MÖBIUS RIGIDITY OF INVARIANT METRICS IN BOUNDARIES OF
SYMMETRIC SPACES OF RANK 1

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Abstract. Let $\mathbf{H}^n_K$ denote the boundary of a symmetric space of rank one and of non compact type and let $d_0$ be the Korányi metric defined in its boundary. We prove that if $d$ is a metric on $\mathbf{H}^n_K$ such that all Heisenberg similarities are $d$–Möbius maps, then under a continuity condition $d$ is a constant multiple of a power of $d_0$.

1. Introduction and Statement of Results

Let $(S, d)$ be a metric space and suppose that there exists a remote point $\infty$ such that $\overline{S} = S \cup \{\infty\}$ is compact. We may extend $d$ to the compactification by agreeing that $d(p, \infty) = +\infty$ for every $p \in S$ and also $d(\infty, \infty) = 0$. A natural metric cross–ratio $|X^d|$ is defined for each four pairwise distinct points $p = (p_1, p_2, p_3, p_4)$ by setting

$$|X^d|(p) = \frac{d(p_1, p_2)}{d(p_1, p_3)} \frac{d(p_3, p_1)}{d(p_3, p_2)},$$

with obvious modifications when one of the points is $\infty$. The Möbius group $\mathcal{M}_d = \mathcal{M}_d(S)$ is the group of homeomorphisms of $\overline{S}$ which leave $|X^d|$ invariant for each quadruple $p$ of pairwise distinct points of $\overline{S}$. The similarity group (or homothety group) $\text{Sim}_d = \text{Sim}_d(S)$ is the subgroup of $\mathcal{M}_d(S)$ comprising homeomorphisms $\phi$ such that there exists a positive constant $K(\phi)$ which satisfies $d(\phi(p), \phi(q)) = K(\phi) \cdot d(p, q)$, for all $p, q \in S$. If $K(\phi) = 1$ then $\phi$ is an isometry. It turns out that $\text{Sim}_d$ is simply $\text{Stab}_{d(\infty)}$, i.e., the stabiliser of $\mathcal{M}_d$ at infinity and the map $K : \text{Sim}_d \to \mathbb{R}_+^*$, $\phi \mapsto K(\phi)$, is a group homomorphism (see Propositions 2.3 and 2.4). We call $K$ the similarity homomorphism.

Given two metrics $d_1$ and $d_2$ on $S$, we say that they define the same Möbius structure on $S$ if $|X^{d_1}| = |X^{d_2}|$. If they have the same remote point, then it is simple to show (see Lemma 2.1 of [2]) that $d_1$ and $d_2$ are homothetic, that is, there exists a $c > 0$ such that $d_1 = c \cdot d_2$. Thus we get as a byproduct that $\mathcal{M}_{d_1} = \mathcal{M}_{d_2}$ and in particular, $\text{Sim}_{d_1} = \text{Sim}_{d_2}$. Conversely, suppose that we are given a Möbius structure on $\overline{S}$ and a representative $d_1$. Suppose also that there exists a metric $d_2$ defined on $S$ which has the same remote point as $d_1$ and also satisfies $\text{Sim}_{d_1} \subseteq \text{Sim}_{d_2}$. Then, what can we say about the relation of $d_1$ and $d_2$? Do they actually belong to the same Möbius structure?

In this paper we give a quite precise answer to this question in the case where $\overline{S}$ is the boundary of a symmetric space of rank one and of non compact type and the given Möbius structure is the canonical one, i.e., the one arising from the Korányi metric. We recall below some known facts about the aforementioned notions; for details we refer to Section 2. Let $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$ be the set of real, complex, quaternionic and octonionic numbers, respectively. With $\mathbf{H}^n_K$ we shall thereafter denote the $n$–dimensional $K$–hyperbolic space (in the case $K = \mathbb{O}$, $n = 2$). Symmetric spaces of rank one and of non compact type are necessarily $K$–hyperbolic spaces. The boundary $\partial \mathbf{H}^n_K$ is a sphere, isomorphic to the one point compactification of the $K$–Heisenberg group $\mathcal{H}_K$: That is, the set $\mathbb{K}^{n-1} \times \mathbb{S}(\mathbb{K})$ endowed with the group multiplication

$$(\zeta, v) \ast (\zeta', v') = (\zeta + \zeta', v + v + 2\omega(\zeta, \zeta')), \quad (1)$$
for each $(\zeta, v), (\zeta', v') \in \mathcal{H}_K$. Where, $\omega$ is the usual symplectic form in $\mathbb{K}^{n-1}$. A natural conjugation $J$ exists in $\mathbb{K}^{n-1} \times \mathbb{R}([\mathbb{K}])$, which is the restriction of the conjugation of $H^\alpha_\mathbb{K}$; that is, $(\zeta, v) \mapsto (\zeta, \mathbf{v})$.

(In case when $\mathbb{K} = \mathbb{R}$ this is of course the identity map). There is a gauge $| \cdot |_K$ on $\mathcal{H}_K$, called the Korányi gauge, which is given for each $(\zeta, v)$ by

$$\|(\zeta, v)|_K = | - \|v\|^2 + v|^{1/2} = (\|v\|^4 + |v|^2)^{1/4}.$$  

The Korányi (or, Korányi-Cygan) metric $d_\mathcal{H}$ is then defined in $\mathcal{H}_K$ by

$$\tag{1.1} d_\mathcal{H} ((\zeta, v) \cdot (\zeta', v')) = |(\zeta', v')^{-1} \ast (\zeta, v)|_{K},$$

for each $(\zeta, v), (\zeta', v') \in \mathcal{H}_K$. Let $G_K = \text{Isom}_h(H^\alpha_\mathbb{K})$ be the group of holomorphic isometries of $G_K$. The similarity group $\text{Sim}(\mathcal{H}_K)$ together with an inversion $I$ generate $G_K$. The full group of isometries comprises elements of $G_K$ followed by $K-$conjugation $J$.

The canonical Möbius structure of $\partial H^\alpha_\mathbb{K}$ is the one arising from $d_\mathcal{H}$. The Möbius group $\mathcal{M}_{d_\mathcal{H}}$ of the metric $d_\mathcal{H}$ consists of the similarities of $d_\mathcal{H}$ and inversion $I$, see Section 2.

As our general question dictates, suppose that we are given a metric $d$ in $\partial H^\alpha_\mathbb{K}$ such that:

(\text{Sim}) $\text{Sim}_{d_\mathcal{H}} \subseteq \text{Sim}_d$.

We wish to find the relation of $d$ and $d_\mathcal{H}$. In our approach, in addition to $(\text{Sim})$ we shall further presuppose a quite plausible continuity condition:

(\text{Cont}) The similarity homomorphism $K : \text{Stab}_{\mathcal{M}_d}(\infty) \to \mathbb{R}_+^*$ is a continuous map.

Continuity here is in the sense of compact–open topology (see Section 2 for details). In our line of proving the below stated results, condition $(\text{Cont})$ is rather indispensable.

Before stating our results, we list a set of conditions for $d$. Those are:

$(\text{Conj})$ $K-$conjugation is in $\mathcal{M}_d$.

$(\text{Inv})$ Inversion $I$ is in $\mathcal{M}_d$.

$(\alpha-$Höl$)$ There exist constants $\beta > 0$ and $\alpha \in (0, 1]$ such that the identity map of $\partial H^\alpha_\mathbb{K}$ is $(\beta \cdot d_\mathcal{H}, d)-$Hölder continuous with exponent $\alpha$.

$(\text{G})$ There exists an $\alpha \in (0, 1]$ such that for each $p = (\zeta, v) \in \mathcal{H}_K$,

$$d^{1/\alpha} (o, p) = d^{1/\alpha} (o, \Pi_{K_{\zeta, 1}} (p)) + d^{1/\alpha} (\Pi_{K_{\zeta, 1}} (p), \Pi_{\mathbb{H}} (p)),$$

where $\Pi_{K_{\zeta, 1}} (p) = \zeta$ and $\Pi_{\mathbb{H}} (p) = v$.

$(\text{Eq})$ The metric $d$ satisfies $d(o, (1, 0)) = d(o, (0, 1)) = d(o, (0, -1))$. Here, $o = (0, \ldots, 0, 0)$, $(1, 0) = (1, \ldots, 0, 1)$ and $(0, 1) = (0, \ldots, 0, 1)$, where in the last case 1 is the first coordinate unit vector of $\mathbb{H}$.

$(\text{biLip})$ The metric $d$ is bi-Lipschitz equivalent to $d_\mathcal{H}$.

$(\alpha-$Met$)$ There exist constants $\beta > 0$ and $\alpha \in (0, 1]$ such that $d = \beta \cdot d_\mathcal{H}$.

Our first theorem assumes the least number of conditions for $d$, that is, $(\text{Sim})$ and $(\text{Cont})$.

**Theorem 1.1.** Let $d$ be a metric in $\mathcal{H}_K$ and denote again by $d$ its extension to $\partial H^\alpha_\mathbb{K} = \mathcal{H} \cup \{\infty\}$. Then the following hold:

1. If $\mathbb{K} = \mathbb{R}$, conditions $(\text{Sim})$ and $(\text{Cont})$ together imply $(\alpha-$Met$)$.
2. If $\mathbb{K} \neq \mathbb{R}$, conditions $(\text{Sim})$ and $(\text{Cont})$ together imply $(\alpha-$Höl$)$. 
One already notes the significant difference between the real case and all the other cases. With the least possible assumptions, the metric $d$ is a power of the Euclidean metric when $\mathbb{K} = \mathbb{R}$. The picture is entirely different in case when $\mathbb{K} \neq \mathbb{R}$; metrics which satisfy ($\alpha$-Höf) may be of nature entirely different from the one of a power of $d_{\mathbb{K}}$ (e.g., the Carnot–Carathéodory metric). It is therefore quite necessary to add more assumptions for $d$ in this case. To that end, the strongest version of our first main result for the case $\mathbb{K} \neq \mathbb{R}$ follows:

**Theorem 1.2.** With the assumptions of Theorem 1.1, suppose $\mathbb{K} \neq \mathbb{R}$. Then conditions (Sim), (Cont), (Conj) and (Inv) together imply ($\alpha$-Met).

But again, adding (Conj) and (Inv) to our basic assumptions looks like an overkill to prove ($\alpha$-Met); in the case $\mathbb{K} = \mathbb{R}$ condition (Conj) is vacuous and (Inv) holds a posteriori. It is natural therefore to question the magnitude of necessity of (Conj) and (Inv). Theorem 1.2 tells us that we obtain a rather weak result if we drop (Conj) and (Inv) entirely. It turns out though that ($\alpha$-Met) follows by replacing (Conj) and (Inv) with (G) and (Eq). In fact, we have:

**Theorem 1.3.** With the assumptions of Theorem 1.1, suppose $\mathbb{K} \neq \mathbb{R}$. Then:

1. Conditions (Sim), (Cont) and (G) together imply (biLip).
2. Conditions (Sim), (Cont), (G) and (Eq) together imply ($\alpha$-Met).

Therefore, (Inv) follows as a side result of Theorem 1.3 and we may further observe that (G) and (Eq) hold vacuously in the real case. Moreover, (G) can be replaced with an equivalent statement which is the closest to parallelogram law in the $\mathbb{K}$–Heisenberg group setting, $\mathbb{K} \neq \mathbb{R}$. For this set for each $p \in \mathcal{F}$,

$$|p| = d(\alpha, p).$$

It turns out that if (Sim) holds, then (G) is equivalent to the following condition, see Proposition 3.5:

(P-L) For $\alpha \in (0, 1]$ and for each $p, q \in \mathcal{F}$,

$$|p * q|^{4/\alpha} + |p^{-1} * q|^{4/\alpha} + |p * q^{-1}|^{4/\alpha} + |p^{-1} * q^{-1}|^{4/\alpha} = 2 \left( |\Pi_{\mathbb{K}^{n-1}} (p * q)|^{4/\alpha} + |\Pi_{\mathbb{K}^{n-1}} (p^{-1} * q)|^{4/\alpha} \right) + |\Pi_{\mathbb{K}} (p * q)|^{4/\alpha} + |\Pi_{\mathbb{K}} (p^{-1} * q)|^{4/\alpha} + |\Pi_{\mathbb{K}} (p * q^{-1})|^{4/\alpha} + |\Pi_{\mathbb{K}} (p^{-1} * q^{-1})|^{4/\alpha},$$

where $\Pi_{\mathbb{K}^{n-1}}$ and $\Pi_{\mathbb{K}}$ are projections of $\mathcal{F}$ to $\mathbb{K}^{n-1}$ and $\mathbb{K}$, respectively.

Thus an equivalent to the second statement of Theorem 1.3 (2) is:

(2') Conditions (Sim), (Cont), (P-L) and (Eq) together imply ($\alpha$-Met).

Recall now that a metric $d$ defined in a space $S$ is Ptolemaean if for each quadruple of points $p = (p, q, r, s)$ of $S$ the following inequality is satisfied for all possible permutations of points in $p$:

$$d(p, r) \cdot d(q, s) \leq d(p, q) \cdot d(r, s) + d(p, s) \cdot d(r, q).$$

A Ptolemaean circle is a subset $\sigma$ of $S$ which is topologically equivalent to the unit circle $S^1$ and for each quadruple of points $p = (p, q, r, s)$ of $\sigma$ such that $p$ and $r$ separate $q$ and $s$, Inequality (1.2) holds as an equality. It is well known that $(\partial \mathbb{H}^n_{\mathbb{K}}; d_{\mathbb{K}})$ is Ptolemaean, see for instance [11]; therefore it is natural to ask which of the metrics $d$ who satisfy ($\alpha$-Met) satisfies also:

(Ptol) The metric $d$ is Ptolemaean.
It turns out that we will also need:

**(Circ)** The metric $d$ has a Ptolemaean circle.

We have the following corollary to Theorem 1.3:

**Theorem 1.4.** Condition (α-Met) implies (Ptol) and (Ptol) together with (Circ) are equivalent to (1-Met). Therefore a metric $d$ in $\partial H^n_K$ which satisfies conditions:

1. (Sim), (Cont) and (Circ) if $K = \mathbb{R}$ and
2. (Sim), (Cont), (P-L) or (G), (Eq) and (Circ) if $K \neq \mathbb{R}$,

is necessarily a constant multiple of the Korányi metric $d_\delta$.

This is in the spirit of the old result of Schoenberg, see [13]. That particular result was on metrics which were derived from semi–norms which also share Ptolemaean property.

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2. Preliminaries

This section is divided in two parts. In the first part (Section 2.1), we state known results about the Korányi metric in $\partial H^n_K$ and its properties. In the second part (Section 2.2), we state in brief some elementary facts concerning Möbius geometry, mainly focussing in properties of the similarity group.

2.1. The $K$–Heisenberg group $\mathfrak{H}_K$ and the Korányi metric $d_\delta$. The following are well known; we refer the reader to the classical book of Mostow, [9], or to [11]. Another useful reference for the case $K = \mathbb{C}$ is the book of Goldman, [7]. For the octonionic case in particular we refer to [1] and to [10].

We shall use hereafter the following notation ($n > 1$):

$$G_K = \begin{cases} 
SO(n,1) & \text{if } K = \mathbb{R}, \\
SU(n,1) & \text{if } K = \mathbb{C}, \\
Sp(n,1) & \text{if } K = \mathbb{H}, \\
F_4(-20) & \text{if } K = \mathbb{O} \ (n = 2).
\end{cases}$$

Also,

$$F(n) = \begin{cases} 
SO(n) & \text{if } K = \mathbb{R}, \\
SU(n) & \text{if } K = \mathbb{C}, \\
Sp(n) & \text{if } K = \mathbb{H}, \\
\text{Spin}_7(\mathbb{R}) & \text{if } K = \mathbb{O} \ (n = 2).
\end{cases}$$

$K$–hyperbolic space $H^n_K$ is $G_K/F(n)$. The metric $d_\delta$ as given in Equation 1.1 is invariant under the following transformations (and their extensions to $\infty$).

1. Left translations which come from the action of $\mathfrak{H}_K$ on itself: For any fixed point $(\zeta', v') \in \mathfrak{H}_K$ let

$$T_{(\zeta', v')}(\zeta, v) = (\zeta', v') \ast (\zeta, v), \quad T_{(\zeta', v')}(\infty) = \infty.$$  

In the particular case where $K = \mathbb{R}$, $d_\delta$ is also invariant under the right translations.

2. Rotations: For the cases $K \neq \mathbb{O}$ these come from the action of $F(n-1)$ on $\mathbb{K}^{n-1}$. Specifically, given a $U \in F(n-1)$, $K \neq \mathbb{O}$, we define

$$S_U(\zeta, v) = (U \cdot \zeta, v), \quad S_U(\infty) = \infty.$$
Only in the case where $\mathbb{K} = \mathbb{H}$ we have the action of $F(1) = \text{Sp}(1)$ given by

$$(\zeta_1, \ldots, \zeta_n-1, v) \mapsto (\mu \zeta_1 \mu^{-1}, \ldots, \mu \zeta_{n-1} \mu^{-1}, \mu v \mu^{-1}), \quad \mu \in \text{Sp}(1);$$

observe that in all other cases this action is vacuous. In the particular case $\mathbb{K} = \mathbb{O}$, for given unit imaginary octonion $\mu$, let

$$S_\mu(\zeta, v) = (\zeta \cdot \overline{\mu}, \mu v \overline{\mu}), \quad S_\mu(\infty) = \infty.$$ 

We stress at this point that in general $S_\mu \circ S_\nu \neq S_{\mu \nu}$ for $\mu$, $\nu$ unit imaginary octonions. The group generated by transformations $S_\mu$ is the compact group $\text{Spin}_7(\mathbb{R})$. All these actions form the group $\text{Isom}_{d_\delta}(\partial(\mathbb{H}_\mathbb{K}^n))$ of $d_\delta$–isometries; this acts transitively on $\mathcal{H}_\delta$. We also consider two other kinds of transformations of $\partial \mathbb{H}_\mathbb{K}^n$.

(3) Dilations: If $\delta \in \mathbb{R}^*_+$ we define

$$D_\delta(\zeta, v) = (\delta \zeta, \delta^2 v), \quad D_\delta(\infty) = \infty.$$ 

(In the bocconionic case, $\delta^2$ is used in [10] instead of $\delta$, but the model for $\mathcal{H}_\delta$ is somewhat different). One verifies that for every $(\zeta, v), (\zeta', v') \in \partial \mathbb{H}_\mathbb{K}^n$ we have

$$d_\delta(D_\delta(\zeta, v), D_\delta(\zeta', v')) = \delta d_\delta((\zeta, v), (\zeta', v')).$$

Thus the metric $d_\delta$ is scaled up to multiplicative constants by the action of dilations. We mention here that together with $d_\delta$–isometries, dilations form the $d_\delta$–similarity group $\text{Sim}_{d_\delta}(\partial \mathbb{H}_\mathbb{K}^n)$.

(4) Inversion $I$ is given by

$$I(\zeta, v) = \left(\zeta(-\|\zeta\|^2 + v)^{-1}, \overline{v} - \|\zeta\|^2 |v|^{-2}\right), \quad \text{if } (\zeta, v) \neq o, \infty, \quad I(o) = \infty, \quad I(\infty) = o.$$ 

Inversion $I$ is an involution of $\partial \mathbb{H}_\mathbb{K}^n$. Moreover, for all $p = (\zeta, v), p' = (\zeta', v') \in \mathcal{H}_\delta \setminus \{o\}$ we have

$$d_\delta(I(p), o) = \frac{1}{d_\delta(p, o)}, \quad d_\delta(I(p), I(p')) = \frac{d_\delta(p, o) d_\delta(o, p')}{d_\delta(p, o')}.$$ 

The similarity group $\text{Sim}_{d_\delta}(\partial \mathbb{H}_\mathbb{K}^n)$ is the semidirect product $\mathbb{R} \rtimes \text{Isom}_{d_\delta}(\partial \mathbb{H}_\mathbb{K}^n)$. The group generated from $\text{Sim}_{d_\delta}(\partial \mathbb{H}_\mathbb{K}^n)$ and inversion $I$ is isomorphic to $G_\mathbb{K}$. Given two distinct points on the boundary, we can find an element of $G_\mathbb{K}$ mapping those points to 0 and $\infty$ respectively; in particular $G_\mathbb{K}$ acts doubly transitively on the boundary. In the exceptional case where $\mathbb{K} = \mathbb{R}$, the action of $G_\mathbb{R}$ is triply transitive; this follows from the fact that we can map three distinct points of the boundary to the points 0, $\infty$ and $(1, 0, \ldots, 0)$ respectively.

2.2. Möbius Group, Similarity Group. Recall from the introduction that in the most general setting we start from a mere metric space $(S, d)$ and we suppose that there is a remote point $\infty$ such that:

1. $S \cup \{\infty\}$ is compact;
2. $d$ is extended to compactification by setting

$$d(p, \infty) = +\infty, \quad \text{for each } p \in S.$$ 

In such a context, the notion of metric cross–ratio is next. The following proposition may be verified straightforwardly, detailed discussions about cross–ratios may be found among others in [5, 6, 8] and [11]:
Proposition 2.1. Let \((S,d)\) be a metric space with a remote point \(\infty\). Denote by \(\mathcal{C}\) the space \(S \times S \times S \times S \setminus \{\text{diagonals}\}\). Then the map \(|X^d| : \mathcal{C} \to \mathbb{R}^+_+\) defined for each \(p = (p_1, p_2, p_3, p_4) \in \mathcal{C}\) by

\[
|X^d|(p) = \frac{d(p_4, p_2)}{d(p_4, p_1)} \cdot \frac{d(p_3, p_1)}{d(p_3, p_2)},
\]

satisfies the following:

1. Symmetries:
   \[|X^d|(p_1, p_2, p_3, p_4) = |X^d|(p_2, p_1, p_4, p_3) = |X^d|(p_3, p_4, p_1, p_2) = |X^d|(p_4, p_3, p_2, p_1).\]

2. Let
   \[|X^d_1|(p) = |X^d|(p_1, p_2, p_3, p_4), \quad \text{and} \quad |X^d_2|(p) = |X^d|(p_1, p_3, p_2, p_4).\]
   Then
   \[|X^d|(p_1, p_3, p_4, p_2) = \frac{1}{|X^d_2|}, \quad |X^d|(p_1, p_4, p_3, p_2) = \frac{|X^d_1|}{|X^d_2|},\]
   \[|X^d|(p_1, p_2, p_4, p_3) = \frac{1}{|X^d_1|}, \quad |X^d|(p_1, p_4, p_2, p_3) = \frac{|X^d_2|}{|X^d_1|}.\]

Thus for any possible permutation \(\tilde{p}\) of points of a given quadruple \(p\), \(|X^d(\tilde{p})|\) depends only on \(|X^d_1|(p)\) and \(|X^d_1|(p)\).

A homeomorphism \(\phi : S \to S\) shall be called a \(d\)-Möbius map if \(|X^d(p)| = |X^d(\phi(p))|\) for each \(p = (p_1, p_2, p_3, p_4) \in \mathcal{C}\). Here, by \(\phi(p)\) we denote the quadruple \((\phi(p_1), \phi(p_2), \phi(p_3), \phi(p_4))\). The set of all Möbius maps of \(S\) form a group \(\mathcal{M}_d = \mathcal{M}_d(S)\), which we shall call the Möbius group of \((S,d)\).

From Proposition 2.1 we have the following elementary but useful corollary.

Corollary 2.2. A homeomorphism \(\phi : S \to S\) is in \(\mathcal{M}_d\) if and only if for each given quadruple \(p\),

\[|X^d_1|(\phi(p)) = |X^d_1|(p) \quad \text{and} \quad |X^d_2|(\phi(p)) = |X^d_2|(p)\]

In particular, we consider the subset \(\text{Sim}_d\) of \(\mathcal{M}_d\) consisting of similarities: An element \(\phi \in \mathcal{M}_d\) is a similarity, if there exists a \(K_\phi > 0\) such that for every \(p, q \in S\)

\[d(\phi(p), \phi(q)) = K_\phi \cdot d(p, q).\]

By definition, \(K_\phi = K(\phi)\) is independent from the choice of points. Now, \(\text{Sim}_d\) is a subgroup of \(\mathcal{M}_d\): If \(\phi, \psi \in \text{Sim}_d\), then for every \(p, q \in S\),

\[d(\phi(p), \phi(q)) = K_\phi \cdot d(p, q) \quad \text{and} \quad d(\psi(p), \psi(q)) = K_\psi \cdot d(p, q).\]

Therefore for each \(p, q \in S\),

\[d((\phi \circ \psi)(p), (\phi \circ \psi)(q)) = d((\phi(\psi(p))), (\phi(\psi(q))) = K_\phi \cdot d(\psi(p), \psi(q)) = K_\phi \cdot K_\psi \cdot d(p, q).\]

Since this shows that \(\phi \circ \psi\) is a similarity, we must also have

\[d(\phi(\psi(p)), (\phi \circ \psi)(q)) = K_{\phi \circ \psi} \cdot d(p, q).\]

We deduce \(K_{\phi^{-1}} = (K_\phi)^{-1}\) and we have proved the following:
Proposition 2.3. The set $\text{Sim}_d$ is a subgroup of $\mathcal{M}_d$. There exists a group homomorphism $K : \text{Sim}_d \to \mathbb{R}_+^*$ given for each $\phi \in \text{Sim}_d$ by

$$K(\phi) = K_\phi.$$ 

We call $K$ the similarity homomorphism. We are interested in the stabiliser of $\infty$ in $\mathcal{M}_d$, that is,

$$\text{Stab}_d(\infty) := \text{Stab}_{\mathcal{M}_d}(\infty) = \{\phi \in \mathcal{M}_d \mid \phi(\infty) = \infty\}.$$ 

Proposition 2.4. $\text{Stab}_d(\infty) = \text{Sim}_d$.

Proof. We show first that $\text{Sim}_d \subseteq \text{Stab}_d(\infty)$. Supposing the contrary, assume that there exists a $\phi \in \text{Sim}_d$ such that $\phi(\infty) = p \in S$. Then

$$+\infty = d(\infty, p) = (K_\phi)^{-1} \cdot d(\phi(\infty), \phi(p)) = (K_\phi)^{-1} \cdot d(p, \phi(p)).$$

Therefore $\phi(p) = \infty$ and thus $\phi^2(p) = p$. But on the other hand,

$$\phi^2(\infty) = \phi(\phi(p)) = \phi(\infty) = p = \phi^2(p),$$

which is a contradiction since $\phi^2$ is a homeomorphism.

To show that $\text{Stab}_d(\infty) \subseteq \text{Sim}_d$ we only have to prove that any element $\phi \in \text{Stab}_d(\infty)$ is a similarity. For this, we consider an arbitrary $\phi \in \text{Stab}_d(\infty)$ and we fix two arbitrary points $p, q \in S$. We also consider the quantity

$$K(\phi, p, q) = \frac{d(\phi(p), \phi(q))}{d(p, q)}.$$ 

Now for any $r \in S$, let $p = (r, q, \infty, p)$. Relations

$$|X^d_1|(p) = |X^d_1|(\phi(p)) \quad \text{and} \quad |X^d_1|(p) = |X^d_1|(\phi(p)),$$

yield

$$K(\phi, p, q) = K(\phi, p, r) \quad \text{and} \quad K(\phi, p, q) = K(\phi, q, r).$$

From the left equality it follows that $K(\phi, p, q)$ does not depend on $q$. But then the right equality shows that it also does not depend on $p$. Since $p, q$ are arbitrary, $K = K(\phi)$ and the proof is complete. \qed

In view of Propositions 2.3 and 2.4 we have the following:

Corollary 2.5. If the similarity homomorphism $K$ is onto, then

$$\text{Sim}_d/\text{Isom}_d \simeq \mathbb{R}_+^*,$$

where $\text{Isom}_d$ is the subgroup of $\mathcal{M}_d$ consisting of isometries and $\mathbb{R}_+^*$ is the multiplicative group of positive real numbers. Thus the short exact sequence

$$1 \to \text{Isom}_d \to \text{Sim}_d \to \mathbb{R}_+^* \to 1$$
Remark 2.6. Note that if there exists an inverse $L : \mathbb{R}_+^* \to \text{Sim}_d$ to the similarity homomorphism $K$ of the short exact sequence \([2.2]\) such that $L \circ K$ is the identity homomorphism when restricted to $\text{Isom}_d$, then by Splitting Lemma (reference?) we have
\[
\text{Sim}_d \cong \mathbb{R}_+^* \rtimes \text{Isom}_d,
\]
the semidirect product of $\mathbb{R}_+^*$ and $\text{Isom}_d$. This is for instance the case where $S = \partial H^2_K$ and $d = d_H$: $L$ maps each positive $\delta$ to $D_\delta$ and $L \circ K = \text{id.}$ when restricted to $\text{Isom}_{d_H}$.

From this point on, we shall assume that $S$ is a regular, locally compact topological space. Then $\mathcal{M}_d$ can be endowed with the compact-open topology and accordingly $\text{Sim}_d$ can be endowed with the relative topology. Condition (Cont) of Theorem [1.2] is that the similarity homomorphism $K : \text{Sim}_d \to \mathbb{R}_+^*$ is a continuous map between topological groups.

We close this section with some remarks concerning inversions. In general, an inversion between two distinct points $r, r' \in S$ is a map $\phi_{r, r'} \in \mathcal{M}_d$ such that $\phi^2 = \text{id.}$ and $\phi(r) = r'$. Of course, an inversion may or may not exist in an arbitrary $\mathcal{M}_d$. But if it does, it satisfies the following properties listed in the next proposition; for simplicity, we only treat the case where $r = \infty$ and $r' = o$ is an arbitrary point in $S$.

Proposition 2.7. Let $\phi \in \mathcal{M}_d$ be an inversion between $o$ and $\infty$. Then there exists a $\beta > 0$ such that
\[
d(o, p) \cdot d(o, \phi(p)) = \beta^2,
\]
for each $p \in S$ other than $o, \infty$. Moreover,
\[
d(\phi(p), \phi(q)) = \frac{d(p, q)}{d(o, p) \cdot d(o, q)} \cdot \beta^2,
\]
for each $p, q \in S$ other than $o, \infty$.

Proof. We consider two arbitrary points $p, q \in S$ other than $o$ and $\infty$ and the quadruple $\mathbf{p} = (\infty, o, p, q)$. Since $\phi \in \mathcal{M}_d$ we have $|X^d_1|(\mathbf{p}) = |X^d_1|(\phi(\mathbf{p}))$, which yields
\[
\frac{d(o, q)}{d(o, p)} = \frac{d(\phi(p), o)}{d(\phi(q), o)}.
\]
Thus the quantity $d(o, p) \cdot d(0, \phi(p))$ is constant for each $p$. By setting $p = p_0$ for some arbitrary $p_0$ other that $o$ and $\infty$ and by letting $\beta$ be the positive square root of $d(o, p_0) \cdot d(0, \phi(p_0))$, we obtain the first relation of our proposition.

On the other hand, from $|X^d_2|((\phi(\mathbf{p}))) = |X^d_2|((\phi(p)))$, we have
\[
\frac{d(p, q)}{d(o, p)} = \frac{d(\phi(p), \phi(q))}{d(o, \phi(q))}.
\]
Therefore,
\[
d(\phi(p), \phi(q)) = d(o, \phi(q)) \cdot \frac{d(p, q)}{d(o, p)} = \frac{d(p, q)}{d(o, p) \cdot d(o, q)} \cdot \beta^2.
\]

Note finally that from Proposition [2.7] it follows that inversion $\phi$ leaves invariant the metric sphere
\[
S^d_\beta = \{ p \in S \mid d(p, o) = \beta \}.
\]
3. Möbius rigidity

In this section we prove our results. In Section 3.1 we prove a lemma which is the key as wellas
the main tool for the proof of Theorems 1.1, 1.2, 1.3 and 1.4 which are in Sections 3.2, 3.3, 3.4 and
respectively.

3.1. The Basic Preparatory Lemma. The following lemma is crucial for the proof of our results.

Lemma 3.1. We refer to the conditions stated in the introduction. Then conditions (Sim) and
(Cont) together imply the following.

1. If $D_\delta$ is a dilation, then $D_\delta \in \mathcal{M}_d$ implies $K(\delta) = \delta^\alpha$, where $\alpha \in (0,1]$.
2. The rotational group $F(n-1)$ is in $\text{Isom}_d(\partial H^n_d)$.
3. The group of left translations of $\mathcal{F}_d$ is in $\text{Isom}_d(\partial H^n_d)$.

Proof. To prove (1), we first show that $K$ cannot be a constant when restricted in the subgroup
of $\text{Sim}_d$ consisting of dilations. For if that was the case, then $K(\delta) \equiv 1$ for all $\delta > 0$ and $K$ would
fit the short exact sequence

$$1 \to \text{Isom}_d \to \text{Sim}_d \to 1 \to 1.$$  

Therefore, $\text{Sim}_d \simeq 1 \rtimes \text{Isom}_d$. But since from (Sim) we have $\text{Sim}_{d_\delta} \subseteq \text{Sim}_d$, that would read

$$\mathbb{R}_+^+ \times \text{Isom}_{d_\delta} \subseteq \{1\} \rtimes \text{Isom}_d,$$

which cannot happen.

Next, we will show that dilations $D_d \in \mathcal{M}_d(\mathcal{F}_d)$ satisfy the following relation: There exists an
$\alpha \in (0,1]$ such that for each $p,q$,

$$d(D_\delta(p), D_\delta(q)) = \delta^\alpha \cdot d(p,q).$$

Since for each $n \in \mathbb{N}$ we have $K(\delta^n) = (K(\delta))^n$ and $K(\delta^{-1}) = (K(\delta))^{-1}$, we obtain for every $q \in \mathbb{Q}$
that $K(\delta^q) = (K(\delta))^q$. It then follows from (Cont), that for every $\delta > 0$ and every $x \in \mathbb{R}$ we
have $K(\delta^x) = (K(\delta))^x$. Pick a positive $\delta_0 \neq 1$ and observe that $K(\delta_0)$ cannot be equal to 1, as this
would lead to $K(\delta) \equiv 1$ for all $\delta$. Indeed, if $\delta > 0$ is arbitrary, then there exists a $y \in \mathbb{R}$ such
that $\delta^y = \delta$. But then,

$$K(\delta) = K(\delta_0^y) = (K(\delta_0))^y = 1,$$

which is a contradiction. Set now $c = K(\delta_0)$; from $K(\delta_0^x) = c^x$ we get $K(z) = z^\alpha$ where $\alpha = \log c \delta_0$.
Hence $K(\delta) = \delta^\alpha$ for some non zero $\alpha$ as desired. Observe that up to this point we have proved
that $K(\delta) = \delta^\alpha$ for $\delta$ for some non zero real $\alpha$. Now $\alpha$ cannot be negative, for if that was the case
then for every $\delta > 0$ we would have $d(\alpha, (\delta, 0)) = \delta^\alpha \cdot d(\alpha, (1, 0))$, where we have written
$(1, 0, \ldots, 0) = (1, 0), (\delta, 0, \ldots, 0) = (\delta, 0)$. Since the function $f(\delta) = d(\alpha, (\delta, 0))$ is continuous (if
$(S, d)$ is a metric space then the mapping $d : S \times S \to \mathbb{R}^+$ is continuous), by letting $\delta$ tending to
$+\infty$ we have a contradiction. The proof that $\alpha$ is in $(0,1]$ lies after the proof of (3).

To prove (2) suppose that $F(n-1) \in \mathcal{M}_d(\mathcal{F}_d)$ and that the restriction of the group homomorphism $K$ to $F(n-1)$ is continuous. Our assertion follows directly from the compactness of the
group $F(n-1)$. Under our hypotheses, $K$ has to be constant and thus equal to 1.

Finally, to prove (3) suppose first that all translations $T_{(0,0)}$ are in $\mathcal{M}_d(\mathcal{F}_d)$ and consider the
restriction of the group homomorphism $K$ to $\mathbb{R}(\mathbb{K})$. When $\mathbb{K} = \mathbb{R}$ we have nothing to prove;
therefore we treat only the cases when $\mathbb{K} \neq \mathbb{R}$. For arbitrary $(0, v) \in \mathcal{S}(\mathbb{K})$ we have:

$$K(v) = \frac{d(T_{(0,v)}(0,-v), T_{(0,v)}(0,0))}{d((0,-v),(0,0))}$$

$$= \frac{d((0,v),(0,0))}{d((0,-v),(0,0))}.$$ 

Write $v = |v|\mu$, where $\mu$ is a unit element of $F(1)$. Then

$$K(v) = \frac{d((0,|v|\mu),(0,0))}{d((0,-|v|\mu),(0,0))}$$

$$= \frac{d\left(D_{|v|^{1/2}}(0,\mu), D_{|v|^{1/2}}(0,0)\right)}{d\left(D_{|v|^{1/2}}(0,-\mu), D_{|v|^{1/2}}(0,0)\right)}$$

$$= \frac{d((0,\mu),(0,0))}{d((0,-\mu),(0,0))}.$$ 

In case where $\mathbb{K} = \mathbb{C}$, $\mu = \pm 1$ and therefore

$$K(v) = \frac{d((0,1),(0,0))}{d((0,-1),(0,0))},$$

which is a constant that does not depend on $v$. Since $K$ is a group homomorphism, it has to be equal to 1. In case where $\mathbb{K} = \mathbb{H}$, $\mu = \mu_1 i + \mu_2 j + \mu_3 k$ with $|\mu| = 1$. Therefore,

$$K(v) = K(\mu) = K(\mu_1 i) \cdot K(\mu_2 j) \cdot K(\mu_3 k)$$

$$= K(\pm i) \cdot K(\pm j) \cdot K(\pm k),$$

which again is a constant that does not depend on $v$. Therefore is has to be equal to 1. An analogous argument holds in the case where $\mu \in \mathcal{S}(\mathbb{O})$.

We now consider translations of the form $T_{(\zeta,0)}$. To prove that they lie in $\text{Isom}_d$, we follow the same path as before:

$$K(\zeta) = \frac{d\left(T_{(\zeta,0)}(-\zeta,0), T_{(\zeta,0)}(0,0)\right)}{d\left((-\zeta,0),(0,0)\right)}$$

$$= \frac{d((\zeta,0),(0,0))}{d((-\zeta,0),(0,0))}.$$ 

Write $\zeta = ||\zeta||\nu$, where $\nu$ is a unit element of $F(n-1)$. Then

$$K(\zeta) = \frac{d\left(||\zeta||\nu,0\right),(0,0))}{d\left(||-\zeta||\nu,0\right),(0,0))}$$

$$= \frac{d(\nu,0),(0,0))}{d((-\nu,0),(0,0))}.$$
Again, from (1) this yields

\[ K(v) = \frac{d((1,0),(0,0))}{d((-1,0),(0,0))} = \frac{d((1,0),(0,0))}{d((1,0),(0,0))} = 1. \]

To finish the proof of (1), we observe that every translation \( T_p = T_{(\zeta,v)} \) may be written as \( T_{(\zeta,0)} \circ T_{(0,v)} \). Therefore

\[ K(p) = K(\zeta) \cdot K(v) = 1, \]

and this proves (3). We finally prove that \( K(\delta) = \delta^\alpha, \alpha \in (0,1] \). For this, observe that for every two arbitrary \( \delta_1, \delta_2 > 0 \) we have

\[ d(o,D_{\delta_1+\delta_2}(1,0)) = (\delta_1 + \delta_2)^\alpha \cdot d(o,(1,0)) \]

and also by invariance of translations and triangle inequality,

\[
\begin{align*}
  d(o,D_{\delta_1+\delta_2}(1,0)) &= d(o,(\delta_1 + \delta_2,0)) \\
  &= d((-\delta_1,0),(\delta_2,0)) \\
  &\leq \delta_1^\alpha \cdot d((-1,0),o) + \delta_2^\alpha \cdot d((1,0),o) \\
  &= \delta_1^\alpha \cdot d((1,0),o) + \delta_2^\alpha \cdot d((1,0),o) \\
  &= (\delta_1^\alpha + \delta_2^\alpha) \cdot d(o,(1,0)).
\end{align*}
\]

But also \( d(o,D_{\delta_1+\delta_2}(1,0)) = (\delta_1 + \delta_2)^\alpha \cdot d(o,(1,0)) \); therefore, \( (\delta_1 + \delta_2)^\alpha \leq \delta_1^\alpha + \delta_2^\alpha \). By putting \( \delta_1 = \delta_2 = \delta \) we have \( 2^\alpha \cdot \delta^\alpha \leq 2 \cdot \delta^\alpha \), i.e., \( 2^{1-\alpha} \geq 1 \) which can happen only if \( \alpha \in (0,1] \). The proof is hereby concluded.

**Remark 3.2.** It is clear from the proof that the following equations hold:

\[
\begin{align*}
  d((\pm 1,0,\ldots,0),o) &= d((0,\pm 1,\ldots,0),o) = \ldots d((0,\ldots,0,\pm 1),o), \\
  d(o,(\pm 1,0,\ldots,0)) &= d(o,(0,\pm 1,\ldots,0)) = \ldots d(o,(0,\ldots,0,\pm 1)).
\end{align*}
\]

The quantities in the first row shall be denoted by \( d((1,0),o) \) whereas the quantities in the second row shall be denoted by \( d((0,1),o) \).
3.2. Proof of Theorem [1.1]. For the case $\mathbb{K} = \mathbb{R}$ (case (1) of the theorem), let $\zeta, \zeta' \in \mathbb{R}^{n-1}$. We have

$$d (\zeta, \zeta') = d (\zeta - \zeta', 0) = d \left(\|\zeta - \zeta'\| \cdot \mu, 0\right), \quad \mu \in O(n - 1),$$

$$= d(0, 1) \cdot \|\zeta - \zeta'\|^\alpha = d(0, 1) \cdot d_S^\alpha(\zeta, \zeta').$$

Where, the penultimate equation follows from the transitive action of $SO(n - 1)$ on $S^{n-1}$: We may map any $r \in S^{n-1}$ to $1 = (1, 0, \ldots, 0)$ via an element of $SO(n - 1)$.

For $\mathbb{K} \neq \mathbb{R}$ (case (2) of the theorem), let

$$\beta_1 = \max \left\{ d \left( (1, 0, o), d \left( (0, 1, o) \right) \right) \right\}, \quad \beta_2 = \min \left\{ d \left( (1, 0, o), d \left( (0, 1, o) \right) \right) \right\}.$$

Since for each $p = (\zeta, v) \in \mathcal{F}_\mathbb{K}$,

$$d ((\zeta, 0), o) = d ((1, 0), o) \cdot \|\zeta\|^\alpha,$$

$$d ((0, v), o) = d ((0, 1), o) \cdot |v|^\alpha/2,$$

from [3.1] and [3.2] we obtain the inequalities:

$$\beta_2 \cdot \|\zeta\|^\alpha \leq d ((\zeta, 0), o) \leq \beta_1 \cdot \|\zeta\|^\alpha,$$

$$\beta_2 \cdot |v|^\alpha/2 \leq d ((0, v), o) \leq \beta_1 \cdot |v|^\alpha/2.$$

Therefore,

$$\beta_2 \cdot d_S^\alpha(o, p) \leq \left( d^{4/\alpha} ((\zeta, 0), o) + d^{4/\alpha} ((0, v), o) \right)^{\alpha/4} \leq \beta_1 \cdot d_S^\alpha(o, p).$$

By triangle inequality and $d$–invariance of translations we have

$$d(p, o) \leq d ((\zeta, 0), o) + d ((0, v), o).$$

We apply Hölder’s inequality with exponent $4/\alpha$ to obtain

$$d(p, o) \leq 2^{(4-\alpha)/4} \cdot \left( d^{4/\alpha} ((\zeta, 0), o) + d^{4/\alpha} ((0, v), o) \right)^{\alpha/4} \leq \beta_1 \cdot 2^{(4-\alpha)/4} \cdot \left( d_\Sigma^\alpha((\zeta, 0), o) + d_\Sigma^\alpha((0, v), o) \right)^{\alpha/4} = \beta_1 \cdot 2^{(4-\alpha)/4} \cdot d_\Sigma^\alpha(p, o).$$

Thus Theorem [1.1] follows.

We wish to address at one important issue at this point. As we have underlined in the introduction, condition $(\alpha$-Hö)l is rather inadequate to describe in full the nature of metrics which satisfy $(\text{Sim})$ and $(\text{Cont})$; there might exist metrics which satisfy $(\alpha$-Hö)l on the one hand but on the other, their nature might be entirely different from that of $d_\Sigma$. We wish to give a concrete example to illustrate this matter and perhaps the most illustrative one is that of the Carnot–Carathéodory metric $d_{cc}$ (for details about $d_{cc}$, see for instance [4]); it suffices only to consider the case $\mathbb{K} = \mathbb{C}$, $n = 2$. Certainly, since $\text{Sim}_{d_{cc}} = \text{Sim}_{d_\Sigma}$, $d_{cc}$ automatically satisfies conditions $(\text{Sim})$ and $(\text{Cont})$ of Theorem [1.1] (but does not satisfy neither $(\text{Inv})$ nor $(\text{G})$). Since $\alpha = 1 = \beta_1$, we have from Theorem [1.1] that for each $p$,

$$d_{cc}(p, o) \leq 2^{3/4} \cdot d_\Sigma(p, 0).$$

The reader is invited to compare this inequality to the well known estimate

$$\pi^{-1/2} \cdot d_\Sigma(p, o) \leq d_{cc}(p, o) \leq d_\Sigma(p, o).$$

We conclude that the Hölder exponent $2^{(4-\alpha)/4}$ deduced from Theorem [1.1] is not optimal.
3.3. Proof of Theorem [1.2]. We need two lemmas for the proof of Theorem 3.3. The first one is an immediate corollary of Proposition 2.7.

Lemma 3.3. If inversion $I$ is in $M_d(H_\mathbb{K})$, then

$$d(o, p) \cdot d(o, I(p)) = d(o, p_0) \cdot d(o, I(p_0)),$$

where $p_0$ is any point other than $o, \infty$ and for each $p \in H_\mathbb{K}$ other than $o, \infty$. Moreover

$$d(I(p), I(q)) = \frac{d(p, q)}{d(o, p) \cdot d(o, q) \cdot d(o, p_0) \cdot d(o, I(p_0))},$$

for each $p, q \in H_\mathbb{K}$ other than $o, \infty$.

The second lemma is about conjugation condition (Conj):

Lemma 3.4. Condition (Conj) implies that $\mathbb{K}$-conjugation $J$ is in $\text{Isom}_d(\partial H_\mathbb{K}^n)$.

Proof. According to Proposition 2.3, $J$ is in $\text{Sim}_d$ and we have $K(J) = 1$. □

We proceed now to the proof of Theorem 1.2. By choosing $p_0 = (1, 0) = 1$ the above formulae also read as:

$$d(o, p) \cdot d(o, I(p)) = d^2(o, 1), \quad d(I(p), I(q)) = \frac{d(p, q)}{d(o, p) \cdot d(o, q) \cdot d^2(o, 1)}.$$

Let now $p = (\zeta, v) \in H_\mathbb{K}$. Then by setting $A(p) = -||\zeta||^2 + v$ we have

$$d(I(p), o) = d\left(\left(\zeta \cdot \frac{A(p)\bar{\tau}}{|A(p)|^2}, \frac{\tau}{|A(p)|^2}\right), o\right)$$

by Lemma 3.4

$$= d\left(D_{|A(p)|^{-1}}\left(\zeta \cdot \frac{A(p)}{|A(p)|}, \bar{\zeta}, v\right), o\right)$$

by Lemma 3.4

$$= |A(p)|^{-\alpha} \cdot d\left((\zeta, v), o\right)$$

This gives

$$d(I(p), o) = \frac{d^2(o, 1)}{d(o, p)} = |A(p)|^{-\alpha} \cdot d(o, p),$$

and by Lemma 3.3 we conclude

$$d^2(o, p) = d^2(o, 1) \cdot |A(p)|^{\alpha} = d^2(o, 1) \cdot d^2_{\beta}(o, p),$$

which proves Theorem 1.2 (ii). □

Note that implicit in the proof lies the following formula for the inversion $I$: For each $p$,

$$I(p) = \left(D_{|A(p)|^{-1}} \circ J \circ R_{\frac{A(p)}{|A(p)|}} \circ J \right)(p^{-1}).$$
This formula only dictates the manner that inversion \( I \) is defined; it cannot be regarded as formula which expresses \( I \) as a compound of similarities, due to the presence of \( p^{-1} \).

3.4. **Proof of Theorem 1.3.** Suppose now that (G) holds, that is,

\[
d(o, p) = \left( d^{4/\alpha} ((\zeta, 0), o) + d^{4/\alpha} ((0, v), o) \right)^{\alpha/4}.
\]

Then \( \beta_2 \cdot d_{\alpha}^\circ (o, p) \leq d(o, p) \leq \beta_1 \cdot d_{\alpha}^\circ (o, p) \). From \( d^- \) and \( d_{\alpha}^- \)-invariance of translations we have:

\[
\beta_2 \cdot d_{\alpha}^\circ (p, q) \leq d(p, q) \leq \beta_1 \cdot d_{\alpha}^\circ (p, q),
\]

therefore \( d \) and \( d_{\alpha}^\circ \) are metrically equivalent. But there exists also a \( \beta \geq 1 \) depending on \( \beta_1, \beta_2 \) such that

\[
\frac{1}{\beta} \cdot d_{\alpha}^\circ (p, q) \leq d(p, q) \leq \beta \cdot d_{\alpha}^\circ (p, q).
\]

If (Eq) holds, then \( \beta_1 = \beta_2 = \beta \) and \( d = \beta \cdot d_{\alpha}^\circ \). In this manner we have proved Theorem 1.3. \( \square \)

We remark here that condition (biLip) is equivalent to that the identity map from \( (\partial \mathcal{H}_K, d) \) to \( (\partial \mathcal{H}_K, d) \) is bi-Lipschtiz.

Finally, we prove:

**Proposition 3.5.** Suppose that condition (Sim) holds. Then conditions (G) is equivalent to the following:

**\( \text{(P-L)} \)** For \( \alpha \in (0, 1] \) and for each \( p, q \in \mathcal{S} \),

\[
|p \ast q|^{4/\alpha} + |p^{-1} \ast q|^{4/\alpha} + |p \ast q^{-1}|^{4/\alpha} + |p^{-1} \ast q^{-1}|^{4/\alpha} = 2 \left( |\Pi_{K^{-1}}(p \ast q)|^{4/\alpha} + |\Pi_{K^{-1}}(p^{-1} \ast q)|^{4/\alpha} \right) + |\Pi_{\mathcal{S}}(p \ast q)|^{4/\alpha} + |\Pi_{\mathcal{S}}(p^{-1} \ast q)|^{4/\alpha} + |\Pi_{\mathcal{S}}(p \ast q^{-1})|^{4/\alpha} + |\Pi_{\mathcal{S}}(p^{-1} \ast q^{-1})|^{4/\alpha},
\]

where \( \Pi_{K^{-1}} \) and \( \Pi_{\mathcal{S}} \) are projections of \( \mathcal{S} \) to \( K^{-1} \) and \( \mathcal{S} \), respectively.

**Proof.** To show that (P-L) implies (G), just set \( q = o \) and use the fact that \( |p^{-1}| = |p| \), due to \( d^- \)-invariance of translations which follows from (Sim) (cf. Lemma 3.1):

\[
d(o, p^{-1}) = d(T_p(o), T_p(p^{-1})) = d(o, p).
\]

To show that (G) implies (P-L), just apply (G) for the points \( p \ast q, p^{-1} \ast q, p \ast q^{-1} \) and \( p^{-1} \ast q^{-1} \). \( \square \)

3.5. **Proof of Theorem 1.4.** Assuming condition (\( \alpha \)-Met) holds, we have for each \( p, q, r, s \) points in \( \partial \mathcal{H}_K \) that

\[
d_{\alpha}^\circ (p, r) \cdot d_{\alpha}^\circ (q, s) \leq (d_{\alpha} (p, q) \cdot d_{\alpha} (r, s) + d_{\alpha} (p, s) \cdot d_{\alpha} (r, q))^\alpha
\]

\[
\leq d_{\alpha}^\circ (p, q) \cdot d_{\alpha}^\circ (r, s) + d_{\alpha}^\circ (p, s) \cdot d_{\alpha}^\circ (r, q),
\]

where the first inequality follows from the Ptolemaean property of \( d_{\alpha} \) and the second inequality holds because \( \alpha \leq 1 \). Thus \( d \) satisfies (Ptol).

Assume now that (Circ) holds and pick any \( p, q, r, s \) lying in the Ptolemaean circle \( \sigma \) and suppose with no loss of generality that \( p \) and \( s \) separate \( q \) and \( r \). We may also normalise so that \( q = o \) and \( r = \infty \). Then

\[
d_{\alpha}^\circ (s, o) + d_{\alpha}^\circ (p, 0) = d_{\alpha}^\circ (p, s).
\]

Therefore,

\[
d_{\alpha} (p, s) = (d_{\alpha}^\circ (s, o) + d_{\alpha}^\circ (p, 0))^{1/\alpha} \geq d_{\alpha} (s, o) + d_{\alpha} (p, 0),
\]
since $1/\alpha \geq 1$. But this contradicts triangle inequality unless $\alpha = 1$. Moreover, we have that then
\( \sigma \) is an \( \mathbb{R} \)-circle and the proof is complete. \( \square \)

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