Interplay between intrinsic and emergent topological protection on interacting helical modes

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(Dated: May 28, 2018)

The interplay between topology and interactions on the edge of a two dimensional topological insulator with time reversal symmetry is studied. We consider a simple non-interacting system of three helical channels with an inherent $\mathbb{Z}_2$ topological protection, and hence a zero-temperature conductance of $G = e^2/h$. We show that when interactions are added to the model, the ground state exhibits two different phases as function of the interaction parameters. One of these phases is a trivial insulator at zero temperature, as the symmetry protecting the non-interacting topological phase is spontaneously broken. In this phase, there is zero conductance $G = 0$ at zero-temperature. The other phase displays enhanced topological properties, with the neutral sector described by a massive version of $\mathbb{Z}_3$ parafermions. In this phase, the system at low energies displays an emergent $\mathbb{Z}_3$ symmetry, which is not present in the lattice model, and has a topologically protected zero-temperature conductance of $G = 3e^2/h$. This state is an example of a dynamically enhanced symmetry protected topological state.

\section{I. INTRODUCTION}

Topology plays a central role in the modern understanding of several physical systems, ranging from superfluid Helium to elementary particles \cite{1, 2, 3, 4, 5, 6}. In the context of solid state physics, one of the first phenomena that were identified as being of topological origin was the integer quantum Hall effect (IQHE). In the IQHE, the existence of protected chiral modes on the edge of the sample is a consequence of the existence of a non-trivial first Chern number \cite{7}. The topological nature of these modes renders them robust against disorder and enforces conductance quantization in units of $e^2/h$. We show that when interactions are added to the model, the ground states exhibit two different phases as function of the interaction parameters. One of these phases is a trivial insulator at zero temperature, as the symmetry protecting the non-interacting topological phase is spontaneously broken. In this phase, there is zero conductance $G = 0$ at zero-temperature. The other phase displays enhanced topological properties, with the neutral sector described by a massive version of $\mathbb{Z}_3$ parafermions. In this phase, the system at low energies displays an emergent $\mathbb{Z}_3$ symmetry, which is not present in the lattice model, and has a topologically protected zero-temperature conductance of $G = 3e^2/h$. This state is an example of a dynamically enhanced symmetry protected topological state.

In general, for non-interacting disordered systems, the topological classification is fully established \cite{20, 21} and is uniquely determined by the symmetry class and dimensionality of the single particle Hamiltonian. Weak interactions can change the topological properties of a non-interacting system in different ways, e.g. by modifying the whole state including the bulk, or by changing the edge degrees of freedom in the system, without changing the overall structure in the bulk. An example of the former corresponds to the interacting Kitaev chain \cite{22}, where the inclusion of interactions allows to connect adiabatically two Hamiltonians belonging to different non-interacting topological states, reducing the non-interacting classification $\mathbb{Z}$ down to $\mathbb{Z}_3$. On the other hand, when the characteristic interaction strength is smaller than the bulk gap energy, interactions can only induce a change at the edge degrees of freedom. In this latter context, it has been recently found that the interactions may lead to an emergence of topologically non-trivial edge states, in systems that are topologically trivial on the bulk according to the non-interacting classification.

The simplest example of this kind of phenomena appears on the edge of a two dimensional TI supporting two parallel helical modes. Generically, in a non-interacting system, these modes can hybridise and be localised by the presence of sufficient density of impurities, making the system topologically trivial. Surprisingly, in the presence of interaction, there is some possibility for these modes to be protected against localisation, by a zero bias anomaly mechanism in the case of vanishing tunneling \cite{23}, or by the emergence of an effective spin gap \cite{24, 25} that suppresses single particle backscattering when tunneling is present. In these cases, the system displays topological signatures like a robust value of conductance, quantized...
in units of $e^2/h$ and fractionalized zero modes in domain wall configurations. This protection has also been predicted to appear in truly one dimensional systems with spin-orbit interaction $\Delta$.

Another mechanism in which interactions can affect the topological properties of a non-trivial SPT state, is by inducing an spontaneous breaking of the protecting symmetry in the groundstate, rendering the state topologically trivial. Recently [27], it has been shown that in a general system of $N$ helical modes, interactions can decrease the conductance of the system to zero at zero temperature, for $N > 2$, by creating a groundstate that spontaneously breaks TR symmetry.

In this work, we focus on a system of three coupled helical modes with inter-channel tunneling, corresponding to the edge structure of an integer TI. Because the number of modes is odd, this system is topologically non-trivial according to the non-interacting classification and disorder can at most localise two modes, leaving one helical mode free to carry the charge. We show that the interactions generate two distinct phases in which each of the effects discussed above can occur: in one phase, the intrinsic topology is destroyed through breaking of TRS; while in the other phase, the intrinsic topological protection is enhanced through a new distinct emergent topological state, which protects all three helical modes against localisation. Both of these states have a number of emergent energy scales with different characteristics, which we summarize below.

A. Summary of main results

Before delving into the technical details of the analysis, it is worth listing the main results that we find in this paper. The model of three coupled helical edges is developed in section II and illustrated schematically in Fig. 2. If one neglects the interactions, the system consists of three helical edge modes, which although we describe it in language most natural for stacking of quantum-spin-Hall insulators (see [29] for related discussions in the context of quantum Hall systems, it could alternatively arise from reconstruction of edge states in a single quantum spin-Hall insulator (which is known to occur also in quantum Hall systems, see e.g. [32]). The essential feature is that in the clean non-interacting system, there are three helical modes, from which one is topologically protected against localisation due to the intrinsic $Z_2$ topology of the model.

Our results consider the fate of this system when weak interactions are introduced. We find that two distinct ground state phases may develop, corresponding to:

1. An emergent topological (ET) state, whose topology differs from the intrinsic topology of three channels. In this ET state all three edge modes are protected against localisation when disorder is added to the system, meaning that the low temperature conductance is $G = 3e^2/h$; and

2. A state that is characterised by time reversal symmetry breaking (TRSB) in the groundstate which destroys the intrinsic topology (that was protected by TR symmetry) and leads to a vanishing low-temperature conductance.

These different phases of the system are determined by the relative strength of the intra- and inter-mode interactions. The phase diagram of the model is displayed in Fig. 4 later in the paper, and shows that the generic scenario of intra chain interactions being repulsive and stronger than inter-chain interactions (which are also repulsive) corresponds to the TRSB phase. However, the phase diagram also shows that even within purely repulsive interactions, either phase is possible in the presence of tunneling between the channels, indicating that details of the edge in any given realisation of the system are crucial to determine the fate of the interacting system.

The TRSB and the ET phases share some commonalities. Their low energy excitations (in the clean system) correspond to a gapless charge plasmon mode, and neutral excitations with a gap $\Delta_n$. Both states display the phenomenon of dynamical symmetry enhancement, whereby the symmetry of the ground state is higher than in the original problem. Both fixed points can be obtained via an adiabatic deformation of the SU(3) Gross-Neveu model, which ultimately has a $Z_3$ symmetry. It is worth stressing that this is true, even through the microscopic model does not possess this symmetry.

We now summarise the physical properties of each of the states in turn. Firstly, in the TRSB state:

1. The ground state can be described by quasi long range order parameters. The dominant one is controlled by details of the interaction and can be either two-particle or trionic. One can picture this state as a sliding charge-density-wave.

2. The TRSB induced by impurities or disorder, arises physically from coherent two-particle backscattering off impurities. This means that the energy scale associated with TRSB, $\Delta_p$ may be much less than the energy scale associated with either the gap $\Delta_n$, or the localisation of two of the three modes by disorder. This suggests that although the ground state is non-topological (as TR symmetry has been spontaneously broken), one still sees remnants of this $Z_2$ topology at finite temperature. In particular one may have a plateaux in conductance of $G = e^2/h$ at intermediate temperatures. This is schematically shown in Fig. 6.

3. TRSB could also occur in a clean system through Umklapp scattering if the Fermi-momenta of the different modes have an appropriate commensurability relationship between them. Like the case of disorder, this may occur at a characteristic energy scale well below the neutral gap, again leaving a wide intermediate temperature regime where phe-
nomena associated with the intrinsic $\mathbb{Z}_2$ topology of the non-interacting system could be observed.

Turning now to the phase with emergent topology

1. The ground state is a $\mathbb{Z}_3$ symmetry protected topological state, where we stress again that the $\mathbb{Z}_3$ symmetry is itself emergent and therefore the lattice model itself is not required to (and in general does not) have this symmetry.

2. The phase boundary between the ET phase and the TRSB phase is described by a critical theory that belongs to the same universality class as the three-state Potts model, corresponding in the continuous limit to a conformal field theory (CFT) with central charge $c = 4/5$ and $\mathbb{Z}_3$ parafermionic low energy modes.

3. At temperatures above the neutral gap, the conductance may drop below $3e^2/h$, while it will recover to the full quantum conductance $G = 3e^2/h$ at low temperature. A schematic diagram of this is plotted in Fig. 6.

All the previous points highlight that while the characterisation of the conductance in the ground state of each phase is an obviously important property to analyse, it does not capture all the physical features of the system.

This article develops as follows: In section II we introduce a simple phenomenological model for three helical states in the clean case that displays the general features, first describing the single particle Hamiltonian, and then introducing generic interactions. In section III we analyse the low energy -or infrared (IR)- description of the system in terms of Abelian and non-Abelian bosonisation. Here we find that the neutral sector is represented by an adiabatic deformation of an emergent SU(3) symmetry. We analyse the structure of all two-particle operators that represent backscattering and introduce the relevant order parameters in the TRSB and ET phases in IV. In section V we discuss the stability of the phases against general interaction terms. Following this analysis, in section VI we discuss the transition between the TRSB and ET phase. To gain further insight we develop an intuition about the structure of the massive degrees of freedom in terms of an effective parafermionic model on the lattice that respects all the symmetries of the continuous model. Here we show that in the transition region between topological to trivial phase along the edge, a parafermionic mode is trapped in the domain wall. In section VII we discuss the fate of disorder in the system, showing the difference between these phases. Finally, in section VIII we discuss the results and present our conclusions.

II. THE MODEL

A. Single particle Hamiltonian

While no symmetry apart from TR symmetry should be expected on the edge of a multichannel TR topological insulator, here we consider a simple model that displays all the features of the generic model, to keep the exposition and the relation to the physical regimes clear. We analyse a generic model in Appendix B. We consider three helical modes, described by the fermion destruction (creation) operator of momentum $k$, $c^\dagger_{k,a}(c^\eta_{k,a})$, where $a = (1,2,3)$ denotes the mode and $\eta = (+,−)$ labels its helicity. For small momenta, the non-interacting single particle Hamiltonian is

$$H_0 = \sum_{k,a,\eta} \eta v_F k c^\dagger_{k,a} c^\eta_{k,a} - t_{\perp} \sum_{k,\eta} c^\dagger_{k,2}(c^\eta_{k,1} + c^\eta_{k,3}),$$

(1)

where $v_F$ is the Fermi velocity and $t_{\perp}$ describes the tunneling between different modes. Here we assume that tunneling only occurs between the modes which are closest in space. A diagram of the arrangement of helical modes and their labellings is given in Fig. 1.

![FIG. 1. (Color online). Three helical modes on the edge of a two dimensional TI. We label the different channels by 1, 2, and 3 and the different interaction strengths $V_0, V_{12}, V_{23}$ as depicted. Tunneling amplitude between mode 1-2 and 2-3 is denoted by $t_\perp$. Tunneling between 1 and 3 is assumed to be negligible.](image)
interacting part of the Hamiltonian is given by the fields around the Fermi points (here we denote a continuous description of the modes and expand \( \psi \) together with the slowly varying fields fields the total density on each site is 

\[
\sum_i \rho_i \eta = \sum_i \sqrt{\rho_i^\eta} \sigma_i^\eta.
\]

The interaction parameter \( \rho_{\eta} = \sqrt{\rho_i^\eta} \). Summing over the helicities we have the total density per band \( \rho \eta = \sum_i \sqrt{\rho_i^\eta} \). The bosonic fields satisfy the bosonized form of the density in the band basis.

B. Interactions

A generic interaction between the three different helical modes is described by the following lattice Hamiltonian

\[
H_{\text{int}} = \sum_{i,a} V_0 n_i a n_{i+1,a} + 2 \sum_{i} (V_{12} n_{i,1} n_{i,2} + V_{23} n_{i,2} n_{i,3}),
\]

where the density at each site \( i \) and channel \( a \) is \( n_{i,a} = \sum_{\alpha} c_{i,a}^\alpha c_{i,a}^{\alpha \dagger} \). The interaction parameter \( V_0 \) denotes the intra-mode interaction, while \( V_{ab} \) denotes the interaction between modes \( a \) and \( b \). For simplicity of the exposition, here we do not consider the interaction between modes 1 and 3, although such interaction is considered in Appendix [B]. In the basis that diagonalizes the Hamiltonian, the density for the band \( a \) and helicity \( \eta \) corresponds to \( \rho_{\eta,i,a} = \sqrt{\rho_i^\eta} c_{i,a}^\eta \). Summing over the helicities we have the total density per band \( \rho_{i,a} = \sum_{\eta} \sqrt{\rho_i^\eta} c_{i,a}^\eta \). The total density on each site is \( \rho_i = \sum_a \rho_{i,a} \).

In the low energy, long wavelength limit, we can introduce a continuous description of the modes and expand the fields around the Fermi points (here \( x = i a_0 \), with \( a_0 \) the lattice spacing)

\[
\frac{\psi_{i,a}^\eta}{\sqrt{a_0}} = \psi_{i,a,\eta}(x),
\]

together with the slowly varying fields \( \psi_{i,a,\eta}(x) = R_{\alpha}(x) e^{i k_{\alpha} x} \) and \( \psi_{i,a,\eta}(x) = L_{\alpha}(x) e^{-i k_{\alpha} x} \). Fixing the chemical potential away from the band crossings, and considering \( k_{F,a} \neq 0 \), the Fermi momenta become \( k_{F,a} = \frac{\mu}{\hbar v_{F,a}} \). In the continuous description, the non-interacting part of the Hamiltonian is given by

\[
H_0 = i v_F \sum_a \int dx (R_{\alpha}^a \partial_x R_{\alpha} - L_{\alpha}^a \partial_x L_{\alpha}).
\]

Collecting processes that conserve momentum, (do not have oscillations with \( k_F \)), the interaction sector of the Hamiltonian becomes (omitting the space dependence of the densities) \( H_{\text{int}} = H_{\rho \rho} + H_{\rho l} \), with

\[
H_{\rho \rho} = \int dx \left( \tilde{V}_0 \rho_i^\eta + g ( \rho_1 + \rho_3 )^2 + g' \rho_i^\eta_1 \rho_i^\eta_3 + 4g \sum_{\eta} \rho_i^\eta_1 \rho_i^\eta_3 \right)
\]

containing the forward scattering interaction terms, and

\[
H_{\rho l} = \tilde{g} \int dx \left( R_{\alpha}^i R_{\alpha}^j L_{\beta}^i L_{\beta}^j + L_{\alpha}^i L_{\alpha}^j R_{\beta}^i R_{\beta}^j + \text{h.c.} \right)
\]

containing the extra interaction terms. Here \( \tilde{V}_0 = \frac{a_0}{\pi} (V_0 + V_{12} + V_{23}) \) and

\[
g = \frac{V_0 a_0}{8}, \quad g' = \frac{(V_0 - V_{23} - V_{12}) a_0}{4}, \quad \tilde{g} = 2(g + g').
\]

Note that the full Hamiltonian is invariant under the operation of permuting the modes 1 ↔ 3, and the interaction strengths \( V_{12} \leftrightarrow V_{23} \).

Taking \( g = 0 \) (or \( g_2 = 0 \) in the general model of Appendix [B], the three helical model reduces to the two helical system studied in Ref. [24] plus a forward scattering interaction with the antisymmetric band mode \( \psi_2^\eta \).

III. BOSONIZATION ANALYSIS

We represent the slow part of the fermionic operators as vertex operators of a bosonic field, as is standard in bosonization [13][53], by \( R_{\alpha}(x) = \frac{a_0}{\sqrt{2\pi a_0}} e^{i k_{\alpha} x} \), and

\[
L_{\alpha}(x) = \frac{\kappa_{\alpha}}{\sqrt{2\pi a_0}} e^{-i k_{\alpha} x} \delta_{\alpha b} \delta_{\eta\eta'} \text{sign}(x-y), \quad \eta = (+, -) = (R, L).
\]

Using these conventions the bosonized form of the density in band \( a \) and with helicity \( \eta \) is \( \rho_{\alpha}^\eta_2 = \frac{1}{\sqrt{6}} \partial_x \varphi_{\alpha,\eta} \).

It is useful to define the following fields

\[
\tilde{\varphi}_{\eta,c}(x) = \frac{3}{a} \sum_{\alpha=1}^{3} \varphi_{\eta,a}(x), \quad \tilde{\varphi}_{\eta,m}(x) = \frac{3}{a} \sum_{\alpha=1}^{3} \varphi_{\eta,m}(x),
\]

together with the inverse relation \( \varphi_{\eta,a}(x) = \frac{1}{\sqrt{3}} \tilde{\varphi}_{\eta,c}(x) + \sum_{m} \varphi_{\eta,m}(x) \). The vectors \( \varphi \) correspond to the three vertices of an equilateral triangle, see Fig. [3] and are explicitly given by

\[
d_1 = \left( \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}} \right), \quad d_2 = \left( 0, -\frac{1}{\sqrt{6}} \right), \quad d_3 = \left( -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}} \right).
\]
They satisfy \( d_a \cdot d_b = \delta_{ab} - \frac{1}{3} \).

We introduce the non-chiral fields \( \tilde{\varphi}_a = \tilde{\phi}_{R,a} - \tilde{\phi}_{L,a} \) and \( \tilde{\theta}_a = \tilde{\phi}_{R,a} + \tilde{\phi}_{L,a} \), with \( a = c,1,2 \). The only non-vanishing commutation relations in this basis are \( [\partial_x \tilde{\varphi}_a(x), \tilde{\theta}_b(y)] = i \delta_{ab} \delta(x - y) \). For future reference we also introduce the basis for neutral fields \( \varphi = (\tilde{\varphi}_1, \tilde{\varphi}_2) \) and \( \theta = (\tilde{\theta}_1, \tilde{\theta}_2) \).

In order to identify the total charge mode we perform a global U(1) transformation on the original fermionic fields \( \psi_{\alpha,a}(x) \rightarrow e^{i \theta} \psi_{\alpha,a}(x) \) which amounts to a shift in the bosonic fields as \( \phi_{\eta,\alpha} \rightarrow \phi_{\eta,\alpha} + \frac{\theta}{\sqrt{4\pi}} \). The fields defined in the transform as \( \tilde{\phi}_{\eta,c} \rightarrow \tilde{\phi}_{\eta,c} + \frac{1}{\sqrt{4\pi}} \Theta \) and \( \tilde{\phi}_{\eta,\mu} \rightarrow \tilde{\phi}_{\eta,\mu} \). This implies that the fields \( \tilde{\theta}_c, \tilde{\varphi}_c \) describe the total charge mode and its conjugate field, while the modes \( \tilde{\theta}_1, \tilde{\theta}_2 \) and their conjugates are neutral with respect to the total U(1) charge.

The Hamiltonian of the system \( H = H_0 + H_{pp} + H_{ai} \) in the bosonized variables splits into \( H = H_c + H_1 + H_2 + H_{\text{mix}} \), where total charge sector \( H_c \) is

\[
H_c = \frac{v_c}{2} \int dx \left( K_c (\partial_x \tilde{\varphi}_c)^2 + \frac{1}{K_c} (\partial_x \tilde{\theta}_c)^2 \right),
\]

while the Hamiltonians for the neutral sectors \( H_1 \) and \( H_2 \) are

\[
H_1 = \frac{v_1}{2} \int dx \left( (\partial_x \tilde{\varphi}_1)^2 + (\partial_x \tilde{\theta}_1)^2 \right) + \frac{g + g'}{\pi a_0} \int dx \cos(\sqrt{8\pi} \tilde{\varphi}_1), \quad \text{and}
\]

\[
H_2 = \frac{v_2}{2} \int dx \left( K_2 (\partial_x \tilde{\varphi}_2)^2 + \frac{1}{K_2} (\partial_x \tilde{\theta}_2)^2 \right) + \frac{4g}{(\pi a_0)^2} \int dx \left( \cos(\sqrt{6\pi} \tilde{\varphi}_2) + \cos(\sqrt{6\pi} \tilde{\theta}_2) \right) \cos(\sqrt{2\pi} \tilde{\varphi}_1).
\]

The renormalized velocities and Luttinger parameters of these modes satisfy \( v_1 = v_F - \frac{g + g'}{2\pi} \), \( v_2 K_2 = v_1 + \frac{2(g' - g)}{3\pi} \), \( v_2 K_2^{-1} = v_1 + \frac{4g'}{3\pi} \), \( v_c K_c = v_F + \frac{g' + 5g}{3\pi} \), \( v_c K_c^{-1} = v_F + \frac{2g' + 3g + V_0}{3\pi} \).

Note that the mode \( \tilde{\theta}_1 \) sees its velocity renormalized, but its Luttinger parameter stays unity, as a consequence of TR symmetry and the fact that the microscopic degrees of freedom are helical. This implies that at all orders in the interaction parameters the scaling dimension \( \Delta_1(\alpha) \equiv \Delta[\cos(\sqrt{2\pi} \tilde{\varphi}_1)] = \alpha \). The remaining part of the Hamiltonian is

\[
H_{\text{mix}} = \frac{\sqrt{7}}{6\pi} \int dx \left[ (g' - g) \partial_x \varphi \partial_x \phi + (3g - g') \partial_x \theta \partial_x \theta_2 \right],
\]

It couples the total charge mode and the second neutral sector. This term is strictly marginal and does not influence the physics in any of the gapped phases, as the field \( \phi_2 (\theta_2) \) is locked by the renormalization of the cosine terms in the ET (TRS) phase. To first order in the interactions parameters the scaling dimensions of the cosine terms are

\[
\Delta\tilde{\varphi}^2 \equiv \Delta[\cos(\sqrt{6\pi} \tilde{\varphi}_2)] = \frac{3}{2} + \frac{g + g'}{\pi v_F},
\]

\[
\Delta\tilde{\theta}^2 \equiv \Delta[\cos(\sqrt{6\pi} \tilde{\theta}_2)] = \frac{3}{2} - \frac{g + g'}{\pi v_F}.
\]

The value of the scaling dimensions determines the fate of the cosine operators under renormalization group (RG). We now consider two limiting cases of purely attractive and purely repulsive interaction. With start with the former, assuming \( g = g' (V_0 = 2(V_{12} + V_{23})) \) for simplicity.

**Attractive Interactions**

In this case \( g < 0 \), and \( \Delta\tilde{\varphi}^2 > \Delta\tilde{\theta}^2 \), so the cosine operator \( \cos(\sqrt{6\pi} \tilde{\varphi}_2) \) grows faster than \( \cos(\sqrt{6\pi} \tilde{\theta}_2) \) under renormalization. Keeping the maximal set of commutating cosine operators with smallest scaling dimensions, the model becomes a marginal deformation of the SU(3) Gross-Neveu model [34] and is given by

\[
H = H_c + H_{\text{SU}(3)} + \frac{\sqrt{2} g}{3\pi} \int dx \partial_x \theta_2 \partial_x \theta_2
\]

\[
+ \frac{4g}{3\pi} \int dx (\partial_x \tilde{\theta}_2)^2 + \frac{2g}{\pi} \sum_a \int dx (\partial_x \tilde{\varphi}_a)^2.
\]
The Hamiltonian well, whenever inter-channel tunneling is present. The dominant order parameters in this phase are odd under TR transformations, indicating the onset of a spontaneous breaking of TR in this phase. This phase is not topologically protected, as disorder or interaction can gap the charge mode.

Considering the general model of Appendix B, the above results are modified slightly, in particular, the definition of the interaction parameters and the specific value of the semi-classical solutions in the repulsive case. Nevertheless, the existence of the two distinct phases remains.

Below we further discuss the main characteristics of these phases in terms of order parameters.

IV. CHARACTERISATION OF THE PHASES

A. Two Particle Normal Order Parameters

The usual order parameters involving two-particle number conserving processes are given by $O^{\text{ord}} = \sum_{ab} c_a L \lambda_{ab} c_b R + c_b R \lambda_{ab} c_a L$, where $\lambda^{(a)}$ are the Gell-Mann matrices. These order parameters can be separated as time reversal even or time reversal odd by $T O^{\text{ord}} = \pm O^{\text{ord}}$. For the even operators we have that $\lambda^{(a)}_{ab} = -\lambda^{(o)}_{ba}$, while for the odd operators $\lambda^{(a)}_{ab} = \lambda^{(o)}_{ab}$. The $3 \times 3$ antisymmetric hermitian matrices can be generated by linear combinations of generators of the SU(3) Lie algebra $\lambda^{(a)}$ in the fundamental representation (with $\alpha \in \alpha_{\text{even}} = \{2, 5, 7\}$ while the symmetric hermitian $3 \times 3$ matrices are generated by linear combinations of $\lambda^{(a)}$, with $\alpha \in \alpha_{\text{odd}} = \{0, 1, 3, 4, 6, 8\}$, where $\lambda^{(0)}$ is the $3 \times 3$ identity matrix.

The odd (even) operators under TR are given by $O^{\text{ord}}$, with $\alpha \in \alpha_{\text{odd}} (\alpha_{\text{even}})$. The even operators describe the processes of electron hopping that are TR invariant, i.e. such terms can be added to the Hamiltonian. The operators that are odd under TR symmetry cannot be included into the Hamiltonian without breaking TR symmetry. Using bosonization, and omitting Fermi momentum contributions, these operators become, in the basis $\tilde{\phi}, \tilde{\theta}$

$$O^{\text{ord}} = \sum_{ab} \lambda^{(a)}_{ab} \cos \left( \sqrt{\pi} D_{ab} + \tilde{\theta} + \frac{2\theta}{\sqrt{3}} \right) e^{i\sqrt{\pi} D_{ab} \cdot \tilde{\phi}},$$

where $D_{ab} = d_a \pm d_b$, see Fig. 3. We have also incorporated the Klein factors $\kappa_a$ in the definition $\lambda^{(a)}_{ab} \to \tilde{\lambda}^{(a)}_{ab} \kappa_a \kappa_b$. In the TRSB phase where $\tilde{\theta}$ is pinned, we observe that these order parameters become quasi long-ranged ordered (QLRO)

$$\langle O^{\text{ord}}(x) \rangle \sim \frac{\lambda^{\alpha}_{ab} \tilde{\lambda}^{\beta}_{ab}}{|x|^{2k_{F,0}}},$$

with a wavevector $2k_{F,0}$. There are constant values of the field $\tilde{\phi}$, which for a repulsive interaction in the special point $g = g' > 0$ are given by $(\tilde{\theta}_a^2, \sqrt{2\pi} \tilde{\phi}_1^2) = (0, \pm \frac{3\pi}{2})$. The largest set of cosine operators that commute with $\tilde{\phi}_1$ lock to the values $(0, \pi, 2\pi)$. This lockings opens a gap in the spectrum of the neutral sector. In general, the sign of the amplitude in front of the cosine terms determines the structure of the ground state. In the case that we are considering here, this amplitude is negative, so the fields $(\tilde{\phi}_1, \tilde{\phi}_2)$ lock to the values $(0, 0)$. As we will show below this phase is topological due to the pinning of the neutral field $\tilde{\phi}_2$. The topological nature of this phase is manifested in two ways: (a) in the stability of a metallic phase against weak disorder; (b) in domain wall configurations, that host localised fractionalized zero modes. In this phase TR symmetry is not broken.

Although for the “oversimplified” model discussed above, this phase appears just for attractive interactions, for a more generic case (see Appendix B) the topological phase can emerge for purely repulsive interactions as well, whenever inter-channel tunneling is present.

Repulsive Interactions

In this regime, $g > 0$, and the scaling dimensions satisfy $\Delta^g < \Delta^0$, so now the cosine operator $\cos(\sqrt{6\pi} \tilde{\theta})$ is the most relevant operator in RG sense. Keeping the largest set of cosine operators that commute with $\tilde{\theta}_2$, the Hamiltonian becomes

$$H = H_c + H_{\text{SU}(3)} + \frac{\sqrt{2g}}{3\pi} \int dx \partial_x \tilde{\theta} \partial_x \tilde{\theta}_2^2 + \frac{10g}{3\pi} \int dx (\partial_x \tilde{\theta}_2^2)^2 + \frac{2g}{\pi} \int dx (\partial_x \tilde{\phi}_1)^2. \tag{20}$$

The Hamiltonian $H_{\text{SU}(3)}$ can be obtained from (19) by the chiral transformation that interchanges $\tilde{\phi}_2 \leftrightarrow \tilde{\theta}_2$.

The cosine operator $\cos(\sqrt{6\pi} \tilde{\theta}_2)$ grows faster under renormalization opening a gap, pinning the value of the field $\tilde{\theta}_2$. The combination of field values $(\tilde{\theta}_2^2, \sqrt{2\pi} \tilde{\phi}_1^2)$ that minimises the energy is given semi-classically by the solutions of the equations

$$\cos(\sqrt{6\pi} \tilde{\theta}_2^2) + 2 \cos(\sqrt{2\pi} \tilde{\phi}_1^2) = 0,$$

$$\sin(\sqrt{6\pi} \tilde{\theta}_2^2) \cos(\sqrt{2\pi} \tilde{\phi}_1^2) = 0,$$

which for a repulsive interaction in the special point $g = g' > 0$ are given by $(\tilde{\theta}_2^2, \sqrt{2\pi} \tilde{\phi}_1^2) = (0, \pm \frac{3\pi}{2})$, or by $(\tilde{\theta}_2^2, \sqrt{2\pi} \tilde{\phi}_1^2) = (\pi, \pm \frac{3\pi}{2})$ with a double degenerate vacua.
On the other hand, for the ET phase where $\hat{\varphi}_2$ is pinned, these order parameters do not exhibit QLRO, decaying exponentially.

**B. Superconducting Order Parameters**

We can also study the superconducting order parameters, given by $S_{\alpha\beta} = \sum_{a,b} (c_{a,L}^\dagger T^{(a)}\tau_{b,R} + c_{b,R} T^{(a)\dagger} c_{a,L})$. These operators do not develop QLRO in any phase as they always contain the field $\theta_1$, dual to $\hat{\varphi}_1$, which is locked in both phases (see also Fig. 3). Correlation functions of these order parameters decay exponentially with distance in the groundstate. This implies that there is no superconducting order in any of the phases.

**C. Trionic order parameters**

As we have discussed, in both phases the low energy Hamiltonian of the model corresponds to an adiabatic deformation of an SU(3) Gross-Neveu model. Based on this structure, we can use the fundamental representation of SU(3) in terms of fermions to construct an order parameter. Starting from the complete antisymmetric Young tableaux corresponding to $\square = c_{1,1}^L c_{1,1}^R$, we define (in the band basis) the order parameter

$$\mathcal{T}_I = \psi_1^\dagger \psi_2^\dagger \psi_3^\dagger + \text{h.c.}$$

(23)

with $\psi_a = \psi_a^+ + \psi_a^-$. In the TRSB phase the order parameter $\mathcal{T}_I$ acquires QLRO, with correlation function satisfying

$$\langle \mathcal{T}_I(x)\mathcal{T}_I(0) \rangle \sim \frac{\sin(k_3 x)}{|x|^{\frac{3}{4}}(\pi^2 + \frac{\pi^2}{3})},$$

(24)

with wavevector $k_3 = k_{F,3} - k_{F,1} + k_{F,2}$. This trionic order parameter is dominant for strong attractive interactions such that $K_c > \sqrt{3} \sim 1.7$. We recall that for the special point $g = g'$ in the model [12], the trionic order parameter is never more dominant than the two particle operator $O_{4,5}^{\text{TD}}$ of Eq. (21). In the general model of Appendix [3] we see that there is a region where the trionic order parameter is dominant, for strong enough interaction.

In contrast, in the ET phase the conjugate field $\hat{\varphi}_2$ is locked. This implies that all two-particle order parameters have exponentially decaying expectation values. In particular, this indicates that the backscattering processes generated by the existence of impurities do not affect the conduction properties in this phase, at least at leading order on the impurity strength.

As there are no two-particle order parameters that dominate in the ET phase, we look for three-particle order parameters. We find that the operator

$$\mathcal{T}_{II} = \frac{\psi_1^\dagger \psi_2^\dagger \psi_3^\dagger + \psi_1^\dagger \psi_2 \psi_3^\dagger + \psi_1^\dagger \psi_3 \psi_3^\dagger}{\sqrt{3}} + \text{h.c.},$$

(25)

has dominant correlation function (discarding the purely right/left contributions $R_1^1 R_2 R_3$, etc.)

$$\langle \mathcal{T}_{II}(x)\mathcal{T}_{II}(0) \rangle \sim \frac{\sin(3k_2 x)}{|x|^{\frac{3}{2}}(\frac{\pi^2}{4} + \frac{\pi^2}{3})}.$$  

(26)

This phase is protected against single particle disorder, and its charge mode cannot be gapped by either two or four fermion terms, regardless of their microscopic origin. This is a feature of the ET phase. In the following section we discuss on general grounds the topological properties of the TRSB and ET phases.

**V. THE STABILITY OF TOPOLOGICAL PHASES AGAINST INTERACTIONS**

So far we have analysed the model of three interacting helical modes, having in mind a microscopic realisation. Now we shift the point of view to a more general perspective. Here we ask: Once the TRSB or ET phases are fully developed, Is it possible to gap their charge mode, without explicitly breaking TR symmetry? We ask this question irrespective of any microscopic realisation. For any given model, some of the terms discussed below will not appear due to momentum conservation or incommensurability. Anyway, they are allowed by the TR symmetry and we consider them.

In the case of two fermion operators, we have already seen that exist terms that can backscatter the helical modes in the trionic phase and do not decay exponentially. These terms are already present in the non-interacting limit and are the responsible for reducing the classification of two dimensional TR invariant systems from $Z$ to $Z_2$. For temperatures comparable with the largest gap in the neutral sector, we can estimate their effect by using the non-interacting Landauer formula [37], replacing the non-interacting parameters with the renormalized ones, given by the flow of the backscattering amplitudes due to the interactions. We do this explicitly in section [11]. Clearly, for lower temperatures, where the gaps in the neutral sector are the largest energy scale, extended backscattering terms can gap the charge mode in the TRSB phase, so this phase is not topologically protected. On the other hand, in the ET phase, all two-fermion operators decay exponentially, so they cannot localise the charge mode.

A general operator allowed by TR symmetry in a system of three helical edges corresponds to a polynomial in the operators

$$O^\theta_n = \cos(\sqrt{4\pi}n \cdot \theta) + \delta, \quad O^\varphi_n = \cos(\sqrt{4\pi}n \cdot \varphi).$$

(27)

with $\theta_a(\varphi_a) = \phi_{R,a} + (-)\phi_{L,a}$. The parameter $\delta$ is an arbitrary real number. The vector $n$ has integer components. Due to TR symmetry, it satisfies $\sum_n n_a = 0 \mod 2$. In the basis of charge and neutral modes, these
In the ET phase, on the other hand, the locked fields are which corresponds to the operator that commutes with the operators that open neutral gaps. The operator that describes backscattering between Kramers wires discussed above are stable under any perturbation phases of the microscopic model of three coupled helical systems with the same topological properties. The model at hand is a representative example for many that does not violate TR symmetry. This suggests that the ET state on the other, we find that a particular underlying microscopic model. The ET phase, on the other hand, does not break spontaneously TR.

\[ \mathcal{O}_n^\theta = \cos \left( \sqrt{4\pi} \left( \frac{2\varphi_c}{\sqrt{3}} + \sum_a n_a d_a \cdot \hat{\theta} \right) \right), \quad (28) \]

\[ \mathcal{O}_n^\varphi = \cos \left( \sqrt{4\pi} \left( \frac{2\varphi_c}{\sqrt{3}} + \sum_a n_a d_a \cdot \hat{\varphi} \right) \right), \quad (29) \]

where we have used \( \sum_a n_a = 2p, \) \( p \in \mathbb{Z}. \) In the TRSB phase, where the pair \( \varphi_1, \varphi_2 \) is locked, it is easy to find an operator that locks the charge mode \( \theta_c. \) A solution (of the infinitely many) is given by \( n_1 = n_3 = 1 \) and \( n_2 = 0, \) which corresponds to the operator

\[ \mathcal{O}_{(1,0,1)}^\theta = R_1 L_1 R_3 L_3 + \text{h.c.} \quad (30) \]

In the ET phase, the other hand, the locked fields are \( \varphi_1, \varphi_2. \) In this phase we can only use the operator \( \mathcal{O}_n^\varphi, \) to lock the (conjugate) charge field, as this is the only operator that commutes with the operators that open the neutral gaps. The operator \( \mathcal{O}_n^\varphi \) does not conserve the overall charge (because to cause the locking of \( \varphi_c, \) it has to have \( p \neq 0 \)). These results can be summarised as:

**Q:** Is it possible to gap the charge mode in a given phase, without explicitly breaking TR symmetry?

**A:** In the TRSB phase, it is possible, so this phase is not topologically protected in the presence of interactions. In the ET phase, on the other hand, it is not possible to gap the charge mode, without also breaking particle number conservation, so this phase is protected by TR symmetry and particle number conservation. This general analysis implies in particular that the different phases of the microscopic model of three coupled helical wires discussed above are stable under any perturbation that does not violate TR symmetry. This suggests that the model at hand is a representative example for many systems with the same topological properties.

### A. Spontaneous breaking of TR in the trivial phase

As its name indicates, the TRSB phase breaks spontaneously the TR symmetry in the groundstate. One way of seeing this is by considering the expectation value of operators that describe backscattering between Kramers pairs. The TR odd hermitian operator

\[ \mathcal{O}_{Kr,a} = \frac{1}{2} \sum_{\eta\eta'} \psi_{a\eta}^\dagger(x) (\sigma_y)_{\eta\eta'} \psi_{a\eta'}(x), \quad (31) \]

in the bosonized form becomes

\[ \mathcal{O}_{Kr,a} \sim \sin \left( 2k_{F,a} x + \sqrt{4\pi} \left( \frac{\theta_c}{\sqrt{3}} + d_a \cdot \hat{\theta} \right) \right). \quad (32) \]

acquires a constant contribution when the charge mode is gapped, which is only possible in the phase where \( \theta_2 \) is locked. In particular, this occurs for the microscopic model of section [11] at \( \mu = 0 \) (which corresponds to an incommensurability condition that allows single particle Umklapp scattering) where the operator \( \mathcal{O}_{(1,0,1)}^\theta \) conserves momentum and locks the charge mode. The presence of a constant order parameter that is odd under TR symmetry indicates the spontaneous breaking of TR symmetry in the groundstate.

We note that due to the coupling to the charge mode this order parameter has QLRO whenever the charge mode is gapless. We stress that this consideration is based purely on general grounds and not associated with a particular model of the microscopic model of section [11] at \( \mu = 0 \) (which corresponds to an incommensurability condition that allows single particle Umklapp scattering) where the operator \( \mathcal{O}_{(1,0,1)}^\theta \) conserves momentum and locks the charge mode. The presence of a constant order parameter that is odd under TR symmetry indicates the spontaneous breaking of TR symmetry in the groundstate.

We observe that the topological protection of the non-interacting system can be absent once we include interactions. It is illustrative to consider some simple limits where the breaking of non-interacting topological protection is clearly appreciated. Taking \( g = 0 \) in our microscopic model of Eqs. [37], the system describes two strongly interacting modes (modes 1 and 3), coupled just through forward scattering with the mode 2. It shouldn’t be surprising that the pair of modes (1,3) can be completely gapped out by disorder, as it is not protected even at the single particle level, (we recall nevertheless, that in the presence of interactions this is possible just for repulsive interactions). Let’s assume that the pair (1,3) is indeed completely gapped out. By turning on a small \( g \) term, the remaining mode is coupled to the (1,3) pair, which is localised and acts like an electron puddle. The interaction induced backscattering with the electrons in this effective puddle breaks the topological protection of the single mode 2, as has been shown in Refs. [38] and [39]. Our model reproduces this behaviour.

In the next section, we discuss the nature of the critical line separating the two neutral massive phases.

### B. Relation with one and two helical modes

We observe that the topological protection of the non-interacting system can be absent once we include interactions. It is illustrative to consider some simple limits where the breaking of non-interacting topological protection is clearly appreciated. Taking \( g = 0 \) in our microscopic model of Eqs. [37], the system describes two strongly interacting modes (modes 1 and 3), coupled just through forward scattering with the mode 2. It shouldn’t be surprising that the pair of modes (1,3) can be completely gapped out by disorder, as it is not protected even at the single particle level, (we recall nevertheless, that in the presence of interactions this is possible just for repulsive interactions). Let’s assume that the pair (1,3) is indeed completely gapped out. By turning on a small \( g \) term, the remaining mode is coupled to the (1,3) pair, which is localised and acts like an electron puddle. The interaction induced backscattering with the electrons in this effective puddle breaks the topological protection of the single mode 2, as has been shown in Refs. [38] and [39]. Our model reproduces this behaviour.

### VI. TRANSITION BETWEEN PHASES

In the transition between the TRSB and ET phases, the gap in the neutral sector of the system vanishes throughout the whole edge. This one-dimensional gapless system is described by a theory at low energies with an emergent \( \mathbb{Z}_3 \) symmetry. By going away from the quantum critical point, a gap in the neutral sector opens. By considering a position dependent interaction that creates the TRSB phase in one sector of the edge, while inducing the ET state on the other, we find that a \( \mathbb{Z}_3 \) parafermion is trapped in the transition region. Below we study the quantum critical point that appears in the transition between these two phases along the edge, and then how this result implies the existence of nontrivial particles trapped in domain wall configurations.
A. $\mathbb{Z}_3$ critical theory at the transition.

The transition between the TRSB and the ET phase happens at $g + g' = 0$. The amplitude of the cosine terms in the Hamiltonian vanishes at the transition in the specific line $g = g'$, indicating that along this line of parameters the critical point is Gaussian. By exploring a more generic state e.g. by considering $g \neq g'$ (see also Appendix B), the amplitude of the cosine terms does remain finite. On the transition line $g + g' = 0$, we find that the Luttinger parameters satisfy $K_1 = K_2 = 1$. We introduce the vertex representation of the SU(2)$_1$ algebra to go across the quantum phase transition between the critical sector, leaving behind a critical Hamiltonian for the $\tilde{\sigma}$ sectors.

Using this representation, it is possible to understand the Hamiltonian (12-13) vanishes at the transition in the ET region. Although it will renormalise to zero at $g + g' = 0$, the diagonal dashed line corresponds to the simple limit $g = g'$, considered in Sec. III.

By doing so, we find a parafermionic zero mode trapped in the transition region. These zero modes are studied in the next section.

B. Parafermionic zero modes

As we have found, the transition between the TRSB and the ET phase is described by a critical theory, whose low energy description is given by a parafermionic CFT of central charge $4/5$, with $\mathbb{Z}_3$ symmetry. Changing the effective interactions between the helical modes along the edge, for example by external gates, it is possible to generate a domain wall configuration, where on one side the system is in the TRSB phase, while on the other is in the ET phase. We can use this result to trap parafermionic quasiparticles in the interface between the two phases, in a mechanism similar to the Jackiw-Rebbi, fractionalization of the electron [16].

Another mechanism to reveal the presence of these parafermionic modes is considering very strong impurity somewhat in the ET region. Although it will renormalise to zero at $T = 0$, there may be an intermediate energy scale below the scale set by the neutral gap $\Delta_n$ where the impurity is still strong and in this intermediate regime one can see the parafermionic edge states (c.f. the equivalent case for two edges discussed in [47]).
rana fermions, as they satisfy the relations in the lattice of parafermions is given by the three-state quantum Potts model, which in terms of lattice sites, the parafermions satisfy
\[ h \sum_{j<k} \chi_j^\dagger \eta_j \bar{\omega} + \eta_j^\dagger \chi_j \omega, \]
where \( h \) is the defining symmetries. The point obtained from the RG flow of the self-dual Hamiltonian \( \mathcal{H}_{\text{top}} \), which correspond to \( \mathbb{Z}_3 \) parafermion CFT.

In general \( \mathbb{Z}_n \) parafermionic modes generalise Majorana fermions, as they satisfy the relations in the lattice
\[ \chi_j^n = \eta_j^n = 1, \quad \chi_j^\dagger = \chi_j^{n-1}, \quad \eta_j^\dagger = \eta_j^{n-1}, \]
where \( j \) denotes a lattice site and \( \omega = e^{2i\pi/n} \). At different lattice sites, the parafermions \( \eta, \chi \) satisfy
\[ \chi_j \chi_k = \omega \chi_k \chi_j, \quad \eta_j \eta_k = \omega \eta_k \eta_j, \quad \chi_j \eta_k = \omega \eta_k \chi_j, \]
for \( j < k \). We are interested in a model that captures the symmetry properties that our system develops in the IR. In particular, the model should display TR and \( \mathbb{Z}_3 \) symmetry. The simplest model that displays both is given by the three-state quantum Potts model, which in terms of parafermions is given by
\[ H_{\text{eff}} = -\sum_{j=1}^{L} \left( h \chi_j^\dagger \eta_j \bar{\omega} + \text{h.c.} \right) + J(\eta_j^\dagger \chi_{j+1} \bar{\omega} + \text{h.c.}) . \]
The parameters \( t, J \) are phenomenological, and represent a description of the original parameters after renormalization. The phase \( h J > 1 \) corresponds to the ordered phase. The spectrum possess a gap and the ground state spontaneously breaks the \( \mathbb{Z}_3 \) and TR symmetry. The opposite limit \( h J \ll 1 \), corresponds to the disordered phase, which is also gapped but does not break spontaneously the defining symmetries. The point \( h J = 1 \) is critical and self-dual. The relation with the microscopic parameters is given by
\[ \left[ g + g' \right]/g = h J, \]
where we denote \([g]_{\text{IR}}\) the renormalized parameter \( g \) in the low energy description. The TRSB phase corresponds to the ordered phase \( h J \gg 1 \). In this phase, the low energy physics is dominated by the Hamiltonian
\[ H_{\text{top}} = -\sum_{j=1}^{N} \left( h \chi_j^\dagger \eta_j \bar{\omega} + \eta_j^\dagger \chi_j \omega \right). \]
On the other hand, the ET phase corresponds to the limit \( h J \ll 1 \), where the Hamiltonian is dominated by
\[ H_{\text{top}} = -\sum_{j=1}^{N} \left( h \chi_j^\dagger \eta_j \bar{\omega} + \eta_j^\dagger \chi_j \omega \right). \]
In this phase, the operators \( (\Psi_{\text{in}}, \Psi_{\text{out}}) \equiv (\chi_1, \eta_N) \) decouple from the Hamiltonian, i.e. \([\Psi_a, H_{jj}] = 0\), but they do not commute with the \( \mathbb{Z}_3 \) symmetry operator \( \Omega \), which has a representation
\[ \Omega = \prod_{j=1}^{N} \eta_j^\dagger \chi_j, \]
thus satisfying \( \Omega \Psi_a = \omega \Psi_a \Omega \). The zero modes map states between different symmetry sectors and are localised at both ends of the topological spatial region.

The TR symmetry \( T \) in this system can be represented as
\[ T \chi_j T^{-1} = \eta_{N+1-j}, \quad T \eta_j T^{-1} = \chi_{N+1-j}, \]
which together with the relation \( T i T^{-1} = -i \) \[49\],
As we have discussed, a main difference between the ET and the TRSB phase that should be readily accessible in experiments is the value of the conductance. It is then important to assess the role of disorder in each system. In the next section we analyse the behaviour of a single impurity and random disorder in each of the phases

VII. DISORDER

For non-interacting electrons, the conductance through the system is given by Landauer formula
\[ G = \sum_{i} T_i / h \]
where the sum runs over all the transport channels. For the clean system the transmission coefficients \( T_i = 1 \) such that the total conductance through the system is \( G = 3e^2/h \). In presence of static disorder the problem can
be solved using the scattering matrix formalism. For a single non-magnetic impurity the electric conductance is given by (see also Appendix C)

\[ G(T) = \frac{e^2}{h} \left[ 1 + 2 \frac{1 - g_{\text{imp}}^2}{1 + g_{\text{imp}}^2} \right]. \tag{47} \]

The first term on the right-hand side of Eq. \(47\) follows from a ballistic propagation along the topologically protected channel.

For an interacting system the Landauer approach is strictly speaking not applicable. Nevertheless, one may still use it as a semi- qualitative approximation. In this case, one needs to replace the values of transmission coefficient by its renormalised value at energy/temperature \(T\) (not to be confused with the transmission coefficients \(T_i\)) dependent scale, \(g_{\text{imp}} \rightarrow g_{\text{imp}}(T)\). However, Eq. \(47\) is valid provided that the system remains in topologically non-trivial state (either inherited or emergent). If topological protection is removed, the conductance will generically go to zero.

The backscattering processes are in general proportional to the Fermi momentum components of the order parameters \(\rho_{\alpha}^{\text{ord}}\) studied previously. In the TRSB phase these operators survive the integration of the massive degrees of freedom, and their amplitude \(g_{\text{imp}}\) scales under renormalization as

\[ \frac{dg_{\text{imp}}}{d\ell} = \left(1 - \frac{K_c}{3}\right) g_{\text{imp}}, \tag{48} \]

with \(\ell = \ln \Lambda_0 / T\), where \(\Lambda_0\) is an ultraviolet cut-off). The inclusion of higher order terms in backscattering amplitude \(g_{\text{imp}}\) into Eq. \(48\) leads to \(g_{\text{imp}}^2\) corrections. Such operators are not relevant in the RG sense for weak interactions and therefore can be neglected. The solution of Eq. \(48\) reads

\[ g_{\text{imp}}(T) = g_{\text{imp}}(\Lambda_0) \left(\frac{\Lambda_0}{T}\right)^{1-K_c/3}. \tag{49} \]

Thus for a system in the TRSB phase, with weak interactions \((K_c < 3)\) this process is RG relevant. Therefore electric conductance vanishes at low temperatures. For larger interactions, such that \(K_c > \sqrt{3}\), the dominant process that makes the conductance vanish at low temperatures corresponds to trion backscattering.

In contrast, in the ET phase, after the massive degrees of freedom are integrated out, these processes (electron and trion backscattering) do not contribute. Therefore the conductance in the topological phase the system at high temperatures is approximately \(3e^2/h\).

We now schematically plot the conductance as function of temperature for the both phases, see Fig. 6. We focus on the limit where the bare value of impurity potential is weak. We assume that the interaction is repulsive and its strength is small, such that all characteristic LL parameters are slightly smaller than one.

Above the two dimensional TI’s gap \(\Delta_b\), the conductance in all phases is a non universal function with a value below (but close to) 3 (in the units of \(e^2/h\)). As temperature decreases towards the gap formed in the neutral sector \(\Delta_n\), the conductance in both phases decrease as \(T^{-\Delta_b/\Delta_n} - 1\). Below \(\Delta_n\) the conductance in the ET phase starts to rise with decreasing temperature, reaching an ideal limit \(G \rightarrow 3\) at \(T \rightarrow 0\). Therefore in this phase the conductance is a non-monotonous function of temperature.

In non-topological TRSB phase, for temperatures below the neutral gap energy the conductance approaches the value \(G = 1\) and stays as \(T^{-K_c/3-1}\) approximately constant until a small energy scale \(\Delta_p\) associated with spontaneous symmetry breaking. This plateau is reminiscent of the topological properties present in the non-interacting limit. What happens below \(\Delta_p\) depends on the type of disorder present. For a weak single impurity and weak interactions \(G = 1\) all the way down to zero temperature, as shown by the blue line in the Fig. 6. For random disorder \(G\) goes to zero as sketched by the red line in Fig. 6. The exponent of the pair-backscattering operator is \(4K_c/3\), which implies that the correction \(G - 1 \propto T^{4K_c/3-1}\). However, due to the possibility of fully gapping the charge mode, it would be more appropriate to perturb around a strong coupling (localised) fixed point. We will explore this point further in a separate work.


**VIII. DISCUSSION AND OUTLOOK**

In this paper we studied the competition of emergent and inherent topological orders. We focused on a system made of three helical wires, that correspond to the edges states three two dimensional topological insulators stacked together. In the non-interacting limit this system is topologically equivalent to a single helical edge state protected against static disorder. We showed that in the presence of electron interaction this picture changes. We now summarise our findings.

In the presence of interaction the system may turn into one of two possible states. In the first case the system acquires new topological order that can not be adiabatically connected to the non-interacting one. In the second case the TR symmetry is spontaneously broken and the system is driven into a topologically trivial state, that in the presence of a static disorder it turns into an Anderson insulator.

To understand the loss of topology one may take the limit where one of the channels is almost decoupled, such that the interaction with it is weak. The remaining two wires may be in the topologically trivial or non-trivial state, depending on the parameter of the interaction constants there \[24\] [27]. If two coupled channels happen to be in a topologically trivial state, they would be localised by any finite amount of disorder. Therefore the system of three helical modes effectively becomes equivalent to a single helical channel coupled by hopping to multiple puddles of electronic fluid. Such system is equivalent to an Anderson insulator \[38\] [59].

The ground state of a topologically trivial state is a strongly correlated one, that develops a QLRO. The character of QLRO depends on the details of interaction. The weak repulsive interaction results in a family of two-particle correlations with power low decay and \[2k_F\] oscillations. For sufficiently strong repulsive interaction \[K_c > \sqrt{3}\] the dominant QLRO is a trionic one.

In the case of small attractive interactions, a new topological order develops. The latter is protected by a gap in the neutral sector, that opens inside the one dimensional system due to many body scattering. This state is robust against Anderson localisation with a total conductance of \(3e^2/h\) for a moderate disorder.

The transition between topological and non-topological phases occurs along the critical line in a multidimensional space of interaction constants. While the neutral sector of the theory is gapped in both phases the gap closes on their boundary. The neutral sector of the theory becomes critical, with a low energy description in the universality class of the \(Z_3\) parafermionic CFT. The latter is manifested by the emergence of parafermionic excitations at the end points of the system.

We also find that the low energy fix point has a higher symmetry with respect to interaction between modes that the original model, signalling dynamically emergent symmetry. This phenomenon was previously observed in the context of three leg ladders \[50\] [61]. In our case, the massive phases are ground states of a Hamiltonian that is obtained by marginal deformations of an emergent SU(3) symmetry, which is not present in the UV, but that manifest itself in the IR. The topological phase corresponds to a deformed SU(3) Hamiltonian \(\tilde{H}_{SU(3)}\) of Eq. (18) that can be obtained from the usual SU(3) Gross-Neveu Hamiltonian by performing a chiral transformation. The emergent topology arises due to a gap in the neutral sector of this Hamiltonian.

Though both symmetry protected topological ordered and dynamically generated symmetries were previously known, the current system is the first example where both effects act together. The interaction enhances the effective symmetry of the problem in the IR limit. The generated symmetry gives rise to the topologically non-trivial state.

The rich physics of this system invites to a further exploration of its different facets. In particular, we consider crucial to find experimental signatures of parafermions that emerge on the boundary between the phases, and to account for strong impurities and random disorder. It is appealing to consider how these results generalise to a larger number of helical modes, exploring the possible connection to the theory of interacting symplectic wires. It remains to be seen if the emergent symmetry allows to find the regimes beyond those predicted within disordered Fermi liquid approach \[62\]. Finally, from a general perspective, it is compelling to study the general criteria for the existence of dynamically emergent symmetry protected states.

*Note added:* When this manuscript was in preparation, we learned about preprints \[28\] [63] with partly overlapping content. The work of Kagalovsky et al. \[28\] discusses TRS breaking in the ground state leading to zero conductance at zero temperature, for any number of channels \(N \geq 3\). Our results are in full agreement with theirs for \(N = 3\). In this specialised case we uncover a number of non-trivial phases as function of interaction and crossovers as a function of temperature, which presumably one would see for any odd \(N\), although this remains work for the future. The work of Keselman et al. \[63\] looks at a different model, concentrating on \(N = 3\) channels in which the non-interacting model is non-topological, and like us finds a phase with TRS breaking, and another phase with an emergent topology. While their TRS breaking phase is the same one we find, they curiously find a different emergent topological phase, in the universality class of the Haldane spin-1 chain as opposed to our \(Z_3\) parafermionic state. This gapless Haldane state relies on a \((Z_2)^3\) symmetry, which we explicitly break by the inter-chain hopping (or equivalently, the splitting of the Fermi-points) in our model. In contrast, our parafermion state explicitly emerges from interaction terms that require the inter-chain tunnelling in the Hamiltonian. It remains work for the future to determine the full phase diagram of a more generic \(N = 3\) channel system, and to see if there are more possibilities...
for emergent topological states beyond these two.

Acknowledgement.- R.S. would like to thank, Eran Sagi, Jinhong Park and Benjamin Béri for stimulating discussions. D. G was supported by ISF (grant 584/14) and Israeli Ministry of Science, Technology and Space.

Appendix A: Zero tunnelling limit

In the limit of zero tunnelling \( t_\perp = 0 \), there is another operator that conserves momentum, given by

\[
O_{t_\perp=0} = \tilde{g} \int dx R_1^1 L_1^1 R_3.
\]

(A1)

The presence of this operator, together with the other operator involving just the modes 1 and 3 (first line of Eq. (7)), modifies the low energy behaviour of the model, preventing the opening of a gap between the modes 1 and 3, as can be observed in the case of two helical modes [24]. This result is in line with the intuition that independent helical modes interacting through their densities, away from commensurate filling, are not gapped by interactions.

Appendix B: General Model

The model considered in the main text correspond to a particularly simple description of a more generic model that we discuss here. We consider three helical modes, described by the fermion destruction (creation) operator of momentum \( k, c^\eta_{k,a} (c^{\dagger \eta}_{k,a}) \), where \( a = (1,2,3) \) denotes the mode and \( \eta = (+, -, \bar{\eta}) \) labels its helicity. For small momenta, the non-interacting Hamiltonian is

\[
H_0 = \sum_{k,a,\eta} \eta v_F k c^{\dagger \eta}_{k,a} c^\eta_{k,a} + \alpha_{so} c^{\dagger \eta}_{k,a} c^\eta_{k,a} + \sum_{k,\eta}(t_L^\eta c^{\dagger \eta}_{k,2} c^\eta_{k,1} + t_R^\eta c^{\dagger \eta}_{k,2} c^\eta_{k,3} + h.c.),
\]

(B1)

where \( v_F \) is the Fermi velocity of the modes, \( \alpha_{so} \) parameterizes a residual spin-orbit coupling along the edge. We assume that tunnelling only occurs between the modes which are closest in space, with amplitudes \( t_L \) and \( t_R \). A diagram of the arrangement of helical modes and their labellings is given in Fig. 7.

The energy dispersion relations in the band basis are

\[
E_\eta^a = \eta v_F k + \lambda_a^\perp, \quad \text{with the new Fermi velocity } \tilde{v}_F = \sqrt{v_F^2 + \alpha_{so}^2}, \quad \text{the perpendicular tunnelling parameter } t_\perp = \sqrt{t_L^2 + t_R^2} \quad \text{and } \lambda_a^\perp = (-1,0,1).
\]

The single particle Hamiltonian is invariant under the symmetry of interchanging the modes 1 \( \leftrightarrow 3 \) and \( t_L \leftrightarrow t_R \).

Going from the original modes to the band modes that diagonalize the Hamiltonian is implemented by the unitary transformation \( [U(\nu)]_{ab}^{\eta\eta'} = (U_c(\nu))_{ab}(U_h)^{\eta\eta'} \) where \( \tan \nu = \frac{t_L}{t_R} \). The unitary transformation \( U_h = e^{i\beta_1 \sigma_y} \) (with \( \tan 2\beta = \frac{2\tilde{g} v}{v_F} \)) acts on the helicities, while \( U_c \) acts in the channel index rotating the modes into the band basis, and is given by

\[
U_c(\nu) = \frac{1}{\sqrt{2}} \begin{pmatrix}
\sin \nu & -1 & \cos \nu \\
\sqrt{2} \cos \nu & 0 & -\sqrt{2} \sin \nu \\
\sin \nu & 1 & \cos \nu
\end{pmatrix}. \quad \text{(B2)}
\]

A generic interaction between the three different helical modes is described by the following Hamiltonian

\[
H_{\text{int}} = \sum_{i,a} V_0 n_i,a n_i,a + \sum_{i,a \neq b} V_{ab} n_i,a n_{i,b}, \quad \text{(B3)}
\]

where the density at each site \( i \) and channel \( a \) is \( n_{i,a} = \sum_i c_{i,a}^{\sigma} c_{i,a}^{\sigma^\dagger} \). This interaction parameters are symmetric \( V_{ab} = V_{ba} \).

After bosonization, using the basis \( |0\rangle \) of the main text, the forward scattering Hamiltonian becomes

\[
H_{\text{fs}} = \sum_{a=c,1,2} \frac{v_a}{2} \int dx \left( K_a (\partial_x \hat{\varphi}_2)^2 + \frac{1}{K_a} (\partial_x \hat{\theta}_2)^2 \right) + \int dx (\zeta_1 \partial_x \hat{\varphi}_c \partial_x \hat{\varphi}_2 + \zeta_2 \partial_x \hat{\theta}_c \partial_x \hat{\theta}_2), \quad \text{(B4)}
\]

where the parameters \( v_a, K_a, \zeta_{1,2} \) satisfy...
The complete Hamiltonian reads

\[ H = H_\text{et} + \tilde{g}_1 \int dx \cos(2\sqrt{2}\pi \varphi_1) + 2\tilde{g}_2 \int dx \left( \cos(\sqrt{6}\pi \varphi_2) + \cos(\sqrt{6}\pi \varphi_3) \right) \cos(\sqrt{2}\pi \varphi_1). \]

with \( \tilde{g} = \frac{g}{2(\pi\sigma_{\text{ms}})} \). The transition line between the ET and TRSB phases corresponds to \( g + g' - g_1 = 0 \). The symmetric limit \( t_L = t_R \) corresponds to \( v = \pi/4 \). For this value the general model reduces to the one we used in the main part of the manuscript.

**Appendix C: Scattering Matrix for non-interacting channels**

The Schrödinger equation for three chiral fermions scattering off an impurity at \( x = 0 \) can be written as

\[ iv_F(1 \otimes \sigma_2)\partial_x \Psi + \mathcal{V}(x)\Psi = E\Psi. \]

Here \( \mathcal{V} \) parameterizes the scatterer and \( \Psi \) is a 6-component spinor that contains the right and left mover part of the chiral fermion. This scatterer potential can be decomposed in the basis \( V = \sum a V_a \otimes \sigma^a \), where \( \sigma^a \) are the Pauli and 2 \times 2 identity matrices. The matrices \( V_a \) act in the channel space, while \( \sigma^a \) acts between the chiralities of the fermions. Without losing generality, the backscattering part of the potential can be written in the form \( V \otimes \sigma_x \), where TR symmetry dictates that the scatterer potential \( V \) in the is such that \( VT = -V \). Taking the determinant of this equation, we find that \( \det(V^T) = (-1)^3 \det(V) = 0 \). Also follows from the antisymmetry of \( V \) that its the trace vanishes. In the basis \( \Psi = (U_{V} \otimes 1_{2 \times 2})\Psi \) that diagonalizes \( V \) Eq. (C1) splits into

\[ iv_F(1 \otimes \sigma_2)\partial_x \Psi_1 + iv \sigma_3 \delta(x)\Psi_1 = E\Psi_1, \]

\[ iv_F(1 \otimes \sigma_2)\partial_x \Psi_2 = E\Psi_2, \]

\[ iv_F(1 \otimes \sigma_2)\partial_x \Psi_3 - iv \sigma_3 \delta(x)\Psi_3 = E\Psi_3. \]

where \( r \) describes the strength of the scattering potential. These equations describe the propagation of three decoupled modes, that constitute independent conducting channels. Due to TR symmetry and the number of channels being odd, there is one mode with zero reflection across the impurity. Solving the previous equations using the regularisation \( \int dx \delta(x)\Psi(0) = \frac{1}{2}(\Psi(0^-) + \Psi(0^-)) \), we find the scattering matrix for the modes 1 and 3 to be

\[ S = \cos \beta \sigma^0 \pm i \sin \beta \sigma^x, \]

where the \((\cdot)\) sign is for the mode 1(3). Here \( \beta = 2\tan^{-1}\frac{r}{2\pi F} \). Defining \( g_{\text{imp}} \equiv \frac{1}{2\pi F} \), it is found [64] that the transmission coefficient \( T_i \) for mode 1 and 3 is

\[ T_1 = T_3 = \frac{1 - g_{\text{imp}}^2}{1 + g_{\text{imp}}^2}. \]

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