Spontaneous symmetry breaking  
on the lattice  
generated by Yukawa interaction∗

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Abstract

We study by numerical simulation a lattice Yukawa model with naive fermions at intermediate values of the Yukawa coupling constant $y$ when the nearest neighbour coupling $\kappa$ of the scalar field $\Phi$ is very weakly ferromagnetic ($\kappa \approx 0$) or even antiferromagnetic ($\kappa < 0$) and the nonvanishing value of $\langle \Phi \rangle$ is generated by the Yukawa interaction. The renormalized Yukawa coupling $y_R$ achieves here its maximal value and this $y$-region is thus of particular importance for lattice investigations of strong Yukawa interaction. However, here the scalar field propagators have a very complex structure caused by fermion loop corrections and by the proximity of phases with antiferromagnetic properties. We develop methods for analyzing these propagators and for extracting the physical observables. We find that going into the negative $\kappa$ region, the scalar field renormalization constant becomes small and $y_R$ does not seem to exceed the unitarity bound, making the existence of a nontrivial fixed point in the investigated Yukawa model quite unlikely.

∗ Supported by the Deutsches Bundesministerium für Forschung und Technologie and the Deutsche Forschungsgemeinschaft

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1 Introduction

The scalar field self-coupling and the Yukawa couplings in the electroweak theory are believed to vanish in the limit of infinite cut-off, as suggested by the signs of the perturbative $\beta$-functions. But the confirmation of this “triviality” of these couplings and a reliable determination of its consequences require nonperturbative methods, because one has to control a very difficult regime – when the renormalized couplings have maximal possible values which could, in principle, be large. A use of nonperturbative lattice methods for these purposes is desirable.

In the pure scalar sector of electroweak interactions these methods have been successful in estimating the upper bound for the Higgs boson mass in the approximation neglecting the gauge and fermion fields \cite{1}–\cite{5}. They also indicate that the renormalized quartic coupling is lower than the upper bound obtained from the tree level unitarity. This result makes a strongly interacting Higgs sector rather improbable and explains the quantitative agreement with calculations using the renormalized perturbation theory. Weakly coupled gauge fields do not seem to have any unexpected effects on the $\Phi^4$ theory \cite{1} and can thus be treated perturbatively.

Possible effects of a strong Yukawa coupling remain relatively unexplored, however. The most important phenomenological questions are:

- What would be the effect of a strong Yukawa coupling on the upper bound for the Higgs boson mass \cite{7}?
- What is the lower bound on the Higgs boson mass which follows from the vacuum stability requirement in the case of a strong Yukawa coupling \cite{7,8}?
- How strong can the Yukawa coupling actually be? What is the maximal fermion mass which can be generated by a Yukawa interaction \cite{9}?

From the general point of view of quantum field theory the investigations of models with strong Yukawa coupling attempt to elucidate the following issues:

- Are the Yukawa theories in 4 dimensions really trivial or do some nontrivial fixed points exist?
- If they are trivial, how far and in what form Yukawa theories can be used as effective quantum field theories?
- What are their relations to the purely fermionic theories with four-fermion coupling \cite{10}–\cite{14}?

Recently numerous explorations of various lattice Yukawa models have been performed \cite{15}–\cite{25} (for a recent review see \cite{26}). In the following discussion, only the models with the so-called lattice parametrization and single-site Yukawa coupling are considered. The studies reveal the existence of three qualitatively different regions of the bare Yukawa coupling $y$. If $y$ is sufficiently small, the lattice Yukawa models behave according to the perturbation theory based on the quasi-classical picture of the spontaneous symmetry breaking (SSB) in the scalar sector. For large $y$ the fermions decouple in the continuum limit so that the physically interesting range of the values of $y$ is restricted from above.

In the intermediate region, the phase with nonvanishing scalar field expectation value $\langle \Phi \rangle$ exists even if the nearest neighbour coupling $\kappa$ of the scalar field is weak or even antiferromagnetic ($\kappa < 0$). The corresponding SSB is then generated by the Yukawa coupling. In
particular, at $\kappa = 0$ the Yukawa models on the lattice correspond to the pure fermion theories with multifermion couplings of the Nambu–Jona-Lasinio type [10]–[14] and one realizes that the SSB in such theories can be understood on the lattice as a special case of the SSB caused by the Yukawa interaction.

In the light of the above-mentioned goals a very important observation [23] is that it is this intermediate region where the renormalized Yukawa coupling achieves its maximal values. Thus there is ample motivation to investigate SSB in Yukawa theories on the lattice in the regime where the Yukawa interaction is its driving mechanism and the standard quasi-classical approximation based on the scalar field potential is not adequate.

Our recent numerical investigations of the $\text{SU}(2)_L \otimes \text{SU}(2)_R$ lattice Yukawa model with naive lattice fermions have shown [22, 23] that in the region of intermediate values of $y$ and negative $\kappa$ the scalar field propagators have a very complex form. As it turns out, it has two reasons: Firstly, the Yukawa coupling causes here sizeable fermion loop corrections to the scalar propagator. These effects can actually be used to estimate the value of the renormalized Yukawa coupling. Secondly, the antiferromagnetic ordering tendency of the negative scalar field coupling $\kappa$ competes with the ferromagnetic ordering effect of the Yukawa interaction. This shows up at large momenta and is therefore a lattice artefact which, however, has to be taken into account particularly on small lattices in the analysis of the scalar propagators.

In this paper we develop reliable methods of analysis of numerical data in Yukawa models and present some results for the renormalized Yukawa and scalar quartic couplings. The main improvements over previous investigations of a similar type are threefold:

(i) We have been able to analyze the scalar propagators in a very satisfactory way by including the 1-fermion loop contribution and as a result the Goldstone wave function renormalization constant is now determined quite precisely.

(ii) To gain experience with finite-size effects we have varied the lattice size upto $12^316$ and have found the main effect to be the Goldstone boson dictated $1/L^2$ dependence.

(iii) We have also covered nearly the full range of bare parameters which includes the maximum possible bare Yukawa coupling, staying within the lattice artefact-free region of the coupling parameter space.

However, the fermion number in our simple model with naive fermions is too large. This probably causes the most serious problem we have found, namely that the fermion mass stays much below the scalar mass even for the largest renormalized Yukawa coupling. Therefore phenomenologically relevant large scale investigations of Yukawa theories should be performed in more realistic models.

After defining the model in sect. 2 we describe in sect. 3 the spectrum in various phases and point out the appearance of the “staggered” scalar states to be interpreted as lattice artefacts occuring on a hypercubic lattice in the vicinity of the phases with antiferromagnetic ordering. In sect. 4 we discuss the physical meaning of the SSB occuring at negative $\kappa$. The complex structure of the scalar propagators and the method of their analysis on finite lattices are described in sects. 5–7. In sect. 8 the fermion and scalar masses are discussed. Some results for the renormalized Yukawa coupling and the renormalized scalar field quartic coupling are presented in sects. 9 and 10, respectively. We summarize our results and conclude in sect. 11. Also a brief appendix elucidating the properties of models with both the ferromagnetic and antiferromagnetic orderings is included.
2 The SU(2)\(_L\)⊗SU(2)\(_R\) model with naive fermions

Our starting point is a fermion-scalar system in the continuum, which is defined in euclidean space-time by the action

\[
S_0 = \int d^4x \left[\frac{1}{2} \text{Tr} \left\{ \left(\partial_\mu \Phi_0\right)^\dagger \left(\partial^\mu \Phi_0\right) \right\} + \frac{m_0^2}{2} \frac{1}{2} \text{Tr} \left\{ \Phi_0^\dagger \Phi_0 \right\} + \frac{g_0}{4!} \frac{1}{2} \text{Tr} \left\{ \left(\Phi_0^\dagger \Phi_0\right)^2 \right\} + \overline{\Psi}_0 \gamma^\mu \partial_\mu \Psi_0 + y_0 \overline{\Psi}_0 \left(\Phi_0 P_R + \Phi_0^\dagger P_L\right) \Psi_0 \right].
\]  

(2.1)

Here \(\Phi_0\) is a 4-component scalar field with a quartic self-coupling and \(\Psi_0\) is a fermionic SU(2) doublet field coupled to the scalar field by an Yukawa interaction with the bare Yukawa coupling parameter \(y_0\). \(P_R\) and \(P_L\) are the right- and left-handed chiral projectors. Because both fermions in the doublet couple with the same strength \(y_0\) to the scalar field, the action has a global SU(2)\(_R\) flavour symmetry. Together with the global SU(2)\(_L\) symmetry, which would turn into a local symmetry when gauge fields are included, the action is invariant under the global chiral SU(2)\(_L\)⊗SU(2)\(_R\) transformations

\[
\begin{align*}
\Psi_0 &\rightarrow (\Omega_L P_L + \Omega_R P_R) \Psi_0, \\
\overline{\Psi}_0 &\rightarrow \overline{\Psi}_0 (\Omega_L^\dagger P_R + \Omega_R^\dagger P_L), \\
\Phi_0 &\rightarrow \Omega_L \Phi_0 \Omega_R^\dagger,
\end{align*}
\]

(2.2) - (2.4)

where \(\Omega_{L,R} \in SU(2)_{L,R}\).

We regularize the model by introducing a 4-dimensional hypercubic lattice with the lattice spacing \(a\). A simple possibility is to keep the continuum parametrization and just to replace the derivatives by the lattice differences in the action (2.1). But for a study of the largest renormalized Yukawa coupling it turns out to be very important to rescale the fields

\[
\Phi_0(x) = \sqrt{2\kappa} \Phi_x / a, \quad \Psi_0(x) = \Psi_x / a^{3/2}
\]

(2.5)

and reparametrize the coupling parameters

\[
(am_0)^2 = \frac{1 - 2\lambda}{\kappa} - 8, \quad g_0 = \frac{6\lambda}{\kappa^2}, \quad y_0 = \frac{y}{\sqrt{2\kappa}},
\]

(2.6)

thus ending up with the model in the lattice parametrization defined by the action

\[
S = -\kappa \sum_{x,\mu} \frac{1}{2} \text{Tr} \left\{ \Phi_x^\dagger \Phi_{x+\mu} + \Phi_{x+\mu}^\dagger \Phi_x \right\} + \sum_{x} \frac{1}{2} \text{Tr} \left\{ \Phi_x^\dagger \Phi_x + \lambda \left(\Phi_x^\dagger \Phi_x - \mathbb{1}\right)^2 \right\} + \sum_{x,\mu} \frac{1}{2} \left( \Psi_{x+\mu}^\dagger \gamma^\mu \Psi_{x+\mu} - \Psi_{x+\mu} \gamma^\mu \Psi_x \right) + y \sum_x \overline{\Psi}_x \left(\Phi_x P_R + \Phi_x^\dagger P_L\right) \Psi_x.
\]

(2.7)

We note that all the fields and the parameters in this expression are dimensionless. In the following we set the bare quartic coupling \(\lambda\) of the scalar field to infinity, which leads to a freezing of the radial mode of the scalar field, \(\frac{1}{2} \text{Tr} \Phi_x^\dagger \Phi_x = 1\), so that \(\Phi_x\) can be represented by an SU(2) matrix. The hopping parameter of the scalar field \(\kappa\) and the bare Yukawa coupling \(y\) are left as free parameters. As usual in statistical mechanics, in the action (2.7) also the values \(\kappa < 0\) are admissible, though the relationships (2.3) and (2.4) to the continuum parametrization are not defined for negative \(\kappa\).

For the sake of simplicity we are using the naive lattice fermions and the model actually involves 16 degenerate fermion doublets. There are some approaches trying to achieve a
more realistic fermion spectrum preserving the chiral symmetry of the action. So far none of these approaches have proved satisfactory (see e.g. [28] and for recent reviews [26, 27]). An analytic treatment of the naive model by the $1/N$ expansion is not feasible in the investigated limit $\lambda = \infty$. In order to use the Hybrid Monte Carlo algorithm in the numerical simulations we further have to double the number of fermions by squaring the fermion determinant. Thus we are actually simulating the model with 32 fermion doublets expecting that qualitative aspects of Yukawa models are not changed by a large fermion number.

One of such properties, found in many lattice Yukawa models with different symmetries and numbers of fermions [16]–[25], is the occurrence of SSB even at negative values of the hopping parameter $\kappa$. Here we face the somewhat puzzling fact that the relation (2.6) between the coupling parameters of the continuum (2.1) and lattice parametrizations (2.7) breaks down for negative $\kappa$. As long as physical (renormalized) quantities are considered, the region $\kappa < 0$ can have a well defined physical meaning. The physically interesting region of the parameter space seems to be extended in the lattice parametrization and the use of this parametrization is thus crucial.

This observation motivates definitions of renormalized quantities which do not use the field $\Phi_0$ nor the transformations (2.5) and (2.6) and are thus applicable also for $\kappa < 0$: The scalar field in lattice parametrization is renormalized in the phase with SSB as

$$\Phi_{x,R} = \frac{\Phi_x}{\sqrt{Z_\pi}}, \quad (2.8)$$

where $Z_\pi$ is the wave function renormalization constant of the Goldstone components of the $\Phi$-field propagator. For $\kappa > 0$ this is equivalent to the renormalized scalar field in the continuum parametrization

$$\Phi_{0,R}(x) = \frac{\Phi_0(x)}{\sqrt{Z_{0,\pi}}}, \quad (2.9)$$

when the dimensions of the fields are taken into account,

$$\Phi_{x,R} = a \, \Phi_{0,R}(x), \quad (2.10)$$

and

$$Z_{0,\pi} = 2\kappa \, Z_\pi. \quad (2.11)$$

The vacuum expectation value of the renormalized scalar field in lattice units is obtained from the magnetization $\langle \Phi \rangle$

$$av_R = \frac{\langle \Phi \rangle}{\sqrt{Z_\pi}}, \quad (2.12)$$

and $v_R$ is the vacuum expectation value in physical units. The renormalized couplings in the broken phase may be defined as

$$y_R = \frac{m_F}{v_R} = \frac{(am_F)}{(am)} \, \sqrt{Z_\pi}, \quad (2.13)$$

and

$$\lambda_R = \frac{m_\sigma^2}{2v_R^2} = \frac{(am_\sigma)^2}{2\langle \Phi \rangle^2} \, Z_\pi, \quad (2.14)$$

where $m_F$ and $m_\sigma$ are the fermion mass and the $\sigma$ boson mass, respectively, in physical units.
3 Spectrum and continuum limit at negative $\kappa$

The phase diagram of the $\text{SU}(2)_L \otimes \text{SU}(2)_R$ model (2.7) with naive fermions shown in fig. 1 consists of several phases and phase regions with ferromagnetic (FM), paramagnetic (PM), antiferromagnetic (AM) and ferrimagnetic (FI) ordering of the scalar field. In addition to the weak coupling phases FM(W), PMW and AM(W) we find at large values of $y$ the strong coupling phases FM(S), PMS and AM(S) with nonperturbative behaviour of several fermionic observables. A similar phase structure was also observed in fermion-scalar models with different symmetry groups and/or different formulations of fermions on the lattice provided the lattice parametrization and single-site Yukawa coupling is used. Common features of all these models are (i) the existence of weak and strong Yukawa coupling phases and phase regions, (ii) the continuation, at intermediate values of $y$, of the FM phase into the negative $\kappa$ region and (iii) the existence of points where several phase transition lines meet.

The width of the funnel in the phase diagram filled by the FM and FI phases becomes smaller if the number of fermions is decreased (compare, e.g., the phase diagrams in refs. [19, 20]) in accordance with mean field calculations [21].

The fermion mass obeys within the error bars the tree level relation $am_F = y \langle \Phi \rangle$ in the FM(W) phase and is zero in the PMW phase where $\langle \Phi \rangle = 0$. It increases when the FM(S)–PMS phase transition is approached from the FM(S) side, does not feel this phase transition and continues to increase when $\kappa$ is lowered within the PMS phase.

The scalar spectrum is quite different in different phases. In the four phases around the point A, where the PMW, AM(W), FM(W) and FI phases meet, the spectrum is indicated in fig. 2. In the phases PMW and AM(W) with zero magnetization $\langle \Phi \rangle$ there exists a scalar field quadruplet of degenerate mass $am_\Phi$. On the other hand, in the phases FM(W) and FI with nonzero magnetization $\langle \Phi \rangle$ there exist three Goldstone bosons called $\pi$ and one massive $\sigma$ boson with masses $am_\pi = 0$ and $am_\sigma$, respectively.

In the $\kappa < 0$ region the scalar propagators show near the momentum $(\pi, \pi, \pi, \pi)$ the presence of further scalar states which we call staggered states in the following. This effect shows up feebly already at small positive $\kappa$ and as $\kappa$ is lowered it results in a gradually increasing curvature in the scalar propagators at large momenta. The existence of such states is obvious at $y = 0$, as here the symmetry of the action (2.7) under the transformation

$$\Phi_x \rightarrow \Phi_x^{st} = (-1)^{x_1+x_2+x_3+x_4} \Phi_x, \kappa \rightarrow -\kappa$$

implies the presence of a $\Phi^{st}$ quadruplet in the PM phase as well as of three $\pi^{st}$ and one $\sigma^{st}$ states in the AM phase with nonvanishing staggered magnetization $\langle \Phi^{st} \rangle = \langle \sum_x (-1)^{x_1+x_2+x_3+x_4} \Phi_x \rangle$. They are visible at low momenta in the 2-point function of the field $\Phi_x^{st}$. We denote the corresponding masses $am_\Phi^{st}$, $am_\pi^{st}$ and $am_\sigma^{st}$, respectively. For $0 < y < \infty$ the symmetry (3.1) is broken explicitly, and these states can appear on the lattice simultaneously with the usual $\Phi$ or $\pi$ and $\sigma$ bosons, as indicated in fig. 2.

The question is, which particles will remain in the various possible continuum limits, or in the large cut-off limits if the theory is an effective one (several possibilities are discussed in ref. [26]). We want of course to recover in the continuum limit the fermions and the usual bosons simultaneously. This includes the renormalized vacuum expectation value of the scalar field $av_R = \langle \Phi_R \rangle$, which is proportional to the gauge boson mass $am_W$. This is possible only in the FM(W) phase in the scaling region of the FM(W)-PMW phase transition. Here the model is of physical relevance from the point of view of the electroweak theory.

\footnote{For simplicity we use the nomenclature “phases” also to denote phase regions with weak (W) or strong (S) bare Yukawa coupling and include a bracket in the abbreviation.}
As $a m_{\Phi}^s$ and $a m_{\sigma}^s$ vanish only on the critical lines where $\langle \Phi^s \rangle$ is vanishing, the staggered states could remain in the continuum limit simultaneously with the fermions and the usual scalars only if the limit is taken at the point A of the phase diagram in fig. 1. The scaling region of this point should therefore for physical reasons probably be avoided. The staggered states are lattice artefacts because they depend substantially on the lattice geometry. But they can have drastic effects on finite lattices and thus have to be taken into consideration in the analysis of the numerical data for the scalar propagator.

4 SSB generated by Yukawa coupling and the relation to four-fermion coupling

The negative $\kappa$ region has no analogue in the continuum parametrization because the transformation equations (2.6) are not defined for $\kappa < 0$. At first glance it seems to be awkward and the question immediately arises whether a sensible continuum limit can be obtained in this region. The fact that for small positive $\kappa$ and for negative $\kappa$ there is a broken symmetry phase FM(W) at large enough Yukawa coupling suggests that the Yukawa coupling must be the driving force for SSB in that region (see also ref. [29]). This is obviously outside the regime of usual perturbation theory in the continuum. It is therefore very important to investigate this region if one is looking for nonperturbative effects of the Yukawa coupling.

Of course, the AM and FI phases appearing only for $\kappa < 0$ are probably lattice artefacts as they depend very much on the lattice geometry. But the scaling regions of the FM(W) and PMW phases down to the point A (fig. 1) are worth of consideration.

The Osterwalder-Schrader (OS) reflection positivity can be proven presumably only for $\kappa \geq 0$. However, it is a sufficient, but not a necessary condition for unitarity, so that unitarity can still hold. In the numerical computations of the propagators of the theory, our failure to detect any state with a negative norm is assuring. Furthermore, as demonstrated later in this article and also in ref. [23], the measured values for all the masses and the renormalized couplings continue analytically from positive to negative $\kappa$ across $\kappa = 0$.

The problems with the transition from $\kappa \geq 0$ to $\kappa < 0$ seem to exist only on the level of bare parameters in the continuum parametrization. One should consider only the renormalized quantities. If one knew the renormalized running coupling $\bar{y}(\mu)$ at all momentum scales $\mu$ one could define also a sensible bare Yukawa coupling $y_B$ as

$$y_B = \bar{y}(\mu \sim 1/a).$$

For small values of $y_0$ and $g_0$, usual perturbation theory is valid and $y_B$ would not differ very much from the coupling parameter $y_0$. At larger values of $y_0$ and in particular in the negative $\kappa$ region the parameter $y_0$ does not any longer have to be close to $y_B$. This so-defined bare coupling $y_B$ would also have in general no simple relation to the bare parameters $\kappa$ and $g$. Fig. 3 schematically illustrates how $y_B$ and $y_0$ may split up from each other as the nonperturbative region is entered along some line of constant physics specified, e.g., by $y_R = \bar{y}(\mu = \mu_{phy}) = const$, where $\mu_{phy}$ is a physical scale.

The stage is therefore well set to plunge into the negative $\kappa$ region with three issues in mind:

(i) Is there a nonperturbative fixed point?

(ii) Even if the theory is trivial, can the couplings be strong at a reasonably large cut-off?
Our previous investigation [23] with more naive methods of analysis of the numerical data has already produced tentatively negative answers to these two questions which we want to confirm in this paper.

(iii) In any case, it is necessary to find out how far the observables in the \( \kappa < 0 \) region differ quantitatively from the \( \kappa > 0 \) region, for determination of bounds on renormalized couplings.

In the rest of this section we want to point out the connection of a Yukawa theory with a Nambu-Jona-Lasinio (NJL) type model. Integrating out the scalar fields in the partition function defined by the action (2.7) with \( \kappa = \lambda = 0 \) produces obviously a pure fermionic theory with local (on-site) four-fermion coupling of strength \( \frac{1}{2}y^2 \) – the NJL model on the lattice. The scalar field is equivalent to the “auxiliary field” used in the context of the four-fermion coupling [10, 11, 30]. Also for \( \lambda > 0 \) the effective fermion interaction is local at \( \kappa = 0 \) and corresponds thus to a multifermion interaction discussed, e.g., in ref. [12]. Thus we conclude that theories of the NJL type are special cases (\( \kappa = 0 \)) of the Yukawa models in the lattice parametrization. But in terms of the renormalized theory, as we have discussed above, \( \kappa = 0 \) is not singular and the same qualitative physics is obtained for the whole FM(W)-PMW scaling region down to the point A unless there is a nonperturbative fixed point somewhere. Recent work [13, 14] using \( 1/N \) expansion also shows an equivalence between Yukawa models and generalized NJL models.

Thus we achieve a unification of concepts and language of the SSB: the Yukawa models treated nonperturbatively using the lattice formulation (2.7) embrace both

- the classical Higgs mechanism, in which the SSB is understood in terms of the quasi-classical approximation for the effective potential of the scalar field and perturbation theory (e.g. the region \( y \ll 1 \) in fig. 1 for any \( \lambda \)), and

- the NJL type mechanism, operating at small (and possibly negative) \( \kappa \) for relatively large values of \( y \), which has to be treated by nonperturbative techniques such as the \( 1/N \) expansion.

Yukawa theories provide a gradual transition between these mechanisms. It seems, therefore, possible to formulate the SSB in the standard model in terms of the NJL mechanism [10, 11, 12]. However, in the light of the above discussion, the distinctions between an elementary and a composite scalar, an auxiliary and a dynamical scalar field and between dynamical symmetry breaking and the usual Higgs mechanism do not seem important.

5 Properties of the scalar propagators

In our numerical simulation we consider \( V = L^3T \) lattices with periodic boundary conditions for the scalar fields. Fermionic fields are periodic in space and antiperiodic in euclidean time directions.

In the symmetric (PM) phase the bosonic spectrum contains the \( \Phi \) quadruplet of mass \( am_\Phi \). The corresponding propagator in the momentum space is

\[
G_\Phi(ap) = \left\langle \frac{1}{iV} \sum_{x,y} \frac{1}{2} \text{Tr} \left\{ \Phi_x \Phi_y \right\} \exp(iap(x - y)) \right\rangle.
\]  

(5.1)
Neglecting the instability of $\Phi$ at the present precision level, the renormalized mass $am_\Phi$ and the wave function renormalization constant $Z_\Phi$ can be defined by means of the limit

$$G_\Phi(ap)|_{p^2 \to 0} = \frac{Z_\Phi}{(am_\Phi)^2 + \tilde{a} p^2},$$

(5.2)

where the quantity $\tilde{a} p^2 = 2 \sum_\mu (1 - \cos(ap_\mu))$ is the dimensionless lattice equivalent of the momentum square in the continuum.

In the *broken (FM)* phase it is useful to introduce the following notation for the scalar field

$$\Phi = \sigma \mathbb{I} + i \sum_{j=1}^{3} \pi^j \tau^j.$$

(5.3)

Here $\tau^j$, $j = 1, \ldots, 3$ are the usual Pauli matrices and $\sqrt{\sigma_x^2 + \sum_{j=1}^{3} (\pi_x^j)^2} = 1$. The components are chosen such that the magnetization is given by $\langle \Phi \rangle = \langle \sigma \rangle$. Then the longitudinal component is associated with the massive $\sigma$ boson whereas the transverse ones with the three massless Goldstone bosons $\pi$. On the lattice the $\pi$ and $\sigma$ propagators in the momentum space are defined by

$$G_\pi(ap) = \left\langle \frac{1}{3V} \sum_{x,y} \pi_x^j \pi_y^j \exp(iap(x-y)) \right\rangle,$$

(5.4)

$$G_\sigma(ap) = \left\langle \frac{1}{V} \sum_{x,y} \sigma_x \sigma_y \exp(iap(x-y)) \right\rangle.$$

(5.5)

The only asymptotic states in the FM phase are the massless $\pi$ bosons. Therefore the wave function renormalization constant $Z_\pi$ for the scalar field is defined through the following limit of the $\pi$ propagator,

$$G_\pi(ap)|_{p^2 \to 0} = \frac{Z_\pi}{\tilde{a} p^2}.$$

(5.6)

Using the so-defined $Z_\pi$ the renormalized field expectation value $v_R$ is then given by eq. (2.12). Again at our present precision level it is presumably sufficient to define the renormalized mass $am_\sigma$ of the unstable $\sigma$ particle by the relation

$$G_\sigma(ap)|_{p^2 \to 0} = \frac{Z_\sigma}{(am_\sigma)^2 + \tilde{a} p^2}.$$

(5.7)

Another point to note is that in the pure $\Phi^4$ theory the renormalized mass defined this way is very close to the physical mass [3, 4].

In a finite system no spontaneous breakdown of the symmetry can occur and during a Monte Carlo simulation in the broken phase the system drifts through the set of degenerate ground states. This causes a vanishing of noninvariant observables like $\langle \Phi \rangle$. To compensate for this drift, each scalar field configuration is rotated so that $\frac{1}{V} \sum_x \Phi_x = \frac{1}{V} \sum_x \sigma_x$. In the pure $\Phi^4$ theory the rotation technique provides a very good approximation of the infinite volume values of the noninvariant quantities [2].

A lot is known about the properties of the scalar propagators in the pure $\Phi^4$ theory which is the limiting case of the model (2.7) both for $y = 0$ and $y = +\infty$ (in the latter case fermions become infinitely heavy and decouple completely from the particle spectrum). When plotting the inverse propagators $G_{\Phi}^{-1}(ap)$, $G_{\pi}^{-1}(ap)$ and $G_{\sigma}^{-1}(ap)$ as functions of $\tilde{a} p^2$ one finds for all propagators and for all possible values of $\tilde{a} p^2$ a straight line behaviour confirming the analysis of the data in terms of free scalar propagators. From straight line fits to the
inverse propagator data, using the relations (5.2), (5.6) and (5.7), one can determine the wave function renormalization constants and the renormalized masses on a finite lattice.

In the Yukawa model we have determined for various values of $\kappa$ and $y$ the momentum space propagator (5.1) in the PMW and PMS phases and the propagators (5.4) and (5.5) in the FM phase close to the critical lines FM(W)-PMW and FM(S)-PMS (see fig. 1). For very small and very large values of the Yukawa coupling $y$ the results for the propagators are very close to those obtained in the $\Phi^4$ theory, i.e., when plotting the inverse propagators as functions of $\hat{ap}^2$ we find approximately a straight line behaviour. However, when entering the intermediate Yukawa coupling region and lowering $\kappa$, where fermions have a strong feedback on the scalar sector and the staggered scalar states (sect. 3) become visible, deviations from the free propagator behaviour at larger values of $\hat{ap}^2$ are observed. These deviations become more and more pronounced when approaching the multicritical points A and B.

In fig. 4 we display for several points in the FM phase the inverse Goldstone propagator $G^{-1}_\pi(ap)$ as a function of the quantity $\hat{ap}^2$. The figures in the left column were obtained at 3 points in the vicinity of the FM(W)-PMW phase transition whereas the figures in the right column correspond to 3 points in the vicinity of the FM(S)-PMS phase transition. The lowest figures were obtained very close to the multicritical points A and B respectively. The figures show that there are three kinds of deviations from a free propagator behaviour:

1. The formation of a second pole in $G_\pi(ap)$ in the corner of the Brillouin zone with $ap = (\pi, \pi, \pi, \pi)$ when approaching the points A or B within the FM phase. This effect is caused by the staggered states $\Phi^{st}$ which are present on the lattice also in the FM phase (see sect. 3).

2. The appearance of dips in $G^{-1}_\pi$ around the momenta $\hat{ap}^2 = 4, 8, 12$ (corresponding to $ap_\mu = 0$ or $\pi$) in the weak coupling region. These dips are already visible at positive $\kappa$ as can be seen from the first figure in the left column.

3. At small $\hat{ap}$ the inverse propagator has in the weak Yukawa coupling region a curvature, which plays a significant role in the data analysis.

Similar structures were also discovered for the propagator $G_\sigma(ap)$ in the FM phase and the propagator $G_\Phi(ap)$ in the PMW and PMS phases near the FM(W)-PMW and FM(S)-PMS phase transitions. As we shall now discuss, the first two effects are actually lattice artefacts, dependent on the geometry of the lattice, but they have to be understood quantitatively in order to extract the physically relevant quantities from the scalar propagator data. The third effect is physical.

6 Staggered scalar states in the FM and PM phases

For an understanding of the two pole phenomenon it is useful to discuss first the situation in the pure $\Phi^4$ theory which is found in the limiting cases $y = 0$ and $y = +\infty$. We use the $\sigma$ propagator $G_\sigma(ap, \kappa)$ as an example and indicate for a moment explicitly the $\kappa$-dependence of the propagator. Using the transformation (3.1) we find the relation:

$$G_\sigma(ap, -\kappa) = G_\sigma(ap + a\tilde{\pi}, \kappa)$$  \hspace{1cm} (6.1)

where $\tilde{\pi} = (\pi, \pi, \pi, \pi)$. As the propagator $G_\sigma(ap, \kappa)$ in the pure scalar theory can for all momenta $ap$ be well described by the free scalar Ansatz eq. (5.7), $G_\sigma(ap, -\kappa)$ has for $\kappa > \kappa_c$...
the form
\[ G_{\sigma}(ap, -\kappa) = \frac{Z_{\sigma}}{(am_{\sigma})^2 + p(p + \tilde{\pi})^2} = \frac{Z_{\sigma}^{st}}{(am_{\sigma}^{st})^2 + 16 - \tilde{ap}^2} \]  
(6.2)

with $m_{\sigma}^{st} = m_{\sigma}$ and $Z_{\sigma}^{st} = Z_{\sigma}$. For $am_{\sigma} = 0$ the propagator $G_{\sigma}(ap, -\kappa)$ has thus a pole at the momentum $p = \tilde{\pi}$. Similar relations can be obtained also for the propagators $G_{\Phi}$ and $G_{\pi}$. Obviously, at no $\kappa$ the normal and staggered states appear simultaneously as their masses are small only close to the FM–PM and PM–AM phase transitions, respectively, which are distant.

For $0 < y < \infty$ the symmetry (3.1) does not hold any more. Nevertheless, for small and large values of $y$ the spectrum is very similar to the pure $\Phi^4$ theory. But when the phase transition lines approach each other and finally meet at the points A and B we expect that both the normal and the staggered state masses are small simultaneously. In particular, $am_{\sigma}^{st}$ can be small also in the FM phase. In the FM(S) phase it is therefore reasonable to try to fit the numerical results for $G_{\sigma}(ap)$ by the two pole Ansatz

\[ G_{\sigma}(ap) = \frac{Z_{\sigma}}{(am_{\sigma})^2 + \tilde{ap}^2} + \frac{Z_{\sigma}^{st}}{(am_{\sigma}^{st})^2 + 16 - \tilde{ap}^2}. \]  
(6.3)

Analogous expressions have been applied also for the propagators $G_{\pi}(ap)$ and $G_{\Phi}(ap)$ in the FM and PM phases respectively. In fig. 5 we show as an example a fit to the Goldstone propagator in the FM(S) phase, where the fit Ansatz for $G_{\pi}(ap)$ is given by eq. (6.3) with $am_{\pi} = 0$ and $Z_{\sigma}$ replaced by $Z_{\pi}$. Fig. 5 shows that the data are described by the two pole Ansatz very well. We furthermore expect that the staggered mass $am_{\Phi}^{st}$ and the wave function renormalization constant $Z_{\Phi}^{st}$ obtained from the fits to $G_{\pi}(ak)$ and $G_{\sigma}(ap)$ should agree. This expectation is indeed confirmed by the numerical results, for example at $\kappa = -0.65$ and $y = 1.8$ the values are $am_{\Phi}^{st} = 1.47(7) / 1.37(4)$ and $Z_{\Phi}^{st} = 1.41(7) / 1.45(4)$ from the $\sigma/\pi$ propagators. As expected, the mass $am_{\Phi}^{st}$ does approach zero when the line AB is approached. It should be also stressed that both terms in the Ansatz (6.3) have positive sign, so that the second term has the usual form of a pole after shifting the momentum by $(\pi, \pi, \pi, \pi)$ and cannot be interpreted as a ghost.

The same formulae describe also the two pole structure of the scalar propagator in the vicinity of the point A. Here, however, a more elaborated Ansatz has to be developed in order to take into account simultaneously also the overlaid finer structure caused by the fermion loop corrections. It is the subject of the next section.

7 One fermion loop contribution to the scalar propagators

The other two features making the scalar propagators different from a free propagator are the appearance of dips at momenta $\tilde{ap}^2 = 4, 8, 12$ and the curvature at small $\tilde{ap}^2$. They occur in the weak coupling phase regions FM(W) and PMW where the fermion masses scale.

According to the definitions of the quantities $Z_{\pi}$, $Z_{\sigma}$ and $am_{\sigma}$ in eqs. (5.6) and (5.7) the scalar propagators have to be analyzed in the limit $\tilde{ap}^2 \to 0$. However, on small lattices with periodic boundary conditions the smallest nonvanishing momentum is $ap = \frac{2\pi}{T}$ (for $T > L$) which is quite large, e.g., $ap = 0.79$ for $T = 8$. In the pure $\Phi^4$ theory this is not a serious problem as the inverse scalar propagator can be fitted with a straight line up to $\tilde{ap}^2 = O(10)$.
In our Yukawa model the situation is much less favourable: in addition to the dips occurring at $\hat{a}p^2 = 4, 8$ and $12$ – for which one might argue that they should simply be ignored as the analysis has to be restricted to the smallest momenta – we have to face the more serious problem of a significant curvature at small $\hat{a}p^2$. The following further observation makes the situation look even worse: Increasing the $T$-extent of the lattice – which is the cheapest possibility to have small momenta – e.g. from $T = 6$ up to $T = 46$, we still have not found an onset of a linear $\hat{a}p^2$ dependence. We conclude from this that for sizeable Yukawa coupling an application of the free particle Ansatz for the scalar propagators on finite lattices produces uncontrollable systematic errors.

Therefore, we have developed a more sophisticated fit Ansatz for the scalar propagator based on the 1-fermion-loop contribution to the self-energy of the $\pi$ or $\sigma$ bosons. The justification for including only the fermion loop comes from the experience in the pure scalar sector of the model where the scalar loop contributions do not change the linear $\hat{a}p^2$-shape of the propagators but only lead to wave function and mass renormalizations. On the other hand, as will be shown below, in the case of naive lattice fermions the 1-fermion-loop contribution will cause deviations from the linear $\hat{a}p^2$-dependence just of the form observed in the data.

Let us discuss the example of the Goldstone boson propagator. On finite lattices we may write

$$G_{\pi,L}^{-1}(ap) = Z_{\pi}^{-1} \left[ (am_{\pi,L})^2 + \hat{a}p^2 - \Sigma_{\pi,L}(ap; am_{F,L}) \right], \quad (7.1)$$

where $\Sigma_{\pi,L}$ denotes the 1-fermion-loop contribution to the self-energy of the Goldstone boson in the renormalized perturbation theory. For the purpose of this section the subscript $L$ points out the possible dependence of various quantities on the spatial lattice size $L$. For instance with regard to the Goldstone bosons we take the finite-size of the lattice into account in a naive but simple way – allowing for a finite mass of the Goldstone boson $am_{\pi,L}$ (see e.g. also [17]). (A treatment based on chiral perturbation theory, successful in the pure $\Phi^4$ theory [2], is in the complex situation with light fermions presumably not applicable.) In the FM phase where the fermions are massive we impose the two necessary normalization conditions on $\Sigma_{\pi}$ in the infinite volume at momentum zero,

$$\left. \Sigma_{\pi,\infty}(ap; am_{F,\infty}) \right|_{p=0} = 0, \quad \left. \frac{\partial}{\partial \hat{a}p^2} \Sigma_{\pi,\infty}(ap; am_{F,\infty}) \right|_{p=0} = 0. \quad (7.2)$$

With this normalization $G_{\pi}$ approaches in the thermodynamic limit the form used for the definition of $Z_{\pi}$, eq. (5.6), provided $(am_{\pi,L})^2 \to 0$ as $L \to \infty$.

The not-yet-normalized $\Sigma'_{\pi,L}$ calculated from the corresponding Feynman diagram on the lattice is

$$\Sigma'_{\pi,L}(ap; am_{F,L}) = (-1) \frac{4}{L^3T} \sum_k \text{Tr} \left\{ (i\gamma_5\tilde{y}_R) \frac{1}{i\tilde{s}(k) + am_{F,L}} (i\gamma_5\tilde{y}_R) \frac{1}{i\tilde{s}(k-ap) + am_{F,L}} \right\}. \quad (7.3)$$

This equation defines the renormalized Yukawa coupling $\tilde{y}_R$ in terms of the 3-point function. Of course, $\tilde{y}_R$ could in principle differ from $y_R$ defined in eq. (2.13). In the above $s_\mu(k) = \sin k_\mu$, the factor $(-1)$ comes from the fermion loop, the trace is to be taken over the Dirac indices. The SU(2) trace together with another factor of 2 from the HMC doubling (see sect. 2) results in the factor 4 whereas the standard fermion doubling is taken into account automatically. The sum over the loop momentum $k$ runs over the set of momenta.
corresponding to a finite $L^3T$ lattice with periodic boundary conditions in the space direction and antiperiodic boundary conditions in the time direction,

$$
\begin{align*}
  k_i &= \frac{2\pi}{L} n_i, \\
  k_4 &= \frac{2\pi}{T} (n_4 + \frac{1}{2}),
\end{align*}
$$

where $n_i, n_4 = 0, 1, 2, \ldots L-1, T-1$. (7.4)

After some simplifications we get

$$
\Sigma'_{\pi,L}(ap; am_{F,L}) = \frac{2^2}{y_R^2 L^3 T} \sum_k \frac{(am_{F,L})^2 + s(k)s(k-ap)}{[(am_{F,L})^2 + s^2(k)][(am_{F,L})^2 + s^2(k-ap)]}
$$

where we introduce the notation $I'_{\pi,L}$ for the (unnormalized) lattice integral after factorizing out $\tilde{y}_R^2$. In order to recognize the 1-fermion-loop as the reason for the deviations in the scalar propagators it is instructive to discuss the dependence of this expression on the external momentum $ap$. For $ap = (0, 0, 0, 0)$ the denominator has minima when the loop momentum components are $k_\mu = 0$ or $\pi$, corresponding to processes involving the physical fermion and its antifermion or some doubler fermion with its own antidoubler. Thus a considerable contribution of $\Sigma_{\pi}$ to $G_{\pi}$ for small momenta $ap$ can be expected, causing the observed strong curvature of $G_{\pi}^{-1}$. In addition, $\Sigma_{\pi}$ also peaks when some components $ap_\mu = \pi$ and others are zero. Then the kinematics allows such intermediate states to be excited which involve e.g. the physical fermion and the antifermion of momentum $ap$, or any other appropriate pair of doubler and antidoubler whose respective positions of poles differ just by $ap$. This is precisely the reason for the dips seen in the inverse scalar propagators near $\tilde{a}p^2 = 4, 8, 12$.

To fit the propagator data using the Ansatz (7.4) we have to perform the following steps: First, $\Sigma_{\pi,L}$ is normalized as

$$
\Sigma_{\pi,L}(ap; am_{F,L}) = \Sigma'_{\pi,L}(ap; am_{F,L}) - \Sigma'_{\pi,\infty}(ap = 0; am_{F,\infty}) + \bar{a}p^2 \left( \frac{\partial}{\partial \bar{a}p^2} \Sigma'_{\pi,\infty}(ap; am_{F,\infty}) \right) |_{p=0}
$$

so that it satisfies the conditions (7.3) in the infinite volume limit. Here the fermion mass $m_{F,L}$ on the given finite lattice is taken from the standard fit to the fermion propagator data. But note that this normalization also requires an estimate of the fermion mass in infinite volume $am_{F,\infty}$ (various attempts to normalize $\Sigma_{\pi,L}$ using only finite volume quantities did not work). In the spontaneously broken phase FM, the major part of the finite-size effects is expected to be due to the massless Goldstone bosons leading to a volume dependence linear in $1/L^2$. So, checking this dependence and then extrapolating $am_{F,L}$ to $am_{F,\infty}$ requires, at a given $(k, y)$-point, simulations on a sequence of at least three lattices. We have performed runs on lattices $L^3 16$ with $L = 6, 8, 10$ and 12 with the result that as long as $am_{F,L}$ itself is not too small the agreement with a linear $1/L^2$ dependence indeed allows an extrapolation to $am_{F,\infty}$ (see next section).

Analogous to eq. (7.3) we define the (normalized) lattice integral $I_{\pi,L}$ with $\tilde{y}_R^2$ factorized out,

$$
\Sigma_{\pi,L}(ap; am_{F,L}) = \frac{2^2}{y_R^2 L^3 T} \bar{a}p^2 I_{\pi,L}(ap; am_{F,L}).
$$

(7.7)

The $\pi$ propagator fit Ansatz is then

$$
G_{\pi,L}^{-1}(ap) = \frac{1}{\bar{a}p^2} \left[ am_{\pi,L}^2 + \bar{a}p^2 - y_R^2 I_{\pi,L}(ap; am_{F,L}) \right]
$$

(7.8)
with the free parameters $am_{\pi,L}$, $Z_\pi$ and $\tilde{y}_R$ (we note that through the normalization conditions (7.2) this expression also depends on $m_{F,\infty}$).

Before describing the results in the next sections let us discuss the quality of the fit. The full Ansatz we use is the superposition of the propagator with the pole at $ap = (0, 0, 0, 0)$ including the 1-fermion-loop contribution and the staggered propagator with the pole at $ap = (\pi, \pi, \pi, \pi)$ as explained in sect. 6 above. This Ansatz is able to describe the scalar propagator data in the complete interval $0 \leq \tilde{a}p^2 \leq 16$. In particular, the curvature at small momenta and also every detail of the peculiar structures near $\tilde{a}p^2 = 4, 8, 12$ are perfectly reproduced. This is demonstrated in figs. 6 and 7 for two typical examples of the fits, one at small positive $\kappa$ and one in the close vicinity of point A where the scalar propagators have the most complex form.

It should be stressed again that the dips at momenta $\tilde{a}p^2 = 4, 8, 12$ are caused by the presence of the doubler fermions and hence should at least look different, if not absent, in models without them. However, in all models the curvature at small momenta will appear for sufficiently strong Yukawa coupling and the second pole at $\tilde{a}p^2 = 16$ will be present near phases with antiferromagnetic ordering.

8 Fermion and scalar masses at small positive $\kappa$ and negative $\kappa$

We have been able to determine reliably both $am_F$ and $am_\sigma$ simultaneously only for $\kappa \geq 0$. Here $am_\sigma$ is always greater than $am_F$ at least by a factor $3-6$. This presumably is a consequence of having a large number (32) of fermion doublets. Estimating that a similar mass-ratio holds also for $\kappa < 0$ we have mostly performed calculations at points with very small fermion mass, $am_F \simeq 0.1 - 0.3$, in order to have the $\sigma$-boson mass at least smaller than 1.

The determination of $am_F$ in this range of values requires only moderate statistics and can be reliably performed at $\kappa < 0$ even in the vicinity of the point A. An important condition is, however, that the long size $T$ of the lattice $L^3T$ is at least 16, otherwise $am_F$ is spuriously small and $T$-dependent when determined from the fit by means of the free fermion propagator. The method of analysis and many results have been presented already in ref. [23]. Here we would like to point out the large spatial volume dependence of $am_F$. For $am_F \simeq 0.3$ the value can decrease by 40% when $L$ increases from 6 to 10. Nevertheless, for $am_F \gtrsim 0.2$ the decrease is linear with $L^{-2}$, so that one can tentatively extrapolate to $L = \infty$. The observable $\langle \Phi \rangle$ is easily measurable everywhere and also has a linear $L^{-2}$ dependence. The $L^{-2}$ dependence and the extrapolation of both the quantities are shown in fig. 8 (we show $av_R$ instead of $\langle \Phi \rangle$ because $Z_\pi$ is $L$-independent as discussed in sect. 9).

The $\sigma$ boson mass has been determined at several points for $\kappa \geq 0$ on lattices of various sizes. The finite-size effects are compatible with the expected $L^{-2}$ dependence but the large error bars prevent a verification. Nevertheless, we have used the same volume dependence to extrapolate $am_\sigma$ to $L = \infty$.

There are two technical reasons making a reliable determination of $am_\sigma$ at negative $\kappa$ very difficult. Firstly, the number of iterations for the fermion matrix inversion needed for the field update increases drastically with decreasing $\kappa$. As the determination of the $\sigma$ propagator requires much higher statistics than of the fermionic propagator, an accumulation of good data in the negative $\kappa$ region, in particular in the vicinity of the point A is prohibitively expensive. Secondly, the maximum of the curve $G^{-1}_\sigma(\tilde{a}p)$ occurs already at rather small momenta (see fig. 7) and only a few data points of the propagator contain the information
about \( am_\sigma \) and \( Z_\sigma \), the rest being dominated by the staggered state. Therefore we have not succeeded to determine reliably \( am_\sigma \) for any of our data points at negative \( \kappa \).

The largest renormalized couplings are expected on the boundary of the scaling region, i.e. relatively far from the critical line. However, at present we do not know the position of this boundary, actually not even its proper definition (e.g. how small \( am_F \) and \( am_\sigma \) should be in order that the lattice model can be used as an effective continuum theory). We are thus not able to extract upper bounds on masses from our results for renormalized couplings.

9 The renormalized Yukawa coupling

The excellent agreement of the fits with the MC data for the \( \pi \) propagator both for positive and negative \( \kappa \) allows to determine \( Z_\pi \) reliably. In comparison to the usual determination of \( Z_\pi \) by a naive free particle fit to the smallest momentum used e.g. in our earlier publication \[23\], the present method yields more precise results (the error bars are reduced by factors 3 – 10). In particular, they are now stable when the \( T \)-size of the lattice is varied, in spite of the strong curvature near \( \hat{p}^2 = 0 \) which is most clearly seen on large-\( T \) lattices. No \( L \)-dependence of \( Z_\pi \) has been found. The actual values are slightly smaller than found in ref. \[23\] confirming the conjecture in that paper that a simple linear fit to the smallest momentum on small lattices gives overestimated values of \( Z_\pi \).

Some values of \( Z_\pi \) obtained in the vicinity of the FM(W)–PMW critical line are plotted in fig. 9. \( Z_\pi \) decreases strongly as \( \kappa \) decreases and appears to vanish at the multicritical point A.

The knowledge of \( Z_\pi \) allows us to determine, from the available very good data for \( am_F \) and \( \langle \Phi \rangle \) for various \( L \), the renormalized Yukawa coupling \( y_R \) by means of the definition (2.13). Both \( am_F \) and \( \langle \Phi \rangle \) are strongly \( L \)-dependent, but their ratio turns out to be practically \( L \)-independent. We use the observed linear \( L^{-2} \) dependence to extrapolate \( am_F \) and \( av_R \) to infinite volume obtaining \( y_R \) in the \( L \to \infty \) limit (see fig. 8). In fig. 10 we plot these results against the fermionic correlation length \( \xi_F = 1/am_F \) at \( L = \infty \). The reason for not using the smaller \( \xi_\sigma = 1/am_\sigma \) at \( L = \infty \) for this purpose is the fact that we do not know its values for \( \kappa < 0 \).

The dotted curve in fig. 10, given by

\[
y_R = \frac{1}{\sqrt{\frac{32}{4\pi^2} \ln a\mu}},
\]

is obtained by choosing infinite bare Yukawa coupling in the 1-loop formula for the running Yukawa coupling and identifying the scale \( \mu \) to be \( m_F \). As pointed out in \[23\], this curve described quite well the results for \( y_R \) at small positive \( \kappa \) and negative \( \kappa \). The dotted error bars associated with this curve in the figure indicate the range of values and the error bars of those former results in \[23\].

The dramatic reduction of the error bars in fig. 10 is mostly due to the refined analysis in this paper. It is now apparent that \( y_R \) is bounded by the dotted curve, suggesting applicability of 1-loop perturbation theory even close to the point A. In addition all the \( y_R \) results are clearly below the \( s \)-wave tree level unitarity bound for 32 fermion doublets. This bound is indicated by the horizontal dashed line in fig. 10. These results suggest the conclusion that in our model there is no strong Yukawa coupling regime at least for \( \xi_F > 5 \). We remark that initial indications from numerical investigations in the mirror fermion model \[25\] are, however, different. The weakness of \( y_R \) in our model in the negative \( \kappa \) region
comes about as follows: the unrenormalized ratio $am_F/\langle \Phi \rangle$ actually increases strongly as $\kappa$ decreases, but this is compensated by the equally strong decrease of $Z_1^{1/2}$ (fig. 9).

The fermion loop correction allows to determine the renormalized Yukawa coupling $\tilde{y}_R$ defined in terms of the vertex function and extracted from the Goldstone propagator data by the Ansatz (7.8). In table 1 we compare $y_R$ and $\tilde{y}_R$ for some typical $(\kappa, y)$-points where we were able to extrapolate $am_F$ to $L = \infty$ in order to satisfy the normalization conditions (7.2). The good agreement found between $y_R$ and $\tilde{y}_R$ indicates that the analysis of the scalar propagator data by taking the 1-fermion loop corrections into account is adequate and that both definitions of the renormalized Yukawa coupling quantitatively agree. This further supports the conclusion that in our model with naive fermions the Yukawa coupling is not strong.

Some caution is due, however. It could be that our linear extrapolation of $am_F$ to $L = \infty$ underestimates $\xi_F$. Furthermore, as we do not know $am_\sigma$ in the vicinity of the point A, we cannot exclude that $\xi_\sigma$ is as large as $\xi_F$ and that our data points there are actually deep in the scaling region.

### Table 1: Comparison of $y_R$ from the tree level definition (2.13) to $\tilde{y}_R$ from the fit to the Goldstone propagator (7.8) at some typical $(\kappa, y)$-points.

| Lattice | $\kappa=0.03$ | $\kappa=0.04$ | $\kappa=-0.65$ | Lattice | $\kappa=0.00$ |
|---------|----------------|----------------|----------------|---------|----------------|
|         | $y=0.60$       | $y=0.60$       | $y=0.98$       |         | $y=0.65$       |
| $\tilde{y}_R$ $y_R$ | $0.427(5)$ | $0.49(4)$ | $0.412(6)$ | $0.55(2)$ | $0.51(1)$ |
| $\tilde{y}_R$ $y_R$ | $0.410(6)$ | $0.451(8)$ | $0.495(8)$ | $0.55(2)$ | $0.51(2)$ |
| $\tilde{y}_R$ $y_R$ | $0.410(6)$ | $0.45(2)$ | $0.49(2)$ | $0.51(1)$ | $0.52(1)$ |
| $\tilde{y}_R$ $y_R$ | $0.40(1)$ | $0.45(1)$ | $0.48(7)$ | $0.51(3)$ | $0.52(4)$ |

### 10 Fermion influence on the scalar mass bound

To study the influence of heavy fermions on the triviality bound for the scalar mass it would be again most interesting to perform this analysis in the vicinity of the multicritical point A and the lack of reliable results for $am_\sigma$ there is deplorable. However, provided the renormalized Yukawa coupling does not attain in the negative $\kappa$ region values significantly larger than at $\kappa$ positive, as suggested by our results, we expect that it is sufficient to investigate the influence of the Yukawa coupling on the scalar sector in the positive $\kappa$ region.

These considerations have motivated our relatively high statistics study of the $\sigma$-propagator at $\kappa \approx 0$. We have fixed $y$ at the value $y = 0.6$, for which the critical $\kappa$ is $\kappa_c = 0.020(5)$. On lattices of three different spatial sizes, $L^316$ with $L = 6, 8, 10$, we have accumulated about 4-5 thousand Hybrid Monte Carlo trajectories at three points $\kappa = 0.03, 0.04, 0.06$. Both scalar propagators have been analyzed by means of the same Ansatz (7.8) and the scalar mass $am_\sigma$ and $Z_\pi$ have been determined.

The results for the ratio $m_\sigma/v_R$ are displayed in fig. 11 as a function of the scalar correlation length $\xi_\sigma = 1/am_\sigma$ (open symbols). The finite-size dependence of $am_\sigma$ can be described
to be linear in $L^{-2}$, though the error bars on $am_\sigma$ leave room for modification. The tentative extrapolation of the results to the infinite volume assuming a $L^{-2}$ dependence is indicated by the full squares. This kind of plot is customary to extract a triviality upper bound for the scalar mass in the pure $\Phi^4$ theory. For comparison the data from this theory ($y = 0$) on a hypercubic lattice [31] are also shown (full circles).

The results in the Yukawa model on finite lattices approach the infinite volume results from below (different from the $\Phi^4$ theory [2]) and the extrapolated results are - within large error bars - consistent with the pure $\Phi^4$ results. As $\xi_\sigma \simeq 1 - 3$ and $\xi_F$ is much larger than $\xi_\sigma$ for all points in fig. 11, one can expect that the edge of the scaling region is contained. So we observe no large influence of the Yukawa coupling on the Higgs mass upper bound in our model for $\kappa > 0$.

11 Summary and conclusions

We have explored the region of the largest renormalized Yukawa coupling in a lattice Yukawa model with naive fermions in the broken symmetry phase. In this region the Yukawa interaction is the driving force of the spontaneous symmetry breaking overwhelming the very weak ferromagnetic or even antiferromagnetic nearest neighbour scalar field coupling. Such competing interactions, together with sizeable fermion loop corrections, result in a complex structure of the scalar propagators. We have demonstrated that this structure can be theoretically understood and even utilized for an extraction of the renormalized Yukawa coupling $y_R$ from the scalar propagator data. The results for $y_R$ showing that the Yukawa coupling is small in the limit of large cutoff are consistent with the triviality of this coupling. In the physically relevant FM(W) phase the lines of constant $y_R$ seem to flow nearly parallel to the FM(W)-PMW phase transition and for $y > 1$ run out of the FM(W) phase, instead of flowing into some point on this phase transition. In particular the suspicious point A does not seem to be a nontrivial fixed point. The values of $y_R$ do not seem to exceed significantly those at $\kappa \geq 0$, indicating that the $\kappa < 0$ region of the FM(W) phase does not add much to the physical content of the model at $\kappa \geq 0$. We expect these results to be generic for various lattice Yukawa models.

Our results, looked at quantitatively, may however be specific to the chosen model with a large number of fermions (32 doublets). The renormalized Yukawa coupling stays below the tree level unitarity bound, thus being never strong. Correspondingly, no influence of the Yukawa interaction on the upper bound for the $\sigma$-mass could be detected. However, a word of caution is warranted because we have not localized reliably the edge of the scaling region, where the largest values of the renormalized couplings should actually be determined. Furthermore, the model suffers from a drawback caused by the large number of fermions: the fermion mass generated by the Yukawa interaction stays substantially lower than the $\sigma$ boson mass. This makes investigations in the scaling region difficult with two very different correlation lengths and is certainly not a generic feature of Yukawa models. There are now suggestions of Yukawa models with a small number (2 and 4) of fermion doublets [32]. The methods developed in this article should be appropriate also in these models. Apart from quantitative differences, it would be interesting to see if any of the qualitative conclusions drawn in this paper change.
Acknowledgement. We thank J. Smit, M. Tsypin and F. Zimmermann for valuable suggestions and H.A. Kastrup for discussions and continuous support. We have also benefited from discussions with K. Jansen, M. Lindner, I. Montvay, G. M"unster, J. Shigemitsu and J. Vink. The numerical computations have been performed on the Cray Y-MP/832 of HLRZ J"ulich and the S-400 of RWTH Aachen.

Appendix A: The metamagnetic Ising model

Integrating out the fermionic fields leads to a scalar model with nonlocal interaction terms. In this appendix we want to discuss a simple type of spin model with a nearest-neighbour (nn)- and a next-to-nearest-neighbour (nnn)-interaction terms, which appears in the literature in the context of metamagnets\[2\]. In this model an overlap of the scaling regions associated with the normal and the staggered magnetizations occurs analogous to the vicinity of the points A and B in Yukawa models.

The model is defined by the action

\[
S = -2\kappa_{nn} \sum_{x,\mu>0} \sigma_x \sigma_{x+\mu} - 2\kappa_{nnn} \sum_{x,\mu,\nu>0} \sigma_x \sigma_{x+\mu+\nu},
\]

(A.1)

where \(\sigma_x\) are Ising spins.

In a mean field approximation the critical lines are given by \[33\]

\[
\kappa_{nn}^c + \kappa_{nnn}^c = \kappa_{Ising}^c \quad \kappa_{nn}^c > 0, \kappa_{nnn}^c > 0 \quad \text{for the FM-PM transition line}
\]

\[
-\kappa_{nn}^c + \kappa_{nnn}^c = \kappa_{Ising}^c \quad \kappa_{nn}^c < 0, \kappa_{nnn}^c > 0 \quad \text{for the AM-PM transition line},
\]

where the constant \(\kappa_{Ising}^c\) is known from the 4-dimensional Ising model to be about 0.0748. At \(\kappa_{nn}^c = 0\) the transition lines meet in a multicritical point. The phase diagram found in a numerical Monte Carlo simulation of the 4-dimensional model shown in fig. 12 has three phases: a ferromagnetic phase (FM) with \(\langle \sigma \rangle > 0, \langle \sigma^{st} \rangle = 0\), a paramagnetic phase (PM) with \(\langle \sigma \rangle = 0, \langle \sigma^{st} \rangle = 0\) and an antiferromagnetic phase (AM) with \(\langle \sigma \rangle = 0, \langle \sigma^{st} \rangle > 0\). The phase transition lines separating the symmetric from the broken phases is of second order and the phase transition separating FM and AM is of first order. Except for the absence of the FI phase, this phase diagram is similar to the phase diagram of the Yukawa model at weak Yukawa coupling, the triple point being analogous to the point A in fig. 1.

In the vicinity of the multicritical point we find by numerical simulation that the inverse propagator of the scalar field in momentum space has two poles, one of them at \(ap = (\pi, \pi, \pi, \pi)\) and some structure for intermediate momenta. An example in the ferromagnetic phase near the multicritical point is shown in fig. 13. This shape is well described by the inverse propagator of a gaussian model with the same (nn)- and (nnn) kinetic terms

\[
S_p = -a_{nn} \sum_{\mu>0} \{(1 - \cos k_\mu) - a_{nnn} \frac{8}{24} \sum_{\mu,\nu>0} \{2 - \cos(k_\mu + k_\nu) - \cos(k_\mu - k_\nu)\}\} + b.
\]

(A.2)

The fit of the Monte-Carlo data with the function (A.2) plotted in fig. 13 demonstrates that one can describe the data very well in spite of a nonlinear dependence on \(\hat{a}p^2\). If one includes further interaction terms, the structure of the propagator at intermediate values of \(\hat{a}p\) changes.

\[2\]See footnote on p. 60 in [33].
Of course, integrating out the fermion field in the Yukawa model produces a scalar theory which has an infinite number of nonlocal interaction terms. Already taking only into account the first two orders of the $1/y$ expansion of the fermion determinant (see for example [2]) leads to a large number of non-single-site terms and also to terms which are no longer bilinear in the scalar fields. A small $y$ expansion leads to infinite range interaction terms. The Yukawa model is therefore much more complicated than the metamagnetic Ising model and so its scalar propagators are analysed in this paper in a different way. In particular, a gaussian model does not describe the data. But the qualitative feature of the propagators, namely the existence of a second pole at $ap = (\pi, \pi, \pi, \pi)$ if two scaling regions overlap, can be understood considering simple gaussian and Ising models.

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Figure Captions

Fig. 1 Phase diagram of the SU(2)$_L \otimes$ SU(2)$_R$ Yukawa model with naive fermions [19].
Fig. 2 Schematic description of the spectrum of scalar states in the phase diagram around point A. The staggered quantities are denoted by the index “st”. We set $a=1$ in the figure.
Fig. 3 Schematic possible evolution of the bare Yukawa coupling constant $y_B$ (its definition is described in the text) along a line of constant physics $y_R=$const in the FM(W) phase. For comparison also $y_0$ is shown.
Fig. 4 Typical forms of the scalar propagators in the FM phase, mostly at negative $\kappa$. Plotted is the inverse $\pi$ propagator in the momentum space as a function of $\tilde{ap}^2 = 2 \sum_\mu (1 - \cos(ap_\mu))$. The left column shows results near the FM(W)–PMW phase transition and the right column near the FM(S)–PMS transition. With $\kappa$ decreasing downwards the points A and B, respectively, are approached. In this figure and in the following, error bars are not shown whenever they are smaller than the symbols.
Fig. 5 An example of the inverse $\pi$ propagator in the FM(S) phase close to the point B. The full line is the fit with the two pole Ansatz analogue to (6.3).
Fig. 6 An example of the inverse $\pi$ propagator in the FM(W) phase at small positive $\kappa$. We show in the upper part the Monte Carlo data and in the lower part the fit with the Ansatz (7.8).
Fig. 7 An example of the inverse $\pi$ propagator in the FM(W) phase at negative $\kappa$ close to the point A. We show in the upper part the Monte Carlo data and in the lower part the fit with the Ansatz (7.8) where a staggered scalar pole is also taken into account.
Fig. 8 Examples of lattice size dependence of $am_F$ and $av_R$. The dotted and dashed lines are fits linear in $L^{-2}$. Extrapolations to infinite volume are also shown.
Fig. 9 The wave function renormalization constant $Z_\pi$ close to the FM(W)–PMW phase transition as a function of decreasing $\kappa$. The point A is situated at $\kappa = -0.75(3)$. The dashed line serves to guide the eye.
Fig. 10 Our present results for $y_R$, as defined by the relation (2.13), obtained from the infinite volume extrapolated values of $am_F$ and $av_R$. The range of the earlier results [23] obtained with a naive analysis of the Goldstone propagators and on finite lattices is indicated by the dotted error bars around the dotted curve (9.1).
Fig. 11 Ratio $m_\sigma/v_R$ as a function of the scalar correlation length $\xi_\sigma$. The full squares represent a tentative infinite volume extrapolation. The circles are Monte Carlo results in the pure $O(4) \Phi^4$ theory [31].
Fig. 12 Phase diagram of an Ising metamagnet. The numerically determined positions of critical lines are indicated by the points with error bars and the dotted lines are mean field results.
Fig. 13 Inverse momentum space propagator in the Ising metamagnet and its fit by means of the gaussian model with a $nnn$-term.