Application of WKB Method in Approximating Wave Functions in Square Potential Wells

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Abstract. This paper introduces a widely used approximation method, the Wentzel-Kramers-Brillouin (WKB) method, for solving the Schrodinger equation, sets up a theoretical framework to derive the general solutions of the method, and carries out the application of the approximation method in specific cases to verify the validity and practicality of the method. Throughout the study, numerical methods, such as the Simpsons method, and programming tools, such as Python, are used to process sophisticated calculations. The final approximated solutions for the Schrodinger equation are able to demonstrate the state of a particle in an isolated quantum system. All the presented studies are based upon the knowledge and skillsets of a high school student.

1. Introduction
It has been almost a century since the Schrodinger equation was formulated to describe the wave function – which demonstrates the probability of finding a particle – of a quantum mechanical system. Solving the Schrodinger equation is one of the essential problems in quantum mechanics since nonlinear second order ordinary differential equation (ODE) has, in general, no analytic solutions. Therefore, the solutions are limited to a list of very few known potential functions, such as inverse power-law potentials, infinite and finite square wells, and parabolic potentials, and due to this fact, all physical systems are modelled with potentials from this list [1]. To overcome this limitation, many techniques have been developed to derive approximate solutions of the Schrodinger equation, and the Wentzel-Kramers-Brillouin (WKB) method is one of the most commonly used practices [2], [3], [4]. It is a power tool and it has many applications, for example, solving waves in an inhomogeneous plasma as shown in the textbook Plasma Waves (D.G. Swanson, Academic, San Diego, CA 1989). Instead of starting with a simplified potential and adding small turns, which leads to perturbation theory, the WKB approximation makes an assumption of a slowly varying potential. Exact solutions for defined potentials are still an ultimate goal in quantum physics and related fields of science. Many researches have been conducted on the justification of the validity of the WKB method, but many of them focus on more complex potential functions. For beginners in the field of quantum mechanics, an examination of the WKB method in simpler potential functions, with numerical codes that allow them to reproduce results, could help them to get a better grip of the essence of the WKB method. Hence, the purpose of this paper is to verify the validity of the WKB method in unsophisticated potential functions numerically. Specifically, we derive the solutions of Schrodinger equations with three well known potentials where their quantized energy levels and wave functions are determined using the WKB method. Three potential wells are studied in this paper: infinite square well, finite symmetric square well, and finite asymmetric square well. Graphical results are demonstrated and analyzed to verify the validity of the WKB method. All the presented studies are based upon the knowledge and skill sets of a high school student.
We will show that some wave functions obtained via the WKB method are discontinuous at the turning points of the potential. These results are not unexpected due to the fact that our potentials have sudden changes at the turning points, so they cannot be approximated to slowly varying potentials at the turning points. After a little survey over standard textbooks, we found that there is no elaboration on the discontinuity of the waves due to the breakdown of the validity of the WKB method. In this case, we have reasons to assume that the restrictions in applying the standard WKB method are not very well known to students. Therefore, our work presented here is valuable to students to gain the knowledge that in the cases where the potential of interest cannot be approximated to a slowly varying potential, the WKB method would not be an optimal approach.

2. Theoretical framework

The WKB Method is an approximation of the solutions for linear differential equations, including the Schrodinger Equation. Its amplitude is taken to vary slowly in respect to the de Broglie wavelength, thus giving the approximation a semiclassical nature [5].

2.1. General solutions

The general solutions of a wave function using the WKB approximation are separated into two cases: $E > V(x)$ and $E < V(x)$.

2.1.1. $E > V(x)$. The region where $E > V(x)$ is called the classically allowed region. As a result, the wave function is oscillatory as expressed below:

$$\psi(x) = \frac{c_+}{\sqrt{p(x)}} \exp \left[ \frac{i}{\hbar} \int p(x') dx' \right] + \frac{c_-}{\sqrt{p(x)}} \exp \left[ -\frac{i}{\hbar} \int p(x') dx' \right]$$  (1)

where $p(x)$ is the momentum of a particle moving in a potential well given as $p(x) = \left[ 2m(E - V(x)) \right]^{\frac{1}{2}}$, and $C_+$ and $C_-$ are two normalization constants.

2.1.2. $E < V(x)$. In the case where $E < V(x)$, or the classically forbidden region, the momentum becomes an imaginary value, and the general solution of the wave function is demonstrated as follows:

$$\psi(x) = \frac{c'_+}{\sqrt{|p(x)|}} \exp \left[ \frac{i}{\hbar} \int |p(x')| dx' \right] + \frac{c'_-}{\sqrt{|p(x)|}} \exp \left[ -\frac{i}{\hbar} \int |p(x')| dx' \right]$$  (2)

where the exponents become real, thus turning the wave function into an exponentially decaying function.

2.2. Non-Rigid Walls Potential Well

In Figure 1, the potential well is divided into three regions by two turning points $x_1$ and $x_2$, where regions 1 and 3 are classified as classically forbidden regions while region 2 is classified as the
classically allowed region. According to (1) and (2), the wave function for the potential well can be written as follows:

\[ \psi_1(x) = \frac{c_1}{\sqrt{p(x)}} \exp \left[ -\frac{1}{\hbar} \int_{x_1}^{x} |p(x')| dx' \right], \quad x < x_1 \]  

\[ \psi_2(x) = \frac{c_2}{\sqrt{p(x)}} \exp \left[ \frac{1}{\hbar} \int_{x_1}^{x} p(x') dx' \right] + \frac{c_2'}{\sqrt{p(x)}} \exp \left[ -\frac{1}{\hbar} \int_{x_1}^{x} p(x') dx' \right], \quad x_1 < x < x_2 \]  

\[ \psi_3(x) = \frac{c_3}{\sqrt{|p(x)|}} \exp \left[ -\frac{1}{\hbar} \int_{x_2}^{x} |p(x')| dx' \right], \quad x > x_2 \]  

2.2.1. Connection of $\psi_3$ and $\psi_2$. The wave function in region 2 can be further written as

\[ \psi_2(x) = \frac{c_2}{\sqrt{p(x)}} \sin \left( \frac{1}{\hbar} \int_{x_1}^{x} p(x') dx' + \alpha \right), \quad x_1 < x < x_2 \]  

where $\alpha$ is an unknown phase factor. However, when $x$ approaches to the turning point $x_2$, the momentum becomes extremely small that the WKB approximation of the wave function fails to hold. As a result, the wave function near the turning point should be solved exactly using the Schrodinger equation. When the distance between $x$ and $x_2$ is close enough, the potential within this region can be represented by a linear function:

\[ V(x) = E + (x - x_2)V' \]  

where $V(x_2)$ is substituted by $E$, and $V'$, the slope of the line at $x = x_2$, is given as $\frac{dV(x)}{dx}|_{x=x_2}$. The Schrodinger equation thereby becomes

\[ \frac{d^2\psi(x)}{dx^2} - \frac{2mV'}{\hbar^2} (x - x_2) \psi(x) = 0 \]  

By applying the change of variable, the equation can be transformed from $y = \left( \frac{2mV'}{\hbar^2} \right)^{\frac{1}{3}} (x - x_2)$ into

\[ \frac{d^2\psi(y)}{dy^2} - y \psi(y) = 0, \]  

which is an airy function. By solving the airy function, we derive the wave function asymptotically [6]

\[ \psi(x) = \begin{cases} \frac{A}{\sqrt{|p(x)|}} \sin \left( \frac{1}{\hbar} \int_{x_1}^{x} p(x') dx' + \frac{\pi}{4} \right), & x << x_2, \\ \frac{A}{\sqrt{|p(x)|}} \exp \left[ -\frac{1}{\hbar} \int_{x_2}^{x} |p(x')| dx' \right], & x >> x_2, \end{cases} \]  

and the connection formulae that connect the wave function at either side of $x_2$

\[ A = 2C_3, A = C_2, \alpha = \frac{\pi}{4}. \]  

Consequently, as we obtain the phase factor $\alpha$, (6) becomes

\[ \psi_2(x) = \frac{c_2}{\sqrt{p(x)}} \sin \left( \frac{1}{\hbar} \int_{x_1}^{x} p(x') dx' + \frac{\pi}{4} \right), \quad x_1 < x < x_2. \]  

2.2.2. Connection of $\psi_1$ and $\psi_2$. Starting from the turning point $x_1$, (6) can be written as

\[ \psi_2(x) = \frac{d}{\sqrt{p(x)}} \sin \left( \frac{1}{\hbar} \int_{x_1}^{x} p(x') dx' + \beta \right), \quad x_1 < x < x_2. \]  

Because the approximation fails near the turning point, the wave function in the vicinity of the turning point can be derived from an exact solution using the Schrodinger equation. After processing, similar to the process in 2.2.1, the wave function is expressed as follows:
\[
\psi(x) = \begin{cases} 
\frac{E}{2|p(x)|} \exp \left( -\frac{1}{\hbar} \int_{x_1}^{x} |p(x')| dx' \right), & x \ll x_1, \\
\frac{E}{\sqrt{p(x)}} \sin \left( \frac{1}{\hbar} \int_{x_1}^{x} p(x') dx' + \frac{\pi}{4} \right), & x \gg x_1. 
\end{cases} 
\] (12)

The other group of connection formulae is:

\[ E = 2C_1, E = D, \beta = \frac{\pi}{4}. \]

Therefore, after plugging in value of the phase factor, \( \psi_2(x) \) becomes

\[ \psi_2(x) = \frac{D}{\sqrt{p(x)}} \sin \left( \frac{1}{\hbar} \int_{x_1}^{x} p(x') dx' + \frac{\pi}{4} \right), \quad x_1 < x < x_2. \] (13)

2.2.3. Quantization of bound state energy levels. Since (10) is equivalent to (13), we come to an equation:

\[ \psi_2(x) = \frac{C_2}{\sqrt{p(x)}} \sin \left( \frac{1}{\hbar} \int_{x_1}^{x} p(x') dx' + \frac{\pi}{4} \right) = \frac{D}{\sqrt{p(x)}} \sin \left( \frac{1}{\hbar} \int_{x_1}^{x} p(x') dx' + \frac{\pi}{4} \right). \] (14)

The solution must adhere to the following relations:

\[ \left( \frac{1}{\hbar} \int_{x_1}^{x} p(x') dx' + \frac{\pi}{4} \right) + \left( \frac{1}{\hbar} \int_{x_1}^{x_2} p(x') dx' + \frac{\pi}{4} \right) = (n + 1)\pi \] (15)

\[ D = (-1)^n C_2 \] (16)

After simplifying (15), the quantized energy levels of the bound states for potential wells with no rigid walls can be calculated using the equation below:

\[ \int_{x_1}^{x_2} p(x) dx = \left( n + \frac{3}{2} \right) \pi \hbar, \quad n = 0, 1, 2, 3, \ldots. \] (17)

To sum up, the WKB approximated wave function for a particle travelling in a potential well with no rigid walls is divided into three sections:

\[
\psi(x) = \begin{cases} 
\psi_1(x) = \frac{(-1)^n A}{2|p(x)|} \exp \left( -\frac{1}{\hbar} \int_{x_1}^{x} |p(x')| dx' \right), & x < x_1, \\
\psi_2(x) = \frac{A}{\sqrt{p(x)}} \sin \left( \frac{1}{\hbar} \int_{x_1}^{x} p(x') dx' + \frac{\pi}{4} \right), & x_1 < x < x_2, \\
\psi_3(x) = \frac{A}{2|p(x)|} \exp \left[ -\frac{1}{\hbar} \int_{x_2}^{x} |p(x')| dx' \right], & x > x_2. 
\end{cases} 
\] (18)

2.3. One Rigid Wall Potential Well

![Figure 2. Potential well with one rigid wall](image)

In Figure 2, a particle is travelling in a potential well and an infinite potential wall exists at \( x = x_1 \). The potential well is divided into classically allowed region 1 \( x \in (x_1, x_2) \) and classically forbidden region 2 \( x \in (x_2, \infty) \). Similar to the derivation of the wave function in 2.2, the wave function within
region 1 exists as an oscillatory form, whereas that of region 2 is presented as an exponentially decaying function:

\[
\psi_1(x) = \frac{B}{\sqrt{p(x)}} \sin \left( \frac{1}{\hbar} \int_{x_1}^{x_2} p(x') dx' + \frac{\pi}{4} \right) = \frac{A}{\sqrt{p(x)}} \sin \left( \frac{1}{\hbar} \int_{x_1}^{x} p(x') dx' \right), \quad x_1 < x < x_2
\]

(19)

\[
\psi_2(x) = \frac{A}{2\sqrt{|p(x)|}} \exp \left[ -\frac{1}{\hbar} \int_{x_2}^{x} |p(x')| dx' \right], \quad x > x_2
\]

(20)

where the second expression for \(\psi_1(x)\) does not have a phase factor because the wave function approaches to zero as it reaches the infinite potential wall at \(x = x_1\).

The following equation that can be applied to calculate the bound state energy levels in this scenario can be derived from (19), as demonstrated in 2.2.3:

\[
\int_{x_1}^{x_2} p(x) dx = \left( n + \frac{3}{4} \right) \pi \hbar, \quad n = 0,1,2,3,\ldots
\]

(21)

2.4. Two rigid walls potential well

As shown in Figure 3, in a potential well restricted by two rigid walls at \(x = x_1\) and \(x = x_2\) respectively, the wave function for the particle is expressed as follows:

\[
\psi(x) = \frac{A}{\sqrt{p(x)}} \sin \left( \frac{1}{\hbar} \int_{x_1}^{x} p(x') dx' \right) = \frac{B}{\sqrt{p(x)}} \sin \left( \frac{1}{\hbar} \int_{x_1}^{x_2} p(x') dx' \right)
\]

(22)

where neither of the expressions has a phase factor because the wave function has to vanish at both \(x = x_1\) and \(x = x_2\).

Similar to the quantization of energy levels in previous two types of potential wells, the quantization of bound state energy levels for potential wells that have two rigid walls can be processed similarly, producing the equation

\[
\int_{x_1}^{x_2} p(x) dx = (n + 1)\pi \hbar, \quad n = 0,1,2,3,\ldots
\]

(23)

3. WKB approximation of wave functions for square potential wells

In this section, the application of the WKB method in solving Schrodinger equations with different square potential wells will be discussed. Specifically, to demonstrate that the WKB method is an effective and straightforward approximation, we will show that the solution of the wave function obtained using the WKB method matches exact the solution obtained by solving the Schrodinger equation with an infinite potential well. In finite symmetric and asymmetric potential wells with one rigid wall where calculations for exact solutions of wave functions often require great efforts, we will show that the implementation of the WKB method in these cases will significantly relaxes the effort required to attain the results. The WKB method presented in this section is realized by Python.
3.1. Variables
As shown in table 1, we assign the following values in Python to demonstrate the application of the WKB method in the aforementioned cases:

| Notation | Description                                          |
|----------|------------------------------------------------------|
| $x_0$    | Location of the infinite potential wall              |
| $x_1$    | Location of the first potential wall                 |
| $x_2$    | Location of the second potential wall                |
| $A$      | Normalization constant                               |
| $E_n$    | Energy at energy level $n$                           |

3.2. Infinite potential well
Table 2 demonstrates the input values for the initial variables:

| Variable | Value                                      |
|----------|--------------------------------------------|
| $n$      | 0, 1, 3, 5                                 |
| $m$      | $9.11 \times 10^{-34}$ [kg]                |
| $x_1$    | 0                                          |
| $x_2$    | 2                                          |
| $V(x_1)$ | $\infty$                                   |
| $V(x_2)$ | $\infty$                                   |

3.2.1. Exact solution by Schrodinger Equation. For a particle of mass $m$ in an infinite potential well, the potential function is expressed as

$$V(x) = \begin{cases} 
0, & x_1 < x < x_2 \\
\infty, & x \leq x_1 \text{ or } x \geq x_2 
\end{cases}. \quad (24)$$

and is visualized as following:

**Figure 4.** Infinite potential well

In Figure 4, because the potential within the potential well is zero, the time independent Schrodinger equation becomes:
\[
d\frac{d^2\psi(x)}{dx^2} + k^2\psi(x) = 0,
\]
where \( k \) is defined as:
\[
k = \left[\frac{2mE_n}{\hbar^2}\right]^{\frac{1}{2}}, \quad n \in \mathbb{Z}^+.
\]
The general solution for (25) is
\[
\psi(x) = A\sin(kx) + B\cos(kx), \quad x_1 < x < x_2.
\]
Because it is impossible to find a particle outside of an infinite potential well, the wave function vanishes at both walls:
\[
\psi(x_1) = \psi(0) = 0 = B.
\]
As a result, (27) becomes
\[
\psi(x) = A\sin(kx), \quad x_1 < x < x_2.
\]
By substituting the value of \( B \) and inserting the value of \( x \) into (29), the solution becomes
\[
\psi(x_2) = \psi(2) = 0 = A\sin(2k).
\]
After solving (29), we obtain the expression for \( k \):
\[
k = \frac{(n+1)\pi}{2}, \quad n \in \mathbb{Z}.
\]
Substituting (31) into (29) leads to
\[
\psi(x) = A\sin\left(\frac{(n+1)\pi x}{2}\right), \quad n \in \mathbb{Z}, \quad x_1 < x < x_2,
\]
where \( A \) is the amplitude of the wave function that serves to normalize the function. In order to calculate the value of \( A \), we need to apply the fact that the probability of finding the particle within the region \( x_1 < x < x_2 \) must be 100%. To express it mathematically,
\[
\int_{x_1}^{x_2} |\psi(x)|^2 dx = 1,
\]
where \( |\psi(x)|^2 \) represents the probability density of the particle.

The value of \( A \) can be obtained by substituting (32) to (33):
\[
A = \sqrt{\frac{2}{x_2-x_1}} = \sqrt{\frac{2}{2-0}} = 1.
\]
Finally, the wave function of a particle travelling in an infinite potential well becomes
\[
\psi(x) = \begin{cases} 
\sin\left(\frac{(n+1)\pi x}{2}\right), & x_1 < x < x_2, \\
0, & x \leq x_1 \text{ or } x \geq x_2
\end{cases}, \quad n \in \mathbb{Z}.
\]

3.2.2. Approximate solution by WKB method.
In an effort to approximate the solution for the wave function, we have to calculate \( E_n \) and substitute it into \( p(x) \). Subsequently, by normalizing the function (the probability of finding the particle within the confined region is 100%), we can derive the normalization constant \( A \) and thereby obtain the final approximate solution of the wave function.

First, since the infinite potential well is a type of a well with two rigid walls, we use (23) to solve for \( E_n \). To express the left side of the equation in terms of \( E_n \), \( p(x) \) is equally divided into 10,000 segments. By applying Simpson's method in Python, we can numerically simplify the integral to represent it in terms of \( E_n \). Finally, we can solve for \( E_n \) based on the equation and substitute it into \( p(x) \).
As demonstrated in the first expression of $\psi(x)$ in (22), we integrate $p(x)$ from $x_1$. To solve for $A$, we normalize the wave function as shown in (33). Substituting the wave function into the equation yields

$$\int_{x_1}^{x_2} \frac{A^2}{p(x)} \sin^2 \left( \frac{1}{h} \int_{x_1}^{x} p(x') dx' \right) = 1. \tag{36}$$

By dividing $|\psi(x)|^2$ into 10,000 segments and applying Simpsons method in Python, we can solve for $A$. The following is an example of the final wave function at ground state $(n = 0)$:

$$\psi(x) = \frac{1.287 \times 10^{-17}}{\sqrt{2.744 \times 10^{-68}}} \sin(1.571 x). \tag{37}$$

After simplifying (37), we find that the coefficient is very close to 1, and the angular frequency of the sine wave approximately equals $\frac{\pi}{2}$. As a result, (37) produces approximately the same result as that of the exact solution in (35) when assigning $n = 0$.

![Figure 5. Exact solution of wave function](image1)

![Figure 6. Approximate solution of wave function](image2)

As presented in Figure 5 and 6, each containing wave functions for energy levels 0, 1, 3, and 5, the approximate solutions of the wave functions are virtually the same as the exact solutions, which implies that the WKB method is an effective approximation approach as it produces results that are very close to the exact ones.

3.3. Finite symmetric potential well

Table 3 demonstrates the inputs for the initial variables:
Table 3. Assigned values of variables

| Variable | Value       |
|----------|-------------|
| $n$      | 0, 1, 3, 5  |
| $m$      | $10^{-67}$ [kg] |
| $x_1$    | -2          |
| $x_2$    | 4           |
| $V(x_1)$ | 2           |
| $V(x_2)$ | 2           |

Figure 7 depicts the finite potential well based on the input values:

The overall calculation process for the wave function in this potential well is similar to that deliberated in section 3.2.2. However, to calculate $E_n$, we use equation (17) since there are no rigid walls. Moreover, the wave function is divided into three portions: $x < x_1$, $x_1 < x < x_2$, and $x > x_2$, represented by region 1, 2, and 3 respectively. The wave function for each region is indicated in equation (18).

In a finite potential well where bound states exist, $E_n < V$. As a result, $\psi(x) \to 0$ when $x \to \pm \infty$, which becomes normalizable. To normalize the piecewise wave function, its complex square is integrated from -50 to 50. Because when the particle traverses into region 1 or 3, the wave function goes asymptotically infinitesimal, so we believe -50 to 50 is a reasonable and insurable limit. After dividing the region within $-50 < x < 50$ into 1,000 segments and applying Simpson’s method to solve the integral (See (33) for equation) numerically, we can derive the value of $A$.

Figure 8. Approximate wave functions at energy levels 1, 3, 5
As shown in Figure 8, $\psi(x)$ appears as an oscillating wave within region 2, i.e., in the classically allowed region, as $x \to \pm \infty$, there exists finite possibilities of finding a particle outside of the potential well, although the wave function decays and quickly approaches 0 as expected. Also, notice that at the turning points between the classically allowed and forbidden regions (i.e., $x = -2, x = 4$), the wave function becomes discontinuous. This is because at turning points, the WKB solutions break down as momentum $p(x)$ becomes 0. In order to establish the connection between the WKB solutions in the allowed region as well as in the forbidden region on either side of the breaking down point (turning point), i.e., $p(x) = 0$, an accurate solution is needed, where the potential can be approximated as a linear potential with the slope equals the tangent of the breaking down point (See (7)), hence linking WKB solutions at a turning point [4]. However, such linear approximation only holds valid for a reasonably smooth potential. In a finite symmetric square potential well, the potential changes sharply at turning points, so linear approximation is no longer valid, and thus the discontinuity inevitably persists.

### 3.4. Finite asymmetric potential well with one rigid wall

In this type of potential well, we will discuss two cases: 1) energy is smaller than both walls and 2) energy is greater than one wall but smaller than the other.

#### 3.4.1. $E < V_1$ and $E < V_2$

Table 4 provides the inputs for the initial variables:

| Variable | Value                  |
|----------|------------------------|
| $n$      | 1, 2, 3                |
| $m$      | $10^{-68} [kg]$        |
| $x_0$    | 0                      |
| $x_1$    | 2                      |
| $x_2$    | 8                      |
| $V_1$    | 3 [J]                  |
| $V_2$    | 5 [J]                  |

Table 4. Assigned values of variables

Figure 9 is the potential well based on the inputs shown in table 4:

![Figure 9. Case 1: Asymmetric potential well with one rigid wall](image)

Since $E_n$ is smaller than both walls, the calculation is the same as that deduced in section 3.3. However, it is noteworthy that the potential walls have different values, and due to the infinite potential wall at $x = 0$, we use 0 rather than -50 as the lower limit of the integral to calculate the normalization constant $A$ in equation (33). The derived wave functions partitioned by regions 1, 2, and 3 are depicted in Figure 10.

![Figure 10. Approximate wave functions at energy levels 1,2,3](image)

In region 2 of Figure 10, we observe that the wave function behaves as an oscillating wave. However, in region 1 and 3, the wave function is decaying at different rates due to distinct values of potential walls.

3.4.2. $V_1 < E < V_2$

Table 5 is the input for initial variables:

| Variable | Value        |
|----------|--------------|
| $n$      | 4,5,6        |
| $m$      | $10^{-68}$ [kg] |
| $x_0$    | 0            |
| $x_1$    | 2            |
| $x_2$    | 8            |
| $V_1$    | 4 [J]        |
| $V_2$    | 12 [J]       |

The potential well is exhibited in Figure 11:
Since $E_n$ is greater than $V_1$, it meets the walls at $x = x_0$ and $x = x_2$, where the potential at $x = x_0$ is infinite. Thus, this potential well can be characterized as that discussed in section 2.3. The wave function is split into two regions: the classically allowed region at $x_0 < x < x_2$ and the classically forbidden region at $x > x_2$, which are represented by equations (20) and (21) respectively. To solve for $E_n$, we use equation (21), where the lower and upper limits of the integral are $x_0$ and $x_2$. To calculate the normalization constant $A$, we apply equation (33) as expressed below:

$$\int_{x_0}^{x_2} |\psi(x)|^2 dx = 1. \quad (38)$$

With both $E_n$ and $A$ obtained, we can derive the wave function as depicted in Figure 12:

As shown in Figure 12, the wave function oscillates at $x_0 < x < x_2$ and decays at $x > x_2$. However, we find that the amplitudes and periods of the wave function differ in regions $x_0 < x < x_1$ and $x_1 < x < x_2$ due to the variation of $p(x)$ with different potentials in these two regions.
We would like to emphasize that there is again a discontinuity showing up in these wave functions from the classically allowed region to the classically forbidden region due to the reason elucidated in the last paragraph of Section 3.3. It is worth pointing out that, the WKB method is a powerful tool dealing with smoothly varying potentials; however, for artificial potentials, such as the symmetric and asymmetric finite potential wells studied here, wave functions derived from the WKB method cannot faithfully establish a particle’s quantum state especially in the forbidden region due to the sudden change in the potential.

4. Conclusion
We present detailed derivations for solving Schrodinger equations with three well known potentials using the WKB method. Although the WKB method is not a universal approach for an arbitrary potential, for smoothly varying potentials without sudden change, it does provide a clear route for good approximations. Our results are straightforward and self-explanatory. The necessary coding technique involved in this work is integration with the Simpsons method. Our work indicates that with a proper and simple training, even a high school student is capable of taking on one of the challenging problems in modern physics – solving Schrodinger equations. This would significantly enhance high school students’ interest in quantum physics and nurture many well-motivated future physicists.

5. References
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