Cryptography and Non Commutative cohomology.

Aristide Tsemo¹, College Boréal, Canada
*Corresponding author: tsemo58@yahoo.ca

Abstract.

In this paper, we study encryption in networks with the notion of Grothendieck site. We use global objects defined in geometry to characterize data conveyed in a network. This approach can be used to define efficiently keys for public encryption.

0. Introduction.

A network is a set $N$ such that for each elements $u$, $v$ of $N$, there exists a set $Hom(u,v)$ called the set of connections between $u$ and $v$. In practice, the set $N$ can be a set of peoples, and $Hom(u,v)$, the communications tools used by $u$ and $v$ to exchange data. Remark that we do not assume that $N$ is a category, that is the Chasles relation is not verified by the elements of the sets $Hom(u,v)$. The network that we are going to consider here is a network whose users are computers, and the routes between two users $u$ and $v$ are wires or wireless communication between $u$ and $v$. The duality principle (The Ying-Yang principle) shows that the existence of a network $N$ implies the existence of a different network $N'$ (this is equivalent to the fact that the set of elements of $N$ is always defined as a subset of a bigger set) whose users are potential opponents to the users of $N$. We assume that each network verifies the life principle, that is its users or manager enforces its characteristics or equivalently reduce its entropy. This implies that the characteristic of the networks are time dependent. To enforce characteristics of a network, its users must develop a science which transforms elements outside of the network for its use. On this purpose they have to develop a graphology to represent the objects of their study. Different networks need to develop themselves, this induces concurrency, and justify the following assertion of Jean Paul Sartre: "Devil are the others", this can also be compared to the Indian Maya philosophy. The concurrency between different networks implies that the results of the knowledge that they develop is often submitted to a secret law. This gives rise to cryptography.

Cryptography is the science of secrecy of communications, that is, the study of secret (crypto) writing (graphy) which may be use to:

- conceal the meaning of a message (plaintext) for all except for the sender and the receiver.
- Verify the correctness of a message (authentication).
Cryptography appeared in the earlier human societies: Ancient Egyptians encrypted their hieroglyphic.
Julius Caesar created the Caesar cipher
Geoffrey Chaucéy, an English author included many ciphers in its work.
The clay of Phaistos were enciphered,...
The development of technologies has generated new types of encryption, like the Jefferson machine.
Electricity and electronic, have introduced a new language: the binary language nowadays used to encrypt texts typed with computers. The widespread development of computer science and internet has provided the need of security for data exchanged with these techniques. In the recent news, we have learnt about crimes perpetrated by hackers.

A language $E$ used to write a text is a finite set. A text is an ordered collection of words, that is a collection of ordered finite subsets of $E$. Let $P(E)$ be the set whose elements are subsets of $E$. An encryption map is a map $h : P(E) \rightarrow P(E')$. Note that the encrypted text can be written in a different alphabet. In practice the encryption map is described by a key or clue called the cipher, and the elements of its images are called the ciphertexts. An user uses a key $L$, that he applies to the plaintext $P$ to obtain the ciphertext $E(L,P) = C$.
The receiver receives the ciphertext $C$ that he decrypts with the key $L'$, and obtain $D(L',C) = P$.

There exists many types of ciphers which can be divided in two categories:
Symmetric ciphers: these are ciphers for which the knowledge of the key used for encryption is equivalent to the knowledge of the key used for decryption. examples of symmetric ciphers are substitutions ciphers like the Caesar cipher, transposition ciphers.
Asymmetric ciphers: these are ciphers for which the knowledge of the key used for the encryption does not imply the knowledge of the key used for the decryption like the R.S.A cipher, the Diffie-Hellman algorithm. They are used to define authentication protocols, digital signatures,.... The first method of public encryption appeared in a classified document published by the Communications-Electronics Security group, the Britain’s counterpart to N.S.A.

In public encryption, each user $U$ has two keys: its private key $L^1_U$ and its public key $L^2_U$, the key $L^2_U$ is known by the others users but not the key $L^1_U$. The secrecy of this protocol is due to the fact that it is infeasible to compute $L^1_U$ with $L^2_U$. To send a message $P$ to $V$, $U$ calculates $E(P,L^2_V) = C$, to decrypt the ciphertext $C$, $V$ calculates $D(L^1_V,C) = P$. In practice asymmetric ciphers are used to exchange symmetric keys between users, since the algorithms which define these ciphers are slow.

A symmetric cipher must encrypt block of large size of the plaintext if the length of the plaintext is big, otherwise the statistical properties (like the frequency of letters) of the language used to write the plaintext is reflected in the ciphertext. It is for these reasons that modern ciphers like D.E.S, A.E.S symmetric ciphers are applied to blocks of at least 64 bits. These ciphers are often the composition of many rounds, each round is roughly the composition of
the following operations: Permutation of the entries of the round, add a round
key, substitution of bits using an S-matrix, the application of a linear map,...
These operations which compose each round have to be elementary to be easily
implemented.

The plaintext block used in modern encryption is often endowed with an
algebraic structure like in A.E.S encryption, where plaintexts block are identified
with elements of a finite field. In this paper, we study plaintexts block endowed
with the structure of a finite algebraic variety or the structure of a finite scheme.

The ciphertexts must resist to cryptanalysis, which is the science of methods
of transforming an unintelligible text, the ciphertext to an intelligible text: the
plaintext. This is the science used by attackers. There exists many different
types of attacks:

Brute force attack: The opponent knows the ciphertext and the algorithm,
he tries every keys to find the plaintext.

Chosen plaintext attacks: the opponent knows the ciphertext, the algorithm,
and he can generates ciphertexts by inserting plaintexts in the encryption ma-
chine.

Chosen ciphertext attacks, the opponent knows the ciphertext, the algo-
rum, and he can generates plaintexts by decrypting ciphertexts.

The main challenge in the organization of a network is the distribution of
dkeys: suppose that two users U and V of a network N want to exchae-
encrypted data, how the keys needed for encryption can be provided to U and
V with secrecy. There exists many solutions to this problem, like the physical
distribution if the users are not physically far each other, they can use of a third
part called the key distribution center, another solution is public encryp-
ion.

The purpose of this paper is to study the geometric properties of cryptogra-
phy. Differential and algebraic geometry are divided in two fields:

The local study, in differential geometry, this is the study of the properties
differentiable maps of $\mathbb{R}^n$, and in algebraic geometry it is the theory of
commutative rings.

The global study, this is the study of geometric objects which are obtained
by gluing local objects.

Cryptography can be thought as a geometry for which the local study is
the study of encryption and decryption maps, the global study is the study of
crypted data conveyed in a network, that is link to link encryption, key distri-
bution center,... The main purpose of this paper is to study the global geometry
defined by cryptography. The natural framework for this study is the theory of
sites, these are categories endowed with a topology. We can endow naturally
a network with a topology, and interpret the global geometry of a network in
terms of torsors and higher non commutative cohomology objects defined on
this site. This point of view allow us to describe the key distribution center as
an initial object in a category. And is well adapted for public encryption. The
public and private keys are defined by a flat connection over a torsor, or a flat
connective structure over an $n$-gerbe. We study also statistical properties of
gerbe encryption.
Plan.

0. Introduction.
I. Topology of categories and torsors.
   Topology defined by a network.
   The groupoid associated to a network.
   The classifying cocycle associated to a torsor.
   Contraction of Torsor.
   The Diffie-Hellman torsor.
I.2. Connection on torsors and encryption.
   Generalization of the Diffie-Hellman torsor.
   Meet in the middle attack.
   The Grothendieck group of the equivalence classes of torsors defined on a
   network.
I.3. Link to Link encryption and torsors.
I.4. Key distribution center and initial object.
   Implementation of link to link encryption.
II. Non Abelian cohomology and end to end encryption.
   II.1. Gerbe and encryption.
   A protocol to define and to end encryption and link to link encryption with
   a gerbe.
   Meet in the Middle attack of encryption defined by gerbes.
   II.2. Connective structure on gerbe and public key encryption.
   II.3. Non commutative cohomology and probabilistic theory of ciphers.
   The entropy cocycle.
   Higher non Abelian cohomology and end to end encryption.
   Encryption with a tower of torsors.
   Attack of an encryption with a tower of torsors.
   Public key encryption and Tower of torsors.
Bibliography.

I. Topology of categories and torsors.

In this part we present the notion of Grothendieck topology that we shall
use to define encryption protocols in a network.

**Definition 1.**
A network is a finite oriented graph.

**Definitions 2.**
Let $E$ be a category, a **sieve** is a subclass $N$ of the class of objects $Ob(E)$
of $E$ such that if $f : X \to Y$ is a map of $E$, such that $Y \in N$, then $X \in N$.
Let $f : E' \to E$ be a functor, and $R$ a sieve of $E$, we denote by $N^f$, the sieve
defined by $N^f = \{ X \in Ob(E') : f(X) \in N \}$. 

4
For each object $T$ of $E$, we denote by $E_T$, the category whose objects are arrows $u : U \to T$, a morphism of $E_T$ between $u_1 : U_1 \to T$, and $u_2 : U_2 \to T$, is a map $h : U_1 \to U_2$ such that $u_2 \circ h = u_1$.

**Definition 3.**

A **topology** on $E$ is defined as follows: to each object $T$ of $E$, we associate a non-empty set $J(T)$ of sieves of the category $E_T$ of $E$, above $T$ such that:

(i) For each map $f : T_1 \to T_2$, and for each element $N$ of $J(T_2)$, $N' \in J(T_1)$.

(The morphism $f$ induces a functor between $E_{T_1}$ and $E_{T_2}$ abusively denoted $f$).

(ii) The sieve $N$ of $E_T$ is an element of $J(T)$, if for every map $f : T' \to T$ of $E$, $N' \in J(T')$.

A category endowed with a topology is called a site.

Examples of sites are:
The category of open subsets of a topological space $E$, for each open subset $U$, a sieve is a family of subsets $(U_i)_{i \in I}$ such that $\bigcup_{i \in I} U_i = U$.

Let $L$ be a field, we consider the category $C_L$ whose objects are finite products of finite extensions of $L$, a morphism $L_1 \to L_2$ induces a $L$-morphism $\text{Spec}(L_2) \to \text{Spec}(L_1)$ between the respective spectrum of $L_2$ and $L_1$. We define a topology on $C_L$ such that for every extension $L_1 \to L_2$, a sieve of $\text{Spec}(L_2)$, is a family of extensions of $L_2 (L_i)_{i \in I}$, such that the Galois group $\text{Gal}(L \mid L_2)$, where $L$ is the algebraic closure of $L$, is the inductive limit of the Galois groups $G(L \mid L_i)$.

**Notations.**

Let $U_{i_1}, \ldots, U_{i_p}$ be objects of a site $E$, we suppose that there exists a final objects. Let $C$ be a presheaf of categories defined on $E$. We will denote by $U_{i_1 \cdots i_p}$ the fiber product of $U_{i_1}, \ldots, U_{i_p}$ over the final object. If $e_i$ is an object of $C(U_{i_1}), e_{i_1 \cdots i_2 \cdots i_p}$ will be the restriction of $e_i$ to $U_{i_1 \cdots i_p}$. For a map $h : e \to e'$ between two objects of $C(U_{i_1 \cdots i_p})$, we denote by $h_{i_1 \cdots i_p}^{i_1+1 \cdots i_n}$ the restriction of $h$ to a morphism between $e_{i_1+1 \cdots i_n} \to e'_{i_1+1 \cdots i_n}$.

**Definitions 4.**

A **sheaf of sets** $L$ defined on the category $E$ endowed with the topology $J$, is a contravariant functor $L : E \to \text{Set}$, where $\text{Set}$ is the category of sets, such that for each object $U$ of $E$, and each element $R$ of $J(U)$, the natural map:

$$L(U) \longrightarrow \lim(L \mid R)$$

is bijective, where $(L \mid R)$ is the correspondence defined on $R$ by $(L \mid R)(f) = L(T)$ for each map $f : T \to U$ in $R$.

Let $h : F \to E$ be a functor, for each object $U$ of $E$, we denote by $F_U$ the subcategory of $F$ defined as follows: an object $T$ of $F_U$ is an object of $F$ such that $h(T) = U$. A map $f : T \to T'$ between a pair of objects $T$ and $T'$ of $F_U$, is a map of $F$ such that $h(f)$ is the identity of $U$. The category $F_U$ is called the **fiber** of $U$. For each objects $X$, and $Y$ of $F_U$, we will denote by $\text{Hom}_U(X, Y)$ the set of morphisms of $F_U$ between $X$ and $Y$.
I.1. The topology defined on a network.

Let $N$ be a network, we define the site defined by $N$ as follows:

First we endow $N$ with the structure of an oriented graph defined as follows: The vertices are the users, there exists an edge from $U$ to $V$ if $V$ can send a message to $U$ without the use of a third part. To this graph we can associate the category abusively denoted $N$, such that $\text{Hom}_N(U,V)$ is the set of paths between the users $U$ and $V$ of the graph.

We can define on the category $N$ the topology such that the covering family of $U$ is the objects $V$ of $N$, such that there exists an arrow $V \to U$.

The topology defined by a network is not always a topos since we are not sure that the fiber products exist.

We shall often consider the graph defined by $N$ to be a the lift to the universal cover of the 1-skeleton of a $\text{CW}$-complex.

Definition 1.

A morphism between the networks $N$ and $N'$ is a morphism between their oriented graphs, that is a map between $N$ and $N'$ which sends an oriented edge of $N$ to an oriented edge of $N'$.

A network $N$ is connected if for each users $U$ and $U'$ of $N$, there exists a path between $U$ and $U'$.

Definition 2.

We can also define the following topology: Consider the category $N_C$, whose objects are networks, we can endow $N$ with the following topology: A sieve of an object $N$, is a family of networks $(N_i)_{i \in I}$, such that there exists an injective map $h_i : N_i \to N$, such that $\bigcup_{i \in I} h_i(\text{objects of } N_i) = \text{objects of } N$, where $\text{objects of } N_i$ is the set of objects of the network $N_i$.

Definition 3.

Let $N$ and $N'$ be two networks, $U$ and $U'$ two respective objects of $N$ and $N'$, the connected sum of $N$ and $N'$ is the network $N \cup N'$ obtained by identifying $U$ and $U'$. The set of users of $N \cup N'$ is $(N \cup N' - \{U, U'\}) \cup \{U''\}$, where $U''$ is an user such that for each user $U_1$ of $N$, the set of edges between $U_1$ and $U''$ is the set of edges between $U_1$ and $U$ in $N$, for every user $U_2$ of $N'$, the set of edges between $U_2$ and $U'$ in $N'$ is the set of edges between $U_2$ and $U''$.

The connected sum depends on the elements $U$ and $U'$ as shows the following example: Let $N$ be the network with objects $U_1, U_2, U_3$ and whose set of arrows contains only the elements $U_1 \to U_2, U_1 \to U_3$, and $N'$ the network whose set of users is $V_1, V_2, V_3$, and whose set of arrows is $V_1 \to V_2, V_1 \to V_3$. The graph of the connected sum of $N$ and $N'$ in $U_1$ and $U_2$ is a graph which has a vertex with 4 adjacent edges. The graph of the connected sum of $N$ and $N'$ in $U_1$ and $V_2$ does not have a user such a vertex.

Thus to endow the set of network with a law, we consider pointed networks.

Definition 4.
A pointed network \((N, U)\) is a network \(N\) with a pointed element \(U\). The connected sum of the pointed networks \((N, U)\) and \((N', U')\), is the connected sum of \(N\) and \(N'\) in \(U\) and \(U'\).

A morphism between two pointed networks \(h: (N, U) \rightarrow (N', U')\) is a morphism \(h: N \rightarrow N'\) such that \(h(U) = U'\).

We can define \([C]\) the set whose elements are isomorphisms classes of pointed networks, we denote by \([(N, U)]\) the class of the pointed network \((N, U)\), the product of pointed networks induces a product on \([C]\) whose neutral element is the class of the network with one user \(U\) and without arrow.

**Definition 5.**
Let \(N\) be a network, and \(h: U \rightarrow U'\) an edge of \(N\), (we suppose that \(h\) is the unique edge between \(U\) and \(U'\)), the retraction or the suppression of the edge \(h\) is the network \(N'\) obtained as follows: the set of users of \(N'\) is \(N - \{U'\}\). Let \(U_1\) and \(U_2\) be two users of \(N\), the set of paths between \(U_1\) and \(U_2\) is the image of the set of paths of \(N\) by the following application: let \((i_1 = U_1, ..., i_n = U_2)\) be a path between \(U_1\) and \(U_2\) of \(N\), if there exists \(l\) such that \(i_l = U'\), we replace \(U'\) by \(U\).

The groupoid associated to a network.

Let \(N\) be a network, we can define the groupoid \(Gr(N)\) associated to \(N\) defined as follows: the set of objects of \(Gr(U)\) is the set of objects of \(N\). Let \(U, V\) be two users of \(N\), \(Hom_{Gr(N)}(U, V)\) is the set of paths between \(V\) and \(U\), and the formal inverse of the path from \(U\) to \(V\). \(Gr(N)\) is the groupoid associated to the category induced by \(N\).

**Definitions 6.**
Let \(h: F \rightarrow E\) be a functor, \(m: x \rightarrow y\) a map of \(F\), and \(f = h(m): T \rightarrow U\) its projection by \(h\). We will say that \(m\) is cartesian, or that \(m\) is the inverse image of \(f\) by \(h\), or \(x\) is an inverse image of \(y\) by \(h\), if for each element \(z\) of \(F_T\), the map
\[
 Hom_T(z, x) \rightarrow Hom_f(z, y)
\]
\[n \rightarrow mn\]
is bijective, where \(Hom_f(z, y)\) is the set of maps \(g: z \rightarrow y\) such that \(h(g) = f\).

A functor \(h: F \rightarrow E\) is a **fibred category** if and only if each map \(f: T \rightarrow U\), has an inverse image, and the composition of two cartesian maps is a cartesian map.

We will say that the category is fibered in groupoids, if for each diagram
\[
x \xrightarrow{f} z \xleftarrow{g} y
\]
of \(F\) above the diagram of \(E\),
\[
U \xrightarrow{\phi} W \xleftarrow{\psi} V
\]
and for each map \( m : U \to V \) such that \( \psi m = \phi \), there exists a unique map \( p : x \to y \), such that \( gp = f \), and \( h(p) = m \).

This implies that the inverse image is unique up to isomorphism.

Consider a map \( \phi : U \to V \) of \( E \), we can define a functor \( \phi^* : F_U \to F_V \), such that for each object \( y \) of \( F_V \), \( \phi^*(y) \) is defined as follows: we consider a cartesian map \( f : x \to y \) above \( \phi \) and set \( \phi^*(y) = x \). Remark that although the definition of \( \phi^*(y) \) depends of the chosen inverse image \( f \), the functors \( (\phi\psi)^* \) and \( \psi^*\phi^* \) are isomorphic.

**Definitions 7.**

A section of a fibered category \( h : F \to E \), is a correspondence defined on the class of arrows of \( E \) as follows: to each map \( f : U \to T \), we define a cartesian map: \( u^f : xu \to yt \) of \( F \), whose image by \( h \) is \( f \) such that: \( u^f, f = u^f \circ u^f \).

**Definition 8.**

Let \( C \) be a site, a torsor \( h : P \to C \) is a fibered category such that there exists a section \( u \).

We suppose that there exists a sheaf \( H \) defined on \( C \), such that \( Hom_U(n_U, n_U) = H(U) \), where \( Hom_U(n_U, n_U) \) is the set of morphisms \( p : n_U \to n_U \) such that \( h(p) = I_{dU} \), and \( n_U \in P_U \).

Every arrow of \( P \) is invertible.

For every object \( e_U, e_V \) of \( P_U \), there exists a map \( h : e_U \to e_V \).

**The classifying cocycle associated to a torsor.**

Let \( (U_i)_{i \in I} \) be a covering family of the topology of the site \( C \), and \( P \to C \), a torsor defined on \( C \), for each object \( U \), we consider the object \( e_U \) of \( P_U \) defined by \( u^{id} \) where \( u \) is the section.

Let \( U \) be an element of \( C \), suppose that there exists a map \( d_i : U_i \to U \), we can define the Cartesian map \( d_i^U : e_i \to e_U \) over \( d_i \), there exists a map \( u_i : e_i \to e_U \), since the fibered category is connected. Suppose that \( U_i \times_U U_j \) exists, then we can define the map \( u_{ij} : e_{U_j} \to e_{U_j}^i \) by \( u_i \circ u_j^{-1} \), this make sense since the fact that \( u \) is a section implies that \( e_{U_j}^i \) is \( e_{U_j}^i \).

The family of maps \( u_{ij} \) verifies \( u_{ij}u_{jl} = u_{il} \).

**Proposition 1.**

Suppose that \( C \) is a site the set of torsors bounded by the sheaf \( H \) is 1 to 1 with \( H^1(C, H) \).

**Definition 9.**

A torsor \( P \to C \) is trivial if and only if for every object \( U \), and every map \( d_i : U_i \to U \), the Cartesian map above \( d_i \) is a map \( u_i : e_i \to e_U \), where \( e_i = u^{id}(U) \).

Let \( (U_i)_{i \in I} \) be a covering family of the site \( C \), a torsor on \( C \) is trivial, if and only if for each object \( U_i \), there exists an element \( d_i \in Aut(P_{U_i}) \) such that \( u_{ij} = u_{ij}^{-1} \). Indeed since for \( d_i : U \to U_i e_{U_i}^i \) is \( e_i \), \( u_i \) is an automorphism of \( e_i \), thus an element of \( H(U_i) \).
Examples of torsors.

Let $N$ be a differentiable manifold, a principal bundle whose structural group is $H$ is an example of torsor defined on the topos defined by the topology of $N$.

Let $N$ be a network, we have seen that in modern encryption plaintexts are encrypted by blocks to create diffusion and confusion. Without restricting the generality, we shall call $N$ the category defined by the network.

We define $P \to N$ a category fibered over $N$ such that for each user $U$, the fiber $P_U$ is a category whose objects are sets of plaintexts for example, the $n$-dimensional $\mathbb{Z}/2\mathbb{Z}$-vector space. A map between two objects $l_U$ and $l'_U$ of $P_U$, is a bijection defined by an encryption/decryption map, that is a bijective map $h : l_U \to l'_U$ such that for each element $C \in l_U$, $h(C) = E(L, C)$. An example of object can be the domain of the D.E.S map which is the 64-dimensional $\mathbb{Z}/2\mathbb{Z}$-vector space.

Let $V$ be another user of the network, suppose that $V$ can send a message to $U$, which is equivalent to saying that there is a map between $h_{UV} : U \to V$ in $N$. A Cartesian map above $h_{UV}$ is a map defined by encryption/decryption map as above $h_{UV}^l : l_V \to l_U$.

This fiber category is a torsor, if for every user $U$ of $N$, there exists an object $l_U$, in the fiber of $U$ an encryption/decryption map $h_{UV}^l : l_V \to l_U$ above each map $h_{UV} : V \to U$, such that $(h_{U_1U_2} \circ h_{U_2U_3})' = h_{U_1U_2}^l \circ h_{U_2U_3}^l$.

Definition 10.

Let $(N, U')$ and $(N', U')$ two pointed networks, supposed that there exist torsors $P \to N$, and $P' \to N'$ such that $P_U$ and $P'_U$, are isomorphic categories, then we can define the torsor $P.P'$ over the connected sum of $N$ and $N' N N'$ as follows: If $U_1$ is an object of $N - \{U\}$, then $P.P'_U$ is $P_{U_1}$, if $U_2$ is an object of $N'$, then $P.P'_U$ is $P'_U$. The fiber of the point $U''$ which is obtained by identifying $U$ with $U'$ is $P_U$.

Let $h_1 : U_1 \to V_1$ be an arrow of $N$, we lift $h_1$ to $P.P'$ to one of its lift defined by $P \to N$. Let $h_2 : U_2 \to V_2$ be an arrow of $N'$, we lift $h_2$ to $P.P'$ to one of its lift defined by $P' \to N'$.

Contraction of torsor.

Let $N$ be a network, and $N_h$, the contraction of the arrow $h : U \to U'$ of $N$, consider the torsor $P \to N$, we can define a torsor $P_h \to N_h$, as follows: consider an edge $h_1 : U_1 \to U_2$, such that $h_1$ is the projection of an edge $h_2 : V_1 \to V_2$ of $N$ by the canonical map $N \to N_h$. Suppose that $V_1$ and $V_2$ are different of $U'$, then the Cartesian map associated to $h_1$ is the Cartesian map of $h_2$, suppose that $V_1 = U'$, then $h$ projects to the arrow $h_1 : U \to U_2$, the Cartesian map above $h_1$ is the cartesian map above $h_2 \circ h : U \to V_2$, suppose that $V_2 = U'$, then the Cartesian map above $h_1$ is $h'^{-1} \circ h'_2$, where $h'_2$ is the Cartesian map above $h_2$, and $h'$ is the Cartesian map above $h$ which is assumed to be invertible.

The Diffie-Hellman torsor.
The following torus can be used in public encryption:

Let $C$ be a finite topos, that is such that the class of objects of $C$ is a finite set. We suppose that there exists a torsor $P \rightarrow C$ such that for each object $U$, the group of automorphisms of $P_U$ is the multiplicative group $\mathbb{Z}/n\mathbb{Z} - \{0\}$, where $n$ is a prime number, consider a generator $\alpha$ of the multiplicative group $\mathbb{Z}/n\mathbb{Z} - \{0\}$, we suppose that for every objects $U, V$ of $C$, there exists $n_U$ and $n_V$ in $\mathbb{Z}$ such that the transition function defined on $U \times_C V$ is $h_{n_U} \circ h_{n_V}^{-1}$, where $h_{n_U}$ is the function defined on $\mathbb{Z}/n\mathbb{Z} - \{0\}$ by $c \mapsto \alpha^{n_U} c$, and the final object of $C$ is abusively denoted $C$. This torsor is trivial.

The public key of the user $U$ is $\alpha^{n_U}$, its private key is $n_U$. Suppose that $U$ want to send a message to $V$, he takes the public key $\alpha^{n_V}$ and calculates $\alpha^{n_U n_V} = (\alpha^{n_V})^{n_U}$. To decrypt the message, $V$ take the public key $\alpha^{n_U}$ of $U$ and calculates $\alpha^{n_U n_V} = (\alpha^{n_V})^{n_U}$. This is the Diffie-Hellman algorithm. The security is due to the fact that it is infeasible to calculate discrete logarithm in reasonable time.

I.2. Connection on torsors and encryption.

In this part we present the theory of torsors defined on a site, and show how it can be used to define public encryption. It is a generalization of the Diffie-Hellman torsor.

Let $N$ be a manifold, $P \rightarrow N$, a principal bundle whose structural group is $H$, defined by the trivialization $(U_i, u_{ij})_{i,j \in I}$ a connection on $H$ is defined by a family of 1-forms $\alpha_i : U_i \rightarrow \mathfrak{H}$, where $\mathfrak{H}$ is the Lie algebra of $H$, which satisfy the relation:

$$\alpha_j - \alpha_i = u_{ij}^{-1} du_{ij}$$

The curvature of the connection $\alpha$ is the 2-form defined locally by $d\alpha_i + \alpha_i \wedge \alpha_i$. The bundle is flat if the curvature vanishes. Suppose that the group $H$ is commutative, then if the curvature vanishes, then $d(\alpha_i) = 0$, we deduce the existence of a 1-chain $u_i : U_i \rightarrow H$ such that $d(u_i) = \alpha_i$. The cocycle $h_{ij} = u_{ij} u_j^{-1} u_i$ is called the holonomy cocycle of the connection. We remark that in this situation the connection is completely characterized by the 0-chain $(u_i)_{i \in I}$. This motivates the following definition:

Definition 1.

A flat connection on $P$ is a 0-chain $(u_i)_{i \in I}$ $u_i : U_i \rightarrow H$. We do not suppose that our group is commutative.

This definition characterizes only flat bundles defined over a manifold when the group $H$ is commutative.

Holonomy map.

Let $U_i$ and $U_j$ be objects of $C$, and $(i_1 = i, ..., i_n = j)$ a path between $U_i$ and $U_j$, we can defined the holonomy map $Hol(\alpha) : P_{U_i} \rightarrow P_{U_j}$ which
is the composition of the following maps: $Hol_{i_l} : P_{U_{i_l}} \rightarrow P_{U_{i_{l+1}}}$ defined by 
$u_{i_{l+1}i_l}^{-1}u_{i_l}$. Thus $Hol(\alpha) = Hol_{i_{l-1}} \circ \cdots \circ Hol_{i_1}$.

The holonomy map will be used to define example of the encryption map between $P_{U_i}$ and $P_{U_j}$.

The holonomy cocycle of the connection is $u_{ij}^{-1}\circ u_{ij}$.

If $P \rightarrow N$ is a principal bundle defined over the manifold $N$, this is similar to the usual definition of connection.

The holonomy map characterizes completely a torsor over a connected site $N$, which can be defined as a representation $Hol : \pi_1(Gr(N)) \rightarrow P_{U_0}$, where $\pi_1(Gr(N))$ is the fundamental group of the groupoid $Gr(N)$, and $P_{U_0}$ the fiber at $U_0$.

**Generalization of the Diffie-Hellmann torsor.**

We consider here networks endowed with the natural topology that we have defined.

Let $P \rightarrow C$ be a torsor defined by the generating family $(U_i)_{i \in I}$, and the transition functions $u_{ij}$ of the topology of $C$, we denote by $H$ the structural group of $P$. We suppose that there exists a commutative group $H$, and a map $exp : H \rightarrow H$, which will play the role of the exponential map of the group $H$. We shall suppose that the exponential map is surjective.

**Definition 2.**

A public encryption defined on the torsor $P \rightarrow C$ is defined by the following data:

A connection $(u_i)_{i \in I}$ defined on the torsor $C$, we denote by $\alpha_i$ an element of $H$ such that $exp(\alpha_i) = u_i$.

A function $L : H \times H \rightarrow V$, where $V$ is a commutative group such that $L(\alpha_i, exp(\alpha_j)) = L(\alpha_j, exp(\alpha_i))$.

The public key of the user $U_i$ is $exp(\alpha_i)$, and its private key is $\alpha_i$.

The key that the users $U_i$ and $U_j$ use to exchange data is $L(\alpha_i, exp(\alpha_j))$.

The security of this problem is related to the fact that it is not feasible to compute the logarithm of $H$.

A particular example of the previous public encryption protocol is the situation when the value of the function $L$ is defined the coordinate changes. This can be realized as follows: if the torsor is trivial, in this situation there exists a 0-chain $u_i : U_i \rightarrow H$ such that $u_{ij} = u_iu_j^{-1}$. We denote $\alpha_i = Log(u_i)$,

$L(\alpha_i, exp(\alpha_j)) = u_j^{-1}$.

The secret key of the user $U_i$ is $\alpha_i$, and its public key is $u_i$.

**Meet in the Middle attack.**

Suppose that an intruder $U_l$ register to the network he can perform the following Meet-in-the-Middle attack, he calculates $L(\alpha_l, u_i) = u_{li}$, and $L(\alpha_l, u_j) = u_{lj}$.
Since the inversion is assumed to be a feasible operation, $U_i$ can calculate $u_{ij}^{-1}u_{ij} = u_{ij}u_{ij} = u_{ij}$, which is the key that share the users $U_i$ and $U_j$.

We shall prove that higher non commutative cohomology can enable to counter this attack.

The Grothendieck group of the equivalence class of torsors over a network.

Let $N$ be a network, consider two torsors $P,P' \to N$ whose fiber are vector spaces defined over a field. We can make the tensor product $P \otimes P'$ of these networks, which is itself a torsor over $N$. If $P$ and $P'$ are respectively defined by the transition functions $u_{ij}$ and $u'_{ij}$, then $P \otimes P'$ is defined by $u_{ij} \otimes u'_{ij}$.

We can define the dual of $P$ to be the torsor over $N$ defined by the transition functions $(u^*_{ij})^{-1}$, where $u^*_{ij}$ is the dual map of $u_{ij}$.

We can define the Grothendieck group of this category, which is the group whose elements are equivalence classes of the previous torsors.

Examples of Grothendieck groups can be defined by cryptography: Consider a class $D$ of cipher maps defined on a category of $L$-vector spaces ($L$ can be thought to be $\mathbb{Z}/2\mathbb{Z}$) stable by addition and tensor product, that is if $u : V \to V$, and $u' : V' \to V'$ are in this class, we suppose also that $u^{-1}$ is isomorphic to an element of $D$, $u + u'$ and $u \otimes u'$ are also isomorphic to elements of $D$. We can consider the category whose objects are torsors $P \to N$, such that the transition functions $u_{ij}$ are elements of $D$. We can define the Grothendieck group of this category.

Many ciphers in cryptography have the following structure: they are a succession of $p$ rounds, and each round is defined as follows: the plaintext is a vector of an even dimensional vector space over $\mathbb{Z}/2\mathbb{Z}$, it is divided in two halves, the left half $LE_0$, and the right half $RE_0$, we have:

$$LE_{i+1} = RE_i, RE_{i+1} = LE_i + H(RE_i, LE_i, L_i)$$

where $L_i$ is a round key. After the $p$-round, the both halves are swapped. The decryption map is the encryption map with the keys used in the inverse order.

This class of cipher is stable by addition, inverse, and tensor product. We can define its Grothendieck group.

If we suppose that the transition functions are linear maps, the category of torsors defined over a network $N$ is a Tannakian category, it is thus equivalent to the category of representations of an affine group scheme. These networks are not useful in practice, since they are vulnerable to a chosen ciphertext attack: If an attacker can obtain ciphertexts from given plaintexts, to retrieve the key he has only to choose a set of plaintexts which is a basis of the vector space $V$. These ciphers are called Hill ciphers and are used in the Mix columns operations of modern ciphers.

I.3 Link to Link encryption and torsors.
Let $C$ be a network, an user $U_i$ who sends a message through the network to $U_j$ often does not encrypt the whole message: the header, that is the part of the message where is recorded the identity of the sender and the identity of the receiver, is either in clear, or encrypted and decrypted at every node of the path between $U_i$ and $U_j$, that is, if $U_i$ wants to send the message $N$ to $U_j$ using the path $(i_1 = i, \ldots, i_n = j)$. An append $N_1$ called the header of the message is added to the message. It cannot be encrypted with the algorithm used to encrypt $N$ since in modern network like internet, the route of the message is not controlled by the sender, but to be sent from $i_l$ to $i_{l+1}$, the header is encrypted, by an encryption function $u_{i_{l+1}i_l}$, we can suppose that this encryption is a symmetric encryption, thus the encryption function used by $U_{i_{l+1}}$ to send messages to $U_i$ is $u_{i_{l+1}i_l} = u_{i_{l+1}i_l}^{-1}$, if we denote by $u_{ij}$ the encryption of the header from $U_i$ to $U_j$, we have $u_{ij} = u_{ij}u_{jl}$ if $U_j$ is an intermediate stage. If we suppose that the header are elements of a set $E$, and the transition functions $u_{ij}$ are automorphisms of $E$, these data define a torsor $P \rightarrow C$ over the site $C$ such that for each object $U_i$, $P_{U_i}$ is a set isomorphic to $E$.

The definition of a torsor over a site used to encrypt the header of a message can be very useful. Practically, the manager of a network has to define or find a simple procedure to define the keys at every nodes for the link to link encryption. Often in mathematics a torsor is defined in a global way, without defining each coordinates changes, for example: the tangent space of a manifold, or of an algebraic variety... this can enable to save a lot of time in the implementations of a network.

We propose the following scheme to define a link to link encryption: We consider a smooth affine algebraic variety $N$ defined over a finite field, we consider a trivialization $(U_i)_{i \in I}$ of one of its canonical bundle like its tangent bundle. We can define a network whose users are the $U_i$, the transition functions of the bundle considered are the keys used for link to link encryption.

Under reasonable conditions on the structure of the encryption algorithm of the header, the previous remark is always true:

**Proposition 1.**

Suppose that the header is written in an alphabet which can be identified with a scheme, and the transition functions are morphisms of scheme, the objects $U_i$ are schemes and the transition functions define an effective descent datum, then there exists a torsor $P \rightarrow C$ of schemes, such that the typical fiber is the alphabet endowed with its scheme structure.

Suppose that the transition functions do not define morphisms of a scheme, the network topos $N$ defines the 1-skeleton of a CW-complex $N$, (the CW-complex is not necessarily unique) the transition functions define a flat bundle on $N$.

**I.4 Key distribution Center and initial object.**

One of the big challenge in symmetric encryption is to establish protocols of distribution of keys in a network. When the participants are not very far each
other, this can be accomplished by physical distribution. When the number of
members of the network is very big, physical distribution is quite impossible, a
solution of this problem is to ask to a third part to distribute keys to participants,
such a third part is called a Key distribution center. The key distribution center
share a master key with each user that he uses to send sessional keys.

Let $D$ be the key distribution center of the network $C$, we assume that $C$ is a
site, and the members of the network are objects of $C$, the key distribution center
must have a connection with every participant, since a connection between $D$
and the object $U$ can be represented by a map $\text{Hom}_C(U, D)$, we shall assume
that $D$ is the initial object in the category $C$. Suppose that $D$ distributes keys
in a link to link network, we have seen that we can represent such a network by
a torsor $P \to C$, for every object $U_i$ of $C$, there exists a map $u_{id} : P_{U_i} \to P_D$,
this map is the master key used by $U_i$.

Suppose that $U_i$ wants to establish a connection with $U_j$, there exist many
protocols in the literature that he can use, for example:

$U_i$ sends to $D$ a message encrypted with $u_{id}$ which contains an identifier
$ID_{U_i}$ of $U_i$ and an identifier $ID_{U_j}$ of $U_j$.

The key distribution center replies to $U_i$ by sending the sessional key $u_{ij}$
encrypted with $u_{id}$.

The key distribution center sends to $U_j$ a message encrypted with $u_{jd}$ which
contains the key $u_{ij}$ and an identifier of $U_i$.

The link to link encryption is defined by a torsor over a site, we have seen
that under reasonable conditions, we can suppose that this torsor can be defined
as a flat bundle over a scheme, or a differentiable manifold. Flat bundle over a
manifold $N$ is determined by a representation of the fundamental group of $N$
which defines the holonomy. Thus it is completely determined by the 2-skeleton
of the manifold, in practice we shall consider differentiable surface, or algebraic
surfaces in link to link encryption.

The knowledge of the genus of the surface used to define keys in a link to
link network can be a very useful information for an attacker because the space
of flat bundles over a surface of a given genus is an object which is well-known
in mathematics.

Suppose that an attacker $L$ wants to make an attack on the link to link en-
cryption network. His purpose is thus to determine keys used to encrypt header
of the messages sent in the network. This can be a very useful information, since
this will enable $L$ to know the identity of the peoples who send messages in the
network. A possibility for $L$ is to perform a brute force attack: We assume that
the messages are written in binary and their length are $n$, and $L$ have a set of
chosen plaintext ciphertext for each couples of users $(U_i, U_j)$, we assume also
that there does not exists spurious keys. thus to determine the key used by $U_i$
and $U_j$, $L$ must try $2^n!$ keys if the header are written with $n$-bits, if there exists
$N$ participants, he has to make $C_N^2 2^n!$ operations. But if $L$ knows the topology
of the network, that is the genus of the surface used, he can determine the car-
dinal of a set of generators of the fundamental group the surface involved, this
number can be small, whenever the number of the participants of the network
is huge, the holonomy is parameterized by the image of the generators of the fundamental group, thus the topological information about the surface can be a crucial information when $N \gg 0$, and the genus is small, since in this situation a brute force attack is impossible.

I.5. Implementation of the link to link encryption.

One of the main challenges in computer science is to define less expensive algorithms, that is an algorithm which can be computed in reasonable time with a computer. It has been shown that every algorithm can be computed with the Turing machine, but to be efficient the implementation must be run with an existing computer.

To define link to link encryption defined on a site $C$, we need a priori to define each couple of keys $u_{ij}$ for any users $U_i$ and $U_j$ of $C$. We shall show how the holonomy representation can reduce the algorithm.

We shall assume that the torsor $P \to C$ which defines the link to link encryption is a bundle over a surface $N_2$. We endow the surface with a CW-structure, and perform a cutting along the 1-skeleton. We assume that the genus of the surface is different of zero. The surface is then the quotient of an hyperbolic or Euclidean polygon. The vertices of the polygon represent the 0-skeleton of the CW-decomposition. Each edge $u_iu_j$ projects to $N_2$ to define an element of $\pi_1(N_2)$.

There can exists in the network elements different of the vertices, these elements can be considered to be element $u_{2n+1}, \ldots, u_{2n+p}$ in the interior of the polygon. Thus the users of the network are the vertices $u_1, \ldots, u_{2n}$ and $u_{2n+1}, \ldots, u_{2n+p}$.

We suppose that the messages conveyed in the network are written in binary, and are encoded in block of length $l$. As before, we shall assume that the fiber is an algebraic variety, and the transition functions are element of the automorphisms group $H$ of this variety.

The bundle $P \to N_2$ is defined by a representation $h : \pi_1(N_2) \to H$.

We can define the algorithm:

Write ”Enter the vertices”
From $i = 1$ to $i = 2n$
Write enter $u_i$, read $u_i$
Write ”Enter the interior points”
For $i = p + 1$ to $i = 2n + p$
Write enter $u_i$, read $u_i$
Write ”Enter the holonomy”
For $i = 1$ to $i = n - 1$ do
Write enter $u_{ii+1}$, read $u_{ii+1}$
Write ”enter $u_{n1}$, read $u_{n1}$
$u_{ii} = 1d$
For $i = 1$ to $i = n$ do
For $j = i + 1$ to $j = n$ do
\[ u_{ij} = u_{i-1,j-1} u_{ij-1} \]

For \( i = 1 \) to \( i = 2n + p \) do

For \( j = 2n + 1 \) to \( j = 2n + p \) do

\[ u_{ij} = \text{Id} \]

This program enter the keys needed to define a link to link encryption defined by a torsor \( P \to C \) isomorphic to a bundle over a surface of genus \( n \).

II. Non abelian cohomology and End to End encryption.

We have seen that the notion of torsor is not a good notion to provide secrecy in End to End encryption. We shall provide a method of encryption using non Abelian cohomology.

II.1. Gerbes and encryption.

The notion of gerbe have been defined by Giraud to study gluing problems in geometry. Let \( h : P \to N \) be a principal bundle defined over the manifold \( N \), whose structural group is \( H \), suppose that there exists an extension \( 1 \to H_1 \to H_2 \to H \), a fundamental question in geometry is to define a principal bundle \( h' : P' \to N \) whose structural is \( H_2 \), such that there exists a map \( l : P' \to P \) such that \( h \circ l = h' \). This problem has been one of the motivation to formulate gerbe theory.

**Proposition-Definition 1.**

Suppose that \( E \) is a site whose topology is generated by a covering family \( (U_i \to U)_{i \in I} \), and \( h : F \to E \) a fibered category in groupoids. For each map \( f : U \to V \) of \( E \), we consider the functor \( r_{U,V}(f) : F_V \to F_U \) defined as follows:

For each object \( y \) of \( F_V \), \( r_{U,V}(f)(y) \) is an object \( x \) of \( F_U \) such that there exists a cartesian map \( n : x \to y \) such that \( h(n) = f \). Consider the maps \( v_1 : U_1 \to U_2 \), and \( v_2 : U_2 \to U_3 \) in \( E \), the functors \( r_{U_1,U_2}(v_1) \circ r_{U_2,U_3}(v_2) \) and \( r_{U_1,U_3}(v_2 v_1) \) are isomorphic. The functor \( h : F \to E \) is a sheaf of categories if and only if the correspondence \( U \to F_U = F(U) \) satisfies the following properties:

(i) Gluing condition for arrows.

Let \( U \) be an object of \( E \), and \( x, y \) objects of \( F(U) \). The functor from \( E_U \), endowed with the restriction of the topology \( J \), to the category of sets which associates to an object \( f : V \to U \) the set \( \text{Hom}_V(r_{V,U}(f)(x), r_{V,U}(f)(y)) \) is a sheaf of sets.

(ii) Gluing condition for objects.

Consider a covering family \( (U_i \to U)_{i \in I} \) of an object \( U \) of \( E \), and for each \( U_i \), an object \( x_i \) of \( F(U_i) \). Let \( t_{ij} : x_j^i \to x_i^i \), a map between the respective restrictions of \( x_j \) and \( x_i \) to \( U_i \times_U U_j \), (we suppose that the fiber product over \( U \) exists) such that on \( U_i \times_U U_i \times_U U_j \), the restrictions of the arrows \( t_{ij} \) and \( t_{ij} t_{ij} \) are equal. There exists an object \( x \) of \( F(U) \) whose restriction to \( F(U_i) \) is \( x_i \).

If moreover the following properties are verified:

16
(iii) There exists a covering family \((U_i \to U)_{i \in I}\) of \(E\) such that \(F(U_i)\) is not empty.

(iv) For each pair of objects \(x\) and \(y\) of \(F(U_i)\), \(\text{Hom}_{U_i}(x, y)\) is not empty (local connectivity).

(v) The elements of \(\text{Hom}_{U_i}(x, y)\) are invertible. The fibered category is called a gerbe.

(vi) We say that the gerbe is bounded by the sheaf \(L_F\) defined on \(E\), or that \(L_F\) is the band of the gerbe, if and only if there exists a sheaf of groups \(L_F\) defined on \(E\) such that for each object \(x\) of \(F(U)\) we have an isomorphism:

\[
L_F(U) \to \text{Hom}_{U}(x, x)
\]

which commutes with restrictions, and with morphisms between objects.

The classifying cocycle of a gerbe.

We suppose that the site \(C\) is defined by a covering family \((U_i)_{i \in I}\), such that \(P_{U_i}\) is not empty.

We consider an object \(u_i\) of \(P_{U_i}\), let \(u_i^j\) be the restriction of \(u_i\) to \(U_i \times_U U_j\), (where \(U\) is an object such that the fiber products over \(U\) exist) the local connectivity implies the existence of a map there exists a map \(u_{ij} : u_i^j \to u_j^i\),

On \(U_i \times_U U_j \times_U U_l = U_{ijl}\), we have the objects \(u_{ijl}^i, u_{ijl}^j, u_{ijl}^l\), we can define the map \(u_{ijl} = u_{ij}u_{ij}u_{ij} : u_{ij}^j \to u_{ij}^j\), this map can be identified with an element of \(L(U_{ijl})\)

**Theorem 1.**

The family of maps \(u_{ijl}\) defines a non commutative \(2\)-Cech cocycle. The set of equivalence classes of gerbes bounded by \(H\) is one to one with \(H^2(C, L)\).

To apply this construction to the problem mentioned at the beginning of this paragraph, we define the following sheaf of categories: for each open subset \(U\) of \(N\), we define the category \(C(U)\) such that the object of \(C(U)\) are bundles whose structural group is \(H_2\), and such that the quotient by \(H_1\) is the restriction of \(P\) to \(U\). The sheaf of category \(U \to C(U)\) is a gerbe defined on \(C\) bounded by the sheaf of \(H_1\) valued functions defined on \(N\).

II.1.2. A protocol to define end to end encryption and link to link encryption with gerbe.

Consider a network \(N\), endowed with the topology that we have defined above. Let \(C \to N\) be a fibered category such that for each object \(U\) of \(N\), \(C_U\) is a category whose objects are isomorphic to a set of plaintext/ciphertext, for example objects of \(C_U\) can be isomorphic to a \(\mathbb{Z}/2\mathbb{Z}\)-vector space. A map between two objects of \(C_U\) is an encryption/decryption map. We suppose that there exists a sheaf \(H_2\) on \(N\), such that for each object \(e_U\) of \(C(U)\), \(\text{Aut}(e_U) = H_2(U)\), and this sheaf is the band of the gerbe defined by \(C\). In practice, this
Gerbe can be defined as follows: we suppose that there exists an exact sequence of sheaves

\[ 1 \rightarrow H_2 \rightarrow H_1 \rightarrow H \rightarrow 1 \]

defined on \( N \). There exists an \( H \)-torsor \( P \rightarrow N \), such that the fiber \( P_U \) is a set of plaintext/ciphertext used to write the header of messages conveyed in the network. Consider the gerbe \( C \) which represents the geometric obstruction to lift the structural group of \( P \) to \( H_21 \), for each object \( U \) of \( C \), the objects of \( C_U \) are plaintexts these are the messages conveyed in the network. The projection of these objects to \( P_U \) is their respective headers.

We can modify the previous example as follows: we suppose that the header of the message are written with an alphabet which has the group structure \( H \), and the main part of the alphabet is written with an alphabet which has the group structure \( H_1 \). In this situation the transition functions are keys of encryption/decryption of monoalphabetic ciphers, since they are applied to each letter. To create confusion and diffusion, we can assume that the encrypted plaintexts are encrypted by blocks, and the transition functions defined polyalphabetic ciphers which are more resistant to the statistical study of the common properties of a set plaintext/ciphertext.

This kind of protocol can be applied to internet in the email distribution, since the route of the message is not defined by the sender, thus the encryption of the header cannot be encrypted using the key used to encrypt the main message: The header is written with \( H \) and the main part of the message with \( H_1 \).

Meet in the Middle attack of encryption defined by gerbes.

The following attack can be performed on the previous encryption protocol:

Suppose that there exists three intruders in the networks, \( U_i, U_j, \) and \( U_l \) suppose that they want to obtain the secret key used by the users \( U_c \) and \( U_d \), they know the keys \( u_{ij}, u_{ic}, u_{id}, u_{jc}, u_{jd}, u_{il}, u_{jl}, u_{cl}, u_{dl} \) thus they know the quantities \( u_{ijc} = u_{ci}u_{ij}u_{jc}, u_{ijd}, u_{icle}, \) and \( u_{ljd}, \) using the fact that the classifying cocycle of the gerbe is trivial (we assume the band to be commutative), we obtain:

\[ u_{jcd} - u_{icd} + u_{ijd} - u_{ijc} = 0, \]

and

\[ u_{jcd} - u_{icd} + u_{ijd} - u_{ijc} = 0 \]

This implies that

\[ u_{ild} - u_{ljd} + u_{lje} - u_{jc} + u_{ijd} - u_{ijc} = 0 \]

Thus the intruders can deduce the value of \( u_{lcd} - u_{icd} \) since they know the values of \( u_{ljd}, u_{lje}, u_{ijd}, u_{ijc} \) we know that \( u_{lcd} = u_{di}u_{lc}u_{cd}, \) and \( u_{icd} = u_{di}u_{lc}u_{cd} \).
since the intruder know $u_{dl}, u_{lc}, u_{di}$ and $u_{ic}$, they can deduce the key $u_{cd}$ if $u_{dl}u_{lc}$ is different of $u_{di}u_{ic}^{-1}$.

This type of attack cannot be performed with two intruder, if we consider the cocycle relation $u_{jcd} - u_{icd} + u_{ijd} - u_{ijc} = 0$, the substraction of the expressions $u_{jcd} = u_{dl}u_{jc}u_{cd}$ and $-u_{icd} = -u_{di}u_{ic}u_{cd}$ implies the cancellation of $u_{cd}$ thus $u_{cd}$ cannot be written as a function of the other keys.

Remark that a network with 3 users whose plaintext/ciphertext are element of a given set, and the encryption/decryption maps are automorphisms of this set is always a gerbe since the cocycle relation is always verified.

We shall define a notion of higher non Abelian cohomology which can enable to counter the previous attack. What can be expected is the fact that there exists a notion of $n$-gerbe such that every network of $n$-users which share information written in the same alphabet is a $n$-gerbe. Unfortunately, the question of the existence of a theory of $n$-gerbe is a deep question in mathematics which is not completely solve nowadays. The first author of this paper has provided a cohomological description of cohomology classes of higher rank, there other theories such as the thesis of Zouhair Tamsamani.

II.1.3. Connective structure on gerbe and public-key encryption.

Let $N$ be a differential manifold, and $C$ a gerbe defined on $N$, we shall suppose that the gerbe $C$ is bounded by a commutative group $L$ and denote by $\mathcal{L}$ the Lie algebra of $L$. The notion of connective structure on gerbe have been defined by Brylinski and Deligne to study the differential geometry of gerbes, and of infinite dimensional bundles. It this the notion analog to the notion of connection defined on manifolds.

Definition 1.

A connective structure defined on the Abelian gerbe $C \to N$, bounded by the commutative group $L$ is defined by:

For each open subset $U$ of $N$, and each $e_U$ of $C(U)$, a torsor $Co(e_U)$ of $\mathcal{L}$ valued 1-forms defined on $U$, which is called the torsor of connections.

We suppose that $Co(e_U)$ behaves naturally in respect of restrictions.

For every maps $u : e_U \to e'_U$ between the objects $e_U$ and $e'_U$ of $C(U)$, there exists a map $u^* : Co(e_U) \to Co(e'_U)$ compatible with the composition and restriction to smaller subsets.

For each morphism $h$ of $e_U$, and each element $\nabla e_U$ of $Co(e_U)$, we have:

$$h^* \nabla e_U = \nabla e_U + h^{-1}dh$$

Let $(U_i)_{i \in I}$ be a trivialization of the gerbe, that its a covering family of $N$ such that $C(U_i)$ is not empty and its objects are isomorphic each others. We consider $e_i$ an object of $C(U_i)$, and an element $\alpha_i \in Co(e_i)$ in $U_i \cap U_j$, we can define $\alpha_{ij} = \alpha_j - u_{ij}^*\alpha_i$, we have the relation:
\[ \alpha_{jl} - \alpha_{il} + \alpha_{ij} = u_{ijl}^{-1} du_{ijl} \]

**Definition 2.**
The curving of a gerbe is defined as follows:

For each object \( e_U \) of \( C(e_U) \), and each connection \( \nabla_{e_U} \) of \( Co(e_U) \), a 2-form \( H(e_U) \) defined on \( U \) such that for each 1-form \( \alpha \) defined on \( U \), \( H(\nabla_{e_U} + \alpha) = \alpha \).

A connective structure is flat if and only if the curving is zero.

Suppose that the curving of a connective structure is zero, and the gerbe is defined by a good cover \( (U_i)_{i \in I} \), we choose an object \( e_i \) of \( C(U_i) \), and an element \( \alpha_i \) of \( Co(e_i) \), we have \( \alpha_j = \alpha_{ij} + \alpha_i \), since the curving is zero, \( H(\alpha_{ij}) = 0 \). This implies that \( d(\alpha_{ij}) = 0 \). Thus there exists \( c_{ij} : U_{ij} \to L \) such that \( dc_{ij} = \alpha_{ij} \). The equality 1 implies that \( d(c_{jl} - c_{il} + c_{ij}) = dLog(u_{ijl}) \).

Denote by \( c'_{ij} = \exp(c_{ij}) \). The 2-Cech cocycle \( u_{ijl}c_{jl}c_{il}^{-1}c_{ij} \) is the holonomy cocycle.

We remark that the flatness of the bundle is completely characterized by the fact that the family of 1-form \( \alpha_{ij} \) are closed, thus by the existence of the cocycle \( c'_{ij} \). This motivates the following definition:

**Definition 3.**
Let \( D \) be a gerbe defined over the site \( C \), bounded by the commutative sheaf, a flat connective structure of \( C \) is a 1-Cech \( L \)-chain.

An example of connective structure defined on a gerbe is a connective structure defined on gerbe \( C \) defined by an extension problem \( 1 \to H_1 \to H_2 \to H \to 1 \) as follows:

Consider the \( H \)-principal bundle \( P \to N \), and a connection \( \nabla \) defined on \( P \), for each object \( e_U \) of \( C(e_U) \), we can consider the set of connections of \( e_U \) which project to the restriction of \( \nabla \) to \( U \).

If the bundle \( P \to N \) is flat, we have seen that a connection is defined by a 0-chain \( u_i \to L \), this motivates the following definition that will be used to define symmetric encryption:

**Definition 4.**
Let \( D \) be a gerbe defined on the site \( C \), we suppose that there exists a flat torsor \( P \to C \) such that \( C \) is the gerbe associated to the lifting problem defined by the exact sequence \( 1 \to H_1 \to H_2 \to H \to 1 \). A connective structure on \( C \), is a 1-H2 Cech chain \( c_1 \) defined on \( C \), such that there exists a 0-chain \( c_0 \), such that \( c_1 \) is the lift of the boundary of \( c_0 \) by the map \( H_2 \to H \). We shall denote this connective structure by \( (c_0,c_1) \), or \( (c_i,c_{ij}) \) in local coordinates.

To define the classifying cocycle of the gerbe \( D \), we consider a trivialization \( U_i, u'_{ij} \) of the torsor \( P \), let \( [c] \) denote the cohomology class of this cocycle, the classifying cocycle is obtained by the image of \( [c] \) by the map \( H^1(C,H) \to H^2(C,H_1) \). We can defined the element \( u_{ij} \) which projection by the map \( H_2 \to H \) is \( u'_{ij} \).
Definition 5.

Let $D$ be a gerbe defined by an exact sequence, $(c_i, c_{ij})$ a connective structure defined on $D$, we consider a lift $c'_i$ of $c_i$ in $H_2$, using the map $H_2 \to H$. A public encryption defined on the gerbe $C$, is defined by the following data:

- A function $J : H_2 \times H_2 \to H_2$, such that for every $J(\Ln(c'_i), c'_j) = u_{ij}$.

This encryption is more secured than the encryption defined by a torsor since the Chasles relation is not satisfied by the family of $u_{ij}$.

The private key of the user $U_i$ is $\Log(c'_i)$, and its public key is $c'_i$, as usual the secrecy is due to the fact that it infeasible to compute the logarithm in a reasonable time.

II.3. Non commutative cohomology and probabilistic theory of ciphers.

The purpose of this part is to study the unconditional security of the ciphers defined with a gerbe.

Let $N$ be a network, $U_i$ and $U_j$ users of $N$, we suppose that $U_i$ and $U_j$ exchanged texts written in an alphabet, and these texts are encrypted by blocks which are element of a set $E$. We suppose that the encryption in the network is defined by a gerbe $D$, and the objects of $Du_i$ are isomorphic to $E$. We shall first study the following question: what is the probability that $U_i$ sends the plaintext $C$ to $U_i$ by following the path $(i_1 = i, \ldots, i_n = j)$.

The local study.

We study first the probability for a given cipher encrypted by $U_i$ to be received by $U_j$ where there exists an edge between $U_i$ and $U_j$. The plaintexts that $U_i$ encrypts or decrypts are elements of an object $E_i$ of the fibers $C_U$. We suppose that $E_i$ is endowed with a probability. To each plaintext $P^i_l$ of $E_i$ we assign the probability $p^i_l$. The keys used by $U_i$ are the transition functions $u_{ji}$, the cardinal of the set of this keys is the cardinal of the band $H_2$ of the gerbe. This is due to the fact that the set of transition functions between $E_i$ and $E_j$ is in bijection with the band. The set of keys $U_{ji}$ used by $U_i$ to send messages to $U_i$ is a probabilistic space we denote the probability of the key $u^i_{ji}, d^i_{ji}$. We assume that the choice of a plaintext is an event independent to the choice of a key.

Consider the probability space $U_{ji} \times E_i$, endowed with the product of the probability of $E_i$ and $U_{ji}$. A cipher $C_j$ is received by $U_j$ is represented by the subspace of $U_{ji} \times E_i$ whose elements $(u^i_{ji}, P^i_l)$ verifies $u^i_{ji}(P^i_l) = C_j$. Thus the probability $p_{C_j}$ to obtain the cipher $C_j$ is:

$$p_{C_j} = \sum_{P^i_l \in E_i, u^i_{ji} \in U_{ji}, u^i_{ji}(P^i_l) = C_j} d^i_{ji} p^i_l$$
The global study.

We consider now the situation when $U_i$ sends a plaintext $P_i^k$ to $U_j$ true the network, by following the path $(i_1 = i, ..., i_n = j)$. When an user $U_{ip}$ receives a plaintext/ciphertext from $U_{ip+1}$, he uses a key $u_{ip+1}ip$ chosen randomly to send a message to $U_{ip+1}$. Thus we consider the product of probability spaces $\Pi_{p=1}^{n-1} U_{ip+1}ip 	imes P_{ip}$. An event of this space is a collection of events $(u_{i_1i_2}, P_i), (u_{i_3i_2}, P_i), ..., (u_{i_ni_{n-1}}, P_{i_{n-1}})$. The probability $p_{C_j}$ represents the probability of the plaintext/ciphertext to be obtained by $U_j$ when a message is sent by $U_i$ conveyed in the path $(i_1 = i, ..., i_n = j)$:

$$p_{C_j} = \sum_{u_{i_1}i_2 ... u_{i_{n-1}}i_1} d_{i_1}^1 d_{i_2}^2 ... d_{i_{n-1}}^{n-1} p_{i_1} p_{i_2} ... p_{i_{n-1}}$$

where $d_{i_1}^1 u_{i_1}i_2 ... u_{i_{n-1}}i_1$ is the probability of the key $u_{i_1}i_2$, and $p_{i_1}$ is the probability of the event $P_{i_1}$.

The probability $p_{C_j}$ depends on the path used by $U_i$ to send a message to $U_j$ as shows the following example: consider the network whose set of users is $\{U_1, U_2, U_3, U_4\}$, and such that there exists a path between $\{U_1, U_2\}, \{U_2, U_3\}, \{U_1, U_3\}, \{U_3, U_4\}$. We assume that each user $U_i$ has a couple of plaintext/ciphertext $P_i^k, P_i^d$. We denote by $u_{ij}$, the unique key between $U_i$ and $U_j$. Suppose that the probability of the plaintext/ciphertext $P_i^k$ to be received by $U_4$ from $U_2$ is 1, the probability of $P_i^d$ to be received by $U_4$ from $U_3$ is 0, a message sent by $U_1$ to $U_4$ using the path $U_1 U_2 U_3 U_4$ is different to a message sent by $U_1$ to $U_4$ using the path $U_1 U_3 U_4$.

Consider a set of plaintext/ciphertext $P_i \in E_1, ..., P_{i_{n-1}} \in E_{i_{n-1}}$. The conditional probability of the plaintext/ciphertext $C_j$ to be realized given $P_1, ..., P_{i_{n-1}}$ is:

$$p_{C_j}(P_1, ..., P_{i_{n-1}}) = \sum_{u_{i_1}i_2 ... u_{i_{n-1}}i_1} d_{i_1}^1 d_{i_2}^2 ... d_{i_{n-1}}^{n-1} p_{i_1} p_{i_2} ... p_{i_{n-1}}$$

The Bayes formula implies that the conditional probability $p(P_1, ..., P_{i_{n-1}} | C_j)$ is

$$p_{C_j}(P_1, ..., P_{i_{n-1}} | C_j) = \frac{p_{C_j}(P_1, ..., P_{i_{n-1}}) | C_j}{p_{C_j}}.$$ 

**Definition 1.**

A path has perfect secrecy if and only if $p(P_1, ..., P_{i_{n-1}} | C_j) = p_i$. That is, if the knowledge of a ciphertext obtained by $U_j$ by an attacker does not give him information about the plaintext/ciphertext chosen at every node.

**Proposition 1.**

Suppose that at every node, $(i, i_2), ..., (i_{n-1}, j)$ there exists perfect secrecy, then the path $(i_1 = i, i_2, ..., i_{n-1}, j)$ has perfect secrecy.
implies that there exists $P$ for any set of plaintexts/ciphertexts $p_i$, there exists a unique key $u_i$ such that $u_i p_{i-1}(P_{i-1}) = P_i$. Thus the cardinal of the set of keys (that is the cardinal of $H_2$) used by $U_{i-1}$ is greater than the cardinal of the set of plaintext/ciphertext used $U_i$ which the cardinal $|E|$ of $E$. The following result is a corollary of the Shannon theorem:

**Theorem 1.**

Suppose that the user $U_i$ sends message to $U_j$ using the path $(i_1 = i, ..., i_n = j)$. Suppose also that $|E| = |H_2|$. Then the path has perfect secrecy if the following conditions are satisfied:

Every key is used with the probability $\frac{1}{|E|}$.

For every set of plaintext/ciphertext $(P_i \in E_i, ..., P_{i-1} \in E_{i-1})$, and every set of plaintext/ciphertext $P_{i-1}' \in E_{i-1}$, and every set of plaintext/ciphertext $P_j' \in E_j$, there exists an unique set of keys $u_{i_1}, ..., u_{i_n}$ such that $u_i p_{i-1}(P_{i-1}) = P_i$.

**Proof.**

The Shannon theorem shows that there exists a perfect secrecy at every node $(i_p-1, i_p)$ if and only if every key is used with probability $\frac{1}{|H_2|}$, and for every plaintext/ciphertext $P_{i-1}$, and every plaintext/ciphertext $P_j'$, there exists a unique key $u_i p_{i-1}$ such that $u_i p_{i-1}(P_{i-1}) = P_j'$.

Thus we have to show that the fact that for every set of plaintext/ciphertext $(P_i \in E_i, ..., P_{i-1} \in E_{i-1})$, and every set of plaintext/ciphertext $P_j' \in E_j$, there exists an unique set of keys $u_{i_1}, ..., u_{i_n}$ such that $u_i p_{i-1}(P_{i-1}) = P_j'$ implies that for every plaintext/ciphertext $P_{i-1}$, and every plaintext/ciphertext $P_j'$, there exists a unique key $u_i p_{i-1}$ such that $u_i p_{i-1}(P_{i-1}) = P_j'$, which is the second hypothesis of the Shannon theorem at every node. This is straightforward.

**The entropy cocycle.**

In this part we are going to study the entropy of data conveyed in a network.

**Definition 2.**

Let $U$ be a random variable defined on the set $\{u_1, ..., u_n\}$, we denote by $p_i$ the probability of the event $U = u_i$. The entropy $H(U) = - \sum_{i=1}^{n} p_i L_n(p_i)$, where $L_n$ is the logarithm.
Let \( V \) be another random variable defined on \( \{v_1, \ldots, v_p\} \), we denote by \( p'_i \) the probability of the event \( V = v_i \). We denote by \( H(U \mid v_j) = -\sum_{i=1}^{v_j} p(u_i \mid v_j) \ln(u_i \mid v_j) \).

We denote by \( H(U \mid V) = \sum_{i=1}^{v_i} H(U \mid v_i) \).

The entropy \( H(U) \) quantifies the information given by the variable \( U \), and the relative entropy \( H(U \mid V) \) quantifies the information given by the variable \( U \) if we know already the information given by the variable \( V \).

**Proposition 2.**

Let \( N \) be a network, \( U_i \), and \( U_j \) users of \( N \), we suppose that the keys are defined by a gerbe \( C \) defined on \( N \), let \( V_{ji} \) the space of keys that \( U_i \) uses to send messages to \( U_j \), the entropy \( H(V_{ji}) \) is a 1-Cech cocycle.

**Proof.**

Let \( U_i, U_j \) and \( U_l \) be users of \( N \), we have to show that \( H(V_{jl}) - H(V_{il}) + H(V_{ij}) = 0 \). The set of keys \( V_{il} \) is the set of automorphisms between \( E_l \rightarrow E_i \) which can be viewed as composition of elements \( u_{ij} \circ u_{jl} \). The probability measure on \( V_{il} \) is thus the product of the probability measures of \( V_{ij} \) and \( V_{jl} \), since we have assume that the choice of keys at each node are independent. Let \( p_{ij} \) be the probability of \( u_{ij} \), and \( p_{jl} \) be the probability of \( u_{jl} \), the probability of \( u_{ij} \circ u_{jl} \) is \( p_{ij}p_{jl} \). We deduce that

\[
H(V_{il}) = \sum_{u_{ij} \in V_{ij}, u_{jl} \in V_{jl}} -p_{ij}p_{jl} \ln(p_{ij}p_{jl})
= -\sum_{u_{ij} \in V_{ij}, u_{jl} \in V_{jl}} p_{ij}p_{jl} \ln(p_{ij}) + \ln(p_{jl})
= (\sum_{u_{jl} \in V_{jl}} p_{jl})(-\sum_{u_{ij} \in V_{ij}} p_{ij} \ln(p_{ij})) + (\sum_{u_{ij} \in V_{ij}} p_{ij})(-\sum_{u_{jl} \in V_{jl}} p_{jl} \ln(p_{jl}))
= H(V_{ij}) + H(V_{jl}).
\]

This implies the result.

The following result is well-known in the theory of information:

**Theorem 2.**

Consider a network, and users \( U_i \) and \( U_j \) of the network, we denote by \( V_{ji} \) the set of keys used by \( U_i \) to send messages to \( U_j \), \( P_i \) the set of plaintext/ciphertext used by \( U_i \), and \( P_j \) the set of plaintext/ciphertext used by \( U_j \), then \( H(V_{ji} \mid P_j) = H(V_{ji}) + H(P_j) - H(P_j) \).

We can show the following result:

**Proposition 3.**

In a network, the quantity \( H(V_{ji} \mid P_j) \) is a 1-Cech cocycle.

**Proof.**
We have to show that for users $U_i, U_j$ and $U_l$ of $N$, $H(V_{lj} | P_l) - H(V_{li} | P_l) + H(V_{ji} | P_j) = 0$. We have:

$$H(V_{lj} | P_l) - H(V_{li} | P_l) + H(V_{ji} | P_j) = H(V_{lj}) + H(P_j) - H(P_l) - (H(V_{li}) + H(P_l) - H(P_j)) + H(V_{ji}) + H(P_l) - H(P_j)$$

$$= H(V_{lj}) - H(V_{li}) + H(V_{ji}) = 0$$

Since we have shown that $H(V_{lj})$ is a 1-Chain cocycle.

**III. Higher non Abelian cohomology and End to End encryption.**

Non commutative geometric cohomologies are needed in geometry to represent higher cohomological classes. These theories are also studied in theoretical physics to interpret the action in string theory. The main difficulty to construct such a theory is to define a theory of $n$-categories, the notion of 2-category has been defined by Benabou, for $n > 2$, the coherence relations needed to define such a theory increases considerably, there exists many attempts to define such a theory, for example the thesis of Zouhair Tamsamani. We shall present now the notion of non Abelian cohomology that we shall use to define end to end encryption. This notion is presented in the paper [5], and does not use a theory of $n$-category. The idea is to define recursively geometric representation of cohomology classes. This theory can be viewed as the geometric representation of the connecting morphism associated to an exact sequence of sheaves.

**Definition 1.**

A tower of torsors defined on the site $C$, is defined by a sequence of functors $F_n \rightarrow F_{n-1} \rightarrow \ldots F_0 \rightarrow C$, such that:
- $F_0 \rightarrow C$ is a torsor bounded by the sheaf $L_0$,
- The projection $p_i : F_{i+1} \rightarrow F_i$ is Cartesian,
- There exists a sheaf $L_i$ defined on $C$ such that for each object $e_i$, $Aut_{p_{i-1}(e_i)}(e_i) = L_i(p_0\ldots p_{i-1}(e_i))$, where $Aut_{p_{i-1}(e_i)}(e_i)$ is the group of automorphisms of $e_i$, which project to the identity map of $p_i(e_i)$

**The classifying cocycle associated to a tower of torsors.**

Suppose that the sheaf $L_i$ are commutative, we shall associate to each tower of torsors $F_n \rightarrow F_{n-1} \ldots F_0 \rightarrow C$ a cohomological cocycles $(c_1, \ldots, c_{n+1})$ defined recursively:

We consider an object $e_i$ of $F_0_{U_i}$, and a map $u_{ij} : e_i^j \rightarrow e_i^j$, the family of maps $(u_{ij})$ define a 1-cocycle, which is the classifying cocycle of the torsor $F_0 \rightarrow C$.

Since the functor $F_1 \rightarrow F_0$ is cartesian, we can find a map $u'_{ij} : e'_i \rightarrow e'_i$ of $F_1$ whose projection to $F_0$ is $u_{ij}$.

**Proposition 1.**
The family of maps $u_{ijl} = u'_{ijl}u''_{ijl}^{-1}$ defined a 2-cocycle, that we denote $c_2$.

The sequence $F_1 \to F_0 \to C$ defines a gerbe over $C$ whose band is $L_1$, $c_2$ is the classifying cocycle of this gerbe, the family $(c_1, c_2)$ is the classifying cocycle of the tower of torsors $F_1 \to F_0 \to C$.

Supposed defined the classifying cocycle of the tower $F_i \to F_{i-1} \to \ldots \to F_0 \to C$, it is a family of cocycles $(c_1, c_2, \ldots, c_{i+1})$, where for $0 < l < i + 1$, $c_l$ is a $l+1$-cocycle. We denote by $u_{i_1 \ldots i_{l+2}}$ the chain which define the $i+1$-cocycle, it is an automorphism of the object $e_{i_1 \ldots i_{l+2}}$ of $F_i$ over $U_{i_1 \ldots i_{l+1}}$, since the map $F_{l+1} \to F_i$ is cartesian, we can lift $u_{i_1 \ldots i_{l+2}}$ to a map $u'_{i_1 \ldots i_{l+2}}$ of the object $e_{i_1 \ldots i_{l+2}}$ of $F_{l+1}$, we suppose that $e_{i_1 \ldots i_{l+2}}$ projects to the restriction of $e_{i_1 \ldots i_{l+2}}$ to $U_{i_1 \ldots i_{l+2}}$.

**Proposition 2.**

The Čech boundary $\delta(u'_{i_1 \ldots i_{l+2}}) = u_{i_1 \ldots i_{l+3}}$ is an $i + 1$-Čech cocycle, which is the classifying cocycle.

We denote by $c_{i+2}$ the $i + 2$-cocycle defined by the chain $u_{i_1 \ldots i_{l+3}}$, the family of cocycles $(c_1, \ldots, c_{i+2})$ is the classifying cocycle of the tower $F_{i+1} \to F_0 \to C$.

Let $U$ be an object of $C$, and $e_U$ an object of $F_0(U)$, we denote by $f_{neU}$, the fiber of $e_U$, it is the $n - 1$ tower $F_{neU} \to F_{1eU} \to e_U$, such that $F_{neU}$ is the subcategory of $F_i$, whose objects project to $e_U$. We endow $e_U$ with the topology of $U$.

Let $U_i$ and $U_j$ be objects of $C$, and $e_{U_i}$ and $e_{U_j}$ two respective objects of $C_{0U_i}$ and $C_{0U_j}$, a map $u_{ij} : e_{i} \to e_{j}$ induces a morphism $u''_{ij} : f_{ne_j} \to f_{ne_i}$ of $n - 1$-towers of torsors between the fibers of $e_j$ and $e_i$.

**Example.**

Let $C$ be a site, consider a torsor $F_0 \to C$, and a family of exact sequences $1 \to L_{i+1} \to L_{i+1} \to L_i \to 1$, $0 \leq i < n$. This sequence induces the following tower of torsors: $F_1$ is the category defined as follows: an object $e_{U_i}$ of $F_1$ is an $L_i$ torsors defined over object $U$ of $C$, such that the quotient of $e_{U_i}$ by $L_i$ is the restriction of $F_0$ to $U$.

Suppose defined the category $F_i$, $i < n - 1$, an object $e_{U_i}^{i+1}$ of $F_{i+1}$ is an $L_{i+1}$-torsors over object $U$ of $C$ such that its quotient by $L_{i+1}$ is the restriction of an object of $F_i$ to $U$.

**III.1. Encryption with a tower of torsors.**

**Definition 1.**

Let $C$ be a tower of torsors, and $F_n \to F_{n-1} \to \ldots F_0 \to C$ a tower of torsor defined on $C$, we suppose that the objects of $C$ are members of a network, and for each object $U$ of $C$, the objects of the category $F_n$ which project to $U$ are isomorphic to an $L'_n$ trivial torsor defined on $U$, where $L'_n$ is a finite (commutative) group that we identify to the alphabet used in the network. The
encryption defined by the tower of torsors \( C \), is the encryption such that the exchange between \( U_i \) and \( U_j \) is defined by a map \( u^n_{ij} : e^n_j \to e^n_i \), where \( e^n_j \) and \( e^n_i \) are objects of \( F^n \) whose respective projection on \( C \) are \( U_j \) and \( U_i \).

This protocol of encryption can be applied to the previous example of tower of torsors. We have seen that in a network, the information is hierarchical, in a network in which encryption is defined by a tower of \( n \)-torsors, the alphabet is \( L'_n \) which can be viewed as a union of the alphabet \( L'_i \), \( 0 \leq i \leq n \). The information written with the alphabet \( L_0 = L'_0 \) is the header, which is encrypted and decrypted at each node. We are going to see, that the most \( n \) is big, the more it is difficult to break information conveyed in the tower of torsors. Thus the natural order of the alphabet \( L'_i \) define, an order of confidentiality on the message.

**Attack on encryption defined by a tower of torsors.**

We have defined an attack for an encryption protocol by defined by a gerbe, we shall generalize this attack to an encryption protocol defined by a tower of torsors. We remark that an encryption defined by a tower of torsor is a priori more secured than an encryption defined with a gerbe, since the relations between the keys are more complicated in the tower of torsor than in the gerbe.

We suppose that we can define an attack for a tower of torsors \( F_n \to F_{n-1} \to ... \to F_0 \to C \), using \( n + 2 \)-intruders, this is equivalent to saying that given \( n + 2 \) intruders \( U_1, ..., U_{n+2} \), and two users \( U_i \) and \( U_j \) of the network, we can define attack which allow \( U_1, ..., U_{n+2} \) to find the key \( u^n_{ij} \).

Let \( f_{n+1} = F_{n+1} \to F_n \to ... \to F_0 \to C \), be a tower of torsors which defines an encryption scheme over the topos \( C \), Suppose that there exists \( n + 3 \) intruders in the network, denoted by \( U_1, ..., U_{n+3} \). We can construct the topos \( C_{U_1} \) whose final object is \( U_1 \), and whose topology is generated by the covering family \((U_i \times C U_j)\), the restriction of \( F_0 \) to \( C_{U_1} \) is trivial, we can thus construct the tower of torsors \( F'_{n} \to ... \to F'_0 \to C_{U_1} \), where \( F'_i \) is the fiber \( F_{i+1 U_1} \) of \( F_{i+1} \) over \( U_1 \). This sequence is an \( n \)-tower of torsors, its classifying cocycle can be constructed using the restriction of the transition functions \( u_{ip} \) of \( F_0 \to C \) to \( C_{U_1} \), the recursive hypothesis implies that the encryption protocol that it induces can be broken with \( n + 2 \)-intruders, since \( U_1 \times C U_2, ..., U_1 \times C U_{n+3} \) are intruders, we deduce that the encryption system can be broken.

The encryption protocol with \( n \)-tower of torsors gives rise to the following problem: Let \( N \) be a network which users are \( U_1, ..., U_n \), we suppose that the \( U_i \) are objects of a topos, and they communicate with plaintexts written in a finite alphabet which can be identified to an algebraic variety \( V \) over a finite field, we suppose that the keys are elements of the group \( H \) of algebraic automorphisms of \( V \). Is there a tower of gerbe, or a notion of \( n \)-gerbe such that the keys are defined using the previous protocol?

**Proposition 1.**
Suppose that there exists an exact sequence $1 \rightarrow H_1 \rightarrow H \rightarrow H_2 \rightarrow 1$ such that $u_{ij}u_{il}u_{li} \in H_1$, then there exists a gerbe $D \rightarrow C$ such that the encryption protocol associated to $D$ defines the keys of the cryptosystem.

**Proof.**

The projection $p : H \rightarrow H_2$ induces a torsor $P \rightarrow C$, whose trivialization is defined by the transition function $p(u_{ij})$, the gerbe associated is the gerbe defined by the extension problem of the exact sequence.

**II.2. Public encryption and tower of torsors.**

We shall define public encryption using tower of torsors, on this purpose we need to define a notion of flat tower of torsors and connective structure on flat tower of torsors.

**Definition 1.**

A connective structure defined on the tower of torsors $F_n \rightarrow F_{n-1} \ldots F_0 \rightarrow C$, is a 0-chain of the sheaf $L_0$.

This definition generalizes the corresponding definition for torsors and gerbes.

In practice as we have seen in the example defined above, the objects of $F_i$ are $H'_{i+1}$-torsors defined over an object of $C$. The map $u_{ij}^n : e_i^n \rightarrow e_j^n$ is induced by a map of the trivial $H'_{n+1}$ torsor defined over $U_i \times_C U_j$. A connection can be defined as a family $(c_i)_{i \in I}$ of elements of the Lie algebra $\mathcal{H}'_{n+1}$.

A public encryption is defined by a function $L : \mathcal{H}_n \times H_n \rightarrow H_n$ such that

$$L(c_i, \exp(c_j)) = u_{ij}^n$$

The private key of the user $U_i$ is $c_i$, its public key is $\exp(c_i)$.

**Bibliography.**

1. A. Grothendieck, Fondements de la geometrie algebrique
2. A. Grothendieck, Recoltes et semaines.
3. C. Shannon, Communication theory of secrecy systems. Bell Syst. Tech. J. vol 28, 656-715 1949
4. W. Stallings, cryptography and network security forth edition, Prentice Hall.
5. A. Tsemo, Non Abelian cohomology the point of view of gerbed towers, to appear in the African Diaspora Journal of Mathematics

Tsemo Aristide,
College Boreal,
1 Yonge Street
M5E 1E5, Toronto Canada.