$. Using the submultiplicativity property of operator norms, we have

$$\| U^{-1} \hat{O}(\epsilon) \| \leq \| U^{-1} \| \times \| \hat{O}(\epsilon) \| = \| \hat{O}(\epsilon) \|$$

(9)

and hence $U^{q-1} = U^{-1} + \hat{O}'(\epsilon)$, where $\| \hat{O}'(\epsilon) \| < \epsilon$.

Now we are in a position to derive a contradiction. We let our histories start with the initial state $\rho_0 = \tilde{\rho}$. Furthermore we choose $\alpha q = \mu', \beta q = \mu''$, and $\alpha q = \beta q = \mu_0$. Since $\mu' \neq \mu''$, condition (8) becomes

$$\forall q \in \mathbb{N} : \text{Tr} \left[ P_{\mu_0} (U^{q-1})^{n} P_{\mu} U^{\tilde{n}} \tilde{\rho} U^{\dagger \tilde{n}} P_{\mu'} (U^{\dagger q-1})^{n} P_{\mu_0} \right] = 0 .$$

(10)

Choosing $q$ such that $\| U^q - \mathbb{1}_H \| < \epsilon$ for a given arbitrarily small $\epsilon > 0$, we get a situation where the expressions $(U^{q-1})^{n}$ and $(U^{\dagger q-1})^{n}$ in Eq. (11) can be replaced by $(U^{\dagger} + \hat{O}'(\epsilon))^{\tilde{n}}$ and $(U + \hat{O}'(\epsilon))^{\tilde{n}}$, respectively. In the following it will be convenient to use the definition

$$A_{r_1, r_2, \ldots, r_{\tilde{n}}} := \prod_{i=1}^{\tilde{n}} \left( U^{\dagger r_i} (\hat{O}'(\epsilon))^{1-r_i} \right)$$

(11)

where the operators inside the product are written out from left to right in the order of increasing index $i$. Using this definition we have:

$$(U^{\dagger} + \hat{O}'(\epsilon))^{\tilde{n}} = \sum_{r_1, \ldots, r_{\tilde{n}} \in \{0,1\}} A_{r_1, \ldots, r_{\tilde{n}}} .$$

(12)
This yields for the left hand side of Eq. (10):\[
\text{Tr} \left[ P_{\mu_0} (U^{q-1}) \tilde{\mu} P_{\mu'} U^{\tilde{\mu}} \tilde{\mu} U^{\dagger \tilde{\mu}} P_{\mu''} (U^{q-1}) \tilde{\mu} P_{\mu_0} \right] =\]
\[
\text{Tr} \left[ P_{\mu_0} \left( \sum_{r_1, \ldots, r_{\tilde{n}} \in \{0,1\}} A_{r_1, \ldots, r_{\tilde{n}}} \right) P_{\mu'} U^{\tilde{\mu}} \tilde{\mu} U^{\dagger \tilde{\mu}} P_{\mu''} \left( \sum_{s_1, \ldots, s_{\tilde{n}} \in \{0,1\}} A^{\dagger}_{s_1, \ldots, s_{\tilde{n}}} \right) P_{\mu_0} \right]
\]
\[
= \sum_{r_1, \ldots, r_{\tilde{n}} \in \{0,1\}} \sum_{s_1, \ldots, s_{\tilde{n}} \in \{0,1\}} \text{Tr} \left[ P_{\mu_0} A_{r_1, \ldots, r_{\tilde{n}}} P_{\mu'} U^{\tilde{\mu}} \tilde{\mu} U^{\dagger \tilde{\mu}} P_{\mu''} A^{\dagger}_{s_1, \ldots, s_{\tilde{n}}} P_{\mu_0} \right]. \tag{13}
\]

According to (10) the left hand side of this equation must be zero. Hence we have:
\[
\text{Tr} \left[ P_{\mu_0} (U^{\dagger})^{\tilde{\mu}} P_{\mu'} U^{\tilde{\mu}} \tilde{\mu} U^{\dagger \tilde{\mu}} P_{\mu''} U^{\tilde{\mu}} P_{\mu_0} \right]
\]
\[
= - \sum_{r_1, \ldots, r_{\tilde{n}} \in \{0,1\}} \sum_{s_1, \ldots, s_{\tilde{n}} \in \{0,1\}} \text{Tr} \left[ P_{\mu_0} A_{r_1, \ldots, r_{\tilde{n}}} P_{\mu'} U^{\tilde{\mu}} \tilde{\mu} U^{\dagger \tilde{\mu}} P_{\mu''} A^{\dagger}_{s_1, \ldots, s_{\tilde{n}}} P_{\mu_0} \right]. \tag{14}
\]

Using the cyclic permutation-invariance property of the trace and the triangle inequality, we obtain
\[
|\text{Tr} \left[ P_{\mu''} (U^{\tilde{\mu}} P_{\mu_0} U^{\dagger \tilde{\mu}}) P_{\mu'} (U^{\tilde{\mu}} \tilde{\mu} U^{\dagger \tilde{\mu}}) P_{\mu''} \right]| \leq \sum_{r_1, \ldots, r_{\tilde{n}} \in \{0,1\}} \sum_{s_1, \ldots, s_{\tilde{n}} \in \{0,1\}} |\text{Tr} \left[ A^{\dagger}_{s_1, \ldots, s_{\tilde{n}}} P_{\mu_0} A_{r_1, \ldots, r_{\tilde{n}}} P_{\mu'} U^{\tilde{\mu}} \tilde{\mu} U^{\dagger \tilde{\mu}} P_{\mu''} \right]|. \tag{15}
\]

Utilizing the inequality $|\text{Tr}[BT]| \leq \| B \| \| \text{Tr}\sqrt{T^\dagger T} \|$ for bounded operators $B : \mathcal{H} \rightarrow \mathcal{H}$ and operators $T : \mathcal{H} \rightarrow \mathcal{H}$ with finite trace norm $\| T \|_1 := \text{Tr}\sqrt{T^\dagger T}$, see Ref. [11], we deduce from Eq. (15):
\[
|\text{Tr} \left[ P_{\mu''} (U^{\tilde{\mu}} P_{\mu_0} U^{\dagger \tilde{\mu}}) P_{\mu'} (U^{\tilde{\mu}} \tilde{\mu} U^{\dagger \tilde{\mu}}) P_{\mu''} \right]| \leq \sum_{r_1, \ldots, r_{\tilde{n}} \in \{0,1\}} \sum_{s_1, \ldots, s_{\tilde{n}} \in \{0,1\}} \| B^{s_1, \ldots, s_{\tilde{n}}} \| \| \text{Tr}\sqrt{T^\dagger T} \|, \tag{16}
\]

where we defined
\[
B^{s_1, \ldots, s_{\tilde{n}}} := A^{\dagger}_{s_1, \ldots, s_{\tilde{n}}} P_{\mu_0} A_{r_1, \ldots, r_{\tilde{n}}}, \tag{17}
\]
\[
T := P_{\mu'} U^{\tilde{\mu}} \tilde{\mu} U^{\dagger \tilde{\mu}} P_{\mu''}. \tag{18}
\]

Using the fact that $\| B^\dagger \| = \| B \|$ for any bounded operator $B$ and it’s adjoint $B^\dagger$ [12], we have $\| \hat{O}^\dagger (\epsilon) \| = \| \hat{O}' (\epsilon) \| < \epsilon$. Utilizing the submultiplicativity property of operator norms
we deduce that the norms of the operators $B_{r_1, \ldots, r_\tilde{n}}^{s_1, \ldots, \tilde{s}_n}$ are all bounded from above by $\epsilon$, except in the case where all $s_1, \ldots, s_\tilde{n}$ and all $r_1, \ldots, r_\tilde{n}$ are equal 1, which is excluded from the sum on the right-hand side of Eq. (16). Indeed we have:

$$
\| B_{r_1, \ldots, r_\tilde{n}}^{s_1, \ldots, \tilde{s}_n} \| \leq \left( \prod_{i=1}^{\tilde{n}} \| U \|^{s_i} \| \mathcal{O}^\dagger (\epsilon) \|^{1-s_i} \right) \| P_{\mu_0} \| \left( \prod_{i=1}^{\tilde{n}} \| U^\dagger \|^{r_i} \| \mathcal{O}' (\epsilon) \|^{1-r_i} \right)
$$

$$
\leq \left( \prod_{i=1}^{\tilde{n}} \epsilon^{1-s_i} \right) \left( \prod_{j=1}^{\tilde{n}} \epsilon^{1-r_j} \right)
$$

$$
\leq \epsilon^2 < \epsilon , \quad \text{if} \quad s_1 + \cdots + s_\tilde{n} < \tilde{n} , \quad r_1 + \cdots + r_\tilde{n} < \tilde{n} ,
$$

where we used $\| P_{\mu_0} \| = \| U \| = \| U^\dagger \| = 1$ and $\epsilon \ll 1$. With the definition $M := \text{Tr} \sqrt{T^\dagger T}$ we finally conclude from Eq. (16):

$$
| \text{Tr} \left[ P_{\mu''} (U^n P_{\mu_0} U^\dagger \tilde{n}) P_{\mu'} (U^n \tilde{\rho} U^\dagger \tilde{n}) P_{\mu''} \right] | < 2^{2\tilde{n}} M \epsilon .
$$

(20)

Since $c_n$ and $M$ are fixed constants, we can always arrange $2^{2\tilde{n}} M \epsilon < |c|$ by choosing a sufficiently small $\epsilon > 0$. This contradicts the assumption (6) and thus proves our proposition (5).

We are now in a position to derive the commutativity condition (b) of the theorem. It is a straightforward consequence of proposition (5) we have just proven. Taking condition (5) and choosing in it the state $\rho \in S_{[P_n]}$ to be proportional to the projector sandwiched between $U^n$ and $U^\dagger n$ within the first bracket,

$$
\rho = \frac{P_{\mu_0}}{\text{Tr} [P_{\mu_0}]},
$$

(21)

where $P_{\mu_0}$ is still arbitrary, we necessarily get the condition

$$
\forall n \in \mathbb{N} \forall \mu_0, \mu', \mu'' \quad \text{with} \quad \mu' \neq \mu'' : \quad \text{Tr} \left[ P_{\mu''} (U^n P_{\mu_0} U^\dagger n) P_{\mu'} (U^n P_{\mu_0} U^\dagger n) P_{\mu''} \right] = 0 .
$$

(22)

With the definition $A := P_{\mu'} (U^n P_{\mu_0} U^\dagger n) P_{\mu''}$ Eq. (22) becomes $\text{Tr} [A^\dagger A] = 0$. Since $A^\dagger A$ is a positive operator, this is possible if and only if $A = 0$. Hence condition (22) is equivalent to

$$
\forall n \in \mathbb{N} \forall \mu_0, \mu', \mu'' \quad \text{with} \quad \mu' \neq \mu'' : \quad P_{\mu'} (U^n P_{\mu_0} U^\dagger n) P_{\mu''} = 0 .
$$

(23)
This condition implies
\[ \sum_{\mu'} P_{\mu'} (U^n P_{\mu_0} U^+)^n P_{\mu'} = P_{\mu''} (U^n P_{\mu_0} U^+)^n P_{\mu''} \] (24)
for any \( \mu_0 \) and \( \mu'' \), and arbitrary \( n \in \mathbb{N} \). But since \( \sum_{\mu'} P_{\mu'} = 1_{\mathcal{H}} \), the left hand side of the last equation must be equal to \( (UP_{\mu_0}U^+)P_{\mu''} \). Hence we obtain
\[ P_{\mu''} (U^n P_{\mu_0} U^+)^n P_{\mu''} = (U^n P_{\mu_0} U^+)^n P_{\mu''} \] (25)
on the one hand and by taking the adjoint of Eq. (25)
\[ P_{\mu''} (U^n P_{\mu_0} U^+)^n P_{\mu''} = (U^n P_{\mu_0} U^+)^n P_{\mu''} \] (26)on the other hand, for any \( n \in \mathbb{N} \) and arbitrary \( \mu_0 \) and \( \mu'' \). Therefore
\[ (U^n P_{\mu_0} U^+)^n P_{\mu''} = (U^n P_{\mu_0} U^+)^n P_{\mu''} \] (27)
for any \( n \in \mathbb{N} \) and arbitrary \( \mu_0, \mu'' \), and so \([U^n P_{\mu_0} U^+^n, P_{\mu''}] = 0 \) for any \( n \in \mathbb{N} \) and all \( P_{\mu_0}, P_{\mu''} \in \{P_{\mu}\} \). □

The implication (a)⇒(c) of the theorem constitutes the main result of this paper: the decoherence of histories of arbitrary length for all initial states from the set \( \mathcal{S}_{\{P_{\mu}\}} \) implies decoherence of such histories for arbitrary initial states \( \rho \in \mathcal{S} \). It should be mentioned that the set \( \mathcal{S}_{\{P_{\mu}\}} \) can be viewed as the smallest natural set of states that is associated with our framework. It is discrete and may consist of just two elements (in the case of “yes-no” propositions). The set \( \mathcal{S} \), on the other hand, contains the continuum of all possible states that are allowed in our framework.

In [1] the notion of classical states with respect to a partition \( \{P_{\mu}\} \) was introduced:

**Definition 4:** A state represented by the density operator \( \rho \) is called classical with respect to (w.r.t.) a partition \( \{P_{\mu}\} \) of the Hilbert space \( \mathcal{H} \), if it is block-diagonal w.r.t. \( \{P_{\mu}\} \), i.e., if \( \rho = \sum_{\mu} P_{\mu} \rho P_{\mu} \). We denote by \( \mathcal{S}_{\{P_{\mu}\}}^{cl} \) the set of all density operators that are classical w.r.t. \( \{P_{\mu}\} \).

In [1] it was shown that in the case of fine-grained partitions sets of histories of arbitrary length decohere for all classical initial states only if the unitary dynamics preserves the classicality of states, i.e. only if
\[ \forall \rho \in \mathcal{S}_{\{P_{\mu}\}}^{cl}, \quad U \rho U^+ \in \mathcal{S}_{\{P_{\mu}\}}^{cl}. \] (28)
It is a single-iteration criterion: to verify that it holds for a particular unitary map $U$, only a single iteration of the map has to be taken into account, which can be much easier than establishing decoherence directly by computing the off-diagonal elements of the decoherence functional. This is especially useful for studying chaotic quantum maps, for which typically only the first iteration is known in a closed analytical form [10]. Unfortunately, condition (28) fails to be necessary in the coarse-grained case. The following simple corollary of our theorem provides a necessary single-iteration condition that applies to arbitrary coarse-grainings and is equivalent to (28) in the fine-grained case.

**Corollary:** Let a projective partition $\{P_\mu\}$ of a finite dimensional Hilbert space $\mathcal{H}$ and a unitary map $U$ on $\mathcal{H}$ be given. The medium decoherence condition is then satisfied for all classical initial states and arbitrarily long histories, i.e.,

$$\forall \rho \in S_{\{P_\mu\}}^{cl} \forall k \in \mathbb{N} \forall h_\alpha, h_\beta \in \mathcal{K}[\{P_\mu\}; k] : D_{U,\rho}[h_\alpha, h_\beta] \propto \delta_{\alpha\beta},$$

(29)

only if the following necessary condition is fulfilled:

$$\forall P_{\mu'}, P_{\mu''} \in \{P_\mu\} : [UP_{\mu'}U^\dagger, P_{\mu''}] = 0.$$

(30)

**Proof:** follows trivially from the implication $(a) \Rightarrow (b)$ of the theorem, as $S_{\{P_\mu\}} \subset S_{\{P_\mu\}}^{cl}$. □

In summary, we investigated decoherence properties of sets of quantum histories constructed from a fixed projective partition $\{P_\mu\}$ of a finite dimensional Hilbert space. We found that if decoherence is established for arbitrary history lengths and all initial states from $S_{\{P_\mu\}}$, which is the smallest natural set induced by $\{P_\mu\}$, then any set of histories constructed from $\{P_\mu\}$ is decoherent for all possible initial states. In addition, we provided a necessary single-iteration criterion for decoherence of arbitrarily long histories that generalizes the condition of [1] to the case of arbitrary coarse-grainings.

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