DUALITY ON FOCK SPACES AND COMBINATORIAL ENERGY FUNCTIONS

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Abstract. We generalize in a combinatorial way the notion of the affine energy function of type $A$ to the case of a more general class of modules over a general linear Lie superalgebra $\mathfrak{g}$ based on a Howe duality of type $(\mathfrak{g}, \mathfrak{gl}_n)$ on various Fock spaces.

1. Introduction

The Kostka-Foulkes polynomials are natural $q$-deformation of Kostka numbers. They appear as the entries of a transition matrix between Schur functions and Hall-Littlewood functions, and also coincide with the Lusztig’s $q$-weight multiplicities of type $A$ (cf. [27]). One of the most important and interesting properties of Kostka-Foulkes polynomials is that they have non-negative integral coefficients. In [25] Lascoux and Schützenberger introduced the notion of charge statistic on semistandard Young tableaux, and proved this positivity of Kostka-Foulkes polynomials in a combinatorial way.

In [28] Nakayashiki-Yamada showed that the energy function on a finite affine crystal associated to a tensor product of symmetric (or exterior) powers of the natural representation of $\mathfrak{gl}_\ell$ is given by the Lascoux and Schützenberger’s charge. This also gives another combinatorial realization of the Kostka-Foulkes polynomials as $q$-deformed decomposition multiplicities. Indeed, if we understand the energy function as a statistic on non-negative integral matrices, then in terms of RSK correspondence the Nakayashiki-Yamada’s result implies that the energy of a given matrix and hence its corresponding insertion tableau (or $P$-tableau) is equal to the charge of its associated recording tableau (or $Q$-tableau).

The purpose of this paper is to generalize the energy function of type $A^{(1)}_{\ell-1}$ to the case of a more general class of modules over a general linear Lie (super)algebra from a viewpoint of duality principle so-called Howe duality [13]. We consider a Lie...
(super)algebra $\mathfrak{g}$ of type $A$, which forms a dual pair with $\mathfrak{gl}_n$ on a Fock space $\mathcal{F}^\otimes n$ for $n \geq 1$, with a family of irreducible $\mathfrak{g}$-modules $L(\lambda)$ satisfying

$$\mathcal{F}^\otimes n \cong \bigoplus_{\lambda \in H_n} L(\lambda) \otimes L_n(\lambda),$$

where $H_n$ is a subset of $\mathbb{Z}_+^n$, the set of generalized partitions of length $n$, and $L_n(\lambda)$ is the finite-dimensional irreducible $\mathfrak{gl}_n$-module corresponding to $\lambda$ (Theorem 2.3). In addition to irreducible polynomial modules over $\mathfrak{gl}$ due to the classical ($\mathfrak{gl}_\ell, \mathfrak{gl}_n$)-duality, it also includes various interesting families of irreducible modules, which forms a semisimple tensor category for $\mathfrak{g}$, for example, the integrable highest weight modules over $\mathfrak{gl}_\infty$, infinite-dimensional unitarizable modules over $\mathfrak{gl}_{p+q}$ called a holomorphic discrete series, the irreducible polynomial modules over a general linear Lie superalgebra $\mathfrak{gl}_{p|q}$, and so on (see [2, 3, 4, 8, 15, 20]). A uniform combinatorial character formula for this family of irreducible $\mathfrak{g}$-module $L(\lambda)$ was given by the first author [22] in terms of a certain pairs of Young tableaux, which we call parabolically semistandard tableaux of shape $\lambda$ (of level 1).

Now, we consider

$$V(\mu) = L(\mu_1) \otimes \cdots \otimes L(\mu_n) \subset \mathcal{F}^\otimes n$$

for $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}_+^n$ with $\mu_i \in H_1$, which is semisimple and decomposes into $L(\lambda)$’s for $\lambda \in H_n$ with finite multiplicity given by Kostka number. Then we define a $q$-deformed character of $V(\mu)$ to be a linear combination of the characters of $L(\lambda)$, where the decomposition multiplicity is replaced by the corresponding Kostka-Foulkes polynomial, which is a natural analogue of modified Hall-Littlewood function. As a main result, we introduce a purely combinatorial statistic called a combinatorial energy function on $n$-tuples of parabolically semistandard tableaux of level 1 associated to $V(\mu)$, and then show that it generalizes the usual energy function of type $A_{\ell-1}^{(1)}$ and also produces the $q$-deformed character of $V(\mu)$ in a bijective way (Theorem 4.5). The main ingredient of our proof is an analogue of RSK algorithm for the decomposition (1.1) of $\mathcal{F}^\otimes n$ as a $(\mathfrak{g}, \mathfrak{gl}_n)$-module [22], which is proved here to be an isomorphism of $\mathfrak{gl}_n$-crystals (Theorem 4.8). Another important one is an intrinsic characterization of the charge statistic on regular $\mathfrak{gl}_n$-crystals, which is deduced by combining the result in [28] and a bicrystal structure on the classical RSK correspondence (Theorem 3.2).

We remark that as in the case of the classical ($\mathfrak{gl}_\ell, \mathfrak{gl}_n$)-duality it would be very interesting to find a representation theoretic meaning of our combinatorial energy function in terms of a representation of a quantum (super)algebra associated to an affinization of $\mathfrak{g}$, especially when $\mathfrak{g} = \mathfrak{gl}_\infty$ with $L(\lambda)$ the integrable highest weight module, or $\mathfrak{g} = \mathfrak{gl}_{p|q}$ with $L(\lambda)$ the irreducible polynomial module [11, 31].
The paper is organized as follows. In Section 2, we recall the notion of parabolically semistandard tableaux and related results. In Section 3, we review the affine energy function of type $\mathfrak{A}_1^{(1)}$ and the charge statistic on regular $\mathfrak{gl}_n$-crystals together with its new intrinsic characterization. In Section 4, we introduce a combinatorial energy function, and then show that the associated $q$-deformed decomposition multiplicities recover the usual Kostka-Foulkes polynomial.

2. Parabolically semistandard tableaux

2.1. Semistandard tableaux. Let us briefly recall necessary background on semistandard tableaux (cf. [71, 22]). Let $\mathcal{P}$ be the set of partitions, where we often identify a partition with its Young diagram as usual. Throughout this paper, $\mathcal{A}$ (or $\mathcal{B}$) denotes a countable $\mathbb{Z}_2$-graded set (that is, $\mathcal{A} = A_0 \cup A_1$) with a total order $\prec$. By convention, $\mathbb{Z}_{>0}$ and $\mathbb{Z}_{<0}$ denote the set of positive and negative integers with the usual total order and even degree, respectively. For a positive integer $n$, we put $[n] = \{1 < 2 < \cdots < n\}$ and $[-n] = \{-n < \cdots < -2 < -1\}$ with even degree. We assume that $\mathcal{A}' = \{a' \mid a \in \mathcal{A}\}$ is the set with the total order $a'_1 < a'_2$ for $a_1 < a_2 \in \mathcal{A}$ and the opposite $\mathbb{Z}_2$-grading.

For a skew Young diagram $\lambda/\mu$, an $\mathcal{A}$-semistandard tableau $T$ of shape $\lambda/\mu$ is a filling $\lambda/\mu$ with entries in $\mathcal{A}$ such that (1) the entries are weakly decreasing from left to right (resp. from top to bottom) in each row (resp. column), (2) the entries in $A_0$ (resp. $A_1$) are strictly increasing in each column (resp. row). Let $\text{sh}(T)$ denote the shape of $T$, and $w_{\text{col}}(T)$ (resp. $w_{\text{row}}(T)$) the word with letters in $\mathcal{A}$ obtained by reading the entries column by column (resp. row by row) from right to left (resp. bottom to top), and in each column (resp. row) from top to bottom (resp. left to right). Let $\text{SST}_\mathcal{A}(\lambda/\mu)$ be the set of all $\mathcal{A}$-semistandard tableaux of shape $\lambda/\mu$. We set $\mathcal{P}_\mathcal{A} = \{\lambda \in \mathcal{P} \mid \text{SST}_\mathcal{A}(\lambda) \neq \emptyset\}$. For example, $\mathcal{P}_n := \mathcal{P}_{[n]} = \{\lambda \in \mathcal{P} \mid \ell(\lambda) \leq n\}$, where $\ell(\lambda)$ is the length of $\lambda$.

Let $P_\mathcal{A} = \bigoplus_{a \in \mathcal{A}} \mathbb{Z} \epsilon_a$ be the free abelian group with the basis $\{\epsilon_a \mid a \in \mathcal{A}\}$, and let $x_\mathcal{A} = \{x_a \mid a \in \mathcal{A}\}$ be the set of commuting formal variables indexed by $\mathcal{A}$. For $T \in \text{SST}_\mathcal{A}(\lambda/\mu)$, let $w_{\mathcal{A}}(T) = \sum_{a \in \mathcal{A}} m_a \epsilon_a \in P_\mathcal{A}$ be the weight of $T$, where $m_a$ is the number of occurrences of $a$ in $T$, and put $x_\mathcal{A}^T = \prod_{a \in \mathcal{A}} x_a^{m_a}$. We define the character of $\text{SST}_\mathcal{A}(\lambda/\mu)$ to be $s_{\lambda/\mu}(x_\mathcal{A}) = \sum_{T \in \text{SST}_\mathcal{A}(\lambda/\mu)} x_\mathcal{A}^T$.

For $a \in \mathcal{A}$ and $T \in \text{SST}_\mathcal{A}(\lambda)$, $(T \leftarrow a)$ denotes the tableau obtained by the Schensted’s column bumping algorithm, and $(a \rightarrow T)$ the tableau obtained by the row bumping algorithm. For $T \in \text{SST}_\mathcal{A}(\lambda)$ and $T' \in \text{SST}_\mathcal{A}(\mu)$, we set $(T \leftarrow T') = (((T \leftarrow c_1) \cdots) \leftarrow c_l)$ and $(T' \rightarrow T) = (r_l \rightarrow (\cdots (r_1 \rightarrow T)))$, where $w_{\text{col}}(T') = c_1 \cdots c_l$ and $w_{\text{row}}(T') = r_1 \cdots r_l$. 

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Let \( \mu = (\mu_1, \ldots, \mu_r) \) be a sequence of non-negative integers. For \((T_1, \ldots, T_r) \in \text{SST}_A(\mu_1) \times \cdots \times \text{SST}_A(\mu_r)\), let \( S_k = (((T_1 \leftarrow T_2) \cdots) \leftarrow T_k) \) for \( k = 1, \ldots, r \). We define \( \varrho_{\text{col}}(T_1, \ldots, T_r) = (S, S_R) \), where \( S = S_1 \) and \( S_R \) is the \([r]\)-semistandard tableau of shape \( \text{sh}(S) \) obtained by filling \( \text{sh}(S_k) / \text{sh}(S_{k-1}) \) with \( k \) for \( 1 \leq k \leq r \). Similarly, we define \( \varrho_{\text{row}}(T_1, \ldots, T_r) = (S', S'_R) \), where \( S' = (T_r \rightarrow (\cdots (T_2 \rightarrow T_1)) \). Then we have bijections

\[
\varrho_{\text{col}}, \varrho_{\text{row}} : \text{SST}_A(\mu_1) \times \cdots \times \text{SST}_A(\mu_r) \rightarrow \bigcup_{\lambda \in \mathcal{P}_A} \text{SST}_A(\lambda) \times \text{SST}_{[r]}(\lambda),
\]

where \( \text{SST}_{[r]}(\lambda) = \{ T \in \text{SST}_{[r]}(\lambda) \mid \text{wt}(T) = \sum_{i=1}^{r} \mu_i \epsilon_i \} \).

Let \( A^\pi \) be \( A \) as a \( \mathbb{Z}_2 \)-graded set with the reverse total order of \( A \). Define \( T^\pi \) to be the tableau obtained after \( 180^\circ \)-rotation of \( T \), which is an \( A^\pi \)-semistandard tableau.

Let \( A * B \) be the \( \mathbb{Z}_2 \)-graded set \( A \uplus B \) with the extended total order defined by \( x < y \) for all \( x \in A \) and \( y \in B \). For \( S \in \text{SST}_A(\mu) \) and \( T \in \text{SST}_B(\lambda/\mu) \), define \( S * T \) to be the tableau of shape \( \lambda \) given by gluing \( S \) and \( T \) so that \( S * T \in \text{SST}_{A*B}(\lambda) \).

2.2. Rational semistandard tableaux. Let us recall the notion of rational semistandard tableaux \([30]\). Let \( \mathbb{Z}_+^n = \{ (\lambda_1, \ldots, \lambda_n) \mid \lambda_i \in \mathbb{Z}, \lambda_1 \geq \ldots \geq \lambda_n \} \) be the set of generalized partitions of length \( n \). We may identify \( \lambda \in \mathbb{Z}_+^n \) with a generalized Young diagram. For example, \( \lambda = (3, 2, 0, -2) \in \mathbb{Z}_+^4 \) corresponds to

\[
\begin{array}{c}
\begin{array}{c|c|c|c}
1 & 1 & 2 \\
2 & 3 \\
-1 & -2 & -1 & 1 & 2 & 3
\end{array}
\end{array}
\]

where the non-zero integers indicate the column indices.

For \( \lambda \in \mathbb{Z}_+^n \), a rational semistandard tableau \( T \) of shape \( \lambda \) is a filling of \( \lambda \) with entries in \( [n] \uplus [-n] \) such that (1) the subtableau with columns of positive indices is \([n]\)-semistandard, (2) the subtableau with columns of negative indices is \([-n]\)-semistandard, (3) if \( b_1 < \cdots < b_s \) (resp. \( -b'_1 < \cdots < -b'_t \)) are entries in the 1st (resp. \(-1\)st) column, then \( b'_i \leq b_i \) for all \( i \), where \( \{ b''_1 < \cdots < b''_{n-1} \} = [n] \setminus \{ b'_1, \ldots, b'_t \} \). Let us call \( n \) the rank of \( T \), and define the weight of \( T \) to be \( \text{wt}_{[n]}(T) = \sum_{i \in [n]} (m_i^+ - m_i^-) \epsilon_i \), where \( m_i^\pm \) is the number of occurrences of \( \pm i \) in \( T \). We denote by \( \text{SST}_{[n]}(\lambda) \) the set of rational semistandard tableaux of shape \( \lambda \). For example,

\[
T = \begin{array}{c|c|c|c}
1 & 1 & 2 \\
2 & 3 \\
-1 & -2 & -1 & 1 & 2 & 3
\end{array}
\in \text{SST}_{[4]}(3, 2, 0, -2)
with \( \text{wt}_{[n]}(T) = 2\varepsilon_1 + 2\varepsilon_2 - \varepsilon_4 \).

For \( 0 \leq t \leq n \), let \( T \) be a tableau in \( SST(0^{n-t}, (-1)^t) \) with the entries \(-b_1, \ldots, -b_t\). We denote by \( \sigma(T) \) the tableau in \( SST(1^{n-t}, 0^t) \) with the entries \([n] \setminus \{b_1, \ldots, b_t\}\).

For an arbitrary tableau \( T \in SST_{[n]}(\lambda) \), by applying \( \sigma \) to the \(-1\)st column of \( T \), we have a bijection

\[
(2.2) \quad \sigma : SST_{[n]}(\lambda) \to SST_{[n]}(\lambda + (1^n)),
\]

where \( \text{wt}_{[n]}(\sigma(T)) = \text{wt}_{[n]}(T) + \sum_{i=1}^n \varepsilon_i \). Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{P}_n \) and \( T \in SST_{[n]}(\lambda) \). For \( d \geq \lambda_1 \), we set \( \delta_d(\lambda) = (d^n) - (\lambda_n, \ldots, \lambda_1) \) and \( \delta_d(T) = (\sigma^{-d}(T))^\pi \in SST_{[-n]^\pi}(\delta_d(\lambda)) \). Identifying \( -k \in [-n]^\pi \) with \( k \in [n] \), we get a bijection

\[
(2.3) \quad \delta_d : SST_{[n]}(\lambda) \to SST_{[n]}(\delta_d(\lambda)).
\]

2.3. Parabolically semistandard tableaux. Now, we review the notion of parabolically semistandard tableaux\(^1\) introduced in [22] to study a combinatorial aspect of Howe dual pairs of type \( A \).

Let \( \lambda \in \mathbb{Z}^n_+ \) be given. A parabolically semistandard tableau of shape \( \lambda \) with respect to \( (A, B) \) is a pair of tableaux \((T^+, T^-)\) such that

\[
T^+ \in SST_A((\lambda + (d^n))/\mu), \quad T^- \in SST_B((d^n)/\mu),
\]

for some integer \( d \geq 0 \) and \( \mu \in \mathcal{P}_n \) satisfying (1) \( \lambda + (d^n) \in \mathcal{P}_n \), (2) \( \mu \subset (d^n), \mu \subset \lambda + (d^n) \). We call \( n \) the level of \( T \) and define the weight of \( T \) to be

\[
\text{wt}_{A/B}(T) = \text{wt}_A(T^+) - \text{wt}_B(T^-) \in P_A \oplus P_B.
\]

We denote by \( SST_{A/B}(\lambda) \) the set of parabolically semistandard tableaux of shape \( \lambda \).

Roughly speaking, \( T \in SST_{A/B}(\lambda) \) is a pair of an \( A \)-semistandard tableau \( T^+ \) and a \( B \)-semistandard tableau \( T^- \), the difference of whose shapes is \( \lambda \). For example, if \( A = B = \mathbb{Z}_{>0} \), then

\[
T^+ = \begin{array}{cccc} 1 & 1 & 2 & 2 \\ 2 & 3 & 3 & 4 \\ 4 & \end{array}, \quad T^- = \begin{array}{c} 1 \\ 2 & 2 & 4 \\ 3 & 3 & 5 \\ \end{array}
\]

belongs to \( SST_{A/B}((3, 2, 0, -2)) \), where the vertical lines in \( T^+ \) and \( T^- \) correspond to the one in the generalized partition \( \lambda = (3, 2, 0, -2) \), and the bold-faced entries denote ones in the overlapping parts of \( \text{sh}(T^+) \) and \( \text{sh}(T^-) \). In this case, we have \( \text{sh}(T^+) = \lambda + (3^4)/(2, 1, 0, 0) \), and \( \text{sh}(T^-) = (3^4)/(2, 1, 0, 0) \).

\(^1\)It was called \( A/B \)-semistandard tableaux in [22].
Let us describe an analogue of RSK correspondence for parabolically semistandard tableaux \(22\). Let
\[ \mathcal{F}_{A/B} = \bigcup_{k \in \mathbb{Z}} \text{SST}_{A/B}(k) \]
be the set of all parabolically semistandard tableaux of level 1, and \( \mathcal{F}_{A/B}^n \) its \( n \)-fold product. Let \( T = (T_1, \ldots, T_n) \in \mathcal{F}_{A/B}^n \) be given with \( T_i = (T_i^+, T_i^-) \). We associate a pair \( (P_T, Q_T) \), where \( P_T \) is a parabolically semistandard tableau of level \( n \) and \( Q_T \) is a rational semistandard tableau of rank \( n \) determined by the following steps:

(k-1) Let \( (P, Q) = \text{col}((T_1^-)^r, \ldots, (T_n^-)^r). \)

Putting \( T^- = P^r \) and write \( \text{sh}(T^-) = (d^\mu)/\mu \) for some \( d \geq 0 \) and \( \mu \in \mathcal{P}_n \).

(k-2) Let \( Q^\nu = \delta_d(Q) \), which is of shape \( \mu \), and let \( \nu = (\nu_1, \ldots, \nu_n) \), where \( \text{wt}_{[n]}(Q^\nu) = \sum \nu_i \epsilon_i \). By \(22\), there exist unique \( S_i \in \text{SST}_{[n]}(\nu_i) \) for \( 1 \leq i \leq n \) such that
\[ \text{row}(S_1, \ldots, S_n) = (H^\mu, Q^\nu) \in \text{SST}_{[n]}(\mu) \times \text{SST}_{[n]}(\mu) \nu, \]
where \( H^\mu \) is the tableau of shape \( \mu \) with weight \( \sum \mu_i \epsilon_i \).

(k-3) For \( 1 \leq i \leq n \), put \( U_i = S_i * T_i^+ \), which is an \([n] * A\)-semistandard tableau. Using \(22\) once again, we let
\[ (U, U_R) = \text{row}(U_1, \ldots, U_n) \in \text{SST}_{[n]+A}(\lambda + (d)^n) \times \text{SST}_{[n]}(\lambda + (d)^n), \]
for some \( \lambda \in \mathbb{Z}^n_\mathcal{A} \).

(k-4) Since \( i < a \) for \( i \in [n] \) and \( a \in \mathcal{A} \) in \([n] * A\), we have \( U = H^\mu * T^+ \) for some \( T^+ \in \text{SST}_A(\lambda + (d)^n/\mu) \). Finally, we define
\[ P_T = (T^+, T^-) \in \text{SST}_{A/B}(\lambda), \]
\[ Q_T = \sigma^{-d}(U_R) \in \text{SST}_{[n]}(\lambda). \]

Example 2.1. Let \( \mathcal{A} = \mathbb{Z}_{\geq 0} \) and \( \mathcal{B} = \mathbb{Z}'_{<0} \). Note that \( \mathcal{A} \) and \( \mathcal{B} \) have only elements of odd degree. We identify \((\mathbb{Z}'_{<0})^n \) with \( \mathbb{Z}_{>0}^n \) and vice versa. Consider the following element
\[ T = (T_1, T_2, T_3) \in \text{SST}_{A/B}(3) \times \text{SST}_{A/B}(1) \times \text{SST}_{A/B}(0) \subset \mathcal{F}_{A/B}^3, \]
where
\[ T_1 = (T_1^+, T_1^-) = (\begin{array}{cccc} 0' & 1' & 3' & 4' \end{array}, \begin{array}{c} -4' -3' \end{array}), \]
\[ T_2 = (T_2^+, T_2^-) = (\begin{array}{cccc} 0' & 2' & 6' \end{array}, \begin{array}{c} -4' -2' -1' \end{array}), \]
\[ T_3 = (T_3^+, T_3^-) = (\begin{array}{c} 4' -5' \end{array}, \begin{array}{c} -2' -1' \end{array}). \]
By computation, we have

\[ \varrho_{\text{col}}((T_1^-)^\pi, (T_2^-)^\pi, (T_3^-)^\pi) = \varrho_{\text{col}}(\begin{pmatrix} 3' & 4' \\ 1' & 2' & 4' \\ 1' & 2' & 4' \end{pmatrix}, \begin{pmatrix} 1' & 2' & 3' & 4' \\ 1' & 1 & 2 & 1 \end{pmatrix}) \]

\[ = \begin{pmatrix} 1' & 2' & 3' & 4' \\ 1' & 2' & 4' \\ 1' & 2' & 4' \end{pmatrix}, \begin{pmatrix} 2 & 3 & 3 \end{pmatrix}, \]

which yield \( T^- = \begin{pmatrix} -4' & -2' & -1' \\ -4' & 3' & -2' & -1' \end{pmatrix} \).

We choose \( d = 4 \) and \( \mu = (4, 1) \). Then,

\[ Q^\vee = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 3 \end{pmatrix} \]

It follows from

\[ \varrho_{\text{row}}(\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \end{pmatrix}) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \]

that

\[ U_1 = \begin{pmatrix} 1 & 1 & 0' & 1' & 3' & 4' & 5' \\ 2 & 0' & 3' & 4' & 5' & 1' \end{pmatrix}, \]
\[ U_2 = \begin{pmatrix} 2 & 0' & 2' & 6' & 7' \end{pmatrix}, \]
\[ U_3 = \begin{pmatrix} 1 & 1 & 4' & 5' \end{pmatrix}. \]

Thus, we have \( \varrho_{\text{row}}(U_1, U_2, U_3) = (U, U_R) \), where

\[ U = \begin{pmatrix} 1 & 1 & 1 & 1 & 2' & 4' & 5' & 6' & 7' \\ 2 & 0' & 3' & 4' & 5' & 1' \end{pmatrix}, \]
\[ U_R = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 3 & 3 \end{pmatrix}. \]

Therefore, \( \lambda = (5, 1, -2) \) and \( U = H^\mu * T^+ \) gives

\[ P_T = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 2' & 4' & 5' & 6' & 7' \\ 0' & 3' & 4' & 5' & 1' \end{pmatrix}, \begin{pmatrix} -4' & -2' & -1' \\ -4' & 3' & -2' & -1' \end{pmatrix} \]

\[ Q_T = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 \\ 3 \\ -3 & -2 \end{pmatrix}. \]
For $T \in T^n_{A/B}$, we set $\text{wt}_{A/B}(T) = \sum_{i \in \mathbb{Z}_+} \text{wt}_{A/B}(T_i)$ and $\text{wt}_{[n]}(T) = \sum_{i \in \mathbb{Z}_+} m_i e_i \in P_{[n]}$, where $T_i \in SST_{A/B}(m_i)$ for $1 \leq i \leq n$. Then we have the following [22, Theorem 4.1].

**Theorem 2.2.** The map $T \mapsto (P_T, Q_T)$ gives a bijection

$$\kappa_{A/B} : T^n_{A/B} \rightarrow \bigcup_{\lambda \in \mathcal{P}_{A/B,n}} SST_{A/B}(\lambda) \times SST_{[n]}(\lambda),$$

which preserves $\text{wt}_{A/B}$ and $\text{wt}_{[n]}$. Here, $\mathcal{P}_{A/B,n} = \{ \lambda \in \mathbb{Z}_+^n \mid SST_{A/B}(\lambda) \neq \emptyset \}$. We define the character of $SST_{A/B}(\lambda)$ to be

$$S^{A/B}_\lambda = \sum_{T \in SST_{A/B}(\lambda)} x^T_{A/B},$$

where $x^T_{A/B} = x^T_A (x^B_A)^{-1}$ for $T = (T^+, T^-) \in SST_{A/B}(\lambda)$. Then Theorem 2.2 establishes the following Cauchy-type identity:

$$\prod_{i \in [n]} \prod_{a \in A_0} (1 + x_a x_i) \prod_{b \in B_0} (1 - x_b^{-1} x_i^{-1}) = \sum_{\lambda \in \mathcal{P}_{A/B,n}} S^{A/B}_\lambda s_\lambda(x_{[n]}).$$

Here $s_\lambda(x_{[n]})$ is a Laurent Schur polynomial corresponding to $\lambda \in \mathbb{Z}_+^n$.

Note that, when $B = \emptyset$, we have $\mathcal{P}_{A/B,n} = \mathcal{P}_A \cap \mathcal{P}_n$, and $S^A_{A/B} = s_\lambda(x_A)$, which is the usual (super) Schur function or polynomial, and [2.5] recovers the well-known Cauchy identity. Hence, a non-trivial generalization of Schur functions or more interesting cases occur when both $A$ and $B$ are non-empty.

### 2.4. Howe duality and irreducible characters.

The notion of parabolically semistandard tableaux and its RSK with rational semistandard tableaux for $\mathfrak{gl}_n$ gives a unified combinatorial interpretation of various dualities of $(\mathfrak{g}, \mathfrak{gl}_n)$, where $\mathfrak{g}$ is a general linear Lie (super)algebras associated to $(A, B)$.

Let us explain it in more detail. For an arbitrary $\mathbb{Z}_+$-graded totally ordered set $S$, let $V_S$ be a superspace with basis $\{ v_s \mid s \in S \}$, and let $\mathfrak{gl}_S$ be the general linear Lie superalgebra spanned by the elementary matrices $E_{ss'}$ for $s, s' \in S$.

Now we consider $\mathfrak{g} = \mathfrak{gl}_C$ with $C = B \ast A$. Let

$$\mathcal{F} = S(V_A \oplus V_B)$$

be the super symmetric algebra generated by $V_A \oplus V_B$, where $V_B$ is the restricted dual space of $V_B$. Recall that $\mathcal{F}$ can be viewed as an irreducible module over a Clifford-Weyl algebra. Following the arguments in [5] Sections 5.1 and 5.4] (cf. [8] [15]), one can define a semisimple action of $\mathfrak{g}$ on $\mathcal{F}$, and a semisimple action of $\mathfrak{gl}_n$ or $GL_n$ on $\mathcal{F}^\otimes n$ for $n \geq 1$ such that $\mathcal{F}^\otimes n$ decomposes into a finite-dimensional $\mathfrak{gl}_n$-modules. Then the actions of $\mathfrak{gl}_n$ and $\mathfrak{g}$ commute with each other, and furthermore...
the image of $\mathfrak{g}$ in $\text{End}_C(\mathcal{F}^\otimes n)$ generates $\text{End}_{\mathfrak{gl}_n}(\mathcal{F}^\otimes n)$. Therefore, we have the following multiplicity-free decomposition as a $(\mathfrak{g}, \mathfrak{gl}_n)$-module,

\begin{equation}
\mathcal{F}^\otimes n \cong \bigoplus_{\lambda \in H_n} L(\lambda) \otimes L_n(\lambda),
\end{equation}

for a subset $H_n$ of $\mathbb{Z}_+^n$, where $L_n(\lambda)$ is an irreducible $\mathfrak{gl}_n$-module with highest weight $\lambda \in H_n$, and $L(\lambda)$ is an irreducible $\mathfrak{g}$-module corresponding to $L_n(\lambda)$. We define the character $\text{ch}L(\lambda)$ to be the trace of the operator $\prod_{c \in C} x_c^{E_{cc}}$ on $L(\lambda)$ for $\lambda \in H_n$.

Then we have the following, which is often referred to as Howe duality (for type $A$) (cf. [2, 3, 8, 13, 15, 20]).

**Theorem 2.3.** Let $\mathcal{A}$ and $\mathcal{B}$ be given. For $n \geq 1$, we have

\begin{equation}
\mathcal{F}^\otimes n \cong \bigoplus_{\lambda \in \mathcal{P}_{\mathcal{A}/\mathcal{B},n}} L(\lambda) \otimes L_n(\lambda),
\end{equation}

as a $(\mathfrak{g}, \mathfrak{gl}_n)$-module, that is, $H_n = \mathcal{P}_{\mathcal{A}/\mathcal{B},n}$, and the irreducible character $\text{ch}L(\lambda)$ is given by $S^\lambda_{\mathcal{A}/\mathcal{B}}$ for $\lambda \in \mathcal{P}_{\mathcal{A}/\mathcal{B},n}$.

**Proof.** Consider the operator $D = \prod_{c \in C} x_c^{E_{cc}} \prod_{i \in [n]} x_i^e_i$, where $e_i$ is the $i$-th elementary diagonal matrix in $\mathfrak{gl}_n$. Taking the trace of $D$ on both sides of (2.6), we have

\[
\prod_{i \in [n]} \prod_{a \in A_1} (1 + x_a x_i) \prod_{b \in B_1} (1 + x_b^{-1} x_i^{-1}) = \sum_{\lambda \in H_n} \text{ch}L(\lambda) s_\lambda(x_{[n]}).
\]

Thus by the Cauchy-type identity (2.5) and the linear independence of Laurent Schur polynomials, we conclude that $H_n = \mathcal{P}_{\mathcal{A}/\mathcal{B},n}$ and $S^\lambda_{\mathcal{A}/\mathcal{B}} = \text{ch}L(\lambda)$ for $\lambda \in \mathcal{P}_{\mathcal{A}/\mathcal{B},n}$.

Note that $L(\lambda)$’s are mutually non-isomorphic irreducible $\mathfrak{g}$-modules for $\lambda \in \bigcup_{n \geq 1} \mathcal{P}_{\mathcal{A}/\mathcal{B},n}$ and the tensor product $L(\mu) \otimes L(\nu)$ for $\mu \in \mathcal{P}_{\mathcal{A}/\mathcal{B},m}$ and $\nu \in \mathcal{P}_{\mathcal{A}/\mathcal{B},n}$ decomposes into a direct sum of $L(\lambda)$’s for $\lambda \in \mathcal{P}_{\mathcal{A}/\mathcal{B},m+n}$ with finite multiplicity given by a Littlewood-Richardson number (see [22, Theorem 4.7]). Also, $L(\lambda)$ is semisimple over a maximal Levi subalgebra $\mathfrak{l} = \mathfrak{gl}_A \oplus \mathfrak{gl}_B$ of $\mathfrak{g}$, and expanding $S^\lambda_{\mathcal{A}/\mathcal{B}}$ as a linear combination of $s_\mu(x_A)s_\nu(x_B^{-1})$ for $\mu, \nu \in \mathcal{P}$ (see [22, Proposition 3.14]) gives a branching rule with respect to $\mathfrak{l}$.

Recall that when $\mathcal{A}$ is finite with $\mathcal{A} = \mathcal{A}_0$ or $\mathcal{A}_1$ and $\mathcal{B} = \emptyset$, the decomposition in Theorem 2.3 is the classical $(\mathfrak{gl}_\ell, \mathfrak{gl}_n)$-Howe duality on symmetric algebra or exterior algebra generated by $\mathbb{C}^\ell \otimes \mathbb{C}^n$, where $\ell = |\mathcal{A}|$ (cf. [13]). Below we list some of important examples where both $\mathcal{A}$ and $\mathcal{B}$ are non-empty, and $\mathfrak{g}$ is a usual general linear Lie algebra (see [22] for more detailed exposition).

**Example 2.4.** (1) If $(\mathcal{A}, \mathcal{B}) = (\mathbb{Z}_{\geq 0}, \mathbb{Z}_{\leq 0})$, then $S^\lambda_{\mathcal{A}/\mathcal{B}} (\lambda \in \mathbb{Z}_+^n)$ is the character of an integrable highest weight module over the general linear Lie algebra $\mathfrak{gl}_\infty$ with
highest weight of positive level $n$. The identity (2.5) corresponds to the \((\mathfrak{gl}_\infty, \mathfrak{gl}_n)\)-duality on the level $n$ fermionic Fock space [8]. In particular, $SST_{A/B}(k)$ ($k \in \mathbb{Z}$) may be identified with the level 1 fermionic Fock space of charge $k$, which is realized by Young diagrams [14 Section 1]. The identification between $SST_{A/B}(k)$ ($k \in \mathbb{Z}$) and the level 1 Fock space of charge $k$ is given by mapping an element

$$T = (i'_s \cdots i'_2 i'_1 - j'_1 - j'_2 \cdots - j'_r),$$

where $i_p \geq 0$, $j_q > 0$ and $s - r = k$, to the following Young diagram

For example, the Young diagrams corresponding to $T_1$, $T_2$ and $T_3$ in Example 2.1 are $\lambda_{T_1} = (3, 3, 3, 2, 2, 2, 2)$, $\lambda_{T_2} = (7, 7, 4, 3, 1)$ and $\lambda_{T_3} = (6, 6)$ respectively.

(2) If $(A, B) = (\mathbb{Z}_{\geq 0}, \mathbb{Z}_{< 0})$, then $S^{A/B}_\lambda$ ($\lambda \in \mathbb{Z}_n^+$) is the character of an irreducible (non-integrable) highest weight module over $\mathfrak{gl}_\infty$ with highest weight of negative level $-n$, which appears in the $(\mathfrak{gl}_\infty, \mathfrak{gl}_n)$-duality on the level $n$ bosonic Fock space [15].

(3) If $(A, B) = ([q], [p])$ for $p, q \in \mathbb{Z}_{> 0}$, then $S^{A/B}_\lambda$ ($\lambda \in \mathbb{Z}_n^+$) is equal to the character of an infinite-dimensional irreducible $\mathfrak{gl}_{p+q}$-module, which is unitarizable. The family of irreducible representations appears in $(\mathfrak{gl}_{p+q}, \mathfrak{gl}_n)$-duality on the symmetric algebra $S(\mathbb{C}^p \otimes \mathbb{C}^q \oplus \mathbb{C}^r \otimes \mathbb{C}^s)$ [20], which is called holomorphic discrete series or oscillator modules.

2.5. Hall-Littlewood functions. Let $q$ be an indeterminate. Fix $n \geq 1$. For $\mu \in \mathcal{P}_n$, let $P_\mu(x_{[n]}, q)$ be the Hall-Littlewood polynomial in $x_{[n]}$ associated to $\mu$ [27 Chapter III.2]. For $\mu \in \mathbb{Z}_n^+$, we put $P_\mu(x_{[n]}, q) = (x_1 \cdots x_n)^{-d}P_{\mu + (d n)}(x_{[n]}, q)$ for some $d \geq 0$ such that $\mu + (d n) \in \mathcal{P}_n$, which is well-defined independent of the choice of $d$.

Consider a formal power series in $x_A$ and $x_B^{-1}$, which is determined by the following Cauchy-type identity:

$$\prod_{i \in [n]} \prod_{a \in A_1} (1 + x_a x_i) \prod_{b \in B_1} (1 + x_b^{-1} x_i^{-1}) = \sum_{\lambda \in \mathbb{Z}_n^+} Q^{A/B}_\lambda P_\lambda(x_{[n]}, q).$$

(2.7)
It is well-known that for $\lambda \in \mathbb{Z}^n_+$

$$s_\lambda(x_{[\mu]}) = \sum_{\mu \in \mathbb{Z}^n_+} K_{\lambda \mu}(q) P_\mu(x_{[\mu]}, q),$$

where $K_{\lambda \mu}(q)$ are the Kostka-Foulkes polynomials or Lusztig’s $q$-weight multiplicities of type $A_{n-1}$. Here we understand $K_{\lambda \mu}(q) = K_{\lambda+(d^n) \mu+(d^n)}(q)$ for $d \geq 1$ with $\lambda+(d^n), \mu+(d^n) \in \mathcal{P}_n$, which is independent of the choice of $d$. By (2.5), (2.7) and (2.8), we have

$$Q_{\mu}^{A/B} = \sum_{\lambda \in \mathbb{Z}^n_+} K_{\lambda \mu}(q) S^{A/B}_\lambda$$

for $\mu \in \mathbb{Z}^n_+$. Since $K_{\lambda \mu}(q)$ has nonnegative integral coefficients with $K_{\lambda \mu}(1) = |SST_{[\mu]}(\lambda)|$, we may view $Q_{\mu}^{A/B}$ as a $q$-analogue of the character of

$$\mathcal{F}_{A/B}^\mu = SST_{A/B}(\mu_1) \times \cdots \times SST_{A/B}(\mu_n)$$

by Theorem 2.2. Recall that $Q_{\mu}^{A/B}$ is a modified Hall-Littlewood function for $\mu \in \mathcal{P}$ when $A = \mathbb{Z}_{>0}$ or $\mathbb{Z}'_{>0}$ and $B = \emptyset$, and it can be realized as a graded character of tensor product of KR crystals with respect to an affine energy function [28].

Our main goal is to introduce a purely combinatorial statistic on $\mathcal{F}_{A/B}^\mu$, which realizes (2.9) as a graded character of $\mathcal{F}_{A/B}^\mu$ for arbitrary $A$ and $B$ generalizing the usual affine energy functions.

3. Affine crystals and charge statistic

3.1. Crystals. Let us give a brief review on crystals (cf. [12] [18]). Let $\mathfrak{g}$ be the Kac-Moody algebra associated to a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$. Let $P^\vee$ be the dual weight lattice, $P = \text{Hom}_{\mathbb{Z}}(P^\vee, \mathbb{Z})$ the weight lattice, $\Pi^\vee = \{ h_i \mid i \in I \}$ the set of simple coroots, and $\Pi = \{ \alpha_i \mid i \in I \}$ the set of simple roots of $\mathfrak{g}$ such that $\langle h_i, \alpha_j \rangle = a_{ij}$ for $i, j \in I$. Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra of $\mathfrak{g}$.

A $\mathfrak{g}$-crystal (or crystal for short) is a set $B$ together with the maps $\text{wt} : B \to P$, $\varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$ and $\bar{e}_i, \bar{f}_i : B \to B \cup \{0\}$ $(i \in I)$ satisfying certain axioms. For a dominant integral weight $\Lambda$ for $\mathfrak{g}$, we denote by $B(\Lambda)$ the crystal associated to an irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight $\Lambda$.

For a crystal $B$, we denote its dual by $B^\vee$, which is a set $B^\vee = \{ b^\vee \mid b \in B \}$ with

$$\text{wt}(b^\vee) = -\text{wt}(b), \quad \varepsilon_i(b^\vee) = \varphi_i(b), \quad \varphi_i(b^\vee) = \varepsilon_i(b),$$

$$\bar{e}_i(b^\vee) = \bar{f}_i(b)^\vee, \quad \bar{f}_i(b^\vee) = \bar{e}_i(b)^\vee,$$
for \( b \in B \) and \( i \in I \). A tensor product \( B_1 \otimes B_2 \) of crystals \( B_1 \) and \( B_2 \) is defined to be a crystal, which is \( B_1 \times B_2 \) as a set with elements denoted by \( b_1 \otimes b_2 \), satisfying

\[
\begin{align*}
\text{wt}(b_1 \otimes b_2) &= \text{wt}(b_1) + \text{wt}(b_2), \\
\varepsilon_i(b_1 \otimes b_2) &= \max\{\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle \text{wt}(b_1), h_i \rangle\}, \\
\varphi_i(b_1 \otimes b_2) &= \max\{\varphi_i(b_1) + \langle \text{wt}(b_2), h_i \rangle, \varphi_i(b_2)\}, \\
\tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} e_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes e_i b_2, & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\
\tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} f_i b_1 \otimes b_2, & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes f_i b_2, & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases}
\end{align*}
\]

for \( i \in I \). Here we assume that \( 0 \otimes b_2 = b_1 \otimes 0 = 0 \).

Given \( b_i \) in crystals \( B_i \) \((i = 1, 2)\), we write \( b_1 \equiv b_2 \) if there is an isomorphism of crystals \( C(b_1) \rightarrow C(b_2) \) mapping \( b_1 \) to \( b_2 \), where \( C(b_i) \) denotes the connected component of \( b_i \) in \( B_i \).

**3.2. \( A_{n-1} \)-crystals.** Fix a positive integer \( n \geq 2 \). Suppose that \( g = A_{n-1} \) with \( I = \{1, \ldots, n-1\} \). We assume that its weight lattice is \( P_n : = P[1] \). We often identify \( \lambda \in \mathbb{Z}_+^n \) with a dominant integral weight \( \sum_{i=1}^{n} \lambda_i \epsilon_i \). Let \( \Delta^+_{n-1} = \{ \epsilon_s - \epsilon_t | 1 \leq s < t \leq n \} \) the set of positive roots. The Weyl group is the symmetric group \( \mathfrak{S}_n \) on \( n \) letters generated by the transposition \( r_j = (j \ j + 1) \) for \( j = 1, \ldots, n - 1 \). From now on, we always denote the associated data of an \( A_{n-1} \)-crystal by \( \tilde{e}_j, \tilde{f}_j, \varepsilon_j, \varphi_j \) \((j = 1, \ldots, n-1)\) and \text{wt}.

We may regard \([n]\) as \( B(\epsilon_1) \) the crystal of the natural representation, and \([-n]\) as its dual. Given \( \lambda \in \mathbb{Z}_+^n \), \( \text{SST}_{[n]}(\lambda) \) has an \( A_{n-1} \)-crystal structure by regarding \( w_{col}(T) \) for \( T \in \text{SST}_{[n]}(\lambda) \) as an element in \([n]^\otimes \otimes [-n]^\otimes q \) for some \( p, q \geq 0 \). Then we have \( \text{SST}_{[n]}(\lambda) \cong B(\lambda) \) (cf. [19]). It is not difficult to see that

\[
\sigma^d(T) \equiv T, \quad \delta_d(S) \equiv S^\vee,
\]

for \( T \in \text{SST}_{[n]}(\lambda) \) with \( \lambda \in \mathbb{Z}_+^n \) and \( S \in \text{SST}_{[n]}(\mu) \) with \( \mu \in P_n \) up to a shift of weight by \( d(\epsilon_1 + \ldots + \epsilon_n) \) \((d \in \mathbb{Z})\) or as elements in \( A_{n-1} \)-crystals with the weight lattice \( P_n / \mathbb{Z}(\epsilon_1 + \ldots + \epsilon_n) \), where \( \sigma \) and \( \delta_d \) are as in [22] and [23], respectively. The crystal equivalence \( \equiv \) is also compatible with row and column insertions, that is, \((T \leftarrow a) \equiv T \otimes a \) and \((a \rightarrow T) \equiv a \otimes T \) for \( a \in [n] \) and \( T \in \text{SST}_{[n]}(\lambda) \) with \( \lambda \in P_n \).

**3.3. Charge statistic.** For \( \lambda, \mu \in P_n \) and \( T \in \text{SST}_{[n]}(\lambda) \), we denote by \( c(T) \) the charge of \( T \), which was introduced by Lascoux and Schützenberger [25]. It is shown
in [20] that
\[
K_{\lambda\mu}(q) = \sum_{T \in SST_{[n]}(\lambda)_{\mu}} q^{c(T)}.
\]

One can naturally induce a charge statistic on a regular \(A_{n-1}\)-crystal \(B\) as follows: First, note that the connected component \(C(b) \subset B\) under \(\tilde{e}_j\) and \(\tilde{f}_j\) \((j = 1, \ldots, n-1)\) is isomorphic to \(B(\lambda)\) for some \(\lambda \in \mathbb{Z}_n^+\). Choose \(d \geq 0\) such that \(\lambda + (d^n) \in \mathcal{P}_n\). Since \(B(\lambda) \cong SST_{[n]}(\lambda + (d^n))\) as a \(\{1, \ldots, n-1\}\)-colored oriented graph by (3.1), \(b\) can be identified with a tableau \(T \in SST_{[n]}(\lambda + (d^n))\). Then we define
\[
charge(b) = c(T'),
\]
where \(T'\) is a unique tableau with dominant weight in the \(\mathfrak{S}_n\)-orbit of \(T\). By definition of Lascoux and Schützenberger’s charge, it is not difficult to see that \(charge(b)\) does not depend on the choice of \(d\). In particular, we define for \(\lambda, \mu \in \mathbb{Z}_n^+\)
\[
K_{\lambda\mu}(q) = \sum_{b \in B(\lambda), \text{ wt}(b) = \mu} q^{charge(b)},
\]
which is equal to the usual Kostka-Foulkes polynomial \(K_{\lambda+(d^n)\mu+(d^n)}(q)\) for some \(d \geq 0\).

Recall that there is an intrinsic characterization of the charge statistic [24], which is described only in terms of the geometry of the crystal graph \(B(\lambda)\) for \(\lambda \in \mathbb{Z}_n^+\). In Section 3.5, we give another intrinsic characterization, which plays a crucial role in this paper. For this, we need the following statistic on a regular \(A_{n-1}\)-crystal \(B\): for \(\alpha = \epsilon_s - \epsilon_t \in \Delta_n^+\) and \(b \in B\)
\[
\epsilon_\alpha(b) = \epsilon_s(S_{s+1}S_{s+2} \cdots S_{t-1}(b)),
\]
\[
\varphi_\alpha(b) = \varphi_s(S_{s+1}S_{s+2} \cdots S_{t-1}(b)),
\]
where \(S_j\) is the \(\mathfrak{S}_n\)-action on \(B\) associated to \(r_j\). Since \(\varphi_\alpha(b) - \epsilon_\alpha(b) = \langle wt(b), \alpha^\vee \rangle\), where \(\alpha^\vee = h_s + \cdots + h_{t-1}\) is the coroot of \(\alpha\), one may think of \(\epsilon_\alpha(b)\) and \(\varphi_\alpha(b)\) as information on an \(\mathfrak{sl}_2\)-string of \(b\) with respect to \(\alpha = r_{t-1} \cdots r_{s+1}(\alpha_s)\). We should remark that they depend on the choice of a simple root conjugate to \(\alpha\). Here we choose it as \(\alpha_s\).

3.4. Affine \(A_{\ell-1}^{(1)}\)-crystals and energy function. Fix a positive integer \(\ell \geq 2\). Suppose that \(\mathfrak{g} = A_{\ell-1}^{(1)}\) with \(I = \{0, \ldots, \ell - 1\}\) and \(\mathfrak{g}_0 = A_{\ell-1}\) is the subalgebra of \(\mathfrak{g}\) corresponding to \(I \setminus \{0\}\). For \(1 \leq r \leq \ell - 1\), let \(\varpi_r\) be the fundamental weight for \(\mathfrak{g}_0\) corresponding to the simple root \(\alpha_r\). For \(s \geq 1\), let \(B^{r,s}\) denote a Kirillov-Reshetikhin crystal (or KR crystal for short) of type \(A_{\ell-1}^{(1)}\), which is isomorphic to \(B(s\varpi_r)\) as an \(A_{\ell-1}\)-crystal [17, 29]. Let \(u_{r,s}\) be the unique element in \(B^{r,s}\) of weight \(s\varpi_r\). For convenience, let us assume that \(B^{0,s}\) and \(B^{\ell,s}\) are trivial crystals.
Let $B_1$ and $B_2$ be two KR crystals with classical highest weight elements $u_1$ and $u_2$, respectively. Let $\sigma = \sigma_{B_1,B_2} : B_1 \otimes B_2 \rightarrow B_2 \otimes B_1$ be a unique $A^{(1)}_{\ell-1}$ crystal isomorphism called the combinatorial $R$-matrix. There exists a function $H = H_{B_1,B_2} : B_1 \otimes B_2 \rightarrow \mathbb{Z}$ such that $H$ is constant on each connected component in $B_1 \otimes B_2$ as an $A^{(1)}_{\ell-1}$-crystal and

$$H(\tilde{e}_0(b_1 \otimes b_2)) =$$

$$\begin{cases} H(b_1 \otimes b_2) + 1, & \text{if } \tilde{e}_0(b_1 \otimes b_2) = \tilde{e}_0(b_1) \otimes b_2 \text{ and } \tilde{e}_0(b_2) = \tilde{e}_0(b_2) \otimes b'_1, \\ H(b_1 \otimes b_2) - 1, & \text{if } \tilde{e}_0(b_1 \otimes b_2) = b_1 \otimes \tilde{e}_0(b_2) \text{ and } \tilde{e}_0(b_2) = b'_2 \otimes \tilde{e}_0(b'_1), \\ H(b_1 \otimes b_2), & \text{otherwise}, \end{cases}$$

for $b_1 \otimes b_2 \in B_1 \otimes B_2$ with $b'_2 \otimes b'_1 = \sigma(b_1 \otimes b_2)$. It is well-known that $H$ is unique up to an additive constant, which is called the local energy function on $B_1 \otimes B_2$ [16].

Suppose that $B = B_1 \otimes \cdots \otimes B_n$ is a tensor product of KR crystals. For $1 \leq \ell \leq n$, let $\sigma_i$ be the $A^{(1)}_{\ell-1}$-crystal isomorphism of $B$, which acts as $\sigma_{B_i,B_{i+1}}$ on $B_i \otimes B_{i+1}$ and as identity elsewhere, and let $H_i$ be the function on $B$ given by $H_i(b_1 \otimes \cdots \otimes b_n) = H_{B_i,B_{i+1}}(b_i \otimes b_{i+1})$. The energy function $D_B : B \rightarrow \mathbb{Z}$ is defined to be

$$D_B(b) = \sum_{1 \leq i < j \leq n} H_i(\sigma_{i+1} \sigma_{i+2} \cdots \sigma_{j-1}(b)) \quad (b \in B),$$

which plays a very important role in the study of finite affine crystals (cf. [9, 10]). Note that $D_B$ is constant on each connected component in $B$ as an $A^{(1)}_{\ell-1}$-crystal, which therefore gives a natural $q$-analogue of the branching multiplicities with respect to $A^{(1)}_{\ell-1} \subset A^{(1)}_{\ell-1}$.

### 3.5. Crystal skew Howe duality

Let $M_{\ell \times n}(\mathbb{Z}_2)$ be the set of $\ell \times n$ matrices $m = (m_{ij})$ such that $m_{ij} = 0,1$ for $1 \leq i \leq \ell$ and $1 \leq j \leq n$.

For $1 \leq i \leq \ell$, let $m(i)$ denote the $i$th row of $m$. We may identify each $m(i)$ with an $[n]$-semistandard tableau of single column by reading the column indices $j$ with $m_{ij} = 1$, and hence regard $M_{\ell \times n}(\mathbb{Z}_2)$ as an $A_{n-1}$-crystal by identifying $m$ with $m(1) \otimes \cdots \otimes m(\ell)$ with respect to $\tilde{e}_j, \tilde{f}_j, \varepsilon_j, \phi_j$ for $1 \leq j \leq n - 1$.

For $1 \leq j \leq \ell$, let $m(j)$ denote the $j$th column of $m$. In the same way, we regard $M_{\ell \times n}(\mathbb{Z}_2)$ as an $A_{\ell-1}$-crystal by identifying $m$ with $m^{(1)} \otimes \cdots \otimes m^{(n)}$ with respect to $\tilde{e}_i, \tilde{f}_i, \varepsilon_i, \phi_i$ for $1 \leq i \leq \ell - 1$.

Then $M_{\ell \times n}(\mathbb{Z}_2)$ is an $(A_{\ell-1}, A_{n-1})$-bicrystal, that is, $\tilde{x}_i \tilde{x}_j = \tilde{x}_j \tilde{x}_i$ for all $i, j$, $x = e, f$ and $x = e, f$, and the well-known (dual) RSK correspondence

$$M_{\ell \times n}(\mathbb{Z}_2) \rightarrow \bigsqcup_{\lambda \in \mathcal{P}} SST_{[\ell]}(\lambda) \times SST_{[n]}(\lambda').$$
is a bicrystal isomorphism \([6, 21]\). Here \(\lambda^\prime\) denotes the conjugate of \(\lambda\). This can be viewed as a crystal version of skew \((\mathfrak{gl}_t, \mathfrak{gl}_n)\)-Howe duality (cf. \([13]\)).

Moreover, \(M_{\ell \times n}(\mathbb{Z}_2)\) is an \(A_{\ell-1}^{(1)}\)-crystal with respect to \(\tilde{e}_i\) and \(\tilde{f}_i\) for \(0 \leq i \leq \ell - 1\), since each column of \(m \in M_{\ell \times n}(\mathbb{Z}_2)\) can be considered as an element in a KR crystal \(B^{r, s}\) for some \(1 \leq r \leq n - 1\) or a trivial crystal. Note that \(\tilde{e}_0\) and \(\tilde{f}_0\) do not commute with \(\tilde{e}_j\) and \(\tilde{f}_j\) for \(1 \leq j \leq n - 1\), in general.

Now we can define the energy function \(D\) on the affine \(A_{\ell-1}^{(1)}\)-crystal \(M_{\ell \times n}(\mathbb{Z}_2)\) as in \([3, 6]\), where we normalize the local energy function by requiring \(H_{B_i, B_j}(u_i \otimes u_{i+1}) = 0\) for KR crystals \(B_i\) with the classical highest weight elements \(u_i \in B_i\) \((1 \leq i \leq n)\). Then by \([3, 7]\), we can rewrite \(D\) in terms of statistics on \(A_{n-1}\)-crystal as follows.

**Proposition 3.1.** For \(m \in M_{\ell \times n}(\mathbb{Z}_2)\), we have

\[
D(m) = - \sum_{\alpha \in \Delta^+_{n-1}} \min\{\varepsilon_\alpha(m), \varphi_\alpha(m)\}.
\]

**Proof.** We may assume that \(m \in B = B_1 \otimes \cdots \otimes B_n\), where \(B_j = SST[\ell](1^t_j) = B^{t_j, 1}\) for some \(t_j\) \((1 \leq j \leq n)\). It is not difficult to check that for \(1 \leq j \leq n - 1\),

\[
\cdot \sigma_{B_j, B_{j+1}} \text{ on } B_j \otimes B_{j+1} \text{ coincides with } S_j \text{ on } m,
\]

\[
\cdot H_{B_j, B_{j+1}}(m[j] \otimes m[j+1]) = - \min\{\varepsilon_j(m), \varphi_j(m)\},
\]

(cf. \([28]\) Section 3.5]). This implies that

\[
H_s(\sigma_s \sigma_{s+2} \cdots \sigma_{t-1}(m)) = - \min\{\varepsilon_\alpha(m), \varphi_\alpha(m)\} \quad (1 \leq s < t \leq n),
\]

where \(\alpha = \varepsilon_s - \varepsilon_t\). Hence, we get \(D(m) = - \sum_{\alpha \in \Delta^+_{n-1}} \min\{\varepsilon_\alpha(m), \varphi_\alpha(m)\}\). \(\square\)

Combining with the result of Nakayashiki and Yamada \([28]\) (see also \([29]\) for its generalization), we obtain the following intrinsic characterization of charge statistic on a regular \(A_{n-1}\)-crystal.

**Theorem 3.2.** Let \(B\) be a regular \(A_{n-1}\)-crystal. For \(b \in B\), we have

\[
\text{charge}(b) = \sum_{\alpha \in \Delta^+_{n-1}} \min\{\varepsilon_\alpha(b), \varphi_\alpha(b)\}.
\]

**Proof.** Given \(b \in B\), we may assume that \(b \in SST[\ell](\lambda)\) for some \(\lambda \in \mathcal{P}_n\) up to a shift of its weight by \(d(\varepsilon_1 + \cdots + \varepsilon_n)\) \((d \geq 0)\), say \(b = T\). Let \(m\) be a unique matrix in \(M_{\ell \times n}(\mathbb{Z}_2)\) such that \(m[i]\) corresponds to the \(i\)th column of \(T\) from the left-most column of \(\lambda\). Since \(\text{charge}(b)\) is invariant under the Weyl group action, we may also assume that the weight of \(b\) is dominant, which corresponds to a partition \(\mu = (\mu_1, \ldots, \mu_n)\). Then \(m \in B^{\mu_1, 1} \otimes \cdots \otimes B^{\mu_n, 1}\) as an \(A_{\ell-1}^{(1)}\)-crystal. Since we have
\[ c(T) = -D(m) \text{ by [28, Section 4.1]}, \text{ we have by Proposition } 3.1, \]
\[ \text{charge}(b) = c(T) = -D(m) = \sum_{\alpha \in \Delta_{n-1}^{+}} \min \{ \varepsilon_{\alpha}(b), \varphi_{\alpha}(b) \}. \]
This completes the proof. \qed

4. A COMBINATORIAL ENERGY FUNCTION

In this section, we introduce a combinatorial energy function \( D \) on \( \mathfrak{F}^{\mu}_{A/B} \) for \( \mu \in \mathbb{Z}^{n}_+ \), which realizes \( Q^{\mu}_{A/B} \) in (2.9) as a graded character of \( \mathfrak{F}^{\mu}_{A/B} \).

4.1. Combinatorial energy function. Consider \( \text{SST}_{A/B}(k_1) \times \text{SST}_{A/B}(k_2) \) for \( k_1, k_2 \in \mathbb{Z} \). Let \( T_j = (T^+_j, T^-_j) \in \text{SST}_{A/B}(k_j) \) be given for \( j = 1, 2 \) with
\[ \text{wt}_{A/B}(T_j) = \sum_{a \in A} m_a \varepsilon_a - \sum_{b \in B} m_b \varphi_b. \]
First, we define a local energy function \( H : \text{SST}_{A/B}(k_1) \times \text{SST}_{A/B}(k_2) \rightarrow \mathbb{Z} \) following the steps below:

(H-1) Choose finite subsets \( A^0 \subset A \) and \( B^0 \subset B \) such that \( T_j \in \text{SST}_{A^0/B^0}(k_j) \) for \( j = 1, 2 \). To each \( i \in A^0 \cup B^0 \), we assign a sequence of \( \pm \) signs as follows:
\[ s_i = \begin{cases} 
- \cdots - + \cdots + & \text{if } i \in A_0, \\
+ \cdots - & \text{if } i \in A_1, \\
- \cdots - + \cdots + & \text{if } i \in B_0, \\
+ \cdots - & \text{if } i \in B_1.
\end{cases} \]

(H-2) Let \( s = s_{T_1, T_2} = (s_{a_{k_1} s_{b_{k_1-1}} \ldots s_{a_1} s_{b_1} s_{b_2} \ldots s_{b_l}}) \) be their concatenation where \( A^0 = \{ a_k > \cdots > a_1 \} \) and \( B^0 = \{ b_1 < \cdots < b_l \} \), and cancel out all possible \((+ -)\) pairs in \( s \) as far as possible to obtain a reduced sequence
\[ s^{\text{red}} = (- \cdots - + \cdots +). \]
Then we define
\[ H(T_1, T_2) = - \min \{ \varepsilon, \varphi \}. \]

Lemma 4.1. With the same notations as above, we have \( \varphi - \varepsilon = k_1 - k_2 \).
Proof. Let $p$ (resp. $q$) be the total number of $+$’s (resp. $-$’s) in $s$. Then
\[ p = \sum_{a \in A} m_{a1} + \sum_{b \in B} m_{b2}, \quad q = \sum_{a \in A} m_{a2} + \sum_{b \in B} m_{b1}. \]
Since $\varphi - \varepsilon = p - q$ and $k_j = \sum_{a \in A} m_{aj} - \sum_{b \in B} m_{bj}$ for $j = 1, 2$, we have $\varphi - \varepsilon = k_1 - k_2$.

Next, we define a combinatorial $R$-matrix
\[ \sigma : \text{SST}_{A/B}(k_1) \times \text{SST}_{A/B}(k_2) \to \text{SST}_{A/B}(k_2) \times \text{SST}_{A/B}(k_1) \]
\[ (T_1, T_2) \quad \mapsto \quad (T'_2, T'_1) \]
where $(T'_2, T'_1)$ is given by moving and rearranging some of the entries in $T_1$ and $T_2$ in the following way:

1. If $k_1 = k_2$, then put $(T'_2, T'_1) = (T_1, T_2)$.
2. If $k_1 > k_2$, then let $y_1, \ldots, y_{k_1-k_2}$ be the entries in $T_1$ or $T_2$ corresponding to the first $k_1 - k_2$ signs of $+$ in $s^{\text{red}}$ from the left (see Lemma 3.1). For each $1 \leq k \leq k_1 - k_2$, if $y_k \in A$ (resp. $y_k \in B$), i.e. $y_k$ appears $T^+_1$ (resp. $T^-_2$), then we move it to $T^+_2$ (resp. $T^-_1$) and rearrange the entries with respect to the total order on $A$ (resp. $B$). Suppose that $x_k \in A$ (resp. $x_k \in B$), i.e. $x_k$ appears $T^+_2$ (resp. $T^-_1$), then we move it to $T^+_1$ (resp. $T^-_2$) and rearrange the entries with respect to the total order on $A$ (resp. $B$).

Example 4.2. (1) If $A = \{1\}^\prime$ and $B = \emptyset$, then we have $\text{SST}_{A/B}(k_j) = B^{k_j,1}$ for $j = 1, 2$, and $H$ on $B^{k_1,1} \times B^{k_2,1}$ recovers the local energy function $H$ on $B^{k_1,1} \otimes B^{k_2,1}$ normalized by $H(u_{k_1,1} \otimes u_{k_2,1}) = 0$ (see Proposition 3.1).

(2) If $A = \{1\}$ and $B = \emptyset$, then $\text{SST}_{A/B}(k_j) = B^{1,k_j}$ for $j = 1, 2$. In this case, we can check that the map $H$ on $B^{1,k_1} \times B^{1,k_2}$ coincides with the local energy function $H$ on $B^{1,k_2} \otimes B^{1,k_1}$ (in reverse order) normalized by $H(u_{1,k_2} \otimes u_{1,k_1}) = -\min\{k_1, k_2\}$ (cf. [28] Section 3). In both of cases (1) and (2), $\sigma$ is equal to the combinatorial $R$-matrix $\sigma$.

(3) Suppose that $A$ is finite with $|A_0| = n$ and $|A_1| = m$ and $B = \emptyset$. Then $\text{SST}_{A/B}(k)$ for $k \geq 1$ can be viewed as a crystal over the quantum superalgebra $U_q(\mathfrak{gl}_{m|n})$ [11]. It would be very nice to find a representation theoretical meaning of $H$ and $\sigma$ from finite-dimensional modules over the quantum affine superalgebra $U_q(\widehat{\mathfrak{s}l}_{m|n})$ [31].
(4) Let $\mathbf{T} = (T_1, T_2, T_3)$ be as in Example 2.1. Since

$$\text{wt}(T_2) = \epsilon_0^2 + \epsilon_0^3 + \epsilon_1^2 - \epsilon_1^3 - \epsilon_2^2 - \epsilon_2^3,$$

$$\text{wt}(T_3) = \epsilon_1^3 - \epsilon_2^3 - \epsilon_2^1,$$

we have $s_{T_2, T_3} = (+ + - + + + + - + -).$ Thus, the reduced sequence $s^\text{red} = (+)$ gives $H(T_2, T_3) = 0$ and

$$\sigma(T_2, T_3) = (T'_2, T'_3) \in \text{SST}_{A/B}(0) \times \text{SST}_{A/B}(1),$$

where

$$T'_2 = \begin{pmatrix} 0 & 6 & 7 \\ -4 & -2 & -1 \end{pmatrix}, \quad T'_3 = \begin{pmatrix} 2 & 4 & 5 \\ -2 & -1 \end{pmatrix}.$$

In the same manner, we compute

$$s_{T_1, T_2} = (- - + + + + - + - + - +),$$

$$s_{T_1, T'_2} = (- - + + + + + - + - +),$$

which yield $H(T_1, T_2) = -2$ and $H(T_1, T'_2) = -2.$

While $\text{SST}_{A/B}(k)$ for $k \in \mathbb{Z}$ produces a character of a level one integrable highest weight module over $U_q(\mathfrak{gl}_\infty)$, it also corresponds to a KR module over $U_q(\hat{\mathfrak{sl}}_\infty)$, where $\hat{\mathfrak{sl}}_\infty$ is an affinization of $\mathfrak{sl}_\infty.$ As in (3), we expect that $H$ and $\sigma$ are closely related with the theory of KR modules over $U_q(\hat{\mathfrak{sl}}_\infty)$.

Now, we fix $n \geq 1.$ For simplicity, we put

$$\mathcal{F}^n = \mathcal{F}_{A/B}^n,$$

$$\mathcal{F}^\mu = \text{SST}_{A/B}(\mu_1) \times \cdots \times \text{SST}_{A/B}(\mu_n)$$

for $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n.$ Clearly, we have $\mathcal{F}^n = \bigsqcup_{\mu \in \mathbb{Z}^n} \mathcal{F}^\mu.$ For $1 \leq i \leq n - 1,$ let $\sigma_i$ be the map on $\mathcal{F}^\mu \subset \mathcal{F}^n,$ which acts as $\sigma$ on $\text{SST}_{A/B}(\mu_i) \times \text{SST}_{A/B}(\mu_{i+1})$ and as identity elsewhere, and let $H_i$ be the map on $\mathcal{F}^n$ given by $H_i(T_1, \ldots, T_n) = H(T_i, T_{i+1})$ for $(T_1, \ldots, T_n) \in \mathcal{F}^n.$

We define a combinatorial energy function $D : \mathcal{F}^n \to \mathbb{Z}$ by

$$D(\mathbf{T}) = \sum_{1 \leq i < j \leq n} H_i(\sigma_{i+1}\sigma_{i+2}\cdots\sigma_{j-1}(\mathbf{T})) \quad (\mathbf{T} \in \mathcal{F}^n).$$

Then we have the following, which is a generalization of [28]. The proof is given in the next section.

**Theorem 4.3.** For $\mathbf{T} \in \mathcal{F}^n$, we have

$$D(\mathbf{T}) = -\text{charge}(Q_\mathbf{T}),$$

where $Q_\mathbf{T}$ is the rational semistandard tableau corresponding to $\mathbf{T}$ under the RSK map $\kappa_{A/B}$ on $\mathcal{F}^n$ in Theorem 2.2. In particular, we have $D(\mathbf{T}) = D(\mathbf{T}')$ for $\mathbf{T}, \mathbf{T}' \in \mathcal{F}^n$ such that $Q_\mathbf{T} = Q_\mathbf{T'}.$
Example 4.4. Continuing Example 4.2 (4), we have

\[ D(T) = H(T_1, T_2) + H(T_1, T_2') + H(T_2, T_3) = -4. \]

Since

\[ \text{charge}(Q_T) = c \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 2 \\ 2 & 3 & 3 \end{bmatrix} = 4, \]

we have \( D(T) = -\text{charge}(Q_T). \)

As a consequence, we obtain a combinatorial realization of (2.9) in terms of \( D. \)

Theorem 4.5. For \( \mu \in \mathbb{Z}_+^n, \) we have

\[ Q^{A/B}_\mu = \sum_{T \in F^\mu} q^{-D(T)} x^{T}_{A/B}, \]

where \( x^T_{A/B} = \prod_{i=1}^n x^T_i_{A/B} \) for \( T = (T_1, \ldots, T_n) \in F^\mu. \)

Proof. Restricting \( \kappa_{A/B} \) to \( F^\mu, \) we have a weight preserving bijection:

\[ \kappa_{A/B} : F^\mu \longrightarrow \bigcup_{\lambda \in \mathcal{P}_{A/B}} \text{SST}_{A/B}(\lambda) \times \text{SST}_{[n]}(\lambda)_\mu. \]

Thus, the assertion follows from (2.9), (3.2) and Theorem 4.3. \( \square \)

4.2. Proof of Theorem 4.3. Let us define an \( A_{n-1} \)-crystal structure on \( F^n. \)

Let \( M_{A/B, n} \) be the set of matrices \( m = (m_{ij}) \) with non-negative integral entries \((i \in A \cup B, j \in [n]) \) satisfying (1) \( m_{ij} \in \{0, 1\} \) if \( i \) is odd, (2) \( \sum_{i,j} m_{ij} < \infty. \) Note that for \( T = (T_1, \ldots, T_n) \in F^n \) with

\[ \text{wt}_{A/B}(T_j) = \sum_{a \in A} m_{aj} \epsilon_a - \sum_{b \in B} m_{bj} \epsilon_b, \]

for \( 1 \leq j \leq n, \) the map sending \( T \) to \( m = (m_{ij}) \) gives a natural bijection from \( F^n \) to \( M_{A/B, n}. \)

Let \( m \in M_{A/B, n} \) be given. For \( i \in A \cup B, \) let \( m_{(i)} = (m_{ij})_{j \in [n]} \) be the \( i \)th row of \( m, \) and set \( |m_{(i)}| = \sum_{j \in [n]} m_{ij}. \) Let \( \lambda^{(i)} \in \mathbb{Z}_+^n \) be given by

\[ \lambda^{(i)} = \begin{cases} (m_{(i)}), 0, \ldots, 0, & \text{if } i \in A_0, \\
(1^{\text{m}_{(i)}}, 0, \ldots, 0), & \text{if } i \in A_1, \\
(0, \ldots, 0, -|m_{(i)}|), & \text{if } i \in B_0, \\
(0, \ldots, 0, -1^{\text{m}_{(i)}}, & \text{if } i \in B_1. \end{cases} \]

We identify \( m_{(i)} \) with a unique rational semistandard tableau \( T^{(i)} \in \text{SST}_{[n]}(\lambda^{(i)}) \) such that

\[ \text{wt}_{[n]}(T^{(i)}) = \begin{cases} \sum_{j \in [n]} m_{ij} \epsilon_j, & \text{if } i \in A, \\
-\sum_{j \in [n]} m_{ij} \epsilon_j, & \text{if } i \in B. \end{cases} \]

For example, if \( a \in A_0 \) and \( b \in B_1, \) then we have
\[ m_{(a)} = (2, 0, 1, 2) \leftrightarrow T^{(a)} = \begin{array}{cccc}
1 & 1 & 3 & 4 \\
\end{array} \in \text{SST}_{[4]}(5, 0, 0, 0), \]
\[ m_{(b)} = (1, 1, 0, 1) \leftrightarrow T^{(b)} = \begin{array}{c}
-4 \\
-2 \\
-1 \\
\end{array} \in \text{SST}_{[4]}(0, -1, -1, -1). \]

Then we define a regular \( A_{n-1} \)-crystal structure on \( M_{A/B, n} \) and hence on \( F^n \) via the correspondence

\[ (4.2) \quad m \leftrightarrow \bigotimes_{a \in A} T^{(a)} \otimes \bigotimes_{b \in B} T^{(b)}. \]

Here we understand \( \bigotimes_{a \in A} T^{(a)} \) as a tenor product with respect to the reverse total order on \( A \). Since \( T^{(a)} \) is an empty tableau except for finitely many \( a \in A \), it is well-defined. Similarly, \( \bigotimes_{b \in B} T^{(b)} \) is a tensor product with respect to the total order on \( B \). One may assume that the row indices of \( m \in M_{A/B, n} \) are parametrized by \( B \pi \ast A \), and we read each row in \( m \) from bottom to top.

Since \( F^n \) is an \( A_{n-1} \)-crystal, one can consider \( \varepsilon_\alpha(T) \) and \( \varphi_\alpha(T) \) for \( T \in F^n \) and \( \alpha \in \Delta^+_n \) as in \((3.5)\). By definitions of \( H \) and \( \sigma \), we can first check that

\[ (4.3) \quad H_i(T) = -\min\{\varepsilon_i(T), \varphi_i(T)\}, \quad \sigma_i(T) = S_i(T), \]

for \( i = 1, \ldots, n - 1 \). In particular \( \sigma_i \)'s satisfy the braid relations. Thus, combining \((4.3)\) with \( \varepsilon_\alpha(T) \) and \( \varphi_\alpha(T) \), we obtain the following, which generalizes Proposition 3.1.

**Proposition 4.6.** For \( T \in F^n \), we have

\[ D(T) = -\sum_{\alpha \in \Delta^+_n} \min\{\varepsilon_\alpha(T), \varphi_\alpha(T)\}. \]

Moreover, the regular \( A_{n-1} \)-crystal structure of \( F^n \) enables us to consider the charge of \( T \). By Theorem 3.2 we have

**Corollary 4.7.** For \( T \in F^n \), we have \( D(T) = -\text{charge}(T) \).

This also immediately implies that \( D \circ \sigma_i = D \) for \( i = 1, \ldots, n - 1 \).

Finally, we interpret the map \( \kappa_{A/B} \)

\[ \kappa_{A/B} : F^n \longrightarrow \bigsqcup_{\lambda \in \mathcal{P}_{A/B, n}} \text{SST}_{A/B}(\lambda) \times \text{SST}_{[n]}(\lambda). \]

from a viewpoint of crystal. We assume that the right-hand side is an \( A_{n-1} \)-crystal, where the operators \( \bar{e}_j \) and \( \bar{f}_j \) act on the second factor \( \text{SST}_{[n]}(\lambda) \).

**Theorem 4.8.** The map \( \kappa_{A/B} \) is an \( A_{n-1} \)-crystal isomorphism.
Proof. Let us recall the bijections \( \varrho_{\text{col}} \) and \( \varrho_{\text{row}} \) given in (2.1). Suppose \( B = \emptyset \) and write \( F_A = \mathcal{F}_{A/B} \). Extending \( \varrho_{\text{col}} \) and \( \varrho_{\text{row}} \) to \( F_A^n \), we have bijections

\[
\varrho_{\text{col}} : F_A^n \to \bigcup_{\lambda \in \mathcal{P}_{A,n}} \text{SST}_A(\lambda) \times \text{SST}_{[n]}(\lambda),
\]

which are indeed the RSK correspondences since \( F_A^n \) can be identified with \( M_{A/\emptyset,n} \).

Let \( T \in F_A^n \) be given with the corresponding matrix \( m \in M_{A/\emptyset,n} \). If we write \( \varrho_{\text{col}}(T) = (P_c, Q_c) \) and \( \varrho_{\text{row}}(T) = (P_r, Q_r) \), then

\[
\bigotimes_{a \in A} m_a \equiv Q_c, \quad \bigotimes_{a \in A} m_a \equiv Q_r.
\]

Indeed, the first equivalence follows from [21, Theorem 3.11] on \( (gl_{|a|n}, gl_{|v|}) \)-bicrystal isomorphism over general linear Lie superalgebras, where we replace \( gl_{|v|} \) with \( gl_{|0|} \) and \( gl_{|a|n} \) with a Lie superalgebra \( gl_A \) associated to \( A \) (see also [23, Lemma 4.9]). Similarly, the second equivalence can be obtained by changing the parity of \( A \) and applying [21, Theorem 4.5].

Let \( T \in F^n \) be given, and \( m \) the corresponding matrix in \( M_{A/B,n} \). Let \( \kappa_{A/B}(T) = (P, Q_T) \). We keep the same notations \( Q, Q^\vee, U_R \) in (\( \kappa-1 \)-(\( \kappa-4 \)) in Section 2.3. It follows from (3.1), (\( \kappa-1 \)), (\( \kappa-2 \)), and the first equivalence in (4.4) that

\[
Q^\vee \equiv \left( \bigotimes_{b \in B^a} m_b^\vee \right) \equiv \left( \bigotimes_{b \in B^a} m_b^\vee \right) \equiv \bigotimes_{b \in B} m_b.
\]

We should remark that on \( M_{\emptyset/B,n} \) the \( A_{n-1} \)-crystal structure is dual to that of \( M_{A/\emptyset,n} \). Moreover, since \( k < a \) in \( [n] * A \) for all \( k \in [n] \) and \( a \in A \), (3.1) and the second equivalence in (4.4) give

\[
Q_T \equiv U_R \equiv \left( \bigotimes_{a \in A} m_a \right) \otimes Q^\vee \equiv \left( \bigotimes_{a \in A} m_a \right) \otimes \left( \bigotimes_{b \in B} m_b \right) \equiv T,
\]

which implies that \( \kappa_{A/B} \) is a morphism of \( A_{n-1} \)-crystal. Thus the assertion follows from Theorem 2.2. \( \square \)

Corollary 4.9. For \( T \in F^n \), \( \text{charge}(T) = \text{charge}(Q_T) \).

Now, Theorem 4.3 follows from Corollaries 4.7 and 4.9. This completes the proof.
Example 4.10. Continuing Example 2.1, we have the matrix \( m \) corresponding to \( T \) as follows:

\[
\begin{array}{c|ccc}
   & 1 & 2 & 3 \\
-4' & \cdot & \cdot & \cdot \\
-3' & \cdot & \cdot & \cdot \\
-2' & \cdot & \cdot & \cdot \\
-1' & \cdot & \cdot & \cdot \\
 0' & \cdot & \cdot & \cdot \\
 1' & \cdot & \cdot & \cdot \\
 2' & \cdot & \cdot & \cdot \\
 3' & \cdot & \cdot & \cdot \\
 4' & \cdot & \cdot & \cdot \\
 5' & \cdot & \cdot & \cdot \\
 6' & \cdot & \cdot & \cdot \\
 7' & \cdot & \cdot & \cdot \\
\end{array}
\]

where \( \cdot \) and \( \bullet \) denote 0 and 1 respectively. This yields

\[
T(0') = \frac{1}{2}, \quad T(1') = T(3') = \frac{1}{4}, \quad T(2') = T(6') = T(7') = \frac{1}{3},
\]
\[
T(-1') = T(-2') = \frac{1}{2}, \quad T(-3') = T(-1') = \frac{1}{3}.
\]

Thus, using the Schensted’s bumping algorithm, we see

\[
T(7') \otimes T(6') \otimes T(5') \otimes T(4') \otimes T(3') \otimes T(2') \otimes T(1') \otimes T(0') \equiv \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{3} \frac{1}{3} \frac{1}{3} 
\]

\[
T(-4') \otimes T(-3') \otimes T(-2') \otimes T(-1') \equiv \frac{1}{3} \otimes \frac{2}{3} \otimes \frac{1}{3} \otimes \frac{1}{3} \equiv \frac{1}{3} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{3} \frac{1}{3} \equiv Q'.
\]

Therefore, we have

\[
T \equiv \bigotimes_{a \in \mathbb{Z}_{\geq 0}} T^{(a)} \otimes \bigotimes_{b \in \mathbb{Z}_{< 0}} T^{(b)} \equiv \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{3} \frac{1}{3} \equiv Q_T.
\]

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