ZERO BIASING AND GROWTH PROCESSES

JASON FULMAN AND LARRY GOLDSTEIN

Abstract. The tools of zero biasing are adapted to yield a general result suitable for analyzing the behavior of certain growth processes. The main theorem is applied to prove central limit theorems, with explicit error terms in the $L^1$ metric, for certain statistics of the Jack measure on partitions and for the number of balls drawn in a Pólya-Eggenberger urn process.

1. Introduction

Zero biasing for the normal approximation of a random variable $W$ using Stein’s method was introduced in Goldstein and Reinert [GR]. One instance in which the zero bias method may be applied is for $W$ for which a Stein pair $W,W'$ may be constructed, that is, for $W$ that may be coupled to a variable $W'$ such that $W,W'$ is exchangeable and satisfies $E(W'|W) = (1 - a)W$ for some $a \in (0,1]$. After giving a brief review of these methods in Section 2, in Section 3 we provide a general result allowing one to apply zero biasing when the statistic $W$ of interest is formed by certain growth processes and can be coupled in a Stein pair.

Section 4 studies a certain statistic $W_\alpha$ under the Jack$_\alpha$ measure on partitions. We defer precise definitions to Section 3 but for now mention that is of interest to study statistical properties of Jack$_\alpha$ measure. The case $\alpha = 1$ corresponds to the actively studied Plancherel measure of the symmetric group. The surveys [AlD], [De], [O2] and the seminal papers [BOO], [J], [O1] indicate how the Plancherel measure of the symmetric group is a discrete analog of random matrix theory, and describe its importance in representation theory and geometry. Okounkov [O2] notes that the study of Jack$_\alpha$ measure is an important open problem, about which relatively little is known. It is a discrete analog of Dyson’s $\beta$ ensembles from random matrix theory [BOT].

The particular statistic $W_\alpha$ under Jack measure which we study is of interest for several reasons. When $\alpha = 1$ it reduces to the character ratio of transpositions under Plancherel measure, or equivalently to the spectrum of the random transposition walk. Also by Corollary 1 of [DIL], there is...
a natural random walk on perfect matchings of the complete graph on \( n \) vertices, whose eigenvalues are precisely \( \frac{W(\lambda)}{\sqrt{n(n-1)}} \), occurring with multiplicity proportional to the Jack\( \text{II} \) measure of \( \lambda \). The proofs to date of central limit theorem for \( W_{\alpha} \) range from combinatorial ones using the method of moments in \([K1],[H],[Sn]\), and the use Stein’s method, which produces an error term (but with no explicit constant) in the Kolomogorov metric \([F1],[F2],[SS]\). Our contribution is to prove a central limit theorem in the \( L^1 \) metric, with a small explicit constant.

Section 5 applies the main result of Section 3 to study a growth process arising from the Pólya-Eggenberger urn model. More precisely, imagine an urn \( U_{A,B} \) containing \( A \) white balls and \( B \) black balls. At each time step one ball is drawn, and returned to the urn along with \( m \) balls of the same color. This is one of the simplest urn models, discussed in detail in the textbooks \([JK]\) and \([M]\). We obtain a central limit theorem with explicit error term for the number of white balls drawn after \( n \) steps. While \([JK]\) and \([M]\) contain many useful results and pointers to the literature, including some central limit theorems in more general settings, to the best of our knowledge the literature does not contain results that provide such error terms for this problem.

2. Stein’s method and zero biasing

Stein’s lemma \([S1]\) states that a random variable \( Z \) has the mean zero normal distribution \( \mathcal{N}(0,\sigma^2) \) if and only if

\[
\sigma^2 E f'(Z) = E[Z f(Z)]
\]

for all absolutely continuous functions \( f \) for which these expectations exist. Motivated by this characterization, for a mean zero, variance \( \sigma^2 \) random variable \( W \) and a given function \( h \) on which to test the difference between \( Eh(W) \) and \( Nh = Eh(Z) \), Stein \([S1]\) considered the differential equation

\[
\sigma^2 f'(w) - wf(w) = h(w) - Nh.
\]

For the unique bounded solution \( h \) of \((2)\), one can evaluate the required difference by substituting \( W \) for \( w \) and taking expectation, to yield

\[
Eh(W) - Nh = E[\sigma^2 f'(W) - W f(W)].
\]

Though it may not be immediately clear why the right hand side may be simpler to evaluate than the left, a variety of techniques have been developed to handle various situations. For instance, the exchangeable pair technique, from \([S2]\) handles the expectation of the right hand side when the given random variable \( W \) can be coupled to \( W' \) so that \( (W,W') \) is an \( a \)-Stein pair, that is, an exchangeable pair that satisfies

\[
E(W'|W) = (1-a)W \quad \text{for some } a \in (0,1).
\]
Other techniques for handling the Stein equation are discussed in detail in [C1] and in the references therein, but of particular relevance here is the zero bias coupling, which we now review.

Though the mean zero normal is the unique distribution satisfying (1), one can ask whether a given variable satisfies a like identity of its own. Indeed, it is shown in [GR] that for every mean zero, variance \( \sigma^2 \) random variable \( X \), there exists a distribution for a random variable \( X^* \), termed the \( X \)-zero biased distribution, such that

\[
\sigma^2 E f'(X^*) = E[X f(X)]
\]

for all absolutely continuous functions \( f \) for which these expectations exist. The mapping of \( \mathcal{L}(X) \), the distribution of \( X \), to \( \mathcal{L}(X^*) \), is known as the zero bias transformation. In particular, Stein’s lemma (1) can be rephrased as the statement that the mean zero normal \( \mathcal{N}(0, \sigma^2) \) is the unique fixed point of the zero bias transformation characterized by (4).

Heuristically, then, if the transformation has a fixed point at the mean zero normal, then an approximate fixed point should be approximately normal. This heuristic has been made precise for a variety of examples in [GR], [G1], [G2], [G3] and [G4] (see also [C1]) in order to yield bounds in both the Kolmogorov and \( L^1 \) metric. For the latter, the following result from [G4] is often useful; we use \( || \cdot ||_1 \) to denote the \( L^1 \) metric.

**Theorem 2.1.** If the mean zero, variance 1 random variable \( W \) can be coupled to \( W^* \) having the \( W \)-zero bias distribution, then

\[
||\mathcal{L}(W) - \mathcal{L}(Z)||_1 \leq 2E|W^* - W|
\]

where \( Z \) is a standard normal variable.

Hence, to obtain \( L^1 \) bounds, the question reduces to finding a way to couple \( W \) and \( W^* \). Lemma 2.2 below of [GR], noting here that the result holds also for \( a = 1 \), shows how the construction of a variable \( W^* \) with the \( W \)-zero bias distribution can be achieved with the help of the distribution \( dF(w, w') \) of a Stein pair. First, it can easily be shown from (3) that if \( W, W' \) is an \( a \)-Stein pair possessing second moments then

\[
E W = 0 \quad \text{and} \quad E(W' - W)^2 = 2a \text{Var}(W),
\]

so in particular,

\[
dF^\dagger(w, w') = \frac{(w' - w)^2}{2a}dF(w, w')
\]

is a bivariate distribution.

**Lemma 2.2.** If \( W^\dagger, W^\ddagger \) have distribution (6) where \( F(w, w') \) is the joint distribution of an \( a \)-Stein pair, and \( U \) is a uniformly distributed variable, independent of \( W^\dagger, W^\ddagger \), then

\[
W^* = UW^\dagger + (1 - U)W^\ddagger
\]

has the \( W \)-zero bias distribution.
In particular, if $W$ and $W^\dagger, W^\ddagger$ can be constructed on a common space, then $W$ and $W^\ast$ can also be. We remark that a number of results are available when $(W, W')$ is only an approximate Stein pair, that is, an exchangeable pair that satisfies the linearity condition (3) with a remainder, see for instance [RR], and [C1]. Correspondingly, here we expect the conclusions of Theorems 2.1 and 3.1 to hold for approximate Stein pairs by including in the bounds the additional terms that arise from such remainders.

In what follows we study processes for which the random variable $W$ of interest can be written as the sum $V + T$, where $V$ is a function of a variable $\tau$ determined by the process run to a penultimate state, and $T$ a function of running the process for one additional step. In our examples, given $\tau$, a Stein pair $(W, W') = (V + T, V + T')$ can be constructed by running two copies of the last step of chain, forming $T$ and $T'$ conditionally independent given $\tau$. In such cases a pair of random variables with distribution (6) can be similarly constructed by forming $(W^\dagger, W^\ddagger) = (V^\dagger + T^\dagger, V^\ddagger + T^\ddagger)$ for $V^\square$ and $T^\dagger, T^\ddagger$ sampled by biasing the distributions of $V$ and $T, T'$ in a certain way. Our first application of Theorem 3.1 to Jack measure, is particularly simple since the biasing factor to form the $V^\bullet$ distribution from that of $V$ is unity, and we may therefore take $V = V^\square$. For our second example, the Pólya-Eggenberger urn, we will see that biasing draws from the urn $U_{A, B}$ in our process results in the urn $U_{A + m, B + m}$.

3. General Result

The purpose of this section is to prove the following theorem.

**Theorem 3.1.** Consider a bivariate distribution $\mathcal{L}(\tau, T)$ on a random object $\tau$ and random variable $T$, and a $\tau$ measurable random variable $V = V_\tau$ such that sampling $\tau$, and then, given $\tau$, sampling $T$ and $T'$ independently from the conditional distribution $\mathcal{L}(T|\tau)$, the random variables

$$W = V + T \quad \text{and} \quad W' = V + T'$$

have variance one and are an $a$-Stein pair. Denoting

$$\mathbb{E}(T|\tau) = \mu_\tau \quad \text{and} \quad \mathbb{E}((T - \mu_\tau)^2|\tau) = \sigma_\tau^2,$$

and the distribution of $\tau$ by $dF(\tau)$, the measure $F^\square(\tau)$ specified by

$$dF^\square(\tau) = \frac{\sigma_\tau^2}{a} dF(\tau)$$

is a probability measure, and for any coupling of $\tau$ to $\tau^\square$ with distribution (8), we have

$$||\mathcal{L}(W) - \mathcal{L}(Z)||_1$$

$$\leq 2 \mathbb{E}|(V^\square - V) + (\mu_\tau^\square - \mu_\tau)| + 2\mathbb{E}|T - \mu_\tau| + \frac{\mathbb{E}|T - \mu_\tau|^3}{\text{Var}(T - \mu_\tau)}.$$
When $\mu_\tau$ equals zero and $\sigma^2_\tau$ is constant almost surely, then

\[
||L(W) - L(Z)||_1 \leq 2E|T| + \frac{E|T^3|}{\text{Var}(T)}.
\]

**Proof.** First consider the case where $\mu_\tau = 0$ a.s.. Since conditional on $\tau$ the pair $T$ and $T'$ are independent, we have $E[T'T|\tau] = E[T'|\tau]E[T|\tau] = 0$, and therefore, from (7) and (8),

\[
E((W' - W)^2|\tau) = E((T' - T)^2|\tau) = 2\sigma^2_\tau.
\]

Taking expectation and applying (13), we have that

\[
E\sigma^2_\tau = a,
\]

verifying that $dF^{\Box}(\tau)$ is a probability measure.

By construction, the joint distribution of $(T, T', \tau)$ is, with some abuse of notation, given by

\[
dF(t, t', \tau) = dF(t'|\tau)dF(t|\tau)dF(\tau),
\]

and therefore the pair $(W, W')$ has distribution

\[
dF(w, w') = \int_{\tau, t, t': v+t=v+t'=w'} dF(t'|\tau)dF(t|\tau)dF(\tau),
\]

where $v = V_\tau$. By Lemma 2.2, with $U$ an independent uniform random variable on $[0, 1]$,

\[
W^* = UW^\dagger + (1 - U)W^\ddagger
\]

has the $W$-zero bias distribution when $(W^\dagger, W^\ddagger)$ has distribution given by

\[
dF^\dagger(w, w') = \frac{(w' - w)^2}{2a}dF(w, w').
\]

For any fixed $\tau$ let $F(t|\tau)$ denote the conditional distribution of $T$ given $\tau$. By (12), for every $\tau$ the measure

\[
dF^{\dagger}_\tau(t, t') = \frac{(t' - t)^2}{2\sigma^2_\tau}dF(t'|\tau)dF(t|\tau),
\]

is a bivariate probability distribution.
Now using (14), (13) and (15)

\[
dF^\dagger(w, w') = \frac{(w' - w)^2}{2a} \int_{\tau, t, \tau': t + t' = w, v + v' = w'} dF(t')dF(t|\tau)dF(\tau)
\]

\[
= \int_{\tau, t, \tau': t + t' = w, v + v' = w'} \frac{(w - w')^2}{2a} dF(t'|\tau)dF(t|\tau)dF(\tau)
\]

\[
= \int_{\tau, t, \tau': t + t' = w, v + v' = w'} \frac{\sigma_1^2 (t' - t)^2}{a} dF(t'|\tau)dF(t|\tau)dF(\tau)
\]

\[
= \int_{\tau} \left( \int_{t, t': t + t' = w, v + v' = w'} \frac{(t' - t)^2}{2\sigma_1^2} dF(t'|\tau)dF(t|\tau) \right) \frac{\sigma_2^2}{a} dF(\tau)
\]

\[
= \int_{\tau} \left( \int_{t, t': t + t' = w, v + v' = w'} dF(t', t) \right) dF(\tau).
\]

The factorization in the integral indicates that given \(\tau^\boxdot\) with distribution \(dF(\tau^\boxdot)\), the pair \((W^\dagger, W^\ddagger)\) can be generated by sampling \(T^\dagger_\tau, T^\ddagger_\tau\) from \(dF^\dagger_\tau(t', t)\), and then setting

\[
W^\dagger = V^\boxdot_\tau + T^\dagger_\tau \quad \text{and} \quad W^\ddagger = V^\boxdot_\tau + T^\ddagger_\tau,
\]

where \(V^\boxdot_\tau\) is the value of \(V\) on \(\tau^\boxdot\). In particular, letting

\[
T^\boxdot_\tau = UT^\dagger_\tau + (1 - U)T^\ddagger_\tau,
\]

we have that

\[
W^* = U(V^\boxdot_\tau + T^\dagger_\tau) + (1 - U)(V^\boxdot_\tau + T^\ddagger_\tau) = V^\boxdot_\tau + T^\boxdot_\tau
\]

has the \(W\)-zero biased distribution.

For a fixed \(\tau\), let \(T_\tau^\dagger\) and \(T_\tau^\ddagger\) denote independent copies of a random variable with distribution \(dF(t|\tau)\). Clearly \(T_\tau^\dagger\) and \(T_\tau^\ddagger\) are exchangeable, and as \(\mu_\tau = 0\), we have \(E(T) = E(E(T|\tau)) = E\mu_\tau = 0\) and therefore \(E(T^\dagger|T) = E(T^\ddagger) = 0\). Hence \((T, T')\) is a 1-Stein pair. In view of (15), Lemma 2.2 yields that when \(T^\dagger_\tau, T^\ddagger_\tau\) have distribution \(F^\dagger_\tau(t, t')\) and \(U\) is an independent uniform random variable,

\[
T^\boxdot_\tau = UT^\dagger_\tau + (1 - U)T^\ddagger_\tau
\]

has the \(T\)-zero biased distribution.

As \(E(T) = 0\), by (13) we obtain

\[
a = E\sigma_2^2 = E(E(T^2|\tau)) = E(T^2) = \text{Var}(T).
\]

Comparing (17) and (18), we see that the distribution \(\mathcal{L}(T^\boxdot_\tau)\) is the mixture of the distributions \(\mathcal{L}(T^\dagger_\tau)\) with mixing measure \(\sigma_2^2/\text{Var}(T)\), by (16). Therefore, by Theorem 2.1 of [G3], \(T^\boxdot_\tau\) has the \(T\)-zero bias distribution. Applying the zero bias identity (4) with \(f(x) = (1/2)x^2\text{sign}(x)\), we have

\[
E[T^\boxdot_\tau] = \frac{E[|T^3_\tau|]}{2\text{Var}(T)}.
\]
Now, with $\tau$ and $\tau^\square$ the given coupling, letting $V = V_\tau$ and $T$ be sampled from $L(T | \tau)$, setting $(W, W^*) = (V + T, V^\square + T^\square)$ yields a coupling of $W$ and $W^*$ on the same space, satisfying
\[
\mathbb{E}|W^* - W| = \mathbb{E}|V^\square - V + T^\square - T| \\
\leq \mathbb{E}|V^\square - V| + \mathbb{E}|T| + \mathbb{E}|T^\square| \\
= \mathbb{E}|V^\square - V| + \mathbb{E}|T| + \frac{\mathbb{E}|T^3|}{2\text{Var}(T)}.
\]

Theorem 2.1 now yields
\[
\|L(W) - L(Z)\|_1 \leq 2\mathbb{E}|V^\square - V| + 2\mathbb{E}|T| + \frac{\mathbb{E}|T^3|}{\text{Var}(T)}.
\] (19)

When $\sigma^2_\tau$ is constant we have that $dF^\square(\tau) = dF(\tau)$, and hence may let $\tau^\square = \tau$; taking $V^\square = V$ in (19) now yields (11).

To obtain the result for general $\mu_\tau$, we reduce to the case $\mu_\tau = 0$ by writing
\[
(W, W') = (V + T, V + T') = ((V + \mu_\tau) + (T - \mu_\tau), (V + \mu_\tau) + (T' - \mu_\tau)).
\]
Replacing $V$ and $T$ in (19) by $V + \mu_\tau$ and $T - \mu_\tau$, respectively, yields (10). \(\square\)

4. The Jack measure

In this section we apply Theorem 3.1 to study a property of the Jack$^\alpha$ measure on the set of partitions of size $n$. For $\alpha > 0$ the Jack$^\alpha$ measure chooses a partition $\lambda$ of size $n$ with probability
\[
\text{Jack}_\alpha(\lambda) = \frac{\alpha^n n!}{\prod_{x \in \lambda}(\alpha a(x) + l(x) + 1)(\alpha a(x) + l(x) + \alpha)},
\]
where in the product over all boxes $x$ in the partition $\lambda$, $a(x)$ denotes the number of boxes in the same row of $x$ and to the right of $x$ (the “arm” of $x$), and $l(x)$ denotes the number of boxes in the same column of $x$ and below $x$ (the “leg” of $x$). For example one calculates that the partition
\[
\lambda = \begin{array}{c}
\Box \\
\Box \\
\Box
\end{array}
\]
of 5 has Jack$^\alpha$ measure
\[
\text{Jack}_\alpha(\lambda) = \frac{60\alpha^2}{(2\alpha + 2)(3\alpha + 1)(\alpha + 2)(2\alpha + 1)(\alpha + 1)}.
\]

With $\lambda$ having the Jack$^\alpha$ distribution, we apply the theory of Section 3 to prove an explicit $L_1$ normal approximation bound for the statistic
\[
W_\alpha(\lambda) = \frac{\sum_{x \in \lambda} c_\alpha(x)}{\sqrt{\alpha \binom{n}{2}}},
\]
where \( c_\alpha(x) \) denotes the "\( \alpha \)-content" of \( x \), defined as
\[
c_\alpha(x) = \alpha(\text{column number of } x - 1) - (\text{row number of } x - 1).
\]
In the diagram below representing a partition of 7, each box is filled with its \( \alpha \)-content:
\[
\begin{array}{c|c|c|c}
0 & \alpha & 2\alpha & 3\alpha \\
\hline
-1 & \alpha - 1 & & \\
\hline
-2 & & & \\
\end{array}
\]

In the Kolmogorov metric, the paper \([F1]\) proved an \( O(n^{-1/4}) \) error term for the normal approximation of \( W_\alpha \); this rate was sharpened in \([F4]\) using martingales to \( O(n^{(-1/2) + \epsilon}) \) for any \( \epsilon > 0 \) and in \([F3]\) to \( O(n^{-1/2}) \) using Bolthausen’s inductive approach to Stein’s method, but without an explicit constant. The text \([HO]\) proves a central limit theorem, with no error term, for \( W_\alpha \) using quantum probability. Here we give an explicit \( L_1 \) bound to the normal with small constants.

To obtain our bound we construct an exchangeable pair using Kerov’s growth process for generating a random partition distributed according to Jack \( \alpha \) measure. Given a box \( x \) in the diagram of \( \lambda \), again letting \( a(x) \) and \( l(x) \) denote the arm and leg of \( x \) respectively, set
\[
c_\lambda(\alpha) = \prod_{x \in \lambda} (\alpha a(x) + l(x) + 1), \quad c'_\lambda(\alpha) = \prod_{x \in \lambda} (\alpha a(x) + l(x) + \alpha)
\]
and, for \( \tau \) a partition obtained from \( \lambda \) by removing a single corner box,
\[
\psi'_{\lambda/\tau}(\alpha) = \prod_{x \in C_{\lambda/\tau} - R_{\lambda/\tau}} \frac{(aa_\lambda(x) + l_\lambda(x) + 1)(aa_\tau(x) + l_\tau(x) + \alpha)}{(aa_\lambda(x) + l_\lambda(x) + \alpha)(aa_\tau(x) + l_\tau(x) + 1)}
\]
where \( C_{\lambda/\tau} \) is the union of columns of \( \lambda \) that intersect \( \lambda - \tau \) and \( R_{\lambda/\tau} \) is the union of rows of \( \lambda \) that intersect \( \lambda - \tau \).

The state of Kerov’s growth process at times \( n = 1, 2, \ldots \) is a partition of size \( n \), starting at time one with the unique partition of 1. If at stage \( n - 1 \) the state of the process is the partition \( \tau \), a transition to the partition \( \lambda \) occurs with probability
\[
\frac{c_\tau(\alpha)}{c_\lambda(\alpha)} \psi'_{\lambda/\tau}(\alpha).
\]
As shown in \([K2], [F4]\), if \( \tau \) is chosen from the Jack \( \alpha \) measure on partitions of size \( n - 1 \), then transitioning according to this rule results in a partition \( \lambda \) of \( n \) distributed according to Jack \( \alpha \) measure.

We now present an \( L_1 \) bound for the normal approximation of \( W_\alpha \).

**Theorem 4.1.** Let
\[
W_\alpha(\lambda) = \sum_{x \in \lambda} c_\alpha(x) \sqrt{\frac{\alpha(n/2)}{}}
\]
and let \( W_\alpha \) be the value of \( W_\alpha(\lambda) \) when \( \lambda \) has the Jack measure distribution for some \( \alpha > 0 \). Then for \( Z \) a standard normal random variable,

\[
\| \mathcal{L}(W_\alpha) - \mathcal{L}(Z) \|_1 \leq \sqrt{\frac{2}{n} \left( 2 + \sqrt{2 + \frac{\max(\alpha, 1/\alpha)}{n-1}} \right)}.
\]

**Proof.** First we show (22) holds for all \( \alpha \geq 1 \). Constructing \( \tau \) from the Jack measure on partitions of size \( n - 1 \) and then taking one step in Kerov’s growth process yields \( \lambda \) with the Jack measure on partitions of size \( n \), and we may write

\[ W_\alpha = V + T \]

where

\[ V = \sum_{x \in \tau} c_\alpha(x) \left( n - 1 \right) \]

and \( T = \frac{c_\alpha(\lambda/\tau)}{\sqrt{\alpha(n-1)}} \),

and \( c_\alpha(\lambda/\tau) \) denotes the \( \alpha \)-content of the box added to \( \tau \) to form \( \lambda \).

It is shown in [F1] that constructing \( \lambda' \) by taking another step in Kerov’s growth process from \( \tau \), independently of \( \lambda/\tau \) given \( \tau \), and then forming \( W_\alpha' \) from \( \lambda' \) as \( W_\alpha \) is formed from \( \lambda \), results in exchangeable variables \( W_\alpha, W_\alpha' \) that satisfy (3) with \( a = 2/n \). Hence, (7) of Theorem 3.1 is satisfied. Corollary 5.3 of [F1] gives that \( \text{Var}(W) = 1 \).

From Section 3 of [F3], one recalls the following three facts:

1. \( E[T|\tau] = 0 \) for all \( \tau \).
2. \( E[T^2|\tau] = \frac{2}{n} \) for all \( \tau \).
3. \( E[T^4] = \frac{\alpha^2(n-1)^2 + \alpha(\alpha-1)^2 + 3\alpha^2(n-1)}{\alpha^2(n-1)^2} \)

As \( V \) is measurable with respect to the \( \sigma \)-algebra generated by \( \tau \), condition (8) is satisfied. From properties (1) and (2) above we have, respectively, that \( \mu_\tau = 0 \) and \( \sigma_\tau^2 \) is a constant, almost surely. Hence the bound (11) of Theorem 3.1 holds.

Applying the Cauchy-Schwarz inequality gives that \( E[T] \leq \sqrt{ET^2} = \sqrt{2/n} \), accounting for the first term in the bound. From property (3), now applying \( \alpha \geq 1 \), we have

\[
E[T^4] \leq \left[ \frac{\binom{n}{3} \binom{n-1}{2} + \alpha(n-1)}{\binom{n}{2}^2} \right] + \frac{\alpha(n-1)}{\binom{n}{2}^2} \leq \frac{8}{n^2} + \frac{4\alpha}{n^2(n-1)}.
\]

The Cauchy-Schwarz inequality gives that \( E[T^3] \leq \sqrt{E[T^2]E[T^4]} \), and properties (1) and (2) give \( \text{Var}(T) = 2/n \), yielding the final term in the bound (22). Thus the result is shown when \( \alpha \geq 1 \).

To obtain a bound for all \( \alpha > 0 \) note first that when taking the transpose \( \lambda^t \) of a partition \( \lambda \) the roles of the arms \( a(x) \) and legs \( l(x) \) become interchanged; hence, letting \( \lambda_\alpha \) be a partition with the Jack measure, from
for all $\alpha > 0$ we have
\[ \mathcal{L}(\lambda_\alpha) = \mathcal{L}(\lambda_{1/\alpha}). \]
Next, as $W_\alpha(\lambda) = -W_{1/\alpha}(\lambda^\prime)$ for all $\lambda$, and $\mathcal{L}(Z) = \mathcal{L}(-Z)$,
\[
||\mathcal{L}(W_\alpha(\lambda_\alpha)) - \mathcal{L}(Z)||_1 \\
= ||\mathcal{L}(-W_{1/\alpha}(\lambda^\prime_\alpha)) - \mathcal{L}(Z)||_1 \\
= ||\mathcal{L}(-W_{1/\alpha}(\lambda_{1/\alpha})) - \mathcal{L}(-Z)||_1 \\
= ||\mathcal{L}(W_{1/\alpha}(\lambda_{1/\alpha})) - \mathcal{L}(Z)||_1.
\]
Hence, as the bound (22) holds for all $\alpha \geq 1$, it holds for all $\alpha > 0$. □

5. Pólya–Eggenberger urn model

For $m, n, A, B > 0$ fixed integers, we define a probability distribution on the set $\{0, 1, \ldots, n\}$ by
\[
M_{n,A,B}(k) = \binom{n}{k} \frac{(A/m)_k(B/m)_{n-k}}{(A/m + B/m)_n}.
\] (23)

Unless clarity demands it, we will simply write $M_n(k)$ for $M_{n,A,B}(k)$. Here $x_r = x(x + 1) \cdots (x + r - 1)$, the rising factorial, where we set $x_0 = 1$.

It is well known [K3, M, JK] that the distribution $M_n(k)$ can be achieved in the following way. Imagine an urn $U_{A,B}$ that initially has $A$ white and $B$ black balls. At each time step, one ball is drawn uniformly from the urn and then returned back along with $m$ balls of the same color. If $S_n$ is the number of white balls drawn in the first $n$ draws, then
\[ P(S_n = k) = M_n(k) \quad \text{for } k = 0, 1, \ldots, n. \]

We note that when $S_n = k$ the urn $U_{A,B}$ contains $A + km$ white balls.

In this section we prove the following $L^1$ normal approximation to the distribution of $S_n$, properly standardized.

**Theorem 5.1.** For $n \in \mathbb{N}$ let $S_n$ be the number of white balls added to $U_{A,B}$ after $n$ time steps, and set
\[
W_n = \sqrt{\frac{(A + B + m)n}{AB(A + B + nm)}} \left[ A - \frac{(A + B)S_n}{n} \right].
\] (24)

Then $W_n$ has mean zero and variance 1, and for $Z$ a standard normal random variable, for $n \geq (A + B + m)/2m$
\[
||\mathcal{L}(W_{n+1}) - \mathcal{L}(Z)||_1 \\
\leq \left( \frac{4mn}{A + B + m} + \frac{A^2 + 6AB + B^2}{AB} \right) \sqrt{\frac{(A + B + m)^3}{AB(A + B + nm + m)(n + 1)}}
\]
while for \( n < (A + B + m)/2m, \)

\[
||\mathcal{L}(W_{n+1}) - \mathcal{L}(Z)||_1 
\leq \left( \frac{A^2 + 8AB + B^2}{AB} \right) \sqrt{\frac{(A + B + m)^3}{AB(A + B + nm + m)(n + 1)}}.
\]

From Theorem 3.2 of [M], we know with \( A, B, m \) fixed and \( n \to \infty, \)

\[
S_n/n \to_d \mathcal{B}(A/m, B/m),
\]

that is, the fraction of white balls drawn converges to the Beta distribution with parameters \( A/m, B/m \). In particular, the limiting value of the bound as \( n \to \infty \), giving an \( L^1 \) bound between the standardized Beta distribution and the normal, is \( 4\sqrt{m(A + B + m)/AB} \); for, say \( A = B \), the bound specializes to \( 4\sqrt{m(2A + m)/A} \), which tends to zero at rate \( 1/\sqrt{A} \) if \( m \) is fixed and \( A \) grows.

For what follows it is useful to relate the distribution \( M_n(k) \) to up and down chains. On the set \( \Gamma_n = \{(n, k) : 0 \leq k \leq n\} \), placing directed edges from \((n-1, k)\) to \((n, k)\) and to \((n, k+1)\) results in what is known as Pascal’s lattice [K3]. It is convenient to define \( d((n, k)) = \binom{n}{k} \), the number of paths from \((0, 0)\) to \((n, k)\). More generally, one defines \( d((n, k)/(m, j)) \) to be the number of paths from \((m, j)\) to \((n, k)\); this is \( \binom{n-m}{k-j} \).

We define an “up” chain that transitions from \((n, k)\) to \((n+1, k)\) with probability \( (B+nm-km)/(A+B+nm) \) and to \((n+1, k+1)\) with probability \( (A+km)/(A+B+nm) \). We also define a “down” chain that transitions from \((n, k)\) to \((n-1, k-1)\) with probability \( k/n \) and to \((n-1, k)\) with probability \( 1 - (k/n) \). One easily checks that if \((n-1, k)\) is distributed according to \( M_{n-1} \), then applying the up chain gives an element of \( \Gamma_n \) distributed according to \( M_n \). Similarly, if \((n, k)\) is distributed according to \( M_n \), one checks that applying the down chain gives an element of \( \Gamma_{n-1} \) distributed according to \( M_{n-1} \).

We denote the up chain from \( \Gamma_{n-1} \) to \( \Gamma_n \) by \( U_{n-1} \) and the down chain from \( \Gamma_n \) to \( \Gamma_{n-1} \) by \( D_n \). A straightforward computation yields that

\[
D_{n+1}U_n = c_nU_{n-1}D_n + (1 - c_n)I_n
\]

with \( c_n = \frac{n(A+B+nm-m)}{(n+1)(A+B+nm)} \), so that the tools of [F3] are in force.

The following lemma shows how to use the up and down chains to construct a Stein pair, that is, a pair of exchangeable random variables satisfying [B].

**Lemma 5.2.** Let \( W_n \) be given by (24) with \( S_n \) the number of white balls added to \( U_{A,B} \) after \( n \) time steps. Now construct \( S'_{n} \) by transitioning down using \( D_n \) and then up using \( U_{n-1} \), and let \( W'_{n} \) be given by (24) with \( S_n \) replaced by \( S'_{n} \). Then \( W_n, W'_{n} \) is an \( a_n \)-Stein pair with

\[
a_n = \frac{A + B}{n(A + B + nm - m)}.
\]
Proof. By Theorem 4.3 of [F5] and equation (25), a left eigenvector with eigenvalue \(1 - a_n\) is obtained by applying the operator \(U^{n-1}\) to \((1, 0) - (1, 1)\). From the general theory of down-up chains (see [F5]), one has that

\[
U^{n-1}(1, 0) = \sum_{k=0}^{n} \frac{M_n(k) d((n, k)/(1, 0))}{M_1(0) d(n, k)} \cdot (n, k)
\]

is a right eigenvector of \(U^{n-1}\) with eigenvalue \(1 - a_n\). Since \(U_n D_n\) is a reversible Markov chain with stationary distribution \(M_n\), its right eigenvectors are obtained from its left eigenvectors by dividing by \(M_n\). Thus

\[
W_n = \sum_{k=0}^{n} M_n(k) d((n, k)/(1, 0)) \cdot (n, k).
\]

Similarly,

\[
U^{n-1}(1, 1) = \sum_{k=0}^{n} \frac{M_n(k) d((n, k)/(1, 1))}{M_1(1) d(n, k)} \cdot (n, k)
\]

is a right eigenvector of \(U^{n-1} D_n\) with eigenvalue \(1 - a_n\). Since \(W_n(k)\) is a scalar multiple of \(\frac{A+B}{A+B} \left[ A - k(A+B) \right] \), the result follows.

The next goal is to compute the mean and variance of \(W_n\) given by (24) with \(S_n\) the number of white balls drawn in the first \(n\) draws. Clearly for all \(n \geq 1\) one may write

\[
S_n = 1_0 + \cdots + 1_{n-1}
\]

where \(1_j = 1\) if a white ball is drawn at time \(j\), and \(1_j = 0\) otherwise. The next lemma computes the mean and covariance of the indicators \(1_j\).

Lemma 5.3. For \(j = 0, \ldots, n - 1\), let \(1_j\) denote the indicator that a white ball is drawn from \(U_{A,B}\) at time \(j\). Then

1. \(E[1_j] = \frac{A}{A+B}\) for all \(j \in \{0, \ldots, n - 1\}\).
2. \(E[1_h 1_j] = \frac{A(A+m)}{(A+B)(A+B+m)}\) for all \(0 \leq h < j \leq n - 1\).
3. \(E[S_n] = \frac{A}{A+B}\).

Proof. It is classical and elementary that the indicators \(1_j, j = 0, \ldots, n - 1\) are an exchangeable sequence (see [JK] or [M] for a proof). Thus \(E[1_j]\) is the probability that the first ball drawn is white, and \(E[1_h 1_j]\) is the probability that the first two balls drawn are white. These observations, and linearity of expectation, yields the lemma.
With the help of Lemma 5.3, we now compute the mean and variance of $W_n$.

**Lemma 5.4.** If $W_n$ is given by (24) where $S_n$ is the number of white balls added to $U_{A,B}$ after $n$ time steps, then

$$E[S_n^2] = \frac{nA}{A+B} + 2 \left( \frac{n}{2} \right) \frac{A(A+m)}{(A+B)(A+B+m)},$$

$$EW_n = 0 \quad \text{and} \quad \text{Var}(W_n) = 1.$$  

**Proof.** Since $W_n, W_n'$ is a Stein pair we have that $EW_n = 0$ by (5). Now, using the fact that $\sum_i = \sum_i$ and both parts of Lemma 5.3, we obtain

$$E[S_n^2] = E[(1_0 + \cdots + 1_{n-1})^2]$$

$$= E\left[ \sum_{i=0}^{n-1} 1_i + 2 \sum_{0 \leq h < j \leq n-1} 1_h 1_j \right]$$

$$= \frac{nA}{A+B} + 2 \frac{n}{2} \frac{A(A+m)}{(A+B)(A+B+m)},$$

yielding the first claim.

Applying the expression for $E[S_n]$ given by Lemma 5.3, it follows that

$$\text{Var}(S_n) = \left[ \frac{nA}{A+B} + 2 \frac{n}{2} \frac{A(A+m)}{(A+B)(A+B+m)} \right] - \frac{n^2A^2}{(A+B)^2}$$

$$= \frac{AB(A+B+nm)n}{(A+B+m)(A+B)^2}.$$  

Hence, from the definition (24) of $W_n$ we conclude that

$$\text{Var}(W_n) = \frac{(A+B+m)(A+B)^2}{AB(A+B+nm)n} \text{Var}(S_n) = 1.$$  

□

We will apply Theorem 3.1 by writing $W_{n+1} = V + T$ where

$$V = \sqrt{\frac{(A+B+m)(n+1)}{AB(A+B+nm+m)} \cdot \left[ A - \frac{(A+B)S_n}{n+1} \right]}$$

(26)

and

$$T = -\sqrt{\frac{(A+B+m)}{AB(A+B+nm+m)(n+1)} \cdot (A+B) \cdot 1_n}.$$  

(27)

and letting $\tau = S_n$. We note that the condition in Theorem 3.1 that $V$ be $\tau$ measurable is here clearly satisfied. The following lemma gives the properties of $T$ needed for computing an $L^1$ bound using Theorem 3.1.

**Lemma 5.5.** Let $T$ be given by (27) and $\tau = S_n$.  

(1) The conditional mean $\mu_\tau = E(T|\tau)$ is given by

$$\mu_\tau = -\sqrt{\frac{(A + B + m)}{AB(A + B + nm + m)(n+1)}} \cdot \frac{(A + B)(A + mS_n)}{A + B + mn}. $$

(2) The conditional variance $\sigma_\tau^2 = E((T - \mu_\tau)^2|\tau)$ is given by

$$\sigma_\tau^2 = \frac{(A + B + m)(A + B)^2}{AB(A + B + nm + m)(n+1)} \cdot \frac{(A + mS_n)(nm - mS_n + B)}{(A + B + mn)^2}. $$

(3) The variance $\text{Var}(T - \mu_\tau)$ satisfies

$$\text{Var}(T - \mu_\tau) = \frac{(A + B)}{(n+1)(A + B + nm)}. $$

(4) The absolute deviation of $T$ about $\mu_\tau$ satisfies

$$E|T - \mu_\tau| \leq \sqrt{\frac{(A + B + m)(A + B)^2}{AB(A + B + nm + m)(n+1)}}. $$

(5) The third order deviation of $T$ about $\mu_\tau$, standardized by $\text{Var}(T - \mu_\tau)$, satisfies

$$\frac{E|T - \mu_\tau|^3}{\text{Var}(T - \mu_\tau)} \leq \sqrt{\frac{(A + B + m)^3}{AB(A + B + nm + m)(n+1)}} \cdot \frac{(A + B)^2}{AB}. $$

Proof. Parts (1) and (2) follow immediately from (27) and that

$$P(1_j = 1|S_j = k) = \frac{A + mk}{A + B + mj} $$

for all $j = 0, \ldots, n-1, k = 0, \ldots, j$.

For part (3), first note that as $E(T - \mu_\tau|\tau) = 0$ we have $\text{Var}(T - \mu_\tau) = E(T - \mu_\tau)^2$. Now again using (28), we have that $E((T - \mu_\tau)^2|S_n)$ equals

$$\left(\frac{(A + B + m)(A + B)^2}{AB(A + B + nm + m)(n+1)}\right)E\left[\left(1_n - \frac{A + mS_n}{A + B + nm}\right)^2 |S_n\right]$$

$$= \left(\frac{(A + B + m)(A + B)^2}{AB(A + B + nm + m)(n+1)}\right)\left(\frac{(A + mS_n)(B + m(n - S_n))}{(A + B + mn)^2}\right).$$

Expanding the product $(A + mS_n)(B + m(n - S_n))$, taking expectation using the expressions for $E[S_n]$ and $E[S_n^2]$ provided by Lemmas 5.3 and 5.4, respectively, the claim follows after some simplification.
For part 4, one has that

\[
E|T - \mu_T| = E\left[ E|T - \mu_T||S_n| \right] = \sqrt{\frac{A + B + m}{AB(A + B + nm + m)(n + 1)}} \cdot (A + B) \cdot E \left[ 1_n - \frac{A + mS_n}{A + B + nm} \right] |S_n| \leq \sqrt{\frac{A + B + m}{AB(A + B + nm + m)(n + 1)}} \cdot (A + B).
\]

The second equality used (28), and the inequality that

\[
E \left[ 1_n - \frac{A + mS_n}{A + B + nm} \right] |S_n| \leq \frac{A + B + m}{AB(A + B + nm + m)(n + 1)} \cdot (A + B).
\]

Part 5 now follows from part 3 by division.

Specializing (9) to the case at hand, with \( M_{n,A,B}(k) \) the distribution of \( S_n \) given by (23), we now consider constructing a coupling of \( S_n \) to a random variable \( S_n \) with distribution \( M_{n,A,B}(k) = \sigma^2 k a_n^{1+k} M_{n,A,B}(k) \).

\[
M_{n,A,B}(k) = \frac{\sigma^2}{a_{n+1}} M_{n,A,B}(k)
\]

where \( a_{n+1} \) is given by Lemma 5.2. The next result shows that one can achieve a variable with distribution \( S_n \) by adding 2m additional balls to the urn at time zero, m white and m black, that is, by using the urn \( U_{A+m,B+m} \).

\[
M_{n,A,B} = M_{n,A+m,B+m}.
\]

Proof. From (2) of Lemma 5.5 and Lemma 5.2 we have

\[
\frac{\sigma^2}{a_{n+1}} = \frac{(A/m + B/m)(A/m + B/m + 1)(A/m + k)(B/m + n - k)}{(A/m)(B/m)(A/m + B/m + n)(A/m + B/m + n + 1)}.
\]
Hence, for all \( k \in \{0, \ldots, n\} \),
\[
M_{n,A,B}^\square(k) = \frac{(A/m + B/m)(A/m + B/m + 1)(A/m + k)(B/m + n - k)}{(A/m)(B/m)(A/m + B/m + n)(A/m + B/m + n + 1)} \cdot \binom{n}{k} \frac{(A/m)_{k}(B/m)_{n-k}}{(A/m + B/m)_{n}} = \frac{n}{n-k} \frac{(A/m + 1)_{k}(B/m + 1)_{n-k}}{(A/m + B/m + 2)_{n}} = M_{n,A+m,B+m}(k).
\]

\[\square\]

Lemma 5.6 shows that for the process \( S_n \) on the urn \( U_{A,B} \), the process \( S_n^\square \) is for the urn \( U_{A+m,B+m} \). As for both processes no additional balls have been added at time zero, we have that \( S_0 = 0 \) and \( S_0^\square = 0 \).

As at times \( n \geq 1 \) both of these chains increase by 1 when a white ball has been selected, if \( S_n = k \) and \( S_n^\square = j \), then \( S_{n+1} = k + 1 \) and \( S_{n+1}^\square = j + 1 \) with respective probabilities

\[ s_n(k) = \frac{A + km}{A + B + mn} \quad \text{and} \quad s_n^\square(j) = \frac{A + jm}{A + B + 2m + mn}. \]

We now couple \( S_n^\square \) and \( S_n^\square \) by coupling, at each stage, the two Bernoulli variables that indicate the drawing of a white ball in each urn. In particular, we couple these two Bernoulli variables so that the chance they are not equal is minimized.

**Theorem 5.7.** Let \( s_n(k) \) and \( s_n^\square(j) \) be given by (32) for \( n,j,k \in \mathbb{N} \). Then the bivariate chain taking values in \( \mathbb{N} \times \mathbb{N} \) characterized by the initial condition \( (S_0, S_0^\square) = (0,0) \) and transitions

\[
p_{n+1,n}(u,v|k,j) = P(S_{n+1} = u, S_{n+1}^\square = v|S_n = k, S_n^\square = j)
\]

at times \( n \geq 0 \) according to

\[
p_{n+1,n}(u,v|k,j) = \begin{cases} 
\min(s_n(k), s_n^\square(j)) & (u,v) = (k+1,j+1) \\
(s_n^\square(j) - s_n(k))^+ & (u,v) = (k,j+1) \\
(s_n(k) - s_n^\square(j))^+ & (u,v) = (k+1,j) \\
1 - \max(s_n(k), s_n^\square(j)) & (u,v) = (k,j)
\end{cases}
\]

is a coupling on a joint space of the urn models \( U_{A,B} \) and \( U_{A+m,B+m} \), respectively.

In addition, letting

\[ N = \inf\{n \geq 1 : S_n \neq S_n^\square\} \]

we have

\[ |S_N - S_N^\square| = 1 \]
and if $S_N = S_N + 1$ then $S_n^\square \geq S_n$ for all $n \geq 0$.

while, otherwise,

if $S_N = S_N^\square + 1$ then $S_n \geq S_n^\square$ for all $n \geq 0$.

Proof. That we must have $(S_0, S_0^\square) = (0, 0)$ is clear by (31). As marginally for $S_n$ we have

$$P(S_{n+1} = k + 1|S_n = k) = \min(s_n(k), s_n^\square(j)) + (s_n(k) - s_n^\square(j)) = s_n(k),$$

and similarly for $S_n^\square$, both marginal transition functions agree with those specified by (32), hence the joint chain is a coupling of the two urn models in question. Further, since $S_0 = S_0^\square$, and at most one white ball is drawn from either of the two urns at each time $n \geq 0$, (33) holds.

Taking the difference between the probabilities of drawing a white ball from either of the two urns yields

$$s_n^\square(j) - s_n(k) = m \left( \frac{(A + mn)(j - k - 1) + B(j - k + 1) + 2m(n - k)}{(A + B + mn)(A + B + 2m + mn)} \right).$$

Suppose now that $S_N^\square = S_N + 1$. We show by induction that $S_n^\square \geq S_n + 1$ for all $n \geq N$. Clearly the claim is true for $n = N$. Assume that $S_n^\square \geq S_n + 1$ for some $n \geq N$, say $(S_n, S_n^\square) = (k, j)$ with $j - k \geq 1$. Then, by (34) we see that $s_n^\square(j) \geq s_n(k)$, and hence $(S_{n+1}, S_{n+1}^\square)$ equals $(k + 1, j + 1), (k, j + 1)$ or $(k, j)$ with respective probabilities $s_n(k), s_n^\square(j) - s_n(k)$ and $1 - s_n^\square(j)$. In particular, $S_{n+1}^\square \geq S_{n+1} + 1$.

As the same argument applies in the case $S_n \geq S_n^\square + 1$, and since $S_n^\square = S_n$ for all $0 \leq n < N$ by the definition of $N$, the final claim of the lemma is shown.

We now compute a bound on $E|S_n^\square - S_n|$ for the coupling provided by Theorem 5.7.

Lemma 5.8. The joint chain $(S_n, S_n^\square)$ as specified in Theorem 5.7 satisfies

$$E|S_n^\square - S_n| \leq \frac{2mn}{A + B + m} \left( n \geq \frac{A + B + m}{2m} \right) + 1 \left( n < \frac{A + B + m}{2m} \right).$$

Proof. By Theorem 5.7 with $N$ as defined there, we have

$$|S_n^\square - S_n| = (S_n^\square - S_n)1_{\{n \geq N, S_N^\square = S_N + 1\}} + (S_n - S_n^\square)1_{\{n \geq N, S_N = S_N^\square + 1\}};$$
For the first expectation,

\[
E \left[ (S_n^\square - S_n)1_{\{n\geq N, S_N^\square = S_N+1\}} \right]
\]

\[
= \sum_{t=1}^{n-1} E \left[ (S_n^\square - S_n)1_{\{N=t, S_N^\square = S_N+1\}} \right] + P(N = n, S_N^\square = S_N + 1)
\]

\[
= \sum_{t=1}^{n-1} \sum_{u\geq 0} E \left[ (S_n^\square - S_n)1_{\{N=t, S_N^\square = S_N+1, S_N = u\}} \right] + P(N = n, S_N^\square = S_N + 1)
\]

\[
= \sum_{t=1}^{n-1} \sum_{u\geq 0} E (S_n^\square - S_n|N = t, S_N^\square = S_N + 1, S_N = u) \cdot P(N = t, S_N^\square = S_N + 1, S_N = u) + P(N = n, S_N^\square = S_N + 1).
\]

For \(1 \leq t \leq n-1\), on the conditioning event, urn \(U_{A,B}\) has \(A + mu\) white balls and \(B + mt - mu\) black balls at time \(t\), and then has been run for time \(n - t\). At each of these time steps, by Lemma 5.3, there is probability \((A + mu)/(A + B + mt)\) that a white ball will be selected from urn \(U_{A,B}\).

Similarly, for \(1 \leq t \leq n-1\), on the conditioning event, urn \(U_{A+m,B+m}\) has \(A + m + (mu + m) = A + mu + 2m\) white balls and \(B + m + mt - (mu + m) = B + mt - mu\) black balls at time \(t\), and then has been run for time \(n - t\). At each of these time steps, by Lemma 5.3, the probability is \((A + mu + 2m)/(A + B + mt + 2m)\) that a white ball is selected from urn \(U_{A+m,B+m}\).

Hence, as it may be that all the balls chosen from \(U_{A,B}\) before time \(N\) are black, that is, we may have \(S_N = u\) for \(u = 0\), we have

\[
E (S_n^\square - S_n|N = t, S_N^\square = S_N + 1, S_N = u)
\]

\[
= (n-t) \left( \frac{A + mu + 2m}{A + B + mt + 2m} - \frac{A + mu}{A + B + mt} \right)
\]

\[
= (n-t) \left( \frac{2m(B + mt - mu)}{(A + B + mt)(A + B + mt + 2m)} \right)
\]

\[
\leq 2m(n-t) \left( \frac{(B + mt)}{(A + B + mt)(A + B + mt + 2m)} \right)
\]

\[
\leq \frac{2m(n-t)(B + mt)}{(A + B + mt)^2}
\]

\[
\leq \frac{2m(n-t)}{A + B + mt}.
\]
Therefore

$$E \left[ (S_n^\square - S_n)1_{\{n \geq N, S_N^\square = S_N + 1\}} \right]$$

$$\leq \sum_{t=1}^{n-1} \frac{2m(n-t)}{A + B + mt} \sum_{u \geq 0} P(N = t, S_N^\square = S_N + 1, S_N = u)$$

$$+ P(N = n, S_N^\square = S_N + 1)$$

$$= \sum_{t=1}^{n-1} \frac{2m(n-t)}{A + B + mt} P(N = t, S_N^\square = S_N + 1) + P(N = n, S_N^\square = S_N + 1).$$

Reversing the roles of $S_n$ and $S_n^\square$, though here noting that it is necessary that $u \leq t-1$ for the event $\{N = t, S_N = S_N^\square + 1, S_N^\square = u\}$ to have positive probability, we similarly obtain

$$E \left[ (S_n - S_n^\square)1_{\{n \geq N, S_N = S_N^\square + 1\}} \right]$$

$$\leq \sum_{t=1}^{n-1} \frac{2m(n-t)}{A + B + mt} P(N = t, S_N = S_N^\square + 1) + P(N = n, S_N = S_N^\square + 1).$$

Now using that $(n-t)/(A + B + mt)$ is a decreasing function of $t \geq 0$, summing yields

$$E|S_n^\square - S_n|$$

$$\leq 2m \sum_{t=1}^{n-1} \frac{n-t}{A + B + mt} P(N = t) + P(N = n)$$

$$\leq \frac{2mn}{A + B + m} P(N \leq n-1) + P(N = n)$$

$$\leq \frac{2mn}{A + B + m} \left( n \geq \frac{A + B + m}{2m} \right) + \left( n < \frac{A + B + m}{2m} \right),$$

as claimed, where in the final inequality we have used the fact that since $\alpha + \beta \leq 1$ for $\alpha = P(N \leq n-1)$ and $\beta = P(N = n)$, then for any nonnegative numbers $a$ and $b$ we have $\alpha a + \beta b \leq \max(a, b)$.

\[\square\]

\textit{Proof of Theorem 5.1.} That $EW_n = 0$ and $\text{Var}(W_n) = 1$ is the content of Lemma 5.4.

We now compute the $L^1$ bound using Theorem 3.1. Applying (1) of Lemma 5.5 with $\tau^\square$ and $\tau$ we obtain

$$|\mu_{\tau^\square} - \mu_{\tau}|$$

$$= \sqrt{\frac{(A + B + m)(A + B)^2}{AB(A + B + mn + m)(n + 1)}} \cdot \frac{(A + m|S_n^\square - S_n|)}{A + B + mn}$$

$$\leq \sqrt{\frac{(A + B + m)(A + B)^2}{AB(A + B + mn + m)(n + 1)}},$$
since both $S_n$ and $S^\square_n$ take values between 0 and $n$.

Applying the definition (26) of $V$ on $\tau^\square$ and $\tau$,

$$|V_{\tau^\square} - V| = \sqrt{(A + B + m)(A + B)^2 \cdot |S_n^\square - S_n|},$$

so that for $n \geq (A + B + m)/2m$ we have

$$E|V_{\tau^\square} - V| = \frac{2mn}{A + B + m} \sqrt{(A + B + m)(A + B)^2 \cdot \frac{AB(A + B + nm + m)}{(n+1)}},$$

while for $n < (A + B + m)/2m$,

$$E|V_{\tau^\square} - V| = \sqrt{(A + B + m)(A + B)^2 \cdot \frac{AB(A + B + nm + m)}{(n+1)}}.$$

The calculation is completed by using (4) and (5) of Lemma 5.5 for the final two terms, and then applying the inequality $(A + B + m)(A + B)^2 \leq (A + B + m)^3$.  

References

[AID] Aldous, D. and Diaconis, P., Longest increasing subsequences: from patience sorting to the Baik-Deift-Johansson theorem, Bull. AMS (N.S.) 36 (1999), 413-432.

[BOO] Borodin, A., Okounkov, A., and Olshanski, G., Asymptotics of Plancherel measure for symmetric groups, J. Amer. Math. Soc. 13 (2000), 481-515.

[BO1] Borodin, A. and Olshanski, G., Z-measures on partitions and their scaling limits, European J. Combin. 26 (2005), 795-834.

[C1] Chen, L., Goldstein, L., and Shao, Q. Normal approximation by Stein’s method. Springer (2010).

[De] Deift, P., Integrable systems and combinatorial theory, Notices Amer. Math. Soc. 47 (2000), 631-640.

[DH] Diaconis, P. and Holmes, S., Random walks on trees and matchings, Elec. J. Probab. 7 (2002), 17 pages (electronic).

[EP] Eggenberger, F. and Pólya, G., Über die Statistik verketteter Vorgänge, Z. Angew. Math. Mech. 1 (1923), 279-289.

[F1] Fulman, J., Stein’s method, Jack measure, and the Metropolis algorithm, J. Combin. Theory Ser. A 108 (2004), 275-296.

[F2] Fulman, J., Stein’s method and Plancherel measure of the symmetric group, Trans. Amer. Math. Soc. 357 (2005), 555-570.

[F3] Fulman, J., An inductive proof of the Berry-Esseen theorem for character ratios, Ann. Comb. 10 (2006), 319-332.

[F4] Fulman, J., Martingales and character ratios, Trans. Amer. Math. Soc. 358 (2006), 4533-4552.

[F5] Fulman, J., Commutation relations and Markov chains, Probab. Theory Related Fields 144 (2009), 99-136.

[G1] Goldstein, L., Berry-Esseen bounds for combinatorial central limit theorems and pattern occurrences, using zero and size biasing, J. Appl. Probab. 42 (2005), 661-683.

[G2] Goldstein, L., $L^1$ bounds in normal approximation, Ann. Probab. 35 (2007), 1888-1930.
[G3] Goldstein, L., Bounds on the constant in the mean central limit theorem, *Ann. Probab.* 38 (2010), 1672-1689.

[G4] Goldstein, L., Normal approximation for hierarchical sequences, *Ann. Appl. Probab.* 14 (2004), 1950-1969.

[GR] Goldstein, L., and Reinert, G., Stein’s method and the zero bias transformation with application to simple random sampling, *Ann. Appl. Probab.*, 7 (1997), 935-952.

[H] Hora, A., Central limit theorem for the adjacency operators on the infinite symmetric group, *Comm. Math. Phys.* 195 (1998), 405-416.

[HO] Hora, A. and Obata, N., Quantum probability and spectral analysis of graphs, Theoretical and Mathematical Physics. Springer, 2007.

[IO] Ivanov, V. and Olshanski, G., Kerov’s central limit theorem for the Plancherel measure on Young diagrams, in *Symmetric Functions 2001: Surveys of developments and perspectives*, Kluwer Academic Publishers, Dodrecht, 2002.

[J] Johansson, K., Discrete orthogonal polynomial ensembles and the Plancherel measure, *Ann. of Math.* 153 (2001), 259-296.

[JK] Johnson, N. and Kotz, S., Urn models and their application, An approach to modern discrete probability theory. John Wiley & Sons, 1977.

[K1] Kerov, S.V., Gaussian limit for the Plancherel measure of the symmetric group, *Compt. Rend. Acad. Sci. Paris, Serie I*, 316 (1993), 303-308.

[K2] Kerov, S.V., Anisotropic Young diagrams and Jack symmetric functions, *Funct. Anal. Appl.* 34 (2000), 41-51.

[K3] Kerov, S. V., The boundary of Young lattice and random Young tableaux, in *Formal power series and algebraic combinatorics (New Brunswick, NJ, 1994)*, 133-158, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 24, Amer. Math. Soc., Providence, RI, 1996.

[M] Mahmoud, H., Pólya urn models, Texts in Statistical Science Series. CRC Press, Boca Raton, FL, 2009.

[O1] Okounkov, A., Random matrices and random permutations, *Internat. Math. Res. Notices* 20 (2000), 1043-1095.

[O2] Okounkov, A., The uses of random partitions, *XIVth International Congress on Mathematical Physics*, 379-403, World Sci. Publ., Hackensack, NJ, 2005.

[RR] Rinott, Y., and Rotar, V. On coupling constructions and rates in the CLT for dependent summands with applications to the antivoter model and weighted U-statistics. *Ann. Appl. Probab.* 7 (1997), 1080-1105.

[SS] Shao, Q., and Su, Z., The Berry-Esseen bound for character ratios, *Proc. Amer. Math. Soc.* 134 (2006), 2153-2159.

[Sn] Sniady, P., Gaussian fluctuations of characters of symmetric groups and of Young diagrams, *Probab. Theory Related Fields* 136 (2006), 263-297.

[S1] Stein, C., A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. *Proc. Sixth Berkeley Symp. Math. Statist. Prob.* 2 (1972) pp. 586-602. Univ. of California Press.

[S2] Stein, C., Approximate computation of expectations. IMS, Hayward, California. 1986

University of Southern California, Los Angeles, CA 90089-2532

E-mail address: fulman@usc.edu

University of Southern California, Los Angeles, CA 90089-2532

E-mail address: larry@math.usc.edu