On a stronger form of hereditary compactness in product spaces

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Abstract
The aim of this paper is to continue the study of sg-compact spaces. The class of sg-compact spaces is a proper subclass of the class of hereditarily compact spaces. In our paper we shall consider sg-compactness in product spaces. Our main result says that if a product space is sg-compact, then either all factor spaces are finite, or exactly one factor space is infinite and sg-compact and the remaining ones are finite and locally indiscrete.

1 Introduction
If a topological space \((X, \tau)\) is hereditarily compact, then under some additional assumptions either \(X\) or \(\tau\) might become finite (or countable). For example, if \((X, \tau)\) is a second countable hereditarily compact space, then \(\tau\) is finite. Hence, if \((X, \tau)\) is a second countable hereditarily compact \(T_0\)-space, then \(X\) must be countable. Moreover, it is well-known that every maximally hereditarily compact space and every hereditarily compact Hausdorff (even \(k\sigma\)-) space is finite. For more information about hereditarily compact spaces we refer the reader to A.H. Stone’s paper [15].

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In 1995 and in 1996, a stronger form of hereditary compactness was introduced independently in three different papers. Caldas [3], Devi, Balachandran and Maki [3] and Tapi, Thakur and Sonwalkar [17] considered topological spaces in which every cover by sg-open sets has a finite subcover. These spaces have been called \textit{sg-compact} and were further studied by the present authors in [7].

As the property sg-compactness is much stronger than hereditary compactness (for even spaces with finite topologies need not be sg-compact), the general behavior of sg-compactness becomes more ‘unusual’ than the one of hereditarily compact spaces. This will be especially the case in product spaces.

It is well-known that the finite product of hereditarily compact spaces is hereditarily compact, and that if a product space is hereditarily compact, then every factor space is hereditarily compact. What we want to show here is the following: If the product space of an arbitrary family of spaces is sg-compact, then all but one factor spaces must be finite and the remaining one must be (at most) sg-compact. Maki, Balachandran and Devi [14, Theorem 3.7] showed (under the additional assumption that the product space satisfies the weak separation axiom \(T_{gs}\)) that if the product of two spaces is sg-compact, then every factor space is sg-compact. Tapi, Thakur and Sonwalkar [17, Theorem 2.7] stated the result for two spaces but their proof is wrong as they claimed that the projection mapping is sg-irresolute. They used a wrong lemma from [16] saying that the product of sg-closed sets is sg-closed (we will show that this is not true even for two sets).

We recall some definitions. A set \(A\) is called \textit{semi-open} if \(A \subseteq \text{cl}(\text{int}(A))\) and \textit{semi-closed} if \(\text{int}(\text{cl}(A)) \subseteq A\). The \textit{semi-interior} (resp. \textit{semi-kernel}) of \(A\), denoted by \(\text{sint}(A)\) (resp. \(\text{sker}(A)\)), is the union (resp. intersection) of all semi-open subsets (resp. supersets) of \(A\). The \textit{semi-closure} of \(A\), denoted by \(\text{scl}(A)\), is the intersection of all semi-closed supersets of \(A\). A subset \(A\) of a topological space \((X, \tau)\) is called \textit{sg-open} [2] (resp. \textit{g-open} [12]) if every semi-closed (resp. closed) subset of \(A\) is included in the semi-interior (resp. interior) of \(A\). A topological space \((X, \tau)\) is called \textit{sg-compact} [3, 6, 17] (resp. \textit{go-compact} [1]) if every cover of \(X\) by sg-open (resp. g-open) sets has a finite subcover.

Complements of sg-open sets are called \textit{sg-closed}. Alternatively, a subset \(A\) of a topological space \((X, \tau)\) is called sg-closed if \(\text{scl}(A) \subseteq \text{sker}(A)\). If every subset of \(A\) is also sg-closed
in \((X, \tau)\), then \(A\) is called *hereditarily sg-closed* (= hsg-closed) \([7]\). Every nowhere dense subset is hsg-closed but not conversely.

Janković and Reilly \([11, \text{Lemma } 2]\) pointed out that in an arbitrary topological space every singleton is either nowhere dense or locally dense. Recall that a set \(A\) is said to be *locally dense* \([5]\) (= preopen) if \(A \subseteq \text{int}(\text{cl}(A))\). We will make significant use of their result throughout this paper.

**Lemma 1.1** For a topological space \((X, \tau)\) the following conditions are equivalent:

(i) \(X\) is locally indiscrete.

(ii) Every singleton is locally dense.

(ii) Every subset is sg-open.

**Lemma 1.2**

(i) Every open continuous surjective function is pre-semi-open, i.e., it preserves semi-open sets.

(ii) Let \((X_i)_{i \in I}\) be a family of spaces and \(\emptyset \neq A_i \subseteq X_i\) for each \(i \in I\). Then, \(\prod_{i \in I} A_i\) is preopen (resp. semi-open) in \(\prod_{i \in I} X_i\) if and only if \(A_i\) is preopen (resp. semi-open) in \(X_i\) for each \(i \in I\) and \(A_i\) is non-dense (resp. \(A_i \neq X_i\)) for only finitely many \(i \in I\).

(iii) If \(f: (X, \tau) \to (Y, \sigma)\) is open and continuous, then the preimage of every nowhere dense subset of \(Y\) is nowhere dense in \(X\), i.e., \(f\) is \(\delta\)-open.

**Lemma 1.3** \([7, \text{Theorem } 2.6]\) For a topological space \((X, \tau)\) the following conditions are equivalent:

(1) \(X\) is sg-compact.

(2) \(X\) is a \(C_3\)-space, i.e., every hsg-closed set is finite.

**Lemma 1.4** \([7, \text{Proposition } 2.1]\) For a subset \(A\) of a topological space \((X, \tau)\) the following conditions are equivalent:

(1) \(A\) is hsg-closed.

(2) \(N(X) \cap \text{int}(\text{cl}(A)) = \emptyset\), where \(N(X)\) denotes the set of nowhere dense singletons in \(X\).
2 Sg-compactness in product spaces

We will start with an example showing that Theorem 2.1 of [17] is not true. There, the authors stated (without proof) that every sg-compact space is go-compact (it is our guess that they assumed that g-open sets are sg-open).

Example 2.1 Let \( \mathbb{N} \) be set of all positive integers. We consider the following topology \( \tau \) on \( \mathbb{N} \) given by \( \tau = \{ \emptyset, \mathbb{N} \} \cup \{ U_n = \{ n, n + 1, n + 2, \ldots \} : n \geq 3 \} \).

We first show that \((\mathbb{N}, \tau)\) is sg-compact. Observe that every singleton of \((\mathbb{N}, \tau)\) is nowhere dense. Since every nonempty semi-open set has finite complement, \((\mathbb{N}, \tau)\) is semi-compact. By [7, Remark 2.7 (i)], \((\mathbb{N}, \tau)\) is sg-compact.

However, every singleton of \((\mathbb{N}, \tau)\) is g-open, and so \((\mathbb{N}, \tau)\) fails to be go-compact.

Lemma 2.2 Let \( X = \prod_{i \in I} X_i \) be a product space. If infinitely many \( X_i \) are not indiscrete, then \( X \) contains an infinite nowhere dense subset.

Proof. Let \( J \) be an infinite subset of \( I \) such that \( X_i \) is not indiscrete for each \( i \in J \). We may choose \( J \) in such a way that \( I \setminus J \) is also infinite. Then, for each \( i \in J \), there exists a closed set \( A_i \subseteq X_i \) distinct from the empty set and from \( X_i \). Now form the product of all \( A_i, i \in J \), and of all \( X_i, i \notin J \), and call it \( A \). Then \( A \) is closed in \( X \), infinite and clearly nowhere dense. \( \square \)

As a consequence of Lemma 2.2 we therefore have:

Corollary 2.3 If a product space \( X = \prod_{i \in I} X_i \) is sg-compact, then only finitely many \( X_i \) are not indiscrete. \( \square \)

Theorem 2.4 Let \((X_i, \tau_i)_{i \in I}\) be a family of topological spaces. If the product space \( X = \prod_{i \in I} X_i \) is sg-compact, then either all factor spaces are finite or exactly one of them is infinite and sg-compact and the rest are finite and locally indiscrete.
Proof. Suppose that two factor spaces, say \( X_i \) and \( X_j \), are infinite. Let \( p_i \) denotes the projection from \( X \) onto \( X_i \) for any \( i \in I \). Let \( k \in I \). If \( x_k \in X_k \), then \( p_k^{-1}(\{x_k\}) \) is infinite, hence cannot be nowhere dense since \( X \) is sg-compact. Thus \( \{x_k\} \) is not nowhere dense in \( X_k \). Consequently, each factor space \( X_k \) must be locally indiscrete. By Corollary 2.3 and Lemma 1.2, each singleton in \( X \) is locally dense and so every subset of \( X \) is sg-open. Since \( X \) is sg-compact, \( X \) must be finite, a contradiction. Hence, at most one factor space can be infinite.

Now suppose that \( X_j \) is infinite and that \( X_i \) is finite for \( i \neq j \). For each \( x_i \in X_i \), where \( i \neq j \), \( p_i^{-1}(\{x_i\}) \) is infinite, therefore \( \{x_i\} \) cannot be nowhere dense in \( X_i \). So \( X_i \) is locally indiscrete for \( i \neq j \). By Corollary 2.3 and Lemma 1.2 it follows that for each \( x \in X \), \( \{x\} \) is nowhere dense in \( X \) if and only if \( \{x_j\} \) is nowhere dense in \( X_j \).

Assume now that \( X_j \) is not sg-compact. Then \( X_j \) contains an infinite hsg-closed subset, say \( A_j \). Let \( A = p_j^{-1}(A_j) \). We want to show that \( N(X) \cap \text{int}(\text{cl}(A)) = \emptyset \), where \( N(X) \) denotes the set of nowhere dense singletons in \( X \). If there exists a point \( x \in N(X) \cap \text{int}(\text{cl}(A)) \), then \( x \) has an open neighbourhood \( W \) contained in \( \text{cl}(A) \). Also, \( \{x_j\} \) is nowhere dense in \( X_j \) and \( x_j \in p_j(W) \subseteq p_j(\text{cl}(A)) \subseteq \text{cl}(A_j) \). So \( x_j \in \text{int}(\text{cl}(A_j)) \), a contradiction to the hsg-closedness of \( A_j \). Hence, by Lemma 1.4, \( A \) is hsg-closed and infinite, a contradiction. Therefore, \( X_j \) is sg-compact. \( \square \)

Tapi, Thakur and Sonwalkar [17, Theorem 2.7] stated our result for two topological spaces but their proof is wrong as they claimed the projection mapping being sg-irresolute. They used the wrong lemma from [16] that the product of sg-closed sets is sg-closed. The following example will correct their claims.

Example 2.5 Let \( X = \{a, b, c\} \) and let \( \tau = \{\emptyset, \{a, b\}, X\} \). Set \( A = \{b, c\} \).

(i) First observe that \( A \) is sg-closed in \( (X, \tau) \) but \( A \times A \) is not sg-closed in \( X \times X \), since \( A \times A \subseteq X \times X \setminus \{(a, c)\} \) and \( \text{scl}(A \times A) = X \times X \).

(ii) If \( p \) is the projection mapping from \( X \times X \) onto \( X \), then \( p^{-1}(A) \) is not sg-closed in \( X \times X \), i.e., the projection map need not be always sg-irresolute.

(iii) We already noted that if \( f : (X, \tau) \to (Y, \sigma) \) is open and continuous, then the preimage of every nowhere dense subset of \( Y \) is nowhere dense in \( X \). There is no similar result for
hsg-closed sets. If \( \sigma \) denotes the indiscrete topology on \( X \), then \( S = \{a, b\} \) is hsg-closed in \((X, \sigma)\) but \( q^{-1}(S) \) is not hsg-closed in \((X, \sigma) \times (X, \tau)\), where \( q \) denotes the projection mapping from \((X, \sigma) \times (X, \tau)\) onto \((X, \sigma)\).

The following result shows when the inverse image of a hsg-closed set is also hsg-closed. Recall that a function \( f: (X, \tau) \to (Y, \sigma) \) is called \textit{almost open} if the image of every regular open set is open. We say that \( f: (X, \tau) \to (Y, \sigma) \) is \textit{anti-\( \delta \)-open} if the image of every nowhere dense singleton is nowhere dense. Observe that if \( Y \) is dense-in-itself and \( T_D \) (= singletons are locally dense), then \( f: (X, \tau) \to (Y, \sigma) \) is always anti-\( \delta \)-open; in particular every real-valued function is anti-\( \delta \)-open.

**Proposition 2.6** If \( f: (X, \tau) \to (Y, \sigma) \) is an almost open, continuous, anti-\( \delta \)-open surjection, then the inverse image of every hsg-closed set is hsg-closed.

**Proof.** Let \( B \) be hsg-closed in \( Y \) and set \( A = f^{-1}(B) \). If for some nowhere dense singleton \( x \) of \( X \) we have \( x \in \text{int}(\text{cl}(A)) \), then \( f(x) \in f(\text{int}(\text{cl}(A))) \subseteq \text{int}(f(\text{cl}(A))) \subseteq \text{int}(\text{cl}(f(A))) = \text{int}(\text{cl}(B)) \). Since \( f(x) \) is nowhere dense in \( Y \), \( B \) is not hsg-closed. By contradiction, \( A \) is hsg-closed. \( \square \)

**Remark 2.7** (i) Let \( A \) be an infinite set with \( p \notin A \). Let \( X = A \cup \{p\} \) and \( \tau = \{\emptyset, A, X\} \). We observed in [7] that \( X \times X \) contains an infinite nowhere dense subset, so even the finite product of sg-compact spaces need not be sg-compact.

(ii) It is rather unexpected that the projection map fails to be sg-irresolute in general, since it is always irresolute and gs-irresolute.

The two examples of infinite sg-compact spaces in [7] and the infinite sg-compact space from Example 2.1 are not even weakly Hausdorff (however one of them is \( T_1 \)). As every hereditarily compact kc-space must be finite, it is natural to ask whether there are any infinite sg-compact semi-Hausdorff spaces (there do exist infinite hereditarily compact semi-Hausdorff spaces). Recall here that a topological space \((X, \tau)\) is called \textit{semi-Hausdorff} [13] if every two distinct points of \( X \) can be separated by disjoint semi-open sets.
Recall additionally that a space \((X, \tau)\) is called \textit{hyperconnected} if every open subset of \(X\) is dense, or equivalently, every pair of nonempty open sets has nonempty intersection. In the opposite case \(X\) is called \textit{hyperdisconnected}. If every infinite open subspace of \(X\) is hyperdisconnected, then we will say that \(X\) is \textit{quasi-hyperdisconnected}. Note that not only Hausdorff spaces but also semi-Hausdorff spaces are quasi-hyperdisconnected (but not vice versa).

**Proposition 2.8** Every quasi-hyperdisconnected sg-compact space \((X, \tau)\) is finite.

**Proof.** Assume that \(X\) is infinite. Let \(U\) and \(V\) be disjoint non-empty open subsets of \(X\). Note that either \(X \setminus U\) or \(X \setminus V\) is infinite. Assume that \(X \setminus U\) is infinite. Since \(\text{cl}(U) \setminus U\) is hsg-closed (in fact even nowhere dense), by Lemma [3], \(\text{cl}(U) \setminus U\) is finite and hence \(X \setminus \text{cl}(U)\) is infinite and open. Set \(A_1 = U\). Since \(X\) is quasi-hyperdisconnected, proceeding as above, we can construct an open subset of \(X \setminus \text{cl}(U)\) and hence of \(X\), say \(U_2\), such that the complement of the closure of \(U_2\) in \(X \setminus \text{cl}(A_1)\) is infinite. Using the method above, we can construct an infinite pairwise disjoint family \(A_1, A_2, \ldots\) of non-empty open subsets of \((X, \tau)\). Since sg-compact spaces are semi-compact and thus satisfy the finite chain condition, \(X\) must be finite. ✷

**Corollary 2.9** Every sg-compact, semi-Hausdorff space is finite.

We have just seen that under some very low separation axioms, sg-compact spaces very easily become finite. If we replace the weak separation axiom with a weaker form of strong irresolvability, we again have finiteness. By definition, a nonempty topological space \((X, \tau)\) is called \textit{resolvable} [10] if \(X\) is the disjoint union of two dense (or equivalently codense) subsets. In the opposite case \(X\) is called \textit{irresolvable}. A topological space \((X, \tau)\) is \textit{strongly irresolvable} [8] if no nonempty open set is resolvable.

**Proposition 2.10** Every sg-compact space \((X, \tau)\) which is the topological sum of a locally indiscrete space and a strongly irresolvable space is finite.
Proof. We will use a result in [3] which states that a space is finite if and only if every cover by \(\beta\)-open sets (i.e., sets which are dense in some regular closed subspace) has a finite subcover. If \(U\) is a cover of \(X\) by \(\beta\)-open sets, then by [3, Theorem 2.1] every element of \(U\) is sg-open. Since \(X\) is sg-compact, \(U\) has a finite subcover. This shows that \(X\) is finite. \(\square\)

We already mentioned in Remark 2.7 that the product of two sg-compact spaces need not be sg-compact. Thus we have the natural question: When is the product of two sg-compact spaces also sg-compact? What turns out is that only in one very special case the product of a sg-compact space with another sg-compact space is also sg-compact. First we note a result whose proof is easy and hence omitted.

**Proposition 2.11** Let \((X_\alpha, \tau_\alpha)_{\alpha \in \Omega}\) be a family of topological spaces. For the topological sum \(X = \sum_{\alpha \in \Omega} X_\alpha\) the following conditions are equivalent:

1. \(X\) is a sg-compact space.
2. Each \(X_\alpha\) is a sg-compact space and \(|\Omega| < \aleph_0\).

**Lemma 2.12** Let \((X, \tau)\) be any space and let \((Y, \sigma)\) be indiscrete. Let \(A \subseteq X \times Y\) and let \(p : X \times Y \to X\) denote the projection. Then \(\text{int}(\text{cl}(A)) = \text{int}(\text{cl}(p(A))) \times Y\).

**Proof.** If \((x, y) \in \text{int}(\text{cl}(A))\), there exists an open neighbourhood \(U_x\) of \(x\) such that \(U_x \times Y \subseteq \text{cl}(A)\). Then \(x \in p(U_x) \subseteq \text{cl}(p(A))\) and so \((x, y) \in \text{int}(\text{cl}(p(A))) \times Y\).

Now, let \(x \in \text{int}(\text{cl}(p(A)))\) and \(y \in Y\). Choose an open set \(U_x \subseteq X\) containing \(x\) such that \(U_x \subseteq \text{cl}(p(A))\). We claim that \(U_x \times Y \subseteq \text{cl}(A)\). Suppose there is a point \((x', y') \in U_x \times Y\) not in \(\text{cl}(A)\). Then there exists an open set \(W_{x'} \subseteq U_x\) containing \(x'\) such that \((W_{x'} \times Y) \cap A = \emptyset\). Consequently, \(W_{x'} \cap p(A) = \emptyset\), a contradiction. Hence, \((x, y) \in \text{int}(\text{cl}(A))\). \(\square\)

**Theorem 2.13** If \((X, \tau)\) is sg-compact and \((Y, \sigma)\) is finite and locally indiscrete, then \(X \times Y\) is sg-compact.
Proof. Since $Y$ is a finite topological sum of indiscrete spaces, by Proposition [2.11] it suffices to assume that $Y$ is indiscrete. Suppose that $A \subseteq X \times Y$ is infinite and hsg-closed. Then, $p(A)$ is infinite and hence, by Lemma [1.3] and Lemma [1.4], we have $N(X) \cap \text{int}(\text{cl}(p(A))) \neq \emptyset$. Pick $x \in N(X) \cap \text{int}(\text{cl}(p(A)))$ and $y \in Y$. Then, $\{(x, y)\}$ is nowhere dense in $X \times Y$ and, by Lemma [2.12], we have $(x, y) \in \text{int}(\text{cl}(A))$, a contradiction to the hsg-closedness of $A$. Thus $X \times Y$ is sg-compact. $\square$

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