Finite element modeling of creep bending of polymer plates

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Abstract. Thin rigid plates are considered in the article and the derivation of the equations of bending of a quadrangular finite element of a plate is given that takes creep into account. The equations are derived using the Lagrange variational principle. The relationships between the integral characteristics of stresses (linear bending and torque moments) and the curvatures of the middle surface are taken as physical relationships. Given as an example, the problem of analyzing a rectangular polymer plate made of secondary PVC, hinged along the contour and loaded with a load uniformly distributed over the area, is reduced to a system of linear algebraic equations. Maxwell-Gurevich equation was chosen as the equation of state between stresses and creep strains. Graphs of stress and deflection changes over time are presented. The stresses in the creep process change insignificantly, the difference between the stresses at the beginning and at the end of the creep process is less than 5%.

1. Introduction

Many structural materials are characterized by the creep phenomenon, i.e. development of deformations under constant loads. At the same time, at the moment there are no general methods for analyzing structures and their elements, taking into account the rheology of the material. However, there are gaps and some issues remain poorly studied, in particular, the problem of analyzing orthotropic plates of variable thickness. Some particular solutions for bar elements [1-4], plates [5] and shells [6] are given in the literature. In [5], a method is considered for analyzing rectangular plates taking into account creep by the finite difference method, but this method is not applicable to plates of arbitrary shape.

In this article, we derive the equations of bending with allowance for creep for a flat quadrangular finite element, which makes it possible to analyze plates of arbitrary shape. Well-established finite element methods are available for solving this problem.

2. Materials and methods

The stress-strain state of the bending of the plates is fully described by one function of the deflections of the middle plane (after deformation - the surface) \( w \). When bending the plates, not only the condition of continuity of displacements, but also of its derivatives must be observed. Therefore, at each node it is necessary to consider at least three unknowns (deflection and two rotation angles).

The end element has 12 possible displacements (one linear offset and two rotation angles at each node). The element's stiffness matrix will be of 12th order. We consider a rectangular bendable element of size \( a \times b \) in a rectangular coordinate system (the \( z \) axis is normal to the median plane). The element is shown in Figure 1.
Taking into account that functional (1) of the total potential energy, which includes the 2nd derivatives of the deflection function, we can conclude that the degree of the approximating polynomial should not be less than 2.

\[ \Pi = \frac{1}{2} \int_{\Omega} \{ e^{\varepsilon} \}^T \{ \sigma \} \, d\Omega - \int_{\Omega} q(x, y) w \, d\Omega \]  

These considerations lead to an approximation of the displacements over the area of a finite element (FE) using an incomplete 4th degree polynomial in two variables:

\[ w(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 y^2 + \alpha_6 xy + \alpha_7 x^2 y + \alpha_8 xy^2 + \alpha_9 x^3 + \alpha_{10} y^3 + \alpha_{11} x^3 y + \alpha_{12} x y^3 \]  

(2)

According to the designations adopted in the technical theory of bending of plates, we will consider:

\[ \varphi_x = \frac{\partial w}{\partial x}; \quad \varphi_y = \frac{\partial w}{\partial y}; \]

Nodal displacements are matched with reactions in additional bonds, i.e. concentrated force \( R_i \) and concentrated moment \( M_{xi} \) and \( M_{yi} \).

Let us form a system of algebraic equations for constant coefficients \( \alpha_i \), substituting the coordinates of the nodes in the displacement functions. The vector of functions describing the accepted displacements is composed of the deflection function (2) and its first derivatives:

\[ \frac{w}{w} \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & x & y & x^2 & y^2 & xy & x^2 y & xy^2 & x^3 & y^3 & x^3 y & xy^3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \\ \alpha_9 \\ \alpha_{10} \\ \alpha_{11} \\ \alpha_{12} \end{bmatrix} \]  

(3)

Substituting the coordinates of the four nodes of the finite element (0,0), (a, 0), (a, b) and (0, b) alternately into expression (3), we get:

\[ \{ U \} = [ C ] \{ \alpha \} \]  

(4)

where

\[ \{ U \} = \{ w_1, \varphi_{x1}, \varphi_{y1}, w_2, \varphi_{x2}, \varphi_{y2}, w_3, \varphi_{x3}, \varphi_{y3}, w_4, \varphi_{x4}, \varphi_{y4} \}^T \]
\( \{U\} \) is a vector of nodal displacements, \([C]\) is a matrix of coefficients for unknowns \( \alpha_i \); \( \{\alpha\} \) is a vector of constant coefficients \( \alpha_i \).

Solving system (4), we determine the coefficients of the displacement function (2):

\[ [\alpha] = [C]^{-1}\{U\} \]  

where

\[
[C]^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{3}{a^2} & -\frac{2}{a^2} & \frac{3}{a^2} & -\frac{1}{a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{3}{b^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{b^2} & -\frac{1}{b} \\
-\frac{1}{ab} & -\frac{1}{ab} & -\frac{1}{ab} & 0 & 1 & -\frac{1}{ab} & 0 & 0 & \frac{1}{ab} & \frac{1}{ab} & 0 & 0 \\
-\frac{3}{ab^2} & 0 & -\frac{3}{ab^2} & 0 & -\frac{2}{ab^2} & 0 & -\frac{1}{ab^2} & -\frac{3}{ab^2} & -\frac{3}{ab^2} & 0 & 1 & 0 \\
2 & 1 & 0 & -\frac{2}{a^2b} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{2}{a^3} & 0 & 1 & -\frac{2}{a^2b} & 0 & -\frac{2}{a^2b} & 0 & \frac{2}{a^2b} & \frac{2}{a^2b} & 0 & 1 & 0 \\
-\frac{2}{ab^3} & 0 & 1 & -\frac{2}{ab^3} & 0 & -\frac{2}{ab^3} & 0 & \frac{2}{ab^3} & \frac{2}{ab^3} & 0 & 1 & 0 \\
\end{bmatrix}
\]

The curvature vector will be written as follows:

\[
\begin{bmatrix}
\frac{\partial^2 w}{\partial x^2} \\
\frac{\partial^2 w}{\partial y^2} \\
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 & 2 & 0 & 0 & 2y & 0 & 6x & 0 & 6xy & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 2x & 0 & 6y & 0 & 6xy \\
\end{bmatrix} [C]^{-1}\{U\} = [\Phi][C]^{-1}\{U\}
\]

The vector of elastic deformations will take the form:

\[ \{\varepsilon^{el}\} = \{\varepsilon\} - \{\varepsilon^*\} = -z[\Phi][C]^{-1}\{U\} - \{\varepsilon^*\} \]  

(6)

where \( \{\varepsilon^*\} = \{\varepsilon_x^* \ v_y^* \ \gamma_z^*\}^T \) is a vector of creep deformations.

We assume that within one element, creep deformations are functions of only \( z \). The stresses can be determined as follows:

\[
\{\sigma\} = \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\end{bmatrix} = [D]\{\varepsilon^{el}\} = [D]\{\varepsilon\} - \{\varepsilon^*\} = [D](-z[C]^{-1}\{U\} - \{\varepsilon^*\})
\]

(7)

where \( [D] = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \frac{1-\nu}{2} \\
\end{bmatrix} \) is a matrix of elastic constants.

After substitution of (6) and (7) in (1) and subsequent differentiation of potential energy by nodal displacements, we get:
\[
\frac{\partial \Pi}{\partial \{U\}} = \frac{h^3}{12} [C]^{-1 \Gamma} \int_0^b \int_0^a [\Phi]^{T} [D] [\Phi] dx dy [C]^{-1} \{U\} + [C]^{-1 \Gamma} \int_0^b [\Phi]^{T} [D] dx dy \cdot \int_{-\frac{h}{2}}^{\frac{h}{2}} \{e^*\} dz;
\]

\[
\frac{\partial \Pi}{\partial \{U\}} = [K] \cdot \{U\} - \{F^*\}
\]

here: \([K] = \frac{h^3}{12} [C]^{-1 \Gamma} \int_0^a \int_0^b [\Phi]^{T} [D] dx dy [C]^{-1}\) is a local stiffness matrix; \([F^*] = -[C]^{-1 \Gamma} \int_0^a \int_0^b [\Phi]^{T} [D] dx dy \cdot \int_{-\frac{h}{2}}^{\frac{h}{2}} \{e^*\} dz\) is a contribution of creep deformations to the load vector.

The stiffness matrix \([K]\), when calculated taking into account creep, does not differ from the elastic one, therefore, it is not presented here.

3. The problem solution and results

The calculation was performed for a rectangular hinge supported along the contour plate loaded with a uniformly distributed load \(q\) (Figure 2).

![Figure 2. Calculation scheme of the plate](image)

Plate material - secondary PVC, elastic modulus \(E = 1480\) MPa, Poisson’s ratio \(\nu = 0.3\), load value \(q = 2\) kPa, plate dimensions: \(a = 0.8\) m, \(b = 0.6\) m, plate thickness \(h = 2\) cm. PVC rheological parameters at \(t = 20^\circ C\): \(\eta_0 = 5590\) Mpa, \(\eta_0 = 9.06 \cdot 10^5\) MPa, \(m^* = 12.6\) MPA.

The nonlinear Maxwell-Gurevich equation was used as the creep law, which in a plane stress state is written in the form:

\[
\frac{\partial \epsilon_{ij}^*}{\partial t} = \frac{f_{ij}}{\eta^*}, \quad i = x, y, \quad j = x, y,
\]

where \(f_{ij}\) – stress function, \(\eta^*\) – relaxation viscosity.

\[
f_{ij} = \frac{3}{2} (\sigma_{ij} - p \delta_{ij}) - E_{\alpha \alpha} \epsilon_{ij}^*,
\]

where \(p = \frac{\sigma_x + \sigma_y}{3}\) – medium stress, \(\delta_{ij}\) – Kronecker symbol, \(E_{\alpha \alpha}\) – high elasticity modulus.

\[
\frac{1}{\eta^*} = \frac{1}{\eta_0} \exp \left( \frac{f_{\text{max}}^*}{m^*} \right),
\]

where \(\eta_0\) – initial relaxation viscosity, \(m^*\) – speed module.

For shear creep deformation: \(\gamma_{xy}^* = 2\epsilon_{xy}^*\).

The resulting graph of deflection growth in the center of the plate is shown in Figure 3. Note that for plates whose material obeys the Maxwell-Gurevich equation, the deflection ratio at \(t \to \infty\) and \(t = 0\) should be equal to:
\[
\frac{w(\infty)}{w(0)} = \frac{\bar{D}}{D_{\infty}}
\]

where \(\bar{D} = \frac{Eh^3}{12(1-\nu^2)}\) is the cylindrical plate stiffness, \(D_{\infty}\) is the long-term cylindrical stiffness:

\[
D_{\infty} = \frac{\alpha h^3}{12(\alpha^2-\beta^2)}
\]

where \(\alpha = \frac{1}{E} + \frac{1}{E_\infty}, \beta = \frac{\nu}{E} + \frac{1}{2E_\infty}\).

Figure 3. Graph of deflection growth in the center of the plate

According to the results of numerical calculations, the ratio \(\frac{w(\infty)}{w(0)}\) was 1.2092, which differs from the exact value by 0.26% and indicates the reliability of the obtained equations and methods. In order to control the correctness of the results, the problem for a rectangular plate was solved using double trigonometric series.

Figure 5 shows graphs of the deflection growth for different numbers of members of the series \(k\). From the presented graphs it can be seen that for \(k = 4\) (\(m = 1, 3\) and \(n = 1, 3\)) and \(k = 9\) (\(m = 1, 3, 5\) and \(n = 1, 3, 5\)) the results practically coincide. Comparison of the results for \(k = 4\) with the solution obtained on the basis of the FEM is given in Table 1.

Figure 4. Time variation of the highest stresses
The largest discrepancy between the results obtained by the two methods is 1.83%, which indicates their reliability.

**Table 1.** Comparison of the deflections in the center of the plate at different points in time, obtained on the basis of the analytical solution $w_1$, with the solution using the FEM $w_2$

| $t$, час | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|---------|----|----|----|----|----|----|----|----|----|
| $w_1$, мм | 1.5813 | 1.6981 | 1.7781 | 1.8334 | 1.8717 | 1.8984 | 1.9171 | 1.9302 | 1.9394 |
| $w_2$, мм | 1.6109 | 1.7260 | 1.8013 | 1.8509 | 1.8836 | 1.9052 | 1.9195 | 1.9291 | 1.9354 |

**Figure 5.** Graphs of deflection growth for different numbers of members of the series

**4. Conclusion**

A method of finite element calculation using quadrangular finite elements has been developed. Comparison of the numerical analytical problem with the use of double trigonometric series and the finite element method is given. On the example of a plate made of recycled polyvinyl chloride, it is shown that the stresses change insignificantly during the creep process.

It has been found that at the beginning of the creep process, stress relaxation occurs, and then there is a return to an elastic solution.

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