GLOBAL MULTIDIMENSIONAL SHOCK WAVES FOR 2-D AND 3-D UNSTEADY POTENTIAL FLOW EQUATIONS

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Abstract

Although local existence of multidimensional shock waves has been established in some fundamental references, there are few results on the global existence of those waves except the ones for the unsteady potential flow equations in \( n \)--dimensional spaces \((n \geq 5)\) or in special unbounded space-time domains with some artificial boundary conditions. In this paper, we are concerned with both the local and global multidimensional conic shock wave problems for unsteady potential flow equations when a pointed piston (i.e., the piston at the initial time degenerates into a single point) or an explosive wave expands fast in 2-D or 3-D static polytropic gases. It is shown that a multidimensional shock wave solution to such a class of quasilinear hyperbolic problems not only exists locally but also exists globally in the whole time-space and tends to a self-similar solution as \( t \to \infty \).

Keywords: Unsteady potential flow equation, multidimensional shock, nonlinear elliptic equation, pseudo-differential operator, improved Hardy-type inequality, modified Klainerman’s vector fields

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§1. Introduction

Although local existence results of multidimensional planar shock waves have been established early in some fundamental references (see [23-25] and the references therein), there are few results on the global existence of those waves except the ones for the unsteady potential flow equations in much higher space dimensions ($n \geq 5$) or in special unbounded space-time domains with some artificial boundary conditions (i.e., the Dirichlet boundary conditions of potentials on some specially chosen boundaries, one can see [10-11] and [36]). This paper concerns both the local and global multidimensional shock wave problems when a pointed piston (i.e., the piston at the initial time degenerates into a single point) or an explosive wave expands in the 2-D or 3-D isentropic irrotational gas. The piston problem is a fundamental one in gas dynamics, in particular, the expansive pointed piston problem is also closely related or similar to the study of explosive waves in physics, one can see the references [1], [4], [7-8], [18], [22], [26], [30-32] and so on. Such a problem is also one of the basic models in establishing the theory of weak solutions to the multidimensional quasilinear hyperbolic equations or systems. As described in pages 120 of [7]: A basic and typical motion of gas is the one caused by a piston in a tube starting from rest and suddenly moving with constant velocity $u_P$ into the quiet gas. No matter how small $u_P$ is, the resulting motion cannot be continuous. Generally speaking, if the piston recedes, then a rarefaction wave will be caused, and otherwise, if the piston is pushed, then a shock wave will be formed. In this paper, we will study the multi-dimensional case of a piston problem as in [1], [4] and so on, but there authors posed some symmetric properties on solutions. That is, we suppose that there is a rest gas filling the whole space outside a given pointed piston with expansive boundary. With the development of time, the pointed piston gradually expands its boundary into the air in two-dimensional spaces or three-dimensional spaces. Subsequently, there will be a multidimensional shock wave moving into the air away from the piston (See the Figures 1-3 below). Mathematically, this is an initial boundary value problem for the 2-D or 3-D compressible Euler system, which contains a free boundary (shock surface) and a moving boundary (surface of expansive piston). For the rapidly expansive piston in the air, we will establish both the local and global existence of a multidimensional (2-D or 3-D) shock wave solution.

Figure 1. A conic shock is formed when a pointed piston expands in polytropic gas
We will use $n$-dimensional ($n = 2, 3$) unsteady potential equation to describe the motion of the polytropic gas in expansive pointed piston problem (this model is also recommended in [4-5], [24], [35], [38] and so on), where polytropic gas means that the pressure $P$ and the density $\rho$ of the gas are described by the state
equation $P = A \rho^\gamma$ with $A > 0$ a constant and the adiabatic constant $\gamma$ satisfying $1 < \gamma < 3$ (for the air, $\gamma \approx 1.4$). Let $\Phi(t,x)$ be the potential of velocity $u = (u_1, \cdots, u_n)$ with $x = (x_1, \cdots, x_n)$, i.e., $u_i = \partial_i \Phi$ $(1 \leq i \leq n)$, then it follows from the Bernoulli’s law that

$$\partial_t \Phi + \frac{1}{2} |\nabla_x \Phi|^2 + h(\rho) = B_0,$$

(1.1)

here $h(\rho) = \frac{c^2(\rho)}{\gamma - 1}$ is the specific enthalpy, $c(\rho) = \sqrt{P'(\rho)}$ is the local sound speed, $\nabla_x = (\partial_1, \cdots, \partial_n)$, $B_0 = \frac{c^2(\rho_0)}{\gamma - 1}$ is the Bernoulli’s constant of static gas with the constant density $\rho_0$.

By (1.1) and the implicit function theorem due to $h'(\rho) = \frac{c^2(\rho)}{\rho} > 0$ for $\rho > 0$, then the density function $\rho(t,x)$ can be expressed as

$$\rho = h^{-1} \left( B_0 - \partial_t \Phi - \frac{1}{2} |\nabla_x \Phi|^2 \right) \equiv H(\nabla \Phi),$$

(1.2)

here $\nabla = (\partial_t, \nabla_x)$.

Substituting (1.2) into the mass conservation equation $\partial_t \rho + \sum_{i=1}^n \partial_i (\rho u_i) = 0$ of gas yields

$$\partial_t (H(\nabla \Phi)) + \sum_{i=1}^n \partial_i (H(\nabla \Phi) \partial_i \Phi) = 0.$$  

(1.3)

More intuitively, for any $C^2$ solution $\Phi$, (1.3) can be rewritten as the following second order quasilinear equation

$$\partial_t^2 \Phi + 2 \sum_{k=1}^n \partial_k \Phi \partial^2_{kk} \Phi + \sum_{i,j=1}^n \partial_i \Phi \partial_j \Phi \partial^2_{ij} \Phi - c^2(\rho) \Delta \Phi = 0,$$

(1.4)

here $c(\rho) = c(H(\nabla \Phi))$, and the Laplace operator $\Delta = \sum_{i=1}^n \partial^2_i$.

It is noted that (1.4) is strictly hyperbolic with respect to the time $t$ when $\rho > 0$ holds.

For convenience to write and compute later on, the following spherical coordinates are often used

$$(t, r, \omega) = (t, |x| / |x|).$$

(1.5)

Under the coordinate transformation (1.5), we suppose that the expansive path of the pointed piston in static gas is $\Sigma$: $r = \sigma(t, \omega)$ and denote that the potential before and behind the resulting shock front $\Gamma$: $r = \zeta(t, \omega)$ with $\zeta(0, \omega) = 0$ are written by $\Phi^-(t,x)$ and $\Phi^+(t,x)$ respectively. And the corresponding domains are denoted by $\Omega_-$ and $\Omega_+$ respectively. Since the gas ahead of the piston is static, then $\Phi^-(t,x) \equiv 0$ can be chosen in $\Omega_-$. In $\Omega_+$, $\Phi^+$ satisfies

$$\partial_t^2 \Phi^+ + 2 \sum_{k=1}^n \partial_k \Phi^+ \partial^2_{kk} \Phi^+ + \sum_{i,j=1}^n \partial_i \Phi^+ \partial_j \Phi^+ \partial^2_{ij} \Phi^+ - c^2(\rho^+) \Delta \Phi^+ = 0,$$

(1.6)

here $c(\rho^+) = c(H(\nabla \Phi^+)).$
On the surface $\Sigma$ of expansive piston, $\Phi^+$ satisfies the following solid boundary condition

$$B_\sigma \Phi^+ \equiv \partial_\sigma \Phi^+ - \frac{1}{r^2} \sum_{i=1}^{2n-3} Z_i \sigma \cdot Z_\sigma \Phi^+ = \partial_\sigma \sigma,$$  \hspace{1cm} (1.7)

where

$$Z_1 = x_1 \partial_2 - x_2 \partial_1, \quad Z_2 = x_2 \partial_3 - x_3 \partial_2, \quad Z_3 = x_3 \partial_1 - x_1 \partial_3,$$

which form a basis of smooth vector fields tangent to the sphere $S^2$. In particular, $Z_1$ is a smooth vector field tangent to the unit circle $S^1 = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$.

Meanwhile, on the shock surface $\Gamma$, by the equation (1.3), the corresponding Rankine-Hugoniot condition is

$$H(\nabla \Phi^+)^2 \partial_\sigma \Phi^+ - (H(\nabla \Phi^+) - \rho_0) \partial_\sigma \zeta = \frac{1}{r^2} H(\nabla \Phi^+) \sum_{i=1}^{2n-3} Z_i \zeta \cdot Z_\sigma \Phi^+ \text{ on } \Gamma.$$  \hspace{1cm} (1.8)

Moreover, the potential function $\Phi(t, x)$ is continuous across $\Gamma$, namely,

$$\Phi^+ = \Phi^- \equiv 0 \text{ on } \Gamma.$$  \hspace{1cm} (1.9)

Additionally, the physical entropy condition holds on the shock surface

$$H(\nabla \Phi^+) > \rho_0 \text{ on } \Gamma.$$  \hspace{1cm} (1.10)

The main result in our paper can be stated as:

**Theorem 1.1.** For $n = 2, 3$, if the equation of $\Sigma$ is $r = \sigma(t, \omega) = tb(t, \omega)$, here $b(t, \omega) \in C^\infty([0, +\infty) \times S^{n-1})$, $|Z^k b(0, \omega) - b_0| \leq C_{k} \varepsilon$ and $|\partial_t^k Z^k b(t, \omega) - b_0| \leq \frac{C_{k_0}}{(1 + t)^{k_0}}$ with $k_0, |k| = k_1 + k_2 + k_3 \in \mathbb{N} \cup \{0\}$, $Z \in \{Z_1, ..., Z_{2n-3}\}$ and a small positive constant $\varepsilon$, then for suitably large constant $b_0$, there exists a positive constant $\varepsilon_0$ depending on $b_0$ and $\gamma$, such that the problem (1.6) together with (1.7)-(1.10) has a global shock solution $(\Phi^+(t, x), \zeta(t, \omega))$ as $0 < \varepsilon < \varepsilon_0$. Moreover, $(\nabla \Phi^+, \zeta(t, \omega) / |x|)$ tend to the corresponding ones, which are formed by the symmetrically expansive pointed piston $r = b(t)$ in static gas, with the decay rate $(1 + t)^{-m_0}$ for any positive numbers $m_0 < \frac{5}{4} - \frac{1}{7} \sqrt{\frac{\gamma + 1}{2}}$ if $n = 2$ and $m_0 < \frac{3}{2} - \frac{1}{4} \sqrt{\frac{\gamma + 7}{2}}$ if $n = 3$ respectively.

**Remark 1.1.** From the expression $r = \sigma(t, \omega) = tb(t, \omega)$ of the equation of $\Sigma$ and the assumptions on $b(t, \omega)$ in Theorem 1.1, we know that the normal expansive velocity of the pointed piston is $\partial_t \sigma(t, \omega) = b(t) + O(\varepsilon)$, which is a small perturbation of the constant expansive speed $b_0$.

**Remark 1.2.** It is noted that the nonlinear hyperbolic equation (1.6) is actually a second order quasi-linear wave equation in three or two space dimensions. By a direct verification, one can know that (1.6) does not fulfill the “null-condition” or admits the cubic order nonlinearity under some structural transformations (one can see [6], [16] or [33]-[34]). Therefore, in terms of the results in [2]-[3], [13], [15], [29], [37] and so on, if there is no main shock for the equation (1.6), then the classical solution of (1.6) must blow up in finite time. Our result in Theorem 1.1 means that the main multidimensional shock can absorb all possible compressions of the flow and prevent the formations of new shocks and other singularities in expansive piston movement as in [19] or [36].

**Remark 1.3.** Since BV spaces fail for the multidimensional hyperbolic equations or systems as shown in [27], then the Glimm scheme method (see [22] and the references therein) can not be used to treat our really multidimensional problem in this paper. On the other hand, there are no any symmetric assumptions on the movement of the piston in our problem, then the equation (1.4) can not be reduced into a second order
nonlinear elliptic equation with a free boundary in a bounded domain as in [4], where the solvability is shown by some techniques from linear elliptic equations.

Remark 1.4. For the solutions of n-dimensional linear wave equations, it is well-known that their optimal decay rates are \( (1 + t)^{-\frac{m_0}{2}} \) as \( t \to \infty \) (see [13]). Compared with this, since one can obtain the decay rates \( \frac{1}{1 + t}^{m_0} \) with \( \frac{1}{2} < m_0 < 1 \) in 2-D case and \( \frac{3}{4} < m_0 < 1 \) in 3-D case respectively in Theorem 1.1, we obtain a better decay in 2-D case but a lower decay in 3-D case with respect to the problem (1.6) together with (1.7)-(1.9). However, due to some special properties of the shock boundary conditions (1.8)-(1.9), such decay rates will be enough to establish the global existence in our nonlinear problem.

Remark 1.5. Since the background solution is self-similar in our problem, then the related nonlinear problem admits largely variable coefficients as in [20] and the related analysis are rather involved.

Remark 1.6. (1.10) naturally holds if we have shown the solution of problem (1.6) together with (1.7)-(1.9) is a small perturbation of the background solution since the background solution satisfies (1.10) by the physical entropy condition. Therefore, we can only focus on problem (1.6) together with (1.7)-(1.9) from now on.

Let us comment on the proof of Theorem 1.1. As the first step, we will establish the local existence of solution to problem (1.6) together with (1.7)-(1.9) in \( \Omega_+ \). For this aim, we have to overcome the difficulties induced by the unknown shock \( \Gamma \) and the double conic point \((t, x) = (0, 0)\) between two conic surfaces \( \Sigma \) and \( \Gamma \). We will apply a partial hodograph transformation as in [25] and [4] to change the unknown domain \( \Omega_+ \) into a fixed cylindrical domain \( \tilde{\Omega}_+ = (0, \infty) \times (1, 2) \times S^{n-1} \), while the equation (1.6) and boundary conditions (1.7)-(1.9) become a rather complicated second order nonlinear equation including the solution \( \psi \) itself as well as the derivatives of \( \psi \) and three involved nonlinear boundary conditions on the new resulting boundaries respectively. Based on this, we start to construct a suitable approximate solution \( \psi_0^{\infty} \) to this resulting nonlinear hyperbolic equation by solving a series of second order elliptic equations, and subsequently consider the related nonlinear equation together with the corresponding boundary conditions on the function \( \psi - \psi_0^{\infty} \), so that the weighted energy estimates are derived and further the local existence of shock solution can be established. In this process, except the construction of approximate solution, we have to treat the well-posedness problems on the linear hyperbolic equation with an inhomogeneous Neumann-type boundary condition and an oblique derivative boundary condition, and then continue to treat the nonlinear hyperbolic equation together with a nonlinear Neumann-type boundary condition and a nonlinear uniform oblique derivative boundary condition. Here we emphasize that some fundamental methods in [5] are not available for our problem. The main reasons are: If we directly use a finite power expansion of \( t \) to look for an approximate solution of (1.6) together with (1.7)-(1.9) as in [5] (see (2.11), (2.1.6) and (2.1.10) in Section 2 of [5]), then a series of equations only in the same domain (see (2.1.4)-(2.1.5) in [5]) are obtained and thus a real approximate solution can not be found since the crucial property of the free boundary \( \Gamma \) is neglected. On the other hand, since the Neumann-type boundary condition on the moving surface \( \Sigma \) does not satisfy the uniform Lopatinski condition, namely, the Local Stability Condition on \( \Sigma \) introduced in [18] is not fulfilled (however, the lines 2-3 from below on pages 177 of [5] give a different assertion), then the well-posedness problem and further the energy estimate on the resulting linearization problem can not be established directly by the results in [18] other than claimed in the proof procedure of Theorem 4 of [5] since we have to treat the well-posedness problem on the second order hyperbolic equations with an inhomogeneous Neumann-type boundary condition and a uniform oblique derivative boundary condition in order to solve the linearized problems from the iteration scheme. Thanks to our delicate analysis and some ideas in [14] and [28], we can finally complete this task.

Based on the local existence result in the above, we will utilize the continuous induction method to prove the global existence as in [20] and so on. To achieve this, we need to derive global weighted energy estimates for the problem (1.6) with (1.7)-(1.9). By such estimates, one then obtains the global existence, stability, and the asymptotic behavior of the shock solution to the perturbed nonlinear problem (1.6). The
key methods in the analysis to obtain weighted energy estimates are to choose the appropriate multipliers and establish a new Hardy-type inequality on the shock surface in terms of the special structures of the shock boundary conditions. Finding such suitable multipliers is much more delicate due to the following reasons: First, in order to obtain the global existence, one needs to establish a global estimate independent of the time $t$, of the potential function and its derivatives on the boundaries as well as in the interior of $\Omega_+$. This yields strict restrictions on the multiplier and makes the computations involved. Second, for the three dimensional case, the Neumann-type boundary condition (1.7) fulfilled by $\Phi^+$ yields additional difficulties compared to [11] and [19], where [11] treats the case of an artificial Dirichlet-type boundary condition for the potential on a multi-dimensionally perturbed conic surface but away from the conic point. The latter plays a key role in the analysis of [11] since the corresponding Poincaré inequalities are available on the shock surface and in the interior of the downstream domain respectively, while this is not the case in the problem treated here. Furthermore, it should be noted that the arbitrary and artificial closeness between the shock surface and the fixed boundary also plays a crucial role in the analysis of [11], which is also not the case for our problem. Meanwhile, the authors in [19] only treat the 2-D shock problem of steady potential flow equation if one takes the supersonic direction as the time direction, and where only the fixed circular cone boundary is considered. Thanks to some careful analysis together with an improved Hardy-type inequality derived by utilizing the special structures of the shock boundary conditions (see (4.11)-(4.12) below), we finally overcome all these difficulties and obtain a uniform estimate of $\|\nabla_{t,x} \Phi^+\|_{L^2(\Omega_+)}$. From this, higher-order energy estimates of $\nabla_{t,x} \Phi^+$ can be established by making full use of modified Klainerman’s vector field and commutator arguments together with a careful verification that some suitably higher-order derivative combinations of the solution satisfy the Neumann-type boundary condition on $\Sigma$. This finally derives Theorem 1.1.

The paper is organized as follows: In §2, we derive some basic estimates on the self-similar background solution which is formed by the symmetric pointed expansion piston in the static polytropic gas. This will be required to simplify the nonlinear problem (1.6) together with (1.7)-(1.9) and look for the multipliers in related energy estimates later on. The local-in-time existence of the problem (1.6) with (1.7)-(1.9) is established in §3. In §4, we reformulate problem (1.6)-(1.9) by decomposing its solution as a sum of the modified background solution and a small perturbation $\dot{\phi}$ so that its main part can be studied in a convenient way. In §5, we first establish a uniform weighted energy estimate for the corresponding 3-D problem, where an appropriate multiplier is also constructed. Based on such an energy estimate, we obtain a uniform weighted energy estimate of $\nabla_{t,x} \dot{\phi}$ for the nonlinear problem through establishing an improved Hardy-type inequality. By the estimates derived in the first step, we continue to establish uniform higher-order weighted energy estimates of $\nabla_{t,x} \dot{\phi}$ in the case of $n = 3$ through looking for the suitably modified Klainerman’s vector fields. Finally, the proof of Theorem 1.1 for the case $n = 3$ is completed by using Sobolev’s embedding theorem and continuous induction. In §6, we give the sketch of the proof on Theorem 1.1 in the case of $n = 2$. Some basic computations are arranged in Appendix A. Additionally, in order to deal with the Neumann-type boundary condition on the curved boundary $r = \sigma(t, \omega)$, we need to modify the self-similar solution and obtain a modified background solution $\Phi_\alpha$ so that the boundary condition (1.7) can be fulfilled. This will be given in Appendix B.

In what follows, we will use the following convention:

- $C$ stands for a generic positive constant which does not depend on any quantity except the adiabatic constant $\gamma$ ($1 < \gamma < 3$).
- $C(\cdot)$ represents a generic positive constant which depends on its argument(s).
- $O(\cdot)$ means that $|O(\cdot)| \leq C |\cdot|$ holds.
- $dS$ stands for the surface measure in the corresponding surface integral.

§2. The analysis on the self-similar background solution

In this section, we will give some detailed properties on the background self-similar solution which is
formed by the symmetric pointed expansive piston \( r = b_0 t \) in the static polytropic gas with suitably large \( b_0 \). These properties will be applied in the later analysis of §3–§6 below.

For the pointed expansive piston \( r = b_0 t \), there will appear a conic shock \( r = s_0 t \ (s_0 > b_0) \) in the static gas. Moreover the solution of (1.3) with (1.1) behind the shock surface is self-similar, that is, the states of density and velocity between the shock front and the surface of piston have such forms: \( \rho = \rho(s), u_i = u(s) \frac{x_i}{r} \) \((i = 1, \cdots, n)\) with \( s = \frac{r}{t} \). In this case, the \( n \)-dimensional potential equation can be reduced to a nonlinear ordinary differential system as follows

\[
\begin{cases}
    \rho'(s) = \frac{(n-1)(s-u)\rho u}{s((s-u)^2 - c^2(\rho))}, \\
    u'(s) = \frac{(n-1)c^2(\rho)u}{s((s-u)^2 - c^2(\rho))}
\end{cases}
\quad \text{for } b_0 \leq s \leq s_0. \tag{2.1}
\]

By [1] or [4], the denominator \((s-u)^2 - c^2(\rho)\) \(\leq 0\) holds for \( b_0 \leq s \leq s_0 \). This means that the system (2.1) makes sense.

On the shock front \( r = s_0 t \), it follows from the Rankine-Hugoniot conditions and Lax’s geometric entropy conditions on the 2-shock that

\[
\begin{cases}
    s_0[\rho] - [pu] = 0, \\
    s_0[u] - [\frac{1}{2}u^2 + h(\rho)] = 0
\end{cases}
\tag{2.2}
\]

and

\[
\begin{cases}
    u(s_0) - c(\rho(s_0)) < s_0 < u(s_0) + c(\rho(s_0)), \\
    c(\rho_0) < s_0.
\end{cases}
\tag{2.3}
\]

Additionally, the flow satisfies the fixed boundary condition on \( s = b_0 \)

\[
u(b_0) = b_0. \tag{2.4}
\]

It has been shown that the boundary value problem (2.1)-(2.4) can be solved by [1] or [4].

For suitably large \( b_0 \), some properties on the background solution can be given as follows:

**Lemma 2.1.** For suitably large \( b_0 \) and \( 1 < \gamma < 3 \), one has for \( b_0 \leq s \leq s_0 \),

(i) \( s_0 = b_0 \left( 1 + O(b_0^{-\frac{2}{\gamma-1}}) + O(b_0^{-2}) \right) \).

(ii) \( u(s) = b_0 \left( 1 + O(b_0^{-\frac{2}{\gamma-1}}) + O(b_0^{-2}) \right) \).

(iii) \( \rho(s) = \left( \frac{\gamma-1}{2A\gamma} \right)^{-\frac{1}{\gamma-1}} b_0^{-\frac{2}{\gamma-1}} \left( 1 + O(b_0^{-\frac{2}{\gamma-1}}) + O(b_0^{-2}) \right) \).

(iv) \( u^2(s) - c^2(\rho(s)) = \frac{3-\gamma}{2} b_0^2 \left( 1 + O(b_0^{-\frac{2}{\gamma-1}}) + O(b_0^{-2}) \right) > 0. \)

(v) \( (s-u(s))^2 - c^2(\rho(s)) = -\frac{\gamma-1}{2} b_0^2 \left( 1 + O(b_0^{-\frac{2}{\gamma-1}}) + O(b_0^{-2}) \right) < 0. \)

(vi) \( u(s) + c(\rho(s)) - s = \sqrt{\frac{\gamma-1}{2} b_0^2 \left( 1 + O(b_0^{-\frac{2}{\gamma-1}}) + O(b_0^{-2}) \right)} > 0. \)

(vii) \( u(s) - c(\rho(s)) - s = -\sqrt{\frac{\gamma-1}{2} b_0^2 \left( 1 + O(b_0^{-\frac{2}{\gamma-1}}) + O(b_0^{-2}) \right)} < 0. \)

(viii) \( \rho'(s) = O(b_0^{-1}). \)

(ix) \( u'(s) = -(n-1) \left( 1 + O(b_0^{-\frac{2}{\gamma-1}}) + O(b_0^{-2}) \right) < 0. \)
Proof. Set $\rho_+ = \lim_{s \to s_0^+} \rho(s), u_+ = \lim_{s \to s_0^-} u(s)$ and $\alpha_0 = \frac{\rho_+}{\rho_0}$.

It follows from (2.2) that

$$\frac{A_\gamma}{\gamma - 1} \left( (\rho_+)^{\gamma+1} - \rho_0^{\gamma-1}(\rho_+)^2 \right) + \frac{1}{2} s_0^2 (\rho_+ - \rho_0)^2 - s_0^2 \rho_+(\rho_+ - \rho_0) = 0. \quad (2.5)$$

Set

$$F(x) = \frac{A_\gamma}{\gamma - 1} \left( x^{\gamma+1} - \rho_0^{\gamma-1} x^2 \right) + \frac{1}{2} s_0^2 (x - \rho_0)^2 - s_0^2 x(x - \rho_0).$$

Then (2.5) implies that $F(\rho_0) = F(\rho_+) = 0$. For $x \in (0, \rho_0)$, a direct computation yields

$$F'(x) = \frac{A_\gamma(\gamma+1)}{\gamma - 1} x^\gamma - \frac{2 A_\gamma}{\gamma - 1} \rho_0^{\gamma-1} x - s_0^2 x \leq x \left( c^2(\rho_0) - s_0^2 \right).$$

Due to condition (2.3), $F'(x) < 0$ holds for $x \in (0, \rho_0)$. This, together with $F(\rho_0) = F(\rho_+) = 0$, yields $\rho_+ > \rho_0$. In this case, it can be derived from (2.5) that

$$\frac{\gamma}{\gamma - 1} A_\alpha^2 \left( \frac{\alpha_0^{\gamma-1} - 1}{\alpha_0^2 - 1} \right) = \frac{1}{2} \rho_0 - \frac{1}{\alpha_0^2} s_0^2, \quad (2.6)$$

where $\alpha > 1$.

Since the left hand side of (2.6) is bounded if $\alpha_0 > 1$ is bounded, then for large $s_0$, $\alpha_0$ is also large. From this fact, one has

$$\alpha_0 = \frac{1}{\rho_0} \left( \frac{\gamma - 1}{2 A_\gamma} \right)^{\frac{1}{\gamma + 1}} s_0^\frac{2}{\gamma + 1} \left( 1 + O\left( b_0 \frac{1}{\gamma - 1} \right) + O\left( b_0^{-2} \right) \right). \quad (2.7)$$

Substituting this into (2.2) yields

$$\begin{cases}
\rho_+ = \left( \frac{\gamma - 1}{2 A_\gamma} \right)^{\frac{1}{\gamma + 1}} s_0^\frac{2}{\gamma + 1} \left( 1 + O\left( b_0 \frac{1}{\gamma - 1} \right) + O\left( b_0^{-2} \right) \right), \\
u_+ = s_0 \left( 1 + O\left( b_0 \frac{1}{\gamma - 1} \right) + O\left( b_0^{-2} \right) \right).
\end{cases} \quad (2.8)$$

Moreover, from $u'(s) \leq 0$ for $s \in [b_0, s_0]$, one has $u_+ \leq u(s)$, namely,

$$s_0 \left( 1 - \frac{1}{\alpha_0} \right) \leq u(s) \leq b_0 \leq s_0.$$

Combining this with (2.7) yields (i) and (ii).

When $s$ is taken a function of $u$, it follows from (2.1) that

$$\frac{dh(\rho)}{du} = s - u,$$

which derives

$$h(\rho) = h(\rho_+) + \int_{u_+}^u (s - \tau) d\tau.$$

This, together with (i)-(ii) and (2.8), yields (iii) and further (iv)-(ix). Therefore, the proof of Lemma 2.1 is completed. □
Remark 2.1. Since the denominator of system (2.1) is negative in the interval \([b_0, s_0]\) (see Lemma 2.1 (v)), one can extend the background solution \((\hat{\rho}(s), \hat{u}(s)\)) of (2.1)-(2.4) to the interval \([b_0 - \tau_0, s_0 + \tau_0]\) for some small positive constant \(\tau_0\) satisfying \(0 < \tau_0 \leq \frac{1}{b_0 - b_0}\). In the following sections, we will still denote this extension of the background solution to \(\{(t, r) : t > 0, (b_0 - \tau_0) \leq r \leq (s_0 + \tau_0)\}\) by \((\hat{\rho}(s), \hat{u}(s))\), where \(s = \frac{r}{t}\). The corresponding extension of the potential will be denoted by \(\hat{\Phi}(t, x)\). Moreover, \(\hat{\Phi}\) can be written as \(\hat{\phi}(\frac{r}{t})\) with \(\hat{\phi}'(s) = \hat{u}(s)\).

§3. Local-in-time existence

In this section, we give the local-in-time existence of the problem (1.6) with (1.7)-(1.9) for \(n = 3\). When the spatial dimensions are two, the existence result can be proved analogously and even much simpler.

§3.1. Reformulation of problem (1.6) with (1.7)-(1.9) under some nonlinear transformations

Let \((\Phi^+, \zeta(t, \omega))\) be the solution of the problem (1.6) with (1.7)-(1.9), and we set

\[
\phi = \frac{\Phi^+}{t}, \quad s = \frac{r}{t}, \quad b(t, \omega) = \frac{\sigma}{t}, \quad \chi(t, \omega) = \frac{\zeta}{t}.
\]

Under the coordinate transformation (1.5) and the notations in (3.1), it follows from a direct but tedious computation that the problem (1.6) with (1.7)-(1.9) for \(n = 3\) can be written as

\[
\begin{cases}
\partial_t^2 \phi + \frac{2(\partial_s \phi - s)}{t} \partial_s^2 \phi + \frac{2}{t^2 s^2} \sum_{i=1}^{3} Z_i \phi \partial_t Z_i \phi + \frac{(\partial_s \phi - s)^2}{t^2} \partial_s^2 \phi + \frac{2(\partial_s \phi - s)}{t^2 s^2} \sum_{i=1}^{3} Z_i \phi \partial_s Z_i \phi \\
+ \frac{1}{t^2 s^4} \sum_{i,j=1}^{3} Z_i \phi Z_j \phi Z_i Z_j \phi - \frac{c^2(\rho)}{t^2} \left( \partial_t^2 \phi + \frac{2}{s} \partial_s \phi + \frac{1}{s^2} \sum_{i=1}^{3} Z_i^2 \phi \right) + \frac{2}{t} \partial_t \phi \\
+ \frac{2s - \partial_s \phi}{t^2 s^3} \sum_{i=1}^{3} (Z_i \phi)^2 = 0, \quad (t, s, \omega) \in (0, +\infty) \times (b(t, \omega), \chi(t, \omega)) \times S^2, \\
\partial_s \phi - \frac{1}{s^2} \sum_{i=1}^{3} Z_i b \cdot Z_i \phi = t \partial_t b + b \quad \text{on} \quad s = b(t, \omega), \\
H(\phi, \nabla \phi)(\partial_s \phi)^2 + (H(\phi, \nabla \phi) - \rho_0)(t \partial_t \phi - s \partial_s \phi) + H(\phi, \nabla \phi) \frac{1}{s^2} \sum_{i=1}^{3} (Z_i \phi)^2 = 0 \quad \text{on} \quad s = \chi(t, \omega), \\
\phi = 0 \quad \text{on} \quad s = \chi(t, \omega)
\end{cases}
\]

with

\[
H(\phi, \nabla \phi) = h^{-1} \left(B_0 - \phi - t \partial_t \phi + s \partial_s \phi - \frac{1}{2}(\partial_s \phi)^2 - \frac{1}{2s^2} \sum_{i=1}^{3} (Z_i \phi)^2\right)
\]

and

\[
c^2(\rho) = (\gamma - 1) \left(B_0 - \phi - t \partial_t \phi + s \partial_s \phi - \frac{1}{2}(\partial_s \phi)^2 - \frac{1}{2s^2} \sum_{i=1}^{3} (Z_i \phi)^2\right).
\]

It is noted that the problem (3.2) is still a free boundary problem with the shock surface as the unknown boundary. In order to prove the local-in-time existence result, we will use the modified partial hodograph
transformation to straighten \( s = b(t, \omega) \) and \( s = \chi(t, \omega) \) simultaneously. To this end, motivated by [5] and [24], we set

\[
\psi = s - b(t, \omega) - \frac{\phi}{b_0},
\]

(3.5)

and take the modified partial hodograph transformation as follows

\[
T = t, \quad R = \frac{s - b}{\psi} + 1, \quad \omega = \omega.
\]

(3.6)

In this case, the boundaries \( s = b(t, \omega) \) and \( s = \chi(t, \omega) \) are changed into \( R = 1 \) and \( R = 2 \) respectively. Additionally, by the continuity condition \( \phi(t, \chi(t, \omega)) = 0 \) in (3.2) and (3.5) we obtain \( \chi(t, \omega) = b(t, \omega) + \psi(t, 2, \omega) \), which means that \( \chi(t, \omega) \) can be determined once the function \( \psi(t, 2, \omega) \) is known.

Next we derive the nonlinear equation and corresponding boundary conditions of \( \psi \).

It follows from (3.5)-(3.6) and a direct computation that

\[
\begin{align*}
 s &= a_0(b, \psi), \\
 \partial_t \phi &= b_0 a_1(b, \psi, \nabla \psi) \cdot a_2(b, \psi, \nabla \psi), \\
 \partial_t \phi &= -b_0 a_1(b, \psi, \nabla \psi) \cdot a_3(b, \psi, \nabla \psi), \\
 Z_1 \phi &= -b_0 a_1(b, \psi, \nabla \psi) \cdot a_3(b, \psi, \nabla \psi), \\
 & \quad i = 1, 2, 3.
\end{align*}
\]

(3.7)

with

\[
\begin{align*}
 a_0(b, \psi) &= b + (R - 1)\psi, \\
 a_1(b, \psi, \nabla \psi) &= \frac{1}{\psi + (R - 1)\partial_R \psi}, \\
 a_2(b, \psi, \nabla \psi) &= \psi + (R - 2)\partial_R \psi, \\
 a_3(b, \psi, \nabla \psi) &= \psi \partial_T \psi + \psi \partial_T b + (R - 2)\partial_T b \partial_R \psi, \\
 a_3^i(b, \psi, \nabla \psi) &= \psi Z_i \psi + \psi Z_i b + (R - 2)Z_i b \partial_R \psi, \\
 & \quad i = 1, 2, 3.
\end{align*}
\]

(3.8)

Set \( \mathbb{H}(b, \psi, \nabla \psi) \triangleq \mathcal{H}(\phi, \nabla \phi) \), then substituting (3.7) into (3.3)-(3.4) yields

\[
\begin{align*}
 & \{ \mathbb{H}(b, \psi, \nabla \psi) = h^{-1}(A_0(b, \psi, \nabla \psi)), \\
 & \quad c^2(\rho) = (\gamma - 1)A_0(b, \psi, \nabla \psi),
\end{align*}
\]

(3.9)

where

\[
A_0(b, \psi, \nabla \psi) = B_0 - b_0(R - 2)\psi + T b_0 a_1 \cdot a_3 + b_0 a_0 \cdot a_1 \cdot a_2 - \frac{b_0^3}{2}(a_1 \cdot a_2)^2 - \frac{b_0^3}{2a_0^2} \sum_{i=1}^{3}(a_1 \cdot a_4^i)^2.
\]

At this time, the problem (3.2) can be rewritten as

\[
\begin{align*}
 & A_1(b, \psi, \nabla \psi) \partial_T^2 \psi + A_2(b, \psi, \nabla \psi) \partial_R^2 \psi + \sum_{i=1}^{3} A_3^i(b, \psi, \nabla \psi) \partial_T Z_i \psi + A_4(b, \psi, \nabla \psi) \partial_R^2 \psi \\
 & + \sum_{i=1}^{3} A_5^i(b, \psi, \nabla \psi) \partial_R Z_i \psi + \sum_{i=1}^{3} \sum_{j=1}^{3} A_6^{ij}(b, \psi, \nabla \psi) Z_i Z_j \psi + A_7(b, \psi, \nabla \psi) = 0,
\end{align*}
\]

(3.10)

\[
\partial_T \psi + \frac{1}{b_0}(T \partial_T b + b_0) \psi - \frac{\psi^3}{b^2} \sum_{i=1}^{3} Z_i b \cdot Z_i \psi - \frac{\psi - \partial_R \psi}{b^2} \sum_{i=1}^{3} (Z_i b)^2 = 0 \quad \text{on} \quad R = 1,
\]

(3.11)

\[
\mathbb{H}(b, \psi, \nabla \psi) \psi - \frac{1}{b_0 a_1} \left( \mathbb{H}(b, \psi, \nabla \psi) - \rho_0 \right)(T \partial_T a_0 + a_0) + \frac{\mathbb{H}(b, \psi, \nabla \psi) \psi}{(b + \psi)^2} \sum_{i=1}^{3} (Z_i a_0)^2 = 0 \quad \text{on} \quad R = 2,
\]

(3.12)
where
\[
\begin{aligned}
A_k(b, \psi, \nabla \psi) &= A_{k,0}(b, \psi, \nabla \psi) + \frac{A_{k,1}(b, \psi, \nabla \psi)}{T} + \frac{A_{k,2}(b, \psi, \nabla \psi)}{T^2}, & k &= 1, 2, 4, 7, \\
A_i^j(b, \psi, \nabla \psi) &= A_{i,0}^j(b, \psi, \nabla \psi) + \frac{A_{i,1}^j(b, \psi, \nabla \psi)}{T} + \frac{A_{i,2}^j(b, \psi, \nabla \psi)}{T^2}, & k &= 3, 5, & 1 \leq i \leq 3, \\
A_{i,j}^k(b, \psi, \nabla \psi) &= A_{i,j,0}^k(b, \psi, \nabla \psi) + \frac{A_{i,j,1}^k(b, \psi, \nabla \psi)}{T} + \frac{A_{i,j,2}^k(b, \psi, \nabla \psi)}{T^2}, & 1 \leq i, j \leq 3,
\end{aligned}
\]
and all expressions of \(A_{k,0}(b, \psi, \nabla \psi), \ldots, A_{i,j,2}^k(b, \psi, \nabla \psi)\) can be found in Lemma A.1-Lemma A.3 of Appendix A since their concrete expressions are required in order to establish related estimates.

Therefore, the problem (1.6) with (1.7)-(1.9) has been reformulated into the problem (3.10)-(3.12). To establish the local-in-time existence result of (1.6) with (1.7)-(1.9), we require to show the local existence of (3.10) with (3.11)-(3.12), which can be stated as

**Theorem 3.1.** Under the assumptions in Theorem 1.1, there exists a positive constant \(T^*\) independent of small \(\varepsilon\), such that the problem (3.10) with (3.11)-(3.12) has a unique solution \(\psi(T, R, \omega) \in C^\infty([0, T^*] \times [1, 2] \times S^2)\), which satisfies for \(m \in \mathbb{N} \cup \{0\}\)

\[
\|\psi - \hat{\psi}\|_{C^m([0, T^*] \times [1, 2] \times S^2)} \leq C(m)\varepsilon,
\]
where the function \(\hat{\psi} = \hat{\psi}(R)\) is obtained when we use \(b_0\) and \(\hat{\phi}(s)\) give in Remark 2.1 instead of \(b(t, \omega)\) and \(\phi(t, s, \omega)\) in (3.5)-(3.6).

**Remark 3.1.** It is noted that there is no initial data information on \(T = 0\) of \(\psi\) in the problem (3.10) with (3.11)-(3.12). Thus, solving (3.10)-(3.12) is not a standard procedure. In addition, by the proof procedure of Theorem 3.1, we can derive that \(T^* \geq \frac{C}{\varepsilon}\) is actually permitted.

For later uses, we now establish some properties of \(\hat{\psi}(R)\).

**Lemma 3.2.** \(\hat{\psi}(R)\) admits the following estimates:

(i). \(\hat{\psi} = (s_0 - b_0)(1 + O(b_0^{-\frac{2}{\varepsilon}})) + O(b_0^{-2}) > 0\),

(ii). \(\hat{\psi}'(R) = \frac{b_0}{b_0}O((s_0 - b_0)^2) > 0\), \(\hat{\psi}'(2) = \frac{b_0}{b_0}(s_0 - b_0)^2(1 + O(b_0^{-\frac{2}{\varepsilon}})) + O(b_0^{-2})\),

(iii). \(\hat{\psi}''(R) = \frac{2}{b_0}(s_0 - b_0)^2(1 + O(b_0^{-\frac{2}{\varepsilon}})) + O(b_0^{-2})\).

**Proof.** Due to \(\hat{\phi}(s_0) = 0\), then by (3.5) and mean value theorem there exists some \(\bar{s} \in (b_0, s_0)\) such that

\[
\hat{\psi} = s - b_0 - \frac{\hat{\phi}}{b_0} = s - b_0 + \frac{1}{b_0} \hat{u}(\bar{s})(s_0 - s).
\]

This, together with Lemma 2.1 (ii), yields Lemma 3.2 (i).

By the second equality in (3.7), one has

\[
(b_0 + (1 - R)(b_0 - \hat{u}(s))) \hat{\psi}'(R) = (b_0 - \hat{u}(s))\hat{\psi}.
\]  

(3.13)

Noting \(\hat{u}(b_0) = b_0\) and \(b_0 - \hat{u}(s) = \hat{u}(b_0) - \hat{u}(s)\), then combining this with mean value theorem, Lemma 2.1 (ix), Lemma 3.2 (i) and (3.13) yields Lemma 3.2 (ii).

Similarly, Lemma 3.2 (iii) can be obtained by taking the first order derivative on two hand sides of (3.13) and applying Lemma 2.1. \(\square\)
Since \( \hat{\psi}(R) \) does not satisfy the fixed wall boundary condition (3.11) for \( b = b(t, \omega) \), this will bring some troubles to study the nonlinear problem (3.10) with Neumann-type boundary condition (3.11). This difficulty can be overcome by choosing a new function \( \hat{\psi}_\sigma(T, R, \omega) \) instead of \( \hat{\psi}(R) \) as follows:

**Lemma 3.3.** Under the assumptions in Theorem 1.1, we define a \( C^\infty \) function \( \hat{\psi}_\sigma(T, R, \omega) \) as follows

\[
\hat{\psi}_\sigma(R, \omega) = \frac{\hat{\psi}(R)}{b^2 + \sum_{i=1}^3 (Z_i b)^2} \left( b^2 + \sum_{i=1}^3 (Z_i b)^2 + (R - 1) \left( \sum_{i=1}^3 (Z_i b)^2 - \frac{b^2}{b_0} (T \partial_T b + b - b_0) \right) \right),
\]

which satisfies

1. \( \partial_R \hat{\psi}_\sigma + \frac{1}{b_0} (T \partial_T b + b - b_0) \hat{\psi}_\sigma - \frac{\hat{\psi}_\sigma}{b^2} \sum_{i=1}^3 Z_i b \cdot Z_i \hat{\psi}_\sigma - \frac{\hat{\psi}_\sigma - \partial_R \hat{\psi}_\sigma}{b^2} \sum_{i=1}^3 (Z_i b)^2 = 0. \)
2. \( \| \hat{\psi}_\sigma - \hat{\psi} \|_{C^m([0,\infty) \times [1,2] \times S^2)} \leq C(m) \varepsilon, \quad \forall m \in \mathbb{N} \cup \{0\}. \)

**Proof.** Since (i) and (ii) can be verified directly with the definition of \( \hat{\psi}_\sigma(R, \omega) \) and the assumptions on \( b \) in Theorem 1.1, then we omit the details here. \( \Box \)

In subsequent part, due to Remark 3.1, we start to construct an approximate solution of (3.10) with (3.11)-(3.12) and give some related estimates.

§3.2. Construction of an approximate solution to problem (3.10) with (3.11)-(3.12)

We will use Taylor’s formula to construct an approximate solution of (3.10) with (3.11)-(3.12). For this end, we set

\[
\psi(T, R, \omega) = \psi_0(R, \omega) + \sum_{l=1}^K T^l \psi_l(R, \omega) + O(T^{K+1})
\]

and

\[
b(T, \omega) = d_0(\omega) + \sum_{l=1}^K T^l b_l(\omega) + O(T^{K+1}),
\]

where \( K \) is a suitably large positive integer, and \( d_0(\omega) = b(0, \omega) \).

Substituting (3.14)-(3.15) into the problem (3.10) with (3.11)-(3.12) and comparing the powers of \( T \) in the resulting equalities, one can get a series of equations and boundary conditions of \( \psi_l \) for \( 0 \leq l \leq K \). This will be illustrated gradually in subsequent parts.

**Part 1. Determination of \( \psi_0(R, \omega) \)**

By comparing the coefficients of \( T^{-2} \) and the coefficients of \( T^0 \) in the resulting equalities from (3.10) and (3.11)-(3.12) respectively by the expressions (3.14) and (3.15), one can arrive at

\[
\begin{aligned}
\left\{
\begin{array}{ll}
A_{4,2}(d_0, \psi_0, \nabla \psi_0) \partial_R^2 \psi_0 + \sum_{i=1}^3 A_{5,2}(d_0, \psi_0, \nabla \psi_0) \partial_R Z_i \psi_0 + \sum_{i=1}^3 \sum_{j=1}^3 A_{6,2}^{ij}(d_0, \psi_0, \nabla \psi_0) Z_i Z_j \psi_0 \\
+ A_{7,2}(d_0, \psi_0, \nabla \psi_0) = 0, & (R, \omega) \in (1, 2) \times S^2, \\
\partial_R \psi_0 + \frac{1}{b_0} (d_0 - b_0) \psi_0 - \frac{\psi_0}{b_0} \sum_{i=1}^3 Z_i d_0 \cdot Z_i \psi_0 - \psi_0 - \partial_R \psi_0 \sum_{i=1}^3 (Z_i d_0)^2 = 0 & \text{on} \ R = 1, \\
\mathbb{H}(d_0, \psi_0, \nabla \psi_0) \psi_0 - \frac{1}{b_0} (\mathbb{H}(d_0, \psi_0, \nabla \psi_0) - \rho_0)(\psi_0 + \partial_R \psi_0)(d_0 + \psi_0) \\
+ \frac{\mathbb{H}(d_0, \psi_0, \nabla \psi_0) \psi_0}{(d_0 + \psi_0)^2} \sum_{i=1}^3 (Z_i d_0 + Z_i \psi_0)^2 = 0 & \text{on} \ R = 2.
\end{array}
\right.
\end{aligned}
\]
Later on, the equation in (3.16) can be shown to be a second order nonlinear elliptic equation (see Lemma 3.4 below).

On the other hand, with respect to the function \( \hat{\psi}(R) \) given in Lemma 3.2, we have

\[
\begin{align*}
\begin{cases}
A_{4,2}(b_0, \hat{\psi}, \nabla \hat{\psi})\hat{\psi}''(R) + A_{7,2}(b_0, \hat{\psi}, \nabla \hat{\psi}) = 0, & R \in (1, 2), \\
\hat{\psi}'(1) = 0, \\
\mathbb{H}(b_0, \hat{\psi}, \nabla \hat{\psi})\hat{\psi} - \frac{1}{b_0} (\mathbb{H}(b_0, \hat{\psi}, \nabla \hat{\psi}) - \rho_0)(\hat{\psi} + \hat{\psi}'(2))(b_0 + \hat{\psi}) = 0 \quad \text{on} \quad R = 2.
\end{cases}
\end{align*}
\]

(3.17)

Set \( \tilde{\psi} = \psi_0 - \hat{\psi} \), then it follows from (3.16)-(3.17) that \( \tilde{\psi} \) satisfies

\[
\begin{align*}
\begin{cases}
A_{4,2}(d_0, \psi_0, \nabla \psi_0)\partial_R^2 \tilde{\psi} + \sum_{i=1}^{3} A_{5,2}^i(d_0, \psi_0, \nabla \psi_0) \partial_R Z_i \tilde{\psi} + \sum_{i=1}^{3} \sum_{j=1}^{3} A_{6,2}^{ij}(d_0, \psi_0, \nabla \psi_0) Z_i Z_j \tilde{\psi} \\
\quad + B_1 \partial_R \tilde{\psi} + \sum_{i=1}^{3} B_2^i Z_i \tilde{\psi} + E_0 \tilde{\psi} = F_0(R, \omega), \quad (R, \omega) \in (1, 2) \times \mathbb{S}^2, \\
D_{11} \partial_R \tilde{\psi} + D_{12} \tilde{\psi} + \sum_{i=1}^{3} D_{13}^i Z_i \tilde{\psi} = G_1(R, \omega) \quad \text{on} \quad R = 1, \\
D_{21}^0 \partial_R \tilde{\psi} + D_{22}^0 \tilde{\psi} + \sum_{i=1}^{3} D_{23}^0 Z_i \tilde{\psi} = G_2(R, \omega) \quad \text{on} \quad R = 2.
\end{cases}
\end{align*}
\]

(3.18)

with

\[
\begin{align*}
F_0(R, \omega) &= \left( A_{4,2}(b_0, \hat{\psi}, \nabla \hat{\psi}) - A_{4,2}(d_0, \psi_0, \nabla \psi_0) \right)\hat{\psi}''(R) + \left( A_{7,2}(b_0, \hat{\psi}, \nabla \hat{\psi}) - A_{7,2}(d_0, \psi_0, \nabla \psi_0) \right), \\
G_1(R, \omega) &= \frac{1}{b_0} (b_0 - d_0) \hat{\psi} + \frac{\hat{\psi} - \partial_R \hat{\psi}}{d_0} \sum_{i=1}^{3} (Z_i d_0)^2, \\
G_2(R, \omega) &= \sum_{i=1}^{3} O(Z_i (d_0 - b_0)) + O(d_0 - b_0)
\end{align*}
\]

(3.19)

and

\[
D_{11} = 1 + \frac{1}{d_0^3} \sum_{i=1}^{3} (Z_i d_0)^2, \quad D_{12} = \frac{1}{b_0} (d_0 - b_0) - \frac{1}{d_0^3} \sum_{i=1}^{3} (Z_i d_0)^2, \quad D_{13}^i(\psi_0) = -\frac{\psi_0}{d_0^2} Z_i d_0
\]

(3.20)

and

\[
\begin{align*}
E_0(R, \omega, \psi_0, \nabla \psi_0) &= \int_0^1 \partial_\psi \left( \hat{\psi}''(R) A_{4,2} + A_{7,2} \right)(d_0, \hat{\psi} + \theta \psi_0, \nabla (\hat{\psi} + \theta \psi_0)) d\theta, \\
(D_{21}^0, D_{22}^0)(R, \omega, \psi_0, \nabla \psi_0) &= \int_0^1 \nabla \partial_\psi \psi_0 \left( \mathbb{H}(\cdot) \psi - \frac{1}{b_0} (\mathbb{H}(\cdot) - \rho_0)(\psi + \partial_R \psi)(d_0 + \psi) \right) \\
&\quad + \mathbb{H}(\cdot) \psi (d_0 + \psi)^2 \sum_{i=1}^{3} (Z_i d_0)^2 \left( d_0, \hat{\psi} + \theta \psi_0, \nabla (\hat{\psi} + \theta \psi_0) \right) d\theta.
\end{align*}
\]

(3.21)

In addition, the coefficients \( B_1, B_2^i, D_{23}^{0,i} \) are smooth functions on the variables \( (R, \omega, \psi_0, \nabla \psi_0) \), whose concrete expressions are not required.
For the requirements to solve the nonlinear problem (3.18), we now give some estimates on the coefficients $A_4$, $A_5^i$ and $A_6^{ij}$ in the equation of (3.18) when their arguments $(d_0(\omega), \psi_0(R, \omega), \nabla \psi_0(R, \omega))$ are replaced by $(b_0, \hat{\psi}(R), \nabla \hat{\psi}(R))$.

**Lemma 3.4.** For large $b_0$, we have

\[
\begin{cases}
A_{4,2}(b_0, \hat{\psi}, \nabla \hat{\psi}) = -\frac{b_0^2}{2(s_0 - b_0)}(1 + O(b_0^{-\frac{2}{3}}) + O(b_0^{-2})) < 0, \\
A_{5,2}^i(b_0, \hat{\psi}, \nabla \hat{\psi}) = 0, \\nA_{6,2}^{ij}(b_0, \hat{\psi}, \nabla \hat{\psi}) = -\left(\frac{\gamma - 1}{s_0 - b_0}\right)(1 + O(b_0^{-\frac{2}{3}}) + O(b_0^{-2}))\delta_{ij}, 
\end{cases}
\] (3.22)

which means that the equation in (3.18) is uniformly elliptic.

**Proof.** By Lemma 3.2 and (3.8), a direct computation yields

\[
\begin{align*}
a_0(b_0, \hat{\psi}) &= b_0(1 + O(b_0^{-\frac{2}{3}}) + O(b_0^{-2})), \\
a_1(b_0, \hat{\psi}, \nabla \hat{\psi}) &= \frac{1}{s_0 - b_0}(1 + O(b_0^{-\frac{2}{3}}) + O(b_0^{-2})), \\
a_2(b_0, \hat{\psi}, \nabla \hat{\psi}) &= (s_0 - b_0)(1 + O(b_0^{-\frac{2}{3}}) + O(b_0^{-2})), \\
a_3(b_0, \hat{\psi}, \nabla \hat{\psi}) &= 0, \\
a_4^i(b_0, \hat{\psi}, \nabla \hat{\psi}) &= 0, 
\end{align*}
\] (3.23)

Then in terms of the concrete expressions of $A_{4,2}$, $A_{5,2}^i$ and $A_{6,2}^{ij}$ in Lemma A.3, with (3.23), we have

\[
\begin{align*}
A_{4,2}(b_0, \hat{\psi}, \nabla \hat{\psi}) &= (c^2(\rho) - (b_0a_1a_2 - a_0^2)((R - 2)a_1 - (R - 1)a_2^2)\left|_{b_0 = b_0; \hat{\psi} = \hat{\psi}}
= -\frac{\gamma - 1}{2(s_0 - b_0)}b_0^2(1 + O(b_0^{-\frac{2}{3}}) + O(b_0^{-2})) < 0, \\
A_{5,2}^i(b_0, \hat{\psi}, \nabla \hat{\psi}) &= 0, \\nA_{6,2}^{ij}(b_0, \hat{\psi}, \nabla \hat{\psi}) &= -\frac{c^2(\rho)}{a_0^2}\delta_{ij}\left|_{b_0 = b_0; \hat{\psi} = \hat{\psi}}
= -\frac{(\gamma - 1)(s_0 - b_0)}{2}\delta_{ij}(1 + O(b_0^{-\frac{2}{3}}) + O(b_0^{-2})), 
\end{align*}
\]

1 \leq i, j \leq 3.

Thus (3.22) is proved. \qed

We now establish the solvability of the problem (3.18).

**Lemma 3.5.** Under the assumptions in Theorem 1.1, the problem (3.18) has a unique smooth solution $\hat{\psi}$ which satisfies

\[
\|\hat{\psi}\|_{C^{3+m,\alpha}([1,2] \times S^2)} \leq C(m)\varepsilon, \tag{3.24}
\]

where $0 < \alpha < 1$ is any fixed constant, and $m \in \mathbb{N} \cup \{0\}$.

**Proof.** At first, we claim that for any $\tilde{\psi}_0$ satisfying $\|\tilde{\psi}_0\|_{C^{3+m,\alpha}([1,2] \times S^2)} \leq C(m)\varepsilon$, there exists a sequence
\[ \{ \tilde{\psi}_i \}_{i=1,2,\ldots} \text{ such that for } V_i \equiv (\hat{\psi} + \tilde{\psi}_{i-1}, \nabla(\hat{\psi} + \tilde{\psi}_{i-1})), \tilde{\psi}_i \text{ satisfies} \]

\[
\begin{aligned}
& A_{4,2}(d_0, V_i) \partial_R^2 \tilde{\psi}_i + \sum_{i=1}^{3} A_{5,2}^i (d_0, V_i) \partial_R Z_i \tilde{\psi}_i + \sum_{i=1}^{3} \sum_{j=1}^{3} A_{6,2}^{ij} (d_0, V_i) Z_i Z_j \tilde{\psi}_i + B_1 (d_0, V_i) \partial_R \tilde{\psi}_i \\
& + \sum_{i=1}^{3} B_2^i (d_0, V_i) Z_i \tilde{\psi}_i + E_0 (d_0, V_i) \tilde{\psi}_i = F_0 (R, \omega), \quad (R, \omega) \in (1, 2) \times S^2, \\
& D_{11} \partial_R \tilde{\psi}_i + D_{12} \tilde{\psi}_i + \sum_{i=1}^{3} D_{13}^i (\hat{\psi} + \tilde{\psi}_{i-1}) Z_i \tilde{\psi}_i = G_1 (R, \omega) \quad \text{on } R = 1, \\
& D_{21}^0 (d_0, V_i) \partial_R \tilde{\psi}_i + D_{22}^0 (d_0, V_i) \tilde{\psi}_i + \sum_{i=1}^{3} D_{23}^{0,i} (d_0, V_i) Z_i \tilde{\psi}_i = G_2 (R, \omega) \quad \text{on } R = 2
\end{aligned}
\]

and admits the following estimate

\[
\| \tilde{\psi}_i \|_{C^{3+m,\alpha}(1,2) \times S^2} \leq C(m) \varepsilon. \tag{3.26}
\]

We will use the induction method to prove the claim (3.26).

Assume that \( \tilde{\psi}_{i-1} \) satisfies (3.26), then it follows from Lemma A.4-Lemma A.5 in Appendix A that there exists a positive constant \( c_0 \) such that

\[
E_0 (d_0, V_i) > c_0 > 0, \quad D_{21}^0 (d_0, V_i) < -c_0 < 0 \quad \text{for } i = 1, 2. \tag{3.27}
\]

Choosing a function \( \vartheta (R) = 2 + c_1 R \) with

\[
c_1 = \frac{1}{4(1 + 2 \| A_4 \|_{L^\infty} + \| B_1 \|_{L^\infty} + \| D_{21}^0 \|_{L^\infty})}
\]

and setting \( \tilde{\Psi}_i (R, \omega) \equiv \vartheta (R) \tilde{\psi}_i \), then \( \tilde{\Psi}_i (R, \omega) \) satisfies

\[
\begin{aligned}
& A_{4,2}(d_0, V_i) \partial_R^2 \tilde{\Psi}_i + \sum_{i=1}^{3} A_{5,2}^i (d_0, V_i) \partial_R Z_i \tilde{\Psi}_i + \sum_{i=1}^{3} \sum_{j=1}^{3} A_{6,2}^{ij} (d_0, V_i) Z_i Z_j \tilde{\Psi}_i + B_1 (d_0, V_i) \partial_R \tilde{\Psi}_i \\
& + \sum_{i=1}^{3} \hat{B}_2^i (d_0, V_i) Z_i \tilde{\Psi}_i + \hat{E}_0 (d_0, V_i) \tilde{\Psi}_i = \vartheta (R) F_0 (R, \omega), \quad (R, \omega) \in (1, 2) \times S^2, \\
& D_{11} \partial_R \tilde{\Psi}_i + \hat{D}_{12} \tilde{\Psi}_i + \sum_{i=1}^{3} \hat{D}_{13}^i (\hat{\psi} + \tilde{\psi}_{i-1}) Z_i \tilde{\Psi}_i = \vartheta (R) G_1 (R, \omega) \quad \text{on } R = 1, \\
& D_{21}^0 (d_0, V_i) \partial_R \tilde{\Psi}_i + \hat{D}_{22}^0 (d_0, V_i) \tilde{\Psi}_i + \sum_{i=1}^{3} \hat{D}_{23}^{0,i} (d_0, V_i) Z_i \tilde{\Psi}_i = \vartheta (R) G_2 (R, \omega) \quad \text{on } R = 2.
\end{aligned}
\]

Then with the expression of \( c_1, (3.20), (3.27) \) and the assumptions in Theorem 3.1, one has

\[
\begin{aligned}
\hat{E}_0 (d_0, V_i) &= \frac{2A_{4,2}}{\vartheta^2} (c_1)^2 - \frac{c_1 B_1}{\vartheta} + E_0 \geq \frac{c_0}{2} > 0, \\
\hat{D}_{12} &= -\frac{c_1}{\vartheta} \hat{D}_{11} + \hat{D}_{12} \leq -\frac{c_0}{2} < 0, \\
\hat{D}_{22}^0 (d_0, V_i) &= \frac{c_1}{\vartheta} \hat{D}_{21}^0 + \hat{D}_{22}^0 \leq -\frac{c_0}{2} < 0.
\end{aligned}
\]
By Lemma 3.4 and Theorem 6.30-Theorem 6.31 in [9] that (3.28) has a unique solution \( \tilde{\psi}_l \) and then (3.25) has a unique solution \( \tilde{\psi}_l \). With the expressions in (3.19) and the assumptions in Theorem 3.1, \( \tilde{\psi}_l \) satisfies

\[
\| \tilde{\psi}_l \|_{C^{3+m,\alpha}} \leq C(m)\| F_0 \|_{C^{2+m,\alpha}} + \| G_1 \|_{C^{2+m,\alpha}} + \| G_2 \|_{C^{2+m,\alpha}} \left( 1 + \frac{1}{b_0} \| \tilde{\psi}_{l-1} \|_{C^{3+m,\alpha}} \right)
\]

\[
\leq C(m)\| d_0 - b_0 \|_{C^{2+m,\alpha}} \leq C(m)\varepsilon,
\]

(3.29)

here we have used the assumption (3.26) in the case of \( l - 1 \) and largeness of \( b_0 \). Thus, the claim (3.26) is proved.

On the other hand, similar to proof of (3.29), we have for small \( \varepsilon \)

\[
\| \tilde{\psi}_l - \tilde{\psi}_{l-1} \|_{C^{2,\alpha}} \leq C\varepsilon \| \tilde{\psi}_{l-1} - \tilde{\psi}_{l-2} \|_{C^{2,\alpha}} \leq \frac{1}{2} \| \tilde{\psi}_{l-1} - \tilde{\psi}_{l-2} \|_{C^{2,\alpha}}.
\]

(3.30)

Combining (3.29) with (3.30) yields that there exists a function \( \tilde{\psi} \in C^{3+m,\alpha} \) such that \( \tilde{\psi}_l \to \tilde{\psi} \) in \( C^{2,\alpha} \) as \( l \to \infty \). Furthermore, (3.24) holds. Thus the proof on Lemma 3.5 is completed. \( \square \)

With Lemma 3.5, then \( \psi_0 = \hat{\psi} + \tilde{\psi} \) is a unique solution of (3.16). Based on this, next we continue to construct the approximate solution of (3.10) with (3.11)-(3.12).

**Part 2. Determination of \( \psi_k \) \( (k \geq 1) \)**

Comparing the coefficients of \( T^{k-2} \) and the coefficients of \( T^k \) in the resulting equalities from the equation (3.10) and (3.11)-(3.12) respectively by the expressions (3.14) and (3.15), then \( \psi_k \) satisfies

\[
\begin{cases}
A_{4,2}(d_0, V_1) \partial^2_R \psi_k + \sum_{i=1}^{3} A_{5,2}^i(d_0, V_1) \partial_R Z_i \psi_k + \sum_{i=1}^{3} \sum_{j=1}^{3} A_{6,2}^{ij}(d_0, \psi_0, V_1) Z_i Z_j \psi_k \\
+ B_{k1}(d_0, V_1) \partial_R \psi_k + \sum_{i=1}^{3} B_{k2}^i(d_0, V_1) Z_i \psi_k + E_k(d_0, V_1) \psi_k = F_k(\psi_l)_{0 \leq l \leq k-1},
\end{cases}
\]

(3.31)

\[
D_{11} \partial_R \psi_k + D_{12} \psi_k + \sum_{i=1}^{3} D_{13}^i(\psi_0) Z_i \psi_k = G_1^k(\psi_l)_{0 \leq l \leq k-1}
\]

on \( R = 1 \),

\[
D_{21}^k(d_0, V_1) \cdot \partial_R \psi_k + D_{22}^k(d_0, V_1) \cdot \psi_k + \sum_{i=1}^{3} D_{23}^{k,i}(d_0, V_1) Z_i \psi_k = G_2^k(\psi_l)_{0 \leq l \leq k-1}
\]

on \( R = 2 \)

with

\[
E_k(d_0, V_1) = (k-1)\psi_0 + \partial_{\psi} (\hat{\psi}''(R) \cdot A_{4,2} + A_{7,2})(b_0, \hat{\psi}, \nabla \hat{\psi}) + k \partial_{\partial_R \psi} (\hat{\psi}''(R) A_{4,1} + A_{7,1} + \frac{\hat{\psi}''(R)}{T} A_{4,2} + \frac{A_{7,2}}{T})(b_0, \hat{\psi}, \nabla \hat{\psi}) + O_k(\varepsilon)
\]

(3.32)

and

\[
\begin{cases}
D_{21}^k(d_0, V_1) = \partial_{\partial_R \psi} \left( \frac{\hat{\psi}(\cdot)}{b_0} - \frac{1}{b_0} (\hat{\psi}(\cdot) - \rho_0) a_1 (T \partial_R a_0 + a_0) \right) (b_0, \hat{\psi}, \nabla \hat{\psi}) + O_k(\varepsilon),
\end{cases}
\]

\[
D_{22}^k(d_0, V_1) = (\partial_{\psi} + k \partial_{\partial_R \psi}) \left( \frac{\hat{\psi}(\cdot)}{b_0} - \frac{1}{b_0} (\hat{\psi}(\cdot) - \rho_0) (T \partial_R a_0 + a_0) a_1 \right) (b_0, \hat{\psi}, \nabla \hat{\psi}) + O_k(\varepsilon).
\]

(3.33)

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Here the error term estimates in (3.31)-(3.33) come from the assumptions on \( b(t, \omega) \) in Theorem 1.1 and (3.24), and \( O_k(\varepsilon) \) stands for a generic quantity satisfying \( |O_k(\varepsilon)| \leq C_k \varepsilon \).

With respect to the problem (3.31), we have

**Lemma 3.6.** The problem (3.31) has a unique smooth solution \( \psi_k \), which satisfies

\[
\| \psi_k \|_{C^{3+m,\alpha}([1,2] \times S^2)} \leq C(m,k)\varepsilon. \tag{3.34}
\]

**Proof.** In terms of Lemma A.4-Lemma A.5 in Appendix A, we have

\[
E_k(d_0,V_1) > c_0 > 0, \quad D^2_{21}(d_0,V_1) < -c_0 < 0, \quad D_{22}^2(d_0,V_1) < -c_0 < 0.
\]

Then similar to the treatment on the problem (3.25), one can derive that the linear problem (3.31) has a unique smooth solution \( \psi_k \) satisfying (3.34). Consequently, the proof of Lemma 3.6 is completed.

\[\Box\]

§3.3. Choice of iteration scheme and proof of local existence

§3.3.1. Choice of iteration scheme

By the construction of \( \{ \psi_l \}_{l \geq 0} \) in §3.2, for any given integer \( \kappa_0 \geq 1 \), we set \( \psi^{\kappa_0}_a := \sum_{l=0}^{\kappa_0} T^l \psi_l \), which can be regarded as the \( \kappa_0 \)-order approximate solution of the problem (3.10) with (3.11)-(3.12). By Lemma 3.5 and Lemma 3.6, there exists \( T^* > 0 \) depending only on \( \kappa_0 \) such that

\[
\| \psi^{\kappa_0}_a - \hat{\psi} \|_{C^m([0,T^*] \times [1,2] \times S^2)} \leq C(m)\varepsilon. \tag{3.35}
\]

Moreover, by (3.14)-(3.16), (3.31) and (3.35), \( \psi^{\kappa_0}_a \) satisfies

\[
\begin{aligned}
A_1(b,V^{\kappa_0}_a)\partial_T^2 \psi^{\kappa_0}_a + A_2(b,V^{\kappa_0}_a)\partial_T \psi^{\kappa_0}_a + \sum_{i=1}^3 A_3^i(b,V^{\kappa_0}_a)\partial_T Z_i \psi^{\kappa_0}_a + A_4(b,V^{\kappa_0}_a)\partial_R^2 \psi^{\kappa_0}_a \\
+ \sum_{i=1}^3 A_5^i(b,V^{\kappa_0}_a)\partial_R Z_i \psi^{\kappa_0}_a + \sum_{i=1}^3 \sum_{j=1}^3 A_6^{ij}(b,V^{\kappa_0}_a)Z_i Z_j \psi^{\kappa_0}_a + A_7(b,V^{\kappa_0}_a) \\
= T^{\kappa_0-1} F^{\kappa_0}, \\
\partial_R \psi^{\kappa_0}_a + \frac{1}{b_0}(T \partial_T b + b - b_0) \psi^{\kappa_0}_a - \frac{\psi^{\kappa_0}_a}{b^2} \sum_{i=1}^3 Z_i b \cdot Z_i \psi^{\kappa_0}_a - \frac{\psi^{\kappa_0}_a - \partial_R \psi^{\kappa_0}_a}{b^2} \sum_{i=1}^3 (Z_i b)^2 = T^{\kappa_0+1} G_1^{\kappa_0} \tag{3.36}
\end{aligned}
\]

on \( R = 1 \),

\[
\begin{aligned}
\mathbb{H}(b,V^{\kappa_0}_a) \psi^{\kappa_0}_a - \frac{1}{b_0}(\mathbb{H}(b,V^{\kappa_0}_a) - \rho_0) a_2(b,V^{\kappa_0}_a)(T \partial_T a_0 + a_0)(b,V^{\kappa_0}_a) \\
+ \frac{\mathbb{H}(b,V^{\kappa_0}_a) \psi^{\kappa_0}_a}{(b + \psi^{\kappa_0}_a)^2} \sum_{i=1}^3 (Z_i a_0(b,V^{\kappa_0}_a))^2 = T^{\kappa_0+1} G_2^{\kappa_0}
\end{aligned}
\]

on \( R = 2 \)

with \( V^{\kappa_0}_a = (\psi^{\kappa_0}_a, \nabla \psi^{\kappa_0}_a) \) and

\[
\| (F^{\kappa_0}, G_1^{\kappa_0}, G_2^{\kappa_0}) \|_{C^m([0,T^*] \times [1,2] \times S^2)} \leq C(m,\kappa_0)\varepsilon. \tag{3.37}
\]
Let \( \dot{\psi} = \psi - \psi_a^{\kappa_0} \), \( X = \ln T \) and \( Y = R \omega \). With the help of (3.36), the problem (3.10) with (3.11)-(3.12) is equivalent to

\[
\begin{align*}
A_1(\dot{\psi}, \nabla \dot{\psi}) \partial_{xx} \dot{\psi} + 2A_2(\dot{\psi}, \nabla \dot{\psi}) \partial_{xx}R \dot{\psi} + 2 \sum_{i=1}^{3} A_{3i}(\dot{\psi}, \nabla \dot{\psi}) \partial_x Z_i \dot{\psi} + A_4(\dot{\psi}, \nabla \dot{\psi}) \partial_{RR} \dot{\psi} \\
+ 2 \sum_{i=1}^{3} A_{3i}(\dot{\psi}, \nabla \dot{\psi}) \partial_R Z_i \dot{\psi} + \sum_{i=1}^{3} \sum_{j=1}^{3} A_{3ij}(\dot{\psi}, \nabla \dot{\psi}) Z_i Z_j \dot{\psi} = \dot{f} \left( e^{(\kappa_0+1)X} F^{\kappa_0}, \psi, \nabla_X \dot{\psi} \right),
\end{align*}
\]

\((X, R, \omega) \in (\infty, X_0) \times (1, 2) \times S^2, \) \hspace{1cm} (3.38)

Here \( X_0 = \ln T^*, \) and the smooth functions \( \dot{f}, \dot{g}_1 \) and \( \dot{g}_2 \) satisfy

\[
\dot{f}(0) = \dot{g}_1(0) = \dot{g}_2(0) = 0, \quad \partial_{\psi} \dot{g}_1(0) = O(\varepsilon), \quad \nabla_{\partial_X \dot{\psi}, \partial_Y \dot{\psi}} \dot{g}_2(0) = 0.
\] \hspace{1cm} (3.39)

In addition,

\[
\begin{align*}
(A_1, A_2, A_3) = & \frac{\psi_{\kappa_0}^{\alpha} + \dot{\psi}}{2(\gamma - 1)A_0(b, V)} (2A_1(b, V), e^X A_2(b, V), e^X A_3(b, V)), \quad i = 1, 2, 3, \hspace{1cm} (3.40) \\
(A_4, A_5, A_6) = & \frac{e^{2X}(\psi_{\kappa_0}^{\alpha} + \dot{\psi})}{2(\gamma - 1)A_0(b, V)} (2A_4(b, V), A_5(b, V), 2A_6(b, V)), \quad i, j = 1, 2, 3
\end{align*}
\]

and

\[
B_{11} = 1 + \frac{1}{b^2} \sum_{i=1}^{3} (Z_i b)^2, \quad B_{12}^i(\psi) = -\frac{1}{b^2} (\psi_{\kappa_0}^{\alpha} + \dot{\psi}) Z_i b \quad \text{for} \quad i = 1, 2, 3
\] \hspace{1cm} (3.41)

and

\[
\begin{align*}
B_{20} = & -\frac{1}{b_0 a_1} \left( \mathcal{H}(\cdot) - \rho_0 \right) \bigg|_{\psi = \psi_{\kappa_0}^{\alpha} + \dot{\psi}} + D_0 \int_{0}^{1} \partial_{\partial_{\psi} \mathcal{H}}(b, \psi_{\kappa_0}^{\alpha} + \dot{\psi}, \nabla (\psi_{\kappa_0}^{\alpha} + \dot{\psi})) d\theta,
\\
B_{21} = & -\frac{(T \partial_T \psi_{\kappa_0}^{\alpha} + \psi_{\kappa_0}^{\alpha})}{b_0} \left( \mathcal{H}(\cdot) - \rho_0 \right) \bigg|_{\psi = \psi_{\kappa_0}^{\alpha} + \dot{\psi}} + D_0 \int_{0}^{1} \partial_{\partial_{\psi} \mathcal{H}}(b, \psi_{\kappa_0}^{\alpha} + \dot{\psi}, \nabla (\psi_{\kappa_0}^{\alpha} + \dot{\psi})) d\theta,
\\
B_{22} = & \frac{\psi_{\kappa_0}^{\alpha}}{(b + \psi_{\kappa_0}^{\alpha})^2} (Z_i a_0(b, \cdot) + Z_i a_0(b, \psi_{\kappa_0}^{\alpha})) \bigg|_{\psi = \psi_{\kappa_0}^{\alpha} + \dot{\psi}} + D_0 \int_{0}^{1} \partial_{z \psi} \mathcal{H}(b, \psi_{\kappa_0}^{\alpha} + \dot{\psi}, \nabla (\psi_{\kappa_0}^{\alpha} + \dot{\psi})) d\theta,
\end{align*}
\] \hspace{1cm} (3.42)

where \( V = (\psi_{\kappa_0}^{\alpha} + \dot{\psi}, \nabla (\psi_{\kappa_0}^{\alpha} + \dot{\psi})) \), and \( D_0 = \left( 1 - \frac{1}{b_0 a_1} (T a_0 + a_0) - \frac{\psi}{(b + \psi)^2} \sum_{i=1}^{3} (Z_i a_0)^2 \right) \bigg|_{\psi = \psi_{\kappa_0}^{\alpha}}. \)

With respect to the boundary condition on \( R = 2 \) in problem (3.38), we have

**Lemma 3.7.** If \( \| \dot{\psi} \|_{C^1([0, T^*] \times [1, 2] \times S^2)} \leq C_5 \), the boundary condition on \( R = 2 \) in (3.38) satisfies the Local Stability Condition which is defined in [24], that is, there exists a constant \( \delta_0 > 0 \) such that
1) $\mathcal{B} = (B_{20}, B_{21}, B_{22}, B_{22}) (\psi, \nabla \dot{\psi})$ is transversal to the boundary $R = 2$, namely, $|B_{21}| > \delta_0 > 0$.

2) Denote $\mathcal{N} = (A_2, A_4, A_5^1, A_5^2, A_5^3) (\psi, \nabla \psi)$, then $\tilde{\mathcal{B}} = \frac{1}{B_{21}} B + \frac{1}{|A_4|} \mathcal{N}$ is a positive time-like direction, namely,

(i) $\frac{1}{B_{21}} B_{20} + \frac{1}{|A_4|} A_2 > \delta_0 > 0$,

(ii) $- \frac{1}{A_4} \tilde{\mathcal{B}} \mathcal{M} \tilde{\mathcal{B}}^T > \delta_0 > 0$ with

$$M = \begin{pmatrix}
A_1 & A_2 & A_5^1 & A_5^2 & A_5^3 \\
A_2 & A_4 & A_5^1 & A_5^2 & A_5^3 \\
A_5^1 & A_5^2 & A_1 & A_3 & A_4 \\
A_5^1 & A_5^2 & A_3 & A_2 & A_4 \\
A_5^1 & A_5^2 & A_6 & A_6 & A_3
\end{pmatrix}.$$ 

(iii)

$$\sum_{i=1}^{\mathcal{M}} (|B_{i22}| + |A_i^4|) \leq C\varepsilon, \quad |A_2| \leq \frac{C}{b_0} (s_0 - b_0)^2.$$

**Remark 3.2.** In terms of the result in [24], if the Local Stability Condition on $R = 2$ holds, then we can look for a suitable multiplier to derive the energy estimates near the boundary $R = 2$. The details can be found in (3.66)-(3.68) below.

**Proof.** It suffices to verify that 1) and 2) hold when the variables $(b, \psi^k, \dot{\psi})$ in the corresponding coefficients of the second boundary condition in (3.38) are replaced by $(b_0, \dot{\psi}, 0)$ respectively.

By Lemma A.6 in Appendix A, we have

$$|B_{21}| = \left( \frac{\gamma - 1}{2A\gamma} \right)^{\frac{1}{\gamma - 1}} b_0^{\frac{\gamma - 2}{\gamma - 1}} (1 + O(b_0^{-\frac{\gamma - 2}{\gamma - 1}}) + O(b_0^{-2})) > 0,$$

which means that 1) is shown.

On the other hand, by Lemma A.1-A.3 in Appendix A and Lemma 3.4, one has

$$\begin{cases}
A_1 = \frac{2(s_0 - b_0)^2}{(\gamma - 1)b_0^2} (1 + O(b_0^{-\frac{\gamma - 2}{\gamma - 1}}) + O(b_0^{-2})) > 0, \\
A_2 = \frac{\psi}{(\gamma - 1)A_0(b, \nabla \psi)} (b_0a_1a_2 - a_0)(1 - (R - 1)a_1\partial_R\psi) \bigg|_{b=b_0, \dot{\psi}=\dot{\psi}} \\
\leq - \frac{C}{b_0} (s_0 - b_0)^2 < 0, \\
A_2 = 0 \quad \text{on} \quad R = 1, \\
A_i^4 = 0, \quad i = 1, 2, 3, \\
A_4 = -(1 + O(b_0^{-\frac{\gamma - 2}{\gamma - 1}}) + O(b_0^{-2})) < 0, \\
A_i^6 = 0, \quad i = 1, 2, 3, \\
A_i^3 = - \frac{1}{b_0^2} \delta_{ij}, \quad i, j = 1, 2, 3.
\end{cases} \quad (3.44)$$

Combining (3.44) with Lemma A.6 yields

$$\frac{1}{B_{21}} B_{20} + \frac{A_2}{|A_4|} = \frac{1}{b_0} (1 + O(b_0^{-\frac{\gamma - 2}{\gamma - 1}}) + O(b_0^{-2})) > 0 \quad (3.45)$$
and

\[- \frac{1}{4} \bar{A}_4 \bar{B} \bar{M} \bar{B}^{-T} \]

\[= - \frac{\bar{A}_1}{\bar{A}_4} \left( - \frac{\bar{A}_4}{\bar{B}_{21}} \bar{B}_{20} + \bar{A}_2 \right) \left( 1 + O(b_0^{-\frac{\gamma - 1}{2}}) + O(b_0^{-2}) \right) \]

\[= \frac{\gamma - 1}{2} (s_0 - b_0)^2 \left( 1 + O(b_0^{-\frac{\gamma - 1}{2}}) + O(b_0^{-2}) \right) > 0. \quad (3.46)\]

Collecting (3.43) and (3.45)-(3.46) yields Lemma 3.7. 1) and (i)-(ii) in 2) if we select \( \delta_0 = \frac{\gamma - 1}{4} (s_0 - b_0)^2 \). Additionally, (iii) in 2) can be derived directly from (3.42) and (3.44).

With respect to the boundary condition on \( R = 1 \) in (3.38), we have

**Lemma 3.8.** The boundary condition on \( R = 1 \) in (3.38) is of inhomogeneous Neumann-type, that is,

\[ A_2(\dot{\psi}) = 0, \quad -A_4(\psi) = B_{11}(\dot{\psi}), \quad -A_5(\ddot{\psi}) = B^i_{12}(\ddot{\psi}) \quad \text{on} \quad R = 1. \]

**Proof.** This can be verified directly by Lemma A.1-Lemma A.3 in Appendix A and the expressions in (3.40)-(3.41), then we omit the details here. \( \Box \)

For \( \| \dot{\psi} \|_{C^1([0, T \cdot] \times [1, 2] \times \mathbb{R}^2)} \leq C \varepsilon \), it follows from (3.44) that the second order quasilinear equation in (3.38) is strictly hyperbolic with respect to the variable \( X \) when \( X \leq X_0 \). Meanwhile, by Lemma 3.7-Lemma 3.8, we know that the first boundary condition in (3.38) is inhomogeneous Neumann-type and the second one admits the Local Stability Condition. Since the Neumann boundary condition does not satisfy the uniform Lopatinski condition and is very sensitive to the perturbation in deriving the well-posedness of solution, we have to choose a suitable iteration scheme to solve the linearized problem of (3.38) and further establish the solvability of nonlinear problem (3.38).

Defining \( \dot{\psi}_{n-1} = \dot{\psi}_0 = \tilde{\psi}_\sigma - \psi^{a}_n \), where \( \tilde{\psi}_\sigma \) is given Lemma 3.3. We assume that \( \dot{\psi}_{n-1}(n \in \mathbb{N}) \) has been constructed. Set \( \dot{\psi}_{n-1} = \tilde{\psi}_{n-1} + \psi^{a}_{n-1} \). Motivated by the expressions in (3.8), we define

\[
\begin{cases}
  a_{0,n} = b + (R - 1) \tilde{\psi}_{n-1}, \\
  a_{1,n} = \frac{1}{\psi_{n-1} + (R - 1) \partial_R \psi_{n-1}}, \\
  a_{2,n} = \tilde{\psi}_{n-1} + (R - 2) \partial_R \psi_{n-1}, \\
  a_{3,n} = \tilde{\psi}_{n-1} \partial_T \tilde{\psi}_{n-1} + \tilde{\psi}_{n-1} \partial_T b + (R - 2) \partial_T b \cdot \partial_R \tilde{\psi}_{n-1}, \\
  a^{i}_{4,n} = \tilde{\psi}_{n-1} Z_i \tilde{\psi}_{n-1} + \tilde{\psi}_{n-1} Z_i b + (R - 2) Z_i b \cdot \partial_R \tilde{\psi}_{n-1}, \quad & i = 1, 2, 3,
\end{cases}
\]

(3.47)

here we emphasize that the appearance of the term \( \tilde{\psi}_{n-2} Z_i \tilde{\psi}_{n-1} \) (other than \( \tilde{\psi}_{n-1} Z_i \tilde{\psi}_{n-1} \)) in the expression of \( a^{i}_{4,n} \) is due to the requirement of Neumann type boundary condition on \( R = 1 \) in the iteration process of solving (3.38) (one can see the concrete explanations in Remark 3.3 below).
Let $\dot{\psi}_n$ be determined by the following problem

\[
\begin{cases}
\mathcal{L}_n(\dot{\psi}_n) = A_{1,n} \partial_X^2 \dot{\psi}_n + 2 A_{2,n} \partial_X \dot{\psi}_n + 2 \sum_{i=1}^{3} A_{i,n} \partial_X Z_i \dot{\psi}_n + A_{4,n} \partial_R^2 \dot{\psi}_n + 2 \sum_{i=1}^{3} A_{i,n} \partial_R Z_i \dot{\psi}_n \\
+ \sum_{i,j=1}^{3} A_{ij,n} Z_i Z_j \dot{\psi}_n = \hat{f}(e^{(\kappa_0+1)X} F^{\kappa_0}, \psi_{n-1}, \nabla \dot{\psi}_{n-1}), \quad (X, R, \omega) \in (-\infty, X_0) \times (1,2) \times S^2, \\
B_{11}^n(\dot{\psi}_n) = B_{11}^n \partial_R \dot{\psi}_n + \sum_{i=1}^{3} B_{i,n} Z_i \dot{\psi}_n = \hat{g}_1(e^{(\kappa_0+1)X} G^{\kappa_0}, \psi_{n-1}) \quad \text{on} \quad R = 1, \\
B_{22}^n(\dot{\psi}_n) = B_{22}^n \partial_X \dot{\psi}_n + \sum_{i=1}^{3} B_{i,n} Z_i \dot{\psi}_n = \hat{g}_2(e^{(\kappa_0+1)X} G^{\kappa_0}, \psi_{n-1}, \nabla \dot{\psi}_{n-1}) \quad \text{on} \quad R = 2,
\end{cases}
\]  

(3.48)

where the coefficients in the operator $\mathcal{L}_n(\dot{\psi}_n)$ and the boundary operators $B_{ii}^n(\dot{\psi}_n)(i = 1, 2)$ are given in terms of the corresponding coefficients in (3.38), whose arguments $(\psi, a_0, a_1, a^2_1)$ $(i = 1, 2, 3)$ are replaced by $(\dot{\psi}_{n-1}, a_{0,n}, a_{i,n}, a^2_{i,n})$ $(i = 1, 2, 3)$ respectively.

**Remark 3.3.** For the expressions in (3.47), it follows from a direct computation that the boundary condition on $R = 1$ in (3.48) is of Neumann type, namely, we have on $R = 1$

\[
A_{2,n} = 0, \quad -A_{4,n} = B_{11}^n, \quad -A_{i,n} = B_{i,n}, \quad i = 1, 2, 3.
\]

§3.3.2. Solvability and energy estimates of problem (3.48)

With some modifications on the notations in [24], we will use some weighted Sobolev spaces in this section. For any smooth function $u(X, R, \omega)$ which vanishes at $X = -\infty$, we define the following norms of $u(X, Y)$ in the domain $\{(X, Y) : X \in (-\infty, a], Y \in (1,2) \times S^2\}$ for the constants $\lambda \in \mathbb{N} \cup \{0\}$, $\eta > 0$ and $a \in \mathbb{R}$:

\[
|u|^2_{\lambda, \eta, X} = \sum_{\tau_0+|\tau| = \lambda \atop \tau_0 + \tau_1 + \tau_2 = \lambda \atop |\tau_1| \leq 2} \int_{-\infty}^{a} \int_{1 \leq |Y| \leq 2} e^{-2\eta X} \eta^{2\tau_0} |\nabla^\tau_{X,Y} u(X, Y)|^2 dY dX, \quad (3.49)
\]

\[
\|u\|^2_{\lambda, \eta, a} = \sum_{\tau_0+|\tau| = \lambda \atop \tau_0 + \tau_1 + \tau_2 = \lambda \atop |\tau_1| \leq 2} \int_{-\infty}^{a} \int_{1 \leq |Y| \leq 2} e^{-2\eta X} \eta^{2\tau_0} |\nabla^\tau_{X,Y} u(X, Y)|^2 Y dX dX. \quad (3.50)
\]

On the boundaries $R = i$ $(i = 1, 2)$, we define the boundary norms as follows

\[
(u)^2_{\lambda, \eta, a} = \int_{-\infty}^{a} \sum_{\tau_0+|\tau| = \lambda \atop \tau_0 + \tau_1 + \tau_2 = \lambda \atop |\tau_1| \leq 2} e^{-2\eta X} \eta^{2\tau_0} \|\nabla^\tau_{X} u(X, \cdot)\|_{H^{\tau_2+1/2}(\partial B(0))}^2 dX \quad (3.51)
\]

and

\[
\ll u \gg^2_{\lambda, \eta, i, a} = \int_{-\infty}^{a} \sum_{\tau_0+|\tau| = \lambda \atop \tau_0 + \tau_1 + \tau_2 = \lambda \atop |\tau_1| \leq 2} e^{-2\eta X} \eta^{2\tau_0} \|\nabla^\tau_{X} u(X, \cdot)\|_{H^{\tau_2}(\partial B(0))}^2 dX, \quad i = 1, 2. \quad (3.52)
\]

Based on the notations given in (3.49)-(3.52), we define the weighted Sobolev spaces $H^2_{\lambda+1, a}$ and $H^2_{\lambda+1, a}$ in the domain $(-\infty, a] \times (1,2) \times S^2$ as

\[
H^2_{\lambda+1, a} = \left\{ u \in H^{2+1} : \|u\|_{\lambda+1, \eta, a} = \sup_{-\infty < X < a} |u|_{\lambda+1, \eta, X} + \eta \|u\|_{\lambda+1, \eta, a} + \sum_{i=1}^{2} \ll u \gg_{\lambda+1, \eta, i, a} + \ll \partial_R u \gg_{\lambda, \eta, i, a} < \infty \right\}
\]
and

\[ H^\alpha_{\lambda+1,a} = \left\{ u \in H^{\lambda+1} : \|u\|_{\lambda+1,a} = \sup_{-\infty < X < a} |u|_{\lambda+1,a} + \eta \right\}, \]

To establish the solvability and energy estimates of problem (3.48), at first, we consider the following initial-boundary problem by some ideas in [14] and [28]:

\[
\begin{cases}
L(u) = e_1 \partial_t^2 u + 2e_2 \partial_t^2 u + 2 \sum_{i=1}^3 e_3^i \partial_t Z_i u - e_4 \partial_t^2 u - 2 \sum_{i=1}^3 e_5^i \partial_t Z_i u - \sum_{i,j=1}^3 e_6^{ij} Z_i Z_j u \\
= f(t,r,\omega) \quad \text{in} \quad \mathcal{D}_0 = (-\infty, 0) \times (1,2) \times S^2, \quad (3.53)
B_i(u) = d_{i1} \partial_t u + \sum_{j=1}^3 d_{i2j} Z_j u + d_{i3} \partial_t u = g_i \quad \text{on} \quad \mathcal{B}_i = (-\infty, 0) \times \{i\} \times S^2, \quad i = 1, 2,
\end{cases}
\]

where we have the following assumptions

(A1) The operator \( L \) is strictly hyperbolic with respect to the time \( t \), and fulfill that there exists two positive constants \( \lambda_1 < \lambda_2 \) such that for any \( (t, x) \in \mathcal{D}_0 \) and \( \xi \in \mathbb{R}^4 \),

\[ \lambda_1 \leq e_1(t,x) \leq \lambda_2, \quad \lambda_1 |\xi|^2 \leq e_4 \xi_0^2 + 2 \sum_{i=1}^3 e_5^i \xi_i \xi_i + \sum_{i,j=1}^3 e_6^{ij} \xi_i \xi_j \leq \lambda_2 |\xi|^2. \]

(A2) The boundary condition on \( \mathcal{B}_1 \) is of Neumann type, namely,

\[ |d_{11} + e_4| + \sum_{i=1}^3 |d_{12i} + e_5^i| + |d_{13} - e_2| = 0 \quad \text{on} \quad \mathcal{B}_1. \]

(A3) The boundary condition on \( \mathcal{B}_2 \) satisfies the Local Stability Condition. Moreover, there exists a positive constant \( \lambda_0 \) which is smaller than \( \lambda_1 \), such that

\[ |d_{21} - e_4| + \sum_{i=1}^3 |d_{22i} - e_5^i| \leq \lambda_0, \quad d_{23} + e_2 \geq \lambda_1 \quad \text{on} \quad \mathcal{B}_2. \]

(A4) For any integer \( \lambda > 5 \),

\[
a(\lambda) = \sum_{i=1,2,4} \|e_i\|_{\lambda,0} + \sum_{l=3,5} \sum_{i=1}^3 \|e_i^l\|_{\lambda,0} + \sum_{i,j=1}^3 \|e_6^{ij}\|_{\lambda,0} + \sum_{i=1,3} \langle d_{1i} \rangle_{\lambda,0} + \sum_{i=1}^3 \langle d_{12} \rangle_{\lambda,0} \]
\[ + \sum_{i=1,3} \ll d_{2i} \gg_{\lambda,2,0} + \sum_{i=1}^3 \ll d_{22i} \gg_{\lambda,2,0} + 1 < \infty. \]

**Proposition 3.9.** Under the assumptions (A1)-(A4), there exists a positive constant \( \eta_0 \), for \( \eta \geq \eta_0 \), the integer \( \lambda > 5 \), and

\[ \|f\|_{\lambda,0} + \langle g_1 \rangle_{\lambda,0} + \ll g_2 \gg_{\lambda,2,0} < +\infty, \]
then the problem (3.53) has a unique solution \( u \in H^\lambda_{\lambda+1,0} \) and satisfies for \( 0 \leq \lambda' \leq \lambda \)

\[
\|u\|_{H^\lambda_{\lambda+1,0}} \leq C \left( \frac{1}{\eta} \|f\|_{H^\lambda_{\lambda,0}} + (g_1)_{H^\lambda_{\lambda,0}} + \ll g_2 \gg_{H^\lambda_{\lambda,2,0}} \right).
\] (3.54)

To establish Proposition 3.9, we now give some necessary preparations.

It is easy to know that there exist constants \( b_l \ (1 \leq l \leq \lambda) \) such that \( \sum_{l=1}^\lambda b_l (-l)^k = 1 \) for \( 0 \leq k \leq \lambda - 1 \).

For \( v(t, \cdot) \in H^\lambda(-\infty, 0] \), we define \( \tilde{v}(t, \cdot) \) as

\[
\tilde{v}(t, \cdot) = \begin{cases} 
  v(t, \cdot) & \text{for } t < 0, \\
  \sum_{l=1}^\lambda b_l v(-lt, \cdot) & \text{for } t > 0.
\end{cases}
\] (3.55)

A direct computation yields

\[
\|\tilde{v}(t, \cdot)\|_{H^\lambda \mathbb{R}} \leq C \|v(t, \cdot)\|_{H^\lambda(-\infty, 0]}.
\] (3.56)

In terms of the definition (3.55), we define the resulting operators \( \tilde{L} \) and \( \tilde{B}_l \ (l = 1, 2) \) from (3.53) as

\[
\begin{aligned}
\tilde{L} &\equiv \tilde{\epsilon}_1 \partial_t^2 + 2 \tilde{\epsilon}_2 \partial_t^2 + 2 \sum_{i=1}^3 \tilde{\epsilon}_3 \partial_t Z_i - \tilde{\epsilon}_4 \partial_t^2 - 2 \sum_{i=1}^3 \tilde{\epsilon}_5 \partial_t Z_i - \sum_{i,j=1}^3 \tilde{\epsilon}_{ij} Z_i Z_j, \\
\tilde{B}_l &\equiv \tilde{a}_{1l} \partial_t + \sum_{j=1}^3 \tilde{d}_{1j} Z_j + \tilde{d}_{3j} \partial_t, \quad i = 1, 2,
\end{aligned}
\] (3.57)

moreover, with (3.56), the corresponding assumptions (A_1)-(A_4) still hold.

A truncated function \( \chi_1(t) \in C^\infty(\mathbb{R}) \) with \( 0 \leq \chi_1(t) \leq 1 \) is defined as

\[
\chi_1(t) = \begin{cases} 
  1 & \text{for } t < \frac{1}{4}, \\
  0 & \text{for } t > \frac{3}{4}.
\end{cases}
\] (3.58)

In order to prove Proposition 3.9, we now study the following modified problem of (3.53) for \( l \geq 0 \)

\[
\begin{aligned}
\tilde{L}(\tilde{u}) &\equiv (\tilde{\epsilon}_1 \partial_t^2 + 2 \tilde{\epsilon}_2 \partial_t^2 + 2 \sum_{i=1}^3 \tilde{\epsilon}_3 \partial_t Z_i - \tilde{\epsilon}_4 \partial_t^2 - 2 \sum_{i=1}^3 \tilde{\epsilon}_5 \partial_t Z_i - \sum_{i,j=1}^3 \tilde{\epsilon}_{ij} Z_i Z_j)(\tilde{u}) \\
&= \chi_1(t) \tilde{f} & \text{in } \tilde{\mathcal{D}}_0 = \mathbb{R} \times (1, 2) \times S^2, \\
\tilde{B}_1(\tilde{u}) &\equiv (\tilde{d}_{11} \partial_t + \sum_{i=1}^3 \tilde{d}_{1i} Z_i + \tilde{d}_{13} \partial_t + l \partial_t) \tilde{u} = \chi_1(t) \tilde{g}_1 & \text{on } \tilde{\mathcal{D}}_1 = \mathbb{R} \times \{1\} \times S^2, \\
\tilde{B}_2(\tilde{u}) &\equiv (\tilde{d}_{21} \partial_t + \sum_{i=1}^3 \tilde{d}_{2i} Z_i + \tilde{d}_{23} \partial_t) \tilde{u} = \chi_1(t) \tilde{g}_2 & \text{on } \tilde{\mathcal{D}}_2 = \mathbb{R} \times \{2\} \times S^2
\end{aligned}
\] (3.59)

with

\[
\begin{aligned}
\tilde{L} &= \chi_1(t) \tilde{L} + (1 - \chi_1(t))(\partial_t^2 - \partial_t^2 - \sum_{i=1}^3 Z_i^2), \\
\tilde{B}_1 &= \chi_1(t) \tilde{B}_1 - (1 - \chi_1(t)) \partial_t + l \partial_t, \\
\tilde{B}_2 &= \chi_1(t) \tilde{B}_2 + (1 - \chi_1(t)) \partial_t.
\end{aligned}
\] (3.60)
With respect to problem (3.59), we have

**Lemma 3.10.** For any fixed \( l > 0 \), if \( \tilde{u}_l \in H_{\lambda+1,\infty}^0 \) is a solution of (3.59), then for any \( T \in \mathbb{R} \) and \( 0 \leq \lambda' \leq \lambda \), we have

\[
\|\|\tilde{u}_l\|\|^2_{\lambda+1,\eta,T} \leq C(l, \lambda')\|\|\chi_1(t)\tilde{f}\|\|^2_{\lambda', \eta, T} + \sum_{i=1}^{2} \|\chi_1(t)\tilde{g}_i\|_{\lambda', \eta, i, T}^2.
\]  

(3.61)

**Proof.** We will look for a suitable differential operator of first order

\[Q = Q_0 \partial_t + Q_1 \partial_r + \sum_{j=1}^{3} Q_j Z_j\]

so that the required norms can be dominated by integrating \(2e^{-2\eta t}Q(\tilde{u}_l)\tilde{L}(\tilde{u}_l)\) over the domain \((-\infty, T] \times (1, 2) \times S^2\).

Choose a \(C^\infty\) cut-off function \(\Upsilon(r) = \begin{cases} 1, & r < \frac{5}{4}, \\ 0, & r > \frac{7}{4}, \end{cases}\) and set

\[V_1 = \Upsilon(r)\tilde{u}_l, \quad V_2 = (1 - \Upsilon(r))\tilde{u}_l.\]  

(3.62)

Then \(V_1\) satisfies

\[
\begin{cases}
\tilde{L}(V_1) = F_1 & \text{in } \tilde{D}_0, \\
\tilde{B}_i(V_1) = \chi_1(t)\tilde{g}_i & \text{on } \tilde{B}_1, \\
\text{supp}V_1 \subset \{r \leq \frac{5}{4}\}
\end{cases}
\]

(3.63)

with \(F_1(t, r, \omega) = \Upsilon(r)\chi_1(t)\tilde{f} + [\tilde{L}, \Upsilon(r)]\tilde{u}_l\).

On the other hand, \(V_2\) satisfies

\[
\begin{cases}
\tilde{L}(V_2) = F_2 & \text{in } \tilde{D}_0, \\
\tilde{B}_2(V_2) = \chi_1(t)\tilde{g}_2 & \text{on } \tilde{B}_2, \\
\text{supp}V_2 \subset \{r \geq \frac{5}{4}\},
\end{cases}
\]

(3.64)

with \(F_2(t, r, \omega) = (1 - \Upsilon(r)\chi_1(t)\tilde{f} + [\tilde{L}, 1 - \Upsilon(r)]\tilde{u}_l\).

With two operators \(Q_i = Q_{i0}\partial_t + Q_{i1}\partial_r + \sum_{j=1}^{3} Q_{ij} Z_j \) \((i = 1, 2)\) to be determined later on, it follows from (3.63)-(3.64) that

\[
\begin{align*}
\int_{-\infty}^{T} \int_{(1, 2) \times S^2} 2e^{-2\eta t}Q_i(V_i)\tilde{L}(V_i)dxdt \\
= e^{-2\eta T} \int_{(1, 2) \times S^2} H_{i0}(T, x)dx + 2\eta \int_{-\infty}^{T} \int_{(1, 2) \times S^2} e^{-2\eta t}H_{i0}(t, x)dxdt \\
+ (-1)^i \int_{-\infty}^{T} \int_{|x|=i} e^{-2\eta t}H_{i1}dSdt + \int_{-\infty}^{T} \int_{(1, 2) \times S^2} e^{-2\eta t}H_{i2}dxdt
\end{align*}
\]

(3.65)
with

\[
H_{i0} = \hat{e}_1 Q_{i0} (\partial_t V_i)^2 + 2\hat{e}_1 Q_{i1} \partial_t V_i \partial_r V_i + 2 \sum_{j=1}^{3} \hat{e}_1 Q_{i2}^j \partial_t V_i Z_j V_i \\
+ (2\hat{e}_2 Q_{i1} + \hat{e}_4 Q_{i0}) (\partial_r V_i)^2 + \sum_{j=1}^{3} (2\hat{e}_2 Q_{i2}^j + 2\hat{e}_3 Q_{i1} + 2\hat{e}_3 Q_{i0}) \partial_r V_i Z_j V_i \\
+ \sum_{j=1}^{3} \sum_{k=1}^{3} (2\hat{e}_3 Q_{i2}^k + \hat{e}_6 Q_{i0}) Z_j V_i Z_k V_i,
\]

\[
H_{i1} = (2\hat{e}_2 Q_{i0} - \hat{e}_1 Q_{i1}) (\partial_t V_i)^2 - 2\hat{e}_4 Q_{i0} \partial_t V_i \partial_r V_i \\
+ \sum_{j=1}^{3} (2\hat{e}_2 Q_{i2}^j - 2\hat{e}_3 Q_{i1} - 2\hat{e}_3 Q_{i0} \partial_r V_i Z_j V_i - \hat{e}_4 Q_{i1} (\partial_r V_i)^2 \\
- \sum_{j=1}^{3} 2\hat{e}_4 Q_{i2}^j \partial_r V_i Z_j V_i - \sum_{j=1}^{3} \sum_{k=1}^{3} (2\hat{e}_3 Q_{i2}^k - \hat{e}_6 Q_{i1}) Z_j V_i Z_k V_i,
\]

and \(H_{i2}\) stands for the quadratic polynomial of \(\nabla_{t,x} V_i\) and is independent of \(\eta\), whose precise expression is not required.

At first, we treat the case of \(i = 2\) in (3.63).

By (A3), (3.57) and (3.60), Lemma 3.7 and Remark 3.2, we can choose a first order operator \(Q_2\) such that

\[
\begin{cases}
H_{20}(t,x) \geq c_0 |\nabla V_2|^2(t,x), \\
H_{21} \geq c_0 |\nabla V_2|^2 - c_1 |\chi_1(t) \tilde{g}_2|^2 \quad \text{on } \mathcal{B}_2,
\end{cases}
\]

(3.66)

where \(c_0, c_1\) are some positive constants.

In addition, with the properties of \(H_{22}\) and \(F_2\) in (3.64), one has

\[
|H_{22}| + |Q_2(V_2) \cdot \hat{L}(V_2)| \leq C (\kappa_0 \eta \nabla \tilde{u}_l + \frac{1}{\kappa_0 \eta} |\chi_1(t) \tilde{f}|^2 + \frac{1}{\kappa_0 \eta} |\tilde{u}|^2),
\]

(3.67)

with the constant \(\kappa_0 > 0\) being determined later on.

Substituting (3.66)-(3.67) into (3.65) yields

\[
|\nabla V_2|^2_{0,\eta,T} + \eta |\nabla V_2|_{0,\eta,T} + \ll \nabla V_2 \gg_{\tilde{D}_{0,2,T}}
\]

\[
\leq C \left( \frac{1}{\kappa_0 \eta} |\chi_1(t) \tilde{f}|^2_{0,\eta,T} + \kappa_0 \eta |\nabla \tilde{u}|_{0,\eta,T}^2 + \frac{1}{\kappa_0 \eta} |\tilde{u}|_{0,\eta,T}^2 + \ll \chi_1(t) \tilde{g}_2 \gg_{\tilde{D}_{0,2,T}} \right).
\]

(3.68)

For the case of \(i = 1\) and \(l > 0\), we also know that the boundary condition on \(r = 1\) in (3.63) also satisfies the Local Stability Condition, then similar to (3.68), we have

\[
|\nabla V_1|^2_{0,\eta,T} + \eta |\nabla V_1|_{0,\eta,T} + \ll \nabla V_1 \gg_{\tilde{D}_{0,1,T}}^2
\]

\[
\leq C(l) \left( \frac{1}{\kappa_0 \eta} |\chi_1(t) \tilde{f}|^2_{0,\eta,T} + \kappa_0 \eta |\nabla \tilde{u}|_{0,\eta,T}^2 + \frac{1}{\kappa_0 \eta} |\tilde{u}|_{0,\eta,T}^2 + \ll \chi_1(t) \tilde{g}_1 \gg_{\tilde{D}_{0,1,T}}^2 \right)
\]

(3.69)

Combining (3.68)-(3.69) with the definition (3.62) yields for \(l > 0\)

\[
|\nabla \tilde{u}|_{0,\eta,T}^2 + \eta |\nabla \tilde{u}|_{0,\eta,T}^2 + \sum_{i=1}^{2} \ll \nabla \tilde{u}_i \gg_{\tilde{D}_{0,i,T}}^2
\]

\[
\leq C(l) \left( \frac{1}{\kappa_0 \eta} |\chi_1(t) \tilde{f}|^2_{0,\eta,1} + \kappa_0 \eta |\nabla \tilde{u}|_{0,\eta,1}^2 + \frac{2}{\kappa_0 \eta} |\chi_1(t) \tilde{g}_1 \gg_{\tilde{D}_{0,1,1}}^2 \right.
\]

\[
+ \spin \ll \tilde{u}_1 \gg_{\tilde{D}_{0,1,1}}^2 + \sum_{i=1}^{2} \ll \tilde{u}_i \gg_{\tilde{D}_{0,i,1}}^2
\]

(3.70)
On the other hand, according to Hardy-type inequality, one has
\[
|\tilde{u}_t|^2_{0,\eta,T} = -2\eta\|\tilde{u}_t\|^2_{0,\eta,T} + 2 \int_{-\infty}^T \int_{|x| \leq 2} e^{-2\eta t} \partial_t \tilde{u}_t \cdot \tilde{u}_t dx dt \\
\leq \frac{1}{\kappa_0 \eta} \|\tilde{u}_t\|^2_{0,\eta,T} + \kappa_0 \eta \|\nabla \tilde{u}_t\|^2_{0,\eta,T}.
\] (3.71)

Similarly,
\[
\eta \|\tilde{u}_t\|^2_{0,\eta,T} + \ll \tilde{u}_t \gg^2_{0,\eta,T} \leq C(l) \left( \kappa_0 \eta \|\nabla \tilde{u}_t\|^2_{0,\eta,T} + \frac{1}{\eta^2} \ll \partial_t \tilde{u}_t \gg^2_{0,\eta,T} \right).
\] (3.72)

Substituting (3.71)-(3.72) into (3.70) yields for \(\kappa_0 = \frac{1}{2^4(l)}\)
\[
\|\tilde{u}_t\|_{2,\eta,T}^2 \leq C(l) \left( \frac{1}{\eta} \|\chi_1(t)\tilde{f}\|^2_{0,\eta,T} + \sum_{i=1}^2 \ll \chi_1(t)\tilde{g}_i \gg^2_{0,\eta,i,T} \right),
\] (3.73)

which means (3.61) holds for \(\lambda' = 0\) with (3.55)-(3.56).

To obtain the higher order energy estimates of \(\tilde{u}_t\), we take the tangential differential operators \(Z_i (i = 1, 2, 3)\) and \(\partial_i\) on each equality in (3.59) and then obtain as in (3.73)
\[
\|\partial \tilde{u}_t\|_{0,\eta,T}^2 \leq C(l) \left( \frac{1}{\eta} \|\partial \left( \chi_1(t)\tilde{f}\right)\|^2_{0,\eta,T} + \sum_{i=1}^2 \ll \partial \left( \chi_1(t)\tilde{g}_i\right) \gg^2_{0,\eta,i,T} \right) \\
+ \frac{1}{\eta} \left[ \left[ \tilde{L}_i, \partial \right] \tilde{u}_t \right]_{0,\eta,T}^2 + \ll \left[ \tilde{B}_1^i, \partial \right] \tilde{u}_t \gg^2_{0,\eta,1,T} + \ll \left[ \tilde{B}_2, \partial \right] \tilde{u}_t \gg^2_{0,\eta,2,T} \right)
\] (3.74)

where \(\partial\) stands for \(Z_i (i = 1, 2, 3)\) or \(\partial_i\).

By the definition of space \(H^\eta_{\lambda+1,T}\), in order to obtain (3.61) for \(\lambda' = 1\), it suffices to estimate
\[
\sup_{t \in (-\infty, T]} |\partial^2_{\tau} \tilde{u}_t|_{0,\eta,t}^2 + \eta \|\partial^2_{\tau} \tilde{u}_t\|^2_{0,\eta,T},
\]

since \(\partial R \tilde{u}_t\) is a linear combination of \(\partial \tilde{u}_t\) and \(\tilde{B}_1^i \tilde{u}_t\) or \(\tilde{B}_2 \tilde{u}_t\) on \(R = i(i = 1, 2)\).

It is noted that we have by the equation in (3.59)
\[
\sup_{t \in (-\infty, T]} |\partial^2_{\tau} \tilde{u}_t|_{0,\eta,t}^2 + \eta \|\partial^2_{\tau} \tilde{u}_t\|^2_{0,\eta,T} \leq C(|||\partial \tilde{u}_t|||^2_{1,\eta,T} + \sup_{t \in (-\infty, T]} \|\chi_1(t)\tilde{f}\|^2_{0,\eta,t} + \eta \|\chi_1(t)\tilde{f}\|^2_{0,\eta,T}).
\] (3.75)

In addition, one has
\[
\sup_{t \in (-\infty, T]} |\chi_1(t)\tilde{f}|_{0,\eta,t}^2 + \eta \|\chi_1(t)\tilde{f}\|^2_{0,\eta,T} \leq \frac{C}{\eta} \|\partial \chi_1(t)\tilde{f}\||^2_{0,\eta,T}.
\] (3.76)

Collecting (3.74)-(3.76) yields (3.61) for \(\lambda' = 1\). Analogously, we can complete the proof of Lemma 3.10 for \(0 \leq \lambda' \leq \lambda\). \(\square\)

In order to solve (3.53), we have to establish a uniform estimate independent of \(l\) in Lemma 3.10. Since the boundary condition on \(\mathcal{B}_1\) does not satisfy the Local Stability Condition for \(l = 0\), then the estimates in
Lemma 3.10 cannot be used directly in this case. To overcome this difficulty, we will apply for some ideas in [28].

**Lemma 3.11.** If \( \tilde{u}_i \in \mathcal{H}^4_{\lambda + 1, \infty} \) is a solution of the problem (3.59), then one has for \( 0 \leq \lambda' \leq \lambda \)

\[
\varpi \tilde{u}_i \|_{\lambda + 1, \eta, \infty}^2 \leq C \left( \| \chi_1(t) \tilde{f} \|_{\lambda', \eta, \infty}^2 + \langle \chi_1(t) \tilde{g}_1 \rangle_{\lambda', \eta, \infty} + \| \chi_1(t) \tilde{g}_2 \|_{\lambda', \eta, \infty}^2 \right), \tag{3.77}
\]

where \( C > 0 \) is independent of \( l \).

**Proof.** For the problem (3.59) and \( l \geq 0 \), we choose \( Q_1 = \partial_t \) in the process of deriving (3.65). At this time, a direct computation yields

\[
H_{10}(t, x) \geq c_0 |\nabla V_i|^2(t, x), \tag{3.78}
\]

where \( c_0 \) is a positive constant.

Additionally, in terms of the boundary condition on \( r = 1 \) in (3.63) and the assumption (\( A_2 \)), we have on \( r = 1 \)

\[
H_{11} = 2 \partial_t V_i (\dd_1 \partial_t V_i + \sum_{i=1}^3 d_{i2} Z_i V_i + d_{i3} \partial_t V_i)
- \partial_t V_i (l \partial_t V_i + \chi_1(t) \tilde{g}_1)
\leq 2 \partial_t V_i (\chi_1(t) \tilde{g}_i).
\]

Thus,

\[
\int_{-\infty}^{T} \int_{|x|=1} e^{-2\eta t} H_{11} dS dt \leq C \left( \kappa_0 \int_{-\infty}^{T} e^{-2\eta t} \| \partial_t V_i(t, \cdot) \|_{\mathcal{H}^{-1/2}(\partial B_1(0))}^2 dt + \frac{1}{\kappa_0} \langle \chi_1(t) \tilde{g}_1 \rangle_{0, \eta, T}^2 \right). \tag{3.79}
\]

In addition, similar to (3.67), we have

\[
|H_{12}| + |Q_1(V_i) \cdot \tilde{L}(V_i)| \leq C (\kappa_0 \eta \| \nabla u_i \|^2 + \frac{1}{\kappa_0 \eta} |\chi_1(t) \tilde{f}|^2 + \frac{1}{\kappa_0 \eta} |\tilde{u}_i|^2). \tag{3.80}
\]

Substituting (3.78)-(3.80) into (3.65) for the case of \( i = 1 \) yields

\[
\| \nabla V_i \|^2_{0, \eta, T} + \eta \| \nabla V_i \|^2_{0, \eta, T}
\leq C \left( \frac{1}{\kappa_0 \eta} \| \chi_1(t) \tilde{f} \|_{0, \eta, T}^2 + \kappa_0 \eta \| \nabla u_i \|^2_{0, \eta, T} + \frac{1}{\kappa_0 \eta} \| \tilde{u}_i \|_{0, \eta}^2
+ \frac{1}{\kappa_0} \langle \chi_1(t) \tilde{g}_1 \rangle_{0, \eta, T} + \kappa_0 \int_{-\infty}^{T} e^{-2\eta t} \| \partial_t V_i(t, \cdot) \|_{\mathcal{H}^{-1/2}(\partial B_1(0))}^2 dt \right).
\]

Combining this with (3.68) and (3.71) shows

\[
\| \nabla u_i \|^2_{0, \eta, \infty} + \eta \| \nabla u_i \|^2_{0, \eta, \infty} + \| \nabla u_i \|^2_{0, \eta, \infty} + \kappa_0 \eta \| \nabla u_i \|^2_{0, \eta, \infty} + \frac{1}{\kappa_0} \langle \chi_1(t) \tilde{g}_1 \rangle_{0, \eta, \infty} + \| \chi_1(t) \tilde{g}_2 \|_{0, \eta, \infty}^2
+ \kappa_0 \int_{-\infty}^{T} e^{-2\eta t} \| \partial_t u_i(t, \cdot) \|^2_{\mathcal{H}^{-1/2}(\partial B_1(0))} dt \right). \tag{3.81}
\]

According to an elementary Proposition 3.17 given in §3.3.4 below, one has

\[
\int_{-\infty}^{T} e^{-2\eta t} \| \partial_t u_i \|^2_{\mathcal{H}^{-1/2}(\partial B_1(0))} dt \leq C \left( \| \chi_1(t) \tilde{f} \|_{0, \eta, \infty}^2 + \| \tilde{u}_i \|_{0, \eta, \infty}^2 \right). \tag{3.82}
\]
Substituting (3.82) into (3.81) with \( \kappa_0 = \frac{1}{y} \) yields (3.77) for \( \lambda' = 0 \).

Analogously, the estimates in the case of \( \lambda' > 0 \) in (3.77) can be obtained as in Lemma 3.10. Thus, we complete the proof of Lemma 3.11.

Based on Lemma 3.10 and Lemma 3.11, we now start to prove the solvability of the problem (3.59).

**Lemma 3.12.** For any \( l > 0 \), the problem (3.59) has a unique solution \( \tilde{u}_l \in \mathcal{H}_{\lambda+1,\eta,\infty} \) which satisfies the estimate (3.77).

**Proof.** The proof is divided into the following three steps.

**Step 1.** First we consider the case which all the coefficients mentioned in (A4) are smooth. Moreover, we assume that there exist two constants \( T_2 < T_3 \) such that

\[
\text{supp} f \subset (T_2, T_3) \times (B_2(0) \setminus B_1(0)), \quad \text{supp} g_i \subset (T_2, T_3) \times \partial B_i(0), \quad i = 1, 2. \tag{3.83}
\]

For \( l > 0 \), we consider the following problem

\[
\begin{cases}
L(\tilde{u}_l) = \chi_1(t) \tilde{f} & \text{in } (T_2 - 1, T_3) \times (1, 2) \times \mathbb{S}^2, \\
\bar{B}_1(\tilde{u}_l) = \chi_1(t) \tilde{g}_1 & \text{on } (T_2 - 1, T_3) \times \{1\} \times \mathbb{S}^2, \\
\bar{B}_2(\tilde{u}_l) = \chi_1(t) \tilde{g}_2 & \text{on } (T_2 - 1, T_3) \times \{2\} \times \mathbb{S}^2, \\
\tilde{v}_l(T_2 - 1, r, \omega) = \partial_r \tilde{v}_l(T_2 - 1, r, \omega) = 0.
\end{cases} \tag{3.84}
\]

By Theorem 3 in Page 142 of [14], the problem (3.84) has a unique solution \( \tilde{v}_l \) which admits

\[
\tilde{v}_l(t, r, \omega) \in H^{\lambda+1}((T_2, T_3) \times (B_2(0) \setminus B_1(0))),
\]

\[
\tilde{v}_l(t, 1, \omega) \in H^{\lambda+1}((T_2, T_3) \times \partial B_1(0)), \quad \tilde{v}_l(t, 2, \omega) \in H^{\lambda+1}((T_2, T_3) \times \partial B_2(0)),
\]

\[
\partial_r \tilde{v}_l(t, 1, \omega) \in H^{\lambda}((T_2, T_3) \times \partial B_1(0)), \quad \partial_r \tilde{v}_l(t, 2, \omega) \in H^{\lambda}((T_2, T_3) \times \partial B_2(0)).
\]

Moreover, it follows from (3.83) and the uniqueness of solution to (3.84) that \( \tilde{u}_l = 0 \) for \( t \in (T_2 - 1, T_2) \).

Let \( \tilde{u}_l \) be defined as

\[
\tilde{u}_l = \begin{cases} 
\tilde{v}_l & \text{for } t \in (T_2, T_3), \\
0 & \text{for } t \leq T_2.
\end{cases}
\]

Then \( \tilde{u}_l \in H^{\lambda+1}((\infty, T_3) \times (B_2(0) \setminus B_1(0))) \) is a solution of (3.59) for \( t \in (\infty, T_3) \) and

\[
\tilde{u}_l(t, 1, \omega) \in H^{\lambda+1}((T_2, T_3) \times \partial B_1(0)), \quad \tilde{u}_l(t, 2, \omega) \in H^{\lambda+1}((T_2, T_3) \times \partial B_2(0)),
\]

\[
\partial_r \tilde{u}_l(t, 1, \omega) \in H^{\lambda}((T_2, T_3) \times \partial B_1(0)), \quad \partial_r \tilde{u}_l(t, 2, \omega) \in H^{\lambda}((T_2, T_3) \times \partial B_2(0)).
\]

With the help of Lemma 3.10, we know that \( \tilde{u}_l \) can be extended to \( t \in \mathbb{R} \) and satisfies (3.59) as well as the estimate (3.61) for any \( l > 0 \).

**Step 2.** We consider the case that the corresponding regularities given by (A4).

In this case, there exist smooth functions \( \tilde{e}_{i,\delta}(i = 1, 2, 4), \tilde{e}_{i,\delta}^j(i = 3, 5; j = 1, 2, 3), \tilde{e}_{6,\delta}^j(i, j = 1, 2, 3) \) and \( \tilde{d}_{i,\delta}^j(i = 1, 2; j = 1, 3), \tilde{d}_{i,\delta}^j(i = 1, 2; j = 1, 3, 5), \tilde{f}_i, \tilde{g}_{i,\delta}(i = 1, 2) \) for \( \delta > 0 \) such that

\[
\sum_{i = 1, 2, 4}^{3} \| \tilde{e}_{i,\delta} - \tilde{e}_i \|_{\lambda, \eta, 1} + \sum_{i = 3, 5}^{3} \| \tilde{e}_{i,\delta}^j - \tilde{e}_i^j \|_{\lambda, \eta, 1} + \sum_{i,j = 1}^{3} \| \tilde{e}_{6,\delta}^j - \tilde{e}_6^j \|_{\lambda, \eta, 1} + \sum_{i = 1, 3}^{3} \| \tilde{d}_{i,\delta}^j - \tilde{d}_i^j \|_{\lambda, \eta, 1}
\]

\[
+ \sum_{i = 1}^{3} \| \tilde{d}_{i,\delta}^j - \tilde{d}_i^j \|_{\lambda, \eta, 1} + \sum_{i = 1, 3}^{3} \| \tilde{f}_i - \tilde{f}_i \|_{\lambda, \eta, 1} + \| \tilde{g}_{i,\delta} - \tilde{g}_i \|_{\lambda, \eta, 1} + \sum_{i = 1}^{3} \| \tilde{g}_{i,\delta} - \tilde{g}_i \|_{\lambda, \eta, 1} \to 0 \quad \text{as } \delta \to 0,
\]

\[
\tag{3.85}
\]
moreover \( \tilde{f}_\delta \) and \( \tilde{g}_{i,\delta} \) \((i = 1, 2)\) have compact support with respect to the variable \( t \).

At this time, there exists a positive constant \( \delta t = \delta(l) \) such that when \( 0 < \delta < \delta t \), \((A_1)\) and \((A_3)-(A_4)\) are also satisfied with each function replaced by the corresponding smooth function and \( \lambda_0, \lambda_1, \lambda_2, a(\lambda) \) replaced by \( 2\lambda_0, 1/2, 2\lambda_2, 2a(\lambda) \) respectively. Meanwhile, \((A_2)\) will be replaced by

\[
(A'_2) \quad |\tilde{d}_{11}^\delta + \tilde{e}_{4,\delta}| + \sum_{i=1}^3 |\tilde{d}_{12}^\delta + \tilde{e}_{5,\delta}| + |\tilde{d}_{13}^\delta - \tilde{e}_{2,\delta}| \ll l \quad \text{for } (t, r, \omega) \in \mathbb{R} \times \{1\} \times \mathbb{S}^2.
\]

For \( l > 0 \) and \( 0 < \delta < \delta t \), we consider the following problem

\[
\begin{align*}
\tilde{L}^\delta_i(\tilde{u}_{i,\delta}) &= \chi_1(t)\tilde{f}_\delta & \text{in } & \mathfrak{D}_0, \\
\tilde{B}_{1i}^l(\tilde{u}_{i,\delta}) &= \chi_1(t)\tilde{g}_{i,\delta} & \text{on } & \mathfrak{B}_1, \\
\tilde{B}_{2i}^l(\tilde{u}_{i,\delta}) &= \chi_1(t)\tilde{g}_{2,\delta} & \text{on } & \mathfrak{B}_2,
\end{align*}
\]

(3.86)

where the operators \( \tilde{L}^\delta, \tilde{B}_{1i}^l, \tilde{B}_{2i}^l \) admit the analogous forms of (3.60), whose coefficients are replaced by the ones in (3.85) respectively.

By Step 1, \((A'_2)\) and Lemma 3.10, the problem (3.86) has a unique solution \( \tilde{u}_{i,\delta} \in H^{\eta}_{\lambda+1,\infty} \), which satisfies for any \( T \in \mathbb{R} \),

\[
||\tilde{u}_{i,\delta}\||_{\lambda+1,\eta,T} \leq C(l)(||\chi_1(t)\tilde{f}_\delta||_{\lambda',\eta,T} + \sum_{i=1}^2 \ll \chi_1(t)\tilde{g}_{i,\delta} \gg^{\eta}_{\lambda',\eta,1,T}).
\]

This, together with (3.85), yields that there exist a sequence \( \{\delta_i\}_{i \in \mathbb{N}} \subset (0, \delta_t) \) with \( \lim_{i \to \infty} \delta_i = 0 \) and a function \( \tilde{u}_i \in H^{\eta}_{\lambda+1,\infty} \) such that

\[
\tilde{u}_{i,\delta_i} \to \tilde{u}_i \quad \text{in } \quad H^{\lambda+1}(\mathfrak{D}_0) \cap H^{\lambda+1}(\mathfrak{B}_1) \cap H^{\lambda+1}(\mathfrak{B}_2), \quad \text{and} \quad \tilde{u}_{i,\delta_i} \to \tilde{u}_i \quad \text{in } \quad C^2(\mathbb{R} \times [1, 2] \times \mathbb{S}^2).
\]

This shows that \( \tilde{u}_i \) is a classical solution to the problem (3.59), which satisfies (3.61) and further admits the uniqueness for any \( l > 0 \).

**Step 3.** Since \( H^{\eta}_{\lambda+1,\infty} \subset H^{\eta}_{\lambda+1,1} \), then \( \tilde{u}_i \) also satisfies (3.77) for any \( l > 0 \). Thus, there exist a positive number sequence \( l_i(i \in \mathbb{N}) \) with \( \lim_{i \to \infty} l_i = 0 \) and a function \( \tilde{u} \in H^{\eta}_{\lambda+1,\infty} \) such that

\[
\tilde{u}_{i,\delta_i} \to \tilde{u} \quad \text{in } \quad H^{\eta}_{\lambda+1,\infty}, \quad \tilde{u}_i \to \tilde{u} \quad \text{in } \quad C^2(\mathbb{R} \times [1, 2] \times \mathbb{S}^2).
\]

Similar to the argument in Step 2, one can derive that \( \tilde{u} \) is a unique solution of problem (3.58) with \( l = 0 \), which satisfies (3.77). Thus, the proof of Lemma 3.12 is finished.

In the end of this section, we start to show Proposition 3.9.

**Proof of Proposition 3.9.** By Lemma 3.11-Lemma 3.12, the problem (3.53) has a solution \( u \in H^{\eta}_{\lambda+1,0} \), which satisfies for \( 0 \leq \lambda' \leq \lambda \)

\[
\begin{align*}
||u||_{\lambda+1,\eta,0} &\leq C \left( \frac{1}{\eta} \||\chi_1(t)f||_{\lambda',\eta,\infty} + (\chi_1(t)\tilde{g}_1)_{\lambda',\eta,\infty} + \ll \chi_1(t)\tilde{g}_2 \gg_{\lambda',\eta,\infty} \right) \\
&\leq C \left( \frac{1}{\eta} \||f||_{\lambda',\eta,0} + (g_1)_{\lambda',\eta,0} + \ll g_2 \gg_{\lambda',\eta,0} \right).
\end{align*}
\]

Namely, (3.54) is proved. The remainder is to show the uniqueness of the solution to the problem (3.53). Suppose that (3.53) has two solutions \( u_i \in H^{\eta}_{\lambda+1,0}(i = 1, 2) \), then \( \tilde{u} = u_1 - u_2 \) satisfies

\[
\begin{align*}
L(\tilde{u}) &= 0 & \text{in } & \mathfrak{D}_0, \\
B_i(\tilde{u}) &= 0 & \text{on } & \mathfrak{B}_i, \quad i = 1, 2.
\end{align*}
\]
Utilizing the notations in Lemma 3.10-Lemma 3.11, we can define \( \dot{u}_1 = Y(r)\dot{u} \) and \( \dot{u}_2 = (1 - Y(r))\dot{u} \) and replace \( V_i \) (\( i = 1, 2 \)) by \( u_i \) in the expressions of \( H_{ij} \) (\( j = 0, 1, 2 \)). Under the assumptions \((A_1)-(A_4)\), it follows from Lemma 3.10 that for \( \lambda' = 4 \) and \( T = 0 \)
\[
\|\dot{u}\|_{5,0,0} = 0,
\]
which implies \( \dot{u} = 0 \). Thus the proof of Proposition 3.9 is completed.

\( \square \)

3.3.3. Solvability of problem (3.10)-(3.12) and proof of Theorem 3.1

In order to solve the nonlinear problem (3.10)-(3.12), we will use the Newton’s iteration. First, we take the approximate solution \( \psi_0^{\kappa_0} \) (mentioned in (3.35)) with large \( \kappa_0 \) as the starting point of the iteration, and set \( \dot{\psi}_1 = \dot{\psi}_0 = \dot{\psi}_\sigma - \psi_\sigma^{\kappa_0} \), then we use the modified Newton’s iteration scheme (see (3.48)) to modify \( \dot{\psi} \) gradually to obtain the precise solution. It is noted that \( \psi_0^{\kappa_0} \) is an approximate solution with error \( e^{(\kappa_0+1)X} \) near \( X = \infty \), and the factor \( e^{(\kappa_0+1)X} \) will play a crucial role in canceling the singularity appeared in the weight of the norm \( H_{\lambda+1}^0 \). Due to (3.37) and (3.39), for the fixed large \( \kappa_0 \), one can select \( T^* \) suitably small such that for \( X \leq X_0 = \ln T^* \) and \( \lambda > 5 \), on has in (3.48)

\[
\frac{1}{\eta} \| \dot{f}(e^{(\kappa_0+1)X}F, 0) \|_{\lambda,0} + \| \dot{g}_1(e^{(\kappa_0+1)X}G_1^{\kappa_0}, 0, 0) \|_{\lambda,0} \\
+ \ll \dot{g}_2(e^{(\kappa_0+1)X}G_2^{\kappa_0}, 0, 0) \gg_{\lambda,2} \leq C \varepsilon \tag{3.87}
\]

and

\[
\|\dot{\psi}_n\|_{\lambda+1,1,n} \leq C \varepsilon, \quad i = -1, 0.
\]

Suppose that \( \|\dot{\psi}_{n-1}\|_{\lambda+1,1,n} \leq \varepsilon \) holds for \( i = 1, 2 \), then it follows from Proposition 3.9 and the smallness of \( \varepsilon \) that the problem (3.48) has a unique solution \( \dot{\psi}_n \in H_{\lambda+1}^0 \), which satisfies

\[
\|\dot{\psi}_n\|_{\lambda+1,1,n} \leq C \varepsilon, \quad n \in \mathbb{N}. \tag{3.88}
\]

To prove the convergence of \( \{\dot{\psi}_n\}_{n\in\mathbb{N}} \), we take

\[
\Delta_n \dot{\psi} = \dot{\psi}_{n+1} - \dot{\psi}_n,
\]

which satisfies

\[
\begin{cases}
\mathcal{L}_{n+1}(\Delta_n \dot{\psi}) = \dot{F}_{n+1}, & (X, R, \omega) \in (-\infty, X_0] \times (1, 2) \times \mathbb{S}^2, \\
\mathcal{B}_1^{n+1}(\Delta_n \dot{\psi}) = \dot{G}_1^{n+1} & \text{on} \ R = 1, \\
\mathcal{B}_2^{n+1}(\Delta_n \dot{\psi}) = (\mathcal{B}_2^n - \mathcal{B}_2^{n+1})(\dot{\psi}_n) + \dot{G}_2^{n+1} & \text{on} \ R = 2,
\end{cases}
\tag{3.89}
\]

with

\[
\begin{aligned}
\dot{F}_{n+1} &= (\mathcal{L}_n - \mathcal{L}_{n+1})(\dot{\psi}_n) + \dot{f}(e^{(\kappa_0+1)X}F^{\kappa_0}, \dot{\psi}, \nabla \dot{\psi}_n) - \dot{f}(e^{(\kappa_0+1)X}F^{\kappa_0}, \dot{\psi}_{n-1}, \nabla \dot{\psi}_{n-1}), \\
\dot{G}_1^{n+1} &= (\mathcal{B}_1^n - \mathcal{B}_1^{n+1})(\dot{\psi}_n) + \dot{g}_1(e^{(\kappa_0+1)X}G_1^{\kappa_0}, \dot{\psi}_n) - \dot{g}_1(e^{(\kappa_0+1)X}G_1^{\kappa_0}, \dot{\psi}_{n-1}), \\
\dot{G}_2^{n+1} &= (\mathcal{B}_2^n - \mathcal{B}_2^{n+1})(\dot{\psi}_n) + \dot{g}_2(e^{(\kappa_0+1)X}G_2^{\kappa_0}, \dot{\psi}_n, \nabla \dot{\psi}_n) - \dot{g}_2(e^{(\kappa_0+1)X}G_2^{\kappa_0}, \dot{\psi}_{n-1}, \nabla \dot{\psi}_{n-1}).
\end{aligned}
\]

By Proposition 3.9 and (3.37), it follows from (3.89) that

\[
\|\Delta_n \dot{\psi}\|_{\lambda,0} \leq C \left( \frac{1}{\eta} \|\dot{F}_{n+1}\|_{\lambda-1,0} + \|\dot{G}_1^{n+1}\|_{\lambda-1,0} \right) \\
\leq C \varepsilon \sum_{i=1}^{2} \|\Delta_n-i\dot{\psi}\|_{\lambda,0} \\
\leq \frac{1}{4} \sum_{i=1}^{2} \|\Delta_n-i\dot{\psi}\|_{\lambda,0},
\tag{3.90}
\]
for small $\varepsilon$.
Combining (3.90) with (3.88) yields that $\{\psi_n\}_{n \in \mathbb{N}}$ is convergent to a function $\psi$ in $H^0_{\lambda+1,X_0}$. Thus, $\psi \in H^0_{\lambda+1,X_0}$ is a solution to (3.38), and $\psi + \psi_{\alpha_0}$ is a solution to (3.10) with (3.11)-(3.12). Consequently, the proof of Theorem 3.1 is completed. \hfill \Box

§3.3.4. A proof of an elementary Proposition

In this part, we will establish an elementary Proposition used in (3.82).

For the second order equation in the domain $D = \mathbb{R} \times \mathbb{R}^3_+ = \{(t,x) : t \in \mathbb{R}, x_1, x_2 \in \mathbb{R}, x_3 > 0\}$

$$P(u) \equiv A_0(t,x)D_t^2u + 2 \sum_{i=1}^3 A_i(t,x)D_iD_tu - \sum_{i,j=1}^3 A_{ij}(t,x)D_{ij}^2u = w(t,x),$$

(3.91)

where $D_t = \frac{\partial}{\partial t}$, $D_i = \frac{\partial}{\partial x_i}$, $A_i, A_{ij} \in C^{2,\alpha}(\bar{D})$, and there exists two positive constants $\Lambda_1 < \Lambda_2$ such that

$$\begin{cases}
A_1 \leq A_0(t,x) \leq A_2, \quad \sum_{i=0}^3 \|A_i\|_{C^{2,\alpha}(\bar{D})} + \sum_{i,j=1}^3 \|A_{ij}\|_{C^{2,\alpha}(\bar{D})} \leq \Lambda_2
\end{cases}
$$

(3.92)

Denote by $x' = (x_1, x_2)$, and introduce the following notations for an integer $\tau \geq 0$, $s \in \mathbb{R}$, $\eta \geq 1$, and functions $u(t,x), v(t,x)$

$$\begin{align*}
|u|_{\tau, \eta}^2 &= \sum_{k+k'|=\tau} \int_{\mathbb{R}} e^{-2\eta t} \eta^{2(\tau-k-|k'|)} \|D_t^kD_x^{k'}u(t,\cdot)\|_{L^2(\mathbb{R}_+)}^2 dt, \\
\langle u \rangle_{s, \eta}^2 &= \int_{\mathbb{R}} e^{-2\eta t} \|u(t,x',0)\|_{L^2(\mathbb{R}^2)}^2 dt, \\
(u,v)_{\eta} &= \int_{D} e^{-2\eta t} u \cdot v dx dt, \\
\langle u, v \rangle_{\eta} &= \int_{\partial D} e^{-2\eta t} u(t,x',0) \cdot v(t,x',0) dx' dt,
\end{align*}
$$

where $\partial D = \{(t,x) : (t,x') \in \mathbb{R}^3, x_3 = 0\}$.

We now prove such a crucial conclusion.

**Lemma 3.13.** Under the assumptions (3.92), if

$$\text{supp} u \subset \{(t,x',x_3) : t \in \mathbb{R},|x'| < M, 0 \leq x_3 < M\}
$$

for some constant $M > 0$, and $|u|_{1, \eta} < +\infty$, then there exists a positive constant $C = C(\Lambda_1, \Lambda_2)$ such that

$$< D_t u >_{-1/2, \eta}^2 \leq C(\eta^{-1}|Pu|_{0, \eta}^2 + \eta |u|_{1, \eta}^2).
$$

To prove Lemma 3.13, we shall use some notations and ideas in micro-local analysis.

Set $X = (t, x'), Y = (s, y'), \Xi = (\tau, \xi')$ with $X, Y, \Xi \in \mathbb{R} \times \mathbb{R}^2$ and $\lambda_\eta(\Xi) = (|\Xi|^2 + \eta^2)^{1/2}$ with $\eta \geq 1$. The symbol class $S^\delta_\eta$ is defined as

$$S^\delta_\eta = \{\theta(x, \Xi, \Psi, \eta) \in C^\infty(\mathbb{R}^1) : |D_X^kD^k_{\Xi}D^\theta_{\Psi}| \leq C_{k_1,k_2,k_3,\lambda_\eta(\Xi)^{\delta-|k_1|}}, \quad \forall k_1,k_2,k_3 \in (\mathbb{N} \cup \{0\})^3\}.$$
The corresponding weighted pseudo-differential operator $\Theta \in \Psi^\delta_\eta$ with symbol $\theta \in S^\delta_\eta$ is defined as

$$\Theta(X, DX, \eta, \eta)u = (2\pi)^{-3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{i(X \cdot \Xi - Y \cdot \Xi)} \theta(X, \Xi, \eta, \eta)e^{(t-s)}u(\eta)d\Xi d\eta,$$

for any $u \in S^\delta_\eta(\mathbb{R}^3) = \{u \in D'(\mathbb{R}^3) : e^{-t\eta}u \in S(\mathbb{R}^3)\}$.

**Lemma 3.14.** Let $\Theta$ be the weighted pseudo-differential operator with the symbol $\theta$.

(i) Let $\sigma > 0$, $\theta \in S^{-\sigma}_\eta$, and put

$$N^0_\sigma(\theta) = \sup_{X, \Xi, \eta, |\beta| \leq n+1} |D^\beta_\Xi \theta(X, \Xi, \eta)| \lambda_\eta(\Xi)^{\sigma + |\beta|},$$

then

$$|\Theta u|_{0, \eta} \leq C(\sigma) N^0_\sigma(\theta)|u|_{0, \eta}.$$

(ii) For $\theta \in S^0_\eta$ and $\beta \in (0, 1)$, put

$$N_\beta(\theta) = \sup_{X, \Xi, \eta, |\beta| \leq n+1} \left\{ \begin{array}{l} |D^\beta_\Xi \theta(X, \Xi, \eta)| \lambda_\eta(\Xi)^{|k|} \\ |D^\beta_\Xi \theta(X, \Xi, \eta) - D^\beta_\Xi \theta(X, \Xi, \eta)| \lambda_\eta(\Xi)^{|k|} \\ |D^\beta_\Xi \theta(X, \Xi, \eta) - D^\beta_\Xi \theta(X, \Xi, \eta)| \lambda_\eta(\Xi)^{|k|} \end{array} \right\},$$

then

$$|\Theta u|_{0, \eta} \leq C(\beta) N_\beta(\theta)|u|_{0, \eta}.$$

**Proof.** One can see page 59 in [28], we omit the proof here. $\Box$

**Lemma 3.15.** Assume $a(X) \in C^{2,\alpha}(\mathbb{R}^3)$ and the function $\phi_0(\Xi, \eta) \in S^\delta_\eta$ with $0 < \delta \leq 1$. Let $\Phi_0$ be the weighted pseudo-differential operator with symbol $\phi_0$ and put the commutator $[a, \Phi_0] = a(\Phi_0) - \Phi_0(a)$.

Then

$$|[a, \Phi_0]|_{0, \eta} \leq C(\delta, \|a\|_{C^{2,\alpha}})|u|_{0, \eta}.$$

**Proof.** Denote by $t = x_0$ and $\tau = \xi_0$, then we have

$$[a, \Phi_0]u = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{i(X \cdot \Xi - Y \cdot \Xi)} \phi_0(\Xi, \eta)(a(X) - a(Y))e^{(t-s)}u(\eta)d\Xi d\eta$$

$$= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{i(X \cdot \Xi - Y \cdot \Xi)} \phi_0(\Xi, \eta)(X - Y) \cdot \left( \int_0^1 \nabla a(Y + \kappa(X - Y)) d\kappa \right) e^{(t-s)}u(\eta)d\Xi d\eta$$

$$= -i \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{i(X \cdot \Xi - Y \cdot \Xi)} \nabla_x \phi_0(\Xi, \eta) \cdot \left( \int_0^1 \nabla a(Y + \kappa(X - Y)) d\kappa \right) e^{(t-s)}u(\eta)d\Xi d\eta$$

$$- \int_{\mathbb{I}_1} \int_{\mathbb{R}^3} e^{i(X \cdot \Xi - Y \cdot \Xi)} \phi_0(\Xi, \eta) \cdot \nabla a(Y) e^{(t-s)}u(\eta)d\Xi d\eta$$

$$- \int_{\mathbb{I}_2} \int_{\mathbb{R}^3} e^{i(X \cdot \Xi - Y \cdot \Xi)} \phi_0(\Xi, \eta) e^{(t-s)}u(\eta)d\Xi d\eta$$

(3.93)
with
\[ \tilde{\phi}_0(\Xi, \eta) = \sum_{l,k=0}^2 \partial^2_{\xi_l \xi_k} \phi_0(\Xi, \eta) \left( \int_0^1 \int_0^1 \partial^2_{y_l y_k} a(Y + \kappa_1 \kappa_2(X - Y)) d\kappa_1 d\kappa_2 \right). \]  \hspace{1cm} (3.94)

Since \( \nabla \xi \phi_0(\Xi, \eta) \in S^d_{\eta} \) and is independent of the variables \( X \) and \( Y \), then by Lemma 3.14 and \( a \in C^2 \), one can arrive at
\[ |I_1| \leq C(\delta) \| \nabla a \cdot u \|_{l,0,\eta} \leq C(\delta) \| a \|_{C^1} |u|_{l,0,\eta}. \]  \hspace{1cm} (3.95)

Due to \( a \in C^{2+\alpha} \), then by Littlewood-Paley decomposition theory, we have
\[ \int_0^1 \int_0^1 \partial^2_{y_l y_k} a(Y + \kappa_1 \kappa_2(X - Y)) d\kappa_1 d\kappa_2 = \sum_{j=-1}^\infty a_{lkj}(X, Y), \quad l, k = 0, 1, 2 \]
with
\[ a_{lkj}(X, Y) \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3), \quad |a_{lkj}| \leq C(\| \nabla^2 a \|_{C^\infty}) 2^{-j \alpha}. \]  \hspace{1cm} (3.96)

Now, with (3.94) and (3.96), \( \tilde{\phi}_0(\Xi, \eta) \) can be rewritten as
\[ \tilde{\phi}_0(\Xi, \eta) = \sum_{j=-1}^\infty \sum_{l,k=0}^2 \tilde{\phi}_{lkj}(\Xi, \eta) \]
with
\[ \tilde{\phi}_{lkj}(\Xi, \eta) = \partial^2_{\xi_l \xi_k} \phi_0(\Xi, \eta) a_{lkj}(X, Y) \in S^d_{\eta}, \quad j = -1, \cdots, \infty; \quad l, k = 0, 1, 2. \]

By Lemma 3.14 and (3.96), for \( j = -1, \cdots, \infty; \quad l, k = 0, 1, 2 \), we have
\[ \left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{i(\Xi - Y) \cdot \xi} \phi_{lkj}(\Xi, \eta) e^{(t-s)\eta} u(Y) dY \right|_{l,0,\eta} \leq C(\delta, \| \nabla^2 a \|_{C^\infty}) 2^{-j \alpha} |u|_{l,0,\eta}. \]

Combining this with (3.94) and (3.97) yields
\[ |I_2|_{l,0,\eta} \leq C(\delta, \| \nabla^2 a \|_{C^\infty}) |u|_{l,0,\eta}. \]  \hspace{1cm} (3.98)

Thus, it follows from (3.93), (3.95) and (3.98) that
\[ |[a, \Phi_0] u|_{l,0,\eta} \leq C(\delta, \| a \|_{C^{2+\alpha}}) |u|_{l,0,\eta}, \]
and then the proof of Lemma 3.15 is completed.

With respect to the equation (3.91), we can easily establish the following a priori estimate.

**Lemma 3.16.** Under the assumptions in (3.92) and Lemma 3.13, if \( |u|_{l,0,\eta} < u >_{l,0,\eta} < +\infty \), then
\[ \langle D_\xi u \rangle_{l,0,\eta} \leq C(\frac{1}{\eta} |P(u)|_{l,0,\eta} + \sum_{j=1}^2 \langle D_j u \rangle_{l,0,\eta}). \]

**Proof.** Since the proof procedure is just only a routine \( L^2 \)-energy estimate and it can be done by following the proof procedure of Lemma 3.11 step by step, then we omit it here.

Next we start to show Lemma 3.13.
Proof of Lemma 3.13. Define $\phi_1(\Xi, \eta) \in C^\infty(\mathbb{R}^4)$ with $0 \leq \phi_1 \leq 1$ and

$$
\phi_1(\Xi, \eta) = \begin{cases} 
1 & \text{for } |\Xi|^2 + \eta^2 \geq 1, \\
0 & \text{for } |\Xi|^2 + \eta^2 \leq \frac{1}{2}.
\end{cases}
$$

For a small number $\nu \in (0, 1)$ which will be determined later on, we set $\phi_2(\Xi, \eta) \in C^\infty(\mathbb{R}^4 \setminus \{0\})$ so that $0 \leq \phi_2 \leq 1$ and

$$
\phi_2(\Xi, \eta) = \begin{cases} 
1 & \text{for } \nu(\tau^2 + \eta^2) \geq 2|\xi'|^2, \\
0 & \text{for } \nu(\tau^2 + \eta^2) \leq |\xi'|^2.
\end{cases}
\quad (3.99)
$$

Let $\Phi_1$, $\Phi_2$ and $\Phi_3$ be the weighted pseudo-differential operators with symbols $1 - \phi_1$, $\phi_1 \phi_2$ and $\phi_1(1 - \phi_2)$ respectively. A direct computation shows that

$$
\Phi_1 \in \Psi^{-\infty}_\eta, \quad \Phi_2 \in \Psi^0_\eta, \quad \Phi_3 \in \Psi^0_\eta.
\quad (3.100)
$$

Choose $\chi_i(x) \in C^\infty(\mathbb{R}^3)(i = 1, 2)$ so that

$$
\chi_1(x) = \begin{cases} 
1 & \text{for } |x| \leq 2M, \\
0 & \text{for } |x| \geq \frac{9}{4}M,
\end{cases} \quad \chi_2(x) = \begin{cases} 
1 & \text{for } |x| \leq \frac{5}{2}M, \\
0 & \text{for } |x| \geq 3M.
\end{cases}
$$

In this case, by the assumption in Lemma 3.12, one has $u = \chi_1 u$ and $u = \sum_{i=1}^3 \Phi_i u$.

At first, we estimate $< D_t (\chi_1 \Phi_2 u) >_{-1/2, \eta}$.

It follows from (3.91) and a direct computation that

$$
P(\chi_1 \Phi_2 u) = w_1 \quad \text{in } D
\quad (3.101)
$$

with

$$
w_1 = \chi_1 \Phi_2(\chi_2 P(u)) - \{ (D_3^2 \chi_1)\Phi_2 u + 2(D_3 \chi_1)\Phi_2(D_3 u) \} \\
- \sum_{i=1}^2 (A_{i} + A_{i3}) \{ (D_{3i} \chi_1)\Phi_2 u + (D_3 \chi_1)D_i(\Phi_2 u) + (D_i \chi_1)\Phi_2(D_3 u) \} \\
- \sum_{i,j=1}^2 A_{ij} \{ (D_i \chi_1)D_j \Phi_2 u + (D_j \chi_1)D_i \Phi_2 u + (D_{ij} \chi_1)\Phi_2 u \} \\
+ \sum_{i=1}^3 A_i (D_i \chi_1)D_t(\Phi_2 u) + \chi_1 \left\{ \sum_{i=1}^2 [\Phi_2 D_{i}, \chi_2 A_{3i}] D_3 u \right\} \\
+ \sum_{i,j=1}^2 [\Phi_2 D_{i}, \chi_2 A_{ij}] D_j u - \sum_{i=1}^3 [\Phi_2 D_{i}, \chi_2 A_{i}] D_i u - [\Phi_2 D_{i}, \chi_2 A_0] D_i u \\
+ \chi_1 \Phi_2 \left\{ - \sum_{i=1}^2 (D_i(\chi_2 A_{3i})) D_3 u - \sum_{i,j=1}^2 (D_i(\chi_2 A_{ij})) D_j u \\
+ \sum_{i=1}^3 (D_i(\chi_2 A_{i})) D_i u + (D_i(\chi_2 A_0)) D_i u \right\}.
$$
Here $[\cdot, \cdot]$ means the commutator.

Then utilizing the results in Lemma 3.14-Lemma 3.15 and (3.100) yields

$$|w_1|_{0, \eta} \leq C(\Lambda_1, \Lambda_2)(|P(u)|_{0, \eta} + |u|_{1, \eta}). \quad (3.102)$$

In addition, by Lemma 3.16 and (3.101)-(3.102), we have

$$< D_t(\chi_1 \Phi_2 u) >_{0, \eta}^2 \leq C(\Lambda_1, \Lambda_2)(\sum_{i=1}^2 < D_i(\chi_1 \Phi_2 u) >_{0, \eta}^2 + \eta |u|_{1, \eta}^2 + \eta^{-1} |P(u)|_{0, \eta}^2). \quad (3.103)$$

On the other hand, it follows from (3.99) that

$$\sum_{i=1}^2 < D_i(\chi_1 \Phi_2 u) >_{0, \eta}^2 \leq C\nu < D_i(\chi \Phi_2 u) >_{0, \eta}^2 + \eta^2 |u|_{1, \eta}^2). \quad (3.104)$$

Substituting this into (3.103) and taking $\nu = \frac{1}{2CC(\Lambda_1, \Lambda_2)\eta}$ yields

$$< D_t(\chi_1 \Phi_2 u) >_{0, \eta}^2 \leq C(\Lambda_1, \Lambda_2)(\eta |u|_{1, \eta}^2 + \eta^{-1} |P(u)|_{0, \eta}^2). \quad (3.105)$$

Next, we estimate $\langle \chi_1 (D')^{-1/2} \Phi_3 u \rangle_{0, \eta}$.

Set

$$\phi_4(\Xi, \eta) = \langle |\Xi|^2 + \eta^2 \rangle^{1/4}(1 + |\xi'|^2)^{-1/4} \phi_t(\Xi, \eta)(1 - \phi_2(\Xi, \eta)),$$

and let $\Phi_4$ be the weighted pseudo-differential operator with symbol $\phi_4$. By (3.99), one has

$$\phi_4 \in S^0_\eta, \quad \langle D' \rangle^{-1/2} \Phi_3 u = \Lambda^{-1/2} \Phi_4 u \quad (3.106)$$

with $\Lambda^{-1/2}$ be the weighted pseudo-differential operator with symbol $(|\Xi|^2 + \eta^2)^{-1/4}$.

In addition, by (3.103), one has

$$P(\chi_1 (D')^{-1/2} \Phi_3 u) = w_2 \text{ in } D,$$

where $w_2$ is a function given by replacing $\Phi_2$ by $\Lambda^{-1/2} \Phi_4$ in the expression of $w_1$.

Applying for Lemma 3.14-Lemma 3.15 with (3.100) and (3.105) to $w_2$ yields

$$|w_2|_{0, \eta} \leq C(\Lambda_1, \Lambda_2)(|P(u)|_{0, \eta} + |u|_{1, \eta}). \quad (3.106)$$

It follows from Lemma 3.16 and (3.106) that

$$\langle D_t(\chi_1 (D')^{-1/2} \Phi_3 u) \rangle_{0, \eta}^2 \leq C(\Lambda_1, \Lambda_2)(\sum_{i=1}^2 \langle D_i(\chi_1 (D')^{-1/2} \Phi_3 u) \rangle_{1, \eta}^2 + \eta |u|_{1, \eta}^2 + \eta^{-1} |P(u)|_{0, \eta}^2). \quad (3.107)$$

By (3.99), one has

$$\sum_{i=1}^2 \langle D_i(\chi_1 (D')^{-1/2} \Phi_3 u) \rangle_{0, \eta}^2 \leq C(u)^2_{1/2, \eta} \leq C|u|_{1, \eta}^2. \quad (3.108)$$
Due to \( \| [x_1, (\mathcal{D}')^{-1/2}] v \|_{L^2(\mathbb{R}^2)} \leq C \| (\mathcal{D}')^{-1/2} v \|_{L^2(\mathbb{R}^2)} \), then we arrive at
\[
\langle D_t (x_1 (\mathcal{D}')^{-1/2} \Phi_3 u) \rangle_{0, \eta} \geq \langle (\mathcal{D}')^{-1/2} (D_t (x_1 \Phi_3 u)) \rangle_{0, \eta} - \langle \| x_1, (\mathcal{D}')^{-1/2}] (D_t (\Phi_3 u)) \rangle_{0, \eta} \\
\geq (D_t (x_1 \Phi_3 u))_{-1/2, \eta} - C \langle (\mathcal{D}')^{-1/2} (D_t (\Phi_3 u)) \rangle_{0, \eta}.
\]
Noticing that \( |x'|^2 \leq \nu (\tau^2 + \eta^2) \) holds on \( \text{Supp} \phi_1 (1 - \phi_2) \), then one obtains for \( \nu = \frac{1}{2CC(A_1, A_2)\eta} \)
\[
\langle (\mathcal{D}')^{-1/2} (D_t (\Phi_3 u)) \rangle_{0, \eta} \leq C(A_1, A_2)\eta \| u \|_{1, \eta}^2.
\]
Substituting these estimates into (3.107) derives
\[
\langle (\mathcal{D}')^{-1/2} (D_t (\Phi_3 u)) \rangle_{0, \eta}^2 \leq C(A_1, A_2)\eta \| u \|_{1, \eta}^2.
\]
Because of \( \text{supp} (1 - \phi_1) \subset \{(\Xi, \eta) \in \mathbb{R}^4 : |\Xi|^2 + \eta^2 \leq 1 \} \) and (3.100), then
\[
\langle D_t (x_1 \Phi_1 u) \rangle_{-1/2, \eta}^2 \leq \langle u \rangle_{1/2, \eta}^2 \leq C \| u \|_{1, \eta}^2.
\]
It is noted that \( D_t u = \sum_{i=1}^{3} D_t (x_1 \Phi_i u) \), then in terms of (3.104) and (3.109)-(3.110), we have
\[
\langle D_t u \rangle_{-1/2, \eta}^2 \leq C(A_1, A_2) \| u \|_{1, \eta}^2.
\]
which means that the proof of Lemma 3.13 is completed. \( \square \)

**Proposition 3.17 (An elementary Proposition)** If \( \Omega \subset \mathbb{R}^3 \) is a smooth bounded domain with compact boundary, with respect to the following equation in \( \mathcal{D} = \mathbb{R} \times \Omega \)
\[
\hat{P}(u) = \hat{A}_0(t, x)D_t^2 u + 2 \sum_{i=1}^{3} \hat{A}_i(t, x)D_i D_t u - \sum_{i=1}^{3} \hat{A}_{ij}(t, x)D_{ij} u = \hat{w}(t, x),
\]
where \( \hat{A}_i (0 \leq i \leq 3) \) and \( \hat{A}_{ij} (1 \leq i, j \leq 3) \) satisfy the assumption (3.92) in \( \mathcal{D} \). Then we have the following a priori estimate
\[
\int_{\mathbb{R}} e^{-2\eta t} \| D_t u(t, \cdot) \|_{H^{-1/2}(\partial \Omega)}^2 \, dt \\
\leq C(A_1, A_2) \left( \frac{1}{\eta} \int_{\mathbb{R}} e^{-2\eta t} \| \hat{P}(u) \|_{L^2(\Omega)}^2 \, dt + \eta \sum_{k+k'=1} \int_{\mathbb{R}} e^{-2\eta t} \| D^k D^{k'} u \|_{L^2(\Omega)}^2 \, dt \right). \tag{3.111}
\]

**Proof.** Since \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^3 \) with compact boundary, (3.111) follows from the skills of partition of unity and local flattening of the boundary of \( \Omega \), and Lemma 3.13, then Proposition 3.17 is proved. \( \square \)

§4. Another reformulation of (1.6) with (1.7)-(1.9) and some preliminaries

In this section, we reformulate the problem (1.6) with (1.7)-(1.9) in another form, which will be required to establish the global weighted energy estimate and further prove the global existence in subsequent sections.
As in [17], we denote certain partial Klainerman’s vector fields by
\[ Z = \{ Z_j : 0 \leq j \leq 2n - 3 \}, \tag{4.1} \]
where \( n = 2,3, Z_0 = t \partial_t + r \partial_r, \) and \( Z_i \) \((1 \leq i \leq 2n - 3)\) is given in (1.7).

In addition, for notational convenience, for \( \nu > 0, \) \( t \in \mathbb{N} \) and \( m \in \mathbb{Z} \), we define the following space
\[ O_m^t(\nu) = \{ u(t,r,\omega) \in C^t(\Omega_+) : |\nabla^{\nu_1}_t Z_1^{\nu_1} Z_2^{\nu_2} Z_3^{\nu_3} u| \leq Ct(1+t)^{m-\nu} \text{ with } |u_0| + \sum_{i=1}^{3} t_i = l \}. \tag{4.2} \]

Under the coordinate transformation (1.5), the equation (1.6) becomes
\[ \partial_t^2 \Phi^+ + 2 \sum_{k=1}^{n} \partial_k \Phi^+ \partial_k^2 \Phi^+ + \sum_{i,j=1}^{n} \partial_i \Phi^+ \partial_j \Phi^+ \partial_i^2 \Phi^+ - c^2(\rho^+) \left( \partial_r^2 + \frac{n-1}{r} \partial_r + \frac{1}{r^2} \Delta_{S^{n-1}} \right) \Phi^+ = 0 \quad \text{in } \Omega_+, \tag{4.3} \]
where \( \Delta_{S^{n-1}} \) is the Laplace-Beltrami operator on \((n-1)-\text{dimensional spherical surface} \). Namely,
\[ \Delta_{S^1} = Z_1^2, \quad \Delta_{S^2} = \sum_{i=1}^{3} Z_i^2. \]

Since the back ground solution \( \hat{\Phi}(t,x) \) given in Remark 2.1 does not satisfy the boundary condition (1.7), we have to modify it so that the new resulting background solution \( \Phi_a(t,x) \) in \( \Omega_+ \) satisfies (1.7) as well as other required properties. This is achieved by choosing a smooth function \( f_a = f_a(t,r,\omega) \) in \( \Omega_+ \) with \( f_a \in O_{2n}^0(\varepsilon) \) and setting \( \Phi_a(t,r,\omega) = (1 + f_a(t,r,\omega))\hat{\Phi}(t,x) \), the details can be found in Appendix B.

Let \((\Phi_+(t,r,\omega), \zeta(t,\omega))\) be the solution of the problem (4.3) with (1.7)-(1.10) and \((\hat{\phi}(t,r,\omega), \zeta(t,\omega))\) be the perturbation of the modified background solution, that is, \( \hat{\phi} = \Phi^+ - \Phi_a, \zeta = \frac{\zeta}{t} - s_0 \). We now start to reformulate the nonlinear problem (4.3) with (1.7)-(1.9). For notational convenience, from now on, we neglect all the superscripts “+”.

By a direct computation, (4.3) can be reduced to
\[ \mathcal{L} \hat{\phi} + \mathcal{P} \hat{\phi} = R_0(t,x) \quad \text{in } \Omega_+, \tag{4.4} \]
where the operators \( \mathcal{L} \) and \( \mathcal{P} \) have the forms
\[ \mathcal{L} = \partial_t^2 + 2P_1(s) \partial_t^2 + P_2(s) \partial_t^2 - \frac{1}{r^2} P_3(s) \Delta_{S^{n-1}} + \frac{1}{r} P_4(s) \partial_t + \frac{1}{r} P_5(s) \partial_r, \]
\[ \mathcal{P} = \sum_{i,j=1}^{n} f_{ij}(t,x) \partial_{ij}^2 + \sum_{i=1}^{n} f_{ii}(t,x) \partial_{ix}^2, \tag{4.5} \]
with
\[ \begin{align*}
P_1(s) &= \hat{u}(s), \\
P_2(s) &= \hat{u}^2(s) - c^2(\hat{\rho}(s)), \\
P_3(s) &= c^2(\hat{\rho}(s)), \\
P_4(s) &= (\gamma - 1) \left( (n-1)\hat{u}(s) + s\hat{u}'(s) \right), \\
P_5(s) &= (n-1)(\gamma - 1)\hat{u}^2(s) - (n-1)c^2(\hat{\rho}(s)) - 2s^2\hat{u}'(s) + (\gamma + 1)s\hat{u}(s)\hat{u}'(s) \\
\end{align*} \tag{4.6} \]
and
\[
\begin{aligned}
f_{ij}(t, x) &= \left( \partial_t \dot{\Phi} \partial_j (\dot{\Phi} f_a + \dot{\phi}) + \partial_j \dot{\Phi} \partial_i (\dot{\Phi} f_a + \dot{\phi}) + \partial_i (\dot{\Phi} f_a + \dot{\phi}) \partial_j (\dot{\Phi} + \dot{\phi}) \right) \\
+ (\gamma - 1) \delta_{ij} \left( \partial_i (\dot{\phi} + \dot{\Phi} f_a) + \sum_{k=1}^{n} (\partial_k (\Phi a + \frac{1}{2} \dot{\phi}) \partial k \dot{\phi} + \partial_k (\dot{\Phi} + \frac{1}{2} \dot{\Phi} f_a) \partial k (\dot{\Phi} f_a)) \right),
\end{aligned}
\tag{4.7}
\]

and
\[
\begin{aligned}
R_0(t, x) &= -2 \sum_{k=1}^{n} \partial^2_{ik} (\dot{\Phi} f_a) \partial k \dot{\phi} - \sum_{i=1}^{n} \left( \partial^2_{ij} (\dot{\Phi} f_a) \partial j \dot{\phi} + \partial_j (\dot{\Phi} f_a) \partial_i \dot{\phi} \right) \\
&\quad + \partial^2_{ij} \dot{\phi} \partial i \dot{\phi} + \partial^2_{ij} (\dot{\Phi} f_a) (\partial_i \partial j \dot{\phi} + \partial_j \partial_i \dot{\phi}) \\
&\quad - (\gamma - 1) \Delta \dot{\Phi} \left( \partial_i (\dot{\Phi} f_a) + \sum_{k=1}^{n} (\partial_k (\dot{\Phi} f_a + \frac{1}{2} \dot{\phi}) \partial k \dot{\phi} + \partial_k (\dot{\Phi} + \frac{1}{2} \dot{\Phi} f_a) \partial k (\dot{\Phi} f_a)) \right) \\
&\quad - (\gamma - 1) \Delta (\dot{\Phi} f_a) \left( \partial_i (\dot{\phi} + \dot{\Phi} f_a) + \sum_{k=1}^{n} (\partial_k (\Phi a + \frac{1}{2} \dot{\phi}) \partial k \dot{\phi} + \partial_k (\dot{\Phi} + \frac{1}{2} \dot{\Phi} f_a) \partial k (\dot{\Phi} f_a)) \right) \\
&\quad - \left( \partial^2_{ij} \dot{\Phi} f_a + 2 \sum_{k=1}^{n} \partial_{i} \Phi a \partial_{j} \Phi a + \sum_{i,j=1}^{n} \partial_i \Phi a \partial_j \Phi a \partial^2_{ij} \Phi a - c^2 (\dot{\rho}) \Delta \Phi a \right).
\end{aligned}
\tag{4.8}
\]

On the fixed boundary \( r = \sigma(t, \omega) \), we have
\[
B_\sigma \dot{\phi} = 0. \tag{4.9}
\]

On the free boundary \( r = \zeta(t, \omega) \), by the continuity condition (1.9), one can rewrite Rankine-Hugoniot condition (1.8) as
\[
B_1 \partial_t \phi + B_2 \partial_x \phi + B_3 \xi = \kappa(\xi, \nabla \phi) + R_1(t, x) \quad \text{on} \quad r = \zeta(t, \omega), \tag{4.10}
\]
where
\[
\begin{aligned}
B_1 &= 2 \dot{\rho}(s_0) \dot{u}(s_0) - \frac{1}{c^2(\dot{\rho}(s_0))} \dot{\rho}(s_0) \dot{u}(s_0) \left( \frac{1}{2} \dot{u}^2(s_0) - h(\dot{\rho})(s_0) + h(\rho_0) \right), \\
B_2 &= \dot{\rho}(s_0) - \rho_0 - \frac{1}{c^2(\dot{\rho}(s_0))} \dot{\rho}(s_0) \left( \frac{1}{2} \dot{u}^2(s_0) - h(\dot{\rho})(s_0) + h(\rho_0) \right), \\
B_3 &= 2 \dot{\rho}(s_0) \dot{u}(s_0) \dot{u}'(s_0) + \dot{\rho}'(s_0) \left( \frac{1}{2} \dot{u}^2(s_0) - h(\dot{\rho})(s_0) + h(\rho_0) \right) \\
&\quad - (\dot{\sigma}(s_0) - \rho_0) \left( \dot{u}(s_0) \dot{u}'(s_0) + \frac{c^2(\dot{\rho}(s_0))}{\dot{\rho}(s_0)} \dot{\rho}'(s_0) \right),
\end{aligned}
\]
the generic function \( \kappa(\xi, \nabla \phi) \) is used to denote the quantity dominated by \( C(b_0)|\xi, \nabla \phi|^2 \).

By Lemma 4.2 below, we know \( B_1 \neq 0 \) in (4.10) for large \( b_0 \). Thus, the equation (4.10) can be rewritten as
\[
B_0 \dot{\phi} + \mu_2 \xi = \frac{1}{B_1} \kappa(\xi, \nabla \phi) + \frac{1}{B_1} R_1(t, x) \quad \text{on} \quad r = \chi(t, \omega), \tag{4.11}
\]
where
\[
B_0 \dot{\phi} = \partial_r \dot{\phi} + \mu_1 \partial_t \dot{\phi} \quad \text{with} \quad \mu_1 = \frac{B_2}{B_1} \quad \text{and} \quad \mu_2 = \frac{B_3}{B_1}.
\]
Besides, (1.9) implies that
\[ \dot{\psi}(t, \chi(t, \omega)) = \Phi(t, \chi(t, \omega), \omega) - \Phi_a(t, s_0 t, \omega) - (\Phi_a(t, \chi(t, \omega), \omega) - \Phi_a(t, s_0 t, \omega)) \]
\[ = - (\Phi_a(t, \chi(t, \omega), \omega) - \Phi_a(t, s_0 t, \omega)) \]
\[ = \mu_3(t, x)\xi(t, \omega) \quad (4.12) \]
with \( \mu_3(t, x) = -\int_0^1 \dot{u}(s_0 + \tau \xi(t, \omega))(1 + f_a(t, \chi(t, \omega), \omega))d\tau < 0 \)

On the other hand, by the local existence and stability result established in §3, we only need to solve problem (4.4) in the domain \( \{(t, r, \omega): t \geq 0, \sigma(t, \omega) \leq r \leq \chi(t, \omega), \omega \in \mathbb{S}^{n-1}\} \) with the boundary conditions (4.9) and (4.11)-(4.12) and small initial data \( \dot{\psi}(t, x)|_{t=1}, \partial_t \dot{\psi}(t, x)|_{t=1}, \) and \( \xi(t, \omega)|_{t=1}. \) Here, the smallness of initial data means that
\[ \sum_{l \leq k_0} |\nabla^l \dot{\psi}(1, x)| + \sum_{l \leq k_0} |\nabla^l \xi(1, \omega)| \leq C\varepsilon, \quad (4.13) \]
where \( k_0 \in \mathbb{N}, k_0 \geq 2n + 3. \)

Under the preparations above, Theorem 1.1 is actually equivalent to

**Theorem 4.1.** For \( n = 2, 3, \) if \( \sigma(t, \omega) = tb(t, \omega) \) satisfies the assumptions in Theorem 1.1, then the problem (4.4) with (4.9) and (4.11)-(4.13) has a unique global \( C^\infty \) shock solution \( (\dot{\psi}, \xi). \) Moreover, \( (\nabla \dot{\psi}, \xi) \) approaches zero as \( t \) tends to infinity with rate \( (1 + t)^{-m_0} \) for any positive number \( m_0 < \frac{5}{4} - \frac{1}{4} \sqrt{\frac{\gamma + 1}{2}} \) if \( n = 2 \) and \( m_0 < \frac{3}{2} - \frac{1}{4} \sqrt{\frac{\gamma + 7}{2}} \) if \( n = 3 \) respectively.

For the later uses, we now list some elementary estimates on the coefficients in (4.6) and (4.10)-(4.11) and the non-homogenous terms \( \mathcal{R}_i(i = 0, 1) \) in (4.4) and (4.11). Since these estimates come from a direct but tedious computation by Lemma 2.1 in §2 and Lemma B.1 in Appendix B, then we omit the details here.

**Lemma 4.2.** For \( \mathcal{R}_i(i = 0, 1) \) given in (4.4) and (4.11) respectively, if \( \nabla \dot{\psi} \in O^l(\varepsilon) \) and \( \xi \in O^l(\varepsilon) \) for any \( l \in \mathbb{N}, \) then
\[ R_0 \in O^l(\varepsilon^2), \quad R_1 \in O^l(\varepsilon). \]

With respect to the coefficients of operator \( \mathcal{L} \) in (4.5), we have

**Lemma 4.3.** If \( b_0 > 0 \) is large enough, \( 1 < \gamma < 3 \) and \( b_0 \leq s \leq s_0, \) then
\[ P_1(s) = b_0(1 + O(b_0^{-\frac{\gamma}{\gamma - 1}}) + O(b_0^{-2})), \]
\[ P_2(s) = \frac{3}{2} - \frac{\gamma}{2} b_0(1 + O(b_0^{-\frac{\gamma}{\gamma - 1}}) + O(b_0^{-2})), \]
\[ P_3(s) = \frac{\gamma - 1}{2} b_0(1 + O(b_0^{-\frac{\gamma}{\gamma - 1}}) + O(b_0^{-2})), \]
\[ P_4(s) = b_0(1 + O(b_0^{-\frac{\gamma}{\gamma - 1}}) + O(b_0^{-2})), \]
\[ P_5(s) = -\frac{\gamma - 1}{2} (n - 1) b_0(1 + O(b_0^{-\frac{\gamma}{\gamma - 1}}) + O(b_0^{-2})), \]
\[ P_6(s) = -2(n - 1)(1 + O(b_0^{-\frac{\gamma}{\gamma - 1}}) + O(b_0^{-2})), \]
\[ P_7(s) = -2(n - 1)(1 + O(b_0^{-\frac{\gamma}{\gamma - 1}}) + O(b_0^{-2})), \]
\[ P_8(s) = b_0(1 + O(b_0^{-\frac{\gamma}{\gamma - 1}}) + O(b_0^{-2})). \]
In addition, $B_i(i = 1, 2, 3)$ in (4.10) and $\mu_j(j = 1, 2)$ in (4.11) admit the following estimates:

**Lemma 4.4.** For large $b_0$, we have

$$B_1 = 2\left(\frac{\gamma - 1}{2A^\gamma}\right)\frac{\gamma + 1}{b_0\gamma + 1}\left(1 + O\left(b_0^{-\frac{\gamma}{\gamma + 1}}\right) + O\left(b_0^{-\frac{1}{\gamma + 1}}\right)\right) > 0,$$

$$B_2 = \left(\frac{\gamma - 1}{2A^\gamma}\right)\frac{\gamma + 1}{b_0\gamma + 1}\left(1 + O\left(b_0^{-\frac{\gamma}{\gamma + 1}}\right) + O\left(b_0^{-\frac{1}{\gamma + 1}}\right)\right),$$

$$B_3 = -(n-1)\left(\frac{\gamma - 1}{2A^\gamma}\right)\frac{\gamma + 1}{b_0\gamma + 1}(1 + O\left(b_0^{-\frac{\gamma}{\gamma + 1}}\right) + O\left(b_0^{-\frac{1}{\gamma + 1}}\right)), $$

$$\mu_1 = \frac{1}{2b_0}\left(1 + O\left(b_0^{-\frac{\gamma}{\gamma + 1}}\right) + O\left(b_0^{-\frac{1}{\gamma + 1}}\right)\right) > 0,$$

$$\mu_2 = -\frac{n-1}{2}\left(1 + O\left(b_0^{-\frac{\gamma}{\gamma + 1}}\right) + O\left(b_0^{-\frac{1}{\gamma + 1}}\right)\right) < 0.$$ 

For the computational requirements later on, we list some properties of the partial Klainerman vector fields, which can be verified directly.

**Lemma 4.5.** The partial Klainerman vector fields given by (4.1) satisfy

(i). $[Z_1, Z_2] = Z_3; [Z_2, Z_3] = Z_1; [Z_3, Z_1] = Z_2; [Z_1, Z_0] = 0, 1 \leq i \leq n.$

(ii). $[Z_i, \partial_r] = 0, 1 \leq i \leq n; [Z_0, \partial_r] = -\partial_r.$

(iii). $Z_i(r) = 0, Z_i\left(\frac{r}{t}\right) = 0, 1 \leq i \leq n; Z_0\left(\frac{1}{r}\right) = -\frac{1}{r}; Z_0\left(\frac{r}{t}\right) = 0.$

(iv). $\nabla_x f \cdot \nabla_x g = \partial_r f \cdot \partial_r g + \frac{1}{r^2} \sum_{i=1}^{2n-3} Z_i f \cdot Z_i g$ for any $C^1$ smooth functions $f$ and $g$.

(v). $|Z_i v| \leq r|\nabla v|$ for any $C^1$ smooth function $v$, here $0 \leq i \leq n.$

(vi). $\partial_1 = \frac{x_1}{r} \partial_r - \frac{x_2}{r^2} Z_1 + \frac{x_3}{r^2} Z_3; \partial_2 = \frac{x_2}{r} \partial_r - \frac{x_3}{r^2} Z_2 + \frac{x_1}{r^2} Z_1; \partial_3 = \frac{x_3}{r} \partial_r - \frac{x_1}{r^2} Z_3 + \frac{x_2}{r^2} Z_2.$

In addition, we now give three basic but important equalities (Lemma 4.6 and Lemma 4.7 below), which will be used in §5 to look for the multipliers in establishing a priori energy estimates.

**Lemma 4.6.** Set $\mathcal{L}_G = \mathcal{L} + \mathcal{P}$ and $\mathcal{M} = A\partial_t + B\partial_r$ with $C^\infty$ smooth coefficients $A$ and $B$, then for any $C^2$ smooth function $G$, one has

$$\mathcal{L}_G \cdot MG = \partial_t N_0 + \sum_{i=1}^{n} \partial_i N_i + E_{00}(\partial_t G)^2 + \sum_{i=1}^{n} E_{0i}\partial_i G\partial_t G + \sum_{i,j=1}^{n} E_{ij}\partial_i G\partial_j G,$$ (4.14)
where

\[
N_0(\nabla G) = \frac{1}{2} A(\partial_t G)^2 + B \partial_t G \partial_r G + B \partial_r G \sum_{k=1}^{n} p_k \partial_k G - \frac{1}{2} A \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j \partial_i G \partial_j G + \frac{1}{2} A p_0 \sum_{k=1}^{n} (\partial_k G)^2,
\]

\[
N_i(\nabla G) = \frac{x_i}{r} \left[ - \frac{1}{2} B (\partial_i G)^2 - B \partial_i G \sum_{k=1}^{n} p_k \partial_k G - \frac{1}{2} B \sum_{j=1}^{n} \sum_{k=1}^{n} p_j p_k \partial_j G \partial_k G + \frac{1}{2} B p_0 \sum_{k=1}^{n} (\partial_k G)^2 \right]_{N^0} + p_i \left[ A (\partial_i G)^2 + B \partial_i G \partial_r G + B \partial_r G \sum_{j=1}^{n} p_j \partial_j G + A \partial_i G \sum_{j=1}^{n} p_j \partial_j G \right]_{N^1} - \partial_i G \left[ A p_0 \partial_i G - B p_0 \partial_r G \right]_{N^2},
\]

\[i = 1, \ldots, n\]

with

\[p = \Phi, \quad p_i = \partial_i \Phi \quad (i = 1, \ldots, n), \quad p_0 = c^2(\rho).\]  

(4.15)

And the coefficients \(E_{00}, E_{0i}\) and \(E_{ij}\) in (4.14) are smooth.

**Proof.** Indeed, in term of (1.6) and (4.15), we have

\[
\mathcal{L}_\Phi = \partial_t^2 + \sum_{k=1}^{n} p_k \partial_k + \sum_{i,j=1}^{n} p_i p_j \partial_{ij} - p_0 \Delta + \text{first order operators}.
\]

This, together with a direct computation, yields (4.14). \(\square\)

**Lemma 4.7.** Under the assumptions of Lemma 4.6, then

\[
\mathcal{L} G \cdot \mathcal{M} G = \partial_t N_0 + \partial_r N_{r0} + \frac{n-1}{r} N_{r0} + \sum_{i=1}^{2n-3} Z_i N_{i0}^i + K_{00} (\partial_t G)^2 + K_{0i} \partial_i G \partial_r G + K_{rn} (\partial_r G)^2 + K_{nn} \sum_{k=1}^{n} (Z_k G)^2 + K_{0n} \partial_i G \sum_{k=1}^{2n-3} Z_k A \cdot Z_k G + K_{rn} \partial_r G \sum_{k=1}^{2n-3} Z_k B \cdot Z_k G
\]

(4.16)

and

\[
\mathcal{P} G \cdot \mathcal{M} G = \partial_t M_0 + \sum_{k=1}^{n} \partial_k M_k + K_{00} (\partial_t G)^2 + \sum_{i=1}^{n} K^{0i} \partial_i G \partial_r G + \sum_{i,j=1}^{n} K^{ij} \partial_i G \partial_j G
\]

(4.17)

with

\[
\begin{aligned}
N_{00}(\nabla G) = \frac{1}{2} A (\partial_t G)^2 + B \partial_t G \partial_r G + B P_1 (\partial_r G)^2 - \frac{1}{2} A P_2 (\partial_r G)^2 + \sum_{i=1}^{2n-3} \frac{1}{2 r^2} A P_3 (Z_i G)^2, \\
N_{0i}(\nabla G) = \frac{1}{2} B (\partial_i G)^2 + A P_1 (\partial_i G)^2 + A P_2 \partial_i G \partial_r G + \frac{1}{2} B P_2 (\partial_r G)^2 + \sum_{i=1}^{2n-3} \frac{1}{2 r^2} B P_3 (Z_i G)^2, \\
N_{i0}(\nabla G) = -\frac{1}{r^2} A P_3 \partial_i G Z_i G - \frac{1}{r^2} B P_3 \partial_i G Z_i G, \\
N_{i0}(\nabla G) = -\frac{1}{r^2} A P_3 \partial_i G Z_i G - \frac{1}{r^2} B P_3 \partial_i G Z_i G, \quad i = 1, \ldots, 2n - 3
\end{aligned}
\]

(4.18)
and

\[
\begin{aligned}
K_{00} &= -\frac{1}{2} \partial_t A + \frac{1}{2} \partial_t B - \partial_t (\Delta t A) + \frac{1}{r} A P_4 + \frac{n-1}{2r} B - \frac{n-1}{r} A P_1, \\
K_{0r} &= -\partial_t B - \partial_r (\Delta r A) + \frac{1}{r} A P_5 + \frac{1}{r} B P_4 - \frac{n-1}{r} A P_2, \\
K_{rr} &= -\partial_r (\Delta r B) + \frac{1}{r} \partial_t (\Delta r A) - \frac{1}{2} \partial_r (\Delta r B) + \frac{1}{r} B P_5 - \frac{n-1}{2r} B P_2, \\
K_{nn} &= -\frac{1}{2r} \partial_t (\Delta n A) - \frac{1}{2} \partial_r (\Delta n B) - \frac{n-1}{2r^3} B P_3, \\
K_{0n} &= \frac{1}{r^2} P_3, \\
K_{rn} &= \frac{1}{r^2} P_3
\end{aligned}
\] (4.19)

and

\[
\begin{aligned}
2K_{0i}^0 &= -\sum_{i=1}^{n} \partial_i (A f_{0i}), \\
2K_{0i}^i &= -\sum_{k=1}^{n} \partial_k (\frac{\partial f_{0i}}{r}) + \sum_{k=1}^{n} \partial_k (\frac{x_k}{r} B f_{0i}) - \sum_{j=1}^{n} \partial_j (A (f_{ji} + f_{ij})), \quad 1 \leq i \leq n, \\
2K_{ij} &= -\partial_i (\frac{x_j}{r} B f_{0j}) - \sum_{k=1}^{n} \partial_k (\frac{x_i}{r} B (f_{kj} + f_{jk})) + \partial_i (A f_{ij}) + \sum_{k=1}^{n} \partial_k (x_j B f_{ij}), \quad 1 \leq i, j \leq n.
\end{aligned}
\] (4.20)

In addition, the explicit expressions of the terms \(M_i\) (0 \leq i \leq n) in (4.17) are not given here since this is not required.

Based on the local existence result given in \(\S\)3, we will use the continuous induction method to prove Theorem 4.1. For this end, a priori estimates on the solution \((\dot{\varphi}, \xi)\) are required to be established. We now introduce some notations in order to fulfill the requirements in \(\S\)5-\(\S\)6 below. For any \(T_0 > 1\), set

\[
\begin{aligned}
D_{T_0} &= \{(t, r, \omega) : 1 < t < T_0, \sigma(t, \omega) < r < \zeta(t, \omega), \omega \in \mathbb{S}^{n-1}\}, \\
B_{T_0} &= \{(t, r, \omega) : 1 < t < T_0, r = \sigma(t, \omega), \omega \in \mathbb{S}^{n-1}\}, \\
\Gamma_{T_0} &= \{(t, r, \omega) : 1 < t < T_0, r = \zeta(t, \omega), \omega \in \mathbb{S}^{n-1}\},
\end{aligned}
\]

where \(B_{T_0}\) and \(\Gamma_{T_0}\) are the lateral boundaries of \(D_{T_0}\).

\section{5. Proof of Theorem 1.1 in the case of \(n = 3\) \label{5}}

In this section, we will establish a uniform weighted energy estimate on \((\dot{\varphi}, \xi)\) for the problem (4.4) together with (4.9) and (4.11)-(4.13) for \(n = 3\) in the domain \(D_{T_0}\), which is defined in the above. In \(\S\)5.1, we will establish the first order weighted energy estimate of \(\nabla \dot{\varphi}\). Subsequently, in \(\S\)5.2, the higher-order weighted energy estimates of \(\nabla^2 \dot{\varphi}\) are derived by utilizing the modified Klainerman’s vector fields. Based on such energy estimates given in \(\S\)5.1-\(\S\)5.2, we can use continuous induction argument given in \(\S\)5.3 to obtain the global existence and behavior at infinity of the solution \((\dot{\varphi}, \xi)\) and then complete the proofs of Theorem 4.1 and Theorem 1.1 in the case of \(n = 3\).

\subsection{5.1. First order weighted energy estimates \label{5.1}}

\textbf{Theorem 5.1.} For \(n = 3\), if \(\varphi \in C^2(D_{T_0})\) is a solution of equation (4.4) with the fixed boundary condition (4.9), and

\[
|\xi| + \sum_{l=0}^{1} t^l (|\nabla^{l+1} \varphi| + |\nabla^{l+1} \xi|) \leq M \varepsilon
\] (5.1)
holds for small $\varepsilon$, $t \in [1, T_0]$ and some positive constant $M$. Then for any fixed constant $\mu < -1 - \frac{1}{2} \sqrt{\frac{2 + \gamma}{2}}$, we have

$$C_1 T_0^{\mu + 1} \int_{\sigma(T_0, \omega) \leq r \leq \zeta(T_0, \omega)} |\nabla \phi|^2 (T_0, x) dx + C_2 \int_{D_{T_0}} t^\mu |\nabla \phi|^2 dt dx$$

$$+ C_3 \int_{\Gamma_{T_0}} t^{\mu + 1} (\partial_t \phi)^2 dS + C_4 \int_{\Gamma_{T_0}} t^{\mu + 1} \frac{1}{r^2} \sum_{k=1}^{3} (Z_k \phi)^2 dS$$

$$\leq C\varepsilon^2 + C_5 \int_{\Gamma_{T_0}} t^{\mu + 1} (B_0 \phi)^2 dS,$$

where $C_i (1 \leq i \leq 5)$ are some positive constants depending on $b_0$ and $\gamma$. In particular,

$$C_3 = \frac{(\gamma - 1)b_0^2}{8} \left(1 + O(b_0^{-\frac{1}{2}}) + O(b_0^{-2})\right),$$

$$C_5 = \frac{(\gamma - 1)b_0^2}{2} \left(1 + O(b_0^{-\frac{1}{2}}) + O(b_0^{-2})\right).$$

**Remark 5.1.** The values of constants $C_3$ and $C_5$ will play an important role in the energy estimates for the problem (4.4) with (4.9) and (4.11)-(4.13) since the most troublesome term $\int_{\Gamma_{T_0}} t^{\mu + 1} (B_0 \phi)^2 dS$ in (5.2) will be shown to be absorbed by the positive integrals in the left hand side of (5.2). The reason which the term $\int_{\Gamma_{T_0}} t^{\mu + 1} (B_0 \phi)^2 dS$ is most troublesome is: due to the Neumann boundary condition (4.11) other than the artificial Dirichlet boundary condition as in [11], the usual Poincaré inequality does not hold for the solution $\phi$ (it is noted that the boundary condition (4.11) contains the function $\xi$, which is roughly equivalent to $\frac{\phi}{t}$ in terms of (4.12), then the estimate on $\phi$ on the shock surface must be done), namely, the $L^2(\Gamma_{T_0})$—estimates of $\nabla \phi$ on the shock surface $\Gamma_{T_0}$ can not be obtained directly.

**Proof.** For $M\phi = A(t, x) \partial_t \phi + B(t, x) \partial_r \phi$, it follows from Lemma 4.6-Lemma 4.7 that

$$\int_{D_{T_0}} R_0(t, x) \cdot M\phi dt dx = \int_{D_{T_0}} (L\phi + P\phi) \cdot M\phi dt dx$$

$$= \int_{D_{T_0}} I_1 dt dx + \int_{\sigma(T_0, \omega) \leq r < \zeta(T_0, \omega)} N_0(\nabla \phi)(T_0, x) dx - \int_{\sigma(1, \omega) \leq r < \zeta(1, \omega)} N_0(\nabla \phi)(1, x) dx$$

$$+ \int_{\Gamma_{T_0}} \left( \sum_{i=1}^{3} \frac{p_i}{r} N_i(\nabla \phi) - \partial_t \chi \cdot N_i(\nabla \phi) - \partial_t \chi N_0(\nabla \phi) \right) dS$$

$$+ \int_{B_{T_0}} \left( \partial_t \sigma N_0(\nabla \phi) - \sum_{i=1}^{3} \frac{p_i}{r} N_i(\nabla \phi) - \partial_i \sigma N_i(\nabla \phi) \right) dS$$

$$I_1 = K_{00}(\partial_t \phi)^2 + K_{0r}(\partial_t \phi) \partial_r \phi + K_{rr}(\partial_r \phi)^2 + K_{33} \frac{1}{r^2} \sum_{i=1}^{3} (Z_i \phi)^2$$

$$+ K_{03} \frac{1}{r^2} \sum_{i=1}^{3} Z_i AZ_i \phi \partial_r \phi + K_{33} \frac{1}{r^2} \sum_{i=1}^{3} Z_i BZ_i \phi \partial_r \phi$$

$$+ K^{0i}(\partial_t \phi)^2 + \sum_{k=1}^{3} K^{0i}(\partial_t \phi) \partial_i \phi + \sum_{i,j=1}^{3} K^{ij} \partial_i \phi \partial_j \phi.$$  

(5.5)
Our purpose is to choose suitable functions \( A(t, x) \) and \( B(t, x) \) so that all integrals on \( D_{T_0}, B_{T_0} \) and \( t = T_0 \) in the right hand side of (5.4) are non-negative and the integral on \( \Gamma_T \) gives a “good” control on \( \dot{\varphi} \). From now on, we will derive some sufficient conditions for the choices of \( A(t, x) \) and \( B(t, x) \) in the process of analyzing each integral, and then \( A(t, x) \) and \( B(t, x) \) can be determined. This process is divided into the following five steps.

**Step 1. The analysis on the term** \( \int_{B_{T_0}} \left( \partial_t \sigma N_0(\nabla \dot{\varphi}) - \sum_{i=1}^{3} \left( \frac{x_i}{r} N_i(\nabla \dot{\varphi}) - \partial_i \sigma N_i(\nabla \dot{\varphi}) \right) \right) dS \)

Due to (1.7) and (4.9), using the notations in (4.14)-(4.15), we have on \( B_{T_0} \)

\[
\partial_t \sigma N_0(\nabla \dot{\varphi}) - \sum_{i=1}^{3} \left( \frac{x_i}{r} N_i(\nabla \dot{\varphi}) - \partial_i \sigma N_i(\nabla \dot{\varphi}) \right) = \partial_t \sigma N_0(\nabla \dot{\varphi}) - N^0(\nabla \dot{\varphi}) - \partial_t \sigma N^1(\nabla \dot{\varphi})
\]

\[
= (B - \partial_t \sigma A) \left[ \frac{1}{2} (\partial_i \dot{\varphi})^2 + \partial_i \dot{\varphi} \sum_{j=1}^{3} p_i \partial_i \dot{\varphi} + \frac{1}{2} p_0 |\nabla \dot{\varphi}|^2 - \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} p_i p_j \partial_i \dot{\varphi} \partial_j \dot{\varphi} \right].
\]

It follows from the assumption (5.1), (4.15) and (iv) in Lemma 2.1 that \( \sum_{i=1}^{3} |p_i|^2 > p_0 \), and thus

\[
p_0 |\nabla \dot{\varphi}|^2 - \frac{3}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} p_i p_j \partial_i \dot{\varphi} \partial_j \dot{\varphi} \leq (p_0 - \sum_{i=1}^{3} |p_i|^2)|\nabla \dot{\varphi}|^2 < 0,
\]

and unfortunately, \( \frac{1}{2} (\partial_i \dot{\varphi})^2 + \partial_i \dot{\varphi} \sum_{j=1}^{3} p_i \partial_i \dot{\varphi} + \frac{1}{2} p_0 |\nabla \dot{\varphi}|^2 - \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} p_i p_j \partial_i \dot{\varphi} \partial_j \dot{\varphi} \) may change its sign on \( B_{T_0} \).

So in order to control the term \( \int_{B_{T_0}} \left( \partial_t \sigma N_0(\nabla \dot{\varphi}) - \sum_{i=1}^{3} \left( \frac{x_i}{r} N_i(\nabla \dot{\varphi}) - \partial_i \sigma N_i(\nabla \dot{\varphi}) \right) \right) dS \) well, \( A \) and \( B \) should satisfy

\[
B = \partial_t \sigma A \quad \text{on} \quad r = \sigma(t, \omega), \quad (5.6)
\]

and then

\[
\int_{B_{T_0}} \left( \partial_t \sigma N_0(\nabla \dot{\varphi}) - \sum_{i=1}^{3} \left( \frac{x_i}{r} N_i(\nabla \dot{\varphi}) - \partial_i \sigma N_i(\nabla \dot{\varphi}) \right) \right) dS = 0. \quad (5.7)
\]

In view of (5.6) and the self-similar property of the background solution \( \hat{\Phi} \), we set

\[
A = t^\mu r a \left( \frac{r}{t} \right), \quad B = t^{\mu+1} b_\sigma(t, r, w), \quad (5.8)
\]

with the functions \( a \) and \( b_\sigma \) to be determined later. By (5.6), \( a \) and \( b_\sigma \) in (5.8) should satisfy the following restriction

\[
b_\sigma = sa \partial_t \sigma \quad \text{on} \quad r = \sigma(t, \omega). \quad (5.9)
\]

**Step 2. Positivity of** \( \int_{\sigma(T_0, \omega) < r < \zeta(T_0, \omega)} N_0(\nabla \dot{\varphi})(T_0, x)dx \)
Under the assumption (5.1), using (vi) in Lemma 4.5, one then has

\[ N_0(\nabla \dot{\varphi}) = N_{00}(\nabla \dot{\varphi}) + t^{\mu+1}O(\varepsilon)|\nabla \dot{\varphi}|^2 \]

\[ = t^{\mu+1} \left[ \frac{1}{2}sa(\partial_t \dot{\varphi})^2 + b_\sigma \partial_t \dot{\varphi} \partial_r \dot{\varphi} + b_\sigma P_1(s)(\partial_r \dot{\varphi})^2 - \frac{1}{2}saP_2(s)(\partial_r \dot{\varphi})^2 \right. \]

\[ + \frac{1}{2r^2}a P_3(s) \sum_{i=1}^{3} (Z_i \dot{\varphi})^2 + O(\varepsilon)|\nabla \dot{\varphi}|^2 \]. \tag{5.10} \]

Due to \( P_3(s) > 0 \) by Lemma 4.3, to ensure the positivity of quadratic polynomial of \( (\partial_t \dot{\varphi}, \partial_r \dot{\varphi}, \frac{1}{r}Z_2 \dot{\varphi}, \frac{1}{r}Z_3 \dot{\varphi}) \) in \( N_0(\nabla \dot{\varphi}) \), \( a \) and \( b_\sigma \) should fulfill

\[
\begin{cases}
a(s) > 0, \\
b_\sigma^2 - 2sP_1(s)a(s)b_\sigma + s^2P_2(s)a^2(s) < 0,
\end{cases}
\]

which is equivalent to

\[ a(s) > 0, \quad \dot{u}(s) - c(\dot{\rho})(s) < \frac{b_\sigma}{sa(s)} < \dot{u}(s) + c(\dot{\rho})(s) \text{ in } \Omega_+. \] \tag{5.11} \]

In this case, we arrive at

\[ \int_{\sigma(T_0, \omega) \leq r \leq \zeta(T_0, \omega)} N_0(\nabla \dot{\varphi})(T_0, x) dS \geq C(b_0, \gamma)T_0^{\mu+1} \int_{\sigma(T_0, \omega) < r < \zeta(T_0, \omega)} |\nabla \dot{\varphi}|^2(T_0, x) dS. \] \tag{5.12} \]

**Step 3. Positivity of the integral on \( D_{T_0} \)**

Under the constraints (5.9) and (5.11), we will choose \( a \) and \( b_\sigma \) such that

\[ I_1 \geq 0, \] \tag{5.13} \]

First, it can be verified directly that under the assumption (5.1), we have

\[ f_{ij} = O_1(\varepsilon), \quad 0 \leq i \leq 3; 1 \leq j \leq 3, \]

then it follow from (4.20) that

\[ \left| K^{00}(\partial_t \dot{\varphi})^2 + \sum_{k=1}^{3} K^{0k} \partial_t \dot{\varphi} \partial_k \dot{\varphi} + \sum_{i,j=1}^{3} K^{ij} \partial_i \dot{\varphi} \partial_j \dot{\varphi} \right| \leq t^{\mu}O(\varepsilon)|\nabla \dot{\varphi}|^2. \] \tag{5.14} \]

On the other hand, according to (5.9) and (5.11), we choose

\[
\begin{cases}
a(s) = 1, \\
b_\sigma(t, \omega) = s^2 \left( 1 + \frac{\varepsilon \partial_t b - \frac{s}{t} \varepsilon b}{s} + \frac{e}{b_0}(s - b_0 - \frac{\varepsilon}{t} b) \right),
\end{cases}
\] \tag{5.15} \]

where the constant \( e \) will be determined later on.
It follows from Lemma 2.1, Lemma 4.3, (4.19), (5.15) and a direct computation that

\[
\begin{align*}
K_{00} &= t^\mu \left( \frac{1}{2}(2 + e - \mu) b_0 (1 + O(b_0^{-2}) + O(b_0^{-2})) \right), \\
K_{0r} &= t^\mu \left( \frac{\gamma + 3}{2} + e - \mu \right) b_0 (1 + O(b_0^{-2}) + O(b_0^{-2})) , \\
K_{rr} &= t^\mu \left( \frac{\gamma + 1}{4} e - \frac{\gamma + 1}{4} \mu + 1 \right) b_0 (1 + O(b_0^{-2}) + O(b_0^{-2})) , \\
K_{33} &= t^\mu \left( - \frac{\gamma - 1}{4} (2 + e + \mu) b_0 (1 + O(b_0^{-2}) + O(b_0^{-2})) \right), \\
K_{03} Z_i A &= 0 , \quad i = 1, 2, 3, \\
K_{r3} Z_i B &= t^\mu O(\varepsilon) , \quad i = 1, 2, 3, \\
K_{0r} - 4K_{00} K_{rr} &= t^2 \mu \left( \frac{\gamma - 1}{4} (\gamma + 7 - 2(e - \mu)^2)(1 + O(b_0^{-2}) + O(b_0^{-2})) \right) .
\end{align*}
\]

(5.16)

Therefore, for large \( b_0 \), with (5.5) and (5.14), the sufficient conditions for (5.13) can be chosen as

\[
2 + e - \mu > 0 , \quad 2 + e + \mu < 0 , \quad \gamma + 7 - 2(e - \mu)^2 < 0 .
\]

(5.17)

If we let

\[
\mu < -1 - \frac{1}{2} \sqrt{\frac{\gamma + 7}{2}}
\]

(5.18)

and

\[
e = \frac{1}{2} \sqrt{\frac{\gamma + 7}{2}} - 1,
\]

(5.19)

then it can be verified from (5.16)-(5.17) directly that

\[
K_{00} > 0 , \quad K_{0r}^2 - 4K_{00} K_{rr} < 0 , \quad K_{33} > 0 .
\]

Combining this with (5.14) shows (5.13).

In this case, we arrive at

\[
\iint_{D_{T_0}} I_1 dt dx \geq C(b_0, \gamma) \iint_{D_{T_0}} t^\mu \left( (\partial_t \tilde{\phi})^2 + (\partial_r \tilde{\phi})^2 + \frac{1}{r^2} \sum_{i=1}^{3} (Z_i \tilde{\phi})^2 \right) dt dx .
\]

(5.20)

**Step 4. Estimate of**

\[
\int_{\Gamma_{T_0}} \left( \sum_{i=1}^{3} \left( \frac{x_i}{r} N_i(\nabla \tilde{\phi}) - \partial_i \chi N_i(\nabla \tilde{\phi}) \right) - \partial_t \chi N_0(\nabla \tilde{\phi}) \right) dS
\]

By (iv) in Lemma 4.5, the assumption (5.1) and the definition of \( \tilde{\phi} \) and \( \xi \) (also see (4.18)),

\[
\sum_{i=1}^{3} \left( \frac{x_i}{r} N_i(\nabla \tilde{\phi}) - \partial_i \chi N_i(\nabla \tilde{\phi}) \right) - \partial_t \chi N_0(\nabla \tilde{\phi}) = N^0(\nabla \tilde{\phi}) + \partial_\xi N^1(\nabla \tilde{\phi}) - \partial_t \phi N^2(\nabla \tilde{\phi})
\]

\[
- \frac{1}{r^2} \sum_{i=1}^{3} Z_i \chi \cdot Z_i \Phi \cdot N^1(\nabla \tilde{\phi}) + \frac{1}{r^2} \sum_{i=1}^{3} Z_i \chi \cdot Z_i \phi N^2(\nabla \tilde{\phi}) - \partial_t \chi N_0(\nabla \tilde{\phi})
\]

\[
= N^0 + P_1(s) N^1 - \partial_t \phi N^2 - s_0 N_0 + t^{\mu + 1} O(\varepsilon) |\nabla \tilde{\phi}|^2 .
\]

(5.21)
By the assumption (5.1), (iv) in Lemma 4.5 and the notations in (4.15), we have

\[
\begin{align*}
N^0 &= t^{\mu+1} \left( \frac{1}{2} b_\sigma (\partial_t \psi)^2 - b_\sigma \dot{u}(s) \partial_t \psi \partial_t \psi + \frac{1}{2} b_\sigma \dot{u}^2(s) (\partial_t \psi)^2 
+ \frac{1}{2} b_\sigma p_0 (\partial_t \psi)^2 + \frac{1}{2\rho^2} b_\sigma p_0 \sum_{k=1}^{3} (Z_k \psi)^2 \right), \\
N^1 &= t^{\mu+1} \left( s(\partial_t \psi)^2 + b_\sigma \partial_t \psi \partial_t \psi + b_\sigma \dot{u}(s) (\partial_t \psi)^2 + s \dot{u}(s) \partial_t \psi \partial_t \psi + O(\varepsilon |\nabla \psi|^2) \right), \\
N^2 &= t^{\mu+1} \left( sp_0 \partial_t \psi + b_\sigma p_0 \partial_t \psi \right).
\end{align*}
\]  

(5.22)

Combining (5.22) with (5.10), (5.15) and (5.21) yields

\[
\sum_{i=1}^{3} \left( \frac{\delta_i}{r} N_i(\nabla \psi) - \partial_t \chi N_i(\nabla \psi) \right) \right) - \partial_t \chi N_0(\nabla \psi) = t^{\mu+1} \left( \beta_{11}(\partial_t \psi)^2 + \beta_{12} \partial_t \psi \partial_t \psi + \beta_{13}(\partial_t \psi)^2 + \beta_{14} \frac{1}{r^2} \sum_{k=1}^{3} (Z_k \psi)^2 \right)
\]  

(5.23)

with

\[
\begin{align*}
\beta_{11} &= b_0^2(1 + O(b_0^{-\frac{1}{2}})) + O(b_0^{-2}), \\
\beta_{12} &= -\frac{\gamma}{2} b_0^2(1 + O(b_0^{-\frac{1}{2}})) + O(b_0^{-2}), \\
\beta_{13} &= -\frac{\gamma}{2} b_0^2(1 + O(b_0^{-\frac{1}{2}})) + O(b_0^{-2}), \\
\beta_{14} &= \frac{\gamma}{4} b_0^2(s_0 - b_0)(1 + O(b_0^{-\frac{1}{2}})) + O(b_0^{-2}).
\end{align*}
\]

Due to \( \partial_t \psi = B_0 \dot{\psi} - \mu_1 \partial_t \psi \), from (5.23) and (5.19) that

\[
\sum_{i=1}^{3} \left( \frac{\delta_i}{r} N_i(\nabla \psi) - \partial_t \chi \cdot N_i(\nabla \psi) \right) = t^{\mu+1} \left( \beta_{11}(\partial_t \psi)^2 + \beta_{12} \partial_t \psi \partial_t \psi + \beta_{13}(\partial_t \psi)^2 + \beta_{14} \sum_{k=1}^{3} (Z_k \psi)^2 \right)
\]  

with

\[
\begin{align*}
\beta_{11} &= \beta_1 - \mu_1 \beta_{12} + \mu_1^2 \beta_{13} = \gamma \frac{1}{2} b_0^2(1 + O(b_0^{-\frac{1}{2}})) + O(b_0^{-2}), \\
\beta_{12} &= \beta_{12} - 2 \mu_1 \beta_{13} = \frac{b_0^2}{4} \left( O(b_0^{-\frac{1}{2}}) + O(b_0^{-2}) \right), \\
\beta_{13} &= \beta_{13} = -\frac{\gamma}{2} b_0^2(1 + O(b_0^{-\frac{1}{2}})) + O(b_0^{-2}), \\
\beta_{14} &= \beta_{14} = \frac{\gamma}{4} b_0^2(s_0 - b_0)(1 + O(b_0^{-\frac{1}{2}})) + O(b_0^{-2}) > 0.
\end{align*}
\]
Consequently, one has
\[
\int_{\Gamma_{t_0}} \left( \sum_{i=1}^{3} \frac{x_i}{r} N_i(\nabla \varphi) - \partial_t \chi N_i(\nabla \varphi) - \partial_t \chi N_0(\nabla \varphi) \right) dS \\
\geq \frac{\gamma - 1}{8} b_0^2 \left( 1 + O(b_0^{-\frac{\gamma-1}{2}}) + O(b_0^{-2}) \right) \int_{\Gamma_{t_0}} t^{\mu+1} |\partial_t \varphi|^2 dS \\
- \frac{\gamma - 1}{2} b_0^2 \left( 1 + O(b_0^{-\frac{\gamma-1}{2}}) + O(b_0^{-2}) \right) \int_{\Gamma_{t_0}} t^{\mu+1} (B_0 \varphi)^2 dS \\
+ C(b_0, \gamma) \int_{\Gamma_{t_0}} t^{\mu+1} \frac{1}{r^2} \sum_{k=1}^{3} (Z_k \varphi)^2 dS. \tag{5.24}
\]

**Step 5. The estimates on** $\int_{\sigma(1, \omega) \leq r \leq \zeta(1, \omega)} N_0(\nabla \varphi)(1, x) dx$ **and** $\int_{D_{t_0}} R_0(t, x) \cdot \mathcal{M}_t \varphi dtdx$

It follows from (4.13) that
\[
\left| \int_{\sigma(1, \omega) \leq r \leq \zeta(1, \omega)} N_0(\nabla \varphi)(1, x) dx \right| \leq C(b_0, \gamma) \int_{\sigma(1, \omega) \leq r \leq \zeta(1, \omega)} |\nabla \varphi|^2(1, x) dx. \tag{5.25}
\]

Moreover, it follows from Lemma 4.2 and $\mu < -2$ in (5.19) that
\[
\left| \int_{D_{t_0}} R_0(t, x) \cdot \mathcal{M}_t \varphi dtdx \right| \leq C(b_0, \gamma) \varepsilon \left( \int_{D_{t_0}} t^{\mu} |\nabla \varphi|^2 dtdx + \varepsilon \right). \tag{5.26}
\]

Finally, substituting the estimates (5.7), (5.12), (5.20) and (5.24)-(5.26) into (5.4), then (5.2) and (5.3) can be obtained. Therefore, Theorem 5.1 is proved.

Based on Theorem 5.1, we will derive the first order uniform energy estimate of $\nabla \varphi$. For this end, we require an improved Hardy-type inequality on $\int_{\Gamma_{t_0}} t^{\mu-1} |\varphi|^2 dS$ in terms of the special structures of (4.11) and (4.12), which is motivated by Theorem 330 in [12] and [19].

**Lemma 5.2. (Improved Hardy-type inequality)** Under the assumptions of Theorem 5.1, for $\mu < -1 - \frac{1}{2} \sqrt{\gamma + \frac{7}{2}}$, we have
\[
\int_{\Gamma_{t_0}} t^{\mu-1} |\varphi|^2 dS \leq C(b_0, \gamma) \varepsilon^2 + \frac{1}{\mu^2} \left( 1 + O(b_0^{-\frac{\gamma-1}{2}}) + O(b_0^{-2}) \right) \int_{\Gamma_{t_0}} t^{\mu+1} (\partial_t \varphi)^2 dS \\
+ C(b_0, \gamma) \varepsilon \left( \int_{\Gamma_{t_0}} t^{\mu+1} (B_0 \varphi)^2 dS + \frac{1}{r^2} \int_{\Gamma_{t_0}} t^{\mu+1} \sum_{k=1}^{3} (Z_k \varphi)^2 dS \right). \tag{5.27}
\]

**Remark 5.2.** Classical Hardy-type inequality in [12] only means that
\[
\int_{\Gamma_{t_0}} t^{\mu-1} |\varphi|^2 dS \leq \frac{2}{\mu^2} \left( 1 + O(b_0^{-\frac{\gamma-1}{2}}) + O(b_0^{-2}) \right) \int_{\Gamma_{t_0}} t^{\mu+1} (\partial_t \varphi)^2 dS + \text{“some small terms”} \tag{5.28}
\]
holds. One should specially notice that the coefficient of \( \int_{\Gamma_{T_0}} t^{\mu+1} (\partial_t \varphi)^2 dS \) is \( \frac{2}{\mu^2} \) in (5.28) other than \( \frac{1}{\mu^2} \) in (5.27). If so, it will completely fail in deriving the first order energy estimates on \( \varphi \) from Theorem 5.1 since the terms on the shock surface can not be absorbed by the left hand side terms in (5.2).

**Proof.** It is noted that

\[
\int_{\Gamma_{T_0}} t^{\mu-1} |\varphi|^2 dS = \int_{\mathbb{R}^2} dt \int_1^{T_0} t^{\mu-1} |\varphi(t, \chi(t, \omega), \omega)|^2 dt.
\]

Set \( m(\omega) \equiv \int_1^{T_0} t^{\mu-1} |\varphi(t, \chi(t, \omega), \omega)|^2 dt \), then by integration by parts

\[
m(\omega) = \frac{1}{\mu} t^{\mu} |\varphi(t, \chi(t, \omega), \omega)|^2 \bigg|_{t=1}^{t=T_0} - \frac{2}{\mu} \int_1^{T_0} t^{\mu} \varphi(t, \chi(t, \omega), \omega) (\partial_t \varphi + \partial_t \chi \partial_x \varphi)(t, \chi(t, \omega), \omega) dt
\]

\[
\leq \frac{1}{|\mu|} |\varphi(1, \chi(1, \omega), \omega)|^2 - \frac{2}{\mu} \int_1^{T_0} t^{\mu} \varphi(t, \chi(t, \omega), \omega) (\partial_t \chi \partial_x \varphi + (1 - \mu_1 \partial_t \chi) \partial_t \varphi)(t, \chi(t, \omega), \omega) dt.
\]

(5.29)

Due to (5.1), Lemma 2.1, Lemma 4.4 and \( \partial_t \chi = b_0 (1 + O(b_0^{-\frac{1}{1-\gamma}}) + O(b_0^2)) \), then

\[
1 - \mu_1 \partial_t \chi = \frac{1}{z} (1 + O(b_0^{-\frac{1}{1-\gamma}}) + O(b_0^2) + O(\varepsilon)) > 0.
\]

(5.30)

It follows from \( \mu_3 < 0 \) in (4.12), \( \mu_2 < 0 \) in Lemma 4.4, Lemma 4.2, (5.18), (5.29) and a direct computation that

\[
m(\omega)
\leq C(b_0, \gamma) \varepsilon^2 - \frac{2}{\mu} \int_1^{T_0} t^{\mu} \varphi(t, \chi(t, \omega), \omega) (1 - \mu_1 \partial_t \chi) \partial_t \varphi(t, \chi(t, \omega), \omega) dt
\]

\[
+ \frac{2\mu_2}{\mu} \int_1^{T_0} t^{\mu} \partial_t \chi \varphi(t, \chi(t, \omega), \omega) \xi dt - \frac{2}{\mu} \int_1^{T_0} t^{\mu} \partial_t \chi \varphi(t, \chi(t, \omega), \omega) \left( \frac{\kappa(\xi, \nabla \varphi) + R_1(t, x)}{B_1} \right) dt
\]

\[
= C(b_0, \gamma) \varepsilon^2 - \frac{2}{\mu} \int_1^{T_0} t^{\mu} \varphi(t, \chi(t, \omega), \omega) (1 - \mu_1 \partial_t \chi) \partial_t \varphi(t, \chi(t, \omega), \omega) dt
\]

\[
+ \frac{2\mu_2}{\mu} \int_1^{T_0} t^{\mu} \partial_t \chi (\mu_3(t, x) t \xi) dt - \frac{2}{\mu} \int_1^{T_0} t^{\mu} \partial_t \chi \varphi(t, \chi(t, \omega), \omega) \left( \frac{\kappa(\xi, \nabla \varphi) + R_1(t, x)}{B_1} \right) dt
\]

\[
\leq C(b_0, \gamma) \varepsilon^2 + \frac{1}{2} \int_1^{T_0} t^{\mu-1} |\varphi(t, \chi(t, \omega), \omega)|^2 dt + \frac{2}{\mu^2} \int_1^{T_0} t^{\mu+1} (1 - \mu_1 \partial_t \chi)^2 |\partial_t \varphi(t, \chi(t, \omega), \omega)|^2 dt
\]

\[
+ C(b_0, \gamma) \varepsilon \left( \int_1^{T_0} t^{\mu-1} |\varphi(t, \chi(t, \omega), \omega)|^2 dt + \int_1^{T_0} t^{\mu+1} |\xi|^2 dt + \int_1^{T_0} t^{\mu+1} (|\nabla \varphi|^2 dt
\]

\[
+ \frac{1}{\mu^2} \sum_{i=1}^3 (Z_i \varphi)^2 dt, \zeta(t, \omega), \omega) dt \right).
\]

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This, together with (5.30), yields
\[
\int_{\Gamma_{T_0}} t^{\mu-1} |\dot{\phi}|^2 dS \leq C(b_0, \gamma)\varepsilon^2 + \left( \frac{4}{\mu^2} + O(\varepsilon) \right) \int_{\Gamma_{T_0}} t^{\mu+1} (1 - \mu_1 \partial_t \chi)^2 (\partial_t \dot{\phi})^2 dS
\]
\[+ C(b_0, \gamma)\varepsilon^2 \int_{\Gamma_{T_0}} t^{\mu+1} (B_0 \dot{\phi})^2 dS + \int_{\Gamma_{T_0}} t^{\mu+1} \frac{1}{\rho_2} \sum_{k=1}^{3} (Z_k \phi)^2 dS \]
\[\leq C(b_0, \gamma)\varepsilon^2 + \frac{1}{\mu^2} (1 + O(b_0^{-2\gamma})) + O(b_0^{-2}) \int_{\Gamma_{T_0}} t^{\mu+1} (\partial_t \dot{\phi})^2 dS
\]
\[+ C(b_0, \gamma)\varepsilon^2 \int_{\Gamma_{T_0}} t^{\mu+1} (B_0 \dot{\phi})^2 dS + \int_{\Gamma_{T_0}} t^{\mu+1} \frac{1}{\rho_2} \sum_{k=1}^{3} (Z_k \phi)^2 dS.
\]
Consequently, Lemma 5.2 is proved. \qed

Lemma 5.2 illustrates that using the special structures in (4.11)-(4.12), we can obtain an improved Hardy-type inequality, which plays a crucial role in establishing the first order weighted energy estimates of $\nabla \dot{\phi}$. With Lemma 5.2, we have

**Theorem 5.3. (First Order Weighted Energy Estimate.)** Under the assumptions of Theorem 5.1, for $\mu < -1 - \frac{1}{2} \sqrt{\gamma + \frac{7}{2}}$, one has
\[
T_0^{\mu+1} \int_{\sigma(T_0, \omega) \leq r \leq \zeta(T_0, \omega)} |\nabla \dot{\phi}|^2(T_0, x) dx + \int_{\mathbb{R}^2} t^\mu |\nabla \dot{\phi}|^2 dtdx + \int_{\Gamma_{T_0}} t^{\mu+1} |\nabla \dot{\phi}|^2 dS \leq C\varepsilon^2.
\]

**Proof.** To obtain (5.31), we require to give a delicate estimate on the term $C_5 \int_{\Gamma_{T_0}} t^{\mu+1} (B_0 \dot{\phi})^2 dS$ in (5.2) so that it can be absorbed by the corresponding positive terms in the left hand side of (5.2).

We now treat the term $\int_{\Gamma_{T_0}} t^{\mu+1} (B_0 \dot{\phi})^2 dS$.

From (4.11) and Lemma 4.2, we have
\[
\int_{\Gamma_{T_0}} t^{\mu+1} (B_0 \dot{\phi})^2 dS = \int_{\Gamma_{T_0}} t^{\mu+1} \left( \kappa(\xi, \nabla \dot{\phi}) + R_1(t, \xi) \right) - \mu_2 \xi \right)^2 dS
\]
\[\leq \mu_2^2 (1 + O(b_0^{-2\gamma})) + O(b_0^{-2}) \int_{\Gamma_{T_0}} t^{\mu-1} |\xi|^2 dS
\]
\[+ C(b_0, \gamma) \int_{\Gamma_{T_0}} t^{\mu+1} \kappa^2(\xi, \nabla \dot{\phi}) dS + C(b_0, \gamma)\varepsilon^2.
\]

Due to the assumption (5.1) and the property of $\kappa(\xi, \nabla \dot{\phi})$ on $\Gamma_{T_0}$, then
\[
\int_{\Gamma_{T_0}} t^{\mu+1} \kappa^2(\xi, \nabla \dot{\phi}) dS
\]
\[\leq C(b_0, \gamma)\varepsilon^2 \int_{\Gamma_{T_0}} t^{\mu+1} \left( |\xi|^2 + |\nabla \dot{\phi}|^2 \right) dS
\]
\[\leq C(b_0, \gamma)\varepsilon^2 \int_{\Gamma_{T_0}} t^{\mu+1} \left( |\xi|^2 + (B_0 \dot{\phi})^2 + (\partial_t \dot{\phi})^2 + \frac{1}{\rho_2} \sum_{i=1}^{3} (Z_i \dot{\phi})^2 \right) dS.
\]

(5.33)
In addition, due to the smallness of $\varepsilon$ and the boundary condition (4.12) together with Lemma 2.1 (ii), then we have from (5.32)-(5.33) that

$$
\int_{\Gamma_{0}} t^{\mu+1} |B_{0}\dot{\varphi}|^2 dS \leq \frac{\mu^2}{b_0^2} \left(1 + O(b_0^{-\frac{2}{\gamma+1}}) + O(b_0^{-2})\right) \int_{\Gamma_{0}} t^{\mu-1} |\dot{\varphi}|^2 dS
+ C(b_0, \gamma)\varepsilon^2 \int_{\Gamma_{0}} t^{\mu+1} \left((\partial_t \dot{\varphi})^2 + \frac{1}{r^2} \sum_{k=1}^{3} (Z_k \dot{\varphi})^2\right) dS + C(b_0, \gamma)\varepsilon^2.
$$

Therefore, combining this with (5.27) yields

$$
\int_{\Gamma_{0}} t^{\mu+1} |B_{0}\dot{\varphi}|^2 dS
\leq C(b_0, \gamma)\varepsilon^2 + \frac{1}{\mu^2 b_0^2} \left(1 + O(b_0^{-\frac{2}{\gamma+1}}) + O(b_0^{-2})\right) \int_{\Gamma_{0}} t^{\mu+1} |\dot{\varphi}|^2 dS
+ C(b_0, \gamma)\varepsilon \int_{\Gamma_{0}} t^{\mu+1} \frac{1}{r^2} \sum_{k=1}^{3} (Z_k \dot{\varphi})^2 dS.
$$

(5.34)

Substituting (5.34) into (5.2), we obtain from the assumption in Theorem 5.1 that

$$
\int_{0}^{T} \int_{\sigma(T_0, \omega) \leq r \leq \zeta(T_0, \omega)} |\nabla \dot{\varphi}|^2 (T_0, x) dx + \int_{\int_{D_{T_0}}} t^{\mu} |\nabla \dot{\varphi}|^2 dt dx
+ \frac{\gamma - 1}{8} (1 - \frac{4}{\mu^2}) b_0^2 \left(1 + O(b_0^{-\frac{2}{\gamma+1}}) + O(b_0^{-2})\right) \int_{\Gamma_{0}} t^{\mu+1} (\partial_t \dot{\varphi})^2 dS
+ \int_{\Gamma_{0}} t^{\mu+1} \frac{1}{r^2} \sum_{k=1}^{3} (Z_k \dot{\varphi})^2 dS \leq C\varepsilon^2.
$$

(5.35)

Due to $\mu^2 > 4$ in (5.18), then (5.35) yields Theorem 5.3.

§5.2. Higher order weighted energy estimate.

In this subsection, we will derive the higher-order energy estimates of $\dot{\varphi}$, so that the decay properties of $\nabla \dot{\varphi}$ and $\xi$ for large $t$ can be established.

Denote

$$
\check{Z} = \{ \check{Z}_j : 0 \leq j \leq 2n-3 \}, \quad \text{where} \quad \check{Z}_0 = t\partial_t + (r + t^2 \partial_r) \partial_r, \quad \check{Z}_k = Z_k + tZ_k \partial_r (1 \leq j \leq 2n-3)
$$

(5.36)

by the vector fields which are tangent to the surface $r = \sigma(t, \omega)$.

Under the assumptions in Theorem 1.1, one has

$$
\check{Z}_i - Z_i = O^\infty(\varepsilon) \nabla, \quad 0 \leq i \leq 2n-3.
$$

(5.37)

**Lemma 5.4.** For $n = 2, 3$, there exist functions $\tau_{ij} = \tau_{ij}(t, r, \omega) (0 \leq i, j \leq 2n-3)$, such that for $i, j = 0, 1, \cdots, 2n-3$,

$$
\tau_{ij} \in O^\infty(1), \quad \text{and} \quad B_{\sigma} (\check{Z}_i + \sum_{j=0}^{2n-3} \tau_{ij} \check{Z}_j) \dot{\varphi} = 0 \quad \text{on} \quad r = \sigma(t, \omega).
$$

(5.38)
Proof. It follows from a direct computation that for $0 \leq i \leq 2n-3$,

$$[B_\sigma, \tilde{Z}_i] = h_{i0} B_\sigma + \sum_{j=1}^{2n-3} h_{ij} \tilde{Z}_j,$$

with the smooth function $h_{ij}(t, r, \omega)(0 \leq i, j \leq 2n-3)$ satisfying

$$h_{i0} = O_0^\infty(1), \quad h_{ij} = O_1^\infty(1), \quad 1 \leq j \leq 2n-3. \quad (5.39)$$

Due to $B_\sigma \dot{\varphi} = 0$ and $\tilde{Z}_i B_\sigma \dot{\varphi} = 0$ ($0 \leq i \leq 2n-3$) on $r = \sigma(t, \omega)$ and the fact that $\tilde{Z}_i$ is tangent to the surface $r = \sigma(t, \omega)$, then for smooth function $\tau_{ij}$, we have on $r = \sigma(t, \omega)$

$$B_\sigma(\tilde{Z}_i + \sum_{j=0}^{2n-3} \tau_{ij} \tilde{Z}_j)\dot{\varphi} = [B_\sigma, \tilde{Z}_i] \dot{\varphi} + \sum_{j=0}^{2n-3} B_\sigma \tau_{ij} \cdot \tilde{Z}_j \dot{\varphi} + \sum_{j=0}^{2n-3} \tau_{ij} [B_\sigma, \tilde{Z}_j] \dot{\varphi}$$

$$= \sum_{j=0}^{2n-3} h_{ij} \tilde{Z}_j \dot{\varphi} + \sum_{j=0}^{2n-3} B_\sigma \tau_{ij} \cdot \tilde{Z}_j \dot{\varphi} + \sum_{j=0}^{2n-3} \tau_{ij} 2n-3 \sum_{k=0}^{2n-3} h_{jk} \tilde{Z}_k \dot{\varphi}$$

$$= \sum_{j=0}^{2n-3} \left( B_\sigma \tau_{ij} + \sum_{k=0}^{2n-3} h_{k \ell} \tau_{ik} + h_{ij} \right) \tilde{Z}_j \dot{\varphi}. \quad (5.40)$$

For $0 \leq i, j \leq 2n-3$, let the functions $\tau_{ij}$ satisfy

$$\begin{cases} 
B_\sigma \tau_{ij} + h_{ij} = 0 & \text{in } \Omega_+, \\
\tau_{ij} = 0 & \text{on } r = \sigma(t, \omega). 
\end{cases} \quad (5.41)$$

By (5.39) and the analogous proof procedure on Lemma B.1 in Appendix B, one can prove that (5.41) has a $C^\infty$ solution $\tau_{ij}(0 \leq i, j \leq 2n-3)$ in $\Omega_+$ which satisfy

$$\tau_{ij} = O_0^\infty(1).$$

Combining this with (5.40)-(5.41) shows (5.38). Therefore, Lemma 5.4 is proved. \qed

In the following, we denote

$$S = \{ S_i : 0 \leq i \leq 2n-3 \}, \text{ where } S_i = \tilde{Z}_i + \sum_{j=0}^{2n-3} \tau_{ij} \tilde{Z}_j, \quad (5.42)$$

which are called as the modified Klainerman’s vector fields.

With (5.36)-(5.37) and (5.39), we have

$$S_i - Z_i = O_1^\infty(1) \nabla, \quad i = 0, 1, \ldots, 2n-3. \quad (5.43)$$

In (5.42), we denote

$$S_T = \{ S_{iT} : 0 \leq i \leq 2n-3 \}, \text{ where } S_{iT} = t \partial t + t \partial t \chi \partial r, \quad S_{IT} = S_i + S_i \chi \cdot \partial_r (1 \leq i \leq 2n-3), \quad (5.44)$$
which are tangential to $\Gamma$.

**Remark 5.3.** From Lemma 5.4, we can also derive that for $m \in \mathbb{N} \cup \{0\}$

$$B_\sigma S^m \hat{\varphi} = 0 \quad \text{on} \quad r = \sigma(t, \omega). \quad (5.45)$$

**Lemma 5.5.** Let $\hat{\varphi}$ be a $C^{k_0}(\overline{D_{T_0}})$ solution to (4.4), where $k_0 \in \mathbb{N}$, and

$$\sum_{0 \leq l \leq \lfloor \frac{n}{2} \rfloor + 1} |\nabla S^l \hat{\varphi}| \leq M \varepsilon,$$

where $M > 0$ is some constant and $\varepsilon$ is sufficiently small. Then

$$C(b_0, \gamma, k_0) \sum_{0 \leq l \leq k_0 - 1} |\nabla_x S^l \hat{\varphi}| \leq \sum_{0 \leq l \leq k_0 - 1} t^l |\nabla_x^{l+1} \hat{\varphi}| \leq C(b_0, \gamma, k_0) \left( \sum_{0 \leq l \leq k_0 - 1} |\nabla_x S^l \hat{\varphi}| + \varepsilon \right) \quad \text{in} \quad D_{T_0}. \quad (5.46)$$

**Proof.** It only comes from a direct computation (or one can see Lemma 5.1 of [19]), we omit it here. \qed

Based on (5.45), (4.13) and Remark 5.3, we can apply Theorem 5.1 for $S^m \hat{\varphi}(0 \leq m \leq k_0 - 1)$ directly to yield

**Lemma 5.6.** Under the assumptions of Theorem 5.1, if $\hat{\varphi}$ is a $C^{k_0}(\overline{D_{T_0}})$ solution ($k_0 \geq 9$) of the problem (4.4) with (4.9) and (4.11)-(4.13), then for $0 \leq m \leq k_0 - 1$ and $\mu < -1 - \frac{1}{2} \sqrt{7 + \frac{7}{2}}$

$$C_1 T_0^{\mu+1} \int_{(T_0, \omega) \leq r \leq (T_0, \omega)} |\nabla S^m \hat{\varphi}|^2 (T_0, x) dx + C_2 \int_{D_{T_0}} t^\mu |\nabla S^m \hat{\varphi}|^2 dt dx$$

$$+ C_3 \int_{I_{T_0}} t^{\mu+1} (\partial_t S^m \hat{\varphi})^2 dS + C_4 \int_{I_{T_0}} t^{\mu+1} \frac{1}{\rho^2} \sum_{k=1}^3 (Z_k S^m \hat{\varphi})^2 dS$$

$$\leq C(b_0, \gamma) \varepsilon^2 + C_5 \int_{I_{T_0}} t^{\mu+1} (B_0 S^m \hat{\varphi})^2 dS + \int_{D_{T_0}} I_{2m} dt dx, \quad (5.47)$$

where the constant $C_i$ ($1 \leq i \leq 5$) is given in Theorem 5.1 and

$$I_{2m} = \left| [S^m, \mathcal{L} + \mathcal{P}] \hat{\varphi} - S^m R_0 \right| \cdot |\mathcal{M} S^m \hat{\varphi}|. \quad (5.48)$$

As in Theorem 5.3, we should control the terms $\int_{I_{T_0}} t^{\mu+1} (B_0 S^m \hat{\varphi})^2 dS$ and $\int_{D_{T_0}} I_{2m} dt dx$ in (5.47) to obtain the related higher-order weighted energy estimates. To this end, as in Lemma 5.2, we can establish the following improved Hardy-type inequality on $\int_{I_{T_0}} t^{\mu-1} |S^m \hat{\varphi}|^2 dS$ due to (4.10)-(4.12).

**Lemma 5.7.** Assume that $\hat{\varphi} \in C^{k_0}(\overline{D_{T_0}})$ and $\xi(t, \omega) \in C^{k_0}([1, T_0] \times \mathbb{S}^2)$ with $k_0 \geq 9$ are the solution of (4.4) with (4.9) and (4.11)-(4.13), and further assume

$$\sum_{0 \leq l \leq \lfloor \frac{n}{2} \rfloor + 1} |S^l \xi| + \sum_{0 \leq l \leq \lfloor \frac{n}{2} \rfloor + 1} |\nabla S^l \hat{\varphi}| \leq M \varepsilon. \quad (5.49)$$
Then

\[
\int_{\Gamma_T} t^{\mu-1} |S^m \varphi|^2 dS \\
\leq C(b_0, \gamma) \varepsilon^2 + \frac{1}{\mu^2} (1 + O(b_0^{-1})) \int_{\Gamma_T} t^{\mu+1} (\partial_t S^m \varphi)^2 dS \\
+ C(b_0, \gamma) \varepsilon \left( \int_{\Gamma_T} t^{\mu+1} (B_0 S^m \varphi)^2 dS + \int_{\Gamma_T} t^{\mu+1} \frac{1}{r^2} \sum_{k=1}^3 (Z_k S^m \varphi)^2 dS \right) \\
+ C(b_0, \gamma) \left( \sum_{0 \leq l \leq m-2} \int_{\Gamma_T} t^{\mu+1} (\nabla S^l \varphi)^2 dS + \int_{\Gamma_T} t^{\mu+1} |\xi|^2 dS \right).
\]

(5.50)

Moreover,

\[
\int_{\Gamma_T} t^{\mu+1} (B_0 S^m \varphi)^2 dS \leq C(b_0, \gamma) \varepsilon^2 + \frac{1}{\mu^2 b_0^2} (1 + O(b_0^{-1})) \int_{\Gamma_T} (\partial_t S^m \varphi)^2 dS \\
+ C(b_0, \gamma) \varepsilon \left( \int_{\Gamma_T} t^{\mu+1} \frac{1}{r^2} \sum_{k=1}^3 (Z_k S^m \varphi)^2 dS \right) \\
+ C(b_0, \gamma) \left( \sum_{0 \leq l \leq m-1} \int_{\Gamma_T} t^{\mu+1} (\nabla S^l \varphi)^2 dS + \int_{\Gamma_T} t^{\mu+1} |\xi|^2 dS \right).
\]

(5.51)

**Proof.** Since the proof procedure is very similar to Lemma 5.2, we just give some necessary descriptions here. It follows from (4.1) and (4.12) that

\[
\begin{align*}
\{ B_0 S^m \varphi + \mu_2 S^m_{T} |\xi| &= \frac{1}{B_1} \sum_{l=1}^m \left( \kappa(\xi, \nabla \varphi) + R_1(t, x) \right) + [B_0, S^m_{T}] \varphi + B_0(S^m - S^m_{T}) \varphi, \\
S^m \varphi &= \mu_3 t : S^m_{T} |\xi| + [S^m_{T}, \mu_3 t] |\xi| + (S^m - S^m_{T}) \varphi \\
\end{align*}
\]

with \([\cdot, \cdot]\) being the commutator.

Then due to Lemma 5.5, the properties of \(\kappa(\xi, \nabla \varphi)\), Lemma 4.2, the assumption (5.49) and (5.44)

\[
\left\{ \begin{array}{l}
B_0 S^m \varphi + \mu_2 S^m_{T} |\xi| = \sum_{0 \leq l \leq m-1} O(b_0) \nabla S^l \varphi + O(\varepsilon) \nabla S^m \varphi + O_{-1}(\varepsilon), \\
S^m \varphi - \mu_3 t : S^m_{T} |\xi| = O(b_0) |\xi| + \sum_{0 \leq l \leq m-2} O(b_0) \nabla S^l \varphi + O(\varepsilon) \nabla S^{m-1} \varphi + O_{-1}(\varepsilon). \\
\end{array} \right.
\]

(5.52)

Since the left sides of the two equalities in (5.52) admit the same forms as in (4.11) and (4.12), similar to Lemma 5.2, one has (5.50). Combining (5.50) with (5.52) and Lemma 5.5 yields (5.51).

**Theorem 5.8.** Assume that \(\varphi \in C^{k_0}(\overline{\Omega_T})\) and \(\xi(t, \omega) \in C^{k_0}([1, T_0] \times S^2)\) with \(k_0 \geq 9\) are the solution of (4.4) with (4.9) and (4.11)-(4.13), and assume

\[
\sum_{0 \leq l \leq \frac{k_0}{2}} |S^l \xi| + \sum_{0 \leq l \leq \frac{k_0}{2} + 1} |\nabla S^l \varphi| \leq M \varepsilon.
\]

(5.53)
Thus with Lemma 5.5,

\[
\int_{\sigma(T_0, \omega) \leq r \leq \zeta(T_0, \omega)} \sum_{0 \leq l \leq k_0 - 1} T_0^{2l+\mu+1} |\nabla\xi^{l+1} \phi|^2(T_0, x) dx + \int_{D_{T_0}} \sum_{0 \leq l \leq k_0 - 1} t^{2l+\mu} |\nabla\xi^{l+1} \phi|^2 dt dx + \int_{\Gamma_{T_0}} \sum_{0 \leq l \leq k_0 - 1} t^{2l+\mu+1} |\nabla\xi^{l+1} \phi|^2 dS \leq C_0 \varepsilon^2. \tag{5.54}
\]

**Proof.** First, by Lemma 5.6-Lemma 5.7 and Theorem 5.3, we have

\[
T_0^{\mu+1} \int_{\sigma(T_0, \omega) \leq r \leq \zeta(T_0, \omega)} |\nabla S^m \phi|^2(T_0, x) dx + \int_{D_{T_0}} t^{\mu} |\nabla S^m \phi|^2 dt dx + \int_{\Gamma_{T_0}} t^{\mu+1} |\nabla S^m \phi|^2 dS \leq C_0 \varepsilon^2 + \int_{D_{T_0}} I_{2m} dt dx + \sum_{0 \leq l \leq m-1} \int_{\Gamma_{T_0}} t^{\mu+1} |\nabla S^l \phi|^2 dS + \int_{\Gamma_{T_0}} t^{\mu+1} |\xi|^2 dS, \tag{5.55}
\]

To prove Theorem 5.8, we should estimate \(\int_{D_{T_0}} I_{2m} dt dx\) and \(\int_{\Gamma_{T_0}} t^{\mu+1} |\xi|^2 dS\) in (5.55).

Due to Lemma 4.5, then \([Z_0, \mathcal{L}] = 2\mathcal{L}, \quad [Z_i, \mathcal{L}] = 0, \quad i = 1, 2, 3.\)

By (5.43), Lemma 5.5 and induction method that

\[
[S^m, \mathcal{L}] \phi = \sum_{0 \leq l \leq m-1} C_{lm} S^l \mathcal{L} \phi + \frac{1}{t} \sum_{k=1}^m O(\varepsilon) \nabla S^k \phi. \tag{5.56}
\]

On the other hand,

\[
[S^m, \mathcal{P}] \phi = \sum_{i,j=1}^2 \left[ S^m, f_{ij} \partial_{ij} \right] \phi + \frac{2}{t} \sum_{i,j=1}^m \left[ S^m, f_{ij} \partial_t \right] \phi.
\]

With the assumption (5.53), it follows from (4.7) that

\[
f_{ij} = O_{b_0}^{k_0-1}(\varepsilon) + O_{b_0}^{k_0-1}(1) \nabla \phi, \quad 0 \leq i \leq 3; 1 \leq j \leq 3,
\]

thus with Lemma 5.5,

\[
||[S^m, \mathcal{P}] \phi|| \leq \frac{1}{1 + t} \sum_{l=0}^m O(\varepsilon) |\nabla S^l \phi|.
\]

Combining this with (5.18), (5.48), (5.56) and Lemma 4.2,

\[
I_{2m} \leq t^{\mu} \left( O(\varepsilon) |\nabla S^m \phi|^2 + \sum_{0 \leq l \leq m-1} \tilde{C}_{lm} |\nabla S^l \phi|^2 + \frac{O(\varepsilon^2)}{(1+t)^2} \right).
\]

This shows that

\[
\int_{D_{T_0}} I_{2m} dt dx \leq C(b_0, \gamma) \left( \varepsilon^2 + \varepsilon \int_{D_{T_0}} t^{\mu} |\nabla S^m \phi|^2 + \sum_{l=0}^{m-1} \int_{D_{T_0}} t^{\mu} |\nabla S^m \phi|^2 dt dx \right). \tag{5.57}
\]
Furthermore, by the boundary condition (4.12) and Lemma 5.2,

$$\int_{\Gamma_{T_0}} t^{\nu+1} |\xi|^2 dS \leq C(b_0, \gamma) \int_{\Gamma_{T_0}} t^{\nu-1} |\dot{\varphi}|^2 dS \leq C(b_0, \gamma) \varepsilon^2. \quad (5.58)$$

Substituting (5.57) and (5.58) into (5.55) and using the induction method and (5.46) in Lemma 5.5 yield (5.46). Thus the proof of Theorem 5.8 is completed. \hfill \Box

**§5.3. Proof of Theorem 1.1 for n = 3**

Based on the higher order energy estimate established in Theorem 5.8, we now prove the global existence of a shock wave in Theorem 1.1 by the local existence result in §3 and the continuous induction method. For any given $t_0 > 0$, the solution of (1.6) with the initial data given on $t = t_0$ and the boundary conditions (1.7)-(1.10) in $[t_0, t_0 + t^*]$ for some $t^* > 0$ can be obtained by the local existence of the solution in §3, provided that the initial data are smooth and satisfy the compatibility conditions. Moreover, if the perturbation of the initial data given on $t = t_0$ is small as $O(\varepsilon)$, then the lifespan of the solution is at least as large as $C \geq \frac{1}{t_0}$ with $C > 0$. Therefore, as long as we can establish that the maximum norm of $\dot{\varphi}, \xi$ and their derivatives decays with a rate in $t$, then the solution can be extended continuously to the whole domain. That is, by the local existence result and the property of decay of the solution we can obtain the uniform bound of $\dot{\varphi}, \xi$ and their derivatives, and then extend the solution continuously from $t_0 \leq t \leq t_0 + \eta^*$ to $t_0 + \eta^* \leq t \leq t_0 + 2\eta^*$ with $\eta^* > 0$ being independent of $t_0$. Hence the key point to prove Theorem 1.1 is to give the decay of the maximum norm of $\dot{\varphi}, \xi$ and their derivatives.

To finish the proof of Theorem 1.1, the following Lemma is required.

**Lemma 5.9.** Under the assumption (5.53) in Theorem 5.8, for $1 \leq t \leq T_0$ we have

$$\sum_{0 \leq l \leq k_0 - 4} |t^l \nabla^{l+1} \dot{\varphi}|^2 \leq C_0 t^{-3} \int_{\sigma(t, \omega) \leq r \leq \tilde{\sigma}(t, \omega)} \sum_{0 \leq l \leq k_0 - 1} |t^l \nabla^{l+1} \dot{\varphi}(t, \omega)|^2 dx. \quad (5.59)$$

**Proof.** We will apply Sobolev’s imbedding theorem to establish (5.59).

For any $t_1 \in [1, T_0]$, set

$$(t', x') = \frac{1}{t_1}(t, x).$$

Then one has

$$\nabla_{t,x'} \dot{\varphi} = \frac{1}{t_1} \nabla_{t',x'} \dot{\varphi}, \quad \forall \, k \in \mathbb{N}. \quad (5.60)$$

Define $D_* = \{(t', x') : t' = 1, \sigma(t, \omega) \leq |x'| \leq \tilde{\sigma}(t, \omega)\}$, then by Sobolev’s imbedding theorem in space dimensions 3 (since $D_*$ has the uniform interior cone condition),

$$|\nabla_{t',x'} \dot{\varphi}|^2(1, x') \leq C \int_{D_*} \sum_{0 \leq l \leq 3} |\nabla^{l+1}_{t',x'} \dot{\varphi}|^2(1, x') dx'. \quad (5.61)$$

With (5.60),

$$|\nabla_{t,x} \dot{\varphi}|^2(t_1, x) = \frac{1}{t_1^2} |\nabla_{t',x'} \dot{\varphi}|^2(1, x') \leq \frac{C}{t_1^4} \int_{D_*} \sum_{0 \leq l \leq 3} |\nabla^{l+1}_{t',x'} \dot{\varphi}|^2(1, x') dx'$$

$$= \frac{C}{t_1^4} \int_{\sigma(t, \omega) \leq r \leq \tilde{\sigma}(t, \omega)} \sum_{0 \leq l \leq 3} |t_1^{l+1} \nabla^{l+1}_{t,x} \dot{\varphi}|^2(t_1, x) \frac{1}{t_1^4} dx \leq \frac{C}{t_1^4} \int_{\sigma(t, \omega) \leq r \leq \tilde{\sigma}(t, \omega)} \sum_{0 \leq l \leq 3} |t_1^{l+1} \nabla^{l+1}_{t,x} \dot{\varphi}|^2(t_1, x) dx \leq \frac{C}{t_1^4} \int_{\sigma(t, \omega) \leq r \leq \tilde{\sigma}(t, \omega)} \sum_{0 \leq l \leq 3} |t_1^{l+1} \nabla^{l+1}_{t,x} \dot{\varphi}|^2(t_1, x) dx.$$
This yields (5.59) for $l = 0$.
In the same way, we can finish the proof of Lemma 5.9 in the case of $1 \leq l \leq k_0 - 4$. □

It follows from (5.54) that

$$\int_{\sigma(t,\omega) \leq r \leq \zeta(t,\omega)} \sum_{0 \leq l \leq k_0 - 1} |t^l \nabla^{l+1} \varphi(t, x)|^2 \, dx \leq C_0 \varepsilon^2 t^{-\mu - 1}.$$ (5.59)

Combining this with (5.59) yields

$$\sum_{0 \leq l \leq k_0 - 4} |t^l \nabla^{l+1} \varphi| \leq C_0 \varepsilon^2 t^{-\mu - 4}$$ for $\sigma(t, \omega) \leq r \leq \chi(t, \omega)$ and $1 \leq t \leq T$. For $k_0 \geq 9$, $k_0 - 4 \geq \left\lfloor \frac{k_0}{2} \right\rfloor + 1$, so one has

$$\sum_{0 \leq l \leq \left\lfloor \frac{k_0}{2} \right\rfloor + 1} |t^l \nabla^{l+1} \varphi| \leq C_0 \varepsilon t^{-\frac{\mu}{2} - 2}.$$ (5.61)

In addition, the equations (4.11) and (4.12) yield

$$\sum_{0 \leq l \leq \left\lfloor \frac{k_0}{2} \right\rfloor + 1} |S^l \xi| \leq M \varepsilon,$$ (5.62)

when $-\frac{\mu}{2} - 2 < 0$.

When we choose $\mu \in (-4, -\tfrac{1}{2} \sqrt{\frac{\gamma + 7}{2}})$, (5.61)-(5.62) shows the induction assumption (5.53) in Theorem 5.8, then the proof of Theorem 4.1 and furthermore Theorem 1.1 can be completed for $n = 3$. □

§6. Sketch of the proof of Theorem 1.1 for $n = 2$

In this section, we only give the sketch on the proof of Theorem 1.1 for $n = 2$ since the main procedure is same as the case $n = 3$ and even simpler. In addition, some related notations in this section admit the same meanings as in 3-D case.

**Theorem 6.1.** For $n=2$, assume that $\varphi \in C^{k_0}(\overline{D_{T_0}})$ and $\xi(t, \omega) \in C^{k_0}([1, T_0] \times [0, 2\pi])$ with $k_0 \geq 7$ are the solution of (4.4) with (4.9) and (4.11)-(4.13), and further assume

$$\sum_{0 \leq l \leq \left\lfloor \frac{k_0}{2} \right\rfloor + 1} |S^l \xi| + \sum_{0 \leq l \leq \left\lfloor \frac{k_0}{2} \right\rfloor + 1} |\nabla S^l \varphi| \leq M \varepsilon.$$ (6.1)

Then for sufficiently small $\varepsilon > 0$ and $\mu < -\tfrac{1}{2} \sqrt{\frac{\gamma + 1}{2}}$,

$$\int_{\sigma(T_0, \omega) \leq r \leq \zeta(T_0, \omega)} \sum_{0 \leq l \leq k_0 - 1} T_0^{2l+\mu+1} |\nabla^{l+1} \varphi(T_0, x)|^2 \, dx + \int_{D_{T_0}} \sum_{0 \leq l \leq k_0 - 1} t^{2l+\mu} |\nabla^{l+1} \varphi|^2 \, dt \, dx$$

$$+ \int_{\Gamma_{T_0}} \sum_{0 \leq l \leq k_0 - 1} t^{2l+\mu+1} |\nabla^{l+1} \varphi|^2 \, dS \leq C(b_0, \gamma) \varepsilon^2.$$ (6.2)

**Sketch of the proof.**

We will divide the proof procedure into the following two steps.

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Step 1. Establishing a priori estimate containing a shock boundary condition

At first, as in Theorem 5.1, we look for such an operator $M$

$$M = A(t, x)\partial_t + B(t, x)\partial_r$$

$$= t^{\mu r} \partial_x + t^{\mu + 1} b_\sigma(t, r, \omega) \partial_r,$$  \hspace{1cm} (6.3)

where

$$b_\sigma(t, r, \omega) = s^2 \left( 1 + \varepsilon \frac{\partial b}{s} - \frac{\varepsilon b}{s} + \varepsilon (e_1 - b_0 - \varepsilon b) \right)$$  \hspace{1cm} (6.4)

with the constant $c_1$ being determined later on.

For $0 \leq m \leq k_0 - 1$, it follows from Lemma 5.4 and Remarks 5.3 that $B_\sigma S^m \bar{\phi} = 0$ holds on $r = \sigma(t, \omega)$.

In this situation, we can claim that for any fixed constant $-3 < \mu < -\frac{1}{2} - \frac{1}{2} \sqrt{\frac{\gamma + 1}{2}}$,

$$C_1 T_0^{\mu + 1} \int_{\sigma(T_0, \omega) \leq r \leq \zeta(T_0, \omega)} |\nabla S^m \bar{\phi}|^2 (T_0, x) dx + C_2 \int \int_{D_{T_0}} t^{\mu} |\nabla S^m \bar{\phi}|^2 dt dx$$

$$+ C_3 \int \int_{\Gamma_{T_0}} t^{\mu + 1} (\partial_t S^m \bar{\phi})^2 dS + C_4 \int \int_{\Gamma_{T_0}} t^{\mu + 1} \frac{1}{r^2} (Z_1 S^m \bar{\phi})^2 dS$$

$$\leq C(b_0, \gamma) \varepsilon^2 + C_5 \int \int_{\Gamma_{T_0}} t^{\mu + 1} (B_0 S^m \bar{\phi})^2 dS + \int \int_{D_{T_0}} I_{3m} dt dx,$$  \hspace{1cm} (6.5)

where $C_i$ ($1 \leq i \leq 5$) are just only the constants given in Theorem 5.1, and

$$I_{3m} = \|[S^m, \mathcal{L} + \mathcal{P}] \bar{\phi} - S^m R_0 \| \cdot |M S^m \bar{\phi}|.$$  \hspace{1cm} (6.6)

Indeed, it follows from the integration by parts that

$$\int \int_{D_{T_0}} (S^m R_0 - [S^m, \mathcal{L} + \mathcal{P}] \bar{\phi}) \cdot M S^m \bar{\phi} dt dx$$

$$= \int \int_{D_{T_0}} (\mathcal{L} + \mathcal{P}) S^m \bar{\phi} \cdot M S^m \bar{\phi} dt dx$$

$$= \int \int_{D_{T_0}} I_{4m} dt dx$$

$$+ \int_{\sigma(T_0, \omega) < r < \zeta(T_0, \omega)} N_0(S^m \bar{\phi})(T_0, x) dx - \int_{\sigma(1, \omega) < r < \zeta(1, \omega)} N_0(S^m \bar{\phi})(1, x) dx$$

$$+ \int_{\Gamma_{T_0}} \left( 2 \sum_{i=1}^{2} \frac{x_i}{r} N_i(S^m \bar{\phi}) - \partial_t \chi \cdot N_i(S^m \bar{\phi}) - \partial_r \chi N_i(S^m \bar{\phi}) \right) dS$$

$$+ \int_{\Gamma_{T_0}} \left( \partial_t \sigma N_0(S^m \bar{\phi}) - 2 \sum_{i=1}^{2} \frac{x_i}{r} N_i(S^m \bar{\phi}) - \partial_t \sigma \cdot N_i(S^m \bar{\phi}) \right) dS$$  \hspace{1cm} (6.7)

with

$$I_{4m} = K_{00}(\partial_t S^m \bar{\phi})^2 + K_{0r} \partial_t S^m \bar{\phi} \partial_r S^m \bar{\phi} + K_{rr} (\partial_r S^m \bar{\phi})^2 + \frac{K_{22}}{r^2} (Z_1 S^m \bar{\phi})^2$$

$$+ K_{01} Z_1 A \cdot \partial_t S^m \bar{\phi} Z_1 S^m \bar{\phi} + K_{r1} Z_1 B \cdot \partial_r S^m \bar{\phi} Z_1 S^m \bar{\phi}$$

$$+ K_{0i} (\partial_t S^m \bar{\phi})^2 + 2 \sum_{i=1}^{2} K_{0i} \partial_t S^m \bar{\phi} \partial_i S^m \bar{\phi} + 2 \sum_{i,j=1}^{2} K_{ij} \partial_i S^m \bar{\phi} \partial_j S^m \bar{\phi}.$$  \hspace{1cm} (6.8)
Similar to Step 3 in the proof of Theorem 5.1, under the assumption (6.1), we can derive from (6.8) that

$$I_{4m} = K_{00}(\partial_t S_m\dot{\varphi})^2 + K_{0r}\partial_t S_m\dot{\varphi}\partial_r S_m\dot{\varphi} + K_{rr}(\partial_r S_m\dot{\varphi})^2 + \frac{K_{22}}{r^2}(Z_1 S_m\dot{\varphi})^2 + t^\mu (O(\varepsilon)|\nabla S_m\dot{\varphi}|)^2$$

with

$$
\begin{align*}
K_{00} &= t^\mu \left( \frac{1}{2}(1 + e_1 - \mu)b_0(1 + O(b_0^{-2})) + O(b_0^{-2}) \right), \\
K_{0r} &= t^\mu \left( \frac{\gamma + 1}{4}(1 + e_1 - \mu)b_0(1 + O(b_0^{-2})) + O(b_0^{-2}) \right), \\
K_{rr} &= t^\mu \left( \frac{\gamma + 1}{4}(1 + e_1 - \mu)b_0^2(1 + O(b_0^{-2})) + O(b_0^{-2}) \right), \\
K_{22} &= t^\mu \left( -\frac{\gamma - 1}{4}(1 + e_1 + \mu)(1 + O(b_0^{-2})) + O(b_0^{-2}) \right), \\
K_{2r} - 4K_{00}K_{rr} &= t^{2\mu} \left( \frac{\gamma - 1}{4}(1 + 2(e_1 - \mu)^2)(1 + O(b_0^{-2})) + O(b_0^{-2}) \right).
\end{align*}
$$

The sufficient conditions for $I_{4m} \geq 0$ are

$$1 + e_1 - \mu > 0, \quad 1 + e_1 + \mu < 0, \quad \gamma + 1 - 2(e_1 - \mu)^2 < 0. \quad (6.9)$$

Consequently, if we select

$$\mu < -\frac{1}{2} - \frac{1}{2} \sqrt{\frac{\gamma + 1}{2}} \quad \text{and} \quad e_1 = \frac{1}{2} \sqrt{\frac{\gamma + 1}{2}} - \frac{1}{2}, \quad (6.10)$$

then (6.9) stands. This implies

$$K_{00} > 0, \quad K_{0r}^2 - 4K_{00}K_{rr} < 0, \quad K_{22} > 0,$$

and then

$$\iint_{D_{T_o}} I_{4m} \, dt \, dx \geq C_2 \iint_{D_{T_o}} t^{\mu}|\nabla \dot{\varphi}|^2 \, dt \, dx. \quad (6.11)$$

Moreover, similar to the proof of Theorem 5.1, under the constrain (6.10), the definitions (6.3)-(6.4) and (4.13), we have

$$
\begin{align*}
\int_{\Gamma_{T_o}} \left( \sum_{k=1}^{2} \frac{x_k}{r} N_i(\nabla S_m\dot{\varphi}) - \partial_i \chi \cdot N_i(\nabla S_m\dot{\varphi}) - \partial_i \chi N_i(\nabla S_m\dot{\varphi}) \right) \, ds \\
\geq C_3 \iint_{\Gamma_{T_o}} t^{\mu+1}(\partial_t S_m\dot{\varphi})^2 \, ds + C_4 \iint_{\Gamma_{T_o}} t^{\mu+1} \frac{1}{r^2}(Z_1 S_m\dot{\varphi})^2 \, ds - C_5 \iint_{\Gamma_{T_o}} t^{\mu+1}(B_0 S_m\dot{\varphi})^2 \, ds, \\
\int_{\sigma(T_0, \omega)} N_i(\nabla S_m\dot{\varphi})(T_0, x) \, dx \geq C_1 \iint_{\sigma(T_0, \omega)} |\nabla S_m\dot{\varphi}|^2(T_0, x) \, dx, \\
\int_{\sigma(1, \omega)} N_i(\nabla S_m\dot{\varphi})(1, x) \, dx \leq C(b_0, \gamma)\varepsilon^2.
\end{align*}
$$

Substituting (6.11)-(6.12) into (6.7) and subsequently combining with (6.6) yield (6.5).
Step 2. Establishing a priori estimate on the solution

In order to obtain (6.2), we should deal with the term \( \int_{\Gamma_{T_0}} t^{\mu+1} (B_0 S^m \phi)^2 dS \) in (6.5).

Due to \( \mu_2 < 0 \) and \( \mu_3 (t, x) < 0 \) in (4.11) and (4.12), we can obtain the following improved Hardy-type inequality

\[
\int_{\Gamma_{T_0}} t^{\mu-1} (S^m \phi)^2 dS \leq C(b_0, \gamma) \varepsilon^2 + \frac{1}{\mu^2} (1 + O(b_0^{-\frac{2}{\mu}})) \int_{\Gamma_{T_0}} t^{\mu+1} (\partial_t S^m \phi)^2 dS \\
+ C(b_0, \gamma) \left( \int_{\Gamma_{T_0}} t^{\mu+1} (B_0 S^m \phi)^2 dS + \int_{\Gamma_{T_0}} \frac{1}{r^2} (Z_1 S^m \phi)^2 dS \right) \\
+ C(b_0, \gamma) \left( \sum_{0 \leq l \leq m} \int_{\Gamma_{T_0}} t^{\mu-1} (\nabla S^l \phi)^2 dS + \int_{\Gamma_{T_0}} t^{\mu+1} |\xi|^2 dS \right).
\]  

(6.13)

So with (6.13), it follows from (5.52) and Lemma 4.4 that

\[
\int_{\Gamma_{T_0}} t^{\mu+1} (B_0 S^m \phi)^2 dS \leq C(b_0, \gamma) \varepsilon^2 + \frac{1}{4 \mu^2 b_0^2} (1 + O(b_0^{-\frac{2}{\mu}})) \int_{\Gamma_{T_0}} t^{\mu+1} (\partial_t S^m \phi)^2 dS \\
+ C(b_0, \gamma) \int_{\Gamma_{T_0}} \frac{1}{r^2} (Z_1 S^m \phi)^2 dS \\
+ C(b_0, \gamma) \left( \sum_{0 \leq l \leq m} \int_{\Gamma_{T_0}} t^{\mu-1} (\nabla S^l \phi)^2 dS + \int_{\Gamma_{T_0}} t^{\mu+1} |\xi|^2 dS \right).
\]  

(6.14)

Substituting (6.13)-(6.14) and (4.12) into (6.5) yields

\[
T_0^{\mu+1} \int_{\sigma(T_0, \omega) \leq r \leq \zeta(T_0, \omega)} |\nabla S^m \phi|^2 (T_0, x) dx + \int_{D_{T_0}} t^{\mu} |\nabla S^m \phi|^2 dtdx \\
+ \frac{\gamma - 1}{8} b_0^2 \left( 1 - \frac{1}{\mu^2} \right) (1 + O(b_0^{-\frac{2}{\mu}})) \int_{\Gamma_{T_0}} t^{\mu+1} (\partial_t S^m \phi)^2 dS \\
+ \int_{\Gamma_{T_0}} \frac{t^{\mu+1}}{r^2} (Z_1 S^m \phi)^2 dS \\
\leq C(b_0, \gamma) \varepsilon^2 + \int_{D_{T_0}} I_{3m} dtdx + \sum_{0 \leq l \leq m-1} \int_{\Gamma_{T_0}} t^{\mu+1} (\nabla S^l \phi)^2 dS.
\]  

(6.15)

Next, we deal with \( \int_{D_{T_0}} I_{3m} dtdx \) in (6.15).

Due to Lemma 4.4,

\[ [Z_0, \mathcal{L}] = 2 \mathcal{L}, \quad [Z_1, \mathcal{L}] = 0. \]

By (5.43), Lemma 5.5 and induction method, one has

\[
[S^m, \mathcal{L}] \phi = \sum_{0 \leq l \leq m-1} C_{lm} S^l \mathcal{L} \phi + \frac{1}{r} \sum_{k=1}^{m} O(\varepsilon) \nabla S^k \phi.
\]  

(6.16)

On the other hand,

\[
[S^m, \mathcal{P}] \phi = \sum_{i,j=1}^{2} \left[ S^m, f_{ij} \partial_{ij} \right] \phi + \frac{2}{r} \sum_{i} \left[ S^m, f_{ij} \partial_i \right] \phi.
\]
With the assumption (5.53), it follows from (4.7) that
\[ f_{ij} = O_0^{k_0-1}(\varepsilon) + O_0^{k_0-1}(1)\nabla \varphi \quad (0 \leq i \leq 2; 1 \leq j \leq 2), \]
thus with Lemma 5.5,
\[ |[S^m, P] \dot{\varphi}| \leq \frac{1}{1 + t} \sum_{l=0}^{m} O(\varepsilon)|\nabla S^m \dot{\varphi}|. \]
Combining this with (5.56), (5.18), (6.16) and Lemma 4.2 yields
\[ I_{3m} \leq t^{\mu} \left( O(\varepsilon)|\nabla S^m \dot{\varphi}|^2 + \sum_{0 \leq l \leq m-1} \tilde{C}_{lm} |\nabla S^l \dot{\varphi}|^2 + \frac{O(\varepsilon^2)}{(1+t)^2} \right). \]
This shows that
\[ \iint_{D_{T_0}} I_{3m} dtdx \leq C(b_0, \gamma) \left( \varepsilon^2 + \varepsilon \iint_{D_{T_0}} t^{\mu}|\nabla S^m \dot{\varphi}|^2 + \sum_{l=0}^{m-1} \iint_{D_{T_0}} t^{\mu}|\nabla S^m \dot{\varphi}|^2 dtdx \right). \]
Substituting this into (6.15) yields (6.2) and then the proof of Theorem 6.1 is completed. \[ \square \]

Based on Theorem 6.1, we can prove Theorem 4.1 and Theorem 1.1 in the case of \( n = 2 \) by the local existence result and the continuous induction method. Similar to the case of \( n = 3 \), we just need to verify the induction assumption (6.2). Similar to Lemma 5.9, it follows from the Sobolev’s imbedding theorem and the assumptions of Theorem 6.1 that for \( \sigma(t, \omega) \leq r \leq \chi(t, \omega) \) and \( 1 \leq t \leq T_0 \), one has
\[ \sum_{0 \leq l \leq k_0-3} |t^l \nabla^{l+1} \dot{\varphi}|^2 \leq C t^{-2} \int_{\sigma(t,\omega) \leq r \leq \chi(t,\omega)} \sum_{0 \leq l \leq k_0-1} |t^l \nabla^{l+1} \dot{\varphi}(t,x)|^2 dx. \]
On the other hand, (6.2) shows that
\[ \int_{\sigma(t,\omega) \leq r \leq \chi(t,\omega)} \sum_{0 \leq l \leq k_0-1} |t^l \nabla^{l+1} \dot{\varphi}(t,x)|^2 dx \leq C_0 \varepsilon^2 t^{-\mu-1}. \]
Hence \[ \sum_{0 \leq l \leq k_0-3} |t^l \nabla^{l+1} \dot{\varphi}|^2 \leq C_0 \varepsilon^2 t^{-\mu-3} \] for \( \sigma(t,\omega) \leq r \leq \chi(t,\omega) \) and \( 1 \leq t \leq T_0 \). For \( k_0 \geq 7 \), then \( k_0 - 3 \geq \left\lceil \frac{k_0}{2} \right\rceil + 1 \), so one has
\[ \sum_{l \leq \left\lceil \frac{k_0}{2} \right\rceil + 1} |t^l \nabla^{l+1} \dot{\varphi}| \leq C(b_0) \varepsilon t^{-\frac{5}{2}} - \frac{3}{2}. \]
(6.17)
In addition, the equations (4.11) and (4.12) yield
\[ \sum_{0 \leq l \leq \left\lceil \frac{k_0}{2} \right\rceil + 1} |S^l \xi| \leq \frac{M}{2} \varepsilon, \]
(6.18)
when \( -\mu - 3 < 0 \). When we choose \( \mu \in (-3, -\frac{3}{2} - \frac{1}{2} \sqrt{\frac{3+1}{2}}) \), (6.17)-(6.18) shows the assumption 6.1 in Theorem 6.1, then the proof of Theorem 4.1 and meanwhile Theorem 1.1 can be completed for \( n = 2 \). \[ \square \]
Appendix A. Some basic computations

In this Appendix, at first, we will give the detailed derivations on the coefficients in the equation (3.10) under the transformations (3.5) and (3.6). In terms of the expressions of $A_{k,0}, A_{k,1}, \ldots, A_{6,2}^i$, which are given in (3.10), we have

**Lemma A.1.**

\[
\begin{align*}
A_{1,0} &= \psi, \\
A_{2,0} &= (R-2)\partial_T b - a_1 [(R-1)a_3 + \psi \partial_T a_0], \\
A_{3,0}^i &= 0, \quad i = 1, 2, 3, \\
A_{4,0} &= \partial_T a_0 \cdot a_1 \left( (R-1)a_1 \cdot a_3 - (R-2)\partial_T b \right), \\
A_{5,0}^i &= 0, \quad i = 1, 2, 3, \\
A_{6,0}^i &= 0, \quad i, j = 1, 2, 3 \\
A_{7,0} &= (\partial_T \psi)^2 + \psi \partial_T b + \partial_T \psi \cdot \partial_T b + (R-2)\partial_T \psi \cdot \partial_T b + (R-2)\partial_T R \psi + 2\partial_T a_0 \cdot a_1 \cdot a_3 \cdot \partial_T \psi, \\
\end{align*}
\]

where $a_0, a_1, a_2$ and $a_3$ are given in (3.8).

**Proof.** Under the transformations (3.5) and (3.6), a direct computation yields

\[
\partial_T^2 \phi = -b_0(\partial_T - a_1 \cdot \partial_T a_0 \partial_T R)(a_1 \cdot a_3)
\]

\[
= b_0 a_1^2 \cdot a_3 (\partial_T \psi + (R - 1)\partial_T^2 R \psi)
\]

\[
- b_0 a_1 (\partial_T \psi)^2 + \psi \partial_T^2 b + \partial_T \psi b + \psi \partial_T b + (R-2)\partial_T \psi \partial_T R \psi + (R-2)\partial_T b \partial_T R \psi
\]

\[
- b_0 a_1^2 \cdot a_3 \cdot \partial_T a_0 (2\partial_T R \psi + (R - 1)\partial_T^2 R \psi)
\]

\[
+ b_0 a_1^2 \partial_T a_0 (\partial_T \psi \partial_T R \psi + \psi \partial_T^2 R \psi + 2\partial_T R \psi \partial_T b + (R-2)\partial_T b \partial_T R \psi).
\]

(A.1)

On the other hand, by comparing the coefficients of power $t^0$ in the equation of (3.2), we also have

\[
\partial_T^2 \phi = -b_0 a_1 \left( A_{1,0} \partial_T^2 \psi + A_{2,0} \partial_T R \psi + \sum_{i=1}^{3} A_{3,0}^i \partial_T Z_i \psi + A_{4,0} \partial_T^2 R \psi + \sum_{i=1}^{3} A_{5,0}^i \partial_T Z_i \psi \\
+ \sum_{i=1}^{3} A_{6,0}^i \partial_T Z_j \psi + A_{7,0} \right).
\]

(A.2)

Thus, by (A.1) and (A.2), we can complete the proof on Lemma A.1. \qed

Analogously, a direct but tedious computation yields
Lemma A.2.

\[
\begin{aligned}
A_{1,1} &= 0, \\
A_{2,1} &= 2(b_0 a_1 a_2 - a_0)(1 - (R - 1)a_1 \partial_R \psi) \\
    &+ \frac{2b_0 a_1}{a_0^2}(R - 1)a_1 \sum_{i=1}^{3} (a_i^4)^2 - (R - 2) \sum_{i=1}^{3} Z_i b \cdot a_i^4, \\
A_{3,1} &= -\frac{2b_0}{a_0^2} a_1 \cdot a_4^4 \cdot \psi, \quad i, j = 1, 2, 3, \\
A_{4,1} &= 2\partial_T a_0 a_1 \left( b_0 a_1 a_2 - a_0 \right) \left( (R - 1)a_1 \partial_R \psi - 1 \right) \\
    &+ \frac{2b_0}{a_0^2} \partial_T a_0 a_1^2 \left( (R - 2) \sum_{i=1}^{3} Z_i b \cdot a_i^4 - (R - 1)a_1 \sum_{i=1}^{3} (a_i^4)^2 \right), \\
A_{5,1} &= \frac{2b_0}{a_0^2} \partial_T a_0 a_1^2 \psi a_i^4, \quad i, j = 1, 2, 3, \\
A_{6,1} &= 0, \quad i, j = 1, 2, 3, \\
A_{7,1} &= 2a_1 \left( \frac{b_0}{a_0^2} a_1 \sum_{i=1}^{3} (a_i^4)^2 - \partial_T \psi (b_0 a_1 a_2 - a_0) \right) \left( \partial_T \psi - 2a_1 \partial_T a_0 \partial_T \psi \right) + 2a_3 \\
    &- \frac{2b_0}{a_0^2} a_1 \sum_{i=1}^{3} a_i^4 \left( \partial_T \psi Z_i \psi + \partial_T \psi Z_i \psi + \psi \partial_T Z_i b + (R - 2) \partial_T Z_i b \partial_T \psi \right) \\
    &- a_1 \partial_T a_0 \left[ \partial_T \psi Z_i \psi + 2\partial_T \psi Z_i \psi \right].
\end{aligned}
\]

Lemma A.3.

\[
A_{1,2} = 0, \quad A_{2,2} = 0, \quad A_{3,2} = 0, \quad i, j = 1, 2, 3,
\]

and

\[
A_{4,2} = a_1 \left( (b_0 a_1 a_2 - a_0)^2 - c^2(\rho) \right) (1 - (R - 1)a_1 \partial_R \psi) \\
    + \frac{2b_0 a_1}{a_0^2} (a_0 - b_0 a_1 a_2) \sum_{i=1}^{3} a_i^4 \left( (R - 2)a_1 Z_i b - (R - 1)a_i^2 a_i^4 \right) \\
    + \frac{1}{a_0^2} \sum_{i=1}^{3} \sum_{j=1}^{3} \left( c^2(\rho) \delta_{ij} - \frac{(b_0 a_1)^2}{a_0^2} a_i^4 a_j^4 \right) \left( (R - 2)Z_j b - (R - 1)a_i^2 \right) a_1 Z_i a_0, \\
A_{5,2} = \frac{2b_0 a_1^2}{a_0^2} (a_0 - b_0 a_1 a_2) a_4^4 \psi \\
    + \frac{1}{a_0^2} \sum_{j=1}^{3} \left( c^2(\rho) \delta_{ij} - \frac{b_0 a_1^2}{a_0^2} a_i^4 a_j^4 \right) \left( a_1 Z_j a_0 \psi + (R - 1)a_i^2 a_4^4 - (R - 2)Z_j b \right), \quad i, j = 1, 2, 3, \\
A_{6,2} = \frac{1}{a_0^2} \left( \frac{b_0^2 a_1^2}{a_0^2} a_i^4 a_j^4 - c^2(\rho) \delta_{ij} \right) \psi, \quad i, j = 1, 2, 3,
\]

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that for $k$

Next, we estimate $E_k$ together with Lemma 3.2, yields for Lemma A.4.

It follows from (3.9) that Under the assumptions in 7.

\[ 21 \left( \sum_{j=1}^{3} (b_0 a_1 a_2 - 2b_0 + \gamma - 1) b_0 \right) (1 + O(b_0^{-\gamma})) + O(b_0^{-2}) \]

& - a_1 Z_i a_0 \partial_R (Z_j b + 2 Z_j b - a_1 a_4 (Z_i \psi + 2 Z_j b)) \right).

Next, we estimate $E_k(d_0, \psi, \nabla \psi)$ for $k = 0, 1, 2, \ldots, K$, which are defined in (3.21) and (3.32) respectively.

Lemma A.4. If $\|\psi - \hat{\psi}\|_{C^2} \leq C \varepsilon$, then for large $b_0$ and small $\varepsilon$

\[ E_k(d_0, \psi, \nabla \psi) \geq \frac{(\gamma - 1) b_0}{2} > 0, \quad k = 0, 1, 2, \ldots, K. \] (A.3)

Proof. It follows from (3.2), Lemma 2.1 and a direct computation that for $k = 0, 1, 2, \ldots, K$

\[ E_k(d_0, \psi, \nabla \psi) = \left( (k + 1)(s_0 - b_0) - 2(\gamma - 1) k s_0 - b_0 + (\gamma - 1) b_0 \right) (1 + O(b_0^{-\gamma})) + O(b_0^{-2}) \]

(\gamma - 1) b_0) + O(b_0^{-1}) \) holds. This, together with (A.4), yields (A.3).

For the functions $D_{21}^k, D_{22}^k$ in (3.21) and $D_{21}^k, D_{22}^k$ in (3.33), we have

Lemma A.5. Under the assumptions in Lemma A.4, one has on $R = 2$ and for $k = 0, 1, \ldots, K$

\[ \left\{ \begin{array}{l}
D_{21}^k = -\rho_0 (1 + O(b_0^{-\gamma})) + O(b_0^{-2}) < 0,
D_{22}^k \leq -\frac{(1 + k)(s_0 - b_0)}{b_0} \rho (1 + O(b_0^{-\gamma})) + O(b_0^{-2}) < 0.
\end{array} \right. \]

Proof. It follows from (3.9) that

\[ \begin{align*}
\partial_{\psi} \mathbb{H}(b, \psi, \nabla \psi) &= \frac{\rho}{c^2(\rho)} \partial_{\psi} A_0 (b, \psi, \nabla \psi),
\partial_{\partial_R \psi} \mathbb{H}(b, \psi, \nabla \psi) &= \frac{\rho}{c^2(\rho)} \partial_{\partial_R \psi} A_0 (b, \psi, \nabla \psi),
\partial_{\partial_T \psi} \mathbb{H}(b, \psi, \nabla \psi) &= \frac{\rho}{c^2(\rho)} \partial_{\partial_T \psi} A_0 (b, \psi, \nabla \psi).
\end{align*} \]

This, together with Lemma 3.2, yields for $k = 0, 1, 2, \ldots, K$ and small $\varepsilon$

\[ D_{21}^k(d_0, \psi, \nabla \psi) = \partial_{\partial_R \psi} \left( \frac{\rho}{c^2(\rho)} \partial_{\partial_R \psi} A_0 - \frac{\rho}{c^2(\rho)} \partial_{\partial_R \psi} A_0 (T \partial_T a_0 + a_0) \right) (b_0, \hat{\psi}, \nabla \hat{\psi}) + O_k(\varepsilon) \]

\[ = -\frac{\rho}{c^2(\rho)} \partial_{\partial_R \psi} A_0 - \frac{\rho}{c^2(\rho)} \partial_{\partial_R \psi} A_0 (T \partial_T a_0 + a_0) \]

\[ + \frac{\rho}{c^2(\rho)} \partial_{\partial_T \psi} A_0 (T \partial_T a_0 + a_0) (b_0, \hat{\psi}, \nabla \hat{\psi}) + O_k(\varepsilon) \]

\[ = -\rho_0 (1 + O(b_0^{-\gamma})) + O(b_0^{-2}) < 0 \]
and

\[ D_{22}^{k}(d_0, \psi, \nabla \psi) = \left( \partial_\psi + \frac{1}{T} k \partial_{\psi} \right) \left( \mathbb{H}(\cdot) \psi - \frac{1}{b_0 a_1} (\mathbb{H}(\cdot) - \rho_0) (T \partial_{T} a_0 + a_0) \right) (b_0, \hat{\psi}) + O_k(\varepsilon) \]

\[ = \left( \frac{\hat{\psi}}{c^2(\hat{\rho})} \partial_\psi A_0 + \mathbb{H}(\cdot) - \frac{1}{b_0 a_1} \frac{\hat{\rho}}{c^2(\hat{\rho})} \partial_\psi A_0 (T \partial_{T} a_0 + a_0) - \frac{1}{b_0} (\mathbb{H}(\cdot) - \rho_0) (T \partial_{T} a_0 + a_0) \right) \]

\[ - \frac{1}{b_0 a_1} (\mathbb{H}(\cdot) - \rho_0) + \frac{k}{T} \left[ \frac{\hat{\psi}}{c^2(\hat{\rho})} \partial_{\psi} \partial_{\varphi} A_0 - \frac{1}{b_0 a_1} \frac{\hat{\rho}}{c^2(\hat{\rho})} \partial_{\psi} \partial_{\varphi} A_0 (T \partial_{T} a_0 + a_0) \right] \]

\[ - \frac{T}{b_0 a_1} (\mathbb{H}(\cdot) - \rho_0) \right) (b_0, \hat{\psi}) + O_k(\varepsilon) \]

\[ \leq - \frac{(1 + k) (s_0 - b_0)}{b_0} \rho_0 (1 + O(b_0^{-\frac{3}{2}})) + O(b_0^{-2}). \]

Thus, we complete the proof on Lemma A.5. \[ \square \]

**Lemma A.6.** In (3.48), if the variables \((b, \psi^k, \hat{\psi}_{n-1})\) are replaced by \((b_0, \hat{\psi}, 0)\) correspondingly, then we have

\[ B_{20}^n = - \left( \frac{\gamma - 1}{2 A \gamma} \right) \frac{1}{b_0} \frac{\hat{\psi}}{b_0} \left( s_0 - b_0 \right) (1 + O(b_0^{-\frac{3}{2}})) + O(b_0^{-2}), \]

\[ B_{21}^n = - \left( \frac{\gamma - 1}{2 A \gamma} \right) \frac{1}{b_0} \frac{\hat{\psi}}{b_0} \left( s_0 - b_0 \right) (1 + O(b_0^{-\frac{3}{2}})) + O(b_0^{-2}), \]

\[ B_{22}^n = O(b_0^{-\frac{3}{2}}) + O(b_0^{-2}), \quad i = 1, 2, 3. \]

**Proof.** For \((b, \psi^k, \hat{\psi}_{n-1}) = (b_0, \hat{\psi}, 0)\), we have

\[ \left\{ \begin{align*}
B_{20}^n &= - \frac{1}{b_0 a_1} (\mathbb{H}(b_0, \hat{\psi}, \nabla \hat{\psi}) - \rho_0) e^{-X(\psi - \frac{1}{b_0 a_1} (\partial_X a_0 + a_0))} \partial_{\partial_{\psi} \psi} \mathbb{H}(\cdot) \bigg|_{b=b_0, \psi=\hat{\psi}}, \\
B_{21}^n &= - \frac{1}{b_0} (\mathbb{H}(\cdot) - \rho_0) (\partial_X a_0 + a_0) + (\hat{\psi} - \frac{1}{b_0 a_1} (\partial_X a_0 + a_0) \partial_{\partial_{\psi} \psi} \mathbb{H}(\cdot) \bigg|_{b=b_0, \psi=\hat{\psi}} \\
B_{22}^i &= 0, & i = 1, 2, 3.
\end{align*} \right. \]

This, together with Lemma 2.1 and Lemma 3.2, yields for \((b, \psi^k, \hat{\psi}_{n-1}) = (b_0, \hat{\psi}, 0)\)

\[ \left\{ \begin{align*}
B_{20}^n &= - \frac{1}{b_0} \hat{\rho}_0 \hat{\psi} (1 + O(b_0^{-\frac{3}{2}})) + O(b_0^{-2}) \\
&= - \left( \frac{\gamma - 1}{2 A \gamma} \right) \frac{1}{b_0} \frac{\hat{\rho}_0 \hat{\psi}}{b_0} (s_0 - b_0) (1 + O(b_0^{-\frac{3}{2}})) + O(b_0^{-2}), \\
B_{21}^n &= - \hat{\rho}_0 \hat{\psi} (1 + O(b_0^{-\frac{3}{2}})) + O(b_0^{-2}) \\
&= - \left( \frac{\gamma - 1}{2 A \gamma} \right) \frac{1}{b_0} \frac{\hat{\rho}_0 \hat{\psi}}{b_0} (s_0 - b_0) (1 + O(b_0^{-\frac{3}{2}})) + O(b_0^{-2}), \\
B_{22}^i &= O(b_0^{-\frac{3}{2}}) + O(b_0^{-2}), & i = 1, 2, 3.
\end{align*} \right. \]

Therefore, we complete the proof of Lemma A.6. \[ \square \]
Appendix B. Modified background solution

In this Appendix, we look for a modified background solution $\Phi_a(t, x)$ such that it satisfies the Neumann-type boundary condition (1.7) since the background solution $\hat{\Phi}(t, x)$ given in Remark 2.1 does not satisfy (1.7), which will be crucial in looking for the multiplier to derive the energy estimate for the problem (4.4) with (4.9) and (4.11)-(4.13).

**Lemma B.1.** Assume $\left| \frac{\zeta}{t} - s_0 \right| \leq \frac{s_0 - b_0}{2}$, then there exists a smooth function $f_a = f_a(t, r, \omega)$ such that

$$f_a \in O^{-1}_\infty(\varepsilon) \quad (B.1)$$

and the function $\Phi_a(t, r, \omega) = (1 + f_a(t, r, \omega))(\hat{\Phi}(t, x))$ satisfies

$$B_\sigma \Phi_a = \partial_t \sigma \quad \text{on} \quad r = \sigma(t, \omega),$$

where the operator $B_\sigma$ is given in (1.7), and the meaning of the notation $O^{-1}_\infty(\varepsilon)$ can be referred to (4.2).

**Proof.** We look for $f_a(t, r, \omega)$ to satisfy

$$
\begin{cases}
B_\sigma f_a \mid_{\Omega_+} = \frac{1}{\Phi(x)} \left( \partial_t \sigma - \hat{u}(\sigma) \right) \\
f_a \mid_{\Gamma} = 0
\end{cases}
$$

which implies that (B.2) holds.

Obviously, under the assumptions in Theorem 1.1, $r = \sigma(t, \omega)$ is not a characteristic surface of the first order linear equation in (B.3). This means that (B.3) has a unique smooth solution $f_a(t, r, \omega)$ in $\Omega_+$. We now analyze the properties of the solution $f_a(t, r, \omega)$.

In terms of Lemma 4.5 (iv), $B_\sigma$ has such a form $B_\sigma = \sum_{i=1}^{n} \frac{x_i r}{\partial_i \sigma} \cdot \partial_i$. We assume that

$$L_{t, r_0, \omega_0} : \quad x = x(r; t, r_0, \omega_0)$$

is an integral curve of $B_\sigma$, which starts from the point $(t, r_0, \omega_0)$ with $r_0 = \sigma(t, \omega_0)$. That is, $x = x(r; t, r_0, \omega_0)$ satisfies

$$
\begin{cases}
\frac{dx_i(r; t, r_0, \omega_0)}{dr} = \left( \frac{x_i}{r} - \partial_i \sigma \right) |_{L_{t, r_0, \omega_0}} , & i = 1, \ldots, n, \\
x(r_0; t, r_0, \omega_0) = r_0 \omega_0.
\end{cases}
$$

Under the assumptions in Theorem 1.1, the ODE system (B.4) has a unique smooth solution in $\Omega_+$, which is written as

$$x = x(r; t, r_0, \omega_0).$$

It follows from (B.3) and (B.4) that

$$
\begin{cases}
\frac{df_a}{dr} = \frac{1}{\Phi(x)} \left( \partial_t \sigma - \hat{u}(\sigma) \right) & \text{on} \quad L_{t, r_0, \omega_0} \cap \Omega_+, \\
f_a(t, r_0, \omega_0) = 0.
\end{cases}
$$

From this, $f_a(t, r, \omega)$ admits an explicit expression as follows

$$f_a(t, r, \omega) = \int_{r_0}^{r} \frac{1}{\Phi(x)} \left( \partial_t \sigma - \hat{u}(\sigma) \right) |_{L_{t, r_0, \omega_0} \cap \Omega_+} d\tau.$$
By the assumptions of Theorem 1.1 and $\hat{u}(b_0) = b_0$, one has
\[ E(t, \omega) \triangleq \frac{1}{\Phi(\frac{\sigma}{t})} (\partial_t \sigma - \hat{u}(\frac{\sigma}{t})) = O^\infty(\varepsilon). \]

This, together with $|\zeta - \sigma| \leq 2(s_0 - b_0)t$ and (B.5), yields
\[ |f_a| \leq \frac{C\varepsilon}{t^2} |\zeta - \sigma| \leq \frac{C\varepsilon}{t}. \quad \text{(B.6)} \]

For any vector field $S \in S$ (here $S$ denotes the modified Klainerman’s vector fields in (5.42)), in terms of Remark 5.3, we have
\[
\begin{cases}
    B_\sigma S f_a = SE(t, \omega) + [B_\sigma, S] f_a & \text{in } \Omega_+,
    \\
    S f_a = 0 & \text{on } r = \sigma(t, \omega).
\end{cases}
\quad \text{(B.7)}
\]

By (B.6) and some related computations in the proof procedure of Lemma 5.4, (B.7) can be reduced into
\[
\begin{cases}
    B_\sigma S f_a = O^\infty_2(\varepsilon) + O^\infty_1(\varepsilon) \tilde{Z} f_a & \text{in } \Omega_+,
    \\
    S f_a = 0 & \text{on } r = \sigma(t, \omega),
\end{cases}
\quad \text{(B.8)}
\]

where $\tilde{Z}$ denotes a linear combination of some components in $S$.

It follows from (B.8) that
\[ |S f_a| \leq \frac{C\varepsilon}{t}. \quad \text{(B.9)} \]

From this, by the equation in (B.3) and the definition of $S$, one has
\[ |\partial_r f_a| \leq \frac{C\varepsilon}{t} |S f_a| + \varepsilon |\partial_r f_a| + \frac{C\varepsilon}{t^2} \]

and further
\[ |\partial_r f_a| \leq \frac{C\varepsilon}{t^2}. \quad \text{(B.10)} \]

Consequently, it follow Lemma 4.5 (vi) and (B.9)-(B.10) that
\[ |\nabla f_a| \leq \frac{C\varepsilon}{t^2}. \]

Analogously, by induction method, we can arrive at
\[ |\nabla^l f_a| \leq \frac{C_l \varepsilon}{t^l+1}, \quad l \in \mathbb{N}. \quad \text{(B.11)} \]

Combining (B.11) with (B.6) shows (B.1) and the proof of Lemma B.1 is completed.

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