Time-Symmetrized Kustaanheimo-Stiefel Regularization

Yoko Funato¹
Department of Earth Science and Astronomy,
College of Arts and Sciences, University of Tokyo

Piet Hut
Institute for Advanced Study

Steve McMillan
Department of Physics and Atmospheric Science,
Drexel University

and

Junichiro Makino
Department of Graphics and Information Sciences,
College of Arts and Sciences, University of Tokyo

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¹JSPS Fellow for Japanese Junior Scientists
ABSTRACT

In this paper we describe a new algorithm for the long-term numerical integration of the two-body problem, in which two particles interact under a Newtonian gravitational potential. Although analytical solutions exist in the unperturbed and weakly perturbed cases, numerical integration is necessary in situations where the perturbation is relatively strong. Kustaanheimo–Stiefel (KS) regularization is widely used to remove the singularity in the equations of motion, making it possible to integrate orbits having very high eccentricity. However, even with KS regularization, long-term integration is difficult, simply because the required accuracy is usually very high. We present a new time-integration algorithm which has no secular error in either the binding energy or the eccentricity, while allowing variable stepsize. The basic approach is to take a time-symmetric algorithm, then apply an implicit criterion for the stepsize to ensure strict time reversibility. We describe the algorithm in detail and present the results of numerical tests involving long-term integration of binaries and hierarchical triples. In all cases studied, we found no systematic error in either the energy or the angular momentum. We also found that its calculation cost does not become higher than those of existing algorithms. By contrast, the stabilization technique, which has been widely used in the field of collisional stellar dynamics, conserves energy very well but does not conserve angular momentum.

Subject headings: Numerical simulation, Regularization, Time-symmetric integration, Two-body problem
1. Introduction

The long-term numerical integration of the two-body problem is important in several areas of astrophysics, particularly in the context of the evolution of star clusters. For example, binaries play a crucial role in the secular evolution of globular clusters (see, e.g. Hut et al. PASP 1992 and references therein). Binary evolution is also important from the observational point of view, since many interesting objects in globular clusters, such as X-ray sources, millisecond pulsars, high-velocity stars, and blue stragglers, are believed to be the result of binary interactions. In order to study the evolution of binaries in globular clusters, self-consistent N-body simulation is a most useful tool. In such simulations, we integrate numerically the orbits of both the stars and the binaries in the cluster.

There are, however, technical difficulties in obtaining useful results from numerical experiments. If we want to study the evolution of globular clusters, we have to follow the orbits of tightly-bound binaries under weak perturbations for, say, $10^{10}$ years, corresponding to several cluster half-mass relaxation times. Since the period of a binary is typically less than one year, this time interval corresponds to more than $10^{10}$ binary orbits, or $\sim 10^{13} - 10^{14}$ numerical integration steps. The computational requirements of such an undertaking would be prohibitive; in addition, both truncation and round-off error would be unacceptably large. Of course, there are several techniques used in order to reduce the number of steps. First of all, in practice, we deal with this difficulty by regarding binaries as “unperturbed” if the relative perturbation is less than, say, $10^{-6}$ or so. By this treatment, the relative calculation cost of integrating of binary orbits greatly reduced (Makino and Hut, 1990). The orbits of binaries with relative perturbation larger than $10^{-6}$ must be integrated numerically. To reduce the number of steps in integrating the orbits of such binaries, another technique — Kustaanheimo–Stiefel regularization — has been used.

Over the last 20 years, the program NBODY5 developed by Aarseth (1963, 1985,
1994) has been the most powerful tool for performing $N$-body simulations of star clusters. One important reason for the success of this program is that it can handle very efficiently the formation and evolution of binaries in $N$-body systems. A key part of NBODY5 is Kustaanheimo–Stiefel (KS) regularization (Kustaanheimo and Stiefel 1965), which is an extension of Levi-Civita’s regularization of the planar Keplerian problem to three dimensions (Levi-Civita, 1956). In KS regularization, a Kepler orbit is transformed into a harmonic oscillator and the number of steps needed for the integration of an orbit is reduced significantly.

In order to apply KS regularization, one first identifies a close pair of particles using some heuristic criterion, then integrates their center-of-mass and relative motion separately. The center-of-mass motion is integrated in the same way as the motion of all other particles in the system; the relative motion of the binary components is integrated using regularized time and coordinates.

To reduce the accumulation of integration error, a technique known as energy stabilization has been used (Baumgarte, 1973; Aarseth, 1985). The basic idea of energy stabilization is to introduce an artificial stabilizing term into the equation of motion so that the energy converges to the “true” value. In the case of an isolated binary, the energy is constant, and the force is simply adjusted so that the calculated energy converges to the original value. For a perturbed binary, the energy is integrated separately, and the force is adjusted so that the calculated energy converges to the integrated quantity. The stabilization technique vastly improves energy conservation; however, it does not conserve angular momentum. Thus, stabilization may give the wrong answer if the evolution of the binary eccentricity is important.

Recently, numerical integrators which have no secular error in the integrals of motions have attracted the interests of both astrophysicists and numerical analysts. The integrators
which had been most extensively studied are the “symplectic” integrators (for an overview, see Sanz–Serna and Calvo, 1994), which have the property of being symplectic (or canonical) transformations in phase space. Since a canonical transformation is one way to define a Hamiltonian system, it is reasonable to suppose that an integration scheme which is itself a canonical transformation might describe the nature of the system better than an integrator which is not. In fact, symplectic integrators have the remarkable property that the errors in the integrals of the motion have no secular term. In the case of the Kepler problem, neither the energy nor the angular momentum change systematically in time.

Time-symmetric integrators have not received as much attention as symplectic integrators. However, it has been demonstrated that some time-symmetric integrators also exhibit no secular errors, regardless of whether or not they are symplectic (Quinlan and Tremaine, 1990). One serious problem with both symplectic and time-symmetric schemes is that they work well only with constant stepsize. If variable stepsize is used, the performance of a symplectic integrator degrades to that of more “usual” schemes, in that it no longer automatically conserves the integrals of motion (Skeel and Gear, 1992; Sanz–Serna and Calvo, 1994).

Recently, Hut et al. (Hut, Makino and McMillan, 1995, hereafter HMM) have developed a novel technique to avoid the accumulation of the error in variable stepsize time-symmetric schemes. They found that the increase of the numerical error in the energy is dramatically reduced if the size of the timestep is determined in a time-symmetric way. When constant stepsize is used, a time-symmetric scheme is time-reversible, in the sense that, if we integrate the system forward for one timestep and then integrate it backward, the system returns precisely to the initial state. In the case of variable stepsize, however, the system generally does not return to the initial state, since the stepsizes used for the forward and backward steps are different. HMM used a time-symmetric criterion for the
timestep, ensuring that the stepsize for the forward and backward steps are exactly the same. They applied this scheme to the numerical integration of the Kepler problem in physical coordinates, and found that there was no secular error in the binding energy.

In this paper, we describe the implementation of this time-symmetric, variable step scheme to an \( N \)-body code using the KS regularization. We have developed a fourth-order integration scheme and performed a series of numerical simulations. We find that our algorithm has the good properties of both symmetrized timesteps and KS regularization. First, like all symmetrized schemes, it shows no secular error in either energy or angular momentum. Second, like the code with KS regularization, it can handle binaries of arbitrary eccentricity with a constant number of timesteps per orbit.

In section 2, we show the equations of motion and KS regularization. In section 3, we describe the integration algorithms used. In section 4, we describe the implementation in more detail. In section 5 we compare the results of our numerical experiments with those of other schemes, considering both isolated and perturbed binaries. As an example of a perturbed system, we consider a hierarchical three-body system in which the inner binary is integrated in KS coordinates and the outer binary is integrated in Cartesian physical coordinates. A summary and discussion are presented in section 6.

2. Equations of Motion

2.1. Cartesian Coordinates

The equations of motions of the two particles to which we apply the KS transformation are expressed as

\[
\frac{d\mathbf{r}_i}{dt} = \mathbf{v}_i, \tag{1}
\]
\[ m_i \frac{d\mathbf{v}_i}{dt} = -\frac{m_i m_j (\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_j - \mathbf{r}_i|^3} + \mathbf{F}_i, \quad (i, j = 1, 2), \]

where \( t \) is physical time and the gravitational constant \( G \) is set to unity. The 3-dimensional vectors \( \mathbf{r}_i \) and \( \mathbf{v}_i \) are the positions and velocities of the particle \( i \) in Cartesian coordinates, \( m_i \) is the mass of particle \( i \), and \( \mathbf{F}_i \) is the force on particle \( i \) due to the rest of the system.

In order to apply the KS transformation, we introduce the motion of the center-of-mass of two particles:

\[ \mathbf{r}_{cm} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}, \]

\[ \mathbf{v}_{cm} = \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2}, \]

and the relative motion:

\[ \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1, \]

\[ \frac{d\mathbf{r}}{dt} \equiv \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1. \]

The accelerations are given by:

\[ \frac{d\mathbf{v}}{dt} = -\frac{(m_1 + m_2)\mathbf{r}}{|\mathbf{r}|^3} + \mathbf{P}, \]

\[ \frac{d\mathbf{v}_{cm}}{dt} = -\frac{\mathbf{F}_1 + \mathbf{F}_2}{m_1 + m_2}, \]

where the second term \( \mathbf{P} \) in equation (9) is the perturbation:

\[ \mathbf{P} = \frac{\mathbf{F}_2}{m_2} - \frac{\mathbf{F}_1}{m_1}. \]

2.2. KS regularization

In KS regularization, the 3-dimensional position in Cartesian coordinates is transformed into a 4-dimensional position in KS coordinates. The physical time \( t \) is also transformed
into the KS time $\tau$. The transformation is expressed as follows:

$$ r = |r| = u_1^2 + u_2^2 + u_3^2 + u_4^2, \quad (10) $$

$$ r = \mathcal{L}(u)u, \quad (11) $$

$$ \mathcal{L} = \begin{pmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \end{pmatrix}, \quad (12) $$

$$ \frac{dt}{d\tau} = r = (u, u), \quad (13) $$

where $(u, u)$ is defined as the inner product of $u$ and $u$. The resulting regularized equations of motion take the form

$$ u'' = \frac{1}{2} h u + \frac{1}{2} r \mathcal{L}^T(u)P, \quad (14) $$

where $'$ denotes differentiation with respect to $\tau$, i.e. $' \equiv d/d\tau$, superscript $T$ denotes matrix transposition, and $h$ is the specific binding energy of the two bodies, which may be expressed in the KS system as follows:

$$ h = \frac{2u' \cdot u' - (m_1 + m_2)}{r}. \quad (15) $$

The time variation of $h$ is described as:

$$ h' = 2u' \cdot \mathcal{L}^T(u)P. \quad (16) $$

For an isolated binary, $P = 0$, so $h$ is constant.

3. Integration Techniques
3.1. The Hermite Scheme

The Hermite scheme (Makino, 1990) is a fourth order predictor-corrector scheme, which may be expressed as follows:

\[
\begin{align*}
    r_e &= r_b + \frac{1}{2}(v_v + v_b)\Delta t - \frac{1}{10}(a_v - a_b)\Delta t^2 + \frac{1}{120}(j_v + j_b)\Delta t^3, \\
    v_e &= v_b + \frac{1}{2}(a_v + a_b)\Delta t - \frac{1}{12}(j_v - j_b)\Delta t^2,
\end{align*}
\]

(17)

(18)

where \(a_X\) and \(j_X\) are the acceleration and the “jerk” (time-derivative of the acceleration), and subscripts \(b\) and \(e\) denote the values at the beginning and the end of the step, respectively.

As can be seen from equations (17) and (18), the Hermite scheme is an implicit scheme. The integration of one step is carried out as follows.

[1] Predict the positions and velocities:

\[
\begin{align*}
    r_{\text{pred}} &= r_b + v_b\Delta t + \frac{1}{2}a_b\Delta t^2 + \frac{1}{6}j_b\Delta t^3, \\
    v_{\text{pred}} &= v_b + a_b\Delta t + \frac{1}{2}j_b\Delta t^2.
\end{align*}
\]

(19)

(20)

[2] Evaluate the acceleration \(a_e\) and the jerk \(j_e\) at the end of the step.

[3] Correct positions and velocities using equations (17) and (18), using the following formula:

\[
\begin{align*}
    r_e &= r_{\text{pred}} + \frac{1}{24}r^{(4)}_b\Delta t^4 + \frac{1}{120}r^{(5)}_b\Delta t^5, \\
    v_e &= v_{\text{pred}} + \frac{1}{6}r^{(4)}_b\Delta t^3 + \frac{1}{24}r^{(5)}_b\Delta t^4,
\end{align*}
\]

(21)

(22)

where \(r^{(i)}_b\) are the \(i\)-th derivatives which are calculated as:

\[
\begin{align*}
    r^{(4)}_b &= \frac{1}{\Delta t^2}[-6(a_b - a_e) - \Delta t(4j_b + 2j_e)], \\
    r^{(5)}_b &= \frac{1}{\Delta t^3}[12(a_b - a_e) + 6\Delta t(j_b + j_e)].
\end{align*}
\]

(23)

(24)
Repeat [2]–[3] until the values of \( r_e \) and \( v_e \) converge.

Equations (21)–(24) are equivalent to equations (17) and (18). However, if the stepsize is calculated from higher order derivatives, equations (21)–(24) are preferred.

The local truncation error of equation (17) is \( O(\Delta t^6) \) and that of equation (18) is \( O(\Delta t^5) \). The global error is \( O(\Delta t^4) \) for both \( r \) and \( v \). One could also use the truncated form,

\[
\begin{align*}
    r_e &= r_b + \frac{1}{2}(v_e + v_b)\Delta t - \frac{1}{12}(a_e - a_b)(\Delta t)^2, \\
    \quad (25)
\end{align*}
\]

which has a local error of \( O(\Delta t^5) \) and a global error (that is, integrated over some fixed time interval) of \( O(\Delta t^4) \) for both \( r \) and \( v \). For the truncated Hermite scheme, the local truncated error in the position during one step is \( O(\Delta t^5) \), while it is \( O(\Delta t^6) \) in the full Hermite scheme. In either case, the global energy error is \( O(\Delta t^4) \).

### 3.2. Time-symmetrization

We now describe the basic idea of the time-symmetric scheme. First, we consider an integration scheme with a constant stepsize. An arbitrary integration scheme with constant stepsize \( \Delta t \) may be expressed compactly as follows:

\[
\xi_e = f(\xi_b, \xi_e, \Delta t), \quad (26)
\]

where \( \xi \) represents the phase-space variables, i.e. \( \xi = (r, v) \). If \( f \) does not depends on \( \xi_e \), the scheme is explicit. Otherwise, the scheme is implicit.

If the scheme is time-symmetric, the following equation holds

\[
\bar{\xi} \equiv f(\xi_e, \xi_b, -\Delta t) = \xi_b. \quad (27)
\]

For example, a leap-frog scheme with constant stepsize, which is a well-known symplectic
scheme, is also time-symmetric. The leap-frog scheme is an explicit scheme. However, many useful time-symmetric schemes are implicit.

The simplest example of an implicit time-symmetric scheme is the implicit trapezoidal rule, which may be expressed as:

$$\xi_e = \xi_b + \frac{1}{2}(\phi(\xi_b) + \phi(\xi_e))\Delta t,$$  \hspace{1cm} (28)

where

$$\phi(\xi) \equiv \frac{d\xi}{dt}. \hspace{1cm} (29)$$

In order to integrate for one step, the implicit equation (28) is solved iteratively until convergence is obtained. There are also time-symmetric integrators of higher order. The Hermite scheme [equations (17) and (18), or (25) and (18)] is an example of a fourth-order symmetric integrator.

Now we consider time-symmetric schemes with variable stepsize. If the scheme is time-symmetric, the following equations are satisfied:

$$\begin{align*}
\xi_e &= f(\xi_b, \xi_e, \Delta t_b), \\
\xi_b &= f(\xi_e, \xi_b, -\Delta t_e),
\end{align*}$$ \hspace{1cm} (30)

where $\Delta t_b$ and $\Delta t_e$ are the stepsize determined at the beginning of the step and the end of the step by some function $\delta(\xi)$, i.e.

$$\Delta t_b = \delta(\xi_b, \xi_e); \hspace{0.5cm} \Delta t_e = \delta(\xi_e, \xi_b). \hspace{1cm} (31)$$

In order to guarantee that equations (30) hold, it is sufficient to make the stepsize criterion time-symmetric:

$$\delta(\xi_b, \xi_e) = \delta(\xi_e, \xi_b). \hspace{1cm} (32)$$
One simple way to construct a timestep criterion that satisfies equation (32) is to take the average of stepsize calculated at the beginning and the end of the step. For example,

$$
\delta(\xi_b, \xi_e) \equiv \frac{1}{2}[s(\xi_b) + s(\xi_e)],
$$

(33)
or

$$
\delta(\xi_b, \xi_e) \equiv \sqrt{\frac{1}{2}[s(\xi_b)^2 + s(\xi_e)^2]}.
$$

(34)

Equations (33) and (34) are implicit equations. The stepsize is determined by solving equation (33) or (34) coupled with equations (17) and (18) [or (25) and (18)].

### 3.3. Energy Stabilization

Energy stabilization has been used in traditional programs to conserve the binding energy of a binary. In a stabilized scheme, energy conservation is enforced by introducing an artificial acceleration into the equation of motion, as follows (Baumgarte 1973):

$$
\frac{d^2 \mathbf{r}}{dt^2} = \frac{(m_1 + m_2)\mathbf{r}}{|\mathbf{r}|^3} + \mathbf{P} - \left(\frac{\alpha}{\Delta t}\right) \frac{(h_{\text{integ}} - h_{\text{direct}})}{v \cdot v} \mathbf{v}.
$$

(35)

Following the prescription by Aarseth (1985), equation (35) is transformed to KS coordinates as:

$$
\frac{d^2 \mathbf{u}}{d\tau^2} = \frac{1}{2} \mathbf{u} + \frac{1}{2} \mathbf{u}^T \mathbf{\lambda}^T \mathbf{P} - \left(\frac{\alpha}{\Delta \tau}\right) \frac{r(h_{\text{integ}} - h_{\text{direct}})}{M} \mathbf{u}'.
$$

(36)

The third term of the right-hand side of equation (36) is the artificial stabilizing acceleration. The terms $h_{\text{integ}}$ and $h_{\text{direct}}$ are the specific binding energy of the binary obtained by integrating equation (16) and evaluating equation (15), respectively. Figure 1 shows the time variation of $h_{\text{direct}}$ for different values of $\alpha$. When $\alpha$ is less than 0.3, the behavior of the energy is simple. When $\alpha$ is larger than 0.3, the behavior of the energy is complex. With small $\alpha$, $h_{\text{direct}}$ cannot follow $h_{\text{integ}}$ fast enough. In fact, for $\alpha = 0.1$, a secular error of
\[ \sim 5 \times 10^{-7}/\text{orbit} \] was observed. For large \( \alpha \), the spikes around the periastron is very large and the error is enough adjusted every period. Aarseth recommended to use \( \alpha = 0.4 \). In this paper, we set \( \alpha \) to 0.4.

4. Implementation of the Symmetrized KS Hermite Scheme

4.1. The KS Hermite scheme

The equations of motion of the KS binary are (14) and (16). We integrate them using the time-symmetrized Hermite scheme.

\[
\begin{align*}
\mathbf{u}_e &= \mathbf{u}_b + \frac{1}{2}(\mathbf{u}_e^{(1)} + \mathbf{u}_b^{(1)}) \Delta \tau - \frac{1}{10}(\mathbf{u}_e^{(2)} - \mathbf{u}_b^{(2)}) \Delta \tau^2 + \frac{1}{120}(\mathbf{u}_e^{(3)} + \mathbf{u}_b^{(3)}) \Delta \tau^3, \\
\mathbf{u}_b^{(1)} &= \mathbf{u}_b^{(1)} + \frac{1}{2}(\mathbf{u}_e^{(2)} + \mathbf{u}_b^{(2)}) \Delta \tau - \frac{1}{12}(\mathbf{u}_e^{(3)} - \mathbf{u}_b^{(3)}) \Delta \tau^2, \\
h_e &= h_b + \frac{1}{2}(h_0^{(1)} + h_1^{(1)}) \Delta \tau - \frac{1}{12}(h_1^{(2)} - h_0^{(2)}) \Delta \tau^2.
\end{align*}
\]

Here and below we denote the \( i \)-th derivative with respect to a variable \( \tau \) by using the operator \( ^{(i)} \). For example,

\[ \frac{d^2 \mathbf{u}}{d\tau^2} = \mathbf{u}^{(2)}. \]

The second and third derivatives are calculated as (Aarseth 1985):

\[
\begin{align*}
\mathbf{u}_b^{(2)} &= \frac{1}{2}h_b \mathbf{u}_b + \frac{1}{2}r_b \mathcal{L}_b^T(\mathbf{u}_b)\mathbf{P}_b, \\
\mathbf{u}_b^{(3)} &= \frac{1}{2}h_b \mathbf{u}_b^{(1)} + \frac{1}{2}h_b^{(1)} \mathbf{u}_b + \frac{1}{2}r_b \mathcal{L}_b^T(\mathbf{u}_b)\mathbf{P}_b + \frac{1}{2}r_b \mathcal{L}_b^T(\mathbf{u}_b) r_b J_{p,b}, \\
h_1^{(1)} &= -2\mathbf{u}^{(1)} \cdot \mathcal{L}^T(\mathbf{u})\mathbf{P}, \\
h_2^{(2)} &= -2[\mathbf{u}^{(2)} \cdot \mathcal{L}^T(\mathbf{u})\mathbf{P} + \mathbf{u}^{(1)} \cdot \mathcal{L}^T(\mathbf{u}^{(1)})\mathbf{P} + \mathbf{u}^{(1)} \cdot \mathcal{L}^T(\mathbf{u}) r J_p].
\end{align*}
\]
The quantities $R_b$ and $L_b$ are, respectively, the separation of the binary in Cartesian coordinates and the transformation matrix at time $\tau = \tau_b$. Because of the numerical difficulties associated with the singularity in equation (15) as $R \to 0$ (Aarseth, 1972), we integrate the binding energy $h$ of the binary instead of calculating it from its definition at each step.

4.2. Timestep Function

The timestep symmetrization technique described in section 3.2 can be applied to an arbitrary algorithm for calculating the stepsize.

The standard Aarseth formula (Aarseth 1985), which is being used in most $N$-body programs, is expressed as

$$
\Delta \tau = s(u) = \eta \sqrt{\frac{|u(4)||u(2)| + |u(3)|^2}{|u(5)||u(3)| + |u(4)|^2}}.
$$

(46)

Here $\eta$ is an accuracy parameter. The stepsize criterion is known to have good properties.

For general $N$-body problem, this criterion works very well. For a perturbed two-body problem with KS regularization, however, a more simpler schemes might be sufficient. We tried two formula.

If we adopt the symmetrized scheme, we may be able to carry out an accurate and fast time integration without the standard formula. We can use simpler functions such as

$$
\Delta \tau = \eta \frac{|u|}{|u(1)|},
$$

(47)

or

$$
\Delta \tau = s(u) = \sqrt{\frac{|u(2)||u| + |u(1)|^2}{|u(5)||u(1)| + |u(2)|^2}},
$$

(48)

as the stepsize function for the KS timestep $\Delta \tau$. 
Here “simpler” means that equations (47) and (48) don’t require the higher-order terms such as \( u^{(4)} \) and \( u^{(5)} \). It makes the convergence faster.

Note that each of the functions (46) and (48) gives a constant stepsizes in KS coordinates for the time integration of a simple binary. For a time-symmetrized scheme, it is better to use the function which gives a nearly constant stepsizes. This is because the estimation of the initial value of iterations becomes easier.

4.3. Implementation in General N-body Systems

4.3.1. For a Few Body Problem

We now describe the implementation of the symmetrized KS Hermite scheme in a full N-body code. We consider here only a shared-timestep scheme, in which the whole system is advanced with a single global timestep. This method is sufficient for 3- or 4-body systems.

The procedure for a single step of the time integration is as follows. At the beginning of the step, the relative position and velocity of the binary components in KS coordinates are known. The positions and velocities of other particles are given in Cartesian coordinates. The binary perturbation \( P \) and its time derivative \( J_p \) are first calculated in Cartesian coordinates. Then the relative position of the binary components is predicted in KS coordinates with KS time step \( \Delta \tau \), while the center of the mass of binary and all other particles are predicted in Cartesian coordinates with physical time step \( \Delta t \), which is a function of \( \Delta \tau \) and \( u \).

The stepsize \( \Delta \tau \) in the regularized system and the corresponding physical stepsizes \( \Delta t \) must be calculated in such a way as to ensure the time-reversibility of the whole N-body
system. We use the following formula to advance the physical time:

\[ t_1 = t_0 + \Delta t, \quad (49) \]

\[ \Delta t = T(\Delta \tau) = t^{(1)}_{1/2} \Delta \tau + t^{(3)}_{1/2} \frac{\Delta \tau^3}{24} + t^{(5)}_{1/2} \frac{\Delta \tau^5}{1920}, \quad (50) \]

Equation (50), which is a 5th order expression, is calculated by using the first through the fifth derivatives of the positions (i.e. all derivatives available during the integration). The formulation is the same as used by Aarseth (1985), except that we use the derivatives at the midpoint of the integration interval, which eliminates all even derivatives. Here \( t^{(i)}_{1/2} \) are the \( i \)-th derivatives at \( \tau = \tau_0 + 0.5\Delta \tau \) in KS time. They are calculated as:

\[ t^{(1)}_{1/2} = r = u \cdot u, \quad (51) \]

\[ t^{(3)}_{1/2} = 2(u \cdot u^{(2)} + u^{(1)} \cdot u^{(1)}), \quad (52) \]

\[ t^{(5)}_{1/2} = 2[u \cdot u^{(4)} + 4(u^{(1)} \cdot u^{(3)}) + 3(u^{(2)} \cdot u^{(2)})], \quad (53) \]

where

\[ u^{(1)}_{1/2} = u_0 + \frac{\Delta \tau}{2} u^{(2)}_0 + \frac{\Delta \tau^2}{8} u^{(3)}_0 + \frac{\Delta \tau^3}{48} u^{(4)}_0 + \frac{\Delta \tau^4}{384} u^{(5)}_0, \quad (54) \]

\[ u^{(1)}_{1/2} = u_0^{(1)} + \frac{\Delta \tau}{2} u^{(2)}_0 + \frac{\Delta \tau^2}{8} u^{(3)}_0 + \frac{\Delta \tau^3}{48} u^{(4)}_0 + \frac{\Delta \tau^4}{384} u^{(5)}_0, \quad (55) \]

\[ u^{(2)}_{1/2} = u_0 + \frac{\Delta \tau}{2} u^{(3)}_0 + \frac{\Delta \tau^2}{8} u^{(4)}_0 + \frac{\Delta \tau^3}{48} u^{(5)}_0, \quad (56) \]

\[ u^{(3)}_{1/2} = u_0^{(3)} + \frac{\Delta \tau}{2} u^{(4)}_0 + \frac{\Delta \tau^2}{8} u^{(5)}_0, \quad (57) \]

\[ u^{(4)}_{1/2} = u_0^{(4)} + u_0^{(5)} \frac{\Delta \tau}{2}, \quad (58) \]

It may be sufficient to use the 4th-order Hermite scheme (the truncated Hermite scheme) to integrate the physical time, since the phase-space variables are globally integrated to 4th-order accuracy. More detailed error analysis will be given elsewhere.
The details of a single integration step are as follows.

[1] Predict the positions and velocities of all particles.
[2] Evaluate all accelerations and jerks using predicted positions and velocities.
[3] Calculate the fourth and fifth derivatives of the relative position of the binary components in KS coordinates at both the beginning and the end of the step using the following formulae:

\[
\begin{align*}
  u_b^{(4)} &= -\frac{1}{\Delta \tau_b^2} \left[ 6(u_b^{(2)} - u_e^{(2)}) + 2\Delta \tau_b (2u_b^{(3)} + u_e^{(3)}) \right], \\
  u_b^{(5)} &= \frac{1}{\Delta \tau_b^3} \left[ 12(u_b^{(2)} - u_e^{(2)}) + 6\Delta \tau_b (u_b^{(3)} + u_e^{(3)}) \right], \\
  u_e^{(4)} &= u_b^{(3)} + \Delta \tau_b u_b^{(4)}, \\
  u_e^{(5)} &= u_b^{(4)}. 
\end{align*}
\]

(59) 

We calculate the higher order derivatives for other particles and the specific energy of the binary in the same way.

[4] Calculate the new stepsize and the physical stepsize at the beginning of the step. For example,

\[
\begin{align*}
  \Delta \tau_{new} &= \frac{1}{2} \left[ s(u_b) + s(u_e) \right], \\
  \Delta t_{new} &= T(\Delta \tau_{new}),
\end{align*}
\]

(63) 

(64) 

[5] Correct the integrated values using the new stepsize \( \Delta \tau_{new} \):

\[
\begin{align*}
  u_e &= u_b + u_b^{(1)} \Delta \tau_{new} + \frac{1}{2} u_b^{(2)} \Delta \tau_{new}^2 + \frac{1}{6} u_b^{(3)} \Delta \tau_{new}^3 + \frac{1}{24} u_b^{(4)} \Delta \tau_{new}^4 + \frac{1}{120} u_b^{(5)} \Delta \tau_{new}^5, \\
  u_e^{(1)} &= u_b^{(1)} + u_b^{(2)} \Delta \tau_{new} + \frac{1}{2} u_b^{(3)} \Delta \tau_{new}^2 + \frac{1}{6} u_b^{(4)} \Delta \tau_{new}^3 + \frac{1}{24} u_b^{(5)} \Delta \tau_{new}^4.
\end{align*}
\]

(65) 

(66) 

[6] Repeat procedures [2]–[5] until both the stepsize and the integrated variables converge.
About the convergence of integrated variables, there is one simple method in which the procedure shall be repeated until the variables of all particles converge. Even if we adopt this simplest method, the number of iterations of the procedure is not so large for a few-body systems. In fact, in an example shown in the following section, the number of iterations was typically 4 for the case of an integration of a hierarchical triplet. The number does not much exceed that required to converge the phase space variables in the ordinary Hermite scheme, which is $2 \sim 3$.

Even for more complicated cases, we may make the number of iterations small if we choose an appropriate criterion formula to determine the stepsize $\Delta\tau$, or if we choose an appropriate accelerator to converge the variables.

4.3.2. For a Large N Body Systems

The outline of the procedure of integration of one step is same as that for the cases of few body systems. However, there are two difficulties. One is the increase of the number of iterations until convergence. Another is how to implement both symmetrized scheme and individual time step scheme. For the system with larger value of $N$, the number of iteration may become larger. For larger values of $N$, individual timesteps become necessary to achieve reasonable efficiency (Aarseth 1985).

Implementation of time-symmetrization for individual timesteps will be discussed elsewhere. Here we only give a comment about this point. In a time integration of a large $N$-body systems, it is not necessary to symmetrize all variables of all particles. To symmetrize the integration of binaries (and its close neighbors) would be sufficient.
5. Numerical Experiments

Figures 2a–2c show the behavior of the errors in the energy, eccentricity, and angular momentum of a binary with initial eccentricity $e = 0.999999$. In figures 2a–2c, the values at the apocenter are plotted. These figures show that there is no secular error for 2000 binary periods. The specific energy is calculated by equation (15).

In the following subsections, we compare (1) the symmetrized KS–Hermite scheme, (2) the “plain” (i.e. neither symmetrized nor stabilized) KS–Hermite scheme, and (3) the stabilized KS–Hermite scheme, for $e = 0.9$. We have carried out other comparisons, with initial eccentricities ranging from $e = 0.0$ to $e = 0.999999$. In all cases we obtain qualitatively the same results as are shown below.

We have investigated the dynamical evolution of both a simple binary and a hierarchical triple. The initial conditions used in our experiments are summarized in Table 1. In the case of the binary the integration was performed for 2000 orbits. For the triple, we integrated the system for 2000 periods of the inner binary.

We used equation (48) as the stepsize function and equation (34) as the symmetric criterion. The coefficient $\eta$ in equation (48) was set to 0.01, corresponding to a stepsize of roughly $1/30$ of the (inner) binary period. The condition to stop the iterations was $|\Delta \tau_e - \Delta \tau_b| < 1.0 \times 10^{-15}$. The total number of timesteps was therefore $\sim 6 \times 10^4$ in all cases. All calculations were done in double precision (16-digit accuracy).

5.1. A Simple Binary

For this case, the number of iterations required by the symmetrization of stepsize $\Delta \tau$ is 2. The number of iterations necessary is small since $\Delta \tau$ should be constant, i.e., $\Delta \tau_b = \Delta \tau_e$ for the criterion (48).
Figures 3 and 4 show the result of an integration of the Kepler two-body problem for 2000 orbital periods of the binary. Figure 3 shows the error in the total energy (i.e. the error in the binding energy of the binary) for the three schemes listed above. Here we plot the error in the energy calculated from equation (15), not the error in the energy integrated using equation (16). The reason is that the integrated $h_{integ}$ is constant, because the binary is unperturbed. The calculated energy thus indicates the accuracy of the orbital integration. We also use this calculated energy for other cases discussed below.

Figure 3 shows that both the symmetrized and the stabilized schemes conserve energy, while the error in the energy increases linearly with time for the plain integrator. In the symmetrized case, the energy error $\Delta E/E$ is less than $10^{-12}$ after $10^3$ orbits, or about $10^5$ steps, indicating that it is mostly due to round-off error.

We have plotted the change of energy at regular time intervals. The error of the energy becomes large at the pericenter, though the error returns back to nearly zero at the apocenter. As a result, the beating of the frequency of the binary orbit and that of our sampling is observed for the symmetrized and stabilized cases. The behavior of energy in the stabilized case in figure 3 is more complicated than that in the symmetrized case, since the time variation of error around the pericenter is complicated (see figure 1d).

Figure 4 is the same as Figure 3, but for the error in the angular momentum $\Delta A$. Figure 4 shows that both the symmetrized and the plain schemes conserve total angular momentum, while the error in the angular momentum increases linearly in the stabilized case. The relative error in the angular momentum in the symmetrized case is again less than $10^{-12}$.

Figure 5 is the same as Figure 3, but for the eccentricity. This figure makes explicit the fact that the eccentricity of the binary is conserved by the symmetrized integrator, but not by either the plain or the stabilized schemes. This result is quite natural since the
eccentricity depends on both the energy and the angular momentum. The plain scheme conserves angular momentum but not energy, while the stabilized scheme conserves energy but not angular momentum. Thus neither scheme preserves the eccentricity. Only the symmetrized scheme conserves both energy and eccentricity up to round-off error.

5.2. A Hierarchical Triple

For this case, the number of iterations required by the symmetrization is about 4. That required by the convergence of the phase space variables is $2 \sim 3$ for the non-symmetrized Hermite scheme. The symmetrized scheme, therefore, is not much expensive than the non-symmetrized scheme.

Figures 6 and 7 show, for our three integration schemes, the relative errors in the total energy $\Delta E/E$ (Figure 6) and the total angular momentum $\Delta A$ (Figure 7) of the three-body system. In this case again, the relative error in the energy using the symmetrized scheme is no more than $10^{-12}$ and the error in the angular momentum is about $10^{-11}$ after $10^5$ steps.

Figures 8-11 show the evolution of the inner binary. From the analytical treatment presented in the Appendix, the time evolution of energy, angular momentum and eccentricity of the (weakly perturbed) inner binary may be expressed as follows.

$$\Delta E/E \cong 0.0, \quad (67)$$

$$\Delta A \cong 0.5 \cdot 10^{-3} [1 - \cos(2\Omega_0 t)], \quad (68)$$

$$\Delta e \cong 0.25 \cdot 10^{-3} [-1 + \cos(2\Omega_0 t)], \quad (69)$$

where subscript 0 indicates unperturbed values.

Figure 8 shows the variation of the specific energy of the binary. The specific energy is conserved in both the symmetrized and the stabilized cases, while it increases linearly in
the plain case. Figure 9 is the same as Figure 8, but for the binary angular momentum. Figure 10 is the same as Figure 8, but for the binary eccentricity. In figures 8, 9, and 10, the bare values at the apocenter of the inner binary are plotted. As shown in figures 8–10, since the amplitude of the variations during one period of the outer binary is much larger than the numerical errors incurred within one period, as shown in equations (64)-(66), the numerical error could not be seen if the bare values were plotted. In order to show the secular variation, we applied the linear regression. Straight lines in figures 8–10 are the results of the least square fitting. Figures 8, 9, and 10 show that there are no significant secular variations in the symmetrized case, and that the eccentricity of the binary is not correctly followed by the stabilizing integrator. Comparing figures 7 and 9, we see that the error in the total angular momentum comes mostly from the integration of the KS binary in the stabilized case.

Figures 11a and b show the time variation of the angular momentum and the eccentricity of the inner binary during the first few periods of the outer binary. The dotted curves correspond to the analytical values, while the solid curve corresponds to the results of numerical experiments in the symmetrized case. Figures 11a and b show that the numerical experiments agree well with the analytical results.

To summarize, our numerical experiments show that both the total system and the inner binary are integrated correctly in the symmetrized case. In other words, the momentum transfer between the outer particle and the inner binary is accurately followed. Thus, our scheme follows the total system consistently, even though part of the system is expressed in KS coordinates and the rest in physical coordinates.
6. Summary and Discussion

We have developed a new algorithm for the integration of perturbed binary motion. The algorithm is constructed using two techniques: KS regularization and time-symmetrization. Our algorithm allows variable stepsize, but produces no secular error in either the binding energy or the eccentricity. We have presented the results of numerical experiments for one case with $e = 0.9$. We have also performed experiments for other eccentricities, and have confirmed the superiority of the method in cases ranging from circular ($e = 0$) to highly elongated ($e = 0.999999$).

Our scheme is composite, in the sense that the internal motion of the binary and the motion of the rest of the system are integrated in separate coordinate systems. This is rather different from the usual applications of symmetric or symplectic integrators, where all degrees of freedom are integrated in the same way. For an unperturbed binary, our algorithm simply applies the symmetrization to the system with a transformed Hamiltonian. Thus it is not surprising that our algorithm works well for unperturbed binaries.

For perturbed systems, the result of numerical experiment shows that the error caused by any inconsistency between the integration of the system in the physical coordinates and the KS coordinates is smaller than the error caused by the integration of KS binary. Furthermore, our numerical experiments shows that the calculation cost of the new scheme is not very high compared with that of ordinary Hermite scheme. This result suggests that our scheme is valid for weakly perturbed cases. An accurate integration of long surviving binary is the main target of the time symmetrized scheme. Since the perturbation on such a binary is weak, our scheme is excellent from a practical point of view.

In conclusion, we have shown that the time-symmetrization scheme, introduced by Hut et al. (1995) can be successfully generalized to include KS regularization. It is clear that our scheme can provide excellent accuracy for individual demanding cases. Under which
circumstances this new scheme will prove to be competitive will have to await a detailed
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A. Appendix: Analytical treatment of a hierarchical triple

We introduce physical coordinates as shown in figure 12. The origin is placed at the
center of mass of the inner binary. The masses of particles 1 and 2 are $m_1 = m_2 = m = 1/2$.
The unperturbed motion of particles 1 and 2 may be expressed as

\begin{align}
\mathbf{r}_1 &= (r_1 \cos \phi, r_1 \sin \phi), \quad (A1) \\
\mathbf{v}_1 &= (\dot{r}_1 \cos \phi - r_1 \dot{\phi} \sin \phi, \dot{r}_1 \sin \phi + r_1 \dot{\phi} \cos \phi), \quad (A2) \\
\mathbf{r}_2 &= -(r_2 \cos \phi, r_2 \sin \phi), \quad (A3) \\
\mathbf{v}_2 &= -(\dot{r}_2 \cos \phi - r_2 \dot{\phi} \sin \phi, \dot{r}_2 \sin \phi + r_2 \dot{\phi} \cos \phi), \quad (A4)
\end{align}

where $\mathbf{r}_i$ and $\mathbf{v}_i$ are the position and velocity and the radius of particle $i$, $r_i = |\mathbf{r}_i|$, and $\phi$ is
the true anomaly of the inner binary.

The unperturbed motion of particle 3 is expressed as a circular orbit around the center
of mass of the binary:

\[ \mathbf{r}_3 = (R \cos \psi, R \sin \psi) = (R \cos \Omega t, R \sin \Omega t), \quad (A5) \]
where $R$ is the distance from the third body to the binary center of mass and $\psi$ is the true anomaly of the outer orbit.

Since $m_1 = m_2$, $r_1 = r_2 = r$, where $r$ is half the distance between particles 1 and 2. The accelerations of particles 1 and 2 due to particle 3 are:

$$a_1 = m_3 \left( R \cos \psi - r \cos \phi, R \sin \psi - r \sin \phi \right) \frac{1}{\sqrt{R^2 + r^2 - 2Rr \cos(\phi - \psi)}}$$ \hspace{1cm} (A6)$$ $$a_2 = m_3 \left( R \cos \psi + r \cos \phi, R \sin \psi + r \sin \phi \right) \frac{1}{\sqrt{R^2 + r^2 + 2Rr \cos(\phi - \psi)}}$$ \hspace{1cm} (A7)$$

For unperturbed motion of the inner binary, the binary separation and time derivatives of the separation and the phase angle are

$$r_1 = r_2 = r = \frac{a(1 - e^2)}{2(1 + e \cos \phi)},$$ \hspace{1cm} (A8)$$ $$\dot{r}_1 = \dot{r}_2 = \dot{r} = \frac{Ae \sin \phi}{2a(1 - e^2)},$$ \hspace{1cm} (A9)$$ $$\dot{\phi} = \frac{A}{4r^2} = \frac{A(1 + e \cos \phi)^2}{a^2(1 - e^2)^2}.$$ \hspace{1cm} (A10)$$

Here $a$, $e$ and $A$ are the semi-major axis, the eccentricity, and the specific angular momentum of the relative motion of the inner binary; $A = \sqrt{a(1 - e^2)}$.

### A.1. Energy Variation

The specific binding energy of the inner binary, and its variation due to infinitesimal changes $\delta x_i$ and $\delta v_i$, are

$$E = \sum_{i=1,2} \frac{1}{2} v_i \cdot v_i - \frac{m}{2r},$$ \hspace{1cm} (A11)$$ $$\delta E = \sum_{i=1,2} v_i \cdot \delta v_i + \frac{m \mathbf{r} \cdot \delta \mathbf{r}}{2r^3} = - \sum_{i=1,2} v_i \cdot \delta v_i,$$ \hspace{1cm} (A12)$$

Expressing the variation of the velocity as $\delta v_i = a_i \delta t$, we obtain

$$\delta E = - \sum_{i=1,2} v_i \cdot a_i \delta t.$$ \hspace{1cm} (A13)$$
From equations (A1)–(A4), we obtain the following equation:

$$\frac{dE}{dt} = m_3 \left\{ \frac{R \ddot{r} \cos(\phi - \psi) - R r \dot{\phi} \sin(\phi - \psi) - r \ddot{r}}{[R^2 + r^2 - 2Rr \cos(\phi - \psi)]^{3/2}} - \frac{R \ddot{r} \cos(\phi - \psi) - R r \dot{\phi} \sin(\phi - \psi) + r \ddot{r}}{[R^2 + r^2 + 2Rr \cos(\phi - \psi)]^{3/2}} \right\}. \tag{A14}$$

Substituting equations (A8), (A9), and (A10) into equation (A14), and expanding with respect to $a/R$, we obtain

$$\frac{dE}{dt} = \frac{m_3}{R^3} \left\{ -2r \ddot{r} + \frac{6r \cos(\phi - \psi)}{R} \left[ R \ddot{r} \cos(\phi - \psi) - R r \dot{\phi} \sin(\phi - \psi) \right] \right\}. \tag{A15}$$

Assuming that the change in the position of particle 3 is negligible, we can integrate equation (A13) along the unperturbed orbit of the inner binary. Integrating it over one period of the inner binary, we obtain:

$$\Delta E = 0. \tag{A16}$$

This result is usually called as adiabatic invariance.

### A.2. Angular Momentum and Eccentricity

The change in the specific angular momentum of the inner binary is

$$A = 2 \sum_{i=1,2} |\mathbf{r}_i \times \mathbf{v}_i|, \tag{A17}$$

$$\delta A = 2 \sum_{i=1,2} |\delta \mathbf{r}_i \times \mathbf{v}_i| + |\mathbf{r}_i \times \delta \mathbf{v}_i| \tag{A18}$$

Again we neglect the variation in $R$ as we did in the previous section and replace $\delta \mathbf{v}$ with $a \delta t$, to find

$$\delta A = 2 \ m_3 \frac{1}{2} a (1 - e^2) \left\{ \frac{\left[ (\cos \phi, \sin \phi) \times (R \cos \psi - r \cos \phi, R \sin \psi - r \sin \phi) \right]}{[R^2 + r^2 - 2Rr \cos(\phi - \psi)]^{3/2}} \right\}.$$
\[\begin{align*}
\frac{\Delta E}{E} &= 0.0, \quad \text{(A24)}
\end{align*}\]
\[ \Delta A(t) = \frac{3m_3}{8\pi} \left( \frac{P_{\text{out}}}{P_{\text{in}}} \right) \frac{a \tilde{A}(1 - e^2) \tilde{a}}{R^3} B(e) \left[ 1 - \cos(2\Omega_0 t) \right], \quad (A25) \]

\[ \Delta e(t) = \frac{3}{8\pi} \left( \frac{P_{\text{out}}}{P_{\text{int}}} \right) \frac{a^3(1 - e^2)^4}{R^3 e} B(e) \left[ -1 + \cos(2\Omega_0 t) \right]. \quad (A26) \]

Here \( P_{\text{in}} \) is the unperturbed period of the inner binary.

For the case of our numerical experiments (i.e. \( e = 0.9 \)), equations (A24)–(A26) yield

\[ \frac{\Delta E(t)}{E} \approx 0.0, \quad (A27) \]

\[ \Delta A(t) \approx 0.46 \cdot 10^{-3} \left( 1 - \cos 2\Omega_0 t \right), \quad (A28) \]

\[ \Delta e(t) \approx 0.23 \cdot 10^{-3} \left( -1 + \cos 2\Omega_0 t \right). \quad (A29) \]
REFERENCES

Aarseth, S. J., 1972, in *Gravitational N-body Problem*, p.380-p.387, Reidel, Holland

Aarseth, S. J., 1985, *Multiple Time Scales*, Academic Press

Aarseth, S. J., 1988, “Integration Methods for Small N-Body Systems” in *The Few Body Problem*, (Valtonen M. J., ed.), 287–306, Kluwer, Dordrecht, Holland

Aarseth, S. J., 1994, “Direct Methods for N-body Simulations”, in *Galactic Dynamics and N-Body Simulations*, G. Contopoulos, N.K. Spyrou and L. Vlahos, eds. (Springer-Verlag), pp.277-312.

Hut, P., McMillan, S., Goodman, J., Mateo, M., Phinney, E. S., Pryor, C., Richer, H. B., Verbunt, F., and Weinberg, M., 1992, PASP, 104, 981

Hut, P., Makino, J., and McMillan, S., 1995, ApJ443, L93

Levi-Civita, T., 1956, Opere Mathematiehe, 2

Makino, J., 1991, ApJ, 369, 200

Makino, J. and Aarseth, S. J., 1992, PASJ, 44, 141

Makino, J. and Hut, P., 1990, ApJ, 365, 208

Quinlan, G. D. and Tremaine, S., 1990, AJ, 100, 1694

Rasio, F. A., and Heggie, D. C., 1995, ApJ, submitted.

Sanz–Serna, J. M., and Calvo, M. P., 1994, *Numerical Hamiltonian Problems*, (Chapman and Hall).

Skeel R. D. and Gear, C. W., 1992, *Physica D*, 60, 311
Table 1: Initial orbit parameters for the simple binary case.

| $m_1$ | $m_2$ | $a$  | $h$   | $e$    | period [$2\pi$] |
|-------|-------|------|-------|--------|-----------------|
| 0.5   | 0.5   | 1.0  | -0.5  | 0.9e+0 | 1               |

Table 2: Initial orbit parameters for the hierarchical triplet case.

|          | $m_1$ | $m_2$ | $a$  | $h$   | $e$    | period [$2\pi$] |
|----------|-------|-------|------|-------|--------|-----------------|
| inner    | 0.5   | 0.5   | 1.0  | -0.5  | 0.9e+0 | 1               |
| outer    | 1.0   | 0.01  | 10.1 | -0.05 | 0.0e+0 | 32              |

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Fig. 1.— Effect of stabilizing parameter \( \alpha \) on time variation of error in binding energy of a binary with eccentricity \( e = 0.9 \). Figures (a) through (f) correspond to the cases of \( \alpha = 0.1, 0.2, 0.3, 0.4, 0.5, \) and 0.6, respectively.

Fig. 2.— Time evolution of errors for the case \( e = 0.999999 \). Unit of time is the period of the binary. 2a). Relative error in the specific energy of the binary. 2b). Error in the angular momentum. 2c). Error in the eccentricity.

Fig. 3.— Time evolution of relative error in binding error for the simple binary case. Unit of the horizontal axis is the period of the binary in KS coordinates. Solid, dashed and long-dashed lines correspond to the cases of symmetrized, “plain” and stabilized case, respectively.

Fig. 4.— Same as Figure 3 but for the angular momentum. Top and bottom are the same figures except for the scale of vertical axis.

Fig. 5.— Same as Figure 3 but for the eccentricity.

Fig. 6.— Time evolution of the error of total energy for the perturbed binary case. The curves are same as those in Figure 3 The unit of horizontal axis are the period of the inner binary in its KS coordinates.

Fig. 7.— Same as Figure 6 but for the angular momentum.

Fig. 8.— Time evolution in the specific energy of the inner binary for the perturbed binary case. The curves and and the unit of horizontal axis are same as those in Figure 6.

Fig. 9.— Same as Figure 8 but for the specific angular momentum of the inner binary.

Fig. 10.— Same as Figure 8 but for the eccentricity of the inner binary.

Fig. 11.— The evolution of the inner binary for the symmetrized case. Unit of time is the period of the inner binary. Solid and dashed curves correspond to the numerical experiment and theoretical, respectively. 11a). Angular momentum. 11b). Eccentricity.

Fig. 12.— Coordinates of the hierarchical triple.