A CHARACTERIZATION OF MULTIPLIER SEQUENCES FOR GENERALIZED LAGUERRE BASES

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Abstract. We give a complete characterization of multiplier sequences for generalized Laguerre bases. We also apply our methods to give a short proof of the characterization of Hermite multiplier sequences achieved by Piotrowski.

1. Introduction

In this paper we study linear operators on real polynomials that preserve the property of having only real zeros (we consider constant polynomials as being real-rooted). Pólya and Schur characterized such linear operators that act diagonally with respect to the standard basis of \( \mathbb{R}[x] \), see [14]. A complete characterization of linear operators preserving real-rootedness was achieved only recently by Borcea and the first author in [3]. However, generalizations of the Pólya–Schur theorem of the following form are still open in many important cases:

**Problem 1** (Problem 4.2 in [2]). Let \( \mathcal{P} = \{ P_n(x) \}_{n=0}^{\infty} \) be a basis for \( \mathbb{R}[x] \). For a sequence \( \{ \lambda_n \}_{n=0}^{\infty} \) of real numbers, define a linear operator \( T: \mathbb{R}[x] \to \mathbb{R}[x] \) by

\[
T(P_n(x)) = \lambda_n P_n(x), \quad \text{for all } n \in \mathbb{N} := \{0, 1, 2, \ldots \}.
\]

Characterize the sequences \( \{ \lambda_n \}_{n=0}^{\infty} \) for which \( T \) preserves real-rootedness.

We call such a sequence a \( \mathcal{P} \)-multiplier sequence, while the term multiplier sequence is reserved for the classical case \( \mathcal{P} = \{ x^n \}_{n=0}^{\infty} \). The case of Problem 1 when \( \mathcal{P} = \{ x^n \}_{n=0}^{\infty} \) goes back to Laguerre and Jensen and was completely solved by Pólya and Schur in [14], see also [7, 12]. Turán [17] and subsequently Bleecker and Csordas [2] provided classes of multiplier sequences for the Hermite polynomials \( \mathcal{H} = \{ H_n(x) \}_{n=0}^{\infty} \), while Piotrowski completely characterized \( \mathcal{H} \)-multiplier sequences in [13]. Recently partial results regarding multiplier sequences for the generalized Laguerre bases [9], and for the Legendre bases [1], were achieved.

Recall that the (generalized) Laguerre polynomials, \( \mathcal{L}_\alpha = \{ L_n^{(\alpha)}(x) \}_{n=0}^{\infty} \), are defined by

\[
L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n + \alpha}{n - k} \frac{(-x)^k}{k!}, \quad \alpha > -1,
\]

see [16, p. 201]. In this paper we give a complete characterization of \( \mathcal{L}_\alpha \)-multiplier sequences for each \( \alpha > -1 \). We say that a sequence \( \{ \lambda_n \}_{n=0}^{\infty} \) is trivial if there is a

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number \( k \in \mathbb{N} \) such that \( \lambda_n = 0 \) for all \( n \notin \{k, k + 1\} \). It is not hard to see that all trivial sequences are \( \mathcal{L}_\alpha \)-multiplier sequences, see [9, Proposition 2.1]. Hence it remains to characterize non-trivial \( \mathcal{L}_\alpha \)-multiplier sequences, which is achieved by the following:

**Theorem 1.1.** Let \( p(y) = \sum_{k=0}^{\infty} (k + 1)\alpha_k y^k \) be a formal power series, where \( \alpha > -1 \), and let \( \{\lambda_n\}_{n=0}^{\infty} \) be a non-trivial sequence defined by

\[
\lambda_n := \sum_{k=0}^{n} a_k \binom{n}{k}.
\]

Then \( \{\lambda_n\}_{n=0}^{\infty} \) is an \( \mathcal{L}_\alpha \)-multiplier sequence if and only if \( p(y) \) is a real-rooted polynomial with all its zeros contained in the interval \([-1, 0]\).

**Remark 1.2.** Note that Theorem 1.1 implies that each non-trivial \( \mathcal{L}_\alpha \)-multiplier sequence is a polynomial in \( n \), i.e., there is a polynomial \( P(x) \) such that \( \lambda_n = P(n) \) for all \( n \in \mathbb{N} \). Hence the corresponding operator \( T \) is a finite order differential operator.

**Remark 1.3.** We may express an arbitrary sequence \( \{\lambda_n\}_{n=0}^{\infty} \) as \( \lambda_n = \sum_{k=0}^{n} a_k \binom{n}{k} \), where the sequence \( \{a_k\}_{k=0}^{\infty} \) is defined by \( a_k = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \lambda_j \), for each \( k \in \mathbb{N} \). It follows (by elementary binomial identities) that the series \( p(y) \), defined in Theorem 1.1, may be expressed in terms of the sequence \( \{\lambda_n\}_{n=0}^{\infty} \) as the formal power series

\[
p(y) = \frac{1}{(1+y)^{\alpha+1}} \sum_{n=0}^{\infty} \lambda_n \binom{n+\alpha}{n} \left( \frac{y}{1+y} \right)^n.
\]

Hence \( \{\lambda_n\}_{n=0}^{\infty} \) is a non-trivial \( \mathcal{L}_\alpha \)-multiplier sequence if and only if (1.2) is a real-rooted polynomial with all its zeros contained in the interval \([-1, 0]\).

Our method of proving Theorem 1.1 is applicable to other bases, and in Section 3 we give a short proof of the characterization of Hermite-multiplier sequences due to Piotrowski [13, Theorem 152].

2. **Proof of Theorem 1.1**

The main tool used to prove Theorem 1.1 is the characterization of linear pre-servers of real–rootedness given in [3], which we now describe. The *Laguerre–Pólya class*, \( \mathcal{L} \mathcal{P}_1(\mathbb{R}) \), consists of all real entire functions that are limits, uniformly on compact subsets of \( \mathbb{C} \), of real–rooted polynomials. Laguerre and Pólya proved that an entire function \( \Phi \) is in the Laguerre–Pólya class if and only it may be expressed in the form

\[
\Phi(x) = cx^n e^{\alpha x - \beta x^2} \prod_{k=1}^{\omega} \left( 1 + \frac{x}{x_k} \right) e^{-x/x_k}, \quad \omega \in \mathbb{N} \cup \{\infty\},
\]

where \( c, \alpha, x_k \in \mathbb{R} \) for all \( k \), \( \beta \geq 0 \), \( n \) is a non-negative integer and \( \sum_{k=1}^{\infty} x_k^{-2} < \infty \). A multivariate polynomial \( P \in \mathbb{C}[x_1, \ldots, x_n] \) is called *stable* if \( P(x_1, \ldots, x_n) \neq 0 \) whenever \( \text{Im}(x_j) > 0 \) for all \( 1 \leq j \leq n \). Hence a real univariate polynomial is stable if and only if it is real–rooted. The *Laguerre–Pólya class* of real entire functions in \( n \) variables, \( \mathcal{L} \mathcal{P}_n(\mathbb{R}) \), consists of all real entire functions that are limits, uniformly on compact subsets of \( \mathbb{C}^n \), of real stable polynomials.
The symbol of a linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ is the formal power series defined by

$$G_T(x, y) := \sum_{n=0}^{\infty} \frac{(-1)^n T(x^n)}{n!} y^n.$$  

**Theorem 2.1** (Theorem 5 in [3]). A linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ preserves real-rootedness if and only if

1. The rank of $T$ is at most two and $T$ is of the form
   
   $$T(P) = \alpha(P)Q + \beta(P)R,$$

   where $\alpha, \beta : \mathbb{R}[x] \to \mathbb{R}$ are linear functionals and $Q + iR$ is a stable polynomial, or;

2. $G_T(x, y) \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$, or;

3. $G_T(-x, y) \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$.

Theorem 2.1 suggests that we should find necessary and sufficient conditions for the symbol, $G_T(x, y)$, of the operator given by $T(L_n^{(\alpha)}(x)) = \lambda_n L_n^{(\alpha)}(x)$ to be in $\mathcal{L}(\mathcal{P}_2(\mathbb{R}))$. By giving an explicit expression for $G_T(x, y)$, the following Lemma is a step towards establishing such conditions. Though the result follows from [9, Proposition 4.2], for the sake of completeness we include a short proof here based on a well known identity for generalized Laguerre polynomials.

**Lemma 2.2.** Let $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of real numbers. The symbol of the operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ defined by $T(L_n^{(\alpha)}(x)) = \lambda_n L_n^{(\alpha)}(x)$, for all $n \in \mathbb{N}$, is given by

$$G_T(x, y) = e^{-xy} \sum_{n=0}^{\infty} a_n y^n L_n^{(\alpha)}(x) + x,$$

where

$$a_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \lambda_k, \quad n \in \mathbb{N}.$$  

**Proof.** Recall that the generalized Laguerre polynomials satisfy the following differential equation (see [16, p. 204]):

$$nL_n^{(\alpha)}(x) = (x - \alpha - 1) \frac{d}{dx} L_n^{(\alpha)}(x) - x \frac{d^2}{dx^2} L_n^{(\alpha)}(x).$$

Consider the operator $\delta := (x - \alpha - 1) d/dx - x d^2/dx^2$ and let

$$S_k := \frac{\delta(\delta - 1) \cdots (\delta - k + 1)}{k!}.$$  

Then $S_k L_n^{(\alpha)}(x) = \binom{n}{k} L_n^{(\alpha)}(x)$, and letting $T$ be the operator corresponding to $\{\lambda_n\}_{n=0}^{\infty}$, we have $T = \sum_{k=0}^{\infty} a_k S_k$. By the change of variables $y = t/(1-t)$ in the generating function for the generalized Laguerre polynomials:

$$\frac{e^{-xt/(1-t)}}{(1-t)^{1+\alpha}} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n,$$

see [16, p. 202], yields

$$G_{S_k}(x, y) = S_k(e^{-xy}) = \sum_{n=0}^{\infty} \binom{n}{k} L_n^{(\alpha)}(x) y^n (1+y)^{-n-\alpha-1}.$$
On the other hand, with the same change of variables as above, identity (9) on page 211 in \[16\] states that

\[
\sum_{n=0}^{\infty} \binom{n}{k} L_n^{(\alpha)}(x)y^n(1+y)^{-n-1-\alpha} = e^{-xy}y^k L_k^{(\alpha)}(xy+x).
\]

Hence

\[
G_T(x, y) = T(e^{-xy}) = \sum_{k=0}^{\infty} a_k S_k(e^{-xy}) = e^{-xy} \sum_{k=0}^{\infty} a_k y^k L_k^{(\alpha)}(xy+x).
\]

The explicit expression (1.1) of the Laguerre polynomials now yields:

\[
G_T(x, y) = e^{-xy} \sum_{n=0}^{\infty} a_n y^n \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x(y+1))^k}{k!}.
\] (2.2)

Setting \(p(y) = \sum_{n=0}^{\infty} \binom{n+\alpha}{n} a_n y^n\) gives

\[
\frac{p^{(k)}(y)}{(\alpha+1)\cdots(\alpha+k)} = \sum_{n=k}^{\infty} \binom{n+\alpha}{n-k} a_n y^{n-k},
\]

which together with changing the order of summation in (2.2) yields the following consequence of Lemma 2.2:

**Corollary 2.3.** The symbol of the operator \(T : \mathbb{R}[x] \to \mathbb{R}[x]\) defined by \(T(L_n^{(\alpha)}(x)) = \lambda_n L_n^{(\alpha)}(x)\), for all \(n \in \mathbb{N}\), is given by

\[
G_T(x, y) = e^{-xy} \sum_{k=0}^{\infty} p^{(k)}(y) \frac{(-xy(y+1))^k}{(\alpha+1)\cdots(\alpha+k)k!},
\]

where \(p(y)\) is defined as in Theorem 1.1.

Before we proceed with the proof of Theorem 1.1 let us collect some fundamental properties of multiplier sequences in a lemma for ease of reference:

**Lemma 2.4.**

(1) (Polya and Schur, [14]). Let \(\{\lambda_n\}_{n=0}^{\infty}\) be a sequence of real numbers, and define a formal power series by

\[
\Phi(x) := \sum_{k=0}^{\infty} \frac{\lambda_k x^k}{k!}.
\]

Then \(\{\lambda_n\}_{n=0}^{\infty}\) is a multiplier sequence if and only if \(\Phi(x)\) or \(\Phi(-x)\) is an entire function that has the form

\[
c x^n e^{sx} \prod_{k=1}^{\omega} \left(1 + \frac{x}{x_k}\right), \quad \omega \in \mathbb{N} \cup \{\infty\}, \quad (2.3)
\]

where \(s \geq 0, n \in \mathbb{N}, c \in \mathbb{R}, x_k > 0\) for all \(k\), and \(\sum_{k=0}^{\omega} x_k^{-1} < \infty\).

(2) If \(\{\lambda_n\}_{n=0}^{\infty}\) is a multiplier sequence and \(\lambda_k \lambda_\ell \neq 0\) for some \(k < \ell\), then \(\lambda_i \neq 0\), for all \(k \leq i \leq \ell\).
(3) Let \( \{\lambda_n\}_{n=0}^{\infty} \) be a sequence of real numbers and let \( T \) be the corresponding diagonal operator. Then \( \{\lambda_n\}_{n=0}^{\infty} \) is a trivial sequence (as defined in the introduction) if and only if \( T \) has rank at most two and \( \{\lambda_n\}_{n=0}^{\infty} \) is a multiplier sequence.

Note that (3) follows easily from (2) which follows easily from (1).

2.1. Proof of Necessity. Before we proceed with the proof we recall a version of the Hermite–Biehler theorem, and the notions of interlacing zeros and proper position. Let \( x_1 \leq x_2 \leq \cdots \leq x_n \) and \( y_1 \leq y_2 \leq \cdots \leq y_m \) be the zeros of two real–rooted polynomials \( f \) and \( g \), where \( \deg f = n \), \( \deg g = m \) and \( |n-m| \leq 1 \). We say that the zeros of \( f \) and \( g \) interlace if they can be ordered so that \( x_1 \leq y_1 \leq x_2 \leq y_2 \leq \cdots \) or \( y_1 \leq x_1 \leq y_2 \leq x_2 \leq \cdots \). By convention we also say that the “zeros” of any constant polynomial interlace the zeros of any polynomial of degree at most one, and that the zeros of the identically zero polynomial interlace the zeros of any real-rooted (or constant) polynomial. If the zeros of two polynomials \( f \) and \( g \) interlace, then the Wronskian

\[
W[f, g] := f'g - fg'
\]

is either non-negative or non-positive on the whole of \( \mathbb{R} \).

We say that an ordered pair \( f, g \) of real–rooted polynomials are in proper position, written \( f \ll g \), if the zeros of \( f \) and \( g \) interlace and \( W[f, g] \leq 0 \) on \( \mathbb{R} \).

Theorem 2.5 (Hermite–Biehler, see e.g. p. 197 in [15]). Let \( f, g \in \mathbb{R}[x] \), not both identically zero. Then \( f \ll g \) if and only if the polynomial \( g + if \) is stable.

We may extend the the notion of proper position to \( L^{-\mathcal{P}}_1(\mathbb{R}) \) by setting \( f \ll g \) if and only if \( g + if \in L^{-\mathcal{P}}_1(\mathbb{C}) \), where \( L^{-\mathcal{P}}_1(\mathbb{C}) \) is the complex Laguerre–Pólya class which is defined to be the set of entire functions that are limits, uniformly on compact subsets of \( \mathbb{C} \), of stable polynomials in \( \mathbb{C}[x] \). In particular if \( g + if \in L^{-\mathcal{P}}_1(\mathbb{C}) \), where \( f \) and \( g \) are real entire functions, then \( W[f, g](x) \leq 0 \) for all \( x \in \mathbb{R} \).

Any \( L^-\alpha \)-multiplier sequence is a multiplier sequence, see [13, Lemma 157]. Assume that \( \{\lambda_n\}_{n=0}^{\infty} \) is a non-trivial \( L^-\alpha \)-multiplier sequence, and let \( T \) be the corresponding operator. Then, by Theorem 2.1, \( G_T(x, y) \in L^\mathcal{P}_2(\mathbb{R}) \) or \( G_T(-x, y) \in L^\mathcal{P}_2(\mathbb{R}) \), since if \( T \) has rank at most two then \( \{\lambda_n\}_{n=0}^{\infty} \) is trivial by Lemma 2.4 (3). Assume \( G_T(x, y) \in L^\mathcal{P}_2(\mathbb{R}) \) and expand its expression given in Corollary 2.3 in powers of \( x \):

\[
G_T(x, y) = p(y) - x \left( yp(y) + \frac{y(y+1)}{1+\alpha} p'(y) \right) + \cdots .
\]

Non-negative multiplier sequences may be extended to act on functions of two variables by the rule \( x^k y^\ell \mapsto \lambda_k x^k y^\ell \) for all \( k, \ell \in \mathbb{N} \). The class \( L^{-\mathcal{P}}_2(\mathbb{R}) \) is preserved under this action (see [4, Lemma 3.7] and [6]). Hence we may truncate the expression above by the multiplier sequence \( \{1, 0, 0, \ldots\} \) and obtain \( p(y) \in L^{-\mathcal{P}}_1(\mathbb{R}) \).

If we instead truncate by the multiplier sequence \( \{1, 1, 0, 0, \ldots\} \) we arrive at the bivariate expression

\[
Q(x, y) = p(y) - xq(y) = p(y) - x \left( yp(y) + \frac{y(1+y)}{(1+\alpha)} p'(y) \right) \quad (2.4)
\]
which belongs to \( L_{-}\mathcal{P}_2(\mathbb{R}) \). Hence \( iQ(i, y) = q(y) + ip(y) \in L_{-}\mathcal{P}_1(\mathbb{C}) \), and thus

\[
W[p, q](y) = -p(y)^2 + \frac{y(y+1)}{1+\alpha}(p'(y)^2 - p(y)p''(y)) - \frac{2y+1}{1+\alpha}p(y)p'(y) \leq 0,
\]

for all \( y \in \mathbb{R} \). We use the above inequality to prove that all zeros of \( p \) are located in the interval \([-1, 0]\).

First suppose that \( y_0 \) is a real and simple zero of \( p(y) \). Evaluating \( W[p, q] \) at \( y_0 \) yields \( y_0(y_0+1)p'(y_0)^2 \leq 0 \), which can only happen if \( y_0 \in [-1, 0] \).

For multiple zeros we proceed as follows. Consider again \( W[p, q](y) \) and a real zero, \( y_0 \), of \( p(y) \) of multiplicity \( M \geq 2 \). If \( y_0 = 0 \) or \( y_0 = -1 \) there is nothing to prove, so assume otherwise. The multiplicity of \( y_0 \) will be \( 2M \) for \( p^2 \), \( 2M - 1 \) for \( pp' \) and \( 2M - 2 \) for \( (p')^2 \) and \( pp'' \). If there is no cancellation the dominating term near \( y_0 \) of \( W[p, q](y) \) is

\[
\frac{y(y+1)}{\alpha+1}(p'(y)^2 - p(y)p''(y)). \tag{2.5}
\]

However, writing \( p(y) = (y-y_0)^M s(y) \) we obtain

\[
p'(y)^2 - p(y)p''(y) = (y-y_0)^{2M-2}(Ms(y)^2 + (s'(y)^2 - s(y)s''(y))(y-y_0)^2),
\]

which proves that (2.5) is the dominating term near \( y_0 \) and from which it follows that \( y_0 \in [-1, 0] \).

We know that \( p(y) \) is an entire function in \( L_{-}\mathcal{P}_1(\mathbb{R}) \) so it has the form (2.1), and we now show that it is in fact a polynomial. Since its zeros lie in the interval \([-1, 0]\), it can only have a finite number of zeros, that is, \( p(y) = e^{ay-by^2}K(y) \), where \( K(y) \) is a real–rooted polynomial with zeros only in \([-1, 0]\), and \( a, b \in \mathbb{R} \) with \( b > 0 \).

Recall the definition of \( Q(x, y) \) in (2.4). Then \( Q(x, y) = e^{ay-by^2}(K(y) - xF(y)) \), where

\[
F(y) = yK(y) + \frac{y(y+1)}{1+\alpha}((a-2by)K(y) + K'(y)).
\]

The zeros of \( F(y) \) and \( K(y) \) interlace by Theorem 2.5 (set \( x = i \)). Notice that \( \deg F \geq \deg K + 2 \), unless \( a = b = 0 \). Hence \( a = b = 0 \) and there is no exponential factor. This completes the proof that \( p(y) \) is a real–rooted polynomial with all its zeros contained in \([-1, 0]\), and finishes the proof of necessity in the case when \( G_T(x, y) \in L_{-}\mathcal{P}_2(\mathbb{R}) \). It remains to prove that we cannot have \( G_T(-x, y) \in L_{-}\mathcal{P}_2(\mathbb{R}) \).

Assume \( G_T(-x, y) \in L_{-}\mathcal{P}_2(\mathbb{R}) \). Then proceeding as in the case when \( G_T(x, y) \in L_{-}\mathcal{P}_2(\mathbb{R}) \), we get \( q(y) \ll p(y) \), where \( q(y) \) is as in (2.4). Thus

\[
W[p, q](y) = -p^2(y) + \frac{y(y+1)}{1+\alpha}(p'(y)^2 - p(y)p''(y)) - \frac{2y+1}{1+\alpha}p(y)p'(y) \geq 0, \tag{2.6}
\]

for all \( y \in \mathbb{R} \). If \( p(-1/2) \neq 0 \), then Laguerre’s inequality (see e.g. [7, Corollary 3.7]) implies that the middle term in (2.6) is non-positive and thus \( W[p, q](-1/2) < 0 \). Suppose \( y = -1/2 \) is a zero of \( p(y) \) of multiplicity \( M \geq 1 \). Then, since

\[
(y+1/2)\frac{p'(y)}{p(y)} \approx M,
\]

near \( y = -1/2 \) we see that also the last term in (2.6) is negative near \( y = -1/2 \). Hence we cannot have \( G_T(-x, y) \in L_{-}\mathcal{P}_2(\mathbb{R}) \).
2.2. Proof of Sufficiency. We now prove that the conditions on \( p(y) \) in Theorem 1.1 imply \( G_T(x, y) \in \mathcal{L} \mathcal{P}_2(\mathbb{R}) \), which will then prove sufficiency by Theorem 2.1. Assume that the zeros of

\[
p(y) = \sum_{k=0}^{n} \binom{k + \alpha}{k} a_k y^k = \prod_{j=1}^{n} (y + \theta_j)
\]

are real and lie in \([-1, 0]\), and consider again the symbol expressed as in Corollary 2.3:

\[
G_T(x, y) = e^{-xy} \sum_{k=0}^{n} p^{(k)}(y) \frac{(-xy(y+1))^k}{(\alpha + 1) \cdots (\alpha + k) k!}
\]

Since \( \{(\alpha + 1) \cdots (\alpha + k)\}^{-1} \) is a non-negative multiplier sequence as proved already by Laguerre [11], and as such preserves \( \mathcal{L} \mathcal{P}_2(\mathbb{R}) \) when acting on \( x \), it is enough to prove that

\[
\sum_{k=0}^{n} p^{(k)}(y) \frac{(-xy(y+1))^k}{k!}
\]

is a stable polynomial in two variables. Now

\[
\sum_{k=0}^{n} p^{(k)}(y) \frac{(-xy(y+1))^k}{k!} = p(y - xy + 1) = \prod_{j=1}^{n} (\theta_j + y - xy + 1),
\]

where \( 0 \leq \theta_j \leq 1 \). Observe that \( -y(y+1) \ll y + \theta \) for all \( 0 \leq \theta \leq 1 \) and thus, by e.g. [5, Lemma 2.8], it follows that each factor is stable. This finishes the proof of Theorem 1.1.

\[ \square \]

3. HERMITE MULTIPLIER SEQUENCES

We will now apply our methods to give a short proof of the characterization of Hermite multiplier sequences achieved by Piotrowski [13]. The Hermite polynomials, \( \mathcal{H} = \{H_n(x)\}_{n=0}^{\infty} \), may be defined by the generating function

\[
\exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n,
\]

see [16]. Since Hermite polynomials are even or odd it is easy to see that \( \{\lambda_n\}_{n=0}^{\infty} \) is an \( \mathcal{H} \)-multiplier sequence if and only if \( \{(-1)^n \lambda_n\}_{n=0}^{\infty} \) is an \( \mathcal{H} \)-multiplier sequence. It is also plain to see that any trivial sequence is an \( \mathcal{H} \)-multiplier sequence, and that all \( \mathcal{H} \)-multiplier sequences are multiplier sequences (see [13, Theorem 158]).

Since the entries of multiplier sequences either have the same sign or alternate in sign (by Lemma 2.4 (1)) it remains to characterize non-negative and non-trivial Hermite multiplier sequences. In [13] a generalization of Pólya’s curve theorem led to the following characterization, which we will now re-prove:

**Theorem 3.1** (Piotrowski, [13]). Let \( \{\lambda_n\}_{n=0}^{\infty} \) be a non-trivial sequence of non-negative numbers. Then \( \{\lambda_n\}_{n=0}^{\infty} \) is a Hermite multiplier sequence if and only if it is a (classical) multiplier sequence with \( \lambda_n \leq \lambda_{n+1} \) for all \( n \geq 0 \).
Let \( \{ \lambda_n \}_{n=0}^{\infty} \) be a non-trivial and non-negative classical multiplier sequence and let \( T \) be the corresponding operator. Note that (3.1) implies
\[
e^{-xy} = e^{y^2/4} \sum_{k=0}^{\infty} \frac{H_k(x)}{k!} \left( \frac{-y}{2} \right)^k,
\]
and thus the symbol of \( T \) has the form
\[
G_T(x,y) = T(e^{-xy}) = e^{y^2/4} \sum_{k=0}^{\infty} \frac{\lambda_k H_k(x)(-y)^k}{2^k k!}.
\]
Since \( T \) is of rank greater than two, by Theorem 2.1 it remains to classify those non-negative sequences \( \{ \lambda_n \}_{n=0}^{\infty} \) for which \( G_T(x,y) \in \mathcal{L} \cdot \mathcal{P}_2(\mathbb{R}) \) or \( G_T(-x,y) \in \mathcal{L} \cdot \mathcal{P}_2(\mathbb{R}) \). First let us prove that \( G_T(-x,y) \) is never in \( \mathcal{L} \cdot \mathcal{P}_2(\mathbb{R}) \). Suppose that \( G_T(-x,y) \in \mathcal{L} \cdot \mathcal{P}_2(\mathbb{R}) \) and let \( M \) be the first index for which \( \lambda_M \neq 0 \). Then, since \( e^{-y^2/4} \in \mathcal{L} \cdot \mathcal{P}_2(\mathbb{R}) \),
\[
y^{-M} e^{-y^2/4} G_T(-x,y) = \frac{\lambda_M H_M(x)}{2^M M!} + \frac{\lambda_{M+1} H_{M+1}(x)}{2^{M+1}(M+1)!} y + \cdots \in \mathcal{L} \cdot \mathcal{P}_2(\mathbb{R}). \tag{3.2}
\]
Since \( \{ \lambda_n \}_{n=0}^{\infty} \) is nonnegative, Lemma 2.4 (2) implies \( \lambda_M, \lambda_{M+1} > 0 \), and as in the previous section we may apply the multiplier sequence \( \{ 1, 1, 0, \ldots \} \) (acting on \( y \)) to (3.2) and conclude
\[
\frac{\lambda_M H_M(x)}{2^M M!} + \frac{\lambda_{M+1} H_{M+1}(x)}{2^{M+1}(M+1)!} y \in \mathcal{L} \cdot \mathcal{P}_2(\mathbb{R}).
\]
Setting \( y = i \) yields \( H_{M+1}(x) \ll H_M(x) \) which is false (although \( H_M(x) \ll H_{M+1}(x) \) is a standard fact about orthogonal polynomials).

It remains to find necessary and sufficient conditions for \( G_T(x,y) \) to belong to the Laguerre–Pólya class. Note that
\[
\sum_{k=0}^{\infty} \frac{H_k(x)(-y)^k}{2^k k!} = e^{-xy} e^{-y^2/4} \in \mathcal{L} \cdot \mathcal{P}_2(\mathbb{R}). \tag{3.3}
\]
Hence for a non-negative multiplier sequence \( \{ \lambda_n \}_{n=0}^{\infty} \),
\[
\sum_{k=0}^{\infty} \frac{\lambda_k H_k(x)(-y)^k}{2^k k!} \in \mathcal{L} \cdot \mathcal{P}_2(\mathbb{R}).
\]
Recall the representation (2.1) of entire functions in \( \mathcal{L} \cdot \mathcal{P}_2(\mathbb{R}) \). A similar representation holds for functions in \( \mathcal{L} \cdot \mathcal{P}_2(\mathbb{K}) \) (see [12, p. 370]):

**Theorem 3.2.** Let \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \). If \( f(x,y) \) is an entire function of two variables, then \( f \) is in \( \mathcal{L} \cdot \mathcal{P}_2(\mathbb{K}) \) if and only if \( f \) has the representation
\[
f(x,y) = e^{-ax^2-by^2} f_1(x,y),
\]
where \( a \) and \( b \) are non-negative numbers, and \( f_1 \) belongs to \( \mathcal{L} \cdot \mathcal{P}_2(\mathbb{K}) \) and has genus at most one in each of its variables under the condition that the other variable is fixed in the open upper half-plane.

Thus we may write
\[
\sum_{k=0}^{\infty} \frac{\lambda_k H_k(x)(-y)^k}{2^k k!} = e^{-ax^2-by^2} g(x,y) \tag{3.4}
\]
for some entire function \( g(x, y) \in \mathcal{L} \mathcal{P}_2(\mathbb{R}) \) of genus at most 1 in each variable under the condition that the other variable is fixed in the open upper half-plane. Hence

\[
G_T(x, y) = e^{\rho y/4} \sum_{k=0}^{\infty} \frac{\lambda_k H_k(x)(-y)^k}{2^k k!} = e^{-ax^2 - (b-1/4)y^2} g(x, y).
\] (3.5)

In light of Theorem 3.2 our task has reduced to establishing when \( b \geq 1/4 \).

To this end, recall that the order \( \rho \) and type \( \sigma \) of an entire function \( f(x) \) may be defined as:

\[
\rho := \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r} \quad \text{and} \quad \sigma := \limsup_{r \to \infty} \frac{\log M(r)}{r^\rho},
\]

where \( M(r) := \max_{|z|=r} |f(z)| \). In the definition of the type it is required that the order is finite and non-zero. In terms of its Taylor coefficients, \( \{c_n\}_{n=0}^\infty \), the order and type of \( f \) are given by

\[
\rho = \limsup_{n \to \infty} \frac{n \log n}{\log |c_n|} \quad \text{and} \quad \sigma \rho \}^{1/\rho} = \limsup_{n \to \infty} n^{1/\rho} |c_n|^{1/n},
\] (3.6)

see e.g. [12, p. 4]. If \( c_n = 0 \), then the quotient is understood to be zero.

**Lemma 3.3.** Let \( \{\lambda_n\}_{n=0}^\infty \) be a non-negative multiplier sequence with exponential generating function expressed in the form (2.3), and let \( g(x) = \sum_{n=0}^{\infty} c_n x^n \) be an entire function in \( \mathcal{L} \mathcal{P}_1(\mathbb{C}) \) of order 2 and type \( c \). Then

\[
T(g)(x) = \sum_{n=0}^{\infty} \lambda_n c_n x^n = \exp(-cs^2 x^2) f(x),
\] (3.7)

where \( f(x) \in \mathcal{L} \mathcal{P}_1(\mathbb{C}) \) has genus at most one.

**Proof.** Note that \( T(g) \in \mathcal{L} \mathcal{P}_1(\mathbb{C}) \), so it may represented as \( T(g) = \exp(-ax^2) f(x) \), where \( f(x) \in \mathcal{L} \mathcal{P}_1(\mathbb{C}) \) has genus at most one (by Theorem 3.2). It remains to prove that \( a = cs^2 \).

Let \( \omega \) and \( \tau \) be the order and type of (2.3), respectively, and let \( \rho \) and \( \sigma \) be the order and type of \( T(g)(x) \), respectively. Then

\[
\rho = \limsup_{n \to \infty} \frac{n \log n}{\log |\lambda_n c_n|} = \limsup_{n \to \infty} \left( \frac{\log n}{n \log n} + \frac{\log |c_n|}{n \log n} - \frac{\log n!}{n \log n} \right)^{-1} = \frac{1}{\omega^{-1} - 1/2}.
\]

Hence if \( \omega < 1 \), then \( \rho < 2 \), which verifies (3.7) in this case (since \( s = 0 \)).

Suppose \( \omega = 1 \). Then \( \rho = 2 \) and by (3.6),

\[
(\sigma e^2)^{1/2} = \limsup_{n \to \infty} n^{1/2} (\lambda_n |c_n|)^{1/n} = \limsup_{n \to \infty} n^{1/2} \left( \frac{\lambda_n}{n!} \right)^{1/n} (n!^{1/n} |c_n|^{1/n}).
\]

Since \( (n!)^{1/n} \sim ne^{-1} \),

\[
(\sigma e^2)^{1/2} = e^{-1} \limsup_{n \to \infty} n^{1/2} |c_n|^{1/n} \left( \frac{\lambda_n}{n!} \right)^{1/n} = e^{-1} (ce^2)^{1/2} \tau e,
\]

that is, \( \sigma = c \tau^2 \). Now

\[
\tau = s + \limsup_{r \to \infty} \sum_{k=1}^{\infty} \frac{\log (1 + r/x_k)}{r/x_k} x_k^{-1} = s,
\]
since \( \sum_{k=0}^{\infty} x^{-1} < \infty \), \( r^{-1} \log(1 + r) \) is bounded on \((0, \infty)\) and tends to zero as \( r \to \infty \). Hence \( \sigma = cs^2 \) and the lemma follows. \( \square \)

We may now establish when \( b \geq 1/4 \) in (3.4) and thus finish our proof of Theorem 3.1. Since the order and type with respect to \( y \) of (3.3) is 2 and 1/4, it follows by Lemma 3.3 that \( b = s^2/4 \). Theorem 3.1 now follows from the following lemma of Craven and Csordas:

Lemma 3.4 (Lemma 2.2, [8]). Let \( \{\lambda_n\}_{n=0}^{\infty} \) be a non-negative multiplier sequence with exponential generating function given by (2.3). Then \( s \geq 1 \) if and only if \( \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \).

\( \square \)

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References

[1] K. Blakeman, E. Davis, T. Forgács, K. Urabe, On Legendre multiplier sequences, Missouri J. Math. Sci. 24 (2012), 7–23.
[2] D. Bleecker, G. Csordas, Hermite expansions and the distribution of zeros of entire functions, Acta Sci. Math. (Szeged), 67 (2001), 177–196.
[3] J. Borcea, P. Brändén, Pólya-Schur master theorems for circular domains and their boundaries, Ann. of Math. (2) 170, No. 1, (2009), 465–492.
[4] J. Borcea, P. Brändén, Multivariate Pólya-Schur classification problems in the Weyl algebra, Proc. London Math. Soc. 101 (2010), 73–104.
[5] J. Borcea, P. Brändén, The Lee-Yang and Pólya-Schur programs. I. Linear operators preserving stability, Invent. Math. 177 (2009), 541–569.
[6] P. Brändén, The Lee-Yang and Pólya-Schur programs. III. Zero-preservation on Bargmann-Fock spaces, Amer. J. Math. (to appear), arXiv:1107.1809
[7] T. Craven, G. Csordas, Composition theorems, multiplier sequences and complex zeros decreasing sequences, Value Distribution Theory and Related Topics, Advances in Complex Analysis and Its Applications, Vol 3, ed. G. Barsegian, I.Laine and C.C Yang, p. 131–166, Kluver Press, 2004.
[8] T. Craven, G. Csordas, The Gauss-Lucas theorem and Jensen polynomials, Trans. Amer. Math. Soc., 278 (1983), 415–429.
[9] P. Piotrowski, Linear operators for generalized Laguerre bases, Rocky Mountain Journal of Math. (to appear), arXiv:1002.0759 [math.CV].
[10] T. Forgács, J. Tipton, B. Wright Multiplier sequences for simple sets of polynomials, Acta Math. Hungar., 137 (2012), 282–295.
[11] E. Laguerre, Oeuvres, Vol. I, Gauther–Villars, Paris, 1898.
[12] B. Levin, Distribution of zeros of entire functions. Translated from Russian by R. P. Boas, J. M. Danskin, F. M. Goodspeed, J. Korevaar, A. L. Shields and H. P. Thielman. Transl. Math. Monogr. Vol. 5. Amer. Math. Soc., Providence, R.I., 1980. p.xii+523.
[13] A. Piotrowski, Linear operators and the distributions of zeros of entire functions, Dissertation, University of Hawaii at Manoa, 2007.
[14] G. Pólya and J. Schur, Über zwei arten von faktorenfolgen in der theorie der algebraischen gleichungen, J. Reine Angew. Math. 144 (1914), 89–113.
[15] Q. I. Rahman, G. Schmeisser, Analytic theory of polynomials, London Math. Soc. Monogr. (N. S.) Vol. 26, Oxford Univ. Press, New York, NY, 2002.
[16] E. D. Rainville, Special functions, The Macmillan Company, New York, 1960
[17] P. Turán, Sur l’algèbre fonctionelle, Compt. Rend. du prem. Congr. des Math. Hongr., 27, Akadémiai Kiadó, Budapest, 1952, 279–290.
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