Dynamics of Intermediate Measurements in Quantum Systems

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We study the dynamics of a quantum system in which an intermediate property $m$ is measured in between initial and final measurements of two different non-commuting properties $a$ and $b$. Since this intermediate measurement must involve an interaction, we use this case to explore in more detail the dynamics of measurement and propose that the physics of this change is described by a unitary transformation parametrised by a random parameter, with the selection of $m$ described by a superposition of these unitaries. We prove a set of conditions which must hold in order for this superposition to remain unitary and then argue that in the case where $U$ is a matrix, it must be a random unitary matrix. We outline a numerical example with matrices and make some predictions based on the conditions we have found which are capable of experimental verification. We finish with a brief discussion of the complex phases which appear in the relations between non-commuting properties and their relevance for the dynamics associated with measurement.

I. INTRODUCTION

The study of measurements of quantum systems has a long history going back to the foundations of the subject [1]. In this article, we consider measurements of a set of three non-commuting physical properties of a quantum system. We examine the details of measurement of an intermediate property and attempt to obtain some information about the dynamics of measurement as interaction (dynamics being taken to mean time evolution under the exponential of some Hamiltonian). The concept of measurement is understood in the usual sense of quantum mechanics and does not imply anything about an observer who is considering the system. The study is a simple theoretical investigation, but it makes predictions which can be verified experimentally and might point towards a more general notion of quantum entanglement.

II. MEASUREMENT-INTERACTION DYNAMICS

We begin by considering the case of $m$ and $(a, b)$, where $m$ represents a third property measured during an intermediate measurement between the pair of properties $a$ and $b$ measured in the initial and final measurements, respectively. The measurement of $m$ interrupts the causality of $a \rightarrow b$ via the dynamics generated by $m$. In general, quantum mechanics would seem to require that the quantum reality of $m$ is conditioned upon a change in the laws of motion which are taking the system from measurement of $a$ at the initial time to measurement of $b$ at the final time and so it would be desirable if the physics of this change could be included in the usual formalism within those same laws of motion.

One possibility would be to describe the physics of this change with a unitary transformation parametrised by a random parameter. The back-action effects due to a measurement of $m$ in a system with a meter can be viewed as random unitary dynamics, hence the representation of the interaction via a random unitary transformation. In order for this to be possible, we would require that the selection of the intermediate value $m$ be described by a non-classical superposition of these random unitary operators (in analogy with quantum superpositions of states). We must establish the mathematical conditions under which a superposition of unitary operators is again unitary. We expect that unit normalisation by itself is no longer sufficient as we are now dealing with a unital algebra over a field rather than a simple normed space.

Proposition: A superposition of $n$ unitary operators is unitary given a number of conditions which is equal to the $(n - 1)$-th central polygonal number. These conditions are unit normalisation, plus the fact that each of the terms in each of the following sums is antihermitian:

$$\Sigma_{i=2}^{n} c_1 \bar{c}_1 U_1^i U_1 + \Sigma_{i=3}^{n} c_2 \bar{c}_2 U_2^i U_1 + \ldots + c_{n-1} \bar{c}_{n-1} U_{n-1}^* U_n. \quad (1)$$

Proof. The proof is a simple induction argument. For the base case, take $n = 2$. We could also consider one unitary operator to be a superposition of one operator where the coefficient is 1. The coefficients are complex numbers and for unitarity it is sufficient to check that $U^* U = I$, since $UU^* = U^* U$ in this case. For $n = 2$ we obtain:

$$(c_1 U_1 + c_2 U_2)^*(c_1 U_1 + c_2 U_2) = (|c_1|^2 + |c_2|^2) I + \bar{c}_1 c_2 U_1^* U_2 + c_1 \bar{c}_2 U_1 U_2^*. \quad (2)$$

The sum of the two operators is unitary if $|c_1|^2 + |c_2|^2 = 1$, and if $c_1 \bar{c}_2 U_1^* U_2$ is antihermitian. The number of conditions required when $n = 2$ is 2, which is the first central polygonal number.
We now consider the case where \( n = j \) and multiply out the brackets.

\[
(c_1 U_1 + c_2 U_2 + \ldots + c_{j-1} U_{j-1} + c_j U_j)^* (c_1 U_1 + c_2 U_2 + \ldots + c_{j-1} U_{j-1} + c_j U_j) = \\
\left( \sum_{j=1}^n |c_j|^2 \right) 1 + \bar{c}_1 c_2 U_1^* U_2 + \ldots + \bar{c}_1 c_{j-1} U_1^* U_{j-1} + \\
+ \bar{c}_1 c_j U_1^* U_j + \ldots + \bar{c}_{j-1} c_j U_{j-1}^* U_1 + \bar{c}_{j-1} c_j U_{j-1}^* U_2 + \\
+ \ldots + \bar{c}_{j-1} c_j U_{j-1}^* U_j + \bar{c}_j c_1 U_1 U_j + \bar{c}_j c_2 U_1 U_2 + \\
+ \ldots + \bar{c}_j c_{j-1} U_1 U_{j-1}. \quad (3)
\]

If we discard the \( \bar{c}_1 c_1 U_1^* U_1 \) term which contributes to the normalisation condition, the number of remaining terms containing \( c_1 \) will be \( 2j - 2 \). Pair these terms as with the base case to get \( j - 1 \) conditions: these are every term in the sum \( \sum_{i=2}^n c_1 U_1^* U_i \) is antihermitian.

We continue with the remaining terms which contain \( c_2 \). Discard the \( \bar{c}_2 c_2 U_2^* U_2 \) term and neglect the \( c_2 \) terms which were already multiplied with \( c_1 \): there were 2 of these so we must have \( 2j - 4 \) terms containing \( c_2 \). Pair these to get a further set of \( j - 2 \) conditions: namely, each term in the sum \( \sum_{i=3}^n \bar{c}_2 c_2 U_2^* U_i \) is antihermitian. Continue in this way until we reach the remaining terms which contain \( c_{j-1} \). Discard the \( \bar{c}_{j-1} c_{j-1} U_{j-1}^* U_{j-1} \) term and neglect the \( c_{j-1} \) terms which were multiplied with \( c_1, \ldots, c_{j-1} \): there were \( 2j - 2 \) of these, so we have \( 2j - (2j - 2) = 2 \) remaining terms containing \( c_{j-1} \). Pair these to get one final condition: \( c_{j-1} \bar{c}_j U_{j-1}^* U_j \) is antihermitian.

We end up with a descending sequence of sums:

\[
\sum_{i=2}^n c_1 U_1^* U_i + \sum_{i=3}^n \bar{c}_1 c_2 U_2^* U_i + \ldots + c_{n-1} \bar{c}_n U_{n-1}^* U_n. \quad (4)
\]

One can see from the previous argument that the number of terms in this sequence is \( \frac{(n-1)n}{2} \). Add 1 for the normalisation condition to get \( \frac{(n-1)n}{2} + 1 \): this is the \( (n-1) \)-th central polygonal number. The fact that the argument holds for \( n = j \) implies that it holds for \( n = j + 1 \), since adding another operator \( U_{j+1} \) to the superposition simply means there are now another two terms containing \( c_1 \), another two terms containing \( c_2 \) and so on, up to a new remaining condition that \( c_n \bar{c}_{n+1} U_{n+1}^* U_{n+1} \) is antihermitian. Every term in the new sequence of sums

\[
\sum_{i=2}^{n+1} c_1 U_1^* U_i + \sum_{i=3}^{n+1} \bar{c}_1 c_2 U_2^* U_i + \ldots + c_n \bar{c}_{n+1} U_{n+1}^* U_{n+1}, \quad (5)
\]

is antihermitian. The number of terms in this sequence is again \( \frac{(n-1)n}{2} \), and so we have the \( (n-1) \)-th central polygonal number once more after adding the normalization condition. This completes the proof. \( \square \)

The fact that the number of necessary conditions for unitarity increases as the number of operators increases makes sense physically, as we are viewing the random operator which selects the value for \( m \) as something which somehow exists simultaneously in a coherent superposition of possible randomizations and adding in more randomizations must increase the opportunities for interference between them which must be accounted for. This idea of simultaneous superposition is similar to the famous Schroedinger’s cat example, where in some philosophical sense an animal can be in a superposition of states such that it is alive and dead at the same time. Of course, a superposition of states is another state, whereas a superposition of unitaries is not another unitary unless the above conditions are satisfied [2].

The form of the interference terms seems reasonable, since there is a product term in the set of conditions for the interaction between every unitary operator with every other unitary. The product terms are also skew-Hermitian and so must have trivial zero eigenvalues or pure imaginary eigenvalues which cannot be measured physically. However, there has been recent research on the imaginary part of weak values linking it to unitary disturbance of an initial state by an observable in the context of weak measurements and so it might be possible to make a similar interpretation here [3].

The link with central polygonal numbers is perhaps not clear physically. Since the central polygonal numbers correspond geometrically to the maximum number of regions which can be constructed inside a circle given intersection with \( n \) straight lines, there might be a phase space interpretation. A circular phase space area for a two-dimensional phase space could be enclosed by states and divided up by a superposition of unitaries acting on each state such that the maximum number of divisions is bounded as we have indicated by the polygonal numbers.

Now that we have conditions for a superposition of unitaries, we can propose a theory for how measurement-interaction dynamics might work. A property \( a \) is measured. An intermediate measurement is made of \( m \) which generates its own random unitary dynamics. The selection of a value for \( m \) is described by a coherent superposition of random unitary transformations. In the process of measurement, the superposition reduces to one of the random unitaries which then transforms \( b \) and effectively interrupts the measurement of \( b \) which had already been determined by the causality of \( a \rightarrow b \). We will attempt to develop this theory with a more concrete example using random matrices.

III. EXAMPLE WITH RANDOM UNITARY MATRICES

Random unitary matrices are used in quantum mechanics where an explicit expression for \( U \) is impossible, but we are suggesting here that they could be used to model the randomness of interaction dynamics during a measurement. Our claim is that the physics of the change due to a measurement is described by a unitary transformation
where $\alpha$ is some arbitrary linear operator corresponding to an observable and $U$ is a matrix or linear operator depending on some random parameter. If $U$ is a matrix, one can fill each entry with the same random parameter such that the entries of the matrix are all independent Gaussian random variables (assuming that the values of that random parameter are being sampled from a Gaussian distribution). The Gaussian random variable assumption is necessary, since we would like to come to conclusions about the density of the eigenvalues of the unitaries, and this density will only be uniform for the Gaussian elements.

This means that we can take a random unitary matrix to define our unitary transformation with a random parameter (provided that we keep the Gaussian assumption). As a simple numerical example, we generate three $3 \times 3$ random unitary matrices and label them:

$$U_1 = \begin{pmatrix} -0.25 + 0.71i & -0.22 + 0.10i & -0.59 - 0.17i \\ -0.04 - 0.13i & -0.90 - 0.22i & 0.047 + 0.36i \\ -0.35 + 0.55i & 0.17 - 0.23i & 0.57 + 0.41i \end{pmatrix},$$

$$U_2 = \begin{pmatrix} 0.37 + 0.62i & -0.46 - 0.26i & 0.11 + 0.42i \\ -0.16 - 0.45i & -0.62 - 0.58i & 0.00 - 0.22i \\ -0.32 + 0.36i & 0.02 - 0.05i & 0.75 - 0.45i \end{pmatrix},$$

$$U_3 = \begin{pmatrix} -0.11 + 0.23i & 0.58 - 0.21i & -0.37 + 0.65i \\ -0.52 + 0.78i & -0.32 - 0.21i & -0.12 - 0.12i \\ -0.07 - 0.34i & -0.68 - 0.12i & -0.48 + 0.41i \end{pmatrix}. $$

In the limit of many repeated measurements, the histogram for the density of the eigenvalues of the unitary matrices involved in the transformation of $b$ should match the theoretical curve for the circular unitary ensemble, given the random nature of the unitary transformations [4]. Theoretical predictions for the eigenvalue density and spacing distribution of the eigenvalues of random unitary matrices are well-documented in the literature [5]. At this point we will adopt the shorthand of denoting the unitary transformation of an observable $b$ by $U(b)$. In terms of experimental verification, we are assuming that we can extrapolate back from the measurement to the unitary transformation which transformed $b$, but this is possible if, for example, the unitary transformation takes the state $|b\rangle$ to an orthogonal state which represents a different experimental outcome [6].

We earlier established a set of conditions for the superposition representing selection of $m$. In our particular case and given that the selection of $m$ is described by a weighted sum of $U_1, U_2$ and $U_3$, we have that the following products are all antihermitian:

$$c_1 c_2 U_1^* U_2, \ c_1 c_3 U_1^* U_3, \ c_2 c_3 U_2^* U_3. \quad (8)$$

To continue with our numerical example, these conditions would be that the following are antihermitian:

One notices immediately that the matrices in the expressions above are not skew-Hermitian, since the entries on the main diagonals are not pure imaginary. Given that the matrices we have found are skew-Hermitian, it follows that the only way to maintain the antihermitian condition is by solving for the complex coefficients $c_i$. We already have a unit normalisation condition and if we interpret this as giving the probabilities for the corresponding unitary operators in the superposition for the selection of $m$ in analogy with the normal coefficient rule for states, we have the conclusion that we could determine the probability of each random unitary operator in the superposition using the associated antihermitian conditions.

Furthermore, since the unitary operators form a superposition, one might speculate on the existence of some kind of interference effect between them during measurement of $m$ which corresponds to the antihermitian conditions we stated earlier. The product unitary matrices (or rather, the matrices multiplied by product complex coefficients) have pure imaginary eigenvalues, but the eigenvalue statistics of an antihermitian matrix are the same as those for a Hermitian matrix up to multiplication by a factor of $i$. Given that the product matrices are both Hermitian and unitary, they must have eigenvalues which are $\pm 1$ (assuming as usual that the unitary matrix has been diagonalized in $U(N)$). In terms of eigenvalues, this gives us $e^{i\theta} = \pm 1$. In the first case, $\theta = n\pi$ for $n = 0$ or $n$ even and in the second $\theta = n\pi$ for odd $n$. This allows us to speculate that in the limit of repeated measurements, the empirical histogram for the density of the eigenvalues of the unitary matrices which transform $m$ during the intermediate measurement matches that of the circular unitary ensemble, but vanishes whenever $\theta \neq n\pi$ for any integer $n$, as these values are forbidden by the conditions we have found. Again, this prediction is capable of experimental verification, since one can work back from the measurement to the unitary transformation and the eigenvalues of the associated unitary operator.

IV. COMPLEX PHASES OF RELATIONS BETWEEN PROPERTIES

One reason for trying to understand the transformation dynamics involved in measurement is that they might shed light on the complex correlations which relate the property $m$ to the pair $(a, b)$. Elsewhere in the literature, it is argued that the fundamental relations between different physical properties of a system can be expressed in terms of complex conditional probabilities [7]. In fact, the argument that complex conditional probabilities describe the relations between three different properties actually originates with Dirac [8]. If we consider the conditional probability for the measurement outcome of the intermediate property $p(m|a, b)$ dependent on the

$$\alpha' = U\alpha U^*, \quad (6)$$
outcomes of the other two properties, we obtain relations which can be written in terms of inner products as:

\[ p(m|a, b) = \frac{\langle b|m \rangle \langle m|a \rangle}{\langle b|a \rangle}. \]  \hfill (9)

Complex phases appear in the relation \( p(m|a, b) \) and describe the action needed to transport the property \( a \) to \( b \) along \( m \) [6]. The exact relation between these complex phases and entanglement is still an open question and it is also unknown why they exhibit interference patterns which originate from the transformation dynamics we have been considering. In the case of the complex conditional probability \( p(x_t|x_0, p_0) \) where the intermediate property is the position of a particle at time \( t \), the rate of change of the complex phase increases as \( x_t \) is shifted away from the solution predicted by classical mechanics [9].

This is similar to the path integral formulation of quantum mechanics, where a small change to a path in phase space for a classical macroscopic system leads to a large change in the phase of \( S/\hbar \) (because \( \Delta S \) is large compared to \( \hbar \)). These phase changes oscillate and cancel out such that the only remaining trajectory is the one for which the phase of \( S/\hbar \) is stationary with respect to infinitesimal variations [10]. In much the same way, if the position of a particle is viewed with classical resolution, the complex phases cause the conditional probabilities to oscillate and cancel out such that the classical trajectory is recovered [9].

It is in fact possible to think of the relations \( p(m|a, b) \) as being actions themselves, with classical causality being recovered if \( p(m|a, b) \) is modelled by a Dirac delta function at the extremum of the action. If we think of the relations as actions in this way with the associated complex phases, we must bear in mind that the properties \( a, b \) and \( m \) only represent effects that are observed as a result of measurement-interaction dynamics. If those effects are not observed, we cannot identify the set of properties with elements of physical reality inside the system [11]. The next step would be to obtain more information about the role which the complex phases play in generation of entanglement and so identify the difference between the classical approximation of causality via stationary action and a notion of causality based on interference patterns associated with measurement and state preparation. Given that the relations are actions, it might be profitable to study the phases of the relations with the same methods which are used in the path integral formulation.

\[ \text{V. CONCLUSIONS} \]

In this article we considered the dynamics associated with an intermediate measurement between the measurement of two different initial and final properties in a system and suggested that it might be possible to represent the change involved in this measurement with a superposition of unitary transformations parametrised by a random parameter. However, we established that there is a set of mathematical conditions which a superposition of such transformations has to satisfy in order to produce another unitary transformation. We suggested a possible physical interpretation for these conditions and speculated on some consequences of this interpretation which might be capable of experimental verification. These predictions rely on being able to view the unitary operator which defines the transformation as a unitary matrix, but in theory this can always be done in some sense, since the existence of a unitary operator implies the existence of a unitary matrix which is the representative matrix of the same unitary transformation with respect to an orthonormal basis for the Hilbert space of the system. In our case, we argued that the unitary transformation with a random parameter can be defined in terms of a unitary matrix whose entries were filled by that parameter. The assumption that we can use unitary matrices is helpful, since there is a body of results for eigenvalues of random unitary matrices which can be compared with experiment.

In general, more explorations of the dynamics of measurements are needed, since they might lead towards a full explanation of the role of entanglement in the description of quantum systems. This is due to the fact that entanglement plays a necessary part in all interactions. It is because of this that we cannot interpret quantum properties as being realities of a system in a simple way. This in turn might help us to solve the long-standing open problem of the correct interpretation of quantum mechanics. Quantum mechanics cannot describe the effects of a system without positing entanglement of the system with the environment, hence that correct interpretation might elude us until we have the correct notion of entanglement.

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