P-Branes, Poisson-Sigma-Models and Embedding Approach to \((p+1)\)-Dimensional Gravity

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Abstract

A generalization of the embedding approach for d-dimensional gravity based upon p-brane theories is considered. We show that the \(D\)-dimensional \(p\)-brane coupled to an antisymmetric tensor field of rank \((p+1)\) provides the dynamical basis for the description of \(d = (p+1)\) dimensional gravity in the isometric embedding formalism. "Physical" matter appears in such an approach as a manifestation of a \(D\)-dimensional antisymmetric tensor (generalized Kalb-Ramond) background.

For the simplest case, the Lorentz harmonic formulation of the bosonic string in a Kalb-Ramond background and its relation to a first order Einstein-Cartan approach for \(d = 2\) dimensional gravity is analysed in some detail. A general Poisson-sigma-model structure emerges. For the minimal choice of \(D = 3\) an interesting “dual” formulation is found which has the structure of a Jackiw-Teitelboim theory, coupled minimally to a massive scalar field. Our approach is intended to serve as a preparation for the study of \(d\)-dimensional supergravity theory, either starting from the generalized action of free supersymmetric \((d-1)\)-branes or \(D_{(d-1)}\)-branes, or from the corresponding geometric equations ('rheotropic' conditions).

PACs: 11.15-q, 11.17+y, 04.60.Kz, 04.50.+h, 11.10.Kk.11.30-j.
1 Introduction

As summarized, e.g. in ref. [1], the embedding approach to gravity has a long history, beginning at least from the famous book of Eisenhart [2]. "From time to time, some interesting results would be derived in this way, but they would also be directly derivable from the Riemannian metric, interest in the embedding method thus subsiding again" [1]. Most results obtained by this approach for General Relativity (GR) can be found in [3, 4].

More recently the problems related to the quantization of GR and some early successes in string theory encouraged Regge and Teitelboim to study the possibility of a 'string-like' description of gravity [3] as an alternative for the 'intrinsic' description in terms of Riemannian geometry. This idea generated renewed interest in the search for a dynamical basis regarding the old embedding method [3, 4]. Since then this approach [3] has been developed in several papers [3, 4, 5, 6], where e.g. in [3] the model for gravity provided by a free bosonic p-brane action in curved (but conformally flat) target space-time was considered.

It should be stressed that at the classical level $p$-brane (and $D_p$-brane) theories in these approaches have only matter (and gauge field) degrees of freedom, with gravity involving only auxiliary non-propagating ones. But a kinetic term for gravity should appear in the effective action of the quantum theory after the integration over the matter fields, yielding propagating gravity as a result of quantum effects (see, for example, the discussion of this point in [6]). Hence all the embedding models could be regarded as particular realizations of the concept of gravity induced by quantum corrections [10, 11].

Recent progress to understand some nonperturbative features of superstring theories [12, 13, 14, 15, 16, 17, 18, 19] (and refs. therein), and especially the concept of 'p-brane democracy' [15] adds reasons to search for an adequate description of Nature in terms of the embedding approach and its supersymmetric generalization. Our present paper intends to prepare the ground for such investigations.

An important motivation for our study was the observation that a close relation of the Lorentz-harmonic formulation of bosonic string theory [20, 21] exists with the Poisson-sigma-models (PSM) [22, 23, 24], in which the unification of all types of matterless 2-dimensional gravity (including dilaton gravity, models with dynamical torsion and spherically symmetric 4d gravity) has been achieved. For our present purposes we mainly need from that approach the fact that vielbeine and spin connection are used systematically as independent variables in a first order formulation. We shall demonstrate here that indeed a PSM-like structure appears naturally in the twistor-like Lorentz harmonic formulation of strings [21, 22] after a change of variables corresponding to the transition to the so-called (Lorentz-) analytical coordinate basis.

The general model appears when the string interacting with a Kalb-Ramond (KR) background is considered. A key point is that the interaction with the bosonic string does not put any restriction on the background at the classical level. This is why the interaction with a KR-background produces an analog of an arbitrary potential involved in the PSM-action [22, 23, 24], whereas "matter" type interactions are induced by the transversal components of the target space coordinates.

In principle, $D = 3$ string theory is enough to describe arbitrary curved 2-dimensional space-time. The deep reason for this is given by the general theorem about local isometric embedding of analytical $d = (p + 1)$-dimensional manifold into $D$-dimensional flat Minkowski space-time $\mathbb{R}^D$. It states that such an embedding is always possible (at least locally) when

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1 Equations of motion for background fields appear only as a condition for the vanishing of the $\beta$-function of the string, i.e. as a condition for conservation of conformal invariance in quantum string theory (see [33] and refs. therein).

2 See [36] for a supersymmetric generalization of the theorem.
\[ D \geq \frac{d(d+1)}{2}. \] (1)

For \( d = 2 \) (\( p = 1 \)) we just have \( D = 3 \). Hence we can consider arbitrary 2-dimensional space-time as bosonic string world-sheet embedded into flat space-time. The interaction with a KR (second rank antisymmetric tensor) field makes such an embedding nonminimal. The main extrinsic curvatures of the world surface become arbitrary functions determined by the \( D = 3 \) dimensional KR field strength. Thus, such a model describes arbitrary curved 2-dimensional space-time and, in this sense, also arbitrary two dimensional gravity. In fact, for \( D = 3 \) even in the absence of a KR field, a complicated PSM structure is found for an action consisting of “coupled” versions of the usual first order Einstein-Cartan action for gravitational theories in \( d = 2 \). A certain “duality” is found between the zweibeine on the world sheet and of the residual transverse components of the spin connection. On the other hand, if the string is embedded in \( D > 3 \), beside the arbitrary “matter” contribution provided by the KR field, the minimal coupling to one “pre-matter” scalar field in \( D = 3 \) generalizes to complicated nonminimal interactions with \( D - 2 \) scalars, \( D - 2 \) one forms and \( (D-2)(D-3)/2 \) gauge fields, but maintaining a general PSM structure, as expected. The point of departure of our paper, however, is the generalization of our string model to high dimensional extended objects (p-branes with \( p > 1 \)) \([37, 21]\) yielding a natural dynamical basis for a description of \( d = (p+1) \)-dimensional gravity (in terms of extrinsic geometry) within the embedding approach \([3, 4]\). We find that general \((p+1)\)-dimensional gravity is produced by interaction of the \( p \)-brane model \([37, 21]\) with a generalized KR (GKR) background.

This seems to be a universal property of \( p \)-brane theory in a GKR background. But, to prove this, we need to develop the extrinsic geometry formalism (i.e. the so-called geometric approach \([14, 13, 12, 14, 21]\) for \( p \)-branes in such a background. The Lorentz harmonic formulation \([20, 37, 21]\) is most suitable for this purpose, because it produces the master equations of the geometric approach (so-called “rheotropic conditions” in the terminology of \([48]\)) as equations of motion (e.o.m.-s) in a straightforward way.

Hence the model can be regarded as a realization of the idea by Regge and Teitelboim for a ‘string-like’ description of gravity. In fact, some realizations of this idea were proposed previously \([6, 7, 8, 9]\). So in \([8]\) a model for gravity was constructed from the free bosonic p-brane action in curved (but conformally flat) space-time. \([3]\)

The basic difference of our result with respect to \([8]\) consists in the possibility to consider flat target space (i.e. without target space gravity), at least at the classical level. Thus quantizing that model we bypass the quantum problem of gravity (or its particular case such as conformal gravity) in the ‘large’, i.e. target space, as well as the embedded, i.e. world-volume space-time. \([3]\)

The fact that the KR field may be the origin of arbitrary gravity in the pure bosonic classical description also seems to be related to some recent developments in superstring theory.

The paper is organized as follows:

In Section 2 we give a short summary of the embedding method of General Relativity (GR). After the metric version we also develop the one for Cartan variables. We consider the general case of \( d = (p+1) \) dimensional gravity in a way similar to the one \([21, 52]\) used in the geometric

\[^{3}\]This result have been briefly reported in \([38]\), where some possible application of the embedding approach for \( F\text{-theory} \) \([12, 11, 12, 43]\) were considered

\[^{4}\]The extrinsic geometry of \( D = 4 \) string in a KR field was studied in \([43]\) (see also \([45, 46]\) ), as well as in recent work \([49]\). In \([50]\) string theory for an anti-de-Sitter background was investigated and it was demonstrated that the minimality condition for the embedding is broken in such a case.

\[^{5}\]From the point of view of supergravity the existence of both possibilities to describe world-volume gravity, namely by considering curved target space with nontrivial gravity as well as the flat one with KR fields only, seems to be very natural. Indeed, at high dimensions \((D = 10, 11)\) the only supermultiplet containing the KR fields is the supergravity multiplet which involves gravity too.
approach to $p$-branes (extended objects with $p$ space like dimensions of the world volume). Moving frame variables (Lorentz harmonics [23, 28, 29, 31, 24, 37, 31, 21, 48] are introduced here.

The moving frame (Lorentz harmonic) formulation of the $p$-brane theory [20, 37, 21] in the $D$-dimensional generalized Kalb-Ramond (GKR) background is described in detail in Section 3. It is proved that it provides the dynamical ground for $d = p + 1$ dimensional gravity in the embedding formalism [3, 4].

In Section 4 the bosonic string interacting with the KR field is studied as a model for $d = 2$ gravity. We shortly describe a Lorentz harmonic formulation of free bosonic strings [20, 21] and discuss the PSM-like structure of this formulation appearing in the world-sheet ('analytical') basis. The integration with the KR background shows that a model for a general type of $d = 2$ gravity with matter is obtained in this manner. As a simplest example we treat the gravity models inspired by free string theory.

Supersymmetric generalizations of our approach and general directions of further research are described in the Conclusion.

2 The Embedding Method for $d$-Dimensional Gravity

2.1 Metric Approach

For GR in $d$ dimensional space-time $M^d$ is obtained by specifying the matter, calculating its energy-momentum tensor $T^{mn}(x) \equiv e_m^a e_n^b T^{ab}(x)$, and use the latter as a source for the Einstein field equation

$$R_{bc}^{ac} - 1/2 \delta_c^a R_{dc}^{dc} = 1/2 T^a_b(x)$$

where the curvature two-form is defined by

$$R^{ab}(d, d) \equiv d \Omega^{ab} - \Omega^{ac} \wedge \Omega_c^b$$

$$\equiv 1/2 dx^m \wedge dx^n R^{ab}_{mn}(x) \equiv 1/2 e^d \wedge e^c R_{cd}^{ab}(x)$$

The spin connections $\Omega^{ab}$ are supposed to be the ones, constructed from the vielbein fields ($e^a = dx^m e^a_m$)

$$e^m_a \Omega_{mbc} =$$

$$= e^m_a e^n_b \partial_{[m} e^c_{n]} + e^m_c e^n_a \partial_{[m} e^c_{n]} + e^m_c e^n_a \partial_{[m} e^c_{n]}$$

as a consequence of vanishing torsion

$$T^a \equiv D e^a \equiv d e^a - e^b \wedge \Omega_b^a = 0,$$

with

$$T^a \equiv 1/2 dx^m \wedge dx^n T^{mn}_{ia} \equiv 1/2 e^b \wedge e^c T^{ab}_{cb}$$

the torsion two-form on $M^d$.

So the curved space-time $M^d$ is described as the solution of Einstein equation (2), (3), (4), for a given matter distribution specified by the expression for energy-momentum tensor $T^{ab}(x)$.  

...
To find exact solutions of the Einstein equations (2) for \( d = 4 \) embedding methods were widely used in the past [3, 4]. They are based on general theorems [2] stating that any analytical \( d \)-dimensional curved space-time \( \mathcal{M}^d \) can be considered as a subspace in flat \( D \)-dimensional pseudo-Euclidean space-time \( R^{1,D-1} \) with \( D \leq d(d+1)/2 \) at least locally [1]. The metric of \( \mathcal{M}^d \) is induced by the embedding

\[
d s^2 = dx^m dx^n g_{mn}(x) = dx^m dx^n \partial_n X^m \partial_m X^n \eta_{mn},
\]

i.e. it is expressed in terms of derivatives \( \partial_n X^m \) of the coordinate functions defining \( \mathcal{M}^d \) parametrically as the subspace in \( R^{1,D-1} \):

\[
X^m = X^m(x^n)
\]

Here \( X^m \) (\( m = 0,1, \ldots, (D - 1) \)) denote Cartesian coordinates of \( R^{1,D-1} \),

\[\eta_{mn} \equiv \text{diag}(+1,-1, \ldots, -1)\]

is the flat Minkowski metric, \( x^m \) (\( m = 0,1, \ldots, (d - 1) \)) represent local holonomic coordinates of \( \mathcal{M}^d \).

This description of \( d \)-dimensional space-time is similar to the one used for the world sheet of strings \( (d = 2) \) and for the world volume of p-brane \( (d = p+1) \) theories. Further steps towards the description of the embedding method of GR imply the introduction of an extrinsic geometry formalism for \( \mathcal{M}^d \). They are similar to the ones performed in the geometric approach for strings \[44, 45, 21\] and will be considered in the same manner [21].

### 2.2 Extrinsic Geometry in Cartan Variables

For our present purpose the vielbein formalism turns out to be more suitable than the metric one. To understand the embedding conditions in this case, let us use an appropriate local Lorentz (\( SO(1,D-1) \)) transformation to adjust to each point of \( \mathcal{M}^d \) a local moving (co-)frame of \( R^{1,D-1} \)

\[
E^a = dX^m u^a_m, \quad a = 0,1, \ldots, D-1
\]

by the matrices \( u^a_m = (u^a_m, u^i_m) \in SO(1,D-1) \)

\[
u^a_m u^a_n u^b_m = \eta^{ab}
\]

in such a way that \( d = p + 1 \) vectors \( u^a_m(x) \) are parallel to \( \mathcal{M}^d \) and \( (D - d) \) vectors \( u^i_m(x) \) are orthogonal to it:

\[
E^a = dX^m u^a_m = \epsilon^a, \quad a = 0,1, \ldots, p \quad p = d-1
\]

\[
E^i = dX^m u^i_m = 0, \quad i = 1, \ldots, (D-d)
\]

\( \epsilon^a = d\xi^m \epsilon^a_m \) in the r.h.s. of \((10)\) is the intrinsic vielbein form of \( \mathcal{M}^d \). Due to \((10)\) it is induced by the embedding \[\]!

The rectangular blocks \( u^a_m, u^i_m \) of the \( SO(1,D-1) \)-valued moving frame matrix \( u^a_m \) are restricted by \[6\] which can be decomposed as

\[\]!

\[\]!

\[\]!

\[\]!
Due to the natural $SO(1,d-1) \otimes SO(D-d)$ gauge invariance of the present construction, the variables $u^a_m$, $u^i_m$ are simply interpreted as homogeneous coordinates of the coset $SO(1,d-1) \otimes SO(D-d)$. We call them (vector) Lorentz harmonic variables [25] (see also [26, 27, 28, 29, 30, 31], [20, 37, 21] and refs. in [24]) using the term 'harmonic' in the sense of the work [22, 25].

The form of the induced metric (6) follows from (10), (12) and the definition $g_{mn} = e^a_m e^b_n$. Eqs. (10), (11) ('rheotropic conditions' [48]) are equivalent to the isometric embedding conditions (3) after the orthonormality relations (9) ((12) - (14)) and the manifest $SO(1,d-1) \otimes SO(D-d)$ gauge symmetry of Eqs. (10), (11) are taken into account.

Passing from Eqs. (10), (11) to their selfconsistency (integrability) conditions

$$dE^a \equiv dX^m \wedge du^a_m = de^a,$$

$$dE^i = dX^m \wedge du^i_m = 0,$$

we can exclude the explicit embedding functions $X^m(x^a)$ from our further considerations.

To obtain the differentials of the moving frame variables $u^a_m = (u^m_a, u^i_m)$ we shall take into account the orthonormality conditions (3) or (12) - (14). Using the unity matrix decomposition

$$\delta^a_m = u^a_m u^b_m = u^a_m u^b_m - u^i_m u^i_m,$$

being equivalent to (3) ((12) - (14)), the differentials of the moving frame vectors $u^a_m = (u^m_a, u^i_m)$ become

$$du^a_m = u^b_m \Omega^a_{\ b}(d)$$

$$\Longleftrightarrow$$

$$\begin{cases} du^a_m = u^b_m \omega^a_{\ b}(d) + u^i_m f^a_i(d), & a = a; \\ du^i_m = -u^i_m A^i + u^a_m f^a_i(d) & a = i \end{cases}$$

Here

$$\Omega^a_{\ b}(d) = -\Omega^b_{\ a}(d) \equiv u^a_m du^b_m = \left( \begin{array}{cc} \omega^a_{\ b}(d) & f^a_i(d) \\ -f^b_{\ j}(d) & A^j_i(d) \end{array} \right)$$

is the $so(1,D-1)$ valued Cartan 1-form. In [49] it is decomposed (in a $SO(1,d-1) \otimes SO(D-d)$ gauge covariant way) into $d \times (D-d)$ covariant forms (forming the basis of the coset $SO(1,d-1) \otimes SO(D-d)$)

$$f^a_i \equiv u^a_m du^m_i,$$

$$\frac{d \times (d-1)}{2} \quad SO(1,d-1) \text{ connection 1-forms}$$

$$\omega^{ab} \equiv u^a_m du^m_b,$$

and (for $D-d > 1$), $\frac{(D-d) \times (D-d-1)}{2}$ internal $SO(D-d)$ connection 1-forms $A^{ij}(d)$, whose pull-backs onto $M^d$, $A^i_j(d) = dx^m A^{ij}_m$, produce world sheet gauge fields $A^i_j_m(x)$

$$A^i_j \equiv u^i_m du^m_j.$$
¿From (19) the Cartan forms $\Omega^a{}_b$ satisfy the Maurer-Cartan equation

$$d\Omega^a{}_b - \Omega^a{}_c \wedge \Omega^c{}_b = 0$$

(23)

which now just reflect the flatness of the D dimensional embedding space.

The $SO(1,d-1) \otimes SO(D-d)$ gauge covariant splitting of the connection form $\Omega^{ab}$ (19) induces the splitting of (23) into the following set of equations for the forms $f^a$, $\omega^{ab}$, $A^{ij}$:

$$D f^a \equiv df^a - f^b \wedge \omega^a_b + f^a \wedge A^{ij} = 0$$

(24)

$$R^{ab} \equiv d\omega^{ab} - \omega^a_c \wedge \omega^b_c = f^a \wedge f^b$$

(25)

$$R^{ij} \equiv dA^{ij} + A^{ik} \wedge A^{kj} = f^i \wedge f^j$$

(26)

Eqs. (24), (25), (26) give rise to Peterson-Codazzi, Gauss and Ricci equations, respectively, of the subspace embedding theory [2, 4] (see also [21]).

The covariant differential $D$ used in Eq. (24) include the form $\omega^{ab}$ and $A^{ij}$ as $SO(1,d-1)$ and $SO(D-p)$ connections. So its pull-back onto the surface $\mathcal{M}^d$ ($D = dx^m \mathcal{D}_m = e^a \mathcal{D}_a$) considered as a covariant differential on $\mathcal{M}^d$ implies that spin connections and gauge fields are being induced by the embedding.

Using this differential we can investigate the integrability conditions for Eqs. (10) and (11) in manifestly $SO(1,d-1) \otimes SO(D-d)$ gauge invariant form

$$D E^a = dX^m \wedge Du^a_m = \mathcal{D} e^a,$$

(27)

$$D E^i = dX^m \mathcal{D} u^i_m = 0.$$  

(28)

The covariant differentials $D$ for the moving frame variables $u^a_m$, $u^i_m$ from (18) are expressed in terms of $SO(1,D-1)/SO(1,d-1) \otimes SO(D-d)$ coset forms $f^a$ alone:

$$Du^a_m = du^a_m - u^b_m \omega^a_b(d) = u^a_m f^a(d),$$

$$Du^i_m = du^i_m + u^j_m A^{ij} = u^i_m f^a(d).$$

(29)

Substituting (29) into (27), (28), we get

$$T^a \equiv D e^a \equiv de^a - e^b \wedge \omega^a_b = E^i \wedge f^a(d).$$

(30)

$$0 = D E^i \equiv dE^i + E^j \wedge A^{ji} = E_a(d) \wedge f^a(d),$$

(31)

The ‘rheotropic conditions’ (11) and (10) result in the vanishing of the r.h.s. of (30) and in the possibility to rewrite the r.h.s. of Eq. (31) in terms of the intrinsic vielbein $e_b$ of $\mathcal{M}^d$. Thus, the selfconsistency conditions for (10), (11) imply the vanishing of torsion (3)

$$T^a \equiv D e^a \equiv de^a - e^b \wedge \omega^a_b = 0,$$

(32)

and of

$$e_a(d) \wedge f^a(d) = 0,$$

(33)

respectively. Eq. (32) determines the induced spin connections $\omega^{ab}$ in terms of the vielbeine. Eq. (33) reflects the symmetry properties of the second fundamental form matrix $K^i{}_m \equiv K^{ij} \equiv \partial_m \partial_n X^{mn} \partial_i, \ L_m \equiv \theta_m \Omega^{mn} \partial_n$ with $\partial_m X^{mn} \partial_i = 0 \ [4].$
$e^a_m e^n_b K^i_{ab} = K^i_{nm}$ appearing in the decomposition of the pull-back of the covariant connection form $f^a_i$ (see [21])

$$f^a_i = e_b K^{abi}. \quad (34)$$

Equations (24)–(26), (32), (33) are known from the Classical Theory of Surfaces [2] and describe an arbitrary surface embedded into flat $D$-dimensional space-time.

The main extrinsic curvatures of the surface $h^i$ are the traces of the second fundamental form matrix $K^{abi}$

$$h^i \equiv K^{abi} \eta_{ab} = e^m_a f^a_i \equiv f^a_i. \quad (35)$$

They can be used to define the embedded surface $\mathcal{M}^d$. In particular, the world volume of the free bosonic $p$-brane ($p = d - 1$) is considered to be a minimal surface, i.e. is defined by

$$h^i = 0$$

In the embedding method of GR the description of the curved space time $\mathcal{M}^d$ is achieved by the use of the extrinsic geometry equations together with the algebraic equation

$$f_c^a f_c^i - f_b^a f_c^i - 1/2 \delta_b^a (f_c^d f^i_c - f_d^i f^c_c) \equiv K^a_i b^c - K^b_i a^c - 1/2 \delta_b^a (K^c_i b^f - K^f_i K^c_i) = T^a_b(x). \quad (36)$$

This can be derived from the Einstein equation (2) when the Gauss equation (25) is taken into account.

3 Bosonic P-Branes with Kalb-Ramond Background and $d = p + 1$ Gravity

The main result of this section is that arbitrary nonvanishing extrinsic curvature $h^i \neq 0$ can be produced by a $D = (p + 1)(p + 2)/2$-dimensional $p$-brane theory interacting with a GKR background, i.e. with an antisymmetric tensor field of rank $(p + 1)$.

This statement seems to be universal, i.e. independent of the special formulation of $p$-brane theory. But for the proof, the extrinsic geometry formalism (i.e. the so-called geometric approach [14, 13, 17, 21, 51, 52]) for $p$-branes in a GKR-background should be developed. The Lorentz harmonic formulation [21, 37, 21] is most suitable to do this, because it produces the master equations of the geometric approach (rheotrophic conditions in the terminology of ref. [48]) as e.o.m.-s (see below).

We will demonstrate that for a $D$-dimensional $p$-brane interacting with an antisymmetric tensor field $B_{m_1...m_{p+1}}(\mathbf{x})$ (GKR field) the main extrinsic curvatures of the $p$-brane world volume indeed are nonvanishing

$$h^i \equiv K^a_i \equiv f^a_i(\mathcal{D}_a) =$$

$$= - \frac{1}{(p+1)!} \varepsilon_{a_0...a_p} u^{a_0 m_0}(x)...u^{a_p m_p}(x) \times$$

$$H_{m_1...m_{p+2}}(\mathbf{x}), \quad (37)$$

and expressed in terms of the field strength of the GKR field

$$H_{m_1...m_{p+2}}(\mathbf{x}) \equiv$$

$$\equiv (\partial_{m_1} B_{m_2...m_{p+2}}(\mathbf{x}) - \partial_{m_2} B_{m_1...m_{p+2}}(\mathbf{x}) + ...). \quad (38)$$

This means that
any curved space-time $M^d$ can be identified locally with some world volume of such a type.

- matter is the manifestation of the GKR background in such an approach.

### 3.1 Action Functional

Let us consider a $p$-brane interacting with an antisymmetric tensor gauge field of the rank $(p+1)$ (GKR field)

$$B_{p+1} = dX^{m_{p+1}} \wedge \ldots \wedge dX^{m_1} B_{m_1 \ldots m_{p+1}}(X)$$

in the twistor-like Lorentz harmonic formulation \[5, 21\].

Then the action functional

$$S_{p+1} = S_{p+1}^0 + S_{p+1}^{\text{int}}$$

is the sum of the free $p$-brane action \[24, 37, 21\]

$$S_{p+1}^0 = -\frac{1}{p!} \int_{M_{p+1}} (E^a \wedge e^{a_1} \wedge \ldots \wedge e^{a_p} - \frac{p}{(p+1)} e^a \wedge e^{a_1} \wedge \ldots \wedge e^{a_p}) \epsilon_{a_1 \ldots a_p}$$

(which represents the bosonic limit of the generalized action for super-$p$-branes \[48\]), and of an interaction term

$$S_{p+1}^{\text{int}} = \int_{M_{p+1}} B_{p+1}.$$ 

In \[11\], \[12\] the Lagrangian $(p+1)$ forms are integrated over a world volume $M^{p+1}$ of the $p$-brane (which we will identify with curved $d = p + 1$ dimensional space-time) whose local coordinates are denoted by $x^m$ $(a = 0, 1, \ldots, p)$ (and will identified with the $x^m$ coordinates from the previous section).

For simplicity in \[11\] we fix the "cosmological constant" (being proportional to the inverse $p$-brane tension, i.e. to the Regge slope parameter $\alpha'$ for the string case) to be equal to one.

The variation of the moving frame vectors $u^a$ and $u^i$ which does not break the orthonormality conditions \[1\] (or \[12\]–\[14\]) reduces to generalized Lorentz ($SO(1, D - 1)$) transformations

$$\delta u^a_m = u^a_i \Omega^i_k \Omega^k_m = u^a_i \Omega^i_m (\delta)$$

$$\Leftrightarrow \begin{cases} \delta u^a_m = u^b_m \omega^a_b (\delta) + u^i_m f^{ai}(\delta), & a = a ; \\ \delta u^i_m = u^i_m A^i_j (\delta) - u^j_m A^{i_j}(\delta), & a = i \end{cases}$$

where the parameters of unconstrained variations $i \Omega^a_k = \Omega^a_k (\delta) = (f^{ai}(\delta), \omega^{ab}(\delta), A^{ij}(\delta))$ can be considered as contractions of the Cartan forms \[13\] (or \[20\] - \[22\]) with the variation symbol $\delta$, or equivalently as Cartan forms depending on the variation symbol $\delta$ instead of the external differential $d$.

Similar expressions for the external differential of the vectors $u^a_m$ are obtained from \[18\] and are equivalent to the definition of the connection forms \[19\] (or \[24\], \[2\], \[2\]).
3.2 Equations of Motion

Taking into account the constrained nature of the harmonic variables \([9] - [12] - [14]\) the variation of the action \([40]\) can be written as

\[
\delta S = \int_{M^{p+1}} \left[ -\frac{1}{(p-1)!} (E^a - e^a) \wedge e^{a_1} \wedge ... \wedge e^{a_{p-1}} \wedge (\delta e^{a_p} - e^b \omega^a_{b} (\delta)) \epsilon_{a_1...a_p} \right]
\]

\[
- \frac{1}{p!} f^{ai}(\delta) \wedge E^i(d)e^{a_1} \wedge ... \wedge e^{a_p} \epsilon_{a_1...a_p}
\]

\[
+ \frac{1}{p!} E^a(\delta) (f^{ai}(d) \wedge e^{a_1} \wedge ... \wedge e^{a_p} \epsilon_{a_1...a_p} - p! i_u H_{p+2})
\]

\[
- \frac{1}{p!} D E^a(\delta)e^{a_1} \wedge ... \wedge e^{a_p} \epsilon_{a_1...a_p} + E^a(\delta)i_u a H_{p+2},
\]

where

\[
H_{p+2} \equiv dB_{p+1} = \frac{1}{(p+2)!} dX^{m \cdot p+2} \wedge ... \wedge dX^{m \cdot p+2} H_{m_1...m_{p+2}}(X)
\]

is a GKR field strength and

\[
i_{c}^a H_{p+2} \equiv \frac{1}{(p+1)!} dX^{m \cdot p+2} \wedge ... \wedge dX^{m \cdot p+2} H_{m_1...m_{p+2}}(X).
\]

The contraction of basic 1-forms with the variation symbol \(\delta\)

\[
E^a(\delta) \equiv i_\delta E^a \equiv \delta X^{m \cdot n_{a \cdot n}}
\]

\[
E^i(\delta) \equiv i_\delta E^i \equiv \delta X^{m \cdot n_{i \cdot n}}
\]

\[
\omega^{ab}(\delta) \equiv i_\delta \omega^{ab} \equiv u^a_{\cdot m} \delta u_{\cdot m}^{b \cdot n}
\]

\[
f^{ai}(\delta) \equiv i_\delta f^{ai} \equiv u^a_{\cdot m} \delta u_{\cdot n}^{i \cdot n} = \delta u^{a \cdot m} u_{\cdot n}^{i \cdot n}
\]

\[
A^{ij}(\delta) \equiv i_\delta A^{ij} \equiv u^i_{\cdot m} \delta u_{\cdot m}^{j \cdot n},
\]

(47)

together with the variation of the world volume vielbein

\[
\delta e^a = d\xi^m \delta e_{a \cdot m}
\]

are taken as a basis in the space of variations.

The absence of the parameters \(A^{ij}(\delta)\) in \([14]\) reflects the \(SO(D - p - 1)\) gauge invariance of the action \([40]\). The \(SO(1, p)\) gauge invariance manifests itself as the presence of the parameter \(\Omega^{ab}(\delta)\) in the combination

\[
\delta e^{a_p} - e^{b} \omega^{a_p}_{b}(\delta)
\]

only. Hence we can compensate \(SO(1, p)\) (pseudo-)rotations of harmonic variables by (pseudo-)rotations of the world volume vielbein

\[
\delta_{SO(1, p)} e^{a_p} = e^{b} \omega^{a_p}_{b}(\delta) = e^{b} i_\delta \omega^{a_p}_{b}.
\]

The latter can be interpreted as world volume Lorentz transformations. Therefore, the \(SO(1, p)\) subgroup of the target space Lorentz group \(SO(1, D - 1)\) is identified as a group of gauge symmetries of the action \([40]\) with the world volume Lorentz group \(SO(1, p)\).
As a consequence, the world volume spin connections \( w^{ab} \) are singled out which are induced by the embedding, i.e.
\[
    w^{ab} = \omega^{ab},
\] (48)
where \( \omega^{ab} \) is the pull-back of the \( SO(1,p) \) connection form (21).

The variation \( E^a(\delta) \equiv \bar{i}_\delta E^a = \delta X \bar{w}^a_{\mu} \) is related to the parameter of general coordinate (reparameterization) invariance. For this purpose the equivalent language of Noether identities is much more convenient, i.e. to analyse the interdependence of the e.o.m.-s as a result of such gauge symmetries (see below).

Now let us consider the e.o.m.-s which follow from (44) in detail starting with the variation of the world volume vielbeins \( \delta e^a \) and the rest of the admissible variations of moving frame variables characterized by parameter \( i_\delta f^{ai} \). From (19) the Cartan forms lead to the nondynamical equations (10), (11) which are the master equations of the geometric approach [44, 45, 21] as well as for the embedding method of GR [3, 4] (cf. section 2): From \( \delta e^a \) follows
\[
    (E^a - e^a) \wedge e^{a_1} \wedge \ldots \wedge e^{a_{p-1}} \epsilon_{a_1 \ldots a_p} = 0
\] or
\[
    E^a = i_\mu \bar{w}_{\mu}^a = e^a,
\] (49)
whereas from \( i_\delta f^{ai} = f^{ai}(\delta) \) we obtain
\[
    E^i \wedge e^{a_1} \wedge \ldots \wedge e^{i_{p-1}} \epsilon_{a_1 \ldots a_p} = 0
\]
or
\[
    E^i = i_\mu \bar{w}_{\mu}^i = 0.
\] (50)
Eqs. (49) and (50) precisely lead to the relations used in section 2: The pull-backs of the \( (p+1) \) basic one forms \( E^a \) of the target space become tangent to the world volume and coincide with world volume vielbeine \( e^a \) (which in turn are introduced by embedding on the shell of the rheotropic conditions), and the pull-backs of the remaining \( (D-p-1) \) basic 1-forms \( E^i \) vanish. Thus the \( (D-p-1) \) vectors being dual to these 1-forms become orthogonal to the world volume.

As demonstrated in section 2, the selfconsistency conditions for the rheotropic relations (49), (50) are identical to eqs. (32), (33), respectively.

The covariant differential \( D \) appearing in eqs. (32) and (33) is the pull-back of the one defined in Eq. (18) (i.e. with \( SO(1,p) \) and \( SO(D-p-1) \) connections induced by embedding \( w^{ab} = \omega^{ab}, B^{ij} = A^{ij} \)).

The variation \( E^a(\delta) \equiv \bar{i}_\delta E^a = \delta X \bar{w}^a_{\mu} \) does not lead to independent e.o.m.-s. In accordance with the second Noether theorem this means that \( E^a(\delta) \) is related to \( (p+1) \) parameters of the gauge symmetry of the \( p \)-brane action, namely reparametrization symmetry (or general coordinate invariance for the world volume).

This can be seen from the e.o.m. for \( E^a(\delta) \) \[11\]
\[
    (-1)^{p+1} \frac{1}{(p-1)!} D e^a \wedge e^{a_1} \wedge \ldots \wedge e^{a_{p-1}} \epsilon_{a_1 \ldots a_p} = i_{\bar{w}^a} H_{p+2}.
\] (51)
The l.h.s. of Eq. (51) vanishes due to (33) and the r.h.s.
\[
    i_{\bar{w}^a} H_{p+2} = \frac{1}{(p+1)!} dX_{m_{p+1}} \ldots dX_{m_1} u^a_\mu \bar{H}_{m_{m_1} \ldots m_{m_{p+1}}} (X(\xi)),
\]
\[10\]And corresponding to the \( \frac{SO(1,D-1)}{SO(1,p) \otimes SO(D-p-1)} \) ("boost") transformations
\[11\]The input from the surface term appearing due to the integration by parts is neglected here for simplicity. Hence we treat the case of closed \( p \)-branes only.
is proportional to

\[ \propto u_{a_{p+2}} \cdots u_{a_2} u_{a_1} H_{mm_2 \cdots m_{p+2}}, \]
on the surface of the rheotropic conditions \([\text{(13)}, (50)]\) (where \(dX^m = e^a u_{a_1}^m\)). The latter expression vanishes also identically as an antisymmetric tensor of rank \((p + 2)\) with \((p + 1)\) valued vector indices.

We emphasize the interesting fact that all the independent equations considered above \((\text{13)}, (50)\) remain the same for the case of a free \(p\)-brane theory.

In the following we drop the rheotropic conditions \((\text{13)}, (50)\) and consider only their integrability conditions \((\text{32)}, (\text{33})\) together with \((\text{23})\) (equivalent to \((\text{24)} - (\text{26)})\) which can be solved by expressions \((\text{19)}\) (or \((\text{20)} - (\text{22})\)) for connection forms in terms of moving frame variables.

The embedding of an arbitrary \((p + 1)\)-dimensional subspace into flat space-time of dimension \(D \geq (p + 1)(p + 2)/2\) is encoded in Eqs. \((\text{32)}, (\text{33}), (\text{24)} - (\text{26})\).

To further specify the subspace under consideration it is necessary to define its main extrinsic curvatures

\[ h^i \equiv \eta_{ab} K_{bai}^\alpha, \]

i.e. the traces of the second fundamental form matrix. This is just implied by the last e.o.m. following from \(i_\delta E^i = E^i(\delta)\) in \((\text{14})\),

\[ f_{a_1}^{ai} \wedge e^a_1 \wedge \cdots \wedge e^a_p \epsilon_{a_{a_1} \cdots a_p} = -\frac{1}{p!} h^i \epsilon_{a_0} \wedge \cdots e^a_p \wedge \epsilon_{a_{a_0} \cdots a_p} = -p!i_\alpha H_{p+2} \]

Thus, the main extrinsic curvatures

\[ h^i \equiv K_{a}^{ai} \equiv f_{a}^{ai}(D_a) = \]

\[ = -\frac{1}{(p + 2)} \epsilon_{a_0 \cdots a_p} u^{a_{a}^{\alpha_0},a_1}(x) \cdots u^{a_{a}^{\alpha_{p}},a_p}(x) \times \]

\[ \times H_{m_2 m_3 \cdots m_{p+2}}(X(x)) \]

of the embedded world volume are defined in terms of the \(D\)-dimensional GKR field strength which may be regarded as an arbitrary function of the coordinate functions \(X^m(\xi)\). This means that our model — at least locally — provides the basis for a description of arbitrary curved space-time of dimension \(d\) satisfying \(D \geq d(d + 1)/2\) as a subspace in \(D\)-dimensional flat space-time.

Of course, to solve these equations in the general case some kind of ’selfconsistency field technique’ need be developed.

From the point of view of recent developments in string theory, the most interesting ones are possibly related to systems with a solitonic solution similar to the generalized magnetic monopoles \([53]\) as a source for generating the main curvature. Further investigations along this line seem to be promising.

### 4 Bosonic String in KR Background and Two-Dimensional Gravity

We now treat the simplest nontrivial example of the models proposed above. Of course most results follow from the general considerations of Section 3. But we take the opportunity to introduce a light-like basis, most useful in \(d = 2\), and to also fix some notations for later
purposes. In an analytical basis of the target space \(39\) we find a close relation to PSM-type models \([22, 23, 24]\) after incorporation of the Maurer Cartan equations into the action by Lagrange multipliers.

For the free string a suitable change of variables leads to the so-called Jackiw-Teitelboim model with an intriguing 'picture-duality' property emerging for such strings in \(D = 3\).

4.1 Twistor-like Action

In light-like notation the action describing the bosonic string interacting with a KR background reads

\[
S = -\frac{1}{2} \int_{\mathcal{M}^2} \left( E^+ (d)e^- (d) - E^- (d)e^+ (d) \right) - e^+ (d)e^- (d)) + \int_{\mathcal{M}^2} B_2 (d, d),
\]

where the second term involves the KR field \(B_{2m} (X(\xi))\)

\[
B_2 \equiv B_2 (d, d) \equiv \frac{1}{2} dX^m dX^n B_{nm} (X) = d\xi^m d\xi^n \partial_m X^u \partial_n X^v B_{uv} (X(\xi)).
\]

\(\mathcal{M}^2\) is a bosonic world sheet with local coordinates \(\{\xi^m\} = \{\tau, \sigma\}\), \(e^\pm = d\xi_m e^\pm_m \equiv e^0 \pm e^1\) are the covariant light cone components of the zweibein 1-form of the string world sheet. The signs in the superscript (+ and −) denote the weight with respect to the action of the world sheet Lorentz group \(SO(1, 1)\).

\[
E^\pm = dX^m u^\pm_m,
\]

\[
E^- = dX^m u^-_m,
\]

are the pull-backs (onto the world sheet) of two basic 1-forms of target space-time written in the light-like notation

\[
E^a = (1/2 (E^+ + E^-), 1/2 (E^+ - E^-)), \quad a = 0, 1.
\]

Together with the other \((D - 2)\) forms \(E^i\) \(11\),

\[
E^i = dX^m u^i_m,
\]

they form the basis \(8\) of the space cotangent to target space-time. This basis differs from the holonomic basis \(dX^m\) by \(SO(1, D - 1)\) Lorentz transformations represented by the components of the orthogonal moving frame matrix \(8\) which reads here

\[
\frac{u^m_m}{u^\pm_m} \equiv (1/2(u^+_m + u^-_m), u^i_m, 1/2(u^+_m - u^-_m)).
\]

The vectors \(u^\pm, u^i\) can be identified with the \(SO(1, D - 1)\) Lorentz harmonics \(25\) \(26, 27, 52\) (cf. section 2). In the present case of \(d = 2\) the orthonormality conditions \(12\) - \(14\) become

\[
u^+_m u^-_m = 0, \quad u^+_m u^-_m = 0, \quad u^i_m u^i_m = 0,
\]

\[
u^-_m u^-_m = 2, \quad u^i_m u^j_m = -\delta^{ij},
\]

the coset forms \(f^ai\) \(21\) are decomposed into two covariant \(SO(D - 2)\) vector forms

\[
f^+ i \equiv u^+_m d u^a_m i.
\]
\[ f^{-i} \equiv u_{m}^{-i} d u^{m}, \tag{63} \]

the \(SO(1,1)\) connection \(\omega^{ab}\) \(\text{(21)}\) having the simple representation \(\omega^{ab} \propto \omega^{ab}\)

\[ \omega \equiv \frac{1}{2} u_{m}^{-i} d u^{m}, \tag{64} \]

and \(SO(D-2)\) connections ('gauge fields') are defined by \(\text{(22)}\)

\[ A^{ij} \equiv u_{m}^{i} d u^{m j}. \tag{65} \]

The Maurer-Cartan equations \(\text{(23)}\) split naturally in terms of \(\text{(62)} \sim \text{(65)} :\)

\[ \mathcal{D} f^{+i} \equiv df^{+i} - f^{+i} \wedge \omega + f^{+j} \wedge A^{ji} = 0 \tag{66} \]
\[ \mathcal{D} f^{-i} \equiv df^{-i} - f^{-i} \wedge \omega + f^{-j} \wedge A^{ji} = 0 \tag{67} \]

\[ \mathcal{R} \equiv d \omega = \frac{1}{2} f^{-i} \wedge f^{+i} \tag{68} \]

\[ f^{ij} \equiv dA^{ij} + A^{ik} \wedge A^{kj} = - f^{-[i} \wedge f^{+j]} \tag{69} \]

Thus the variations of moving frame vectors \(\delta u_{m}^{a} = u_{m}^{b} \Omega_{b}^{a} = u_{m}^{b} \Omega_{b}^{a}(\delta)\), being necessary for the derivation of the e.o.m.-s of the action \(\text{(55)}\), become

\[ \delta u_{m}^{\pm} = \pm 1/2 u_{m}^{\pm} \omega(\delta) + u_{m}^{i} f^{\pm i}(\delta), \tag{70} \]

The nondynamical equations of motion ('rheotropic conditions' \(\text{(48)}\)), which appear as a result of variations with respect to the moving frame variables \((i \delta f^{ai} \equiv f^{ai}(\delta))\) and the vielbeine \(e^{\pm}\)

\[ E^{i} = 0 \tag{71} \]
\[ E^{+} = e^{+} \tag{72} \]
\[ E^{-} = e^{-} \tag{73} \]

could be collected in one equation

\[ dX^{m} = 1/2 e^{\mp} u^{\pm m} \tag{74} \]

which can be solved, in particular, with respect to \(u^{\pm m}\). Substituting this expression into Eqs. \(\text{(62)}, \text{(63)}\) we find that the coset forms \(f^{\pm i}\) are related to the second fundamental form of the embedded surface

\[ K_{mn}^{i} \propto \partial_{m} \partial_{n} X^{m} u^{i}_{m} (\partial_{n} X^{m} u^{i}_{m} = 0) \]

(whose traces define the main curvatures \(\text{(3)}\)) by

\[ f^{\pm i} = d \xi^{m} \Omega^{\pm i}_{m}, \quad \Omega^{\pm i}_{m} \propto K_{mn}^{i} e^{n \pm}. \tag{75} \]

The evident symmetry of the second fundamental form \(K_{mn}^{i} = K_{nm}^{i}\) with respect to permutations of the world sheet vector indices implies that the \(f^{\pm i}\) satisfy

\[ f^{-i} \wedge e^{+} + f^{+i} \wedge e^{-} = 0 \quad \Rightarrow \quad f_{-i} = f_{+i} \equiv 1/2 h^{i} \tag{76} \]

which indeed also follows from selfconsistency conditions of the rheotropic relations \(\text{(71)}\). On the other hand, \(\text{(72)}\) and \(\text{(73)}\) yield vanishing world sheet torsion

\[ T^{\pm} \equiv \mathcal{D} e^{\pm} = de^{\pm} \mp e^{\pm} \omega = 0. \tag{77} \]
The only proper dynamical equation appears as a result of the variation with respect to the $X$ variable. Its independent part

$$f^+ - f^- = e^+ \wedge e^- \equiv 2i_u H_3$$

$$\equiv dX^\mu(\xi) \wedge dX^\nu(\xi) u^\mu H_1 \wedge u^\nu m(X(\xi))$$

(78)

can be extracted by contraction with $u_i^j$ and expresses the main curvature of the world sheet

$$h^i \equiv f^+ - f^- = 2f^+ = \frac{1}{2} u^\mu u^\nu - u^k H_k \wedge u^\mu m(X(\xi))$$

(79)

through the pull-back on the KR field strength

$$H_3 \equiv dB_2 = \frac{1}{3!} dX^\mu \wedge dX^\nu \wedge dX^\kappa H_{\kappa \mu \nu}.$$  

(80)

Thus, the main curvature of the world volume can be assumed to be an arbitrary function of the world volume coordinates by appropriately choosing the KR field. In accordance with the general theorem [3, 2] mentioned in the previous Section, this means that this model even describes arbitrary two-dimensional surfaces already for the case of $D = 3$ target space dimensions.

4.2 Bosonic String and Poisson-Sigma-Models

During recent years it became evident that all 2d covariant models of gravity, including generalized dilaton models and models with gauge fields etc. are special cases of so-called Poisson-Sigma-Models (PSM) [22, 23]

$$S_{PSM} = \int \mathcal{M}^2 (Y^C dA_C + \mathcal{P}^{BC}(Y) A_B \wedge A_C)$$

(81)

where $\mathcal{P}^{BC}(Y)$ is a Poisson-structure on $\mathcal{M}^2$, the (zero-forms) $Y$ represent PSM target space coordinates, $A_B$ are one-forms. Generally valid properties like the existence of absolute conservation laws [22, 54], special behaviors of quantized systems of this type [22, 55], etc. are well-known by now. Here we shall argue that that the considered bosonic string action acquires the structure of an action functional for $d = 2$ gravity in the PSM approach. This structure becomes evident after a change of variables is carried out and an integration by parts is performed. We recall that the PSM-action for $d = 2$ gravity has the form of first order Einstein-Cartan theory [23]

$$S_{PSM} =$$

$$= -1/2 \int \left( \tilde{X}^+ \tilde{D} e^- - \tilde{X}^- \tilde{D} e^+ + \tilde{X}^\perp d\tilde{\omega} 
- e^+ e^- V(\tilde{X}^\perp, \tilde{X}^+ \tilde{X}^-) \right)$$

(82)

where $X^\pm$, $\tilde{X}^\perp$ are Lagrange multipliers, $\tilde{\omega}$ is a world sheet spin connection (regarded as independent 1-form variable) and, defining a covariant differential $\tilde{D}$,

$$\tilde{D} e^\pm = de^\pm \mp e^\pm \wedge \tilde{\omega}.$$  

where $V(\tilde{X}^\perp, X^+ X^-)$ is an arbitrary function. By the e.o.m. for $\delta X^\pm$ the derivative of $V$ defines world-sheet torsion

$$\tilde{T}^\pm \equiv \tilde{D} e^\pm = de^\pm \mp e^\pm \tilde{\omega} = e^+ e^- \frac{\partial V}{\partial X^\pm},$$

(83)
and from $\delta \tilde{X}^\perp$ Riemannian curvature

$$R \equiv \ast d\tilde{\omega} = \frac{\partial V(\tilde{X}^\perp, \tilde{X}^+\tilde{X}^-)}{\partial X^\perp}$$  \hspace{1cm} (84)$$

By elimination of $X^\perp$ and $X^\pm$ evidently any 2d-theory with arbitrary powers in curvature and torsion can be constructed. By a conformal transformation $e^\pm = e^{\phi(\tilde{X}^\perp)}\tilde{e}^\pm$ also the generalized dilaton theories can be obtained, comprising, e.g. spherically reduced gravity [57] and the string-inspired dilaton black-hole [57].

Recently it has been proved [24] that $d = 2$ gravity with torsion locally is equivalent to torsionless generalized dilaton gravity. Hence, we restrict our considerations to cases with a potential $V$ being independent of $X^\pm$, i.e. $V = V(\tilde{X}^\perp)$.

In order to exhibit a similar structure in the bosonic string formulation (55) a change of variables in the free part of the string action

$$\begin{pmatrix} X^m; u^+_m, u^-_m \end{pmatrix} \rightarrow \begin{pmatrix} X^\pm; u^+_m, u^-_m \end{pmatrix}$$  \hspace{1cm} (85)$$

should be made. It corresponds to a transition to an analytical basis [32, 25, 28] of the target Minkowski space-time. The pull-backs of the target space vielbein forms $E^\pm$ are expressed in terms of these variables are expressed as

$$E^+ = D X^+ - X^f f^+, \hspace{1cm} E^- = -D X^- + X^f f^-.$$  \hspace{1cm} (86)$$

In (86) $D$ is the world sheet covariant derivatives involving the pull back of the Cartan form $\omega$ (64) as the spin connection (this is possible due to the identification of the $SO(1,1)$ subgroup of the target space structure group $SO(1, D - 1)$ with the world sheet Lorentz group, which corresponds to the property of the Lorentz harmonic formulation [20, 37, 48]):

$$D X^\pm \equiv d X^\pm - X^\pm \omega, \hspace{1cm} D X^- \equiv d X^- + X^- \omega.$$  \hspace{1cm} (87)$$

For fields with $SO(D - 2)$ indices the pull-backs of (65) are used as "internal" $SO(D - 2)$ connections in $D$

$$D X^i \equiv d X^i + X^j A^{ij}.$$  \hspace{1cm} (88)$$

Substituting expressions (86), (87) into Eq. (55) and integrating by parts (neglecting the surface terms for simplicity [57]) we obtain the action functional for the bosonic string in the form

$$S = -1/2 \int \left( X^+ D e^- - X^- D e^+ + X^i (f^- i \wedge e^+ - f^+ i \wedge e^-) - e^+ \wedge e^- \right) + \int B_2.$$  \hspace{1cm} (89)$$

Extracting the world sheet volume two-form $e^+ \wedge e^-$ in the last two terms we arrive at

$$S = -1/2 \int \left( X^+ D e^- - X^- D e^+ + e^- e^+ [1 + X^i ((f^- i + f^+ i))] \right) + \int B_2.$$  \hspace{1cm} (90)$$

which is similar to but not identical to the PSM-action (82).

\textsuperscript{12}So, we consider closed string theory here.
The first (free) part of this action involves the target space coordinate fields $X^\pm, X^i$ (the latter with $SO(1, D - 1)$ Lorentz indices) and the pull-backs of one-forms

$$f^\pm = e^+ f^\pm_+ + e^- f^\pm_- = d\zeta^m f^\pm_m \quad (91)$$

$$\omega = e^+ \omega_+ + e^- \omega_- = d\zeta^m \omega_m \quad (92)$$

only. In this approach the latter variables are still constrained to satisfy (23) (or equivalently (30) - (33)), i.e. we do not need any reference to the explicit expressions (19) (or (62) - (65)) for them in terms of harmonic variables. Varying the action with respect to $f$ of (91) and $\omega$ this must be taken into account. $\delta \Omega^{ab}(d)$ can be determined from (23) contracted with the variation symbol

$$i_\delta (d\Omega^{ab} - \Omega^{ac}_\delta \wedge \Omega^{cb}_\delta) = 0, \quad (93)$$

i.e.

$$\delta \Omega^{ab}(d) = d\Omega^a_{\delta}(d) + \Omega^{ac}_\delta(d) \Omega^{cb}_\delta(d) - \Omega^{bc}_\delta(d) \Omega^{ac}_\delta(d), \quad (94)$$

and thus

$$\delta f^\pm_i = D f^\pm_i(\delta) - f^\pm_i \omega(\delta) + f^\pm_j A^{ij}(\delta),$$

$$\delta \omega = d\omega(\delta) - 1/2 f^{-1} f^+(\delta) + 1/2 f^{-1}(\delta) f^+, \quad dA^{ij} = D A^{ij}(\delta) - f^{-[i} f^+[j](\delta) + f^{-[i}(\delta) f^+[j] \quad (95)$$

The independent variations are produced by the contractions of the corresponding components of the spin connection one-forms with the variation symbol $\delta$

$$i_\delta f^\pm_i = f^\pm_i(\delta), \quad i_\delta \omega = \omega(\delta), \quad i_\delta A^{ij} = A^{ij}(\delta), \quad (96)$$

which parameterise the Lorentz group algebra $so(1, D - 1)$.

To vary the second term of the action (involving the KR field which is assumed to be dependent on the coordinates $X^{a}(\xi)$ only) in the analytical basis we use the expression for the variation of the differential form and completeness of the set of moving frame variables

$$\int \delta B_2 = \int \left( i_\delta (dB_2) + d(i_\delta B_2) \right) = \int i_\delta (dB_2) = \int 1/2 dX^m \wedge dX^n \wedge \delta X^k H_{kmn} = \int 1/2 E^a \wedge E^b \wedge E^c(\delta) H_{cba} \quad (97)$$

The forms $E^\pm, E^i$ and their contractions with the variation symbol

$$i_\delta E^a \equiv E^a(\delta) \equiv (E^a(\delta), E^i(\delta)) = \delta X^m \nu^a E_m$$

are now expressed in terms of the analytical basis coordinates by (86) and

$$E^i = D X^i - 1/2 X^+ f^{+i} - 1/2 X^i f^{-i} \quad (98)$$

The action (99) expressed in that basis contains $X^\pm$ which play the role of Lagrange multipliers for the condition of vanishing torsion for the induced connection $\omega$ (77)

$$T^\pm \equiv De^\pm = de^\pm \pm e^\pm \wedge \omega = 0, \quad (99)$$

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just as in Einstein-Cartan action \( (82) \) \([22, 23, 24]\) for torsionless gravity.

The field \( X^i \) is a Lagrange multiplier for the condition defining the main curvatures of the world sheet. In the case of the free string \( (B_2 = 0) \) they simply vanish

\[
h^i \equiv f^{-i} e^+ - f^{+i} e^- = 0 \Rightarrow h^i \equiv f^{-i} + f^{+i} = 0
\]

and this means that the world sheet is embedded into the flat \( D \)-dimensional Minkowski space-time as a minimal surface.

In the corresponding case of a nonvanishing KR background one obtains in the analytical basis by \( (97), (78) \) or \( (79) \)

\[
h^i \equiv f^{+i} + f^{-i} = 2 f^{+i} = \frac{1}{2} H^{+i},
\]

where the function \( H^{+i} \)

\[
H^{+i} = u^{mi} u^{-m} u^{k} H_{k} u^{m}(X^l(\xi))
\]

depends not only on the coordinates \( X^\pm, X^i \), but also on the moving frame degrees of freedom \( (X^m = 1/2X^+ u^{-m} + 1/2X^- u^m - X^i u^{-m}) \) which can be regarded as related to 1-form variables \( f^{\pm i}, \omega, A^{ij} \) constrained to satisfy \( (23) \).

The e.o.m.-s appearing as a result of variations with respect to the Cartan forms and the vielbeine \( e^\pm \) are nondynamical

\[
E^i \equiv DX^i - 1/2X^\pm f^{\mp i} = 0 \quad (102)
\]

\[
E^+ \equiv DX^+ - X^i f^{+i} = e^+
\]

\[
E^- \equiv DX^- - X^i f^{-i} = e^- .
\]

These 'rheotropic' conditions are similar to the algebraic equations appearing in the PSM approach \([22, 23]\). Finally also \( (66) \)–\( (68) \) must be taken into account. In addition the forms \( f^{\pm i} \) satisfy \( (76) \)

\[
f^{-i} \wedge e^+ + f^{+i} \wedge e^- = 0
\]

or

\[
f^{-i} = f^{+i} \equiv 1/2 h^i
\]

which follows from selfconsistency conditions of the rheotropic relations \( (71) \).

The Riemann curvature is determined in terms of \( SO(1, D - 1)/(SO(1, 1) \times SO(D - 2)) \) connection forms \( f^{\pm i} \) \( (i = \perp \ for \ D = 3) \) by the Gauss equation \( (85) \)

\[
\mathcal{R} = d\omega = 1/2 e^+ \wedge e^- (f^{-i} f^{+i} - (h^i)^2)
\]

(Here Eq. \( (74) \) is taken into account, the Ricci tensor which is proportional to \( ^*\mathcal{R} \) should not be confused with \( \mathcal{R} \) in \( (106) \)).

In several respects the KR field assumes the role of the potential \( V \) in the Einstein-Cartan approach for \( d = 2 \) \([22, 23]\). Its dependence on moving frame variables (related to the connection forms) gives the possibility to describe not only matterless gravity, but any two-dimensional gravity in accordance with the general theorems about local isometric embedding \([2, 4]\) mentioned above. Still, this approach so far did not achieve an action with fully unconstrained variables and with the expected full PSM structure.
4.3 Unconstrained Variables and ’Picture Duality’

In 4.2 we identified the analytical basis coordinates $X^\pm, X^i$ with the Lagrange multipliers $\tilde{X}$ of the Einstein-Cartan approach \[23, 22\]. Now we show that the picture could be changed to a ‘dual’ one containing more straightforward counterparts of the $\tilde{X}$ field. In fact this ‘dual’ picture appears when the Einstein-Cartan approach is incorporated at a more basic level.

We now introduce also the Maurer-Cartan Equations (66)–(68) with Lagrange multipliers into the free string action ((89) with $B_2 = 0$)

\[
S = \int \left( -1/2X^+D\epsilon^- + 1/2X^-D\epsilon^+ + 1/2\epsilon^+ \wedge \epsilon^- + \right.
\]

\[
+ X^i(f^- \wedge \epsilon^+ - f^+ \wedge \epsilon^-) +
\]

\[
+ Y^iDf^-i + 1/2Y^-iDf^+i + Y(\omega_0 - 1/2f^- \wedge f^+ + 
\]

\[
+ Y^{ij}(dA^{ij} + A^{ik} \wedge A^{kj} + f^-[i \wedge f^+j]) \right) . \tag{107}
\]

The last two lines of (107) have the form of Einstein-Cartan-like model (82) without potential, but for the vielbein replaced by the pull-back of the $SO(1, D-1)/[SO(1,1) \times SO(D-2)]$ coset covariant forms $f^{\pm i}$. The first and the third line are just the actions for two Einstein-Cartan gravity models, and the second line and the second line produces an interaction of these two models, gluing them together.

In fact, the entire action (107) represents a general PSM model (81) with one forms $A_b = (e^\pm, f^{\pm i}, \omega, A^{ij})$.

For the simplest case $D = 3$ the number of the forms $f^\pm$ is the same as of vielbein forms $e^\pm$, the gauge fields $A$ disappear, and the structure of the action functional

\[
S = \int \left( -1/2X^+D\epsilon^- + 1/2X^-D\epsilon^+ + 1/2\epsilon^+ \wedge \epsilon^- + 
\]

\[
X^\pm(f^- \wedge \epsilon^+ - f^+ \wedge \epsilon^-) +
\]

\[
+ Y^\pmDf^-i + 1/2Y^-iDf^+i + Y(\omega_0 - 1/2f^- \wedge f^+) \right) . \tag{108}
\]

becomes almost symmetric with respect to replacement $e$ by $f$ and coordinate fields $X$ by Lagrange multipliers $Y$.

Hence, supposing the forms $f^\pm$ to be linear independent, we could consider them in the cotangent space as an alternative basis to the $e^\pm$. Passing from the $e$-basis to the $f$-basis we thus obtain another picture of the model, which in some sense is dual to the initial one. As a consequence of this we will be able to exclude vielbeins $e$ and variables $X^\pm$ from the action.

4.4 Dual picture for free string theory and Jackiw-Teitelboim model

The e.o.m.-s from variation with respect to the vielbein $e^\pm$ are purely algebraic

\[
e^- = DX^- - X^if^-i, \quad e^+ = DX^+ - X^if^+i, \tag{109}
\]

and, hence, can be substituted back into the action so that $e^\pm$ are eliminated. The same is true for the variations with respect to the coordinate fields $X^\pm$

\[
f^-i \wedge (DX^i - 1/2X^- f^+i) = 0 \tag{110}
\]

\[
f^+i \wedge (DX^i - 1/2X^+ f^-i) = 0 \tag{111}
\]
Eqs. (110), (111) are solved by

\[ X^- = \frac{(2D X^i \wedge f^{-i})}{(f^{+i} \wedge f^{-i})} = \frac{1}{\mathcal{R}}(D_+ X^i f^{-i} - D_- X^i f^{+i}) \]  

(112)

\[ X^+ = \frac{(2D X^i \wedge f^{+i})}{(f^{+i} \wedge f^{-i})} = \frac{1}{\mathcal{R}}(D_- X^i f^{+i} - D_+ X^i f^{-i}) \]  

(113)

where

\[ \mathcal{R} \equiv 1/2(f^{+i} f^{-i} - f^{-i} f^{+i}) \]

can be regarded as the Ricci curvature scalar divided by the square root of the metric, or as some counterpart of the determinant of the metric in the dual picture. E.g. in the simplest \( D = 3 \) case indeed

\[ \mathcal{R} \equiv 1/2(f^{+i} f^{-i} - f^{-i} f^{+i}) = \det\left(\begin{array}{cc} f^{+} & f^{-} \\ f^{-} & f^{+} \end{array}\right) \]

follows. Substituting (109), (110), (111) into (107) we arrive at

\[ S' = \int (Y^{ij} (dA^{ij} + A^{ik} \wedge A^{kj} + f^{-[i} \wedge f^{+j]})) + 1/2\tilde{Y}^{ij} X^{-i} X^{-j} + 1/2\tilde{Y}^{-i} X^{+i} X^{-i} + 1/2\tilde{Y}^{+i} X^{+i} X^{+i} \]

(114)

\[ + \frac{1}{2\mathcal{R}} (D_+ X^j f^{-i} - D_- X^j f^{+i}) (D X^i \wedge f^{+i}) \]

\[ - \frac{1}{2\mathcal{R}} (D_- X^j f^{+i} - D_+ X^j f^{-i}) (D X^i \wedge f^{-i}) \]

\[ + \frac{1}{4\mathcal{R}^2} (D_+ X^j f^{-i} - D_- X^j f^{+i}) \]  

(115)

\[ \times (D_- X^j f^{+i} - D_+ X^j f^{-i}) f^{+i} \wedge f^{-i} \]

\[ + 1/2 X^i X^j f^{-i} f^{+j} \]

where the redefinitions

\[ \tilde{Y} = Y - 1/2X^+ X^-, \]

\[ \tilde{Y}^+ = Y^+ + 1/2X^+ X^{-}, \]

\[ \tilde{Y}^- = Y^- + 1/2X^- X^{-} \]

of the Lagrange multipliers \( Y \) have been performed.

The action (114) even with GKR background describes complicated (not very illuminating) nonminimal gravity interactions of "pre-matter" scalar fields \( X^i \) and natural gauge fields \( A^{ij} \).

The action for the dual situation in the \( D=3 \) case, however, becomes simple:

\[ S' = \int \frac{1}{2}\tilde{Y}^{+i} \mathcal{D} f^{-i} + \frac{1}{2}\tilde{Y}^{-i} \mathcal{D} f^{+i} + \sum d\omega + \]

(116)

\[ + \frac{1}{2}\tilde{Y} f^{+i} f^{-i} \]

Here the \( \nabla^f_\pm \) are covariant derivatives appearing in the decomposition of the differential on the basis of the cotangent space provided by the forms \( f^\pm \)

\[ d = d\xi^m \partial_m = e^{\pm} \nabla_\pm = f^\pm \nabla^f_\pm. \]

The first line of the action (116) coincides with the Einstein-Cartan action with constant potential for \( d = 2 \) gravity described by the vielbein \( f^\pm \). The second line is the minimally
coupled action for a massive 'pre-matter' scalar field interacting with 'f-gravity'. Thus this action describes a Jackiw-Teitelboim model \[58\] coupled to matter which has provided a very useful laboratory for the study of the quantization for gravity.

It is important that the dependence of the action functional on the field \(X^\perp\) is bilinear. So, in a quantum theory, the integration over this field still would provide a relatively simple effective action.

It should be stressed that in the derivation of the action \(114\) from \(107\) we have used only nondynamical equations. Hence the same can be done for the more general case of arbitrary KR background. The result will be the sum of the action \(114\) with the same interaction term as in \(55\). However, to simply integrate out the \(X^\perp\) field for an arbitrary KR background will not be possible in general.

As a particular example, the complete solution of the e.o.m.-s for the action \(116\) in a conformal gauge is presented in Appendix A.

5 Conclusion and Outlook

In this paper we have shown that \(D\)-dimensional \(p\)-brane theory interacting with a \((p + 1)\)-rank antisymmetric tensor (GKR) field represents a dynamical system providing a model for description of a general type of \(d = (p + 1)\) dimensional gravity in the frame of the isometric embedding formalism \([2, 3, 4]\), if the number of dimensions \(D\) of the target flat space-time satisfies \(D \geq (p + 1)(p + 2)/2\).

This has been done using the moving frame (Lorentz harmonic) formulation \([20, 37, 21]\) of the bosonic \(p\)-brane theories which produces the master equations of the so-called geometric approach \([44, 45, 46, 47]\), as e.o.m.-s. \(d\)-dimensional "physical" matter appears in such models as a manifestation of the GKR background.

As a simple example we have studied a model for \(d = 2\) gravity provided by \(D\)-dimensional string theory in more detail. We found that the model possesses a PSM structure \([22, 23, 24, 54]\) in the general case of arbitrary \(D\). Also a deeper relation between a \(D = 3\) string model with the PSM action for 2-dimensional matterless gravity \([22, 23, 24]\) appeared. The simplest model of a free bosonic string was shown to be equivalent to a Jackiw-Teitelboim model \([58]\).

For \(d = 4\) our model realizes the idea of Regge and Teitelboim for a 'string-like' description of gravity \([3]\) and provides a dynamical ground for description of General Relativity within the embedding approach \([3, 4]\). In our framework the Universe can be considered as a 3-brane in \(D = 10\) dimensional space-time with a rank-4 antisymmetric tensor background. Matter in this Universe appears as a manifestation of \(D = 10\) GKR field.

It seems to be more than a coincidence that the number of target space-time dimensions \(D = 10\) is distinguished as a critical dimension of superstring theory too, inspiring speculations about a relation of the model considered here with string theory.

As it is well known \([4]\), in the type IIB superstring spectrum a self-dual four-form gauge field appears. Moreover, among the so-called string solitons in IIB superstring theory there is a 3-brane \([6]\). In accordance with the Mantonen-Olive conjecture \([7]\), the dual theory, where solitons become fundamental objects, should exist. Such a dual theory is just the one of a (Dirichlet \(N = 2\) super-)3-brane, being under active investigation now \([82, 63, 84, 75, 26, 67]\). A 4-form GKR gauge field can be coupled naturally to this 3-brane. For nontrivial GKR background the embedding of the 3-brane into flat 10-dimensional Minkowski space–time should be nonminimal and should describe arbitrary curved 4-dimensional Einstein space-time which may be suitable as a model for the Universe. Thus a (simplified) model of the effective action for such a 'solitonic' Universe seems to be covered by our approach.

It is interesting that the idea of the embedded Universe, explored here seems to attract renewed interest also from the field theoretical point of view. Recently a new study \([58]\) devoted
to a dynamical compactification mechanism appeared. The idea of a dynamical generation of the Universe as a 4-dimensional topological (or nontopological) defect in higher dimensional space-time is its central subject. The authors of Ref. [68] deal with the embedding of low dimensional models of the Universe in the form of a domain wall and of a cosmic string and have found that this mechanism can even help to solve the problem of supersymmetry breaking in the Universe.

A natural next step could be to consider gravity models inspired by superstrings and supermembranes. The existence of supersymmetric generalizations of the extrinsic geometry approach [21] and of our model [48] make supersymmetric generalizations of the 'string-like' description of gravity rather straightforward.

In high dimensional space-time the only multiplets are the supergravity ones. So, there is some hope to overcome the difficulties from quantum gravity when free super-$p$-branes and super-$D_p$-branes are taken as point of departure. Recent progress in studying $D = 10$ Dirichlet super-$p$-branes [69, 63, 54, 55, 66, 67], 11-dimensional 5-branes [70] and F-theory [39, 40, 71] opens the possibility of new applications of our approach for "physical" supergravity. This would correspond to the search for an adequate variant of the 'preon' model [72]. Models of that type were very popular after the realization that nontrivial counterterms for $N = 8$ supergravity may be produced in higher loops (which destroyed, at least for a time, the hope for finiteness of that theory) and that the field contents of $N = 8$ supergravity is not sufficient to provide physical gauge fields at low energies (see [72] and refs. therein).

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Appendix A: Free string in $D = 3$. Complete solution of the equations of motion.

The free bosonic string in embedding dimension $D$ is the simplest model for $d = 2$ gravity within this approach. It allows a complete solution. The e.o.m.-s following from the action (89) split into a set of one-form equations

$$
\begin{align*}
    dX^+ &= X^+ \omega + X^\perp f^+ + e^+, \\
    dX^- &= -X^- \omega + X^\perp f^- + e^-, \\
    dX^\perp &= \frac{1}{2} X^+ f^- + \frac{1}{2} X^- f^+,
\end{align*}
$$

which naturally produce relations similar to conservation laws in PSM-models \[22, 23\]

$$
\begin{align*}
    d(X^+ X^-) &= X^+(e^- + X^\perp f^-) + X^-(e^+ + X^\perp f^+), \\
    ds^2 &\equiv d(X^+ X^- - (X^\perp)^2) = X^+ e^- + X^- e^+.
\end{align*}
$$

The set of two-form equations comprises

$$
\begin{align*}
    e^+ \wedge f^- + e^- \wedge f^+ &= 0, \\
    f^+ \wedge e^- - f^- \wedge e^+ &= e^+ \wedge e^- h(X, u), \\
    D e^+ &= de^+ - e^+ \wedge \omega = 0, \\
    D e^- &= de^- + e^- \wedge \omega = 0.
\end{align*}
$$

From variation of $X^\perp$ follows the 'Gauss' equation

$$
d\omega = \frac{1}{2} f^- \wedge f^+
$$

and from $\delta Y^\perp$ the Peterson-Codazzi equations

$$
\begin{align*}
    D f^+ &= df^+ - f^+ \wedge \omega = 0, \\
    D f^- &= df^- + f^- \wedge \omega = 0,
\end{align*}
$$

A.1 Two-form equations

The two-form equations contain the integrability conditions for the one-form equations and the additional proper dynamical equation (122). We follow the line of argument presented in \[52, 59\].

In this case (121) and (122) result in (cf. the notation (91))

$$
f^+ = f^- = 0,
$$

and hence

$$
\begin{align*}
    f^+ &= e^- f^-, \\
    f^- &= e^+ f^-.
\end{align*}
$$

From (128), (129), (126), (127) the expression for the spin connection form

$$
\omega = \frac{1}{2} e^+ \nabla_+ ln(f^+) - \frac{1}{2} e^- \nabla_- ln(f^-),
$$

23
can be obtained. Substituting (130) into (123), (124), one could write the latter as condition for some invariant forms to be closed:

\[ d(e^+ (f_+^-)^{1/2}) = 0 \]
\[ d(e^- (f_-^+)^{1/2}) = 0 \]

For trivial topology of the world sheet eqs. (131), (132) are simply solved by

\[ e^+ = (f_+^-)^{-1/2} g(+) (\xi^+) d\xi^+ \]
\[ e^- = (f_-^+)^{-1/2} g(-) (\xi^-) d\xi^- \]

where, in general, \( \xi^{(\pm)} = \xi^{(\pm)}(\xi^m) \) are some functions of the world sheet coordinates \( \xi^m \) and \( g(-) = g(-)(\xi^-), g(+) = g(+)(\xi^+) \) are arbitrary functions of the \( \xi^- \) and \( \xi^+ \) respectively.

It is useful to choose local world sheet coordinates coincident with the functions \( \xi^{(\pm)} \):

\[ \xi^m = (\xi^+, \xi^-). \]

This (gauge-)choice breaks general coordinate (reparametrization) invariance up to the conformal transformations, whose parameters are expressed by the arbitrary chiral functions \( g(+) \) and \( g(-) \)

\[ \partial(-)g(+) \equiv \frac{\partial}{\partial \xi^-} g(+) = 0, \quad \partial(+)g(-) \equiv \frac{\partial}{\partial \xi^+} g(-) = 0. \]

Now the expressions (128), (129), (130) for the forms \( f^\pm, \omega \) become

\[ f^+ = e^- f^+_- = d\xi^- g(-)(\xi^-) (f_-^+)^{1/2}, \]
\[ f^- = e^+ f^+_- = d\xi^+ g(+) (\xi^+) (f_+^-)^{-1/2}, \]
\[ \omega = d\xi^+ \partial(+) \ln(f_+^-)^{1/2} - d\xi^- \partial(-) \ln(f_-^+)^{1/2}. \]

The Gauss equation (125) for the forms (137), (136), (133) produces the relation

\[ \partial(-)\partial(+) (f_+^- f_+^+)^{1/2} = \]
\[ = \frac{1}{2} (f_+^- f_+^+)^{1/2} g(+)(\xi^+) g(-)(\xi^-) . \]

Denoting

\[ (f_+^- f_+^+)^{1/2} g(+)(\xi^+) g(-)(\xi^-) \equiv e^{2W}, \]

eq. (138) turns into the nonlinear Liouville equation

\[ \partial(-)\partial(+) W = \frac{1}{4} e^{2W} \]

whose general solution is well-known (see, for example, [73])

\[ W = \frac{1}{2} \ln \frac{4\partial(+) A \partial(-) B}{(A + B)^2}, \quad \partial(-) A = 0 = \partial(+) B. \]

Since this equation is typical for the 2d gravity with constant curvature its appearance is not surprising, because the related Jackiw-Teitelboim model (without matter) precisely has this property.
A.2: Solution of One-Form Equations

Eq. (119) may be rewritten as

\[
\frac{dX}{\perp} = \frac{1}{2} \frac{1}{X} + f - \frac{1}{2} X - f + \frac{1}{2} = \frac{1}{2} \frac{d\xi}{(+) g(+) (f(+) - 1/2 X^+)} + 1/2 \frac{d\xi}{(-) g(-) (f(-) - 1/2 X^-)}
\]

which means that the \(X^\pm\) fields are expressed through the derivatives of \(X^\perp\) field

\[
X^+ = 2(f^+)^{-1/2} (g(+))^{-1} \partial(+) X^\perp
\]

\[
X^- = 2(f^-)^{-1/2} (g(-))^{-1} \partial(-) X^\perp.
\]

Since Eq. (117) expressed in terms of coordinates \(\xi^\pm\) becomes

\[
(f^+)^{1/2} \partial(+) ((f^+)^{-1/2} X^+)) = g(+) (f^+)^{-1/2}
\]

\[
(f^-)^{-1/2} \partial(-) ((f^-)^{1/2} X^-)) = g(-) (f^-)^{1/2} X^\perp
\]

substituting in (143) yields

\[
\partial(+) (e^{-2W} \partial(+) X^\perp) = 1/2 (g(+))^2 e^{-2W}
\]

\[
\partial(-) \partial(+) X^\perp = 1/2 X^\perp e^{-2W}
\]

where the definition (134) of \(W\) has been used.

Eq. (141) for \(W\) in eq. (148) yields

\[
\partial(-) \partial(+) X^\perp = 2 X^\perp \frac{\partial(+) A \partial(-) B}{(A + B)^2},
\]

and the general solution of the linear equation (149) is simply

\[
X^\perp = C_1 \frac{1}{A + B} + C_2 (A + B)^2, \quad \partial(-) A = 0 = \partial(+) B
\]

where \(C_1, C_2\) are two integration constants. Substituting the solution (150) and the expression (141) for the superfield \(W\) into (145) we reduce the latter to the expression for the chiral function \(g(+) = g(+) (\xi^{(+)}))\) in terms of the chiral function \(A = A(\xi^{(+)})\) which was the parameter of the general solution of the nonlinear Liouville equation

\[
g(+) = \pm (3C_2)^{1/2} \partial(+) A.
\]

The same procedure can be carried through for the equations involving the \(X^-\) coordinates.

As a result all the coordinates become expressed in terms of the chiral functional parameters \(A(\xi^{(-)}), B(\xi^{(+)}))\) of the general solution (141) of the Liouville equation, two integration constants \(C_{1,2}\) and the functional parameter \(L = L(\xi^{(\pm)})\) of the gauge \(SO(1,1)\) (world sheet Lorentz) transformations.

So the complete solution of the free string model involves two constants ('Casimir-functions') of the PSM structure [22]

\[
C_1 = \text{const} \quad C_2 = \text{const}
\]

and two chiral (left-moving and right-moving) functions \(A(\xi^{(+)}, B(\xi^{(-)}))\)

\[
\partial(-) A = 0 = \partial(+) B
\]

involved in the general solution (141) of the Liouville equation.

The complete solution is represented by the following set of relations:
• Coordinates of the analytical basis:

\[
X^+ = e^{-L}(\partial_{(+)A}/\partial_{(-)B})^{1/2}(-\frac{C_1}{A + B} + 2C_2(A + B)^2),
\]

\[
X^- = e^L(\partial_{(-)B}/\partial_{(+)A})^{1/2}(-\frac{C_1}{A + B} + 2C_2(A + B)^2),
\]

\[
X^\perp = \frac{C_1}{A + B} + C_2(A + B)^2,
\]

• Vielbeine:

\[
e^+ = 6e^{-L}C_2(A + B)(\partial_{(+)A}/\partial_{(-)B})^{1/2}d\xi^{(+)}
\]

\[
e^- = 6e^L C_2(A + B)(\partial_{(-)B}/\partial_{(+)A})^{1/2}d\xi^{(-)}
\]

• Pull-backs of the vielbeine of the coset: \text{SO}(1, 2)/\text{SO}(1, 1)

\[
f^+ = 2e^{-L}(\partial_{(+)A}/\partial_{(-)B})^{1/2}(A + B)d\xi^{(-)}
\]

\[
f^- = 2e^L(\partial_{(-)B}/\partial_{(+)A})^{1/2}(A + B)d\xi^{(+)}
\]

• Riemann curvature two-form:

\[
R = d\omega = 1/2 f^- \wedge f^+ =
\]

\[
= 2d\xi^{(+)} \wedge d\xi^{(-)} (\partial_{(+)A}/\partial_{(-)B})(A + B)
\]

• Metric:

\[
ds^2 = e^+ \otimes e^- = 36(C_2)^2 d\xi^{(+)} \otimes d\xi^{(-)}(A + B)^2
\]
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