Preliminary results on the homogenization of thin piezoelectric perforated shells

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Abstract

We consider a composite piezoelectric material whose reference configuration is a thin shell with fixed thickness. In this work, we give a new approach based on the periodic unfolding method to justify the modelling of a thin piezoelectric perforated shells and we establish the limit constitutive law by letting the size of holes is supposed to go to zero. This allows to use the homogenization technique to derive the limiting equations and the homogenized coefficients are explicitly described.

Key words. Homogenization; Piezoelectricity; Perforations; Shells.

1 Introduction

The shells is a three dimensional continous medium, where its thickness is small compared to other dimensions. Its geometry is characterized by two small parameters : the tickness of the shells and the size of perforations. The behavior of piezoelectric shells when the tickness goes to zero has been studied by Haenel [5].

In this work, we consider a periodically perforated piezoelectric shells. For two dimensional limit equations obtained by Haenel [5], we study the behavior for the elastic displacement and electric potential as the size of perforations becomes smaller and smaller. We are concerned with two independent problems : the membrane and bending problems and we give the convergence results based on new periodic unfolding method, recently introduced by Cioranescu, Damlamian and Griso [2].

We find explicitly the overall homogenized tensors and we study their properties in order to give a corrector results associated for membrane and bending problems.

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The periodic unfolding method

The Periodic Unfolding Method is a novel approach to periodic homogenization problems that applies as well to problems with holes and truss-like structures or in linearized elasticity. The periodic unfolding method is equivalent to two-scale convergence method but is both simpler and more efficient.

First we briefly introduce this method which was developped in [2].

Let \( \Omega \subset \mathbb{R}^3 \) be a domain with a smooth boundary \( \partial \Omega \) and let \( Y = [0,1]^3 \) be the reference cell. We consider that \( S \) is an open smooth boundary subset of \( Y \) such that \( \overline{S} \subset \overline{Y} \) and set \( Y^* = Y \setminus S \). \( S \) plays of the reference hole while \( Y^* \) is the part of \( Y \) occupied by the material. We also set \( S^\varepsilon = \varepsilon(S + k) \cap \Omega, k \in \mathbb{Z}^3 \).

The perforated domain \( \Omega^\varepsilon \) is defined as the set \( \Omega^\varepsilon = \Omega \setminus S^\varepsilon \). We assume that \( \Omega^\varepsilon \) is connected and that the holes do not intersect the boundary \( \partial \Omega \).

For \( z \in \mathbb{R}^3 \), we denote by \( \lfloor z \rfloor_{Y^*} \) the unique integer combination such that \( z - \lfloor z \rfloor_{Y^*} \in Y^* \) and set \( \{z\}_{Y^*} = z - \lfloor z \rfloor_{Y^*} \in Y^* \).

For any \( x \in \mathbb{R}^3 \) and \( \varepsilon > 0 \) we have
\[
x = \varepsilon \left( \lfloor x \rfloor_{Y^*} + \{x\}_{Y^*} \right)
\]
Define \( T^\varepsilon : L^2(\Omega^\varepsilon) \to L^2(\Omega^\varepsilon \times Y^*) \) with
\[
T^\varepsilon(w)(x,y) = w(\varepsilon \lfloor x \rfloor_{Y^*} + \varepsilon y), \quad \text{for all } (x,y) \in \Omega^\varepsilon \times Y^*.
\]

Obviously, for any \( v, w \in L^2(\Omega^\varepsilon) \) we have
\[
T^\varepsilon(vw) = T^\varepsilon(v)T^\varepsilon(w) \quad \text{(2.1)}
\]
\[
T^\varepsilon(v + w) = T^\varepsilon(v) + T^\varepsilon(w) \quad \text{(2.2)}
\]

For our purpose, all functions defined in \( L^2(\Omega^\varepsilon) \) are extended by zero outside \( \Omega^\varepsilon \).

**Proposition 1.** (Properties of \( T^\varepsilon \)) (see [2])

(a) For all \( w \in L^1(\Omega^\varepsilon) \) we have
\[
\int_{\Omega^\varepsilon} w dx = \frac{1}{|Y^*|} \int_{\Omega^\varepsilon \times Y^*} T^\varepsilon(w) dxdy; \quad \text{(2.3)}
\]

(b) For any \( w \in L^2(\Omega^\varepsilon) \) we have
\[
T^\varepsilon(w) \to w \quad \text{strongly in } L^2(\Omega \times Y^*); \quad \text{(2.4)}
\]

(c) If \( (w^\varepsilon) \subset L^2(\Omega^\varepsilon) \), then
\[
T^\varepsilon(w^\varepsilon) \to w \quad \text{weakly in } L^2(\Omega) \implies T^\varepsilon(w^\varepsilon) \to w \quad \text{weakly in } L^2(\Omega \times Y^*);
\]
\[
T^\varepsilon(w^\varepsilon) \to \tilde{w} \quad \text{weakly in } L^2(\Omega \times Y^*) \implies w^\varepsilon \rightharpoonup \frac{1}{|Y^*|} \int_{Y^*} \tilde{w} dy \quad \text{weakly in } L^2(\Omega).
\]
Proposition 2. (see [2]) Let \((w^\varepsilon) \subset L^2(\Omega^\varepsilon)\) be a bounded sequence. Then

\[ T^\varepsilon(w^\varepsilon) \rightharpoonup w \text{ weakly in } L^2(\Omega \times Y^*) \iff \text{two-scales converges to } w. \]

Theorem 2.1. (see [2]) Let \((w^\varepsilon) \subset H^1(\Omega^\varepsilon)\) be a bounded sequence that weakly converges to \(w\) in \(H^1(\Omega)\). Then there exists \(u^1 \in L^2(\Omega; H_{\text{per}}^1(Y))\) such that

\[ T^\varepsilon(w^\varepsilon) \rightharpoonup u \text{ weakly in } L^2(\Omega \times Y^*), \quad (2.5) \]

\[ T^\varepsilon(\nabla_x w^\varepsilon) \rightharpoonup \nabla_x w + \nabla_y w^1 \text{ weakly in } L^2(\Omega \times Y^*). \quad (2.6) \]

Theorem 2.2. Let \((u^\varepsilon)\varepsilon be a bounded sequence in \(H^2(\Omega^\varepsilon)\). Then there exists \(u \in H^2(\Omega)\) and \(u^2 \in L^2(\Omega; H_{\text{per}}^2(Y)/\mathbb{R})\) such that

\[ T^\varepsilon(u^\varepsilon) \rightharpoonup u \text{ weakly in } L^2(\Omega \times Y^*), \quad (2.7) \]

\[ T^\varepsilon(\nabla_x u^\varepsilon) \rightharpoonup \nabla_x u + \nabla_y u^2 \text{ weakly in } L^2(\Omega \times Y^*). \]

Proof. Since \((u^\varepsilon)\) is bounded in \(H^2(\Omega^\varepsilon)\), it follows that it weakly converges to some \(u \in H^2(\Omega)\). According to Theorem 2.1, there exists \(u^1 \in L^2(\Omega; H_{\text{per}}^1(Y^*))\) such that

\[ T^\varepsilon(u^\varepsilon) \rightharpoonup u \text{ weakly in } L^2(\Omega \times Y^*), \quad (2.8) \]

and

\[ T^\varepsilon(\nabla_x u^\varepsilon) \rightharpoonup \nabla_x u + \nabla_y u^2 \text{ weakly in } L^2(\Omega \times Y^*). \quad (2.9) \]

Moreover, by the boundedness of \((u^\varepsilon)\) in \(H^2(\Omega^\varepsilon)\), it follows that \(T^\varepsilon(\nabla_x^2 u^\varepsilon)\) is bounded in \(L^2(\Omega \times Y^*)\). Hence, there exists \(\varrho \in L^2(\Omega \times Y^*)\) such that

\[ T^\varepsilon(\nabla_x^2 u^\varepsilon) \rightharpoonup \varrho \text{ weakly in } L^2(\Omega \times Y^*). \quad (2.10) \]

Let \(\psi \in D(\Omega; C_{\text{per}}^\infty(Y^*))\). Then we have

\[ \int_{\Omega} \nabla_x u^\varepsilon \cdot \psi(x, x/\varepsilon) \, dx = - \int_{\Omega} \frac{\partial u^\varepsilon}{\partial x_j} \left( \frac{\partial \psi}{\partial x_i} + \frac{1}{\varepsilon} \frac{\partial \psi}{\partial y_j} \right)(x, x/\varepsilon) \, dx \quad (2.11) \]

Using the unfolding operator and \((2.6)\) in the above relation we have

\[ \varepsilon \int_{\Omega \times Y^*} T^\varepsilon \left( \frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j} \right) \psi(x, y) \, dxdy = - \int_{\Omega \times Y^*} T^\varepsilon \left( \frac{\partial u^\varepsilon}{\partial x_j} \right) \left( \frac{\partial \psi}{\partial x_i} + \frac{\partial \psi}{\partial y_j} \right)(x, y) \, dxdy. \quad (2.12) \]

Passing to the limit in \((2.12)\) and using \((2.5), (2.7)\) we get

\[ 0 = \int_{\Omega \times Y^*} \left( \frac{\partial u}{\partial x_j} + \frac{\partial u^1}{\partial y_j} \right) \psi(x, y) \, dxdy \quad \forall \psi \in D(\Omega; C_{\text{per}}^\infty(Y^*)). \]

This yields that \(\frac{\partial u}{\partial x_j} + \frac{\partial u^1}{\partial y_j}\) does not depend on \(y\). Since \(u^1\) is \(Y^*\)-periodic in the second variable we conclude that \(u^1(x, y) = u^1(x)\) and by \((2.9)\) we deduce

\[ T^\varepsilon(\nabla_x u^\varepsilon) \rightharpoonup \nabla_x u \text{ weakly in } L^2(\Omega \times Y^*). \]

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Let now $\psi \in \mathcal{D}(\Omega; C^\infty_{\text{per}}(Y^*))$ with $\nabla_y \psi(x,y) = 0$. From (2.11) we obtain
\[
\int_\Omega \frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j} \psi \left(x, \frac{x}{\varepsilon}\right) dx = - \int_\Omega \frac{\partial u^\varepsilon}{\partial x_j} \frac{\partial \psi}{\partial x_i} \left(x, \frac{x}{\varepsilon}\right) dx.
\]
Using again the unfolding operator we have
\[
\int_{\Omega \times Y^*} T^\varepsilon \left( \frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j} \right) \psi (x,y) dxdy = - \int_{\Omega \times Y^*} T^\varepsilon \left( \frac{\partial u^\varepsilon}{\partial x_j} \right) \frac{\partial \psi}{\partial x_i} (x,y) dxdy. \tag{2.13}
\]
Passing to the limit we get
\[
\int_{\Omega \times Y^*} \left[ \varrho_{ij} (x,y) \psi (x,y) \right] dxdy = - \int_{\Omega \times Y^*} \frac{\partial u}{\partial x_j} (x,y) \frac{\partial \psi}{\partial x_i} (x,y) dxdy
\]
\[
= \int_{\Omega \times Y^*} \frac{\partial^2 u}{\partial x_i \partial x_j} (x,y) \psi (x,y) dxdy.
\]
The above relations lead us to
\[
\int_{\Omega \times Y} \left[ \varrho_{ij} (x,y) - \frac{\partial^2 u}{\partial x_i \partial x_j} (x,y) \right] \psi (x,y) dxdy = 0,
\]
for all $\psi \in \mathcal{D}(\Omega; C^\infty_{\text{per}}(Y^*))$ with $\nabla_y \psi(x,y) = 0$.
Then, there exists $\tilde{\varrho} \in \left[ L^2(\Omega; H^1_{\text{per}}(Y^*)/\mathbb{R}) \right]^N$ such that
\[
\varrho - \nabla^2_x u = \nabla_y \tilde{\varrho}. \tag{2.14}
\]
Taking into account the symmetry of the left side member in the above equality, we may conclude that there exists $u^2 \in L^2(\Omega; H^2_{\text{per}}(Y^*)/\mathbb{R})$ such that $\tilde{\varrho} = \nabla_y u^2$. Now (2.14) becomes
\[
\varrho = \nabla^2_x u + \nabla^2_y u^2.
\]
The proof of Theorem 2.2 is now complete. \hfill \square

We introduce now the averaging operator
\[
\mathcal{U}^\varepsilon : L^2(\Omega^\varepsilon \times Y^*) \to L^2(\Omega^\varepsilon), \quad \mathcal{U}^\varepsilon(\Phi)(x) = \frac{1}{|Y^*|} \int_{Y^*} \Phi \left( \frac{x}{\varepsilon} + \varepsilon z, \left\{ \frac{x}{\varepsilon} \right\}_{Y^*} \right) dz.
\]

**Proposition 3.** (see [2])
(i) $\mathcal{U}^\varepsilon(\phi) \to \phi$ strongly in $L^2(\Omega)$, for all $\phi \in L^2(\Omega)$;
(ii) $\mathcal{U}^\varepsilon(T^\varepsilon(\phi)) = \phi$, for all $\phi \in L^2(\Omega^\varepsilon)$;
(iii) $T^\varepsilon(\mathcal{U}^\varepsilon(\Psi))(x,y) = \frac{1}{|Y^*|} \int_{Y^*} \Phi \left( \frac{x}{\varepsilon} + \varepsilon z, \left\{ \frac{x}{\varepsilon} \right\}_{Y^*} \right) dz$, for all $\Phi \in$
Latin indices take their values in the set \{1, 2, 3\} and Greek indices take their values in \{1, 2\}. The summation convention is also used. Boldface letters represent vector-valued functions.

Let \( \omega \subset \mathbb{R}^2 \) be an open bounded and connected set with a Lipschitz-continuous boundary. Let \( \varphi : \overline{\omega} \to \mathbb{R}^3 \) be a \( C^2(\overline{\omega}) \) one-to-one function such that the vectors \( \mathbf{a}_1 = \partial \varphi / \partial x_1 \) and \( \mathbf{a}_2 = \partial \varphi / \partial x_2 \) are linearly independent at all points in \( \overline{\omega} \). Denote

\[
\mathbf{a}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|} \quad \text{and} \quad a = \det(\mathbf{a}_\alpha \cdot \mathbf{a}_\beta).
\]

We consider the membrane problem

\[
\begin{cases}
\int_{\omega^c} e^\varepsilon(\mathbf{u}, \mathbf{v}) + e^\varepsilon(\varphi^\varepsilon, \mathbf{v}) = \int_{\omega^c} F_i v_i \sqrt{a^\varepsilon} + \int_{\Gamma^\varepsilon_+ \cup \Gamma^\varepsilon_-} q^i v_i \sqrt{a^\varepsilon} + \int_{\omega^c} h^\alpha \beta,\varepsilon \gamma^\alpha \beta(\mathbf{v}) \sqrt{a^\varepsilon} \\
\int_{\omega^c} e^\varepsilon(\psi, \mathbf{u}) + \mathbf{d}^\varepsilon(\varphi^\varepsilon, \psi) = \int_{\omega^c} h^\alpha \partial_\alpha \psi \sqrt{a^\varepsilon},
\end{cases}
\]

for all \((\mathbf{v}, \psi) \in V(\omega^c) \times W(\omega^c)\), where

\[
\begin{aligned}
V(\omega^c) &= \{ \mathbf{v} \in H^1(\omega^c) \times H^1(\omega^c) \times L^2(\omega^c); \quad v_0 = 0 \quad \text{on} \quad \gamma_0^M \}, \quad \text{with} \quad \text{meas} \gamma_0^M > 0, \\
W(\omega^c) &= \{ \psi \in H^1(\omega^c); \quad \psi = 0 \quad \text{in} \quad \omega^c \setminus \omega_0, \quad \psi = 0 \quad \text{on} \quad \gamma_0^E \}. \\
\end{aligned}
\]

\[
\begin{aligned}
e^\varepsilon(\mathbf{u}, \mathbf{v}) &= e^{\alpha \beta \mu,\varepsilon} \sqrt{a^\varepsilon} \gamma_{\alpha \beta}(\mathbf{u}) \gamma_{\lambda \mu}(\mathbf{v}), \\
e^\varepsilon(\mathbf{v}, \psi) &= e^{\lambda \alpha \beta,\varepsilon} \sqrt{a^\varepsilon} \partial_\lambda \psi \gamma_{\alpha \beta}(\mathbf{v}), \\
\mathbf{d}^\varepsilon(\psi, \varphi) &= \mathbf{d}^{\alpha \lambda,\varepsilon} \sqrt{a^\varepsilon} \partial_\alpha \psi \partial_\lambda \varphi, \\
\gamma^\alpha \beta(\mathbf{v}) &= s^\alpha \beta(\mathbf{v}) - \Gamma^\alpha \beta_{\lambda \mu} u_\lambda, \\
s^\alpha \beta(\mathbf{v}) &= \frac{1}{2} (\partial_\alpha v_\beta + \partial_\beta v_\alpha), \\
F^i = F^i(x_1, x_2) &= \int_{-1}^{1} f^i(x_1, x_2, t) dt, \quad \text{where} \quad f = (f^i) \in L^2(\omega \times [-1, 1]),
\end{aligned}
\]
The local functions \((\omega, \varphi)\) and \((\sigma, \eta)\) verifies the local problems

\[
\begin{aligned}
&\int_{Y^*} \left( c_y (\Sigma^{\tau\theta} + w^{\tau\theta}, \psi) + e_y (\psi, \zeta^{\tau\theta}) \right) \, dy = 0, \\
&\int_{Y^*} \left( -e_y (\Sigma^{\tau\theta} + w^{\tau\theta}, \psi) + d_y (\zeta^{\tau\theta}, \psi) \right) \, dy = 0,
\end{aligned}
\]

for all \((\psi, \varphi) \in H^1_{\text{per}}(Y^*)/\mathbb{R} \times H^1_{\text{per}}(Y^*)/\mathbb{R}, \) where \(\Sigma^{\alpha\beta} = y_\alpha e_\beta + y_\beta e_\alpha\) and

\[
\begin{aligned}
c_y(w, \varphi) &= e^{\alpha\beta\gamma\delta} \sqrt{\alpha} s_{\alpha\beta\gamma}(w) s_{\gamma\delta}(\varphi), \\
e_y(v, \psi) &= e^{\lambda\alpha\beta} \sqrt{\alpha} s_{\alpha\beta}(v) \partial_{\lambda}(\psi), \\
d_y(\psi, \phi) &= d^{\alpha\lambda} \sqrt{\alpha} \partial_{\alpha}(\psi) \partial_{\lambda}(\phi).
\end{aligned}
\]
Proof. To obtain an a priori estimate, we take $v = u^\varepsilon$ and $\psi = \varphi^\varepsilon$ in (3.15). By Korn and Poincaré’s inequalities for perforated domains, it follows

$$\|u^\varepsilon\|_{V(\omega^\varepsilon)} + \|\varphi^\varepsilon\|_{W(\omega^\varepsilon)} \leq C,$$

where $C$ is a positive constant that depends only on $\omega$ (but not on $\varepsilon$).

By Theorem 2.1, there exist $(u, \varphi) \in V(\omega) \times W(\omega)$ and two fields of correctors $u^1 = (u_1^1, u_2^1) \in L^2(\omega; H^1_{per}(Y)/\mathbb{R})$, $\varphi^1 \in L^2(\omega; H^1_{per}(Y^*))$ such that, up to a sequence we have

$$T^\varepsilon(u^\varepsilon) \rightharpoonup u \quad \text{weakly in } L^2(\omega \times Y^*; \mathbb{R}^3), \quad (3.23)$$

$$T^\varepsilon(\nabla_x(u^\varepsilon)) \rightharpoonup \nabla_x(u) + \nabla_y u^1 \quad \text{weakly in } L^2(\omega \times Y^*; \mathbb{R}^2), \quad (3.24)$$

$$T^\varepsilon(\nabla_x(\varphi^\varepsilon)) \rightharpoonup \nabla_x(\varphi) + \nabla_y \varphi^1 \quad \text{weakly in } L^2(\omega \times Y^*; \mathbb{R}^2). \quad (3.25)$$

The linearity of $T^\varepsilon$ implies

$$T^\varepsilon(\gamma_{\alpha\beta}(u^\varepsilon)) \rightharpoonup \gamma_{\alpha\beta,x}(u) + s_{\alpha\beta,y}(u^1) \quad \text{weakly in } L^2(\omega \times Y^*; \mathbb{R}^2) \quad (3.26)$$

and

$$T^\varepsilon(\partial_{y}\varphi^\varepsilon) \rightharpoonup \partial_{\alpha,x}\varphi + \partial_{\alpha,y}\varphi^1 \quad \text{weakly in } L^2(\omega \times Y^*; \mathbb{R}^2). \quad (3.27)$$

We chose as a test function in (3.15)

$$v^\varepsilon(x) = v_1(x) + \varepsilon v_2\left(x, \frac{x}{\varepsilon}\right), \quad v_1 \in D(\omega); v_2 \in D(\omega; C^\infty_{per}(Y^*)), \quad (3.28)$$

It follows that

$$v^\varepsilon(x) \rightharpoonup v_1(x) \quad \text{strongly in } L^2(\omega), \quad (3.29)$$

$$\nabla_x v^\varepsilon(x) \rightharpoonup \nabla_x v_1(x) + \nabla_y v_2(x, y) \quad \text{weakly in } L^2(\omega \times Y^*), \quad (3.30)$$

Using the linearity of $T^\varepsilon$ and the fact that $\int_\omega v = \frac{1}{|Y^*|} \int_{\omega \times Y^*} T^\varepsilon(v)$ for all $v \in L^2(\omega)$, we can pass to the limit in the variational form (3.15). We get

$$\begin{align*}
\int_{\omega \times Y^*} \left[ e^{\alpha\beta} \sqrt{\gamma} \left( \gamma_{\alpha\beta,x}(u) + s_{\alpha\beta,y}(u^1) \right) + e^{\lambda \alpha\beta} \sqrt{\gamma} \left( \partial_{\lambda} \varphi + \partial_{\lambda,y} \varphi^1 \right) \right] (\gamma_{\alpha\beta,x}(v_1) + s_{\alpha\beta,y}(v_2)) \, dx \, dy \\
= |Y^*| \int_\omega F^\varepsilon v_1 \, dx + |Y^*| \int_\Gamma_+ \int_{\Gamma_- \cup \Gamma_+} q^\varepsilon v_1 \, d\tau + \int_{\omega \times Y^*} h^{\alpha\beta} \sqrt{\alpha} \left( \gamma_{\alpha\beta,x}(v_1) + s_{\alpha\beta,y}(v_2) \right) \, dx \, dy, \\
\int_{\omega \times Y^*} [- e^{\alpha\beta} \sqrt{\gamma} \left( \gamma_{\alpha\beta,x}(u) + s_{\alpha\beta,y}(u^1) \right) + d^{\alpha\beta} \sqrt{\gamma} \left( \partial_{\lambda} \varphi + \partial_{\lambda,y} \varphi^1 \right) ] (\partial_{\alpha,x} \psi_1 + \partial_{\alpha,y} \psi_2) \, dx \, dy \\
= \int_{\omega \times Y^*} I^\varepsilon \sqrt{\alpha} \left( \partial_{\alpha,x} \psi_1 + \partial_{\alpha,y} \psi_2 \right) \, dx \, dy \quad (3.31)
\end{align*}$$
Letting \( v_2 = 0, \psi_2 = 0 \) in the above equalities, by density we have

\[
\begin{align*}
\int_{\omega \times Y^*} \left[ e^{\alpha \beta \gamma \lambda} \sqrt{a} \left( \gamma_{\lambda \mu, x}(u) + s_{\lambda \mu, y}(u^1) \right) + e^{\lambda \alpha \beta} \sqrt{a} \left( \partial_\lambda \varphi + \partial_{\lambda, y} \theta^1 \right) \right] \gamma_{\alpha \beta, x}(v) dx dy \\
= |Y^*| \int_{\omega} F^i v_i dx + |Y^*| \int_{\Gamma_- \cup \Gamma_+} q^i v_i d\Gamma + \int_{\omega \times Y^*} h^{\alpha \beta} \sqrt{a} \gamma_{\alpha \beta, x}(v) dx dy \\
\int_{\omega \times Y^*} \left[ -e^{\alpha \lambda \mu} \sqrt{a} \left( \gamma_{\lambda \mu, x}(u) + s_{\lambda \mu, y}(u^1) \right) + d^{\alpha \lambda} \sqrt{a} \left( \partial_\lambda \varphi + \partial_{\lambda, y} \theta^1 \right) \right] \partial_{\alpha, x} \psi dx dy \\
= \int_{\omega \times Y^*} l^\alpha \sqrt{a} \partial_{\alpha, x} \psi dx dy,
\end{align*}
\]

for all \((v, \psi) \in V(\omega) \times W(\omega)\). By linearity, we take

\[
\begin{align*}
u^1(x, y) &= \gamma_{\tau \sigma}(u(x, y))w^{\tau \sigma}(y) + \partial_\tau \varphi(x)z^{\tau}(y) + q(x, y), \\
\varphi^1(x, y) &= \gamma_{\tau \sigma}(u(x, y))\zeta^{\tau \sigma}(y) + \partial_\tau \varphi(x)\eta^{\tau}(y) + \xi(x, y),
\end{align*}
\]

where \(w^{\tau \sigma}, z^{\tau \sigma}, \zeta^{\tau \sigma}, \eta^{\tau} \) are \(Y^*\) periodic in \(y\) and

\[
e^{\lambda \beta \gamma \lambda} s_{\lambda \mu, y}(q(x, y)) + e^{\lambda \alpha \beta} \partial_{\lambda, y} \xi(x, y) = \iota^{\alpha \beta}(x, y),
\]

\[
-e^{\lambda \alpha \beta} s_{\lambda \mu, y}(q(x, y)) + d^{\alpha \lambda} \partial_{\lambda, y} \xi(x, y) = \iota^\alpha(x, y).
\]

Replacing \((3.33)\) and \((3.34)\) in \((3.32)\) and taking into account the properties of \(q(x, y)\) and \(\xi(x, y)\) we obtain

\[
\begin{align*}
\begin{cases}
\int_{\omega \times Y^*} \left[ e^{\alpha \beta \gamma \lambda} \gamma_{\tau \sigma}(u) + e^{\alpha \beta \gamma \lambda} \gamma_{\tau \sigma}(v) \right] \gamma_{\alpha \beta}(v) dx dy \\
\int_{\omega \times Y^*} \left[ -e^{\alpha \beta \gamma \lambda} \gamma_{\tau \sigma}(u) + e^{\alpha \beta \gamma \lambda} \gamma_{\tau \sigma}(v) \right] \gamma_{\alpha \beta}(v) dx dy = 0,
\end{cases}
\end{align*}
\]

for all \((v, \psi) \in V(\omega) \times W(\omega)\), where

\[
\begin{align*}
\overline{\tau}^{\alpha \beta \gamma \lambda} &= \int_{Y^*} \left[ e^{\alpha \beta \gamma \lambda} \sqrt{a} \left( \delta_{\lambda \mu}^{\tau \sigma} + s_{\lambda \mu, y}(w^{\tau \sigma}) \right) \right] dy, \\
\overline{\overline{\tau}}^{\alpha \beta} &= \int_{Y^*} \left[ e^{\alpha \beta \gamma \lambda} \sqrt{a} \left( s_{\lambda \mu, y}(z^{\tau \sigma}) + e^{\lambda \alpha \beta} \sqrt{a} \left( \delta_{\lambda}^{\tau \sigma} + \partial_{\lambda, y} \eta^{\tau \sigma} \right) \right) \right] dy, \\
\overline{\overline{\overline{\tau}}}^{\alpha \beta \gamma \lambda} &= \int_{Y^*} \left[ e^{\alpha \beta \gamma \lambda} \sqrt{a} \left( s_{\lambda \mu, y}(z^{\tau \sigma}) + d^{\alpha \lambda} \sqrt{a} \left( \delta_{\lambda}^{\tau \sigma} + \partial_{\lambda, y} \eta^{\tau \sigma} \right) \right) \right] dy,
\end{align*}
\]

We will prove that \(\overline{\overline{\overline{\tau}}}^{\alpha \beta} = \overline{\overline{\tau}}^{\alpha \beta}\). In that sense we first determine the local
problems verified by the functions \( w^\tau, \zeta^\tau, z^\tau, \eta^\tau \). From (3.38)-(3.41) we have

\[
\sigma^{\alpha\beta\theta} = \int_{Y^*} \left[ c^{\alpha\beta\gamma} \sqrt{a} s_{\lambda\gamma}(y(\Sigma^\theta + w^\tau\theta)) + e^{\lambda\alpha \beta} \sqrt{a} \partial_{\lambda,y} c^{\tau\theta} \right] dy, \tag{3.42}
\]

\[
\sigma^{\alpha\beta} = \int_{Y^*} \left[ c^{\alpha\beta\gamma} \sqrt{a} s_{\lambda\gamma}(e^\tau) + e^{\lambda\alpha \beta} \sqrt{a} \partial_{\lambda,y} (y_\sigma + \eta^\sigma) \right] dy, \tag{3.43}
\]

\[
\bar{\nabla}^{\tau\theta} = \int_{Y^*} \left[ e^{\alpha\mu\lambda} \sqrt{a} s_{\lambda\mu\gamma}(y(y(\Sigma^\tau + w^\tau\theta)) - d^{\alpha\lambda} \sqrt{a} \partial_{\lambda,y} c^\tau \right] dy, \tag{3.44}
\]

\[
\bar{\nabla}^{\tau} = \int_{Y^*} \left[ -e^{\alpha\lambda\mu} \sqrt{a} s_{\lambda\mu\gamma}(e^\tau) + d^{\alpha\lambda} \sqrt{a} \partial_{\lambda,y} (y_\sigma + \eta^\sigma) \right] dy. \tag{3.45}
\]

We let \( v_1 = 0 \) and \( \psi_1 = 0 \) in (3.31). By density we deduce

\[
\left\{ \begin{array}{l}
\int_{\omega \times Y^*} \left[ c^{\alpha\beta\gamma} \sqrt{a} (\gamma_{\mu,x}(u) + s_{\lambda\mu\gamma}(u^1)) + e^{\lambda\alpha \beta} \sqrt{a} (\partial_{\lambda,y} \varphi + \partial_{\lambda,y} \varphi^1) \right] s_{\alpha\beta\gamma}(v) dxdy \\
= \int_{\omega \times Y^*} h^{\alpha\beta} \sqrt{a} s_{\alpha\beta\gamma}(v) dxdy,
\end{array} \right. \tag{3.46}
\]

for all \((v, \psi) \in L^2(\omega; H^1_{\text{per}}(Y^*/\mathbb{R}) \times L^2(\omega; H^1_{\text{per}}(Y^*/\mathbb{R})\). Using again (3.33) and (3.34) we deduce

\[
\left\{ \begin{array}{ll}
\int_{Y^*} (c_y(\Sigma^\tau + w^\tau\theta, v) + e_y(v, \zeta^\tau)) \, dy &= 0, \\
\int_{Y^*} (c_y(\Sigma^\tau + w^\tau\theta, \psi) + d_y(\zeta^\tau, \psi)) \, dy &= 0,
\end{array} \right. \tag{3.47}
\]

\[
\left\{ \begin{array}{ll}
\int_{Y^*} (c_y(z^\tau, v) + e_y(v, y_\sigma + \eta^\sigma)) \, dy &= 0, \\
\int_{Y^*} (c_y(z^\tau, \psi) + d_y(y_\sigma + \eta^\sigma, \psi)) \, dy &= 0,
\end{array} \right. \tag{3.48}
\]

for all \((v, \psi) \in H^1_{\text{per}}(Y^*)/\mathbb{R} \times H^1_{\text{per}}(Y^*)/\mathbb{R})\), where \( c_y, e_y, d_y \) are given by (3.20)-(3.22).

Next, we prove that \( \sigma^{\tau\theta} = \bar{\nabla}^{\tau\theta} \). By taking \((v, \psi) = (z^\tau, \eta^\tau)\) in (3.47), we get

\[
\left\{ \begin{array}{l}
\int_{Y^*} (c_y(w^\tau\theta, z^\tau) + e_y(z^\tau, \zeta^\tau)) \, dy = -\int_{Y^*} c^{\tau\theta\lambda\mu} \sqrt{a} s_{\lambda\mu}(z^\tau), \\
\int_{Y^*} (c_y(w^\tau\theta, \eta^\tau) + d_y(\zeta^\tau, \eta^\tau)) \, dy = \int_{Y^*} e^{\lambda\tau\theta} \sqrt{a} \partial_{\lambda,y} \eta^\tau dy,
\end{array} \right.
\]

and so, by (3.39) it follows

\[
\sigma^{\tau\theta} = \int_{Y^*} \left( e^{\tau\theta} - c_y(w^\tau\theta, z^\tau) \right) - e_y(w^\tau\theta, \eta^\tau) - e_y(z^\tau, \zeta^\tau) + d_y(\eta^\tau, \zeta^\tau) \right) \, dy. \tag{3.49}
\]
In the same manner, by taking \((v, \psi) = (z^\sigma, \eta^\sigma)\) as a test function in (3.48), we get

\[
\begin{align*}
\int_{Y^*} (c_y(w^{\tau\theta}, z^\sigma) + e_y(w^{\tau\theta}, \eta^\sigma)) \, dy &= -\int_{Y^*} e^{\sigma\lambda\mu} \sqrt{a} \, s_{\lambda\mu}(w^{\tau\theta}), \\
\int_{Y^*} \left( -e_y(z^\sigma, \zeta^{\tau\theta}) + d_y(\eta^\sigma, \zeta^{\tau\theta}) \right) \, dy &= -\int_{Y^*} d^{\tau\lambda} \sqrt{a} \, \partial_{\lambda,y} \zeta^{\tau\theta} \, dy,
\end{align*}
\]

and by (3.40) it follows

\[
\begin{align*}
\int_{Y^*} Y^* \left[ e_y(w^{\tau\theta}, \eta^\sigma) - e_y(w^{\tau\theta}, \zeta^{\tau\theta}) + d_y(\eta^\sigma, \zeta^{\tau\theta}) \right] \, dy &= \int_{Y^*} Y^* d_y(\zeta^{\tau\theta}) \, dy.
\end{align*}
\]

Now, by (3.40) and (3.50) we can conclude that \(e^{\sigma\tau\theta} = f^{\sigma\tau\theta}\).

To prove that the homogenized problem is well posed, it remains only to show the symmetry and coercivity of \((c^{\alpha\beta\tau\theta})\) and \((d^{\alpha\beta})\).

(i) The ellipticity and symmetry of the tensor \((c^{\alpha\beta\tau\theta})\).

Let us prove first the symmetry. From (3.38) we deduce

\[
\begin{align*}
c^{\alpha\beta\tau\theta} &= c^{\beta\alpha\tau\theta} = c^{\alpha\beta\theta\tau}.
\end{align*}
\]

It remains to show that \(c^{\alpha\beta\tau\theta} = c^{\tau\theta\alpha\beta}\). From (3.42) we have

\[
\begin{align*}
c^{\alpha\beta\tau\theta} &= \int_{Y^*} c_y(\Sigma^{\tau\theta} + w^{\tau\theta}, \Sigma^{\alpha\beta}) + e_y(\Sigma^{\alpha\beta}, \zeta^{\tau\theta}) \, dy.
\end{align*}
\]

But

\[
\begin{align*}
\int_{Y^*} e_y(\Sigma^{\alpha\beta}, \zeta^{\tau\theta}) \, dy &= -\int_{Y^*} e_y(w^{\alpha\beta}, \zeta^{\tau\theta}) \, dy + \int_{Y^*} e_y(\Sigma^{\alpha\beta} + w^{\alpha\beta}, \zeta^{\tau\theta}) \, dy.
\end{align*}
\]

On the other hand, by taking \((v, \psi) = (w^{\alpha\beta}, \zeta^{\alpha\beta})\) in (3.47), we get

\[
\begin{align*}
\left\{ \begin{array}{l}
-\int_{Y^*} e_y(w^{\alpha\beta}, \zeta^{\tau\theta}) \, dy = \int_{Y^*} c_y(\Sigma^{\tau\theta} + w^{\tau\theta}, w^{\alpha\beta}) \, dy, \\
\int_{Y^*} e_y(\Sigma^{\alpha\beta} + w^{\alpha\beta}, \zeta^{\tau\theta}) \, dy = \int_{Y^*} d_y(\zeta^{\alpha\beta}, \zeta^{\tau\theta}) \, dy.
\end{array} \right.
\end{align*}
\]

Replacing now (3.52)-(3.53) in (3.51) we deduce

\[
\begin{align*}
c^{\alpha\beta\tau\theta} &= \int_{Y^*} c_y(\Sigma^{\tau\theta} + w^{\tau\theta}, \Sigma^{\alpha\beta} + w^{\alpha\beta}) \, dy + \int_{Y^*} d_y(\zeta^{\alpha\beta}, \zeta^{\tau\theta}) \, dy.
\end{align*}
\]

From the above relations we can easily conclude the symmetry of the tensor \((c^{\alpha\beta\tau\theta})\).

Let us prove the coercivity. Let \((X_{\alpha\beta})\) be a symmetric tensor (i.e. \(X_{\alpha\beta} = X_{\beta\alpha}\)). First we note that by (3.38) we have

\[
\begin{align*}
c^{\alpha\beta\lambda\mu} \sqrt{a}(s_{\lambda\mu,y}(W) + X_{\lambda\mu})X_{\alpha\beta} + \int_{Y^*} e^{\lambda\alpha\beta} \sqrt{a}(\partial_{\lambda,y} \Lambda)X_{\alpha\beta} \, dy.
\end{align*}
\]
where \( W = w^{\tau \theta} X_{\tau \theta} \) and \( \Lambda = \zeta^{\tau \theta} X_{\tau \theta} \). On the other hand, \((W, \Lambda)\) is a solution of the following problem

\[
\begin{align*}
\int_{Y^*} (e_y(X_{\tau \theta} \Sigma^{\tau \theta} + W, v) + d_y(v, \Lambda)) dy &= 0, \\
\int_{Y^*} (-e_y(X_{\tau \theta} \Sigma^{\tau \theta} + W, \psi) + d_y(\Lambda, \psi)) dy &= 0,
\end{align*}
\]

for all \((v, \psi) \in H^1_{\text{per}}(Y^*)/\mathbb{R} \times H^1_{\text{per}}(Y^*)/\mathbb{R}\).

Thus \((W, \Lambda)\) is a saddle point of the following functional

\[
I : H^1_{\text{per}}(Y^*)/\mathbb{R} \times H^1_{\text{per}}(Y^*)/\mathbb{R} \to \mathbb{R},
\]

defined by

\[
I(v, \psi) = \frac{1}{2} \int_{Y^*} c^{\alpha \beta \lambda \mu} \sqrt{a}(s_{\lambda \mu, y}(v) + X_{\lambda \mu})(s_{\alpha \beta, y}(v) + X_{\alpha \beta}) dy + \int_{Y^*} e^{\lambda \alpha \beta} \sqrt{a}(s_{\alpha \beta, y}(v) + X_{\alpha \beta}) \partial_{\lambda, y} \psi dy - \frac{1}{2} \int_{Y^*} d^{\alpha \lambda} \sqrt{a} \partial_{\alpha, y} \psi \partial_{\lambda, y} \psi dy.
\]

This yields

\[
I(W, \psi) \leq I(W, \Lambda) \leq I(v, \Lambda),
\]

for all \((v, \psi) \in H^1_{\text{per}}(Y^*)/\mathbb{R} \times H^1_{\text{per}}(Y^*)/\mathbb{R}\).

Consequently, for \( \psi = 0 \) we get

\[
I(W, \Lambda) \geq I(W, 0) = \frac{1}{2} \int_{Y^*} c^{\alpha \beta \lambda \mu} \sqrt{a}(s_{\lambda \mu, y}(v) + X_{\lambda \mu})(s_{\alpha \beta, y}(v) + X_{\alpha \beta}) dy > 0.
\]

Moreover, by taking \((v, \psi) = (W, \Lambda)\) in (3.55) we obtain

\[
\tau^{\alpha \beta \tau \theta} X_{\alpha \beta} X_{\tau \theta} = 2I(W, \Lambda) > 0.
\]

Let us define the function \( \Phi : \mathbb{R}^4 \to \mathbb{R} \) by

\[
\Phi(\xi_{\alpha \beta}) = \sqrt{C^{\alpha \beta \gamma \eta}} \xi_{\alpha \beta} \xi_{\gamma \eta}.
\]

It is easy that \( \Phi \) is continuous in \( \mathbb{R}^4 \) endowed with to the norm \( \| \xi \| = (\xi_{\alpha \beta} \xi_{\alpha \beta})^{\frac{1}{2}} \). Let

\[
B = \{ \xi \in \mathbb{R}^4; \xi \text{ symmetric}, \| \xi \| = 1 \}.
\]

Since \( B \) is compact, \( \Phi \) attains its minimum in \( B \). Then, there exists \( c > 0 \) such that \( \Phi \geq c \) in \( B \), that is

\[
\Phi \left( \frac{\xi_{\alpha \beta}}{\| \xi_{\alpha \beta} \|} \right) \geq c, \quad \text{for all symmetric tensor} \quad \xi = (\xi_{\alpha \beta}).
\]

This means that \( \tau^{\alpha \beta \tau \theta} \xi_{\alpha \beta} \xi_{\tau \theta} \geq c \xi_{\alpha \beta} \xi_{\tau \theta} \). The coercivity of \( (c^{\alpha \beta \tau \theta}) \) is now proved.

(ii) The ellipticity and symmetry of the tensor \( (d^{\alpha \tau \theta}) \).
First we prove the symmetry. According to (3.48) we have
\begin{equation}
\overline{\tau}^{\alpha\sigma} = \int_{Y^*} \left[ -e_y(z^\sigma, y_\alpha) + d_y(y_\sigma + \eta^\sigma, y_\alpha) \right] dy,
\end{equation}
and
\begin{equation}
-\int_{Y^*} e_y(z^\sigma, y_\alpha) dy = -\int_{Y^*} e_y(z^\sigma, y_\alpha + \eta^\alpha) dy + \int_{Y^*} e_y(z^\sigma, \eta^\alpha) dy.
\end{equation}
Now, we take \((v, \psi) = (z^\alpha, \eta^\alpha)\) in (3.48). We get
\begin{equation}
\begin{cases}
-\int_{Y^*} e_y(z^\sigma, y_\alpha + \eta^\alpha) dy = \int_{Y^*} c_y(z^\sigma, z^\alpha) dy, \\
\int_{Y^*} e_y(z^\sigma, \eta^\alpha) = \int_{Y^*} d_y(y_\sigma + \eta^\sigma, \eta^\alpha) dy.
\end{cases}
\end{equation}
Replacing (3.57) and (3.58) in (3.56) we obtain
\begin{equation}
\overline{\tau}^{\alpha\sigma} = \int_{Y^*} c_y(z^\alpha, z^\sigma) + \int_{Y^*} d_y(y_\sigma + \eta^\sigma, y_\alpha + \eta^\alpha) dy.
\end{equation}
From the above relation, we deduce the symmetry of the tensor \((\overline{\tau}^{\alpha\sigma})\).
Let us prove the coercivity of \((\overline{\tau}^{\alpha\sigma})\). Let \((X_\alpha)\) be a vector. From (3.41) we have
\begin{equation}
\overline{\tau}^{\alpha\sigma} X_\alpha X_\sigma = -\int_{Y^*} e^{\alpha\lambda\mu\nu} \sqrt{a} s_{\lambda\mu,\nu}(Z) X_\sigma dy + \int_{Y^*} d^{\alpha\lambda\nu} \sqrt{\alpha} (X_\lambda + \partial_\lambda \Theta) X_\sigma dy,
\end{equation}
where \(Z = z^\sigma X_\sigma\), and \(\Theta = \eta^\sigma X_\sigma\). It is easy to see that \((Z, \Theta)\) is the solution of the following variational problem
\begin{equation}
\begin{cases}
-\int_{Y^*} e_y(Z, \psi) + e_y(v, X_\sigma y_\sigma + \Theta)) dy = 0, \\
\int_{Y^*} (-e_y(Z, \psi) + d_y(y_\sigma + \eta^\sigma, \psi)) dy = 0,
\end{cases}
\end{equation}
for all \((v, \psi) \in H^1_{\text{per}}(Y^*)/\mathbb{R} \times H^1_{\text{per}}(Y^*)/\mathbb{R}\). Moreover, \((Z, \Theta)\) is a saddle point of the following functional
\begin{equation}
J : H^1_{\text{per}}(Y^*)/\mathbb{R} \times H^1_{\text{per}}(Y^*)/\mathbb{R} \to \mathbb{R},
\end{equation}
defined by
\begin{equation}
J(v, \psi) = \frac{1}{2} \int_{Y^*} c^{\alpha\beta\lambda\mu\nu} \sqrt{\alpha} s_{\alpha\beta\lambda\mu\nu}(v)s_{\lambda\mu,\nu}(v) dy + \int_{Y^*} e^{\lambda\alpha\beta\nu} \sqrt{\alpha} s_{\lambda\alpha\beta\nu}(v)(X_\lambda + \partial_\lambda \psi) dy \\
-\frac{1}{2} \int_{Y^*} d^{\alpha\lambda\nu} \sqrt{\alpha} (X_\alpha + \partial_\alpha \psi)(X_\lambda + \partial_\lambda \psi) dy.
\end{equation}
This yields
\begin{equation}
J(Z, \psi) \leq I(Z, \Theta) \leq I(v, \Theta),
\end{equation}
for all \((v, \psi) \in H^1_{\text{per}}(Y^*)/\mathbb{R} \times H^1_{\text{per}}(Y^*)/\mathbb{R}\).

By taking \(\psi = 0\) in the above inequality, we obtain

\[
J(Z, \Theta) \geq J(Z, 0) = \frac{1}{2} \int_{Y^*} d^{\alpha\sigma} \sqrt{a}(X_\alpha + \partial_{\alpha, y}\Theta)(X_\sigma + \partial_{\sigma, y}\Theta)dy > 0.
\]

On the other hand, by taking \((v, \psi) = (Z, \Theta)\) in (3.59) we obtain

\[
\int_{Y^*} d^{\alpha\sigma} X_\alpha X_\sigma = 2J(Z, \Theta) > 0.
\]

With the same proof as for the coercivity of \((c_{\alpha\beta\tau\theta})\) we deduce the existence of \(d > 0\) such that

\[
\int_{Y^*} d^{\alpha\sigma} X_\alpha X_\sigma \geq d|X_\alpha X_\sigma|.
\]

The uniqueness of \((u, \varphi)\) follows by the Lax-Milgram Theorem.

The proof of Theorem 3.1 is now complete. □

### 3.2 Corrector result

We have the following convergence

\[
T^\varepsilon(\gamma_{\alpha\beta}(u^\varepsilon)) - \gamma_{\alpha\beta,x}(u) + s_{\alpha\beta,y}(u^1) \quad \text{weakly in } L^2(\omega \times Y^*),
\]

\[
T^\varepsilon(\nabla\varphi^\varepsilon) - \nabla_x\varphi + \nabla_y\varphi^1 \quad \text{weakly in } L^2(\omega \times Y^*).
\]

The convergence of energies follow easily from the above relations. Moreover, the weak convergences in (3.60) and (3.61) are actually strong

\[
T^\varepsilon(\gamma_{\alpha\beta,x}(u^\varepsilon)) - \gamma_{\alpha\beta,x}(u) - s_{\alpha\beta,y}(u^1) \to 0 \quad \text{strongly in } L^2(\omega \times Y^*),
\]

\[
T^\varepsilon(\nabla_x\varphi^\varepsilon) - \nabla_x\varphi - \nabla_y\varphi^1 \to 0 \quad \text{strongly in } L^2(\omega \times Y^*).
\]

Now, we can state the following corrector result:

**Theorem 3.2. (correctors).** One has the following strong convergences:

\[
\gamma_{\alpha\beta,x}(u^\varepsilon) - \gamma_{\alpha\beta,x}(u) - U^\varepsilon(s_{\alpha\beta,y}(u^1)) \to 0 \quad \text{strongly in } L^2(\omega),
\]

\[
\nabla_x\varphi^\varepsilon - \nabla_x\varphi - U^\varepsilon(\nabla_y\varphi^1) \to 0 \quad \text{strongly in } L^2(\omega).
\]

**Proof.** Using convergences (3.60)-(3.61) and Theorem 2.2, we have

\[
\gamma_{\alpha\beta,x}(u^\varepsilon) - U^\varepsilon(\gamma_{\alpha\beta,x}(u)) - U^\varepsilon(s_{\alpha\beta,y}(u^1)) \to 0 \quad \text{strongly in } L^2(\omega),
\]

\[
\nabla_x\varphi^\varepsilon - U^\varepsilon(\nabla_x\varphi) - U^\varepsilon(s_{\alpha\beta,y}(u^1)) \to 0 \quad \text{strongly in } L^2(\omega).
\]

But \(\gamma_{\alpha\beta,x}(u) \in L^2(\omega)\) and \(\nabla_x\varphi \in L^2(\omega)\) so, by Proposition 3(i) we get

\[
U^\varepsilon(\gamma_{\alpha\beta,x}(u)) \to \gamma_{\alpha\beta,x}(u) \quad \text{strongly in } L^2(\omega),
\]

\[
U^\varepsilon(\nabla_x\varphi) \to \nabla_x\varphi \quad \text{strongly in } L^2(\omega).
\]

From (3.66)-(3.69) we deduce the convergence (3.64) and (3.65). □
4 The bending problem

4.1 The convergence results

We now consider the variational bending problem

\[
\frac{2}{3} \int_{\omega^\varepsilon} C^{\alpha\beta\mu\nu,\varepsilon} Y_{\alpha\beta}(u^\varepsilon) Y_{\mu\nu}(v) \sqrt{a^\varepsilon} \, dx = \int_{\omega^\varepsilon} f^i(x_1, x_2, z) dx + \int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} q^i v^3 \sqrt{a^\varepsilon} \, d\Gamma, \tag{4.70}
\]

for all \( v \in W(\omega^\varepsilon) \), where

\[
C^{\alpha\beta\mu\nu,\varepsilon} = C^{\alpha\beta\mu\nu}(x^\varepsilon), \quad a^\varepsilon = a(x^\varepsilon)
\]

\[
W(\omega^\varepsilon) = \left\{ w \in H^1(\omega^\varepsilon) \times H^1(\omega^\varepsilon) \times H^2(\omega^\varepsilon); \gamma_{\alpha\beta}(w) = 0 \text{ and } w^i = \partial_{\nu} w^3 = 0 \text{ on } \gamma_0^\varepsilon \subset \partial \omega^\varepsilon \right\}
\]

We have

\[
f = (f^i) \in L^2(\omega^\varepsilon \times [-1, 1]), \quad q = (q^i) \in L^2(\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon),
\]

\[
\Upsilon_{\alpha\beta}(v) = -\left( \partial^2_{\alpha} v_3 - v_\rho \left( -\partial_{\beta} b_\rho^\varepsilon + b_\rho^\varepsilon \Gamma_{\alpha\gamma}^\varepsilon + \gamma_{\alpha\beta}^\varepsilon b_\delta^\varepsilon \right) - c_{\alpha\beta} v_3 + b_\rho^\varepsilon \partial_{\beta} v_\nu - \Gamma_{\alpha\beta}^\varepsilon \partial_{\delta} v_3 \right).
\tag{4.71}
\]

Due to the ellipticity and symmetry of the bending tensor, by using the Lax-Milgram Theorem, we can deduce the existence and uniqueness to solution \( u^\varepsilon \) of the variational problem (4.70).

Let us denote

\[
F^i(x_1, x_2) = \int_{-1}^{1} f^i(x_1, x_2, z) \, dz.
\]

Thus, the equation (4.70) takes the form

\[
\frac{2}{3} \int_{\omega^\varepsilon} \sqrt{a^\varepsilon} Y_{\alpha\beta}(u^\varepsilon) Y_{\mu\nu}(v) dx = \int_{\omega^\varepsilon} F^i v^i \sqrt{a^\varepsilon} \, dx + \int_{\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon} h^i v^i \sqrt{a^\varepsilon} \, d\Gamma, \tag{4.72}
\]

for all \( v \in W(\omega^\varepsilon) \).

**Theorem 4.1.** The sequence \( (T^\varepsilon(u^\varepsilon))_\varepsilon \) weakly converges to \( u \in W(\omega) \) which is the unique solution of the homogenized problem

\[
\frac{2}{3} \int_{\omega} \sqrt{a(y)} Y_{\alpha\beta}(u) Y_{\mu\nu}(v) \, dx = |Y^*|_a \int_{\omega} F^i v^i \, dx + |Y^*|_a \int_{\Gamma_+ \cup \Gamma_-} h^i v^i \, d\Gamma, \tag{4.73}
\]

for all \( v \in W(\omega) \), where

\[
|Y^*|_a = \int_{Y^*} \sqrt{a(y)} \, dy.
\]
\[ C^{\alpha\beta\rho\sigma} = \int_{Y^*} C^{\tau\rho\sigma} \sqrt{a} \left[ \delta_{\alpha\tau} \delta_{\beta\gamma} + \partial^2_{\tau\theta, y} w^{\alpha\beta} \right] dy, \quad (4.74) \]

and the local functions \( w^{\tau\theta} \) verifies the local problems

\[
\begin{cases}
\frac{\partial}{\partial y_{\rho}} \left\{ C^{\alpha\beta\rho\sigma} \sqrt{a} \left[ \delta_{\alpha\tau} \delta_{\beta\theta} + \partial^2_{\alpha\beta, y} w^{\tau\sigma} \right] \right\} = 0 \quad \text{in } \omega \times Y^*, \quad (4.75)
\end{cases}
\]

**Remark.** We can give another expression for the homogenized bending coefficients, that as form:

\[ C^{\alpha\beta\rho\sigma} = \int_{Y^*} C^{\tau\rho\sigma} \sqrt{a} \left[ \partial^2_{\alpha\beta, y} w^{\tau\sigma} \right] \left[ \Pi^{\alpha\beta} + w^{\alpha\beta} \right] dy, \]

where \( \Pi^{\alpha\beta} = \frac{1}{2} y_{\alpha} y_{\beta} \).

**Proof.** To obtain an a priori estimate for \( u^\varepsilon \), we choose \( v = u^\varepsilon \) in (4.72). It follows

\[ \int_{\omega} C^{\alpha\beta\rho\sigma, \varepsilon} \sqrt{a^\varepsilon} \left[ \Pi^{\alpha\beta} \right] \left( u^\varepsilon \right) d\omega \leq M \| u^\varepsilon \|^2_{L^2(\omega)}, \]

where \( M > 0 \) is a positive constant that does not depend on \( \varepsilon \).

Using the ellipticity of the tensor \( C^{\alpha\beta\rho\sigma, \varepsilon} \) we get

\[ c \int_{\omega} \Pi^{2\alpha\beta} \left( u^\varepsilon \right) d\omega \leq M \| u^\varepsilon \|^2_{L^2(\omega)}. \]

Now, the Korn and Poincaré’s inequalities in perforated domains imply

\[ \| u^\varepsilon \|_{W(\omega)} \leq C, \]

where \( C \) does not depend on \( \varepsilon \). Then, up to a subsequence, \( (u^\varepsilon) \) weakly converges to some \( u \in W(\omega) \). By Theorem 1.2 we deduce that there exists \( u^2 \in L^2(\omega; H^2_{\text{per}}(Y^*)) \) such that

\[ T^\varepsilon(u^\varepsilon) \rightarrow u \quad \text{weakly in } L^2(\omega \times Y^*), \quad (4.76) \]

\[ T^\varepsilon(\nabla u^\varepsilon) \rightarrow \nabla u \quad \text{weakly in } L^2(\omega \times Y^*), \quad (4.77) \]

and

\[ T^\varepsilon(\nabla^2 u^\varepsilon) \rightarrow \nabla^2 u + \nabla^2 u^2 \quad \text{weakly in } L^2(\omega \times Y^*). \quad (4.78) \]

The linearity of \( T^\varepsilon \) implies

\[ T^\varepsilon(\nabla^2 u^\varepsilon) \rightarrow \nabla^2 u + \nabla^2 u^2 \quad \text{weakly in } L^2(\omega \times Y^*). \quad (4.79) \]

Using now the properties of the unfolding operator \( T^\varepsilon \), in (4.72), we get

\[
\frac{2}{3} \int_{\omega^* \times Y^*} T^\varepsilon \left( C^{\alpha\beta\rho\sigma, \varepsilon} \sqrt{a^\varepsilon} \right) T^\varepsilon(\nabla^2 u^\varepsilon) d\omega \text{d}y \text{d}y
\]

\[
= \int_{\omega^* \times Y^*} T^\varepsilon(\Omega^i v^i \sqrt{a^\varepsilon}) d\omega \text{d}y + \int_{(\Omega^*_+ \cup \Omega^*_-) \times Y^*} T^\varepsilon(h^i v^i \sqrt{a^\varepsilon}) d\Gamma \text{d}y, \]

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that is
\[
\frac{2}{3} \int_{\omega \times Y^*} C^{\alpha \beta \rho \sigma} \sqrt{a} T^\varepsilon (T_{\alpha \beta}(u^\varepsilon)) T^\varepsilon (T_{\rho \sigma}(v)) \, dx \, dy \\
= \int_{\omega \times Y^*} T^\varepsilon (F^i v^i \sqrt{a^\varepsilon}) \, dx \, dy + \int_{(\Gamma_+^+ \cup \Gamma_-^+)} T^\varepsilon (h^i v^i \sqrt{a^\varepsilon}) \, d\Gamma, \\
\tag{4.80}
\]

We chose as a test function in \( 4.80 \)

\[ v^\varepsilon (x) = v_1 (x) + \varepsilon^2 v_2 \left( x, \frac{x}{\varepsilon} \right), \quad v_1 \in D(\omega); \quad v_2 \in D(\omega; C^\infty_{\text{per}}(Y^*)) . \]

Then
\[
\nabla_x v^\varepsilon (x) = \nabla v_1 (x) + \varepsilon^2 \nabla_x v_2 \left( x, \frac{x}{\varepsilon} \right) + \varepsilon \nabla_y v_2 \left( x, \frac{x}{\varepsilon} \right), \\
\nabla_x^2 v^\varepsilon (x) = \nabla^2 v_1 (x) + \varepsilon^2 \nabla_x^2 v_2 \left( x, \frac{x}{\varepsilon} \right) + 2 \varepsilon \nabla_x \nabla_y v_2 \left( x, \frac{x}{\varepsilon} \right) + \nabla_y^2 v_2 \left( x, \frac{x}{\varepsilon} \right) \|
\]

It follows that
\[
\begin{align*}
\nabla_x v^\varepsilon (x) & \to v_1 (x) & \text{strongly in} \ L^2 (\omega), \\
\nabla_x^2 v^\varepsilon (x) & \to \nabla^2 v_1 (x) + \nabla_y^2 v_2 (x, y) & \text{weakly in} \ L^2 (\omega \times Y^*). \\
\end{align*}
\tag{4.82}
\]

And
\[
T^\varepsilon (T_{\rho \sigma}(v^\varepsilon)) \to T_{\rho \sigma}(v_1) + \partial^2_{\rho \sigma, y} v_2 \quad \text{weakly in} \ L^2 (\omega \times Y^*). \tag{4.83}
\]

Using \( 4.82 \) and \( 4.83 \), and passing to the limit in \( 4.80 \) with \( \varepsilon \searrow 0 \), we get
\[
\frac{2}{3} \int_{\omega \times Y^*} C^{\alpha \beta \rho \sigma} \sqrt{a} \left[ T_{\alpha \beta}(u) + \partial_{\alpha \beta, y}^2 u^2 \right] T_{\rho \sigma}(v_1) \, dx \, dy \\
= |Y^*|_a \int_{\omega} F^i v^i_1 \, dx + |Y^*|_a \int_{(\Gamma_+^+ \cup \Gamma_-^+)} h^i v^i_1 \, d\Gamma. \tag{4.84}
\]

We now let \( v_2 = 0 \) in \( 4.83 \). We obtain
\[
\frac{2}{3} \int_{\omega \times Y^*} C^{\alpha \beta \rho \sigma} \sqrt{a} \left[ T_{\alpha \beta}(u) + \partial_{\alpha \beta, y}^2 u^2 \right] T_{\rho \sigma}(v_1) \, dx \, dy \\
= |Y^*|_a \int_{\omega} F^i v^i_1 \, dx + |Y^*|_a \int_{(\Gamma_+^+ \cup \Gamma_-^+)} h^i v^i_1 \, d\Gamma. \tag{4.85}
\]

By density, we deduce
\[
\frac{2}{3} \int_{\omega \times Y^*} C^{\alpha \beta \rho \sigma} \sqrt{a} \left[ T_{\alpha \beta}(u) + \partial_{\alpha \beta, y}^2 u^2 \right] T_{\rho \sigma}(v) \, dx \, dy \\
= |Y^*|_a \int_{\omega} F^i v^i_1 \, dx + |Y^*|_a \int_{(\Gamma_+^+ \cup \Gamma_-^+)} h^i v^i_1 \, d\Gamma, \tag{4.86}
\]
for all \( v \in L^2(\omega; H^2_{\text{per}}(Y^*)/\mathbb{R}) \). In what follows we chose
\[
\mathbf{u}^2 = \mathbf{\tau}_{\theta}(\mathbf{u})w^\tau, \quad w^\tau \in L^2(\omega; H^2_{\text{per}}(Y^*)/\mathbb{R}). \tag{4.87}
\]
Then
\[
\partial_{\alpha\beta,y} \mathbf{u}^2 = \mathbf{\tau}_{\theta}(\mathbf{u})\partial_{\alpha\beta,y}w^\tau. \tag{4.88}
\]
Replacing (4.88) in (4.86) we obtain
\[
\frac{2}{3} \int_{\omega \times Y^*} C^{\alpha\beta\rho\sigma} \sqrt{a} \left[ \mathbf{\tau}_{\theta}(\mathbf{u}) + \mathbf{\tau}_{\theta}(\mathbf{u})\partial_{\alpha\beta,y}w^\tau \right] \mathbf{\tau}_{\rho\sigma}(v) \, dx dy
\]
for all \( v \in L^2(\omega; H^2_{\text{per}}(Y^*)/\mathbb{R}) \). This yields
\[
\frac{2}{3} \int_{\omega \times Y^*} C^{\alpha\beta\rho\sigma} \sqrt{a} \left[ \delta_{\alpha\tau\beta\theta} + \partial_{\alpha\beta,y}w^\alpha \right] \mathbf{\tau}_{\rho\sigma}(v) \, dx dy = |Y^*|_a \int_{\omega} F^iy^i \, dx + |Y^*|_a \int_{\Gamma_+ \cup \Gamma_-} h^iy^i \, d\Gamma, \tag{4.90}
\]
for all \( v \in L^2(\omega; H^2_{\text{per}}(Y^*)/\mathbb{R}) \).

If we denote
\[
\mathbf{\tau}^{\alpha\beta\rho\sigma} = \int_{Y^*} C^{\alpha\beta\rho\sigma} \sqrt{a(y)} \left[ \delta_{\alpha\tau\beta\theta} + \partial_{\alpha\beta,y}w^\alpha \right] \, dy, \tag{4.91}
\]
by (4.90) we get the homogenized equation which corresponds to (4.72):
\[
\frac{2}{3} \int_{\omega} \mathbf{\tau}^{\alpha\beta\rho\sigma} \mathbf{\tau}_{\alpha\beta}(\mathbf{u}) \mathbf{\tau}_{\rho\sigma}(v) \, dx = |Y^*|_a \int_{\omega} F^iy^i \, dx + |Y^*|_a \int_{\Gamma_+ \cup \Gamma_-} h^iy^i \, d\Gamma, \tag{4.92}
\]
for all \( v \in L^2(\omega; H^2_{\text{per}}(Y^*)/\mathbb{R}) \).

Let us find now the equations verified by the local functions \( w^\tau \).

Letting \( \phi = 0 \) in (4.83) we get
\[
\int_{\omega \times Y^*} C^{\alpha\beta\rho\sigma} \sqrt{a} \left[ \mathbf{\tau}_{\alpha\beta}(\mathbf{u}) + \partial_{\alpha\beta,y}^2 \mathbf{u}^2 \right] \partial_{\rho\sigma,y}^2 \mathbf{v}_2(x,y) \, dx dy = 0.
\]
By density it follows that
\[
\int_{\omega \times Y^*} C^{\alpha\beta\rho\sigma} \sqrt{a} \left[ \mathbf{\tau}_{\alpha\beta}(\mathbf{u}) + \partial_{\alpha\beta,y}^2 \mathbf{u}^2 \right] \partial_{\rho\sigma,y}^2 \mathbf{v}(x,y) \, dx dy = 0, \tag{4.93}
\]

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for all \( v \in L^2(\omega; H^2_{\text{per}}(Y^*)/\mathbb{R}) \).

Using now (4.88) in (4.93) we have

\[
\frac{\partial}{\partial y_p} \left\{ \Upsilon_{\alpha\beta}(u) C^{\alpha\beta\rho\sigma} \sqrt{a} \left[ \delta_{\alpha\tau} \delta_{\beta\rho} + \partial_{\alpha\beta,y}^2 w^\tau \right] \right\} = 0 \quad \text{in} \quad \omega \times Y^*
\]

and so

\[
\frac{\partial}{\partial y_p} \left\{ C^{\alpha\beta\rho\sigma} \sqrt{a} \left[ \delta_{\alpha\tau} \delta_{\beta\rho} + \partial_{\alpha\beta,y}^2 w^\tau \right] \right\} = 0 \quad \text{in} \quad \omega \times Y^*. \tag{4.94}
\]

In order to establish the existence and uniqueness of the solution of (4.73), if suffices to prove the coercivity of \( C^{\alpha\beta\gamma\eta} \) in the following sense

\[
\exists \Lambda_C \neq \Lambda_C(\varepsilon) > 0, \quad \forall (\xi_{\alpha\beta})_{\alpha\beta} : \xi_{\alpha\beta} = \xi_{\beta\alpha}, \quad C_{\alpha\beta\gamma\eta} \xi_{\alpha\beta} \xi_{\gamma\eta} \geq \Lambda_C \xi_{\alpha\beta} \xi_{\alpha\beta}
\]

**Symmetry**

It is easy to check that

\[
C^{\alpha\beta\gamma\eta} = C^{\beta\alpha\gamma\eta} = C^{\alpha\beta\eta\gamma}
\]

It suffices to prove

\[
C^{\alpha\beta\gamma\eta} = C^{\gamma\alpha\beta}\eta
\]

Starting from the definition (4.91) of the \( C = (C^{\alpha\beta\gamma\eta}) \), the homogenized bending tensor is evaluated by

\[
C^{\alpha\beta\gamma\eta} = \int_{Y^*} C_{\alpha\beta\delta\tau} \sqrt{a} \partial_{\delta\tau,y}^2 \left[ \Pi^{\gamma\eta} + w^{\gamma\eta} \right] dy
\]

By multiplying (4.75) by \( w^{\alpha\beta} \) and integrating by parts, we prove

\[
\int_{Y^*} C^{\zeta\xi\delta\tau} \sqrt{a} \partial_{\delta\tau,y}^2 \left[ \Pi^{\gamma\eta} + w^{\gamma\eta} \right] \partial_{\xi\zeta,y}^{\alpha\beta} w^{\alpha\beta} dy = 0.
\]

It follows that

\[
C^{\zeta\xi\gamma\eta} = \int_{Y^*} C_{\delta\eta\zeta\xi} \sqrt{a} \partial_{\zeta\xi,y}^2 \left( \Pi^{\gamma\eta} + w^{\gamma\eta}(y) \right) \partial_{\delta\tau,y}^2 \left( \Pi^{\tau\eta} + w^{\tau\eta}(y) \right) dy. \tag{4.96}
\]

From (4.96), we deduce \( C^{\zeta\xi\gamma\eta} = C^{\gamma\alpha\beta} \).
Ellipticity

Let \((\xi_{\alpha\beta})_{\alpha\beta}\) be a symmetric tensor \((\xi_{\alpha\beta} = \xi_{\beta\alpha})\) and set

\[
\tau_{\delta\eta} = \xi_{\alpha\beta} \partial^2_{\delta\eta, y} \left( \Pi^{\alpha\beta} + w^{\alpha\beta} \right).
\]

Using now the coercivity of tensor \(C^{\lambda\mu\nu}(x, y)\) and the fact that \(a \neq 0\), we can write

\[
C^{\alpha\beta\gamma\eta} \xi_{\alpha\beta} \xi_{\gamma\eta} \geq \int_{Y^*} C^{\delta\eta\zeta\nu} \sqrt{a} \tau_{\delta\eta} \tau_{\zeta\nu} \, dy \geq c \int_{Y^*} \tau_{\delta\eta} \tau_{\delta\eta} \, dy. \tag{4.97}
\]

We claim that the second integral in (4.97) is positive. Assume the contrary. Then

\[
\forall \ (\delta, \eta) \in \{1, 2\}^2, \quad \tau_{\delta\eta} = \xi_{\alpha\beta} \partial^2_{\delta\eta, y} (\Pi^{\alpha\beta} - w^{\alpha\beta}) = 0. \tag{4.98}
\]

It follows that

\[
\partial^2_{\delta\eta, y} (\xi_{\alpha\beta} (\Pi^{\alpha\beta} - w^{\alpha\beta})) = 0.
\]

This implies that

\[
\xi_{\alpha\beta} (\Pi^{\alpha\beta} - w^{\alpha\beta}) = a_\epsilon y_{\epsilon} + b,
\]

for some constants \(a_\epsilon\) and \(b\), \(\epsilon = 1, 2\). This yields

\[
w^{\alpha\beta} \xi_{\alpha\beta} = \Pi^{\alpha\beta} \xi_{\alpha\beta} + a_\epsilon y_{\epsilon} + b.
\]

Since \(\xi \neq 0\), we can find an indice \((\alpha, \beta)\) such that \(\xi_{\alpha\beta} \neq 0\). In this case, the left-hand side of the above equality is \(Y^*\)-periodic, but the right-hand side is not, this is clearly a contradiction. Then the second integral of (4.97) is positive and so

\[
C^{\alpha\beta\gamma\eta} \xi_{\alpha\beta} \xi_{\gamma\eta} > 0 \quad \forall \ (\xi_{\alpha\beta}) \neq 0 \text{ symmetric}.
\]

Let us define the function \(\Psi : \mathbb{R}^4 \rightarrow \mathbb{R}\) by

\[
\Psi(\xi_{\alpha\beta}) = C^{\alpha\beta\gamma\eta} \xi_{\alpha\beta} \xi_{\gamma\eta}.
\]

It is easy to see that \(\Psi\) is continuous in \(\mathbb{R}^4\) endowed with to the norm

\[
\| \tau \| = (\tau_{\alpha\beta} \tau_{\alpha\beta})^{1/2}.
\]

Since \(\Phi\) attains its minimum on the unit sphere in \(\mathbb{R}^4\) and \(\Psi > 0\) for all symmetric tensor \((\xi_{\alpha\beta}) \neq 0\), we can conclude that there exists \(M > 0\) such that

\[
\Psi \left( \frac{\xi_{\alpha\beta}}{\xi} \right) \geq M, \quad \text{for all symmetric tensor} \ (\xi_{\alpha\beta}) \neq 0.
\]

From the above inequality we deduce

\[
C^{\alpha\beta\gamma\eta} \xi_{\alpha\beta} \xi_{\gamma\eta} \geq M \xi_{\alpha\beta} \xi_{\alpha\beta}.
\]

The uniqueness of the solution of (4.73) follows now by using the Lax-Milgram Theorem. \(\Box\)
4.2 Corrector result

We have the following convergence:

\[ T^\varepsilon(\Upsilon_{\rho\sigma}(\mathbf{u}^\varepsilon)) \rightharpoonup \Upsilon_{\rho\sigma} x(\mathbf{u}) + \partial^2_{\rho\sigma,y} \mathbf{u}_2 \quad \text{weakly in } L^2(\omega \times Y^*). \] (4.99)

The convergence of energies is also proved easily, and implies in particular that the weak convergences in (4.99) is actually strong:

\[ T^\varepsilon(\Upsilon_{\rho\sigma} x(\mathbf{u}^\varepsilon)) \rightarrow \Upsilon_{\rho\sigma} x(\mathbf{u}) + \partial^2_{\rho\sigma,y} \mathbf{u}_2 \quad \text{strongly in } L^2(\omega \times Y^*). \] (4.100)

**Theorem 4.2. (correctors).** One has the following strong convergence:

\[ \Upsilon_{\rho\sigma} x(\mathbf{u}^\varepsilon) - \Upsilon_{\rho\sigma} x(\mathbf{u}) - \mathcal{U}^\varepsilon(\partial^2_{\rho\sigma,y} \mathbf{u}_2) \rightarrow 0 \quad \text{strongly in } L^2(\omega). \]

**Proof.** We have already seen (see (4.100)) that

\[ T^\varepsilon(\Upsilon_{\rho\sigma} x(\mathbf{u}^\varepsilon)) - \Upsilon_{\rho\sigma} x(\mathbf{u}) - \partial^2_{\rho\sigma,y} \mathbf{u}_2 \rightarrow 0 \quad \text{strongly in } L^2(\omega \times Y^*), \]

which, by Theorem 2.3 is equivalent to

\[ \Upsilon_{\rho\sigma} x(\mathbf{u}^\varepsilon) - \mathcal{U}^\varepsilon(\Upsilon_{\rho\sigma} x(\mathbf{u})) - \partial^2_{\rho\sigma,y} \mathbf{u}_2 \rightarrow 0 \quad \text{strongly in } L^2(\omega). \]

But \( \Upsilon_{\rho\sigma} x(\mathbf{u}) \in L^2(\omega) \), so from (i) of Proposition 3, one has \( \mathcal{U}^\varepsilon(\Upsilon_{\rho\sigma} x(\mathbf{u})) \rightarrow \Upsilon_{\rho\sigma} x(\mathbf{u}) \) strongly in \( L^2(\omega) \), whence the desired result. \( \square \)

5 Conclusion

In this paper we have rigorously established the limiting equations modelling the behavior of a thin piezoelectric perforated shells, i.e., we have explicitly described the *homogenized coefficients* of the elastic, dielectric and coupling tensors (for details, see [9]).

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