Snapshots on Vortex Dynamics

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Abstract

Salient features of vortex dynamics in super media are summarized. Recent examples are: the demonstration of prominent role of topology in vortex dynamics; the solution to the Hall anomaly which once bothered Bardeen, de Gennes and many others; the unified microscopic treatment of both transverse and frictional forces on moving vortex. The fundamental dynamical equation of vortex matter can now be casted into the elegant form of quantum dissipative dynamics of Leggett. Together with the Kosterlitz-Thouless transition, we have finally reached a coherent picture on both thermodynamic and dynamical roles played by vortices. The key historical progresses are discussed with a broader perspective, to move into the post high $T_c$ superconductor era, the quantum era.

References mentioned in the text and given at the end, though very incomplete, along with a list of a few outstanding open problems, may provide a reader a useful guidance and an interesting perspective.

Puzzle:

By 1999 Kopnin and Vinokur reached the conclusion that the anomalous Hall effect can be compatible with the Magnus force, though the present author reached the same conclusion 4 years earlier but was not cited by them. By 2001 Blatter and Geshkenbein and their coworkers reached the conclusion that in discussion of vortex interference effect only the vortex velocity part of Magnus force is needed, though again same conclusion was reached by the presented author 5 years earlier but was not cited by them. It is, however, very comforting that those important physics have been explored by very different groups of able physicists.

The puzzle here is not on their inability to cite relevant prior works, for an analysis of such behaviors the readers are referred to the Kirby-Houle article in Nov. (2004) Physics Today. Instead, the puzzle is on their ability to maintain (Kopnin, 2001; Blatter and Geshkenbein, 2003) that there is reduced and/or sign-reversed transverse force without giving any discussion on the contradiction to their own as well as other related works.
I. FUNDAMENTAL VORTEX DYNAMICS EQUATION

A. Fundamental equation for vortex matter

All the fundamental features regarding to vortex dynamics are already present in two dimensions. The generalization to three dimensions is straightforward. After long and strenuous efforts by Ao, Geller, Niu, Rhee, Tang, Thouless, Wexler, Zhu, and many others, elegant formulation of vortex dynamics and its proper physical understanding have apparently been reached.

In two dimensions, the fundamental equation of motion for a vortex reads:

\[ m_v \frac{d^2 \mathbf{r}(t)}{dt^2} = -\nabla V(\mathbf{r}(t)) - 2\pi \hbar q_v \rho_s(\mathbf{r}) \frac{d\mathbf{r}(t)}{dt} \times \hat{z} - \int_{-\infty}^{t} dt' \alpha(t-t') \frac{d\mathbf{r}(t')}{dt'} + \xi(t) \]  

(1)

with the correlation function

\[ \alpha(t) = \frac{2}{\pi} \int_{0}^{\infty} d\omega \frac{J(\omega)}{\omega} \cos(\omega t) \]  

(2)

and the spectral function

\[ J(\omega) = \eta \omega^s \exp\left(-\frac{\omega}{\omega_c}\right) \]  

(3)

Here \( \mathbf{r} \) is the vortex position vector, \( \hat{z} \) is the unit vector perpendicular to the plane; \( \rho_s(\mathbf{r}) \) is the superfluid density at the vortex, the Planck constant \( \hbar \), and the vorticity \( q_v \) which is an integer. The cutoff frequency \( \omega_c \) will be chosen to be larger than any characteristic frequency in the problem. All other terms, the potential \( V \), the transverse force, the frictional force, and the noise, will be explained below.

Ao and Thouless (1993, [3]) and Thouless, Ao, Niu (1996,[5]) are two most important theoretical progresses during 1990’s in the understanding of vortex dynamics. They have been served as the light houses in the navigation through the vortex dynamics rough water. To my knowledge Ao and Zhu (Ao and Zhu, 1999 [6]) is the best place to get into vortex dynamics in the formulation of Eq.(1) from a microscopic point of view. The general physics behind such equation, system plus environment, in the context of dissipative quantum dynamics can be found, for example, in Leggett (1992) or in Feynman and Vernon (Ann. Phys. 1963).

The well-known linear friction case is a special case of Eq.(1):

\[ m_v \frac{d^2 \mathbf{r}(t)}{dt^2} = -\nabla V(\mathbf{r}(t)) - 2\pi \hbar q_v \rho_s(\mathbf{r}) \frac{d\mathbf{r}(t)}{dt} \times \hat{z} - \eta \frac{d\mathbf{r}(t)}{dt} + \xi(t) \]

(4)
It is the Ohmic case that $s = 1$ in the spectral function, the cutoff frequency goes to infinite ($\omega_c \to \infty$), the Planck constant is zero ($\hbar = 0$) or the high temperature limit.

### B. Nonlinear Schrödinger equation

The most physically transparent and mathematically elegant way to derive Eq.(1) is to from the nonlinear Schrödinger equation (NLSE): Every term in Eq.(2) can be obtained from NLSE. For topological quantities, such as the Magnus force, NLSE is sufficient. But some contributions to friction, such as the core and extended quasi-particle states, have to be calculated from microscopic theories. The advantage of starting from NLSE is that the connection to hydrodynamics and to quantization is direct. For example, for the neutral case, $s = 2$, a superohmic case according to Leggett’s formulation of dissipative dynamics, while for the charged case, $s = \infty$, that is, there is a gap, the Coulomb gap, in the elementary excitations.

The NSLE in both neutral and charged cases can be derived from microscopic theories. The simple form of NLSE reads:

$$i\hbar \frac{\partial \psi(r, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(r, t) - \mu_0 \psi(r, t) + U_0 |\psi(r, t)|^2 \psi(r, t)$$

(5)

With $\rho_s(r, t) = |\psi(r, t)|^2$ the superfluid density at time $t$ and position $r$, $m$ the effective mass of the Cooper pairs or bosons, $\mu_0$ the chemical potential determined the mean superfluid density, and $U_0$ the effective strength of the short range repulsive interaction. Phonons and vortices are automatically included in this formulation. The macroscopic slow dynamics of NLSE is completely exhausted by dynamics of phonons and vortices.

The consistence of NLSE with microscopic theory for superfluid Helium was pointed out by Demircan, Ao, Niu (1996). Such connection was already implicitly known to Feynman and to Anderson. NLSE can also be obtained from Kohn’s density functional approach. Josephson relation can follows directly from this NLSE, as shown by Feynman.

The short length scale in NLSE is the healing length or coherence length determined by $U_0$, the important short length scale for a macroscopic description. The introduction of coupling to electromagnetic field is straightforward: the standard minimum coupling. In this case another length scale, the London penetration depth connected to the superfluid density, enters into the description. Therefore, the even the usual two types of superconductors, type
I and type II, can be effectively described by NLSE.

There should be no confusion of NLSE with Gross-Pitaevskii equation (GPE): GPE is about the off-diagonal part, the condensate part (first clearly conceived by London in 1948), which is highly sensitive to the strength of the interaction among the fluid particles, bosons or fermions. Instead, NLSE is about the superfluid density, which is always the total fluid density (for simple fluid) at zero temperature regardless of the interaction strength. For example, for a strongly interacting bosons, such as He II, the condensate can be a small fraction of the superfluid density at zero temperature.

The current description of BEC at zero temperature is in a happy situation: at zero temperature, the GPE and NLSE are almost identical, because the interaction is weak. Nevertheless, for the discussion of vortex dynamics, physical it is the NLSE not GPE which one should use and keep in mind to avoid confusions.

The derivation of NLSE in superconductor from BCS theory was given by Aitchison, Ao, Thouless, Zhu (1995). There has been a consideration amount of confusion between NLSE and time dependent Ginzburg-Landau equation (TDGL) till these days. TDGL is essentially a GPE equation (and vice versa) in fermionic superfluid: strictly it is about the condensate fraction, not the superfluid density. Again, at zero temperature, there is no simple relation between the gap function in TDGL and the superfluid function $\psi$. For example, the gap can be exponentially small but the superfluid density will be the total free electron density. One should not be surprised that TDGL can take a complete different form from that of NLSE. Nevertheless, we have another happy situation in superconductors: near transition temperature $T_c$ the super fluid wavefunction $\psi$ was shown by Gorkov to be proportional to the gap function in TDGL.

In the present of weak disorder in superconductors, NLSE will retain its form of Eq.(5), with the same superfluid density implied by Anderson’s dirty superconductors theorem and justified by Green’s function approach by many others, but with a different effective mass known to Pippard.

C. Vortex mass $m_v$

Vortex mass is perhaps the first example of the acquiring mass from the environment, discussed more than 100 years ago. It is the first example of the renormalization of mass.
However, it is also interesting to point out that it is perhaps the least experimentally tested quantity in this category.

It is effectively the mass of the fluid excluded by the vortex core, for the ideal incompressible fluid.

This mass can be calculated. The hydrodynamics case can be found in H Lamb’s classical book. The superconductor case can be found, for example, in Han et al (Han, Kim, Kim, Ao, 2005).

In the slow dynamics limit the left hand side of Eq.(1) is a higher order contribution to dynamics. It may be negligible. Then the dynamics would be dominated by the Lorentz force like transverse force and/or the correlation function which contains the dissipation.

This may explain the difficulty in experimental measurement of vortex mass: For slow dynamics, it’s contribution is of higher order, and for a relative fast dynamics, the dissipative effect becomes large. Hence, a very precise measurement should be needed in order to have reliable number on the vortex mass. This implies that a different type of experimental design, other than those to measure the potential, transverse force, and friction, is needed.

D. Vortex potential $V(r)$ and its gradient

The potential includes all the contributions which are not dependent on the vortex velocity. More precisely, all the positional dependence in this term is instantaneous.

It contains a term coming from the fluid velocity generated by others vortices, including those from the image of the vortex under consideration. It’s gradient has the form, the superfluid velocity part of the Magnus force:

$$F_{\text{Magnus,\nu_s}} = 2\pi \hbar q_\nu \rho_s(r) \mathbf{v}_s(r) \times \hat{z}$$  \hspace{1cm} (6)

If there is no other terms such as pinning in $\nabla V$, trapping potential in in BEC, and no frictional force and noise, this term together with the transverse force is the known Magnus force in fluid dynamics. It makes the vortex moving along the superfluid flow stream line.

Some famous results have obtained from this term which describes the vortex-vortex interaction:

a) Critical velocity. Feynman (1954), Anderson (e.g., Basic notions of condensed matter physics, 1984), Leggett (Physica Fennica, 1973). The meaning of critical velocity is firmly
placed on the topology, not of Landau critical velocity type of quasiparticle with no topology.

There is, however, another happy situation. In many cases the numerical values of critical velocity due to Landau and due to topological consideration are the same, or, very close to each other, though in general it has been shown by Anderson and by Leggett that there is no relation between them.

b) Abrikosov vortex lattice: vortex form lattices.
This force leads to logarithmic interaction in neutral case and a short range (on the scale of London penetration depth) in the charged case. An equilibrium lattice structure almost follows immediate this way.

c) Kosterlitz-Thouless transition: the unbinding of vortex-antivortex pairs.
This transition is extremely important in the understanding of the topological stability of condensed phase, and resulting in the name of Kosterlitz-Thouless-Halperin-Nelson-Young transition.

d) Quark confinement and asymptotic freedom.
Kosterlitz-Thouless transition is also an elementary (2D) illustration of the quark confinement (The phases of quantum chromodynamics: from confinement to extreme environments. JB Kogut and MA Stephanov. Cambridge University Press, 2004).

E. Transverse force: the vortex velocity part of Magnus force

This transverse force is the second term at the right hand of Eq.(1), identical in form to the Lorentz force:

\[ F_{\text{Magnus},dr/dt} = -2\pi \hbar q_v \rho_s(r) \frac{dr(t)}{dt} \times \hat{z} \]  

(7)

It’s derivation from microscopic theory (Ao and Thouless, 1993) is one of the nontrivial applications of Berry phase to obtain important physical results. The topological structure of a vortex had been discussed by London (1948), Onsager (1949), and Feynman (1954).

The first macroscopic derivation of Eq.(7) was given by Nozieres and Vinen (1966). See also Fetter, PR 163, 1967.

It is another expression for the Josephson-Anderson relation. Anderson, RMP, 1966; ME Fisher and Langer, PRL, 1968.

The full detailed microscopic derivation in superconductors was given by Ao and Zhu (Ao and Zhu, 1999), including both the contributions from the vortex core and extended states,
as well as in both clean and dirty limits. The feasibility of such derivation is guaranteed by the Anderson’s dirty superconductor theorem.

This force has rich physics consequences in addition to the Josephson-Anderson relation, for example:

- **a)** turbulence (Onsager, 1949);
- **b)** anomalous Hall effect in superconductor (Ao, 1995; Kopnin and Vinokur, 1999);
- **c)** vortex interference (van Wees, 1990; MPA Fisher, 1991; Ao and Zhu, 1995);
- **d)** quantum Hall effect in Josephson junction arrays (Zhu, Tan, and Ao, 1996);
- **e)** vortex processing in BEC (Lundh and Ao, 2000)
- **f)** interference effect (Ivanov, Ioffe, Geshkenbein, Blatter, 2001)

Experimental evidences are numerous to support the above theoretical proposals. It is clear that by 1999 theoretically there exists an agreement that the anomalous Hall effect is consistent with the transverse force as given by Eq.(7).

It is also clear by 2001 that in order to consider the transverse effect on vortex motion in Josephson junctions arrays, Eq.(7) is the only transverse force responsible for various quantum effects. No other transverse effects introduced by various authors are needed in such discussions.

**F. Frictional force**  

\[- \int_{-\infty}^{t} dt' \alpha(t - t') \frac{d\mathbf{r}(t')}{dt'} \]

For \(2 \geq s \geq 0\), if one perform the usual effective energy calculation with constant vortex velocity, infinite vortex mass correction will be resulted:

For \(s = 2\), the effective mass correction will diverge algorithmically with systems size, a fact elegantly discussed by Duan and Leggett (1995) and confirmed by Niu, Ao, Thouless (1996) via a dynamical and many-body wavefunction consideration.

From the microscopic derivation, one contribution to \(s = 1\) was first found by Bardeen and Stephen (1965) from the vortex core in the dirty limit. \(s = 1\) was also found by Ao and Zhu (1999) from the extended state contribution. Such contributions are the Ohmic type:

\[- \int_{-\infty}^{t} dt' \alpha(t - t') \frac{d\mathbf{r}(t')}{dt'} \rightarrow - \eta \frac{d\mathbf{r}(t)}{dt} \]

(8)

For \(s < 0\) the system is thermodynamically unstable. For \(s > 2\), vortex mass renormalization due to \(\alpha(t)\) and \(\xi(t)\) is finite. The diverging mass encountered here \((2 \geq s \geq 0)\) is
closely related to those diverging quantities in non-Fermi liquid theory.

The regime $s \geq 0$ also makes the adiabatic consideration of vortex motion possible, though in the regime $2 \geq s \geq 0$ a strict Landau quasiparticle type picture (finite effective mass etc) is not valid.

The very existence of this friction force implies, in addition to the effective mass, that vortices can be independent variables: it will not necessary move along the superfluid flow stream line, and can cut through the streamlines. Thus, the vortex motion can generate dissipation, even when the fluid is “super”, a common knowledge now in superfluid and superconductors, after several Nobel prizes.

G. Noise $\xi$

The noise is related to the friction by the fluctuation-dissipation theorem, derivable from microscopic theories:

$$\langle \xi(t)\xi^\tau(t') \rangle = \frac{h}{\pi} \int_0^\infty d\omega J(\omega) \coth \left( \frac{\hbar \omega}{2k_B T} \right) \cos(\omega(t-t'))$$

(9)

and $\langle \xi \rangle = 0$. Here superscript $\tau$ denotes the transpose. For simplicity we have assumed the friction matrix to be a constant matrix. No anisotropic frictional effect will be considered here.

Such an expression can be derived either starting from NLSE or from microscopic theories: we already mentioned that the vortex-phonon interaction corresponds to $s = 2$ and core and extended states contributions correspond to $s = 1$.

In the zero $\hbar$ or high temperature limit, we have for $s = 1$,

$$\langle \xi(t)\xi^\tau(t') \rangle = 2k_B T \eta \delta(t-t')$$

(10)

This corresponds to Eq.(4).

II. SOME HIGH AND LOW POINTS

Here are snapshots on the progress in vortex dynamics, emphasizing on superfluids and superconductors.
A. Pre-high $T_c$ superconductor era ( $< 1989$ )

Vortices were not in Landau’s original formulation of two fluid model of Helium II. In fact, Landau initially opposed the existence of the vortices. This “absence of vorticity” might be the origin of confusing on vortex dynamics from the theoretical side.

1965, Bardeen and Stephen. Microscopic calculation of vortex friction on core contribution in the dirty limit $[1]$. An elegant paper perhaps has not been widely read, though widely cited. The misunderstanding on the origin of friction still exists.

1966, Nozieres and Vinen. Macroscopic derivation of Magnus force $[2]$. Very insightful paper. A. Fetter’s 1967 PR paper is also helpful.

1976, Noto, Shinzawa, Muto. Summarizing the Hall anomaly experiments in superconductors: the Hall effect is usually small and often change signs, in an apparent contradiction to the transverse force as given by Eq.(7) if using the independent vortex dynamics model to calculate the Hall effect.

Similar effect has been observed in superfluids.

This “anomalous” effect might be the origin of confusing on vortex dynamics from the experimental side.

1976, Kopnin and Kratsov. In response to the small Hall effect in the mixed state, relaxation time approximation was conceived by Kopnin and Kratsov to derive the core friction contribution with vanishing small transverse force in the dirty limit.

In the hindsight, this approximation is not applicable in this case. The physical and mathematical reasons for such a invalid approximation have been discussed at least since 1940’s. In particular, R. Kubo had extensively discussed such approximation (Statistical physics, M. Toda, R. Kubo, and N. Saito, v.1 and 2, second edition, 1992). See also Zubarev of Bogoliubov school (Nonequilibrium statistical thermodynamics, D. N. Zubarev, 1974) for appropriate time scales in the problem.

1976, Sonin. Approximated calculation of additional transverse force due to phonons. There is no clear interpretation of the additional force by Sonin, such as whether add or subtract to the transverse force as given in Eq.(7). The direct contradiction of such result with Vinen’s experiment has never been discussed.

Theoretically, NLSE gives a complete description at zero temperature: the superfluid density is the total fluid density, there are vortices and phonons, and vortex and phonons
interact. There is no additional contribution from phonons to the super (total) fluid density.

**B. High $T_c$ superconductor era ( > 1989 )**

1993, Ao and Thouless. Berry phase derivation of the Magnus force based on topology and symmetry of many-body wavefunction. A nontrivial application of Berry’s method. The topological aspect of vortex dynamics was emphasized.

1993, Volovik. The absence of transverse force in the dirty limit was interpreted as the cancellation between the topological contribution from the core stated, the spectral flow, and the topological effect of Berry phase.

This is an erroneous conclusion. The fact is that there are two equivalent ways to compute the transverse force on a moving vortex: one from core regime and one away from core. They are equal according to Stokes theorem.

The spectral flow is independent of the impurity because of its topological nature, not something continuously tunable by a non-topological parameters such as the relaxation time.

Hence, two mistakes would not make it right.

1995, Kopnin and Lapatin; van Otterlo, Feigelman, Geshkenbein, Blatter. Repeating the relaxation time approximation to Helium 3 by the first group author and extended to superconductor with under path integral formulation by the second group authors. Confirmed their old conclusion that there is no transverse force in the dirty limit.

Again, the mistake is the invalidity of the relaxation time approximation.

1995, Feigelman, Geshkenbein, Larkin, Vinokur. Trying to demonstration an additional Berry phase term from the vortex core to cancel the Berry phase computed by Ao and Thouless (1993).

Their calculation is in clear violation of the basic requirement from quantum mechanics: at the phase singularity the amplitude of the wave function must be zero. Hence, there is no additional Berry phase term as they claimed.

However, since their result apparently reproduced what obtained by Kopnin and Krastsov, by Volovik, and the results of their other collaborators based on erroneous approximation schemes, they believe their cancellation should be right. Hence, they found that not only the transverse force is usually small, it occasionally changes signs, controlled by relaxation time, etc.
1995, Ao. First explicit proposal that the large transverse force is consistent with Hall anomaly if vortex many-body and pinning effects are considered. Several quantitative predictions were made here.

The main conclusion is that, the anomalous Hall effect in the mixed state, the small Hall angle and sign change, can be explained by the universal Magnus force derived from the Berry phase. This may not be a surprising result for people familiar with the Hall effect in semiconductors: there we see small and zero Hall angle, size changes, etc, and they are all consistent with the universal Lorentz force.

1995, Ao and Zhu. Vortex interference by controlling the number of particles in the superfluid enclosed by the vortex trajectory loop.

Since the transverse force is similar to the Lorentz force, this is just another form of Aharonov-Bohm effect for vortices.

1996, Thouless, Ao, and Niu. Extension of Berry phase formulation to include the friction. No relaxation time approximation is needed. But a proper thermodynamical limit is required: the dissipative energy has been carried out of the system, preferably to infinite in an explicit manner.

This is a nontrivial extension of Berry’s method. Mistakes have often been committed in such an extension. The discussions of R. Kubo mentioned above as well as those by Zubarev are useful here for a better understanding of physics.

1996 Zhu, Tan, and Ao. Quantum Hall effect in Josephson junction arrays from the view of vortices.

1997, Sonin. Same approximated calculation as his 1976 was repeated. It is clear that even within such approximation, the linear correction term wanted by Sonin cannot be rigorously obtained. But this mathematical inconsistency was completely ignored by Sonin in order to generate result he wanted.

The present of phonons and the total superfluid density is equal to the total fluid density at zero temperature implied in NLSE (Eq.(5)) clearly suggests Sonin’s concept here is completely wrong.

By a careful analysis, it should be concluded that what Sonin discussed was a different phenomena other than what he thought. After all it cannot produce what he wanted in a mathematically consistent manner.

1997, Zhu, Brandstrom, and Sundqvist. First direct confirmation of transverse
force on vortex in superconductor [11]. This elegant experiment was done in the tradition of directly measuring the Lorentz force for electron in the magnetic field (1890’s) and vortices in superfluid (1960’s).

It is very surprising that despite over 30 years controversies on the transverse force, this is the only systematic experiment to directly measure the force in superconductors.

**1999, Ao and Zhu.** Detailed and microscopic implementation [6] of framework developed in 1996 by Thouless, Ao, Niu.

The results of Bardeen and Stephen and of Nozieres and Vinen were unified and extended. Detailed calculation showed how to obtain the vortex friction without the relaxation time approximation, consistent with what Bardeen and Stephen did.

It is interesting to note that there is no controversy at all on Eq.(6) (Wexler, PRL, 1997). From the macroscopic hydrodynamical point of view Eq.(6) and (7) are just the two sides of same coin.

Thermodynamically, it was demonstrated by Ao and Zhu that the reduction of total transverse from Eq.(7) as fiercely argued by Blatter, Feigelman, Geshkenbein, Kopnin, Larkin, Vinokur, Volovik, and others (their conclusions are all based on uncontrolled approximations) would lead to the violation of the second law of thermodynamics.

To summarize what done by Ao and Zhu, the invalidity of relaxation time approximation was carefully considered from both critical and constructive points of views, from both macroscopic and microscopic points of views:

a) An elementary kinetic model was devised in Ao and Zhu (PRB 1999), adapted from Kubo and others, to show how the seemly simple use of relaxation time approximation lead to wrong result.

The essence of the demonstration is that, in the calculation of transport coefficients, there are usually two different starting points for systematic approximation, though rigorously they are equivalent in the linear regime. The first one is to treat the force as perturbation and calculate the response velocity (current):

\[ \text{small force} \implies \text{velocity} , \ (I) . \]

In this case there is usually a well-defined expression for the force to begin with, and the velocity-velocity (current-current) correlation is the one subjected to systematic approxi-
The well-known example in this category is the conductivity. The relaxation time approximation is usually OK, and one can simply start from a typical kinetic equations such as the Boltzmann equation.

The second starting point is to treat the current, or velocity, as the perturbation and calculate the response force.

\[
\text{small velocity} \Rightarrow \text{force}, \ (II) .
\]

This method also has other names, such as the force-balance equation. It is the force-force correlation subjected to systematic approximation. The well-known example here is the computation of resistivity. Unfortunately, the usual relaxation time approximation is problematic here, documented over past 50 years in literature.

In the case of derivation of vortex dynamics microscopically, we do NOT know that form of the force on a moving vortex at beginning: It is precisely this force needed to be found out. Hence, we cannot use the usual approach of starting from using the force as perturbation. We are compelled to deal with the second one, using the vortex velocity as the perturbation. This is what has been used by all of us: Thouless \textit{et al}, Kopnin \textit{et al}, van Otterlo \textit{et al}, and so on. As it is known in literature, one should avoid the problematic relaxation time approximation in this case. But Kopnin \textit{et al}, van Otterlo \textit{et al} etc have not.

A sophisticated and clear demonstration can be found in Kubo’s book as well as in the book of Zuburev, mentioned above: there are several time scales important at the microscopic level, but not apparent at the macroscopic level. One has to be carefully on the limiting procedure. Kubo himself had complained about the blind and wrong use of relaxation time approximation in transport problems, which appears periodically in literature.

One may put it in following way: It is the relaxation time approximation which needs to be justified here: Boltzmann recognized this long long time ago. It arises from the interaction between different parts of the whole system. Here, in the context of vortex dynamics, the friction of vortex directly comes from the interaction of vortex with the quasiparticles, and can and have been calculated without the relaxation time approximation. If one wants an expert understanding of this issue, Leggett’s formulation of dissipative quantum dynamics and Kubo’s book are among the must readings. We may summarize what has been known in transport theory in the following table:
b) A thermodynamical demonstration was also devise to show that the change in superfluid kinetic energy must come from the transverse force on the moving vortex, since the entropy of superfluid is zero. Any reduction of the magnitude of this transverse force, as would be the case for the present relaxation time approximation, will violate the second law of thermodynamics. This gives a thermodynamical reason to abandon the relaxation time approximation in this case.

c) A full microscopic derivation of the transverse was provided in Ao and Zhu (PR B 1999). It was a detailed implementation of the formulation developed by Thouless, Ao, and Niu (PRL, 1996). It is very important to point out that this microscopic formulation is similar to what used by Kopin et al, by van Otterlo et al, and by many others. The only major difference is the absence of relaxation time approximation in the context of vortex dynamics in Ao and Zhu.

It was found that there are many contributions to the vortex friction: core states, extended states, etc. The contribution of core states is due to the mixing of core levels by impurity scattering under an appropriate time scale. This mixing contribution to friction has been known for a long time, reminiscent to Thouless energy, at least since 1980’s, and has been made very clear and explicit in the recent study of chaotic contribution to friction.

**1999, Kopnin and Vinokur.** The compactibility of large transverse force in Eq.(7) with Hall anomaly was argued [12], though no citation to Ao and/or Thouless.

Experiment of Zhu et al on transverse force was cited by Kopnin and Vinokur.

It is very comforting that the same result obtained by Ao four years earlier on Hall anomaly was reached by a very different group of able physicists.
At least as late as in 1999, one may be able to conclude that the Hall anomaly is compatible with the large transverse force.

2001, Ivaon, Ioffe, Geshkenbein, Blatter. Vortex interference and its effect in Josephson junction arrays were discussed solely based on Eq.(7), with no citation to Ao and/or Thouless. There is however no presence of other transverse forces discussed by Geshkenbein, Blatter, and others in their earlier works. Indeed, such additional transverse forces are not needed, either.

Even if those authors still believe in the existence of other extra transverse forces, from the professional point of view they should state that either the extra forces are not needed or they do not exist in this situation. Those authors should also state that there is an alternative theory by Ao and Thouless, and by others that no such extra forces at all.

Again, it is very comforting same physics explored a few years earlier was done by a very different group of able physicists. NO other transverse forces, such as discussed by kopnin et al, by van oterrio et al, by Feigelman et al, by Volovik, etc, on a moving vortex is needed.

Summary: We may be able to conclude that the controversy on the transverse force of Eq.(7) may be finally behind us. There is no reduction from Eq.(7).

C. Post high $T_c$ superconductor era ( > 1998 )

The post high $T_c$ superconductors era study is marked by Bose-Einstein condensation, topological controlled quantum computation, quantum turbulence, etc. It is the explicitly exploration of quantum behaviors of vortices, hence the quantum era. It looks that we have finally, by 1999 if earlier or by 2001 if later, reached a firm understanding on vortex dynamics, in the form given by Eq.(1), and in particular the transverse force in the form of Eq.(7).

However, it appears that rest of us are all too naive.

2001, Kopnin. In 2001 Kopnin still presented the situation as based on his erroneous theory of vanishing transverse force in the dirty limit. There is no citation to Ao and/or Thouless.

It appears that he has never consulted the discussion by R. Kubo on the invalidity of relaxation time approximation in certain approach, including his case. No discussion on his
violation of second law of thermodynamics as demonstrated by Ao and Zhu in 1999.

**2003, Blatter and Geshkenbein.** In 2003 Blatter and Geshkenbein misled the community by systematic suppressing the literature on the existence of transverse force and by emphasizing their works on the vanishing of transverse force in the dirty limit [13].

There are some questions here before moving on.

Should the new era start from erroneous results which has been demonstrated? Let’s put aside the question on Kopnin, Vinokur, Blatter, Geshkenbein systematic omission to relevant prior work. What is the logic behind Kopnin as well as Blatter and Geshkenbein that when they need the transverse force, it is there, and when they don’t, they simply announce that it would not exist?

### III. FUTURE

First of all, it seems that the research community deserves explanations (none so far) from Kopnin, Vinokur, Blatter, Geshkenbein, and their collaborators, on their inconsistent behaviors regarding to the use of Eq.(7), the transverse force. This is science. Researchers deserve honest answers.

It is difficult to predict what will be the exciting results coming out of BEC, quantum computation, and quantum turbulence and other related fields. Here I would rather focus on the unsolved problems along the more traditional line. Their solutions will undoubtedly help us understand other problems.

**treatment of boundary layer.** On the phenomenological level of two fluid model, it would be nice to further extend the result obtained by Thouless, Geller, Vinen, Fortin, Rhee (2001) (Also, Rhee, PhD Thesis, 2003, University of Washington). The approach will very likely be based on a method on the treatment of boundary layer. This will not only gain an understanding on vortex dynamics, it may also lead to a new insight on boundary layer problem in general. However, this may be a rather hard problem.

**microscopic derivation of fundamental equation in bosonic superfluid.** Though mathematical framework to calculate the transverse force and friction on a moving vortex has been set up by Thouless et al (Thouless, Ao, Niu, 1996), and such a calculation has been performed for superconductors (Ao and Zhu, 1999), strangely enough there is no full
range calculation yet for bosonic superfluid even based on Bogoliubov theory. There is an apparent difficulty due to the existence of both infrared and ultraviolet divergences built into the conventional approximation. In calculation of the friction and transverse force, a consistent consideration of all degrees of freedom is needed, because of the topology of vortex.

Hence, one needs to develop a consistent microscopic superfluid theory at finite temperatures based on Bogoliubov formulation. The rest of calculation on vortex dynamics would strongly resemble what has been done by Ao and Zhu (1999). A major difference may be that there is now no localized core states. I expect to see a progress soon along this direction, and I would be delighted to hear it as soon as possible (E-mail: aoping@u.washington.edu).

**phonons and quasiparticles.** For fermionic superfluid, the different between phonons and quasiparticles who carry super current is as clear as blue sky. This is not so in bosonic superfluid. We know that phonons exist in both super and normal phases, but supercurrent exists only in one phase. Even in super phase, the receiver of Packard can get phonons in Helium II, not supercurrent. A clear understanding of this issue may deepen our understanding of the microscopic theory of bosonic superfluid, and may help understand some issues in quantum turbulence, too.

**measurement of transverse force and friction.** There is no doubt that more precise as well as further measurements on both transverse force Eq.(7) and friction Eq.(8) are needed, for both bosonic and fermionic superfluids. Such experiments really require courage and talent: Courage to initiate experiments and talent to design good experiments. This has already been demonstrated [9, 10, 11]. It is surprising that how little has been done experimentally to test the fundamental vortex equation.

Though it is impossible in this short note to give a complete reference list, I do hope that the works mentioned in the text should give a reader a useful guidance to literature with an interesting perspective. From my own point of view, a good entry point to have an overview may be Ao and Zhu (1999) [6]. Some of additional relevant comments on the theoretical
side may be found in Ref.[16, 17].

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