INTERTWINING DIFFUSIONS AND WAVE EQUATIONS

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Abstract. We develop a general theory of intertwined diffusion processes of any dimension. Our main result gives an SDE characterization of all possible intertwinings of diffusion processes and shows that they correspond to nonnegative solutions of hyperbolic partial differential equations. For example, solutions of the classical wave equation correspond to the intertwinings of two Brownian motions. The theory allows us to unify many older examples of intertwinings, such as the process extension of the beta-gamma algebra, with more recent examples such as the ones arising in the study of two-dimensional growth models. We also find many new classes of intertwinings and develop systematic procedures for building more complex intertwinings by combining simpler ones. In particular, ‘orthogonal waves’ combine unidimensional intertwinings to produce multidimensional ones. Connections with duality, time reversals, and Doob’s h-transforms are also explored.

1. Introduction

We start with the definition of intertwining of two Markov semigroups that is reminiscent of a similarity transform of two finite dimensional matrices.

Definition 1. Let \((Q_t, t \geq 0), (P_t, t \geq 0)\) be two Markov semigroups on measurable spaces \((E_1, B_1), (E_2, B_2)\), respectively. Suppose \(L\) is a stochastic transition operator that maps bounded measurable functions on \(E_2\) to those on \(E_1\). We say that the ordered pair \((Q, P)\) is intertwined with link \(L\) if for all \(t \geq 0\) the relation \(Q_t L = LP_t\) holds. If this is the case, we write \(Q \langle L \rangle P\).

It is clear that intertwinings are special constructions which transfer a lot of spectral information from one semigroup to the other. Naturally one is interested in two kinds of broad questions: (a) Given two semigroups can we determine if they are intertwined via some link? (b) Can we find a coupling of two Markov processes, with transition semigroups \((Q_t)\) and \((P_t)\), respectively, such that the coupling construction naturally reflects the intertwining relationship? One should also ask what influence the analytic definition of intertwining has on the path properties of this coupling.

Question (a) is known to have an affirmative answer when the transition probabilities of a Markov process have symmetries. One can then intertwine this process with another process running on the quotient space. Other criteria were given based on the explicit knowledge of eigenvalues of the semigroup. Neither symmetries nor eigenvalues are generally available, and, hence, the answer to question (a) for general Markov processes is unknown. In the next subsection we outline briefly the development in this area over the last few decades.

On the other hand, Diaconis and Fill \cite{DF90} initiated a program of constructing couplings of two Markov chains whose semigroups \((Q_t)\) and \((P_t)\) satisfy \(Q \langle L \rangle P\). Such couplings lead to remarkable objects called strong stationary times which can be then used to determine the convergence rate of the Markov chain with transition semigroup \((P_t)\).

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Our main result settles both questions (a) and (b) when the semigroups are diffusion semigroups and we insist on the coupling to be a joint diffusion. We provide a general theory of intertwinings in the setting of diffusion processes allowing also for (possibly oblique) reflection at the boundary of their domains and on each other. In fact, our result provides a complete characterization of intertwinings for diffusion processes in terms of their joint stochastic differential equations. This, on the one hand, allows us to reprove all the intertwining relations known so far, as well as to produce several large classes of new examples.

It turns out that in this setup the link kernels are solutions to hyperbolic partial differential equations, such as the classical wave equation in the case of intertwinings of two Brownian motions (see Theorems 1 and 2 below for the details). This is interesting in itself since, to the best of our knowledge, solutions of hyperbolic equations (or, wave equations) have not had any probabilistic representation so far.

Throughout the paper we consider diffusion semigroups on finite-dimensional Euclidean spaces. Here, by a diffusion semigroup we mean a semigroup generated by a second order elliptic partial differential operator with no zero-order terms and either no boundary conditions or (possibly oblique) Neumann boundary conditions. Before we describe our coupling construction we recall a key concept in the Diaconis-Fill construction, namely the commutative diagram in Figure 1, which we have extended to the continuous time setting.

We consider two Markov process in continuous time, \( Z_1 \) and \( Z_2 \), with transition semi-groups \((P_t)\) and \((Q_t)\), respectively. The direction of arrows represents the action on measures (as opposed to that on functions). The diagram captures the following equivalence of sampling schemes: starting from \( Z_2(s) \) it is possible to generate a sample of \( Z_1(s+t) \) in two equivalent ways. Either sample \( Z_2(s+t) \), conditionally on \( Z_2(s) \) and then sample \( Z_1(s+t) \) according to \( L \). Or, sample \( Z_1(s) \), conditionally on \( Z_2(s) \), via \( L \), and follow \( Z_1 \) to time \( (s+t) \). It is a part of the construction that both \((Z_2(s), Z_2(s+t), Z_1(s+t))\) and \((Z_2(s), Z_1(s), Z_1(s+t))\) are three step Markov chains. This insistence produces a coupling with nice path properties that can be further exploited.

The above discussion motivates the following definition of a coupling realization of \( Q \langle L \rangle P \) in terms of random processes. Let \((X(t), t \geq 0)\) and \((Y(t), t \geq 0)\) represent two time-homogeneous diffusions with state spaces \( \mathcal{X} \subset \mathbb{R}^m \), \( \mathcal{Y} \subset \mathbb{R}^n \) and transition semigroups \((P_t, t \geq 0)\), \((Q_t, t \geq 0)\), respectively. We abuse the notation slightly. Although, \( X \) and \( Y \) are diffusions, their laws are unspecified because we do not specify their initial distributions. They are merely processes with the correct transition semigroup. We also suppose that \( L \) is a probability transition operator.

**Definition 2.** We call a \( \mathcal{X} \times \mathcal{Y} \)-valued diffusion process \( Z = (Z_1, Z_2) \) an intertwining of the diffusions \( X \) and \( Y \) with link \( L \) (we say \( Z = Y \langle L \rangle X \)) if the following hold.
(i) $Z_1 \overset{d}{=} X$ and $Z_2 \overset{d}{=} Y$, where $\overset{d}{=}$ refers to identity in law, and
\[
\mathbb{E} [ f (Z_1(0)) \mid Z_2(0) = y] = (Lf)(y),
\]
for all bounded Borel measurable function $f$ on $\mathcal{X}$.
(ii) The transition semigroups are intertwined: $Q \langle L \rangle P$.
(iii) The process $Z_1$ is Markovian with respect to the joint filtration generated by $(Z_1, Z_2)$.
(iv) For any $s \geq 0$, conditional on $Z_2(s)$, the random variable $Z_1(s)$ is independent of 
$(Z_2(u), 0 \leq u \leq s)$, and is conditionally distributed according to $L$.

The conditional independence condition (iii) captures the idea that $Z_2$ is obtained from $Z_1$ with possible excess independent noise, and condition (iv) is the notion that the process that concatenates $Z_1(s)$ to $\{Z_2(u), 0 \leq u \leq s\}$ is Markov. See Figure 1 for an illustration.

Our primary results Theorem 1 and Theorem 2 answer the questions (b) and (a), respectively, raised in the beginning of the introduction. Suppose we are given the two generators
\[
\mathcal{A}^X = \sum_{i=1}^{m} b_i(x) \partial_{x_i} + \frac{1}{2} \sum_{k,j=1}^{m} a_{ij}(x) \partial_{x_k} \partial_{x_j} \quad \text{and}
\]
\[
\mathcal{A}^Y = \sum_{k=1}^{n} \gamma_k(y) \partial_{y_k} + \frac{1}{2} \sum_{k,l}^{n} \rho_{kl}(y) \partial_{y_k} \partial_{y_l},
\]
where $(b_i)_{i=1}^{m}$ is an $\mathbb{R}^m$-valued function, $(\gamma_k)_{k=1}^{n}$ is an $\mathbb{R}^n$-valued function, $(a_{ij})_{1 \leq i,j \leq m}$ and $(\rho_{kl})_{1 \leq k,l \leq n}$ are functions taking values in the set of positive semidefinite $m \times m$ and $n \times n$ matrices, respectively. We make the following assumption.

**Assumption 1.** Assume that each $X$ and $Y$ satisfy either one of the following two conditions.

(a) **No boundary conditions.** The martingale problem corresponding to $\mathcal{A}^X$ on $\mathcal{X}$ with no boundary conditions is well-posed in the sense of [SV79]. Moreover, the Cauchy problem for $\mathcal{A}^X$ has a unique generalized solution in the space $C_b([0, \infty) \times \mathcal{X})$ of continuous bounded functions for any initial condition in $C_b(\mathcal{X})$. For the $Y$ diffusion replace $\mathcal{A}^X$ by $\mathcal{A}^Y$, $\mathcal{X}$ by $\mathcal{Y}$, and so on.

(b) **Neumann boundary conditions.** The domain $\mathcal{X}$ is smooth. Moreover, for some smooth and nowhere vanishing vector field $U_1 : \partial \mathcal{X} \to \mathbb{R}^m$ the submartingale problem corresponding to $\mathcal{A}^X$ with Neumann boundary conditions with respect to $U_1$ is well-posed in the sense of [SV71]. Moreover, the initial-boundary value problem for $\mathcal{A}^X$ with these boundary conditions has a unique generalized solution in the space $C_b([0, \infty) \times \mathcal{X})$ for any initial condition in $C_b(\mathcal{X})$. For $Y$ replace $\partial \mathcal{X}$ by $\partial \mathcal{Y}$, $U_1$ by $U_2$, and so on.

**Theorem 1.** Let $X$, $Y$ be the (reflected) diffusions given by the solutions of the above martingale (submartingale resp.) problems. Suppose that $L$ is given by an integral operator
\[
(Lf)(y) = \int_{\mathcal{X}} f(x) \Lambda(y, x) \, dx
\]
such that the density function $\Lambda$ is differentiable in $y$ in the interior of $\mathcal{Y}$ for every fixed $x$.
Set $V = \log \Lambda$ and let $V_{y_k}$ denote the partial derivative of $V$ with respect to $y_k$.
Consider $z \in \mathbb{R}^{m+n}$ as $z = (x, y)$, where $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$. Let $Z = (Z_1, Z_2)$ be a diffusion process on $\mathcal{X} \times \mathcal{Y}$ with generator
\[
\mathcal{A}^Z = \mathcal{A}^X + \mathcal{A}^Y + \sum_{k,l=1}^{n} \rho_{kl}(y) (V_{y_k}(y, x)) \partial_{y_l}
\]
and boundary conditions on $\partial \mathcal{X} \times \mathcal{Y}$ ($\mathcal{X} \times \partial \mathcal{Y}$ resp.) coinciding with those of $\mathcal{X}$ on $\partial \mathcal{X}$ ($\mathcal{Y}$ on $\partial \mathcal{Y}$ resp.). Suppose that $\mathcal{A}^Z$ is of one of the two types described in Assumption 1. Moreover, let the initial condition of the diffusion $Z$ satisfy

$$P(Z_1(0) \in B \mid Z_2(0) = y) = \int_B \Lambda(y, x) \, dx, \quad \text{for all Borel } B \subseteq \mathbb{R}^m.$$  

If $\Lambda$ is a distributional solution of the hyperbolic PDE

$$(A^X)^* \Lambda = A^Y \Lambda$$

on $\mathcal{X} \times \mathcal{Y}$ with boundary conditions of $(A^X)^*$ on $\partial \mathcal{X} \times \mathcal{Y}$ and of $A^Y$ on $\mathcal{X} \times \partial \mathcal{Y}$ (with $(A^X)^*$ being the adjoint of $A^X$ in the distributional sense), then $Z = Y \langle L \rangle \mathcal{X}$.

As a quick example, consider the Cauchy density kernel

$$\Lambda(y, x) = \frac{1}{\pi (1 + (y - x)^2)}.$$  

It satisfies the one-dimensional wave equation. Consider the diffusion given by

$$dZ_1(t) = d\beta_1(t), \quad dZ_2(t) = d\beta_2(t) - \left( \frac{2 (Z_2(t) - Z_1(t))}{1 + (Z_2(t) - Z_1(t))^2} \right) \, dt,$$

where $\beta_1, \beta_2$ are two independent one-dimensional standard Brownian motions. Then, by Theorem 1, for appropriate initial conditions the marginal law of $Z_2$ is that of a standard Brownian motion and the conditional law of $Z_1(t)$ given $Z_2(t)$ is Cauchy for every $t \geq 0$.

**Theorem 2.** Suppose that the generators $A^X, A^Y$ of (1.1), (1.2) satisfy Assumption 1 and let $\mathcal{X}, \mathcal{Y}$ be the corresponding diffusion processes. Suppose there is a diffusion process $Z$ satisfying conditions (i), (ii) in Definition 2 and the following conditional independence conditions.

(iii)’ Given $Z_1(0)$ and any $t > 0$, the random variables $Z_2(0)$ and $Z_1(t)$ are conditionally independent.

(iv)’ Given $Z_2(t)$, the random variables $Z_2(0)$ and $Z_1(t)$ are conditionally independent.

Then the commutativity relation

$$(L A^X) f = (A^Y L) f$$

holds for all functions $f$ such that $f \in \text{dom}(L) \cap \text{dom}(A^X)$ and $(A^X f) \in \text{dom}(L)$. Moreover, the generator of $Z$ is given by (1.3) with the boundary conditions as in Theorem 1 and $\Lambda$ is a distributional solution of the hyperbolic PDE (1.4) with the boundary conditions as in Theorem 1.

In the analytic literature the commutativity relation (1.5) is usually referred to as transmutation of the operators $A^X$ and $A^Y$. The latter is a classical concept in the study of partial differential equations and goes back to Euler, Poisson and Darboux in the case that $A^X$ is the Laplacian and $A^Y$ is its radial part (or, in other words, the generator of a Bessel process). An excellent introduction to this area is the book [Car82b] by Carroll which, in particular, stresses the role that special functions play in the theory of transmutations.

The rest of the paper is structured as follows.

(i) We end the introduction with the following subsection that reviews the literature that has led to the development of the subject so far.

(ii) In Section 2 we give the proofs of Theorems 1 and 2. We also prove a generalization to diffusions reflecting on moving boundaries and establish an important connection to harmonic functions and Doob’s h-transforms.
In Section 3 we explore the Markov chain of diffusions induced by intertwinings. We also explore the deep connection of intertwining with duality which demonstrates how the direction of intertwining reverses with time-reversal. We also construct simultaneous intertwining that allows us to couple multiple duals with the same diffusion.

Section 4 is in two parts. The first collects most known examples and shows that they are all covered by our results. This includes recent examples such as the 2d-Whittaker growth model (related to the Hamiltonian of the quantum Toda lattice). In the second part, we produce classes of new examples by solving the corresponding hyperbolic partial differential equations.

In Section 5 we cover diffusions reflected on a moving boundary. A major example is the Warren construction of interlacing Dyson Brownian motions on the Gelfand-Tsetlin cone for which we give two new proofs.

Section 6 explains how intertwinings can be used to give bounds on the rate of convergence to equilibrium for diffusion semigroups.

Finally, an appendix has been added on the literature on common hyperbolic PDEs for the benefit of a reader with a probability background.

1.1. A brief review of the literature. The study of intertwinings started with the question of when a function of a Markov process is again a Markov process. General criteria were given by Dynkin (see [Dyn65]), Kemeny and Snell (see [KS76]), and Rosenblatt (see [Ros11]). In [RP81], Rogers and Pitman derived a new criterion of this type and used it to reprove the celebrated $2M - B$ Theorem of Pitman (see [Pit75] for the original result and [JY79] by Jeulin and Yor for yet another proof). These examples have been reviewed in detail in Section 4.

Pitman’s Theorem triggered an extensive study of functionals of Brownian motion (and, more generally, of Lévy processes) through intertwining relations. Notable examples include the articles by Matsumoto and Yor (see [MY00], [MY01]) which extend Pitman’s Theorem to exponential functionals of Brownian motion by exploiting the fact that the latter are intertwined with the Brownian motion itself (see also Baudoin and O’Connell [BO11] for an extension to higher dimensions); the paper [CPY98] by Carmona, Petit, and Yor presents a new class of intertwining relations between Bessel processes of different dimensions, which can be viewed as the process extension of the well-known Beta-Gamma algebra; the article [Dub04] by Dubédat shows that a certain reflected Brownian motion in a two-dimensional wedge is intertwined with a 3-dimensional Bessel process and uses this fact to derive formulas for some hitting probabilities of the former; and the paper [Yor94] extends the results in [MY00], [MY01] further to exponential functionals of Lévy processes.

More recently, intertwining relations were discovered in the study of random matrices and related particle systems. In [DMDMY04], the authors Donati-Martin, Doumerc, Matsumoto, and Yor give a matrix version of the findings in [CPY98], namely an intertwining relation between Wishart processes of different parameters. The works by Warren [War07], Warren and Windridge [WW09], O’Connell [O’C12], Borodin and Corwin [BC13] and Gorin and Shkolnikov [GS13] exploit the idea that one can concatenate multiple finite-dimensional Markov processes, each viewed as a particle system on the real line given by its components, to a multilevel process provided that any two consecutive levels obey an intertwining relation. This program was initiated by Warren in [War07] who constructed a multilevel process in which the particle systems on the different levels are given by Dyson Brownian motions of varying dimensions with parameter $\beta = 2$ (corresponding to the evolution of eigenvalues of a Hermitian Brownian motion). Related dynamics were studied in [WW09] and an extension to arbitrary positive $\beta$ is given in [GS13]. Such processes arise as diffusive limits of continuous time Markov chains defined in terms of symmetric polynomials (Schur
polynomials in the case of $\beta = 2$ and, more generally, Jack polynomials, see [GS12], [GS13] and the references therein). The articles [BC13], [O'C12] explore (among other things) the multilevel diffusion processes corresponding to a class of Macdonald polynomials.

In many situations, intertwining relations arise as the result of deep algebraic structures. Biane (see [Bia95]) gives a group theoretic construction that produces intertwinings based on Gelfand pairs. In Diaz and Weinberger [DW53] the construction of intertwinings is based on the determinantal (Karlin-McGregor) form of the transition semigroups involved. The paper by Gallardo and Yor [GY06] exploits the intertwining of Dunkl processes with Brownian motion and the link operator there is an algebraic isomorphism on the space of polynomials which preserves the subspaces of homogeneous polynomials of any fixed degree. Another example is the deep connection of the Robinson-Schensted correspondence with the intertwining relation between a Dyson Brownian motion and a standard Brownian motion of the same dimension established by O’Connell (see [O'C03]). An example of intertwining given by an underlying branching structure appears in Johnson and Pal [JP13].

Originally intertwining relations have been used to derive explicit formulas for the more complicated of two intertwined processes from the simpler of the two processes (see the references above). However, there are other interesting applications of intertwinings. Diaconis and Fill [DF90] show that intertwinings of two Markov chains can be used to understand the convergence to equilibrium of one of the chains by understanding the hitting times of the other chain. This method relies on the fact that the latter hitting times are strong stationary times of the former Markov chain and, thus, give sharp control on its convergence to equilibrium in the separation distance as explained by Aldous and Diaconis [ADS7].

Another application of intertwinings lies in the construction of new Markov processes, typically ones with non-standard state spaces (such as a number of copies of $\mathbb{R}_+$ glued together at 0 in the case of Walsh’s spider), from existing ones (see Barlow and Evans [BE04], Evans and Sowers [ES03] for a collection of such constructions).

There is another notion of duality, originally due to Holley and Stroock [HS79], and prevalent in areas of probability such as interacting particle systems and population biology models. See the book by Liggett [Lig85, Definition 2.3.1] for numerous applications. This concept is sometimes called $h$-duality, a particular case of which is Siegmund duality [Sie76]. Two Markov semigroups $\{Q_t\}$ and $\{P_t\}$ are dual with respect to a function $h : \mathcal{Y} \times \mathcal{X} \to [0, \infty)$ if for every $(y,x) \in \mathcal{Y} \times \mathcal{X}$ we have

$$Q_t (h_x) (y) = P_t (h_y) (x),$$

where $h_x (y) = h^y (x) = h(y,x)$. When $\mathcal{X} = \mathcal{Y} = \mathbb{R}$ and $h(y,x) = \text{sgn}(y-x)$ this is called Siegmund duality. The notions of $h$-duality and intertwining are to some extent equivalent, in that the function $h$, suitably normalized, acts as an intertwining kernel between $Q$ and the time-reversal of $P$ under a Doob’s $h$-transform. This has been shown in [CPY98, Proposition 5.1] and in various results in [DF90, Section 5.2]. Please consult these references for an exact statement. For more on the role of $h$-transforms in the context of intertwinings please see Section 2.

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2. Proofs of the main results, extensions, and generalizations

We start with the proof of Theorem 1.
Proof of Theorem 1. We only deal with the case that \( X, Y \) are reflected diffusions as in part (b) of Assumption 1. All other cases are similar, but simpler (in fact, one only needs to change the respective initial-boundary problems below to Cauchy problems in the case of no boundary conditions). The proof is broken down into several steps. Throughout the proof we will assume that the underlying filtered probability space is given by the canonical space of continuous paths, \( C([0,\infty), X \times Y) \), from \([0,\infty)\) to \( X \times Y\), along with the standard Borel \( \sigma \)-algebra and a probability measure \( \mathbb{P} \), the law of the process \( Z \). This space is then equipped with the right continuous filtration \( \{\mathcal{F}_t, t \geq 0\} \) generated by the coordinates and augmented with the common null sets under \((\mathbb{P}_z, z \in X \times Y\)\), the set of solutions of the martingale problem for \( \mathcal{A}^Z \) starting at \( z \in X \times Y \). The notation \( \mathbb{E} \) will refer to a generic expectation.

We will also need two sub-filtrations. Let \( \{\mathcal{F}_t^X, t \geq 0\} \) and \( \{\mathcal{F}_t^Y, t \geq 0\} \) denote the right-continuous complete sub-filtrations of \( \{\mathcal{F}_t, t \geq 0\} \) generated by the first \( m \) and the next \( n \) coordinate processes in \( C([0,\infty), X \times Y) \), respectively.

**Step 1.** We first prove that the process \( Z_1 \) is a Markov process with respect to its own filtration. By applying Itô’s formula to functions of \( Z_1 \) it is easy to see that \( Z_1 \) solves the (sub)martingale problem for \( \mathcal{A}^X \). Since the initial distributions of \( Z_1 \) and \( X \) match, we must have \( Z_1 \overset{d}{=} X \). Thus, \( Z_1 \) is Markov with respect to \( \{\mathcal{F}_t^X, t \geq 0\} \).

**Step 2.** We now claim the following.

**Claim.** Take any test function \( f \in C_0^\infty (Y) \), the space of smooth functions vanishing on \( \partial Y \) together with all their derivatives. Then the function

\[
(2.1) \quad u(t, y) = \mathbb{E}[f(Z_2(t)) \mid Z_2(0) = y], \quad (t, y) \in [0, \infty) \times Y
\]

is a generalized solution of the initial-boundary value problem

\[
(2.2) \quad \partial_t u = \mathcal{A}^Y u, \quad u(0, \cdot) = f, \quad \langle (\nabla_y u)(\cdot, y), U_1(y) \rangle = 0, \quad y \in \partial Y.
\]

To prove the above claim we define

\[
(2.3) \quad v(t, x, y) = \mathbb{E}[f(Z_2(t)) \mid Z_1(0) = x, Z_2(0) = y].
\]

Thanks to the assumption on the conditional distribution of \( Z_1(0) \) given \( Z_2(0) \) the expectation in (2.1) can be rewritten as

\[
(2.4) \quad \int_X \Lambda(y, x) v(t, x, y) \, dx.
\]

Moreover, the Feynman-Kac formula implies that \( v \) is a generalized solution of the initial-boundary value problem

\[
(2.5) \quad \partial_t v = \mathcal{A}^X v + \mathcal{A}^Y v + (\nabla_y V)' \rho (\nabla_y v), \quad v(0, x, y) = f(y),
\]

\[
(2.6) \quad \langle (\nabla_x v)(\cdot, x, \cdot), U_1(x) \rangle = 0, \quad x \in \partial X, \quad \langle (\nabla_y v)(\cdot, \cdot, y), U_2(y) \rangle = 0, \quad y \in \partial Y,
\]

where the superscript ‘\( \prime \)’ denotes the transpose.

We see that \( u \) inherits the initial and boundary conditions from those of \( v \) and \( \Lambda \). In addition, the following identities hold in the distributional sense:

\[
(2.7) \quad \partial_t u = \int_X \Lambda \partial_t v \, dx = \int_X \Lambda \left( \mathcal{A}^X v + \mathcal{A}^Y v + (\nabla_y V)' \rho (\nabla_y v) \right) \, dx
\]

\[
= \int_X \left( (\mathcal{A}^X) \Lambda \right) v + \Lambda (\mathcal{A}^Y v) + (\nabla_y \Lambda)' \rho (\nabla_y v) \, dx
\]

\[
= \int_X \left( (\mathcal{A}^Y) \Lambda \right) v + \Lambda (\mathcal{A}^Y v) + (\nabla_y \Lambda)' \rho (\nabla_y v) \, dx,
\]
where in the fourth identity we have used the hyperbolic PDE (1.4).

The rest follows by expanding $A^Y (A(y,x)v(t,x,y))$ and exchanging $\int_{\mathcal{X}}$ with $A^Y$ to get

$$A^Y u = \int_{\mathcal{X}} ((A^Y \Lambda) v + \Lambda (A^Y v) + (\nabla_y \Lambda') \rho (\nabla_y v)) \, dx.$$  

This completes the proof of the claim.

**Step 3.** We now prove condition (iii) in Definition 2. To this end, it suffices to show that for any test function $f \in C_c^\infty (\mathcal{X})$ and $t \geq 0$ it holds

$$(2.8) \quad Q_t L f = L P_t f.$$  

In other words, we need to prove

$$(2.9) \quad \mathbb{E} \left[ \int_{\mathcal{X}} \Lambda (Z_2(t),x) f(x) \, dx \bigg| Z_2(0) = y \right] = \int_{\mathcal{X}} \Lambda (y,x) \mathbb{E} [f(Z_1(t)) \mid Z_1(0) = x] \, dx.$$  

Define

$$u(t, x) = \mathbb{E} [f(Z_1(t)) \mid Z_1(0) = x],$$

$$v(t, y) = \mathbb{E} \left[ \int_{\mathcal{X}} \Lambda (Z_2(t),x) f(x) \, dx \bigg| Z_2(0) = y \right].$$

From the previous step we know that $v$ is the unique generalized solution of the initial boundary value problem

$$(2.10) \quad \partial_t v = A^Y v, \quad v(0, y) = \int_{\mathcal{X}} \Lambda (y,x) f(x) \, dx, \quad \langle (\nabla_y v)(\cdot, y), U_2(y) \rangle = 0, \quad y \in \partial \mathcal{Y}.$$  

Therefore, to prove (2.9) it is enough to demonstrate that the right-hand side of (2.9), given by $\int_{\mathcal{X}} \Lambda (y,x) u(t,x) \, dx$, also solves (2.10). The boundary condition is satisfied due to the boundary condition on $\Lambda$ and, moreover, we have (in the distributional sense)

$$\partial_t \int_{\mathcal{X}} \Lambda (y,x) u(t,x) \, dx = \int_{\mathcal{X}} \Lambda (y,x) (\partial_t u)(t,x) \, dx$$

$$= \int_{\mathcal{X}} \Lambda (y,x) \left( A^X u \right) (t,x) \, dx = \int_{\mathcal{X}} \left( (A^Y)^* \Lambda \right) (y,x) u(t,x) \, dx$$

$$= \int_{\mathcal{X}} (A^Y \Lambda)(y,x) u(t,x) \, dx = A^Y \int_{\mathcal{X}} \Lambda (y,x) u(t,x) \, dx.$$  

All in all, it follows that

$$v(t,y) = \int_{\mathcal{X}} \Lambda (y,x) u(t,x) \, dx,$$

which yields (2.9).

**Step 4.** We now prove condition (iv) of Definition 2. The main claim is an iteration of the previous step.

**Claim.** Fix $k \in \mathbb{N}$, and let $0 = t_0 < t_1 < \ldots < t_k = t$ be distinct time points. Let $\mathcal{G}$ denote the sub-$\sigma$-algebra of $\mathcal{F}_t$ generated by $(Z_2(t_i), i = 0, 1, \ldots, k)$. Then, for all nonnegative Borel measurable function $f$ on $\mathcal{X}$, we have

$$(2.11) \quad \mathbb{E} [f(Z_1(t)) \mid \mathcal{G}] = (L f)(Z_2(t)).$$  

The proof of the claim proceeds by induction. First consider the case of $k = 1$. Take any $f \in C_c^\infty (\mathcal{X})$ and any $g \in C_c^\infty (\mathcal{Y})$ and consider the functions

$$u(t,y) = \mathbb{E} [f(Z_1(t)) g(Z_2(t)) \mid Z_2(0) = y], \quad v(t,y) = \mathbb{E} [(L f)(Z_2(t)) g(Z_2(t)) \mid Z_2(0) = y].$$

We will show that the two functions $u, v$ are identical for every choice of $f$ and $g$ which proves our claim for $k = 1$. To show that they are identical we will demonstrate that they satisfy the same initial-boundary value problem and appeal to the uniqueness of the solution of the latter.
That $u$ and $v$ have identical initial value follows from our assumption on the initial conditional distribution of $Z_1(0)$ given $Z_2(0)$. Next, recall from Step 2 that $v$ satisfies the initial-boundary value problem
\begin{equation}
\frac{\partial}{\partial t} v = A^Z v, \quad v(0, \cdot) = (L f)(y) g(y), \quad \langle \nabla_y v(\cdot, y), U_2(y) \rangle = 0, \ y \in \partial Y.
\end{equation}

Now, as before, if we define
\[
w(t, x, y) = \mathbb{E} \left[ f(Z_1(t)) g(Z_2(t)) \mid Z_1(0) = x, \ Z_2(0) = y \right],
\]
then $u(t, y) = \int_X w(t, x, y) \Lambda(y, x) \, dx$. However, $w$ solves the initial-boundary value problem
\begin{align}
\frac{\partial}{\partial t} w &= A^Z w, \quad w(0, x, y) = f(x) g(y), \\
\langle \nabla_x w(\cdot, x, \cdot), U_1(x) \rangle &= 0, \ x \in \partial X, \quad \langle \nabla_y w(\cdot, \cdot, y), U_2(y) \rangle = 0, \ y \in \partial Y.
\end{align}

Now, identical calculations to the ones in (2.7) show that $u$ is a solution of the problem (2.12), and we are done.

Now, suppose the claim holds true for some $k \in \mathbb{N}$. Then, the conditional expectation operator of $Z_1(t_k)$ given $(Z_2(0), \ldots, Z_2(t_k))$ is again $L$. To show that the claim holds true for $(k + 1)$, one can repeat the argument for $k = 1$ after shifting the homogenous Markov process $Z$ by time $t_k$ and conditioning on $(Z_2(0), \ldots, Z_2(t_k))$. This completes the proof of the claim.

We have shown so far that, for any $k \in \mathbb{N}$ and any bounded Borel function $g$ on $\mathcal{Y}^{k+1}$, we have
\[
\mathbb{E} \left[ f(Z_1(t_k)) g(Z_2(t_0), \ldots, Z_2(t_k)) \right] = \mathbb{E} \left[ (L f)(Z_1(t_k)) g(Z_2(t_0), \ldots, Z_2(t_k)) \right].
\]
Since the Borel $\sigma$-algebra $\mathcal{F}_t^Y$ is generated by the coordinate projections, an application of the Monotone Class Theorem shows condition (iv).

**Step 5.** We now argue that $Z_2 \overset{d}{=} Y$. To see this it is enough to argue that, for any $k \in \mathbb{N}$, and for any choice of $0 = t_0 < t_1 < \ldots < t_k = t$, we have the correct transition probability:
\begin{equation}
\mathbb{E} \left[ f(Z_2(t)) \mid Z_2(t_0), \ldots, Z_2(t_{k-1}) \right] = Q_{t-t_{k-1}} f(Z_2(t_{k-1})).
\end{equation}

The argument proceeds by induction. The case of $k = 1$ follows from Step 2 and the uniqueness for the initial-boundary problem in the claim there. Next, suppose (2.15) holds for some $k$. To show the corresponding statement for $(k + 1)$ we proceed similarly to our arguments in Step 2, by considering
\[
\mathbb{E} \left[ f(Z_2(t)) \mid Z_1(t_{k-1}), Z_2(t_{k-1}), \ldots, Z_2(t_0) \right].
\]
By the Markov property of $Z$, the above is only a function of $Z_1(t_{k-1})$ and $Z_2(t_{k-1})$. Also, the conditional law of $Z_1(t_{k-1})$, given $\mathcal{F}_{t_{k-1}}^Y$, has been identified in Step 4 as $\Lambda(Z_2(t_{k-1}), \cdot)$. Now, an identical argument as in Step 2 completes the proof.

**Step 6.** Finally, we show condition (iii) in Definition 2. Fix any $s < t$. We need to show that $Z_1(t)$, conditioned on $Z_1(s)$, is independent of the $\sigma$-algebra $\mathcal{F}_s^Z$. Since $Z$ is assumed to be Markovian, it is enough to show that for any $s < t$, given $Z_1(s)$, $Z_1(t)$ is independent of $Z_2(s)$. To this end, we observe that due to the time-homogeneity of all semigroups involved it is sufficient to consider $s = 0$. In this setting, condition (iii) in Definition 2 can be written as
\begin{equation}
\mathbb{E} [f(Z_1(t)) \mid Z_1(0) = x, Z_2(0) = y] = \mathbb{E} [f(Z_1(t)) \mid Z_1(0) = x], \quad (t, x, y) \in [0, \infty) \times \mathcal{X} \times \mathcal{Y}
\end{equation}
for all test functions \( f \in C_0^{\infty}(\mathcal{X}) \). Writing \( v(t,x,y) \) for the left-hand side of (2.16) and \( u(t,x) \) for the right-hand side of (2.16) we see that \( v, u \) are the unique generalized solutions of the initial-boundary value problems

\[
\partial_t v = A^X v, \quad v(0,x,y) = f(x), \\
(\nabla_x v(\cdot, x, \cdot), U_1(x)) = 0, \quad x \in \partial\mathcal{X}, \quad (\nabla_y v(\cdot, \cdot, y), U_2(y)) = 0, \quad y \in \partial\mathcal{Y},
\]

and

\[
\partial_t u = A^Y u, \quad u(0,x) = f(x), \quad (\nabla_x u(\cdot, x), U_1(x)) = 0, \quad x \in \partial\mathcal{X},
\]

respectively. One verifies directly that \( u \) is a solution of (2.17), (2.18), so that \( v = u \) as desired. This completes the proof of the theorem. \( \blacksquare \)

We now turn to the proof of Theorem 2.

**Proof of Theorem 2.** Step 1. We start with the proof of the commutativity relation (1.5). To this end, let \( f \) be a test function as described below (1.5). Our starting point is the intertwining identity

\[
LP_t f = Q_t L f, \quad t \geq 0.
\]

In particular,

\[
\frac{LP_t f - L f}{t} = \frac{Q_t L f - L f}{t}, \quad t > 0.
\]

Taking the limit \( t \downarrow 0 \), we note that the left-hand side converges and, hence, so does the right-hand side as well, and we end up with (1.5).

**Step 2.** To derive the formula for the generator of \( Z \) we pick a function \( f \in C_0^{\infty}(\mathcal{X} \times \mathcal{Y}) \) with compact support contained in the interior of \( \mathcal{X} \times \mathcal{Y} \), and note first that the properties (i), (iii), (iv) of Definition 2 and an application of the Bayes’ rule allow us to write for any \( t > 0 \):

\[
\mathbb{E}[f(Z_1(t), Z_2(t)) \mid Z(0) = (x_0, y_0)] = \int_{\mathcal{X} \times \mathcal{Y}} \frac{\Lambda(y_1, x_1)f(x_1, y_1)}{\int_{\mathcal{Y}} Q_t(y_0, d\tilde{y}_1)\Lambda(\tilde{y}_1, x_1)} P_t(x_0, dx_1) Q_t(y_0, dy_1).
\]

Consequently, for all \( t > 0 \):

\[
\frac{1}{t} \left( \mathbb{E}[f(Z_1(t), Z_2(t)) \mid Z(0) = (x_0, y_0)] - f(x_0, y_0) \right)
\]

\[
\frac{1}{t} \left( \int_{\mathcal{X} \times \mathcal{Y}} \frac{\Lambda(y_1, x_1)f(x_1, y_1)}{\int_{\mathcal{Y}} Q_t(y_0, d\tilde{y}_1)\Lambda(\tilde{y}_1, x_1)} - \frac{\Lambda(y_1, x_1)f(x_1, y_1)}{\Lambda(y_0, x_1)} P_t(x_0, dx_1) Q_t(y_0, dy_1) \right)
\]

\[
+ \frac{1}{t} \left( \int_{\mathcal{X} \times \mathcal{Y}} \frac{\Lambda(y_1, x_1)f(x_1, y_1)}{\Lambda(y_0, x_1)} P_t(x_0, dx_1) Q_t(y_0, dy_1) - f(x_0, y_0) \right)
\]

To proceed we assume that \( \Lambda \) belongs to \( C^\infty(\mathcal{X} \times \mathcal{Y}) \); the general case can be then dealt with by approximating the kernel \( \Lambda \) by smooth kernels. Taking limits \( t \to 0 \) of the latter two summands separately and applying the chain rule in the evaluation of the limit of the first summand we get

\[
-\frac{\Lambda(y_0, x_0)f(x_0, y_0)}{\left( \int_{\mathcal{Y}} Q_t(y_0, d\tilde{y}_1)\Lambda(\tilde{y}_1, x_1) \right)^2} (A^Y_{y_0} \Lambda)(y_0, x_0)
\]

\[
+(A^X_{x_1} + A^Y_{y_1}) \left( \frac{\Lambda(y_1, x_1)f(x_1, y_1)}{\Lambda(y_0, x_1)} \right) \bigg|_{x_1=x_0, y_1=y_0}.
\]
Applying the product rule twice (first for $A^X_{x_1}$, then for $A^Y_{y_1}$) we can simplify the latter expression to

$$-\frac{\Lambda(y_0,x_0) f(x_0,y_0)}{\left(\int_X Q_0(y_0,dy_1) \Lambda(y_1,x_1)\right)^2} (A^V_{y_0}\Lambda)(y_0,x_0)$$

$$+ \left( A^X_{x_1} \frac{\Lambda(y_1,x_1)}{\Lambda(y_0,x_1)} \right) f(x_1,y_1) + \frac{\Lambda(y_1,x_1)}{\Lambda(y_0,x_1)} A^X_{x_1} f(x_1,y_1)$$

$$+ \left( \nabla_x \left( \frac{\Lambda(y_1,x_1)}{\Lambda(y_0,x_1)} \right) \right)' a(x_1) \nabla_x f(x_1,y_1) + \frac{(A^Y_{y_1}\Lambda)(y_1,x_1) f(x_1,y_1)}{\Lambda(y_0,x_1)}$$

$$+ \frac{\Lambda(y_1,x_1)(A^Y_{y_1} f)(x_1,y_1)}{\Lambda(y_0,x_1)} + \frac{(\nabla_y \Lambda(y_1,x_1))' \rho(y_1) \nabla_y f(x_1,y_1)}{\Lambda(y_0,x_1)}$$

$$= (A^X_{x_1} f)(x_0,y_0) + (A^Y_{y_1} f)(x_0,y_0) + (\nabla_y \log \Lambda)(x_0,y_0)' \rho(y_0) \nabla_y f)(x_0,y_0).$$

Recalling that $V = \log \Lambda$ by definition we can conclude that the generator of $Z$ is given by (1.3). Lastly, applying Itô’s formula to $C^\infty$ functions of $Z_1$ and $Z_2$, respectively, and using condition (2) in Definition 2 we see that the boundary conditions on $\partial(\mathcal{X} \times \mathcal{Y}) = (\partial \mathcal{X} \times \mathcal{Y}) \cup (\mathcal{X} \times \partial \mathcal{Y})$ are as described below (1.3).

**Step 3.** Finally, we show that $\Lambda$ solves the PDE (1.4) in the distributional sense. We only consider the case that $X$, $Y$ are as in part (b) of Assumption 1, since all other cases can be dealt with in a similar fashion (in fact, it suffices to drop the respective boundary terms in the computations below). We need to show that for all $f \in C^\infty(\mathcal{X} \times \mathcal{Y})$:

$$\int_{\mathcal{X} \times \mathcal{Y}} \Lambda(y,x) (A^X f)(x,y) \, dx \, dy + \int_{\partial \mathcal{X} \times \mathcal{Y}} \Lambda(y,x) \langle (\nabla_x f)(x,y), U_1(x) \rangle d\theta(x) \, dy$$

$$= \int_{\mathcal{X} \times \mathcal{Y}} \Lambda(y,x) ((A^Y\ast f)(x,y)) \, dx \, dy + \int_{\partial \mathcal{X} \times \mathcal{Y}} \Lambda(y,x) \langle (\nabla_y f)(x,y), U_2\ast(y) \rangle d\theta(y) \, dx$$

with $\theta$ being the surface measure on $\partial(\mathcal{X} \times \mathcal{Y})$ and $U_2\ast$ being the vector field corresponding to the Neumann boundary condition of the adjoint operator $(A^Y)^\ast$. We may assume that $f(x,y)$ is of the product form $\phi(x)\psi(y)$ with $\phi \in C^\infty(\mathcal{X})$, $\psi \in C^\infty(\mathcal{Y})$, since any function $f \in C^\infty(\mathcal{X} \times \mathcal{Y})$ can be approximated in the uniform norm together with all its partial derivatives by linear combinations of product form functions. Moreover, for $f(x,y) = \phi(x)\psi(y)$ the left-hand side of (2.25) can be rewritten as

$$\int_{\mathcal{X} \times \mathcal{Y}} \phi(x) \, dx \, dy = \int_{\mathcal{X}} \phi(x) \, dx + \int_{\mathcal{Y}} \phi(y) \, dy$$

$$\int_{\partial \mathcal{Y}} \psi(y) \, d\theta(y) = \int_{\partial \mathcal{Y}} \langle \nabla_y (L\phi)(y), U_2\ast(y) \rangle \, d\theta(y)$$

$$\int_{\mathcal{Y}} \langle (A^Y)^\ast \psi \rangle (y) \, dy = \int_{\mathcal{Y}} \langle (A^Y)^\ast \psi \rangle (y) \, dy + \int_{\partial \mathcal{Y}} \langle \nabla_y \psi(y), U_2\ast(y) \rangle \, d\theta(y)$$

$$= \int_{\mathcal{X} \times \mathcal{Y}} \Lambda(y,x) ((A^Y)^\ast f)(x,y) \, dx \, dy + \int_{\mathcal{X} \times \partial \mathcal{Y}} \Lambda(y,x) \langle (\nabla_y f)(x,y), U_2\ast(y) \rangle d\theta(y) \, dx,$$

where we have used (1.5) in the first identity, integration by parts in the second identity and Fubini’s Theorem in the third identity. This finishes the proof.

A major restriction of Theorem 1 is that the kernel $\Lambda$ is assumed to be stochastic and to satisfy (1.3) on the entire space $\mathcal{X} \times \mathcal{Y}$. This leaves out situations, where the domain of $Z$ is not of product form and $\Lambda$ is a solution of (1.4) on that domain. Our next results
relax these constraints and will allow us to cover several important examples. For the sake of clarity we keep the following theorem restricted to the case when the state space of $Z$ is polyhedral and the components of $Z$ are driven by independent standard Brownian motions. This covers all known examples, although it is not hard to see that the scope of the theorem can be enlarged significantly.

Consider the same set-up as in Assumption 1 with $a_{ij} = \delta_{ij}$ and $\rho_{kl} = \delta_{kl}$. As before, we write $z \in \mathbb{R}^{m+n}$ as $z = (x, y)$, where $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$. Let $D \subset \mathbb{R}^{m+n}$ be a domain and $\Lambda$ be a function on $D$ taking nonnegative values such that

(i) The projection of $D$ on $\mathbb{R}^m$, given by $\cup_{y \in \mathbb{R}^n} D(\cdot, y)$, is $\mathcal{X}$, and the projection of $D$ on $\mathbb{R}^n$, given by $\cup_{x \in \mathbb{R}^m} D(x, \cdot)$ is $\mathcal{Y}$.

(ii) For every $y \in \mathcal{Y}$, the domain $D(y) := D(\cdot, y)$ has a boundary $\partial D(y)$ such that the Divergence Theorem and Green’s second identity hold for $D(y)$. For example, piecewise smooth boundaries suffice.

(iii) At each point $x \in \partial D(y)$ the directional derivatives $\Psi_j$ of that boundary point with respect to changes in the coordinates $y_j$ exist and are piecewise constant in $(x, y)$. In addition, $\eta = \sum_{j=1}^n \Psi_j(\Psi_j, \eta)\big|_{\partial D(y)}$ on $\partial D(y)$, and $\sum_{j=1}^n \langle \Psi_j, \eta \rangle U_{2,j} = 0$ on $\partial D(y)$ for $y \in \partial \mathcal{Y}$, where $\eta$ is the unit outward normal vector on $\partial D(y)$ and $U_{2,j}$ are the coordinates of the vector field $U_2$.

(iv) For every $y \in \mathcal{Y}$, $\Lambda(y, \cdot)$ is a probability density on $\mathcal{Y}$. Moreover, $\Lambda$ is twice continuously differentiable on $\overline{D}$ and solves (1.4) pointwise with boundary conditions $\langle \nabla_y \Lambda, U_2 \rangle = 0$ on $\partial \mathcal{Y}$ and

\begin{equation}
(2.26) \quad \Lambda \langle \eta, \nabla_x \Lambda \rangle = \sum_{j=1}^m \langle \Psi_j, \eta \rangle \left( \gamma_j \Lambda + \partial_y \Lambda \right) \text{ on } \partial D(y) \text{ for each } y \in \mathcal{Y}.
\end{equation}

**Theorem 3.** Assume the well-posedness of the submartingale problem for the diffusion process $Z = (Z_1, Z_2)$ with a generator given by (1.3) acting on compactly supported smooth functions in $D$, with boundary conditions as in Theorem 1 and reflecting boundary conditions on $\partial D(y)$ given by

\begin{equation}
(2.27) \quad \sum_{j=1}^n \langle \partial_{y_j}, v \rangle \langle \Psi_j, \eta \rangle = 0, \quad x \in \partial D(y), \ y \in \mathcal{Y}.
\end{equation}

More specifically, recall the vector fields $U_1$, $U_2$ of Assumption 1. For $f \in C_0^\infty(D)$, we consider the initial-boundary value problem

\begin{equation}
(2.28) \quad \begin{aligned}
\partial_t v &= A^X v + A^Y v + \langle \nabla_y V, \nabla_y v \rangle, \quad v(0, x, y) = f(x, y), \\
\langle \nabla_x v, U_1 \rangle &= 0, \quad x \in \partial \mathcal{X}, \quad \langle \nabla_y v, U_2 \rangle = 0, \quad y \in \partial \mathcal{Y}, \\
\sum_{j=1}^n \langle \partial_{y_j}, v \rangle \langle \Psi_j, \eta \rangle &= 0, \quad x \in \partial D(y), \ y \in \mathcal{Y},
\end{aligned}
\end{equation}

and assume that the unique generalized solution to this problem is given by

\begin{equation*}
\begin{aligned}
v(t, x, y) &= \mathbb{E} \left[ f(Z_1(t), Z_2(t)) \mid Z_1(0) = x, Z_2(0) = y \right].
\end{aligned}
\end{equation*}

Then, provided that $Z(0)$ is as in condition 1 of Definition 2, we have $Z = Y \langle L \rangle X$.

**Proof.** The proof is an extension to that of Theorem 1. Step 1 remains the same. We verify Step 2. Define $u, v$ as in (2.1), (2.3) for some $f \in C_0^\infty(D)$. The representation (2.4) for $u$ continues to hold. By assumption, $v$ is the solution of the initial-boundary value problem
Hence, we have (in the distributional sense)

\[
\partial_t u = \int_{D(y)} \Lambda (\partial_t v) \, dx = \int_{D(y)} \Lambda (A^X v + A^Y v + \langle \nabla_y V, \nabla_y v \rangle) \, dx
\]

\[
= \int_{D(y)} \Lambda (A^X v) \, dx + \int_{D(y)} \Lambda (A^Y v + \langle \nabla_y V, \nabla_y v \rangle) \, dx.
\]

Next, we apply the Divergence Theorem and Green’s second identity to evaluate the first summand in the latter expression:

\[
\int_{D(y)} \Lambda (A^X v) \, dx = \int_{D(y)} \Lambda \langle b, \nabla_x v \rangle \, dx + \frac{1}{2} \int_{D(y)} \Lambda (\Delta_x v) \, dx
\]

\[
= \int_{\partial D(y)} \Lambda v \langle b, \eta \rangle \, d\theta(x) - \int_{D(y)} \text{div}_x (\Lambda b) \, v \, dx
\]

\[
+ \frac{1}{2} \int_{\partial D(y)} \Lambda \langle \nabla_x v - v \nabla_x V, \eta \rangle \, d\theta(x) + \frac{1}{2} \int_{D(y)} (\Delta_x \Lambda) \, v \, dx
\]

\[
= \frac{1}{2} \int_{\partial D(y)} \Lambda \langle 2v b + \nabla_x v - v \nabla_x V, \eta \rangle \, d\theta(x) + \int_{D(y)} ((A^X)^* \Lambda) \, v \, dx
\]

\[
= \frac{1}{2} \int_{\partial D(y)} \Lambda \langle 2v b + \nabla_x v - v \nabla_x V, \eta \rangle \, d\theta(x) + \int_{D(y)} (A^Y \Lambda) \, v \, dx.
\]

On the other hand, the multidimensional Leibniz rule yields

\[
\partial_{y_j} \int_{D(y)} \Lambda v \, dx = \int_{D(y)} \text{div}_x (\Lambda v \Psi^j) + \partial_{y_j} (\Lambda v) \, dx
\]

\[
\partial_{y_j y_j} \int_{D(y)} \Lambda v \, dx = \int_{D(y)} \left( \text{div}_x (\text{div}_x (\Lambda v \Psi^j) \Psi^j) + \partial_{y_j} (\text{div}_x (\Lambda v \Psi^j)) \right.
\]

\[
+ \text{div}_x (\partial_{y_j} (\Lambda v \Psi^j) + \partial_{y_j y_j} (\Lambda v) \right) \, dx.
\]

Therefore,

\[
A^Y u = A^Y \int_{D(y)} \Lambda v \, dx = \int_{D(y)} A^Y (\Lambda v) \, dx + \sum_{j=1}^n \gamma_j \int_{D(y)} \text{div}_x (\Lambda v \Psi^j) \, dx
\]

\[
+ \frac{1}{2} \sum_{j=1}^n \int_{D(y)} \left( \text{div}_x (\text{div}_x (\Lambda v \Psi^j) \Psi^j) + \partial_{y_j} (\text{div}_x (\Lambda v \Psi^j)) \right)
\]

\[
+ \text{div}_x (\partial_{y_j} (\Lambda v \Psi^j) + \partial_{y_j y_j} (\Lambda v) \Psi^j) \, dx.
\]

Applying the Divergence Theorem we get

\[
A^Y u = \int_{D(y)} A^Y (\Lambda v) \, dx + \sum_{j=1}^m \int_{\partial D(y)} \left( \gamma_j \Lambda v \langle \Psi^j, \eta \rangle + \frac{1}{2} \text{div}_x (\Lambda v \Psi^j) \langle \Psi^j, \eta \rangle \right.
\]

\[
+ \frac{1}{2} \langle \partial_{y_j} (\Lambda v \Psi^j), \eta \rangle + \frac{1}{2} \partial_{y_j y_j} (\Lambda v) \langle \Psi^j, \eta \rangle \right) \, d\theta(x).
\]

Let us now verify that the boundary terms in (2.29) and (2.30) match. We use the fact that each \( \Psi^j \) is piecewise constant, \( \eta = \sum_{j=1}^n \Psi^j \langle \Psi^j, \eta \rangle \), (2.26) and (2.27). We group the terms in the boundary integrand in (2.30) into those which contain \( v \) and those which
contain derivatives of $v$. The terms containing $v$ are
\[ v \sum_{j=1}^{n} \left( \Lambda \gamma_j \langle \Psi^j, \eta \rangle + \frac{1}{2} \langle \nabla_x \Lambda, \Psi^j \rangle \langle \Psi^j, \eta \rangle + \partial_{y_j} \Lambda \langle \Psi^j, \eta \rangle \right) \]
\[ = v \left( \sum_{j=1}^{n} \langle \Psi^j, \eta \rangle \left( \gamma_j \Lambda + \partial_{y_j} \Lambda \right) + \frac{1}{2} \langle \nabla_x \Lambda, \eta \rangle \right) = v \Lambda \langle b, \eta \rangle - \frac{1}{2} v \langle \nabla_x \Lambda, \eta \rangle. \]

This matches the terms containing $v$ in the boundary integrand in (2.29).

Next, we move on to Step 3. As in the proof of Theorem 1, consider $f \in C_0^\infty(Y)$ and define
\[ u(t, x) = \mathbb{E} [f (Z_1(t)) \mid Z_1(0) = x], \quad w(t, y) = \int_{D(y)} \Lambda(y, x) u(t, x) \, dx. \]

Then, we need to show that $w$ satisfies $\partial_t w = A^Y w$ with boundary condition $\langle \nabla_y w, U_2 \rangle = 0$ on $\partial Y$ if any. The boundary condition is a consequence of the multidimensional Leibniz rule in the form above, the boundary condition on $\Lambda$ and $\sum_{j=1}^{n} \langle \Psi^j, \eta \rangle U_{2,j} = 0$ on $\partial D(y)$ for $y \in \partial Y$. In addition, performing the same steps as in the proof of Theorem 1 and using the Divergence Theorem we get
\[ \partial_t u = \int_{D(y)} \Lambda \left( A^X u \right) \, dx = \int_{D(y)} \left( A^Y \Lambda \right) u \, dx \]
\[ + \frac{1}{2} \int_{\partial D(y)} \left( 2 \Lambda \langle b, \eta \rangle + \Lambda \langle \nabla_x u, \eta \rangle - u \langle \nabla_x \Lambda, \eta \rangle \right) \, dx = A^Y \int_{D(y)} \Lambda u \, dx. \]

The final equality is due to the multidimensional Leibniz rule in the form above and to the equality of the resulting boundary terms (see Step 2 and replace $v$ by $u$ there). This completes Step 3.

Finally, to establish Steps 4 through 6 one can proceed exactly as in the proof of Theorem 1, taking into account the boundary conditions on $\partial D(y)$ as in the previous steps. \(\square\)

In Assumption 1 we impose that $\Lambda(y, \cdot)$ is a probability density for each $y$. Suppose $\Lambda$ is a nonnegative solution of the PDE (1.4) such that $\Lambda(y, \cdot)$ is integrable for each $y$, but has a nontrivial normalizing constant $\tau(y)$. Then, we can define the normalized kernel according to
\[ (2.31) \quad \xi(y, x) = \frac{\Lambda(y, x)}{\tau(y)}, \quad \tau(y) = \int_{\mathcal{X}} \Lambda(y, x) \, dx. \]

Let $\Xi$ denote the Markov transition operator corresponding to $\xi$. Our next theorem shows that $\Xi$ intertwines the semigroup $(P_t, t \geq 0)$ with a version of the Doob’s $h$-transform of the semigroup $(Q_t, t \geq 0)$.

**Theorem 4.** Let $\xi, \tau$ and $\Xi$ be as in the preceding paragraph. Then, the function $\tau$ is a positive harmonic function for the infinitesimal generator $A^Y$, that is, $\tau(Y(t)), t \geq 0$ is a positive local martingale for the diffusion $Y$ with semigroup $(Q_t, t \geq 0)$ generated by $A^Y$. Suppose that the generator
\[ A^\tau \phi = \tau^{-1} A^Y (\tau \phi) \]
with no boundary conditions, if \( \mathcal{A}^Y \) is as in part (a) of Assumption \([1]\) or with boundary condition

\[
\langle \nabla_y \phi, U_2 \rangle - \langle \nabla_y \log \tau, U_2^* \rangle \phi = 0, \quad y \in \partial \mathcal{Y},
\]

if \( \mathcal{A}^Y \) is as in part (b) of Assumption \([1]\) is as in part (a) or (b) of Assumption \([1]\) respectively \((U_2^* \) being the vector field corresponding to the Neumann boundary condition of the adjoint operator \((\mathcal{A}^Y)^*\)).

Write \((Q^t_\tau, t \geq 0)\) for the corresponding semigroup. Then, \(Q^\tau (\xi) \mathcal{P}\). In the case of no boundary conditions and for any fixed \( T < \infty \), one can realize \((Q^t_\tau, t \in [0,T])\) by a stochastic process via a change of measure with density \(\tau(Y(T))\) with respect to the law of \(Y(t), t \in [0,T]\).

**Proof.** We only consider Neumann boundary conditions for both \(\mathcal{A}^X\) and \(\mathcal{A}^Y\), since all other cases are simpler. We start by recalling that \(\Lambda\) solves the PDE \((\text{1.4})\) in the generalized sense. Therefore, for any \(\psi \in C^\infty(\mathcal{Y})\), we have

\[
\int_{\mathcal{Y}} \langle (A^Y_\tau)^* \psi \rangle \tau \, dy + \int_{\partial \mathcal{Y}} \langle (\nabla_y \psi), U_2^* \rangle \tau \, d\theta(y)
\]

\[
= \int_{X \times \mathcal{Y}} \langle (A^Y_\tau)^* \psi \rangle \Lambda \, dx \, dy + \int_{X \times \partial \mathcal{Y}} \langle \nabla_y \psi, U_2^* \rangle \Lambda \, dx \, d\theta(y)
\]

\[
= \int_{X \times \mathcal{Y}} (A^X_\tau \psi) \Lambda \, dx \, dy + \int_{\partial X \times \mathcal{Y}} \langle \nabla_x \psi, U_1 \rangle \Lambda \, d\theta(x) \, dy = 0.
\]

In other words, \(\tau\) is a distributional solution of the equation \(A^Y_\tau \tau = 0\) with Neumann boundary conditions given by \(U_2\). It follows that \(\tau(Y(t)), t \geq 0\) is a positive local martingale, hence, a supermartingale. Moreover, for any \(f(x,y) = \phi(x)\psi(y)\) with \(\phi \in C^\infty(X), \psi \in C^\infty(Y)\) we compute

\[
\int_{X \times \mathcal{Y}} \xi (A^X_\tau \psi) \psi \, dx \, dy + \int_{\partial X \times \mathcal{Y}} \xi \langle (\nabla_x \phi), U_1 \rangle \psi \psi \, dx \, d\theta(x) \, dy
\]

\[
= \int_{X \times \mathcal{Y}} \Lambda (A^X_\tau \phi) \psi \psi \, dx \, dy + \int_{\partial X \times \mathcal{Y}} \Lambda \langle (\nabla_x \phi), U_1 \rangle \psi \psi \, dx \, d\theta(x) \, dy
\]

\[
= \int_{X \times \mathcal{Y}} \Lambda \phi (A^Y_\tau)^* \psi \psi \, dx \, dy + \int_{\partial X \times \mathcal{Y}} \Lambda \phi \langle \nabla_y \tau, U_2^* \rangle \psi \psi \, dx \, d\theta(x) \, dy.
\]

Plugging in \(\xi \tau\) for \(\Lambda\) in the latter expression, we see that \(\xi\) is a generalized solution of \((A^X)^* \xi = A^T \xi\) with boundary conditions corresponding to the ones of \((A^X)^*\) and \(A^T\). One can now proceed as in step 3 of the proof of Theorem \([1]\) to deduce \(Q^\tau (\xi) \mathcal{P}\). Finally, in the case of no boundary conditions and for any fixed \( T < \infty \), it is evident that \((Q^t_\tau, t \geq 0)\) can be realized as a Doob h-transform as described in the theorem.

If \(\mathcal{A}^Y\) is the generator of a one-dimensional homogeneous diffusion, then there are only two linearly independent choices for \(\tau\), the constant function and the scale function of \(\mathcal{A}^Y\). See Remark \([2]\) in Section \([4]\) below and the proposition preceding it for more details. In general, suppose \(\mathcal{A}^Y\) satisfies the Liouville property, that is, any bounded function \(\tau\) satisfying \(\mathcal{A}^Y \tau = 0\) has to be constant. Then, once we show \(\tau\) is bounded, a further h-transform is unnecessary. The Liouville property is satisfied by many natural operators. For example, if \(\mathcal{A}^Y\) is strictly elliptic with \(p\) being bounded and \(\gamma \equiv 0\), the Liouville property holds (see e.g. Corollary 9.25 in \([GT01]\) for the corresponding Harnack’s inequality). A sharper condition that allows for a drift can be found in \([PW10]\).
3. On various properties of intertwined diffusions

We prove several results on properties of intertwined processes and semigroups. We start with an obvious lemma. See Figure 2 for a commutative diagram representation.

**Lemma 5.** Suppose $Q, P, R$ are three semigroups defined on three measurable spaces. Suppose there exists two positive link operators $L_1, L_2$, defined on appropriate spaces, such that $Q \langle L_1 \rangle P$ and $P \langle L_2 \rangle R$. Then, $Q \langle L_1 L_2 \rangle R$.

We show that the representation Theorem 1 respects this product operation. Consider three diffusions $S, X, Y$ with state spaces given by subsets of $\mathbb{R}^k, \mathbb{R}^m, \mathbb{R}^n$, respectively, for some $k, m, n \in \mathbb{N}$. Denote their semigroups by $P, Q, R$, respectively, and let $\Lambda_1$ and $\Lambda_2$ be kernels of stochastic transition operators from the state space of $Y$ to the state space of $X$ and from the state space of $X$ to the state space of $S$, respectively. Finally, set $V = \log \Lambda_1$, $U = \log \Lambda_2$, define $\Lambda$ by

$$
\Lambda(y, s) = \int_{\mathbb{R}^m} \Lambda_1(y, x) \Lambda_2(x, s) \, dx
$$

and let $T = \log \Lambda$.

Suppose $A^S, A^X, A^Y$ are the generators of $S, X, Y$ with the latter two being given by (1.1), (1.2), and

$$
A^S = \sum_{i=1}^{k} \eta_i(s) \partial_{s_i} + \frac{1}{2} \sum_{i,j=1}^{k} \sigma_{ij}(s) \partial_{s_i} \partial_{s_j},
$$

(3.1)
each with boundary conditions either as in part (a) or as in part (b) of Assumption 1.

**Theorem 6.** For any $z \in \mathbb{R}^{k+m+n} = \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^n$, we write $z = (s, x, y)$. Consider a diffusion $Z = (Z_1, Z_2, Z_3)$ with state space being given by the product of the state spaces of $S, X, Y$, generator

$$
A^Z = A^S + A^X + A^Y + \sum_{i,j=1}^{m} a_{ij}(x) U_{x_i}(x, s) \partial_{x_j} + \sum_{k,l=1}^{n} \rho_{kl}(y) V_{y_k}(y, x) \partial_{y_l}
$$

and boundary conditions corresponding to those of $S, X, Y$. Suppose that all generators involved are as in Assumption 1 and that the vector $(Z_1(0), Z_2(0), Z_3(0))$ satisfies the constraint that the conditional density of $Z_2(0)$ at $x$, given $Z_3(0) = y$, is $\Lambda_1(y, x)$, and the conditional density of $Z_1(0)$ at $s$, given $Z_2(0) = x, Z_3(0) = y$, is $\Lambda_2(x, s)$ (in particular, it is independent of $y$).
If $\Lambda_1, \Lambda_2$ satisfy the hyperbolic PDEs
\[(A^X)^* \Lambda_1 = A^Y \Lambda_1, \quad (A^S)^* \Lambda_2 = A^X \Lambda_2,\]
in the generalized sense with boundary conditions corresponding to those of $A^S, A^X, A^Y$, then $(Z_1,Z_3)$ is a diffusion with generator
\[A^S + A^Y + \sum_{k,l=1}^{n} \rho_{kl}(y)T_{yl}(y,s)\partial_{yl}\]
and boundary conditions corresponding to those of $S,Y$. Moreover, $Y \langle L \rangle S$, where $L$ is the operator corresponding to $\Lambda$.

Proof. Define the transition kernel $\Gamma((x,y),s) = \Lambda_2(x,s)$. It is clear that
\[(A^S)^* \Gamma = A^{(Z_2,Z_3)} \Gamma,\]
where $A^{(Z_2,Z_3)}$ is the joint generator corresponding to the intertwining $Q \langle L \rangle P$ (see (1.3) in Theorem 1). Hence, it holds $(Z_2,Z_3) \langle \Gamma \rangle Z_1$ by Theorem 1. In particular, $Z_1 \overset{d}{=} S$ and $Z_3 \overset{d}{=} Y$. Therefore, it is enough to show that $Z_3 \langle L \rangle Z_1$ according to Definition 2, since then the theorem will follow from Theorem 2.

The first part of condition (i) in Definition 2 has already been checked and the conditional distribution of $Z_1(0)$ given $Z_3(0)$ is readily obtained from the conditional distributions of $Z_2(0)$ given $Z_3(0)$ and of $Z_1(0)$ given $(Z_2(0),Z_3(0))$. Condition (ii) can be verified by the same line of argument as in steps 2 and 3 in the proof of Theorem 1. Condition (iii) follows immediately from the corresponding condition for the intertwining $(Z_2,Z_3) \langle \Gamma \rangle Z_1$. Lastly, condition (iv) can be obtained in the same way as in step 4 in the proof of Theorem 1.

Remark 1. It is clear that the above theorem can be extended to any number of diffusions $(Z_1,Z_2,\ldots,Z_k)$. The links create a Markov chain taking values in the spaces of paths of the respective diffusions. A particularly important example is explored in Section 5.1.

Duality and time-reversal. Our next result is a version of Bayes’ rule. Suppose $Q \langle L \rangle P$ for some appropriate triplet $(Q,L,P)$. Is there a transition kernel $\hat{L}$ such that $P \langle \hat{L} \rangle Q$?

We show that this is true when both $Q$ and $P$ are reversible with respect to their respective invariant measures. This has interesting consequences for the time reversal of the diffusion with generator given by (1.3). See Figure 3.
Definition 3. We say that two semigroups \( P \) and \( \hat{P} \) on \( \mathbb{R}^d \) are in duality with respect to a Radon measure \( \nu \) if they satisfy

\[
\int_{\mathbb{R}^d} (P_t f) g \, d\nu = \int_{\mathbb{R}^d} f(\hat{P}_t g) \, d\nu \quad \text{for all nonnegative bounded } f, g \text{ and all } t \geq 0.
\]

We say \( P \) is reversible with respect to \( \nu \) if the above holds with \( \hat{P} = P \).

The above definition states that the Markov process with semigroup \( P \) and initial distribution \( \nu \), looked at backwards in time, is Markovian with transition semigroup \( \hat{P} \).

Consider two diffusion semigroups \((P_t, t \geq 0)\) and \((Q_t, t \geq 0)\) as in Assumption 1 and a stochastic transition operator \( L \) such that \( Q \langle L \rangle P \). Suppose there exists semigroups \( \hat{P} \), \( \hat{Q} \) and two measures \( \nu_1 \) and \( \nu_2 \) such that

(i) \( P \) and \( \hat{P} \) are in duality with respect to \( \nu_1 \), and \( Q \) and \( \hat{Q} \) are in duality with respect to \( \nu_2 \).

(ii) \( \nu_1 \) and \( \nu_2 \) are absolutely continuous with respect to the Lebesgue measure with continuous density functions \( h_1 \) and \( h_2 \), respectively.

(iii) \( P \) and \( Q \) are ergodic in the sense that for any probability measures \( \mu, \nu \) on \( \mathcal{X}, \mathcal{Y} \), respectively,

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t (\mu P_s f) \, ds = \int_{\mathcal{X}} f \, d\nu_1, \quad \lim_{t \to \infty} \frac{1}{t} \int_0^t (\nu Q_s g) \, ds = \int_{\mathcal{Y}} g \, d\nu_2
\]

for all continuous bounded functions \( f, g \).

Theorem 7. Let \( \Lambda(y, x) \) denote the transition kernel corresponding to \( L \) and suppose that it is jointly continuous in \( y, x \). Define

\[
\hat{\Lambda}(x, y) = \Lambda(y, x) \frac{h_2(y)}{h_1(x)},
\]

and write \( \hat{L} \) for the corresponding transition operator. Then, we have the following conclusions.

(i) \( \hat{\Lambda} \) is a stochastic transition kernel, and \( \hat{P} \langle \hat{L} \rangle \hat{Q} \).

(ii) Suppose the diffusion corresponding to this relation and given by Theorem 1 exists and is unique in law. Let \( \hat{R}_t, t \geq 0 \) denote its transition semigroup. Similarly, let \( (R_t, t \geq 0) \) be the semigroup of the diffusion corresponding to the relation \( Q \langle L \rangle P \) via Theorem 1. Define a measure \( \rho \) on \( \mathcal{X} \times \mathcal{Y} \) given by its density \( \Lambda(y, x) h_2(y) \) with respect to the Lebesgue measure. Then, the semigroups \( R \) and \( \hat{R} \) are in duality with respect to \( \rho \).

(iii) If \( P = \hat{P} \) and \( Q = \hat{Q} \), then the SDEs corresponding to \( R \) and \( \hat{R} \), under the initial condition \( \rho \), are valid representations of the same diffusion. Conditions (iii) and (iv) in Definition 2 become in this case symmetric with respect to \( Z_1 \) and \( Z_2 \) (modulo changing \( L \) to \( \hat{L} \)).

Proof. The proof is broken down into several steps.

Step 1. We first argue that \( \hat{\Lambda} \) is a stochastic transition kernel (and, thus, \( \hat{L} \) is a stochastic transition operator). We need to show that

\[
\int_{\mathcal{Y}} \Lambda(y, x) h_2(y) \, dy = h_1(x).
\]
To see this, let $f$ be a continuous bounded function on $X$. Consider the intertwining $Z = (Z_1, Z_2)$ with $Z_2$ started according to an initial distribution $\nu$ and $Z_1$ started according to $\mu = \nu L$. Using $Q(L) P$, the ergodicity condition (3.3) and Fubini’s Theorem we derive
\[
\int_X f(x) h_1(x) \, dx = \lim_{t \to \infty} \frac{1}{t} \int_0^t (\mu \, P_s \, f) \, ds = \lim_{t \to \infty} \frac{1}{t} \int_0^t (\nu \, Q_s \, (L f)) \, ds = \int_Y (L f)(y) \, h_2(y) \, dy
\]
\[
= \int_X f(x) \int_Y \Lambda(y, x) \, h_2(y) \, dy \, dx.
\]
This clearly proves our claim.

**Step 2.** Next, we show $\widehat{P} \, (\widehat{L}) \, \widehat{Q}$. To this end, consider continuous bounded functions $f$, $g$ on $X, Y$, respectively. For any fixed $t > 0$, the duality relation (3.2), Fubini’s Theorem and $Q(L) P$ yield
\[
(3.6)
\]
\[
\int_X (\widehat{P}_t \widehat{L} g)(x) \, f(x) \, d\nu_1(x) = \int_X (\widehat{L} g)(x) \, (P_t f)(x) \, h_1(x) \, dx
\]
\[
= \int_X \left( \int_Y \Lambda(y, x) \, g(y) \, h_2(y) \, dy \right) (P_t f)(x) \, dx = \int_Y \left( \int_X \Lambda(y, x) \, (P_t f)(x) \, dx \right) g(y) \, h_2(y) \, dy
\]
\[
= \int_Y (L P_t f)(y) \, g(y) \, h_2(y) \, dy = \int_Y (Q_t L f)(y) \, g(y) \, d\nu_2(y).
\]
On the other hand, a similar calculation shows
\[
(3.7)
\]
\[
\int_X (\widehat{L} \widehat{Q}_t g)(x) \, f(x) \, d\nu_1 = \int_X \left( \int_Y \Lambda(y, x) \, (\widehat{Q}_t g)(y) \, d\nu_2(y) \right) f(x) \, dx
\]
\[
= \int_X \left( \int_Y (Q_t \Lambda)(y, x) \, g(y) \, d\nu_2(y) \right) f(x) \, dx = \int_Y \left( \int_X (Q_t \Lambda)(y, x) \, f(x) \, dx \right) g(y) \, d\nu_2(y)
\]
\[
= \int_Y (Q_t L f)(y) \, g(y) \, d\nu_2(y).
\]
A comparison of the final expressions in (3.6) and (3.7) proves $\widehat{P} \, (\widehat{L}) \, \widehat{Q}$.

**Step 3.** We now move on to showing that $R$ and $\widehat{R}$ are in duality with respect to $\rho$. Since $X \times Y$ is equipped with the product $\sigma$-algebra, it is enough to verify (3.2) for functions of the type $u_i(x, y) = f_i(x) \, g_i(y), i = 1, 2$, where $f_i, i = 1, 2$ are continuous bounded functions on $X$ and $g_i, i = 1, 2$ are continuous bounded functions on $Y$. By the same calculation as in (2.22), we get
\[
(R_t \, u_1)(x, y) = \int_{X \times Y} \Lambda(y', x') \, u_1(x', y') \, P_t(x, dx') \, Q_t(y, dy'), \quad \text{and}
\]
\[
(\widehat{R}_t \, u_2)(x, y) = \int_{X \times Y} \widehat{\Lambda}(x', y') \, u_2(x', y') \, \widehat{P}_t(x, dx') \, \widehat{Q}_t(y, dy').
\]
Consider the intertwining $Z_2 \, (L) \, Z_1$ with initial distribution $\rho$ and fix a $t > 0$. Since $Z_2(0)$ has distribution $\nu_2$, it follows from condition (i) of Definition 2 that $Z_2(t)$ has also distribution $\nu_2$. Since the law of $Z_1(t)$ conditional on $Z_2(t)$ is given by $L$, the joint law of $(Z_1(t), Z_2(t))$ must be given by $\rho$. Thus, $\rho$ is an invariant distribution of $(Z_1, Z_2)$.

Consequently, to show the desired duality it suffices to argue that $\widehat{R}_t$ gives the conditional law of $(Z_1(0), Z_2(0))$ given $(Z_1(t), Z_2(t))$. Since $Z_1$ is Markovian with respect to the joint filtration (see condition (iii) in Definition 2), it is clear from duality that the conditional law
Simultaneous intertwining. Constructing explicit intertwining relationships among multidimensional processes is difficult. One needs to solve the PDE (1.4) explicitly. The next result displays a systematic method of constructing higher dimensional intertwining relationships starting with one-dimensional ones. An application of this construction will be demonstrated on an important example in Section 5.1.

We ask the following question. Suppose there exist three diffusions $X, Y, S$ and two link operators $L_1$ (with density $\Lambda_1$) and $L_2$ (with density $\Lambda_2$) such that $X \langle L_1 \rangle S$ and $Y \langle L_2 \rangle S$. Suppose we construct a suitably enlarged probability space where we have a realization of $(X, Y, S)$ such that $(X, Y)$ are conditionally independent given $S$. One can integrate out $S$ to get the joint law of $(X, Y)$. When can we claim that $(X, Y) \langle L \rangle S$ for some link function $L$? See Figure 3 for a commutative diagram representation.

One can take simple examples and check that this is not true in general. A priori it is not even obvious that $(X, Y)$ is Markovian. Consistency conditions require an assumption on the ‘base measure’ for the process $S$, and we restrict ourselves to the set-up in Theorem 7 of reversible diffusions. Suppose $A_X, A_Y, A_S$ given by (1.1), (1.2), (3.1) are the respective generators of $X, Y, S$. We assume that all the diffusions are reversible and ergodic, and that their reversible measures have densities $h_1(x), h_2(y)$, and $h_3(s)$, respectively, with respect to the corresponding Lebesgue measures. It is not important that the reversible measures are finite ($\sigma$-finite measures suffice). A slightly more general result for non-reversible diffusions $X, Y$ can be formulated similarly.

Before we state the theorem we need to introduce the so-called carré-du-champ operator for $S$:

$$\Gamma^S(f, g) = A^S(fg) - g A^S(f) - f A^S(g).$$

Figure 4. Simultaneous intertwining.

of $Z_1(0)$ given $(Z_1(t), Z_2(t))$ is governed by $\hat{P}_t$. So, it is left to argue that the conditional law of $Z_2(0)$ given $(Z_1(0), Z_1(t), Z_2(t))$ is consistent with the expression for $\hat{R}_t$ in (3.8). This is a standard Bayes’ rule calculation on the four random variables involved. We skip the details.

Finally, the last claim of the theorem is a consequence of the fact that, under the conditions $P = \hat{P}$ and $Q = \hat{Q}$, the transition probabilities of the stationary version of the process $(Z_1, Z_2)$ are governed by $R$ both forward and backward in time. Indeed, this is due to the characterization of intertwinings in Theorem 2 and the symmetry of the assumptions there with respect to time-reversal. In view of the duality of $R$ and $\hat{R}$ with respect to $\rho$, the SDEs corresponding to the two semigroups are valid representations of the stationary version of the process $(Z_1, Z_2)$. □
This is an operator of fundamental geometric and probabilistic importance. An introduction can be found in Chapter VIII, Section 3 of [RY99]. The following theorem says that the question above has an affirmative answer if the following orthogonality relationship holds:

\[(3.10)\quad \Gamma^S \left( \frac{\Lambda_1(x,\cdot)}{h_3}, \frac{\Lambda_2(y,\cdot)}{h_3} \right) = 0, \quad \text{for each } (x,y) \in \mathcal{X} \times \mathcal{Y}.\]

**Theorem 8.** Let \( z = (x,y) \in \mathbb{Z} := \mathcal{X} \times \mathcal{Y} \). Define a function \( \eta \) on \( \mathbb{Z} \) and a kernel \( \Psi \) on \( \mathbb{Z} \times \mathcal{S} \) by

\[(3.11)\quad \eta(z) = \int_{\mathcal{S}} \frac{\Lambda_1(x,s)\Lambda_2(y,s)}{h_3(s)} \, ds, \quad \Psi(z,s) = \frac{\Lambda_1(x,s)\Lambda_2(y,s)}{h_3(s)\eta(z)}.\]

Then, \( \eta \) is well-defined and \( \Psi \) is a stochastic kernel.

Under the assumption \((3.10)\) the function \( \eta \) is harmonic for \( \mathcal{A}^X + \mathcal{A}^Y \). Suppose that the corresponding \( h \)-transform of the generator \( \mathcal{A}^X + \mathcal{A}^Y \):

\[(3.12)\quad \mathcal{A} f = \mathcal{A}^X f + \mathcal{A}^Y f + \langle \nabla_x \log \eta, a \nabla_x f \rangle + \langle \nabla_y \log \eta, \rho \nabla_y f \rangle,
\]

with appropriate boundary conditions as in Theorem 4, satisfies Assumption 1.

If \( Z \) is a diffusion with generator \( \mathcal{A} \), with boundary conditions on \( \partial Z \) as dictated above, then \( Z \langle G \rangle S \), where \( G \) is the operator corresponding to \( \Psi \). Moreover,

(i) In the realization of \( Z \langle G \rangle S \) given by Theorem 4, the \( x \)-components and the \( y \)-components of \( Z \) are independent given \( S \), and the marginal laws are identical to those of \( X \langle L_1 \rangle S \) and \( Y \langle L_2 \rangle S \), respectively.

(ii) \( Z \) is reversible with a reversible density given by \( h_1(x)h_2(y)\eta(z) \).

**Proof.** We claim that the facts ‘\( \eta \) is well-defined’ and ‘\( \Psi \) is stochastic’ are due to Bayes’ rule. Consider the random vector \((X_0, Y_0, S_0)\) such that \( S_0 \) has density \( h_3 \); \( X_0, Y_0 \) are conditionally independent given \( S_0 \); and the conditional density of \( S_0 \), given \( X_0 \), is \( \Lambda_1(x,s) \), while the conditional density of \( S_0 \), given \( Y_0 \), is \( \Lambda_2(y,s) \). As shown in the proof of Theorem 7, the ergodicity condition implies that the marginal density of \( X_0 \) is then \( h_1 \), and the marginal density of \( Y_0 \) is \( h_2 \).

We are interested in the conditional density of \( S_0 \) given \((X_0, Y_0)\). By Bayes’ rule the conditional densities of \( X_0 \) and \( Y_0 \), given \( S_0 \), are given respectively by

\[
\frac{h_1(x)\Lambda_1(x,s)}{h_3(s)} \quad \text{and} \quad \frac{h_2(y)\Lambda_2(y,s)}{h_3(s)}.
\]

By conditional independence, the joint density of \((X_0, Y_0, S_0)\) is

\[
\frac{h_1(x)h_2(y)\Lambda_1(x,s)\Lambda_2(y,s)}{h_3(s)}.
\]

Integrating the latter with respect to \( s \) we get \( h_1(x)h_2(y)\eta(z) \) as the joint density of \((X_0, Y_0)\), and \( \Psi \) to be the conditional density of \( S_0 \) given \((X_0, Y_0)\). This proves our claim.

For simplicity of exposition we assume from now on the absence of boundary conditions. Neumann boundary conditions can be dealt with exactly as in the proof of Theorem 4. We show next that \( \eta \) is a harmonic function for the generator \( \mathcal{A}^X + \mathcal{A}^Y \). To see this, observe that for any \( g, f \in C_0^\infty(S) \), we have

\[
\int_S g \, (\mathcal{A}^S)^* (fh_3) \, ds = \int_S (\mathcal{A}^S g) \, fh_3 \, ds = \int_S g \, (\mathcal{A}^S f) \, h_3 \, ds
\]

by reversibility. In other words,

\[(3.13)\quad (\mathcal{A}^S)^* (fh_3) = h_3 (\mathcal{A}^S f).\]
We use this identity repeatedly to see
\[ (A^S)^* \left( \frac{\Lambda_1(x,s)\Lambda_2(y,s)}{h_3(s)} \right) = (A^S)^* \left( \frac{\Lambda_1}{h_3} \right) \left( \frac{\Lambda_2}{h_3} \right) = h_3 A^S \left( \frac{\Lambda_1}{h_3} \right) \left( \frac{\Lambda_2}{h_3} \right), \]
by (3.10),
\[ = \frac{\Lambda_1}{h_3} (A^S)^* (\Lambda_2) + \frac{\Lambda_2}{h_3} (A^S)^* (\Lambda_1), \]
by (3.13),
\[ = \frac{\Lambda_1}{h_3} A^Y \Lambda_2 + \frac{\Lambda_2}{h_3} A^X \Lambda_1, \]
by intertwining,
\[ = (A^X + A^Y) \left( \frac{\Lambda_1(x,s)\Lambda_2(y,s)}{h_3(s)} \right). \]
Therefore,
\[ (A^X + A^Y) \eta = \int_S (A^S)^* \left( \frac{\Lambda_1(x,s)\Lambda_2(y,s)}{h_3(s)} \right) \, ds = 0, \]
which proves the harmonicity of \( \eta \).

Now, the proof that \( Z^T GS \) is identical to that of Theorem 4. Moreover, Assertion (i) follows from the construction, whereas Assertion (ii) is due to the consistency of the Bayes’ rule calculation that we started with. This finishes the proof. □

The above theorem can be easily generalized to simultaneous intertwinings with any number of diffusions, provided they are pairwise orthogonal in the sense of (3.10).

4. On various old and new examples

4.1. Some examples from [CPY98]. In [CPY98] the authors discuss various examples of intertwinings of Markovian semigroups in continuous time. The perspective is somewhat different from ours and worth comparing. The set-up in [CPY98] is that of filtering. Let us briefly describe their approach below. The presentation is deliberately kept somewhat informal.

Consider two filtrations \( (\mathcal{F}_t, \ t \geq 0) \) and \( (\mathcal{G}_t, \ t \geq 0) \) such that \( \mathcal{G}_t \) is a sub-\( \sigma \)-algebra of \( \mathcal{F}_t \) for every \( t \). We now consider two processes: \( (X(t), \ t \geq 0) \), which is \( (\mathcal{F}_t) \)-adapted, and \( (Y(t), \ t \geq 0) \), which is \( (\mathcal{G}_t) \)-adapted. Assume that \( X \) is Markovian with respect to \( (\mathcal{F}_t) \) with transition semi-group \( (P_t, \ t \geq 0) \), and \( Y \) is Markovian with respect to \( (\mathcal{G}_t) \) with transition semigroup \( (Q_t, \ t \geq 0) \). Suppose that there is a Markovian kernel \( L \) such that
\[ \mathbb{E} [f(X(t)) \mid \mathcal{G}_t] = (L f)(Y(t)) \quad \text{for every } t \geq 0 \]
and all continuous bounded functions \( f \). It is then shown in Proposition 2.1 of [CPY98] that the intertwining relation \( Q_t L = L P_t \) holds for every \( t \geq 0 \). In [CPY98] the authors do not explicitly require the conditional independence conditions (iii) and (iv) in Definition 2, although condition (iv) always holds since \( X \) is \( (\mathcal{F}_t) \)-Markovian. However, the diffusion examples covered in [CPY98] all satisfy condition (iii). The rest of the subsection proves conditional independence for three major examples treated in [CPY98].

Example 1. The following is a well-known example following Dynkin’s criterion for when a function of a Markov process is itself Markovian with respect to the same filtration. Take \( Y \) to be an \( n \)-dimensional standard Brownian motion and let \( X \) be its Euclidean norm. The filtration in both cases is the one generated by \( Y \). The law of \( X \) is that of a Bessel process of dimension \( n \). The link \( L \) is simply the map \((L f)(y) = f(|y|)\). As such it intertwines the two semigroups by definition. The conditional independence properties are also true. Condition (iii) can be verified by considering two different values of \( Y(s) \) with the same
norm and an orthogonal linear transformation $A$ that maps one to the other. The process $AY$ is again a Brownian motion with the same norm process. This validates condition (iii).

Notice that $\Lambda$ does not have a density with respect to the Lebesgue measure and, hence, Theorem 2 fails to apply.

\textit{Example 2.} The following example is due to Pitman (see also [RPSI] for similar ones). Let $(B(t), \ t \geq 0)$ be a standard one-dimensional Brownian motion, and set $X(t) = |B(t)|$ and $Y(t) = |B(t)| + \Theta(t)$, where $\Theta$ is the local time at zero of $B$. We take $(\mathcal{F}_t)$ and $(\mathcal{G}_t)$ to be the filtrations generated by $X$ and $Y$, respectively. Then, $X$ is a reflecting Brownian motion and $Y$ is a Bessel process of dimension 3. The link $L$ is given by

$$
\mathbb{E}[f(X(t)) \mid \mathcal{G}_t] = \int_0^1 f(xY(t)) \, dx \quad \text{for all continuous bounded functions } f.
$$

In other words, the conditional law of $X(t)$ given $\mathcal{G}_t$ is uniform on the interval $[0, Y(t)]$.

To see condition (iii) of Definition 2 in this example, we use Pitman’s theorem (see [RY99]). Let $R$ be a 3-dimensional Bessel process, starting from zero, and let

$$
J(t) = \inf_{s \geq t} R(s), \quad t \geq 0.
$$

Then, the joint law of the processes $(R, R - J)$ is identical to that of $(Y, X)$. Now, by the Markov property of $R$, for any $0 \leq s < t$, conditional on $R(t)$, the random variable $J(t)$ is independent of $\sigma(R(u), 0 \leq u \leq s)$. Translating this to the pair $(Y, X)$ shows condition (iii).

The pair $(X, Y)$ is not a joint diffusion and Theorem 2 does not apply. However, (1.4) continues to hold. To wit, the transition kernel corresponding to $L$ is given by $\Lambda(y, x) = y^{-1}$ on its domain $\{y, x \in \mathbb{R}^2 : 0 < x < y\}$. The function $y^{-1}$ is harmonic for the generator of $Y$. In other words, $A^Y \Lambda = 0$. Moreover, $\Lambda$ does not depend on $x$, so that trivially $(A^X)^* \Lambda = 0$. This shows (1.4).

\textit{Example 3. Process extension of Beta-Gamma algebra.} The primary example in [CPY98] is a process extension of the well-known Beta-Gamma algebra. Let $(Q^t, \ t \geq 0)$ denote the semigroup of the squared-Bessel process of dimension $2\alpha > 0$. For $\alpha, \beta > 0$, let $Z_{\alpha,\beta}$ denote a Beta($\alpha, \beta$) random variable. Define $L_{\alpha, \beta}$ as the following stochastic kernel:

$$
(L_{\alpha, \beta} f)(y) = \mathbb{E}[f(yZ_{\alpha, \beta})] = \frac{1}{B(\alpha, \beta)} \int_0^1 f(yz) \, z^{\alpha-1} (1-z)^{\beta-1} \, dz.
$$

Here, $B(\cdot, \cdot)$ is the Beta function. Clearly the transition kernel corresponding to $L$ is given by

$$
\Lambda(y, x) = \frac{y^{-1}}{B(\alpha, \beta)} \left( \frac{x}{y} \right)^{\alpha-1} \left( 1 - \frac{x}{y} \right)^{\beta-1}, \quad 0 < x < y.
$$

(4.1)

Theorem 3.1 in [CPY98] proves that

$$
Q_t^{\alpha+\beta} L_{\alpha, \beta} = L_{\alpha, \beta} Q_t^\alpha, \quad t \geq 0.
$$

(4.2)

In our notation, $Q_t = Q_t^{\alpha+\beta}$ and $P_t = Q_t^\alpha$.

To put this into the filtering setup consider a product space that supports two independent squared Bessel processes $X_{\alpha}$, $X_{\beta}$ of dimensions $2\alpha$, $2\beta$, respectively, both starting from zero. Let $(\mathcal{F}_t)$ be the filtration generated by the pair $(X_{\alpha}, X_{\beta})$, and let $(\mathcal{G}_t)$ be the filtration generated by $Y = X_{\alpha} + X_{\beta}$. Setting $X = X_{\alpha}$, one ends up with a process realization of the intertwining relationship.
The pair \((X,Y)\) is jointly a diffusion and \(L\) has a differentiable density. Thus, Theorem 2 should hold. Indeed, \(\Lambda\) is a classical solution of the PDE (1.4) on the domain \(\{(y,x) \in \mathbb{R}^2 : 0 < x < y\}\). The verification is computational. Notice that
\[
\frac{1}{2} (A^X)^* = -\alpha \partial_x + 2 \partial_x + x \partial_{xx} \quad \text{and} \quad \frac{1}{2} A^Y = (\alpha + \beta) \partial_y + y \partial_{yy}.
\]
We first compute
\[
\frac{1}{2} (A^X)^* \left( x^{\alpha-1} (y-x)^{\beta-1} \right) = \frac{\beta - 1}{B(\alpha, \beta)} x^{\alpha-1} (y-x)^{\beta-3} ((\alpha + \beta - 2)x - \alpha y).
\]
This and the definition of \(\Lambda\) (see (4.1)) yield
\[
\frac{1}{2} (A^X)^* \Lambda(y,x) = \frac{\beta - 1}{B(\alpha, \beta)} x^{\alpha-1} y^{1-\alpha-\beta} (y-x)^{\beta-3} ((\alpha + \beta - 2)x - \alpha y).
\]
On the other hand, we compute
\[
\frac{1}{2} A^Y (y^{1-\alpha-\beta} (y-x)^{\beta-1}) = \frac{\beta - 1}{B(\alpha, \beta)} y^{1-\alpha-\beta} (y-x)^{\beta-3} ((\alpha + \beta - 2)x - \alpha y).
\]
Putting this together with (1.1), we get
\[
A^Y \Lambda(y,x) = \frac{\beta - 1}{B(\alpha, \beta)} x^{\alpha-1} y^{1-\alpha-\beta} (y-x)^{\beta-3} ((\alpha + \beta - 2)x - \alpha y).
\]
A comparison of (4.3) and (4.4) yields the PDE (1.4).

4.2. Whittaker 2d-growth model. The following is an example of intertwined diffusions that appeared in the study of a semi-discrete polymer model in [O'C12]. The resulting processes were investigated further in [BC13] under the name Whittaker 2d-growth model.

In the latter article, it is shown that such processes arise as diffusive limits of certain intertwined Markov chains which are constructed by means of Macdonald symmetric functions.

Fix some \(N \in \mathbb{N}\) and \(a = (a_1, a_2, \ldots, a_N) \in \mathbb{R}^N\) and consider the diffusion process \(R = (R_i^{(k)}; 1 \leq i \leq k \leq N)\) defined recursively by
\[
dR_1^{(1)}(t) = dW_1^{(k)}(t) + a_1 dt,
dR_1^{(k)}(t) = dW_1^{(k)}(t) + \left( a_k + e^{R_1^{(k-1)}(t) - R_1^{(k)}(t)} \right) dt,
dR_2^{(k)}(t) = dW_2^{(k)}(t) + \left( a_k + e^{R_1^{(k-1)}(t) - R_2^{(k)}(t)} - e^{R_2^{(k)}(t) - R_1^{(k-1)}(t)} \right) dt,
\]
\[
\vdots
\]
\[
dR_{k-1}^{(k)}(t) = dW_{k-1}^{(k)}(t) + \left( a_k + e^{R_{k-1}^{(k-1)}(t) - R_{k-1}^{(k)}(t)} - e^{R_{k-1}^{(k)}(t) - R_{k-2}^{(k-1)}(t)} \right) dt,
dR_k^{(k)}(t) = dW_k^{(k)}(t) + \left( a_k - e^{R_k^{(k)}(t) - R_{k-1}^{(k-1)}(t)} \right) dt,
\]
where \((W_i^{(k)}; 1 \leq i \leq k \leq N)\) are independent standard Brownian motions.

Given a doubly indexed sequence \(\tilde{x} := (x_i^{(k)}; 1 \leq i \leq k \leq N)\) define two functions:
\[
T_1(\tilde{x}) = \sum_{k=1}^{N} a_k \left( \sum_{i=1}^{k} x_i^{(k)} - \sum_{i=1}^{k-1} x_i^{(k-1)} \right),
T_2(\tilde{x}) = \sum_{1 \leq i \leq k \leq N-1} \left[ \exp \left( x_i^{(k)} - x_i^{(k+1)} \right) + \exp \left( x_{i+1}^{(k+1)} - x_i^{(k)} \right) \right].
\]
Let $X$ be the diffusion process comprised by the coordinates $R_i^{(k)}$, $1 \leq i \leq k \leq N - 1$, write $A^X$ for its generator and let $Y$ be the diffusion on $\mathbb{R}^N$ with generator

\begin{equation}
A^Y = \frac{1}{2} \Delta + (\nabla \log \psi_a(y)) \cdot \nabla,
\end{equation}

where

\begin{equation}
\psi_a(y) = \int_{\mathbb{R}^{(N-1)N/2}} \exp \left( T_1(\tilde{x}) - T_2(\tilde{x}) \right) \, dx_1^{(1)} \cdots dx_{N-1}^{(N-1)} \bigg|_{x_1^{(N)} = \cdots = x_N^{(N)} = y_N}.
\end{equation}

As remarked in [O'C12], the generator $A^Y$ can be rewritten as

\begin{equation}
\frac{1}{2} \psi_a(y)^{-1} \left( H - \sum_{i=1}^{N} a_i^2 \right) \psi_a(y),
\end{equation}

where $H = \Delta - 2 \sum_{i=1}^{N-1} e^{y_{i+1} - y_i}$ is the operator known as the Hamiltonian of the quantum Toda lattice (see [O'C12] and the references therein for more details on the latter).

Let $x = (x_i^{(k)})$, $1 \leq i \leq k \leq N - 1$ be a doubly indexed sequence up to level $(N - 1)$, and let $y$ be a sequence of length $N$. One can naturally concatenate $y$ ‘above’ $x$ to get a doubly indexed sequence up to level $N$. Call this sequence $\tilde{x}$. Consider the stochastic transition kernel

\begin{equation}
\Lambda(y, x) = \frac{1}{\psi_a(y)} \exp \left( T_1(\tilde{x}) - T_2(\tilde{x}) \right).
\end{equation}

It was shown in [O'C12] (see equation (12) there and the paragraph following it) that $\Lambda$ is a classical solution of the PDE (1.4) with $A^X$, $A^Y$ defined as above. By computing $\nabla_y \log \Lambda$ it is now straightforward to verify that the generator of $R$ is of the form (1.3). Hence, the Whittaker $2d$-growth model is an instance of the general construction in Theorem [4].

4.3. Constructing new examples. The difficulty in constructing intertwining relationships consists in solving the hyperbolic PDE (1.4) explicitly, within the class of stochastic solutions. Although abstract semigroup methods can be employed to prove existence of solutions of hyperbolic equations, showing nonnegativity is not easy. Below we display some classes of nonnegative solutions, without caring whether they integrate to one, which can then be turned into proper intertwining relations using Theorem [4]. These examples can be further combined using Theorems [6] and [7].

**Diffusions on multidimensional tori.** Suppose the coefficients of the generators $A^X$, $A^Y$ in (1.1), (1.2) are defined on multidimensional tori and extended to the whole space by periodicity. Then, one can think of the corresponding diffusion processes as processes on the respective tori. Simple examples are Brownian motions and Brownian bridges on the circle. Let $u$ be any classical solution of the hyperbolic PDE (1.4) on the (compact) product of the two tori. Then, there is a large enough constant $M$ such that $u + M$ is a positive classical solution of (1.4).

One might wonder how the choice of $M$ affects the intertwining relationship resulting from $u + M$ via Theorem [4]. Suppose that $A^Y$ is strictly elliptic with $\gamma \equiv 0$ and $\rho$ being bounded. Moreover, without loss of generality, assume that the tori above are of unit volume. Write $\tau(y)$ for the integral of $u$ with respect to $x$ for every fixed $y$. Then $\tau$ is a bounded harmonic function for the operator $A^Y$ on the torus. Extending it to the entire space by periodicity, one obtains a bounded harmonic function for $A^Y$ on $\mathbb{R}^n$. Hence, by a version of Liouville’s Theorem based on the Harnack’s inequality for $A^Y$ (see e.g. Corollary 9.25 in [GT01]), it must be a constant. Therefore, the choice of $M$ does not affect $A^Y$ and
only enters into the generator $\mathcal{A}^Z$ of the intertwining through the term
\[(\nabla_y \log \Lambda)' \rho \nabla_y = (\nabla_y \log \frac{u + M}{\tau + M})' \rho \nabla_y = (\nabla_y \log (u + M))' \rho \nabla_y.\]

For example, let $F$ be any twice continuously differentiable function on the circle of circumference 1 that integrates to some $c \in \mathbb{R}$. Then, for $M$ large enough, the kernel $\Lambda(y, x) = \frac{F(y-x)+M}{c+M}$ is stochastic and intertwines two Brownian motions on that circle. For a more general class of solutions of one-dimensional wave equations, we refer to d’Alembert’s formula (7.2) below.

### Intertwinings of multidimensional Brownian motions with $h$-transforms of Bessel processes.

The following lemma is well-known and is usually used to solve classical multidimensional wave equations. For its proof, see the proof of Lemma 1 on page 71 in [Eva10].

**Lemma 9.** Let $u$ be a probability density on $\mathbb{R}^m$ with $m > 1$. Let $\gamma_m = \pi^{m/2}/\Gamma(1 + m/2)$ denote the volume of the unit ball in dimension $m$. For $r > 0$, define the spherical means of $u$ by
\[(4.7) \quad u(r, x) = \frac{1}{m \gamma_m \int_{\partial B(0,1)}} u(x + rz) \, d\theta(z).\]

Here, $B(0,1)$ is the unit ball centered at $y$ and $\theta$ is its boundary measure. Then, $u(r, x)$ is nonnegative and satisfies
\[(4.8) \quad \frac{m-1}{2r} \partial_r u(r, x) + \frac{1}{2} \partial_{rr} u(r, x) = \frac{1}{2} \Delta_x u(r, x).\]

By Fubini’s Theorem the kernel $u(r, x)$ is stochastic. This allows us to use Theorem [1] to construct intertwinings of multidimensional Brownian motions with Bessel processes of the same dimension. Note that such intertwinings are in general different from the one in Example [1] since for any given $r > 0$ the density $u(r, \cdot)$ may be supported on the entire $\mathbb{R}^m$.

The above analysis extends naturally to the following interesting proposition which entails a conservation principle for solutions of (4.8).

### Proposition 10.

Let $u(r, x)$ be a classical solution of (4.8). Suppose that the integral $\tau(r) := \int_{\mathbb{R}^m} u(r, x) \, dx$ exists and is finite for every $r > 0$. Then, there exist constants $a, b \in \mathbb{R}$ such that $\tau(r) = a + br^{2-m}$. In particular, if $\limsup_{r \to 0} \tau(r) < \infty$, then $\tau(r)$ is a constant.

**Proof.** It is known that the function $s(r) = -r^{2-m}$ is a scale function for a Bessel process $Y$ of dimension $m$. We refer to chapter XI in [RY99] for the definitions and proofs. It follows that $R := s(Y)$ is a Markov process, which is a local martingale. Let $h = \tau \circ s^{-1}$. Since $\tau$ is harmonic with respect to the generator of $Y$, the process $h(R)$ is a local martingale. Applying Itô’s formula, we see that this is only possible if $h'' \equiv 0$. In other words, $h$ must be affine on $(-\infty, 0)$. The proposition readily follows. \qed

**Remark 2.** It is clear that the proof of Proposition [10] can be generalized to a generic one-dimensional diffusion process instead of a Bessel process. All possible harmonic functions with respect to its generator are then given by affine transformations of the scale function of the process. For more details on scale functions we refer the reader to Chapter VII, Section 3 in [RY99].

### Product form solutions.

We now restrict our attention to intertwinings of one-dimensional diffusion processes with generators $\mathcal{A}^X, \mathcal{A}^Y$. Suppose that $(\mathcal{A}^X)^*$ and $\mathcal{A}^Y$ admit common
eigenvalues \((\lambda_l)_{l \in I}\), where \(I\) is some countable index set. Let \(f_l, g_l\) be eigenfunctions with eigenvalue \(\lambda_l\) for \((A^X)^*, A^Y\), respectively. Then, for any collection of constants \(\alpha, (c_l)_{l \in I}\), the function
\[
\Lambda(y, x) = \alpha + \sum_{l \in I} c_l f_l(x) g_l(y)
\]
is a solution of (1.4), provided that the series on the right-hand side converges together with the corresponding series for its (formal) first and second partial derivatives.

This formulation is particularly useful for orthogonal polynomials. As an example, consider the Chebyshev orthogonal polynomials \(T_l(y), l = 1, 2, \ldots\) of type 1. These are polynomials on \([-1, 1]\), which are orthogonal in \(L^2\) with respect to the arcsine law. Moreover, each \(T_l(y)\) satisfies the ODE
\[
2 A^Y u := -x u' + (1 - x^2) u'' = -l^2 u.
\]
It is known that the solutions are bounded in absolute value by 1 on \([-1, 1]\). The second order operator \(A^Y\) is the generator of a particular Jacobi (or, Wright-Fisher) diffusion on \([-1, 1]\). The latter is a diffusion process on \([-1, 1]\) that does not reach \([-1, 1]\) in finite time, well-studied due to its importance in population genetics.

On the other hand, take \(A^X = \frac{1}{2} \partial_{xx}\) on the unit circle. A solution of
\[
u''(x) = -l^2 \nu(x)
\]
is given by \(f(x) = \cos(l x)\), which is clearly also bounded by 1 in absolute value.

Hence, for any choice of \((c_l)_{l \in I}\) such that \(\sum_{l \in I} |c_l| \leq 1\) and \(\sum_{l \in I} |c_l| l^2 < \infty\) the function
\[
\Lambda(y, x) = 1 + \sum_{l \in I} c_l T_l(y) \cos(l x)
\]
is well-defined and a nonnegative classical solution of the PDE (1.4). Moreover, since \(\int_0^{2\pi} \Lambda(y, x) \, dx = 2\pi\), the kernel \(\frac{1}{2\pi} \Lambda\) gives rise to an intertwining relationship between the two processes: the Brownian motion on the circle and the Jacobi diffusion.

A similar analysis extends to all classical orthogonal polynomials: Laguerre (squared Bessel processes), Hermite (Ornstein-Uhlenbeck processes) and Jacobi polynomials (Wright-Fisher processes). Similar eigenfunction expansions might be also possible for other one-dimensional diffusions via the Sturm-Liouville theory.

**σ-finite link measures.** It is often useful to consider a kernel \(\Lambda\) that is not stochastic, but instead \(\sigma\)-finite. As an example, we show how products of orthogonal waves can be combined using Theorem 8. Let \(\zeta_1, \zeta_2, \ldots, \zeta_d\) be an orthonormal basis of \(\mathbb{R}^d\). Consider \(d\) many twice continuously differentiable probability densities on \(\mathbb{R}\): \(f_1, f_2, \ldots, f_d\). Then, each function
\[
u_i(t, x) = f_i(t + \langle x, \zeta_i \rangle) \quad \text{satisfies} \quad \partial_{tt} u_i = \Delta_x u_i.
\]
Also, by construction \(\langle \nabla_x u_i, \nabla_x u_j \rangle = 0\) for all \(i \neq j\).

Each \(u_i\), being a classical solution of the wave equation, could be interpreted as an intertwining of a multidimensional and a unidimensional Brownian motion. However, \(u_i\) is not integrable. Nonetheless, the product \(u(t, x) := \prod_{i=1}^d u_i(t_i, x)\) is integrable with a total integral of one. Thus, a simple extension of Theorem 8 shows that \(u\) intertwines a \(d\)-dimensional Brownian motion with a \(d\)-dimensional process whose components, conditional on the former Brownian motion, are independent unidimensional Brownian motions.
5. Intertwinings of Diffusions with Reflections

5.1. An Intertwining of Dyson Brownian Motions. The following intertwining of Dyson Brownian motions was recently constructed in [War07]. Consider the unique weak solution $R$ of the system of SDEs

\[
dR_i^k(t) = dW_i^k(t) + dL_i^k(t) - dL_i^{k-1}(t), \quad 1 \leq i \leq k \leq N,
\]

in the Gelfand-Tsetlin cone

\[
\overline{G}^N = \left\{ r = (r_i^k : 1 \leq i \leq k \leq N) \in \mathbb{R}^{N(N+1)/2} : r_{i-1}^{k-1} \leq r_i^k \leq r_i^{k-1} \right\}
\]

starting from $R(0) = 0 \in \overline{G}^N$ and with entrance laws given by suitable multiples of

\[
\prod_{1 \leq i < j \leq N} \left( r_j^{(N)} - r_i^{(N)} \right) \prod_{i=1}^N \exp \left( -\frac{(r_i^{(N)})^2}{2t} \right).
\]

Here, $L_i^{k,\pm}$ are the local times at zero of the processes $R_i^k - R_{i-1}^{k-1}$, $R_i^{k-1} - R_i^k$, respectively. The probability distribution on $\overline{G}^N$ given by the density (5.3) with $t = 1$ describes the joint law of the eigenvalues of top left $k \times k$ corners of a $N \times N$ random matrix from the Gaussian unitary ensemble (GUE). In particular, given the eigenvalues of the whole matrix, the joint distribution of the eigenvalues of the top left $k \times k$ corners for $k = 1, 2, \ldots, N - 1$ is uniform on the polytope determined by the interlacing relations in (5.2). One usually refers to coordinates $R_1^k, R_2^k, \ldots, R_k^k$ as the processes (or particles) on level $k$ and plots a fixed time configuration $R(t)$ of $R$ by plotting the dots $(R_i^k(t), k), 1 \leq i \leq k \leq N$ in the plane (see Figure 2).

Next, we pick an $1 \leq M < N$ and set

\[
X := (X_i : i = 1, 2, \ldots, M) := \left( R_i^M : i = 1, 2, \ldots, M \right),
\]

\[
Y := (Y_j : j = 1, 2, \ldots, M + 1) := \left( R_j^{M+1} : j = 1, 2, \ldots, M + 1 \right),
\]

\[
Z := (Z_1, Z_2) := (X, Y).
\]
Then, the findings in [War07] show that $X$, $Y$ and $Z$ are Markovian and satisfy the stochastic differential equations

$$
\begin{align*}
\mathrm{d}X_i(t) &= \mathrm{d}\tilde{W}_i(t) + \sum_{j \neq i} \frac{\mathrm{d}t}{X_i(t) - X_j(t)}, \quad i = 1, 2, \ldots, M, \\
\mathrm{d}Y_j(t) &= \mathrm{d}\tilde{W}_j^{M+1}(t) + \sum_{j' \neq j} \frac{\mathrm{d}t}{Y_j(t) - Y_{j'}(t)}, \quad j = 1, 2, \ldots, M + 1, \\
\mathrm{d}(Z_1)_i(t) &= \mathrm{d}\tilde{W}_i^{M}(t) + \sum_{i' \neq i} \frac{\mathrm{d}t}{(Z_1)_i(t) - (Z_1)_{i'}(t)}, \quad i = 1, 2, \ldots, M, \\
\mathrm{d}(Z_2)_j(t) &= \mathrm{d}\tilde{W}_j^{M+1}(t) + \mathrm{d}L_j^{M+1,+}(t) - \mathrm{d}L_j^{M+1,-}(t), \quad j = 1, 2, \ldots, M + 1.
\end{align*}
$$

In particular, $X$ corresponds precisely to the normal reflection of the components of $Z_2$ in the SDE above. The well-posedness of the submartingale problem for $Z$ (in fact, even the pathwise uniqueness for the corresponding SDE) was established in [War07], so that Theorem 3 applies here and yields $Z = Y\langle L \rangle X$ as claimed.

We show now that the process $Z$ fits into the framework of our Theorem 3 and, hence, the main result of [War07] becomes a corollary of that more general result. The appropriate link function for the case at hand turns out to be

$$\Lambda(y, x) = M! \prod_{1 \leq i < m \leq M} (x_m - x_i) \prod_{1 \leq j < n \leq M+1} (y_n - y_j)^{-1}. \tag{5.5}$$

The latter function usually goes by the name Dixon-Anderson conditional probability density function (see [For09], [Dix05], [And91], where in particular it is shown that each $\Lambda(y, \cdot)$ is a probability density). It is evident that the conditions (i), (ii) in Theorem 3 are satisfied in the present situation with

$$D(y) = [y_1, y_2] \times [y_2, y_3] \times \ldots \times [y_M, y_{M+1}]. \tag{5.6}$$

Moreover, a straightforward computation shows that $(A^X)^* \Lambda = 0 = A^Y \Lambda$ pointwise. In addition, for a face of $D(y)$ of the form $\{x : x_i = y_i\}$ the outward normal vector $\eta$ is the negative $i$-th canonical basis vector in $\mathbb{R}^M$, $\Psi^i = \eta$ and $\Psi^j = 0$ for all $j \neq i$; and for a face of $D(y)$ of the form $\{x : x_i = y_{i+1}\}$ the outward normal vector $\eta$ is the $i$-th canonical basis vector in $\mathbb{R}^M$, $\Psi^{i+1} = \eta$ and $\Psi^j = 0$ for all $j \neq i + 1$. Therefore, $\eta = \sum_{j=1}^n \Psi^j \langle \Psi^j, \eta \rangle$ and (2.20) reads

$$
\begin{align*}
-b_i \Lambda - \partial_{x_i} \Lambda &= \gamma_i \Lambda + \partial_{y_i} \Lambda, \quad \text{if} \quad x_i = y_i, \\
b_i \Lambda - \partial_{x_i} \Lambda &= \gamma_{i+1} \Lambda + \partial_{y_{i+1}} \Lambda, \quad \text{if} \quad x_i = y_{i+1}.
\end{align*} \tag{5.7}
$$

Using the explicit formula for $\Lambda$ it is easy to check that in both equations both sides are equal to zero. In addition,

$$\nabla_y \log \Lambda = \left( -\sum_{j' \neq j} \frac{1}{y_j - y_{j'}} \right)_{j=1, 2, \ldots, M+1}, \tag{5.8}
$$

so that the generator $A^Z$ is of the form (1.3). Moreover, the boundary condition (2.27) corresponds precisely to the normal reflection of the components of $Z_2$ on the components of $Z_1$ in the SDE above. The well-posedness of the submartingale problem for $Z$ (in fact, even the pathwise uniqueness for the corresponding SDE) was established in [War07], so that Theorem 3 applies here and yields $Z = Y\langle L \rangle X$ as claimed.
5.2. Using infinite link measures and simultaneous intertwining. Given two independent standard Brownian motions $B$, $W$ consider the following process $Z = (Z_1, Z_2)$ taking values in $\mathbb{R}^2$. Start with a $\mathbb{R}^2$-valued random variable $Z(0) = (Z_1(0), Z_2(0))$, and let $Z_1$ evolve according to the Brownian motion $B$ and $Z_2$ according to the Brownian motion $W$ reflected on $B$. More explicitly, $Z$ is given by the unique strong solution of the system of SDEs
\begin{align}
(5.9) & \quad \mathrm{d}Z_1(t) = \mathrm{d}B(t), \\
(5.10) & \quad \mathrm{d}Z_2(t) = \mathrm{d}W(t) + \mathrm{d}L(t),
\end{align}
where $L$ is defined as the local time at zero of $Z_2 - Z_1$ if $Z_2(0) \geq Z_1(0)$ and as minus the local time at zero of $Z_1 - Z_2$ if $Z_1(0) > Z_2(0)$.

In this context, the main result of [SW02] (see also [STW00]) can be stated as follows: there exists random variables $Z(0)$ such that the process $Z_2$ is a standard Brownian motion started from $Z_2(0)$. Indeed, in [SW02] the authors consider a time-reversed standard Brownian motion reflected on a regular standard Brownian motion. Looking at the time-reversed Brownian motion forward in time and recalling the reversibility of Brownian motion as well as the fact that there is a unique way to reflect a continuous path in a time-dependent interval whose boundary changes continuously (see [BKR09]), one concludes that the process in [SW02] solves the system (5.9), (5.10) (with the initial condition being difficult to determine explicitly).

One might think that the solution of (5.9), (5.10) can be obtained as the $\epsilon \to 0$ limit of intertwining of standard Brownian motions of the form
\begin{align}
(5.11) & \quad \mathrm{d}Z_1^{(\epsilon)}(t) = \mathrm{d}B(t), \\
(5.12) & \quad \mathrm{d}Z_2^{(\epsilon)}(t) = \mathrm{d}W(t) + \frac{f_\epsilon(Z_1^{(\epsilon)}(t) - Z_1^{(\epsilon)}(t))}{f_\epsilon(Z_2^{(\epsilon)}(t) - Z_1^{(\epsilon)}(t))} \, \mathrm{d}t.
\end{align}
Here, the links $\Lambda_\epsilon$ are given by $\Lambda_\epsilon(y, x) = f_\epsilon(y - x)$. In the following we explain why this is not the case.

We argue by contradiction. For the solutions of (5.11), (5.12) to be an approximating sequence of the solution of (5.9), (5.10), one needs to have
\begin{align}
(5.13) & \quad \lim_{\epsilon \to 0} \frac{f_\epsilon}{f_\epsilon} = 0 \quad \text{uniformly on compact subsets of } (-\infty, 0) \cup (0, \infty).
\end{align}
Noting $\log \frac{f_\epsilon(r_2)}{f_\epsilon(r_1)} = \int_{r_1}^{r_2} \frac{f_\epsilon(r)}{f_\epsilon(r)} \, \mathrm{d}r$ we conclude that we must also have $\lim_{\epsilon \to 0} \frac{f_\epsilon(r_2)}{f_\epsilon(r_1)} = 1$ uniformly on compact subsets of $(-\infty, 0) \cup (0, \infty)$. Writing $F_\epsilon$ for the cumulative distribution function corresponding to the probability density $f_\epsilon$, we deduce further
\begin{align}
(5.14) & \quad \lim_{\epsilon \to 0} \frac{F_\epsilon(r_2) - F_\epsilon(r_1)}{(r_2 - r_1)f_\epsilon(r_1)} = 1 \quad \text{uniformly on compact subsets of } (-\infty, 0) \cup (0, \infty).
\end{align}
Now, the weak convergence $\Lambda_\epsilon \to \Lambda$ shows that the numerator on the left-hand side of (5.14) converges to $F(r_2) - F(r_1)$ for Lebesgue a.e. $r_1, r_2$, where $F$ is the cumulative distribution function corresponding to $\Lambda$. Consequently, the denominator on the left-hand side of (5.14) must also converge almost everywhere and we write $(r_2 - r_1)f(r_1)$ for the limit. Finally, the simple observation
\begin{align}
(r_3 - r_2)f(r_2) + (r_2 - r_1)f(r_1) = F(r_3) - F(r_2) + F(r_2) - F(r_1) = F(r_3) - F(r_1) = (r_3 - r_1)f(r_1)
\end{align}
shows that \( f(r_2) = f(r_1) \) for Lebesgue a.e. \( r_1, r_2 \in (0, \infty) \) (or \( (-\infty, 0) \)). Consequently, there are constants \( 0 \leq f \leq F < \infty \) such that
\[
F(r_2) - F(r_1) = f(r_2 - r_1) \quad \text{for a.e.} \quad r_1, r_2 \in (-\infty, 0),
\]
\[
F(r_2) - F(r_1) = f(r_2 - r_1) \quad \text{for a.e.} \quad r_1, r_2 \in (0, \infty).
\]
Since \( F \) is a cumulative distribution function (and, in particular, takes values in \([0, 1]\)), it follows \( F \equiv 0 \) on \((-\infty, 0)\) and \( F \equiv 1 \) on \((0, \infty)\). Consequently, \( \Lambda(y, x) = \delta_0(y - x) \).

However, for a solution of (5.9), (5.10) it cannot hold \( \mathbb{P}(Z_1(t) \in \cdot | Z_2(t) = y) = \delta_y(\cdot) \) for all \( t \geq 0 \) and \( y \in \mathbb{R} \). This is the desired contradiction.

Nonetheless, the results of \([SW02]\) and \([STW00]\) can be easily derived if we can make sense of an intertwining with the link function \( \Lambda(y, x) = \text{sgn}(y - x) \) which is a generalized solution of the one-dimensional wave equation. That is, suppose the following statement remains true if one replaces \( ψ \) by \( \text{sgn}(y - x) \) in condition (iii) of Definition 2, so that the contribution of the drift term to (2.28) is zero. Thus,
\[
\mathbb{E}[f(Z_1(s)) \mid Z_2(u), 0 \leq u \leq s] = (Lf)(Z_2(s)).
\]

It is easy to check that appropriate modifications of all our results continue to hold for infinite link measures. In particular, we have the following.

**Definition 4.** Suppose \( \mathcal{X}, \mathcal{Y} \) are domains in Euclidean spaces. Consider a link operator \( L \) whose density \( \Lambda \) is continuous with infinite total mass. Then, as in Definition 2 we say that \( Z = Y \langle L \rangle X \) if the following hold.

1. Condition (i) in Definition 2 holds for functions \( f \) that are compactly supported.
2. \( Q \langle L \rangle P \) is satisfied with all operators acting on compactly supported functions.
3. Condition (iii) in Definition 2 holds as stated there.
4. For any \( s \geq 0 \) and any compactly supported function \( f \),
   \[
   \mathbb{E}[f(Z_1(s)) \mid Z_2(u), 0 \leq u \leq s] = (Lf)(Z_2(s)).
   \]

**Proposition 11.** Let \( X, Y \) be two one-dimensional Brownian motions. Consider the process \( Z = (Z_1, Z_2) \) where \( Z_1 \overset{d}{=} X, Z_2 \overset{d}{=} Y, Z_2(0) \) is Lebesgue distributed on \( \mathbb{R}, Z_1(0) \) is Lebesgue distributed on \((-\infty, 0) \) and \( Z_2 \) reflects on \( Z_1 \). Then, \( Z = Y \langle L \rangle X \) in the sense of Theorem 3 where the density corresponding to \( L \) is \( \mathbf{1}_{(-\infty, 0)}(x) \). The statement remains true if one replaces \((-\infty, 0)\) by \([Z_2(0), \infty)\) and \( \mathbf{1}_{(-\infty, 0)}(x) \) by \( \mathbf{1}_{(y, \infty)}(x) \).

**Proof.** By symmetry it suffices to consider the link with density \( \mathbf{1}_{(-\infty, 0)}(x) \). In the notation of Theorem 3 we have \( D(y) = (-\infty, y) \). This satisfies conditions (i) and (ii) there. The directional derivative \( \Psi \) in condition (iii) is constantly equal to 1 for the case at hand, so that condition (iii) also holds. Condition (iv) is trivially satisfied since every term in (2.28) is zero. Moreover, \( V \equiv 0 \), so that the contribution of the drift term to (2.28) is zero. Thus, Proposition \( 11 \) is a consequence of Theorem 3.

Now, the kernels \( \Lambda_1(y, s) = \mathbf{1}_{(-\infty, y)}(s) \) and \( \Lambda_2(x, s) = \mathbf{1}_{(x, \infty)}(s) \) are orthogonal in the sense of (2.10). A straightforward extension of Theorem 8 for moving boundaries then...
yields a simultaneous intertwining in the sense of Theorem 8 with
\[ \eta(x, y) = 1_{\{x < y\}} \int_x^y ds = (y - x) 1_{\{x < y\}}. \]
Plugged into (3.12), this gives the generator of a two-dimensional Dyson Brownian motion.
In other words, Theorem 8 shows that a two-dimensional Dyson Brownian motion \((X, Y)\)
is intertwined with a one-dimensional Brownian motion \(S\) that starts uniformly between
the two coordinates \(X\) and \(Y\). The coordinates \(X, Y\) are independent given \(S\) and
the conditional law is that of a Brownian motion reflecting on \(S\). This is precisely the Warren
construction.

One can go from level \(N\) to level \((N + 1)\) in the Warren construction in a similar
fashion. We have \((N + 1)\) many pairwise orthogonal links with constant densities on the polygonal
domains prescribed by the intertwining relations in (5.2). Combining them by means of
Theorem 8 one obtains the Dyson Brownian motion on the \((N + 1)\)-st level of the Warren
construction. Notice also that the sequence of harmonic functions generated via Theorem
8 is the sequence of Vandermonde determinants of increasing dimensions.

6. INTERTWINING AND CONVERGENCE RATE OF DIFFUSION SEMIGROUPS

In this section we show how intertwinings can be used to translate questions on rate of
convergence of diffusion semigroups to controllability questions for hyperbolic PDEs of the
type (1.4). We measure the distance to stationarity using the continuous version of the
Aldous-Diaconis separation distance:

\[
d(\alpha, \nu) = \sup_{A \subset \mathcal{X}} \left( 1 - \frac{\alpha(A)}{\nu(A)} \right).
\]

Here, the supremum is taken over all Borel measurable subsets of \(\mathcal{X}\). Our approach is based
on the following theorem which might be particularly useful for non-reversible diffusions.

**Theorem 12.** Let \(X, Y\) be two diffusions as in Theorem 1 and such that \(X\) is positively
recurrent with invariant distribution \(\nu\). Suppose that \(\nu\) has a density \(f\) on \(X\), let \(\alpha\) be
another probability measure on \(X\) with density \(g\), and consider the two point boundary
value problem

\[
(\mathcal{L}^X)^* \Lambda = \mathcal{L}^Y \Lambda,
\]

\[
\Lambda(0, \cdot) = f,
\]

\[
\Lambda(y^*, \cdot) = g
\]

for some fixed \(y \in \mathcal{Y}\). If the problem (6.2) - (6.4) has a distributional solution \(\Lambda\) given
by a positive function satisfying \(\int_{\mathcal{X}} \Lambda(y, x) dx = 1\) for any \(y \in \mathcal{Y}\), which is differentiable in \(y\)
on the interior of \(\mathcal{Y}\) for any fixed \(x\) with \(\nabla_y \log \Lambda\) being bounded, then

\[
d((\alpha P_t), \nu) \leq \mathbb{P}(\tau_0 \geq t),
\]

where \(\tau_0\) is the first hitting time of 0 by the process \(Y\), started from \(y^*\).

**Proof.** Consider the diffusion \(Z\) with generator given by (1.3), \(Z_2(0) = y^*\) and \(Z_1(0)\)
distributed according to \(\Lambda(y^*, \cdot)\). In view of the boundedness of \(\nabla_y \log \Lambda\) the martingale
problem for \(\mathcal{A}Z\) is well-posed due to Girsanov’s Theorem (see e.g. Propositions 5.3.6 and
5.3.10 in [KS91] for a similar argument). Hence, we can apply Theorem 1 to conclude that
\(Z_1 \overset{d}{=} X, Z_2 \overset{d}{=} Y\) (by property (i) in Definition 2), that \(\tau_0\) is independent of \(X(\tau_0)\) (by
property (iv) in Definition 2) and that \(X(\tau_0)\) is distributed according to \(\nu\) (by property (ii)
in Definition 2).
Now, (6.5) follows from
\[ \mathbb{P}(X(t) \in A) \geq \mathbb{P}(X(t) \in A, \tau_0 \leq t) = \mathbb{P}(\tau_0 \leq t) \nu(A) = (1 - \mathbb{P}(\tau_0 > t)) \nu(A) \]
(the latter estimates for finite state space Markov chains are due to Aldous and Diaconis, see the proof of Proposition 3.2 (a) in [AD87]).

For example, if \( \mathcal{L}Y = \frac{1}{2} \partial_{yy} - \kappa \partial_y \) for some \( \kappa \geq 0 \) (that is, \( Y \) is a Brownian motion with non-positive drift), then (6.5) leads to
\[ d((\alpha P_t), \nu) \leq \int_t^\infty \frac{y^*}{\sqrt{2\pi t^3}} \exp \left( - \frac{(y^* - \kappa s)^2}{2s} \right) ds \leq e^{\kappa y^*} y^* \sqrt{\frac{2}{\pi t}} e^{-\kappa^2 t/2}. \]

The first inequality in (6.6) follows from the explicit formula for the first passage times of a Brownian motion with drift (see e.g. section 3.5C in [KS91]), whereas the second inequality in (6.6) is a consequence of the following chain of estimates:
\[ \int_t^\infty \frac{y^*}{\sqrt{2\pi t^3}} \exp \left( - \frac{(y^* - \kappa s)^2}{2s} \right) ds \leq e^{\kappa y^*} y^* \frac{1}{\sqrt{2\pi}} \int_t^\infty s^{-3/2} e^{-\kappa^2 s/2} ds \]
\[ = e^{\kappa y^*} y^* \frac{1}{\sqrt{2\pi}} \left( 2 \sqrt{2\pi} \kappa \Phi(\kappa \sqrt{t}) - 2 \sqrt{2\pi} \kappa + \frac{2}{\sqrt{t}} e^{-\kappa^2 t/2} \right) \leq e^{\kappa y^*} y^* \sqrt{\frac{2}{\pi t}} e^{-\kappa^2 t/2}. \]

Here, \( \Phi \) is the cumulative distribution function of a standard normal random variable.

The mechanism behind Theorem 12 can be understood better through the following example.

**Example 4.** Let \( X \) be a reflected Brownian motion on \([0, 1]\), that is, a process taking values in \([0, 1]\), which evolves as a standard Brownian motion while in \((0, 1)\) and reflects when it reaches 0 or 1. In addition, let \( Y \) be a Brownian motion on \( \mathbb{R} \) with drift \( -\kappa \) for a \( \kappa > 0 \) to be fixed later. Clearly, \( X \) is positively recurrent and its invariant distribution is the uniform distribution on \([0, 1]\). Hence, the two point boundary value problem in Theorem 12 becomes
\[ \Lambda_{yy} - 2\kappa \Lambda_y = \Lambda_{xx} \quad \text{on} \quad \mathbb{R} \times (0, 1) \]
\[ \Lambda(0, \cdot) = 1, \quad \Lambda(y^*, \cdot) = g, \]
\[ \Lambda_x(\cdot, 0) = 0, \quad \Lambda_x(\cdot, 1) = 0. \]

Next, we fix some \( g \in L^2([0, 1]) \) which is a probability density, restrict \( \Lambda \) to \([0, y^*] \times [0, 1]\) and expand the resulting function in a basis of \( L^2([0, y^*] \times [0, 1]) \) of the form \( \eta(y; p) \theta(x; q) \), \( p, q \in \mathbb{N} \), where \( \eta(y; p), p \in \mathbb{N} \) are eigenfunctions of \( \partial_{yy} - 2\kappa \partial_y \) and \( \theta(x; q), q \in \mathbb{N} \) are eigenfunctions of \( \partial_{xx} \) satisfying the Neumann boundary conditions in (6.9). We conclude that the solution of the two point boundary value problem must satisfy
\[ \Lambda(y, x) = 1 + \sum_{1 \leq p \leq \frac{y}{2}} c_p e^{\kappa y} \sinh(\kappa^2 - p^2 \pi^2 y) \cos(p\pi x) \]
\[ + \sum_{p > \frac{y}{2}} c_p e^{\kappa y} \sin(p^2 \pi^2 - \kappa^2 y) \cos(p\pi x) \]
on \([0, y^*] \times [0, 1]\), where \( c_p, p = 1, 2, \ldots \) are determined by the coefficients of \( \cos(p\pi x) \), \( p = 1, 2, \ldots \) in the expansion of \( g \). Since \( \Lambda \) is smooth on \([0, y^*] \times (0, 1)\) and \( \int_0^1 \Lambda(y, x) \, dx = 1 \) for all \( y \in [0, 1] \), we see that Theorem 12 can be applied whenever \( \Lambda \) can be extended to a smooth strictly positive function on \( \mathbb{R} \times (0, 1) \) with a bounded logarithmic derivative in \( y \). This
can be verified for specific choices of the function \( g \). Take for example \( g(x) = 1 + \cos(p\pi x) \) for some \( p > \frac{\pi}{2} \). Then, (6.11) becomes
\[
(6.11) \quad \Lambda(y, x) = 1 + \frac{e^{\kappa y} \sin(\sqrt{p^2 \pi^2 - \kappa^2 y})}{e^{\kappa y^*} \sin(\sqrt{p^2 \pi^2 - \kappa^2 y^*})} \cos(p\pi x), \quad (y, x) \in [0, y^*] \times (0, 1)
\]
and the latter function can be extended to a smooth function on \( \mathbb{R} \times (0, 1) \) in the obvious way. Moreover, one observes that the resulting function is everywhere positive iff \( \kappa = 0 \) and \( y^* = 1/(2p) \). Applying Theorem 12 one gets the bound
\[
(6.12) \quad d((\alpha P_t), \nu) \leq \frac{1}{p\sqrt{2\pi t}},
\]
where \( \alpha \) is the probability measure on \([0, 1] \) with density \( g(x) = 1 + \cos(p\pi x) \), \( (P_t, t \geq 0) \)
is the semigroup of \( X \) and \( \nu \) is the uniform measure on \([0, 1] \). The bound (6.12) becomes better as \( p \) grows, which corresponds to the fact that \( \alpha \) approaches \( \nu \) as \( p \to \infty \).

7. Appendix: Some solutions of hyperbolic PDEs

In view of Theorem 11 the question of finding intertwinings of diffusions amounts to finding solutions of second-order hyperbolic PDEs. In this section, we give a list of examples of operators \( \mathcal{A}^X, \mathcal{A}^Y \), for which explicit solutions of the PDE (1.4) are known, as well as a set of general existence results.

Example 5. Classical wave equations. We start with the simplest example of \( \mathcal{A}^X = \partial_{xx} \), \( \mathcal{A}^Y = \Delta_y = \sum_{k=1}^n \partial_{y_k y_k} \), in which case (1.4) turns into the classical wave equation
\[
(7.1) \quad \partial_{xx} \Lambda = \Delta_y \Lambda.
\]
In the case \( n = 1 \), the classical solutions of (7.1) are given by the well-known d’Alembert’s formula:
\[
(7.2) \quad \Lambda(y, x) = \frac{1}{2} \left( g(y - x) + g(y + x) \right) + \frac{1}{2} \left( h(y + x) - h(y - x) \right).
\]
Clearly, for any \( g, h \geq 0 \) with \( g, h \in L^1(\mathbb{R}) \cap C^1_b(\mathbb{R}) \) and \( \int_{\mathbb{R}} g(q) \, dq = 1 \) the corresponding \( \Lambda \) yields an intertwining of two standard Brownian motions via the construction described in Theorem 11. For \( n \geq 2 \), the classical solutions of (7.1) are given by the following formulas (see e.g. [Eva10])
\[
(7.3) \quad \Lambda(y, x) = \partial_x \left( \frac{1}{x} \partial_x \right)^{\frac{n-2}{2}} \left( \frac{1}{x} \int_{\partial B(y, x)} \phi(q) \, d\theta(q) \right) + \left( \frac{1}{x} \partial_x \right)^{\frac{n-2}{2}} \left( \frac{1}{x} \int_{\partial B(y, x)} \psi(q) \, d\theta(q) \right)
\]
if \( n \) is odd, and by
\[
(7.4) \quad \partial_x \left( \frac{1}{x} \partial_x \right)^{\frac{n-2}{2}} \left( \int_{B(y, x)} \frac{\phi(q)}{(x^2 - |q - y|^2)^{1/2}} \, dq \right) + \left( \frac{1}{x} \partial_x \right)^{\frac{n-2}{2}} \left( \int_{B(y, x)} \frac{\psi(q)}{(x^2 - |q - y|^2)^{1/2}} \, dq \right)
\]
if \( n \) is even. Here, \( B(y, x) \) is the ball of radius \( x \) around \( y \), \( \partial B(y, x) \) is its boundary and \( \theta \) is the boundary measure on \( \partial B(y, x) \).

Example 6. Radial part of the Laplace-Beltrami operator on symmetric spaces. Next, we consider the situation where \( \mathcal{A}^X = \partial_{xx} \) and \( \mathcal{A}^Y = \frac{1}{v(y)} \partial_y v(y) \partial_y \) with a potential \( v \geq 0 \). Note that if \( v \) is a smooth probability density, \( \mathcal{A}^Y \) is the generator of a reversible diffusion with invariant measure \( v(y) \, dy \). For some examples, in which \( \mathcal{A}^Y \) corresponds to the radial part of the Laplace-Beltrami operator on a suitable noncompact symmetric space
of rank 1, explicit solutions of (1.4) can be obtained by a procedure described in [Car82a] and the references therein. Consider the eigenfunctions

\begin{align*}
(7.5) & \quad A^X \phi_\lambda = \lambda \phi_\lambda \quad \text{with} \quad \phi_\lambda(0) = 1, \ \phi_\lambda'(0) = 0, \\
(7.6) & \quad A^Y \psi_\lambda = \lambda \psi_\lambda \quad \text{with} \quad \psi_\lambda(0) = 1, \ \psi_\lambda'(0) = 0,
\end{align*}

where \( \lambda \) varies over the spectrum of \( A^Y \). Then, functions of the form

\begin{equation}
(7.7) \quad \Lambda(y, x) = v(y) \int_0^\infty \phi_\lambda(x) \psi_\lambda(y) \, d\alpha(\lambda)
\end{equation}

solve (7.1), where \( \alpha \) is a positive measure supported on the spectrum of \( A^Y \).

One case, in which this procedure leads to explicit solutions, is that of \( \nu = 2 \) solve (7.1). where \( \nu \text{ and } [DW53] \). When \( 2 \nu + 1 = n - 1 \), the solution reads (see [Asg37], [Wei52], [Wei54] and [DW53]). When \( 2\nu + 1 = n - 1 \), the solution reads (see [Asg37])

\begin{equation}
(7.9) \quad \Lambda(y, x) = \frac{1}{c_{n-1}} \int_{\partial B(0,1)} f(x + yq) \, d\theta(q),
\end{equation}

where \( c_{n-1} \) is the volume of the \( (n-1) \)-dimensional unit sphere \( \partial B(0, 1) \) and \( \theta \) is the surface measure on the latter. When \( 2\nu + 1 > n - 1 \), the solution is (see [Wei52])

\begin{equation}
(7.10) \quad \Lambda(y, x) = \frac{c_{2\nu+2-n}}{c_{2\nu+2}} \int_{B(0,1)} f(x + yq)(1 - |q|^2)^{\nu - n/2} \, dq.
\end{equation}

Finally, when \( 2\nu + 1 < n \) and \( 2\nu + 1 \neq -1, -3, -5, \ldots \), a family of solutions is given by (see [Wei54])

\begin{equation}
(7.11) \quad \Lambda(y, x) = y^{-2\nu} \left( \frac{1}{y} \partial_y \right)^n y^{2\nu + 2n} \Lambda(y, x),
\end{equation}

where \( \Lambda(y, x) \) is the solution of the EPD equation with \( (2\nu + 1) \) replaced by \( (2\nu + 2n + 1) \). Moreover, the solution of the Cauchy problem is unique if \( 2\nu + 1 > 0 \) and is not unique for \( 2\nu + 1 < 0 \). In particular, another set of solutions in the latter case (including \( (2\nu + 1) \in \{-1, -3, -5, \ldots\} \)) was obtained in [DW53] by analytic continuation of the solutions in (7.10) in the \( \nu \) variable.

We supplement the explicit solutions above by a set of general existence results for equations of the type (1.4) taken from section 7.2 in [Eva10].

**Proposition 13.** In each of the following cases classical solutions of the PDE (1.4) exist.

(a) \( m = 1 \), \( A^X = \partial_{xx} \), \( n \) is arbitrary and (1.4) is uniformly elliptic.

(b) \( m \) is arbitrary, (1.4) is uniformly elliptic, \( n = 1 \) and \( A^Y = \partial_{yy} \).

To the best of our knowledge, conditions for nonnegativity of these solutions have not been studied in this generality.
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