A Short Proof of the Bernstein Inequality for Formal Power Series

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Abstract

Let $k$ be a field of characteristic zero, let $R$ be the ring of formal power series in $n$ variables over $k$ and let $D(R,k)$ be the ring of $k$–linear differential operators in $R$. If $M$ is a finitely generated $D(R,k)$–module then $d(M) \geq n$ where $d(M)$ is the dimension of $M$. This inequality is called the Bernstein inequality. We provide a short proof.

1 Introduction

Throughout this paper $k$ is a field of characteristic zero. Let $R = k[[x_1, \cdots, x_n]]$ be the ring of formal power series in $n$ variables over $k$ and let $D = D(R,k)$ be the ring of $k$–linear differential operators of $R$. A celebrated result, known as the Bernstein inequality, says the following.

Theorem 1.1. If $M$ is a finitely generated left $D$–module, then

$$d(M) \geq n,$$

where $d(M)$ is the dimension of $M$.

For the ring of polynomials there is now a very short and simple proof of this result [3 9.4.2] due to A. Joseph. But for formal power series the only known (to the author) proof, due to Bjork [2 2.7.2], is much more complicated. It uses some sophisticated homological algebra including spectral sequences.

Since the formal power series case is important in applications of $D$-modules to commutative algebra, it would be useful to have a simple proof of the Bernstein inequality in this case. This is accomplished in this paper. To emphasize the elementary nature of the proof this paper has been made self-contained modulo some basic commutative algebra.

Our proof is inspired by [4] and is from the author’s thesis. I would like to thank my adviser, Professor Gennady Lyubeznik for guidance and support.

2 Preliminaries

Keeping the notation of the Introduction, $R$ is a subring of $\text{Hom}_k(R,R)$, the $k$-linear endomorphisms of $R$, via the map that sends $r \in R$ to the map $R \to R$ which is the multiplication by $r$ on $R$. Let

*NSF for support through grant DMS-1500264 is gratefully acknowledged
$d_i = \frac{\partial}{\partial x_i} : R \to R$ for $1 \leq i \leq n$ be the $k$-linear partial differentiation with respect to $x_i$. The ring $D$ of $k$-linear differential operators of $R$ is the subring of the ring of $k$-linear endomorphisms $\text{Hom}_k(R, R)$ of $R$ generated by $R$ and $d_1, \ldots, d_n$.

Standard theorems of calculus imply that $d_i$ commutes with $d_j$ and $x_j$ for $j \neq i$. The product formula says that $d_i(fg) = d_i(f)g + f d_i(g)$ for every $f, g \in R$. Fixing $f$ and viewing this as equality between two differential operators applied to $g$, we get the following equality of differential operators

$$d_i f = fd_i + \frac{\partial f}{\partial x_i}. \quad (1)$$

An element of $D$ is a finite sum with coefficients in $k$ of products of $x_1, \ldots, x_n, d_1, \ldots, d_n$. Using commutativity of $d_i$ with $x_j$ and $d_j$ for $j \neq i$ and \( \frac{\partial}{\partial x_i} \) with $f = x_i$ one can rewrite every product of $x_1, \ldots, x_n, d_1, \ldots, d_n$ as a sum of monomials $x_1^{a_1} \cdots x_n^{a_n} d_1^{b_1} \cdots d_n^{b_n}$ with coefficients in $k$.

Collecting similar terms one can write every element of $D$ as a sum $\Sigma_{i_1, \ldots, i_n} r_{i_1, \ldots, i_n} d_1^{i_1} \cdots d_n^{i_n}$ where $r_{i_1, \ldots, i_n} \in R$. Let $\{i_1, \ldots, i_n\}$ be the index with smallest total degree $i_1 + \cdots + i_n$ in such a sum.

Applying this sum to $x_1^{i_1} \cdots x_n^{i_n}$ one gets

$$\Sigma_{i_1, \ldots, i_n} r_{i_1, \ldots, i_n} d_1^{i_1} \cdots d_n^{i_n} (x_1^{i_1} \cdots x_n^{i_n}) = r_{i_1, \ldots, i_n} i_1! \cdots i_n!.$$ 

Hence $\Sigma_{i_1, \ldots, i_n} r_{i_1, \ldots, i_n} d_1^{i_1} \cdots d_n^{i_n} = 0$ if and only if every $r_{i_1, \ldots, i_n} = 0$. Thus the ring $D$ is a free left $R$-module on the monomials $d_1^{i_1} \cdots d_n^{i_n}$ with $\alpha_j \geq 0$.

There is a filtration $\Sigma = \Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \cdots$ on $D$ where each $\Sigma_i$ is the free $R$-submodule of $D(R, k)$ on the monomials $d_1^{i_1} \cdots d_n^{i_n}$ with $\sum \alpha_j \leq i$. Clearly

$$\text{gr}_\Sigma D = \Sigma_0 \oplus (\Sigma_1/\Sigma_0) \oplus (\Sigma_2/\Sigma_1) \oplus \cdots = R[\bar{d}_1, \ldots, \bar{d}_n]$$

is the ring of polynomials over $R$ in $\bar{d}_1, \ldots, \bar{d}_n$ where $\bar{d}_i$ is the image of $d_i$ in $\Sigma_1/\Sigma_0$ for $1 \leq i \leq n$. This ring $\text{gr}_\Sigma D$ is graded in the usual way, i.e. $\deg r = 0$ for $r \in R$ and $\deg \bar{d}_i = 1$ for all $i$.

Now let $M$ be a left $D(R, k)$-module. A good filtration $\Gamma$ for $M$ is an ascending chain $\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$ of $R$–submodules of $M$, such that $\bigcup_{i=0}^\infty \Gamma_i = M$, $\Sigma_i \Gamma_j \subseteq \Gamma_{i+j}$ and $\text{gr}_\Gamma(M) = \Gamma_0 \oplus (\Gamma_1/\Gamma_0) \oplus (\Gamma_2/\Gamma_1) \oplus \cdots$ is a finitely generated $\text{gr}_\Sigma D$–module. A good filtration exists provided $M$ is finitely generated. For example, if $M$ is generated by elements $m_1, \ldots, m_s \in M$, then the filtration with $\Gamma_i = \Sigma_i m_1 + \cdots + \Sigma_i m_s$ is a good filtration.

The following proposition is well-known [2 1.3.4] and [3 11.1.1].

**Proposition 2.1.** Let $M$ be a finitely generated left $D(R, k)$–module. Let $\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$ and $\Omega = \Omega_0 \subseteq \Omega_1 \subseteq \Omega_2 \subseteq \cdots$ be two good filtrations on $M$. Then:

(i) There exists a number $w$ such that $\Omega_j - w \subseteq \Gamma_j \subseteq \Omega_{j+w}$ for all $j \geq 0$.

(ii) $\sqrt{\text{Ann}_R[\bar{d}_1, \ldots, \bar{d}_n]}(\text{gr}_\Gamma(M)) = \sqrt{\text{Ann}_R[\bar{d}_1, \ldots, \bar{d}_n]}(\text{gr}_\Omega(M))$.

**Proof.** (i) Let $\sigma_i : \Sigma_i \to \Sigma_i/\Sigma_{i-1}$ and $\mu_i : \Gamma_i \to \Gamma_i/\Gamma_{i-1}$ be the natural surjections. Since $\text{gr}_\Gamma(M)$ is a finitely generated $R[\bar{d}_1, \ldots, \bar{d}_n]$–module, there exist integers $k_1, \ldots, k_s$ and elements $u_i \in \Gamma_{k_i}$ \ $\Gamma_{k_i-1}$ with $1 \leq i \leq s$ such that $\mu_{k_1}(u_1), \ldots, \mu_{k_s}(u_s)$ generate $\text{gr}_\Gamma(M)$ as an $R[\bar{d}_1, \ldots, \bar{d}_n]$–module.

Let $k_0 = \max\{k_1, \ldots, k_s\}$. For $v \geq k_0$ we set $R_v = \Sigma_v \Gamma_0 + \cdots + \Sigma_{v-k_0} \Gamma_{k_0}$. We show that $\Gamma_v \subseteq R_v$.
for \( v \geq k_0 \) by induction. Clearly \( \Gamma_{k_0} = R_{k_0} \). Assume that \( v > k_0 \) and \( \Gamma_{v-1} \subseteq R_{v-1} \). Let \( \gamma \in \Gamma_v \). There exist \( b_{v-k_1} \in \Sigma_{v-k_1}, \ldots, b_{v-k_s} \in \Sigma_{v-k_s} \) such that

\[
\mu_v(\gamma) = \sigma_{v-k_1}(b_{v-k_1})\mu_{k_1}(u_1) + \cdots + \sigma_{v-k_s}(b_{v-k_s})\mu_{k_s}(u_s)
\]

thus

\[
\gamma \in \Sigma_{v-k_1}\Gamma_{k_1} + \cdots + \Sigma_{v-k_s}\Gamma_{k_s} + \Gamma_{v-1} \subseteq R_v + \Gamma_{v-1} \subseteq R_v + R_{v-1} = R_v
\]

whence \( \Gamma_v \subseteq R_v \). This completes the induction.

Clearly \( \{\Omega_j \cap \Gamma_{k_0}\}_{j \geq 0} \) forms an increasing chain of the finitely generated \( R \)-module \( \Gamma_{k_0} \). Since \( \bigcup_{j \geq 0} \Omega_j = \Omega \) and \( R \) is Noetherian we have \( \Gamma_{k_0} \subseteq \Omega_{w'} \) for some number \( w' \). If \( 0 \leq j \leq k_0 \) and \( v \geq k_0 \) we have

\[
\Sigma_{v-j}\Gamma_j \subseteq \Sigma_{v-j}\Gamma_{k_0} \subseteq \Sigma_{v-j}\Omega_{w'} \subseteq \Sigma_v\Omega_{w'} \subseteq \Omega_{v+w'}
\]

This shows that \( \Gamma_v \subseteq R_v \subseteq \Omega_{v+w} \) for all \( v \geq k_0 \). If \( v < k_0 \) then \( \Gamma_v \subseteq \Gamma_{k_0} \subseteq \Omega_w \subseteq \Omega_{v+w} \). Thus \( \Gamma_v \subseteq \Omega_{v+w} \) for all \( v \geq 0 \).

Swapping \( \Gamma \) and \( \Omega \) and repeating the above proof gives a number \( w'' \) with \( \Gamma_v \subseteq \Gamma_{v+w''} \) for all \( v \geq 0 \). Set \( w = \max\{w', w''\} \). This proves (i).

(ii) Let \( f \in \sqrt{\text{Ann}_{R[\bar{d}_1, \ldots, \bar{d}_n]}(\text{gr}_\Gamma(M))} \) be homogeneous of degree \( s \). There exists an integer \( m \geq 1 \) with \( f^m \in \text{Ann}_{R[\bar{d}_1, \ldots, \bar{d}_n]}(\text{gr}_\Gamma(M)) \) and an element \( \beta \in \Sigma_s \) with \( f = \sigma_s(\beta) \). Thus \( \beta^m\Gamma_i \subseteq \Gamma_{ms+i-1} \) for every \( i \geq 0 \). By induction on \( q \) this implies that

\[
\beta^{mq}\Gamma_i \subseteq \Gamma_{i+msq-q} \tag{2}
\]

for every \( q \geq 1 \). By (i) there exists an integer \( w \) such that \( \Gamma_{i-w} \subseteq \Omega_i \subseteq \Gamma_{i+w} \) for all \( i \geq 0 \). Together with (2) for \( q = 2w + 1 \) this implies

\[
\beta^{m(2w+1)}\Omega_i \subseteq \beta^{m(2w+1)}\Gamma_{i+w} \subseteq \Gamma_{i+ms(2w+1)-w-1} \subseteq \Omega_{i+ms(2w+1)-1}.
\]

Thus \( \beta^{m(2w+1)}\Omega_i \subseteq \Omega_{i+ms(2w+1)-1} \) for all \( i \geq 0 \). Therefore \( f^{m(2w+1)} \in \sqrt{\text{Ann}_{R[\bar{d}_1, \ldots, \bar{d}_n]}(\text{gr}_\Omega(M))} \), i.e. \( f \in \sqrt{\text{Ann}_{R[\bar{d}_1, \ldots, \bar{d}_n]}(\text{gr}_\Omega(M))} \). Hence \( \sqrt{\text{Ann}_{R[\bar{d}_1, \ldots, \bar{d}_n]}(\text{gr}_\Gamma(M))} \subseteq \sqrt{\text{Ann}_{R[\bar{d}_1, \ldots, \bar{d}_n]}(\text{gr}_\Omega(M))} \). The opposite inclusion follows similarly.

Recall that if \( A \) is a graded commutative Noetherian ring and \( N \) is a finitely generated graded graded \( A \)-module, then \( \text{ann}(N) \), the annihilator of \( N \), is a homogeneous ideal of \( A \) and the dimension of \( N \) is defined to be the Krull dimension of the graded ring \( A/\text{ann}(M) \), i.e. the maximum length of a chain of homogeneous prime ideals \( p_1 \subset p_2 \subset \cdots \subset p_d \) in the ring \( A/\text{ann}(M) \).

The above proposition shows, in particular, that the dimension of \( \text{gr}_\Gamma(M) \) as a graded \( \text{gr}_\Sigma D \)-module, is independent of the good filtration \( \Gamma \).

**Definition 2.2.** The dimension of a finitely generated \( D \)-module \( M \), denoted \( d(M) \), is the dimension of \( \text{gr}_\Gamma(M) \) as a graded \( \text{gr}_\Sigma D \)-module, where \( \Gamma \) is any good filtration on \( M \).

**Lemma 2.3.** Let \( M \) be a finitely generated \( D \)-module and let \( M' \subset M \) be a submodule. Then \( d(M') \leq d(M) \).
Proof. A good filtration $\Gamma$ on $M$ induces the filtration $\Gamma'_i = \Gamma_i \cap M'$ on $M'$. The $R$-module injections $\Gamma'_i \subset \Gamma_i$ and $\Gamma'_{i-1} \subset \Gamma_{i-1}$ induce an $R$-module injection $\Gamma'_i/\Gamma'_{i-1} \subset \Gamma_i/\Gamma_{i-1}$. Hence $\mathfrak{gr}_{\Gamma_i}(M')$ is a graded $D$-submodule of $\mathfrak{gr}_{\Gamma_i}(M)$ and so $\text{ann}(\mathfrak{gr}_{\Gamma_i}(M')) \supset \text{ann}(\mathfrak{gr}_{\Gamma_i}(M))$. □

**Lemma 2.4.** [11.2] Let $A = A_0 \oplus A_1 \oplus A_2 \oplus \ldots$ be a commutative graded Noetherian ring with $A_0 = k$ a field, generated by $A_1$ as an $A_0$-algebra. Let $N = N_0 \oplus N_1 \oplus N_2 \oplus \ldots$ be a finitely generated graded $A$-module. For sufficiently large $t$ the function $p(t) = \dim_k(N_0 \oplus \cdots \oplus N_t)$ is a polynomial in $t$ of degree $d = \dim N$.

**Lemma 2.5.** (Noether normalization [5] VII, Theorem 31) Let $p$ be a prime ideal of $R$ of height $h$. Variables $x_1, \ldots, x_n$ can be chosen so that $p \cap k[[x_{h+1}, \cdots, x_n]] = 0$ and $R/p$ is finite over $k[[x_{h+1}, \cdots, x_n]]$.

### 3 Proof of Theorem 1.1

Let $p \subset R$ be an associated prime of $M$, let $z \in M$ be an element of $M$ with $\text{Ann}_R(z) = p$, let $h = \text{height}p$ and assume that the variables $x_1, \ldots, x_n$ satisfy the conditions in Lemma 2.5. Let $N = Dz \subset M$ be the $D$-submodule of $M$ generated by $z$. Lemma 2.3 implies that it is enough to show that $\dim N \geq n$.

Let $\mathfrak{gr}(N) = \Sigma_0 z \oplus (\Sigma_1 z/\Sigma_0 z) \oplus (\Sigma_2 z/\Sigma_1 z) \oplus \cdots$ be the associated graded $\mathfrak{gr}(D)$-module. Clearly $\mathfrak{gr}(N)$ is a cyclic $\mathfrak{gr}(D)$-module, generated by $z \in \Sigma_0 z$. Hence $\mathfrak{gr}(N) \cong \mathfrak{gr}(D)/J$ where $J \subset \mathfrak{gr}(D)$ is the annihilator of $z \in \mathfrak{gr}(N)$. Since $\mathfrak{gr}(N)$ is a graded module, $J$ is a homogeneous ideal of $\mathfrak{gr}(D)$.

Set $\tilde{D}$ to be the ring $\mathfrak{gr}(D)/J$. Since $\mathfrak{gr}(N) \cong \mathfrak{gr}(D)/J$, we need to show that the dimension of $\mathfrak{gr}(D)/J$ is at least $n$. Let $\tilde{D}_i$ be the degree $i$ piece of $\tilde{D}$, i.e. $\tilde{D} = \tilde{D}_0 \oplus \tilde{D}_1 \oplus \ldots$. Let $\tilde{D}_+ \subset \tilde{D}$ be the ideal generated by the elements of positive degrees. Clearly, $\tilde{D}_+$ is a homogeneous prime ideal of $\tilde{D}$, hence $\dim \tilde{D} \geq \dim \tilde{D}/\tilde{D}_+ + \text{height} \tilde{D}_+$. Since $\text{Ann}_R(z) = p$, it follows that $\tilde{D}_0 \cong R/p$. and so $\dim \tilde{D}/\tilde{D}_+ = \dim R/p = n - h$. It follows that it is enough to prove that

$$\text{height} \tilde{D}_+ \geq h.$$

Let $S \subset \tilde{D}_0$ be the non-zero elements of $\tilde{D}_0$. Since $\tilde{D}_0 \cong R/p$ and $p$ is a prime ideal, $S$ is a multiplicatively closed set. Let $S^{-1}\tilde{D} = S^{-1}\tilde{D}_0 \oplus S^{-1}\tilde{D}_1 \oplus \ldots$ be the ring obtained from $\tilde{D}$ by inverting every element of $S$. Since $\text{height} \tilde{D}_+ \geq \text{height} S^{-1}\tilde{D}_+$ it is enough to prove that

$$\text{height} S^{-1}\tilde{D}_+ \geq h.$$

Clearly, $S^{-1}\tilde{D}_0 \cong K$, the fraction field of $R/p$, and $S^{-1}\tilde{D}$ is a finitely generated graded $K$-algebra. By Lemma 2.4 the function $p(t) = \dim_K(S^{-1}\tilde{D}_0 \oplus S^{-1}\tilde{D}_1 \oplus \cdots \oplus S^{-1}\tilde{D}_t)$ is, for sufficiently big $t$, a polynomial in $t$ which we denote $\bar{p}(t)$, and $\text{deg} \bar{p}(t) = \dim S^{-1}\tilde{D}$. It is enough to prove that $\text{deg} \bar{p}(t) = h$.

Let $T = k[[x_{h+1}, \cdots, x_n]]$ and let $K$ be the field of fractions of $T$. By Lemma 2.5 $K$ is a finite field extension of $K$. Let $d$ be the degree of this extension. Since $\dim_K L = d(\dim_K L)$ for every $K$-vector space $L$, we conclude that

$$q(t) = \dim_K(S^{-1}\tilde{D}_0 \oplus S^{-1}\tilde{D}_1 \oplus \cdots \oplus S^{-1}\tilde{D}_t) = dp(t).$$
Hence \(q(t)\), for sufficiently high \(t\), is a polynomial in \(t\) of the same degree as \(\hat{p}(t)\). We denote this polynomial \(\hat{q}(t)\). It is enough to show that
\[
\deg \hat{q}(t) \geq h.
\]

Since \(K\) is a finite field extension of \(\mathbb{K}\), \(S^{-1}L = \mathbb{K} \otimes_T L\) for every \((\mathbb{D}_0 = R/P)\)-module \(L\). In particular, \(S^{-1}\mathbb{D}_0 \otimes S^{-1}\mathbb{D}_1 \oplus \cdots \oplus S^{-1}\mathbb{D}_t = \mathbb{K} \otimes_T \mathbb{D}_0 \oplus \mathbb{K} \otimes_T \mathbb{D}_1 \oplus \cdots \oplus \mathbb{K} \otimes_T \mathbb{D}_t\). But \(\mathbb{D}_t \cong \Sigma_{t+1} z/\Sigma_{t} z\), hence
\[
\hat{q}(t) = \dim_{\mathbb{K}}(\mathbb{K} \otimes_T \mathbb{D}_0 \oplus \mathbb{K} \otimes_T \mathbb{D}_1 \oplus \cdots \oplus \mathbb{K} \otimes_T \mathbb{D}_t) = \dim_{\mathbb{K}}(\mathbb{K} \otimes_T (\mathbb{D}_0 \oplus \mathbb{D}_1 \oplus \cdots \oplus \mathbb{D}_t)) = \dim_{\mathbb{K}}(\mathbb{K} \otimes_T (\Sigma_{t} z)).
\]

The last equality and the next lemma use the crucial fact that \(\Sigma_{t} z\) and \(N\) are \(T\)-modules, hence \(\mathbb{K} \otimes_T (\Sigma_{t} z)\) and \(\mathbb{K} \otimes_T N\) exist (in contrast, \(\Sigma_{t} z\) and \(N\) are not \((\mathbb{D}_0 = R/P)\)-modules).

**Lemma 3.1.** The set \(\{d_1^i \cdots d_h^i z\} \subset \mathbb{K} \otimes_T N\), as \(t_1, \cdots, t_h\) range over all non-negative integers, is linearly independent over \(\mathbb{K}\) (by a slight abuse of notation we identify the elements \(d_1^i \cdots d_h^i z\) of \(N\) with their images in \(\mathbb{K} \otimes_T N\) under the natural localization map \(N \to \mathbb{K} \otimes_T N\) that sends every \(n \in N\) to \(1 \otimes n\)).

**Proof.** Since \(K\) is a finite extension of \(K\), let \(f_i\), for every \(i\) with \(1 \leq i \leq h\), be the monic minimal polynomial of \(\bar{x}_i\) over \(K\), where \(\bar{x}_i\) is the image of \(x_i\) in \(K\). Clearly, \(f_i(x_i) \in \mathbb{p}\) and therefore \(f_i(x_i) z = 0\) while \(f_i'(x_i)\), where \(f_i'\) is the derivative of \(f_i\), is non-zero in \(K\) and therefore \(f_i'(x_i) z \neq 0\).

We claim that if \(s > t\) then
\[
f_i(x_i)^s d_i^t z = 0 \quad (3)
\]
If \(t = 0\) (hence \(s \geq 1\)), there is nothing to prove since \(f_i(x_i) z = 0\). By (1)
\[
f_i(x_i)^s d_i = d_i f_i(x_i)^s - s f_i'(x_i) f_i(x_i)^{s-1},
\]
therefore for \(t > 0\) we have that
\[
f_i(x_i)^s d_i^t z = d_i f_i(x_i)^s d_i^{t-1} z - s f_i'(x_i) f_i(x_i)^{s-1} d_i^{t-1} z = 0, \quad (4)
\]
where both summands in the middle vanish by induction on \(t\). This proves the claim.

Equalities (3) and (4) imply by induction on \(t\) that
\[
f_i(x_i)^t d_i^t z = (-1)^t t! f_i'(x_i)^t z \neq 0. \quad (5)
\]

Now let
\[
\gamma = \Sigma_{t_1, \ldots, t_h} c_{t_1, \ldots, t_h} d_1^{t_1} \cdots d_h^{t_h} z,
\]
where \(c_{t_1, \ldots, t_h} \in \mathbb{K}\) be a linear combination of finitely many elements of the set \(\{d_1^i \cdots d_h^i z\}\). Let \(\{\tau_1, \ldots, \tau_h\}\) be an index of highest total degree \(\tau_1 + \cdots + \tau_h\). Every other \(c_{t_1, \ldots, t_h} d_1^{t_1} \cdots d_h^{t_h} z\) in this linear combination has some \(t_j\) with \(t_j < \tau_j\), hence \(f_j(x_j)^\tau c_{t_1, \ldots, t_h} d_1^{t_1} \cdots d_h^{t_h} z = 0\) and
\[
f_1(x_1)^{\tau_1} \cdots f_h(x_h)^{\tau_h} c_{t_1, \ldots, t_h} f_1(x_1)^{\tau_1} \cdots f_h(x_h)^{\tau_h} d_1^{t_1} \cdots d_h^{t_h} z
\]
\[
= (-1)^{\tau_1 + \cdots + \tau_h} \cdot t_1! \cdots t_h! c_{t_1, \ldots, t_h} f_1'(x_1)^{\tau_1} \cdots f_h'(x_h)^{\tau_h} z \neq 0,
\]
where we use (3), (5) and the fact that \(f_i(x_i)^{\tau_i}\) and \(f_j'(x_i)^{\tau_j}\) commute with every \(d_j^{t_j}\) with \(j \neq i\). Therefore \(\gamma \neq 0\). \(\square\)
The number of the elements \( \{d_1^{d_1} \cdots d_h^{d_h} z\} \), as \( d_1 + \cdots + d_h \leq t \), is the number of monomials in \( h \) variables of total degree at most \( t \), which equals \( \left( \begin{array}{c} t \+ h \\ h \end{array} \right) \). Since these elements are in \( \Sigma_t z \) and are linearly independent, \( \bar{q}(t) \geq \left( \begin{array}{c} t \+ h \\ h \end{array} \right) \) for sufficiently high \( t \). But \( \bar{q}(t) \) is a polynomial in \( t \) and \( \left( \begin{array}{c} t \+ h \\ h \end{array} \right) \) is a polynomial in \( t \) of degree \( h \). Hence the degree of \( \bar{q}(t) \) is at least \( h \). This completes the proof of Theorem 1.1.

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