The Bedrock of Quantum Nonlocality

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Abstract

Nonlocality is a distinct feature of quantum theory, and the questions of where does it originate from and how is it different from the classical theory are still far from settled. By means of the generalized uncertainty principle, we find that different degrees of nonlocality of non-separable state, ranging from superquantum to quantum, and to classical theory, may be well characterized. Various quantum nonlocal phenomena will emerge in different order of observables’ mutual dependence. We exhibit the third order “skewness nonlocality”, indicating that the Bell inequality violation being only the second order nonlocality, the “variance nonlocality”.

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1 Introduction

In classical physics, observables are represented by real numbers (or vectors composed of real numbers), which implies that all properties of real numbers should be respected. While confronting the micro-world, not taken it for granted, Heisenberg questioned this underlying prerequisite by dint of a Gedanken experiment where the canonically conjugate quantities, $x$ and $p$, can only be determined simultaneously with a characteristic indeterminacy \cite{1}. Explicit uncertainty (indeterminacy) relations for incompatible observables are derivable from the fundamental postulates of quantum mechanics (QM). Unsatisfied with the measurement uncertainties, Einstein together with his colleagues ever attempted to exemplify the QM being incomplete via an entangled bipartite system, viz. the renowned EPR paradox \cite{2}. We now know that any refutation of the EPR paradox would invoke certain forms of nonlocality.

One may restore the completeness and locality of QM by introducing additional local hidden variables. However in 1964, Bell derived a set of inequalities by which all local hidden variable theories (LHVTs) abide, while quantum theory not. Among the various Bell inequalities (BIs), one of the most eminent ones is the Clauser-Horne-Shimony-Holt (CHSH) inequality \cite{3}

$$|E(X, Y) - E(X, Y') + E(X', Y) + E(X', Y')| \leq 2. \quad (1)$$

Here the four terms denote the correlation functions between observables $X, X'$ and $Y, Y'$ in a bipartite qubit system. Nevertheless, in quantum theory the left side of equation (1) may reach $2\sqrt{2}$ \cite{4} violating the bound. A heuristic question regarding to this violation may arise: why quantum limit is $2\sqrt{2}$ but not more \cite{5}? Some principles beyond the QM were proposed to address this question, e.g., communication complexity \cite{6,7} and
information causality [8], however the physical meaning of them are still vague [9]. On the other hand, up to now people have derived numerous types of Bell inequalities [10], in which both linear and nonlinear correlations may exist [11]. Hence, it is tempting to think what constrains the regime of nonlocal and hidden variable theory, the Bell inequalities, and whether we can find the physical mechanism governing the violation degrees of BI or not.

Here we provide a solution to the above question, proposing a scenario for the study of nonlocal phenomena through the generalized quantum uncertainty relations. It will be quantitatively demonstrated that the uncertainty relations may determine the strengths of nonlocalities ranging from the superquantum to quantum ones, and from the Bell local to non-steering ones, etc. Exploring the high order dependency revealed by the generalized uncertainty relations [12], we find a mechanism which may generate some novel inequalities to witness the fundamentally different nonlocal phenomena other than Bell nonlocality. Explicit example of “skewness nonlocality” is constructed, and the Bell nonlocality turns out to be only the “variance nonlocality”.

2 The physics of nonlocality

2.1 The origin of the quantum nonlocality

The fundamental postulates of QM tell that physical obervables are represented by Hermitian matrices, and measurement results of an observable can only be the eigenvalues of the Hermitian matrix. Two observables represented by $N \times N$-dimensional Hermitian matrices $X$ and $Y$ may sum to $X + Y = Z$, and there exists

$$\sum_{i=1}^{l} \alpha_i + \sum_{j=1}^{l} \beta_j \geq \sum_{k=1}^{l} \gamma_k , \quad 1 \leq l \leq N , \tag{2}$$

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where $\alpha_i$, $\beta_j$, and $\gamma_k$ are eigenvalues of $X$, $Y$, and $Z$, arranged in descending order (see Ref. [13] for more general forms and Ref. [14] for its application in quantum information theory). If $Y$ and $Y'$ are two-dimensional observables, the qubit, with eigenvalues $\pm 1$, the summation $(Y - Y') + (Y + Y') = 2Y$ gives

$$\alpha_1 + \beta_1 \geq \gamma_1 = 2 .$$

(3)

Here $\alpha_1$, $\beta_1$, and $\gamma_1$ are the largest eigenvalues of $Y - Y'$, $Y + Y'$, and $2Y$, respectively. When $Y$ and $Y'$ are taken to be orthogonal qubit observables, i.e., Pauli matrices $Y = \sigma_x$ and $Y' = \sigma_z$, we have $\alpha_1 = \beta_1 = \sqrt{2}$ and equation (3) turns to $\sqrt{2} + \sqrt{2} \geq 2$.

Let $y_i$ and $y'_j$ be the eigenvalues of $Y$ and $Y'$ with the observing probabilities $p_i(y)$ and $p_j(y')$, the expectation value of their sum can be written as

$$\langle Y + Y' \rangle = (\vec{y} \oplus \vec{y}') \cdot (\vec{p}_y \oplus \vec{p}_{y'}) ,$$

(4)

where $\vec{y}$ and $\vec{y}'$ are vectors composed of the eigenvalues and $\vec{p}_y, \vec{p}_{y'}$ are the corresponding probability distributions. For qubit observables, we have the following

$$\langle Y + Y' \rangle \leq (\vec{y} \oplus \vec{y}')^\downarrow \cdot \vec{s} .$$

(5)

Here $\downarrow$ denotes that the components are rearranged in descending orders and $\vec{s}$ is the optimal bound from majorization uncertainty relation $\vec{p}_y \oplus \vec{p}_{y'} \prec \vec{s}$ [15]. New progress in understanding the uncertainty principle interprets the generalized uncertainty relations as the dependence between the measurements [12]. In this sense, the expectation value $\langle Y + Y' \rangle$ would reach 2 if $Y$ and $Y'$ are independent observables with boundaries of $\pm 1$. While the uncertainty relation (dependence) between $Y$ and $Y'$ limit the value to be less than 2 (see Appendix A).

With the above preparations we are ready to examine how the quantum nonlocality
arises and behaves. We start by considering the correlation terms in CHSH inequality

\[ E(X, Y) - E(X, Y') + E(X', Y) + E(X', Y') \]  \[1\]  

The LHVT and quantum evaluations of the correlation, i.e., \( E(X, Y) \), are respectively

\begin{align*}
\text{LHVT : } & E(X, Y) = \int \xi(\lambda) A(\lambda, X) B(\lambda, Y) \, d\lambda, \\
\text{QM : } & E(X, Y) = \langle X \otimes Y \rangle ,
\end{align*}

where \( \xi(\lambda) \) is an unknown distribution of the hidden variables, positive and normalized. In LHVT, both \( A(\lambda, X) \) and \( B(\lambda, Y) \) take the values in between \([-1, 1]\) that are determined by \( \lambda \) together with the observables \( X \) or \( Y \).

1. LHVT: There is the following inequality related to equation \(1\)

\[-2 \leq A(\lambda, X)[B(\lambda, Y) - B(\lambda, Y')] + A(\lambda, X')[B(\lambda, Y) + B(\lambda, Y')] \leq 2 . \]

The bounds \( \pm 2 \) are obtained from the following arguments: 1. The values of \( A(\lambda, X) \) and \( A(\lambda, X') \) are independent and can be \( \pm 1 \); 2. The values of \( B(\lambda, Y) \) and \( B(\lambda, Y') \) are also independent, but the sum of \( B(\lambda, Y) - B(\lambda, Y') \) and \( B(\lambda, Y) + B(\lambda, Y') \) is bounded to \([-2, 2]\). Therefore, integrating over the distribution \( \xi(\lambda) \), we have

\[-2 \leq E(X, Y) - E(X, Y') + E(X', Y) + E(X', Y') \leq 2 . \]

This is the CHSH inequalities for LHVTs.

2. Non-steering: If \( A \) cannot steer \( B \), then the states of \( B \) conditioned on the measurement result \( i \) of any observable \( X \) are represented by the following assemblages

\[ \sigma_{i|x} = \sum_{\lambda} \xi(\lambda) p_i(\lambda)(x) \sigma(\lambda) . \]
Here $\xi_\lambda$ is an unknown normalized distribution and $\sum_i p_i(\lambda)(x) = 1$. In evaluating the correlation terms in equation (6) from the assemblages in equation (11), we would get similar terms as that in equation (9) where the differences lie in $B$

$$B(\lambda, Y) - B(\lambda, Y') = \text{Tr}[\sigma(\lambda)(Y - Y')] ,$$  \hspace{1cm} (12)  

$$B(\lambda, Y) + B(\lambda, Y') = \text{Tr}[\sigma(\lambda)(Y + Y')] .$$  \hspace{1cm} (13)  

We can see that: 1. The values of $A(\lambda, X)$ and $A(\lambda, X')$ remain independent and can be ±1; 2. The values of $B(\lambda, Y)$ and $B(\lambda, Y')$ are not independent any more due to the uncertainty relation [17]; 3. The uncertainty relation in equation (5) limits each value of equations (12) and (13) to be $\sqrt{2}$. And there is the following (see Appendix A)

$$[E(X, Y) - E(X, Y')]^2 +$$  

$$[E(X', Y) + E(X', Y')]^2 \leq 2 .$$  \hspace{1cm} (14)  

This is an inequality that non-steering correlations must satisfy.

3. Quantum correlation: The steering and uncertainty relation have been shown to be responsible for improving the correlation from LHVT to QM [18]. For the sake of demonstration, we present the following toy steering model

$$\sigma_{i1|x} = \sum_\lambda \xi_\lambda p_{i1}(\lambda)(x)\sigma(\lambda)(x) ,$$  \hspace{1cm} (15)  

$$\sigma_{i2|x'} = \sum_\lambda \xi_\lambda p_{i2}(\lambda)(x')\sigma(\lambda)(x') .$$  \hspace{1cm} (16)  

Here the different assemblages resulted from different measurements $X$ and $X'$ are independent (not from a common refined states $\{\sigma(\lambda)\}$ as that of equation (11)), and we have

$$B(\lambda, Y) - B(\lambda, Y') = \text{Tr}[\sigma(\lambda)(x)(Y - Y')] ,$$  \hspace{1cm} (17)  

$$B(\lambda, Y) + B(\lambda, Y') = \text{Tr}[\sigma(\lambda)(x')(Y + Y')] .$$  \hspace{1cm} (18)
Now there exist the following: 1. The values of $A(\lambda, X)$ and $A(\lambda, X')$ remain independent and can be $\pm 1$; 2. The values of $B(\lambda, Y)$ and $B(\lambda, Y')$ are dependent due to the uncertainty relation [17]; 3. Unlike equations (12) and (13), equations (17) and (18) are independent and may reach the maximum values $\{2 \cos \frac{\theta}{2}, 2 \sin \frac{\theta}{2}\}$ simultaneously (see Appendix A)

$$\begin{align*}
[E(X, Y) - E(X, Y')]^2 + \\
[E(X', Y) + E(X', Y')]^2 &\leq 4 . \\
\end{align*}$$

(19)

Considering the inequality $(a + b)^2 \leq 2(a^2 + b^2)$, equation (19) is just the QM prediction for the CHSH and can be regarded as the direct corollary of equations (3) and (5).

4. **Superquantum correlation:** Now we make a further assumption regarding the quantum mechanical results of equations (17) and (18): Let $Y$ and $Y'$ be independent observables, i.e., there is no uncertainty relation for them. Without the constraint imposed on expectation value of $Y + Y'$ (discussions after equation (5)), we have

$$\begin{align*}
-4 &\leq E(X, Y) - E(X, Y') + \\
E(X', Y) + E(X', Y') &\leq 4 . \\
\end{align*}$$

(20)

With the above discussions, the nonlocalities of different theoretical models are summarized in Figure 1. Some of the transitions between the different types of nonlocalities in Figure 1 have been reported in literatures: the arrow of $\odot$ [18]; the arrows of $\oplus$ and $\ominus$ [17]. It is an interesting question whether the nonlocal hidden variable theory (NHVT), i.e., the Leggett model [19], can be embedded in to the nonlocal structure in Figure 1. In the next we shall show how the nonlocality behaves when high order dependence in the generalized uncertainty relations is involved.
Figure 1. The nonlocalities of different models. Interpreting the uncertainty relation (UR) as the dependence between observables, different types of nonlocalities are listed with the potential transitions denoted by the arrows, where the arrows of ①-③ have been studied in literatures [17, 18]. An interesting question is how the NHVT (the Leggett model) is related to others via UR.

### 2.2 Higher order quantum nonlocalities

In any bipartite system, we may define a joint measurement operator $S = \sum_{i,j} m_{ij} X_i \otimes Y_j$ with $m_{ij} \in \mathbb{R}$. The statistical quantity cumulant $\kappa_n(S)$ exists up to $N$, if the observables $X_i$ and $Y_j$ have the moments up to order $N$ [12]. We present the main result of our work as the following theorem

**Theorem 1** In bipartite system, a physical theory has nth order nonlocality if it predicts a violation of the following inequality

$$M_- \leq \kappa_n(S) \leq M_+ ,$$  \hspace{1cm} (21)
where $M_+$ and $M_-$ are the maximum and minimum values obtained from LHVT.

For the sake of demonstration, we consider the most representative joint observable for bipartite qubit states

$$S = X \otimes Y - X \otimes Y' + X' \otimes Y + X' \otimes Y'. \quad (22)$$

Here

$$X = \sigma_z, \quad X' = \sigma_x, \quad Y = \sin \theta \sigma_x + \cos \theta \sigma_z, \quad Y' = \cos \theta \sigma_x - \sin \theta \sigma_z. \quad (23)$$

The observable pair $X$ and $X'$ are orthogonal observables, and so are $Y$ and $Y'$.

The second order cumulant is the variance $\kappa_2(S) = \langle S^2 \rangle - \langle S \rangle^2$. In LHVT, the cumulant $\kappa_2(S)$ has the limits of $M_- = 0$ and $M_+ = 4$, and we have

**Corollary 1** The bipartite system contains second order nonlocality if the following Bell inequality is violated

$$\kappa_2(S) \geq 0 \Rightarrow |\langle S \rangle| \leq 2. \quad (24)$$

This is just the CHSH inequality $|E(X,Y) - E(X,Y') + E(X',Y) + E(X',Y')| \leq 2$.

The Corollary agrees with results of [20], and the QM prediction of $|\langle S \rangle|$ is plotted in Figure 2(a) for the singlet state $|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$ of two spin 1/2 particles.

The third order cumulant is the skewness $\kappa_3(S) = \langle S^3 \rangle - 3\langle S^2 \rangle \langle S \rangle + 2\langle S \rangle^3$. In LHVT, the cumulant $\kappa_3(S)$ has the limits of $M_- = -\frac{32\sqrt{3}}{9}$ and $M_+ = \frac{32\sqrt{3}}{9}$, and we have

**Corollary 2** The bipartite system contains third order nonlocality if the following “skewness” inequality is violated

$$\left| \frac{\kappa_3(S)}{2} \right| = |\langle S \rangle^3 - 8\langle S \rangle| \leq \frac{16\sqrt{3}}{9}. \quad (25)$$

Here $\langle S \rangle = E(X,Y) - E(X,Y') + E(X',Y) + E(X',Y')$. 
Figure 2. The Bell and skewness nonlocalities in spin singlet state. The second order cumulant (variance) gives the CHSH inequality that witnessing Bell nonlocality. The third order cumulant gives the skewness nonlocality that is fundamentally different to Bell locality. Here \( \langle S \rangle = E(X,Y) - E(X,Y') + E(X',Y) + E(X',Y') \) and \( \theta \) is the angle between the observables \( X \) and \( Y \).

The QM prediction of equation (25) for the spin singlet state is plotted in Figure 2(b). The derivations of the two Corollaries are presented in Appendix B.

In Figure 2(b), inequality (25) is violated for a wide range of the parameter \( \theta \) in equation (23). The violation indicates that the correlation terms in \( S \) induce an abnormally high skewness than any classical theory would admit, i.e., \( X \otimes Y \), \( X \otimes Y' \), \( X' \otimes Y \), and \( X' \otimes Y' \) have high order dependencies [12]. As the skewness measures the asymmetry of a distribution, the anomalous asymmetry in cumulant \( \kappa_3(S) \) signifies the non-classical correlations among the local physical observables in composed systems. Comparing (a) and (b) in Figure 2, it is interesting to observe that the value \( \theta = \frac{\pi}{4} \) giving the maximal violation of the CHSH inequality predicts \( \langle S \rangle^3 - 8\langle S \rangle = 0 \). This is because \( \theta = \frac{\pi}{4} \) is a symmetric position for observables \( X-X' \) and \( Y-Y' \) where there is no skewness. The \( \theta \) for violating either Bell nonlocality or skewness nonlocality covers nearly the whole parameter
space, i.e., there is always a violation of classical results via either Bell or skewness nonlocality. Considering the existence of different order nonlocalities, we may conclude that, it is not some of the QM predictions contradict the classical theories, but the existence of QM contradicts the classical theories.

3 Conclusions

We have demonstrated that the generalized uncertainty relations can be used to explain the nonlocalities ranging from superquantum to classical ones. In a physical world having entangled states but no uncertainty relation, the correlations in CHSH inequality may reach the superquantum value of 4. The postulate of operator form observables in QM generates the uncertainty relation and limit the value of CHSH inequality to be $2\sqrt{2}$. With classically correlated (separable) states and no uncertainty relation for real number observables, the correlation gives the LHVT value of 2. Novel steering and separability criteria are also derived as byproducts of the above arguments.

Moreover, the higher order dependence revealed by the generalized uncertainty relations are shown to be able to exhibit higher order nonlinear quantum nonlocality. By dint of an explicit example of “skewness nonlocality”, we show that the Bell nonlocality in fact is the “variance nonlocality”. There will be no principle difficulties to obtain even higher order nonlocality results, such as the fourth order “Kurtosis nonlocality”, etc. It is noteworthy that the higher order nonlocalities may unveil some yet unknown quantum phenomena and unique applications in quantum information processing.
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Appendix

A  The constraints from uncertainty relations

Consider the following qubit observables

\[ Y = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y' = \cos \theta \sigma_z + \sin \theta \sigma_x = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}. \]  \hfill (S1)

Here \( \theta \in [0, \pi/2] \). There exists the following direct sum majorization uncertainty relation

\[ \vec{p}_y \oplus \vec{p}_{y'} \preceq \begin{pmatrix} 1 \\ \cos \theta \\ 1 \\ -\cos \theta \\ 0 \end{pmatrix}. \]  \hfill (S2)

The expectation values of \( Y + Y' \) and \( Y - Y' \) have the following constraints

\[ \langle \psi | Y + Y' | \psi \rangle \leq (\vec{y} \oplus \vec{y}')^\dagger \cdot (\vec{p}_y \oplus \vec{p}_{y'}) \leq 2 \cos \frac{\theta}{2}, \]  \hfill (S3)

\[ \langle \phi | Y - Y' | \phi \rangle \leq (\vec{y} \oplus -\vec{y}')^\dagger \cdot (\vec{p}_y \oplus \vec{p}_{-y'}) \leq 2 \sin \frac{\theta}{2}, \]  \hfill (S4)

where the eigenvalue vectors satisfy \((\vec{y} \oplus \vec{y}')^\dagger = (\vec{y} \oplus -\vec{y}')^\dagger = (1, 1, -1, -1)^T\).

For orthogonal observables \( Y \) and \( Y' \) where \( \theta = \pi/2 \), we would have

\[ \text{Tr}[\rho(Y + Y')] = a\sqrt{2}, \quad \text{Tr}[\rho(Y - Y')] = a'\sqrt{2}. \]  \hfill (S5)

Here density matrix \( \rho = \frac{1}{2} + \frac{1}{2} \vec{r} \cdot \vec{\sigma} \); \( a^2 + a'^2 \leq 1 \) with \( a \) and \( a' \) being related to the angles between the Bloch vectors of \( \vec{r} \) and those of \( Y + Y' \) and \( Y - Y' \).

B  The variance and skewness nonlocalities

The definition of operator \( S \) for qubit observables in the main text is

\[ S = X \otimes Y - X \otimes Y' + X' \otimes Y + X' \otimes Y', \]  \hfill (S6)
Figure S1. The configurations of the observables $X$-$X'$ on $A$ and $Y$-$Y'$ on $B$. Here $X$ and $X'$ in Bloch vector form represent orthogonal spin observables, so are $Y$ and $Y'$. The relative angle between $X$ and $Y$ is $\theta \in [0, \pi]$.

where $X = \sigma_z$, $X' = \sigma_x$, $Y = \sin \theta \sigma_x + \cos \theta \sigma_z$, and $Y' = \cos \theta \sigma_x - \sin \theta \sigma_z$, see Figure S1. The powers of $S$ in QM can be evaluated as

$$S^2 = 4(1 \otimes 1) + [X, X'] \otimes [Y, Y'],$$  \hspace{1cm} (S7)

$$S^3 = 4(X \otimes Y - X \otimes Y' + X' \otimes Y + X' \otimes Y') +$$

$$+ [X, X'X] \otimes [Y, Y'Y] - [X, X'X] \otimes [Y', Y'Y] +$$

$$+ [X', X'X] \otimes [Y, Y'Y] + [X', X'X] \otimes [Y', Y'Y].$$  \hspace{1cm} (S8)

For orthogonal spin observables $X \perp X'$ ($Y \perp Y'$) we may further get

$$S^3 = 8(X \otimes Y - X \otimes Y' + X' \otimes Y + X' \otimes Y')$$  \hspace{1cm} (S9)

Here we have used $X'XX' = -X$, $XX'X = -X'$, $Y'YY' = -Y$, and $YY'Y = -Y'$. 

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We first consider the second order cumulant $\kappa_2(S) = \langle S^2 \rangle - \langle S \rangle^2$. In the LHVT, the second order cumulant is evaluated in the following

$$\kappa_2(S) = 4 - \langle S \rangle^2 \geq 0,$$  \hspace{1cm} (S10)

where both $[X, X']$ and $[Y, Y']$ in equation (S7) give zero expectation values in LHVT, and we get the Bell inequality $|\langle S \rangle| \leq 2$. In QM, for the spin singlet state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |\rangle),$$  \hspace{1cm} (S11)

the expectation value of equation (S7) is $\langle S^2 \rangle = 8$, and we have

$$|\langle S \rangle| \leq 2\sqrt{2}.$$  \hspace{1cm} (S12)

This is just the CHSH inequality $|E(X,Y) - E(X,Y') + E(X',Y) + E(X',Y')| \leq 2\sqrt{2}$.

Then we consider the third order cumulant of $\kappa_3(S) = \langle S^3 \rangle - 3\langle S^2 \rangle\langle S \rangle + 2\langle S \rangle^3$, where the LHVT prediction is

$$\kappa_3(S) = 4\langle S \rangle - 12\langle S \rangle^2 + 2\langle S \rangle^3 = -8\langle S \rangle + 2\langle S \rangle^3.$$  \hspace{1cm} (S13)

Equation (S13) is obtained by noticing that all the expectation values of commutators are zero in equations (S7) and (S8) in LHVT. Because we also have $|\langle S \rangle| \leq 2$ for LHVT, the maximum value is

$$|\kappa_3(S)|_{\text{LHVT}} \leq \frac{32\sqrt{3}}{9}.$$  \hspace{1cm} (S14)

Here the maximum and minimum values happens when $\langle S \rangle = \pm \sqrt{4/3}$. For the singlet state, the QM prediction of $\kappa_3(S)$ is

$$\kappa_3(S) = 8\langle S \rangle - 24\langle S \rangle^2 + 2\langle S \rangle^3 = -16\langle S \rangle + 2\langle S \rangle^3.$$  \hspace{1cm} (S15)

This is the result of the Corollary 2.
Figure S2. The third order cumulant $\kappa_3(S)$ in different models. The horizontal axis represent the values of $\langle S \rangle$, while the vertical axis represent the values of $\kappa_3(S)$ for different models, i.e., QM with singlet state, LHVT, and QM with direct product states.

We may further consider the third order cumulant $\kappa_3(S)$ for pure product states $|\psi\rangle = |\varphi\rangle \otimes |\phi\rangle$. In order to get the maximal values of $|\langle S \rangle|$ within the direct product states, the Bloch vectors of $|\varphi\rangle$ and $|\phi\rangle$ shall lie in the plane of the Bloch vector spaces of $X$-$Y$. Now we would have

$$\kappa_3(S) = 8\langle S \rangle - 12\langle S \rangle^2 + 2\langle S \rangle^3 = -4\langle S \rangle + 2\langle S \rangle^3 .$$

(S16)

The values for the cumulant $\kappa_3(S)$ for LHVT, QM singlet state, QM product states are plotted in Figure S2.

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