FINITE GROUPS WITH SOME RESTRICTION ON THE VANISHING SET

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Abstract. Let $x$ be an element of a finite group $G$ and denote the order of $x$ by $\text{ord}(x)$. We consider a finite group $G$ such that $\gcd(\text{ord}(x), \text{ord}(y)) \leq 2$ for any two vanishing elements $x$ and $y$ contained in distinct conjugacy classes. We show that such a group $G$ is solvable. When $G$ with the property above is supersolvable, we show that $G$ has a normal metabelian 2-complement.

1. Introduction

Let $G$ be a finite group. An element $g \in G$ is a vanishing element if there exists an irreducible character $\chi$ of $G$ such that $\chi(g) = 0$. The set of all vanishing elements of $G$ is denoted by $\text{Van}(G)$. A classical theorem of Burnside [15, Theorem 3.15] implies that $\text{Van}(G)$ is non-empty when $G$ is non-abelian. Note that $\text{Van}(G) = \bigcup_{i=1}^{r} vC_i$, where each $vC_i$ is a vanishing conjugacy class. We denote the order of the elements contained in a vanishing conjugacy class $vC_i$ by $\text{ord}(vC_i)$. Many authors have studied finite groups $G$ with certain restrictions on the set $\text{Van}(G)$ (see [9, 6, 10, 11, 17, 22]). We shall discuss some of that work here. Let $p$ be a fixed prime. Dolfi, Pacifici, Sanus and Spiga in [9] studied finite groups $G$ such that $\text{ord}(vC_i) \neq p^n$ for some $n$, for all $i \in \{1, 2, \ldots, r\}$. They showed that $G$ has a normal Sylow $p$-subgroup [9, Theorem A]. On the other hand, in [6], the authors studied finite groups $G$ such that $\text{ord}(vC_i) = p^n$ for some $n$, for all $i \in \{1, 2, \ldots, r\}$, and proved that $G$ is either a $p$-group or $G$ has a homomorphic image which is a Frobenius group with a complement of $p$-power order.

Robati [22] recently proved that if $\text{Van}(G)$ contains three conjugacy classes of $G$, then the group $G$ is solvable.

In this article, we investigate finite groups $G$ with the property below:

$$\gcd(\text{ord}(vC_i), \text{ord}(vC_j)) = 1 \text{ for } i \neq j, \ i, j \in \{1, 2, \ldots, r\}.$$ (⋆)

We also investigate finite groups $G$ with a more general property:

$$\gcd(\text{ord}(vC_i), \text{ord}(vC_j)) \leq 2 \text{ for } i \neq j, \ i, j \in \{1, 2, \ldots, r\}.$$ (⋆⋆)

In particular, using the classification of finite simple groups we show that if $G$ has property (⋆⋆), then $G$ is solvable:

Theorem A. Let $G$ be a finite group. If $G$ satisfies property (⋆⋆), then $G$ is solvable.

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Remark. If \( \gcd(\text{ord}(vC_i), \text{ord}(vC_j)) \leq 3 \), then \( G \) is not necessarily solvable for \( S_5 \) satisfies this property. Let \( \text{Vo}(G) \) be the set of orders of vanishing elements of \( G \). Then if for every \( a, b \in \text{Vo}(G) \), \( \gcd(a, b) = 1 \), then \( G \) is also not necessarily solvable: \( A_5 \) is a counterexample.

A theorem of Thompson [15, Corollary 12.2] states that, given a prime number \( p \), if every character degree of a non-linear character of \( G \) is a multiple of \( p \), then the group \( G \) has a normal \( p \)-complement. In [9, Corollary B], it was shown that if \( G \) is a finite group and if \( p \mid a \) for all \( a \in \text{Vo}(G) \) for some fixed prime \( p \), then \( G \) has a normal nilpotent \( p \)-complement. This does not necessarily hold when \( G \) satisfies property (\( \star \star \)).

An example is \( S_4 \), since \( \gcd(\text{ord}(vC_i), \text{ord}(vC_j)) \leq 2 \) for all \( vC_i, vC_j \subseteq \text{Van}(S_4) \), that is, \( S_4 \) satisfies property (\( \star \star \)) but \( S_4 \) does not have a normal 2-complement or 3-complement. However, if \( G \) is supersolvable or \( O_2(G) = 1 \), then \( G \) has a normal 2-complement with one exception: some Frobenius groups with a homomorphic image isomorphic to \( S_4 \), as the following result states:

**Theorem B.** Let \( G \) be a finite non-abelian group satisfying property (\( \star \star \)).

(a) If \( G \) is supersolvable, then \( G \) has a normal metabelian 2-complement

(b) If \( O_2(G) = 1 \), then either

(i) \( G \) has a normal 2-complement of Fitting height at most 3, or

(ii) \( G \) is a Frobenius group which has an abelian kernel and a Frobenius complement isomorphic to \( S_4 \).

In [7, Proposition 2.7], Chillag showed that if \( G \) is a non-abelian group, then \( G \) is a Frobenius group with an abelian odd order kernel and a complement of order 2 if and only if every irreducible character of \( G \) vanishes on at most one conjugacy class. In this article, we prove a new characterisation of these Frobenius groups:

**Corollary C.** Let \( G \) be a finite non-abelian group. Then \( G \) has property (\( \star \)) if and only if \( G \) is a Frobenius group with an abelian kernel and complement of order two.

## 2. Preliminaries

In this section we shall list some properties of vanishing elements needed to prove our results.

**Lemma 2.1.** Let \( G \) be an finite group and let \( N \) be a normal subgroup of \( G \). Then the following statements hold:

(a) If \( G \) satisfies property (\( \star \)), then \( G/N \) satisfies property (\( \star \)).

(b) If \( O_2(G) = 1 \), then either

(i) \( G \) has a normal 2-complement of Fitting height at most 3, or

(ii) \( G \) is a Frobenius group which has an abelian kernel and a Frobenius complement isomorphic to \( S_4 \).

**Proof.** The result follows by the standard observation that \( xN \in \text{Van}(G/N) \) implies that \( xN \subseteq \text{Van}(G) \). \( \square \)

**Lemma 2.2.** [21, Lemma 2] Let \( G \) be a finite solvable group. Suppose \( M, N \) are normal subgroups of \( G \).

(a) If \( M \setminus N \) is a conjugacy class and \( \gcd(|M:N|, |N|) = 1 \), then \( M \) is a Frobenius group with kernel \( N \) and prime order complement.

(b) If \( G \setminus N \) is a conjugacy class, then \( G \) is a Frobenius group with an abelian kernel and complement of order two.

For a positive integer \( m \), set \( \pi(m) := \{ p \mid p \text{ divides } m, \text{ where } p \text{ is prime} \} \).
Corollary 2.3. [10] Corollary 2.6] Let $G$ be a finite group and let $K$ be a nilpotent normal subgroup of $G$. If $K \cap \text{Van}(G) \neq \emptyset$, then there exists $g \in K \cap \text{Van}(G)$ whose order is divisible by every prime in $\pi(|K|)$.

Lemma 2.4. [16] Theorem D] Let $G$ be a finite solvable group. If $x$ is a non-vanishing element of $G$, then $xF(G)$ is a 2-element of $G/F(G)$. If $G$ is not nilpotent, then $x$ lies in the penultimate term of the Fitting series.

A non-linear irreducible character $\chi$ of $G$ is said to be of $p$-defect zero if $p$ does not divide $|G|/\chi(1)$. By a result of Brauer (see [15, Theorem 8.17]), if $\chi$ is an irreducible character of $p$-defect zero of $G$, then $\chi(g) = 0$ whenever $p$ divides the order of $g$ in $G$.

The existence of $p$-defect zero characters is guaranteed in finite simple groups $G$ for almost all primes $p \geq 5$ dividing $|G|$ as the following result shows:

Lemma 2.5. [14] Corollary 2.2] Let $G$ be a non-abelian finite simple group and $p$ be a prime. If $G$ is a finite group of Lie type, or if $p \geq 5$, then there exists $\chi \in \text{Irr}(G)$ of $p$-defect zero.

Lemma 2.6. [4] Lemma 2.2] Let $G$ be a finite group, $N$ a normal subgroup of $G$ and $p$ be a prime. If $N$ has an irreducible character of $p$-defect zero, then every element of $N$ of order divisible by $p$ is a vanishing element in $G$.

Lemma 2.7. [3] Lemma 5] Let $G$ be a finite group, and $N = S_1 \times \cdots \times S_k$ a minimal normal subgroup of $G$, where every $S_i$ is isomorphic to a non-abelian simple group $S$. If $\theta \in \text{Irr}(S)$ extends to $\text{Aut}(S)$, then $\phi = \theta \times \cdots \times \theta \in \text{Irr}(N)$ extends to $G$.

Lemma 2.8. [19] Theorem 1.1] Suppose that $N$ is a minimal normal non-abelian subgroup of a finite group $G$. Then there exists an irreducible character $\theta$ of $N$ such that $\theta$ is extendible to $G$ with $\theta(1) \geq 5$.

The number theory result below follows easily.

Lemma 2.9. Let $p$ be and $f$ be a positive integer. If $q = p^f \geq 32$, then $f < (q - 2)/2$.

We end this section by stating a result on groups in which every irreducible character vanishes on at most two conjugacy classes.

Theorem 2.10. [2] Theorem 1] Let $G$ be a non-abelian finite group in which every irreducible character vanishes on at most two conjugacy classes. Then one of the following holds:

(a) $G \cong A_5$ or $G \cong \text{PSL}_2(7)$;
(b) $G$ is solvable and one of the following holds:
   (i) $G$ has a subgroup $Z$ with $|Z| \leq 2$ such that $G/Z$ is Frobenius group with a Frobenius complement of order 2 and an abelian Frobenius kernel of odd order.
   (ii) $G/Z = FA$ is a semidirect product, where $|A| \leq 2$, $|Z| \leq 2$ and $F$ is a Frobenius group with a Frobenius complement of order 3 and a nilpotent Frobenius kernel of class at most 2.

3. Theorem A

Proof of Theorem A. We prove the result by induction on $|G|$. Let $N$ be a non-trivial normal subgroup of $G$. Then $G/N$ satisfies property (**) by Lemma 2.1(b) and hence $G/N$ is a solvable group. If $N_1$ and $N_2$ are two minimal normal subgroups of
$G$, then $G/N_1$ and $G/N_2$ are solvable. Hence $G$ is solvable. We may assume that $G$ has a unique non-abelian minimal normal subgroup $N$. If $N = S_1 \times S_2 \times \cdots \times S_k$, where $S_i \cong S$, $S$ is a simple group and $i = 1, 2, \ldots, k$, then by Lemma 2.8 there exists $\theta \in \text{Irr}(N)$ which is extendible to $G$. Note that $\theta = \phi_1 \times \phi_2 \times \cdots \times \phi_k$ with $\phi_i \in \text{Irr}(S_i)$ for each $i \in \{1, 2, \ldots, k\}$. Suppose that $k \geq 2$. Since $\phi_1$ is non-linear, we may assume that $\phi_1$ vanishes on a $p$-element $x_1 \in S_1$ for some prime $p$ by [20, Theorem B]. Suppose that $p$ is odd. Note that $2 \not| |N|$ and let $y_2 \in S_2$ be a 2-element. Then $\theta(x_1y_2) = \phi_1(x_1)\phi_2(y_2) \cdots \phi_k(1) = 0$ and $p \not| \text{gcd}(\text{ord}(x_1), \text{ord}(x_1y_2))$. Hence $G$ does not satisfy (**)}. Suppose that $p$ is even. Then there is a prime $q \geq 5$ such that $q \not| |N|$ since by [15, Theorem 3.10], $\pi(|G|) \geq 3$. Let $y_2 \in S_2$ be a $q$-element. Note that $x_1y_2$ and $y_2$ are vanishing elements of $G$ by Lemma 2.5 and Lemma 2.6. Since $q \not| \text{gcd}(\text{ord}(y_2), \text{ord}(x_1y_2))$, the result follows.

We may assume that $N$ is a simple group. Since $N$ is the unique minimal normal subgroup of $G$, $C_G(N) = 1$ and so $G$ is almost simple. Let $N$ be a sporadic simple group or $2^F(2)'$. Table 3 below contains an irreducible character $\theta$ of $N$ of $p$-defect zero for some odd prime $p$ and two elements of distinct orders divisible by $p$. The result that $G$ does not satisfy property (**) follows from Lemma 2.6. We shall use the character tables and notation in the Atlas [8].
Suppose that \( N \) is an alternating group \( A_n, n \geq 5 \). For \( N \cong A_5 \) and \( N \cong A_7 \), our result follows by consulting the \textit{Atlas} \cite{Atlas}. Assume that \( N \cong A_6 \cong PSL_2(9) \). Then for an almost simple \( G \) such that \( |G/N| \leq 2 \), our result follows by consulting the explicit character tables in the \textit{Atlas} \cite{Atlas}. For the case when \( |G/N| = 4 \), we obtain the character table in \textit{GAP} \cite{GAP} and our result follows.

Let \( N \cong A_n, n \geq 8 \). By Lemma 2.5, \( N \) has a 5-defect zero character. Note that \( N \) has two elements \((12345)\) and \((12345)(678)\) of orders 5 and 15, respectively. These two elements are vanishing elements of \( G \) by Lemma 2.6 and hence \( G \) does not satisfy property \((\ast\ast)\).

Suppose that \( N \) has an element of order \( 2r, r \) an odd prime. Then \( N \) has an irreducible character \( \theta \) of \( r \)-defect zero by Lemma 2.5. Since \( N \) has elements of order \( r \) and \( 2r \), the result follows since these two elements are vanishing elements by Lemma 2.6. We may assume that \( N \) has no element of order \( 2r, r \) an odd prime. Then the centralizer of each involution contained in \( N \) is a 2-group. It follows from \cite[Theorem 5]{Atlas} that \( N \) is isomorphic to one of the following: \( PSL_2(p) \), where \( p \) is a Fermat or Mersenne prime; \( PSL_2(9); N \cong PSL_3(4); N \cong B_2(2^{2n+1}), n \geq 1 \).

Thus far, we have dealt with the cases when \( N \cong PSL_2(5) \cong A_5 \) and \( N \cong PSL_2(9) \cong A_6 \). For \( N \cong PSL_2(7) \) the result follows by checking the character tables in the \textit{Atlas} \cite{Atlas}. Suppose that \( N \cong PSL_2(p), \) where \( p \geq 17 \), is a Fermat or Mersenne prime. Then the centralizer of an involution in \( N \) is a dihedral group of order \( 2^m, n \geq 4 \). Hence \( N \) contains elements of order 4 and 8. Using Lemmas 2.5 and 2.6 we conclude that \( G \) does not satisfy property \((\ast\ast)\).

Suppose \( N \cong PSL_3(4) \). Then for an almost simple \( G \) such that \( |G/N| \leq 6 \), our result follows by checking the explicit character tables in the \textit{Atlas} \cite{Atlas}. For the case when \( |G/N| = 12 \), we obtain the character table from \textit{GAP} \cite{GAP} and by checking the pertinent information, our result follows.

We may assume that \( N \cong B_2(2^{2n+1}) \) with \( n \geq 1 \). The result basically follows from \cite[Proposition 3.13]{Atlas} but we shall prove it here for completeness. Now \( N \) has two conjugacy classes of elements of order 4 by \cite[Proposition 18]{Atlas}. Since the outer automorphism group is cyclic of odd order \( 2n + 1 \), the outer automorphisms cannot fuse these two conjugacy classes to one conjugacy class in \( G \). Hence \( G \) has two conjugacy classes of order 4 and so \( G \) does not satisfy property \((\ast\ast)\). This concludes our argument. \( \square \)

### 4. Normal 2-complements

Given a finite set of positive integers \( Y \), the prime graph \( \Pi(Y) \) is defined as the undirected graph whose vertices are the primes \( p \) such that there exists an element of \( Y \) divisible by \( p \), and two distinct vertices \( p, q \) are adjacent if and only if there exists an element of \( Y \) divisible by \( pq \). The vanishing prime graph of \( G \), denoted by \( \Gamma(G) \), is the prime graph \( \Pi(\text{Vo}(G)) \). We shall state a result on solvable groups with disconnected vanishing prime graphs. We first recall two definitions:

A group \( G \) is said to be a \textit{2-Frobenius group} if there exists two normal subgroups \( F \) and \( L \) of \( G \) such that \( G/F \) is a Frobenius group with kernel \( L/F \) and \( L \) is a Frobenius group with kernel \( F \).

A group \( G \) is said to be a \textit{nearly 2-Frobenius group} if there exist two normal subgroups \( F \) and \( L \) of \( G \) with the following properties: \( F = F_1 \times F_2 \) is nilpotent, where \( F_1 \) and \( F_2 \) are normal subgroups of \( G \). Furthermore, \( G/F \) is a Frobenius group with kernel \( L/F \), \( G/F_1 \) is a Frobenius group with kernel \( L/F_1 \), and \( G/F_2 \) is a 2-Frobenius group.
Theorem 4.1. \cite{11} Theorem A] Let $G$ be a finite solvable group. Then $\Gamma(G)$ has at most two connected components. Moreover, if $\Gamma(G)$ is disconnected, then $G$ is either a Frobenius group or a nearly 2-Frobenius group.

The following is a classification of Frobenius complements.

Theorem 4.2. \cite{5} Theorem 1.4] Let $G$ be a Frobenius group with Frobenius complement $M$. Then $M$ has a normal subgroup $N$ such that all Sylow subgroups of $N$ are cyclic and one of the following holds:

(a) $M/N \cong 1$;
(b) $M/N \cong V_4$, the Sylow 2-subgroup of the alternating group $A_4$;
(c) $M/N \cong A_4$;
(d) $M/N \cong S_4$;
(e) $M/N \cong A_5$;
(f) $M/N \cong S_5$.

Proof. Theorem B. We first assume that $G$ satisfies property \((\mathcal{F})\). Since $G$ is solvable, $G$ contains at most two vanishing conjugacy classes by Theorem 4.1. If $G$ has one vanishing class, then every irreducible character of $G$ vanishes on at most one conjugacy class and by \cite{7}, Proposition 2.7, $G$ is a Frobenius group with a Frobenius complement of order 2 and an odd order kernel. Hence $G$ has an abelian normal 2-complement. Suppose that $G$ has exactly two conjugacy classes. Note that $\text{ord}(vC_i) \neq \text{ord}(vC_j)$. Then every irreducible character of $G$ vanishes on at most two conjugacy classes. Using Theorem 2.10, we have two cases. Suppose that Theorem 2.10(b)(ii) holds. Then $G$ has at least two vanishing conjugacy classes of order 3, a contradiction. Assume that Theorem 2.10(b)(i) holds. Then $G$ has one vanishing conjugacy class with elements of order 2 contained in the Frobenius complement of $G/Z$. By Lemma 2.2(b), $G$ is a Frobenius group with an abelian kernel and complement of order two, that is, $Z = 1$ and the result follows.

Assume that $G$ satisfies property \((\mathcal{F})\) and suppose $\text{gcd}(\text{ord}(vC_i), \text{ord}(vC_j)) = 2$ for some $i \neq j$. Let $G$ be nilpotent. Then $G = P_2 \times H$ with $P_2$, the Sylow 2-subgroup of $G$ and $H$, a nilpotent group of odd order. If $H$ is non-abelian, then $H$ has a vanishing $h$ of $G$ by \cite{16} Theorem B]. This means that $\chi(h) = \theta_1(1) \times \theta_2(h) = \theta_2(h) = 0$ for some $\chi = \theta_1 \times \theta_2 \in \text{Irr}(P_2 \times H)$ with $\theta_1 \in \text{Irr}(P_2)$ and $\theta_2 \in \text{Irr}(H)$. It follows that for any $x \in P_2$, $\chi(xh) = \theta_1(x) \times \theta_2(h) = 0$ and so $xh$ and $h$ are vanishing elements of $G$, a contradiction. Thus $H$ is abelian which implies that $G$ has a normal abelian 2-complement.

Let $2 = p_1 < p_2 < \cdots < p_n$ and let $P_i$ be a Sylow $p_i$-subgroup of $G$ for $i \in \{1, 2, \ldots, n\}$. Suppose that $G$ is a non-nilpotent supersolvable group and consider $F(G)$. If $\pi(|F(G)|) = 1$, then by \cite{11} Theorems 6.2.5 and 6.2.2], $F(G)$ is the $p_n$-subgroup $P_n$ of $G$. Let $M$ be a normal subgroup of $G$ such that $M/Z(F(G))$ is a chief factor of $G$. Then $M \setminus Z(F(G))$ is a conjugacy class of $G$ since $M \setminus Z(F(G)) \subseteq \text{Van}(G)$ and $\text{gcd}(\text{ord}(vC_i), \text{ord}(vC_j)) \leq 2$. Hence $M/Z(F(G))$ is cyclic and so $M$ is abelian. Thus $M = F(G) = P_n$.

Suppose that $p_{n-1}$ is an odd prime. Then $P_{n-1}F(G)/F(G)$ is a normal subgroup of $G/F(G)$. Thus $P_{n-1}F(G)/F(G)$ is a conjugacy class since $\text{gcd}(\text{ord}(vC_i), \text{ord}(vC_j)) \leq 2$. By Lemma 2.2(b), $F(G)P_{n-1}P_{n-2}$ is a Frobenius group of kernel $F(G)P_{n-1}$ and complement of order $p_{n-2}$. This means that the kernel $F(G)P_{n-1}$ is nilpotent, that is, $F(G)P_{n-1} = F(G) \times P_{n-1}$. Since $P_{n-1}P_{n-2} \cdots P_1$ is supersolvable and since by \cite{11} Theorems 6.2.5 and 6.2.2], $P_{n-1}$ is normal in $P_{n-1}P_{n-2} \cdots P_1$, we obtain that $P_{n-1}$ is
a normal subgroup of $G$, a contradiction since $F(G) = P_n$. Then $P_{n-2} = P_1$ is a Sylow 2-subgroup of $G$ and so $\pi(|G|) \leq 3$. Hence $F(G)P_{n-1}$ is a metabelian normal 2-complement of $G$.

Suppose that $\pi(|F(G)|) \geq 2$. Let $F(G) = Q_1 \times Q_2 \times \cdots \times Q_n$, where $2 = q_1 < q_2 < \cdots < q_n$ and let $Q_i$ be the Sylow $q_i$-subgroup of $F(G)$ for $i \in \{1, 2, \ldots, n\}$. Note that $Q_i \cap Z(Q_i) \subseteq \text{van}(F(G))$ by [16 Theorem B] since $G$ is supersolvable and so all the non-vanishing elements of $G$ are contained in $Z(F(G)) = Z(Q_1) \times Z(Q_2) \times \cdots \times Z(Q_n)$. If $q_i$ is an odd prime and there exists a $q_i$-element $m$ which is a vanishing element, then by Corollary 2.3 there exists a vanishing element $mn$ whose order is divisible by every prime in $\pi(|F(G)|)$, a contradiction since $\gcd(\text{ord}(m), \text{ord}(mn)) > 2$. Hence $Q_i$ is abelian for all odd $q_i$’s. Consider $Q_i$, the Sylow 2-subgroup of $F(G)$. If $Q_1 \setminus Z(Q_1)$ has an element of order greater than 2, then by Corollary 2.3 and the above argument, we obtain a contradiction. Then $Q_1 \setminus Z(Q_1)$ consists only of involutions. By [6. Theorem C], $Q_1$ is a direct product of an elementary abelian 2-group and a Frobenius complement of order 2, a contradiction. Hence $Q_1$ is abelian and therefore $F(G)$ is abelian. Thus $G$ is metabelian.

Finally suppose $2 = p_1 < p_2 < \cdots < p_n$ and let $P_i$ be a Sylow $p_i$-subgroup of $G$ for $i \in \{1, 2, \ldots, n\}$. By [11 Theorems 6.2.5 and 6.2.2], $H = P_nP_{n-1} \cdots P_2$ is a normal subgroup of $G$ and therefore $H$ is a metabelian 2-complement as required.

We may now assume that $G$ is not supersolvable and $O_2(G) = 1$. Suppose that $\Gamma(G)$ consists of a single vertex of an odd prime. Since $G$ satisfies property (\textsc{iv}), we have that $G$ has one vanishing conjugacy class. This means that every irreducible character of $G$ vanishes on at most one conjugacy class and by [11 Proposition 2.7], $G$ is a Frobenius group with a Frobenius complement of order 2, a contradiction. So if $\Gamma(G)$ is connected, then property (\textsc{iv}) implies that every element of $\text{Vo}(G)$ is divisible by 2. By [9 Corollary B], $G$ has a normal nilpotent 2-complement as required. We may assume that $\Gamma(G)$ is disconnected. Then by Theorem 4.1, $G$ is either a Frobenius group or a nearly 2-Frobenius group.

Suppose that $G$ is a Frobenius group. Assume further that the Frobenius complement of $G$ has odd order. Let $H$ be a maximal subgroup of $G$ that contains $G'$. Then $|G/H| = p$ for some odd prime $p$. Note that $F(G) \leq H$. Hence $G' \setminus H \subseteq \text{van}(G)$. Since $p$ divides the order of every element in $G' \setminus H$. By Lemma 2.2(a), $G' \setminus H$ has at least two conjugacy classes, $vC_1$ and $vC_2$ say, and $p \mid \gcd(\text{ord}(vC_1), \text{ord}(vC_2))$, contradicting our hypothesis. We may assume that the Frobenius complement has even order. Denote it by $M$. Then the Frobenius kernel $K$ is abelian. Using Theorem 4.2, we have that $M$ has a unique normal subgroup $N$ such that all the Sylow subgroups of $N$ are cyclic and $M/N \in \{1, V_4, A_4, S_4, A_5, S_5\}$. Since $G$ is solvable we need not consider $A_5, S_5$. Note that $N$ is metacyclic and supersolvable by [23 p. 290]. Suppose that $M/N \in \{1, V_4\}$. Then since $N$ is supersolvable, the Hall 2′-subgroup $R$ of $N$ is normal in $N = M$. Now $KR$ is a normal 2-complement of $G$. Also note that $KR$ is of derived length at most 3 and thus of Fitting height at most 3, as required. Suppose $M/N \cong A_4$. Then there exists a normal subgroup $T$ of $G$ such that $|G/T| = 3$. The result follows using the argument above in the case when the Frobenius complement is of odd order. We now suppose that $M/N \cong S_4$. Note that $\Gamma(G)$ has two connected components and vertex 3 is isolated.

A Sylow 2-subgroup $T$ of $G$ is a generalized quaternion. This means that $|T| = 8$ and $|N|$ is of odd order, otherwise $G$ does not satisfy property (\textsc{iv}). If there is a prime $r \neq 3$ such that $r \mid |N|$, then using [11 Proposition 3.2], there exists an vanishing element $g$ such that $\text{ord}(g)$ is either divisible by $3s$ or $rs$ for some prime $s$ such that
s | |M|, a contradiction. But that means the Frobenius complement has a cyclic Sylow 3-subgroup of order k greater than 3. Then G has vanishing elements of orders 3 and k, a contradiction. Hence N = 1 and therefore G/K ∼= S4. The result follows.

Suppose that G is a nearly 2-Frobenius group. Then there exist two normal subgroups F and L of G with the following properties: F = F1 × F2 is nilpotent, where F1 and F2 are normal subgroups of G. Furthermore, G/F is a Frobenius group with kernel L/F, G/F1 is a Frobenius group with kernel L/F1, and G/F2 is a 2-Frobenius group. Since G/F2 is a 2-Frobenius group and G/F is a Frobenius group with kernel L/F, it follows that G/F2 is a Frobenius group with kernel F/F2. By [11, Remark 1.2], G/L is cyclic and L/F is cyclic with |L/F| odd. If |G/L| is odd, then using the argument in the first part of the previous paragraph, we obtain a contradiction. Hence we may assume that |G/L| is even. Since G/F1 is a Frobenius group with kernel L/F1, we conclude that L/F1 is nilpotent and gcd(|G/L|, |L/F1|) = 1. Note that |F| is odd since O2(G) = 1. We consider G/F, a Frobenius group with a cyclic kernel L/F and a cyclic Frobenius complement G/L. It follows that the Hall 2'-subgroup J/L of G/L is cyclic. Hence J/L is cyclic, L/F1 is nilpotent and F1 is nilpotent, that is J is a normal 2-complement of G with Fitting height at most 3. This concludes our proof. □

Proof of Corollary C. If G satisfies property (7), then G is a Frobenius group with an abelian kernel and complement of order two by the argument in the first paragraph of the proof of Theorem B. The converse holds because if G is a Frobenius group with an abelian kernel and complement of order two, then Van(G) contains only one conjugacy class of elements of order two. □

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