Interval type-2 fuzzy brain emotional control design for the synchronization of 4D nonlinear hyperchaotic systems

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Abstract
This research provides a novel intelligent control structure for 4D nonlinear hyperchaotic systems. This is a hybrid design containing a new interval type-2 fuzzy fourfold function-link brain emotional controller and a smooth robust controller. It comprises a fuzzy inference system and three subnetworks. The subnetworks are a new fourfold function-link network, a type-2 fuzzy amygdala network and a type-2 fuzzy prefrontal cortex network that decrease the synchronization errors efficiently, follow the reference signal well, and achieve good performance. Two Lyapunov stability functions are utilized to get the adaptive laws, and they are applied to online tune the parameters of the system. The proposed design is used to synchronize two 4D nonlinear hyperchaotic systems and the simulation results are given to demonstrate its superiority and effectiveness.

Keywords Interval type-2 fuzzy system · Fourfold function-link network · Fuzzy brain emotional controller · 4D nonlinear hyperchaotic system

1 Introduction
Over the years, chaos studies had a strong influence on the development of global science and technology, in which chaotic synchronization is one of the interesting topics that attract many scholars (Hsu et al. 2009; Sothmann et al. 2012; Sun et al. 2013; Vaidyanathan and Rasappan 2014; Wang et al. 2019a; Wu et al. 2017). Synchronization of chaotic systems is described as the phenomenon that happens when a master system controls a slave system by tuning a given characteristic of their motion (Lin and Huynh; Wang et al. 2019b). Nowadays, various chaotic and hyperchaotic systems have been investigated in many fields (Chen et al. 2018; Panahi et al. 2019; Pham et al. 2017). Particularly, the 4D hyperchaotic systems comprising complex shapes of equilibrium points are studied in recent years (Rakheja et al. 2019; Sambas et al. 2018; Vaidyanathan et al. 2018). The first Lyapunov exponent is generally used to define the disorders of chaotic systems. Moreover, the studies of chaos with hidden attractions are significant since they can create undesirable and harmful problems with minor changes in dynamics such as engine system (Marzbanrad and Babalooei 2016), airplane system (Andrievsky et al. 2018), transportation system (Adeli and Jiang 2008), electromechanical system (Xue et al. 2019), radar system (Beal et al. 2016), and bridge system (Ni et al. 2019).

Recently, many control systems have been proposed for nonlinear chaotic systems to achieve good control performance such as an adaptive fuzzy control (Sambas et al. 2020), a passive control (Sambas et al. 2019a), an active backstepping control (Sambas et al. 2021), an adaptive control (Sambas et al. 2019b), an integral sliding mode control (Sambas et al. 2020), a robust control (Pham et al. 2017), and a robust adaptive control (Sambas et al. 2020). The proposed design is used to synchronize two 4D nonlinear hyperchaotic systems and the simulation results are given to demonstrate its superiority and effectiveness.

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control (Vaidyanathan et al. 2019), a double function-link brain emotional control (Huynh et al. 2020c), a modified grey wolf-based multilayer type-2 asymmetric fuzzy control (Le et al. 2020b), a self-organizing interval type-2 fuzzy asymmetric cerebellar model articulation control (Le et al. 2020a), a wavelet interval type-2 fuzzy brain emotional control (Huynh et al. 2020a), and a brain-imitated neural network control (Lin et al. 2021a, b).

A function-link network (FLN) is a kind of feed-forward network model that is efficiently applied for function approximation with quick convergence speed and less computational load (Patra and Pal 1995). In the past year, many scholars have used the FLN in their researches to achieve better results (Huynh et al. 2019a, b; Lin and Huynh 2018; Zhou et al. 2018). A single FLN was used to adjust the weights for two independent networks in a brain emotional learning network (BELN) that are the amygdala and orbitofrontal cortex networks. However, it is necessary to attain the BELN’s weights precisely and their values separately. Recently, a dual FLN was proposed to improve this drawback (Huynh and Lin 2019; Lin et al. 2021a, b). In this research, a new fourfold FLN is designed for the proposed type-2 fuzzy BELN’s structure.

LeDoux proposed a brain emotional learning network (BELN), which is a computational simulation system describing the data processing scheme of the mammal brain (LeDoux 1991). A BELN connects a stimulus to the equivalent emotional reaction appearing in an amygdala of a brain. The brain has an amygdala and an orbitofrontal cortex so that an output of the BELN is associated among the two networks, which influence each other. For that reason, BELN still works well with system uncertainty with fast learning speeds and good approximation capabilities, and it can reduce tracking errors effectively. Over the years, some remarkable studies have applied BELN in different fields (Dashiti et al. 2017; Hsu et al. 2016; Kong et al. 2019; Le et al. 2018).

So far, intelligent controllers based on type 1 (T1) and type 2 (T2) platforms have been developed for different applications in different fields (Boubellouta et al. 2019; Lin and Huynh 2019; Mendel et al. 2020; Zhao and Lin 2019). Since T1-fuzzy logic systems (FLSs) are required to have membership functions that are well defined, it cannot thoroughly handle the large uncertainty of inputs and parameters of nonlinear systems well. To overcome this drawback, T2-FLS and interval T2-FLS are typically used as they have general expanded features than T1-FLS, more freely generated for better control performance and improved response to uncertain input of membership functions (Le 2019; Rong et al. 2018). In order to design effective networks, many researches have come up with solutions that combine different efficient techniques (Lin et al. 2018; Wang et al. 2018), additional functional networks (Ding et al. 2019; Huynh et al. 2020b; Huynh et al. 2019a, b) and extra algorithms (Rahmani et al. 2018; Ravi et al. 2017) to build networks automatically.

This study develops a new more efficient interval type-2 fuzzy fourfold function-link brain emotional controller (IT2FFFLBC) for 4D nonlinear hyperchaotic systems. The proposed IT2FFFLBC control system includes an IT2FFFLBC and a smooth robust controller. The IT2FFFLBC is used as the main controller and a smooth robust controller is used to eliminate the approximate error term and to warrant the system stability. The main contributions of this research are summarized as follows:

1. The proposed IT2FFFLBC comprises a set of fuzzy inference rules and three subnetworks that are the fourfold function-link network, the type-2 fuzzy prefrontal cortex network, and the type-2 fuzzy amygdala network to efficiently reduce the synchronization error and achieve good performance.
2. A new fourfold FLN is designed to tune the particular weights for the type-2 fuzzy structures of the orbitofrontal cortex and amygdala of the proposed IT2FFFLBC.
3. Effective adaptive learning laws for updating the system parameters effectively are obtained from two Lyapunov functions, and they are also used to prove the stability of the system.
4. Finally, the simulation results and some comparisons in root mean square error with former studies for a 4D hyperchaotic Lorenz–Lu system and a 4D hyperchaotic Rikitake two-wing dynamo system have shown the effectiveness and advantage of the proposed control system.

2 Theoretical problems

The master system (MS) for an nth-order 4D nonlinear hyperchaotic system is defined as:

\[
x_{MS}^{(n)}(t) = f_{MS}(x_{MS}(t))
\]

The slave system (SS) is given as:

\[
y_{SS}^{(n)}(t) = f_{SS}(y_{SS}(t) + u_{SS}(t) + n_{SS}(t))
\]

where \(x_{MS}(t) \in \mathbb{R}^m\) is the output for the MS, \(y_{SS}(t) \in \mathbb{R}^m\) is the output for the SS, \(x_{MS}^{(n)}(t) \in \mathbb{R}^m\) is the state for the MS, \(x_{MS}^{(n-1)T}(t) \in \mathbb{R}^m\) is the state for the SS, \(u_{SS}(t) \in \mathbb{R}^m\) is the control input for the SS, \(n_{SS}(t) \in \mathbb{R}^m\) is the unknown bounded external noise for the SS, where \(m\) is the number of outputs and inputs of the system, and the
subscript letters SS and MS indicate the slave system and master system, respectively.

The synchronization error state is defined as follows:
\[
e_{\text{Sync}}(t) \triangleq y_{\text{SS}}(t) - x_{\text{MS}}(t) = [e_1(t), e_2(t), \ldots, e_m(t)]^T \in \mathbb{R}^m
\]  

(3)

The error vector is then defined as:
\[
\bar{e}_{\text{SS}}(t) = [e_{\text{Sync}}^T, e_{\text{Sync}}^T, \ldots, e_{\text{Sync}}^{(m-1)}]^T \in \mathbb{R}^{mn}
\]  

(4)

If the unknown bounded nonlinear functions \( f_{\text{MS}}(x_{\text{MS}}(t)) \), \( f_{\text{SS}}(y_{\text{SS}}(t)) \), and the external noise \( n_{\text{SS}}(t) \) of the SS are known, the ideal controller is then determined as:
\[
u_{\text{idc}}(t) = -f_{\text{SS}}(y_{\text{SS}}(t)) + n_{\text{SS}}(t) + \bar{K}^T \bar{e}_{\text{SS}}
\]  

(5)

where \( \bar{K} = [K_1, K_2, \ldots, K_n]^T \in \mathbb{R}^{mn \times m} \) is the gain matrix with real values.

Inserting (5) into (2), attain the following error dynamic condition
\[
e_{\text{Sync}} + \bar{K}^T \bar{e}_{\text{SS}} = 0
\]  

(6)

If \( \bar{K} \) is suitably determined to satisfy the Hurwitz stability criterion to produce the roots on the left side of the complex plane, i.e., \( \lim_{t \to \infty} \bar{e}_{\text{SS}}(t) = 0 \). Due to \( f_{\text{MS}}(x_{\text{MS}}(t)) \), \( f_{\text{SS}}(y_{\text{SS}}(t)) \) and \( n_{\text{SS}}(t) \) are unknown, \( u_{\text{idc}}(t) \) in (5) is unattainable. As a result, an IT2FFFLBC is used to imitate \( u_{\text{idc}}(t) \).

### 3 Interval type-2 fuzzy fourfold function-link brain emotional controller

#### 3.1 Fourfold function-link network

This study proposes a new fourfold FLN (FFLN) to expand the processing ability of the IT2FFFLBC. The main FFLN duty is to enhance accuracy for lower and upper weights of the type-2 fuzzy orbitofrontal cortex and amygdala networks of the proposed structure. The FFLN structure is displayed in Fig. 1, where the FFLN operates on input variables by producing a linearly independent set. The proposed FFLN uses the cosine and sine functions because of their simple and clear functions and rapidly computed. Define \( I = [I_1, I_2, \ldots, I_N]^T \in \mathbb{R}^N \), the inputs are then allocated in the extended space as follows:
\[
\Phi = [\Phi_1, \Phi_2, \ldots, \Phi_M]^T \in \mathbb{R}^M, \quad \text{where } M \text{ and } N \text{ are, respectively, the number of the outputs and the total number of input signals.}
\]

Particularly, if the input is \( I = [I_1, I_2]^T \), then \( \Phi = (1, 1_1, \cos(\pi \times I_1), \sin(\pi \times I_1), I_2,\cos(\pi \times I_2), \sin(\pi \times I_2), I_1 \times I_2) \). Next, the FFLN outputs are determined as:
\[
\begin{align*}
\hat{\omega}_k &= q_{1k} \Phi_1 + \cdots + q_{mk} \Phi_m + \cdots + q_{MK} \Phi_M = \sum_{m=1}^{M} q_{mk} \Phi_m \\
&= \Phi_k^T \Phi \\
\hat{\omega}_k &= q_{1k} \Phi_1 + \cdots + q_{mk} \Phi_m + \cdots + q_{MK} \Phi_M = \sum_{m=1}^{M} q_{mk} \Phi_m \\
&= \Phi_k^T \Phi \\
\end{align*}
\]  

(7)

(8)

(9)

for \( m = 1, 2, \ldots, M, k = 1, 2, \ldots, L \), where \( \Phi_m \) is the \( m \)-th function expansion output, \( \hat{\omega}_k, \hat{\omega}_k, \hat{\omega}_k, \hat{\omega}_k \) and \( \hat{\omega}_k \) are the output of FFLN, \( \tilde{P}_k, \tilde{q}_k, \tilde{p}_k, \tilde{m}_k \) and \( \tilde{q}_k \) are the connective weight among \( \hat{\omega}_k, \hat{\omega}_k, \hat{\omega}_k, \hat{\omega}_k \) and \( \Phi_m \), and connective weight vectors are defined as:
\[
\begin{align*}
\tilde{P}_k &= [\tilde{p}_{1k}, \cdots, \tilde{p}_{mk}, \cdots, \tilde{p}_{MK}] \in \mathbb{R}^M \\
\tilde{P}_k &= [\tilde{p}_{1k}, \cdots, \tilde{p}_{mk}, \cdots, \tilde{p}_{MK}] \in \mathbb{R}^M \\
\tilde{q}_k &= [\tilde{q}_{1k}, \cdots, \tilde{q}_{mk}, \cdots, \tilde{q}_{MK}] \in \mathbb{R}^M \\
\tilde{q}_k &= [\tilde{q}_{1k}, \cdots, \tilde{q}_{mk}, \cdots, \tilde{q}_{MK}] \in \mathbb{R}^M \\
\end{align*}
\]  

(10)

(11)

(12)

(13)

(14)

#### 3.2 Interval type-2 fuzzy fourfold function-link brain emotional controller

The structure of IT2FFFLBC is shown in Fig. 2, which is composed of five layers: input layer, fuzzy membership function layer, output weight layer using FFLN, amygdala–orbitofrontal layer and output layer. The signal propagation in each layer is described below:

**Layer 1: Input layer**
Fig. 1 Fourfold FLN

Fig. 2 Structure of IT2FFFLBC
Interval type-2 fuzzy brain emotional control design for the synchronization of 4D nonlinear systems with the application of the proposed control methods. The design is based on the incorporation of type-2 fuzzy control, which is effective for handling uncertainties and non-linearities in complex systems. The proposed approach involves the use of type-2 fuzzy brain emotional control, designed to improve the synchronization of 4D nonlinear systems. By applying the proposed control methods, the design aims to achieve effective synchronization and control in complex systems.

Mathematically, the design is represented as follows:

\[ I = [I_1, \ldots, I_i, \ldots, I_N]^T \in \mathbb{R}^N, \text{ for } i = 1, 2, \ldots, N \]  

where \( I_i \) is the \( i \)th input.

**Layer 2: Fuzzy membership function layer**

\[ \overline{a}_{ik} = \overline{a}_{ik} = \exp \left[ \frac{-(I_i - m_k)^2}{2\sigma^2_{ik}} \right], \text{ for } k = 1, 2, \ldots, L \]  

\[ a_{ik} = a_{ik} = \exp \left[ \frac{-(I_i - m_k)^2}{2\sigma^2_{ik}} \right], \text{ for } k = 1, 2, \ldots, L \]

where \( \overline{a}_{ik}, \overline{a}_{ik} \) and \( a_{ik}, a_{ik} \) are the upper and lower values for the type-2 Gaussian membership function. \( \sigma_{ik} \) is the variance, \( m_k \) is the mean, and \( L \) is the number of layers.

\[ \overline{a}_k = \prod_{i=1}^{N} \overline{a}_{ik} \]  

\[ a_k = \prod_{i=1}^{N} a_{ik} \]

\[ \overline{u}_k = \prod_{i=1}^{N} \overline{u}_{ik} \]  

\[ u_k = \prod_{i=1}^{N} u_{ik} \]

The proposed structure utilizes the following fuzzy inference rules:

\[ R^k : \begin{cases} 
\text{If } I_i \text{ is } a_{1k}, I_i \text{ is } a_{2k}, \ldots, \text{and } I_N \text{ is } a_{Nk}, & \\
\text{then } \overline{v}_k = p_i^T \Phi, \text{ for } i = 1, 2, \ldots, N & \\
\text{If } I_i \text{ is } o_{1k}, I_i \text{ is } o_{2k}, \ldots, \text{and } I_N \text{ is } o_{Nk}, & \\
\text{then } \overline{v}_k = q_i^T \Phi, \text{ for } i = 1, 2, \ldots, L & 
\end{cases} \]

**Layer 3: Output weight layer using FFLN**

\[ \overline{w} = [\overline{w}_1, \ldots, \overline{w}_k, \ldots, \overline{w}_L]^T \in \mathbb{R}^L \]  

\[ w = [w_1, \ldots, w_k, \ldots, w_L]^T \in \mathbb{R}^L \]

\[ \overline{v} = [\overline{v}_1, \ldots, \overline{v}_k, \ldots, \overline{v}_L]^T \in \mathbb{R}^L \]  

\[ v = [v_1, \ldots, v_k, \ldots, v_L]^T \in \mathbb{R}^L \]

where \( \overline{w}, w \) and \( \overline{v}, v \) are, respectively, the weights of orbitofrontal and amygdala; and they are defined as:

\[ p^T = [p_1, \ldots, p_k, \ldots, p_L]^T \]

\[ p^T = [p_1, \ldots, p_k, \ldots, p_L]^T \]

where \( \Phi = [\Phi_1, \ldots, \Phi_m, \ldots, \Phi_M]^T \).
Layer 4: Amygdala–orbitofrontal layer

\begin{align}
A_k &= a_k z_k^a \\
O_k &= a_k z_k^o
\end{align}

where $z_k^a$ is the $k$th output weight of amygdala and $A_k$ is the $k$th amygdala output.

\begin{align}
O_k &= a_k z_k^o
\end{align}

where $z_k^o$ is the $k$th output weight of orbitofrontal cortex and $O_k$ is the $k$th orbitofrontal output.

Layer 5: Output layer

The proposed IT2FFFLBC is shown in Fig. 2, which is determined as:

\begin{align}
u_{IT2FFFLBC} &= A - O = az^a - oz^o
\end{align}

4 Online learning laws and convergence analysis

The ideal controller in (5) theoretically makes the system stable. However, the nonlinear functions $f_{MS}(\tilde{X}_{MS}(t))$ and the external noise $n_{SS}(t)$ are unobtainable accurately in general. As a result, (5) is unobtainable, so the proposed IT2FFFLBC is used for 4D nonlinear hyperchaotic systems as presented in Fig. 3. The total control effort is defined as:

\begin{align}
u = u_{IT2FFFLBC} + u_{SRC}
\end{align}

where $u_{IT2FFFLBC}$ is the main controller that imitates $u_{IDC}$. $u_{SRC}$ is a smooth robust compensator which spurs the dissimilarity between $u_{IDC}$ and the proposed IT2FFFLBC. Suppose that we obtain an optimal controller, $u^*_{IT2FFFLBC}$, that mimics an ideal controller $u^*_{IDC}$ in (5), then

\begin{align}
u^*_{IDC} &= u^*_{IT2FFFLBC} + \varepsilon = a^* z^a + o^* z^o + \varepsilon
\end{align}

where $\varepsilon$ is the minimum error between $u_{IDC}$ and $u^*_{IT2FFFLBC}$, and assume it is bounded; $a^*$, $o^*$, $z^a$ and $z^o$ are, respectively, optimal parameters of $a$, $o$, $z^a$ and $z^o$. However, $u^*_{IT2FFFLBC}$ cannot be obtained, and thus, an online estimation of IT2FFFLBC, $\hat{u}_{IT2FFFLBC}$, is applied to estimate $u^*_{IT2FFFLBC}$. Using (43), the control law (44) becomes

\begin{align}
u = \hat{u}_{IT2FFFLBC} + u_{SRC} = \hat{a} z^a - \hat{o} z^o + u_{SRC}
\end{align}

where $\hat{a}$, $\hat{o}$, $\hat{z}^a$, and $\hat{z}^o$ are, respectively, estimations of the optimal parameters $a^*$, $o^*$, $z^a$, and $z^o$. The estimation error, $\hat{u}$, is then calculated by subtracting (46) from (45):
\[
\dot{\mathbf{u}} = \mathbf{u}^*_{\text{DC}} - \mathbf{u} = {\mathbf{z}}^{20T} \mathbf{a}^* + {\mathbf{z}}^{20T} \dot{\mathbf{a}} - {\mathbf{z}}^{20T} \mathbf{a}^* + {\mathbf{z}}^{20T} \dot{\mathbf{a}} + \mathbf{e} - \mathbf{u}_{\text{SRC}}
\]  

(47)

where \( \dot{\mathbf{a}} = \mathbf{a}^* - \dot{\mathbf{a}}, \ \dot{\mathbf{z}} = \mathbf{a}^{20} - \dot{\mathbf{a}}^2, \ \dot{\mathbf{o}} = \mathbf{a}^* - \dot{\mathbf{o}} \) and \( \mathbf{z}^{20} = \mathbf{a}^{20} - \dot{\mathbf{a}}^{20} \). The expansion of \( \dot{\mathbf{a}} \) and \( \dot{\mathbf{o}} \) in the Taylor series is attained as (Slotine and Li 1991):

\[
\begin{align*}
\dot{\mathbf{a}} & = \alpha_{r}^{e} \dot{\mathbf{m}} + \alpha_{r}^{e} \dot{\mathbf{q}} + \alpha_{r}^{e} \dot{\mathbf{p}} + \alpha_{r}^{e} \dot{\mathbf{q}} + \alpha_{r}^{e} \dot{\mathbf{q}} + \mathbf{H}_a \\
\dot{\mathbf{o}} & = \alpha_{m}^{e} \dot{\mathbf{m}} + \alpha_{m}^{e} \dot{\mathbf{q}} + \alpha_{m}^{e} \dot{\mathbf{p}} + \alpha_{m}^{e} \dot{\mathbf{q}} + \alpha_{m}^{e} \dot{\mathbf{q}} + \mathbf{H}_o
\end{align*}
\]  

(48)

where \( \mathbf{H}_a \) and \( \mathbf{H}_o \in \mathbb{R}^{n_l} \) are vectors with high-order terms, and \( \frac{\partial \mathbf{u}}{\partial \mathbf{m}}, \frac{\partial \mathbf{u}}{\partial \mathbf{q}}, \frac{\partial \mathbf{u}}{\partial \mathbf{p}}, \frac{\partial \mathbf{u}}{\partial \mathbf{q}}, \frac{\partial \mathbf{u}}{\partial \mathbf{q}}, \frac{\partial \mathbf{u}}{\partial \mathbf{q}} \) are determined as:

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial \mathbf{m}} &= \begin{bmatrix} \frac{\partial \mathbf{u}}{\partial m_k} & \cdots & \frac{\partial \mathbf{u}}{\partial m_k} \end{bmatrix} \quad \left( k = 1, \ldots, n_k \right) \\
\frac{\partial \mathbf{u}}{\partial \mathbf{q}} &= \begin{bmatrix} \frac{\partial \mathbf{u}}{\partial q_k} & \cdots & \frac{\partial \mathbf{u}}{\partial q_k} \end{bmatrix} \quad \left( k = 1, \ldots, n_k \right) \\
\frac{\partial \mathbf{u}}{\partial \mathbf{p}} &= \begin{bmatrix} \frac{\partial \mathbf{u}}{\partial p_k} & \cdots & \frac{\partial \mathbf{u}}{\partial p_k} \end{bmatrix} \quad \left( k = 1, \ldots, n_k \right)
\end{align*}
\]  

(49)

Fig. 3 Block diagram of synchronization for 4D nonlinear hyperchaotic systems using the proposed IT2FFLBC
Rewriting (48) gives
\[
a^* = a + a_m^T \tilde{m} + a_\rho^T \tilde{\rho} + a_p^T \tilde{p}_k + a_{\tilde{q}}^T \tilde{\tilde{q}}_k + H_a
\]
\[
\begin{align*}
o^* &= \dot{o} + o_m^T \tilde{m} + o_\rho^T \tilde{\rho} + o_p^T \tilde{p}_k + o_{\tilde{q}}^T \tilde{\tilde{q}}_k + o_{\dot{\tilde{q}}}^T \dot{\tilde{\tilde{q}}}_k \\
&\quad + o_q^T \dot{\tilde{q}}_k + H_o
\end{align*}
\]

Inserting (48), (63), and (64) into (47) yields
\[
\dot{\tilde{u}} = \tilde{z}^T \dot{\tilde{a}} + \tilde{z}^T (a_m^T \tilde{m} + a_\rho^T \tilde{\rho} + a_p^T \tilde{p}_k + a_{\tilde{q}}^T \tilde{\tilde{q}}_k + H_a)
\]
\[
+ \tilde{z}^T (a_p^T \dot{\tilde{p}} + a_p^T \tilde{p}_k + a_q^T \dot{\tilde{q}} + a_{\tilde{q}}^T \tilde{\tilde{q}}_k)
\]
\[
- \tilde{z}^T \dot{a} - \tilde{z}^T (o_m^T \dot{\tilde{m}} + o_\rho^T \dot{\tilde{\rho}} + o_p^T \dot{\tilde{p}}_k + o_q^T \dot{\tilde{q}}_k)
\]
\[
- \tilde{z}^T (o_p^T \dot{\tilde{p}} + o_p^T \dot{\tilde{p}}_k + o_q^T \dot{\tilde{q}}_k + o_{\tilde{q}}^T \dot{\tilde{q}}_k)
\]
\[
+ \Pi_{AE}(t) - u_{SRC}
\]

where the approximation error,
\[
\Pi_{AE}(t) = \tilde{z}^T (a_m^T \dot{\tilde{m}} + a_\rho^T \dot{\tilde{\rho}} + a_p^T \dot{\tilde{p}}_k + a_{\tilde{q}}^T \dot{\tilde{\tilde{q}}}_k + o_{\dot{\tilde{q}}}^T \dot{\tilde{\tilde{q}}}_k)
\]
\[
+ \tilde{z}^T \dot{H}_a + \tilde{z}^T (a_p^T \dot{\tilde{p}} + a_p^T \dot{\tilde{p}}_k + a_q^T \dot{\tilde{q}} + a_{\tilde{q}}^T \dot{\tilde{q}}_k)
\]
\[
+ o_p^T \dot{\tilde{p}}_k + o_q^T \dot{\tilde{q}}_k + o_{\tilde{q}}^T \dot{\tilde{q}}_k) + \tilde{z}^T \dot{H}_a + \epsilon
\]

is supposedly bounded by \(|\Pi_{AE}(t)| < \tilde{\Delta}^*\), where \(\tilde{\Delta}^*\) is a positive constant.

This research uses the high-order sliding mode for improving the control system. Define the sliding surface as:
\[
s(t) \triangleq K_1 \dot{e}_{Sync} + K_2 e_{Sync}^{(1)} + \cdots + K_{n-1} e_{Sync}^{(n-3)} + K_n e_{Sync}^{(n-2)} + e_{Sync}^{(n-1)} + e_{Sync}^{(n)}
\]

Then, taking the derivative of (66), gives
\[
\dot{s}(t) = K_1 \ddot{e}_{Sync} + K_2 e_{Sync}^{(2)} + \cdots + K_{n-1} e_{Sync}^{(n-2)} + K_n e_{Sync}^{(n-1)} + e_{Sync}^{(n)}
\]
\[
= e_{Sync}^{(n)} + \hat{K}^T \dot{e}_{SS}
\]  

Equation (67) is represented by using (45) and (47) as:

\[
\dot{s}(t) = e_{Sync}^{(n-1)} + \hat{K}^T \dot{e}_{SS} = u_{IDC} - u
\]
\[ V_A(t) = \frac{1}{2} s^T(t) s(t) + \frac{1}{2\lambda_p} tr(\hat{p}^T \hat{p}) + \frac{1}{2\lambda_q} tr(\hat{q}^T \hat{q}) + \frac{1}{2\lambda_e} tr(\hat{z}^T \hat{z}) \]

\[ + \frac{1}{2\lambda_{\hat{m}}} tr(\hat{z}^{\hat{m}} \hat{z}^{\hat{m}}) + \frac{1}{2\lambda_{\hat{a}}} \hat{a}^T \hat{a} + \frac{1}{2\hat{\lambda}} \hat{\Delta} \]

Taking the derivative of (79), then using (65) and (68), gives

\[ \dot{V}_A(t) = s^T(t) \dot{s}(t) + \frac{1}{\lambda_p} tr(\dot{p}^T \hat{p}) + \frac{1}{\lambda_q} tr(\dot{q}^T \hat{q}) + \frac{1}{\lambda_e} tr(\dot{z}^T \hat{z}) \]

\[ + \frac{1}{\lambda_{\hat{m}}} tr(\hat{z}^{\hat{m}} \dot{m}) + \frac{1}{\lambda_{\hat{a}}} \dot{a}^T \hat{a} + \frac{1}{\hat{\lambda}} \hat{\Delta} \]

Then, (80) becomes

\[ \frac{\dot{z}^{\hat{m}}}{\hat{m}} = -\frac{\hat{z}^{\hat{a}}}{\hat{a}} = \hat{m}, \frac{\dot{z}^{\hat{a}}}{\hat{a}} = -\hat{z}^{\hat{m}} = -\hat{m}, \]
\[
\dot{V}_A(t) = \left[ \sum_{k=1}^{L} p_k^T \left( s_k(t) \left( a_{p_k} z^a - o_{p_k} z^a \right) - \frac{\dot{p}_k}{\lambda_p} \right) \right] + \left[ \sum_{k=1}^{L} q_k^T \left( s_k(t) \left( a_{q_k} z^a - o_{q_k} z^a \right) - \frac{\dot{q}_k}{\lambda_q} \right) \right] \\
+ \left[ \sum_{k=1}^{L} r_k^T \left( s_k(t) \left( a_{r_k} z^a - o_{r_k} z^a \right) - \frac{\dot{r}_k}{\lambda_r} \right) \right] + \left[ \sum_{k=1}^{L} \zeta^{20T} \left( s(t) \tilde{a} - \frac{\zeta^{20}}{\lambda_{\zeta}} \right) \right] + \left[ \sum_{k=1}^{L} \zeta^{20T} \left( -s(t) \dot{\tilde{a}} - \frac{\zeta^{20}}{\lambda_{\zeta}} \right) \right] + \bar{m}^T \left[ s^T(t) \left[ a_z z^a - o_z z^a \right] - \frac{\bar{m}}{\lambda_m} \right] \\
+ \bar{\sigma}^T \left[ s^T(t) \left[ a_{\bar{\sigma}} z^a - o_{\bar{\sigma}} z^a \right] - \frac{\bar{\sigma}}{\lambda_{\bar{\sigma}}} \right] + \bar{\sigma}^T \left[ s^T(t) \left[ a_{\bar{\sigma}} z^a - o_{\bar{\sigma}} z^a \right] - \frac{\bar{\sigma}}{\lambda_{\bar{\sigma}}} \right] \\
+ s^T(t) [\Pi_{AE}(t) - u_{SRB}] + \frac{\Delta \Lambda}{\Lambda} \\
\]

(81)

Via the adaptive laws in (69)–(77) and the robust controller in (78a), (81) is then rewritten as:

\[
\dot{V}_A(t) = s^T(t) [\Pi_{AE}(t) - u_{SRB}] + \frac{\Delta \Lambda}{\Lambda} \\
= s^T(t) [\Pi_{AE}(t) - \Lambda \text{sgn}(s(t)))] + \frac{\Delta \Lambda}{\Lambda} \\
= s^T(t) [\Pi_{AE}(t) - \overline{\Lambda} s(t)] + \frac{\Delta \Lambda}{\Lambda} \\
\]

(82)

If the error bound is updated as:

\[
\dot{\overline{\Lambda}} = -\overline{\Lambda} = -\lambda_{\overline{\Lambda}} |s(t)| ,
\]

(83)

where \( \lambda_{\overline{\Lambda}} \) is the positive learning-rate constant, then (82) is rewritten as:

\[
\dot{V}_A(t) = s^T(t) [\Pi_{AE}(t) - \overline{\Lambda} s(t)] - \left( \overline{\Lambda} - \overline{\Lambda} \right) |s(t)| \leq \left( |\Pi_{AE}(t)||s(t)| - \overline{\Lambda} |s(t)| \right) \\
= -\left( \overline{\Lambda} - |\Pi_{AE}(t)| \right) |s(t)| \leq 0 \\
\]

(84)

Since \( \dot{V}_A(t) \) is negative semi-definite, \( \dot{V}_A(t) \leq 0 \), it points out that \( s(t) \) and \( \overline{\Lambda} \) are bounded. Define \( \Gamma(t) \triangleq \left( \overline{\Lambda} - |\Pi_{AE}(t)| \right) s(t) \leq \left( |\Pi_{AE}(t)| \right) |s(t)| \leq -\dot{V}_A(t) \), then taking the integral \( \Gamma(t) \) with respect to time gives

\[
\int_{0}^{t} \Gamma(\tau) d\tau \leq \dot{V}_A(0) - \dot{V}_A(t) \\
\]

(85)

Since \( \dot{V}_A(0) \) is bounded, and \( \dot{V}_A(t) \) does not increase and bound, then obtain

\[
\lim_{t \to \infty} \int_{0}^{t} \Gamma(\tau) d\tau < \infty \\
\]

(86)

Furthermore, \( \dot{\Gamma}(t) \) is bounded, then \( \lim_{t \to \infty} \dot{\Gamma}(t) = 0 \). It means \( s \to 0 \) when \( t \to \infty \) (Slotine and Li 1991). Consequently, the IT2FFFLBC control system is asymptotically stable for the case \( |s| > \psi \).

The robust compensator in (78a) employs a \( \text{sgn}(.) \) function to warrant the system stability. However, the robust compensation controller is normally intermittent across \( s \). It implies that the control input will occur the chattering phenomenon. To prove the stability for the case \( |s| \leq \psi \), the second Lyapunov function is selected as:

\[
\dot{V}_d(t) = \frac{1}{2} s^T(t) s(t) + \frac{1}{2r_p} \text{tr}(\dot{p}^T \dot{p}) + \frac{1}{2r_q} \text{tr}(\dot{q}^T \dot{q}) + \frac{1}{2} (\dot{q}^T \dot{q}) \\
+ \frac{1}{2} \text{tr}(\dot{\tilde{a}}^T \dot{\tilde{a}}) + \frac{1}{2} \text{tr}(\dot{\tilde{a}}^T \dot{\tilde{a}}) + \frac{1}{2} \text{tr}(\dot{\tilde{a}}^T \dot{\tilde{a}}) \\
+ \frac{1}{2\lambda_m} \bar{m}^T \bar{m} + \frac{1}{2\lambda_{\bar{\sigma}}} \bar{\sigma}^T \bar{\sigma} + \frac{1}{2\lambda_{\bar{\sigma}}} \bar{\sigma}^T \bar{\sigma} + \frac{\bar{\Delta}^2}{2\lambda_{\bar{\sigma}}} \\
\]

(87)

where \( \lambda_{\overline{\Lambda}} \) is the positive learning-rate constant.

In theory, there is an optimal constant \( \Delta^* \) that satisfies the robust stability for (78b) as follows:

\[
\Delta^* |s| > |\Pi_{AE}(t)| \\
\]

(88)

Taking the derivative of (87) and using (69)–(77) and (78b) gives

\[
\dot{V}_d(t) \\
\]

(89)

\[
\dot{V}_d(t) < 0 \\
\]

(90)
\( \dot{V}_B(t) = s^T[\Pi_{AE}(t) - \Delta] = s^T[\Pi_{AE}(t) - \Delta s] + \frac{\Delta s}{\lambda} \)
\[ \Rightarrow s^T[\Pi_{AE}(t) - \Delta s] + \frac{\Delta s}{\lambda} = s^T[\Pi_{AE}(t) - \Delta s] + \frac{\lambda}{\lambda} = s^T[\Pi_{AE}(t) - \Delta s] + \frac{\lambda}{\lambda} \tag{89} \]

The parameter estimation law is selected as
\[ \dot{\Delta} = -\hat{\Delta} = -\lambda s^T s, \tag{90} \]

Thus, (89) becomes
\[ \dot{V}_B(t) = s^T[\Pi_{AE}(t) - \Delta s] - \left( \Delta^* - \hat{\Delta} \right) s^T s \]
\[ = (s^T[\Pi_{AE}(t) - \Delta^* s] s \leq (s^T[\Pi_{AE}(t)] ||s| - \Delta^* s| ||s||) \]
\[ \leq -\Delta^* ||s| - [\Pi_{AE}(t)]||s|| \leq 0 \tag{91} \]

Therefore, the IT2FFFLBC control system is asymptotically stable for the case \( ||s|| \leq \psi \). As a result of two cases, the proof is complete.

5 Simulation results

This study uses two 4D nonlinear chaotic systems, a 4D hyperchaotic Lorenz–Lu system and a 4D hyperchaotic Rikitake dynamo system, to demonstrate the effectiveness and superiority of the proposed structure.

Fig. 4 External noises using for synchronization of the 4D hyperchaotic Lorenz–Lu system

5.1 Chaos synchronization for 4D hyperchaotic Lorenz–Lu system (Chen et al. 2006, 2011)

The MS for the 4D hyperchaotic Lorenz’s system is given as:
\[ \begin{align*}
\dot{x}_{MS1}(t) &= a_{MS} \times (x_{MS2}(t) - x_{MS1}(t)) \\
\dot{x}_{MS2}(t) &= b_{MS} \times x_{MS1}(t) + x_{MS2}(t) x_{MS3}(t) - x_{MS4}(t) \\
\dot{x}_{MS3}(t) &= x_{MS1}(t) x_{MS2}(t) - c_{MS} \times x_{MS4}(t) \\
\dot{x}_{MS4}(t) &= d_{MS} \times x_{MS2}(t) x_{MS3}(t)
\end{align*} \tag{92} \]

The SS of the 4D hyperchaotic Lu’s system is given as:
\[ \begin{align*}
\dot{y}_{SS1}(t) &= a_{SS} \times (y_{SS2}(t) - y_{SS1}(t)) + y_{SS4}(t) + u_{SS1}(t) + n_{SS1}(t) \\
\dot{y}_{SS2}(t) &= b_{SS} \times y_{SS2}(t) - y_{SS1}(t) y_{SS3}(t) + u_{SS2}(t) + n_{SS2}(t) \\
\dot{y}_{SS3}(t) &= y_{SS1}(t) y_{SS2}(t) - c_{SS} \times y_{SS3}(t) + u_{SS3}(t) + n_{SS3}(t) \\
\dot{y}_{SS4}(t) &= d_{SS} \times y_{SS4}(t) + y_{SS1}(t) y_{SS3}(t) + u_{SS4}(t) + n_{SS4}(t)
\end{align*} \tag{93} \]

where \( x_{MS1}(t), x_{MS2}(t), x_{MS3}(t), x_{MS4}(t), y_{SS1}(t), y_{SS2}(t), y_{SS3}(t), \) and \( y_{SS4}(t) \) are, respectively, the states of MS and SS. The parameters of the MS and SS are set as \( a_{MS} = 10, b_{MS} = 28, c_{MS} = 8/3, d_{MS} = 0.1, a_{SS} = 36, b_{SS} = 20, c_{SS} = 3, d_{SS} = 1 \), respectively. In this example, the external noises (see Fig. 4) are assumed as \( n_{SS}(t) = [n_{SS1}(t), n_{SS2}(t), n_{SS3}(t), n_{SS4}(t)] = [0.2 \times \cos(\pi t), 0.1 \times \sin(t), 0.3 \times \sin(2t), 0.1 \times \cos(t)] \), the initial position states for the MS \( x_{MS1}(0) = -1.0, x_{MS2}(0) = -1.0, x_{MS3}(0) = 1.0, x_{MS4}(0) = 1.0 \) and the SS \( y_{SS1}(0) = 5.0, y_{SS2}(0) = 2.0, y_{SS3}(0) = -5.0, y_{SS4}(0) = -2.0 \) are
used. The initial parameters for learning rates are selected as
\( \lambda_p = \lambda_q = 0.2, \lambda_r = \lambda_s = 0.1, \lambda_w = \lambda_m = 0.5, \lambda_\delta = 0.05, \) and \( \lambda_\lambda = 0.5, \) and \( K_1 = 3.8I_{4 \times 4}, \) \( K_2 = 0.8I_{4 \times 4} \). The simulation results for this example are displayed in Figs. 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17. Particularly, the 3D trajectory phase portraits of the 4D synchronizations are manifested in Figs. 5, 6, 7, 8. The 2D state trajectories are plotted in Figs. 9, 10, 11, 15. In addition, the synchronization errors are displayed in Fig. 16, which points out that the tracking errors are quickly driven to zero. Finally, the control efforts are presented in Fig. 17. The simulation results of synchronization for the 4D hyperchaotic Lorenz–Lu system attain good performance and rapid response regardless of the influence of external noises and uncertainty parameters of the system. Moreover, the total RMSE is measured to demonstrate that the proposed IT2FFFLBC control system

\[ \text{Fig. 5} \quad \text{3D trajectory phase portrait of master (X1, X2, X3) and slave (Y1, Y2, Y3)} \]

\[ \text{Fig. 6} \quad \text{3D trajectory phase portrait of master (X1, X2, X4) and slave (Y1, Y2, Y4)} \]
synchronizes the master–slave systems well with smaller tracking errors than other control systems (see Table 1). The results for the proposed IT2FFFLBC control system are also compared with some former methods such as adaptive PID (Chen et al. 2011), radial basis function neural network (RBFNN) (Chen et al. 2011), neuro-wavelet control (NWC) using integral (I)-type training method (Chen et al. 2011), and NWC using (proportional–integral) PI-type training method (Chen et al. 2011). From that, we can see the IT2FFFLBC control system is qualified to handle well the noises and uncertainties than the other control systems. Table 1 points out that the proposed IT2FFFLBC control system has a smaller root mean square error (RMSE) and attains better in synchronizing the 4D hyperchaotic Lorenz–Lu system with smaller tracking errors than other methods. Our findings indicate that the IT2FFFLBC control system can work well with the impact of external noises and uncertainties of system parameters.
Fig. 9 The state trajectory for master (X1, X2) and slave (Y1, Y2)

Fig. 10 The state trajectory for master (X1, X3) and slave (Y1, Y3)
Fig. 11 The state trajectory for master (X1, X4) and slave (Y1, Y4)

Fig. 12 The state trajectory for master (X2, X3) and slave (Y2, Y3)
Fig. 13 The state trajectory for master (X2, X4) and slave (Y2, Y4)

Fig. 14 The state trajectory for master (X3, X4) and slave (Y3, Y4)
Fig. 15 State trajectories for master–slave (X1, Y1), (X2, Y2), (X3, Y3), and (X4, Y4)

Fig. 16 Errors for synchronization of the 4D hyperchaotic Lorenz–Lu system
Fig. 17 Control efforts for synchronization of the 4D hyperchaotic Lorenz–Lu system

![Control efforts for synchronization of the 4D hyperchaotic Lorenz–Lu system](image)

Table 1 The comparison of RMSE between IT2FFFLBC control system and the former methods

|                      | RMSE1 | RMSE2 | RMSE3 | RMSE4 | Average RMSE |
|----------------------|-------|-------|-------|-------|--------------|
| Adaptive PID         | 8.2922| 5.8457| 1.9369| 9.7788| 6.4634       |
| RBFNN                | 2.6765| 1.9455| 1.0070| 3.1995| 2.2071       |
| NWC with I-type training method | 1.5102| 1.0407| 0.6064| 1.6916| 1.2122       |
| NWC with PI-type training method | 0.3828| 0.2693| 0.1324| 0.4483| 0.3082       |
| IT2FFFLBC control system | 0.1219| 0.0924| 0.0671| 0.1842| 0.1164       |

Fig. 18 External noises using for synchronization of the 4D hyperchaotic Rikitake dynamo system

![External noises using for synchronization of the 4D hyperchaotic Rikitake dynamo system](image)
5.2 Chaos synchronization for 4D hyperchaotic Rikitake two-wing dynamo system (Vaidyanathan et al. 2018)

The MS is given as:

\[
\begin{align*}
\dot{x}_{MS1}(t) &= -\alpha_{MS} x_{MS1}(t) + x_{MS2}(t) x_{MS3}(t) - x_{MS4}(t) \\
\dot{x}_{MS2}(t) &= -\alpha_{MS} x_{MS2}(t) + (\beta_{MS} x_{MS1}(t) - \gamma_{MS} x_{MS3}(t) - x_{MS4}(t) \\
\dot{x}_{MS3}(t) &= 1 - x_{MS1}(t) x_{MS2}(t) \\
\dot{x}_{MS4}(t) &= \gamma_{MS} x_{MS2}(t)
\end{align*}
\]

and the SS is given as:

\[
\begin{align*}
\dot{y}_{SS1}(t) &= -\alpha_{SS} y_{SS1}(t) + y_{SS2}(t) y_{SS3}(t) - y_{SS4}(t) + u_{SS1}(t) + n_{SS1}(t) \\
\dot{y}_{SS2}(t) &= -\alpha_{SS} y_{SS2}(t) + (\beta_{SS} y_{SS1}(t) - \gamma_{SS} y_{SS3}(t) - y_{SS4}(t) + u_{SS1}(t) + n_{SS1}(t) \\
\dot{y}_{SS3}(t) &= 1 - y_{SS1}(t) y_{SS2}(t) + u_{SS1}(t) + n_{SS1}(t) \\
\dot{y}_{SS4}(t) &= \gamma_{SS} y_{SS2}(t) + u_{SS1}(t) + n_{SS1}(t)
\end{align*}
\]

where \( x_{MS1}(t), x_{MS2}(t), x_{MS3}(t), x_{MS4}(t), y_{SS1}(t), y_{SS2}(t), y_{SS3}(t), \) and \( y_{SS4}(t) \) are the states of the MS and SS. The parameters of the MS and SS are set as \( \alpha_{MS} = \alpha_{SS} = 1.0, \beta_{MS} = \beta_{SS} = 1.0, \gamma_{MS} = \gamma_{SS} = 0.7 \). In this example, the external noises (see Fig. 18) are given as
\( n_{SS}(t) = [n_{SS1}(t), n_{SS2}(t), n_{SS3}(t), n_{SS4}(t)] = [0.2 \times \cos(\pi \times t), 0.1 \times \cos(t), 0.3 \times \cos(2 \times t), 0.1 \times \cos(t)] \),

the initial position states of the MS \( x_{MS1}(0) = 0.4, x_{MS2}(0) = 0.4, x_{MS3}(0) = 0.4, x_{MS4}(0) = 0.4 \) and the SS \( y_{SS1}(0) = -1.0, y_{SS2}(0) = -1.0, y_{SS3}(0) = -1.0, y_{SS4}(0) = -1.0 \) are utilized. The initial parameters of the IT2FFFLBC control system are \( \hat{\lambda}_p = \hat{\lambda}_\rho = 0.5, \hat{\lambda}_q = \hat{\lambda}_\varphi = 0.1, \hat{\lambda}_v = \hat{\lambda}_\epsilon = 0.2, \hat{\lambda}_m = 0.5, \hat{\lambda}_a = \hat{\lambda}_\eta = 0.1, \) and \( \lambda_\Delta = 0.5 \). The simulation results using the proposed IT2FFFLBC control system for the 4D hyperchaotic Rikitake two-wing dynamo system are plotted in Figs. 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31. Specifically, the 3D results are plotted in Figs. 19, 20, 21, 22. The 2D results of state trajectories are illustrated in Figs. 23, 24, 25, 26, 27, 28, 29, respectively. Subsequently, the synchronization errors are shown in Fig. 30, which indicate that they are rapidly converged to zero. Then, the control efforts are shown in Fig. 31. Initially, the control efforts are increased to suitable values, then they are reduced to nearly zero after achieving the synchronization. According to the simulation results, our findings point out that the synchronization of the 4D hyperchaotic Rikitake two-wing dynamo system attains good performance and quick response even with the impact of external noises and uncertain parameters.

Fig. 21 3D trajectory phase portrait of master (X1, X3, X4) and slave (Y1, Y3, Y4)

Fig. 22 3D trajectory phase portrait of master (X2, X3, X4) and slave (Y2, Y3, Y4)
results in RMSE of some recent controllers such as a recurrent cerebellar model articulation controller (RCMAC) (Huynh et al. 2020a, b, c, d), a fuzzy brain emotional learning controller (Lin and Chung 2015), a brain-imitated neural network controller (Lin et al. 2021a, b), and the proposed IT2FFFLBC control system are compared and shown in Table 2. The proposed IT2FFFLBC control system synchronizes well the master–slave systems with smaller tracking errors than other controllers. In summary, from the simulation results of two
Fig. 25 The state trajectory of master \((X_1, X_4)\) and slave \((Y_1, Y_4)\)

Fig. 26 The state trajectory of master \((X_2, X_3)\) and slave \((Y_2, Y_3)\)
examples, our findings show that the IT2FFLBC control system can work well for 4D nonlinear hyperchaotic systems with the impact of external noises.

6 Conclusion

In this research, we design the interval type-2 fuzzy four-fold function-link brain emotional controller for 4D nonlinear hyperchaotic systems. The principal novelty of this research is the successful design of the new fourfold function-link for the IT2FFLBC that can adjust efficiently
Fig. 29 The state trajectory of master–slave (X1, Y1), (X2, Y2), (X3, Y3), and (X4, Y4).

Fig. 30 The errors of synchronization for the 4D hyperchaotic Rikitake dynamo system.
the lower and upper weights for orbitofrontal cortex and amygdala networks. A smooth robust compensator is used to eliminate undesired approximate errors and to avoid the chattering phenomenon. Two Lyapunov functions are utilized to determine the online learning laws for tuning network parameters and to prove the stability of the system. Subsequently, the simulation results of two 4D hyperchaotic Lorenz–Lu and 4D hyperchaotic Rikitake two-wing dynamo systems show that the IT2FFFLBC control system efficiently achieves good synchronization. In summary, the proposed controller can deal with system uncertainty and external noise with small tracking errors. However, the major limitation of the proposed scheme is that the learning rates of the adaptive laws are selected by trial-and-error to improve the control performance. The learning rates are very important and they will change the control performance significantly. Our future work will apply some optimization algorithms to select the optimal learning rates of the proposed control system. The other future study is to apply the proposed control system to some practical nonlinear systems.

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**Declaration**

**Conflict of interest** The authors declare that they have no conflict of interest.

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