On the analytic Birkhoff normal form of the Benjamin-Ono equation and applications

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Abstract

In this paper we prove that the Benjamin-Ono equation admits an analytic Birkhoff normal form in an open neighborhood of zero in $H^s_0(T, \mathbb{R})$ for any $s > -1/2$ where $H^s_0(T, \mathbb{R})$ denotes the subspace of the Sobolev space $H^s(T, \mathbb{R})$ of elements with mean 0. As an application we show that for any $-1/2 < s < 0$, the flow map of the Benjamin-Ono equation $S_t^0 : H^s_0(T, \mathbb{R}) \to H^s_0(T, \mathbb{R})$ is nowhere locally uniformly continuous in a neighborhood of zero in $H^s_0(T, \mathbb{R})$.

Keywords: Benjamin–Ono equation, analytic Birkhoff normal form, well-posedness, solution map, nowhere locally uniformly continuous maps

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1 Introduction

In this paper we study the Benjamin-Ono equation on the torus \( \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z} \),
\[
\partial_t u = \partial_x (|\partial_x| u - u^2),
\]
where \( u \equiv u(x, t), x \in \mathbb{T}, t \in \mathbb{R} \), is real valued and \( |\partial_x| : H^\beta_r \rightarrow H^{\beta-1}_r, \beta \in \mathbb{R} \), is the Fourier multiplier
\[
|\partial_x| : \sum_{n \in \mathbb{Z}} \hat{v}(n)e^{inx} \mapsto \sum_{n \in \mathbb{Z}} |n| \hat{v}(n)e^{inx}
\]
where \( \hat{v}(n), n \in \mathbb{Z} \), are the Fourier coefficients of \( v \in H^\beta_r \) and \( H^\beta_r := H^\beta(\mathbb{T}, \mathbb{C}) \) is the Sobolev space of complex valued distributions on the torus \( \mathbb{T} \). The equation (1) was introduced in 1967 by Benjamin [1] and Davis & Acrivos [5] as a model for a special regime of internal gravity waves at the interface of two fluids. It is well known that (1) admits a Lax pair representation (cf. [19]) that leads to an infinite sequence of conserved quantities (cf. [19], [4]) and that it can be written in Hamiltonian form with Hamiltonian
\[
\mathcal{H}(u) := \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{1}{2} |\partial_x|^{1/2}u \right)^2 - \frac{1}{3} u^3 \, dx
\]
by the use of the Gardner bracket
\[
\{ F, G \}(u) := \frac{1}{2\pi} \int_0^{2\pi} (\partial_x F) \nabla_u G \, dx
\]
where \( \nabla_u F \) and \( \nabla_u G \) are the \( L^2 \)-gradients of \( F, G \in C^1(H^s_r, \mathbb{R}) \) at \( u \in H^s_r \) where \( H^s_r \equiv H^s(\mathbb{T}, \mathbb{R}) \) is the Sobolev space of real valued distributions on \( \mathbb{T} \). By the Sobolev embedding \( H^{1/2}\mathbb{R} \hookrightarrow L^4(\mathbb{T}, \mathbb{R}) \), the Hamiltonian (2) is well defined and real analytic on \( H^{1/2}_r \), the energy space of (1). The problem of the existence and uniqueness of the solutions of the Benjamin-Ono equation is well studied – see [7], [21] and references therein. We refer to [21] for an excellent survey and a derivation of (1).

By using the Hamiltonian formalism for (1), it was recently proven in [6, 7] that for any \( s > -1/2 \), the Benjamin-Ono equation has a Birkhoff map
\[
\Phi : H^s_{r,0} \rightarrow \mathfrak{h}_{r,0}^{1+s}, \, u \mapsto \left( \left( \Phi_n(u) \right)_{n \leq -1}, (\Phi_n(u))_{n \geq 1} \right),
\]
where
\[
H^\beta_{r,0} := H^\beta_r \cap H^\beta_c, \quad H^\beta_c := \left\{ u \in H^\beta_c \mid \hat{u}(0) = 0 \right\},
\]
and
\[
\mathfrak{h}_{r,0}^\beta := \left\{ z \in \mathfrak{h}_{c,0} \mid z_{-n} = \pi_n \forall n \geq 1 \right\}
\]
is a real subspace in the complex Hilbert space
\[
\mathfrak{h}_{c,0}^\beta := \left\{ (z_n)_{n \in \mathbb{Z}} \mid z_0 = 0 \text{ and } \sum_{n \in \mathbb{Z}} |n|^{2\beta} |z_n|^2 < \infty \right\}, \quad \langle n \rangle := \max\{1, |n|\}.
\]
By [6, Theorem 1] and [7, Theorem 6], the Birkhoff map (4) is a \textit{homeomorphism} that transforms the trajectories of the Benjamin-Ono equation (1) into straight lines, which are winding around the underlying invariant torus, defined by the action variables. Furthermore, these trajectories evolve on the isospectral sets of potentials of the corresponding Lax operator (see (15) below). In this sense, the Birkhoff map can be considered as a non-linear Fourier transform that significantly simplifies the construction of solutions of (1). This fact allows us to prove that for any $-1/2 < s < 0$, (1) is globally $C^9$-well-posed on $H_{r,0}^s$ ([7, Theorem 1]) improving in this way the previously known well-posedness results (see [17, 18]). This result is sharp by [7, Theorem 2]. Additional applications of the Birkhoff map include the proof of the almost periodicity of the solution $s$ of the Benjamin-Ono equation and the orbital stability of the Benjamin-Ono traveling waves (see [7, Theorem 3 and Theorem 4]). Finally, since the Hamiltonian (2) is well defined on $H_r^s$ for any $s \geq 1/2$, it follows from [6, Proposition 8.1] that for any $z$ in $h_{r,0}^{2+s}$, $s \geq 1/2$, 

$$\mathcal{H} \circ \Phi^{-1}(z) = \sum_{n=1}^{\infty} n^2 I_n - \sum_{n=1}^{\infty} \left( \sum_{k=n}^{\infty} I_k \right)^2,$$  

(7)

where $I_n(z) := |z_n|^2/2$ are action variables for any $n \geq 1$ (cf. (5)). In addition, by [6, Corollary 7.1], the Birkhoff coordinate functions $\zeta_n(u) \equiv \Phi_n(u)$, $n \geq 1$, satisfy the (well defined) canonical relations

$$\{\zeta_n, \zeta_m\} = 0, \quad \{\zeta_n, \overline{\zeta}_m\} = -i \delta_{nm}, \quad \forall n, m \geq 1,$$  

(8)

with respect to the Gardner bracket (3) on $H_{r,0}^s$ for $s \geq 0$.

In the present paper we prove that for any $s > -1/2$ the Benjamin-Ono equation admits an analytic Birkhoff normal form in an open neighborhood of zero in $H_{r,0}^s$ (see Theorem 1.1 below). Let us first recall the notion of an analytic Birkhoff normal form in the case of a Hamiltonian system on $\mathbb{R}^{2N}$ with coordinates $\{(x_1, y_1, ..., x_N, y_N)\}$ and canonical symplectic structure $\Omega = \sum_{n=1}^{N} dx_n \wedge dy_n$. Assume that the Hamiltonian $H \equiv H(x, y)$ is real analytic and

$$H(x, y) = \sum_{n=1}^{N} \omega_n \frac{x_n^2 + y_n^2}{2} + O(r^3), \quad r := \sqrt{x^2 + y^2},$$  

(9)

where $O(r^3)$ stands for terms of order $\geq 3$ in the Taylor expansion of $H$ at zero and $\omega_n \in \mathbb{R}$, $1 \leq n \leq N$. Note that then the Hamiltonian vector field $X_H$ of $H$ has a singular point at zero and its linearization $d_0X_H : T_0\mathbb{R}^{2N} \to T_0\mathbb{R}^{2N}$ at zero has imaginary eigenvalues, $\text{Spec } d_0X_H = \{\pm i \omega_1, ..., \pm i \omega_N\}$. By definition, the Hamiltonian system corresponding to $X_H$ has an analytic Birkhoff normal form in an open neighborhood of zero if there exist open neighborhoods $U$ and $V$ of zero in $\mathbb{R}^{2N}$ and a real analytic canonical diffeomorphism

$$\mathcal{F} : U \to V, \quad (x_1, y_1, ..., x_N, y_N) \mapsto (q_1, p_1, ..., q_N, p_N),$$

\newpage
such that the Hamiltonian $H \circ F^{-1} : V \to \mathbb{R}$ has the form
\[
H \circ F^{-1}(q_1, p_1, ..., q_N, p_N) = H(I_1, ..., I_N) \quad I_n := \frac{p_n^2 + q_n^2}{2}, \quad 1 \leq n \leq N.
\]

In this case, the quantities $I_1, ..., I_N$ are Poisson commuting integrals and the Hamiltonian equations can be easily solved in terms of explicit formulas. A classical theorem of Birkhoff [3] states that in the case when the coefficients $\omega_1, ..., \omega_N$ in (9) are rationally independent then $F : U \to V$ can be constructed in terms of a formal power series. However, this formal power series generically diverges because of small denominators. In the case when the Hamiltonian system is completely integrable in an open neighborhood of zero and the integrals satisfy a non-degeneracy condition at zero, then the Hamiltonian system admits an analytic Birkhoff normal form (Vey [22]). Vey’s result was subsequently improved – see [23] and the references therein. We remark that typically, the Hamiltonians of infinite dimensional Hamiltonian systems appearing in applications such as the Benjamin-Ono equation, the Korteweg-de Vries equation, or the NLS equation, are only defined on a continuously embedded proper subspace of the corresponding phase space. The following theorem can be considered as an instance of a Hamiltonian system of infinite dimension that admits an analytic Birkhoff normal form in an open neighborhood of zero.

**Theorem 1.1.** For any $s > -1/2$ there exists an open neighborhood $U \equiv U^s$ of zero in $H^s_{r,0}$ such that the Birkhoff map of the Benjamin-Ono equation
\[
\Phi : H^s_{r,0} \to h^{s+\frac{1}{2}}_{r,0},
\]
introduced in [6, 7], extends to an analytic map $\Phi : U \to h^{s+\frac{1}{2}}_{r,0}$. The restriction $\Phi|_{U \cap H^s_{r,0}} : U \cap H^s_{r,0} \to h^{s+\frac{1}{2}}_{r,0}$ is a real analytic diffeomorphism onto its image. In particular, in view of (7) and (8), for $s > -1/2$ the Benjamin-Ono equation (1) has an analytic Birkhoff normal form in the open neighborhood $U \cap H^s_{r,0}$ of zero in $H^s_{r,0}$ such that the Hamiltonian $H \circ \Phi^{-1}$ given by (7) is well defined and real analytic on $h^{s}_{r,0}$.

**Remark 1.1.** Since, prior to this work, the Benjamin-Ono equation was not known to be integrable in terms of an appropriate infinite dimensional real analytic setup, Theorem 1.1 should not be viewed as an instance of Vey’s result for the infinite dimensional Hamiltonian system (1). For the same reason, the techniques developed by Kuksin & Perelman in [16] do not apply in this case.

We apply Theorem 1.1 to further analyze regularity properties of the solution map of the Benjamin-Ono equation. Assume that $s > -1/2$. For $u_0 \in H^s_{r,0}$, denote by $t \mapsto u(t) \equiv u(t, u_0)$ the solution of the Benjamin-Ono equation (1) with initial data $u_0$, constructed in [7]. For given $t \in \mathbb{R}$ and $T > 0$ consider the flow map
\[
S^t_0 : H^s_{r,0} \to H^s_{r,0}, \quad u_0 \mapsto u(t, u_0),
\]
as well as the solution map
\[ S_{0,T} : H^s_{r,0} \to C([-T,T], H^s_{r,0}), \quad u_0 \mapsto u|_{[-T,T]}, \]
of the Benjamin-Ono equation. To state our results, we need to introduce one
more definition. A continuous map \( F : X \to Y \) between two Banach spaces \( X \)
and \( Y \) is called nowhere locally uniformly continuous in an open neighborhood
\( U \) in \( X \) if the restriction \( F|_V : V \to Y \) of \( F \) to any open neighborhood \( V \subseteq U \)
is not uniformly continuous. In a similar way one defines the notion of a nowhere
locally Lipschitz map in an open neighborhood \( U \subseteq X \).

**Theorem 1.2.** (i) For any \(-1/2 < s < 0 \) and \( t \neq 0 \), the flow map \( S^s_{0,0} : H^s_{r,0} \to H^s_{r,0} \)
of the Benjamin-Ono equation \((1)\) is nowhere locally uniformly continuous in an open neighborhood \( U \) of zero in \( H^s_{r,0} \). In particular, \( S^s_{0} : H^s_{r,0} \to H^s_{r,0} \) is nowhere locally Lipschitz in \( U \).

(ii) For any \( s \geq 0 \) there exists an open neighborhood \( U \equiv U^s \) of zero in \( H^s_{r,0} \) so
that for any \( T > 0 \), the solution map \( S^s_{0,T} : U \cap H^s_{r,0} \to C([-T,T], H^s_{r,0}) \)
is real analytic.

**Addendum to Theorem 1.2(ii).** For any \( k \geq 1 \), \( s > -1/2 + 2k \), and \( T > 0 \),
the solution map
\[ S_{0,T}^s : H^s_{r,0} \to \bigcap_{j=0}^k C^j([-T,T], H^{s-2j}_{r,0}) \]
is well defined and real analytic in an open neighborhood of zero in \( H^s_{r,0} \).

**Remark 1.2.** Item (i) of Theorem 1.2 improves on the result by Molinet in \([17, \text{Theorem 1.2}]\), saying that for any \( s < 0 \), \( t \in \mathbb{R} \), the flow map \( S^s_{0} \) (if it exists at all) is not of class \( C^{1+\alpha} \) for any \( \alpha > 0 \). Item (ii) of Theorem 1.2 improves on
the result by Molinet, saying that for any \( s \geq 0 \), \( t \in \mathbb{R} \), the flow map \( S^s_{0} \) is real
analytic near zero \(([17, \text{Theorem 1.2}]\)).

**Remark 1.3.** Any solution \( u \) of the Benjamin-Ono equation in \( H^s_{r,0} \), \( s > -1/2 \),
constructed in \([7]\), has the property that for any \( c \in \mathbb{R} \), \( u(t,x-2ct)+c \) is again
a solution with constant mean value \( c \). It is straightforward to see that for any \( c \in \mathbb{R} \), Theorem 1.2 holds on the affine space \( \{u \in H^s_{r} \mid \hat{u}(0) = c \} \).

**Remark 1.4.** Note that in Theorem 1.2 we restrict our attention to solutions
with mean value zero. In case when the mean value of solutions is not prescribed,
the flow map is no longer locally uniformly continuous on \( H^s_{r} \) with \( s \geq 0 \) as can
be seen by considering families of solutions \( c + u(t,x-2ct) \) of the Benjamin-Ono
equation, parametrized by \( c \in \mathbb{R} \), where \( c \) tends to zero and \( u \) is oscillating. \(^1\)

\(^1\)This fact can also be proved in a very transparent way using Birkhoff coordinates (cf. \([13, \text{Appendix A}]\)).
on bounded subsets of initial data in $H^s(\mathbb{R}, \mathbb{R})$ with $s > 0$ (cf. [14]). We point out that Theorem 1.2(i) does not follow from the lack of uniform continuity on bounded subsets of initial data of the solution map of the Benjamin-Ono equation on $H^s_r$, $s > -1/2$, and its proof requires new arguments. We remark that the Birkhoff coordinates allow to explain in clear terms why on any neighborhood of zero in $H^s_{r,0}$ with $-1/2 < s < 0$, the solution map of the Benjamin-Ono equation is nowhere locally uniformly continuous: the reason is that on these neighborhoods, the frequencies $\omega_n$ of the Benjamin-Ono equation are not locally Lipschitz continuous uniformly in $n \geq 1$. It follows from the analyticity of the Birkhoff map on all of $H^s_{r,0}$, established in [9], that these properties of the solution map of the Benjamin-Ono equation actually hold on the corresponding entire Sobolev spaces.

Ideas of the proofs. The Birkhoff map $\Phi : H^s_{r,0} \to B^s_{r,0}$, $u \mapsto (\Phi_n(u))_{n \geq 1}$, is defined in terms of the Lax operator $L_u = -i \partial_x - T_u$ where $T_u$ denotes the Toeplitz operator with potential $u \in H^s_{r,0}$, $s > -1/2$. We refer to Section 2 for a review of terminology and results concerning this operator, established in the previous papers [6], [7], and [8]. At this point we only mention that the spectrum of $L_u$ is discrete and consists of a sequence of simple, real eigenvalues, bounded from below, $\lambda_0(u) < \lambda_1(u) < \cdots$. Furthermore, there exist $L^2-$normalized eigenfunctions $f_n(u)$ of $L_u$, corresponding to the eigenvalues $\lambda_n(u)$, which are uniquely determined by the normalization conditions

$$\langle f_0(u) | 1 \rangle > 0, \quad \langle e^{ix} f_{n-1}(u) | f_n(u) \rangle > 0, \quad \forall n \geq 1. \quad (11)$$

The components of the Birkhoff map $\Phi$ are then defined as

$$\Phi_n(u) = \frac{\langle 1 | f_n(u) \rangle}{\sqrt{\kappa_n(u)}}, \quad \forall n \geq 1, \quad (12)$$

where $\kappa_n(u) > 0$ are scaling factors (cf. Section 4). We point out that the normalization conditions (11) are defined inductively. It is this fact which makes it more difficult to prove that $\Phi$ extends to an analytic map.

For $u = 0$, one has

$$\lambda_n(0) = n, \quad f_n(0) = e^{inx}, \quad \kappa_n(0) = 1, \quad \forall n \geq 0.$$  

We then use perturbation arguments to show that for any $s > -1/2$, $\Phi$ extends to an analytic map on a neighborhood of zero in $H^s_{c,0}$. Let us outline the main steps of the proof of this result.

In [8] we proved that there exists a neighborhood $U \equiv U^s$ of zero in $H^s_{c,0}$ so that for any $u \in U$, the spectrum of $L_u$ consists of simple eigenvalues $\lambda_n(u)$, $n \geq 0$. These eigenvalues are analytic maps on $U$ and so are the Riesz projectors $P_n(u)$ onto the one dimensional eigenspaces, corresponding to these eigenvalues. See Proposition 2.1 and Proposition 2.2 in Section 2 for a review of these results. It then follows that for any $n \geq 0$, $h_n(u) = P_n(u)e^{inx}$ is an analytic function $U \to H^1_{r,s}$ (see (13) below).
In Section 3 we introduce the pre-Birkhoff map

\[ \Psi(u) = (\Psi_n(u))_{n \geq 1}, \quad \Psi_n(u) = \langle h_n(u) \rangle 1, \quad \forall n \geq 1. \]

With the help of the Taylor expansion of \( P_n(u) \), \( n \geq 1 \), we show that for any \( u \in U \), \( \Psi(u) \in h^{1+s} \) and that \( \Psi : U \rightarrow h^{1+s} \) is analytic (see (14) below). The arguments developed allow to prove that these results hold for any \( s > -1/2 \), not just for \(-1/2 < s \leq 0\).

In Section 4, we relate the Birkhoff map \( \Phi \) to the pre-Birkhoff map \( \Psi \) by using in addition to \( \nu_n(u) \), \( n \geq 0 \), the scaling factors \( \mu_n(u) \), \( n \geq 1 \), introduced in [6] and further analyzed in [8]. The main ingredient for proving that \( \Phi \) extends to an analytic map on a neighborhood of zero in \( H^s_{r,0} \) is a novel Vanishing Lemma (cf. Lemma 5.1), which is discussed and proved in Section 5.

Theorem 1.1 and Theorem 1.2 are proved in Section 6. To prove Theorem 1.2 we use arguments developed in [12] to show a corresponding result for the Korteweg-de Vries equation.

**Related work.** In [9] we prove an extension of Theorem 1.1, saying that for any \( s > -1/2 \), the Birkhoff map \( \Phi : H^s_{r,0} \rightarrow h^{1+s}_{r,0} \) is a real analytic diffeomorphism. The proof of this result uses, in broad terms, the same strategy developed to prove Theorem 1.1, but it is more technical, relying on the approximation of arbitrary elements in \( H^s_{r,0} \) by finite gap potentials and on properties of the spectrum of the Lax operator, associated to such potentials. As in the proof of Theorem 1.1, one of the main ingredients of the proof of the analytic extension is Lemma 5.1 (Vanishing Lemma) of Section 5.

The result saying that the Birkhoff map \( \Phi : H^s_{r,0} \rightarrow h^{1+s}_{r,0} \) is a real analytic diffeomorphism for the appropriate range of \( s \), shows that similarly as for the Korteweg-de Vries (KdV) equation (cf. [11], [13]) and the defocusing nonlinear Schrödinger (NLS) equation (cf. [10]), the Benjamin-Ono equation on the torus is integrable in the strongest possible sense. We point out that the proof of the analyticity of the Birkhoff map in the case of the Benjamin-Ono equation significantly differs from the one in the case of the KdV and NLS equations due to the fact the Benjamin-Ono equation is not a differential equation.

The above result on the Birkhoff map \( \Phi \) can be applied to prove in a straightforward way that Theorem 1.2 extends to all of \( H^s_{r,0} \). We remark that a result of this type has been first proved for the KdV equation in [12, Theorem 3.10]. It turns out that the analysis of the solution map of the KdV equation, expressed in Birkhoff coordinates, can also be used to prove Theorem 1.2 and its extension.

Similarly as in the case of the KdV equation (cf. [11], [15]) and the NLS equation (cf. [2] and references therein), a major application of the result, saying that \( \Phi : H^s_{r,0} \rightarrow h^{1+s}_{r,0} \) is a real analytic diffeomorphism for any \( s > -1/2 \), concerns its use to study (Hamiltonian) perturbations of the Benjamin-Ono equation by KAM methods near finite gap solutions of arbitrary large amplitude.

**Notation.** In this paragraph we summarize the most frequently used notations in the paper. For any \( \beta \in \mathbb{R} \), \( H^{\beta}_\mathbb{C} \) denotes the Sobolev space \( H^{\beta}(T, \mathbb{C}) \) of complex
valued functions on the torus \( T = \mathbb{R}/2\pi \mathbb{Z} \) with regularity exponent \( \beta \). The norm in \( H^\beta_c \) is given by
\[
\|u\|_\beta := \left( \sum_{n \in \mathbb{Z}} |\hat{u}(n)|^2 \right)^{1/2}
\]
where \( \hat{u}(n) \), \( n \in \mathbb{Z} \), are the Fourier coefficients of \( u \in H^\beta_c \). For \( \beta = 0 \) and \( u \in H^0_c \equiv L^2(T, \mathbb{C}) \) we set \( \|u\| \equiv \|u\|_0 \). By \( H^\beta_{c,0} \) we denote the complex subspace in \( H^\beta_c \) of functions with mean value zero,
\[
H^\beta_{c,0} = \{ u \in H^\beta_c \mid \hat{u}(0) = 0 \}.
\]
For \( f \in H^\beta_c \) and \( g \in H^{-\beta}_c \), define the sesquilinear and bilinear pairing,
\[
\langle f | g \rangle := \sum_{n \in \mathbb{Z}} \hat{f}(n) \overline{\hat{g}(n)}, \quad \langle f, g \rangle := \sum_{n \in \mathbb{Z}} \hat{f}(n) \hat{g}(-n).
\]
The positive Hardy space \( H^\beta_+ \) with regularity exponent \( \beta \in \mathbb{R} \) is defined as
\[
H^\beta_+ := \{ f \in H^\beta_c \mid \hat{f}(n) = 0 \ \forall n < 0 \}.
\]
For \( 1 \leq p < \infty \) denote by \( \ell^p_+ \equiv \ell^p(\mathbb{Z}_{\geq 1}, \mathbb{C}) \) the Banach space of complex valued sequences \( z = (z_n)_{n \geq 1} \) with finite norm
\[
\|z\|_{\ell^p_+} := \left( \sum_{n \geq 1} |z_n|^p \right)^{1/p} < \infty.
\]
Similarly, denote by \( \ell^\infty_+ \equiv \ell^\infty(\mathbb{Z}_{\geq 1}, \mathbb{C}) \) the Banach space of complex valued sequences with finite supremum norm \( \|z\|_{\ell^\infty_+} := \sup_{n \geq 1} |z_n| \). More generally, for \( 1 \leq p < \infty \) and \( m \in \mathbb{Z} \), we introduce the Banach space
\[
\ell^p_{\geq m} \equiv \ell^p(\mathbb{Z}_{\geq m}, \mathbb{C}), \quad Z_{\geq m} := \{ n \in \mathbb{Z} \mid n \geq m \},
\]
as well as the space \( \ell^p_c \equiv \ell^p(Z, \mathbb{C}) \), defined in a similar way. Furthermore, denote by \( h^\beta_+ \), \( \beta \in \mathbb{R} \), the Hilbert space of complex valued sequences \( z = (z_n)_{n \geq 1} \) with
\[
\|z\|_{h^\beta_+} := \left( \sum_{n \geq 1} |n|^{2\beta} |z_n|^2 \right)^{1/2} < \infty.
\]
In a similar way, we define the space of complex valued sequences \( h^\beta_{\geq m} \) for any \( m \in \mathbb{Z} \) and \( \beta \in \mathbb{R} \). Finally, for \( z \in \mathbb{C} \setminus (-\infty, 0] \), we denote by \( \sqrt{z} \) the principal branch of the square root of \( z \), defined by \( \Re(\sqrt{z}) > 0 \).

2 The Lax Operator

In this section we review results from [8] on the Lax operator \( L_u \) of the Benjamin-Ono equation with potentials \( u \) in \( H^s_c \) with \( s > -1/2 \), used in this paper. Throughout this section we assume that \( -1/2 < s \leq 0 \).
For \( u \in H^s_c \) with \(-1/2 < s \leq 0\), consider the pseudo-differential expression
\[
L_u := D - T_u
\]
where \( D = -i\partial_x\), \( T_u : H^{1+s}_+ \to H^s_+ \) is the Toeplitz operator with potential \( u \),
\[
T_u f := \Pi(uf), \quad f \in H^{1+s}_c,
\]
and \( \Pi \equiv \Pi^+ : H^s_c \to H^s_+ \) is the Szegő projector
\[
\Pi : H^s_c \to H^s_+, \quad \sum_{n \in \mathbb{Z}} \hat{v}(n)e^{inx} \to \sum_{n \geq 0} \hat{v}(n)e^{inx},
\]
onto the (positive) Hardy space \( H^s_+ \), introduced in (13). Note that when restricted to \( H^{1+s}_+ \), \( D \) coincides with the Fourier multiplier \( |\partial_x| \),
\[
|\partial_x| : H^{1+s}_c \to H^s_c, \quad \sum_{n \in \mathbb{Z}} \hat{f}(n) \to \sum_{n \in \mathbb{Z}} |n|\hat{f}(n),
\]
It follows from Lemma 1 in [8] that \( L_u \) defines an operator in \( H^s_+ \) with domain \( H^{1+s}_+ \) so that the map \( L_u : H^{1+s}_+ \to H^s_+ \) is bounded. The following result follows from [8, Theorem 1] and [7].

**Proposition 2.1.** For any \(-1/2 < s \leq 0\), there exists an open neighborhood \( W \equiv W^s \) of \( H^s_{r,0} \) in \( H^s_{c,0} \) so that for any \( u \in W \) the operator \( L_u \) is a closed operator in \( H^s_+ \) with domain \( H^{1+s}_+ \). The operator has a compact resolvent and all its eigenvalues are simple. When appropriately listed, \( \lambda_n = \lambda_n(u), n \geq 0 \), satisfy \( \text{Re}(\lambda_n) < \text{Re}(\lambda_{n+1}) \) for any \( n \geq 0 \) and \( |\lambda_n - n| \to 0 \) as \( n \to \infty \). For \( u \in H^s_{r,0} \) the eigenvalues are real valued and \( \gamma_n(u) := \lambda_n(u) - \lambda_{n-1}(u) - 1 \geq 0 \) for \( n \geq 1 \).

For \( u \) in an open neighborhood of zero in \( H^s_{r,0} \), \(-1/2 < s \leq 0\), the proposition above can be specified as follows (see [8, Corollary 2, Proposition 1]). For \( \varrho > 0 \) consider the sets in \( \mathbb{C} \),
\[
\text{Vert}_n(\varrho) := \{ \lambda \in \mathbb{C} \mid |\lambda - n| \geq \varrho, |\text{Re}(\lambda) - n| \leq 1/2 \}, \quad n \geq 0,
\]
\[
D_n(\varrho) := \{ \lambda \in \mathbb{C} \mid |\lambda - n| < \varrho \}, \quad n \geq 0,
\]
and let \( \partial D_n(\varrho) \) be the counterclockwise oriented boundary of \( D_n(\varrho) \) in \( \mathbb{C} \). Denote by \( \text{Vert}_n^0(\varrho) \) the interior of \( \text{Vert}_n(\varrho) \) in \( \mathbb{C} \).

**Proposition 2.2.** For any \(-1/2 < s \leq 0\), there exists an open neighborhood \( U \equiv U^s \) of zero in \( H^s_{r,0} \) so that for any \( u \in U \) the operator \( L_u \) is a closed operator in \( H^s_+ \) with domain \( H^{1+s}_+ \). The operator has a compact resolvent and all its eigenvalues are simple. When listed appropriately, they satisfy
\[
\lambda_n \in D_n(1/4), \quad n \geq 0.
\]
Moreover, for any \( n \geq 0 \) the resolvent map
\[
U \times \text{Vert}_n^0 (1/4) \to \mathcal{L}(H^s_+, H^{1+s}_+), \quad (u, \lambda) \mapsto (L_u - \lambda)^{-1},
\]
is analytic. In addition, \( (L_u - \lambda)^{-1} = (D - \lambda)^{-1} (I - T_u (D - \lambda)^{-1})^{-1} \) and for any \( n \geq 0 \), the Neumann series
\[
(I - T_u (D - \lambda)^{-1})^{-1} = \sum_{m \geq 0} [T_u (D - \lambda)^{-1}]^m
\]
converges in \( \mathcal{L}(H^s_+) \) absolutely and uniformly for \((u, \lambda) \in U \times \text{Vert}_n^0 (1/4)\).

**Remark 2.1.** By [8, Corollary 2] the neighborhood \( U \) in Proposition 2.2 can be chosen so that for any \( n \geq 0 \),
\[
\left\| T_u (D - \lambda)^{-1} \right\|_{\mathcal{L}(H^s_+^n)} < 1/2, \quad \forall (u, \lambda) \in U \times \text{Vert}_n^0 (1/4),
\]
where \( H^s_+^n \) stands for the space \( H^s_+ \) equipped with the equivalent “shifted” norm
\[
\left\| f \right\|_{s,n} := \left( \sum_{k \geq 0} (n - k)^{2s} |\hat{f}(k)|^2 \right)^{1/2} \quad \text{(cf. [8, Lemma 3]).}
\]
In particular, for any \( n \geq 0 \) the series in (18) converges uniformly on \( U \times \text{Vert}_n^0 (1/4) \) with respect to an operator norm which is equivalent to the operator norm in \( \mathcal{L}(H^s_+) \).

**Remark 2.2.** For any \( u \in H^s_{+,0}, -1/2 < s \leq 0 \), an analogue of Proposition 2.2 holds in an open neighborhood \( U_u \) of \( u \) in \( H^s_c \) – see [8, Section 5] for details.

Given any \( -1/2 < s \leq 0 \), Proposition 2.2 implies that for any \( u \in U \) and \( n \geq 0 \), the Riesz projector
\[
P_n(u) := -\frac{1}{2\pi i} \oint_{\partial D_n} (L_u - \lambda)^{-1} d\lambda \in \mathcal{L}(H^s_+, H^{1+s}_+), \quad D_n := D_n(1/3),
\]
is well defined and that the map
\[
P_n : U \to \mathcal{L}(H^s_+, H^{1+s}_+), \quad u \mapsto P_n(u).
\]
is analytic.

## 3 Analytic extension of the pre-Birkhoff map

In this section we introduce the pre-Birkhoff map and study its properties. Throughout this section, we assume that \( s > -1/2 \) and define
\[
\sigma \equiv \sigma(s) := \min(s, 0).
\]

It follows from Proposition 2.2 (cf. (20)) that for any \( s > -1/2 \), there exists an open neighborhood \( U^\sigma \) of zero in \( H^s_{+,0} \) so that for any \( u \in U^\sigma \) and \( n \geq 0 \) the Riesz projector
\[
P_n(u) = -\frac{1}{2\pi i} \oint_{\partial D_n} (L_u - \lambda)^{-1} d\lambda \in \mathcal{L}(H^s_+, H^{1+s}_+),
\]

is analytic.
is well defined and the map $U \to \mathcal{L}(H^s_\sigma, H^{1+\sigma}_1)$, $u \mapsto P_n(u)$, is analytic. Here $\partial D_n$ is the counterclockwise oriented boundary of $D_n \equiv D_n(1/3)$ (see (17)). Hence, for any $n \geq 0$ the map

$$U^\sigma \to H^{1+\sigma}_1, \quad u \mapsto h_n(u),$$

is analytic, where

$$h_n(u) := P_n(u) e_n \in H^{1+\sigma}_1, \quad e_n := e^{inx}.$$ (23)

The main objective in this section is to prove the following

**Proposition 3.1.** For any $s > -1/2$ there exists an open neighborhood $U^s$ of zero in $H^s_{c,0}$ so that the following holds:

(i) The map,

$$\Psi : U^s \to \mathfrak{h}^{1+s}_1, \quad u \mapsto (\langle 1, h_n(u) \rangle)_{n \geq 1},$$ (24)

is analytic. We refer to $\Psi$ as the pre-Birkhoff map (near zero).

(ii) For any $n \geq 0$, the map $U^s \to \mathbb{C}$, $u \mapsto \langle h_n(u) | e_n \rangle$, is analytic and

$$|\langle h_n(u) | e_n \rangle| \geq 1/2, \quad \forall u \in U^s.$$ (25)

**Proof of Proposition 3.1.** Take $s > -1/2$ and let $U^\sigma$ be an open neighborhood of zero in $H^s_{c,0}$ so that the statement of Proposition 2.2 holds. Then, by the discussion above, the map (22) is analytic for any $n \geq 0$. In particular, the components $\langle 1, h_n(u) \rangle$, $n \geq 1$, of the map (24) are analytic on $U^\sigma$. Hence, item (i) will follow from [11, Theorem A.5] once we prove that the restriction of the map (24) to a sufficiently small neighborhood $U^s \subseteq U^\sigma \cap H^s_{c,0}$ of zero in $H^s_{c,0}$ is bounded. Let us show this. It follows from Proposition 2.2 that for any $n \geq 1$ and for any $u \in U^\sigma$ we have

$$\Psi_n(u) := \langle 1, h_n(u) \rangle = -\frac{1}{2\pi i} \oint_{\partial D_n} \langle (L_u - \lambda)^{-1} e_n, 1 \rangle d\lambda$$

$$= \frac{1}{2\pi i} \sum_{m \geq 1} \oint_{\partial D_n} \langle [T_u(D - \lambda)^{-1}]^m e_n, 1 \rangle d\lambda$$

$$= \Psi_n^{(1)}(u) + \sum_{m \geq 1} \Psi_n^{(m+1)}(u)$$ (25)

where

$$\Psi_n^{(1)}(u) := -\frac{\hat{u}(-n)}{n},$$ (26)

$$\Psi_n^{(m+1)}(u) := \sum_{k_1 \geq 0, \ 1 \leq i \leq m} \Psi_{n,u}(k_1, ..., k_m), \quad m \geq 1,$$ (27)

and

$$\Psi_{n,u}(k_1, ..., k_m) := \frac{1}{2\pi i} \oint_{\partial D_n} \frac{\hat{u}(-k_m) \hat{u}(k_m - k_{m-1}) \cdots \hat{u}(k_1 - n)}{n - \lambda} d\lambda.$$ (28)
By Proposition 2.2 and Remark 2.1, the series in (25) and (27) converge absolutely and uniformly on $U^\sigma$. Moreover, by Remark 2.1, for any $n \geq 1$ and for any $m \geq 0$, $\Psi_n^{(m+1)} : H^\sigma_+ \to \mathbb{C}$ is a bounded polynomial map of order $m + 1$. Hence, for any $n \geq 1$ the series (25) is the Taylor’s expansion of $\Psi_n : U^\sigma \to \mathbb{C}$ at zero $u = 0$. For $m = 1$, one obtains from (27), (28), and Cauchy’s formula that for any $n \geq 1$,
\[
\Psi_n^{(2)}(u) = -\frac{1}{n} \sum_{k_1 \geq 0, k_1 \neq n} \frac{\hat{u}(-k_1) \hat{u}(k_1 - n)}{k_1 - n} = -\frac{1}{n} \sum_{l_1 \geq -n, l_1 \neq 0} \hat{u}(-n - l_1) \frac{\hat{u}(l_1)}{l_1}.
\]
Let us now consider the general case $m \geq 1$. By passing to the variable $\mu := \lambda - n$ in the contour integral (28) and then setting $l_j := k_j - n$, $1 \leq j \leq m$, we obtain from (27) and (28) that
\[
\Psi_n^{(m+1)}(u) = \sum_{\substack{l_j \geq -n \\|l_j \leq m}} H_{u,n}(l_1, \ldots, l_m), \quad m \geq 2,
\]
where $H_{u,n}(l_1, \ldots, l_m)$ is given by
\[
-\frac{1}{2\pi i} \oint_{\partial D_0} \frac{1}{n + \mu} \frac{\hat{u}(-n - l_m) \hat{u}(l_m - l_{m-1})}{l_m - \mu} \cdots \frac{\hat{u}(l_2 - l_1)}{l_1 - \mu} \hat{u}(l_1) \frac{d\mu}{\mu}. \tag{30}
\]
Note that the latter integral and hence $H_{u,n}$ is defined for any $l_1, \ldots, l_m$ in $\mathbb{Z}$.

To estimate the term $H_{u,n}(l_1, \ldots, l_m)$, note that for any $\mu \in \partial D_0$, $n \geq 1$, and $l_1, \ldots, l_m \in \mathbb{Z}$
\[
|n + \mu| \geq \frac{n}{2}, \quad |l_j - \mu| \geq \frac{|l_j| + 1}{5}, \quad 1 \leq j \leq m, \tag{31}
\]
implying that
\[
|H_{u,n}(l_1, \ldots, l_m)| \leq \frac{2}{5^n} \sum_{l_j \in \mathbb{Z}} \frac{|\hat{u}(-n - l_m)| |\hat{u}(l_m - l_{m-1})| \cdots |\hat{u}(l_2 - l_1)| |\hat{u}(l_1)|}{|l_m| + 1 |l_{m-1}| + 1 \cdots |l_1| + 1} \tag{32}
\]
By Lemma 3.1 below, it then follows that for a $y \in U^\sigma \cap H^s_{c,0}$,
\[
\sum_{n \geq 1} |n|^{2(1+s)} \left( \sum_{l_j \geq -n \\|l_j \leq m} |H_{u,n}(l_1, \ldots, l_m)|^2 \right) \leq (5^{m+1})^2 \sum_{n \geq 1} |n|^{2s} \left( \sum_{l_j \in \mathbb{Z}} \frac{|\hat{u}(-n - l_m)| |\hat{u}(l_m - l_{m-1})| \cdots |\hat{u}(l_2 - l_1)| |\hat{u}(l_1)|}{|l_m| + 1 |l_{m-1}| + 1 \cdots |l_1| + 1} \right)^2 \leq \left( 5^{m+1} C_s^{m+1} \|u\|_s^{m+1} \right)^2 \leq \left( 5C_s \|u\|_s \right)^{2(m+1)} \tag{33}
\]
where $C_s \geq 1$ is the constant given by Lemma 3.1. Hence, by shrinking $U^\sigma \cap H^s_{c,0}$ to a (bounded) neighborhood $U^s \subseteq H^s_{c,0}$, the latter estimate implies that
\[
\|\Psi(u)\|_{H^{1+s}_c} \leq \sum_{m \geq 0} \|\Psi^{(m+1)}(u)\|_{H^{1+s}_c} \leq 1, \quad \forall u \in U^s.
\]
This proves item (i). The proof of item (ii) is similar and hence omitted. □
The following estimate follows directly from [8, Lemma 1].

**Lemma 3.1.** For any \( z = (z_n)_{n \in \mathbb{Z}} \in h_s^\kappa \) with \( s > -1/2 \), the linear operator

\[
Q(z) : h_s^\kappa \to h_s^\kappa, \quad y = (y_k)_{k \in \mathbb{Z}} \mapsto Q(z)[y] := \left( \sum_{k \in \mathbb{Z}} \frac{z_{k-n}}{|k|+1} y_k \right)_{n \in \mathbb{Z}},
\]

is well defined and there exists a constant \( C_s \geq 1 \) so that

\[
\| Q(z) y \|_s \leq C_s \| z \|_s \| y \|_s, \quad \forall z, y \in h_s^\kappa.
\]

### 4 Analytic extension of the Birkhoff map in a neighborhood of zero

The goal of this section is to extend for any \( s > -1/2 \) the Birkhoff map, defined on \( H_{r,0}^s \) (cf. [6] \( s = 0 \), [7] \( s > -1/2 \)), to an analytic map on an open neighborhood of zero in \( H_{r,0}^s \).

Recall from [7] that for any \( s > -1/2 \), the Birkhoff map is given by

\[
\Phi : H_{r,0}^s \to h_{r+s}^+, \quad u \mapsto (\Phi_n(u))_{n \geq 1}, \quad \Phi_n(u) \equiv \zeta_n(u) := \frac{\langle 1|f_n(u) \rangle}{\|f_n(u)\|}, \quad (32)
\]

where the sequence space \( h_{r+s}^+ \) is defined in (14), the eigenfunctions \( f_n \equiv f_n(u) \), \( n \geq 0 \), of the Lax operator \( L_u \), corresponding to the eigenvalues \( \lambda_n \equiv \lambda_n(u) \), are normalized so that \( \|f_n\| = 1 \) and (cf. [6, Definition 2.1], [7, Lemma 6]),

\[
\langle 1|f_0 \rangle > 0, \quad \langle e^{iz} f_{n-1}|f_n \rangle > 0, \quad \forall n \geq 1,
\]

(33)

and the norming constants \( \kappa_n \equiv \kappa_n(u) > 0, n \geq 0 \), are given by (cf. [6, Corollary 3.1], [7, (29)])

\[
\kappa_0 := \prod_{k \geq 1} \left( 1 - \frac{\gamma_k}{\lambda_k - \lambda_0} \right), \quad \kappa_n := \frac{1}{\lambda_n - \lambda_0} \prod_{n \neq k \geq 1} \left( 1 - \frac{\gamma_k}{\lambda_k - \lambda_n} \right), \quad n \geq 1, \quad (34)
\]

with \( \gamma_k \equiv \gamma_k(u) \) defined as in Proposition 2.1. To construct the analytic extension of \( \Phi \), we express the normalisation conditions (33) of the eigenfunctions \( (f_n)_{n \geq 0} \) in terms of \( (\kappa_n)_{n \geq 0} \), and the norming constants \( \mu_n \equiv \mu_n(u) > 0, n \geq 1 \), defined for \( u \in H_{r,0}^s \), \( s > -1/2 \), by (cf. [6, Remark 4.1], [8, formula (28)]) \( s = 0 \), [7, Section 3] \((-1/2 < s < 0))

\[
\mu_n := \left( 1 - \frac{\gamma_n}{\lambda_n - \lambda_0} \right) \prod_{n \neq k \geq 1} \left( 1 - \frac{\gamma_n}{\lambda_k - \lambda_n} \right) \frac{\gamma_k}{(\lambda_k - \lambda_{n-1})(\lambda_k - \lambda_n)}, \quad (35)
\]

For any \( u \in H_{r,0}^s \) and \( n \geq 0 \), the projection \( P_n \equiv P_n(u) \), introduced in (19), coincides with the orthogonal projection

\[
H_s^{\kappa} \to H_s^{r+1/2}, \quad f \mapsto \langle f|f_n \rangle f_n.
\]
Expressed in terms of \( P_n \), the normalisation conditions (33) of \( f_n \) read (cf. [6, Remark 4.1, Corollary 3.1] \( s = 0 \), [7, Lemma 6] \(-1/2 < s \leq 0\))

\[
f_n(u) = \frac{1}{\sqrt{\mu_n(u)}} P_n(Sf_{n-1}(u)), \quad n \geq 1, \quad (f_0(u), 1) = \sqrt{\kappa_0(u)},
\]

where \( S \) denotes the shift operator \( S : H_+^{1+s} \rightarrow H_+^{1+s}, u(x) \mapsto u(x)e^{ix} \). We note that for any \( u \in H_+^s \), \(-1/2 < s \leq 0\), the infinite products in (34) and (35) are well defined and converge absolutely (cf. [7, Section 3]).

**Remark 4.1.** Using that for any \( s > -1/2 \), the inclusion

\[
\mathbb{R}^{\frac{1}{s} + s} \rightarrow \mathbb{R}^{\frac{1}{s} + s}, \quad (z_n)_{n \geq 1} \mapsto ((\bar{z}_n)_{n \leq -1}, (z_n)_{n \geq 1}),
\]

is an \( \mathbb{R} \)-linear isomorphism\(^2\), it follows from [6, Theorem 1] \( s = 0 \) and [7, Theorem 6] \(-1/2 < s < 0\) and [7, Proposition 5, Appendix A] \( s > 0 \) that for any \( s > -1/2 \), the map

\[
H^s_{r,0} \rightarrow \mathbb{R}^{\frac{1}{s} + s}, \quad u \mapsto ((\Phi_n - u)_{n \leq -1}, (\Phi_n(u))_{n \geq 1})
\]

is a homeomorphism. For notational convenience, we denote it also by \( \Phi \) and also refer to it as Birkhoff map.

**Remark 4.2.** We record that for \( u \in H^s_{r,0}, s > -1/2, \kappa_n(u) > 0, n \geq 0, \) and \( \mu_n(u) > 0, n \geq 1 \). If \( u = 0 \) then \( f_n(u) = h_n(u) = e_n \) (cf. (23)) and \( \lambda_n(u) = n \) for any \( n \geq 0 \), implying that \( \gamma_n(u) = 0 \) for any \( n \geq 1, \kappa_0(u) = 1 \) and \( n\kappa_n(u) = 1, \mu_n(u) = 1 \) for any \( n \geq 1 \).

For any \(-1/2 < s \leq 0\), let \( U \equiv U^s \) be an open neighborhood of zero in \( H^s_{r,0} \), chosen so that the statement of Proposition 2.2 and Proposition 3.1 hold. In the course of our argument, we will shrink \( U \) several times, but continue to denote it by \( U \). It follows from (23) and Proposition 2.2 that for any \( n \geq 0 \) the map

\[
U \rightarrow H_+^{1+s}, \quad u \mapsto h_n(u), \tag{38}
\]

is analytic. In addition, \( h_n(u) \neq 0 \) for any \( n \geq 0 \) by Proposition 3.1 (ii). In particular, we obtain that

\[
P_n\left(Sh_{n-1}(u)\right) = \nu_n(u)h_n(u), \quad n \geq 1, \tag{39}
\]

where \( \nu_n : U \rightarrow \mathbb{C} \) is analytic. It follows from [8, Theorem 3] and Proposition 2.2 that for any \( n \geq 1 \), the infinite product in (35) converges absolutely for \( u \in U \). Hence \( \mu_n, n \geq 1 \), extend as analytic functions to the neighborhood \( U \) of zero in \( H^s_{r,0} \). By shrinking the neighborhood \( U \) if needed, we then obtain from [8, Proposition 5] that

\[
|\mu_n(u) - 1| < \frac{1}{2}, \quad u \in U, n \geq 1, \tag{40}
\]

\(^2\)We ignore the complex structure of \( \mathbb{R}^{\frac{1}{s} + s} \) and consider the space as a real Banach space.
implying that for any \( n \geq 1 \), the map
\[
\sqrt[n]{\mu_n} : U \to \mathbb{C}
\]
(41)
is well defined and analytic. By shrinking the neighborhood \( U \) once more if necessary, we obtain from [8, Corollary 6] that \( \kappa_n, n \geq 0 \), in (34) extend as analytic functions to the neighborhood \( U \). Similarly, by shrinking \( U \) further if necessary, we obtain from [8, Proposition 4] and the fact that \( \kappa_0(0) = 1 \) that for any \( u \in U \),
\[
|\kappa_0(u) - 1| < \frac{1}{2} \quad \text{and} \quad |n\kappa_n(u) - 1| < \frac{9}{10} \quad n \geq 1,
\]
and, by [8, Proposition 3], we have the following lemma.

**Lemma 4.1.** For any \(-1/2 < s \leq 0\) there exists an open neighborhood \( U^s \) of zero in \( H_{<0}^s \) so that the map
\[
\kappa : U^s \to \ell_+^\infty, \quad u \mapsto (\kappa_n(u))_{n \geq 1}, \quad \kappa_n(u) := 1/\sqrt[n]{n\kappa_n(u)},
\]
(42)
is analytic.

Proposition 2.2, (40), and (41), then allow us to extend \( f_n \) to a neighborhood of zero in \( H_{<0}^s \), \(-1/2 < s \leq 0\), so that the extension is analytic and the normalization conditions (36) are satisfied. We have the following

**Lemma 4.2.** For any \(-1/2 < s \leq 0\), there exists an open neighborhood \( U^s \) of zero in \( H_{<0}^s \) so that for any \( n \geq 0 \), the map \( f_n : U^s \cap H_{<0}^s \to H_{+}^{1+s} \) extends to an analytic map
\[
U^s \to H_{+}^{1+s}, \quad u \mapsto f_n(u).
\]
(43)

**Proof of Lemma 4.2.** Let \(-1/2 < s \leq 0\) and choose the neighborhood \( U \equiv U^s \) of zero in \( H_{<0}^s \) so that Proposition 2.2 and Proposition 3.1 hold and so that the properties, discussed above, are satisfied. First we extend the eigenfunction \( f_0 \) to \( U \). Since by (22), \( h_0 : U \to H_{+}^{1+s} \) is analytic and by Proposition 3.1, \( \langle h_0, 1 \rangle \) does not vanish on \( U \), the map
\[
U \to \mathbb{C}, \ u \mapsto a_0(u) := \frac{\sqrt[n]{\kappa_0(u)}}{\langle h_0(u), 1 \rangle},
\]
(44)
and hence \( U \to H_{+}^{1+s}, \ u \mapsto f_0(u) := a_0(u)h_0(u) \), are well defined and analytic. Using that for any \( n \geq 1 \), \( U \to \mathcal{L}(H_{+}^s, H_{+}^{1+s}), \ u \mapsto P_n(u) \) (cf. (20)) is analytic and \( \mu_n : U \to \mathbb{C} \) satisfies (40) and (41) one then concludes by induction that for any \( n \geq 1 \),
\[
U \to H_{+}^{1+s}, \ u \mapsto f_n(u) := \frac{1}{\sqrt[n]{\mu_n(u)}} P_n(Sf_{n-1}(u))
\]
is analytic as well. We thus have proved that for any \( u \in U \), \( f_n : U \to H_{+}^{1+s} \), \( n \geq 0 \), are analytic, satisfying \( L_n f_n(u) = \lambda_n(u) f_n(u) \) and the normalization conditions (36). \( \square \)
Let us now study for any \( u \in U \) and \( n \geq 0 \) the relation between \( f_n(u) \) and \( h_n(u) \) where \( U \equiv U^s \) is the neighborhood of zero in \( H^s_{1,0} \) of Lemma 4.2. Recall that by the proof of Lemma 4.2, \( f_0(u) = a_0(u) h_0(u) \) where \( a_0 : U \to \mathbb{C} \) is the analytic map given by (44). Let us now turn to the case \( n \geq 1 \). Since \( f_n(u) \) and \( h_n(u) \) belong to the one dimensional (complex) eigenspace of \( L_u \) corresponding to the simple eigenvalue \( \lambda_n(u) \) and both do not vanish we have that

\[
f_n(u) = a_n(u) h_n(u), \quad n \geq 1,
\]

where the map

\[
a_n : U \to \mathbb{C}
\]

is analytic. It now follows from (45), (39), and the normalization conditions (36), extended to complex valued potentials \( u \in U \) as explained above, that for any \( u \in U \) and \( n \geq 1 \),

\[
\sqrt{\mu_n} a_n h_n = \sqrt{\mu_n} f_n = P_n(S f_{n-1}) = P_n(S(a_{n-1} h_{n-1})) = a_{n-1} P_n(S h_{n-1}) = a_{n-1} \nu_n h_n.
\]

In view of (40), (45), and the fact that \( \langle h_0(u)|1 \rangle \neq 0 \) (cf. Proposition 3.1 (ii)), one then infers that

\[
a_n(u) = \frac{\nu_n(u)}{\sqrt{\mu_n(u)}} a_{n-1}(u), \quad \forall n \geq 1.
\]

Hence, for any \( u \in U \) and \( n \geq 1 \) we have that

\[
a_n(u) = a_0(u) \prod_{k=1}^{n} \frac{\nu_k(u)}{\sqrt{\mu_k(u)}} = a_0(u) \left( \prod_{k=1}^{n} \nu_k(u) \right) \left( \prod_{k=1}^{n} \frac{1}{\sqrt{\mu_k(u)}} \right).
\]

It follows from (39) that for any \( u \in U \) and \( n \geq 1 \),

\[
\nu_n(u) = \frac{\langle P_n(S h_{n-1}(u)|e_n) \rangle}{\langle h_n(u)|e_n \rangle}.
\]

Hence,

\[
\nu_n(u) = 1 + \frac{\delta_n(u)}{\alpha_n(u)}, \quad \alpha_n(u) := \langle P_n e_n|e_n \rangle
\]

where

\[
\delta_n(u) := \beta_n(u) - \alpha_n(u), \quad \beta_n(u) := \langle P_n S P_{n-1} e_{n-1}|e_n \rangle.
\]

Recall from Proposition 3.1 (ii) that

\[
|\alpha_n(u)| \geq 1/2
\]

for any \( u \in U \) and \( n \geq 1 \). In Section 5 we prove the following important proposition, which will be a key ingredient into the proof of Theorem 1.1.
Proposition 4.1. For any $-1/2 < s \leq 0$, there exists an open neighborhood $U^s$ of zero in $H^s_{c,0}$ so that the map

$$\delta : U^s \to \ell^1_+, \ u \mapsto (\delta_n(u))_{n \geq 1},$$

with $\delta_n$ defined in (50), is analytic and bounded.

Remark 4.3. By Remark 4.2, $\delta_n(0) = 0$ for any $n \geq 1$.

Proposition 4.1 allows us to prove the following lemma.

Lemma 4.3. For any $-1/2 < s \leq 0$ there exists an open neighborhood $U^s$ of zero in $H^s_{c,0}$ so that the map

$$a : U^s \to \ell\infty_+, \ u \mapsto (a_n(u))_{n \geq 1},$$

with $a_n$ defined by (45), is analytic.

Proof of Lemma 4.3. For any given $-1/2 < s \leq 0$, choose a neighborhood $U \equiv U^s$ of zero in $H^s_{c,0}$ so that the results discussed above (in particular Proposition 4.1) hold. First, note that by the analyticity of (46), any of the components of the map (53) is an analytic function of $u \in U$. Hence, the lemma will follow if we prove that the map (53) is bounded (see e.g. [11, Theorem A.3]). To this end, we prove that the two products appearing on the right side of (48) are bounded uniformly in $u \in U$ and $n \geq 1$. By shrinking the neighborhood $U$ if necessary, we can ensure from Proposition 4.1, Remark 4.3, and (51) that there exists $C > 0$ such that for any $u \in U$ and $n \geq 1$ we have that

$$\sum_{k=1}^{n} |\delta_k(u)| \leq C \quad \text{and} \quad \left| \frac{\delta_n(u)}{\alpha_n(u)} \right| \leq \frac{1}{2}.$$  

This together with (49) and (51) then implies that there exists $C > 0$ such that for any $u \in U$ and $n \geq 1$,

$$\sum_{k=1}^{n} \left| \log \nu_k(u) \right| = \sum_{k=1}^{n} \left| \log \left( 1 + \frac{\delta_k(u)}{\alpha_k(u)} \right) \right| \leq C,$$

where log $\lambda$ denotes the standard branch of the (natural) logarithm on $\mathbb{C} \setminus (-\infty, 0]$, defined by log $\lambda := \log |\lambda| + i \text{arg} \lambda$ and $-\pi < \text{arg} \lambda < \pi$. Hence, in view of (54), for any $u \in U$ and $n \geq 1$ we obtain

$$\left| \prod_{k=1}^{n} \nu_k(u) \right| = \left| \exp \left( \sum_{k=1}^{n} \log \nu_k(u) \right) \right| \leq \exp \left( \sum_{k=1}^{n} \left| \log \nu_k(u) \right| \right) \leq \exp C.$$  

Let us now prove the boundedness of the second product on the right side of (48). It follows from (40) that log $\mu_n(u) = \log \left( 1 - (1 - \mu_n(u)) \right)$ is well defined
for any $u \in U$ and $n \geq 1$. By shrinking the neighborhood $U$ if necessary, we obtain from [8, Theorem 3 and Remark 4] that there exists $C > 0$ such that for any $u \in U$ and $n \geq 1$ we have that

$$\sum_{k=1}^{n} |\log \mu_k(u)| \leq C.$$  

This implies that for any $u \in U$ and $n \geq 1$ we obtain

$$\prod_{k=1}^{n} \frac{1}{\sqrt[\mu_k(u)]} = \exp \left( -\frac{1}{2} \sum_{k=1}^{n} \log \mu_k(u) \right) \leq \exp \left( \frac{1}{2} \sum_{k=1}^{n} \left| \log \mu_k(u) \right| \right) \leq \exp \left( C/2 \right),$$

which completes the proof of the lemma.

As a consequence from Lemma 4.1, Proposition 4.1, and Lemma 4.3, we obtain the following more general statement.

**Corollary 4.1.** For any $s > -1/2$ there exists an open neighborhood $U^s$ of zero in $H^s_{c,0}$ such that the maps (42), (52), and (53) are well defined and analytic. The neighborhood $U^s$ can be chosen invariant with respect to the complex conjugation of functions.

**Proof of Corollary 4.1.** Take $s > -1/2$ and denote $\sigma \equiv \sigma(s) := \min(s,0)$. Then, it follows from Lemma 4.1, Proposition 4.1, and Lemma 4.3 that there exists an open neighborhood $U^\sigma$ of zero in $H^\sigma_{c,0}$ such that the maps (42), (52), and (53) (with $U^s$ replaced by $U^\sigma$) are analytic. Denote by $U^\sigma$ the open neighborhood $U^\sigma \cap H^s_{c,0}$ of zero in $H^s_{c,0}$. The analyticity of the maps (42), (52), and (53) then follow from the boundedness of the inclusion $U^\sigma \hookrightarrow U^\sigma$. Finally, by taking $U^\sigma$ above to be an open ball in $H^\sigma_{c,0}$, centered at zero, we obtain the last statement in Corollary 4.1.

Our last step is to rewrite formula (37), which defines the Birkhoff map $\Phi : H^s_{r,0} \to H^{1+s}_{r,0}$ for real valued $u \in H^s_{r,0}$, $s > -1/2$, in a form that will allow us to extend $\Phi$ to an analytic map in a (complex) neighborhood $U^s$ of zero in $H^s_{c,0}$. To this end, we use the symmetries of the Lax operator $L_u$ established in Corollary A.2 in Appendix A and argue as follows: First, we note that if the map $F : U \to Y$, where $U \subseteq X$ is an open neighborhood and $X$ and $Y$ are (complex) Banach spaces, is analytic then so is the map

$$F^* : U \to Y, \quad x \mapsto F^*(x) := \overline{F(x)},$$

where $\overline{\cdot}$ denotes complex conjugation. With this in mind, we note that for any $u \in H^s_{r,0}$ with $s > -1/2$ and $n \geq 0$ we have that

$$\langle 1, F_n(u) \rangle = \langle 1, \overline{F_n(u)} \rangle.$$
For any $u \in H^s_{r,0}$ with $s > -1/2$, we obtain from Corollary A.2 that $\lambda_n(u) = \lambda_n(\overline{u})$, $n \geq 0$. This, together with the definition (34) of $\kappa_n$ implies that for any $u \in H^s_{r,0}$, $s > -1/2$,

$$\kappa_n(u) = \overline{\kappa_n(\overline{u})}, \quad n \geq 0. \quad (57)$$

We choose the neighborhood $U^s$ of zero in $H^s_{c,0}$, $s > -1/2$, so that Corollary 4.1 and Proposition 3.1 hold and so that it is invariant with respect to complex conjugation. It then follows from (32), (45), (56), and (57), that for any $u \in U^s \cap H^s_{r,0}$, $s > -1/2$, and $n \geq 1$,

$$\Phi_n(u) = \sqrt{n} \frac{a_n(\overline{u})}{\sqrt{n} \kappa_n(u)} \langle 1, h^*_n(\overline{u}) \rangle \quad (58)$$

and

$$\overline{\Phi_n(u)} = \sqrt{n} \frac{a_n(u)}{\sqrt{n} \kappa_n(u)} \langle 1, h^*_n(u) \rangle, \quad (59)$$

where we used the notation, introduced in (55). Hence, the Birkhoff map (37) satisfies for any $u \in U^s \cap H^s_{r,0}$, $s > -1/2$,

$$\Phi(u) = \left( (\Phi_{-n}(u))_{n \leq -1}, (\Phi_n(u))_{n \geq 1} \right) \in h^s_{r,0}. \quad (60)$$

Note that by the discussion above, the right hand sides of the identities in (58) and (59) are well defined and analytic (cf. Corollary 4.1) for any $u \in U^s$, $s > -1/2$, and $n \geq 1$. The main result of this section is the following proposition.

**Proposition 4.2.** For any $s > -1/2$, there exists an invariant with respect to the complex conjugation open neighborhood $U^s$ of zero in $H^s_{c,0}$ so that the right hand side of the identity in (60) is well defined for any $u \in U^s$ and the map

$$\Phi : U^s \to h^s_{c,0}, \quad u \mapsto \left( (\Phi_{-n}(u))_{n \leq -1}, (\Phi_n(u))_{n \geq 1} \right), \quad (61)$$

is analytic.

**Remark 4.4.** Since $H^s_{r,0}$ is a real space in the complex space $H^s_{c,0}$ and, similarly, $h^s_{r,0}$ is a real space in the complex space $h^s_{c,0}$, we conclude from Proposition 4.2 that the Birkhoff map (37) is real analytic on $U^s \cap H^s_{r,0}$.

**Proof of Proposition 4.2.** For a given $s > -1/2$ we choose the open neighborhood $U^s$ of zero in $H^s_{c,0}$ as in Corollary 4.1 and Proposition 3.1 and assume that $U^s$ is invariant with respect to the complex conjugation. Then, the map (61) is well defined on $U^s$ and takes values in $h^s_{c,0}$. The map (61) is analytic if the map

$$\Phi^{(2)} : U^s \to h^s_{c,0}, \quad u \mapsto (\Phi_n(u))_{n \geq 1},$$

is analytic.
with Φₙ(u) given by (58), is analytic. Let Λ₁/² denote the linear multiplier,

$$\Lambda^{1/2} : \mathfrak{h}_+^s \to \mathfrak{h}_+^{s - \frac{1}{2}}, \quad (x_n)_{n \geq 1} \mapsto \left(\sqrt{n} x_n\right)_{n \geq 1}.$$  

The map Φ(²) is then represented by the commutative diagram

$$\begin{array}{ccc}
U^s & \xrightarrow{\Phi^{(2)}} & \mathfrak{h}_+^{1+s} \\
\downarrow{(a^*, x^*, \Psi^*)} & & \downarrow{M} \\
\ell^\infty_+ \times \ell^\infty_+ \times \mathfrak{h}_+^{1+s} & \xrightarrow{\Lambda^{1/2}} & \mathfrak{h}_+^{1+s}
\end{array}$$

where

$$(a^*, x^*, \Psi^*) : U^s \to \ell^\infty_+ \times \ell^\infty_+ \times \mathfrak{h}_+^{1+s}, \quad u \mapsto (a^*(u), x(u), \Psi^*(u)), \quad (63)$$

$$M : \ell^\infty_+ \times \ell^\infty_+ \times \mathfrak{h}_+^{1+s} \to \mathfrak{h}_+^{1+s}, \quad (x_n)_{n \geq 1}, (y_n)_{n \geq 1}, (z_n)_{n \geq 1} \mapsto (x_n y_n z_n)_{n \geq 1}, \quad (64)$$

and $a : U^\sigma \to \ell^\infty_+, \ x : U^\sigma \to \ell^\infty_+$, and $\Psi : U^s \to \mathfrak{h}_+^{1+s}$ are given respectively in (53), (42), and (24). By Corollary 4.1 and Proposition 3.1 the map (63) is analytic, whereas the map (64) is analytic since it is a bounded (complex) trilinear map. This, together with the analyticity of the multiplier operator $\Lambda^{1/2}$ and the commutative diagram above implies that (62) is analytic. This completes the proof of the proposition. □

5 Analyticity of the delta map

In this section we prove Proposition 4.1 of Section 4, which plays a significant role in the proof of Proposition 4.2. Our proof of Proposition 4.1 is based on a vanishing lemma – see Lemma 5.1 below. Throughout this section we assume that $-1/2 < s \leq 0$.

First we make some preliminary considerations. Assume that $-1/2 < s \leq 0$ and let $U \equiv U^s$ be an open neighborhood of zero in $H^s_0$ chosen so that the statement of Proposition 2.2 holds. Then, the Lax operator $L_u = D - T_u$ is a closed operator in $H^s_0$ with domain $H^{1+s}_0$, has a compact resolvent and its spectrum consists of countably many simple eigenvalues, $\lambda_n \in D_n(1/4)$, $n \geq 0$. Recall that the $\delta$-map is defined on $U$ and for any $u \in U$, one has

$$\delta(u) = (\delta_n(u))_{n \geq 1}, \quad \delta_n(u) = \beta_n(u) - \alpha_n(u), \quad (65)$$

where $\beta_n(u) = \langle P_n S P_{n-1} e_{n-1} | e_n \rangle$ (cf. (50)) and $\alpha_n(u) = \langle P_n e_n | e_n \rangle$ (cf. (49)).

It follows from Proposition 2.2 and Remark 2.1 that the Neumann series expansion of the resolvent of $L_u$,

$$(L_u - \lambda)^{-1} = \sum_{m \geq 0} (D - \lambda)^{-1} [T_u (D - \lambda)^{-1}]^m, \quad (66)$$

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converges uniformly in $\mathcal{L}(H^1)$ for $(u, \lambda) \in U \times \bigcup_{n \geq 0} \text{Vert}_0(1/4)$. By the definition of $P_{n-1}$ in (20), this implies that $\beta_n \equiv \beta_n(u)$, $n \geq 1$, satisfies

$$\beta_n = -\sum_{m \geq 0} \frac{1}{2\pi i} \oint_{\partial D_{n-1}} \langle P_n S(D - \lambda)^{-1} [T_u(D - \lambda)^{-1}]^m e_{n-1} | e_n \rangle \, d\lambda,$$  \hspace{1cm} (67)

where the series converges absolutely and where $D_{n-1} = D_{n-1}(1/3)$ and $\partial D_{n-1}$ is counterclockwise oriented. The first term in the latter series is

$$-\frac{1}{2\pi i} \oint_{\partial D_{n-1}} \langle P_n S(D - \lambda)^{-1} e_{n-1} | e_n \rangle \, d\lambda = \langle P_n S e_{n-1} | e_n \rangle \frac{1}{2\pi i} \oint_{\partial D_{n-1}} \frac{d\lambda}{\lambda - (n-1)} = \langle P_n e_n | e_n \rangle = \alpha_n,$$  \hspace{1cm} (68)

where we used that $S e_{n-1} = e_n$. By combining (65) with (67) and (68) we obtain that for $n \geq 1$,

$$\delta_n \equiv \delta_n(u) = -\sum_{m \geq 1} \frac{1}{2\pi i} \oint_{\partial D_{n-1}} \langle P_n S(D - \lambda)^{-1} [T_u(D - \lambda)^{-1}]^m e_{n-1} | e_n \rangle \, d\lambda$$

$$= -\sum_{m \geq 1} \sum_{k_j \geq 0 \atop 1 \leq j \leq m} C_u(k_1, \ldots, k_m) \langle P_n S e_{k_m} | e_n \rangle$$  \hspace{1cm} (69)

where

$$C_u(k_1, \ldots, k_m) := \frac{1}{2\pi i} \oint_{\partial D_{n-1}} \frac{\hat{u}(k_1 - (n-1)) \hat{u}(k_2 - k_1) \cdots \hat{u}(k_m - k_{m-1})}{(n-1) - \lambda} \frac{\hat{u}(1)}{k_1 - \lambda} \cdots \frac{\hat{u}(k_{m-1})}{k_{m-1} - \lambda} \frac{\hat{u}(m)}{k_m - \lambda} \, d\lambda.$$

By passing to the variable $\mu := \lambda - (n-1)$ in the contour integral above and then setting $l_j := k_j - (n-1)$, $1 \leq j \leq m$, we obtain from (69) that

$$\delta_n = \sum_{m \geq 1} \sum_{l_j \geq -n+1 \atop 1 \leq j \leq m} A(l_1, \ldots, l_m) B_u(l_1, \ldots, l_m) \langle P_n e_{l_m+n} | e_n \rangle$$  \hspace{1cm} (70)

where

$$A(l_1, \ldots, l_m) := \frac{1}{2\pi i} \oint_{\partial D_l} \frac{1}{\mu} \prod_{j=1}^m \frac{1}{l_j - \mu} \, d\mu,$$  \hspace{1cm} (71)

and

$$B_u(l_1, \ldots, l_m) := \begin{cases} \hat{u}(l_1) & \text{if } m = 1, \\ \hat{u}(l_1) \hat{u}(l_2 - l_1) \cdots \hat{u}(l_m - l_{m-1}) & \text{if } m \geq 2. \end{cases}$$  \hspace{1cm} (72)

Consider the term $\langle P_n e_{l_m+n} | e_n \rangle$ that appears in (70). It follows from (18) and the formula for the Riesz projector that

$$\langle P_n e_{l_m+n} | e_n \rangle = -\sum_{r \geq 0} \frac{1}{2\pi i} \oint_{\partial D_n} \langle (D - \lambda)^{-1} [T_u(D - \lambda)^{-1}]^r e_{l_m+n} | e_n \rangle \, d\lambda.$$
For the first term in the latter series we have
\[
-\frac{1}{2\pi i} \oint_{\partial D_n} ((D - \lambda)^{-1} e_{l_m+n} | e_n) d\lambda = \langle e_{l_m+n} | e_n \rangle \frac{1}{2\pi i} \oint_{\partial D_n} \frac{d\lambda}{\lambda - (l_m + n)} = \delta_{l_m0}.
\] (73)

Hence
\[
\langle P_n e_{l_m+n} | e_n \rangle = \delta_{l_m0} + \sum_{r \geq 1} \sum_{l_j \geq 0} \sum_{1 \leq j \leq r-1} C_u(k_1, ..., k_{r-1})
\] (74)

where
\[
C_u(k_1, ..., k_{r-1}) := \begin{cases} 
-\frac{1}{2\pi i} \oint_{\partial D_n} \frac{\hat{u}(-l_m)}{\lambda} d\lambda & \text{if } r = 1, \\
-\frac{1}{2\pi i} \oint_{\partial D_n} \frac{\hat{u}(k_1 - (l_m+n)) \hat{u}(k_2-k_1) \cdots \hat{u}(n-k_{r-1})}{(l_m+n) - \lambda} \frac{d\lambda}{\lambda - n-\lambda} & \text{if } r \geq 2.
\end{cases}
\]

By passing to the variable \( \mu := \lambda - n \) in the contour integral and then setting \( l_{m+j} := k_j - n, 1 \leq j \leq r-1, \) we obtain from (74) that
\[
\langle P_n e_{l_m+n} | e_n \rangle = \delta_{l_m0} + \sum_{r \geq 1} \sum_{l_j \geq 0} \sum_{m+r \leq l_j \leq m+r-1} A(l_m, ..., l_{m+1}) B'_u(l_m, ..., l_{m+r-1})
\]
\[
= \delta_{l_m0} + \sum_{r \geq 0} \sum_{l_j \geq 0} \sum_{m+r \leq l_j \leq m+r} A(l_m, ..., l_{m+r}) B'_u(l_m, ..., l_{m+r})
\] (75)

where
\[
B'_u(l_m, ..., l_{m+r}) := \begin{cases} 
\hat{u}(l_{m+1} - l_m) \cdots \hat{u}(l_{m+r} - l_{m+r-1}) \hat{u}(-l_m) & \text{if } r \geq 1, \\
\hat{u}(-l_m) & \text{if } r = 0.
\end{cases}
\]

In view of the expansion (75), we split \( \langle P_n e_{l_m+n} | e_n \rangle \), with one of the terms being \( \delta_{l_m0} \), and then split \( \delta_n \) accordingly. To this end, it turns out to be useful to introduce
\[
E_u(l_1, ..., l_d) := \begin{cases} 
\hat{u}(l_1) \hat{u}(-l_1) & \text{if } d = 1, \\
\hat{u}(l_1) \hat{u}(l_2 - l_1) \cdots \hat{u}(l_d - l_{d-1}) \hat{u}(-l_d) & \text{if } d \geq 2.
\end{cases}
\] (77)

By (72) and (76) one has
\[
B_u(l_1, ..., l_m) B'_u(l_m, ..., l_{m+r}) = E_u(l_1, ..., l_{m+r}).
\]

By (70) and (75), we then have
\[
\delta_n = \delta_n^{(1)} + \delta_n^{(2)}
\] (78)

where
\[
\delta_n^{(2)} := \sum_{m \geq 1} \sum_{l_j \geq m+1} \sum_{1 \leq j \leq m+1} A(l_1, ..., l_{m+1}, 0) B_u(l_1, ..., l_{m+1}, 0)
\]
\[
= -\sum_{d \geq 1} \sum_{l_j \geq m+1} \sum_{1 \leq j \leq d} A(l_1, ..., l_d, 0) E_u(l_1, ..., l_d)
\] (79)
and

$$
\delta_n^{(1)} := \sum_{m \geq 1} \sum_{r \geq 0} \sum_{l_j \geq n+1 \atop 1 \leq j \leq m} \sum_{l_j \geq n \atop m+1 \leq j \leq m+r} A(l_1, ..., l_m)A(l_m, ..., l_{m+r}) E_u(l_1, ..., l_{m+r}) = \sum_{d \geq 1} \sum_{l_j \geq n+1 \atop 1 \leq j \leq m} \sum_{l_j \geq n \atop m+1 \leq j \leq d} A(l_1, ..., l_m)A(l_m, ..., l_d) E_u(l_1, ..., l_d).
$$

Since the range of \( l_j, 1 \leq j \leq m \), and the one of \( l_j, m + 1 \leq j \leq d \), are different, we split the latter sum into two parts

$$
\delta_n^{(1)} = \sum_{d \geq 1} \sum_{l_j \geq n+1 \atop 1 \leq j \leq d} \left( \sum_{1 \leq m \leq d} A(l_1, ..., l_m)A(l_m, ..., l_d) \right) E_u(l_1, ..., l_d)
+ \sum_{d \geq 2} \sum_{l_j \geq n+1 \atop 1 \leq j \leq d} \sum_{m+1 \leq k \leq d} \sum_{l_j \geq n \atop m+1 \leq j \leq k} \sum_{l_j \geq n, k \leq d} A(l_1, ..., l_m)A(l_m, ..., l_d) E_u(l_1, ..., l_d).
$$

By combining this with (79) we conclude from (78) that

$$
\delta_n = \sum_{d \geq 1} \sum_{l_j \geq n+1 \atop 1 \leq j \leq d} D(l_1, ..., l_d) E_u(l_1, ..., l_d) + R_n(u)
$$

where by (71),

$$
D(l_1, ..., l_d) := \left( \sum_{1 \leq m \leq d} A(l_1, ..., l_m)A(l_m, ..., l_d) \right) - A(l_1, ..., l_d, 0)
= \sum_{1 \leq m \leq d} \left( \frac{1}{2\pi i} \oint_{\partial D_0} \frac{1}{\mu} \prod_{j=1}^{m} \frac{1}{l_j - \mu} d\mu \right) \left( \frac{1}{2\pi i} \oint_{\partial D_0} \frac{1}{\mu} \prod_{j=m+1}^{d} \frac{1}{l_j - \mu} d\mu \right)
- \frac{1}{2\pi i} \oint_{\partial D_0} \frac{1}{\mu} \prod_{j=1}^{d} \frac{1}{l_j - \mu} d\mu,
$$

and where the remainder \( R_n(u) \) equals

$$
\sum_{d \geq 2} \sum_{l_j \geq n+1 \atop 1 \leq j \leq d} \sum_{m+1 \leq k \leq d} \sum_{l_j \geq n \atop m+1 \leq j \leq k} \sum_{l_j \geq n, k \leq d} A(l_1, ..., l_m)A(l_m, ..., l_d) E_u(l_1, ..., l_d).
$$

We have the following important Vanishing Lemma.

**Lemma 5.1.** \( D(l_1, ..., l_d) = 0 \) for any \( l_1, ..., l_d \in \mathbb{Z}, \ d \geq 1 \).

**Proof of Lemma 5.1.** Let \( d \geq 1 \) be given. We show a slightly stronger result than the one claimed by the lemma. Note that \( D(l_1, ..., l_d) \) is well defined for any \( (l_1, ..., l_d) \in (\mathbb{C} \setminus \overline{D_0}) \cup \{0\} \). We prove that

$$
D(l_1, ..., l_d) = 0, \ \forall (l_1, ..., l_d) \in (\mathbb{C} \setminus \overline{D_0}) \cup \{0\}.
$$

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For any given \((\ell_1, \ldots, \ell_d) \in (\mathbb{C} \setminus \overline{D_0}) \cup \{0\}\), set

\[
J := \{ j \in \{1, \ldots, d\} \mid \ell_j = 0\}, \quad K := \{ k \in \{1, \ldots, d\} \mid \ell_k \neq 0\}.
\]

In the case where \(K = \emptyset, I := \sum_{1 \leq m \leq d} A(l_1, \ldots, l_m) A(l_m, \ldots, l_d)\) and \(II := A^{(2)}(l_1, \ldots, l_d)\) both vanish by the residue theorem and hence by the definition (81), the claimed identity (83) holds. For the remaining part of the proof we assume that \(K \neq \emptyset\). Notice that the terms \(I\) and \(II\) are holomorphic functions of the variable \((\ell_k)_{k \in K} \in (\mathbb{C} \setminus \overline{D_0})^K\), so we may assume that the \(\ell_k, k \in K\), are pairwise distinct complex numbers in \(\mathbb{C} \setminus \overline{D_0}\). By Cauchy's theorem and Leibniz's rule, \(A^{(2)}(\ell_1, \ldots, \ell_d)\) can be computed as

\[
(-1)^{|J|} \left( \prod_{k \in K} \frac{1}{\ell_k - \mu} \right)_{|\mu| = 0} = (-1)^{|J|} \sum_{q \in Q(K, |J| + 1)} \prod_{k \in K} \frac{1}{\ell_k^{q_k + 1}},
\]

where for any integer \(p \geq 1\), we set

\[
Q(K, p) := \left\{ q = (q_k)_{k \in K} \in \mathbb{Z}_{\geq 0}^K \mid \sum_{k \in K} q_k = p \right\}.
\]

Given \(m \in \{1, \ldots, d\}\), define

\[
J_m := J \cap \{1, m\}, \quad J'_m := J \cap [m, d], \quad K_m := K \cap \{1, m\}, \quad K'_m := K \cap [m, d].
\]

One then obtains, in a similar way,

\[
A(\ell_1, \ldots, \ell_m) = (-1)^{|J_m|} \sum_{r \in Q(K_m, |J_m|)} \prod_{k \in K_m} \frac{1}{\ell_k^{r_k + 1}},
\]

\[
A(\ell_m, \ldots, \ell_d) = (-1)^{|J'_m|} \sum_{s \in Q(K'_m, |J'_m|)} \prod_{k \in K'_m} \frac{1}{\ell_k^{s_k + 1}}.
\]

Since

\[
|J_m| + |J'_m| = \begin{cases} |J| + 1 & \text{if } m \in J \\ |J| & \text{if } m \in K \end{cases}
\]

one infers that \(\sum_{m=1}^d A(\ell_1, \ldots, \ell_m) A(\ell_m, \ldots, \ell_d)\) equals

\[
\begin{align*}
(-1)^{|J|} \sum_{m \in K} \left( \sum_{r \in Q(K_m, |J_m|)} \prod_{k \in K_m} \frac{1}{\ell_k^{r_k + 1}} \right) \left( \sum_{s \in Q(K'_m, |J'_m|)} \prod_{k \in K'_m} \frac{1}{\ell_k^{s_k + 1}} \right) \\
(-1)^{|J|} \sum_{m \in J} \left( \sum_{r \in Q(K_m, |J_m|)} \prod_{k \in K_m} \frac{1}{\ell_k^{r_k + 1}} \right) \left( \sum_{s \in Q(K'_m, |J'_m|)} \prod_{k \in K'_m} \frac{1}{\ell_k^{s_k + 1}} \right).
\end{align*}
\]

For any \(m \in \{1, \ldots, d\}\), \(r \in Q(K_m, |J_m|)\), and \(s \in Q(K'_m, |J'_m|)\), the corresponding term in the latter expression is of the form \(\prod_{k \in K} \frac{1}{\ell_k^{q_k + 1}}\), where

\[
q_k := r_k \ (k < m), \quad q_k := s_k \ (k > m),
\]

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and in case \( m \in K \), \( q_m := r_m + s_m + 1 \). It then follows from (84) that \((q_k)_{k \in K}\) belongs to \( Q(K, |J| + 1) \). In order to describe \( \sum_{m=1}^d A(\ell_1, \ldots, \ell_m) A(\ell_m, \ldots, \ell_d) \) in more detail, we define for \( q \) in \( Q(K, |J| + 1) \),

\[
J_{ad}(q) := \left\{ m \in J \left| \sum_{k \in K_m} q_k = |J_m| \right. \right\},
\]

\[
K_{ad}(q) := \left\{ m \in K \left| \sum_{k \in K_m \setminus \{ m \}} q_k \leq |J_m|, \sum_{k \in K_m \setminus \{ m \}} q_k \leq |J'_m| \right. \right\}.
\]

We remark that \( J_{ad}(q) \) might be empty and that by (84), for any \( m \in J_{ad}(q) \) and \( q \in Q(K, |J| + 1) \), the identity \( \sum_{k \in K_m} q_k = |J'_m| \) is automatically satisfied.

In view of these definitions, one has

\[
\sum_{m=1}^d A(\ell_1, \ldots, \ell_m) A(\ell_m, \ldots, \ell_d)
= (-1)^{|J|} \sum_{q \in Q(K, |J| + 1)} (|K_{ad}(q)| - |J_{ad}(q)|) \prod_{k \in K} \frac{1}{q_{k+1}}
\]

and (83) follows from the following combinatorial statement,

\[
|K_{ad}(q)| = |J_{ad}(q)| + 1, \quad \forall q \in Q(K, |J| + 1).
\]

(Recall that we consider the case \( \ell_1, \ldots, \ell_d \in (C \setminus \{0\}) \cup \{0\} \) where \( K \neq \emptyset \).) To prove identity (85), let us fix \( q \in Q(K, |J| + 1) \). For notational convenience, we write \( J_{ad}, K_{ad} \) instead of \( J_{ad}(q), K_{ad}(q) \). Furthermore, define

\[
S(E) := \sum_{k \in E} q_k, \quad E \subseteq K.
\]

In a first step we prove that \( K_{ad} \neq \emptyset \). (Recall that we assume \( K \neq \emptyset \).) Denote by \( m \) is the smallest element of \( K \). Then \( S(K_m \setminus \{ m \}) = 0 \leq |J_m| \). Hence the largest number among all \( m \) in \( K \), satisfying \( S(K_m \setminus \{ m \}) \leq |J_m| \), exists. We denote it by \( \overline{m} \) and claim that \( \overline{m} \in K_{ad} \), i.e., that \( S(K_{\overline{m}} \setminus \{ \overline{m} \}) \leq |J'_{\overline{m}}| \).

Indeed, this clearly holds if \( \overline{m} \) is the largest element of \( K \). Otherwise, let \( j \) be the successor of \( \overline{m} \) in \( K \). Then

\[
S(K_{\overline{m}}) = S(K_j \setminus \{ j \}) \geq |J_j| + 1,
\]

and \( S(K_{\overline{m}} \setminus \{ \overline{m} \}) = (|J| + 1) - S(K_{\overline{m}} \setminus \{ \overline{m} \}) \) can be estimated by (84) as

\[
S(K_{\overline{m}} \setminus \{ \overline{m} \}) \leq (|J| + 1) - (|J_j| + 1) = |J'_j| \leq |J'_{\overline{m}}|.
\]

We thus have proved that \( \overline{m} \in K_{ad} \). Using the same type of arguments, one verifies that the following properties are satisfied.

\[\text{(P1) } \forall n \in J_{ad}, \exists m_-, m_+ \in K_{ad} \text{ with } m_- < n < m_+\]
Property (P1) is proved by verifying that \( m_- \), defined as the largest \( m \in K \) satisfying \( m < n \) and \( S(K_m \setminus \{m\}) \leq |J_m| \), and that \( m_+ \), defined as the smallest \( m \in K \) satisfying \( m > n \) and \( S(K_m' \setminus \{m\}) \leq |J'_m| \), are both in \( K_{ad} \). In more detail, one argues as follows. To prove that \( m_- \in K_{ad} \), recall that \( m \in K \), introduced in step 1, satisfies \( S(K_m \setminus \{m\}) = 0 \). Hence the largest number of all \( m \) in \( K \), satisfying \( S(K_m \setminus \{m\}) \leq |J_m| \) and \( m < n \), exists. We denote it by \( m_- \) and claim that \( m_- \in K_{ad} \). For this to be true, it remains to verify that \( S(K_{m_-} \setminus \{m_-\}) \leq |J'_{m_-}| \). If \( m_- \) is the largest element in \( K_n \), then

\[
S(K_{m_-}' \setminus \{m_-\}) = S(K_n') = |J'_m| \leq |J'_{m_-}|,
\]

implying that \( m_- \in K_{ad} \). Otherwise, let \( j \) be the successor of \( m_- \) in \( K \). Since by the definition of \( m_- \), \( S(K_{m_-}) = S(K_j \setminus \{j\}) \geq |J_j| + 1 \) one concludes

\[
S(K_{m_-}' \setminus \{m_-\}) = (|J| + 1) - S(K_{m_-}) \leq (|J| + 1) - (|J_j| + 1) = |J'_j| \leq |J'_{m_-}|,
\]

implying that \( m_- \in K_{ad} \). (Here we used that by (84), \(|J| - |J_j| = |J'_j|\).)

To prove that \( m_+ \in K_{ad} \), recall that \( m \in K \), introduced in step 1, is in \( K_{ad} \). In particular, one has \( S(K_m \setminus \{m\}) \leq |J'_m| \). Hence the smallest number among all \( m \) in \( K \), satisfying \( S(K_m \setminus \{m\}) \leq |J'_m| \) and \( n < m \), exists. We denote it by \( m_+ \). If \( m_+ \) is the smallest element in \( K_n' \), then

\[
S(K_{m_+} \setminus \{m_+\}) = S(K_n) = |J_n| \leq |J_{m_+}|,
\]

implying that \( m_+ \in K_{ad} \). Otherwise let \( j \) be the predecessor of \( m_+ \) in \( K \). By the definition of \( m_+ \), \( S(K_{m_+}) = S(K_j \setminus \{j\}) \geq |J'_j| + 1 \) and thus

\[
S(K_{m_+}' \setminus \{m_+\}) = (|J| + 1) - S(K_{m_+}) \leq (|J| + 1) - (|J'_j| + 1) \leq |J_j| \leq |J_{m_+}|,
\]

implying that \( m_+ \in K_{ad} \) also in this case.

Similarly, (P2) is proved by verifying that \( m \), defined as the largest \( p \in K \) satisfying \( n_1 < p < n_2 \) and \( S(K_p \setminus \{p\}) \leq |J_p| \), is in \( K_{ad} \). In more detail, one argues as follows. First we verify that \((n_1, n_2) \cap K \neq \emptyset \). Indeed, otherwise one has

\[
|J| + 1 = S(K_{n_1}) + S(K'_{n_1}) = S(K_{n_2}) + S(K'_{n_2}) = |J_{n_2}| + |J'_{n_2}|,
\]

implying that \(|J| + 1 \geq |J_{n_1}| + 1 + |J'_{n_1}| = |J| + 2 \), which is a contradiction. Denote by \( m_- \) the smallest element in \((n_1, n_2) \cap K \). Since \( S(K_{n_1}) = |J_{n_1}| \) it then follows that

\[
S(K_{m_-} \setminus \{m_-\}) = S(K_{n_1}) = |J_{n_1}| \leq |J_{m_-}|.
\]

Hence the largest number among all \( m \) in \( K \), satisfying \( S(K_m \setminus \{m\}) \leq |J_m| \) and \( m < n_2 \), exists. We denote it by \( m_+ \) and claim that \( m_+ \in K_{ad} \). If \( m_+ \) is the maximal element in \( K_{n_2} \), then

\[
S(K_{m_+}' \setminus \{m_+\}) = S(K_{n_2}) = |J'_{n_2}| \leq |J'_{m_+}|
\]
and hence \( m_+ \in K_{ad} \). Otherwise, let \( j \) be the successor of \( m_+ \) in \( K_{m_2} \). Then by the definition of \( m_+ \), \( S(K_j \setminus \{j\}) \geq |J_j| + 1 \). Therefore

\[
S(K_{m_+} \setminus \{m_+\}) = (|J| + 1) - S(K_{m_+}) \leq (|J| + 1) - S(K_j \setminus \{j\})
\]
yields

\[
S(K_{m_+} \setminus \{m_+\}) \leq (|J| + 1) - (|J_j| + 1) = |J_j| \leq |J_{m_+}^\prime|.
\]

Hence also in this case, \( m_+ \in K_{ad} \).

It remains to prove (P3). Let \( m_1, m_2 \in K_{ad} \) with \( m_1 < m_2 \). First we observe that \( (m_1, m_2) \cap J \neq \emptyset \). Otherwise, \( |J_{m_2}| = |J_{m_1}| \), implying that

\[
|J| + 1 = S(K_{m_1}) + S(K_{m_1}^\prime \setminus \{m_1\}) \leq S(K_{m_2} \setminus \{m_2\}) + S(K_{m_1} \setminus \{m_1\})
\]

\[
\leq |J_{m_2}| + |J_{m_1}^\prime| = |J_{m_1}| + |J_{m_1}^\prime| = |J|,
\]

which is a contradiction. Denote by \( \underline{n} \) the smallest element in \( J \) which is larger than \( m_1 \). We then have three alternatives.

(A1) \( S(\underline{n}) = |J_\underline{n}| \). Then \( \underline{n} \in J_{ad} \) and (P3) holds with \( n := \underline{n} \).

(A2) \( S(\underline{n}) \geq |J_\underline{n}| + 1 \). Since \( \underline{n} \in J \) and \( |J| + 1 - |J_\underline{n}| = |J_{\underline{n}}^\prime| \) (cf. (84)), one has

\[
S(K_{\underline{n}}) = (|J| + 1) - S(\underline{n}) \leq (|J| + 1) - (|J_\underline{n}| + 1) = |J_{\underline{n}}^\prime| - 1.
\]

Hence the largest number among the \( n \in J \cap (m_1, m_2) \) satisfying \( S(K_n) \leq |J_n| \) exists. We denote it by \( \overline{n} \). In the case \( (\overline{n}, m_2) \cap J = \emptyset \), one has

\[
|J_{m_2}| = |J_{\overline{n}}|\] and hence

\[
S(K_{\overline{n}}) \leq S(K_{m_2} \setminus \{m_2\}) \leq |J_{m_2}| = |J_{\overline{n}}|
\]

and thus \( \overline{n} \in J_{ad} \). In the case \( (\overline{n}, m_2) \cap J \neq \emptyset \), denote by \( j \) the smallest element of \( J \cap (\overline{n}, m_2) \). Since by the definition of \( \overline{n} \), \( S(K_{\overline{n}}) \geq |J_{\overline{n}}^\prime| + 1 \) and \( |J| + 1 - |J_{\overline{n}}^\prime| = |J_j| = |J_{\overline{n}}| + 1 \) one has

\[
S(K_{\overline{n}}) \leq S(K_j) \leq (|J| + 1) - (|J_{\overline{n}}^\prime| + 1) = |J_j| - 1 = |J_{\overline{n}}|,
\]

and we conclude again that \( \overline{n} \in J_{ad} \).

(A3) \( S(\underline{n}) \leq |J_\underline{n}| - 1 \). We claim that this case does not occur. Indeed,

\[
S(K_{\underline{n}}) \leq S(K_{m_1} \setminus \{m_1\}) \leq |J_{m_1}| = |J_{\underline{n}}|.
\]

Since \( |J_\underline{n}| + |J_{\underline{n}}^\prime| = |J| + 1 \) (cf. (84)) it then follows that

\[
|J| + 1 = S(K_{\underline{n}}) + S(K_{\underline{n}}^\prime) \leq |J_{\underline{n}}| - 1 + |J_{\underline{n}}^\prime| = |J|,
\]

which is a contradiction.

This proves (P3).

Finally, properties (P1), (P2), (P3) above together with the fact that \( K_{ad} \) is not empty, imply identity (85).
We are now ready to prove Proposition 4.1.

Proof of Proposition 4.1. For any given $-1/2 < s \leq 0$, choose $U \equiv U^*$ as at the beginning of this section. Lemma 5.1 and (80) imply that for any $u \in U$,

\[ \delta_n(u) = R_n(u), \quad n \geq 1. \] (86)

Moreover, for any $n \geq 1$ and for any $u \in U$ we obtain from (82) that

\[ \sum_{n \geq 1} |R_n(u)| \leq \sum_{n \geq 1} \sum_{d \geq 2} \sum_{1 \leq m \leq d-1} \sum_{m+1 \leq k \leq d} \sum_{l_k = \cdots = l_{j+1} < \infty \atop j \in \{1, \ldots, d\} \setminus \{k\}} |A(l_1, \ldots, l_m)||A(l_m, \ldots, l_d)||E_u| \]

\[ \leq \sum_{d \geq 2} \sum_{-\infty < l_j < \infty \atop 1 \leq j \leq d} \left( \sum_{1 \leq m \leq d-1} (d - m)|A(l_1, \ldots, l_m)||A(l_m, \ldots, l_d)| \right)|E_u| \]

where we write $|E_u|$ for $|E_u(l_1, \ldots, l_d)|$. Proposition 4.1 now follows from (86) and Lemma 5.2, Lemma 5.3, stated below.

Lemma 5.2. For any $-1/2 < s \leq 0$ there exists an open neighborhood $U^*$ of zero in $H^s_{c,0}$ and a constant $C > 0$ so that for any $u \in U^*$

\[ \sum_{d \geq 2} \sum_{-\infty < l_j < \infty \atop 1 \leq j \leq d} \left( \sum_{1 \leq m \leq d-1} (d - m)|A(l_1, \ldots, l_m)||A(l_m, \ldots, l_d)| \right)|E_u| \leq C\|u\|_{s}^3. \] (87)

Proof of Lemma 5.2. Let $U \equiv U^*$ be an open neighborhood of zero in $H^s_{c,0}$ so that the statement of Proposition 2.2 holds. Let $d \geq 2$ and any $1 \leq m \leq d-1$. By (31), one has

\[ |A(l_1, \ldots, l_m)| \leq 5^m \prod_{j=1}^{m} \frac{1}{|l_j| + 1}, \quad |A(l_m, \ldots, l_d)| \leq 5^{d-m+1} \prod_{j=m}^{d} \frac{1}{|l_j| + 1}. \]

Hence $|A(l_1, \ldots, l_m)||A(l_m, \ldots, l_d)||E_u| \leq 5^{d+1}F_u(l_1, \ldots, l_m)F'_u(l_m, \ldots, l_d)$ where

\[ F_u(l_1, \ldots, l_m) := \frac{1}{|l_m| + 1} \frac{|\tilde{u}(l_m - l_{m-1})|}{|l_{m-1}| + 1} \cdots \frac{|\tilde{u}(l_1)|}{|l_1| + 1}, \]

\[ F'_u(l_m, \ldots, l_d) := \frac{1}{|l_m| + 1} \frac{|\tilde{u}(-l_m - l_{m+1})|}{|l_{m+1}| + 1} \cdots \frac{|\tilde{u}(-l_d)|}{|l_d| + 1}. \]

By Lemma 3.1 it then follows that for any $u \in U$

\[ \left\| \sum_{-\infty < l_j < \infty \atop 1 \leq j \leq m-1} F_u(l_1, \ldots, l_m) \right\|_{s+1} \leq C_s^{m-1}\|u\|_{s}^m, \]

\[ \left\| \sum_{-\infty < l_j < \infty \atop m+1 \leq j \leq d} F'_u(l_m, \ldots, l_d) \right\|_{s+1} \leq C_s^{d-m}\|u\|_{s}^{d-m+1}. \]
where \( C_s \geq 1 \) is the constant of Lemma 3.1. One then concludes that the left hand side of the inequality in (87) is bounded by

\[
\sum_{d \geq 2} 5^{d+1} d^2 C^{d-1}_s \| u \|^{d+1}_s \leq \sum_{d \geq 2} (C'_s \| u \|_s)^{d+1}
\]

where the constant \( C'_s \) satisfies \( 5^{d+1} d^2 C^{d-1}_s \leq (C'_s)^{d+1}. \) By shrinking \( U, \) if needed, the claimed estimate (87) follows.

\[ \square \]

**Lemma 5.3.** Let \( U \) be an open neighborhood in a complex Banach space \( X. \) Assume that the map \( f : U \to \ell^1_+, x \mapsto (f_n(x))_{n \geq 1} \) is locally bounded and for any \( n \geq 1 \) the “coordinate” function \( f_n : X \to \mathbb{C} \) is analytic. Then \( f : X \to \ell^1_+ \) is analytic.

**Proof of Lemma 5.3.** The lemma follows if we prove that \( f : U \to \ell^1_+ \) is weakly analytic (cf. [11, Appendix A]). To this end, fix \( x_0 \in U, h \in X, \| h \|_X = 1, \) and \( c = (c_n)_{n \geq 1} \in \ell^\infty_+ = (\ell^1_+)^\vee. \) For any \( \lambda \in \mathbb{C} \) with \( |\lambda| < r \) and \( r > 0 \) sufficiently small, consider the complex valued function

\[
g(\lambda) := \sum_{n \geq 1} c_n f_n(x_0 + \lambda h)
\]

and the partial sums \( g_N(\lambda) := \sum_{n=1}^N c_n f_n(x_0 + \lambda h), N \geq 1. \) Since \( f : U \to \ell^1_+ \) is locally bounded, there exist \( M > 0, r > 0 \) so that for any \( |\lambda| < r \) and \( N \geq 1, \)

\[
|g_N(\lambda)| \leq \|c\|_{\ell^\infty_+} \|f(x_0 + \lambda h)\|_{\ell^1_+} \leq M.
\]

Hence, the holomorphic functions \( (g_N)_{N \geq 1} \) are uniformly bounded in the disk \( D_0(r) = \{ \lambda \in \mathbb{C} \mid |\lambda| < r \}. \) By Montel’s theorem, there exists a subsequence \( (g_{N_k})_{k \geq 1} \) of \( (g_N)_{N \geq 1}, \) converging to \( g \) uniformly on any compact subset of \( D_0(r). \) Hence, by the Weierstrass theorem, \( g \) is holomorphic. Since \( x_0 \in U, h \in X \) with \( \|h\|_X = 1, \) and \( (c_n)_{n \geq 1} \in \ell^\infty_+ \) are arbitrary, \( f \) is weakly analytic. \[ \square \]

6  **Proof of Theorem 1.1 and Theorem 1.2.**

In this section we prove Theorem 1.1 and Theorem 1.2, stated in Section 1.

**Proof of Theorem 1.1.** Recall from Proposition 4.2 that for any \( s > -1/2, \) there exists a neighborhood \( U \equiv U^s \) of zero in \( H^s_{\epsilon,0} \) so that the Birkhoff map extends from \( U \cap H^s_{\epsilon,0} \) to an analytic map

\[
\Phi : U \to \mathfrak{h}^{s+1/2}_{\epsilon,0}, \quad u \mapsto \left( (\Phi^s_{\epsilon,0}(u))_{n \leq -1}, (\Phi^s_{\epsilon,0}(u))_{n \geq 1} \right)
\]

where, by (58),

\[
\Phi_n(u) = v^n \frac{a^n_{\epsilon,0}(u)}{\sqrt{n \kappa_n(u)}} \Psi^s_{\epsilon,0}(u), \quad n \geq 1,
\]

(88)
with \( \Psi : U \rightarrow \mathbb{h}_+^{1+s} \) being the pre-Birkhoff map (24), studied in Section 3.

Let us compute the differential \( d_0\Phi \) of \( \Phi \) at \( u = 0 \). To this end note that the differential of \( \Psi \) at \( u = 0 \) is given by the weighted Fourier transform (cf. (26)), \( d_0\Psi : H_0^s \rightarrow \mathbb{h}_+^{1+s}, \ u \mapsto \left( -\frac{\hat{u}(n)}{n} \right)_{n \geq 1} \). Hence, in view of (55),

\[
d_0\Psi^* : H_{c,0}^s \rightarrow \mathbb{h}_+^{1+s}, \ u \mapsto \left( \frac{\hat{u}(n)}{n} \right)_{n \geq 1}.
\]

Recall from Remark 4.2 that for \( u = 0 \) one has

\[
\kappa_0(0) = 1, \ n\kappa_n(0) = 1, \ n \geq 1, \quad \hat{f}_n(0) = e_n, \quad \hat{h}_n(0) = e_n, \quad n \geq 0. \tag{91}
\]

The definition (45) of \( a_n(u) \) then implies that for \( u = 0 \),

\[
a_n^*(0) = a_n(0) = 1, \quad n \geq 1. \tag{92}
\]

From (88)–(92), the fact that \( \Psi_n(u) = 0, \ n \geq 1, \) and Leibniz’s rule one then infers that

\[
d_0\Phi : H_{c,0}^s \rightarrow \mathbb{h}_+^{1+s}, \ u \mapsto \left( -\frac{\hat{u}(n)}{\sqrt{|n|}} \right)_{n \in \mathbb{Z} \setminus \{0\}}. \tag{93}
\]

Hence, being a weighted Fourier transform, \( d_0\Phi : H_{c,0}^s \rightarrow \mathbb{h}_+^{1+s} \) is a linear isomorphism of complex linear spaces. Moreover, \( d_0\Phi|_{H_{r,0}^s} : H_{r,0}^s \rightarrow \mathbb{h}_+^{1+s} \) is a linear isomorphism of the corresponding real subspaces. Theorem 1.1 now follows from the inverse function theorem in Banach spaces together with the fact that \( \Phi|_{H_{r,0}^s} : H_{r,0}^s \rightarrow \mathbb{h}_+^{1+s} \) is a homeomorphism (cf. [7, Theorem 6]). \( \square \)

Let us now turn to the proof of Theorem 1.2. To prove item (i), we first need to make some preliminary considerations. Recall that the Benjamin-Ono equation is globally well-posed in \( H_{r,0}^s \) for any \( s > -1/2 \) (cf. [7] for details and references). For any given \( t \in \mathbb{R} \), denote by \( S_0^t \) the flow map of the Benjamin-Ono equation on \( H_{r,0}^s \), \( S_0^t : H_{r,0}^s \rightarrow H_{r,0}^s \) and by \( S_B^t : \mathbb{h}_+^{1/2+s} \rightarrow \mathbb{h}_+^{1/2+s} \) the version of \( S_0^t \) obtained, when expressed in the Birkhoff coordinates \( (\zeta_n)_{n \geq 1} \) (cf. Remark 4.1). To describe \( S_B^t \) more explicitly, recall that the \( n \)th frequency \( \omega_n \), \( n \geq 1 \), of the Benjamin-Ono equation is the real valued, affine function defined on \( \mathbb{h}_+^{1/2+s} \) (cf. [6], [7]),

\[
\omega_n(\zeta) = n^2 - 2 \sum_{k=1}^{n} k|\zeta_k|^2 - 2n \sum_{k=n+1}^{\infty} |\zeta_k|^2. \tag{94}
\]

For any initial data \( \zeta(0) \in \mathbb{h}_+^{1/2+s} \), \( s > -1/2 \), \( S_B^t(\zeta(0)) \) is given by

\[
S_B^t(\zeta(0)) := \left( \zeta_n(0)e^{it\omega_n(\zeta(0))} \right)_{n \geq 1}.
\]

The key ingredient into the proof of Theorem 1.2 (i) is a corresponding result for the flow map \( S_B^t \). More precisely, one has the following
Lemma 6.1. For any $t \neq 0$ and any $-1/2 < s < 0$, $S^*_B : h^{1/2+s}_+ \to h^{1/2+s}_+$ is nowhere locally uniformly continuous. In particular, it is not locally Lipschitz.

Proof of Lemma 6.1. We argue as in the proof of a corresponding result for the KdV equation in [12, Theorem 3.10]. Let $U = U^s$ be an arbitrary non-empty open subset of $h^{1/2+s}_+$ with $-1/2 < s < 0$ and $t \neq 0$. Choose $\zeta^{(0)} \in U$ so that there exists $N \geq 1$ with the property that $\zeta^{(0)}(n) = 0$ for any $n > N$. For any $\delta > 0$ and $m > N$, let $\zeta^{(m,\delta)} := (\zeta^{(m,\delta)}(n))_{n \geq 1}$ and $\xi^{(m,\delta)} := (\xi^{(m,\delta)}(n))_{n \geq 1}$, where

$$\zeta^{(m,\delta)}(n) = \zeta^{(0)}(n), \quad \forall n \neq m, \quad \zeta^{(m,\delta)}(m) = \frac{\delta}{m^{1/2+s}},$$

and

$$\xi^{(m,\delta)}(n) = \zeta^{(0)}(n), \quad \forall n \neq m, \quad \xi^{(m,\delta)}(m) = \frac{\delta(1 + ims/2)}{m^{1/2+s}}.$$ Then

$$\|\zeta^{(m,\delta)} - \zeta^{(0)}\|_{h^{1/2+s}_+} = \delta, \quad \|\zeta^{(m,\delta)} - \zeta^{(0)}\|_{h^{1/2+s}_+} = \delta \sqrt{1 + ms}.$$

Choose $\delta_0 > 0$ so small that $\zeta^{(m,\delta)}$ and $\xi^{(m,\delta)}$ are elements of $U$ for any $m > N$ and $0 < \delta \leq \delta_0$. Furthermore, one has for any $m > N$,

$$\|\zeta^{(m,\delta)} - \xi^{(m,\delta)}\|_{h^{1/2+s}_+} = \delta m^{s/2}, \quad (96)$$

and by (94),

$$\left|\omega_m(\zeta^{(m,\delta)}) - \omega_m(\xi^{(m,\delta)})\right| = 2m\delta^2 m^s m^{1+2s} = \frac{1}{m^s} 2\delta^2.$$

For any given $t \neq 0$, choose an integer $k \geq 1$ so large that

$$\delta \equiv \delta(t) := \left(\frac{\pi k^s}{2|t|}\right)^{1/2} < \delta_0$$

and hence $2|t|\delta^2 = \pi k^s$. It then follows that

$$\left|\omega_m(\zeta^{(m,\delta)})t - \omega_m(\xi^{(m,\delta)})t\right| = \left(\frac{k}{m}\right)^s \pi.$$

Since $0 < |s| < 1/2$ and

$$|(m + 1)^{|s|} - m^{|s|}| \leq |s| \int_m^{m+1} \frac{1}{x^{|s|}} dx \leq |s|,$$

there exists a subsequence $(m_j)_{j \geq 1}$ of the sequence $(kn)_{n \geq N+1}$ so that (with $N = \{1, 2, 3, \ldots\}$),

$$\text{dist} \left(\left(\frac{k}{m_j}\right)^s, 2N - 1\right)^s < 1/2, \quad \forall j \geq 1.$$
Since for any $x \in \mathbb{R}$ with $\text{dist}(x, 2\mathbb{N} - 1) < 1/2$ one has $|\exp(\pm ix\pi) - 1| > 1$, one concludes that

$$|\exp(it\omega_mz(\zeta(m,\delta)) - it\omega_mz(\zeta(m,\delta))) - 1| > 1$$

and hence

$$m_j^{1/2+s}|\xi_{m_j}(\zeta(m,\delta)) - \xi_{m_j}(\zeta(m,\delta))| \cdot |\exp(it\omega_mz(\zeta(m,\delta))) - \exp(it\omega_mz(\zeta(m,\delta)))|$$

$$= m_j^{1/2+s}|\xi_{m_j}(\zeta(m,\delta)) - \xi_{m_j}(\zeta(m,\delta))| \cdot |\exp(it\omega_mz(\zeta(m,\delta))) - \exp(it\omega_mz(\zeta(m,\delta)))| - 1$$

$$\geq \sqrt{1 + m_j^s\delta}.$$ (97)

In view of the estimate (cf. (96))

$$m_j^{1/2+s}|\xi_{m_j}(\zeta(m,\delta)) - \xi_{m_j}(\zeta(m,\delta))| \cdot |\exp(it\omega_mz(\zeta(m,\delta)))| \leq \delta m_j^{s/2},$$ (98)

one then concludes, by comparing only the $m_j$th component of $S^t_B(\zeta(m,\delta))$ with the one of $S^t_B(\zeta(m,\delta))$ (cf. (95)) that

$$\|S^t_B(\zeta(m,\delta)) - S^t_B(\zeta(m,\delta))\|_{h^{1/2+s}_+} \geq m_j^{1/2+s}|\xi_{m_j}(\zeta(m,\delta)) - \zeta_{m_j}(\zeta(m,\delta))|$$

$$\geq m_j^{1/2+s}|\xi_{m_j}(\zeta(m,\delta)) - \zeta_{m_j}(\zeta(m,\delta))| - m_j^{1/2+s}|\zeta_{m_j}(\zeta(m,\delta)) - \zeta_{m_j}(\zeta(m,\delta))|.$$

This together with (97) and (98) yields

$$\|S^t_B(\zeta(m,\delta)) - S^t_B(\zeta(m,\delta))\|_{h^{1/2+s}_+} \geq ((1 + m_j^s)^{1/2} - m_j^{s/2})\delta,$$

with the latter expression converging to $\delta > 0$ as $j \to \infty$, whereas by (96),

$$\|\zeta(m,\delta) - \zeta(m,\delta)\|_{h^{1/2+s}_+} = \delta m_j^{s/2},$$

which converges to 0 as $j \to \infty$. This completes the proof of Lemma 6.1.

\[ \square \]

Proof of Theorem 1.2(i). The claimed result follows directly from Lemma 6.1 and Theorem 1.1.

\[ \square \]

To prove Theorem 1.2(ii), we first need to make some preliminary considerations. Denote by $\ell^\infty_{c,0}$ the space $\ell^\infty(\mathbb{Z} \setminus \{0\}, \mathbb{C})$ of bounded complex valued sequences $z = (z_n)_{n \neq 0}$, endowed with the supremum norm. It is a Banach space and can be identified with the subspace of $\ell^\infty$, consisting of sequences $(z_n)_{n \in \mathbb{Z}}$ with $z_0 = 0$. Note that $\ell^\infty_{c,0}$ is a Banach algebra with respect to the multiplication

$$\ell^\infty_{c,0} \times \ell^\infty_{c,0} \to \ell^\infty_{c,0}, \quad (z, w) \mapsto z \cdot w := (z_nw_n)_{n \neq 0}.$$ (99)
Similarly, for any \( s \in \mathbb{R} \), the bilinear map
\[
\mathfrak{h}^s_{c,0} \times \ell^\infty_{c,0} \to \mathfrak{h}^s_{c,0}, \quad (z, w) \mapsto z \cdot w,
\] (100)
is bounded. For any given \( T > 0 \) and \( s \in \mathbb{R} \) introduce the \( \mathbb{C} \)-Banach spaces
\[
C_{T,s} := C([-T, T], \mathfrak{h}^s_{c,0}) \quad \text{and} \quad C_{T,\infty} := C([-T, T], \ell^\infty_{c,0}),
\]
endowed with supremum norm. Elements of these spaces are denoted by \( \xi \), or more explicitly, \( \xi(t) = (\xi_n(t))_{n \neq 0} \). The multiplication in (99) and (100) induces in a natural way the following bounded bilinear maps
\[
C_{T,\infty} \times C_{T,\infty} \to C_{T,\infty}, \quad C_{T,s} \times C_{T,\infty} \to C_{T,s}, \quad C_{T,s} \times \ell^\infty_{c,0} \to C_{T,s}. \quad (101)
\]
For example, the boundedness of the first map in (101) follows from the Banach algebra property of the multiplication in \( \ell^\infty_{c,0} \) and the estimate
\[
\|
\xi^{(1)} \cdot \xi^{(2)}\|_{C_{T,\infty}} = \sup_{t \in [-T,T]} \|
\xi^{(1)}(t) \cdot \xi^{(2)}(t)\|_{\ell^\infty_{c}} \leq \sup_{t \in [-T,T]} \|
\xi^{(1)}(t)\|_{\ell^\infty_{c}} \|
\xi^{(2)}(t)\|_{\ell^\infty_{c}}
\]
\[
\leq \|
\xi^{(1)}\|_{C_{T,\infty}} \|
\xi^{(2)}\|_{C_{T,\infty}}, \quad \xi^{(1)}, \xi^{(2)} \in C_{T,\infty}.
\]
Hence, \((C_{T,\infty}, \cdot)\) is a Banach algebra. The following lemma can be shown in a straightforward way and hence we omit its proof.

**Lemma 6.2.** The map \( C_{T,\infty} \to C_{T,\infty}, \xi \mapsto e^\xi := (e^{\xi_n})_{n \neq 0} \), is analytic.

For any \( s > -1/2 \) and any \( n \geq 1 \), the \( n \)th frequencies (94) of the Benjamin-Ono equation extends from \( \mathfrak{h}^{\frac{1}{2}+s}_{r,0} \) (cf. Remark 4.1) to an analytic function on \( \mathfrak{h}^{\frac{1}{2}+s}_{c,0} \) given by
\[
\omega_n(\zeta) = n^2 + \Omega_n(\zeta),
\] (102)
where
\[
\Omega_n(\zeta) := -2 \sum_{k=1}^{n} k \zeta \zeta - 2n \sum_{k=n+1}^{\infty} \zeta \zeta, \quad \zeta \in \mathfrak{h}^{\frac{1}{2}+s}_{c,0}.
\] (103)
For \( n \leq -1 \) we set
\[
\Omega_n(\zeta) := -\Omega_{-n}(\zeta), \quad \zeta \in \mathfrak{h}^{\frac{1}{2}+s}_{c,0}.
\] (104)
We have the following

**Lemma 6.3.** For any \( s \geq 0 \) the map
\[
\Omega : \mathfrak{h}^{\frac{1}{2}+s}_{c,0} \to \ell^\infty_{c,0}, \quad \zeta \mapsto (\Omega_n(\zeta))_{n \neq 0},
\] (105)
is analytic.
Proof of Lemma 6.3. Let $s \geq 0$. Then for any $n \geq 1$, $\zeta \in h_{c,0}^{1/2+s}$, one has by (103)

$$|\Omega_n(\zeta)| \leq 2 \left| \sum_{k=1}^{n} k \zeta_k \zeta_k \right| + 2 \left| \sum_{k=n+1}^{\infty} k \zeta_k \zeta_k \right| \leq 2 \left( \sum_{k=1}^{\infty} k \right) |\zeta|_{1/2+s}^2 \leq 2\|\zeta\|_{n_0^{1/2+s}}^2.$$ 

This, together with (104) then implies that for any $s \geq 0$, the map (105) is well defined and locally bounded. The estimate above also shows that for any $n \neq 0$ the $n$th component of (105) is a (continuous) quadratic form in $\zeta \in h_{c,0}^{1/2+s}$ and hence analytic. The lemma then follows from [11, Theorem A.3].

We now consider the curves $\xi^{(1)}, \xi^{(2)} \in C_{T,\infty}$, defined by

$$\xi^{(1)} : [-T, T] \to \ell_{c,0}^\infty, \quad \xi^{(1)}_n(t) := e^{i \text{sign}(n) n^2 t}, \quad n \neq 0,$$

and, respectively,

$$\xi^{(2)} : [-T, T] \to \ell_{c,0}^\infty, \quad \xi^{(2)}_n(t) := t, \quad n \neq 0. \quad (106)$$

The following corollary follows from Lemma 6.2, Lemma 6.3, and the continuity of the bilinear maps in (101).

**Corollary 6.1.** For any $s \geq 0$, the map (cf. (95))

$$S_{B,T} : h_{c,0}^{1/2+s} \to C_{T,\frac{1}{2}+s}, \quad \zeta \mapsto \zeta \cdot \xi^{(1)} \cdot e^{i \Omega(\cdot) \cdot \xi^{(2)}}, \quad (108)$$

is analytic.

**Proof of Theorem 1.2(ii).** Let $s \geq 0$ and let $U^s$ be the neighborhood of zero in $H_{c,0}^s$ of Theorem 1.1, so that $\Phi : U^s \to \Phi(U^s) \subseteq h_{c,0}^{1/2+s}$ is a diffeomorphism. According to Corollary 6.1 there exists a neighborhood $W$ of zero in $h_{c,0}^{1/2+s}$, which is contained in $\Phi(U^s)$, so that for any $\zeta \in W$, one has $S_B' e(H(\zeta)) \in \Phi(U^s)$ for any $t \in [-T, T]$. Let $V^s := \Phi^{-1}(W)$. By Corollary 6.1, Theorem 1.1, Lemma 6.4 below, and the fact $S_B'(u_0) = (\Phi^{-1} \circ S_B \circ \Phi)(u_0)$ for any $u_0 \in V^s$ and $t \in [-T, T]$, it then follows that $S_{B,T}$ extends to an analytic map $V^s \to C([-T, T], H_{c,0}^s)$. \hfill \Box

The following lemma is used in the proof of Theorem 1.2(ii). To state it, we first need to introduce some more notation. Let $X$ be a complex Banach space with norm $\| \cdot \| \equiv \| \cdot \|_X$. For any $T > 0$, denote by $C([-T, T], X)$ the Banach space of continuous functions $x : [-T, T] \to X$, endowed with the supremum norm $\|x\|_{T,X} := \sup_{t \in [-T,T]} \|x(t)\|$. Furthermore, for any open neighborhood $U$ of $X$, denote by $C([-T, T], U)$ the subset of $C([-T, T], X)$, consisting of continuous functions $[-T, T] \to X$ with values in $U$.

**Lemma 6.4.** Let $f : U \to Y$ be an analytic map from an open neighborhood $U$ in $X$ where $X$ and $Y$ are complex Banach spaces. Then for any $T > 0$, the associated push forward map

$$f_* : C([-T, T], U) \to C([-T, T], Y), \quad [t \mapsto x(t)] \mapsto [t \mapsto f(x(t))],$$

is analytic.
Proof of Lemma 6.4. Let $x_0 \in C([-T,T], U)$ be given. Since $x_0([-T,T])$ is compact in $U$ and $f: U \rightarrow Y$ is locally bounded there exist $M > 0$ and $r > 0$ so that for any $x \in C([-T,T], U)$ with $\|x - x_0\|_{T,X} < 2r$ one has
\[
\|f_* x\|_{T,Y} := \sup_{t \in [-T,T]} \|f(x(t))\|_Y \leq M,
\tag{109}
\]
where $\| \cdot \|_Y$ denotes the norm of $Y$. For any given $t \in [-T,T]$ we obtain from the analyticity of the map $f: U \rightarrow Y$ and Cauchy’s formula that for any $x$, $h \in C([-T,T], U)$ such that
\[
\|x - x_0\|_{T,X} < r/2, \quad \|h\|_{T,X} < r,
\tag{110}
\]
we have that for any $0 < |\mu| < 1/2$,
\[
\frac{f(x(t) + \mu h(t))}{\mu} - f(x(t)) = \frac{1}{2\pi i} \oint_{|\lambda| = 1} \frac{f(x(t) + \lambda h(t))}{\lambda(\lambda - \mu)} d\lambda
\]
and
\[
d_{x(t)} f(h(t)) = \frac{1}{2\pi i} \oint_{|\lambda| = 1} \frac{f(x(t) + \lambda h(t))}{\lambda^2} d\lambda
\]
These identities imply that for any $x$, $h \in C([-T,T], U)$ satisfying (110), and for any $t \in [-T,T]$, $0 < |\mu| < 1/2$, one has
\[
\left\| \frac{f(x(t) + \mu h(t))}{\mu} - f(x(t)) - d_{x(t)} f(h(t)) \right\|_Y \leq \| \mu \frac{1}{2\pi i} \oint_{|\lambda| = 1} \frac{f(x(t) + \mu h(t))}{\lambda^2(\lambda - \mu)} d\lambda \|_Y \leq 2M|\mu|
\]
where we used (109). Expressed in terms of the push forward map $f_*$, it means that for any $x$, $h \in C([-T,T], U)$ satisfying (110),
\[
\left\| \frac{f_*(x + \mu h)}{\mu} - f_*(x) - d_{x} f_*(h) \right\|_{T,Y} \leq 2M|\mu|, \quad 0 < |\mu| < 1/2.
\]
Therefore, $f_*: C([-T,T], U) \rightarrow C([-T,T], Y)$ is a $C^1$-map between $C$-Banach spaces (see e.g. [20, Problem 3]) and hence analytic.

It remains to prove the Addendum to Theorem 1.2 (ii), stated in Section 1.

Proof of Addendum to Theorem 1.2 (ii). Corollary 6.1 continues to hold when $C_{T,\frac{1}{2}+s}$ is replaced by the Banach space $C^k([-T,T], h^{\frac{1}{2}+s-2k}_{c,0})$, $k \geq 1$, of $k$ times continuously differentiable functions $\xi: [-T,T] \rightarrow h^{\frac{1}{2}+s-2k}_{c,0}$. For any $k \geq 1$, this follows from the proof of Theorem 1.2(ii) and the fact that the curve $\zeta \cdot \xi^{(1)}$ [resp. $e^{i\Omega(t)\cdot \xi^{(2)}}$], which appears as a factor on the right side of (108), belongs to $C^k([-T,T], h^{\frac{1}{2}+s-2k}_{c,0})$ [resp. $C^k([-T,T], \ell^{\infty}_{c,0})$]. By combining this with Theorem 1.1 one obtains the claimed result.  

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A Symmetries of the Lax operator

In this appendix we study symmetries of the Lax operator \( L_u \). Recall that in [8], we defined and studied the Lax operator \( L_u = D - T_u : H^{1+s}_+ \to H^{1+s}_+ \) for \( u \in H^s_c, -1/2 < s \leq 0 \). In a similar way one defines

\[
L_u^\ast := -D - T_u^\ast : H^{1+s}_- \to H^-_s
\]

where \( T_u^\ast : H^{1+s}_- \to H^-_s \) is the Toeplitz operator with potential \( u \),

\[
T_u f := \Pi^-(uf), \quad f \in H^{1+s}_-,
\]

and \( \Pi^- : H^-_c \to H^-_s \) is the Szegő projector

\[
\Pi^- : H^s_c \to H^-_s, \quad \sum_{n \in \mathbb{Z}} \hat{v}(n)e^{inx} \mapsto \sum_{n \leq 0} \hat{v}(n)e^{inx},
\]

onto the (negative) Hardy space \( H^-_s = \{ f \in H^s \mid \hat{f}(k) = 0 \ \forall k > 0 \} \). Indeed, it follows from Lemma 1 in [8] that \( T_u^\ast : H^{1+s}_- \to H^s_- \) is a well bounded linear operator and hence \( L_u^\ast \) defines an operator on \( H^-_s \) with domain \( H^{1+s}_- \) so that the map \( L_u : H^{1+s}_- \to H^s_- \) is bounded. For \( \beta \in \mathbb{R} \), denote by \((\cdot)_\ast : H^\beta_c \to H^\beta_c\) the involution, defined for \( v \in H^\beta_c \) by

\[
v_\ast(x) := v(-x), \quad x \in \mathbb{T}.
\]

Clearly, \((\cdot)_\ast\) is a \( \mathbb{C}\)-linear isometry. The Fourier coefficients of \( v_\ast \) satisfy

\[
\hat{v}_\ast(k) = \hat{v}(-k), \quad k \in \mathbb{Z}.
\]

**Lemma A.1.** For any \( u \in H^s_c, -1/2 < s \leq 1/2 \), we have the commutative diagrams

\[
\begin{array}{ccc}
H^{1+s}_+ & \xrightarrow{L_u} & H^s_+ \\
(\cdot)_\ast & \downarrow & (\cdot)_\ast \\
H^{1+s}_- & \xrightarrow{L_u^\ast} & H^-_s \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
H^{1+s}_+ & \xrightarrow{L_u} & H^s_+ \\
(\overline{\cdot}) & \downarrow & (\overline{\cdot}) \\
H^{1+s}_- & \xrightarrow{L_u^\ast} & H^-_s \\
\end{array}
\]

where \((\cdot)_\ast\) denotes the involution defined by (112) and \((\overline{\cdot}) : H^\beta_+ \to H^\beta_-, \beta \in \mathbb{R},\) is the complex conjugation of functions.

**Proof of Lemma A.1.** Since the proof of the commutativity of the second diagram is similar to the proof of the one of the first diagram we will prove only the first one. For any \( u \in H^s_c \) and \( f \in H^{1+s}_- \) we have

\[
(\Pi(uf))_\ast = \sum_{n \geq 0} \left( \sum_{k \geq 0} \hat{u}(n-k)\hat{f}(k) \right)e^{-inx} = \sum_{n \leq 0} \left( \sum_{k \leq 0} \hat{u}(k-n)\hat{f}(-k) \right)e^{inx}
\]

\[
= \sum_{n \leq 0} \left( \sum_{k \leq 0} \hat{u}_\ast(n-k)\hat{f}_\ast(k) \right)e^{inx} = \Pi^- (u_\ast f_\ast).
\]
By combining this with the fact that \((Df)_* = -D(f_*)\) we conclude that

\[(Lu)f_* = Luf_*.\]

This completes the proof of the lemma. \(\square\)

By combining Lemma A.1 with Proposition 2.1 we obtain

**Corollary A.1.** For any \(-1/2 < s \leq 0\), there exists an open neighborhood \(W \equiv W^s\) of \(H^s_{e,0}\) in \(H^s_{e,0}\), invariant under the involution (112) and the complex conjugation of functions, so that for any \(u \in W\), the operator \(L_n^-\) given by (111) is a closed operator in \(H^s\) with domain \(H^{1+s}\). The operator has a compact resolvent and all its eigenvalues are simple. When appropriately listed, \(\lambda_n^-\), \(n \geq 0\), satisfy \(\text{Re}(\lambda_n^-) < \text{Re}(\lambda_{n+1}^-)\) for \(n \geq 0\) and \(|\lambda_n^- - n| \to 0\) as \(n \to \infty\).

For \(u \in H^s_{e,0}\), \(-1/2 < s \leq 0\), let \((f_n(u))_{n \geq 0}\), be the eigenfunctions of \(L_u\), corresponding to the eigenvalues \((\lambda_n(u))_{n \geq 0}\), normalized as

\[
\|f_n(u)\| = 1, \quad n \geq 0, \quad (1|f_0(u)) > 0, \quad (f_n(u)|Sf_{n-1}(u)) > 0, \quad n \geq 1 \quad (114)
\]

(cf. [6, Definition 2.1]). Similarly, denote by \((f_n^-(u))_{n \geq 0}\), the eigenfunctions of \(L_n^-\), corresponding to the eigenvalues \((\lambda_n^-(u))_{n \geq 0}\), normalized as

\[
\|f_n^-(u)\| = 1, \quad n \geq 0, \quad (1|f_0^-(u)) > 0, \quad (Sf_n^-(u)|f_{n-1}^-(u)) > 0, \quad n \geq 1. \quad (115)
\]

It follows from Proposition 2.2 (cf. (20)) and Lemma A.1 that for any \(-1/2 < s \leq 0\), there exists an open neighborhood \(U^s\) of zero in \(H^s_{e,0}\) so that for any \(u \in U^s\) and \(n \geq 0\), the Riesz projector

\[
P_n^- (u) = -\frac{1}{2\pi i} \oint_{\partial D_n} (L_n^- - \lambda)^{-1} d\lambda \in \mathcal{L}(H^s, H^{1+s}_-),
\]

is well defined and the map \(U^s \to \mathcal{L}(H^s, H^{1+s}_-), u \mapsto P_n^- (u),\) is analytic. Here \(\partial D_n\) is the counterclockwise oriented boundary of \(D_n \equiv D_n(1/3)\) (see (17)). Hence for any \(n \geq 0\), the map

\[
U^s \to H^{1+s}_-, \quad u \mapsto h_n^- (u) := P_n^- (u)e_--n \in H^{1+s}_-.
\]

is analytic. Since the map \((\cdot ,\cdot )\), and the complex conjugation are isometries, without of loss of generality, we can choose \(U^s\) to be invariant under these two maps. With this notation established, we can state the following

**Corollary A.2.** Let \(-1/2 < s \leq 0\). Then for \(u \in H^s_{e,0}\) and \(n \geq 0\),

\[
\lambda_n^-(u) = \lambda_n^-(\overline{u}) = \lambda_n(u), \quad \lambda_n^-(\overline{u}) = \lambda_n(u), \quad (117)
\]

and

\[
f_n^- (u) = \overline{f_n(u)} = (f_n(u*))_*, \quad (118)
\]

Similarly, for any \(u \in H^s_{e,0} \cap U^s\) and \(n \geq 0\),

\[
h_n^- (u) = \overline{h_n(u)} = (h_n(u*))_*, \quad (119)
\]

where \(h_n(u)\) is given by (23) and \(h_n^- (u)\) by (116).

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Remark A.1. In a straightforward way it follows from Proposition 2.1, Proposition 2.2, and Remark 2.2 that for any $-1/2 < s \leq 0$ and $n \geq 0$, the identities $\lambda_n(u) = \lambda_n(u_\ast)$, $\lambda_n(\overline{u}) = \lambda_n(u)$ (cf. (117)) and $f_n(u) = (f_n(u_\ast))_\ast$ (cf. (118)) extend by analyticity to a neighborhood $W^s$ of $H^s_{r,0}$ in $H^s_{c,0}$. Similarly, the identity $h_n^-(u) = (h_n(u_\ast))_\ast$ (cf. (119)) extends to the neighborhood $U^s$.

Proof of Corollary A.2. Let $u \in H^s_{r,0}$ with $-1/2 < s \leq 0$ and $n \geq 0$. The identities (117) and the fact that
\begin{equation}
\overline{f_n(u)} \quad \text{and} \quad (f_n(u_\ast))_\ast \tag{120}
\end{equation}
are eigenfunctions of $L_u^-$ with eigenvalue $\lambda_n^-(u)$ follow directly from the commutative diagrams in Lemma A.1. Since for any $f, g \in H^1_{c,0}$,
\begin{align*}
\|f\| &= \|\overline{f}\| = \|f_\ast\|, \quad \langle 1\bar{f} \rangle = \langle 1\bar{f} \rangle, \quad \langle 1f \rangle = \langle 1f \rangle,
\end{align*}
and
\begin{align*}
(\overline{Sf}g) &= (\overline{f}Sg), \quad (Sf\bar{g}) = (f_\ast Sg_\ast),
\end{align*}
the normalization conditions (114) then imply that the eigenfunctions in (120) satisfy the normalization condition (115). By the simplicity of the eigenvalue $\lambda_n^-(u)$ we then conclude that $f_n^-(u) = \overline{f_n(u)}$ and $f_n^-(u) = (f_n(u_\ast))_\ast$.

The last statement of the corollary follows from the definition of $h_n(u)$ in (23) and Lemma A.1. Indeed, for any $u \in H^s_{r,0}$ and any $\lambda$ in the resolvent set of $L_u$, Lemma A.1 implies that
\begin{align*}
(L_u - \lambda)^{-1} &= C^{-1}(L_{\overline{u}} - \overline{\lambda})^{-1}C \quad \text{and} \quad (L_u - \lambda)^{-1} = \mathcal{I}^{-1}(L_{u_\ast} - \lambda)^{-1}\mathcal{I},
\end{align*}
where $C : H^s_{\ast} \to H^s_{\ast}$ denotes the restriction of the complex conjugation of functions to $H^s_{\ast}$ and $\mathcal{I} : H^s_{\ast} \to H^s_{\ast}$ the one of the map $(\cdot)_\ast$. This, together with (23) then implies that for any $u \in H^s_{r,0} \cap U^s$ and $n \geq 0$ we have
\begin{align*}
h_n(u) &= -\frac{1}{2\pi i} \oint_{\partial D_n} \mathcal{C}^{-1}((L_{\overline{u}} - \overline{\lambda})^{-1}e_{-n}) \, d\lambda \\
&= \mathcal{C}^{-1}\left(-\frac{1}{2\pi i} \oint_{\partial D_n} (L_{\overline{u}} - \overline{\lambda})^{-1}e_{-n} \, d\lambda\right) = \overline{h_n(u)}, \tag{121}
\end{align*}
where we used the definition (116) of $h_n^-$. By similar arguments one shows that for any $u \in H^s_{r,0} \cap U^s$ and $n \geq 0$ we have
\begin{align*}
h_n(u) &= -\frac{1}{2\pi i} \oint_{\partial D_n} \mathcal{I}^{-1}((L_{u_\ast} - \lambda)^{-1}e_{-n}) \, d\lambda \\
&= \mathcal{I}^{-1}\left(-\frac{1}{2\pi i} \oint_{\partial D_n} (L_{u_\ast} - \lambda)^{-1}e_{-n} \, d\lambda\right) = (h_n^-)(u), \tag{122}
\end{align*}
where we used $\lambda_n^-(u) = \lambda_n(u_\ast)$ (cf. (117)), the assumption that $U^s$ is invariant under $\mathcal{I}$, and the definition (116) of $h_n^-$. The formulas in (119) now follow from (121) and (122). \[\square\]
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