Gauge fields of the matrix Dirac equation *

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Abstract

We introduce an equation named matrix Dirac equation which can be considered as a generalization of Dirac equation for an electron. The liaison between matrix Dirac equation and standard Dirac equation is discussed. We write a lagrangian from which matrix Dirac equation can be derived. This lagrangian is invariant under global unitary transformations of variables. The requirement of a local (gauge) invariance of lagrangian leads us to lagrangian with gauge fields.

Introduction.

After the famous Dirac equation for an electron was found in 1928 [1], works of H.Weil [2], V.A.Fock [3] and others appeared, in which electromagnetic field was described as gauge field of the Dirac equation appearing from the demand of local gauge invariance with respect to phase transformation (an abelian U(1) gauge group ) of Dirac’s lagrangian. Further development of this approach for the non-Abelian gauge fields was made in the work of Yang and Mills (1954) [4] considering the group of isotopic transformations (gauge group SU(2)). Their work was soon generalized for the wide class of Lie groups. Non-Abelian gauge fields began to be named Yang-Mills fields, whereas the equations describing them were named Yang-Mills equations. In modern physics such fields are used in models of electroweak and strong interactions.

In the present article an equation named matrix Dirac equation is introduced. This equation can be considered as a generalization of Dirac equation for an electron. Certain features of matrix Dirac equation are investigated (currents, canonical forms). Further on the basis of local gauge invariance regarding unitary group a system of equations is introduced consisting of matrix Dirac equation and equations of Yang-Mills or Maxwell. This system of equations describes Dirac’s field interacting with the gauge field of Yang-Mills or Maxwell.

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Some ideas proposed in \cite{5, 6, 16} are used in the article. Certain elements of the construction proposed could be found in the works of Hestenes \cite{7}, Kähler \cite{8}, Pestov \cite{9}, Pezzaglia and Differ \cite{10}.

1 A standard Dirac equation.

Let us consider a Dirac equation and its generalizations. A vector $x = (x^0, x^1, x^2, x^3) \in \mathbb{R}^4$ defines a point in space-time, $x^0$ – time coordinate, $x^1, x^2, x^3$ – space coordinates, $\partial_{\mu} = \partial/\partial x^\mu$, $\mu = 0, 1, 2, 3$ are the partial derivatives and $\Box = \partial_\mu \partial^\mu = \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2$. A Klein-Gordon equation for a function $\phi = \phi(x)$

$$\Box \phi = 0,$$

where $m$ – nonnegative real number, describes particles with spin 0 and mass $m$. For a description of spin 1/2 particles (electrons) P. A. M. Dirac suggested a system of equations of first order that he received by factorizing the Klein-Gordon operator

$$(i\gamma^\mu \partial_\mu + m)(i\gamma^\nu \partial_\nu - m) = -(\Box + m^2),$$

where $\gamma^\mu$, $\mu = 0, 1, 2, 3$ are algebraic objects that satisfy the relations

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3$$

with Minkowski tensor $g = (g^{\mu\nu}) = \text{diag}(1, -1, -1, -1)$. $\gamma^\mu$ can be represented by the matrices of order no less than four. We shall use the following representation for $\gamma^\mu$:

$$\gamma^0 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & -\sigma^k \\ \sigma^k & 0 \end{pmatrix}, \quad k = 1, 2, 3$$

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which is called the Dirac representation ($\sigma^k$ – Pauli matrices). Let $1$ denote the identity $4 \times 4$-matrix and $\gamma^{\mu\nu} = \gamma^\mu \gamma^\nu$ for $0 \leq \mu < \nu \leq 3$, $\gamma^{\mu\nu\lambda} = \gamma^\mu \gamma^\nu \gamma^\lambda$ for $0 \leq \mu < \nu < \lambda \leq 3$, $\gamma^5 = \gamma^{0123} = \gamma^0 \gamma^1 \gamma^2 \gamma^3$. The 16 matrices

$$1, \quad \gamma^\mu, \quad \gamma^{\mu\nu}, \quad \gamma^{\mu\nu\lambda}, \quad \gamma^5$$

are linear independent and form a basis of $\mathcal{M}(4, \mathbb{C})$ – algebra of four dimensional complex matrices.

The Dirac equation has a form

$$(i\gamma^\mu \partial_\mu - m1)\psi = 0,$$
and if we consider the aggregate in brackets as a \(4 \times 4\)-matrix, then \(\psi = \psi(x)\) must be a matrix with four lines and an arbitrary number of columns. The identity (2) asserts that all the components of the matrix \(\psi\) (if they are smooth enough) satisfy the Klein-Gordon equation.

We shall call the equation (1) with one column matrix \(\psi\) the standard Dirac equation. In that case \(\psi\) is called bispinor or Dirac spinor. One can also consider the equations (1) with \(l > 1\) columns in the matrix \(\psi\). For the different purposes physicists use the Dirac equation (1) with the different numbers of columns in \(\psi\). The following list does not lay claim for completeness or indisputability:

\(l = 1\): Quantum electrodynamics (a gauge field theory with \(U(1)\) symmetry group).

\(l = 2\): Theory of electroweak interactions (\(SU(2)\) gauge field theory).

\(l = 3\): Theory of strong interactions - quantum chromodynamics (\(SU(3)\) gauge field theory).

\(l = 4\): The Dirac equation (1) with \(l = 4\) is called a Rarita-Schwinger equation. It is used for a description of spin \(3/2\) particles.

\(l = 5\): Georgi and Glashow \cite{11} have suggested \(SU(5)\) gauge field theory as Grand Unified Theory (GUT).

\(l \geq 6\): \(SU(l)\) gauge field theories are developed by physicists as GUT.

Gelfand, Minlos and Shapiro \cite{12} considered all relativistic invariant systems of equations of the form

\[
L^\mu \partial_\mu \psi + i\kappa \psi = 0,
\]

where \(L^\mu\) are square matrices, \(\psi\) - vector, \(\kappa\) - real constant.

There is an evident generalization of the identity (2)

\[
(i\gamma^\mu \partial_\mu + m(z1 - y\gamma^5))(i\gamma^\nu \partial_\nu - m(z1 + y\gamma^5)) = -(\Box + m^2),
\]

where \(z, y\) are complex constants and \(z^2 + y^2 = 1\). It leads to the equation

\[
(i\gamma^\mu \partial_\mu - m(z1 + y\gamma^5))\psi = 0,
\]

We can consider the factorizations (3), (8) of the Klein-Gordon operator as one of possible methods of a reduction of the Klein-Gordon equation of second order (1) to a system of equations of first order. There is another method of such a reduction that leads to the following system of equations of first order:

\[
i\gamma^\mu \partial_\mu \psi - m(\psi N + \gamma^5 \psi K) = 0,
\]

which depends on two matrices \(N, K \in \mathcal{M}(l, \mathbb{C})\). We shall call the equation (10) the matrix Dirac equation, emphasizing that unknown (wave) function
ψ = ψ(x) in (10) is a 4×l-matrix. If matrix ψ has only one column, then we get an equation (9), or standard Dirac equation (when N = 1, K = 0). The equation (10), generally speaking, can’t be reduced to the equation of the form (7).

**Theorem 1** If a 4×l-matrix ψ = ψ(x) with twice continuously differentiable elements is a solution of (10), where the matrices N, K ∈ M(l, C) do not depend on x and satisfy the relations

\[ [N, K] = NK - KN = 0, \quad N^2 + K^2 = 1_l, \]  

where 1_l is identity l×l-matrix, then the matrix ψ is also a solution of the Klein-Gordon equation.

**Proof.** Let us consider an action of the operator \( iγ^μ∂_μ \) from left to the equation (10)

\[ (iγ^μ∂_μ)^2ψ - m(iγ^μ∂_μψ)N + mγ^5(iγ^μ∂_μψ)K = 0, \]

and use the relations (11) and

\( (iγ^μ∂_μ)^2 = −□, \quad iγ^μ∂_μψ = m(ψN + γ^5ψK). \)

As a result we get the Klein-Gordon equation \( □ + m^2ψ = 0 \). Theorem is proved.

The formula (10) gives us a set of equations that depend on two matrices N, K with the relations (11). How to describe all matrix pairs N, K that satisfy (11)? If we write the matrix N using a Jordan normal form, then we can find all the corresponding matrices K by doing an elementary calculation [13]. For example, when l = 4 we get 15 classes of pairs N, K that depend on several parameters.

**Example 1.** The matrices

\[ N = V \text{diag}(z_1, \ldots, z_l)V^{-1}, \quad K = V \text{diag}(y_1, \ldots, y_l)V^{-1}, \]

where \( z_k^2 + y_k^2 = 1 \), V− nondegenerate matrix from \( \mathcal{M}(l, C) \), satisfy (11).

**Example 2.** Let l = 4. The matrices

\[ N = V \begin{pmatrix} z & 1 & 0 & 0 \\ 0 & z & 1 & 0 \\ 0 & 0 & z & 1 \\ 0 & 0 & 0 & z \end{pmatrix} V^{-1}, \quad K = V \begin{pmatrix} y & a & b & c \\ 0 & y & a & b \\ 0 & 0 & y & a \\ 0 & 0 & 0 & y \end{pmatrix} V^{-1}, \]

where \( z^2 + y^2 = 1, \ y \neq 0, \ a = -z/y, \ b = -1/(2y^3), \ c = -z/(2y^5) \) and V− nondegenerate matrix from \( \mathcal{M}(4, C) \), satisfy (11).
Let us consider a solution $\psi$ of (10) with the matrices $N, K$ from the example 1 and connect it with the solutions of the standard Dirac equation. The columns of the matrix $V$ denote by $v_k \ (k = 1, \ldots, l)$. Then $v_k$ are the eigenvectors of $N$ corresponding to the eigenvalues $z_k$. Simultaneously, $v_k$ are the eigenvectors of $K$ corresponding to the eigenvalues $t_k$. So, if we multiply (10) from right by $v_k$ and denote $\psi_k = \psi v_k$ then we come to $l$ equations

$$i \gamma^\mu \partial_\mu \psi_k - m(z_k \mathbf{1} + y_k \gamma^5)\psi_k = 0, \quad k = 1, \ldots, l$$

that have the form (9), or (6).

2 A lagrangian of the matrix Dirac equation.

The standard Dirac equation (1) can be derived from Dirac’s lagrangian

$$\frac{i}{2}(\bar{\psi} \gamma^\mu \partial_\mu \psi - \partial_\mu \bar{\psi} \gamma^\mu \psi) - m \bar{\psi} \psi$$

with the aid of variational principle [14]. If $N, K$ are hermitian matrices from $\mathcal{M}(l, \mathbb{C})$, then the matrix Dirac equation (10) can be derived with the aid of variational principle from the lagrangian

$$\frac{1}{2} \text{tr} \left( i(\bar{\psi} \gamma^\mu \partial_\mu \psi - \partial_\mu \bar{\psi} \gamma^\mu \psi) - m(\bar{\psi} \psi N + N \bar{\psi} \psi + \bar{\psi} \gamma^5 \psi K + K \bar{\psi} \gamma^5 \psi) \right),$$

in which $\text{tr}$ is a trace of matrix and $\bar{\psi} = \psi^\dagger \gamma^0$.

Let us introduce a commutator algebra of matrices $N, K$:

$$\text{com}(N, K) = \{ V \in \mathcal{M}(l, \mathbb{C}) : [V, N] = [V, K] = 0 \}$$

and a group

$$\mathcal{G}(l, N, K) = \text{com}(N, K) \cap \mathcal{U}(l),$$

where $\mathcal{U}(l)$ is a group of unitary matrices from $\mathcal{M}(l, \mathbb{C})$. The group $\mathcal{G}(l, N, K)$ is a compact Lie group. A set of all antihermitian matrices ($V^\dagger = -V$) commuting with $N$ and $K$

$$\mathcal{L}(l, N, K) = \text{com}(N, K) \cap \mathfrak{u}(l).$$

can be considered as a real Lie algebra of the Lie group $\mathcal{G}(l, N, K)$.

The lagrangian (14) is invariant under global (not dependent on $x$) transformations

$$\psi \rightarrow V \psi, \quad \bar{\psi} \rightarrow V^{-1} \bar{\psi}, \quad \text{with } V \in \mathcal{G}(l, N, K),$$

This fact is evident from the identity $\text{tr}(V^{-1}BV) = \text{tr}B$.

In order to get a lagrangian which is invariant under a local (gauge) transformations (15), where $V = V(x)$ is a function of $x$ with values in $\mathcal{G}(l, N, K)$, we must replace in (14) partial derivatives $\partial_\mu \psi, \partial_\mu \bar{\psi}$ by the respective covariant derivatives $D_\mu \psi = \partial_\mu \psi - \frac{i}{2} \gamma_\mu \psi$. 

5
\[
\partial_\mu \psi - \psi a_\mu, \quad \bar{D}_\mu \bar{\psi} = \partial_\mu \bar{\psi} + a_\mu \bar{\psi},
\]
that depend on functions \(a_\mu = a_\mu(x)\) with its values in the Lie algebra \(L(l, N, K)\). A transformation rule for \(a_\mu\) has a form
\[
a_\mu \rightarrow V^{-1} a_\mu V + V^{-1} \partial_\mu V, \quad V \in G(l, N, K).
\]

There is a complete gauge invariant lagrangian, which describes a field \(\psi\) interacting with the gauge field \(a\).

\[
L = \frac{1}{2} \text{tr} \left( i \bar{\psi} \gamma^\mu (\partial_\mu \psi - \psi a_\mu) - (\partial_\mu \bar{\psi} + a_\mu \bar{\psi}) \gamma^\mu \psi \right) - m \bar{\psi} \psi N + N \bar{\psi} \psi + \bar{\psi} \gamma^5 \psi K + K \bar{\psi} \gamma^5 \psi \right) + \frac{1}{4} \text{tr} (f_{\mu \nu} f^{\mu \nu}),
\]

where \(f_{\mu \nu} = \partial_\mu a_\nu - \partial_\nu a_\mu - [a_\mu, a_\nu]\). If \(N = N^\dagger, \ K = K^\dagger\), then the lagrangian \(L\) leads to the system of equations
\[
\begin{align*}
\partial_\mu \psi - \psi a_\mu - m \bar{\psi} \psi N &= 0, \\
\partial_\mu a_\nu - \partial_\nu a_\mu - [a_\mu, a_\nu] - f_{\mu \nu} &= 0, \\
\partial_\mu f^{\mu \nu} - [f^{\mu \nu}, a_\mu] &= -\frac{\ell}{\pi} (i \bar{\psi} \gamma^\nu \psi),
\end{align*}
\]

\(\ell, u(l) \rightarrow \mathcal{L}(l, N, K)\) is a projector operator to the Lie algebra \(\mathcal{L} = \mathcal{L}(l, N, K)\) which can be considered as a linear subspace of the vector space \(u(l)\) of antihermitean matrices. The second and the third equations in (18) are called Yang-Mills equations.

**Definition.** If \(V = V(x)\) is a function of \(x\) with the value in a Lie group \(G \subseteq G(l, N, K)\), then the transformation
\[
\begin{align*}
\psi & \rightarrow \psi' = \psi V, \\
a_\mu & \rightarrow a'_\mu = V^{-1} a_\mu V + V^{-1} \partial_\mu V, \\
f_{\mu \nu} & \rightarrow f'_{\mu \nu} = V^{-1} f_{\mu \nu} V
\end{align*}
\]
is called a gauge transformation of fields \(\psi, a_\mu, f_{\mu \nu}\) with the group \(G\) (note, that the third relation from (19) is a consequence of the second).

The lagrangian (17) and the system of equations (18) are invariant under a gauge transformation (19). That means \(L(\psi, a_\mu) = L(\psi', a'_\mu)\), and if \(\psi, a_\mu, f_{\mu \nu}\) satisfy (18), then \(\psi', a'_\mu, f'_{\mu \nu}\) from (19) also satisfy (18).

**Theorem 2** Let \(N, K \in \mathcal{M}(l, \mathcal{C})\) be such, that
\[
\begin{align*}
N &= \alpha_1 \mathbf{1}_l + \beta_1 P_1, \quad P_1 P_1^\dagger = \mathbf{1}_l, \quad P_1^\dagger = P_1, \quad \alpha_1, \beta_1 \in \mathcal{R} \\
K &= \alpha_2 \mathbf{1}_l + \beta_2 P_2, \quad P_2 P_2^\dagger = \mathbf{1}_l, \quad P_2^\dagger = P_2, \quad \alpha_2, \beta_2 \in \mathcal{R},
\end{align*}
\]
and \(\psi = \psi(x)\) is a solution of the matrix Dirac equation (10). Let us denote \(J^\nu = \frac{\ell}{\pi} (i \bar{\psi} \gamma^\nu \psi)\), where \(\mathcal{L} = \mathcal{L}(l, N, K) = \text{com}(N, K) \cap u(l)\). Then
\[
\partial_\nu J^\nu (\psi) = 0.
\]
In other words, if we pose additional conditions (20) on $N, K$, then a right hand part of Yang-Mills equations is a current of matrix Dirac equation.

**Proof.** A solution $\psi$ of the matrix Dirac equation (10) also satisfies an identity

$$\partial_\mu (i\bar{\psi}\gamma^\mu \psi) - m(\bar{\psi}\psi N - N^\dagger \bar{\psi}\psi + \bar{\psi}\gamma^5 \psi K - K^\dagger \bar{\psi}\gamma^5 \psi) = 0. \quad (22)$$

which is consequence of (10). To show this we must multiply (10) from left on $\bar{\psi}$ and subtract hermitian conjugated equation

$$-i\partial_\mu \psi^\dagger (\gamma^\mu)^\dagger - m(N^\dagger \psi^\dagger - K^\dagger \psi^\dagger \gamma^5) = 0,$$

multiplied from right on $\gamma^0 \psi$. The result can be written in the form (22).

Let us prove that if $N, K$ satisfy (20), then

$$\frac{c}{\pi} (\bar{\psi}\psi N - N^\dagger \bar{\psi}\psi + \bar{\psi}\gamma^5 \psi K - K^\dagger \bar{\psi}\gamma^5 \psi) = 0. \quad (23)$$

It is easy to check, that the matrices

$$B_1 = \bar{\psi}\psi N - N^\dagger \bar{\psi}\psi, \quad B_2 = \bar{\psi}\gamma^5 \psi K - K^\dagger \bar{\psi}\gamma^5 \psi,$$

anticommute with the matrices $P_1, P_2$ respectively:

$$B_1 P_1 = -P_1 B_1, \quad B_2 P_2 = -P_2 B_2.$$

We can rewrite this fact as

$$B_1 \in \mathcal{M}(l, \mathbb{C}) \setminus \text{com}(P_1), \quad B_2 \in \mathcal{M}(l, \mathbb{C}) \setminus \text{com}(P_2), \quad (24)$$

where $\text{com}(P) \subseteq \mathcal{M}(l, \mathbb{C})$ is a subspace of all matrices commuting with the matrix $P \in \mathcal{M}(l, \mathbb{C})$. It follows from (24), that

$$B_1 + B_2 \in \mathcal{M}(l, \mathbb{C}) \setminus (\text{com}(P_1) \cap \text{com}(P_2)). \quad (25)$$

Taking into account the definition of the Lie algebra $\mathcal{L}$, we get from (25) the relation $\frac{c}{\pi} (B_1 + B_2) = 0$. Acting by the projector operator $\frac{c}{\pi}$ on the right hand and left hand parts of the identity (22), and using the formula (23), we get

$$\frac{c}{\pi} (\partial_\mu (i\bar{\psi}\gamma^\mu \psi)) = \partial_\mu J^\mu = 0.$$

Theorem is proved.

### 3 A general form of the matrices $N, K$.

The matrix Dirac equation (10) depends on two matrices $N, K$. Let us compile all conditions on the matrices $N, K$. 


If \( N, K \) satisfy (11), then the solution \( \psi \) of equation (10) also satisfies Klein-Gordon equation.

If \( N, K \) are hermitian, then the matrix Dirac equation can be derived from the lagrangian (14) with the aid of variational principle.

If \( N, K \) satisfy (20), then the right hand part of Yang-Mills equations (18) is a current of matrix Dirac equation.

Evidently, if \( N, K \) satisfy (20), then \( N, K \) are hermitian matrices.

**Theorem 3** The matrices \( N, K \in \mathcal{M}(l, \mathbb{C}) \) satisfy (11) and (20) if and only if

\[
N = U^\dagger \text{diag}(\cos \xi, \ldots, \cos \xi, \cos \eta, \ldots, \cos \eta)U,
\]

\[
K = U^\dagger \text{diag}(\sin \xi, \ldots, \sin \xi, \sin \eta, \ldots, \sin \eta)U,
\]

or

\[
N = U^\dagger \text{diag}(\pm 1, \ldots, \pm 1)U \cos \xi,
\]

\[
K = U^\dagger \text{diag}(\pm 1, \ldots, \pm 1)U \sin \xi,
\]

where \( p + q = l \), \( U \)-unitary matrix and \( 0 \leq \xi, \eta < 2\pi \).

**Proof.** The matrix \( N \) in (20) depends on a hermitian and simultaneously unitary matrix \( P_1 \). So, \( P_1 \) can be written with the aid of a unitary matrix \( U \):

\[
P_1 = U^\dagger \text{diag}(\pm 1, \ldots, \pm 1)U.
\]

Hence

\[
N = U^\dagger \text{diag}(\lambda_1, \ldots, \lambda_l)U,
\]

where \( \lambda_k = \lambda^+ = \alpha_1 + \beta_1 \), or \( \lambda_k = \lambda^- = \alpha_1 - \beta_1 \). The same is true for the matrix \( K \):

\[
K = V^\dagger \text{diag}(\epsilon_1, \ldots, \epsilon_l)V,
\]

where \( V^\dagger V = 1 \), \( \epsilon_k = \epsilon^+ = \alpha_2 + \beta_2 \), or \( \epsilon_k = \epsilon^- = \alpha_2 - \beta_2 \). Matrices \( N, K \) commute with each other and can be simultaneously reduced to a diagonal form with the same similarity transformation (13). That means, that we can take \( U = V \). The condition \( N^2 + K^2 = 1 \) gives \( \lambda_k^2 + \epsilon_k^2 = 1 \), \( k = 1, \ldots, l \). There are two possibilities. The first: \( (\lambda^+)^2 \neq (\lambda^-)^2; (\epsilon^+)^2 \neq (\epsilon^-)^2 \) and so

\[
\lambda_k = \lambda^+ = \cos \xi, \quad \epsilon_k = \epsilon^+ = \sin \xi, \quad k = 1, \ldots, p,
\]

\[
\lambda_k = \lambda^- = \cos \eta, \quad \epsilon_k = \epsilon^- = \sin \eta, \quad k = p + 1, \ldots, l.
\]

The second: \( (\lambda^+)^2 = (\lambda^-)^2; (\epsilon^+)^2 = (\epsilon^-)^2 \) and so

\[
\lambda_k = \pm \cos \xi, \quad \epsilon_k = \pm \sin \xi, \quad k = 1, \ldots, l.
\]

Theorem is proved.
4 A polar gauge.

There is a theorem about the polar decomposition of matrix.

**Theorem 4** ([15]) An arbitrary matrix $M \in \mathcal{M}(l, \mathbb{C})$ can be written in a form $M = PU$, where $P \in \mathcal{M}(l, \mathbb{C})$ is a hermitian nonnegative defined matrix with the same rank as $M$, and a matrix $U \in \mathcal{M}(l, \mathbb{C})$ is unitary.

Let us consider a system of equations (18) where $l = 4$, $N = 1 \cos \xi$, $K = 1 \sin \xi$ with the gauge group $\mathcal{G} = \text{U}(4)$. And let $\psi, a_{\mu}, f_{\mu \nu}$ be a continuously differentiable functions of $x$ which satisfy (18) and such that $\psi \in \mathcal{M}(4, \mathbb{C})$ and $a_{\mu}, f_{\mu \nu} \in \mathcal{L} = \text{u}(4)$. Using the theorem 4, we get that in every point $x \in \mathbb{R}^4$ the matrix $\psi = \psi(x)$ can be written in a form $\psi = PU$, where $P \in \mathcal{M}(l, \mathbb{C})$ is a hermitian nonnegative defined matrix and $U \in \mathcal{M}(l, \mathbb{C})$ is a unitary matrix.

Let us suppose, that the matrix $\psi = \psi(x)$ can be written in a form $\psi(x) = P(x)U(x)$ in some region $\Omega \subset \mathbb{R}^4$ and $P(x), U(x)$ have continuously differentiable elements for all $x \in \Omega$. In that case the solution $\psi = \psi(x)$ of (18) in the region $\Omega$ defines a unitary matrix $U = U(x)$ with continuously differentiable elements. We can take $V = V(x) = U^{-1}$. The system of equations (18) is invariant under a gauge group $\text{U}(4)$, and after a gauge transformation with $V \in \text{U}(4)$, we come to the solution of (18)

$$\psi' = \psi V, \quad a'_{\mu} = V^{-1} a_{\mu} V + V^{-1} \partial_{\mu} V, \quad F' = V^{-1} F V,$$

where $\psi' = \psi'(x)$ is a hermitian nonnegative defined matrix and we name it solution of (18) in polar gauge.

In quantum mechanics particles are described by the wave functions which belong to some complex finite or infinite dimensional vector (hilbert) space. Observables are hermitian operators on that space. It will be naturally to suppose, that the solution $\psi$ of (18) is a wave function of fermion.

If we take $\psi$ in the polar gauge (in the case of gauge group $\text{U}(4)$), then $\psi$ is a hermitian $4 \times 4$-matrix equivalent to the hermitian $16 \times 16$-matrix $\Psi \otimes 1$. That means, that we can consider $\psi$ not only as a wave function, but simultaneously as some observable value.

Let us make two final remarks.

It can be shown, that the matrix Dirac equation (10) is invariant under the Lorentz transformations.

In case $l = 4$, the wave function $\psi$, which satisfy (10), is a $4 \times 4$-matrix and so, it can be represented as a linear combination of basis vectors $\vec{F}$ of Dirac’s algebra (Clifford’s algebra). Substituting it into (10), we get an equation written in Clifford’s algebra terms and not dependent on matrix representations. Such an equation can be naturally generalized to the curved space-time with an arbitrary pseudorimianian metric.

This questions will be considered in the next publications.
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