Error estimates of the Godunov method for the multidimensional compressible Euler system

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Abstract

We derive a priori error of the Godunov method for the multidimensional Euler system of gas dynamics. To this end we apply the relative energy principle and estimate the distance between the numerical solution and the strong solution. This yields also the estimates of the $L^2$-norm of errors in density, momentum and entropy. Under the assumption that the numerical density and energy are bounded, we obtain a convergence rate of 1/2 for the relative energy in the $L^1$-norm. Further, under the assumption – the total variation of numerical solution is bounded, we obtain the first order convergence rate for the relative energy in the $L^1$-norm. Consequently, numerical solutions (density, momentum and entropy) converge in the $L^2$-norm with the convergence rate of 1/2. The numerical results presented for Riemann problems are consistent with our theoretical analysis.

Keywords: compressible Euler system, error estimates, relative energy, Godunov method, consistency formulation, strong solution

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1 Introduction

We consider the Euler system governing the motion of a compressible gas
\[
\partial_t U + \text{div}_x F(U) = 0, \quad (t, x) \in (0, T) \times \Omega. \tag{1.1}
\]
Here \( \Omega \subset \mathbb{R}^d (d = 1, 2, 3) \) is a bounded computational domain, \( U = (\varrho, m, E)^T \) represents the fluid density, momentum and total energy, while \( F \) is the flux function given by
\[
F = (m, u \otimes m + pI, u(E + p))^T.
\]
Here for positive \( \varrho \), \( u = \frac{m}{\varrho} \) is the velocity of the fluid and \( p \) is the pressure satisfying the state equation of perfect gas
\[
p = (\gamma - 1)\varrho e, \quad \gamma \in (1, 2] \tag{1.2}
\]
with the specific internal energy \( e = \frac{E}{\varrho} - \frac{1}{2}|u|^2 \).

We close the system with initial data
\[
U(0, x) = U_0 = (\varrho_0, m_0 := \varrho_0 u_0, E_0) \tag{1.3a}
\]
satisfying
\[
\varrho_0 > 0 \quad \text{and} \quad E_0 \in L^1(\Omega) \tag{1.3b}
\]
and impermeability boundary condition
\[
|u|_{\partial \Omega} = 0, \tag{1.4}
\]
where \( n \) is the outer normal vector on the boundary \( \partial \Omega \). Taking the Second law of Thermodynamics into account we further require that the entropy inequality holds, i.e.
\[
\partial_t \eta(U) + \text{div}_x q(U) \geq 0. \tag{1.5}
\]
Here $(\eta, q)$ is the physical entropy pair given by

\[ \eta = C_v \varrho S, \quad q = \eta u \quad \text{with} \quad C_v = \frac{1}{\gamma - 1} \quad \text{and} \quad S = \ln \left( \frac{p}{\varrho^\gamma} \right). \]  

(1.6)

During the past few decades numerical simulation of the Euler system has been a hot topic in the field of computational mechanics and physics, cf. Toro [23], Feistauer et al. [10], Li et al. [15], LeVeque [14]. Despite the success in practical simulations, a rigorous convergence analysis of the numerical methods still remains open in general. Most literature results were focused on scalar conservation laws. Kuznetsov [13] showed that the (upper) $L^1$ error bound is $O(h^{1/2})$ for multi-dimensional scalar conservation laws under the assumptions on the boundedness of the total variation and continuity in time of numerical solutions, where $h$ is the mesh parameter. Further, Cockburn et al. [2] and Vila [24] extended the result of Kuznetsov and obtained the $L^1$-error bounds of $O(h^{1/4})$ without the assumptions of bounded total variation and continuity in time. The convergence rate of some specific waves was also studied in one dimension. Concerning the linear advection equation, Tang and Teng [20] showed the sharpness of the $O(\sqrt{\Delta x})$ $L^1$-error for monotone difference schemes with BV initial data. For the nonlinear scalar equation Teng and Zhang [22] showed the optimal convergence rate of 1 in the $L^1$-norm for the viscosity method and monotone schemes if a solution is piecewise constant with finitely many shocks. Moreover, for the piecewise smooth entropy solution with finitely many rarefaction waves, Tang and Teng [21] showed that the error of viscosity solution to the inviscid solution is bounded by $O(\varepsilon \log \varepsilon + \varepsilon)$ in the $L^1$-norm, where $\varepsilon$ denotes the viscosity coefficient. Furthermore, Tadmor and Tang [19] studied the pointwise error estimates and showed that the thicknesses of the shock and rarefaction layers are of order $O(\varepsilon)$ and $O(\varepsilon \log^2 \varepsilon)$, respectively. We point out that the error estimates for scalar hyperbolic conservation laws are typically given in terms of the $L^1$-norm in space.

When considering the multidimensional nonlinear system of hyperbolic conservation laws, to our best knowledge, the only result was done by Jovanović and Rohde [12], where the convergence rate of 1/2 was presented in terms of the $L^2$-errors between the numerical solutions and the classical solution $(U \in C^1)$ under the assumption of uniform boundedness of numerical solutions and their $H^1$ semi-norm. In this paper we estimate the error between the numerical solutions and the strong solution $(U \in W^{1,\infty})$ assuming that the total variation of the numerical solution is bounded. Comparing with [12], we obtain the same convergence rate under a weaker assumption. Moreover, without the assumption of bounded total variation, we still have the convergence rate of 1/4.

The main tool used in the paper is the so-called relative energy functional originally introduced by Dafermos [3]. This technique has been largely used in the analysis of the weak–strong uniqueness and singular limit of the compressible fluid flows, see the monograph of Feireisl and Novotný [9], Březina and Feireisl [1], and Feireisl et al. [7, 8]. Recently, this technique has also been successfully applied to the convergence analysis of numerical solutions of compressible viscous fluids, see Feireisl et al. [4] and Mizerová and She [17]. Here we adapt the technique to the Euler system and estimate the corresponding relative energy, which yields the control of the $L^2$-error of density, momentum and entropy, too.
The rest of the paper is organized as follows. In Section 2 we introduce some preliminaries. More precisely, we recall the Godunov method and its consistency formulation proved in Lukáčová and Yuan [16]. We define the strong solution of the Euler system and the relative energy. Further, we prove the relative energy inequality in Section 3 and estimate its error in the $L^1$-norm. Finally, in Section 4 we present some numerical experiments to validate theoretical results.

2 Preliminaries

In this section we introduce the preliminaries, including the formulation of the Godunov method, its consistency formulation, and the definitions of the strong solution and relative energy.

To begin, we define the following notations for the later use

- $a \lesssim b$ if $a \leq cb$ with a positive constant $c$,
- $a \approx b$ if $a \lesssim b$ and $b \lesssim a$.

2.1 Godunov method

The computational domain $\Omega$ consists of rectangular meshes $\overline{\Omega} := \bigcup_K \overline{K}$. We denote the set of all mesh cells as $T_h$ and the set of all interior faces of $T_h$ as $\Sigma_{\text{int}}$. We consider the space of piecewise constant functions

$$Q_h(\Omega) = \{ v : v|_K = \text{constant}, \text{ for all } K \in T_h \}$$

and define the projection operator

$$\Pi_h : L^1(\Omega) \to Q_h(\Omega), \quad \Pi_h[\phi]_K = \frac{1}{|K|} \int_K \phi(x) \, dx,$$

where $|K|$ is the Lebesgue measure of $K$.

Let $U_h \in Q_h(\Omega; \mathbb{R}^{d+2})$. Then the semi-discrete form of the finite volume method with the Godunov flux, i.e. the Godunov method can be described as

$$\int_\Omega \phi \frac{d}{dt} U_h \, dx - \sum_{\sigma \in \Sigma_{\text{int}}} \int_\sigma F(U_{\sigma}^{RP}) \cdot n \, [[\phi]] \, dS_x = 0,$$

$$U_{h0} = \Pi_h[U_0].$$

Here $\phi \in Q_h(\Omega)$ is the test function, $U_{\sigma}^{RP}$ is the exact solution of a local Riemann problem along the interface $\sigma$, and the notation $[[\cdot]]$ denotes the jump along the interface.

2.2 Consistency formulation

We recall the consistency formulation of the Godunov method derived by Lukáčová and Yuan [16]. We start with the following assumption.
Assumption 2.1. We assume that the solution to (2.3) satisfies
\[ 0 < \varrho \leq \varrho_h, \quad 0 < E_h \leq \bar{E} \] uniformly for \( h \to 0 \) (2.4)
for all \( t \in [0, T] \), where \( \varrho, \bar{E} \) are some positive constants.

Lemma 2.2. Under Assumption 2.1 there hold
\[ 0 < \varrho \leq \varrho_h \leq \bar{\varrho}, \quad |u_h| \leq \bar{u}, \quad 0 < p \leq p_h \leq \bar{p}, \]
\[ |m_h| \leq \bar{m}, \quad 0 < E \leq E_h \leq \bar{E}, \quad 0 < \vartheta \leq \vartheta_h \leq \bar{\vartheta} \] (2.5)
uniformly for \( h \to 0, t \in [0, T] \) with positive constants \( \varrho, \bar{u}, p, \bar{p}, \bar{m}, E, \vartheta, \bar{\vartheta} \) depending on \( \varrho, \bar{E} \), where \( \vartheta := \frac{\varrho}{\bar{\varrho}} \) is the absolute temperature.

Theorem 2.3. (Consistency formulation) Let \((\varrho_h, m_h, \eta_h)\) be the numerical solutions obtained by the Godunov method (2.3) on the time interval \([0, T]\) satisfying Assumption 2.1. Then for any \( \tau \in (0, T) \) the following hold:
- for all \( \phi \in W^{1,\infty}((0, T) \times \Omega) \)
  \[ \left[ \int_{\Omega} \varrho_h \phi \ dx \right]_{t=0}^{t=\tau} = \int_{t=0}^{t} \int_{\Omega} \left( \varrho_h \partial_t \phi + m_h \cdot \nabla \phi \right) \ dx \ dt + \int_{0}^{\tau} e_{\varrho,h}(t, \phi) \ dt; \] (2.7)
- for all \( \phi \in W^{1,\infty}((0, T) \times \Omega; \mathbb{R}^d) \)
  \[ \left[ \int_{\Omega} m_h \cdot \phi \ dx \right]_{t=0}^{t=\tau} = \int_{t=0}^{t} \int_{\Omega} \left( \frac{m_h \otimes m_h}{\varrho_h} : \nabla \phi + p_h \text{div} \phi \right) \ dx \ dt + \int_{0}^{\tau} e_{m,h}(t, \phi) \ dt; \] (2.8)
- for all \( \phi \in W^{1,\infty}((0, T) \times \Omega), \phi \geq 0 \)
  \[ \left[ \int_{\Omega} \eta_h \phi \ dx \right]_{t=0}^{t=\tau} \geq \int_{t=0}^{t} \int_{\Omega} \left( \eta_h \partial_t \phi + q_h \cdot \nabla \phi \right) \ dx \ dt + \int_{0}^{\tau} e_{\eta,h}(t, \phi) \ dt; \] (2.9)
- \[ \int_{\Omega} E_h(\tau) \ dx = \int_{\Omega} E_{0,h} \ dx \] (2.10)
with bounded errors \( e_{j,h}, (j = \varrho, m, \eta) \) satisfying
\[ \| e_{j,h} \|_{L^1(0, T)} \lesssim h \| \phi \|_{W^{1,\infty}((0, T) \times \Omega)} \left( \int_{0}^{T} \sum_{\sigma \in \Sigma_{\text{int}}} \int_{\sigma} \| [U_h] \| \ dS_x \ dt \right)^{1/2} \] (2.11)
2.3 Strong solution

Our aim is to analyze the convergence rate of the Godunov method when approximating the strong solution of the Euler system (1.1)–(1.4).

Definition 2.4 (Strong solution). Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a boundary $\partial \Omega$ of class $C^1$. We say that a trio $[\tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{\eta}]$ is the strong solution of the Euler system (1.1)–(1.4) if

\[ \tilde{\varrho} \in W^{1,\infty}((0,T) \times \Omega), \]
\[ \tilde{\mathbf{u}} \in W^{1,\infty}((0,T) \times \Omega; \mathbb{R}^d), \]
\[ \tilde{\eta} \in W^{1,\infty}((0,T) \times \Omega), \]
\[ \tilde{\varrho} > 0 \text{ and } \vartheta(\tilde{\varrho}, \tilde{\eta}) > 0 \text{ for any } (t,x) \in [0,T] \times \Omega \]

and the equations (1.1)–(1.4) are satisfied for almost everywhere.

Let us point out that we consider $\varrho$ and $\eta$ as the independent thermodynamical variables throughout the paper, meaning that all other thermodynamical variables are functions of $(\varrho,\eta)$. Accordingly, we write $\tilde{v} = v(\tilde{\varrho}, \tilde{\eta})$, $v \in \{p,e,\vartheta,S\}$, for the strong solution. Moreover, we denote $\tilde{\mathbf{m}} = \tilde{\varrho}\tilde{\mathbf{u}}$ and $\tilde{\mathbf{U}} = \mathbf{U}(\tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{\eta})$.

Since the domain is bounded and $(\tilde{\varrho}, \tilde{\vartheta})$ is continuous and positive, we have

\[ 0 < \varrho \leq \tilde{\varrho}, \quad 0 < \tilde{\vartheta} \leq \tilde{\vartheta}. \tag{2.12} \]

Remark 2.5. According to the definition of strong solution, we know that an entropy solution only containing finitely many rarefaction waves is also a strong solution.

We recall Gibbs’ relation

\[ \tilde{\vartheta} d\tilde{S} = d\tilde{\varrho} + \tilde{p} \, d(1/\tilde{\varrho}). \tag{2.13} \]

Consequently, for any strong solution $(\tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{\eta})$ we obtain the following identities which will be used in Section 3

\[ \partial_t \tilde{\varrho} + \tilde{\mathbf{u}} \cdot \nabla_x \tilde{\varrho} + \tilde{\varrho} \text{div}_x \tilde{\mathbf{u}} = 0, \]
\[ \partial_t \tilde{\mathbf{u}} + \tilde{\mathbf{u}} \cdot \nabla_x \tilde{\mathbf{u}} + \frac{1}{\tilde{\varrho}} \nabla_x \tilde{p} = 0, \]
\[ \partial_t \tilde{\eta} + \tilde{\mathbf{u}} \cdot \nabla_x \tilde{\eta} + \tilde{\eta} \text{div}_x \tilde{\mathbf{u}} = 0, \]
\[ \partial_t \tilde{p} + \tilde{\mathbf{u}} \cdot \nabla_x \tilde{p} + \gamma \tilde{p} \text{div}_x \tilde{\mathbf{u}} = 0, \]
\[ \partial_t \tilde{S} + \tilde{\mathbf{u}} \cdot \nabla_x \tilde{S} = 0, \]
\[ \partial_t \tilde{\vartheta} + \tilde{\mathbf{u}} \cdot \nabla_x \tilde{\vartheta} + (\partial_{\tilde{\varrho} \tilde{p}}) \tilde{\varrho} \text{div}_x \tilde{\mathbf{u}} = 0, \] \tag{2.14}

see [7, 23] for more details.
2.4 Relative energy

In this part we introduce the relative energy and show the relationship between the relative energy and the $L^2$-error of $(\varrho, \mathbf{m}, \eta)$ for numerical solutions.

Let $(\varrho, \mathbf{m}, \eta)$ and $(\tilde{\varrho}, \tilde{\mathbf{m}}, \tilde{\eta})$ be two vectors consisting of density, momentum and velocity, respectively, and total entropy. In the context of the compressible Euler system, the relative energy reads

$$
\mathbb{E}\left(\varrho, \mathbf{m}, \eta \mid \tilde{\varrho}, \tilde{\mathbf{m}}, \tilde{\eta}\right) = \frac{1}{2} \varrho \left| \frac{\mathbf{m}}{\varrho} - \tilde{\mathbf{u}} \right|^2 + \varrho e - \frac{\partial(\varrho e)}{\partial \varrho} \mid_{\tilde{\varrho}, \tilde{\eta}} (\varrho - \tilde{\varrho}) - \frac{\partial(\varrho e)}{\partial \eta} \mid_{\tilde{\varrho}, \tilde{\eta}} (\eta - \tilde{\eta}) - \tilde{\varrho} e \quad (2.15)
$$

for $\varrho > 0$.

**Lemma 2.6.** Let $(\tilde{\varrho}, \tilde{\mathbf{m}}, \tilde{\eta})$ be the strong solution of the Euler system in the sense of Definition 2.4 and let $(\varrho_h, \mathbf{m}_h, \eta_h)$ be a numerical solution of the Euler system obtained by (2.3) satisfying Assumption 2.1. Then we have the following equivalence

$$
\mathbb{E}\left(\varrho_h, \mathbf{m}_h, \eta_h \mid \tilde{\varrho}, \tilde{\mathbf{m}}, \tilde{\eta}\right) \approx |\mathbf{m}_h - \tilde{\mathbf{m}}|^2 + |\eta_h - \tilde{\eta}|^2 + |\varrho_h - \tilde{\varrho}|^2. \quad (2.16)
$$

**Proof.** First, taking the derivatives of $\varrho e$ with respect to $(\varrho, \eta)$ we obtain

$$
\varrho \varrho_e \varrho = (1 + C_v) \varrho - \eta \frac{\varrho}{\varrho} \varrho, \quad \varrho \varrho = \varrho. \quad (2.17)
$$

Further, by the product rule and Gibbs’ relation (2.13) we derive

$$
\frac{d (1 + C_v) \varrho}{\varrho} = (1 + C_v) \left(1 - \frac{\varrho}{\varrho}\right) d\varrho + \frac{\varrho}{\varrho} \varrho d\eta, \quad \text{(2.18)}
$$

and

$$
\frac{d \left(1 - \frac{\varrho}{\varrho}\right)}{\varrho} = \left(1 + C_v \left(1 - \frac{\varrho}{\varrho}\right) d\varrho + \frac{\varrho}{\varrho} \varrho d\eta + \varrho \varrho d\varrho, \quad \text{(2.19)}
$$

which leads to

$$
\nabla^2_{(\varrho, \eta)}(\varrho e) = \frac{\varrho}{\varrho} C_v \left(1 - \frac{\varrho}{\varrho} \right) \left(1 + \left(1 - \frac{\varrho}{\varrho}\right) \right)^2. \quad (2.20)
$$

As $(\varrho, \mathbf{u}, \eta)$ is the strong solution we know that $\nabla^2_{(\varrho, \eta)}(\varrho e)\mid_{\tilde{\varrho}, \tilde{\eta}}$ is symmetric positive definite and bounded from below and above, which implies

$$
\mathbb{E}\left(\varrho_h, \mathbf{m}_h, \eta_h \mid \tilde{\varrho}, \tilde{\mathbf{m}}, \tilde{\eta}\right) \approx |\mathbf{u}_h - \tilde{\mathbf{u}}|^2 + |\eta_h - \tilde{\eta}|^2 + |\varrho_h - \tilde{\varrho}|^2. \quad (2.21)
$$

Next, we recall Assumption 2.1 and the uniform upper bound of $\mathbf{u}_h$ due to Lemma 2.2 to conclude that

$$
|\mathbf{m}_h - \tilde{\mathbf{m}}|^2 \leq |\varrho_h(\mathbf{u}_h - \tilde{\mathbf{u}})|^2 + |(\varrho_h - \tilde{\varrho})\tilde{\mathbf{u}}|^2 \leq |\mathbf{u}_h - \tilde{\mathbf{u}}|^2 + |\varrho_h - \tilde{\varrho}|^2,
$$

$$
|\mathbf{u}_h - \tilde{\mathbf{u}}|^2 \leq |(\varrho_h - \varrho)(\mathbf{u}_h - \tilde{\mathbf{u}})|^2 \leq |\mathbf{m}_h - \tilde{\mathbf{m}}|^2 + |\mathbf{u}_h(\varrho_h - \varrho)|^2 \leq |\mathbf{m}_h - \tilde{\mathbf{m}}|^2 + |\varrho_h - \tilde{\varrho}|^2.
$$

Substituting the above two inequalities into (2.21) we finish the proof. □
Lemma 2.6 means that the $L^1$-norm of $\mathbb{E}\left(\varrho_h, \mathbf{m}_h, \eta_h \bigg| \tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{\eta} \right)$ is equivalent to the $L^2$-norm of the errors in $(\varrho_h - \tilde{\varrho}, \mathbf{m}_h - \tilde{\mathbf{m}}, \eta_h - \tilde{\eta})$ as long as the entropy stable numerical solution $(\varrho_h, \mathbf{m}_h, \eta_h)$ satisfies Assumption 2.1 and $(\tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{\eta})$ is the strong solution of the Euler system in the sense of Definition 2.4.

3 Error estimates

Equipped with consistency formulation of the Godunov method we are now ready to estimate the relative energy in the $L^1$-norm and error between the numerical solution $(\varrho_h, \mathbf{m}_h, \eta_h)$ and the strong solution $(\tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{\eta})$ in the $L^2$-norm.

Theorem 3.1 (Error estimates). Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, be a bounded domain with a boundary $\partial \Omega \in C^1$. Let $(\tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{\eta})$ be the strong solution of the complete Euler system (1.1) in the sense of Definition 2.4 with initial data (1.3) satisfying

$$
\|U_{h0} - U_0\|_{L^2(\Omega)} \lesssim h^{1/2}
$$

and the impermeability boundary condition (1.4).

Suppose that $(\varrho_h, \mathbf{m}_h, \eta_h)$ is the numerical solution obtained by the Godunov method (2.3). Let Assumption 2.1 hold. Then the following estimate of the relative energy holds for any $\tau \in (0, T]$

$$
\int_{\Omega} \mathbb{E}\left(\varrho_h, \mathbf{m}_h, \eta_h \bigg| \tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{\eta} \right) (\tau, \cdot) \, dx \lesssim \exp \left( \tau \, c \left( \Omega, \|\tilde{U}\|_{W^{1,\infty}(\Omega; \mathbb{R}^d)} \right) \right) \|\tilde{U}\|_{W^{1,\infty}(\Omega; \mathbb{R}^d)} \right) \right)^{1/2}.
$$

(3.1)

Proof. We prove (3.1) in two steps:

- Viewing $(\tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{\eta})$ as the test function in the consistency formulation, we derive the relative energy inequality between $(\varrho_h, \mathbf{m}_h, \eta_h)$ and $(\tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{\eta})$;

- Approximating the above inequality such that all terms on the right hand side can be bounded by the discretization parameter $h$ or by the relative energy, we finally estimate the relative energy by Gronwall’s lemma.

Step 1. Rewriting the relative energy (2.15) into a more convenient form we obtain

$$
E \left(\varrho_h, \mathbf{m}_h, \eta_h \bigg| \varrho, \tilde{\mathbf{u}}, \tilde{\eta} \right) = \frac{1}{2} \varrho_h | \mathbf{m}_h |^2 + \varrho_h \mathbf{e}_h - \left( 1 + C_v \right) \mathbf{\bar{\varrho}} - \mathbf{\bar{\eta}} \tilde{\mathbf{u}} + \mathbf{\bar{\eta}} \tilde{\mathbf{u}} \mathbf{\bar{\varrho}} - \mathbf{\bar{\eta}} \mathbf{\bar{\varrho}} + \mathbf{\bar{\eta}} \mathbf{\bar{\varrho}}
$$

$$
= \left[ \frac{1}{2} | \mathbf{m}_h |^2 + \varrho_h \mathbf{e}_h \right] + \varrho_h \left[ \frac{1}{2} | \mathbf{\bar{\mathbf{u}}} |^2 - \left( 1 + C_v \right) \mathbf{\bar{\varrho}} + \mathbf{\bar{\eta}} \mathbf{\bar{\varrho}} \right] - \mathbf{\bar{\mathbf{m}}}_h \cdot \mathbf{\bar{\mathbf{u}}} - \eta_h \mathbf{\bar{\varrho}} \cdot \mathbf{\bar{\mathbf{u}}} + \mathbf{\bar{\varrho}}.
$$

(3.2)
First, we take \( \frac{1}{2} |\tilde{u}|^2 - (1 + C_v) \tilde{\vartheta} + \frac{\tilde{\vartheta} \tilde{\eta}}{\vartheta} \) as the test function in consistency formulation of the density equation (2.7) to derive

\[
\left[ \int_{\Omega} g_{h} \left( \frac{1}{2} |\tilde{u}|^2 - (1 + C_v) \tilde{\vartheta} + \frac{\tilde{\vartheta} \tilde{\eta}}{\vartheta} \right) \right]_{t=0}^{t=\tau} = \int_0^{\tau} \int_{\Omega} \left( g_{h} \partial_t \left( \frac{1}{2} |\tilde{u}|^2 - (1 + C_v) \tilde{\vartheta} + \frac{\tilde{\vartheta} \tilde{\eta}}{\vartheta} \right) \right) dx dt + \int_0^{\tau} e_{\vartheta,h}(t, \tilde{U}) dt.
\]

Then, we combine the above three formulae together with the energy equality (2.10) and find

\[
\left[ \int_{\Omega} \mathbf{m} \cdot \tilde{u} \ dx \right]_{t=0}^{t=\tau} = \int_0^{\tau} \int_{\Omega} \left( \mathbf{m} \cdot \partial_t \tilde{u} + \mathbf{m} \otimes \mathbf{m} \frac{\varrho}{\varrho} : \nabla_x \tilde{u} + p_{h} \text{div}_x \tilde{u} \right) dx dt + \int_0^{\tau} e_{\mathbf{m},h}(t, \tilde{U}) dt,
\]

and

\[
\left[ \int_{\Omega} \eta_{h} \tilde{\vartheta} \ dx \right]_{t=0}^{t=\tau} \geq \int_0^{\tau} \int_{\Omega} \left( \eta_{h} \partial_t \tilde{\vartheta} + \eta_{h} \mathbf{m} \cdot \nabla_x \tilde{\vartheta} \right) dx dt + \int_0^{\tau} e_{\eta,h}(t, \tilde{U}) dt.
\]

Analogously, we set \( \tilde{\vartheta} \) and \( \tilde{\vartheta} \) respectively as the test functions in consistency formulations of the momentum equation (2.8) and entropy inequality (2.9) to get

\[
\left[ \int_{\Omega} \mathbb{E} \left( g_{h} , \mathbf{m} , \eta_{h} \big| \tilde{\vartheta} \right) \tilde{u} \tilde{\vartheta} \tilde{\eta} \big( t, \cdot \big) \ dx \right]_{t=0}^{t=\tau} \leq - \int_0^{\tau} \int_{\Omega} \left( \frac{g_{h} \tilde{u} - \mathbf{m} \otimes \mathbf{m} \frac{\varrho}{\varrho}}{\varrho} : \nabla_x \tilde{u} \right) dx dt + \int_0^{\tau} \int_{\Omega} \left( g_{h} \tilde{p} - p_{h} \right) \text{div}_x \tilde{u} + (\partial_t \tilde{p} + \tilde{\vartheta} \cdot \nabla_x \tilde{p}) \ dx dt + \int_0^{\tau} e_{\vartheta,h}(t, \tilde{U}) dt.
\]

where we have used the following identities

\[
\mathbf{u} \otimes \mathbf{u} : \nabla_x \tilde{u} = \mathbf{u} \cdot (\mathbf{u} \cdot \nabla_x \tilde{u}), \quad \int_{\Omega} \tilde{u} \cdot \nabla_x \tilde{p} \ dx = - \int_{\Omega} \tilde{p} \text{div}_x \tilde{u} \ dx,
\]

\[
\left( \frac{g_{h} \tilde{u} - \mathbf{m} \otimes \mathbf{m} \frac{\varrho}{\varrho}}{\varrho} : \nabla_x \tilde{u} \right) = \frac{g_{h} \tilde{u} - \mathbf{m} \otimes \mathbf{m} \frac{\varrho}{\varrho}}{\varrho} : \nabla_x \tilde{u} = \frac{g_{h} \tilde{u} - \mathbf{m} \otimes \mathbf{m} \frac{\varrho}{\varrho}}{\varrho} : \nabla_x \tilde{u}
\]
Further, employing the relations (2.18) and (2.19) we can reformulate (3.3) as

\[-m_h \cdot (\tilde{u} \cdot \nabla_x)\tilde{u} - \tilde{u} \cdot (m_h \cdot \nabla_x)\tilde{u}.
\]

Further, employing the relations (2.18) and (2.19) we can reformulate (3.3) as

\[
\left[ \int_\Omega \mathbb{E} \left( \varrho_h, m_h, \eta_h \left| \tilde{\varrho}, \tilde{\tilde{u}}, \tilde{\eta} \right. \right) (t, \cdot) \, dx \right]_{t=0}^{t=T}
\leq - \int_0^T \int_\Omega \left( \varrho_h \tilde{u} - m_h \right) \otimes \left( \varrho_h \tilde{u} - m_h \right) : \nabla_x \tilde{u} \, dx \, dt
\]

\[- \int_0^T \int_\Omega \left[ p_h - \tilde{\tilde{p}} - \partial_{\tilde{\tilde{\varrho}}} (\varrho_h - \tilde{\varrho}) - \partial_{\tilde{\tilde{\eta}}} (\eta_h - \tilde{\eta}) \right] \text{div}_x \tilde{u} \, dx \, dt
\]

\[+ \int_0^T \int_\Omega \left( \varrho_h \tilde{u} - m_h \right) \cdot \left[ \partial_t \tilde{\tilde{u}} + \tilde{\tilde{u}} \cdot \nabla_x \tilde{\tilde{u}} + \frac{1}{\varrho} \nabla_x \tilde{p} \right] \, dx \, dt
\]

\[+ \int_0^T \int_\Omega \left[ (\varrho_h - \tilde{\tilde{\varrho}} - \eta_h - \varrho) \partial_{\tilde{\tilde{\varrho}}} \left( \eta_h - \varrho \right) \partial_{\tilde{\tilde{\eta}}} \left( \eta_h - \eta \right) \left( \varrho_h - \tilde{\tilde{\varrho}} - \eta_h - \eta \right) \partial_{\tilde{\tilde{\eta}}} \left( \eta_h - \eta \right) \right] \left( m_h \varrho_h - \tilde{\tilde{u}} \right) \cdot \nabla_x \tilde{\tilde{u}} \, dx \, dt
\]

\[+ \int_0^T \left( e_{\varrho,h}(t, \tilde{\tilde{U}}) - e_{m,h}(t, \tilde{\tilde{U}}) - e_{\eta,h}(t, \tilde{\tilde{U}}) \right) dt,
\]

where we have denoted \( \partial_{\tilde{\tilde{\varrho}}} : = \frac{\partial \tilde{\tilde{p}}}{\partial \varrho} \) and the definitions of \( \partial_{\tilde{\tilde{\varrho}}} \) and \( \partial_{\tilde{\tilde{\eta}}} \) are analogous.

Then applying the equalities stated in (2.14) to (3.4) we have

\[
\left[ \int_\Omega \mathbb{E} \left( \varrho_h, m_h, \eta_h \left| \tilde{\varrho}, \tilde{\tilde{u}}, \tilde{\eta} \right. \right) (t, \cdot) \, dx \right]_{t=0}^{t=T}
\leq - \int_0^T \int_\Omega \left( \varrho_h \tilde{u} - m_h \right) \otimes \left( \varrho_h \tilde{u} - m_h \right) : \nabla_x \tilde{u} \, dx \, dt
\]

\[- \int_0^T \int_\Omega \left[ p_h - \tilde{\tilde{p}} - \partial_{\tilde{\tilde{\varrho}}} (\varrho_h - \tilde{\varrho}) - \partial_{\tilde{\tilde{\eta}}} (\eta_h - \tilde{\eta}) \right] \text{div}_x \tilde{u} \, dx \, dt
\]

\[+ \int_0^T \int_\Omega \left( \varrho_h \tilde{u} - m_h \right) \cdot \left( m_h \varrho_h - \tilde{\tilde{u}} \right) \cdot \nabla_x \tilde{\tilde{u}} \, dx \, dt
\]

\[+ \int_0^T \left( e_{\varrho,h}(t, \tilde{\tilde{U}}) - e_{m,h}(t, \tilde{\tilde{U}}) - e_{\eta,h}(t, \tilde{\tilde{U}}) \right) dt.
\]

**Step 2.** In this step we shall estimate the right hand side of the inequality (3.5) and complete the proof by Gronwall’s lemma. We begin with the following observation owing to the uniform bounds on \( \tilde{\varrho}, \tilde{\tilde{\varrho}} \) and \( \tilde{\eta} \), as well as (2.21)

\[
\left| \left( \eta_h - \frac{\varrho_h}{\tilde{\tilde{\varrho}}} \right) \cdot \left( \frac{m_h}{\varrho_h} - \tilde{\tilde{u}} \right) \right| \leq \left| \eta_h - \frac{\varrho_h}{\tilde{\tilde{\varrho}}} \right|^2 + \left| \frac{m_h}{\varrho_h} - \tilde{\tilde{u}} \right|^2
\]
we conclude the proof, i.e.

\[ \| \eta - \tilde{\eta} \| + \| \frac{\partial}{\partial t} \| + \| \frac{\partial}{\partial x} \| = | \eta - \tilde{\eta} | + \left| \frac{\partial}{\partial x} \right| (\tilde{\eta})^2 + \| \frac{\partial}{\partial x} \| \]

Hence, we may estimate (3.5) in the following way

\[
\left[ \int_\Omega E \left( \varrho_h, m_h, \eta, \tilde{\varrho}, \tilde{u}, \tilde{\eta} \right) \right]_{t=0}^{t=\tau} \leq c(\| \tilde{U} \|_{W^{1,\infty}(0,T)} \| \tilde{\varrho} \|_{L^\infty}) h^{1/2}
\]

Further, applying Gronwall's lemma and recalling the projection error for piecewise constant functions

\[ \| \tilde{U} - \Pi_h \tilde{U} \|_{L^\infty} \lesssim c(\| \tilde{U} \|_{W^{1,\infty}(0,T)} \| \tilde{\varrho} \|_{L^\infty}) h \]

we conclude the proof, i.e.

\[
\int_\Omega E \left( \varrho_h, m_h, \eta, \tilde{\varrho}, \tilde{u}, \tilde{\eta} \right) (\tau, \cdot) \ dx \leq c(\| \tilde{U} \|_{W^{1,\infty}(0,T)} \| \tilde{\varrho} \|_{L^\infty}) h^{1/2}
\]

In what follows we prove the first order convergence rate in terms of the relative energy under an additional assumption of bounded total variation for numerical solutions.

**Proposition 3.2.** Under the same condition as Theorem 3.1 it holds for any \( \tau \in (0,T) \)

\[
\| \varrho_h - \tilde{\varrho} \|_{L^2(\Omega)} \lesssim h^{1/4}, \quad \| m_h - \tilde{m} \|_{L^2(\Omega)} \lesssim h^{1/4}, \quad \| \eta_h - \tilde{\eta} \|_{L^2(\Omega)} \lesssim h^{1/4}. \tag{3.7}
\]

We directly obtained the following a priori error estimates in the \( L^2 \)-norm.

**Theorem 3.3.** In addition to the assumptions of Theorem 3.1, we assume that

\[
\sum_{\sigma \in \Sigma_{int}} \int_{\sigma} \mu |S_x| \ dS_x \lesssim 1. \tag{3.8}
\]

Then there hold

\[
\int_\Omega E \left( \varrho_h, m_h, \eta, \tilde{\varrho}, \tilde{u}, \tilde{\eta} \right) (\tau, \cdot) \ dx \lesssim h \exp \left( \tau c(\Omega, \| \tilde{U} \|_{W^{1,\infty}(0,T)} \| \tilde{\varrho} \|_{L^\infty}) \right).
\]
Proof. With (3.8) the consistency error can be estimated and improved by
\[ \|e_{j,h}(\phi)\|_{L^1(0,T)} \lesssim h\|\phi\|_{W^{1,\infty}((0,T)\times\Omega)}, \]
which concludes the proof.

Remark 3.4. Here we point out that the assumption (3.8) is slightly weaker than the assumption used in [12]
\[ \sum_{\sigma \in \Sigma_{int}} \int_{\sigma} \left| \frac{[U_h]}{h} \right|^2 dS_x \lesssim 1. \tag{3.9} \]
Moreover, for the case of \( d = 1 \) the assumption (3.8) is exactly the TVB condition, which is a known property for the Godunov method.

Remark 3.5. Let us consider piecewise constant initial data which generate finitely many rarefaction waves. It is obvious that such kind of initial data fulfills the condition \( \|U_{h_0} - U_0\|_{L^2(\Omega)} \lesssim h^{1/2} \) assumed in Theorem 3.1. Moreover, we can expect (3.8) or (3.9) to hold, which consequently implies Theorem 3.3. Thus, \( \|U_h - \tilde{U}\|_{L^2(\Omega)} \lesssim h^{1/2} \) holds for any \( \tau \in (0,T) \).

4 Numerical experiments

In this section we simulate several one- and two-dimensional Riemann problems. The examples only containing rarefaction waves are used to validate our theoretical results. In addition, we also test examples containing contact waves or shock waves or both and compute experimentally convergence rates. We point out that in our simulations there is no projection error of initial data due to these simple Riemann problems and good uniform meshes.

In the following we calculate the relative energy in the \( L^1 \)-norm and the errors of \((\rho, m, \eta)\) in the \( L^2 \)-norm. In addition to the Godunov method, we also test the convergence rates of the viscosity finite volume (VFV) method originally introduced and studied by Feireisl et al. [5]. In our numerical tests, we take \( \gamma = 1.4 \) and \( \text{CFL} = 0.9 \) for the Godunov method while \( \text{CFL} = 0.3 \) is used for the VFV method. Unless otherwise specified, the errors of \((\rho, m, \eta), E\) mean the \( L^2 \)-error of \((\rho, m, \eta)\) and the \( L^1 \)-norm of the relative energy \( E\); the convergence rates of \((\rho, m, \eta), E\) mean the convergence rate of the \( L^2 \)-error of \((\rho, m, \eta)\) and the \( L^1 \)-norm of \( E\).

4.1 One dimensional experiments

We start with one dimensional Riemann problems in the computational domain \( \Omega = [0,1] \). Here, the solution \( \tilde{U} \) in the relative energy is taken as the reference (exact) solution computed on the uniform mesh with 20480 cells.

Example 4.1 (1D single wave). This example is used to measure the convergence rate of three different types of waves – a single contact (C) wave, a single rarefaction (R) wave and a single shock (S) wave.
Table 1: Initial data of 1D single wave.

|   | ϱ  | u   | p   |   | ϱ  | u   | p   |   | ϱ  | u   | p   |
|---|---|---|---|---|---|---|---|---|---|---|---|
| C | left | 0.5 | 0.5 | 5 |   | left | 0.5197 | -0.7259 | 0.4 |   | left | 1 | 0.7276 | 1 |
|   | right | 1 | 0.5 | 5 |   | right | 1 | 0 | 1 |   | right | 0.5313 | 0 | 0.4 |

Given the initial data in Table 1, we compute the contact, rarefaction and shock wave till $T = 0.2, 0.2$ and 0.25, respectively. Figure 1 (resp. Figure 2) shows the density $\rho$ (resp. the entropy $\eta$) obtained on different meshes with $n(= 1/h) = 32, 64, \ldots, 1024$ cells. Moreover, we present in Figure 3 the errors of $(\rho, m, \eta)$ in $L^2$-norm and $E$ in $L^1$-norm, see the details in Table 2 and 3.

The numerical results show that

- the Godunov method and the VFV method have similar convergence rates;
- for single rarefaction wave the convergence rate of $(\rho, m, \eta)$ (resp. $E$) is slightly greater than $1/2$ (resp. 1), which is consistent to our theoretical results;
- for single contact wave the convergence rate of $(\rho, m, \eta)$ (resp. $E$) is around $1/4$ (resp. $1/2$);
- for single shock wave the convergence rate of $(\rho, m, \eta)$ (resp. $E$) is around $1/2$ (resp. 1).

**Remark 4.2.** Here we compare the above observation with the result of Tadmor and Tang [19] for the rarefaction wave and the shock wave.

- Directly applying the pointwise error estimate for scalar equation in [19], i.e.
  $$|(u^\varepsilon - u)(x, t)| \approx \operatorname{dist}(x, R(t))^{-1}\varepsilon \log^2 \varepsilon$$
  with rarefaction set $R(t)$, we obtain that the $L^2$-error is bounded by $\varepsilon^{1/2} \log^2 \varepsilon$. Setting the vanishing viscosity coefficient $\varepsilon \approx h$ means that our analysis gives a better convergence rate.
- Applying the pointwise error estimate for scalar equation in [19], i.e.
  $$|(u^\varepsilon - u)(x, t)| \approx \operatorname{dist}(x, S(t))^{-1}\varepsilon,$$
  where $S(t)$ is the streamline of shock discontinuities, we obtain that $L^2$-convergence rate is $1/2$, which is consistent with our observations.

**Example 4.3.** This experiment is used to further test our theoretical analysis. It describes left-going and right-going rarefaction waves, whose initial data are given by

$$(\rho, u, p)(x, 0) = \begin{cases} (1, -2, 0.4), & x < 0.5, \\ (1, 2, 0.4), & x > 0.5. \end{cases}$$
Figure 1: Example 4.1: density $\rho$ obtained by the Godunov method (top) and the VFV method (bottom).
Figure 2: Example 4.1: entropy $\eta$ obtained by the Godunov method (top) and the VFV method (bottom).
Table 2: Example 4.1: errors and convergence rates of $\varrho, \eta, E$ of the Godunov method.

| $n = 1/h$ | **Contact** | **Rarefaction** | **Shock** |
|-----------|-------------|-----------------|-----------|
|           | error       | order           | error     | order     | error     | order     |
| Density   |             |                 |           |           |           |           |
| 32        | 0.0569      | -               | 0.0292    | -         | 0.0459    | -         |
| 64        | 0.0479      | 0.2497          | 0.0201    | 0.5380    | 0.0297    | 0.6252    |
| 128       | 0.0395      | 0.2753          | 0.0135    | 0.5743    | 0.0224    | 0.4072    |
| 256       | 0.0332      | 0.2504          | 0.0089    | 0.6098    | 0.0160    | 0.4886    |
| 512       | 0.0278      | 0.2565          | 0.0057    | 0.6398    | 0.0112    | 0.5183    |
| 1024      | 0.0234      | 0.2501          | 0.0036    | 0.6656    | 0.0080    | 0.4805    |
| Entropy   |             |                 |           |           |           |           |
| 32        | 0.1038      | -               | 0.0150    | -         | 0.0139    | -         |
| 64        | 0.0869      | 0.2563          | 0.0105    | 0.5138    | 0.0085    | 0.7083    |
| 128       | 0.0719      | 0.2732          | 0.0073    | 0.5335    | 0.0054    | 0.6689    |
| 256       | 0.0603      | 0.2551          | 0.0049    | 0.5524    | 0.0034    | 0.6375    |
| 512       | 0.0504      | 0.2580          | 0.0033    | 0.5687    | 0.0021    | 0.6807    |
| 1024      | 0.0423      | 0.2522          | 0.0022    | 0.5818    | 0.0014    | 0.5978    |
| Relative energy |     |                 |           |           |           |           |
| 32        | 0.069415    | -               | 0.001640  | -         | 0.004126  | -         |
| 64        | 0.048272    | 0.5241          | 0.000752  | 1.1246    | 0.001771  | 1.2207    |
| 128       | 0.032676    | 0.5630          | 0.000330  | 1.1871    | 0.000998  | 0.8274    |
| 256       | 0.022931    | 0.5109          | 0.000139  | 1.2519    | 0.000504  | 0.9858    |
| 512       | 0.015997    | 0.5195          | 0.000056  | 1.3075    | 0.000247  | 1.0266    |
| 1024      | 0.011269    | 0.5054          | 0.000022  | 1.3554    | 0.000126  | 0.9697    |
Table 3: Example 4.1: errors and convergence rates of $\varrho, \eta, \mathcal{E}$ of the VFV method.

| $n = 1/h$ | Contact | Rarefaction | Shock |
|-----------|---------|-------------|-------|
|           | error   | error       | error       | order | order       | order       | order       |
| Density   |         |             |             |       |             |             |             |
| 32        | 0.0751  | 0.0440      | 0.0575      | -     |             |             |             |
| 64        | 0.0619  | 0.0307      | 0.0391      | 0.2784| 0.5185      | 0.5584      | -           |
| 128       | 0.0507  | 0.0205      | 0.0268      | 0.2877| 0.5805      | 0.5440      | -           |
| 256       | 0.0418  | 0.0131      | 0.0178      | 0.2784| 0.6479      | 0.5882      | -           |
| 512       | 0.0345  | 0.0081      | 0.0120      | 0.2799| 0.7021      | 0.5752      | -           |
| 1024      | 0.0285  | 0.0048      | 0.0082      | 0.2759| 0.7436      | 0.5465      | -           |
| Entropy   |         |             |             |       |             |             |             |
| 32        | 0.1356  | 0.0270      | 0.0214      | -     |             |             |             |
| 64        | 0.1117  | 0.0182      | 0.0124      | 0.2791| 0.5696      | 0.7956      | -           |
| 128       | 0.0916  | 0.0120      | 0.0068      | 0.2862| 0.6043      | 0.8582      | -           |
| 256       | 0.0755  | 0.0077      | 0.0037      | 0.2792| 0.6295      | 0.8719      | -           |
| 512       | 0.0622  | 0.0049      | 0.0020      | 0.2799| 0.6510      | 0.8713      | -           |
| 1024      | 0.0513  | 0.0031      | 0.0011      | 0.2764| 0.6675      | 0.8287      | -           |
| Relative energy |         |             |             |       |             |             |             |
| 32        | 0.116107| 0.004119    | 0.006390    | -     |             |             |             |
| 64        | 0.078778| 0.001906    | 0.003002    | 0.5596| 1.1118      | 1.0899      | -           |
| 128       | 0.052828| 0.000818    | 0.001409    | 0.5765| 1.2196      | 1.0911      | -           |
| 256       | 0.035875| 0.000327    | 0.000613    | 0.5583| 1.3248      | 1.2011      | -           |
| 512       | 0.024321| 0.000122    | 0.000271    | 0.5608| 1.4157      | 1.1768      | -           |
| 1024      | 0.016577| 0.000044    | 0.000123    | 0.5530| 1.4865      | 1.1391      | -           |
Figure 4(a) and (c) show the density \( \rho \) obtained at \( T = 0.15 \) by the Godunov method and the VFV method, respectively. Moreover, the corresponding \( L^2 \)-error of \((\rho, m, \eta)\) as well as the \( L^1 \)-norm of \( E \) are shown in Figure 4(b) and (d), see also Table 4.

Our numerical results show that the converge rate is approximately 1/2 (resp. 1) for \((\rho, m, \eta)\) (resp. \( E \)), which is consistent with our theoretical analysis.

**Example 4.4.** This experiment is devoted to the 1D Sod problem, in order to test the convergence rate for the solution consisting of the left rarefaction, contact and right shock waves. Although the exact solution is not smooth we can still test corresponding convergence rates. In this example the final time is set to \( T = 0.15 \) and the initial data are given by

\[
(\rho, u, p)(x, 0) = \begin{cases} 
(1, 0, 1), & x < 0.5, \\
(0.125, 0, 0.1), & x > 0.5.
\end{cases}
\]

Figure 5(a) and (c) show the density obtained with the Godunov and VFV methods on different meshes. Moreover, errors of \((\rho, m, \eta)\) and \( E \) are shown in Figure 5(b) and (d), respectively, see also Table 5 for more details.

These numerical results indicate that the convergence rates of \((\rho, m, \eta)\) (resp. \( E \)) seem to be between 1/4 and 1/2 (resp. between 1/2 and 1).
Figure 4: Example 4.3: density \( \rho \) and errors at \( T = 0.15 \) of the Godunov method and the VFV method.
Table 4: Example 4.3: errors and convergence rates of \( \rho, m, \eta, \mathcal{E} \) of the Godunov and VFV methods.

| \( n \) | density | momentum | entropy | relative energy |
|---|---|---|---|---|
| \( n \) | error | order | error | order | error | order | error | order |
| 32 | 0.0523 | - | 0.1299 | - | 0.1271 | - | 0.008792 | - |
| 64 | 0.0346 | 0.5987 | 0.0869 | 0.5803 | 0.0864 | 0.5566 | 0.003810 | 1.2064 |
| 128 | 0.0230 | 0.5865 | 0.0579 | 0.5853 | 0.0605 | 0.5135 | 0.001641 | 1.2148 |
| 256 | 0.0152 | 0.6012 | 0.0380 | 0.6090 | 0.0418 | 0.5354 | 0.000706 | 1.2162 |
| 512 | 0.0098 | 0.6303 | 0.0244 | 0.6392 | 0.0280 | 0.5755 | 0.000305 | 1.2101 |
| 1024 | 0.0062 | 0.6531 | 0.0153 | 0.6671 | 0.0186 | 0.5944 | 0.000134 | 1.1928 |

Godunov

| \( n \) | density | momentum | entropy | relative energy |
|---|---|---|---|---|
| \( n \) | error | order | error | order | error | order | error | order |
| 32 | 0.1019 | - | 0.2310 | - | 0.3146 | - | 0.047945 | - |
| 64 | 0.0639 | 0.6723 | 0.1602 | 0.5279 | 0.1824 | 0.7866 | 0.019004 | 1.3350 |
| 128 | 0.0433 | 0.5616 | 0.1126 | 0.5091 | 0.1153 | 0.6617 | 0.007298 | 1.3808 |
| 256 | 0.0307 | 0.4950 | 0.0792 | 0.5072 | 0.0807 | 0.5151 | 0.002798 | 1.3830 |
| 512 | 0.0213 | 0.5301 | 0.0543 | 0.5453 | 0.0560 | 0.5261 | 0.001086 | 1.3660 |
| 1024 | 0.0142 | 0.5878 | 0.0361 | 0.5904 | 0.0373 | 0.5884 | 0.000427 | 1.3453 |

VFV

| \( n \) | density | momentum | entropy | relative energy |
|---|---|---|---|---|
| \( n \) | error | order | error | order | error | order | error | order |
| 32 | 0.0378 | - | 0.0376 | - | 0.0615 | - | 0.005135 | - |
| 64 | 0.0273 | 0.4693 | 0.0269 | 0.4819 | 0.0484 | 0.3481 | 0.002642 | 0.9587 |
| 128 | 0.0203 | 0.4260 | 0.0206 | 0.3855 | 0.0400 | 0.2735 | 0.001561 | 0.7594 |
| 256 | 0.0151 | 0.4268 | 0.0154 | 0.4217 | 0.0328 | 0.2865 | 0.000913 | 0.7741 |
| 512 | 0.0114 | 0.4025 | 0.0117 | 0.4003 | 0.0268 | 0.2895 | 0.000554 | 0.7202 |
| 1024 | 0.0088 | 0.3773 | 0.0087 | 0.4153 | 0.0221 | 0.2831 | 0.000342 | 0.6978 |

Table 5: Example 4.4: errors and convergence rates of \( \rho, m, \eta, \mathcal{E} \) of the Godunov and VFV methods.

| \( n \) | density | momentum | entropy | relative energy |
|---|---|---|---|---|
| \( n \) | error | order | error | order | error | order | error | order |
| 32 | 0.0491 | - | 0.0525 | - | 0.0929 | - | 0.010427 | - |
| 64 | 0.0381 | 0.3658 | 0.0377 | 0.4779 | 0.0703 | 0.4023 | 0.005392 | 0.9514 |
| 128 | 0.0285 | 0.4183 | 0.0270 | 0.4802 | 0.0546 | 0.3641 | 0.002844 | 0.9231 |
| 256 | 0.0205 | 0.4774 | 0.0192 | 0.4958 | 0.0430 | 0.3441 | 0.001514 | 0.9090 |
| 512 | 0.0148 | 0.4650 | 0.0140 | 0.4524 | 0.0343 | 0.3274 | 0.000859 | 0.8176 |
| 1024 | 0.0110 | 0.4293 | 0.0104 | 0.4337 | 0.0276 | 0.3153 | 0.000508 | 0.7595 |
Figure 5: Example 4.4: density $\rho$ and errors obtained on different meshes.
4.2 Two dimensional experiments

In this section we present four two-dimensional Riemann problems. The computational domain is taken as \([0,1]^2\). Here the exact solution \(\tilde{U}\) used in the relative energy is taken as the reference solution computed on the uniform mesh of 4096\(^2\) cells.

**Example 4.5.** The first 2D Riemann problem describes the interaction of four rarefaction waves. The initial data are given by

\[
(q, u, v, p)(x, 0) = \begin{cases} 
(1, 0, 0, 1), & x > 0.5, y > 0.5, \\
(0.5197, -0.7259, 0, 0.4), & x < 0.5, y > 0.5, \\
(1, -0.7259, -0.7259, 1), & x < 0.5, y < 0.5, \\
(0.5197, 0, -0.7259, 0.4), & x > 0.5, y < 0.5.
\end{cases}
\]

In this example the final time is set to \(T = 0.2\). Figure 6(a) and (c) show the density \(q\) obtained by the Godunov and VFV method on a mesh with 1024\(^2\) cells. Moreover, Figure 6(b) and (d) show the \(L^2\)-errors of \(q, m, \eta\) and \(L^1\)-norm of \(E\) on different meshes, see also Table 6.

The numerical results show that the convergence rates of \(q, m, \eta\) (resp. \(E\)) are slightly better than \(1/2\) (resp. 1). This may indicate that our rigorous error estimates are suboptimal in the case of finitely many rarefaction waves.

Table 6: Example 4.5: errors and convergence rates of \(q, m, \eta, E\) of the Godunov and VFV methods.

| \(n\)  | density error | momentum error | entropy error | relative energy error |
|-------|--------------|----------------|--------------|-----------------------|
|       | order        | order          | order        | order                 |
|-------|--------------|----------------|--------------|-----------------------|
| Godunov |              |                |              |                       |
| 16    | 0.0572       | -              | 0.0365       | 0.007821              |
|       | -            | 0.0749         | -            | 0.04021               |
| 32    | 0.0541       | 0.4408         | 0.0267       | 0.01921               |
|       | 0.4475       | 0.4482         | 0.4808       | 0.001952              |
| 64    | 0.0298       | 0.4975         | 0.0192       | 0.001952              |
|       | 0.4950       | 0.4729         | 0.4808       | 0.001952              |
| 128   | 0.0202       | 0.5636         | 0.0132       | 0.001952              |
|       | 0.5567       | 0.5454         | 0.5354       | 0.001952              |
| 256   | 0.0129       | 0.6402         | 0.0087       | 0.001952              |
|       | 0.6316       | 0.6061         | 0.6026       | 0.001952              |
| 512   | 0.0077       | 0.7434         | 0.0054       | 0.001952              |
|       | 0.7353       | 0.8063         | 0.6973       | 0.001952              |
|       |              |                |              |                       |
| VFV   |              |                |              |                       |
| 16    | 0.0751       | -              | 0.0515       | 0.014156              |
|       | -            | 0.0946         | -            | 0.07097               |
| 32    | 0.0541       | 0.4729         | 0.0353       | 0.003257              |
|       | 0.4823       | 0.5454         | 0.5451       | 0.003257              |
| 64    | 0.0375       | 0.5276         | 0.0235       | 0.003257              |
|       | 0.5454       | 0.6195         | 0.5868       | 0.003257              |
| 128   | 0.0247       | 0.6061         | 0.0151       | 0.003257              |
|       | 0.6195       | 0.6347         | 1.1237       | 0.003257              |
| 256   | 0.0152       | 0.6976         | 0.0093       | 1.2666                |
|       | 0.7026       | 0.6997         | 1.4263       | 1.2666                |
| 512   | 0.0087       | 0.8063         | 0.0054       | 1.6287                |
|       | 0.8093       | 0.7938         | 1.6287       |                       |
Figure 6: Example 4.5: density on a mesh with $1024^2$ cells and errors at $T = 0.2$. 
Example 4.6. The initial data of the second 2D Riemann problem are given by
\[
(g, u, v, p)(x, 0) = \begin{cases}
(0.5, 0.5, -0.5, 5), & x > 0.5, y > 0.5, \\
(1, 0.5, 0.5, 5), & x < 0.5, y > 0.5, \\
(2, -0.5, 0.5, 5), & x < 0.5, y < 0.5, \\
(1.5, -0.5, -0.5, 5), & x > 0.5, y < 0.5.
\end{cases}
\]
The exact solution consists of four interacting contact discontinuities yielding vortex sheets with negative signs. We simulate till \( T = 0.2 \). Figure 7(a) and (c) show the density obtained by the Godunov method and the VFV method on a mesh with 1024\(^2\) cells. The \( L^2 \)-errors of \((g, m, \eta)\) as well as the \( L^1 \)-norm of \( E \) are shown in Figure 7(b) and (d), see also Table 7.

Numerical results indicate that \((g, m, \eta)\) converges with the convergence rate about 1/2 and the convergence rate for \( E \) is approximately 1. It seems that our theoretical results for the convergence rates obtained for the strong exact solutions practically holds also for some discontinuous (weak) solutions.

Table 7: Example 4.6: errors and convergence rates of \( g, m, \eta, E \) of the Godunov and VFV methods.

| \( n \) | density error | momentum error | entropy error | relative energy error | \( n \) | density order | momentum order | entropy order | relative energy order |
|--------|---------------|----------------|---------------|-----------------------|--------|---------------|-------------------|---------------|----------------------|
| 16     | 0.1534        | -              | 0.2355        | -                     | 0.31123 | -              | 0.1177           | 0.3816        |                      |
| 32     | 0.1177        | 0.3816         | 0.1780        | 0.4040                | 0.1599  | 0.3543        | 0.1280           | 0.3180        | 0.122058            | 0.6168         |                      |
| 64     | 0.0958        | 0.2979         | 0.1419        | 0.3267                | 0.1283  | 0.3180        | 0.1012           | 0.3425        | 0.075685            | 0.6895         |                      |
| 128    | 0.0757        | 0.3390         | 0.1096        | 0.3724                | 0.1012  | 0.3425        | 0.0773           | 0.3881        | 0.043366            | 0.8034         |                      |
| 256    | 0.0578        | 0.3903         | 0.0816        | 0.4266                | 0.0773  | 0.3881        | 0.0569           | 0.4761        | 0.021832            | 0.9901         |                      |
| 512    | 0.0414        | 0.4792         | 0.0569        | 0.5190                | 0.0556  | 0.4761        | 0.021832         | 0.9901        |                      |                |                      |

Example 4.7. The initial data of third 2D Riemann problem are given by
\[
(g, u, v, p)(x, 0) = \begin{cases}
(1.5, 0, 0, 1.5), & x > 0.5, y > 0.5, \\
(0.5323, 1.206, 0, 0.3), & x < 0.5, y > 0.5, \\
(0.138, 1.206, 1.206, 0.029), & x < 0.5, y < 0.5, \\
(0.5323, 0, 1.206, 0.3), & x > 0.5, y < 0.5.
\end{cases}
\]
Figure 7: Example 4.6: density on a mesh with $1024^2$ cells and errors obtained on different meshes.
which describes the interaction of four shock waves. In this example the final time is set to $T = 0.35$. Figure 8 shows the density on a mesh with $1024^2$ cells and errors of $(\varrho, m, \eta)$ and $E$ obtained on different meshes. Table 8 lists the errors and convergence rate.

From these numerical results we see that $(\varrho, m, \eta)$ converges with a ratio between $1/4$ and $1/2$ and $E$ converges to a ratio between $1/2$ and $1$.

Figure 8: Example 4.7: density on a mesh with $1024^2$ cells and errors obtained on different meshes.
Table 8: Example 4.7: errors and convergence rates of $\rho$, $m$, $\eta$, $E$ of the Godunov and VFV methods.

| $n$ | density error | momentum error | entropy error | relative energy error |
|-----|---------------|----------------|---------------|-----------------------|
|     | order | order | order | order |
| **Godunov** | | | | |
| 16  | 0.1589 | - | 0.1764 | - | 0.2013 | - | 0.061809 | - |
| 32  | 0.1284 | 0.3077 | 0.1404 | 0.3204 | 0.1639 | 0.2964 | 0.038141 | 0.6965 |
| 64  | 0.0963 | 0.4160 | 0.1133 | 0.3098 | 0.1337 | 0.2940 | 0.022547 | 0.7584 |
| 128 | 0.0739 | 0.3806 | 0.0926 | 0.2917 | 0.1078 | 0.3106 | 0.014170 | 0.6701 |
| 256 | 0.0576 | 0.3590 | 0.0777 | 0.2523 | 0.0867 | 0.3137 | 0.009518 | 0.5740 |
| 512 | 0.0466 | 0.3084 | 0.0650 | 0.2577 | 0.0734 | 0.2409 | 0.006632 | 0.5212 |

| **VFV** | | | | |
| 16  | 0.2075 | - | 0.2018 | - | 0.2840 | - | 0.107017 | - |
| 32  | 0.1566 | 0.4063 | 0.1765 | 0.1938 | 0.2090 | 0.4420 | 0.061465 | 0.8000 |
| 64  | 0.1246 | 0.3290 | 0.1471 | 0.2626 | 0.1647 | 0.3441 | 0.038455 | 0.6766 |
| 128 | 0.0975 | 0.3546 | 0.1168 | 0.3325 | 0.1342 | 0.2957 | 0.022700 | 0.7605 |
| 256 | 0.0719 | 0.4397 | 0.0912 | 0.3576 | 0.1044 | 0.3614 | 0.012882 | 0.8173 |
| 512 | 0.0519 | 0.4707 | 0.0706 | 0.3689 | 0.0790 | 0.4018 | 0.007407 | 0.7985 |

Example 4.8. The initial data of the fourth 2D Riemann problem are given by

$$(g, u, v, p)(x, 0) = \begin{cases} 
(0.5313, 0, 0, 0.4), & x > 0.5, y > 0.5, \\
(1, 0.7276, 0, 1), & x < 0.5, y > 0.5, \\
(0.8, 0, 0, 1), & x < 0.5, y < 0.5, \\
(1, 0, 0.7276, 1), & x > 0.5, y < 0.5.
\end{cases}$$

This experiment describes the interaction of four discontinuities (the left and bottom discontinuities are two contact discontinuities and the top and right are two shock waves). The final time is set to $T = 0.25$. Figure 9 shows the density obtained by the Godunov and VFV methods on a mesh with $1024^2$ cells, respectively. The $L^2$-errors of $g$, $m$, $\eta$, and the $L^1$-norm of $E$ obtained on different meshes are presented in Figure 9 and Table 9.

These numerical results indicate the convergence rate around $1/2$ for the $L^2$-errors in $(g, m, \eta)$ and rates around 1 for the $L^1$-norm in the relative energy $E$. Similarly as in the previous experiments, it seems that the VFV method converges faster than the Godunov method.
Figure 9: Example 4.8: density on a mesh with $1024^2$ cells and errors obtained on different meshes.
Table 9: Example 4.8: errors and convergence rates of $\rho, m, \eta, E$ of the Godunov and VFV methods.

| $n$  | density error | momentum error | entropy error | relative energy error |
|------|----------------|----------------|---------------|-----------------------|
|      | order          | order          | order         | order                 |
| Godunov | 0.0791 - | 0.1351 - | 0.0557 - | 0.010648 - |
| 16    | 0.0604 0.3891 | 0.1055 0.3567 | 0.0479 0.2184 | 0.006658 0.6775 |
| 32    | 0.0458 0.4012 | 0.0821 0.3619 | 0.0413 0.2134 | 0.004084 0.7050 |
| 64    | 0.0344 0.4103 | 0.0643 0.3538 | 0.0356 0.2145 | 0.002519 0.6971 |
| 128   | 0.0258 0.4152 | 0.0507 0.3426 | 0.0296 0.2664 | 0.001537 0.7128 |
| 256   | 0.0191 0.4382 | 0.0391 0.3724 | 0.0242 0.2932 | 0.000896 0.7786 |
| 512   | 0.1013 - | 0.1992 - | 0.1343 - | 0.023507 - |
| VFV   | 0.0764 0.4069 | 0.1556 0.3559 | 0.1066 0.3340 | 0.014532 0.6938 |
| 16    | 0.0559 0.4522 | 0.1186 0.3921 | 0.0837 0.3493 | 0.008488 0.7757 |
| 32    | 0.0404 0.4676 | 0.0891 0.4126 | 0.0650 0.3647 | 0.004795 0.8240 |
| 64    | 0.0293 0.4640 | 0.0667 0.4174 | 0.0493 0.3992 | 0.002630 0.8664 |
| 128   | 0.0207 0.5035 | 0.0484 0.4632 | 0.0355 0.4719 | 0.001340 0.9725 |

5 Conclusion

In this paper we have analyzed a priori errors between numerical solutions obtained by the Godunov method and the strong exact solution for the multidimensional Euler system via the relative energy. Assuming that there exist a uniform lower bound on the density and an upper bound on the energy, we showed that the $L^1$-norm of the relative energy is equivalent to the $L^2$-norm of errors of the numerical solutions, see (2.16). Recalling the consistency formulation proved in [16] and applying Gronwall’s lemma, we have derived the estimates for the relative energy in Theorem 3.1. Specifically, the relative energy converges at least at the rate of $1/2$ in the $L^1$-norm. At the same time, the density, momentum and entropy converge at least at the rate of $1/4$ in the $L^2$-norm. Being inspired by the fact that the Godunov method for scalar conservation laws has bounded total variations we have formulated additional hypothesis (3.8). If we assume that (3.8) holds, the convergence rate of density, momentum and entropy (resp. relative energy) can be improved to at least $1/2$ (resp. 1), see Theorem 3.3. Finally, we pointed out that our theoretical analysis rigorously holds only for strong solutions, e.g. for a solution that contains finitely many rarefaction waves.

We have experimentally computed convergence rates for several one- and two-dimensional Riemann problems. From Example 4.1 and Example 4.3 containing only rarefaction waves, we observed that the convergence rate of density, momentum and entropy (resp. relative energy) is slightly higher than $1/2$ (resp. 1), which is consistent with the theoretical results presented in Theorem 3.3. Our numerical experiments for the Riemann problems with discontinuous solutions show
that the convergence rate of the Godunov method are about 1/4 for the contact wave and about 1/2 for the shock wave. In future it will be interesting to analyze theoretically the convergence rates towards a weak exact solution containing shock and contact wave.

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**Availability of data and materials**

The datasets supporting the conclusions of this article are included within the article.

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