FACTORIZATION OF SUPERTROPICAL MATRICES

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Abstract. In contrast to the situation in classical linear algebra, we establish that not every nonsingular matrix can be factored into a product of elementary matrices. We prove the factorizability of any nonsingular $2 \times 2$ matrix and classify the factorizability of $3 \times 3$ matrices. Nevertheless, the quasi-inverse of the quasi-inverse is always factorizable.

1. Introduction

The work in [2] shows that even though the semiring of matrices over the supertropical semiring lacks negation, it satisfies many of the classical matrix theory properties when using the ghost ideal $G$ in supertropical semifields, by defining the ghost surpasses relation $a \models_{gs} b$, which means $a = b$, or $a \in G$ and $a^\nu > b^\nu$. We say $a$ is $\nu$-equivalent to $b$, denoted by $a \trianglelefteq_\nu b$, if $a^\nu = b^\nu$.

We establish some fundamental definitions for our work.

Definition 1.1. Let $R^n$ be the free module of rank $n$ over the supertropical semiring $R = T \cup G \cup \{\infty\}$ where $\forall a \in T$, $a^\nu \in G$,

as defined in [2]. We define the standard base of $R^n$ to be $e_1, \ldots, e_n$ where

$$e_i = \begin{cases} 1_R, & \text{in the } i^{th} \text{ coordinate} \\ 0_R, & \text{otherwise} \end{cases}$$

Looking at the theorem of supertropical determinants (defined as in [3] to be the usual permanent) which satisfies $|AB| \models_{gs} |A||B|$ ([1, Theorem 3.5]), one might wonder why the product of two nonsingular matrices maybe singular. In this work we attempt to understand the reason by investigating the elementary matrices as the "generators" of matrices, in analogy to the well known classical fact that over a field $R$, $GL_n(R)$ is generated by elementary matrices. This situation is subtler for matrix semirings over a supertropical semifield. Whereas every $2 \times 2$ nonsingular matrix is factorizable, this fails for $3 \times 3$ matrices. However, we salvage a positive result in Corollary 6.6 by passing to $\text{adj}(\text{adj}(A))$.

Definition 1.2. The identity matrix in the supertropical matrix semiring is the $n \times n$ matrix with the standard base for its columns. We denote this matrix as $I_R$ or $I$.

Definition 1.3. We define a matrix $A \in M_n(R)$ to be tropically singular if $|A| \in G$, strictly singular if $|A| = 0_R$, and nonsingular if $|A| \in T$.

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Definition 1.4. A matrix $A \in M_n(R)$ is **invertible** if there exists a matrix $B \in M_n(R)$ such that

$$AB = BA = I.$$ 

Definition 1.5. Corresponding to the three elementary row matrix operations, we define respectively three types of tropical **elementary matrices** obtained by applying one such operation to the identity matrix. We denote these matrices as follows:

$$E_{i,j} = (a_{t,l}), \text{ where } a_{t,l} = \begin{cases} 1_R, & \text{where } t = l \neq i, j \\ 1_R, & \text{where } i = t \neq l = j \text{ or } j = t \neq l = i \\ 0_R, & \text{otherwise} \end{cases}$$

which means switching the $i^{th}$ and $j^{th}$ rows.

$$E_{k \cdot (i^{th} \text{ row})} = (b_{t,l}), \text{ where } b_{t,l} = \begin{cases} 1_R, & \text{where } t = l \neq i \\ k, & \text{where } t = l = i \\ 0_R, & \text{where } t \neq l \end{cases}$$

which means multiplying the $i^{th}$ row by an invertible $k \in T$.

$$E_{i+k \cdot (j^{th} \text{ row})} = (c_{t,l}), \text{ where } c_{t,l} = \begin{cases} 1_R, & \text{where } t = l \\ k, & \text{where } i = t \neq l = j \\ 0_R, & \text{otherwise} \end{cases}$$

which means adding the $j^{th}$ row, multiplied by $k$, to the $i^{th}$ row, where $k \in T$. (We can define these matrices for $k \in R \setminus \{0_R\}$, but since applying $E_{i+k \cdot (j^{th} \text{ row})}$ for some $k \in G$, would be the same as applying $E_{i+a \cdot (j^{th} \text{ row})}$ twice for some $a \in T$ such that $a^\nu = k$, we can reduce our set of elementary matrices to these definitions.)

We refer to the matrices $E_{i,j}$ as elementary matrices of type 1, to the matrices $E_{k \cdot (i^{th} \text{ row})}$ as elementary matrices of type 2 and to the matrices $E_{i+k \cdot (j^{th} \text{ row})}$ as elementary matrices of type 3.

Remark 1.6. A supertropical matrix $A$ is invertible if and only if it is a product of elementary matrices of type 1 and 2.

**Proof.** See [4, Proposition 3.9].

Calculating the determinants of the elementary matrices, one can easily conclude that the product of elementary matrices over a supertropical semifield, might yield a singular matrix only when there is an elementary matrix of type 3 involved in the product. This means that inequality in the rule of determinates arises from elementary matrices of type 3. But is it always possible to generate a matrix as a product of elementary matrices.
2. Nonfactorizable matrices

**Definition 2.1.** A tropically **factorizable matrix** is defined to be a matrix that can be written as a product of tropical elementary matrices.

It is important to pay attention to the difference of the factorization process in our case from the classical case. In matrix theory the factorization of a matrix is achieved by applying elementary row operations to the matrix in order to transform it to the identity matrix (a process known as Gaussian elimination or reduction of the matrix); then, multiplying the inverses to the corresponding elementary matrices in the opposite order would yield our matrix. In the supertropical semifield \( R \), we cannot reduce a nonzero element to zero using elementary operations. Therefore, we are approaching this construction by applying elementary row operations to the identity matrix in order to transform it to our matrix (an expansion of the matrix instead of reduction of the matrix).

**Claim 2.2.** For every elementary matrix \( E_1 \) of type 1 or 2 and elementary matrix \( E_2 \) of type 3 there exist an elementary matrix \( E_4 \) of type 1 or 2 respectively and an elementary matrix \( E_3 \) of type 3 such that \( E_1 E_2 = E_3 E_4 \).

**Proof.** This is a well known property. We provide the proof here for the convenience of the reader.

Let \( E_2 = E_{u+k, (m^{th} \text{row})} \) be an elementary matrix of type 3.

If \( E_1 = E_{i,j} \) is an elementary matrix of type 1, then

\[
E_1 E_2 = \begin{cases} 
E_2 E_1, & \text{where } u, m \neq i, j \\
E_{j+k, (i^{th} \text{row})} E_{i,j}, & \text{where } u = i \text{ and } m = j \\
E_{j+k, (m^{th} \text{row})} E_{i,j}, & \text{where } u = i \text{ and } m \neq j \\
E_{u+k, (i^{th} \text{row})} E_{i,j}, & \text{where } u \neq i \text{ and } m = j 
\end{cases}
\]

If \( E_1 = E_{h, (i^{th} \text{row})} \) is an elementary matrix of type 2, then

\[
E_1 E_2 = \begin{cases} 
E_2 E_1, & \text{where } u, m \neq i \\
E_{i+kh, (m^{th} \text{row})} E_{h, (i^{th} \text{row})}, & \text{where } u = i \\
E_{u+kh, (i^{th} \text{row})} E_{h, (i^{th} \text{row})}, & \text{where } m = i 
\end{cases}
\]

Therefore, once a factorization has been achieved one may relocate all of its elementary matrices of type 3 to the end of the factorization.

**Definition 2.3.** A **track of a permutation** \( \pi \in S_n \) is the sequence \( a_{1, \pi(1)} a_{2, \pi(2)} \cdots a_{n, \pi(n)} \) of \( n \) entries of the matrix \( A = (a_{i,j}) \in M_n(R) \).

In the next proposition we prove that not every supertropical nonsingular matrix is factorizable.

**Proposition 2.4.** Let \( \pi \) and \( \sigma \) be two different permutations in \( S_n \) such that there exists \( t \in \mathbb{Z}_n \setminus \{ \frac{n}{2} \} \) (if \( \frac{n}{2} \) is not in \( \mathbb{Z}_n \) then \( t \in \mathbb{Z}_n \)) so that \( \pi(i) = \sigma(i) + t \mod n \) \( \forall i \)
(i.e. $\pi$ is a shift of $\sigma$, but not by 0 or $\frac{n}{2}$). Over a supertropical semifield, for $n > 2$, any $n \times n$ matrix $A = (a_{i,j})$, such that $a_{i,j}$ \begin{align*}
\ne 0_R, \text{ whenever } j = \pi(i) \text{ or } \sigma(i) \\
= 0_R, \text{ otherwise}
\end{align*}
is not factorizable.

**Proof.** First we notice some important facts regarding the process of constructing a factorization of $A$, if it is to terminate at $A$:

1) The minimum value of each entry in the matrix can be $0_R = -\infty$, and any elementary row operation of type 3 that changes a $0_R$ entry would raise it beyond correction, due to the lack of additive inverses.

2) Since the construction starts with the identity matrix, throughout the process, every row and column has one non-zero entry, and because of (1) every row and column will never have more than two non-zero entries.

3) Elementary matrices of types 1 and 2 do not change the number of zeros in the matrix.

4) The requirement $t \neq \frac{n}{2}$ implies that if $\pi(i) = \sigma(j)$ for some $i, j$, then $\pi(j) \neq \sigma(i)$.

*Proof. We know that $\pi(i) = \sigma(j)$. Assume $\sigma(i) = \pi(j)$ then*

\[
\pi(i) = \sigma(j) = \sigma(i) + t \pmod{n}
\]

and

\[
\sigma(i) = \pi(j) = \sigma(j) + t \pmod{n}
\]

*therefore*

\[
\sigma(j) = \sigma(i) + t = \sigma(j) + 2t \pmod{n},
\]

*which means $2t = 0 \pmod{n}$ and we get $t = 0$ or $t = \frac{n}{2}$, contrary to the assumption on $t$.*

Assume that such a matrix can be factorized. According to Claim 2.2 we may relocate the last elementary matrix of type 3 to the end of the factorization. Let us look at the matrix we received one step before applying this last elementary matrix of type 3. Without loss of generality we may assume it yields the last entry on the track of the permutation $\sigma$. Therefore we now have a matrix with $2n - 1$ non-zero entries:

$a_{i_1, \sigma(i_1)}, \ldots, a_{i_{n-1}, \sigma(i_{n-1})}$ and $a_{i_1, \pi(i_1)}, \ldots, a_{i_{n-1}, \pi(i_{n-1})}, b$, where $b$ is in the $i_n, \pi(i_n)$ position.

We will show that we cannot produce the last non-zero entry under our assumptions.

The last elementary matrix in the factorization would change the zero in the $i_n, \sigma(i_n)$ position to $a_{i_n, \sigma(i_n)}$ by adding a row to the $i_n$ row. In order to do so, we must use $a_{k, \pi(k)}$ where $\pi(k) = \sigma(i_n)$, since it is the only non-zero entry in this column. The entry in the $k, \pi(i_n)$ position must be $0_R$ since $\pi(k) = \sigma(i_n) \Rightarrow \pi(i_n) \neq \sigma(k)$ and the only other non-zero entry in the $k^{th}$ row is in position $k, \sigma(k)$. Therefore $b = a_{i_n, \pi(i_n)}$. We already produced the $a_{k, \sigma(k)}$ entry in the $k^{th}$ row which would influence the $i_n, \sigma(k)$ position in the row $i_n$. However, the only other non-zero entry we want in the $i_n$ row is $a_{i_n, \pi(i_n)}$, that would require once again $\sigma(k) = \pi(i_n)$ which cannot occur.

$\square$
Example. The $3 \times 3$ matrix

$$A = \begin{pmatrix} 1_R & a_1 & 0_R \\ 0_R & 1_R & a_2 \\ a_3 & 0_R & 1_R \end{pmatrix}$$

is not factorizable, where $a_1, a_2, a_3 \neq 0_R$. If it were factorizable then it would have a factorization such that the last elementary matrix is of type 3. We may assume we already received the first and second rows, the general case is being proved analogously by writing $a_{ji}$ instead of $a_i \; \forall i = 1, 2, 3$.

In order to obtain $a_3$ we must use the only non-zero entry in its column which is in position $(1, 1)$. That is, applying $E_{3+a_3(1^{st\,row})}$ (the $(1, 3)$ position is $0_R$, therefore the $(3, 3)$ position has already been obtained at this point). However, this operation would change the zero in the $(3, 2)$ position beyond correction.

3. Factorization of supertropical $2 \times 2$ nonsingular matrices

**Proposition 3.1.** Over a semifield $R$, nonsingular matrices in $M_2(R)$ are factorizable.

**Proof.** Let $A = (a_{i,j})$ be a $2 \times 2$ nonsingular matrix.

Step 1. Relocate the dominant permutation track to the diagonal (elementary matrices of type 1), yields $A$ or $E_{1,2}A$.

Step 2. Dividing each row by the entry in the $(i, i)$ position (elementary matrices of type 2) yields

$$\bar{A} = \begin{pmatrix} 1_R & a \\ b & 1_R \end{pmatrix}.$$ 

Step 3. Notice that $1_R > ab$ because the diagonal remains the dominant permutation track in $\bar{A}$. Thus

$$\bar{A} = \begin{pmatrix} 1_R & a \\ b & 1_R \end{pmatrix} = \begin{pmatrix} 1_R & a \\ 0_R & 1_R \end{pmatrix} \begin{pmatrix} 1_R & 0_R \\ b & 1_R \end{pmatrix}. \qed$$

**Example 3.2.**

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$$

We can easily calculate that $|A| = 7$ and therefore $A$ is factorizable. Indeed

$$A = E_{1,1}(1^{st\,row})E_{6,2}(2^{nd\,row})E_{1,2,1}(2^{nd\,row})E_{2,3,6}(1^{st\,row}) =$$

$$= \begin{pmatrix} 1 & - \\ - & 0 \end{pmatrix} \begin{pmatrix} 0 & - \\ - & 6 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ - & 0 \end{pmatrix} \begin{pmatrix} 0 & - \\ -3 & 0 \end{pmatrix},$$
where $-$ denotes $-\infty = 0_R$.

4. Factorization of supertropical $3 \times 3$ nonsingular matrices

The fact that the determinant of a nonsingular matrix $A$ is tangible means that the matrix has one dominant permutation track. By using elementary matrices of type 1 we can relocate the corresponding permutation to the diagonal and by using elementary matrices of type 2 we can change the diagonal entries to $1_R = 0$, receiving a nonsingular matrix with one dominant permutation track equals $1_R$ on the diagonal. That is, $A = EA$ where $E$ is an invertible matrix (see Remark 1.6) and $|A| = 1_R$. We denote $\bar{A}$ as the normal form of $A$.

Let $A$ be a $3 \times 3$ nonsingular matrix over a supertropical semifield. We analyze $3 \times 3$ matrices in normal form displayed as:

\[
\bar{A} = \begin{pmatrix}
1_R & a_1 & b_1 \\
 b_2 & 1_R & a_2 \\
 a_3 & b_3 & 1_R
\end{pmatrix}
\]

where the diagonal is the dominant permutation track.

Claim 4.1. A nonsingular matrix $A$ is factorizable if and only if its normal form $\bar{A}$ is factorizable.

Proof. Let $\bar{E}$ be the product of elementary matrices of type 1 and 2 such that $A = \bar{E}\bar{A}$. According to Remark 1.6 $\bar{E}$ is invertible and we can conclude that $\bar{A} = \bar{E}^{-1}A$. Of course $\bar{E}^{-1}$ is also invertible. Therefore by using Remark 1.6 again we have that $\bar{E}^{-1}$ is also a product of elementary matrices of type 1 and 2. Hence the factorizability of $A$ and the factorizability of $\bar{A}$ are equivalent. $\square$

Lemma 4.2. Given any nondiagonal entry $a_{i,j}$ of a $3 \times 3$ matrix, there exists precisely one permutation track in which this nondiagonal entry appears and for which all other entries are also nondiagonal.

Proof. A permutation track is of the form $a_{1,\pi(1)}a_{2,\pi(2)}a_{3,\pi(3)}$ for some $\pi \in S_3$. If we want all the entries to be nondiagonal then $\pi(i) \neq i$, $\forall i = 1, 2, 3$, which means $\pi = (1\ 2\ 3)$ or $(1\ 3\ 2)$ and therefore the possible permutation tracks are $a_{1,2}a_{3,1}$ and $a_{1,3}a_{2,1}$, which consist of all of the nondiagonal entries, exactly one time each. $\square$

Noticing that every row and column in the normal form of a nonsingular $3 \times 3$ matrix has two nondiagonal entries, we introduce two helpful notions regarding these entries:

**Permutation track of a nondiagonal entry**: The product of the entries of the track that includes the given nondiagonal entry and no diagonal entries. For example, the permutation track of the nondiagonal entry $a_{1}$ is $a_{1}a_{2}a_{3}$.

**Entry condition**: The relation ($>$, $<$, $=$) of a nondiagonal entry to the product of the other nondiagonal entry in its row with the other nondiagonal entry in its column. For example, we denote the entry condition of $a_{1}$ as $<$, $>$, or $=$ depending on whether $a_{1}$
There exists a row $i$, or $= b_1b_3$ respectively. We refer to the matrix
\[
\begin{pmatrix}
1_R & a_1 + b_1b_3 & b_1 + a_1a_2 \\
b_2 + a_2a_3 & 1_R & a_2 + b_1b_2 \\
a_3 + b_2b_3 & b_3 + a_1a_3 & 1_R
\end{pmatrix}
\]
as the matrix of entry conditions.

**Lemma 4.3.** A $3 \times 3$ nonsingular matrix over a supertropical semifield is not factorizable, if and only if there exists a permutation track of a nondiagonal entry in its normal form all of whose entries conditions satisfy $<.$

**Proof.** We denote $A = (a_{ij}).$ We can obtain any $2 \times 2$ minor, by using Proposition 3.1 as $2 \times 2$ matrices embedded to rows and columns $i_1$ and $i_2$ in the $3 \times 3$ identity matrix. Then we can recover the third column $i_3$, obtaining a $2 \times 3$ minor, by applying
\[
E_{i_2+a_{i_2,i_3}(i_3^{th\text{row}})} \cdot E_{i_1+a_{i_1,i_3}(i_3^{th\text{row}})} \cdot E_{i_2+a_{i_2,i_3}(i_3^{th\text{row}})} \cdot E_{i_1+a_{i_1,i_2}(i_2^{th\text{row}})}.
\]

For example, when $i_j = j \forall j = 1, 2, 3$:
\[
\begin{pmatrix}
1_R & 0 & b_1 \\
0 & 1_R & 0 \\
0 & 0 & 1_R
\end{pmatrix}
\begin{pmatrix}
1_R & 0 & 0 \\
0 & 1_R & 0 \\
0 & 0 & 1_R
\end{pmatrix}
\begin{pmatrix}
1_R & 0 & 0 \\
b_2 & 1_R & 0 \\
0 & 0 & 1_R
\end{pmatrix}
\begin{pmatrix}
1_R & a_1 & 0 \\
0 & 1_R & 0 \\
0 & 0 & 1_R
\end{pmatrix}
\begin{pmatrix}
1_R & a_1 & b_1 \\
b_2 & 1_R & a_2 \\
0 & 0 & 1_R
\end{pmatrix}.
\]

Therefore we can conclude that obtaining the third row $i_3$ is the only obstruction to the factorization process of a $3 \times 3$ nonsingular matrix.

We notice that in (4.1) the entry $a_i$ is in the $(i, i+1 \pmod{3})$ position and the entry $b_i$ is in the $(i, i+2 \pmod{3})$ position for every $i = 1, 2, 3$. Since $a_i \cdot b_{i+1} \cdot 1_R$ and $a_{i+2} \cdot b_i \cdot 1_R$ are permutation tracks, we get $a_i \cdot b_{i+1}, a_{i+2} \cdot b_i < 1_R$ for every $i$. Therefore if the entry condition of a nondiagonal entry is $<$, then it forces the entry conditions of the nondiagonal entries in its row and column to be $>$:

\[
\begin{align*}
b_i < a_i & \Rightarrow b_i \cdot b_{i+2} < a_i \cdot a_i \cdot b_{i+2} < a_i \\
& \Rightarrow b_i \cdot b_{i+1} < a_i \cdot a_i \cdot b_{i+1} < a_i \\
da_i & \Rightarrow a_i \cdot a_i \cdot b_{i+1} < b_i \\
& \Rightarrow a_i \cdot b_{i+1} < a_i \cdot b_{i+2} < a_i \\
& \Rightarrow a_i \cdot a_i \cdot b_{i+1} < b_i \\
& \Rightarrow a_i \cdot b_{i+2} < a_i \cdot b_{i+2} < a_i
\end{align*}
\]

where $i = 1, 2, 3$, $i + 1$ stands for $i + 1 \pmod{3}$ and $i + 2$ stands for $i + 2 \pmod{3}$.

Therefore, we may assume that the row left to be constructed is row number three, the general case is being proved analogously by writing $a_j$, instead of $a_i \ \forall i = 1, 2, 3$.

We return to the proof of the Lemma

$(\Rightarrow)$ There exists a row $j$ such that both of its entry conditions satisfy $\geq$.

Assuming $j = 3$, by applying $E_{3+a_{3,i}(i^{th\text{row}})}$ for $i = 1, 2$ to the $2 \times 3$ minor of rows one and two, we obtain $A$: 

\[
\begin{pmatrix}
1_R & 0 & b_1 \\
0 & 1_R & 0 \\
0 & 0 & 1_R
\end{pmatrix}
\begin{pmatrix}
1_R & 0 & 0 \\
0 & 1_R & 0 \\
0 & 0 & 1_R
\end{pmatrix}
\begin{pmatrix}
1_R & 0 & 0 \\
b_2 & 1_R & 0 \\
0 & 0 & 1_R
\end{pmatrix}
\begin{pmatrix}
1_R & a_1 & 0 \\
0 & 1_R & 0 \\
0 & 0 & 1_R
\end{pmatrix}
\begin{pmatrix}
1_R & a_1 & b_1 \\
b_2 & 1_R & a_2 \\
0 & 0 & 1_R
\end{pmatrix}.
\]
\[
\begin{pmatrix}
1_R & 0 & 0 \\
0 & 1_R & 0 \\
a_3 & 0 & 1_R
\end{pmatrix}
\begin{pmatrix}
1_R & 0 & 0 \\
0 & 1_R & 0 \\
b_3 & 1_R
\end{pmatrix}
\begin{pmatrix}
1_R & a_1 & b_1 \\
b_2 & 1_R & a_2 \\
a_3 & b_3 & 1_R
\end{pmatrix} = \\
\begin{pmatrix}
1_R & a_1 & b_1 \\
b_2 & 1_R & a_2 \\
a_3 + b_3 b_2 & b_3 + a_3 a_1 & 1_R
\end{pmatrix} = \\
\begin{pmatrix}
1_R & a_1 & b_1 \\
b_2 & 1_R & a_2 \\
a_3 & b_3 & 1_R
\end{pmatrix}.
\]

Likewise, if at least one entry condition satisfies \(=\), then \(A\) is factorizable by applying only the operation that does not correspond to the \(=\) entry condition. Therefore, in any of these cases a matrix with such a normal form would also be factorizable.

\((\Leftarrow)\) If there is no row whose entry conditions both satisfy \(\ge\), then each row has exactly one \(<\) condition (since we showed two conditions \(<\) cannot occur in the same row). Moreover, since the condition \(<\) forces conditions \(>\) in the other nondiagonal entries of its row and column, we get one permutation track of nondiagonal entries with condition \(>\) and the other permutation track of nondiagonal entries with condition \(<\). We show that in this case the normal form \(\bar{A}\) is not factorizable and therefore any matrix with such a normal form is not factorizable.

Assume \(\bar{A}\) is factorizable. By Claim 2.2 we can relocate all the elementary matrices of type 3 to the end of the factorization. We now look at the last elementary matrix of type 3 in this factorization (which is the last elementary matrix in this factorization). As before, we may assume that this elementary matrix of type 3 operates on the third row, whose entries satisfy either \(a_3 < b_2 b_3\) or \(b_3 < a_3 a_1\) (but not both), meaning we are starting with

\[
\begin{pmatrix}
1_R & a_1 & b_1 \\
b_2 & 1_R & a_2 \\
a & b & c
\end{pmatrix}
\]

where at least one entry in the third row is different from the entry we want to produce by applying elementary matrix of type 3: \(E_{3+m_2,(2^{nd\text{row}})}\) or \(E_{3+m_1,(1^{st\text{row}})}\) where \(m_1, m_2 \in T\), to obtain the third row: \((a_3 \ b_3 \ 1_R)\).

Therefore we have reduced to four cases:

\begin{center}
\begin{tabular}{c|c|c}
& \(E_{3+m_2,(2^{nd\text{row}})}\) & \(E_{3+m_1,(1^{st\text{row}})}\) \\
\hline
\(a_3 < b_2 b_3\) & (so \(a_3 = a\) or \(m_2 b_2\), \(b_3 = b\) or \(m_2\), \(1_R = c\) or \(m_2 a_2\)) & (so \(a_3 = a\) or \(m_1\), \(b_3 = b\) or \(m_1 a_1\), \(1_R = c\) or \(m_1 b_1\)) \\
\(b_3 < a_3 a_1\) & Therefore \(m_2 < b_3\) so \(b_3 = b, 1_R = c\) & Therefore \(m_1 \leq a_3\) so \(m_1 a_1 < b_3 \Rightarrow b_3 = b\) and \(m_1 b_1 < 1_R \Rightarrow 1_R = c\) \\
\hline
\end{tabular}
\end{center}

(4.2)

Hence we get either
and the remaining entry must be different from the desired one.

If the remaining entry is bigger than the desired one then clearly we cannot produce the desired entry with an elementary matrix of type 3. Thus, it must be smaller, which means we are back to the same entry conditions: one permutation track of nondiagonal entries satisfy $<$. Therefore, looking at all the previous elementary matrices of type 3, we notice that it would again yield a matrix changed only at the permutation track of nondiagonal entries satisfying $<$. By reducing them:

$$
\begin{pmatrix}
1 & R & c & b_1 \\
 b_2 & 1 & R & b \\
 a & b_3 & 1 & R
\end{pmatrix}
\text{ or }
\begin{pmatrix}
1 & R & a_1 & c \\
 a & 1 & R & a_2 \\
 a_3 & b & 1 & R
\end{pmatrix}
$$

If $a, b$ or $c$ never reaches $0_R$ in the string of elementary matrices of type 3, then the factorization does not terminate, which is not possible. That leads us to the conclusion that $a, b$ and $c$ reach $0_R$ in the string of elementary matrices of type 3, which means at some point of the factorization we get either

$$
\begin{pmatrix}
1 & R & 0_R & b_1 \\
 b_2 & 1 & R & 0_R \\
 0_R & b_3 & 1 & R
\end{pmatrix}
\text{ or }
\begin{pmatrix}
1 & R & a_1 & 0_R \\
 0_R & 1 & R & a_2 \\
 a_3 & 0_R & 1 & R
\end{pmatrix}
$$

where the entries of the permutation track of entry conditions $>$ are strictly bigger than $0_R$, since the $0_R$ in the nondiagonal positions in their rows and columns are determining their entry conditions by definition. These matrices are not factorizable according to Proposition 2.4 and therefore cannot appear as a part of any factorization.

\[\square\]

**Example 4.4.**

$$
A = \begin{pmatrix}
0 & -3 & 0 \\
1 & 5 & 0 \\
3 & 1 & 6
\end{pmatrix}
$$

We easily calculate that $|A| = 11$ and that the diagonal is the dominant permutation track. Therefore, by applying $E_{5-(2^{nd}\text{ row})}E_{6-(3^{rd}\text{ row})}$ to the normal form

$$
\bar{A} = \begin{pmatrix}
0 & -3 & 0 \\
-4 & 0 & -5 \\
-3 & -5 & 0
\end{pmatrix}
$$

we might achieve a factorization of $A$.

The next step would be to check the entry conditions of $\bar{A}$, which may be displayed as:

$$
\begin{pmatrix}
0 & > & > \\
> & 0 & < \\
> & > & 0
\end{pmatrix}
$$
We do not have a permutation track of entry conditions < (as could be seen after checking the first row) and therefore $A$ is factorizable, indeed:

$$A = \begin{pmatrix} 0 & - & - \\ - & 5 & - \\ - & - & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & - & - \\ - & 0 & - \\ - & - & 6 \end{pmatrix} \cdot \begin{pmatrix} 0 & - & - \\ - & 0 & - \\ - & - & 5 \end{pmatrix} \cdot \begin{pmatrix} 0 & - & - \\ - & 0 & - \\ - & - & 3 \end{pmatrix} \cdot \begin{pmatrix} 0 & - & - \\ - & - & 1 \\ - & - & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & - & - \\ - & 0 & - \\ - & - & 5 \end{pmatrix} \cdot \begin{pmatrix} 0 & - & - \\ - & - & 0 \\ - & - & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & - & - \\ - & - & 0 \\ - & - & 3 \end{pmatrix} \cdot \begin{pmatrix} 0 & - & - \\ - & - & 0 \\ - & - & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & - & - \\ - & - & 0 \\ - & - & 0 \end{pmatrix}$$

$A \rightarrow A \rightarrow A \rightarrow A \rightarrow A \rightarrow A \rightarrow A \rightarrow A \rightarrow A \rightarrow A$.

Example 4.5.

$$A = \begin{pmatrix} 4 & 3 & 3 \\ 4 & 5 & 2 \\ 5 & 7 & 6 \end{pmatrix}$$

We easily calculate that $|A| = 15$ and that the diagonal is the dominant permutation track. Therefore, by applying $E_4(1^{st\,row})E_5(2^{nd\,row})E_6(3^{rd\,row})$ to the normal form

$$\bar{A} = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -3 \\ -1 & 1 & 0 \end{pmatrix}$$

we might achieve a factorization of $A$.

The next step would be to check the entry conditions of $\bar{A}$, which may be displayed as:

$$\begin{pmatrix} 0 & < & > \\ > & 0 & < \\ < & > & 0 \end{pmatrix}.$$

We have a permutation track of entry conditions <. Therefore, by table 4.1, looking at all the previous elementary matrices of type 3 would yield the matrix

$$\begin{pmatrix} 0 & a & -1 \\ -1 & 0 & b \\ c & 1 & 0 \end{pmatrix}$$

where either $a, b, c = 0_R$, which is not a factorizable matrix, or $a < -1$, $b < -3$, $c < -1$, but at least one is different then $0_R$, from which we conclude that the factorization does not terminate. Therefore $A$ is not factorizable.

This classification would be rather hard to generalize for $n \times n$ matrices since the required number of conditions will increase significantly. In the next section we present a $\nu$-equivalent approach to supertropical matrices that would help us in constructing a general tropical factorization for nonsingular matrices.
5. 3 × 3 Quasi-factorization

In order to recover factorization result for supertropical nonsingular 3 × 3 matrices, we follow the terminology in [3] when extending the classical definitions by considering the supertropical ghost ideal.

**Definition 5.1.** A quasi-zero matrix $Z_G$ is a matrix equal to 0 on the diagonal, and whose off-diagonal entries are ghosts or 0. A quasi-identity matrix $I_G$ is a nonsingular, multiplicatively idempotent matrix equal to $I + Z_G$, where $Z_G$ is a quasi-zero matrix.

**Definition 5.2.** The $t, l$-minor $A_{t,l}$ of a matrix $A = (a_{i,j})$ is obtained by deleting the $t$th row and $l$th column of $A$. The adjoint matrix $\text{adj}(A)$ of $A$ is defined as the matrix $\begin{bmatrix} a'_{i,j} \end{bmatrix}$, where $a'_{i,j} = \begin{vmatrix} A_{j,i} \end{vmatrix}$. The matrix $A^\nabla$ denotes $\frac{\text{adj}(A)}{|A|}$.

Notice that $|A_{j,i}| = \sum_{\sigma \in S_n : \sigma(j) = i} a_{1,\sigma(1)} \cdots a_{j-1,\sigma(j-1)} a_{j+1,\sigma(j+1)} \cdots a_{n,\sigma(n)}$, which means the $a_{j,i}$ element is being canceled from the permutation tracks.

**Lemma 5.4.**
(i) $A^\nabla$ is a quasi-inverse of $A$.
(ii) $A$ is a quasi-inverse of $A^\nabla$.

**Proof.** See [5, Theorem 2.8] □

We denote $I_A = AA^\nabla$ and $I'_A = A^\nabla A$, which are quasi-identity matrices.

**Theorem 5.5.**
(i) $|A\text{adj}(A)| = |A|^n$.
(ii) $|\text{adj}(A)| = |A|^{n-1}$.

**Proof.** [4, Theorem 4.9].

**Proposition 5.6.** $\text{adj}(AB) \equiv_{g_s} \text{adj}(B)\text{adj}(A)$.

**Proof.** [4, Proposition 4.8].

**Lemma 5.7.**
(i) $E^\nabla = E^{-1}$ where $E$ is an invertible matrix.
(ii) $(EA)^\nabla = A^\nabla E^\nabla$ where $A$ is nonsingular and $E$ is an invertible matrix.
(iii) Let $\bar{A}$ be the normal form of a nonsingular matrix $A$ (i.e. $A = \bar{E} \bar{A}$ where $\bar{E}$ is the invertible matrix that normalizes the dominant permutation track of $A$ to the diagonal). Then $A^\nabla = \bar{A}^\nabla \bar{E}^{-1}$.

**Proof.** We will show that properties (i) and (ii) hold where $E$ is an elementary matrix of type 1 or 2, and the general case follows by induction.
Let $E = E_{i,j}$ be an elementary matrix of type 1. Then $|E| = 1_R$ and $\text{adj}(E)$ differs from $E$ by exchanging the entries in positions $(i, i)$ and $(j, j)$. The entries in these positions are both $0_R$, which means

$$E^\nabla = E = E^{-1}.$$ 

If $E = E_{k,i^{th\text{ row}}}$ is an elementary matrix of type 2 then $|E| = k$ and $\text{adj}(E)$ is the matrix with $k$ on the diagonal except for the $(i, i)$ position which is $1_R$, and $0_R$ off the diagonal. Therefore

$$E^\nabla = E_{k^{-1},i^{th\text{ row}}} = E^{-1}.$$ 

We notice that where $A$ is nonsingular and $E$ is an elementary matrix of type 1 or 2 we have $|EA| = |A||E|$. Thus it suffices to show that $\text{adj}(EA) = \text{adj}(A)\text{adj}(E)$ for such $A$ and $E$. Denote $\text{adj}(A) = (|A_{s,t}|)$, the matrix $\text{adj}(A)\text{adj}(E)$, where $E$ is of type 1 (resp. of type 2), is being obtained by switching columns $i$ and $j$ (resp. multiplying column $i$ by $k$) in the transpose of the matrix $(|A_{s,t}|)$. This process is identical to switching rows $i$ and $j$ (resp. multiplying row $i$ by $k$) in the matrix $A$ and then applying the adjoint, which obtains $\text{adj}(EA)$.

Assuming (i) and (ii) for an invertible matrix $E$, we have

$$(E'E)^\nabla = E^\nabla E'^\nabla = E^{-1}E'^{-1} = (E'E)^{-1}$$

and

$$(E'EA)^\nabla = (EA)^\nabla E'^\nabla = A^\nabla E^{-1}E'^{-1} = A^\nabla (E'E)^{-1},$$

where $E'$ is an elementary matrix of type 1 or 2.

(iii) Using the arguments in (i) and (ii) we have

$$A^\nabla = (\bar{E}\bar{A})^\nabla = \bar{A}^\nabla E^\nabla = \bar{A}^\nabla \bar{E}^{-1}$$

as required. \[\square\]

This last result allows us to approach the factorization in two stages, the “well-behaved” invertible part, consist of elementary matrices of type 1 and 2, and the remaining, quasi-invertible part, includes elementary matrices of type 3, obtaining the normal form. That is, we preserve the invertible part and aspire to obtain factorization to matrices that are, in a way, equivalent to normal forms.

**Lemma 5.8.** The following assertions hold for any nonsingular $3 \times 3$ matrix $A$ in normal form.

1. $A^\nabla$ is the matrix of entry conditions.
2. $A^\nabla$ is always factorizable.
3. $A^\nabla A^\nabla$ is $\nu$-equivalent to $A^\nabla$.
4. $A^\nabla A^\nabla$ is always factorizable.

**Proof.** By hypothesis, writing $A = (a_{i,j})$ where $a_{i,j} = \begin{cases} 1_R, & i = j \\
1, & j = i + 1 \pmod{3} \\
 b_i, & j = i + 2 \pmod{3} \end{cases}$,

we have $1_R > b_i a_{i+2} \pmod{n}, a_i b_{i+1} \pmod{n}, a_1 a_2 a_3, b_1 b_2 b_3 \forall i = 1, 2, 3.$
1. By definition of $A^\nlor$ and the matrix of entry conditions we get:

$$A^\nlor = \begin{pmatrix} 1_R & a_1 & b_1 \\ b_2 & 1_R & a_2 \\ a_3 & b_3 & 1_R \end{pmatrix}^{\nlor} = \begin{pmatrix} 1_R & a_1 + b_1b_3 & b_1 + a_1a_2 \\ b_2 + a_2a_3 & 1_R & a_2 + b_1b_2 \\ a_3 + b_2b_3 & b_3 + a_1a_3 & 1_R \end{pmatrix}.$$  

2. According to Lemma 4.3 we can factor the matrix iff there exists a row with two entry conditions $\geq$. Looking at the third row of $A^\nlor$ we can see that

$$a_3 + b_2b_3 \geq b_2b_3 + b_2a_1a_3 + a_2a_3b_3 + a_2a_3a_1a_3 = (b_2 + a_2a_3)(b_3 + a_1a_3)$$

and

$$b_3 + a_1a_3 \geq a_3a_1 + a_3b_1b_3 + b_2b_3a_1 + b_2b_3b_1b_3 = (a_3 + b_2b_3)(a_1 + b_1b_3)$$

and therefore $A^\nlor$ is factorizable.

3. By definition

$$A^{\nlor\nlor} = \begin{pmatrix} 1_R & a_1 + (b_1b_3)^{\nu} & b_1 + (a_1a_2)^{\nu} \\ b_2 + (a_2a_3)^{\nu} & 1_R & a_2 + (b_1b_2)^{\nu} \\ a_3 + (b_2b_3)^{\nu} & b_3 + (a_1a_3)^{\nu} & 1_R \end{pmatrix} \cong_\nu A^\nlor$$

4. Looking at the third row of $A^{\nlor\nlor}$ we can see that this matrix is factorizable by the same argument as in (2).

In the next section we are generalizing parts (3) and (4) of the last Lemma for non-singular $n \times n$ matrices.

6. $n \times n$ Quasi-factorization

Having established factorizability for the quasi-inverses of the quasi-inverses of non-singular $3 \times 3$ matrices, we would like to obtain this result for $n \times n$ nonsingular matrices. In order to do so we achieve the same ghost-equivalence as in Lemma 5.8, part 3 which guarantees that the ghost value of the entries of a matrix in normal form are preserved under $\nlor$. In view of Lemma 5.7 we may assume that every nonsingular matrix $A$ is a normal form.

Claim 6.1. If $A$ is in normal form, then $A^\nlor A \cong_\nu A^\nlor \cong_\nu AA^\nlor$.

Proof. $AA^\nlor$, $A^\nlor A$ are quasi-identities and therefore equal to $A^\nlor$ on the diagonal. We check $AA^\nlor$ outside the diagonal. The proof to $A^\nlor A$ would be obtained analogously by exchanging $a_{i,k}a'_{k,j}$ by $a'_{i,k}a_{k,j}$, where $A = (a_{i,j})$ and $A^\nlor = (a'_{i,j}) = (|A_{j,i}|)$.

Clearly $A^\nlor A$, $AA^\nlor \cong_\nu A^\nlor$, since $a'_{i,j}$ is a summand in the $i, j$ position of $AA^\nlor$ and $A^\nlor A$. So it suffices to prove that $A^\nlor \geq_\nu AA^\nlor$. For $i \neq j$, the $i, j$ entry of $AA^\nlor$ is a sum of the form

$$\sum_{k=1}^n a_{i,k}a'_{k,j} = a_{i,i}a'_{i,j} + a_{i,j}a'_{j,j} + \sum_{k \neq i,j} a_{i,k}a'_{k,j}.$$  

Since $A^\nlor, A$ are in normal form their diagonal entries are $1_R$, yielding
(6.2) \[ a'_{i,j} + a_{i,j} + \sum_{k \neq i,j} a_{i,k} a'_{k,j} \] 

By definition \( a'_{k,j} = |A_{j,k}| \) and \( a'_{i,j} = |A_{j,i}| \):

\[
\begin{align*}
&\left( \sum_{\pi \in S_n : \pi(j) = i} a_{1,\sigma(1)} \cdots a_{j-1,\sigma(j-1)} a_{j+1,\sigma(j+1)} \cdots a_{n,\sigma(n)} \right) + a_{i,j} + \\
&+ \sum_{k \neq i,j} a_{i,k} \left( \sum_{\sigma \in S_n : \sigma(j) = k} a_{1,\sigma(1)} \cdots a_{j-1,\sigma(j-1)} a_{j+1,\sigma(j+1)} \cdots a_{n,\sigma(n)} \right).
\end{align*}
\]

We will show that the ghost value of the first sum in (6.3):

\[
\sum_{\pi \in S_n : \pi(j) = i} a_{1,\sigma(1)} \cdots a_{j-1,\sigma(j-1)} a_{j+1,\sigma(j+1)} \cdots a_{n,\sigma(n)}
\]

is equal to or bigger than the ghost value of the last sum in (6.3):

\[
\sum_{k \neq i,j} a_{i,k} \left( \sum_{\sigma \in S_n : \sigma(j) = k} a_{1,\sigma(1)} \cdots a_{j-1,\sigma(j-1)} a_{j+1,\sigma(j+1)} \cdots a_{n,\sigma(n)} \right).
\]

We call every disjoint cycle of a permutation a simple cycle of the permutation, and notice that composing a simple cycle with disjoint Id simple cycles would yield a permutation. For example, in the permutation: \((1 2)(3 4)\), the transpositions \((1 2)\) and \((3 4)\) are simple cycles, and \((1 2)(3)(4)\), \((1)(2)(3 4)\) are also permutations.

We notice that using this definition we can factor a permutation track to simple cycle tracks, and compose new permutation tracks using disjoint Id tracks. For example, the permutation track: \(a_{1,2}a_{2,1}a_{3,4}a_{4,3}\) may be factor into the following simple cycle tracks: \(a_{1,2}a_{2,1}\), \(a_{3,4}a_{4,3}\), and indeed \(a_{1,2}a_{2,1}a_{3,3}a_{4,4}\), \(a_{1,1}a_{2,2}a_{3,4}a_{4,3}\) are permutation tracks.

It is important to notice that when computing the determinant of the minor \(A_{j,i}\) we receive deficient permutation tracks with one deficient cycle track lacking \(a_{j,i}\). We refer to this cycle track as the deficient track of \(j\), and observe that all the other cycle tracks are simple cycles, since they do not lack any element. For example, the permutation track in the 4, 3-minor of a 4 \(\times\) 4 matrix: \(a_{1,2}a_{2,1}a_{3,4}\) (lacking the element \(a_{4,3}\)) has the deficient cycle track of 4: \(a_{3,4}\), composed with the Id it would yield another deficient permutation track in the 4, 3-minor of the 4 \(\times\) 4 matrix: \(a_{1,1}a_{2,2}a_{3,4}\). The other track is a simple cycle: \(a_{1,2}a_{2,1}\), composed with the Id it would yield a permutation track in the 4 \(\times\) 4 matrix: \(a_{1,2}a_{2,1}a_{3,3}a_{4,4}\).
We return to the proof by looking at (6.5). Considering the last definitions, it suffices to show we can factor each summand to a permutation track and a deficient permutation track from (6.4). For every $k \neq i, j$ and $\sigma \in S_n$ such that $\sigma(j) = k$ we look at
\[
a_{i,k}(a_{1,\sigma(1)} \cdots a_{j-1,\sigma(j-1)}a_{j+1,\sigma(j+1)} \cdots a_{n,\sigma(n)})
\]
and distinguish between two cases in which $i$ is in the same simple cycle as $j$ in $\sigma$ or not.

Case I: $i$ is in the same simple cycle of $j$:
\[
(6.6) \quad a_{i,k}(a_{k,\sigma(k)} \cdots a_{\sigma^{-1}(i),i}a_{i,\sigma(i)} \cdots a_{\sigma^{-1}(j),j}) P,
\]
The deficient track of $j$ where $P$ is the product of the remaining simple cycle tracks in $\sigma$. By factoring this expression into the following disjoint terms: $(a_{i,k}a_{k,\sigma(k)} \cdots a_{\sigma^{-1}(i),i}) P$ and $(a_{i,\sigma(i)} \cdots a_{\sigma^{-1}(j),j})$, and composing them with disjoint $\text{Id}$ tracks, we obtain $(a_{i,k}a_{k,\sigma(k)} \cdots a_{\sigma^{-1}(i),i}) P \cdot 1_R \cdots 1_R$, which is a permutation track and therefore smaller then $1_R$, and $(a_{i,\sigma(i)} \cdots a_{\sigma^{-1}(j),j}) 1_R \cdots 1_R$, which is a summand in (6.4) for a permutation $\pi \in S_n$ such that
\[
\pi(v) = \begin{cases} 
\sigma(v), & v = i, \sigma(i), ..., \sigma^{-1}(j) \\
i, & v = j \\
v, & \text{otherwise}.
\end{cases}
\]
Consequently it is clear that (6.6) is dominated by (6.4).

Case II: $i$ is not in the same simple cycle of $j$:
\[
(6.7) \quad a_{i,k}(a_{k,\sigma(k)} \cdots a_{\sigma^{-1}(j),j}) (a_{i,\sigma(i)} \cdots a_{\sigma^{-1}(i),i}) P,
\]
The deficient track of $j$ The simple cycle track of $i$ where $P$ is the product of the remaining simple cycle tracks in $\sigma$. By factoring this expression into the disjoint terms: $a_{i,k}(a_{k,\sigma(k)} \cdots a_{\sigma^{-1}(j),j})$ and $(a_{i,\sigma(i)} \cdots a_{\sigma^{-1}(i),i}) P$, and composing each one with disjoint $\text{Id}$ tracks, we obtain $(a_{i,k}a_{k,\sigma(k)} \cdots a_{\sigma^{-1}(j),j}) 1_R \cdots 1_R$, which is a summand in (6.4) for a permutation $\pi \in S_n$ that satisfies
\[
\pi(v) = \begin{cases} 
\sigma(v), & v = k, \sigma(k), ..., \sigma^{-1}(j) \\
k, & v = i \\
i, & v = j \\
v, & \text{otherwise}
\end{cases},
\]
and $(a_{i,\sigma(i)} \cdots a_{\sigma^{-1}(i),i}) P \cdot 1_R \cdots 1_R$ which is a permutation track and therefore smaller then $1_R$. Consequently it is clear that (6.7) is dominated by (6.4).

Hence, we get:
\[
(6.8) \quad a'_{i,j} + a_{i,j} + \sum_{k \neq i,j} a_{i,k}a'_{k,j} \simeq_\nu a'_{i,j} + a_{i,j}.
\]
To conclude the proof we will show that $a'_{i,j} + a_{i,j} \simeq_\nu a'_{i,j}$. Indeed $a_{i,j} \cdot 1_R \cdots 1_R$ is a summand in $a'_{i,j} = |A_{i,j}|$ where $\pi$ is the transposition $(i,j)$, thus
\[
(6.9) \quad a'_{i,j} + a_{i,j} + \sum_{k \neq i,j} a_{i,k}a'_{k,j} \simeq_\nu a'_{i,j} + a_{i,j} \simeq_\nu a'_{i,j}.
\]
Corollary 6.2. Let $A$ be a nonsingular matrix with normal form $\bar{A}$, i.e. $A = \bar{E}\bar{A}$ for some invertible matrix $\bar{E}$.

a. $A^\triangledown \cong_{\nu} \bar{A}^\triangledown$.
b. $A^\triangledown \cong_{\nu} \bar{E}A^\triangledown \bar{E}$.

Proof. We can obtain any $2 \times 2$ minor $M_{i_1,\ldots,i_n}$ of a matrix $A = (a_{i,j})$ from $\bar{A}$ by deleting the $i_1,\ldots,i_n$ rows of $A$ and their corresponding columns.

Definition 6.3. The $(i_1,\ldots,i_k)$-minor $M_{i_1,\ldots,i_k}$ of a matrix $A = (a_{i,j})$ is obtained by deleting the $i_1,\ldots,i_k$ rows of $A$ and their corresponding columns.

Proposition 6.4. Let $A$ be an $n \times n$ matrix in normal form. If an $n - 1 \times n - 1$ minor $M_{i_n}$ of $A$ is in $\mathcal{E}$, then $A \in \mathcal{E}$ if

$$a_{i_n,j} > a_{i_n,k}a_{k,j} \quad \forall i_n \neq j \text{ and } k \neq i_n,j,$$

and a matrix $B \cong_{\nu} A$ is in $\mathcal{E}$ if

$$a_{i_n,j} \geq a_{i_n,k}a_{k,j} \quad \forall i_n \neq j \text{ and } k \neq i_n,j.$$

Proof. The $i_n$th column may be obtained by applying $E_{i_t+(a_{i_n,i_n})^{-1}i_n}$, $\forall t \neq n$. Then, by applying $E_{i_n+(a_{i_n,k})^{-1}k}$, $\forall k \neq i_n$, we obtain

$$a_{i_n,j} \geq \sum_{k \neq i_n,j} a_{i_n,k}a_{k,j}$$

in the $i_n,j$ position, for every $i_n \neq j$, which is $a_{i_n,j}$ if $a_{i_n,j} > a_{i_n,k}a_{k,j}$ and $\nu$-equivalent to $a_{i_n,j}$ if $a_{i_n,j} \geq a_{i_n,k}a_{k,j}$. \hfill \Box

Lemma 6.5. If $A$ is of normal form, then any $m \times m$-minor $M_{i_{m+1},\ldots,i_n}$ of $A^\triangledown$ is in $\mathcal{E}$.

Proof. We can obtain any $2 \times 2$ minor $M_{i_1,\ldots,i_n}$ by embedding the matrices in the algorithm of Proposition 3.1 into rows and columns $i_1$ and $i_2$ in $A^\triangledown$.

Inductively, we assume that any $m - 1 \times m - 1$-minor $M_{i_{m+1},\ldots,i_n}$ of $A^\triangledown$ is in $\mathcal{E}$, and show that $M_{i_{m+1},\ldots,i_n} \in \mathcal{E}$. \hfill \Box
By Proposition 6.4, in order to recover row $i_m$, we need to verify that
\[ a'_{i_m,j} \geq a'_{i_m,k} a'_{k,j} \quad \forall i_m \neq j \text{ and } k \neq j, i_m, \ldots, i_n \]
where $a'_{r,s}$ are the entries of $A^\nabla$ and the index elements of our minor are $i_1, \ldots, i_m$. By Corollary 6.2 the entries of $A^\nabla$ are $\nu$-equivalent to the entries $a_{r,s}$ of $A^\nabla$. Hence, we may use the entries of $A^\nabla$ and need to show that
\[
(6.10) \quad \sum_{\pi \in S_m : \pi(j) = i_m} a_{1,\sigma(1)} \cdots a_{j-1,\sigma(j-1)} a_{j+1,\sigma(j+1)} \cdots a_{i_m,\sigma(i_m)}
\]
dominate the product between
\[
(6.11) \quad \left( \sum_{\sigma \in S_m : \sigma(k) = i_m} a_{1,\sigma(1)} \cdots a_{k-1,\sigma(k-1)} a_{k+1,\sigma(k+1)} \cdots a_{i_m,\sigma(i_m)} \right)
\]
and
\[
\left( \sum_{\tau \in S_m : \tau(j) = k} a_{1,\sigma(1)} \cdots a_{j-1,\sigma(j-1)} a_{j+1,\sigma(j+1)} \cdots a_{i_m,\sigma(i_m)} \right)
\]
for every $i_m \neq j$ and $k \neq j, i_m, \ldots, i_n$.

Indeed, every summand in the product (6.11) can be factored into a product of a term from (6.10) and permutation tracks of $A$ which are $\nu$-smaller than $1_R$, causing each summand in the product (6.11) to be dominated by some summand from (6.10), as desired. We prove this property, similarly to Claim 6.1, by looking at the simple cycles of $i_m$ and $j$ in the terms of $\sigma$ and $\tau$ in the product (6.11) for every $\sigma, \tau \in S_m$ such that $\sigma(k) = i_m, \tau(j) = k$. For both permutations we need to distinguish between cases in which $i_m$ and $j$ are in the same simple cycle or not, where $P$ and $P'$ are the products of the remaining simple cycle tracks in $\sigma$ and $\tau$ respectively. Hence, there are two possible types of summands:

I- Where $i_m$ and $j$ share the same simple cycle in at least one of the permutations:
\[
\begin{align*}
\left[ (a_{k,\tau(k)} \cdots a_{\tau^{-1}(i_m),i_m} a_{i_m,\tau(i_m)} \cdots a_{\tau^{-1}(j),j}) P' \right] & \left[ (a_{i_m,\sigma(i_m)} \cdots a_{\sigma^{-1}(j),j} a_{j,\sigma(j)} \cdots a_{\sigma^{-1}(k),k}) P \right], \\
\left[ (a_{k,\tau(k)} \cdots a_{\tau^{-1}(i_m),i_m} a_{i_m,\tau(i_m)} \cdots a_{\tau^{-1}(j),j}) P' \right] & \left[ (a_{i_m,\sigma(i_m)} \cdots a_{\sigma^{-1}(k),k} (a_{j,\sigma(j)} \cdots a_{\sigma^{-1}(j),j}) P \right]
\end{align*}
\]
or
\[
\begin{align*}
\left[ (a_{k,\tau(k)} \cdots a_{\tau^{-1}(j),j}) (a_{i_m,\tau(i_m)} \cdots a_{\tau^{-1}(i_m),i_m}) P' \right] & \left[ (a_{i_m,\sigma(i_m)} \cdots a_{\sigma^{-1}(j),j} a_{j,\sigma(j)} \cdots a_{\sigma^{-1}(k),k}) P \right].
\end{align*}
\]

The marked sequence, when composed with disjoint Id cycle tracks, is a deficient track of $j$, which is a summand from (6.10) (where the permutation image of $j$ is $i_m$). By excluding it we "ignore" all the index elements of these entries once. Since each index element appears the same number of times in the first and second indices, the rest of the monomial is composed from simple cycle tracks, each can be viewed as a permutation track of $A$ when composed with the Id, and therefore is $\nu$-smaller than $1_R$. 
II- Where \( i_m \) and \( j \) do not share the same simple cycle, neither in \( \sigma \) or in \( \tau \):

\[
\left[ (a_{i_m,\sigma(i_m)} \cdots a_{\sigma^{-1}(k),k}) (a_{j,\sigma(j)} \cdots a_{\sigma^{-1}(j),j}) P \right] \left[ (a_{k,\tau(k)} \cdots a_{\tau^{-1}(j),j}) (a_{i_m,\tau(i_m)} \cdots a_{\tau^{-1}(i_m),i_m}) P' \right].
\]

We compose the two marked sequences, which when composed with disjoint Id cycle tracks, would include a deficient track of \( j \), which is a summand from (6.10) (where the permutation image of \( j \) is \( i_m \)). If the two sequences include common index elements, then since they appear the same number of times in the first and second indices this composite would include simple cycle tracks as well. The rest of the monomial is composed once again from simple cycle tracks, and therefore \( \nu \)-smaller than \( 1_R \).

\[ \square \]

**Corollary 6.6.** If \( A \) is a nonsingular matrix, then \( A^{\nabla \nabla} \) is factorizable.

**Proof.** Let \( \tilde{A} \) be the normal form of \( A \). By Lemma 6.5 we may factor any \( k \times k \) minor of \( A^{\nabla \nabla} \), including \( A^{\nabla \nabla} \) itself. Using Lemma 5.7 where \( A = \tilde{E} \tilde{A} \), we get that \( A^{\nabla \nabla} = \tilde{E} A^{\nabla \nabla} \) is factorizable as well. \[ \square \]

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