The \((k, \ell)\)-Rainbow Index of Random Graphs

Qingqiong Cai\(^1\) · Xueliang Li\(^1\) · Jiangli Song\(^1\)

Received: 7 November 2013 / Revised: 19 February 2014 / Published online: 30 December 2015 © Malaysian Mathematical Sciences Society and Universiti Sains Malaysia 2015

Abstract A tree in an edge-colored graph \(G\) is said to be a rainbow tree if no two edges on the tree share the same color. Given two positive integers \(k, \ell\) with \(k \geq 3\), the \((k, \ell)\)-rainbow index \(r_{x_k, \ell}(G)\) of \(G\) is the minimum number of colors needed in an edge-coloring of \(G\) such that for any set \(S\) of \(k\) vertices of \(G\), there exist \(\ell\) internally disjoint rainbow trees connecting \(S\). This concept was introduced by Chartrand et. al., and there have been very few known results about it. In this paper, we establish a sharp threshold function for \(r_{x_k, \ell}(G_{n,p}) \leq k\) and \(r_{x_k, \ell}(G_{n,M}) \leq k\), respectively, where \(G_{n,p}\) and \(G_{n,M}\) are the usually defined random graphs.

Keywords Rainbow index · Random graphs · Threshold function

Mathematics Subject Classification 05C05 · 05C15 · 05C80 · 05D40

1 Introduction

All graphs in this paper are undirected, finite, and simple. We follow [3] for graph theoretical notation and terminology not defined here. Let \(G\) be a nontrivial connected graph.
graph with an edge-coloring \( c : E(G) \to \{1, 2, \ldots, t\} \), \( t \in \mathbb{N} \), where adjacent edges may be colored the same. A path of \( G \) is said to be a \textit{rainbow path} if no two edges on the path have the same color. An edge-colored graph \( G \) is called \textit{rainbow connected} if for every pair of distinct vertices of \( G \) there exists a rainbow path connecting them.

The \textit{rainbow connection number} of a graph \( G \), denoted by \( rc(G) \), is defined as the minimum number of colors that are needed in order to make \( G \) rainbow connected. For any two vertices \( u \) and \( v \) of \( G \), a rainbow \( u \rightarrow v \) path of length \( d(u, v) \), where \( d(u, v) \) is the distance between \( u \) and \( v \). The graph \( G \) is \textit{strongly rainbow connected} if there exists a rainbow \( u \rightarrow v \) geodesic for any pair of vertices \( u \) and \( v \) in \( G \). Similarly, we define the \textit{strong rainbow connection number} of a connected graph \( G \), denoted by \( src(G) \), as the smallest number of colors that are needed in order to make \( G \) strongly rainbow connected. Clearly, we have \( diam(G) \leq rc(G) \leq src(G) \leq m \), where \( diam(G) \) denotes the diameter of \( G \) and \( m \) is the number of edges of \( G \). The \textit{rainbow \( k \)-connectivity} of \( G \), denoted by \( rc_k(G) \), is defined as the minimum number of colors in an edge-coloring of \( G \) such that every two distinct vertices of \( G \) are connected by \( k \) internally disjoint rainbow paths. These concepts were introduced by Chartrand et al. in [7, 8]. Recently, a lot of relevant results have been published; see [5, 6, 10–12, 14]. The interested readers can see [13, 16] for a survey on this topic.

Here we recall the concept of generalized connectivity. Let \( G \) be a connected graph of order \( n \) and size \( m \). For \( S \subseteq V(G) \), an \( S \)-tree is a tree connecting the vertices of \( S \). Suppose that \( \{T_1, T_2, \ldots, T_\ell\} \) is a set of \( S \)-trees. They are called \textit{internally disjoint} if \( E(T_i) \cap E(T_j) = \emptyset \) and \( V(T_i) \cap V(T_j) = S \) for every pair of distinct integers \( i, j \) with \( 1 \leq i, j \leq \ell \) (note that the trees are vertex disjoint in \( G \setminus S \)). For a set \( S \) of \( k \) vertices of \( G \), let \( \kappa(S) \) denote the maximum number of internally disjoint \( S \)-trees in \( G \). The \( k \)-connectivity \( \kappa_k(G) \) of \( G \) is defined by \( \kappa_k(G) = \min\{\kappa(S)\} \), where the minimum is taken over all \( k \)-element subsets \( S \) of \( V(G) \). We refer to [9, 15, 17, 18] for more details about the generalized connectivity.

A tree \( T \) in an edge-colored graph \( G \) is called a \textit{rainbow tree} if no two edges of \( T \) have the same color. Given two positive integers \( k, \ell \) with \( 2 \leq k \leq n \) and \( 1 \leq \ell \leq \kappa_k(G) \), the \((k, \ell)\)-\textit{rainbow index} \( rx_{k,\ell}(G) \) of \( G \) is the minimum number of colors needed in an edge-coloring of \( G \) such that for any set \( S \) of \( k \) vertices of \( G \), there exist \( \ell \) internally disjoint rainbow \( S \)-trees. In particular, for \( \ell = 1 \), we often write \( rx_k(G) \) rather than \( rx_{k,1}(G) \) and call it the \( k \)-\textit{rainbow index}. It is easy to see that \( rx_{2,\ell}(G) = rc_\ell(G) \). So the \((k, \ell)\)-rainbow index can be viewed as a generalization of the rainbow connectivity. In the sequel, we always assume \( k \geq 3 \).

The concept of \((k, \ell)\)-rainbow index was also introduced by Chartrand et al.; see [9]. They determined the \( k \)-rainbow index of all unicyclic graphs and the \((3, \ell)\)-rainbow index of complete graphs for \( \ell = 1, 2 \). In [4], we investigated the \((k, \ell)\)-rainbow index of complete graphs for every pair of integers \( k, \ell \). We proved that for every pair of positive integers \( k, \ell \) with \( k \geq 3 \), there exists a positive integer \( N = N(k, \ell) \) such that \( rx_{k,\ell}(K_n) = k \) for every integer \( n \geq N \), which settled down the two conjectures in [9].

In this paper, we study the \((k, \ell)\)-rainbow index of random graphs and establish a sharp threshold function for the property \( rx_{k,\ell}(G_{n, p}) \leq k \) and \( rx_{k,\ell}(G_{n, M}) \leq k \), respectively, where \( G_{n, p} \) and \( G_{n, M} \) are defined as usual; see [2].
2 Basic Notation on Random Graphs

The two most frequently occurring probability models of random graphs are $\mathcal{G}(n, p)$ and $\mathcal{G}(n, M)$. The model $\mathcal{G}(n, p)$ consists of all graphs on $n$ vertices, in which the edges are chosen independently and randomly with probability $p$, whereas the model $\mathcal{G}(n, M)$ consists of all graphs on $n$ vertices and $M$ edges, in which each graph has the same probability. Let $G_{n,p}$ and $G_{n,M}$ stand for random graphs from the models $\mathcal{G}(n, p)$ and $\mathcal{G}(n, M)$, respectively. We say that an event $E = E(n)$ happens almost surely (or a.s. for short) if $\lim_{n \to \infty} \Pr[E(n)] = 1$. Let $K, G, H$ be three graphs on $n$ vertices.

A property $Q$ is said to be monotone if whenever $G \subseteq H$ and $G$ satisfies $Q$, then $H$ also satisfies $Q$. Moreover, we call a property $Q$ convex if whenever $K \subseteq G \subseteq H$, and $K$ satisfies $Q$ and $H$ satisfies $Q$, then $G$ also satisfies $Q$. For a graph property $Q$, a function $p(n)$ is called a threshold function of $Q$ if

- $\frac{p'(n)}{p(n)} \to 0$, then $G_{n,p'(n)}$ almost surely does not satisfy $Q$;
- $\frac{p''(n)}{p(n)} \to \infty$, then $G_{n,p''(n)}$ almost surely satisfies $Q$.

Furthermore, $p(n)$ is called a sharp threshold function of $Q$ if there are two positive constants $c$ and $C$ such that

- for every $p'(n) \leq cp(n)$, $G_{n,p'(n)}$ almost surely does not satisfy $Q$; and
- for every $p''(n) \geq Cp(n)$, $G_{n,p''(n)}$ almost surely satisfies $Q$.

Similarly, we can define $M(n)$ as a threshold function of $Q$ in the model $\mathcal{G}(n, M)$; see [2].

It is well known that all monotone graph properties have a threshold function [2]. Obviously, for every pair of positive integers $k, \ell$, the property that the $(k, \ell)$-rainbow index is at most $k$ is monotone and thus has a threshold.

3 Main Results

As Caro et al. pointed out, the random graph setting poses several intriguing questions. In [5], Caro et al. proved that $p = \sqrt{\log n}/n$ is a sharp threshold for the property $rc(G_{n,p}) \leq 2$. This was generalized by Fujita et al. [10] who obtained that $p = \sqrt{\log n}/n$ is a sharp threshold for the property $rc_k(G_{n,p}) \leq 2$ and $M = \sqrt{n^3 \log n}$ is a sharp threshold for the property $rc_k(G_{n,M}) \leq 2$ for all integer $k \geq 1$. In this section, we employ similar methods to study the $(k, \ell)$-rainbow index of random graphs $G_{n,p}$ and $G_{n,M}$.

**Theorem 1** For every pair of positive integers $k, \ell$ with $k \geq 3$, $\frac{k! \log n}{n}$ is a sharp threshold function for the property $rx_{k,\ell}(G_{n,p}) \leq k$, where $a = \frac{k^k}{k!}$.

**Proof** The proof will be two-fold. For the first part, we show that there exists a positive constant $c_1$ such that for every $p \geq c_1 \frac{k! \log n}{n}$, almost surely $rx_{k,\ell}(G_{n,p}) \leq k$, which can be derived from the following two claims.
Claim 1 For any $c_1 \geq 3$, if $p \geq c_1^{k/k\log an}$, then almost surely any $k$ vertices in $G_{n,p}$ have at least $2k\log an$ common neighbors.

For any $S \subseteq V(G_{n,p})$ with $|S| = k$, let $D(S)$ denote the event that the vertices in $S$ have at least $2k\log an$ common neighbors. Then it suffices to prove that, for $p = c_1^{k/k\log an}$, $Pr[ \bigcap_S D(S) ] \rightarrow 1$, as $n \rightarrow \infty$. Define $X$ as the number of common neighbors of all the vertices in $S$. Then $X \sim Bin\left(n - k, \left(c_1^{k/k\log an}\right)^k\right)$ and $E(X) = \frac{n-k}{n} c_1^k \log an$. Assume that $n > \frac{c_1^k}{c_1^k - 2k}$. Using the Chernoff Bound [1], we get that

$$Pr[\overline{D(S)}] = Pr[X < 2k\log an]$$

$$= Pr\left[X < \frac{c_1^k(n-k)}{n} \log an \left(1 - \frac{(c_1^k - 2k)n - c_1^k}{c_1^k(n-k)}\right)\right]$$

$$\leq e^{-\frac{\frac{c_1^k(n-k)}{n} \log an \left(\frac{(c_1^k - 2k)n - c_1^k}{c_1^k(n-k)}\right)^2}{2}}$$

$$< n^{-\frac{\frac{c_1^k(n-k)}{n} \log an \left(\frac{(c_1^k - 2k)n - c_1^k}{c_1^k(n-k)}\right)^2}{2}}.$$

Note that the assumption $n > \frac{c_1^k}{c_1^k - 2k}$ ensures $\frac{(c_1^k - 2k)n - c_1^k}{c_1^k(n-k)} > 0$. So we can apply the Chernoff Bound to scaling the above inequalities. The last inequality holds, since $1 < a = \frac{k^k}{k^k - k!} < e$ and then $\log an > \ln n$.

It follows from the union bound that

$$Pr\left[\bigcap_S D(S)\right] = 1 - Pr\left[\bigcup_S \overline{D(S)}\right]$$

$$\geq 1 - \sum_S Pr\left[\overline{D(S)}\right]$$

$$> 1 - \left(\frac{n}{k}\right)^n \frac{\frac{c_1^k(n-k)}{n} \log an \left(\frac{(c_1^k - 2k)n - c_1^k}{c_1^k(n-k)}\right)^2}{2}$$

$$> 1 - n^{-\frac{\frac{c_1^k(n-k)}{n} \log an \left(\frac{(c_1^k - 2k)n - c_1^k}{c_1^k(n-k)}\right)^2}{2}}.$$

It is not hard to see that $c_1 > 3$ can guarantee $k - \frac{\frac{c_1^k(n-k)}{n} \log an \left(\frac{(c_1^k - 2k)n - c_1^k}{c_1^k(n-k)}\right)^2}{2} < 0$ for sufficiently large $n$. Then $\lim_{n \rightarrow \infty} 1 - n^{-\frac{\frac{c_1^k(n-k)}{n} \log an \left(\frac{(c_1^k - 2k)n - c_1^k}{c_1^k(n-k)}\right)^2}{2}} = 1$, which implies that $\lim_{n \rightarrow \infty} Pr\left[\bigcap_S D(S)\right] = 1$ as desired.
Claim 2 If any $k$ vertices in $G_{n,p}$ have at least $2k\log_{a}n$ common neighbors, then there exists a positive integer $N = N(k)$ such that $rx_{k,\ell}(G_{n,p}) \leq k$ for every integer $n \geq N$.

Let $C = \{1, 2, \ldots, k\}$ be a set of $k$ different colors. We color the edges of $G_{n,p}$ with the colors from $C$ randomly and independently. For $S \subseteq V(G_{n,p})$ with $|S| = k$, define $F(S)$ as the event that there exist at least $\ell$ internally disjoint rainbow $S$-trees. It suffices to prove that $\Pr[\bigcap_{S} F(S)] > 0$.

Suppose $S = \{v_{1}, v_{2}, \ldots, v_{k}\}$. For any common neighbor $u$ of the vertices in $S$, let $T(u)$ denote the star with $V(T(u)) = \{u, v_{1}, v_{2}, \ldots, v_{k}\}$ and $E(T(u)) = \{uv_{1}, uv_{2}, \ldots, uv_{k}\}$. Set $T = \{T(u)|u$ is a common neighbor of the vertices in $S\}$. Then $T$ is a set of at least $2k\log_{a}n$ internally disjoint $S$-trees. It is easy to see that $q := \Pr[T \in T]$ is a rainbow tree $= \frac{k!}{2^{k}} < \frac{1}{2}$. So $1 - q > q$. Define $Y$ as the number of rainbow $S$-trees in $T$. Then we have

$$
Pr[\overline{F(S)}] \leq Pr[Y \leq \ell - 1] \\
\leq \sum_{i=0}^{\ell - 1} \binom{2k\log_{a}n}{i} q^{i} (1 - q)^{2k\log_{a}n - i} \\
\leq (1 - q)^{2k\log_{a}n} \sum_{i=0}^{\ell - 1} \binom{2k\log_{a}n}{i} \\
\leq (1 - q)^{2k\log_{a}n (1 + 2k\log_{a}n)^{\ell - 1}} \\
= \frac{(1 + 2k\log_{a}n)^{\ell - 1}}{n^{2k}}.
$$

It yields that

$$
Pr\left[ \bigcap_{S} F(S) \right] = 1 - Pr\left[ \bigcup_{S} F(S) \right] \\
\geq 1 - \sum_{S} Pr\left[ F(S) \right] \\
\geq 1 - \frac{n}{k} \frac{(1 + 2k\log_{a}n)^{\ell - 1}}{n^{2k}} \\
> 1 - \frac{(1 + 2k\log_{a}n)^{\ell - 1}}{n^{k}}.
$$

Obviously, $\lim_{n \to \infty} 1 - \frac{(1 + 2k\log_{a}n)^{\ell - 1}}{n^{k}} = 1$, and then $\lim_{n \to \infty} Pr[\bigcap_{S} F(S)] = 1$. Thus, there exists a positive integer $N = N(k)$ such that $Pr[\bigcap_{S} F(S)] > 0$ for every integer $n \geq N$. 

\(\square\) Springer
For the other direction, we show that there exists a positive constant $c_2$ such that for every $p \leq c_2 \frac{k \log a n}{n}$, almost surely $r_{x,k,\ell}(G_{n,p}) \geq k + 1$.

It suffices to prove that for a sufficiently small constant $c_2$, the random graph $G_{n,p}$ with $p = c_2 \frac{k \log a n}{n}$ almost surely contains a set $S$ of $k$ vertices satisfying that

(i) $S$ is an independent set;

(ii) the vertices in $S$ have no common neighbors.

Clearly, for such $S$ there exist no rainbow $S$-trees in any $k$-edge-coloring, which implies that $r_{x,k,\ell}(G_{n,p}) \geq k + 1$.

Fix a set $H$ of $n^{1/(2k+1)}$ vertices in $G_{n,p}$ (we may and will assume that $n^{1/(2k+1)}/k$ is an integer). Let $E_1$ be the event that $H$ is an independent set. Then

$$\Pr[E_1] = \left(1 - c_2 \frac{\log a n}{n}\right)^{n^{1/(2k+1)}} = 1 - o(1),$$

where $o(1)$ denotes a function tending to 0 as $n$ tends to infinity.

Partition $H$ into $t$ subsets $H_1, H_2, \ldots, H_t$ arbitrarily, where $t = n^{1/(2k+1)}/k$ and $|H_1| = |H_2| = \cdots = |H_t| = k$. Let $E_2$ be the event that there exists some $H_i$ without common neighbors in $V(G_{n,p})\setminus H$. Then, for sufficiently small $c_2$,

$$\Pr[E_2] = 1 - \left(1 - \left(1 - c_2 \frac{k \log a n}{n}\right)^{n - n^{1/(2k+1)}}\right)^{n^{1/(2k+1)}/k} = 1 - o(1).$$

So, almost surely there exists some set $H_i$ of $k$ vertices satisfying properties (i) and (ii).

Thus, for sufficiently small $c_2$ and every $p \leq c_2 \frac{k \log a n}{n}$, almost surely $r_{x,k,\ell}(G_{n,p}) \geq k + 1$. The proof is thus complete. \QED

Next, we will turn to another well-known random graph model $\mathcal{G}(n, M)$. We start with a useful lemma which reveals the relationship between $\mathcal{G}(n, p)$ and $\mathcal{G}(n, M)$. Set $N = \binom{n}{2}$.

**Lemma 2** [2] If $Q$ is a convex property and $p(1 - p)N \to \infty$, then $G_{n,p}$ almost surely has $Q$ if and only if for every fixed $x$, $G_{n,M}$ almost surely has $Q$, where $M = \lfloor pN + x (p(1 - p)N)^{1/2} \rfloor$.

Clearly, the property that the $(k, \ell)$-rainbow index of a given graph is at most $k$ is a convex property. By Theorem 1 and Lemma 2, we get the following result.

**Corollary 3** For every pair of positive integers $k, \ell$ with $k \geq 3$, $M(n) = \frac{k^{k} \log a n}{\sqrt{n^{2k-1}}}n^{2k-1}$ is a sharp threshold function for the property $r_{x,k,\ell}(G_{n,M}) \leq k$, where $a = \frac{k^k}{k^{k-1}}$.

**Remark** If $p$ is a threshold function for a given property $Q$, then so is $\lambda p$ for any positive constant $\lambda$. It follows that $p(n) = \sqrt{n \log a n}$ ($M(n) = \sqrt{n^{2k-1}}$) is also a sharp threshold function for the property $r_{x,k,\ell}(G_{n,p}) \leq k$ ($r_{x,k,\ell}(G_{n,M}) \leq k$), which corresponds to the results in [10].
Acknowledgments  The authors are very grateful to the reviewers for their helpful comments and suggestions. Supported by NSFC No. 11371205 and 11071130.

References

1. Alon, N., Spencer, J.H.: The Probabilistic Method. Wiley, New York (2004)
2. Bollobás, B.: Random Graphs. Cambridge University Press, Cambridge (2001)
3. Bondy, J.A., Murty, U.S.R.: Graph Theory, GTM 244. Springer, New York (2008)
4. Cai, Q., Li, X., Song, J.: Solutions to conjectures on the $(k, \ell)$-rainbow index of complete graphs. Networks 62, 220–224 (2013)
5. Caro, Y., Lev, A., Roditty, Y., Tuza, Z., Yuster, R.: On rainbow connection. Electron. J. Combin. 15(1), R57 (2008)
6. Chandran, L., Das, A., Rajendraprasad, D., Varma, N.: Rainbow connection number and connected dominating sets. J. Graph Theory 71(2), 206–218 (2012)
7. Chartrand, G., Johns, G., McKeon, K., Zhang, P.: Rainbow connection in graphs. Math. Bohem. 133, 85–98 (2008)
8. Chartrand, G., Johns, G., McKeon, K., Zhang, P.: The rainbow connectivity of a graph. Networks 54(2), 75–81 (2009)
9. Chartrand, G., Okamoto, F., Zhang, P.: Rainbow trees in graphs and generalized connectivity. Networks 55, 360–367 (2010)
10. Fujita, S., Liu, H., Magnant, C.: Rainbow k-connection in dense graphs. Electron. Notes Discrete Math. 38, 361-366 (2011), or, J. Combin. Math. Combin. Comput., to appear
11. Huang, X., Li, X., Shi, Y.: Note on the hardness of rainbow connections for planar and line graphs. Bull. Malays. Math. Sci. Soc. 38(3), 1235–1241 (2015)
12. Krivelevich, M., Yuster, R.: The rainbow connection of a graph is (at most) reciprocal to its minimum degree. J. Graph Theory 63(3), 185–191 (2010)
13. Li, X., Sun, Y.: Rainbow Connections of Graphs. Springer Briefs in Math., Springer, New York (2012)
14. Li, X., Sun, Y.: On the strong rainbow connection of a graph. Bull. Malays. Math. Sci. Soc. (2) 36(2), 299–311 (2013)
15. Li, S., Li, W., Li, X.: The generalized connectivity of complete bipartite graphs. Ars Combin. 104, 65–79 (2012)
16. Li, X., Shi, Y., Sun, Y.: Rainbow connections of graphs: a survey. Graphs Combin. 29(1), 1–38 (2013)
17. Li, S., Li, W., Li, X.: The generalized connectivity of complete equipartition 3-partite graphs. Bull. Malays. Math. Sci. Soc. (2) 37(1), 103–121 (2014)
18. Li, H., Li, X., Mao, Y.: On extremal graphs with at most two internally disjoint Steiner trees connecting any three vertices. Bull. Malays. Math. Sci. Soc. (2), in press