Pre-freezing of multifractal exponents in random energy models with a logarithmically correlated potential

Yan V Fyodorov

School of Mathematical Sciences, University of Nottingham, Nottingham NG7 2RD, UK
E-mail: yan.fyodorov@nottingham.ac.uk

Received 20 March 2009
Accepted 10 June 2009
Published 10 July 2009

Online at stacks.iop.org/JSTAT/2009/P07022
doi:10.1088/1742-5468/2009/07/P07022

Abstract. Boltzmann–Gibbs measures generated from logarithmically correlated random potentials are multifractal. We investigate the abrupt change (‘pre-freezing’) of multifractality exponents extracted from the averaged moments of the measure—the so-called inverse participation ratios. The pre-freezing can be identified with termination of the disorder-averaged multifractality spectrum. The naive replica limit employed to study a one-dimensional variant of the model is shown to break down at the pre-freezing point. Further insights are possible when employing zero-dimensional and infinite-dimensional versions of the problem. In particular, the latter version allows one to identify the pattern of the replica symmetry breaking responsible for the pre-freezing phenomenon.

Keywords: cavity and replica method, disordered systems (theory), energy landscapes (theory)

ArXiv ePrint: 0903.2502
1. Introduction

Investigations of multifractal measures of diverse origins has been a very active field of research in various branches of physics for several decades [1]. To set the notation, consider a certain (e.g. hypercubic) lattice of linear extent $L$ in $N$-dimensional space, with $M \sim L^N$ standing for the total number of sites in the lattice. The measures of interest are usually defined via weights $0 \leq p_i \leq 1$ associated with every lattice site $i = 1, 2, \ldots, M$ and normalized as $\sum_{i=1}^{M} p_i = 1$. One can imagine a few different spatial arrangements of weights $p_i$ across the lattice sites. In the case of simply extended measures the weights are of similar magnitude at each lattice site, the normalization condition then implying the scaling $p_i \sim M^{-1}$ in the large-$M$ limit. As a generalization of the above example one can imagine the non-zero weights $p_i$ supported evenly on a fractal subset of lattice sites of effective dimension $0 \leq N_{\text{ef}} < N$. In the limiting case of $N_{\text{ef}} = 0$ we then deal with localized measures characterized by weights $p_i$ essentially different from zero only inside one or a few blobs of finite total volume. In such a situation weights stay finite even when $M \to \infty$, that is $p_i = O(M^0)$. Finally, in the most interesting case of multifractal measures the weights scale differently at different sites: $p_i \sim M^{-\alpha_i}$.

The full set of exponents $0 \leq \alpha_i < \infty$ can be conveniently characterized by the density $\rho(\alpha) = \sum_{i=1}^{M} \delta(\alpha - \alpha_i)$ whose scaling behaviour in the large-$M$ limit is expected to be non-trivial: $\rho(\alpha) \sim M^{f(\alpha)}$, with the (convex) function $f(\alpha)$ known in this context as the multifractality spectrum. Note that the total number $m(\alpha) = \int_{0}^{\alpha} \rho(\alpha) \, d\alpha$ of sites of the lattice characterized by the scaling exponents $\alpha_i < \alpha$ must satisfy the inequality $m(\alpha) \gtrsim M^{f(\alpha)} \geq 1$; hence $f(\alpha) \geq 0$. The condition $f(\alpha) = 0$ defines generically the minimal $\alpha_-$ and maximal $\alpha_+$ threshold values of the exponents which can be observed in a typical realization of disorder. Note that the constraint $p_i \leq 1$ implies $\alpha_- \geq 0$.

An alternative, frequently more practical way of describing multifractality is via the set of exponents $\tau_q$ characterizing the large-$M$ behaviour of the so-called inverse participation ratios (IPRs) $P_q$ which are simply the moments of the corresponding

1 Usually one defines exponents via the relation $p_i \sim L^{-N\alpha_i}$, i.e. by reference to the linear scale $L$ instead of the total number of sites $M \sim L^N$. We however find it more convenient to get rid of trivial spatial dimension factor $N$, and concentrate only on essential parameter behaviour.

doi:10.1088/1742-5468/2009/07/P07022

2
Pre-freezing in a logarithmically correlated potential

$$P_q = \sum_{i=1}^{M} p_i^q = \int M^{-q\alpha} \rho(\alpha) \, d\alpha.$$  \hspace{1cm} (1)

Substituting in the above definition the relation $\rho(\alpha) \sim M^{f(\alpha)}$ one can evaluate the integral in the large-$M$ limit by saddle-point method. One then finds the relation between $\tau_q$ and $f(\alpha)$ given by the Legendre transform:

$$P_q \sim M^{-\tau_q}, \quad \tau_q = q\alpha - f(\alpha) \quad \text{where} \quad q = \frac{df}{d\alpha}. \hspace{1cm} (2)$$

The relation is valid as long as the $\alpha$-integral is dominated by the saddle point. It is easy to see however that at large enough $|q|$ the integral should be dominated rather by the vicinity of the thresholds $\alpha_{\pm}$, resulting in linear behaviour of the exponents with $q$, i.e. $\tau_q = q\alpha_{\pm}$.

The above description is valid for multifractal measures of any nature. In recent years important insights were obtained for disorder-generated multifractality; see [2] and also [3] for a comprehensive discussion in the context of Anderson localization transitions, and [4] for an example related to polymers in disordered media. One of the specific features of multifractality in the presence of disorder is the possibility of the existence of two different sets of exponents, $\tau_q$ and $\tilde{\tau}_q$, governing the scaling behaviour of typical $P_q$ versus disorder-averaged IPRs, $\langle P_q \rangle \sim M^{-\tilde{\tau}_q}$. Here and henceforth the brackets stand for the averaging over different realizations of the disorder. The difference is related to a possibility for disorder-averaged moments to be dominated by exponentially rare configurations. A related aspect of the problem is that the ‘annealed’ multifractality spectrum recovered from the disorder-averaged multifractal exponents $\tilde{\tau}_q$ via the Legendre transform (1) can be negative: $\tilde{f}(\alpha) < 0$. Indeed, those values reflect events which are exponentially rare [5] and need exponentially many realizations of disorder to be observed experimentally or numerically. In the context of Anderson localization the disorder-averaged moments of wavefunction intensities are readily available via standard techniques in the nonlinear $\sigma$-model framework; see [2,3] and references therein. At the same time, extracting typical values of the multifractality exponents in that case is a much more challenging task which has been successfully accomplished only very recently [3]. In the present paper we would like to concentrate on a different type of models where, in contrast, calculating disorder-averaged moments in the full parameter range is more difficult, whereas the typical values of IPR exponents are readily accessible.

Arguably the simplest model with disorder-induced multifractality which has attracted considerable interest in recent years is the case of a single classical particle subject to a random Gaussian potential $V(x)$ logarithmically correlated in space:

$$\langle V(x_1) V(x_2) \rangle = -g^2 \ln \left[ \frac{(x_1 - x_2)^2 + a^2}{L^2} \right], \quad a \ll L, \quad x \in \mathbb{R}^N, \hspace{1cm} (3)$$

where we assume $|x| < L$, and the parameter $a$ stands for a small-scale cut-off. From the point of view of equilibrium statistical mechanics the model is characterized by the Gibbs–Boltzmann measure $p_\beta(x) = (1/Z(\beta)) \exp -\beta V(x)$, as a function of the inverse temperature $\beta = 1/T$, and the sample size $L$. The normalization $\int_{|x|\leq L} p_\beta(x) \, dx = 1$.

doi:10.1088/1742-5468/2009/07/P07022
implies the value of the partition function to be given by
\[ Z(\beta) = \int_{|x| \leq L} \exp -\beta V(x) \, dx. \] (4)

According to the general discussion, the multifractal structure of the Gibbs–Boltzmann measure can be extracted from the knowledge of moments
\[ P_q = \int_{|x| \leq L} p^q(x) \, dx = \frac{Z(\beta q)}{[Z(\beta)]^q} \sim L^{-N\tau_q} \quad \text{as} \quad L \to \infty. \] (5)

Identifying \( M \sim (L/a)^N \), the equations (5) and (4) imply the following expression for the typical exponents \( \tau_q \) in terms of the appropriately normalized free energy of the system:
\[ \tau_q = \frac{|q|\beta F(|q|/\beta) - q\beta F(\beta)}{F(\beta)} = -\lim_{M \to \infty} \frac{\langle \ln Z(\beta) \rangle}{\beta \ln M}. \] (6)

Although the model is well defined in any \( N \)-dimensional space [6,7] it is the two-dimensional situation which attracts most attention, with equation (3) describing the correlations of the Gaussian free field. In particular, for \( N = 2 \) the corresponding statistics of Gibbs–Boltzmann weights is known to be deeply related to a variety of interesting physical problems, ranging from quantum mechanics of Dirac particles in a random magnetic field [8,9] to the Liouville model of quantum gravity [10] and the theory of self-gravitating particles [11]; see a detailed discussion in [6]. Actually, the dimensionality of space plays in many respects only a secondary role and many (although not all) essential features of the model are expected to be independent of \( N \). The latter point of view is amply supported by the renormalization group arguments [6] and by explicit computations for \( N = \infty \) [7] and \( N = 1 \) [12]. It is also worth mentioning that various one-dimensional versions of the problem attracted considerable interest recently in the context of multifractal random walks [13,14], extreme value statistics [12,6,15] and quantum gravity-related [16] probabilistic questions; see [17] and references therein.

2. The random energy model as a toy model for disorder-induced multifractality

A particular extreme ‘toy model’ case of the problem is represented by the famous random energy model (REM) of Derrida [18,19], which amounts to replacing the random potential by a collection of \( M \) independent Gaussian variables, after the natural identification \( M \sim (L/a)^N \) and with the variances scaled with \( M \) in the same way as in the logarithmic case: \( \langle V_i^2 \rangle = 2g^2 \ln M \). The only control parameter for the model is \( \gamma = \beta^2 g^2 \), and the REM is simple enough to allow explicit calculation of the free energy [18,19]. The typical multifractality exponents turned out to be given by [9]
\[ \tau_{q>1}(\gamma) = \begin{cases} 
(q - 1)(1 - \gamma q), & 0 \leq \gamma < \frac{1}{q^2}, \\
q(1 - \sqrt{q})^2, & \frac{1}{q^2} < \gamma < 1, \\
0, & \gamma > 1.
\end{cases} \] (7)

The phenomenon of vanishing of the exponents \( \tau_{q>1} \) in the low-temperature phase \( \gamma > 1 \) is called freezing and is qualitatively interpreted in terms of the Boltzmann measure being essentially localized on a few sites for low enough temperature or strong enough disorder.
The typical multifractality spectrum corresponding to the above exponents is [9]

\[
    f(\alpha) = \begin{cases} 
        1 - \frac{1}{4\gamma} [\alpha - (1 + \gamma)]^2, & \text{for } \gamma < 1, \\
        -\frac{1}{4\gamma} [\alpha^2 - 4\sqrt{\gamma}\alpha], & \text{for } \gamma > 1,
    \end{cases}
\]  

(8)

where the expression in the first line assumes the range of exponents \( \alpha_- = (1 - \sqrt{\gamma})^2 \leq \alpha \leq 1 + \gamma \), whereas in the second line \( 0 \leq \alpha \leq 2\sqrt{\gamma} \). Thinking in terms of the multifractality spectrum it is easy to see that the freezing phenomenon at \( \gamma > 1 \) is related to \( \alpha_- = 0 \), when the leftmost end of the curve \( f(\alpha) \) hits the vertical axis precisely at zero level: \( f(0) = 0 \). Similarly, the change of behaviour of the typical exponent \( \tau_q \) for \( \gamma > 1/q^2 \) is induced by dominance of the point \( \alpha_- \) in the integration over \( \alpha \), equation (1).

Although obtained in the framework of REM approximation, the above features of the typical spectrum are expected to be shared by all the logarithmic models for any \( N \geq 1 \) [9, 6], which is indeed confirmed by explicit calculations for \( N = \infty \) [7] and \( N = 1 \) cases [12]. In contrast to the case of typical exponents \( \tau_q \), extracting the ‘annealed’ exponents \( \bar{\tau}_q \) from disorder-averaged moments in the logarithmic models poses a serious technical challenge. The only systematic attempt in this direction was undertaken for \( N = 2 \) in the framework of mapping to the Liouville model of quantum gravity [10] where it was found that \( \bar{\tau}_{q=1} = (q-1)(1 - \gamma q) \) for \( 0 \leq \gamma < \gamma_q = 1/(2q - 1) < 1 \). However, for \( \gamma > \gamma_q \) the Liouville theory was observed to develop unsurmountable singularities and yielded no reliable value of the exponents \( \bar{\tau}_q \). This state of matter clearly calls for reconsidering the problem within the general framework of logarithmic models.

The natural starting point is again the standard REM representing in many respects a zero-dimensional limit of the logarithmic models. For such a ‘toy model’ case the disorder-averaged moments (IPRs) \( \langle P_q \rangle \) can be evaluated by well-controlled methods [19, 20]. Surprisingly, explicit expressions for the IPRs seem to be available in the literature only in the low-temperature phase \( \gamma > 1 \) [20]–[22]. Extending the analysis of [19, 20] we find for \( M \gg 1 \) and \( q > 1 \)

\[ \langle P_q \rangle = \langle Z(\beta q) / |Z(\beta)|^q \rangle \]

\[
    = \begin{cases} 
        M^{-(q-1)(1-\gamma q)} / 2\sqrt{\pi \gamma} \ln M, & 0 \leq \gamma < 1/(2q - 1), \\
        M^{-(1-\gamma q)^2/(4\gamma)} \Gamma ((1 + \gamma)/(2\gamma)) (q - (1 + \gamma)/(2\gamma)), & \frac{1}{2q-1} < \gamma < 1, \\
        \Gamma(q - (1/\sqrt{\gamma})) / \Gamma(q) \Gamma (1 - (1/\sqrt{\gamma})), & \gamma > 1,
    \end{cases}
\]

(9)

where we included in the last line the well-known low-temperature results of [20]–[22], with \( \Gamma(x) \) standing for the Euler gamma-function. The fact that for \( \gamma > 1 \) the moments remain finite in the limit \( M \to \infty \) reflects the quasi-localized nature of the Boltzmann–Gibbs measure in the low-temperature phase. As a consequence, the ‘annealed’ multifractal exponents remain frozen: \( \bar{\tau}_q = 0 \). In the high-temperature phase the exponents \( \bar{\tau}_q \) are non-vanishing and in the range \( 0 < \gamma < (1/q^2) \) typical and annealed exponents coincide. The annealed exponents actually keep that common value up to \( \gamma = \gamma_q = 1/(2q - 1) \). Both the value \( \bar{\tau}_q \) for \( \gamma < \gamma_q \) and the value of the threshold \( \gamma_q \) are in full agreement with the Liouville model analysis [10]. Finally, in the range \( \gamma_q < \gamma < 1 \) the annealed exponents

doi:10.1088/1742-5468/2009/07/P07022
are changed drastically by acquiring the $q$-independent value which relates to $\gamma$ in a non-polynomial way, $\tilde{\tau}_q = (1 - \gamma)^2 / 2\gamma$. The behaviour of typical and annealed multifractal exponents for various values of the control parameter $\gamma$ is summarized in the diagram 1, and is further discussed below.

Using equation (9) one can restore the corresponding mean density of multifractal exponents:

$$\langle \rho(\alpha) \rangle = \left\langle \sum_{i=1}^{M} \delta(\alpha - \alpha_i) \right\rangle \approx C(M, \alpha) M \check{f}(\alpha),$$

where the ‘annealed’ multifractality spectrum turned out to be given by

$$\check{f}(\alpha) = \begin{cases} 
1 - \frac{1}{4\gamma} [\alpha - (1 + \gamma)]^2, & \text{for } 0 \leq \gamma < 1 \text{ and } 0 \leq \alpha \leq 1 + \gamma, \\
-\frac{1}{4\gamma} [\alpha^2 - 4\sqrt{\gamma} \alpha], & \text{for } \gamma > 1 \text{ and } 0 \leq \alpha \leq 2\sqrt{\gamma}. 
\end{cases} \quad (10)$$

These expressions show that the disorder-averaged (or ‘annealed’) spectrum is precisely the same as the typical spectrum corresponding to equation (8), with the only essential difference being that annealed spectrum in the first line of equation (10) becomes negative in the range $0 \leq \alpha < \alpha_- = (1 - \sqrt{\gamma})^2$. These values of $\alpha$ correspond to exponentially rare events in full agreement with our earlier discussion and the picture developed in [2,5]. The value $\alpha = 0$ is the lowest possible value of the exponent $\alpha$, and for this reason is frequently called the ‘termination point’ of the (disorder-averaged) multifractality spectrum [2]. Further substituting the corresponding $\langle \rho(\alpha) \rangle$ into the integral over $2\alpha$ (see equation (1)), we find that in the range $1/(2q - 1) < \gamma < 1$ the integral for $\langle P_{q>1} \rangle$ is actually dominated

---

**Figure 1.** Regimes of behaviour for two kinds of multifractal exponents for the REM: typical exponent $\tau_q$ versus annealed $\tilde{\tau}_q$ at different values of effective disorder $\gamma$. 

---

2 It is worth mentioning that for the REM one can extract not only the multifractality spectrum, but also the expressions for the pre-exponential factors $C(M, \alpha)$, which goes beyond the standard precision of the multifractality analysis. For example, in the high-temperature phase $0 \leq \gamma < 1$ one finds $C(M, \alpha) = \frac{1}{\gamma} \sqrt{\frac{M}{\pi \gamma}} (1 - M^{-\alpha})^{-(\alpha - 1 - \gamma)/(2\gamma)}$. 

doi:10.1088/1742-5468/2009/07/P07022
in the limit $M \gg 1$ by the vicinity of the lower integration limit $\alpha = 0$ rather than by the stationary point of the integrand. It is in this way that $\tilde{\tau}_q$ acquires the $q$-independent value $(1 - \gamma)^2/2\gamma$ (cf equation (9)), as was indeed anticipated in [2] on heuristic grounds. We see that the actual mechanism behind the observed drastic change of the annealed exponents is very close to that forcing the exponent freezing at $\gamma > 1$, and in a sense is the precursor of the latter behaviour. For this reason it is natural to suggest calling this phenomenon pre-freezing. We shall also see later on that the replica approach gives additional support to associating the observed behaviour with a partial freezing of a certain kind.

3. Annealed multifractality exponents for the circular logarithmic model

After achieving a detailed understanding of the toy REM limit it is natural to try to extract annealed exponents for a more realistic case of a one-dimensional model with logarithmic correlations. The most promising instance of a 1D system of that type is arguably the ‘circular logarithmic model’ (CLM) introduced in [12]. To define it, consider the lattice of coordinates of the points are given by $\theta_k = (2\pi/M)k$, $k = 1, 2, \ldots, M$. Then it is easy to see that the values of the free field $V_i$ associated with those points are characterized by covariances

$$ C_{kl} = \langle V_k V_l \rangle = -g^2 \ln \left\{ 4 \sin^2 \frac{\theta_k - \theta_l}{2} \right\} + 2g^2 \ln \left( \frac{L}{R} \right), \quad k \neq l. \tag{11} $$

The first term in (11) defines precisely the CLM as described in [12], with the constant second term playing a somewhat trivial role and thus omitted henceforth. It turns out that the consistency of the procedure requires one to choose the variance $\langle V_k^2 \rangle = V^2$ to satisfy $V^2 \geq 2g^2 \ln M$, and following [12] we choose simply $V^2 = 2g^2 \ln M$. The partition function for our model is defined in the standard way through $Z_\beta = \sum_{i=1}^M e^{-\beta V_i}$, with the goal of evaluating the IPR moments $P_q = \langle Z(\beta q)/[Z(\beta)]^q \rangle$ in the limit $M \gg 1$ for various values of the control parameter $\gamma = \beta g^2$.

In the rest of the paper we restrict our explicit calculations to the simplest representative case $q = 2$, and employ a variant of the replica trick:

$$ \langle P_2 \rangle = \langle Z(2\beta)/[Z(\beta)]^2 \rangle = \lim_{n \to 0} \langle Z(2\beta) [Z(\beta)]^{n-2} \rangle, \tag{12} $$

implying a kind of continuation from integer values $n \geq 2$ to $n = 0$. The key point is that for integer $n \geq 2$ the disorder averaging in (12) can be performed by the method described in detail in [12], which gives, in particular

$$ \left\langle Z(2\beta) [Z(\beta)]^k \right\rangle |_{M \gg 1} = M^{1+k+\gamma(k+4)} J_k(\gamma), \quad 0 \leq k < \frac{1}{\gamma} - 3, \tag{13} $$

where $J_k(\gamma)$ is the Dyson–Morris integral [23] given by a product of gamma-functions:

$$ J_k(\gamma) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} d\theta_1 \cdots d\theta_k \prod_{p < q} \left| e^{i\theta_p} - e^{i\theta_q} \right|^{-2\gamma} \prod_{l=1}^k \left| 1 - e^{i\theta_l} \right|^{-4\gamma} \tag{14} $$

$$ = \frac{1}{[\Gamma(1-\gamma)]^{k-1}} \frac{1}{\Gamma(1-\gamma)(k+2)\Gamma(1-\gamma(k+3))} \tag{15} $$

doi:10.1088/1742-5468/2009/07/P07022
Taking in the above expression the ‘naive’ replica limit $k \to -2$ we arrive at the following expression for the IPR:

$$\langle P_2 \rangle = M^{-(1-2\gamma)} \frac{[\Gamma(1 - \gamma)]^4}{\Gamma(1 + \gamma)\Gamma(1 - 2\gamma)\Gamma(1 - 3\gamma)}.$$  (16)

We see that the value of the annealed multifractality exponent $\tilde{\tau}_2 = 1 - 2\gamma$ which emerges from our calculation coincides with the ‘simple scaling’ value $\tilde{\tau}_q = (q - 1)(1 - q\gamma)$, $q = 2$, discussed by us earlier. Moreover, the consistency of the above procedure obviously requires $0 \leq \gamma < 1/3$, with the upper limit being precisely the threshold $\gamma_{q=2} = 1/3$ of validity of the above ‘simple scaling’ regime. We conclude that a simple-minded replica limit could be employed to produce meaningful results only as long as the pre-freezing phenomenon responsible for the change of multifractality exponent is not operative.

4. The infinite-dimensional limit: pre-freezing via replica symmetry breaking

To make some progress in the understanding of the mechanisms behind the failure of the simple scaling in the replica approach, we turn from now on to another exactly solvable limit of the logarithmic model, that is to the infinite-dimensional case. The free energy, and hence the typical multifractality spectrum, was calculated in [7] in the framework of the replica trick, and very recently confirmed by rigorous mathematical methods [24]. The system was found to display the REM-type freezing transition at $\gamma = \beta^2 g^2 = 1$, with the low-temperature phase $\gamma > 1$ described by the standard one-step replica symmetry breaking pattern. The meaning of the freezing could be elucidated by invoking the probability for two independent particles distributed in such a random potential according to the Boltzmann–Gibbs measure to end up at a distance of order of the small cut-off scale $a^2$. The probability was found [7] to tend to zero in the thermodynamic limit $L/a \to \infty$ everywhere in the high-temperature phase $0 \leq \gamma < 1$, confirming the particle delocalization over the sample. In contrast, in the lower-temperature phase $\gamma > 1$ two particles have a finite probability of being trapped at the small-scale distance even in the infinite sample.

Our starting point here is again the identity (12). Employing it, one can easily perform the disorder average for any integer number of replicas $n \geq 2$. After appropriate rescaling of the coupling constant $g \to g\sqrt{N}$ and length scales $L \to L\sqrt{2N}$ and $a \to a\sqrt{2N}$, manipulations similar to those described in detail in [7] yield a convenient representation for the IPR in terms of an integral over a positive definite matrix $Q$ of the size $(n-1) \times (n-1)$ with entries $q_{\mu,\nu}$. We have

$$\langle P_2 \rangle = \lim_{n \to 0} C_{N,n}(a) L^{n^2} \int_{D_Q} (\det Q)^{-n/2} e^{-N\Phi_n(Q)} \, dQ,$$  (17)

where

$$\Phi_n(Q) = -\frac{1}{2} \ln (\det Q) + \gamma \sum_{1 \leq \mu < \nu \leq n-2} \ln \left[ \frac{1}{2}(q_{\mu,\mu} + q_{\nu,\nu}) - q_{\mu,\nu} + a^2 \right] + 2\gamma \sum_{\mu=1}^{n-2} \ln \left[ \frac{1}{2}(q_{\mu,\mu} + q_{n-1,n-1}) - q_{\mu,n-1} + a^2 \right], \quad \gamma = \beta^2 g^2.$$  (18)
and the integration domain $D_Q$ in the above expression is simply $D_Q = \{ Q > 0, q_{\mu,\mu} \leq L^2, \mu = 1, \ldots, n - 1 \}$. The proportionality constant $C_{N,n}(a)$ is also explicitly known, but its value is inessential for the subsequent calculation.

The shape of the integrand in (17) is suggestive of application of the saddle-point method for evaluation of the integral in the large-$N$ limit. The corresponding saddle-point equations for the entries of the matrix $Q$ amount to $\partial \Phi / \partial q_{\mu,\nu} = 0$ for any choice of the indices $1 \leq \mu \leq \nu \leq (n - 1)$. A closer inspection of the replica limit $n \to 0$ reveals however that solutions to the saddle-point equations do not actually exist unless one fixes all the diagonal entries $q_{\mu,\mu}, 1 \leq \mu \leq (n - 1)$, of the matrix $Q$ to the boundary of the integration domain by setting $q_{\mu,\mu} = L^2$, and excluding them from the variational procedure (cf a similar result in [7]). The remaining off-diagonal entries should be found from the system of equations

\[
\begin{align*}
\left[ Q^{-1} \right]_{\mu,\nu} + \gamma \frac{1}{L^2 - q_{\mu,\nu} + a^2} &= 0, \quad 1 \leq \mu < \nu \leq n - 2, \\
\left[ Q^{-1} \right]_{\mu,\mu-1} + 2\gamma \frac{1}{L^2 - q_{\mu,\mu-1} + a^2} &= 0, \quad 1 \leq \mu \leq n - 2.
\end{align*}
\]

The only solution of such a system existing in the high-temperature phase in the thermodynamic limit $L \gg a$ has the following form:

\[
q_0 = \frac{\gamma (1 + 3\gamma)}{(1 + \gamma)^2} L^2 + O(a^2), \quad v = \frac{2\gamma}{1 + \gamma} L^2 + O(a^2).
\]

The condition $q_0 < L^2$ which is readily seen to be satisfied everywhere in the high-temperature phase $0 \leq \gamma < 1$ ensures that $Q^{(0)} > 0$.

Substituting these values back to the relation (18) and using the identity

\[
\det Q_{\text{r.s.}} = L^2 (L^2 - q_0)^{n-3} \left[ L^2 - v^2/L^2 + (n-3)(q_0 - v^2/L^2) \right]
\]

we find to the leading order $\Phi_{n-0}(Q_{\text{r.s.}}) = (1 - 2\gamma) \ln L$ which in turn yields the asymptotic behaviour for the averaged inverse participation ratio $\langle P_2 \rangle \propto e^{-N\Phi_{n-0}} \propto M^{-(1-2\gamma)}$ where

doi:10.1088/1742-5468/2009/07/P07022
we have identified $M \sim L^N$. We arrive at an important conclusion of the replica-symmetric ansatz equation (21) implying necessarily the ‘simple scaling’ result $\tilde{\gamma}_2 = 1 - 2\gamma$ for the annealed multifractality exponent; see the discussion after (12). As is clear, the latter value cannot have any meaning at least for $\gamma > 1/2$, which in turn implies that the replica-symmetric solution of the problem cannot be valid over the whole high-temperature phase $0 \leq \gamma < 1$. The only way out is therefore to look for an appropriate scheme of replica symmetry breaking which occurs already in the high-temperature phase.

To get a guiding idea in our search for a solution to the saddle-point equations (19) and (20) which goes beyond the replica-symmetric ansatz, equation (21), it is useful to recall that the singled-out replica indexed with $\nu = n - 1$ has its origin in representing the factor $Z(2\beta)$ in the averaged inverse participation ratio; cf equation (12). As a result, that special replica is effectively ‘colder’ than all other equivalent replicas indexed with $1, 2, \ldots, n - 2$ which originated from the factors $Z(\beta)$. It is then natural to suppose that by interacting with other replica the colder one could eventually ‘pre-freeze’ a certain group of replicas around it. The manifestation of the freezing mechanism within the replica approach was discussed for the present model in detail in [7]. Employing it one should expect such a ‘pre-freezing’ to manifest itself via existence of a diagonal block inside the matrix $\tilde{Q}$ having in the thermodynamic limit $L \gg a$ all entries equal, up to the leading order, to $L^2$.

The simplest ansatz for the matrix $Q$ compatible with a possibility of such pre-frozen group of $m$ replicas, $0 \leq m \leq n - 2$, would be of the following structure:

$$Q_{r,s.b.} = \begin{pmatrix} Q^{(1)} & q_3 E^T & v_1^T \\ q_3 E & Q^{(2)} & v_2^T \\ v_1 & v_2 & L^2 \end{pmatrix},$$

$$Q^{(1)}_{\mu,\nu} = (L^2 - q_1)\delta_{\mu,\nu} + q_1, \quad 1 \leq \mu, \nu \leq n - m - 2,$$

$$Q^{(2)}_{\mu,\nu} = (L^2 - q_2)\delta_{\mu,\nu} + q_2, \quad 1 \leq \mu, \nu \leq m,$$

where all entries of the matrix $E$ of the size $m \times (n - m - 2)$ are equal to 1, the $(n - m - 2)$-component vector $v_1$ has all components equal to $v_1$ and $m$-component vector $v_2$ is of the same structure: $v_2 = v_2(1, \ldots, 1)$. The values of five parameters $q_1, q_2, q_3, v_1, v_2$ for a given size $m$ of the pre-frozen block are to be found from the saddle-point conditions (19) and (20) in the replica limit $n \rightarrow 0$. Finally, the parameter $m$ satisfying in the replica limit inequality $-2 \leq m \leq 0$ should itself be chosen so as to extremize the resulting $\Phi(Q)$, equation (18), which is evaluated with the help of the identity

$$\det \tilde{Q}_{r,s.b.} = L^2(L^2 - q_1)^{n-m-3}[L^2 - v_1^2/L^2 + (n - m - 3)(q_1 - v_1^2/L^2)]$$

$$\times \frac{(L^2 - q_2)^{m-1}[L^2 - v_2^2/L^2 + (m - 1)(q_2 - v_2^2/L^2)]}{r m(q_3 - v_1 v_2/L^2)},$$

$$r = \frac{n - m - 2}{L^2 - v_1^2/L^2 + (n - m - 3)(q_1 - v_1^2/L^2)}.$$
whereas for $0 \leq \gamma < 1$ the extremum is at an internal point of the interval $m = m_\ast = \frac{1}{2} (1 - 3\gamma)/\gamma$, whereas for $0 \leq \gamma \leq 1/3$ the extremum is at the boundary point $m = 0$. Substituting
those values into the functional leads in the thermodynamic limit to the value \( \Phi_{n \to 0}(Q) \) given in such a scheme to the leading order by

\[
\Phi_{\text{extr}} = \ln L \begin{cases} 
1 - 2\gamma, & 0 \leq \gamma \leq 1/3, \\
(1 - \gamma)^2, & 1/3 \leq \gamma < 1.
\end{cases}
\]  

(36)

The above result immediately implies the values for the corresponding multifractality exponent \( \tilde{\tau}_2 \) coinciding everywhere in the high-temperature phase \( 0 \leq \gamma < 1 \) with that exemplified by the standard REM; cf (9).

To complete the picture, one can perform the standard de Almeida–Thouless-like analysis [25] and verify that the replica-symmetric solution of equations (24) does not show any local instability at the point of the pre-freezing transition \( \gamma \to 1/3 - 0 \) (although it does become unstable for a certain value \( \gamma = \gamma_{\text{inst}} \in (1/3, 1/2) \)). Absence of local instability is consistent with the aforementioned observation that in spite of having \( m \to 0 \) when approaching the pre-freezing point the solution with broken replica symmetry (34) remains formally different from the replica-symmetric solution (24). Above the pre-freezing temperature, that is for \( \gamma < 1/3 \), the two solutions share the same value of the functional \( \Phi_{\text{extr}} \), and therefore produce exactly the same simple scaling for the multifractality exponent \( \tilde{\tau}_2 \). In the pre-freezing domain \( 1/3 < \gamma < 1 \) the solution with broken replica symmetry ‘wins’.

In conclusion, we have performed systematic analysis of the multifractality exponents extracted from the averaged moments of the Boltzmann–Gibbs measure generated from logarithmically correlated random potentials. In particular, using zero-dimensional and infinite-dimensional versions of the model we have identified a pattern of the replica symmetry breaking responsible for the abrupt change (‘pre-freezing’) of those exponents in the high-temperature phase. Implementing such a pattern in explicit calculations for one- and two-dimensional versions of the models remains a challenging open problem.

Acknowledgments

This work was supported by Leverhulme Research Fellowship project ‘A single particle in random energy landscapes’. The major part of the research was completed during the author’s participation in the programme on Mathematics and Physics of the Anderson Localization at the Newton Institute (Cambridge) whose support and hospitality are gratefully acknowledged. The author is grateful to A Mirlin for stimulating initial discussions on the problem of annealed multifractality exponents, and to C Mudry, P Le Doussal and M Skvortsov for encouraging interaction at various stages of the work.

References

[1] Paladin G and Vulpiani A, 1987 Phys. Rep. 156 147
[2] Evers F and Mirlin A D, 2008 Rev. Mod. Phys. 80 1355
[3] Foster M S, Ryu S and Ludwig A W W, 2009 arXiv:0901.0284
[4] Monthus C and Garel T, 2007 Phys. Rev. E 75 051122
[5] Mandelbrot B, 1990 Physica A 163 306
[6] Chabra A B and Sreenivasan K R, 1990 Phys. Rev. A 43 1114
[7] Halsey T C, Honda K and Duplantier B, 1996 J. Stat. Phys. 85 681
[8] Carpentier D and Le Doussal P, 2001 Phys. Rev. E 63 026110
[9] Fyodorov Y V and Bouchaud J P, 2008 J. Phys. A: Math. Theor. 41 324009
Pre-freezing in a logarithmically correlated potential

Fyodorov Y V and Sommers H-J, 2007 Nucl. Phys. B 764 128
[8] Ludwig A, Fisher M P A, Shankar R and Grinstein G, 1994 Phys. Rev. B 50 7526
[9] Chamon C, Mudry C and Wen X-G, 1996 Phys. Rev. Lett. 77 4194
[10] Castillo H E, Chamon C C, Fradkin E, Goldbart P M and Mudry C, 1997 Phys. Rev. B 56 10668
[11] Kogan I I, Mudry C and Tewelik A M, 1996 Phys. Rev. Lett. 77 707
[12] Abdulla E and Tabar M R R, 1998 Phys. Lett. B 440 339
[13] Fyodorov Y V and Bouchaud J P, 2008 J. Phys. A: Math. Theor. 41 372001
[14] Muzy J-F, Delour J and Bacry E, 2000 Eur. Phys. J. B 17 537
[15] Kogan I I, Mudry C and Tsvelik A M, 1996 Phys. Rev. Lett. 77 707
[16] Kogan I I, Mudry C and Tsvelik A M, 1996 Phys. Rev. Lett. 77 707
[17] Fyodorov Y V and Bouchaud J P, 2008 J. Phys. A: Math. Theor. 41 372001
[18] Bacry E, Delour J and Muzy J-F, 2001 Phys. Rev. E 64 026103
[19] Ostrovsky D, 2008 Lett. Math. Phys. 83 265
[20] Fyodorov Y V, Le Doussal P and Rosso A, 2009 in preparation
[21] Duplantier B and Sheffield S, 2008 arXiv:0808.1560 [math.PR]
[22] Rhodes R and Vargas V, 2008 arXiv:0807.1036 [math.PR]
[23] Derrida B, 1981 Phys. Rev. B 24 2613
[24] Gardner E and Derrida B, 1989 J. Phys. A: Math. Gen. 22 1975
[25] Derrida B and Toulouse T, 1985 J. Phys. Lett. 46 L223
[26] Delort J and Muzy J-F, 2001 Phys. Rev. E 64 026103
[27] Klimovsky A, Parisi landscapes in high-dimensional Euclidean spaces, 2009 Talk at the Workshop ‘Mathematical Models from Physics and Biology’ (Bonn, Germany, April 2009)
[28] de Almeida J R L and Thouless D J, 1978 J. Phys. A: Math. Gen. 11 983