Abstract The pressure correction scheme is combined with interior penalty discontinuous Galerkin method to solve the time-dependent Navier-Stokes equations. Optimal error estimates are derived for the velocity in the $L^2$ norm in time and in space. Error bounds for the discrete time derivative of the velocity and for the pressure are also established.

Keywords Discontinuous Galerkin · Pressure correction · Time-dependent Navier–Stokes · Error estimates

Mathematics Subject Classification (2020) 65M12 · 65M15 · 65M60

1 Introduction

This paper contains the derivation of a priori error estimates for the velocity and the pressure of a splitting scheme for solving the incompressible Navier-Stokes equations. This work follows up [17] where the scheme is introduced and stability is proved. In [17], a priori error estimates for the velocity in $L^2$ in time and in the broken $H^1$ space are obtained; these error bounds are optimal in space but suboptimal in time. The main goal of the current paper is to obtain optimal error estimates for
the velocity in the $L^2$ norm in time and space. After deriving error bounds for the discrete derivative of the velocity, we also show error estimates for the pressure that are optimal in space and suboptimal in time.

The type of pressure-correction splitting scheme used in this work, belongs to the class of operator splitting schemes that decouple the nonlinearity in the momentum equation from the incompressibility constraint. An overview of such splitting schemes can be found in the work by Guermond, Minev, and Shen [11] (see also the comprehensive text [10]). Splitting schemes are known to be computationally efficient for large scale problems. Error analysis of pressure correction schemes combined with continuous finite element method has been carried out in several papers [13,12,18].

While there are several computational works on discontinuous Galerkin schemes combined with the pressure correction splitting technique [14,19,16], the only theoretical convergence paper in the literature is [17]. In this second paper, we improve upon the error estimate obtained for the velocity by employing a duality argument and assuming convexity of the domain. The discrete velocities belong to the space of discontinuous polynomials of degree $k$ and the discrete pressure and potential belong to the space of discontinuous polynomials of degree $k - 1$. The theoretical rates of convergence for the velocity are shown to be order one in time and order $k + 1$ in space. For the pressure error bounds, the rate is $k$ in space and $1/2$ in time, which is consistent with the results obtained for the continuous finite element method [18].

This paper is organized as follows. Section 2 introduces the model problem and the fully discrete scheme. The improved error estimate for the velocity is obtained in Section 3. We show error estimates for the discrete time derivative of the velocity in Section 4 and for the pressure in Section 5.

## 2 Model problem and fully discrete scheme

Let $\Omega$ be an open bounded and convex polyhedral domain in $\mathbb{R}^d$ where $d=2$ or $3$. Consider the incompressible Navier-Stokes equations in $\Omega$ over the time interval $[0,T]$.

\[
\partial_t u - \mu \Delta u + u \cdot \nabla u + \nabla p = f, \quad \text{in } \Omega \times (0,T],
\]

\[
\nabla \cdot u = 0, \quad \text{in } \Omega \times (0,T],
\]

\[
u = u^0, \quad \text{in } \Omega \times \{0\},
\]

\[
\nu = 0, \quad \text{on } \partial \Omega \times (0,T],
\]

\[
\int_{\Omega} p(t) = 0, \quad \forall t \in (0,T].
\]

In the above system, $\nu$ is the fluid velocity, $p$ is the pressure, $f$ is the external force, and $\mu > 0$ is the viscosity. For a given non-negative integer $m$ and real number $r \geq 1$, the Sobolev space $W^{m,r}(\Omega)$ on a domain $\Omega \subset \mathbb{R}^d$ is equipped with the usual Sobolev norms and semi-norms $\| \cdot \|_{W^{m,r}(\Omega)}$ and $| \cdot |_{W^{m,r}(\Omega)}$ respectively. If $r = 2$, we denote
Let $\delta_h = \{ E_h \}$ denote a family of regular and uniform partitions of the domain $\Omega$.

We define the following broken Sobolev spaces.

$$X = \{ v \in L^2(\Omega)^d : \forall E \in \delta_h, \ v|_E \in W^{2,4/3}(E)^d \},$$  \hspace{0.5cm} (6)  

$$M = \{ q \in L^2(\Omega) : \forall E \in \delta_h, \ q|_E \in W^{1,4/3}(E) \}. $$  \hspace{0.5cm} (7)

Let $h = \max_{E \in \delta_h} h_E$ where $h_E = \text{diam}(E)$. Denote by $I_h$ the set of all interior faces of the subdivision $\delta_h$. For an interior edge $e \in I_h$, we associate a normal $n_e$ and we denote by $E^1_e$ and $E^2_e$ the two elements that share $e$, such that $n_e$ points from $E^1_e$ to $E^2_e$.

Define the average and jump in this case are extended as such,

$$\{ \theta \} = \frac{1}{2} (\theta|_{E^1_e} + \theta|_{E^2_e}), \quad [\theta] = \theta|_{E^1_e} - \theta|_{E^2_e}, \quad \forall e \in \partial E^1_e \cap \partial E^2_e.$$  \hspace{0.5cm} (8)

For a boundary face, $e \in \partial \Omega$, the vector $n_e$ is chosen as the unit outward vector to $\partial \Omega$. The definition of the average and jump in this case are extended as such,

$$\{ \theta \} = \theta|_{E_e}, \quad [\theta] = \theta|_{E_e} \cdot n_e \cdot \nabla \cdot \theta.$$  \hspace{0.5cm} (9)

Similar definitions are used for scalar valued functions, $q \in M$. For the convection term, we use the same discretization form, $a_{\psi}$, as in [8]. Denote by $n_E$ the outward normal to $E$, and denote by $\psi^\text{int}$ and $\psi^\text{ext}$ the trace of a function $v$ on the boundary of $E$ coming from the interior (resp. exterior) of $E$. By convention, $\psi^\text{ext}|_{n_e} = \mathbf{0}$ if $e$ is a boundary face ($e \subset \partial \Omega$). We also introduce the notation for the inflow boundary of $E$ with respect to a function $z$:

$$\partial E_E = \{ x \in \partial E : \{ z(x) \} \cdot n_E < 0 \}, \quad z \in X.$$  \hspace{0.5cm} (10)

With this notation, we have for $z, w, v, \theta \in X$,

$$a_{\psi}(z,w,v,\theta) = \sum_{E \in \delta_h} \left( \int_E (w \cdot \nabla v) \cdot \theta + \frac{1}{2} \int_E (\nabla \cdot w) v \cdot \theta \right)$$

$$- \frac{1}{2} \sum_{e \in I_h \cup \partial \Omega} \int_e [w] \cdot n_e \{ \theta \} + \sum_{E \in \delta_h} \int_{\partial E_E} [w] \cdot n_E \{ | v^\text{int} - v^\text{ext} | \cdot \theta \}.$$  \hspace{0.5cm} (11)

The form $a_{\psi}$ admits an “integration by parts” formula [8,9]. To this end, define for $z, w, v, \theta \in X$, the form $\tilde{a}_{\psi}$ as:

$$\tilde{a}_{\psi}(z,w,\theta,v) = \sum_{E \in \delta_h} \left( \int_E (w \cdot \nabla \theta) \cdot v + \frac{1}{2} \int_E (\nabla \cdot w) \theta \cdot v \right)$$

$$- \frac{1}{2} \sum_{e \in I_h \cup \partial \Omega} \int_e [w] \cdot n_e \{ v \} \cdot \theta + \sum_{E \in \delta_h} \int_{\partial E_E} [w] \cdot n_E \{ | \theta^\text{int} - \theta^\text{ext} | \cdot v^\text{ext} \}$$

$$- \frac{1}{2} \sum_{e \in \partial \Omega} \int_e ([w] \cdot n_e - w \cdot n_e) \theta \cdot v.$$  \hspace{0.5cm} (12)
The following holds (see (1.16) and (1.17) in [9] and Lemma 6.1 in [8]):

$$a_{\theta}(w, v, \theta) = -\bar{a}_{\theta}(w, v, \theta), \quad \forall w, v, \theta \in X.$$  \hspace{1cm} (13)

We recall the positivity property satisfied by $a_{\theta}$ (see (1.18) in [8]):

$$a_{\theta}(w, v, v, v) \geq 0, \quad \forall w, v \in X.$$  \hspace{1cm} (14)

In the analysis below, we will use properties of the form $a_{\theta}$ and it helps to define the following forms. For $z, w, v, \theta \in X$, we write:

$$\mathcal{C}(w, v, \theta) = \sum_{E \in \partial_{h}} \left( \int_{E} (w \cdot \nabla v) \cdot \theta + \frac{1}{2} \int_{E} (\nabla \cdot w) \theta \right)$$

$$- \frac{1}{2} \sum_{e \in \Gamma \cup \partial_{h}} \int_{e} |w| n_{e} [v \cdot \theta],$$

$$\mathcal{W}(z; w, v, \theta) = \sum_{E \in \partial_{h}} \int_{\partial_{E}} |w| n_{E} (v^{\text{int}} - v^{\text{ext}}) \cdot \theta^{\text{int}}.$$  \hspace{1cm} (15)

Therefore, we have

$$a_{\theta}(w, v, \theta) = \mathcal{C}(w, v, \theta) - \mathcal{W}(w; w, v, \theta).$$   \hspace{1cm} (16)

With this notation, for any $u, w, v_{1}, v_{2}, \theta \in X$, we have:

$$a_{\theta}(u, u, v_{1}, \theta) - a_{\theta}(w; w, v_{2}, \theta) = a_{\theta}(w, w, v_{1} - v_{2}, \theta) + \mathcal{C}(u - w, v_{1}, \theta)$$

$$- \mathcal{W}(w, u - w, v_{1}, \theta) - (\mathcal{W}(u, u, v_{1}, \theta) - \mathcal{W}(w, w, v_{1}, \theta)).$$  \hspace{1cm} (17)

We use the symmetric interior penalty dG for the elliptic operator $-\Delta v$ [20]. For $v, \theta \in X$,

$$a_{\theta}(v, \theta) = \sum_{E \in \partial_{h}} \int_{E} \nabla v \cdot \nabla \theta - \sum_{e \in \Gamma \cup \partial_{h}} \int_{e} [\nabla_{b} v] n_{e} \cdot [\theta]$$

$$- \sum_{e \in \Gamma \cup \partial_{h}} \int_{e} [\nabla_{b} \theta] n_{e} \cdot [v] + \sum_{e \in \Gamma \cup \partial_{h}} \frac{\sigma}{h_{e}} \int_{e} |v| \cdot |\theta|. $$  \hspace{1cm} (18)

In the above form, $h_{e} = |e|^{1/(d-1)}, \sigma > 0$ is a user specified penalty parameter, and $\nabla_{b}$ denotes the broken gradient operator. The discretization for the term $-\nabla p$ is given as follows. For $\theta \in X$ and $q \in M$, define

$$b(\theta, q) = \sum_{E \in \partial_{h}} \int_{E} (\nabla \cdot \theta) q - \sum_{e \in \Gamma \cup \partial_{h}} \int_{e} [q] \cdot n_{e}$$

$$- \sum_{E \in \partial_{h}} \int_{E} \theta \cdot \nabla q + \sum_{e \in \Gamma \cup \partial_{h}} \int_{e} [\theta] \cdot n_{e} [q].$$  \hspace{1cm} (19)
To approximate $u$ and $p$, we introduce discrete function spaces $X_h \subset X$ and $M_{h0} \subset M_h \subset M$. For any integer $k \geq 1$:

\[ X_h = \{ v_h \in (L^2(\Omega))^d : \forall E \in \mathcal{E}_h, \quad v_h|_E \in (\mathcal{P}_k(E))^d \}, \]
\[ M_h = \{ q_h \in L^2(\Omega) : \forall E \in \mathcal{E}_h, \quad q_h|_E \in \mathcal{P}_{k-1}(E) \}, \]
\[ M_{h0} = \{ q_h \in M_h : \int_{\Omega} q_h = 0 \}. \]

In the above, for $n \in \mathbb{N}$, $\mathcal{P}_n(E)$ denotes the space of polynomials of degree at most $n$.

To discretize the elliptic operator $-\Delta \phi$, we define for $\phi_h, q_h \in M_h$,

\[ a_{\text{clip}}(\phi_h, q_h) = \sum_{E \in \mathcal{E}_h} \int_E \nabla \phi_h \cdot \nabla q_h - \sum_{e \in \mathcal{I}_h} \int_e \{ \nabla_h \phi_h \} \cdot n_e[q_h] \]
\[ - \sum_{e \in \mathcal{I}_h} \int_e \{ \nabla_h q_h \} \cdot n_e[\phi_h] + \sum_{e \in \mathcal{I}_h} \int_{\Gamma_e} \sigma[\phi_h][q_h]. \quad (24) \]

Here, $\sigma > 0$ is a penalty parameter that will be specified later.

We now present the fully discrete scheme by partitioning the time interval $(0, T)$ into $N_T$ subintervals with equal size $\tau$. Throughout the paper, we use the notation $g^n = g(t^n)$ and $q^n = q(t^n)$ for given functions $g$ and $q$. We start by setting $p_0^h = 0$ and letting $u_0^h$ be the $L^2$ projection of $u^0$ onto $X_h$. For $n = 1, \ldots, N_T$, given $(u_{n-1}^h, p_{n-1}^h) \in X_h \times M_h$, compute $\phi_n^h \in X_h$ such that for all $\phi_h \in X_h$,

\[ \langle \phi_n^h, \theta_h \rangle + \tau a_{\phi}(u_{n-1}^h; u_{n-1}^h, \phi_n^h, \theta_h) + \tau \mu a_{\phi}(\phi_n^h, \theta_h) = \langle u_{n-1}^h, \theta_h \rangle \]
\[ + \tau b(\theta_h, p_{n-1}^h) + \tau (f^n, \theta_h). \quad (25) \]

Next, compute $\phi_n^h \in M_{h0}$ such that for all $q_h \in M_{h0}$,

\[ a_{\text{clip}}(\phi_n^h, q_h) = -\frac{1}{\tau} b(\phi_n^h, q_h). \quad (26) \]

Finally, compute $p_n^h \in M_h$ and $u_n^h \in X_h$ such that for all $q_h \in M_h$ and for all $\theta_h \in X_h$,

\[ \langle p_n^h, q_h \rangle = \langle p_{n-1}^h, q_h \rangle + \langle \phi_n^h, q_h \rangle - \delta a_{\phi}(\phi_n^h, \theta_h), \quad (27) \]
\[ \langle u_n^h, \theta_h \rangle = \langle \phi_n^h, \theta_h \rangle + \tau b(\theta_h, \phi_n^h). \quad (28) \]

Here $\delta$ is a positive constant to be determined in the subsequent sections. The unique solvability of above algorithm is proved in [17].

For $\theta \in X$, define the energy norm as follows:

\[ \| \theta \|^2_{\text{DG}} = \sum_{E \in \mathcal{E}_h} \| \nabla \theta \|^2_{L^2(E)} + \sum_{e \in \mathcal{I}_h \cup \partial \Omega} \sigma_{\text{h.e.}} \| \theta \|^2_{L^2(h)} \cdot (29) \]

For $q \in M$, the energy semi-norm is defined as such:

\[ |q|^2_{\text{DG}} = \sum_{E \in \mathcal{E}_h} \| \nabla q \|^2_{L^2(E)} + \sum_{e \in \mathcal{I}_h} \sigma_{\text{h.e.}} \| q \|^2_{L^2(h)}. \quad (30) \]
Clearly, $|\cdot|_{DG}$ is a norm for the space $M_{DG}$. We recall the following coercivity properties which hold if $\sigma$ and $\tilde{\sigma}$ are large enough [20]. For all $\theta_h \in X_h$ for all $q_h \in M_h$:

$$a_{\sigma}(\theta_h, \theta_h) \geq \frac{1}{2}||\theta_h||^2_{DG}, \quad a_{\text{ellip}}(q_h, q_h) \geq \frac{1}{2}||q_h||^2_{DG}. \quad (31)$$

In addition, we have the following continuity bound:

$$|a_{\sigma}(\theta_h, v_h)| \leq C||\theta_h||_{DG}||v_h||_{DG}, \quad \forall \theta_h, v_h \in X_h. \quad (32)$$

We also recall the following lift operators [17], $R_h : X_h \rightarrow M_h$ and $G_h : M_h \rightarrow X_h$:

$$(R_h([\theta_h]), q_h) = \sum_{e \in I_h \cup \partial \Omega} \int_e \{q_h\} \cdot n_e, \quad \theta_h \in X_h, \quad q_h \in M_h, \quad (33)$$

$$(G_h([q_h]), \theta_h) = \sum_{e \in I_h} \int_e \{\theta_h\} \cdot n_e[q_h], \quad \theta_h \in X_h, \quad q_h \in M_h. \quad (34)$$

There exist constants $M_{k-1}, M_k > 0$ independent of $h$ but depending on the polynomial degree $k$, such that for $q_h \in M_h$ and $\theta_h \in X_h$, the following bounds hold.

$$||R_h([\theta_h])||^2 \leq M_{k-1} \sum_{e \in I_h \cup \partial \Omega} h_e^{-1} ||\theta_h||^2_{L^2(e)}, \quad (35)$$

$$||G_h([q_h])||^2 \leq M_k \sum_{e \in I_h} h_e^{-1} ||q_h||^2_{L^2(e)}. \quad (36)$$

Considering the definitions of the lift operators (33)-(34) and (19)-(20), we have

$$b(\theta_h, q_h) = (\nabla_h \cdot \theta_h, q_h) - (R_h([\theta_h]), q_h), \quad \forall \theta_h \in X_h, \forall q_h \in M_h, \quad (37)$$

$$b(\theta_h, q_h) = -(\nabla_h q_h, \theta_h) + (G_h([q_h]), \theta_h), \quad \forall \theta_h \in X_h, \forall q_h \in M_h. \quad (38)$$

Consequently, using Cauchy-Schwarz’s inequality and (36), it is easy to show that

$$b(\theta_h, q_h) \leq C||\theta_h|| ||q_h||_{DG}, \quad \forall \theta_h \in X_h, \forall q_h \in M_h. \quad (39)$$

### 3 Improved error estimate for the velocity

Throughout the paper, we denote by $C$ a generic constant independent of $\mu, h$, and $\tau$ and by $C_\mu$ a generic constant independent of $h$ and $\tau$ but depending on $\epsilon^{1/\mu}$. These constants may take different values when used in different places. For a function $v \in (W^{1,3}(\Omega) \cap L^\infty(\Omega))^d$, define the norm $||v|| = ||v||_{L^\infty(\Omega)} + ||v||_{W^{1,3}(\Omega)}$. We begin by recalling the interpolation operators, error equations, and the error and stability estimates derived in [17]. We will make use of the operator $\Pi_h : H^1_0(\Omega)^d \rightarrow X_h$ that satisfies the following [20, 2].

$$b(\Pi_h u(t), q_h) = b(u(t), q_h) = 0, \quad \forall q_h \in M_h, \forall 0 \leq t \leq T. \quad (40)$$
This operator satisfies the following stability and approximation properties [2]. For $0 \leq t \leq T$, if $u(t) \in (W^{r,s}(\Omega) \cap H^1_0(\Omega))^d$, we have the global estimates:

$$
\|\Pi_h u(t)\|_{L^r(\Omega)} \leq C\|u(t)\|, \quad r = 3, s = 1, \ u(t) \in L^\infty(\Omega)^d, \quad (41)
$$

$$
\|\Pi_h u(t) - u(t)\|_{L^r(\Omega)} \leq C h^r|u(t)|_{W^{r,s}(\Omega)}, \quad 1 \leq r \leq \infty, \ 1 \leq s \leq k + 1, \quad (42)
$$

$$
\|\Pi_h u(t) - u(t)\|_{DG} \leq Ch^{r-1}|u(t)|_{H^r(\Omega)}, \quad 1 \leq r \leq \infty, \ 1 \leq s \leq k + 1. \quad (43)
$$

For $E \in \mathcal{E}_h$ and $u(t) \in X \cap (W^{r,s}(E) \cap H^1_0(\Omega))^d$

$$
\|\nabla(\Pi_h u(t) - u(t))\|_{L^r(E)} \leq C h^{r-1}|u(t)|_{W^{r,s}(\Delta E)} \quad (44)
$$

where $\Delta E$ is a macro element that contains $E$. As a corollary, we have

$$
\|u - \Pi_h u\| \leq C\|u\|. \quad (45)
$$

Let $\pi_h : L^2(\Omega) \rightarrow M_h$ denote the $L^2$ projection. For $0 \leq t \leq T$ and $p(t) \in H^1(\Omega)$, the following estimate holds.

$$
\|\pi_h p(t) - p(t)\|_{L^2(E)} + h_E \|\nabla h(\pi_h p(t) - p(t))\|_{L^2(E)} \leq C h^{\min(k,i)}|p(t)|_{H^1(p(E))}. \quad (46)
$$

We also recall that the $L^2$ projection is stable in the dG norm [6]. For $p(t) \in H^1(\Omega)$,

$$
\|\pi_h p(t)\|_{DG} \leq C|p(t)|_{H^1(\Omega)}. \quad (47)
$$

Further, we recall the following Poincare inequality [8, 15].

$$
\|\theta\|_{L^2(\Omega)} \leq C_{DG}\|\theta\|_{DG}, \quad \forall \theta \in X, \quad (48)
$$

where $2 \leq q < \infty$ in 2D ($d = 2$) and $2 \leq q \leq 6$ in 3D ($d = 3$). We proceed by stating the error equations. To this end, define the following discretization errors, $\hat{v}_h^n, \ e_h^n \in X_h$;

$$
\hat{v}_h^n = v_h^n - \Pi_h u^n, \quad e_h^n = u_h^n - \Pi_h u^n, \quad \forall n \geq 0. \quad (49)
$$

We define $v_h^n = \Pi_h u^n$, therefore $\hat{v}_h^n = 0$. To simplify the writeup, we also define

$$
R_n^\theta(\theta_h) = a_{\theta'}(u_{h}^{n-1}, u_{h}^{n-1}, \Pi_h u^n, \theta_h) - a_{\theta'}(u^n, u^n, \theta_h), \quad (50)
$$

$$
R_i(\theta_h) = g(\Pi_h u^n, \theta_h) + (\Pi_h u^n - \Pi_h u^0, \theta_h). \quad (51)
$$

The error equations are, for $n \geq 1$ [17].

$$
(\hat{v}_h^n, \theta_h) + h_t a_{\theta'}(u_{h}^{n-1}, u_{h}^{n-1}, \hat{v}_h^n, \theta_h) + R_n^\theta(\theta_h) + h_t \mu a_{\theta'}(e_h^n, \theta_h) = (e_h^{n-1}, \theta_h) - \mu a_{\theta'}(\Pi_h u^n - u^n, \theta_h) + \tau b(\theta_h, p_h^{n-1} - p^n) + R_i(\theta_h), \quad \forall \theta_h \in X_h, \quad (52)
$$

$$
(\hat{v}_h^n, \theta_h) = (\hat{v}_h^n, \theta_h) + \tau b(\theta_h, \phi_h^n), \quad \forall \theta_h \in X_h, \quad (53)
$$

$$
a_{\text{elip}}(\phi_h^n, q_h) = \frac{1}{\tau} b(\phi_h^n, q_h), \quad \forall q_h \in M_{h0}, \quad (54)
$$

$$
(\tau p_h^n, q_h) = (p_h^{n-1}, q_h) + (\phi_h^n, q_h) - \delta \mu (\nabla_h e_h^n - R_h(\hat{v}_h^n), q_h), \quad \forall q_h \in M_{h0}. \quad (55)
$$

We now recall and establish useful properties for the error functions and for the forms $a_{\theta'}$ and $b$. The error function $\hat{v}_h^n$ satisfies the following important property [17].
Lemma 1 For all $q_h \in M_h$ and $n \geq 1$, the following holds.

$$b(e_h^n, q_h) = b(e_h^n, q_h) + \tau a_{\text{clip}}(\phi_h^n, q_h) - \tau \sum_{e \in T_h} \frac{\partial}{\partial n} \int_e [\phi_h^n] [q_h]$$

$$+ \tau (G_h([\phi_h^n]), G_h([q_h])), \quad (56)$$

$$b(e_h^n, q_h) = -\tau \sum_{e \in T_h} \frac{\partial}{\partial n} \int_e [\phi_h^n] [q_h] + \tau (G_h([\phi_h^n]), G_h([q_h])). \quad (57)$$

The forms $a_{\phi}$ and $b$ admit the following bounds.

Lemma 2 There exists a constant $C$, independent of $h$, $\tau$, $w_h$, $z$, $v$ and $\theta_h$, such that the following estimates hold.

(i) If $v \in (W^{1,3}(\Omega) \cap H_0^1(\Omega)) \cap L^\infty(\Omega)^d$, then for any $z \in X$ and $\theta_h, w_h \in X_h$:

$$|a_{\phi}(z, w_h, v, \theta_h) + |b(w_h, v \cdot \theta_h))| \leq C\|w_h\|\|v\|\|\theta_h\|_{DG}. \quad (58)$$

(ii) If $v \in (H^{k+1}(\Omega) \cap H_0^1(\Omega))^d$, then for any $z \in X$ and $\theta_h, w_h \in X_h$:

$$\|\phi(w_h, v - I_h v, \theta_h)\| + |b(w_h, (v - I_h v) \cdot \theta_h))| \leq C\|w_h\|\|v\|_{H^{k+1}(\Omega)}\|\theta_h\|_{DG}. \quad (59)$$

Proof (i) Since $v$ belongs to $H^1(\Omega)^d$ and vanishes on the boundary, we have:

$$a_{\phi}(z; w_h, v, \theta_h) = \sum_{E \in \partial_h} \int_E (w_h \cdot \nabla v) \cdot \theta_h + \frac{1}{2} b(w_h, v \cdot \theta_h). \quad (60)$$

With Hölder’s inequality and (48), we have

$$\left| \sum_{E \in \partial_h} \int_E (w_h \cdot \nabla v) \cdot \theta_h \right| \leq C_p \|w_h\|\|v\|_{W^{1,1}(\Omega)}\|\theta_h\|_{DG}. \quad (61)$$

To bound the second term in (60), we use (20). We have

$$\frac{1}{2} b(w_h, v \cdot \theta_h) = -\frac{1}{2} \sum_{E \in \partial_h} \int_E w_h \cdot \nabla (v \cdot \theta_h) + \frac{1}{2} \sum_{e \in \partial_h} \int_e (w_h) \cdot n_e (v \cdot [\theta_h]) = A_1 + A_2.$$

The term $A_1$ is bounded as follows. With Hölder’s inequality and (48),

$$|A_1| \leq \|w_h\|\left(\|\nabla v\|_{L^\infty(\Omega)}\|\theta_h\|_{L^1(\Omega)} + \|v\|_{L^\infty(\Omega)}\|\nabla h_\theta\|_{L^2(\Omega)}\right) \leq C\|w_h\|\|v\|\|\theta_h\|_{DG}.$$

For $A_2$, consider a face $e \in T_h$ and let $E^1_e$ and $E^2_e$ denote the elements sharing $e$.

$$\left| \int_{E^1_e} w_h [E^1_e] \cdot n_e (v \cdot [\theta_h]) \right| \leq C\|v\|_{L^\infty(\Omega)}|e|^{1/2} |E^1_e|^{-1/2} \|w_h\|_{E^1_e} \|E^2_e\| \|\theta_h\|_{L^2(e)}.$$

Since $|e|^{1/2} |E^1_e|^{-1/2} h_e^{1/2} \leq C$, we apply Hölder’s inequality and we obtain

$$\sum_{e \in T_h} \left| \int_{E^1_e} w_h [E^1_e] \cdot n_e (v \cdot [\theta_h]) \right| \leq C\|v\|_{L^\infty(\Omega)}\|w_h\|_{L^2(\Omega)}\|\theta_h\|_{DG}.$$
Lemma 3

The first term is bounded exactly like in the proof of (i) (see the derivation of bound (61)) with \( v \) replaced by \( v - \Pi_h v \). By the stability of the interpolant (44), we have

\[
\sum_{E \in \mathcal{D}_h} \left| \int_E (w_h \cdot \nabla (v - \Pi_h v)) \cdot \theta_h \right| \leq C \| w_h \| \| v \|_{W^{1,1}([\Omega])} \| \theta_h \|_{\mathcal{DG}}.
\]

For the second term, we use (20) and the following identity.

\[
[(v - \Pi_h v) \cdot \theta_h] = [v - \Pi_h v] \cdot \theta_h + [v - \Pi_h v] \cdot \theta_h, \quad \forall \varepsilon \in \mathcal{D}_h.
\]

We obtain:

\[
b(w_h, (v - \Pi_h v) \cdot \theta_h) = -\sum_{E \in \mathcal{D}_h} \int_E w_h \cdot \nabla ((v - \Pi_h v) \cdot \theta_h)
+ \sum_{\varepsilon \in \mathcal{D}_h} \int_{\varepsilon} \{ w_h \} \cdot n_\varepsilon [(v - \Pi_h v) \cdot \theta_h]
+ \sum_{\varepsilon \in \mathcal{D}_h} \int_{\varepsilon} \{ w_h \} \cdot n_\varepsilon (v - \Pi_h v) \cdot \theta_h
\]

The terms \( \tilde{A}_1 \) and \( \tilde{A}_2 \) are handled in a similar way to \( A_1 \) and \( A_2 \) with \( v \) being replaced with \( v - \Pi_h v \). With the stability property (41) and (45), we have

\[
|\tilde{A}_1| + |\tilde{A}_2| \leq C \| w_h \| \| v - \Pi_h v \|_{\mathcal{DG}} \| \theta_h \|_{\mathcal{DG}} \leq C \| w_h \| \| v \| \| \theta_h \|_{\mathcal{DG}}.
\]

To handle \( \tilde{A}_3 \), we refer to the proof of bound (60) in Lemma 6.5 in [17]. We have

\[
|\tilde{A}_3| \leq C \| w_h \| \| v \|_{H^{k+1}([\Omega])} \| \theta_h \|_{\mathcal{DG}}.
\]

We conclude that (59) holds.

In addition, we have the following bounds on \( a_{(e)}', b, \mathcal{W}', \) and \( \mathcal{C} \) [17].

**Lemma 3** There exists a constant \( C \), independent of \( h, \varepsilon, w, z, v, \) and \( \theta_h \), such that the following bounds hold.

(i) If \( w \in (H^{k+1}(\Omega) \cap H_0^0(\Omega))^d \) satisfies \( b(w, q_h) = 0, \forall q_h \in M_h \) and if \( v \) belongs to \( (W^{1,3}(\Omega) \cap H_0^0(\Omega))^d \), then for any \( z \in \mathcal{X} \) and \( \theta_h \in \mathcal{X}_h \):

\[
|a_{(e)}(z; \Pi_h w - w, v, \theta_h)| + |b(\Pi_h w - w, v \cdot \theta_h)| \leq C h^{k+1} \| w \|_{H^{k+1}([\Omega])} \| v \| \| \theta_h \|_{\mathcal{DG}}.
\]

(ii) If \( w \in (H^{k+1}(\Omega) \cap H_0^0(\Omega))^d \) and \( v \in (W^{1,3}(\Omega) \cap H_0^0(\Omega))^d \), then for any \( z \in \mathcal{X} \) and \( \theta_h \in \mathcal{X}_h \):

\[
|\mathcal{C}(\Pi_h v, \Pi_h w - w, \theta_h)| + |\mathcal{W}(z; \Pi_h v, \Pi_h w - w, \theta_h)| \leq C h^{k} \| w \|_{H^{k+1}([\Omega])} \| v \| \| \theta_h \|_{\mathcal{DG}}.
\]
The following error and stability estimates were shown in [17].

**Lemma 4** Assume that \( \sigma \geq \frac{M_{k-1}^2}{d} \), \( \sigma \geq \frac{4M_{k}^2}{d} \), and \( \delta \leq 1/(4d) \). We have

\[
\|u_h\| + \mu \frac{1}{4} \sum_{n=1}^{m} \|v_h^n\|_{DG} \leq \|u^0\| + \frac{4C_{DG}^2}{\mu} \sum_{n=1}^{m} \|f^n\|^2.
\]

Assume that \( u \in L^\infty(0,T;H^{k+1}(\Omega)^d) \), \( \partial_t u \in L^2(0,T;H^k(\Omega)^d) \), \( \partial_t u \in L^2(0,T;\Omega) \), and \( p \in L^\infty(0,T;H^k(\Omega)) \). Then, there exists a constant \( \gamma \) independent of \( \tau, h, \) and \( \mu \) such that if \( \tau \leq \gamma \mu \),

\[
\|e^n_h\|^2 + \sum_{n=1}^{m} \|e^n_h - e^{n-1}_h\|^2 + \frac{1}{16} \sum_{n=1}^{m} \|v^n_h\|^2_{DG} + \frac{\mu}{8} \sum_{n=1}^{m} \|e^n_h\|^2_{DG} \leq C_{DG} \left( 1 + \mu + \frac{1}{\mu} \right) (\tau + h^{2k}).
\]

We will make use of the following inverse estimates, see Theorem 4.5.11 in [1] for (68)-(69) and see Lemma 3.8 in [20] for (70).

\[
\|\theta_h\|_{L^1(\Omega)} \leq Ch^{-d/6}, \forall \theta_h \in X_h,
\]

\[
\|\theta_h\|_{L^\infty(\Omega)} \leq Ch^{-d/2}, \forall \theta_h \in X_h,
\]

\[
\|\theta_h\|_{DG} \leq C h^{-1} \|\theta_h\|, \forall \theta_h \in X_h, \text{ and } \|q_h\|_{DG} \leq C h^{-1} \|q_h\|, \forall q_h \in M_h.
\]

We will also use the following trace inequalities, see Section 2.1.3 in [20] and Lemma 1.5.2 in [4]. Let \( E \in \delta E_h \). For all \( e \in E \),

\[
\|v\|_{L^2(E)} \leq C h^{-1/2} \|v\|_{L^2(E)} + h \|\nabla v\|_{L^2(E)}, \forall v \in H^1(E)^d
\]

\[
\|\theta_h\|_{L^r(E)} \leq C h^{-1/r} \|\theta_h\|_{L^r(E)}, \forall \theta_h \in X_h, r \geq 1.
\]

One important step in deriving improved error estimates is to carefully construct a dual problem and its discretization. To this end, define the error functions

\[
\chi(t) = u_h^n - u(t), \quad \forall t < t^n, \quad \forall n \geq 1, \quad \chi(0) = u_h^0 - u^0.
\]

For all \( t \geq 0 \), let \( (U(t), P(t)) \in H^1_0(\Omega)^d \times L^2_0(\Omega) \) be the solution of the following dual Stokes problem.

\[
-\Delta U(t) + \nabla P(t) = \chi(t), \quad \text{in } \Omega, \quad (73)
\]

\[
\nabla \cdot U(t) = 0, \quad \text{in } \Omega, \quad (74)
\]

\[
U(t) = 0, \quad \text{on } \partial \Omega. \quad (75)
\]

Since \( \chi(t) \) belongs to \( L^2(\Omega)^d \), we can assume that the solution to the above problem satisfies the following, if the domain is convex.

\[
\|U(t)\|_{H^1(\Omega)} + \|P(t)\|_{H^1(\Omega)} \leq C ||\chi(t)||. \quad (76)
\]
Further, for $t \geq 0$, define $(U_h(t), P_h(t)) \in X_h \times M_{60}$ as the dG solution to the auxiliary problem (73)-(75).

$$a_{\theta}(U_h(t), \theta_h) - b(\theta_h, P_h(t)) = (\chi(t), \theta_h), \quad \forall \theta_h \in X_h,$$

$$b(U_h(t), q_h) = 0, \quad \forall q_h \in M_{60}. \quad (77)$$

The proofs for existence and uniqueness of $(U_h(t), P_h(t))$ are found in [20]. We now state stability and error bounds for the dG solution of the dual problem.

**Lemma 5** For $t \in [0, T]$, there exists a constant $C$ independent of $h$ such that

$$||U(t)|| \leq C||\chi(t)||, \quad (79)$$

$$||U(t) - U_h(t)|| + h||U(t) - U_h(t)||_{DG} + h||P(t) - P_h(t)|| \leq C h^2 ||\chi(t)||, \quad (80)$$

$$||U_h(t)||_{L^2(\Omega)} \leq C||\chi(t)||. \quad (81)$$

**Proof** Fix $t \in [0, T]$. Note a Sobolev embedding result, see Theorem 1.4.6 in [1], and Theorem 2 in Section 5.6.1 in [5], and (76) yield

$$||U(t)||_{L^2(\Omega)} + ||U(t)||_{W^{1,3}(\Omega)} \leq C||U(t)||_{H^2(\Omega)} \leq C||\chi(t)||. \quad (82)$$

For $k \geq 1$, since the domain is convex, we have an error estimate for the dG error in the $L^2$ and energy norms [20,8]. Using (76), we obtain

$$||U(t) - U_h(t)|| + h||U(t) - U_h(t)||_{DG} + h||P(t) - P_h(t)|| \leq C h^2 ||\chi(t)||. \quad (83)$$

To show (81), let $U_h(t) \in X_h$ be the Lagrange interpolant of $U(t)$. As a result of (79), (80), inverse estimate (69), and approximation properties, we obtain

$$||U_h(t)||_{L^2(\Omega)} \leq ||U_h(t) - U_h(t)||_{L^2(\Omega)} + ||U_h(t)||_{L^2(\Omega)} \leq C h^{-d/2}(||U_h(t) - U(t)|| + ||U(t) - U_h(t)||) + C||U(t)||_{L^2(\Omega)} \leq C||\chi(t)||. \quad (84)$$

**Theorem 1** Assume that if $k \geq 2$, the condition $h^2 \leq \tau$ holds. Under the same assumptions as Lemma 4, with $\partial_t u \in L^2(0, T; H^{k+1}(\Omega)^d)$, we have, with $K_\mu = \sum_{i=-4}^2 \mu_i$:

$$\mu \tau \sum_{i=1}^m ||u_h^n - u^n||^2 + \mu \tau \sum_{i=1}^m ||v_h^n - u^n||^2 \leq C_\mu K_\mu (\tau^2 + h^{2k+2}), \quad \forall 0 \leq m \leq N_T. \quad (85)$$

**Proof** By consistency of the dG discretization, we have for $\theta_h \in X_h$ and $n \geq 1$:

$$((\partial_t u)^n, \theta_h) + a_{\theta}(u^n; u^n, u^n, \theta_h) + \mu a_{\theta}(u^n, \theta_h) = b(\theta_h, p^n) + (f^n, \theta_h). \quad (86)$$

Multiplying (85) by $\tau$, subtracting it from (25), and choosing $\theta_h = U_h^n = U_h(t^n)$ yields

$$(v_h^n - u^n - \chi^{n-1}, U_h^n) + \tau \dot{K}_\theta(U_h^n) + \tau \mu a_{\theta}(v_h^n - u^n, U_h^n) \tau b(U_h^n, p_h^n - p^n) + (\tau(\partial_t u)^n - (u^n - u^{n-1}), U_h^n), \quad (86)$$
where
\[ R_{\mathcal{D}}(U^n_h) = a_{\mathcal{D}}(u^{n-1}_h; u^n, v^n_h; U^n_h) - a_{\mathcal{D}}(u^n; u^n, U^n_h). \]
Throughout the proof, \( \varepsilon \) denotes a small positive constant to be determined later. Let us begin with the last two terms in (86). We use the fact that \( U^n_h \) satisfies (78) and the definition of \( \pi_h p^n \) to obtain the following.
\[
b(U^n_h, p^n_h - p^n) = -b(U^n_h, p^n - \pi_h p^n) = \sum_{e \in \Gamma_n} \int_e \{ p^n - \pi_h p^n \} \cdot n_e.
\]
Let \( \Delta_e \) denote the union of the two elements sharing a face \( e \). By a trace inequality, approximation property (46), (80), and the fact that \( [U^n_h] = 0 \) a.e on any \( e \in \Gamma_n \cup \partial \Omega \) since \( U^n \in H^2_0(\Omega)^d \), we obtain
\[
|b(U^n_h, p^n_h - p^n)| \leq Ch^{k+1} |p^n|_{H^k(\Omega)} \| \chi^n \| \leq \varepsilon \mu \| \varepsilon^n_h \|^2 + Ch^{2k+2} \left( 1 + \frac{1}{\varepsilon \mu} \right). 
\]
In the above, we used that
\[
\| \chi^n \| \leq C(h^{k+1} |u^n|_{H^{k+1}(\Omega)} + \| \varepsilon^n_h \|),
\]
which is obtained by applying the triangle inequality and approximation property (42). This bound will be used repeatedly in this proof. For the last term in (86), we simply have:
\[
|\tau(\partial_t u^n - (u^n - u^{n-1}), U^n_h)| \leq C \tau^2 \int_{\tau_{n-1}}^{\tau_n} \| \partial_t u^n \|^2 + \tau \| U^n_h \|^2_{1DG}.
\]
We now consider the terms on the left hand side of (86). With (28) and (78), we have
\[
(v^n_h - u^n - \chi^{n-1}, U^n_h) = (\chi^n - \chi^{n-1}, U^n_h) - \tau b(U^n_h, \phi^n_h) = (\chi^n - \chi^{n-1}, U^n_h).
\]
Note that from (77), (78), the above equality, and the symmetry of \( a_{\mathcal{D}}(\cdot, \cdot) \), we have
\[
(v^n_h - u^n - \chi^{n-1}, U^n_h) = a_{\mathcal{D}}(U^n_h - U^{n-1} h, U^n_h) = \frac{1}{2} \left( a_{\mathcal{D}}(U^n_h, U^n_h) - a_{\mathcal{D}}(U^n_h, U^{n-1}_h) \right) + \frac{1}{2} a_{\mathcal{D}}(U^n_h - U^{n-1}_h, U^n_h - U^{n-1}_h).
\]
In addition, we write
\[
a_{\mathcal{D}}(v^n_h - u^n, U^n_h) = a_{\mathcal{D}}(v^n_h - e^n_h, U^n_h) + a_{\mathcal{D}}(e^n_h, U^n_h) + a_{\mathcal{D}}(\Pi_h u^n - u^n, U^n_h).
\]
To handle the last term in the equality above, let \( Q_h u^n \) be the elliptic projection of \( u^n \) onto the space \( X_h \). Since the domain is convex, this projection satisfies [20]:
\[
\forall \theta_h \in X_h, a_{\mathcal{D}}(u^n - Q_h u^n, \theta_h) = 0, \quad \text{and} \quad \| u^n - Q_h u^n \| \leq C(h^{k+1} |u^n|_{H^{k+1}(\Omega)}).
\]
Let \( \theta_h = \Pi_h u^n - Q_h u^n \) in (77). We obtain
\[
a_{\mathcal{D}}(\Pi_h u^n - u^n, U^n_h) = a_{\mathcal{D}}(\Pi_h u^n - Q_h u^n, U^n_h) = (\chi^n, \Pi_h u^n - Q_h u^n) + b(\Pi_h u^n - Q_h u^n, p^n_h).
\]

Using (39), we obtain
\[ |b(\Pi_h u^\Pi - Q_h u^\Pi, P_n^\Pi)\| \leq C\|\Pi_h u^\Pi - Q_h u^\Pi\|P_n^\Pi|_{DG}. \]

Note that with (47), the inverse estimate (70), (46), (80), (76), and triangle inequality:
\[ |P_n^\Pi|_{DG} \leq |P_n^\Pi - \tau_n P_n^\Pi|_{DG} + |\tau_n P_n^\Pi|_{DG} \leq C h^{-1} |P_n^\Pi - \tau_n P_n^\Pi| + C |P_n^\Pi|_{H^1(\Omega)}. \]  
\[ \leq C(\|\theta_n^\Pi\| + h^{k+1}\|u^\Pi\|_{H^{k+1}(\Omega)}). \]  

Hence, Cauchy-Schwarz’s inequality, the above bounds, the assumption that \( u \in L^\infty(0,T;H^{k+1}(\Omega)^d) \), and Young’s inequality yield:
\[ |a_{\theta}(\Pi_h u^\Pi - u^\Pi, U_n^\Pi)| \leq C \|\Pi_h u^\Pi - Q_h u^\Pi\|\|u^\Pi\|_{H^{k+1}(\Omega)} \leq C |\Pi_h u^\Pi - Q_h u^\Pi\|_{H^{k+1}(\Omega)} \leq C \|\theta_n^\Pi\| + C h^{k+1}\|u^\Pi\|_{H^{k+1}(\Omega)}. \]  

Consider now the second term in (91). Letting \( \theta_h = \theta_n^\Pi \) in (77), we obtain
\[ a_{\theta}(\theta_n^\Pi, U_n^\Pi) = \|\theta_n^\Pi\|^2 + (\chi^\Pi - \theta_n^\Pi, \theta_n^\Pi) + b(\theta_n^\Pi, P_n^\Pi). \]  

With (87), (89),(90), (91), (96), and (97), the equality (86) becomes
\[ \frac{1}{2} \left( a_{\theta}(U_n^\Pi, U_n^\Pi) - a_{\theta}(U_n^{\Pi-1}, U_n^{\Pi-1}) + a_{\theta}(U_n^\Pi - U_n^{\Pi-1}, U_n^\Pi - U_n^{\Pi-1}) \right) + \tau \mu \|\theta_n^\Pi\|^2 \leq 2 \epsilon \tau \mu \|\theta_n^\Pi\|^2 + C \tau \mu \|U_n^\Pi\|^2_{DG} + C \left( \frac{1 + 1}{\epsilon} \right) + 2 \frac{1}{\mu} \|\theta_n^\Pi\|^2 + C(1 + 1\frac{1}{\epsilon}). \]
\[ \leq \tau \mu \|\theta_n^\Pi\|^2 + C \tau \mu \|U_n^\Pi\|^2_{DG} + C(1 + 1\frac{1}{\epsilon}). \]

We now handle the last three terms in the above bound. Let \( \theta_h = \theta_n^\Pi - \theta_n^\Pi \) in (77).
\[ a_{\theta}(\theta_n^\Pi - \theta_n^\Pi, U_n^\Pi) = (\theta_n^\Pi, \theta_n^\Pi - \theta_n^\Pi) + (\Pi_h u^\Pi - u^\Pi, \theta_n^\Pi - \theta_n^\Pi) + b(\theta_n^\Pi - \theta_n^\Pi, P_n^\Pi). \]  

Recall that by (53), (57), (36), and the assumption that \( \tilde{\sigma} \geq M_h^2 \), we have
\[ (\theta_n^\Pi, \theta_n^\Pi - \theta_n^\Pi) = -\tau b(\theta_n^\Pi, \theta_n^\Pi) = \tau^2 \sum_{e \in h_e} \tilde{\sigma}(\bar{\theta}_n^\Pi)_{LB} - \tau^2 |G_n(\bar{\theta}_n^\Pi)|^2 \geq 0. \]

Using Cauchy-Schwarz’s inequality, (39), and (42), we have the following bound.
\[ |(\Pi_h u^\Pi - u^\Pi, \theta_n^\Pi - \theta_n^\Pi)| + |b(\theta_n^\Pi - \theta_n^\Pi, P_n^\Pi)| \leq C \|\theta_n^\Pi - \theta_n^\Pi\|_{H^{k+1}(\Omega)} + |P_n^\Pi|_{DG}. \]  

With (53) and (39), we immediately obtain:
\[ \|\theta_n^\Pi - \theta_n^\Pi\|_{DG} \leq C \tau |\phi_n^\Pi|_{DG}. \]

Therefore with (95), the assumption that \( u \in L^\infty(0,T;H^{k+1}(\Omega)^d) \), and Young’s inequality, the bound (101) becomes
\[ |(\Pi_h u^\Pi - u^\Pi, \theta_n^\Pi - \theta_n^\Pi)| + |b(\theta_n^\Pi - \theta_n^\Pi, P_n^\Pi)| \leq C \tau |\phi_n^\Pi|_{DG}(h^{k+1}\|u^\Pi\|_{H^{k+1}(\Omega)} + \|\theta_n^\Pi\|) \]
\[ \leq \epsilon \|\theta_n^\Pi\|^2 + C \left( \frac{1}{\epsilon} \right) \tau^2 |\phi_n^\Pi|^2_{DG} + C h^{k+1}. \]
With (42), we have

\[ |(T_h u^n - u^n, e_h^n)| \leq \epsilon \|e_h^n\|^2 + C \epsilon h^{2k+2} |u^n|_{H^{k+1}(\Omega)}. \]

To handle the last term in (98), we use (57), (36), and (95).

\[
|b(e_h^n, P_h^n)| = -\tau \sum_{e \in \mathcal{E}_h} \int_e [\phi_h^n]|P_h^n| + \tau (G_h(|\phi_h^n|), G_h([P_h^n]))
\leq C\tau \|\phi_h^n\|_{DG} |P_h^n|_{DG} \leq \epsilon \|e_h^n\|^2 + C \left(1 + \frac{1}{\epsilon}\right) \tau \|\phi_h^n\|^2_{DG} + C h^{2k+2}. \tag{104}
\]

With the above bounds combined, (98) becomes

\[
\frac{1}{2} (a_{\varphi}(U^n_h, U^n_h) - a_{\varphi}(U^{n-1}_h, U^{n-1}_h)) + a_{\varphi}(U^n_h - U^{n-1}_h, U^n_h - U^{n-1}_h) + \tau \mu \|e_h^n\|^2
\leq 5\epsilon \tau \mu \|e_h^n\|^2 + C \left(1 + \frac{1}{\epsilon}\right) \tau \|\phi_h^n\|^2_{DG} + C \tau \|U_h^n\|^2_{DG} + C \left(1 + \frac{1}{\epsilon\mu}\right) \tau h^{2k+2}. \tag{105}
\]

We now want to handle the nonlinear term \( \bar{R}_{\varphi}(U^n_h) \). We write

\[
\bar{R}_{\varphi}(U^n_h) = a_{\varphi}(u_h^{n-1}; u_h^{n-1}, v_h^n - u^n, U^n_h) - a_{\varphi}(u^n; u^n - u^{n-1}, u^n, U^n_h) = T^n_1 + T^n_2. \]

To handle \( T^n_1 \), we resort to (13). We have,

\[
T^n_1 = -\bar{a}_{\varphi}(u_h^{n-1}; u_h^{n-1}, U^n_h - U^n_h, v_h^n - u^n) - \bar{a}_{\varphi}(u_h^{n-1}; u_h^{n-1}, U^n_h, v_h^n - u^n) - \bar{a}_{\varphi}(u_h^{n-1}; U^n_h, U^n_h, v_h^n - u^n).
\]

The upwind term in the second term in the expression for \( T^n_1 \) vanishes and it becomes linear in the second argument. Thus, we can further split it in the expression for \( T^n_1 \):

\[
T^n_1 = -\bar{a}_{\varphi}(u_h^{n-1}; U^n_h, U^n_h, v_h^n - u^n) - \bar{a}_{\varphi}(u_h^{n-1}; U^n_h, U^n_h, v_h^n - u^n) - \bar{a}_{\varphi}(u_h^{n-1}; U^n_h, U^n_h, v_h^n - u^n) = \sum_{i=1}^{d} A_{i1}^n.
\]

We postpone handling \( A_{i1}^n \) till the end as it is more intricate than the other terms. We begin with \( A_{i2}^n \). With (13), we write

\[
A_{i2}^n = -\bar{a}_{\varphi}(e_h^{n-1}; e_h^{n-1}, U^n_h, v_h^n - u^n) = a_{\varphi}(e_h^{n-1}; e_h^{n-1}, v_h^n - u^n, U^n_h)
= a_{\varphi}(e_h^{n-1}; e_h^{n-1}, U^n_h) + a_{\varphi}(e_h^{n-1}; e_h^{n-1}, \Pi_h u^n - u^n, U^n_h) = A_{i2,1}^n + A_{i2,2}^n.
\]

The term \( A_{i2,1}^n \) reads

\[
A_{i2,1}^n = \sum_{E \in \mathcal{D}_h} \int_E (e_h^{n-1}, \nabla \bar{e}_h^{n}) \cdot U^n + \frac{1}{2} b(e_h^{n-1}, \bar{e}_h^{n}) \cdot U^n
+ \sum_{E \in \mathcal{D}_h} \int_{E_{h,i}} [e_h^{n-1}] \cdot n_E \left[\left((\bar{e}_h^n)^{\text{int}} - (\bar{e}_h^n)^{\text{ext}}\right) \cdot (U^n)^{\text{int}}.\right.
\]

\[
\]
With Hölder’s inequality and (79), we bound the first term:
\[
\sum_{E \in \mathcal{E}_h} \left| \int_{E} (e_h^{n-1} \cdot \nabla e_h^n) \cdot U^n \right| \leq \|e_h^{n-1}\| \|\nabla_h e_h^n\| \|U^n\|_{L^\infty(\Omega)} \leq C \|e_h^{n-1}\| \|e_h^n\|_{DG} \|X^n\|.
\]
We use (58) to bound the second term in \(A_{2,1}^h\). With (79), we have
\[
\frac{1}{2} |b(e_h^{n-1}, e_h^n, U^n)| \leq C \|e_h^{n-1}\| \|U^n\| \|e_h^n\|_{DG} \leq C \|e_h^{n-1}\| \|X^n\| \|e_h^n\|_{DG}.
\]
With trace estimate (72) and (79), the upwind term in \(A_{2,1}^h\) is bounded by:
\[
C \|e_h^{n-1}\| \|e_h^n\|_{DG} \|U^n\|_{L^\infty(\Omega)} \leq C \|e_h^{n-1}\| \|e_h^n\|_{DG} \|X^n\|.
\]
Combining the above bounds and applying Young’s and triangle inequality yields:
\[
|A_{2,1}^h| \leq \varepsilon \mu \|e_h^n\|^2 + C \left( 1 + \frac{1}{\varepsilon \mu} \right) \|e_h^{n-1}\|^2 \|e_h^n\|_{DG}^2 + Ch^{2k+2}. \tag{106}
\]
We now handle \(A_{2,2}^h\), it reads
\[
A_{2,2}^h = \sum_{E \in \mathcal{E}_h} \int_{E} (e_h^{n-1} \cdot \nabla (\Pi_h u^n - u^n)) \cdot U^n + \frac{1}{2} b(e_h^{n-1}, (\Pi_h u^n - u^n) \cdot U^n)
\]
\[
+ \sum_{E \in \mathcal{E}_h} \int_{\partial E} \langle e_h^n \rangle \cdot n_E \left( ((\Pi_h u^n - u^n)^{\text{inte}} - (\Pi_h u^n - u^n)^{\text{ext}}) \cdot (U^n)^{\text{inte}} \right).
\]
With Hölder’s inequality, approximation property (43), and (79):
\[
\sum_{E \in \mathcal{E}_h} \int_{E} (e_h^{n-1} \cdot \nabla (\Pi_h u^n - u^n)) \cdot U^n \leq \|e_h^{n-1}\| \|\nabla_h (\Pi_h u^n - u^n)\| \|U^n\|_{L^\infty(\Omega)} \leq Ch^k |u^n|_{H^{k+1}(\Omega)} \|e_h^{n-1}\| \|X^n\|.
\]
For the second term in \(A_{2,2}^h\), applying Hölder’s inequality, (42)-(43), (79), inverse estimate (70) and trace estimates (71)-(72), we obtain:
\[
\frac{1}{2} |b(e_h^{n-1}, (\Pi_h u^n - u^n) \cdot U^n)| \leq Ch^{-1} \|e_h^{n-1}\| \|\Pi_h u^n - u^n\| \|U^n\|_{L^\infty(\Omega)} + C \|U^n\|_{L^\infty(\Omega)} (h^{-1} \|\Pi_h u^n - u^n\| + \|\nabla_h (\Pi_h u^n - u^n)\|) \|e_h^{n-1}\| \leq Ch^k |u^n|_{H^{k+1}(\Omega)} \|e_h^{n-1}\| \|X^n\|. \tag{107}
\]
The upwind term in \(A_{2,2}^h\) is bounded similarly by
\[
C \|e_h^{n-1}\| \|U^n\|_{L^\infty(\Omega)} (h^{-1} \|u^n - \Pi_h u^n\| + \|\nabla_h (u^n - \Pi_h u^n)\|) \leq Ch^k |u^n|_{H^{k+1}(\Omega)} \|e_h^{n-1}\| \|X^n\|.
\]
Hence, with the assumption that \(u \in L^n(0, T; H^{k+1}(\Omega)^d)\) and Young’s inequality, we attain the following bound for \(A_{2,2}^h\):
\[
|A_{2,2}^h| \leq \varepsilon \mu \|e_h^n\|^2 + C \left( \frac{1}{\varepsilon \mu} + 1 \right) h^k \|e_h^{n-1}\|^2 + Ch^{2k+2}. \tag{108}
\]
With (106) and (108), we obtain a bound on $A_3^n$:

$$\|A_3^n\| \leq 2\varepsilon\mu\|e_h^n\|^2 + C \left(\frac{1}{\varepsilon\mu} + 1\right) \left(h^{2k}\|e_h^{n-1}\|^2 + \|e_h^{n-1}\|^2\|e_h^n\|_{DG}\right) + Ch^{2k+2}. \quad (109)$$

We now focus on $A_3^n$ and write

$$A_3^n = -\sum_{E \in D_h} \int_E \left( (\Pi_h u^{n-1} - u^{n-1} \cdot \nabla ((U^n - U_h^n) + U_h^n)) \cdot (v_h^n - u^n) \right)$$

$$- \frac{1}{2} \sum_{E \in D_h} \int_E (\nabla \cdot (\Pi_h u^{n-1} - u^{n-1})) (U^n \cdot (v_h^n - u^n))$$

$$+ \frac{1}{2} \sum_{E \in D_h \cup \partial \Omega} \int_E (\Pi_h u^{n-1} - u^{n-1}) \cdot n_E ((U^n \cdot \pi_h^n) + (U^n \cdot (\Pi_h u^n - u^n))) = \sum_{j=1}^3 A_{3,j}.$$  

Applying Hölder’s inequality and trace estimates yields:

$$|A_{3,1}^n| \leq C|\Pi_h u^{n-1} - u^{n-1}|_{L^\infty(\Omega)} \|\nabla_h (U^n - U_h^n)\| + \|U_h^n\|_{DG}\|v_h^n - u^n\|,$$

$$|A_{3,2}^n| \leq C\|\nabla_h (\Pi_h u^{n-1} - u^{n-1})\| \|U^n\|_{L^\infty(\Omega)} \|v_h^n - u^n\|,$$

$$|A_{3,3}^n| \leq C(h^{-1}|\Pi_h u^{n-1} - u^{n-1}||\nabla_h (\Pi_h u^{n-1} - u^{n-1})||\nabla_h (\Pi_h u^{n-1} - u^{n-1})||\nabla_h (U^n - U_h^n)\|v_h^n - u^n\| + C(h^{-1/2}|\Pi_h u^{n-1} - u^{n-1}||h^{-1/2}\|\nabla_h (\Pi_h u^{n-1} - u^{n-1})||U^n\|_{L^\infty(\Omega)}\|v_h^n - u^n\|\nabla_h (\Pi_h u^n - u^n)\|\nabla_h (U^n - U_h^n)\|v_h^n - u^n\| + Ch^{k+1}|u^n|_{H^{k+1}(\Omega)}\|\nabla_h (U^n - U_h^n)\|v_h^n - u^n\|.\quad (110)$$

In view of (102) and the triangle inequality, we have

$$\|\pi_h^n\| \leq \|\pi_h^n - e_h^n\| + \|e_h^n\| \leq C|\phi_h^n|_{DG} + \|e_h^n\|. \quad (110)$$

Similarly, with (28), (39) and (42), we obtain

$$\|v_h^n - u^n\| \leq C|\phi_h^n|_{DG} + \|e_h^n\| + Ch^{k+1}|u^n|_{H^{k+1}(\Omega)}. \quad (111)$$

Using (110), (111), and the assumption that $u \in L^\infty(0,T;H^{k+1}(\Omega)^d)$, we have:

$$|A_{3}^n| \leq C(h\|\pi_h^n\|_{DG}) (\|e_h^n\| + \|\phi_h^n\|_{DG} + h^{k+1}) + Ch^{2k+1}\|\pi_h^n\|.\quad (111)$$

With (88), this yields

$$|A_{3}^n| \leq C(h\|e_h^n\| + h^{k+1} + \|\phi_h^n\|_{DG}) (\|e_h^n\| + h^{k+1} + \|\phi_h^n\|_{DG}) + Ch^{3k+2} + Ch^{2k+1}\|e_h^n\|.\quad (111)$$

To handle $A_4^n$, we note that it reduces to

$$A_4^n = -\sum_{E \in D_h} \int_E (u^{n-1} \cdot \nabla U^n) \cdot (v_h^n - u^n).$$
Thus, with Hölder’s inequality, (80), (88), (111), and \( u \in L^\infty(0, T; H^{k+1}(\Omega)) \):

\[
|A^n_1| \leq \|u^{n-1}\|_{L^2(\Omega)} (\|\nabla_h(U^n_h - U_h^n)\| + \|\nabla_h(U_h^n)\|) v_h^n - u^n \| \\
\leq C(h\|\mathbf{e}_h^n\| + h^{k+2} + \|U_h^n\|_{DG})(\|\mathbf{e}_h^n\| + h^{k+1} + \tau|\phi_h^n|_{DG}).
\]

Applying Young’s inequality, we obtain the following bound for \( |A^n_1| + |A^n_2| \):

\[
|A^n_1| + |A^n_2| \leq \varepsilon \mu \|\mathbf{e}_h^n\|^2 + C \left( \frac{1}{\varepsilon \mu} + 1 \right) \|U_h^n\|_{DG}^2 + C \left( \frac{h^2}{\varepsilon \mu} + 1 \right) h^{2k+2} \\
+ C \frac{h^2}{\varepsilon \mu} \|\mathbf{e}_h^n\|^2 + C \tau^2 \left( \frac{h^2}{\varepsilon \mu} + 1 \right) |\phi_h^n|_{DG}. \quad (112)
\]

We now consider \( A^n_1 \). It reads:

\[
A^n_1 = - \sum_{E \in \partial \Omega} \int_E (u_h^{n-1} \cdot \nabla_h (U_h^n - U_h^n)) \cdot (v_h^n - u^n) - \frac{1}{2}b(u_h^{n-1}, (U_h^n - U_h^n) \cdot (v_h^n - u^n)) \\
- \sum_{E \in \partial \Omega} \int_{\partial E} (u_h^{n-1} \cdot n_E) (v_h^n - u^n)(U_h^n - U_h^n) \cdot (v_h^n - u^n) \\
+ \frac{1}{2} \sum_{E \in \partial \Omega} \int_E (u_h^{n-1} \cdot n_E - u_h^{n-1} \cdot n_E) (U_h^n - U_h^n) \cdot (v_h^n - u^n) = \sum_{i=1}^4 A^n_{1,i}.
\]

Writing \( u_h^{n-1} = e_h^{n-1} + \Pi_h u^{n-1} \) and using Hölder’s inequality, we obtain

\[
|A^n_{1,i}| \leq \|\nabla_h(U_h^n - U_h^n)\| \|\mathbf{e}_h^{n-1}\|_{L^2(\Omega)} \|v_h^n - u^n\|_{L^2(\Omega)} + \|\Pi_h u^{n-1}\|_{L^2(\Omega)} \|v_h^n - u^n\|_{L^2(\Omega)}.
\]

By using (48),(42), and a Sobolev embedding result, we obtain

\[
\|v_h^n - u^n\|_{L^2(\Omega)} \leq \|v_h^n - \Pi_h u^n\|_{L^2(\Omega)} + \|\Pi_h u^n - u^n\|_{L^2(\Omega)} \\
\leq C\|\mathbf{e}_h^n\|_{DG} + C \varepsilon \mu \|\mathbf{e}_h^n\|_{DG} + C h^k \|\mathbf{e}_h^n\|_{DG} + C h^k \|\mathbf{e}_h^n\|_{DG}. \quad (113)
\]

With bounds (113), (111), (68), (41), and (80), we have

\[
|A^n_{1,i}| \leq C \|\mathbf{e}_h^{n-1}\| \|\mathbf{e}_h^n\|_{DG} + C \varepsilon \mu \|\mathbf{e}_h^n\|_{DG} + C \varepsilon \mu \|\mathbf{e}_h^n\|_{DG} + h^{k+1} + \tau|\phi_h^n|_{DG}.
\]

We apply (88), the assumption that \( u \in L^\infty(0, T; H^{k+1}(\Omega)) \) and Young’s inequality.

\[
|A^n_{1,1}| \leq \varepsilon \mu \|\mathbf{e}_h^n\|^2 + C \left( \frac{1}{\varepsilon \mu} + 1 \right) h^2 \|\mathbf{e}_h^n\|^2 + h^{2k+2} + \tau^{2}|\phi_h^n|_{DG} \\
+ C \left( \frac{1}{\varepsilon \mu} + 1 \right) \left( \|\mathbf{e}_h^{n-1}\|^2 \|\mathbf{e}_h^n\|^2 + h^{2k+2} \|\mathbf{e}_h^{n-1}\|^2 \right) + C h^{2k+2}. \quad (114)
\]

To handle \( A^n_{1,2} \), we split it into:

\[
A^n_{1,2} = - \frac{1}{2}b(u_h^{n-1}, (U_h^n - U_h^n) \cdot \mathbf{e}_h^n) - \frac{1}{2}b(u_h^{n-1}, (U_h^n - U_h^n) \cdot (\Pi_h u^n - u^n)).
\]
With Holder’s inequality and trace estimate (72), we have the bound:

\[
\frac{1}{2} |b'(\mathbf{u}^{n-1}_h, (\mathbf{U}^n_h - \mathbf{U}^n) \cdot \mathbf{e}^{n-1}_h)| \leq C \|\nabla_h \mathbf{u}^{n-1}_h\|_{L^3(\Omega)} \|\mathbf{U}^n_h - \mathbf{U}^n\|_{L^3(\Omega)} \|\mathbf{e}^{n-1}_h\|_{L^3(\Omega)} \\
+ C \|\mathbf{e}^{n-1}_h\|_{L^3(\Omega)} \|\mathbf{u}^{n-1}_h\|_{L^3(\Omega)} h^{-1} (\|\mathbf{U}^n_h - \mathbf{U}^n\| + h \|\nabla_h (\mathbf{U}^n_h - \mathbf{U}^n)\|). 
\]

Note that with Poincare’s inequality (48) and (80), we have

\[
\|\mathbf{U}^n_h - \mathbf{U}^n\|_{L^3(\Omega)} \leq C_T \|\mathbf{U}^n_h - \mathbf{U}^n\|_{DG} \leq C h \|\mathcal{X}^n\|. 
\]

With (115), (80), (48), triangle inequality, (70), and (68), we obtain:

\[
\frac{1}{2} |b'(\mathbf{u}^{n-1}_h, (\mathbf{U}^n_h - \mathbf{U}^n) \cdot \mathbf{e}^{n-1}_h)| \leq C (h^{-1} \|\mathbf{e}^{n-1}_h\| + \|h^{-1/6} (\mathbf{u}^{n-1}_h)\|) h \|\mathcal{X}^n\| \|\mathbf{e}^{n-1}_h\|_{DG} \\
+ C \|\mathbf{e}^{n-1}_h\|_{DG} (h^{-d/6} \|\mathbf{e}^{n-1}_h\| + \|h^{1/2} (\mathbf{u}^{n-1}_h)\|) h \|\mathcal{X}^n\| \\
\leq C (h^{-1} \|\mathbf{e}^{n-1}_h\| + h^{k+1} \|\mathbf{u}^{n-1}_h\|_{W^{1,3}(\Omega)}) \left( (h^{-1} \|\mathbf{e}^{n-1}_h\| + h \|\mathbf{u}^{n-1}_h\|_{W^{1,3}(\Omega)}) \right) \|\mathbf{e}^{n-1}_h\|_{DG}. 
\]

For the second term in the expression for $A^{n}_{1,2}$, we write $\mathbf{u}^{n-1}_h = \mathbf{e}^{n-1}_h + \Pi_h \mathbf{u}^{n-1}$. With Holder’s inequality, inverse estimate (70) and trace estimates (71)-(72), (80), (115) and (42) - (43), this term is bounded by

\[
C (h^{-1} \|\mathbf{e}^{n-1}_h\| \|\mathbf{U}^n_h - \mathbf{U}^n\|_{L^3(\Omega)} + \|\nabla_h \Pi_h \mathbf{u}^{n-1}\|_{L^3(\Omega)} \|\mathbf{U}^n_h - \mathbf{U}^n\|) \|\Pi_h \mathbf{u}^{n-1} - \mathbf{u}^{n-1}\|_{L^3(\Omega)} \\
C \|\Pi_h \mathbf{u}^{n-1} - \mathbf{u}^{n-1}\|_{L^3(\Omega)} (h^{-1} \|\mathbf{U}^n_h - \mathbf{U}^n\| + h^{1/2} \|\nabla_h (\mathbf{U}^n_h - \mathbf{U}^n)\|) \\
C \|\Pi_h \mathbf{u}^{n-1} - \mathbf{u}^{n-1}\|_{L^3(\Omega)} (h^{-1/2} \|\Pi_h \mathbf{u}^{n-1} - \mathbf{u}^{n-1}\| + h^{1/2} \|\nabla_h (\Pi_h \mathbf{u}^{n-1} - \mathbf{u}^{n-1})\|) \\
\leq C h^{k+2} \|\mathcal{X}^n\| \|\mathbf{u}^{n-1}\|_{W^{1,3}(\Omega)} + C h^{k+2} \|\mathbf{u}^{n-1}\|_{W^{1,3}(\Omega)} \|\mathcal{X}^n\| \\
+ C h \|\mathbf{u}^{n-1}\| \|\mathcal{X}^n\| \|\mathbf{e}^{n-1}_h\|_{DG} + C h^{k+2} \|\mathbf{u}^{n-1}\|_{L^3(\Omega)} \|\mathcal{X}^n\|. 
\]

With (116), (117), Young’s inequality, and that $\mathbf{u} \in L^\infty(0,T;H^{k+1}(\Omega)^d)$, we obtain a bound for $A^{n}_{1,2}$:

\[
|A^{n}_{1,2}| \leq \varepsilon \mu \|\mathbf{e}^{n-1}_h\|^2 + C \left( \frac{1}{\varepsilon \mu} + 1 \right) (\|\mathbf{e}^{n-1}_h\|^2 + h^2) \|\mathbf{e}^{n-1}_h\|_{DG}^2 \\
+ C \left( \frac{1}{\varepsilon \mu} + 1 \right) h^2 \|\mathbf{e}^{n-1}_h\|^2 + C \left( \frac{h^2}{\varepsilon \mu} + 1 \right) h^{2k+2}. 
\]

It remains to handle $A^{n}_{1,3}$ and $A^{n}_{1,4}$. We will focus on $A^{n}_{1,3}$. We use the definition of $\partial h \mathbf{u}^{n-1}$, see (10), and write $\mathbf{u}^{n-1}_h = \mathbf{e}^{n-1}_h + \Pi_h \mathbf{u}^{n-1}$. With trace estimates (71)-(72), we obtain

\[
|A^{n}_{1,3}| \leq C \|\mathbf{e}^{n-1}_h\|_{L^3(\Omega)} (h^{-1} \|\mathbf{U}^n_h - \mathbf{U}^n\| + \|\nabla_h (\mathbf{U}^n_h - \mathbf{U}^n)\|) \|\mathbf{e}^{n-1}_h\|_{L^3(\Omega)} \\
+ C \|\Pi_h \mathbf{u}^{n-1} - \mathbf{u}^{n-1}\|_{L^3(\Omega)} \|\mathbf{e}^{n-1}_h\|_{DG} + C \|\Pi_h \mathbf{u}^{n-1}\|_{L^3(\Omega)} \|\mathbf{U}^n_h\|_{DG} \|\mathbf{e}^{n-1}_h\| \\
+ C \|\Pi_h \mathbf{u}^{n-1}\|_{L^3(\Omega)} \|\mathbf{U}^n_h\|_{DG} (\|\Pi_h \mathbf{u}^{n-1} - \mathbf{u}^{n-1}\| + h \|\nabla_h (\Pi_h \mathbf{u}^{n-1} - \mathbf{u}^{n-1})\|). 
\]
A similar technique is used to bound $A_{n,d}^4$ and the above bound also holds for $A_{n,d}^4$. Using (80), (68), (48), (110), and the assumption $u \in L^\infty(0,T;H^{k+1}(\Omega)^d)$, we obtain

$$|A_{n,d}^4| + |A_{n,d}^4| \leq C h^{-d/2} \| e_h^{n-1} \|_H \| e_h^n \|_H \| e_h^n \|_D + C \| u^n \| \| e_h^{n-1} - e_h^n \|_H + \| U_h^n \|_D$$

$$+ C \| u^{n-1} \| \| U_h^n \|_D (\| e_h^n \|_H + \| e_h^n \|_H + h^{k+1} \| u^n \|_{H^{k+1}(\Omega)})$$

$$\leq \varepsilon \mu \| e_h^n \|^2 + \varepsilon \| e_h^{n-1} - e_h^n \|^2 + C \left( \frac{1}{\varepsilon \mu} + 1 \right) \| e_h^{n-1} \|^2 \| e_h^n \|^2_D + C \varepsilon \mu \| e_h^n \|^2 + C \left( \frac{1}{\varepsilon \mu} + 1 \right) \| U_h^n \|^2_D. \quad (119)$$

With bounds (114), (118), and (119), we obtain

$$|A_{n}^1| \leq 3 \varepsilon \mu \| e_h^n \|^2 + \varepsilon \| e_h^{n-1} - e_h^n \|^2 + C \left( \frac{h^2}{\varepsilon \mu} + 1 \right) (\varepsilon \mu \| e_h^n \|^2 + h^{k+2})$$

$$+ C \left( \frac{1}{\varepsilon \mu} + 1 \right) (h^2 (\| e_h^n \|^2 + \| e_h^{n-1} \|^2)) + ((\| e_h^{n-1} \|^2 + h^2) \| e_h^n \|^2_D + \| U_h^n \|^2_D). \quad (120)$$

To handle $T_2$, we write

$$T_2^n = -\alpha_e(u^n, u^n - u^{n-1}), u^n, U_h^n) - \alpha_e(u^n, u^{n-1} - \Pi_h u^{n-1}, u^n, U_h^n)$$

$$+ \alpha_e(u^n, e_h^{n-1}, u^n, U_h^n) = T_{2,1}^n + T_{2,2}^n + T_{2,3}^n. \quad (121)$$

The first term simply reads:

$$T_{2,1}^n = -\sum_{E \in T_h} \int_E ((u^n - u^{n-1}) \cdot \nabla u^n) \cdot U_h^n. \quad (122)$$

By Hölder’s inequality, (48), and a Taylor’s expansion, we have:

$$|T_{2,1}^n| \leq C \| u^n \|_{w^{k+1}(\Omega)} \| \partial u^n \|^2 + C \| U_h^n \|^2_D. \quad (123)$$

The term $T_{2,2}^n$ is bounded by (63) in Lemma 3, and the term $T_{2,3}^n$ is bounded by (58) in Lemma 2. With the assumption $u \in L^\infty(0,T;H^{k+1}(\Omega)^d)$, we have

$$|T_{2,2}^n| + |T_{2,3}^n| \leq C (h^{k+1} |u^{n-1}|_{H^{k+1}(\Omega)} + \| e_h^{n-1} \|_H) \| u^n \| \| U_h^n \|_D$$

$$\leq \varepsilon \mu \| e_h^n \|^2 + C h^{2k+2} + \varepsilon \| e_h^{n-1} - e_h^n \|^2 + C \left( \frac{1}{\varepsilon \mu} + 1 \right) \| U_h^n \|^2_D. \quad (124)$$

Combining the above bounds, (109), (112),(120),(123), and (124), yield the following bound on $|R_e(U_h^n)|$.

$$|R_e(U_h^n)| \leq 7 \varepsilon \mu \| e_h^n \|^2 + 2 \varepsilon \| e_h^{n-1} - e_h^n \|^2 + C \left( \frac{h^2}{\varepsilon \mu} + 1 \right) (\varepsilon \mu \| e_h^n \|^2 + h^{2k+2})$$

$$+ C \left( \frac{1}{\varepsilon \mu} + 1 \right) (h^2 (\| e_h^n \|^2 + \| e_h^{n-1} \|^2) + ((\| e_h^{n-1} \|^2 + h^2) \| e_h^n \|^2_D)$$

$$+ C \varepsilon \mu \| e_h^n \|^2 + C \left( \frac{1}{\varepsilon \mu} + 1 \right) \| U_h^n \|^2_D. \quad (125)$$
We use the bounds above in (105), use the coercivity property (31), and choose $\varepsilon = 1/24$. We sum the resulting equation, from $n = 1$ to $n = m$, use the regularity assumptions, and obtain the following.

$$
\frac{1}{2} a_\varphi(U^n_h, U^n_h) - \frac{1}{2} a_\varphi(U^0_h, U^0_h) + \frac{1}{4} \sum_{n=1}^m \|U^n_h - U^{n-1}_h\|_{DG}^2 + \frac{\tau \mu}{2} \sum_{n=1}^m \|e^n_h\|^2
\leq C \left( 1 + \frac{1}{\mu} + \mu \right) h^{2k+2} + C \tau^2 + C \tau \left( \frac{h^2}{\mu} + 1 + \mu \right) |\phi^n_0|_{DG}^2
+ \frac{\tau}{12} \sum_{n=1}^m \|e^n_h - e^{n-1}_h\|^2 + C \tau \left( \frac{1}{\mu} + 1 \right) \sum_{n=1}^m \|U^n_0\|_{DG}^2
+ C \left( \frac{1}{\mu} + 1 \right) \tau \sum_{n=1}^m (h^2(\|e^n_0\|^2 + \|e^{n-1}_0\|^2) + (\|e^n_0\|^2 + h^2)(\|e^n_0\|_{DG}^2)).
$$

(126)

Note that by (77)-(78) and by definition of the $L^2$ projection, we have

$$
a_\varphi(U^0_h, U^0_h) = (\mathbf{x}^0, U^0_h) = 0.
$$

(127)

Hence, $U^0_h = 0$ since $a_\varphi$ is coercive. With Lemma 4, coercivity of $a_\varphi$ (31), we have

$$
\|U^m_h\|_{DG}^2 + \sum_{n=1}^m \|U^n_h - U^{n-1}_h\|_{DG}^2 + 2 \tau \mu \sum_{n=1}^m \|e^n_h\|^2 \leq C \left( 1 + \frac{1}{\mu} + \mu \right) h^{2k+2} + C \tau^2
+ C \mu (\tau + h^2) \sum_{i=-4}^2 \mu^i + C \tau \left( \frac{1}{\mu} + 1 \right) \sum_{n=1}^m \|U^n_0\|_{DG}^2.
$$

Under the assumption that $C(1/\mu + 1) \tau < 1$, we use Gronwall's inequality and obtain

$$
\tau \mu \sum_{n=1}^m \|e^n_h\|^2 \leq C \mu (\tau^2 + \tau h^2 + h^{2k+2}) \sum_{i=-4}^2 \mu^i.
$$

(128)

Since $\tau h^2 \leq (\tau^2 + h^4)/2$, the result is obtained if $k = 1$. If $k \geq 2$, we use the assumption $h^2 \leq \tau$. To obtain a bound on $e^n_h$, we use (53), (36) and (67).

$$
\tau \mu \sum_{n=1}^m \|e^n_h\|^2 \leq 2 \tau \mu \sum_{n=1}^m \|e^n_h - e^n_0\|^2 + 2 \tau \mu \sum_{n=1}^m \|e^n_0\|^2 \leq C \mu \tau \sum_{n=1}^m \|\phi^n_0\|_{DG}^2
+ 2 \tau \mu \sum_{n=1}^m \|e^n_0\|^2 \leq C \mu (\tau^2 + \tau h^2 + h^{2k+2}) \sum_{i=-4}^2 \mu^i.
$$

(129)

For $k \geq 2$, we use the assumption $h^2 \leq \tau$. The result follows by the triangle inequality.
4 Stability and error estimates for the discrete time derivative of the velocity

In this section, we establish estimates for the discrete time derivative of the velocity. For a given function $W \in X$, define the discrete time derivative as

$$\delta_t W^{n+1} = \frac{W^{n+1} - W^n}{\tau}, \quad n \geq 0. \quad (130)$$

The same notation is used for the discrete time derivative of a scalar function in $C([0, T], M)$.

**Lemma 6** Assume $\partial_t u \in L^2(0, T; H^2(\Omega)^d)$ and $\partial_t p \in L^2(0, T; H^1(\Omega))$. Fix $0 < \gamma \leq 1$ and assume that there exist positive constants $c_1, c_2$ such that $\tau$ satisfies

$$c_1 h^2 \leq \tau \leq c_2 h^{(1+\gamma)d/3}. \quad (131)$$

Under the same assumptions of Theorem 1, we have the following stability bound. For $1 \leq m \leq N_T - 1$,

$$\|\delta_t e_h^{m+1}\|^2 + \sum_{n=1}^m \|\delta_t e_h^n - \delta_t e_h^n\|^2 + \tau^2 \sum_{n=1}^m \|\delta_t \phi_h^{n+1}\|^2_{DG} + \mu \tau \sum_{n=1}^m \|\delta_t e_h^{n+1}\|^2_{DG} \leq C_{\gamma, \mu}. \quad (132)$$

The constant $C_{\gamma, \mu}$ is independent of $h$ and $\tau$ but depends nonlinearly on $\gamma$ and $\mu$:

$$C_{\gamma, \mu} = C\mu \hat{K}_\mu + (C\mu \hat{K}_\mu)^{\beta/2} \mu^{-\beta/2},$$

where $\beta$ is an integer such that $\beta \geq 1/\gamma$, $C\mu$ depends on $e_h^\beta$ and $\hat{K}_\mu = \sum_{j=-3}^\infty \mu^j$.

**Proof** Subtracting (52) at time $t^n$ from (52) at time $t^{n+1}$, we obtain for all $\theta_h \in X_h$,

$$(\delta_t e_h^{n+1} - \delta_t e_h^n, \theta_h) + \tau \mu a_{DG}(\delta_t e_h^{n+1}, \theta_h) = \tau b(\theta_h, \delta_t e_h^{n+1} - \delta_t e_h^n)$$

$$- \tau \mu a_{DG}(\delta_t \Pi_h u^{n+1} - \delta_t u^{n+1}, \theta_h) + A_1(\theta_h) + A_2(\theta_h) + \tilde{R}(\theta_h). \quad (132)$$

Here $A_1(\theta_h), A_2(\theta_h)$ and $\tilde{R}(\theta_h)$ are defined as follows. For $\theta_h \in X_h$,

$$A_1(\theta_h) = a_{DG}(u^{n+1}; u^{n+1}, \theta_h) - a_{DG}(u^n_h; u^n_h, \theta_h),$$

$$A_2(\theta_h) = a_{DG}(u^n; u^n, \theta_h) - a_{DG}(u^n; u^n, \theta_h),$$

$$\tilde{R}(\theta_h) = (\partial_t u)^n - (\partial_t u)^n, \theta_h) - \frac{1}{\tau} (\Pi_h u^{n+1} - 2 \Pi_h u^n + \Pi_h u^{n-1}, \theta_h).$$

Choosing $\theta_h = \delta_t e_h^{n+1}$ in (132) and using the coercivity of $a_{DG}$ (31) yields

$$\frac{1}{2} (\|\delta_t e_h^{n+1}\|^2 - \|\delta_t e_h^n\|^2 + \|\delta_t e_h^{n+1} - \delta_t e_h^n\|^2) + \frac{1}{2} \|\delta_t e_h^{n+1}\|^2_{DG} \leq \tau b(\delta_t e_h^{n+1}, \delta_t e_h^n) - \tau \mu a_{DG}(\delta_t \Pi_h u^{n+1} - \delta_t u^{n+1}, \delta_t e_h^{n+1})$$

$$+ A_1(\delta_t e_h^{n+1}) + A_2(\delta_t e_h^{n+1}) + \tilde{R}(\delta_t e_h^{n+1}). \quad (133)$$

From (53), we have for $n \geq 1$:

$$(\delta_t e_h^{n+1} - \delta_t e_h^n, \theta_h) = \tau b(\theta_h, \delta_t \phi_h^{n+1}), \quad \forall \theta_h \in X_h. \quad (134)$$
In addition, from (56), and (57), we obtain \( \forall q_h \in M_h \) and \( n \geq 1 \)

\[
b(\delta_r e_h^{n+1}, q_h) = b(\delta_r e_h^{n+1}, q_h) + \tau a_{\text{clip}}(\delta_r \phi_h^{n+1}, q_h) - \tau \sum_{e \in I_h} \frac{\tilde{\sigma}}{h_e} \int_e [\delta_r \phi_h^{n+1}] [q_h] \\
+ \tau (G_h([\delta_r \phi_h^{n+1}]), G_h([q_h]))
\]

\[
= - \tau \sum_{e \in I_h} \frac{\tilde{\sigma}}{h_e} \int_e [\delta_r \phi_h^{n+1}] [q_h] + \tau (G_h([\delta_r \phi_h^{n+1}]), G_h([q_h])).
\]  

(135)

From the above properties and (134), observe that

\[
\| \delta_r e_h^{n+1} - \delta_r e_h^n \|^2 = \tau b(\delta_r e_h^{n+1} - \delta_r e_h^n, \delta_r \phi_h^{n+1}) = \tau^2 a_{\text{clip}}(\delta_r \phi_h^{n+1}, \delta_r \phi_h^{n+1}) \\
- \tau^2 \sum_{e \in I_h} \frac{\tilde{\sigma}}{h_e} \|[\delta_r \phi_h^{n+1}]\|_{L^2(e)}^2 + \tau^2 \|G_h([\delta_r \phi_h^{n+1}])\|^2.
\]  

(136)

Hence, by letting \( \theta_h = \delta_r e_h^{n+1} \) in (134) and using (135), we obtain

\[
\frac{1}{2} \left( \| \delta_r e_h^{n+1} \|^2 - \| \delta_r e_h^n \|^2 \right) + \frac{\tau^2}{2} a_{\text{clip}}(\delta_r \phi_h^{n+1}, \delta_r \phi_h^{n+1}) \\
+ \frac{\tau^2}{2} \sum_{e \in I_h} \frac{\tilde{\sigma}}{h_e} \|[\delta_r \phi_h^{n+1}]\|_{L^2(e)}^2 = \frac{\tau^2}{2} \|G_h([\delta_r \phi_h^{n+1}])\|^2.
\]  

(137)

In addition, note that from (134) and (38), we have

\[
\delta_r e_h^{n+1} - \delta_r e_h^n = - \tau \nabla_h \delta_r \phi_h^{n+1} + \tau G_h(\delta_r \phi_h^{n+1}).
\]

This implies that

\[
\| \delta_r e_h^{n+1} - \delta_r e_h^n \|^2 = \| \tau \nabla_h \delta_r \phi_h^{n+1} - \tau G_h(\delta_r \phi_h^{n+1}) + \delta_r e_h^{n+1} - \delta_r e_h^n \|^2.
\]

Expanding the norm and using (38) with \( \theta_h = \delta_r e_h^{n+1} - \delta_r e_h^n \) and \( q_h = \delta_r \phi_h^{n+1} \) yields

\[
\| \delta_r e_h^{n+1} - \delta_r e_h^n \|^2 = \| \delta_r e_h^{n+1} - \delta_r e_h^n \|^2 + 2 \tau \| \nabla_h \delta_r \phi_h^{n+1} \|^2 - 2 \tau \| G_h(\delta_r \phi_h^{n+1}) \|^2
\]

\[
- 2 \tau b(\delta_r e_h^{n+1} - \delta_r e_h^n, \delta_r \phi_h^{n+1}) - 2 \tau^2 \| \nabla_h \delta_r \phi_h^{n+1} \|^2 - 2 \tau^2 \| G_h(\delta_r \phi_h^{n+1}) \|^2.
\]  

(138)

With (135) and (57), we find (see Lemma 8 in Appendix)

\[
- 2 b(\delta_r e_h^{n+1} - \delta_r e_h^n, \delta_r \phi_h^{n+1}) = \tau (\Delta_1^n - \Delta_1^0) + \tau \sum_{e \in I_h} \frac{\tilde{\sigma}}{h_e} \|[\delta_r \phi_h^{n+1} - \delta_r \phi_h^n]\|_{L^2(e)}^2 \\
- \tau \|G_h([\delta_r \phi_h^{n+1} - \delta_r \phi_h^n])\|^2 - \frac{2}{\tau} \Delta_1 b(e_h^n, \delta_r \phi_h^n),
\]

(139)

where for \( n \geq 1 \)

\[
\Delta_1^n = \sum_{e \in I_h} \frac{\tilde{\sigma}}{h_e} \left( \|[\delta_r \phi_h^{n+1}]\|_{L^2(e)}^2 - \|[\delta_r \phi_h^n]\|_{L^2(e)}^2 \right),
\]

\[
\Delta_2^n = \|G_h([\delta_r \phi_h^{n+1}])\|^2 - \|G_h([\delta_r \phi_h^n])\|^2.
\]
Using (36) and assuming $\sigma \geq 4M_h^2$, we have

$$
\sum_{c \in F_h} \frac{\sigma}{h_c} \| | \delta_t \phi^{n+1}_h - \phi^n_h | |_{L^2(c)}^2 - \| \mathbf{G}_h( | \delta_t \phi^{n+1}_h - \phi^n_h | ) \|_{L^2(c)}^2 \\
\geq \frac{1}{2} \sum_{c \in F_h} \frac{\sigma}{h_c} \| | \delta_t \phi^{n+1}_h - \phi^n_h | |_{L^2(c)}^2,
$$

\begin{align*}
| | (\nabla_h \delta_t \phi^{n+1}_h, \mathbf{G}_h( | \delta_t \phi^{n+1}_h | ) ) | & \leq \frac{1}{4} \| \nabla_h \delta_t \phi^{n+1}_h \|_{L^2(c)}^2 + \frac{1}{2} \sum_{c \in F_h} \frac{\sigma}{h_c} \| | \delta_t \phi^{n+1}_h | |_{L^2(c)}^2.
\end{align*}

With the above expressions and (137), (133) becomes:

$$
\frac{1}{2} ( \| \delta_t e^{n+1}_h \|_{L^2(c)}^2 - \| \delta_t e^n_h \|_{L^2(c)}^2 + \| \delta_t e^{n+1}_h - \delta_t e^n_h \|_{L^2(c)}^2 ) + \frac{1}{2} \tau \mu \| \delta_t e^{n+1}_h \|_{L^2(c)}^2 + \frac{1}{4} \| \delta_t \phi^{n+1}_h \|_{DG}^2 + \frac{1}{2} \tau^2 \| \delta_t \phi^{n+1}_h \|_{DG}^2 \\
+ \frac{\tau^2}{4} a_{ellip}( | \delta_t \phi^{n+1}_h, \delta_t \phi^n_h | ) + \frac{\tau^2}{4} \sum_{c \in F_h} \frac{\sigma}{h_c} \| | \delta_t \phi^{n+1}_h - \phi^n_h | |_{L^2(c)}^2 + \frac{\tau^2}{2} ( \hat{A}_{h1} - \hat{A}_{h2} ) \\
\leq \tau b( | \delta_t e^{n+1}_h, \delta_t e^n_h - \delta_t e^{n+1}_h | ) - \mu a_{ellip}( | \delta_t \Pi_h u^{n+1}_h - \delta_t u^{n+1}_h, \delta_t e^{n+1}_h | \\
+ \lambda_1 ( | \delta_t e^{n+1}_h | + \lambda_2 ( | \delta_t e^{n+1}_h | + \hat{R}_h( | \delta_t e^{n+1}_h | + \delta_h b( | e^n_h, \delta_t \phi^n_h ) | ) ) ) (140)
$$

We begin by handling the first two terms on the right hand side of (140). We write:

$$
b( | \delta_t e^{n+1}_h, \delta_t e^n_h - \delta_t e^{n+1}_h | ) = b( | \delta_t e^{n+1}_h, \delta_t e^n_h | ) - b( | \delta_t e^{n+1}_h, \pi_h( | \delta_t e^{n+1}_h | ) | ) \\
+ b( | \delta_t e^{n+1}_h, \pi_h( | \delta_t e^{n+1}_h | ) | ) - b( | \delta_t e^{n+1}_h, \delta_t e^{n+1}_h | )
$$

Since $\delta_t e^{n+1}_h$ has a zero average and $\pi_h$ preserves cell averages, $\pi_h( | \delta_t e^{n+1}_h | )$ belongs to $M_0$. Hence, by (54) and the stability of the $L^2$ projection (47), we obtain

$$
| b( | \delta_t e^{n+1}_h, \pi_h( | \delta_t e^{n+1}_h | ) | ) | = | \tau a_{ellip}( | \delta_t \phi^{n+1}_h, \pi_h( | \delta_t e^{n+1}_h | ) | ) |
\leq C \tau | \delta_t \phi^{n+1}_h |_{DG} | \delta_t e^{n+1}_h |_{H^1(\Omega)}.
$$

Using Young’s inequality and Taylor’s theorem, we obtain

$$
| b( | \delta_t e^{n+1}_h, \pi_h( | \delta_t e^{n+1}_h | ) | ) | \leq \frac{\tau}{16} | \delta_t \phi^{n+1}_h |_{DG}^2 + C \int_{\Omega}^{n+1} | \partial_t p |_{H^1(\Omega)}^2.
$$

By (19), the definition of $\pi_h( | \delta_t e^{n+1}_h | )$, a trace inequality, and (46), we have:

$$
| b( | \delta_t e^{n+1}_h, \pi_h( | \delta_t e^{n+1}_h | ) - \delta_t e^{n+1}_h | ) |
= | \sum_{c \in F_h \cup \partial \Omega} \int_{c} | \pi_h( | \delta_t e^{n+1}_h | - \delta_t e^{n+1}_h | ) | | n_c |
\leq C \| | \delta_t e^{n+1}_h | |_{DG} \| | \delta_t \phi^{n+1}_h | |_{DG} + C \frac{1}{\epsilon \mu} \pi \tau | \partial_t p |_{H^1(\Omega)}^2.
$$
To handle \( b(\delta, \hat{e}^{n+1}, \delta_c p^n) \), we introduce the auxiliary functions:

\[
\hat{S}_h^n = \delta \mu \sum_{i=1}^n (\nabla_h \cdot \delta_c \hat{e}_h^i - R_h(\delta_c \hat{e}_h^i)), \quad \hat{\xi}_h^n = \delta_c p^n + \hat{S}_h^n, \quad n \geq 1.
\]

From (55) and the above definitions, it is implied that \( \hat{e}_h^{n+1} - \hat{\xi}_h^n = \delta_c \phi_h^{n+1} \). With this expression, we write:

\[
b(\delta_c \hat{e}_h^{n+1}, \delta_c p^n) = b(\delta_c \hat{e}_h^{n+1}, \hat{\xi}_h^n) - b(\delta_c \phi_h^{n+1}, \hat{\xi}_h^n).
\]

Observe that \( \hat{\xi}_h^n \in M_{h0} \) since \( \hat{S}_h^n \in M_{h0} \) for \( n \geq 1 \). Hence, we use (54) to deduce that

\[
b(\delta_c \hat{e}_h^{n+1}, \hat{\xi}_h^n) = -\tau a_{\text{clip}}(\delta_c \phi_h^{n+1}, \hat{\xi}_h^n) = -\tau a_{\text{clip}}(\hat{e}_h^{n+1}, \hat{\xi}_h^n)
\]

\[
\quad = -\frac{\tau}{2} \left( a_{\text{clip}}(\hat{e}_h^{n+1}, \hat{\xi}_h^n) - a_{\text{clip}}(\hat{e}_h^n, \hat{\xi}_h^n) - a_{\text{clip}}(\delta_c \phi_h^{n+1}, \delta_c \phi_h^{n+1}) \right).
\]

With (37), we have

\[
b(\delta_c \hat{e}_h^{n+1}, \hat{\xi}_h^n) = \frac{1}{\delta \mu} (\hat{S}_h^{n+1} - \hat{\xi}_h^n, \hat{\xi}_h^n) = \frac{1}{2\delta \mu} (\|\hat{S}_h^{n+1}\|^2 - \|\hat{\xi}_h^n\|^2 - \|\hat{\xi}_h^{n+1} - \hat{\xi}_h^n\|^2).
\]

With the assumption that \( \delta \leq 1/(4d) \) and \( \sigma \geq M_{h^{-1}}^2 / d \), we use (35) and the bound \( \|\nabla_h \cdot \theta\| \leq d^{1/2} \|\nabla_h \theta\| \) for \( \theta \in X \). We obtain

\[
\frac{1}{2\delta \mu} \|\hat{\xi}_h^{n+1} - \hat{\xi}_h^n\|^2 \leq \frac{\mu}{4} \|\nabla_h \delta_c \hat{e}_h^{n+1}\|^2 + \frac{\mu}{4} \sum_{e \in \mathcal{E}\cap \partial \Omega} \sigma h_e^{-1} \|\delta_c \hat{e}_h^{n+1}\|^2_{L^2(e)}.
\]

(141)

Following a similar technique as the one used in [9] and in [17], we also have

\[
|a_{\phi}(\delta_c \Pi_h u^{n+1} - \delta_c u^{n+1}, \delta_c \hat{e}_h^{n+1})| = |a_{\phi}(\Pi_h \delta_c u^{n+1} - \delta_c u^{n+1}, \delta_c \hat{e}_h^{n+1})| \\
\leq C h \|\delta_c u^{n+1}\|_{H^2(\Omega)} \|\delta_c \hat{e}_h^{n+1}\|_{DG} \leq C \|\delta_c \hat{e}_h^{n+1}\|_{DG}^2 + \frac{C}{h} \tau h^{-1} \int_{\Omega} |\partial_\theta u^2|_{H^2(\Omega)}.
\]

With the above expressions and bounds, (140) reads:

\[
\frac{1}{2} \left( \|\delta_c \hat{e}_h^{n+1}\|^2 + \|\delta_c \phi_h^{n+1} - \delta_c \phi_h^{n}\|^2 \right) + \frac{\tau}{4} \|\delta_c \phi_h^{n+1}\|^2_{DG} \\
+ \frac{3}{2} \left( a_{\text{clip}}(\hat{e}_h^{n+1}, \hat{\xi}_h^n) - a_{\text{clip}}(\hat{\xi}_h^n, \hat{\xi}_h^n) - a_{\text{clip}}(\delta_c \phi_h^{n+1}, \delta_c \phi_h^{n+1}) \right) \\
+ \frac{3}{2} \left( \delta \mu \sqrt{h} \|\delta_c \phi_h^{n+1} - \delta_c \phi_h^n\|_{L^2(e)} + \frac{\tau}{2} (\delta_\ell - \delta_\xi) + \frac{\tau}{2 \delta \mu} (\|\hat{S}_h^{n+1}\|^2 - \|\hat{\xi}_h^n\|^2) \right) \\
\leq 2 \epsilon \tau h \|\delta_c \hat{e}_h^{n+1}\|_{DG}^2 + C \frac{h^2}{\epsilon} \int_{\Omega} |\hat{e}_h^{n+1}|^2_{H^2(\Omega)} + C \left( \tau + \frac{h^2}{\epsilon \mu} \right) \int_{\Omega} |\partial_\theta u^2|_{H^2(\Omega)} \\
+ M_1(\delta_c \hat{e}_h^{n+1}) + M_2(\delta_c \hat{e}_h^{n+1}) + R(\delta_c \hat{e}_h^{n+1}) + \delta_{h1} b(e^i, \delta_c \phi_h^n) + \delta_{h2} b(e^i, \delta_c \phi_h^n).
\]

(142)
We proceed by noting the following splitting on the nonlinear terms.

\[
M_1(\theta_h) = a\phi(u^{n+1}, u^{n+1} - u^n, \theta_h) + a\phi(u^n_h, u^n - u^n_h, \theta_h)
+ a\phi(u^n_h, u^{n+1} - v^{n+1}_h, \theta_h) = \xi_1^n(\theta_h) + \xi_2^n(\theta_h) + \xi_3^n(\theta_h),
\]

\[
M_2(\theta_h) = -a\phi(u^n; u^n - u^{n-1}, \theta_h) - a\phi(u^{n-1}, u^{n-1} - u^n, \theta_h)
- a\phi(u^{n-1}_h; u^n - v^n_h, \theta_h) = \vartheta_1^n(\theta_h) + \vartheta_2^n(\theta_h) + \vartheta_3^n(\theta_h).
\]

(143)

To further simplify the writeup, we write:

\[
M_1(\delta_t \tilde{e}_h^{n+1}) + M_2(\delta_t \tilde{e}_h^{n+1}) = \sum_{i=1}^3 (\xi_i^n(\delta_t \tilde{e}_h^{n+1}) + \vartheta_i^n(\delta_t \tilde{e}_h^{n+1})) = \sum_{i=1}^3 Q_i^n.
\]

The term \(Q_i^n\) can be handled as follows.

\[
Q_i^n = \sum_{\tau \in \mathcal{T}} \int_{\tau} ((u^{n+1} - u^n) \cdot \nabla u^{n+1} - (u^n - u^{n-1}) \cdot \nabla u^n) \cdot \delta_t \tilde{e}_h^{n+1}.
\]

With Hölder’s inequality, (48), and the assumption that \(u \in L^\infty(0; \; T; H^{k+1}(\Omega)^d)\):

\[
|Q_i^n| \leq C(\|u^{n+1} - u^n\|_w^{H^1(\Omega)} + \|u^n - u^{n-1}\|_w^{H^1(\Omega)}) \|\delta_t \tilde{e}_h^{n+1}\|_{\text{DG}}
\leq \varepsilon \tau \mu \|\delta_t \tilde{e}_h^{n+1}\|_{\text{DG}}^2 + \frac{C}{\varepsilon \mu} \int_{t_{n-1}}^{t_n} \|\delta_t \tilde{u}\|^2.
\]

Since the upwind terms in \(Q_2^n\) vanish, we write

\[
Q_2^n = \tau \delta_t(u^n - \Pi_h u^n, \delta_t \tilde{e}_h^{n+1}, \delta_t \tilde{e}_h^{n+1}) - \tau \delta_t(e^n_h, \delta_t \tilde{e}_h^{n+1}, \delta_t \tilde{e}_h^{n+1}) + \tau \delta_t(\delta_t(u^n - \Pi_h u^n), u^n, \delta_t \tilde{e}_h^{n+1}) - \tau \delta_t(e^n_h, u^n, \delta_t \tilde{e}_h^{n+1}).
\]

(145)

The first and third terms are bounded by (63). The second and fourth terms are bounded by (58).

\[
|Q_2^n| \leq C \tau \delta_t \|u^n - \Pi_h u^n\|_{H^1(\Omega)} \|\delta_t \tilde{e}_h^{n+1}\|_{\text{DG}}
+ C \tau \|\delta_t e^n_h\| \|\tilde{u}\| \|\delta_t \tilde{e}_h^{n+1}\|_{\text{DG}} + C h^2 \sqrt{T} \left( \int_{t_{n-1}}^{t_n} \|\tilde{u}\|_{H^2(\Omega)}^2 \right)^{1/2} \|u^n\| \|\tilde{e}_h^{n+1}\|_{\text{DG}}.
\]

With the assumption that \(\delta_t \tilde{u} \in L^2(0; \; T; H^2(\Omega)^d)\), a Sobolev embedding result and a Taylor expansion, we have

\[
\|\delta_t \tilde{u}\|^2 \leq C \tau^{-2} \|u^n - u^{n-1}\|_{H^2(\Omega)}^2 \leq C \tau^{-1} \int_{t_{n-1}}^{t_n} \|\delta_t \tilde{u}\|^2_{H^2(\Omega)}, \quad n \geq 1.
\]

(146)

With the assumption that \(u \in L^\infty(0; \; T; H^{k+1}(\Omega)^d)\), (146), (131), and Young’s inequality, we obtain

\[
|Q_2^n| \leq \varepsilon \tau \mu \|\delta_t \tilde{e}_h^{n+1}\|_{\text{DG}} + \frac{C}{\varepsilon \mu} \left(\tau^2 + h^{2k+2} + \|e^n_h\|^2\right) \int_{t_{n-1}}^{t_n} \|\delta_t \tilde{u}\|^2_{H^2(\Omega)}
+ \frac{C}{\varepsilon \mu} \tau^{-1} \|e^n_h - e^{n-1}_h\|^2.
\]

(147)
We write $Q^3_n$ as follows.

\[ Q^3_n = a \phi(u_h^n; u_h^{n+1} - \Pi_h u^{n+1}, \delta \varepsilon_h^{n+1}) - a \phi(u_h^n; \delta \varepsilon_h^{n+1}) \]

\[ - a \phi(u_h^{n-1}; u_h^{n-1} - \Pi_h u^n, \delta \varepsilon_h^{n+1}) + a \phi(u_h^{n-1}; u_h^{n-1}, \delta \varepsilon_h^{n+1}) = \sum_{i=1}^{4} \eta_i^n. \quad (148) \]

To handle $\eta^n_i$, we use the forms $\mathcal{W}$ and $\mathcal{C}$ given in (15)-(16). Using (17), we have:

\[ \eta^n_1 + \eta^n_2 = \tau a \phi(u_h^{n-1}; u_h^{n-1}, \delta \varepsilon_h^{n+1}) + \tau \mathcal{W}(u_h^{n-1}; u_h^{n+1}, \delta \varepsilon_h^{n+1}) + U^n_1 \]

\[ + \tau \mathcal{C}(\delta \varepsilon_h^{n+1} - \Pi_h u^{n+1}, \delta \varepsilon_h^{n+1}) - \tau \mathcal{W}(u_h^{n-1}; u_h^{n+1}, \delta \varepsilon_h^{n+1}), \]

where

\[ U^n_1 = \mathcal{W}(u_h^{n-1}; u_h^{n+1} - \Pi_h u^{n+1}, \delta \varepsilon_h^{n+1}) - \mathcal{W}(u_h^{n-1}; u_h^{n+1} - \Pi_h u^{n+1}, \delta \varepsilon_h^{n+1}). \]

The term $U^n_1$ is bounded by Lemma 9 in the Appendix, by the inverse inequality (68) and by the approximation property (43).

\[ |U^n_1| \leq C \tau \| \delta \varepsilon_h^{n+1} \|_{H^1(\Omega)} \| \delta \varepsilon_h^{n+1} \|_{DG} \]

\[ \leq C \tau \| \delta \varepsilon_h^{n+1} \|_{H^1(\Omega)} \| \delta \varepsilon_h^{n+1} \|_{DG}. \]

We use (16), (59) and (65) to bound the remaining terms in $\eta^n_1 + \eta^n_2$.

\[ |\eta^n_1 + \eta^n_2| \leq C \left( \| u_h^{n-1} \|_{H^2(\Omega)} + \| \delta \varepsilon_h^{n+1} \|_{H^1(\Omega)} \right) \| \delta \varepsilon_h^{n+1} \|_{DG}. \]

Further, with the triangle inequality, (42), and a Taylor expansion, we have:

\[ \| \delta \varepsilon_h^{n+1} \|^2 \leq 2 \| \delta \varepsilon_h^{n+1} \|^2 + 4 \| \delta \varepsilon_h^{n+1} \|^2 + 4 \| \delta \varepsilon_h^{n-1} \|^2 \]

\[ \leq 2 \tau \| \delta \varepsilon_h^{n-1} \|^2 + C \tau \| \delta \varepsilon_h^{n+1} \|_{DG} + C \tau \| \delta \varepsilon_h^{n+1} \|^2. \quad (149) \]

Hence, (146), (149), Lemma 4, Young’s inequality, and the assumption that $u \in L^\infty(0, T; H^{k+1}(\Omega)^d)$ yield the following bound.

\[ |\eta^n_1 + \eta^n_2| \leq \varepsilon \tau \| \delta \varepsilon_h^{n+1} \|_{DG} + \frac{C}{\varepsilon \mu \tau} \| \delta \varepsilon_h^{n-1} \|^2 \]

\[ + \frac{C}{\varepsilon \mu} \left( 1 + \frac{1}{\mu} \right) \int_{t_{n-1}}^{t_n} \| \delta \varepsilon_h^{n+1} \|_{DG}^2. \]

Similarly, we use (17) and write:

\[ \eta^n_3 + \eta^n_4 = - \tau a \phi(u_h^{n-1}; u_h^{n-1}, \delta \varepsilon_h^{n+1}) + U^n_2, \quad (150) \]

with

\[ U^n_2 = \mathcal{W}(u_h^{n+1}; u_h^{n+1}, \delta \varepsilon_h^{n+1}) - \mathcal{W}(u_h^{n-1}; u_h^{n+1}, \delta \varepsilon_h^{n+1}) \]

\[ - \mathcal{C}(u_h^{n-1}; u_h^{n-1}, \delta \varepsilon_h^{n+1}) + \mathcal{W}(u_h^{n-1}; u_h^{n-1}, \delta \varepsilon_h^{n+1}) + \mathcal{W}(u_h^{n-1}; u_h^{n+1}, \delta \varepsilon_h^{n+1}). \]
By the positivity property of \( \alpha_{\ell} \) (14), the first term in the right-hand side of (150) is non-positive. The first two terms of \( U_h^n \) are bounded by Lemma 9 and the last two terms by Lemma 10.

\[
|U_h^n| \leq C |u_h^n - u_h^{n-1}|_{L^2(\Omega)} \|\varphi_{h+1}^{n+1}\|_{DG} \|\partial_t \varphi_{h+1}^{n+1}\|_{DG}.
\]

Note that, with a triangle inequality, (48), and the stability of the interpolant (43),

\[
|u_h^n - u_h^{n-1}|_{L^2(\Omega)} \leq 2 \tau^2 \|\delta_t \varphi_{h+1}^{n+1}\|_{L^2(\Omega)} + C \tau \int_{t_{n-1}}^{t_n} |\partial_t u_h|^2_{H^1(\Omega)}.
\]

With Young's inequality and the above bounds, we attain

\[
|U_h^n| \leq \varepsilon \tau \mu \|\partial_t \varphi_{h+1}^{n+1}\|_{DG} + \frac{C}{\varepsilon \mu} \tau \|\partial_t \varphi_{h+1}^{n+1}\|_{L^2(\Omega)} \|\varphi_{h+1}^{n+1}\|_{DG} \tau \int_{t_{n-1}}^{t_n} |\partial_t u_h|^2_{H^1(\Omega)}.
\]

\[
(151)
\]

Therefore, we obtain the following bound on the nonlinear terms:

\[
\mathcal{N}_1(\partial_t \varphi_{h+1}^{n+1}) + \mathcal{N}_2(\partial_t \varphi_{h+1}^{n+1}) \leq 4 \varepsilon \tau \mu \|\partial_t \varphi_{h+1}^{n+1}\|_{DG} + \frac{C}{\varepsilon \mu} \tau \|\partial_t \varphi_{h+1}^{n+1}\|_{L^2(\Omega)} \|\varphi_{h+1}^{n+1}\|_{DG} \tau \int_{t_{n-1}}^{t_n} |\partial_t u_h|^2_{H^1(\Omega)}.
\]

\[
(152)
\]

We now handle \( \mathcal{R}_i(\partial_t \varphi_{h+1}^{n+1}) \). For details, we refer to Lemma 11 in the Appendix. With Taylor expansions, approximation properties (42) - (43), and (131), we have for \( n \geq 1 \):

\[
\mathcal{R}_i(\partial_t \varphi_{h+1}^{n+1}) \leq \varepsilon \tau \mu \|\partial_t \varphi_{h+1}^{n+1}\|_{DG} + \frac{C}{\varepsilon \mu} \int_{t_{n-1}}^{t_n} |\partial_t u_h|^2 + \frac{C}{\varepsilon \mu} \int_{t_{n-1}}^{t_n} |\partial_t u_h|^2_{H^1(\Omega)}.
\]

\[
(153)
\]

With the above bounds and with \( \varepsilon = 1/56 \), (142) reads:

\[
\frac{1}{2} \left( \|\partial_t \varphi_{h+1}^{n+1}\|^2 - \|\partial_t \varphi_{h+1}^{n-1}\|^2 + \|\partial_t \varphi_{h+1}^{n+1} - \partial_t \varphi_{h+1}^{n-1}\|^2 \right) + \frac{\tau \mu}{8} \|\partial_t \varphi_{h+1}^{n+1}\|_{DG} + \frac{\tau^2}{16} |\partial_t \varphi_{h+1}^{n+1}|_{DG}^2
\]

\[
+ \left( \frac{a_{\text{slip}}(\varphi_{h+1}^{n+1} - \varphi_{h+1}^{n-1}) - a_{\text{slip}}(\varphi_{h}^{n+1} - \varphi_{h}^{n-1})}{2} \right) + \frac{\tau^2}{2} (A_1 - A_2) + \frac{\tau}{2 \delta \mu} (|S_{h+1}^n|^2 - |S_{h}^n|^2)
\]

\[
\leq \frac{C}{\mu} \int_{t_{n-1}}^{t_n} |\partial_t u_h|^2 + C(\tau + h^2 \mu^{-1}) \int_{t_{n-1}}^{t_n} |\partial_t p|_{H^1(\Omega)}^2 + \frac{C}{\mu} \|\partial_t \varphi_{h+1}^{n+1}\|_{L^2(\Omega)} \|\varphi_{h+1}^{n+1}\|_{DG}^2
\]

\[
+ \frac{C}{\mu} (1 + \mu^{-1} + \|\varphi_{h+1}^{n+1}\|^2 + \|\varphi_{h}^{n+1}\|^2_{DG}) \int_{t_{n-1}}^{t_n} |\partial_t u_h|^2_{H^1(\Omega)}
\]

\[
+ \frac{C}{\mu} \tau^{-1} |\varphi_{h}^{n+1} - \varphi_{h-1}^{n-1}|^2 + \hat{\delta}_{\Omega} [b(\varphi_{h+1}^{n+1}, \partial_t \varphi_{h+1}^{n+1})].
\]

\[
(154)
\]
Note the following
\[
\frac{\tau^2}{2} \sum_{n=1}^{m} (\tilde{A}_n^2 - \tilde{A}_n^2) = \frac{\tau^2}{2} \left( \sum_{e \in I_h} h_e^2 \left\| [\partial_{\tau} \phi_b^{m+1}] \right\|^2 - \left\| G_h(\partial_{\tau} \phi_b^{m+1}) \right\|^2 \right) \\
+ \frac{\tau^2}{2} \left( \left\| G_h(\partial_{\tau} \phi_b^1) \right\|^2 - \sum_{e \in I_h} h_e^2 \left\| [\partial_{\tau} \phi_b^1] \right\|^2 \right). \quad (155)
\]

Using (36), the first term in the above expression is non-negative since \( \bar{\sigma} \geq \bar{M} \) and the second term is bounded in absolute value by \( C \tau^2 \| \partial_{\tau} \phi_b^1 \|_{DG} \). We sum (154) from \( n = 1 \) to \( n = m \) and use the inverse estimate (68). The continuity of \( \partial_{\tau} \), Lemma 4, (67), the lower bound on \( \bar{\mu} \), and (131), the regularity assumptions yield
\[
\| \partial_{\tau} \phi_b^{m+1} \|^2 + \sum_{n=1}^{m} \| \partial_{\tau} \phi_b^{n+1} - \partial_{\tau} \phi_b^n \|^2 + \frac{1}{8} \sum_{n=1}^{m} \tau^2 \| \partial_{\tau} \phi_b^{n+1} \|_{DG}^2 + \frac{\mu}{4} \sum_{n=1}^{m} \tau \| \partial_{\tau} \phi_b^{n+1} \|_{DG}^2 \\
\leq C_{\mu} \bar{K}_{\mu} + \| \partial_{\tau} \phi_b^1 \|^2 + \tau \| \phi_b^0 \|^2 + C \tau^2 \left( \| \phi_b^1 \|_{DG} + \| \partial_{\tau} \phi_b^1 \|_{DG} \right) \\
+ \frac{C}{\mu} \tau \sum_{n=1}^{m} h^{-d/3} \| \partial_{\tau} \phi_b^n \|_{DG}^2 \| \partial_{\tau} \phi_b^{n+1} \|_{DG}^2 + \| b(e_b^0, \partial_{\tau} \phi_b^2) \|. \quad (156)
\]

where \( \bar{K}_{\mu} = \sum_{i=1}^{3} \mu_i \). To handle the last term, we apply (39), the approximation properties, and (131),
\[
\| b(e_b^0, \partial_{\tau} \phi_b^2) \| \leq C \| \phi_b^0 \| \| \partial_{\tau} \phi_b^2 \|_{DG} \leq C \tau \| w^0 \|_{H^2(\Omega)} \| \partial_{\tau} \phi_b^2 \|_{DG} \leq C + \frac{\tau^2}{32} \| \partial_{\tau} \phi_b^2 \|_{DG}^2. \quad (157)
\]

We now derive a bound for the initial error \( \| e_b^1 - e_b^0 \| \) by following an argument in [17]. We start with the equation (6.89) in [17] and choose \( n = 1 \). All the terms in the right-hand side of (6.89) are bounded exactly in [17] except for the time error term \( R_t(\phi_b^1) \) defined in (6.15) or (51). Under enough regularity for the exact solution, namely \( \partial_{\tau} u \in L^2(0, T; H^2(\Omega)^d) \), we have
\[
\| R_t(\phi_b^1) \| \leq C \frac{\tau^2}{\mu} \int_{t_{n-1}}^{t_n} \| \partial_{\tau} u \|^2 + \frac{C}{\mu} h^4 \int_{t_{n-1}}^{t_n} \| \partial_{\tau} u \|_{H^2(\Omega)}^2 + \frac{\tau \mu}{32} \| e_b^0 \|_{DG}^2.
\]

This yields the following estimate for the initial error
\[
\frac{1}{2} \| e_b^1 - e_b^0 \|^2 + \frac{\tau^2}{16} \left( \| \phi_b^1 \|_{DG}^2 + \| p_b^1 \|_{DG}^2 + \| S_b^1 \|_{DG}^2 \right) + \frac{\tau}{2 \bar{\mu}} \| S_b^1 \|^2 + \| e_b^1 \|_{DG}^2 \\
\leq C \left( \frac{1}{\mu} + \mu + 1 \right) \tau^2 + \tau | b(e_b^0, \phi_b^1) | + \frac{1}{2} \tau | b(e_b^0, \Pi_h u^1 - e_b^1) |.
\]

where \( S_b^1 = \delta \mu (\nabla_{h} \cdot \nabla_{h} - R_h(\| v_{h}^1 \|)) \). Recall that by (40) and (37), we have for \( n \geq 0 \) :
\[
b(\Pi_h u^1, q_h) = (\nabla_{h} \cdot \Pi_h u^1 - R_h(\| v_{h}^1 \|), q_h) = 0, \forall q_h \in M_h
\]
Thus, $\tilde{S}_h^1 = \delta \mu (\nabla \tau \cdot \delta \tau \psi^1 + R_s(\delta \tau \psi^1))$. By recalling that $\psi^0 = \Pi_h u^0$ and by using the above equality, we obtain
$$
S_h^1 = \frac{1}{\tau} S_h^1, \quad \hat{S}_h^1 = \frac{1}{\tau} (p_h^1 + S_h^1).
$$

With (39) and approximation property (42), we have
$$
b(e_h^0, \delta \mu) \leq C\|e_h^0\|_\text{DG}^2 \leq \frac{\tau}{32} \|\phi_h^0\|_{\text{DG}}^2 + C \tau^{-1} h^{2k+2} \|u^0\|_{H^{k+1}(\Omega)}^2.
$$

We split the last term, and we use (58), (59), (42), and the regularity assumption.

$$
|b(e_h^0, \Pi_b u^1, \delta \tau \psi^1)| \leq |b(e_h^0, (\Pi_b u^1 - u^1), \delta \tau \psi^1)| + |b(e_h^0, u^1, \delta \tau \psi^1)|
$$

$$
\leq C\|e_h^0\|\|(u^1)_{H^{k+1}(\Omega)} + \|u^1\|\|\delta \tau \psi^1\|_{\text{DG}} \leq \frac{C}{\mu} h^{2k+2} + \frac{\mu}{16} \|\delta \tau \psi^1\|_{\text{DG}}.
$$

With (131), the above bounds, and with recalling that $\phi_h^0 = p_h^0 = 0$, we obtain
$$
\|\delta \tau \psi^1\|^2 + \frac{\tau}{\delta \mu} \|\delta \psi^1\|^2 + \frac{\tau^2}{16} (\|\delta \tau \psi^1\|^2_{\text{DG}} + \|\delta \tau \psi^1\|_{\text{DG}}^2 + \frac{\tau}{16} \|\delta \psi^1\|_{\text{DG}}^2
$$

$$
\leq C \left( \frac{1}{\mu} + \mu + 1 \right). \quad (158)
$$

Thus, with (157) and (158), the bound (156) becomes
$$
\|\delta \tau \psi^{m+1}\|^2 + \sum_{n=1}^m \|\delta \tau \psi^{n+1}\|^2 + \frac{\tau^2}{32} \sum_{n=1}^m \|\delta \tau \psi^n\|_{\text{DG}}^2 + \frac{\mu \tau}{4} \sum_{n=1}^m \|\delta \tau \psi^n\|_{\text{DG}}^2
$$

$$
\leq C \mu \tilde{K}_\mu + \frac{C}{\mu} \tau \sum_{n=1}^m \|\delta \tau \psi^n\|_{\text{DG}}^2 \|\delta \tau \psi^n\|_{\text{DG}}^2 (160)
$$

It remains to handle the last term. We will follow a similar technique as the one used in [18]. From Lemma 4 and condition (131), we have:

$$
\|\delta \tau \psi^n\|^2 \leq \frac{1}{\tau^2} \|\psi^n - \psi^{n-1}\|^2 \leq C \mu (1 + \mu + 1) \tau^{-1}, \quad \sum_{n=1}^m \|\psi^{n+1}\|_{\text{DG}}^2 \leq C \mu \tilde{K}_\mu.
$$

Using the above estimates in (159) and for $h$ small enough, we obtain:

$$
\|\delta \tau \psi^{m+1}\|^2 \leq C \mu \tilde{K}_\mu + C \mu^2 \tilde{K}_\mu h^{-d/3} \leq C \mu^2 \tilde{K}_\mu h^{-d/3}, \quad m \geq 0.
$$

We will iteratively apply this bound in (159). Applying it once, and using Lemma 4:

$$
\|\delta \tau \psi^{m+1}\|^2 \leq C \mu \tilde{K}_\mu + C \mu^2 \tilde{K}_\mu h^{-2d/3} \sum_{n=1}^m \|\psi^{n+1}\|_{\text{DG}}^2 \leq C \mu \tilde{K}_\mu + C \mu^2 \tilde{K}_\mu h^{-2d/3}.
$$

Noting that by (131), we have that $\tau h^{-2d/3} \leq c_2 h^{(\gamma-1)d/3}$. Since $\gamma < 1$, for $h$ small enough, we have:

$$
\|\delta \tau \psi^{m+1}\|^2 \leq C \mu \tilde{K}_\mu + C \mu^2 \tilde{K}_\mu h^{(\gamma-1)d/3} \leq C \mu^2 \tilde{K}_\mu h^{(\gamma-1)d/3}.
$$
Select $\beta$ to be the smallest integer such that $\beta \gamma \geq 1$ and iteratively apply above bound in (159) $\beta$ times. With the condition on $\tau$ (131), we obtain:

$$\|\delta_t \varepsilon_t^{n+1}\|^2 \leq C \mu \Theta + (C \beta \mu)^{\beta} \mu^{-\beta} \beta \gamma \frac{d}{d \tau^{m/3}} \leq C \mu \Theta + (C \beta \mu)^{\beta} \mu^{-\beta} \beta \gamma ^{1/3}.$$ 

The result then follows.

We will now use Lemma 6 and the dual problem (73) - (75) to obtain an $\ell^2$ error estimate for $\delta_t \varepsilon_t^{n+1}$.

**Lemma 7** We assume that the hypothesis of Lemma 6 hold. If $\tau$ is small enough ($\tau \leq \tau_0$), we have the following error estimate. For $1 \leq m \leq N_T - 1$,

$$\frac{\mu}{2} \tau \sum_{n=1}^{m} \|\delta_t \varepsilon_t^{n+1}\|^2 \leq \bar{C}_y \mu (\tau + h^2).$$ 

(161)

Here, $\bar{C}_y = C \mu \Theta + C \mu (1 + K \mu)$ is independent of $h$ and $\tau$.

**Proof** From (77), we have for $a \geq 0$

$$a \varphi(\delta_t U_h^{n+1}, \theta_h) - b(\theta_h, \delta_t P_h^{n+1}) = (\delta_t \chi^{n+1}, \theta_h), \quad \forall \theta_h \in X_h.$$ 

(162)

Let $\theta_h = \delta_t U_h^{n+1}$ in (132) and use (78). After some rearranging, we have

$$(\delta_t (v_h^{n+1} - u^{n+1}) - \delta_t \chi^{n+1}, \delta_t U_h^{n+1}) + \tau \mu a \varphi(\delta_t \varepsilon_t^{n+1}, \delta_t U_h^{n+1})$$

$$= -\tau b(\delta_t U_h^{n+1}, \delta_t \varepsilon_t^{n+1}) + \tau \mu a \varphi(\delta_t U_h^{n+1}, \delta_t \varepsilon_t^{n+1})$$

$$+ \bar{N}_1(\delta_t U_h^{n+1}) + \bar{N}_2(\delta_t U_h^{n+1}) + \bar{R}(\delta_t U_h^{n+1}).$$ 

(163)

Here,

$$\bar{R}(\delta_t U_h^{n+1}) = ((\partial_t u)^{n+1} - (\partial_t u)^n - (\delta_t u^{n+1} - \delta_t u^n), \delta_t U_h^{n+1}).$$

The difficulty is in bounding the last three terms. For completeness, we provide an overview of the way the other terms are handled. We first observe that from (134), (78), and (162) we have:

$$(\delta_t (v_h^{n+1} - u^{n+1}) - \delta_t \chi^{n+1}, \delta_t U_h^{n+1}) = (\delta_t \chi^{n+1}, \delta_t U_h^{n+1})$$

$$- \tau b(\delta_t U_h^{n+1}, \delta_t \varepsilon_t^{n+1}) = a \varphi(\delta_t U_h^{n+1} - \delta_t U_h^{n}, \delta_t U_h^{n+1})$$

$$= \frac{1}{2} a \varphi(\delta_t U_h^{n+1} - \delta_t U_h^{n}, \delta_t U_h^{n+1} - \delta_t U_h^{n}).$$

In addition, with choosing $\theta_h = \delta_t \varepsilon_t^{n+1}$ in (162), we have

$$a \varphi(\delta_t \varepsilon_t^{n+1}, \delta_t U_h^{n+1}) = (\delta_t \chi^{n+1}, \delta_t \varepsilon_t^{n+1}) + b(\delta_t \varepsilon_t^{n+1}, \delta_t \varepsilon_t^{n+1})$$

$$= (\delta_t \chi^{n+1}, \delta_t \varepsilon_t^{n+1} - \delta_t \varepsilon_t^{n}) + (\delta_t U_h^{n+1}, \delta_t \varepsilon_t^{n+1}) + (\delta_t \varepsilon_t^{n+1})^2$$

$$+ b(\delta_t \varepsilon_t^{n+1} - \delta_t \varepsilon_t^{n}, \delta_t \varepsilon_t^{n+1}) + b(\delta_t \varepsilon_t^{n+1}, \delta_t \varepsilon_t^{n+1}).$$
From (73), we have for $n \geq 1$
\[-\Delta (\delta_t \mathbf{U}^{n+1}) + \nabla (\delta_t P^{n+1}) = \delta_t \mathbf{X}^{n+1}.\]

(164)

Since the domain is assumed to be convex and similar to (76) and (94), we have
\[
\|\delta_t \mathbf{U}_h^{n+1}\|_{DG} + |\delta_t P_h^{n+1}|_{DG} \leq C(\|\delta_t \mathbf{U}_h^{n+1}\|_{H^1(\Omega)} + |\delta_t P_h^{n+1}|_{H^1(\Omega)})
\]
\[
\leq C\|\delta_t \mathbf{X}^{n+1}\| \leq C\|\delta_t \mathbf{e}_h^{n+1}\| + C\tau^{-1/2}h^{k+1} \left( \int_{\mathcal{P}_n} |\partial \mathbf{u}|_{H^{k+1}(\Omega)}^2 \right)^{1/2}. \tag{165}
\]

From (134), (39) and (165), we have
\[
|\delta_t \mathbf{X}^{n+1}, \delta_t \mathbf{e}_h^{n+1} - \delta_t \mathbf{e}_h^{n+1}| \leq \varepsilon \|\delta_t \mathbf{e}_h^{n+1}\|^2 + C \left( \frac{1}{\varepsilon} + 1 \right) \tau^2 |\delta_t \mathbf{X}^{n+1}|_{DG}^2
\]
\[+ C\tau^{-1}h^{2k+2} \int_{\mathcal{P}_n} |\partial \mathbf{u}|_{H^{k+1}(\Omega)}^2. \]

Similarly with (39) and (165), we have
\[
|b(\delta_t \mathbf{e}_h^{n+1} - \delta_t \mathbf{e}_h^{n+1}, \delta_t P_h^{n+1})| \leq \varepsilon \|\delta_t \mathbf{e}_h^{n+1}\|^2 + C \left( \frac{1}{\varepsilon} + 1 \right) \tau^2 |\delta_t \mathbf{X}^{n+1}|_{DG}^2
\]
\[+ C\tau^{-1}h^{2k+2} \int_{\mathcal{P}_n} |\partial \mathbf{u}|_{H^{k+1}(\Omega)}^2. \]

In addition, from (135), Cauchy-Schwarz’s inequality and (36), we obtain:
\[
(\delta_t (H^1 u^{n+1}) - \delta_t \mathbf{U}_h^{n+1}, \delta_t \mathbf{e}_h^{n+1}) + |b(\delta_t \mathbf{e}_h^{n+1}, \delta_t P_h^{n+1})| \leq Ch^{2k+1} |\delta_t \mathbf{X}^{n+1}|_{DG}^2 + C|\delta_t \mathbf{e}_h^{n+1}|_{DG} + C|\delta_t P_h^{n+1}|_{DG}
\]
\[
\leq \varepsilon \|\delta_t \mathbf{e}_h^{n+1}\|^2 + C \left( \frac{1}{\varepsilon} + 1 \right) \tau^2 |\delta_t \mathbf{X}^{n+1}|_{DG}^2
\]
\[+ C\tau^{-1}h^{2k+2} \int_{\mathcal{P}_n} |\partial \mathbf{u}|_{H^{k+1}(\Omega)}^2. \]

Further, note that by using (78), the definition of $L^2$ projection, and a bound similar to (80), we obtain:
\[
b(\delta_t \mathbf{U}_h^{n+1}, \delta_t \mathbf{X}^{n+1}) = b(\delta_t \mathbf{U}_h^{n+1}, \delta_t P^{n+1} - \mathbf{p}_0(\delta_t P^{n+1}))
\]
\[
\leq \sum_{\Gamma \in \Gamma_{DG} \cup \Omega} \left| \int_{\Gamma} \left( \delta_t \mathbf{p}^{n+1} - \mathbf{p}_0(\delta_t P^{n+1}) \right) [\delta_t \mathbf{U}_h^{n+1}] \cdot \mathbf{n}_e \right| \leq Ch^2 |\delta_t \mathbf{X}^{n+1}|_{DG} + C\tau^{-1}h^{2k+2} \int_{\mathcal{P}_n} |\partial \mathbf{u}|_{H^{k+1}(\Omega)}^2.
\]
By taking $\theta_h = \Pi_h \delta_t u^{n+1} - \delta_t Q_h u^{n+1}$ in (162) (where we recall $Q_h$ is the elliptic projection operator defined in (92)), using (39) and (163), we obtain:

$$
|a_{\phi}(\delta_t \Pi_h u^{n+1} - \delta_t u^{n+1}, \delta_t U_h^{n+1})| \leq |(\delta_t X^{n+1}, \delta_t \Pi_h u^{n+1} - \delta_t Q_h u^{n+1})| \\
+ |b(\delta_t \Pi_h u^{n+1} - \delta_t Q_h u^{n+1}, \delta_t P_h^n)| \leq C |\delta_t \Pi_h u^{n+1} - \delta_t Q_h u^{n+1}| \|\delta_t X^{n+1}\| \\
\leq \varepsilon \|\delta_t e_h^{n+1}\|^2 + C \left(\frac{1}{\varepsilon} + 1\right) \tau^{-\mu + 2} \int_{t^n}^{t^{n+1}} \|\partial_t u_h^2\|^2_{H^1(\Omega)}. \quad (166)
$$

To handle $\bar{R}_i(\delta_t U_{h, n+1})$, introduce the function $G^n = \tau(\partial_t u)^n - (u^n - u^{-1})$ for $n \geq 1$. Clearly, $G^n$ belongs to $X$. We then write:

$$
\bar{R}_i(\delta_t U_{h, n+1}) = \frac{1}{\tau}(G^{n+1}, \delta_t U_{h, n+1}) - \frac{1}{\tau}(G^n, \delta_t U_{h}^n) - \frac{1}{\tau}(G^n, \delta_t U_{h}^n - \delta_t U_{h}^n).
$$

We use Cauchy-Schwarz’s inequality, (48), and Taylor expansions.

$$
|G^n, \delta_t U_{h, n+1} - \delta_t U_{h}^n| \leq C \|G^n\| \|\delta_t U_{h, n+1} - \delta_t U_{h}^n\|_{DG} \\
\leq C \tau^2 \int_{t^n}^{t^{n+1}} \|\partial_t u_h^2\|^2 + \frac{1}{8} \tau \|\delta_t U_{h, n+1} - \delta_t U_{h}^n\|^2_{DG}. \quad (167)
$$

With the above bounds and the coercivity of $a_{\phi}$ (31), (163) reads:

$$
\frac{1}{2}(a_{\phi}(\delta_t U_{h, n+1}, \delta_t U_{h, n+1}) - a_{\phi}(\delta_t U_{h}^n, \delta_t U_{h}^n)) + \frac{1}{8} \|\delta_t U_{h, n+1} - \delta_t U_{h}^n\|^2_{DG} \\
+ (1 - 5\varepsilon) \tau \mu \|\delta_t e_h^{n+1}\|^2 \leq C \mu \left(\frac{1}{\varepsilon} + 1\right) \tau^{-\mu + 2} |\delta_t \phi_{h, n+1}^n|_{DG} + C \tau \int_{t^n}^{t^{n+1}} \|\partial_t u_h^2\|^2 \\
+ C \left(\frac{\mu}{\varepsilon} + \mu + 1\right) \tau^{-\mu + 2} \int_{t^n}^{t^{n+1}} \|\partial_t u_h^{n+1}\|^2_{H^1(\Omega)} + C \left(\frac{1}{\varepsilon} + 1\right) \tau^{-\mu + 2} \int_{t^n}^{t^{n+1}} \|\partial_t p_h^{n+1}\|^2_{H^1(\Omega)} \\
+ A_1(\delta_t U_{h, n+1}) + A_2(\delta_t U_{h, n+1}) + \frac{1}{\tau}(G^{n+1} + \delta_t U_{h, n+1} - \delta_t U_{h}^n). \quad (168)
$$

It remains to handle the nonlinear terms. To this end, we use (143)-(144) and write:

$$
A_1(\delta_t U_{h, n+1}) + A_2(\delta_t U_{h, n+1}) = \sum_{i=1}^{3} (\xi_{i, h}^{n}(\delta_t U_{h}^n) + \partial_t^n(\delta_t U_{h}^n)) = \sum_{i=1}^{3} \beta_i. \quad (169)
$$

We begin with bounding $\beta_1^n$, we have

$$
\beta_1^n = \mathcal{C}(u^{n+1} - u^n, u^{n+1} - u^n, \delta_t U_{h}^{n+1}) + \mathcal{C}(u^{n+1} - 2u^n + u^{-1}, u^n, \delta_t U_{h}^{n+1}) \\
= \sum_{E \in \mathcal{H}_E} \left(\int_{E} ((u^{n+1} - u^n) \cdot \nabla (u^{n+1} - u^n)) + ((u^{n+1} - 2u^n + u^{-1}) \cdot \nabla u^n) \cdot \delta_t U_{h}^{n+1} \right) \\
\leq (\|u^{n+1} - u^n\|_{L^1(\Omega)} \|\nabla (u^{n+1} - u^n)\| + \|u^{n+1} - 2u^n + u^{-1}\|_{L^1(\Omega)} \|\delta_t U_{h}^{n+1}\|_{L^1(\Omega)}). \quad (169)
$$
Since $H^1(\Omega)$ is embedded into $L^3(\Omega)$, with the above bound, (48), the assumption that $u \in L^\infty(0,T;H^{k+1}(\Omega)^d)$, Taylor expansions and Young’s inequality, we have

$$\|\beta_1^s\| \leq C_\tau \|\delta_t U_h^{n+1}\|_{DG}^2 + C\tau^{-1}\|u_h^{n+1} - u_h^n\|_{H^1(\Omega)}^2 + C\tau^{-1}\|u_h^{n+1} - 2u_h^n + u_h^{n-1}\|^2$$

$$\leq C_\tau \|\delta_t U_h^{n+1}\|_{DG}^2 + C\tau \int_{t}^{t+\tau} \|\partial_t u_h^n(\Omega)\|_0^2 \|\partial_t u_h^n(\Omega)\| + C\tau^2 \int_{t}^{t+\tau} \|\partial_t u_h\|^2.$$

Taking the sum of the above inequality from $n=1$ to $n=m$ yields:

$$\sum_{n=1}^{m} |\beta_1^s| \leq C_\tau \sum_{n=1}^{m} \|\delta_t U_h^{n+1}\|_{DG}^2 + C\tau. \quad (170)$$

We write $\beta_2^s$ as such:

$$\beta_2^s = \mathcal{C}(u^n - u^n_h, u^{n+1}_h - u^n, \delta_t U_h^{n+1}) + \mathcal{C}(u^n - u^n_h, u^{n-1}_h - u^n_h, \delta_t U_h^{n+1})$$

$$= \tau\mathcal{C}(u^n - \Pi_h u^n, \delta_t u^{n+1}_h, \delta_t U_h^{n+1}) - \tau\mathcal{C}(\epsilon_h^n, \delta_t u^{n+1}_h, \delta_t U_h^{n+1})$$

$$+ \tau\mathcal{C}(\delta_t(u^n - \Pi_h u^n), u^n, \delta_t U_h^{n+1}) - \tau\mathcal{C}(\delta_t \epsilon_h^n, u^n, \delta_t U_h^{n+1}).$$

We apply (63) to bound the first and third terms. We use (58) to bound the second and fourth terms.

$$\|\beta_2^s\| \leq \tau C_\mu \|\delta_t \epsilon_h^{n+1}\|^2 + C \left( 1 + \frac{1}{\varepsilon \mu} \right) \tau \|\delta_t U_h^{n+1}\|_{DG}^2 + \tau \|\delta_t \epsilon_h^n - \delta_t \epsilon_h^{n+1}\|^2$$

$$+ C(h^{2k+2} + |\epsilon_h^n|) \int_{t}^{t+\tau} \|\partial_t u_h^n\|^2_{H^1(\Omega)} + C h^{2k+2} \int_{t}^{t+\tau} \|\partial_t u_h^n\|^2_{H^1(\Omega)}.$$ 

(171)

Taking the sum of $\beta_2^s$ and using the results of Lemma 6 and Lemma 4, we obtain (recall that the constant $C_\mu$ is a generic constant that depends on $\mu$)

$$\sum_{n=1}^{m} \|\beta_2^s\| \leq \tau C_\mu \sum_{n=1}^{m} \|\delta_t \epsilon_h^{n+1}\|^2 + C \left( 1 + \frac{1}{\varepsilon \mu} \right) \tau \sum_{n=1}^{m} \|\delta_t U_h^{n+1}\|_{DG}^2$$

$$+ C \mu(1 + \mu^{-1} + \mu)(\tau + h^{2k}) + C_\gamma \mu \tau. \quad (172)$$

To handle $\beta_3^s$, recall that with (17), we have:

$$\beta_3^s = \alpha(\epsilon^n_h, \epsilon^{n+1}_h, u^n_h - v^n_h, \delta_t U_h^{n+1}) + P_2^n$$

$$+ \mathcal{C}(u^n_h - u^n_h - v^{n+1}_h, \delta_t U_h^{n+1}) - \mathcal{C}(u^{n+1}_h, u^n_h - u^n_h, v^{n+1}_h, \delta_t U_h^{n+1})$$

$$= P_1^n + P_2^n + P_3^n,$$

where

$$P_1^n = -\mathcal{C}(u^n_h, u^n_h - v^n_h, \delta_t U_h^{n+1} + \mathcal{C}(u^{n+1}_h; u^n_h - v^n_h, \delta_t U_h^{n+1}).$$
We have the following bound on $P_2^n$, see Lemma 9 in the Appendix.

\[|P_2^n| \leq C\|u^n_h - u^{n-1}_h\|\|\delta_t U^{n+1}_h\|_{L^\infty(\Omega)}\|u^{n+1}_h - v^{n+1}_h\|_{DG}.\]

The terms $P_1^n + P_2^n$ are handled as follows. Note that $u^{n+1}_h - v^{n+1}_h = u^{n+1}_h - \Pi_h u^{n+1} - \hat{e}^{n+1}_h$. Hence, we apply Lemma 10 estimate (210) in the appendix:

\[|P_1^n| + |P_2^n| \leq C\|u^n_h - u^{n-1}_h\|\|\delta_t U^{n+1}_h\|_{L^\infty(\Omega)} + \|\nabla_h \delta_t U^{n+1}_h\|_{L^1(\Omega)}\|u^{n+1}_h - v^{n+1}_h\|_{DG} + C\|\delta_t U^{n+1}_h\|_{L^2(\Omega)^2} + C\|\delta_t U^{n+1}_h\|_{L^2(\Omega)^2} + C\|\delta_t U^{n+1}_h\|_{L^2(\Omega)^2} + C\|\delta_t U^{n+1}_h\|_{L^2(\Omega)^2} + C\|\delta_t U^{n+1}_h\|_{L^2(\Omega)^2} + C\|\delta_t U^{n+1}_h\|_{L^2(\Omega)^2}.

Similar to Lemma 5, by the linearity of the PDE (73)-(75), we have:

\[\|\delta_t U^{n+1}_h\| + \|\delta_t U^{n+1}_h\|_{L^\infty(\Omega)} \leq C\|\delta_t U^{n+1}_h\|, \quad (173)\]
\[\|\delta_t U^{n+1}_h - \delta_t U^{n+1}_h\|_{DG} \leq C\|\delta_t U^{n+1}_h\|, \quad (174)\]

With the above bounds, triangle inequality, inverse estimate (68), and (43), we find

\[\|\nabla_h \delta_t U^{n+1}_h\|_{L^1(\Omega)} \leq Ch^{-d/6}\|\nabla_h (\delta_t U^{n+1}_h - \Pi_h (\delta_t U^{n+1}_h))\| + \|\nabla_h (\Pi_h (\delta_t U^{n+1}_h))\|_{L^1(\Omega)} \leq Ch^{-d/6}\|\delta_t U^{n+1}_h\| + C\|\nabla \delta_t U^{n+1}_h\|_{L^1(\Omega)} \leq C\|\delta_t U^{n+1}_h\|, \quad (175)\]

Hence, with (68), the triangle inequality and (42)-(43), we obtain

\[\sum_{i=2}^4 |P_i^n| \leq C\tau\|\delta_t e^n_h\| + \|\delta_t \Pi_h u^n\|\|\delta_t U^{n+1}_h\|_{H^{k+1}(\Omega)} + \|e^{n+1}_h\|_{DG}.\]

With a Taylor’s expansion, (42), (131), and the regularity assumptions we have

\[\|\delta_t \Pi_h u^n\|^2 \leq 2\|\delta_t (\Pi_h u^n - u^n)\|^2 + 2\|\delta_t u^n\|^2 \leq Ch^{2k}\int_{t^{n-1}}^{t^n} \|\delta_t u^n\|^2_{H^{k+1}(\Omega)} + 4\|\delta_t u^n\|^2_{L^2(0,T;L^2(\Omega))} + \frac{4\tau}{3}\|\delta_t u^n\|^2_{L^2(0,T;L^2(\Omega))} \leq C.\]

We use the following bound, which is easy to obtain from the definition of $\chi$.

\[\|\delta_t \chi^{n+1}\| \leq \|\delta_t e^{n+1}_h\| + Ch^{k+1}\|\delta_t U^{n+1}_h\|_{H^{k+1}(\Omega)}.\]

Hence, with Young’s inequality and the assumption that $u \in L^\infty(0,T;H^{k+1}(\Omega)^d)$:

\[\sum_{i=2}^4 |P_i^n| \leq C\tau\|\delta_t e^{n+1}_h\| + C \left(1 + \frac{1}{\epsilon\mu}\right) \tau\|\delta_t e^{n+1}_h\|^2 + C \left(1 + \frac{1}{\epsilon\mu}\right) \tau\|\delta_t e^{n+1}_h\|^2 + C\|\delta_t U^{n+1}_h\|^2_{H^{k+1}(\Omega)}.\]

\[\sum_{i=2}^4 |P_i^n| \leq C\|\delta_t e^{n+1}_h\|^2 + C\tau\|\delta_t e^{n+1}_h\|^2 + C\|\delta_t U^{n+1}_h\|^2_{H^{k+1}(\Omega)}.\]

\[\sum_{i=2}^4 |P_i^n| \leq C\|\delta_t e^{n+1}_h\|^2 + C\tau\|\delta_t e^{n+1}_h\|^2 + C\|\delta_t U^{n+1}_h\|^2_{H^{k+1}(\Omega)}.\]
It remains to handle \( I_1^1 \). It is helpful to define the function \( E^n \in X, E^n = \Pi_h u^n - u^n \) for \( n \geq 1 \). We have from (41), (42), (43):

\[
\| \delta_t E^{n+1} \|^2 + h^2 \| \delta_t E^{n+1} \|^2_{DG} \leq C h^{2^k+2} \| \delta_t u^{n+1} \|^2_{H^{k+1}(\Omega)},
\]

\[
\| \delta_t E^{n+1} \|^2_{L^2(\Omega)} \leq C \| \delta_t u^{n+1} \|^2_{H^2(\Omega)}.
\]

With the definition of \( E^n \) and (13),

\[
I_1^1 = \tau \delta_t (u^n_h - u^n - \delta_t U^{n+1}_h + \delta_t e^{n+1}_h)
\]

\[
= \tau \delta_t (u^n_h - u^n - \delta_t U^{n+1}_h + \delta_t e^{n+1}_h)
\]

\[
+ \tau \delta_t (u^n_h - u^n - \delta_t U^{n+1}_h + \delta_t e^{n+1}_h)
\]

\[
+ \tau \delta_t (u^n_h - u^n - \delta_t U^{n+1}_h + \delta_t e^{n+1}_h)
\]

\[
+ \tau \delta_t (u^n_h - u^n - \delta_t U^{n+1}_h + \delta_t e^{n+1}_h)
\]

\[
+ \tau \delta_t (u^n_h - u^n - \delta_t U^{n+1}_h + \delta_t e^{n+1}_h)
\]

\[
+ \tau \delta_t (u^n_h - u^n - \delta_t U^{n+1}_h + \delta_t e^{n+1}_h)
\]

To handle \( I_1^1 \), we closely follow the derivation of the bound on \( A_1 \), see the derivation of bound (120) in the proof of Theorem 1. We provide the details in the Appendix: Lemma 12.

\[
| I_1^1 | \leq \epsilon \| \delta_t e^{n+1}_h \|^2
\]

\[
+ C \left( 1 + \frac{1}{\epsilon \mu} \right) \left( \| e^{n+1}_h \|^2 + h^2 \| \delta_t u^{n+1} \|^2_{DG} \right)
\]

\[
+ C \left( 1 + \frac{1}{\epsilon \mu} \right) \left( \| \delta_t U^{n+1}_h \|^2_{DG} \right)
\]

\[
+ C \left( 1 + \frac{1}{\epsilon \mu} \right) \left( \| \delta_t e^{n+1}_h \|^2_{DG} \right)
\]

\[
+ C \left( 1 + \frac{1}{\epsilon \mu} \right) \left( h^2 (1 + \frac{1}{\mu}) + \| e^{n+1}_h \|^2 \right) \| \delta_t U^{n+1} \|^2_{DG}.
\]

For \( I_2^1 \), we have:

\[
I_2^1 = \sum_{E \in \mathcal{O}} \int_E \left( (u^n_h - u^n - \delta_t U^{n+1}_h) \cdot (\delta_t E^{n+1} + \delta_t e^{n+1}_h) \right)
\]

\[
+ \frac{1}{2} \left( (\delta_t U^{n+1}_h - u^n - \delta_t e^{n+1}_h) \right)
\]

\[
+ \frac{1}{2} \left( (\delta_t U^{n+1}_h - u^n - \delta_t e^{n+1}_h) \right).
\]

With (173), (178), Holder’s inequality, triangle inequality, approximation property (42), and (48), we bound the first term by:

\[
\| u^n - u^n - (\delta_t U^{n+1}_h - \delta_t e^{n+1}_h) \|_{L^2(\Omega)} \| \delta_t E^{n+1} + \delta_t e^{n+1}_h \|_{L^2(\Omega)}
\]

\[
\leq C (\| e^{n+1}_h \|^2 + h^2 \| u^n - u^n \|^2 |_{H^k(\Omega)}) \| \delta_t e^{n+1}_h \|_{L^2(\Omega)}
\]

\[
\leq C (\| e^{n+1}_h \|^2 + h^2 \| u^n - u^n \|^2 |_{H^k(\Omega)}) \| \delta_t e^{n+1}_h \|_{L^2(\Omega)} + \| \delta_t e^{n+1}_h \|^2_{DG}.
\]

To bound the second term in \( I_2^1 \), we refer to the derivation of bounds (107) and (58). We use (173), (178) and obtain:

\[
| b(\delta_t U^{n+1}_h - \delta_t e^{n+1}_h) | \leq C (\| e^{n+1}_h \|^2 + \| \delta_t e^{n+1}_h \|_{DG}).
\]
Bounds (63) and (173) yield:
\[ |b(\Pi_h u^{n-1} - u^{n-1}, \delta_t U^{n+1}, \delta_t \epsilon_h^{n+1})| \leq Ch^{k+2} |u^{n-1}|_{H^{k+1}(\Omega)} \| \delta_t \mathbf{X}^{n+1} \| \| \delta_t \epsilon_h^{n+1} \|_{DG}. \]

Approximation properties, trace estimates (71)-(72), and (173) yield:
\[ |b(\Pi_h u^{n-1} - u^{n-1}, \delta_t U^{n+1}, \delta_t E^{n+1})| \]
\[ \leq C \| \delta_t U^{n+1} \|_{L^2(\Omega)} (h^k |u^{n-1}|_{H^{k+1}(\Omega)} \| \delta_t E^{n+1} \| + h^{k+2} |u^{n-1}|_{H^{k+1}(\Omega)} \| \delta_t U^{n+1} \|_{H^2(\Omega)}) \]
\[ \leq Ch^{k+2} |\delta_t U^{n+1}|_{L^1(\Omega)} |u^{n-1}|_{H^{k+1}(\Omega)} \| \delta_t \mathbf{X}^{n+1} \|. \]

Hence, Young’s inequality and the assumption that \( u \in L^\infty(0, T; H^{k+1}(\Omega)^d) \) yield
\[ |I_{1,2}^{n} | \leq \epsilon \mu \| \delta_t \epsilon_h^{n+1} \|^2 \]
\[ + C \left( \frac{1}{\epsilon \mu} + 1 \right) \left( \| \delta_t \epsilon_h^{n+1} \| + h^2 \right)(h^2 |\delta_t U^{n+1}|_{H^1(\Omega)} + \| \delta_t \epsilon_h^{n+1} \|_{DG}) \]
\[ + Ch^{2k+2} \| \delta_t U^{n+1} \|_{H^1(\Omega)} \]  
(181)

To handle \( I_{1,3}^{n} \), we have:
\[ I_{1,3}^{n} = \sum_{k \in \mathbb{N}^n} \int_E (u^{n-1} \nabla((\delta_t U^{n+1} - \delta_t U_h^{n+1}) + \delta_t U_h^{n+1}) \cdot (\delta_t E^{n+1} + \delta_t \epsilon_h^{n+1}), \]

Note that by the triangle inequality and (134), we have:
\[ \| \delta_t \epsilon_h^{n+1} \| \leq \| \delta_t \epsilon_h^{n+1} - \delta_t \epsilon_h^{n+1} \| + \| \delta_t \epsilon_h^{n+1} \| \leq C \tau |\delta_t \phi_h^{n+1}|_{DG} + \| \delta_t \epsilon_h^{n+1} \|. \]  
(182)

Hence, with the help of (174), (178) and Holder’s inequality, we obtain
\[ |I_{1,3}^{n} | \leq C \| u^{n-1} \|_{L^1(\Omega)} (h \| \delta_t \mathbf{X}^{n+1} \| + \| \delta_t U_h^{n+1} \|_{DG}) \]
\[ \times ((\| \delta_t E^{n+1} \| + \| \delta_t \epsilon_h^{n+1} \| + \tau |\delta_t \phi_h^{n+1}|_{DG}) \]
\[ \leq \epsilon \mu \| \delta_t \epsilon_h^{n+1} \|^2 \]
\[ + C \left( \frac{1}{\epsilon \mu} + 1 \right) \left( \frac{h^2}{\epsilon \mu} \right)(h^2 |\delta_t U^{n+1}|_{H^1(\Omega)} + \tau^2 |\delta_t \phi_h^{n+1}|_{DG}) \]
\[ + C \left( \frac{1}{\epsilon \mu} \right) \| \delta_t U_h^{n+1} \|^2_{DG}. \]  
(183)

With (177), (180), (181), and (183), we obtain a bound on \( \beta_3^n \):
\[ |\beta_3^n | \leq 4 \epsilon \mu \| \delta_t \epsilon_h^{n+1} \|^2 \]
\[ + C \left( \frac{1}{\epsilon \mu} \right) \tau (\| \delta_t \epsilon_h^{n+1} \|^2 \| \delta_t \epsilon_h^{n+1} \|^2_{DG}) \]
\[ + C \left( \frac{1}{\epsilon \mu} \right) \sigma (\| \delta_t \epsilon_h^{n+1} \|^2 + C \left( \frac{1}{\epsilon \mu} \right) \| \delta_t U_h^{n+1} \|^2_{DG}) \]
\[ + C \left( \frac{1}{\epsilon \mu} \right) \tau (h^2 \left( \frac{1}{\epsilon \mu} \right) \| \delta_t \epsilon_h^{n+1} \|^2 \| \delta_t U_h^{n+1} \|^2_{DG}) \]
\[ + C \left( \frac{1}{\epsilon \mu} \right) \tau (h^2 \left( \frac{1}{\mu} \right) + \| \delta_t \epsilon_h^{n+1} \|^2 \| \delta_t U_h^{n+1} \|^2_{DG}) \]  
(184)
We take the sum of $\beta^n$, choose $\varepsilon = 1/18$, and use the results of Lemma 6, Lemma 4, and the condition on $\tau$ (131). We obtain
\[
\sum_{n=1}^{m} |\beta^n| \leq \frac{2}{9} \tau \mu \sum_{n=1}^{m} ||\delta_t e^n||^2 + C \left( 1 + \frac{1}{\mu} \right) \tau \sum_{n=1}^{m} ||\delta_t U_{h}^{n+1}||_{DG}^2 \\
+ (C_{\mu} K_{\mu} + C_{\gamma_{DG}} K_{\mu} (1 + C_{\mu})) (\tau + h^{2k}).
\] (185)

Further, observe that the sum of the last two terms in (168) yields
\[
\frac{1}{\tau} (G_{h}^{m+1}, \delta_t U_{h}^{m+1}) - \frac{1}{\tau} (G^1, \delta_t U_{h}^1) \\
\leq \frac{1}{8} (||\delta_t U_{h}^{m+1}||_{DG}^2 + ||\delta_t U_{h}^1||_{DG}^2) + C \tau \int_{t_{m}}^{t_{m+1}} ||\partial_t u||^2 + C \tau \int_{C}^{1} ||\partial_t u||^2.
\] (186)

We sum (168) from $n = 1$ to $n = m$ and use Lemma 6, Lemma 4, the bounds (170), (172), (185), (186), the coercivity and continuity of $a_{DG}$, the condition (131). We obtain
\[
\frac{1}{8} (||\delta_t U_{h}^{m+1}||_{DG}^2 + ||\delta_t U_{h}^1||_{DG}^2) \leq (C_{\mu} K_{\mu} + C_{\gamma_{DG}} K_{\mu} (1 + C_{\mu})) (\tau + h^{2k}) \\
+ C ||\delta_t U_{h}^1||_{DG}^2 + C \left( 1 + \frac{1}{\mu} \right) \tau \sum_{n=1}^{m} ||\delta_t U_{h}^{n+1}||_{DG}^2.
\]

To handle $||\delta_t U_{h}^1||_{DG}^2$, we take $n = 1$ in (86) and use the same expressions and bounds as in the proof of Theorem 1 except for bound (89) and the bound on $T_{2,1}^1$ (123). Instead, we use the following alternative bounds which result from a different application of Young's inequality:
\[
||T_{2,1}^1|| \leq C \tau^2 \int_{\rho}^{\tau} ||\partial_t u||^2 + \frac{1}{16} \tau^{-1} ||U_{h}^1||_{DG}^2, \\
||T_{2,1}^1 - \langle u^1 - u^0, U_{h}^1 \rangle|| \leq C \tau^3 \int_{\rho}^{\tau} ||\partial_t u||^2 + \frac{1}{16} ||U_{h}^1||_{DG}^2.
\]

We use (131), (158) and its derivation, and the observation that $U_{h}^0 = 0$, see (127). With assuming that $C \tau (1 + 1/\mu) \leq 1/16$, we have:
\[
\frac{1}{16} ||U_{h}^1||_{DG}^2 \\
\leq C \left( \frac{1}{\mu} + 1 + \mu \right) \tau \left( h^{2k+2} + \tau \langle \delta_t U_{h}^1 ||DG, ||\delta_t U_{h}^1 ||DG + ||e_{h}^1 - e_{h}^0 \rangle \right) + h^2 ||\delta_t U_{h}^1 ||DG + ||e_{h}^1 ||^2 ||U_{h}^1 ||_{DG}^2 \leq C \left( \frac{1}{\mu} + 1 + \mu \right)^3 \tau^3.
\]

Hence $||\delta_t U_{h}^1 ||_{DG} \leq CK_{\mu} \tau$. With $\tau$ small enough, say $C \tau (1 + 1/\mu) < 1$, the final result is obtained by Gronwall's and triangle inequalities.
5 Error estimate for the pressure

With the results of the previous section, we show the following a priori estimate for the pressure, optimal in space and sub-optimal in time.

**Theorem 2** Under the same assumptions of Lemma 7, we have the following error estimate. There exists a constant $C_{\gamma, \mu}$ independent of $h$ and $\tau$ but inversely dependent on $\gamma$ and $\mu$ such that, for $1 \leq m \leq N_T$:

$$\tau \sum_{n=1}^{m} \|p^n - p_h^n\|^2 \leq \hat{C}_{\gamma, \mu}(\tau + h^{2k}).$$

(187)

Here, $\hat{C}_{\gamma, \mu} = C\mu^{-1}\hat{C}_{\gamma, \mu} + C\mu K_\mu$ where $C\mu$ depends on $e^n_\tau$.

**Proof** Let $\hat{X}_h$ be a subspace of $X_h$ satisfying

$$\hat{X}_h = \{ \theta_h \in X_h : \forall e \in G_h \cup \partial \Omega, [\theta_h] \cdot n_e = 0 \}. \quad (188)$$

There exists a positive constant $\beta^*$, independent of $h$, such that [8, 20]

$$\text{inf}_{q_h \in M_h} \text{sup}_{\psi_h \in \hat{X}_h} \frac{-b(\psi_h, q_h)}{\|\psi_h\| \|q_h\|} \geq \beta^*. \quad (189)$$

As a result of this inf-sup condition, for $n \geq 0$, there exists a unique $\xi^n_h \in \hat{X}_h$ with

$$b(\xi^n_h, p^n_h - \pi_h p^n) = \|p^n_h - \pi_h p^n\|^2, \quad \|\xi^n_h\|_{DG} \leq \frac{1}{2\beta^*} \|p^n_h - \pi_h p^n\|. \quad (190)$$

We substitute (53) in (52). For $\theta_h \in \hat{X}_h$,

$$(\epsilon^n_h - \epsilon_h^{n-1}, \theta_h) + \tau a_{\phi}(u_h^{n-1}; u_h^{n-1}, \epsilon_h^0, \theta_h) + \tau R_{\phi}(\theta_h)
+ \tau a_{\phi}(\epsilon_h^0, \theta_h) = tau(\theta_h, p^n_h - \pi_h p^n) - \tau a_{\phi}(\Pi_h u^e - u^e, \theta_h) + R_t(\theta_h).$$

Using (55), we obtain for all $\theta_h \in \hat{X}_h$,

$$(\epsilon^n_h - \epsilon_h^{n-1}, \theta_h) + \tau a_{\phi}(u_h^{n-1}; u_h^{n-1}, \epsilon_h^0, \theta_h) + \tau R_{\phi}(\theta_h)
+ \tau a_{\phi}(\epsilon_h^0, \theta_h) = tau(\theta_h, p^n_h - \pi_h p^n) - \tau a_{\phi}(\Pi_h u^e - u^e, \theta_h)
+ \tau \delta \mu b(\theta_h, \nabla \epsilon_h^0 R_h(\Pi_h u^e)) + R_t(\theta_h).$$

Choosing $\theta_h = \xi^n_h$ and using (190) result in

$$\tau \|p^n_h - \pi_h p^n\|^2 = (\epsilon_h^n - \epsilon_h^{n-1}, \xi^n_h) + \tau a_{\phi}(u_h^{n-1}; u_h^{n-1}, \xi^n_h, \xi^n_h) + \tau R_{\phi}(\xi^n_h)
+ \tau a_{\phi}(\xi^n_h, \xi^n_h) + \tau a_{\phi}(\Pi_h u^e - u^e, \xi^n_h)
- \tau \delta \mu b(\xi^n_h, \nabla \epsilon_h^0 R_h(\Pi_h u^e)) - R_t(\xi^n_h). \quad (191)$$

The first term is bounded using Cauchy-Schwarz’s inequality, (190) and Young’s inequality.

$$|(\epsilon_h^n - \epsilon_h^{n-1}, \xi^n_h)| \leq C \tau \| \delta \epsilon_h^n \|^2 + \frac{\tau}{16} \|p^n_h - \pi_h p^n\|^2. \quad (192)$$
With Lemma 10 bound (209) in the appendix, we obtain:

\[
|a_h(u_h^{n-1}; u_h^{n-1}, \bar{e}_h^n, \xi_h^n)| \leq C \|u_h^{n-1}\|_{L^2(\Omega)} \|\bar{e}_h^n\|_{DG} \|\xi_h^n\|_{DG}.
\]  

(193)

With Lemma 4, (42), (68), the assumption that \( u \in L^\infty(0,T; H^{k+1}(\Omega)^d) \), and the CFL condition (131), we have

\[
\|u_h^n\|_{L^2(\Omega)}^2 \leq 2 \|e_h^{n-1}\|_{L^2(\Omega)}^2 + 2 \|\Pi_h u^{n-1}\|_{L^2(\Omega)}^2
\]

\[
\leq C\mu \left(1 + \frac{1}{\mu} + \mu \right) h^{-d/3}(\tau + h^2k) + C \|u^{n-1}\|_{W^{1,1}(\Omega)}^2 \leq C\mu \left(1 + \frac{1}{\mu} + \mu \right).
\]

Hence, using Young’s inequality and (190), we have:

\[
|a_h(u_h^{n-1}; u_h^{n-1}, \bar{e}_h^n, \xi_h^n)| \leq C\mu \left(1 + \frac{1}{\mu} + \mu \right) \|\bar{e}_h^n\|_{DG}^2 + \frac{1}{16} \|p_h^n - p^n\|^2.
\]  

(194)

For the remaining nonlinear terms, we split them as follows:

We further split \( L_1 \) in the following way.

\[
R_h(\xi_h^n) = a_h(u_h^{n-1}; u_h^{n-1}, \Pi_h u^n - u^n, \xi_h^n) + a_h(u_h^{n-1}; u_h^{n-1}, \xi_h^n)
\]

\[
+ a_h(u_h^{n-1}; \Pi_h u^n - u^n, \xi_h^n) + a_h(u_h^{n-1}; u_h^{n-1} - u^n, \xi_h^n) = \sum_{i=1}^4 L_i^n
\]

We apply Lemma 2: (59) for the first term, (64) for the second and fourth terms, and (65) for the third term.

\[
|L_i^n| \leq C(\|e_h^{n-1}\| + h^k \|u^n\|_{H^{k+1}(\Omega)}) \|\xi_h^n\|_{DG}.
\]

The term \( L_1^n \) is bounded by (58), \( L_2^n \) is bounded by (63), and \( L_4^n \) is bounded as follows:

\[
L_4^n = \int_\Omega (u^{n-1} - u^n) \cdot \nabla u^n \cdot \xi_h^n \leq C \|u^n - u^{n-1}\|_{W^{1,1}(\Omega)} \|\xi_h^n\|_{DG}.
\]

With the assumption that \( u \in L^\infty(0,T; H^{k+1}(\Omega)^d) \) and (190), we obtain:

\[
|R_h(\xi_h^n)| \leq C(\|e_h^{n-1}\|^2 + h^2k) + C\tau \int_{n-1}^n \|\partial_t u\|^2 + \frac{1}{16} \|p_h^n - p^n\|^2.
\]  

(195)

Following similar arguments to bound (6.90) in [17], we have

\[
|R_h(\xi_h^n)| \leq \frac{1}{16} \|p_h^n - p^n\|^2 + C\tau^2 \int_{n-1}^n \|\partial_t u\|^2 + C\tau^{2k+2} \int_{n-1}^n \|\partial_t u\|^2_{H^{k+1}(\Omega)}.
\]
The remaining terms are handled via similar arguments as before. The details are skipped for brevity. We have:

\[ |a_0(\hat{e}_h^n, \xi_h^m)| \leq C|\hat{e}_h^n||\hat{e}_h^n|DG|\xi_h^m||DG \leq C|\hat{e}_h^n||DG||p_h^n - \pi_h p^n|, \]
\[ |b(\xi_h^m, p_h^n - \pi_h p^n)| \leq Ch^k|\xi_h^m||DG|p_h^n|H^k(\Omega) \leq Ch^k|p_h^n||H^k(\Omega) ||p_h^n - \pi_h p^n|, \]
\[ |a_0(\Pi_h u^\tau - u^\tau, \xi_h^m)| \leq Ch^k|u^\tau||H^{k+1}(\Omega)||\xi_h^m||DG \leq Ch^k|u^\tau||H^{k+1}(\Omega) ||p_h^n - \pi_h p^n|, \]
\[ |b(\xi_h^m, \nabla_h \cdot \hat{e}_h^n - R_h(\hat{e}_h^n))| \leq C|\xi_h^m||DG||\hat{e}_h^n||DG \leq C|\xi_h^m||DG||p_h^n - \pi_h p^n|. \]

Making use of Young’s inequality for the above bounds with (190), substituting the bounds in (191), and summing the resulting inequality from \( n = 1 \) to \( n = m \) we obtain:

\[ \frac{T}{2} \sum_{n=1}^m \| p_h^n - \pi_h p^n \|^2 \leq C \mu \left( \sum_{i=1}^m \mu^i \right) \sum_{n=1}^m \| \hat{e}_h^n \|_{DG}^2 + C T^2 \int_0^T (\| \partial_t u^\tau \|^2 + \| \partial_t u \|^2) \]
\[ + C T \sum_{n=1}^m (\| e_h^{n-1} \|^2 + h^{2k} + \| \delta e_h^n \|^2) + Ch^{2k+2} \int_0^T \| \partial_t u \|^2_{L^{k+1}(\Omega)} + C (\mu^2 + 1) h^{2k}. \]

The result is concluded by invoking Lemma 4, Lemma 7, the bound on the initial error (158), and the triangle inequality.

6 Appendix

**Lemma 8** [Equality (139)]

**Proof** For \( n = 1 \), with (135), we have

\[ A = -2 b(\delta e_h^1 - \delta e_h^1, \delta e_h^1, \delta e_h^1) = -2 b(\delta e_h^1, \delta e_h^1, \delta e_h^1) + 2 b(\delta e_h^1, \delta e_h^1, \delta e_h^1) \]
\[ = 2 \tau \sum_{c \in I_h} \hat{h}_h^{-1} \| \delta e_h^1 \|_{L^2(e)}^2 - 2 \tau \| G_h(\delta e_h^1) \|_{L^2(e)}^2 + 2 b(\delta e_h^1, \delta e_h^1, \delta e_h^1). \]

We can write the last term as follows. Since \( \phi_h^0 = 0, \phi_h^1 = \tau \delta e_h^1 \), we write

\[ 2 b(\delta e_h^1, \delta e_h^1, \delta e_h^1) = \frac{2}{\tau} b(e_h^1, \delta e_h^1, \delta e_h^1) - \frac{2}{\tau} b(e_h^0, \delta e_h^1, \delta e_h^1) \]
\[ = -2 \tau \sum_{c \in I_h} \hat{h}_h^{-1} \int [\delta e_h^1, \delta e_h^1, \delta e_h^1] + 2 \tau (G_h(\delta e_h^1)), \| G_h(\delta e_h^1) \|_{L^2(e)}^2 - \frac{2}{\tau} b(e_h^0, \delta e_h^1, \delta e_h^1). \]

Thus, we have

\[ A = \tau \sum_{c \in I_h} \hat{h}_h^{-1} \| (\delta e_h^1) \|_{L^2(e)}^2 - 2 \tau \| (\delta e_h^1) \|_{L^2(e)}^2 + 2 \tau (G_h(\delta e_h^1)), \| G_h(\delta e_h^1) \|_{L^2(e)}^2 - \frac{2}{\tau} b(e_h^0, \delta e_h^1, \delta e_h^1) \]
\[ = \tau (\hat{A}_1 - \hat{A}_2) + \tau \sum_{c \in I_h} \hat{h}_h^{-1} \| (\delta e_h^1 - \delta e_h^1) \|_{L^2(e)}^2 + \| G_h(\delta e_h^1) \|_{L^2(e)}^2 - \frac{2}{\tau} b(e_h^0, \delta e_h^1, \delta e_h^1) \]
\[ = \| G_h(\delta e_h^1 - \delta e_h^1) \|_{L^2(e)}^2 - \frac{2}{\tau} b(e_h^0, \delta e_h^1, \delta e_h^1). \]
For \( n \geq 2 \), we use (135) for both terms.

\[
-2b(\delta \varepsilon_h^{n+1} - \delta \varepsilon_h^n, \delta \varepsilon_h^n = 2\tau \sum_{e \in I_h} \delta h_e^{-1} \int_{e} (\delta \varepsilon_h^{n+1} - [\delta \varepsilon_h^n])
\]

Using the fact that:

\[
(f, f - g) = \frac{1}{2}||f||^2 - \frac{1}{2}||g||^2 + \frac{1}{2}||f - g||^2,
\]

we obtain the result.

**Lemma 9** Let \( u_h, \theta_h, w_h \in X_h \) and \( v \in X \). There exists a constant \( C \) independent of \( h \) such that the following bounds hold for \( I = W(u_h; u_h, v, w_h) - W(\theta_h; u_h, v, w_h) \).

\[
|I| \leq C||u_h - \theta_h||_{L^2(\Omega)} \left( \sum_{e \in I_h \cup \partial \Omega} \delta h_e^{-1} ||[v]||_{L^2(e)}^2 \right)^{1/2} ||w_h||_{DG},
\]

\[
|I| \leq C||u_h - \theta_h|| \left( \sum_{e \in I_h \cup \partial \Omega} \delta h_e^{-1} ||[v]||_{L^2(e)}^2 \right)^{1/2} ||w_h||_{L^2(\Omega)}.
\]

**Proof** The proof of this Lemma closely follows the proof of Proposition 4.10 in [7]. The main idea of the proof is to write the expression of \( W \) in terms of the faces’ contributions [7]. Take \( z_h \in X_h \). Fix a face \( e \in I_h \) adjacent to the elements \( E^1_e \) and \( E^2_e \) with \( n_e = n_{E^1_e} \). The contribution of \( e \) to the expression \( W(z_h; u_h, v, w_h) \) is given by:

\[
\int_e \{u_h\} \cdot n_e [v] \cdot w_h^e,
\]

where for \( e \in I_h \), \( w_h^e \) is defined as follows.

\[
w_h^e|e = \begin{cases} w_h|E^1_e, & \{z_h\} \cdot n_e < 0, \\ 0, & \{z_h\} \cdot n_e = 0, \\ w_h|E^2_e, & \{z_h\} \cdot n_e > 0. \end{cases}
\]

If \( e \in \partial \Omega \cap E_e \), then \( n_e = n_{\partial \Omega} \). Then, the contribution of \( e \) is given by:

\[
\int_e u_h \cdot n_e v \cdot w_h^e,
\]

where for \( e \in \partial \Omega \), \( w_h^e \) is defined as follows.

\[
w_h^e|e = \begin{cases} w_h|E_e, & \{z_h\} \cdot n_e < 0, \\ 0, & \{z_h\} \cdot n_e > 0. \end{cases}
\]

Hence, we have the following expression for \( I = W(u_h; u_h, v, w_h) - W(\theta_h; u_h, v, w_h) \):

\[
I = \sum_{e \in I_h \cup \partial \Omega} \int_e \{u_h\} \cdot n_e [v] (w_h^e - w_h^\theta),
\]
We now split the domain of integration as follows [7],

\[ I_h \cup \partial \Omega = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3, \]  
\[ \text{(203)} \]

where

\[ \mathcal{F}_1 = \{ e : \{ u_h \} \cdot \mathbf{n}_e \neq 0 \text{ and } \{ \theta_h \} \cdot \mathbf{n}_e \neq 0 \text{ a.e. on } e \}, \]
\[ \mathcal{F}_2 = \{ e : \{ u_h \} \cdot \mathbf{n}_e \neq 0 \text{ and } \{ \theta_h \} \cdot \mathbf{n}_e = 0 \text{ a.e. on } e \}, \]
\[ \mathcal{F}_3 = (I_h \cup \partial \Omega) \setminus (\mathcal{F}_1 \cup \mathcal{F}_2). \]

The contribution of \( \mathcal{F}_3 \) to \( I \) is zero. Consider \( \mathcal{F}_1 \) and split \( e \) into \( e_1 \) where \( \{ u_h \} \cdot \mathbf{n}_e \) and \( \{ \theta_h \} \cdot \mathbf{n}_e \) have the same sign and into \( e_2 \) where they have opposite signs. On \( e_1 \), \( w_h^\theta - w_h^\theta = 0 \). On \( e_2 \), \( w_h^\theta - w_h^\theta = |w_h| \) up to the sign. Note that, due to the opposite signs on \( e_2 \), we have

\[ |\{ u_h \} \cdot \mathbf{n}_e| \leq |\{ \theta_h \} \cdot \mathbf{n}_e - \{ u_h \} \cdot \mathbf{n}_e|. \]
\[ \text{(204)} \]

We apply Hölder’s inequality, trace estimate (72), and (48). We obtain:

\[ \sum_{e \in \mathcal{F}_1} \left| \int_e \{ u_h \} \cdot \mathbf{n}_e [v](w_h^\theta - w_h^\theta) \right| \]
\[ \leq \sum_{e \in \mathcal{F}_1} \| \{ \theta_h \} \cdot \mathbf{n}_e - \{ u_h \} \cdot \mathbf{n}_e \|_{L^3(e)} \| [v] \|_{L^2(e)} \| |w_h| \|_{L^6(e)} \]
\[ \leq \sum_{e \in \mathcal{F}_1} \sum_{i,j=1}^2 C h^{-1/3} \| \theta_h - u_h \|_{L^3(e)} \| [v] \|_{L^2(e)} \| |w_h| \|_{L^6(e)} h^{-1/6} \]
\[ \leq C \| \theta_h - u_h \|_{L^3(\Omega)} \left( \sum_{e \in I_h \cup \partial \Omega} \sigma h_e^{-1} \| [v] \|_{L^2(e)}^2 \right)^{1/2} \| w_h \|_{DG}. \]
\[ \text{(205)} \]

An alternative bound reads:

\[ \sum_{e \in \mathcal{F}_1} \left| \int_e \{ u_h \} \cdot \mathbf{n}_e [v](w_h^\theta - w_h^\theta) \right| \]
\[ \leq C \| w_h \|_{L^\infty(\Omega)} \| \theta_h - u_h \|_{L^3(\Omega)} \left( \sum_{e \in I_h \cup \partial \Omega} \sigma h_e^{-1} \| [v] \|_{L^2(e)}^2 \right)^{1/2}. \]
\[ \text{(206)} \]

The contribution of \( \mathcal{F}_2 \) to \( I \) is bounded similarly by:

\[ \sum_{e \in \mathcal{F}_2} \left| \int_e \{ u_h \} \cdot \mathbf{n}_e [w_h^\theta] \right| \]
\[ \leq C \| \theta_h - u_h \|_{L^3(\Omega)} \left( \sum_{e \in I_h \cup \partial \Omega} \sigma h_e^{-1} \| [v] \|_{L^2(e)}^2 \right)^{1/2} \| w_h \|_{DG}. \]  
\[ \text{(207)} \]
Alternatively, we have
\[
\sum_{e \in \mathcal{E}_h} \left| \int_E (\mathbf{u}_h) \cdot n_e \mathbf{w}_h \right| \leq \| \mathbf{\theta}_h - \mathbf{u}_h \| \left( \sum_{e \in \mathcal{E}_h} \sigma h e^{-1} \| \mathbf{\nu} \|_{L^2(e)}^2 \right)^{1/2} \| \mathbf{w}_h \|_{L^\infty(\Omega)}. \tag{208}
\]

Bounds (205) and (207) yield (196). Bounds (206) and (208) yield (197).

**Lemma 10** Let \( \mathbf{u}_h, \mathbf{\theta}_h, \mathbf{v}_h \), and \( \mathbf{w}_h \in \mathbf{X}_h \). There exists a constant \( C \) independent of \( h \) such that:
\[
|\mathcal{G}(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h)| + |\mathcal{W}(\mathbf{\theta}_h; \mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h)| \leq C \| \mathbf{u}_h \|_{L^1(\Omega)} \| \mathbf{w}_h \|_{DG} \| \mathbf{v}_h \|_{DG}. \tag{209}
\]

In addition, let \( \mathbf{v} = \mathbf{z} + \mathbf{z}_h \) where \( \mathbf{z} \in \mathbf{X} \) and \( \mathbf{z}_h \in \mathbf{X}_h \). We have,
\[
|\mathcal{G}(\mathbf{u}_h, \mathbf{v}, \mathbf{w}_h)| + |\mathcal{W}(\mathbf{\theta}_h; \mathbf{u}_h, \mathbf{v}, \mathbf{w}_h)| \leq C \| \mathbf{u}_h \| \left( \| \nabla \mathbf{w}_h \|_{L^2(\Omega)} + \| \mathbf{w}_h \|_{L^\infty(\Omega)} \right) \| \mathbf{v} \|_{DG} + C \| \mathbf{u}_h \| \| \mathbf{w}_h \|_{L^\infty(\Omega)} (h^{-1} \| \mathbf{z} \| + \| \nabla \mathbf{z} \|). \tag{210}
\]

**Proof** We have
\[
\mathcal{G}(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h) = \sum_{E \in \mathcal{T}_h} \int_E (\mathbf{u}_h - \nabla \mathbf{v}_h) \cdot \mathbf{w}_h + \frac{1}{2} b(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h).
\]

The first term is bounded by Cauchy-Schwarz inequality and (48). We have
\[
\sum_{E \in \mathcal{T}_h} \left| \int_E (\mathbf{u}_h - \nabla \mathbf{v}_h) \cdot \mathbf{w}_h \right| \leq C \| \mathbf{u}_h \|_{L^1(\Omega)} \| \nabla \mathbf{v}_h \| \| \mathbf{w}_h \|_{DG}. \tag{211}
\]

For the second term, we use (20) and the following identity:
\[
[\mathbf{v}_h, \mathbf{w}_h] = [\mathbf{v}_h] \cdot \{ \mathbf{w}_h \} + [\mathbf{w}_h] \cdot \{ \mathbf{v}_h \}, \quad \forall \mathbf{e} \in \mathcal{E}_h \tag{212}
\]

We obtain
\[
\frac{1}{2} b(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h) = -\frac{1}{2} \sum_{E \in \mathcal{T}_h} \int_E \mathbf{u}_h \cdot ((\nabla \mathbf{v}_h)^T \mathbf{w}_h + (\nabla \mathbf{w}_h)^T \mathbf{v}_h)
+ \frac{1}{2} \sum_{e \in \mathcal{E}_h} \int_E \{ \mathbf{u}_e \} \cdot n_e (\{ \mathbf{v}_h \} \cdot \{ \mathbf{w}_h \} + [\mathbf{w}_h] \cdot \{ \mathbf{v}_h \}).
\]

With Holder’s inequality, trace estimate (72) and (48), we have
\[
\frac{1}{2} |b(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}_h)| \leq C \| \mathbf{u}_h \|_{L^1(\Omega)} \| \mathbf{v}_h \|_{DG} \| \mathbf{w}_h \|_{DG}. \tag{213}
\]
The upwind term is bounded via similar arguments:
\[
\| \mathcal{B}(\boldsymbol{\theta}; \mathbf{u}_h, \mathbf{v}, \mathbf{w}_h) \| \leq C \| \mathbf{u}_h \|_{L^2(\Omega)} \| \mathbf{v} \|_{DG} \| \mathbf{w}_h \|_{DG}.
\] (214)

Bounds (211), (213) and (214) yield estimate (209). Let \( \mathbf{v}_h \) be replaced by \( \mathbf{v} = \mathbf{z} + \mathbf{z}_h \) where \( \mathbf{z} \in X \) and \( \mathbf{z}_h \in X_h \). We replace bound (211) by:
\[
\sum_{E \in \delta_h} \left| \int_{E} (\mathbf{u}_h \cdot \nabla \mathbf{v}) \cdot \mathbf{w}_h \right| \leq \| \mathbf{u}_h \| \| \nabla_h \mathbf{v} \| \| \mathbf{w}_h \|_{L^2(\Omega)}.
\] (215)

We also write:
\[
\begin{align*}
\frac{1}{2} b(\mathbf{u}_h, \mathbf{v}, \mathbf{w}_h) &= -\frac{1}{2} \sum_{E \in \delta_h} \int_{E} \mathbf{u}_h \cdot ((\nabla \mathbf{v})^T) \mathbf{w}_h + (\nabla \mathbf{w}_h)^T \mathbf{v} \\
&\quad + \frac{1}{2} \sum_{e \in I_h} \left( \mathbf{u}_h \cdot \mathbf{n}_e \{ \mathbf{v} \cdot \{ \mathbf{w}_h \} \} + |\mathbf{w}_h| \cdot \{ \mathbf{z} \} + |\mathbf{w}_h| \cdot \{ \mathbf{z}_h \} \right).
\end{align*}
\]

With Hölder’s inequality and trace estimate (72), we obtain
\[
\begin{align*}
\frac{1}{2} |b(\mathbf{u}_h, \mathbf{v}, \mathbf{w}_h)| &\leq C |\mathbf{u}_h| \| \nabla_h \mathbf{v} \| \| \mathbf{w}_h \|_{L^2(\Omega)} + C |\mathbf{u}_h| \| \nabla_h \mathbf{w}_h \|_{L^2(\Omega)} \| \mathbf{v} \|_{L^2(\Omega)} \\
&\quad + C \| \mathbf{w}_h \|_{L^2(\Omega)} |\mathbf{u}_h| \left( \sum_{e \in I_h} \sigma h^{-1}_e \| \mathbf{v} \|_{L^2(e)}^2 \right)^{1/2} \\
&\quad + C \| \mathbf{w}_h \|_{L^2(\Omega)} |\mathbf{u}_h| (h^{-1} \| \mathbf{z} \| + \| \nabla_h \mathbf{z} \|) \\
&\quad + C \| \mathbf{u}_h \|_{L^2(\Omega)} \left( \sum_{e \in I_h} \sigma h^{-1}_e \| \mathbf{w}_h \|_{L^2(e)}^2 \right)^{1/2} \| \mathbf{z}_h \|_{L^2(\Omega)}.
\end{align*}
\] (216)

An alternative bound for the upwind term reads:
\[
|b(\mathbf{u}_h, \mathbf{v}, \mathbf{w}_h)| \leq C \| \mathbf{w}_h \|_{L^2(\Omega)} \| \mathbf{u}_h \| \| \mathbf{v} \|_{DG}.
\] (217)

Bounds (215), (216), (217), and (48) yield estimate (210).

**Lemma 11 [Bound (153)]**

**Proof** For \( n \geq 0 \), recall that \( \mathbf{E}^n \in X \cdot \mathbf{E}^n = \Pi_h \mathbf{u}^n - \mathbf{u}^n \). We write:
\[
\begin{align*}
\hat{R}_t(\delta_t \mathbf{e}^{n+1}_h) &= \frac{1}{\tau} (\tau (\partial_t \mathbf{u})^{n+1} - \tau (\partial_t \mathbf{u})^n - \mathbf{u}^{n+1} + 2 \mathbf{u}^n - \mathbf{u}^{n-1}, \delta_t \mathbf{e}^{n+1}_h) \\
&\quad - \frac{1}{\tau} (\mathbf{E}^{n+1} - 2 \mathbf{E}^n + \mathbf{E}^{n-1}, \delta_t \mathbf{e}^{n+1}_h).
\end{align*}
\]

With Cauchy-Schwarz’s and Young’s inequality, we have
\[
\begin{align*}
|\hat{R}_t(\delta_t \mathbf{e}^{n+1}_h)| \leq \ell \tau \mu \| \delta_t \mathbf{e}^{n+1}_h \|_{DG}^2 + C \tau^{-3} \| \mathbf{u}^n - \mathbf{u}^{n+1} + \tau (\partial_t \mathbf{u})^{n+1} \| \| \mathbf{E}^{n+1} - \mathbf{E}^n \|^2 \\
&\quad + \frac{C}{\mu} \tau^{-3} \| \mathbf{u}^n - \mathbf{u}^{n-1} + \tau (\partial_t \mathbf{u})^n \|^2 + \frac{C}{\mu} \tau^{-3} \| \mathbf{E}^{n+1} - \mathbf{E}^n \|^2 + \frac{C}{\mu} \tau^{-3} \| \mathbf{E}^n - \mathbf{E}^{n-1} \|^2.
\end{align*}
\]
With Taylor expansions and Cauchy-Schwarz’s inequality, we have
\[
\|u^n - u^{n+1} + \tau(\partial_t u)^{n+1}\|^2 \leq \tau^3 \int_{\Omega} \|\partial_t u\|^2,
\]
\[
\|u^n - u^{n-1} - \tau(\partial_t u)^{n}\|^2 \leq \tau^3 \int_{\Omega} \|\partial_t u\|^2.
\]
Similarly, with a Taylor expansion and approximation properties (42) - (43), for \( n \geq 1 \):
\[
\|E^n - E^{n-1}\|^2 \leq \tau \int_{\Omega} \|\partial_t E\|^2 \leq C \tau h^4 \int_{\Omega} \|\partial_t u\|^2.
\]
With the CFL condition (131), we know that \( h^4 \leq c_2^2 \tau^2 \). This implies that for \( n \geq 1 \):
\[
\|E^n - E^{n-1}\|^2 \leq C \tau^3 \int_{\Omega} \|\partial_t u\|^2.
\]
Thus, by using these bounds, we obtain
\[
\hat{R}_i(\delta_\tau \tilde{e}_b^{n+1}) \leq \tau \mu \|\delta_\tau \tilde{e}_b^{n+1}\|^2 + \frac{C}{\mu} \int_{\Omega} \|\partial_t u\|^2 \leq \frac{C}{\mu} \int_{\Omega} \|\partial_t u\|^2.
\]

**Lemma 12 [Bound (180)]**

**Proof** The expression for \( l_{1n} \) reads:
\[
l_{1n} = \sum_{E \in \partial \partial \Omega} \left( u_n^{n-1} \cdot \nabla (\delta_\tau U^{n+1} - \delta_\tau U^{n}) (\delta_\tau E^{n+1} + \delta_\tau \tilde{e}_b^{n+1}) + \frac{1}{2} b(u_n^{n-1}, (\delta_\tau U^{n+1} - \delta_\tau U^{n}) (\delta_\tau E^{n+1} + \delta_\tau \tilde{e}_b^{n+1})ight)
\]
\[
+ \sum_{E \in \partial \partial \Omega} \left( (u_n^{n-1} \cdot n_x) (\delta_\tau U^{n+1} - \delta_\tau U^{n}) (\delta_\tau E^{n+1} + \delta_\tau \tilde{e}_b^{n+1})ight) + \frac{1}{2} \sum_{E \in \partial \partial \Omega} \left( (u_n^{n-1} \cdot n_x, (\delta_\tau U^{n+1} - \delta_\tau U^{n}) (\delta_\tau E^{n+1} + \delta_\tau \tilde{e}_b^{n+1}) = \sum S_i^n.
\]
We handle each term separately. For \( S_1 \), we have
\[
S_1^n = \sum_{E \in \partial \partial \Omega} \left( (\delta_\tau E^{n+1} + \delta_\tau \tilde{e}_b^{n+1}) \cdot \nabla (\delta_\tau U^{n+1} - \delta_\tau U^{n}) (\delta_\tau E^{n+1} + \delta_\tau \tilde{e}_b^{n+1} \cdot n_x) = \sum S_i^n.
\]
We use Holder’s inequality, (48), (173), (174), (175), and (41).
\[
|S_i^n| \leq \|e_n^{n-1}\| \|\nabla (\delta_\tau U^{n+1} - \delta_\tau U^{n})\| L^2(\Omega) \|\delta_\tau E^{n+1} + \delta_\tau \tilde{e}_b^{n+1}\| L^2(\Omega)
\]
\[
+ \|\Pi_\tau u_n^{n-1}\| L^2(\Omega) \|\nabla (\delta_\tau U^{n+1} - \delta_\tau U^{n})\| L^2(\Omega) \|\delta_\tau E^{n+1} + \delta_\tau \tilde{e}_b^{n+1}\| L^2(\Omega)
\]
\[
\leq C \|e_n^{n-1}\| \|\delta_\tau X^{n+1}\| \|\delta_\tau E^{n+1}\| \|\delta_\tau \tilde{e}_b^{n+1}\|.
\]

(219)
From (134) and (39), note that
\[
\|\delta_t e_h^{n+1}\| \leq \|\delta_t e_h^{n+1}\| + C\tau|\delta_t \phi_{h}^{n+1}|_{DG}.
\] (220)

Hence, with (178), we obtain
\[
\begin{align*}
|S_1^n| & \leq C\|e_h^{n-1}\||\delta_t \chi^{n+1}||h|^2|\delta_t u^{n+1}|_{H^1(\Omega)} + \|\delta_t e_h^{n+1}\|_{DG} \\
+ C\|u^{n-1}\||h||\delta_t \chi^{n+1}||h|^2|\delta_t u^{n+1}|_{H^1(\Omega)} + \tau|\delta_t \phi_{h}^{n+1}|_{DG} + \|\delta_t e_h^{n+1}\|. 
\end{align*}
\] (221)

For $S_2^n$, we have
\[
S_2^n = \frac{1}{2} b(u_h^{n-1}, (\delta_t U_h^{n+1} - \delta_t U^{n+1}) \cdot \delta_t E^{n+1}) \\
+ \frac{1}{2} b(u_h^{n-1}, (\delta_t U_h^{n+1} - \delta_t U^{n+1}) \cdot \delta_t e_h^{n+1}) = S_{2,1}^n + S_{2,2}^n. 
\] (222)

The term $S_{2,1}^n$ reads:
\[
S_{2,1}^n = \frac{1}{2} \sum_{E \in \delta_k} \int_E (\nabla \cdot u_h^{n-1})(\delta_t U_h^{n+1} - \delta_t U^{n+1}) \cdot \delta_t E^{n+1} \\
- \frac{1}{2} \sum_{e \in \partial_k \Omega \cap e} \int_e \{(\delta_t U_h^{n+1} - \delta_t U^{n+1}) \cdot \delta_t e_h^{n+1}\}[u_h^{n-1}] \cdot n_e. 
\] (223)

We use Holder’s inequality, trace inequalities (71)-(72), and inverse estimates (70). We obtain:
\[
|S_{2,1}^n| \leq C h^{-1}\|u_h^{n-1}\||\delta_t U_h^{n+1} - \delta_t U^{n+1}||\delta_t e^{n+1}||_{L^\infty(\Omega)} \\
+ C|\delta_t e^{n+1}||_{L^\infty(\Omega)}|h^{-1}|\delta_t U_h^{n+1} - \delta_t U^{n+1}|| + ||\nabla_h(\delta_t U_h^{n+1} - \delta_t U^{n+1})||u_h^{n-1}||. 
\]

Similar arguments to (80) yield:
\[
\|\delta_t U_h^{n+1} - \delta_t U^{n+1}\| \leq C h^2||\delta_t \chi^{n+1}|| 
\]

With (174), the above bound, and the stability result in Lemma 4, we obtain
\[
|S_{2,1}^n| \leq C(1 + 1/\mu)^{1/2}h||\delta_t \chi^{n+1}||||\delta_t e^{n+1}||_{L^\infty(\Omega)} \\
\leq C(1 + 1/\mu)^{1/2}h||\delta_t \chi^{n+1}||||\delta_t u^{n+1}||_{H^2(\Omega)}. 
\] (224)

The expression for $S_{2,2}$ is:
\[
S_{2,2}^n = \frac{1}{2} \sum_{E \in \delta_k} \int_E (\nabla \cdot u_h^{n-1})(\delta_t U_h^{n+1} - \delta_t U^{n+1}) \cdot \delta_t e_h^{n+1} \\
- \frac{1}{2} \sum_{e \in \partial_k \Omega \cap e} \int_e \{(\delta_t U_h^{n+1} - \delta_t U^{n+1}) \cdot \delta_t e_h^{n+1}\}[u_h^{n-1}] \cdot n_e. 
\] (225)
With Holder’s, trace estimates (71)-(72), and inverse inequalities (68), we obtain

\[
|S_{2,2}^n| \leq C(h^{-1}||\varepsilon_h^{n-1}|| + ||\nabla_h P_1 u^{n-1}||)\|\delta_t U_h^{n+1} - \delta_t U_h^{n+1}||_{L^2(\Omega)} + \delta_t \varepsilon_h^{n+1}||_{L^2(\Omega)}^2 \\
+ C(h^{-1/2}||\varepsilon_h^{n}|| + ||\Pi_h u^{n-1}||_{L^2(\Omega)}) (h^{-1}||\delta_t U_h^{n+1} - \delta_t U_h^{n+1}||) \\
+ ||\nabla_h (\delta_t U_h^{n+1} ||_{L^2(\Omega)}^2 ||\delta_t \varepsilon_h^{n+1}||_{L^2(\Omega)}^2)
\]

With Poincare’s inequality, we have

\[
||\delta_t U_h^{n+1} - \delta_t U_h^{n+1}||_{L^2(\Omega)} \leq C||\delta_t U_h^{n+1} - \delta_t U_h^{n+1}||_{DG} \leq Ch ||\delta_t \chi^{n+1}||
\]

Hence, with (48) and (174), we have

\[
|S_{2,2}^n| \leq C||\varepsilon_h^{n-1}||^2 ||\delta_t \chi^{n+1}||^2 ||\delta_t \varepsilon_h^{n+1}||_{DG} \\
+ C(||\varepsilon_h^{n-1}||^2 ||h^{1/2}|| ||\delta_t \chi^{n+1}|| + ||\delta_t \varepsilon_h^{n+1}||_{DG}) (h^{1/2} ||\delta_t \chi^{n+1}|| + ||\delta_t \varepsilon_h^{n+1}||_{DG}).
\]

With the assumption that \( u \in L^2(0, T; H^{k+1}(\Omega)) \), we obtain a bound for \( S_{2}^n \):

\[
|S_{2}^n| \leq C(||\varepsilon_h^{n-1}||^2 + ||h|| ||\delta_t \chi^{n+1}||^2 ||\delta_t \varepsilon_h^{n+1}||_{DG} \\
+ C(1 + 1/\mu)^{1/2} h ||\delta_t \chi^{n+1}|| ||\delta_t \varepsilon_h^{n+1}||_{H^2(\Omega)}.
\]

For \( S_{3}^n \), we use the definition (10) and write \( u_h^{n-1} = \varepsilon_h^{n-1} + \Pi_h u^{n-1} \). We also use that

\[
||\delta_t U_h^{n+1}||_{DG} = 0 \text{ a.e. on } \Gamma_k \cup \partial \Omega. \text{ Hence, we bound } S_{3}^n \text{ as follows.}
\]

\[
|S_{3}^n| \leq C(||\varepsilon_h^{n-1}||_{L^2(\Omega)} ||\delta_t U_h^{n+1} - \delta_t U_h^{n+1}||_{DG} ||\delta_t \varepsilon_h^{n+1}||_{L^2(\Omega)} \\
+ C||\delta_t U_h^{n+1}||_{L^2(\Omega)} ||\varepsilon_h^{n}|| ||\delta_t \varepsilon_h^{n+1}||_{DG} + C||\Pi_h u^{n-1}||_{L^2(\Omega)} (||\varepsilon_h^{n}|| + ||\delta_t \varepsilon_h^{n+1}||_{DG}) + (1 + 1/\mu)^{1/2} h ||\delta_t \varepsilon_h^{n+1}||_{DG}).
\]

With (178), (179), (200), (68), (48), (41), (42)-(43), and (174), we obtain

\[
|S_{3}^n| \leq C(||\varepsilon_h^{n-1}|| ||\delta_t \chi^{n+1}|| ||\delta_t \varepsilon_h^{n+1}||_{DG} + C||\delta_t \varepsilon_h^{n+1}||_{H^2(\Omega)} (||\varepsilon_h^{n-1}|| ||\delta_t \varepsilon_h^{n+1}||_{DG} \\
+ Ch||\delta_t \chi^{n+1}|| ||\delta_t \varepsilon_h^{n+1}||_{DG} + (1 + 1/\mu)^{1/2} ||\delta_t \varepsilon_h^{n+1}||_{H^2(\Omega)} + ||\delta_t \varepsilon_h^{n+1}|| + \tau ||\delta_t \varepsilon_h^{n+1}||_{DG}).
\]

The term \( S_{3}^n \) is treated in a similar way to \( S_{3}^n \) and the above bound also holds for \( S_{3}^n \).

We now combine the above bounds, use the assumption that \( u \in L^2(0, T; H^{k+1}(\Omega)) \) and obtain:

\[
|I_{1,1}^n| \leq C(||\varepsilon_h^{n-1}|| ||\delta_t \chi^{n+1}|| (h^{1/2} ||\delta_t U_h^{n+1}||_{H^1(\Omega)} + ||\delta_t \varepsilon_h^{n+1}||_{DG}) \\
+ Ch||\delta_t \chi^{n+1}|| ||\delta_t U_h^{n+1}||_{H^1(\Omega)} + ||\delta_t \varepsilon_h^{n+1}||_{DG}) \\
+ Ch||\delta_t \chi^{n+1}|| ||\delta_t \varepsilon_h^{n+1}||_{DG} + (1 + 1/\mu)^{1/2} ||\delta_t U_h^{n+1}||_{H^2(\Omega)} + ||\delta_t \varepsilon_h^{n+1}||_{DG} \\
+ C||\delta_t U_h^{n+1}||_{DG} (||\varepsilon_h^{n-1}|| ||\delta_t U_h^{n+1}||_{H^2(\Omega)} + ||\delta_t \varepsilon_h^{n+1}||_{DG} \\
+ ||\delta_t \varepsilon_h^{n+1}||_{DG} + (1 + 1/\mu)^{1/2} ||\delta_t U_h^{n+1}||_{H^2(\Omega)}).
\]

Bound (180) is obtained by using (176) and applying Young’s inequality in the above bound.
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