Exponential laws for spaces of differentiable functions on topological groups

Natalie Nikitin

Abstract
Smooth functions \( f : G \to E \) from a topological group \( G \) to a locally convex space \( E \) were considered by Riss (1953), Boseck, Czichowski and Rudolph (1981), Beltită and Nicolae (2015), and others, in varying degrees of generality. The space \( C^\infty(G, E) \) of such functions carries a natural topology, the compact-open \( C^\infty \)-topology. For topological groups \( G \) and \( H \), we show that \( C^\infty(G \times H, E) \cong C^\infty(G, C^\infty(H, E)) \) as a locally convex space, whenever both \( G \) and \( H \) are metrizable or both \( G \) and \( H \) are locally compact. Likewise, \( C^k(G, C^l(H, E)) \) can be identified with a suitable space of functions on \( G \times H \).

1 Introduction

Exponential laws of the form \( C^\infty(M \times N, E) \cong C^\infty(M, C^\infty(N, E)) \) for spaces of vector-valued smooth functions on manifolds are essential tools in infinite-dimensional calculus and infinite-dimensional Lie theory (cf. works by Kriegl and Michor [10], Kriegl, Michor and Rainer [11], Alzaareer and Schmeding [1], Glöckner [5], Glöckner and Neeb [6], Neeb and Wagemann [12], and others). Stimulated by recent research by Beltită and Nicolae [2], we provide exponential laws for function spaces on topological groups.

Let \( G \) be a topological group, \( U \subseteq G \) be an open subset, \( f : U \to E \) be a function to a locally convex space and \( \mathcal{L}(G) := \text{Hom}_{cts}(\mathbb{R}, G) \) be the set of continuous one-parameter subgroups \( \gamma : \mathbb{R} \to G \), endowed with the compact-open topology. For \( x \in U \) and \( \gamma \in \mathcal{L}(G) \) let us write \( D_\gamma f(x) := \lim_{t \to 0} \frac{1}{t} (f(x \cdot \gamma(t)) - f(x)) \) if the limit exists. Following Riss [14] and Boseck et al. [3], we say that \( f \) is \( C^k \) (where \( k \in \mathbb{N}_0 \cup \{\infty\} \)) if \( f \) is continuous, the iterated derivatives

\[
d^{(i)} f(x, \gamma_1, \ldots, \gamma_i) := (D_{\gamma_i} \cdots D_{\gamma_1} f)(x)
\]

exist for all \( x \in U, i \in \mathbb{N} \) with \( i \leq k \) and \( \gamma_1, \ldots, \gamma_i \in \mathcal{L}(G) \), and the maps \( d^{(i)} f : U \times \mathcal{L}(G)^i \to E \) so obtained are continuous. We endow the space \( C^k(U, E) \) of all \( C^k \)-maps \( f : U \to E \) with the compact-open \( C^k \)-topology (recalled in Definition 2.3). If \( G \) and \( H \) are topological groups and \( f : G \times H \to E \) is \( C^\infty \), then...
\[ f^\vee(x) := f(x, \bullet) \in C^\infty(H,E) \text{ for all } x \in G. \] With a view towards universal enveloping algebras, Beltitǎ and Nicolae \cite{2} verified that \( f^\vee \in C^\infty(G,C^\infty(H,E)) \) and showed that the linear map

\[ \Phi : C^\infty(G \times H, E) \to C^\infty(G, C^\infty(H,E)), \quad f \mapsto f^\vee \]

is a topological embedding.

Recall that a Hausdorff space \( X \) is called a \( k_R \)-space if functions \( f : X \to \mathbb{R} \) are continuous if and only if \( f|_K \) is continuous for each compact subset \( K \subseteq X \).

We obtain the following criterion for surjectivity of \( \Phi \):

**Theorem (A).** Let \( U \subseteq G, V \subseteq H \) be open subsets of topological groups \( G \) and \( H \), and \( E \) be a locally convex space. If \( U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j \) is a \( k_R \)-space for all \( i, j \in \mathbb{N}_0 \), then

\[ \Phi : C^\infty(U \times V,E) \to C^\infty(U, C^\infty(V,E)), \quad f \mapsto f^\vee \]

is an isomorphism of topological vector spaces.

The condition is satisfied, for example, if both \( G \) and \( H \) are locally compact or both \( G \) and \( H \) are metrizable (see Corollary \cite{3.5}).

Generalizing the case of open subsets \( U \) and \( V \) in locally convex spaces treated by Alzaareer and Schmeding \cite{11} and Glöckner and Neeb \cite{6}, we introduce \( C^{k,l} \)-functions \( f : U \times V \to E \) on open subsets \( U \subseteq G \) and \( V \subseteq H \) of topological groups with separate degrees \( k,l \in \mathbb{N}_0 \cup \{\infty\} \) of differentiability in the two variables, and a natural topology on the space \( C^{k,l}(U \times V,E) \) of such maps (see Definition \cite{2.4} for details). **Theorem (A)** is a consequence of the following result:

**Theorem (B).** Let \( U \subseteq G, V \subseteq H \) be open subsets of topological groups \( G \) and \( H \), let \( E \) be a locally convex space and \( k,l \in \mathbb{N}_0 \cup \{\infty\} \). If \( U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j \) is a \( k_R \)-space for all \( i, j \in \mathbb{N}_0 \) with \( i \leq k, j \leq l \), then

\[ \Phi : C^{k,l}(U \times V,E) \to C^{k}(U, C^l(V,E)), \quad f \mapsto f^\vee \]

is an isomorphism of topological vector spaces.

**Notation:** All topological spaces are assumed Hausdorff. We call a map \( f : X \to Y \) between topological spaces \( X \) and \( Y \) a topological embedding if \( f \) is a homeomorphism onto its image (it is known that an injective map \( f \) is a topological embedding if and only if the topology on \( X \) is initial with respect to \( f \), that is, \( X \) carries the coarsest topology making \( f \) continuous).

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## 2 Differentiability of mappings on topological groups

**Definition 2.1.** Let \( G \) be a topological group, a one-parameter subgroup is a group homomorphism \( \gamma : \mathbb{R} \to G \). We denote by \( \mathfrak{L}(G) := \text{Hom}_{cts}(\mathbb{R}, G) \) the
set of all continuous one-parameter subgroups, endowed with the compact-open topology.

**Remark 2.2.** If $\gamma \in \mathcal{L}(G)$ and $\phi : G \to H$ is a continuous group homomorphism, then $\phi \circ \gamma \in \mathcal{L}(H)$ and the map $\mathcal{L}(\phi) : \mathcal{L}(G) \to \mathcal{L}(H), \gamma \mapsto \phi \circ \gamma$ is continuous (cf. Appendix A.5, see also Appendix B).

Further, for $\psi = (\gamma, \eta) \in C(\mathbb{R}, G \times H)$ it is easy to see that $\psi \in \mathcal{L}(G \times H)$ if and only if $\gamma \in \mathcal{L}(G)$ and $\eta \in \mathcal{L}(H)$. Moreover, the natural map $(\mathcal{L}(pr_1), \mathcal{L}(pr_2)) : \mathcal{L}(G \times H) \to \mathcal{L}(G) \times \mathcal{L}(H)$ (where $pr_1 : G \times H \to G$, $pr_2 : G \times H \to H$ are the coordinate projections) is a homeomorphism (cf. Appendix A.5, Appendix B).

Now, we define the notion of differentiability along one-parameter subgroups of vector-valued functions on topological groups:

**Definition 2.3.** Let $U \subseteq G$ be an open subset of a topological group $G$ and $E$ be a locally convex space. For a map $f : U \to E$, $x \in U$ and $\gamma \in \mathcal{L}(G)$ we define

$$d^{(i)} f(x, \gamma) := df(x, \gamma) := D_\gamma f(x) := \lim_{t \to 0} \frac{1}{t}(f(x \cdot \gamma(t)) - f(x))$$

if the limit exists.

We call $f$ a $C^k$-map for $k \in \mathbb{N}$ if $f$ is continuous and for each $x \in U$, $i \in \mathbb{N}$ with $i \leq k$ and $\gamma_1, \ldots, \gamma_i \in \mathcal{L}(G)$ the iterative derivatives

$$d^{(i)} f(x, \gamma_1, \ldots, \gamma_i) := (D_{\gamma_i} \cdots D_{\gamma_1} f)(x)$$

exist and define continuous maps

$$d^{(i)} f : U \times \mathcal{L}(G)^i \to E, \quad (x, \gamma_1, \ldots, \gamma_i) \mapsto (D_{\gamma_i} \cdots D_{\gamma_1} f)(x).$$

If $f$ is $C^k$ for each $k \in \mathbb{N}$, then we call $f$ a $C^\infty$-map or smooth. Further, we call continuous maps $C^0$ and denote $d^{(0)} f := f$.

The set of all $C^k$-maps $f : U \to E$ will be denoted by $C^k(U, E)$ and we endow it with the initial topology with respect to the family $(d^{(i)})_{i \in \mathbb{N}, i \leq k}$ of maps

$$d^{(i)} : C^k(U, E) \to C(U \times \mathcal{L}(G)^i, E)_{c.o.}, \quad f \mapsto d^{(i)} f$$

(where the right-hand side is equipped with the compact-open topology) turning $C^k(U, E)$ into a Hausdorff locally convex space. (This topology is known as the compact-open $C^k$-topology.)

**Definition 2.4.** Let $U \subseteq G$ and $V \subseteq H$ be open subsets of topological groups $G$ and $H$, let $E$ be a locally convex space. For a map $f : U \times V \to E$, $x \in U$, $y \in V$, $\gamma \in \mathcal{L}(G)$ and $\eta \in \mathcal{L}(H)$ we define

$$d^{(1,0)} f(x, y, \gamma) := D_{(\gamma, 0)} f(x, y) := \lim_{t \to 0} \frac{1}{t}(f(x \cdot \gamma(t), y) - f(x, y))$$

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and

\[ d^{(0,1)}f(x, y, \eta) := D_{(0, \eta)}f(x, y) := \lim_{t \to 0} \frac{1}{t} (f(x, y \cdot \eta(t)) - f(x, y)) \]

whenever the limits exist.

We call a continuous map \( f : U \times V \to E \) a \( C^{k,l} \)-map for \( k, l \in \mathbb{N}_0 \cup \{ \infty \} \) if the derivatives

\[ d^{(i,j)}f(x, y, \gamma_1, \ldots, \gamma_i, \eta_1, \ldots, \eta_j) := (D_{(\gamma_1, 0)} \cdots D_{(\gamma_i, 0)}D_{(0, \eta_1)} \cdots D_{(0, \eta_j)}f)(x, y) \]

exist for all \( x \in U, y \in V, i, j \in \mathbb{N}_0 \) with \( i \leq k, j \leq l \) and \( \gamma_1, \ldots, \gamma_i \in \mathcal{L}(G), \eta_1, \ldots, \eta_j \in \mathcal{L}(H) \), and define continuous functions

\[ d^{(i,j)} : C^{k,l}(U \times V, E) \to C(U \times V \times \mathcal{L}(G)^i \times \mathcal{L}(H)^j, E)_{c.o.}, \quad f \mapsto d^{(i,j)}f, \]

where the right-hand side is equipped with the compact-open topology. (The so obtained topology on \( C^{k,l}(U \times V, E) \) is called the compact-open \( C^{k,l} \)-topology.)

**Remark 2.5.** If \( k = 0 \) or \( l = 0 \), then the definition of \( C^{k,l} \)-maps \( f : U \times V \to E \) also makes sense if \( U \) or \( V \), respectively, is any Hausdorff topological space. All further results for \( C^{k,l} \)-maps on topological groups carry over to this situation.

**Remark 2.6.** Simple computations show that for \( k \geq 1 \) a map \( f : U \to E \) is \( C^k \) if and only if \( f \) is \( C^1 \) and \( df : U \times \mathcal{L}(G) \to E \) is \( C^{k-1,0} \), in this case we have \( d^{(i,0)}(df) = d^{(i+1)}f \) for all \( i \in \mathbb{N} \) with \( i \leq k - 1 \).

Similarly, we can show that a map \( f : U \times V \to E \) is \( C^{k,0} \) if and only if \( f \) is \( C^{1,0} \) and \( d^{(1,0)}(f : U \times (V \times \mathcal{L}(G))) \to E \) is \( C^{k-1,0} \) with differentials \( d^{(i,0)}(d^{(1,0)}f) = d^{(i+1,0)}f \) for all \( i \) as above.

Further, if a map \( f : U \times V \to E \) is \( C^{k,l} \), then for each \( i, j \in \mathbb{N}_0 \) with \( i \leq k, j \leq l \) and fixed \( \gamma_1, \ldots, \gamma_i \in \mathcal{L}(G), \eta_1, \ldots, \eta_j \in \mathcal{L}(H) \) the map

\[ D_{(\gamma_1, 0)} \cdots D_{(\gamma_i, 0)}D_{(0, \eta_1)} \cdots D_{(0, \eta_j)}f : U \times V \to E \]

is \( C^{k,l-i-j} \) if \( i = 0 \), and \( C^{k-i,0} \) otherwise.

We warn the reader that the full statement of Schwarz’ Theorem does not carry over to non-abelian topological groups; for a function \( f : G \to \mathbb{R} \) and \( \gamma, \eta \in \mathcal{L}(G) \) such that \( D_\gamma f, D_\eta f, D_\gamma D_\eta f : U \to \mathbb{R} \) are continuous functions it may happen that \( D_\gamma D_\eta f \neq D_\eta D_\gamma f \) (see Example 3.1). Nevertheless, we can prove the following restricted version of Schwarz’ Theorem for \( C^{k,l} \)-maps:
Proposition 2.7. Let $U \subseteq G$ and $V \subseteq H$ be open subsets of topological groups $G$ and $H$, let $E$ be a locally convex space and $f : U \times V \to E$ be a $C^{k,l}$-map for some $k,l \in \mathbb{N} \cup \{\infty\}$. Then the derivatives

$$(D_{(0,\eta_j)} \cdots D_{(0,\eta_1)} D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)} f)(x,y)$$

exist for all $(x,y) \in U \times V$, $i,j \in \mathbb{N}$ with $i \leq k$, $j \leq l$ and $\gamma_1,\ldots,\gamma_i \in \mathcal{L}(G)$, $\eta_1,\ldots,\eta_j \in \mathcal{L}(H)$ and we have

$$(D_{(0,\eta_j)} \cdots D_{(0,\eta_1)} D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)} f)(x,y) = (D_{(\gamma_i,0)} \cdots D_{(\gamma_1,0)} D_{(0,\eta_j)} \cdots D_{(0,\eta_1)} f)(x,y).$$

Proof. First we prove the assertion for $j = 1$ by induction on $i$.

**Induction start:** Let $(x,y) \in U \times V$, $\gamma \in \mathcal{L}(G)$ and $\eta \in \mathcal{L}(H)$. For suitable $\varepsilon,\delta > 0$ we define the continuous map

$$h : [\varepsilon,\varepsilon] \times [-\delta,\delta] \to E, \quad (s,t) \mapsto f(x \cdot \gamma(s), y \cdot \eta(t)),$$

and obtain the partial derivatives of $h$ via

$$\frac{\partial h}{\partial s}(s,t) = \lim_{r \to 0} \frac{1}{r}(h(s + r, t) - h(s, t))$$

$$= \lim_{r \to 0} \frac{1}{r}(f(x \cdot \gamma(s) \cdot \gamma(r), y \cdot \eta(t)) - f(x \cdot \gamma(s), y \cdot \eta(t)))$$

$$= D_{(\gamma,0)} f(x \cdot \gamma(s), y \cdot \eta(t)),$$

and analogously,

$$\frac{\partial h}{\partial t}(s,t) = D_{(0,\eta)} f(x \cdot \gamma(s), y \cdot \eta(t))$$

and

$$\frac{\partial^2 h}{\partial s \partial t}(s,t) = (D_{(\gamma,0)} D_{(0,\eta)} f)(x \cdot \gamma(s), y \cdot \eta(t)).$$

The obtained maps $\frac{\partial h}{\partial s}$, $\frac{\partial h}{\partial t}$ and $\frac{\partial^2 h}{\partial s \partial t}$ are continuous, hence we apply [6, Lemma 1.3.18], which states that in this case also the partial derivative $\frac{\partial^2 h}{\partial s \partial t}$ exists and coincides with $\frac{\partial^2 h}{\partial s \partial t}$. Therefore, we have

$$(D_{(\gamma,0)} D_{(0,\eta)} f)(x,y) = \frac{\partial^2 h}{\partial s \partial t}(0,0) = \frac{\partial^2 h}{\partial s \partial t}(0,0) = \lim_{r \to 0} \frac{1}{r} \left( \frac{\partial h}{\partial s}(0, r) - \frac{\partial h}{\partial s}(0, 0) \right)$$

$$= \lim_{r \to 0} \frac{1}{r} \left( D_{(\gamma,0)} f(x, y \cdot \eta(r)) - D_{(\gamma,0)} f(x, y) \right)$$

$$= (D_{(0,\eta)} D_{(\gamma,0)} f)(x,y).$$
Thus the assertion holds for $i = 1$.

**Induction step:** Now, let $2 \leq i \leq k$, $(x, y) \in U \times V$, $\gamma_1, \ldots, \gamma_i \in \mathcal{L}(G)$ and $\eta \in \mathcal{L}(H)$. Consider the map

$$g_1 : U \times V \to E, \quad (x, y) \mapsto (D_{(\gamma_1,0)} \cdots D_{(\gamma_i,0)} f)(x, y),$$

which is $C^{1,0}$ (see Remark 2.6). Further, $g_1$ is $C^{0,1}$, because

$$D_{(0,\eta)} g_1(x, y) = (D_{(0,\eta)} D_{(\gamma_1,0)} \cdots D_{(\gamma_i,0)} f)(x, y)$$

$$= (D_{(\gamma_1,0)} \cdots D_{(\gamma_i,0)} D_{(0,\eta)} f)(x, y),$$

by the induction hypothesis, and we see that

$$g_j(y) = (D_{(\gamma_1,0)} \cdots D_{(\gamma_i,0)} g_j)(x, y),$$

whence $g_1$ is $C^{1,1}$. By the induction start, the derivative $(D_{(0,\eta)} D_{(\gamma_1,0)} g_1)(x, y)$ exists and equals $(D_{(\gamma_1,0)} D_{(0,\eta)} g_1)(x, y)$, thus we get

$$D_{(\gamma_1,0)} \cdots D_{(\gamma_i,0)} D_{(0,\eta)} f(x, y) = (D_{(\gamma_1,0)} \cdots D_{(\gamma_i,0)} g_1)(x, y)$$

$$= (D_{(0,\eta)} D_{(\gamma_1,0)} g_1)(x, y)$$

$$= (D_{(\gamma_1,0)} \cdots D_{(\gamma_i,0)} D_{(0,\eta)} f)(x, y).$$

Hence the assertion holds for $j = 1$.

Now, let $2 \leq j \leq l$, $1 \leq i \leq k$, $\gamma_1, \ldots, \gamma_i \in \mathcal{L}(G)$, $\eta_1, \ldots, \eta_j \in \mathcal{L}(H)$ and $(x, y) \in U \times V$. By Remark 2.6 the map

$$g_2 : U \times V \to E, \quad (x, y) \mapsto (D_{(0,\eta_{j-1})} \cdots D_{(0,\eta_1)} f)(x, y)$$

is $C^{k,1}$, whence we have

$$D_{(0,\eta_j)} D_{(\gamma_1,0)} \cdots D_{(\gamma_i,0)} D_{(0,\eta_{j-1})} \cdots D_{(0,\eta_1)} f(x, y)$$

$$= (D_{(0,\eta_j)} \cdots D_{(\gamma_i,0)} g_2)(x, y)$$

$$= (D_{(\gamma_1,0)} \cdots D_{(\gamma_i,0)} D_{(0,\eta_j)} g_2)(x, y)$$

$$= (D_{(\gamma_1,0)} \cdots D_{(\gamma_i,0)} D_{(0,\eta_j)} \cdots D_{(0,\eta_1)} f)(x, y),$$

(1)

using the first part of the proof. But we also have

$$D_{(0,\eta_j)} D_{(\gamma_1,0)} \cdots D_{(\gamma_i,0)} D_{(0,\eta_{j-1})} \cdots D_{(0,\eta_1)} f(x, y)$$

$$= (D_{(0,\eta_j)} \cdots D_{(\gamma_i,0)} D_{(0,\eta_{j-1})} \cdots D_{(0,\eta_1)} f)(x, y),$$

(2)

by induction, whence (2) equals (1), that is

$$D_{(0,\eta_j)} \cdots D_{(\gamma_i,0)} D_{(\gamma_1,0)} \cdots D_{(\gamma_1,0)} f(x, y)$$

$$= (D_{(\gamma_1,0)} \cdots D_{(\gamma_1,0)} D_{(0,\eta_j)} \cdots D_{(0,\eta_1)} f)(x, y),$$

and the proof is finished.
Corollary 2.8. Let $U \subseteq G$ and $V \subseteq H$ be open subsets of topological groups $G$ and $H$, let $E$ be a locally convex space and $k, l \in \mathbb{N}_0 \cup \{\infty\}$. A map $f : U \times V \to E$ is $C^{k,l}$ if and only if the map

$$g : V \times U \to E, \quad (y, x) \mapsto f(x, y)$$

is $C^{l,k}$. Moreover, we have

$$d^{(i,j)}g(y,x,\eta_1,\ldots,\eta_j,\gamma_1,\ldots,\gamma_i) = d^{(i,j)}f(x,y,\gamma_1,\ldots,\gamma_i,\eta_1,\ldots,\eta_j)$$

for each $x \in U$, $y \in V$, $i, j \in \mathbb{N}_0$ with $i \leq k$, $j \leq l$ and $\gamma_1,\ldots,\gamma_i \in \Sigma(G)$, $\eta_1,\ldots,\eta_j \in \Sigma(H)$.

Proof. First, we assume that $l = 0$, that is, $f : U \times V \to E$ is $C^{k,0}$. Then for $x \in U$, $y \in V$ and $\gamma \in \Sigma(G)$ we have

$$d^{(1,0)}f(x,y,\gamma) = \lim_{t \to 0} \frac{1}{t} (f(x \cdot \gamma(t), y) - f(x, y))$$

and similarly we get $d^{(0,i)}g(y,x,\gamma_1,\ldots,\gamma_i) = d^{(i,0)}f(x,y,\gamma_1,\ldots,\gamma_i)$ for each $i \in \mathbb{N}$ with $i \leq k$ and $\gamma_1,\ldots,\gamma_i \in \Sigma(G)$. The obtained maps $d^{(0,i)}g : V \times U \times \Sigma(G)^i \to E$ are obviously continuous, hence $g$ is $C^{0,k}$. The other implication, as well as the case $k = 0$, can be proven analogously.

If $k, l \geq 1$, then the assertion follows immediately from Proposition 2.6. \qed

Remark 2.9. Using Remark 2.6 and Corollary 2.8, we can easily show that if $f : U \times V \to E$ is $C^{k,l}$, then for all $i, j \in \mathbb{N}_0$ with $i \leq k$, $j \leq l$ and fixed $\gamma_1,\ldots,\gamma_i \in \Sigma(G)$, $\eta_1,\ldots,\eta_j \in \Sigma(H)$ the maps

$$D_{(\gamma_1,0)} \cdots D_{(\gamma_1,0)} D_{(0,\eta_j)} \cdots D_{(0,\eta_j)} f : U \times V \to E$$

are $C^{k-i,l-j}$.

The following lemma will be useful:

Lemma 2.10. Let $U \subseteq G$, $V \subseteq H$ be open subsets of topological groups $G$ and $H$, let $E$, $F$ be locally convex spaces, $\lambda : E \to F$ be a continuous and linear map and $k, l \in \mathbb{N}_0 \cup \{\infty\}$.

(a) If $f : U \to E$ is a $C^k$-map, then the map $\lambda \circ f : U \to F$ is $C^k$.

(b) If $f : U \times V \to E$ is a $C^{k,l}$-map, then the map $\lambda \circ f : U \times V \to F$ is $C^{k,l}$.

Proof. To prove (a), let $x \in U$, $\gamma \in \Sigma(G)$ and $t \neq 0$ small enough, then we have

$$\frac{\lambda(f(x \cdot \gamma(t))) - \lambda(f(x))}{t} = \lambda \left( \frac{f(x \cdot \gamma(t)) - f(x)}{t} \right) \to \lambda(df(x, \gamma)), \quad t \to 0, \quad (x, \gamma) \in U \times \Sigma(G).$$
as $t \to 0$, because $\lambda$ is assumed linear and continuous. Therefore, the derivative $d(\lambda \circ f)(x, \gamma)$ exists and we have $d(\lambda \circ f)(x, \gamma) = (\lambda \circ df)(x, \gamma)$.

Proof. First, assume that the map $x \in U$ as such that $d \circ f(x, \gamma, t, y) ∈ t$ as $t → 0$. Hence $d^0(\lambda \circ f)(x, \gamma) = (\lambda \circ d^0 f)(x, \gamma)$, since each of the obtained maps $d^0(\lambda \circ f) = \lambda \circ d^0 f : U × E \to F$ is continuous, we see that the map $\lambda \circ f$ is $C^k$.

Analogously, assertion (b) can be proved showing that for each $i, j \in N_0$ with $i ≤ k$, $j ≤ l$ we have $d^{(i, j)}(\lambda \circ f) = \lambda \circ d^{(i, j)} f$.

Let us introduce the following notation (the analogue for $C^1$-maps is Lemma A.3):

**Lemma 2.11.** Let $U ⊆ G$, $V ⊆ H$ be open subsets of topological groups $G$ and $H$, let $E$ be a locally convex space. A continuous map $f : U × V → E$ is $C^{1, 0}$ if and only if there exists a continuous map

$$f^{[1, 0]} : U^{[1]} × V → E,$$

where

$$U^{[1]} := \{(x, γ, t) ∈ U × E : x : γ(t) ∈ U\},$$

such that

$$f^{[1, 0]}(x, γ, t, y) = \frac{1}{t}(f(x, γ(t), y) - f(x, y))$$

for each $(x, γ, t, y) ∈ U^{[1]} × V$ with $t ≠ 0$.

In this case we have $d^{[1, 0]} f(x, y, γ) = f^{[1, 0]}(x, γ, 0, y)$ for all $x ∈ U$, $y ∈ V$ and $γ ∈ E$.

**Proof.** First, assume that the map $f^{[1, 0]}$ exists and is continuous. Then for $x ∈ U$, $y ∈ V$, $γ ∈ E$ and $t ≠ 0$ small enough we have

$$\frac{1}{t}(f(x, γ(t), y) - f(x, y)) = f^{[1, 0]}(x, γ, t, y) → f^{[1, 0]}(x, γ, 0, y)$$

as $t → 0$. Hence $d^{[1, 0]} f(x, y, γ)$ exists and is given by $f^{[1, 0]}(x, γ, 0, y)$, whence the map

$$d^{[1, 0]} f : U × V × E → E, \quad (x, y, γ) → f^{[1, 0]}(x, γ, 0, y)$$

is continuous. Thus $f$ is $C^{1, 0}$.

Conversely, let $f$ be a $C^{1, 0}$-map. Then we define

$$f^{[1, 0]} : U^{[1]} × V → E, \quad f^{[1, 0]}(x, γ, t, y) := \begin{cases} \frac{f(x, γ(t), y) - f(x, y)}{t} & \text{if } t ≠ 0 \\ d^{[1, 0]} f(x, y, γ) & \text{if } t = 0. \end{cases}$$

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Since $f$ is continuous, the map $f^{[1,0]}$ is continuous at each $(x, \gamma, t, y)$ with $t \neq 0$. Given $x_0 \in U$ and $\gamma_0 \in \mathcal{L}(G)$, we have $(x_0, \gamma_0, 0) \in U^{[1]}$: let $W := U_{x_0} \times U_{\gamma_0} \times \epsilon > 0$. Then for $t \in [1, 0]$ be an open neighborhood of $(x_0, \gamma_0, 0)$ in $U^{[1]}$, where $U_{x_0} \subseteq U$ and $U_{\gamma_0} \subseteq \mathcal{L}(G)$ are open neighborhoods of $x_0$ and $\gamma_0$, respectively, and $\epsilon > 0$. Now, for fixed $(x, \gamma, y) \in U_{x_0} \times U_{\gamma_0} \times V$ define the continuous curve

$$h : [-\epsilon, \epsilon] \to E, \quad h(t) := f(x \cdot \gamma(t), y).$$

Then for $t \in [-\epsilon, \epsilon], s \neq 0$ with $t + s \in [-\epsilon, \epsilon]$ we have

$$\frac{h(t + s) - h(t)}{s} = \frac{f(x \cdot \gamma(t + s), y) - f(x \cdot \gamma(t), y)}{s} = \frac{f(x \cdot \gamma(t) \cdot \gamma(s), y) - f(x \cdot \gamma(t), y)}{s} \to d^{(1,0)} f(x \cdot \gamma(t), y, \gamma)$$

as $s \to 0$. Thus, the derivative $h'(t)$ exists and is given by $d^{(1,0)} f(x \cdot \gamma(t), y, \gamma)$. The so obtained map $h' : [-\epsilon, \epsilon] \to E$ is continuous, hence $h$ is a $C^1$-curve (see [6] for details on $C^1$-curves with values in locally convex spaces and also on weak integrals which we use in the next step). We use the Fundamental Theorem of Calculus ([6 Proposition 1.1.5]) and obtain for $t \neq 0$

$$f^{[1,0]}(x, \gamma, t, y) = \frac{1}{t} \left( f(x \cdot \gamma(t), y) - f(x, y) \right) = \frac{1}{t} \left( h(t) - h(0) \right)$$

$$= \frac{1}{t} \int_0^t h'(\tau) d\tau = \frac{1}{t} \int_0^t d^{(1,0)} f(x \cdot \gamma(\tau), y, \gamma) d\tau$$

$$= \frac{1}{t} \int_0^1 t d^{(1,0)} f(x \cdot \gamma(tu), y, \gamma) du = \int_0^1 d^{(1,0)} f(x \cdot \gamma(tu), y, \gamma) du.$$ 

But if $t = 0$, then

$$\int_0^1 d^{(1,0)} f(x \cdot \gamma(0), y, \gamma) du = d^{(1,0)} f(x, y, \gamma) = f^{[1,0]}(x, \gamma, 0, y),$$

hence

$$f^{[1,0]}(x, \gamma, t, y) = \int_0^1 d^{(1,0)} f(x \cdot \gamma(tu), y, \gamma) du$$

for all $(x, \gamma, t, y) \in W \times V$. Since the map

$$W \times V \times [0, 1] \to E, \quad (x, \gamma, t, y, u) \mapsto d^{(1,0)} f(x \cdot \gamma(tu), y, \gamma)$$

is continuous, also the parameter-dependent integral

$$W \times V \to E, \quad (x, \gamma, t, y) \mapsto f^{[1,0]}(x, \gamma, t, y) = \int_0^1 d^{(1,0)} f(x \cdot \gamma(tu), y, \gamma) du$$

is continuous (by [6 Lemma 1.1.11]), in particular in $(x_0, \gamma_0, 0, y)$. Consequently, $f^{[1,0]}$ is continuous. \[\square\]
The following two propositions provide a relation between $C^k$- and $C^{k,1}$-maps on products of topological groups (a version can also be found in [3]), in particular, we will conclude that $C^{\infty,\infty}(U \times V, E) \cong C^{\infty}(U \times V, E)$ as topological vector spaces (Corollary 2.14).

**Proposition 2.12.** Let $U \subseteq G, V \subseteq H$ be open subsets of topological groups $G$ and $H$, let $E$ be a locally convex space and $k \in \mathbb{N}_0 \cup \{\infty\}$. If $f : U \times V \to E$ is $C^{k,k}$, then $f$ is $C^k$.

Moreover, the inclusion map

$$\Psi : C^{k,k}(U \times V, E) \to C^k(U \times V, E), \quad f \mapsto f$$

is continuous and linear.

**Proof.** The case $k = 0$ is trivial. For $k \geq 1$, we show by induction on $i \in \mathbb{N}$ with $i \leq k$ that for all $(x, y) \in U \times V$, $(\gamma_1, \eta_1), \ldots, (\gamma_i, \eta_i) \in \mathcal{L}(G \times H)$ the derivatives of $f$ are given by

$$d^i f((x, y), (\gamma_1, \eta_1), \ldots, (\gamma_i, \eta_i)) = \sum_{j=0}^i \sum_{I_j: I_j \supseteq \{1, \ldots, i\}} \left( \sum \prod \right)$$

where $I_j := \{r_1, \ldots, r_j\} \cup \{s_1, \ldots, s_{i-j}\} = \{1, \ldots, i\}$.

**Induction start:** Let $(x, y) \in U \times V$ and $(\gamma, \eta) \in \mathcal{L}(G \times H)$, that is, $\gamma \in \mathcal{L}(G)$ and $\eta \in \mathcal{L}(H)$, see Remark 2.2. For $t \neq 0$ small enough we have

$$
\begin{align*}
\frac{f((x, y) \cdot (\gamma(t), \eta(t)) - f(x, y)}{t} &= \frac{f(x \cdot \gamma(t), y \cdot \eta(t)) - f(x, y)}{t} \\
&= \frac{f(x \cdot \gamma(t), y \cdot \eta(t)) - f(x \cdot \gamma(t), y)}{t} + \frac{f(x \cdot \gamma(t), y) - f(x, y)}{t} \\
&= \frac{g(y \cdot \eta(t), x \cdot \gamma(t)) - g(y, x \cdot \gamma(t))}{t} + \frac{f(x \cdot \gamma(t), y) - f(x, y)}{t},
\end{align*}
$$

where $g$ is the map $g : V \times U \to E, (y, x) \mapsto f(x, y)$. By Corollary 2.8 the map $g$ is $C^{1,1}$, whence the map $g^{[1,0]}$ is defined and continuous, as well as $f^{[1,0]}$ (see Lemma 2.7). Thus we have

$$
\begin{align*}
g(y \cdot \eta(t), x \cdot \gamma(t)) - g(y, x \cdot \gamma(t)) &= \frac{f(x \cdot \gamma(t), y) - f(x, y)}{t} \\
&= g^{[1,0]}(y, \eta, t, x \cdot \gamma(t)) + f^{[1,0]}(x, \gamma, t, y) \\
&\to g^{[1,0]}(y, \eta, 0, x) + f^{[1,0]}(x, \gamma, 0, y)
\end{align*}
$$

as $t \to 0$. Therefore, the derivative $df((x, y), (\gamma, \eta))$ exists and is given by

$$
\begin{align*}
df((x, y), (\gamma, \eta)) &= g^{[1,0]}(y, \eta, 0, x) + f^{[1,0]}(x, \gamma, 0, y) \\
&= d^{(1,0)} g(y, x, \eta) + d^{(1,0)} f(x, y, \gamma) \\
&= d^{(0,1)} f(x, y, \eta) + d^{(1,0)} f(x, y, \gamma).
\end{align*}
$$
Induction step: Now, let \(2 \leq i \leq k\), \((x, y) \in U \times V\), \((\gamma_1, \eta_1), \ldots, (\gamma_i, \eta_i) \in \mathcal{L}(G \times H)\). Then for \(t \neq 0\) small enough we have

\[
\frac{1}{t} \left( d^{(i-1)} f((x \cdot \gamma_i(t), y \cdot \eta_i(t)), (\gamma_1, \eta_1), \ldots, (\gamma_{i-1}, \eta_{i-1})) - d^{(i-1)} f((x, y), (\gamma_1, \eta_1), \ldots, (\gamma_{i-1}, \eta_{i-1})) \right)
\]

\[
= \sum_{j=0}^{i-1} \sum_{I_{j,i-1}} \frac{1}{t} \left( d^{(j,i-j-1)} f(x, \gamma_j(t), \gamma_{j+1}(t), \ldots, \gamma_{i-1}, \eta_j, \ldots, \eta_{i-1}) \right)
\]

\[
= \sum_{j=0}^{i-1} \sum_{I_{j,i-1}} \frac{1}{t} \left( D_{(\gamma_j,0)} \cdots D_{(\gamma_{j-i},0)} D_{(0,\eta_{i-j-1})} \cdots D_{(0,\eta_1)} f(x, \gamma_j(t), \gamma_{j+1}(t), \ldots, \gamma_{i-1}, \eta_j, \ldots, \eta_{i-1}) \right)
\]

Each of the maps

\[
D_{(\gamma_j,0)} \cdots D_{(\gamma_{j-i},0)} D_{(0,\eta_{i-j-1})} \cdots D_{(0,\eta_1)} f : U \times V \to E
\]

is \(C^1,1\) (see Remark 2.9), hence \(C^1\) and we have

\[
\sum_{j=0}^{i-1} \sum_{I_{j,i-1}} \frac{1}{t} \left( D_{(\gamma_j,0)} \cdots D_{(\gamma_{j-i},0)} D_{(0,\eta_{i-j-1})} \cdots D_{(0,\eta_1)} f(x, \gamma_j(t), \gamma_{j+1}(t), \ldots, \gamma_{i-1}, \eta_j, \ldots, \eta_{i-1}) \right)
\]

\[
= \sum_{j=0}^{i-1} \sum_{I_{j,i-1}} d^{(j,i-j)} f(x, y, \gamma_j, \ldots, \gamma_{j-i}, \eta_j, \ldots, \eta_{i-1})
\]

as \(t \to 0\) (using Proposition 2.7). Thus (3) holds, and we have

\[
d^{(i)} f = \sum_{j=0}^{i} \sum_{I_{j,i}} d^{(j,i-j)} f \circ g_{t,j,i},
\]

where

\[
g_{t,j,i} : U \times V \times \mathcal{L}(G \times H)^j \to U \times V \times \mathcal{L}(G)^j \times \mathcal{L}(H)^{i-j},
\]

\((x, y, (\gamma_1, \eta_1), \ldots, (\gamma_i, \eta_i)) \mapsto (x, y, \gamma_j, \ldots, \gamma_{j-i}, \eta_j, \ldots, \eta_{i-1})\)
are continuous maps (see Remark 2.2). Hence $f$ is $C^k$.
The linearity of the map $\Psi$ is clear. Further, each of the maps

$$g^*_{I_{j,i}} : C(U \times V \times \mathcal{L}(G)^j \times \mathcal{L}(H)^{i-j}, E)_{c.o} \to C(U \times V \times \mathcal{L}(G \times H)^{i-j}, E)_{c.o},$$

$h \mapsto h \circ g^*_{I_{j,i}}$ is continuous (see [6, Appendix A.5] or [4, Lemma B.9]), whence each of the maps

$$d^{(i)} \circ \Psi = \sum_{j=0}^{i} \sum_{I_{j,i}} g^*_{I_{j,i}} \circ d^{(i,i-j)}$$

is continuous. Since the topology on $C^k(U \times V, E)$ is initial with respect to the maps $d^{(i)}$, the continuity of $\Psi$ follows.

**Proposition 2.13.** Let $U \subseteq G$, $V \subseteq H$ be open subsets of topological groups $G$ and $H$, let $E$ be a locally convex space and $k, l \in \mathbb{N}_0$. If $f : U \times V \to E$ is a $C^{k+l}$-map, then $f$ is $C^{k,l}$.

Moreover, the inclusion map

$$\Psi : C^{k+l}(U \times V, E) \to C^{k,l}(U \times V, E), \quad f \mapsto f$$

is continuous and linear.

**Proof.** We denote by $\varepsilon_G \in \mathcal{L}(G)$ the constant map $\varepsilon_G : \mathbb{R} \to G, t \mapsto e_G$, where $e_G$ is the identity element of $G$, and $\varepsilon_H \in \mathcal{L}(H)$ is defined analogously.

Let $x \in U$, $y \in V$, $\gamma_1, \ldots, \gamma_i \in \mathcal{L}(G)$ and $\eta_1, \ldots, \eta_j \in \mathcal{L}(H)$ for some $i, j \in \mathbb{N}_0$ with $i \leq k$, $j \leq l$. Then we obviously have

$$d^{(i,j)}f(x, y, \gamma_1, \ldots, \gamma_i, \eta_1, \ldots, \eta_j) = d^{(i+j)}f((x, y, (\gamma_1, \varepsilon_H), \ldots, (\gamma_i, \varepsilon_H), (\varepsilon_G, \eta_1), \ldots, (\varepsilon_G, \eta_j))).$$

Each of the maps

$$\rho_{i,j} : U \times V \times \mathcal{L}(G)^i \times \mathcal{L}(H)^{j} \to U \times V \times \mathcal{L}(G \times H)^{i+j}$$

$$(x, y, \gamma_1, \ldots, \gamma_i, \eta_1, \ldots, \eta_j) \mapsto (x, y, (\gamma_1, \varepsilon_H), \ldots, (\gamma_i, \varepsilon_H), (\varepsilon_G, \eta_1), \ldots, (\varepsilon_G, \eta_j))$$

is continuous (see Remark 2.2) and we have

$$d^{(i,j)}f = d^{(i+j)}f \circ \rho_{i,j}.$$  

Therefore, $f$ is $C^{k,l}$.

The linearity of the map $\Psi$ is clear. Further, by [6, Appendix A.5] (see also [4, Lemma B.9]), each of the maps
\[ \rho_{i,j}^* : C(U \times V \times \mathcal{L}(G \times H)^{i+j}, E)_{c.o} \to C(U \times V \times \mathcal{L}(G)^i \times \mathcal{L}(H)^j, E)_{c.o} \]

\[ h \mapsto h \circ \rho_{i,j} \]

is continuous, whence each of the maps

\[ d^{(i,j)} \circ \Psi = \rho_{i,j}^* \circ d^{(i+j)} \]

is continuous. Hence, the continuity of \( \Psi \) follows, since the topology on the space \( C^{k,l}(U \times V, E) \) is initial with respect to the maps \( d^{(i,j)}. \)

**Corollary 2.14.** Let \( U \subseteq G, V \subseteq H \) be open subsets of topological groups \( G \) and \( H \), let \( E \) be a locally convex space. A map \( f : U \times V \to E \) is \( C^\infty \) if and only if \( f \) is \( C^{\infty, \infty} \). Moreover, the map

\[ \Psi : C^\infty(U \times V, E) \to C^{\infty, \infty}(U \times V, E), \quad f \mapsto f \]

is an isomorphism of topological vector spaces.

**Proof.** The assertion is an immediate consequence of Propositions 2.12 and 2.13.

3 The exponential law

We recall the classical Exponential Law for spaces of continuous functions, which can be found, for example, in [6, Appendix A.5]:

**Proposition 3.1.** Let \( X_1, X_2, Y \) be topological spaces. If \( f : X_1 \times X_2 \to Y \) is a continuous map, then also the map

\[ f^\vee : X_1 \to C(X_2, Y)_{c.o}, \quad x \mapsto f^\vee(x) := f(x, \bullet) \]

is continuous. Moreover, the map

\[ \Phi : C(X_1 \times X_2, Y)_{c.o} \to C(X_1, C(X_2, Y))_{c.o}, \quad f \mapsto f^\vee \]

is a topological embedding. If \( X_2 \) is locally compact or \( X_1 \times X_2 \) is a \( k \)-space, or \( X_1 \times X_2 \) is a \( kR \)-space and \( Y \) is completely regular, then \( \Phi \) is a homeomorphism.

The following terminology is used here:

**Remark 3.2.** (a) A Hausdorff topological space \( X \) is called a \( k \)-space if functions \( f : X \to Y \) to a topological space \( Y \) are continuous if and only if the restrictions \( f|_K : K \to Y \) are continuous for all compact subsets \( K \subseteq X \). All locally compact spaces and all metrizable spaces are \( k \)-spaces.

(b) A Hausdorff topological space \( X \) is called a \( kR \)-space if real-valued functions \( f : X \to \mathbb{R} \) are continuous if and only if the restrictions \( f|_K : K \to \mathbb{R} \) are
continuous for all compact subsets $K \subseteq X$. Each $k$-space is a $kR$-space, hence also each locally compact and each metrizable space is a $kR$-space.

(c) A Hausdorff topological space $X$ is called completely regular if its topology is initial with respect to the set $C(X, \mathbb{R})$. Each Hausdorff locally convex space (moreover, each Hausdorff topological group) is completely regular, see [7].

**Theorem 3.3.** Let $U \subseteq G$, $V \subseteq H$ be open subsets of topological groups $G$ and $H$, let $E$ be a locally convex space and $k, l \in \mathbb{N}_0 \cup \{\infty\}$. Then the following holds:

(a) If a map $f : U \times V \to E$ is $C^{k,l}$, then the map

$$f^\vee(y) := f(x, y) : V \to E, \quad y \mapsto f^\vee(x)(y) := f(x, y)$$

is $C^l$ for each $x \in U$ and the map

$$f^\vee : U \to C^l(V, E), \quad x \mapsto f^\vee(x)$$

is $C^k$.

(b) The map

$$\Phi : C^{k,l}(U \times V, E) \to C^k(U, C^l(V, E)), \quad f \mapsto f^\vee$$

is linear and a topological embedding.

**Proof.** (a) We will consider the following cases:

The case $k = l = 0$ : This case is covered by the classical Exponential Law [3,4]

The case $k = 0$, $l \geq 1$ : Let $x \in U$; the map $f^\vee(x) = f(x, \bullet)$ is obviously continuous, and for $y \in V$, $\eta \in \mathfrak{L}(H)$ and $t \neq 0$ small enough we have

$$\frac{1}{t}(f^\vee(y \cdot \eta(t)) - f^\vee(y)) = \frac{1}{t}(f(x, y \cdot \eta(t)) - f(x, y)) \to D_{(0,\eta)}f(x, y)$$

as $t \to 0$. Thus the derivative $D_{\eta}(f^\vee(x))(y)$ exists and equals $D_{(0,\eta)}f(x, y) = (D_{(0,\eta)}f^\vee)(x)(y)$. Proceeding similarly, for each $j \in \mathbb{N}$ with $j \leq l$ and $\eta_1, \ldots, \eta_j \in \mathfrak{L}(H)$, we obtain the derivatives

$$(D_{\eta_j} \cdots D_{\eta_1}(f^\vee(x))) (y) = (D_{(0,\eta_j)} \cdots D_{(0,\eta_1)}f^\vee(x))(y)$$

(4)

The obtained differentials $d^{(j)}(f^\vee(x)) = (d^{(0,j)}f^\vee)(x) : V \times \mathfrak{L}(H)^j \to E$ are continuous, therefore $f^\vee(x)$ is $C^l$. Further, by the classical Exponential Law [3,4] each of the maps

$$f^\vee : U \to C(V, E)_{c.o.}, \quad x \mapsto f^\vee(x),$$

$$(d^{(0,j)}f^\vee) : U \to C(V \times \mathfrak{L}(H)^j, E)_{c.o.}, \quad x \mapsto (d^{(0,j)}f^\vee)(x)$$

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is continuous, and we have $d^{(j)} \circ f^\vee = (d^{(0,j)} f)^\vee$ for all $j \in \mathbb{N}_0$ with $j \leq l$. Thus, the continuity of $f^\vee$ follows from the fact that the topology on $C^l(V,E)$ is initial with respect to the maps $d^{(j)}$.

The case $k \geq 1$, $l \geq 0$ : By the preceding steps, the map $f^\vee(x)$ is $C^l$ for each $x \in U$ (with derivatives given in (3)). Now we show by induction on $i \in \mathbb{N}$ with $i \leq k$ that

$$(D_{\gamma_i} \cdots D_{\gamma_1} (f^\vee))(x) = (D_{\gamma_1,0} \cdots D_{\gamma_i,0} f)^\vee (x)$$

for all $x \in U$ and $\gamma_1, \ldots, \gamma_i \in \mathcal{L}(G)$.

Induction start: Since $f$ is $C^{1,0}$, by Lemma 2.11 the map $f^{[1,0]} : U^{[1]} \times V \to E$ is continuous, hence so is the map $(f^{[1,0]})^\vee : U^{[1]} \to C(V,E)_{c.o}$ (see Proposition 3.11). Let $(x, \gamma, t) \in U^{[1]}$ such that $t \neq 0$ and let $y \in V$, then we have

$$\frac{1}{t} (f^\vee(x \cdot \gamma(t))(y) - f^\vee(x)(y)) = \frac{1}{t} (f(x \cdot \gamma(t), y) - f(x, y)) = f^{[1,0]}(x, \gamma, t, y) = (f^{[1,0]})^\vee(x, \gamma, t)(y).$$

Therefore

$$\frac{1}{t} (f^\vee(x \cdot \gamma(t)) - f^\vee(x)) = (f^{[1,0]})^\vee(x, \gamma, t)$$

$$\to (f^{[1,0]})^\vee(x, \gamma, 0) = (D_{\gamma,0} f)^\vee(x)$$

as $t \to 0$. Thus, $D_{\gamma}(f^\vee)(x)$ exists and is given by $(D_{\gamma,0} f)^\vee(x)$.

Induction step: Now, let $2 \leq i \leq k$, $x \in U$ and $\gamma_1, \ldots, \gamma_i \in \mathcal{L}(G)$. For $t \neq 0$ small enough we have

$$\frac{1}{t} ((D_{\gamma_{i-1}} \cdots D_{\gamma_1} (f^\vee))(x \cdot \gamma_i(t)) - (D_{\gamma_{i-1}} \cdots D_{\gamma_i} (f^\vee))(x))$$

$$= \frac{1}{t} ((D_{\gamma_1,0} \cdots D_{\gamma_i,0} f)^\vee(x \cdot \gamma_i(t)) - (D_{\gamma_1,0} \cdots D_{\gamma_i,0} f)^\vee(x))$$

by the induction hypothesis. But the map $D_{\gamma_{i-1},0} \cdots D_{\gamma_1,0} f : U \times V \to E$ is $C^{1,0}$ (see Remark 2.13), hence by the induction start we have

$$\frac{1}{t} ((D_{\gamma_{i-1},0} \cdots D_{\gamma_1,0} f)^\vee(x \cdot \gamma_i(t)) - (D_{\gamma_{i-1},0} \cdots D_{\gamma_1,0} f)^\vee(x))$$

$$\to D_{\gamma_i}((D_{\gamma_{i-1},0} \cdots D_{\gamma_1,0} f)^\vee(x)) = (D_{\gamma_i,0} \cdots D_{\gamma_1,0} f)^\vee(x),$$

which shows that the derivative $(D_{\gamma_1,0} \cdots D_{\gamma_i,0} f)^\vee(x)$ exists and is given by $(D_{\gamma,0} \cdots D_{\gamma_i,0} f)^\vee(x)$, thus (5) holds.

From Remark 2.13 we know that each of the maps

$$D_{\gamma_1,0} \cdots D_{\gamma_i,0} f : U \times V \to E$$

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is $C^{0,l}$, hence $(D_{(\gamma_0, 0)} \cdots D_{(\gamma_l, 0)} f)^\gamma(x) \in C^l(V, E)$ for each $x \in U$. Now, it remains to show that each of the maps

\[d^{(i)}(f^\gamma) : U \times \mathcal{L}(G)^l \to C^l(V, E),\]

\[(x, \gamma_1, \ldots, \gamma_l) \mapsto (D_{\gamma_1} \cdots D_{\gamma_l}(f^\gamma))(x) = (D_{(\gamma_0, 0)} \cdots D_{(\gamma_l, 0)} f)^\gamma(x)\]

is continuous. To this end, let $y \in V$, $j \in \mathbb{N}_0$ with $j \leq l$ and $\eta_1, \ldots, \eta_j \in \mathcal{L}(H)$. Then we have

\[(d^{(i)} \circ d^{(i)}(f^\gamma))(x, \gamma_1, \ldots, \gamma_l)(y, \eta_1, \ldots, \eta_j)\]

\[= d^{(j)}(d^{(i)}(f^\gamma))(x, \gamma_1, \ldots, \gamma_l)(y, \eta_1, \ldots, \eta_j)\]

\[= [D_{\eta_j} \cdots D_{\eta_1}[(D_{\gamma_l} \cdots D_{\gamma_1}(f^\gamma))(x)](y)\]

Using $(5)$ and $(4)$ in turn we obtain

\[[D_{\eta_j} \cdots D_{\eta_1}[(D_{\gamma_l} \cdots D_{\gamma_1}(f^\gamma))(x)](y)\]

\[= (D_{(0, \eta_j)} \cdots D_{(0, \eta_1)} D_{(\gamma_l, 0)} \cdots D_{(\gamma_1, 0)} f)^\gamma(x)(y)\]

Finally, from Proposition 2.7 we conclude

\[(D_{(0, \eta_j)} \cdots D_{(0, \eta_1)} D_{(\gamma_l, 0)} \cdots D_{(\gamma_1, 0)} f)^\gamma(x)(y)\]

\[= d^{(i,j)} f(x, y, \gamma_1, \ldots, \gamma_l, \eta_1, \ldots, \eta_j)\]

\[= (d^{(i,j)} f \circ \rho_{i,j})(x, \gamma_1, \ldots, \gamma_l, y, \eta_1, \ldots, \eta_j)\]

\[= (d^{(i,j)} f \circ \rho_{i,j})^\gamma(x, \gamma_1, \ldots, \gamma_l)(y, \eta_1, \ldots, \eta_j),\]

where each $\rho_{i,j}$ is the continuous map

\[\rho_{i,j} : U \times \mathcal{L}(G)^l \times V \times \mathcal{L}(H)^j \to U \times V \times \mathcal{L}(G)^l \times \mathcal{L}(H)^j,\]

\[(x, \gamma, y, \eta) \mapsto (x, y, \gamma, \eta)\]

Now, from the classical Exponential Law \[6.1\] follows that the maps

\[(d^{(i,j)} f \circ \rho_{i,j})^\gamma : U \times \mathcal{L}(G)^l \to C(V \times \mathcal{L}(H)^j, E)_{c.o}\]

are continuous, and we have shown that

\[d^{(j)} \circ d^{(i)}(f^\gamma) = (d^{(i,j)} f \circ \rho_{i,j})^\gamma,\]

(6)

thus the continuity of $d^{(i)}(f^\gamma)$ follows from the fact that the topology on $C^l(V, E)$ is initial with respect to the maps $d^{(i)}$, whence $f^\gamma$ is $C^k$. 16
(b) The linearity and injectivity of \( \Phi \) is clear. To show that \( \Phi \) is a topological embedding we will prove that the given topology on \( C^{k,l}(U \times V, E) \) is initial with respect to \( \Phi \). We define the functions

\[
\rho^*_{i,j} : C(U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j, E)_{c.o} \to C(U \times \mathfrak{L}(G)^i \times V \times \mathfrak{L}(H)^j, E)_{c.o},
\]

\[ g \mapsto g \circ \rho_{i,j}, \]

and

\[
\Psi_{i,j} : C(U \times \mathfrak{L}(G)^i \times V \times \mathfrak{L}(H)^j, E)_{c.o} \to C(U \times \mathfrak{L}(G)^i, C(V \times \mathfrak{L}(H)^j, E)_{c.o})_{c.o},
\]

\[ g \mapsto g^\vee \]

for \( i, j \in \mathbb{N}_0 \) such that \( i \leq k, j \leq l \). Then we have

\[
(d^{(i,j)} f \circ \rho_{i,j})^\vee = (\Psi_{i,j} \circ \rho^*_{i,j} \circ d^{(i,j)})(f).
\]

On the other hand, we have

\[
d^{(j)} \circ d^{(i)}(f^\vee) = (C(U \times \mathfrak{L}(G)^i, d^{(j)}(d^{(i)}(f) \circ \Phi))(f),
\]

where \( (C(U \times \mathfrak{L}(G)^i, d^{(j)}) \) are the maps

\[
C(U \times \mathfrak{L}(G)^i, C^l(V, E))_{c.o} \to C(U \times \mathfrak{L}(G)^i, C(V \times \mathfrak{L}(H)^j, E)_{c.o})_{c.o},
\]

\[ g \mapsto d^{(j)} \circ g. \]

Thus, from (6) follows the equality

\[
C(U \times \mathfrak{L}(G)^i, d^{(j)}) \circ d^{(i)} \circ \Phi = \Psi_{i,j} \circ \rho^*_{i,j} \circ d^{(i,j)}.
\]

The maps \( d^{(i,j)}, \rho^*_{i,j} \) and \( \Psi_{i,j} \) are topological embeddings (see definition of the topology on \( C^{k,l}(U \times V, E) \), Appendix A.5, and Proposition 5.1 respectively), hence by the transitivity of initial topologies Appendix A.2 the given topology on \( C^{k,l}(U \times V, E) \) is initial with respect to the maps \( \Psi_{i,j} \circ \rho^*_{i,j} \circ d^{(i,j)} \).

But by the above equality, this topology is also initial with respect to the maps \( C(U \times \mathfrak{L}(G)^i, d^{(j)}) \circ d^{(i)} \circ \Phi \). Since \( d^{(i)} \) and \( C(U \times \mathfrak{L}(G)^i, d^{(j)}) \) are topological embeddings (see definition of the topology on \( C^l(V, E) \) and Appendix A.5), respectively) we conclude from Appendix A.2 that the topology on the space \( C^{k,l}(U \times V, E) \) is initial with respect to \( \Phi \). This completes the proof.

Now, we go over to the proof of Theorem (B):

**Proof of Theorem (B).** We need to show that if \( g \in C^k(U, C^l(V, E)) \), then the map

\[
g^\wedge : U \times V \to E, \quad g^\wedge(x,y) := g(x)(y)
\]
(which is continuous, since the locally convex space \( E \) is completely regular and we assumed that \( U \times V \) is a \( k_\mathbb{R} \)-space, see Proposition \ref{prop:continuous}). Since \( \Phi(g^\wedge) = (g^\wedge)' = g \), the map \( \Phi \) will be surjective, hence a homeomorphism (being a topological embedding by Theorem \ref{thm:topological_embedding}).

To this end, we fix \( x \in U \), then \( g(x) \in C^l(U, E) \) and for \( y \in V, \eta \in \mathcal{L}(H) \) and \( t \neq 0 \) small enough we have

\[
\frac{1}{t}(g^\wedge(x, y \cdot \eta(t)) - g^\wedge(x, y)) = \frac{1}{t}(g(x)(y \cdot \eta(t)) - g(x)(y)) \to d(g(x))(y, \eta)
\]

as \( t \to 0 \). Consequently \( d^{(0,1)}(g^\wedge)(x, y, \eta) \) exists and equals \( d(g(x))(y, \eta) = (d^{(1)} \circ g)(x, y, \eta) = (d^{(1)} \circ g)^\wedge(x, y, \eta) \). Analogously, for \( j \in \mathbb{N}_0 \) with \( j \leq l \) and \( \eta_1, \ldots, \eta_j \in \mathcal{L}(H) \) we obtain the derivatives

\[
d^{(0,j)}(g^\wedge)(x, y, \eta_1, \ldots, \eta_j) = (d^{(j)} \circ g)^\wedge(x, y, \eta_1, \ldots, \eta_j).
\]

But for fixed \((y, \eta_1, \ldots, \eta_j)\) we have

\[
(d^{(j)} \circ g)^\wedge(x, y, \eta_1, \ldots, \eta_j) = (d^{(j)} \circ g)(x)(y, \eta_1, \ldots, \eta_j)
\]

\[
= (\text{ev}_{(y, \eta_1, \ldots, \eta_j)} \circ d^{(j)} \circ g)(x),
\]

where \( \text{ev}_{(y, \eta_1, \ldots, \eta_j)} \) is the continuous linear map

\[
\text{ev}_{(y, \eta_1, \ldots, \eta_j)} : C(V \times \mathcal{L}(H)^j, E)_{c.o} \to E, \quad h \mapsto h(y, \eta_1, \ldots, \eta_j).
\]

Since also \( d^{(j)} : C^l(V, E) \to C(V \times \mathcal{L}(H)^j, E)_{c.o} \) is continuous and linear, the composition \( \text{ev}_{(y, \eta_1, \ldots, \eta_j)} \circ d^{(j)} \circ g : U \to E \) is \( C^k \), by Lemma \ref{lem:composition}. Thus for \( \gamma \in \mathcal{L}(G) \) and \( t \neq 0 \) small enough we obtain

\[
\frac{1}{t}(d^{(0,j)}(g^\wedge)(x \cdot \gamma(t), y, \eta_1, \ldots, \eta_j)) - d^{(0,j)}(g^\wedge)(x, y, \eta_1, \ldots, \eta_j))
\]

\[
= \frac{1}{t}((\text{ev}_{(y, \eta_1, \ldots, \eta_j)} \circ d^{(j)} \circ g)(x \cdot \gamma(t)) - (\text{ev}_{(y, \eta_1, \ldots, \eta_j)} \circ d^{(j)} \circ g)(x))
\]

\[
\to d((\text{ev}_{(y, \eta_1, \ldots, \eta_j)} \circ d^{(j)} \circ g)(x, \gamma),
\]

as \( t \to 0 \). Thus \( d^{(1,j)}(g^\wedge)(x, y, \gamma, \eta_1, \ldots, \eta_j) \) is given by

\[
d((\text{ev}_{(y, \eta_1, \ldots, \eta_j)} \circ d^{(j)} \circ g)(x, \gamma) = (\text{ev}_{(y, \eta_1, \ldots, \eta_j)} \circ d^{(j)} \circ dg)(x, \gamma)
\]

\[
= (d^{(j)} \circ dg)(x, \gamma)(y, \eta_1, \ldots, \eta_j)
\]

\[
= (d^{(j)} \circ dg)^\wedge(x, \gamma, y, \eta_1, \ldots, \eta_j).
\]

Analogously, for each \( i \in \mathbb{N}_0 \) with \( i \leq k \) and \( \gamma_1, \ldots, \gamma_i \in \mathcal{L}(G) \) we obtain

\[
d^{(i,j)}(g^\wedge)(x, y, \gamma_1, \ldots, \gamma_i, \eta_1, \ldots, \eta_j) = (d^{(j)} \circ d^{(i)} g)^\wedge(x, \gamma_1, \ldots, \gamma_i, y, \eta_1, \ldots, \eta_j).
\]
To see that \( g^\wedge \) is \( C^{k,l} \) we need to show that the maps

\[
d^{(i,j)}(g^\wedge) : U \times V \times \mathcal{L}(G)^i \times \mathcal{L}(H)^j \to E, \quad (x, y, \gamma_1, \ldots, \gamma_i, \eta_1, \ldots, \eta_j) \mapsto (d^{(j)} \circ d^{(i)} g)^\wedge(x, \gamma_1, \ldots, \gamma_i, y, \eta_1, \ldots, \eta_j)
\]

are continuous for all \( i, j \in \mathbb{N}_0 \) with \( i \leq k, j \leq l \). To this end, consider the continuous maps

\[
d^{(j)} \circ d^{(i)} g : U \times \mathcal{L}(G)^i \to C(V \times \mathcal{L}(H)^j, E_{c.o}).
\]

By Proposition 3.1 the maps \((d^{(j)} \circ d^{(i)} g)^\wedge : U \times \mathcal{L}(G)^i \times V \times \mathcal{L}(H)^j \to E\) are continuous, since \( E \) is completely regular and we assumed that \( U \times V \times \mathcal{L}(G)^i \times \mathcal{L}(H)^j \) is a \( k_{\mathbb{R}} \)-space, hence the maps \( d^{(i,j)}(g^\wedge) \) are continuous and \( g^\wedge \) is \( C^{k,l} \).

\[\square\]

Remark 3.4. Theorem (A) follows from Theorem (B), since \( C^\infty \infty(U \times V, E) \cong C^\infty(U \times V, E) \) as a topological vector space, by Corollary 2.14.

Corollary 3.5. Let \( U \subseteq G, V \subseteq H \) be open subsets of topological groups \( G \) and \( H \), let \( E \) be a locally convex space and \( k, l \in \mathbb{N}_0 \cup \{\infty\} \). Assume that at least one of the following conditions is satisfied:

(a) \( l = 0 \) and \( V \) is locally compact,

(b) \( k, l < \infty \) and \( U \times V \times \mathcal{L}(G)^k \times \mathcal{L}(H)^l \) is a \( k_{\mathbb{R}} \)-space,

(c) \( G \) and \( H \) are metrizable,

(d) \( G \) and \( H \) are locally compact.

Then the map

\[
\Phi : C^{k,l}(U \times V, E) \to C^k(U, C^l(V, E)), \quad f \mapsto f^\circ
\]

is a homeomorphism.

Proof. (a) As in the proof of Theorem (B), we need to show that if \( g \in C^k(U, C(V, E)) \), then \( g^\wedge \in C^{k,0}(U \times V, E) \). The computations of the derivatives of \( g^\wedge \) carry over (with \( j = 0 \)), hence it remains to show that the maps \( d^{(i,0)}(g^\wedge) \) in (4) are continuous for all \( i \in \mathbb{N}_0 \) with \( i \leq k \). But since \( V \) is assumed locally compact, each of the maps \( (d^{(i,0)} \circ d^{(i)} g)^\wedge : U \times \mathcal{L}(G)^i \times V \to E \) is continuous by Proposition 3.1, hence so is each of the maps \( d^{(i,0)}(g^\wedge) \), as required.

(b) By [5] Proposition, p.62, if \( U \times V \times \mathcal{L}(G)^k \times \mathcal{L}(H)^l \) is a \( k_{\mathbb{R}} \)-space, then so is \( U \times V \times \mathcal{L}(G)^i \times \mathcal{L}(H)^j \) for each \( i, j \in \mathbb{N}_0 \) with \( i \leq k, j \leq l \). Hence, Theorem (B) holds and \( \Phi \) is a homeomorphism.

(c) Since \( G \) is metrizable, the space \( C(\mathbb{R}, G) \) is metrizable (see [6] Appendix A.5 or [3] Lemma B.21), whence so is \( \mathcal{L}(G) \subseteq C(\mathbb{R}, G) \) as well as \( U \times \mathcal{L}(G)^i \) for each \( i \in \mathbb{N}_0, i \leq k \) as a finite product of metrizable spaces. With a similar
argumentation we conclude that also $V \times \mathfrak{L}(H)^J$ is metrizable for each $j \in \mathbb{N}_0$ with $j \leq l$, whence so is $U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j$. But each metrizable space is a $k$-space, hence a $k_\mathcal{R}$-space. Therefore, Theorem (B) holds in this case and $\Phi$ is a homeomorphism.

(d) As $G$ is locally compact, it is known that the identity component $G_0$ of $G$ (being a connected locally compact subgroup of $G$) is a pro-Lie group (in the sense that $G_0$ is complete and every identity neighborhood of $G_0$ contains a normal subgroup $N$ such that $G/N$ is a Lie group, see [5, Definition 3.25]). Hence, by [5, Theorem 3.12], $\mathfrak{L}(G)$ is a pro-Lie algebra, and from [5, Proposition 3.7] follows that $\mathfrak{L}(G) \cong \mathbb{R}^I$ for some set $I$ as a topological vector space. Since also $H$ is assumed locally compact, for each $i, j \in \mathbb{N}_0$ with $i \leq k, j \leq l$ we have $U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j \cong U \times V \times (\mathbb{R}^I)^i \times (\mathbb{R}^J)^j$ for some set $J$. Now, from [13, Theorem 5.6 (ii)] follows that $U \times V \times \mathfrak{L}(G)^i \times \mathfrak{L}(H)^j$ is a $k_\mathcal{R}$-space (being isomorphic to a product of completely regular locally compact spaces), whence Theorem (B) holds and $\Phi$ is a homeomorphism.

\[\square\]

A Some properties of $C^k$- and $C^{k,l}$-functions on topological groups

First, we prove a simple chain rule for compositions of continuous group homomorphisms and $C^k$-functions:

**Lemma A.1.** Let $G$ and $H$ be topological groups, $E$ be a locally convex space. Let $\phi : G \to H$ be a continuous group homomorphism and $f : V \to E$ be a $C^k$-map ($k \in \mathbb{N} \cup \{\infty\}$) on an open subset $V \subseteq H$. Then for $U := \phi^{-1}(V)$ the map

$$f \circ \phi|_U : U \to E, \quad x \mapsto f(\phi(x))$$

is $C^k$.

**Proof.** Obviously, the map $f \circ \phi|_U$ is continuous. Now, let $x \in U$ and $\gamma \in \mathfrak{L}(G)$. For $t \neq 0$ small enough we have

$$\frac{f(\phi(x \cdot \gamma(t))) - f(\phi(x))}{t} = \frac{f(\phi(x) \cdot \phi(\gamma(t))) - f(\phi(x))}{t} \to df(\phi(x), \phi \circ \gamma)$$

as $t \to 0$, since $\phi \circ \gamma \in \mathfrak{L}(H)$, see Remark 2.2. Therefore $df(\phi \circ \phi|_U)(x, \gamma)$ exists and is given by $df(\phi(x), \phi \circ \gamma)$. Repeating the above steps, we obtain for $i \in \mathbb{N}$ with $i \leq k$, $\gamma_1, \ldots, \gamma_i \in \mathfrak{L}(G)$ the derivatives $d^{(i)}(f \circ \phi|_U)(x, \gamma_1, \ldots, \gamma_i) = d^{(i)}f(\phi(x), \phi \circ \gamma_1, \ldots, \phi \circ \gamma_i)$.

Now, recall that the map $\Sigma(\phi) : \mathfrak{L}(G) \to \mathfrak{L}(H), \eta \mapsto \phi \circ \eta$ is continuous (Remark 2.2), whence also each of the maps

$$d^{(i)}(f \circ \phi|_U) := (d^{(i)}f) \circ (\phi|_U \times \mathfrak{L}(\phi) \times \cdots \times \mathfrak{L}(\phi)) : U \times \mathfrak{L}(G)^i \to E$$

is continuous. Hence $f \circ \phi|_U$ is $C^k$. \[\square\]
Lemma A.2. Let $U \subseteq G$, $V \subseteq H$ be open subsets of topological groups $G$ and $H$, let $(E_α)_{α \in A}$ be a family of locally convex spaces with direct product $E := \prod_{α \in A} E_α$ and the coordinate projections $pr_α : E \to E_α$. For $k, l \in \mathbb{N}_0 \cup \{\infty\}$ the following holds:

(a) A map $f : U \to E$ is $C^k$ if and only if all of its components $f_α := pr_α \circ f$ are $C^k$.

(b) A map $f : U \times V \to E$ is $C^{k,l}$ if and only if all of its components $f_α := pr_α \circ f$ are $C^{k,l}$.

Proof. To prove (a), first recall that because each of the projections $pr_α$ is continuous and linear, the compositions $pr_α \circ f$ are $C^k$ if $f$ is $C^k$, by Lemma 2.10 (a).

Conversely, assume that each $f_α$ is $C^k$ and let $x \in U$, $γ \in \Sigma(G)$ and $t \neq 0$ small enough. Then we have

$$
\frac{1}{t} \left( f(x \cdot γ(t)) - f(x) \right) = \left( \frac{1}{t} (f_α(x \cdot γ(t)) - f_α(x)) \right)_{α \in A}.
$$

Since $\frac{1}{t} (f_α(x \cdot γ(t)) - f_α(x))$ converges to $df_α(x, γ)$ as $t \to 0$ for each $α \in A$, the derivative $df(x, γ)$ exists and is given by $(df_α(x, γ))_{α \in A}$.

Repeating the above steps, we obtain for $i \in \mathbb{N}$ with $i \leq k$ and $γ_1, \ldots, γ_i \in \Sigma(G)$ the derivatives $d^{(i)} f(x, γ_1, \ldots, γ_i) = (d^{(i)} f_α(x, γ_1, \ldots, γ_i))_{α \in A}$, which define continuous maps

$$
d^{(i)} f = \left( d^{(i)} f_α \right)_{α \in A} : U \times \Sigma(G)^i \to E.
$$

Therefore, $f$ is $C^k$.

The assertion (b) can be proven similarly, by using Lemma 2.10 (b) and showing that for all $i, j \in \mathbb{N}_0$, with $i \leq k$, $j \leq l$ we have $d^{(i,j)} f = (d^{(i,j)} f_α)_{α \in A}$.

The following lemma is a special case of Lemma 2.11.

Lemma A.3. Let $U \subseteq G$ be an open subset of a topological group $G$, and $E$ be a locally convex space. A continuous map $f : U \to E$ is $C^1$ if and only if there exists a continuous map

$$
f^{[1]} : U^{[1]} \to E
$$

on the open set

$$
U^{[1]} := \{(x, γ, t) \in U \times \Sigma(G) \times \mathbb{R} : x \cdot γ(t) \in U\}
$$

such that

$$
f^{[1]}(x, γ, t) = \frac{1}{t}(f(x \cdot γ(t)) - f(x))
$$

for each $(x, γ, t) \in U^{[1]}$ with $t \neq 0$.

In this case we have $df(x, γ) = f^{[1]}(x, γ, 0)$ for all $x \in U$ and $γ \in \Sigma(G)$.
We use this lemma, as well as the analogue for \( C^4 \)-maps on locally convex spaces (which can be found in \([5\), Lemma 1.2.10\]), for the proof of a chain rule for compositions of \( C^k \)-functions \( f : G \to E \) and \( g : E \to F \), which will be provided after the following version:

**Lemma A.4.** Let \( G \) be a topological group, \( P \) be a topological space and \( E, F \) be locally convex spaces. Let \( U \subseteq G, V \subseteq E \) be open subsets, and \( k \in \mathbb{N} \cup \{ \infty \} \). If \( f : U \times P \to E \) is a \( C^{k,0} \)-map such that \( f(U \times P) \subseteq V \), and \( g : V \to F \) is a \( C^k \)-map (in the sense of differentiability on locally convex spaces), then

\[
g \circ f : U \times P \to F
\]

is a \( C^{k,0} \)-map.

**Proof.** We may assume that \( k \) is finite and prove the assertion by induction.

**Induction start:** Assume that \( f \) is \( C^{1,0} \), \( g \) is \( C^1 \) and let \( x \in U, p \in P \) and \( \gamma \in \mathfrak{L}(G) \). For \( t \neq 0 \) small enough we have

\[
g(f(x \cdot (t), p)) - g(f(x, p)) = \frac{g(f(x, p) + \frac{t(f(x, p))}{2} - f(x, p))}{t} - g(f(x, p))
\]

\[
= \frac{g(f(x, p) + t \cdot f^{[1,0]}(x, \gamma, t, p))}{t} - g(f(x, p))
\]

where \( g^{[1]} \), \( f^{[1,0]} \) are the continuous maps from \([6\), Lemma 1.2.10\] and Lemma \( 2.11 \). As \( t \to 0 \) we consequently have

\[
g(f(x \cdot (t), p)) - g(f(x, p)) \to g^{[1]}(f(x, p), f^{[1,0]}(x, \gamma, 0, p), 0)
\]

\[
= dg(f(x, p), d^{[1,0]} f(x, p, \gamma)).
\]

Therefore, the derivative \( d^{[1,0]}(g \circ f)(x, p, \gamma) \) exists and is given by the directional derivative \( dg(f(x, p), d^{[1,0]} f(x, p, \gamma)) \).

Consider the continuous map

\[
h : U \times P \times \mathfrak{L}(G) \to E, \quad (x, p, \gamma) \mapsto f(x, p).
\]

Since \( d^{[1,0]}(g \circ f)(x, p, \gamma) = (dg \circ (h, d^{[1,0]} f))(x, p, \gamma) \), the map

\[
d^{[1,0]}(g \circ f) = dg \circ (h, d^{[1,0]} f) : U \times P \times \mathfrak{L}(G) \to F
\]

is continuous, whence \( g \circ f \) is \( C^{1,0} \).

**Induction step:** Now, assume that \( f \) is \( C^{k,0} \) and \( g \) is \( C^k \) for some \( k \geq 2 \). By Remark \( 23 \) the map \( d^{[1,0]} f : U \times (P \times \mathfrak{L}(G)) \to E \) is \( C^{k-1,0} \), and it is easily seen that the map \( h : U \times (P \times \mathfrak{L}(G)) \to E \) defined in the induction start is \( C^{k,0} \). Hence, using Lemma \( A.2 \)(b), we see that \( (h, d^{[1,0]}) : U \times (P \times \mathfrak{L}(G)) \to E \times E \)

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is a $C^{k-1,0}$-map. Since $dg : V \times E \to F$ is $C^{k-1}$ (see [6, Definition 1.3.1]), the map

$$d^{(1,0)}(g \circ f) = dg \circ (h, d^{(1,0)}) : U \times (P \times \mathcal{L}(G)) \to F$$

is $C^{k-1,0}$, by the induction hypothesis, and from Remark 2.6 follows that $g \circ f$ is $C^{k,0}$.

**Lemma A.5.** Let $G$ be a topological group, $E$, $F$ be locally convex spaces and $k \in \mathbb{N} \cup \{\infty\}$. Let $U \subseteq G$, $V \subseteq E$ be open subsets. If $f : U \to E$ is a $C^k$-map with $f(U) \subseteq V$ and also $g : V \to F$ is a $C^k$-map, then the map

$$g \circ f : U \to F$$

is $C^k$.

**Proof.** We may assume that $k$ is finite and prove the assertion by induction.

**Induction start:** Assume that $f$ and $g$ are $C^1$-maps. Analogously to the preceding lemma, for $x \in U$, $\gamma \in \mathcal{L}(G)$ and $t \neq 0$ small enough we have

$$\frac{1}{t}(g(f(x \cdot \gamma(t))) - g(f(x))) = g^{[1]}(f(x), f^{[1]}(x, \gamma, t), t),$$

with continuous maps $f^{[1]}$ as in Lemma A.3 and $g^{[1]}$ as in [6, Lemma 1.2.10]. Thus, the derivative $d(g \circ f)(x, \gamma)$ exists and we have

$$d(g \circ f)(x, \gamma) = g^{[1]}(f(x), f^{[1]}(x, \gamma, 0), 0) = dg(f(x), df(x, \gamma)).$$

Using the continuous function

$$h : U \times \mathcal{L}(G) \to E, \quad (x, \gamma) \mapsto f(x),$$

we see that

$$d(g \circ f) = dg \circ (h, df) : U \times \mathcal{L}(G) \to F$$

is continuous, hence $g \circ f$ is a $C^1$-map.

**Induction step:** Now, let $f$ and $g$ be $C^k$-maps for some $k \geq 2$. Then the map $df : U \times \mathcal{L}(G) \to E$ is $C^{k-1,0}$, by Remark 2.6 and the map $h : U \times \mathcal{L}(G) \to E$ is obviously $C^{k-0}$. We use Lemma A.2 (b) and see that $(h, df) : U \times \mathcal{L}(G) \to E \times E$ is a $C^{k-1,0}$-map. By [6, Definition 1.3.1], the map $dg : V \times E \to F$ is $C^{k-1}$, hence by Lemma A.4, the composition

$$d(g \circ f) = dg \circ (h, df) : U \times \mathcal{L}(G) \to F$$

is $C^{k-1,0}$, whence $g \circ f$ is $C^k$, by Remark 2.6. \qed
Finally, the following example illustrates that the statement of Schwarz’ Theorem does not hold for maps on non-abelian topological groups.

**Example A.6.** Consider the following subgroup $G$ of $GL_3(\mathbb{R})$:

\[
G := \left\{ x = \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}
\]

(known as the Heisenberg group) and $\gamma, \eta \in \mathcal{L}(G)$ defined as

\[
\gamma(t) := \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \eta(t) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \quad (\forall t \in \mathbb{R}).
\]

Then $G \cong \mathbb{R}^3$ via

\[
\phi : G \to \mathbb{R}^3, \quad x := \begin{pmatrix} 1 & x_1 & x_2 \\ 0 & 1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} \mapsto (x_1, x_2, x_3).
\]

Let $g : \mathbb{R}^3 \to \mathbb{R}$ be a partially $C^2$-map in the usual sense and define

\[
f := g \circ \phi : G \to \mathbb{R}.
\]

Then for each $x \in G$, the derivatives $D_\gamma f(x)$, $D_\eta f(x)$, $(D_\eta D_\gamma f)(x)$ and $(D_\gamma D_\eta f)(x)$ can be expressed using the partial derivatives of $g$.

First, we have

\[
D_\gamma f(x) = \lim_{t \to 0} \frac{1}{t} (f(x \cdot \gamma(t)) - f(x)) = \lim_{t \to 0} \frac{1}{t} (g(\phi(x \cdot \gamma(t))) - g(\phi(x)))
\]

\[
= \lim_{t \to 0} \frac{1}{t} (g(x_1 + t, x_2, x_3) - g(x_1, x_2, x_3))
\]

\[
= \lim_{t \to 0} \frac{1}{t} (g((x_1, x_2, x_3) + t(1, 0, 0)) - g(x_1, x_2, x_3)) = \frac{\partial}{\partial x_1} g(x_1, x_2, x_3).
\]

Further,

\[
D_\eta f(x) = \lim_{t \to 0} \frac{1}{t} (f(x \cdot \eta(t)) - f(x)) = \lim_{t \to 0} \frac{1}{t} (g(\phi(x \cdot \eta(t))) - g(\phi(x)))
\]

\[
= \lim_{t \to 0} \frac{1}{t} (g(x_1, x_2 + tx_1, x_3 + t) - g(x_1, x_2, x_3))
\]

\[
= x_1 \frac{\partial}{\partial x_2} g(x_1, x_2, x_3) + \frac{\partial}{\partial x_3} g(x_1, x_2, x_3).
\]

Now,
\[(D_\eta D_\gamma f)(x) = \lim_{t \to 0} \frac{1}{t} (D_\gamma f(x \cdot \eta(t)) - D_\gamma f(x))\]
\[= \lim_{t \to 0} \frac{1}{t} \left( \frac{\partial}{\partial x_1} g(x_1, x_2 + tx_1 + t, x_3 + t) - \frac{\partial}{\partial x_1} g(x_1, x_2, x_3) \right)\]
\[= x_1 \cdot \frac{\partial^2}{\partial x_1 \partial x_2} g(x_1, x_2, x_3) + \frac{\partial^2}{\partial x_1 \partial x_3} g(x_1, x_2, x_3).\]

And, finally
\[(D_\gamma D_\eta f)(x) = \lim_{t \to 0} \frac{1}{t} (D_\eta f(x \cdot \gamma(t)) - D_\eta f(x))\]
\[= \lim_{t \to 0} \frac{1}{t} \left( (x_1 + t) \cdot \frac{\partial}{\partial x_2} g(x_1 + t, x_2, x_3) + \frac{\partial}{\partial x_3} g(x_1, x_2, x_3) \right)\]
\[= \lim_{t \to 0} \frac{x_1}{t} \left( \frac{\partial}{\partial x_2} g(x_1 + t, x_2, x_3) - \frac{\partial}{\partial x_3} g(x_1, x_2, x_3) \right)\]
\[+ \lim_{t \to 0} \frac{1}{t} \left( \frac{\partial}{\partial x_3} g(x_1 + t, x_2, x_3) - \frac{\partial}{\partial x_2} g(x_1, x_2, x_3) \right) + \lim_{t \to 0} \frac{\partial}{\partial x_2} g(x_1 + t, x_2, x_3)\]
\[= x_1 \cdot \frac{\partial^2}{\partial x_1 \partial x_2} g(x_1, x_2, x_3) + \frac{\partial^2}{\partial x_1 \partial x_3} g(x_1, x_2, x_3) + \frac{\partial}{\partial x_2} g(x_1, x_2, x_3)\]
\[= (D_\eta D_\gamma f)(x) + \frac{\partial}{\partial x_2} g(x_1, x_2, x_3).\]

Thus we see that if \(\frac{\partial}{\partial x_2} g(x_1, x_2, x_3) \neq 0\), then \((D_\gamma D_\eta f)(x) \neq (D_\eta D_\gamma f)(x)\).

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