Gabber’s presentation lemma for finite fields

By Amit Hogadi at Pashan and Girish Kulkarni at Pashan

Abstract. We give a proof of Gabber’s presentation lemma for finite fields. We first prove this lemma in the special case of open subsets of the affine plane using ideas from Poonen’s proof of Bertini’s theorem over finite fields. We then reduce the case of general smooth varieties to this special case.

1. Introduction

A presentation lemma proved by Gabber in [2] (see also [1]) plays a foundational role in $\mathbb{A}^1$-algebraic topology as developed by Morel in [4]. This lemma can be thought of as an algebro-geometric analogue of tubular neighborhood theorem in differential geometry. The current published proof of Gabber’s presentation lemma works only over infinite fields. In a private communication to Morel, Gabber has pointed out that the proof of this theorem also holds for finite fields. Unfortunately, there is no published proof for this case.

The goal of this paper is to prove the following version of Gabber’s presentation lemma over finite fields.

Theorem 1.1. Let $X$ be a smooth variety of dimension $d \geq 1$ over a finite field $F$ and let $Z \subset X$ be a closed subvariety. Let $p \in Z$ be a point. Let $\mathbb{A}^d_F \xrightarrow{\pi} \mathbb{A}^{d-1}_F$ denote the projection onto the first $d-1$ coordinates. Then there exist

(i) an open neighborhood $U \subset X$ of $p$,
(ii) a map $\Phi : U \to \mathbb{A}^d_F$,
(iii) an open neighborhood $V \subset \mathbb{A}^{d-1}_F$ of $\Psi(p)$, where $\Psi : U \to \mathbb{A}^{d-1}_F$ denotes the composition $U \xrightarrow{\Phi} \mathbb{A}^d_F \xrightarrow{\pi} \mathbb{A}^{d-1}_F$

such that the following hold:

(1) $\Phi$ is étale.
(2) $\Psi|_{Z_V} : Z_V \to V$ is finite, where $Z_V := Z \cap \Psi^{-1}(V)$.
(3) $\Phi|_{Z_V} : Z_V \to \mathbb{A}^1_V = \pi^{-1}(V)$ is a closed immersion.

Remark 1.2. Without loss of generality, we may (and will) assume henceforth that $X$ is affine. Moreover, by [1, Section 3.2], we may also assume that $Z$ is a principal divisor defined by $f \in \mathfrak{O}(X)$ and $p$ is a closed point.
Remark 1.3. If one finds $U, \Phi, V, \Psi$ satisfying conditions (1)–(3) of Theorem 1.1, one can also arrange, by shrinking $U$ if necessary, the following additional condition:

(4) $Z_V = \Phi^{-1}(Z_V)$.

To see this, let $\tilde{Z}$ denote the image of $Z_V$ in $\mathbb{A}^1_V$. The morphism $\Phi^{-1}(Z_V) \to \tilde{Z}$ is étale, and it admits a section as $Z_V$ maps isomorphically onto $\tilde{Z}$. Thus $\Phi^{-1}(Z_V)$ is a disjoint union of $Z_V$ and a closed subset $T$ of $\Psi^{-1}(V)$. Replacing $U$ by $U \setminus T$, one sees that the additional condition (4) is satisfied.

The proof of Gabber’s presentation lemma for infinite fields (see [2, Section 3.1] or [1, Section 3.1]) shows that for the maps $\Phi, \Psi$ appearing in the statement of the lemma, suitable generic choices work. The problem in making this proof work over a finite field is very similar to the problem of making Bertini’s theorem work over a finite field. Bertini’s theorem for finite fields was proved by Poonen in [6] using an extremely clever counting argument. Because of the broad similarities of the issues involved, it is natural to try to use Poonen’s argument to prove Gabber’s presentation lemma over finite fields. However, Poonen’s counting argument, in our opinion, is easier to apply in the case of subvarieties of an affine space. Thus, the first step of the proof of Theorem 1.1 is reduction to the case where $X$ is an open subset of $\mathbb{A}^d_F$. This is done in Section 2. Unfortunately, we found even this case too complex to directly apply Poonen’s ideas from [6]. Fortunately, we are able to reduce this complexity by using induction on $d$ to reduce to the case where $d = 2$, i.e. $X$ is an open subset of $\mathbb{A}^2_F$. This is done in Section 3. This induction argument, although short, was one of the most time-taking tasks for us in proving Theorem 1.1. An important ingredient of this induction is a slightly modified version of Noether normalization trick (see Lemma 3.1).

The case of open subsets of $\mathbb{A}^2_F$ is now ideal for using Poonen’s counting argument. Indeed, the handling of points of small degree is very similar to that of [6]. However, we are unable to handle the error term for ‘high degree points’ as is done in [6]. We fix this with a small trick (see Lemma 4.10).

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2. Reduction to open subsets of $\mathbb{A}^d_F$

The goal of this section is to prove Lemma 2.4, which reduces Theorem 1.1 to the case where $X$ is an open subset of $\mathbb{A}^d_F$ and $p \in Z \subset X$ is a closed point with first $d - 1$ coordinates equal to 0.

Notation 2.1. Throughout this paper we work over a fixed finite field $F$. We further fix the following notation.

1. Let $Y$ be a subset of a scheme $X / F$. We let $Y_{\leq r} := \{ x \in Y \mid \deg(x) \leq r \}$ and similarly $Y_{< r} := \{ x \in Y \mid \deg(x) < r \}$ and $Y_{= r} := \{ x \in Y \mid \deg(x) = r \}$. 


(2) For \( f_1, \ldots, f_i \in F[X_1, \ldots, X_n] \) we let \( Z(f_1, \ldots, f_i) \) denote the closed subscheme of \( \mathbb{A}_F^n \) defined by the ideal \((f_1, \ldots, f_i)\).

We first start by recalling the following standard trick (see [5]) used in the proof of Noether’s normalization lemma.

**Lemma 2.2** ([5, p. 2]). Let \( k \) be any field and let \( n \geq 1 \) be any integer. Let \( Z / k \) be a finitely generated affine scheme of dimension at most \( n - 1 \). Let

\[
Z \xrightarrow{(\phi_1, \ldots, \phi_n)} \mathbb{A}_k^n
\]

be a finite map. Let \( Q(T) \in k[T] \) be a non-constant monic polynomial and \( Q = Q(\phi_n) \). Then for \( \ell \gg 0 \), the map

\[
Z \xrightarrow{(\phi_1 - Q^{\ell+1}, \ldots, \phi_{n-1} - Q^\ell)} \mathbb{A}_k^{n-1}
\]

is finite.

**Remark 2.3.** We claim that finiteness of

\[
Z \xrightarrow{(\phi_1, \ldots, \phi_n)} \mathbb{A}_k^n
\]

implies that of

\[
Z \xrightarrow{(\phi_1, \ldots, Q)} \mathbb{A}_k^n.
\]

This is because the later map is a composition of the following two finite maps:

\[
Z \xrightarrow{(\phi_1, \ldots, \phi_n)} \mathbb{A}_k^n \xrightarrow{(Y_1, \ldots, Q(Y_n))} \mathbb{A}_k^n.
\]

One can thus easily reduce the proof of the above general case to the case where \( Q(T) = T \). Unless explicitly mentioned, we will usually assume \( Q(T) = T \) while applying the lemma. As in the proof of Noether normalization, the above lemma is usually applied repeatedly until one gets a map from \( Z \) to \( \mathbb{A}^{\dim(Z)}_k \).

**Lemma 2.4.** Let \( p \in Z \subset X \) be as in Theorem 1.1. Further, assume that \( X \) is affine, \( Z \) is a principal divisor and \( p \) is a closed point (see Remark 1.2). Then there exists a map \( \varphi : X \to \mathbb{A}_F^d \) and an open neighborhood \( W \) of \( \varphi(p) \) such that the following hold:

1. \( \varphi^{-1}(W) \to W \) is étale.
2. \( Z_W := Z \cap \varphi^{-1}(W) \to W \) is a closed immersion.
3. The first \( d - 1 \) coordinates of \( \varphi(p) \) are 0.

In particular, it suffices to prove Theorem 1.1 where \( X \) is an open subset of \( \mathbb{A}_F^d \) and the first \( d - 1 \) coordinates of \( p \) are zero.

**Proof.** We fix the following notation:

- Let \( X = \text{Spec}(A) \).
- Let \( Z = \text{Spec}(A/(f)) \) and let \( \overline{A} := A/(f) \).
- Let \( \mathfrak{m} \subset A \) be the maximal ideal of the closed point \( p \).
- Let \( F(p) \) denote the residue field of \( p \).
Step 1. As $X/F$ is smooth, $\dim F(p)(\mathfrak{m}/\mathfrak{m}^2) = d$. Choose $\{x_1, \ldots, x_{d-1}\} \subset \mathfrak{m}$ such that they span a $(d - 1)$-dimensional $F(p)$-subspace of $\mathfrak{m}/\mathfrak{m}^2$. In this step we claim that there exists $y \in A$ such that the following hold:

1. $y \bmod \mathfrak{m}$ is a primitive element of $F(p)/F$.
2. The set $\{x_1, \ldots, x_{d-1}, h(y)\}$ (modulo $\mathfrak{m}^2$) gives an $F(p)$-basis of $\mathfrak{m}/\mathfrak{m}^2$, where $h$ is the minimal polynomial of $y$ mod $\mathfrak{m}$.
3. The map $(x_1, \ldots, x_{d-1}, y) : X \to \mathbb{A}^d$ is étale at $p$.
4. The map $\eta$ induces an isomorphism on residue fields $F(\eta(p)) \to F(p)$.

Let $w \in \mathfrak{m}$ be an element such that $\{x_1, \ldots, x_{d-1}, w\}$ span $\mathfrak{m}/\mathfrak{m}^2$ as an $F(p)$-vector space. Let $c$ be a primitive element of $F(p)/F$ and let $h$ be its minimal polynomial. Choose $\hat{y} \in A$ such that

$$\hat{y} \equiv c \bmod \mathfrak{m}.$$

Since $c$ is separable over $F$, $h'(c) \neq 0$. Thus $h'(\hat{y}) \notin \mathfrak{m}$ or equivalently $h'(\hat{y})$ is a unit in the ring $A/\mathfrak{m}^2$. Choose $\epsilon \in \mathfrak{m}$ such that

$$\epsilon \equiv \frac{w - h(\hat{y})}{h'(\hat{y})} \bmod \mathfrak{m}^2.$$

Thus the $F(p)$-span of $\{x_1, \ldots, x_{d-1}, h(\hat{y}) + \epsilon h'(\hat{y})\}$ is $\mathfrak{m}/\mathfrak{m}^2$. Let

$$y = \hat{y} + \epsilon.$$

We note that

$$h(y) = h(\hat{y} + \epsilon) \equiv h(\hat{y}) + \epsilon h'(\hat{y}) \bmod \mathfrak{m}^2.$$

Hence $\{x_1, \ldots, x_{d-1}, h(y)\}$ gives an $F(p)$-basis for $\mathfrak{m}/\mathfrak{m}^2$.

Let $\eta$ be the map $(x_1, \ldots, x_{d-1}, y) : X \to \mathbb{A}^d_F$. Since $y \bmod \mathfrak{m}$ is a primitive element of $F(p)$, one observes that $F(\eta(p)) \to F(p)$ is an isomorphism. It remains to show that $\eta$ is étale at $p$. The maximal ideal of $\eta(p)$ in $F[X_1, \ldots, X_d]$ is $\mathfrak{n} = (X_1, \ldots, X_{d-1}, h(X_d))$. As $\{x_1, \ldots, x_{d-1}, h(y)\}$ is an $F(p)$-basis for $\mathfrak{m}/\mathfrak{m}^2$, that $\eta$ is étale at $p$ follows from the surjectivity of

$$\mathfrak{n}/\mathfrak{n}^2 \xrightarrow{\eta^*} \mathfrak{m}/\mathfrak{m}^2.$$

Step 2. Let $U$ be an open neighborhood of $p$ in $X$ such that $\eta|_U$ is étale. Let

$$B = (X \setminus U) \sqcup Z.$$

In this step we modify $x_1, \ldots, x_{d-1}$ to $z_1, \ldots, z_{d-1}$ so that the following hold:

1. The map $\tilde{\eta} = (z_1, \ldots, z_{d-1}, y) : X \to \mathbb{A}^d_F$ is étale on $U$.
2. The set $\{z_1, \ldots, z_{d-1}, h(y)\}$ is an $F(p)$ basis for $\mathfrak{m}/\mathfrak{m}^2$.
3. The map $B \xrightarrow{(z_1, \ldots, z_{d-1})} \mathbb{A}^{d-1}_F$ is finite.

Let $\widetilde{A} := A/I(B)$ and let $\tilde{\mathfrak{m}}$ denote the image of $\mathfrak{m}$ in $\widetilde{A}$. For any element $\alpha \in A$, let $\tilde{\alpha}$ denote its image in $\widetilde{A}$. Choose $y_1, \ldots, y_m \in A$ which generate $A$ as an $F$ algebra. We expand this generating set to include the elements $x_i$. In particular,

$$A = F[x_1, \ldots, x_{d-1}, y_1, \ldots, y_m],$$

$$\widetilde{A} = F[\widetilde{x}_1, \ldots, \widetilde{x}_{d-1}, \widetilde{y}_1, \ldots, \widetilde{y}_m].$$
The image of $y_i$ in $A/m$ satisfies a non-constant monic polynomial, say $f_i$, over $F$. Let

$$y_{i,0} := f_i(y_i) \in m,$$

$$x_{i,0} := x_i,$$

$$A_0 := F[x_{1,0}, \ldots, x_{d-1,0}, y_{1,0}, \ldots, y_{m,0}],$$

$$\tilde{A}_0 := F[\tilde{x}_{1,0}, \ldots, \tilde{x}_{d-1,0}, \tilde{y}_{1,0}, \ldots, \tilde{y}_{m,0}].$$

Clearly, $\tilde{A}$ is finite over $\tilde{A}_0$.

For $0 \leq r \leq m - 1$, we inductively define $A_{r+1}$ and elements $x_{i,r+1}, y_{i,r+1}$ as follows.

By Lemma 2.2, we choose an integer $\ell_r > 1$ such that the following definitions make $\tilde{A}_r$ a finite $\tilde{A}_{r+1}$-algebra. Since any sufficiently large choice of $\ell_r$ works, we assume that $\ell_r$ is a multiple of the char($F$). Let

$$x_{i,r+1} := x_{i,r} - (y_{m-r,r})^{\ell_r}$$

for all $1 \leq i \leq d - 1$,

$$y_{i,r+1} := y_{i,r} - (y_{m-r,r})^{\ell_r^{r-i}}$$

for all $1 \leq i \leq m - r - 1$,

$$A_{r+1} := F[x_{1,r+1}, \ldots, x_{d-1,r+1}, y_{1,r+1}, \ldots, y_{m-r-1,r+1}],$$

$$\tilde{A}_{r+1} := F[\tilde{x}_{1,r+1}, \ldots, \tilde{x}_{d-1,r+1}, \tilde{y}_{1,r+1}, \ldots, \tilde{y}_{m-r-1,r+1}].$$

Since $x_{i,0}$ and $y_{i,0}$ belong to $m$, one can inductively observe

$$y_{i,r} \in m, \quad x_{i,r} \in m, \quad x_{i,r+1} \equiv x_{i,r} \mod m^2.$$

For ease of notation, let us rename

$$z_i := x_{i,m}.$$

Note that for all $i \leq d - 1$, $z_i - x_i$ is of the form $\beta_i^k$ for $\beta_i \in m$ and an integer $k_i$ divisible by char($F$). This ensures requirements (1) and (2) of Step 2. Recall that $m$ is an integer such that $\{y_1, \ldots, y_m\}$ are the chosen generators of $A$ as an $F$ algebra. It is now straightforward to see that $\{z_1, \ldots, z_{d-1}\} \subset m$ such that $\tilde{A}$ is a finite algebra over $\tilde{A}_m = F[\tilde{z}_1, \ldots, \tilde{z}_{d-1}]$.

**Step 3.** In this step we will further modify $y$ while ensuring that (1) and (2) of the above step continue to hold. Since the map $\tilde{\eta}_B : B \to A^{d-1}_F$ is finite, there exists finitely many points $\{p, p_1, \ldots, p_t\} \subset B$ which are contained in the zero locus $Z(z_1, \ldots, z_{d-1})$. Let $m_i$ be the maximal ideal corresponding to $p_i$ for $1 \leq i \leq t$. By the Chinese Remainder Theorem, choose $\delta \in A$ such that

$$\delta \equiv 0 \mod m,$$

$$\delta^{\text{char}(F)} \equiv -y \mod m_i$$

for all $1 \leq i \leq t$ (note that $A/m_i$ is perfect). Let

$$z = y + \delta^{\text{char}(F)}.$$

For later use, we note that

$$z \equiv 0 \mod m_i$$

for all $1 \leq i \leq t$.

By using the fact that $z - y$ is the char($F$)-th power of an element of $m$, it is straightforward to deduce the following from (1) and (2) of the above step.

1. The map $\varphi : X \to A^{d}_F$ defined by $(z_1, \ldots, z_{d-1}, z)$ is étale at $p$.
2. $\{z_1, \ldots, z_{d-1}, h(z)\}$ is an $F(p)$-basis of $m/m^2$.
3. $z \mod m$ is the primitive element $c$ of $F(p)/F$. 
We further claim that we have the following equality of ideals of \( \widetilde{A} = A/(I(B)) \):
\[
\sqrt{(\widetilde{z}_1, \ldots, \widetilde{z}_{d-1}, h(\widetilde{z}))} = \widetilde{m}.
\]
To see the claim, we first observe
\[
h(z) \in \widetilde{m}, \quad h(z) \notin \widetilde{m}_i \quad \text{for all } 1 \leq i \leq t.
\]
The first containment follows as \( h \) is the irreducible polynomial of \( z \mod m \). Moreover, since \( h(0) \neq 0 \), the second statement follows from the fact that \( z \equiv 0 \mod m_i \).

As \( \{\widetilde{m}, \widetilde{m}_1, \ldots, \widetilde{m}_t\} \) are the only prime ideals of \( \widetilde{A} \) containing the ideal \( (\widetilde{z}_1, \ldots, \widetilde{z}_{d-1}) \), and \( h(z) \notin \widetilde{m}_i \) for all \( i \), we conclude that \( \widetilde{m} \) is the unique prime ideal of \( \widetilde{A} \) containing the ideal \( (\widetilde{z}_1, \ldots, \widetilde{z}_{d-1}, h(\widetilde{z})) \). Therefore
\[
\sqrt{(\widetilde{z}_1, \ldots, \widetilde{z}_{d-1}, h(\widetilde{z}))} = \widetilde{m}.
\]

**Step 4.** We claim that in fact
\[
(\widetilde{z}_1, \ldots, \widetilde{z}_{d-1}, h(\widetilde{z})) = \widetilde{m}.
\]
Note that both are \( \widetilde{m} \)-primary ideals and hence it is enough to show the equality in the localizaton \( \widetilde{A}_{\widetilde{m}} \). But the equality holds in this local ring by Nakayama’s lemma since it holds modulo \( \widetilde{m}^2 \) as \( \{z_1, \ldots, z_{d-1}, h(z)\} \mod \widetilde{m}^2 \) gives a basis of \( \widetilde{m}/\widetilde{m}^2 \) (see condition (2) of the above step).

**Step 5.** Recall that \( \varphi : X \to \mathbb{A}^d_F \) is the map defined by \( (z_1, \ldots, z_{d-1}, z) \). We claim that \( p \) is the unique point in \( \varphi^{-1}(\varphi(p)) \cap Z \). In fact, we have that \( p \) is the unique point of \( \varphi^{-1}(\varphi(p)) \cap B \). This is a direct consequence of Step 3, since the ideal defining \( \varphi^{-1}(\varphi(p)) \cap B \) in \( B = \text{Spec}(\widetilde{A}) \) is equal to \( (\widetilde{z}_1, \ldots, \widetilde{z}_{d-1}, h(\widetilde{z})) = \widetilde{m} \). Indeed, what we have observed is that the scheme \( \varphi^{-1}(\varphi(p)) \cap B \) is reduced and has \( p \) as the only underlying point. Thus the same holds for \( \varphi^{-1}(\varphi(p)) \cap Z \). If \( n \) denotes the maximal ideal in the coordinate ring of \( \mathbb{A}^d_F \) of the point \( \varphi(p) \), then \( n A = \widetilde{m} \). Recall that \( \widetilde{A} := A/(f) \) and \( Z = \text{Spec}(\widetilde{A}) \).

**Step 6.** In this step we prove the rest of the theorem using a trick used in the proof of [1, Theorem 3.5.1]. In fact, the argument in this step has been taken directly from [1, Theorem 3.5.1]. Note that the map \( \varphi : Z \to \mathbb{A}^d_F \) is finite. Let \( \pi \) be the maximal ideal of \( \varphi(p) \) in \( F[X_1, \ldots, X_d] \). By Step 5, \( \pi \widetilde{A} = \overline{\pi} \widetilde{A} \) and the map
\[
\frac{F[X_1, \ldots, X_d]}{\pi} \xrightarrow{\varphi^*} \frac{\widetilde{A}}{\overline{\pi} \widetilde{A}}
\]
is an isomorphism, in particular surjective. By Nakayama’s lemma, there exists an element \( g \in F[X_1, \ldots, X_d] \setminus \pi \) such that the map
\[
F[X_1, \ldots, X_d]_g \to \widetilde{A}_g
\]
is surjective. In particular, if \( V = \mathbb{A}^d_F \setminus Z(g) \), then
\[
Z \cap \varphi^{-1}(V) \to V
\]
is a closed immersion. Note that \( V \) is an open neighborhood of \( \varphi(p) \).
Let $D \subset X$ be the maximal closed subset on which the map $\varphi$ is not étale. Clearly $p \notin D$. Also, since $D$ is a subset of $B$ (see Step 2) and $p$ is the only point in $\varphi^{-1}(\varphi(p)) \cap B$, we must have $\varphi(p) \notin \varphi(D)$. However, the map $\varphi|_B$ is finite, we have that $\varphi(D)$ is a closed subset of $\mathbb{A}_F^d$. Let

$$W := (\mathbb{A}_F^d \setminus \varphi(D)) \cap (\mathbb{A}_F^d \setminus Z(g)).$$

Thus $\varphi^{-1}(W) \to W$ is étale. Moreover, $\varphi^{-1}(W)$ is an open neighborhood of $p$. It is now clear that $\varphi$ and $W$ satisfy conditions (1) and (2) of the lemma. Condition (3) is also immediate since the map $\varphi$ is defined by $(z_1, \ldots, z_{d-1}, z)$ and $z_i$ vanish on $p$ for $1 \leq i \leq d - 1$. □

### 3. Reduction to open subsets of $\mathbb{A}_F^2$

The previous section reduces the general case of Theorem 1.1 to the case where $X$ is an open subset of $\mathbb{A}_F^d$. The goal of this section is to further reduce to the case where $d = 2$ (see Lemma 3.8). This reduction, which is an induction argument, is an important step in the proof of Theorem 1.1. One of the ingredients required for this induction argument to work is the following variation of the standard Noether normalization trick (see Lemma 2.2).

**Lemma 3.1.** Let $n \geq 2$ be any integer, let $k$ be any field and let $Z/k$ be an affine variety of dimension $n - 1$. Let

$$Z \xrightarrow{\phi_1, \ldots, \phi_n} \mathbb{A}_k^n$$

be a finite map. Let $Q \in k[\phi_n]$ be a non-constant monic polynomial. Then for an integer $\ell \gg 0$, the map

$$Z \xrightarrow{\phi_1 - Q_1^\ell \cdots \phi_{n-1} - Q_{n-1}^\ell} \mathbb{A}_k^{n-1}$$

is finite, where the polynomials $Q_i$ are inductively defined by

$$Q_{n-1} := Q,$$

$$Q_i := \phi_{i+1} - Q_{i+1}^\ell \quad \text{for all } 1 \leq i \leq n - 2.$$

**Proof.** The proof is similar to that of Lemma 2.2 (see [5, p. 2]) and hence we only give a sketch. Since $\dim(Z) = n - 1$, $\phi_1, \phi_2, \ldots, \phi_n$ cannot be algebraically independent. Thus there exists a non-zero polynomial $f \in k[Y_1, \ldots, Y_n]$ such that $f(\phi_1, \phi_2, \ldots, \phi_n) = 0$. Let $\ell$ be any integer greater than $n \deg(f)$, where $\deg(f)$ is the total degree of $f$. Let $\widetilde{Q} \in k[Y_n]$ be a polynomial such that $Q = \widetilde{Q}(\phi_n)$. Inductively define $\widetilde{Q}_i$ for $1 \leq i \leq n - 1$ as follows:

$$\widetilde{Q}_{n-1} := \widetilde{Q},$$

$$\widetilde{Q}_i := Y_{i+1} - \widetilde{Q}_{i+1}^\ell \quad \text{for all } 1 \leq i \leq n - 2.$$

Notice that the polynomials $\widetilde{Q}_i$ are defined such that

$$\widetilde{Q}_i(\phi_1, \ldots, \phi_n) = Q_i.$$

Moreover, we note that if $d = Y_n$-degree of $\widetilde{Q}$, then each $\widetilde{Q}_i$ is monic in $Y_n$ of degree $\ell^{n-i-1} d$. Consider the elements $Z_1, \ldots, Z_{n-1} \in k[Y_1, \ldots, Y_n]$ defined as follows:

$$Z_i := Y_i - \widetilde{Q}_i^\ell \quad \text{for all } 1 \leq i \leq n - 1.$$
We leave it to the reader to check that 
\[ k[Z_1, \ldots, Z_{n-1}, Y_n] = k[Y_1, \ldots, Y_n]. \]
For future reference, we note that the map 
\[ \eta : \mathbb{A}^n_F \rightarrow \mathbb{A}^n_F \]
is an automorphism. It is enough to show that the polynomial \( f \), expressed in the variables \( Z_1, \ldots, Z_{n-1}, Y_n \) is monic in \( Y_n \). Let us write \( f \) as 
\[ f = \sum_{I=(i_1, \ldots, i_n)} \alpha_I \cdot m_I, \]
where \( m_I \) are monomials in \( Y_1, \ldots, Y_n \) and \( \alpha_I \in k \). We leave it to the reader to verify that when expressed in new coordinates \( Z_1, \ldots, Z_{n-1}, Y_n \), each monomial \( m_I \) becomes a polynomial which is monic in \( Y_n \) of degree equal to \( i_n + \sum_{k=1}^{n-1} i_k \cdot \ell^{n-k} \cdot d \). Since \( \ell > n \deg(f) \), one can show that these \( Y_n \)-degrees are distinct. Thus in the coordinates \( Z_1, \ldots, Z_{n-1}, Y_n \), \( f \) remains monic in \( Y_n \).

**Notation 3.2.** Let \( d \geq 2 \) be an integer and let \( f, g \in F[X_1, \ldots, X_d] \) be nonzero polynomials with no common irreducible factors (see Remark 3.5). Let \( X := \mathbb{A}^d_F \setminus \mathbb{Z}(g) \) and let \( Z := \mathbb{Z}(f) \cap X \). Let \( p \in Z \) be a closed point (see Remark 1.2) whose first \( d-1 \) coordinates are 0.

Recall that by Lemma 2.4 it is enough to prove Theorem 1.1 for \((X, Z, p)\) as above. In order to prove this, we have to first come up with a map from \( \Phi : X \rightarrow \mathbb{A}^d_F \). Indeed, we will look for maps \( \Phi \) which are defined on the whole of \( \mathbb{A}^d_F \). In other words, we will look for suitable polynomials \( \{\phi_1, \ldots, \phi_d\} \subset F[X_1, \ldots, X_d] \). The goal of the following definition is to list necessary conditions on these polynomials which will ensure (see Lemma 3.4) that the resulting map \( \Phi \) is as desired in Theorem 1.1.

**Definition 3.3.** Let \( f, g, X, Z, p \) be as in Notation 3.2. For 
\[ \{\phi_1, \ldots, \phi_d\} \subset F[X_1, \ldots, X_d], \]
let 
\[ (i) \quad \Phi : \mathbb{A}^d_F \rightarrow \mathbb{A}^d_F, \]
\[ (ii) \quad \Psi : \mathbb{A}^{d-1}_F \rightarrow \mathbb{A}^{d-1}_F. \]
We say that \( \{\phi_1, \ldots, \phi_d\} \) presents \((X, Z(f), p)\) if 
\[ (1) \quad \Psi|_{Z(f)} \text{ is finite and } \Psi(p) = (0, \ldots, 0), \]
\[ (2) \quad \Psi^{-1}(\psi(p) \cap Z(f)) \subset Z, \]
\[ (3) \quad \Phi \text{ is étale at } S := \psi^{-1}(\psi(p) \cap Z), \]
\[ (4) \quad \Phi \text{ is radicial at } S. \]

Recall that \( \Phi \) is said to be radicial [7, Tag 01S2] if \( \Phi|_S \) is injective and for all \( x \in S \) the residue field extension \( F(x)/F(\Phi(x)) \) is trivial.

The following lemma shows that in order to prove Theorem 1.1 for \( X, Z, p \) as in Notation 3.2, it is enough to find \( \phi_1, \ldots, \phi_d \) which presents \((X, Z(f), p)\).
Lemma 3.4. Let \( X, Z, p \) be as above. Assume that there exists \( \{\phi_1, \ldots, \phi_d\} \) which presents \( (X, Z(f), p) \) and let \( \Phi, \Psi \) be as in Definition 3.3. Then there exist open neighborhoods \( V \subset \mathbb{A}^{d-1}_F \) of \( \psi(p) \) and \( U \subset X \) of \( p \) such that \( \Phi|_U, \Psi|_U, U, V \) satisfy conditions (1)–(3) of Theorem 1.1. Moreover, \( \Psi^{-1}(V) \cap Z(f) \subset U \).

Proof. The argument here is directly taken from [1, Theorem 3.5.1]. We construct an open neighborhood \( V \) of \( \psi(p) \) in \( \mathbb{A}^{d-1}_F \) such that if \( Z \subset V = \psi^{-1}(V) \cap Z(f) \), then

(i) \( Z \subset Z \),

(ii) \( \Phi \) is étale at all points in \( Z \),

(iii) \( \Phi|_Z : Z \rightarrow \mathbb{A}^1 \) is closed immersion.

Let \( B \) be the smallest closed subset of \( Z(f) \) containing all points of \( Z(f) \) at which \( \Phi \) is not étale and also containing \( Z(f) \setminus X \). Since \( \Psi|_{Z(f)} \) is a finite map, \( \Psi(B) \) is closed in \( \mathbb{A}^{d-1}_F \). Moreover, because of conditions (2) and (3) of Definition 3.3, we have \( \Psi(p) \notin \Psi(B) \). Thus, we can choose affine open subset \( W \subset \mathbb{A}^{d-1}_F \) such that

\[
\Psi(p) \in W \subset \mathbb{A}^{d-1}_F \setminus \Psi(B).
\]

Let \( Z = Z \cap \Psi^{-1}(W) \). We have following commutative diagram of affine schemes and consequently their coordinate rings:

\[
\begin{array}{ccc}
Z_W & \xrightarrow{\Phi} & \mathbb{A}^1_W \\
\downarrow{\psi} & & \downarrow{\pi} \\
W & \xrightarrow{\Phi^*} & F[Z_W] \\
\end{array}
\]

\[
\begin{array}{ccc}
F[\mathbb{A}^1_W] & \xrightarrow{\Phi^*} & F[W] \\
\downarrow{\psi^*} & & \downarrow{\pi} \\
F[Z_W] & \xrightarrow{\Phi^*} & F[W].
\end{array}
\]

Let \( \Psi(p) = q \) and let \( m_q \) be the maximal ideal in \( F[W] \) corresponding to \( q \). Thus the ideal corresponding to \( S = \Psi^{-1}(q) \cap Z \) in \( F[Z_W] \) is \( m_q \cdot F[Z_W] \). Since \( \Phi \) is radicial as well as étale at \( S \), it follows that \( \Phi|_{S} : S \hookrightarrow \mathbb{A}^1_W \) is a closed immersion. Thus the map on the coordinate rings

\[
F[\mathbb{A}^1_W] \twoheadrightarrow F[Z_W] \otimes_{m_q F[Z_W]} F[W]
\]

is surjective. The surjectivity of the above map is equivalent to

\[
C \otimes_{F[W]} F[W]/m_q F[Z_W] = 0,
\]

where

\[
C := \text{Coker}(F[\mathbb{A}^1_W] \rightarrow F[Z_W]).
\]

But \( C \) is a finite \( F[W] \) module. Hence by Nakayama’s lemma \( C_{m_q} = 0 \). Thus there exists \( h \in F[W]/m_q \) such that \( C_h = 0 \) or equivalently

\[
F[\mathbb{A}^1_W]_h \twoheadrightarrow F[Z_W]_h
\]

is surjective. Let \( V := W \setminus Z(h) \). The coordinate ring of \( Z_V := \Psi^{-1}(V) \cap Z(f) \) is \( F[Z_W]_h \) and that of \( \pi^{-1}V \) is \( F[\mathbb{A}^1_W]_h \). Thus the surjectivity of the above map implies that

\[
Z_V \hookrightarrow \mathbb{A}^1_V
\]

is a closed immersion as required.
Let $U \subset X$ be the maximal open subset containing points at which $\Phi$ is étale. To finish the proof, we need to show that $U, V, \Phi|_U, \Psi_U$ satisfy conditions (1)-(3) of Theorem 1.1. Condition (1) is clearly satisfied by the definition of $U$. To see (2), note that $\Psi|_{Z(f)}$ is finite, and hence, as $Z_V = \Psi^{-1}(V) \cap Z(f)$, $\Psi|_{Z_V} : Z_V \to V$ is finite. Condition (3) is precisely condition (iii) mentioned at the beginning of the proof. By the construction of $W$, subsequently $V$, it follows that $\Psi^{-1}(V) \cap Z(f) \subset U$. 

**Remark 3.5.** Since our main goal is to prove Theorem 1.1 for $(X, Z, p)$, we may change $Z(f)$ as long as it does not change $Z$. If $f$ and $g$ have common irreducible factors, dividing $f$ by the g.c.d. of $f$ and $g$ does not change $Z(f) \setminus Z(g)$. This justifies our assumption in Definition 3.2 that $f$ and $g$ have no common irreducible factors.

The following lemma is proved using a simple coordinate change argument. It will be used in the proof of Lemma 3.8, which is the main result of this section.

**Lemma 3.6.** Let $(\phi_1, \ldots, \phi_d)$ present $(X, Z(f), p)$ as in Lemma 3.4. Then there exist $(\widetilde{\phi}_1, \ldots, \widetilde{\phi}_d)$ which present $(X, Z(f), p)$ such that there exists an open subset $V \subset \mathbb{A}^{d-1}_F$ satisfying the conclusion of Lemma 3.4 for $(\widetilde{\phi}_1, \ldots, \widetilde{\phi}_d)$ and which satisfies the following additional condition:

$$\dim(Z(\widetilde{\phi}_1, \ldots, \widetilde{\phi}_{d-2}) \cap \Psi^{-1}_{Z(f)}(\mathbb{A}^{d-1}_F \setminus V)) = 0.$$ 

**Proof.** We note that if $d = 2$, by convention,

$$Z(\widetilde{\phi}_1, \ldots, \widetilde{\phi}_{d-2}) \cap \Psi^{-1}_{Z(f)}(\mathbb{A}^{1}_F \setminus V) = \Psi^{-1}_{Z(f)}(\mathbb{A}^{1}_F \setminus V),$$

which is of zero dimension since $V$ is non-empty. Thus we may assume $d \geq 3$. For an integer $\ell$, consider the automorphism $\rho : \mathbb{A}^{d-1}_F \to \mathbb{A}^{d-1}_F$ induced by

$$(X_1, \ldots, X_{d-1}) \mapsto (X_1 - X^\ell_{d-1}, X_2 - X^\ell_{d-1} - 2, \ldots, X_{d-2} - X^\ell_{d-1}, X_{d-1}).$$

We choose $\ell \gg 0$ such that by Lemma 2.2, $(X_1, \ldots, X_{d-2})|_{V_\rho(\mathbb{A}^{d-1}_F \setminus V)}$ is a finite map. Let

$$\widetilde{\phi}_i := \phi_i - \phi^\ell_{d-1} \quad \text{for } i \leq d - 2,$$

$$\widetilde{\phi}_i := \phi_i \quad \text{for } i = d - 1, d.$$

It is then straightforward to check that $(\widetilde{\phi}_1, \ldots, \widetilde{\phi}_d)$ presents $(X, Z(f), p)$ (since it is obtained by a coordinate change from the original polynomials $\phi_i$) and moreover

$$\dim(Z(\widetilde{\phi}_1, \ldots, \widetilde{\phi}_{d-2}) \cap \Psi^{-1}_{Z(f)}(\mathbb{A}^{d-1}_F \setminus \rho(V))) = 0. \quad \Box$$

**Lemma 3.7.** Let $d \geq 3$, and let $f, g \in F[X_1, \ldots, X_d]$ be two non-zero polynomials with no common factors. Let $p$ be a closed point of $\mathbb{A}^d_F$ such that $X_i(p) = 0$ for all $i \leq d - 1$. Then there exists a coordinate change of $F[X_1, \ldots, X_d]$, i.e., elements $Y_i \in F[X_1, \ldots, X_d]$ with

$$F[X_1, \ldots, X_d] = F[Y_1, \ldots, Y_d]$$

such that $f(0, Y_2, \ldots, Y_d)$ and $g(0, Y_2, \ldots, Y_d)$ are nonzero polynomials with no common irreducible factors and $Y_i(p) = 0$ for all $i \leq d - 1$. 

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Proof. The condition that \( f(0, Y_2, \ldots, Y_d) \) and \( g(0, Y_2, \ldots, Y_d) \) are nonzero polynomials with no common irreducible factors is equivalent to the condition that no irreducible component of \( Z(f) \cap Z(g) \) is contained in \( Z(Y_1) \).

By the Noether normalization trick (Lemma 2.2), we may assume, by a suitable coordinate change, that the projection

\[
(X_2, \ldots, X_d) : Z(f) \cap Z(g) \xrightarrow{\eta} \mathbb{A}^{d-1}_F
\]

is finite. Note that since \( d \geq 3 \), the image of every irreducible component of \( Z(f) \cap Z(g) \) under \( \eta \) is of dimension at least one. Thus we may choose closed points \( z_1, \ldots, z_\tau \), one in each irreducible component of \( Z(f) \cap Z(g) \) such that \( \eta(z_i) \) are pairwise distinct and also different from \( \eta(p) \). For every closed point \( x \) of \( \mathbb{A}^{d-1}_F \), either \( X_1 \) or \( X_1 + 1 \) is non-vanishing on \( x \). Thus for each \( z_i \), we choose \( \epsilon_i = 0 \) or \( 1 \) such that \( X_1 + \epsilon_i \) does not vanish on \( z_i \). By the Chinese Remainder Theorem, there exists a polynomial \( \gamma \in F[X_2, \ldots, X_d] \) such that

\[
\gamma(\eta(z_i)) = \epsilon_i \quad \text{and} \quad \gamma(p) = 0.
\]

It is now straightforward to check that

\[
Y_1 := X_1 - \gamma \quad \text{and} \quad Y_i := X_i \quad \text{for all} \quad 2 \leq i \leq d
\]

satisfies our requirement. \( \square \)

**Lemma 3.8** (Reduction to \( d = 2 \)). Assume that for \( d = 2 \) and every \( f, g, X, Z, p \) as in Notation 3.2, there exists \( \phi_1, \phi_2 \subset F[X_1, X_2] \) which presents \( (X, Z(f), p) \). Then the same holds for every \( d \geq 2 \).

**Proof.** We prove this lemma by induction on \( d \). Assume \( d \geq 3 \).

**Step 0.** As before, we let \( F[X_1, \ldots, X_d] \) be the coordinate ring of \( \mathbb{A}^d_F \). Let

\[
\overline{f}(X_2, \ldots, X_d) := f(0, X_2, \ldots, X_d), \quad \overline{g}(X_2, \ldots, X_d) := g(0, X_2, \ldots, X_d).
\]

By Lemma 3.7, we may assume that then \( \overline{f} \) and \( \overline{g} \) are non-zero and have no common factors. We let

- \( \overline{X} := X \cap Z(X_1) \),
- \( \overline{Z} := Z \cap \overline{X} \).

Note that \( p \in \overline{Z} \) and

\[
\overline{X} = Z(X_1) \setminus Z(g),
\]

where \( Z(X_1) \cong \mathbb{A}^{d-1}_F \) with coordinate ring \( F[X_2, \ldots, X_d] \). By induction, there exists

\[
\{\overline{\phi}_2, \ldots, \overline{\phi}_d \} \subset F[X_2, \ldots, X_d]
\]

which presents \( (\overline{X}, Z(\overline{f}), p) \). Let

\[
\overline{\Phi} := (\overline{\phi}_2, \ldots, \overline{\phi}_d) \quad \text{and} \quad \overline{\Psi} := (\overline{\phi}_2, \ldots, \overline{\phi}_{d-1}).
\]

By Lemma 3.4, there exist neighborhoods \( \overline{V} \subset \mathbb{A}^{d-1}_F \) and \( \overline{U} \subset \overline{X} \) of \( \overline{\Psi}(p) \) and \( p \) respectively such that if

\[
\overline{Z}_{\overline{V}} := \overline{Z} \cap \overline{\Psi}^{-1}(\overline{V}),
\]
then the following conditions of Theorem 1.1 are satisfied:

1. \( \overline{\Phi}|_{\mathcal{U}} \) is étale.
2. \( \overline{\Psi}|_{\mathcal{Z}} : Z_{\mathcal{V}} \to \mathcal{V} \) is finite.
3. \( \overline{\Phi}|_{\mathcal{Z}} : Z_{\mathcal{V}} \to \mathbb{A}^1_{\mathcal{V}} \) is a closed immersion.

Further, by Lemma 3.6, we also assume (without loss of generality) that if \( E \) is the closed subset of \( Z(\tilde{f}) \) defined by

\[
E := Z(\tilde{f}) \setminus \overline{\Psi}^{-1}(\mathcal{V}),
\]

then

4. \( \dim(E \cap Z(\tilde{\phi}_2, \ldots, \tilde{\phi}_{d-2})) = 0. \)

Note that (4) is vacuously satisfied unless \( d \geq 4 \). Indeed, for \( d = 3, \mathbb{A}^{d-2} \setminus \mathcal{V} \) is a finite set, and since \( \overline{\Psi}|_{Z(\tilde{f})} : Z(\tilde{f}) \to \mathbb{A}^{d-2} \) is finite, \( E \) is thus a finite set.

**Step 1.** Since the map

\[
Z(\tilde{f}) \xrightarrow{\tilde{\phi}_2, \ldots, \tilde{\phi}_{d-1}} \mathbb{A}^{d-2}_{/\mathcal{F}}
\]

is finite (see Definition 3.3 (1)), for \( 2 \leq i \leq d \), the image of \( X_i \) in \( F[X_2, \ldots, X_d]/(\tilde{f}) \) satisfies a monic polynomial

\[
P_i(T) := T^{m_i} + a_{m_i-1,i}T^{m_i-1} + \cdots + a_{0,i},
\]

where each \( a_{i,j} \in F[\tilde{\phi}_2, \ldots, \tilde{\phi}_{d-1}] \). So \( P_i(X_i) \) is zero in \( F[X_2, \ldots, X_d]/(\tilde{f}) \). Note that each \( \tilde{\phi}_i \) is an element of \( F[X_2, \ldots, X_d] \). Thus we have a map of algebras

\[
F[\tilde{\phi}_2, \ldots, \tilde{\phi}_{d-1}][T] \rightarrow F[X_1, \ldots, X_d][T]/(f).
\]

We let \( \tilde{P}_i(T) \) be the image of the polynomial \( P_i(T) \) under this map. Since \( P_i(X_i) \) is zero in \( F[X_2, \ldots, X_d]/(\tilde{f}) \), it follows that \( \tilde{P}_i(X_i) \) maps to zero via the map

\[
F[X_1, \ldots, X_d]/(f) \xrightarrow{X_i \mapsto 0} F[X_2, \ldots, X_d]/(f).
\]

Therefore

\[
\tilde{P}_i(X_i) = X_1g_i
\]

for some \( g_i \in F[X_1, \ldots, X_d]/(f) \). We claim that the map

\[
Z(\tilde{f}) \xrightarrow{\tilde{\phi}_2, \ldots, \tilde{\phi}_d, X_1g_2, \ldots, X_1g_d} \mathbb{A}^{2d-1}_{/\mathcal{F}}
\]

is finite. This is clear because for \( i \geq 2 \), each \( X_i \) satisfies the monic polynomial \( \tilde{P}_i(T) - X_1g_i \) with coefficients which are polynomial expressions in the functions defining the above map. Applying Lemma 2.2 repeatedly to this map (see Remark 2.3), we get

\[
\phi_2, \ldots, \phi_d \in F[X_1, \ldots, X_d]
\]

such that

\[
\phi_i \equiv \tilde{\phi}_i \mod X_1
\]

and the map \( (\phi_2, \ldots, \phi_d)|_{Z(\tilde{f})} \) is finite.
Step 2. Consider the maps
\[ \widehat{\Phi} : \mathbb{A}^d_F \to \mathbb{A}^d_F, \quad \widehat{\Psi} : \mathbb{A}^d_F \to \mathbb{A}^{d-1}_F. \]

Note that for all points \( x \in Z(X_1) \), \( \Phi \) is étale at \( x \) if and only if \( Z(X_1) \) is étale at \( x \). Let \( E \) be the closed subset of \( Z(f) \subset Z(f) \) defined in Step 0. We have the following:

1. \( \Phi|_{Z(f)} \) is finite. In fact, the map \( (\phi_2, \ldots, \phi_d)|_{Z(f)} \) is finite.
2. \( \Psi(p) \neq \Psi(E) \) (this follows from the definition of \( E \)).
3. \( \Phi \) restricted to \( Z(f) \setminus E \) is a locally closed immersion.
4. \( \Phi \) is étale at all points in \( Z(f) \setminus E \).

By condition (4) of Step 0, we have that\(^1\)
\[ E \cap Z(\phi_2, \ldots, \phi_{d-2}) = E \cap Z(\phi_2, \ldots, \phi_{d-2}) \]
is finite. Let \( Q \) be any non-constant polynomial expression in \( \phi_d \) which vanishes on the finite set
\[ (E \cap Z(\phi_2, \ldots, \phi_{d-2})) \cup \{p\}. \]
Let \( \ell \) be a large enough integer which is divisible by \( \text{char}(F) \). Let \( \phi_1 = X_1 \) and as in Lemma 3.1, let \( Q_{d-1} := Q \) and
\[ Q_i := \phi_i + 1 - Q_i^\ell \quad \text{for all } i \leq d - 2. \]
Let
\[ \Phi := (\phi_1 - Q_1^\ell, \ldots, \phi_{d-1} - Q_{d-1}^\ell : \phi_d) : \mathbb{A}^d_F \to \mathbb{A}^d_F \]
and
\[ \Psi := (\phi_1 - Q_1^\ell, \ldots, \phi_{d-1} - Q_{d-1}^\ell) : \mathbb{A}^d_F \to \mathbb{A}^{d-1}_F. \]

By Lemma 3.1, \( \Psi|_{Z(f)} \) is finite. We let \( S \) be the finite set of points in \( \Psi^{-1}(\Psi(p) \cap Z(f)) \). To finish the proof, it suffices to verify conditions (2)–(4) of Definition 3.3. We first note that \( S \subset Z(\phi_1, \ldots, \phi_{d-2}) \). This is because if \( x \in S \), then by the definition of \( S \),
\[ \phi_{i+1} - Q_{i+1}^\ell(x) = Q_i(x) = 0 \quad \text{for all } i \leq d - 2. \]
Thus
\[ \phi_i - Q_i^\ell(x) = \phi_i(x) = 0 \quad \text{for all } i \leq d - 2. \]

We show that \( S \) is disjoint from \( E \). First note that \( S \subset Z(\phi_1) = Z(X_1) \). Also \( \Psi(p) = 0 \) since \( Q(p) = 0 \) and \( \phi_i(p) = 0 \) for \( 1 \leq i \leq d - 1 \). Let \( x \in S \cap E \) if possible. Hence \( x \) is necessarily in \( E \cap Z(\phi_2, \ldots, \phi_{d-2}) \) by the above argument. In particular, we have \( \phi_{d-2}(x) = 0 \). Now we claim that
\[ \phi_{d-1}(x) = 0. \]
Since \( \Psi(x) = 0 \), we have \( (\phi_{d-2} - Q_{d-2}^\ell)(x) = 0 \). But as \( \phi_{d-2}(x) = 0 \), we conclude that
\[ Q_{d-2}(x) = 0. \]

\(^1\) Here by convention \( Z(\phi_2, \ldots, \phi_{d-3}) \) is the whole of \( \mathbb{A}^d_F \) if \( d \leq 3 \).
Thus

$$\phi_{d-1}(x) = (Q_{d-2} - Q_1)(x) = 0.$$  

This proves the claim. Consequently, \(x \in Z(\phi_2, \ldots, \phi_{d-1})\). By the definition of \(E\), \(x \in E\) implies \(\bar{\Psi}(x) \notin \bar{V}\), where \(\bar{V}\) is as defined in Step 0. As \(\bar{V}\) is a neighborhood of \(0 = \Psi(p)\), it follows that \(\bar{\Psi}(x) \neq 0\). But as \(x \in E \subset Z(X_1)\), we have

$$\bar{\Psi}(x) = (\phi_2, \ldots, \phi_{d-1})(x) = (\phi_2, \ldots, \phi_{d-1})(x).$$

Hence we obtain \(\phi_i(x) \neq 0\) for some \(i\) with \(2 \leq i \leq d - 1\). This is a contradiction to the fact that \(x \in Z(\phi_2, \ldots, \phi_{d-1})\). Hence \(S\) must be disjoint from \(E\). It follows that \(\Phi\) is a locally closed immersion on \(S\) by property (3) of Step 2.

As in the proof of Lemma 3.1, we let

$$\mathbb{A}_F^{d-2} \to \mathbb{A}_F^{d-2}$$

be the automorphism defined by

$$\eta = (Y_1 - \bar{Q}_1^\ell, \ldots, Y_{d-1} - \bar{Q}_{d-1}^\ell, Y_d),$$

where \(\bar{Q}_i \in F[Y_1, \ldots, Y_d]\) are polynomials satisfying \(Q_i = \bar{Q}_i(\phi_1, \ldots, \phi_d)\). It is straightforward to check that

$$\Phi = \eta \circ \bar{\Phi}.$$  

Hence \(\Phi\) is a locally closed immersion on \(S\), this proves condition (4) of Definition 3.3.

From Lemma 3.4 we have \(Z(f) \cap \Psi^{-1}(V) \subset X\). This with the fact that \(Z = Z(f) \cap X\) implies conditions (2) of Definition 3.3. For checking condition (3), i.e. to check \(\Phi\) is étale at all points in \(S\), we note that since \(\ell\) is divisible by \(\text{char}(F)\), \(\Phi\) is étale precisely at those points where \(\bar{\Phi}\) is étale. In particular, \(\Phi\) is étale at all points of \(Z(f) \setminus E\).  

\[\square\]

4. Open subsets of \(\mathbb{A}_F^2\)

In this section, we finish the proof of Theorem 1.1. By Lemmas 2.4 and 3.8 we only have to deal with the case of open subsets of \(\mathbb{A}_F^2\). While the handling of low degree points is similar, in spirit, to that of [6], for high degree points we proceed differently (see Lemma 4.10).

**Lemma 4.1.** Let \(F\) be a finite field as before, and let \(C \subset \mathbb{A}_F^2\) be a closed curve such that the projection onto the \(Y\)-coordinate \(Y_{[C]} : C \to \mathbb{A}_F^1\) is finite. Let \(C^{(1)}\) denote the set of closed points of \(C\). Then the set of points \(\{x \in C^{(1)} : \deg_F(Y(x)) = \deg_F(x)\}\) is dense in \(C\).

**Proof.** Without loss of generality, we may assume \(C\) is irreducible and hence we simply have to show that the set \(\{x \in C^{(1)} : \deg_F(Y(x)) = \deg_F(x)\}\) is infinite. Let \(x_1, \ldots, x_q\) be the \(F\)-rational points of \(\mathbb{A}_F^1\). Let \(C' := C \setminus Y_{[C]}^{-1}(\{x_1, \ldots, x_q\})\). We see that \(C'\) is a dense open subset of \(C\) as \(Y_{[C]}\) is finite. Any point \(x \in C'\) of prime degree satisfies

$$\deg_F(Y(x)) = \deg_F(x).$$

By the Lang–Weil estimates [3], for all large enough prime number \(\ell\), there is a point \(x \in C^{(1)}\) of degree \(\ell\). Hence, since \(\ell\) is a prime, we must have \(\deg_F(Y(x)) = \deg_F(x)\). This proves the lemma.  

\[\square\]
Notation 4.2. We fix the following notation.

1. Let $A = F[X, Y]$ and for $d \geq 0$ let $A_{\leq d} = \{ h \in A \mid \deg(h) \leq d \}$. Here $\deg(h)$ denotes the total degree.

2. Let $f, g \in A$ be two non-constant polynomials, with no common irreducible factors. By performing a change of coordinates if necessary, we will assume that $f$ is monic in $X$ of degree $m$.

3. Let $W := \mathbb{A}_F^2 \setminus Z(g)$. In this section, we call our variety $W$ instead of $X$, since the latter will denote a coordinate function on $\mathbb{A}_F^2$.

4. Let $Z := Z(f) \cap W$. Note that $Z(f) \setminus Z$ is finite as $f$ and $g$ have no common irreducible components.

5. Let $p \in Z$ be a closed point such that its $X$-coordinate is 0. We also choose a set of closed points $\{p_1, \ldots, p_t\}$ in $Z$ such that the set $T := \{ p, p_1, \ldots, p_t \}$ satisfies the following:
   
   (a) $T$ contains at least one point from each irreducible component of $Z$.
   
   (b) No two points in $T$ have same degrees and for all $p_i \in T$, $\deg(Y(p_i)) = \deg(p_i)$.

   This can be ensured by Lemma 4.1. Note that since $X$-coordinate of $p$ is 0, we also have $\deg(Y(p)) = \deg(p)$.

6. Let $D = \{ q_1, \ldots, q_s \}$ be a finite set of closed points in $Z(f)$ satisfying:
   
   (a) $D$ contains all points in $Z(f) \setminus Z$.
   
   (b) $D$ contains at least one point from each irreducible component of $Z(f)$.
   
   (c) $D$ does not contain any point of $\{ p, p_1, \ldots, p_t \}$.

Moreover, for a point $x$ in $Z(f)$, the notation $O_x$ (resp. $m_x$) will denote $O_{\mathbb{A}_F^2, x}$ (resp. $m_{\mathbb{A}_F^2, x}$) i.e. the local ring (resp. maximal ideal) of $x$ as a point of $\mathbb{A}_F^2$.

The main result of this section is the following.

Theorem 4.3. There exists $(\phi_1, \phi_2) \in F[X, Y]$ which presents $(W, Z(f), p)$.

This is enough to prove Theorem 1.1.

Proof of Theorem 1.1. This statement follows from Lemmas 2.4, 3.4 and 3.8 and Theorem 4.3. □

To prove Theorem 4.3, we will find $\phi_1$ by using Lemma 4.4 and $\phi_2$ by Lemma 4.10. We heavily use the counting techniques by Poonen [6] to prove these lemmas.

Recall from Notation 2.1 that, for $Y$ a subset of a scheme $\bar{X}/F$,

$$Y_{\leq r} := \{ x \in Y \mid \deg(x) \leq r \}.$$ 

Lemma 4.4. Let the notation be as in Notation 4.2. There exists $c \in \mathbb{N}$ such that for every $d > 0$, there exists a $\phi \in A_{\leq d}$ satisfying the following:

1. $\phi(p) = \phi(p_i) = 0$ for all $i = 1, \ldots, t$ and $\phi(q_i) \neq 0$ for all $i = 1, \ldots, s$.

2. $(\phi, Y)$ is étale at all $x \in S$, where $S := Z(\phi) \cap Z$.

3. The projection $Y : \mathbb{A}_F^2 \to \mathbb{A}_F^1$ is radicial at $S_{\leq d}(d-c)$. 


**Remark 4.5.** The above lemma is motivated by writing down conditions for \( \phi \) such that \((\phi, Y)\) presents \((W, Z(f), p)\), and then keeping only those which we can prove. Indeed, if \( \phi|_{Z(f)} \) is a finite map and \( Y \) is radicial at whole of \( S \) (as opposed to \( S_{\frac{1}{2} (d - c)} \) above), then \((\phi, Y)\) would present \((W, Z, p)\) thereby proving Theorem 4.3.

**Remark 4.6.** The set \( S = Z(\phi) \cap Z \) appearing in the statement of the above lemma is necessarily finite. This is because, in each irreducible component of \( Z \), there is at least one \( q_i \) (see Notation 4.2 (6) (b)) on which \( \phi \) does not vanish. Since \( T \) intersects each irreducible component of \( Z(f) \) (see Notation 4.2 (5) (a)), we know that any open neighborhood of \( S \) is dense in \( Z(f) \).

Following [6], define the density of a subset \( C \subset A \) by

\[
\mu(C) := \lim_{d \to \infty} \frac{\#(C \cap A_{\leq d})}{\#A_{\leq d}}
\]

provided the limit exists. Similarly, the upper and lower densities of \( C \), denoted by \( \overline{\mu}(C) \) and \( \underline{\mu}(C) \), are defined by replacing limit in the above expression by lim sup and lim inf, respectively.

To prove the existence of \( \phi \) in Lemma 4.4, we will show that the density of such \( \phi \) is positive. We prove Lemma 4.4 in two steps. First, we show (Lemma 4.8) that \( \phi \) satisfying conditions (1), (3) and condition (2) for points up to certain degree, exists. Next, we show (Lemma 4.9) that the set of \( \phi \) which does not satisfy condition (2) for points of higher degrees has zero density.

Let \( \phi \in A \) and let \( r \geq 1 \) be an integer. Consider the following conditions on \( \phi \), which are closely related to conditions (1), (2) and (3) of Lemma 4.4.

(a) \( \phi(p) = \phi(p_i) = 0 \) for all \( 1 \leq i \leq t \) and \( \phi(q_i) = 1 \) for all \( 1 \leq i \leq s \).

(b) For all \( x \in Z(f)_{\leq r} \) such that \( \phi(x) = 0 \), \( \frac{\partial \phi}{\partial x} (x) \neq 0 \).

(c) For all \( x_1, x_2 \in Z(f)_{\leq r} \) such that \( \deg(x_1) = \deg(x_2) = \deg(Y(x_1)) = \deg(Y(x_2)) \) and \( \phi(x_1) = \phi(x_2) = 0 \), we have \( Y(x_1) \neq Y(x_2) \).

(d) For all \( x \in Z(f)_{\leq r} \) such that \( \phi(x) = 0 \), \( \deg(Y(x)) = \deg(x) \).

**Remark 4.7.** The main motivation for introducing the above conditions, are the following straightforward implications between them and the conditions of Lemma 4.4

- \( \phi \) satisfies Lemma 4.4 (1) if \( \phi \) satisfies (a).
- \( \phi \) satisfies Lemma 4.4 (2) if and only if \( \phi \) satisfies (b) for all \( r \geq 1 \).
- \( \phi \) satisfies Lemma 4.4 (3) if and only if \( \phi \) satisfies (c) and (d) for all \( r \leq \frac{1}{3} (d - c) \).

**Lemma 4.8.** There exist integers \( r_0, c \in \mathbb{N} \), with

\[
r_0 > \max \{ \deg(p), \deg(p_1), \ldots, \deg(p_t), \deg(q_1), \ldots, \deg(q_s) \}\]

such that the lower density of the following set is positive:

\[
P := \bigcup_{d > c + 2r_0} \left\{ \phi \in A_{\leq d} \mid \phi \text{ satisfies (a), (b)}_{\frac{1}{3} (d - c)}, (c), (d), (d)_{\frac{1}{3} (d - c)} \text{ and } \phi(x) = 1 \text{ for all } x \in Z(f)_{\leq r_0} \setminus T \right\}.
\]
Proof. By the Lang–Weil estimates [3] there exists \( c' \in \mathbb{N} \) such that for all \( n \geq 1 \),
\[
\#(Z(f)=n) \leq c' \cdot q^n.
\]
For reasons which be clear during the course of the calculations below, we choose \( r_0 \) and \( c \) as follows. Recall that \( m \) is the \( X \)-degree of \( f \). Let \( r_0 \) be any integer satisfying
\[
(i) \quad r_0 > \max \{ \deg(p), \deg(p_1), \ldots, \deg(p_t), \deg(q_1), \ldots, \deg(q_s) \},
\]
\[
(ii) \quad \left( \sum_{i > \frac{r_0}{m}} \frac{1}{q^i} \right) \cdot \left( c' + \left( \frac{m}{2} \right) + \frac{m}{2} \right) < 1 - \sum_{x \in T} q^{-\deg(x)}.
\]
Note that it is always possible to ensure (ii) as \( \sum_{i > \frac{r_0}{m}} \frac{1}{q^i} \to 0 \) as \( r_0 \to \infty \) and as degrees of points in \( T \) are distinct we have
\[
\sum_{x \in T} q^{-\deg(x)} < \sum_{i=1}^{\infty} q^{-i} \leq 1.
\]
Let
\[
c = \sum_{x \in Z(f) \leq r_0} \deg(x).
\]
Let \( d \geq c + 2r_0 \) be any integer and \( r := \frac{1}{3}(d - c) \). Let
\[
\mathcal{T} := \{ \phi \in A_{\leq d} \mid \phi \text{ satisfies (a) and } \phi(x) = 1 \text{ for all } x \in Z(f) \leq r_0 \setminus T \},
\]
\[
\mathcal{T}_b := \{ \phi \in \mathcal{T} \mid \phi \text{ does not satisfy } (b_r) \},
\]
\[
\mathcal{T}_c := \{ \phi \in \mathcal{T} \mid \phi \text{ does not satisfy } (c_r) \},
\]
\[
\mathcal{T}_d := \{ \phi \in \mathcal{T} \mid \phi \text{ does not satisfy } (d_r) \}.
\]
Let
\[
\delta := \frac{\#\mathcal{T}}{\#A_{\leq d}}, \quad \delta_b := \frac{\#\mathcal{T}_b}{\#A_{\leq d}}, \quad \delta_c := \frac{\#\mathcal{T}_c}{\#A_{\leq d}}, \quad \delta_d := \frac{\#\mathcal{T}_d}{\#A_{\leq d}}.
\]
In the following steps we will estimate \( \delta, \delta_b, \delta_c, \delta_d \).

**Step 1: Estimation for \( \delta \).** Note that the condition that \( \phi \) belongs to \( \mathcal{T} \) depends solely on the image of \( \phi \) in the zero-dimensional ring
\[
\prod_{x \in Z(f) \leq r_0} (\Theta_x / m_x).
\]
Since the dimension over \( F \) of the above ring is \( c \) and since \( d \geq c \), by [6, Lemma 2.1] the map
\[
A_{\leq d} \xrightarrow{\rho} \prod_{x \in Z(f) \leq r_0} (\Theta_x / m_x)
\]
is surjective. One can easily see that \( \mathcal{T} \) is a coset of \( \text{Ker}(\rho) \). Therefore
\[
\delta = \prod_{x \in Z(f) \leq r_0} q^{-\deg(x)}.
\]
**Step 2: Estimation for \( \delta_b \).** Let \( x \in Z(f)_{\leq r} \) (recall that \( r = \frac{1}{3} (d - c) \)). The following are equivalent:

(i) \( \phi \in \mathcal{T} \) and \( \phi(x) = 0 \) and \( \frac{\partial \phi}{\partial X} (x) = 0 \).

(ii) \( \phi \in \mathcal{T} \) and \( \phi \mod m_x^2 \) lies in the kernel of the linear map \( \frac{\partial}{\partial X} : m_x/m_x^2 \to F(x) \).

Let us first consider the case when \( \deg(x) > r_0 \). In this case, each of the above condition for \( \phi \) depends only on its image in the zero-dimensional ring

\[
\left( \prod_{q \in Z(f)_{\leq r_0}} (\mathcal{O}_q/m_q) \right) \times (\mathcal{O}_x/m_x^2).
\]

The cardinality of the above ring is

\[
\left( \prod_{y \in Z(f)_{\leq r_0}} q^{\deg(y)} \right) \cdot q^{3\deg(x)}.
\]

Let us call an element \( \xi \) in the above ring as a favorable value if and only if all \( \phi \) mapping to \( \xi \) satisfy the above conditions. It is an easy exercise to check that the set of all favorable values has cardinality \( q^{\deg(x)} \). Thus the ratio of the number of favorable values to the cardinality of the ring is nothing but \( \delta q^{-2\deg(x)} \). The dimension over \( F \) of this ring is \( c + 3 \deg(x) \). Since \( d \geq c + 3 \deg(x) \), by [6, Lemma 2.1], \( A_{\leq \mathcal{A}} \) surjects onto this ring. Due to this, the ratio of \( \phi \in A_{\leq \mathcal{A}} \) satisfying the above two conditions to the \# \( A_{\leq \mathcal{A}} \) is nothing but \( \delta q^{-2\deg(x)} \). In other words,

\[
\frac{\# \{ \phi \in \mathcal{T} \mid \phi(x) = 0, \frac{\partial \phi}{\partial X}(x) = 0 \}}{\# A_{\leq \mathcal{A}}} = \delta \cdot q^{-2\deg(x)}.
\]

Now let us consider the case where \( \deg(x) \leq r_0 \). We claim that unless \( x \in T \), there is no \( \phi \in \mathcal{T} \) which vanishes on \( x \). This follows from the definition of \( \mathcal{T} \). So let us assume \( x \in T \). In this case, the above two conditions for \( \phi \) depend solely on the image of \( \phi \) in the ring

\[
\left( \prod_{q \in Z(f)_{\leq r_0} \atop q \neq x} (\mathcal{O}_q/m_q) \right) \times (\mathcal{O}_x/m_x^2).
\]

Proceeding in a manner similar to the case where \( \deg(x) > r_0 \), we find that for \( x \in T \),

\[
\frac{\# \{ \phi \in \mathcal{T} \mid \phi(x) = 0, \frac{\partial \phi}{\partial X}(x) = 0 \}}{\# A_{\leq \mathcal{A}}} = \delta \cdot q^{-\deg(x)}.
\]

Since

\[
\mathcal{T}_{b} = \bigcup_{x \in Z(f)_{\leq r} \text{ such that } x \in T \text{ or } \deg(x) > r_0} \left\{ \phi \in \mathcal{T} \mid \phi(x) = 0, \frac{\partial \phi}{\partial X}(x) = 0 \right\},
\]

we get an estimate

\[
\delta_b \leq \sum_{x \in T} \delta q^{-\deg(x)} + \sum_{x \in Z(f)_{\leq r} \text{ such that } \deg(x) > r_0} \delta q^{-2\deg(x)}
\]

\[
\leq \delta \left( \sum_{x \in T} q^{-\deg(x)} + \sum_{r_0 < i \leq r} c'^{-i} q^{-i} \right),
\]

where recall that \( c' \) was the constant in the Lang–Weil estimates such that \( \# Z(f)_{=n} \leq c' q^n \).
Step 3: Estimation for \( \delta_c \). Let \( y \in \mathbb{A}^1_F \) with \( i := \deg(y) \leq r \). Let

\[
\mathcal{T}_c^y := \{ \phi \in \mathcal{T} \mid \text{there exist distinct } x_1, x_2 \in \mathbb{Z}(f)_{\equiv i} \text{ with } Y(x_1) = Y(x_2) = y \\
\text{and } \phi(x_1) = \phi(x_2) = 0 \}\.
\]

First, note that \( \mathcal{T}_c^y \) is empty unless \( i > r_0 \). This is because the only points of degree \( \leq r_0 \) on which a \( \phi \in \mathcal{T} \) vanishes are the points in \( T \). However, by choice, all points in \( x \in T \) have different degrees and satisfy \( \deg(x) = \deg(Y(x)) \). Thus, let us assume \( i > r_0 \). In this case, we claim that

\[
\frac{\# \mathcal{T}_c^y}{\# \mathcal{A}_{\leq d}} \leq \delta \cdot \left( \frac{m}{2} \right) \cdot q^{-2i}.
\]

For fixed \( x_1, x_2 \) with \( Y(x_1) = Y(x_2) = y \), the set \( \{ \phi \in \mathcal{T} \mid \phi(x_1) = \phi(x_2) = 0 \} \) is a coset of the kernel of the map

\[
\mathcal{A}_{\leq d} \rightarrow \left( \prod_{q \in \mathbb{Z}(f)_{\leq r_0}} (\Theta_q/m_q) \right) \times (\Theta_{x_1}/m_{x_1}) \times (\Theta_{x_2}/m_{x_2}).
\]

which is surjective by [6, Theorem 2.1]. Thus

\[
\frac{\{ \phi \in \mathcal{T} \mid \phi(x_1) = \phi(x_2) = 0 \}}{\# \mathcal{A}_{\leq d}} \leq \delta \cdot q^{-2i}.
\]

To prove the claim, we now simply observe that since \( f \) is monic in \( X \) of degree \( m \), there are at most \( \left( \frac{m}{2} \right) \) possible choices for a pair \( \{x_1, x_2\} \) as above.

As discussed above, since \( \mathcal{T}_c^y \) is empty unless \( i > r_0 \), we have

\[
\mathcal{T}_c = \bigcup_{y \in \mathbb{A}^1_F} \mathcal{T}_c^y.
\]

For a fixed \( i \),

\[
\# \{ y \in \mathbb{A}^1_F \mid \deg(y) = i \} \leq q^i.
\]

From this, it is elementary to deduce

\[
\delta_c = \frac{\# \mathcal{T}_c}{\# \mathcal{A}_{\leq d}} \leq \delta \left( \sum_{r_0 < i \leq r} \left( \frac{m}{2} \right) q^{-i} \right).
\]

Step 4: Estimation for \( \delta_d \). As in the above step, let \( y \in \mathbb{A}^1_F \) with \( i := \deg(y) \leq r \). Let

\[
\mathcal{T}_d^y := \{ \phi \in \mathcal{T} \mid \text{there exists } x \in \mathbb{Z}(f)_{\leq r} \text{ with } \phi(x) = 0, Y(x) = y \text{ and } \deg(x) > i \}.
\]

We claim that \( \mathcal{T}_d^y \) is empty unless \( \deg(y) > \frac{r_0}{m} \). Otherwise, there would exist a \( \phi \in \mathcal{T} \) and an \( x \in \mathbb{Z}(f)_{\leq r} \) with \( Y(x) = y, \phi(x) = 0 \) and \( \deg(x) > \deg(y) \). But as \( f \) is monic in \( X \) of degree \( m \), the maximum degree of a point \( x \) lying over \( y \) is \( m \cdot \deg(y) \leq r_0 \), which means \( x \in \mathbb{Z}(f)_{\leq r_0} \). However, as \( \phi \in \mathcal{T} \), the only points in \( \mathbb{Z}(f)_{\leq r_0} \) on which \( \phi \) vanishes are those in \( T \). Thus \( x \in T \). But by Notation 4.2 (5) (c), for such \( x \), \( \deg(Y(x)) = \deg(y) = \deg(x) \) which is a contradiction.
We will now estimate \( \frac{\#T_y}{\#A_{\leq d}} \).

Fix a point \( x \in \mathbb{Z}(f)_{\leq r} \) with \( \deg(x) > i \) and \( Y(x) = y \). For this \( x \), we first note that because of Notation 4.2 (5) (c), \( x \notin T \).

For \( \deg(x) > r_0 \) we note that
\[
\frac{\#\{\phi \in T \mid \phi(x) = 0\}}{\#A_{\leq d}} \leq \delta \cdot q^{-\deg(x)} \leq \delta \cdot q^{-2i}.
\]

This is deduced, as before, from the surjectivity of
\[
A_{\leq d} \rightarrow \left( \prod_{q \in \mathbb{Z}(f)_{\leq r_0}} (\mathbb{F}_q/m_q) \right) \times (\mathbb{F}_x/m_x).
\]

If \( \deg(x) \leq r_0 \), then \( \{\phi \in T \mid \phi(x) = 0\} \) is empty as there are no points in \( \mathbb{Z}(f) \setminus T \) on which a \( \phi \in T \) vanishes. And hence, the above estimate trivially holds in this case also.

As \( f \) is monic in \( X \) of degree \( r_0 \), then \( \{\phi \in T \mid \phi(x) = 0\} \) vanishes. And hence, the above estimate trivially holds in this case also.

For a fixed \( i \),
\[
\#\{y \in A_{\frac{1}{p}} \mid \deg(y) = i\} \leq q^i.
\]

Thus
\[
\delta_d = \frac{\#T_d}{\#A_{\leq d}} \leq \delta \left( \sum_{r_0 < i \leq r} \frac{m}{2} q^{-i} \right).
\]

**Step 5: Estimation for \( \mathcal{P} \).** If we let
\[
\mathcal{P}_d := \{\phi \in A_{\leq d} \mid \phi \text{ satisfies (a), (b)_{1(d-c)}, (c)_{1(d-c)}, (d)_{1(d-c)} and } \phi(x) = 1 \text{ for all } x \in \mathbb{Z}(f)_{\leq r_0} \setminus T \},
\]
then
\[
\mathcal{P}_d = T \setminus (T_b \cup T_c \cup T_d).
\]

Therefore
\[
\frac{\#\mathcal{P}_d}{\#A_{\leq d}} \geq \delta - \delta_b - \delta_c - \delta_d
\]
\[
\geq \delta \left[ 1 - \sum_{x \in T} q^{-\deg(x)} - \sum_{r_0 < i \leq r} c' q^{-i} - \sum_{r_0 < i \leq r} \left( \frac{m}{2} \right) q^{-i} - \sum_{r_0 < i \leq r} \frac{m}{2} q^{-i} \right]
\]
\[
\geq \delta \left[ 1 - \sum_{x \in T} q^{-\deg(x)} - \left( \sum_{\frac{r_0}{m} < i \leq r} \frac{1}{q^i} \right) \left( c' + \left( \frac{m}{2} + \frac{m}{2} \right) \right) \right].
\]
Note that in the above expression \( r = \frac{1}{2} (d - c) \). As \( d \to \infty \), so does \( r \). Hence we observe that

\[
\inf \left( \frac{\# P_d}{\# A \leq d} \right) \geq \delta \left[ 1 - \sum_{x \in T} q^{-\deg(x)} - \left( \sum_{i > \frac{r}{m}} \frac{1}{q^i} \right) \left( \epsilon' + \left( \frac{m}{2} \right) + \frac{m}{2} \right) \right],
\]

which is positive, thanks to the definition of \( r_0 \). Thus the lower density of

\[
P = \bigcup_d P_d
\]

is positive as required. \( \square \)

**Lemma 4.9.** Let \( c \) be as in Lemma 4.8 and let

\[
Q := \bigcup_{d \geq 0} \left\{ \phi \in A \leq d \mid \text{there exists } x \in \mathbb{Z}(f) > \frac{1}{2} (d-c) \text{ such that } \phi(x) = \frac{\partial \phi}{\partial X}(x) = 0 \right\}.
\]

Then \( \mu(Q) = 0 \).

**Proof.** The proof is identical to that of [6, Theorem 2.6]. We reproduce the argument verbatim here for the convenience of the reader. We will bound the probability of \( \phi \) constructed as

\[
\phi = \phi_0 + g^p X + h^p
\]

and for which there is a point \( x \in \mathbb{Z}(f) > \frac{1}{2} (d-c) \) with \( \phi(x) = \frac{\partial \phi}{\partial X}(x) = 0 \). Note that if \( \phi \) is of the above form, then

\[
\frac{\partial \phi}{\partial X} = \frac{\partial \phi_0}{\partial X} + g^p.
\]

Further, if \( \phi_0 \in A \leq d, g \in A \leq d - \frac{1}{p} \) and \( h \in A \leq d \), then \( \phi \in A \leq d \). Define

\[
W_0 := \mathbb{Z}(f) \quad \text{and} \quad W_1 := \mathbb{Z} \left( f, \frac{\partial \phi}{\partial X} \right).
\]

Note that \( \dim(W_0) = 1 \).

Let

\[
\gamma := \left\lfloor \frac{d - 1}{p} \right\rfloor \quad \text{and} \quad \eta := \left\lfloor \frac{d}{p} \right\rfloor.
\]

**Claim 1.** The probability (as a function of \( d \)) of choosing \( \phi_0 \in A \leq d \) and \( g \in A \leq \frac{1}{p} (d-1) \) such that \( \dim(W_1) = 0 \) is \( 1 - o(1) \) as \( d \to \infty \).

Let \( V_1, \ldots, V_\ell \) be \( F \) irreducible components of \( W_0 \). Clearly \( \ell \leq \deg(f) \) (where \( \deg(f) \) is the total degree). Since the projection onto the \( Y \) coordinate is finite on \( Z(f) \) (by Notation 4.2 (2)), we know that \( Y(V_k) \) is of dimension one for all \( k \). We will now bound the set

\[
G_k^{bad} := \left\{ g \in A \leq \gamma \mid \frac{\partial \phi}{\partial X} = \frac{\partial \phi_0}{\partial X} + g^p \text{ vanishes identically on } V_k \right\}.
\]

If \( g, g' \in G_k^{bad} \), then \( g - g' \) vanishes on \( V_k \). Thus if \( G_k^{bad} \) is non-empty, it is a coset of the subspace of functions in \( A \gamma \) which vanish identically on \( V_k \). The codimension of that subspace, or equivalently the dimension of the image of \( A \gamma \) in the regular functions on \( V_k \) is at least \( \gamma + 1 \), since no polynomial in \( Y \) vanishes on \( V_k \). Thus the probability that \( \frac{\partial \phi}{\partial X} \) vanishes on \( V_k \) is at
most \( q^{-\gamma - 1} \). Thus, the probability that \( \frac{\partial \phi}{\partial x} \) vanishes on some \( V_k \) is at most \( \ell q^{-\gamma - 1} = o(1) \). Since \( \dim(W_1) = 0 \) if and only if \( \frac{\partial \phi}{\partial x} \) does not identically vanish on any component \( V_k \), the claim follows.

We will now estimate the probability of choosing \( h \) such that there is no bad point in \( Z(f) \), i.e. a point in \( Z(f)_{> \frac{1}{3}(d-c)} \) where both \( \phi \) and \( \frac{\partial \phi}{\partial x} \) vanish. Note that the set of such bad points is precisely

\[ Z(\phi) \cap W_1 \cap Z(f)_{> \frac{1}{3}(d-c)}. \]

Claim 2. Conditioned on the choice of \( \phi_0 \) and \( g \) such that \( W_1 \) is finite, the probability of choosing \( h \) such that

\[ Z(\phi) \cap W_1 \cap Z(f)_{> \frac{1}{3}(d-c)} = \emptyset \]

is \( 1 - o(1) \) as \( d \to \infty \).

It is clear by the Bezout theorem that \( \#W_1 = O(d) \). For a given \( x \in W_1 \), the set

\[ H_{\text{bad}} = \{ h \in A_q \mid \phi = \phi_0 + g^p X + h^p \text{ vanishes on } x \} \]

is either \( \emptyset \) or a coset of \( \text{Ker}(A_q \xrightarrow{ev_x} F(x)) \) where \( F(x) \) is the residue field of \( x \). For the purpose of this claim, we only need to consider \( x \) such that \( \deg(x) > \frac{1}{3}(d - c) \). In this case, [6, Lemma 2.5] implies that

\[ \frac{\#H_{\text{bad}}}{\#A_q} \leq q^{-v}, \text{ where } v = \min\left( \eta + 1, \frac{1}{3}(d - c) \right). \]

Thus, the probability that both \( \phi \) and \( \frac{\partial \phi}{\partial x} \) vanish at such \( x \) is at most \( q^{-v} \). There are at most \( \#W_1 \) many possibilities for \( x \). Thus the probability that there exists a “bad point”, i.e. a point in \( x \in W_1 \) with \( \deg(x) > \frac{1}{3}(d - c) \) such that both \( \phi \) and \( \frac{\partial \phi}{\partial x} \) vanish at such \( x \) is at most \( (\#W_1)q^{-v} = O(dq^{-v}) \). Since as \( d \to \infty \), \( v \) grows linearly in \( d \), we have \( O(dq^{-v}) = o(1) \). In other words, the probability of choosing \( h \) such that there is no bad point is \( 1 - o(1) \).

By combining the above two claims, it follows that the probability of choosing

\[ \phi = \phi_0 + g^p X + h^p \]

such that

\[ Z(\phi) \cap W_1 \cap Z(f)_{> \frac{1}{3}(d-c)} = \emptyset \]

is equal to \( (1 - o(1))(1 - o(1)) = 1 - o(1) \). This shows that \( \mu(Q) = 0 \). \( \Box \)

Proof of Lemma 4.4. Let \( \overline{Q} \) denote the complement of \( Q \) in \( A \), and let \( \mathcal{P} \) be as in Lemma 4.8. To prove Lemma 4.4, we need to show that \( \mathcal{P} \cap \overline{Q} \) is non-empty. However, combining the above two lemmas, we in fact get that \( \mu(\mathcal{P} \cap \overline{Q}) > 0 \). This finishes the proof. \( \Box \)

Condition (3) of Lemma 4.4 ensures that \( Y : \mathbb{A}^2_F \to \mathbb{A}^1_F \) is radical at \( S_{\leq \frac{1}{3}(d-c)} \). We would have ideally liked to have \( S \) instead of \( S_{\leq \frac{1}{3}(d-c)} \) here. If this was the case, and if \( \phi_{|Z(f)} \) was finite, as mentioned in Remark 4.5, we would be able to deduce that \( (\phi, Y) \) presents \((W, Z(f), p)\). However, we are unable to handle points in \( S \) of degree greater than \( \frac{1}{3}(d-c) \). In order to rectify that, we replace the map \((\phi, Y)\) with a map \((\phi, h)\) for a suitable \( h \) as found by the following lemma. Finiteness of \( \phi \) will be handled later using a Noether normalization argument.
Lemma 4.10. Let \( c \in \mathbb{N} \) be as in Lemma 4.4. Let \( d \gg 0 \) be an integer such that for every \( i > \frac{1}{2}(d - c) \),
\[
\#(A_{\mathbb{F}}^{1})_{=i} > dm.
\]
Let \( \phi \in A_{\leq d} \) be as given by Lemma 4.4 and \( S := \mathcal{Z}(\phi) \cap \mathbb{Z} \). Then there exists \( h \in F[X, Y] \) such that the following hold:

1. \( h_{|S} : S \to \mathbb{A}^{1} \) is radical, i.e. injective and preserves the degree.
2. The map \( \mathbb{A}_{\mathbb{F}}^{2} \xrightarrow{(\phi, h)} \mathbb{A}_{\mathbb{F}}^{2} \) is étale at all \( x \in S \).
3. \( h_{|Z(f)} : Z(f) \to \mathbb{A}_{\mathbb{F}}^{1} \) is a finite map.

Proof. We divide the proof into three steps.

Step 1. In this step we will show that with the given choice of \( d \), it is possible to choose \( h_{1} \) which satisfies condition (1) of the lemma. We claim that
\[
\#S_{=i} \leq \#(A_{\mathbb{F}}^{1})_{=i} \quad \text{for all } i \geq 1.
\]
As explained in Remark 4.6, \( \mathcal{Z}(\phi) \cap \mathcal{Z}(f) \) is finite. By Bezout theorem,
\[
\#S \leq \deg(\phi)\deg(f) = dm.
\]
Thus the above claim holds for all \( i > \frac{1}{2}(d - c) \) by the choice of \( d \). On the other hand, the claim also holds for \( i \leq \frac{1}{2}(d - c) \) since by Lemma 4.4, \( Y \) is radicial at \( S_{\leq \frac{1}{2}(d - c)} \). Thus we can choose a set theoretic map
\[
S \xrightarrow{\tilde{h}} \mathbb{A}_{\mathbb{F}}^{1}
\]
which is injective and preserves degree of points. By the Chinese Remainder Theorem, there exists an \( h_{1} \in F[X, Y] \) such that for all \( x \in S \), \( h_{1}(x) = \tilde{h}(x) \).

Step 2. Now, using the \( h_{1} \) from above step, we will find a \( h_{2} \in F[X, Y] \) which satisfies conditions (1) and (2) of the lemma. It is sufficient to find an \( h_{2} \in F[X, Y] \) such that for all \( x \in S \),

\[
\begin{align*}
(i) & \quad h_{2} \equiv h_{1} \mod m_{x}, \\
(ii) & \quad \frac{\partial h_{2}}{\partial X}(x) = 0, \\
(iii) & \quad \frac{\partial h_{2}}{\partial Y}(x) = 1.
\end{align*}
\]
First, we claim that for any closed point \( x \in \mathbb{A}_{\mathbb{F}}^{2} \), there exists an \( h_{x} \in F[X, Y] \) such that
\[
h_{x} \equiv h_{1} \mod m_{x}, \quad \frac{\partial h_{x}}{\partial X}(x) = 0, \quad \frac{\partial h_{x}}{\partial Y}(x) = 1.
\]
We choose a polynomial \( f_{1} \in F[X] \) such that \( f_{1}(x) = 0 \) and \( \frac{\partial f_{1}}{\partial X}(x) \neq 0 \). To see that such a choice is possible, let \( \pi_{1} : \mathbb{A}_{\mathbb{F}}^{2} \to \mathbb{A}_{\mathbb{F}}^{1} \) be the projection on to the \( X \)-coordinate. The minimal polynomial of any primitive element of the residue field of \( \pi_{1}(x) \) satisfies our requirement. Similarly, we choose \( f_{2} \in F[Y] \) such that \( f_{2}(x) = 0 \) and \( \frac{\partial f_{2}}{\partial Y}(x) \neq 0 \). Using the Chinese Remainder Theorem and the fact that the residue field \( F(x) \) is perfect, we choose two polynomials \( g_{1}, g_{2} \in F[X, Y] \) such that
\[
g_{1}^{P}(x) = -\frac{\partial h_{1}}{\partial X}(x), \quad g_{2}^{P}(x) = \frac{(1 - \frac{\partial h_{1}}{\partial Y}(x))}{\frac{\partial f_{2}}{\partial Y}(x)}.
\]
We leave it to the reader that
\[ h_x = h_1 + g_1 f_1 + g_2 f_2 \]
satisfies the requirement of our claim. Now, by the Chinese Remainder Theorem, there exists \( h_2 \in F[X,Y] \) such that
\[ h_2 \equiv h_x \mod \mathfrak{m}_x^2 \]
for all \( x \in S \).

It is straightforward to see that \( h_2 \) satisfies conditions (1) and (2) of the lemma.

**Step 3.** Choose a non-constant polynomial \( \beta \in F[Y] \) such that \( \beta(x) = 0 \) for all \( x \in S \). Since \( f \) is monic in \( X \), it follows that \( Z(f) \) is a finite map. Thus \( \beta : Z(f) \to \mathbb{A}_F^1 \) is also a finite map. As \( \dim(Z(f)) = 1 \), for a sufficiently large integer \( \ell \),
\[ h := h_2 - \beta P \]
defines a finite map \( Z(f) \to \mathbb{A}^1_F \) by Noether normalization trick (see Lemma 2.2). Clearly \( h \) continues to satisfy conditions (1) and (2) of the lemma since \( \beta P \equiv m_x^2 \) for all \( x \in S \).

**Proof of Theorem 4.3.** Let \( \phi \) and \( h \) be as in Lemmas 4.4 and 4.10, respectively. Let \( \Phi \) be the map
\[ \mathbb{A}_F^2 \to \mathbb{A}_F^2, \]
and \( \bar{\Phi} := \phi \). Recall that \( \bar{S} := \phi^{-1}(0) \cap Z(f) \) (with reduced scheme structure). By Remark 4.6 it is finite.

**Step 1.** We claim that there exists a \( g \in F[X,Y] \) such that if \( W_g := \mathbb{A}_F^2 \setminus Z(g) \), then \( \Phi(S) \subseteq W_g \) and
\[ \Phi^{-1}(W_g) \cap Z(f) \to W_g \]
is a closed immersion. The proof of this claim is a repetition of the argument in [1, Theorem 3.5.1] (see also Lemma 3.4). Let \( \{p, x_1, \ldots, x_n\} \) be the set of points in \( \bar{S} \). Since \( \Phi \) is étale and radicial at all points of \( \bar{S} \) (see Lemma 4.4 (2) and Lemma 4.10 (1)), it follows that \( \Phi^{-1}(\bar{S}) \to \mathbb{A}_F^2 \) is a closed immersion. Let \( y_0, \ldots, y_n \) be the points in \( \Phi(S) \). Let \( \eta_i \) be the maximal ideal in \( F[X,Y] \) corresponding to the closed point \( y_i \). Thus the above closed immersion gives us a surjective map
\[ F[X,Y] \to F[Z(f)]_{\eta_0 \cdots \eta_n}. \]
Here \( F[Z(f)] \) is the coordinate ring of \( Z(f) \). If \( C \) denotes the cokernel of \( F[X,Y] \to F[Z(f)] \) (as \( F[X,Y] \) modules), then the above surjective map implies that
\[ C \otimes_{F[X,Y]} = 0. \]
Note that \( \Phi[Z(f)] \) is a finite map, since \( h \) is a finite map (Lemma 4.10 (3)). Thus \( F[Z(f)] \) is a finite \( F[X,Y] \) module. Thus, by Nakayama’s lemma, there exists an element \( g \in F[X,Y] \) such that \( g \not\in \eta_0 \cdots \eta_n \) and \( C_g = 0 \). In other words, the map
\[ F[X,Y]_g \to F[Z(f)]_g \]
is surjective. This proves the claim since if \( W_g := \mathbb{A}_F^2 \setminus Z(g) \), the above surjectivity is equivalent to the following being a closed immersion:
\[ \Phi : Z(f) \cap \Phi^{-1}(W_g) \to W_g. \]
Step 2. Let $E$ be the smallest closed subset of $Z(f)$ satisfying the following three conditions:

(i) $x \in E$ if $x \in Z(f)$ and $\Phi$ is not étale at $x$.

(ii) $Z(f) \setminus Z \subset E$.

(iii) $Z(f) \setminus (\overline{T}^{-1}(W) \cap Z(f)) \subset E$.

Since $\overline{S}$ contains at least one point in each irreducible component of $Z(f)$, condition (iii) implies that $E$ is finite (see also Remark 4.6). Moreover, by the above step and condition (iii) we have

$$Z(f) \setminus E \to \mathbb{A}^2_F$$

is a locally closed immersion. Moreover, $\overline{S}$ and $E$ are disjoint, and hence $\phi(p) \notin \phi(E)$. Since $E$ is finite, we choose a non-constant polynomial expression $Q$ in $h$ which vanishes on $p$ as well as $E$. For an integer $\ell \gg 0$ and divisible by $\text{char}(F)$, we claim that $(\phi - Q^\ell, h)$ presents $(W, Z(f), p)$. Let

$$\Phi := (\phi - Q^\ell, h) \quad \text{and} \quad \Psi := \phi - Q^\ell.$$

To prove the claim, we need to verify conditions (1)–(4) of Definition 3.3. Condition (1), i.e. the finiteness of $\Psi_{|Z(f)}$, follows by Lemma 2.2 since $\ell$ is large, and $h_{|Z(f)}$ is finite. As $Q$ vanishes on $p$ and $E$, $\Psi(p) \notin \Psi(E)$ follows from $\phi(p) \notin \phi(E)$. Thus if $S := \Psi^{-1}(p) \cap Z$, then $S \subset Z(f) \setminus E$. Conditions (2)–(4) of Definition 3.3 follow from conditions (i)–(iii) of $E$ in the beginning of this step. \qed

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Amit Hogadi, IISER Pune, Dr. Homi Bhabha Road, Pashan, Pune 411008, India
e-mail: amit@iiserpune.ac.in

Girish Kulkarni, IISER Pune, Dr. Homi Bhabha Road, Pashan, Pune 411008, India
e-mail: girish.kulkarni@students.iiserpune.ac.in

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