Boundary condition for D-brane from Wilson loop, and gravitational interpretation of eigenvalue in matrix model in AdS/CFT correspondence

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Abstract

We study the supersymmetric Wilson loops in the four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory in the context of AdS/CFT correspondence. In the gauge theory side, it is known that the expectation value of the Wilson loops of circular shape with winding number $k$, $W_k(C)$, is calculable by using a Gaussian matrix model. In the gravity side, the expectation value of the loop is conjectured to be given by the classical value of the action $S_{D3}$ for a probe D3-brane with $k$ electric fluxes as $\langle W_k(C) \rangle = e^{-S_{D3}}$. Given such correspondence, we pursue the interpretation of the matrix model eigenvalue density, or more precisely the resolvent, from the viewpoint of the probe D3-brane. We see that the position of an eigenvalue appears as an integrated flux on the D3-brane. In the course of our analysis, we also clarify the boundary condition on the D3-brane in terms of the Wilson loop.

PACS codes: 11.15.Pg, 11.25.Tq, 11.25.Uv.

Keywords: AdS/CFT correspondence, Wilson loops, matrix models.

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1 Introduction

Duality between string theory and gauge theory has been an important concept for theoretical particle physics. In particular Wilson loop would play a unique role there, since originally it has been introduced in the context of stringy behavior of the strong coupling gauge theory, i.e., the area law behavior in the confining gauge theory.

In the recent development of the duality, the “area law” of the Wilson loop has been rediscovered [1] in the context of the AdS/CFT correspondence. Hence in AdS/CFT, Wilson loop is useful for concrete realization of the duality. There, we usually begin with the $N$ D3-branes located on top of each other and put another D3-brane in parallel with them at a large distance. Then we evaluate the amplitude for the propagation of a string stretched between the $N$ D3-branes and the single separated brane from two different points of view, from gauge theory and from gravity picture.

From the viewpoint of the gauge theory, the stretched string corresponds to a bi-fundamental matter of the $SU(N) \times U(1)$ gauge theory. By evaluating the amplitude for the string propagation along a loop $C$ on the isolated brane, we can introduce a Wilson loop for $C$ in the bi-fundamental representation of the $SU(N) \times U(1)$ gauge group.

In order to discuss the gravity side, we replace the $N$ D3-branes with the near horizon geometry of the extremal black 3-brane solution. Then the isolated brane is recognized as a single probe D3-brane in the near horizon region and we evaluate the propagation amplitude for the string attached to the loop $C$ on the probe D3-brane.

The AdS/CFT correspondence claims that these two different points of view give the
same result. Therefore we reach the following conjecture:

\[
\left\langle \frac{1}{N} \text{tr} \exp \left( \int ds (iA_\mu \dot{x}^\mu + |\dot{x}|^2 \Phi_i) \right) \right\rangle_{\text{CFT}} = \int_C dX e^{-S_{\text{string}}}. \tag{1.1}
\]

Here on the left hand side, not only the gauge field but the scalar fields are included since a string attached to D-branes is coupled to all of them. Also, the U(1) part of the Wilson loop has been omitted since we discuss the limit in which the U(1) brane is separated by a large distance from the remaining \(N\) D3-branes and thus its dynamics decouples from the SU(\(N\)) part we are interested in. Actually in the gravity side we consider the string world sheet which is attached to the loop \(C\) on the AdS boundary. Hence the right hand side only takes account of the SU(\(N\)) contribution in terms of the gauge theory side. Also we need to take care of the definition of the functional \(S_{\text{string}}\) on the right hand side. In the paper [2] it has been proposed that in addition to the usual Nambu-Goto type action, we need to add appropriate boundary terms which eliminate divergence due to the infinite scale factor from the vicinity of the AdS boundary. Introducing these boundary terms was recognized as performing the Legendre transformation to change the boundary condition of the world sheet.

The conjecture (1.1) has been checked mainly in the case of the straight Wilson line or the circular Wilson loop which preserves some global supersymmetry. In the study of the circular Wilson loop, a Gaussian matrix model plays an important role. It was proposed as a technical tool for summing up all the planar ladder diagrams [4] and a further argument for the matrix model including the non-planar diagrams was given in [5]. In the more recent paper [7], an argument based on the topological twist and localization technique is discussed. A relation to the Gaussian matrix model has also been discussed by using the mirror symmetry and the geometric transition [8].

In the study of the validity of the conjecture (1.1), an important work has been done in the paper [9]. There the authors considered a circular Wilson loop with winding number \(k\). This winding number corresponds to the string charge because \(k\) stretched strings between \(N\) D3-branes and the single D-brane mentioned above yield the Wilson loop with winding number \(k\). In the gauge theory side the expectation value of the operator can be evaluated by the matrix model of the same Gaussian action as in the single winding case. On the other hand, in the gravity side, they considered a spike D3-brane solution carrying a non-trivial electric flux by \(k\) units of the electric charge, because the electric charge on the D-brane corresponds to the string charge. In the gravity side the computation is essentially the same as the right hand side of (1.1) with the string action being replaced by the D3-brane action. In [9], it was found that the relevant D3-brane solution has the geometry

\[1\] For a recent quantitative test of a similar conjecture to (1.1) in the case with less symmetry, see the paper [3]. There a similar relation in the case of finite temperature D0-brane system is tested by using the Monte Carlo simulation.

\[2\] See also [6] for a discussion on the relation between the Wilson loop and the matrix model in the beta-deformed super Yang-Mills theory.

\[3\] It has been proposed that the D3-brane actually corresponds to the Wilson loop in the symmetric representation [10], and has been argued that in the strong coupling limit of the gauge theory, a multiply wound Wilson loop and a symmetric one give an identical expectation value. See the paper [11], for these issues. We however consider that there is a subtlety here. See the concluding section.
of AdS$_2 \times$S$^2$ where the radius of the S$^2$ is given by the parameter $\kappa \equiv k\sqrt{\lambda}/(4N)$. This suggests that the large-$N$ limit with fixed $\kappa$ in the gauge theory side is a quite interesting limit. Actually it was found that in this limit, the expectation value of the Wilson loop computed by using the Gaussian matrix model agrees with the prediction of the gravity side. A very important point in this work was that the large-$N$ limit is not the planar limit but the result contains a class of the non-planar contributions since the parameter $\kappa$ depends on $N$ inversely.

One of the interesting developments which followed the paper [9] is the study of the geometric aspects of the eigenvalue in the Gaussian matrix model. Stimulated by the case of a local operator [12], the correspondence between the eigenvalue distribution and a certain aspects of the geometry has been discussed in [13, 14]. In general, gravitational interpretation of eigenvalues in matrix models would be an interesting and essential problem for gauge/string duality, or even for nonperturbative formulation of string theory. For example, in the IIB matrix model [15] the eigenvalues of matrices are interpreted as the space-time points and their dynamics is discussed as emergent geometry [16]. Therefore, even if we concentrate on a particular eigenvalue of a specific matrix model, it would be intriguing to find its gravitational interpretation clearly in the context of the AdS/CFT correspondence.

Information on the eigenvalue of a matrix model is well packed in the expectation value of an operator called the resolvent. In this paper we thus consider the resolvent of the matrix model from the viewpoint of the probe D3-brane. Our main goal is to discuss a gravitational interpretation of the eigenvalue distribution, or more precisely the resolvent, in the Gaussian matrix model in the presence of the Wilson loop with a large winding number $k$. In particular we are interested in results which follow directly from the basic correspondence (1.1) using the D3-brane action. This is in contrast to the “bubbling geometry” approach taken in [12–14]. On the way of the analysis, we also identify what kind of boundary conditions should be imposed on the D3-brane configuration so that it will correspond to the Wilson loop with winding number $k$.

The rest of the paper is organized as follows. In section 2, we review the matrix model computation and discuss the resolvent in the $1/N$-expansion with fixed $\kappa$. From this result, we will see what kind of quantity in the gravity side should reproduce the matrix model resolvent. In section 3, we begin with a brief review of the setup and the results of [9]. Next we perform the computation in the gravity side which yields the same result as in the matrix model, and then we find a gravitational interpretation of an eigenvalue. In the course of the analysis we will give some interesting aspects of the work [9], especially identification of appropriate boundary conditions for D3-brane configurations. We then conclude the paper in section 4 with some discussions. A couple of appendices are devoted to filling in some technical details in the main part of the paper.

## 2 Multiply-wound Wilson loops in gauge theory

In this section we discuss the gauge theory side and its relationship to the matrix model. In subsection 2.1 we start with a brief review of the expectation value of the Wilson
loop in the large-$N$ matrix model. We also mention the eigenvalue distribution derived in [13] by solving the saddle point equation of the matrix model. For our present purpose, however, it will turn out to be more useful to discuss the resolvent through the Laplace transformation. In subsection 2.2 we discuss the Laplace transformation that motivates us to proceed to the gravitational interpretation.

2.1 Matrix model calculation

We consider the Wilson loop in the four-dimensional $N = 4$ super Yang-Mills theory, which is defined by

$$W_k(C) = \frac{1}{N} \text{tr} \text{Pexp} \left( \int ds \left( iA_\mu(x(s)) \dot{x}^\mu(s) + \Phi_i(x(s)) |\dot{x}(s)| \theta^i(s) \right) \right). \quad (2.1)$$

Here $A_\mu(x)$ ($\mu = 1 \ldots 4$) and $\Phi_i(x)$ ($i = 1 \ldots 6$) are the gauge field and the scalar fields, respectively. The coordinates $x^\mu$ span the Euclidean four-dimensional space and $\theta^i$ is the coordinate on the unit $S^5$. We consider the operator in which the path $C: \{ x^\mu(s) | 0 \leq s < 2k\pi \}$ is a multiply winding circle and $\theta^i$ is constant. The subscript $k$ on the left hand side indicates that the loop $C$ goes around the circle $\{ x^\mu(s) | 0 \leq s < 2\pi \}$ $k$ times.

The expectation value of the operator (2.1) is calculable by means of the Gaussian matrix model [4, 5, 7]:

$$\langle W_k(C) \rangle_{N=4 \text{SYM}} = \left\langle \frac{1}{N} \text{tr} e^{kM} \right\rangle_{\text{matrix model}} = \frac{1}{Z} \int dM \frac{1}{N} \text{tr} e^{kM} e^{-\frac{2N}{k} \text{tr} M^2} = \frac{1}{N} e^{\frac{k^2}{2L_1^{(1)}}} L_{N-1}^{(1)}(-k^2) \equiv f(k', N), \quad (2.2)$$

$$Z \equiv \int dM e^{-\frac{2N}{k} \text{tr} M^2}. \quad (2.3)$$

Here $k' \equiv \sqrt{N/(4N)}k$ and $L_n^{(\alpha)}(\zeta) \equiv (e^\zeta \zeta^{-\alpha}/n!) d^n/d\zeta^n (e^{-\zeta} \zeta^{n+\alpha})$ is the Laguerre polynomial. We introduced the function $f(k', N)$ for later convenience. Hereafter, the expectation value of the Wilson loop (2.1) is always understood as this matrix model expectation value.

In [9], the large-$N$ limit of $f(k', N)$ with fixed $\kappa \equiv k\sqrt{N}/(4N)$ was derived by using a differential equation. Here we follow their steps and introduce $\mathcal{F}(\kappa, N) \equiv -N^{-1} \log f(k', N)$. Then $\mathcal{F}$ satisfies the following differential equation:

$$(\partial_\kappa \mathcal{F})^2 - \frac{1}{\kappa N}(\kappa \partial^2_\kappa \mathcal{F} + 3\partial_\kappa \mathcal{F}) - 16(1 + \kappa^2) = 0. \quad (2.4)$$

In the large-$N$ limit with fixed $\kappa$, the differential equation (2.4) can be solved perturbatively and the solution is given by

$\mathcal{F}_\pm(\kappa, N)$

\[\text{The constant term can be fixed, for example, by comparing the small $\kappa$ limit of (2.5) with the modified Bessel function. For the relation between the function $f(k', N)$ with small $\kappa$ and the modified Bessel function, see the discussion around (2.28).}\]
\[ = \pm 2\left( \kappa \sqrt{1 + \kappa^2} + \text{arcsinh}\kappa \right) + \frac{1}{2N} \left( \log \kappa^3 \sqrt{1 + \kappa^2} + \log(32\pi N^3) \right) + \mathcal{O}(N^{-2}). \quad (2.5) \]

We take \( \mathcal{F}_- \) since it dominates in the large-\( N \) limit.

Now let us see that the leading term in (2.5) originates from an eigenvalue apart from the cut of the standard semi-circle distribution of the Gaussian matrix model [19]. In terms of eigenvalues, (2.2) can be written as

\[
\langle W_k(C) \rangle = \frac{1}{Z} \prod_i dm_i \exp(-V_{\text{eff}}),
\]

\[
V_{\text{eff}} \equiv N \sum_{i=1}^{N-1} V(m_i) - \sum_{i>j}^{N-1} \log(m_i - m_j)^2 + NV(m_N) - \sum_{j=1}^{N-1} \log(m_N - m_j)^2 - km_N,
\]

\[
V(m) = \frac{2}{\lambda} m^2,
\]

which tells us that the Wilson loop gives rise to force only on a single eigenvalue (say \( m_N \)). Therefore let us introduce the eigenvalue distribution as

\[
\rho(m) = \rho^{(0)}(m) + \frac{1}{N} \rho^{(1)}(m),
\]

\[
\rho^{(0)}(m) = \frac{1}{N} \sum_{i=1}^{N-1} \delta(m - m_i), \quad \rho^{(1)}(m) = \delta(m - m_N). \quad (2.7)
\]

Then \( V_{\text{eff}} \) can be rewritten as

\[
V_{\text{eff}} = N^2 \int dm \rho^{(0)}(m)V(m) - \frac{N^2}{2} \int dm dm' \rho^{(0)}(m)\rho^{(0)}(m') \log(m - m')^2
\]

\[
+ N \int dm \rho^{(1)}(m)V(m) - N \int dm dm' \rho^{(0)}(m)\rho^{(1)}(m') \log(m - m')^2 - k \int dm \rho^{(1)}(m)m.
\]

By considering the variation of \( \rho^{(0)}(m) \), the saddle point equation reads

\[
V'(m) - 2 \int dm' \frac{\rho^{(0)}(m')}{m - m'} - \frac{2}{N} \int dm' \frac{\rho^{(1)}(m')}{m - m'} = 0. \quad (2.9)
\]

In order for the subleading distribution \( \rho^{(1)}(m) \) to make sense, we need to discuss the distribution \( \rho^{(0)}(m) \) to the same order. Hence we further expand it with respect to \( 1/N \) as

\[
\rho^{(0)}(m) = \rho^{(0,0)}(m) + \frac{1}{N} \rho^{(0,1)}(m) + \mathcal{O}(N^{-2}). \quad (2.10)
\]

Here we note that the distribution functions \( \rho^{(0,0)}(m) \) and \( \rho^{(0,1)}(m) \) satisfy the following conditions:

\[
\int dm \rho^{(0,0)}(m) = 1, \quad \int dm \rho^{(0,1)}(m) = -1, \quad (2.11)
\]
which can be seen from (2.7) and the expansion (2.10). Hence, formally, \( \rho^{(0,0)}(m) \) corresponds to the distribution of \( N \) eigenvalues while \( (1/N)\rho^{(0,1)}(m) \) subtracts a single eigenvalue. By taking terms of order \( N^0 \) in (2.9), we find that \( \rho^{(0,0)}(m) \) satisfies the saddle point equation for the semi-circle distribution \( 2\pi\rho^{(0,0)}(m) = \sqrt{\frac{16}{\lambda} - \frac{16}{\lambda^2}m^2} \) with support \((-\sqrt{\lambda}, \sqrt{\lambda})\).

Next, before discussing \( \rho^{(0,1)}(m) \), we shall study the saddle point equation which will be derived by considering the variation of \( \rho^{(1)}(m) \). In order to solve this, it is reasonable to assume that \( m_N \) is isolated from the other \( N-1 \) eigenvalues due to the extra force coming from the Wilson loop and to make an ansatz \( \rho^{(1)}(m) = \delta(m - m^*) \) with \( m^* \) outside the range \((-\sqrt{\lambda}, \sqrt{\lambda})\). Then the saddle point equation for positive \( m^* \) becomes

\[
V'(m^*) - \left( V'(m^*) - \sqrt{V'(m^*)^2 - \frac{16}{\lambda}} \right) - \frac{k}{N} = 0,
\tag{2.12}
\]

from which we find

\[
m^* = \sqrt{\lambda}\sqrt{1 + \kappa^2},
\tag{2.13}
\]

which is indeed outside the cut\(^5\).

Finally, we discuss \( \rho^{(0,1)}(m) \). Let us first study the behavior of \( \rho^{(0,1)}(m) \) in the limit \( \kappa \to \infty \) by a physical argument without using the saddle point equation. In this limit, the isolated eigenvalue goes to infinity and its effect on the remaining \( N-1 \) eigenvalues should vanish. Then the resulting distribution function \( \rho^{(0)}(m) \) becomes the semi-circle distribution for \( N-1 \) eigenvalues which is given by

\[
\lim_{\kappa \to \infty} \rho^{(0)}(m) = \frac{N-1}{N} \times \frac{2}{\pi} \frac{1}{\lambda_{N-1}} \sqrt{\lambda_{N-1} - m^2} + O(N^{-2})
\tag{2.14}
\]

\[
= \frac{2}{\pi} \frac{1}{\lambda} \sqrt{\lambda - m^2} - \frac{1}{N} \frac{1}{\pi} \frac{1}{\sqrt{\lambda - m^2}} + O(N^{-2}).
\tag{2.15}
\]

Here we have introduced \( \lambda_{N-1} = (N-1)/N \times \lambda \) and the overall factor \( (N-1)/N \) on the right hand side of (2.14) reflects the fact that \( \rho^{(0)}(m) \) is defined with the overall factor \( 1/N \) instead of \( 1/(N-1) \) (see (2.7)). So, from this simple argument we find that, in the limit \( \kappa \to \infty \), \( (1/N)\rho^{(0,1)}(m) \) is given by the second term of (2.15) (including the minus sign).

For a generic value of \( \kappa \), we need to solve the saddle point equation for \( \rho^{(0,1)}(m) \), i.e., the terms of \( O(N^{-1}) \) in (2.9), which is now given by

\[
- \int dm' \rho^{(0,1)}(m') \frac{1}{m - m'} = 0.
\tag{2.16}
\]

Physically, this equation shows that the subleading distribution \( \rho^{(0,1)}(m) \) is determined by the following manner: the repulsive force from the isolated eigenvalue at \( m = m^* \)

\(^5\)This is the unique solution for \( k > 0 \). For negative \( k \), we would have the same solution with the opposite sign. In this paper we assume that \( k > 0 \) without loss of generality.
changes the subleading distribution $\rho^{(0,1)}(m)$ of the $N-1$ eigenvalues from its form in the limit $\kappa \to \infty$ discussed above. This change gives rise to additional force among the $N-1$ eigenvalues themselves. Then the distribution $\rho^{(0,1)}(m)$ is determined by the force balance condition among these effects. Therefore, $\rho^{(0,1)}(m)$ describes both effects of subtracting a single eigenvalue and distortion on the distribution of the remaining eigenvalues due to the isolated eigenvalue. It is evident that they are of $O(1/N)$. Since $\rho^{(0,1)}(m)$ subtracts an eigenvalue, it is natural to assume that it has its support only inside the range $(-\sqrt{\lambda}, \sqrt{\lambda})$ or at $m = m_\ast$. This is because the total eigenvalue density $\rho(m)$ must be positive everywhere. In fact, as we have shown above, the support of $\rho^{(0,1)}(m)$ in the limit $\kappa \to \infty$ is given by $(-\sqrt{\lambda}, \sqrt{\lambda})$. Furthermore, since the distortion due to the isolated eigenvalue should reach the whole distribution of the remaining eigenvalues, it is natural again that the support of $\rho^{(0,1)}(m)$ is $(-\sqrt{\lambda}, \sqrt{\lambda})$ for a generic value of $\kappa$. In the rest of this subsection we assume this property.

Let us solve the saddle point equation (2.16) based on this physical argument. For this purpose, we introduce the resolvent $R^{(0,1)}(m)$ as

$$R^{(0,1)}(m) = \frac{1}{N} \int dm' \rho^{(0,1)}(m') \frac{m - m'}{m - m'}.$$  (2.17)

Then $R^{(0,1)}(m)$ is determined by the following conditions:

- The real part of $R^{(0,1)}(m)$ at the support of $\rho^{(0,1)}(m)$ is given by (2.16).

- For the generic value of $\kappa$, the resolvent has one cut $(-\sqrt{\lambda}, \sqrt{\lambda})$ while it has no pole. This is due to the assumption for the support of $\rho^{(0,1)}(m)$ mentioned above.

- In the large-$m$ limit, the condition $R^{(0,1)}(m) \to -\frac{1}{N} \frac{1}{m}$ should be satisfied. This is due to the second equation of (2.11).

By taking account of these conditions, we can fix the resolvent and the eigenvalue distribution completely as

$$R^{(0,1)}(m) = \frac{1}{N} \left( -\frac{1}{m - m_\ast} + \frac{\sqrt{\lambda} \kappa}{(m - m_\ast) \sqrt{m^2 - \lambda}} \right),$$  (2.18)

$$\rho^{(0,1)}(m) = -\frac{N}{2\pi i} \left( R^{(0,1)}(m + i\epsilon) - R^{(0,1)}(m - i\epsilon) \right)$$  (2.19)

$$= \begin{cases} \frac{1}{\pi (m - m_\ast) \sqrt{\lambda - m^2}} & (-\sqrt{\lambda} \leq m \leq \sqrt{\lambda}) \quad (\kappa \neq 0), \\ -\delta(m - \sqrt{\lambda}) & (\kappa = 0). \end{cases}$$  (2.20)

Here in the case of $\kappa = 0$, we have the delta function distribution which exactly cancels the another subleading distribution $\rho^{(1)}(m)$. This is indeed the expected behavior since

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6 In the next subsection and appendix A.1 we give a rigorous discussion of this model by using the orthogonal polynomial method without making any assumption. In particular, in appendix A.1 we derive the explicit form of the resolvent independently, which also provides a proof of this property.
for $\kappa = 0$, the distribution should reduce to the leading one $\rho^{(0,0)}(m)$. It is also easy to check that in the limit $\kappa \to \infty$, the above distribution shows the behavior of $(N \text{ times})$ the second term in (2.15).

Having studied the eigenvalue distribution, we now consider the saddle point value of $V_{\text{eff}}$. The leading term of order $N^2$ is just canceled by the same term from the denominator $Z$. The subleading term of order $N$ which comes from the first line of (2.8) is given by

$$V^{(1),1\text{st line}}_{\text{eff}} = N \int dm \rho^{(0,1)}(m) \left( V(m) - \int dm' \rho^{(0,0)}(m') \log(m - m')^2 \right).$$

(2.21)

By using the saddle point equation, it is easy to show that the combination in the round bracket, which is obviously $k$-independent, is also $m$-independent. From this, we find that (2.21) is just a $k$-independent constant, since $m$-integral of $\rho^{(0,1)}(m)$ is $k$-independent.

Next, the order $N$ term coming from the second line of (2.8) is given by

$$V^{(1),2\text{nd line}}_{\text{eff}} = N \left( -2\kappa \sqrt{1 + \kappa^2} - 2\text{arcsinh}\kappa + \text{constant} \right),$$

(2.22)

where the last term is a $k$-independent constant. We can check that the constant term in (2.22) is canceled by (2.21) due to the relation $\rho^{(0,1)}(m) + \rho^{(1)}(m) = 0$ for $\kappa = 0$, which has been mentioned after (2.20). Therefore, we have derived the first term of (2.5) with confirming again that we should take the minus sign there. An interesting point in this analysis is that, since we consider the large $k \sim N$ limit, the effects of the Wilson loop on the saddle point are not negligible; the standard semi-circle eigenvalue distribution is corrected by the $1/N$ terms and these corrections affect the leading behavior of the Wilson loop expectation value. In particular, the important contribution of (2.22) originates from the isolated eigenvalue.

### 2.2 Resolvent through the Laplace transformation

The eigenvalue distribution discussed in the previous subsection is a kind of master field for the correlators in the presence of the Wilson loop. If AdS/CFT can be regarded as realizing the idea of the master field, it is natural to expect that there should be a counterpart of the eigenvalue distribution in the gravity side. Here we note that in general multi-matrix models like the four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory would not have a master field, but in a certain BPS sector like circular Wilson loops a kind of master field may exist due to higher supersymmetries by which the system can be effectively described by a one-matrix model. As we have already used in the previous section, the eigenvalue distribution is given by the imaginary part of the resolvent, $R(z)$, which is more tractable as a complex function, and the position of the eigenvalue is the pole of $R(z)$. We thus study how the isolated pole arises when we discuss the resolvent based on its standard definition:

$$R(z) = \frac{1}{Z_V} \int dM \frac{1}{N} \text{tr} \frac{1}{z - M} e^{-V(M)}$$

(2.23)

Such isolated eigenvalues play a crucial role in nonperturbative effects in noncritical string theories [17, 18]. It would be interesting if the Wilson loop with winding number of $O(N)$ is shown to play a similar role in critical string theories.
\[ Z_V = \int dM e^{-V(M)}. \] (2.25)

Here \( V(M) \) is a potential whose saddle point we are interested in. Such a question may be interesting even purely from the matrix model viewpoint. What is more, it will guide us to a gravitational interpretation, as we shall see.

In the integral representation (2.24), we should pay attention to the range of the variable \( p \). Since we are interested in the saddle point which is defined solely by the potential \( V(M) \), we need to drop effects of the inserted operator \( \text{tr} e^{\rho M} \) on the saddle point. This means that although the upper limit of the \( p \)-integral is set to be infinity, we still assume the relation \( p \ll N \). In order to make this point clear, let us recall the case of the Gaussian potential, \( V(M) = (2N/\lambda) \text{tr} M^2 \). In this case, the resolvent (2.24) is given by

\[ R_G(z) = \int_0^{\infty} dp e^{-pz} f(p', N), \quad \left( p' = \sqrt{\lambda/4Np} \right). \] (2.26)

The function \( f(p', N) \), which is defined in (2.2), satisfies the following differential equation:

\[ p' \partial_{p'}^2 f(p', N) + 3 \partial_{p'} f(p', N) - p'(4N + p'^2) f(p', N) = 0. \] (2.27)

By using the relation \( p' = \sqrt{\lambda/(4Np)} \) and by taking the large-\( N \) limit with \( p \ll N \), (2.27) is reduced to the Bessel differential equation and then we have

\[ f(p', N) \to \frac{2}{p\sqrt{\lambda}} I_1(p\sqrt{\lambda}), \quad (N \to \infty, \ p \ll N). \] (2.28)

Here \( I_1 \) is the modified Bessel function and the overall constant can be fixed by requiring \( f(p', N) \to 1 \) in the limit \( p \to 0 \). Inserting this into (2.26), and performing the \( p \)-integral, we find the resolvent for the Gaussian potential:

\[ R_G(z) = \int_0^{\infty} dp e^{-pz} \frac{2}{p\sqrt{\lambda}} I_1(p\sqrt{\lambda}) = \frac{2}{\lambda} (z - \sqrt{z^2 - \lambda}). \] (2.29)

Next we turn to the case with the insertion of a Wilson loop with a large winding number \( k \) of \( \mathcal{O}(N) \). In this case, we need to take account of effects of the Wilson loop on the saddle point, i.e., we should consider the following potential:

\[ e^{-V(M)} = \frac{1}{N} \text{tr} e^{kM} e^{-2N\text{tr} M^2}. \] (2.30)

Hence the resolvent in the presence of the Wilson loop with winding number \( k \), which we call \( R_k(z) \), is defined by

\[ R_k(z) = \int_0^{\infty} dp e^{-pz} W(p)_k, \] (2.31)
\[ W(p)_k \equiv \frac{1}{Z_k} \int dM \frac{1}{N} \text{tr} e^{pM} \frac{1}{N} \text{tr} e^{kM} e^{-\frac{2N}{N} \text{tr} M^2}, \]  
\[ Z_k \equiv \int dM \frac{1}{N} \text{tr} e^{kM} e^{-\frac{2N}{N} \text{tr} M^2}. \]  

Here we note that the computation in (2.31) and (2.32) should be done in the following order: first we calculate the “two loop correlator” (2.32) for finite \( N \) and then we take the large-\( N \) and large-\( k \) limit with \( \kappa \) and \( p \) kept finite. Finally we perform the Laplace transformation (2.31).

An explicit computation of the two loop correlator \( W(p)_k \) is given in appendix A by going to the eigenvalue integral and using the orthogonal polynomial method. In the eigenvalue integral, we observe that the following decomposition,
\[ \text{tr} e^{pM} \text{tr} e^{kM} = \sum_{i=1}^{N} \sum_{j \neq i} e^{p m_i} e^{k m_j} + \sum_{i=1}^{N} e^{(p+k)m_i}, \]  

naturally arises and then these two terms are analyzed separately. A physical interpretation of these terms will be discussed shortly. Here we just call the expectation value of the first term \( w(k, p) \), and the second \( w(k + p) \):
\[ W(p)_k = w(k, p) + w(k + p). \]  

The explicit results of \( w(k, p) \) and \( w(k + p) \) are given in (A.30) and (A.31):
\[ w(k, p) = f(p', N) - \frac{1}{N} e^{\frac{p^2}{2}} \sum_{ij} k^{i-j} L_{j-1}^{(i-j)} (-k^2) p^{i-j} L_{i-1}^{(j-i)} (-p'^2), \]  
\[ w(k + p) = \frac{1}{N} f(k' + p', N) f(k', N). \]  

The first term on the right hand side of (2.36) gives the resolvent which is identical to the one for the Gaussian potential (2.29) in the large-\( N \) limit after the Laplace transformation. The second term gives the \( \mathcal{O}(N^{-1}) \) resolvent (2.18), as shown in appendix A.1. Here note that in the previous subsection, we assumed that the \( \mathcal{O}(N^{-1}) \) resolvent (2.18) has the cut at \((-\sqrt{\lambda}, \sqrt{\lambda})\) and has no pole (see the second condition after (2.17)). However in appendix A.1 we explicitly derive the resolvent (2.18) by taking the large-\( N \), fixed \( \kappa \) and \( p \) limit for the second term in (2.36) and performing the Laplace transformation, which justifies the assumption.

Next let us consider the Laplace transformation of (2.37). We should take the large-\( N \) limit with assuming the relation \( p \ll N \) before the Laplace transformation. Furthermore, this term depends on the additional parameter \( k \) and the parameter region is chosen so that the combination \( \kappa = k \sqrt{\lambda}/(4N) \) will be kept finite. The relevant terms in this limit are already given in (2.25). We just define \( \tilde{\kappa} \equiv (k+p) \sqrt{\lambda}/(4N) \) and then (2.37) is evaluated in the large-\( N \) limit with \( \kappa \) fixed and \( p \ll N \) as
\[ \frac{1}{N} \frac{f(k' + p', N)}{f(k', N)} = \frac{1}{N} \exp \left( -N \left( \mathcal{F}_{-}((k+p)/4N) - \mathcal{F}_{-}(\kappa/N) \right) \right), \]  

\[ \text{Here we note that the computation in (2.31) and (2.32) should be done in the following order: first we calculate the “two loop correlator” (2.32) for finite } \]  
\[ \text{and then we take the large-}\( N \) and large-\( k \) limit with \( \kappa \) and \( p \) kept finite. Finally we perform the Laplace transformation (2.31).} \]  
\[ \text{An explicit computation of the two loop correlator } W(p)_k \text{ is given in appendix A by going to the eigenvalue integral and using the orthogonal polynomial method. In the} \]  
\[ \text{eigenvalue integral, we observe that the following decomposition,} \]  
\[ \text{tr} e^{pM} \text{tr} e^{kM} = \sum_{i=1}^{N} \sum_{j \neq i} e^{p m_i} e^{k m_j} + \sum_{i=1}^{N} e^{(p+k)m_i}, \]  

\[ \text{naturally arises and then these two terms are analyzed separately. A physical interpretation of these terms will be discussed shortly. Here we just call the expectation value of the first term } w(k, p) \text{, and the second } w(k + p): \]  
\[ W(p)_k = w(k, p) + w(k + p). \]  

\[ \text{The explicit results of } w(k, p) \text{ and } w(k + p) \text{ are given in (A.30) and (A.31)}: \]  
\[ w(k, p) = f(p', N) - \frac{1}{N} e^{\frac{p^2}{2}} \sum_{ij} k^{i-j} L_{j-1}^{(i-j)} (-k^2) p^{i-j} L_{i-1}^{(j-i)} (-p'^2), \]  
\[ w(k + p) = \frac{1}{N} f(k' + p', N) f(k', N). \]  

\[ \text{The first term on the right hand side of (2.36) gives the resolvent which is identical to the one for the Gaussian potential (2.29) in the large-}\( N \) limit after the Laplace transformation. The second term gives the } \mathcal{O}(N^{-1}) \text{ resolvent (2.18), as shown in appendix A.1. Here note that in the previous subsection, we assumed that the } \mathcal{O}(N^{-1}) \text{ resolvent (2.18) has the cut at } (-\sqrt{\lambda}, \sqrt{\lambda}) \text{ and has no pole (see the second condition after (2.17)). However in appendix A.1 we explicitly derive the resolvent (2.18) by taking the large-}\( N \), fixed } \kappa \text{ and } p \text{ limit for the second term in (2.36) and performing the Laplace transformation, which justifies the assumption.} \]  
\[ \text{Next let us consider the Laplace transformation of (2.37). We should take the large-}\( N \) limit with assuming the relation } p \ll N \text{ before the Laplace transformation. Furthermore, this term depends on the additional parameter } k \text{ and the parameter region is chosen so that the combination } \kappa = k \sqrt{\lambda}/(4N) \text{ will be kept finite. The relevant terms in this limit are already given in (2.25). We just define } \tilde{\kappa} \equiv (k+p) \sqrt{\lambda}/(4N) \text{ and then (2.37) is evaluated in the large-}\( N \) limit with } \kappa \text{ fixed and } p \ll N \text{ as} \]  
\[ \frac{1}{N} \frac{f(k' + p', N)}{f(k', N)} = \frac{1}{N} \exp \left( -N \left( \mathcal{F}_{-}((k+p)/4N) - \mathcal{F}_{-}(\kappa/N) \right) \right) \]
By performing the Laplace transformation, we obtain the following contribution to the resolvent of $\mathcal{O}(1/N)$:

$$R^{(2)}(z) \equiv \int_0^\infty dp e^{-pz} \frac{1}{N} \frac{f(k'+p',N)}{f(k',N)} \rightarrow \frac{1}{N} \frac{1}{z - z_*}, \quad (z_* = \sqrt{\lambda \sqrt{1 + \kappa^2}}).$$  \hspace{1cm} (2.40)

This is exactly the pole corresponding to the isolated eigenvalue \ref{2.13} found in the papers [13, 19]. Since the other $\mathcal{O}(N^{-1})$ contribution, the second term in $w(k,p)$, is shown to give only the cut, these two $\mathcal{O}(N^{-1})$ contributions therefore give clearly different contributions to the resolvent or eigenvalue distribution.

We have thus succeeded in identifying which correlator gives rise to the isolated pole, but, of course, the result \ref{2.40} itself is not quite new since we have just calculated the position of the isolated eigenvalue by a different method. However, the above procedure implies the following interpretation of the Laplace transformation and the resolvent.

First note that, as we shall review in subsection 3.1, the Wilson loop $\text{tr} e^{i k M}$ corresponds to the D3-brane with $k$ units of the electric charge in the dual gravity picture in the AdS/CFT correspondence (1.1). Then the two loop correlator \ref{2.32} may be regarded as treating the D3-brane with string charge $k$ as “a part of the background” and then introducing additional small number of string charge $p$, by which the total background is probed.

Now we may advocate the following physical interpretation of each term in the decomposition \ref{2.34}:

- The second term can be regarded as a situation where the additional string charge $p$ is dissolved in the D3-brane with charge $k$, now altering the number of the charge from $k$ to $k + p$. On the other hand, the first term may correspond to the case in which the additional string charge $p$ is described by a probe D3-brane with charge $p$ that is separated from the “background” D3-brane with charge $k$, but has an interaction with it. At the leading order of the large-$N$ limit, we may expect that the interaction between them is not taken into account and that the first term of \ref{2.36} describes two non-communicating D3-branes. Actually the first term in \ref{2.36} is blind to $k$ as a result of cancellation of the effect from the D3-brane with charge $k$ between the numerator and the denominator in \ref{2.32}, while the second term in \ref{2.36} describes the interaction between these D-branes in higher order of the string coupling constant. It then seems natural that the first term of \ref{2.35} reproduces the semi-circle distribution, which would carry information on the AdS$_5 \times $S$^5$, while the second term gives the pole, which is information of the D3-brane with $k$ charge, in the leading contribution with respect to $1/N$ of each term.

Based on the above derivation of $z_*$, we may also expect that, in the gravity side, $z_*$ appears as an object which is conjugate to the string charge in some sense. We will further investigate these points from the gravity side in the following section.

\textsuperscript{8}In [19], a similar decomposition of the exponential operator is discussed in terms of world sheet nonperturbative effects, in a slightly different context.
3 Matrix model resolvent from D3-brane picture

In this section we pursue the gravitational interpretation of the matrix model resolvent. In particular we concentrate on the pole corresponding to the D3-brane with string charge $k$. We will discuss the cut in the concluding section.

Since we found that the isolated pole is derived by the Laplace transformation of the expectation value of the operator $\text{tr} e^{(k+p)M}$, it is rather easy to derive the pole itself in the D3-brane picture once the correspondence between the operator $\text{tr} e^{kM}$ and the D3-brane solution with $k$ flux is established. Indeed the expectation value of the operator $\text{tr} e^{(k+p)M}$ in the large-$N$ and large-$k$ limit, or equivalently the function $F(\tilde{\kappa}, N)$ is exactly reproduced by using the D3-brane solution as reviewed in the following subsection\textsuperscript{9}. So, what we need to do is just to expand it with respect to $p$ and perform the Laplace transformation. This computation is just a repetition of the matrix model case\textsuperscript{10}. However, as we pointed out in the previous section, we should be able to extract more information about the role of the resolvent in the gravitational point of view. In the rest of the paper, we will investigate this issue.

3.1 A review of the Drukker-Fiol D3-brane solution

Let us give a brief review of the D3-brane solution derived in [9], which will be useful for later discussions. Their proposal is that the expectation value of the Wilson loop with winding number $k$ is given by the amplitude of a D3-brane carrying string charge $k$ and attached to the loop on the AdS boundary:

$$\left\langle \frac{1}{N} \text{tr} e^{kM} \right\rangle = \int_{\text{b.c.}} e^{-(S_{\text{D3}}+S_b)}.$$  \hfill (3.1)

Here $S_{\text{D3}}$ consists of the Dirac-Born-Infeld action $S_{\text{DBI}}$ and the Wess-Zumino term $S_{\text{WZ}}$ for a D3-brane:

$$S_{\text{D3}} = S_{\text{DBI}} + S_{\text{WZ}} = \int d\sigma^1 d\sigma^2 d\sigma^3 d\sigma^4 L,$$  \hfill (3.2)

$$S_{\text{DBI}} = T_{\text{D3}} \int d\sigma^1 d\sigma^2 d\sigma^3 d\sigma^4 \sqrt{\det(g_{ab} + 2\pi \alpha' F_{ab})},$$  \hfill (3.3)

$$S_{\text{WZ}} = -T_{\text{D3}} \int \mathcal{P}[C_4].$$  \hfill (3.4)

\textsuperscript{9} More precisely, only the leading term $F_0$, i.e. the leading term of $F_-$, has been reproduced from the D3-brane solution. Note that in the previous section, the pole has been derived by using the explicit form of $F_0$ and also the fact that $F_-$ can be expanded as $F_- = F_0(\kappa) + (1/N) F_1(\kappa) + \cdots$, but the explicit form of higher terms, including $F_1$ are not needed.

\textsuperscript{10} Here we should mention that this argument does not fix the residue of the pole. In order to do this, it is necessary to fix a normalization factor in AdS/CFT correspondence up to the subleading order in $1/N$. There may exist another difficulty in determining a relative normalization between the cut and the pole. This is because, based on the viewpoint discussed at the end of the previous section, these two contributions correspond to different setups in the gravity side, namely different perturbative vacua in string theory.
Here $\mathcal{P}[C_4]$ is the pullback of the four-form potential $C_4$ and the D3-brane tension is given by $T_{D3} = \frac{1}{(2\pi)^2 g_s}$. The path integral on the right hand side of (3.1) is taken over the D3-brane configurations satisfying appropriate boundary conditions which are specified in terms of the Wilson loop. The boundary term $S_b$ is introduced in order to flip the boundary conditions for the world volume gauge field and the scalar fields. In the semi-classical regime, the path integral may be well estimated by a saddle point value.

The authors of the paper started with the following metric for the $\text{AdS}_5$ part in the Poincaré coordinate:

$$ds^2 = \frac{L^2}{y^2}(dy^2 + dr_1^2 + r_1^2 d\psi^2 + dr_2^2 + r_2^2 d\phi^2),$$

with the curvature radius $L = (4\pi g_s N)_3^{1/4} l_s = \lambda^{1/4} l_s$, and considered a circle $r_1 = R$, $r_2 = 0$ on the AdS boundary $y = 0$ and then discussed the D3-brane solution which is attached to the circle.

The actual D3-brane solution and the evaluation of the action were discussed by using the following metric:

$$ds^2 = \frac{L^2}{\sin^2 \eta} \left( d\eta^2 + \cos^2 \eta d\psi^2 + d\rho^2 + \sinh^2 \rho (d\theta^2 + \sin^2 \theta d\phi^2) \right).$$

Here, $\eta$, $\rho$ and $\theta$ are related to $r_1$, $r_2$ and $y$ by the coordinate transformation

$$r_1 = \frac{R \cos \eta}{\cosh \rho - \sinh \rho \cos \theta}, \quad r_2 = \frac{R \sinh \rho \sin \theta}{\cosh \rho - \sinh \rho \cos \theta}, \quad y = \frac{R \sin \eta}{\cosh \rho - \sinh \rho \cos \theta}.$$

The range of these new variables are taken to be $0 \leq \rho < \infty$, $0 \leq \theta \leq \pi$ and $0 \leq \eta \leq \pi/2$. They chose $\psi$, $\rho$, $\theta$ and $\phi$ as the world volume coordinates by $\sigma^1 = \psi$, $\sigma^2 = \rho$, $\sigma^3 = \theta$ and $\sigma^4 = \phi$ and assumed an $S^2$ symmetric and also $\psi$-translational invariant ansatz, $\eta = \eta(\rho)$ and $F_{\psi \rho} = F_{\psi \rho}(\rho)$. Then $S_{\text{DBI}}$ and $S_{\text{WZ}}$ are written down as

$$S_{\text{DBI}} = 2N \int d\rho d\theta \frac{\sin \theta \sin^2 \rho}{\sin^4 \eta} \rho \sqrt{\cos^2 \eta (1 + \eta^2) + (2\pi \alpha')^2 \frac{\sin^4 \eta}{L^4} F_{\psi \rho}^2},$$

$$S_{\text{WZ}} = -2N \int d\rho d\theta \frac{\cos \eta \sin \theta \sin^2 \rho}{\sin^4 \eta} \left( \cos \eta + \eta' \sin \eta \frac{\sinh \rho - \cosh \rho \cos \theta}{\cosh \rho - \sinh \rho \cos \theta} \right),$$

and the solution for the equations of motion is given by

$$\sin \eta = \kappa^{-1} \sinh \rho, \quad F_{\psi \rho} = \frac{i k \lambda}{8\pi N \sinh^2 \rho}, \quad (\kappa = k \sqrt{\lambda} / 4N).$$

Here $k$ is the string charge defined as

$$k = i \Pi \equiv i \int d\theta d\phi \frac{\partial L}{\partial F_{\rho \psi}},$$

In this article we follow the notation of [9] for the D3-brane solution except this II which differs in the factor $i$. 

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where $\mathcal{L}$ is defined as in (3.2). The solution is attached to the circle on the AdS boundary at $\rho = 0$.

The boundary terms are introduced in order to take account of the appropriate boundary conditions; the D3-brane carries a fixed number of the string charge $k$, and also the scalar fields transverse to the AdS horizon should satisfy the Neumann boundary condition. The latter boundary condition is motivated by the T-duality argument for the case with the fundamental string world sheet description instead of the D3-brane [2]. The boundary condition for the gauge field is taken into account by adding the following boundary term:

$$S_A = \int d\psi d\theta d\phi \frac{\partial \mathcal{L}}{\partial (\partial_\rho A_\psi)} A_\psi \bigg|_{\rho=0}. \tag{3.12}$$

On the other hand, the relevant boundary term for the scalar fields is the one for the radial direction of AdS. Here we need to take care of the fact that the boundary terms for the radial coordinate differ for different choices of the coordinates system. In [20], it was argued that a natural choice is to set the Neumann boundary condition for the coordinate $u = 1/y$:

$$S_u = \int d\psi d\theta d\phi \frac{\partial \mathcal{L}}{\partial (\partial_\rho u)} u \bigg|_{\rho=0}. \tag{3.13}$$

One of the reasons for this choice was that the conjugate momentum of $u$ is finite at the AdS boundary, while that of $y$ is infinite. At the same time, this choice is necessary for the correct cancellation of divergence. In the following subsections we will see this choice is natural in our case as well.

Summing up all the terms (3.8), (3.9), (3.12) and (3.13), the classical value of $S_{D3} + S_b$ is evaluated as

$$S_{D3} + S_b = -2N \left( \kappa \sqrt{1 + \kappa^2} + \text{arcsinh} \kappa \right), \tag{3.14}$$

which indeed agrees with the matrix model result (2.5) [9]. Therefore, as mentioned at the beginning of this section, by replacing $\kappa$ with $\tilde{\kappa} = (k + p) \sqrt{\lambda}/(4N)$ and performing the Laplace transformation for $e^{-(S_{D3} + S_b)}$ with respect to $p \ll N$, the pole at $z = z_* = \sqrt{\lambda \sqrt{1 + \kappa^2}}$ can be reproduced.

In the above discussion, we used the explicit value of the total action with the solution substituted, which results in a function of $k$. As a result, the origin of the pole is not very clear. Since the string charge carried by the solution should have been set by the boundary condition before solving the equations of motion, we should first change the boundary condition as $k \rightarrow k + p$ and then observe a response of the action. It would allow us to discuss the Laplace transformation without using the explicit form of the solution from the beginning.

Before doing this analysis, here it is useful to take much care of the boundary condition in the above analysis. In the above discussion, we imposed the boundary condition for the conjugate momentum of the gauge field as $i\Pi = k$. On the other hand, we did not mention the condition on the boundary value of the conjugate momentum of $u$; it is just given by inserting the solution. In principle, the boundary condition for the conjugate momentum of $u$ should be specified independently before we discuss the solutions or the equations of motion. Moreover, based on the spirit of the AdS/CFT correspondence,
the boundary condition should be specified in terms of the information contained in the Wilson loop.

In the case of the correspondence between the Wilson loops in fundamental representation and string world sheet, a detailed discussion on the boundary condition was given in [2]. In the next subsection we consider a boundary condition for the D3-brane. This analysis plays an important role in subsection 3.3.

3.2 D3-brane boundary condition imposed by Wilson loop

As we have mentioned, the boundary condition for the D3-brane should be specified in terms of the information carried by the Wilson loop. Before discussing the connection with the Wilson loop, let us take a look at what boundary condition is satisfied by the actual D3-brane solution (3.10). Near the AdS boundary, the radial coordinate \( u \) can be approximated by \( u \sim 1/(R\eta) \). Hence the conjugate momentum of \( u \), at the D3-brane boundary, is given by

\[
\left. \frac{\partial \mathcal{L}}{\partial (\partial_\rho u)} \right|_{\rho=0} = -R\eta^2 \left. \frac{\partial \mathcal{L}}{\partial (\partial_\rho \eta)} \right|_{\rho=0} = -\frac{L^2}{2\pi \alpha'} Rk \frac{\sin \theta}{4\pi}.
\]  

(3.15)

Here we introduce a new coordinate \( U \) as

\[
U \equiv \frac{L^2}{2\pi \alpha'} u = \frac{L^2}{2\pi \alpha'} \frac{1}{y}.
\]  

(3.16)

The coordinate \( U \) is a natural choice for the scalar field, since \( L^2/y \) corresponds to the radial coordinate of the flat six dimensions in the asymptotic region of the original extremal black 3-brane solution.

We define \( P_U \) as the conjugate momentum of \( U \) integrated over \( S^2 \):

\[
P_U \equiv \int d\theta d\phi \frac{\partial \mathcal{L}}{\partial (\partial_\rho U)},
\]  

then the boundary condition is simply given by

\[
\Pi = -ik, \quad P_U = -Rk.
\]  

(3.18)

Note that the factor \( \sin \theta/(4\pi) \) in (3.15) originates from the \( S^2 \) symmetric property of the solution. Hence if we assume \( S^2 \) symmetry, we lose no information by integrating over the \( S^2 \).

In the case of string, the connection between the boundary value of the momentum and the information on the Wilson loop is proposed and its validity is supported by an argument using the Hamiltonian constraint in [2]. In our present case for the D3-brane, we consider the other way round: we examine what kind of conditions such a constraint implies on the boundary value of the momenta for the world volume fields, from which we can guess appropriate boundary conditions.

For this purpose, first we note that the D3-brane solution in the previous subsection can be locally approximated by the solution corresponding to the straight Wilson line.
For later reference, we clarify this point. Let us consider a small region around a point on the loop, say \( \psi = \psi_0 \) and \( r_1 = R \), and expand the coordinate as

\[
\psi = \psi_0 + \epsilon t, \quad r_1 = R(1 + \epsilon \rho \cos \theta), \quad r_2 = R\epsilon \rho \sin \theta, \quad y = R\epsilon \eta.
\]

Here \( \epsilon \) is a small parameter and we assume that the coordinates \( t, \rho, \) and \( \eta \) take values of order 1. The angular coordinate \( \theta \) can take value in the original range as \( 0 \leq \theta \leq \pi \).

Then the AdS\(_5\)\(\times\)S\(_5\) metric is reduced to the following one:

\[
ds^2 = \frac{L^2}{\eta^2}(d\eta^2 + dt^2 + d\rho^2 + \rho^2(d\theta^2 + \sin^2 \theta d\phi^2)) + L^2 d\Omega^2_5,
\]

and the Wilson loop is extended along the \( t \)-direction. The solution (3.19) becomes the one corresponding to the straight Wilson line in \( \epsilon \to 0 \) limit. In the following we discuss a kind of Hamiltonian constraint only for the case of the straight line. However, from this observation, we expect that the analysis can also be applied for a local patch of a Wilson loop of an arbitrary (smooth) shape. Here we note that as we will see, when we interpret our analysis as the local version of the analysis for a circle, we must take the scale factor \( R \) and \( \epsilon \) into account, which is lost in the case of the straight line.

Moreover, it should be stressed that in the following argument, although we use the \( S^2 \) symmetric ansatz, we do not use information on the classical solution reviewed in the previous subsection (nor its reduction to that for the straight line) in the gravity side. In our semi-classical analysis, it seems natural to allow any field fluctuation around a classical configuration on the right hand side in (3.11) as long as it is consistent with the boundary condition. From such a point of view, the \( S^2 \) symmetric ansatz is just an assumption for allowed fluctuations.\(^{12}\) Of course requiring global symmetries including the supersymmetries is sometimes so strong that an allowed configuration will be uniquely a classical solution, especially when the Wilson loop is highly symmetric as for the straight line and the circular loop; see appendix \(^{13}\). However here we emphasize that the \( S^2 \) symmetric ansatz still allows a wide range of fluctuations, and the following argument should hold for such field fluctuations.

We use the following coordinates for the AdS\(_5\)\(\times\)S\(_5\) geometry:

\[
ds^2 = \left(\frac{2\pi \alpha'}{L}\right)^2 U^2(dt^2 + d\rho^2 + \rho^2(d\theta^2 + \sin^2 \theta d\phi^2)) + L^2 \frac{(dU_i)^2}{U^2} + G_{\mu\nu} dx^\mu dx^\nu.
\]

The four directions \((t, \rho, \theta, \phi)\) are identified as the four-dimensional space where the gauge theory lives. The Wilson loop is located at \( \rho = 0 \) and extended into \( t \)-direction. We are

\(^{12}\)This corresponds to \( \rho \to \epsilon \rho, \eta \to \epsilon \eta \).

\(^{13}\) It may also be possible that, as a definition of the correspondence, the functional integral on the right hand side of (3.11) should be constrained in a domain where the world volume fields preserve some symmetries of the Wilson loop. In such a case, the \( S^2 \) symmetric ansatz becomes more important than in the case where it is just an assumption. Although it is an important problem to give a precise definition for the integration region on the right hand side of (3.11), it is beyond the scope of the present article.
now discussing the D3-brane configuration which is pinched to the line $\rho = 0$ on the AdS boundary, $U = \infty$. We introduce the world volume coordinate $\sigma^a$ ($a = 1 \sim 4$) and, as mentioned above, assume the $S^2$ symmetric ansatz\(^\text{14}\)

$$
\theta = \sigma^3, \quad \phi = \sigma^4,
$$

$$
t = t(\sigma^1, \sigma^2), \quad \rho = \rho(\sigma^1, \sigma^2), \quad U^i = U^i(\sigma^1, \sigma^2), \quad F_{12} = F_{12}(\sigma^1, \sigma^2). \tag{3.25}
$$

We take the coordinate $\sigma^2$ in such a way that the D3-brane boundary is at $\sigma^2 = 0$, i.e., $\rho(\sigma^1, \sigma^2 = 0) = 0$ and $U(\sigma^1, \sigma^2 = 0) = \infty$. Then on the boundary $\sigma^2 = 0$, the parameter $\sigma^1$ should parameterize the loop on which the D3-brane is attached.

Now the D3-brane action is given by

$$
S_{\text{DBI}} + S_{\text{WZ}} = \int d\sigma^1 d\sigma^2 L_{\text{DBI}} = \int d\sigma^1 d\sigma^2 (L_{\text{DBI}} + L_{\text{WZ}}), \tag{3.26}
$$

$$
L_{\text{DBI}} = 4\pi T_{D3} G_{\theta \theta} \sqrt{g_{11} g_{22} - g_{12}^2 + (2\pi \alpha')^2 F_{12}^2}, \tag{3.27}
$$

$$
L_{\text{WZ}} = -4\pi T_{D3} G_{\theta \theta} \sqrt{g_{11} g_{22} - g_{12}^2} \rho \rho (\partial_t \partial_2 \rho - \partial_1 \rho \partial_2 t). \tag{3.28}
$$

Here the Lagrangian $L_{D3}$ and $L_{\text{DBI,WZ}}$ are defined with the $S^2$ part integrated.

We define the conjugate momentum $P_{X^M}$ and $\Pi^a$ for $X^M$ and $A_a$, respectively as

$$
P_{X^M} = \frac{\partial L_{D3}}{\partial (\partial_2 X^M)}, \quad \Pi^a = \frac{\partial L_{D3}}{\partial (\partial_2 A_a)} \tag{3.29}
$$

Then, the following identity holds:

$$
G^{tt}(P_t - \mathcal{P}_t)^2 + G^{\rho \rho}(P_{\rho} - \mathcal{P}_{\rho})^2 + G^{U_1 U_1}(P_{U_1})^2 + \frac{1}{(2\pi \alpha')^2}(\Pi^1)^2 g_{11} = (4\pi T_{D3} G_{\theta \theta})^2 g_{11}, \tag{3.30}
$$

where $g_{11} = G_{tt}(\partial_t t)^2 + G_{\rho \rho}(\partial_1 \rho)^2 + G_{U_1 U_1}(\partial_1 U_1)^2$. The “momenta” $\mathcal{P}_{X^M}$ are defined as $\mathcal{P}_{X^M} \equiv \partial L_{\text{WZ}}/\partial (\partial_2 X^M)$, whose explicit forms are given by

$$
\mathcal{P}_t = 4\pi T_{D3} G_{\theta \theta} \sqrt{G_{tt} G_{\rho \rho}} \partial_1 \rho, \quad \mathcal{P}_{\rho} = -4\pi T_{D3} G_{\theta \theta} \sqrt{G_{tt} G_{\rho \rho}} \partial_t t. \tag{3.31}
$$

These do not include any derivative with respect to $\sigma^2$ and thus (3.30) can be regarded as a constraint in the phase space. By using the explicit form of the metric, the constraint (3.30) can be rewritten as

$$
0 = (P_{U_1})^2 + (\Pi^1)^2 ((\partial_t t)^2 + (\partial_1 \rho)^2) - \frac{16\pi}{\lambda} N (P_t \partial_1 \rho - P_{\rho} \partial_1 t) \rho^2
$$

$$
- \left( 64\pi^2 \lambda^{-2} N^2 (U \rho)^4 - \frac{\lambda}{4\pi^2} (\Pi^1)^2 \right) \left( \frac{\partial_1 U_1}{U^2} \right)^2 + \frac{\lambda}{4\pi^2} (P_t^2 + P_{\rho}^2) \frac{1}{U^4}. \tag{3.32}
$$

\(^{14}\)There are several $S^2$ in this $\text{AdS}_5 \times S^5$ geometry to be identified with the world volume $S^2$, for example, some $S^2 \subset S^5$. Here we consider configurations which are trivial in the $S^5$ direction. This is also an assumption.
Let us assume that the D3-brane configuration asymptotically satisfies the following conditions in the limit $\sigma^2 \to 0$:

\[ U \to \infty, \quad U\rho < \infty, \quad P_t, P_\rho \lesssim P_U, \quad \Pi^1 (\sim N). \quad (3.33) \]

The first condition has already been imposed as a boundary condition, which requires that the D3-brane should be attached to the AdS boundary. The second means that the radius of the $S^2$ part of the D3-brane is finite at the boundary. The last one refers to the behavior of the fields at the boundary with respect to $N$. Although the last two inequalities are new assumptions, we expect that (3.33) picks up a reasonable set of fluctuations.

Under the conditions (3.33), the equation (3.30), in the limit $\sigma^2 \to 0$, is reduced to a simple form:

\[ (P_{U^i})^2 + (\Pi^1)^2(\partial_1 t)^2 = 0. \quad (3.34) \]

Thus we find that the world volume fields on the D-brane attached to the Wilson loop at the AdS boundary are subject to this constraint, at least under the $S^2$ symmetric ansatz.

Now it is natural to impose the boundary condition on $\Pi^1$ as

\[ \Pi^1(\sigma^1, \sigma^2 = 0) = -ik, \quad \text{for each } \sigma^1, \quad (3.35) \]

since the boundary condition imposed by the straight Wilson line, with constant $\theta^i$, should not depend on the parameter $\sigma^1$. Then since the conjugate momentum of $U^i$ and that of $U$, which are defined by (3.29), are related through $P_{U^i} = (U^i / U)P_U$ on the boundary, the equation (3.34) tells us that at $\sigma^2 = 0$,

\[ \Pi^1 = -ik, \quad P_{U^i} = P_U \Theta_i = -k|\partial_1 t|\Theta_i, \quad \Theta_i \equiv U^i / U. \quad (3.36) \]

Note that there should be a sign ambiguity for $P_{U^i}$. Here we took the minus sign so that it will be consistent with the classical solution. This is the only input from the classical solution.

As an application, in order to reinterpret (3.36) in the case of the circular loop, we need to take account of the scale factor $R$ and $\epsilon$ as pointed out in the discussion after (3.20). Since the first equation of (3.36) does not have any uncontracted target indices, it does not need to be changed. On the other hand, the second equation should be reinterpreted by taking account of the scale factor as $U(\text{straight}) = R\epsilon \times U(\text{circle})$ as implied by the last equation in (3.19), and $t = \epsilon^{-1}\psi$. Hence we have

\[ P_U = -k|\partial_1 t| \quad (\text{for straight line}) \quad \to \quad P_U = -kR|\partial_1 \psi| \quad (\text{for circular loop}). \quad (3.37) \]

This is precisely the relation we found at (3.18) (note that the (3.37) does not depend on the choice of the parameter $\sigma^1$).

So far we have considered the local straight line (3.36) or the circular loop (3.37). In more general case, we thus expect following boundary conditions:

\[ \Pi^1 = -ik, \quad P_{U^i} = -k|\partial_1 X|\Theta_i. \quad (3.38) \]

Here $X^\mu$ is the four-dimensional Cartesian coordinate, i.e., $ds^2_{\text{AdS}_5} = Y^{-2}(dY^2 + dX^\mu dX^\mu)$ which is identified with the space where the gauge theory lives.
In the spirit of the AdS/CFT correspondence, the boundary condition should be given in terms of the Wilson loop:

\[ W_k(C) = \text{tr} P \exp \left( \int ds \left( iA_\mu \dot{x}^\mu + \Phi_i |\dot{x}\theta^i \right) \right). \]  

(3.39)

Thus we find that natural boundary conditions are

\[ X^\mu (\sigma^1 = \sigma^1(s), \sigma^2 = 0) = x^\mu (s), \]  

(3.40)

\[ \Pi^1 (\sigma^1, \sigma^2 = 0) = -ik, \quad P_{U^i} (\sigma^1 = \sigma^1(s), \sigma^2 = 0) = -k |\dot{x}(s)| \theta^i (s), \]  

(3.41)

where we omitted the trivial $S^2$-dependence in (3.40) and $\sigma^1(s)$ determines a relation between two parameters of the loop (or the boundary of the world volume) representing the same point. Hereafter we assume $\sigma^1 = s$ by using the reparametrization invariance of the Wilson loop. We find that the Wilson loop gives the Dirichlet boundary conditions (3.40) on the scalar fields along the world volume direction of the N D3-branes and the Neumann boundary conditions (3.41) on the gauge field and the scalar fields perpendicular to the world volume direction, as expected [9].

We would make some remarks. First we notice that in (3.41) the integration over $S^2$ in AdS$_5$ is done in $\Pi^1$ and $P_{U^i}$, which is seen from the definitions (3.26) and (3.29). We expect that this integration is trivial even for loops of arbitrary shape if the loop is locally approximated by a straight line with the $S^2$ symmetric ansatz. Next, we note that our boundary conditions are natural generalization of the ones given in [2] in the sense that (3.40) and the second boundary condition in (3.41) amounts to them in the case of the string world sheet, namely when we neglect the $\sigma^3, \sigma^4$-dependence and use the Nambu-Goto action.

Finally let us make a remark on the most important aspect of our boundary conditions. By use of the embedding coordinates $X^\mu$, we can convert the world volume indices of $\Pi^a$ into the space-time indices along the world volume like $\Pi^\mu \equiv \partial_a X^\mu \Pi^a$. Since $\Pi^a$ has the only non-vanishing component for $a = 1$, $\Pi^\mu$ satisfies the following boundary condition

\[ \Pi^\mu = \partial_1 X^\mu \Pi^1 = -ik \dot{x}^\mu, \quad \text{at } \sigma^2 = 0, \]  

(3.42)

where we have used (3.40). Using the second boundary condition in (3.41), this gives the following relation

\[ (\Pi^\mu)^2 + (P_{U^i})^2 = 0. \]  

(3.43)

It is worth noting that the boundary condition (3.41) imposed by the Wilson loop (3.39) thus corresponds to the BPS condition in [21], i.e., force balance between the electric charge $\Pi^1$ and the deformation of the D3-brane which is characterized by $P_{U^i}$, in the case of the spike solution in the flat space. The force balance equation becomes simple and symmetric form for our choice of the coordinate, i.e., $X^\mu$ and $U^i$. This may be understood by the fact that the coordinate system $\{X^\mu, X^{i+4} = 2\pi \alpha' U^i\}$ corresponds to the Cartesian coordinates of the flat ten dimensional space in the asymptotic region of the black 3-brane solution. Thus it is natural that the equation (3.43) also implies a local BPS condition for the Wilson loop. In fact, in gauge theory side, we can introduce a
loop of an arbitrary shape in the internal space by replacing \(|\dot{x}|\theta^i\) in (3.39) with some function \(\dot{y}^i(s)\). For a corresponding D3-brane solution our boundary condition becomes the following symmetric form:

\[
\Pi^\mu(\sigma^1, \sigma^2 = 0) = -ik\dot{x}^\mu(\sigma^1), \quad P_U(\sigma^1, \sigma^2 = 0) = -k\dot{y}_i(\sigma^1),
\]  

(3.44)

and then the force balance condition is equivalent to the relation

\[
\dot{x}^2 = \dot{y}^2,
\]  

(3.45)

which is nothing but the local BPS condition for Wilson loops in the gauge theory side [2, 22]. Therefore we find our boundary condition quite natural because once it is assumed, the local BPS conditions in both sides become equivalent\(^{15}\).

Another interesting point is that the Gauss law constraint implies that the coordinates \(\sigma^1\) and \(\sigma^2\), i.e. normal direction and tangential direction of the loop, are mutually independent. Indeed, from (3.41) and (3.44)

\[
0 = \Pi^{a=2} = \frac{\partial \sigma^2}{\partial X^\mu} (-ik\dot{x}^\mu), \quad -ik\dot{x}^\mu = \Pi^\mu = \frac{\partial X^\mu}{\partial \sigma^1} \Pi^1 = \frac{\partial X^\mu}{\partial \sigma^1} (-ik),
\]  

(3.46)

and hence \(\partial \sigma^2/\partial \sigma^1 = 0\). We would also like to note that (3.44) makes the boundary terms (3.12) and (3.13) manifestly invariant under the reparametrization\(^{16}\).

3.3 Laplace transformation and matrix model resolvent revisited

Let us reconsider the Laplace transformation of the circular Wilson loop taking account of the following boundary condition

\[
\Pi = -ik, \quad P_U = -Rk.
\]  

(3.47)

We now evaluate the right hand side of (3.1) under this boundary condition. In a region like \(\lambda \gg 1\) the saddle point approximation will be valid, and then we can estimate the summation \(S_{D3} + S_b\) by its saddle point value. Since adding the boundary terms amounts to the Legendre transformation, this saddle point value is given by a function of the boundary values of momenta as \((S_{D3} + S_b)|_{\text{saddle point}} = \mathcal{R}(\Pi, P_U)\), where \(\Pi\) and \(P_U\) express the boundary values of the momenta\(^{17}\). Then the AdS/CFT correspondence for the Wilson loop in a semi-classical limit claims

\[
\left\langle \frac{1}{N} \text{tr} e^{iM} \right\rangle = e^{-\mathcal{R}(-ik, -Rk)}. \tag{3.48}
\]

\(^{15}\)However this equivalence may be a peculiar feature of AdS\(_5\) \(\times\) S\(_5\) background [23].

\(^{16}\)It might be interesting to notice that the boundary term (3.12) and (3.13) takes the form of the exponent of the \(U(1)\) Wilson loop under our boundary conditions, though we should not be confused about the gauge field in the super Yang-Mills theory and that on the D-brane world volume.

\(^{17}\)Here we still assume the S\(^2\) symmetric ansatz. Hence only the conjugate momentum of \(\eta(\rho)\) and \(F_{\rho\psi}(\rho)\) are considered.
As mentioned at the beginning of this section, in order to derive the matrix model resolvent associated with the isolated eigenvalue, we change the boundary condition as $k \rightarrow k + p$ and make the Laplace transformation with respect to $p$, with taking account of the relation $p \ll N$. Now let us follow simply this procedure without using other information like the explicit form of the saddle point configuration in contrast to the derivation described below (3.14). In fact, this approach enables us to clarify a gravitational interpretation of the isolated eigenvalue as we will see shortly.

Recalling that the variation of $\mathcal{R}$ with respect to change of the boundary value originates only from that of the boundary terms \[\delta \mathcal{R}(\Pi, P_U) = \int d\psi \delta(\Pi) A_\psi \bigg|_{\rho=0} + \int d\psi \delta(P_U) U \bigg|_{\rho=0}. \tag{3.49}\]

By neglecting the terms which vanish in the limit $p \ll N$, $\mathcal{R}(-i(k + p), -R(k + p))$ is rewritten as

$$\mathcal{R}(-ik, -Rk) - p \left( \int d\psi (iA_\psi + RU) \right). \tag{3.50}$$

Then by performing the Laplace transformation, we can derive the following pole term:

$$\int_0^\infty dp e^{-pz} e^{-\mathcal{R}(-i(k+p), -R(k+p))} \rightarrow \frac{e^{-\mathcal{R}(-ik, -Rk)}}{z - \int d\psi (iA_\psi + RU)}. \tag{3.51}$$

As a check, we shall plug the solution (3.10) into this, and then the location of the pole reproduces the position of the isolated eigenvalue correctly:

$$\int d\psi (iA_\psi + RU) = \sqrt{\lambda} \sqrt{1 + \kappa^2}. \tag{3.52}$$

As we have noted a couple of times in this paper, it is not so striking a result in itself that the position of the isolated eigenvalue is reproduced. Here we claim that the position of the pole, namely the position of the eigenvalue, has been identified with the electric flux $F_{\rho\psi}$ integrated with respect to $\rho$ and $\psi$ directions of the D3-brane (plus contribution from scalar field $U$) as far as the isolated eigenvalue is concerned. This is exactly what we anticipated in the last of section 2, because this value is conjugate to the electric flux, namely the string charge, in the sense of (3.49) and (3.50).

Here it is important to recognize that in deriving the gravitational interpretation of the eigenvalue (3.52), our boundary condition (3.47) plays a crucial role. This is because each boundary term (3.12) and (3.13) diverges at the boundary, but after an appropriate regularization, dependence on the regularization cancels out between them as shown in [9]. One can easily find that under our boundary condition (3.47), the divergence again cancels because of the fact that both of boundary values of $\Pi$ and $P_U$ are given in terms of the same $k$. In particular, if we impose the boundary condition only on $\Pi$, this is not the case. Finally, it is interesting to see that from (3.52) the gravitational counterpart of the isolated eigenvalue takes the form of the exponent of the $U(1)$ Wilson loop (see the footnote at the end of subsection 3.2).

\[\text{It is important to notice that this is true because of the saddle point equation. Hence this is not generally true for general configurations.}\]
4 Conclusions and discussions

In this paper, we have analyzed the gauge theory Wilson loop winding around a circular loop $k$ times, by using a D3-brane carrying $k$ units of string charge in the context of AdS/CFT correspondence. It is known that the calculation of the expectation value of this Wilson loop, thanks to its symmetry, boils down to considering a Gaussian matrix model with an exponential operator insertion. We have then aimed our goal at establishing a gravitational interpretation of the eigenvalue in this matrix model based on the D3-brane version of the correspondence (1.1). This point presents a contrast to the preceding papers on “bubbling Wilson loops” [12–14].

We started with analyzing the resolvent of the Gaussian matrix model using the orthogonal polynomials. In the calculation, we first derive the resolvent in its inverse Laplace transformed form, by use of $e^{pz}$. Then after the Laplace transformation we obtained the usual resolvent. In the inverse Laplace transformed form, we derived the expression which is valid to all orders in $1/N$, and we identified a term that corresponds to the eigenvalue which is isolated due to a large $k \sim \mathcal{O}(N)$ effect. The remaining terms are responsible for the rest of the resolvent. The part corresponding to the isolated eigenvalue has the structure in which $p$ is included in the shift of $k$ as $\mathrm{tr} e^{kM} \rightarrow \mathrm{tr} e^{(k+p)M}$. In accordance with the D3-brane description of the matrix model operator $\frac{1}{N} \mathrm{tr} e^{kM}$, we were naturally led to consider the D3-brane with $k+p$ string charge and the Laplace transformation of its amplitude. Eventually we identify the position of the isolated eigenvalue, observed as an isolated pole of the resolvent, with an integrated flux (and scalar fluctuation) on the D3-brane. We therefore succeed in providing, at least in part, a gravitational interpretation of the eigenvalue in the Gaussian matrix model.

As a by-product, we have proposed natural boundary conditions for the D3-brane configuration with fluxes in terms of the Wilson loop. These boundary conditions also provide a direct relationship between local BPS conditions in the gauge theory side and that of the effective theory on the probe D3-brane.

Let us now discuss the cut of the resolvent. In the present paper, we have mainly studied the gravitational description of the isolated pole of the resolvent. It is surely nice if the cut of the resolvent can also be discussed based on our gravitational description. In the gauge theory side we saw in subsection 2.2 that the cut of the leading semi-circle originates from the leading term in $w(k,p)$. In the gravity side, we argued that this term corresponds to the configuration with a D3-brane carrying string charge $p (\ll N)$ in addition to the one with the string charge $k (\sim N)$ which is now treated as a part of the background. So, we may expect that by performing the Laplace transformation of the contribution from such configuration, the cut would be reproduced. However, here we should recall that the saddle point value of the gravity side gives only the leading term of the gauge theory observables. In fact, the saddle point value of the D3-brane action with string charge $p (\ll N)$ naively gives the result $e^{\sqrt{\lambda}}$. This is just the leading contribution of the Bessel function (2.28) in the large ’t Hooft coupling limit, and it does not lead to the cut after the Laplace transformation. Hence, it is clear that in order to reproduce the cut, we need to take account of the quantum $\alpha'$ correction of the D3-brane with string charge $p$. 

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Another point we should also mention is the case of the correspondence between anti-symmetric Wilson loop and D5-brane \cite{10,24}. It would be interesting to examine whether our gravitational interpretation is valid also in this case. However the situation is not straightforward. This is because the multiplication of the $k$-th anti-symmetric trace operator $\text{Tr}_A e^M$ and the probe operator $\text{tr} e^{\rho M}$ does not include the $(k+p)$-th anti-symmetric trace operator. It seems to suggest that we need to consider the configuration with a D3-brane carrying $p$ flux around the background D5-brane with $k$ flux, instead of only considering a single D5-brane with $(k+p)$-flux.

Here we also point out a subtlety concerning the relation between a Wilson loop with winding number $k$ and that in the $k$-th symmetric representation. Let $u_i$ be $i$-th eigenvalue of $e^M$ where $M$ is the matrix variable used in the matrix model analysis in section 2.2. The Wilson loop with winding number $k$ can be calculated by $\text{Tr} e^{kM} = \sum_i u_i^k$ while the Wilson loop in the $k$-th symmetric representation corresponds to

$$\text{Tr}_{S_k} e^M \equiv \sum_i u_i^k + \sum_i \sum_{j \neq i} u_i^{k-1} u_j + \cdots,$$

where $\cdots$ includes the other combinations of the powers of the eigenvalues. See \cite{10,19} for the details. Here we note that the decomposition that appeared in our matrix model calculation \cite{22,35} is equivalent to the terms here when we write them as

$$w((k-1) + 1) + w(k - 1, 1)$$

with $k$ being $k - 1$ and $p$ being 1. It has been argued that the expectation values of these two Wilson loops coincide under the large 't Hooft coupling limit, namely in (4.1), compared to the first term, the rests are exponentially suppressed in large $\lambda$. However our explicit calculation shows that $w(k)$ is of order $1/N$ compared to $w(k-1,1)$. Note that $k$ and $p$ have been assumed to be of $\mathcal{O}(N)$ and $\mathcal{O}(1)$ respectively, and then the calculation is still valid. The equivalence thus holds only when the large-$\lambda$ limit overcomes the difference in $1/N$, that is, $\lambda$ needs to be larger than $\log N$. This is therefore out of the usual limit such as first taking $N \to \infty$ with fixed $\lambda$ and then taking $\lambda$ to be large. Note that in this article we employ only the fact that in the strong coupling limit the D3-brane solution with $k$-flux agrees with $\text{tr} e^{kM}$, and then this subtlety does not matter.

Finally, it would be also important and interesting future work to clarify the relation between our viewpoint and the bubbling picture \cite{13,14}. We believe our viewpoint leads to a deep understanding of the gravitational interpretation of the eigenvalue, or moreover the connection between gauge theory and gravity.

**Acknowledgements**

The authors would like to thank S. Yahikozawa, S. Yamaguchi, T. Yoneya. They also thank Dimitrios Giataganas and Nadav Drukker for helpful comments on the preprint version of the article. The work of A. M. is supported in part by JSPS Research Fellowships for Young Scientists. We thank the Yukawa Institute for Theoretical Physics at Kyoto University. Discussions during the YITP workshop YITP-W-08-04 on “Development of Quantum Field Theory and String Theory” were useful to complete this work.
A Two loop correlator in Gaussian matrix model

In this appendix we compute the “two loop correlator” of the Wilson loop:

\[ W(p_k) = \frac{1}{Z_k} \int dM \frac{1}{N} \text{tr} e^{pM} \frac{1}{N} \text{tr} e^{kM} e^{-\frac{2N}{\lambda} \text{tr} M^2}, \quad (A.1) \]

\[ Z_k \equiv \int dM \frac{1}{N} \text{tr} e^{kM} e^{-\frac{2N}{\lambda} \text{tr} M^2}, \quad (A.2) \]

in order to examine the resolvent in the presence of the Wilson loop with a large winding number. For this purpose, we have to evaluate (A.1) for finite \( N \) as noticed below (2.33). Therefore, we calculate (A.1) by means of the orthogonal polynomials. We first change the variables as

\[ k' = \sqrt{\frac{\lambda}{4N}} k, \quad p' = \sqrt{\frac{\lambda}{4N}} p, \quad M' = \sqrt{\frac{4N}{\lambda}} M, \quad (A.3) \]

then \( W(p_k) \) becomes

\[ W(p_k) = \frac{1}{Z_k} \int dM' \frac{1}{N} \text{tr} e^{p'M'} \frac{1}{N} \text{tr} e^{k'M'} e^{-\frac{1}{2} \text{tr} M'^2}, \quad (A.4) \]

\[ \tilde{Z}_k = \left( \frac{4N}{\lambda} \right)^\frac{N^2}{2} Z_k = \int dM' \frac{1}{N} \text{tr} e^{k'M'} e^{-\frac{1}{2} \text{tr} M'^2}. \quad (A.5) \]

Using the Hermite polynomial \( P_i(m) \) satisfying

\[ \int dm e^{-\frac{1}{2} m^2} P_i(m) P_j(m) = h_i \delta_{ij}, \quad h_i = \sqrt{2\pi i}!, \quad P_i(m) = m^i + O(m^{i-1}), \quad (A.6) \]

the Vandermonde determinant can be written as

\[ \Delta(m)^2 = \left( \det_{ij} P_j-1(m_i) \right)^2 = \sum_{\sigma\tau} \text{sgn}(\sigma\tau) \prod_{\ell} P_{\sigma(\ell)-1}(m_\ell) P_{\tau(\ell)-1}(m_\ell), \quad (A.7) \]

and the relevant integral is given by

\[ \tilde{Z}_k W(p)_k = \int \prod_n dm_n e^{-\frac{1}{2} m_n^2} \sum_{\sigma\tau} \text{sgn}(\sigma\tau) \prod_{i} P_{\sigma(i)-1}(m_i) P_{\tau(i)-1}(m_i) \sum_{ij} \frac{1}{N} e^{p'_m + k'_m} \frac{1}{N} e^{k'_m}. \quad (A.8) \]

We decompose the sum over \( i, j \) in (A.8) into two types according to \( i = j \) or \( i \neq j \)

\[ (A.8) \]

\[ = \frac{1}{N^2} \sum_{\sigma\tau} \text{sgn}(\sigma\tau) \sum_{i} \sum_{i \neq j} \int \prod_k dm_k e^{-\frac{1}{2} m_k^2} \prod_{i} P_{\sigma(i)-1}(m_i) P_{\tau(i)-1}(m_i) e^{p'_m + k'_m} \quad (A.9) \]

\[ + \frac{1}{N^2} \sum_{\sigma\tau} \text{sgn}(\sigma\tau) \sum_i \int \prod_k dm_k e^{-\frac{1}{2} m_k^2} \prod_{i} P_{\sigma(i)-1}(m_i) P_{\tau(i)-1}(m_i) e^{(p'_i + k'_i)}. \quad (A.10) \]
Each term (A.9) and (A.10) corresponds to $\tilde{Z}_k w(k, p)$ and $\tilde{Z}_k w(k + p)$, respectively where $w(k, p)$ and $w(k + p)$ are given in (2.35).

The first term can be rewritten as

(A.9)

\[
\frac{1}{N^2} \sum_{\sigma \tau} \text{sgn}(\sigma \tau) \sum_i \sum_{j \neq i} \prod_{k \neq i,j} \int dm_k e^{-\frac{1}{2}m_k^2} P_{\sigma(k)-1}(m_k) P_{\tau(k)-1}(m_k)
\]

\[
\times \int dm_j e^{-\frac{1}{2}m_j^2} P_{\sigma(j)-1}(m_j) P_{\tau(j)-1}(m_j) e^{k' m_j}
\]

\[
\times \int dm_i e^{-\frac{1}{2}m_i^2} P_{\sigma(i)-1}(m_i) P_{\tau(i)-1}(m_i) e^{p' m_i}
\]

(A.11)

$$= \frac{1}{N^2} \sum_{\sigma \tau} \text{sgn}(\sigma \tau) \sum_i \sum_{j \neq i} \left( \prod_{k \neq i,j} \delta_{\sigma(k) \tau(k)} h_{\sigma(k)-1} \right) I_{\sigma(j)-1, \tau(j)-1}(k'; p').$$

(A.12)

Here we defined

$$I_{i,j}(k) \equiv \int dm e^{-\frac{1}{2}m^2} P_i(m) P_j(m) e^{km}.\quad (A.13)$$

On the other hand the second term (A.10) can be rewritten as

(A.10)

\[
\frac{1}{N^2} \sum_{\sigma \tau} \text{sgn}(\sigma \tau) \sum_i \int \prod_{j \neq i} \left( dm_j e^{-\frac{1}{2}m_j^2} P_{\sigma(j)-1}(m_j) P_{\tau(j)-1}(m_j) \right)
\]

\[
\times \int dm_i e^{-\frac{1}{2}m_i^2} P_{\sigma(i)-1}(m_i) P_{\tau(i)-1}(m_i) e^{(p' + k')m_i}
\]

(A.14)

$$= \frac{1}{N^2} \sum_{\sigma \tau} \sum_i \left( \prod_{j \neq i} \delta_{\sigma(j) \tau(j)} h_{\sigma(j)-1} \right) \int dm_i e^{-\frac{1}{2}m_i^2} P_{\sigma(i)-1}(m_i) P_{\tau(i)-1}(m_i) e^{(p' + k')m_i}
\]

(A.15)

$$= \frac{1}{N^2} \sum_{\sigma \tau} \sum_i \left( \prod_{j \neq i} h_{\sigma(j)-1} \right) I_{\sigma(i)-1}(p' + k'),\quad (A.16)$$

with $I_i(k) \equiv I_{i,i}(k)$. From (A.15) to (A.16), we used the fact that the Kronecker delta $\prod_{j \neq i} \delta_{\sigma(j) \tau(j)}$ implies that two permutations $\sigma$ and $\tau$ are identical.

Next we perform the integral in (A.13). We first note that from the generating function of the Hermite polynomial

$$e^{tm - \frac{t^2}{2}} = \sum_{i=0}^{\infty} P_i(m) \frac{t^i}{i!},\quad (A.17)$$

we have

$$P_i(m) = \frac{\partial_i e^{tm - \frac{t^2}{2}}}{t} \bigg|_{t=0}.\quad (A.18)$$

Plugging this equation into (A.13), we have

$$I_{i,j}(k) = \int dm e^{-\frac{1}{2}m^2 + km} \partial_i e^{tm - \frac{t^2}{2}} \partial_j e^{sm - \frac{s^2}{2}} \bigg|_{s=t=0}.$$
Here the Kronecker delta in the last expression implies that only two cases are possible:

\[
\frac{\partial^k}{\partial t^k} e^{-\frac{1}{2}t^2} e^{-\frac{1}{2}s^2} \int dt e^{-\frac{1}{2}m^2+(k+t+s)m} \bigg|_{s=t=0} = \frac{\sqrt{2\pi}}{\partial^k} e^{kt} (t + k)^j \bigg|_{t=0}.
\]

By differentiating the generating function of Laguerre polynomial \( L_{i}^{(\alpha)}(x) \)

\[
(1 + t)^{\alpha} e^{-xt} = \sum_{i=0}^{\infty} L_{i}^{(\alpha-i)}(x)t^i,
\]

with respect to \( t \) we obtain

\[
\frac{d^k}{dt^k} (1 + t)^{\alpha} e^{-xt} \bigg|_{t=0} = i! L_{i}^{(\alpha-i)}(x)\,.
\]

By using this equation with replacing \( x \to -k^2 \) and \( t \to t/k \), we can rewrite (A.19) as

\[
I_{i,j}(k) = h_i e^{\frac{1}{2}k^2} L_{i}^{(j-i)}(-k^2)\,.
\]

Notice that by definition

\[
I_{i,j}(k) = I_{j,i}(k),
\]

which can also be proved explicitly from (A.22), and

\[
I_{i}(k) \equiv I_{i,i}(k) = h_i e^{\frac{1}{2}k^2} L_{i}^{(0)}(-k^2), \quad L_{i}(x) \equiv L_{i}^{(0)}(x)\,.
\]

Substituting (A.22) into (A.12) yields

\[
\text{(A.9)}
\]

\[
\frac{1}{N^2} \sum_{\sigma \tau} \operatorname{sgn}(\sigma \tau) \sum_{i} \sum_{j \neq i} \left( \prod_{k \neq i,j} \delta_{\sigma(k)\tau(k)} h_{\sigma(k)-1} \right) \times h_{\sigma(j)-1} e^{\frac{1}{2}k^2} k^{\tau(j)-\sigma(j)} L_{\sigma(j)-1}^{(\tau(j)-\sigma(j))} (-k^2) h_{\sigma(i)-1} e^{\frac{1}{2}p^2} p^{\tau(i)-\sigma(i)} L_{\sigma(i)-1}^{(\tau(i)-\sigma(i))} (-p^2)
\]

\[
= \frac{1}{N^2} e^{\frac{1}{2}(k^2+p^2)} \sum_{\sigma \tau} \operatorname{sgn}(\sigma \tau) \left( \prod_{l} h_{\sigma(l)-1} \right) \sum_{i} \sum_{j \neq i} \left( \prod_{k \neq i,j} \delta_{\sigma(k)\tau(k)} \right) \times k^{\tau(j)-\sigma(j)} L_{\sigma(j)-1}^{(\tau(j)-\sigma(j))} (-k^2) p^{\tau(i)-\sigma(i)} L_{\sigma(i)-1}^{(\tau(i)-\sigma(i))} (-p^2)
\]

\[
= \frac{1}{N^2} e^{\frac{1}{2}(k^2+p^2)} \left( \prod_{l} h_{l-1} \right) \sum_{\sigma \tau} \operatorname{sgn}(\sigma \tau) \sum_{i} \sum_{j \neq i} \left( \prod_{k \neq i,j} \delta_{\sigma(k)\tau(k)} \right) \times k^{\tau(j)-\sigma(j)} L_{\sigma(j)-1}^{(\tau(j)-\sigma(j))} (-k^2) p^{\tau(i)-\sigma(i)} L_{\sigma(i)-1}^{(\tau(i)-\sigma(i))} (-p^2)\,.
\]

Here the Kronecker delta in the last expression implies that only two cases are possible:
1. $\sigma = \tau$,

2. $\sigma(k) = \tau(k)$ for $k \neq i, j$ and $\sigma(i) = \tau(j)$, $\sigma(j) = \tau(i)$.

According to this, we get

\[ \text{(A.9)} \]
\[
= \frac{1}{N^2} e^\frac{i}{2} (p' + k')^2 \left( \prod_l h_{l-1} \right) \sum_{\sigma} \sum_i \sum_{j \neq i} \left( L_{\sigma(j)-1} (-k'^2) L_{\sigma(i)-1} (-p'^2) - k'^{\sigma(i)-\sigma(j)} L^{(\sigma(i)-\sigma(j))}_{\sigma(j)-1} (-k'^2) p'^{\sigma(j)-\sigma(i)} L^{(\sigma(j)-\sigma(i))}_{\sigma(i)-1} (-p'^2) \right)
\]
\[
= \frac{N!}{N^2} e^\frac{i}{2} (p' + k')^2 \left( \prod_l h_{l-1} \right) \sum_{ij} \left( L_{j-1} (-k'^2) L_{i-1} (-p'^2) - k'^{i-j} L^{(i-j)}_{j-1} (-k'^2) p'^{j-i} L^{(j-i)}_{i-1} (-p'^2) \right)
\]
\[
= \frac{N!}{N^2} e^\frac{i}{2} (p' + k')^2 \left( \prod_l h_{l-1} \right) \left( \sum_{ij} \left( L_{N-1}^{(i)} (-k'^2) L_{N-1}^{(j)} (-p'^2) - \sum_{ij} k'^{i-j} L^{(i-j)}_{N-1} (-k'^2) p'^{j-i} L^{(j-i)}_{N-1} (-p'^2) \right) \right), \quad \text{(A.26)}
\]

where in the last step we have used an identity of the Laguerre polynomial, $L_n^{(\alpha+1)}(x) = \sum_{j=0}^n L_j^{(\alpha)}(x)$.

On the other hand, using (A.24) in (A.16), we obtain

\[ \text{(A.10)} \]
\[
= \frac{1}{N^2} \sum_{\sigma} \sum_i \left( \prod_{j \neq i} h_{\sigma(j)-1} \right) h_{\sigma(i)-1} e^\frac{i}{2} (p' + k')^2 L_{\sigma(i)-1} (-p' + k')^2
\]
\[
= \frac{1}{N^2} e^\frac{i}{2} (p' + k')^2 \sum_{\sigma} \sum_i \left( \prod_j h_{\sigma(j)-1} \right) L_{\sigma(i)-1} (-p' + k')^2
\]
\[
= \frac{1}{N^2} e^\frac{i}{2} (p' + k')^2 \left( \prod_j h_{j-1} \right) \sum_{\sigma} \sum_i L_{\sigma(i)-1} (-p' + k')^2
\]
\[
= \frac{N!}{N^2} e^\frac{i}{2} (p' + k')^2 \left( \prod_j h_{j-1} \right) \sum_i L_{i-1} (-p' + k')^2
\]
\[
= \frac{N!}{N^2} e^\frac{i}{2} (p' + k')^2 \left( \prod_j h_{j-1} \right) L_{N-1}^{(1)} (-p' + k')^2. \quad \text{(A.27)}
\]

From (A.26) and (A.27), (A.8) can be rewritten as

\[
\tilde{W}_k(p) = \frac{N!}{N^2} e^\frac{i}{2} (p'^2 + k'^2) \left( \prod_l h_{l-1} \right)
\]

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For this purpose, we recall its integral representation (A.13) and rescale the variable as (A.30) and (A.31) come from (A.9) and (A.10), so give the explicit form of where we have multiplied the additional factor $e^{2k'}$ and, for notational simplicity, we have changed $i \to i + 1, j \to j + 1$. $h_i$ and $I_{i,j}$ are defined as (A.6) and (A.13), respectively.

Let us start with the large-$N$ expansion of $I_{j,i}(k')$ for arbitrary $i, j$ and finite $\kappa$. For this purpose, we recall its integral representation (A.13) and rescale the variable as $\tilde{m} = m/\sqrt{N}$ such that the Gaussian potential becomes proportional to $N$;

$$I_{j,i}(k') = N^{i+j+1} \int d\tilde{m} e^{-\frac{N}{2} \tilde{m}^2 + k' \tilde{m}} \tilde{P}_j(\tilde{m}) \tilde{P}_i(\tilde{m}).$$

(A.35)

Here $\tilde{k}' = \sqrt{N} k'$ and we have also rescaled the Hermite polynomials as $\tilde{P}_i(\tilde{m}) = N^{-i/2} P_i(m)$ to make it satisfy the normalization $\tilde{P}_i(\tilde{m}) = \tilde{m}^i + \cdots$. The large-$N$ behavior of the generic
orthogonal polynomial in this normalization is addressed in [18]. Because of the large linear term \( \tilde{k}' \tilde{m} = 2N \tilde{m} \), integral \([A.35]\) has a saddle point \( \tilde{m} = \tilde{m}_* \) at a non-oscillating region \( \sqrt{4i/N} < \tilde{m}_* \) for the Hermite polynomial \( \tilde{P}_i(\tilde{m}) \) (see [18] for generic behavior of the orthogonal polynomial). Hence we concentrate on the expression for that region:

\[
\tilde{P}_i(\tilde{m}) = \exp \left( N \int_0^{\tilde{m}} d\xi \log k^{(0)}(\tilde{m}, \xi) + \frac{1}{2} \log k^{(0)}(\tilde{m}, \frac{i}{N}) - \frac{1}{2} \log q(\tilde{m}, \frac{i}{N}) + \mathcal{O}(N^{-1}) \right),
\]

(A.36)

where

\[
k^{(0)}(\tilde{m}, \xi) = \tilde{m} + \sqrt{\tilde{m}^2 - 4\xi}, \quad q(\tilde{m}, \xi) = \sqrt{\tilde{m}^2 - 4\xi}.
\]

(A.37)

By using this expression, we find the saddle point of the integral \([A.35]\) as

\[
\tilde{m}_* = \frac{\bar{k}^2}{N^2} + \frac{2(i + j)}{N^2} + \frac{(i - j)^2}{k^2}.
\]

(A.38)

Evaluating \([A.35]\) semi-classically including the Gaussian integral around the saddle point, we obtain the following large-\( N \) behavior

\[
I_{i,j}(k') = N^{\frac{i+j+1}{2}} \exp \left[ \frac{1}{2} \tilde{k}' \tilde{m}_* + i \log \frac{\tilde{m}_* + \frac{k'}{N}}{2} + j \log \frac{\tilde{m}_* + \frac{k'}{N}}{2} - \frac{i}{2} - \frac{j}{2} \right.
\]

\[
+ \frac{1}{2} \log \left( \frac{\tilde{m}_* + \frac{k'}{N}}{4} \right) - \frac{1}{2} \log(\bar{k}' \tilde{m}_*) + \frac{1}{2} \log(2\pi) + \mathcal{O}(N^{-1}) \right].
\]

(A.39)

Next we study the large-\( N \) behavior of \( I_{i,j}(p') \) for finite \( p = p' \sqrt{4N/\lambda} \). In this case, the linear term \( \bar{p}' \tilde{m} = p' \tilde{m} \) is small and we need to consider an oscillating region. Here, instead of studying it, we derive the large-\( N \) behavior of \( I_{i,j}(p') \) by solving the differential equation satisfied by \( f(i, j, p') = e^{\frac{p'^2}{2}} L_i^{(j-i)}(-p'^2) \):

\[
\partial_{p'}^2 f + \frac{2(j - i) + 1}{p} \partial_p f - \frac{\lambda}{4N} \left( \frac{\lambda}{4N} p^2 + 2(i + j + 1) \right) = 0.
\]

(A.40)

The first term \( (\lambda/4N)p^2 \) in the round bracket is small compared to the second term in the same bracket. By neglecting this term, we can reduce the the equation to Bessel’s differential equation and we obtain

\[
I_{i,j}(p') = \sqrt{2\pi} \max(i, j)! \left( \frac{2}{i + j + 1} \right)^{\frac{|i-j|}{2}} I_{i-j}(p \sqrt{\Lambda}) \left( 1 + \mathcal{O}(N^{-1}) \right),
\]

(A.41)

where \( \Lambda = \lambda(i + j + 1)/2N \). \( \max(i, j) \) represents the larger value and \( I_{i-j} \) is the modified Bessel function. The overall constant can be fixed by comparing the asymptotic behavior of the Bessel function with that of the Laguerre polynomial in the small \( p \) limit. This expression is again valid for any values of \( i \) and \( j \).
Now we discuss the summation (A.32) using these ingredients. We should be careful that the subleading $O(N^{-1})$ terms in (A.39) and (A.41) possibly contribute to the leading behavior of $A$, since $A$ is defined by the summation over $N^2$ terms. However, as we will see later, only the terms with $N - i = O(1)$ and $N - j = O(1)$ actually contribute to the second term of (A.30). Then the effective number of terms being summed is of order $O(1)$, and we can consistently neglect the $O(N^{-1})$ terms. In the following, we call the range $N - i = O(1)$, $N - j = O(1)$ “$C$”. We first concentrate on the terms in the range $C$, and then we study the contributions of the terms outside $C$.

In order to discuss the terms in $C$, we change variables as $m \equiv 2N - (i + j)$ and $d \equiv i - j$ and expand (A.39) and (A.41) with assuming $m = O(1)$, $d = O(1)$. Then we obtain the following simple expression for $F_{i,j}(k', p')$:

$$F_{i,j}(k', p') = \left[ 2N \left( \kappa \sqrt{\kappa^2 + 1} + \log(\sqrt{\kappa^2 + 1} + \kappa) \right) + \log(\sqrt{\kappa^2 + 1} + \kappa) - \frac{1}{2} \log(\kappa \sqrt{\kappa^2 + 1} + 1) - \frac{1}{2} \log(8\pi N) \right] \frac{I_{d|p\sqrt{\lambda}}}{(\sqrt{\kappa^2 + 1} + \kappa)^m} \left( 1 + O(N^{-1}) \right). \quad (A.43)$$

We set the range of summation as $d = 0, \pm 1, \pm 2, \cdots, \pm D$ and $m = |d| + 2, |d| + 4, \cdots, |d| + 2M$ for a fixed value of $d$. Here $M$ and $D$ define the upper limits of the summation which should be large but still of $O(N)$ in the present assumption of the range $C$. In the large-$N$ limit, both of the upper limits can be consistently taken to be infinity. After the $m$-summation, we obtain the following large-$N$ expression for $A$:

$$A = \sum_{d=-D}^{D} \sum_{(m-|d|)/2=1}^{M} F_{i,j}(k', p') + (\cdots) \quad (A.44)$$

$$\rightarrow e^{\frac{k'^2}{2}} L_{N-1}^{(1)}(-k'^2) \sum_{d=-\infty}^{\infty} \frac{I_{|d|p\sqrt{\lambda}}}{(\sqrt{\kappa^2 + 1} + \kappa)^{|d|}} \left( 1 + O(N^{-1}) \right) + (\cdots). \quad (A.45)$$

Here for simplicity we have taken the limit $D \to \infty$, $M \to \infty$ which is consistent in the large-$N$ limit as noted above. The extracted factor $e^{\frac{k'^2}{2}} L_{N-1}^{(1)}(-k'^2)$ is exactly canceled by the denominator of (A.30) taking account of the extra factor $e^{\frac{k'^2}{2}}$.

The dots $(\cdots)$ in (A.45) express the terms outside $C$. We will shortly see that these terms actually do not give finite contribution to the resolvent in the large-$N$ limit. Before that we shall derive the resolvent corresponding to the first term of (A.45) by performing the Laplace transformation (recall the overall factor $-1/N$ in the second term of (A.30)):

$$R_{2nd \ term}(z) = -\frac{1}{N} \int_{0}^{\infty} dp e^{-pz} \sum_{d=-\infty}^{\infty} \frac{I_{|d|p\sqrt{\lambda}}}{(\sqrt{\kappa^2 + 1} + \kappa)^{|d|}} \left( 1 + 2 \sum_{d=1}^{\infty} \left( \frac{z - \sqrt{z^2 - \lambda}}{\sqrt{\lambda} (\kappa + \sqrt{\kappa^2 + 1})} \right)^d \right) \quad (A.46)$$

$$= -\frac{1}{N} \frac{1}{\sqrt{z^2 - \lambda}} \left( 1 + 2 \sum_{d=1}^{\infty} \left( \frac{z - \sqrt{z^2 - \lambda}}{\sqrt{\lambda} (\kappa + \sqrt{\kappa^2 + 1})} \right)^d \right) \quad (A.47)$$
Therefore, the only singularity of $R_{2\text{nd term}}(z)$ is the cut $-\sqrt{\lambda} \leq z \leq \sqrt{\lambda}$. In fact, it is easy to see that (A.48) is just another form of (2.18). By checking the residue of the pole at infinity it is clear that the resolvent corresponds to “minus one” eigenvalue, i.e., it subtracts one eigenvalue from the leading semi-circle distribution. Hence we have derived the resolvent (2.18) without assuming the conditions mentioned after (2.17).

Finally let us show that the terms outside $C$ do not contribute to the resolvent. In fact these terms are exponentially suppressed with respect to $N$ compared to the terms in the range $C$. Since such exponential suppressions will not be compensated even by summing up all the $N^2$ terms, we can conclude that these terms do not contribute in the large-$N$ limit.

In order to clarify this point, it is helpful to start with the following simple ordering property of $F_{ij}(k', p')$:

$$F_{ij}(k', p') \leq F_{i+1,j+1}(k', p'), \quad (A.49)$$

which follows directly from the definition of the Laguerre polynomial: $L_i^{(j-i)}(-x^2) = \sum_{r=0}^j jC_{i-r}x^{2r}/r!$. Here $jC_{i-r}$ is a binomial coefficient and we define $jC_{i-r} = 0$ for $j - i + r < 0$. Because of this ordering property and also the symmetric property $F_{ij}(k', p') = F_{ji}(k', p')$, it is sufficient to show the following two claims: (I) among the terms $F_{i,N-1}(k', p')$, those in $C$ are exponentially larger than the other terms, and (II) among the terms $F_{i,j}(k', p')$ with $i - j = O(1)$, those in $C$ are exponentially larger than the other terms. By showing these two claims and considering the ordering property (A.49), one can easily see that the terms in $C$ are exponentially larger than all the other terms.

Let us begin with the first claim (I). We use the expressions (A.39) and (A.41) with setting $j = N - 1$. Keeping terms of up to $O(N)$, we have

$$F_{i,N-1}(k', p') = \exp \left[ \frac{N}{2}(\xi - 1) \log N + \frac{N}{2}(1 - \xi) + \frac{N}{2}R + N\log \frac{K_-}{2} + N\xi \log \frac{K_+}{2} + N(1 - \xi) \log p' - N\xi \log \xi - N(1 - \xi) \log (1 - \xi) + O(\log N) \right], \quad (A.50)$$

where $\xi = i/N$ and $K_{\pm}$ and $R$ are defined as

$$K_{\pm} = \frac{R}{2\kappa} + 2\kappa \pm \frac{1}{2\kappa}(1 - \xi), \quad R = \sqrt{16\kappa^4 + 8\kappa^2(1 + \xi) + (1 - \xi)^2}. \quad (A.51)$$

The first and the second line is the asymptotic form of $h_{N-1}^{-1}I_{N-1,i}(k')$ and $h_i^{-1}I_{i,N-1}(p')$, respectively. We have used the asymptotic form of the Bessel function which follows from the saddle point approximation of the integral:

$$h_i^{-1}I_{i,N-1}(p') = p'^{N-1-i}(N-1)!(\frac{2}{p\sqrt{\Lambda}})^{N-1-i}I_{N-1-i}(p\sqrt{\Lambda})(1 + O(N^{-1})) \quad (A.52)$$
\[
\frac{1}{i!} \frac{p^{\nu}}{\sqrt{\pi} \Gamma(\nu + 1/2)} \int_{-1}^{1} dt (1 - t^2)^{\nu - \frac{1}{2}} e^{-p\sqrt{t}} \left(1 + \mathcal{O}(N^{-1})\right), \quad (A.53)
\]

where \( \nu = N - 1 - i \). The second line of \((A.50)\) follows by assuming \( \nu \gg 1 \), and applying the saddle point approximation for the integral and also Stirling’s formula for \( \Gamma(\nu + 1/2) \).

For the range \( \nu = \mathcal{O}(1) \), in the sense of large-\( N \) limit, both of these approximations break down. However, even in this region, we can find the regime where \( p \) is still large compared to 1 and also \( p \sqrt{X} \). In such a regime, the asymptotic behavior \((A.50)\) is still valid. In the second line of \((A.50)\), we have also used Stirling’s formula for \( \Gamma(1) \).

Here we have written down only the leading \( \nu \)-dependence. So, for the range \( 0 < \gamma \leq 1 \), \( F_{i,N-1}(k',p') \) increases quite rapidly with respect to \( \xi \) as \( N^\gamma N^\xi \).

The dominant \( \xi \)-dependence of \((A.50)\) arises from the first term on the first line and the terms on the second line. Let us take the derivative of the summation of these dominant terms with respect to \( \xi \):

\[
\frac{\partial}{\partial \xi} N \left( \frac{1}{2} \log N - \xi \log p' - \xi \log \xi - (1 - \xi) \log(1 - \xi) \right) = N \log \frac{N^\gamma}{1 - N^{\gamma - 1}} + \mathcal{O}(N). \quad (A.54)
\]

In the right hand side, we introduced \( \gamma \) by \( N - i = N^\gamma \), i.e., \( 1 - \xi = N^{\gamma - 1} \) and used \( p' \approx N^{-1/2} \). From this expression, we see that for the range \( 0 < \gamma \leq 1 \), \( F_{i,N-1}(k',p') \) increases quite rapidly with respect to \( \xi \) as \( N^\gamma N^\xi \).

Let us also check the \( \xi \)-dependence for the range \( 1 - \xi = \mathcal{O}(N^{-1}) \), i.e., \( N - i = \mathcal{O}(1) \). For this purpose we expand \((A.50)\) with respect to the small parameter \( \epsilon = 1 - \xi \). Then we have

\[
F_{i,N-1}(k',p') = \exp N \left( -\epsilon \log(\kappa + \sqrt{1 + \kappa^2}) - \frac{1}{2} \epsilon \log N + \epsilon \log p' + \epsilon \log \epsilon + \cdots \right). \quad (A.55)
\]

Here we have written down only the leading \( \epsilon \)-dependence. So, dots (\( \cdots \)) include the subleading terms and also the leading but \( \epsilon \)-independent terms. By taking the derivative of the exponent with respect to \( \xi = 1 - \epsilon \), we obtain the following result:

\[
N \log(\kappa + \sqrt{\kappa^2 + 1}) + N \log \left( \frac{2}{p \sqrt{X}} \right). \quad (A.56)
\]

So, for the range \( \epsilon > p \sqrt{X}/2N \), the function \( F_{i,N-1}(k',p') \) increases with \( \xi \) at least as \( e^{\xi N \log(\kappa + \sqrt{\kappa^2 + 1})} \). The \( \mathcal{O}(\log N) \) terms in \((A.50)\) do not change this conclusion. Hence we have shown the first claim (I).

Next we turn to the second claim (II), i.e., the parameter range \( i - j = \mathcal{O}(1) \). For this range, \( h_{i,j}^{-1}(p') \) behaves, at most, like power in \( N \) as can be seen from \((A.41)\). Hence the dominant behavior is determined by the following asymptotic form of \((A.39)\):

\[
e^{\frac{\kappa^2}{2} k'^{i-j} L_j^{(i-j)}(-k'^2)} = \exp \left( 2N \left( \kappa \sqrt{\kappa^2 + \zeta} + \zeta \log \frac{\kappa + \sqrt{\kappa^2 + \zeta}}{\sqrt{\zeta}} \right) + \mathcal{O}(\log N) \right), \quad (A.57)
\]

where \( \zeta = j/N \). The dependence on \( i - j \) appears only in the subleading terms. By taking derivative of the leading exponent with respect to \( \zeta \), we find that \( F_{i,j}(k',p') \) in this range grows at least as \( e^{2N \xi \log(\kappa + \sqrt{\kappa^2 + 1})} \). This shows that the second claim (II) indeed holds.
B  BPS conditions and the uniqueness of the solution

In AdS/CFT correspondence, one of the most important guiding principles to find out a corresponding object to its holographic counterpart is the symmetry preserved by the object. For example, the circular and the straight line Wilson loops preserve $SL(2, \mathbb{R}) \times SO(3)$ symmetry as a part of the Euclidean conformal group $SO(5, 1)$ of the $\mathcal{N} = 4$ super Yang-Mills (SYM) theory. In accordance with this preserved symmetry, the D3-brane solutions found in [9] have the structure of AdS$_2 \times S^2$ whose isometry is indeed $SL(2, \mathbb{R}) \times SO(3)$. The circular loop and the straight line also preserve a part of the supersymmetry of SYM, namely they are BPS objects, and therefore corresponding D3-brane solutions should preserve some of global supersymmetry of type IIB supergravity and, as also shown in [9], so do they.

In this appendix, we see that the BPS condition for D3-brane solutions, together with the $S^2$ symmetric ansatz, suffices to determine the solution (virtually) uniquely, at least in the circular loop and the straight line cases. In [9], the authors checked that, for the straight line, the BPS equation is satisfied by their solution. Here we do not assume the D3-brane solution, but just postulate the $S^2$ symmetric ansatz $\eta = \eta(\rho)$ and $F_{\rho \psi}(\rho)$, and observe how the BPS condition restricts the form of the solution.

We examine the circular loop case with a brief summary of the BPS condition for D-brane solutions. We start with the Euclidean AdS$_5$ metric (3.6),

$$ds^2 = \frac{L^2}{\sin^2 \eta} \left( d\eta^2 + \cos^2 \eta d\psi^2 + d\rho^2 + \sinh^2 \rho (d\theta^2 + \sin^2 \theta d\phi^2) \right), \quad (B.1)$$

and the $S^5$ part is omitted in the analysis since the solutions we are interested in are trivial on $S^5$. By a coordinate transformation, this metric is mapped into

$$ds^2 = \frac{L^2}{y^2} \left( dy^2 + dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right), \quad (B.2)$$

where Wick rotation to Lorentzian metric, $t_E = it$, is easily understood. One can see that this Wick rotated Lorentzian metric is mapped into the metric (B.1) with $\psi$ replaced with $\psi_E = i\psi$, and then we can consider $\psi_E = i\psi$ as “Wick rotation” of our metric. From (B.1), we define the vielbeins,

$$e^1 = \frac{L}{\sin \eta} d\eta, \quad e^2 = \frac{L}{\sin \eta} \cos \eta d\psi, \quad e^3 = \frac{L}{\sin \eta} d\rho, \quad e^4 = \frac{L}{\sin \eta} \sinh \rho d\theta, \quad e^5 = \frac{L}{\sin \eta} \sinh \rho \sin \theta d\phi. \quad (B.3)$$

We then consider the Killing spinor equation,

$$\left( D_\mu + \frac{1}{2L} \Gamma_\star \gamma_\mu \right) \epsilon = 0, \quad (B.4)$$

19The small fluctuation around the loop also obeys the classification with respect to this symmetry. See, for example, [25].
where $\Gamma_\star = \Gamma_{12345}$ and the gamma matrices in the flat tangent space are denoted by capital $\Gamma_a$ and those in the curved background by $\gamma_\mu = e^a_\mu \Gamma_a$, and $\epsilon$ is a complex combination of two chiral Majorana-Weyl spinors of type IIB supergravity. We find the Killing spinor of this background,

$$\epsilon = \tan^{1/2} \left( \frac{\eta}{2} \right) \left( M_+ \epsilon_1^- + M_- \epsilon_2^- \right) + i \cot^{1/2} \left( \frac{\eta}{2} \right) \Gamma_{12}(M_+ \epsilon_1^- - M_- \epsilon_2^-), \quad (B.5)$$

where $\epsilon_{1,2}$ are constant spinors satisfying $\tilde{\Gamma}_1 \epsilon_{1,2} = -\epsilon_{1,2}, \tilde{\Gamma} = \Gamma_{2345}$, and

$$M_{\pm}(\psi, \rho, \theta, \phi) = e^{\pm \frac{i}{2} \psi} e^{\pm \frac{i}{2} \gamma_{23}} e^{\frac{i}{2} \gamma_{34}} e^{\frac{i}{2} \gamma_{45}}. \quad (B.6)$$

The BPS condition for a D-brane solution is given by a compatibility condition of the $\kappa$-symmetry of the D-brane action and a part of the global super symmetry of the background. See, for example, [26]. The $\kappa$-symmetry projector with Lorentzian signature is defined by

$$d^{\mu+1} \xi \Gamma = - e^{-\phi} \mathcal{L}^{-1}_{DBI} e^F \wedge X|_{vol}, \quad (B.7)$$

$$X \equiv \oplus \Gamma_{(2n)} K^n I, \quad (B.8)$$

$$\Gamma_{(n)} = \frac{1}{n!} d\xi^{i_1} \cdots d\xi^{i_n} \partial_{i_1} X^{\mu_1} \cdots \partial_{i_n} X^{\mu_n} \gamma_{\mu_1 \cdots \mu_n}, \quad (B.9)$$

where $\xi$ denotes the world volume coordinates, $X^\mu(\xi)$ are the embedding coordinates, $\mathcal{F} = 2\pi \alpha' F$ is the two-form field strength of the world volume $U(1)$ gauge field, and $K$ and $I$ act on spinors as $K \psi = \psi^*, I \psi = -i\psi$. This projector is traceless and equipotent,

$$\text{Tr} \Gamma = 0, \quad \Gamma^2 = 1, \quad (B.10)$$

and the D-brane action enjoys the $\kappa$-symmetry

$$\delta \theta(\xi) = (1 + \Gamma(\xi)) \kappa(\xi), \quad (B.11)$$

where $\theta$ is the fermionic partner of $X^\mu$. $\theta$ is fermionic coordinate of the target space-time as well and then it transforms under the space-time supersymmetry as $\delta \theta = \epsilon$. Together with the $\kappa$-symmetry, $\theta$ transforms as

$$\delta \theta = (1 + \Gamma) \kappa + \epsilon, \quad (B.12)$$

and therefore if a constant $\epsilon$ satisfies

$$(1 - \Gamma) \epsilon = 0, \quad (B.13)$$

then this global space-time supersymmetry is compatible with the $\kappa$-symmetry, hence the embedded D-brane is BPS.

We choose $\rho, \psi, \theta, \phi$ as the world volume coordinates as before, and then with the $S^2$ symmetric ansatz, the projector takes the following form after Wick rotation,

$$\Gamma = if(\eta, F)^{-1} \left[ \cos \eta (1 + \eta \Gamma_{13}) - 2\pi \alpha' \frac{\sin^2 \eta}{L^2} F_{\psi \rho} \Gamma_{23} K \right] \tilde{\Gamma} I, \quad (B.14)$$
where

\[ f = \sqrt{\cos^2 \eta (1 + \eta^2) + (2\pi\alpha')^2 \frac{\sin^4 \eta}{L^4} \Gamma_{\psi \rho}^2}. \]  

(B.15)

Now we try to solve the BPS equation

\[ \Gamma \epsilon = \epsilon. \]  

(B.16)

First we rewrite the Killing spinor \((B.5)\),

\[ \epsilon = T(\eta)(M_+ \epsilon_1^- + M_- \epsilon_2^-) + i \cosh \rho C(\eta)(M_+ \Gamma_{12} \epsilon_1^- - M_- \Gamma_{12} \epsilon_2^-) + \sinh \rho C(\eta)(M_+ g(\theta, \phi) \Gamma_1 \epsilon_1^- + M_- g(\theta, \phi) \Gamma_1 \epsilon_2^-), \]  

(B.17)

where

\[ g(\theta, \phi) \equiv \Gamma_3 \cos \theta + \Gamma_4 \sin \theta \cos \phi + \Gamma_5 \sin \theta \sin \phi, \]  

(B.18)

and \(C(\eta) \equiv \cot^{1/2}(\frac{\eta}{2}), T(\eta) \equiv \tan^{1/2}(\frac{\eta}{2})\) have been introduced for simplicity. The action of the projector \(\Gamma\) on the Killing spinor is

\[ if(\eta, F)\Gamma \epsilon = -i \cos \eta T(\eta)(M_+ \epsilon_1^- + M_- \epsilon_2^-) - \cos \eta (\cosh \rho C(\eta) + \eta' \sinh \rho T(\eta)) (M_+ \Gamma_{12} \epsilon_1^- - M_- \Gamma_{12} \epsilon_2^-) + i \cos \eta (\sinh \rho C(\eta) + \eta' \cosh \rho T(\eta)) (M_+ g(\theta, \phi) \Gamma_1 \epsilon_1^- + M_- g(\theta, \phi) \Gamma_1 \epsilon_2^-) + \eta' \cos \eta C(\eta)(M_+ g(\theta, \phi) \Gamma_2 \epsilon_1^- - M_- g(\theta, \phi) \Gamma_2 \epsilon_2^-) + i 2\pi \alpha \frac{\sin^2 \eta}{L^2} F_{\psi \rho} (-T(\eta)(M_+ g(\theta, \phi) \Gamma_2 \epsilon_1^{-*} + M_- g(\theta, \phi) \Gamma_2 \epsilon_2^{-*}) - i C(\eta) \cosh \rho (M_+ g(\theta, \phi) \Gamma_1 \epsilon_1^{-*} + M_- g(\theta, \phi) \Gamma_1 \epsilon_2^{-*}) + C(\eta) \sinh \rho (M_+ \Gamma_1 \Gamma_2 \epsilon_1^{-*} + M_- \Gamma_1 \Gamma_2 \epsilon_2^{-*})). \]  

(B.19)

In order for a solution to exist, \(\epsilon_{1,2}^-\) have to be related to their complex conjugates \(\epsilon_{1,2}^+\) in a certain way, say,

\[ \Gamma_{1,2} \Gamma_{1,2}^{-*} \propto \Gamma_{1,2} \epsilon_{1,2}^-, \]  

(B.20)

and at this stage we need to consider all possible combinations of 1, 2 indices. Since \(\epsilon_1^-\) and \(\epsilon_2^-,\) and their complex conjugations, always appear with \(M_+\) and \(M_-\) respectively, flipping the index, say like \(\epsilon_1^{-*} \leftrightarrow \epsilon_2^-\), is not allowed. So there remain two possibilities:

Case I: \(\Gamma_{1} \Gamma_{1}^{-*} = \alpha_1 \Gamma_{2} \epsilon_{1}^-\), \(\Gamma_{1} \Gamma_{2}^{-*} = \alpha_2 \Gamma_{2} \epsilon_{2}^-\) where \(\alpha_{1,2} \in \mathbb{C}\)

First by looking at the signature of \(\epsilon_2^-\), \(\alpha_1 = \alpha_2\) is concluded, and thus we take \(\alpha = \alpha_1 = \alpha_2\) and obtain

\[ if(\Gamma - 1) \epsilon = \left[ -i \cos \eta T(\eta) - i 2\pi \alpha \frac{\sin^2 \eta}{L^2} F_{\psi \rho} \alpha C(\eta) \sinh \rho - if T(\eta) \right] \]  

(B.21)
\[ \times (M_+ \epsilon_1^- + M_- \epsilon_2^-) \]
\[ + \left[ - \cos \eta \cosh \rho C(\eta) + \eta' \sinh \rho T(\eta) + f \cosh \rho C(\eta) \right] \times (M_+ \Gamma_{12} \epsilon_1^- - M_- \Gamma_{12} \epsilon_2^-) \]
\[ + \left[ i \cos \eta \sinh \rho C(\eta) \right] \times (M_+ g(\theta, \phi) \Gamma_1 \epsilon_1^- + M_- g(\theta, \phi) \Gamma_1 \epsilon_2^-) \]
\[ + \left[ \eta' \cos \eta C(\eta) + 2 \pi \alpha' \frac{\sin^2 \eta}{L^2} F_{\psi \rho} \alpha C(\eta) \cosh \rho \right] \times (M_+ g(\theta, \phi) \Gamma_2 \epsilon_1^- - M_- g(\theta, \phi) \Gamma_2 \epsilon_2^-) \]
\[ = 0. \quad (B.25) \]

Each coefficient of \( \epsilon \) terms has to vanish independently, and by eliminating \( F_{\psi \rho} \) and \( f(\rho) \) from these four equations we have

\[ \eta' \cot \eta = \coth \rho, \quad (B.26) \]

which can be integrated to be

\[ \sin \eta = \kappa^{-1} \sinh \rho. \quad (B.27) \]

Here \( \kappa^{-1} \) is a constant of integration. By inserting this solution for the condition which follows from the term \((B.24)\), we have

\[ F_{\psi \rho} = -\alpha^{-1} \frac{\kappa L^2}{2 \pi \alpha' \sinh^2 \rho}, \quad (B.28) \]

and also from \((B.22)\), we have

\[ f(\rho) = \sqrt{1 + \frac{1}{\kappa^2} (1 + \alpha^{-2})} = 1, \quad (B.29) \]

which leads \( \alpha = \pm i \). It is easy to see that these solve all conditions.

Thus a half BPS solutions in this case are

\[ \sin \eta = \kappa^{-1} \sinh \rho, \quad (B.30) \]

\[ F_{\psi \rho} = \pm i \frac{\kappa L^2}{2 \pi \alpha' \sinh^2 \rho}, \quad (B.31) \]

\[ \Gamma_1 \Gamma_{1,2}^{\epsilon_1^-} = \pm i \Gamma_2 \epsilon_2^- \], \quad (B.32) \]

where the signatures are taken to be same. The solution found in [9] corresponds to the plus sign. In order to determine \( \kappa \), one needs to solve the equations of motion for the D3-brane action, and we have already known that these BPS solutions solve the equations of motion as well.
Case II: \( \tilde{\Gamma}_1^{-*} = \alpha_1 \epsilon_1^{-*} , \tilde{\Gamma}_2^{-*} = \alpha_2 \epsilon_2^{-*} \) where \( \alpha_{1,2} \in \mathbb{C} \)

The analysis goes in parallel with the case I, and one finds

\[
 f(\rho) = -\cos \eta,
\]

which does not have the solution since \( 0 \leq \eta \leq \pi/2 \). So this projection does not provide a BPS solution.

We therefore conclude that the BPS condition for D3-brane solutions with the \( \mathbb{S}^2 \) symmetric ansatz is sufficient to determine the classical solution. In the main part of the text, we have implicitly assumed that as for a small deformation of the boundary condition \( k \rightarrow k + p \) there exists a unique classical solution associated with the new boundary condition. The result of this appendix justifies this prescription, since the solution depends only on the parameter \( \kappa \).

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