Negative spectrum in Harmonic oscillator under simultaneous 
Non-hermitian transformation of co-ordinate and momentum with Real 
wave function.

Biswanath Rath 
Department of Physics, North Orissa University, Takatpur, Baripada -757003, 
Odisha, INDIA E.mail: biswanathrath10@gmail.com

We notice that PT symmetric non-Hermitian one dimensional simple Harmonic 
Oscillator under simultaneous transformation of co-ordinate and momentum with 
proper selection of wave function can also reflect real negative energy eigen spectra 
provided the associated wave function is well behaved, square integrable and nor-
malised to unity.

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, wave function, simultaneous transformation, co-ordinate, momentum. Perturbation theory.

I. INTRODUCTION

In Quantum Physics two important things[1] namely on (i) well behaved wave 
function (ii) invariance in commutation relation between co-ordinate(x) and momentum (p) are necessary for understanding and development of the subject. Mathematically one can write it as

\[ <\psi_i|\psi_j> = \delta_{i,j} \quad (1) \]

and

\[ [x, p] = i \quad (2) \]
In above if $i = j$ then $\delta_{i,i} = 1$ otherwise zero. Further the wave function satisfies the eigenvalue relation as

$$H\psi_n(x) = E_n\psi_n(x) \tag{3a}$$

where $E_n$ is the real energy corresponding to Hamiltonian $H$. the $\psi_n$ satisfies the condition

$$\psi_n(x \to \infty) \to 0 \tag{3b}$$

Here the Hamiltonian $H$ is the Hermitian in nature as per the basic understanding of Quantum Mechanics. Sometime back Bender and Boettcher [2] (BB) relaxed this condition by imposing a simpler $PT$ symmetry condition on real eigenvalue. In $PT$ symmetry $P$ stands for parity operator i.e $x \to -x$; $p \to -p$ and $T$ stands for time reversal i.e $i \to -i$. One check that under $PT$ symmetry condition, the above commutation relation Eq(2) is satisfied [3,4]. However Mostafadazeh [5] proposed a pseudo-Hermiticity condition

$$\eta H \eta^{-1} = H^+ \tag{4}$$

for real eigenvalue. Further Bender, Hook and Klevansky [6] (BVK) showed that in a complex plane one can observe negative eigenvalue

$$E_n^{(0)} = -(2n + 1) \tag{5}$$

of the Harmonic oscillator

$$H = p^2 + x^2 \tag{7}$$

In order to give a satisfactory explanation on negative energy BHK [6] put forwarded the idea of complex transformation as

$$x = ix \tag{8}$$
Of course one can very well see that the Hamiltonian in Eq-7, becomes

$$H(x \to ix) = -[p^2 + x^2]$$  \hspace{1cm} (9)

From Eq(6) the concept of negative energy is well understood. However authors BVK[6] do not discuss anything regarding wave function and its behaviour. In this context the work of Hatano and Nelson [7] is worth reading. According to HN[7] any Hamiltonian

$$H = \frac{(p + ig)^2}{2} + V(x)$$  \hspace{1cm} (10)

with arbitrary $V(x)$ can lead to real eigenvalues for $g < g_c$ where $g_c$ is a critical value.

The work of HN[7] inspired Ahmed [8] to propose a slightly deviated transformation on momentum $P \to p + i\beta x$ on Harmonic Oscillator Hamiltonian

$$H_{SO} = \frac{p^2}{2} + \frac{(\beta^2 + \alpha^2)x^2}{2}$$  \hspace{1cm} (10)

leading to Non-Hermitian Harmonic Oscillator Hamiltonian as

$$H_{\beta} = \frac{(p + i\beta x)^2}{2} + \frac{(\alpha^2 + \beta^2)x^2}{2}$$  \hspace{1cm} (11a)

having the iso-spectral behaviour i.e

$$E_n = \sqrt{(\alpha^2 + \beta^2)(n + \frac{1}{2})}$$  \hspace{1cm} (11b)

Recently Rath and Mallick [9] using zero perturbation correction method(ZPCM) proved that non-Hermitian Hamiltonian

$$H = \frac{(p + i\beta x)^2}{2(1 + \lambda \beta)} + \frac{(x + i\lambda p)^2}{2(1 + \lambda \beta)}$$  \hspace{1cm} (12)

and simple oscillator Hamiltonian

$$H_{SO} = \frac{p^2}{2} + \frac{x^2}{2}$$  \hspace{1cm} (13a)

are iso-spectral i.e

$$E_n = (n + \frac{1}{2})$$  \hspace{1cm} (13b)
Further RM[10] argued that if one can use variational method in stead of ZPCM the calculation become tedious. Very shortly after the work of RM[9], Fernandez [10] in a rigorous calculation proved the iso-spectral nature between Eq(12) and EQ(13a) using similarity transformation.

However the neither RM[9], Fernandez [10], Ahmed [8] nor even BVK[6] discussed the negative energy concept in any of the above Hamiltonian considered using well behaved wave function.

Hence the aim of the present paper is to discuss the concept of negative energy in Harmonic oscillator under simultaneous non-Hermitian transformation of co-ordinate and momentum alongwith the proper selection of real wave function.

II. NON-HERMITIAN TRANSFORMATION OF CO-ORDINATE AND MOMENTUM

Now use the transformation as

\[
x \to \frac{x + i\lambda p}{\sqrt{(1 + \beta \lambda)}}
\]

and

\[
p \to \frac{p + i\beta x}{\sqrt{(1 + \beta \lambda)}}
\]

Hence the new Hamiltonian becomes non-Hermitian in nature and is

\[
H = \frac{(p + i\beta x)^2}{2(1 + \lambda \beta)} + \frac{(x + i\lambda p)^2}{2(1 + \lambda \beta)}
\]

One can handle this Hamiltonian in many ways, however perturbation theory [9] is a nice way for understanding the physics as well as mathematics. So we deal with perturbation theory as developed earlier [9]. Further it easy to understand the theory using second quantization formalism.

III. SECOND QUANTIZATION, HAMILTONIAN AND NEGATIVE ENERGY
In order to solve the above Hamiltonian (Eqn. (12)), we use the second quantization formalism as

\[ x = \frac{(a + a^+)}{\sqrt{2}\omega} \]  

(16a)

and

\[ p = i\sqrt{\omega} (a^+ - a) \]  

(16b)

where the creation operator, \( a^+ \) and annihilation operator \( a \) satisfy the commutation relation

\[ [a, a^+] = 1 \]  

(17)

and \( \omega \) is an unknown parameter. The Hamiltonian in Eqn. (12) can be written as

\[ H = H_D + H_N \]  

(18)

where

\[ H_D = [(1 - \lambda^2)\omega + (1 - \beta^2)\omega] \frac{(2a^+ a + 1)}{4(1 + \lambda \beta)} \]  

(19a)

\[ H_N = U \frac{a^2}{4(1 + \lambda \beta)} + V \frac{(a^+)^2}{4(1 + \lambda \beta)} \]  

(19b)

\[ V = [-\omega(1 - \lambda^2) + \omega(1 - \beta^2) - 2(\lambda + \beta)] \]  

(20)

\[ U = [-\omega(1 - \lambda^2) + \omega(1 - \beta^2) + 2(\lambda + \beta)] \]  

(21)

Case-I(A) Selection of \( \omega_1 \) and real Wave Function for Negative Energy

Let the coefficient of \( a^2 \) is zero i.e.

\[ U = [-\omega(1 - \lambda^2) + \frac{(1 - \beta^2)}{\omega} + 2(\lambda + \beta)] = 0 \]  

(22)

considering negative sign we get \( \omega = \omega_1 \)

\[ \omega_1 = \frac{(\beta - 1)}{(1 + \lambda)} \]  

(23)
In this case, the wave function is

$$\psi_n = \left( \frac{\sqrt{w_1}}{\sqrt{\pi} 2^{n!}} \right)^{1/2} H_n(\sqrt{w_1}x) e^{-w_1x^2/2}$$  \hspace{1cm} (24) $$

Here for any value of $\beta > 1$, the wave function is well behaved. Now substituting the value of $\omega_1$ in $H_D$ and taking the expectation value we get

$$<n|H_D|n>_{w_1} = -(n + 1/2)$$ \hspace{1cm} (25) $$

Here we get the negative energy behaviour of the transformed the Hamiltonian in Eq(12).

**Case-II(B) Selection of $\omega_2$ and real Wave function for Negative Energy behaviour.**

Let the coefficient of $(a^+)^2$ is zero i.e.

$$V = [-\omega(1 - \lambda^2) + \frac{(1 - \beta^2)}{\omega} - 2(\lambda + \beta)] = 0 \hspace{1cm} (26)$$

considering negative sign we get $w = w_2$

$$\omega_2 = \frac{(1 + \beta)}{(\lambda - 1)} \hspace{1cm} (27)$$

In this case, the wave function is

$$\psi_n = \left( \frac{\sqrt{w_2}}{\sqrt{\pi} 2^{n!}} \right)^{1/2} H_n(\sqrt{w_2}x) e^{-w_2x^2/2}$$ \hspace{1cm} (28) $$

Here for any value of $\lambda > 1$, the wave function is well behaved.

Now substituting the value of $\omega_1$ in $H_D$ and taking the expectation value we get

$$<n|H_D|n>_{w_2} = -(n + 1/2)$$ \hspace{1cm} (29) $$

**VII. Conclusion**

In this paper, we suggest a simpler procedure for calculating the negative energy levels of the non-Hermitian harmonic oscillator under simultaneous transformion of co-ordinate and momentum with appropriate choice of real wave function. It should
be borne in mind that if one will try with variational method [1] to visualise the negative energy concept, it will not be possible. In other words, this work is not only motivate many to perform experiment to realise the negative energy but also reflect the superiority of perturbation theory over variational method.

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