Magnetic unit vector fields

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Abstract
We show that a unit vector field on an oriented Riemannian manifold is a critical point of the
Landau Hall functional if and only if it is a critical point of the Dirichlet energy functional.
Therefore, we provide a characterization for a unit vector field to be a magnetic map into its
unit tangent sphere bundle. Then, we classify all magnetic left invariant unit vector fields on
3-dimensional Lie groups.

Keywords
Dirichlet energy · Magnetic field · Harmonic map · Harmonic unit vector field · Magnetic map

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1 Introduction

Smooth vector fields appear in physics or technology as mathematical models of “fields”.
For instance, a static magnetic field is a smooth vector field \( B \) on a region of the Euclidean
space 3-space \( \mathbb{E}^3 \) satisfying the Gauss’ law \( \text{div} \, B = 0 \). A static magnetic field \( B \) is uniform
if it is covariantly constant. Thus parallel vector fields are mathematical models of uniform
static magnetic field.

Now let us turn our attention to smooth vector fields on (oriented) Riemannian manifolds.
The bending energy (biegung) \( B \) of a vector field \( X \) on an oriented Riemannian manifold
(\( M, g \)) is

\[
B(X) = \int_M |\nabla X|^2 \, dv_g.
\]

Dedicated to professor Kazumi Tsukada on the occasion of his retirement.

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The bending energy of vector fields measures to what extent globally defined vector fields fail to be parallel.

Additionally, we may consider Dirichlet energy

$$E(X) = \int_M \frac{1}{2} |dX|^2 \, dv_g$$

of a vector field $X$. Here we regard $X$ as a smooth map from $M$ to its tangent bundle $TM$ (equipped with a natural Riemannian metric $g_s$ derived from $g$). Critical points of the Dirichlet energy are called harmonic maps [2]. These two variational problems are closely related each other. In fact, when $(M, g)$ is compact and oriented, we have

$$2E(X) = \dim M \cdot \text{Vol}(M) + B(X).$$

To investigate these functionals, we should be nervous and careful about mapping space and variations. In fact, the bending energy is considered on the space $\mathcal{X}(M)$ of all smooth vector fields. To deduce the Euler–Lagrange equation we compute the first variation through variations $\{X(s); |s| < \varepsilon \} \subset \mathcal{X}(M)$.

On the other hand, usually Dirichlet energy is considered on the space $C^\infty(M, TM)$ of all smooth maps from $M$ into $TM$. The first variation is computed for variations $\{X(s); |s| < \varepsilon \} \subset C^\infty(M, TM)$.

Unfortunately, the later setting is quite discouraging. Actually, Ishihara and Nouhaud showed that the only vector field which are harmonic maps are parallel vector fields if $M$ is compact [15, 19]. Even if we restrict the variations in $\mathcal{X}(M)$, the same conclusion holds (showed by Gil-Medrano [3]).

One of the appropriate mapping space for Dirichlet or bending energy is the space $C^\infty(M, UM)$ of all smooth maps form $M$ into its unit tangent sphere bundle $UM$ or the space $\mathcal{X}_1(M)$ of all smooth unit vector fields. In fact, a unit vector field $X$ is a critical point of $B$ through (compactly supported) variations in $\mathcal{X}_1(M)$ if and only if

$$\bar{\Delta}_g X = |\nabla X|^2 X,$$

where $\bar{\Delta}_g$ is the rough Laplacian (see (3.1)). Unit vector fields satisfying (1.1) are referred as to harmonic unit vector fields [1].

Next, $X$ is a harmonic map into $UM$, that is, critical point of $E$ through (compactly supported) variations in $C^\infty(M, UM)$ if and only if $X$ satisfies (1.1) and, in addition,

$$\text{tr}_g R(\nabla X, X) = 0,$$

where $R$ is the Riemannian curvature. Harmonic unit vector fields have been studied extensively. See a monograph by Dragomir and Perrone [1]. For example, González-Dávila and Vanhecke [6] classified left invariant unit harmonic vector fields on 3-dimensional Lie groups. Accordingly, in the non-unimodular case, the left-invariant harmonic unit vector fields determining harmonic maps are very rare. In [5], the same authors study the stability and instability of harmonic and minimal unit vector fields on three-dimensional compact manifolds, in particular on compact quotients of unimodular Lie groups.

Furthermore, every unit vector field is an immersion of $M$ into $UM$. Thus we may develop submanifold geometry of unit vector fields in $UM$. For example, the study of minimal vector fields was initiated by Gluck and Ziller [4]. They studied optimal unit vector fields on the unit 3-sphere $S^3$. In [7], Han and Yim proved that Hopf vector fields on odd dimensional spheres $S^{2m+1}$ are harmonic maps into $US^{2m+1}$. In the same paper [7], a converse is proved for $m = 1$, that is a unit vector field on $S^3$ is a harmonic map into $US^3$ if and only if it is a
Hopf vector field on $\mathbb{S}^3$. In fact, Hopf vector fields on $\mathbb{S}^{2m+1}$ are precisely the unit Killing vector fields [25]. Later on, Tsukada and Vanhecke studied minimal unit vector fields on Lie groups. In particular, they classified left invariant minimal unit vector fields on 3-dimensional Lie groups [22]. It should be remarked that model spaces of Thurston geometries are realized as 3-dimensional Lie groups with left invariant metric except the product space $\mathbb{S}^2 \times \mathbb{R}$. These observations suggest us to introduce a new variational problem to look for “optimal” unit vector fields.

Motivated by static magnetic theory, we have started our investigation on magnetized harmonic vector fields (magnetic vector fields, in short) [11, 12]. A vector field $X$ of $(M, g)$ is said to be magnetic if it is a critical point of the Landau-Hall functional, which is a magnetization of Dirichlet energy (see Sect. 2.3 for the precise definition).

In this paper we give a detailed investigation on magnetized unit vector fields on Riemannian manifolds. In particular, we classify magnetic left invariant unit vector fields on 3-dimensional Lie groups.

## 2 Magnetic maps

### 2.1 Magnetic fields

Let $(N, h)$ be an $n$-dimensional Riemannian manifold. A magnetic field is a closed 2-form $F$ on $N$ and the Lorentz force of a magnetic field $F$ on $(N, h)$ is the unique $(1, 1)$-tensor field $L$ given by

$$g(LX, Y) = F(X, Y), \quad \forall X, Y \in \mathfrak{X}(N).$$

### 2.2 Energy and tension

Let $f : (M, g) \to (N, h)$ be a smooth maps between two Riemannian manifolds $(M, g)$ of dimension $m$ and $(N, h)$ of dimension $n$. Assume that $M$ is oriented. The Dirichlet energy of a smooth map $f : M \to N$ over a closed region $D \subset M$ is defined by

$$E(f; D) = \int_D \frac{1}{2} |df|^2 dv_g.$$

When $M$ is compact, we use an abbreviation $E(f) = E(f; M)$.

The second fundamental form $\nabla df$ of $f$ is defined by

$$(\nabla df)(X, Y) = \nabla^f_X f^*_h Y - f^*_h (\nabla_X Y), \quad X, Y \in \mathfrak{X}(M).$$

We adopt the notations $\nabla$ and $\nabla^N$ for the Levi-Civita connections of $M$ and $N$, respectively and $\nabla^f$ for the connection in the pull-back bundle $f^*TN$. The tension field $\tau(f)$ is a vector field along $f$, that is a section of $f^*TN$ defined by

$$\tau(f) = \text{tr}_g(\nabla df) = \sum_{i=1}^m \langle \nabla df)(e_i, e_i),$$

where $\{e_1, e_2, \ldots, e_m\}$ is a local orthonormal frame field of $M$.

A smooth map $f : M \to N$ is said to be a harmonic map if it is a critical point of the Dirichlet energy over all compactly supported regions in $M$ [2]. The Euler–Lagrange
equation of the variational problem of the Dirichlet energy is

$$\tau(f) = 0.$$  

In case when $f$ is an isometric immersion, then $\tau(f) = mH$, where $H$ is the mean curvature vector field. Thus an isometric immersion is a harmonic map if and only if it is a minimal immersion.

### 2.3 Magnetic maps

Let $(M, g)$ and $(N, h)$ be Riemannian manifolds as before. Take a magnetic field $F$ on $N$ and a vector field $\Xi$ on $M$. A smooth map $f : (M, g) \to (N, h, F)$ is said to be a magnetic map (or magnetized harmonic map) of charge $q$ with the directional vector field $\Xi$, if it satisfies

$$\tau(f) = q L f^* \Xi,$$

for some constant $q$. The prescribed vector field $\Xi$ controls the direction of the movement of the image of the magnetic map $f$ in $N$ under the influence of magnetic field $F$. See e.g. [10].

Under the assumption $\Xi$ is divergence free and $F$ is exact, magnetic maps are characterized as critical points of the Landau-Hall functional:

$$LH(f; D) = E(f; D) + q \int_D A(f^* \Xi) dv_g$$

under all compactly supported variations [10]. Here $A$ is a potential 1-form of $F$ (called the magnetic potential).

### 3 Magnetic vector fields

#### 3.1 The tangent bundles

As is well known, the tangent bundle of a Riemannian manifold admits an almost Kähler structure, naturally induced from the Riemannian structure of the base manifold. In particular, the canonical symplectic form of the tangent bundle, is a magnetic field. We briefly present a collection of basic materials about the geometry of tangent bundles for the study of vector fields [13].

Let $TM$ be the tangent bundle of a Riemannian manifold $(M, g)$ with the projection $\pi : TM \to M$. The vertical distribution $V$ is defined by

$$V_{(p; u)} = \text{Ker}(\pi^*(p; u)), \quad u = (p; u) \in T_p M \subset TM.$$  

The Levi-Civita connection $\nabla$ induces a complementary distribution $\mathcal{H}$, that is a distribution satisfying $T(TM) = \mathcal{H} \oplus V$. The distribution $\mathcal{H}$ is called a horizontal distribution determined by $\nabla$. In addition there exists a linear isomorphism $h = h_{(p; u)} : T_p M \to \mathcal{H}_{(p; u)}$. The isomorphism $h$ is called the horizontal lift operation. There exists also a linear isomorphism $v = v_{(p; u)} : T_p M \to V_{(p; u)}$. The isomorphism $v$ is called the vertical lift operation. These operations are naturally extended to vector fields.

The Riemannian metric $g$ of $(M, g)$ induces an almost Kähler structure $(g_\mathcal{V}, J)$ on the tangent bundle $TM$ of $M$:

$$g_\mathcal{V}(X^h, Y^h) = g_\mathcal{V}(X^\mathcal{V}, Y^\mathcal{V}) = g(X, Y) \circ \pi, \quad g_\mathcal{V}(X^h, Y^\mathcal{V}) = 0,$$

$$JX^h = X^\mathcal{V}, \quad JX^\mathcal{V} = -X^h.$$
for all \( X, Y \in \mathfrak{X}(M) \). The metric \( g_s \) is the so-called Sasaki lift metric of \( TM \). The fundamental 2-form \( F \) of \((TM, g_s, J)\) defined by \( F(\cdot, \cdot) = g_s(J\cdot, \cdot) \) is a symplectic form on \( TM \) and called the canonical 2-form or canonical symplectic form of \( TM \). Since \( F \) is closed, we regard \( F \) as a canonical magnetic field on \( TM \). It should be remarked that the canonical magnetic field \( F \) is exact. Indeed, \( F \) is represented as \( F = -d\omega \). Here the 1-form \( \omega \) (called the canonical 1-form) is defined by

\[
\omega(p;u)(X^h) = g(u, X_p), \quad \omega(p;u)(X^v) = 0.
\]

Here we use the Det convention. Hereafter, we equip \( TM \) with the magnetic field \( F = -d\omega \).

### 3.2 Tension fields

Take a vector field \( X \) on \( M \). Regarding \( X \) as a smooth map from \((M, g)\) into \((TM, g_s, F)\), then the tension field \( \tau(X) \) of \( X \in \mathcal{C}^\infty(M, TM) \) is given by [15]:

\[
\tau(X) = -\left\{ (\text{tr} g R(\nabla X, X)) \cdot h + (\Delta g X)^v \right\} \circ X,
\]

where \( h \) and \( v \) are horizontal lift and vertical lift from \( M \) to \( TM \), respectively. The operator \( \Delta g \) acting on \( \mathfrak{X}(M) \) is called the rough Laplacian and it is defined by

\[
\Delta g = -\sum_{i=1}^m \left( \nabla_{e_i} \nabla_{e_i} - \nabla_{\nabla_{e_i} e_i} \right), \quad (3.1)
\]

where \( \{e_1, e_2, \ldots, e_m\} \) is a local orthonormal frame field of \( M \) as before.

**Theorem 3.1** [3] A vector field \( X : M \to TM \) is a critical point of the Dirichlet energy with respect to all compactly supported variations in \( \mathfrak{X}(M) \) if and only if \( \Delta g X = 0 \).

Note that vector fields which are critical points of the Dirichlet energy with respect to all compactly supported variations in \( \mathfrak{X}(M) \) are called harmonic vector field in some literature; see e.g. [1] and references therein.

Let \( \Delta g \) be the Laplace-Beltrami operator of \((M, g)\) acting on 1-forms. Via the musical isomorphisms between \( TM \) and \( T^*M \), we graft \( \Delta g \) for vector fields (denoted by the same symbol \( \Delta g \)). A vector field \( X \) is said to be \( \Delta g \)-harmonic if \( \Delta g X = 0 \).

The following Weitzenböck formula measures the difference between \( \Delta g \) and \( \Delta g \):

\[
\Delta g X = \overline{\Delta g} X + SX,
\]

where \( S \) is the Ricci operator of \( M \).

**Corollary 3.1** A vector field \( X : M \to TM \) is a critical point of the Dirichlet energy with respect to all compactly supported variations in \( \mathfrak{X}(M) \) if and only if it satisfies \( \Delta g X = SX \).

For a vector field, the condition of being harmonic map is much stronger than that of being harmonic vector field.

**Theorem 3.2** [3] A vector field \( X : (M, g) \to (TM, g_s) \) is a harmonic map if and only if

\[
\text{tr}_g R(\nabla X, X) = 0, \quad \Delta g X = 0.
\]

See also [1, Proposition 2.12].

As we have also mentioned in the Introduction, harmonicity of vector field is a too strong restriction for vector fields. Actually, Gil-Medrano [3] showed the following fact.
Theorem 3.3  Every harmonic vector field on a compact oriented Riemannian manifold \( M \) is a parallel vector field.

Corollary 3.2  (Ishihara, Nouhaud) Every vector field on a compact oriented Riemannian manifold \( M \), which is a harmonic map into \( TM \), is parallel.

3.3 Magnetic vector fields

Since the harmonicity is a too strong restriction for vector fields, we weaken this property to “magnetic”. Namely, we consider vector fields which are magnetic maps into \( TM \). A vector field \( X \) on \( M \) is a magnetic map with charge \( q \) if it satisfies \( \tau(X) = q J(X^* X) \).

In our previous paper [11] we obtained the following fundamental formula:

\[
X_s p Y_p = Y^h_{X_p} + (\nabla_Y X)^v_{X_p} \tag{3.2}
\]

for any vector fields \( X \) and \( Y \) on \( M \).

From this formula, we have

\[
J(X_s X) = X^v_Y - (\nabla_X X)^h_Y.
\]

Hence the magnetic map equation \( \tau(X) = q J(X^* X) \) is the system [11]:

\[
\text{tr}_s R(\nabla X, X) = q \nabla_X X, \quad \Delta_g X = -q X.
\]

In [11], some examples of vector fields satisfying this system are exhibited.

3.4 LH-critical vector fields

Next, we consider again a vector field preserving variation \( \{X(s)\} \) through a vector field \( X \). More precisely \( \{X(s)\} \) is a smooth map from the product manifold \((-\varepsilon, \varepsilon) \times M\) to \( TM \) satisfying

\[
(-\varepsilon, \varepsilon) \times M \ni (s, p) \mapsto X_p^{(s)} \in T_p M; \quad X_p^{(0)} = X_p.
\]

Here \( \varepsilon \) is a sufficiently small positive number. The variation \( \{X^{(s)}\} \) is interpreted as a smooth map

\[
\chi : (-\varepsilon, \varepsilon) \times M \to TM; \quad \chi(s, p) = X_p^{(s)}.
\]

It should be remarked that for every \( s \in (-\varepsilon, \varepsilon) \), the correspondence \( p \mapsto \chi(s, p) = X_p^{(s)} \) is a vector field on \( M \). In other words, \( X^{(s)} \) is understood as a vector field \( X^{(s)} : p \mapsto X_p^{(s)} \).

Note that \( X^{(0)}(p) = X_p \) for any \( p \in M \). Next, since every \( X^{(s)} \) is a vector field,

\[
\pi(X^{(s)}(p)) = p
\]

holds for any \( s \) and \( p \in M \).

The variational vector field \( V \) of the variation \( \{X^{(s)}\} \) is defined by

\[
V_p = \chi_{s(0,p)} \frac{\partial}{\partial s} \bigg|_{(0,p)}.
\]

By definition \( V \) is a section of the pull-back bundle \( X^* T(TM) \).
Since $X$ is an immersion into $TM$, one can introduce the fiber metric on the pull-back bundle $X^*T(TM)$ by $X^*g_S$. Then the first variation for the Dirichlet energy is given by
\[
\frac{d}{ds}\bigg|_{s=0} E(X^{(s)}; D) = -\int_D (X^*g_S)(\nabla, \tau(X)) \, dv_g.
\]
On the other hand, the tangential vector field $V$ of $\{X^{(s)}\}$ is defined by (see [1, p. 57]):
\[
V_p = \frac{d}{ds}\bigg|_{s=0} X^{(s)}(p) = \lim_{s \to 0} \frac{1}{s} \left( X^{(s)}(p) - X_p \right) \in T_p M.
\]
The tangential vector field and the variational vector field are related by [1, p. 58]
\[
\nabla = V^\nabla \circ X.
\]
Hence, the first variation for the Dirichlet energy may be expressed as
\[
\frac{d}{ds}\bigg|_{s=0} E(X^{(s)}; D) = -\int_D g_S(V^\nabla \circ X, \left\{ (\text{tr}_g R(\nabla X, X))^{\nabla h} + (\nabla g X)^\nabla \right\} \circ X) \, dv_g
\]
\[
= \int_D g(V, \nabla g X) \, dv_g.
\]
Now let us compute the first variation for the magnetic term of the Landau-Hall functional, namely
\[
\frac{d}{ds}\bigg|_{s=0} \int_D \omega_X((X^{(s)})\ast_p X_p) \, dv_g.
\]
Since $X^{(s)}$ is a vector field, by using (3.2) we get
\[
\left( X^{(s)} \right)^\ast_p X_p = X^h_{X^{(s)}} + \left( \nabla_{X^{(s)}} X^\nabla \right)^\nabla_{X^{(s)}}.
\]
From this formula we have
\[
\omega_X((X^{(s)})) \ast_p X_p = \omega_{X^{(s)}}(X^h_{X^{(s)}}) = g_p(X^{(s)}_p, X_p).
\]
Hence
\[
\frac{d}{ds}\bigg|_{s=0} \omega_X((X^{(s)})) \ast_p X_p = g_p(V, X).
\]
The first variation of the magnetic term $\int_D \omega_X((X^{(s)})) \ast X) \, dv_g$ is computed as
\[
\frac{d}{ds}\bigg|_{s=0} \int_D \omega((X^{(s)})) \ast X) \, dv_g = \int_D \frac{d}{ds} \omega\left( (X^{(s)})) \ast X \right) \bigg|_{s=0} \, dv_g
\]
\[
= \int_D g(V, X) \, dv_g.
\]
Thus we arrive at the following result:

\[
\left. \frac{d}{ds} \right|_{s=0} \text{LH}(X^{(s)}; D) = \left. \frac{d}{ds} \right|_{s=0} \left( E(X^{(s)}; D) + q \int_D \omega \left( \left( X^{(s)} \right)_* X \right) d\nu_g \right) \\
= \int_D g(V, \overline{\Delta}_g X + qX) d\nu_g.
\]

Thus we obtain the Euler–Lagrange equation for Landau-Hall functional on \( \mathcal{X}(M) \).

**Theorem 3.4** A vector field \( X \) on an oriented Riemannian manifold \( (M, g) \) is a critical point of the Landau-Hall functional under compact supported variations in \( \mathcal{X}(M) \) if and only if

\[
\overline{\Delta}_g X = -qX.
\]

### 4 Magnetic unit vector fields

Let us denote by \( UM \) the unit tangent sphere bundle of \( (M, g) \). Then \( UM \) is a hypersurface of \( TM \) with unit normal vector field \( U \). Here \( U \) is the so-called canonical vertical vector field of \( TM \). The key formula we use for \( U \) is

\[
U_{(p;w)} = w^v_w
\]

for any \( w \in T_p M \).

The almost Kähler structure \( (g_S, J) \) induces an almost contact metric structure \( (\phi, \xi, \eta, g_S) \) on \( UM \) by

\[
JV = \phi V + \eta(V)U, \quad \xi = -JU.
\]

The 1-form \( \eta \) is a contact form on \( UM \). Moreover \( \phi \) is skew-adjoint with respect to the induced metric and hence \( F_U(\cdot, \cdot) = g_S(\phi \cdot, \cdot) \) is a magnetic field with Lorentz force \( \phi \). Moreover \( F_U \) is exact; in fact, \( F_U = -d\eta \).

For any vector field \( Y \) on \( M \) and a unit tangent vector \( u = (p; u) \in T_p M \), its horizontal lift \( Y^h_u \) is tangent to \( UM \), i.e., \( Y^h_u \in T_u(UM) \). On the other hand, \( Y^v_u \) is not always tangent to \( UM \).

The tangential lift \( Y^v_u \) is defined by

\[
Y^v_u = Y^h_u - g(u, Y_p)U_u \in T_u(UM).
\]

The contact form \( \eta \) is given explicitly by

\[
\eta_{(p;u)}(X^h_u) = g(p, X_p, u), \quad \eta_{(p;u)}(X^v_u) = 0.
\]

#### 4.1 Tension field

Let us take now a unit vector field \( X \) on \( M \) and regard it as a smooth map \( X : M \to UM \). Then its tension field \( \tau_1(X) := \tau(X; UM) \) is computed as (see [1, p. 59, [7]]):

\[
\tau_1(X) = -\left\{ (\text{tr}_g R(\nabla X, X) \cdot \cdot) + (\overline{\Delta}_g X)^h \right\} \circ X.
\]

Next, we compute \( \phi(X_\ast X) \). We know that

\[
J(X_\ast X)_p = \phi(X_\ast X)_p + \eta(X_\ast X)_p X^v_p X^v_p.
\]
On the other hand,
\[ J(X_{*}X)_{X} = X_{*}^v - (\nabla X)^{h}_{X} = X_{_X}^v + g(X, X)U_{X} - (\nabla X)^{h}_{X}. \]
It follows that
\[ \phi(X_{*}X)_{X} = X_{_X}^t - (\nabla X)^{h}_{X}, \quad \eta(X_{*}X)_{X} = g(X, X) = 1. \]
Hence the magnetic map equation \( \tau_{*}(X) = q\phi(X_{*}X) \) is the system
\[ \text{tr}_{g}(R(\nabla X, X)\cdot)^{h}_{X} = q(\nabla X)^{h}_{X}, \quad (\nabla g)^{1}_{X} = -qX_{_X}^t. \]
The first equation is equivalent to
\[ \text{tr}_{g}R(\nabla X, X)\cdot = -q\nabla X. \]
The second equation is rewritten as
\[ (\nabla g)^{v}_{X} - g(X, \nabla g)U_{X} = -q(X_{_X}^v - g(X, X)U_{X}). \]
This equation is equivalent to
\[ \nabla gX - g(X, \nabla g)X = -q(X - g(X, X)X) = 0. \]
Thus we obtain
\[ \nabla gX = g(X, \nabla g)X = |\nabla X|^{2} X. \]

**Theorem 4.1**  A unit vector field \( X \) on \( (M, g) \) is a magnetic map into \( UM \) if and only if
\[ \text{tr}_{g}R(\nabla X, X)\cdot = q\nabla X, \quad \nabla gX = |\nabla X|^{2} X. \tag{4.1} \]

Next, we consider a unit vector field preserving variation \( \{X^{(s)}\} \) through a unit vector field \( X \). About Dirichlet energy the following result is known.

**Theorem 4.2**  [7, 24, 26] A unit vector field \( X \) on an oriented Riemannian manifold \( (M, g) \) is a critical point of the Dirichlet energy with respect to compactly supported variations in \( X_{1}(M) \) if and only if
\[ \nabla gX = |\nabla X|^{2} X. \]

A unit vector field \( X \) satisfying \( \nabla gX = |\nabla X|^{2} X \) is referred to as a harmonic unit vector field [1].

In order to compute the first variation for the Landau-Hall functional, let us make some remarks. Using (3.2) we write \( X_{*}^{(s)}X^{(0)} = (X^{(0)})_{X_{(s)}}^{h} + (\nabla X^{(0)})_{X_{(s)}}^{v}. \) But we have also
\[ g_{s}\left(U, (\nabla X^{(0)})_{X_{(s)}}^{v}ight)_{X_{(s)}} = 0, \] which means that \( (\nabla X^{(0)})_{X_{(s)}}^{v} = (\nabla X^{(0)})_{X_{(s)}}^{h}. \) As a consequence, \( \eta_{X_{(s)}}(X_{*}^{(s)}X^{(0)}) = \eta_{X_{(s)}}((X^{(0)})_{X_{(s)}}^{h}) = g(X^{(s)}, X). \) Thus the variation of the magnetic term is given by
\[ \frac{d}{ds}\bigg|_{s=0} \int_{D} \eta(X_{*}^{(s)}X^{(0)}) \text{d}v_{g} = \int_{D} g(V, X) \text{d}v_{g}. \tag{4.2} \]

Here \( V \) is the vector field on \( D \) previously defined by \( V(p) = \lim_{s \to 0} \frac{1}{s}(X^{(s)}(p) - X(p)) \).

We will prove the following result.
Theorem 4.3 A unit vector field $X$ on an oriented Riemannian manifold $(M, g)$ is a critical point of the Landau-Hall functional under compact support variations in $\mathcal{X}_1(M)$ if and only if it is a critical point of the Dirichlet energy under compact support variations in $\mathcal{X}_1(M)$.

Before giving a proof, it seems to be better to add some helpful remarks.

Lemma 4.1 Let $X \in \mathcal{X}_1(M)$ be a unit vector field and $X^{(s)}$ be a variation of $X$ through unit vector fields. Then, the vector field $V$ obtained from the variation $X^{(s)}$ of $X$ is orthogonal to $X$. Conversely, let $V$ be a vector field orthogonal to $X$. Then there exists a variation $X^{(s)}$ of $X$ through unit vector fields whose variational vector field leads to $V$.

Proof Since $X^{(s)}$ is a variation of $X$ through unit vector fields, that is $g(X^{(s)}, X^{(s)}) = 1$, we immediately obtain $0 = \frac{d}{ds} \bigg|_{s=0} g(X^{(s)}, X^{(s)}) = 2g(V, X)$. See also [1, p. 65]. Hence the conclusion.

To prove the converse, we set $W^{(s)} = X + sV$ and $X^{(s)} = \frac{1}{f^{(s)}} W^{(s)}$, where $f^{(s)} = |W^{(s)}|$. Obviously, $f(0) = 1$ and

$$ f(s)^2 = |W^{(s)}|^2 = 1 + s^2 |V|^2. $$

This implies $f'(0) = 0$. Consequently, the variational vector field of $W^{(s)}$ is

$$ \frac{d}{ds} \bigg|_{s=0} X^{(s)} = \left( -\frac{f'(s)}{f(s)^2} W^{(s)} + \frac{1}{f(s)} \frac{d}{ds} W^{(s)} \right)_{s=0} = V. $$

We conclude that $X^{(s)}$ is a required variation. (See e.g. [1, p. 65].) \qed

Lemma 4.2 Let $X \in \mathcal{X}_1(M)$ be a unit vector field and $X^{(s)}$ be a variation of $X$ through unit vector fields whose variational vector field is $V$. Let

$$ \text{LH}(s) := \text{LH}(X^{(s)}; D) = E(X^{(s)}; D) + q \int_{D} \eta_{X^{(s)}}(X^{(s)}X^{(0)}) dv_{g} $$

be the Landau-Hall functional under compact support variations in $\mathcal{X}_1(M)$. Then

$$ \text{LH}'(0) = \int_{D} g(V, \Delta g X + qX) dv_{g}. $$

Proof The first variation of the Dirichlet energy is given by [24, 26] (see also [1, p. 65]):

$$ \frac{d}{ds} \bigg|_{s=0} E(X^{(s)}; D) = \int_{D} g(V, \Delta g X) dv_{g}. $$

Combining with (4.2) we obtain that the first variation formula of the Landau-Hall functional is given by

$$ \frac{d}{ds} \bigg|_{s=0} \text{LH}(X^{(s)}; D) = \int_{D} g(V, \Delta g X + qX) dv_{g}. $$

\qed

Lemma 4.3 Let $X$ be a unit vector field and $V$ a vector field orthogonal to $X$. Assume that $X$ is a critical point of the Landau-Hall functional under unit vector field preserving variations. Then we have

$$ \int_{D} g(V, \Delta g X + qX) dv_{g} = 0. $$
Proof It is enough to choose \( X(s) = W(s)/|W(s)| \) (see Lemma 4.1) in the variational formula obtained in the previous Lemma.

Proof of the Theorem 4.3 Let \( X \) be a magnetic unit vector field with charge \( q \). Then, from the first variational formula, we deduce that

\[
\int_D g(V, \Delta_g X + qX) dv_g = 0
\]

for all vector fields \( V \) orthogonal to \( X \).

Let us decompose the tangent space \( T_p M \) as \( T_p M = \mathbb{R}X_p \oplus (\mathbb{R}X_p)^\perp \). Accordingly, we consider

\[
\Delta_g X + qX = \lambda X + V,
\]

for some smooth function \( \lambda \). This equation implies \( g(\Delta_g X + qX, V) = |V|^2 \). Thus we get

\[
\int_D |V|^2 dv_g = \int_D g(\Delta_g X + qX, V) dv_g = 0.
\]

Hence \( V = 0 \). Thus \( \Delta_g X + qX = \lambda X \). From this we have

\[
\lambda = g(\Delta_g X + qX, X) = |\nabla X|^2 + q,
\]

which implies

\[
\Delta_g X = |\nabla X|^2 X.
\]

Conversely, assume that \( X \) satisfies \( \Delta_g X = |\nabla X|^2 X \). Take a variation \( X(s) \) with variational vector field \( V \), which is orthogonal to \( X \). See Lemma 4.1. We compute

\[
\left. \frac{d}{ds} \right|_{s=0} LH(X(s)) = \int_D g(\Delta_g X + qX, V) dv_g = \int_D g(|\nabla X|^2 + qX, V) dv_g = 0.
\]

Hence \( X \) is a critical point of the Landau-Hall functional under unit vector field preserving variations.

It should be remarked that the Euler–Lagrange equation of this variational problem is nothing but the second equation of magnetic map equations (4.1) for unit vector fields. Therefore, we state the following.

Corollary 4.1 A unit vector field \( X \) is a magnetic map into \( UM \) with charge \( q \) if and only if it is a critical point of the Landau-Hall functional under compact support variations in \( X_1(M) \) and in addition satisfies \( \text{tr}_g R(\nabla X, X) \cdot X = q \nabla X X \).

Because of the Theorem 4.3 we give the following definition: A magnetic unit vector field on \( (M, g) \) is a unit vector field on \( M \) such that it is a magnetic map from \( M \) to \( UM \).

In the next two sections we look for explicit examples of unit vector fields satisfying (4.1).

5 Unimodular Lie groups

5.1 Unimodular basis

Let \( G \) be a 3-dimensional unimodular Lie group with a left invariant metric \( g = \langle \cdot, \cdot \rangle \). Then there exists an orthonormal basis \( \{e_1, e_2, e_3\} \) of the Lie algebra \( \mathfrak{g} \) such that

\[
[e_1, e_2] = c_3 e_3, \quad [e_2, e_3] = c_1 e_1, \quad [e_3, e_1] = c_2 e_2, \quad c_i \in \mathbb{R}.
\]

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Three-dimensional unimodular Lie groups are classified by Milnor as follows (signature of structure constants are given up to numeration) [17]:

| Signature of \((c_1, c_2, c_3)\) | Simply connected Lie group | Property |
|----------------------------------|-----------------------------|----------|
| \(+, +, +\)                      | SU(2)                       | Compact and simple |
| \(+, +, −\)                      | \(\widetilde{S}L_2\mathbb{R}\) | Non-compact and simple |
| \(+, +, 0\)                      | \(\widetilde{E}(2)\)       | Solvable |
| \(+, −, 0\)                      | \(E(1, 1)\)                | Solvable |
| \(+, 0, 0\)                      | Heisenberg group           | Nilpotent |
| \(0, 0, 0\)                      | \((\mathbb{R}^3, +)\)      | Abelian |

Denote by the same letters also the left translated vector fields determined by \(\{e_1, e_2, e_3\}\). To describe the Levi-Civita connection \(\nabla\) of \(G\), we introduce the following constants:

\[
\mu_i = \frac{1}{2} (c_1 + c_2 + c_3) - c_i. 
\]

**Proposition 5.1** The Levi-Civita connection is given by

\[
\begin{align*}
\nabla_{e_1} e_1 &= 0, & \nabla_{e_1} e_2 &= \mu_1 e_3, & \nabla_{e_1} e_3 &= -\mu_1 e_2 \\
\nabla_{e_2} e_1 &= -\mu_2 e_3, & \nabla_{e_2} e_2 &= 0, & \nabla_{e_2} e_3 &= \mu_2 e_1 \\
\nabla_{e_3} e_1 &= \mu_3 e_2, & \nabla_{e_3} e_2 &= -\mu_3 e_1, & \nabla_{e_3} e_3 &= 0.
\end{align*}
\]

The Riemannian curvature \(R\) is given by

\[
\begin{align*}
R(e_1, e_2) e_1 &= (\mu_1 \mu_2 - c_3 \mu_3) e_2, & R(e_1, e_2) e_2 &= -(\mu_1 \mu_2 - c_3 \mu_3) e_1, \\
R(e_2, e_3) e_2 &= (\mu_2 \mu_3 - c_1 \mu_1) e_3, & R(e_2, e_3) e_3 &= -(\mu_2 \mu_3 - c_1 \mu_1) e_2, \\
R(e_1, e_3) e_1 &= (\mu_3 \mu_1 - c_2 \mu_2) e_3, & R(e_1, e_3) e_3 &= -(\mu_3 \mu_1 - c_2 \mu_2) e_1.
\end{align*}
\]

The basis \(\{e_1, e_2, e_3\}\) diagonalizes the Ricci tensor field. The principal Ricci curvatures are \(\rho_1 = 2 \mu_2 \mu_3, \rho_2 = 2 \mu_3 \mu_1, \rho_3 = 2 \mu_1 \mu_2\).

**Example 5.1** (The space Nil\(_3\)) The model space Nil\(_3\) of nilgeometry in the sense of Thurston is realized as a nilpotent Lie group

\[
G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \bigg| x, y, z \in \mathbb{R} \right\}
\]

equipped with left invariant metric \(dx^2 + dy^2 + (dz - xy)^2\). The Lie algebra \(\mathfrak{g}\) of \(G\) is given by

\[
\mathfrak{g} = \left\{ \begin{pmatrix} 0 & u & w \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix} \bigg| u, v, w \in \mathbb{R} \right\}
\]

Take an orthonormal basis \(\{e_1, e_2, e_3\}\) of \(\mathfrak{g}\):

\[
e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.
\]
Then \( \{e_1, e_2, e_3\} \) is a unimodular basis satisfying \((c_1, c_2, c_3) = (1, 0, 0)\). The left invariant vector fields determined by \( e_1, e_2 \) and \( e_3 \) are given by

\[
e_1 = \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial x}, \quad e_3 = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}.
\]

**Example 5.2** (The space Sol3) The model space Sol3 of solvegeometry in the sense of Thurston is realized as a solvable Lie group

\[
G = \left\{ \begin{pmatrix} e^{-z} & 0 & x \\ 0 & e^{z} & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}
\]

equipped with left invariant metric \( e^{2z}dx^2 + e^{-2z}dy^2 + dz^2 \). The Lie algebra \( \mathfrak{g} \) of \( G \) is given by

\[
\mathfrak{g} = \left\{ \begin{pmatrix} -w & 0 & u \\ 0 & w & v \\ 0 & 0 & 0 \end{pmatrix} \middle| u, v, w \in \mathbb{R} \right\}
\]

Take an orthonormal basis \( \{e_1, e_2, e_3\} \) of \( \mathfrak{g} \):

\[
e_1 = \frac{1}{\sqrt{2}} (\hat{e}_1 + \hat{e}_2), \quad e_2 = \frac{1}{\sqrt{2}} (\hat{e}_1 - \hat{e}_2), \quad e_3 = \hat{e}_3,
\]

where

\[
\hat{e}_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{e}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{e}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

Then \( \{e_1, e_2, e_3\} \) is a unimodular basis satisfying \((c_1, c_2, c_3) = (1, -1, 0)\). The left translated vector fields of \( e_1, e_2 \) and \( e_3 \) are

\[
e_1 = \frac{1}{\sqrt{2}} \left( e^{-z} \frac{\partial}{\partial x} + e^{z} \frac{\partial}{\partial y} \right), \quad e_2 = \frac{1}{\sqrt{2}} \left( e^{-z} \frac{\partial}{\partial x} - e^{z} \frac{\partial}{\partial y} \right), \quad e_3 = \frac{\partial}{\partial z}.
\]

### 5.2 Magnetic equation

Consider a left invariant unit vector field \( X = x_1e_1 + x_2e_2 + x_3e_3 \) on \( G \), where \( x_1, x_2, x_3 \) are constants such that \( x_1^2 + x_2^2 + x_3^2 = 1 \).

We have to develop the two magnetic equations \((4.1)\) provided in Theorem 4.1. 

(1) \( \text{tr}_\mathfrak{g} R(\nabla X, X) \cdot = g \nabla X \)

The left side of the equation above can be developed as follows

\[
\text{tr} R(\nabla X, X) \cdot = R(\nabla_{e_1} X, X) e_1 + R(\nabla_{e_2} X, X) e_2 + R(\nabla_{e_3} X, X) e_3
\]

\[
= \sum_{(1,2,3)} \left[ \mu_1 (\mu_2^2 - \mu_3^2) + (c_3 - c_2) \mu_2 \mu_3 \right] x_2 x_3 e_1,
\]

where \( \sum_{(1,2,3)} \) denotes the cyclic summation over the permutation \((1, 2, 3)\).

On the other hand, we have \( \nabla X = \sum_{(1,2,3)} (\mu_2 - \mu_3) x_2 x_3 e_1 \).

Using that \( \mu_2 - \mu_3 = c_3 - c_2 \) and \( \mu_2 + \mu_3 = c_1 \), together with other four identities obtained by cyclic permutations, the first magnetic equation yields

\[
\begin{cases}
\mu_1 c_1 - \mu_2 \mu_3 + q \right) (c_2 - c_3) x_2 x_3 = 0, \\
\mu_2 c_2 - \mu_3 \mu_1 + q \right) (c_3 - c_1) x_3 x_1 = 0, \\
\mu_3 c_3 - \mu_1 \mu_2 + q \right) (c_1 - c_2) x_1 x_2 = 0.
\end{cases}
\]

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(2) \( \Delta g X = |\nabla X|^2 X \)
We compute first
\[
\Delta g X = -\sum_{i=1}^{3} (\nabla_{e_i} \nabla_{e_i} X - \nabla_{\nabla_{e_i} e_i} X) = \sum_{(1,2,3)} (\mu_1^2 + \mu_2^2) x_1 e_1
\]
and
\[
|\nabla X|^2 = g(X, \Delta g X) = x_1^2(\mu_2^2 + \mu_3^2) + x_2^2(\mu_3^2 + \mu_1^2) + x_3^2(\mu_1^2 + \mu_2^2).
\]
The second magnetic equation yields
\[
\phi(\phi, \xi, \eta) = 0
\]
Then one can check that
\[
\text{on the other hand, the equation (5.3) implies }
\]
\[
\text{Then the unit magnetic vector fields on } G \text{ belong to the list given in Table 1.}
\]
\[
\text{Theorem 5.1}
\]
Let \( G \) be a 3-dimensional unimodular Lie group with a left invariant metric and let \( \{e_1, e_2, e_3\} \) be an orthonormal basis of the Lie algebra \( g \) satisfying (5.1) with \( c_1 \geq c_2 \geq c_3 \). Then the unit magnetic vector fields on \( G \) belong to the list given in Table 1. \( S \) is the unit sphere in the Lie algebra \( g \) of \( G \) centered at the origin.

\textbf{Remark 5.1} (Contact metric structure) On a 3-dimensional non-abelian unimodular Lie group \( G = G(c_1, c_2, c_3) \) with \( c_1 = 2 \), we can introduce a left invariant almost contact structure \( (\varphi, \xi, \eta) \) compatible with the left invariant metric \( (\cdot, \cdot) \) by [8, 20]:
\[
\xi = e_1, \quad \eta = g(\xi, \cdot), \quad \varphi e_1 = 0, \quad \varphi e_2 = e_3, \quad \varphi e_3 = -e_2.
\]
Then one can check that \( (\varphi, \xi, \eta, (\cdot, \cdot)) \) is a left invariant contact metric structure on \( G \). The \( \varphi \)-sectional curvature of \( G \) is constant
\[
-3 + \frac{1}{4}(c_2 - c_3)^2 + c_2 + c_3.
\]
Table 1  Unit magnetic vector fields on a unimodular Lie group

| Conditions for $c_i$  | $G$                   | The set of all unit magnetic vector fields | $q$     |
|------------------------|-----------------------|------------------------------------------|---------|
| $c_1 = c_2 = c_3 \neq 0$ | $S^3$                 | $S$                                      | $\forall q$ |
| $= 0$                  | $\mathbb{R}^3$        | $S$                                      | $\forall q$ |
| $c_1 > c_2 = c_3 \neq 0$ | $SU(2), SL_2\mathbb{R}$ | $\pm e_1, S \cap \{e_1, e_3\}_\mathbb{R}$ | $\forall q$ |
| $= 0$                  | Heisenberg group      | $\pm e_1, S \cap \{e_1, e_3\}_\mathbb{R}$ | $\forall q$ |
| $S$                    | $-\frac{c_1^2}{4}$   | $\forall q$                              |         |
| $0 \neq c_1 = c_2 > c_3$ | $SU(2), SL_2\mathbb{R}$ | $\pm e_3, S \cap \{e_1, e_2\}_\mathbb{R}$ | $\forall q$ |
| $0 = $                 | Heisenberg group      | $\pm e_1, S \cap \{e_1, e_3\}_\mathbb{R}$ | $\forall q$ |
| $S$                    | $-\frac{c_3^2}{4}$   | $\forall q$                              |         |
| $c_1 > c_2 > c_3 > 0$  | $SU(2)$               | $\pm e_1, \pm e_2 \pm e_3$              | $\forall q$ |
| $c_1 > c_2 > 0 > c_3$  | $SL_2\mathbb{R}$      | $\pm e_1, \pm e_2 \pm e_3$              | $\forall q$ |
| $c_1 > 0 > c_2 > c_3$  | $SL_2\mathbb{R}$      | $\pm e_1, \pm e_2 \pm e_3$              | $\forall q$ |
| $0 > c_1 > c_2 > c_3$  | $SU(2)$               | $\pm e_1, \pm e_2 \pm e_3$              | $\forall q$ |
| $c_1 > c_2 > c_3 = 0$  | $E(2)$                | $\pm e_1, \pm e_2 \pm e_3$              | $\forall q$ |
|                        |                       | $\pm e_3, S \cap \{e_1, e_2\}_\mathbb{R}$ | $-\frac{(c_1-c_2)^2}{4}$ |
| $c_1 > c_2 = 0 > c_3$  | $E(1, 1)$             | $\pm e_1, \pm e_2 \pm e_3$              | $\forall q$ |
|                        |                       | $\pm e_2, S \cap \{e_1, e_3\}_\mathbb{R}$ | $-\frac{(c_3-c_1)^2}{4}$ |
| $0 = c_1 > c_2 > c_3$  | $E(2)$                | $\pm e_1, \pm e_2 \pm e_3$              | $\forall q$ |
|                        |                       | $\pm e_1, S \cap \{e_2, e_3\}_\mathbb{R}$ | $-\frac{(c_2-c_3)^2}{4}$ |

In particular $G$ is Sasakian when and only when $c_2 = c_3$.

Moreover if $c_2 \neq c_3$, $G$ is a contact $(\kappa, \mu)$-space with

$$\kappa = 1 - \frac{1}{4}(c_2 - c_3)^2, \quad \mu = 2 - (c_2 + c_3).$$

Perrone [21] investigated stability of the Reeb vector fields on 3-dimensional compact contact $(\kappa, \mu)$-spaces with respect to the Dirichlet energy. For example, in [21, Example 3.1] the author provides (probably) the first examples of non-Killing harmonic vector fields which are stable.

6 Non-unimodular Lie groups

Let $G$ be a non-unimodular 3-dimensional Lie group with a left invariant metric. Then the unimodular kernel $u$ of $\mathfrak{g}$ is defined by

$$u = \{ X \in \mathfrak{g} \mid \text{tr} \text{ad}(X) = 0 \}. $$

Here $\text{ad} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$ is a homomorphism defined by

$$\text{ad}(X)Y = [X, Y].$$

One can see that $u$ is an ideal of $\mathfrak{g}$ which contains the ideal $[\mathfrak{g}, \mathfrak{g}]$.  

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On \( g \), we can take an orthonormal basis \( \{e_1, e_2, e_3\} \) such that

1. \( \langle e_1, X \rangle = 0, \) \( X \in u, \)
2. \( \{[e_1, e_2], [e_1, e_3]\} = 0. \)

Then the commutation relations of the basis are given by

\[
[e_1, e_2] = a_{11}e_2 + a_{12}e_3, \quad [e_2, e_3] = 0, \quad [e_1, e_3] = a_{21}e_2 + a_{22}e_3, \tag{6.1}
\]

with \( a_{11} + a_{22} \neq 0 \) and \( a_{11}a_{21} + a_{12}a_{22} = 0. \) Under a suitable homothetic change of the metric, we may assume that \( a_{11} + a_{22} = 2. \) Moreover, we may presume that \( a_{11}a_{22} + a_{12}a_{21} = 0 \) [16, §2.5]. Then the constants \( a_{11}, a_{12}, a_{21} \) and \( a_{22} \) are represented as

\[
a_{11} = 1 + \alpha, \quad a_{12} = (1 + \alpha)\beta, \quad a_{21} = -(1 - \alpha)\beta, \quad a_{22} = 1 - \alpha. \tag{6.2}
\]

If necessarily, by changing the sign of \( e_1, e_2 \) and \( e_3 \), we may assume that the constants \( \alpha \) and \( \beta \) satisfy the condition \( \alpha, \beta \geq 0. \) We note that for the case that \( \alpha = \beta = 0, \) \( G \) is of constant negative curvature (see Example 6.1). We refer (\( \alpha, \beta \)) as the structure constants of the non-unimodular Lie algebra \( g. \) Non-unimodular Lie algebras \( g = g(\alpha, \beta) \) are classified by the Milnor invariant \( D = (1 - \alpha^2)(1 + \beta^2). \)

**Proposition 6.1** For any \( (\alpha, \beta) \neq (0, 0) \), two Lie algebras \( g(\alpha, \beta) \) and \( g(\alpha', \beta') \) are isomorphic if and only if their Milnor invariants \( D \) and \( D' \) agree.

Under this normalization, the Levi-Civita connection of \( G \) is given by the following table:

**Proposition 6.2**

\[
\begin{align*}
\nabla_{e_1}e_1 & = 0, & \nabla_{e_1}e_2 & = \beta e_3, & \nabla_{e_1}e_3 & = -\beta e_2, \\
\nabla_{e_2}e_1 & = -(1 + \alpha)e_2 - \alpha \beta e_3, & \nabla_{e_2}e_2 & = (1 + \alpha)e_1, & \nabla_{e_2}e_3 & = \alpha \beta e_1, \\
\nabla_{e_3}e_1 & = -\alpha \beta e_2 - (1 - \alpha)e_3, & \nabla_{e_3}e_2 & = \alpha \beta e_1, & \nabla_{e_3}e_3 & = (1 - \alpha)e_1.
\end{align*}
\]

The Riemannian curvature \( R \) is given by

\[
\begin{align*}
R(e_1, e_2)e_1 & = (\alpha \beta^2 + (1 + \alpha)^2 + \alpha \beta^2(1 + \alpha))e_2, \\
R(e_1, e_2)e_2 & = -[(\alpha \beta^2 + (1 + \alpha)^2 + \alpha \beta^2(1 + \alpha))]e_1, \\
R(e_1, e_3)e_1 & = -(\alpha \beta^2 - (1 - \alpha)^2 + \alpha \beta^2(1 - \alpha))e_3, \\
R(e_1, e_3)e_3 & = (\alpha \beta^2 - (1 - \alpha)^2 + \alpha \beta^2(1 - \alpha))e_1, \\
R(e_2, e_3)e_2 & = (1 - \alpha^2(1 + \beta^2))e_3, \\
R(e_2, e_3)e_3 & = -(1 - \alpha^2(1 + \beta^2))e_2.
\end{align*}
\]

The basis \( \{e_1, e_2, e_3\} \) diagonalizes the Ricci tensor field. The principal Ricci curvatures are given by

\[
\rho_1 = -2[1 + \alpha^2(1 + \beta^2)] < -2, \quad \rho_2 = -2[1 + \alpha(1 + \beta^2)] < -2, \quad \rho_3 = -2[1 - \alpha(1 + \beta^2)].
\]

The scalar curvature is

\[
-2[3 + \alpha^2(1 + \beta^2)] < 0.
\]
The simply connected Lie group \( \tilde{G} = \tilde{G}(\alpha, \beta) \) corresponding to the non-unimodular Lie algebra \( g(\alpha, \beta) \) is given explicitly by [14]:

\[
\tilde{G}(\alpha, \beta) = \left\{ \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & \alpha_{11}(x) & \alpha_{12}(x) & y \\ 0 & \alpha_{21}(x) & \alpha_{22}(x) & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\},
\]

where

\[
\begin{pmatrix} \alpha_{11}(x) & \alpha_{12}(x) \\ \alpha_{21}(x) & \alpha_{22}(x) \end{pmatrix} = \exp\left( x \begin{pmatrix} 1 + \alpha & (1 + \alpha)\beta \\ -(1 - \alpha)\beta & 1 - \alpha \end{pmatrix} \right).
\]

This shows that \( \tilde{G}(\alpha, \beta) \) is the semi-direct product \( \mathbb{R} \ltimes \mathbb{R}^2 \) with multiplication

\[
(x, y, z) \cdot (x', y', z') = (x + x', y + \alpha_{11}(x)y' + \alpha_{12}(x)z', z + \alpha_{21}(x)y' + \alpha_{22}(x)z').
\]

The Lie algebra of \( \tilde{G}(\alpha, \beta) \) is is spanned by the basis

\[
e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 + \alpha & -(1 - \alpha)\beta \\ 0 & (1 + \alpha)\beta & 1 - \alpha \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

This basis satisfies the commutation relation

\[
[e_1, e_2] = (1 + \alpha)(e_2 + \beta e_3), \quad [e_2, e_3] = 0, \quad [e_3, e_1] = (1 - \alpha)(\beta e_2 - e_3).
\]

Thus the Lie algebra of \( \tilde{G}(\alpha, \beta) \) is the non-unimodular Lie algebra \( g = g(\alpha, \beta) \). The left invariant vector fields corresponding to \( e_1, e_2 \) and \( e_3 \) are

\[
e_1 = \frac{\partial}{\partial x}, \quad e_2 = \alpha_{11}(x) \frac{\partial}{\partial y} + \alpha_{12}(x) \frac{\partial}{\partial z}, \quad e_3 = \alpha_{21}(x) \frac{\partial}{\partial y} + \alpha_{22}(x) \frac{\partial}{\partial z}.
\]

**Example 6.1** (\( \alpha = 0, \mathcal{D} \geq 1 \)) The simply connected Lie group \( \tilde{G}(0, \beta) \) is isometric to the hyperbolic 3-space \( \mathbb{H}^3(-1) \) of curvature \(-1\) and given explicitly by

\[
\tilde{G}(0, \beta) = \left\{ \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & e^{x} \cos(\beta x) & -e^{y} \sin(\beta x) & y \\ 0 & e^{y} \sin(\beta x) & e^{x} \cos(\beta x) & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}.
\]

The left invariant metric is \( dx^2 + e^{-2x}(dy^2 + dz^2) \). Thus \( \tilde{G}(0, \beta) \) is the warped product model of \( \mathbb{H}^3(-1) \). In fact, setting \( w = e^x \), the left invariant metric of \( \tilde{G} \) can be rewritten as the Poincaré metric

\[
\frac{dy^2 + dz^2 + dw^2}{w^2}.
\]

The Milnor invariant of \( \tilde{G}(0, \beta) \) is \( \mathcal{D} = 1 + \beta^2 \geq 1 \).
Example 6.2 \((\beta = 0, \mathcal{D} \leq 1)\) For each \(\alpha \geq 0\), \(\widetilde{G}(\alpha, 0)\) is given by:

\[
\widetilde{G}(\alpha, 0) = \left\{ \begin{pmatrix}
1 & 0 & 0 & x \\
0 & e^{(1+\alpha)x} & 0 & y \\
0 & 0 & e^{(1-\alpha)x} & z \\
0 & 0 & 0 & 1
\end{pmatrix} \bigg| x, y, z \in \mathbb{R} \right\}.
\]

The left invariant Riemannian metric is given explicitly by

\[
dx^2 + e^{-2(1+\alpha)x}dy^2 + e^{-2(1-\alpha)x}dz^2.
\]

The Milnor invariant is \(\mathcal{D} = 1 - \alpha^2 \leq 1\).

This family of Riemannian homogenous spaces has been studied in \([9, 18]\). The Lie group \(\widetilde{G}(\alpha, 0)\) is realized as a warped product of the hyperbolic plane \(\mathbb{H}^2(-(1+\alpha)^2)\) and the real line \(\mathbb{R}\) with warping function \(e^{(\alpha-1)x}\). In fact, via the coordinate change \((u, v) = ((1+\alpha)y, e^{(1+\alpha)x})\), the metric is rewritten as

\[
d\frac{u^2 + dv^2}{(1+\alpha)^2v^2} + f_{\alpha}(u, v)^2dz^2, \quad f_{\alpha}(u, v) = \exp\left(\frac{\alpha - 1}{\alpha + 1}\log v\right).
\]

Here we observe locally symmetric examples:

- If \(\alpha = 0\) then \(\widetilde{G}(0, 0)\) is a warped product model of hyperbolic 3-space \(\mathbb{H}^3(-1)\).
- If \(\alpha = 1\) then \(\widetilde{G}(1, 0)\) is isometric to the Riemannian product \(\mathbb{H}^2(-4) \times \mathbb{R}\).

Note that \(\mathbb{H}^3\) does not admit any other Lie group structure.

Example 6.3 \((\alpha = 1, \mathcal{D} = 0)\) Assume that \(\alpha = 1\). Then \(\widetilde{G}(1, \beta)\) is given explicitly by

\[
\widetilde{G}(1, \beta) = \left\{ \begin{pmatrix}
1 & 0 & 0 & x \\
0 & e^{2x} & 0 & y \\
0 & \beta(e^{2x} - 1) & 1 & z \\
0 & 0 & 0 & 1
\end{pmatrix} \bigg| x, y, z \in \mathbb{R} \right\}.
\]

The left invariant metric is

\[
dx^2 + \{e^{-4x} + \beta^2(1 - e^{-2x})^2\}dy^2 - 2\beta(1 - e^{-2x})dydz + dz^2.
\]

The non-unimodular Lie group \(\widetilde{G}(1, \beta)\) has sectional curvatures

\[
K_{12} = -3\beta^2 - 4, \quad K_{13} = K_{23} = \beta^2,
\]

where \(K_{ij}\) (\(i \neq j\)) denote the sectional curvatures of the planes spanned by vectors \(e_i\) and \(e_j\).

One can check that \(\widetilde{G}(1, \beta)\) is isometric to the so-called Bianchi-Cartan-Vranceanu space \(M^3(-4, \beta)\) with 4-dimensional isometry group and isotropy subgroup \(SO(2)\):

\[
M^3(-4, \beta) = \left\{ (u, v, w) \in \mathbb{R}^3 \mid u^2 + v^2 < 1, g_\beta \right\},
\]

with metric

\[
g_\beta = \frac{du^2 + dv^2}{(1 - u^2 - v^2)^2} + \left( dw + \frac{\beta(vdu - udv)}{1 - u^2 - v^2} \right)^2, \quad \beta \geq 0.
\]

The family \(\{\widetilde{G}(1, \beta)\}_{\beta \geq 0}\) is characterized by the condition \(\mathcal{D} = 0\). In particular \(M^3(-4, \beta)\) with positive \(\beta\) is isometric to the universal covering \(\tilde{SL}_2\mathbb{R}\) of the special linear group equipped with the above metric (cf. \([23]\)), but not isomorphic to \(\tilde{SL}_2\mathbb{R}\) as Lie groups. We here note that \(\tilde{SL}_2\mathbb{R}\) is a unimodular Lie group, while \(\widetilde{G}(1, \beta)\) is non-unimodular.
6.1 Magnetic equations

We first recall that \( \alpha, \beta \geq 0 \) and make some notations in order to simplify some other complicated expressions

\[
\begin{align*}
  u &= \alpha \beta^2 + (1 + \alpha)^2 + \alpha \beta^2 (1 + \alpha), \\
  v &= \alpha \beta^2 - (1 - \alpha)^2 + \alpha \beta^2 (1 - \alpha), \\
  w &= 1 - \alpha^2 (1 + \beta^2).
\end{align*}
\]

Take a left invariant unit vector field \( X = x_1 e_1 + x_2 e_2 + x_3 e_3 \) on \( G \), where \( x_1, x_2, x_3 \) are constants such that \( x_1^2 + x_2^2 + x_3^2 = 1 \).

We have to develop the two magnetic equations provided in Theorem 4.1.

\[(1) \ \text{tr}_g R(\nabla X, X) = q \nabla_X X \]

After a straightforward computation we obtain

\[
\begin{align*}
  - [x_2 (1 + \alpha) + x_3 \alpha \beta] x_2 u - x_1^2 (1 + \alpha) u + [x_2 \alpha \beta + x_3 (1 - \alpha)] x_3 v \\
  + x_1^2 (1 - \alpha) v &= q \left[ x_2^2 (1 + \alpha) + x_3^2 (1 - \alpha) + x_2 x_3 \alpha \beta \right] \\
  x_1 x_3 \beta u + x_1 x_3 \beta w &= - q \left[ x_1 x_3 \beta (1 + \alpha) + x_1 x_2 (1 + \alpha) \right] \\
  x_1 x_2 \beta v - x_1 x_3 (1 + \alpha) w + x_1 x_2 \alpha \beta w + x_1 x_2 (1 - \alpha) w \\
  &= q \left[ x_1 x_2 \beta (1 - \alpha) - x_1 x_3 (1 - \alpha) \right].
\end{align*}
\]

\[(2) \ \overline{\Delta}_g X = |\nabla X|^2 X \]

Again, after a straightforward computations, we get:

\[
\begin{align*}
  2 x_1 (1 + \alpha^2 + \alpha^2 \beta^2) &= |\nabla X|^2 x_1 \\
  x_2 \left[ \beta^2 + (1 + \alpha)^2 + \alpha^2 \beta^2 \right] - 2 x_3 \beta (1 - \alpha) &= |\nabla X|^2 x_2 \\
  x_3 \left[ \beta^2 + (1 - \alpha)^2 + \alpha^2 \beta^2 \right] + 2 x_2 \beta (1 + \alpha) &= |\nabla X|^2 x_3,
\end{align*}
\]

where

\[
|\nabla X|^2 = g(\overline{\Delta}_g X, X) = 2 x_1^2 (1 + \alpha^2 + \alpha^2 \beta^2) + x_2^2 \left[ \beta^2 + (1 + \alpha)^2 + \alpha^2 \beta^2 \right] + x_3^2 \left[ \beta^2 + (1 - \alpha)^2 + \alpha^2 \beta^2 \right] + 4 x_2 x_3 \alpha \beta.
\]

Recall that \( \pm e_1, \pm e_2 \) and \( \pm e_3 \) are all unit magnetic vector fields in the unimodular Lie groups. Contrary, the non-unimodular case is more rigid. For example, we have the following:

| When \( e_i \) is magnetic? | Condition for \( q \) |
|-----------------------------|------------------|
| \( e_1 \) \( \neq \) magnetic | \( q = -(1 + \alpha)^2 \) |
| \( e_2 \) \( \beta = 0 \) | \( q = -(1 - \alpha)^2 \) |
| \( e_3 \) \( \alpha = 1 \) | \( q = -(1 - \alpha)^2 \) |
| \( \alpha \neq 1 \) \( \text{and} \ \beta = 0 \) | \( q = -(1 + \alpha)^2 \) |
| Otherwise, never \( \alpha \neq 1 \) \( \text{and} \ \beta = 0 \) | \( q = -(1 + \alpha)^2 \) |

Let us continue our investigation with some particular situations:

(1) \( \alpha = 0 \) and \( \beta \in \mathbb{R} \) \( G \) is isometric to \( \mathbb{H}^3 (-1) \) \( D \geq 1 \)
We have \( u = 1, v = -1, w = 1 \) and \(|\nabla X|^2 = 1 + x_1^2 + \beta^2(1 - x_1^2)\).

The two magnetic equations simplify and we obtain

\[
\begin{align*}
1 + x_1^2 &= -q(1 - x_1^2), \\
0 &= -q x_1 (x_2 + \beta x_3), \\
-x_1 x_2 \beta - x_1 x_3 \beta + x_1 x_2 &= q x_1 (\beta x_2 - x_3), \\
2x_1 &= |\nabla X|^2 x_1, \\
(1 + \beta^2)x_2 - 2\beta x_3 &= |\nabla X|^2 x_2, \\
(1 + \beta^2)x_3 + 2\beta x_2 &= |\nabla X|^2 x_3.
\end{align*}
\]

This system has solution if and only \( \beta = 0 \). Hence we can state the following.

**Proposition 6.3** Let \( G(0, \beta) \) be a 3-dimensional non-unimodular Lie group with left invariant metric and let \( \{e_1, e_2, e_3\} \) be an orthonormal basis of the Lie algebra \( \mathfrak{g} \) as described before. Then

(i) if \( \beta \neq 0 \) there do not exist unit magnetic vector fields on \( G \);

(ii) if \( \beta = 0 \) then the unit magnetic vector fields on \( G \) are \( x_2 e_2 + x_3 e_3 \), with \( x_2^2 + x_3^2 = 1 \), case when the strength \( q = -1 \).

(II) \( \alpha = 1 \) and \( \beta = 0 \) \([G \text{ is isometric to } \mathbb{H}^2(-4) \times \mathbb{R}] \) \( \mathcal{D} = 0 \)

We have \( u = 4, v = 0, w = 0 \) and \(|\nabla X|^2 = 4x_1^2 + x_2^2\).

The two magnetic equations reduce to

\[
\begin{align*}
-8x_2^2 - 8x_1^2 &= 2q x_2^2, \\
0 &= -2q x_1 x_2, \\
4x_1 &= |\nabla X|^2 x_1, \\
4x_2 &= |\nabla X|^2 x_2, \\
0 &= |\nabla X|^2 x_3.
\end{align*}
\]

The following result holds.

**Proposition 6.4** Let \( G(1, 0) \) be a 3-dimensional non-unimodular Lie group with left invariant metric and let \( \{e_1, e_2, e_3\} \) be an orthonormal basis of the Lie algebra \( \mathfrak{g} \) as described before. Then the unit magnetic vector fields on \( G \) are

(i) \( \pm e_3 \), for arbitrary \( q \)

(ii) \( \pm e_2 \), for \( q = -4 \).

(III) \( \alpha = 1 \) and \( \beta \neq 0 \) \([G \text{ is isometric to the BCV space } \mathbb{M}^3(-4, \beta)] \) \( \mathcal{D} = 0 \)

We have \( u = 3\beta^2 + 4, v = \beta^2, w = -\beta^2 \) and \(|\nabla X|^2 = 2\beta^2 + 4(x_1^2 + x_2^2) + 4x_2x_3\beta \).

The two magnetic equations reduce to

\[
\begin{align*}
-(2x_2 + \beta x_3)x_2(3\beta^2 + 4) - 2x_1^2(3\beta^2 + 4) + x_2x_3x_3^3 &= q(2x_2^2 + \beta x_2x_3), \\
2x_1 x_3(\beta^2 + 2) &= -2q x_1 (\beta x_3 + x_2), \\
2x_1 x_3 \beta^2 &= 0, \\
2x_1(\beta^2 + 2) &= |\nabla X|^2 x_1, \\
x_2(\beta^2 + 2) &= |\nabla X|^2 x_2, \\
x_3 \beta^2 + 4x_2 \beta &= |\nabla X|^2 x_3.
\end{align*}
\]

We obtain the following.
Proposition 6.5 Let $G(1, \beta)$ with $\beta \neq 0$, be a 3-dimensional non-unimodular Lie group with left invariant metric and let $\{e_1, e_2, e_3\}$ be an orthonormal basis of the Lie algebra $\mathfrak{g}$ as described before. Then the unit magnetic vector fields on $G$ are

1. $\pm e_3$, case when the strength $q$ is arbitrary,
2. $\pm \frac{1}{\sqrt{\beta^2 + 1}} (e_2 + \beta e_3)$, case when the strength $q = \frac{4}{\beta^2 + 2} - 2\beta^2 - 6. (q$ never vanishes.)

Before discussing the general situation, we ask, as in the case when $G$ is unimodular, the following question:

When $\pm e_1, \pm e_2$ and $\pm e_3$ are, respectively unit magnetic vector fields in $G(\alpha, \beta)$?

Q1. When $\pm e_1$ is magnetic?

Equation (6.3a) implies $-(1 + \alpha)u + (1 - \alpha)v = 0$, that is equivalent to $1 + 3\alpha^2 + 3\alpha^2\beta^2 = 0$. But this is a contradiction.

A1. The unit vector field $\pm e_1$ is never magnetic.

Q2. When $\pm e_2$ is magnetic?

The two magnetic equations (6.3) and (6.4) imply $q = -u$ and $\beta = 0$.

We have $D = 1 - \alpha^2 \leq 1$.

A2. The unit vector field $\pm e_2$ is magnetic only on $G(\alpha, 0)$ and in this case the strength is $q = -(1 + \alpha)^2$.

Q3. When $\pm e_3$ is magnetic?

A3. The unit vector field $\pm e_3$ is magnetic on $G(\alpha, \beta)$ if and only if

- either $G = G(1, \beta)$ (case when $D = 0$ and we do not have any condition for $q$),
- or $G = G(\alpha, 0)$ with $\alpha \neq 1$ and in this case $D \leq 1$ and the strength is $q = -(1 - \alpha)^2$.

Finally, we analyze the general situation.

We are looking first for solutions $X = x_1e_1 + x_2e_2 + x_3e_3$ with $x_1 \neq 0$.

From (6.4a) we find $|\nabla X|^2 = 2(1 + \alpha^2 + \alpha^2\beta^2)$. Replacing $|\nabla X|^2$ in equations (6.4b) and (6.4c) we obtain

$$\begin{align*}
\begin{cases}
\beta^2 - (1 - \alpha)^2 - \alpha^2\beta^2 & x_2 - 2\beta(1 - \alpha)x_3 = 0, \\
2\beta(1 + \alpha)x_2 + [\beta^2 - (1 + \alpha)^2 - \alpha^2\beta^2]x_3 = 0.
\end{cases}
\end{align*}$$

This is a homogeneous system of two linear equations whose determinant is $D^2$.

So, if $D \neq 0$, we have only the trivial solution, i.e. $x_2 = 0$ and $x_3 = 0$. This implies that $X = \pm e_1$. Bringing the question Q1 to the present moment, we note that this situation cannot occur. Therefore, a necessary condition to have a solution $X$ (with $x_1 \neq 0$) is $D = 0$, that is $\alpha = 1$. This situation is explained in details in Propositions 6.4 and 6.5. Subsequently, we find that the solutions do not exist if $x_1 \neq 0$.

We are looking now for solutions $X = x_2e_2 + x_3e_3$. The only non-trivial equations are obtained from (6.3a), (6.4b) and (6.4c). The last two yield

$$\begin{align*}
\begin{cases}
2\alpha x_2x_3(x_3 - \beta x_2) - \beta(1 - \alpha)x_3 = 0, \\
2\alpha x_2x_3(x_2 + \beta x_3) - \beta(1 + \alpha)x_2 = 0.
\end{cases}
\end{align*}$$

The cases $x_2 = 0$ and $x_3 = 0$ are discussed in Q3 and Q2, respectively. Moreover, the case $\alpha = 0$ is presented in Proposition 6.3. Therefore, from now on we consider $\alpha \neq 0$ and we suppose that both $x_2$ and $x_3$ do not vanish.

An immediate consequence is that $\beta$ cannot vanish.

The equations (6.7) imply

$$\beta(1 + \alpha)x_2^2 - 2\alpha x_2x_3 + \beta(1 - \alpha)x_3^2 = 0.$$
This equation has solutions if and only if \( \mathcal{D} \leq 1 \), namely

\[
x_2 = \frac{\varepsilon t}{\sqrt{1 + t^2}}, \quad x_3 = \frac{\varepsilon}{\sqrt{1 + t^2}}, \quad \text{where} \quad \varepsilon = \pm 1 \quad \text{and} \quad t = \alpha \pm \sqrt{1 - \mathcal{D}} \quad \text{with} \quad \mathcal{D} = \frac{\beta(1 + \alpha)}{\beta^2 + 1}.
\] (6.8)

They are solutions also for (6.7). Finally, the strength \( q \) is obtained from equation (6.3a):

\[
q = -\frac{2(1 + \beta^2)(1 + \alpha^2 + \alpha^2 \beta^2 \pm 2 \sqrt{1 - \mathcal{D}})}{2 + \beta^2}.
\] (6.9)

**Remark 6.1** The conclusion of Proposition 6.5 is a particular case of the previous discussion, that is choosing the non vanishing solution for \( t \) in the case \( \alpha = 1 \).

**Remark 6.2** In the case when \( \mathcal{D} = 1 \), the unit magnetic vector field is

\[
\pm \left( \frac{\sqrt{1 - \alpha}}{2} e_2 + \frac{\sqrt{1 + \alpha}}{2} e_3 \right)
\]

and the strength is \( q = -\frac{2}{(1 - \alpha^2)(2 - \alpha^2)} = -\frac{2(\beta^2 + 1)^2}{\beta^2 + 2} \).

**Theorem 6.1** Let \( G \) be a 3-dimensional non-unimodular Lie group with a left invariant metric and let \( \{e_1, e_2, e_3\} \) be an orthonormal basis of the Lie algebra \( \mathfrak{g} \) satisfying (6.1) with (6.2). Then the unit magnetic vector fields on \( G \) belong to the list given in Table 2.

We conclude this section by noting that, according to [6, Table III] in non-unimodular case, the list of left-invariant harmonic unit vector fields determining harmonic maps is poor, contrary to the list obtained in Table 2. This argument represented a good motivation for us to study unit magnetic vector fields.

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References

1. Dragomir, S., Perrone, D.: Harmonic Vector Fields. Variational Principles and Differential Geometry. Elsevier, Amsterdam (2012)
2. Eells, J., Lemaire, L.: Selected Topics in Harmonic Maps, CBMS Regional Conference Series in Mathematics 50. American Mathematical Society, Providence, RI (1983)
3. Gil-Medrano, O.: Relationship between volume and energy of unit vector fields. Differ. Geom. Appl. 15, 137–152 (2001)
4. Gluck, H., Ziller, W.: On the volume of a unit vector field on the three sphere. Comment. Math. Helv. 61(1), 177–192 (1986)
5. González-Dávila, J.C., Vanhecke, L.: Energy and volume of unit vector fields on three-dimensional Riemannian manifolds. Differ. Geom. Appl. 16, 225–244 (2002)
6. González-Dávila, J.C., Vanhecke, L.: Invariant harmonic unit vector fields on Lie groups. Boll. Un. Mat. Ital. 52–B, 377–403 (2002)
7. Han, S.D., Yim, J.W.: Unit vector fields on spheres which are harmonic maps. Math. Z. 227, 83–92 (1998)
8. Inoguchi, J.: On homogeneous contact 3-manifolds. Bull. Fac. Edu. Utsunomiya Univ. Sect. 2(59), 1–12 (2009)
9. Inoguchi, J., Lee, S.: A Weierstrass representation for minimal surfaces in Sol. Proc. Am. Math. Soc. 136, 2209–2216 (2008)
10. Inoguchi, J., Munteanu, M.I.: Magnetic maps. Int. J. Geom. Methods Mod. Phys. 11(6), 1450058 (2014)
11. Inoguchi, J., Munteanu, M.I.: New examples of magnetic maps involving tangent bundles. Rend. Semin. Mat. Univ. Politec. Torino 73(1–3–4), 101–116 (2015)
12. Inoguchi, J., Munteanu, M.I.: Magnetic vector fields: New examples. Publ. Inst. Math. Beograd. 103(117), 91–102 (2018)
13. Inoguchi, J., Munteanu, M.I.: Magnetic curves on tangent sphere bundles. RACSAM 113(3), 2087–2112 (2019)
14. Inoguchi, J., Naitoh, H.: Grassmann geometry on the 3-dimensional non-unimodular Lie groups. Hokkaido Math. J. 48(2), 385–406 (2019)
15. Ishihara, T.: Harmonic sections of tangent bundles. J. Math. Tokushima Univ. 13, 23–27 (1979)
16. Meeks, W.H., III., Pérez, J.: Constant mean curvature surfaces in metric Lie groups. Contemp. Math. 570, 25–110 (2012)
17. Mihor, J.: Curvatures of left invariant metrics on Lie groups. Adv. Math. 21, 293–329 (1976)
18. Nistor, A.I.: Constant angle surfaces in solvable Lie groups. Kyushu J. Math. 68, 315–332 (2014)
19. Nouhaud, O.: Applications harmoniques d’une variété Riemannienne dans son fibré tangent. C. R. Acad. Sci. Paris 284, 815–818 (1977)
20. Perrone, D.: Homogeneous contact Riemannian three-manifolds. Ill. J. Math. 42(2), 243–256 (1998)
21. Perrone, D.: Stability of the Reeb vector field of $H$-contact manifolds. Math. Z. 263, 125–147 (2009)
22. Tsukada, K., Vanhecke, L.: Invariant minimal unit vector fields on Lie groups. Per. Math. Hung. 40(2), 123–133 (2000)
23. Thurston, W.M.: Three-dimensional Geometry and Topology I. In: Levy, S. (ed.) Princeton Mathematical Series, vol. 35. Princeton University Press, Princeton (1997)
24. Wiegmink, G.: Total bending of vector fields on Riemannian manifolds. Math. Ann. 303(2), 325–344 (1995)
25. Wiegmink, G.: Total bending of vector fields on the sphere $S^3$. Differ. Geom. Appl. 6(3), 219–236 (1996)
26. Wood, C.M.: On the energy of a unit vector field. Geom. Dedicata 64, 319–330 (1997)

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