WT IDENTITIES FOR PROPER VERTICES AND RENORMALIZATION IN A SUPERSPACE FORMULATION OF GAUGE THEORIES

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Abstract

We formulate the WT identity for proper vertices in a simple and compact form $\frac{\partial \Gamma}{\partial \theta} = 0$ in a superspace formulation of gauge theories proposed earlier. We show this WT identity (together with a subsidiary constraint) lead, in transparent way, the superfield superspace multiplet renormalizations formulated earlier (and shown to explain symmetries of Yang-Mills theory renormalization).

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I. INTRODUCTION

Yang-Mills theories have acquired a central place in the theoretical formulation of strong, weak and electro-magnetic interactions (the Standard Model) [1]. Central to Yang-Mills theories is the local gauge invariance of the basic action. This basic gauge invariance manifests, in particular, as the global BRS invariance of the effective action [2]. The consequences of BRS invariance, formulated as WT identities, are central to the discussion of renormalizability, unitarity, gauge independence of the theory [2][3][4]. Any attempt that sheds light on, offer a reformulation of, and leads to simplified understanding of BRS symmetry and Yang-Mills theory is therefore of significance to particle physics.

With the above aim in mind, viz. to simplify the expression of BRS invariance, of WT identities and to simplify the discussion of renormalization of gauge theories, a superspace formulation of gauge theories was proposed [5]. It was an improvement upon a number of earlier superfield /superspace formulations [6][7][8] in a number of ways. Firstly, unlike the earlier formulations where structure of superfields was, in effect, determined by hand, the superfield here [5] had all their components free. The 6-dimensional superspace was also one in which superspace rotations could be freely carried out [unlike Ref. 7]. BRS transformations were effectively generated from within. The sources for fields and BRS composite operators arose out of a multiplet sources structure. This has been summarized in Ref. 5.

The superspace formulation of Ref. 5 was further developed in a number of ways. It was shown that the WT identities in Yang-Mills theories could be cast in a neat and compact form, \( \frac{\partial \bar{W}}{\partial \theta} = o \) [9]. It was further shown that this compact form can be derived by simply considering an \( OSp(3, 1|2) \) rotation in superspace which is an approximate symmetry of the superspace generating functional \( \bar{W} \) [10]. Thus in Ref. 10 it was shown that, a coordinate rotation in superspace contains all the consequences of the field transformation (BRS). It was later shown that this coordinate rotation is equivalent to a set of local BRS transformations [11]. The superspace of Ref. 5 was generalized to include the anti-BRS
symmetry [12] scalars and gauge invariant operators [13].

Of particular interest to the present work are results of Ref. 14, where it was shown that renormalization transformations in Yang-Mills theories take a simple form when expressed in the superspace formalism. Interrelations between renormalizations of all sources for fields and composite operators is shown to arise from the fact that they belong together in source supermultiplets which transform as whole under renormalization. The OSp-symmetry breaking piece of the Lagrangian, $L_1$ (that generates a large numbers of terms: sources for the fields and composite operators, gauge-fixing and ghost term) was shown to be form invariant under renormalization transformations.

One drawback of the result of Ref. 14, despite their attractiveness, has been that they were not directly "derived" from the superspace WT identity but shown to be true by invoking the correspondence between superspace generating functional and the ordinary Yang-Mills theory established in Ref. 5. The main aim of this work is to fill up this technical gap and complete the derivation of the superspace renormalization of gauge theories.

A natural tool for discussing renormalization is the WT identities for proper vertices. This was not done in Ref. 9 and 10, where the aim was to formulate the WT identity for $\bar{W}$ in a simple form. As done in section II of this work sources now have to be introduced for all elementary fields (including of those called $A_{i,\lambda}$) and this was not necessary in Ref. 9, 10. These modifications allow one to Legendre transform and cast the WT identity for $\Gamma$ in a equally simple form $\frac{\partial \Gamma}{\partial \theta} = 0$ (section III). This simplifies the discussion of renormalization of gauge theories as seen in section IV. The simplification is seen to arise from the fact that the nilpotent operator $\frac{\partial}{\partial \theta}$ independent of the variables (fields, coupling constant etc) that get renormalized.

**II. PRELIMINARY**

In this section, we shall introduce our notations, review past results and introduce slightly modified generating functional of superspace formulation.
A. The Superspace Formulation

We shall work in the context of a pure Yang-Mills theory with simple gauge group given by the Lie algebra of generators.

\[ [T^\alpha, T^\beta] = i f^{\alpha\beta\gamma} T^\gamma \]  \hspace{1cm} (2.1)

and the covariant derivative

\[ D_\mu c^\beta = \left( -\partial_\mu \delta^\alpha_{\beta} + g f^{\alpha\beta\gamma} A_\mu^\gamma \right) c^\beta \]  \hspace{1cm} (2.2)

\( f^{\alpha\beta\gamma} \) are totally anti-symmetric.

The generating functional for the Green’s functions in linear gauges, (with additional sources for BRS variations introduced) is given by

\[ W = \int [dA dc d\zeta] \exp \left( iS + i \int d^4x \left( j_\mu^\alpha A^\alpha_\mu + \bar{\xi}^\alpha c^\alpha + \zeta^\alpha \xi^\alpha + \kappa^\alpha \mu D^\alpha_\mu c^\beta + \frac{g}{2} l^\alpha f^{\alpha\beta\gamma} c^\beta c^\gamma \right) \right) \equiv W[j, \bar{\xi}, \xi, \kappa, -l, t] \]  \hspace{1cm} (2.3)

Where the action \( S \) is given by,

\[ S = S_0 + S_g + S_{gf} \]  \hspace{1cm} (2.4)

With

\[ S_0 = \int d^4x \left\{ -\frac{1}{4} F^\alpha_\mu F^{\alpha\mu} \right\} \]
\[ S_g = \int d^4x \left\{ -\partial_\mu \xi^\alpha D^\alpha_\mu c^\beta \right\} \]
\[ S_{gf} = \int d^4x \left\{ -\frac{1}{2\eta_0} \left( \partial \cdot A^\alpha + t^\alpha \right)^2 \right\} \]  \hspace{1cm} (2.5)

We shall briefly introduce the superspace formulation first introduced in Ref. 5. It was a six dimensional superspace which has two anti-commuting dimensions \( \lambda \) and \( \theta \) such that the 6-dimensional coordinate vector is \( \bar{x}^i \equiv (x^\mu, \lambda, \theta) \). \( \lambda \) and \( \theta \) are scalars under the Lorentz transformations. In this space, a metric is introduced whose non-vanishing components are \( g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{45} = g_{54} = 1 \). The group of linear homogeneous transformations that preserves \( \bar{x}^i g_{ij} \bar{x}^j \) is \( OSp(3, 1|2) \).
The superspace formulation of Ref. 5 introduces superfields $\bar{A}_\alpha^i(\bar{x})$ and $\zeta^\alpha(\bar{x})$ transforming as a covariant vector and a scalar under $OSp(3,1|2)$. The superfield $\bar{A}_\mu^\alpha(\bar{x})$ contains a component $A^\alpha_\mu(x)$ which is identified with the the usual Yang-Mills field; $A_5^\alpha(x) \equiv c^\alpha_5(x)$ is identified with the usual ghost field and $\zeta^\alpha(x)$ with anti-ghost field. The remaining fields in the expression of $A^\alpha_\mu(\bar{x})$ and $\zeta^\alpha(\bar{x})$ in terms of $\lambda$ and $\theta$ are certain auxiliary fields whose role is primarily to generate the BRS constraints from within the superfield action. We also introduce two vector supersources $K^{\alpha i}(\bar{x}), L^{\alpha i}(\bar{x})$ and a scalar supersource $t^\alpha(\bar{x})$. As was shown in Ref. 5, the source $K^{\alpha i}(\bar{x})$ contains compactly in it sources for the gauge field, the ghost field and also the sources for the BRS variation composite operators ($\kappa$ and $l$) all in a single supermultiplet of $OSp(3,1|2)$; and a similar statement holds for $t(\bar{x})$.

In this work, we use a superspace formulation which is somewhat modified compared to Ref.5. We introduce the superspace action

$$\bar{S} = \int d^4x \left\{ -\frac{1}{4} g^{ij} g^{kl} F_{ij}^\alpha F_{kl}^\alpha \right\} + S_1 \equiv S_0 + S_1$$

(2.6)

with the symmetry breaking piece,

$$S_1 = \int d^4x \left[ \frac{\partial}{\partial \theta} \left( K^{\alpha i} \bar{A}_i^\alpha(\bar{x}) + \zeta^\alpha \left[ \partial^\mu A_\mu^\alpha(\bar{x}) + \frac{1}{2\eta_0} \zeta_\theta^\alpha(\bar{x}) + t^\alpha(\bar{x}) \right] \right) + \frac{\partial}{\partial \lambda} [L^{\alpha i} \bar{A}_i^\alpha] + \partial \cdot L\zeta \right]$$

(2.7)

and $F_{ij}^\alpha$, superspace field strength tensor, is defined as

$$F_{ij}^\alpha \equiv \partial_i \bar{A}_j^\alpha(\bar{x}) - \bar{A}_i^\alpha(\bar{x}) \partial_j + g F^{\alpha \beta \gamma} \bar{A}_i^\beta(\bar{x}) \bar{A}_j^\gamma(\bar{x})$$

(2.8)

where $i, j$ takes values $0, 1 \cdots 5$.

We, further, introduce the generating functional

$$\bar{W}[K, L, t] = \int \left\{ d\bar{A} \right\} \left\{ d\zeta \right\} e^{iS[\bar{A}, \zeta, K, L, t]}$$

(2.9)

Where the measure is defined as

$$\left\{ d\bar{A} \right\} = \left\{ dA \right\} \left\{ dc_4 \right\} \{ dc_5 \}$$

$$\left\{ dA \right\} = \prod_{\alpha \mu x} dA_\mu^\alpha(\bar{x}) \; dA_{\mu,\lambda}^\alpha(\bar{x}) \; dA_{\mu,\theta}^\alpha(\bar{x}); \quad 0 \leq \mu \leq 3$$

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The generating functional in Eq. 2.9 differs from the one in Ref. 5 only by the addition of a new sources $\tilde{L}^{\alpha i}(\bar{x})$ in Eq. 2.7. This source has been added to provide a source for the field $A_{i,\lambda}$ [this is necessary before a Legendre transformation can be carried out to generate $\Gamma$, as a source for every field is needed]. The last term in $S_1$ viz. $\partial \cdot L\zeta$ is added not so much out of necessity, but for the sake of certain anticipated simplifications.

We note that it is enough to consider $\tilde{W}$ at $L^{i,\lambda} = 0 = L^i_{,\theta}$ because $L^{i,\lambda}$ is only a part of a source for $A_i$ (and hence can be dropped) and $L^i_{,\theta}$ is not a source for any field. Hence, in future we shall assume that $L^{i,\lambda} = 0 = L^i_{,\theta}$.

**B. Review of some of the earlier results**

Here, we review some of the relevant earlier results. The generating functional of Eq. 2.9, at $L = 0$, has been shown to be related to the generating functional of Yang-Mills theory of Eq. 2.3 via

$$\int [dK^4][dK_{,\theta}^4]W[\bar{K}, L, t] \bigg|_{L=0} = W [K^{\alpha\mu}; K_{,\theta}^{\alpha5}; t_{,\theta}; K^{\alpha\mu}; K^{\alpha5}; t^{\alpha}] \quad (2.10)$$

The WT identities of gauge theories have been cast in a simple form, at $L = 0$ [3]

$$\frac{\partial \tilde{W}}{\partial \theta} = 0 \quad (2.11)$$

The underlying Lagrangian possesses broken $OSp(3,1|2)$ invariance in the superspace. It has been shown that a superrotation in superspace takes the place of BRS transformations in this formulation and this has been used to derived 2.11 directly [10].

It has been shown [14] that the renormalization transformations of sources for fields and composite BRS variation operators, when cast in the superspace formulation [using the correspondence in Eq. 2.10 and 2.3], take a very simple form. They consist of (i) a supermultiplet renormalization as a whole (ii) renormalizations of $\lambda$ and $\theta$ (iii) an optional $Sp(2)$ rotation applied uniformly everywhere [to $(\lambda, \theta)$ to components of a vector].
nontrivial relation between the renormalization constants of all sources for fields and composite operators are shown to arise from the fact that they belonged to supermultiplets. A “nonrenormalization theorem” was proved for the symmetry breaking piece of the action $S_1$ (which generates sources for fields and composite operators, gauge-fixing term and the ghost term) that $S_1$ is left form invariant under renormalization.

The results mentioned above from Ref. [14], were not “derived” directly but shown to hold using the correspondence between the superspace formulation and the usual Yang-Mills theory as given in Eq. 2.10 and 2.3. One of the purposes of the present work is to derive them and show how simply they arise once the WT identity for generating functional $\Gamma$ for proper vertices is formulated. The task of obtaining this WT identity for $\Gamma$ taken up in the next section.

III. DERIVATION OF WT IDENTITIES FOR PROPER VERTICES

The purpose of this section is to derive the WT identity for the generating functional for proper vertices. The procedure is generally straightforward. We derive the WT identity for $\tilde{W}$ (or rather a related quantity $\tilde{W}'$ of Eq. 3.1 below); and use Legendre transform of $\tilde{Z}$ to obtain the desired result. Only a few subtleties not encountered in the treatment in the ordinary space have to be dealt with carefully.

We shall find it convenient to deal not with $\tilde{W}$ of Eq. 2.10 but rather with

$$\tilde{W}' \equiv \int [dK^4][dK^4_{\bar{\partial}}]\tilde{W}[^K, ^L, t]$$

(3.1)
as this is the quantity related to generating functional of ordinary Yang-Mills theory via an equation like 2.10.

Finally, we shall derive the WT identity satisfied by $\tilde{W}'[^K, ^L, t]$. The derivation here follows the method of Ref. 9 as it is more direct. We shall find it convenient to use the form of $\tilde{W}'[^K, ^L, t]$ that is obtained by integrating out the auxiliary fields. We obtain $\tilde{W}'$, after these integrations as,
\[ \bar{W}' = \int [d \text{Ad}c_5 d \zeta] \exp \left[ i \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (K^\mu - \partial^\mu \zeta) Dc_5 - \frac{1}{2} K^5 g f c_5 c_5 
- \frac{\eta_0}{2} (\partial \cdot A + t)^2 + K^\mu_\theta A_\mu + K^5_\theta c_5 + t_\theta \zeta - K^\mu L^\mu + \frac{1}{2} K^5 L^4 - \frac{1}{2} (L^5)^2 \right\} \right] \] (3.2)

We then proceed in the following steps as in Ref. 9. (i) Evaluate \( \frac{\partial \bar{W}'}{\partial \theta} \) explicitly using Eq. 3.2 as

\[ \frac{\partial W'}{\partial \theta} = K^\mu_\theta < Dc_5 > - \frac{1}{2} K^5_\theta < g f c_5 c_5 > - < \eta_0 (\partial \cdot A + t) > t_\theta 
+ i \frac{\partial}{\partial \theta} \left[ -K^\mu L^\mu + \frac{1}{2} K^5 L^4 - \frac{1}{2} (L^5)^2 \right] \bar{W}' \] (3.3)

(ii) Obtain a WT identity using BRS invariance of the action appearing on the right hand side of Eq. 3.2.

\[ K^\mu_\theta < Dc_5 > - \frac{1}{2} K^5_\theta < g f c_5 c_5 > - < \eta_0 (\partial \cdot A + t) > t_\theta = 0 \] (3.4)

(iii) We next simplify the result for \( \frac{\partial \bar{W}'}{\partial \theta} \) in Eq. 3.3 using Eq. 3.4 to obtain the WT identity.

\[ \frac{\partial \bar{W}'}{\partial \theta} = -i \frac{\partial}{\partial \theta} \left[ K^\mu L^\mu - \frac{1}{2} K^5 L^4 + \frac{1}{2} (L^5)^2 \right] \bar{W}' \] (3.5)

Note that Eq. 3.3 is consistent with the simple result \( \frac{\partial \bar{W}}{\partial \theta} = 0 \) of Ref. [10] at \( L = 0 \).

The generating functional of constructed Green’s functions \( \bar{Z}[\bar{K}, \bar{L}, t] \) is given by

\[ \bar{Z}[\bar{K}, \bar{L}, t] = -i \ln \bar{W}'[\bar{K}, \bar{L}, t] \] (3.6)

[Here we note that \( \bar{W}' \) is a function of \( \lambda \) and \( \theta \) (through sources). The logarithm of a function of a Grassmannian defined as \( \log(a + b \theta) = \log(a(1 + \frac{b \theta}{a})) = \log(ae^{\frac{b \theta}{a}}) = \log a + \frac{b \theta}{a} \).

Similarly, \( \log(a + b \theta + c \lambda) = \log a + \frac{b \theta}{a} + \frac{c \lambda}{a} + \frac{bc \lambda \theta}{a} \) etc.]

Next we define the generating functional for proper vertices

\[ \Gamma = \bar{Z} - \sum_i \int S_i(x) \frac{\delta \bar{Z}}{\delta S_i(x)} d^4x \] (3.7)

Where \( S_i(x) \) generally denotes all sources (as function of \( x \) only) that appear for elementary fields viz. \( S_i(x) \equiv \{ K^\mu, K^\mu_\theta, K^5, K^5_\theta, L^\mu, L^5, L^4, t, t_\theta \} \). From now on we shall find it
convenient to drop explicit $\lambda$ occurring in $Z$ and $\Gamma$ though it is not necessary. Then Eq. 3.7 can alternatively be written as

$$\Gamma = \bar{Z} - \sum_{S_i(x)} \int S_i(x) \frac{\delta' \bar{Z}}{\delta S_i(x)} d^4x$$  \hspace{1cm} (3.8)$$

Here, now, $S_i(x)$ are function of $x$ and $\theta$ and $\frac{\delta'}{\delta S_i(x)}$ denotes differentiating $S_i(x)$ taking into account only the explicit $S_i(x)$ dependence. [ e.g. while differentiating with respect to $K^\mu$, we don’t consider the $K^\mu_\theta$ dependence in $K^\mu(x)$]. This is easily verified using the identities such as

$$\frac{\delta \bar{Z}}{\delta K^\mu(x)} = \frac{\delta' \bar{Z}}{\delta K^\mu(x)}$$

$$\frac{\delta \bar{Z}}{\delta K^\mu_\theta(x)} = \frac{\delta' \bar{Z}}{\delta K^\mu_\theta(x)} - \theta \frac{\delta' \bar{Z}}{\delta K^\mu(x)}$$  \hspace{1cm} (3.9)$$

$\Gamma$ defined in Eq. 3.7 is a functional of expectation values as usual. By rearrangement of variables, it can also re-expressed as a functional of variables $\frac{\delta' \bar{Z}}{\delta S_i(x)}$. Thus $\Gamma$ can be looked upon as function of

$$\Gamma = \Gamma [\langle A^\mu_\theta (\bar{x}) \rangle, \langle A^\mu_\theta \rangle, \langle c_5(\bar{x}) \rangle, \langle c_{5,\theta} \rangle, \langle A'^\mu_{\mu,\lambda} \rangle, \langle c_{5,\lambda} \rangle, \langle c_{4,\lambda} \rangle, \langle \zeta \rangle, \langle \zeta_\theta \rangle]$$  \hspace{1cm} (3.10)$$

Now, just as $\bar{W}'$ had no explicit dependence on $\theta$, when looked up on as a functional of $\bar{K}(\bar{x}), K_\theta(x), L(x), t(x), t_\theta(x)$ [i.e. all its $\theta$-dependence arose from its dependence on sources $\bar{K}(\bar{x}), t(\bar{x})$], it can be shown [ See Appendix] that $\Gamma$ seen as functional of variables listed in Eq. 3.10 has no explicit dependence on $\theta$ either.

Now from Eq. 3.3, $\bar{Z}$ can be shown to satisfy

$$\frac{\partial \bar{Z}}{\partial \theta} = -\frac{\partial}{\partial \theta} \left[ K^\mu L^\mu - \frac{1}{2} K^5 L^4 + \frac{1}{2} (L^5)^2 \right]$$  \hspace{1cm} (3.11)$$

This then leads, straightforwardly, using Eq. 3.8 to the WT identity for $\Gamma$.

$$\frac{\partial \Gamma}{\partial \theta} = L^\mu K^\mu_\theta - \frac{1}{2} L^4 K^5_\theta$$  \hspace{1cm} (3.12)$$

Here, we understand that $L^\mu, L^4, K^\mu_\theta$ and $K^5_\theta$ are to be expressed back in terms of expectation values. Here, we note that the arguments of $\Gamma$ are, as mentioned earlier,
\[ \langle A_\mu(x) \rangle = \frac{\delta' \bar{Z}}{\delta K_{\mu,\theta}(x)} ; \quad \langle A_{\mu,\theta}(x) \rangle = \frac{\delta' \bar{Z}}{\delta K_{\mu}(x)} \]

\[ \langle A'_{\mu,\lambda} \rangle = \langle A_{\mu,\lambda} + \partial_\mu \zeta \rangle \equiv -\frac{\delta' Z}{\delta L^\mu} \]

(3.13)

etc.

Now, we note that (we drop brackets around expectation values)

\[ L^\mu = -\frac{\delta \Gamma}{\delta A_{\mu,\lambda}} ; \quad K_{\mu,\theta}^\mu = -\frac{\delta \Gamma}{\delta A_\mu} \]

\[ L^4 = -\frac{\delta \Gamma}{\delta c_{4,\lambda}} ; \quad K_{\mu,\theta}^5 = \frac{1}{2} \frac{\delta \Gamma}{\delta c_5} \]

(3.14)

We thus have the WT identity

\[ \frac{\partial \Gamma}{\partial \theta} = \frac{\delta \Gamma}{\delta A_{\mu,\lambda}} \frac{\delta \Gamma}{\delta A_\mu} + \frac{1}{2} \frac{\delta \Gamma}{\delta c_{4,\lambda}} \frac{\delta \Gamma}{\delta c_5} \]

(3.15)

We can deal with the renormalization of gauge theories from this equation itself. However we prefer to use a somewhat alternate approach using a somewhat unusual change of variables.

Firstly, we denote collectively

\[ \Phi_i(\bar{x}) \equiv \{ < A_\mu(\bar{x}) >, < c_5(\bar{x}) > \} \]

(3.16)

\[ \langle A'_{i,\lambda} \rangle \equiv \{ < A'_{\mu,\lambda} >, < c_{4,\lambda} > \} \]

(3.17)

We now want to calculate \( \frac{\partial}{\partial \theta} \Phi_\mu(\bar{x}) \). We note

\[ \frac{\partial}{\partial \theta} \Phi_\mu(\bar{x}) = \frac{\partial}{\partial \theta} \left[ \frac{\delta' \bar{Z}}{\delta K_{\mu,\theta}^\mu} \right] \]

\[ = \frac{\partial}{\partial \theta} \left[ \frac{\delta \bar{Z}}{\delta K_{\theta}^\mu} + \theta \frac{\delta \bar{Z}}{\delta K_{\mu}} \right] \]

\[ = \frac{\delta}{\delta K_{\theta}^\mu} \left[ \frac{\partial \bar{Z}}{\partial \theta} \right] + \theta \frac{\delta}{\delta K_{\mu}} \left[ \frac{\partial \bar{Z}}{\partial \theta} \right] + \frac{\delta \bar{Z}}{\delta K_{\mu}} \]

\[ = -L^\mu + < A_{\mu,\theta} > \]

(3.18)

Similarly

\[ \frac{\partial}{\partial \theta} \Phi_5(\bar{x}) = -\frac{1}{2} L^4 + < c_{5,\theta} > \]

(3.19)

\[ \frac{\partial}{\partial \theta} < \zeta > = < \zeta_{\theta} > \]

(3.20)
Now if we define
\[
\Phi'_\mu(\bar{x}) = \Phi_\mu(\bar{x}) + \theta L^\mu \\
\Phi'_5(\bar{x}) = \Phi_5(\bar{x}) + \frac{1}{2} L^4
\]
(3.21)

Then, Eq. 3.18 and 3.19 can be written in nice form
\[
\frac{\partial}{\partial \theta} \Phi'_\mu(\bar{x}) = < A_{\mu,\theta} > \\
\frac{\partial}{\partial \theta} \Phi'_5(\bar{x}) = < c_{5,\theta} >
\]
(3.22)

In terms of these redefined variables the WT identity for \( \Gamma \) reads
\[
\frac{\partial}{\partial \theta} \left[ \Phi'_\mu(\bar{x}) - \theta L^\mu, \Phi'_5(\bar{x}) - \frac{\theta}{2} L^4, < A_{i,\theta}(\bar{x}) >, < A'_{i,\lambda}(\bar{x}) >, < \zeta(\bar{x}) >, < \zeta_{\theta} > \right] = L^\mu K^\mu_{\theta} - \frac{1}{2} L^4 K^5_{\theta}
\]
(3.23)
i.e.
\[
\frac{\partial}{\partial \theta} \left[ \Gamma [ \Phi'_\mu, \Phi'_5, < A_{i,\theta} >, < A'_{i,\lambda} >, < \zeta(\bar{x}) >, < \zeta_{\theta} > ] - \theta L^\mu \frac{\delta \Gamma}{\delta \Phi'_{\mu}} - \frac{\theta}{2} L^4 \frac{\delta \Gamma}{\delta c_5} \right] = L^\mu K^\mu_{\theta} - \frac{1}{2} L^4 K^5_{\theta}
\]
(3.24)

Now using the expression 3.14 for \( L^\mu, K^\mu_{\theta} \) etc, we obtain
\[
\frac{\partial}{\partial \theta} \left[ \Phi'_\mu(\bar{x}), \Phi'_5(\bar{x}), < A_{i,\theta}(\bar{x}) >, < A'_{i,\lambda}(\bar{x}) >, < \zeta(\bar{x}) >, < \zeta_{\theta} > \right] = 0
\]
(3.25)

Noting the relation 3.22 for \(< A_{i,\theta} >\), we need not write these as separate variables. Then the WT identity for \( \Gamma \) can be compactly written as,
\[
\frac{\partial}{\partial \theta} \left[ \Phi'_i(\bar{x}), < A'_{i,\lambda}(\bar{x}) >, < \zeta(\bar{x}) > \right] = 0
\]
(3.26)

[ We should clarify that while our attempt to put the WT identity in the neat form of Eq. 3.26 by change of variables of 3.21 may seem a bit artificial, it is not quite so. As we shall see in the next section, under renormalization, not only the old variables \( \Phi_i \) but also the new variables \( \Phi'_i \) of 3.21 become multiplicatively renormalized. More on this later. ]

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Also the new variables are more natural since it is the new variables which directly contain 
\(< A_\mu(x) > \) and \(< A_{\mu,\theta} > \) as components: \( \Phi'(\vec{x}) = < A_\mu(x) > + \theta < A_{\mu,\theta} > + \cdots \)

In addition to Eq. (3.26), there is further restriction on the expectation value \(< A'_{\mu,\lambda} >\). This is seen, for example, as follows:

\[
\frac{\partial}{\partial \theta} < A'_{\mu,\lambda} > = \frac{\partial}{\partial \theta} \frac{\delta Z}{\delta L^\mu} = \frac{\delta}{\delta L^\mu} \frac{\partial Z}{\partial \theta} = -K^\mu_{\rho}(x) = \frac{\delta \Gamma}{\delta A_\mu} \tag{3.27}
\]

In general it is easy to see that

\[
\frac{\partial}{\partial \theta} < A'_{i,\lambda} > = \eta_i \frac{\delta \Gamma}{\delta A_i} \tag{3.28}
\]

with \( \eta_\mu = 1, \eta_4 = \frac{1}{2}, \eta_5 = 0 \). [ The index ‘i’ on the right hand side of the Eq. 3.28 in not summed over]. Equations (3.26) and (3.28) constitute the basis for discussion of renormalization.

**IV. SOLUTION TO WT IDENTITIES AND RENORMALIZATION**

In previous section, we had arrived at the WT identity for proper vertices

\[
\frac{\partial}{\partial \theta} \Gamma \left[ \Phi'_i, < A'_{i,\lambda} >, < \zeta > \right] = 0 \tag{4.1}
\]

with the subsidiary constraint

\[
\frac{\partial}{\partial \theta} < A'_{i,\lambda} > = \eta_i \frac{\delta \Gamma}{\delta A_i} \tag{4.2}
\]

Before we proceed with the discussion of renormalization of gauge theories, we shall draw attention to the advantages of our formulation.

(1) Our formulation does not contain any composite operators explicitly, it contains only the “elementary fields” and as such our discussion of renormalization centers on only the renormalization of these elementary fields.

(2) The form of the WT identity in 4.1 is extremely simple, compared to the earlier formulations [3,4]. In particular, the nilpotent operator \( \mathcal{G} \) of Ref. 3 or its analogies are related by a very simple operator \( \frac{\partial}{\partial \theta} \). More importantly this latter operator is independent of \( g \) and fields unlike the nilpotent operator \( \mathcal{G} \) [3].
\[ G = \int \left[ D_\mu c \frac{\delta}{\delta A_\mu} - \frac{1}{2} g_0 f_{cc} \frac{\delta}{\delta c} + \frac{\delta \tilde{S}}{\delta A_\mu^\alpha} \frac{\delta}{\delta \kappa_\mu^\alpha} + \frac{\delta \tilde{S}}{\delta c^\alpha} \frac{\delta}{\delta l^\alpha} \right] \] (4.3)

(used in the original discussion of renormalizability) which depends in a complicated fashion on the coupling constant \( g \) and the fields and sources. This simplifies greatly and makes it very transparent the kind of renormalization allowed by (4.1) and (4.2) as seen below in (3).

(3) In the original formulation of renormalization using BRS \[3\] it does require some work to prove that rescaling of \( g \) is an allowed renormalization. This requires showing that the operator \( O = g \frac{\partial \tilde{S}}{\partial g} \) is a solution of

\[ G O = 0 \] (4.4)

To deduce this form

\[ G \tilde{S} = 0 \] (4.5)

requires some work because \( G \) itself depends on \( g \) and \( G(g \frac{\partial \tilde{S}}{\partial g}) = 0 \) can’t be deduced by differentiating Eq. (4.3) with respect to \( g \). A similar statement holds about alternate formulations \[4\]. In the present formulation, however, the operator \( g \frac{\partial}{\partial g} \) commutes with \( \frac{\partial}{\partial \theta} \). Hence if \( \Gamma \) satisfies (4.1) and (4.2), then \( \Gamma + \epsilon g \frac{\partial}{\partial g} \Gamma \) also satisfies the same equations is trivially established. In fact, it is easy to see that as the operator \( \frac{\partial}{\partial \theta} \) is independent of fields also, any multiplicative renormalization of \( A_\mu, c_5, A_\mu, \theta, c_5, \theta, c_4, \theta \) etc are also compatible with Eq. (4.1), the form of Eq. (4.2) determine the restrictions to be placed on these rescalings.

Having brought out the advantages of our formulations, we now proceed to determine the renormalization transformations and show that they agree with those of Ref. 14

At first, we shall assume a multiplicative renormalization on superfields, \( \lambda, \theta \) and coupling constants and seek a solution that is consistent with it. The justification for this procedure will be given at the end of this section.

We imagine rescalings

\[ \lambda = Z_\lambda \lambda^R; \quad \theta = Z_\theta \theta^R \]

\[ A_\mu = Z^{\frac{1}{2}} A_\mu^R; \quad c_4 = Z^{\frac{1}{2}} Z(4) c_4^R; \quad c_5 = Z^{\frac{1}{2}} Z(5) c_5^R \] (4.6)
All such rescaling are trivially compatible with Eq. 4.1. Thus, if they are also to satisfy Eq. 4.2, all that is needed is

\[ Z\lambda Z\theta = Z = Z_{(4)} Z_{(5)} Z \]  

(4.7)

Now define

\[ \frac{Z\theta}{Z_{(4)}} = \tilde{Z} \]  

(4.8)

Then equation 4.7 yields

\[ Z\lambda = \frac{Z}{\tilde{Z} Z_{(4)}} \]  

(4.9)

We relabel

\[ Z_{(4)} = Z^{(5)} \text{ and } Z_{(5)} = Z^{(4)} \]  

(4.10)

and parameterize

\[ Z^{(5)} = Z_1 \frac{Z^\frac{1}{2}}{\tilde{Z}} \]  

(4.11)

Then we have the following set of transformations

\[ Z\lambda = \frac{Z}{ZZ^{(5)}} = \frac{Z}{\tilde{Z}} Z_{(4)} = Z^{\frac{1}{2}} Z_1^{-1} \]  

(4.12)

\[ Z\theta = \tilde{Z} Z^{(5)} \]  

(4.13)

\[ A_\mu(\bar{x}) = Z^{\frac{1}{2}} A^R_\mu(\bar{x}^R); \quad c_4(\bar{x}) = Z^{\frac{1}{2}} Z_{(4)} c_4^R(\bar{x}^R); \quad c_5(\bar{x}) = Z^{\frac{1}{2}} Z_{(5)} c_5^R(\bar{x}^R) \]  

(4.14)

In view of Eq. 4.2, it follows that as \( A_\mu \), and \( \theta \) multiplicatively renormalized, so is \( A'_{\mu,\lambda} = A_{\mu,\lambda} - \partial_\mu \zeta \). Hence the renormalization of \( \zeta \) is the same as that of \( A_{\mu,\lambda} \) viz.

\[ A_{\mu,\lambda} = Z^{\frac{1}{2}} Z_\lambda^{-1} A^R_{\mu,\lambda} = Z_1 A^R_{\mu,\lambda} \]  

(4.15)

Hence

\[ \zeta(\bar{x}) = Z_1 \zeta^R(\bar{x}^R) \]  

(4.16)
In view of the fact that $K_\mu^\mu, t, \theta$ are sources for $A_\mu$ and $\zeta$ and must transform contragradiently, we further have

$$K^i(\bar{x}) = Z_1 K^i R(\bar{x} R); \quad t(\bar{x}) = Z_2^\frac{1}{2} t R(\bar{x} R)$$

(4.17)

The equations 4.12-4.17 in fact represents the renormalization transformations written down in Ref. 14. And that they represent solution to 4.1 and 4.2 has been seen almost trivially.

This then would complete the proof of the superfield renormalization transformation except that we have assumption of multiplicative renormalization of superfields as a whole. We shall now turn to justifying this.

In 4.1 we expect $\Gamma = \Gamma^R$. Hence the only freedom we have is for transforming $\theta$ such that

$$\frac{\partial}{\partial \theta} = Z_{\theta}^{-1} \frac{\partial}{\partial \theta^R}$$

(4.18)

for then Eq. 4.1 will read $\frac{\partial}{\partial \theta^R} \Gamma^R = 0$ i.e. it will remain form invariant. Now $A_\mu(x), c_5(x)$ are all dimension 1 operators and hence cannot mix with any other field on account of dimension, Lorentz transformation and ghost number. Hence they must be multiplicatively renormalizable. [ Hence we are talking only of $\theta, \lambda$ independent piece of $A_\mu(\bar{x})$ and $c_5(\bar{x})$]. Taking into account Eq. 4.18 Eq. 4.12 implies that $A'_{i,\lambda}$ should be multiplicatively renormalizable. Writing

$$A'_{i,\lambda}(x) = A'_{i,\lambda} R Z_\theta^{-1}$$

(4.19)

1 The renormalization transformation of Eq. 4.12-4.17 differ slightly from those written down in Ref. 14 in that the renormalization of $\lambda$ is different here. In Ref. 14, no sources were introduced for $A_{i,\lambda}$; and hence renormalization of $A_{i,\lambda}$ (and therefore of $\lambda$) were arbitrary there. $Z_\lambda$ was fixed by convention in Ref. 14 so that $\lambda$ and $\theta$ renormalize symmetrically. But this assumption is not necessary there. It is only in this work that $Z_\lambda$ is fixed uniquely and is given by 4.12.
we note

\[
\frac{\delta}{\delta A'_{i,\lambda}} = Z_i Z^{-1}_\theta \frac{\delta}{\delta A_{i,\lambda}^R}
\]  

(4.20)

Now note that using 4.12,

\[
\frac{\partial \Gamma}{\partial \theta} = \frac{\delta \Gamma}{\delta \Phi_i} \Phi'_i + \frac{\delta \Gamma}{\delta \delta A'_{i,\lambda}} \delta A_{i,\lambda}^R = Z_i Z^{-1}_\theta \frac{\delta \Gamma}{\delta A_{i,\lambda}^R} \delta A_{i,\lambda}
\]

(4.21)

Hence for \( \frac{\delta \Gamma}{\delta \Phi_i} \Phi_i,\theta \) also we must have

\[
\frac{\delta \Gamma}{\delta A_{i,\lambda}^R} \delta A_{i,\lambda} = Z_i Z^{-1}_\theta \frac{\delta \Gamma}{\delta A_{i,\lambda}^R} \delta A_{i,\lambda}^R \eta_i
\]

(4.22)

This implies that \( \Phi_i(x) \) and \( \Phi_i,\theta(x) \) get renormalized in such a fashion that

\[
\Phi_i(x, \theta) = \Phi_i(x) + \theta \Phi_i,\theta(x) = Z \Phi \left( \Phi_i^R(x) + \theta^R \Phi_i,\theta^R(x) \right)
\]

\[
= Z \Phi \left( \Phi_i^R(x) + \theta^R \Phi_i,\theta^R(x) \right)
\]

(4.23)

This justifies the assumption that \( \Phi_i(x, \theta) \) is multiplicatively renormalized as a whole. Now note that the \( \Gamma \) in Eqs 4.1 and 4.2 is \( \lambda \) independent as \( \lambda \) has been set to zero. The only reference to \( \lambda \) comes from \( A'_{i,\lambda} \). Hence we can always choose \( Z_\lambda \) such that \( A_{i,\lambda} \) is renormalized as

\[
A_{i,\lambda}(x) = Z_i Z^{-1}_\lambda A^R_{i,\lambda}(x)
\]

(4.25)

This justifies the assumptions made earlier.

V. CONCLUSIONS

In conclusion, we have, in this work, generalized and carried forward our earlier work on superspace formulation of gauge theories by introducing the generating functional for proper
vertices in this formulation and have shown that the WT identities can be cast in the same simple form [as for $\bar{W}$] viz. $\frac{dW}{d\theta} = 0$. We have further shown how this simplifies the discussion of superspace renormalization of gauge theories and how it leads to the supermultiplet renormalization [and hence other elegant results] that were formulated in Ref.14. The simplification has been due to the fact that the nilpotent operator $\frac{d}{d\theta}$ occurring in the WT identities for $\Gamma$ is independent of the variables such as fields and coupling constants.
APPENDIX A:

Let $S_i(\bar{x})$ denotes collectively all “whole” sources [such as $\bar{K}(\bar{x}), \bar{K}_\theta(\bar{x}) \cdots$ etc]. Then,

$$\bar{Z} = \bar{Z}[S_i(\bar{x})] \quad (A1)$$

and has no explicit dependence on $\theta$. We define the modified expectation values denoted collectively by $\Psi_i$

$$\Psi_j(\bar{x}) \equiv \frac{\delta'\bar{Z}[S]}{\delta S_j(\bar{x})} \quad (A2)$$

Thus $\Psi_j(\bar{x})$ when expressed as a functional of “whole” sources $S_i(\bar{x})$ has no explicit $\theta$ dependence. Now

$$\frac{\partial}{\partial \theta} \Psi_j(\bar{x}) \bigg|_{\Psi_j=0} = 0 = \int d^4x \frac{\partial}{\partial \theta} S_k(\bar{y}) |_{\Psi_j} \frac{\delta^2 \bar{Z}[S]}{\delta S_k(\bar{y}) \delta S_j(\bar{x})} \quad (A3)$$

Assuming $\frac{\delta^2 \bar{Z}}{\delta S_k(\bar{y}) \delta S_j(\bar{x})}$ to be an invertible matrix as is needed for the invertibility & to obtain $S_i(\bar{x})$ in terms of $\Psi_j$, we hence obtain

$$\frac{\partial}{\partial \theta} S_k(\bar{y}) \bigg|_{\Psi_j=0} \quad (A4)$$

hence when $S_k$ are expressed as functionals of $\Psi_j$, they contain no explicit $\theta$ dependence.

$$S_k = S_k[\Psi_j] \quad (A5)$$

We substitute this expression for $S_k$ in

$$\Gamma = \bar{Z} - \sum_i \int S_i(\bar{x}) \frac{\delta'\bar{Z}}{\delta S_i(\bar{x})} d^4x \quad (A6)$$

to obtain

$$\Gamma = \Gamma[\Psi_j] \quad (A7)$$

i.e. $\Gamma$ has no explicit $\theta$ dependence when expressed as functional of $\Psi_j$. 

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