Graphs of plural cuts

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1. Introduction

Plural cut is a structural inference rule introduced by Gentzen in [7] for his plural sequent system of classical logic. A plural sequent (more often called multiple-conclusion sequent, or something like that) is a sequent $\Gamma \vdash A, \Delta$ where $\Delta$, as $\Gamma$, may be a collection (sequence, multiset or set) of formulae with more than one member (see [13], Theorem 1.2, [14], Chapters 1, 2, 5, [6], Chapter 1.1, Theorem 13, and [4] for results about the relationship between singular and plural consequence relations). Plural cut as formulated by Gentzen with sequents based on sequences of formulae $\Gamma, \Theta, \Delta$ and $\Lambda$ is the following rule:

$$
\begin{align*}
\Gamma, \Theta, A & \vdash A, \Delta & A, \Delta & \vdash \Lambda, \\
\end{align*}
\Gamma, \Delta, \Theta & \vdash A, \Lambda
$$

A sequent $\Gamma \vdash A$ is singular when the collection of formulae $\Delta$ cannot have more than one member, and singular cut is obtained from Gentzen’s plural cut by assuming that $\Theta$ is empty and that $\Lambda$ cannot have more than one member.

Gentzen assumed his rule of plural cut together with the structural rule of permutation, on both the left and right of the turnstile $\vdash$, so that the exact place of the formula $A$ in his formulation of plural cut is not essential. Besides the plural cut rule as stated by Gentzen, the following plural cut rules:

$$
\begin{align*}
\Gamma & \vdash A, \Theta & \Delta, A & \vdash \Lambda, \\
\Delta, \Gamma & \vdash A, \Theta
\end{align*}
\Gamma & \vdash A & \Delta_1, A, \Delta_2 & \vdash \Lambda, \\
\Delta_1, \Gamma, \Delta_2 & \vdash \Lambda
$$

$$
\Gamma & \vdash \Theta_1, A, \Theta_2 & \Lambda, \\
\Gamma & \vdash \Theta_1, \Lambda, \Theta_2
$$

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were considered in [1,12] as appropriate for plural sequent systems where one does not assume the structural rule of permutation. A detailed study of cut elimination in the context of these rules may be found in [9]. (Something analogous in a different area may be found in the definition of the literal shuffle of [2], Part 1, p. 29.) Let us call these four kinds of plural cuts planar plural cuts (as the literature suggests).

Planar plural cuts are found in the polycategories of [3], which were called planar polycategories in [10]. These polycategories differ from the polycategories of [15] where we have the following plural cut rule:

\[
\begin{array}{c}
\Gamma \vdash \Theta_1, A, \Theta_2 \\
\Delta_1, A, \Delta_2 \vdash A \\
\Delta_1, \Gamma, \Delta_2 \vdash \Theta_1, \Lambda, \Theta_2
\end{array}
\]

This rule involves a kind of permutation, which is manifested in the crossings of the following diagram:

\[
\begin{array}{c}
\Theta_1 \\
\Gamma \\
A \\
A \\
\Lambda \\
\Theta_2 \\
\Delta_2 \\
\Delta_1
\end{array}
\]

These crossings require that we have the structural rule of permutation on the left and on the right in order to state the equations implicit in the definition of polycategory of [15] (see P3 in Section 2: in the first of these equations, which are analogous to the equations that stand behind our Propositions 2.2 and 2.3, we must permute \( \Gamma_2 \) with \( \Delta_2 \) and \( \Gamma_3 \) with \( \Delta_3 \), and in the second we must permute \( \Delta_1 \) with \( \Phi_1 \) and \( \Delta_2 \) with \( \Phi_2 \)). Planar plural cuts are obtained from (PC) by requiring that either \( \Theta_1 \) or \( \Delta_1 \) be empty and that either \( \Theta_2 \) or \( \Delta_2 \) be empty, so that the crossings do not arise.

In this paper our main goal is to characterize in a graph-theoretical, combinatorial, manner the planarity involved in planar plural cuts. To achieve that, we define in three different manners a kind of oriented graph, which we call K-graph. (The notion of oriented graph, and other notions we need concerning these graphs, and directed graphs in general, are defined in Section 2.) The name of K-graphs is derived from the form of these graphs, that may resemble up to a point a rotated K:

\[
\begin{array}{c}
\Theta_1 \\
\Gamma \\
A \\
A \\
\Lambda \\
\Theta_2 \\
\Delta_2 \\
\Delta_1
\end{array}
\]

(see the picture below; this form resembles equally a rotated X, but we need X, as a variable, for other purposes later on; K was chosen faute de mieux).

Our first definition, given in Section 2, is inductive. With it, K-graphs are obtained from some basic K-graphs by applying operations that correspond to planar plural cuts. This definition yields our notion of global K-graph, which is closest to planar polycategories.

It corresponds actually to a notion somewhat more general than the notion of planar polycategory, which we could call compass polycategory. Compass polycategories would be defined like planar polycategories, but instead of having polyarrows with sources and targets made of sequences of objects, in compass polycategories we would have these sources and targets made of multisets of objects with two distinguished objects, if the multiset is not a singleton. We refer to these distinguished objects by \( N \) and \( S \) (which stand for north and south respectively; we take inspiration from the compass because in \( \Gamma \vdash \Delta \) we have \( \Gamma \) on the west and \( \Delta \) on the east). In sequences of objects, the \( N \) and \( S \) object are the first and last object. The collections of objects in the polyarrows of compass polycategories need not however be sequences. We need \( N \) and \( S \) to characterize the operations on global K-graphs that correspond to planar plural cuts, and we do not need anything else.

The assumption that we have sequences is not necessary to characterize these operations.

The polyarrows of a freely generated compass polycategory may be identified with global K-graphs where the inner vertices (a vertex of a directed graph is inner when an edge ends in it and another one begins in it; see Section 2) are labelled by the free generators of the polycategory, and the remaining vertices are labelled by objects of the polycategory. Our Propositions 2.4 and 2.5 (see Section 2) contain the essence of a completeness proof of our notion of global K-graph with respect to compass polycategories, which as planar polycategories are characterized by three equations that stand behind our Propositions 2.1–2.3 (see the notion of \( \rho \)-equivalence before Proposition 2.4), and by additional equations involving the identity polyarrows. To simplify the exposition, we deal separately in Section 6 with matters involving these identities. This section brings a mathematically not very essential addition to the preceding exposition in the main body of the paper.

The presence of identity polyarrows is however expected if our polyarrows are interpreted as proofs of plural sequents in a plural sequent system. This would be a very simple kind of such a system, where we have only plural cut as a rule, and axiomatic sequents corresponding either to the free generators of our compass polycategory or to the identity polyarrows. The equations we have just mentioned would then be equations that arise out of an equivalence relation on derivations of sequents in such a sequent system (an equivalence analogous to a relation extending \( \rho \)-equivalence with equations involving identities). Two derivations can be equivalent only if they are derivations of the same plural sequent. A proof of a sequent would be an equivalence class of such derivations with respect to this equivalence of derivations.
In the main body of this paper, after this introduction, we will however not mention compass polycategories any more. Planar polycategories will be mentioned in passing in a comment before Proposition 2.1, and nothing in our exposition later on depends on knowing this notion (a reader not acquainted with the notion may skip the comment). It is not our intention to concentrate further on the notion of compass polycategory. A more detailed study of that topic is left for another place.

We will speak about our second definition of K-graph within a moment, and will first say something about the third definition. This third definition of K-graph, given in Section 5, is non-inductive and it does not mention N and S any more. It is purely graph-theoretical, and by showing the equivalence of the notion the third definition gives with the notion of global K-graph we have achieved the main goal of the paper.

Our second definition, in Section 3, gives the notion of local K-graph, which is intermediary between the notion of global K-graph given by the first definition and the notion of K-graph given by the third definition. The notion of local K-graph is of an instrumental value: it will help us to prove the equivalence mentioned in the preceding paragraph, which is our main goal. All the three notions, global K-graph, local K-graph and the notion of K-graph given by the third definition will be shown equivalent at the end.

The second definition comes in two variants. In the main variant, the notion of local K-graph is defined non-inductively, as in the third definition, but it still involves N and S, as the first definition does. We give however also another variant, which is an inductive definition of local K-graph. The notion of local K-graph, which is equivalent to the notion of global K-graph, as proved in Section 4, helps us to prove in Section 5 the equivalences mentioned in the preceding two paragraphs.

With the third definition of K-graph, the planarity of planar plural cuts, or rather their compass character, is characterized in a way that can be compared to Kuratowski’s way of characterizing the planarity of graphs (see [8], Chapter 11, and the first part of the proof of Proposition 5.5). The two approaches may be compared, but the results involved are different. In our case, we do not deal in fact with planarity, but with a related notion involving N and S. We deal also with a special kind of oriented graph, whereas Kuratowski was concerned with the planarity of ordinary, non-directed, graphs.

The third definition yields the following picture. An arbitrary K-graph may very roughly be described as having in the middle a non-circular line of edges with changing directions, which we call the transversal. Together with the transversal we have two sets of trees, each set of a different kind: the first set consists of trees oriented towards the root, and the second of trees oriented towards the leafs. Both kinds of trees are planted with their roots in the transversal. Here is an example:

In the middle, drawn with dotted lines, is the transversal, on the left of which, growing westward, we have trees oriented towards the root, and on the right of which, growing eastward, we have trees oriented towards the leafs (for details see Section 5). The combinatorial essence of the planarity of K-graphs is that the transversal is non-circular (more precisely, asemicyclic; see Section 2) and linear (more precisely, non-bifurcating; see Section 5).

With singular cuts we would obtain just trees, oriented towards the root. One bases on such trees derivations in ordinary natural deduction, and also the notion of multicategory. A limit case of singular cut is ordinary composition in categories, which yields as graphs just chains. With K-graphs we do not have trees, but we have not gone very far away from the notion of tree.

With plural cuts in general, which are based on the cut rule (PC), we are further removed from trees, and we obtain a notion of oriented graph, which we call Q-graph, simpler to define than our notion of K-graph, both inductively and non-inductively. We investigate this notion, which when defined non-inductively reduces essentially to a weak form of connectedness and a weak form of non-circularity, in Section 7, the last section of the paper. An arbitrary Q-graph may be pictured as an arbitrary K-graph, with the transversal and two sets of trees, but the transversal is not linear any more.

The notion of global K-graph is the notion that should be used to prove by induction that every K-graph can be geometrically realized in the plane in the following special manner. A point that realizes a vertex a that is not inner has the first coordinate 0 if an edge begins in a, and it has the first coordinate 1 if an edge ends in a. We require moreover in this realization that for every edge (a, b) of our K-graph the first coordinate of the point that realizes the vertex a is strictly smaller than the first coordinate of the point that realizes the vertex b.

Conversely, for an oriented graph of a special kind, which is connected and non-circular in a weak sense, and satisfies moreover a condition concerning its vertices that are not inner (see conditions (1)–(3) in Section 3), we should be able to prove that if it is realized in the plane in the special manner above, then it is a K-graph. The proof of that would be inductive too, and would rely on the notion of global K-graph. We will not go here into this rather geometrical matter, which however would not improve significantly our mathematical perception of the geometrical planarity of K-graphs. We suppose that the notion of global K-graph suffices for that. The accent in this paper is put on other matters, like our third definition of K-graph, which characterizes the planarity of these graphs in a combinatorial way.
2. Global K-graphs

In this section we deal with our first definition of K-graph, which yields the notion of global K-graph (the reason for using the term “global” is given at the beginning of Section 3). We establish for this notion a completeness result in Propositions 2.1–2.5, which will help us for the equivalence proofs in later sections. We start first with some elementary notions of graph theory.

A digraph \( D \) is an irreflexive binary relation on a finite nonempty set, called the set of vertices of \( D \). The ordered pairs in \( D \) are its edges. An edge \((a, b)\) begins in \( a \) and ends in \( b \).

An oriented graph is an antisymmetric digraph.

A vertex of a digraph \( D \) is a \( W \)-vertex (\( W \) stands for west) of \( D \) when in \( D \) there are no edges ending in this vertex. It is an \( E \)-vertex (\( E \) stands for east) of \( D \) when in \( D \) there are no edges beginning in this vertex (which means that it is a \( W \)-vertex of the digraph converse to \( D \)). It is an inner vertex of \( D \) when it is neither a \( W \)-vertex nor an \( E \)-vertex of \( D \).

An edge of \( D \) is a \( W \)-edge of \( D \) when it begins in a \( W \)-vertex of \( D \), and it is an \( E \)-edge of \( D \) when it ends in an \( E \)-vertex of \( D \). It is an inner edge of \( D \) when it begins in an inner vertex of \( D \) and ends in an inner vertex of \( D \).

Intuitively, in logical terms, the \( W \)-vertices should be understood as labelled by formulae from the left-hand side of sequents, while the \( E \)-vertices are labelled by formulae from the right-hand side of sequents. This is because we write from west to east. Otherwise, we could as well understand everything in the opposite way. The inner vertices should be understood in logical terms as codes of axiomatic sequents.

Throughout the paper we use \( X \) as a variable standing for \( W \) or \( E \), and sometimes instead of \( X \) we also use \( Z \) for the same purpose. We assume that \( W \) is \( E \) and \( E \) is \( W \). We reserve the variable \( Y \) for \( N \) or \( S \) (which stand for north and south respectively).

A \( W \)-edge \((a, b)\) of \( D \) is functional when \((a, c) \in D \) implies \( b = c \). An \( E \)-edge \((b, a)\) of \( D \) is functional when \((c, a) \in D \) implies \( b = c \) (i.e., it is functional as a \( W \)-edge of the digraph converse to \( D \)).

A basic K-graph \( B \) is an oriented graph of the form

\[
\begin{array}{c}
\bullet \\
\vdots \\
\bullet \\
a \circ \\
\vdots \\
c \circ
\end{array}
\]

for \( k_W, k_E \geq 1 \), together with the distinguished \( W \)-edges \( NW(B) \) and \( SW(B) \) and the distinguished \( E \)-edges \( NE(B) \) and \( SE(B) \), which satisfy the following condition for every \( X \in \{W, E\} \):

\( (XYB) \) if \( k_X \geq 2 \), then \( NX(B) \neq SX(B) \).

Let \( D_X \) be an oriented graph with a functional \( X \)-edge \( e_X \). Here \( X \) can be \( W \) or \( E \), and we assume that \( D_W \) and \( D_E \) are disjoint digraphs, by which we mean that their sets of vertices are disjoint. We assume also that \( e_W \) is \((a, b)\), \( e_E \) is \((c, d)\) and \( e \) is \((a, d)\).

Then let \( D_W[e_W - e_E]D_E \) be the oriented graph

\( (D_W - \{e_W\}) \cup (D_E - \{e_E\}) \cup \{e\} \)

on the union of the vertices of \( D_W \) and \( D_E \) with the vertices \( b \) and \( c \) omitted. This is illustrated by the following picture:

\[
\begin{array}{c}
D_W \quad D_W[e_W - e_E]D_E \\
\quad \downarrow \\
\quad \downarrow \\
D_E \\
\end{array}
\]

The oriented graphs \( D_W \) and \( D_E \) may be conceived as obtained from \( D_W[e_W - e_E]D_E \) by cutting the edge \( e \) into the two pieces \( e_W \) and \( e_E \), which may justify calling cut the corresponding inference rule.

We define now by induction the notion of construction of a global K-graph, which for short we call just construction. A construction will be a finite binary tree in whose nodes we have an oriented graph together with some distinguished edges of this graph.

The oriented graph at the root of a construction \( G \) will be called the root graph of \( G \), and we say that \( G \) is a construction of its root graph. For \( X \in \{W, E\} \) and \( Y \in \{N, S\} \), we write \( YX(G) \) for the distinguished edges of the root graph of \( G \), which are at the root of \( G \) together with the root graph.
Here are the two clauses of our definition of construction:

(1) The single-node tree in whose single node, which is both the root and the unique leaf, we have the underlying oriented graph of a basic $K$-graph, together with the distinguished edges $X Y D$, is a construction.

(2) For every $X \in \{ W, E \}$, let $G_X$ be a construction of the oriented graph $D_X$, and let $e_X$ be a functional $X$-edge of $D_X$. The tree of the construction $G = G_W[e_W - e_X]G_E$ is obtained by adding to the trees of the constructions $G_W$ and $G_E$ a new node, which will be the root of $G$, whose successors are the roots of the trees of $G_W$ and $G_E$. The oriented graph at the root of $G$, i.e. the root graph of $G$, is $D = D_W[e_W - e_X]D_E$ provided the following is satisfied for every $Y \in \{ N, S \}$:

\[(XYC) \quad e_X = Y \bar{X}(G_X) \text{ or } e_X = YX(G_X)\]

(Note that this condition for $X$ being $W$ is the same as this condition for $X$ being $E$.) The distinguished edges of $D$ at the root of $G$ are obtained as follows:

\[(XYD) \quad YX(G) = \begin{cases} YX(G_X) & \text{if } e_X = YX(G_X), \\ YX(G_E) & \text{otherwise.} \end{cases}\]

At the other nodes of the tree of $G$, which are not the root of $G$, we have in $G$ the same oriented graphs and the same distinguished edges that we had in $G_X$.

This concludes our definition of construction.

A global $K$-graph is an oriented graph that is the root graph of a construction.

Let $G_W$ and $G_E$ be respectively constructions of the global $K$-graphs $D_W$ and $D_E$ in the example for $D_W[e_W - e_X]D_E$ given above, and let $G = G_W[e_W - e_X]G_E$. We may take, as the picture suggests, that $SE(G_W) = e_W$ and $NW(G_E) = e_E$. So (XYC) would be satisfied. To illustrate how (XYD) is applied, we have the following picture:

Note that $e_X = N \bar{X}(G_X) = S \bar{X}(G_X)$ for $X$ being $W$ is the same as $e_X = N \bar{X}(G_X) = S \bar{X}(G_X)$ for $X$ being $E$, and $e_X = N \bar{X}(G_X) = S \bar{X}(G_X)$ for $X$ being $E$ is the same as $e_X = N \bar{X}(G_X) = S \bar{X}(G_X)$ for $X$ being $W$.

If $e_W = NE(G_W) = SE(W)$, then (XYC) certainly holds. According to (XYD), we then have that $NE(G) = NE(G_E)$ and $SE(G) = SE(G_E)$, while $YW(G)$ depends on whether $e_X$ is $YW(G_E)$ or not.

(1) If both $NX(G)$ and $SX(G)$ are from $D_X$, then $e_X = NX(G_X) = SX(G_X)$. By the induction hypothesis we know that $D_X$ has no other $X$-edge save $e_X$. So all the $X$-edges of $D$ are $X$-edges of $D_X$, and then we apply the induction hypothesis to $G_X$.

(2) If both $NX(G)$ and $SX(G)$ are from $D_X$, then we apply the induction hypothesis to $G_X$.

(3) If one of $NX(G)$ and $SX(G)$ is from $D_X$, while the other is from $G_X$, then (XYG) is trivial because $D_X$ and $D_X$ are disjoint digraphs. □

**Proof of (XYG).** We proceed by induction on the number of inner edges of $D$. In the basis, when $G$ is a basic $K$-graph, we have (XYB). In the induction step we have three cases.

(1) If both $NX(G)$ and $SX(G)$ are from $D_X$, then $e_X = NX(G_X) = SX(G_X)$. By the induction hypothesis we know that $D_X$ has no other $X$-edge save $e_X$. So all the $X$-edges of $D$ are $X$-edges of $D_X$, and then we apply the induction hypothesis to $G_X$.

(2) If both $NX(G)$ and $SX(G)$ are from $D_X$, then we apply the induction hypothesis to $G_X$.

(3) If one of $NX(G)$ and $SX(G)$ is from $D_X$, while the other is from $G_X$, then (XYG) is trivial because $D_X$ and $D_X$ are disjoint digraphs. □

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Our purpose next is to find conditions equivalent with \((XYD)\) of clause (2) of the definition of construction above. These equivalent conditions will come handy for proofs later on. Note first that \((XYD)\) amounts to the following two implications:

\[
\begin{align*}
\text{(XYD1)} & \quad \text{if } e_{X} = YX(G_{X}), \text{ then } YX(G) = YX(G_{X}). \\
\text{(XYD2)} & \quad \text{if } YX(G) \neq YX(G_{X}), \text{ then } e_{X} = YX(G_{X}).
\end{align*}
\]

We infer easily the following from these two implications:

\[
\begin{align*}
\text{(D)} & \quad \text{if } YX(G) = YX(G_{X}) \text{ or } YX(G) = YX(G_{X}), \text{ then } e_{X} = YX(G_{X}).
\end{align*}
\]

Then from (D) we infer easily for every \(Z \in \{W, E\}\) that

\[
\begin{align*}
\text{(XY1)} & \quad \text{if } YX(G) \in D_{Z}, \text{ then } YX(G) = YX(G_{Z}).
\end{align*}
\]

From (XYD2) we also infer easily that

\[
\begin{align*}
\text{(XY2)} & \quad \text{if } YX(G) \in D_{X}, \text{ then } e_{X} = YX(G_{X}).
\end{align*}
\]

So we have deduced (XY1) and (XY2) from (XYD).

We will now show that, conversely, we may deduce (XYD) from (XY1) and (XY2). Here is how we obtain (XYD):

\[
\begin{align*}
\text{if } YX(G) \neq YX(G_{X}), \text{ then } YX(G) \notin D_{X}, \text{ by (XY1)},
\end{align*}
\]

\[
\begin{align*}
\text{then } YX(G) \in D_{X},
\end{align*}
\]

\[
\begin{align*}
\text{then } e_{X} = YX(G_{X}), \text{ by (XY2)}.
\end{align*}
\]

We infer (D) from (XY1):

\[
\begin{align*}
\text{if } YX(G) \neq YX(G_{X}), \text{ then } YX(G) \notin D_{X}, \text{ by (XY1)},
\end{align*}
\]

\[
\begin{align*}
\text{then } YX(G) \in D_{X},
\end{align*}
\]

\[
\begin{align*}
\text{then } YX(G) = YX(G_{X}), \text{ by (XY1)},
\end{align*}
\]

and we infer (XYD1) from (D):

\[
\begin{align*}
\text{if } e_{X} = YX(G_{X}), \text{ then } YX(G) \neq YX(G_{X}),
\end{align*}
\]

\[
\begin{align*}
\text{then } YX(G) = YX(G_{X}), \text{ by (D)}.
\end{align*}
\]

So (XY1) and (XY2) have the same force as (XYD).

The remainder of this section is devoted to proving for our notion of global K-graph a completeness result, which will help us for the equivalence proofs in later sections. We start with three propositions that involve the equations that are assumed for planar polycategories (see the Introduction). The equations involved in Propositions 2.1 and 2.2 are like the equations of multicategories (see [11], Section 3; analogous equations are also assumed for operads), while the equation involved in Proposition 2.3 is dual to that involved in Proposition 2.2.

Let \(P, Q\) and \(R\) be constructions, and let \(e_{W}\) and \(f_{W}\) be \(E\)-edges of the root graphs of \(P\) and \(Q\), respectively, while \(e_{E}\) and \(f_{E}\) are \(W\)-edges of the root graphs of \(Q\) and \(R\) respectively. Let \(G_{1}\) be \(P[e_{W}-e_{E}\mid Q][f_{W}-f_{E}\mid R]\) and let \(G_{2}\) be \(P[e_{W}-e_{E}\mid Q][f_{W}-f_{E}\mid R]\). We can prove the following.

**Proposition 2.1.** We have that \(G_{1}\) is a construction iff \(G_{2}\) is a construction. The root graphs of these constructions are the same and the distinguished edges of these root graphs at these roots are the same.

**Proof.** In this proof we write \([-]\) for both \([e_{W}-e_{E}]\) and \([f_{W}-f_{E}]\), since it is clear from the context which we have in mind. We show first that if \((P[-]\ Q)[-]R\) is a construction, then \((P[-]\ Q)[-]R\) is a construction.

We have, by (XYC), that

\[
P([-]Q)[-]R \text{ is a construction iff for every } Y \in \{N, S\} \text{ we have } e_{W} = YE(P) \text{ or } e_{E} = YYW(Q).
\]

\[
(P[-]Q)[-]R \text{ is a construction iff for every } Y \in \{N, S\} \text{ we have } f_{W} = YE(P[-]Q) \text{ or } f_{E} = YYW(R).
\]

We have that \(f_{W} = YE(P[-]Q)\) and the fact that \(f_{W}\) is in the root graph of \(Q\) imply \(f_{W} = YE(Q)\), by (XY1). Since \((P[-]Q)[-]R\) is a construction, we can conclude that \((Q[-]R)\) is a construction.

To show that \((P[-]\ Q)[-]R\) is a construction it remains to verify that we have \(e_{W} = YE(P)\) or \(e_{E} = YYW(Q[-]R)\). We have the implication

\[
\text{if } e_{W} \neq YE(P), \text{ then } e_{E} = YYW(Q),
\]

since \((P[-]Q)\) is a construction. We also have

\[
\text{if } e_{W} \neq YE(P), \text{ then } YE(P[-]Q) = YE(P), \text{ by (XYD2)},
\]

\[
\text{then } f_{W} \neq YE(P[-]Q), \text{ since } f_{W} \text{ is not an edge of the root graph of } P,
\]

\[
\text{then } f_{E} = YYW(R), \text{ since } (P[-]Q)[-]R \text{ is a construction,}
\]

\[
\text{then } YYW(Q[-]R) = YYW(Q), \text{ by (XYD1)},
\]

\[
\text{then } e_{E} = YYW(Q[-]R),
\]

by the implication established above. So \((P[-]\ Q)[-]R\) is a construction.
We proceed analogously to show that if $P[\neg](Q[\neg]R)$ is a construction, then $(P[\neg]Q)[\neg]R$ is a construction. It is clear that the root graphs of these two constructions are the same. It remains to establish that the distinguished edges of these root graphs at these roots are the same.

We have

$$YE((P[\neg]Q)[\neg]R) = \begin{cases} YE(R) & \text{if } f_W = YE(P[\neg]Q), \\ YE(P[\neg]Q) & \text{otherwise,} \end{cases}$$

$$YE(P[\neg]Q) = \begin{cases} YE(Q) & \text{if } e_W = YE(P), \\ YE(P) & \text{otherwise.} \end{cases}$$

Since $f_W$ is an edge of the root graph of $Q$, we have that

- if $f_W = YE(P[\neg]Q)$, then $f_W = YE(Q)$, by (XY1),
- if $f_W = YE(P[\neg]Q)$, then $e_W = YE(P)$, by (XY2),

and we have that

- if $f_W = YE(Q)$ and $e_W = YE(P)$, then $f_W = YE(P[\neg]Q)$.

because if $e_W = YE(P)$, then $YE(Q) = YE(P[\neg]Q)$, as stated above. So we have

$$YE((P[\neg]Q)[\neg]R) = \begin{cases} YE(R) & \text{if } f_W = YE(Q) \text{ and } e_W = YE(P), \\ YE(Q) & \text{if } f_W \neq YE(Q) \text{ and } e_W = YE(P), \\ YE(P) & \text{if } f_W \neq YE(Q) \text{ and } e_W \neq YE(P). \end{cases}$$

On the other hand,

$$YE(P[\neg](Q[\neg]R)) = \begin{cases} YE(Q[\neg]R) & \text{if } e_W = YE(P), \\ YE(P) & \text{otherwise,} \end{cases}$$

$$YE(Q[\neg]R) = \begin{cases} YE(Q) & \text{if } f_W = YE(Q), \\ YE(R) & \text{otherwise,} \end{cases}$$

which implies

$$YX((P[\neg]Q)[\neg]R) = YX(P[\neg](Q[\neg]R))$$

when $X$ is $E$. We proceed in a dual manner when $X$ is $W$. □

Let $P$, $Q$, and $R$ be constructions, and let $e_W$ and $f_W$ be different $E$-edges of the root graph of $P$, while $e_X$ and $f_X$ are $W$-edges of the root graphs of $Q$ and $R$ respectively. Let $G_1$ be $(P[e_W-e_X]Q)[f_W-f_X]R$ and let $G_2$ be $(P[f_W-f_X]Q)[e_W-e_X]R$.

**Proposition 2.2.** It is formulated exactly as **Proposition 2.1** for these new constructions $G_1$ and $G_2$.

**Proof of Proposition 2.2.** As in the preceding proof, we use the abbreviation $[\neg]$. We show that if $(P[\neg]Q)[\neg]R$ is a construction, then $(P[\neg]R)[\neg]Q$ is a construction. We have that $P[\neg]Q$ and $(P[\neg]Q)[\neg]R$ are constructions under the same conditions concerning $e_X$ and $f_X$ displayed at the beginning of the proof of **Proposition 2.1**.

We have that $f_W = YE(P[\neg]Q)$ and the fact that $f_W$ is in the root graph of $P$ imply $f_W = YE(P)$, by (XY1). Since $(P[\neg]Q)[\neg]R$ is a construction, we can conclude that $P[\neg]R$ is a construction.

To show that $(P[\neg]R)[\neg]Q$ is a construction, it remains to verify that we have $e_W = YE(P[\neg]R)$ or $e_X = YW(Q)$. We have

- if $e_X \neq YW(Q)$, then $e_W = YE(P)$, since $P[\neg]Q$ is a construction,
  - then $f_W \neq YE(P)$, since $e_W \neq f_W$,
  - then $YE(P[\neg]R) = YE(P)$, by (XYD2),
  - then $e_W = YE(P[\neg]R)$.

So $(P[\neg]R)[\neg]Q$ is a construction.

We proceed in exactly the same manner to show that if $(P[\neg]R)[\neg]Q$ is a construction, then $(P[\neg]Q)[\neg]R$ is a construction. It is clear that the root graphs of these two constructions are the same. It remains to establish that the distinguished edges of these root graphs at these roots are the same.

We can conclude that

$$YE((P[\neg]Q)[\neg]R) = \begin{cases} YE(R) & \text{if } f_W = YE(P) \text{ and } e_W \neq YE(P), \\ YE(Q) & \text{if } f_W \neq YE(P[\neg]Q) \text{ and } e_W = YE(P), \\ YE(P) & \text{if } f_W \neq YE(P[\neg]Q) \text{ and } e_W \neq YE(P). \end{cases}$$

Since $f_W$ is an edge of the root graph of $P$, we have that

- if $f_W = YE(P[\neg]Q)$, then $f_W = YE(P)$ and $e_W \neq YE(P)$,
by using \((XY1)\) and \(e_W \neq f_W\). We also have that

\[
\text{if } f_W = YE(P) \text{ and } e_W \neq YE(P), \text{ then } f_W = YE(P[-]Q),
\]

by the clause for \(YE(P[-]Q)\).

We can conclude analogously that

\[
YE((P[-]R)[-]Q) = \begin{cases} 
YE(Q) & \text{if } e_W = YE(P) \text{ and } f_W \neq YE(P), \\
YE(R) & \text{if } e_W \neq YE(P[-]R) \text{ and } f_W = YE(P), \\
YE(P) & \text{if } e_W \neq YE(P[-]R) \text{ and } f_W \neq YE(P).
\end{cases}
\]

We show first that

\[
(P) \quad (f_W \neq YE(P[-]Q) \text{ and } e_W \neq YE(P)) \iff (f_W \neq YE(P) \text{ and } e_W \neq YE(P[-]R)).
\]

By \((XY1)\), we have that

\[
\text{if } f_W = YE(P[-]Q), \text{ then } f_W = YE(P),
\]

and the converse implications hold by the clauses for \(YE(P[-]Q)\) and \(YE(P[-]R)\) because \(e_W \neq f_W\). This is enough to establish \((P)\).

We establish that

\[
(Q) \quad (f_W \neq YE(P[-]Q) \text{ and } e_W = YE(P)) \iff (f_W \neq YE(P) \text{ and } e_W = YE(P)),
\]

\[
(R) \quad (f_W = YE(P) \text{ and } e_W \neq YE(P)) \iff (f_W = YE(P) \text{ and } e_W \neq YE(P[-]R)),
\]

by using \((XY1)\) and \(e_W \neq f_W\). So we have that

\[
YE((P[-]Q)[-]R) = YE((P[-]R)[-]Q).
\]

We have that

\[
YW((P[-]Q)[-]R) = \begin{cases} 
YW(P) & \text{if } f_e = YW(R) \text{ and } e_e = YW(Q), \\
YW(Q) & \text{if } f_e = YW(R) \text{ and } e_e \neq YW(Q), \\
YW(R) & \text{if } f_e \neq YW(R).
\end{cases}
\]

\[
YW((P[-]R)[-]Q) = \begin{cases} 
YW(P) & \text{if } f_e = YW(R) \text{ and } e_e = YW(Q), \\
YW(R) & \text{if } f_e = YW(Q) \text{ and } f_e \neq YW(R), \\
YW(Q) & \text{if } e_e \neq YW(Q).
\end{cases}
\]

We have that

\[
\text{if } e_e \neq YW(Q), \text{ then } e_W = YE(P), \text{ since } P[-]Q \text{ is a construction},
\]

\[
\text{then } f_W \neq YE(P), \text{ since } e_W \neq f_W,
\]

\[
\text{then } f_e = YW(R), \text{ since } P[-]R \text{ is a construction},
\]

and by contraposition we have that if \(f_e \neq YW(R)\), then \(e_e = YW(Q)\). This, together with what we have established previously, shows that

\[
YX((P[-]Q)[-]R) = YX((P[-]R)[-]Q)
\]

for every \(X \in \{W, E\}\) and every \(Y \in \{N, S\}\).

Let \(P, Q\) and \(R\) be constructions, and let \(e_e\) and \(f_e\) be different \(W\)-edges of the root graph of \(P\), while \(e_w\) and \(f_w\) are \(E\)-edges of the root graphs of \(Q\) and \(R\) respectively. Let \(G_1\) be \(R[f_w - f_e][Q[e_w - e_e]P]\) and let \(G_2\) be \(Q[e_w - e_e][R[f_w - f_e]P]\).

**Proposition 2.3.** It is formulated exactly as Proposition 2.1 for these new constructions \(G_1\) and \(G_2\). It is proved in a manner dual to what we had for the proof of Proposition 2.2.

Consider the relations between constructions that exist between the constructions \(G_1\) and \(G_2\) of Propositions 2.1–2.3. We call these relations \(\rho_1, \rho_2\) and \(\rho_3\) respectively.

Let \(\rho\)-equivalence be the equivalence relation between constructions that is the reflexive, symmetric and transitive closure of \(\rho_1 \cup \rho_2 \cup \rho_3\), and which is closed moreover under \(\rho\)-congruence:

\[
\text{if } G_1 \text{ is } \rho\text{-equivalent with } G_2 \text{ and } H_1 \text{ is } \rho\text{-equivalent with } H_2, \text{ then}
\]

\[
G_1[e_w - e_e][H_1] \text{ is } \rho\text{-equivalent with } G_2[e_w - e_e][H_2],
\]

provided the last two constructions are defined. We can prove the following.

**Proposition 2.4.** If \(e\) is an inner edge of the root graph of a construction \(G\), then there are two constructions \(H_W\) and \(H_E\) such that \(G\) is \(\rho\)-equivalent to \(H_W[e_w - e_e]H_E\).
Proof. We proceed by induction on the number \( n \) of inner edges in the root graph of \( G \). If \( n = 1 \), then \( G \) is of the form \( G_W[e_w - e_e]G_E \) and we take \( H_k \) to be \( G_X \). If \( n \geq 2 \), and \( G \) is again of that form, then again we choose \( H_k \) to be \( G_X \).

Suppose \( n \geq 2 \), and \( G \) is of the form \( G_W[f_w - f_e]G_E \) for \( f \) different from \( e \). If \( e \) is in the root graph of \( G_W \), then by the induction hypothesis \( G \) is \( \rho \)-equivalent to a construction

\[
(G_W[e_w - e_e]G_W)[f_w - f_e]G_E,
\]

which is \( \rho \)-equivalent to either

\[
(G_W[f_w - f_e]G_E)[e_w - e_e]G_W,
\]

because of \( \rho_1 \), or

\[
(G_W[f_w - f_e]G_E)[e_w - e_e]G_W,
\]

because of \( \rho_2 \). We proceed analogously if \( e \) is in the root graph of \( G_E \), by appealing to \( \rho_1 \) and \( \rho_3 \). \( \square \)

We say that a basic K-graph \( B \) determines a leaf of a construction \( G \) when \( B \) occurs in an application of clause (1) for the definition of \( G \). For a given construction \( G \), let \( [G] \) be the set of all the constructions that have leaves determined by the same basic K-graphs as \( G \), and that have the same root graph as \( G \). We can prove the following for every pair of constructions \( G \) and \( H \).

**Proposition 2.5.** We have that \( G \) and \( H \) are \( \rho \)-equivalent iff \([G] = [H]\).

**Proof.** For the implication from left to right we have essentially just an easy application of Propositions 2.1–2.3. For the other direction, suppose \([G] = [H]\). We proceed by induction on the number \( n \) of inner edges in the root graph \( D \) of \( G \) and \( H \), which they share. If \( n = 0 \), then \( G \) and \( H \) are the same construction, given by the same basic K-graph.

Let \( n \geq 1 \), and let \( e \) be an inner edge of \( D \). Then by Proposition 2.4 we have that \( G \) and \( H \) are \( \rho \)-equivalent to respectively \( G_W[e_w - e_e]G_E \) and \( H_W[e_w - e_e]H_E \). We apply the induction hypothesis to \( G_W \) and \( H_W \), and then we appeal to \( \rho \)-congruence. \( \square \)

With that we have proved the completeness result we set ourselves as a goal in this section. As a consequence of Propositions 2.1–2.3 we also have the following for every pair of constructions \( G \) and \( H \), for every \( X \in \{W, E\} \) and every \( Y \in \{N, S\} \).

**Proposition 2.6.** If \( G \) and \( H \) are \( \rho \)-equivalent, then \( YX(G) = YX(H) \).

We conclude this section with some terminological matters, which we need for the exposition later on. In a construction \( G \) let the root vertices of \( G \) be the vertices of the root graph of \( G \). The other vertices that may occur in the oriented graph at a node of \( G \) that is not the root, which are not root vertices, will be called secondary vertices.

Two constructions are said to be \( \sigma \)-equivalent when they are in all respects the same, save that they may differ in the choice of secondary vertices. One could say that they are the same construction up to renaming of secondary vertices.

For a construction, consider \([G]\), and let \([G]\) be the set of all the constructions \( \sigma \)-equivalent to a construction in \([G]\). We call \([G]\) a **global compass graph**.

### 3. Local K-graphs

In this section we deal with our second definition of K-graph (see the Introduction), which yields the notion of local K-graph. We call them local because for every inner we will specify the edges that stand at the \( NW, NE, SW \) and \( SE \) of it. With global K-graphs, we had such a specification only for the whole graph.

For \( D \) a digraph and \( a \) an inner vertex of \( D \) consider for \( \gamma \in \{N, S\} \) the two functions \( Y\gamma(a) \) such that \( Y\gamma(a) \) is an edge of \( D \) of the form \( (b, a) \), and consider the two functions \( YE(a) \) such that \( YE(a) \) is an edge of \( D \) of the form \( (a, b) \). For every inner vertex \( a \) of \( D \) let \( k_w^a \geq 1 \) be the number of edges of \( D \) of the form \( (b, a) \), while \( k_e^a \geq 1 \) is the number of edges of \( D \) of the form \( (a, b) \).

Let \( L \) be a set of such four functions. Then we say that \( (D, L) \) separates \( N \) from \( S \) when the following condition (analogous to \( X(YB) \) of Section 2) holds for every inner vertex \( a \) of \( D \) and every \( X \in \{W, E\} \):

\[
\text{if } k^a_e \geq 2, \text{ then } NX(a) \neq SX(a).
\]

We say that a path \( a_1, \ldots, a_n \), with \( n \geq 1 \), of \( D \) is \( Y \)-decent in \( (D, L) \) when either \( n = 1 \) or if \( n \geq 2 \), then \( YE(a_1) = (a_1, a_2) \) or \( YW(a_n) = (a_{n-1}, a_n) \) (see Section 2 for the notion of path). A path of \( D \) is \( Y \)-decent in \( (D, L) \) when it is both \( N \)-decent and \( S \)-decent in \( (D, L) \).

For example, if \( (D, L) \) is such that \( NE(a) = (a, d) \) and \( NW(c) = (e, c) \), as in the following picture of \( D \), then the path \( a, b, c \) is not \( N \)-decent in \( (D, L) \):

![Diagram of K-graph](image-url)
Proposition 4.2. We have that $h = YX(G)$ if and only if $h$ is an $X$-edge of the root graph $D$ of $G$ such that every path of $D$ that covers $h$ is a $YX$-path in $\lambda(G)$.
Proof. Suppose X is W. From left to right we proceed by induction on the number n of inner edges of D. In the basis, when n = 0, we deal with a basic K-graph, and the proposition holds trivially. If n ≥ 1, consider a path a₁, . . . , aₘ with m ≥ 2, that covers YW(G). If for (aᵢ, aᵢ₊₁) = e, for i ∈ {1, . . . , m − 1}, we have that YW(aᵢ₊₁) = (c, aᵢ₊₁) ≠ e, by Proposition 2.4 we have that G is ρ-equivalent to H = Hₑ YW(eₑ − eₑ)Hₑ. (Note that e must be an inner edge of D, by (W) of Proposition 4.4.) Since eₑ ≠ YW(aₑ₊₁), we have eₑ ≠ YW(Hₑ), by (W) of Proposition 4.1. So YW(H) = YW(Hₑ), by (XYD), which, together with Proposition 2.6, contradicts the assumption that h = YW(G).

From right to left we make again an induction on the number n of inner edges of D. The basis, when n = 0, is again trivial. For the induction step, when n ≥ 1, suppose h ≠ YW(G). We want to show that if h is a W-edge of D, then there is a path a₁, . . . , aₘ with m ≥ 2, such that (a₁, a₂) = h and this path is not a YW-path. Since D is weakly connected, there is a semipath a₁, . . . , aₘ, b₁, . . . , bₖ of D, with m ≥ 2, aₘ = b₁ and k ≥ 2, such that (a₁, a₂) = h, (bₖ, bₖ₊₁) = YW(G), a₁, . . . , aₘ is a path of D and (b₁, b₂) = e ∈ D. We use here the assumption that h is a W-edge of D; otherwise, YW(G) could, for example, be of the form (c, a₁).

If e = YW(G), then, by (W) of Proposition 4.1, we have e = YW(aₘ) in λ(G), and (aₘ₊₁, aₘ) is not a YW-edge. If e ≠ YW(G), then, with the help of the assumption that D is W-E-functional, we conclude that e is an inner edge of D, and then, by Proposition 2.4, we have that G is ρ-equivalent to H = Hₑ YW(eₑ − eₑ)Hₑ. We must have that YW(G), which, by Proposition 2.6, is equal to YW(H), is in the root graph of Hₑ (because we have a semipath b₁, . . . , bₖ in this root graph). By (XY2), we conclude that eₑ = YW(G), and, by (W) of Proposition 4.1, we have eₑ = YW(aₑ) ≠ (aₑ₊₁, aₑ). So a₁, . . . , aₘ is not a YW-path in λ(Hₑ), which implies that it is not a YW-path in λ(G). We proceed analogously when X ∈ E. □

We can now prove the following for every construction G.

Proposition 4.3. We have that λ(G) is a local compass graph.

Proof. As we noted at the beginning of the section, it remains to verify condition (S) of the definition of local compass graph. Suppose we have a path a₁, . . . , aₘ of the root graph D of G that is not decent. So n ≥ 2, and for some Y ∈ {N, S} we have YE(a₁) ≠ (a₁, a₂) and YW(aₘ) ≠ (aₘ₋₁, aₘ). Any edge covered by this path must be an inner edge of D, and, since n ≥ 2, there is such an edge; let us call it e. By Proposition 2.4, we have that G is ρ-equivalent to Hₑ YW(eₑ − eₑ)Hₑ. By Proposition 4.2, we conclude that eₑ ≠ YE(Hₑ) and eₑ ≠ YW(Hₑ), but this contradicts the fact that Hₑ YW(eₑ − eₑ)Hₑ is a construction. □

The following two propositions serve to prove that there is a bijection between global and local compass graphs

Proposition 4.4. For every local compass graph (D, L) there is a construction G such that λ(G) = (D, L).

Proof. We proceed by induction on the number n of inner edges of D. If n = 0, then (D, L) determines a basic K-graph, and the proposition holds trivially. If n ≥ 1, then D is of the form Dₑ YW(eₑ − eₑ)Dₑ for the local compass graphs (Dₑ, Lₑ) and (Dₑ, Lₑ). This follows from the inductive definition of local compass graphs, which gives an equivalent notion.

By the induction hypothesis, for X ∈ {W, E} we have the constructions Gₓ such that λ(Gₓ) = (Dₓ, Lₓ). We show first that G = Gₓ YW(eₑ − eₑ)Gₓ is a construction of a global K-graph. For that we have to check (XYC). Suppose for some Y ∈ {N, S} we have eₑ ≠ YE(Gₓ) and eₑ ≠ YW(Gₓ). By Proposition 4.2, there is a path a₁, . . . , aₘ, where n ≥ 2, in Dₑ that is not a YE-path with (aₘ₋₁, aₘ) = eₑ and there is a path b₁, . . . , bₘ, where m ≥ 2, in Dₑ that is not a YW-path with (b₁, b₂) = eₑ. We may assume that (a₁, a₂) ≠ YE(a₁) and (bₘ₋₁, bₘ) ≠ YW(bₘ). The path a₁, . . . , aₘ₋₁, b₂, . . . , bₘ of D is not a decent path. So (XYC) holds.

To finish the proof we have to check that λ(G) = (D, L). It is clear that the root graph of Gₑ YW(eₑ − eₑ)Gₑ is D, while the definition of L in terms of Lₑ and Lₑ involved in the definition of λ(G) is in accordance with the clause for L in the inductive definition of local compass graph. □

Proposition 4.5. If λ(G) = λ(H), then ||G|| = ||H||.

Proof. Let λ(G) = λ(H) = (D, L). The oriented graph D together with the functions in L determines the basic K-graphs that enter into the inductive definitions of G and H up to renaming of secondary vertices (see the end of Section 2). Since D is the root graph of both G and H, we may conclude that [G] and [H] are the same up to renaming of these secondary vertices, which means that ||G|| = ||H||. □

If we define Λ ||G|| as λ(G), as we did at the beginning of this section, then from Propositions 4.4 and 4.5 we infer that Λ is a bijection between global and local compass graphs. From the definition of this bijection, we may conclude that the notions of global and local K-graphs coincide.

5. K-graphs

In this section we deal with our third definition of K-graph (see the Introduction). For the notion this definition gives we establish that it is equivalent with the notion of local K-graph, and hence, by the results of Section 4, with both notions given by the preceding two definitions.

The following definitions are for oriented graphs, and build upon notions defined in Section 2. A proper semipath is a semipath such that neither it nor its cognate is a path. Intuitively, there must be a change of direction in a proper semipath. An edge (a, b) is transversal when there is a proper semipath a, b, . . . , c and a proper semipath b, a, . . . , d.
A bifurcation is a triple of different edges that have a common vertex. The following four kinds of bifurcations are possible:

\[ \xrightarrow{\text{in-going}} \quad \xrightarrow{\text{out-going}} \quad \xleftarrow{\text{in-going}} \quad \xleftarrow{\text{out-going}} \]

A bifurcation is called transversal when all the three edges in it are transversal.

A K-graph is an oriented graph $D$ such that we have (1)-(3) from the definition of local K-graph of Section 3, and we have moreover (instead of (4) and (5)) the following condition:

No bifurcation is transversal.

In a semipath $a_1, \ldots, a_n$ with $n \geq 2$, of an oriented graph $D$ we have for $i \in \{1, \ldots, n-1\}$ that either $(a_i, a_{i+1})$ or $(a_{i+1}, a_i)$ is an edge of $D$, but not both. We call this edge of $D$ the edge that connects $a_i$ and $a_{i+1}$. We can now prove the following.

**Proposition 5.1.** If in a semipath $a_1, \ldots, a_n$, with $n \geq 2$, of an asemicyclic oriented graph $D$ the edge that connects $a_1$ and $a_2$ and the edge that connects $a_{n-1}$ and $a_n$ are transversal, then for every $i \in \{1, \ldots, n-1\}$ the edge that connects $a_i$ and $a_{i+1}$ is transversal.

**Proof.** If the edge that connects $a_1$ and $a_2$ is transversal, then there is a proper semipath $a_2, a_3, \ldots, c$, and if the edge that connects $a_{n-1}$ and $a_n$ is transversal, then there is a proper semipath $a_{n-1}, a_n, \ldots, d$. For every $i \in \{1, \ldots, n-1\}$ we have that

\[
\begin{align*}
& a_{i+1}, a_i, \ldots, a_2, a_1, \ldots, c \\
& a_i, a_{i+1}, \ldots, a_{n-1}, a_n, \ldots, d
\end{align*}
\]

are proper semipaths. They are semipaths because $D$ is asemicyclic, and hence all their members are mutually distinct, and they are proper because they extend proper semipaths. We can conclude that the edge that connects $a_i$ and $a_{i+1}$ is transversal. $\square$

For every K-graph $D$, if $D$ has transversal edges, by relying on Proposition 5.1, we conclude that all the transversal edges of $D$ make a semipath $a_1, \ldots, a_n$, with $n \geq 2$, which we will call the transversal of $D$. The transversal is unique up to cognation; the transversal is either a semipath or its cognate (see Section 2). The vertices in the transversal, which must all be inner, are called transversal vertices.

All the edges of $D$ that share a single vertex with the transversal of $D$ are either in-going, when for some $i \in \{1, \ldots, n\}$ they are of the form $(b, a_i)$, or they are out-going, when they are of the form $(a_i, b)$, where $a_i$ is a transversal vertex. For an in-going edge $(b, a_i)$ we have in $D$ a tree oriented from the leaves towards the root $a_i$:

\[ \xrightarrow{\text{in-going}} \quad \xrightarrow{\text{out-going}} \]

which we call an in-going tree.

For an out-going edge $(a_i, b)$ we have in $D$ a tree oriented from the root $a_i$ towards the leaves, which we call an out-going tree. These trees cannot share an edge with the transversal of $D$; all the vertices in these trees except $a_i$ are not in the transversal of $D$. The orientation is imposed because no transversal edge of $D$ is in these trees. If our K-graph does not have transversal edges, then it has no transversal, and is made only of trees analogous to in-going and out-going trees that share a root.

The following proposition establishes that the notions of K-graph and local K-graph are equivalent.

**Proposition 5.2.** An oriented graph is a local K-graph iff it is a K-graph.

**Proof.** Let $D$ be an oriented graph that satisfies (1)-(3) of Section 3. To prove the proposition from left to right, suppose that there is a transversal bifurcation in $D$. This bifurcation can be of the four kinds mentioned above, which will produce in $D$ subgraphs of the following four patterns (a subgraph of a digraph is given by a subset of its edges on a subset of its vertices):
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(These four subgraphs play here a role analogous to Kuratowski’s graphs $K_5$ and $K_{3,3}$, one of which must be found in nonplanar graphs; see [8], Chapter 11. Actually, if in the graphs where these subgraphs occur there is a single $W$-vertex and a single $E$-vertex, then in these graphs we have an oriented version of $K_{3,3}$. On the other hand, $K_5$ is related to asemicylicity. We intend to deal with these matters one another occasion.)

In all the four cases we go through all possible functions that could make $L$ to show that there must be a path of $D$ that is not decent in $(D, L)$. For example, with the first picture we will find the path that is not decent in the middle, while with the second picture we would look for that path at the top. If instead we had

\[
\begin{array}{c}
\text{Path 1} \\
\text{Path 2} \\
\end{array}
\]

then we would look for the path that is not decent at the bottom. This establishes the proposition from left to right.

To prove the proposition from right to left, assume we are given a $K$-graph $D$. We define the functions in $L$ by giving their value first for non-transversal inner vertices $b$. We can do it in many ways, provided we take care to guarantee that $(D, L)$ separates $N$ from $S$ (see Section 3). For $X \in \{W, E\}$ we choose $NX(b)$ and $SX(b)$ as the same edge when there is no $X$-ward branching in $b$; otherwise, $NX(b)$ and $SX(b)$ are arbitrarily chosen different edges ending in $b$ when $X$ is $W$, and beginning in $b$ when $X$ is $E$. (Note that for a non-transversal inner vertex that belongs to an in-going tree there is no $E$-ward branching, and for one that belongs to an out-going tree there is no $W$-ward branching.)

It remains to define the values of the functions in $L$ for the transversal vertices, if there are such vertices in $D$. Let $a_1, \ldots, a_n$ for $n \geq 2$, be our transversal of $D$. For $i \in \{1, \ldots, n-1\}$, if $(a_i, a_{i+1})$ is an edge of $D$, then $SE(a_i) = NW(a_{i+1}) = (a_i, a_{i+1})$, and if $(a_{i+1}, a_i)$ is an edge of $D$, then $SW(a_i) = NE(a_{i+1}) = (a_{i+1}, a_i)$. (The other possibility would be to take that if $(a_i, a_{i+1})$ is an edge of $D$, then $NE(a_i) = SW(a_{i+1}) = (a_i, a_{i+1})$, and if $(a_{i+1}, a_i)$ is an edge of $D$, then $NW(a_i) = SE(a_{i+1}) = (a_{i+1}, a_i).$)

For example, $a, b, c, d, e, f, g$ is the transversal of the K-graph given below, and we define $SW(a) = NE(b) = (b, a)$, $SW(b) = NE(c) = (c, b)$, $SE(c) = NW(d) = (c, d)$, etc., as it is suggested by the following picture:

\[
\begin{array}{c}
\text{Path 1} \\
\text{Path 2} \\
\end{array}
\]

The remaining values of the functions in $L$ for transversal vertices may be chosen freely provided we take care to guarantee that $(D, L)$ separates $N$ from $S$.

It is clear that $(D, L)$ so defined separates $N$ from $S$. It remains to verify that every path of $D$ is decent in $(D, L)$. If there were a path $b_1, \ldots, b_m$, with $m \geq 2$, of $D$ that is not decent in $(D, L)$, then all the vertices in this path would be transversal. This path coincides either with $a_{j+1}, \ldots, a_{j+m}$ or with $a_{j+m}, \ldots, a_{j+1}$, where $0 \leq j$ and $j+m \leq n$. In the first case,
(b₁, b₂) = SE(b₁), while (bₘ₋₁, bₘ) = NW(bₘ), which yields that the path b₁, ..., bₘ is decent, contrary to our assumption. In the second case, (b₁, b₂) = NE(b₁), while (bₘ₋₁, bₘ) = SW(bₘ), which yields again that the path b₁, ..., bₘ is decent. So every path of D is decent in (D, ℒ).

6. Adding the identity graphs to K-graphs

Our notion of K-graph, and the equivalent notions of global and local K-graph, could be extended a little bit by allowing as K-graphs oriented graphs of the form

![Diagram of oriented graph]

with two vertices, one a W-vertex and the other an E-vertex; these oriented graphs have a single edge made of these two vertices, and they have no inner vertex. These additional K-graphs would serve to represent identity deductions, which are related to the sequents A ⊢ A, and we will call them identity graphs.

For every oriented graph D and every identity graph I we will have that the oriented graphs D[−]I and I[−]D, with [−] replaced by an appropriate [e₁ − eₙ], are both equal to D up to replacement of vertices. The construction I′ of an identity graph I would be a single-node tree with I in this unique node, and the distinguished edges all being the unique edge of I. The definition of construction involves now an appropriate modification of (XYD). The notion of ρ-equivalence would be extended so that for every construction G we would have that GI′[−] is ρ-equivalent to G[−]I, which is ρ-equivalent to G.

In the definition of local compass graph of Section 3 and in the definition of K-graph of Section 5, in condition (3) we would just replace the requirement that D has an inner vertex by the requirement that it has an edge, while everything else in these definitions would remain the same.

7. Q-graphs

If we determined the graphs produced by the rule (PC) of the Introduction in the same manner as we determined in this paper the graphs produced by planar plural cuts, we would obtain something more general and more simple to characterize.

For the definition of the new notion of global K-graph one possibility is to reject in the definition of basic K-graph the requirement (XYB). Everything else in the definition of construction and global K-graph of Section 2 would remain unchanged. Let the new global K-graphs be called global Q-graphs.

The new global Q-graphs can however be characterized more simply. Let a Q-graph be defined as an oriented graph D that satisfies conditions (1)–(3) of the definition of local compass graph (see Section 3). The same three conditions are also found in the definition of K-graph of Section 5. A notion of graph associated with plural cuts in a context with the structural rule of permutation, which, as our notion of Q-graph, is based essentially on connectedness and non-circularity, may be found in [14].

As for constructions of global K-graphs, we say that G is a construction of a global Q-graph D when D is the root graph of G. One can show the following for every X ∈ {W, E}.

**Proposition 7.1.** For every Q-graph D and every X-edge d of D there is a construction G of D such that NX(G) = SX(G) = d.

**Proof.** We proceed by induction on the number n of inner edges of D. In the basis, when n = 0, we rely on the new definition of basic Q-graph (i.e. basic K-graph without (XYB)). In the induction step we have D = DW[e₁ − eₙ]DX. By the induction hypothesis, we have two constructions Gₓ and GX with root graphs DX and GX respectively such that if d is in DX, then NX(Gₓ) = SX(Gₓ) = d and NX(Gₓ) = SX(Gₓ) = eₓ, and if d is in GX, then NX(Gₓ) = SX(Gₓ) = d and NX(Gₓ) = SX(Gₓ) = eₓ. One can then verify that (XYC) is satisfied, and that NX(G) = SX(G) = d, according to (XYD).

As a corollary of this proposition we have every Q-graph is a global Q-graph. The converse being trivial, we have that the two notions are equivalent.

This means that global Q-graphs could be defined by constructions G that do not involve at all the distinguished edges YX(G). For two arbitrary Q-graphs DW and DX, an arbitrary W-edge e₁ of DW and an arbitrary E-edge eₓ of DX, the oriented graph DW[e₁ − eₓ]DX is a Q-graph. We need not pay attention to (XYC) any more.

One could envisage the notion of Q-graph enlarged with identity graphs, as in Section 6. The Q-graphs could be described in the manner in which we have described K-graphs after **Proposition 5.1**, which should still be applied (see also the introduction). As a K-graph, a Q-graph is made of a transversal and in-going and out-going trees rooted in it. The difference is only that the transversal need not be linear any more.

8. Conclusion

A natural interpretation of singular (single-conclusion) sequents is that they state that there is a natural deduction derivation with un-cancelled hypotheses on the left-hand side of the sequent and the conclusion of the derivation on the right-hand side of the sequent. Such derivations have the form of trees.
Plural (multiple-conclusion) sequents yield graphs, related to trees, of a more complicated kind, which were investigated in this paper with respect to the rule of plural cut. Besides the inductive definition of these oriented graphs, which is based on plural sequent systems and plural cut, a non-inductive, graph-theoretical, combinatorial, definition is given, and to reach that other definition for plural sequent systems without the structural rule of permutation was the main goal of the paper, achieved in Section 5. (The graphs when permutation is present are characterized much more simply; see [14], and Section 7 above.) Our result gives a combinatorial characterization of the planarity of the graphs involved.

One could expect applications for our study in less theoretical areas to the extent that plural-sequent variants of substructural logics related to the Lambek calculus find such applications (see [5]). These applications could be motivated not by strictly logical concerns only, but also by theoretical linguistics.

In Section 6 we consider briefly the additions needed when besides plural cut we have in our sequent systems identity axiomatic sequents. These additions do not bring much mathematical novelty. The last section of the paper, Section 7, contains remarks about plural sequent systems where permutation is not completely banned any more. This context is simpler, and is covered by the literature up to a point (as we mentioned in the preceding paragraph.)

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