A systematic and complete proof of the existence and uniqueness of self-descriptive numbers

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Abstract

A self-descriptive number in a base \( b \geq 2 \) is an integer \( n \) of \( b \) digits in which the digit \( j_i \), \( 0 \leq j_i \leq b - 1 \) at the position \( p_i \), \( 0 \leq i \leq b - 1 \), counts how many times the digit \( i \) occurs in the number. It’s known that self-descriptive numbers don’t exist the bases \( b = 2, 3, 6 \) and that they exist and which they are in the bases \( b = 4, 5 \). Also, that at least one defined self-descriptive number exists in each base \( b \geq 7 \), while it’s unknown if others do, apart from direct negative checks for smaller bases \( b \geq 7 \). All these results, together with a demonstration of the uniqueness for \( b \geq 7 \), are here obtained through a systematic scheme of proof. The proof is also complete for all the possible cases had been taken into account.

1 Introduction

The subject of self-descriptive numbers as it will be defined and discussed in the following has received some interest by mathematicians [1], [2], [3], [4]. A list of these numbers expressed in the base \( b = 10 \) is found on the OEIS [5].

Part of the mathematical interest in the subject is that although the existence and numerosness of these numbers is very well known for bases \( b \leq 6 \), and the existence of at least one defined self-descriptive number had been recognized in each base \( b \geq 7 \), the eventual uniqueness in these greater bases had remained unproven.

The acknowledged results seem however to have been obtain’d mostly through direct checks and trial-and-error procedures, easy for smaller bases. On the internet they can be found several amateur algorithms to check the eventual existence of multiple self-descriptive numbers in greater bases.

In this work instead, the already known results, together with the demonstration of the still unproven uniqueness, are all obtain’d through a systematic scheme of proof that applies the idea of restricted partition of an integer.
1.1 Self-descriptive numbers

Definition 1.1 A number $n$ of $b$ digits in some base $b \geq 2$, $n = \sum_{i=0}^{b-1} j_i b^{(b-1-i)}$, $0 \leq j_i \leq b - 1$; represented by the ordered list of its digits $j_i$ at the $b$ positions $p_i$, $i = 0, 1, \ldots, b - 1$; is self-descriptive iff the digit $j_i$ counts how many times the digit $i$ occurs in $n$.

Examples

* In the base $b = 4$ the number 2020 is self-descriptive, for there are in it:
  - 2 instances of the digit “0”;
  - 0 instances of the digit “1”;
  - and so on

* In the base $b = 10$ the number 6210001000 is self-descriptive and unique.

The following trivially holds

Lemma 1.1 In any base $b$ the sum of all the digits of an eventual self-descriptive number is $b$.

Related to self-descriptive numbers also autobiographical numbers in a base $b$ are considered in literature ([3]). Those are just all the self-descriptive numbers in bases up to $b$ as included, expressed in the base $b$.

1.2 State of the art

It’s known that

- there are no self-descriptive numbers in the bases $b = 2, 3, 6$;
- in $b = 4$ the numbers 2020 and 1210 are self-descriptive and unique;
- in $b = 5$ the number 21200 is self-descriptive and unique;
- in any $b \geq 7$ it’s self-descriptive the number which has similar entries as 6210001000, it is the number
  \[(b - 4)b^{b-1} + (2)b^{b-2} + (1)b^{b-3} + (1)b^{3}\]; (1)

Leaving aside direct checks, proving negative, for smaller bases $\geq 7$, it has been hitherto unknown if the numbers of the kind of (1) are the unique self-descriptive numbers in any $b > 7$.

1.3 Results here proven

All the former results are here originally proven in a systematic way, it is not through trial-and-error procedures. Through the same approach it’s also here proven that

A number of the kind of (1) is the unique self-descriptive number in each base $b \geq 7$. 

2
2 Theorems and proofs

Given some first entry \( J \equiv j_0 \), it is the number of instances of “0” in \( n \),
\[
J = (b - m), \quad 1 \leq m < b \iff (J = 0 \text{ is inconsistent } \rightarrow m \neq b)
\]
there are then \( (b - J - 1) = (m - 1) \) empty positions left in the list of the digits, it is \( (m - 1) \)
parts in which some restricted partition of the integer \( m \) is to be had.

It’s then immediate that \( m \neq 1 \) for there would be no empty position left.

This already implies that

**Theorem 2.1** There are no self-descriptive numbers in the base \( b = 2 \).

There is then only one kind of restricted partition of \( m \geq 2 \) in \( m - 1 \) parts, it is
\[
m = 2 \ ( +1)^{m-2 \text{ times}} \iff m \geq 3. \tag{2}
\]
Thus apart from “\( J \)” at the position \( p_0 \), and the \( J \) instances of “0”,
all the other digits in a self-descriptive number must be:

“2” in 1 only instance; and
“1” in eventual instances, yet anyway not more than 2.

Thus \( j_1 = 0, 1, 2 \), and these three and only cases will be now discussed.

2.1 Case 1: \( j_1 = 0 \)

There are no instances of “1” in the number, thus because of formula (2)
\[
m = 2, \text{ and there is only “2” as a nonzero entry at the positions } p_{i(>0)}.
\]
Consequently \( j_2 = 2 \), as “\( J \)” must have at least 1 instance. This means that “\( J \)” occurs twice,
and, as the only nonzero entries are “\( J \)” and “2”, so \( J = 2 \). Thus \( b = 4 \), and

**Theorem 2.2** The number 2020 is self-descriptive in the base \( b = 4 \).

2.2 Case 2: \( j_1 = 1 \)

Again \( (j_1 = 2 \rightarrow J = 2) \) as in case 1.

This implies

**Theorem 2.3** The number 21200 is self-descriptive in the base \( b = 5 \).

2.3 Case 3: \( j_1 = 2 \)

Then \( j_2 = 1 \), for \( p_2 \) has a nonzero entry, and it can be only the digit “1”; thus

Subcase 3.1: \( J = 1 \),
then \( p_J \equiv p_1 \) and, as all the nonzero entries have already been considered, \( b = 4 \) and

**Theorem 2.4** The number 1210 is self-descriptive in the base \( b = 4 \).
Subcase 3.2: \[ J \neq 1 \],
then \( p_J \neq p_1 \) and \( J \neq 2 \) for \( j_2 = 1 \) and “2” already occurs at the position \( p_1 \);
thus \( J = (b - 4) \), for \( (j_1 = 2, j_2 = 1, j_{j,J \neq (1.2)} = 1 \rightarrow j_1 + j_2 + j_J = m = 4) \),
and \( b > 4 \), yet \( (b \neq 5 \leftrightarrow J \neq 1) \), \( (b \neq 6 \leftrightarrow J \neq 2) \).

Together with the previously deduced results, this implies

**Theorem 2.5** There are no self-descriptive numbers in the bases \( b = 3, 6 \).

And

**Theorem 2.6** The number \((b - 4)b^{b-1} + (2)b^{b-2} + (1)b^{b-3} + (1)b^3\) is self-descriptive in each base
\( b \geq 7 \).

2.4 Concluding uniqueness

Finally, as \( j_1 \) cannot assume any other value than \( (0,1,2) \) and all the cases had already been
considered, it’s so proven that

**Theorem 2.7** The number

\[
(b - 4)b^{b-1} + (2)b^{b-2} + (1)b^{b-3} + (1)b^3
\]

is unique as self-descriptive number in each base \( b \geq 7 \).

3 Conclusions

A systematic scheme of proof that makes a simple use of the idea of restricted partition of an
integer has been applied in a complete proof of the existences and uniquenesses of self-descriptive
numbers.

The results are here listed and their proves in the text referred to.

- [2.1,2.5] There are no self-descriptive numbers in the bases \( b = 2, 3, 6 \).
- [2.2,2.4] In the base \( b = 4 \) there are the two, and only, self-descriptive numbers 2020 and 1210.
- [2.6,2.7] In the base \( b = 5 \) there is the one and only self-descriptive number 21200.
- [2.6,2.7] In each base \( b \geq 7 \) is self-descriptive and unique the number

\[
(b - 4)b^{b-1} + (2)b^{b-2} + (1)b^{b-3} + (1)b^3
\]
References

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