TOPOLOGICAL CLASSIFICATION OF ORIENTED CYCLES OF LINEAR MAPPINGS

We consider oriented cycles of linear mappings over the fields of real and complex numbers. The problem of their classification to within the homeomorphisms of spaces is reduced to the problem of classification of linear operators to within the homeomorphisms of spaces studied by N. Kuiper and J. Robbin in 1973.

1. Introduction. We consider the problem of topological classification of oriented cycles of linear mappings.

Let

\[ A: \ V_1 \underset{A_1}{\rightarrow} V_2 \underset{A_2}{\rightarrow} \cdots \underset{A_{t-1}}{\rightarrow} V_{t-1} \underset{A_t}{\rightarrow} V_t \]  

and

\[ B: \ W_1 \underset{B_1}{\rightarrow} W_2 \underset{B_2}{\rightarrow} \cdots \underset{B_{t-1}}{\rightarrow} W_{t-1} \underset{B_t}{\rightarrow} W_t \]

be two oriented cycles of linear mappings of the same length \( t \) over a field \( \mathbb{F} \). We say that a system \( \varphi = \{ \varphi_i: V_i \rightarrow W_i \}_{i=1}^t \) of bijections transforms \( A \) to \( B \) if all squares in the diagram

\[ V_1 \underset{A_1}{\rightarrow} V_2 \underset{A_2}{\rightarrow} \cdots \underset{A_{t-1}}{\rightarrow} V_{t-1} \underset{A_t}{\rightarrow} V_t \]

\[ W_1 \underset{B_1}{\rightarrow} W_2 \underset{B_2}{\rightarrow} \cdots \underset{B_{t-1}}{\rightarrow} W_{t-1} \underset{B_t}{\rightarrow} W_t \]

are commutative; that is,

\[ \varphi_2 A_1 = B_1 \varphi_1, \quad \ldots, \quad \varphi_{t-1} A_{t-1} = B_{t-1} \varphi_{t-1}, \quad \varphi_1 A_t = B_t \varphi_t. \]
**Definition 1.** Let \( A \) and \( B \) be cycles of linear mappings of the form (1) and (2) over a field \( F \).

(i) \( A \) and \( B \) are isomorphic if there exists a system of linear bijections that transforms \( A \) to \( B \).

(ii) \( A \) and \( B \) are topologically equivalent if \( F = \mathbb{C} \) or \( \mathbb{R} \), and there exists a system of homeomorphisms that transforms \( A \) to \( B \).

The direct sum of cycles (1) and (2) is the cycle

\[
A \oplus B: \quad V_1 \oplus W_1 \xrightarrow{A_1 \oplus B_1} V_2 \oplus W_2 \xrightarrow{A_2 \oplus B_2} \ldots \xrightarrow{A_{t-1} \oplus B_{t-1}} V_t \oplus W_t.
\]

The vector \( \dim A := (\dim V_1, \ldots, \dim V_t) \) is the dimension of \( A \). A cycle \( A \) is indecomposable if its dimension is nonzero and \( A \) cannot be decomposed into a direct sum of cycles of smaller dimensions.

A cycle \( A \) is regular if all \( A_1, \ldots, A_t \) are bijections, and singular otherwise. Each cycle \( A \) possesses a regularizing decomposition

\[
A = A_{\text{reg}} \oplus A_1 \oplus \ldots \oplus A_r,
\]

in which \( A_{\text{reg}} \) is regular and all \( A_1, \ldots, A_r \) are indecomposable singular. An algorithm that constructs a regularizing decomposition of a nonoriented cycle of linear mappings over \( \mathbb{C} \) and uses only unitary transformations was given in [3].

The following theorem reduces the problem of topological classification of oriented cycles of linear mappings to the problem of topological classification of linear operators.

**Theorem 1.** (a) Let \( F = \mathbb{C} \) or \( \mathbb{R} \), and let

\[
A: \quad \mathbb{F}^{m_1} \xrightarrow{A_1} \mathbb{F}^{m_2} \xrightarrow{A_2} \ldots \xrightarrow{A_{t-1}} \mathbb{F}^{m_{t-1}} \xrightarrow{A_t} \mathbb{F}^{m_t}
\]

and

\[
B: \quad \mathbb{F}^{n_1} \xrightarrow{B_1} \mathbb{F}^{n_2} \xrightarrow{B_2} \ldots \xrightarrow{B_{t-1}} \mathbb{F}^{n_{t-1}} \xrightarrow{B_t} \mathbb{F}^{n_t}
\]

be topologically equivalent. Let

\[
A = A_{\text{reg}} \oplus A_1 \oplus \ldots \oplus A_r, \quad B = B_{\text{reg}} \oplus B_1 \oplus \ldots \oplus B_s
\]

be their regularizing decompositions. Then their regular parts \( A_{\text{reg}} \) and \( B_{\text{reg}} \) are topologically equivalent, \( r = s \), and after a suitable renumbering their indecomposable singular summands \( A_i \) and \( B_i \) are isomorphic for all \( i = 1, \ldots, r \).

---

1By [1] (Corollary 19.10) or [2] (Section 11) \( m_1 = n_1, \ldots, m_t = n_t \).
(b) Each regular cycle \( A \) of the form (6) is isomorphic to the cycle

\[
A': \quad \mathbb{F}^{m_1} \xrightarrow{1} \mathbb{F}^{m_2} \xrightarrow{1} \ldots \xrightarrow{1} \mathbb{F}^{m_{t-1}} \xrightarrow{A_t \ldots A_2 A_1} \mathbb{F}^{m_t}.
\]

If cycles (6) and (7) are regular, then they are topologically equivalent if and only if the linear operators \( A_t \ldots A_2 A_1 \) and \( B_t \ldots B_2 B_1 \) are topologically equivalent (as the cycles \( \mathbb{F}^{m_1} \xrightarrow{A_t \ldots A_2 A_1} \mathbb{F}^{m_2} \xrightarrow{A_t \ldots A_2 A_1} \ldots \xrightarrow{A_t \ldots A_2 A_1} \mathbb{F}^{m_t} \) and \( \mathbb{F}^{m_1} \xrightarrow{B_t \ldots B_2 B_1} \mathbb{F}^{m_2} \xrightarrow{B_t \ldots B_2 B_1} \ldots \xrightarrow{B_t \ldots B_2 B_1} \mathbb{F}^{m_t} \) of length 1).

Kuiper and Robbin [4, 5] gave a criterion for topological equivalence of linear operators over \( \mathbb{R} \) without eigenvalues that are roots of 1. Budnitska [6] (Theorem 2.2) found a canonical form with respect to topological equivalence of linear operators over \( \mathbb{R} \) and \( \mathbb{C} \) without eigenvalues that are roots of 1. The problem of topological classification of affine operators was studied in [6, 13 – 16]. The topological classifications of pairs of counter mappings \( V_1 \leftrightarrow V_2 \) (i.e., oriented cycles of length 2) and of chains of linear mappings were given in [17] and [18].

2. Oriented cycles of linear mappings up to isomorphism. This section is not topological; we construct a regularizing decomposition of an oriented cycle of linear mappings over an arbitrary field \( \mathbb{F} \).

A classification of cycles of length 1 (i.e., linear operators \( V \subset \mathbb{F} \)) over any field is given by the Frobenius canonical form of a square matrix under similarity. The oriented cycles of length 2 (i.e., pairs of counter mappings \( V_1 \leftrightarrow V_2 \)) are classified in [19, 20]. The classification of cycles of arbitrary length and with arbitrary orientation of its arrows is well known in the theory of representations of quivers; see [21] (Section 11.1).

For each \( c \in \mathbb{Z} \), we denote by \([c]\) the natural number such that

\[ 1 \leq [c] \leq t, \quad [c] \equiv c \pmod{t}. \]

By the Jordan theorem, for each indecomposable singular cycle \( V \subset \mathbb{F} \) there exists a basis \( e_1, \ldots, e_n \) of \( V \) in which the matrix of \( A \) is a singular Jordan block. This means that the basis vectors form a Jordan chain

\[ e_1 \underbrace{A} \rightarrow e_2 \underbrace{A} \rightarrow e_3 \underbrace{A} \rightarrow \ldots \underbrace{A} \rightarrow e_n \underbrace{A} \rightarrow 0. \]

In the same manner, each indecomposable singular cycle \( A \) of an arbitrary length \( t \) also can be given by a chain

\[ e_p \underbrace{A_p} \rightarrow e_{p+1} \underbrace{A_{p+1}} \rightarrow e_{p+2} \underbrace{A_{p+2}} \rightarrow \ldots \underbrace{A_{q-1}} \rightarrow e_q \underbrace{A_q} \rightarrow 0 \]

in which \( 1 \leq p \leq q \leq t \) and for each \( l = 1, 2, \ldots, t \) the set \( \{ e_i \mid i \equiv l \pmod{t} \} \) is a basis of \( V_l \); see [21] (Section 11.1). We say that this chain ends in \( V_{[q]} \) since \( e_q \in V_{[q]} \). The number \( q - p \) is called the length of the chain.

For example, the chain

\[ e_6 \leftrightarrow e_7 \rightarrow e_8 \rightarrow e_9 \rightarrow e_{10} \rightarrow e_{11} \leftrightarrow e_{12} \rightarrow 0 \]
of length 8 gives an indecomposable singular cycle on the spaces \( V_1 = \mathbb{F}e_6 \oplus \mathbb{F}e_{11}, V_2 = \mathbb{F}e_7 \oplus \mathbb{F}e_{12}, V_3 = \mathbb{F}e_8, V_4 = \mathbb{F}e_4 \oplus \mathbb{F}e_9, V_5 = \mathbb{F}e_5 \oplus \mathbb{F}e_{10}. \)

**Lemma 1.** Let

\[
A : V_1 \xrightarrow{A_1} V_2 \xrightarrow{A_2} \cdots \xrightarrow{A_{t-1}} V_{t-1} \xrightarrow{A_t} V_t
\]

be an oriented cycle of linear mappings, and let (5) be its regularizing decomposition.

(a) Write

\[
\hat{A}_i := A_{[i+t-1]} \cdots A_{[i+1]} A_i : V_i \to V_i
\]

and fix a natural number \( z \) such that

\[
\tilde{V}_i := \hat{A}_i^z V_i = \hat{A}_i^{z+1} V_i \quad \text{for all } i = 1, \ldots, t.
\]

Let

\[
\tilde{A} : \tilde{V}_1 \xrightarrow{\hat{A}_1} \tilde{V}_2 \xrightarrow{\hat{A}_2} \cdots \xrightarrow{\hat{A}_{t-1}} \tilde{V}_{t-1} \xrightarrow{\hat{A}_t} \tilde{V}_t
\]

be the cycle formed by the restrictions \( \tilde{A}_i : \tilde{V}_i \to \tilde{V}_{i+1} \) of \( A_i : V_i \to V_{i+1} \). Then \( A_{\text{reg}} = \tilde{A} \) (and so the regular part is uniquely determined by \( A \)).

(b) The numbers

\[
k_{ij} := \dim \ker(A_{[i+j]} \cdots A_{[i+1]} A_i), \quad i = 1, \ldots, t \quad \text{and} \quad j \geq 0,
\]

determine the singular summands \( A_1, \ldots, A_r \) of regularizing decomposition (5) up to isomorphism since the number \( n_{ij} \) \((l = 1, \ldots, t\) and \( j \geq 0)\) of singular summands given by chains of length \( j \) that end in \( V_l \) can be calculated by the formula

\[
n_{ij} = k_{[l-j],j} - k_{[l-j],j-1} - k_{[l-j-1],j+1} + k_{[l-j-1],j}
\]

in which \( k_{i,-1} := 0. \)

**Proof.** (a) Let (5) be a regularizing decomposition of \( A \). Let

\[
V_i = V_{i,\text{reg}} \oplus V_{ir} \oplus \cdots \oplus V_{ir}, \quad i = 1, \ldots, t,
\]

be the corresponding decompositions of its vector spaces. Then \( \hat{A}_i^z V_{i,\text{reg}} = V_{i,\text{reg}} \) (since all linear mappings in \( A_{\text{reg}} \) are bijections) and \( \hat{A}_i^z V_{ir} = \cdots = \hat{A}_i^z V_{ir} = 0. \) Hence \( V_{i,\text{reg}} = \tilde{V}_i \), and so \( A_{\text{reg}} = \tilde{A} \).

(b) Denote by

\[
\sigma_{ij} := n_{ij} + n_{i,j+1} + n_{i,j+2} + \cdots
\]

the number of chains of length \( \geq j \) that end in \( V_l \). Clearly, \( k_{i0} = \sigma_{i0}, k_{i1} = \sigma_{i0} + \sigma_{i+1,1}, \ldots, \) and

\[
k_{ij} = \sigma_{i0} + \sigma_{[i+1],1} + \cdots + \sigma_{[i+j],j}
\]

for each \( 1 \leq i \leq t \) and \( j \geq 0 \). Therefore,

\[
\sigma_{ij} = k_{ij} - k_{i,j-1}, \quad l := [i+j]
\]

(recall that \( k_{i,-1} = 0 \)). This means that \( l \equiv i + j \) \((\text{mod } t)\), \( i \equiv l - j \) \((\text{mod } t)\), \( i = [l-j] \), and so

\[
\sigma_{ij} = k_{[l-j],j} - k_{[l-j],j-1}.
\]

We get

\[
n_{ij} = \sigma_{ij} - \sigma_{i,j+1} = k_{[l-j],j} - k_{[l-j],j-1} - k_{[l-j-1],j+1} + k_{[l-j-1],j}.
\]

Lemma 1 is proved.
3. Proof of Theorem 1. In this section, \( F = \mathbb{C} \) or \( \mathbb{R} \).

(a) Let \( A \) and \( B \) be cycles (6) and (7). Let them be topologically equivalent; that is, \( A \) is transformed to \( B \) by a system \( \{ \varphi_i : \mathbb{F}^m_i \to \mathbb{F}^n_i \}_{i=1}^t \) of homeomorphisms. Let (8) be regularizing decompositions of \( A \) and \( B \).

First we prove that their regular parts \( A_{\text{reg}} \) and \( B_{\text{reg}} \) are topologically equivalent. In notation (10),

\[
\hat{A}_i = A_{[i+t-1]} \ldots A_{[i+1]} A_i, \quad \hat{B}_i = B_{[i+t-1]} \ldots B_{[i+1]} B_i.
\]

Let \( z \) be a natural number that satisfies both \( \hat{A}_i^z \mathbb{F}^{m_i} = \hat{A}_i^{z+1} \mathbb{F}^{m_i} \) and \( \hat{B}_i^z \mathbb{F}^{n_i} = \hat{B}_i^{z+1} \mathbb{F}^{n_i} \) for all \( i = 1, \ldots, t \). By (3), the diagram

\[
\begin{array}{ccc}
\mathbb{F}^{m_i} & \overset{\hat{A}_i}{\longrightarrow} & \mathbb{F}^{m_i} \\
\varphi_i & \downarrow & \varphi_i \\
\mathbb{F}^{n_i} & \overset{\hat{B}_i}{\longrightarrow} & \mathbb{F}^{n_i}
\end{array}
\]

is commutative. Then \( \varphi_i \text{ Im } \hat{A}_i^z = \text{ Im } \hat{B}_i^z \) for all \( i \). Therefore, the restriction \( \hat{\varphi}_i : \text{ Im } \hat{A}_i^z \to \text{ Im } \hat{B}_i^z \) is a homeomorphism. The system of homeomorphisms \( \hat{\varphi}_1, \ldots, \hat{\varphi}_t \) transforms \( A \) to \( B \), which are the regular parts of \( A \) and \( B \) by Lemma 1(a).

Let us prove that the numbers \( k_{ij} \) are invariant with respect to topological equivalence.

In the same manner as \( k_{ij} \) is constructed by \( A \), we construct \( k'_{ij} \) by \( B \). Let us fix \( i \) and \( j \) and prove that \( k_{ij} = k'_{ij} \). Write

\[
A := A_{[i+j]} \ldots A_{[i+1]} A_i, \quad B := B_{[i+j]} \ldots B_{[i+1]} B_i, \quad q := [i + j + 1]
\]

and consider the commutative diagram

\[
\begin{array}{ccc}
\mathbb{F}^{m_i} & \overset{A}{\longrightarrow} & \mathbb{F}^{m_q} \\
\varphi_i & \downarrow & \varphi_q \\
\mathbb{F}^{n_i} & \overset{B}{\longrightarrow} & \mathbb{F}^{n_q}
\end{array}
\]

which is a fragment of (3). We have

\[
k_{ij} = \dim \text{ Ker } A = m_i - \dim \text{ Im } A, \quad k'_{ij} = n_i - \dim \text{ Im } B.
\]

Because \( \varphi_i : \mathbb{F}^{m_i} \to \mathbb{F}^{n_i} \) is a homeomorphism, \( m_i = n_i \) (see [1], Corollary 19.10, or [2], Section 11). Since the diagram (12) is commutative, \( \varphi_q(\text{ Im } A) = \text{ Im } B \). Hence, the vector spaces \( \text{ Im } A \) and \( \text{ Im } B \) are homeomorphic, and so \( \dim \text{ Im } A = \dim \text{ Im } B \), which proves \( k_{ij} = k'_{ij} \).

(b) Each regular cycle \( A \) of the form (6) is isomorphic to the cycle \( A' \) of the form (9) since the diagram

\[
\text{ISSN 1027-3190. Укр. мат. журн., 2014, т. 66, № 10}
\]
is commutative.

Let \( A \) and \( B \) be regular cycles of the form (6) and (7). Let them be topologically equivalent; that is, \( A \) is transformed to \( B \) by a system \( \varphi = (\varphi_1, \ldots, \varphi_t) \) of homeomorphisms; see (3). By (4),

\[
\varphi_1A_tA_{t-1}\ldots A_1 = B_t\varphi_tA_{t-1}\ldots A_1 = B_tB_{t-1}\varphi_{t-1}A_{t-2}\ldots A_1 = \ldots = B_tB_{t-1}\ldots B_1\varphi_1,
\]

and so the cycles \( F^{m_1} \circ A_1 \ldots A_2A_1 \) and \( F^{m_1} \circ B_1 \ldots B_2B_1 \) are topologically equivalent via \( \varphi_1 \).

Conversely, let \( F^{m_1} \circ A_1 \ldots A_2A_1 \) and \( F^{m_1} \circ B_1 \ldots B_2B_1 \) be topologically equivalent via some homeomorphism \( \varphi_1 \), and let \( A' \) and \( B' \) be constructed by \( A \) and \( B \) as in (9). Then \( A' \) and \( B' \) are topologically equivalent via the system of homeomorphisms \( \varphi = (\varphi_1, \ldots, \varphi_t) \). Let \( \varepsilon \) and \( \delta \) be systems of linear bijections that transform \( A' \) to \( A \) and \( B' \) to \( B \); see (13). Then \( A \) and \( B \) are topologically equivalent via the system of homeomorphisms \( \delta \varphi \varepsilon^{-1} \).

1. Bredon G. E. Topology and geometry. – New York: Springer-Verlag, 1993. – 587 p.
2. McCleary J. A first course in topology: continuity and dimension. – Providence: Amer. Math. Soc., 2006. – 141 p.
3. Sergeichuk V. V. Computation of canonical matrices for chains and cycles of linear mappings // Linear Algebra and Appl. – 2004. – 376. – P. 235–263.
4. Robbin J. W. Topological conjugacy and structural stability for discrete dynamical systems // Bull. Amer. Math. Soc. – 1972. – 78. – P. 923–952.
5. Kuiper N. H., Robbin J. W. Topological classification of linear endomorphisms // Invent. Math. – 1973. – 19, № 2. – P. 83–106.
6. Budnitska T. Topological classification of affine operators on unitary and Euclidean spaces // Linear Algebra and Appl. – 2011. – 434. – P. 582–592.
7. Cappell S. E., Shaneson J. L. Linear algebra and topology // Bull. Amer. Math. Soc. (N. S.). – 1979. – 1. – P. 685–687.
8. Cappell S. E., Shaneson J. L. Nonlinear similarity of matrices // Bull. Amer. Math. Soc. (N. S.). – 1979. – 1. – P. 899–902.
9. Cappell S. E., Shaneson J. L. Non-linear similarity // Ann. Math. – 1981. – 113, № 2. – P. 315–355.
10. Cappell S. E., Shaneson J. L. Non-linear similarity and linear similarity are equivariant below dimension 6 // Contemp. Math. – 1999. – 231. – P. 59–66.
11. Cappell S. E., Shaneson J. L., Steinberger M., West J. E. Nonlinear similarity begins in dimension six // Amer. J. Math. – 1989. – 111. – P. 717–752.
12. Hsiang W. C., Pardon W. When are topologically equivalent orthogonal transformations linearly equivalent? // Invent. Math. – 1982. – 68, № 2. – P. 275–316.
13. Ephraimovitsch W. Topologische Klassifikation affiner Abbildungen der Ebene // Mat. Sb. – 1935. – 42, № 1. – P. 23–36.
14. Blanc J. Conjugacy classes of affine automorphisms of \( \mathbb{R}^n \) and linear automorphisms of \( \mathbb{P}^n \) in the Cremona groups // Manuscr. Math. – 2006. – 119, № 2. – P. 225–241.
15. Budnitska T. V. Classification of topological conjugate affine mappings // Ukr. Math. J. – 2009. – 61, № 2. – P. 164–170.
16. Budnitska T., Budnitska N. Classification of affine operators up to biregular conjugacy // Linear Algebra and Appl. – 2011. – 434. – P. 1195–1199.

ISSN 1027-3190. Укр. мат. журн., 2014, т. 66, № 10
17. Rybalkina T. Topological classification of pairs of counter linear maps // Mat. Stud. – 2013. – 39, № 1. – P. 21–28 (in Ukrainian).

18. Rybalkina T., Sergeichuk V. V. Topological classification of chains of linear mappings // Linear Algebra and Appl. – 2012. – 437. – P. 860–869.

19. Dobrovolskaya N. M., Ponomarev V. A. A pair of counter-operators // Uspekhi Mat. Nauk. – 1965. – 20, № 6. – P. 80–86 (in Russian).

20. Horn R. A., Merino D. I. Contragredient equivalence: a canonical form and some applications // Linear Algebra and Appl. – 1995. – 214. – P. 43–92.

21. Gabriel P., Roiter A. V. Representations of finite-dimensional algebras // Encycl. Math. Sci. – Berlin: Springer-Verlag, 1992. – 73. – 177 p.

Received 16.07.13