COMPARISON BETWEEN $W_2$ DISTANCE AND $\dot{H}^{-1}$ NORM, AND LOCALIZATION OF WASSERSTEIN DISTANCE

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Abstract. It is well known that the quadratic Wasserstein distance $W_2(\cdot,\cdot)$ is formally equivalent, for infinitesimally small perturbations, to some weighted $H^{-1}$ homogeneous Sobolev norm. In this article I show that this equivalence can be integrated to get non-asymptotic comparison results between these distances. Then I give an application of these results to prove that the $W_2$ distance exhibits some localization phenomenon: if $\mu$ and $\nu$ are measures on $\mathbb{R}^n$ and $\varphi: \mathbb{R}^n \to \mathbb{R}_+$ is some bump function with compact support, then under mild hypotheses, you can bound above the Wasserstein distance between $\varphi \cdot \mu$ and $\varphi \cdot \nu$ by an explicit multiple of $W_2(\mu,\nu)$.

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1. Foreword

This article is divided into two sections, each of which having its own introduction. Section 2 deals with general results of comparison between Wasserstein distance and homogeneous Sobolev norm, while Section 3 handles an application to localization of $W_2$ distance.

2. Non-asymptotic equivalence between $W_2$ distance and $\dot{H}^{-1}$ norm

2.1. Introduction

In all this section, $M$ denotes a connected Riemannian manifold endowed with its distance $\text{dist}(\cdot,\cdot)$ and its standard measure $\lambda$ provided by the volume form (so, in the case $M = \mathbb{R}^n$, $\lambda$ is the Lebesgue measure). Let us give a few standard definitions which will be at the core of our work:

- For $\mu, \nu$ two positive measures on $M$, denoting by $\Pi(\mu, \nu)$ the set of (positive) measures on $M \times M$ whose respective marginals are $\mu$ and $\nu$, for $\pi \in \Pi(\mu, \nu)$ one defines

$$I(\pi) := \int_{M \times M} \text{dist}(x,y)^2 \pi(dx,dy)$$ (2.1)

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and then
\[
W_2(\mu, \nu) := \inf \{ I(\pi) | \pi \in \Pi(\mu, \nu) \}^{1/2}. \tag{2.2}
\]

$W_2$ is a (possibly infinite) distance, called the quadratic Wasserstein distance ([13], Sect. 7.1). Note that this distance is finite only between measures having the same total mass.

- On the other hand, for $\mu$ a (positive) measure on $M$, if $f$ is a $C^1$ real function on $M$, one denotes
\[
\|f\|_{\dot{H}^1(\mu)} := \left( \int_M |\nabla f(x)|^2 \mu(dx) \right)^{1/2}, \tag{2.3}
\]
which defines a semi-norm; for $\nu$ a signed measure on $M$, one then denotes
\[
\|\nu\|_{\dot{H}^{-1}(\mu)} := \sup \{ |\langle f, \nu \rangle| | \|f\|_{\dot{H}^1(\mu)} \leq 1 \}, \tag{2.4}
\]
where the duality product $\langle f, \nu \rangle$ denotes the integral of the function $f$ against the measure $\nu$.\footnote{The rationale behind the use of duality notation in this article is that we cannot use the notation “$d\mu$” to refer to the measure of a small volume; see indeed Footnote 3 below.}

We observe that $\|\cdot\|_{\dot{H}^{-1}(\mu)}$ defines a (possibly infinite) norm, which we will call the $\dot{H}^{-1}(\mu)$ weighted homogeneous Sobolev norm. Note that this norm is finite only for measures having zero total mass. In the case $\mu$ is the standard measure, we will merely write “$H^{-1}$” for “$\dot{H}^{-1}(\lambda)$”.

The $W_2$ Wasserstein distance is an important object in analysis; but it is non-linear, which makes it harder to study. For infinitesimal perturbations however, the linearized behaviour of $W_2$ is well known: if $\mu$ is a positive measure on $M$ and $d\mu$ is an infinitesimally small perturbation of this measure,\footnote{Beware that here $d\mu$ denotes a small measure on $M$, not the value of $\mu$ on a small volume.} one has formally (see [13], Sect. 7.6, or [9], Sect. 7)
\[
W_2(\mu, \mu + d\mu) = \|d\mu\|_{\dot{H}^{-1}(\mu)} + o(\|d\mu\|). \tag{2.5}
\]

More precisely, one has the following equality, known as the Benamou–Brenier formula ([2], Prop. 1.1) (see [10] when $M$ is a general Riemannian manifold): for two positive measures $\mu, \nu$ on $M$,
\[
W_2(\mu, \nu) = \inf \left\{ \int_0^1 \|d\mu\|_{\dot{H}^{-1}(\mu_t)} \bigg| \mu_0 = \mu, \mu_1 = \nu \right\}. \tag{2.6}
\]

Then, a natural question is the following: are there non-asymptotic comparisons between the $W_2$ distance and the $\dot{H}^{-1}$ norm? Concretely, we are looking for inequalities like
\[
C_a \|\mu - \nu\|_{\dot{H}^{-1}(\mu)} \leq W_2(\mu, \nu) \leq C_b \|\mu - \nu\|_{\dot{H}^{-1}(\mu)} \tag{2.7}
\]
for constants $0 < C_a \leq C_b < \infty$, under mild assumptions on $\mu$ and $\nu$.

### 2.2. Controlling $W_2$ by $\dot{H}^{-1}$

**Theorem 2.1.** For any positive measures $\mu, \nu$ on $M$,
\[
W_2(\mu, \nu) \leq 2 \|\mu - \nu\|_{\dot{H}^{-1}(\mu)}. \tag{2.8}
\]

**Proof.** We suppose that $\|\mu - \nu\|_{\dot{H}^{-1}(\mu)} < \infty$, otherwise there is nothing to prove. For $t \in [0,1]$, let
\[
\mu_t := (1-t)\mu + t\nu, \tag{2.9}
\]

This formula has to be understood in the sense that, for every measure $\nu$, one has $W_2(\mu, \mu + \varepsilon \nu) \leq \varepsilon \|\nu\|_{\dot{H}^{-1}(\mu)} + o(\varepsilon)$. As explained in the references cited, some regularity assumptions on $\nu$ shall be required for that property to hold rigorously: in particular, one must have $\nu \ll \mu$ with a bounded and smooth enough density.
so that \( \mu_0 = \mu, \mu_1 = \nu \) and \( d\mu_t = (\mu - \nu)dt \). Then, by the Benamou–Brenier formula (2.6):

\[
W_2(\mu, \nu) \leq \frac{1}{\rho} \|\mu - \nu\|_{\dot{H}^{-1}(\mu)} dt.
\] (2.10)

Now, we use the following key lemma, whose proof is postponed:

**Lemma 2.2.** If \( \mu, \mu' \) are two measures such that \( \mu' \geq \rho \mu \) for some \( \rho > 0 \), then \( \|\cdot\|_{\dot{H}^{-1}(\mu')} \leq \rho^{-1/2} \|\cdot\|_{\dot{H}^{-1}(\mu)}. \)

Here obviously \( \mu_t \geq (1 - t)\mu \), so

\[
W_2(\mu, \nu) \leq \int_0^1 (1 - t)^{-1/2} \|\mu - \nu\|_{\dot{H}^{-1}(\mu)} dt = 2 \|\mu - \nu\|_{\dot{H}^{-1}(\mu)}. \quad (2.11)
\]

**Corollary 2.3.** If \( \mu \geq \rho \lambda \) for some \( \rho > 0 \), then

\[
W_2(\mu, \nu) \leq 2\rho^{-1/2} \|\mu - \nu\|_{\dot{H}^{-1}}. \quad (2.12)
\]

**Proof.** Just use that \( \|\cdot\|_{\dot{H}^{-1}(\mu)} \leq \rho^{-1/2} \|\cdot\|_{\dot{H}^{-1}} \) by Lemma 2.2. \( \square \)

**Proof of Lemma 2.2.** Take \( \mu' \geq \rho \mu \) and let \( \nu \) be a signed measure on \( M \) such that \( \mu + \nu \) is positive; then \( \mu' + \rho \nu \) is also positive. For \( m \) a measure on \( M \), we denote by \( \text{diag}(m) \) the measure on \( M \times M \) supported by the diagonal whose marginals (which are equal) are \( m \), i.e.:

\[
(\text{diag}(m))(A \times B) := m(A \cap B); \quad (2.13)
\]

with that notation,

\[
\pi \in \Pi(\mu, \mu + \nu) \Rightarrow \rho \pi + \text{diag}(\mu' - \rho \mu) \in \Pi(\mu', \mu' + \rho \nu), \quad (2.14)
\]

and

\[
I(\rho \pi + \text{diag}(\mu' - \rho \mu)) = \rho I(\pi). \quad (2.15)
\]

Therefore, taking infima,

\[
W_2(\mu', \mu' + \rho \nu)^2 = \inf \left\{ I(\pi') \mid \pi' \in \Gamma(\mu', \mu' + \rho \nu) \right\} \\
\leq \inf \left\{ I(\rho \pi + \text{diag}(\mu' - \rho \mu)) \mid \pi \in \Gamma(\mu, \mu + \nu) \right\} \\
= \rho \inf \left\{ I(\pi) \mid \pi \in \Gamma(\mu, \mu + \nu) \right\} = \rho W_2(\mu, \mu + \nu)^2. \quad (2.16)
\]

For infinitesimally small \( \nu \),\(^6\) it follows by equation (2.5) that \( \|\rho \nu\|_{\dot{H}^{-1}(\mu')}^2 \leq \rho \|\nu\|_{\dot{H}^{-1}(\mu)}^2 \), hence \( \|\nu\|_{\dot{H}^{-1}(\mu')} \leq \rho^{-1/2} \|\nu\|_{\dot{H}^{-1}(\mu)}. \) This relation remains true even for non-infinitesimal \( \nu \), by linearity, which ends the proof. \( \square \)

**Remark 2.4.** Lemma 2.2 could also be proved very quickly by using the definition (2.3)–(2.4) of the \( \dot{H}^{-1}(\mu) \) norm. The proof above, however, has the advantage that it does not need the precise expression of \( \|\cdot\|_{\dot{H}^{-1}(\mu)} \), but only the fact that it is the linearized \( W_2 \) distance.\(^5\)

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\(^5\)Beware that here ‘:’ stands for a measure, not for a function: otherwise the formula would be false.— When \( f \) is a function, \( \|f\|_{\dot{H}^{-1}(\mu)} \) stands for the \( \dot{H}^{-1}(\mu) \) norm of the measure having density \( f \) w.r.t. \( \mu \).

\(^6\)To make rigorous the formal argument of taking an infinitesimally small \( \nu \), according to Footnote 4 above, one would have to replace \( \nu \) by \( \nu_{1} \), where \( \nu_{1} \) is a regular enough measure, and to let \( \varepsilon \) tend to 0; then the regularity assumption on \( \nu_{1} \) would be relaxed by a classical approximation argument. Anyway, Lemma 2.2 can also be proved easily and rigorously without referring to optimal transportation at all, cf. Remark 2.4 below.
2.3. Controlling $\dot{H}^{-1}$ by $W_2$

**Theorem 2.5.** Assume $M$ has nonnegative Ricci curvature. Then for any positive measures $\mu, \nu$ on $M$ such that $\mu \leq \rho_0 \lambda$ and $\nu \leq \rho_1 \lambda$,

$$\|\mu - \nu\|_{\dot{H}^{-1}} \leq \frac{2(\rho_0^{1/2} - \rho_1^{1/2})}{\ln(\rho_0 / \rho_1)} W_2(\mu, \nu).$$  \hspace{1cm} (2.17)

(For $\rho_1 = \rho_0$, the right-hand side of (2.17) is to be taken as $\rho_0^{1/2} W_2(\mu, \nu)$ by continuity).

**Remark 2.6.** For $M = \mathbb{R}^n$ a similar result was already stated in ([7], Prop. 2.8), with a different proof.

**Proof.** Assume that $W_2(\mu, \nu) < \infty$, otherwise there is nothing to prove. Let $(\mu_t)_{0 \leq t \leq 1}$ be the displacement interpolation between $\mu$ and $\nu$ (cf. [14], Chapt. 7), which is such that $\mu_0 = \mu$, $\mu_1 = \nu$ and the infimum in (2.6) is attained with $\|d\mu_t\|_{\dot{H}^{-1}(\mu_t)} = W_2(\mu, \nu) \forall t$. Since Ricci curvature is nonnegative, the Lott–Sturm–Villani theory tells us that, denoting by $\|\mu\|_{\infty}$ the essential supremum of the density of $\mu$ w.r.t. $\lambda$, one has $\|\mu\|_{\infty} \leq \|\mu_0\|_{\infty}^{1-t} \|\mu_1\|_{\infty}^{t}$ (see [14], Cor. 17.19 or [5], Lem. 6.1); so that $\|\mu\|_{\dot{H}^{-1}} \leq \rho_0^{1-t} \rho_1^t W_{2}(\mu)$ by Lemma 2.2.

Then, by the integral triangle inequality for normed vector spaces,

$$\|\mu - \nu\|_{\dot{H}^{-1}} = \int_0^1 \|d\mu_t\|_{\dot{H}^{-1}} \leq \int_0^1 \|d\mu_t\|_{\dot{H}^{-1}} \leq \int_0^1 \rho_0^{(1-t)/2} \rho_1^{t/2} \|d\mu_t\|_{\dot{H}^{-1}(\mu_t)} \leq \int_0^1 \rho_0^{(1-t)/2} \rho_1^{t/2} \frac{2(\rho_0^{1/2} - \rho_1^{1/2})}{\ln(\rho_0 / \rho_1)} W_2(\mu, \nu).$$

$$\hspace{1cm} \hspace{1cm} (2.18)$$

**Remark 2.7.** Taking into account the dimension $n$ of the manifold $M$, the bound on $\|\mu\|_{\infty}$ could be refined into

$$\|\mu\|_{\infty} \leq ((1 - t) \|\mu_0\|_{\infty}^{1/n} + t \|\mu_1\|_{\infty}^{1/n} )^{n}$$

(cf. [8], Thm. 2.3), which would yield a slightly sharper bound in equation (2.17), namely:

$$\|\mu - \nu\|_{\dot{H}^{-1}} \leq \left( \int_0^1 ((1 - t)\rho_0^{-1/n} + t\rho_1^{-1/n})^{n/2} dt \right) W_2(\mu, \nu) = \begin{cases} \frac{\rho_0^{1/2-1/n} - \rho_1^{1/2-1/n}}{\ln(\rho_1 / \rho_0)} W_2(\mu, \nu) & n \geq 2; \\ \frac{(n/2 - 1)(\rho_1^{-1/n} - \rho_0^{-1/n})}{\log(\rho_1 / \rho_0)} W_2(\mu, \nu) & n = 2. \end{cases}$$

$$\hspace{1cm} \hspace{1cm} (2.20)$$

For $n = 1$ it turns out that one can let tend $\rho_1 \to \infty$ in (2.20) without making the integral diverge; which leads to a much more powerful result:

**Theorem 2.8.** When $M$ is an interval of $\mathbb{R}$, then under the sole assumption that $\mu \leq \rho_0 \lambda$, one has for all positive measures $\nu$ on $M$:

$$\|\mu - \nu\|_{\dot{H}^{-1}} \leq 2\rho_0^{1/2} W_2(\mu, \nu).$$

$$\hspace{1cm} \hspace{1cm} (2.21)$$

**Remark 2.9.** For $n \geq 2$ there is no hope to get a bound valid for all $\nu$, because then it can occur that $W_2(\mu, \nu) < \infty$ but $\|\mu - \nu\|_{\dot{H}^{-1}} = \infty$: for instance, take $\mu$ to be the uniform measure on the 2-dimensional sphere and $\nu$ a Dirac mass.
3. APPLICATION TO LOCALIZATION OF WASSERSTEIN DISTANCE

3.1. Introduction

In all this section, we work in the Euclidean space $\mathbb{R}^n$, whose norm is denoted by $|\cdot|$. $\text{dist}(x, A) := \inf \{|x - y| : y \in A\}$ denotes the distance between a point $x$ and a set $A$; $A^c$ denotes the complement of $A$; $\lambda$ denotes the Lebesgue measure. We will use the following notation to handle measures:

- For $\mu$ a measure on $\mathbb{R}^n$ and $f: \mathbb{R}^n \to \mathbb{R}^n$ a measurable map, $f_* \mu$ denotes the pushforward of $\mu$ by $f$, that is, $(f_* \mu)(A) := \mu(f^{-1}(A))$.
- For $\mu$ a measure on $\mathbb{R}^n$ and $\varphi: \mathbb{R}^n \to \mathbb{R}_+$ a nonnegative measurable function, $\varphi \cdot \mu$ denotes the measure such that $(\varphi \cdot \mu)(dx) := \varphi(x)\mu(dx)$.

We will also use the following norms on measures:

- $\|\mu\|_{H^{-1}(\nu)}$ has the same definition as in Section 2;
- $\|\mu\|_1 := \int_{\mathbb{R}^n} |\mu(dx)|$ is the total variation norm of $\mu$;
- For $\nu$ a positive measure with $\mu \ll \nu$, we define
  \[ \|\mu\|_{L^2(\nu)} := \left( \int_{\text{supp} \nu} \frac{\mu(dx)}{\nu(dx)} \nu(dx) \right)^{1/2}. \] (3.1)

For $A \subset \mathbb{R}^n$, we also denote $\|\cdot\|_{L^2(A)}$ for $\|\cdot\|_{L^2(1_A \lambda)}$.

The goal of this section is to give an application of Theorem 2.1 to the problem of localization of the quadratic Wasserstein distance. Morally, the question is the following: take two measures $\mu, \nu$ on $\mathbb{R}^n$ being close to each other in the sense of $W_2$ distance; is it true that $\mu$ and $\nu$ remain close when you consider their restrictions to a subset of $\mathbb{R}^n$? Concretely, if $\varphi$ is a non-negative real function on $\mathbb{R}^n$ with compact support (plus some technical assumptions to be specified later), we want to bound above $W_2(a \varphi \cdot \mu, \varphi \cdot \nu)$ by some multiple of $W_2(\mu, \nu)$ — where, in the former expression, $a$ is a constant factor ensuring that $a \varphi \cdot \mu$ and $\varphi \cdot \nu$ have the same mass (for otherwise the distance between $\varphi \cdot \mu$ and $\varphi \cdot \nu$ is generically infinite).

This question, which was my initial motivation for the results of Section 2, was asked to me by Xavier Tolsa, who needed such a result for his paper [12] on characterizing uniform rectifiability in terms of mass transport. Actually Xavier managed to devise a proof of his own ([12], Thm. 1.1), but it was quite long (about thirty pages) and involved arguments of multi-scale analysis. With Theorem 2.1 at hand, however, the reasoning becomes far more direct; moreover we will be able to relax some of the assumptions of Xavier’s theorem.

3.2. Statement of the theorem

**Theorem 3.1.** Let $\mu, \nu$ be (positive) measures on $\mathbb{R}^n$ having the same total mass; let $B$ be a ball of $\mathbb{R}^n$ (whose radius will be denoted by $R$ when needed). Assume that on $B$, the density of $\mu$ w.r.t. the Lebesgue measure is bounded above and below:

$$ \exists \quad 0 < m_1 \leq m_2 < \infty \quad \forall x \in B \quad m_1 \lambda(dx) \leq \mu(dx) \leq m_2 \lambda(dx). \quad (3.2) $$

Let $\varphi: \mathbb{R}^n \to \mathbb{R}_+$ be a function such that:

- (i) $\varphi$ is zero outside $B$;
- (ii) There exists $0 < c_1 \leq c_2 < \infty$ such that for all $x \in B$, $c_1 \text{dist}(x, B^c)^2 \leq \varphi(x) \leq c_2 \text{dist}(x, B^c)^2$.
- (iii) $\varphi$ is $k$-Lipschitz for some $k < \infty$.

---

7Note that in the case $\mu$ is a positive measure on $\mathbb{R}^n$, then $\|\mu\|_1$ is nothing but $\mu(\mathbb{R}^n)$.
8What we denote here by $\mu(dx)/\nu(dx)$ here is what is commonly called $\langle d\mu/d\nu \rangle(x)$; indeed, as we already told, in this article we reserve the use of “$d\mu$” to denote a mass distribution of infinitesimally small magnitude, rather than for the mass of an infinitely small volume.
Then, denoting \( a := \|\varphi \cdot \nu\|_1 / \|\varphi \cdot \mu\|_1 \),
\[
W_2(a \varphi \cdot \mu, \varphi \cdot \nu) \leq C(n) \frac{c_2^{3/2} m_2^{3/2}}{c_1^{3/2} m_1^{3/2}} k c_1^{-1/2} W_2(\mu, \nu),
\] (3.3)
for \( C(n) < \infty \) some absolute constant only depending on \( n \). Moreover, one can bound explicitly \( C(n) \) in such a way that \( C(n) = O(n^{1/2}) \) when \( n \to \infty \).

**Remark 3.2.** Theorem 3.1 relaxes the assumptions of Theorem 1.1 of [12] on the following points: first, Tolsa’s theorem required that \( |\nabla \varphi| \) was bounded by a multiple of \( \text{dist}(\cdot, B^c) \), while ours does not impose any specific control on \( |\nabla \varphi| \) near the boundary of \( B \); second, Tolsa’s theorem worked only for radially symmetric \( \varphi \). Also, contrary to [12], our conclusions state explicitly how the bound on \( W_2(a \varphi \cdot \mu, \varphi \cdot \nu) \) depends on the constants \( k, c_1, c_2, m_1, m_2 \) and on the dimension \( n \).

**Remark 3.3.** Actually the constraint that the support of \( \varphi \) is a ball is of little importance: we could assume as well that it would be a cube, a simplex, or many other shapes, as the corollary below shows:

**Corollary 3.4.** Make the same assumptions as in Theorem 3.1, except that \( B \) need not be a ball: instead, we only assume that, denoting by \( B_0 \) the (true) ball having the same volume as \( B \), there exists a bijection \( \Phi: B \leftrightarrow B_0 \) mapping the uniform measure on \( B \) onto the uniform measure on \( B_0 \) (i.e. such that \( \Phi_\ast (1_{B} : \lambda) = 1_{B_0} : \lambda \) ) such that \( \Phi \) is bi-Lipschitz (i.e. such that both \( \Phi \) and \( \Phi^{-1} \) are Lipschitz). Denote by \( \|\Phi\|_{\text{Lip}} \) and \( \|\Phi^{-1}\|_{\text{Lip}} \) the optimal Lipschitz constants for \( \Phi \) and \( \Phi^{-1} \). Then, the conclusion of Theorem 3.1 remains true, except that now you have to replace the factor \( C(n) \) by
\[
(\|\Phi\|_{\text{Lip}} \|\Phi^{-1}\|_{\text{Lip}})^5 C(n).
\] (3.4)

**Proof.** Consider the measures \( \mu_0 := \Phi_\ast \mu \) and \( \nu_0 := \Phi_\ast \nu \), and the bump function \( \varphi_0 := \varphi \circ \Phi^{-1} \); then, \( \mu_0, \nu_0 \) and \( \varphi_0 \) satisfy the original assumptions of Theorem 3.1, the roles of \( \cdot m_1 \) and \( \cdot m_2 \) (in the ball situation) being held by \( m_1 \) and \( m_2 \) (in the general situation) themselves, the role of \( \cdot k \) being held by \( \|\Phi^{-1}\|_{\text{Lip}} k \), and the roles of \( \cdot c_1 \) and \( \cdot c_2 \) being held by \( c_1 / \|\Phi\|_{\text{Lip}}^2 \) and \( c_2 \|\Phi^{-1}\|_{\text{Lip}}^2 \). Therefore, applying (3.3):
\[
W_2(a \varphi_0 \cdot \mu_0, \varphi_0 \cdot \nu_0) \leq C(n) \|\Phi\|_{\text{Lip}}^4 \|\Phi^{-1}\|_{\text{Lip}}^{-1} \frac{c_2^{3/2} m_2^{3/2}}{c_1^{3/2} m_1^{3/2}} W_2(\mu_0, \nu_0).
\] (3.5)

But the optimal transportation plan from \( \mu \) to \( \nu \), with cost \( W_2(\mu, \nu)^2 \), can be pushed forward by \( \Phi \) into a (not optimal in general) transportation plan from \( \mu_0 \) to \( \nu_0 \), whose cost will then be \( \leq \|\Phi\|_{\text{Lip}} W_2(\mu, \nu)^2 \); so \( W_2(\mu_0, \nu_0) \leq \|\Phi\|_{\text{Lip}} W_2(\mu, \nu) \). Similarly \( W_2(a \varphi \cdot \mu, \varphi \cdot \nu) \leq \|\Phi^{-1}\|_{\text{Lip}} W_2(a \varphi_0 \cdot \mu_0, \varphi_0 \cdot \nu_0) \). The announced result follows.

### 3.3. Proof of the main theorem

In the sequel we will shorthand \( W_2(\mu, \nu) =: w \), and also \( \varphi \cdot \mu =: \tilde{\mu} \), resp. \( \varphi \cdot \nu =: \tilde{\nu} \). Let \( g =: \text{id} + S \) be a map achieving optimal transportation from \( \nu \) to \( \mu \), i.e. such that \( \mu = g_\ast \nu \) with \( \int_{\mathbb{R}^n} |S(y)|^2 \nu(dy) = w^2 \).

Our strategy will consist in transforming \( \tilde{\nu} \) into \( a \tilde{\mu} \) according to the following procedure:

1. We apply the transportation plan \( g \) to \( \tilde{\nu} \); this transforms \( \tilde{\nu} \) into some measure \( \tilde{\mu}^* \). The measure \( \tilde{\mu}^* \) is not supported by \( B \) a priori, so we split it into \( \tilde{\mu}_B^* + \tilde{\mu}_B^* := 1_B : \tilde{\mu}^* + 1_{B^c} : \tilde{\mu}^* \).

\[9\] For instance, with the estimates of this article, one finds that \( C(n) := 47n^{1/2} \) fits—though this may be strongly suboptimal.

\[10\] Actually such an \( g \) does not always exist, as it can occur that the optimal transportation plan from \( \nu \) to \( \mu \) “splits points” if \( \nu \) is not regular enough. However it would suffice to use the general formalism of transportation plans to handle that case: we do not do it here to keep notation light, but this is straightforward. Also note that it is not obvious that the infimum in (2.2) is attained: again, that is not a real problem as our proof still works by considering a sequence of transportation plans approaching optimality.
Denoting \( a_\varepsilon := \| \hat{\mu}_\varepsilon^* \|_1 / \| \hat{\mu} \|_1 \), we then transform \( \hat{\mu}_\varepsilon^* \) into \( a_\varepsilon \hat{\mu} \) according to an arbitrary transference plan. Finally, denoting \( a_B := \| \hat{\mu}_B^* \|_1 / \| \hat{\mu} \|_1 \),\(^{11}\) we transform \( \hat{\mu}_B^* \) into \( a_B \hat{\mu} \) according to the optimal transference plan: the cost of this operation is \( W_2(\hat{\mu}_B^*, a_B \hat{\mu}) \), which we bound above by \( 2 \| \hat{\mu}_B^* - a_B \hat{\mu} \|_{H^{-1}(a_B \hat{\mu})} \) thanks to Theorem 2.1.

Then, denoting by \( W_2(\text{#1}), W_2(\text{#2}), W_2(\text{#3}) \) the respective Wasserstein distances of these steps, we shall have \( W_2(\hat{\nu}, a \hat{\mu}) \leq W_2(\text{#1}) + (W_2(\text{#2}) + W_2(\text{#3}))^{1/2} \).

Let us begin with bounding the cost of Step \( \text{#1} \). The squared cost of this step is

\[
W_2(\text{#1})^2 = \int |S(y)|^2 \hat{\nu}(dy) = \int |S(y)|^2 \varphi(y) \nu(dy) \leq \sup \varphi \times \int |S(y)|^2 \nu(dy) = \sup \varphi \times w^2 \leq c_2 R^2 w^2, \tag{3.6}
\]

whence \( W_2(\text{#1}) \leq c_2^{1/2} R w \).

Now consider Step \( \text{#2} \). As \( a_\varepsilon \hat{\mu} \) is supported by \( B \), one has obviously

\[
W_2(\text{#2})^2 \leq \left( \text{dist}(x, B) + 2R \right)^2 \hat{\mu}_B^*(dx) = \int_{B^c} \left( \text{dist}(x, B) + 2R \right)^2 \hat{\mu}^*(dx). \tag{3.7}
\]

From that we deduce that \( W_2(\text{#2}) \leq 2c_2^{1/2} R w \) by the following computation:

\[
\int_{B^c} \left( \text{dist}(x, B) + 2R \right)^2 \hat{\mu}^*(dx) = \int_{g(y) \notin B} \left( \text{dist}(g(y), B) + 2R \right)^2 \varphi(y) \nu(dy)
\leq c_2 \int_{g(y) \notin B} \left( \text{dist}(g(y), B) + 2R \right)^2 \varphi(y) \nu(dy)
\leq c_2 \int_{g(y) \notin B} \left( R \text{dist}(g(y), B) + 2R \text{dist}(y, B^c) \right)^2 \nu(dy)
\leq 4c_2 R^2 \int_{g(y) \notin B} \left( \text{dist}(g(y), B) + \text{dist}(y, B^c) \right)^2 \nu(dy)
\leq 4c_2 R^2 \int |y - g(y)|^2 \nu(dy) = 4c_2 R^2 w^2. \tag{3.8}
\]

Step \( \text{#3} \) is the difficult one. We begin with observing that it is easy to bound the \( L^2(B) \) distance between \( \hat{\mu}_B^* \) and \( \hat{\mu} \): indeed, denoting by \( f =: \text{id} + T \) the inverse map of \( g \),\(^{12}\)

\[
\| \hat{\mu}_B^* - \hat{\mu} \|_{L^2(1_B \mu)}^2 = \int_B \left( \hat{\mu}^*(dx) - \varphi(x) \mu(dx) \right)^2 \mu(dx) = \int_B \left( \varphi(f(x)) - \varphi(x) \right)^2 \mu(dx)
\leq k_2 \int_{\mathbb{R}^n} |x - f(x)|^2 \mu(dx) = k_2 \int |T(x)|^2 \mu(dx) = k_2 w^2, \tag{3.9}
\]

(where we used that \( \hat{\mu}^*(dx) = \hat{\nu}(d(f(x))) = \varphi(f(x)) \nu(d(f(x))) = \varphi(f(x)) \mu(dx) \), so that

\[
\| \hat{\mu}_B^* - \hat{\mu} \|_{L^2(1_B \mu)}^2 \leq k_2^2 m_2 w^2. \tag{3.10}
\]

\(^{12}\)Observe that \( a_B + a_\varepsilon = a \).

\(^{11}\)For \( f \) to exist, \( g \) should be bijective, which is not always true \textit{stricto sensu}; but we can safely carry out the reasoning with pretending so, by the same argument as in Footnote 10 on page 1494.

\(^{13}\)Remember that when \( \nu \) stands for a \textit{measure}, \( \| \nu \|_{L^2(\mu)} \) means what is more commonly denoted by \( \| d\nu / d\mu \|_{L^2(\mu)} \), so that the relation \( \mu \leq m \lambda \) implies that \( \| \nu \|_{L^2(\mu)} \leq m \| \nu \|_{L^2(\lambda)} \)—while on the other hand, when \( f \) stands for a \textit{function}, one has \( \| f \|_{L^2(\mu)} \leq m \| f \|_{L^2(\lambda)} \).
Now we have to link $\|\cdot\|_{L^2(B)}$ with $\|\cdot\|_{H^{-1}(\mu)}$. This is achieved by the following lemma, whose proof is postponed:

**Lemma 3.5.** Define $\hat{\lambda}$ to be the measure on $B$ such that $\hat{\lambda}(dx) := \text{dist}(x, B^c)^2 \lambda(dx)$. Then, for any signed measure $m$ on $B$ having total mass zero:

$$\|m\|_{H^{-1}(\hat{\lambda})} \leq C_1(n)^{1/2} \|m\|_{L^2(B)},$$

where $C_1(n)$ is some absolute constant only depending on $n$. Moreover, taking $C_1(n) := ((2e + 1)n - 1) \vee 8e$ fits.

Thanks to Theorem 2.1 and Lemma 3.5, we have that

$$W_2(\mathfrak{A}) \leq 2a_B \hat{\mu} - \hat{\mu}^* B \leq 2(a_B c_1 m_1)^{-1/2} \|a_B \hat{\mu} - \hat{\mu}^*_B\|_{H^{-1}(\hat{\lambda})} \leq 2C_1(n)^{1/2}(a_B c_1 m_1)^{-1/2} \|a_B \hat{\mu} - \hat{\mu}^*_B\|_{L^2(B)}.$$

Next, we compute

$$\|a_B \hat{\mu} - \hat{\mu}^*_B\|_{L^2(B)} = \left\| \frac{\hat{\mu}^*_B - \hat{\mu}}{\|\hat{\mu}_1^*\|_1} \right\|_{L^2(B)} \leq \frac{\|\hat{\mu}^*_B - \|\hat{\mu}_1^*\|_1 \|\hat{\mu}_1\|_1}{\|\hat{\mu}_1^*\|_1} \|\hat{\mu}_1\|_{L^2(B)} + \|\hat{\mu}^*_B - \hat{\mu}\|_{L^2(B)} \leq \left( \frac{\|\hat{\mu}\|_{L^2(B)} \lambda(B)^{1/2}}{\|\hat{\mu}_1^*\|_1} \right) \|\hat{\mu}^*_B - \hat{\mu}\|_{L^2(B)} \leq \left( \frac{c_2 m_2}{c_1 m_1} \right)^{1/2} \|\hat{\mu}^*_B - \hat{\mu}\|_{L^2(B)} \leq (\sqrt{6} + 1) \frac{c_2 m_2}{c_1 m_1} \|\hat{\mu}\|_{L^2(B)}^{1/2} w,$$

so that, combining (3.12) and (3.13), we have got:

$$W_2(\mathfrak{A}) \leq (2\sqrt{6} + 2)C_1(n)^{1/2}a_B^{-1/2} \frac{c_2 m_2^{3/2}}{c_1 m_1} \frac{k}{\sqrt{12}} w.$$  (3.14)

Equation (3.14) is the kind of bound we were looking for, provided $a_B \leq 1$. Though this will be the case in practice (since we are mainly interested in cases where $\nu$ is close to $\mu$ and thus $\hat{\mu}^*$ is close to $\hat{\mu}$), this is not quite satisfactory yet. So, what can we do when $a_B \ll 1$, that is, when $\|\hat{\mu}^*_B\|_1 \ll \|\hat{\mu}\|_1$? In fact that case is easier, because transportation between small measures has low cost, while $w$ has to be large to make $\hat{\mu}^*_B$ very different from $\hat{\mu}$.

The computations are the following. First, it is obvious that

$$W_2(\mathfrak{A}) = W_2(\hat{\mu}, a_B \hat{\mu}) \leq 2R \|\hat{\mu}^*_B\|_{H^{-1}(\lambda)}^{1/2}.$$  (3.15)

---

14This step comes from the computation $\lambda(B)^{1/2} \|\hat{\lambda}\|_{L^2(B)} / \|\hat{\lambda}\|_1 = (\int_0^1 r^{n-1} dr)^{1/2} (\int_0^1 (1 - r)^4 r^{n-1} dr)^{1/2} / (\int_0^1 (1 - r)^2 r^{n-1} dr) = (6(1 + n)(2 + n) / (3 + n)(4 + n))^{1/2} \leq \sqrt{6} \forall n.$
Next, observing that $\varphi(f(x)) \geq \frac{c_1}{c_2} \varphi(x) - 2c_1 \operatorname{dist}(x,B^c)|T(x)|$, we compute that

$$\|\hat{\mu}_B\|_1 = \int_B \varphi(f(x)) \mu(dx) \geq \int_B \left( \frac{c_1}{c_2} \varphi(x) - 2c_1 \operatorname{dist}(x,B^c)|T(x)| \right) \mu(dx)$$

$$\geq \frac{c_1}{c_2} \|\hat{\mu}\|_1 - 2c_1 \left( \int_B \operatorname{dist}(x,B^c)^2 \mu(dx) \right)^{1/2} \left( \int_B |T(x)|^2 \mu(dx) \right)^{1/2}$$

$$= \frac{c_1}{c_2} \|\hat{\mu}\|_1 - 2c_1 \|\operatorname{dist}(\cdot,B^c)^2 \cdot \mu\|_1^{1/2} \geq \frac{c_1}{c_2} \|\hat{\mu}\|_1 - 2c_1 m_2^{1/2} \|\lambda\|_1^{1/2} w,$$

whence

$$w \geq \frac{\left( \frac{c_1}{c_2} \|\hat{\mu}\|_1 - \|\hat{\mu}_B\|_1 \right)^+}{2c_1 m_2^{1/2} \|\lambda\|_1^{1/2}} = \frac{\left( \frac{c_1}{c_2} - a_B \right)^+}{2c_1 m_2^{1/2} \|\lambda\|_1^{1/2}} \geq \frac{m_1^{1/2}}{2c_1 m_2^{1/2}} \left( \frac{c_1}{c_2} - a_B \right)^+ \|\hat{\mu}\|_1^{1/2}.$$  \hfill (3.17)

So,

$$W_2(3) \leq 2R \|\hat{\mu}_B\|_1^{1/2} = 2Ra_B \|\hat{\mu}\|_1^{1/2} \leq 4Rc_1^{1/2} m_2^{1/2} \left( \frac{c_1}{c_2} - a_B \right)^+ w.$$  \hfill (3.19)

In the end, choosing either (3.14) if $a_B \geq c_1 / 2c_2$ or (3.19) if $c_1 / 2c_2$, and observing that $c_1 \leq kR^{-1}$, one has always:

$$W_2(3) \leq ((4\sqrt{3} + 2\sqrt{2})C_1(n)^{1/2} \vee 4\sqrt{2}) \frac{3/2}{c_1^{3/2}} \frac{3/2}{m_1^{3/2}} \frac{k}{c_1^{3/2}} m_2^{3/2} c_1^{1/2} w.$$  \hfill (3.20)

**Remark 3.6.** To bound $W_2(3)$ in the situation where $a_B \ll 1$, we could also have started from "$\varphi(f(x)) \geq \varphi(x) - k|T(x)|$" (instead of "$\varphi(f(x)) \geq \frac{c_1}{c_2} \varphi(x) - 2c_1 \operatorname{dist}(x,B^c)|T(x)|$") to get another bound analogous to (3.17). Following such an approach, the factor $(c_2 / c_1)^{3/2}$ in (3.19) would be improved into $(c_2 / c_1)$ in the analogous formula; however the dimensional factor would behave in $O(n)$ rather than in $O(n^{1/2})$.

**3.4. Proof of Lemma 3.5**

It still remains to prove Lemma 3.5, whose statement we recall to be:

**Lemma 3.7.** Denoting $\hat{\lambda} := \operatorname{dist}(\cdot,B^c)^2 \cdot \lambda$, one has, for any signed measure $m$ on $B$ having total mass zero:

$$\|m\|_{H^{-1}(\hat{\lambda})} \leq \left( \left((2e + 1)n - 1 \right) \vee 8e \right)^{1/2} \|m\|_{L^2(B)}.$$  \hfill (3.21)

---In the sequel, "$(2e + 1)n - 1 \vee 8e$" will be shorthanded into "$C_1(n)$".

**Remark 3.8.** The bound (3.21) is within a constant factor of being optimal, uniformly in $n$, as one sees by taking a linear function $f$ in (3.24).

**Proof of the lemma.** We begin by translating the lemma into a functional analysis statement by a duality argument. Recall the duality definition of $\|m\|_{H^{-1}(\hat{\lambda})}$ from Section 2:

$$\|m\|_{H^{-1}(\hat{\lambda})} := \sup\{ \|f\| \; | \; \|f\|_{H^1(\hat{\lambda})} \leq 1 \}.$$  \hfill (3.22)

---

This follows from the computation:

$$\varphi(f(x)) \geq c_1 \operatorname{dist}(f(x),B^c)^2 \geq c_1 \left( \operatorname{dist}(x,B^c) - |T(x)| \right)^2 \geq c_1 \operatorname{dist}(x,B^c)^2 - 2c_1 \operatorname{dist}(x,B^c)|T(x)| \geq \frac{c_1}{c_2} \varphi(x) - 2c_1 \operatorname{dist}(x,B^c)|T(x)|.$$  \hfill (3.16)
There is a similar duality formula for \(\|m\|_{L^2(B)}\):
\[
\|m\|_{L^2(B)} = \sup\{\langle f, m \rangle \mid \|f\|_{L^2(B)} \leq 1\},
\]
(3.23)
where, for \(f\) a function, \(\|f\|_{L^2(B)}\) has its usual meaning, namely \(\|f\|_{L^2(B)} := (\int_B f(x)^2 \lambda(dx))^{1/2}\). Since \(m\) is assumed to have total mass zero, \(|\langle f, m \rangle|\) does not change when one adds a constant to \(f\). On the other hand, when \(f\) describes the set \(\{\|f_u + a\| \mid a \in \mathbb{R}\}\), \(\|f\|_{L^2(B)}\) is minimal when \(a\) is such that \(f\) has zero mean on \(B\), while the value of \(\|f\|_{\dot{H}^1(\hat{\lambda})}\) remains constant.\(^{16}\) As a consequence, we can restrict the supremum in (3.22) and (3.23) to those \(f\) having zero mean on \(B\). Thus, the lemma will be implied\(^{17}\) by proving that
\[
\langle f, 1_B \cdot \lambda \rangle = 0 \implies \|f\|_{L^2(B)} \leq C_1(n)^{1/2} \|f\|_{\dot{H}^1(\hat{\lambda})}.
\]
(3.24)
Going back to the definitions of \(\|\cdot\|_{\dot{H}^{-1}(\hat{\lambda})}\) and \(\|\cdot\|_{L^2(B)}\), relaxing the condition on \(f\) to be centred by projecting it orthogonally in \(L^2(B)\) onto the subspace of centred functions, and denoting by \(P\) the uniform probability measure on \(B\), Equation (3.24) turns into:
\[
\forall f \quad \text{Var}_P(f) \leq C_1(n) \int \text{dist}(x, B^c)^2 |\nabla f(x)|^2 P(dx),
\]
(3.25)
which we recognize to be a so-called “improved Poincaré inequality” \([3, 6]\). In general, Poincaré inequalities, bounding the variance of \(f\) by a quadratic integral of its first derivative, are linked with the exponential convergence of a certain diffusion Markov process towards equilibrium (cf. [1], Chap. 2): that probabilistic vision initially guided me to tackle Equation (3.25), although this will not be apparent in the sequel.

To prove (3.25), the first key idea (inspired by [4]) is to separate radial and spherical coordinates. This is, considering the bijection
\[
\varphi: (0, R) \times S^{n-1} \to B \setminus \{0\}
\]
(3.26)
\[(r, \theta) \mapsto r\theta\]
(the origin of space being set at the center of \(B\)), we introduce the measure \(\tilde{P} := \varphi^{-1}_* P\), which is obviously the product measure \(\tilde{P}_r \otimes \tilde{P}_\theta\), where \(\tilde{P}_r\) is the probability measure on \((0, R)\) such that \(\tilde{P}_r(dr) := nR^{-n}r^{n-1}dr\), resp. \(\tilde{P}_\theta\) is the uniform measure on the sphere \(S^{n-1}\). With this notation, we perform can a change of variables to see that (3.25) is equivalent to proving that, for all \(g \in L^2(\tilde{P})\):
\[
C_1(n)^{-1} \text{Var}_{\tilde{P}}(g) \leq \int_0^R \int_{S^{n-1}} (R - r)^2 (|\nabla_r g(r, \theta)|^2 + r^{-2} |\nabla_\theta g(r, \theta)|^2) \tilde{P}_r(dr) \tilde{P}_\theta(d\theta),
\]
(3.27)
where \(\nabla_r\) and \(\nabla_\theta\) denote the gradient along resp. the \(r\) coordinate and the \(\theta\) coordinate.\(^{18}\) We will denote the right-hand side of (3.27) by \(\mathcal{E}(g, g)\).

Because \(P = \tilde{P}_r \otimes \tilde{P}_\theta\), we know that \(L^2(\tilde{P})\) can be seen as (the closure of) the tensor product of \(L^2(\tilde{P}_r)\) and \(L^2(\tilde{P}_\theta)\):
\[
L^2(\tilde{P}) = \text{cl}(L^2(\tilde{P}_r) \otimes L^2(\tilde{P}_\theta)),
\]
(3.28)
where the symbol \(\otimes\) means that the Hilbertian structure of \(L^2(\tilde{P})\) is compatible with the Hilbertian structures of \(L^2(\tilde{P}_r)\) and \(L^2(\tilde{P}_\theta)\)—i.e., that \(\langle h_a \otimes u_a, h_b \otimes u_b \rangle_{L^2(\tilde{P})} = \langle h_a, h_b \rangle_{L^2(\tilde{P}_r)} \times \langle u_a, u_b \rangle_{L^2(\tilde{P}_\theta)}\). Now consider the spherical harmonics \(Y_0, Y_1, \ldots\), which by definition are an orthonormal basis, in \(L^2(\tilde{P}_\theta)\), of eigenfunctions of the

\(^{16}\)Here we implicitly assume that \(\int_B |f(x)| \lambda(dx) < \infty\), which is legitimate since an approximation argument allows to restrict the suprema in (3.22) and (3.23) to those \(f\) having a \(C^\infty\) continuation on \(\text{cl}(B)\).

\(^{17}\)Actually there is even equivalence.

\(^{18}\)In the latter case, we have to use the Riemannian definition of the gradient on \(S^{n-1}\).
Laplace–Beltrami operator \( \Delta \) on \( \mathbb{S}^{n-1} \); and call \( \ell_0, \ell_1, \ldots \) the associated eigenvalues, which are known to be such that (up to permuting indices) \( Y_0 \equiv 1 \) with \( \ell_0 = 0 \), and \( \ell_i \leq -(n - 1) \forall i \neq 0 \) (see for instance [11]). By construction, \( L^2(\tilde{P}) = \text{cl} \left( \bigoplus_{i \in \mathbb{N}} (\mathbb{R} \cdot Y_i) \right) \); therefore, one has that

\[
L^2(\tilde{P}) = \text{cl} \left( \bigoplus_{i \in \mathbb{N}} L^2(\tilde{P}_r) \cdot Y_i \right); \tag{3.29}
\]

in other words, the functions of \( L^2(\tilde{P}) \) are those of the form

\[
g(r, \theta) = \sum_{i \in \mathbb{N}} h_i(r)Y_i(\theta), \tag{3.30}
\]

with \( \sum_i \|h_i\|^2_{L^2(\tilde{P}_r)} < \infty \), and the correspondence is bijective. An interesting point is that, then, one has:

\[
\text{Var}_{\tilde{P}}(g) = \text{Var}_{\tilde{P}_r}(h_0) + \sum_{i \neq 0} \|h_i\|^2_{L^2(\tilde{P}_r)}. \tag{3.31}
\]

On the other hand, one has

\[
\mathcal{E}(g, g) = -\langle Lg, g \rangle_{L^2(\tilde{P})}, \tag{3.32}
\]

where

\[
(Lg)(r, \theta) := (R - r)^2 \Delta_r g + \left( n - 1 \frac{(R - r)^2}{r} - 2(R - r) \right) e_r \cdot \nabla_r g + \frac{(R - r)^2}{r^2} \Delta_{\theta} g. \tag{3.33}
\]

From (3.33) we see that, since the \( Y_i \) are eigenfunctions of \( \Delta_g \), all the \( L^2(\tilde{P}_r) \cdot Y_i \) are invariant by \( L \), and that one has:

\[
\mathcal{E}(g, g) = \sum_{i \in \mathbb{N}} \int_0^R \left( (R - r)^2 |\nabla h_i(r)|^2 - \ell_i \frac{(R - r)^2}{r^2} h_i(r) \right) \tilde{P}_r(dr). \tag{3.34}
\]

So, proving (3.27) becomes equivalent to proving that both following formulas hold for all \( h \in L^2(\tilde{P}_r) \):

\[
\text{Var}_{\tilde{P}_r}(h) \leq C_1(n) \int_0^R (R - r)^2 |\nabla h(r)|^2 \tilde{P}_r(dr); \tag{3.35}
\]

\[
\|h\|^2_{L^2(\tilde{P}_r)} \leq C_1(n) \int_0^R \left( (R - r)^2 |\nabla h(r)|^2 + (n - 1) \frac{(R - r)^2}{r^2} h(r)^2 \right) \tilde{P}_r(dr). \tag{3.36}
\]

Let us start with (3.35). In all the sequel of the proof, we introduce

\[
b := 1 - n^{-1}. \tag{3.37}
\]

By the Cauchy–Schwarz inequality, one has, for all \( r \in (bR, R) \):

\[
(h(r) - h(bR))^2 = \left( \int_{bR}^r h'(s)ds \right)^2 \leq \left( \int_{bR}^r (R - s)^{-3/2} ds \right) \times \int_{bR}^r (R - s)^{3/2} |\nabla h(s)|^2 ds
\]

\[
\leq 2 \left( (R - r)^{-1/2} - (R - bR)^{-1/2} \right) \int_{bR}^r (R - s)^{3/2} |\nabla h(s)|^2 ds
\]

\[
\leq 2 (R - r)^{-1/2} \int_{bR}^r (R - s)^{3/2} |\nabla h(s)|^2 ds. \tag{3.38}
\]
Integrating and using Fubini’s formula, it follows that

\[ \int_{bR}^{R} (h(r) - h(bR))^{2} \tilde{P}_r(dr) \leq 2 \int_{s=bR}^{R} \left( \int_{r=s}^{R} nR^{-n}(R - r)^{-1/2}r^{n-1} dr \right) (R - s)^{3/2} |\nabla h(s)|^2 ds \]

\[ \leq 2 \int_{s=bR}^{R} \left( \int_{r=s}^{R} nR^{-n}(b^{-1}s)^{-n}(R - r)^{-1/2}dr \right) (R - s)^{3/2} |\nabla h(s)|^2 ds \]

\[ = 2b^{-(n-1)} \int_{s=bR}^{R} \left( \int_{r=s}^{R} (R - r)^{-1/2}dr \right) (R - s)^{3/2} |\nabla h(s)|^2 \tilde{P}_r(ds) \]

\[ = 4b^{-(n-1)} \int_{s=bR}^{R} (R - s)^2 |\nabla h(s)|^2 ds. \quad (3.39) \]

One can apply the same line of reasoning for \( r \in (0, bR) \): the (unweighted this time) Cauchy–Schwarz inequality then yields \((h(r) - h(bR))^2 \leq (bR - r) \int_{r}^{bR} |\nabla h(s)|^2 ds\), whence:

\[ \int_{0}^{bR} (h(r) - h(bR))^{2} \tilde{P}_r(dr) \leq \int_{s=0}^{bR} \left( \int_{r=0}^{s} nR^{-n}(bR - r)^{n-1} dr \right) |\nabla h(s)|^2 ds \]

\[ \leq bR^{-(n-1)} \int_{s=0}^{bR} \left( \int_{r=0}^{s} nr^{n-1} dr \right) |\nabla h(s)|^2 ds = bR \int_{0}^{bR} |\nabla h(s)|^2 s^{n} ds \]

\[ \leq bn^{-1}R^{2} \int_{0}^{bR} |\nabla h(s)|^2 \tilde{P}_r(ds) \leq b(1 - b)^{-2n-1} \int_{0}^{bR} (R - s)^{2} |\nabla h(s)|^2 \tilde{P}_r(ds). \quad (3.40) \]

Summing (3.39) and (3.40), we get that

\[ \int_{0}^{R} (h(r) - h(bR))^{2} \tilde{P}_r(dr) \leq (4b^{-(n-1)} \lor b(1 - b)^{-2n-1}) \int_{0}^{bR} (R - s)^{2} |\nabla h(s)|^2 \tilde{P}_r(ds), \quad (3.41) \]

where \((4b^{-(n-1)} \lor b(1 - b)^{-2n-1})\) can itself be bounded by \(((n - 1) \lor 4e)\). The left-hand-side of (3.41) being an upper bound for \(\text{Var}_{\tilde{P}_r}(h)\), this proves (3.35).

Now we turn to (3.36). For \( r \in (bR, R) \) we have, similarly to (3.38), that

\[ (h(r) - h(br))^{2} \leq 2(R - r)^{-1/2} \int_{br}^{r} (R - s)^{3/2} |\nabla h(s)|^2 ds, \]

so that

\[ h(r)^2 \leq 2h(br)^2 + 4(R - r)^{-1/2} \int_{br}^{r} (R - s)^{3/2} |\nabla h(s)|^2 ds. \quad (3.42) \]

Then, integrating and applying Fubini’s formula:

\[ \int_{bR}^{R} h(r)^2 \tilde{P}_r(dr) \leq 2 \int_{bR}^{R} h(br)^2 \tilde{P}_r(dr) + 4 \int_{s=bR}^{R} \left( \int_{r=s}^{b^{-1}s} nR^{-n}(R - r)^{-1/2}dr \right) (R - s)^{3/2} |\nabla h(s)|^2 ds. \quad (3.44) \]

By change of variables, the first term of the right-hand side of (3.44) is equal to \(2b^{-(n-2)} \int_{b^{-1}R}^{bR} h(s)^2 \tilde{P}_r(ds)\), which we can bound by

\[ 2b^{-(n-2)} \frac{(1 - b)^{-2}}{n - 1} \int_{b^{-1}R}^{bR} (n - 1) \frac{(R - r)^2}{r^2} h(s)^2 \tilde{P}_r(ds) \leq 2ne \int_{0}^{R} (n - 1) \frac{(R - r)^2}{r^2} h(s)^2 \tilde{P}_r(ds). \quad (3.45) \]
The second term of the right-hand side of (3.44) is itself bounded by

\[
4b^{-(n-1)} \int_{s=b}^{bR} \left( \int_{r=s}^{R} (R-r)^{-1/2} \, dr \right) (R-s)^{3/2} |\nabla h(s)|^2 \tilde{P}_r(ds) \leq 8c \int_{0}^{R} (R-s)^2 |\nabla h(s)|^2 \tilde{P}_r(ds).
\]  

(3.46)

This way, we have bounded \( \int_{b}^{bR} h(r)^2 \tilde{P}_r(dr) \).

On the other hand, it is trivial that, for \( r \leq bR \),

\[
h(r)^2 \leq \frac{b^2}{(n-1)(1-b)^2} \times (n-1) \frac{(R-r)^2}{r^2} h(r)^2,
\]

whence:

\[
\int_{0}^{bR} h(r)^2 \tilde{P}_r(dr) \leq (n-1) \int_{0}^{R} \frac{(R-r)^2}{r^2} h(r)^2 \tilde{P}_r(dr).
\]

(3.48)

Combining (3.45), (3.46) and (3.48), we finally get the wanted bound (3.36).

\( \square \)

**Remark 3.9.** At the time I wrote that proof I was not aware of the already existing results on improved Poincaré inequalities, in particular ([6], Thm. 1.3), which equation (3.25) is actually a particular case of; nor were the people whom I had asked about such inequalities. Compared to the result of [6] however, my equation (3.25) is actually a particular case of; nor were



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