Incorporation of measurement models in the IHCP: validation of methods for computing correction kernels

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Abstract. Thermocouples or other measuring devices are often imbedded into a solid to provide data for an inverse calculation. It is well-documented that such installations will result in erroneous (biased) sensor readings, unless the thermal properties of the measurement wires and surrounding insulation can be carefully matched to those of the parent domain. Since this rarely can be done, or doing so is prohibitively expensive, an alternative is to include a sensor model in the solution of the inverse problem. In this paper we consider a technique in which a thermocouple model is used to generate a correction kernel for use in the inverse solver. The technique yields a kernel function with terms in the Laplace domain. The challenge of determining the values of the correction kernel function is the focus of this paper. An adaptation of the sequential function specification method [1] as well as numerical Laplace transform inversion techniques are considered for determination of the kernel function values. Each inversion method is evaluated with analytical test functions which provide simulated “measurements”. Reconstruction of the undisturbed temperature from the “measured” temperature and the correction kernel is demonstrated.

1. Introduction

Thermocouples are often used in experiments in which the allowable range of accuracy is very small. It is well-documented [2][8] that thermocouples imbedded into a solid will yield erroneous (biased) readings. In this paper, a method set forth by Beck [4] for correcting thermocouple conduction error is revisited. This method can be used to estimate the temperature which would exist if the thermocouple were not present. In Beck’s work [4], Laplace transform analysis yielded a form of Duhamel’s integral which contained a correction kernel. Knowledge of the kernel values as a function of time allows the correction of temperature measurements. The kernel values are determined from the computed temperatures obtained from a computational heat transfer model. The model used in this work is a simple axisymmetric model, but this investigation is a means to evaluate the methodologies for future use in conjunction with a more detailed three-dimensional thermocouple model to study the effect of thermocouple error on interfacial heat transfer calculations via the inverse heat conduction problem.

Terms in the equation for the correction kernel are in Laplace space, thus an inversion method is necessary to determine the kernel values. Beck reported an inversion technique for obtaining the kernel values [9]. In the current work, various inversion techniques which were previously unexplored are considered. First, Beck’s original formulation of the correction kernel function is reviewed. Brief descriptions are provided for the various inversion techniques, which include an inverse solution of
Duhamel’s integral and numerical Laplace transform inversion methods. Finally, the various inversion techniques are evaluated with test functions. It is shown that the undisturbed temperatures can be obtained from the “measured” temperature history and knowledge of the correction kernel as a function of time.

2. Correction Kernel Formulation

A Laplace transform analysis described by Beck\textsuperscript{[4]} is the basis for the development of the correction kernel function used to determine the undisturbed transient temperature based on temperature measurements. The mathematical formulation results in a kernel function in the Laplace domain which must be inverted to the time domain. Beck describes a method for handling the task of inverting the kernel\textsuperscript{[9]}. The method that Beck used involves the numerical inversion of a convolution integral. This inversion method is different from the inversion of a Laplace transform. In this section, the correction kernel is found by numerical inversion of the convolution integral as well as by various techniques of numerical Laplace transform inversion.

Following is the correction kernel formulation by Laplace transform analysis as described by Beck\textsuperscript{[4]}. The complete derivation is too lengthy to include here, therefore only a brief overview of the formulation is provided. A problem description is provided which includes the geometry considered (Figure 1). The problem considered by Beck\textsuperscript{[4]} was a single wire positioned perpendicular to and at a distance $E$ from a heated surface. The domain is semi-infinite in the $z$ and $r$ directions. The variables pertinent to the formulation include the conductivity, $k$, the wire radius, $R$, the distance of the wire tip from the heat surface, $E$, time, $t$, and the thermal diffusivity, $\alpha$. The initial temperature of the system is $T_0$. The temperature at the tip of the wire is considered the “measured” temperature, $T_p$. The “undisturbed” temperature, $T_{pr}$, is the temperature at a point sufficiently far away from the wire that it is not affected by the presence of the wire.

![Figure 1. Problem geometry for a wire in a dissimilar low conductivity material normal to a heated surface.](image)

The temperature rise, $T_p(t) - T_0$, at a point $p$ (located along the axis of the wire at a distance $E$ from the surface) is related to the surface heat flux $\dot{q}(t)$ by the convolution integral

$$T_p(t) - T_0 = \int_0^t \dot{q}(\lambda) \frac{\partial h(0,E,t-\lambda)}{\partial t} d\lambda \quad (1)$$

where $h(r,z,t)$ is the temperature rise at a point $(r,z)$ due to a unit step increase in the surface heat flux $\dot{q}$. The convolution integral can be inverted to find $\dot{q}(t)$ which can then be used to calculate the undisturbed temperature rise $T_{pr} - T_0$ with
\[ T_{p_{0}} - T_{0} = \frac{R}{k} \int_{0}^{\infty} \dot{q}(\lambda) \frac{\partial \theta_{c}(t-\lambda)}{\partial t} d\lambda \]  

(2)

where \( \theta_{c} \) is the dimensionless temperature rise for the undisturbed temperature \( T_{p_{0}} \) due to a unit step increase in the surface heat flux \( \dot{q} \). The \( \theta_{c} \) term is expressed as

\[ \theta_{c} = \frac{T_{p_{0}} - T_{0}}{\dot{q}R/k} = \sqrt{\frac{2}{\pi}} \text{erf} \left[ \sqrt{\frac{2}{R}} \frac{E}{2R} \right] \]  

(3)

where \( \tau \) is nondimensional time and is equal to \( \alpha t / R^{2} \). Equation (1) can be inverted to solve for \( \dot{q} \), which can then be used to find the undisturbed temperature rise with equation (2). A Laplace transform analysis is used to derive an expression for the measurement error. The procedure involves taking the Laplace transform of equations (1) and (2) and then subtracting and manipulating the resulting equations. The result is the convolution integral

\[ T_{p_{0}}(t) - T_{p}(t) = \int_{0}^{\infty} H(\lambda) \frac{\partial T_{p}(t-\lambda)}{\partial t} d\lambda \]  

(4)

where

\[ H(\lambda) = \mathcal{L}^{-1} \left[ \frac{\bar{m}}{s \bar{h}} \right] - 1 \]  

(5)

Here the bar indicates that Laplace transform has been taken and the prime on \( h \) and \( m \) indicates differentiation with respect to time. The term \( m \) is given as

\[ m(t) = \frac{R}{k} \theta_{c}(t) \]  

(6)

It should be noted that for the dimensionless problem considered here and due to the properties of Laplace transforms, the ratio of \( \bar{m} \) to \( \bar{h} \) is equivalent to the ratio of \( \bar{m} \) to \( \bar{h} \).

With this procedure, the repeated inversion of the convolution integral equation (4) can be avoided each time that a new experimental temperature history \( T_{p}(t) \) is provided. Also, the function \( H(t) \) is independent of the time-variation of the surface heat flux and, therefore, can be determined for a constant heat flux. Beck described the application of a method for numerically inverting the convolution integral in equation (4). In the current work, two general approaches are used to numerically obtain the correction kernel: an adaptation of the sequential function specification method to invert the convolution (4) and numerical Laplace transform inversion of equation (5).

2.1. Function Specification by Sequential Estimation

The inverse problem solution considered in this work is the sequential estimation method of the function specification procedure. This procedure is used to evaluate the integral in equation (4) to estimate \( \hat{H}_{M} \) based on numerical data.

\[ \hat{H}_{M} = \sum_{i=1}^{M} K_{i} \left( T_{p_{0}} - T_{p} \right)_{M+i-1} - \hat{T}_{M+i-1} \mid_{H_{M-1} = \ldots = H_{0} = 0} \phi_{i} \]  

(7)

where \( M = 1, 2, \ldots, n \), \( n \) = number of time steps, \( r \) is the number of future times, \( \phi_{i} = T_{pi} - T_{0} \), \( K_{i} \) is the gain coefficient given by

\[ K_{i} = \frac{\phi_{i}}{\sum_{j=1}^{n} \phi_{j}} \]  

(8)

and \( \hat{T}_{M+i-1} \mid_{H_{0} = 0} \) represents the calculated temperature for the model at time \( t_{M} \) for the estimated kernel components \( \hat{H} \) to \( \hat{H}_{M-1} \), but \( \hat{H}_{M} \) is set equal to zero.
The use of future times data has a regularizing effect on the sequential estimation of the correction kernel values. Since the data used in this work is extracted from either an exact solution or a computational model, the necessary number of future times is relatively small.

2.2. Numerical Laplace Transform Inversion

Various numerical Laplace transform inversion methods were considered to evaluate the kernel function equation (5) in order to determine the correction kernel values. It is necessary to briefly introduce the analytic Laplace transform inversion before addressing the considerable challenges of numerical Laplace transform inversion. For a time domain function \( f(t) \) with the Laplace transform

\[
F(s) = \int_0^\infty e^{-st} f(t) dt \tag{10}
\]

the inverse Laplace transform

\[
f(t) = \frac{1}{2\pi i} \int_{\Lambda} e^{st} F(s) ds \tag{11}
\]

can be evaluated by integration along a contour \( \Lambda \). The Bromwich contour is the standard approach. The Bromwich contour is a contour running parallel to the imaginary axis in the complex-\( s \) plane. The position of the Bromwich contour along the real axis is known as the abscissa of convergence, \( \sigma \), and is conventionally chosen such that it lies to the right of all singularities of \( F(s) \).

The problem of the numerical inversion of the Laplace transform is to obtain approximations for \( f(t) \) when numerical values of the transform function, \( F(s) \), have been computed. Numerical Laplace transform inversion is an inherently ill-posed problem. Considering the need for multiplication by a potentially increasing large exponential \( e^\sigma \), the inherent sensitivity and instability of the numerical inversion procedure can be understood. Algorithmic and precision errors may prevent convergence of numerical solutions. The numerical inversion of a function in the Laplace domain is the most challenging aspect of determining the correction kernel values from the method described by Beck \[4\], which is the reason Beck’s inversion of the convolution integral offers so much appeal.

Several methods of widely used numerical Laplace inversion algorithms can be generalized into four categories: the Post-Widder formula, the Fourier Series Expansion, Talbot’s method, and Weeks method. Each method has found a field of application that is based on the capabilities of the algorithm to invert certain classes of Laplace space functions. Various Laplace transform inversion methods have been reviewed and evaluated in the past \[10\]-\[12\]. Following are descriptions of only the numerical Laplace transform inversion methods considered in this study. These methods include the Gaver-Stehfest algorithm, Weeks method, and Schapery’s simplified approach.

2.2.1. Gaver-Stehfest Method. The Gaver-Stehfest \[13\],[14\] formula for the approximation of the time domain solution from \( F(s) \) is

\[
f(t) = \frac{\ln 2}{t} \sum_{i=1}^{N} V_i F \left( \frac{\ln 2}{t} i \right) \tag{12}
\]

where \( V_i \) is given by
This formulation requires the optimization of parameter $N$, the number of terms in the summation. Different values of $N$ are optimal for different types of functions. The optimal choice for the current problem was found by trial and error to be $N = 4$.

#### 2.2.2. Weeks Method

The formula for the inverse Laplace transform by Weeks method\cite{15} is

$$f(t) = e^{\sigma t} \sum_{n=0}^{N-1} a_n e^{-bt} L_n(2bt)$$ \hspace{1cm} (14)

where the functions $L_n(x)$ for $n \geq 0$ are defined on $x \in (0, \infty)$ by

$$L_n(x) = e^x \frac{a^n}{n!} \int_0^x e^{-t} t^n \, dt$$ \hspace{1cm} (15)

The time independent coefficients $a_n$ contain information particular to the Laplace space function $F(s)$ and may be complex if $f(t)$ is complex. The time independence of the coefficients allows $f(t)$ to be evaluated at multiple times from a single set of coefficients. The two free scaling parameters $\sigma$ and $b$ in the expansion must be selected such that $b > 0$ and $\sigma > \sigma_0$ where $\sigma_0$ is the abscissa of convergence. The selection of these two parameters is critical to the accuracy of the Weeks method. While there are no set criteria for selecting these parameters, Weideman\cite{16} has proposed two algorithms which compute the optimal parameters $\sigma$ and $b$. One algorithm combines numerical and analytical approaches while the other is a strictly numerical approach.

#### 2.2.3. Schapery’s Simplified Laplace Transform Inversion

Beck, Schisler, and Keltner\cite{17} utilize an approximate inverse Laplace transform relation given by Schapery\cite{18} to obtain functional forms of solutions to transient heat conduction problems. While the method is not universally applicable, it works well for some problems where Laplace transform inversion using tables is difficult. The method involves two “rules” for Laplace transforms and their inversion. One rule is for the direct Laplace transform and will be called the Direct Rule while other is for the inverse transform and will be called the Inverse Rule. The Direct Rule is

$$\psi(s) = \frac{1}{s} \left. \psi(t) \right|_{t \to (C^0)}$$ \hspace{1cm} (16)

The Inverse Rule is

$$\psi(t) = \left. s \psi(s) \right|_{s \to (C^0)}$$ \hspace{1cm} (17)

In order to apply this simplified inversion to the correction kernel in equation (5), the Direct Rule must first be applied to both $m'$ and $h'$. The new expressions for $m'$ and $h'$ must be substituted into equation (5) before the Inverse Rule is applied. The resulting kernel expression in the time domain is

$$H^D(s) = m'(\tau) \frac{h'(\tau)}{h(\tau)} - 1$$ \hspace{1cm} (18)

where the superscript $D$ indicates that “dual” rules (both Direct and Inverse Rules) have been applied.

#### 2.2.4. Numerical Differentiation

Regardless of which Laplace transform inversion method is used, the numerical differentiation of values of $h$ and $m$ is necessary to determine $m'$ and $h'$. It is well-known that the numerical differentiation of inexact data is an ill-posed problem\cite{19}. Murio presented a discrete mollification method and computer code for the stable reconstruction of the derivative of a function that is known approximately as a discrete set of data points\cite{20}. A general introduction to the subject of numerical differentiation and a bibliography is provided by Murio\cite{19} and several numerical
examples for the discrete mollification method are presented by Murio et al.\cite{21}. The mollification method is used to handle the ill-posedness of the numerical differentiation in this work.

3. Validation of Inversion Methods with Test Functions

In order to assess the applicability of the various inversion methods, test functions which could be evaluated analytically were used. Since different numerical Laplace transform inversion techniques are best suited only for functions of a specific form, the applicability of the inversion methods is evaluated with functions of form similar to the temperature rise of a semi-infinite heated body. The test functions which represent the temperature curves for the “undisturbed” temperature, \( T_{p\infty} \), and the “measured” temperature, \( T_p \), were used to check the validity of the various inversion methods.

The numerical data was obtained with an axisymmetric heat transfer model of the system illustrated in Figure 1. The nondimensional parameters used to generate the \( T_p \) and \( T_{p\infty} \) histories were \( R/E = 0.5 \), \( \rho/c_p/(\rho c_p)_w = 0.5 \), \( k/k_w = 0.1 \), and \( \tau = 0.05 \) (the subscript \( w \) indicates a quantity associated with the wire). The results are seen as the thin black and gray lines in Figure 2. Analytic expressions which approximate these simulated temperatures are

\[
T_{p\infty} = A\left(1 - \exp\left(-\frac{\tau}{100}\right)\right)
\]

\[
T_p = A\left(1 - \exp\left(-0.75\frac{\tau}{100}\right)\right)
\]

where \( A \) is simply a scale factor (here \( A = 15 \)) used to get the test function values on the same scale as the simulated temperatures. The advantage of using these test functions is that exact analytical expressions can be determined for every step that would otherwise be determined numerically. Temperatures generated from the test functions are shown for dimensionless times from \( \tau = 0 \) to 100 in Figure 2 as thick black and gray lines.

The correction kernel function was estimated using the Sequential Estimation method, the Weeks algorithm, the Gaver-Stehfest method, and Schapery’s simplified approach. The kernel estimates are shown with the exact solution as a function of dimensionless time in Figure 3. Note that the Sequential Estimation solution very nearly overlaps the exact solution.

The error for the various inversion methods is shown in Figure 4. The kernel error is determined by comparison to the exact kernel solution. All of the methods are shown to be reasonably accurate up to about \( \tau = 5 \). The Weeks method is reliable for up to about \( \tau = 25 \). Only the Sequential Estimation
method is reliable for the entire time. Also, the sensitivity of the Laplace transform inversions to noise demands data smoothing while the regularizing effect of Sequential Estimation handles noise nicely.

Using the kernel function estimates and the “measured” temperature test function values, the convolution integral equation (4) was solved to reconstruct the “undisturbed” temperature values using each of the four results. The reconstructed temperatures for all four methods are shown in Figure 5 with the exact values for $T_p$. Despite the error in the kernel values (Figure 4), the reconstructed temperatures came quite close to the exact values. The reconstructed temperature error, Figure 6, shows that the sequential estimation method for determining the correction kernel results in the least error for the overall time domain. However, all of the errors are comparable when considering only the early times, as shown in Figure 7.

**Figure 4.** The correction kernel error for the four inversion techniques determined by comparison to the exact solution.

**Figure 5.** The reconstructed undisturbed temperatures calculated with the exact solution as well as all four inversion techniques.

**Figure 6.** The reconstructed temperature error for the four inversion techniques, full time scale.

**Figure 7.** The reconstructed temperature error for the four inversion techniques, smaller time scale.

Each technique was considered for dimensionless times from $\tau = 0$ to 100 at increments of $\Delta \tau = 0.05$. The computations were performed using Matlab®. The computational expense was greatest for the Gaver-Stehfest algorithm, which took 18.5 seconds. The sequential estimation procedure took 1.3
seconds and the Weeks method took only 0.06 seconds. Schapery’s simplified inversion was the least computationally expensive, taking less than 0.006 seconds to complete.

4. Conclusions
In this paper, a thermocouple temperature measurement error correction method was considered. The method, originally reported by Beck[4], involves a Laplace transform analysis which yields a convolution integral containing a correction kernel. An inversion procedure is necessary to obtain the values of the correction kernel function. It has been shown in this paper that, in addition to the solution reported by Beck[4], various other inversion techniques may be useful in determining the correction kernels. Test functions were used to illustrate the determination of the kernel functions and the reconstruction of the undisturbed temperature from the correction convolution equation (4).

It has been demonstrated that the sequential estimation procedure results in the most reliable overall correction kernel values. The other methods are suitable for shorter times. The kernel values generated from the Weeks algorithm yielded good reconstructed temperatures for dimensionless time $\tau = 0.1$ to 25. All of the methods yielded good reconstructed undisturbed temperatures up to $\tau = 10$. The Gaver-Stehfest method was demonstrated to provide excellent correction up to $\tau = 1$. Schapery’s simplified Laplace transform inversion method yields a simple analytical expression that can be implemented without numerical calculations.

The scope of this paper is the evaluation of the correction kernel method and the various inversion techniques. The results of this investigation will contribute to future work with a more detailed thermocouple model. All of the inversion techniques were useful for determining the correction kernels. The resultant kernel functions all successfully reconstructed the undisturbed temperature histories, thus it may be worthwhile to consider any of the inversion methods with the detailed model.

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5. Appendix – Complete Correction Kernel Formulation

The Laplace transform analysis involved in the derivation of the correction kernel is described in detail in this appendix. This analysis involves the geometry shown in Figure 1. Taking the Laplace transform of the convolution integrals in equations (1) and (2), the following equations are obtained respectively

\[ \mathcal{T}_p - \mathcal{T}_0 = \bar{q} h' \]  \hspace{1cm} (21) \\
and

\[ \mathcal{T}_{ps} - \mathcal{T}_0 = \bar{q} m' \]  \hspace{1cm} (22)

where \( m \) is given by equation (6). The right sides of these equations are obtained using the following general Laplace transformation rules for convolution integrals. For \( \mathcal{L}[f(x)] = F(s) \) and \( \mathcal{L}[g(x)] = G(s) \)

\[ F(s)G(s) = \mathcal{L}\left[ \int_0^\infty f(\lambda)g(t-\lambda) d\lambda \right] \]  \hspace{1cm} (23)

Subtracting equation (21) from (22) yields

\[ \mathcal{T}_{ps} - \mathcal{T}_p = \bar{q} [m' - h'] \]  \hspace{1cm} (24)

In the next step, equation (21) is solved for \( \bar{q} \) which is used to replace \( \bar{q} \) in equation (24).

\[ \mathcal{T}_{ps} - \mathcal{T}_p = \frac{\mathcal{T}_p - \mathcal{T}_0}{h'} [m' - h'] \]  \hspace{1cm} (25)

This simplifies to

\[ \mathcal{T}_{ps} - \mathcal{T}_p = \left( \mathcal{T}_p - \mathcal{T}_0 \right) \frac{m'}{h'} - 1 \]  \hspace{1cm} (26)

This is divided and multiplied by \( s \), the variable associated with the Laplace domain.

\[ \mathcal{T}_{ps} - \mathcal{T}_p = \left[ \frac{m'}{sh'} - \frac{1}{s} \right] \left( s\mathcal{T}_p - s\mathcal{T}_0 \right) \]  \hspace{1cm} (27)

Since \( T_0 \) is a constant, we can substitute \( \mathcal{T}_0 = \frac{T_p}{s} \) to get

\[ \mathcal{T}_{ps} - \mathcal{T}_p = \left( \mathcal{T}_p - \mathcal{T}_0 \right) \left[ \frac{m'}{sh'} - \frac{1}{s} \right] \]  \hspace{1cm} (28)

Note that, in general, the Laplace transform of the first derivative of a function is \( \mathcal{L}[f'(x)] = sF(s) - f(0) \). Apply this and the inverse to the case of \( F(s) = T_p \), where \( T_p(0) = T_0 \), and the following relations result

\[ \mathcal{L}\left[ \frac{\partial T_p(t)}{\partial t} \right] = s\mathcal{T}_p - T_0 \]  \hspace{1cm} (29)

Using the inverse of the relations shown in equations (23) and (29), the inverse Laplace transform of equation (28) can be taken to get the following basic correction equation used to determine the undisturbed temperature, \( T_{ps}(t) \).

\[ T_{ps}(t) - T_p(t) = \int_0^t H(\lambda) \frac{\partial T_p(t - \lambda)}{\partial t} d\lambda \]  \hspace{1cm} (30)

where \( H(\lambda) \) is the correction kernel given in equation (5).