NORM AND ESSENTIAL NORM OF A WEIGHTED COMPOSITION OPERATOR ON THE BLOCH SPACE

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Abstract. We give some new estimates for the norm and essential norm of a weighted composition operator on the Bloch space. As corollaries, we obtain some new characterizations of the boundedness and compactness of a weighted composition operator on the Bloch space.

1. Introduction

Let \( D = \{ z : |z| < 1 \} \) be the unit disk in the complex plane and \( H(D) \) be the space of all analytic functions on \( D \). For \( a \in D \), let \( \sigma_a \) be the automorphism of \( D \) exchanging 0 for \( a \), namely \( \sigma_a(z) = a - \frac{z - a}{1 - \bar{a}z} \), \( z \in D \). For \( 0 < p < \infty \), the Bergman space \( A^p \) consists of all \( f \in H(D) \) such that
\[
\| f \|_{A^p}^p = \int_D |f(z)|^p dA(z) < \infty,
\]
where \( dA(z) = \frac{1}{\pi} dx dy \) denote the normalized area Lebesgue measure. The Bloch space, denoted by \( B = B(D) \), is the space of all \( f \in H(D) \) such that
\[
\| f \|_B = \sup_{z \in D} \left( 1 - |z|^2 \right) |f'(z)| < \infty.
\]
Under the norm \( \| f \|_B = \| f(0) \| + \| f \|_B \), the Bloch space is a Banach space. From Theorem 1 of \cite{1}, we see that
\[
\| f \|_B \approx \sup_{a \in \mathbb{D}} \| f \circ \sigma_a - f(a) \|_{A^2}.
\]
See \cite{23} for more information of the Bloch space.

For \( 0 < p < \infty \), let \( H^p \) denote the Hardy space of functions \( f \in H(D) \) such that
\[
\| f \|_{H^p}^p = \sup_{0 < r < 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\} < \infty.
\]
We say that an \( f \in H(D) \) belongs to the \( BMOA \) space, if
\[
\| f \|_{BMOA}^2 = \sup_{t \subseteq \partial D} \frac{1}{|t|} \int_t |f(\zeta) - f(t)|^2 \frac{d\zeta}{2\pi} < \infty,
\]
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for any \(\{10, 12, 18, 20, 22\}\). It is well known that \(BMOA\) is a Banach space under the norm \(\|f\|_{BMOA} = |f(0)| + \|f\|_\ast\). From [6], we have
\[
\|f\|_\ast \approx \sup_{w \in \mathbb{D}} \|f \circ \sigma_w - f(w)\|_{H^2}.
\]
Throughout the paper, \(S(\mathbb{D})\) denotes the set of all analytic self-maps of \(\mathbb{D}\). Let \(u \in H(\mathbb{D})\) and \(\varphi \in S(\mathbb{D})\). For \(f \in H(\mathbb{D})\), the composition operator \(C_\varphi\) and the multiplication operator \(M_u\) are defined by
\[
(C_\varphi f)(z) = f(\varphi(z)) \quad \text{and} \quad (M_u f)(z) = u(z) f(z),
\]
respectively. The weighted composition operator \(uC_\varphi\), induced by \(u\) and \(\varphi\), is defined as follows.
\[
(uC_\varphi f)(z) = u(z) f(\varphi(z)), \quad f \in H(\mathbb{D}).
\]
It is clear that the weighted composition operator \(uC_\varphi\) is the generalization of \(C_\varphi\) and \(M_u\).

It is well known that \(C_\varphi\) is bounded on \(BMOA\) for any \(\varphi \in S(\mathbb{D})\) by Littlewood’s subordination theorem. The compactness of the operator \(C_\varphi : BMOA \to BMOA\) was studied in \([2, 5, 17, 19, 20]\). Based on results in \([2]\) and \([17]\), Wulan in \([19]\) showed that \(C_\varphi : BMOA \to BMOA\) is compact if and only if
\[
\lim_{n \to \infty} \|\varphi^n\|_\ast = 0 \quad \text{and} \quad \lim_{z \to \infty} \|\sigma_a \circ \varphi\|_\ast = 0.
\]
In \([20]\), Wulan, Zheng and Zhu further showed that \(C_\varphi : BMOA \to BMOA\) is compact if and only if \(\lim_{n \to \infty} \|\varphi^n\|_\ast = 0\). In \([8]\), Laitila gave some function theoretic characterizations for the boundedness and compactness of the operator \(uC_\varphi : BMOA \to BMOA\). In \([4]\), Colonna used the idea of \([20]\) and showed that \(uC_\varphi : BMOA \to BMOA\) is compact if and only if
\[
\lim_{n \to \infty} \|u \varphi^n\|_\ast = 0 \quad \text{and} \quad \lim_{z \to \infty} \left(\log \frac{2}{1 - |\varphi(a)|^2}\right) \|u \circ \sigma_a - u(a)\|_{H^2} = 0.
\]
Motivated by results in \([4]\), Laitila and Lindström gave the estimates for norm and essential norm of the weighted composition \(uC_\varphi : BMOA \to BMOA\) in \([9]\), among others, they showed that, under the assumption of the boundedness of \(uC_\varphi\) on \(BMOA\),
\[
\|uC_\varphi\|_{c, BMOA \to BMOA} \approx \lim \sup_{n \to \infty} \|u \varphi^n\|_\ast + \lim \sup_{|\varphi(a)| \to 1} \left(\log \frac{2}{1 - |\varphi(a)|^2}\right) \|u \circ \sigma_a - u(a)\|_{H^2}.
\]
Recall that the essential norm of a bounded linear operator \(T : X \to Y\) is its distance to the set of compact operators \(K\) mapping \(X\) into \(Y\), that is,
\[
\|T\|_{c, X \to Y} = \inf \{\|T - K\|_{X \to Y} : K \text{ is compact}\},
\]
where \(X, Y\) are Banach spaces and \(\|\cdot\|_{X \to Y}\) is the operator norm.

By Schwarz-Pick lemma, it is easy to see that \(C_\varphi\) is bounded on the Bloch space \(B\) for any \(\varphi \in S(\mathbb{D})\). The compactness of \(C_\varphi\) on \(B\) was studied in for example \([10, 12, 18, 20, 22]\). In \([20]\), Wulan, Zheng and Zhu proved that \(C_\varphi : B \to B\) is compact if and only if \(\lim_{n \to \infty} \|\varphi^n\|_B = 0\). In \([22]\), Zhao obtained the exact value for the essential norm of \(C_\varphi : B \to B\) as follows.
\[
\|C_\varphi\|_{c, B \to B} = \left(\frac{e}{2}\right) \lim \sup_{n \to \infty} \|\varphi^n\|_B.
\]
Lemma 2.3.\[16\]
\[
\|uC_\varphi\|_{C,B \rightarrow B} \\
\approx \max \left( \limsup_{|\varphi(a)| \rightarrow 1} \frac{|u(z)| |\varphi'(z)|(1 - |z|^2)}{1 - |\varphi(z)|^2}, \limsup_{|\varphi(a)| \rightarrow 1} \frac{e}{1 - |\varphi(a)|^2} |u'(z)|(1 - |z|^2) \right)
\]
In [7], the authors obtained a new estimate for the essential norm of \(uC_\varphi : B \rightarrow B\), i.e., they showed that
\[
\|uC_\varphi\|_{e,B \rightarrow B} \approx \max \left( \limsup_{j \rightarrow \infty} \|I_u(\varphi^j)\|_B, \limsup_{j \rightarrow \infty} n \|J_u(\varphi^j)\|_B \right),
\]
where \(I_u f(z) = \int_0^1 f'(\zeta) u(\zeta) d\zeta, J_u f(z) = \int_0^1 f(\zeta) u'(\zeta) d\zeta\).

Motivated by the work of [3, 4, 9, 20], the aim of this article is to give some new estimates for the norm and essential norm of the operator \(uC_\varphi : B \rightarrow B\). As corollaries, we obtain some new characterizations for the boundedness and compactness of the operator \(uC_\varphi : B \rightarrow B\).

Throughout this paper, constants are denoted by \(C\), they are positive and may differ from one occurrence to the other. The notation \(a \lesssim b\) means that there is a positive constant \(C\) such that \(a \leq Cb\). Moreover, if both \(a \lesssim b\) and \(b \lesssim a\) hold, then one says that \(a \approx b\).

2. Norm of \(uC_\varphi\) on the Bloch space

In this section we give some estimates for the norm of the operator \(uC_\varphi : B \rightarrow B\). For this purpose, we need some lemmas which stated as follows. The following lemma can be found in [23].

**Lemma 2.1.** Let \(f \in B\). Then
\[
|f(z)| \lesssim \log \frac{2}{1 - |z|^2} \|f\|_B, \quad z \in \mathbb{D}.
\]

**Lemma 2.2.** For \(2 \leq p < \infty\) and \(f \in B\),
\[
\sup_{a \in \mathbb{D}} \|f \circ \sigma_a - f(a)\|_{A^p} \approx \sup_{a \in \mathbb{D}} \|f \circ \sigma_a - f(a)\|_{A^p}.
\]

**Proof.** Using Hölder inequality, we get
\[
(2.1) \quad \sup_{a \in \mathbb{D}} \|f \circ \sigma_a - f(a)\|_{A^2} \leq \sup_{a \in \mathbb{D}} \|f \circ \sigma_a - f(a)\|_{A^p},
\]
for \(2 \leq p < \infty\).

On the other hand, there exists a constant \(C > 0\) such that (see [21, p.38])
\[
(2.2) \quad \sup_{a \in \mathbb{D}} \|f \circ \sigma_a - f(a)\|_{A^p} \leq C \|f\|_B \lesssim \sup_{a \in \mathbb{D}} \|f \circ \sigma_a - f(a)\|_{A^2},
\]
which, combined with (2.1), implies the desired result. \(\square\)

**Lemma 2.3.** [16] For \(f \in A^2\),
\[
\|f\|_{A^2}^2 \approx |f(0)|^2 + \int_{\mathbb{D}} |f'(w)|^2 (1 - |w|^2)^2 dA(w).
\]
The classical Nevanlinna counting function \( N_\varphi \) and the generalized Nevanlinna counting functions \( N_{\varphi,\gamma} \) for \( \varphi \) are defined by (see [15])

\[
N_\varphi(w) = \sum_{z \in \varphi^{-1}\{w\}} \log \frac{1}{|z|} \quad \text{and} \quad N_{\varphi,\gamma}(w) = \sum_{z \in \varphi^{-1}\{w\}} \left( \log \frac{1}{|z|} \right)^\gamma,
\]

respectively, where \( \gamma > 0 \) and \( w \in \mathbb{D}\setminus\{\varphi(0)\} \).

**Lemma 2.4.** [16] Let \( \varphi \in S(\mathbb{D}) \) and \( f \in A^2 \). Then\[
\|f \circ \varphi\|_{A^2}^2 \approx |f(\varphi(0))|^2 + \int_{\mathbb{D}} |f'(w)|^2 N_{\varphi,2}(w) dA(w).
\]

**Lemma 2.5.** [15] Let \( \varphi \in S(\mathbb{D}) \) and \( \gamma > 0 \). If \( \varphi(0) \neq 0 \) and \( 0 < r < |\varphi(0)| \), then\[
N_{\varphi,\gamma}(0) \leq \frac{1}{r^2} \int_{r\mathbb{D}} N_{\varphi,\gamma} dA.
\]

**Lemma 2.6.** Let \( \varphi \in S(\mathbb{D}) \) such that \( \varphi(0) = 0 \). If \( \sup_{0 < |w| < 1} |w|^2 N_{\varphi,2}(w) < \delta \), then
\[
N_{\varphi,2}(w) \leq \frac{4\delta}{(\log 2)^2} \left( \log \frac{1}{|w|} \right)^2
\]
when \( \frac{1}{2} \leq |w| < 1 \).

**Proof.** See the proof of Lemma 2.1 in [17]. \( \Box \)

**Lemma 2.7.** For all \( g \in A^2 \) and \( \phi \in S(\mathbb{D}) \) such \( g(0) = \phi(0) = 0 \), we have\[
\|g \circ \phi\|_{A^2} \lesssim \|\phi\|_{A^2} \|g\|_{A^2}.
\]
In particular, for all \( f \in B \), \( a \in \mathbb{D} \) and \( \varphi \in S(\mathbb{D}) \),\[
\|f \circ \varphi \circ \sigma_a - f(\varphi(a))\|_{A^2} \lesssim \|\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a\|_{A^2} \|f \circ \sigma_a - f(a)\|_{A^2}.
\]

**Proof.** Let \( \phi \in S(\mathbb{D}) \) such that \( \phi(0) = 0 \). Then,
\[
\|\sigma_z \circ \phi - \sigma_z(\phi(0))\|_{A^2} = \int_{\mathbb{D}} \frac{(1 - |z|^2)^2 |\phi(w)|^2}{1 - z\phi(w)^2} dA(w) \leq 4 \|\phi\|_{A^2}^2.
\]

From Lemmas 2.3 and 2.4, and (2.5) we obtain
\[
\|\sigma_z \circ \phi - \sigma_z(\phi(0))\|_{A^2} = \int_{\mathbb{D}} |(\sigma_z \circ \phi)'|^2 (\log \frac{1}{|w|})^2 dA(w)
= \int_{\mathbb{D}} N_{\sigma_z \circ \phi,2} dA(w) \leq 4 \|\phi\|_{A^2}^2.
\]

For \( z \in \mathbb{D} \setminus \{0\} \), from Lemma 4.2 in [16] and Lemma 2.5, we have
\[
|z|^2 N_{\phi,2}(z) = |z|^2 N_{\sigma_z \circ \phi,2}(0) \leq \int_{|z|\mathbb{D}} N_{\sigma_z \circ \phi,2}(w) dA(w) \leq 4 \|\phi\|_{A^2}^2.
\]

So, by Lemma 2.6 we get
\[
N_{\phi,2}(z) \leq \frac{16}{(\log 2)^2} \|\phi\|_{A^2}^2 (\log \frac{1}{|z|})^2,
\]
for \( z \in \mathbb{D} \setminus \frac{1}{2}\mathbb{D} \). Thus,
\[
\int_{\mathbb{D} \setminus \frac{1}{2}\mathbb{D}} |g'(z)|^2 N_{\phi,2}(z) dA(z) \leq \frac{16}{(\log 2)^2} \|\phi\|_{A^2}^2 \|g\|_{A^2}^2.
\]
In addition, for $z \in \mathbb{D}$ and $g \in A^2$, from Theorems 4.14 and 4.28 of [23], we have $|g'(z)| \leq (1 - |z|^2)^{-2} \|g\|_{A^2}$. Then,

$$\int_{\mathbb{D}} |g'(z)|^2 N_{\phi,a}(z)dA(z) \leq 16\|g\|^2_{A^2} \int_{\mathbb{D}} N_{\phi,a}(z)dA(z) \leq 16\|\phi\|^2_{A^2} \|g\|^2_{A^2}.$$  

(2.10) \hspace{1cm} \int_{\mathbb{D}} |g'(z)|^2 N_{\phi,a}(z)dA(z) \leq 16\|g\|^2_{A^2} \int_{\mathbb{D}} N_{\phi,a}(z)dA(z) \leq 16\|\phi\|^2_{A^2} \|g\|^2_{A^2}.

Since $g(0) = 0$, by Lemma 2.4 we have

$$\|g \circ \phi\|^2_{A^2} \approx \int_{\mathbb{D}} |g'(z)|^2 N_{\phi,a}(z)dA(z).$$

(2.11) \hspace{1cm} \|g \circ \phi\|^2_{A^2} \approx \int_{\mathbb{D}} |g'(z)|^2 N_{\phi,a}(z)dA(z).

Combine with (2.9), (2.10) and (2.11), we obtain

$$\|g \circ \phi\|_{A^2} \lesssim \|\phi\|_{A^2} \|g\|_{A^2},$$

as desired. In particular, for all $f \in B$, $a \in \mathbb{D}$ and $\varphi \in S(\mathbb{D})$, if we set

$$g = f \circ \sigma_{\varphi(a)} - f(\varphi(a)), \quad \phi = \sigma_{\varphi(a)} \circ \varphi \circ \sigma_a,$$

we get

$$\|f \circ \varphi \circ \sigma_a - f(\varphi(a))\|_{A^2} \lesssim \|\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a\|_{A^2} \|f \circ \sigma_a - f(a)\|_{A^2}.$$  

The proof is complete. \hfill \qed

For the simplicity of the rest of this paper, we introduce the following abbreviation. Set

$$\alpha(u, \varphi, a) = |u(a)| \cdot \|\sigma_{\varphi(a)} \circ \varphi \circ \sigma_a\|_{A^2},$$

$$\beta(u, \varphi, a) = \log \frac{2}{1 - |\varphi(a)|^2} \|u \circ \sigma_a - u(a)\|_{A^2},$$

where $a \in \mathbb{D}$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$.

**Theorem 2.8.** Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then

$$\|uC_\varphi\|_{B \to B} \approx |u(0)| \log \frac{2}{1 - |\varphi(0)|^2} + \sup_{a \in \mathbb{D}} \alpha(u, \varphi, a) + \sup_{a \in \mathbb{D}} \beta(u, \varphi, a).$$

**Proof.** First we give the upper estimate for $\|uC_\varphi\|_{B \to B}$. For all $f \in B$, using the triangle inequality, we get

$$\|(uC_\varphi f) \circ \sigma_a - (uC_\varphi f)(a)\|_{A^2}$$

$$= \|(u \circ \sigma_a - u(a)) \cdot (f \circ \varphi \circ \sigma_a - f(\varphi(a))) + u(a)(f \circ \varphi \circ \sigma_a - f(\varphi(a))) + (u \circ \sigma_a - u(a))f(\varphi(a))\|_{A^2}$$

$$\leq \|(u \circ \sigma_a - u(a)) \cdot (f \circ \varphi \circ \sigma_a - f(\varphi(a)))\|_{A^2}$$

$$+ |u(a)||f \circ \varphi \circ \sigma_a - f(\varphi(a))\|_{A^2} + |f(\varphi(a))||u \circ \sigma_a - u(a)||_{A^2}.$$  

(2.12) \hspace{1cm} \|(uC_\varphi f) \circ \sigma_a - (uC_\varphi f)(a)\|_{A^2}$$

By Lemmas 2.1 and 2.7, we have

$$|u(a)||f \circ \varphi \circ \sigma_a - f(\varphi(a))\|_{A^2} + |f(\varphi(a))||u \circ \sigma_a - u(a)||_{A^2}$$

$$\lesssim \alpha(u, \varphi, a)||f \circ \sigma_a - f(a)||_{A^2} + \frac{2}{1 - |\varphi(a)|^2} \|u \circ \sigma_a - u(a)\|^2_{A^2} \|f\|_B$$

(2.13) \hspace{1cm} \lesssim \alpha(u, \varphi, a) + \beta(u, \varphi, a)\|f\|_B.$$
From Lemmas 2.1 and 2.2, we get

\[ \sup_{a \in \mathbb{D}} \| (u \circ \sigma_a - u(a)) \cdot (f \circ \varphi \circ \sigma_a - f(\varphi(a))) \|_{A^2} \]

\[ \leq \sup_{a \in \mathbb{D}} \log 2 \| u \circ \sigma_a - u(a) \|_{A^2} \| f \circ \varphi \circ \sigma_a - f(\varphi(a)) \|_{A^2} \]

\[ \leq \sup_{a \in \mathbb{D}} \log \frac{2}{1 - |\varphi(a)|^2} \| u \circ \sigma_a - u(a) \|_{A^2} \| f \circ \varphi \circ \sigma_a - f(\varphi(a)) \|_{A^2} \]

(2.14) \[ \leq \sup_{a \in \mathbb{D}} \beta(u, \varphi, a) \| f \circ \varphi \|_B \leq \sup_{a \in \mathbb{D}} \beta(u, \varphi, a) \| f \|_B. \]

Then, by (2.12), (2.13) and (2.14), we have

\[ \sup_{a \in \mathbb{D}} \| (uC_\varphi f) \circ \sigma_a - (uC_\varphi f)(a) \|_{A^2} \leq \left( \sup_{a \in \mathbb{D}} \alpha(u, \varphi, a) + \sup_{a \in \mathbb{D}} \beta(u, \varphi, a) \right) \| f \|_B. \]

In addition, by Lemma 2.1, \( \| (uC_\varphi f)(0) \| \leq \| u(0) \| \log \frac{2}{1 - |\varphi(0)|^2} \| f \|_B \), we get

\[ \| uC_\varphi f \|_B \approx \| (uC_\varphi f)(0) \| + \sup_{a \in \mathbb{D}} \| (uC_\varphi f) \circ \sigma_a - (uC_\varphi f)(a) \|_{A^2} \]

\[ \leq \| u(0) \| \log \frac{2}{1 - |\varphi(0)|^2} \| f \|_B + \sup_{a \in \mathbb{D}} \alpha(u, \varphi, a) \| f \|_B + \sup_{a \in \mathbb{D}} \beta(u, \varphi, a) \| f \|_B, \]

which implies

(2.15) \[ \| uC_\varphi \|_{B \to B} \leq \| u(0) \| \log \frac{2}{1 - |\varphi(0)|^2} + \sup_{a \in \mathbb{D}} \alpha(u, \varphi, a) + \sup_{a \in \mathbb{D}} \beta(u, \varphi, a). \]

Next we find the lower estimate for \( \| uC_\varphi \|_{B \to B} \). Let \( f = 1 \). It is easy to see that \( \| u \|_B \leq \| uC_\varphi \|_{B \to B} \). For any \( a \in \mathbb{D} \), set

(2.16) \[ f_a(z) = \sigma_{\varphi(a)}(z) - \varphi(a), \quad z \in \mathbb{D}. \]

Then, \( f_a(0) = 0 \), \( f_a(\varphi(a)) = -\varphi(a) \), \( \| f_a \|_B \leq 4 \) and \( \| f_a \|_\infty \leq 2 \). Using triangle inequality, we get

\[ \alpha(u, \varphi, a) = |u(a)| \cdot \| \sigma_{\varphi(a)} \circ \varphi \circ \sigma_a - \varphi(a) + \varphi(a) \|_{A^2} \]

\[ = |u(a)| \cdot \| f_a \circ \varphi \circ \sigma_a = f_a(\varphi(a)) \|_{A^2} \]

\[ \leq \| (u \circ \sigma_a - u(a)) \cdot f_a \circ \varphi \circ \sigma_a \|_{A^2} \]

\[ + \| (u \circ \sigma_a) \cdot f_a \circ \varphi \circ \sigma_a - u(a) f_a(\varphi(a)) \|_{A^2} \]

\[ \leq 2 \| u \circ \sigma_a - u(a) \|_{A^2} + \| (uC_\varphi f_a) \circ \sigma_a - (uC_\varphi f_a)(a) \|_{A^2} \]

\[ \leq 2 \| u \|_B + 4 \| uC_\varphi \|_{B \to B} \leq 6 \| uC_\varphi \|_{B \to B}. \]

Set

(2.18) \[ h_a(z) = \log \frac{2}{1 - \varphi(a)z}, \quad z \in \mathbb{D}. \]
Then, \( h_a \in \mathcal{B} \), \( h_a(\varphi(a)) = \log \frac{2}{1 - |\varphi(a)|^2} \) and \( \sup_{a \in \mathbb{D}} \| h_a \|_{\mathcal{B}} \leq 2 + \log 2 \). Using triangle inequality and Lemma 2.7, we obtain

\[
\beta(u, \varphi, a) = \left\| \log \frac{2}{1 - |\varphi(a)|^2} \cdot (u \circ \sigma_a - u(a)) \right\|_{A^2} \\
= \left\| h_a(\varphi(a)) (u \circ \sigma_a - u(a)) \right\|_{A^2} \\
\leq \left\| (h_a \circ \varphi \circ \sigma_a - h_a(\varphi(a))) \cdot (u \circ \sigma_a - u(a)) \right\|_{A^2} \\
+ \left\| u(a) (h_a \circ \varphi \circ \sigma_a - h_a(\varphi(a))) \right\|_{A^2} \\
\approx \left\| (h_a \circ \varphi \circ \sigma_a - h_a(\varphi(a))) \cdot (u \circ \sigma_a - u(a)) \right\|_{A^2} \\
+ 2 + \log 2 \| u \|_{B \rightarrow B} \leq \left\| u \right\|_{B \rightarrow B}.
\]

By Lemmas 2.2 and 2.7, we have

\[
\left\| (h_a \circ \varphi \circ \sigma_a - h_a(\varphi(a))) \cdot (u \circ \sigma_a - u(a)) \right\|_{A^2} \leq \left\| h_a \circ \varphi \circ \sigma_a - h_a(\varphi(a)) \right\|_{A^2} \\| u \circ \sigma_a - u(a) \|_{A^2} \\
\approx \left\| (h_a \circ \varphi \circ \sigma_a - h_a(\varphi(a))) \cdot (u \circ \sigma_a - u(a)) \right\|_{A^2} + \alpha(u, \varphi, a) \| h_a \circ \varphi \circ \sigma_a - h_a(\varphi(a)) \|_{A^2} \\
\approx \left\| (h_a \circ \varphi \circ \sigma_a - h_a(\varphi(a))) \cdot (u \circ \sigma_a - u(a)) \right\|_{A^2} \\
+ (2 + \log 2) \| u \|_{B \rightarrow B} + (2 + \log 2) \alpha(u, \varphi, a).
\]

Combining (2.17), (2.19) and (2.20), we have

\[
\sup_{a \in \mathbb{D}} \alpha(u, \varphi, a) + \sup_{a \in \mathbb{D}} \beta(u, \varphi, a) \approx \| u \|_{B \rightarrow B}.
\]

Moreover, since

\[
|u(0)| \log \frac{2}{1 - |\varphi(0)|^2} = |(u C_\varphi h_0)(0)| \leq (2 + \log 2) \| u \|_{B \rightarrow B} \leq \| u \|_{B \rightarrow B}.
\]

Therefore,

\[
|u(0)| \log \frac{2}{1 - |\varphi(0)|^2} + \sup_{a \in \mathbb{D}} \alpha(u, \varphi, a) + \sup_{a \in \mathbb{D}} \beta(u, \varphi, a) \approx \| u \|_{B \rightarrow B}.
\]

We complete the proof of the theorem. \( \square \)

As a corollary, we obtain the following new characterization of the boundedness of \( u C_\varphi : \mathcal{B} \rightarrow \mathcal{B} \).

**Corollary 2.9.** Let \( u \in H(\mathbb{D}) \) and \( \varphi \in S(\mathbb{D}) \). Then \( u C_\varphi : \mathcal{B} \rightarrow \mathcal{B} \) is bounded if and only if

\[
\sup_{a \in \mathbb{D}} |u(a)| \cdot \| \sigma_\varphi(a) \circ \varphi \circ \sigma_a \|_{A^2} < \infty
\]

and

\[
\sup_{a \in \mathbb{D}} \log \frac{2}{1 - |\varphi(a)|^2} \| u \circ \sigma_a - u(a) \|_{A^2} < \infty.
\]

In particular, when \( \varphi(z) = z \), we obtain the estimate of the norm of the multiplication \( M_u : \mathcal{B} \rightarrow \mathcal{B} \).

**Corollary 2.10.** Let \( u \in H(\mathbb{D}) \). Then

\[
\| M_u \|_{B \rightarrow B} \approx |u(0)| \log 2 + \sup_{a \in \mathbb{D}} \log \frac{2}{1 - |a|^2} \| u \circ \sigma_a - u(a) \|_{A^2}.
\]
Lemma 2.11. Suppose that $uC_\varphi : \mathcal{B} \to \mathcal{B}$ is bounded. Then

$$\sup_{a \in \mathbb{D}} \|uC_\varphi(\sigma_{\varphi(a)} - \varphi(a))\|_\mathcal{B} \leq \sup_{n \geq 0} \|u\varphi^n\|_\mathcal{B} \tag{2.21}$$

and

$$\limsup_{|\varphi(a)| \to 1} \|uC_\varphi(\sigma_{\varphi(a)} - \varphi(a))\|_\mathcal{B} \leq \limsup_{n \to \infty} \|u\varphi^n\|_\mathcal{B}. \tag{2.22}$$

Proof. From Corollary 2.1 of [3], we see that

$$\sup_{a \in \mathbb{D}} \|uC_\varphi\varphi(a)|_\mathcal{B} \approx \sup_{n \geq 0} \|u\varphi^n\|_\mathcal{B}.$$

Then (2.21) follows immediately.

The Taylor expansion of $\sigma_{\varphi(a)} - \varphi(a)$ is

$$\sigma_{\varphi(a)} - \varphi(a) = -\sum_{n=0}^{\infty} \left(\frac{\varphi(a)}{1 - |\varphi(a)|^2}\right)^n z^{n+1}.$$

Then, by the boundedness of $uC_\varphi : \mathcal{B} \to \mathcal{B}$ we have

$$\|uC_\varphi(\sigma_{\varphi(a)} - \varphi(a))\|_\mathcal{B} \leq (1 - |\varphi(a)|^2) \sum_{n=0}^{\infty} |\varphi(a)|^n \|u\varphi^{n+1}\|_\mathcal{B}$$

For each $N$, set

$$M_1 := \sum_{n=0}^{N} |\varphi(a)|^n \|u\varphi^{n+1}\|_\mathcal{B}.$$

Then we get

$$\|uC_\varphi(\sigma_{\varphi(a)} - \varphi(a))\|_\mathcal{B} \leq (1 - |\varphi(a)|^2) \sum_{n=0}^{N} |\varphi(a)|^n \|u\varphi^{n+1}\|_\mathcal{B} + ((1 - |\varphi(a)|^2) \sum_{n=N+1}^{\infty} |\varphi(a)|^n \|u\varphi^{n+1}\|_\mathcal{B}$$

$$\leq M_1 (1 - |\varphi(a)|^2) + ((1 - |\varphi(a)|^2) \sum_{n=N+1}^{\infty} |\varphi(a)|^n \sup_{n \geq N+1} \|u\varphi^{n+1}\|_\mathcal{B}$$

$$\leq M_1 (1 - |\varphi(a)|^2) + 2 \sup_{n \geq N+1} \|u\varphi^{n+1}\|_\mathcal{B}.$$

Taking $\limsup_{|\varphi(a)| \to 1}$ to the last inequality and then letting $N \to \infty$, we get the desired result. $\square$

Proposition 2.12. Let $\varphi \in S(\mathbb{D})$ and $u \in H(\mathbb{D})$. The following statements hold.

(i) For $a \in \mathbb{D}$, let $f_a(z) = \sigma_{\varphi(a)} - \varphi(a)$. Then

$$\alpha(u, \varphi, a) \leq \frac{\beta(u, \varphi, a)}{\log \frac{1 - |\varphi(a)|^2}{1 - |\varphi(a)|^2}} + \|(uC_\varphi f_a) \circ \sigma_a - (uC_\varphi f_a)(a)\|_{A^2}.$$

(ii) For $a \in \mathbb{D}$, let $g_a = \frac{h^2_a}{\sigma_a(\varphi(a))}$, where $h_a(z) = \log \frac{2}{1 - |\varphi(a)|^2}$. Then

$$\beta(u, \varphi, a) \leq \alpha(u, \varphi, a) + \|(g_a \circ \sigma_a - g_a(\varphi(a))) \cdot (u \circ \sigma_a - u(a))\|_{A^2} + \|(uC_\varphi g_a) \circ \sigma_a - (uC_\varphi g_a)(a)\|_{A^2}.$$
(iii) For all $f \in B$ and $a \in \mathbb{D}$,
\[
\|uC_{\varphi}f \circ \sigma_a - (uC_{\varphi}f)(a)\|_{A^2} \lesssim \|f \circ \sigma_a - u(a)\|_{A^2} \cdot \|f \circ \varphi \circ \sigma_a - f(\varphi(a))\|_{A^2} + \left(\alpha(u, \varphi, a) + \beta(u, \varphi, a)\right)\|f\|_{B}.
\]

(iv) For all $f \in B$ and $a \in \mathbb{D}$,
\[
\|f\|_{B} \min \left\{ \sup_{w \in \mathbb{D}} \beta(u, \varphi, w), \frac{\|uC_{\varphi}\|_{B \rightarrow B}}{\sqrt{\log \frac{2}{1 - |\varphi(a)|^2}}} \right\}.
\]

Proof. (i) It is easy to see that $\|f_a \circ \varphi \circ \sigma_a\|_{\infty} \leq 2$. For any $a \in \mathbb{D}$, we get
\[
\alpha(u, \varphi, a) = |u(a)||f \circ \varphi \circ \sigma_a - f(\varphi(a))|_{A^2}
\]
\[
= \|(u \circ \sigma_a - u(a)) \cdot f_a \circ \varphi \circ \sigma_a - (uC_{\varphi}f_a)(a)\|_{A^2} + \|(uC_{\varphi}f_a)(a)\|_{A^2}
\]
\[
\lesssim \|u \circ \sigma_a - u(a)\|_{A^2} + \|(uC_{\varphi}f_a)(a)\|_{A^2}.
\]

(ii) It is obvious that $g_a(\varphi(a)) = \log \frac{2}{1 - |\varphi(a)|^2}$. Since $(g_a \circ \sigma_a - g_a(\varphi(a)))(0) = 0,$
\[
g_a \circ \varphi \circ \sigma_a - g_a(\varphi(a)) = g_a \circ \sigma_a \circ (g_a \circ \varphi \circ \sigma_a - g_a(\varphi(a)),
\]
by Lemma 2.7 and the fact that $\sup_{a \in \mathbb{D}} \|g_a\|_{B} < \infty$ we obtain
\[
|u(a)||g_a \circ \varphi \circ \sigma_a - g_a(\varphi(a))|_{A^2} \lesssim \alpha(u, \varphi, a) \sup_{a \in \mathbb{D}} \|g_a\|_{B} \lesssim \alpha(u, \varphi, a).
\]

By the triangle inequality we get
\[
\beta(u, \varphi, a) = \|g_a(\varphi(a)) \cdot (u \circ \sigma_a - u(a))\|_{A^2}
\]
\[
= \|(g_a \circ \varphi \circ \sigma_a - g_a(\varphi(a))) \cdot (u \circ \sigma_a - u(a))
\]
\[
+ u(a)(g_a \circ \varphi \circ \sigma_a - g_a(\varphi(a))) - (u(a)g_a \circ \varphi \circ \sigma_a - u(a)g_a(\varphi(a)))\|_{A^2}
\]
\[
\lesssim \|(g_a \circ \varphi \circ \sigma_a - g_a(\varphi(a))) \cdot (u \circ \sigma_a - u(a))\|_{A^2}
\]
\[
+ \|u(a)||g_a \circ \varphi \circ \sigma_a - g_a(\varphi(a))\|_{A^2} + \|(uC_{\varphi}g_a)(a)\circ \sigma_a - (uC_{\varphi}g_a)(a)\|_{A^2}
\]
\[
\lesssim \alpha(u, \varphi, a) + \|g_a \circ \varphi \circ \sigma_a - g_a(\varphi(a))\|_{A^2} + \|(uC_{\varphi}g_a)(a)\circ \sigma_a - (uC_{\varphi}g_a)(a)\|_{A^2},
\]
as desired.

(iii) See the proof of Theorem 2.8.

(iv) Using the fact that $\log 2 \leq \log \frac{2}{1 - |\varphi(a)|^2}$ and Theorem 2.8, we have
\[
\sup_{a \in \mathbb{D}} \|u \circ \sigma_a - u(a)\|_{A^2} \lesssim \sup_{a \in \mathbb{D}} \beta(u, \varphi, a) \lesssim \|uC_{\varphi}\|_{B \rightarrow B}.
\]

By Lemma 2.2 and Hölder inequality, we obtain
\[
\|f \circ \varphi \circ \sigma_a - f(\varphi(a))\|_{A^2}^2
\]
\[
= \|(u \circ \sigma_a - u(a)) \cdot (f \circ \varphi \circ \sigma_a - f(\varphi(a)))\|_{A^2}^2
\]
\[
\leq \|u \circ \sigma_a - u(a)\|_{A^2}^2 \|f \circ \varphi \circ \sigma_a - f(\varphi(a))\|_{A^2}^2
\]
\[
\lesssim \beta(u, \varphi, a) \sup_{a \in \mathbb{D}} \|u \circ \sigma_a - u(a)\|_{A^2} \sup_{a \in \mathbb{D}} \|f \circ \varphi \circ \sigma_a - f(\varphi(a))\|_{A^2}^2 / \log \frac{2}{1 - |\varphi(a)|^2}.
\]
Then, by the boundedness of $C_\varphi$ on $\mathcal{B}$ and (2.23), we obtain
\[
\beta(u, \varphi, a) \sup_{a \in \mathcal{D}} \| u \circ \sigma_a - u(a) \|_{A^2} \sup_{a \in \mathcal{D}} \| f \circ \varphi \circ \sigma_a - f(\varphi(a)) \|_{A^2} / \log \frac{2}{1 - |\varphi(a)|^2} 
\lesssim (\sup_{a \in \mathcal{D}} \beta(u, \varphi, a))^2 \sup_{a \in \mathcal{D}} \| f \circ \varphi \circ \sigma_a - f(\varphi(a)) \|_{A^2} / \log \frac{2}{1 - |\varphi(a)|^2} 
\lesssim \sup_{a \in \mathcal{D}} \| f \circ \varphi \circ \sigma_a - f(\varphi(a)) \|_{A^2} \min \left\{ \sup_{a \in \mathcal{D}} \beta(u, \varphi, a), \frac{\| uC_\varphi \|_{\mathcal{B} \to \mathcal{B}}}{\sqrt{\log \frac{2}{1 - |\varphi(a)|^2}}} \right\}^2 
\lesssim \| f \|_{A^2}^2 \min \left\{ \sup_{a \in \mathcal{D}} \beta(u, \varphi, a), \frac{\| uC_\varphi \|_{\mathcal{B} \to \mathcal{B}}}{\sqrt{\log \frac{2}{1 - |\varphi(a)|^2}}} \right\}^2.
\]

The proof is complete. \qed

**Theorem 2.13.** Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Suppose that $uC_\varphi$ is bounded on $\mathcal{B}$. Then
\[
\| uC_\varphi \|_{\mathcal{B} \to \mathcal{B}} \approx |u(0)| \log \frac{2}{1 - |\varphi(0)|^2} + \sup_{a \in \mathcal{D}} \| u\varphi^n \|_{\mathcal{B}} + \beta(u, \varphi, a).
\]

**Proof.** For any $f \in \mathcal{B}$, by (iii) and (iv) of Proposition 2.12, we get
\[
\| uC_\varphi f \|_{\beta} \lesssim \sup_{a \in \mathcal{D}} (\alpha(u, \varphi, a) + \beta(u, \varphi, a)) \| f \|_{\mathcal{B}}.
\]

By Lemma 2.11 and (i) of Proposition 2.12, we have
\[
\alpha(u, \varphi, a) \lesssim \beta(u, \varphi, a) / \log \frac{2}{1 - |\varphi(a)|^2} + \sup_{a \in \mathcal{D}} \| uC_\varphi f_a \|_{\mathcal{B}} 
\lesssim \beta(u, \varphi, a) + \sup_{n \geq 0} \| u\varphi^n \|_{\mathcal{B}}.
\]

Thus,
\[
\| uC_\varphi f \|_{\beta} \lesssim \left( \sup_{a \in \mathcal{D}} \beta(u, \varphi, a) + \sup_{n \geq 0} \| u\varphi^n \|_{\mathcal{B}} \right) \| f \|_{\mathcal{B}}.
\]

In addition, $(uC_\varphi f)(0) = |u(0)||f(\varphi(0))| \lesssim |u(0)| \log \frac{2}{1 - |\varphi(0)|^2} \| f \|_{\mathcal{B}}$. Therefore,
\[
\| uC_\varphi \|_{\mathcal{B} \to \mathcal{B}} \lesssim |u(0)| \log \frac{2}{1 - |\varphi(0)|^2} + \sup_{n \geq 0} \| u\varphi^n \|_{\mathcal{B}} + \beta(u, \varphi, a).
\]

On the other hand, let $f(z) = z^n$. Then $f \in \mathcal{B}$ for all $n \geq 0$. Thus
\[
\sup_{n \geq 0} \| u\varphi^n \|_{\mathcal{B}} = \sup_{n \geq 0} \| (uC_\varphi)z^n \|_{\mathcal{B}} \leq \| uC_\varphi \|_{\mathcal{B} \to \mathcal{B}} < \infty,
\]

which, together with Theorem 2.8, implies
\[
|u(0)| \log \frac{2}{1 - |\varphi(0)|^2} + \sup_{n \leq \beta(u, \varphi, a) \lesssim \| uC_\varphi \|_{\mathcal{B} \to \mathcal{B}}.
\]

The proof of the theorem is complete. \qed
Corollary 2.14. Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then $uC_\varphi : \mathcal{B} \to \mathcal{B}$ is bounded if and only if

$$\sup_{n \geq 0} \|u \varphi^n\|_B < \infty \text{ and } \sup_{a \in \mathbb{D}} \log \frac{2}{1 - |\varphi(a)|^2} \|u \circ \sigma_n - u(a)\|_A^2 < \infty.$$ 

3. Essential norm of $uC_\varphi$ on the Bloch space

In this section we characterize the essential norm of the weighted composition operator $uC_\varphi : \mathcal{B} \to \mathcal{B}$ by using various forms, especially we will use the Bloch norm $\|\cdot\|_B$. For $t \in (0, 1)$, we define

$$E(\varphi, a, t) = \{z \in \mathbb{D} : |(\sigma_\varphi(a) \circ \varphi \circ \sigma_a)(z)| > t\}.$$ 

Similarly to the proof of Lemma 9 of [9], we get the following result. Since the proof is similar, we omit the details.

Lemma 3.1. Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then

$$\widetilde{\gamma} := \limsup_{r \to 1} \limsup_{t \to 1} \sup_{|\varphi(a)| \leq r} \left( \int_{E(\varphi, a, t)} |u(\sigma_a(z))|^4 dA(z) \right)^{1/4} \lesssim \limsup_{n \to \infty} \|u \varphi^n\|_B.$$ 

Theorem 3.2. Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$ such that $uC_\varphi : \mathcal{B} \to \mathcal{B}$ is bounded. Then

$$\|uC_\varphi\|_{e, \mathcal{B} \to \mathcal{B}} \approx \limsup_{n \to \infty} \|u \varphi^n\|_B + \limsup_{|\varphi(a)| \to 1} \|uC_\varphi g_a\|_B$$

$$\approx \widetilde{\alpha} + \widetilde{\beta} + \widetilde{\gamma}$$

$$\approx \widetilde{\alpha} + \limsup_{|\varphi(a)| \to 1} \|uC_\varphi g_a\|_B + \widetilde{\gamma}$$

$$\approx \limsup_{n \to \infty} \|u \varphi^n\|_B + \widetilde{\beta},$$

where $\widetilde{\alpha} = \limsup_{|\varphi(a)| \to 1} \alpha(u, \varphi, a)$, $\widetilde{\beta} = \limsup_{|\varphi(a)| \to 1} \beta(u, \varphi, a)$ and

$$g_a(z) = \left( \log \frac{2}{1 - \varphi(a)z} \right)^2 \left( \log \frac{2}{1 - |\varphi(a)|^2} \right)^{-1}.$$

Proof. Set $f_n(z) = z^n$. It is well known that $f_n \in \mathcal{B}$ and $f_n \to 0$ weakly in $\mathcal{B}$ as $n \to \infty$. Then

$$\|uC_\varphi\|_{e, \mathcal{B} \to \mathcal{B}} \gtrsim \limsup_{n \to \infty} \|uC_\varphi f_n\|_B = \limsup_{n \to \infty} \|u \varphi^n\|_B.$$ 

Choose $a_n \in \mathbb{D}$ such that $|\varphi(a_n)| \to 1$ as $n \to \infty$. It is easy to check $g_{a_n}$ are uniformly bounded in $\mathcal{B}$ and converges weakly to zero in $\mathcal{B}$ (see [14]). By these facts we obtain

$$\|uC_\varphi\|_{e, \mathcal{B} \to \mathcal{B}} \gtrsim \limsup_{n \to \infty} \|uC_\varphi g_{a_n}\|_B = \limsup_{|\varphi(a)| \to 1} \|uC_\varphi g_{a_n}\|_B.$$ 

By (3.1) and (3.2), we obtain

$$\|uC_\varphi\|_{e, \mathcal{B} \to \mathcal{B}} \gtrsim \limsup_{n \to \infty} \|u \varphi^n\|_B + \limsup_{|\varphi(a)| \to 1} \|uC_\varphi g_{a_n}\|_B.$$ 

From (i) of Proposition 2.12, we see that

$$\alpha(u, \varphi, a) \lesssim \frac{\beta(u, \varphi, a)}{\log \frac{2}{1 - |\varphi(a)|^2}} + \|uC_\varphi f_a\|_B,$$
which together with Lemma 2.11 implies that

\[
(3.4) \quad \alpha = \limsup_{|\varphi(a)| \to 1} \alpha(u, \varphi, a) \leq \limsup_{|\varphi(a)| \to 1} \|uC_{\varphi}f_a\|_B \leq \limsup_{n \to \infty} \|u_{\varphi^n}\|_B.
\]

From (ii) and (iv) of Proposition 2.12, we see that

\[
\beta(u, \varphi, a) \leq \alpha(u, \varphi, a) + \|g_a \circ \varphi \circ \sigma_a - g_a(\varphi(a))\|_{A^2} + \|\sum_{k=1}^{\infty} (u \circ \sigma_a - u(a))\|_{A^2}
\]

\[
\leq \alpha(u, \varphi, a) + \|g_a\|_B \frac{\|uC_{\varphi}\|_{g \to B}}{\sqrt{\log \frac{1}{1 - |\varphi(a)|^r}}} + \|uC_{\varphi}g_a\|_B,
\]

which implies that

\[
(3.5) \quad \beta = \limsup_{|\varphi(a)| \to 1} \beta(u, \varphi, a) \leq \bar{\alpha} + \limsup_{|\varphi(a)| \to 1} \|uC_{\varphi}g_a\|_B.
\]

By Lemma 3.1, (3.3), (3.4) and (3.5), we have

\[
\|uC_{\varphi}\|_{e, B \to B} \geq \alpha + \gamma + \limsup_{|\varphi(a)| \to 1} \|uC_{\varphi}g_a\|_B
\]

\[
\geq \bar{\alpha} + \bar{\gamma} + \bar{\beta},
\]

and

\[
\|uC_{\varphi}\|_{e, B \to B} \geq \limsup_{n \to \infty} \|u_{\varphi^n}\|_B + \bar{\alpha} + \limsup_{|\varphi(a)| \to 1} \|uC_{\varphi}g_a\|_B
\]

\[
\geq \limsup_{n \to \infty} \|u_{\varphi^n}\|_B + \bar{\beta}.
\]

Next we give the upper estimate for \(\|uC_{\varphi}\|_{e, B \to B}\). For \( n \geq 0 \), we define the linear operator on \( B \) by \( (K_n f)(z) = f(\frac{n}{n+1}) \). It is easy to check that \( K_n \) is a compact operator on \( B \). Thus

\[
\|uC_{\varphi}\|_{e, B \to B} \leq \limsup_{n \to \infty} \sup_{\|f\|_B \leq 1} \|uC_{\varphi}(I - K_n)f\|_B,
\]

where \( I \) is the identity operator. Let \( S_n = I - K_n \). Then,

\[
(3.6) \quad \|uC_{\varphi}\|_{e, B \to B} \leq \liminf_{n \to \infty} \|uC_{\varphi}S_n\|_B
\]

\[
= \liminf_{n \to \infty} \sup_{\|f\|_B \leq 1} (\|u(0)(S_n f)(\varphi(0))\| + \|uC_{\varphi}S_n f\|_B)
\]

\[
= \liminf_{n \to \infty} \sup_{\|f\|_B \leq 1} \|uC_{\varphi}S_n f\|_B.
\]

Let \( f \in B \) such that \( \|f\|_B \leq 1 \). Fix \( n \geq 0 \), \( r \in (0, 1) \) and \( t \in (\frac{1}{2}, 1) \). Then

\[
\|uC_{\varphi}S_n f\|_B \approx \sup_{a \in B} \|u_{\varphi^n}(uC_{\varphi}S_n f) \circ \sigma_a - (uC_{\varphi}S_n f)(a)\|_{A^2}
\]

\[
\leq \sup_{|\varphi(a)| \leq r} \|u_{\varphi^n}(uC_{\varphi}S_n f) \circ \sigma_a - (uC_{\varphi}S_n f)(a)\|_{A^2}
\]

\[
+ \sup_{|\varphi(a)| > r} \|u_{\varphi^n}(uC_{\varphi}S_n f) \circ \sigma_a - (uC_{\varphi}S_n f)(a)\|_{A^2}.
\]

\[
(3.7) \quad \|uC_{\varphi}S_n f\|_B \leq \sup_{|\varphi(a)| \leq r} \|u_{\varphi^n}(uC_{\varphi}S_n f) \circ \sigma_a - (uC_{\varphi}S_n f)(a)\|_{A^2}
\]

\[
+ \sup_{|\varphi(a)| > r} \|u_{\varphi^n}(uC_{\varphi}S_n f) \circ \sigma_a - (uC_{\varphi}S_n f)(a)\|_{A^2}.
\]
By (iii) and (iv) of Proposition 2.11, we have
\[
\sup_{|\varphi(a)| > r} \|(uC_\varphi S_n f) \circ \sigma_a - (uC_\varphi S_n f)(a)\|_{A^2}
\leq \sup_{|\varphi(a)| \leq r} \left( \alpha(u, \varphi, a) + \beta(u, \varphi, a) + \frac{\|uC_\varphi\|_{B \rightarrow B}}{\sqrt{\log 1/|\varphi(a)|^2}} \right).
\]  
(3.8)

In addition,
\[
\sup_{|\varphi(a)| \leq r} \|(uC_\varphi S_n f) \circ \sigma_a - (uC_\varphi S_n f)(a)\|_{A^2}
\leq \sup_{|\varphi(a)| \leq r} \left( \|S_n f\|_B \sup_{|\varphi(a)| > r} \left( \alpha(u, \varphi, a) + \beta(u, \varphi, a) + \frac{\|uC_\varphi\|_{B \rightarrow B}}{\sqrt{\log 1/|\varphi(a)|^2}} \right) \right.
\]
\[
+ \left. \|u \circ \sigma_a \cdot ((C_\varphi S_n f) \circ \sigma_a - (C_\varphi S_n f)(a))\|_{A^2} \right)
\leq \|u\|_B \max_{|w| \leq r} \|S_n f(w)\| + I_1^{1/2} + I_2^{1/2},
\]
(3.9)

where
\[
I_1 = \sup_{|\varphi(a)| \leq r} \int_{D \setminus E(\varphi, a, t)} |(u \circ \sigma_a)(z)(z) \cdot ((S_n f) \circ \varphi \circ \sigma_a(z) - (S_n f)(\varphi(a)))|^2 dA(z),
\]
\[
I_2 = \sup_{|\varphi(a)| \leq r} \int_{E(\varphi, a, t)} |(u \circ \sigma_a)(z) \cdot ((S_n f) \circ \varphi \circ \sigma_a(z) - (S_n f)(\varphi(a)))|^2 dA(z).
\]

Let \( \varphi_a = \sigma_a \circ \varphi \circ \sigma_a \). Then by (3.19) in [8, p.37], we have
\[
|S_n f(\varphi_a)(z) - (S_n f \circ \varphi)(a)| \leq \sup_{|w| \leq t} \|S_n(f \circ \sigma_{\varphi(a)}(w) - (S_n f)(\varphi(a))|)
\]
for \( z \in D \setminus E(\varphi, a, t) \). Since
\[
\|u \circ \sigma_a \cdot \varphi_a\|_{A^2} \leq \|u \circ \sigma_a - u(a)\|_{A^2} \|\varphi_a\|_{\infty} + |u(a)||\varphi_a|_2
\]
\[
\lesssim \sup_{a \in D} \|u \circ \sigma_a - u(a)\|_{A^2} + \alpha(u, \varphi, a_n)
\]
\[
\lesssim \|uC_\varphi\|_{B \rightarrow B},
\]
we have
\[
I_1 \lesssim \sup_{|\varphi(a)| \leq r} \sup_{|w| \leq t} [(S_n f(\varphi_a)(w) - (S_n f)(\varphi(a)))^2]\|u \circ \sigma_a \cdot \varphi_a\|_{A^2}
\]
\[
\lesssim \|uC_\varphi\|_{B \rightarrow B}^2 \sup_{|z| \leq r/2} |S_n f(z)|^2.
\]

By Lemma 2.2, we get
\[
\|S_n f \circ \varphi \circ \sigma_a - (S_n f)(\varphi(a))\|_{A^2} \leq \sup_{a \in D} \|S_n f \circ \varphi \circ \sigma_a - (S_n f)(\varphi(a))\|_{A^2}
\]
\[
\lesssim \sup_{a \in D} \|f \circ \sigma_a - f(a)\|_{A^2} \leq 1,
\]
which implies that
\[
I_2 \leq \sup_{|\varphi(a)| \leq r} \left( \int_{E(\varphi, a, t)} |u(\sigma_a(z))|^4 dA(z) \right)^{1/2} \|S_n f \circ \varphi \circ \sigma_a - (S_n f)(\varphi(a))\|_{A^2}
\]
\[
\leq \sup_{|\varphi(a)| \leq r} \left( \int_{E(\varphi, a, t)} |u(\sigma_a(z))|^4 dA(z) \right)^{1/2}.
\]
By combining the above estimates, for \( r \in (0, 1) \) and \( t \in (\frac{1}{2}, 1) \), we obtain

\[
\|u_{C_{\varphi}} S_n f\|_{B} \lesssim \sup_{|\varphi(a)| > r} \left( \alpha(u, \varphi, a) + \beta(u, \varphi, a) + \frac{\|u_{C_{\varphi}}\|_{B=B}}{\sqrt{\log \frac{1}{1-|\varphi(a)|^2}}} \right) \\
+ \sup_{|\varphi(a)| \leq r} \left( \int_{E(\varphi, a, t)} |u(\sigma_a(z))|^4 dA(z) \right)^{1/4} + \sup_{|z| \leq \frac{r}{1-|\varphi(a)|^2}} |(S_n f)(z)||u_{C_{\varphi}}\|_{B=B}.
\]

Taking the supremum over \( \|f\|_B \leq 1 \) and letting \( n \to \infty \), we obtain

\[
\|u_{C_{\varphi}}\|_{e,B \to B} \lesssim \sup_{|\varphi(a)| > r} \left( \alpha(u, \varphi, a) + \beta(u, \varphi, a) + \frac{\|u_{C_{\varphi}}\|_{B=B}}{\sqrt{\log \frac{1}{1-|\varphi(a)|^2}}} \right) \\
+ \sup_{|\varphi(a)| \leq r} \left( \int_{E(\varphi, a, t)} |u(\sigma_a(z))|^4 dA(z) \right)^{1/4},
\]

which implies that

\[
\|u_{C_{\varphi}}\|_{e,B \to B} \lesssim \tilde{\alpha} + \tilde{\beta} + \tilde{\gamma},
\]

By (3.4) (3.5) and Lemma 3.1, we get

\[
\|u_{C_{\varphi}}\|_{e,B \to B} \lesssim \tilde{\beta} + \limsup_{n \to \infty} \|u_{\varphi^n}\|_B \\
\lesssim \tilde{\alpha} + \limsup_{n \to \infty} \|u_{C_{\varphi}g_a}\|_B + \limsup_{n \to \infty} \|u_{\varphi^n}\|_B \\
\lesssim \limsup_{n \to \infty} \|u_{C_{\varphi}g_a}\|_B + \limsup_{n \to \infty} \|u_{\varphi^n}\|_B.
\]

By (ii), (iv) of Proposition 2.12, we have

\[
\|u_{C_{\varphi}}\|_{e,B \to B} \lesssim \sup_{|\varphi(a)| > r} \left( \alpha(u, \varphi, a) + \beta(u, \varphi, a) + \frac{\|u_{C_{\varphi}}\|_{B=B}}{\sqrt{\log \frac{1}{1-|\varphi(a)|^2}}} \right) \\
+ \sup_{|\varphi(a)| \leq r} \left( \int_{E(\varphi, a, t)} |u(\sigma_a(z))|^4 dA(z) \right)^{1/4} \\
\leq \sup_{|\varphi(a)| > r} \left( \alpha(u, \varphi, a) + \|u_{C_{\varphi}g_a}\|_B + \|g_a\|_B + \frac{\|u_{C_{\varphi}}\|_{B=B}}{\sqrt{\log \frac{1}{1-|\varphi(a)|^2}}} \right) \\
+ \left( \int_{E(\varphi, a, t)} |u(\sigma_a(z))|^4 dA(z) \right)^{1/4} \\
\lesssim \sup_{|\varphi(a)| > r} \left( \alpha(u, \varphi, a) + \|u_{C_{\varphi}g_a}\|_B + \|g_a\|_B + \frac{\|u_{C_{\varphi}}\|_{B=B}}{\sqrt{\log \frac{1}{1-|\varphi(a)|^2}}} + \frac{\|u_{C_{\varphi}}\|_{B=B}}{\sqrt{\log \frac{1}{1-|\varphi(a)|^2}}} \right) \\
+ \left( \int_{E(\varphi, a, t)} |u(\sigma_a(z))|^4 dA(z) \right)^{1/4},
\]
which implies that
\[ \| uC\varphi \|_{\ell^\infty, B \to B} \lesssim \tilde{\alpha} + \limsup_{\| \varphi(a) \| \to 1} \| uC\varphi g_a \|_B + \tilde{\gamma}. \]

We complete the proof of the theorem. \qed

From Theorem 3.2, we immediately get the following characterizations for the compactness of the operator \( uC\varphi : B \to B \).

**Corollary 3.3.** Let \( u \in H(\mathbb{D}) \) and \( \varphi \in S(\mathbb{D}) \) such that \( uC\varphi \) is bounded on \( B \). Then the following statements are equivalent.

(i) The operator \( uC\varphi : B \to B \) is compact.

(ii) \[ \limsup_{n \to \infty} \| u\varphi^n \|_B = 0 \text{ and } \limsup_{\| \varphi(a) \| \to 1} \| uC\varphi g_a \|_B = 0. \]

(iii) \[ \limsup_{n \to \infty} \| u\varphi^n \|_B = 0 \text{ and } \limsup_{\| \varphi(a) \| \to 1} \beta(u, \varphi, a) = 0. \]

(iv) \[ \limsup_{\| \varphi(a) \| \to 1} \alpha(u, \varphi, a) = 0, \limsup_{\| \varphi(a) \| \to 1} \beta(u, \varphi, a) = 0, \]
\[ \text{and} \]
\[ \limsup_{r \to 1} \limsup_{t \to 1} \sup_{\| \varphi(a) \| \leq r} \left( \int_{E(\varphi,a,t)} |u(\sigma_a(z))|^4 dA(z) \right)^{1/4} = 0. \]

(v) \[ \limsup_{\| \varphi(a) \| \to 1} \alpha(u, \varphi, a) = 0, \limsup_{\| \varphi(a) \| \to 1} \| uC\varphi g_a \|_B = 0, \]
\[ \text{and} \]
\[ \limsup_{r \to 1} \limsup_{t \to 1} \sup_{\| \varphi(a) \| \leq r} \left( \int_{E(\varphi,a,t)} |u(\sigma_a(z))|^4 dA(z) \right)^{1/4} = 0. \]

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