JET SCHEMES OF DETERMINANTAL VARIETIES

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Abstract. This article studies the scheme structure of the jet schemes of determinantal varieties. We show that in general, these jet schemes are not irreducible. In the case of the determinantal variety $X$ of $r \times s$ matrices of rank at most one, we give a formula for the dimension of each of the components of its jet schemes. As an application, we compute the log canonical threshold of the pair $(\mathbb{A}^{rs}, X)$.

1. Introduction

Let $X$ be a scheme of finite type over an algebraically closed field $k$ of characteristic zero. An arc on an algebraic variety $X$ is an “infinitesimal curve” on it. Formally, this is a morphism, defined over $k$, from the curve germ scheme Spec $k[[t]]$ into $X$. The set of all arcs of $X$ carries the structure of a scheme, called the arc space of $X$ and denoted by $J_\infty(X)$.

An $m$-jet on $X$ is a truncated arc on $X$, that is, a $k$-morphism

$$\text{Spec } k[t]/(t^{m+1}) \to X.$$  

The set of all $m$-jets on $X$ also forms a scheme in a natural way. This is the $m$th jet scheme $J_m(X)$.

The surjection $k[t]/(t^{m+1}) \twoheadrightarrow k[t]/(t^m)$ induces a morphism $\pi_m^{m-1} : J_m(X) \to J_{m-1}(X)$, and composition gives a morphism $\pi_m : J_m(X) \to J_0(X) = X$. Taking the inverse limit of these jet schemes gives the arc space $J_\infty(X)$.

In [5], Mustață proved that a locally complete intersection variety has rational singularities if and only if all its jet schemes are irreducible. Combining this result with the fact that determinantal varieties have rational singularities, we know that a determinantal variety of singular square matrices is a hypersurface and so all its jet schemes are irreducible. Though determinantal varieties always have rational singularities, they are rarely complete intersections, so Mustață’s theorem is not applicable to more general determinantal varieties. This leads to the natural question: Are the jet schemes of all determinantal varieties irreducible?

In this paper, we will see that the answer to this question is no. In particular, we will show that the second jet scheme and all the odd jet schemes of essentially all determinantal varieties are reducible. In the special case of the variety of matrices

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of rank at most one, we will give a formula for the number of irreducible components of its jet schemes and their dimensions.

2. BACKGROUND AND NOTATION

Let \( X_c \subseteq \mathbb{A}^{rs} \) be the determinantal variety of \( r \times s \) matrices of rank at most \( c \), that is, \( X_c \) is defined by \( I_{c+1} \), the ideal of \((c + 1)\)-minors\(^1\) of a generic \( r \times s \) matrix \((x_{ij})\). So an \( m \)-jet of \( X_c \) corresponds to a \( k \)-algebra homomorphism

\[
\phi : k[x_{ij}] \to k[t]/(t^{m+1})
\]

\[
\phi(x_{ij}) = x_{ij}^{(0)} + x_{ij}^{(1)}t + \ldots + x_{ij}^{(m)}t^m,
\]

subject to the condition \( \phi(I_{c+1}) = 0 \).

For a generator \( \Delta \in I_{c+1} \), we write

\[
\phi(\Delta) = \Delta^{(0)} + \Delta^{(1)}t + \ldots + \Delta^{(m)}t^m
\]

where \( \Delta^{(l)} \in k[x_{ij}^{(k)}] \). These polynomials \( \Delta^{(l)} \) can be obtained by calculating the corresponding \((c + 1)\)-minors of the matrix whose entries are \( x_{ij}^{(0)} + x_{ij}^{(1)}t + \ldots + x_{ij}^{(m)}t^m \), and extracting the coefficients of each power of \( t \). Therefore, \( \mathcal{J}_m(X_c) \) is the closed subscheme of \( \mathcal{J}_m(\mathbb{A}^{rs}) = \text{Spec} \ k[x_{ij}^{(k)}] \) defined by the ideal \((\Delta^{(0)}, \Delta^{(1)}, \ldots, \Delta^{(m)} : \Delta)^r \) are \((c + 1)\)-minors of \((x_{ij})\).

Before we present the results, let us outline the general strategy.

A general irreducible scheme \( X \) over a field \( k \) is the disjoint union of its singular locus \( X^{\text{sing}} \) and its smooth locus \( X^{\text{reg}} \). So to understand the jet scheme \( \mathcal{J}_m(X) \), we can study the preimage of these two loci under the natural projection \( \pi_m : \mathcal{J}_m(X) \to X \). Since \( X^{\text{reg}} \) is smooth and irreducible, \( \pi_m \) is an affine bundle over \( X^{\text{reg}} \), and therefore \( \pi_m^{-1}(X^{\text{reg}}) \) is an irreducible component of \( \mathcal{J}_m(X) \) of dimension \((m + 1)\dim X \). On the other hand, \( \pi_m^{-1}(X^{\text{sing}}) \) is a closed subset of \( \mathcal{J}_m(X) \). So it will contribute components to \( \mathcal{J}_m(X) \) except when it is contained in \( \pi_m^{-1}(X^{\text{reg}}) \) (as is the case when \( X \) is a local complete intersection with rational singularities). Therefore, if we want to show \( \mathcal{J}_m(X) \) is reducible, it suffices to show \( \dim \pi_m^{-1}(X^{\text{sing}}) \geq \dim \pi_m^{-1}(X^{\text{reg}}) \).

**Remark 2.1.** There are examples when \( \mathcal{J}_m(X) \) is reducible even though \( \dim \pi_m^{-1}(X^{\text{sing}}) < \dim \pi_m^{-1}(X^{\text{reg}}) \), see Remark [SM].

3. ODD JET SCHEMES ARE REDUCIBLE

Our first result says that all the odd jet schemes of essentially all determinantal varieties are reducible. More precisely,

**Theorem 3.1.** Let \( X_c \) be the determinantal variety of \( r \times s \) matrices of rank at most \( c, \ c \geq 1 \). If \( r, s \geq c + 2 \) and \( r + s \geq 2c + 5 \), then the jet scheme \( \mathcal{J}_m(X_c) \) is reducible for \( m \) odd.

To prove this theorem, we need the following general result:

\(^1\)We define a \( d \)-minor of a matrix to be a \( d \times d \) subdeterminant of that matrix.
Lemma 3.2. Let $X$ be a smooth scheme and $Z \subseteq Y \subseteq X$ so that $Y$ is closed in $X$ and $Z$ is closed in $Y$. Then $\dim \pi_{mp}^{-1}(Z) \geq p \dim \pi_{m-1}^{-1}(Z)$ for all $m, p \geq 1$, where $\pi_i : J_i(Y) \to Y$ are the natural projections as described in the Introduction.

To prove this lemma, we will need a general fact (Proposition 3.7 below) about contact loci whose statement and proof were provided by Mustata. But first let us look at some basics of contact loci (see for example [1, §2.4 and §5]).

Definition 3.3. Let $Y \subseteq X$ be a closed subscheme of a smooth scheme $X$ defined by the sheaf of ideals $I_Y$. We define the function

$$\text{ord}_Y : J_\infty(X) \to \mathbb{N} \cup \{\infty\}$$

sending an arc $\gamma : \mathcal{O}_X \to k[[t]]$ of $X$ to the order of vanishing of $\gamma$ along $Y$, that is, $\text{ord}_Y(\gamma)$ is the integer $e$ such that the ideal $\gamma(I_Y) \subseteq k[[t]]$ is exactly the ideal $(t^e)$.

Definition 3.4. With $Y \subseteq X$ as above, the contact locus $\text{Cont}_\geq(Y)$ is the set of arcs of $X$ whose order of vanishing along $Y$ is at least $m$. In other words,

$$\text{Cont}_\geq(Y) = \{\gamma \in J_\infty(X) | \text{ord}_Y(\gamma) \geq m\} = \mu_{m-1}^{-1}(J_{m-1}(Y))$$

where $\mu_{m-1} : J_\infty(X) \to J_{m-1}(X)$ is the natural projection. In this case, we define

$$\text{codim}(J_\infty(X), \text{Cont}_\geq(Y)) := \text{codim}(J_{m-1}(X), \mu_{m-1}(\text{Cont}_\geq(Y)))$$

$$= \text{codim}(J_{m-1}(X), J_{m-1}(Y)).$$

We can also define

$$\text{Cont}_m(Y) = \text{Cont}_\geq(Y) \setminus \text{Cont}_\geq(Y)^{m+1}$$

$$= \mu_{m-1}^{-1}(J_{m-1}(Y)) \setminus \mu_{m-1}(J_{m}(Y)),$$

and $\text{codim}(J_\infty(X), \text{Cont}_m(Y)) = \text{codim}(J_\infty(X), \text{Cont}_\geq(Y))$, because $\text{Cont}_m(Y)$ is a non-empty open subset of $\text{Cont}_\geq(Y)$.

To understand the statement of Proposition 3.7, we also need to recall the definition of a log resolution.

Definition 3.5. Let $X$ be a smooth variety over a field $k$ of characteristic zero, and $Y \subseteq X$ a closed subscheme. A map $f : \tilde{X} \to X$ is called a log resolution of the pair $(X, Y)$ if $f$ is proper birational such that

1. $\tilde{X}$ is smooth and
2. writing $F := f^{-1}(Y) = \sum_{i=1}^r a_i D_i$ and $K_{\tilde{X}/X} = \sum_{i=1}^r k_i D_i$ for some $a_i, k_i \in \mathbb{Q}$ and prime divisors $D_i$, the divisors $F$, $K_{\tilde{X}/X}$ and $F + K_{\tilde{X}/X}$ have simple normal crossing support.

Remark 3.6. The existence of log resolutions follows from Hironaka’s resolution of singularities [3]. Note also that one can find a simultaneous log resolution for any finite set $Y_1, \ldots, Y_r$ of closed subschemes of $X$. 
Proposition 3.7. Let \( \{Y_i\}_{i=1}^{r} \) be a collection of closed subschemes of a smooth scheme \( X \). Let \( f : X' \to X \) be a log resolution of \( Y_1, \ldots, Y_r \), and denote
\[
f^{-1}(Y_i) = \sum_{j=1}^{n} a_{ij} E_j \text{ and } K_{X'/X} = \sum_{j=1}^{n} k_j E_j,
\]
where \( \sum E_j \) is a simple normal crossing divisor. Then
\[
\text{codim} \left( \bigcap_{i=1}^{r} \text{Cont}^{\geq m_i}(Y_i) \right) = \min_{\nu \in \mathbb{N}^n} \left\{ \sum_{j=1}^{n} \nu_j (k_j + 1) \bigg| \bigcap_{j \geq 1} E_j \neq \emptyset \text{ and } \sum_{j=1}^{n} a_{ij} \nu_j \geq m_i \forall i \right\}.
\]
Proof. Let \( C = \bigcap_{i=1}^{r} \text{Cont}^{\geq m_i}(Y_i) \) and \( f_\infty : \mathcal{J}_\infty(X') \to \mathcal{J}_\infty(X) \) be the induced map on arc spaces. Then
\[
f^{-1}_\infty(C) = \{ \gamma \in \mathcal{J}_\infty(X') | f_\infty(\gamma) \in C \}
= \{ \gamma \in \mathcal{J}_\infty(X') | \text{ord}_{Y_i}(f_\infty(\gamma)) \geq m_i \text{ for } 1 \leq i \leq r \}
= \{ \gamma \in \mathcal{J}_\infty(X') | \sum_{j} a_{ij} \text{ord}_{E_j}(\gamma) \geq m_i \text{ for } 1 \leq i \leq r \}.
\]
Now for an arc \( \gamma \) of \( X' \) with order of vanishing \( \nu_j \) along \( E_j \) for all \( j \), \( \gamma \in f^{-1}_\infty(C) \) if and only if \( \sum_{j=1}^{n} a_{ij} \nu_j \geq m_i \) for \( 1 \leq i \leq r \). So we have
\[
f^{-1}_\infty(C) = \bigcup_{\nu} \left( \bigcap_{j=1}^{n} \text{Cont}^{\nu_j}(E_j) \right)
\]
where the disjoint union is taken over all \( \nu \in \mathbb{N}^n \) such that \( \sum_{j} a_{ij} \nu_j \geq m_i \) for all \( i = 1, \ldots, r \). This implies
\[
C = \bigcup_{\nu} f_\infty \left( \bigcap_{j=1}^{n} \text{Cont}^{\nu_j}(E_j) \right),
\]
and thus
\[
\text{codim } C = \min_{\nu} \left\{ \text{codim } f_\infty \left( \bigcap_{j=1}^{n} \text{Cont}^{\nu_j}(E_j) \right) \right\}
\]
where the minimum is taken over all \( \nu \in \mathbb{N}^n \) such that \( \bigcap_{j \geq 1} E_j \neq \emptyset \) and \( \sum_{j} a_{ij} \nu_j \geq m_i \) for all \( i = 1, \ldots, r \). By [2] Theorem 2.1,
\[
\text{codim } f_\infty \left( \bigcap_{j=1}^{n} \text{Cont}^{\nu_j}(E_j) \right) = \sum_{j=1}^{n} \nu_j (k_j + 1).
\]
As a result,
\[
\text{codim } C = \min_{\nu \in \mathbb{N}^n} \left\{ \sum_{j=1}^{n} \nu_j (k_j + 1) \bigg| \bigcap_{j \geq 1} E_j \neq \emptyset \text{ and } \sum_{j=1}^{n} a_{ij} \nu_j \geq m_i \forall i \right\}.
\]
\[\square\]
Proof of Lemma 3.2. Taking $Y_1 = Y$, $Y_2 = Z$, $m_1 = m$ and $m_2 = 1$ in Proposition 3.1, we have

$$\text{codim} \left( \text{Cont}^{\geq m}(Y) \cap \mu^{-1}(Z) \right) = \min_{\nu \in \mathbb{N}^n} \left\{ \sum_{j=1}^n \nu_j(k_j + 1) \left| \bigcap_{\nu_j \geq 1} E_j \neq \emptyset \right. \right. \left. \left. + \sum_{j=1}^n a_j \nu_j \geq m \right\},$$

where $\mu : J_\infty(Y) \to Y$ is the natural projection. Say $\nu \in \mathbb{N}^n$ is the $n$-tuple achieving the minimum value above. Then the $n$-tuple $p\nu$ satisfies the conditions $\bigcap_{\nu_j \geq 1} E_j \neq \emptyset$ and $\sum_{j=1}^n a_j (p\nu_j) \geq mp$. So we obtain

$$(\dagger) \quad \text{codim} \left( \text{Cont}^{\geq mp}(Y) \cap \mu^{-1}(Z) \right) \leq p \cdot \text{codim} \left( \text{Cont}^{\geq m}(Y) \cap \pi^{-1}(Z) \right).$$

On the other hand,

$$\text{codim} \left( \text{Cont}^{\geq m}(Y) \cap \pi^{-1}(Z) \right) = \text{codim}(J_{m-1}(X), J_{m-1}(Y) \cap \pi_{m-1}^{-1}(Z)) = \dim J_{m-1}(X) - \dim \pi_{m-1}^{-1}(Z) = m \dim X - \dim \pi_{m-1}^{-1}(Z).$$

Therefore, the inequality $(\dagger)$ gives us

$$mp \dim X - \dim \pi_{mp-1}^{-1}(Z) \leq p(m \dim X - \dim \pi_{m-1}^{-1}(Z)),$$

or equivalently, $\dim \pi_{mp-1}^{-1}(Z) \geq p \dim \pi_{m-1}^{-1}(Z)$, as desired. \[\square\]

Proof of Theorem 3.7. Because $m$ is odd, we can write $m = 2p - 1$ for some $p \geq 1$. According to our discussion prior to Remark 2.1, to show that $J_m(X_c)$ is reducible, it suffices to show $\dim \pi_{m-1}^{-1}(X_c^{\text{sing}}) \geq \dim \pi_{m-1}^{-1}(X_c^{\text{reg}})$. By Lemma 3.2, we have $\dim \pi_{m-1}^{-1}(X_c^{\text{sing}}) \geq p \dim \pi_{1}^{-1}(X_c^{\text{sing}})$. So we need to compute the dimension of $\pi_{1}^{-1}(X_c^{\text{sing}})$.

Recall that $X_c^{\text{sing}}$ is the set of all $r \times s$ matrices of rank at most $c - 1$, and hence is the subvariety of $X_c$ defined by the $c$-minors of the generic matrix $(x_{ij})$. Let $B = (x_{ij}^{(0)})$ be an $r \times s$ matrix of indeterminates. Then

$$\pi_{1}^{-1}(X_c^{\text{sing}}) = \text{Spec} k[x_{ij}^{(0)}, x_{ij}^{(1)}]/(J_1(X_c) + (c\text{-minors of } B))$$

$$= \text{Spec} k[x_{ij}^{(0)}, x_{ij}^{(1)}]/(c\text{-minors of } B),$$

since $J_1(X_c) \subseteq (c\text{-minors of } B)$. So $\pi_{1}^{-1}(X_c^{\text{sing}})$ has dimension $rs + (c-1)(r+s-c+1)$.

Since $X_c$ has dimension $c(r+s-c)$, the component of $J_m(X_c)$ over the smooth part of $X_c$ has dimension $(m+1) \dim X_c$, which is $2pc(r+s-c)$. Using our hypotheses that $r, s \geq c + 2$ and $r + s \geq 2c + 5$, we see that $(r-c-1)(s-c-1) \geq 2$, or equivalently,

$$rs + (c-1)(r+s-c+1) \geq 2c(r+s-c).$$

Therefore, $\dim \pi_{m}^{-1}(X_c^{\text{sing}}) \geq \dim \pi_{m}^{-1}(X_c^{\text{reg}})$. Thus, the preimage of the singular locus of $X_c$ in $J_m(X_c)$ has dimension at least as large as the dimension of $J_m(X_c)$
over the generic point of $X_c$. As explained in the paragraph prior to Remark 2.1, therefore, $\mathcal{J}_m(X_c)$ is not irreducible.

**Remark 3.8.** The argument above tells us that

$$\dim \mathcal{J}_1(X_c) = rs + (c - 1)(r + s - c + 1)$$

provided that $c \geq 1$, $r, s \geq c + 2$ and $r + s \geq 2c + 5$.

**Remark 3.9.** Although our dimension analysis draws no conclusion to the irreducibility of odd jet schemes of $X_c$ when $c \geq 1$ and $r = s = c + 2$, there are examples demonstrating that $\mathcal{J}_m(X_c)$ is still reducible. For example, take $X$ to be the variety of $3 \times 3$ matrices of rank at most one. Then the proof of Theorem 3.1 tells us that

$$\dim \pi_1^{-1}(X^{\text{sing}}) = 9 < 10 = \dim \pi_1^{-1}(X^{\text{reg}}).$$

However, a Macaulay calculation says the ideal $J_1(X) \subset k[x_{ij}^{(0)}, x_{ij}^{(1)}]$ for $1 \leq i, j \leq 3$, has two minimal primes:

$$J_1(X) + \left( x_{11}^{(1)} x_{22}^{(1)} x_{33}^{(1)} - x_{11}^{(1)} x_{33}^{(1)} x_{22}^{(1)} - x_{12}^{(1)} x_{21}^{(1)} x_{33}^{(1)} + x_{12}^{(1)} x_{33}^{(1)} x_{21}^{(1)} + x_{13}^{(1)} x_{21}^{(1)} x_{32}^{(1)} - x_{13}^{(1)} x_{32}^{(1)} x_{21}^{(1)} \right)$$

and

$$\left( x_{11}^{(0)}, x_{12}^{(0)}, x_{13}^{(0)}, x_{21}^{(0)}, x_{22}^{(0)}, x_{23}^{(0)}, x_{31}^{(0)}, x_{32}^{(0)}, x_{33}^{(0)} \right).$$

That is, the jet scheme $\mathcal{J}_1(X)$ has two irreducible components and is therefore reducible.

### 4. Second Jet Scheme is Reducible

In this section, we investigate the second jet scheme of a determinantal variety.

We denote by $A_k$ the $r \times s$ matrix with a $k \times k$ identity submatrix in the upper left corner and zero entries everywhere else.

**Theorem 4.1.** Let $X_c$ be the variety of $r \times s$ matrices of rank at most $c$, $c \geq 2$. Suppose $r, s \geq c + 2$ and $r + s \geq 2c + 6$. Then $\mathcal{J}_2(X_c)$ is reducible.

**Proof.** As outlined in the paragraph prior to Remark 2.1, our goal is to show that

$$\dim \pi_2^{-1}(X_c^{\text{sing}}) \geq \dim \pi_2^{-1}(X_c^{\text{reg}}).$$

First notice that the singular locus of $X_c$ can be stratified, according to the rank of the singular points. So to understand $\pi_2^{-1}(X_c^{\text{sing}})$, we can study the preimage of matrices of a fixed rank. Notice that the group $GL(r) \times GL(s)$ acts transitively on matrices of a fixed rank. So when we consider the fiber of $\pi_2$ over a singular point of rank $k < c$, we may pick the representative $A_k$. Now we describe the fiber over $A_k$.

The ideal $J_2(X_c)$ is homogeneous. Its generators are the coefficients of $t^0$, $t^1$ and $t^2$ in the $(c + 1)$-minors of the $r \times s$ matrix

$$\left( x_{ij}^{(0)} + x_{ij}^{(1)} t + x_{ij}^{(2)} t^2 \right)_{1 \leq i \leq r, 1 \leq j \leq s}.$$
Thus, every generator has degree \( c + 1 \) and every term of each of its generators has at least \( c - 1 \) variables of the form \( x_{ij}^{(0)} \)'s. So over the singular point \( A_k \), for \( k \leq c - 2 \), we have
\[
\pi_2^{-1}(A_k) = \mathcal{J}_2(X_c) \times \text{Spec } k[x_{ij}^{(0)}]/m_{A_k}
\]
where \( m_{A_k} \) is the maximal ideal of the point \( A_k \in X_{c-2} \). Since
\[
m_{A_k} = (x_{pp}^{(0)} - 1 : 1 \leq p \leq k, x_{mn}^{(0)} : m = n > k \text{ or } m \neq n),
\]
we have
\[
k[x_{ij}^{(0)}, x_{ij}^{(1)}, x_{ij}^{(2)}]/(J_2(X_c) + (x_{pp}^{(0)} - 1 : 1 \leq p \leq k, x_{mn}^{(0)} : m = n > k \text{ or } m \neq n))
\]
\[
\cong k[x_{ij}^{(0)}, x_{ij}^{(1)}, x_{ij}^{(2)}]/(x_{pp}^{(0)} - 1 : 1 \leq p \leq k, x_{mn}^{(0)} : m = n > k \text{ or } m \neq n).
\]
This means that the fiber over any singular point of rank at most \( c - 2 \) is isomorphic to \( \mathbb{A}^{2rs} \), and therefore
\[
\dim \pi_2^{-1}(X_{c-2}) = 2rs + (c - 2)(r + s - c + 2).
\]
Over the singular point \( A_{c-1} \), a surviving term of a generator of \( J_2(X_c) \) has the form
\[
x_{11}^{(0)}x_{22}^{(0)} \cdots x_{c-1,c-1}^{(0)}x_{ij}^{(1)}x_{kl}^{(1)}
\]
where \( x_{ij}^{(1)} \) and \( x_{kl}^{(1)} \) are two distinct entries in the lower right \((r - c + 1) \times (s - c + 1)\) submatrix of the \( r \times s \) matrix \((x_{pq}^{(1)})\). So over \( A_{c-1} \), we obtain
\[
k[x_{ij}^{(0)}, x_{ij}^{(1)}, x_{ij}^{(2)}]/(J_2(X_c) + (x_{pp}^{(0)} - 1 : 1 \leq p \leq c - 1, x_{mn}^{(0)} : m = n \geq c \text{ or } m \neq n))
\]
\[
\cong k[x_{ij}^{(1)}, x_{ij}^{(2)}]/(2\text{-minors of the matrix } (x_{kl}^{(1)})_{1 \leq i, j \leq r - c + 1}).
\]
This implies that the fiber over \( A_{c-1} \) has dimension
\[
(r + s - 2c + 1) + (rs - (r - c + 1)(s - c + 1)) + rs.
\]
As a result, the preimage of the set of rank \( c - 1 \) matrices under the map \( \pi_2 \) has dimension
\[
r + s - 2c + 1 + 2rs - (r - c + 1)(s - c + 1) + (c - 1)(r + s - c + 1).
\]
Now, to compare
\[
\dim \pi_2^{-1}(X_c^{\text{sing}}) = \max\{\dim \pi_2^{-1}(X_{c-2}), \dim \pi_2^{-1}(X_{c-1} \setminus X_{c-2})\}
\]
and
\[
\dim \pi_2^{-1}(X_c^{\text{reg}}) = 3c(r + s - c),
\]
we observe that
\[
\dim \pi_2^{-1}(X_{c-2}) \geq \dim \pi_2^{-1}(X_c^{\text{reg}}) \text{ if and only if } (r - c - 1)(s - c - 1) \geq 3
\]
and
\[
\dim \pi_2^{-1}(X_{c-1} \setminus X_{c-2}) \geq \dim \pi_2^{-1}(X_c^{\text{reg}}) \text{ if and only if } (r - c - 1)(s - c - 1) \geq 2.
\]
But our hypotheses \( r, s \geq c + 2 \) and \( r + s \geq 2c + 6 \) are equivalent to the condition 
\((r - c - 1)(s - c - 1) \geq 3\). This completes the proof that \( J_2(X_c) \) is reducible with 
our assumptions on \( r, s \) and \( c \).

\[ \square \]

5. Varieties of Matrices of Rank at Most One

While the analysis on the scheme structure of the jet schemes of a general determinantal variety 
remains incomplete, the case when \( t = 1 \) is much better understood. This is largely due to the 
fact that the singular locus of this type of determinantal varieties is an isolated origin and that we have 
a very nice description of the preimage of this singular set under the map \( \pi_m \).

Mustaţă showed that the higher jet schemes of the determinantal variety of \( 2 \times n \) matrices of 
rank at most one are all irreducible [5, Example 4.7]. However, this result does not hold for larger matrices. 
In fact, we have a complete understanding of the number of components of the jet schemes in this case, and a 
formula for the dimension of each of the components.

**Theorem 5.1.** Let \( X \) be the variety of \( r \times s \) matrices of rank at most one. Assume 
\( r > s \geq 3 \). Then \( J_m(X) \) has precisely \( \lfloor \frac{m+1}{2} \rfloor + 1 \) irreducible components and these 
components have dimensions \( qrs + (m + 1 - 2q) \dim X \) where \( q = 0, \ldots, \lfloor \frac{m+1}{2} \rfloor \). In 
particular, the dimension of \( J_m(X) \) is \( = \lfloor \frac{m+1}{2} \rfloor rs + (m \mod 2) \dim X \).

As preparation for the proof of this theorem, let us first examine the preimage of the origin under the natural 
projection \( \pi_m : J_m(X) \to X \) in a slightly more general context.

**Proposition 5.2.** Let \( X \subseteq \mathbb{A}^n \) be a closed subscheme defined over \( k \) by a set of 
homogeneous polynomials, all of the same degree \( d \). Let \( \pi_m : J_m(X) \to X \subseteq \mathbb{A}^n \) be 
the natural surjection. Then for \( m \geq d \geq 2 \), 
\[ \pi_m^{-1}(0) \cong J_{m-d}(X) \times \mathbb{A}^{n(d-1)}. \]

**Proof.** Let \( I \) be the defining ideal of \( X \). Then an \( m \)-jet of \( X \) corresponds to a ring homomorphism 
\[ k[x_1, \ldots, x_n]/I \to k[t]/(t^{m+1}) \] 
\[ x_i \mapsto x_i^{(0)} + x_i^{(1)} t + \ldots + x_i^{(m)} t^m \]
where \( x_i^{(j)} \in k \) are arbitrary. This \( m \)-jet lies in \( \pi_m^{-1}(0) \) if and only if \( x_1^{(0)} = \ldots = x_n^{(0)} = 0 \). Thus such a map of rings gives a well-defined \( m \)-jet centered at the origin of \( X \) if and only if we have 
\[ f(t(x_1^{(1)} + \ldots + x_1^{(m)} t^{m-1}), \ldots, t(x_n^{(1)} + \ldots + x_n^{(m)} t^{m-1})) \in (t^{m+1}) \]
for each generator \( f \) of \( I \). Since \( I \) is generated by homogeneous degree \( d \) elements, this is equivalent to 
\[ f(x_1^{(1)} + \ldots + x_1^{(m)} t^{m-1}, \ldots, x_n^{(1)} + \ldots + x_n^{(m)} t^{m-1}) \in (t^{m+1-d}). \]
But this is the same as saying that the ring map
\[ k[x_1, \ldots, x_n]/I \to k[t]/(t^{m-d+1}) \]
\[ x_i \mapsto x_i^{(1)} + x_i^{(2)} t + \ldots + x_i^{(m-d+1)} t^{m-d} \]
is an \((m-d)\)-jet of \(X\). Since there are no constraints on the variables \(x_i^{(m-d+2)}, \ldots, x_i^{(m)}\) for all \(i = 1, \ldots, n\), we see that \(\pi_m^{-1}(0) \cong \mathbb{A}^{n(d-1)}\).

\[ \square \]

**Proof of Theorem 5.1.** We will proceed by induction on \(m\). Because we will use Proposition 5.2 to relate \(J_m\) to \(J_{m-2}\), we will need base cases for \(m = 0\) and \(m = 1\). If \(m = 0\), then \(\left\lfloor \frac{m+1}{2} \right\rfloor = 0\) and \(J_0(X) \cong X\). The theorem predicts one component of dimension same as that of \(X\), which is obvious. If \(m = 1\), the closed subset \(\pi_1^{-1}(X^{reg})\) is an irreducible component of \(J_1(X)\) and has dimension twice that of \(X\). Note that the singular locus of \(X\) is simply the origin, and
\[ k[x_{ij}^{(k)}]/(J_1(X) + (x_{ij}^{(0)})) \cong k[x_{ij}^{(1)}], \]
as is easy to see that \(J_1 \subseteq (x_{ij}^{(0)} : 1 \leq i \leq r, 1 \leq j \leq s)\). So \(\pi_1^{-1}(0) \cong \text{Spec} \mathbb{A}^{rs}\). The condition \(r > s \geq 3\) is equivalent to
\[ (*) \quad rs \geq 2 \dim X. \]
Thus, \(J_1(X)\) has two components: \(\pi_1^{-1}(X^{reg})\) of dimension \(2 \dim X\) and \(\pi_1^{-1}(X^{sing})\) of dimension \(rs\), as predicted by the theorem. This completes the \(m = 1\) base case.

For general \(m\), the closed subset \(\pi_m^{-1}(X^{reg})\) is irreducible and has dimension \((m+1)\dim X\). On the other hand, Proposition 5.2 tells us \(\pi_m^{-1}(0) \cong J_{m-2}(X) \times \mathbb{A}^{rs}\). So by induction, \(\pi_m^{-1}(0)\) has \(\left\lfloor \frac{m-1}{2} \right\rfloor + 1\) components, where the \(q^{th}\) component has dimension
\[ qrs + (m-1-2q) \dim X + rs \quad \text{for } q = 0, \ldots, \left\lfloor \frac{m-1}{2} \right\rfloor. \]
Because \(\dim X = r + s - 1\), it follows that \(\pi_m^{-1}(0)\) has \(\left\lfloor \frac{m+1}{2} \right\rfloor\) components, where the \(q^{th}\) component has dimension
\[ qrs + (m+1-2q) \dim X \quad \text{where } q = 1, \ldots, \left\lfloor \frac{m+1}{2} \right\rfloor. \]
Notice that by our assumptions on \(r\) and \(s\), the minimum value is \(rs + (m-1) \dim X\), which is greater than or equal to the dimension of the component of \(J_m(X)\) over the smooth part by \((*)\). Therefore, each of these \(\left\lfloor \frac{m+1}{2} \right\rfloor\) components of \(\pi_m^{-1}(0)\) is a component of \(J_m(X)\). Thus, \(J_m(X)\) has \(\left\lfloor \frac{m+1}{2} \right\rfloor + 1\) components of dimensions \(qrs + (m+1-2q) \dim X\) where \(q\) ranges from 0 to \(\left\lfloor \frac{m+1}{2} \right\rfloor\).

\[ \square \]

**Corollary 5.3.** With the same assumptions as in Theorem 5.1, the log canonical threshold of the pair \((\mathbb{A}^{rs}, X)\) is exactly \(\frac{1}{2}rs\).
Proof. By applying a result of Mustaţă [6, Corollary 0.2], we obtain
\[
lct(A^{rs}, X) = \dim A^{rs} - \sup_{m \geq 0} \frac{\dim J_m(X)}{m + 1}
\]
\[
= rs - \sup_{m \geq 0} \frac{\left\lfloor \frac{m+1}{2} \right\rfloor rs + (m \mod 2) \dim X}{m + 1}
\]
\[
= rs - \sup_{m \geq 0} \frac{\left\lfloor \frac{m+1}{2} \right\rfloor rs}{m + 1}
\]
\[
= rs - \frac{1}{2} rs
\]
\[
= \frac{1}{2} rs.
\]
\[\square\]

Remark 5.4. In her thesis [4], Johnson produced log canonical thresholds of other determinantal varieties by direct calculations of their log resolutions.

Remark 5.5. The previous three theorems give a host of examples of Gorenstein varieties with rational singularities whose jet schemes are not irreducible. In particular, they illustrate that Mustaţa’s result that locally complete intersection varieties have rational singularities if and only if their jet schemes are irreducible cannot be weakened: we may not replace the local complete intersection hypothesis with a Gorenstein hypothesis. For example, taking \( r = s = c + 3 \geq 4 \) in Theorem 3.1 gives a rationally singular Gorenstein variety whose odd jet schemes are not irreducible. Mustaţa himself also gave an example of a toric variety to illustrate this fact [5, Example 4.6].

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