QUIVER HECKE ALGEBRAS FOR BORCHERDS-CARTAN DATUM

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Abstract. We introduce a family of quiver Hecke algebras which give a categorification of quantum Borcherds algebra associated to an arbitrary Borcherds-Cartan datum.

Introduction

Quiver Hecke algebras, also known as Khovanov-Lauda-Rouquier algebras, were discovered independently by Khovanov-Lauda [4, 5] and Rouquier [6], and their representation theory is shown to be closely related to quantum groups. In Kac-Moody type, the category of finitely generated graded projective modules over quiver Hecke algebras give a categorification of corresponding quantum groups. Varagnolo-Vasserot [9] and Rouquier [7] proved that, under this connection, the indecomposable projective modules correspond to the Lusztig’s canonical basis, and their irreducible modules correspond to the dual canonical basis.

In this paper, we apply Khovanov-Lauda’s categorification theory to the quantum Borcherds algebras, which were introduced by Kang in [1]. Given a Borcherds-Cartan datum consisting of an index set $I$ and a symmetrizable Borcherds-Cartan matrix $A = (a_{ij})_{i,j \in I}$, we construct a family of graded algebras $R(\nu)$ ($\nu \in \mathbb{N}[I]$) associated to it, using the braid-like planar diagrams, and give a faithful polynomial representation for each $R(\nu)$. When $i$ is a real index in $I^+$, the degenerated algebras $R(ni)$ for $n \in \mathbb{N}$ are exactly the nil-Hecke algebras $NH_n$ as usual. When $i$ is an imaginary index in $I^-$, $R(ni)$ is generated by ‘dots’ $x_1, \ldots, x_n$ and ‘intersections’ $\tau_1, \ldots, \tau_{n-1}$, with local relations expressed diagrammatically:

\[
\begin{align*}
\begin{gathered}
\includegraphics{dot_relations.png}
\end{gathered}
\end{align*}
\]

We show that $R(ni)$ has a unique graded irreducible module in this case, which is a one-dimensional trivial module denoted by $V(i^n)$. The induction of two irreducible modules, $\text{Ind}V(i^n) \otimes V(i^m)$, has an irreducible head isomorphic to $V(i^{n+m})$.

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We then form the Grothendieck group $K_0(R) = \bigoplus_{\nu \in \mathbb{N}[I]} K_0(R(\nu))$ of the category of finitely generated graded projective modules. Let $U^-$ be the negative part of the quantum Borcherds algebra associated to the given Borcherds-Cartan datum. A classical framework given in [4, 5] leads to an injective homomorphism $\Gamma_{\mathbb{Q}}(q) : U^- \rightarrow \mathbb{Q}[q, q^{-1}] \otimes K_0(R)$. The surjectivity of $\Gamma_{\mathbb{Q}}(q)$ follows from the arguments in [3, Chapter 5] and [4, Section 3.2]. But we need to modify some proofs there since the $R((n + m)i)$-module $\text{Ind}_{V^-}(i^m) \otimes V(i^n)$ is not irreducible again when $i \in I^-$. Finally, the map $\Gamma_{\mathbb{Q}}(q)$ induces a $\mathbb{Z}[q, q^{-1}]$-algebra isomorphism $\Gamma : A U^- \rightarrow K_0(R)$, where $A U^-$ is the $A$-form of $U^-$. In [2], Kang, Oh and Park gave a categorification of this algebra with the condition $a_{ii} \neq 0$ in the Borcherds-Cartan matrix $A$. Our construction of the quiver Hecke algebras is different from their and applies to categorifying the quantum group with an arbitrary Borcherds-Cartan datum.

1. Preliminaries

1.1. $\mathbb{Z}$-gradings.

We fix an algebraically closed field $K$. Let $A$ be a $\mathbb{Z}$-graded algebra over $K$. For a graded $A$-module $M = \bigoplus_{n \in \mathbb{Z}} M_n$, its graded dimension is defined by

$$\text{Dim} M := \sum_{n \in \mathbb{Z}} (\text{dim} M_n) q^n,$$

where $q$ is a formal variable. For $m \in \mathbb{Z}$, we denote by $M\{m\}$ the graded $A$-module obtained from $M$ by putting $(M\{m\})_n = M_{n-m}$. For $f(q) = \sum_{n \in \mathbb{Z}} a_n q^n \in \mathbb{Z}[q, q^{-1}]$, define $M^f := \bigoplus_{n \in \mathbb{Z}} (M\{n\})^{a_n}$, we have $\text{Dim} M^f = f(q) \cdot \text{Dim} M$.

Given two graded $A$-modules $M$ and $N$, we denote by $\text{Hom}_A(M, N)$ the $K$-vector space of grading-preserving homomorphisms and define the $\mathbb{Z}$-graded vector space $\text{HOM}_A(M, N)$ by

$$\text{HOM}_A(M, N) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_A(M\{n\}, N) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_A(M, N\{-n\}).$$

1.2. Negative parts of quantum Borcherds algebras.

Let $I$ be a finite index set. A Borcherds-Cartan datum $(I, A, \cdot)$ consists of

(a) an integer-valued matrix $A = (a_{ij})_{i,j \in I}$ satisfying

(i) $a_{ii} = 2, 0, -2, -4, \ldots$,

(ii) $a_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$,

(iii) there is a diagonal matrix $D = \text{diag}(r_i \in \mathbb{Z}_{>0} \mid i \in I)$ such that $DA$ is symmetric.
(b) a symmetric bilinear form \(\nu,\nu' \mapsto \nu \cdot \nu'\) on \(\mathbb{Z}[I]\) taking values in \(\mathbb{Z}\), such that

\[i \cdot j = r_ia_{ij} = r_ja_{ji}\quad\text{for all } i, j \in I.\]

For such a datum, we assign a graph \(\Lambda\) with vertices set \(I\) and an edge between \(i\) and \(j\) if \(i \cdot j \neq 0\).

We set \(I^+ = \{ i \in I \mid a_{ii} = 2 \}\) and \(I^- = \{ i \in I \mid a_{ii} \leq 0 \}\). Let \(q\) be an indeterminate. For each \(i \in I\), let \(q_i = q^n_i\). For \(i \in I^+\) and \(n \in \mathbb{N}\), we define

\[[n]_i = \frac{q^n_i - q_i^n}{q_i - q_i^{-1}}\quad\text{and } [n]_i! = [n]_i[n - 1]_i \cdots [1]_i.\]

The negative part \(U^-\) of the quantum Borcherds algebra associated to a Borcherds-Cartan datum \((I,A,\cdot)\) is an associative algebra over \(\mathbb{Q}(q)\) with generators \(f_i\) \((i \in I)\) and the defining relations

\[
\sum_{r+s=1-i-j} (-1)^r f_i^{(r)} f_{ij} f_j^{(s)} = 0 \quad\text{for } i \in I^+, j \in I \text{ and } i \neq j,
\]

\[f_i f_j = f_j f_i \quad\text{for } i \in I, j \in I \text{ and } i \cdot j = 0.\]

Here we denote \(f_i^{(n)} = f_i^n/[n]_i!\) for \(i \in I^+\) and \(n \in \mathbb{N}\). The algebra \(U^-\) is \(\mathbb{N}[I]\)-graded by assigning \(\deg(f_i) = i\).

Define a twisted multiplication on \(U^- \otimes U^-\) by

\[(x_1 \otimes x_2)(y_1 \otimes y_2) = q^{-|x_2| |y_1|} x_1 y_1 \otimes x_2 y_2,\]

for homogeneous \(x_1, x_2, y_1, y_2\). By [8, Proposition 2.4], we have an algebra homomorphism \(\rho : U^- \to U^- \otimes U^-\) given by \(\rho(f_i) = f_i \otimes 1 + 1 \otimes f_i\) \((i \in I)\) with respect to the above algebra structure on \(U^- \otimes U^-\), and a nondegenerate symmetric bilinear form \{ , \} : \(U^- \times U^- \to \mathbb{Q}(q)\) satisfying the following properties

(i) \(\{x, y\} = 0\) if \(|x| \neq |y|\),

(ii) \(\{1, 1\} = 1\),

(iii) \(\{f_i, f_j\} = (1 - q_i^2)^{-1}\) for all \(i \in I\),

(iv) \(\{x, yz\} = \{\rho(x), y \otimes z\}\) for \(x, y, z \in U^-\).

Let \(A = \mathbb{Z}[q, q^{-1}]\) be the ring of Laurent polynomials. The \(A\)-form \(A U^-\) is the \(A\)-subalgebra of \(U^-\) generated by the divided powers \(f_i^{(n)}\) for \(i \in I^+, n \in \mathbb{Z}_{\geq 0}\) and \(f_i\) for \(i \in I^-\).
2. Algebras $R(\nu)$ for Borcherds-Catan datum

As in [4], we construct $\mathbb{K}$-algebras $R(\nu)$ ($\nu \in \mathbb{N}[I]$) for Borcherds-Catan datum by braid-like planar diagrams, in which each strand is labelled by an element of $I$ and can carry dots. These diagrams are invariant when planar isotropy is considered.

2.1. Definition and polynomial representation.

Given a Borcherds-Catan datum $(I, A, \cdot)$. We fix a $\nu = \sum_{i \in I} \nu_i i \in \mathbb{N}[I]$ with $ht(\nu) = n$. Let $\text{Seq}(\nu)$ be the set of all sequences $i = i_1 i_2 \ldots i_n$ in $I$ such that $\nu = i_1 + i_2 \cdots + i_n$. We define the homogeneous generators of $R(\nu)$ by diagrams:

$1_i = \begin{array}{ccc} \vdots & \cdots & \vdots \\ i_1 & i_k & i_n \end{array}$ for $i = i_1 i_2 \ldots i_n \in \text{Seq}(\nu)$ with $\text{deg}(1_i) = 0$,

$x_{k,i} = \begin{array}{ccc} \vdots & \cdots & \vdots \\ i_1 & i_k & i_n \end{array}$ for $i \in \text{Seq}(\nu), 1 \leq k \leq n$ with $\text{deg}(x_{k,i}) = 2r_{ik}$,

$\tau_{k,i} = \begin{array}{ccc} \vdots & \cdots & \vdots \\ i_1 & i_k & i_{k+1} \end{array}$ for $i \in \text{Seq}(\nu), 1 \leq k \leq n-1$ with $\text{deg}(\tau_{k,i}) = -i_k \cdot i_{k+1}$.

The multiplication $A \cdot B$ of two diagrams $A, B$ is given by concatenation if the bottom sequence of $A$ coincides with the top sequence of $B$, and otherwise is zero. The local relations of $R(\nu)$ are defined as follows:

\begin{align*}
(2.1) \quad & \begin{array}{ccc} \vdots & \cdots & \vdots \\ i & j \end{array} = \begin{cases} 0 & \text{if } i = j, \\ -a_{ij} & \text{if } i \neq j \text{ and } i \cdot j = 0, \\ -a_{ji} & \text{if } i \neq j \text{ and } i \cdot j \neq 0, \end{cases} \\
(2.2) \quad & \begin{array}{ccc} \vdots & \cdots & \vdots \\ i & i & i \end{array} = \begin{array}{ccc} \vdots & \cdots & \vdots \\ i & i & i \end{array}, \quad \begin{array}{ccc} \vdots & \cdots & \vdots \\ i & i & i \end{array} = \begin{array}{ccc} \vdots & \cdots & \vdots \\ i & i & i \end{array} \quad \text{if } i \in I^+, \\
(2.3) \quad & \begin{array}{ccc} \vdots & \cdots & \vdots \\ i & i & i \end{array} = \begin{array}{ccc} \vdots & \cdots & \vdots \\ i & i & i \end{array}, \quad \begin{array}{ccc} \vdots & \cdots & \vdots \\ i & i & i \end{array} = \begin{array}{ccc} \vdots & \cdots & \vdots \\ i & i & i \end{array} \quad \text{if } i \in I^-, 
\end{align*}
\[ i \cdot j = \begin{cases} 1 & \text{if } i \neq j, \end{cases} \]

\[ (2.4) \]

\[ i \cdot j = \begin{cases} -a_{ij} - 1 & \text{if } i \in I^+, i \neq j \text{ and } i \cdot j \neq 0, \end{cases} \]

\[ (2.5) \]

\[ i \cdot j = \begin{cases} 0 & \text{otherwise}, \end{cases} \]

\[ (2.6) \]

For \( i, j \in \text{Seq}(\nu) \), we set \( jR(\nu)i = 1jR(\nu)1i \), \( P_i = R(\nu)1i \) and \( jP = 1jR(\nu) \). We have \( R(\nu) = \bigoplus_{i,j} jR(\nu)i \) and \( P_i \) (resp. \( jP \)) is a gr-projective left (resp. right) \( R(\nu) \)-module.

Choose an orientation for each edge of the graph \( \Lambda \). For \( i \in \text{Seq}(\nu) \), we set \( \mathcal{P}_i = \mathbb{K}[x_1(\dot{i}), \ldots, x_n(\dot{i}), y_1(\dot{i}), \ldots, y_n(\dot{i})] \)

and form the \( \mathbb{K} \)-vector space \( \mathcal{P}_\nu = \bigoplus_{i \in \text{Seq}(\nu)} \mathcal{P}_i \). Let \( S_n = \langle s_1, \ldots, s_{n-1} \rangle \) be the symmetric group. For each \( \omega \in S_n \), define the operators

\[ \omega : x_a(\dot{i}) \mapsto x_{\omega(a)}(\omega(\dot{i})), \ y_a(\dot{i}) \mapsto y_{\omega(a)}(\omega(\dot{i})), \]

\[ \tilde{\omega} : x_a(\dot{i}) \mapsto x_{\omega(a)}(\omega(\dot{i})), \ y_a(\dot{i}) \mapsto y_a(\omega(\dot{i})). \]

We then define an action of \( R(\nu) \) on \( \mathcal{P}_\nu \) as follows. \( jR(\nu)i \) acts by 0 on \( \mathcal{P}_k \) if \( i \neq k \). For \( f \in \mathcal{P}_i \), \( 1_i \cdot f = f \), \( x_{k,i} \cdot f = x_k(\dot{i})f \) and

\[ \tau_{k,i} \cdot f = \begin{cases} s_kf & \text{if } i_k \neq i_{k+1} \text{ and } i_k \cdot i_{k+1} = 0, \\ \frac{s_kf - \tilde{s}_kf}{x_k(\dot{i}) - x_{k+1}(\dot{i})} & \text{if } i_k = i_{k+1} \in I^+, \\ \frac{\tilde{s}_kf - s_kf}{y_k(\dot{i}) - y_{k+1}(\dot{i})} & \text{if } i_k = i_{k+1} \in I^-, \\ s_kf & \text{if } i_k \leftarrow i_{k+1}, \\ (x_k(s_k\dot{i}) - a_{ij} + x_{k+1}(s_k\dot{i}) - a_{ij})s_kf & \text{if } i_k \rightarrow i_{k+1}. \end{cases} \]

\[ (2.7) \]

Proposition 2.1. \( \mathcal{P}_\nu \) is a \( R(\nu) \)-module with the action defined above.

Proof. This can be obtained immediately by checking the relations of \( R(\nu) \). \( \square \)
2.2. Algebras $R(ni)$ for $i \in I^-$.

In this section, we consider the graph $\Lambda$ with one vertex $i$ and the corresponding algebras $R(ni)$ for $n \in \mathbb{N}$. If $i \in I^+$, $R(ni)$ is isomorphic to the nil-Hecke algebra $NH_n$, its algebraic structure and graded representations are well-known (cf. [4, Example 2.2]). So we consider $i \in I^-$ only. In this case, $R(ni)$ is isomorphic to the $\mathbb{K}$-algebra $R^i_n$ with generators $x_1, \ldots, x_n$ of degree $2r_i$ and $\tau_1, \ldots, \tau_n$ of degree $-i \cdot i$, subject to the following relations:

- $x_kx_t = x_tx_k$ for all $1 \leq k, t \leq n$,
- $\tau_k^2 = 0$, $\tau_k\tau_{k+1}\tau_k = \tau_{k+1}\tau_k\tau_{k+1}$, $\tau_k\tau_t = \tau_t\tau_k$ if $|k - t| > 1$,
- $x_k\tau_k = \tau_kx_{k+1}$, $\tau_kx_k = x_{k+1}\tau_k$,
- $\tau_kx_t = x_t\tau_k$ if $t \neq k, k + 1$.

We simply write $R_n$ for $R^i_n$ if there is no ambiguity. Since $\tau_k$ ($1 \leq k \leq n - 1$) satisfy the braid relations, for each $\omega \in S_n$, we can define $\tau_\omega = \tau_{k_1} \cdots \tau_{k_r}$ if $\omega$ has a reduced expression $\omega = s_{k_1} \cdots s_{k_r}$.

Let $\mathcal{P}_n = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_n]$, and let $\partial_k : \mathcal{P}_n \to \mathcal{P}_n$ ($1 \leq k \leq n - 1$) be the linear operators given by

$$f \mapsto \tilde{s}_k f - s_kf \frac{y_k - y_{k+1}}{y_{k} - y_{k+1}}.$$

Here $\tilde{s}_k$ acts on $f$ by interchanging $x_k$ and $x_{k+1}$, $s_k$ acts on $f$ by interchanging $x_k$ and $x_{k+1}$ and interchanging $y_k$ and $y_{k+1}$ simultaneously. According to Proposition 2.1, $\mathcal{P}_n$ is a left $R_n$-module with the action of $x_k$ by multiplication and the action of $\tau_\omega$ by $\partial_\omega$.

**Proposition 2.2.** The algebra $R_n$ has a basis $\{x_1^{r_1} \cdots x_n^{r_n} \tau_\omega \mid \omega \in S_n, r_1, \ldots, r_n \geq 0\}$.

**Proof.** We show that these elements act on $\mathcal{P}_n$ independently. Suppose that we have a non-trivial linear combination $\sum_{\omega; r_1, \ldots, r_n} k_{\omega; r_1, \ldots, r_n} x_1^{r_1} \cdots x_n^{r_n} \tau_\omega$ acts by zero on $\mathcal{P}_n$. Choose a minimal length element $\omega$ such that $k_{\omega; r_1, \ldots, r_n} \neq 0$ for some $r_1, \ldots, r_n$. Let $\omega_0 = s_1(s_1s_2) \cdots (s_1 \cdots s_{n-1})$ be the longest element in $S_n$ and write $\omega_0 = \omega\omega'$. By applying this linear combination to $\partial_{\omega'}(y_1^{n-1}y_2^{n-2} \cdots y_{n-1})$, we get

$$\sum_{\omega; r_1, \ldots, r_n} k_{\omega; r_1, \ldots, r_n} x_1^{r_1} \cdots x_n^{r_n} = 0,$$

which implies $k_{\omega; r_1, \ldots, r_n} = 0$ for all $r_1, \ldots, r_n \geq 0$, a contradiction. \hfill $\square$

Since there is an anti-automorphism of $R_n$ taking $x_k$ to $x_k$ and $\tau_k$ to $\tau_k$, we see that $\{\tau_\omega x_1^{r_1} \cdots x_n^{r_n} \mid \omega \in S_n, r_1, \ldots, r_n \geq 0\}$ is also a basis of $R_n$. We identify the polynomial algebra $P_n = \mathbb{K}[x_1, \ldots, x_n]$ with the subalgebra of $R_n$ generated by $x_1, \ldots, x_n$. Let $P_n^{S_n}$ be the subalgebra consisting of all symmetric polynomials in $P_n$. 
Proposition 2.3. The center of $R_n$ is $P_n^{S_n}$.

Proof. The proof is an analogue of Theorem 3.3.1 in [3]. Let $z = \sum_{\omega \in S_n} f_{\omega} \tau_\omega$ be a center element. Assume that $\omega \neq 1$ with $f_{\omega} \neq 0$, then there exists $k \in \{1, \ldots, n\}$ such that $\omega(k) \neq k$. But this implies $x_kz - z\tau_k = \sum_{\omega \in S_n} f_{\omega}(x_k - x(\omega(k)))\tau_\omega \neq 0$, a contradiction. Thus $z \in P_n$. Write $z = \sum i, j \geq 0 p_{ij} x_i^1 x_j^2$ with $p_{ij} \in \mathbb{K}[x_3, \ldots, x_n]$. Now $\tau_1z = z\tau_1$ implies $p_{ij} = p_{ji}$ for each $i, j$. Hence $z$ is symmetric in $x_1$ and $x_2$. Similarly, we can show that $z$ is symmetric in $x_k$ and $x_{k+1}$ for all $1 \leq k \leq n - 1$.

We denote by $\mathcal{L}$ the one-dimensional trivial module over $P_n$. Note that $\mathcal{L}$ is the unique gr-irreducible $P_n$-module, up to a degree shift. Let

$$\mathcal{L} = R_n \otimes_{P_n} L = \bigoplus_{\omega \in S_n} \tau_\omega \otimes L,$$

which is a graded left $R_n$-module. Since $x_k\tau_\omega = \tau_\omega x_{\omega^{-1}(k)}$ for any $k$ and $\omega$, we have

$$x_1 \cdot \mathcal{L} = x_2 \cdot \mathcal{L} = \cdots = x_n \cdot \mathcal{L} = 0.$$

Fix a nonzero $v \in L$, then $\{\tau_\omega \otimes v \mid \omega \in S_n\}$ is a basis of $\mathcal{L}$.

Lemma 2.4. $\mathcal{L}$ has a unique (graded) irreducible submodule $V = \text{Span}\{\tau_{\omega_0} \otimes v\}$ with the action of $R_n$ trivially.

Proof. Let $W$ be a nonzero submodule of $\mathcal{L}$. Assume $m = \sum_{\omega} k_{\omega} \tau_\omega \otimes v$ is a nonzero element of $W$. Choose a minimal length element $\omega$ such that $k_\omega \neq 0$ and write $\omega = \omega' \omega_0$, we have $\tau_{\omega'} m = k_\omega \tau_{\omega_0} \otimes v \in W$. This shows that each nonzero submodule contains $V$. Moreover, $V$ itself is a (graded) $R_n$-module.

Lemma 2.5. $\mathcal{L}$ has a unique (graded) maximal submodule $M = \text{Span}\{\tau_\omega \otimes v \mid \omega \neq 1\}$. In particular, $\mathcal{L}/M \cong V$ as $R_n$-modules.

Proof. It’s obvious that $M$ is a (graded) maximal submodule of $\mathcal{L}$. For any nonzero submodule $W$ of $\mathcal{L}$, if $W$ contains an element $z$ of the form

$$z = 1 \otimes v + \sum_{\omega \in S_n, \omega \neq 1} k_{\omega} \tau_\omega \otimes v,$$

then we choose a minimal length element $\omega \neq 1$ with $k_\omega \neq 0$ and obtain $z_1 = z - k_\omega \tau_\omega z \in W$. Note $z_1$ is of the form

$$z_1 = 1 \otimes v + \sum_{l(\omega') \geq l(\omega), \omega' \neq \omega} c_{\omega'} \tau_{\omega'} \otimes v$$

for some $c_{\omega'} \in \mathbb{K}$. By repeating this process, one can deduce that $1 \otimes v \in W$. Therefore, if a submodule $W \neq \mathcal{L}$, then $W \subseteq M$. The lemma is proved. \hfill \Box
Theorem 2.6. $V$ is the unique gr-irreducible module over $R_n$, up to a degree shift.

Proof. Let $N$ be a gr-irreducible $R_n$-module, then $N$ contains a $P_n$-submodule isomorphic to $L\{r\}$ for some $r \in \mathbb{Z}$. Since we have the graded isomorphism

$$\text{HOM}_{R_n}(\mathcal{L}, N) \simeq \text{HOM}_{P_n}(L, \text{HOM}_{R_n}(R_n, N)) \simeq \text{HOM}_{P_n}(L, N) \neq 0,$$

there exist a surjective graded homomorphism $\mathcal{L} \to N\{-r\}$ by the irreducibility of $N$. By Lemma 2.5 we have $N\{-r\} \simeq \mathcal{L}/M \simeq V$. \qed

We shall denote by $V(i^n)$ the unique gr-irreducible $R(i^n)$-module for $i \in I$, which is a one-dimensional trivial module for $i \in I^-$ by arguments above. Recall that, for $i \in I^+$, $V(i^n)$ is isomorphic to $\text{Ind}L = NH_n \otimes_{P_n} L$, up to a degree shift.

3. Basic properties and representation theory of $R(\nu)$

This section follows [4] and [5] closely. We list our main results without proof as they can be proved step by step according to [4] and [5] with appropriate deformations.

3.1. Basis and center.

For $i, j \in \text{Seq}(\nu)$, set $jS_i = \{\omega \in S_n \mid \omega(i) = j\}$. We fix a reduced expression for each $\omega \in jS_i$, which determines a unique element $\tilde{\omega}_i \in jR(\nu)$, and set

$$jB_i = \{\tilde{\omega}_i \cdot x_{1,i}^{r_1} \cdots x_{n,i}^{r_n} \mid \omega \in jS_i, r_1, \ldots, r_n \in \mathbb{N}\}.$$

Proposition 3.1. $jB_i$ is a basis of $jR(\nu)_i$. Moreover, $\mathcal{P}_\nu$ is a faithful $R(\nu)$-module with the actions given in (2.7).

Proof. This proposition follows from Proposition 2.2 and the standard arguments in [4] Theorem 2.5. \qed

Assume $\nu = \nu_1 i_1 + \cdots + \nu_t i_t$ such that $i_1, \ldots, i_t$ are all distinct and $\nu_k > 0$. By Proposition 2.3 and Theorem 2.9 in [4], we describe the center $Z(R(\nu))$ of $R(\nu)$ as follows.

Proposition 3.2. $Z(R(\nu)) \simeq \bigotimes_{k=1}^t \mathbb{K}[z_1, \ldots, z_{\nu_k}]^{S_{\nu_k}}$.

$R(\nu)$ is a free $Z(R(\nu))$-module of rank $(n!)^2$. It is also a graded free $Z(R(\nu))$-module of finite rank. We have

$$\text{Dim}Z(R(\nu)) = \prod_{k=1}^t \left( \prod_{c=1}^{\nu_k} \frac{1}{1 - q_{ik}^{2c}} \right),$$

and $\text{Dim}R(\nu) \in \mathbb{Z}[q, q^{-1}] \cdot \text{Dim}Z(R(\nu))$.\[\]
3.2. Grothendieck groups and bilinear pairings.

Let $R(\nu)\text{-Mod}$ be the category of finitely generated graded $R(\nu)$-modules, and let $R(\nu)\text{-fMod}$ (resp. $R(\nu)\text{-pMod}$) be the full subcategory of $R(\nu)\text{-Mod}$ of finite-dimensional (resp. finitely generated projective) $R(\nu)$-modules.

Since $R(\nu)$ is Laurentian by Proposition 3.1, there are only finitely many gr-irreducible $R(\nu)$-module, up to isomorphism and degree shifts. All gr-irreducible $R(\nu)$-module are finite-dimensional. Moreover, if $S$ is a gr-irreducible $R(\nu)$-module, then $S$ is an irreducible $R(\nu)$-module by forgetting the grading.

Let $\mathbb{B}_\nu$ be the set of equivalence classes (under isomorphism and degree shifts) of gr-irreducible $R(\nu)$-modules. The Grothendieck group $G_0(R(\nu))$ of $R(\nu)$-fMod is a free $\mathbb{Z}[q, q^{-1}]$-module with the basis $\{S_b\}_{b \in \mathbb{B}_\nu}$, where $q[M] = [M\{1\}]$ for $[M] \in G_0(R(\nu))$. Each $S_b$ has a unique gr-indecomposable projective cover $P_b$. The Grothendieck group $K_0(R(\nu))$ of $R(\nu)$-pMod is a free $\mathbb{Z}[q, q^{-1}]$-module with the basis $\{P_b\}_{b \in \mathbb{B}_\nu}$.

Let $\psi : R(\nu) \to R(\nu)$ be the anti-involution of $R(\nu)$ by flipping the diagrams about horizontal axis. For $P \in R(\nu)$-pMod, let $\overline{P} = \text{HOM}(P, R(\nu))^\psi$ be the gr-projective left $R(\nu)$-module with the action twisted by $\psi$. This gives a self-equivalence of $R(\nu)$-pMod and induces a $\mathbb{Z}[q, q^{-1}]$-antilinear involution of $K_0(R(\nu))$ denoted again by $\overline{\cdot}$.

Define the $\mathbb{Z}[q, q^{-1}]$-bilinear pairing $(\ , \ ) : K_0(R(\nu)) \times G_0(R(\nu)) \to \mathbb{Z}[q, q^{-1}]$ by

$$([P], [M]) = \text{Dim}(P^{\overline{\psi}} \otimes_{R(\nu)} M) = \text{DimHOM}_{R(\nu)}(\overline{P}, M).$$

Since $\mathbb{K}$ is algebraically closed, $G_0(R(\nu))$ and $K_0(R(\nu))$ are dual $\mathbb{Z}[q, q^{-1}]$-module under this pairing. There is also a symmetric $\mathbb{Z}[q, q^{-1}]$-bilinear form $(\ , \ ) : K_0(R(\nu)) \times K_0(R(\nu)) \to \mathbb{Z}((q))$ defined in the same way.

3.3. Character and quantum Serre relations.

For $M \in R(\nu)$-Mod, define the character of $M$ as

$$\text{Ch}M = \sum_{i \in \text{Seqd}(\nu)} \text{Dim}(1_i M)i.$$

We denote by $\text{Seqd}(\nu)$ the set of sequences $i$ of $\nu$ with the ‘divided powers’ for $i \in I^+$. Such a sequence is of the form

$$i = j_1 \ldots j_{p_t} i_1^{(n_1)} k_1 \ldots k_{p_1} i_2^{(n_2)} \ldots i_t^{(n_t)} l_1 \ldots l_{p_t},$$

where $i_1, \ldots, i_t \in I^+$ and $i$ is of weight $\nu$.

For $i \in I^+$ and $n > 0$, let $e_{i,n}$ be the primitive idempotent of $R(ni)$ corresponding to the element $x_1^{n-1}x_2^{n-2} \ldots x_{n-1}\partial_{\omega_0}$ of $NH_n$. For each $i \in \text{Seqd}(\nu)$, we assign the following
idempotent of $R(\nu)$

$$1_i = 1_{i_0 \ldots i_p} \otimes e_{i_1,n_1} \otimes 1_{i_1 \ldots i_{\nu}} \otimes e_{i_2,n_2} \otimes \cdots \otimes e_{i_t,n_t} \otimes 1_{i_t \ldots i_{\nu}}.$$ 

We abbreviate $i = \ldots i_1^{(n_1)} \ldots i_2^{(n_2)} \ldots i_t^{(n_t)}$ and denote

$$\tilde{i} = \ldots i_1 \ldots i_2 \ldots i_t \in \text{Seq}(\nu).$$

Let $\mathcal{I} = [n_1]_{i_1} \cdots [n_t]_{i_t}$ and $\langle \tilde{i} \rangle = \sum_{k=1}^{t} n_k(n_k-1)!r_{ik}$, we have by the structure of nil-Hecke algebra

$$\text{Dim}(1_iM) = q^{-\langle \tilde{i} \rangle} \mathcal{I} \cdot \text{Dim}(1_iM).$$

For $i \in \text{Seqd}(\nu)$, let $\mathbb{I}P = 1_i R(\nu) \{ - \langle \tilde{i} \rangle \}$ and $P_i = R(\nu)\psi(1_i) \{ - \langle \tilde{i} \rangle \}$. We have the following proposition which gives a categorification of quantum Serre relations in $U^-$. 

**Proposition 3.3.** For $i \in I^+, \ j \in I$ and $i \neq j$. Let $m = 1 - a_{ij}$. We have an isomorphism of graded left $R(\nu)$-modules

$$\bigoplus_{c=0}^{m/2} P_{ij}(2c) j_1^{(m-2c)} \cong \bigoplus_{c=0}^{m-1} P_{ij}(2c+1) j_1^{(m-2c-1)}.$$ 

Moreover, for $i, j \in I$ and $i \cdot j = 0$, we have an isomorphism $P_{ij} \cong P_{ji}$.

**Proof.** The proof is the same as the ‘Box’ calculations in [5].

**\qed**

### 3.4. Induction and Restriction.

As in [4] Section 2.6, we define the induction and restriction functors as

$$\text{Ind}_{\nu,\nu'}^{\nu+\nu'} : R(\nu) \otimes R(\nu')\text{-Mod} \to R(\nu + \nu')\text{-Mod}, \ M \mapsto R(\nu + \nu')1_{\nu,\nu'} \otimes R(\nu) \otimes R(\nu') \otimes M,$$

$$\text{Res}_{\nu,\nu'}^{\nu+\nu'} : R(\nu + \nu')\text{-Mod} \to R(\nu) \otimes R(\nu')\text{-Mod}, \ M \mapsto 1_{\nu,\nu'}M,$$

where $1_{\nu,\nu'} = 1_{\nu} \otimes 1_{\nu'}$. Since $R(\nu + \nu')1_{\nu,\nu'}$ is a free graded right $R(\nu) \otimes R(\nu')$-module, the functors $\text{Ind}_{\nu,\nu'}^{\nu+\nu'}$ and $\text{Res}_{\nu,\nu'}^{\nu+\nu'}$ are both exact and take projective modules to projective modules. For $i \in \text{Seqd}(\nu)$ and $j \in \text{Seqd}(\nu')$, we have by the definition $\text{Ind}_{\nu,\nu'} P_i \otimes P_j \simeq P_{ij}$.

For $i \in \text{Seq}(\nu)$, $j \in \text{Seq}(\nu')$ and $k \in \text{Seq}(\nu + \nu')$, we denote by $\text{Sh}(i, j; k)$ the set of all shuffles $u \in_{ij} R(\nu + \nu')k$ from $i, j$ to $k$.
The gr-projective $R(\nu) \otimes R(\nu')$-module $\text{Res}_{\nu, \nu'} P_k$ has the following decomposition
\[
\text{Res}_{\nu, \nu'} P_k \simeq \bigoplus_{i \in \nu, j \in \nu', u \in \text{Sh}(i, j; k)} P_i \otimes P_j \{u\}.
\]
For $M \in R(\nu)$-Mod, $N \in R(\nu')$-Mod and $k \in \text{Seq}(\nu + \nu')$, we have the following equality, so called the ‘Quantum Shuffle Lemma’
\[
\text{Dim}(1_k \text{Ind}_{\nu, \nu'} M \otimes N) = \sum_{i \in \nu, j \in \nu', u \in \text{Sh}(i, j; k)} q^{|u|} \text{Dim}(1_i M) \cdot \text{Dim}(1_j N).
\]

**Proposition 3.4.** (‘Mackey Theorem’) Let $\nu, \nu', \mu, \mu' \in \mathbb{N}[I]$ with $\nu + \nu' = \mu + \mu'$. For $M \in R(\mu)$-Mod, $N \in R(\mu')$-Mod, we have a filtration of $\text{Res}_{\nu, \nu'} \text{Ind}_{\mu, \mu'} M \otimes N$ with subquotients over all $\lambda \in \mathbb{N}[I]$ such that $\nu - \lambda, \mu' - \lambda, \nu' + \lambda - \mu' \in \mathbb{N}[I]$, which are isomorphic to
\[
\text{Ind}_{\nu, \nu'} M^{\nu - \lambda, \lambda, \mu'} \mu' - \lambda M \otimes N \{ - \lambda \cdot (\nu' + \lambda - \mu') \}.
\]
Here if $\text{Res}M \otimes N = Q_1 \otimes Q_2 \otimes Q_3 \otimes Q_4$, then $\circ(\text{Res}M \otimes N) = Q_1 \otimes Q_3 \otimes Q_2 \otimes Q_4$.

**Proof.** The proof is the same as [4], Proposition 2.8.

### 3.5. Bialgebra $K_0(R)$

Let $R = \bigoplus_{\nu \in \mathbb{N}[I]} R(\nu)$ and form the following categories of $R$-modules
\[
\text{R-fMod} = \bigoplus_{\nu \in \mathbb{N}[I]} R(\nu)\text{-fMod}, \quad \text{R-pMod} = \bigoplus_{\nu \in \mathbb{N}[I]} R(\nu)\text{-pMod}.
\]
The Grothendieck groups of $\text{R-fMod}$ (resp. $\text{R-pMod}$) is given by $G_0(R) = \bigoplus_{\nu \in \mathbb{N}[I]} G_0(R(\nu))$ (resp. $K_0(R) = \bigoplus_{\nu \in \mathbb{N}[I]} K_0(R(\nu))$). By summing up all $\nu, \nu'$, the induction and restriction functors induce the following $\mathbb{Z}[q, q^{-1}]$-linear maps
\[
\widetilde{\text{Ind}} : K_0(R) \otimes K_0(R) \to K_0(R), \quad \widetilde{\text{Res}} : K_0(R) \to K_0(R) \otimes K_0(R).
\]
Now, $K_0(R)$ becomes a $\mathbb{Z}[q, q^{-1}]$-algebra with the multiplication given by $xy := \widetilde{\text{Ind}}(x \otimes y)$ for all $x, y \in K_0(R)$. If we equip $K_0(R) \otimes K_0(R)$ with a twisted algebra structure via
\[
(x_1 \otimes x_2)(y_1 \otimes y_2) = q^{-|x_2|} y_1 x_1 y_1 \otimes x_2 y_2,
\]
then $\widetilde{\text{Res}}$ is a $\mathbb{Z}[q, q^{-1}]$-algebra homomorphism by Mackey’s Theorem given in Proposition 3.4.

Extend the bilinear pairings in Section 3.2 to $K_0(R) \times K_0(R)$ and to $K_0(R) \times G_0(R)$ by requiring $([M], [N]) = 0$ if $M \in R(\nu)$-pMod, $N \in R(\mu)$-pMod (or $R(\nu)$-fMod) with $\nu \neq \mu$. We have the following proposition from the definition.

**Proposition 3.5.** The symmetric bilinear form on $K_0(R)$ satisfies
(i) \((1,1) = 1,\) 
(ii) \(\langle [P_i], [P_j] \rangle = \delta_{ij}(1 - q_i^2)^{-1}\) for all \(i, j \in I,\)
(iii) \((x, yz) = (\text{Res}(x), y \otimes z)\) for \(x, y, z \in K_0(R),\)

where \(1 = \mathbb{K}\) as a module over \(R(0) = \mathbb{K}.\)

4. Categorification of \(U^-\) and \(\mathcal{A}U^-\)

As in [3 Proposition 3.4], we connect the Grothendieck group \(K_0(R)\) with the half part of quantum Borcherds algebra \(U^-\) as follows. Let \(K_0(R)_{\mathbb{Q}(q)} = \mathbb{Q}(q) \otimes_{\mathbb{Z}[q, q^{-1}]} K_0(R).\) By Proposition 3.3, we have a well-defined \(\mathbb{Q}(q)\)-algebra homomorphism

\[ \Gamma_{\mathbb{Q}(q)} : U^- \to K_0(R)_{\mathbb{Q}(q)} \]

given by \(\Gamma_{\mathbb{Q}(q)}(f_i) = [P_i]\) for all \(i \in I.\) By Proposition 3.5, the bilinear form \(\{ \ , \ \}\) on \(U^-\) and the form \((\ , \ )\) on \(K_0(R)_{\mathbb{Q}(q)}\) take same values under \(\Gamma_{\mathbb{Q}(q)}\), that is

\[ (\Gamma_{\mathbb{Q}(q)}(x), \Gamma_{\mathbb{Q}(q)}(y)) = \{x, y\} \text{ for } x, y \in U^- . \]

Thus \(\Gamma_{\mathbb{Q}(q)}\) is injective by the non-degeneracy of \(\{ \ , \ \}\). Moreover, it induces an injective \(\mathbb{Z}[q, q^{-1}]-\)algebra homomorphism \(\Gamma : \mathcal{A} U^- \to K_0(R).\)

In the rest of this section, we shall prove the surjectivity of \(\Gamma_{\mathbb{Q}(q)}\) and \(\Gamma\) using the frameworks given in [3 Chapter 5] and in [4 Section 3.2]. Recall that the one-dimensional trivial module \(V(i^n)\) is the unique gr-irreducible \(R(mi)-\)module for \(i \in I^- .\)

Lemma 4.1. Let \(i \in I^-\) and let \((m_1, \ldots, m_r)\) be a composition of \(n.\)

(i) \(\text{Res}_{m_1, \ldots, m_r}^n V(i^n) \cong V(i^{m_1}) \otimes \cdots \otimes V(i^{m_r}),\)
(ii) \(\text{Res}_{n-1}^n V(i^n) = V(i^{n-1}),\)
(iii) \(\text{Ind}_{n,m}^{n+m} V(i^n) \otimes V(i^m)\) has a unique (graded) maximal submodule. In particular, the graded head \(\text{hdInd}_{n,m}^{n+m} V(i^n) \otimes V(i^m)\) is irreducible.

Proof. The assertions (i) and (ii) are obvious. We shall prove (iii). Let \(D_{(n,m)}\) be the set of minimal length left \(S_n \times S_m\)-coset representatives in \(S_{n+m},\) then \(\text{Ind}_{n,m}^{n+m} V(i^n) \otimes V(i^m)\) has a basis \(\{\tau_\omega \otimes v \mid \omega \in D_{n,m}\}\) for a nonzero \(v \in V(i^n) \otimes V(i^m).\) The \(\mathbb{K}\)-vector space \(\text{Span}\{\tau_\omega \otimes v \mid \omega \in D_{n,m}, \omega \neq 1\}\) is a maximal submodule of \(\text{Ind}_{n,m}^{n+m} V(i^n) \otimes V(i^m)\) since for \(\lambda \in S_{n+m}\) and \(\omega \in D_{n,m} (\omega \neq 1),\) if \(\lambda \omega \in S_n \times S_m,\) we must have \(l(\lambda \omega) < l(\lambda) + l(\omega).\) The uniqueness follows from the same argument in Lemma 2.5. \(\square\)

For \(i \in I\) and \(n \geq 0,\) define the functor

\[ \Delta_{i^n} : R(\nu)\text{-Mod} \to R(\nu - ni) \otimes R(ni)\text{-Mod}, \quad M \mapsto (1_{\nu - ni} \otimes 1_{ni})M. \]
By Frobenius reciprocity, we have for $M \in R(\nu)-\text{Mod}$ and $N \in R(\nu - ni)-\text{Mod}$

\begin{equation}
\text{HOM}_{R(\nu)}(\text{Ind}_{\nu-ni,n} N \otimes V(i^n), M) \simeq \text{HOM}_{R(\nu-ni) \otimes R(ni)}(N \otimes V(i^n), \Delta_{\nu} M).
\end{equation}

Let $\varepsilon_i(M) = \max\{n \geq 0 \mid \Delta_{\nu} M \neq 0\}$ be the number of the largest $i$-tail in sequence $k$ such that $1_k M \neq 0$.

**Lemma 4.2.** Let $i \in I$ and $M \in R(\nu)-\text{fMod}$ be a gr-irreducible module. If $N \otimes V(i^n)$ is a gr-irreducible submodule of $\Delta_{\nu} M$ for some $0 \leq n \leq \varepsilon_i(M)$, then $\varepsilon_i(N) = \varepsilon_i(M) - n$.

**Proof.** Let $\varepsilon_i(N) = a$ and $\varepsilon_i(M) = b$, there exists a sequence $\nu^a \in \text{Seq}(\nu - ni)$ such that $1_{\nu^a} N \neq 0$. Hence $1_{\nu^a} \otimes 1_{\nu^a}(N \otimes V(i^n)) = 1_{\nu+a+n}(N \otimes V(i^n)) \neq 0$. It follows that $b \geq a + n$.

By Frobenius reciprocity \ref{lemma:1} and the irreducibility of $M$, $M$ is a quotient of $\text{Ind}_{\nu-ni,n} N \otimes V(i^n)$. The exactness of $\Delta_{\nu}$ implies $\Delta_{\nu} M$ is a quotient of $\Delta_{\nu} \text{Ind}_{\nu-ni,n} N \otimes V(i^n)$, we get $\Delta_{\nu} \text{Ind}_{\nu-ni,n} N \otimes V(i^n) \neq 0$.

On the other hand, we have $\varepsilon_i(\text{Ind}_{\nu-ni,n} N \otimes V(i^n)) = a + n$ by the Shuffle Lemma. Therefore $b \leq a + n$. \hfill \Box

**Lemma 4.3.** Let $i \in I$ and $N \in R(\nu)-\text{fMod}$ be a gr-irreducible module with $\varepsilon_i(N) = 0$. Set $M = \text{Ind}_{\nu-ni,n} N \otimes V(i^n)$. Then

(i) $\Delta_{\nu} M \simeq N \otimes V(i^n)$,

(ii) $\text{hd} M$ is gr-irreducible with $\varepsilon_i(\text{hd} M) = n$,

(iii) all other composition factors $L$ of $M$ have $\varepsilon_i(L) < n$.

**Proof.** In the case of $i \in I^+$, the lemma has been proved in [\ref{lemma:3}] Lemma 3.7. We now consider the cases $i \in I^-$. \hfill 

(i) By Frobenius reciprocity \ref{lemma:1} and the irreducibility of $N \otimes V(i^n)$, we have $N \otimes V(i^n) \hookrightarrow \Delta_{\nu} M$ as a graded submodule. Assume $\text{ht}(\nu) = m$, then $\text{Ch} V(i^n) = i^n$ and $\text{Ch} N = \sum_{j \in \nu, j_m \neq i} \text{Dim}(1_j N) j$. By Shuffle Lemma, we have

$$\text{Ch} M = \sum_{k \in \nu + ni} \left( \sum_{j \in \nu, j_m \neq i, u \in \text{Sh}(j,i^n;k)} q^{[u]} \text{Dim}(1_j N) \right) k.$$ 

It follows that

$$\text{Ch} (\Delta_{\nu} M) = \sum_{j \in \nu, j_m \neq i} \text{Dim}(1_j N) j^{i^n} = \text{Ch}(N \otimes V(i^n)).$$

Hence $\Delta_{\nu} M \simeq N \otimes V(i^n)$.

(ii) For any nonzero quotient $Q$ of $M$, we have $N \otimes V(i^n) \hookrightarrow \Delta_{\nu} Q$ by Frobenius reciprocity \ref{lemma:1}. Assume we have the decomposition

$$\text{hd} M = M / J_{\nu}^{gr} M = M / M_1 \oplus M / M_2 \oplus \cdots \oplus M / M_s,$$
such that each $M/M_k$ is gr-irreducible. Then $N \otimes V(i^n)$ is embedded into each $\Delta_{\nu}(M/M_k)$ and $\Delta_{\nu}(\text{hd}M)$, which are quotients of $\Delta_{\nu}M$ by the exactness of $\Delta_{\nu}$. It follows from (i) that $\Delta_{\nu}(\text{hd}M) \cong \Delta_{\nu}(M/M_k) \cong N \otimes V(i^n)$. Hence $\text{hd}M$ must be gr-irreducible. Moreover, we have $\varepsilon_i(\text{hd}M) = \varepsilon_i(M) = n$.

(iii) Since we have proved $\Delta_{\nu}(\text{hd}M) \cong \Delta_{\nu}M$. Our assertion follows from the exactness of $\Delta_{\nu}$.

\begin{proof}
Choose a gr-irreducible submodule $K \otimes V(i^n)$ of $\Delta_{\nu}M$, then we have $\varepsilon_i(K) = 0$ by Lemma 4.2. By Frobenius reciprocity (4.1) and the irreducibility of $K$, $M$ is a quotient of $\text{Ind}_{\nu-\eta,ni}K \otimes V(i^n)$, which is isomorphic to $K \otimes V(i^n)$ according to Lemma 4.3 (i). Now, $\Delta_{\nu}M \cong K \otimes V(i^n)$ since $K \otimes V(i^n)$ is gr-irreducible.

Since we have a surjective map $\text{Ind}_{\nu-\eta,ni}K \otimes V(i^n) \twoheadrightarrow M$ and since $\text{hdInd}_{\nu-\eta,ni}K \otimes V(i^n)$ is gr-irreducible by Lemma 4.3 (ii), we see that $M \cong \text{hdInd}_{\nu-\eta,ni}K \otimes V(i^n)$.
\end{proof}

\begin{corollary}
Let $i \in I$ and $M, M' \in R(\nu)$-fMod be gr-irreducible modules with $\varepsilon_i(M) = \varepsilon_i(M') = n$. Assume $M \not\cong M'$ and

$$\Delta_{\nu}M \cong K \otimes V(i^n), \quad \Delta_{\nu}M' \cong K' \otimes V(i^n)$$

for gr-irreducible $K, K' \in R(\nu-\eta)$-fMod with $\varepsilon_i(K) = \varepsilon_i(K') = 0$. Then $K \not\cong K'$.

\begin{proof}
If $K \cong K'$, then $M \cong \text{hdInd}_{\nu-\eta,ni}K \otimes V(i^n) \cong \text{hdInd}_{\nu-\eta,ni}K' \otimes V(i^n) \cong M'$ by Proposition 4.4. This proves our claim.
\end{proof}

\begin{theorem}
The map $\text{Ch} : G_0(R(\nu)) \rightarrow \mathbb{Z}[q, q^{-1}]\text{Seg}(\nu)$ is injective.

\begin{proof}
We prove the characters of gr-irreducible $R(\nu)$-modules in $\mathbb{B}_\nu$ are linearly independent over $\mathbb{Z}[q, q^{-1}]$ by induction on $\text{ht}(\nu)$. The case of $\text{ht}(\nu) = 0$ is trivial. Assume for $\text{ht}(\nu) < n$, our assertion is true. Now, suppose $\text{ht}(\nu) = n$ and we are given a non-trivial linear composition

\begin{equation}
\sum_{M} c_M \text{Ch}M = 0
\end{equation}

for some $M \in \mathbb{B}_\nu$ and some $c_M \in \mathbb{Z}[q, q^{-1}]$. Choose an $i \in I$. We show by downward induction on $k = n, \ldots, 1$ that $c_M = 0$ for all $M$ with $\varepsilon_i(M) = k$.
\end{proof}

If \( k = n \) and \( M \in \mathcal{B}_\nu \) such that \( \varepsilon_i(M) = n \), we must have \( \nu = ni \) and \( M = V(i^n) \), our assertion is trivial. Assume for \( 1 \leq k < n \), we have \( c_M = 0 \) for all \( L \) with \( \varepsilon_i(L) > k \). Taking out the terms with \( i^k \)-tail in the rest of \( \{1, 2\} \), we obtain

\[
\sum_{M: \varepsilon_i(M) = k} c_M \text{Ch}(\Delta, M) = 0.
\]

If \( \Delta, M \cong K \otimes V(i^k) \) for a gr-irreducible \( K \in R(\nu - ki)\text{-fMod} \) with \( \varepsilon_i(K) = 0 \), then

\[
\text{Ch}(\Delta, M) = \text{Dim}V(i^k) \cdot \text{Ch}K \cdot i^k.
\]

By the inductive hypothesis and the Corollary \[4.5\], we get \( c_M = 0 \) for all \( M \) with \( \varepsilon_i(M) = k \). Since each gr-irreducible \( R(\nu) \)-modules \( M \) has \( \varepsilon_i(M) > 0 \) for at least one \( i \in I \), the theorem has been proved. \( \square \)

For each \( \nu \in \mathbb{N}[I] \), ‘Ch’ induces an injective map of \( \mathbb{Q}(q) \)-vector space \( \text{Ch} : \mathbb{Q}(q) \otimes \mathbb{Z}[q, q^{-1}] G_0(R(\nu)) \to \mathbb{Q}(q)\text{Seq}(\nu) \), which is dual to

\[
\mathbb{Q}(q)\text{Seq}(\nu) \to U^- \xrightarrow{\Gamma_{\mathbb{Q}(q)}} K_0(R(\nu))_{\mathbb{Q}(q)}.
\]

It follows that \( \Gamma_{\mathbb{Q}(q)} \) is surjective. Combine with the injectivity of \( \Gamma_{\mathbb{Q}(q)} \), we obtain the following categorification of \( U^- \).

**Proposition 4.7.** \( \Gamma_{\mathbb{Q}(q)} : U^- \to K_0(R)_{\mathbb{Q}(q)} \) is an isomorphism.

We next consider the surjectivity of \( \Gamma : A U^- \to K_0(R) \). The following several results can be proved by the same manner in \[3\] Chapter 5).

**Lemma 4.8.** Let \( i \in I \) and \( M \in R(\nu)\text{-fMod} \) be a gr-irreducible module. Then for any \( 0 \leq n \leq \varepsilon_i(M) \), the graded socle \( \text{soc}(\Delta, M) \) is a gr-irreducible \( R(\nu - ni) \otimes R(ni) \)-module of the form \( L \otimes V(i^n) \) with \( \varepsilon_i(L) = \varepsilon_i(M) - n \).

**Proof.** Let \( \varepsilon_i(M) = a \) and \( \Delta, M \cong K \otimes V(i^a) \) for some gr-irreducible \( K \in R(\nu - ai)\text{-fMod} \). For each constituent \( L \otimes V(i^n) \) of \( \text{soc}(\Delta, M) \) with \( \varepsilon_i(L) = a - n \), we have

\[
\text{Res}^{\nu - ni, ni}_{\nu - ai, (a-n)i, ni} L \otimes V(i^n) \cong \text{Res}^{\nu - ni, ni}_{\nu - ai, (a-n)i, ni} \Delta, M.
\]

On the other hand, by the transitivity of the Res, we obtain

\[
\text{Res}^{\nu - ni, ni}_{\nu - ai, (a-n)i, ni} \Delta, M \cong \text{Res}^{\nu - ai, ai}_{\nu - ai, (a-n)i, ni} \text{Res}^{\nu - ai, ai}_{\nu - ai, ai} M \cong K \otimes V(i^{a-n}) \otimes V(i^n).
\]

Hence \( \text{soc}(\Delta, M) \) must equal \( L \otimes V(i^n) \). \( \square \)

Define the functor \( \varepsilon_i = \text{Res}^{\nu - i, i}_{\nu - i} \circ \Delta, : R(\nu)\text{-fMod} \to R(\nu - i)\text{-fMod} \). Then for \( M \in R(\nu)\text{-fMod} \), \( \varepsilon_i(M) = \max\{n \geq 0 \mid \varepsilon_i^n M \neq 0\} \).
Lemma 4.9. Let \( i \in I \) and \( M \in R(\nu)-\text{fMod} \) be a gr-irreducible module with \( \varepsilon_i(M) > 0 \). Then \( \text{soc}(e_iM) \) is a gr-irreducible \( R(\nu - i) \)-module with \( \varepsilon_i(\text{soc}(e_iM)) = \varepsilon_i(M) - 1 \).

Proof. Let \( L \) be a gr-irreducible submodule of \( e_iM \). Since \( e_iM = \bigoplus_{j \in \text{Seq}(\nu - i)} 1_j \otimes 1_iM \), we have \( (1_{\nu - i} \otimes x_i^1)e_iM = 0 \) for \( l \gg 0 \). By Schur's Lemma and Proposition 3.2, \( z = \sum_{i \in \text{Seq}(\nu), 1 \leq k \leq m} x_k,i \ (m = \text{ht}(\nu)) \) acts on \( M \) by a scalar. Similarly, \( z' = \sum_{i \in \text{Seq}(\nu - i), 1 \leq k \leq m - 1} x_k,i \) acts on \( L \) by scalar and so \( z - z' \) acts on \( L \) by a scalar \( c \). Since \( L \subseteq 1_{\nu - i} \otimes 1_iM \), for every \( m \in L \), we get
\[
(z - z')m = \left( \sum_{j \in \text{Seq}(\nu - i)} 1_j \otimes x_i m = 1_{\nu - i} \otimes x_i m = cm.
\]
Now \( (1_{\nu - i} \otimes x_i^1)m = 0 \) for \( l \gg 0 \) yields \( c = 0 \), and so \( (1_{\nu - i} \otimes x_i)L = 0 \). Hence \( L \) is a gr-irreducible \( R(\nu - i) \otimes R(i) \)-submodule of \( \Delta_iM \), which is isomorphic to \( L \otimes V(i) \). By Lemma 4.8, \( \text{soc}(\Delta_iM) \) is gr-irreducible. It follows that \( \text{soc}(e_iM) = L \) is gr-irreducible.

Let \( i \in I \). For a gr-irreducible \( M \in R(\nu)-\text{fMod} \), define \( \tilde{e}_iM = \text{soc}(e_iM) \). If \( \varepsilon_i(M) > 0 \), \( \tilde{e}_iM \) is gr-irreducible with \( \varepsilon_i(\tilde{e}_iM) = \varepsilon_i(M) - 1 \).

Proposition 4.10. For a gr-irreducible \( M \in R(\nu)-\text{fMod} \) and \( n \geq 0 \), we have
\[
\text{soc}(\Delta_{\nu}M) \simeq \tilde{e}_i^nM \otimes V(i^n)^{\{r\}}.
\]
for some \( r \in \mathbb{Z} \).

Proof. The case of \( i \in I^+ \) has been proved in [11 Lemma 3.13]. Assume \( i \in I^- \), since \( \tilde{e}_iM \otimes V(i) \) is a graded submodule of \( \Delta_iM \), we see that \( \tilde{e}_i^nM \otimes V(i)^\otimes n \) is a graded submodule of \( \text{Res}^\nu_{\nu - ni, \ldots , \Delta_{\nu}M} \). By the following Frobenius reciprocity
\[
\text{HOM}((\text{Ind}^\nu_{\nu - ni, \ldots , \Delta_{\nu}M} \otimes V(i)^\otimes n, \Delta_{\nu}M) \simeq \text{HOM}(\tilde{e}_i^nM \otimes V(i)^\otimes n, \text{Res}^\nu_{\nu - ni, \ldots , \Delta_{\nu}M}),
\]
we have a nonzero homomorphism from \( \tilde{e}_i^nM \otimes \text{Ind}^n_{\nu - ni, \ldots , i}V(i)^\otimes n \) to \( \Delta_{\nu}M \). The composition factors of \( \tilde{e}_i^nM \otimes \text{Ind}^n_{\nu - ni, \ldots , i}V(i)^\otimes n \) can only be \( \tilde{e}_i^nM \otimes V(i^n) \), up to degree shifts. So we obtain \( \tilde{e}_i^nM \otimes V(i^n)^{\{r\}} \rightarrow \Delta_{\nu}M \) for some \( r \in \mathbb{Z} \). Now our assertion follows from Lemma 4.8.

Lemma 4.11. Let \( i \in I \) and \( M \in R(\nu)-\text{fMod} \) be a gr-irreducible module with \( \varepsilon_i(M) = n \). We have \( M \simeq \text{hdInd}^n_{\nu - ni, \nu} \tilde{e}_i^nM \otimes V(i^n) \), up to a degree shift.

Proof. The lemma follows from Proposition 4.4 and 4.10.
\[ c_0 + c_1 + \cdots = \text{ht}(\nu) \] and only finitely many terms in the sequence are nonzero. Note that if \( b \neq b' \), then \( W_b \neq W_{b'} \) by Lemma 4.11.

Introduce a lexicographic order on sequences of nonnegative integers: \( c_0c_1 \cdots > d_0d_1 \cdots \) if for some \( t \), \( c_0 = d_0, c_1 = d_1, \ldots, c_{t-1} = d_{t-1} \) and \( c_t > d_t \). We set \( b > b' \) in \( \mathbb{B}_\nu \) if and only if \( W_b > W_{b'} \). To each \( b \in \mathbb{B}_\nu \), assume \( W_b = c_0c_1 \cdots \), assign the projective \( R(\nu) \)-module \( P_{W_b}^\bullet \) associated to the sequence \( W_b^\bullet = \cdots i_{k}^c i_{k-1}^{(c_k-1)} \cdots i_{p+1}^{(c_p+1)} i_p^{c_p} \cdots i_0^{c_0} \).

**Proposition 4.12.** \( \text{HOM}(P_{W_b}^\bullet, S_\nu) = 0 \) if \( b > b' \) and \( \text{HOM}(P_{W_b}^\bullet, S_b) \simeq \mathbb{K} \).

**Proof.** For \( i \in I^+ \), we have \( \text{HOM}(P_{e^{(i)}}(\nu), V(i^\nu)) \simeq \mathbb{K} \) since \( P_{e^{(i)}}(\nu) \) is the graded projective cover of \( V(i^\nu) \). For \( i \in I^- \), \( \text{HOM}(R_{ni}; V(i^\nu)) \simeq V(i^\nu) \simeq \mathbb{K} \) as graded vector spaces. The results follow immediately from the Frobenius reciprocity and Proposition 4.10. \( \square \)

By proposition above, each \([P] \in K_0(R(\nu))\) can be written as a \( \mathbb{Z}[q, q^{-1}] \)-linear combination of \([P_{W_b}^\bullet]\) for \( b \in \mathbb{B}_\nu \). Therefore, \( \Gamma \) is surjective. We obtain

**Theorem 4.13.** \( \Gamma : A^- \to K_0(R) \) is an isomorphism.

For \( M \in R(\nu) \)-fMod, let \( M^* = \text{HOM}_\mathbb{K}(M, \mathbb{K})^\bullet \) be the dual module in \( R(\nu) \)-fMod with the action given by

\[
(zf)(m) := f(\psi(z)m) \quad \text{for} \quad z \in R(\nu), f \in \text{HOM}_\mathbb{K}(M, \mathbb{K}), m \in M.
\]

As proved in [3 Section 3.2], for each gr-irreducible \( R(\nu) \)-module \( S \), there is a unique \( r \in \mathbb{Z} \) such that \( (L\{r\})^* \simeq L\{r\} \), and the graded projective cover of \( L\{r\} \) is stable under the bar-involution \( \bar{\cdot} \).

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