A new fictitious domain approach for Stokes equation

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Abstract. The purpose of this paper is to present a new fictitious domain approach based on the Nitsche’s method combining with a penalty method for the Stokes equation. This method allows for an easy and flexible handling of the geometrical aspects. Stability and a priori error estimate are proved. Finally, a numerical experiment is provided to verify the theoretical findings.

1. Introduction

In many fields, such as fluid dynamics and elasticity, problems occur making it necessary to solve problem on domains being geometrically complex or time-dependent. As the generation of boundary-fitted meshes of good quality is a rather complex, often time consuming high costly, the fictitious domain method has been developed in order to overcome the meshing and re-meshing problem. One method for describing the essential Dirichlet boundary conditions, fitting into the context, is the boundary penalty method in [1]. The real advantage of it is the combination with Nitsche’s method in a fictitious domain in order to impose the essential boundary condition accurately in a weak sense. This approach was first proposed for an elliptic interface problem in [2] and later for a Stokes problem in [3]. In [4] and later in [5, 6] a stabilization of the classical Nitsche’s method for the imposition of Dirichlet boundary conditions on a boundary not fitted to the mesh was considered for the Poisson problem and for the Stokes problem. In these methods the stabilization is applied in the boundary region and optimal convergence order and well-conditioned system matrices are ensured.

In this paper we propose a new fictitious domain approach for Stokes equations based on the Nitsche’s method by using the lowest velocity-pressure projection penalty parameter. Rather than requiring calculation of higher order derivatives or edge based on data structures, we use a local projection to construct our scheme. We also show that the new method is inf-sup stable and optimally accurate. Moreover, its advantage is its ability to easily treat complex boundary conditions compared to existing ones. This method also can be of interest for computational domains having moving boundaries or boundaries with a complex geometry and various conditions on them. In this paper only Dirichlet and Neumann boundary conditions are considered. An extension to more complex boundary data is straightforward.

The paper is organized as follows. We describe the model problem which is represented by a linear scalar model problem with Neumann and Dirichlet boundary conditions. In Section 2 we describe the new method for the new model problem without any stabilization. In Section 3 we present new stabilization technique with the theoretical convergence analysis. We also give an error estimate in Section 4. Finally, some numerical experiments are presented in Section 5.
2. The model problem
In this paper we consider the following stationary Stokes equation in $\mathbb{R}^2$:

$$
\begin{aligned}
-\nabla \cdot (\gamma \nabla u) + \sigma u + \nabla p &= f & \text{in } & \Omega \\
\nabla \cdot u &= 0 & \text{in } & \Omega \\
u &= g_d & \text{on } & \Gamma_d \\
\rho \partial_n u &= g_N & \text{on } & \Gamma_N
\end{aligned}
$$

(1)

Where $\Omega$ is an open, bounded subset of $\mathbb{R}^2$, $u$ denotes the average fluid velocity, $p$ the pressure, $\gamma$ the viscosity coefficient of the flow, $\sigma$ the viscosity divided by the permeability, and $f$ is a given external force. We assume that $\Omega$ has polygonal boundary $\partial \Omega$ and that the boundary is divided into two non-overlapping sets $\partial \Omega = \Gamma_d \cup \Gamma_N$. Then the weak formulation of (1) now takes the form:

Finding $(u, p) \in V(\Omega) \times Q(\Omega)$ such that

$$a(u, p; v, q) = F(v), \quad \forall (v, q) \in V(\Omega) \times Q(\Omega)$$

where

$$a(u, p; v, q) = (\gamma \nabla u, \nabla v)_\Omega + (\sigma u, v)_\Omega - (p, \nabla v)_\Omega + (\nabla u, q)_\Omega;$$

$$F(v) = (f, v)_\Omega + (g_N, v)_{\Gamma_N}$$

$$V(\Omega) = \{ v \in (H^1_0(\Omega))^2 \mid v|_{\Gamma_d} = g_d \}; \quad Q(\Omega) = L^2(\Omega) = \{ q \in L^2(\Omega) \mid \int_\Omega q \, dx = 0 \}$$

3. A stabilized formulation and its stability
We embed the domain $\Omega$ into the larger rectangular domain $\tilde{\Omega}$, where $\Omega \subset \tilde{\Omega}$. In a standard finite element method the mesh is fitted to the boundary or interpolates the boundary to some suitable order. Instead we propose to solve problem (1) approximately on a family of conforming triangulations $T_h$.

Let $h_\Gamma$ be a curvilinear triangulation of $\Omega$, such that $\Omega \subset T_h$, but $T_h \not\subset \Omega$. For all triangles $K \in T_h$ holds $K \cap \Omega \neq \emptyset$, and the domain covered by $T_h$ and $\Omega_T := \bigcup_{K \in T_h} K$ and $F_h$ denotes the set of interior faces of the triangles in $T_h$. Let $h_K$ be the diameter of $K$ which is less or equal to the grid-parameter $h$ and $h_0 = \max_{K \in T_h} h_K > 0$. By $G_h := \{ K \in T_h \mid K \cap \Gamma \neq \emptyset \}$ we denote the set of elements that are intersected by the boundary. For an element $K \in G_h$, let $\Gamma_K := \Gamma \cap K$ be the part of $\Gamma$ intersecting $K$. Similarly, $\Gamma_{D, K} := \Gamma_D \cap K$ and $\Gamma_{N, K} := \Gamma_N \cap K$. $F_G$ will denote the faces of triangles in $G_h$ and not on the mesh boundary $\Gamma$. For each $F \in F_G$ we introduce the pair of element $K$ and $K'$ such that $F = K \cup K'$ and set $K_F := K \cup K'$. For details one can refer to [6]. Furthermore, the assumptions A1-A3 and definitions regarding the domain $\Omega$ and its boundary $\Gamma$ are made as in [6]. Associated with $T_h$ we have the finite element spaces

$$V_h = \{ v_h \in (C^0(\tilde{\Omega}))^2 : v_h|_K \in (P_1(K))^2, \forall K \in T_h \};$$

$$Q_h = \{ q_h \in C^0(\tilde{\Omega}) : q_h|_K \in P_1(K), \forall K \in T_h \}$$
Here $P_1(K)$ denotes the linear polynomials on $T_h$. Let $J_f$ be the $L^2$-projection onto $P_1(K_F)$ and let $\pi_K$ be the $L^2$-projection onto $P_0(K)$ for each $K \in T_h$. Now we propose our stabilized Nitsche method: finding $(u_h, p_h) \in V_h \times Q_h$ such that

$$A_h(u_h, p_h; v_h, q_h) + G_1(u_h, v_h) - G_2(p_h, q_h) = L_h(v_h), \quad \forall (v_h, q_h) \in V_h \times Q_h$$

(2)

Where $\lambda > 0$ is a penalty parameter and

$$A_h(u, p; v, q) = (\nabla u, \nabla v)_\Omega + (\sigma u, v)_\Omega - (p, \nabla \cdot v)_\Omega - \gamma \delta_h u, v > \Gamma_p - \gamma \nu, \nu > \Gamma_n$$

$$+ \sum_{I \in \Gamma} \gamma \delta_{h, I}^- < u, v > \Gamma_{I, n} + < p, n \cdot v > \Gamma_p - < n \cdot u, q > \Gamma_n$$

$$L_h(v) = (f, v)_\Omega + (g_{\partial \Omega}, v)_{\partial \Omega} - (\gamma g_{\partial \Omega}, \nu, v)_{\partial \Omega} + \sum_{I \in \Gamma} \gamma \delta_{h, I}^- < g_{\partial \Omega}, v > \Gamma_{I, n} - < n \cdot g_{\partial \Omega}, q > \Gamma_n$$

$$G_1(u, v) = \sum_{F \in \mathcal{T}_h} \int_F h_{-2}^2 (u - J_f u) (v - J_f v) dx; G_2(p, q) = \sum_{F \in \mathcal{T}_h} \int_F (p - \pi_k p) (q - \pi_k q) dx$$

On the discrete space $W_h = V_h \times Q_h$ we define the following expression:

$$\|v, q\|^2_h = \|\nabla v\|^2_\Omega + \|\sigma v\|^2_\Omega + \sum_{I \in \Gamma} \gamma \delta_{h, I}^- \|v\|^2_{0, \Gamma_{I, n}} + \sum_{I \in \Gamma} \gamma \delta_{h, I}^- \|\nabla v\|^2_{\Gamma, I} + \sum_{F \in \mathcal{T}_h} \|q - \pi_k q\|^2_{K}$$

In order to analyze the method the following versions of trace/inverse inequalities are needed.

**Lemma 3.1.** (Trace inequality)There exists $C \in R$ such that for all $v_h \in V_h$, and $K \in T_h$ there holds

$$h^{-1} \|v_h\|_{K} \leq C(\|v_h\|_{K} + h_{K} \|\nabla v_h\|_{K} - h^{1/2} \|\nabla v_h \cdot n_h\|_{\Gamma_{K}} \leq C\|\nabla v_h\|_{K} , \quad h^{1/2} \|\nabla v_h \cdot n_h\|_{\Gamma_{K}} \leq C\|\nabla v_h\|_{K}$$

**Lemma 3.2.** (Inverse inequality)There exists $C \in R$ such that for all $v_h \in V_h$, and $K \in T_h$ there holds

$$\|\nabla v_h\|_{K} \leq C h^{-1} \|v_h\|_{K}.$$
By applying the Cauchy-Schwarz, Young’s inequality and using Lemma 3.1 we can get that
\[
\left| 2\gamma \partial_u u_h, u_h > \Gamma_{p,x} \right| \leq 2\gamma Ch_k^2 \| \nabla u_h \|_0, \| u_h \|_{0, \Gamma_{p,x}} \leq \frac{\gamma}{2} \| \nabla u_h \|^2_0 + \frac{2\gamma C^2}{h_k} \| u_h \|^2_{0, \Gamma_{p,x}}
\]

Then we have with \( \lambda \geq \lambda_0 \geq \frac{1}{2} + 2C^2 \)
\[
A_h(u_h, p_h; u_h, -p_h) + G_1(u_h, u_h) - G_2(p_h, -p_h) \geq \frac{\gamma}{2} \| \nabla u_h \|^2_0 + \gamma \| u_h \|^2 + \sum_{p, x} \gamma \left( \frac{\lambda - 2C^2}{h_k} \right) \| u_h \|^2_{0, \Gamma_{p,x}}
+ G_1(u_h, u_h) + G_2(p_h, p_h)
\]
\[
\geq \frac{\gamma}{2} \| \nabla u_h \|^2_0 + \gamma \| u_h \|^2 + \sum_{p, x} \frac{\gamma \lambda}{2h_k} \| u_h \|^2_{0, \Gamma_{p,x}} + G_1(u_h, u_h) + G_2(p_h, p_h)
\]
\[
(4)
\]

Secondly, as \( p_h \in H^1(\Omega) \), it exists a unique real constant \( c_0 \) with \( p_h \|_{\Omega} = p_{0,h} + c_0 \), \( p_{0,h} \in L^2(\Omega) \).
From Lemma 3.3 it follows the existence of \( w \in (H^1_0(\Omega))^2 \) and constants such that
\[
\| w \|_{1, \Omega} \leq C_1 \| p \|_{0, \Omega} \quad \| w \|_{1, \Omega} \leq C_2 \| p \|_{0, \Omega} \quad (\nabla \cdot \cdot w, \| p \|_{0, \Omega} = \| p \|_{0, \Omega} \cdot
\]

Let now \( w_h := \Pi \in V_h \), \( \Pi \) be \( L^2 \) interpolation operator, fulfilling the standard interpolation properties. Also we have \( \| w - w_h \|_{1, \Omega} \leq C \| w \|_{0, h, \Omega} \). Then there clearly exist positive constants \( C_1', C_2' > 0 \) with \( \| w_h \|_{1, \Omega} \leq C_1' \| p \|_{0, \Omega} \); \( \| w_h \|_{1, \Omega} \leq C_2' \| p \|_{0, \Omega} \);

Note that \( -\int_{\Omega} p_h \nabla \cdot \cdot (w_h - w) dx = -\int_{\Omega} p_h \nabla \cdot \cdot (w_h - w) dx + \| p_h \|^2_{0, \Omega} \)

Testing now with setting \( (v_h, q_h) = (w_h, 0) \) in the second argument, we get
\[
A_h(u_h, p_h; w_h, 0) + G_2(p_h, 0) = \gamma \nabla u_h \cdot \nabla w_h + (\sigma u_h, w_h - (p_h, \nabla \cdot \cdot (w_h - w))) + \| p_h \|^2_{0, \Omega} - \gamma \partial_h u_h, w_h - w > \Gamma_{p, \omega} - \gamma u_h, \partial_h w_h > \Gamma_{p, \omega} + \sum_{p, x} \frac{\gamma \lambda}{h_k} < u_h, w_h - w > \Gamma_{p, \omega} + \gamma p_h, n \cdot (w_h - w) > \Gamma_{p, \omega}
\]

Using Schwarz inequality and Young’s inequality, we have
\[
\gamma \nabla u_h \cdot \nabla w_h \geq -\gamma \nabla u_h \cdot \nabla w_h \geq -C_3 \gamma \| u_h \|_{1, \Omega} \| p_h \|_{0, \Omega} \geq -C_3 \left( \frac{\gamma}{2\epsilon} \| u_h \|^2_{1, \Omega} + \frac{\epsilon}{2} \| p_h \|^2_{0, \Omega} \right)
\]

Similarly, \( (\sigma u_h, w_h, \| p \|_{0, \Omega} \geq -C_4 \left( \frac{\gamma}{2\epsilon} \| u_h \|^2_{1, \Omega} + \frac{\epsilon}{2} \| p_h \|^2_{0, \Omega} \right) \)

Using intergration by parts, the interpolation estimate and the standard inverse inequality, we have
\[
-\int_{\Omega} p_h \nabla \cdot \cdot (w_h - w) dx + \int_{\Omega} p_h \nabla \cdot \cdot (w_h - w) dx = \sum_{\Gamma_{p, \omega}} \int_{\Gamma_{p, \omega}} (w_h - w) \nabla p_h dx \geq -\sum_{\Gamma_{p, \omega}} \int_{\Gamma_{p, \omega}} (w_h - w) \nabla p_h dx 
\]
\[
\geq -\sum_{\Gamma_{p, \omega}} \int_{\Gamma_{p, \omega}} (w_h - w) \nabla p_h dx \geq -C_5 \| \nabla w_h \|_{0, \omega} \| h^{-1} \| p_h \|_{0, \Omega} \geq -C_5 \| p_h \|^2_{0, \Omega}
\]

Then we focus on the boundary terms:
\[
\sum_{\Gamma_{p, \omega}} \frac{\gamma \lambda}{h_k} < u_h, w_h - w > \Gamma_{p, \omega} \geq \sum_{\Gamma_{p, \omega}} \frac{\gamma \lambda}{h_k} \| u_h \|^2_{0, \Gamma_{p, \omega}} \| w_h - w > \Gamma_{p, \omega} \| w_h - w > \Gamma_{p, \omega} 
\]
\[
\geq -C_6 \left( \sum_{\Gamma_{p, \omega}} \frac{\lambda}{h_k} \| u_h \|^2_{0, \Gamma_{p, \omega}} \right)^2 \left( \sum_{\Gamma_{p, \omega}} \frac{\lambda}{h_k} \| w_h - w > \Gamma_{p, \omega} \|^2 \right)^2 
\]
\[ L_{0}(\sum_{\Gamma,p,x} h_{K}^{i} \|u_{h}\|_{0,\Gamma,p,x}^{2})^{1/2} (\sum_{\Gamma,p,x} \mathcal{W}_{h}^{2})^{1/2} \geq -C_{0} \left( \sum_{\Gamma,p,x} \|h_{K}^{i} u_{h}\|_{0,\Gamma,p,x}^{2} + \frac{\varepsilon}{2} \|p_{h}\|_{0,\Omega}^{2} \right) \]

Similarly as the above, we can get
\[ < \partial_{n} u_{h}, w_{h} - w >_{\Gamma,x} \geq -C_{7} \left( \frac{1}{2\varepsilon} \sum_{\Gamma,p,x} h_{K}^{i} \|u_{h}\|_{0,\Gamma,p,x}^{2} + \frac{\varepsilon}{2} \|p_{h}\|_{0,\Omega}^{2} \right) \]

This yields the following inequality
\[ A_{h}(u_{h}, p_{h}; w_{h}, 0) + G_{1}(u_{h}, w_{h}) - G_{2}(p_{h}, 0) \geq -C_{3} \gamma \|u_{h}\|_{1,\Omega}^{2} - C_{5} \frac{\varepsilon}{2} \sigma \|u_{h}\|_{0,\Omega}^{2} + (1 - C_{5} \frac{\varepsilon}{C_{10}}) \|p_{h}\|_{0,\Omega}^{2} - C_{11} \frac{\varepsilon}{2} \sum_{\Gamma,p,x} h_{K}^{i} \|u_{h}\|_{0,\Gamma,p,x}^{2} \]

Choosing now \( \varepsilon = \frac{1-2C_{5}}{C_{10}} \) and with an appropriate constant \( C_{13} \) then we can get that
\[ A_{h}(u_{h}, p_{h}; w_{h}, 0) + G_{1}(u_{h}, w_{h}) - G_{2}(p_{h}, 0) \geq \frac{1}{2} \|p_{h}\|_{0,\Omega}^{2} - C_{13} \frac{\varepsilon}{2} (\gamma \|u_{h}\|_{1,\Omega} + \sigma \|u_{h}\|_{0,\Omega}^{2} + \sum_{\Gamma,p,x} h_{K}^{i} \|u_{h}\|_{0,\Gamma,p,x}^{2}) \]

Finally, taking \( (v_{h}, q_{h}) = (\partial_{n} u_{h} + w_{h}, -p_{h}) \) where \( \delta \geq 0 \) is a parameter
\[ A_{h}(u_{h}, p_{h}; \partial_{n} u_{h} + w_{h}, -p_{h}) + G_{1}(u_{h}, \partial_{n} u_{h} + w_{h}) - G_{2}(p_{h}, -p_{h}) \geq \frac{1}{2} \|p_{h}\|_{0,\Omega}^{2} + \frac{\delta}{2} \|u_{h}\|_{1,\Omega}^{2} + (\delta - C_{13} \frac{\varepsilon}{2}) \sigma \|u_{h}\|_{0,\Omega}^{2} + \sum_{\Gamma,p,x} \frac{\delta}{2} - C_{13} \frac{\varepsilon}{2} h_{K}^{i} \|u_{h}\|_{0,\Gamma,p,x}^{2} \]

Fixing the parameter \( \delta \) by \( \delta := \max\{1 + C_{13}, \frac{1}{2}\} \) yields
\[ A_{h}(u_{h}, p_{h}; \partial_{n} u_{h} + w_{h}, -p_{h}) + G_{1}(u_{h}, \partial_{n} u_{h} + w_{h}) - G_{2}(p_{h}, -p_{h}) \geq \frac{1}{2} \|(u_{h}, p_{h})\|_{h}^{2} \quad (5) \]

On the other hand we have
\[ \|(\partial_{n} u_{h} + w_{h}, -p_{h})\|_{h} \leq \delta \|(u_{h}, p_{h})\|_{h} + \|(w_{h}, 0)\|_{h} \leq (1 + \delta) \|(u_{h}, p_{h})\|_{h} \]

With (5) we can prove the stability estimate.

4. Error analysis

**Lemma 4.1.** Let \( (u, p) \in V \times Q \) be the solution of (1) and \( (u_{h}, p_{h}) \in V_{h} \times Q_{h} \) be the solution of (2), respectively, for all \( (v_{h}, q_{h}) \in V_{h} \times Q_{h} \)
\[ A_{h}(E_{h} u - u_{h}, E_{h} p - p_{h}; v_{h}, q_{h}) = G_{1}(u_{h}, v_{h}) - G_{2}(p_{h}, q_{h}) \]
Where $E_i, i = 1, 2$ are two extension operators $E_1 : H^1(\Omega) \rightarrow H^1(\tilde{\Omega})$; $E_2 : H^2(\Omega) \rightarrow H^2(\tilde{\Omega})$ such that $(E_1 v)v|_{\partial\Omega} = v$ and $\|E_1 v\|_{H^1(\tilde{\Omega})} \leq C\|v\|_{H^1(\Omega)}$ for all $v \in H^1(\Omega), i = 1, 2$.

In order to give a priori error estimate we define:

$$e^h := (e_u, e_p) = (E_2 u - u_h, E_1 p - p_h) = (E_2 u - I^* u, E_1 p - J^* p) + (I^* u - u_h, J^* p - p_h)$$

Where $I^* := I_h E_2$ with $I_h$ being the standard nodal interpolation and $J^* := J_h E_1$ with $J_h$ the $L^2$ - interpolation operator. For simply we set $\gamma = 1, \sigma = 1$.

**Lemma 4.2.** There exists a constant $C > 0$ not depending on $h$ such that

$$\left\|(E_2 u - I^* u, E_1 p - J^* p)\right\|_h \leq C h \left(\frac{\|u\|_{L^2(\Omega)}}{1} + \|p\|_{L^2(\Omega)}\right) \tag{7}$$

Proof: As $\left\|(E_2 u - I^* u, E_1 p - J^* p)\right\|_h = \left\|E_2 u - I^* u\right\|_{L^2(\Omega)}^2 + \left\|E_1 p - J^* p\right\|_{L^2(\Omega)}^2$

$$+ \sum_{i,F} h_k^{-1} \left\|E_2 u - I^* u\right\|_{L^2(\Omega)}^2 + \sum_{i,F} h_k^{-1} \left\|E_2 u - I^* u - J_i (E_2 u - I^* u)\right\|_{L^2(\Omega)}^2$$

$$+ \sum_{i,F} \left\|E_1 p - J^* p - \pi_k (E_1 p - J^* p)\right\|_{L^2(\Omega)}^2$$

The individual terms can be handled in the following way

$$\left\|E_2 u - I^* u\right\|_{L^2(\Omega)}^2 \leq C h^2 \left\|E_2 u\right\|_{L^2(\Omega)}^2 \leq C h^2 \left\|E_1 p - J^* p\right\|_{L^2(\Omega)}^2 \leq C h^2 \left\|E_1 p\right\|_{L^2(\Omega)}^2$$

$$h_k^{-2} \left\|E_2 u - I^* u - J_i (E_2 u - I^* u)\right\|_{L^2(\Omega)}^2 \leq h_k^{-2} \left(\left\|E_2 u - J_i E_2 u\right\|_{L^2(\Omega)}^2 + \left\|I^* u - J_i I^* u\right\|_{L^2(\Omega)}^2\right)$$

$$\leq C h_k^{-2} \left(\left\|E_2 u\right\|_{L^2(\Omega)}^2 + \left\|I^* u\right\|_{L^2(\Omega)}^2\right) \leq C h_k^{-2} \left\|E_2 u\right\|_{L^2(\Omega)}^2 + C h_k^{-2} \left\|I^* u\right\|_{L^2(\Omega)}^2$$

The boundary term is treated by using trace inequalities and inverse inequality

$$h_k^{-1} \left\|E_2 u - I^* u\right\|_{L^2(\Omega)}^2 \leq C h_k^{-1} \left\|E_2 u - I^* u\right\|_{L^2(\Omega)}^2 \leq C h_k^{-1} \left\|E_2 u\right\|_{L^2(\Omega)}^2$$

Hence by squaring and summing together we can get that

$$\sum_{i,F} h_k^{-1} \left\|E_2 u - I^* u\right\|_{L^2(\Omega)}^2 \leq C h^2 \left\|u\right\|_{L^2(\Omega)}^2$$

Finally, with the above analysis we can get the result.

$$\left\|(E_2 u - I^* u, E_1 p - J^* p)\right\|_h \leq C h^2 \left(\left\|u\right\|_{L^2(\Omega)} + \left\|p\right\|_{L^2(\Omega)}\right)$$

**Theorem 4.1.** Let $(u, p) \in V \times Q$ be the solution of (1) and $(u_h, p_h) \in V_h \times Q_h$ be the solution of (2), Then there exists a constant $C > 0$ not depending on $h$ such that

$$\left\|(u - u_h, p - p_h)\right\|_h \leq C h^2 \left(\left\|u\right\|_{L^2(\Omega)} + \left\|p\right\|_{L^2(\Omega)}\right) \tag{8}$$

Proof: Firstly, $\left\|(u - u_h, p - p_h)\right\|_h \leq \left\|(E_2 u - u_h, E_1 p - p_h)\right\|_h$
\[ \leq \left\| (E_2 u - I' u, E_1 p - J^* p) \right\|_h + \left\| (I' u - u_h, J^* p - p_h) \right\|_h \]

Secondly, by applying Theorem 3.1 and Lemma 4.1 we have

\[
\leq \beta \sup_{(v_h, q_h) \in V_h \times Q_h} \beta A_h(I^* u - u_h, J^* p - p_h; v_h, q_h) + G_1(I^* u - u_h, v_h) - G_2(J^* p - p_h, q_h)
\]

With the definition of \( A_h(\cdot, \cdot) \) and \( G_i(\cdot, \cdot), i = 1,2 \) and the proof of Lemma 4.1 we have

\[
\begin{align*}
A_h(I^* u - E_2 u, J^* p - E_1 p; v_h, q_h) & \leq C h \left( \| u \|_{L^2(\Omega)} + \| p \|_{L^2(\Omega)} \right) \| (v_h, q_h) \|_h \\
G_i(I^* u - E_2 u, v_h) & - G_2(J^* p, q_h) \\
& = G_1(I^* u - E_2 u, v_h) + G_2(E_1 p - J^* p, q_h) + G_1(E_2 u, v_h) - G_2(E_1 p, q_h) \\
& \leq C h \left( \| u \|_{L^2(\Omega)} + \| p \|_{L^2(\Omega)} \right) \| (v_h, q_h) \|_h 
\end{align*}
\]

Finally, with Lemma 4.2 we can get the result.

5. A numerical experiment

Let \( \Omega \) is a circle centered in \((0, 0)\) with the radius 0.6 and \( \gamma = 1, \sigma = 1 \). \( \widetilde{\Omega} = [-1,1] \times [-1,1] \) is the unit square with pure Dirichlet boundary conditions computed from the known exact solution. The analytical solution of the problem is \( u = -\nabla p = (\cos(x) \sinh(y), \sin(x) \cosh(y)) \), \( p = -\sin(x) \sinh(y) - (\cos(1) - 1)(\cosh(1) - 1) \). Since the pressure \( p \) is harmonic, the solution is independent of the viscosity. In the subsequent computations Nietzsche stability parameters are \( \lambda = 0.1 \). A direct solver based on the UMFPACK library has been employed for solving the resulting system. The results regarding the usual error analysis are showed in Table 1.

Table 1. Relative error-reduction for the velocity and pressure component

| h    | \( \| e_u \|_{0,\Omega_h} / \| u \|_{0,\Omega_h} \) | rate | \( \| e_p \|_{0,\Omega_h} / \| p \|_{0,\Omega_h} \) | rate |
|------|---------------------------------|------|---------------------------------|------|
| 0.2022 | 4.63e-02 | - | 2.89e-01 | - |
| 0.1011 | 3.47e-03 | 1.24 | 7.76e-02 | 1.22 |
| 0.0505 | 4.15e-04 | 1.67 | 2.45e-02 | 1.56 |

6. Conclusion

We proposed a new stabilized fictitious domain method to solve the Stokes problem. The method is enforced based on the local projection and Nietzsche’s method. We use the lowest equal order finite element spaces to approximate the velocity and the pressure. The optimal order error estimates are obtained. The numerical test also confirmed the theoretical analysis. In this paper only Dirichlet and Neumann boundary conditions are considered. An extension to more complex boundary data and other problem will also be one of our future subjects.

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