The radial gauge propagators in quantum gravity.*

P. Menotti, G. Modanese
Dipartimento di Fisica della Università, Pisa 56100, Italy and
INFN, Sezione di Pisa

D. Seminara
Scuola Normale Superiore, Pisa 56100, Italy and
INFN, Sezione di Pisa

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Abstract

We give a general procedure for extracting the propagators in gauge theories in presence of a sharp gauge fixing and we apply it to derive the propagators in quantum gravity in the radial gauge, both in the first and in the second order formalism in any space-time dimension. In the three dimensional case such propagators vanish except for singular collinear contributions, in agreement with the absence of propagating gravitons.

1 Introduction.

The radial gauge (or Poincaré gauge) was introduced by Fock and Schwinger \[1, 2\] in the context of quantum electrodynamics and often used in problems connected with QCD at the non perturbative level \[3\]. The main feature of such a gauge is the possibility of expressing correlation functions in terms of v.e.v. of physical fields like the field strength \(F_{\mu\nu}\). Recently an extension of the radial gauge to general relativity has been given by Modanese and Toller \[4\]. Here the meaning of the gauge is more profound than in the simpler case of gauge theories, because it involves also the reparametrization group. In fact the radial coordinates in which such a gauge is realized have a well defined physical meaning as geodesic or Riemann-Eisenhart coordinates that radiate from a given origin. The hamiltonian approach in Riemann normal coordinates has been developed by Nelson and Regge \[5\]. Here we shall look at the problem from the lagrangian point of view, i.e. starting from the functional integral. In the functional integral approach the correlation functions are computed by averaging the products of fields like \(O(x)\bar{O}(y)\) on all geometries weighted by the exponential of the gravitational action. Fixing the gauge to the radial gauge gives \(x, y, \ldots\) a well defined meaning as the points that acquire geodesic coordinates \(x, y, \ldots\) in each of the geometries we are summing over.

The problem of computing physical correlation functions has become recently acute when people started performing Monte Carlo simulations in general relativity models \[6\]. In the usually adopted method of the Regge calculus, one resorts to computing expectation values of products of fields defined on the sample geometry at a given geodesic distance \[7\]. This amounts in the continuum to fixing the gauge as the geodesic gauge. A similar approach developed from the Mandelstam path space formulation \[8\] has been recently given \[9, 10\]. The aim of the present paper is to compute the propagators in such a gauge. As we shall see such a problem is not at all trivial because a simple minded computation of the propagator gives meaningless, i.e. infinite results.

It is well known that gravity can be formulated in the second or first order approach and we shall derive the propagators in both formulations. Formally in the second order approach one starts from the functional integral

\[
\int e^{\int \sqrt{g}R(x)d^nx} \mu[g_{\mu\nu}] D[g_{\mu\nu}]
\]

where the fundamental variable is the metric tensor \(g_{\mu\nu}\) and \(\mu[g_{\mu\nu}]\) is the integration measure which we do not need to specify here \[11\]. We shall show in section 2 that the radial gauge condition
in the second order formalism can be written as \( x^\mu g_{\mu\nu} = x^\mu \eta_{\mu\nu} \). Thus the sharp gauge fixing will be given by \( \delta(x^\mu(g_{\mu\nu} - \eta_{\mu\nu})) \) which is to be accompanied by the relative Faddeev-Popov terms by adding to the lagrangian the expression

\[
\bar{c}^\nu(x) x^\mu (\nabla_\mu c_\nu + \nabla_\nu c_\mu)(x).
\]

Note that contrary to what happens in the usual gauge theories formulated in the radial gauge, the ghosts do not decouple. In the first order formalism on the other hand, one considers as fundamental variables the vierbeins \( \tau^a_\mu \) and the connections \( \Gamma^{ab}_\mu \). The functional integral becomes

\[
\int e^{-\frac{1}{2\kappa^2}} \int (d\Gamma + \Gamma \wedge \Gamma)_{ab} \wedge e_{c} \ldots \epsilon_{abc...} \mu \left[ e^a_{\mu} \right] \mathcal{D}[\Gamma^{ab}_\mu] \mathcal{D}[e^a_\mu].
\]

The natural way to introduce the radial gauge fixing without breaking the symmetry between the connections and the vierbeins in obtained by the \( N + N(N-1)/2 \) gauge conditions \( \delta(x^\mu \Gamma^{ab}_\mu) \) \( \delta(x^\mu (e^a_\mu - \delta^a_\mu)) \) and supplying the relative Faddeev-Popov terms. As it happens in Yang-Mills theory, the ghost associated to the local Lorentz symmetry formally decouple, while those related to the diffeomorphisms survive as in the second order case.

The technique for deriving the propagator in the radial gauge in a generalization of the gauge projection method developed in \([12]\) for the usual gauge theory case. We recall that given a generic field one can define several projections to the radial gauge that differ by the behavior of the projected field at the origin and at infinity. One has to keep in mind that all the problem is to give a solution to the radial Green’s equation, i.e. the equation arising when one performs in the gravitational field lagrangian in presence of sources, an arbitrary variation of the gravitational field subject to the radial gauge conditions; such a solution has to be symmetric in the field argument and radial.

In section 2 we give a general construction of the propagator in any “sharp gauge” of which the radial gauge is an example, starting from the propagator in a generic gauge (e.g. a gauge of the Feynman-Landau type). Care however has to be exerted in the projection process, because due to the singular nature of the correlation function at short distances, spurious infinite solutions of the homogeneous equation can develop. In fact the most natural propagator in the radial gauge would be the correlator

\[
\langle P^0[h]_{\mu\nu}(x) P^0[h]_{\rho\sigma}(y) \rangle
\]

that corresponds to the selection, in the projection procedure, of the field which is regular at the origin. In practice such a field is given by a proper average of the Feynman field along the geodesics joining the points \( x \) and \( y \) to the origin and due to the singular nature of the Riemann correlators at short distances, \( d^{-N-2} \), it makes the written correlation function divergent in all dimensions.

Similarly the radial projected field that is regular at infinity, \( P^\infty[h]_{\mu\nu} \), leads to a propagator divergent in all \( N \leq 4 \), for infrared problems. On the other hand it is shown in section 2 that one can consider a projected equation that treats the origin and the infinity in symmetrical way.
singular behavior of the field $g_{\mu\nu}$ at the origin and/or at infinity is unavoidable as can be seen by considering a special family of Wilson loops described in section 4. A formal solution to the radial $P^s$ projected Green’s function is given by

$$\langle P^s[h]_{\mu\nu}(x)P^s[h]_{\rho\sigma}(y) \rangle$$

(5)

where $P^s = (P^0 + P^\infty)/2$, which however still contains an infinite gauge term, solution of the homogeneous equation that has to be subtracted away. The result of such a subtraction procedure gives for the solution of the radial $P^s$ projected Green’s equation

$$\frac{1}{2}\langle P^0[h]_{\mu\nu}(x)P^\infty[h]_{\rho\sigma}(y) \rangle + \frac{1}{2}\langle P^\infty[h]_{\mu\nu}(x)P^0[h]_{\rho\sigma}(y) \rangle$$

(6)

which is symmetric in the field arguments and finite for all $N > 2$ and explicitly solves the problem of finding the propagator in the radial gauge. In addition it is immediately verified that the given propagator satisfies the radiality condition $x^\mu G_{\mu\nu,\rho\sigma}(x,y) = 0$. A similar procedure works successfully also in the first order formalism. From the technical viewpoint the transition from the propagator in a gauge of the Feynman-Landau type to the radial gauge propagator in obtained by expressing the radial fields in terms of the linearized Riemann and torsion tensors which are invariant under linearized gauge transformations; analytically the radial propagators are naturally expressed in terms of hypergeometric functions. In three dimensions the propagators acquire a particularly simple structure: in the first order approach the correlators of two connections vanish identically while the correlator of two dreibeins or of a dreibein and a connection reduce to collinear contributions. In the second order formalism the metric-metric correlation function is also zero except for collinear contributions. This is clearly related to the absence of propagating gravitons in three dimensions and to the physical nature of the considered gauge.

The paper is structured as follows: in section 2 we develop the general procedure for deriving the propagators in sharp gauges from the ones in a generic gauge. In section 3 we formulate the radial condition in the second order formalism and relate it to the usual definition of the Riemann-Eisenhart coordinates; special attention is paid to the allowed singularities of the metric at the origin and behavior at infinity. Then we write down the radial projectors in terms of the linearized Riemann and torsion tensors and finally we derive the propagator. In section 4 we repeat the same procedure in first order case, underlining the most important differences between the two approaches. Section 5 is devoted to the conclusions. In Appendix A we show how to express the propagators in terms of hypergeometric functions and work out explicitly the behavior of the propagators at the origin and infinity. Appendix B outlines the $\beta \neq 0$ (i.e. non sharp) radial gauge in the second order formalism.
2 Propagators.

In this section we shall develop a general procedure for the construction of the propagators in a class of “sharp” gauge conditions starting from the propagator given in a generic gauge. Subsection 2.A contains the general treatment; in subsections 2.B, 2.C, 2.D we display the concrete form of the various operators in the case of Electrodynamics and linearized Einstein theory in the second and first order formalism, respectively.

2.1 General formalism.

Let the equation of motion for a generic gauge field $A(x)$ be written in the form

$$K_x A(x) = J(x),$$

where $K$ is a linear, non-invertible, hermitean “kinetic” operator and $J(x)$ is an external source coupled to $A$. The gauge transformations of $A$ have the form

$$A(x) \rightarrow A(x) + C_x f(x).$$

Here $C_x$ is another linear operator, which has the properties

$$K_x C_x = 0 \quad \text{(gauge invariance)}$$

and

$$C_x^\dagger K_x = 0 \quad \text{which implies} \quad C_x^\dagger J(x) = 0 \quad \text{ (“source conservation”).}$$

We assume that we can always add to the kinetic operator $K$ an operator $K^F_x$ that makes $K$ invertible, as it happens, for example, in the Feynman gauge. We assume $K^F_x$ to be of the form $F^\dagger F_x$, as deriving from a quadratic gauge fixing $\int dx F^2(A(x))$. $F$ is meant to be a linear operator on $A$. The propagator $G^F_x$ corresponding to this gauge fixing satisfies

$$(K_x + K^F_x) G^F_x(x, y) = \delta(x - y)$$

and has the following property

$$\int dy K^F_x G^F_x(x, y) J(y) = 0 \quad \text{if} \quad C_y^\dagger J(y) = 0. $$

In other words, $K^F_x$ vanishes when applied to the fields generated by physical sources. In fact, applying $C^\dagger$ to (11) we get

$$C^\dagger_x K_x G^F_x(x, y) + C^\dagger_x F^\dagger x F_x G^F_x(x, y) = C^\dagger_x \delta(x - y).$$
But using (10) and integrating on a conserved source \( J \) we obtain

\[
\int dy \, C_x^\dagger \mathcal{F}_x \mathcal{F}_y \mathcal{F}(x, y) J(y) = 0. \tag{14}
\]

We notice that \( \mathcal{F} C \) is the kinetic ghost operator and as such invertible. Then from (14) we get

\[
\int dy \, \mathcal{F}_x \mathcal{F}(x, y) J(y) = 0 \tag{15}
\]

and multiplying by \( \mathcal{F}_x^\dagger \) we finally prove (12).

Let us now impose on \( A \) a generic "sharp" gauge condition \( \mathcal{G} \)

\[
A^\mathcal{G} = \{ A(x) : \mathcal{G}(A(x)) = 0 \}. \tag{16}
\]

\( \mathcal{G} \) is meant to be a linear function of \( A \) and of its derivatives. The field \( A^\mathcal{G}(x) \) can be obtained from a generic field \( A(x) \) through a (generally non-local) projector \( P^\mathcal{G} \)

\[
A^\mathcal{G}(x) = P^\mathcal{G}[A](x) = A(x) + C_x \mathcal{F}^\mathcal{G}[A](x). \tag{17}
\]

We require this projector to be insensitive to any "previous gauge" of the field, namely to satisfy

\[
P^\mathcal{G}[C_x f](x) = 0, \quad \text{for any } f(x). \tag{18}
\]

We consider now the adjoint projector \( P^\mathcal{G}^\dagger \) that in general does not coincide with \( P^\mathcal{G} \); an exception is given by Feynman-Landau type of gauges. We shall now prove the following properties of the adjoint projector.

(1) \( P^\mathcal{G}^\dagger \) produces conserved sources \( \mathcal{G} \).

For, integrating (18) on a current \( J(x) \) we have

\[
\int dx \, P^\mathcal{G}[C_x f](x) J(x) = 0; \tag{19}
\]

by definition of the adjoint projector, this means that

\[
\int dx \, f(x) \, C_x^\dagger P^\mathcal{G}^\dagger [J](x) = 0 \tag{20}
\]

and due to the arbitrariness of \( f(x) \)

\[
C_x^\dagger P^\mathcal{G}^\dagger [J](x) = 0 \quad \text{for any } J(x). \tag{21}
\]

(2) \( P^\mathcal{G}^\dagger \) leaves a conserved source unchanged.
Let us suppose that $J$ is conserved. Using (17) we have for a generic $A$

$$\int dx \, A(x) \, P^{g\dagger}[J](x) = \int dx \, A(x) \, J(x) + \int dx \, \{C_x F^g[A](x)\} \, J(x).$$

(22)

Integrating by parts the second term on the r.h.s. and using (10) and the arbitrariness of $A$ we have

$$P^{g\dagger}[J](x) = J(x) \quad \text{if} \quad C_x^\dagger J(x) = 0.$$  

(23)

The equation of motion obtained varying the action under the constraint (16) is

$$P^{g\dagger}[K_x A^g](x) = P^{g\dagger}[J](x).$$

(24)

From (10) and Property 2 we have

$$K_x A^g(x) = P^{g\dagger}[J](x),$$

(25)

or for the propagator

$$K_x G^g(x, y) = P^{g\dagger}[\delta(x - y)];$$

(26)

where the meaning of the r.h.s. is

$$\int dy \, P^{g\dagger}[\delta(x - y)] \, J(y) = P^{g\dagger}[J](x).$$

(27)

Next we show that a solution of (26) is

$$G^g(x, y) = i \langle P^g[A^F](x) \, P^g[A^F](y) \rangle_0,$$

(28)

where $A^F$ denotes the field in the original gauge. Integrating on a source $J(y)$ we have

$$i \int dy \, K_x \langle P^g[A^F](x) \, P^g[A^F](y) \rangle_0 \, J(y) =$$

$$= i \int dy \, K_x \langle P^g[A^F](x) \, A^F(y) \rangle_0 \, P^{g\dagger}[J](y) =$$

$$= i \int dy \, K_x \langle \{A^F(x) + C_x F^g[A^F](x)\} \, A^F(y) \rangle_0 \, P^{g\dagger}[J](y) =$$

$$= i \int dy \, K_x \langle A^F(x) \, A^F(y) \rangle_0 \, P^{g\dagger}[J](y) =$$

$$= \int dy \, (K_x + K_x^F) G^F(x, y) \, P^{g\dagger}[J](y) - \int dy \, K_x^F G^F(x, y) \, P^{g\dagger}[J](y) =$$

$$= \int dy \, \delta(x - y) \, P^{g\dagger}[J](y).$$

(29)
In the last step we have used (12) and Property 1. Note however that $G$ defined in (28) remains unchanged if we replace the original field with any other gauge equivalent field.

Given two different projectors $P_1$ and $P_2$ that project on the same gauge (which as a rule differ for different boundary conditions), one has $P_1 P_2 = P_1$ and $P_2 P_1 = P_2$ due to eq. (18)\(^1\). Thus also $P_{12} = \alpha P_1 + (1 - \alpha) P_2$ is a projector on the considered gauge and one can write down the $P_{12}$-projected Green function equation (we omit the suffix $\mathcal{G}$)

$$K_x G(x, y) = P_{12}^\dagger [\delta(x - y)].$$  \quad (30)

It is immediate to verify that a solution of (30) is also given by

$$\alpha \langle P_2[A](x) P_1[A](y) \rangle_0 + (1 - \alpha) \langle P_1[A](x) P_2[A](y) \rangle_0.$$  \quad (31)

In fact

$$K_x \alpha \langle P_2[A](x) P_1[A](y) \rangle_0 = \alpha K_x \langle A(x) P_1[A](y) \rangle_0 = \alpha P_1^\dagger [\delta(x - y)],$$  \quad (32)

repeating the same procedure of eq. (29). Acting similarly with the $(1 - \alpha)$ term in (31), we get (30). We notice that for $\alpha = \frac{1}{2}$, (31) is symmetric in the exchange of the field arguments. In the following we write the explicit form of the operators appearing in the Electrodynamics and linearized Einstein theory

### 2.2 Electrodynamics and linearized Yang-Mills theory.

This is the most simple case. We have the following identifications

$$A \rightarrow A_\mu;$$  \quad (33)
$$J \rightarrow J_\mu;$$  \quad (34)
$$Cf \rightarrow \partial_\mu f;$$  \quad (35)
$$K \rightarrow \eta_{\mu\nu} \Box - \partial_\mu \partial_\nu;$$  \quad (36)
$$K^F \rightarrow \partial_\mu \partial_\nu,$$  \quad (37)

where $K^F$ is the operator produced by the usual Feynman gauge fixing $\frac{1}{2}(\partial^\mu A_\mu)^2$.

### 2.3 Linearized Einstein gravity in the second-order formalism.

In this case we have

$$A \rightarrow h_{\mu\nu};$$  \quad (38)

\(^1\)The definition of $P^0$ and $P^{\infty}$ is different from one adopted in ref. [12], and they obey a different algebra. The definition adopted here privileges the invariance under gauge transformations on the original fields while the ones given in [12] privileges the radiality. These projectors give the same results when applied to regular fields.
\[ J \rightarrow T_{\mu\nu}; \quad (39) \]

\[ Cf \rightarrow (\delta_{\alpha\sigma} \partial_{\rho} + \delta_{\alpha\rho} \partial_{\sigma}) f_\alpha; \quad (40) \]

\[ K \rightarrow K_{\mu\nu\rho\sigma} = \frac{1}{4} \left[ 2 \eta_{\rho\sigma} \partial_{\mu} \partial_{\nu} + 2 \eta_{\mu\nu} \partial_{\rho} \partial_{\sigma} + \right. \]
\[ \left. - (\eta_{\mu\nu} \partial_{\rho} \partial_{\sigma} + \eta_{\mu\rho} \partial_{\nu} \partial_{\sigma} + \eta_{\nu\rho} \partial_{\mu} \partial_{\sigma} + \eta_{\mu\sigma} \partial_{\rho} \partial_{\nu} + \eta_{\nu\sigma} \partial_{\rho} \partial_{\mu}) + \right. \]
\[ \left. + (\eta_{\rho\sigma} \eta_{\nu\mu} + \eta_{\nu\rho} \eta_{\sigma\mu} - 2 \eta_{\mu\nu} \eta_{\rho\sigma}) \Box \right]; \quad (41) \]

\[ K^F \rightarrow K^F_{\mu\nu\rho\sigma} = \frac{1}{4} \left[ -(2 \eta_{\rho\sigma} \partial_{\mu} \partial_{\nu} + 2 \eta_{\mu\nu} \partial_{\rho} \partial_{\sigma}) + \right. \]
\[ \left. + (\eta_{\nu\rho} \partial_{\mu} \partial_{\sigma} + \eta_{\mu\rho} \partial_{\nu} \partial_{\sigma} + \eta_{\nu\sigma} \partial_{\mu} \partial_{\rho} + \eta_{\mu\sigma} \partial_{\nu} \partial_{\rho}) \right]. \quad (42) \]

Here \( K^F \) is the operator produced by the harmonic gauge fixing

\[ \frac{1}{2} \left( \partial^\mu h_{\mu\nu} - \frac{1}{2} \partial^\nu h^\mu_{\mu} \right)^2. \quad (43) \]

### 2.4 Linearized Einstein gravity in the first-order formalism.

The quadratic part of the lagrangian has the form

\[ L^{(2)} = \left( \delta_{\alpha\gamma} \partial_{\beta \gamma} \partial_{\alpha \beta} + \delta_{\alpha\beta} \Gamma^{\gamma}_{\alpha} \Gamma^{\beta}_{\gamma} + T^\alpha_{\mu} \tau^\mu_{\alpha} + \Sigma^{\mu}_{\alpha\beta} \Gamma^\mu_{\alpha\beta} \right), \quad (44) \]

where

\[ \tau^\mu_{\alpha} = e^\alpha_{\mu} - \delta^\alpha_{\mu} \quad (45) \]

and \( T^\mu_{\alpha} \) and \( \Sigma^{\mu}_{\alpha\beta} \) are the energy-momentum source and the spin-torsion source, respectively. The gauge transformations have the form

\[ Cf \rightarrow \begin{pmatrix} 0 & \delta_{cd} \partial_{\mu} \\ \partial_{\mu} & -\eta_{ab} \delta_{cd} \end{pmatrix} \begin{pmatrix} \Lambda^a \\ \theta^{cd} \end{pmatrix}. \quad (46) \]

The field equations are given by

\[ K A = J \rightarrow \begin{pmatrix} \frac{1}{2} (\eta_{\alpha\sigma} \delta_{\mu\mu}^{\nu\sigma} - \eta_{\sigma\mu} \delta_{\mu\nu}^{\nu\sigma}) & -\delta_{\mu\nu} \partial_{\lambda} \\ \delta_{\mu\nu} \partial_{\lambda} & 0 \end{pmatrix} \begin{pmatrix} \Gamma^\mu_{\nu} \\ \tau_{\gamma} \end{pmatrix} = \begin{pmatrix} \Sigma^{\mu}_{\alpha\beta} \\ T^\mu_{\alpha} \end{pmatrix} \quad (47) \]

and the gauge-fixing term has the form

\[ K^F A \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Gamma^\mu_{\nu} \\ \tau_{\gamma} \end{pmatrix}, \quad (48) \]
and is produced by the harmonic gauge fixing \([43]\) with \(h_{\mu\nu} = \tau_{\mu}^a \eta_{a\nu} + \tau_{\nu}^a \eta_{a\mu}\), to which the symmetric gauge fixing \(\frac{\beta}{2}(\tau_{\mu}^a \eta_{a\nu} - \tau_{\nu}^a \eta_{a\mu})^2\) has been added.

3 Second order formalism.

3.1 Radial gauge.

We first give a discussion at the classical level. In the second order formalism we define the radial gauge through the condition

\[\xi^\mu g_{\mu\nu}(\xi) = \xi^\mu \eta_{\mu\nu}\]  

(\(\eta_{\mu\nu} = \text{diag}(1,-1,-1,\ldots)\), and \(N=\text{space-time dimensions}\)). Taking the derivative of (49) with respect to \(\xi^\lambda\) we have

\[g_{\lambda\mu}(\xi) + \xi^\mu \partial_\lambda g_{\mu\nu}(\xi) = \eta_{\lambda\nu}.\]  

Under regularity hypothesis on \(g_{\mu\nu}(\xi)\) we obtain

\[g_{\lambda\nu}(0) = \eta_{\lambda\nu}.\]  

Contracting the Levi-Civita connection on \(\xi^\alpha\) and \(\xi^\beta\) we get, using (49) and (50)

\[\xi^\alpha \xi^\beta \Gamma_{\mu,\alpha\beta}(\xi) = \frac{1}{2} \xi^\alpha \xi^\beta [\partial_\beta g_{\mu\alpha}(\xi) + \partial_\alpha g_{\mu\beta}(\xi) - \partial_\mu g_{\alpha\beta}(\xi)] =
\]

\[= \frac{1}{2} \xi^\beta [\eta_{\beta\mu} - g_{\beta\mu}(\xi)] + \frac{1}{2} \xi^\alpha [\eta_{\alpha\mu} - g_{\alpha\mu}(\xi)] - \frac{1}{2} \xi^\beta [\eta_{\beta\mu} - g_{\beta\mu}(\xi)] = 0.\]

Thus coordinates satisfying (49) are Riemann’s normal coordinates, in the sense that the lines \(\xi^\mu = \lambda n^\mu\) are autoparallel lines [14]. Furthermore (52) combined with (49) tells us that \(\xi^\mu = \lambda n^\mu\) are also geodesic in the sense that they are extrema of the distance between two events. In fact from (49) we have that

\[(ds)^2 = d\xi^\mu g_{\mu\nu}(\xi) d\xi^\nu = (d\lambda)^2 n^\mu g_{\mu\nu}(\lambda) n^\nu = (d\lambda)^2 n^\mu g_{\mu\nu}(\xi)|_{\xi=0} n^\nu\]

and thus \(\xi^\mu(s)\) satisfies the equation

\[\frac{d^2 \xi^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}(\xi) \frac{d\xi^\alpha}{ds} \frac{d\xi^\beta}{ds} = 0.\]

The usual definition of the geodesic gauge is [13]

\[\xi^\alpha \xi^\beta \Gamma_{\mu,\alpha\beta}(\xi) = 0,\]  

(55)
as given by eq. (52). We want to prove that eq. (49) follows from (55) (apart from a global linear transformation). In fact from (55) we have

\[
\frac{1}{2} \xi^\alpha \xi^\beta \left[ \partial_\beta g_{\mu \alpha}(\xi) + \partial_\alpha g_{\mu \beta}(\xi) - \partial_\mu g_{\alpha \beta}(\xi) \right] = \\
= \xi^\alpha \partial_\alpha [\xi^\beta g_{\beta \mu}(\xi)] - \frac{1}{2} \partial_\mu [\xi^\alpha \xi^\beta g_{\alpha \beta}(\xi)] = 0. \tag{56}
\]

Multiplying (56) by \( \xi^\mu \) we obtain

\[
\frac{1}{2} \xi^\alpha \xi^\beta \partial_\alpha [\xi^\beta \xi^\mu g_{\beta \mu}(\xi)] - \xi^\beta \xi^\mu g_{\beta \mu}(\xi) = 0, \tag{57}
\]
i.e. \( \xi^\mu \xi^\beta g_{\mu \beta}(\xi) \) is a homogeneous function of degree 2 in \( \xi \). In the following (see subsection 3.C) we shall consider metrics \( g_{\mu \nu} \) which a priori are not regular at the origin but such that \( \xi^\mu g_{\mu \beta}(\xi) \) is a regular \( (C^2) \) function in a domain containing the origin and which vanish for \( \xi^\mu = 0 \). Under such assumption we have

\[
\xi^\beta \xi^\mu g_{\beta \mu}(\xi) = c_{\mu \beta} \xi^\beta \xi^\mu. \tag{58}
\]

In fact taking the second derivative we have

\[
\partial_\rho \partial_\lambda [\xi^\beta \xi^\mu g_{\beta \mu}(\xi)] = H^0_{\rho \lambda}(\xi) = c_{\rho \lambda}, \tag{59}
\]
because the only continuous homogeneous function of degree 0 is the constant. Substituting this relation into (56) we have

\[
\xi^\alpha \partial_\alpha [\xi^\beta g_{\beta \mu}(\xi)] - \xi^\beta c_{\beta \mu} = 0, \tag{60}
\]
i.e. \( \xi^\beta g_{\beta \mu}(\xi) - \xi^\beta c_{\beta \mu} \) must be an homogeneous function of degree 0, that has to vanish for \( \xi = 0 \) and is thus identically zero

\[
\xi^\beta g_{\beta \mu}(\xi) = \xi^\beta c_{\beta \mu}. \tag{61}
\]

In conclusion we have found that (55) implies (49) up to a global linear transformation of the coordinates. In the following we shall adopt (49) as the definition of the radial gauge in the second order formalism.

### 3.2 Radial Projectors.

In this section we shall derive, given the linearized Riemann tensor, radial fields \( h^0_{\mu \nu}(x) \) and \( h^\infty_{\mu \nu}(x) \) which generate such a Riemann tensor. \( h^0_{\mu \nu}(x) \) and \( h^\infty_{\mu \nu}(x) \) differ for their regularity properties at the origin and infinity and they will be the ingredients out of which we construct the Green’s function.
We now want to express the radial metric \(g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)\) in terms of the Riemann tensor, at the linearized level. Let us start from the expression of the linearized Riemann tensor
\[
R^L_{\mu\nu,\alpha\beta}(x) = \frac{1}{2} \left[ \partial_{\alpha} \partial_{\nu} h_{\mu\beta}(x) - \partial_{\alpha} \partial_{\mu} h_{\nu\beta}(x) - \partial_{\beta} \partial_{\nu} h_{\mu\alpha}(x) + \partial_{\beta} \partial_{\mu} h_{\alpha\nu}(x) \right].
\] (62)

\(R^L_{\mu\nu,\alpha\beta}(x)\) is an invariant under the linearized reparametrization transformations. Using the radial condition (49) in the form
\[
G(h)_\nu(x) = x^\mu h_{\mu\nu}(x) = 0 \quad (63)
\]
we can rewrite the contraction of (62) with \(x^\mu x^\alpha\) as follows
\[
2 x^\mu x^\alpha R^L_{\mu\nu,\alpha\beta}(x) = -x^\mu \partial_\mu [x^\alpha \partial_\alpha h_{\nu\beta}(x)] - x^\alpha \partial_\alpha h_{\nu\beta}(x).
\] (64)

Applying the usual change of variables \(x \to \lambda x\) we get
\[
-2 \lambda^2 x^\mu x^\alpha R^L_{\mu\nu,\alpha\beta}(\lambda x) = \lambda x^\mu \partial_\mu [\lambda x^\alpha \partial_\alpha h_{\nu\beta}(\lambda x)] + \lambda x^\alpha \partial_\alpha h_{\nu\beta}(\lambda x) = \frac{d}{d\lambda} \left[ \lambda^2 x^\alpha \partial_\alpha h_{\nu\beta}(\lambda x) \right].
\] (65)

Integrating (65) in \(\lambda\) from 0 to 1 we obtain
\[
x^\alpha \partial_\alpha h_{\nu\beta}(x) = -2 x^\alpha x^\mu \int_0^1 d\lambda \lambda^2 R^L_{\mu\nu,\alpha\beta}(\lambda x),
\] (66)

provided the metric is such that \(|R^L(x)| \leq |x|^{-3+\varepsilon}\) as \(x \to 0\). With \(|x|^\alpha\) we understand here a quantity that for \(x^\mu = \lambda x^\mu\) behaves like \(\lambda^\alpha\) for \(\lambda \to 0\). We notice again that
\[
\tau x^\alpha \partial_\alpha h_{\nu\beta}(\tau x) = \tau \frac{d}{d\tau} [h_{\nu\beta}(\tau x)]
\] (67)
and thus we have
\[
h^0_{\nu\beta}(x) = -2 x^\alpha x^\mu \int_0^1 d\tau \tau \int_0^1 d\lambda \lambda^2 R^L_{\mu\nu,\alpha\beta}(\lambda \tau x) =
= -2 x^\alpha x^\mu \int_0^1 d\lambda \lambda (1 - \lambda) R^L_{\mu\nu,\alpha\beta}(\lambda x) =
= P^0[h]_{\nu\beta}(x),
\] (68)

provided \(|R^L(x)| < |x|^{-2+\varepsilon}\) for \(x \to 0\). The superscript “0” on \(h_{\nu\beta}(x)\) specifies the integration limits.

Another possibility to solve equation (65) is to integrate twice from 1 to \(\infty\), provided the metric is
such that $|R^L(x)| \leq |x|^{-3-\varepsilon}$ as $x \to \infty$. Then we have

$$h_{\nu \beta}^\infty(x) = -2 x^\alpha x^\mu \int_1^\infty d\tau \int_1^\infty d\lambda \lambda^2 R^L_{\mu \nu, \alpha \beta}(\lambda \tau x) =$$

$$= -2 x^\alpha x^\mu \int_1^\infty d\lambda \lambda (\lambda - 1) R^L_{\mu \nu, \alpha \beta}(\lambda x) =$$

$$= P^\infty[h]_{\nu \beta}(x).$$

Eqs. (68), (69) define projectors from an arbitrary gauge to the radial gauge. The projector nature of $P^0$ and $P^\infty$ is proven by showing that e.g. $h$ and $P^0[h]$ give rise to the same Riemann tensor. In fact by using the linearized Bianchi identities

$$\partial_\lambda R_{\mu \nu, \alpha \beta}(x) + \partial_\mu R_{\nu \lambda, \alpha \beta}(x) + \partial_\nu R_{\lambda \mu, \alpha \beta}(x) = 0$$

and the familiar symmetry properties of the Riemann tensor, one reduces $R^L_{\mu \nu, \alpha \beta}(x)$ to

$$R^L_{\mu \nu, \alpha \beta}(x) = \int_0^1 d\lambda \frac{d}{d\lambda} [\lambda^2 R^L_{\mu \nu, \alpha \beta}(\lambda x)].$$

Whenever $P^0$ is defined on $h$ we have from (71)

$$R^L_{\mu \nu, \alpha \beta}(x) = R^L_{\mu \nu, \alpha \beta}(x).$$

Furthermore from the gauge invariance of $R^L_{\mu \nu, \alpha \beta}(x)$ appearing in the definitions (68), (69) we have the properties

$$P^0 P^\infty = P^0, \quad P^\infty P^0 = P^\infty.$$  

de (73)

The propagator

$$\langle P^0[h]_{\mu \nu}(x) P^0[h]_{\rho \sigma}(y) \rangle_0$$

computed by means of (68) is divergent in all dimensions due to the $(x - y)^{-N-2}$ behavior of the correlation of two Riemann’s tensors at short distances. On the other hand

$$\langle P^\infty[h]_{\mu \nu}(x) P^\infty[h]_{\rho \sigma}(y) \rangle_0$$

diverges for $N \leq 4$ due to the infrared behavior of the correlator of two Riemann’s tensors. For this reason we shall adopt a different gauge projector given by

$$P^S = \frac{1}{2}(P^0 + P^\infty)$$

(76)

whose projector nature is immediately seen from (73). We elucidate the nature of the boundary condition at 0 and $\infty$ of the projector $P^S$ at the end of section 3.C.
As we have shown in section 2 the solution of the projected Green’s function equation is given by
\[
\frac{1}{2} \langle P_0[h]_{\mu\nu}(x) P_\infty[h]_{\rho\sigma}(y) \rangle_0 + \frac{1}{2} \langle P_\infty[h]_{\mu\nu}(x) P_0[h]_{\rho\sigma}(y) \rangle_0
\]
which as we shall show in the next section is convergent for \( N > 2 \).

In order to write the source term in the projected Green’s function equation we must compute the adjoint of \( P_S \), i.e. the adjoints of \( P_0 \) and \( P_\infty \). We have
\[
P_0^\dagger[T]_{\mu\nu} = T_{\mu\nu}(x) + \int_1^\infty d\tau \tau^N \left( x^\rho \partial_\rho T_{\mu\nu}(\tau x) + x^\mu \partial_\nu T_{\mu\nu}(\tau x) \right) + \int_1^\infty d\tau \tau^N (\tau - 1) \partial_\alpha \partial_\beta T_{\alpha\beta}(\tau x),
\]
\[
P_\infty^\dagger[T]_{\mu\nu} = T_{\mu\nu}(x) - \int_0^1 d\tau \tau^N \left( x^\rho \partial_\rho T_{\mu\nu}(\tau x) + x^\mu \partial_\nu T_{\mu\nu}(\tau x) \right) + \int_0^1 d\tau \tau^N (\tau - 1) \partial_\alpha \partial_\beta T_{\alpha\beta}(\tau x)
\]
and
\[
P^S = \frac{1}{2} (P_0^\dagger + P_\infty^\dagger).
\]

### 3.3 Radial propagators.

We shall now derive the propagators using the general procedure explained in section 2. The kinetic operator \( K \) is given by formula (41). The equation for the propagator is
\[
(K_x)_{\mu\nu}^{\rho\sigma} G_{\rho\sigma\alpha\beta}(x, y) = P^\dagger[\Delta_{\mu\nu\alpha\beta}(x - y)],
\]
where
\[
\Delta_{\mu\nu\alpha\beta}(x - y) = \frac{1}{2} (\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha}) \delta^N (x - y).
\]

As a preliminary step we must compute the correlator of two Riemann tensors; this is most simply obtained by using the harmonic gauge. We find
\[
\langle R_{\mu\nu,\alpha\beta}(x) R_{\rho\sigma,\lambda\gamma}(y) \rangle_0 = \frac{1}{4} \delta^{\mu\nu'} \eta_{\nu'\nu'} (\delta^{\rho\sigma'} \delta^{\alpha'\beta'} \delta^{\lambda'\gamma'} \eta_{\beta'\beta'') + \delta^{\lambda'\gamma'} \delta^{\alpha'\beta'} \delta^{\rho'\sigma'} \eta_{\beta'\beta'') + \frac{2}{N-2} \delta^{\lambda'\gamma'} \delta^{\alpha'\beta'} \delta^{\rho'\sigma'} \delta^{\lambda'\gamma'} \eta_{\sigma'\sigma''} \frac{\eta_{\alpha'\beta'}}{\eta_{\alpha\beta}} \partial_\mu \partial_\alpha \partial_\rho \partial_\lambda D(x - y)
\]
being \( D(x - y) \) the usual Feynman propagator
\[
D(x) = \frac{\Gamma(N/2 - 1)}{4\pi^2} \frac{i}{(x^2 - i\varepsilon)^{N/2 - 1}}.
\]
If we choose $P = P^0$ and substitute in (28) eq. (82) we find that the propagator can be expressed as a linear combination of derivatives of integrals of the type

$$F_{\alpha\beta}^0(x, y) = \int_0^1 d\lambda (1 - \lambda) \int_0^1 d\tau (1 - \tau) \partial_\alpha \partial_\beta D(\lambda x - \tau y).$$  \hspace{1cm} (84)

With the method explained below one sees that such integral diverges in any dimension due to the bad ultraviolet behavior of $\partial_\alpha \partial_\beta D(\lambda x - \tau y)$. Similarly

$$G_{\mu\nu\rho\sigma}^\infty(x, y) = \langle P^\infty[h]_{\mu\nu}(x) P^\infty[h]_{\rho\sigma}(y) \rangle_0$$ \hspace{1cm} (85)

computed along the same lines is expressed in terms of derivatives of

$$F_{\alpha\beta}^\infty(x, y) = \int_1^\infty d\lambda (1 - \lambda) \int_1^\infty d\tau (1 - \tau) \partial_\alpha \partial_\beta D(\lambda x - \tau y)$$ \hspace{1cm} (86)

and as shown below it diverges for $N \leq 4$. On the other hand the correlator

$$\langle P^0[h]_{\mu\nu}(x) P^\infty[h]_{\rho\sigma}(y) \rangle_0$$ \hspace{1cm} (87)

used to construct the solution of the $PS$ projected Green’s equation converges for $N > 2$.

We prove now the above stated results: the correlator (87) can be expressed as linear combination of derivatives of the double integral

$$F_{\alpha\beta}^S(x, y) = \int_0^1 d\lambda (1 - \lambda) \int_1^\infty d\tau (\tau - 1) \partial_\alpha \partial_\beta D(\lambda x - \tau y)$$ \hspace{1cm} (88)

where

$$\partial_\alpha \partial_\beta D(x) = -(N - 2) \frac{\eta_{\alpha\beta}}{(x^2)^{N/2}} + N(N - 2) \frac{x_\alpha x_\beta}{(x^2)^{N/2+1}}.$$ \hspace{1cm} (89)

The first term of (89) substituted in (88) gives

$$\int_0^1 d\lambda \lambda^{-N+1} (1 - \lambda) \int_0^\lambda d\rho \left( \frac{\lambda}{\rho} - 1 \right) \frac{\rho^{N-2}}{(\rho x - y)^N}$$ \hspace{1cm} (90)

and converges for $N > 2$. Similarly one proves the convergence for $N > 2$ of the second term of (89) substituted in (88). Thus the solution of the $PS$ projected Green’s equation

$$(K_x)^{\rho\sigma}_{\mu\nu} G_{\rho\sigma\alpha\beta}^S(x, y) = P^S \Delta_{\mu\nu\alpha\beta}(x - y)$$ \hspace{1cm} (91)

given by

$$- i G_{\mu\nu\rho\sigma}^S(x, y) = \frac{1}{2} \langle P^0[h]_{\mu\nu}(x) P^\infty[h]_{\rho\sigma}(y) \rangle_0 + \frac{1}{2} \langle P^\infty[h]_{\mu\nu}(x) P^0[h]_{\rho\sigma}(y) \rangle_0$$ \hspace{1cm} (92)
converges for all \( N > 2 \).

With regard to the \( G^\infty(x, y) \) given by (86) one reaches for the analogue \( F^\infty_{\alpha\beta}(x, y) \) of \( F^S_{\alpha\beta}(x, y) \) the integral

\[
(N - 2) \int_1^\infty d\lambda \, \lambda^{-N+1} (1 - \lambda) \int_0^\lambda d\rho \, \left( \frac{\lambda}{\rho} - 1 \right) \frac{\rho^{N-2}}{(\rho x - y)^N} \left( -\eta_{\alpha\beta} + N \frac{x_\alpha x_\beta}{(\rho x - y)^2} \right)
\]

which converges only for \( N > 4 \). Finally the function \( F^0_{\alpha\beta}(x, y) \) given by

\[
F^0_{\alpha\beta}(x, y) = \int_0^1 d\lambda \, (1 - \lambda) \int_0^\lambda d\tau \, (1 - \tau) \partial_\alpha \partial_\beta D(\lambda x - \tau y)
\]

leads to integrals of the form

\[
\int_0^1 d\lambda \, (1 - \lambda) \lambda^{-N+1} \int_0^\infty d\rho \, \left( \frac{\lambda}{\rho} - 1 \right) \frac{\rho^{N-2}}{(\rho x - y)^N}
\]

which diverge for all \( N \geq 2 \).

In conclusion, we produced a solution of the radial Green’s function equation that is radial, symmetric and finite for \( N > 2 \). Such a solution treats the infinity and the origin in symmetrical way in the sense that the field

\[
h^S_{\mu\nu}(x) = \int G^S_{\mu\nu\rho\sigma}(x, y) \, t^{\rho\sigma}(y) \, d^N y,
\]

computed on a physical (i.e. conserved) source \( t^{\rho\sigma} \), behaves like

\[
h^S(x) = K^0(x) + \frac{H^0(x)}{r} + O(r^\varepsilon) \quad \text{for} \quad r \to 0
\]

and

\[
h^S(x) = -K^0(x) - \frac{H^0(x)}{r} + O(r^{-1-\varepsilon}) \quad \text{for} \quad r \to \infty,
\]

where \( K^0 \) and \( H^0 \) are homogeneous functions of \( x \) of degree 0, and \( r = |x| \). In fact, given a conserved source \( t^{\rho\sigma}(y) \) we have

\[
-i \int G^S_{\mu\nu\rho\sigma}(x, y) \, t^{\rho\sigma}(y) \, d^N y = \frac{1}{2} \int \left\langle P^0[h^F]_{\mu\nu}(x) \, h^F_{\rho\sigma}(y) \right\rangle_0 \, P^{\infty \dagger}[t]^{\rho\sigma}(y) \, d^N y + \frac{1}{2} \int \left\langle P^{\infty}[h^F]_{\mu\nu}(x) \, h^F_{\rho\sigma}(y) \right\rangle_0 \, P^0[t]^{\rho\sigma}(y) \, d^N y
\]

and using the fact that \( P^{\infty \dagger} \) and \( P^0 \) act like the identity on conserved sources we have that the l.h.s. of (99) is given by \( P^S[h^F]_{\mu\nu}(x) \) being \( h^F_{\mu\nu} \) the field, computed in the harmonic gauge, associated to
the conserved source $i^{[\sigma}$. For $r \to \infty$, $h^0_{\mu\nu}(x)$ behaves from (18) like

$$h^0_{\mu\nu}(x) = -\frac{2x^\alpha x^\beta}{r^2} \int_0^r dr' r' R^{L}_{\alpha\mu,\beta\nu}(r' \hat{x}) + \frac{2x^\alpha x^\beta}{r^3} \int_0^r dr' r'^2 R^L_{\alpha\mu,\beta\nu}(r' \hat{x}) =$$

$$= K^0_{\mu\nu}(x) + \frac{H^0_{\mu\nu}(x)}{r} + O(r^{-1-\epsilon})$$

while $h^\infty_{\mu\nu}(x)$ from (19) is given by

$$h^\infty_{\mu\nu}(x) = -\frac{2x^\alpha x^\beta}{r^2} \int_r^\infty dr' r' R^{L}_{\alpha\mu,\beta\nu}(r' \hat{x}) + \frac{2x^\alpha x^\beta}{r^3} \int_r^\infty dr' r'^2 R^L_{\alpha\mu,\beta\nu}(r' \hat{x})$$

and is regular in the same limit, in the sense that it vanishes at least as quickly as $h^F_{\mu\nu}$ itself. Similarly in the limit $r \to 0$ we find that $h^0_{\mu\nu}(x)$ vanishes and

$$h^\infty_{\mu\nu}(x) = -K^0_{\mu\nu}(x) - \frac{H^0_{\mu\nu}(x)}{r} + O(r^\epsilon),$$

from which one obtains the two relations (97) and (98). In Appendix A it is shown how one can express the propagators $G^S$ in terms of the familiar hypergeometric functions, while in Appendix B we treat the $\beta \neq 0$ (i.e. non sharp) radial gauge in the second order formalism.

### 4 First order formalism

#### 4.1 Radial gauge

We recall for completeness the main formulae for the radial gauge in the first order formalism given by [4]. The gauge is defined by

$$\xi^\mu \Gamma^a_{\mu}(\xi) = 0$$

$$\xi^\mu e^a_{\mu}(\xi) = \xi^\mu \delta^a_{\mu}$$

and $\Gamma^a_{\mu}(\xi)$ and $e^a_{\mu}(\xi)$ are expressed in term of the radial Riemann and torsion two forms as follows

$$\Gamma^a_{b,\mu}(\xi) = \xi^\nu \int_0^1 d\lambda \lambda R^a_{b,\nu\mu}(\lambda \xi)$$

$$e^a_{\mu}(\xi) = \delta^a_{\mu} + \xi^\nu \xi^b \int_0^1 d\lambda \lambda (1 - \lambda) R^a_{b,\nu\mu}(\lambda \xi) + \xi^\nu \int_0^1 d\lambda \lambda S^a_{\nu\mu}(\lambda \xi)$$

for the derivation of which we refer to [4]. These formulae hold under the hypothesis of regularity at the origin for the field $(\Gamma^a_{\mu}(x), e^a_{\mu}(x))$. Radial projections corresponding to different assumption about the behavior of the field at the origin will be considered in the following subsection.
4.2 Radial Projectors

Similarly as done in section 3, at the linearized level we can use relations (105) and (106) to project the general field \((\Gamma, e)\) into the radial field \((\Gamma^R, e^R)\) as the r.h.s of eqs. (105) and (106) are invariant under linearized gauge transformations

\[
\begin{pmatrix}
\Gamma_{ab}^\mu(x) \\
\tau^a_\mu(x)
\end{pmatrix} \rightarrow \begin{pmatrix}
\Gamma_{ab}^\mu(x) + \partial_\mu \Theta_{ab}(x) \\
\tau^a_\mu(x) - \Theta_{ab}(x)\eta_{b\mu} + \partial_\mu \Lambda^a(x)
\end{pmatrix}
\]

(107)

where \(\tau^a_\mu(x)\) is related to \(e^a_\mu(x)\) by

\[
e^a_\mu(x) = \delta^a_\mu + \tau^a_\mu(x).
\]

(108)

\(\Theta_{ab}(x)\) is the infinitesimal Lorentz-transformation and \(\Lambda^a(x)\) the infinitesimal translation. We shall be interested in the projectors \(P^0\) and \(P^\infty\) to construct the projector

\[
P^S = \frac{1}{2} (P^0 + P^\infty).
\]

(109)

\(P^0\) is given by (105) and (106) substituting to \(R^ab_\mu\) and \(S^a_\mu\) their linearized expression

\[
R_{\mu\nu}^{Lab}(x) = \partial_{\mu} \Gamma_{\nu}^{ab}(x) - \partial_{\nu} \Gamma_{\mu}^{ab}(x)
\]

(110)

\[
S_{\mu\nu}^{La}(x) = \partial_{\mu} \tau_{\nu}^{a}(x) - \partial_{\nu} \tau_{\mu}^{a}(x) + \Gamma_{\mu}^{ab}(x)\eta_{b\nu} - \Gamma_{\nu}^{ab}(x)\eta_{b\mu}
\]

(111)

which are invariant under linearized gauge transformation; i.e.

\[
P^0 \begin{pmatrix}
\Gamma_{b,\mu}^a(x) \\
\tau_{\mu}^a(x)
\end{pmatrix} = \begin{pmatrix}
x^\nu \int_0^1 d\lambda \lambda R_{b,\nu}^{La}(\lambda x) \\
x^\nu x^b \int_0^1 d\lambda \lambda (1 - \lambda) R_{b,\nu}^{La}(\lambda x) + x^\nu \int_0^1 d\lambda \lambda S_{\nu\mu}^{La}(\lambda x)
\end{pmatrix}
\]

(112)

\[
P^\infty \begin{pmatrix}
\Gamma_{b,\mu}^a(x) \\
\tau_{\mu}^a(x)
\end{pmatrix} = \begin{pmatrix}
-x^\nu \int_1^\infty d\lambda \lambda R_{b,\nu}^{La}(\lambda x) \\
-x^\nu x^b \int_1^\infty d\lambda \lambda (1 - \lambda) R_{b,\nu}^{La}(\lambda x) - x^\nu \int_1^\infty d\lambda \lambda S_{\nu\mu}^{La}(\lambda x)
\end{pmatrix}.
\]

(113)

\(P^0\) projects on radial fields that are regular at the origin (i.e. behaving like the original field) and give a connection \(\Gamma_{\mu}^{ab}(x)\) behaving like \(1/r\) at \(\infty\). On the other hand \(P^\infty\) projects on a radial field that is regular at \(\infty\) and give a connection \(\Gamma_{\mu}^{ab}(x)\) behaving like \(1/r\) at the origin. As we noticed in the introduction the radial connection has to behave like \(1/r\) at the origin and/or at \(\infty\), otherwise
the Wilson loop given by two radii and closed by two arcs one going to the origin and the other
going to infinity (see Fig. 1) would be identically 1, due to the radial nature of the connection; but
this would be contradictory as one cannot fix at the kinematical level the value of a gauge invariant
quantity. \(P^S\) which allows us to construct a finite propagator in \(N > 2\), treats the origin and infinity
in symmetrical way by giving the same \(1/r\) behavior in the two limits with opposite coefficients.
This is seen in the same way as shown at the end of section 3 and in Appendix A in the second
order formalism. Eq. (109) together with eqs. (112) and (113) will allow us to compute the radial
propagator associated to the \(P^S\) projector, in terms of the correlation functions of the linearized
Riemann and torsion two forms, which are invariant under linearized gauge transformations and as
such can be computed in any gauge (e.g. the harmonic gauge). First we must derive the action of
the adjoint projectors on the torsion source \(\Sigma^\mu_{ab} = -\Sigma_{ba}^\mu\) and on the energy momentum tensor \(T^\mu_a\).

\[
P^0(\Sigma)^\mu_{ab} = \Sigma^\mu_{ab}(x) + x^\mu \int_1^\infty d\lambda \lambda^{N-1} \left( \partial_\mu \Sigma_{ab}(\lambda x) + \frac{1}{2} \left( T^p_a(\lambda x) \eta_{pb} - T^p_b(\lambda x) \eta_{pa} \right) \right)
\]

\[
+ \frac{x^\mu}{2} \int_1^\infty d\lambda (\lambda N - \lambda^{N-1}) (x_b \partial_\mu T^p_a - x_a \partial_\mu T^p_b)
\]

\[
P^0(\Sigma)_{\mu a} = \Sigma_{\mu a}(x) + x^\mu \int_1^\infty d\lambda \lambda^{N-1} \partial_\mu T^\rho_a(\lambda x)
\]

The \(P^0\) are given by changing in (114) and (115) the integration limit from \((1, \infty)\) to \((1, 0)\).
One easily verifies from (114) and (115) that the projected sources \(P^0(\Sigma, T)\) satisfy the linearized
Bianchi identities, as proved on general grounds in section 2 eq. (2.17)

\[
\partial_\mu \Sigma_{ab} + \frac{1}{2} (T^p_{a \eta_{pb}} - T^p_{b \eta_{pa}}) = 0
\]

\[
\partial_\mu T^\rho_a = 0.
\]

In addition one notices that the r.h.s of (114) and (115) are integrals of the linearized Bianchi
identities (116) and (117) and thus \(P^0\) and \(P^\infty\) leave unchanged the sources satisfying the Bianchi
identities, in agreement with the general argument of section 2.

4.3 Radial Propagators

Similarly to what happens in the second order formalism, the propagators constructed from \(\langle P^0(\Gamma, \tau)P^0(\Gamma, \tau)\rangle_0\) are divergent while \(\langle P^\infty(\Gamma, \tau)P^\infty(\Gamma, \tau)\rangle_0\) diverge for \(N \leq 4\). Thus we shall construct the solution
for the the \(P^S\) projected Green’ s function equation

\[
\left( \begin{array}{cc}
\frac{1}{2} (\delta^\mu_{am} \eta_{bn} - \delta^\mu_{an} \eta_{bm}) & \delta^\mu_{ab} \partial_\lambda \\
\delta^\mu_{pmn} \partial_\lambda & 0
\end{array} \right)
\left( \begin{array}{cc}
G^{mn,rs}_{\nu,\gamma} & G^{mn,g}_{\nu,\beta} \\
G^{d,rs}_{\sigma,\gamma} & G^{d,g}_{\sigma,\beta}
\end{array} \right)
= P^S \left( \begin{array}{cc}
\delta^\mu_{\gamma} \delta^\rho_{ab} & 0 \\
0 & \delta^\mu_{\beta} \delta^g_{\rho}
\end{array} \right) \delta^N(x - y)
\]

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by means of general technique described in section 2 and 3. One proves that
\[
-iG^S(x, y) = \frac{1}{2} \left< P^0 \left( \frac{\tau^a(x)}{\tau^a} \right) P^\infty \left( \frac{\Gamma^a_{\mu} (x)}{\tau^a} \right) \right> + \frac{1}{2} \left< P^\infty \left( \frac{\tau^a(x)}{\tau^a} \right) \right> P^0 \left( \frac{\Gamma^a_{\mu} (y)}{\tau^a} \right)
\]
is a convergent symmetric radial solution of (118) for all \( N > 2 \). The explicit form of solution (119) of the Green’s function equation (118) is easily computed by using (112) and (113) where the correlators between Riemann and torsion two-forms which are invariant under linearized gauge transformations, can be obtained using e.g. the usual symmetric harmonic gauge \((\beta = \infty)\) in which
\[
\left< \tau_{\alpha \mu}(x) \tau_{\nu \sigma}(y) \right> = -i \left( \eta_{\alpha \nu} \eta_{\mu \sigma} - \frac{2}{N - 2} \eta_{\alpha \mu} \eta_{\nu \sigma} \right) D(x - y).
\]
We get
\[
\left< R_{\mu \nu}^{La}(x) R_{\mu \nu}^{Lc}(y) \right> = \left< R_{\mu \nu}^{La}(x) R_{\mu \nu}^{Lc}(y) \right>^{II} + \left[ \partial_{\mu} \partial_{\nu} M_{ab,cd} - \partial_{\mu} \partial_{\nu} M_{ab,cd} \right] - (\mu \leftrightarrow \nu)
\]
where \( \left< \right>^{II} \) means the correlator in the second order formalism (see section 4) and
\[
M_{ab,cd} = -\frac{i}{4} \left( \delta_{\nu \rho} \delta_{\mu \rho} \eta_{c\delta} - \frac{2}{N - 2} \delta_{\mu \rho} \eta_{c\delta} \right) \delta^{N}(x - y)
\]
is the usual ultralocal part of the propagator of \( \Gamma \) in the first order formalism [10];
\[
\left< S_{\mu \rho}^{La}(x) S_{\mu \rho}^{Lb}(y) \right> = \left[ (M_{ac,bd}^{\mu,\rho} \eta_{cd} - M_{ac,bd}^{\mu,\rho} \eta_{cd}) \right] - (\mu \leftrightarrow \nu)
\]
\[
\left< R_{\mu \nu}^{La}(x) S_{\mu \rho}^{Lc}(y) \right> = \left[ (\partial_{\mu} M_{ab,cd}^{\mu,\rho} \eta_{cd} - \partial_{\nu} M_{ab,cd}^{\mu,\rho} \eta_{cd}) \right] - (\rho \leftrightarrow \sigma)
\]
The local nature of the correlators (123) and (124) reflects the well known fact that the torsion does not propagate in the Einstein-Cartan theory. The propagators are obtained by substituting the correlators of the Riemann and torsion tensors given by (121), (123) and (124) into (119). The integrals over \( \lambda \) and \( \tau \) of the \( \langle RR \rangle^{II} \) correlators are the same as those given in the second order formalism. The contact terms generated by the expression of \( M_{ab,cd} \) and appearing on the r.h.s. eqs. (121),(123) and (124) lead to integrals in \( \lambda \) and \( \tau \) which are convergent. In fact the generic integral is of the form
\[
\int_0^1 d\lambda A \int_1^\infty d\tau B \delta^N(\lambda x - \tau y) = \\
\int_0^1 d\lambda A \int_1^\infty d\tau B \delta(\lambda |x| - \tau |y|)(\lambda |x|)^{-N+1} \delta^{N-1}(\lambda x - \lambda y) = \\
\delta^{N-1}(\lambda x - \lambda y) \Theta(|x| - |y|) \frac{1}{A + B - N + 2} \left( |y|^{-B-1} |x|^{-B-N+1} - |y|^{-A-N+1} |x|^{-A-1} \right).
\]
We notice that in three dimensions the Riemann-Riemann correlator (121) is identically zero as can be explicitly verified substituting (120) and (122) into (121) or alternatively, observing that the (2,1) component of eq. (118) in three dimensions reads
\[ \langle R^L(x) \Gamma(y) \rangle \equiv 0 \] which implies that \[ \langle R^L(x) R^L(y) \rangle \equiv 0 \]. This means through (112) and (113) that \[ \langle \Gamma(x) \Gamma(y) \rangle \equiv 0 \] and that \[ \langle \tau(x) \tau(y) \rangle \] is zero except for singular contributions that arise when the origin, \( x \) and \( y \) are collinear; such contributions are given by (121), (123) and (124). Also in the second order approach the correlator \( \langle h(x) h(y) \rangle \) in three dimensions reduces to the collinear singular terms. Thus in the radial gauge one reads directly the non existence of propagating gravitons, while in the harmonic gauge one propagates a pure gauge. The surviving collinear singularity of the radial propagator in three dimension reflects the topological nature of the theory.

5 Conclusions

In the present paper we considered the quantization of the gravitational field in the radial gauge. The main advantage of this gauge is to provide physical correlation functions, i.e. correlation functions defined at points described invariantly in terms of a geodesic system of coordinates radiating from a given origin. Given a generic field we developed general projection formulae for extracting from it a radial field that is gauge equivalent to the given one; the projection is obviously not unique due to the presence of the residual gauge. For reasons intrinsic to the gauge nature of the considered theory and that can be easily understood by considering a special family of Wilson loops, such projected fields develop some kind of singular behavior at the origin or a very slow decrease at infinity. The best choice is to treat the origin and infinity in symmetrical way by sharing equally such non regular behavior and this fixes completely the gauge. Technically the projection procedure is obtained through the Riemann and torsion tensors which are invariant at the linearized level. Still the direct computation of the v.e.v. of the symmetrically projected field contains an infinite solution of the homogeneous radially projected Green’s function, due to the singular behavior of the Riemann-Riemann correlator, which has to be subtracted away. In this way one obtains an explicit solution of the radially projected Green’s function equation that is symmetric in the field arguments, radial and finite for all dimensions larger than 2. In three dimensions the propagators vanish identically except for collinear contributions; this is in agreement with the absence of propagating gravitons in three dimensions, while the collinear contributions are remnants at the perturbative level of topological effects (conical defects) introduced by matter in three dimensional gravity [17].

In this Appendix we show how the radial propagators discussed in the text can be written in terms of the hypergeometric functions. We start for simplicity sake from the case of Yang-Mills theory or quantum electrodynamics in the euclidean formulation, in which the propagator is given
by [12]

\[ G^{S}_{\mu\nu}(x, y) = \frac{1}{2} x^{\rho} y^{\sigma} D_{\mu\rho\nu\sigma}[D(x, y) + D(y, x)] \]  

with

\[ D_{\mu\rho\nu\sigma} = \delta^{\mu}_{\mu'} \delta^{\rho}_{\nu'} \delta^{\nu}_{\sigma'} \frac{\partial}{\partial x^{\mu'}} \frac{\partial}{\partial y^{\nu'}} \]  

and

\[ D(x, y) = \frac{\Gamma\left(\frac{N}{2} - 1\right)}{4\pi^{N/2}} \int_0^1 d\lambda \int_1^\infty d\tau \frac{1}{\left[\lambda x - \tau y\right]^{N/2 - 1}}. \]  

D(x, y) can be easily rewritten performing the change of variable \( \rho = \lambda/\tau \) and then integrating in \( \lambda \), as follows

\[ D(x, y) = \frac{\Gamma\left(\frac{N}{2} - 1\right)}{4\pi^{N/2}} \int_0^1 d\rho \frac{1}{\left[\left(\rho x - y\right)^2\right]^{N/2 - 1}} \left[\rho^{N-4} - 1\right]. \]  

Writing

\[ \int_0^1 d\rho \frac{1}{\left[\left(\rho x - y\right)^2\right]^{N/2 - 1}} = \int_0^\infty d\rho \frac{1}{\left[\left(\rho x - y\right)^2\right]^{N/2 - 1}} - \int_0^1 d\rho \frac{\rho^{N-4}}{\left[\left(x - \rho y\right)^2\right]^{N/2 - 1}} \]  

we obtain

\[ D(x, y) = \frac{\Gamma\left(\frac{N}{2} - 1\right)}{4\pi^{N/2}} \int_0^1 d\rho \frac{1}{\left[\left(\rho x - y\right)^2\right]^{N/2 - 1}} \left[\rho^{N-4} \left\{ \int_0^1 d\rho \frac{1}{\left[\left(\rho x - y\right)^2\right]^{N/2 - 1}} + \rho^{N-4} \right\} \right] \]  

\[ \quad + \frac{1}{\left[\left(x - \rho y\right)^2\right]^{N/2 - 1}} - \int_0^\infty d\rho \frac{1}{\left[\left(x - \rho y\right)^2\right]^{N/2 - 1}} \]  

The first integral appearing in \([131]\) can be rewritten as follows

\[ I_1 = \int_0^1 d\rho \frac{\rho^{N-4}}{\left[\rho^2 - 2\rho x y \cos \theta + y^2\right]^{N/2 - 1}} \]  

where \( X = |x|, Y = |y| \) and performing the change of variable \( \rho = \frac{Y \zeta}{d \zeta + 1} \) with \( d = \sqrt{X^2 + Y^2 - 2XY \cos \theta} \) obtains [18]

\[ Y^{-1} d^{3-N} \int_0^\infty d\zeta \frac{\zeta^{N-4}}{\left[\zeta^2 + 2\zeta \left(\frac{Y - X \cos \theta}{d}\right) + 1\right]^{N/2 - 1}} = \]  

\[ = \frac{1}{N - 3} Y^{-1} d^{3-N} \left(4 \sin^2 \phi_x\right)^{\frac{3-N}{4}} \Gamma \left(\frac{N - 1}{2}\right) P_{\frac{3-N}{2}} \left(\cos \phi_x\right) \]  

(133)
\[ \cos \phi = \frac{Y - X \cos \theta}{d}. \]

While \( \theta \) is the angle between \( x \) and \( y \), \( \phi_x \) is the angle opposite to \( x \) in the euclidean triangle of sides \( x \), \( y \), \( d \). Expressing \( P^{(3-N)}_3 \) in terms of hypergeometric functions \[18\], we have

\[ I_1 = \frac{1}{N-3} d^{3-N} Y^{-1} \left( \cos^2 \frac{\phi_x}{2} \right)^{\frac{3-N}{2}} 2F_1 \left( \frac{N-3}{2}, \frac{5-N}{2}, \frac{N-1}{2}; \sin^2 \frac{\phi_x}{2} \right). \] (134)

Similarly one computes

\[ I_2 = \int_0^\infty d\rho \frac{1}{\rho^2 X^2 - 2\rho X Y \cos \theta + Y^2} \] (135)

to obtain

\[ I_2 = \frac{1}{N-3} Y^{3-N} X^{-1} \left( \sin^2 \frac{\theta}{2} \right)^{\frac{3-N}{2}} 2F_1 \left( \frac{N-3}{2}, \frac{5-N}{2}, \frac{N-1}{2}; \cos^2 \frac{\theta}{2} \right). \] (136)

The behavior of \( D(x, y) + D(y, x) \) for \( X \to 0 \), \( y \) and \( \theta \) fixed, is obtained from \( I_1 \) and \( I_2 \) keeping in mind that in such limit \( \phi_x \to 0 \) and \( \phi_y \to \pi - \theta \) thus giving

\[ D(x, y) + D(y, x) \simeq \frac{1}{N-3} \left( \sin^2 \frac{\theta}{2} \right)^{\frac{3-N}{2}} 2F_1 \left( \frac{N-3}{2}, \frac{5-N}{2}, \frac{N-1}{2}; \cos^2 \frac{\theta}{2} \right) (Y^{3-N} X^{-1} - X^{3-N} Y^{-1}). \] (137)

This is also the behavior for \( x \) and \( \theta \) fixed and \( Y \to \infty \). Due to the symmetry of \( D(x, y) + D(y, x) \) we have that for \( Y \to 0 \), \( x \) and \( \theta \) fixed

\[ D(x, y) + D(y, x) \simeq -\frac{1}{N-3} \left( \sin^2 \frac{\theta}{2} \right)^{\frac{3-N}{2}} 2F_1 \left( \frac{N-3}{2}, \frac{5-N}{2}, \frac{N-1}{2}; \cos^2 \frac{\theta}{2} \right) (Y^{3-N} X^{-1} - X^{3-N} Y^{-1}) \] (138)

and also for \( X \to \infty \), \( y \) and \( \theta \) fixed. We see that the behavior for \( X \to \infty \) is just the opposite of that for \( X \to 0 \). Being \( D_{\mu\rho\sigma\alpha} \) a zero degree operator the same holds for the propagator \( G_{\mu\nu}^S(x, y) \).

The addition to the computed propagator of a residual gauge term, i.e. \( \frac{\partial F\nu(x, y)}{\partial x^\mu} \) with

\[ x^\mu \frac{\partial F\nu(x, y)}{\partial x^\mu} = 0 \] (139)

which obviously satisfies the homogenous equation

\[ \left( \Box x^\alpha \delta_{\mu\alpha} - \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\alpha} \right) \frac{\partial F\nu(x, y)}{\partial x^\alpha} = 0 \] (140)
destroys such a opposite behavior. In fact eq. (139) tells us that $F_{\nu}(x, y)$ is a homogeneous function of $x$ of degree zero and as such $\frac{\partial F_{\nu}(x, y)}{\partial x^\mu}$ behaves for $X \to 0$ and $X \to \infty$ like $a(\theta, y)/X$ with the same coefficient $a(\theta, y)$. Thus the imposition of the opposite behavior at the origin and infinity fixes the radial gauge completely.

In the case of gravity the integrals of the text can be written as combinations of

$$\int_0^1 d\rho \frac{\rho^{c-b-2}}{[(\rho x - y)^2]^{\frac{3}{2}}} \tag{141}$$

and

$$\int_0^\infty d\rho \frac{\rho^a}{[(\rho x - y)^2]^{\frac{3}{2}}} \tag{142}$$

with $c = N$ and where $a$ and $b$ take the values 0 or 1; or with $c = N + 2$ where $a$ and $b$ take the values 0,1,2,3. Eq.(142) is treated as (135) [18], while (141) can be written as a combination of hypergeometric functions using [18]

$$\int_0^1 d\rho \frac{\rho^{c-n-2}(1 - \rho)^n}{[(\rho x - y)^2]^{\frac{3}{2}}} = d^{1+n-c}Y^{-1-n} \int_0^\infty d\zeta \frac{\zeta^{c-n-2}}{[\zeta^2 + 2\zeta \cos \phi_x + 1]^{\frac{3}{2}}} = B(n + 1, c - n - 1) \times$$

$$d^{1+n-c}Y^{-1-n} \left(\cos^2 \phi_x \right)^{1-n} \frac{1}{2} F_1 \left(c - 2n - 1, \frac{2n + 3 - c}{2}; \frac{c + 1}{2}; \sin^2 \phi_x \right). \tag{143}$$

With a method similar to that explained above one proves that the behavior of

$$F_{\alpha\beta}^S(x, y) + F_{\beta\alpha}^S(y, x) \tag{144}$$

for $X \to 0$ is of the form

$$f_0(x, y)(X^{-2}Y^{-N+2} - Y^{-2}X^{-N+2}) + g_0(x, y)(X^{-1}Y^{-N+1} - Y^{-1}X^{-N+1}) \tag{145}$$

and for $X \to \infty$

$$- f_0(x, y)(X^{-2}Y^{-N+2} - Y^{-2}X^{-N+2}) - g_0(x, y)(X^{-1}Y^{-N+1} - Y^{-1}X^{-N+1}) \tag{146}$$

where $f_0$ and $g_0$ are homogeneous functions of degree zero in $x$ and $y$. One reaches the propagator by applying to (144) the operator obtained from (82) and (88)

$$D_{\nu,\beta;\sigma,\gamma;\alpha,\lambda} = \frac{1}{4} x^\mu x^\alpha y^\beta y^\lambda \delta_{\mu\nu} \delta_{\sigma,\lambda} \delta_{\alpha,\beta} +$$

$$+ \delta_{\lambda,\gamma} \delta_{\alpha,\beta} \delta_{\nu,\beta} \delta_{\rho,\sigma} \delta_{\sigma,\gamma} \delta_{\alpha,\lambda} \delta_{\mu,\nu} +$$

$$2 \delta_{\lambda,\gamma} \delta_{\alpha,\beta} \delta_{\nu,\beta} \delta_{\rho,\sigma} \delta_{\sigma,\gamma} \delta_{\lambda,\mu} \delta_{\alpha,\lambda} \delta_{\beta,\sigma} \delta_{\sigma,\alpha} \delta_{\gamma,\mu} +$$

$$\frac{1}{2} \delta_{\lambda,\gamma} \delta_{\alpha,\beta} \delta_{\nu,\beta} \delta_{\rho,\sigma} \delta_{\sigma,\gamma} \delta_{\alpha,\lambda} \delta_{\mu,\nu} \tag{147}$$

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Being this an operator of dimension 1 in $x$ and $y$ it generates a propagator behaving for $X \to 0$ like
\[ \tilde{f}_0(x, y)(X^{-1}Y^{-N+3} - Y^{-1}X^{-N+3}) + \tilde{g}_0(x, y)(Y^{-N+2} - X^{-N+2}) \]  
(148)
and the opposite for $X \to \infty$. Similarly to what happens in the Yang-Mills case, addition of a residual gauge term destroys the opposite behavior between the origin and infinity. In fact a residual gauge term has the form
\[ \frac{\partial F_{\nu,\alpha\beta}(x, y)}{\partial x^\mu} + \frac{\partial F_{\mu,\alpha\beta}(x, y)}{\partial x^\nu} \]
(149)
with
\[ x^\mu \frac{\partial F_{\nu,\alpha\beta}(x, y)}{\partial x^\mu} + x^\nu \frac{\partial F_{\mu,\alpha\beta}(x, y)}{\partial x^\nu} = 0. \]
(150)
Multiplying by $x^\nu$ one easily proves that $x^\nu F_{\nu,\alpha\beta}(x, y)$ is a homogeneous function of degree 1 in $x$ and using this fact one shows that
\[ F_{\nu,\alpha\beta}(x, y) = H^0_{\nu,\alpha\beta}(x, y) + H^1_{\nu,\alpha\beta}(x, y) \]
(151)
being $H^0$ and $H^1$ homogenous functions of $x$ of degree 0 and 1. Thus (149) has the same $a(\theta, y)/X + b(\theta, y)$ behavior at the origin and infinity thus violating the opposite behavior of the propagator.

In the main text we computed the propagator in the “sharp” radial gauge both in the first and in the second order formalism. Now we give a general technique to derive the propagator in presence of a radial gauge-fixing term in the lagrangian. As an example we consider the second order formalism with the following gauge-fixed lagrangian
\[ \mathcal{L}(x) = \mathcal{L}^{II}(x) + \frac{1}{2\beta}(x^\mu h_{\mu\nu}(x)x^\nu h^\nu_x(x)). \]
(152)
The sharp case can be recovered in the limit $\beta \to 0$. The addition of the term in $\beta$ modifies the equation for the propagator in the following way
\[ K_{x\mu
u}^{\rho\sigma} G_{\rho\sigma,\alpha\beta}(x, y) + K_{x\mu\nu}^{R\rho\sigma} G_{\rho\sigma,\alpha\beta}(x, y) = \frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha})\delta^N(x - y) \]
(153)
where $K_{x\mu\nu}^{\rho\sigma}$ is the usual kinematical term defined in the subsection 2.C and $K_{x\mu\nu}^{R\rho\sigma}$ is given by
\[ K_{x\mu\nu}^{R\rho\sigma} = \frac{1}{4\beta}(x^\mu x^\rho \delta^\nu_\sigma + x^\nu x^\rho \delta^\mu_\sigma + x^\mu x^\sigma \delta^\rho_\nu + x^\nu x^\sigma \delta^\rho_\mu). \]
(154)
We shall solve this equation writing $G = G^{(1)} + G^{(2)}$ with
\[ K_{x\mu\nu}^{R\rho\sigma} G^{(1)}_{\alpha\beta,\rho\sigma}(x, y) = -P^\dagger \left( \frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha})\delta^N(x - y) \right), \]
(155)
\[
K_{x\mu}^{R\rho\sigma} G^{(2)}_{\alpha\beta,\rho\sigma}(x, y) = -(1 - P^\dagger) \left( \frac{1}{2} (\eta_{\mu\alpha}^l \eta_{\rho\beta}^l + \eta_{\mu\beta}^l \eta_{\rho\alpha}^l) \delta^N(x - y) \right)
\]  
(156)

and at the same time

\[
K_{x\mu}^{R\rho\sigma} G^{(1)}_{\alpha\beta,\rho\sigma}(x, y) = 0,
\]

(157)

\[
K_{x\mu}^{R\rho\sigma} G^{(2)}_{\alpha\beta,\rho\sigma}(x, y) = 0.
\]

We know already a solution of eq.(155) and eq.(157). In fact the “sharp” radial propagators that we found in the paper satisfy both equations. We look now for a solution of the system eq.(156) and eq.(158). If \( G^{(2)} \) is a pure gauge term, i.e.

\[
\frac{\partial}{\partial x^\rho} F_{\nu\alpha\beta}(x, y) + \frac{\partial}{\partial x^\nu} F_{\rho\alpha\beta}(x, y),
\]

(159)

it automatically solves eq.(158). Now substituting (159) into eq.(156) we find the following equation

\[
\frac{1}{4\beta} \left( x_\mu x^\rho \left( \frac{\partial}{\partial x^\rho} F_{\nu\alpha\beta}(x, y) + \frac{\partial}{\partial x^\nu} F_{\rho\alpha\beta}(x, y) \right) + x_\nu x^\rho \left( \frac{\partial}{\partial x^\rho} F_{\rho\alpha\beta}(x, y) + \frac{\partial}{\partial x^\nu} F_{\rho\alpha\beta}(x, y) \right) \right) = -(1 - P^\dagger) \left( \frac{1}{2} (\eta_{\mu\alpha}^l \eta_{\rho\beta}^l + \eta_{\mu\beta}^l \eta_{\rho\alpha}^l) \delta^N(x - y) \right).
\]

(160)

Using the explicit form of the projectors one can see that the tensor structure of the r.h.s. of eq.(160) is the following

\[
x_\mu j_{\nu,\alpha\beta}(x, y) + x_\nu j_{\rho,\alpha\beta}(x, y)
\]

(161)

where \( j_{\mu,\alpha\beta}(x, y) \) changes according to the projector chosen. Thus equating the same tensor structures we obtain

\[
\frac{x^\rho}{4\beta} \left( \frac{\partial}{\partial x^\rho} F_{\nu\alpha\beta}(x, y) + \frac{\partial}{\partial x^\nu} F_{\rho\alpha\beta}(x, y) \right) = j_{\nu,\alpha\beta}(x, y).
\]

(162)

This equation can be solved with the usual technique described in the paper. We notice that the inversion of the dilatator that appears in (162) has to be done taking in account the boundary condition that the propagator must satisfy. In this way the solution is automatically symmetric. For example if we choose the case of \( P^\infty \) the solution is given by

\[
\frac{1}{4\beta} F^{\infty}_{\nu,\alpha\beta}(x, y) = - \int_1^{\infty} \frac{d\alpha}{\alpha^2} j^{\infty}_{\nu,\alpha\beta}(\alpha x, y) - \frac{1}{2} \frac{\partial}{\partial x^\rho} \int_1^{\infty} \frac{d\alpha}{\alpha} \left( 1 - \frac{1}{\alpha} \right) x^\rho j^{\infty}_{\rho,\alpha\beta}(\alpha x, y)
\]

(163)

where

\[
j^{\infty}_{\nu,\alpha\beta}(x, y) = -\frac{1}{2} \eta_{\nu\beta} \frac{\partial}{\partial y^\alpha} \int_1^{\infty} \frac{d\alpha}{\alpha^2} \delta(x - \alpha y) - \frac{1}{2} \eta_{\nu\alpha} \frac{\partial}{\partial y^\beta} \int_1^{\infty} \frac{d\alpha}{\alpha^2} \delta(x - \alpha y) - \frac{1}{2} \frac{\partial}{\partial y^\alpha} \frac{\partial}{\partial y^\beta} \int_1^{\infty} \frac{d\alpha}{\alpha^2} (\alpha - 1) y_c \delta(x - \alpha y).
\]

(164)
The final result is
\[
\frac{1}{4\beta} F_{\nu, \alpha \beta}^\infty (x, y) = \frac{1}{2} \eta_{\nu \beta} \frac{\partial}{\partial y^{\alpha}} \int_1^\infty \frac{d\alpha}{\alpha^2} \int_1^\infty \frac{d\lambda}{\lambda^2} \delta(\lambda x - \alpha y) + \frac{1}{2} \eta_{\nu \alpha} \frac{\partial}{\partial y^{\beta}} \int_1^\infty \frac{d\alpha}{\alpha^2} \int_1^\infty \frac{d\lambda}{\lambda^2} \delta(\lambda x - \alpha y)
\]
\[+ \frac{1}{2} \frac{\partial}{\partial y^{\alpha}} \frac{\partial}{\partial y^{\beta}} \int_1^\infty \frac{d\alpha}{\alpha^2} \int_1^\infty \frac{d\lambda}{\lambda^2} (\alpha - 1) \delta(\lambda x - \alpha y) + \]
\[+ \frac{1}{4} \frac{\partial}{\partial x^{\nu}} \left( x^{\beta} \frac{\partial}{\partial y^{\alpha}} \int_1^\infty \frac{d\alpha}{\alpha^2} \int_1^\infty \frac{d\lambda}{\lambda^2} (\lambda - 1) \delta(\lambda x - \alpha y) \right) \]
\[+ x^{\alpha} \frac{\partial}{\partial y^{\beta}} \int_1^\infty \frac{d\alpha}{\alpha^2} \int_1^\infty \frac{d\lambda}{\lambda^2} (\lambda - 1) \delta(\lambda x - \alpha y) \]
\[- \frac{1}{4} \frac{\partial}{\partial x^{\nu}} \frac{\partial}{\partial y^{\alpha}} \frac{\partial}{\partial y^{\beta}} \int_1^\infty \frac{d\alpha}{\alpha^2} \int_1^\infty \frac{d\lambda}{\lambda^2} (\lambda - 1)(\alpha - 1)(x \cdot y) \delta(\lambda x - \alpha y) \quad (165)\]

We have not performed the integrals to enlighten the symmetry of the result between \(x\) and \(y\) when one substitutes eq. (165) into eq. (159). The complete integration of the previous expression can be executed with the help of the general formula given at the end of section 4. The case \(P^S\) can be treated similarly. The generalization to the first order formalism is trivial. For Yang-Mills theory see [12].

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Figure Captions

Fig.1: Wilson loop restricting the behavior of the connection at the origin and infinity