Properties and Transformations of Weingarten Surfaces

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Abstract

This paper explores Weingarten relations satisfied by surfaces of revolution in Euclidean 3-space $E^3$. Firstly, we establish that the local geometry of a surface around umbilic points restricts its possible Weingarten relations. We demonstrate that the rate at which the surface becomes spherical at umbilic points imposes bounds on the slope of any satisfied Weingarten relation, extending previous research by a number of authors.

Secondly, we investigate transformations between Weingarten relations through the action of $SL_2(\mathbb{R})$, acting as fractional linear transformations on the surface’s curvatures. We integrate this action, which splits into three natural geometric actions on surfaces in $E^3$, providing a method of generating rotationally symmetric solutions to a transformed Weingarten relation. This technique is applied to a class of Weingarten relations known as semi-quadratic. We prove the action is transitive on such relations and give a classification result on their solutions.

1 Introduction

Introduced by J. Weingarten in 1861 [30], Weingarten surfaces are a topic of classical differential geometry and have found applications in architectural design [26, 27, 29]. An oriented surface in Euclidean 3-space $E^3$ is Weingarten when its principal curvatures $k_1$ and $k_2$ satisfy a differentiable functional relationship expressed as

$$W(k_1, k_2) = 0.$$  \hspace{1cm} (1.1)

$W$ is called the Weingarten relationship and is a non-linear second-order PDE satisfied by the surface. Recent work has focused on understanding how Weingarten relations determine geometric...

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properties of their rotationally symmetric solutions [3, 5]. In conjunction with this theme, this paper investigates the possible Weingarten relations for surfaces of revolution. This is done through two approaches. Firstly, obstruction criteria for Weingarten relations are given in terms of local surface geometry around umbilic points. Secondly, given an initial surface of revolution and its Weingarten relation, a family surface transformations are applied, generating solutions to transformed Weingarten relationships. A classification result of certain Weingarten surfaces is then given. Further details are now given.

Obstructions to Weingarten Relations for Surfaces of Revolution

Our first topic explores how the geometric behaviour of a surface near its umbilic points affects the supported Weingarten relations. This is done in terms of a surface’s curvature diagram, denoted as \( \mathcal{F}(S) \), which represents the set of curvatures \( (k_1, k_2) \) attained by points on the surface \( S \) as a subset of the \( k_1k_2 \)-plane. In the literature, \( k_1 \) and \( k_2 \) are typically labelled by the condition \( k_2 \leq k_1 \). A surface \( S \) is Weingarten with relation (1.1) if and only if \( \mathcal{F}(S) \subseteq W^{-1}\{0\} \), hence \( \mathcal{F}(S) \) strongly determines the supported Weingarten relationships. Examples are depicted in Figure 1. Points of \( S \) with equal principal curvatures are called umbilic points and the diagonal \( k_1 = k_2 \) in the \( k_1k_2 \)-plane is called the umbilic axis. The study of umbilic points is a classical yet still active area of research [8, 13, 14]. They are guaranteed to exist on closed \( C^2 \) surfaces of zero genus, and thus the curvature diagram of such surfaces must intersect the umbilic axis.

Various authors have described the possible shapes of curvature diagrams, near the umbilic axis [4, 6, 16, 19]. Different assumptions on the surface \( S \) are made, see [4], Lemma 2 or [6], Theorem 1.1 for examples, however the general conclusion is that if \( \mathcal{F}(S) \) intersects the umbilic axis, it does so with a non-negative slope in the \( k_1k_2 \)-plane, or, is a point on the umbilic axis (and thus \( S \) is congruent to a subset of the round sphere or a plane). Directions of negative slope are shaded in Figure 1. Due to the above, the orange curve in Figure 1 cannot be the curvature diagram of a surface, as it meets the umbilic axis from a direction of negative slope. Thus surfaces congruent to round spheres are the only surface homeomorphic to \( S^2 \) which can satisfy the relationship \( k_1 + k_2 = c \), \( c > 0 \), as is well known [1].

Figure 1: Left: Pairs \((k_1, k_2)\) satisfying a CMC relationship \( k_1 + k_2 = c \) (orange) and the curvature diagrams of a generic surface (blue) and a generic Weingarten surface (red). Right: The directions of negative slope from the umbilic axis.
In this paper we give strictly positive lower bounds on the slope at which $\mathcal{F}(S)$ intersects the umbilic axis for $S$ a $C^2$-smooth surface of revolution. Throughout the rest of the paper, $S$ will denote such a surface, in which case $\mathcal{F}(S)$ is generically a curve (being the continuous image of the profile curve of $S$).

- If $p \in S$ is an umbilic point, the umbilic slope at $p$, denoted $\mu_p$, is the slope at which $\mathcal{F}(S)$ meets the umbilic axis. We remark that $\mu_p$ may not be well defined for every $S$.
- A surface is said to be totally umbilic around $p$ if there exists a neighbourhood of $p$ in $S$ which contains only umbilic points.
- $S$ will be called non-flat at the point $q \in S$ if $K(q) \neq 0$, and non-flat if it is non-flat at every point.
- $S$ will be called convex at the point $q \in S$ if $K(q) \geq 0$, and convex if it is convex at every point.
- $S$ will be called strictly convex at the point $q \in S$ if $K(q) > 0$, and strictly convex if it is strictly convex at every point.

We remark that our usage of the word ‘non-flat’ here does not coincide with the idea of a surface being distinct from a plane, locally. A point is non-flat if and only if it is not a parabolic and planar point.

We first consider an umbilics points which lie off the axis of rotational symmetry. With a minor technical assumption on the umbilic $p$, we show that if $S$ is non-flat at $p$, then if $\mathcal{F}(S)$ has a tangent line at $p$, it must be vertical. (Theorem 3.2). The behaviour of $\mu_p$ for an umbilic point $p$ lying on the axis of rotational symmetry is then considered. Let $r_1$ and $r_2$ be the radii of curvature of $S$ and $\theta$ the angle formed between $S$’s (oriented) axis of rotational symmetry and its oriented normal vector. We may assume without loss of generality that $\theta = 0$ at $p$.

**Theorem 3.8.** Let $S$ be strictly convex at an isolated umbilic point $p$ on the axis of rotational symmetry. Suppose $\mu_p \in \mathbb{R}$.

(A) If the radii of curvature satisfy $\lim_{\theta \to 0} \left( \frac{r_2 - r_1}{\sin \theta} \right) = \gamma$ for some $\alpha \geq 0$, $\gamma \in \mathbb{R}$, then $\mu_p \geq \alpha + 1$, with equality if $\gamma \neq 0$.

(B) Conversely if $\mu_p > \alpha + 1$ then $\lim_{\theta \to 0} \left( \frac{r_2 - r_1}{\sin \theta} \right) = 0$.

Theorem 3.8 is a corollary of Theorem 3.7 which bounds the limit superior and limit inferior of the average slope of $\mathcal{F}(S)$ near an umbilic point, covering cases where $\mu_p$ may not be well defined. As a corollary of Theorem 3.8 we show the following

**Corollary 3.10.** Let $p$ be an isolated umbilic point on the axis of rotational symmetry of $S$, a strictly convex and $C^3$-smooth surface. Then if $\mu_p$ exists, $\mu_p \geq 2$. If in addition $S$ is $C^4$-smooth then $\mu_p \geq 3$.

The above theorems hence characterise $\mu_p$ as a measure of the rate at which $S$ becomes umbilic.
SL$_2(\mathbb{R})$ Transformations

Our second topic concerns a family of transformations which, for each Weingarten relation (1.1) having a rotationally symmetric solution, produces a new relation which also admits a rotationally symmetric solution. SL$_2(\mathbb{R})$ acts on the $k_1k_2$-plane by real fractional linear transformations, which coincide with the isometries of the geometrised $k_1k_2$-plane considered in [10]. Our main theorem is

**Theorem 4.8.** If $T$ is a real fractional linear transformation of the $k_1k_2$-plane with $S$ non-flat, then there exists a rotationally symmetric and possibly non-regular surface $\tilde{S}$ such that $\tilde{S}(\tilde{S}) = T(S(S))$.

The action of SL$_2(\mathbb{R})$ is then described geometrically in terms of transformations of $S$ in $\mathbb{E}^3$ (Theorem 4.14). A class of surfaces called *semi-quadratic Weingarten surfaces* satisfying a Weingarten relation of the form

$$\alpha k_1 k_2 + \beta k_1 + \gamma k_2 + \delta = 0 \quad \alpha, \beta, \gamma, \delta \in \mathbb{R},$$

are investigated. This class contains well-known subclasses of surfaces, such as ones linear in $k_1$ and $k_2$, ones linear in the mean and Gauss curvature, $H$ and $K$, and ones linear in the radii of curvature, $r_1 = \frac{1}{k_1}$ and $r_2 = \frac{1}{k_2}$, investigated in [22, 24], [7, 23] and [12], respectively. The quantities

$$\Lambda_1 = \beta - \gamma, \quad \Lambda_2 = (\beta + \gamma)^2 - 4\alpha\delta,$$

are introduced which characterise when the PDE (1.2) is elliptic, namely $\Lambda_2 > \Lambda_1^2$ (Proposition 4.19). When $\Lambda_1 = 0$ semi-quadratic surfaces become LW-surfaces. LW-surfaces are classified into three types, *elliptic* when $\Lambda_2 > 0$ [7], *hyperbolic* when $\Lambda_2 < 0$ [23] and a border case $\Lambda_2 = 0$ which describe subsets of spheres, tubular surfaces or planes. This motivates a generalisation of the nomenclature:

**Definition 1.1.** A semi-quadratic Weingarten surface satisfying $\Lambda_2 > \Lambda_1^2$ is said to be *elliptic*. If $\Lambda_2 < \Lambda_1^2$ it is said to be *hyperbolic*.

Semi-quadratic relations form an invariant set under the action of SL$_2(\mathbb{R})$ and the ratio $\Lambda_1^2/\Lambda_2$ is shown to be an invariant (Proposition 4.21). The SL$_2(\mathbb{R})$ transformations are then shown to be transitive on all semi-quadratic relations satisfying $\Lambda_2 > 0$ and sharing the same invariant (Proposition 4.23), therefore such semi-quadratic surfaces can be transitively related by induced transformations in $\mathbb{E}^3$. This is used to show the following.

**Theorem 4.24.** Let $S$ be a connected rotationally symmetric semi-quadratic Weingarten surface for which $\Lambda_1^2 = \Lambda_2$. Then $S$ is a subset of a round sphere, tubular surface or plane.

**Theorem 4.28.** Any non-flat rotationally symmetric, connected semi-quadratic Weingarten surface with $\Lambda_2 > 0$ is the image under a composition of homotheties, parallel translations and reciprocal transformations of a Weingarten surface satisfying the relation

$$k_2 = \lambda k_1,$$

for $\lambda > 0$ when the surface is elliptic, or for $\lambda < 0$ when the surface is hyperbolic.

Surfaces of revolution satisfying relation (1.3) were classified in [24]. Theorem 4.28 therefore extends this classification to rotationally symmetric semi-quadratic surfaces with $\Lambda_2 > 0$.

The paper is organised as follows. Section 2 details our method of describing surfaces of revolution and introduces the radius of curvature equivalent of the curvature diagram, termed the RoC.
Section 3 explores the possible RoC diagrams for surfaces of revolution while Section 4 investigates the effect of the $\text{SL}_2(\mathbb{R})$ mappings on Weingarten relations and the transformations they induce on surfaces.

2 Background

2.1 The Curvature of Surfaces of Revolution.

In this short subsection we define a coordinate system on $\mathcal{S}$ in the special case $\mathcal{S}$ is non-flat. By continuity of the Gauss curvature, if a $C^2$-smooth surface is non-flat at a point it is also non-flat in a neighbourhood of that point - hence the constructed coordinates will be used to describe surfaces of revolution locally around non-flat points. Position $\mathcal{S}$ in $\mathbb{R}^3$ with the axis of rotational symmetry aligned with the $z$ axis. The principal foliations of $\mathcal{S}$ are given by the parallels and profile curves of $\mathcal{S}$ whose respective principal curvatures we denote by $k_1$ and $k_2$. Note because $\mathcal{S}$ is assumed non-flat, both $k_1$ and $k_2$ are non-zero. Let $\alpha$ be the profile curve of $\mathcal{S}$ which lies in the $yz$-plane (i.e. the generating curve of $\mathcal{S}$). Since $k_2 \neq 0$ the Gauss map $N : \alpha \rightarrow S^1$ is a local diffeomorphism and we may thus use the Gauss angle $\theta \in (-\pi, \pi]$ on $S^1$ to locally parameterise $\alpha$. Here $\theta$ is the angle made between the normal vector of $\alpha$, denoted $\hat{n}$, and the positive $z$ axis. We orient $\alpha$ so that $\hat{n}(\theta) = (\sin(\theta), \cos(\theta))$. Points of $\alpha$ on the $z$ axis such that $\theta = 0$ or $\theta = \pi$ will be called north and south poles respectively. If $\rho$ and $h$ denote the respective $y$ and $z$ components of $\alpha$, they satisfy the relationship

$$\frac{dh}{d\rho} = -\tan \theta,$$

and $\mathcal{S}$ may be described via $\vec{X}(\theta, \phi) = (\rho(\theta) \sin \phi, \rho(\theta) \cos \phi, h(\theta))$ for $(\theta, \phi) \in (-\pi, \pi] \times [0, \pi)$. This is illustrated in Figure 2.

![Figure 2](image_url)

Figure 2: The profile curve $\alpha$ (blue) in the $yz$-plane is parameterised by the angle $\theta$ and revolved by an angle of $\phi$ around the axis of rotation (black) to generate $\mathcal{S}$. 
Throughout the paper we assume that $S$ is fully revolved around the $z$-axis, so due to symmetry, if $\theta \in (-\pi, \pi)$ is a Gauss angle of $S$ so is $-\theta$, with $\rho$ and $h$ satisfying $\rho(-\theta) = -\rho(\theta)$ and $h(-\theta) = h(\theta)$. Therefore we will often assume $\theta \in I$ where $I \subset [0, \pi]$ is the range of positive Gauss angles attained by $S$. One may also describe $S$ by its support function:

$$r = \vec{X} \cdot \vec{N},$$

with $\vec{N} = (\sin \theta \sin \phi, \sin \theta \cos \phi, \cos \theta)$ being the unit normal vector of $S$. One can check that

$$r = \rho(\theta) \sin \theta + h(\theta) \cos \theta$$

and so $r$ depends only on $\theta$ and $r(\theta) = r(-\theta)$. We also have

$$\rho = r \sin \theta + \frac{dr}{d\theta} \cos \theta, \quad h = r \cos \theta - \frac{dr}{d\theta} \sin \theta. \tag{2.2}$$

In this paper the radii of curvature of $S$, namely $r_1 : S \to \mathbb{R}$ and $r_2 : S \to \mathbb{R}$, will be understood through their restrictions $r_1 : \alpha \to \mathbb{R}$ and $r_2 : \alpha \to \mathbb{R}$ to the profile curve $\alpha$. The corresponding coordinate expressions are denoted as $r_1(\theta)$ and $r_2(\theta)$ and by abuse of notation, often just as $r_1$ and $r_2$.

**Proposition 2.1.** The radii of curvature of $S$ can be expressed as

$$r_1 = r + \frac{dr}{d\theta} \cot \theta, \quad r_2 = r + \frac{d^2r}{d\theta^2}, \tag{2.3}$$

for $\theta \in (0, \pi)$. Conversely, the support function $r$ and the coordinates $(\rho, h)$ are given by

$$r(\theta) = \frac{r(\theta_0)}{\cos \theta_0} \cos \theta + \cos \theta \int_{\theta_0}^{\theta} \frac{r_1(\theta) \sin \theta}{\cos^2 \theta} d\theta, \tag{2.4}$$

$$\rho = r_1 \sin \theta, \quad \frac{dh}{d\theta} = -r_2 \sin \theta. \tag{2.5}$$

**Proof.** Firstly the pair of relations in (2.5) are derived from equation (2.1) and the standard formula for the curvatures of a surface of revolution in terms of their generating curve [2, p.120]. Relations (2.3) then follow from equations (2.2) and (2.5). Finally equation (2.4) follows from integration of the equation for $r_1$ in (2.3).

We remark that as a corollary of equations (2.5), $\rho$ takes the same sign as $r_1$ when $\theta \in [0, \pi]$ and

$$r_1(-\theta) = r_1(\theta) \quad \text{and} \quad r_2(-\theta) = r_2(\theta).$$

Thus whenever $r_1$ and $r_2$ are differentiable functions at $\theta = 0$ we have $r_1'(0) = r_1'(0) = 0$.

In the general scenario, the radii of curvature of a $C^3$-smooth surface cannot be assumed differentiable at an isolated umbilic point. However in the case considered in this section, where $S$ is rotationally symmetric and strictly convex, the regularity of the radii of curvatures is implied by the regularity of $S$. 

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Lemma 2.2. If $S$ is a strictly convex, rotationally symmetric and $C^3$-smooth surface, then $r_1(\theta)$ and $r_2(\theta)$ are $C^1$-smooth on $[0, \pi]$.

Proof. If $S$ is $C^3$-smooth then the support function $r$ is also $C^3$-smooth. The claimed regularity of $r_2(\theta)$ follows trivially from equations (2.3). To show the claimed regularity of $r_1(\theta)$, note that for $\theta \in (0, \pi)$, equations (2.3) show $r_1(\theta)$ has regularity $C^2$, thus $C^1$ regularity is implied on $(0, \pi)$. We only need to show the required regularity at $\theta = 0$ and $\theta = \pi$. We show the $\theta = 0$ case first. Write by differentiation of equations (2.3)

$$r'_1 = \frac{r'' \sin \theta \cos \theta - r' \cos^2 \theta}{\sin^2 \theta}.$$  \hfill (2.6)

The numerator of this quotient vanishes as $\theta \to 0$ since $r'(0) = 0$. Thus a straightforward application of L’Hopital’s rule gives

$$\lim_{\theta \to 0} r'_1(\theta) = \frac{1}{2} r'''(0) + r''(0) = 0,$$

since $r'''(0) = 0$ also. The claim at $\theta = \pi$ follows similarly. \hfill $\square$

Lemma 2.3. If $S$ is a strictly convex, rotationally symmetric and $C^4$-smooth surface, then $r_1(\theta)$ and $r_2(\theta)$ are $C^2$-smooth on $[0, \pi]$.

Proof. This lemma follows almost identically to the proof of the previous lemma, except this time we must check if $\lim_{\theta \to 0} r''_1(\theta)$ exists. Differentiating equation (2.6) yields for all $\theta \in (0, \pi)$;

$$r''_1 = \frac{r''' \cos \theta \sin^2 \theta - r'' \sin \theta (1 + \cos^2 \theta) + 2r' \cos(\theta)}{\sin^3 \theta}.$$

An application of L’Hopital yields

$$\lim_{\theta \to 0} r''_1(\theta) = \lim_{\theta \to 0} \left( \frac{1}{3} r^{(4)}(\theta) - \frac{2r''' \sin \theta}{3 \cos \theta} + r'' - \frac{2}{3} r' \cot \theta \right).$$

Note one quickly checks $r' \cot \theta \to r''(0)$ as $\theta \to 0$ giving

$$\lim_{\theta \to 0} r''_1(\theta) = \frac{1}{3} \left( r^{(4)}(0) + r''(0) \right).$$

Proposition 2.4. The radii of curvature of $S$ satisfy an integrability condition called the derived Codazzi-Mainardi equation

$$\frac{dr_1}{d\theta} = (r_2 - r_1) \cot \theta, \quad \theta \in I \setminus \{0, \pi\}. \hfill (2.7)$$

Conversely, (2.7) is sufficient for a continuous map $J : I \to \mathbb{R}^2 \setminus \{(0, 0)\}$, $\theta \mapsto (r_1(\theta), r_2(\theta))$, with $I \subset [0, \pi]$ and $r_1(\theta)$ differentiable except possibly at $\theta = 0$ or $\pi$, to parameterise the radii of curvature of a rotationally symmetric $C^2$-smooth surface.
Proof. To derive the Codazzi-Mainardi equation, multiply the difference between equations (2.3) by \( \cot \theta \) and integrate between \( \theta_1 \) and \( \theta_2 \), for \( (\theta_1, \theta_2) \subset (0, \pi) \), to derive the integral relationship

\[
r_1(\theta_2) - r_1(\theta_1) = \int_{\theta_1}^{\theta_2} (r_2 - r_1) \cot \theta \, d\theta.
\]

This is referred to as the integrated Codazzi-Mainardi relationship. Dividing by \( \theta_2 - \theta_1 \) and letting \( \theta_2 \to \theta_1 \) yields the result. Conversely, if \( J(\theta) = (r_1(\theta), r_2(\theta)) \) is as stated, then define a \( C^2 \)-smooth function \( r : I \to \mathbb{R} \) by

\[
r_2(\theta) = r + \frac{d^2r}{d\theta^2}.
\]

From the Codazzi-Mainardi equation it is easy to show \( r \) satisfies

\[
r_1(\theta) = r + \cot(\theta) \frac{dr}{d\theta},
\]

for \( \theta \in I \). Letting \( \vec{X} = (\rho \sin \phi, \rho \cos \phi, h) \) where \( \rho \) and \( h \) are given by equation (2.2) gives a parametrisation for a \( C^2 \) rotationally symmetric surface whose support function is \( r \) (given by equation (2.4) explicitly) with the functions \( r_1(\theta) \) and \( r_2(\theta) \) as its radii of curvature. Since \( r_1 \neq 0 \) and \( r_2 \neq 0 \) by assumption, it can be quickly checked via equation (2.5) that \( \vec{X}_\theta \times \vec{X}_\phi \neq 0 \) and so the surface is regular.

The equations presented in this section also allow us to describe non-regular surfaces, i.e. images of \( C^2 \) homeomorphisms from an open subset of \( \mathbb{R}^2 \) to an open subset of \( \mathbb{R}^3 \) whose tangent vectors are linearly dependent at points. In present setting, one can check that \( \vec{X}_\theta = 0 \) iff \( r_2 = 0 \) and \( \vec{X}_\phi = 0 \) iff \( r_1 = 0 \), we call points with at least one vanishing radii of curvature cusps.

### 2.2 The Radii of Curvature Diagram

We will now again permit \( S \) to be possibly flat (i.e. to have vanishing Gauss curvature) at points. The curvature diagram \( \mathcal{F}(S) \) of \( S \) has already been introduced. We now introduce its radii of curvature equivalent for surfaces of revolution. Let \( r_2 \) be the radius of curvature of the profile curve of \( S \), and \( r_1 \) the radius of curvature of curves of constant \( \theta \). Let \( k_2 \) and \( k_1 \) be their respective reciprocals.

**Definition 2.5.** The radii of curvature (RoC) diagram of \( S \), \( \mathcal{R}(S) \), is the set

\[
\mathcal{R}(S) = \left\{ (r_1, r_2) \in \mathbb{R} \times \mathbb{R} \mid r_1, r_2 \text{ are radii of curvature attained at a point of } S \right\},
\]

where \( \mathbb{R} \) is the projectively extended real line, so \( \mathbb{R} \times \mathbb{R} \cong S^1 \times S^1 \).

The ambient space \( \mathbb{R} \times \mathbb{R} \) is called RoC space and compactifies \( \mathbb{R}^2 \) by gluing in two copies of \( S^1 \) along the lines \( r_1 = \infty \) and \( r_2 = \infty \). The umbilic axis remains the diagonal line and surfaces that have zero Gauss curvature at points have RoC diagrams which are unbounded.

**Definition 2.6.** Given a (possibly non-regular) surface \( S \), a point \( p \in S \) is said to be an umbilic point of \( S \) if the pair \( (r_1|_p, r_2|_p) \) lies on the diagonal of \( \mathbb{R} \times \mathbb{R} \), where \( r_i|_p, i = 1, 2 \) are the radii of curvature of \( S \) at \( p \).
Hence these definitions account for umbilics at flat points, when both \( r_1 \) and \( r_2 \) are non-finite, and also umbilics at cusps, when both \( r_1 \) and \( r_2 \) are zero. To study the slope of \( \mathfrak{R}(S) \) at the point \((r_0, r_0)\), with \( r_0 < \infty \) we will consider the limiting value of the function \( \mu(\theta) \):

\[
\mu(\theta) = \frac{r_2(\theta) - r_0}{r_1(\theta) - r_0},
\]

which gives the average slope of \( \mathfrak{R}(S) \) between the points \((r_1(\theta), r_2(\theta))\) and \((r_0, r_0)\).

\[\text{Definition 2.7.}\] Let \( p \in S \) be a point of \( S \) with Gauss angle \( \theta_0 \). If \( r_1(\theta_0) = r_2(\theta_0) = r_0 \) so that \( p \) is an umbilic point, we define the umbilic slope of \( S \) at \( p \) as

\[
\mu_p = \lim_{\theta \to \theta_0} \left( \frac{r_2(\theta) - r_0}{r_1(\theta) - r_0} \right),
\]

(2.10)

Note when \( r_0 \) is finite, the above definition of \( \mu_p \) gives the slope of the tangent line to \( \mathfrak{R}(S) \) in the \( r_1r_2 \)-plane at the point \((r_0, r_0)\), when such a tangent line exists. When in addition \( r_0 \neq 0 \), \( \mu_p \) is equal to the slope of \( \mathfrak{F}(S) \) at \((k_0, k_0)\) in the \( k_1k_2 \)-plane:

\[
\mu_p = \lim_{\theta \to \theta_0} \left( \frac{k_2(\theta) - k_0}{k_1(\theta) - k_0} \right),
\]

(2.11)

where \( k_0 = 1/r_0 \).

The limit (2.10) is, in general, not well defined for all choices of \( S \) or for all choices of \( p \in S \). For example when \( S \) is a subset of a round sphere, both \( r_1 \) and \( r_2 \) are equal to \( r_0 \) at every point. It is also possible that umbilic points of \( S \) accumulate at \( p \), leading \( \mu(\theta) \) to be ill-defined on any open neighbourhood of \( \theta_0 \). In Section 3 we will make assumptions on \( S \) around \( p \) to avoid this scenario. We remark that when a Weingarten relationship is given, namely if \( W(k_1(\theta), k_2(\theta)) = 0 \) in a punctured neighbourhood of \( \theta_0 \), then \( \mu_p \) can be given in terms of the relationship \( W \):

\[
\mu_p = -\lim_{\theta \to \theta_0} \left( \frac{\partial W}{\partial k_1} / \frac{\partial W}{\partial k_2} \right),
\]

(2.12)
a standard condition sufficient for the existence of $\mu_p$ is therefore that the gradient of $W(k_1, k_2)$ is non-vanishing at $p$ so the above limit is well defined [18, 19]. A Weingarten surface is said to be elliptic, if equation (1.1) is an elliptic PDE. It can be shown [19] that equation (1.1) is elliptic at $q \in S$ if and only if
\[
\left( \frac{\partial W}{\partial k_1} \cdot \frac{\partial W}{\partial k_2} \right)_q > 0.
\] (2.13)
In particular, for a relation elliptic at the umbilic point $p \in S$ we have that both $\frac{\partial W}{\partial k_1}_p$ and $\frac{\partial W}{\partial k_2}_p$ are non-zero, hence by equation (2.12), elliptic Weingarten relations always have $\mu_p < 0$.

3 Obstructions to Weingarten Relations for Surfaces of Revolution

In this section consequences of rotational symmetry are derived in terms of $R(S)$. The class of Weingarten surfaces solving a given relation can be large, as exemplified by the Weierstrass-Enneper representation for minimal surfaces [17]. The requirement of rotational symmetry restricts the possible solutions to a relation greatly and often allows one to explicitly integrate the Weingarten relation (via equation (2.7)) to find a 1-parameter family of rotationally symmetric solutions.

Example 3.1. The linear Hopf surfaces are defined as surfaces satisfying the Weingarten relation
\[
r_2 = \lambda r_1 + C,
\] (3.1)
for $\lambda, C \in \mathbb{R}$. They are the stationary solutions to the linear Hopf curvature flow studied in [12] and [15]. Note that radii of curvature pairs $(r_1, r_2) = (0, C)$ and $(r_1, r_2) = (-C/\lambda, 0)$ satisfy the linear Hopf relation, and so surfaces satisfying this relationship can potentially be non-regular. Inserting the linear Hopf relation into the Codazzi-Mainardi equation derives the separable ODE for $r_1$:
\[
\frac{dr_1}{d\theta} = ((\lambda - 1) r_1 + C) \cot \theta,
\]
which is solved to give
\[
r_1(\theta) = \frac{C}{1-\lambda} + \frac{A_0 \sin^{\lambda-1} \theta}{\lambda - 1}, \quad A_0 \in \mathbb{R}.
\] (3.2)
We can then use equation (2.4) to recover the support function for $\theta \in (\theta_1, \theta_2)$:
\[
r(\theta) = \frac{C}{1-\lambda} + \left( r_{\text{Hopf}}(\theta_1) - \frac{C}{1-\lambda} \right) \cot \theta + \frac{A_0 \cos \theta}{\lambda - 1} \int_{\theta_1}^{\theta} \frac{\sin^{\lambda-1} \theta}{\cos^2 \theta} d\theta.
\]
One can check that when $A_0 > C$, linear Hopf surfaces possess cusps so are non-regular. △

Our first point of discussion is to explore how the Codazzi-Mainardi equation controls the behaviour of $R(S)$. For example, it is quickly observed from equation (2.7) that for $\theta \in [0, \pi/2]$ if $R(S)$ is above the umbilic axis, i.e. $r_2 > r_1$, $r_1$ must be increasing, and vice-versa when $R(S)$ is below the axis. These roles are reversed when $\theta \in [\pi/2, \pi]$, illustrated in Figure 4. Equation (2.7) will be used to relate the slope of $R(S)$ at the umbilic axis to the rate at which $r_2 - r_1$ vanishes as one approaches an umbilic point on $S$. $S$ is necessarily convex around umbilic points and thus our
arguments will be local in nature - taking place in a convex subset of \( S \) in which \( \theta \) may be used to parameterise the radii of curvature. We consider first the case when \( p \) is an umbilic point of \( S \) which lies off the axis of rotational symmetry, in this case the Gauss angle of \( p \), namely \( \theta_0 \), satisfies \( \theta_0 \neq 0, \pi \) and, due to the rotational symmetry of \( S \), \( p \) lies in a curve consisting only of umbilic points, so is a non-isolated umbilic.

**Theorem 3.2.** Suppose \( p \in S \) is an umbilic which lies off the axis of symmetry of \( S \) and that \( S \) is non-flat at \( p \). In addition assume that there is a punctured neighbourhood \( U \) around \( \theta_0 \) such that \( r_2(\theta) \neq r_1(\theta) \) on \( U \). Then \( \mu(\theta) \) is unbounded on \( U \).

**Proof.** We may assume \( U \subset (0, \pi) \) since \( p \) lies off the axis of symmetry. Let \( s(\theta) = r_2(\theta) - r_1(\theta) \). By assumption there is a \( \delta > 0 \) such that \( s \) has no zeros on the interval \( (\theta_0, \theta_0 + \delta) \subset U \). We may also assume wlog that \( \frac{\pi}{2} \not\in (\theta_0, \theta_0 + \delta) \) so \( \cot \theta \) has no zeros in this interval either. Hence by continuity both \( s(\theta) \) and \( \cot \theta \) take a fixed sign on \( (\theta_0, \theta_0 + \delta) \). By taking \( \theta_1 = \theta_0 \) and \( \theta_2 = \theta_0 + \delta \) in the integrated Codazzi-Mainardi equation (2.8) we find that for \( \theta \in (\theta_0, \theta_0 + \delta) \)

\[
|r_1(\theta) - r_0| = \int_{\theta_0}^{\theta} |s(\tau)| \cot \tau d\tau > 0,
\]

hence \( \mu(\theta) = \frac{r_2(\theta) - r_0}{r_1(\theta) - r_0} \) is well defined and may be written as

\[
\mu(\theta) = 1 + \frac{s(\theta)}{\int_{\theta_0}^{\theta} s(\tau) \cot \tau d\tau},
\]

which gives the following equation for \( s(\theta) \):

\[
s(\theta) = (\mu(\theta) - 1) \int_{\theta_0}^{\theta} s(\tau) \cot \tau d\tau. \tag{3.3}
\]

Suppose for contradiction that \( \mu(\theta) \) is bounded on \( (\theta_0, \theta_0 + \delta) \). Then

\[
|s(\theta)| \leq M \int_{\theta_0}^{\theta} |s(\tau)| |\cot \tau| d\tau,
\]
for some $M > 0$. Performing the substitution $J(\theta) = \int_{\theta_0}^{\theta} |s(\tau)| \cot \tau \, d\tau$ and applying Grönwall’s inequality shows $s \equiv 0$ on $(\theta_0, \theta_0 + \delta)$, which is a contradiction. Hence $\mu$ is unbounded. \qed

**Corollary 3.3.** If $\Re(S)$ has a well defined tangent line at $(r_0, r_0)$, it must be vertical.

Now we consider when $p$ is isolated and must therefore lie on the axis of rotational symmetry. Without loss of generality we may assume $p$ is at the north pole of $S$ so that $\theta_0 = 0$. In [6] it was shown that when a surface has an isolated umbilic point, the slope of $\Re(S)$ must be positive as it meets the umbilic axis. Surfaces are necessarily convex around umbilic points - we show with the stronger assumption of strict convexity, a better bound can be given in the rotationally symmetric setting.

**Lemma 3.4.** Let $S$ be strictly convex at an isolated umbilic point $p$ on the axis of rotational symmetry. Then $\lim_{\theta \to 0} \mu(\theta) \geq 1$.

**Proof.** Since $p$ is isolated there exists an interval $(0, \delta)$, with $\delta < \frac{\pi}{2}$ on which $s(\theta) = r_2(\theta) - r_1(\theta)$ is non-zero. As before $\mu(\theta)$ can be given as

$$\mu(\theta) = 1 + \frac{s(\theta)}{\int_{\theta_0}^{\theta} s(\tau) \cot \tau \, d\tau}, \tag{3.4}$$

which is well defined in $(0, \delta)$ since $p$ is isolated. For all $\theta \in (0, \delta)$, $s(\theta)$ is of a fixed sign and $\cot \theta > 0$, thus $\int_{\theta_0}^{\theta} s(\tau) \cot \tau \, d\tau$ is of the same sign as $s(\theta)$. It follows that $\mu(\theta) \geq 1$ for all $\theta \in (0, \delta)$. \qed

**Corollary 3.5.** Let $S$ be strictly convex at an isolated umbilic point $p$ on the axis of rotational symmetry. When $\mu_p$ exists, $\mu_p \geq 1$.

We now go on to bound $\mu_p$ below by bounds larger than 1, the bounds depending on the geometry of $S$ near $p$. First a technical lemma is established.

**Lemma 3.6.** Let $\beta \geq 0$ and let $J : [0, c) \to \mathbb{R}$ be a continuous function on $[0, c)$ and differentiable on $(0, c)$ for some $c \geq 0$. Furthermore, suppose that $J$ is not identically 0 on any punctured neighbourhood of 0 and that

$$\liminf_{x \to 0} \left( \frac{\tan x J'(x)}{J(x)} \right) > -\infty. $$

(A) If $\lim_{x \to 0} \left( \frac{J(x)}{\sin^2 x} \right)$ is finite then $\limsup_{x \to 0} \left( \frac{\tan x J'(x)}{J(x)} \right) \geq \beta$.

(B) If $\liminf_{x \to 0} \left( \frac{\tan x J'(x)}{J(x)} \right) > \beta$, then $\lim_{x \to 0} \left( \frac{J(x)}{\sin^2 x} \right) = 0$.

**Proof.** First we prove (A). Note

$$\limsup_{x \to 0} \left( \frac{\tan x J'(x)}{J(x)} \right) \geq \liminf_{x \to 0} \left( \frac{\tan x J'(x)}{J(x)} \right) > -\infty,$$

and if $\limsup_{x \to 0} \left( \frac{\tan x J'(x)}{J(x)} \right) = \infty$ we are done. Hence the case $\limsup_{x \to 0} \left( \frac{\tan x J'(x)}{J(x)} \right) = M$ for some $M \in \mathbb{R}$ is all that need be considered. Let $\varepsilon > 0$. There exists some $\delta$ satisfying $0 < \delta < c$ (which we may assume to be less than $\pi/2$) such that

$$\frac{\tan x J'(x)}{J(x)} \leq M + \varepsilon, \quad \text{for all } x \in (0, \delta).$$
If $0 < z < y < \delta$, dividing through the above inequality by $\tan x$ and integrating between $x = z$ and $x = y$ gives the inequality $|J(y)| \leq |J(z)| \left( \frac{\sin y}{\sin z} \right)^{M+\varepsilon}$ implying

$$|J(y)| \leq \left| \frac{J(z)}{\sin^\beta z} \right| (\sin^{\beta-(M+\varepsilon)} z)(\sin^{M+\varepsilon} y), \quad 0 < z < y < \delta. \tag{3.5}$$

If we assume for a contradiction that $M < \beta$, taking $\varepsilon$ such that $M + \varepsilon < \beta$ and letting $z \to 0$ inequality (3.5) implies implying $J(y) = 0$ for all $y \in (0, \delta)$ which is a contradiction. Hence $M \geq \beta$. Now (B) is proven. Let $N \in \mathbb{R}$ be such that $\lim \inf_{x \to 0} \left( \frac{\tan x J'(x)}{J(x)} \right) \geq N > \beta$.

Let $\varepsilon > 0$. There exists some $\delta$ satisfying $0 < \delta < c$ (which we may again assume to be less than $\pi/2$) such that

$$N - \varepsilon \leq \frac{\tan x J'(x)}{J(x)} \quad \text{for all } x \in (0, \delta).$$

If $0 < z < y < \delta$, dividing through the above inequality by $\tan x$ and integrating between $x = z$ and $x = y$ gives the inequality $|J(z)| \left( \frac{\sin y}{\sin z} \right)^{N-\varepsilon} \leq |J(y)|$, or after re-arrangement

$$|J(z)| \sin^\beta z \leq |J(y)| \sin^{N-\varepsilon} y \cdot \sin^{N-\varepsilon-\beta} z \tag{3.6}$$

Taking $\varepsilon > 0$ such that $N > \beta + \varepsilon$ and letting $z \to 0$ in inequality (3.6) gives the claimed result. \(\square\)

**Theorem 3.7.** Let $\mathcal{S}$ be strictly convex at an isolated umbilic point $p$ on the axis of rotational symmetry.

(A) If the limit $\lim_{\theta \to 0} \left( \frac{r_1 - r_2}{\sin^\alpha \theta} \right)$ is finite for some $\alpha \geq 0$, then $\lim \sup_{\theta \to 0} \mu(\theta) \geq \alpha + 1$.

(B) Conversely if $\lim \inf_{\theta \to 0} \mu(\theta) > \alpha + 1$ then $\lim_{\theta \to 0} \left( \frac{r_2 - r_1}{\sin^\alpha \theta} \right) = 0$.

**Proof.** First consider the $\alpha = 0$ case separately. Since $\mathcal{S}$ is assumed strictly convex

$$\lim_{\theta \to 0} \sup_{\theta \to 0} \mu(\theta) \geq \lim_{\theta \to 0} \inf_{\theta \to 0} \mu(\theta) \geq 1,$$

with the last inequality holding by Lemma 3.4. Thus the conclusion of (A) holds. Also, again by strict convexity, $r_1(0) = r_2(0) = r_0 < \infty$, where $r_0$ is the finite radii of curvature of $\mathcal{S}$ at $p$. Hence $\lim_{\theta \to 0} \left( \frac{r_2 - r_1}{\sin^\alpha \theta} \right) = \lim_{\theta \to 0} (r_2 - r_1) = 0$. Hence the conclusion of (B) holds.

Now assume $\alpha > 0$. In line with the notation of Lemma 3.6, let

$$J(\theta) = r_1(\theta) - r_0.$$

It follows from equation (3.4) that

$$\mu(\theta) = 1 + \frac{\tan \theta J'(\theta)}{J(\theta)}.$$
Note that since \( p \) is an isolated umbilic, \( J(\theta) \) is non-zero in an interval \((0, c)\) for some \( c > 0 \) and by the Codazzi-Mainardi equation (2.7) \( J(\theta) \) is differentiable on \((0, c)\). Note we may take \( c < \pi/2 \) so that \( \tan \theta > 0 \) and the Codazzi–Mainardi equation is not singular on \((0, c)\). Furthermore by Lemma 3.4

\[
\liminf_{\theta \to 0} \left( \frac{\tan \theta J'(\theta)}{J(\theta)} \right) = \liminf_{\theta \to 0} (\mu(\theta) - 1) > 0.
\]

Hence \( J(\theta) \) satisfies the prerequisites of Lemma 3.6. Statements (A) and (B) now follow from the respective statements in Lemma 3.6.

In the special case we have the umbilic slope \( \mu_p \) existing we have the following corollary

**Theorem 3.8.** Let \( S \) be strictly convex at an isolated umbilic point \( p \) on the axis of rotational symmetry. Suppose at \( p \), \( S \) has an umbilic slope of \( \mu_p \in \mathbb{R} \).

(A) If the radii of curvature satisfy \( \lim_{\theta \to 0} \left( \frac{r_2 - r_1}{\sin^\alpha \theta} \right) = \gamma \) for some \( \alpha \geq 0, \gamma \in \mathbb{R} \), then \( \mu_p \geq \alpha + 1 \), with equality if \( \gamma \neq 0 \).

(B) Conversely if \( \mu_p \in \mathbb{R} \) satisfies \( \mu_p > \alpha + 1 \) then \( \lim_{\theta \to 0} \left( \frac{r_2 - r_1}{\sin^\alpha \theta} \right) = 0 \).

**Proof.** Since \( \mu_p \) exists,

\[
\mu_p = \lim_{\theta \to 0} \mu(\theta) = \limsup_{\theta \to 0} \mu(\theta) = \liminf_{\theta \to 0} \mu(\theta).
\]

Statement (A) almost follows by (A) of Theorem 3.7, we just need to show that \( \mu_p = \alpha + 1 \) when \( \gamma \neq 0 \). If we assume this is the case then by L’Hôpital’s rule

\[
\mu_p = 1 + \lim_{\theta \to 0} \left( \frac{s(\theta)}{\int_0^\theta s(\tau) \cot \tau d\tau} \right),
\]

\[
= 1 + \lim_{\theta \to 0} \left( \frac{s(\theta)}{\sin^\alpha \theta} \right) \cdot \lim_{\theta \to 0} \left( \frac{\sin^\alpha \theta}{\int_0^\theta s(\tau) \cot \tau d\tau} \right),
\]

\[
= \gamma \cdot \frac{\alpha}{\gamma} + 1,
\]

\[
= \alpha + 1.
\]

Statement (B) follows directly from (B) of Theorem 3.7.

We remark that \( \mu_p = \alpha + 1 \) is not sufficient to determine the behaviour of \( \frac{r_2 - r_1}{\sin^\alpha \theta} \) as \( \theta \to 0 \). In particular, the strict inequality in Theorem 3.8 part (B) is tight, as the next example shows.

**Example 3.9.** Consider the family of surfaces of revolution satisfying

\[
r_2(\theta) - r_1(\theta) = \sin^\alpha \theta \cdot \ln(2 \csc \theta)^\beta
\]

for \( \beta \in \mathbb{R} \). The existence of these surfaces is shown by solving the above equation for \( r_2 - r_1 \) together with the Codazzi-Mainardi equation (2.7) to find \( r_1(\theta) \) and \( r_2(\theta) \), and then applying Proposition 2.4.

We remark that \( \mu_p = \alpha + 1 \) for each of these surfaces but

\[
\lim_{\theta \to 0} \left( \frac{r_2(\theta) - r_1(\theta)}{\sin^\alpha \theta} \right) = \begin{cases} 0 & \beta = -1 \\ 1 & \beta = 0 \\ \infty & \beta = +1 \end{cases}.
\]
Corollary 3.10. Let \( p \) be an isolated umbilic point on the axis of rotational symmetry of \( S \), a strictly convex and \( C^3 \)-smooth surface. Then if \( \mu_p \) exists, \( \mu_p \geq 2 \). If in addition \( S \) is \( C^4 \)-smooth then \( \mu_p \geq 3 \).

Proof. Since \( S \) is \( C^3 \)-smooth, Lemma (2.2) implies the radii of curvature are \( C^1 \)-smooth and in particular have vanishing derivative at \( \theta = 0 \). By L'Hôpital's rule

\[
\lim_{\theta \to 0} \frac{r_2 - r_1}{\sin \theta} = \lim_{\theta \to 0} \frac{r_2' - r_1'}{\cos \theta} = \frac{r_2'(0) - r_1'(0)}{0} = 0,
\]

and hence by Theorem 3.8 \( \mu_p \geq 2 \). If \( S \) is \( C^4 \)-smooth then by a similar argument, applying Lemma (2.3);

\[
\lim_{\theta \to 0} \frac{r_2 - r_1}{\sin^2 \theta} = \frac{r_2''(0) - r_1''(0)}{2}.
\]

Since this limit exists Theorem 3.8 implies \( \mu_p \geq 3 \). \( \square \)

4 \( \text{SL}_2(\mathbb{R}) \) Transformations

We first motivate the study of \( \text{SL}_2(\mathbb{R}) \) transformations of RoC space by discussing their geometrical significance. Consider the anti de-Sitter metric on RoC space, given in \((\psi, s)\) coordinates:

\[ g = \frac{d\psi^2 - ds^2}{s^2}, \quad (\psi, s) = \left(\frac{r_2 + r_1}{2}, \frac{r_2 - r_1}{2}\right). \]  

The quantities \((\psi, s)\) are natural when discussing the set of oriented normal lines of a surface, which is a subset of the space of oriented lines in \( \mathbb{R}^3 \), denoted as \( \mathbb{L} \). \( \mathbb{L} \) is a 4 dimensional real manifold which is diffeomorphic to \( T\mathbb{S}^2 \) [9] and carries a canonical Kähler structure with metric \( G \), of signature \((2,2)\) induced from the round metric on \( S^2 \) [10]. Given a surface in \( \mathbb{R}^3 \), its oriented normal lines form a surface in \( \mathbb{L} \) which inherits a sub-manifold geometry from \( G \). The anti de-Sitter metric for RoC space arises from pushing forward this geometry to RoC space. See [11], Theorem 9, page 8 for further details. There is a simple description of the isometries of \((\mathbb{R}^2, g)\) once we have complexified \( \mathbb{R}^2 \) with the split complex variable \( j \) satisfying \( j^2 = +1 \).

Proposition 4.1 ([20], Lemma 9.3, page 118). The fractional linear transformations

\[ z \mapsto \frac{az + b}{cz + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}), \]

where \( z = \psi + js \) are isometries of \((\mathbb{R}^2, g)\).

From this point on-wards, the above transformations are referred to as \( \text{SL}_2(\mathbb{R}) \) transformations and the symbols \( a, b, c, d \in \mathbb{R} \) will exclusively denote the entries of a general element of \( \text{SL}_2(\mathbb{R}) \) as done in Proposition 4.1. The geodesics of this metric correspond to the RoC diagrams of surfaces which are linear in their Gauss and mean curvature, i.e. LW-surfaces.
Proposition 4.2. The geodesics of \((g, \mathbb{R}^2)\) are the curves satisfying
\[ \alpha H + \beta K = \gamma \]  
where \(\alpha, \beta, \gamma \in \mathbb{R}\) and \(H\) and \(K\) are the mean and Gauss curvature respectively.

Proof. The 2D anti de-Sitter space is maximally symmetric, hence there are three linearly independent Killing vectors of \(g\) denoted \(U_i, i = 1, 2, 3\) which we give in \((\psi, s)\) coordinates:
\[ U_1 = (\psi^2 + s^2) \partial_\psi + 2\psi s \partial_s, \quad U_2 = \psi \partial_\psi + s \partial_s, \quad U_3 = \partial_\psi. \]

If \(V\) is a geodesic tangent vector, the quantities
\[ \lambda_i = g(U_i, V), \quad i = 1, 2, 3, \]
are constant along geodesics and therefore may be used to algebraically describe them. A short calculation shows that
\[ \lambda_1 = \psi^2 + s^2 \frac{d\psi}{d\theta} - 2\psi s \frac{ds}{d\theta}, \quad \lambda_2 = \frac{\psi}{s^2} \frac{d\psi}{d\theta} - \frac{1}{s} \frac{ds}{d\theta}, \quad \lambda_3 = \frac{1}{s^2} \frac{d\psi}{d\theta}, \]
where \(V = \left( \frac{d\psi}{d\theta}, \frac{ds}{d\theta} \right)\). Eliminating \(V\) from these equations gives an algebraic equation defining the geodesics of \(g\):
\[ \left( \psi - \frac{\lambda_2}{\lambda_3} \right)^2 - s^2 = \frac{\lambda_2^2 - \lambda_1 \lambda_3}{\lambda_3^2}, \quad \lambda_3 \neq 0, \quad (4.3) \]
\[ \psi = \text{constant}, \quad \lambda_3 = 0. \quad (4.4) \]
Writing the above equations in terms of the principal curvatures \(k_1\) and \(k_2\) via equation (4.1) gives the stated relationship (4.2) for some constants \(\alpha, \beta, \gamma \in \mathbb{R}\).

The action of the \(\text{SL}_2(\mathbb{R})\) transformations on RoC space can be given in \((r_1, r_2)\) coordinates, denote the corresponding coordinate transformation as \(T_M : \mathbb{R}^2 \rightarrow \mathbb{R}^2\), for a given \(M \in \text{SL}_2(\mathbb{R})\).

Proposition 4.3. Under a general \(\text{SL}_2(\mathbb{R})\) transformation the radii of curvature are mapped to
\[ T_M(r_1, r_2) = \left( \frac{ar_1 + b}{cr_1 + d}, \frac{ar_2 + b}{cr_2 + d} \right), \]
\(M\) a general element of \(\text{SL}_2(\mathbb{R})\).

Proof. Denote the image of \(z\) under the fractional linear transformation given in Proposition 4.1 as \(\tilde{z}\). Denote the real and imaginary parts of \(\tilde{z}\) as \(\psi\) and \(\tilde{s}\) respectively, in accordance with equation (4.1), \(\tilde{r}_1\) and \(\tilde{r}_2\) are then given by
\[ \tilde{r}_1 = \tilde{\psi} + \tilde{s}, \quad \tilde{r}_2 = \tilde{\psi} - \tilde{s}. \]
The rest of the proof is now an exercise in writing \(\tilde{r}_i\) in terms of \(r_i, i = 1, 2\). In terms of \(\psi\) and \(s\), we have
\[ \tilde{\psi} = \text{Re}(\tilde{z}) = \text{Re} \left( \frac{az + b}{cz + d} \right) = \frac{ac(\psi^2 - s^2) + (ad + bc)\psi + bd}{(c\psi + d)^2 - c^2s^2}, \]
and
\[ s = \text{Im}(z) = \text{Im} \left( \frac{az + b}{cz + d} \right) = \frac{s}{(c\psi + d)^2 - c^2s^2}. \]
Solving for \((\bar{r}_1, \bar{r}_2)\) in terms of \((r_1, r_2)\) using equation (4.1) finishes the proof.

**Remark 4.4.** The map \(T_M\) is extended to all of RoC space, i.e. to \(\hat{\mathbb{R}} \times \hat{\mathbb{R}}\) by extending the domain and range of each of its component maps from \(\mathbb{R}\) to \(\hat{\mathbb{R}}\) as is often done with fractional linear transformations.

**Remark 4.5.** We will abbreviate by \(\bar{r}_i\) the image of \(r_i\) under a \(\text{SL}_2(\mathbb{R})\) transformation as in Proposition 4.3 when the transformation in question is clear. We let \(\bar{k}_i = 1/\bar{r}_i\) for \(i=1,2\), which are the transformed principal curvatures. It is easy to show that
\[ \bar{k}_i = \frac{dk_i + c}{bk_i + a}. \]
which is the corresponding transformation on curvature space \(\mathfrak{g}(S)\) stated in the introduction.

### 4.1 Induced Surface Transformations

The Codazzi-Mainardi equation (2.7) will now be used as an integrability condition to find a (possibly non-regular) surface \(\bar{S}\) such that \(\mathfrak{R}(\bar{S})\) is the image, under a \(\text{SL}_2(\mathbb{R})\) transformation, of an initial RoC diagram \(\mathfrak{R}(S)\) for some \(S\). In what follows let \(a, b, c, d\) represent the coefficients of a general \(M \in \text{SL}_2(\mathbb{R})\).

**Lemma 4.6.** If \(J : I \to \mathbb{R}^2\) is parameterised by \(\theta \in I \subseteq [0, \pi]\) and satisfies the Codazzi-Mainardi equation (2.7), then if \(M \in \text{SL}_2(\mathbb{R})\), and \(r_i \neq -d/c\) for \(i = 1, 2\), \(T_M(\mathcal{J}(I))\) can be parameterised to satisfy the Codazzi-Mainardi equation. In particular, if \(\tilde{\theta}\) satisfies
\[ \sin \tilde{\theta} = A \sin \theta \cdot (c r_1(\theta) + d), \quad A \in \mathbb{R}\backslash \{0\}, \]
then \(\tilde{\theta}\) is such a parametrisation of \(T_M(\mathcal{J}(I))\).

**Proof.** We have that
\[ T_M(\mathcal{J}(I)) = \left\{ (\bar{r}_1, \bar{r}_2) \in \hat{\mathbb{R}} \times \hat{\mathbb{R}} \mid \bar{r}_i = \frac{a r_i + b}{c r_i + d}, \quad i = 1, 2, \quad (r_1, r_2) \in \mathcal{J}(I) \right\}. \]
Assume that \(\bar{\theta}\) parameterises a part of \(T_M(\mathcal{J}(I))\) for which \(c r_i + d \neq 0\). Suppose \(\bar{\theta}\) satisfies equation (4.6). Then
\[ \frac{d\bar{r}_1}{d\bar{\theta}} = \frac{d\theta}{d\bar{\theta}} \cdot \frac{d\bar{r}_1}{d\theta} = \frac{\cos \bar{\theta}}{A \cos \theta \cdot (c r_2 + d)} \frac{d}{d\bar{\theta}} \left( \frac{a r_1 + b}{c r_1 + d} \right), \]
where we have implicitly differentiated equation (4.6) by \(\theta\) to find \(d\theta/d\bar{\theta}\), and used the Codazzi-Mainardi equation to remove any derivatives of \(r_1\). A short calculation, removing any derivatives of \(r_1\) by Codazzi-Mainardi, and removing occurrences of \(\sin \theta\) by equation (4.6) yields the result
\[ \frac{d\bar{r}_1}{d\bar{\theta}} = (\bar{r}_2 - \bar{r}_1) \cot \bar{\theta}. \]
\[ \square \]
Remark 4.7. In fact, away from points of $\mathcal{J}(I)$ for which $r_1 = r_2$, parameters $\tilde{\theta}$ satisfying relation (4.6) are the only parameters for which $T_M(\mathcal{J}(I))$ satisfies the Codazzi-Mainardi equation. Indeed if
\[
\frac{d r_1}{d \theta} = (r_2 - r_1) \cot \theta, \quad \text{and} \quad \frac{d \tilde{r}_1}{d \theta} = (\tilde{r}_2 - \tilde{r}_1) \cot \tilde{\theta},
\]
hold on $\mathcal{J}(I)$ and $T_M(\mathcal{J}(I))$ respectively, then the quotient of these equations is a separable ODE which solves to give $\tilde{\theta}$ implicitly by equation (4.6).

We now prove the main theorem of Section 4, stated in terms of $\Re(S)$ instead of $\Re(S)$ as was done in the introduction.

**Theorem 4.8.** If $T_M$ is a real fractional linear transformation and $S$ is non-flat, then there exists a rotationally symmetric and possibly non-regular surface $\tilde{S}$ such that $\Re(\tilde{S}) = T(\Re(S))$.

**Proof.** Using the notation of Lemma 4.6 we first consider the case where $r_1 = r_2$, parameters $\tilde{\theta}$ satisfying relation (4.6) are the only parameters for which $T_M(\mathcal{J}(I))$ satisfies the Codazzi-Mainardi equation. Indeed if
\[
\frac{d r_1}{d \theta} = (r_2 - r_1) \cot \theta, \quad \text{and} \quad \frac{d \tilde{r}_1}{d \theta} = (\tilde{r}_2 - \tilde{r}_1) \cot \tilde{\theta},
\]
hold on $\mathcal{J}(I)$ and $T_M(\mathcal{J}(I))$ respectively, then the quotient of these equations is a separable ODE which solves to give $\tilde{\theta}$ implicitly by equation (4.6).

We now prove the main theorem of Section 4, stated in terms of $\Re(S)$ instead of $\Re(S)$ as was done in the introduction.

**Theorem 4.8.** If $T_M$ is a real fractional linear transformation and $S$ is non-flat, then there exists a rotationally symmetric and possibly non-regular surface $\tilde{S}$ such that $\Re(\tilde{S}) = T(\Re(S))$.

**Proof.** Using the notation of Lemma 4.6 we first consider the case where $r_1 = r_2$, parameters $\tilde{\theta}$ satisfying relation (4.6) are the only parameters for which $T_M(\mathcal{J}(I))$ satisfies the Codazzi-Mainardi equation. Indeed if
\[
\frac{d r_1}{d \theta} = (r_2 - r_1) \cot \theta, \quad \text{and} \quad \frac{d \tilde{r}_1}{d \theta} = (\tilde{r}_2 - \tilde{r}_1) \cot \tilde{\theta},
\]
hold on $\mathcal{J}(I)$ and $T_M(\mathcal{J}(I))$ respectively, then the quotient of these equations is a separable ODE which solves to give $\tilde{\theta}$ implicitly by equation (4.6).

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\]
hold on $\mathcal{J}(I)$ and $T_M(\mathcal{J}(I))$ respectively, then the quotient of these equations is a separable ODE which solves to give $\tilde{\theta}$ implicitly by equation (4.6).
general $S$, the map $\theta \mapsto \bar{\theta}(\theta)$ given by equation (4.8) will not always be well defined since the RHS may not lie between 1 and $-1$ for all $\theta \in I$. If $\rho$ is bounded however, setting

$$A = \left( \max_S |c\rho + d\sin \theta| \right)^{-1},$$

ensures that the surface transformation can be defined for all $\theta \in I$. We now decompose the $SL_2(\mathbb{R})$ transformations into a composition of simpler transformations which have a clearer geometric interpretation. Consider the following subgroups of $SL_2(\mathbb{R})$

$$N = \left\{ \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} : v \in \mathbb{R} \right\} \quad \text{and} \quad A = \left\{ \begin{pmatrix} \omega & 0 \\ 0 & \frac{1}{\omega} \end{pmatrix} : \omega \neq 0 \right\}.$$

For these subgroups there is a natural choice of $A$ which allows $S$ and $\bar{S}$ to be parameterised by the same Gauss angle, furthermore with this choice of $A$ the induced transformations on surfaces are very geometric.

**Proposition 4.10.** The induced transformations of a surface in $E^3$ by the subgroup $N$ with $A = 1$ are parallel translations.

**Proof.** Let the transformation $T_M, M = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \in N$ for $v \in \mathbb{R}$ act on RoC space. Then we have by Proposition 4.9

$$\sin \bar{\theta} = A \sin \theta, \quad \bar{\rho}(\bar{\theta}) = A (\rho(\theta) + v \sin \theta).$$

Therefore taking $A = 1$, we can take $\bar{\theta} = \theta$ and $\bar{\rho} = \rho + v \sin \theta$, furthermore by equations (2.2) we find $\bar{h} = h + v \cos \theta$. Since $\sin \theta$ and $\cos \theta$ are the radial and axial components of the unit normal vector of $S$ at a point with Gauss angle $\theta$, the transformation translates every point of $S$ a distance $v$ in the normal direction. \qed

**Proposition 4.11.** The action on a surface in $E^3$ of the subgroup $A$ with $A = \omega$ is homothety.

**Proof.** Proceeding as in the proof of Proposition 4.10, let $M = \begin{pmatrix} \omega & 0 \\ 0 & \frac{1}{\omega} \end{pmatrix} \in A, \omega \neq 0$. Then $\bar{\theta}$ and $\bar{\rho}$ satisfy

$$\sin \bar{\theta} = \frac{A}{\omega} \sin \theta, \quad \bar{\rho} (\bar{\theta}) = \omega A \rho(\theta).$$

Taking $A = \omega$, gives $\bar{\theta} = \theta, \bar{\rho} = \omega^2 \rho$ and again by equations (2.2) $\bar{h} = \omega^2 h$. Hence the transformation is a homothety with a scale factor of $\omega^2$. \qed

Elements of $SL_2(\mathbb{R})$ have the following decomposition, if $c = 0$ the general element can be written as

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} 1 & ba \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix},$$

(4.9)
on the other hand if $c \neq 0$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & a/c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/c & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix}.$$

(4.10)
Hence any element of $SL_2(\mathbb{R})$ can be constructed from composition of elements from $A$, $N$ and the matrix
\[
Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\] (4.11)

We now describe the action of $Q$ in more detail. As a corollary of Proposition 4.9 we have

**Proposition 4.12.** The matrix $Q$ induces a 1-parameter family of transformations mapping $S$ to a surface of revolution $\tilde{S}_A$ satisfying
\[
\sin \tilde{\theta} = A \rho(\theta), \quad \tilde{\rho}(\tilde{\theta}) = -A \sin \theta.
\] (4.12)

Unlike with elements in the subgroups $A$ and $N$ there is no obvious canonical choice of $A$ to associate with $Q$ in the case of a general $S$. If $S$ is a $C^2$-smooth, closed, strictly convex and regular surface, we have the following.

**Proposition 4.13.** If $S$ is a $C^2$-smooth, closed, strictly convex and regular surface, then with $A = \rho(\pi/2)^{-1}$, the transformation induced by $Q$ sends $S$ to a closed, strictly convex and regular surface, $\tilde{S}$.

**Proof.** Take $\theta \in [0, \pi]$. Since $S$ is strictly convex and regular, $r_1$ and $r_2$ are bounded, have the same sign and are non-zero, hence by continuity do not change sign on $S$. Furthermore from relations (2.5) and (2.1) we see
\[
\frac{d \rho}{d \theta} = r_2 \cos \theta,
\] (4.13)

implying $\theta = \pi/2$ is a stationary point of $\rho$. Furthermore by taking a difference quotient of the above equation we see that $\rho'((\pi/2) = -r_2(\pi/2) \neq 0$ hence $\rho$ attains a local maxima (or minima) at $\theta = \pi/2$, depending on if $r_2(\theta)$ is positive (or negative) on $S$. However $\rho$ takes the same sign as $r_1$ when $\theta \in [0, \pi]$ (via equation 2.5) which takes the same sign as $r_2$, thus $\theta = \pi/2$ is a maxima when $\rho(\theta) > 0$ on $[0, \pi]$ and a minima when $\rho < 0$ on $[0, \pi]$. Hence $|\rho|$ attains a maximum of $|\rho(\pi/2)|$ implying $-1 \leq \frac{\rho(\theta)}{\rho(\pi/2)} \leq 1$. Letting $A = \rho(\pi/2)^{-1}$ we can define the bijective and continuous map $\theta \mapsto \tilde{\theta}$ satisfying equation (4.12):
\[
\tilde{\theta}(\theta) = \begin{cases} 
\sin^{-1}\left( \frac{\rho(\theta)}{\rho(\pi/2)} \right), & 0 \leq \theta \leq \frac{\pi}{2} \\
\pi - \sin^{-1}\left( \frac{\rho(\theta)}{\rho(\pi/2)} \right), & \frac{\pi}{2} \leq \theta \leq \pi.
\end{cases}
\] (4.14)

By Proposition 4.8, $\tilde{\theta}$ is the Gauss angle of a surface of revolution $\tilde{S}$. Note that since for $i = 1, 2$, $r_i$ is bounded, $\tilde{r}_i$ is not zero, hence $\tilde{S}$ is regular. We now show that the profile curve of $\tilde{S}$ is the continuous image of the compact set $[0, \pi]$. It then follows that $\tilde{S}$ is compact. Note that from relation (4.12), $\tilde{\rho}$ is a bounded and continuous function of $\theta$ and from relations (2.5),(4.12) and the Codazzi-Mainardi equation, up to the addition of some constant of integration,
\[
\hat{h} = -\int \tilde{r}_2 \sin \tilde{\theta} d\tilde{\theta} = \int \frac{\rho(\theta)}{\rho(\pi/2) r_2(\theta)} \frac{d\tilde{\theta}}{d\theta} d\theta = \int \frac{\text{sgn}(\pi/2 - \theta) \rho(\theta) \cos \theta}{|\rho(\pi/2)| \sqrt{\rho(\pi/2)^2 - \rho(\theta)^2}} d\theta,
\] (4.15)

where $d\tilde{\theta}/d\theta$ is calculated from equation (4.14). Hence to show $\hat{h}$ is continuous on $[0, \pi]$, it is shown that the integrand in equation (4.15) is continuous and bounded, in particular at $\theta = \pi/2$. Since $\rho$
is \( C^2 \) in the variable \( \theta \), equation (4.13) Taylor’s theorem with remainder \( \omega \) gives;

\[
\rho(\theta) = \rho \left( \frac{\pi}{2} \right) + \frac{1}{2} \rho'' \left( \frac{\pi}{2} \right) \left( \theta - \frac{\pi}{2} \right)^2 + \omega(\theta) \left( \theta - \frac{\pi}{2} \right)^2 ,
\]

where \( \omega(\theta) \to 0 \) as \( \theta \to \pi/2 \). This implies the asymptotic behaviour of the integrand

\[
\frac{\text{sgn}(\pi/2 - \theta)\rho(\theta) \cos \theta}{|\rho(\pi/2)|} \to \frac{\text{sgn}(\rho(\pi/2)) \sqrt{K(\pi/2)}}{|\rho(\pi/2)|^{3/2} - \rho(\theta)^2} ,
\]

as \( \theta \to \pi/2 \), where \( K \) is the Gauss curvature of \( S \). Hence the profile curve of \( \bar{S} \) is the continuous image of the set \([0, \pi]\). Since \( \tilde{\rho} \to 0 \) as \( \theta \to 0, \pi \), we have that \( \bar{S} \) is without boundary and therefore closed.

The surface transformations induced by \( Q \) are called reciprocal transformations as their action on RoC space is \((r_1, r_2) \mapsto (-1/r_1, -1/r_2)\). The action of the reciprocal transformations on a surface in \( \mathbb{E}^3 \) may be understood as exchanging the functions \( \sin \theta \) and \( \rho(\theta) \), up to a scalar multiple, as illustrated by Figure 5. The following theorem ties together the results of this section.

**Theorem 4.14.** Any surface transformation induced by an element of \( \text{SL}_2(\mathbb{R}) \) is a composition of parallel translations, homotheties and reciprocal mappings.

### 4.2 General Properties of the \( \text{SL}_2(\mathbb{R}) \) Transformations.

The \( \text{SL}_2(\mathbb{R}) \) transformations generate examples of surfaces satisfying transformed Weingarten relations. If \( S \) has principal curvatures \((k_1, k_2)\), then its image \( \bar{S} \), has principal curvatures \((\bar{k}_1, \bar{k}_2)\) given by Remark 4.5. If \( \eta \) denotes the map \( \eta : (k_1, k_2) \mapsto (\bar{k}_1, \bar{k}_2) \) and \( S \) satisfies a Weingarten relation \( W(k_1, k_2) = 0 \) we have

\[
(W \circ \eta^{-1})(\bar{k}_1, \bar{k}_2) = W(k_1, k_2) = 0 ,
\]

hence dropping the tildes on the principal curvatures, \( \bar{S} \) satisfies the Weingarten relationship \( \bar{W}(k_1, k_2) = 0 \), where \( \bar{W} = W \circ \eta^{-1} \). Furthermore, it allows us to relate surfaces satisfying different
Weingarten relations through induced transformations in $\mathbb{E}^3$. Now we give some general properties of the $\text{SL}_2(\mathbb{R})$ transformations in terms of how they transform surfaces and their Weingarten relations.

**Theorem 4.15.** Let $\bar{S}$ be the image of a surface of revolution $S$ under the transformation of surfaces by $M \in \text{SL}_2(\mathbb{R})$.

1. Umbilic points of $S$ are mapped to umbilic points of $\bar{S}$.
2. When $S$ is Weingarten, elliptic points of $S$ are mapped to elliptic points of $\bar{S}$, providing the principal curvatures at this point, $k_i$, satisfy $k_i \neq -a/b$.
3. If $p \in S$ and $\bar{p} \in \bar{S}$ are both non-flat and isolated umbilic points with $\bar{p}$ being the image of $p$ under an $\text{SL}_2(\mathbb{R})$ transformation then $\tilde{\mu}_\bar{p} = \mu_p$.

**Proof.** These claims follow from the curvature transformations given in Proposition 4.3/Remark 4.5.

Let $S$ and $\bar{S}$ have principal curvatures $(k_1, k_2)$ and $(\bar{k}_1, \bar{k}_2)$ respectively. The map $\eta : (k_1, k_2) \rightarrow (\bar{k}_1, \bar{k}_2)$ as in equation (4.16) is given explicitly as

$$\eta(x, y) = \left(\frac{dx + c}{bx + a}, \frac{dy + c}{by + a}\right).$$

As long as $k_i \neq -a/b$, $k_1 = k_2$ if and only if $\bar{k}_1 = \bar{k}_2$. If Claim 2 follows by checking that

$$\frac{\partial \bar{W}(\bar{k}_1, \bar{k}_2)}{\partial k_i} = (bk_i + a)^2 \cdot \frac{\partial W(k_1, k_2)}{\partial k_i},$$

for $i = 1, 2$. Hence, recalling that for a Weingarten surface, ellipticity is equivalent to condition (2.13), it follows that $S$ is elliptic at the point $q$ if and only if $\bar{S}$ is elliptic at the point $\bar{q}$, the image of $q$ under an $\text{SL}_2(\mathbb{R})$ transformation. To show claim 3, first note that as $\bar{p}$ is a non-flat point, $\bar{r}_0 \neq \infty$ and so $r_0 \neq -d/c$. Note that after applying an $\text{SL}_2(\mathbb{R})$ transformation to the curvatures

$$\frac{\bar{r}_2 - \bar{r}_0}{\bar{r}_1 - \bar{r}_0} = \frac{\frac{a r_2 + b}{cr_2 + d} - \frac{a r_0 + b}{cr_0 + d}}{\frac{a r_1 + b}{cr_1 + d} - \frac{a r_0 + b}{cr_0 + d}} = \left(\frac{cr_1 + d}{cr_2 + d}\right) \left(\frac{r_2 - r_0}{r_1 - r_0}\right).$$

Hence taking the limit of the above as $(r_1, r_2) \rightarrow (r_0, r_0)$ gives $\mu_\bar{p} = \mu_p$. □

### 4.3 Application to Semi-Quadratic Weingarten Surfaces

A *quadratic Weingarten surface* is any $C^2$-smooth surface satisfying the Weingarten relationship

$$\tau k_1^2 + \nu k_1^2 + \alpha k_1 k_2 + \beta k_1 + \gamma k_2 + \delta = 0, \quad (\tau, \nu, \alpha, \beta, \gamma, \delta) \in \mathbb{R}^6 \setminus \{0\},$$

where $k_i$ are the principal curvatures. Special cases of this Weingarten relationship have been studied previously in the rotationally symmetric setting [21]. We study the following subfamily.

**Definition 4.16.** The subfamily of quadratic Weingarten surfaces satisfying

$$\alpha k_1 k_2 + \beta k_1 + \gamma k_2 + \delta = 0 \quad \alpha, \beta, \gamma, \delta \in \mathbb{R},$$

we call *semi-quadratic Weingarten surfaces*. 22
This subfamily contains well studied classes of surfaces:

\[ r_2 = \lambda r_1 + C, \quad k_2 = \lambda k_1 + C, \quad \lambda H + \Upsilon K + C = 0, \]

\[(\alpha \neq 0, \delta = 0) \quad (\alpha = 0, \delta \neq 0) \quad (\beta = \gamma)\]

for \(\lambda, \Upsilon, C \in \mathbb{R}\), which are from left to right the linear Hopf surfaces which satisfy relation (3.1), k-linear surfaces and LW-surfaces, studied in [12, 25], [22, 24] and [7, 23] respectively. Consider the following quantities:

\[ \Lambda_1 = \beta - \gamma, \quad \Lambda_2 = (\beta + \gamma)^2 - 4\alpha\delta. \quad (4.20) \]

When \(\Lambda_1 = 0\) semi-quadratic surfaces become LW-surfaces. LW-surfaces are said to be hyperbolic (elliptic) when \(\Lambda_2 < 0\) (\(> 0\)) [7, 23] or satisfy \(\Lambda_2 = 0\) which characterises tubular surfaces. This motivates the following nomenclature

**Definition 4.17.** A semi-quadratic Weingarten surface satisfying \(\Lambda_2 > \Lambda_1^2\) is said to be elliptic. If \(\Lambda_2 < \Lambda_1^2\) it is said to be hyperbolic.

The relative sizes and signs of \(\Lambda_1\) and \(\Lambda_2\) strongly control a semi-quadratic surface’s behaviour:

**Proposition 4.18.** A semi-quadratic Weingarten surface cannot have umbilic points unless \(\Lambda_2 \geq 0\).

*Proof.* First assume \(\alpha \neq 0\), otherwise \(\Lambda_2 \geq 0\) and we are done. If we have such a surface, the curvatures at the umbilic point must satisfy the relation (4.19) when \(k_1 = k_2 = k\), i.e.

\[ \alpha k^2 + (\beta + \gamma)k + \delta = 0. \]

Solving the above quadratic implies the curvatures at the umbilic point satisfy

\[ k = \frac{1}{2\alpha} (-\beta + \gamma) \pm \sqrt{\Lambda_2}, \quad (4.21) \]

and therefore \(k \in \mathbb{R}\) if and only if \(\Lambda_2 \geq 0\). \(\square\)

**Proposition 4.19.** A semi-quadratic Weingarten surface’s Weingarten relation is an elliptic PDE at an umbilic point if and only if

\[ \Lambda_2 > \Lambda_1^2. \quad (4.22) \]

*Proof.* Let \(S\) be a semi-quadratic Weingarten surface with an umbilic point \(p\). Also let

\[ W(k_1, k_2) = \alpha k_1 k_2 + \beta k_1 + \gamma k_2 + \delta. \quad (4.23) \]

Let \(k\) be the common value of \(k_1(p)\) and \(k_2(p)\), then \(k\) is given by either one of the values in equation (4.21) and necessarily \(\Lambda_2 \geq 0\). In which case one can calculate that

\[ \left. \left( \frac{\partial W}{\partial k_1} \cdot \frac{\partial W}{\partial k_2} \right) \right|_p = \frac{1}{4}(\Lambda_2 - \Lambda_1^2). \quad (4.24) \]

Therefore the result follows by the definition of a Weingarten relation being elliptic (2.13). \(\square\)
Proposition 4.20. If $\Lambda_2 \neq \Lambda_1^2$, the umbilic slope at an isolated umbilic point $p$ of a semi-quadratic Weingarten surface takes one of the values

$$\mu_p = \frac{\Lambda_1 \pm \sqrt{\Lambda_2}}{\Lambda_1 \mp \sqrt{\Lambda_2}}, \quad (4.25)$$

If $\Lambda_2 = \Lambda_1^2$, then at such a point $p$ either $\mu_p = 0$ or $\mu_p$ is unbounded.

Proof. The possible values of $\mu_p$ are simply the possible slopes the algebraic curve $W(k_1, k_2) = \alpha k_1 k_2 + \beta k_1 + \gamma k_2 + \delta = 0$ intersects the diagonal line $k_1 = k_2$. Let $k_0$ be given by equation (4.21) so that $W(k_0, k_0) = 0$. If $\Lambda_2 \neq \Lambda_1^2$, then equation (4.24) implies $\partial W / \partial k_2 \neq 0$. Then if $k_1 = k_2 = k_0$,

$$\mu_p = \Lambda_1 \pm \sqrt{\Lambda_2} \frac{\partial W}{\partial k_1} \frac{\partial W}{\partial k_2} = \frac{\Lambda_1 \pm \sqrt{\Lambda_2}}{\Lambda_1 \mp \sqrt{\Lambda_2}}.$$

If $\Lambda_2 = \Lambda_1^2$ then $\beta \gamma = \alpha \delta$, forcing $W^{-1}\{0\}$ to be either a line of constant $k_1$, a line of constant $k_2$, or a union of the two. In which case $\mu_p$ may be either 0 or unbounded. \qed

We will study $SL_2(\mathbb{R})$ transformations between semi-quadratic Weingarten surfaces that are rotationally symmetric and satisfy $\Lambda_2 > 0$. The case $\Lambda_2 < 0$ has been studied in the particular scenario $\Lambda_1 = 0$ with a classification result being obtained in the rotationally symmetric setting [23]. To make the following discussion simpler we assume w.l.o.g. that $\Lambda_2 = 1$ by dividing equation (4.19) through by $\sqrt{\Lambda_2}$. In such a case we say relationship (4.19) is normalised.

Proposition 4.21. The $SL_2(\mathbb{R})$ transformations map normalised quadratic Weingarten relations to normalised quadratic Weingarten relations. If the initial relation has coefficients $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, the coefficients of the target relationship are given by

$$\alpha' = \alpha d^2 + \beta b^2 + (\beta + \gamma)bd, \quad (4.26)$$
$$\beta' = \alpha cd + \delta ab + (\beta + \gamma)bc + \beta, \quad (4.27)$$
$$\gamma' = \alpha cd + \delta ab + (\beta + \gamma)bc + \gamma, \quad (4.28)$$
$$\delta' = \alpha c^2 + \delta a^2 + (\beta + \gamma)ac. \quad (4.29)$$

Furthermore $\Lambda_1^2$ is an invariant.

Proof. We substitute the $SL_2(\mathbb{R})$ transformations from Proposition 4.3 in the form

$$k_i \mapsto \frac{c + dk_i}{a + bk_i},$$

into the initial quadratic relationship (4.19). If $b \neq 0$, multiplying through by any denominators gives a relationship of the form

$$\alpha' k_1 k_2 + \beta' k_1 + \gamma' k_2 + \delta' = 0, \quad (4.30)$$

with $\alpha', \beta', \gamma'$ and $\delta'$ satisfying equations (4.26)-(4.29). It is immediate from these equations that $\Lambda_1^2 = \Lambda_1^2$. Furthermore, a calculation verifies that $\Lambda_2' = \Lambda_2 = 1$ and so the target relationship (4.30) is normalised. If $b = 0$, the target relation is already of the form (4.30), however with $\Lambda_1' = \frac{d^2}{b} \Lambda_1$ and $\Lambda_2' = \frac{d^2}{b^2}$. Dividing equation (4.30) through by $d/a$ normalises the relationship so that $\Lambda_1^2 = 1$ implying $\Lambda_1^2 = \Lambda_1^2$. \qed
**Remark 4.22.** When the semi-quadratic relationship is not normalised, the corresponding invariant is $\Lambda_2^2/\Lambda_2$, shown by multiplying through a normalised equation by $\sqrt{\Lambda_2}$.

**Proposition 4.23.** When $\Lambda_2 > 0$, the $\text{SL}_2(\mathbb{R})$ transformations act transitively on the families of normalised semi-quadratic Weingarten relations with the same $\Lambda_2^2$.

*Proof.* Fix the value of $\Lambda_1$ and let $\Lambda_2 = 1$. Note that by re-arrangement of equation (4.19), $\Lambda_1$ can always be assumed positive and therefore it can be assumed that $\Lambda_1$ is conserved under the $\text{SL}_2(\mathbb{R})$ transformations rather than $\Lambda_1^2$. Let $(\alpha, \beta, \gamma, \delta)$ be the parameters of the initial relation, and $(\alpha', \beta', \gamma', \delta')$ be the parameters of the target relation. To prove transitivity one just needs to show that the system of equations (4.26)-(4.29) has a solution $(a, b, c, d) \in \mathbb{R}^4$, with $ad - bc = 1$, for any choice of the two 4-tuples $(\alpha, \beta, \gamma, \delta), (\alpha', \beta', \gamma', \delta') \in \mathbb{R}^4$ satisfying $\Lambda_1 = \Lambda_1'$ and $\Lambda_2 = \Lambda_2' = 1$. First substitute $\Lambda_1 = \beta' - \gamma' = \beta - \gamma$ into equations (4.26)-(4.29) to remove the parameters $\beta$ and $\beta'$. If one considers $\text{SL}_2(\mathbb{R})$ transformations for which $c \neq 0$, the constraint $ad - bc = 1$ can be used to eliminate $b$ giving a system of equations relating the initial coefficients with the target coefficients:

\[
\begin{align*}
\alpha' &= \frac{1}{c^2} \left[ ac^2d^2 + \delta(ad-1)^2 + (\Lambda_1 + 2\gamma)dcd(ad-1) \right], \\
c\gamma' &= \frac{a\alpha^2d + (\Lambda_1 + 2\gamma)c(ad-1) + \gamma c}{\delta'}, \\
\delta' &= \frac{a\alpha^2 + 2\alpha^2 + (\Lambda_1 + 2\gamma)ac}{\delta'}.
\end{align*}
\]  

The equations (4.31), (4.32) and (4.33) correspond to (4.26), (4.27-4.28), and (4.29) respectively. Transitivity is proven by considering 2 separate cases.

**Case:** $\delta' \neq 0$.

Taking $c \neq 0$, solving the equation $\Lambda_2 = \Lambda_2'$ for $\alpha'$ gives

\[
\alpha' = \frac{(\Lambda_1 + 2\gamma')^2 - (\Lambda_1 + 2\gamma)^2 + 4\alpha\delta}{4\delta'},
\]

Substituting equations (4.32) and (4.33) into the above implies equation (4.31), hence equations (4.32) and (4.33) form an under-determined system

\[
\begin{align*}
c\gamma' &= \delta'd - \delta a - (\Lambda_1 + \gamma)c, \\
\delta' &= \delta a^2 + (\Lambda_1 + 2\gamma)ac,
\end{align*}
\]  

where we have re-written the equation for $\gamma'$, removing the $a\alpha^2d$ term by virtue of equation (4.33) to make calculations easier. This system can be solved to give $a$ and $d$ in terms of $c$: If $\delta \neq 0$, then the solutions are given by

\[
\begin{align*}
a &= \frac{1}{2\delta'} \left( -(\Lambda_1 + 2\gamma)c \pm \sqrt{c^2 + 4\delta^2} \right), \\
d &= \frac{1}{2\delta'} \left( (\Lambda_1 + 2\gamma)c \pm \sqrt{c^2 + 4\delta^2} \right).
\end{align*}
\]

The parameters $a$ and $d$ can always be taken to be real by taking a sufficiently large $c$, and $b$ is determined by $ad - bc = 1$. If $\delta = 0$, then the solutions are

\[
\begin{align*}
a &= \frac{\delta' - \alpha\gamma^2}{c(\Lambda_1 + 2\gamma)}, \\
d &= \frac{c}{\delta'}(\Lambda_1 + \gamma + \gamma'),
\end{align*}
\]

noting that when $\delta = 0$, $(\Lambda_1 + 2\gamma)^2 = (\beta + \gamma)^2 = \Lambda_2 \neq 0$. Hence we have found an $\text{SL}_2(\mathbb{R})$ transformation taking $(\alpha, \beta, \gamma, \delta)$ to $(\alpha', \beta', \gamma', \delta')$ when $\delta' \neq 0$.  

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Case: $\delta' = 0$.

Note that if $\delta \neq 0$, by the above case there exists an $\text{SL}_2(\mathbb{R})$ transformation sending $(\alpha', \beta', \gamma', 0)$ to $(\alpha, \beta, \gamma, \delta)$. Taking the inverse transformation proves transitivity when $\delta \neq 0$. Now assume $\delta = 0$. Any normalised relationship for which $\delta = 0$ must be of the form

$$\alpha k_1 k_2 + \frac{1}{2} (\Lambda_1 + 1) k_1 + \frac{1}{2} (-\Lambda_1 + 1) k_2 = 0,$$

since $\beta$ and $\gamma$ must solve $\beta - \gamma = \Lambda_1$ and $(\beta + \gamma)^2 = \Lambda_2 = 1$ simultaneously. Thus $\delta = \delta' = 0$ implies that $\gamma$ and $\gamma'$ take either of the values $\frac{1}{2}(-\Lambda_1 + 1)$ and so for this sub-case, all initial and target relations must be of the respective forms

$$(\alpha, \frac{1}{2}(\Lambda_1 + 1), \frac{1}{2}(-\Lambda_1 + 1), 0), \quad (\alpha', \frac{1}{2}(\Lambda_1 + 1), \frac{1}{2}(-\Lambda_1 + 1), 0).$$

If an $\text{SL}_2(\mathbb{R})$ transformation is taken with $c \neq 0$, equations (4.31) and (4.33) are solved for $a$ and $d$ to give

$$a = -\frac{\alpha c}{\Lambda_1 + 2\gamma}, \quad d = -\frac{\alpha' c}{\Lambda_1 + 2\gamma}.$$

The remaining equation (4.32) implies that $\gamma = \frac{1}{2}(-\Lambda_1 + 1)$ and $\gamma' = \frac{1}{2}(-\Lambda_1 + 1)$. On the other hand, taking $c = 0$ fixes $\gamma = \gamma'$ which can be seen by equation (4.28). Equation (4.26) can then be solved for $d$ in terms of $b$, if $\alpha \neq 0$, then;

$$d = \frac{1}{2\alpha} \left( -\Lambda_1 + 2\gamma \right) b \pm \sqrt{b^2 + 4\alpha a'},$$

taking $b$ sufficiently large implies $d \in \mathbb{R}$, $a$ is then determined by $ad - bc = 1$. Alternatively, if $\alpha = 0$ then solving equation (4.26) gives

$$d = -\frac{\alpha'}{b(\Lambda_1 + 2\gamma)}.$$

Therefore we have found $\text{SL}_2(\mathbb{R})$ transformations taking

$$(\alpha, \frac{1}{2}(\Lambda_1 + 1), \frac{1}{2}(-\Lambda_1 + 1), \delta) \mapsto (\alpha', \frac{1}{2}(\Lambda_1 + 1), \frac{1}{2}(-\Lambda_1 + 1), \delta'),$$

if $c \neq 0$, or

$$(\alpha, \frac{1}{2}(\Lambda_1 + 1), \frac{1}{2}(-\Lambda_1 + 1), \delta) \mapsto (\alpha', \frac{1}{2}(\Lambda_1 + 1), \frac{1}{2}(-\Lambda_1 + 1), \delta'),$$

if $c = 0$, which together describe all possible transformations between relations of type $\delta' = \delta = 0$.

Since any pair of initial and target relationships fall into one of the above cases, transitivity has been proven. \[ \square \]

The above result is useful in classifying semi-quadratic Weingarten surfaces based on their $\Lambda_1$ and $\Lambda_2$ values. We first consider a special case.
Theorem 4.24. Let $S$ be a connected rotationally symmetric semi-quadratic Weingarten surface for which $\Lambda_1^2 = \Lambda_2$. Then $S$ is a subset of a torus of revolution, round sphere, plane, cone or cylinder.

Proof. Note that since $\Lambda_2 = \Lambda_1^2 \geq 0$, either $\Lambda_2 = 0$ or $\Lambda_2 > 0$. Taking $\Lambda_2 = 0$ implies $\Lambda_1 = 0$, which gives a LW relationship:

$$\lambda K + \Upsilon H + C = 0,$$

for some constants $\lambda, \Upsilon, C \in \mathbb{R}$. It has been remarked in [23] that LW relationships for which $\Upsilon^2 - 4\lambda C = 0$ characterise either tubular surfaces or planes (that is in the rotational case, cylinders, tori of revolution or planes). It is observed that for LW surfaces, $\Lambda_2 = \Upsilon^2 - 4\lambda C$ and so the $\Lambda_2 = 0$ case has already been proven. Now consider the $\Lambda_2 > 0$ case.

We first recall the following fact: Every connected component of a surface with a constant principal curvature is a subset of either a round sphere, a tube over a curve (if the constant principal curvature is non-zero) or a developable surface (if a principal curvature is constantly zero). A connected surface of revolution with constant principal curvatures must therefore be a subset of a torus of revolution or a round sphere (if $K$ is not identically zero) or a subset of a plane, cone or cylinder (if $K \equiv 0$).

Our strategy of proof will be to, assuming $S$ is semi-quadratic, show that one of the principal curvatures of $S$ is constant. We first assume $K$ is nowhere vanishing on $S$, i.e. $S$ is non-flat. Assume for contradiction that both $k_1$ and $k_2$ are non constant. Take a $\text{SL}_2(\mathbb{R})$ transformation of $S$, whose relation we may assume w.l.o.g. to be normalised, to another semi-quadratic surface $\tilde{S}$ sharing the same invariant $\Lambda_1$ and satisfying either of the Weingarten relations

$$\frac{1}{2}(\Lambda_1 \pm 1)\tilde{k}_1 + \frac{1}{2}(-\Lambda_1 \pm 1)\tilde{k}_2 = 0,$$

(4.36)

where the principal curvatures $\tilde{k}_i$ of $\tilde{S}$ are related to the principal curvatures $k_i$ of $S$ by

$$\tilde{k}_i = \frac{c + dk_i}{a + bk_i},$$

for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R})$.

Note that by assumption $k_1$ and $k_2$ are not identically equal to $-a/b$, hence equation (4.36) is satisfied on all finite parts of $\mathbb{R}(\tilde{S})$. Since $\Lambda_1^2 = 1$ however, equation (4.36) implies at least one of the principal curvatures of $\tilde{S}$ must be identically zero, implying one of the principal curvatures of $S$ takes the constant value $-c/d$ which is a contradiction. Now consider the more general case that $K$ vanishes on parts of $S$ and let

$$S_0 = K^{-1}\{0\}, \quad S_\pm = K^{-1}(\mathbb{R}\setminus\{0\}),$$

so that $S$ can be partitioned as $S = S_0 \cup S_\pm$. We will show that $S_0 \neq \emptyset$ and $S_\pm \neq \emptyset$ cannot hold simultaneously. Assume otherwise, then since $S_\pm$ is open it is a sub-surface so has a countable number of connected components which we denote as $V_n, n \in \mathbb{N}$. Each $V_n$ is connected and non-flat.
and so the repeating the argument previously given shows that \( k_2 \) takes a constant value on each \( V_n \), denoted by \( c_n \). Therefore

\[
k_2(S) = k_2(S_0) \cup k_2(S_{\pm}) = k_2(S_0) \cup \bigcup_{n=1}^{\infty} k_2(V_n) = \{0\} \cup \bigcup_{n=1}^{\infty} \{c_n\},
\]

implying \( k_2(S) \) is disconnected, contradicting the connectedness of \( S \) or the continuity of \( k_2 \). Hence one of \( S_0 \) or \( S_{\pm} \) is empty. The \( S_0 = \emptyset \) case was considered above. On the other hand if \( S_{\pm} = \emptyset \) then \( K \equiv 0 \) on \( S \) and \( S \) is a plane, cone or cylinder.

The main theorem of this section is motivated by the classification result given in [24] for surfaces of revolution satisfying the relation

\[
k_2 = \lambda k_1.
\]

**Theorem 4.25** ([24], Theorem 4.1, page 15).

1. If \( \lambda > 0 \), the surfaces satisfying (4.37) are either planes, closed surfaces with a convex profile curve, or subsets thereof. In the special case \( \lambda = 1 \), the only solutions are round spheres or subsets thereof.

2. If \( \lambda < 0 \), the surfaces satisfying (4.37) are either planes or open, catenoid-like surfaces with a convex profile curve, or subsets thereof.

In the case \( \lambda = 0 \), the above Weingarten surfaces are developable, i.e. planes, cones or cylinders. Surfaces satisfying equation (4.37) are related in \( E^3 \) as follows.

**Lemma 4.26.** All rotationally symmetric, non-planar, surfaces satisfying the relation (4.37) for a fixed \( \lambda \neq 0 \) are related by a homothety.

**Proof.** The radii of curvature of a non-planar solution of relation (4.37) may be assumed to be finite as \( \lambda \neq 0 \). Thus, the \( r_1 \) radius of curvature of such a surface is given by equation (3.2) in Example 3.1 with a slight change in notation \( \lambda \mapsto 1/\lambda \) and by setting \( C = 0 \).

\[
\begin{align*}
r_1(\theta) &= \frac{\lambda A_0 \sin^{\frac{1}{\lambda}-1} \theta}{\lambda - 1}, \quad \lambda \neq 0, 1. \quad (4.38) \\
r_1(\theta) &= \text{constant}, \quad \lambda = 1, \lambda \neq 0. \quad (4.39)
\end{align*}
\]

where \( A_0 \) is a constant of integration and has the effect of scaling \( r_1 \). Since \( r_2 = \frac{1}{\lambda} r_1 \), the curvatures of every possible surface of revolution satisfying relation (4.37) for a fixed \( \lambda \neq 0 \) are related by a map

\[
(r_1, r_2) \mapsto (Cr_1, Cr_2),
\]

\( C \in \mathbb{R}\{0\} \). Therefore such surfaces are related by a homothety. \( \square \)
Corollary 4.27. Any two rotationally symmetric linear Hopf surfaces satisfying a given relation (3.1) may be related to each other geometrically by conjugating a homothety \( h \) with a parallel translation \( p \), i.e. by the map \( p \circ h \circ p^{-1} \).

Proof. Given any two linear Hopf surfaces \( S_1 \) and \( S_2 \) satisfying
\[
r_2 = \lambda r_1 + C,
\]
after applying a suitable parallel translation denoted \( p \), their Weingarten relation is mapped to relation (4.37). Let \( \tilde{S}_1 \) and \( \tilde{S}_2 \) be the images of \( S_1 \) and \( S_2 \) under \( p \). Proposition 4.26 implies \( \tilde{S}_1 \) and \( \tilde{S}_2 \) are related by a homothety \( h \), hence \( S_1 \) and \( S_2 \) are related by
\[
S_1 \xrightarrow{p} \tilde{S}_1 \xrightarrow{h} \tilde{S}_2 \xrightarrow{p^{-1}} S_2,
\]
\( p^{-1} \) being a parallel translation. \( \square \)

Corollary 4.27 says that any rotationally symmetric linear Hopf surface satisfying a fixed relation is a composition of a parallel translation and a homothety of a single, surface of revolution satisfying relation (4.37). The following theorem generalises this idea to semi-quadratic Weingarten surfaces.

Theorem 4.28. Any non-flat rotationally symmetric, connected semi-quadratic Weingarten surface with \( \Lambda_2 > 0 \) is the image under a composition of homotheties, parallel translations and reciprocal transformations of a Weingarten surface satisfying the relation
\[
k_2 = \lambda k_1,
\]
for \( \lambda > 0 \) when the surface is elliptic, or for \( \lambda < 0 \) when the surface is hyperbolic.

Proof. Let \( S \) be as such with invariant \( \Lambda_2 \) and assume \( \Lambda_2 = 1 \). In the case that \( \Lambda_2 = 1 \), Theorem 4.24 implies that \( S \) is a round sphere, torus of revolution, plane, cone or cylinder. Of these, only the round sphere is non-flat, in which case relationship (4.37) is already satisfied for \( \lambda = 1 \) so the required \( \text{SL}_2(\mathbb{R}) \) transformation can be taken to be the identity, hence the result is proven when \( \Lambda_2 = 1 \). Now assume \( \Lambda_2 \neq 1 \). By Proposition 4.23 we can map the Weingarten relation of \( S \) to either of the relations
\[
\frac{1}{2} (\Lambda_1 \pm 1) k_1 + \frac{1}{2} (-\Lambda_1 \pm 1) k_2 = 0, \tag{4.40}
\]
with a \( \text{SL}_2(\mathbb{R}) \) transformation. The relations (4.40) are equivalent to
\[
k_2 = \lambda^{\pm} k_1, \quad \lambda^{\pm} = \frac{\Lambda_1 \pm 1}{\Lambda_1 \mp 1}. \tag{4.41}
\]
Hence \( S \) can be mapped by a composition of parallel translations, a homothety and a reciprocal transformation to either of two surfaces of revolution satisfying the Weingarten relation (4.41) with \( \lambda = \lambda^+ \) or \( \lambda = \lambda^- \). If \( S \) is hyperbolic so that \( \Lambda_1 > 1 \) then \( \lambda^+ > 0 \). On the other hand if \( S \) is elliptic then \( \Lambda_1 < 1 \) and \( \lambda^- < 0 \), which is as expected since \( \text{SL}_2(\mathbb{R}) \) transformations preserve ellipticity. \( \square \)

Remark 4.29. Since \( \lambda^+ = 1/\lambda^- \), both of the possible classes of target surfaces given in Theorem 4.28 are reciprocal transformations of one another given by Proposition 4.12 with \( \mathcal{A} = 1 \).
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