RELATIVE DGA AND MIXED ELLIPTIC MOTIVES

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1. Introduction and convention

1.1. Introduction. Let $K$ be a field. In the paper [VSF], Voevodsky defined a triangulated category $D_{MM,K}$ of mixed motives. Under the assumption of Beilinson-Soule vanishing conjecture for varieties over $K$, there exists a reasonable t-structure $\tau$ on $D_{MM,K}$, and the abelian category $A_{MM,K}$ of mixed motives is defined as the heart of $D_{MM,K}$ with respect to the t-structure $\tau$.

Let $E$ be an elliptic curve over $K$ and we assume Beilinson-Soule vanishing conjectures for varieties over $K$. The category of elliptic motives is defined as the smallest full subcategory of $A_{MM,K}$ containing $h^1(E)$ closed under taking direct sums, direct summands, and tensor products. The category of mixed elliptic motives is the smallest full subcategory of $A_{MM,K}$ containing elliptic motives and closed under extensions. That is, an object $M$ in $A_{MM,K}$ is a mixed elliptic motif if there exists a filtration $M = F^0M \supset F^1M \supset \cdots \supset F^nM = 0$ such that $F^iM/F^{i+1}M$ are elliptic motives for $i = 0, \ldots, n-1$. In other words, the category of mixed elliptic motives is the relative completion of the category of mixed motives with respect to the category of elliptic motives ([H]). For example, the category of mixed Tate motives is the relative completion of the category of mixed motives with respect to the category of pure Tate motives, which is the smallest full subcategory of $A_{MM,K}$ containing Tate objects $\mathbb{Q}(i)$ closed under taking direct sums.

In the paper [BK], Bloch and Kriz construct an abelian category of mixed Tate motives as the category of comodules over a Hopf algebra obtained by the bar construction of the DGA of cycle complexes. One advantage of their construction is that Beilinson-Soule vanishing conjectures is not necessary for their construction. In this paper, we construct a DG category $(VMEM)$ of DG complexes of elliptic motives and prove that it is homotopy equivalent to the DG category of $\text{Bar}(C_{VEM})$-comodules. As a consequence, the homotopy category of the subcategory of $(VMEM)$ consisting of objects concentrated at degree zero is equivalent to the abelian category of $H^0(\text{Bar}(C_{VEM}))$-comodules. Another generalization of Bloch-Kriz construction in the context of elliptic curves is also studied in the earlier work by Bloch [B] and Patashnick [P]. Hain and Matsumoto ([HM]) studies Hodge and $l$-adic counterpart part of the mixed elliptic motives.

Before going into this subject, it will be helpful to recall a similar construction for the category of local systems over a manifold $X$ after R. Hain. Let $F$ a field of characteristic zero, $x$ a point in $X$ and $V$ be a $F$-local system on $X$. Let $G = \pi_1(X,x)$ and $\rho : G \rightarrow \text{Aut}(V)$ be the monodromy representation and $S$ be the Zariski closure of the image of $\rho$. Assume that $S$ is a reductive group over $F$.

**Definition 1.1.** Let $W$ be a finite dimensional representation of $G$. If there exists a finite filtration $W = W^0 \supset W^1 \supset \cdots \supset W^n = 0$ such that $Gr^i(W) = \ldots$
$W^i/W^{i+1}$ is isomorphic to the pull back of an algebraic representation of $S$, the representation $W$ is called a successive extension of algebraic representations of $S$. The full subcategory of $G$ consisting of successive extensions of algebraic representations of $S$ is denoted as $(\text{Rep}_G)^S$.

Then objects in $(\text{Rep}_G)^S$ are stable under taking duals, direct products and tensor products, and $(\text{Rep}_G)^S$ becomes a Tannakian category. When $F = \mathbb{R}$, Hain constructed a Hopf algebra $H$, in [H], using differential forms on $X$ and the connection associated to the local system $V$ such that the category of comodules over $H$ is equivalent to $(\text{Rep}_G)^S$. In other words, the Tannaka fundamental group $\pi_1((\text{Rep}_G)^S)$ of $(\text{Rep}_G)^S$ is isomorphic to $\text{Spec}(H)$. His construction is called the relative bar construction.

In this paper, we reformulate Hain’s construction using relative DGA’s so that it can be applied to the motivic context. We give a general homotopical framework of relative bar complexes and DG-categories in §2 and §3. In §2 we define a relative DGA, and introduce the relative bar complex $\text{Bar}(A/\mathcal{O}, \epsilon)$ and the relative simplicial bar complex $\text{Bar}_{\text{simp}}(A/\mathcal{O}, \epsilon)$ of a relative DGA $A$ with a relative augmentation $\epsilon$. We show that these two bar complexes are quasi-isomorphic. The simplicial bar complex $\text{Bar}_{\text{simp}}(A/\mathcal{O}, \epsilon)$ is defined in order to establish the equivalence of the category of bar comodules to the category of DG complexes in $C(A/\mathcal{O})$ defined in the next section. In §3, we introduce a DG category $C(A/\mathcal{O})$ arising from a relative DGA $A$ over $\mathcal{O}$. In this section, we prove that the DG category $KC(A/\mathcal{O})$ of DG complexes in $C(A/\mathcal{O})$ is homotopy equivalent to that of $\text{Bar}(A/\mathcal{O}, \epsilon)$-comodules. The main theorem in this section is Theorem 3.14.

In §§3.4 and 3.5 we recover Hain’s construction in our formulation. Although this is not necessary to construct mixed elliptic motives, it will give an evidence that our formulation of relative bar complex is a right one. By the homomorphism $\rho : G \to S(k)$, the coordinate ring $\mathcal{O}_S$ of the algebraic group $S$ becomes a bi-$G$ module. The comodule $\mathcal{O}_S$ as a left $G$-module is written as $L\mathcal{O}_S$. The $G$-cochain complex $A = \text{Hom}_{LG}(L\mathcal{O}_S, L\mathcal{O}_S)$ becomes a bi-$\mathcal{O}_S$ comodule and is equipped with a multiplication arising from Yoneda pairing. The complex $A$ is called the relative DGA associated to the map $\rho$. In §§3.5 we prove that the Hopf algebra $\mathcal{O}(\pi_1((\text{Rep})^S))$ is isomorphic to the 0-th homology of relative bar complex defined in §2.2. Our proof is based on DG categories and DG complexes introduced in [BoK]. (See also [T].)

In §4 we construct a DG category $(\text{MEM})$ of naive mixed elliptic motives as the category of DG complexes of pure elliptic motives. We expect that the homotopy category $H^0(\text{MEM})$ of $(\text{MEM})$ is isomorphic to the full sub-triangulated tensor category of $D_{\text{MM}, K}$ generated by $h^1(E)$ and their tensors. We define a relative quasi-DGA $A_{EM}$ from algebraic cycles and then apply the general result in §2 and §3 to show that the category $(\text{MEM})$ is homotopy equivalent to the category of $\text{Bar}(A_{EM})$-comodules. As a consequence, an object concentrated at degree zero defines an $H^0(\text{Bar}(A_{EM}))$-comodule. The main theorem in this section is Theorem 4.8.
The product structure on $H^0(\text{Bar}(A_{EM}))$ is given by shuffle product. This comes from a tensor structure on the category $(MEM)$. However, as is explained in §4.6.2, the category of naive elliptic motives does not have a tensor structure with distributive property which gives rise to a shuffle product structure on the bar complex $\text{Bar}(A_{EM})$. So we introduce a category of virtual mixed elliptic motives $(VMEM)$ which is homotopy equivalent to $(MEM)$ and equipped with a tensor structure with distributive property. To show the homotopy equivalence of categories $(EM)$ and $(VEM)$, the injectivity of linear Chow group (Proposition 4.1) is necessary. Though there is a proof of this proposition also in [BL], we will give a more direct proof. Using this properties, we construct a quasi-DG category $(VMEM)$ which has distributive tensor structure, and is homotopy equivalent to $(MEM)$. (Theorem 4.30.) By the shuffle product induced by the tensor structure, the coalgebra $H^0(\text{Bar}(A_{EM})) = H^0(\text{Bar}(C_{VEM}))$ becomes a Hopf algebra. (Theorem 4.34.) The definition of the Tannakian category of mixed elliptic motives is defined in Definition 4.35.

In §5 we construct an elliptic polylog motif $Pl_n$ as an example of object concentrated at degree zero in $(MEM)$. To define the polylog motif, we first recall the elliptic polylog class $P_n$ as an element of a higher Chow group introduced in [BL]. Using the elliptic polylog class, we construct an elliptic polylog motif $Pl_n$. Using the bijection of objects of $(MEM)$ and $\text{Bar}(A_{EM})$, we write down the comodule structure on $Pl_n$ over $H^0(\text{Bar}(A_{EM}))$. In this paper, we assume that the elliptic curve $E$ has no complex multiplication.

1.2. Convention. The subset $\{i \in \mathbb{Z} \mid p \leq i \leq q\}$ of $\mathbb{Z}$ is denoted by $[p, q]$. Let $k$ be a field of characteristic zero. The tensor products mean those over the base field $k$.

1.2.1. Let $\mathcal{O}$ be a counitary coalgebra. The counit $\mathcal{O} \to k$ is denoted by $u$ and the comultiplication of $\mathcal{O}$ is denoted by $\Delta_{\mathcal{O}} : \mathcal{O} \to \mathcal{O} \otimes \mathcal{O}$. Let $M$ and $N$ be right and left $\mathcal{O}$ comodules. We define the cotensor product $M \otimes_{\mathcal{O}} N$ by

$$\text{Ker}(M \otimes N \xrightarrow{d} M \otimes \mathcal{O} \otimes N),$$

where the map $d$ is defined by $\Delta_M \otimes id_N - id_M \otimes \Delta_N$. Let $V_1, V_2$ be left $\mathcal{O}$-comodules. We define $Hom_{\mathcal{O}}(V_1, V_2)$ by

$$\text{Ker}(Hom_k(V_1, V_2) \xrightarrow{d} Hom_k(V_1, \mathcal{O} \otimes V_2))$$

where the map $d$ is the difference of

$$\Delta_{V_2,*} : Hom_k(V_1, V_2) \to Hom_k(V_1, \mathcal{O} \otimes V_2)$$

and the composite

$$Hom_k(V_1, V_2) \to Hom_k(\mathcal{O} \otimes V_1, \mathcal{O} \otimes V_2) \xrightarrow{\Delta_{V_2,*}} Hom_k(V_1, \mathcal{O} \otimes V_2).$$

Let $A$ be a bimodule over the coalgebra $\mathcal{O}$, and $\varphi \in Hom_{\mathcal{O}}(V_1, V_2)$. Then $1_A \otimes \varphi$ becomes an element in $Hom_{\mathcal{O}}(A \otimes \mathcal{O} V_1, A \otimes \mathcal{O} V_2)$. 

1.2.2. Let $M$ be an $O$-comodule. Then we have the following complex
\[ 0 \rightarrow M \xrightarrow{\Delta_M} O \otimes M \xrightarrow{\Delta_O \otimes 1 - 1 \otimes \Delta_M} O \otimes O \otimes M \]
by the coassociativity of $M$. This is exact since the maps
\[ u \otimes 1_M : O \otimes M \rightarrow M, \]
\[ u \otimes 1_O \otimes u \otimes 1_M : O \otimes O \otimes M \rightarrow O \otimes M \]
give homotopy. As a consequence, the natural map $M \rightarrow O \otimes O M$ is an isomorphism.

1.2.3. Let $S$ be an algebraic group and $O_S$ the coordinate ring. Then $O_S$ becomes a Hopf algebra whose coproduct $O_S \rightarrow O_S \otimes O_S$ is obtained by
\[ S \times S \rightarrow S : (g, h) \mapsto hg. \]
There is natural one to one correspondence between left algebraic representations of $S$ and left comodules over $O_S$. The correspondence is given as follows. Let $g$ be an element in $S(k)$. Then the evaluation at $g$ defines an algebra homomorphism $ev_g : O_S \rightarrow k$. Let $V$ be a left $O_S$-comodule and $\Delta_V : V \rightarrow O_S \otimes V$ be the comodule structure on $V$. The action of $g$ on $V$ is given by the composite of
\[ V \rightarrow O_S \otimes V \xrightarrow{ev_g} k \otimes V = V. \]
The left $S$ module $O_S$ is written by $L O_S$. A reductive group $S$ is said to be split if all irreducible representations over $k$ are absolutely irreducible. Let $Irr(S)$ be the set of isomorphism classes of irreducible representations. If $S$ is split, then we have
\[ O_S = \bigoplus_{\alpha \in Irr(S)} V^\alpha \otimes {}^\alpha V, \]
where $\alpha$ runs through the isomorphism classes of irreducible representations of left $S$-modules and $V^\alpha$ be the corresponding left $S$-module. The dual vector space $^\alpha V$ of $V^\alpha$ becomes a right $S$-module. The function $\varphi$ corresponding to $(v \otimes v^*)$ for $v \in V^\alpha, v^* \in ^\alpha V$ is defined by $\varphi(g) = (v^*, gv) = (v^*g, v)$. Therefore the dual $Hom_k(O_S, k)$ of $O_S$ is naturally isomorphic to $O_S$. Here for an $S$ representation $W$, $Hom_k(\ast, \ast)$ is defined by
\[ Hom_k(W_1, W_2) = \bigoplus_{\alpha \in Irr(S)} Hom_k(V^\alpha \otimes Hom_{O_S}(V^\alpha, W_1), W_2). \]
The natural pairing is given by
\[ (v \otimes v^*) \otimes (w \otimes w^*) = (w^*, v)(v^*, w) \]
for $v, w \in V^\alpha, w^*, w^* \in {}^\alpha V$. This isomorphism does not depend on the choice of representative of the isomorphism classes. By descent theory, there exists a natural isomorphism between $O_S$ and $Hom(O_S, k)$. 
1.2.4. ¿From now on, we assume that \( S \) is split. We set \( \mathcal{O} = \mathcal{O}_S \). The multiplication \( \mathcal{O} \otimes \mathcal{O} \to \mathcal{O} \) on \( \mathcal{O} \) is defined by the diagonal embedding \( S \to S \times S \). If functions \( \varphi \) and \( \psi \) correspond to elements \( v \otimes v^* \) and \( w \otimes w^* \), then the product function \( \varphi \psi \) is the function \( g \mapsto (v^* g, v) \cdot (w^* g, w) \).

This map is described by the duality of intertwining spaces. Let \( V \) be an irreducible representation of \( S \) and \( I_V \) and \( I^*_V \) be covariant and contravariant intertwining spaces defined by

\[
I_V(W) = \text{Hom}_S(V, W), \quad I^*_V(W) = \text{Hom}_S(W, V).
\]

Then the composite \( g \circ f \in \text{Hom}_S(V, V) \) of \( f \in I_V(W) \) and \( g \in I^*_V(W) \) is regarded as an element of \( k \) via the isomorphism \( \text{Hom}_G(V, V) = k \). It is called the contraction and is denoted as \( \text{con}(f, g) : I_V(W) \otimes I^*_V(W) \to k \).

The multiplication map is the sum of the composite

\[
\text{con}^\gamma_{\alpha, \beta} : (V^\alpha \otimes \alpha V) \otimes (V^\beta \otimes \beta V) \simeq (V^\gamma \otimes I_V(1) \otimes V^\beta)) \otimes (\gamma V \otimes I^*_V(1) \otimes V^\beta)) \to V^\gamma \otimes \gamma V
\]

1.2.5. We introduce copies of \( S \) indexed by some set \( X \). For an element \( x \in X \) the copy is denoted as \( S_x \). To distinguish the right and left actions of \( S \), we use the notation \( xS_y \) for the algebraic group \( S \). On the group \( xS_y \), \( S_x \) and \( S_y \) acts from the left and the right. The coordinate ring of \( xS_y \) is denoted by \( x\mathcal{O}_y \). For an element \( \alpha \in \text{Irr}(S) \), the copy of \( V^\alpha \) considered as a left \( S_x \)-module is denoted by \( xV^\alpha \). The dual of \( xV^\alpha \) as a right \( S_x \)-module is denoted by \( \alpha V_x \). Thus we have an identity \( x\mathcal{O}_y = \oplus_x V^\alpha \otimes \alpha V_y \). The direct sum is taken over the set \( \text{Irr}(S) \).

1.2.6. Let \( M \) and \( N \) be complexes of \( k \)-vector spaces. The set of homogeneous maps of degree \( n \) from \( M \) to \( N \) is denoted by \( \text{Hom}^n_{\text{Vect}_k}(M, N) \). For an element \( f \) in \( \text{Hom}^n_{\text{Vect}_k}(M, N) \), the differential \( \partial(f) \) of \( f \) is defined by

\[
(\partial(f)(x)) = d(f(x)) - (-1)^{\text{deg}(f)} f(dx).
\]

Let \( e^n \) be the degree \((-n)\)-element in the complex \( k[n] \) corresponding to \( 1 \). We define the degree \((-n)\) map \( t^n \) in \( \text{Hom}^{(-n)}(k, k[n]) \) by \( t(1) = e^n \). We set \( t = t^1 \) and \( e = e^1 \). The complex \( K \otimes k[n] \) is denoted by \( Ke^n \).

For two complexes \( K \) and \( L \), we define the complex \( K \otimes L \) by usual sign rule,

\[
d_{L \otimes K}(x \otimes y) = dx \otimes y + (-1)^{\text{deg}(x)} x \otimes dy.
\]

This rule is applied to a complex object in an abelian category with tensor product.

For two homogeneous map of complexes \( f_1 \in \text{Hom}_i(K_1, L_1) \) and \( f_2 \in \text{Hom}_j(K_2, L_2) \), we define the homogeneous map \( f_1 \otimes f_2 \in \text{Hom}_{i+j}(K_1 \otimes K_2, L_1 \otimes L_2) \) by

\[
(f_1 \otimes f_2)(k_1 \otimes k_2) = (-1)^{\text{deg}(f_2) \cdot \text{deg}(k_1)} f_1(k_1) \otimes f_2(k_2)
\]

for \( k_1 \in K_1, k_2 \in K_2 \). Thus we have an isomorphism of complexes:

\[
\text{Hom}^*(K_1, L_1) \otimes \text{Hom}^*(K_2, L_2) \to \text{Hom}^*(K_1 \otimes K_2, L_1 \otimes L_2).
\]
For example, we have the following isomorphism

\[ \text{Hom}^{+n-m}(K, L) \otimes \mathbb{k}t^{n-m} = \text{Hom}^*(K, L) \otimes \text{Hom}^*(\mathbb{k}e^m, \mathbb{k}e^n) \rightarrow \text{Hom}^*(K^e, L^n). \]

Using this rule, the differential of \( K \otimes L \) is written as \( d \otimes 1 + 1 \otimes d \).

A complex object \( \{ K^i = (K^i, \delta), d^i : K^i \rightarrow K^{i+1} \} \) in the category of complex is called a double complex. Then \( d^i \otimes t^{-1} : K^i e^{-i} \rightarrow K^{i+1} e^{-i-1} \) is a homogeneous map of degree one, and the sum \( \oplus_i K^i e^{-i} \) becomes a complex with the differential \( \delta \otimes 1 + d \otimes t^{-1} \), which is called the associated simple complex and denoted as \( s(K^*) \). Using the above sign rules, we have a natural isomorphism

\[ s(K \otimes L) \simeq s(K) \otimes s(L) : (K^i \otimes L^j) e^{-i-j} \simeq K^i e^{-i} \otimes L^{-j} e^{-j} \]

for two double complexes \( K \) and \( L \).

1.2.7. Let \( K \) and \( L \) be complexes. Then we define an isomorphism of complex \( \sigma : K \otimes L \rightarrow L \otimes K \) by

\[ \sigma(x \otimes y) = (-1)^{\deg(x) \deg(y)} y \otimes x. \]

This is called a transposition homomorphism. On a complex \( K^\otimes_n = K \otimes \cdots \otimes K \)

the transposition of \( i \)-th and \( (i+1) \)-th component is denoted as \( \sigma_{i,i+1} \). It is easy to show that this action can be extended to the action of \( S_n \). As a consequence, \( K^\otimes_n \) becomes a \( \mathbb{k}[S_n] \)-module. If \( \text{char}(\mathbb{k}) > n \), then the symmetric part is the image of \( \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \).

1.2.8. We assume that \( \text{char}(\mathbb{k}) = 0 \). Let \( A \) be a finite set and \( n = \# A \). For each element \( a \in A \), we prepare element \( e_a \) of degree \(-1\) and we consider a complex \( K = \oplus_{a \in A} \mathbb{k} e_a \). The symmetric part \( \Lambda(A) \) of \( K^\otimes_n \) under the action of \( S_n \) is isomorphic to \( \mathbb{k}[n] \) by choosing an “orientation of \( A \)”. If \( A = [1, n] \), then \( \mathbb{k}[n] \) is generated by \( e_1 \wedge \cdots \wedge e_n \). The group of bijection of \( A \) is denoted as \( S[A] \). On the complex \( \Lambda(A) \), the group \( S[A] \) acts via the sign. Let \( K^* = \oplus_{a \in A} \mathbb{k} f_a \) be a complex generated by degree 1-element \( f_a \) indexed by \( A \). The symmetric part of \( (K^*)^\otimes_n \) is denoted as \( \Lambda^*(A) \). We have the canonical isomorphism \( \Lambda(A \bigsqcup B) = \Lambda(A) \otimes \Lambda(B) \). The complex \( \Lambda([1, n]) \) is denoted as \( \Lambda_n \).

2. Relative DGA and relative bar construction

In this section, we define a relative DGA and relative bar constructions. For a relative DGA \( A \) and its relative augmentation \( \epsilon \), we define two bar complexes \( \text{Bar}(A/O, \epsilon) \) and \( \text{Bar}(A/O, \epsilon)^{\text{simp}} \). The latter one is called the simplicial relative bar complex. In \( \S 2.3 \) we prove that they are quasi-isomorphic. In the last subsection, we define coproduct structures on bar complexes.

2.1. Relative DGA.
2.1.1. Let $S$ be a reductive group over $k$ and $\mathcal{O}_S$ be the coordinate ring of $S$. The coproduct structure on $\mathcal{O}_S$ is denoted by $\Delta_S : \mathcal{O}_S \to \mathcal{O}_S \otimes \mathcal{O}_S$. We define a relative DGA’s over $\mathcal{O}_S$ as follows.

**Definition 2.1.** Let $A$ be a complex. A data of 5-ple $(A, \Delta_A^l, \Delta_A^r, i, \mu_A)$ consisting of

1. (the coactions) a homomorphisms of complexes $\Delta_A^l : A \to \mathcal{O}_S \otimes A$ and $\Delta_A^r : A \to A \otimes \mathcal{O}_S$,
2. a homomorphism of complexes compatible with the bi-$\mathcal{O}_S$ comodule structures $i : \mathcal{O}_S \to A$, and
3. (multiplication) a homomorphism of complexes $\mu_A : A \otimes \mathcal{O}_S A \to A$ is called a relative DGA over $\mathcal{O}_S$, if the following conditions are satisfied.

1. the multiplication $\mu_A$ is associative, and
2. the left and the right coactions of $\mathcal{O}_S$ are coassociative and counitary.

2.1.2. We set $\mathcal{O} = \mathcal{O}_S$. We use notations $V^\alpha$ for irreducible representations as in [1,2,3]. We set $A^{\alpha\beta} = \text{Hom}_\mathcal{O}(V^\alpha, A \otimes \mathcal{O} V^\beta)$. Since $S$ is split, we have

$$A = \oplus_{\alpha, \beta} V^\alpha \otimes A^{\alpha\beta} \otimes \beta V.$$ 

Under this decomposition, we have

$$A \otimes \mathcal{O} A \simeq \oplus_{\alpha, \beta, \gamma} V^\alpha \otimes A^{\alpha\beta} \otimes A^{\beta\gamma} \otimes \gamma V$$

and the multiplication map is obtained by

$$V^\alpha \otimes A^{\alpha\beta} \otimes A^{\beta\gamma} \otimes V^\gamma \xrightarrow{1 \otimes \eta \otimes 1} V^\alpha \otimes A^{\alpha\gamma} \otimes V^\gamma$$

where $\eta : A^{\alpha,\beta} \otimes A^{\beta,\gamma} \to A^{\alpha,\gamma}$ is the map defined by

$$\text{Hom}_\mathcal{O}(V^\alpha, A \otimes \mathcal{O} V^\beta) \otimes \text{Hom}_\mathcal{O}(V^\beta, A \otimes \mathcal{O} V^\gamma) \xrightarrow{\sigma} \text{Hom}_\mathcal{O}(V^\beta, A \otimes \mathcal{O} V^\gamma) \otimes \text{Hom}_\mathcal{O}(V^\alpha, A \otimes \mathcal{O} V^\beta) \xrightarrow{\text{Hom}_\mathcal{O}(A \otimes \mathcal{O} V^\beta, A \otimes \mathcal{O} A \otimes \mathcal{O} V^\gamma) \otimes \text{Hom}_\mathcal{O}(V^\alpha, A \otimes \mathcal{O} V^\beta)} \xrightarrow{\text{Hom}_\mathcal{O}(V^\alpha, A \otimes \mathcal{O} V^\gamma)} \text{Hom}_\mathcal{O}(V^\alpha, A \otimes \mathcal{O} V^\gamma).$$

Here $\sigma$ is the transposition $(1,2)$. The left $\mathcal{O}$-structure $\Delta^l$ is the direct sum of the map

$$A^{\alpha\beta} \to \alpha V \otimes V^\alpha \otimes A^{\alpha\beta} : x \mapsto \delta_\alpha \otimes x,$$

where $\{b_i\}$ and $\{b_i^*\}$ are dual bases of $\alpha V$ and $V^\alpha$, and $\delta_\alpha = \sum_i b_i \otimes b_i^*$. The map $\Delta^r$ can be written similarly. The natural map $A \otimes \mathcal{O} A \to A \otimes A$ is identified with

$$V^\alpha \otimes A^{\alpha\beta} \otimes A^{\beta\gamma} \otimes \gamma V \to V^\alpha \otimes A^{\alpha\beta} \otimes \beta V \otimes V^\beta \otimes A^{\beta\gamma} \otimes \gamma V \xrightarrow{x_1 \otimes x_2 \otimes y_1 \otimes y_2 \mapsto x_1 \otimes x_2 \otimes \delta_\beta \otimes y_1 \otimes y_2.}$$

This natural map is written as $[x \otimes y] \mapsto [x] \otimes [y]$. 
2.1.3. Example. Relative DGA associated to $\rho : G \to S(k)$. Let $G$ be a group, $S$ a reductive group over a field $k$ and $\rho : G \to S(k)$ a group homomorphism whose image is Zariski dense. We give an example of relative DGA associated to $\rho$. Let $V_1, V_2$ be algebraic representations of $S$. The following complex is called the canonical cochain complex:

$$\text{Hom}_G(V_1, V_2) : 0 \to \text{Hom}_k(V_1, V_2) \to \text{Hom}_k(k[G] \otimes V_1, V_2)$$

$$\to \text{Hom}_k(k[G] \otimes k[G] \otimes V_1, V_2) \to \cdots$$

where the differential is given by

$$d(\varphi)(g_1 \otimes \cdots \otimes g_{p+1} \otimes v_1)$$

$$= g_1 \varphi(g_2 \otimes \cdots \otimes g_{p+1} \otimes v_1) - \varphi(g_1 g_2 \otimes \cdots \otimes g_{p+1} \otimes v_1)$$

$$+ \varphi(g_1 \otimes g_2 g_3 \otimes \cdots \otimes g_{p+1} \otimes v_1) - \cdots$$

$$\pm \varphi(g_1 \otimes \cdots \otimes g_p g_{p+1} \otimes v_1) \mp \varphi(g_1 \otimes \cdots \otimes g_p \otimes g_{p+1} v_1).$$

Then the extension group $\text{Ext}_i^G(V_1, V_2)$ of $V_1$ and $V_2$ as $G$-modules is the $i$-th cohomology group of $\text{Hom}_G(V_1, V_2)$. We define the multiplication map

$$\text{Hom}_G(V_1, V_2) \otimes \text{Hom}_G(V_2, V_3) \to \text{Hom}_G(V_1, V_3).$$

by

$$(\varphi \otimes \psi) \mapsto \left[ g_1 \otimes \cdots \otimes g_j \otimes g_{j+1} \otimes \cdots \otimes g_{i+j} \otimes v_1$$

$$\mapsto (-1)^{\text{deg}(\varphi) \text{deg}(\psi)} \varphi(g_1 \otimes \cdots \otimes g_j \otimes \varphi(g_{j+1} \otimes \cdots \otimes g_{i+j} \otimes v_1)) \right]$$

Let $O_S$ be the coordinate ring of $S$. The module $O_S$ as a left $S$-module is written as $LO_S$. We set $A = \text{Hom}_G(LO_S, LO_S)$. Using the right $O_S$-comodule structure of $LO_S$, the complex $A$ becomes a bi-$O_S$-comodule. The multiplication map $A \otimes A \to A$ induces a map $\eta : A \otimes_O A \to A$, which is also called the multiplication map of $A$. Then $A$ becomes a relative DGA over $O$. The natural map $\epsilon : A \to \text{Hom}_k(LO, LO)$ is called the relative augmentation of $A$.

2.2. Relative bar complex and simplicial relative bar complex.

2.2.1. Let $A$ be a relative DGA over $O$. We use notations for copies of a reductive group in §1.2.5. A homomorphism of complexes

$$\chi A_y \xrightarrow{\varphi} \text{Hom}_k(\chi O_x^{(1)}, \chi O_y^{(2)})$$

is called a relative augmentation if

1. it is a homomorphism of bi-$O$ comodules, and
2. the multiplication $A \otimes_O A \to A$ is compatible with the multiplication homomorphism of $\text{Hom}_k(O, O)$ which is defined by the composite of

$$\text{Hom}_k(O, O) \otimes_O \text{Hom}_k(O, O)$$

$$\xrightarrow{\varphi} \text{Hom}_k(O, O) \otimes \text{Hom}_k(O, O) \to \text{Hom}_k(O, O),$$
where \( \sigma \) is the transposition defined in (1.2).

Using the relative augmentations, we have two homomorphisms \( \epsilon^l \) and \( \epsilon^r \) by
\[
e^l : x A_y \to x A_y \otimes y O_y^{(2)} \xrightarrow{\zeta} \text{Hom}_k(\mu O_x^{(1)}, y O_y^{(2)}) \otimes y O_y^{(2)} \xrightarrow{\text{pair}} y O_y^{(1)} \sim x O_y^{(1)}
\]
\[
e^r : x A_y \to x O_x^{(1)} \otimes x A_y \xrightarrow{\zeta} x O_x^{(1)} \otimes \text{Hom}_k(x O_x^{(1)}, x O_y^{(2)}) \xrightarrow{\text{ev}} x O_y^{(2)}.
\]
The map \( \epsilon^l \) and \( \epsilon^r \) are called the left and right augmentations.

2.2.2. Since \( S \) is split, we have \( x A_y = \bigoplus_{\alpha, \beta} (x V^\alpha \otimes A^{\alpha \beta} \otimes \beta V_y) \). Then the relative augmentation induces a family of homomorphisms
\[
A^{\alpha \beta} \to \text{Hom}_O(V^\alpha, \text{Hom}_k(O, O) \otimes_O V^\beta)
\]
indexed by \( \alpha, \beta \). Via these homomorphisms, the map \( \eta : A^{\alpha, \beta} \otimes A^{\beta, \gamma} \to A^{\alpha, \gamma} \) is compatible with the map
\[
\text{Hom}_k(V^\alpha, V^\beta) \otimes \text{Hom}_k(V^\beta, V^\gamma) \to \text{Hom}_k(V^\alpha, V^\gamma) : x \otimes y \mapsto y \circ x.
\]
The map \( \epsilon^l \) is written as follows.
\[
\epsilon^l : x V^\alpha \otimes A^{\alpha \beta} \otimes \beta V_y \to x V^\alpha \otimes \alpha V_y \otimes y V^\beta \otimes \beta V_y
\]
\[
\to x V^\alpha \otimes \alpha V_y.
\]

2.2.3. We introduce a relative bar complex \( \text{Bar}(A/O, \epsilon) \). For an integer \( n \geq 1 \), we define \( \text{Bar}_n = \text{Bar}(A/O, \epsilon)_n \) by
\[
\text{Bar}(A/O, \epsilon)_n = A \otimes_O \cdots \otimes_O A.
\]
and \( \text{Bar}_0 = O \). We have a sequence of homomorphisms
\[
(2.5) \quad \text{Bar}(A/O, \epsilon) : \cdots \to \text{Bar}_n \to \cdots \to \text{Bar}_1 \to \text{Bar}_0 \to 0.
\]
given by
\[
x_1 \otimes \cdots \otimes x_n \mapsto \epsilon^r(x_1) \otimes x_2 \otimes \cdots \otimes x_n + \sum_{i=1}^{n-1} (-1)^i x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n
\]
\[
+ (-1)^n x_1 \otimes \cdots \otimes \epsilon^l(x_n).
\]
Here the multiplication map \( \mu : A \otimes_O A \to A \) is denoted by \( x \otimes y \mapsto xy \).

**Proposition 2.2.** The sequence (2.5) is a complex.

**Proof.** This is a consequence of the associativity of the multiplication of \( A \) and the following commutativity arising from the axiom (3).

\[
\begin{array}{cccc}
O \otimes_O A & \otimes_O A & 1 \otimes \mu & O \otimes_O A \\
\epsilon^r \otimes 1 & \downarrow & \downarrow & \epsilon^r \\
O \otimes_O A & \otimes_O A & 1 & \epsilon^l \otimes 1 \\
\otimes_O A & \epsilon^r & O & \otimes_O A \\
A \otimes_O O & \otimes_O A & \epsilon^l & O
\end{array}
\]

\( \square \)
The complex $\text{Bar}(A/\mathcal{O}, \epsilon)$ is defined by the associate simple complex of $\text{Bar}(A/\mathcal{O}, \epsilon)$.

### 2.2.4 We define a relative simplicial bar complex $\text{Bar}^{\text{simp}} = \text{Bar}^{\text{simp}}(A/\mathcal{O}, \epsilon)$. For a sequence of integers $\alpha = (\alpha_0 < \alpha_1 < \cdots < \alpha_n)$, we define a complex $\text{Bar}^{\alpha}_{\text{simp}} = \text{Bar}^{\alpha}_{\text{simp}}(A/\mathcal{O}, \epsilon)$ by $A \otimes \mathcal{O} \cdots \otimes A \otimes \mathcal{O}$.

It will be denoted as $\mathcal{O}^{\alpha_0} A^{\alpha_1} \cdots A^{\alpha_n-1} \otimes A^{\alpha_n} \mathcal{O}$ to distinguish the index $\alpha$. If $\beta = (\#\beta = \#\alpha + 1$ and $\alpha$ is a subsequence of $\beta$, we define a homomorphism of complexes $\text{Bar}^{\beta}_{\text{simp}} \to \text{Bar}^{\alpha}_{\text{simp}}$ as follows.

1. If $\beta = (\alpha_0 < \cdots < \alpha_i < \beta < \alpha_{i+1} \cdots < \alpha_n)$, the map $\partial_{\beta,i+1}$ is given by
   
   \[ y_0 \otimes x_1 \otimes \cdots \otimes x_i \otimes y \otimes x_{i+1} \otimes x_{i+2} \cdots \otimes x_n \otimes y_{n+1}. \]

2. If $\beta = (b < \alpha_0 < \cdots < \alpha_n)$ (resp. $\beta = (\alpha_0 < \cdots < \alpha_n < b$)), the map $\partial_{\beta,0}$ (resp. $\partial_{\beta,n+1}$) is given by
   
   \[ y_0 \otimes x_1 \otimes \cdots \otimes x_n \otimes y_{n+1} \mapsto y_0 \otimes x_1 \otimes \cdots \otimes x_n \otimes y_{n+1}. \]

We define $\text{Bar}^{\text{simp},n}$ by

\[ \text{Bar}^{\text{simp},n} = \bigoplus_{\alpha=(\alpha_0<\cdots<\alpha_n)} \text{Bar}^{\alpha}_{\text{simp}} \]

and the map $\partial_n : \text{Bar}^{\text{simp},n} \to \text{Bar}^{\text{simp},n-1}$ by

\[ \partial_n = \sum_{\alpha_0<\cdots<\alpha_n} \sum_{i=0}^{n} (-1)^i \partial_{\alpha,i}. \]

We can prove the following proposition as in Proposition 2.2.

**Proposition 2.3.** The sequence

\[ \text{Bar}^{\text{simp}} : \cdots \to \text{Bar}^{\text{simp},n} \to \cdots \to \text{Bar}^{\text{simp},1} \to \text{Bar}^{\text{simp},0} \to 0 \]

is a double complex.

**Definition 2.4.** The associate simple complex of $\text{Bar}^{\text{simp}}(A/\mathcal{O}, \epsilon)$ is called the relative simplicial bar complex and denoted as $\text{Bar}^{\text{simp}}(A/\mathcal{O}, \epsilon)$.

### 2.3. Comparison of two complexes. In this subsection, we compare the relative bar complex and the relative simplicial bar complex for a relative DGA $A$ over $\mathcal{O}$. 
2.3.1. We introduce two double complexes \( \widetilde{\text{Bar}} = \widetilde{\text{Bar}}(A/O) \) and \( \widetilde{\text{Bar}}^+ = \widetilde{\text{Bar}}^+(A/O) \) as follows. For \( n \geq 0 \), we define

\[
\widetilde{\text{Bar}}_n = A \otimes \cdots \otimes A_{n+2}
\]

and \( d_n : \widetilde{\text{Bar}}_n \to \widetilde{\text{Bar}}_{n-1} \) for \( n \geq 1 \) by

\[
d_n(x_0 \otimes x_1 \otimes \cdots \otimes x_n \otimes x_{n+1}) = \sum_{i=0}^{n} (-1)^i x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1}.
\]

The free bar complex \( \widetilde{\text{Bar}} \) is defined by

\[
\cdots \to \widetilde{\text{Bar}}_n \to \cdots \to \widetilde{\text{Bar}}_1 \to \widetilde{\text{Bar}}_0 \to 0.
\]

We define the augmentation morphism \( d_0 : \widetilde{\text{Bar}}_0 = A \otimes A \to A \) by the multiplication map \( \mu \). We define an augmented free bar complex by

\[
\widetilde{\text{Bar}}^+ = (\widetilde{\text{Bar}} d_0 \to A).
\]

**Proposition 2.5.** The double complex \( \widetilde{\text{Bar}}^+ \) is exact.

**Proof.** We prove the proposition by constructing a homotopy. We define

\[
\theta_n : \widetilde{\text{Bar}}_n = \underbrace{A \otimes \cdots \otimes A}_{n+2} = \underbrace{O \otimes \cdots \otimes A}_{n+2} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad

Thus we have the proposition. \( \square \)

**Corollary 2.6.** The complex \( \widetilde{\text{Bar}}(A/O) \xrightarrow{d_0} A \) is a free \( A \otimes A^o \)-resolution of \( A \).

2.3.2. We define a simplicial free bar complex \( \widetilde{\text{Bar}}_{\text{simp}} = \widetilde{\text{Bar}}_{\text{simp}}(A/O) \) and an augmented free simplicial bar complex \( \widetilde{\text{Bar}}^+_{\text{simp}} = \widetilde{\text{Bar}}^+_{\text{simp}}(A/O) \).

Let \( \alpha = (\alpha_0 < \cdots < \alpha_n) \) be a sequence of integers. We define \( \text{Bar}_{\text{simp}} \) by

\[
\widetilde{\text{Bar}}_{\text{simp}} \alpha = A_{\alpha_0} \otimes \cdots \otimes A_{\alpha_n}.
\]
Let $0 \leq p \leq n$. We define $\partial_\alpha : \widetilde{\text{Bar}}_{\text{simp}}^\alpha \to \widetilde{\text{Bar}}_{\text{simp}}^\alpha$ by

$$\partial_\alpha(x_0^\alpha_0 \otimes x_1^\alpha_1 \ldots x_n^\alpha_n x_{n+1}) = \sum_{p=0}^{n}(-1)^p(x_0^\alpha_0 \otimes \ldots \otimes x_p^\alpha_p \otimes \ldots \otimes x_n^\alpha_n x_{n+1}).$$

We set

$$\widetilde{\text{Bar}}_{\text{simp},n} = \bigoplus_{\alpha_0 < \ldots < \alpha_n} \widetilde{\text{Bar}}_{\text{simp}}^\alpha.$$

By taking the summation on $\alpha$, we have a sequence

$$\widetilde{\text{Bar}}_{\text{simp}} : \cdots \to \widetilde{\text{Bar}}_{\text{simp}}^n \xrightarrow{d_n} \cdots \xrightarrow{d_2} \widetilde{\text{Bar}}_{\text{simp}}^1 \xrightarrow{d_1} \widetilde{\text{Bar}}_{\text{simp},0} \to 0.$$

We can check that $\widetilde{\text{Bar}}_{\text{simp}}$ becomes a complex. We define the augmentation map $d_0 : \widetilde{\text{Bar}}_{\text{simp}}^0 \to A$ by the sum of the multiplication map $\mu : A \otimes A \to A$. We define the double complex

$$\widetilde{\text{Bar}}_{\text{simp}}^+ = (\widetilde{\text{Bar}}_{\text{simp}} \xrightarrow{d_0} A).$$

**Proposition 2.7.** The double complex $\widetilde{\text{Bar}}_{\text{simp}}^+$ is exact.

**Proof.** To prove the proposition, we define a subcomplex $\widetilde{\text{Bar}}_{N<,\text{simp}}$ by

$$\widetilde{\text{Bar}}_{N<,\text{simp},n} = \bigoplus_{N<\alpha_0 < \ldots < \alpha_n} \widetilde{\text{Bar}}_{\text{simp}}^{\alpha_0 \ldots \alpha_n}.$$

We define a map $\theta_n^\alpha$ by

$$\theta_n^\alpha : \widetilde{\text{Bar}}_{\text{simp}}^{\alpha_0 \ldots \alpha_n} = A \otimes \ldots \otimes A = A \otimes \ldots \otimes A$$

$$i \otimes 1 \otimes \ldots \otimes 1 \to A \otimes \ldots \otimes A = \text{Bar}_{\text{simp}}^{N,\alpha_0 \ldots \alpha_n} \subset \text{Bar}_{N-1<,\text{simp}}^{n+1}.$$

By taking the summation on $\alpha$, we have a map

$$\theta_n : \widetilde{\text{Bar}}_{N<,\text{simp},n} \to \widetilde{\text{Bar}}_{N-1<,\text{simp},n+1}.$$

We define a map $\theta_{-1} : A = \mathcal{O} \otimes A \to A = \mathcal{O} \otimes A$ by $i \otimes 1 = \Delta^l$. Then we have the homotopy relation (2.8). Since

$$\text{Bar}_{\text{simp}}^+ = \lim_{N} \text{Bar}_{N<,\text{simp}}^+,$$

we proved the proposition. □

**Corollary 2.8.** The complex $\text{Bar}_{\text{simp}}(A/\mathcal{O}) \xrightarrow{d_0} A$ is a free $A \otimes A^\circ$-resolution of $A$. 
2.3.3. We define a homomorphism \( \sigma : \widetilde{\text{Bar}}_{\text{simp}}(A/\mathcal{O}) \to \text{Bar}(A/\mathcal{O}) \) of double complexes by
\[
\sigma(x_0 \otimes \cdots \otimes x_{n+1}) = x_0 \otimes \cdots \otimes x_{n+1}.
\]
Then the homomorphism \( \sigma \) commutes with the augmentations \( d_0 \). By Corollary 2.6 and 2.8, the homomorphism
\[
\mathcal{O} \otimes \epsilon^r, A \widetilde{\text{Bar}}_{\text{simp}}(A/\mathcal{O}) \otimes \epsilon^l, A \mathcal{O} \to \mathcal{O} \otimes \epsilon^r, A \text{Bar}(A/\mathcal{O}) \otimes \epsilon^l, A \mathcal{O}
\]
is a quasi-isomorphism. Since
\[
\mathcal{O} \otimes \epsilon^r, A \widetilde{\text{Bar}}_{\text{simp}}(A/\mathcal{O}) \otimes \epsilon^l, A \mathcal{O} \cong \text{Bar}_{\text{simp}}(A/\mathcal{O}, \epsilon)
\]
and
\[
\mathcal{O} \otimes \epsilon^r, A \widetilde{\text{Bar}}(A/\mathcal{O}) \otimes \epsilon^l, A \mathcal{O} \cong \text{Bar}(A/\mathcal{O}, \epsilon),
\]
we have the following theorem

**Theorem 2.9.** The natural map
\[
\sigma : \text{Bar}_{\text{simp}}(A/\mathcal{O}, \epsilon) \to \text{Bar}(A/\mathcal{O}, \epsilon)
\]
is a quasi-isomorphism.

2.4. **Coproducts on bar complexes.** In this subsection, we introduce a coproduct structure on bar complexes.

2.4.1. We define a homomorphism
\[
\underbrace{A \otimes \cdots \otimes \mathcal{O}}_{n+m} A \to (\underbrace{A \otimes \cdots \otimes \mathcal{O}}_n A) \otimes (\underbrace{A \otimes \cdots \otimes \mathcal{O}}_m A)
\]
by applying the natural map \( A \otimes \mathcal{O} A \to A \otimes A \) for \( n \)-th and \( n+1 \)-th tensor components. This map is written as
\[
[x_1 \otimes \cdots \otimes x_{n+m}] \mapsto [x_1 \otimes \cdots \otimes x_n] \otimes [x_{n+1} \otimes \cdots \otimes x_{n+m}]
\]
As for this notation, see also (2.2). The map
\[
A \otimes \cdots \otimes \mathcal{O} A \to \mathcal{O} \otimes (A \otimes \cdots \otimes \mathcal{O} A)
\]
(resp. \( A \otimes \cdots \otimes \mathcal{O} A \to (A \otimes \cdots \otimes \mathcal{O} A) \otimes \mathcal{O} \))

obtained by the left (resp. right) \( \mathcal{O} \) coproduct of \( A \) at the first (resp. the last) factor is written as
\[
x_1 \otimes \cdots \otimes x_{n+m} \mapsto \Delta^l(x_1) \otimes \cdots \otimes x_{n+m}
\]
(resp. \( x_1 \otimes \cdots \otimes x_{n+m} \mapsto x_1 \otimes \cdots \otimes \Delta^r(x_{n+m}) \)).

We introduce a coalgebra structure
\[
(2.9) \quad \Delta_B : \text{Bar}(A/\mathcal{O}, \epsilon) \to \text{Bar}(A/\mathcal{O}, \epsilon) \otimes \text{Bar}(A/\mathcal{O}, \epsilon)
\]
by
\[
\Delta_B([x_1 \otimes \cdots \otimes x_n]) = \Delta^l(x_1) \otimes \cdots \otimes x_n \\
+ \sum_{i=1}^{n-1} [x_1 \otimes \cdots \otimes x_i] \otimes [x_{i+1} \otimes \cdots \otimes x_n] \\
+ x_1 \otimes \cdots \otimes \Delta^r(x_n).
\]

The right hand side of (2.9) becomes a double complex using the tensor product of complexes.

**Proposition 2.10.** This homomorphism \(\Delta_B\) is a homomorphism of double complexes.

**Proof.** Since the differential of \(\text{Bar}(A/\mathcal{O}, \epsilon)\) is defined by the multiplications and left and right augmentations, it commute with that of \(\text{Bar}(A/\mathcal{O}, \epsilon) \otimes \text{Bar}(A/\mathcal{O}, \epsilon)\) by the following commutative diagrams.

This commutativity follows from the identification (2.4).

\[\square\]

2.4.2. We define a coproduct

\[\Delta^\alpha_{\text{simp}} : \text{Bar}_{\text{simp}}^\alpha \to \text{Bar}_{\text{simp}}^\alpha \otimes \text{Bar}_{\text{simp}}^\alpha\]

by the formula

\[
\Delta^\alpha_{\text{simp}}([y_0 \otimes x_1 \otimes \cdots \otimes x_n \otimes y_{n+1}]) \\
= \Delta^l(y_0 \otimes x_1) \otimes \cdots \otimes x_n \otimes y_{n+1} \\
+ \sum_{i=1}^{n-1} [y_0 \otimes x_1 \otimes \cdots \otimes x_i \otimes \Delta^r(x_i)] \otimes \Delta^l(x_{i+1}) \otimes \cdots \otimes x_n \otimes y_{n+1} \\
+ y_0 \otimes x_1 \otimes \cdots \otimes x_n \otimes \Delta^r(x_n \otimes y_{n+1}).
\]

Here \(\Delta^l(y_0 \otimes x_1)\) and \(\Delta^r(x_n \otimes y_{n+1})\) are considered as maps

\[
\mathcal{O} \otimes \mathcal{O} A = A \to \mathcal{O} \otimes A = (\mathcal{O} \otimes \mathcal{O}) \otimes (\mathcal{O} \otimes \mathcal{O} A)
\]

\[
A \otimes \mathcal{O} A = A \to A \otimes \mathcal{O} = (A \otimes \mathcal{O} A) \otimes (\mathcal{O} \otimes \mathcal{O}).
\]
Proposition 2.11. (1) The sum of the map $\Delta_{\text{simp}} = \sum_{\alpha} \Delta_{\text{simp}}^\alpha$ becomes a homomorphism of double complexes

$$\text{Bar}_{\text{simp}}(A/O, \epsilon) \rightarrow \text{Bar}_{\text{simp}}(A/O, \epsilon) \otimes \text{Bar}_{\text{simp}}(A/O, \epsilon).$$

(2) The coproduct structure of $\text{Bar}(A/O, \epsilon)$ and $\text{Bar}_{\text{simp}}(A/O, \epsilon)$ are compatible with the quasi-isomorphism $\sigma$ defined in [2.3.3].

The proof is similar to Proposition 2.10 and we omit it.

2.4.3. The homomorphism $O \rightarrow k$ and the sum of $O \otimes_O O \rightarrow k$ defines counit $\text{Bar}(A/O, \epsilon) \rightarrow k$ and $\text{Bar}_{\text{simp}}(A/O, \epsilon) \rightarrow k$.

3. Relative DGA and DG category

Let $S$ be a split reductive group, $O$ the coordinate ring of $S$, $A$ a relative DGA over $O$, and $\epsilon$ a relative augmentation of $A$. In this section, we introduce the DG category associated to a relative DGA $A$ over $O$. Moreover, we prove that DG category $KC(A/O)$ of DG complexes in $\mathcal{C}(A/O)$ is homotopy equivalent to that of $\text{Bar}(A/O, \epsilon)$-comodules (Theorem 3.14).

3.1. DG category associated to a relative DGA.

3.1.1. We define a DG category $\mathcal{C}(A/O)$ for a relative DGA $A$ over the Hopf algebra $O$. An object $V = V^\bullet$ of $\mathcal{C}(A/O)$ is a complex $V = V^\bullet$ of finite dimensional left $O$-comodules such that $V^\bullet \rightarrow \sigma \otimes V^\bullet$ is a homomorphism of complexes. For objects $V_1$ and $V_2$ in $\mathcal{C}(A/O)$, we define a complex of homomorphisms $\text{Hom}_{\mathcal{C}(A/O)}(V_1, V_2)$ by

$$\text{Hom}_{\mathcal{C}(A/O)}(V_1, V_2) = \text{Hom}_O(V_1, A \otimes_O V_2).$$

Then the composite of the complexes of homomorphisms is defined by the composite of the following maps:

$$(3.1) \quad \text{Hom}_O(V_2, A \otimes_O V_3) \otimes \text{Hom}_O(V_1, A \otimes_O V_2) \rightarrow \text{Hom}_O(A \otimes_O V_2, A \otimes_O A \otimes_O V_3) \otimes \text{Hom}_O(V_1, A \otimes_O V_2) \rightarrow \text{Hom}_O(V_1, A \otimes_O A \otimes_O V_3) \xrightarrow{\sigma} \text{Hom}_{\mathcal{C}(A/O)}(V_1, V_2).$$

Then we have $A \simeq \text{Hom}_{\mathcal{C}(A/O)}(LO, LO)$ and the multiplication map is equal to

$$(3.2) \quad A \otimes_O A \rightarrow A \otimes A \xrightarrow{\sigma} A \otimes A \simeq \text{Hom}_{\mathcal{C}(A/O)}(LO, LO) \otimes \text{Hom}_{\mathcal{C}(A/O)}(LO, LO) \rightarrow \text{Hom}_{\mathcal{C}(A/O)}(LO, LO).$$

Here $\sigma$ denotes the transposition $(1, 2)$.

The DG category of finite DG complexes in $\mathcal{C}(A/O)$ is denoted as $KC(A/O)$. An object in $KC(A/O)$ is written as $(V^i, d_{ij})$ where

$$d_{ij} \in \text{Hom}_{\mathcal{C}(A/O)}^1(V^j e^{-j}, V^i e^{-i}) \xrightarrow{(\star)} \text{Hom}_{\mathcal{C}(A/O)}^{1+j-i}(V^j, V^i).$$
Here the map $(\ast)$ is given in (1.1). (See also [T], §2.2.) By the definition of DG complex, we have

\[
\partial d_{ik} + \sum_{i<j<k} d_{ij} \circ d_{jk} = 0.
\]

3.1.2. Let $C_1$ and $C_2$ be DG categories. A pair $F = (ob(F), mor(F))$ of map $ob(C_1) \to ob(C_2)$ and $mor(C_1) \to mor(C_2)$ is called a DG functor, if it is compatible with the composite, and preserves identity morphisms.

3.1.3. If $A_{pre} = Hom_k(\mathcal{O}, \mathcal{O})$, then $C(A_{pre}/\mathcal{O})$ is nothing but the full sub-DG category of complex of finite dimensional $k$-vector spaces consisting of $\mathcal{O}$-comodules. For an object $M, N \in C(A_{pre}/\mathcal{O})$, we have

\[
Hom_{C(A_{pre}/\mathcal{O})}(M, N) = Hom_{\mathcal{O}}(M, A_{pre} \otimes_{\mathcal{O}} N) = Hom_k(M, N).
\]

Let $\epsilon : A \to Hom_k(LO, LO)$ be a relative augmentation. We define a DG functor $\rho_\epsilon : C(A/\mathcal{O}) \to C(k)$ by forgetting left $\mathcal{O}$-comodule structure for objects. For a morphism $\varphi \in Hom_{C(A/\mathcal{O})}(M, N)$, we define $\rho_\epsilon(\varphi)$ by the image of $\varphi$ under the map

\[
Hom_{C(A/\mathcal{O})}(M, N) \to Hom_{C(A_{pre}/\mathcal{O})}(M, N) = Hom_{Vect_k}(M, N).
\]

3.1.4. DG category ($B - com$). Let $B$ be a counitary and coassociative differential graded coalgebra over $k$. The comultiplication and the counit is written as $\Delta_B$ and $\epsilon$. A complex $M$ with a homomorphism $M \to B \otimes M$ of complexes is called a $B$-comodule if it is coassociative and counitary. For two $B$-comodules $M$ and $N$, we define the double complex $RHom_B(M, N)$ by

\[
Hom_{\text{Vect}_k}(M, N) \to Hom_{\text{Vect}_k}(M, B \otimes N) \to Hom_{\text{Vect}_k}(M, B \otimes B \otimes N) \to \cdots
\]

where the differential is given by

\[
d\varphi = (1_B \otimes 1_B \otimes \cdots \otimes 1_B \otimes \Delta_N) \circ \varphi - (1_B \otimes 1_B \otimes \cdots \otimes \Delta_B \otimes 1_N) \circ \varphi + \cdots + (-1)^n(\Delta_B \otimes 1_B \otimes \cdots \otimes 1_B \otimes 1_N) \circ \varphi + (-1)^{n+1}(1_B \otimes \varphi) \circ \Delta_M
\]

for an element $\varphi \in Hom_{\text{Vect}_k}(M, B^\otimes n \otimes N)$. We introduce a composite map

\[
c : RHom_B(M, N) \otimes RHom_B(L, M) \to RHom_B(L, N).
\]

For elements $\varphi$ and $\psi$ in $Hom_{\text{Vect}_k}(M, B^\otimes n \otimes N)$ and $Hom_{\text{Vect}_k}(L, B^\otimes m \otimes M)$, we define $c(\varphi \otimes \psi)$ in $Hom_{\text{Vect}_k}(L, B^\otimes (n+m) \otimes N)$ by

\[
c(\varphi \otimes \psi) = (1_B^\otimes m \otimes \varphi) \circ \psi.
\]

We can check that this composite is associative. The associate simple complex of $RHom_B(M, N)$ is written as $RHom_B(M, N)$. 

Definition 3.1. We define a DG category \((B - \text{com})\) for a differential graded coalgebra \(B\) as follows. The class of objects of \((B - \text{com})\) consists of finite dimensional \(B\)-comodules, and for \(B\)-comodules \(M, N\), the complex of homomorphisms is defined by \(R\text{Hom}_B(M, N)\). The composite of homomorphisms is defined by the homomorphism \([3, 4]\).

Proposition 3.2. Let \(M\) be a cofree \(B\)-comodule. Then the functor \(N \mapsto R\text{Hom}_B(N, M)\) is exact.

Proof. We consider the stupid filtration on \(R\text{Hom}_B(N, M)\) and reduce to the exactness of \(\text{Hom}_{\text{KVect}_k}(N, B \otimes \cdots \otimes B \otimes M)\).

3.1.5. Let \(N\) be a complex of \(B\)-comodules. We define a standard cofree resolution \(F(N)\) of \(N\) by

\[
F(N) : B \otimes N \rightarrow B \otimes B \otimes N \rightarrow B \otimes B \otimes B \otimes N \rightarrow \cdots
\]

where

\[
d(b_n \otimes \cdots b_0 \otimes n) = \sum_{i=0}^{n} (-1)^{i+1} b_n \otimes \cdots \otimes \Delta_B(b_i) \otimes \cdots \otimes b_0 \otimes n + b_n \otimes \cdots b_0 \otimes \Delta_N(n)
\]

Then the associate simple complex \(F(N)\) becomes a complex of \(B\)-comodules.

Let \(N_1, N_2\) be \(B\)-comodules and \(\varphi : N_1 \rightarrow N_2\) be a \(B\)-homomorphism, i.e. the following diagram commutes:

\[
\begin{align*}
N_1 & \xrightarrow{\varphi} N_2 \\
\Delta_{N_1} \downarrow & \downarrow \Delta_{N_2} \\
B \otimes N_1 & \xrightarrow{1 \otimes \varphi} B \otimes N_2
\end{align*}
\]

Note that we do not assume that \(\varphi\) commutes with the differentials. The set of \(B\)-homomorphism from \(N_1\) to \(N_2\) is denoted by \(\text{Hom}_B(M, N)\). This space becomes a sub complex of \(\text{Hom}_{\text{KVect}_k}(M, N)\), since the differentials and the comodule structures on \(N_1\) and \(N_2\) commute.

Lemma 3.3. (1) Let \(M\) be a \(B\)-comodule, and \(N\) a complex of \(k\)-vector spaces. Then \(B \otimes N\) becomes a \(B\)-comodule with the product complex structure. By attaching \(\varphi \in \text{Hom}_B(M, B \otimes N)\) to the element \(\bar{\varphi}\) defined by

\[
(\bar{\varphi} : M \xrightarrow{\varphi} B \otimes N \xrightarrow{\otimes 1} N) \in \text{Hom}_k(M, N),
\]

we have an isomorphism \(\text{Hom}_B(M, B \otimes N) \rightarrow \text{Hom}_k(M, N)\). The inverse map is given by

\[
\varphi : M \xrightarrow{\Delta_M} B \otimes M \xrightarrow{1 \otimes \bar{\varphi}} B \otimes N.
\]

(2) Using the isomorphism of (1), we have a natural isomorphism of complexes

\[
R\text{Hom}_B(M, N) \simeq \text{Hom}_B(M, F(N)).
\]
We define a homomorphism of complexes
\[ \alpha : R\text{Hom}_B(M, N) \to \text{Hom}_B(F(M), F(N)). \]

Let \( \varphi \) be an element of \( R\text{Hom}_B(M, N)^p = \text{Hom}_k(M, B \otimes \cdots \otimes B \otimes N). \) Then the map
\[
(3.5) \quad \alpha^q(\varphi) : B \otimes \cdots \otimes B \otimes M \xrightarrow{1 \otimes \varphi} B \otimes \cdots \otimes B \otimes B \otimes \cdots \otimes B \otimes N
\]
is an element of \( \text{Hom}_B(F(M)^q, F(N)^{p+q}) \) for \( q \geq 0. \) By taking the sum of \( \alpha^q(\varphi), \) we have a map \( \alpha(\varphi) \in \text{Hom}_B(F(M), F(N)). \) By Proposition 3.2, we have the following lemma.

**Lemma 3.4.**

1. The homomorphism \( \alpha \) is a homomorphism of complexes and a quasi-isomorphism.

2. Let
\[
\mu : \text{Hom}_B(F(M), F(N)) \otimes \text{Hom}_B(F(L), F(M)) \to \text{Hom}_B(F(L), F(N))
\]
be a homomorphism of complexes defined by the composite. The map \( \mu \) commute with the composite map \( c \) in (3.4) via the homomorphism \( \alpha. \)

### 3.2. Bijection on objects.

#### 3.2.1. Construction of a bijection \( \varphi. \)

In this subsection, we construct a map
\[
\varphi : \text{ob}(KC(A/\mathcal{O})) \to \text{ob}(\text{Bar}_{\text{simp}}(A/\mathcal{O}, e) - \text{com}).
\]

Let \((V^i, d_{ij})\) be an object of \( KC(A/\mathcal{O}). \) For an index \( \alpha = (\alpha_0 < \cdots < \alpha_n), \) we define the following degree \( n \) left \( \mathcal{O}\)-homomorphism
\[
\Delta^\alpha : V^{\alpha_0}e^{-\alpha_0} \to \text{Bar}_{\text{simp}}^{\alpha_0, \ldots, \alpha_n} \otimes V^{\alpha_n}e^{-\alpha_n},
\]
where
\[
\text{Bar}_{\text{simp}}^{\alpha_0, \ldots, \alpha_n} = \mathcal{O}^{\alpha_0} \otimes \mathcal{O}^{\alpha_1} \otimes \cdots \otimes \mathcal{O}^{\alpha_{n-1}} \otimes \mathcal{O}^{\alpha_n} \otimes \mathcal{O}.
\]

For \( n = 0, \) the map
\[
\Delta^{\alpha_0} : V^{\alpha_0}e^{-\alpha_0} \to \mathcal{O} \otimes V^{\alpha_0}e^{-\alpha_0} = \text{Bar}_{\text{simp}}^{\alpha_0} \otimes V^{\alpha_0}e^{-\alpha_0}
\]
is defined as the coproduct structure on \( V^{\alpha_0}e^{-\alpha_0}. \) We define \( \Delta^{\alpha_0, \ldots, \alpha_n} \) for \( n \geq 1 \) by the induction on \( n. \) For \( n = 1, \) by the definition of \( KC(A/\mathcal{O}), \) we have a left \( \mathcal{O}\)-homomorphism \( d_{\alpha_1, \alpha_0} : V^{\alpha_0}e^{-\alpha_0} \to A \otimes V^{\alpha_1}e^{-\alpha_1} \) of degree one. By composing the following \( \mathcal{O}\)-homomorphisms
\[
\Delta^{\alpha_0, \alpha_1} : V^{\alpha_0}e^{-\alpha_0} \xrightarrow{d_{\alpha_1, \alpha_0}} A \otimes V^{\alpha_1}e^{-\alpha_1}
\]
\[
\to A \otimes V^{\alpha_1}e^{-\alpha_1} = (\mathcal{O}^{\alpha_0} \otimes A^{\alpha_1} \otimes \mathcal{O}) \otimes V^{\alpha_1}e^{-\alpha_1},
\]
we have the required homomorphism. Suppose that a left \( \mathcal{O}\)-homomorphism
\[
\Delta^{\alpha_1, \ldots, \alpha_n} : V^{\alpha_1}e^{\alpha_1} \to \text{Bar}_{\text{simp}}^{\alpha_1, \ldots, \alpha_n} \otimes V^{\alpha_n}e^{-\alpha_n}
\]
of degree \((n-1)\) is given. Using the inductive definition of \(\Delta^{\alpha_1,\ldots,\alpha_n}\), we have the following composite map
\[
V^{\alpha_0}e^{-\alpha_0} \xrightarrow{d_{\alpha_1,\alpha_0}} A \otimes_\mathcal{O} V^{\alpha_1}e^{-\alpha_1} \xrightarrow{\otimes \Delta^{\alpha_1,\ldots,\alpha_n}} A \otimes_\mathcal{O} \text{Bar}_{\text{simp}}^{\alpha_1,\ldots,\alpha_n} \otimes V^{\alpha_n}e^{-\alpha_n} = \text{Bar}_{\text{simp}}^{\alpha_1,\ldots,\alpha_n} \otimes V^{\alpha_n}e^{-\alpha_n},
\]
and we have a required degree \(n\)-homomorphism. The map \(\Delta^\alpha\) is written as
\[
\Delta^\alpha = (1 \otimes d_{\alpha_n,\alpha_{n-1}}) \circ (1 \otimes d_{\alpha_{n-1},\alpha_{n-2}}) \circ \cdots \circ d_{\alpha_1,\alpha_0}.
\]
We set \(V = \bigoplus_i V^i e^{-i}\). Then it is a finite dimensional vector space. We define a homogeneous map \(\Delta_V : V \rightarrow \text{Bar}_{\text{simp}}(A/\mathcal{O}, \epsilon) \otimes V\) of degree zero by
\[
\Delta_V = \sum_{0 \leq n} \sum_{|\alpha|=n} (1 \otimes t^n \otimes 1) \circ \Delta^\alpha,
\]
where
\[
1 \otimes t^n \otimes 1 : \text{Bar}_{\text{simp}}^{\alpha_1,\ldots,\alpha_n} \otimes V^{\alpha_n}e^{-\alpha_n} \rightarrow \text{Bar}_{\text{simp}}^{\alpha_1,\ldots,\alpha_n} e^n \otimes V^{\alpha_n}e^{-\alpha_n}.
\]
We define a differential \(\delta\) on the vector space \(V\). We set \(\delta_i = \delta_i \otimes 1\) on \(V^i e^{-i}\), where \(\delta_i\) is the differential on \(V^i\), and \(\delta_{ji} : V^i \rightarrow V^j\) is the image of \(d_{ji}\) under the map
\[
\overline{\text{Hom}}^1(A/\mathcal{O})(V^i e^{-i}, A \otimes_\mathcal{O} V^j e^{-j}) \rightarrow \overline{\text{Hom}}^1(A/\mathcal{O})(V^i e^{-i}, A_{\text{pre}} \otimes_\mathcal{O} V^j e^{-j}) = \overline{\text{Hom}}^1_{\text{Vect}}(V^i e^{-i}, V^j e^{-j})
\]
induced by the relative augmentation. We define \(\delta_V\) on \(V\) by \(\delta_V = \sum_{i < j} \delta_{ji}\).

**Lemma 3.5.** The map \(\delta_V\) defines a differential on \(V\).

**Proof.** The map \(\delta_{ji} (i < j)\) is the composite of the following map.
\[
V^i e^{-i} \rightarrow A \otimes_\mathcal{O} V^j e^{-j} \xrightarrow{\epsilon^r \otimes 1} \mathcal{O} \otimes_\mathcal{O} V^j e^{-j} = V^j e^{-j}.
\]
Let \(i : V^i e^{-i} \rightarrow V\) and \(p_j : V \rightarrow V^j e^{-j}\) be the inclusion and the projection. It is enough to show that \(p_j \delta_V^2 i = 0\) for \(i \leq j\). If \(i = j\), then the equality holds, because \(\delta_{ii}\) is a differential of \(V^i\). By the commutative diagram
\[
\begin{array}{c}
V^i e^{-i} \xrightarrow{\delta_{ki}} A \otimes_\mathcal{O} V^k e^{-k} \xrightarrow{1 \otimes d_{jk}} A \otimes_\mathcal{O} A \otimes_\mathcal{O} V^j e^{-j} \xrightarrow{\mu \otimes 1} A \otimes_\mathcal{O} V^j e^{-j} \\
e^{-k} \xrightarrow{\epsilon^r \otimes 1} A \otimes_\mathcal{O} V^j e^{-j} \xrightarrow{\epsilon^r \otimes 1} V^j e^{-j}
\end{array}
\]
we have
\[
(e^r \otimes 1)(d_{jk} \circ d_{ki}) = (e^r \otimes 1)(\mu \otimes 1)(1 \otimes d_{jk})d_{ki} = \delta_{jk} \delta_{ki},
\]
\[
(e^r \otimes 1)(\delta d_{ji}) = \delta_{jj} \delta_{ji} + \delta_{ji} \delta_{ii}
\]
for \(i < k < j\). Therefore by the condition \(\text{33}\) for DG complex, we have the lemma. \(\square\)

**Proposition 3.6.** The homomorphism \(\Delta_V\) defines a \(\text{Bar}_{\text{simp}}(A/\mathcal{O}, \epsilon)\)-comodule structure on \(V\).
Proof. We can easily check the coassociativity and the counitarity. We show that the homomorphisms \( \Delta_V \) is a homomorphism of complexes, in other words, the homomorphism \( \Delta_V \) commutes with the differential \( \delta_V \) on \( V \) and differential \( d_{\text{Bar}_{\text{simp}}}, \otimes 1 + 1 \otimes \delta_V \) on \( \text{Bar}_{\text{simp}} \otimes V \).

The outer differential defined in (2.6) for \( \text{Bar}_{\text{simp}} \) is denoted by \( \partial_{\alpha,i} \), and the inner differential \( \text{Bar}^\alpha_{\text{simp}} \to \text{Bar}^\alpha_{\text{simp}} \) is denoted by \( d_{in} \). We compute

\[
(d_{\text{Bar}_{\text{simp}}} \otimes 1 + 1 \otimes \delta_V) \Delta_V - \Delta_V \circ \delta_V
\]
on the component from \( V^\alpha e^{-\alpha} \) to \( \text{Bar}^\alpha_{\text{simp}} e^n \otimes V^\beta e^{-\beta} \) for \( \alpha \leq \alpha_0 < \cdots < \alpha_n \leq \beta \). It is the sum of the following terms:

1. \( - \partial_{\gamma,i} (1 \otimes t^n \otimes 1) \Delta^\gamma \delta_{\gamma,i} \alpha \) if \( \alpha \leq \alpha_0 < \cdots < \alpha_n \leq \beta \).
2. \( - \partial_{\gamma,i} (1 \otimes t^n \otimes 1) \Delta^\gamma \delta_{\gamma,i} \alpha \) if \( \alpha_0 = \beta < \alpha_n < \alpha_0 \).
3. \( - \partial_{\gamma,i} (1 \otimes t^n \otimes 1) \Delta^\gamma \delta_{\gamma,i} \alpha \) if \( \alpha_0 \) and \( \alpha_n < \beta \).
4. \( - \partial_{\gamma,i} (1 \otimes t^n \otimes 1) \Delta^\gamma \delta_{\gamma,i} \alpha \) if \( \alpha = \alpha_0 < \alpha_n = \beta \).
5. \( - \partial_{\gamma,i} (1 \otimes t^n \otimes 1) \Delta^\gamma \delta_{\gamma,i} \alpha \) if \( \alpha = \alpha_0 < \alpha_n = \beta \).
6. \( - \partial_{\gamma,i} (1 \otimes t^n \otimes 1) \Delta^\gamma \delta_{\gamma,i} \alpha \) if \( \alpha = \alpha_0 < \alpha_n = \beta \).
7. \( - \partial_{\gamma,i} (1 \otimes t^n \otimes 1) \Delta^\gamma \delta_{\gamma,i} \alpha \) if \( \alpha = \alpha_0 < \alpha_n < \beta \).
8. \( - \partial_{\gamma,i} (1 \otimes t^n \otimes 1) \Delta^\gamma \delta_{\gamma,i} \alpha \) if \( \alpha = \alpha_0 < \alpha_n < \beta \).

We compute (3.6) for all components.

(a) The case \( \alpha < \alpha_0 \) and \( \alpha_n = \beta \). The terms (1) and (7) contribute. They cancel by the definition of \( \partial_{\gamma,0} \) and \( \delta_{\gamma,0} \).

(b) The case \( \alpha = \alpha_0 \) and \( \alpha_n < \beta \). The terms (3) and (8) contribute. They cancel by the following commutative diagrams.

\[
A \otimes V^{\alpha_n} e^{-\alpha_n} \xrightarrow{1 \otimes \partial_{\gamma,i}} A \otimes V^\beta e^{-\beta} \quad \xrightarrow{(A \otimes 0 A) \otimes V^\beta e^{-\beta}} \quad (1 \otimes e^i) \otimes 1 \downarrow \quad A \otimes V^\beta e^{-\beta}.
\]

(c) The case \( \alpha = \alpha_0 \) and \( \alpha_n = \beta \). The summation of the term in (3) for a fixed \( i \) is equal to the summation \( (-1)^i \) times the composites of

\[
V^{\alpha_0} e^{-\alpha_0} \to \text{Bar}^\alpha_{\text{simp}} \otimes V^{\alpha_1} e^{-\alpha_1} \to \text{Bar}^\alpha_{\text{simp}} \otimes V^{\alpha_1} e^{-\alpha_1} \to \text{Bar}^\alpha_{\text{simp}} \otimes V^{\alpha_1} e^{-\alpha_1} \to \text{Bar}^\alpha_{\text{simp}} \otimes V^{\alpha_1} e^{-\alpha_1} \to \text{Bar}^\alpha_{\text{simp}} \otimes V^{\alpha_1} e^{-\alpha_1} \to \text{Bar}^\alpha_{\text{simp}} \otimes V^{\alpha_1} e^{-\alpha_1} \to \text{Bar}^\alpha_{\text{simp}} \otimes V^{\alpha_1} e^{-\alpha_1} \to \text{Bar}^\alpha_{\text{simp}} \otimes V^{\alpha_1} e^{-\alpha_1} \to \text{Bar}^\alpha_{\text{simp}} \otimes V^{\alpha_1} e^{-\alpha_1} \to \text{Bar}^\alpha_{\text{simp}} \otimes V^{\alpha_1} e^{-\alpha_1} \to \text{Bar}^\alpha_{\text{simp}} \otimes V^{\alpha_1} e^{-\alpha_1}
\]

over \( \alpha_{i-1} < \gamma_i < \alpha_i \). By the relation (3.3), the morphism (3) is \(-1\)-times the summation of
Thus

(A) the summation for (c-3) cancels with the term (5),

(B) the term (c-1-(i+1)) and (c-2-i) cancel for $i = 1, \ldots, n - 1$,

(C) the term (c-1-1) and (2) cancel, and

(D) the term (c-2-n) and (4) cancel.

Therefore (3.6) is equal to zero and $\Delta_V$ is a homomorphism of complexes. □

**Definition 3.7.** By associating $(V, \Delta_V)$ to $(V^i, d_{ij})$, we have a map $\varphi : ob(KC(A/O_S)) \to ob(Bar_{simp} - com)$.

3.2.2. Construction of the functor $\psi$. We construct the inverse $\psi : ob(Bar_{simp}(A/O_S) - com) \to ob(KC(A/O_S))$ of the map $\varphi$. Let $V$ be a bounded complex of finite dimensional vector spaces and

$$\Delta_V : V \to Bar_{simp} \otimes V$$

be a $Bar_{simp}$-comodule structure on $V$. Let

$$\pi_i : Bar_{simp} \to O \otimes O,$$

$$\pi_{ij} : Bar_{simp} \to O \otimes O A \otimes O e 1 \otimes t^{-1} \otimes O \otimes O A \otimes O$$

be the projection to the $Bar_{simp}^{(i)}$ and $Bar_{simp}^{(ij)}$ components. Let $\Delta_i$ be the composite

$$V \xrightarrow{\Delta_V} Bar_{simp} \otimes V \xrightarrow{1 \otimes \pi_i} (O \otimes O S) \otimes V = O \otimes V$$

and $pr_i = (e \otimes 1_V) \Delta_i : V \to V$. Let $d_{ji}$ be the composite

$$V \to Bar_{simp} \otimes V \xrightarrow{\pi_{ij} \otimes 1} (O \otimes O S A \otimes O S O S) \otimes V = A \otimes V.$$

**Proposition 3.8.** (1) The maps $pr_i$ ($i \in \mathbb{Z}$) define complete orthogonal idempotents of $V$. 
We have
\[ \Delta_i pr_i = \Delta_i = (1 \otimes pr_i)\Delta_i, \quad d_{ij} pr_i = d_{ji} = (1 \otimes pr_j)d_{ji} \]
As a consequence \( V^i \) becomes an \( \mathcal{O} \)-comodule.

(3) The differential on \( V \) commute with the projection \( pr_i \) and the coaction of \( \mathcal{O} \). As a consequence, \( V^i \) becomes an object in \( \mathcal{C}(A/\mathcal{O}) \).

(4) The homomorphism \( d_{ji} \) defines an element in \( \text{Hom}_{\mathcal{O}}(V^i, A \otimes_{\mathcal{O}} V^j) = \text{Hom}_{\mathcal{C}(A/\mathcal{O})}(V^i, V^j) \).

**Proof.** (1) By the counitarity of the coaction of \( \text{Bar}_{\text{simp}} \) on \( V \), the composite map
\[ V \xrightarrow{\Delta_V} \text{Bar}_{\text{simp}} \otimes V \xrightarrow{\psi \otimes 1} V \]
is the identity map on \( V \). Therefore we have \( \sum_i pr_i = 1_V \). The statement (2),(3) follows from the coassociativity.

By the compatibility of \( \Delta_V \) for the differentials of \( V \) and \( \text{Bar}_{\text{simp}} \otimes V \), we have the following proposition.

**Proposition 3.9.**

1. The map \( d_{ij} \) defines a DG complex in \( \mathcal{C}(A/\mathcal{O}) \)
2. The maps \( \varphi \) and \( \psi \) are inverse to each other.

We define the functor \( \psi \) by associating the \( \text{Bar} \)-comodule \( (V, \Delta_V) \) to the system \( \{V^i, d_{ij}\} \) in \( \mathcal{K}(A/\mathcal{O}_S) \).

**Remark 3.10.** Without the finite dimensionality of the \( \text{Bar}_{\text{simp}} \)-comodule \( V \), we can define the above decomposition \( V = \oplus_i V^i \) and the map \( \Delta_i \) by the counitarity and the coassociativity.

### 3.3. Homotopy equivalence of \( (\mathcal{K}(A/\mathcal{O})) \) and \( (\text{Bar}_{\text{simp}} - \text{com}) \).

**3.3.1.** We set \( B = \text{Bar}_{\text{simp}}(A/\mathcal{O}, \epsilon) \). Let \( N_1 \) and \( N_2 \) be \( B \)-comodules. We set \( \psi(N_1) = (N_1^i, d_{ji}), \psi(N_2) = (N_2^i, d_{ji}) \) and let \( i_i : N_1^i \to N_1 \) and \( p_j : N_2 \to N_2^j \) be the natural inclusion and the projection. Via the bijections \( \varphi \) and \( \psi \), we identify the class of objects in \( \mathcal{K}(A/\mathcal{O}) \) and that of \( (B - \text{com}) \) defined in §3.2.

Let \( f \) be an element of \( \text{Hom}_B(N_1, N_2) \). Since the coaction of \( B \) is compatible with the map \( f \), we have \( f(N_1^i) \subseteq N_2^j \) and the restriction \( f^i \) of \( f \) to \( N_1^i \) is a \( \mathcal{O} \)-homomorphism. Let \( \beta(f)^i \in \text{Hom}_\mathcal{O}(N_1^i, A \otimes_{\mathcal{O}} N_2^j) \) be the following composite homomorphism

\[ N_1^i \xrightarrow{f^i} N_2^j = \mathcal{O} \otimes_{\mathcal{O}} N_2^j \to A \otimes_{\mathcal{O}} N_2^j. \]

Then we have \( \beta(f) \in \text{Hom}_{\mathcal{K}(A/\mathcal{O})}(\psi(N_1), \psi(N_2)) \).

**Definition 3.11.** The category of DG complexes in \( \mathcal{C} \) without assuming the finiteness of \( \{i \mid V^i \neq 0\} \) for \( \{V^i, d_{ij}\} \) is denoted as \( \mathcal{K}' \). In this category, the morphism \( \{\varphi_{ij}\} \) is assumed to be bounded from below, i.e. there exists \( m \) such that \( \varphi_{i,j} = 0 \) for \( j < i + m \), so that the composite of two morphisms are well defined.
Lemma 3.12.  

1. The homomorphism
\[ \beta : \text{Hom}_B(N_1, N_2) \to \text{Hom}_{KC(A/O)}(\psi(N_1), \psi(N_2)) \]
commutes with the differentials and is compatible with the composites.

2. Moreover if \( N_2 \) is a cofree \( B \)-comodule, the map \( \beta \) is a quasi-isomorphism.

3.3.2. Let \( M \) be an object of \( KC(A/O) \). Then the free resolution \( F(\varphi(M)) \) defined in \( (3.7) \) of \( \varphi(M) \) is a cofree \( B \)-comodule. The object \( \psi(F(\varphi(M))) \in KC(A/O) \) corresponding to \( F(\varphi(M)) \) is denoted by \( F(M) \). By the above lemma, we have the following corollary.

Lemma 3.13. Let \( N_1, N_2 \) be \( B \)-comodules.

1. Then the homomorphism
\[ (3.7) \quad \beta : \text{Hom}_B(F(N_1), F(N_2)) \to \text{Hom}_{KC'(A/O)}(\psi(F(N_1)), \psi(F(N_2))) \]
is a quasi-isomorphism and compatible with the composites.

2. The natural homomorphisms of complexes
\[ (3.8) \quad \text{Hom}_{KC'(A/O)}(N_1, N_2) \to \text{Hom}_{KC'(A/O)}(N_1, F(N_2)) \]
\[ (3.9) \quad \text{Hom}_{KC'(A/O)}(F(N_1), F(N_2)) \to \text{Hom}_{KC'(A/O)}(N_1, F(N_2)) \]
induced by the natural homomorphism \( N_1 \to F(N_1) \) and \( N_2 \to F(N_2) \) are quasi-isomorphisms.

3.3.3. We define a DG category \( BK(A/O) \). The class of objects of \( BK(A/O) \) is that of \( B - \text{com} \). Let \( M, N \) be objects in \( BK(A/O) \). The complex of homomorphism \( \text{Hom}_{BK(A/O)}(M, N) \) is defined by the cone of
\[ \text{Hom}_{KC(A/O)}(\psi(M), \psi(N)) \]
\[ \downarrow \eta (3.8) \]
\[ R\text{Hom}_B(M, N) \xrightarrow{\xi} \text{Hom}_{KC'(A/O)}(\psi(M), F(\psi(N))). \]

Here the map \( \xi \) is the composite
\[ R\text{Hom}_B(M, N) \xrightarrow{\alpha (3.5)} \text{Hom}_B(F(M), F(N)) \]
\[ \xrightarrow{\beta (3.7)} \text{Hom}_{KC'(A/O)}(\psi(F(M)), \psi(F(N))) \]
\[ \xrightarrow{(3.9)} \text{Hom}_{KC'(A/O)}(\psi(M), F(\psi(N))). \]

Then the homomorphisms \( \xi \) and \( \eta \) are quasi-isomorphism. We introduce a composite structure
\[ \circ : \text{Hom}_{BK(A/O)}(M, N) \otimes \text{Hom}_{BK(A/O)}(L, M) \to \text{Hom}_{BK(A/O)}(L, M) \]
by the rule
\[ (a + b + c) \circ (a' + b' + c') = (a \circ a') + (b \circ b') + (c \circ \eta(a')) + \xi(b \circ c') \]
for
\[ a \in \text{Hom}_{KC(A/O)}(\psi(M), \psi(N)), \quad b \in R\text{Hom}_B(M, N) \]
\[ c \in \text{Hom}_{KC'(A/O)}(\psi(M), F(\psi(N))). \]
Then this composite is associative. Thus we have the following theorem.

**Theorem 3.14.**

(1) The natural projections

\[ \text{Hom}_{BK(A/O)}(M, N) \rightarrow \text{Hom}_{KC}(\psi(M), \psi(N)) \]

\[ \text{Hom}_{BK(A/O)}(M, N) \rightarrow \text{RHom}_B(M, N) \]

defines a DG functor.

(2) The DG functors \( BK(A/O) \rightarrow KC(A/O) \) and \( BK(A/O) \rightarrow (B_{simp-com}) \) are homotopy equivalent.

### 3.4. Integrable \((A/O)\)-connection

In this and the next subsection, we show that the category \((\text{Rep}_G)^S\) and that of \(H^0(\text{Bar}(A/O), \epsilon)\)-comodules are equivalent when \(A\) is the relative DGA introduced in \(\S 2.1.3\). The contents of \(\S 3.4\) and \(\S 3.5\) are not used for the definition of the category of mixed elliptic motives.

We define the category of integrable \((A/O)\)-connections.

**Definition 3.15.**

(1) Let \(V = (V^i e^i, \{d_{ij}\})\) be an object of \(KC(A/O_S)\). The object \(V\) is said to be concentrated at degree zero if and only if \(V^i\) is an \(O_S\)-comodule put at degree zero. The full subcategory of \(KC(A/O_S)\) consisting of objects concentrated at degree zero is denoted by \(KC(A/O_S)^0\).

(2) We define the homotopy category \(H^0(KC(A/O_S)^0)\) of \(KC(A/O_S)^0\) whose class of objects is that in \(KC(A/O_S)^0\). The set of morphisms from \(M\) to \(N\) in \(H^0(KC(A/O_S)^0)\) is defined by \(H^0(\text{Hom}_{KC(A/O)}(M, N))\). The category \(IC(A/O) = H^0(KC(A/O_S)^0)\) is called the category of integrable \((A/O)\)-connections.

**Proposition 3.16.** An object in \(H^0(KC(A/O_S)^0)\) is equivalent to the following data:

(1) A finite set of comodules \(V^i\). The comodule structure on \(V^i\) is denoted by \(\Delta_i : V^i \rightarrow O_S \otimes_k V^i\).

(2) \(O\)-homomorphisms \(\nabla_{ji} : V^i \rightarrow A^1 \otimes_k V^j\) for each \(i < j\).

We impose the following conditions for data.

(1) The composite

\[ V^i \rightarrow A^1 \otimes_k V^j \xrightarrow{\Delta_i \otimes 1 - 1 \otimes \Delta_j} A^1 \otimes_k O_S \otimes_k V^j \]

is zero.

(2) Let

\[ \nabla_{kji}^2 = (1_A \otimes \nabla_{kj}) \circ \nabla_{ji} : V^i \rightarrow A^1 \otimes_k A^1 \otimes_k V^k \]

be the composite map which defines an element in \(\text{Hom}_{O_S}(V^i, A^1 \otimes_{O_S} A^1 \otimes_{O_S} V^k)\). The image of \(\nabla_{kji}^2\) under the map

\[ \text{Hom}_{O_S}(V^i, A^1 \otimes_{O_S} A^1 \otimes_{O_S} V^k) \rightarrow \text{Hom}_{O_S}(V^i, A^2 \otimes_{O_S} V^j) \]
can be written as $\nabla_{k_j} \circ \nabla_{j_i}$ using the composite in $\mathcal{C}(A/O)$. Then the equality
\begin{equation}
\partial \nabla_{k_i} + \sum_{i<j<k} \nabla_{k_j} \circ \nabla_{j_i} = 0
\end{equation}
holds.

3.4.1. Let $V$ and $W$ be objects in $K\mathcal{C}(A/O)^0$. A closed homomorphism $\varphi$ in $Z^0\text{Hom}_{K\mathcal{C}(A/O)}(V,W)$ of degree zero is a set of homomorphisms $\{\varphi_{j_i}\}$ with $\varphi_{j_i} \in \text{Hom}_{\mathcal{O}_S}(V^i, A^0 \otimes_{\mathcal{O}_S} W^j)$ such that
\begin{equation}
\partial \varphi_{k_i} + \sum_{k>j} \nabla_{k_j} \circ \varphi_{j_i} - \sum_{j>i} \varphi_{k_j} \circ \nabla_{j_i} = 0,
\end{equation}
using differentials and composites in $\mathcal{C}(A/O)$.

3.4.2. Let $A \to A'$ be a quasi-isomorphism between relative DGA’s over $\mathcal{O}_S$. An $A$-connection $V$ relative to $S$ can be regarded as an $A'$-connection.

**Proposition 3.17.** (1) Let $V$ be an $A$-connection relative to $S$. Then the set of homomorphisms $\text{Hom}_{\mathcal{IC}(A/O)}(V,W)$ and $\text{Hom}_{\mathcal{IC}(A'/O)}(V,W)$ are naturally isomorphic.

(2) For any $A'$-connection $W$, there exists an $A$-connection $V$ which is isomorphic to $W$. As a consequence, the categories $\mathcal{IC}(A/O)$ and $\mathcal{IC}(A'/O)$ of $A$-connections and $A'$-connections are equivalent.

**Proof.** (1) We can introduce a natural complex structure on $\text{Hom}_{\mathcal{O}_S}(V, A^\bullet \otimes_{\mathcal{O}_S} W)$ and can show that $\text{Hom}_{\mathcal{O}_S}(V, A^\bullet \otimes_{\mathcal{O}_S} W)$ and $\text{Hom}_{\mathcal{O}_S}(V, A'^\bullet \otimes_{\mathcal{O}_S} W)$ are quasi-isomorphic by taking a suitable filtrations on $V$ and $W$ and considering the associate graded objects.

(2) We set $W = \bigoplus_{i=0}^n W^i$ and prove that there exists an $A$-connection $V$ such that $V^i = W^i$ and a closed homomorphism $\varphi : V \to A^0 \otimes W$ such that
\[
\varphi_{j_i} = 0 \text{ for } j < i \text{ and } \varphi_{ii} = id. \quad (P)
\]
We set $F^1 W = \bigoplus_{i \geq 1} W^i$, $F^1 V = \bigoplus_{i \geq 1} V^i$ and assume that there exists a closed isomorphism $\varphi : F^1 V \to A^0 \otimes F^1 W$ by induction. We extend this isomorphism to $V \to A^0 \otimes W$ with the same properties (P). Since $\varphi_{00} = id$ and $\varphi_{0i} = 0$ for $i > 0$, it is enough to define $\varphi_{j0} \in \text{Hom}_{\mathcal{O}_S}(V^0, A^0 \otimes_{\mathcal{O}_S} W^j)$ for $j > 0$ and $\nabla_{j0} \in \text{Hom}_{\mathcal{O}_S}(V^0, A^1 \otimes_{\mathcal{O}_S} V^j)$ for $j > 0$ such that
\begin{equation}
\partial \varphi_{j0} - \varphi_{jj} \circ \nabla_{j0} = - \nabla_{j0} \circ \varphi_{00} - \sum_{0<k<j} \nabla_{j0} \circ \varphi_{k0} + \sum_{0<l<j} \varphi_{jl} \circ \nabla_{j0}
\end{equation}
in $\text{Hom}_{\mathcal{O}_S}(V^0, A^1 \otimes_{\mathcal{O}_S} W^j)$. We define $\varphi_{j0}$ by the induction on $j$. By the assumption of induction, the right hand side of (3.12) is defined. Then we
have
\[
\partial(- \sum_{0 \leq k < j} \nabla^W_{jk} \circ \varphi_{k0} + \sum_{0 < l < j} \varphi_{jl} \circ \nabla^V_l) = \\
\sum_{0 \leq k < l < j} \nabla^W_{jk} \circ \nabla^W_{lk} \circ \varphi_{k0} + \sum_{0 < l < j} \nabla^W_{jl} \circ (\partial \varphi_l) \\
+ \sum_{0 < k < j} (\partial \varphi_{jk}) \circ \nabla^V_{k0} - \sum_{0 < k < l < j} \varphi_{jl} \circ \nabla^V_{lk} \circ \nabla^V_{k0}
\]
\[
= \sum_{0 \leq k < l < j} \nabla^W_{jk} \circ \nabla^W_{lk} \circ \varphi_{k0} - \sum_{0 \leq p < l < j} \nabla^W_{jl} \circ \nabla^W_{lp} \circ \varphi_{p0} + \sum_{0 < p < l < j} \nabla^W_{jl} \circ \varphi_{lp} \circ \nabla^V_{p0} \\
+ \sum_{0 < k < p < j} \varphi_{jp} \circ \nabla^V_{pk} \circ \nabla^V_{k0} - \sum_{0 < k < l < j} \varphi_{jl} \circ \nabla^V_{lk} \circ \nabla^V_{k0} \\
= \sum_{0 < k < j} \varphi_{jj} \circ \nabla^V_{jk} \circ \nabla^V_{k0}.
\]

Since \(\text{Hom}_{\O_S}(V^0, A^* \otimes_{\O_S} W^j)\) and \(\text{Hom}_{\O_S}(V^0, A^* \otimes_{\O_S} W^j)\) are quasi-isomorphic, we can choose \(\nabla_{j0}' \in \text{Hom}^1_{\O_S}(V^0, A^* \otimes_{\O_S} W^j)\) such that
\[
\partial \nabla_{j0}' + \sum_{0 < k < j} \nabla^V_{jk} \circ \nabla^V_{k0} = 0.
\]

Therefore (3.13)
\[
- \sum_{0 \leq k < j} \nabla^W_{jk} \circ \varphi_{k0} + \sum_{0 < l < j} \varphi_{jl} \circ \nabla^V_{l0} + \varphi_{jj} \circ \nabla_{j0}' \in Z^1 \text{Hom}^1_{\O_S}(V^0, A^* \otimes_{\O_S} W^j)
\]
and there exist elements
\[
\nabla_{j0}'' \in Z^1 \text{Hom}^1_{\O_S}(V^0, A^* \otimes_{\O_S} W^j) \quad \text{and} \quad \varphi_{j0} \in \text{Hom}^0_{\O_S}(V^0, A^* \otimes_{\O_S} W^j)
\]
such that the left hand side of (3.13) is equal to \(-\nabla_{j0}'' + d\varphi_{j0}\). Therefore the equation (3.12) holds for \(\nabla^V_{j0} = \nabla_{j0}' + \nabla_{j0}''\).

**Definition 3.18** (Rigid relative DGA). Let \(A\) be a relative DGA over the coalgebra \(\O\), \(i : \O \to A\) be the homomorphism of Definition [2.7] [2], and \(\epsilon : A \to \text{Hom}_k(L\O, L\O)\) be a relative augmentation. The relative DGA \(A\) is said to be rigid if

1. \(A^i = 0\) for \(i < 0\),
2. \(i : \O \to A^0\) is an isomorphism, and
3. the differential \(A^0 \to A^1\) is a zero map.

If \(A\) is rigid, then the map \(i\) induces an isomorphism \(\O \simeq H^0(A^0)\).

Let \(A\) be a rigid relative DGA. We define the augmentation ideal \(I_\epsilon\) by the kernel of the relative augmentation \(\epsilon\). The reduced bar complex \(\text{Bar}_{\text{red}}(A/\O, \epsilon)\) is defined by the following sub-complex of \(\text{Bar}(A/\O, \epsilon)\).
\[
\cdots \to I_\epsilon \otimes_{\O} I_\epsilon \otimes_{\O} I_\epsilon \to I_\epsilon \otimes_{\O} I_\epsilon \to I_\epsilon \to \O \to 0.
\]
The associate simple complex of $\text{Bar}(A/\mathcal{O}, \epsilon)$ is denoted by $\text{Bar}(A/\mathcal{O}, \epsilon)$. The proof of the following lemma is similar to [T], Theorem 5.2, and we omit it.

**Lemma 3.19.**

1. The inclusion $\text{Bar}_{\text{red}}(A/\mathcal{O}, \epsilon) \to \text{Bar}(A/\mathcal{O}, \epsilon)$ is a quasi-isomorphism.
2. $\text{Bar}_{\text{red}}(A/\mathcal{O}, \epsilon)^n = 0$ for $n < 0$. As a consequence, we have the following inclusion

$$H^0(\text{Bar}(A/\mathcal{O}, \epsilon)) \to \text{Bar}_{\text{red}}(A/\mathcal{O}, \epsilon)^0$$

Using the above lemma, we have the following proposition.

**Proposition 3.20.** Let $A$ be a DGA relative to $\mathcal{O}$, and $\epsilon$ be a relative augmentation of $A$. Assume that there is a sub-DGA $A_{\text{rig}} \to A$ of $A$ which is rigid and quasi-isomorphic to $A$. Then the category $IC(A/\mathcal{O})$ is isomorphic to the category of $H^0(\text{Bar}(A/\mathcal{O}, \epsilon))$-comodules.

**Proof.** By Proposition 3.17, we may assume that $A$ is a rigid relative DGA over $\mathcal{O}$. Let $M$ be an object of $IC(A/\mathcal{O})$. Then we have a $\text{Bar}(A/\mathcal{O}, \epsilon)$-comodule $\mathcal{M}$ corresponding to $M$. By taking 0-th cohomology of $\mathcal{M} \to \text{Bar}(A/\mathcal{O}, \epsilon) \otimes \mathcal{M}$, we have a $H^0(\text{Bar}(A/\mathcal{O}, \epsilon))$-comodule $\tilde{\mathcal{M}} = H^0(\mathcal{M})$, since $H^1(\mathcal{M}) = 0$ for $i \neq 0$.

Conversely, let $\tilde{\mathcal{M}}$ be a $H^0(\text{Bar}(A/\mathcal{O}, \epsilon))$-comodule. Using the inclusion (3.14), we have the following map.

$$\tilde{\mathcal{M}} \to H^0(\text{Bar}(A/\mathcal{O}, \epsilon)) \otimes \tilde{\mathcal{M}} \to \text{Bar}_{\text{red}}(A/\mathcal{O}, \epsilon)^0 \otimes \tilde{\mathcal{M}}.$$ 

By this map, we have an object $M$ in $IC(A/\mathcal{O})$. By this bijections, we have a category equivalence of $IC(A/\mathcal{O})$ and the category of $H^0(\text{Bar}(A/\mathcal{O}, \epsilon))$-comodules. \hfill \Box

### 3.5. The case associated to $G \to S(k)$

Let $G$ be a group and $G \to S(k)$ be a Zariski dense homomorphism. Let $A = \text{Hom}_G(L\mathcal{O}, L\mathcal{O})$ be the relative DGA with respect to $\mathcal{O} = \mathcal{O}_S$ introduced in [2.1.3].

#### 3.5.1. We consider the category of integrable $(A/\mathcal{O})$-connections. For two $G$-modules $V_1, V_2$, the extension group of $\text{Ext}_G(V_1, V_2)$ is equal to the cohomology of the complex

$$\text{Hom}_G(V_1, V_2) = \text{Hom}_\mathcal{O}(V_1, A \otimes_{\mathcal{O}} V_2).$$

An object in $H^0(KC(A/\mathcal{O})^0)$ is equivalent to the following data.

1. Algebraic representations $(V^1, \rho_1, V^2, \rho_2, \ldots, V^n, \rho_n)$ of $S$,
2. $\nabla_{i,j}$ an element of $\text{Hom}^1_G(V_i, V_j)$ for $i < j$.

The element in $\text{Hom}_k(V_i, V_j)$ determined by

$$v_i \mapsto \nabla_{i,j}(g \otimes v_i)$$

is denoted as $\nabla_{i,j}(g)$. Since

$$(\partial \nabla_{i,k})(g_1 \otimes g_2 \otimes v) = \rho_k(g_1) \nabla_{i,k}(g_2 \otimes v) - \nabla_{i,k}(g_1 g_2 \otimes v) + \nabla(g_1 \otimes \rho_i(g_2)v),$$

$$\nabla_{j,k} \circ \nabla_{i,j}(g_1 \otimes g_2 \otimes v) = \nabla_{j,k}(g_1 \otimes \nabla_{i,j}(g_2 \otimes v)), $$

$$\nabla_{i,k}(g_1 \otimes g_2 \otimes v) = \nabla_{i,k}(g_1 \otimes g_2 \otimes g_3 \otimes v) $$

we have the following properties:

1. $\nabla_{i,j}(g_1 \otimes g_2 \otimes v) = \nabla_{i,j}(g_1 \otimes g_2 \otimes v)$,
2. $\nabla_{i,j}(g_1 \otimes g_2 \otimes v) = \nabla_{i,j}(g_1 \otimes g_2 \otimes v)$,
3. $\nabla_{i,j}(g_1 \otimes g_2 \otimes v) = \nabla_{i,j}(g_1 \otimes g_2 \otimes v)$.
by the condition (3.10) is equivalent to the relation
\begin{equation}
\nabla_{i,k}(g_1g_2) = \nabla_{i,k}(g_1)\rho_i(g_2) + \rho_k(g_1)\nabla_{i,k}(g_2) + \sum_{i<j<k} \nabla_{j,k}(g_1)\nabla_{i,j}(g_2)
\end{equation}
for all \(g_1, g_2 \in G\). Therefore the map
\[ \rho : G \to \text{Aut}(V) : g \mapsto \sum_i \rho_i(g) + \sum_{i<j} \nabla_{i,j}(g) \]
is a homomorphism of groups if and only if the condition (3.15) holds.

For an object \(V = (V^i, \nabla_{ij})\) in \(H^0(KC(A/O)^0)\), the \(G\)-module \((\rho, \sum_i V^i)\) obtained from \(V\) as in the last proposition is denoted as \(\rho(V)\).

**Proposition 3.21.** There exists a rigid relative sub-DGA \(A_{\text{rig}}\) of \(A\), which is quasi-isomorphic to \(A\). As a consequence, \(IC(A/O)\) is equivalent to the category of \(H^0(\text{Bar}(A/O, \epsilon))\)-comodules.

**Proof.** Since the image \(G \to S(k)\) is Zariski dense, we have
\[ H^0(A^{\alpha,\beta}) = \text{Hom}_\mathcal{O}(V^\alpha, V^\beta) = \begin{cases} k & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta. \end{cases} \]
Let \(C^{\alpha,\beta}\) be a complement of the image of the differential \(A^{\alpha,\beta,0} \to A^{\alpha,\beta,1}\). We set
\[ A_{\text{rig}}^i = \begin{cases} 0 & \text{for } i < 0 \\ \oplus_{\alpha} V^\alpha \otimes V & \text{for } i = 0 \\ \oplus_{\alpha} V^\alpha \otimes C^{\alpha,\beta} \otimes V & \text{for } i = 1 \\ A^i & \text{for } i \geq 2. \end{cases} \]
Then \(A_{\text{rig}}\) satisfies the required properties. The latter part is a consequence of Proposition 3.20. \(\square\)

3.5.2. Then for two connections, \(V\) and \(W\), the set \(\text{Hom}_{IC(A/O)}(V,W)\) is a subset of \(\text{Hom}_k(V,W)\) and can be identified with the set of \(G\)-homomorphisms from \(\rho(V)\) to \(\rho(W)\) by the closedness condition (3.11). Thus we get a fully faithful functor from \(IC(A/O)\) to \((\text{Rep}_G)^S\). We have the following proposition.

**Proposition 3.22.**

1. The essential image of \(\rho : IC(A/O) \to (\text{Rep}_G)^S\) is equal to \((\text{Rep}_G)^S\). As a consequence, \((\text{Rep}_G)^S\) is equivalent to \(IC(A/O)\).

2. The category \((\text{Rep}_G)^S\) is equivalent to the category of \(H^0(\text{Bar}(A/O))\)-comodules.

**Proof.** Let \(V\) be an element of \((\text{Rep}_G)^S\) and \(F^\bullet W\) be a filtration whose associated graded modules come from \(\mathcal{O}\)-comodules. We choose a splitting \(\{W^p\}\) of \(F^\bullet W\) in \(W\) as \(k\)-vector spaces. Then we have \(F^p W = \oplus_{k \geq p} W^p\). By the isomorphism \(W^p \to \text{Gr}^p_W(W)\) of \(k\) vector spaces, we introduce a
O module structure on $W^p$. The action of $G$ on $W$ defines a map $\nabla_{ji}$ in $\text{Hom}_G(W^i, W^j) = \text{Hom}_O(W^i, A^1 \otimes O W^j)$. Since $W$ is a $G$-module, $(W^p, \nabla_{ij})$ defines a $G$-connection $W_A$ relative to $S$. By Proposition 3.1, the natural functor $IC(A_{\text{rig}}/O) \to IC(A/O)$ is an equivalence of category, we have an object $W_{A'}$ in $IC(A_{\text{rig}}/O)$ such that the image is isomorphic to $W_A$. Then one can check that $\rho(W_{A'})$ is isomorphic to the given $W$.

4. Mixed elliptic motif

In this section, we define quasi-DG categories of naive mixed elliptic motives ($MEM$) and virtual mixed elliptic motives ($VMEM$). Roughly speaking, the DG category of virtual mixed elliptic motives is obtained by adding objects which are homotopy equivalent to zero. Therefore ($MEM$) and ($VMEM$) are weak homotopy equivalent.

We can not introduce a natural tensor structure on $(MEM)$ with the distributive property as is explained in §4.6.2. On the other hand, ($VMEM$) has a distributive tensor structure, which is necessary to obtain a shuffle product on the bar complex $\text{Bar}(\mathcal{C}_{VEM})$. Using this shuffle product, $H^0(\text{Bar}(\mathcal{C}_{VEM}))$ becomes a Hopf algebra. In this paper, Bloch cycle groups, higher Chow groups are $Q$-coefficients.

4.1. Injectivity of linear Chow group. In this subsection, we prove the injectivity of linear Chow group, which is also proved in [BL]. The proof given here is more direct.

**Proposition 4.1.** Let $E$ be an elliptic curve over a field $k$ which does not have complex multiplication, that is $\text{End}_k(E) = \mathbb{Z}$. Let $CH_{\text{lin}}^*(E^n)$ be the subring of $CH^*(E^n)$ generated by $f^*([0]) \in CH^1(E^n)$, where $f$ is a homomorphism of abelian varieties

$$f : E^n \to E.$$ 

The scalar extension of $E$ to its algebraic closure is written as $\overline{E}$. Then the cycle map

$$cl : CH_{\text{lin}}^*(E^n) \to H^2_{\text{et}}(\overline{E}^n, Q_l)$$

is injective.

Let $\pi_i : E^n \to E$ and $\pi_{ij} : E^n \to E \times E$ be the projection to $i$-th and $ij$-th components, and the diagonal in $E \times E$ is denoted by $\Delta$. The classes $\pi_i^*([0])$ and $\pi_{ij}^*(\Delta)$ in $CH^1(E^n)$ are denoted by $p_i$ and $\Delta_{ij}$, respectively. Let $f : E^n \to E$ be a homomorphism defined by $f(x_1, \ldots, x_n) = \sum_{i=1}^n a_i x_i$ for $a_i \in \mathbb{Z}$. Let $\alpha, \beta$ be symplectic basis of $H^1(\overline{E}) = H^1_{\text{et}}(\overline{E}, Q_l)$. For the copy of $E_i$, the corresponding symplectic basis in $H^1(\overline{E}_i)$ are denoted by $\alpha_i, \beta_i$. Then the class $cl(f^*([0])) \in H^2(\overline{E}^n)$ is equal to

$$f^*(\alpha)f^*(\beta) = \left( \sum_i a_i \alpha_i \right) \left( \sum_i a_i \beta_i \right) = \sum_i a_i^2 \alpha_i \beta_i + \sum_{i < j} a_i a_j (\alpha_i \beta_j - \beta_i \alpha_j)$$

$$= \sum_i a_i^2 cl(p_i) - \sum_{i < j} a_i a_j cl(\Delta_{ij} - p_i - p_j).$$
Since $\text{Pic}^0(E^n)$ component of $f^*([0])$ is zero, we have
\[
f^*([0]) = \sum_i a_i^2 p_i - \sum_{i<j} a_i a_j (\Delta_{ij} - p_i - p_j)
\]
in $\text{CH}^1(E^n)$. Therefore $\text{CH}^*_{\text{lin}}(E^n)$ is generated by $p_i$ and $D_{ij} = -\Delta_{ij} + p_i + p_j$ for $i \neq j$. We have $D_{ij} = D_{ji}$. By the map $\sigma : E_1 \times E_2 \rightarrow E_1 \times E_2 : (x, y) \mapsto (x, -y)$, we have $\sigma(D_{12}) = -D_{12}$.

**Lemma 4.2.** We have
\begin{align*}
(1) & \quad p_i D_{ij} = 0, \\
(2) & \quad D_{ij} D_{ik} + p_i D_{jk} = 0, \quad \text{and} \\
(3) & \quad D_{ij} D_{kl} + D_{ik} D_{jl} + D_{il} D_{jk} = 0
\end{align*}
in $\text{CH}^2(E^n)$ for $\#\{i, j\} = 2$, $\#\{i, j, k\} = 3$ or $\#\{i, j, k, l\} = 4$, respectively.

**Proof.** The equality $p_i D_{12} = p_1 p_2 - p_1 \Delta = 0$ is trivial.

In $\text{CH}^2(E_1 \times E_2 \times E_3)$, we have $\Delta_{12} \cap \Delta_{13} = \Delta_{12} \cap \Delta_{23}$. Therefore
\[
(p_1 + p_2 - D_{12})(p_1 + p_3 - D_{13}) = (p_1 + p_2 - D_{12})(p_2 + p_3 - D_{23}),
\]

\[
p_1 p_2 + p_2 p_3 + p_3 p_1 - p_3 D_{12} - p_2 D_{13} + D_{12} D_{13} = 0
\]
and we have
\[
-p_2 D_{13} + D_{12} D_{13} = -p_1 D_{23} + D_{12} D_{23}.
\]
Applying an automorphism $(x_1, x_2, x_3) \mapsto (x_1, -x_2, x_3)$, we have
\[
-p_2 D_{13} - D_{12} D_{13} = p_1 D_{23} + D_{12} D_{23}
\]
and we get the equality
\[
p_2 D_{13} = -D_{12} D_{23}.
\]
(Chow groups are considered as $\mathbb{Q}$-coefficient.) We consider the map $\varphi : E_1 \times E_2 \times E_3 \times E_4 \rightarrow E_1 \times E_2 \times E_3$ defined by $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2 - x_4, x_3)$. Then applying $\varphi$ to the equality (1.1) and using the equality
\[
\varphi^*(p_2) = p_2 + p_4 - D_{42}, \quad \varphi^*(D_{12}) = D_{12} - D_{14}, \quad \varphi^*(D_{23}) = D_{23} - D_{34},
\]
we have
\[
(p_2 + p_4 - D_{42}) D_{13} = -(D_{12} - D_{14})(D_{23} - D_{34}),
\]
which implies the third equality of the lemma. \hfill $\square$

**Lemma 4.3.** $\text{CH}^m_{\text{lin}}(E^n)$ is generated by the set of elements of the form
\[
p_i \cdots p_i r D_{j_1, k_1} \cdots D_{j_q, k_q}
\]
such that
\begin{enumerate}
\item $i_1, \ldots, i_p, j_1, \ldots, j_q, k_1, \ldots, k_q$ are distinct and
\item (Gelfand-Zetlin condition)
\[
j_1 < j_2 < \cdots < j_q \quad \Lambda \quad \Lambda \quad \Lambda \quad .
\]
\[
k_1 < k_2 < \cdots < k_q
\]
\end{enumerate}
Proof. By the relations (4.1) and (4.2), $CH_{fin}^*(E^n)$ is generated by the monomial of the form (4.5) with the condition (1) of Lemma 4.3. Let $W$ be the subspace of $CH^*(E^n)$ generated by monomials for the form (4.5) with the conditions (1) and (2) of Lemma 4.3. Assume that there exists a monomial of the form (4.5) which is not an element of $W$. We set $S = \{j_1, \ldots, j_q, k_1, \ldots, k_q\}$. We consider the set $\mathcal{M}$ of monomials of the form

$$M' = p_{i_1} \cdots p_{i_p} D_{j'_1, k'_1} \cdots D_{j'_q, k'_q}, \tag{4.6}$$

which is not contained in $W$ such that $\{j'_1, \ldots, j'_q, k'_1, \ldots, k'_q\} = S$ and

$$j'_1 < j'_2 < \cdots < j'_q \quad \bigwedge_j k'_1 < k'_2 < \cdots < k'_q.$$ 

By the assumption, $\mathcal{M}$ is not an empty set. Since the monomial (4.6) is not contained in $W$, there exists a $t$ such that $k'_t > k'_{t+1}$. The minimal of such $t$ is denoted by $t(M')$. We introduce a total order of $\mathcal{M}$ by the lexicographic order of

$$-t(M'), j'_1, k'_1, \ldots, j'_q, k'_q. \tag{4.7}$$

Let $M'$ be the minimal element in $\mathcal{M}$ and set $t = t(M') < q$. Then we have the equality

$$j'_t < j'_1, k'_t < k'_1, \ldots \quad \bigwedge_j k'_t > k'_1.$$ 

By the equality (4.3), we have

$$D_{j'_t, k'_t} D_{j'_{t+1}, k'_{t+1}} + D_{j'_t, j'_{t+1}} D_{k'_{t+1}, k'_t} + D_{j'_t, k'_{t+1}} D_{j'_{t+1}, k'_t} = 0.$$

Let $M''$ and $M'''$ be monomial obtained by replacing the factor $D_{j'_t, k'_t} D_{j'_{t+1}, k'_{t+1}}$ by $D_{j'_t, j'_{t+1}} D_{k'_{t+1}, k'_t}$ and $D_{j'_t, k'_{t+1}} D_{j'_{t+1}, k'_t}$. Then $t(M'') \geq t(M') + 1$ and $t(M''') \geq t(M') + 1$. Therefore $M'', M''' \in W$. Since $M' + M'' + M''' = 0$, we have $M' \in W$ and a contradiction to the choice of $M'$. \qed

**Lemma 4.4.** The set of the images $cl(p_{i_1} \cdots p_{i_p} D_{j'_1, k'_1} \cdots D_{j'_q, k'_q})$ with the conditions (1) and (2) of Lemma 4.3 are linearly independent in $H^*(E^n)$.

**Proof.** Since $cl(D_{ij}) = \alpha_i \beta_j - \beta_i \alpha_j$, it is enough to prove the linear independence for the set of monomial for fixed $\{i_1, \ldots, i_p\}$ and $S = \{j_1, k_1, \ldots, j_q, k_q\}$. For a monomial

$$\alpha_{i_1} \beta_{i_1} \cdots \alpha_{i_p} \beta_{i_p} \alpha_{j'_1} \beta_{j'_1} \cdots \alpha_{j'_q} \beta_{j'_q}, \tag{4.8}$$

with $j'_1 < \cdots < j'_q, k'_1 < \cdots < k'_q$, we introduce lexicographic order by deleting “$-t(M')$"-part of (4.7). Let $p_{i_1} \cdots p_{i_p} D_{j'_1, k'_1} \cdots D_{j'_q, k'_q}$ be an element with the condition (1) and (2) of Lemma 4.3. Then the lowest monomial appeared in $cl(p_{i_1} \cdots p_{i_p} D_{j'_1, k'_1} \cdots D_{j'_q, k'_q})$ of the above form is equal to (4.8). Therefore they are independent. \qed
**Proof of Proposition 4.1.** The proof of the proposition is a consequence of Lemma 4.3 and 4.4. □

**Corollary 4.5.** Let $p_1, \ldots, p_m$ be elements in the correspondence algebra $\text{CH}^n(E^n \times E^n)$ of $E^n$, which are contained in $\text{CH}_{\text{lin}}^n(E^n \times E^n)$. The image of $p_i$ in $H^{2n}(E^n \times E^n, \mathbb{Q})$ is written by $\text{cl}(p_i)$.

(1) If $\text{cl}(p_1) = \text{cl}(p_2)$, then $p_1 = p_2$ in $\text{CH}^n(E^n \times E^n, \mathbb{Q})$. In particular, the images $p^*_1 \text{CH}^i(E^n, j)$ and $p^*_2 \text{CH}^i(E^n, j)$ of the correspondences $p^*_1$ and $p^*_2$ in the higher Chow group $\text{CH}^i(E^n, j)$ are equal.

(2) The element $p_i$ is a projector if and only if the class $H^{2n}(E^n \times E^n, \mathbb{Q})$ is a projector in the cohomological correspondence ring.

(3) The correspondences $\{p_i\}$ are orthogonal (resp. complete set of projectors) if and only if $\{\text{cl}(p_i)\}$ are orthogonal (resp. complete set of projectors).

**Corollary 4.6.** Let $G = S_2 \rtimes \langle \sigma \rangle$ and $\rho$ be the character of $G$ defined by $\rho((1, 2)) = 1$ and $\rho(\sigma) = -1$. Here the element $\sigma$ is the inversion of $E$. Let $\Delta^+$ and $\Delta^-$ be the diagonal divisor of $E \times E$ and the divisor defined by $x + y = 0$, where $(x, y)$ are the coordinates of $E \times E$. The maps $Z^i(E \times E \times X, j) \rightarrow Z^i(E \times E \times X, j)$ on Bloch cycle groups induced by

$$
\varphi_1 : Z \mapsto -\frac{1}{4}(\Delta^+ - \Delta^-) \times \text{pr}_X(Z \cap \{(\Delta^+ - \Delta^-) \times X\})
$$

and

$$
\varphi_2 : Z \mapsto \frac{1}{\#G} \sum_{g \in G} \rho(g) g^* Z
$$

induce the same maps in the higher Chow group $\text{CH}^i(E \times E \times X, j)$.

**Proof.** The pull back of the divisors $\Delta^+$ and $\Delta^-$ by the $(i, j)$-component are denoted by $\Delta^+_{ij}$ and $\Delta^-_{ij}$, respectively. Since the maps $\varphi_1$ and $\varphi_2$ are induced by the algebraic correspondences

$$
-\frac{1}{4}(\Delta^+_{12} - \Delta^-_{12}) \times (\Delta^+_{34} - \Delta^-_{34})
$$

and

$$
\frac{1}{8}(\Delta^+_{13} \Delta^+_{24} - \Delta^-_{13} \Delta^+_{24} - \Delta^+_{13} \Delta^-_{24} + \Delta^-_{13} \Delta^-_{24}) \\
+ \Delta^+_{14} \Delta^+_{23} - \Delta^-_{14} \Delta^+_{23} - \Delta^+_{14} \Delta^-_{23} + \Delta^-_{14} \Delta^-_{23})
$$

in $\text{CH}^2_{\text{lin}}(E_1 \times E_2 \times E_3 \times E_4)$. By computing the images of two cycles in $H^i(E_1 \times E_2 \times E_3 \times E_4)$, we can check the identity. □

4.2. Naive mixed elliptic motives.
4.2.1. Let $S = GL(2)$ and $\mathcal{O} = \mathcal{O}_S$. The natural two dimensional representation is written as $V$. The set of isomorphism classes of irreducible representations is written by $Irr_2$. Then using alternating and symmetric tensor products, we have

$$Irr_2 = \{(Alt^2)^{\otimes n} \otimes Sym^m | n \in \mathbb{Z}, m \in \mathbb{N}\}.$$  

Let $E$ be an elliptic curve over a field $K$. We assume that $E$ does not have complex multiplication, i.e. $End_K(E) = \mathbb{Z}$. Let $Z^i(X, j) = Z^{i, j}(X)$ be the cubical anti-symmetric Bloch higher cycle group. This becomes a complex which is denoted by $Z^{i, \bullet}(Z)$. Let $A$ be a finite set. The $A$-power of the elliptic curve $E$ is denoted by $E^A$. The group $(\mathbb{Z}/2\mathbb{Z})^A$ acts on $Z^{i, \bullet}(E^A)$ by the inversions of elliptic curves for each component. The $(-, \ldots, -)$-part of $Z^{i, \bullet}(E^A)$ under the action of $(\mathbb{Z}/2\mathbb{Z})^A$ is denoted by $Z^{i, \bullet}_-(E^A)$. The group $\mathcal{G}[A]$ acts on $Z^{i, \bullet}(E^A)$. We define the complex $\mathcal{H}^\bullet(E^A, E^B, k)$ by

$$\mathcal{H}^\bullet(E^A, E^B, k) = \Lambda^\bullet(A) \otimes Z^\bullet_{-k, A}(E^A \times E^B) \otimes \Lambda(B)[-2k],$$

where $a = \# A$. The complexes $\Lambda^\bullet(A)$ and $\Lambda(B)$ are defined in §1.2.8.

Then the groups $\mathcal{G}[A]$ and $\mathcal{G}[B]$ act on this complex. If $A = [1, a], B = [1, b]$, then we have

$$\mathcal{H}^i(E^A, E^B, k) = f_a \wedge \cdots \wedge f_1 \cdot Z^\bullet_{-k}(E^a \times E^b, 2k + a - b - i) \cdot e_1 \wedge \cdots \wedge e_b$$

4.2.2. Object and morphisms of $(EM)$. For a natural number $n$, $E^{[1, n]}$ are denoted by $E^n$. We define quasi-DG category of naive elliptic motives $(EM)$. As for the definition of quasi-DG category, see §4.2.4. The objects and complexes of homomorphisms of $(EM)$ are defined as follows:

1. An object of $(EM)$ is a finite dimensional complex of $\mathcal{O}$-comodule.
2. $Sym^a \otimes (Alt^2)^{\otimes (-p)}$ is denoted by $Sym^a(p)$. Let $Sym^a(s), Sym^b(t)$ be elements in $Irr_2$. We set

$$\text{Hom}_{EM}^\bullet(Sym^a(s), Sym^b(t)) = sym^a \mathcal{H}^\bullet(E^a, E^b, t - s) sym^b.$$  

3. Let $U_1, U_2$ be finite dimensional complexes of $\mathcal{O}$-comodules. The complex of homomorphisms is defined by

$$\text{Hom}_{EM}(U_1, U_2) = \oplus_{V_1, V_2 \in Irr_2} \text{Hom}_{\mathcal{O}}(U_1, V_1) \otimes \text{Hom}_{EM}(V_1, V_2) \otimes \text{Hom}_{\mathcal{O}}(V_2, U_2).$$

4.2.3. Multiplication map for $(EM)$. Here $\pi$ is the projector to the $\chi$-part for the action of group and $Z^\bullet(X, \bullet)$ denotes the Bloch cycle group of cubical type of $X$. The intersection theory for Bloch cycle complexes we have a “homomorphism” of complex

$$\Pi : Z^{p_1}(E^{m_1} \times E^{m_2}, q_1) \otimes Z^{p_2}(E^{m_2} \times E^{m_3}, q_2) \rightarrow Z^{p_1 + p_2 - m_2}(E^{m_1} \times E^{m_3}, q_1 + q_2).$$
Roughly speaking, this intersection homomorphism $\Pi$ is defined by

$$z \otimes w \mapsto pr_{13*}((z \times w) \cap (E^{m_1} \times \Delta_{m_2} \times E^{m_3} \times \square^{q_1+q_2})).$$

Here $\Delta_m$ is the image of the diagonal map $E^m \to E^m \times E^m$ and $pr_{13*}$ is the push forward for the map

$$(E^{m_1} \times E^{m_2} \times E^{m_3} \times \square^{q_1+q_2}) \to (E^{m_1} \times E^{m_3} \times \square^{q_1+q_2}).$$

4.2.4. To get the correct definition of $\Pi$, it is necessary to take a quasi-isomorphic subcomplex of $Z^{p_1}(E^{m_1} \times E^{m_2}, q_1) \otimes Z^{p_2}(E^{m_2} \times E^{m_3}, q_2)$ consisting of elements which intersect properly to all the boundary stratification of $(E^{m_1} \times \Delta_{m_2} \times E^{m_3} \times \square^{q_1+q_2})$. See [Han], Proposition 1.3, p.112, and [Le], Corollary 4.8, p.297, for details. A homomorphism of complex which is defined only on a quasi-isomorphic subcomplex is called a quasi-morphism. Similarly, a DGA whose “multiplication” is defined only on a quasi-isomorphic subcomplex is called a quasi-DGA. We can similarly define quasi-DG category, quasi-comodule over a quasi-DG Hopf algebra.

4.2.5. Thus by taking projector and the above intersection pairing, we have a quasi-morphism of complexes.

$$(4.10) \quad \text{Hom}_EM^i(S\text{ym}^{m_1}(q_1), \text{Sym}^{m_2}(q_2)) \otimes \text{Hom}_EM^j(\text{Sym}^{m_2}(q_2), \text{Sym}^{m_3}(q_3)) \to \text{Hom}_EM^{i+j}(\text{Sym}^{m_1}(q_1), \text{Sym}^{m_3}(q_3))$$

Here the sign rule for the pairing of “orientations” are given by

$$(e_1 \wedge \cdots \wedge e_a) \otimes (f_a \wedge \cdots \wedge f_1) \mapsto 1.$$

Using the above pairing, we define quasi-DGA’s $(EM)$, $(MEM)$ as follows.

**Definition 4.7.**

1. We define a relative quasi-DGA $A_{EM}$ by

$$A_{EM} = \bigoplus_{V_1, V_2 \in Irr_2} V_1 \otimes \text{Hom}_EM(V_1, V_2) \otimes V_2^*$$

The multiplication map $\mu$ is given by

$$A_{EM} \otimes_O A_{EM} = \bigoplus_{V_1, V_2, V_3 \in Irr_2} V_1 \otimes \text{Hom}_EM(V_1, V_2) \otimes \text{Hom}_EM(V_2, V_3) \otimes V_3^* \to \bigoplus_{V_1, V_3 \in Irr_2} V_1 \otimes \text{Hom}_EM(V_1, V_3) \otimes V_3^*,$$

where the last arrow is induced by the multiplication map of $(MEM)$.

2. We define the quasi-DG category $(EM)$ by $C(A_{EM}/O)$ defined in [3.1].

3. The quasi-DG category $K(EM)$ of DG complexes in $(EM)$ is called the category of mixed elliptic motives and is denoted by $(MEM)$.

Let $U_1$, $U_2$, and $U_3$ be an object in $(EM)$. Then we have an identification

$$\text{Hom}_EM(U_1, U_2) \simeq \bigoplus_{V_1, V_2 \in Irr_2} \text{Hom}_O(U_1, V_1) \otimes \text{Hom}_EM(V_1, V_2) \otimes \text{Hom}_O(V_2, U_2).$$
The multiplication is given by the following composite map:
\begin{equation}
\Hom_{EM}(U_1, U_2) \otimes \Hom_{EM}(U_2, U_3) = \oplus_{V_1, V_2, V_3} \Hom_{O}(U_1, V_1) \otimes \Hom_{EM}(V_1, V_2) \otimes \Hom_{O}(V_2, U_2) \\
\otimes \Hom_{O}(U_2, V_2') \otimes \Hom_{EM}(V_2', V_3) \Hom_{O}(V_3, U_3) \\
= \oplus_{V_1, V_2, V_3} \Hom_{O}(U_1, V_1) \otimes \Hom_{EM}(V_1, V_2) \otimes \Hom_{O}(V_2, U_2) \\
\otimes \Hom_{O}(U_2, V_2') \otimes \Hom_{O}(V_3, U_3)
\end{equation}
where $\alpha$ is induced by the multiplication in $O$-homomorphisms
\[ \Hom_{O}(V_2, U_2) \otimes \Hom_{O}(U_2, V_2) \to \Hom_{O}(V_2, V_2) \simeq k \]
for $V_2 = V_2'$ and the map $\beta$ is defined in (4.10). The composite map for $(EM)$ is defined by the rule (3.1).

4.2.6. Augmentation. We introduce an augmentation $\epsilon_{EM} : A_{EM} \to k$. The augmentation is defined by the composite map
\[ \Hom_{EM}^0(Sym^n(q), Sym^n(q)) \]
\[ \simeq \text{sym}^n(f_1 \wedge \cdots \wedge f_1)Z^n(E^n \times E^n, 0)(e_1 \wedge \cdots e_n)\text{sym}^n \]
\[ \to \text{sym}^n(f_1 \wedge \cdots \wedge f_1)CH^n(E^n \times E^n, 0)(e_1 \wedge \cdots e_n)\text{sym}^n \]
\[ \to \text{sym}^n(f_1 \wedge \cdots \wedge f_1)CH^\text{hom, -}(E^n \times E^n, 0)(e_1 \wedge \cdots e_n)\text{sym}^n \]
\[ \simeq k \]
where $CH^\text{hom, -}$ is the $(\cdot)$-part of the Chow group modulo homological equivalence. Here we use the assumption that the elliptic curve $E$ has no complex multiplication. On the component $\Hom_{EM}^i(Sym^{n_1}(q_1), Sym^{n_2}(q_2))$, where either $i \neq 0$, $n_1 \neq n_2$ or $q_1 \neq q_2$, the augmentation is set to the zero map. By Theorem 3.14 we have the following theorem.

**Theorem 4.8.** The quasi-DG category $(MEM)$ of naive mixed elliptic motives is homotopy equivalent to the quasi-DG category $(\text{Bar}(A_{EM}/O, \epsilon_{EM}) - \text{com})$ of $\text{Bar}(A_{EM}/O, \epsilon_{EM})$-comodules.

4.3. An application of Schur-Weyl reciprocity. In this section, we review some properties of group ring of symmetric group. For a finite set $A$, the symmetric group of the set $A$ is denoted by $\mathfrak{S}[A]$. Assume that a finite set $A$ is equipped with a total order. A sequence of integers $(\lambda_1, \ldots, \lambda_n)$ is called a partition if $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. A tableau $Y$ consisting of a set $A$ is defined in [M]. Then the support of $Y$ is a partition.

Let $W$ be a vector space and $W_i$ be a copy of $W$ indexed by $i \in A$. The tensor product $\otimes_{i \in A} W_i$ is denoted by $W \otimes A$. The set of tableaux consisting of a finite set $A$ is denoted by $\text{Tab}(A)$.

For a tableau $Y$ consisting of $A$, we can define a projector $e_Y \in \mathbb{Q}[\mathfrak{S}[A]]$ by $e_Y = e\text{sym}_Y e\text{alt}_Y$, where $e\text{sym}_Y$ and $e\text{alt}_Y$ are symmetric and anti-symmetric
projectors for horizontal and vertical symmetric subgroups. (See [M].) The projector $e_Y$ defines a $GL(W)$-homomorphism $W^{\otimes A} \to W^{\otimes A}$ and the image $e_Y W^{\otimes A} = M_Y(W)$ is an irreducible representation of $GL(W)$ and $M_Y$ becomes a functor from $(\text{Vec}_k)$ to $(\text{Vec}_k)$. The isomorphism class of the functor $M_Y$ depends only on the support of the tableaux $Y$ of $A$, and the set of isomorphism class of irreducible representation contained in $W^{\otimes n}$ corresponds in one to one with the set of partition of $n$.

On the other hand, for any projector $p : W^{\otimes A} \to W^{\otimes A}$ of $GL(W)$-modules, there exists an idempotent $e_p \in Q[\mathfrak{S}[A]]$ which induces the projector $p$. Moreover, if $#A < \dim(W)$, the idempotent $e_p$ which induces the projector $p$ is unique by Schur-Weyl reciprocity [W], p.150.

**Definition 4.9.** The linear map $Q[\text{Isom}(S,S')] \to Q[\text{Isom}(S',S)]$ defined by $g \mapsto g^{-1}$ is called the adjoint map. The adjoint of $x$ is denoted by $x^*$. An element $x$ in $Q[\mathfrak{S}[A]]$ is called self-adjoint if $x = x^*$. We have $e_Y^* = e_{\text{alt}_Y e_{\text{sym}_Y}}$.

For three sets $S, S'$ and $S''$ with the same cardinality, and $x \in Q[\text{Isom}(S,S')]$ and $y \in Q[\text{Isom}(S',S'')]$, we have $(yx)^* = x^*y^*$.

We can reformulate for the set of isomorphisms between $A$ and $A'$ instead of automorphism of $A$. Let $A$ and $A'$ be a finite set such that $#A = #A' < \dim(W)$. Then any $GL(W)$-equivariant homomorphism $\varphi : W^{\otimes A} \to W^{\otimes A'}$ is induced by a unique element $e_\varphi$ in $Q[\text{Isom}(A,A')]$. This action is written from the left. An element $e$ of $Q[\text{Isom}(A',A)]$ acts on $W^{\otimes A}$ from the right via the conjugate $e^*$ of $e$. Thus properties for elements in $Q[\mathfrak{S}[A]]$ and $Q[\text{Isom}(A,A')]$ is reduced to that of $GL(W)$ equivariant homomorphisms between $W^{\otimes A}$ and $W^{\otimes A'}$. In other words, the natural map

$$Q[\text{Isom}(S,S')] \to \text{Hom}_{GL(W)}(W^{\otimes S}, W^{\otimes S'})$$

is an isomorphism. Via this isomorphism, we have the following isomorphism

$$\text{Hom}_{GL(W)}(M_{Y_1}, M_{Y_2}) \simeq e_{Y_1}Q[\text{Isom}(\text{Supp}(Y_1), \text{Supp}(Y_2))]e_{Y_1}.$$ 

The symmetric product $\text{Sym}_A(W)$ and alternating product $\text{Alt}_A(W)$ are defined as subspaces of $W^{\otimes A}$ corresponding to the Young tableaux with supports $(n)$ and $(1,1,\ldots,1) = (1^n)$, where $n = #A$. The associate idempotents are written as $\text{sym}_A$ and $\text{alt}_A$. For a subset $B$ of $A$, the element $\text{sym}_B$ and $\text{alt}_B$ can be regarded as elements in $Q[\mathfrak{S}[A]]$.

**Definition 4.10 (Category $(GL_\infty)$).** Let $Y_i$ be Young tableaux and $V_i$ complexes of vector spaces. The direct sum $\oplus_i M_{Y_i} \otimes V_i$ becomes a functor from $(\text{Vect})$ to $(K\text{Vect})$. A functor which is isomorphic to this form is called a Schur functor. The full subcategory of $(\text{Vect}) \to (K\text{Vect})$ consisting of Schur functors is denoted by $(GL_\infty)$.

**Lemma 4.11.** The category $(GL_\infty)$ is closed under tensor product.
The set of partitions is denoted by $\mathcal{P}$. We choose a tableau $Y$ for each partition $|Y|$ and the set of chosen tableaux is denoted by $\tilde{\mathcal{P}}$.

**Proof of Lemma 4.11.** Let $Y_1$ and $Y_2$ be Young tableaux. The tensor product $M_{Y_1} \otimes M_{Y_2}$ is isomorphic to

$$\bigoplus_{Y \in \tilde{\mathcal{P}}} \text{Hom}(GL_\infty)(M_Y, M_{Y_1} \otimes M_{Y_2}) \otimes M_Y.$$ 

The subspace $\text{Hom}(GL_\infty)(M_Y, M_{Y_1} \otimes M_{Y_2})$ is finite dimensional and it is zero for finite number of partitions $|Y| \in \mathcal{P}$. $\square$

4.4. **Virtual mixed elliptic motives.** We use the set $\tilde{\mathcal{P}}$ of Young tableaux chosen in the last subsection.

4.4.1. In this section, we define the quasi-DG category of virtual elliptic motives (VEM) for the elliptic curve $E$.

**Definition 4.12.**

1. An object of $(VEM)$ is a direct sum of symbolic Tate twisted object $M(p)$, where $M$ is an object in $\text{ob}(GL_\infty)$.
2. Let $M_1(p_1)$ and $M_2(p_2)$ be objects in $(VEM)$. We define the complex of homomorphisms by

$$\text{Hom}_{VEM}^i(M_1(p_1), M_2(p_2)) = \bigoplus_{Y_1, Y_2} \text{Hom}_{GL_\infty}(M_1, M_{Y_1}) \otimes e_{Y_1} \mathcal{H}^*(E^s(Y_1), E^s(Y_2), p_2 - p_1) e_{Y_2} \otimes \text{Hom}_{GL_\infty}(M_{Y_2}, M_2),$$

where $\mathcal{H}^*(\ast, \ast)$ is defined in (4.9).
3. The multiplication is defined by the intersection theory and the rule (4.11). Composite of complex of homomorphisms is defined by

$$f \circ g = (-1)^{\text{deg}(f) \cdot \text{deg}(g)} g \cdot f,$$

where $\cdot$ is the multiplication.
4. We define the quasi-DG category of virtual mixed elliptic motives (VMEM) by $K(VEM)$.

4.4.2. We define Young tableaux $Y_{p,n}$ as follows.

$$Y_{p,m} = \begin{pmatrix}
1 & 3 & \cdots & 2p - 1 & 2p + 1 & \cdots & 2p + m \\
2 & 4 & \cdots & 2p
\end{pmatrix}.$$ 

**Proposition 4.13.**

1. The objects $\text{Sym}^n$ and $M_{Y,p}(p)$ of $(VEM)$ are homotopy equivalent in the sense of Definition 4.14 (2).
2. If either the depth of $Y_1$ or that of $Y_2$ is greater than 2, then the complex $\text{Hom}^i(M_{Y_1}(s), M_{Y_2}(t))$ is acyclic.

**Proof.**

1. We denote the set $\{1, \cdots, 2p + n\}$ by $A$. Let $p_A \in \mathbb{Q}[S[A]]$ be the projector

$$\text{alt}(1, 2)\text{alt}(3, 4)\cdots\text{alt}(2p - 1, 2p)\text{sym}^n.$$ 

The projector $p_A^*$ induces the isomorphism

$$e_{Y_{p,n}}^* H^1(E)^{\otimes A} \simeq p_A^* H^1(E)^{\otimes A}$$
of the cohomology groups. By Corollary 4.5(1) and the Schur-Weyl reciprocity, it induces a homotopy equivalence between the objects $M_{p,i}(p)$ and $M_{Y,p,n}(p)$. So it suffices to show that the objects $\text{Sym}^n$ and $M_{p,i}(p)$ are homotopy equivalent. Let $pr_2 : E^n \times E^{2p+n} = E^n \times E^{2p} \times E^n \to E^{2p}$ be the projection to the second factor and $pr_{13} : E^n \times E^{2p+n} = E^n \times E^{2p} \times E^n \to E^n \times E^n$ be the projection to the product of the first and the third factors.

Let

\[(4.13) \quad I \in \text{Hom}^0(\text{Sym}^n, M_{p,i}(p))\]

be the closed homomorphism of degree zero defined by

\[\text{sym}^n f_n \wedge \cdots \wedge f_1 \{ (pr_2^*(-\frac{1}{4}(\Delta^+ - \Delta^-))^p) \cap pr_{13}^* \Delta_{E^n} \} - e_1 \wedge \cdots \wedge e_{2p+n} p_i,\]

where the subscript $-$ means the $(\cdot)$-part by the action of the group $(\mathbb{Z}/2\mathbb{Z})^\mathbb{A}$, and let

\[(4.14) \quad J \in \text{Hom}^0(M_{p,i}(p), \text{Sym}^n)\]

be the element defined by

\[p_i f_{2p+n} \wedge \cdots \wedge f_1 \{ \Delta^p \times \Delta_{E^n} \} - e_1 \wedge \cdots \wedge e_n \text{sym}^n.\]

The composite class $[J] \circ [I] \in H^0(\text{Hom}(\text{Sym}^n, \text{Sym}^n))$ is equal to the class of the diagonal by the intersection equality

\[[-\frac{1}{4}(\Delta^+ - \Delta^-)] \cap [\Delta] = \{0\}.\]

By changing the cycles $I$ and $J$ to their rationally equivalent ones $[I']$ and $[J']$, the class $[I' \circ J'] \in H^0(\text{Hom}(M_{Y,p,n}(V)(p), M_{Y,p,n}(V)(p))$ is homologous to the diagonal from Corollary 4.6.

(2) Let $p_1 = \|Y_1\|$. If the depth of $Y_1$ is greater than 2, then the cohomology class of

\[1_{M_{Y_2}} = e_{Y_1} \{ \Delta_{E^{p_1}} \} - e_{Y_1} \in Z^0_{p_1}(E^{p_1} \times E^{p_1})\]

is zero. It follows from Proposition 4.1 that this class is rationally trivial. □

**Definition 4.14.** Let $F : C_1 \to C_2$ be a functor of two DG-categories $C_1$ and $C_2$.

1. The functor $F$ is said to be homotopy fully faithful if the map

\[\text{Hom}^{\bullet}_1(A, B) \to \text{Hom}_2^{\bullet}(F(A), F(B))\]

is a quasi-isomorphism for all $A, B \in \text{ob}(C_1)$.

2. Two objects $M, N$ in $C_2$ are said to be homotopy equivalent if there exist closed homomorphisms $I, J$ of degree zero

\[I \in \text{Hom}^0(M, N), \quad J \in \text{Hom}^0(N, M),\]

such that the cohomology classes $I \circ J$ and $J \circ I$ are equal to identities.

3. The functor $F$ is said to be homotopy essentially surjective if for any object $M$ in $C_2$, there exists an object $N$ in $C_1$ such that $F(N)$ is homotopy equivalent to $M$. 
The functor $F$ is said to be weak homotopy equivalent if it is homotopy fully faithful and homotopy essentially surjective.

The following proposition is a consequence of Proposition 4.13.

**Proposition 4.15.** The natural functor $(MEM) \to (VMEM)$ is weak homotopy equivalent.

**4.5. Bar complex of a small DG-category.**

**4.5.1.** Let $T_2$ be a set. We consider a small DG category $C_2$ whose class of objects is the set $T_2$. The complex of homomorphisms $\text{Hom}_{C_2}(V_1, V_2)$ is denoted by $H(V_1, V_2)$. The multiplication

$$\eta : H(V_1, V_2) \otimes H(V_2, V_3) \to H(V_1, V_3)$$

is defined by the composite of the transposition and the composite of the complex of homomorphisms. Let $ev : C_2 \to \text{Vect}_k$ be a functor of DG categories. Here the category $\text{Vect}_k$ of vector spaces is a DG category in obvious way. The functor $ev$ is called an augmentation of $C_2$.

In this subsection, we define $\text{Bar}(C_2, ev)$. For a finite sequence $\alpha = (\alpha_0 < \cdots < \alpha_n)$ of integers, we define $\text{Bar}_\alpha(C_2, ev)$ by

$$\bigoplus_{v_0, \ldots, v_n \in T_2} \left( ev(V_0) \otimes H(V_0, V_1) \otimes \cdots \otimes H(V_{n-1}, V_n) \otimes ev(V_n)^* \right).$$

We define a complex $\text{Bar}_n(C_2, ev)$ by

$$\bigoplus_{|\alpha|=n} \text{Bar}_\alpha(C_2, ev).$$

For $\beta = (\alpha_0, \ldots, \alpha_k, \ldots, \alpha_n)$, we define

$$d_{\beta \alpha} : \text{Bar}_\alpha(C_2, ev) \to \text{Bar}_\beta(C_2, ev)$$

by

$$d_{\beta \alpha}(v_0 \otimes \varphi_{01} \otimes \cdots \otimes \varphi_{n-1n} \otimes v_n^*)$$

$$= (-1)^k \begin{cases} ev(\varphi_{01})(v_0) \otimes \varphi_{12} \otimes \cdots \otimes \varphi_{n-1n} \otimes v_n^* & \text{if } k = 0 \\ v_0 \otimes \cdots \otimes (\varphi_{k-1,k} \cdot \varphi_{k,k+1}) \otimes \cdots \otimes v_n^* & \text{if } 1 \leq k \leq n \\ v_0 \otimes \varphi_{01} \otimes \cdots \otimes \varphi_{n-2,n-1} \otimes (v_n^* \circ ev(\varphi_{n-1n})) & \text{if } k = n. \end{cases}$$

By taking summation for $\alpha$ and $\beta$, we have a double complex

$$\cdots \to \text{Bar}_n(C_2, ev) \to \text{Bar}_{n-1}(C_2, ev) \to \cdots$$

$$\to \text{Bar}_1(C_2, ev) \to \text{Bar}_0(C_2, ev) \to 0.$$

**Definition 4.16.** The bar complex $\text{Bar}(C_2, ev)$ is defined by the associate simple complex of $\text{Bar}(C_2, ev)$. 

4.5.2. Let $T_1$ and $T_2$ be sets and we consider a surjective map $\varphi : T_1 \to T_2$ and a section $\psi : T_2 \to T_1$, i.e. $\varphi \circ \psi = id_{T_2}$. We introduce a DG category $\varphi^* C_2$ as follows.

1. The class of objects is defined to be $T_1$.
2. For elements $V_1, V_2$ in $T_1$, the complex of homomorphism $\text{Hom}_{\varphi^* C_2}(V_1, V_2)$ is defined to be $\text{Hom}_{C_2}(\varphi(V_1), \varphi(V_2))$.
3. The composite of the complex of homomorphisms are defined to be those in $C_2$.

Note that if $\varphi(V_1) = \varphi(V_2) = W$, then $V_1$ and $V_2$ are canonically isomorphic in $\varphi^* C_2$. The composite of functors $ev \circ \varphi : \varphi^* C_2 \to C_2 \to Vect_k$ defines an augmentation of $\varphi^* C_2$.

Then we also have a bar complex $\text{Bar}(\varphi^* C_2, ev \circ \varphi)$. The natural functor $\varphi : \varphi^* C_2 \to C_2$ defines a homomorphism of bar complexes

$$\text{Bar}(\varphi) : \text{Bar}(\varphi^* C_2, ev \circ \varphi) \to \text{Bar}(C_2, ev).$$

**Proposition 4.17.** The homomorphism $\text{Bar}(\varphi)$ is a quasi-isomorphism.

**Proof.** The map $\psi : T_2 \to T_1$ defines a homomorphism of complex

$$\text{Bar}(\psi) : \text{Bar}(C_2, ev) \to \text{Bar}(\varphi^* C_2, ev \circ \varphi).$$

One can check that the composite $\text{Bar}(\varphi) \circ \text{Bar}(\psi)$ is the identity. We will show that $\text{Bar}(\psi) \circ \text{Bar}(\varphi)$ induces the identity on the cohomologies.

Let $N$ be an integer. By taking summations for $\alpha = (\alpha_0 < \cdots < \alpha_n)$ such that $N < \alpha_0$, we obtain a subcomplex $\text{Bar}(C_2, ev)_{N<}$ of $\text{Bar}(C_2, ev)$. (In the proof of Proposition 2.7, we used the similar notation.) We define a map

$$\theta : \text{Bar}(\varphi^* C_2, ev \circ \varphi)_{N<} \to \text{Bar}(\varphi^* C_2, ev \circ \varphi)_{N-1<}$$

of degree $-1$ such that

$$(4.15) \quad d\theta + \theta d = \text{Bar}(\psi) \circ \text{Bar}(\varphi) - 1$$

on $V\text{Bar}(\varphi^* C_2, ev \circ \varphi)_{N<}$. The composite map $\psi \circ \varphi$ is denoted by $r$ and $H(V_i, V_{i+1}), H(V_i, r(V_i))$ and $H(r(V_i), r(V_{i+1}))$ are denoted by $H_{i,i+1}, H_{i,i}$ and $H_{i,i+1}$. The element in $H_{i,i+1}$ corresponding to $\varphi_{i,i+1}$ in $H_{i,i+1}$ is denoted by $\varphi_{i,i+1}^{-1}$. The element in $H_{i,i}$ corresponding to the identity is denoted by $id_{i,i}$. For $i = 0, \ldots, n$, we define $\theta_1^{(i)}$ as follows.

$$\theta_1^{(i)} : ev(V_0)^* \otimes H_{0,1} \otimes \cdots \otimes H_{n-1, n} \otimes ev(V_n)$$

$\quad \to ev(V_0)^n \otimes H_{0,1} \otimes \cdots \otimes H_{i-1,i} \otimes H_{i,i} \otimes H_{i+1,i} \otimes \cdots \otimes ev(r(V_n))$

$\quad \quad \quad = v_0^{* \otimes \varphi_{0,1} \otimes \cdots \otimes \varphi_{n-1,n}^{* \otimes v_n}$

$\quad \quad \quad \quad \mapsto v_0^n \otimes \varphi_{0,1} \otimes \cdots \otimes \varphi_{i-1,i} \otimes id_{i,i} \otimes \varphi_{i,i+1} \otimes \cdots \otimes v_n$
The map $\theta_{2}^{(i)}$

$\theta_{1}^{(i)}: ev(V_0)^* \otimes H_{0,1} \otimes \cdots \otimes H_{n-1,n} \otimes ev(V_n)$

$\rightarrow ev(V_0)^* \otimes H_{0,1} \otimes \cdots \otimes H_{i-1,i} \otimes H_{i,i} \otimes H_{i,i+1} \cdots \otimes ev(V_n)$

for $i = 0, \ldots, n$ is defined similarly. Then one can check that the identity (4.15) holds for $\theta = \sum_{i=0}^{n} (-1)^i (\theta_{1}^{(i)} - \theta_{2}^{(i)})$. By taking the inductive limit for $N$, we have the proposition. \hfill \Box

4.5.3. Let $TT = \{(Y,p)\}$ the set of twisted tableaux, i.e. pairs of $Y \in \tilde{P}$ and $p \in \mathbb{Z}$. We define a small DG category $C_{VEM}$ as follows.

(1) The class of objects is $TT$.

(2) For elements $V_1 = (Y_1, p_1), V_2 = (Y_2, p_2) \in TT$, the complex of homomorphism is given by

$H(V_1, V_2) = \text{Hom}_{VEM}(M_{Y_1}(p_1), M_{Y_2}(p_2))$.

For a two dimensional vector space $V$ let $ev_V$ be a functor from $(VEM)$ to $(KVect)$ defined by

$M_Y(p) \mapsto M_Y(V) \otimes (Alt^2(V))^\otimes(-p)$.

We have defined the bar complex $Bar(C_{VEM}, ev_V)$. Let $TT_{\leq 1}$ be the set of twisted tableaux of depth smaller than or equal to one and $C_{EM}$ be the full sub DG category of $M_Y$ with $Y \in TT_{\leq 1}$. If $V$ is the standard representation of the group $GL_2$ then we have

$Bar(A_{EM}, \epsilon) = Bar(C_{EM}, ev_V)$.

By Proposition 4.13 and Proposition 4.17 we have the following proposition.

**Proposition 4.18.** The natural homomorphism

$Bar(C_{EM}, ev_V) \rightarrow Bar(C_{VEM}, ev_V)$

is a quasi-isomorphism.

**Proof.** Let $TT_{\leq 1}$ be the set of twisted tableaux of depth smaller than or equal to one. We define a surjective map $\varphi: TT \rightarrow TT_{\leq 1}$ by

$\varphi(Y) = \begin{cases} Y_{0,m}(q-p) & \text{if } Y = Y_{p,m}(q) \\ 0 & \text{otherwise.} \end{cases}$

Then the natural inclusion $\psi: TT_{\leq 1} \rightarrow TT$ is a section of $\varphi$. By Proposition 4.17 the map $Bar(\varphi): Bar(\varphi^*C_{EM}) \rightarrow Bar(C_{EM})$ is a quasi-isomorphism. As a consequence, the natural inclusion

(4.16) $Bar(\psi): Bar(C_{EM}) \rightarrow Bar(\varphi^*C_{EM})$

is also a quasi-isomorphism. We define a DG functor $\alpha: \varphi^*C_{EM} \rightarrow C_{VEM}$ as follows.
(1) The map on the set of objects are identity.
(2) For objects $Y_1, Y_2$ with $Y_1, Y_2 \in TT$, the map

$$\alpha_{Y_1,Y_2}: \text{Hom}_{\mathcal{C}_{EM}}(Y_1,Y_2) \to \text{Hom}_{\mathcal{C}_{VEM}}(Y_1,Y_2)$$

is obtained by the homotopy equivalence given in Proposition 4.12.

Then the composite of the functor $\alpha$ and the augmentation map $ev_V : \mathcal{C}_{VEM} \to \text{Vect}$ is equal to the augmentation map $ev_V$ of $\varphi^*\mathcal{C}_{EM}$. Therefore we have the following homomorphism of bar complexes:

\begin{equation}
\text{Bar}(\varphi^*\mathcal{C}_{EM}, ev_V) \to \text{Bar}(\mathcal{C}_{VEM}, ev_V).
\end{equation}

Since the maps $\alpha_{Y_1,Y_2}$ are quasi-isomorphisms, the above map is a quasi-isomorphism. By the quasi-isomorphisms (4.16) and (4.17), we have the required quasi-isomorphism. \hfill \Box

By the similar argument as in 3.3, we have the following proposition.

**Proposition 4.19.** The quasi-DG category of comodules over the bar complex $\text{Bar}(\mathcal{C}_{VEM}, ev_V)$ is homotopy equivalent to the quasi-DG category $(VEM)$ of virtual mixed elliptic motives.

Before giving the outline of the proof, we define the contraction map $\text{con}$.

4.5.4. **Definition of the map con.** Let $N_0, N_1, N_2$ be objects in $(VEM)$ and $p$ be an integer. We assume that $N_2 = \oplus_i M_Y(p) \otimes V_i$. We introduce a contraction homomorphism $\text{con}$ by the composite of

$$H(N_1, N_2) \otimes H(N_2, N_3)$$

$$= \oplus_{Y_1(p), Y_2(p) \in TT} H(N_1, M_Y(p)) \otimes \text{Hom}_{GL\infty}(M_Y(p), N_2)$$

$$\otimes \text{Hom}_{GL\infty}(N_2, M_Y(p)) \otimes H(M_Y(p), N_3)$$

$$\cong \oplus_{Y(p) \in TT} H(N_1, M_Y(p)) \otimes H(M_Y(p), N_3)$$

Here the map $\tau$ is the map induced by the multiplication:

$$\tau: \text{Hom}_{GL\infty}(M_Y, N_1) \otimes \text{Hom}_{GL\infty}(N_1, M_Y)$$

$$\to \text{Hom}_{GL\infty}(M_Y, M_Y) \simeq \begin{cases} k & (Y_1 = Y_2) \\ 0 & (Y_1 \neq Y_2) \end{cases}$$

Similarly, we can define homomorphisms $\text{con}$:

$$ev_V(N_0) \otimes H(N_0, N_1) \to \oplus_{Y(p) \in TT} ev_V(M_Y(p)) \otimes H(M_Y(p), N_1),$$

$$H(N_0, N_1) \otimes ev_V(N_1) \to \oplus_{Y(p) \in TT} H(N_0, M_Y(p)) \otimes ev_V(M_Y(p))^*.$$  

By composing the above contraction, we define the following map, which is also called a contraction:

\begin{equation}
H(U_0, U_1) \otimes \cdots \otimes H(U_{n-1}, U_n)
\end{equation}

$$\to \oplus_{Y_0, \ldots, Y_n} \text{Hom}_{GL\infty}(U_0, M_{Y_0}) \otimes H(M_{Y_0}, M_{Y_1}) \otimes \cdots$$

$$\otimes H(M_{Y_{n-1}}, M_{Y_n}) \otimes \text{Hom}_{GL\infty}(M_{Y_n}, U_n).$$
4.5.5. Outline of the proof of Prop 4.19

Proof. The proof is similar to Theorem 3.14. We only give the correspondence on objects. Let $W = (W^i, d_{ji})$ be a DG-complex in $(VEM)$. We introduce a $\text{Bar}(C_{VEM}, ev_V)$-comodule structure

$$\Delta_W : ev_V(W) \to \text{Bar}(C_{VEM}, ev_V) \otimes ev_V(W)$$

on $ev_V(W) = \bigoplus_i ev_V(W^i)e^{-i}$. For an index $\alpha = (\alpha_0 < \cdots < \alpha_n)$ such that $\alpha_0 = i, \alpha_n = j$, the component

$$\Delta_\alpha W : ev_V(W^i) \to \text{Bar}(C, ev_V)^\alpha \otimes ev_V(W^j)$$

can be described as follows. By the data $d_{ji} \in H(W^i, W^j)$ of the DG complex $W$, we define an element

$$D^\alpha = d_{\alpha_0, \alpha_1} \otimes \cdots \otimes d_{\alpha_{n-1}, \alpha_n} \in H(W^{\alpha_0}, W^{\alpha_1}) \otimes \cdots \otimes H(W^{\alpha_{n-1}}, W^{\alpha_n}).$$

By the contraction map (4.18), we have the following map

$$H(W^{\alpha_0}, W^{\alpha_1}) \otimes \cdots \otimes H(W^{\alpha_{n-1}}, W^{\alpha_n}) \to \bigoplus_{Y_0, \ldots, Y_n \in TT} H(M_{Y_0}, M_{Y_1}) \otimes H(M_{Y_{n-1}}, M_{Y_n}) \otimes \ldots$$

$$\otimes H(M_{Y_0}, M_{Y_1}) \otimes H(M_{Y_{n-1}}, M_{Y_n})$$

The image of $D^\alpha$ under the map defines the required map

$$\Delta_W : ev_V(W^{\alpha_0}) \to \left[ ev_V(M_{Y_0})^{\alpha_0} \otimes H(M_{Y_0}, M_{Y_1}) \otimes \ldots \right.$$  

$$\left. \otimes H(M_{Y_{n-1}}, M_{Y_n})^{\alpha_n} \otimes ev_V(M_{Y_n})^* \right] \otimes ev_V(W^{\alpha_n}).$$

□

Definition 4.20. The quasi-DG category of comodule over $\text{Bar}(C_{VEM})$ is called a quasi-DG category of mixed elliptic motives. The homotopy category becomes a triangulated category and it is called the triangulated category of virtual mixed elliptic motives.

Remark 4.21. In this section, we assume that $E$ is an elliptic curve over a field $K$. In general we can modify the definition of relative quasi-DG category and relative quasi-DG algebra for an elliptic curve $X \to S$ over an arbitrary scheme $S$ over a field $k$. Then $\text{Bar}(C_{VEM})(E/S)$ and $\text{VMEM}(E/S)$ become a contravariant functor and a fibered quasi-DG category over non-CM points $(\text{Sch}/S)_{\text{non-CM}}$ of $(\text{Sch}/S)$.

4.6. Tensor and antipodal structure on DG category.
4.6.1. Let $C$ be a DG category.

**Definition 4.22.**

1. A tensor structure $(\mathcal{C}, \otimes)$ on $C$ consists of
   (a) The biadditive correspondence
   \[
   \bullet \otimes \bullet : \text{ob}(C) \times \text{ob}(C) \to \text{ob}(C) : (M, N) \mapsto M \otimes N
   \]
   on pairs of objects, and
   (b) a natural homomorphism of complexes
   \[
   \text{Hom}^\bullet(M_1, M_2) \otimes \text{Hom}^\bullet(N_1, N_2) \to \text{Hom}^\bullet(M_1 \otimes N_1, M_2 \otimes N_2).
   \]
2. A tensor structure $(\mathcal{C}, \otimes)$ satisfies the distributive law if the relation
   \[
   (f_1 \otimes f_2) \circ (g_1 \otimes g_2) = (-1)^{\deg(f_2) \deg(g_1)}(f_1 \circ g_1) \otimes (f_2 \circ g_2),
   \]
   is satisfied for $f_i \in \text{Hom}^\bullet(M_i, N_i)$, $g_i \in \text{Hom}^\bullet(L_i, M_i)$ for $i = 1, 2$.
3. Let $(\mathcal{C}, \otimes)$ be a tensor structure on the category $\mathcal{C}$. A system $\{c_{A,B}\}$ of closed degree zero isomorphisms $c_{A,B} : A \otimes B \to B \otimes A$ is called the commutativity constrain if $c_{A,B}$ and $c_{B,A}$ are inverse to each other and they satisfies the relation:
   \[
   (f_N \otimes f_M) \circ c_{M_1,N_1} = (-1)^{\deg(f_M) \deg(f_N)}c_{M_2,N_2} \circ (f_M \otimes f_N)
   \]
   \[
   \in \text{Hom}(M_1 \otimes N_1, N_2 \otimes M_2)
   \]
   for $f_M \in \text{Hom}(M_1, M_2), f_N \in \text{Hom}(N_1, N_2)$.
4. Let $(\mathcal{C}, \otimes)$ be a tensor structure on the category $\mathcal{C}$. A system $\{c_{A,B,C}\}$ of closed degree zero isomorphisms $c_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ is called the associativity constrain if the following holds:
   \[
   c_{L_2,M_2,N_2} \circ ((f_L \otimes f_M) \otimes f_N) = (f_L \otimes (f_M \otimes f_N)) \circ c_{L_1,M_1,N_1}
   \]
   \[
   \in \text{Hom}((L_1 \otimes M_1) \otimes N_1, L_2 \otimes (N_2 \otimes M_2))
   \]
   for $f_L \in \text{Hom}(L_1, L_2), f_M \in \text{Hom}(M_1, M_2), f_N \in \text{Hom}(N_1, N_2)$.

4.6.2. The following example illustrates the category of naive elliptic motives does not have a natural tensor structure with the distributive property. On the category of $\text{Rep}_{\text{GCL}}(V)$, we have $V \otimes V \simeq \text{Sym}^2(V) \oplus \mathbb{Q}(-1)$. Let $V_1, \ldots, V_6$ be copies of $V$ and choose an isomorphism $V_3 \otimes V_4 = \text{Sym}^2(V) \oplus \mathbb{Q}(-1)$. We consider the composites of
   \[
   id_3 \otimes id_4 \in \text{Hom}^0(V_3 \otimes V_4, V_5 \otimes V_6), \quad id_1 \otimes id_2 \in \text{Hom}^0(V_1 \otimes V_2, V_3 \otimes V_4)
   \]
   and consider a decomposition
   \[
   id_3 \otimes id_4 = S_1 + A_1, \quad id_1 \otimes id_2 = S_2 + A_2
   \]
   according to the decomposition
   \[
   id_3 \otimes id_4 \in \text{Hom}^0(V_3 \otimes V_4, V_5 \otimes V_6) = \text{Hom}^0(\text{Sym}^2 \oplus \mathbb{Q}(-1), V_5 \otimes V_6),
   \]
   \[
   id_1 \otimes id_2 \in \text{Hom}^0(V_1 \otimes V_2, V_3 \otimes V_4) = \text{Hom}^0(V_1 \otimes V_2, \text{Sym}^2 \oplus \mathbb{Q}(-1)).
   \]
   Let $\Delta^+_ij$ and $\Delta^-ij$ be divisors defined by $x_i = x_j$ and $x_i + x_j = 0$, where $x_i$ is the coordinate of the copy $E_i$. Thus $A_1$ and $A_2$ are equal to $\frac{1}{2}(\Delta^+_56 + \Delta^-56)$ and
\[ \frac{1}{2}(\Delta_{12}^+ + \Delta_{12}^-) \] as elements in \( Z^*(pt \times (E_5 \times E_6), \bullet) \) and \( Z^*((E_1 \times E_2) \times pt, \bullet) \). Therefore the \( A_1 \circ A_2 \) composite is equal to
\[ \frac{1}{4}(\Delta_{56}^+ - \Delta_{56}^-) \cap (\Delta_{12}^+ - \Delta_{12}^-) \]
On the other hand, the anti-symmetric part \( A \) of \( (id_3 \circ id_1) \otimes (id_4 \circ id_2) \) is equal to the intersection
\[ \frac{1}{8}\left[(\Delta_{15}^+ - \Delta_{15}^-) \cap (\Delta_{26}^+ - \Delta_{26}^-) + (\Delta_{16}^+ - \Delta_{16}^-) \cap (\Delta_{25}^+ - \Delta_{25}^-)\right] \]
Though \( A \) and \( A_1 \circ A_2 \) are homotopy equivalent, they are different as cycles.

4.6.3. Using the inverse of \( S \), we have a transpose representation \( M^t \) on \( M^* = \text{Hom}_k(M, k) \), which is a left \( S \)-comodule on the underlying right \( S \)-comodule \( M^* \). In general, we introduce an antipodal structure on DG categories.

**Definition 4.23.** Let \( \mathcal{C} \) be a DG category with a tensor structure. A pair of
1. contravariant functor \( \mathcal{C} \to \mathcal{C} : M \to M^t \), and
2. a system of degree zero closed isomorphisms
\[ \theta : (M \otimes N)^t \to M^t \otimes N^t \]
is called an antipodal structure on \( \mathcal{C} \) if the following diagram commutes.
\[
\begin{array}{ccc}
\text{Hom}_C(M_1, N_1) \otimes \text{Hom}_C(M_2, N_2) & \xrightarrow{\otimes} & \text{Hom}_C(M_1 \otimes M_2, N_1 \otimes N_2) \\
\downarrow & & \downarrow \\
\text{Hom}_C(N_1^t, M_1^t) \otimes \text{Hom}_C(N_2^t, M_2^t) & \xrightarrow{\otimes} & \text{Hom}_C(N_1^t \otimes N_2^t, M_1^t \otimes M_2^t)
\end{array}
\]

\[ \text{Hom}_C((N_1 \otimes N_2)^t, (M_1 \otimes M_2)^t). \]

4.6.4. Assume that \( \mathcal{C} \) has a tensor structure \((\mathcal{C}, \otimes)\). We extend a tensor structure \((K\mathcal{C}, \otimes)\) on the category \( K\mathcal{C} \) as follows. Let \((V^i, \{d_{ij}\})\) and \((W^j, \{e_{ij}\})\) be DG complexes in \( \mathcal{C} \). Then \( V^i \otimes W^j \) is an object in \( \mathcal{C} \). The relative DG complex structure \((\oplus_{i+j=k}(V^i \otimes W^j), f_{kl})\) is given by
\[ f_{kl} = \sum_{i+j=k, p+j=l} \tau(d_{ip} \otimes 1_{W^j}) + \sum_{i+j=k, i+q=l} \tau(1_{V^i} \otimes e_{jq}). \]

We can check the following lemma.

**Lemma 4.24.** If the tensor structure \( \otimes \) on \( \mathcal{C} \) is distributive (resp. commutative), the induced tensor structure on \( K\mathcal{C} \) is also distributive (resp. commutative).

4.6.5. Let \( \mathcal{C} \) be a small DG-category and \( ev : \mathcal{C} \to \text{Vect}_k \) be an augmentation. This tensor structure on \( K\mathcal{C} \) gives rise to the shuffle product on the relative bar complex \( \text{Bar} = \text{Bar}(\mathcal{C}, ev) \) via the equivalence of categories as follows. Let \( \text{Bar} \) be the object in \( K\mathcal{C} \) corresponding to the left \( \text{Bar} \) comodule \( \text{Bar} \). By the tensor structure, the object \( \text{Bar} \otimes \text{Bar} \) in \( K\mathcal{C} \) is defined. Therefore we have the corresponding object \( \text{Bar} \otimes \text{Bar} \) in \( \text{Bar} \)-comodule. Let
\[ \Delta_{\text{Bar} \otimes \text{Bar}} : \text{Bar} \otimes \text{Bar} \to \text{Bar} \otimes (\text{Bar} \otimes \text{Bar}) \]
be the $\text{Bar}$-comodule structure on $\text{Bar} \otimes \text{Bar}$. By composing the counit $u \otimes u : \text{Bar} \otimes \text{Bar} \to k$, we have a shuffle product

$$\text{Bar} \otimes \text{Bar} \to \text{Bar}.$$ 

One can check the following properties.

**Proposition 4.25.** If there is a commutative constrain (resp. associativity constrain), the shuffle product on $\text{Bar}$ is commutative (resp. associative). Moreover, an antipodal structure on $C$ gives an antipodal of $\text{Bar}$.

### 4.7. Tensor product for virtual mixed elliptic motif.

We define a tensor structure on $(\text{VEM})$ in this subsection. Let $M_{Y_1}(p_1), M_{Y_2}(p_2), M_{Y_3}(p_3)$ and $M_{Y_4}(p_4)$ be objects of $(\text{VEM})$. We define a homomorphism

(4.19)

$$\text{Hom}(M_{Y_1}(p_1), M_{Y_3}(p_3)) \otimes \text{Hom}(M_{Y_2}(p_2), M_{Y_4}(p_4))$$

$$= e_{Y_1} \mathcal{H}^*(E^{s(Y_1), E^{s(Y_3)}, p_3 - p_1} e_{Y_3} \otimes e_{Y_2} \mathcal{H}^*(E^{s(Y_2), E^{s(Y_4)}, p_4 - p_2}) e_{Y_4}$$

$$\to \text{Hom}(M_{Y_1}(p_1) \otimes M_{Y_2}(p_2), M_{Y_3}(p_3) \otimes M_{Y_4}(p_4))$$

$$= \oplus_{Z_1, Z_2} \text{Hom}_{GL_\infty}(M_{Y_1} \otimes M_{Y_2}, M_{Z_1})$$

$$\otimes \text{Hom}(M_{Z_1}(p_1 + p_2), M_{Z_2}(p_3 + p_4))$$

$$\otimes \text{Hom}_{GL_\infty}(M_{Z_2}, M_{Y_3} \otimes M_{Y_4})$$

We choose bases $\{\varphi_{Z_1, i}\}$ and $\{\psi_{Z_2, j}\}$ of

$$\text{Hom}_{GL_\infty}(M_{Z_1}, M_{Y_1} \otimes M_{Z_2})$$

and $\text{Hom}_{GL_\infty}(M_{Z_2}, M_{Y_3} \otimes M_{Y_4})$.

Their dual bases in

$$\text{Hom}_{GL_\infty}(M_{Y_1} \otimes M_{Z_2}, M_{Z_1})$$

and $\text{Hom}_{GL_\infty}(M_{Y_3} \otimes M_{Y_4}, M_{Z_2})$.

under the composite paring are written as $\{\varphi^*_{Z_1, i}\}$ and $\{\psi^*_{Z_2, j}\}$. Let $(i, j, k) = (1, 1, 3)$ or $(2, 2, 4)$. For

$$f_j = e_{Y_1} \Lambda(s_i) Z_j \Lambda(s_k) e_{Y_k} \in e_{Y_1} \mathcal{H}^*(E^{s(Y_1), E^{s(Y_k)}, p_k - p_i}) e_{Y_k},$$

we set

$$f_1 \times f_2 = (-1)^{\# s_1(\# s_2 + \deg(Z_2)) + \# s_2 \deg(Z_1)} \Lambda(s_1) \Lambda(s_3) (Z_1 \times Z_2) \Lambda(s_2) \Lambda(s_4).$$

The map (4.19) is defined by

$$e_{Y_1} f_1 e_{Y_3} \otimes e_{Y_2} f_2 e_{Y_4} \mapsto \sum_{Z_1, Z_2, i, j} \varphi^*_{Z_1, i} \otimes (\varphi_{Z_1, i}(f_1 \times f_2) \psi^*_{Z_2, j}) \otimes \psi_{Z_2, j}.$$ 

for Here $s_i = s(Y_i)$ and $Z_1 \times Z_2$ is the product of algebraic cycles in $E^{s(Y_1) \cup \cdots \cup s(Y_4)} \times \mathcal{A}^*$ and $\varphi_{Z_1, i}$ and $\psi^*_{Z_2, j}$ acts on the space $\mathcal{H}^*(E^{s(Y_1) \cup s(Y_3)}, E^{s(Y_2) \cup s(Y_4)}, \bullet)$ from the left and the right via the Schur-Weyl reciprocity (4.12).
Proposition 4.26 (Distributive relation). This tensor structure satisfies distributive law.

To prove the above proposition, we use the following easy lemma.

Lemma 4.27. Let $S$ be a finite set and 

$$(\ast,\ast)_S : Z^i(E^a \times E^S, p) \otimes Z^j(E^S \times E^b, q) \to Z^{i+j}(E^a \times E^b, p+q)$$

be the composite with the push forward $E^a \times \Delta_{E^S} \times E^b \to E^a \times E^b$ and the intersection with $E^a \times \Delta_{E^S} \times E^b$. Let $\psi \in \mathcal{Q}[\text{Isom}(S', S)]$. The left and right action of $\psi$ defines a homomorphisms

$$Z^i(E^{S'}, E^b, q) \to Z^i(E^S \times E^b, q),$$
$$Z^i(E^a \times E^S, p) \to Z^i(E^a \times E^{S'}, p).$$

Then we have $(g\psi, f)_{S'} = (g, \psi f)_S$ for $f \in Z^i(E^{S'} \times E^b, q)$ and $g \in Z^i(E^a \times E^S, p)$.

Proof. The lemma follows from the identity:

$$(x, w) \in E^a \times E^b \mid (x, \sigma(y)) \in \sigma(f), (z, w) \in g, \sigma(y) = z$$

$$= \{(x, w) \in E^a \times E^b \mid (x, y) \in f, (\sigma^{-1}(z), w) \in \sigma^{-1}(g), y = \sigma^{-1}(z)\},$$

for $\sigma \in \text{Isom}(S, S')$. □

Proof of Proposition 4.26. Let $f_i \in \text{Hom}^\bullet(L_i, M_i), g_i \in \text{Hom}^\bullet(M_i, N_i)$ for $i = 1, 2$. We check the distributive law for the multiplication map $\mu$. We fix a bases $\{\varphi_{Z_1,i}\}, \{\phi_{Z_2,j}\}$ and $\{\psi_{Z_3,k}\}$ of $\text{Hom}_{GL_{\infty}}(M_{Z_1}, L_1 \otimes L_2), \text{Hom}_{GL_{\infty}}(M_{Z_2}, M_1 \otimes M_2)$ and $\text{Hom}_{GL_{\infty}}(M_{Z_3}, N_1 \otimes N_2)$. The dual bases are written by $\{\varphi^*_{Z_1,i}\}, \{\phi^*_{Z_2,j}\}$ and $\{\psi^*_{Z_3,k}\}$. We use the same notations for $f_1, f_2$ etc. as before. Then

$$\mu((f_1 \otimes f_2) \otimes (g_1 \otimes g_2))$$

$$= \sum_{Z_1, Z_2, Z_3, i, j, k} \mu\left(\varphi^*_{Z_1,i} \otimes (\varphi_{Z_1,i}(f_1 \times f_2)\phi^*_{Z_2,j}) \otimes \phi_{Z_2,j}\right)$$

$$\otimes \left(\phi^*_{Z_2,j} \otimes (\phi_{Z_2,j}(g_1 \times g_2)\psi^*_{Z_3,k}) \otimes \psi_{Z_3,k}\right)$$

$$= \sum_{Z_1, Z_2, Z_3, i, j, k} \varphi^*_{Z_1,i} \otimes \mu\left(\varphi_{Z_1,i}(f_1 \times f_2)\phi^*_{Z_2,j} \otimes (\phi_{Z_2,j}(g_1 \times g_2)\psi^*_{Z_3,k})\right) \otimes \psi_{Z_3,k}$$

$$= \sum_{Z_1, Z_3, i, k} \varphi^*_{Z_1,i} \otimes \varphi_{Z_1,i} \mu\left((f_1 \times f_2) \otimes (g_1 \times g_2)\right) \otimes \psi_{Z_3,k}$$

$$= (-1)^{\deg(f_2) + \deg(g_1)} \sum_{Z_1, Z_3, i, k} \varphi^*_{Z_1,i} \otimes \varphi_{Z_1,i} \mu(f_1 \otimes g_1) \times \mu(f_2 \otimes g_2) \psi^*_{Z_3,k} \otimes \psi_{Z_3,k}$$

$$= (-1)^{\deg(f_2) + \deg(g_1)} \mu(f_1 \otimes g_1) \otimes \mu(f_2 \otimes g_2).$$
4.7.1. Commutativity constraint and associativity constraint. We define a system of closed homomorphisms of degree zero \(c_{M,N}\) and \(c_{L,M,N}\) for \(L, M, N \in (VEM)\) by

\[
c_{M,N} = \sum_{Y,i} \alpha_i \otimes \Delta_Y \otimes \alpha_i^* \in \text{Hom}_{VEM}(M \otimes N, N \otimes M)
\]

\[
= \oplus_Y \text{Hom}_{GL_\infty}(M \otimes N, M_Y) \otimes \text{Hom}_{VEM}(M_Y, M) \otimes \text{Hom}_{GL_\infty}(M_Y, N \otimes M)
\]

\[
c_{L,M,N} = \sum_{Y,i} \beta_i \otimes \Delta_Y \otimes \beta_i^* \in \text{Hom}_{VEM}((L \otimes M) \otimes N, L \otimes (M \otimes N))
\]

\[
= \oplus_Y \text{Hom}_{GL_\infty}((L \otimes M) \otimes N, M_Y) \otimes \text{Hom}_{VEM}(M_Y, M) \otimes \text{Hom}_{GL_\infty}(M_Y, L \otimes (M \otimes N))
\]

where \(\{\alpha_i\}\) (resp. \(\{\beta_i\}\)) and \(\{\alpha_i^*\}\) (resp. \(\{\beta_i^*\}\)) are dual bases under the contraction pairing induced by the natural isomorphism

\[
M \otimes N \simeq N \otimes M, \quad (\text{resp.} (L \otimes M) \otimes N \simeq L \otimes (M \otimes N))
\]

in \(GL_\infty\). The following proposition is direct from the definitions.

**Proposition 4.28.** The systems of the above maps \(c_{M,N}\) and \(c_{L,M,N}\) satisfy the commutativity axiom and the associativity axiom, respectively.

4.7.2. Antipodal for \((VEM)\). We define a self-contravariant functor \(a : (VEM) \to (VEM)\) as follows. Let \(Y\) be a Young tableaux. We define \(\text{Hom}_{VEM}(M_Y(p), n)\)

\[
(M_Y(p))^t = e_Y^*(V^{\otimes s(Y)})(\#s(Y) - p),
\]

where \(e_Y^*\) is the adjoint of \(e_Y\) defined in Definition 4.9. We have the following isomorphism of complexes:

\[
\mathcal{H}^*(E^A, E^B, k) = \Lambda^*(A) \otimes Z^{a+k}(E^A \times E^B, \bullet) \otimes \Lambda(B)[-2k]
\]

\[
\tau : \Lambda^*(B) \otimes Z^{b+k-(k-b+a)}(E^A \times E^B, \bullet) \otimes \Lambda(A)[-2k+2b-2a]
\]

\[
= \mathcal{H}^*(E^B, E^A, k-b+a),
\]

where \(a = \#A, b = \#B\). Here \(\tau\) is defined by

\[
\tau(f_a \wedge \cdots \wedge f_1 \otimes z \otimes e_1 \wedge \cdots e_b \cdot e^{-2k})
\]

\[
= (-1)^{(a+b) \deg(z)+ab} f_b \wedge \cdots \wedge f_1 \otimes z \otimes e_1 \wedge \cdots e_a \cdot e^{-2k+2b-2a}.
\]
Using the map $\tau$, the functor $a$ for homomorphisms is defined by the following composite map:

$$
\begin{align*}
\text{Hom}_{VEM}((M_{Y_1}(p_1))^t, (M_{Y_2}(p_2))^t) \\
&= c_{Y_1} \text{Hom}_{VEM}(V^{\otimes S_1} (s_1 - p_1), V^{\otimes S_2} (s_2 - p_2)) c_{Y_2} \\
&= c_{Y_1} \mathcal{H}^* (E^S_1, E^S_2, s_2 - s_1 + p_1 - p_2) c_{Y_2} \\
&\xrightarrow{\tau} c_{Y_1} \mathcal{H}^* (E^S_2, E^S_1, p_1 - p_2) c_{Y_1} \\
&\simeq \text{Hom}_{VEM}(M_{Y_2}(p_2), M_{Y_1}(p_1)).
\end{align*}
$$

Here $S_1 = s(Y_1)$ and $S_2 = s(Y_2)$. We can check the following proposition.

**Proposition 4.29.** The map $a$ defines a contravariant functor from $(VEM)$ to $(VEM)$. Moreover $a$ defines an antipodal structure.

Thus we have the following theorem.

**Theorem 4.30.**

1. The natural DG functors $(EM) \to (VEM)$, $(MEM) \to (VMEM)$ are weak homotopy equivalent.

2. The quasi-DG category $(VMEM)$ has distributive, commutative and associative tensor structure with an antipodal.

4.8. **Shuffle product of virtual bar complex** $\text{Bar}(VEM)$. In this subsection, we give an explicit description of the shuffle product of $\text{Bar}(VEM)$ introduced in [16.4]. Let

$$
\sqcup(m, n) = \{(p^{(0)}, \ldots, p^{(m+n)}) \in (\mathbb{N}^2)^{m+n+1} | p^{(0)} = (0, 0), p^{(m+n)} = (m, n), p^{(i+1)} - p^{(i)} \in \{(0, 1), (0, 1)\}\}
$$

be the set of shuffles. Let $\sigma = (p^{(0)}, \ldots, p^{(m+n)})$ be an element in $\sqcup$. For $V_0, \ldots, V_m, W_0, \ldots, W_n \in TT$, we define $U^{\sigma}_i = V_{a_i} \otimes W_{b_i}$, where $p^{(k)} = (a_k, b_k)$. For $\alpha = (\alpha_0 < \cdots < \alpha_m)$ and $\beta = (\beta_0 < \cdots < \beta_n)$, we define $\gamma^\sigma = (\gamma^\sigma_0 < \cdots < \gamma^\sigma_{m+n})$ by $\gamma^\sigma_k = \alpha_k + \beta_k$. The complex of homomorphisms $\text{Hom}_{VEM}(A, B)$ is denoted as $\text{H}(A, B)$. Let $\varphi_{i,j+1} \in \text{H}(V_i, V_{i+1})$ and $\psi_{j,j+1} \in \text{H}(W_j, W_{j+1})$. We define $\tau_{k,k+1} \in \text{H}(U^{\sigma}_k, U^{\sigma}_{k+1})$ by

$$
\tau_{k,k+1} = \begin{cases} 
\varphi_{a_k, a_{k+1}} \otimes 1 & \text{if } p^{(k+1)} - p^{(k)} = (1, 0) \\
1 \otimes \psi_{b_k, b_{k+1}} & \text{if } p^{(k+1)} - p^{(k)} = (0, 1).
\end{cases}
$$

For elements

$$
v = v_0^{\alpha_0} \varphi_{01}^{\alpha_1} \cdots \varphi_{n-1,n}^{\alpha_{n-1}} v_n^* \\
\in \text{ev}(V_0) \otimes H(V_0, V_1) \otimes \cdots \otimes H(V_{n-1}, V_n) \otimes \text{ev}(V_n)^*,
$$

$$
w = w_0^{\beta_0} \psi_{01}^{\beta_1} \psi_{m-1,m}^{\beta_m} w_m^* \\
\in \text{ev}(W_0) \otimes H(W_0, W_1) \otimes \cdots \otimes H(W_{m-1}, W_m) \otimes \text{ev}(W_m)^*,
we define \((v \square w)^\sigma\) by
\[
(v \square w)^\sigma = (v_0 \otimes w_0) \otimes \tau_{01}^{\gamma_0} \otimes \cdots \otimes \tau_{m+n-1,m+n}^{\gamma_{m+n-1}} \otimes (v_m^* \otimes w_m^*)
\]
\[
\in ev(U_0^\sigma) \otimes H(U_0^\sigma, U_0^\sigma) \otimes \cdots \otimes H(U_{m+n-1}, U_{m+n}^\sigma) \otimes ev(U_{m+n}^\sigma).
\]

4.8.1. Using the maps \(\con\) defined in [4.5.4], we have the following map:
\[
ev(U_0^\sigma) \otimes H(U_1^\sigma, U_0^\sigma) \otimes \cdots \otimes H(U_{m+n}, U_{m+n}^\sigma) \otimes ev(U_{m+n}^\sigma)
\]
\[
\rightarrow \oplus_{Y_0, \ldots, Y_{m+n} \in TT} \ev(M_{Y_0})^\sigma \otimes H(M_{Y_1}, M_{Y_0}) \otimes \cdots
\]
\[
\otimes H(M_{Y_{m+n}}, M_{Y_{m+n-1}}) \otimes ev(M_{Y_{m+n}}).
\]

The image of \((v \square w)^\sigma\) is denoted as \((v \sqcup w)^\sigma\). The shuffle product of the bar complex is defined similarly.

**Example 4.31.** If \(\sigma = \{(0, 0), (1, 0), (1, 1), (2, 1)\}\), then \(U_1^\sigma\) and \(\tau_{j,j+1}^\sigma\) look as follows:
\[
V_1 \otimes W_1 \xrightarrow{\varphi_1 \otimes 1} V_2 \otimes W_1
\]
\[
V_0 \otimes W_0 \xrightarrow{\varphi_0 \otimes 1} V_1 \otimes W_0.
\]

For bar complex, we have and
\[
([v_0 | \varphi_0 | \varphi_{12} | v_2] \sqcup [w_0 | \tau_{01} | w_1])^\sigma
\]
\[
= \con( (v_0 \otimes w_0) \otimes (\varphi_0 \otimes 1) \otimes (1 \otimes \psi_0) \otimes (\varphi_{12} \otimes 1) \otimes (v_2^* \otimes w_1^*) )
\]

**Definition 4.32.** We define the shuffle product \(v \sqcup w\) of \(v, w \in \text{Bar}(\mathcal{C}_{VEM}, ev_V)\) by
\[
(4.20)\quad v \sqcup w = \sum_{\sigma \in \mathcal{S}(m, n)} (v \sqcup w)^\sigma.
\]

We can check the following proposition by definition.

**Proposition 4.33.** The shuffle product \((4.20)\) coincides with that defined in [4.6.5].

By Proposition [4.28, 4.29] we have the following theorem.

**Theorem 4.34.** The shuffle product on \(\text{Bar}(\mathcal{C}_{VEM}, ev_V)\) is commutative and associative. Therefore \(\text{Bar}(\mathcal{C}_{VEM}, ev_V)\) is a differential graded quasi-Hopf algebra and \(H^0(\text{Bar}(\mathcal{C}_{VEM}, ev_V)) = H^0(\text{Bar}(A_{EM}, ev_V))\) is a Hopf algebra. Therefore \(\text{Spec}(H^0(\text{Bar}(A_{EM}, ev_V)))\) is a pro-algebraic group.

**Definition 4.35 (Algebraic group \(G_{MEM}\)).**

1. We define the pro-algebraic group \(G_{MEM} = \text{Spec}(H^0(\text{Bar}(A_{EM}, ev_V)))\)
2. We define the Tannakian category \(A_{MEM}\) of mixed elliptic motives as the category of algebraic representations of \(G_{MEM}\). It is equivalent to the category of comodules over \(\text{Spec}(H^0(\text{Bar}(A_{EM}, ev_V)))\).
5. Mixed elliptic motif associated to elliptic polylogarithm

In the paper of [BL], they introduced an elliptic polylog motif $Pl_n$. In this section, we construct an object $Pl_n$ in $(MEM)$ concentrated at degree zero. Therefore the corresponding object defines a comodule over $H^0(Bar_{MEM})$. We write down the explicit comodule structure of $Pl_n$ in §5.5.

5.1. Simplicial, Cubical and Cubical-simplicial log complexes. In this section, we consider three double complexes $K_S$, $K_E$, and $K_{CS}$. Let $E$ be an elliptic curve over a field $K$ and $E_B$ the constant family of $E$ over the base scheme $B = E$. Let $s$ be the section of $E_B$ defined by the diagonal map $B \to E_B$. The map $\epsilon : x \mapsto s - x$ defines an action of $\mathbb{Z}/2\mathbb{Z}$ on $E_B$.

5.1.1. By using localization sequences for higher Chow group [B2], we have the following proposition.

**Proposition 5.1** (Levin [L] Proposition 1.1). Let $G$ be a finite group and $\chi : G \to \{\pm 1\}$ be a character of $G$. Let $X$ be a variety with an action $\rho : G \to \text{Aut}(X)$ of $G$. Let $Z_i \subset X$ ($i = 1, \ldots, k$) be a closed subset and $G_i = \{g \in G \mid \rho(x) = x \text{ for all } x \in Z_i\}$ be the stabilizer of points in $Z_i$. Assume that $\chi(G_i) = \{\pm 1\}$. Then we have

\[ CH^p(X - \cup_i Z_i, q)_\chi = CH^p(X, q)_\chi \]

5.1.2. Simplicial log complex $K_{S,n}$. We set $I_{S,p} = \{I \subseteq [1, n + 1] \mid \#I = p\}$ and $G_S = \mathbb{S}_{n+1}$. We set

\[ (E_{B}^{n+1})_s = \{p = (p_1, \ldots, p_{n+1}) \in E_{B}^{n+1} \mid \sum_{i=1}^{n+1} p_i = s\} \subset E_{B}^{n+1} \]

and define divisors $D_{S,i} = \{p \in (E_{B}^{n+1})_s \mid p_i = 0\}$ for $i = 1, \ldots, n + 1$ and $D_{S,I} = \cap_{i \in I} D_i$. The complex $K_{S,n}$ is defined by

\[ \cdots \to \oplus_{I \in I_{S,2}} \mathbb{Z}^{n-1}(E \times D_{S,I}, \bullet) \to \oplus_{I \in I_{S,1}} \mathbb{Z}^n(E \times D_{S,I}, \bullet) \]

\[ \to \mathbb{Z}^{n+1}(E \times (E_{B}^{n+1})_s, \bullet) \to 0. \]

The group $G_S$ acts naturally on the complex $K_{S,n}$. We define an open set $(U_{B}^{n+1})_s$ of $(E_{B}^{n+1})_s$ by

\[ (U_{B}^{n+1})_s = (E_{B}^{n+1})_s - \cup_{i=1}^{n+1} D_{S,i}. \]

By the localization sequence [B], we have

\[ H^j(K_{S,n}) \simeq CH^{n+1}(E \times (U_{B}^{n+1})_s, n + 1 - j). \]

and $H^j(K_{S,n, sgn}) \simeq CH^{n+1}(U_{n+1}^{n+1}, n + 1 - j)_{sgn}$, where $K_{S,n, sgn}$ is the alternating part of the complex $K_{S,n}$ for the action of $\mathfrak{S}_{n+1}$.
5.1.3. Cubical log complex $\mathbf{K}_{C,n}$. We define an index set
\[
\mathcal{I}_{C,p} = \{(I, \varphi) \mid I \subset [1, n], \#I = p, \varphi : I \to \{0, s\}\},
\]
and a group
\[
G_{C,n} = G_C = \mathfrak{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n = \{(\tau, \sigma_1, \ldots, \sigma_n)\}.
\]
Let $\psi$ be the non-trivial character of $\mathbb{Z}/2\mathbb{Z}$, and $\chi$ be a character of $G_C$ defined by
\[
\chi_n(\tau, \sigma_1, \ldots, \sigma_n) = \chi(\tau, \sigma_1, \ldots, \sigma_n) = sgn(\tau)\psi(\sigma_1) \cdots \psi(\sigma_n).
\]
We define a family of subvarieties $\{D_{C,J}\}$ of $E^n_B$ of codimension $p$ indexed by $J \in \mathcal{I}_{C,p}$ by
\[
D_{C,J} = \{(x_1, \ldots, x_n) \in E^n_B \mid x_i = \varphi(i) \text{ for all } i \in I\},
\]
where $J = (I, \varphi)$. The complex $\mathbf{K}_{C,n}$ is defined by
\[
\cdots \to \oplus_{J \in \mathcal{I}_{C,2}} \mathbb{Z}^{n-1}(E \times D_{C,J}, \bullet) \to \oplus_{J \in \mathcal{I}_{C,1}} \mathbb{Z}^n(E \times D_{C,J}, \bullet) \to Z^{n+1}(E \times E^n_B, \bullet) \to 0.
\]
The group $(\mathbb{Z}/2\mathbb{Z})^n$ acts on the complex $\mathbf{K}_{C,n}$ by the action
\[
(\sigma_1, \ldots, \sigma_n)(x_1, \ldots, x_n) = (\sigma_1(x_1), \ldots, \sigma_n(x_n)),
\]
where
\[
\sigma_i(x_i) = \begin{cases} x_i & \text{if } \sigma = id \\ s - x_i & \text{if } \sigma \neq id \end{cases}
\]
using the action $\epsilon : x \mapsto s - x$. The $\chi$ part of the complex is denoted by $\mathbf{K}_{C,n,\chi}$.

5.1.4. Cubical-simplicial log complex $\mathbf{K}_{CS,n}$. We use the same notations for the group $G_C$ and the action of $G_C$ on $E^n_B$ and the set of indices $\mathcal{I}_{C,p}$. For $1 \leq i \neq j \leq n$, we define a divisor $D_{ij}^\pm$ of $E^n_B$ by
\[
D_{ij}^+ = \{x_i = x_j\}, \quad D_{ij}^- = \{x_i = \sigma_j(x_j)\}.
\]
The group $G_C$ acts on the configuration $\{D_{ij}^\pm\}_{0 \leq i < j \leq n}$. We set $\mathcal{D} = \cup_{0 \leq i < j \leq n} D_{ij}^\pm$ and $(E^n)^0 = E^n - \mathcal{D}$. We define a family of subvarieties $\{D_{C,J}^0\}$ of $E^n_B$ of codimension $p$ ($J \in \mathcal{I}_{C,p}$) by $D_{C,J}^0 = D_{C,J} \cap (E^n)^0$. The complex $\mathbf{K}_{CS,n}$ is defined by
\[
\cdots \to \oplus_{J \in \mathcal{I}_{C,2}} \mathbb{Z}^{n-1}(E \times D_{C,J}^0, \bullet) \to \oplus_{J \in \mathcal{I}_{C,1}} \mathbb{Z}^n(E \times D_{C,J}^0, \bullet) \to Z^{n+1}(E \times (E^n_B)^0, \bullet) \to 0.
\]
The $\chi$-part of the complex is denoted by $\mathbf{K}_{CS,n,\chi}$.

**Proposition 5.2.** Let $x \in E^n$ be a point in $\mathcal{D}$. The the restriction of $\chi$ to the stabilizer $\text{Stab}_x(G_C)$ of $x$ in $G_C$ is non-trivial.

**Proof.** If $x$ is contained in $D_{ij}^+$, then the permutation $(ij)$ fixes $x$ and $\chi((ij), 1) = -1$. If $x$ is contained in $D_{ij}^-$, then the element $((ij), \delta)$ fixes the point $x$ and $\chi((ij), \delta) = -1$, where $\delta = (0, \ldots, i, \ldots, 1, \ldots, 0)$. \qed
Corollary 5.3. The homomorphism of complexes $K_{C,n,\chi} \to K_{CS,n,\chi}$ is a quasi-isomorphism.

Proof. By the argument in Proposition 5.1 and Proposition 5.2, the complex $K_{CS,n,\chi}$ is quasi-isomorphic to the $\chi$-part of the complex

$$
\cdots \to \bigoplus_{j \in \mathcal{I}_{n,1}} \mathbb{Z}^{n-1}(E \times D_{C,j}, \bullet) \to \bigoplus_{j \in \mathcal{I}_{n,1}} \mathbb{Z}^n(E \times D_{C,j}, \bullet) \to \mathbb{Z}^{n+1}(E \times E_B^n, \bullet) \to 0.
$$

Therefore the natural restriction map $K_{C,n,\chi} \to K_{CS,n,\chi}$ is a quasi-isomorphism. □

5.2. Linear correspondences and their translations. For an element $s \in E^n(K)$, the algebraic correspondence associated to the translation by $s$ is denoted by $T_s \in CH^n(E^n \times E^n)$. Let $F/K$ be a Galois extension of $K$, $N$ a positive integer. Assume that all points $\tilde{s}$ such that $N\tilde{s} = s$ are defined over $F$. Then

$$
T_{s,N} = \frac{1}{N^{2n}} \sum_{N\tilde{s}=s} T_{\tilde{s}}
$$

defines a correspondence in $CH^n(E^n \times E^n)$. Note that the coefficient of Chow groups are $\mathbb{Q}$. We have a natural inclusion:

$$
CH^i(E^n \times E^n, j) \to CH^i(X_F^n \times_F E_F^n, j).
$$

Lemma 5.4. (1) Under the above notations, the image of $T_{s,N} \in CH^n(E^n \times E^n)$ in $CH^n(E_F^n \times E_F^n)$ is equal to $T_{\tilde{s}}$.

(2) Let $a_1, \ldots, a_p, b_1, \ldots, b_p$ be points in $E^n(K)$. Assume that $\sum_i a_i = \sum_j b_j$. Then $\sum_i T_{a_i}$ is equal to $\sum_j T_{b_j}$ as correspondences in $CH^n(E^n \times E^n)$.

Let $\rho$ be an action of a finite group $G$ on the variety $E^n$, preserving origin, $\chi$ be a character of $G$ to $\{\pm 1\}$. Let $P(\chi) = \frac{1}{\# G} \sum_{\sigma \in G} \chi(\sigma)\sigma^*$ be the projector to $\chi$-part. Let

$$
i_\chi : P(\chi)CH^i(X \times E^n, j) \to CH^i(X \times E^n, j),

\pi_\chi : CH^i(X \times E^n, j) \to P(\chi)CH^i(X \times E^n, j)
$$

be the natural inclusion and projection for the projector $P(\chi)$.

Proposition 5.5. Under the above notations, the composite

$$
P(\chi)CH^i(X \times E^n, j) \xrightarrow{i_\chi} CH^i(X \times E^n, j) \xrightarrow{T_{s,N}} CH^i(X \times E^n, j) \xrightarrow{\pi_\chi} P(\chi)CH^i(X \times E^n, j)
$$

is equal to $T_{tr(s),M}$ where $M = N \cdot \# G$ and $tr(s) = \sum_{\sigma \in G} \sigma(s)$.

Proof. We may assume that all $N$-torsion points of $s$ and $M$-torsion points of $tr(s)$ are defined over $K$. Let $x = \frac{1}{\# G} \sum_{\sigma} \chi(\sigma)\sigma^*Z$ be an element in
Thus, we have
\[ P(\chi)CH^i(X \times E^n, j). \] We choose \( s \) and \( t \) such that \( Ns = s \) and \( \# G \cdot t = \text{tr}(s) \). Thus, we have
\[
P(\chi)T_{s,N}(x) = \frac{1}{(\# G)^2} \sum_{\sigma, \tau \in G} \chi(\tau \sigma)[\tau^*\sigma^*Z + \tau^*s]
\]
\[
= \frac{1}{(\# G)^2} \sum_{\sigma, \tau \in G} \chi(\sigma)[\sigma^*Z + \tau^*s]
\]
\[
\sim \frac{1}{(\# G)} \sum_{\sigma \in G} \chi(\sigma)[\sigma^*Z + t] = T_t(x).
\]

\[ \square \]

5.2.1. Quasi-isomorphism \( K_{S,n,sgn} \to K_{CS,n,\chi} \). We define an isomorphism
\[
\tilde{\alpha} : (E_B^{n+1})_s \to E_B^n : (p_1, \ldots, p_{n+1}) \mapsto (x_1, \ldots, x_n)
\]
by
\[
x_1 = p_1, \ x_2 = p_1 + p_2, \ldots, x_n = p_1 + \cdots + p_n.
\]
Let \( I \in \mathcal{I}_{S,p} \). If \( \tilde{\alpha}(D_{S,I}) \not\subset D \), then there exists a unique \( J \in \mathcal{I}_{C,p} \) such that \( \tilde{\alpha}(D_{S,I}) = D_{C,J} \). Therefore the map \( \tilde{\alpha} \) defines a homomorphism
\[
Z^i(E \times D_{S,I}, j) \to Z^i(E \times D_{C,J}^0, j),
\]
and by taking summations, we have a map of complexes \( \tilde{\alpha} : K_{S,n} \to K_{CS,n} \).
By composing inclusion \( K_{S,n,sgn} \to K_{S,n} \) and projector \( K_{CS,n} \to K_{CS,n,\chi} \), we have the map of complexes \( \beta : K_{S,n,sgn} \to K_{CS,n,\chi} \).

**Proposition 5.6.** The map \( \beta : K_{S,n,sgn} \to K_{CS,n,\chi} \) is a quasi-isomorphism. As a corollary, we have
\[
CH^{n+1}(E \times (U_B^{n+1})_s, n + 1 - j)_{sgn} \simeq H^i(K_{C,n,\chi})
\]

**Proof.** Let \( I \in \mathcal{I}_{S,p} \) and \( J \in \mathcal{I}_{C,p} \), \( \mathcal{G}_I \) and \( G_{C,J} \) be the stabilizer of the component \( D_{S,I} \) and \( D_{C,J} \) in \( \mathcal{G}_{n+1} \) and \( G_C \), respectively. The restriction of the characters \( sgn \) and \( \chi \) to \( \mathcal{G}_I \) and \( G_{C,J} \) are denoted by \( (sgn, I) \) and \( (\chi, J) \).

The proposition is reduced to the quasi-isomorphism of
\[
\tilde{\alpha} : P(sgn, I)Z^i(E \times D_{S,I}, j)_{sgn,I} \to P(\chi, J)Z^i(E \times D_{C,J}, j)_{\chi,J}
\]
for \( \tilde{\alpha}(D_{S,I}) = D_{C,J} \). Therefore it is enough to show that the induced homomorphism
\[
\tilde{\alpha} : P(sgn, I)CH^i(E \times D_{S,I}, j)_{sgn,I} \to P(\chi, J)CH^i(E \times D_{C,J}, j)_{\chi,J}
\]
is an isomorphism. This is reduced to the case where \( I = \emptyset \) and \( J = \emptyset \).
We consider the algebraic correspondence
\[
\Gamma_\Sigma = \{(x_1, \ldots, x_n; p_1, \ldots, p_n) = (p_1, p_1 + p_2, \ldots, p_1 + \cdots + p_n; p_1, \ldots, p_n) \in E_B^n \times_B E_B^n \in CH^n(E_B^n \times_B E_B^n).
\]
The action of \( \sigma \in \mathcal{G}_{n+1} \) and \( \tau \in G_C \) is obtained by the graph \( \Gamma(\sigma) \) and \( \Gamma(\tau) \) of \( \sigma \) and \( \tau \).
Let $\tilde{B}$ be a finite flat Galois extension of $B$. Let $p_0$ and $q_0$ be sections in $E_B$ such that $(n+1)p_0 = s$ and $2q_0 = s$. We set $p = (p_0, \ldots, p_0)$ and $q = (q_0, \ldots, q_0)$. The translations $T_p$ and $T_q$ by $p$ and $q$ is obtained by the algebraic correspondences $\Gamma(T_p)$ and $\Gamma(T_q)$. Then
\[
\Gamma(\sigma_{lin}) = \Gamma(T_{-p}) \circ \Gamma(\sigma) \circ \Gamma(T_p)
\]
\[
\Gamma(\tau_{lin}) = \Gamma(T_{-q}) \circ \Gamma(\tau) \circ \Gamma(T_q)
\]
are linear algebraic correspondences. By considering cohomological classes, we have the following lemma.

**Lemma 5.7.** We have the following equality in $CH^*(E_B \times_B E^n_B)$ for projectors:
\[
\frac{1}{\#\mathcal{S}_{n+1}} \sum_{\sigma \in \mathcal{S}_{n+1}} \text{sgn}(\sigma) \Gamma(\sigma_{lin}) \circ \Gamma(\tau_{lin})
\]
\[
= \frac{1}{\#G_C} \sum_{\tau \in G_C} \chi(\tau) \Gamma(\tau_{lin})
\]
We set $CH = CH^*(E \times E^n_B)$ and $\Xi = CH^*(E \times E^n_B, j)$. We have the following commutative diagram:
\[
P(sgn_{lin})CH \xrightarrow{T_p} P(sgn)CH \quad \xrightarrow{(\ast)} P(\chi)CH \quad \xrightarrow{T_{-q}} P(\chi_{lin})CH
\]
\[
i_{sgn_{lin}} \downarrow \quad \downarrow \quad \Gamma_{\Sigma} \quad \uparrow \quad \uparrow \pi_{\chi_{lin}}
\]
\[
CH \xrightarrow{T_p} CH \xrightarrow{\Gamma_{\Sigma}} CH \xrightarrow{T_{-q}} CH
\]
By this commutative diagram, $(\ast)$ is an isomorphism, since we have
\[
\pi_{\chi_{lin}} \circ T_{-q} \circ \Gamma_{\Sigma} \circ T_p \circ i_{sgn_{lin}} = \pi_{\chi_{lin}} \circ T_{-q+\Sigma p} \circ \Gamma_{\Sigma} \circ i_{sgn_{lin}}
\]
\[
= \pi_{\chi_{lin}} \circ T_{-q+\Sigma p} \circ i_{\chi_{lin}} \circ \Gamma_{\Sigma}
\]
\[
= T_{tr(-q+\Sigma p), G} \circ \Gamma_{\Sigma},
\]
by Proposition 5.5.

**Remark 5.8.** In this section, we treat an elliptic curve over a field $K$. One can define the notion of mixed elliptic motives for a family of elliptic curve $E \to B$ over a base scheme $B$.

### 5.3. Definition of LogMEM
First we construct a mixed elliptic motif $\text{LogMEM}$ in the category $\mathcal{K}(A_{MEM}/O_S)^0$. There are two constructions of $\text{LogMEM}$ in [BL], one is in §1 and the other is in §6. In 5.2.1 we show that two double complexes associated to two constructions are quasi-isomorphic.

Let $E$ be an elliptic curve over $K$ and $B$ be a copy of $E$. Let $E_B$ be the trivial family $E \times E$ of $E$ over $B$ and the inversion of $E$ is written as $i$. Then the diagonal subvariety $s$ defines a section of $E_B$ over $B$.

Let $s$ be the section in $E_B$ defined by the diagonal map. Let $\tilde{B}$ be a flat finite variety over $B$ such that 2-torsion points of $s$ are defined in $\tilde{B}$. We
define a cycle $\tilde{\gamma}_{s,n}$ in $\mathbb{Z}^{n+1}(E^n_B \times E^{n+1}_B, 0)$ by
\[
\frac{1}{q} \sum_{2s = s} \left[ (x_1, \ldots, x_n : \tilde{s}, x_1, \ldots, x_n) \right].
\]
Then this correspondence is defined over $B$, which is also denoted as $\tilde{\gamma}_{s,n}$. An element $(\sigma, e_1, \ldots, e_n)$ in the groups $\mathbb{Z}/2\mathbb{Z}$ acts on $E^n_B$ by
\[
(x_1, \ldots, x_n) \mapsto (\sigma x_1, \ldots, \sigma x_n)
\]
which is extended to the action of the group $G_{C,n}$ defined in (5.3). Let $\chi_n$ be the character of $G_{C,n}$ defined in (5.3). Then the group $G_{C,n} \times G_{C,n+1}$ acts on the group $\mathbb{Z}^{n+1}(E^n_B \times E^{n+1}_B, 0)$. By applying the projector $P(\chi_n) \cdot P(\chi_{n+1})$ to $\tilde{\gamma}_{s,n}$, we get an element $i_{s,n}$ in
\[
P(\chi_n)Z^{n+1}(E^n_B \times E^{n+1}_B, 0)P(\chi_{n+1})
\]
By restricting to the open set $B^0 = B - \{0\}$, we have
\[
i_{s,n} \in \text{Hom}_{MEM/B^0}^1(Sym^n(V)(n), Sym^{n+1}(V)(n+1))
\]
which is also denoted by $i_{s,n}$. We introduce an object $Log_{MEM,n}$ in $K(C_{MEM}/O_S)$ over $B^0$ by setting
\[
Log_{MEM,n} = Sym^n(V)(p) \quad (0 \leq p \leq n).
\]
**Lemma 5.9.** We have
\[
i_{s,p+1} \circ i_{s,p} = 0.
\]
By the above lemma, by setting $d_{p+k,p} = 0$ for $k \geq 2$, we have a DG complex $Log_{MEM,n}$
\[
0 \rightarrow Q \rightarrow Sym^1(V)(1) \rightarrow \cdots \rightarrow Sym^n(V)(n) \rightarrow 0
\]
\[
0 \rightarrow Log_{MEM,n}^0 \rightarrow Log_{MEM,n}^1 \rightarrow \cdots \rightarrow Log_{MEM,n}^n \rightarrow 0,
\]
using
\[
i_s \in \text{Hom}_{MEM}^1(Log_{MEM,n}^s, Log_{MEM,n}^{s+1}).
\]
The morphism $Log_{MEM,n+1} \rightarrow Log_{MEM,n}$ defines a projective system in the category of mixed elliptic motives. Using the homomorphisms $i_{s,q}$, we get the following double complex
\[
\text{Hom}_{MEM}(V, Log_{MEM,n}) :
\]
\[
0 \rightarrow Z^1_1(E \times B, \bullet) \xrightarrow{i_{s,0}} Z^2_1(E \times E_B, \bullet) \xrightarrow{i_{s,1}} \cdots
\]
\[
\cdots i_{s,n-2} \xrightarrow{} Z^n_1(E \times E^{n-1}_B, \bullet) alt^{n-1} \xrightarrow{i_{s,n-1}} Z^{n+1}_1(E \times E^n_B, \bullet) alt^n \rightarrow 0
\]
The associate simple complex of $\text{Hom}_{MEM}(V, Log_{MEM,n})$ is denoted by $\text{Hom}_{MEM}(V, Log_{MEM,n})$. The relation between the differential $d_{n+1,n}$ and the complex $K_{C,n,\chi}$ is given by the following proposition.

The $(-1)$-actions $\iota$ on the left most factor $E \times E^n_B$ and $E \times (U^{n+2}_B)_s$ are compatible, where the subscript $-$ means the $-1$ part for the involution $\iota$. 

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Proposition 5.10. (1) The map
\[
Z^{n+1-i}(E \times D_{C,J}, \bullet) \rightarrow Z^{n+1-i}(E \times E_{B}^{n-i}, \bullet) \text{alt}^{n-i}
\]
defined by \(\frac{1}{4} \sum_{2^{j}=s} T_{i} \) induces a homomorphism
\[
K_{C,n,\chi} \rightarrow \text{Hom}_{MEM}(V, \text{Log}_{MEM}, n)
\]
of double complexes.

(2) The \((-1)\)-actions \(i\) on the left most factor \(E \times E_{B}^{i}\) and \(E \times (U_{B}^{n+1})_{s}\) are compatible. The above homomorphism is a quasi-isomorphism.

As a consequence, we have the following isomorphism:

\[
CH^{n+1}(E \times (U_{B}^{n+1})_{s}, n + 1 - j)_{-,-,sgn} \simeq H^{j} \text{Hom}_{MEM}(V, \text{Log}_{MEM}, n).
\]

5.4. Polylog object as a mixed elliptic motif. We recall the definition of elliptic polylog class in \(CH^{n+1}(E \times (U_{B}^{n+1})_{s}, n)_{-,sgn}\) defined in [BL] using the eigen decomposition of the higher Chow group.

Let \([a]_{B}^{n+1} : (E_{B}^{n+1})_{s} \rightarrow (E_{B}^{n+1})_{1}\) be a map defined by
\[
(p_{1}, \ldots, p_{n+1}, s) \mapsto ([a]_{B}^{n+1}) p_{1}, \ldots, [a]_{B}^{n+1}, [a]_{B}^{n+1}
\]
We set \((U_{a,B}^{n+1})_{s} = ([a]_{B}^{n+1})^{-1}(U_{B}^{n+1})_{s}\) and define a homomorphism \(Nm\) by the composite of the following homomorphisms:

\[
CH^{n+1}(E \times (U_{B}^{n+1})_{s}, n)_{-,sgn} \xrightarrow{j^{*}} CH^{n+1}(E \times (U_{a,B}^{n+1})_{s}, n)_{-,sgn} \xrightarrow{Nm} CH^{n+1}(E \times (U_{B}^{n+1})_{s}, n)_{-,sgn}.
\]

Here \(j^{*}\) is induced by the open immersion \((U_{a,B}^{n+1})_{s} \rightarrow (U_{B}^{n+1})_{s}\) and \(Nm\) is the norm map for the map \([a]_{B}^{n+1}\).

Definition 5.11. We define \(CH^{n+1}(U \times (U_{B}^{n+1})_{s}, n)_{-,sgn}^{(1)}\) by the \(a\)-eigen space for the action of the map \(Nm\).

The following proposition is proved in [BL].

Proposition 5.12 ([BL]). The residue map
\[
CH^{n+1}(U \times (U_{B}^{n+1})_{s}, n)_{-,sgn}^{(1)} \rightarrow CH^{n}(U \times (U_{B}^{n})_{s}, n - 1)_{-,sgn}^{(1)}
\]
is an isomorphism.

Definition 5.13 ([BL]).

(1) By applying Proposition 5.12 successively, we have an isomorphism
\[
CH^{n+1}(U \times (U_{B}^{n+1})_{s}, n)_{-,sgn}^{(1)} \rightarrow CH^{1}(U \times (U_{B}^{1})_{s}, 0)_{-,sgn}^{(1)} \simeq \mathbb{Q}\Delta,
\]
where \(\Delta\) is the diagonal in \(U \times (U_{B}^{1})_{s}\). We define the polylog class \(P_{n}\) by the element in \(CH^{n+1}(U \times (U_{B}^{n+1})_{s}, n)_{-,sgn}^{(1)}\) corresponding to \(\Delta\).

(2) The element in \(H^{1}\text{Hom}_{MEM}(V, \text{Log}_{MEM}, n)\) corresponding to \(P_{n}\) via the isomorphism (5.3) is also denoted as \(P_{n}\) and called the elliptic polylog class.
Let $\tilde{P}_{n}$ be a representative of the cohomology class $P_{n}$ in the following associate simple complex $\mathcal{H}om_{MEM}(V, \text{Log}_{MEM,n})$:

\[
\begin{align*}
\partial \downarrow & \quad z^{n}(E \times E_{B}^{n-1}, n)_{sgn}e^{-n+1} \quad \partial \downarrow & \quad z^{n+1}(E \times E_{B}^{n}, n)_{sgn}e^{-n} \rightarrow 0 \\
\partial \downarrow & \quad z^{n}(E \times E_{B}^{n-1}, n-1)_{sgn}e^{-n+1} \quad \partial \downarrow & \quad z^{n+1}(E \times E_{B}^{n}, n-1)_{sgn}e^{-n} \rightarrow 0 \\
\partial \downarrow & \quad z^{n}(E \times E_{B}^{n-1}, n-2)_{sgn}e^{-n+1} \quad \partial \downarrow & \quad z^{n+1}(E \times E_{B}^{n}, n-2)_{sgn}e^{-n} \rightarrow 0
\end{align*}
\]

Then $\tilde{P}_{n}$ is a direct sum of elements

\[pl_{i} = pl_{i}^\# e^{-i} \in Z^{i+1}(E \times E_{B}^{i}, i)_{sgn}e^{-i} = \mathcal{H}om_{MEM}^{1}(V, \text{Log}_{MEM,n}^{i} e^{-i}).\]

which is called an elliptic polylog extension data. The closedness of $\tilde{P}_{n}$ is equivalent to the relation: $\partial(pl_{i}) + (-1)^{i}i_{s}(pl_{i-1}) = 0$. Then we have

\[\partial pl_{i} + (-1)^{i-1}pl_{i} \cdot i_{s} = 0,
\]

where $\cdot$ is the multiplication.

**Definition 5.14.** We define an object $P_{i,n} = \{P_{i,n}^{i}, d_{ji}\}$ in the quasi-DG category $(MEM) = K(EM)$ of DG complex in $\mathcal{C}_{EM}$ as follows.

\[P_{i,n}^{i} = \begin{cases} V & (i = -1) \\
\text{Log}_{MEM,n}^{i} & (0 \leq i \leq n) \\
0 & (\text{otherwise}) \end{cases} \]

The maps $d_{ji} \in \mathcal{H}om_{MEM}^{1}(P_{i,n}^{i}, P_{j,n}^{j})$ are defined by

\[d_{ji} = \begin{cases} d_{j,-1} = pl_{j} & (0 \leq j \leq n) \\
\quad d_{j+1,j} = (-1)^{j+1}i_{s} & (0 \leq j \leq n-1) \\
0 & (\text{otherwise}) \end{cases} \]

5.5. Description of the comodule associated to polylog motif. The polylog motif is concentrated at degree zero. By the functor $\varphi$, the object $P_{i,n} = \{P_{i,n}^{i}, d_{ji}\}$ corresponds to an $H^{0}(\text{Bar}(A_{EM}/\mathcal{O}), \epsilon)$-comodule. In this subsection, we write down the $H^{0}(\text{Bar}) = H^{0}(\text{Bar}(A_{EM}/\mathcal{O}), \epsilon)$-comodule structure of $\varphi(P_{i,n})$. As a vector space, we have

\[P_{n} = \varphi(P_{i,n}) = \oplus_{i=-1}^{n}P_{n}^{i},
\]

\[P_{n}^{i} = \begin{cases} V & (i = -1) \\
\text{Sym}^{i}(V) & (0 \leq i \leq n). \end{cases} \]

Let $M(P_{n}, H^{0}(\text{Bar}))$ be the endomorphism of $P_{n}$ obtained by the scalar extension to the commutative ring $H^{0}(\text{Bar})$. By the isomorphism

\[\mathcal{H}om_{k}(P_{n}, H^{0}(\text{Bar}) \otimes P_{n}) \simeq \oplus_{i,j=-1}^{n}(P_{n}^{i})^{*} \otimes H^{0}(\text{Bar}) \otimes P_{n}^{i},\]
the coproduct $\Delta_{P^i_{MEM,n}}$ is given by the sum of $a_{ij} \in (P^j_n)^* \otimes H^0(Bar) \otimes P^i_n$, where

$$a_{ij} = \begin{cases} 
\delta_V \otimes \sum_{u=0}^i (-1)^{(i+1)(u+1)}p^{(i-u)}_i \otimes s^{(i-u)}_i \otimes \delta S^1V(i) & j = -1, \ i \geq 0 \\
(-1)^{(i-j)(j-1)} \delta S^0V(j) \otimes s^{(i-j)}_i \otimes \delta S^1V(i) & j \geq 0, \ i > j \\
\Delta V & i = j = -1 \\
\Delta S^1V(i) & i = j \geq 0 \\
0 & \text{otherwise},
\end{cases}$$

and $\delta_V$ is an element in $V^* \otimes V$ corresponding to the identity element in $V^* \otimes V = Hom_k(V, V)$. 

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