Entropy Admissibility of the Limit Solution for a Nonlocal Model of Traffic Flow

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Abstract

We consider a conservation law model of traffic flow, where the velocity of each car depends on a weighted average of the traffic density $\rho$ ahead. The averaging kernel is of exponential type: $w_\varepsilon(s) = \varepsilon^{-1} e^{-s/\varepsilon}$. For any decreasing velocity function $v$, we prove that, as $\varepsilon \to 0$, the limit of solutions to the nonlocal equation coincides with the unique entropy-admissible solution to the scalar conservation law $\rho_t + (\rho v(\rho))_x = 0$.

1 Introduction

We consider a nonlocal PDE model for traffic flow, where the traffic density $\rho = \rho(t,x)$ satisfies a scalar conservation law with nonlocal flux

$$\rho_t + (\rho v(q))_x = 0. \quad (1.1)$$

Here $\rho \mapsto v(\rho)$ is a decreasing function, modeling the velocity of cars depending on the traffic density, while the integral

$$q(x) = \int_{+\infty}^{x} \varepsilon^{-1} e^{(x-y)/\varepsilon} \rho(y) \, ds \quad (1.2)$$

computes a weighted average of the density of cars ahead. As in [2], we shall assume

(A1) The velocity function $v : [0,\rho_{jam}] \mapsto \mathbb{R}_+$ is $C^2$, and satisfies

$$v(\rho_{jam}) = 0, \quad v'(\rho) \leq -\delta_* < 0, \quad \text{for all } \rho \in [0,\rho_{jam}]. \quad (1.3)$$

One can think of $\rho_{jam}$ as the maximum possible density of cars along the road, when all cars are packed bumper-to-bumper and nobody moves. The conservation equation (1.1) will be solved with initial data

$$\rho(0,x) = \bar{\rho}(x) \in [0,\rho_{jam}] \quad (1.4)$$
As \( \varepsilon \to 0^+ \), the weight function \( w_\varepsilon(s) = \varepsilon^{-1} e^{-s/\varepsilon} \) converges to a Dirac mass at the origin, and the nonlocal equation (1.1)-(1.2) formally converges to the scalar conservation law

\[
\rho_t + (\rho v(\rho))_x = 0.
\]  

(1.5)

Assuming that the initial datum \( \bar{\rho} \) has bounded total variation and takes uniformly positive values, the recent analysis in [2] has established:

(i) For every \( \varepsilon > 0 \), the Cauchy problem with non-local flux (1.1), (1.2), (1.4), has a unique solution \( \rho = \rho_\varepsilon(t,x) \). Its total variation satisfies a uniform bound

\[
\text{Tot.Var.}\{\rho_\varepsilon(t,\cdot)\} \leq M
\]  

where the constant \( M \) is independent of \( t,\varepsilon \).

(ii) As \( \varepsilon \to 0 \), by possibly taking a subsequence, one obtains the convergence \( \rho_\varepsilon \to \rho \) in \( L^1_{\text{loc}} \). The limit function \( \rho = \rho(t,x) \) provides a weak solution to the Cauchy problem (1.4)-(1.5).

A major issue, which was not fully resolved in [2], is the entropy admissibility of the limit solution \( \rho \). Aim of the present note is to resolve this question in the affirmative. Namely, we prove:

**Theorem.** Let \( v \) satisfy the assumptions (A1), and let \( \rho_\varepsilon \) be a sequence of solutions to the nonlocal Cauchy problem (1.1), (1.2) and (1.4), satisfying the uniform BV bounds (1.6). Assume that, as \( \varepsilon \to 0 \), we have the convergence \( \rho_\varepsilon \to \rho \) in \( L^1_{\text{loc}} \).

Then \( \rho \) is the unique entropy admissible solution to the Cauchy problem (1.4)-(1.5).

The above result was proved in [2] in the special case where the velocity is affine: \( v(\rho) = a - b\rho \). The earlier proof was based on the Hardy-Littlewood inequality. In the next section we give a simpler proof, valid for a general class of velocity functions \( v \).

For a more general class of averaging kernels, assuming that the initial datum \( \bar{\rho} \) satisfies a one-sided Lipschitz condition, the convergence to the unique entropy admissible solution was recently proved in [3]. Our result requires an exponential kernel, but it applies to any BV initial data. In particular, \( \bar{\rho} \) can be piecewise constant.

For the general theory of conservation laws we refer to [1, 5, 6]. A brief review of literature on hyperbolic conservation laws with nonlocal flux can be found in [2].

## 2 Proof of the theorem

1. According to [4, 7], to prove uniqueness it suffices to prove that the limit solution dissipates one single strictly convex entropy. We thus consider the entropy and entropy flux pair

\[
\eta(\rho) = \frac{\rho^2}{2}, \quad \psi(\rho) = \int_0^\rho [sv(s) + s^2v'(s)] \, ds.
\]

(2.1)
For future use, we observe that (1.2) implies
\[ \rho = q - \varepsilon q_x. \] (2.2)
Moreover, we introduce the function
\[ W(\rho) = \int_0^\rho s^2 v'(s) \, ds. \] (2.3)
The equation (1.1) can now be written as
\[ \rho_t + (\rho v(\rho))_x = \left( \rho(v(\rho) - v(q)) \right)_x. \]
Multiplying both sides by \( \eta'(\rho) = \rho \), we obtain
\[ \eta(\rho)_t + \psi(\rho)_x = \rho \left( \rho(v(\rho) - v(q)) \right)_x. \] (2.4)

2. Given a test function \( \varphi \in C^1_c(\mathbb{R}), \varphi \geq 0 \), using (2.2) we estimate the quantity
\[ J = \int \rho \left( \rho(v(\rho) - v(q)) \right)_x \varphi \, dx \]
\[ = \int (\rho^2)_x (v(\rho) - v(q)) \varphi \, dx + \int 2 \rho^2 (v(\rho) - v(q)) x \varphi \, dx \]
\[ = - \int \rho^2 (v(\rho) - v(q)) \varphi_x \, dx + \int \rho^2 (v(\rho) - v(q)) x \varphi \, dx \]
\[ = J_1 + J_2. \] (2.5)
Concerning the second integral, using (2.2) we obtain
\[ J_2 = \int \rho v'(\rho) \rho_x \varphi \, dx - \int \rho v'(q)q_x \varphi \, dx + \int \rho \varepsilon (q_x)^2 \varphi \, dx \]
\[ = J_{21} + J_{22} + J_{23}. \] (2.6)
Using (2.2) once again, we now compute
\[ J_{21} + J_{22} = \int \rho^2 v'(\rho) \rho_x \varphi \, dx - \int q^2 v'(q)q_x \varphi \, dx + \int q \varepsilon (q_x)^2 \varphi \, dx \]
\[ \leq J_3 + J_4 + J_5. \] (2.7)
Since \( \rho, q, \varphi \geq 0 \) while \( v' \leq 0 \), from (2.6) and (2.7) we immediately see that
\[ J_{23} \leq 0, \quad J_5 \leq 0. \] (2.8)
On the other hand, integrating by parts and recalling (2.3), we obtain
\[ J_3 + J_4 = \int [W(\rho)]_x \varphi \, dx - \int [W(q)]_x \varphi \, dx \]
\[ = - \int [W(\rho) - W(q)] \varphi_x \, dx. \] (2.9)
3. To conclude, consider a sequence of solutions \( \rho_\varepsilon \) to (1.1)-(1.2), (1.4). Assume that, as \( \varepsilon \to 0 \), we have the convergence \( \rho_\varepsilon \to \rho \) in \( L^1_{loc} \). Notice that this implies \( q_\varepsilon \to \rho \) in \( L^1_{loc} \) as well. Hence, the integrals \( J_1 \) and \( J_3 + J_4 \) both approach zero. By the previous analysis,

\[
2 \int_0^T \int \left\{ \eta(\rho_\varepsilon)\varphi_t + \psi(\rho_\varepsilon)\varphi_x \right\} \, dx \, dt \\
\geq \int_0^T \int \rho_\varepsilon^2 (v(\rho_\varepsilon) - v(q_\varepsilon)) \varphi_x \, dx \, dt + \int_0^T \int \left[ W(\rho_\varepsilon) - W(q_\varepsilon) \right] \varphi_x \, dx \, dt.
\]

Letting \( \varepsilon \to 0 \), since the right hand side converges to zero, we obtain

\[
\int_0^T \int \left\{ \eta(\rho)\varphi_t + \psi(\rho)\varphi_x \right\} \, dx \, dt \geq 0.
\]

This proves that the limit solution \( \rho \) is entropy admissible. In particular, by [4, 7], \( \rho \) is the unique entropy weak solution to the Cauchy problem (1.4)-(1.5).

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