Quantum-like Representation Algorithm: Transformation of Probabilistic Data into vectors on Bloch’s Sphere

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Abstract

In this paper we present a simple algorithm for representation of statistical data of any origin by complex probability amplitudes. Numerical simulation with Mathematica-6 is performed. The Bloch’s sphere is used for visualization of results of numerical simulation. On the one hand, creation of such a quantum-like (QL) representation and its numerical approval is an important step in clarification of extremely complicated interrelation between classical and quantum randomness. On the other hand, it opens new possibilities for application the mathematical formalism of QM in other domains of science.

Keywords: probabilistic data, complex amplitudes, Bloch’s sphere, classical and quantum randomness, applications of quantum formalism

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1 Introduction

The problem of interrelation between classical and quantum randomness is extremely complicated, see von Neumann [1] for the pioneer study. It has not yet been solved completely, see e.g. Svozil [2], [3] for discussions. By the orthodox Copenhagen viewpoint these are two
completely different types of randomness. The gap between them is huge. However, we can mention a number of attempts to minimize this gap, e.g. introduction of Wigner distribution and later development of quantum tomography, see e.g. [1], [5] and references hereby, see also [6] on recent numerical simulation for quantum correlations. Mentioned studies were oriented toward classical probabilistic interpretation of quantum randomness.

In this paper we would like to formulate and study ”inverse Born’s rule problem” – IBP: to construct a representation of probabilistic data by probability amplitudes which match Born’s rule. In [7], [8] we proposed an algorithm – quantum-like representation algorithm (QLRA) – which transforms probabilistic data of any origin in probability amplitudes (complex and hyperbolic). In this paper we simplify QLRA essentially (in particular, by restricting its domain of application). We performed extended numerical simulation based on Mathematica-6. The Bloch’s sphere is used for visualization of results of numerical simulation. On the one hand, creation QLRA and its numerical approval is an important step clarification of extremely complicated interrelation between classical and quantum randomness. On the other hand, it opens new possibilities for application the mathematical formalism of QM in other domains of science, see e.g. [9], [12]. By representing probabilistic data by QL-states one might try to apply methods of quantum information theory to study such data. Of course, only the first step has been done in this direction.

2 Inversion of Born’s rule

We consider the simplest situation. There are given two dichotomous observables $a = \alpha_1, \alpha_2$ and $b = \beta_1, \beta_2$. They can be physical (classical or quantum) observables or e.g. two questions which are used for tests in psychology, cognitive or social science and so on. We set $X_a = \{\alpha_1, \alpha_2\}$ and $X_b = \{\beta_1, \beta_2\}$ – ”spectra of observables.”

We suppose that there is given the matrix of transition probabilities $P^{b|a} = (p_{\beta\alpha})$, where $p_{\beta\alpha} \equiv P(b = \beta|a = \alpha)$ is the probability to obtain the result $b = \beta$ under the condition that the result $a = \alpha$ has been obtained.

There are also given probabilities $p_{\alpha}^a \equiv P(a = \alpha), \alpha \in X_a$, and $p_{\beta}^b \equiv P(b = \beta), \beta \in X_b$. Probabilistic data $C = \{p_{\alpha}^a, p_{\beta}^b\}$ is related to some experimental context (in physics preparation procedure). It is
not assumed that both observables can be measured simultaneously. Thus in general two samples (prepared under the same complex of conditions) are used to collect the probabilistic data for observations of $a$ and $b$, respectively.

Our aim is to represent this data by a probability amplitude $\psi$ (in the simplest case it is complex valued) such that Born’s rule holds for both observables:

$$p^b_\beta = |\langle \psi, e^b_\beta \rangle|^2, \quad p^a_\alpha = |\langle \psi, e^a_\alpha \rangle|^2,$$

where $\{e^b_\beta\}_{\beta \in X_b}$ and $\{e^a_\alpha\}_{\alpha \in X_a}$ are orthonormal bases (which are also produced by QLRA) for observables $b$ and $a$, respectively. These observables are represented by operators $\hat{b}$ and $\hat{a}$ which are diagonal in these bases.

### 3 QLRA

In [7], [8] we derived the following formula for interference of probabilities:

$$p^b_\beta = \sum_\alpha p^a_\alpha p_{\beta\alpha} + \frac{2\lambda_\beta}{\prod_\alpha p^a_\alpha p_{\beta\alpha}},$$

where the "coefficient of interference"

$$\lambda_\beta = \frac{p^b_\beta - \sum_\alpha p^a_\alpha p_{\beta\alpha}}{2\sqrt{\prod_\alpha p^a_\alpha} p_{\beta\alpha}}.$$

This is a trivial mathematical identity. To prove it, one should just put $\lambda$ given by (3) into (2). A similar representation we have for the $a$-probabilities. To simplify considerations, we shall proceed under the condition:

**DS:** The matrix of transition probabilities $P^{b/a}$ is doubly stochastic.

We also suppose that probabilistic data $C = \{p^a_\alpha, p^b_\beta\}$ consists of strictly positive probabilities. We proceed under the basic assumption (specifying the type of representation):

**RC:** Coefficients of interference $\lambda_\beta, \beta \in X_b$, are bounded by one:

$$|\lambda_\beta| \leq 1.$$

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1 In a doubly stochastic matrix all entries are nonnegative and all rows and all columns sum to 1.
Probabilistic data \( C \) such that \( RC \) holds is called trigonometric, because in this case we have the conventional formula of trigonometric interference:

\[
p^b_\beta = \sum_\alpha p^a_\alpha p^c_\beta + 2 \cos \phi_\beta \sqrt{\prod_\alpha p^a_\alpha p^c_\beta},
\]  

(4)

where

\[
\lambda_\beta = \cos \phi_\beta.
\]

This is simply a new parametrization: a new parameter \( \phi \) is used, instead of \( \lambda \). Parameters \( \phi_\beta \) are said to be \( b|a \)-relative phases for the data \( C \). We defined these phases purely on the basis of probabilities.

From the probabilistic viewpoint the formula for interference of probabilities is nothing else than perturbation of the classical formula of total probability (FTP). We recall this law in the simplest case of dichotomous random variables

\[
P(b = \beta) = P(a = \alpha_1)P(b = \beta|a = \alpha_1) + P(a = \alpha_2)P(b = \beta|a = \alpha_2)
\]

(5)

Thus the probability \( P(b = \beta) \) can be reconstructed on the basis of conditional probabilities \( P(b = \beta|a = \alpha) \). FTP plays the fundamental role in classical statistics and decision making. However, it is violated in experiments with quantum systems.

We denote the collection of trigonometric probabilistic data by the symbol \( C^{tr} \). By using the elementary formula: \( D = A + B + 2\sqrt{AB} \cos \theta = |\sqrt{A} + e^{i\phi} \sqrt{B}|^2 \), for real numbers \( A, B > 0, \theta \in [0, 2\pi] \), we can represent the probability \( p^b_\beta \) as the square of the complex amplitude (Born’s rule):

\[
p^b_\beta = |\psi(\beta)|^2.
\]

(6)

Here

\[
\psi(\beta) = \sqrt{p^a_\alpha, p^c_\beta, \alpha_1} + e^{i\phi_\beta} \sqrt{p^a_\alpha, p^c_\beta, \alpha_2}, \beta \in X_b.
\]

(7)

The formula (7) gives the quantum-like representation algorithm – QLRA. For any trigonometric probabilistic data \( C \) QLRA produces the complex amplitude \( \psi \). This algorithm can be used in any domain of science to create the QL-representation of probabilistic data.

\[^{2}\text{We have not started with any linear space; in contrast we shall define geometry from probability. In the conventional quantum formalism the formula of interference of probabilities is derived starting directly with the Hilbert space. We recall that in QM interference of probabilities is derived via transition from the basis for the } a\text{-observable to the basis for the } b\text{-observable. From the very beginning observables are given by self-adjoint operators.}\]
We denote the space of functions: $\psi : X_b \to \mathbb{C}$ by the symbol $\Phi = \Phi(X_b, \mathbb{C})$. Since $X = \{\beta_1, \beta_2\}$, the $\Phi$ is the two dimensional complex linear space. By using QLRA we construct the map $J_{b|a} : C^{tr} \to \Phi(X, \mathbb{C})$ which maps probabilistic data into complex amplitudes. The representation (6) of probability is nothing else than the famous Born rule. The complex amplitude $\psi(\beta)$ can be called a wave function of the data $C$ or a (pure) state.

By using the terminology of quantum information theory we can say that QLRA represents probabilistic data (of a special sort, namely, trigonometric) by qubits.

We set $e_{b\beta}(\cdot) = \delta(\beta - \cdot)$ – Dirac delta-functions concentrated in points $\beta = \beta_1, \beta_2$. The Born’s rule for complex amplitudes (6) can be rewritten in the following form:

$$p_{b\beta} = |\langle \psi, e_{b\beta} \rangle|^2,$$

where the scalar product in the space $\Phi(X_b, \mathbb{C})$ is defined by the standard formula:

$$\langle \psi_1, \psi_2 \rangle = \sum_{\beta \in X_b} \overline{\psi_1(\beta)} \psi_2(\beta). \quad (8)$$

The system of functions $\{e_{b\beta}\}_{\beta \in X_b}$ is an orthonormal basis in the Hilbert space $H = (\Phi, \langle \cdot, \cdot \rangle)$.

Let now $X_b \subset \mathbb{R}$ (in general $\beta$ is just a label for a result of observation). By using the Hilbert space representation of the Born’s rule we obtain the Hilbert space representation of the expectation of the observable $b$:

$$E_b = \sum_{\beta \in X_b} \beta |\psi(\beta)|^2 = \sum_{\beta \in X_b} \beta \langle \psi, e_{b\beta} \rangle \langle e_{b\beta}, \psi \rangle = \langle \hat{b}\psi, \psi \rangle,$$

where the (self-adjoint) operator $\hat{b} : H \to H$ is determined by its eigenvectors: $\hat{b}e_{b\beta} = \beta e_{b\beta}, \beta \in X_b$. This is the multiplication operator in the space of complex functions $\Phi(X_b, \mathbb{C}) : b\psi(\beta) = \beta \psi(\beta)$.

To solve IBP completely, we would like to have Born’s rule not only for the $b$-variable, but also for the $a$-variable: $p^a_{a\alpha} = |\langle \psi, e_{a\alpha} \rangle|^2$, $\alpha \in X_a$. How can we define the basis $\{e_{a\alpha}\}$ corresponding to the $a$-observable? Such a basis can be found starting with interference of probabilities. We have:

$$\psi = \sqrt{p^a_{a1}} f^a_{a1} + \sqrt{p^a_{a2}} f^a_{a2}, \quad (9)$$

where

$$f^a_{a1} = \left( \frac{\sqrt{p_{b1\alpha1}}}{\sqrt{p_{b2\alpha1}}} \right), \quad f^a_{a2} = \left( \frac{e^{i\phi_{b1}} \sqrt{p_{b1\alpha2}}}{\sqrt{P_{b2\alpha2}}} \right) \quad (10)$$

\[5\]
The condition $\mathbf{DS}$ implies that the system of vectors $\{f^\beta_{\alpha i}\}$ is an orthonormal basis iff the probabilistic phases satisfy the constraint:

$$\phi_{\beta_2} - \phi_{\beta_1} = \pi \mod 2\pi,$$

Thus, instead of the $a$-basis (10) which depends on phases, we can consider a new $a$-basis which depends only on the matrix of transition probabilities:

$$e^a_{\alpha_1} = \left( \frac{\sqrt{p_{\beta_1\alpha_1}}}{\sqrt{p_{\beta_2\alpha_1}}} \right), \quad e^a_{\alpha_2} = \left( \frac{\sqrt{p_{\beta_1\alpha_2}}}{-\sqrt{p_{\beta_2\alpha_2}}} \right) \quad (11)$$

The $a$-observable is represented by the operator $\hat{a}$ which is diagonal with eigenvalues $\alpha_1, \alpha_2$ in the basis $\{e^a_{\alpha}\}$.

4 Numerical simulation and visualization on Bloch’s sphere

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Transition probability $P = 0.1$}
\end{figure}
We recall that at the moment we work under condition RC. First of all our computer program checks this condition. If RC is violated, then the program’s output is empty – no point on Bloch’s sphere. We use spherical coordinates: $x = \sin 2\theta \cos \phi, y = \sin 2\theta \sin \phi, z = \cos 2\theta$. A vector on Bloch’s sphere is given by

$$\psi = \cos\theta |0\rangle + \sin\theta e^{i\phi} |1\rangle \quad (12)$$

It is convenient to use the QLRA-output in the $a$-basis. Thus we make identification: $|0\rangle = e_{a_1}^a, |1\rangle = e_{a_2}^a$. We have: $p_{a_1}^a = \cos \theta, p_{a_2}^a = \sin \theta; \lambda_{\beta_1} = \cos \phi, \pm \sqrt{1 - \lambda_{\beta_1}^2} = \sin \phi$. Finally:

$$x = 2\sqrt{p_{a_1}^a p_{a_2}^a} \lambda_{\beta_1}$$

$$y = \pm 2\sqrt{p_{a_1}^a p_{a_2}^a} \sqrt{1 - \lambda_{\beta_1}^2}$$

$$z = p_{a_1}^a - p_{a_2}^a$$

Figure 2: Transition probability $P = 0.5$
In the program we make the parametrization of probabilities:

\[ p_{a_1} = q, p_{a_2} = 1 - q, p_{b_1} = p, p_{b_2} = 1 - p. \]

Bigger values of \( q \) give more red color, larger values of \( p \) give more green. If both \( q \) and \( p \) are rather small, then picture is dark. If both \( q \) and \( p \) are rather big, then picture is yellow. Since we proceed under condition DS, the elements of the matrix of transition probabilities can be parameterized as

\[ p_{\beta_1 \alpha_1} = p_{\beta_2 \alpha_2} = P, \quad p_{\beta_1 \alpha_2} = p_{\beta_2 \alpha_1} = 1 - P. \]

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3Simulation was done with help (in programming and visualization) of I. Basieva during her visit to International Center for Mathematical Modelling of University of Växjö in December 2007.