ON A THEOREM OF BANACH
AND KURATOWSKI AND \(K\)-LUSIN SETS

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ABSTRACT. In a paper of 1929, Banach and Kuratowski proved—assuming the continuum hypothesis—a combinatorial theorem which implies that there is no non-vanishing \(\sigma\)-additive finite measure \(\mu\) on \(\mathbb{R}\) which is defined for every set of reals. It will be shown that the combinatorial theorem is equivalent to the existence of a \(K\)-Lusin set of size \(2^{\aleph_0}\) and that the existence of such sets is independent of ZFC + \(\neg\)CH.

0. Introduction. In [1], Stefan Banach and Kazimierz Kuratowski investigated the following problem in measure theory:

**Problem.** Does there exist a non-vanishing finite measure \(\mu\) on \([0,1]\) defined for every \(X \subseteq [0,1]\), which is \(\sigma\)-additive and such that for each \(x \in [0,1]\), \(\mu(\{x\}) = 0\)?

They showed that such a measure does not exist if one assumes the continuum hypothesis, denoted by CH. More precisely, assuming CH, they proved a combinatorial theorem [1, Théorème II] and showed that this theorem implies the nonexistence of such a measure. The combinatorial result is as follows:

**Banach-Kuratowski theorem.** Under the assumption of CH, there is an infinite matrix \(A^i_k \subseteq [0,1]\) (where \(i,k \in \omega\)) such that:

(i) For each \(i \in \omega\), \([0,1] = \bigcup_{k \in \omega} A^i_k\).

(ii) For each \(i \in \omega\), if \(k \neq k'\), then \(A^i_k \cap A^i_{k'} = \emptyset\).
(iii) For every sequence \( k_0, k_1, \ldots, k_i, \ldots \) of \( \omega \), the set \( \cap_{i \in \omega} (A^i_0 \cup A^i_1 \cup \cdots \cup A^i_{k_i}) \) is at most countable.

In the following we call an infinite matrix \( A^i_k \subseteq [0, 1] \) (where \( i, k \in \omega \)) for which (i), (ii) and (iii) hold, a BK-matrix.

Wacław-Sierpiński proved—assuming CH—in [11] and [12] two theorems involving sequences of functions on \([0, 1]\) and showed in [11] and [13] that these two theorems are equivalent to the Banach-Kuratowski theorem, or equivalently, to the existence of a BK-matrix.

Remark. Concerning the problem in measure theory mentioned above, we like to recall the well-known theorem of Stanisław Ulam (cf. [15] or [7, Theorem 5.6]), who showed that each \( \sigma \)-additive finite measure \( \mu \) on \( \omega_1 \), defined for every set \( X \subseteq \omega_1 \) with \( \mu(\{x\}) = 0 \) for each \( x \in \omega_1 \), vanishes identically. This result implies that if CH holds, then there is no non-vanishing \( \sigma \)-additive finite measure on \([0, 1]\).

In the sequel we show that even if CH fails, a BK-matrix—which will be shown to be equivalent to the existence of a \( K \)-Lusin set of size \( 2^{\aleph_0} \)—may still exist.

Our set-theoretical terminology (including forcing) is standard and may be found in textbooks like [2], [4] and [6].

1. The Banach-Kuratowski theorem revisited. Before we give a slightly modified version of the Banach-Kuratowski proof of their theorem, we introduce some notation.

For two functions \( f, g \in \omega^\omega \), let \( f \preceq g \) if and only if for each \( n \in \omega \), \( f(n) \leq g(n) \).

For \( \mathcal{F} \subseteq \omega^\omega \), let \( \lambda(\mathcal{F}) \) denote the least cardinality such that, for each \( g \in \omega^\omega \), the cardinality of \( \{ f \in \mathcal{F} : f \preceq g \} \) is strictly less than \( \lambda(\mathcal{F}) \). If \( \mathcal{F} \subseteq \omega^\omega \) is a family of size \( \lambda \), where \( \lambda \) is the cardinality of the continuum, then we obviously have \( \aleph_1 \leq \lambda(\mathcal{F}) \leq \lambda^+ \). This leads to the following definition:

\[
\mathcal{l} := \min \{ \lambda(\mathcal{F}) : \mathcal{F} \subseteq \omega^\omega \land |\mathcal{F}| = \lambda \}.
\]

If one assumes CH, then one can easily construct a family \( \mathcal{F} \subseteq \omega^\omega \) of cardinality \( \lambda \) such that \( \lambda(\mathcal{F}) = \aleph_1 \) and therefore CH implies that \( \mathcal{l} = \aleph_1 \).
The crucial point in the Banach-Kuratowski proof of their theorem is [1, Théorème II']. In our notation it reads as follows:

**Proposition 1.1.** The existence of a BK-matrix is equivalent to \( l = \aleph_1 \).

For the sake of completeness and for the reader's convenience, we give the Banach-Kuratowski proof of Proposition 1.1.

**Proof.** (\( \Leftarrow \)). Let \( F \subseteq \omega^\omega \) be a family of cardinality \( \mathfrak{c} \) with \( \lambda(F) = \aleph_1 \). In particular, for each \( g \in \omega^\omega \), the set \( \{ f \in F : f \preceq g \} \) is at most countable. Let \( f_\alpha, \alpha < \mathfrak{c} \), be an enumeration of \( F \). Since the interval \( [0, 1] \) has cardinality \( \mathfrak{c} \), there is a one-to-one function \( \Xi \) from \( [0, 1] \) onto \( F \).

For \( x \in [0, 1] \), let \( n_x^i := \Xi(x)(i) \). Now for \( i, k \in \omega \), define the sets \( A^i_k \subseteq [0, 1] \) as follows:

\[
x \in A^i_k \quad \text{if and only if} \quad k = n_x^i.
\]

It is easy to see that these sets satisfy the conditions (i) and (ii) of a BK-matrix. For (iii), take any sequence \( k_0, k_1, \ldots, k_i, \ldots \) of \( \omega \) and pick an arbitrary \( x \in \bigcap_{i \in \omega} (A^i_0 \cup A^i_1 \cup \cdots \cup A^i_{k_i}) \). By definition, for each \( i \in \omega \), we get \( n_x^i \leq k_i \), which implies that for \( g \in \omega^\omega \) with \( g(i) := k_i \) we have \( \Xi(x) \preceq g \). Now, since \( \lambda(F) = \aleph_1 \), \( \Xi(x) \in F \) and \( x \) was arbitrary, the set \( \{ x \in [0, 1] : \Xi(x) \preceq g \} = \bigcap_{i \in \omega} (A^i_0 \cup A^i_1 \cup \cdots \cup A^i_{k_i}) \) is at most countable.

(\( \Rightarrow \)). Let \( A^i_k \subseteq [0, 1] \) (where \( i, k \in \omega \)), be a BK-matrix, and let \( F \subseteq \omega^\omega \) be the family of all functions \( f \in \omega^\omega \) such that \( \bigcap_{i \in \omega} A^i_{f(i)} \) is nonempty. It is easy to see that \( F \) has cardinality \( \mathfrak{c} \). Now, for any sequence \( k_0, k_1, \ldots, k_i, \ldots \) of \( \omega \), the set \( \bigcap_{i \in \omega} (A^i_0 \cup A^i_1 \cup \cdots \cup A^i_{k_i}) \) is at most countable, which implies that for \( g \in \omega^\omega \) with \( g(i) := k_i \), the set \( \{ f \in F : f \preceq g \} \) is at most countable. Hence, \( \lambda(F) = \aleph_1 \).

2. \( K \)-Lusin sets. In this section we show that \( l = \aleph_1 \) is equivalent to the existence of a \( K \)-Lusin set of size \( \mathfrak{c} \).

We work in the Polish space \( \omega^\omega \).
Fact 2.1. A closed set $K \subseteq \omega^\omega$ is compact if and only if there is a function $f \in \omega^\omega$ such that $K \subseteq \{g \in \omega^\omega : g \preceq f\}$.

(See [2, Lemma 1.2.3].)

An uncountable set $X \subseteq \omega^\omega$ is a Lusin set if, for each meager set $M \subseteq \omega^\omega$, $X \cap M$ is countable.

An uncountable set $X \subseteq \omega^\omega$ is a $K$-Lusin set if, for each compact set $K \subseteq \omega^\omega$, $X \cap K$ is countable.

Lemma 2.2. Every Lusin set is a $K$-Lusin set.

Proof. By Fact 2.1 every compact set $K \subseteq \omega^\omega$ is meager (even nowhere dense), and therefore, every Lusin set is a $K$-Lusin set. $\square$

Lemma 2.3. The following are equivalent:

(a) $\mathfrak{l} = \aleph_1$.

(b) There is a $K$-Lusin set of cardinality $\mathfrak{c}$.

Proof. This follows immediately from the definitions and Fact 2.1. $\square$

Remark. Concerning Lusin sets we would like to mention that Sierpiński gave in [14] a combinatorial result which is equivalent to the existence of a Lusin set of cardinality $\mathfrak{c}$.

For $f, g \in \omega^\omega$, define $f \preceq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. The cardinal numbers $\mathfrak{b}$ and $\mathfrak{d}$ are defined as follows:

$$\mathfrak{b} := \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \text{ and } \forall g \in \omega^\omega \exists f \in \mathcal{F}(f \nsubseteq^* g)\}$$

$$\mathfrak{d} := \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \text{ and } \forall g \in \omega^\omega \exists f \in \mathcal{F}(g \nsubseteq^* f)\}.$$ 

Lemma 2.4. $\mathfrak{l} = \aleph_1$ implies $\mathfrak{b} = \aleph_1$ and $\mathfrak{d} = \mathfrak{c}$. Moreover, $K$-Lusin sets are exactly those (uncountable) subsets of $\omega^\omega$ whose all uncountable subsets are unbounded. (Families like that are also called strongly unbounded and they play an important role in preserving unbounded families in iterations, see, e.g., [2] for details.)
Proof. Assume \( l = \aleph_1 \), then, by Lemma 2.3, there exists a \( K \)-Lusin set \( X \subseteq \omega^\omega \) of cardinality \( c \). It is easy to see that every uncountable subset of \( X \) is unbounded, so \( b = \aleph_1 \). On the other hand, every function \( g \in \omega^\omega \) dominates only countably many elements of \( X \). Hence no family \( \mathcal{F} \subseteq \omega^\omega \) of cardinality strictly less than \( c \) can dominate all elements of \( X \), and thus \( d = c \).  

Proposition 2.5. Adding \( \kappa \) many Cohen reals produces a Lusin set of size \( \kappa \).

(See [2, Lemma 8.2.6].)

Theorem 2.6. The existence of a \( K \)-Lusin set of cardinality \( c \) is independent of \( \text{ZFC} + \neg \text{CH} \).

Proof. By Proposition 2.5 and Lemma 2.2 it is consistent with \( \text{ZFC} \) that there is a \( K \)-Lusin set of cardinality \( c \).

On the other hand, it is consistent with \( \text{ZFC} \) that \( b > \aleph_1 \) or that \( d < c \) (cf. [2]). Therefore, by Lemma 2.4, it is consistent with \( \text{ZFC} \) that there are no \( K \)-Lusin sets of cardinality \( c \).  

By Lemma 2.3 and Proposition 1.1, as an immediate consequence of Theorem 2.6, we get the following.

Corollary 2.7. The existence of a \( \text{BK} \)-matrix is independent of \( \text{ZFC} + \neg \text{CH} \).

3. Odds and ends. An uncountable set \( X \subseteq [0, 1] \) is a Sierpiński set if, for each measure zero set \( N \subseteq [0, 1] \), \( X \cap N \) is countable.

Proposition 3.1. The following are equivalent:

(a) \( \text{CH} \).

(b) There exists a Lusin set of cardinality \( c \) and an uncountable Sierpiński set.
There exists a Sierpiński set of cardinality ω and an uncountable Lusin set.

(See [9, p. 217].)

Proposition 3.2. It is consistent with ZFC that there exists a K-Lusin set of cardinality ω, but there are neither Lusin nor Sierpiński sets.

Proof. Let Mω denote the ω2-iteration of Miller forcing—also called “rational perfect set forcing”—with countable support. Let us start with a model V in which CH holds, and let Gω2 = ⟨mι : ι < ω2⟩ be the corresponding generic sequence of Miller reals. Then, in V[Gω2], Gω2 is a K-Lusin set of cardinality ω = ℵ2. For this we have to show the following property:

For all f ∈ ω ∩ V[Gω2], the set {ι : mι $\prec$ f} is countable.

Suppose not, and let f ∈ ω ∩ V[Gω2] be a witness. Further, let p be an Mω-condition such that

\[ p \Vdash_{M\omega} \text{"for some } n_0 \in \omega, \text{ the set } \{ι : \forall k \geq n_0 (m_ι(k) < \dot{f}(k))\} \text{ is uncountable."} \]

We can assume that these dominated reals are among \( m_α : α < β < \omega_2 \) and that β is minimal. This way, f is added after step β of the iteration. Let \( a^* := \text{cl}(\dot{f}) \) be the (countable) set of ordinals such that, if we know \( \{m_ι : ι \in a\} \), then we can compute \( \dot{f} \). (Notice that \( a^* \) is much more than just the support of \( \dot{f} \), since it contains all supports of all conditions that are involved in conditions involved in \( \dot{f} \), and so on.) Let N be a countable model such that \( p, f \in N, a^* \subseteq N \), and let Mα be the iteration of Miller forcing, where we put the empty forcing at stages \( α \notin a^* \) (essentially, Mα is the same as Mβ(...)).

The crucial lemma—which is done in [10, Lemma 3.1] for Mathias forcing, but also works for Miller forcing—is the following: If \( N \models p \in M_{α^*} \), then there exists a q ∈ Mω2 which is stronger than p such that \( \text{cl}(q) = a^* \) and q is (N,Mα)-generic over N. In particular, if \( \{m_ι : ι < \omega_2\} \) is a generic sequence of Miller reals consistent with q, then \( \{m_ι : ι \in a^*\} \) is Mα-generic over N (consistent with p).
So, fix such a \( q \). Now we claim that for \( \gamma \in \beta \setminus N \), \( q \) forces that \( \dot{f}(k) > m_\gamma(k) \) for some \( k \geq n_0 \): Take any \( \gamma \in \beta \setminus N \) and let \( q^* \) be a condition stronger than \( q \). Let \( q^*_1 = q_1 | \beta \), and let \( q^*_2 = q^* | a \). Without loss of generality, we may assume that \( q^*_2 = q \). Now, first we strengthen \( q^*_1 \) to determine the length of stem of \( q^*_1(\gamma) \) and make it equal to some \( k > n_0 \). Next we shrink \( q^*_2 \) to determine the first \( k \) digits of \( \dot{f} \). Finally we shrink \( q^*_1(\gamma) \) such that \( q^*_1(\gamma)(k) > \dot{f}(k) \). Why can we do this? Although \( f \) is added after \( m_\gamma \), from the point of view of model \( N \), it was added before. So, working below condition \( q^*_2 \) (in \( M_{\alpha^*} \)) we can compute as many digits of \( \dot{f} \) as we want without making any commitments on \( m_\gamma \) and vice versa. Even though the computation is in \( N \), it is absolute. This completes the first part of the proof.

On the other hand, it is known (cf. [5]) that in \( V[G_{\omega_2}] \), there are neither Lusin nor Sierpiński sets of any uncountable size, which completes the proof.

Proposition 3.3. It is consistent with ZFC that \( b = \aleph_1 \) and \( d = \omega \), but there is no \( K \)-Lusin set of cardinality \( \omega \).

Proof. Take a model \( M \) in which we have \( c = \aleph_2 \) and in which Martin’s Axiom MA holds. Let \( G = \langle c_\beta : \beta < \omega_1 \rangle \) be a generic sequence of Cohen reals of length \( \omega_1 \). In the resulting model \( M[G] \) we have \( b = \aleph_1 \) (since the set of Cohen reals forms an unbounded family) and \( d = \aleph_2 \). On the other hand, there is no \( K \)-Lusin set of cardinality \( c \) in \( M[G] \). Why? Suppose \( X \subseteq \omega \) has cardinality \( \aleph_2 \). Take a countable ordinal \( \alpha \) and a subset \( X' \subseteq X \) of cardinality \( \aleph_2 \) such that \( X' \subseteq M[G_\alpha] \), where \( G_\alpha := \langle c_\beta : \beta \leq \alpha \rangle \). Now \( M[G_\alpha] = M[c] \) (for some Cohen real \( c \)) and \( M[c] \models \text{MA (} \sigma \text{-centered)} \) (cf. [8] or [2, Theorem 3.3.8]). In particular, since MA (\( \sigma \)-centered) implies \( p = c \) and \( p \leq b \), we have \( M[c] \models b = \aleph_2 \). Thus there is a function which bounds uncountably many elements of \( X' \). Hence, by Lemma 2.4, \( X \) cannot be a \( K \)-Lusin set.

Let \( Q \) be a countable dense subset of the interval \([0, 1]\). Then \( X \subseteq [0, 1] \) is concentrated on \( Q \) if every open set of \([0, 1]\) containing \( Q \) contains all but countably many elements of \( X \).
Proposition 3.4. The following are equivalent:

(a) There exists a $K$-Lusin set of cardinality $\mathfrak{c}$.

(b) There exists a concentrated set of cardinality $\mathfrak{c}$.

Proof. (b) $\rightarrow$ (a). Let $Q$ be a countable dense set in $[0,1]$, and let $\varphi : [0,1] \setminus Q \rightarrow \omega$ be a homeomorphism. If $U \subseteq \omega$ is compact, then $\varphi^{-1}[U]$ is compact, so closed in $[0,1]$ and $[0,1] \setminus \varphi^{-1}[U]$ is an open set containing $Q$. Hence, the image under $\varphi$ of an uncountable set $X \subseteq [0,1]$ concentrated on $Q$ is a $K$-Lusin set of the same cardinality as $X$.

(a) $\rightarrow$ (b). The preimage under $\varphi$ of a $K$-Lusin set of cardinality $\mathfrak{c}$ is a set concentrated on $Q$ of the same cardinality.

Remark. A Lusin set is concentrated on every countable dense set, and concentrated sets always have strong measure zero. However, the existence of a strong measure zero set of size $\mathfrak{c}$ does not imply the existence of a concentrated sets of size $\mathfrak{c}$. In fact, the existence of a strong measure zero set of size $\mathfrak{c}$ is consistent with $\mathfrak{d} = \aleph_1$ (see [3]).

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