A Morse theoretic description of the Goresky-Hingston coproduct

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Abstract

The Goresky-Hingston coproduct was first introduced by D. Sullivan and later extended by M. Goresky and N. Hingston. In this article we give a Morse theoretic description of the coproduct. Using the description we prove homotopy invariance property of the coproduct. We describe a connection between our Morse theoretic coproduct and a coproduct on Floer homology of cotangent bundle.

Let $M$ be closed Riemannian manifold of dimension $n$. In [Su03], D. Sullivan described various operations on chains of free loop space also known as string topology operations. One of them is a non-trivial coproduct defined on homology of free loop space relative to the constant loops

$$\vee : H_*(\Lambda, M) \to H_*(\Lambda, M) \otimes H_*(\Lambda, M), \text{ degree } \vee = -\dim(M) + 1.$$  

In [GH09] M. Goresky and N. Hingston reinterpreted and extended the dual of the coproduct, and discovered many of its algebraic and geometric properties.

In [CS08], R. Cohen and M. Schwarz gave a Morse theoretic description of the Chas-Sullivan product. In this article we give a Morse theoretic description of the Goresky-Hingston coproduct. Our construction is done using the Morse complex of gradient flow of perturbed energy functional. The chain homomorphisms between the Morse complexes are described by counting the isolated points in the intersection of stable manifold and the image of the unstable of manifold. Using the Morse theoretic description we prove homotopy invariance properties of the dual of the coproduct with $\mathbb{Z}_2$ coefficient.

We consider only homologies with $\mathbb{Z}_2$ coefficient in order to avoid technicalities involved in dealing with orientation. However, when our manifold is oriented then all our constructions work also with $\mathbb{Z}$ coefficients.
1 The free loop space

Let $M$ be an $n$-dimensional connected closed Riemannian manifold with metric $g$. Let $\Lambda(M)$ be the free loop space which is a completion of all piecewise smooth curves with same initial and end point,

$$\Lambda(M) = \{ \alpha \in H^1([0, 1], M) | \alpha(0) = \alpha(1) \}.$$

We denote the free loop space of $M$ simply by $\Lambda$ when there is no confusion about the underlying manifold. It admits a structure of a Hilbert manifold (see [CS08]). We also the space of constant loops in $M$ by $\Lambda_0$. The energy and length of an element $\alpha \in \Lambda$ are defined respectively by

$$E(\alpha) = \int_0^1 g(\alpha'(t), \alpha'(t))dt \quad \text{and} \quad L(\alpha) = \int_0^1 \sqrt{g(\alpha'(t), \alpha'(t))}dt \quad (1.0.1)$$

We have the natural evaluation mapping

$$ev_s : \Lambda \to M, ev_s(\alpha) = \alpha(s).$$

Let $\Theta = \{ \alpha, \beta \in \Lambda \times \Lambda | \alpha(0) = \beta(0) \}$ be the space of figure-eight loops in $M$. For $s \in (0, 1)$ let

$$\Theta_s = \{ \alpha \in \Lambda | \alpha(0) = \alpha(s) \}$$

be the space of self-intersecting loops intersecting at time $s$. We will denote the space $\Theta_s \downarrow$ also by $\Theta$ since they are diffeomorphic. For any $s \in (0, 1)$ the space $\Theta_s$ is homotopic to the space $\Theta$.

Let

$$\phi_{1/2}(\alpha, \beta) : \Theta \in \Lambda \times \Lambda \to \Lambda$$

be the usual concatenation of loops and

$$c : \Theta \to \Lambda \times \Lambda$$

be the cutting map, $c(\gamma) = (\alpha, \beta)$ if $\phi_{1/2}(\alpha, \beta) = \gamma$.

1.1 The Goresky-Hingston coproduct

Let us denote by $I$ the interval $[0, 1]$ . Let

$$\Gamma : \Lambda \times I \to \Lambda \text{ given by } \Gamma(\alpha, s) = \alpha \circ \theta_{1/2 \to s}$$

where $\theta = \theta_{1/2 \to s} : I \to I$ the reparametrization function that is linear on $[0, 1/2]$, linear on $[1/2, 1]$, has $\theta(0) = 0, \theta(1) = 1$, and $\theta(1/2) = s$, see figure 1.1.1. Let $\Gamma_1 : \Lambda \to \Lambda$ be the map defined by $\Gamma_1(\alpha) = \Gamma(\alpha, t)$.

We have the following pullback diagram

$$\begin{array}{ccc}
\Lambda & \xrightarrow{ev_0 \times ev_0} & M \times M \\
\downarrow & & \downarrow \quad D \\
\Theta & \longrightarrow & M
\end{array}$$
where $D$ is the diagonal embedding. So $\Theta$ has tubular neighborhood and a normal bundle of fibre dimension $n$ in $\Lambda$. We also have the following map on chains:

$$P : C_i(\Lambda) \xrightarrow{\hat{\Gamma}} C_{i+1}(\Lambda) \xrightarrow{\pi} C_{i-n+1}(\Theta) \xrightarrow{c} C_{i-n+1}(\Lambda \times \Lambda)$$

(1.1.1)

where $\hat{\Gamma} : C_i(\Lambda) \to C_{i+1}(\Lambda)$ is the prism operator (See [AH]) induced by the homotopy $\Gamma$ and $\pi$ is the Gysin homomorphism for the co-dimension $n$ embedding $\Theta \hookrightarrow \Lambda$.

$P$ is not a chain map, however it defines a chain map on chains relative to the constant loops $C_*(\Lambda, \Lambda_0)$. This results in a coproduct known as the Goresky-Hingston coproduct defined by the following composition:

$$H_i(\Lambda, \Lambda_0) \to H_i(\Lambda, \Lambda_0) \otimes H_1(I, \partial I) \cong H_{i+1}(\Lambda \times I, \Lambda_0 \times I \cup \Lambda \times \partial I)$$

$$\xrightarrow{\Gamma} H_{i+1}(\Lambda, \Lambda - \Theta^{>,>0}) \xrightarrow{\tau} H_{i-n+1}(\Theta, \Theta - \Theta^{>,>0})$$

$$\xrightarrow{c} H_{i-n+1}(\Lambda \times \Lambda, \Lambda \times \Lambda_0 \cup \Lambda_0 \times \Lambda),$$

(1.1.2)

where $\kappa$ is the Künneth map, $\bar{I}$ is the generator of $H^1(I, \partial I)$, $\tau$ is the relative Thom isomorphism.

## 2 Morse theory on Hilbert manifolds

In the article [AM06] A. Abbondandolo and P. Majer constructed Morse homology of gradient-like flows on Banach manifolds. Their construction naturally gives Morse homology of Hilbert manifolds and in particular of free loop space. We recall their construction in brief.
2.1 The Morse complex

Let \((M, g)\) be a Hilbert manifold (possibly infinite dimensional) with a complete Riemannian \(C^1\) metric \(g\) on \(M\) and \(A\) be an open subset of \(M\). Let \(\mathcal{F}(M, A, g)\) be the set of \(C^2\) functions \(f : M \to \mathbb{R}\) such that

- \((C1)\) \(f\) is bounded below on \(M \setminus A\);
- \((C2)\) each critical point of \(f\) in \(M \setminus A\) is non-degenerate and has finite Morse index;
- \((C3)\) \(f\) satisfies the Palais-Smale condition with respect to \(g\) on \(M \setminus \overline{A}\);
- \((C4)\) \(A\) is positively invariant for the flow of \(-\nabla f\), and this flow is positively complete with respect to \(A\) (meaning that the orbits that never enter \(A\) are defined for every \(t \geq 0\)).

We denote by \(\text{Crit}(f)_{\mid \mathcal{M}}\) the set of critical points of \(f\) in \(M \setminus A\), and by \(\text{Crit}_k(f)_{\mid \mathcal{M}}\) the critical points of \(f\) in \(M \setminus A\) with Morse index \(k\). Let \(\phi : \mathbb{R} \times M \to M\) be the flow determined by the vector field \(-\nabla f\). Note that local gradient flow extends to entire real line due to our assumptions. The limit \(\lim_{t \to +\infty} \phi(t, \cdot)\) exist and is a critical point of \(f\). Let \(x, y \in \text{Crit}(f)\). The unstable manifold of \(x\) defined by

\[ W^u(x) := \left\{ p \in M \mid \lim_{t \to -\infty} \phi(t, p) = x \right\} \]

and the stable manifold of \(x\) defined by

\[ W^s(x) := \left\{ p \in M \mid \lim_{t \to +\infty} \phi(t, p) = x \right\} \]

are submanifolds of dimension \(\text{ind}(x)\) and codimension \(\text{ind}(x)\) respectively. We denote the space of gradient trajectories from \(x\) to \(y\) by

\[ \mathcal{M}(x, y) = W^u(x) \cap W^s(y) \]

and

\[ \widehat{\mathcal{M}}(x, y) = \mathcal{M}(x, y)/\mathbb{R} \]

Let us assume that \(-\nabla f\) satisfies the Morse-Smale condition on \(M \setminus A\). Then compactness and transversality implies that for each \(x, y \in \text{Crit}(f)\) with \(\text{ind}(x) - \text{ind}(y) = 1\) the moduli space \(\widehat{\mathcal{M}}(x, y)\) consists of finite number of points, we denote by \(\eta_y(x, y)\) the parity.

Let \(M_k(f_A, g)\) be the \(\mathbb{Z}_2\)-module generated by critical points of index \(k\) in \(M \setminus A\), then we can construct a chain complex \((M_k(f_A, g), \partial_k(f_A, g))\) where the boundary homomorphism \(\partial_k(f, g)\) is defined by

\[ \partial_k(f, g) : M_k(f_A, g) \to M_{k-1}(f_A, g), \quad \partial_k(f_A, g)(x) = \sum_{y \in \text{crit}_{k-1}(f)} \eta_y(x, y)y, \]

for each \(y \in \text{crit}_{k-1}(f)\). The chain complex \(M^*(f_A, g)\) is known as Morse complex of \((f_A, g)\). The homology of the chain complex \(M_k(f_A, g)\) is isomorphic to singular homology of the pair \((M, A)\).

\[ H_k M(f_A, g) \cong H_k(M, A; \mathbb{Z}_2). \]

Different choice of metric yields isomorphic homology groups. When our manifold is additionally oriented then moduli space \(\widehat{\mathcal{M}}(x, y)\) inherits a natural orientation. If we count the zero dimension moduli space together with orientation then the above isomorphism also holds for \(\mathbb{Z}\) coefficients.
2.2 Functoriality

Let \((M_1, g_1)\) and \((M_2, g_2)\) be complete Riemanian manifolds. Let \(f_1 \in F(M_1, g_1)\) and \(f_2 \in F(M_2, g_2)\) be such that \(-\text{grad} f_1\) and \(-\text{grad} f_2\) satisfy the Morse-Smale condition. Let \(\varphi : M_1 \to M_2\) be a smooth map, we may assume that upon generic choices of metrices \(g_1, g_2\) and \(f_1, f_2\) the map \(\varphi\) satisfies the following conditions:

\((F1)\) each \(y \in \text{crit}(f_2)\) is regular value of \(\varphi\).
\((F2)\) \(x \in \text{crit}(f_1)\) \(\implies \varphi(x) \in \text{crit}(f_2) \implies \text{ind}(x; f_1) \geq \text{ind}(\varphi(x); f_2)\).

Given \(x \in \text{crit} f_1, y \in \text{crit} f_2\), we define \(W_\varphi(x, y)\) to be the set \(\varphi(W^u(x) \cap W^s(y))\)

The conditions \(F1 - F2\) above ensure that by generic choices of metrices \(g_1, g_2\) the above intersection is transversal and consequently the set \(W_\varphi(x, y)\) is a manifold of dimension \(\text{ind}(x; f_1) - \text{ind}(y; f_2)\).

In particular, when \(\text{ind}(x; f_1) = \text{ind}(y; f_2)\), the set \(W_\varphi(x, y)\) is discreet. The fact that the closure of \(W^u(x; -\text{grad} f_1)\) in \(M_1\) is compact imply that the discrete sets \(W_\varphi(x, y)\) is compact, hence finite. We denote by \(\eta_\varphi(x, y)\) to be the parity of the set \(W_\varphi(x, y)\).

We can define the homomorphism \(M_k(\varphi) : M_k(f_1, g_1) \to M_k(f_2, g_2), M_k(\varphi)x = \sum_{y \in \text{crit}_k(f_2)} \eta_\varphi(x, y)y\)

for every \(x \in \text{crit}_k(f_1)\).

In [AM06] and [AS08] the authors show that \(M_k(\phi)\) is a chain homomorphism and induces the homomorphism \(\varphi_* : H_*(M_1) \to H_*(M_2)\) on homology.

2.3 Morse theoretic description of Gysin map in the relative case

Let \((X, g_X)\) be a Hilbert manifold, \(E \to X\) be a vector bundle of real dimension \(n\) ( the dimension of the fibre) then we have the Thom isomorphism

\(H_i(X) \cong H_{i-n}(E, E - X)\)

induced by cap product with the Thom class \(\mu \in H^n(E, E - X)\) of the vector bundle \(E \to X\).

Let \(A \subset X\) be closed subsets of Hilbert manifold \((Y, g_Y)\). Assume that \(X\) has a tubular neighborhood \(N\) in \(Y\), meaning that there exist a real vector bundle \(\pi : E \to X\) of fiber dimension \(n\) and a diffeomorphism \(F : E \to N\) such that restriction to zero section is the natural identification of zero section with \(X\). Let us denote by \(E^0\) the zero section of \(E \to X\) and \(E^A = \pi^{-1}(X - A)\).

Then we have the following Gysin isomorphism in singular homology:

\(H_i(X, X - A; \mathbb{Z}_2) \cong H_{i-n}(Y, Y - A; \mathbb{Z}_2)\)
The isomorphism can be seen as the composition of the following isomorphisms

\[
\mu_T : H_i(X, X - A) \cong H_i(E, E^A) \xrightarrow{\partial_n} H_{i-n}(E, E^0 \cup E^A) \\
\cong H_{i-n}(N, N - A) \cong H_{i-n}(Y, Y - A), \quad (2.3.1)
\]

where \( \mu \) is the Thom class of the vector bundle \( \pi : E \to X \). The last isomorphism in the sequence is induced by excision.

We first consider the Morse theoretic description of the isomorphism

\[
h_\# : H_i(X, X - A) \cong H_i(E, E^A) \to H_{i-n}(E, E^0 \cup E^A). \quad (2.3.2)
\]

We endow the vector bundle \( \pi : E \to X \) with a bundle metric \( \pi^* g_x \). Let \( q \) be the associated positive quadratic form. We choose a \( f \in \mathcal{F}(X, A; g_x) \) as defined in section 2.1. We consider the \( C^2 \) function on \( E \) defined by

\[
f_q(e) = f(\pi(e)) - q(e),
\]

then \( f_q \in \mathcal{F}(E, E^0 \cup E^A; \pi^* q) \).

Now, we can identify the critical points of \( f \) on \( X \setminus (X \setminus A) = A \) and \( f_q \) on \( E \setminus (E^0 \cup E^A) = A \). This gives us a homomorphism

\[
Mh_\# : M_\ast(X, X - A; f) \to M_{\ast-n}(E, E^0 \cup E^A; f_q). \quad (2.3.3)
\]

Now, we have the following diagram which commute up to chain homotopy.

\[
\begin{array}{ccc}
M_i(X; f) & \longrightarrow & M_i(X, X - A; f) \\
\downarrow \quad M_{i-n}(E, E^0, f) & & \downarrow \quad M_{i-n}(E, E^0 \cup E^A; \tilde{f})
\end{array}
\]

In \cite{CS08}, it was shown that \( Mh_\# \) in the left hand column induces an isomorphism on homology. Therefore, \( Mh_\# \) induces the isomorphism \( h_\# \) on homology.

We further choose a \( \tilde{f} \in \mathcal{F}(Y, Y-A; g_y) \) such that \( \tilde{f}|_N = f|_N \) where \( f : E \to N \) is the diffeomorphism defined above.

A further identification of critical points of \( f_q \) in \( A \) and \( \tilde{f} \) in \( A = Y \setminus (Y \setminus A) \) gives a homomorphism

\[
M_\ast(E, E^0 \cup E^A; f_q) \to M_\ast(Y, Y - A; \tilde{f}). \quad (2.3.4)
\]

The induced homomorphism in Morse homology is an isomorphism and it describes the excision isomorphism used in the equation 2.3.4

\[
H_\ast(E, E^0 \cup E^A; f_q) \cong H_\ast(Y, Y - A; \tilde{f}).
\]

Now if we choose any \( k \in \mathcal{F}(Y, Y - A; g_y) \) then for generic choices of Riemannian metrics \( g_y \) on \( Y \) and \( g_x \) on \( X \), we have the following homomorphism

\[
M_\ast(Y, Y - A; \tilde{f}) \to M_\ast(Y, Y - A; k), \quad m \to \sum_{i(m') = i(m)} n(m, m')m', \quad (2.3.5)
\]

with \( n(m, m') = \#(W^u(m; f) \cap W^s(m'; \tilde{k}_q)) \mod \mathbb{Z}_2 \).
which induces the canonical isomorphism
\[ \text{H}^\ast_m(Y, Y - A; \bar{f}) \cong \text{H}^\ast_m(Y, Y - A; k), \]
also known as continuation isomorphism.

Thus, composing the Morse chain homomorphism one obtains a Morse theoretic description of the Gysin map \( \mu_T \). Below we describe a direct Morse chain map defined in terms of intersection number which gives \( \mu_T \) on homology.

Let \( k \in F(Y, Y - A; g_Y) \) and \( f \in F(X, X - A; g_X) \). For generic choices of metrics on \( X \) and \( Y \) the unstable manifold of \( y \in \text{crit} f |_A \) and the stable manifold of \( x \in \text{crit} k |_A \) intersect transversally
\[ W^u(y; f, Y) \cap W^s(x; k, X). \]

If \( \text{ind}(y; k) = \text{ind}(x; f) + n \), then \( W^u(y; f, Y) \cap W^s(x; k, X) \) is finite dimensional manifold. Let us denote by \( \eta_T(y, x) \) to be the parity of such set.

We have the following proposition.

**Proposition 2.1.** The chain homomorphism
\[ M\mu_T : M_i(Y, Y - A, k) \to M_{i-n}(X, X - A, f), \quad M\mu_T(y) = \sum_{\text{ind}(x; f) = \text{ind}(y; k) - n} \eta_T(x, y)x \]
induces relative Gysin map \( \mu_T \) on homology.

**Proof.** Using a standard compactness and gluing arguments of Morse theory one can show that the homomorphism \( M\mu_T \) is chain homotopic to the composition of the homomorphisms \( \text{ 2.3.2, 2.3.4, 2.3.5} \) \( \square \)

### 2.4 Morse theoretic interpretation of the G-H coproduct

In this section we proceed to give a Morse theoretic description of the coproduct defined above in 1.1.2.

Consider the perturbed energy functional \( S_L \) defined by
\[ S_L(\gamma) = \int_T L(t, \gamma(t), \gamma'(t))dt, \quad \gamma \in \Lambda, \]
where \( L = |p|^2 - V(t, q) \) for some \( V(t, q) > 0 \) with \( ||V||_\infty \) smaller than the smallest length of a closed geodesic of \( M \).

Let us denote by \( \mathcal{P}(L) = \text{Crit}(S_L) \) consisting of solutions of the Lagrange equation
\[ \frac{d}{dt} \partial_v L(t, \gamma(t), \gamma'(t)) = \partial_q L(t, \gamma(t), \gamma'(t))dt. \]

In order to define Morse homology of \( \Theta \) we consider the Lagrangian of type \( L_1 \# L_2 \) which was already introduced in [AS08]. Given two Lagrangian \( L_1, L_2 \in C^\infty([0, 1] \times TM) \) of the above form such that \( L_1(1, .) = L_2(0, .) \) with all the time derivatives, we define the Lagrangian \( L_1 \# L_2 \in C^\infty([0, 1], TM) \) as
\[ L_1 \# L_2(t, q, v) = \begin{cases} 2L_1(2t, q, v/2) & \text{if } 0 \leq t \leq 1/2 \\ 2L_1(2t - 1, q, v/2) & \text{if } 1/2 \leq t \leq 1 \end{cases} \quad (2.4.1) \]
The curve $\gamma : [0,1] \to M$ is a solution of the Lagrangian equation with $L_1 \# L_2$ if and only if the rescaled curves $t \mapsto \gamma(t/2)$ and $t \mapsto \gamma((t+1)/2)$ solves the corresponding equation given by the Lagrangians $L_1$ and $L_2$ on $[0,1]$.

Let $M^\geq_\epsilon(S_L, g), M^\leq_\epsilon(S_L, g)$ denote the subcomplex of $M(S_L, g)$ restricted to the space $\Lambda \setminus \Lambda^{\leq \epsilon}, \Lambda \setminus \Lambda^{\geq \epsilon}$ respectively. Then the action functional $S_L$ is $C^2$ on $\Lambda$ and satisfies conditions $C1 - C3$ and condition $C4$ above for $A = \Lambda^{\leq \epsilon}$. Since $\epsilon$ is smaller than length of the smallest closed geodesic, we have

$$HM^{\geq_\epsilon}_*(S_L, g) \cong HM^{\geq_\epsilon}_*(S_L, g) \cong H_*(\Lambda, A_0)\]$$

Let $\epsilon > 0$ be smaller than the smallest length of a closed geodesic of $M$. We denote by $\Lambda^{\leq \epsilon}$ the set of loops with $\alpha \in \Lambda$ such that $S_L(\alpha) \leq \epsilon$.

Let $S^\epsilon_L$ denote the Lagrangian $S_{L_1 \# L_2}$ restricted to the space $\Theta$. Let $\Theta^{\leq \epsilon, \leq \epsilon}$ be the space of loops with $\theta \in \theta$ such that $S_{L_1}(\theta|[0,1]) \leq \epsilon$ or $S_{L_2}(\theta|[1,2]) \leq \epsilon$. Let $M^\leq_\epsilon, \geq_\epsilon(S_{L_1 \# L_2}, g)$ and $M^\leq_\epsilon, \geq_\epsilon(S^\epsilon_{L_1 \# L_2}, g)$ be the subcomplexes of $M(S_{L_1 \# L_2}, g)$ and $M(S^\epsilon_{L_1 \# L_2}, g)$ respectively restricted to the spaces $\Lambda \setminus \Theta^{\leq \epsilon, \leq \epsilon}$ and $\Theta \setminus \Theta^{\leq \epsilon, \leq \epsilon}$ respectively. They are indeed subcomplexes due to our choice of $\epsilon$ and $L_i$'s.

Let $\tilde{\Gamma}$ be a homotopic smooth approximation of $\Gamma$ with $\tilde{\Gamma}(\alpha, 0) = \Gamma(\alpha, 0)$ and $\tilde{\Gamma}(\alpha, 1) = \Gamma(\alpha, 1)$. Then $\Gamma$ and $\tilde{\Gamma}$ define the same map on homology.

We consider Morse homology of manifold with boundary for $I = [0,1]$. Let $\phi_I : I \to \mathbb{R}$, be given by the below function in the figure with $\frac{1}{2}$ its only critical point of index 1.

![Figure 2.4.1](image)

**Figure 2.4.1:** The function $\phi_I$

Then we have an isomorphism of Morse homology of $\phi_I$ and singular homology of the pair $(I, \partial I)$, $HM_*(\phi_I) \cong H_*(I, \partial I)$. We can identify the Morse complex of $S_{L_1 \# L_2}$ with the Morse complex of $(S_{L_1 \# L_2} \times \phi_I)$ with a dimension shift by 1.

Let $L_i = |p|^2 - V_i(t, q)$ for some $V_i(t, q) > 0$ with $||V||_\infty < \epsilon/2$, $i = 1, 2, 3$. Let $g_1, g_2$ be complete metrics on $\Lambda$ such that $-\text{grad}_g S_{L_1 \# L_2}$ satisfy the Palais-Smale condition on $\Lambda \setminus \Lambda^{\leq \epsilon}$. Then, up to perturbing $g_1, g_2$ and for generic choice of $V_1, V_2$, we can assume that for every $\gamma_1 \in \mathcal{P}(L_1 \# L_2)$, with $S_{L_1 \# L_2}(\gamma_1) > \epsilon$, and for every $\gamma_2 \in \mathcal{P}^e(L_1 \# L_2)$ with $S_{L_1}(\gamma_2|[0, \frac{1}{2}]) > \epsilon$ or $S_{L_2}(\gamma_2|[\frac{1}{2}, 1]) > \epsilon$ we have transversal intersection

$$W_\epsilon(\gamma_1, \gamma_2) := \tilde{\Gamma}(W^u(\gamma_1) \times (0, 1)) \cap W^s(\gamma_2).$$

Then $W_\epsilon(\gamma_1, \gamma_2)$ is a manifold of dimension $\text{Ind}(\gamma_1) - \text{Ind}(\gamma_2) + 1$. If $\text{Ind}(\gamma_2) = \text{Ind}(\gamma_1) + 1$ then $W_\epsilon(\gamma_1, \gamma_2)$ is finite set of points, we denote by $\sigma(\gamma_1, \gamma_2)$ its parity.

**Proposition 2.2.** Let $\epsilon$ is smaller than the length of the smallest closed geodesic. For generic choices of metrics $g_1, g_2$ and $V_1, V_2$ the homomorphism

$$M_\Gamma : M^\geq_\epsilon(S_{L_1 \# L_2}, g_1) \to M^\geq_{\epsilon+1}_*(S_{L_2 \# L_2}, g_2)$$

given by

$$\gamma_1 \mapsto \sum_{\text{Ind}(\gamma_2) = \text{Ind}(\gamma_1) + 1} \sigma(\gamma_1, \gamma_2)\gamma.$$
is a chain map, and induces the composition $H_{*-n}(\Lambda, \Lambda_0) \to H_{*-n}(\Lambda \times I, \Lambda \times \partial I \cup I \times \Lambda_0) \xrightarrow{\Gamma} H_{*-n}(\Lambda, \Lambda - \Theta^{0,>0})$ on homology.

Proof. By transversality, and standard compactness cobordism argument in Morse theory it follows that $M_\Gamma$ is a chain map. Since $\epsilon$ is smaller than smallest length of a closed geodesic in $M$, we have

$$HM_{*-\epsilon}(S_{L_1 \# L_2}, g) \cong H_*(\Lambda, \Lambda_0) \text{ and } HM^{*-\epsilon, > \epsilon}_{*-\epsilon}(S_{L_1 \# L_2}, g) = H_*(\Lambda, \Lambda - \Theta^{0,>0})$$

The proposition then follows from functoriality in Morse homology discussed in section 2.2, and that $\Gamma$ is homotopic to $\Gamma$. \hfill \Box

Let $ev_I : \Lambda \times (0,1) \times M \times M$ be given by $(\alpha, t) \mapsto (\alpha(0), \alpha(t))$. We have the following pullback square:

\[
\begin{array}{ccc}
\Lambda \times (0,1) & \xrightarrow{ev_I} & M \times M \\
\downarrow & & \downarrow D \\
\{\Theta_s, s\} \times (0,1) & \longrightarrow & M
\end{array}
\]

The map $ev_I$ is transverse to the diagonal, so $\{\Theta_s, s\}_{s \in (0,1)}$ is submanifold of $\Lambda \times (0,1)$ and it has tubular neighborhood and normal bundle in $\Lambda \times (0,1)$ of fibre dimension $n$. Consequently, $\{\Theta_s\}_{s \in (0,1)}$ is a codimension $n - 1$ submanifold of $\Lambda$.

Let $g_1, g_2, g_3$ be complete metrics on $\Lambda$ such that $-\text{grad}_{g_i}S_{L_1}, -\text{grad}_{g_2}S_{L_2}$ and $-\text{grad}_{g_3}S_{L_1 \# L_2}$ satisfy the Palais-Smale condition on $\Lambda \setminus \Lambda^{\leq \epsilon}$. Then, up to perturbing $g_i$ and for generic choice of $V_i$, we can assume that for every $\gamma_i \in \mathcal{P}(L_i)$ with $S_{L_i}(\gamma_i) > \epsilon$ for $i = 1, 2$ and $\gamma_3 \in \mathcal{P}(L_1 \# L_2)$ with $S_{L_1 \# L_2}(\gamma_3) > \epsilon$ we have transversal intersection

\[
W_{GH}(\gamma_3, \bar{\gamma}) := \phi_2\left( (W^*(\gamma_1; -\text{grad}_{g_1}S_{L_1}) \times W^*(\gamma_2; -\text{grad}_{g_2}S_{L_2}) \cap \Theta ) \\
\cap \{\Theta_s\}_{s \in (0,1)} \cap W^u(\gamma_3; -\text{grad}_{g_3}S_{L_1 \# L_2}) \right)
\]

\[\simeq \tilde{\Gamma}(W^u(\gamma_3; -\text{grad}_{g_3}S_{L_1 \# L_2}) \times I^0) \cap (\phi_2((W^*(\gamma_1; -\text{grad}_{g_1}S_{L_1}) \\
\times W^*(\gamma_2; -\text{grad}_{g_2}S_{L_2}) \cap \Theta)) \right) \tag{2.4.3}
\]

So $W_{GH}$ is a manifold of dimension $\text{ind}(\gamma_3) + \text{ind}(\gamma_1) + \text{ind}(\gamma_2) - n + 1$.

Here we note that in the first expression the space $W_{GH}$ is described independent of the reparametrization function $\Gamma$ or $\Gamma$ chosen.

If $\text{ind}(\gamma_1) + \text{ind}(\gamma_2) = \text{ind}(\gamma_3) - n + 1$, then the space $W_{GH}(\gamma, \bar{\gamma})$ is a finite set of points, and we denote by $\sigma(\gamma_3, \bar{\gamma})$ its parity. We have the following theorem which gives a Morse theoretic interpretation of the Goresky-Hingston coproduct.

\[\text{Figure 2.4.2: An element of } W_{GH}(\gamma, \bar{\gamma})\]
Theorem 2.3. Let $\epsilon$ is smaller than the length of the smallest closed geodesic. For generic choices of metrics $g_1, g_2, g_3$ and potentials $V_1, V_2$ the homomorphism

$$M_P : M_h^\epsilon(S_{L_1 \# L_2}, g_3) \to (M_h^\epsilon(S_{L_1}, g_1) \otimes M_h^\epsilon(S_{L_2}, g_2))_{h-n+1} \quad (2.4.4)$$

$$\gamma_3 \mapsto \sum_{\bar{\gamma} \in \mathcal{P}(L_1) \times \mathcal{P}(L_2)} \sigma(\gamma_3, \bar{\gamma}) \bar{\gamma}$$

is a chain map, and induces the Goresky-Hingston coproduct on homology.

Proof. Above transversality and standard compactness and cobordism argument in Morse theory shows that $M_P$ is a chain map.

We have described a Morse chain map $M_\Gamma$ in the proposition 2.2 which induces the composition

$$\hat{\Gamma} : H_s-n(\Lambda, \Lambda_0) \to H_s-n(\Lambda \times I, \Lambda \times \partial I \cup I \times \Lambda_0) \xrightarrow{\Gamma} H_s-n(\Lambda, \Theta - \Theta^{0,0})$$
on homology.

We also have a Morse chain map $M_{\mu T}$ described in the proposition 2.1 which induces the Gysin map in the relative case. We apply the proposition with Morse functions $S_{L_1 \# L_2}, S_{L_1 \# L_2}^\Theta$ on $\Lambda$ and $\Theta$ respectively, and with $X = \Theta, Y = \Lambda$ and $A = \Theta - \Theta^{0,0}$. Then $M_{\mu T}$ induces the Gysin map $H_s(\Theta, \Theta - \Theta^{0,0}) \to H_s-n(\Lambda, \Lambda - \Theta^{0,0})$ on homology.

The homomorphism $c_* : H_*(\Theta, \Theta - \Theta^{0,0}) \to (\Lambda \times \Lambda, \Lambda \times \Lambda_0 \cup \Lambda_0 \times \Lambda)$ is induced by the smooth map $c : \Theta \to \Lambda \times \Lambda$. So, as in section 2.2, we can define a Morse chain map

$$M_c : M^\epsilon(S_{L_1 \# L_2}, g_3) \to M^\epsilon(S_{L_1}, g_1) \times M^\epsilon(S_{L_2}, g_2)$$
defined in terms of intersection number such that $M_c$ induces the homomorphism $c_*$ on homology.

Now using standard gluing and compactness argument in Morse theory one can show that $M_P$ is chain homotopic to the composition of $M_\Gamma, M_{\mu T}$ and $c_I$. So, $M_P$ induces composition of homomorphisms $\hat{\Gamma}, \tau$ and $c_*$ on homology. The Goresky-Hingston coproduct is composition of the homomorphisms $\hat{\Gamma}, \tau$ and $c_*$. This concludes the proof of the theorem.

\[\blacksquare\]

- Homotopy invariance of the G-H coproduct

We have the following corollary of the above theorem 2.4.4.

Corollary 2.4. Let $M$ and $M'$ be closed oriented manifolds. $f : M \to M'$ be a homotopy equivalence. Then the induced homotopy equivalence of loop spaces, $\Lambda f : \Lambda(M) \to \Lambda(M')$ induces a ring isomorphism.

$$(\Lambda f)^* : (H^*(\Lambda(M), M; \mathbb{Z}_2), \otimes) \cong (H^*(\Lambda(M'), M'; \mathbb{Z}_2), \otimes).$$
Proof. Let $\epsilon > 0$ be smaller than the smallest length of a closed geodesic on $M$. Let $L_i$ and $L'_i$ be Lagrangian of the above form on $\Lambda(M)$ and $\Lambda(M')$ respectively for $i = 1, 2, 3$. Let $g_1, g_2, g_3$ and $g'_1, g'_2, g'_3$ complete metrics on $M$ and $M'$ respectively.

With the above notations it is enough to show that the following diagram commutes.

$$
\begin{array}{ccc}
M^>^>^>^\epsilon\epsilon(S_{L_1\#L_2}, g_3) & \xrightarrow{M_P} & M^>^>^>^\epsilon\epsilon(S_{L_1}, g_1) \times M^>^>^>^\epsilon\epsilon(S_{L_2}, g_2) \\
\Bigg| & & \Bigg| \downarrow M_{f\times Mf} \\
M^>^>^>^\epsilon\epsilon(S_{L'_1\#L'_2}, g'_3) & \xrightarrow{M_P} & M^>^>^>^\epsilon\epsilon(S_{L'_1}, g'_1) \times M^>^>^>^\epsilon\epsilon(S_{L'_2}, g'_2)
\end{array}
$$

In the same manner of defining the above homomorphism $M_P$ [2.4.4]. For every $\gamma_3 \in \mathcal{P}(L_3)$, $\gamma'_1 \in \mathcal{P}(L'_1)$, $\gamma'_2 \in \mathcal{P}(L'_2)$ with $S_{L_3}(\gamma_3) > \epsilon$, $S_{L'_1}(\gamma'_1) > \epsilon$, $S_{L'_2}(\gamma'_2) > \epsilon$, for generic choices of metrics $g_3$, $g'_1, g'_2$ and potentials $V_1, V_2, V'_1, V'_2$, we define a homomorphism

$$K_1 : M^>^>^>^\epsilon\epsilon(S_{L_1\#L_2}, g_3) \to (M^>^>^>^\epsilon\epsilon(S_{L'_1}, g'_1) \otimes M^>^>^>^\epsilon\epsilon(S_{L'_2}, g'_2))_{h-n+1}$$

by counting isolated points in the transversal intersection

$$f(\tilde{f}(W^n(\gamma_3; -\text{grad}_g S_{L_1\#L_2}) \times I^0) \cap ((W^n(\gamma'_1; -\text{grad}_g S_{L'_1}) \otimes W^n(\gamma'_2; -\text{grad}_g S'_{L_2}) \cap \Theta(M')))$$

Using standard compactness and gluing argument one can show that $K_1$ is a chain map and, that $M_P \circ M_f$ is homotopic to $K_1$.

For generic choices of metric $g_3$ on $M$ and $g'_1, g'_2$ on $M'$ we also define another homomorphism

$$K_2 : M^>^>^>^\epsilon\epsilon(S_{L_1\#L_2}, g_3) \to (M^>^>^>^\epsilon\epsilon(S_{L'_1}, g'_1) \otimes M^>^>^>^\epsilon\epsilon(S_{L'_2}, g'_2))_{h-n+1}$$

by counting isolated points in the transversal intersection

$$\tilde{f}(f(W^n(\gamma_3; -\text{grad}_g S_{L_1\#L_2}) \times I^0) \cap ((W^n(\gamma'_1; -\text{grad}_g S_{L'_1}) \otimes W^n(\gamma'_2; \text{grad}_g S'_{L_2}) \cap \Theta(M')))$$

Again using standard compactness and gluing argument one can show that $M_f \circ M_P$ is homotopic to $K_2$.

Now since $f \circ \tilde{f}(\alpha, s) = \tilde{f}(f(\alpha), s)$ the above two intersections are just the same, consequently $K_1 = K_2$. This concludes the proof.

We note that if our manifold is oriented then a $Z$ coefficient version of the theorem [2.4.4] will give us homotopy invariance of the Goresky-Hingston coproduct with $Z$ coefficients.

### 2.5 Coproduct on Floer homology

In [AS08], the authors gave a Morse theoretic description of the Chas-Sullivan product. They also described an isomorphism between Morse homology of above types of Lagrangians(sum of kinetic and potential) on free loop space and the Floer homology.
of corresponding Hamiltonians, which are Legendre dual to such Lagrangians, on
cotangent bundle. The authors used the isomorphism to show that the pair-of-pants
product on Floer homology is equivalent to the Chas-Sullivan product on free loop
space homology via the Morse theoretic Chas-Sullivan product.

A moduli space obtained by restricting \( s = \frac{1}{2} \) in \( W_{GH} \) above i.e. taking
the transversal intersection with \( \Theta_1 = \Theta \) instead of \( \{ (\Theta_s)_{s \in (0,1)} \} \)
in fact gives the Chas-Sullivan product on Morse homology. In that case our figure 2.4.2
becomes a pair-of-pants surface with a waist shape of figure eight loops \( \Theta \). This corresponds to
the Moduli space of Floer trajectories (pseudo-holomorphic maps with Hamiltonian
perturbation) from pair-paints surface to the cotangent bundle. The moduli space
gives rise to the pair-of-pants product on Floer homology.

However, our space \( W_{GH} \) and figure 2.4.3 corresponds to a Moduli space
of Floer trajectories from pair-of-paints with varying waist \( \Theta_s \) with \( s \in (0,1) \)
to the cotangent bundle. Such Moduli space also gives rise to a coproduct on relative Floer
homology of cotangent bundle which is already described in [AS10]. The authors also
expected the coproduct to be equivalent to the Goresky-Hingston coproduct. Thus
one expect that a method, analogous to the method used to prove the equivalence
between the Chas-Sullivan product and the pair-of-pants product, can be used to
prove the equivalence between the Goresky-Hingston coproduct and coproduct on
relative Floer homology described in [AS10].

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