FINITE GENERATION TYPE PROPERTIES FOR FUCHSIAN GROUP VON NEUMANN ALGEBRAS

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In this paper we prove the existence of finite systems $G$ of generators, with special properties, for the von Neumann algebras $L(\Gamma) \otimes B(H)$, obtained by tensoring the group von Neumann algebra of a fuchsian group $\Gamma$ with the bounded linear operators on an infinite dimensional, separable, Hilbert space $H$. The results show that the linear span of ordered monomials that are products of powers of adjoints of elements in $G$ with powers of elements in $G$, is weakly dense in the algebra. In addition the systems of generators are a commuting family, having non-trivial, joint invariant subspaces affiliated to the von Neumann algebra. If $\Gamma = \text{PSL}(2, \mathbb{Z})$, we can find such a generating set consisting of a single element.

The construction in this paper may also be used to provide examples for Toeplitz operator with unbounded symbol with unexpected behaviour and non-closability for multiplication operators, with bounded symbol, in Sobolev type Hilbert spaces associated to differential operators.

We recall some of the notations that were used in [Ra3]. We let $H_t$ be the Hilbert space of square summable, analytic functions on $\mathbb{H}$, with respect to the measure $d\nu_t = (\text{Im } z)^{t-2}dz$. In [Pu], [Sa] it was proven that there exist a one parameter family of irreducible, projective unitary representations of $\text{PSL}(2, \mathbb{R})$ on $H_t$ that extends the analytic discrete series of representations for $\text{PSL}(2, \mathbb{R})$. Moreover the methods in [VFR] along with the trace formula in [Pu], were used in ([Ra3]) to show that for any fuchsian group $\Gamma$, the von Neumann algebra $\{\pi_t(\Gamma)\}^\prime\prime$ is a type II factor acting on $H_t$. The Murray von Neumann dimension for this algebra acting on $H_t$ is proportional to $t$ and the covolume of $\Gamma$. In particular the commutant algebras $A_t = \{\pi_t(\Gamma)\}^\prime$ are isomorphic to the twisted group von Neumann algebra $\mathcal{L}(\Gamma, \sigma^t)/\text{covolume } \Gamma$, with $\sigma^t$ the cocycle corresponding to the projective representation $\pi_t$.

As shown in [GHJ], automorphic forms $f$ of weight $k$ for $\Gamma$ correspond to bounded intertwining operators $S^t_j$ from $H_t$ into $H_{t+k}$, by multiplication with $f$ on $H_t$. The intertwining property means that

$$\pi_{t+k}(\gamma)S^t_j = S^t_j \pi_t(\gamma), \gamma \in \Gamma.$$
Let \( f \) be any measurable, bounded \( \Gamma \)-invariant function on \( \mathbb{H} \). Let \( M_f \) be the multiplication operator by \( f \) on \( H \) and let \( P_t \) be the projection operator from bounded square summable functions on \( \mathbb{H} \) onto \( H_t \). The Toeplitz operator \( T^f = P_t M_f P_t \) clearly commutes with \( \pi_t(\Gamma) \) and hence \( T^f \) belongs to \( \mathcal{A}_t \).

A natural object to consider is the (unbounded) Toeplitz operator with symbol \( \Gamma \) invariant, analytic function on \( \mathbb{H} \). Unfortunately, as we show below, though affiliated with the commutant algebras, such an operator is not densely defined, nor is its adjoint closable. One possible symbol function, for \( \Gamma = \text{PSL}(2, \mathbb{Z}) \), is the classical modular invariant function \( j = \frac{24}{c^2} \).

**Lemma.** Let \( \Gamma \) be a fuchsian group and let \( j \) be an analytic, \( \Gamma \)-invariant function on \( \mathbb{H} \). Assume that \( j = \frac{24}{z^2} \) is the quotient of two automorphic forms for \( \Gamma \) having the same weight \( m \). Assume that the functions \( z \to |a(z)|^2(\text{Im } z)^{m-2} \), \( z \to |b(z)|^2(\text{Im } z)^{m-2} \), are bounded. Also we assume that \( a, b \) have not all their zeroes and poles in common.

For a bounded, measurable, \( \Gamma \)-invariant function \( f \) on \( \mathbb{H} \) we let \( T^f \) be the Toeplitz operator on \( H_t \) with symbol \( f \). Then, clearly, \( T^f \) belongs to the commutant \( \mathcal{A}_t = \{ \pi_t(\text{PSL}(2, \mathbb{Z})) \} \) and the set of all such operators is a weakly dense subspace of the Hilbert space \( L^2(\mathcal{A}_t) \) associated with the trace on \( \mathcal{A}_t \hspace{0.2cm} \text{(see \cite{Ra1})} \).

Then the operator \( T^f \to T^j \) is not closable on \( L^2(\mathcal{A}_t) \) for any \( t > 1 \) (and hence \( T^j \to T^j \) is not densely defined). Moreover the same holds true if we replace \( j \) by \( |j|^2 \) or by \( j^2 \).

**Proof.** The hypothesis shows that the bounded (see \cite{GHJ}) operators \( S^t_a, S^t_b \) on \( H_t \) into \( H_{t+k} \), defined by by multiplication with \( a \) and respectively \( b \) have non equal ranges. Let \( c \) be any other automorphic form form \( \Gamma \) having the same order as \( a, b \). Let \( \Psi \) be the operator in the statement. Then for any \( \Gamma \)-invariant, bounded, measurable function \( h \) we will have that

\[
\Psi((S^t_a) T^t_{h+k} S^t_c) = (S^t_b) T^t_{h+k} S^t_c.
\]

Let \( e_a \) be the projections onto the range of \( S^t_a \), which is a subspace of \( H_{t+k} \). As the set of all \( T^t_{h+k} \), when \( h \) runs through the bounded, measurable, \( \Gamma \)-equivariant functions on \( \mathbb{H} \) is weakly dense in the commutant \( \mathcal{A}_{t+k} \) it follows that for any \( x \) in \( \mathcal{A}_{t+k} \) we may find a sequence of such functions \( h_n \) on \( \mathbb{H} \) such that \( T^t_{h_n} \) converges weakly to \( (1 - e_a)x \) and hence \( (S^t_a) T^t_{h+k} S^t_c \) converges weakly to \( 0 \).

Then \( (S^t_a) T^t_{h+k} S^t_c \) converges weakly to \( (S^t_a) (1 - e_a)x S^t_c \). If \( \Psi \) was closable then it would follow that \( (S^t_a) (1 - e_a)x S^t_c \) is zero for all \( x \) in \( \mathcal{A}_t \). Since \( x \) is arbitrary and since the operators \( S^t_a, S^t_b \) have non equal ranges, this is impossible. This completes the proof for \( j \). The statement for the other two functions is proved in a similar way: multiplication by \( |j|^2 \) maps \( (S^t_a) T^t_{h+k} S^t_a \) into \( (S^t_a) T^t_{h+k} S^t_b \) (and this is a positive map) and multiplication by \( j^2 \) maps \( (S^t_a) T^t_{h+k} S^t_b \) into \( (S^t_b) T^t_{h+k} S^t_a \).

**Remark.** With the notations in the above statement let \( F \) be a fundamental domain for the action of \( \Gamma \) on \( \mathbb{H} \). Let \( B_t = B_t(\Delta) \) be the Berezin operator
commuting with the invariant laplacian, formally defined by

$$B_t f(z) = \int_H f(w) \frac{(\text{Im } z)(\text{Im } w)}{z-w} t^{-2} (\text{Im } w)^{-2} d\bar{w} dw, \quad z \in \mathbb{H}.$$  

Then \(L^2(A_t, \tau)\) is identified with the completion of a dense subset of \(L^2(F, (\text{Im } w)^{-2} d\bar{w} dw)\) with the scalar product

\[(f, g) = (B_t(\Delta) f, g)_{L^2(F)}.
\]

With this scalar product, the operator of multiplication by \(j\) on a dense subspace of \(L^2(F, (\text{Im } w)^{-2} d\bar{w} dw)\) is non closable. Moreover the same holds true if we replace \(j\) by \(|j|^2\) or by \(j\).

The following corollary shows that one of the properties that is valid for Toeplitz operators with bounded, antianalytic symbol, fails for unbounded symbols (see also [Ja], [Saf]).

**Corollary.** Let \(\Gamma\) be a fuchsian group of finite covolume, and \(j = \frac{a}{b}\) an analytic, \(\Gamma\) invariant function on \(\mathbb{H}\) as above. Let \(P_t\) be the projection from \(L^2(\mathbb{H}, (\text{Im } z)^{-2} d\bar{z} dz)\) onto \(H_t = H^2(\mathbb{H}, (\text{Im } z)^{t-2} d\bar{z} dz)\). Let \(M_T\) be the multiplication operator with \(\frac{a}{b}\) which is defined on a dense subset of the Hilbert space \(L^2(\mathbb{H}, (\text{Im } z)^{t-2} d\bar{z} dz)\).

Then \(P_t M_T (1 - P_t)\) is nonzero (in particular has nonzero domain).

**Proof.** Assume the contrary. Let \(h, h_1\) be any \(\Gamma\)-invariant, bounded, measurable, real valued functions \(h, h_1\) on \(\mathbb{H}\), such that the operators \(T_h^t, T_{h_1}^t\) are injective. Assume that the supports of \(h, h_1\) are so that the functions \(\bar{j}h, \bar{j}h_1\) are bounded.

Let \(Z_h, Z_{h_1}\) be the closable operators ([MvN]), defined by

\[Z_h = T_{\bar{j}h}^t (T_h^t)^{-1}, \quad Z_{h_1} = T_{\bar{j}h_1}^t (T_{h_1}^t)^{-1}.
\]

Our assumption implies that

\[T_{\bar{j}h}^t \zeta = T_{\bar{j}h_1}^t \zeta_1 \text{ whenever } T_h^t \zeta = T_{h_1}^t \zeta_1.
\]

This implies that the closable, unbounded operators \(Z_h, Z_{h_1}\) coincide on a densely defined core and hence that they are equal.

Hence there exists a unique, closed operator, affiliated with \(A_t = \{\pi_t(\Gamma)\}'\) such that for any real valued, bounded, measurable functions \(h, h_1\) with \(T_h^t\) invertible, we have (by [MvN])

\[ZT_h^t = T_{\bar{j}h}^t.
\]

But this is impossible by the previous statement.

**Corollary.** With the notations in the previous statement the same conclusion holds if \(j\) is replaced by \(h \circ j\), where \(h\) is any univalent entire function \(j\).

**Proof.** By examining the above argument, we see that in fact we proved that if \(K\) is any compact subset of the interior of \(F\) such that \(j\) is bounded when restricted to \(K\) and \(\chi_K\) is the characteristic function of \(K\) then

\[P_t M_T [(\text{Id} - P_t) \wedge \chi_K] \neq 0.
\]
Now assume that the (bounded) linear operator $M_{f \circ j}$ would have the property that
\[ M_{f \circ j}(\text{Id} - P_t) \wedge \chi_{K}(H_t) \subseteq (\text{Id} - P_t)(H_t). \]
It would follow that $M_{f \circ j}$ would also have the property that
\[ M_{f \circ j}(\text{Id} - P_t) \wedge \chi_{K}(H_t) \subseteq (\text{Id} - P_t) \wedge \chi_{K}(H_t). \]
Since $f$ is univalent the same would then hold true about $j$ instead of $f \circ j$, and this we know to be false.

Questions.

(i). Is the above statement true if one drops the univalence condition on $f$?

(ii). If $t > 24$, $\Gamma = \text{PSL}(2, \mathbb{Z})$ then if $j = \Delta_2^G \Delta_3^G = \Delta_{2,3}^G$ it is clear that the domain of $M_j$ intersects $H_t$ non trivially. Does this hold for smaller values of $t$?

Assume that $\Gamma$ is a fuchsian group (necessary of infinite covolume). Assume that $H^\infty(\mathbb{H}/\Gamma)$ has a sufficiently rich structure so that there are a finite number of bounded analytic functions that separate points on $\mathbb{H}/\Gamma$. In this case, by using the methods developed in [Ra3] we can show that $\mathcal{L}(\Gamma) \otimes B(K)$ has a set of generators with the properties in the following proposition.

**Proposition 2.** Let $\Gamma$ be a fuchsian group such that $H^\infty(\mathbb{H}/\Gamma)$ contains functions $h_1, ..., h_k$ that separate the points on $\mathbb{H}$. Let $K$ be an infinite dimensional Hilbert space. Then there exists commuting, bounded operators $Z_1, ..., Z_k$ in $B = \mathcal{L}(\Gamma) \otimes B(K)$ such that
\[
\mathcal{B} = \overline{\text{Sp} \{(Z^n_i) Z^m_j | n, m \in \mathbb{N}, i, j = 1, 2, ..., k\}}^{\text{weak}}.
\]

Proof. Let $Z_i = T_{h_i}, i = 1, ..., k$. Let $c$ be an arbitrary invertible element in $B \cap L^1(B, \tau)$. Assume that $a \in B$ is orthogonal to the following subspace of $L^1(B, \tau)$
\[
\overline{\text{Sp} \{ c(Z^n_i) Z^m_j | n, m \in \mathbb{N}, i = 1, ..., k\}}^{\text{weak}}.
\]

We use the notations in [Ra3]. Let $\hat{(ac)}(z, \bar{z}) z \in \mathbb{H}$ be the Berezin’s symbol of $ac \in B \cap L^1(B, \tau)$. The trace formula in [Ra3] shows that
\[
\int_F \hat{(ac)}(z, \bar{z}) \hat{h}_j(z)(\text{Im } z)^t d\tau dz = 0.
\]

Hence $ac = 0$ and hence $a = 0$. Consequently
\[
\text{Sp} \{ c(Z^n_i) Z^m_j | n, m \in \mathbb{N}, i = 1, ..., k\}
\]

is a weakly dense subspace of $L^1(B, \tau)$ and consequently, since $c$ was invertible we get that
\[
\overline{\text{Sp} \{(Z^n_i) Z^m_j | n, m \in \mathbb{N}, i, j = 1, ..., k\}}^{\text{weak}} = \mathcal{B}.
\]
Proposition. Let $E$ be an open $\Gamma$-invariant subset of $\mathbb{H}$, such that $j$ (or $\frac{1}{j}$) is bounded on $E$. Let $H_t(E)$ be the subspace of $L^2(\mathbb{H}, \nu_t)$, ($\Gamma$-invariant, with respect to the unitary representation of $\text{PSL}(2, \mathbb{R})$ on $L^2(\mathbb{H}, \nu_t)$), consisting of square integrable, analytic functions on $E$ (that are extended with 0 outside $E$). Let $Z$ be the Toeplitz operator on $H_t(E)$ with symbol $j|_E$. Then $H_t(E)$ is a (finite or infinite) Hilbert module over $\mathcal{L}(\Gamma)$, as a submodule of $L^2(\mathbb{H}, \nu_t)$.

Then the commutant of $\Gamma$ in $\mathcal{B}(H_t(E))$ is the weak closure of the linear span of the set $\{(Z^*)^nZ^m | n, m = 0, 1, 2...\}$.

Proof. Clearly $H_t(E)$ is a submodule of $L^2(\mathbb{H}, \nu_t)$ over $\mathcal{L}(\Gamma)$. If $k_E$ is the reproducing kernel for $H_t(E)$ and $F(E)$ is a fundamental domain for $\Gamma$ acting on $E$, then the Murray von Neumann dimension of $H_t(E)$ over $\mathcal{L}(\Gamma)$ is equal to $\int_{F(E)} k_E(z, z)$. Moreover, as in [Ra3], if $A$ is an operator acting on $H_t(E)$, we consider its Berezin kernel (with respect $H_t(E)$) to be

$$k_A^E(z, \zeta) = \frac{\langle A z, t, e_{z, t}^E \rangle}{\langle e_{z, t}^E, e_{\zeta, t}^E \rangle}, z, \zeta \in E,$$

where $e_{z, t}^E$, for $z$ in $E$ is the evaluation vector in $H_t(E)$ at $z$. Then, if $A$ is trace class in the commutant of $\Gamma$, the trace is $\int_{F(E)} k_A^E(z, z) d\nu_t(z)$. This trace comes from the trace on the commutant of $\Gamma$ on $L^2(\mathbb{H}, \nu_t)$ that is normalized by giving value $\frac{1}{2\pi}$ to $H^2(\mathbb{H}, \nu_t)$. As in [Ra3], if an operator is orthogonal on all Toeplitz operators on $H_t(E)$, with $\Gamma$-equivariant symbol it follows that $A = 0$. Hence the linear span of Toeplitz operators, with $\Gamma$-equivariant symbols, is weakly dense in the commutant and since the symbol of $(Z^*)^nZ^m$ is $(j|_E)^n(j|_E)^m$ the statement follows.

Corollary. In the algebra $\mathcal{A} = \mathcal{L}(\text{PSL}(2, \mathbb{Z}) \otimes B(H) = \mathcal{L}(F_N) \otimes B(H)$, $N$ finite, there exists a bounded subnormal operator $Z$, such that $\mathcal{A}$ is the weak closure of linear span of the set $\{(Z^*)^nZ^m | n, m = 0, 1, 2...\}$.

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