COMPLEX ZERO STRIP DECREASING OPERATORS

DAVID A. CARDON

Abstract. Let \( \phi(z) \) be a function in the Laguerre-Pólya class. Write \( \phi(z) = e^{-\alpha z^2} \phi_1(z) \) where \( \alpha \geq 0 \) and where \( \phi_1(z) \) is a real entire function of genus 0 or 1. Let \( f(z) \) be any real entire function of the form \( f(z) = e^{-\gamma z^2} f_1(z) \) where \( \gamma \geq 0 \) and \( f_1(z) \) is a real entire function of genus 0 or 1 having all of its zeros in the strip \( S(r) = \{ z \in \mathbb{C} : -r \leq \text{Im} z \leq r \} \), where \( r > 0 \). If \( \alpha \gamma < 1/4 \), the linear differential operator \( \phi(D) f(z) \), where \( D \) denotes differentiation, is known to converge to a real entire function whose zeros also belong to the strip \( S(r) \). We describe several necessary and sufficient conditions on \( \phi(z) \) such that all zeros of \( \phi(D) f(z) \) belong to a smaller strip \( S(r_1) = \{ z \in \mathbb{C} : -r_1 \leq \text{Im} z \leq r_1 \} \), where \( 0 \leq r_1 < r \) and \( r_1 \) depends on \( \phi(z) \) but is independent of \( f(z) \). We call a linear operator having this property a complex zero strip decreasing operator or CZSDO. We examine several relevant examples, in certain cases we give explicit upper and lower bounds for \( r' \), and we state several conjectures and open problems regarding complex zero strip decreasing operators.

1. Introduction

An important problem in the theory of the distribution of zeros of a collection of entire functions is to understand the effect of linear operators that act on the collection. It is particularly interesting when the operators preserve a nice property about the location of the zeros. The linear operators we will study in this paper are differential operators \( \phi(D) \) where \( \phi(z) \) is a function in the Laguerre-Pólya class and \( D \) is differentiation. If \( f(z) \) is a real entire function satisfying appropriate technical requirements whose zeros belong to the strip \( S(r) = \{ z \in \mathbb{C} : -r \leq \text{Im} z \leq r \} \), we study the problem of when all zeros of \( \phi(D) f(z) \) belong to a smaller strip \( S(r') = \{ z \in \mathbb{C} : -r' \leq \text{Im} z \leq r' \} \) where \( 0 \leq r' < r \). The main results in the paper are stated in Theorems 1.5 and 1.6.

Before stating these theorems we will need a few definitions and a technical lemma that defines the linear differential operator \( \phi(D) \) and tells us when the expression \( \phi(D) f(z) \) makes sense.

Definition 1.1 (\( \mathcal{LP} \) and \( \mathcal{LP}_1 \)). The Laguerre-Pólya class, denoted \( \mathcal{LP} \), consists of the real entire functions whose Weierstrass product representations are of the form

\[
(c z^m e^{\alpha z - \beta z^2} \prod_k \left(1 - \frac{z}{\alpha_k}\right) e^{z/\alpha_k},
\]

Date: January 3, 2014.

2010 Mathematics Subject Classification. 30C15, 47B38.

Key words and phrases. zeros of entire functions, Laguerre-Pólya class, complex zero strip decreasing operators.
where $c, \alpha, \beta, \alpha_k$ are real, $\beta \geq 0$, $m$ is a nonnegative integer, and $\sum_k |\alpha_k|^{-2} < \infty$. The subclass $LP_1$ of $LP$ consist of those functions in $LP$ with $\beta = 0$ in equation (1).

The class $LP$ consists of the entire functions obtained as uniform limits on compact sets of sequences of real polynomials having only real zeros. See Levin [12, Thm. 3, p. 331]. Motivation for why this class of functions naturally arises in relation to differential operators is given in [2].

**Definition 1.2** ($LP(r)$ and $LP_1(r)$). For $r \geq 0$, the extended Laguerre-Pólya class, denoted $LP(r)$, consists of the real entire functions having the Weierstrass product representation in equation (1) except that the zeros belong to the strip

$$S(r) = \{z \in \mathbb{C} : -r \leq \text{Im} \, z \leq r\}.$$ 

Thus, the zeros of a function $f(z) \in LP(r)$ are either real or occur in complex conjugate pairs. The subclass $LP_1(r)$ of $LP(r)$ consists of those functions in $LP_1(r)$ with $\beta = 0$ in equation (1). If $r < 0$ or $r$ is imaginary, we define $LP(r) = LP$ and $S(r) = \mathbb{R}$.

The following lemma shows how functions in $LP$ define linear differential operators on functions in $LP(r)$. A trivial modification to the proof of a theorem in Levin [12] gives:

**Lemma 1.3** (Levin [12], Thm.8, p.360). Assume

$$\phi(z) = e^{-\gamma_1 z^2} \phi_1(z) = \sum_{k=0}^{\infty} a_k z^k \in LP$$

where $\gamma_1 \geq 0$ and $\phi_1(z) \in LP_1$. Also let $r \geq 0$ and assume $f(z) = e^{-\gamma_2 z^2} f_1(z) \in LP(r)$ where $\gamma_2 \geq 0$ and $f_1(z) \in LP_1(r)$. If $\gamma_1 \gamma_2 < 1/4$, the linear differential operator $\phi(D)$ is defined by

$$\phi(D) f(z) = \sum_{k=0}^{\infty} a_k f^{(k)}(z),$$

where $D$ denoted differentiation. The sum converges uniformly on every compact subset of $\mathbb{C}$ and $\phi(D)f(z) \in LP(r)$.

The assumption $\gamma_1 \gamma_2 < 1/4$ is essential. Levin [12, p.361] gives the explicit example $\phi(z) = e^{-\gamma_1 z^2}$ and $f(z) = e^{-\gamma_2 z^2}$ to show that $\phi(D)f(z)$ diverges at $z = 0$ when $\gamma_1 \gamma_2 = 1/4$.

In the lemma the zeros of $f(z)$ are in the strip $S(r)$ as are the zeros of $\phi(D)f(z)$. So, $\phi(D)$ is an operator that preserves the strip $S(r)$ containing the zeros. However, our main interest in this paper is to study the operators $\phi(D)$ such that the zeros of $\phi(D)f(z)$ belong to a strictly smaller strip $S(r_1)$ where $0 \leq r_1 < r$.

**Definition 1.4** (Complex zero strip decreasing operator or CZSDO).

(a) Given a function $\phi(z) = e^{-\gamma_1 z^2} \phi_1(z) \in LP$ where $\phi_1(z) \in LP_1$, we define $LP^{\gamma_1}(r)$ to be the subclass of $LP(r)$ of functions of the form $f(z) = e^{-\gamma_2 z^2} f_1(z)$
where $f_1(z) \in \mathcal{LP}_1(r)$ and $\gamma_2$ is any nonnegative real number such that $\gamma_1 \gamma_2 < 1/4$.

(b) The linear differential operator $\phi(D)$ in part (a) is called a complex zero strip decreasing operator if for each $r > 0$ there exists a corresponding $r_1$ with $0 \leq r_1 < r$ such that $\phi(D)f(z) \in \mathcal{LP}(r_1)$ for all $f(z) \in \mathcal{LP}^{\gamma_1}(r)$. For short, we will say $\phi(D)$ is a CZSDO.

Lemma 1.3 implies $\phi(D)f(z) \in \mathcal{LP}(r)$ for all $f(z) \in \mathcal{LP}^{\gamma_1}(r)$, which is why we defined $\mathcal{LP}^{\gamma_1}(r)$ in Definition 1.4(a).

In this paper we will prove two main theorems. Theorem 1.5 provides a sufficient condition for $\phi(D)$ to be a CZSDO. Theorem 1.6 gives a necessary condition for $\phi(D)$ to be a CZSDO.

**Theorem 1.5.** Assume $\phi(z) = e^{-\alpha z^2/2} \phi_1(z)$ where $\phi_1(z) \in \mathcal{LP}_1$ and $\alpha > 0$. If $f(z) \in \mathcal{LP}^{\alpha^2/2}(r)$, then $\phi(D)f(z) \in \mathcal{LP}(\sqrt{r^2 - \alpha^2})$. Therefore $\phi(D)$ is a CZSDO.

**Theorem 1.6.**

(a) If $\phi(z) \in \mathcal{LP}_1$ has order $\rho < 1$, then $\phi(D)$ is not a CZSDO.

(b) If $\phi(z) \in \mathcal{LP}_1$ has order $\rho = 1$ but is of minimal type, then $\phi(D)$ is not a CZSDO.

(c) If $\phi(z) = e^{\alpha z}$ where $\alpha \in \mathbb{R}$ and $f(z) \in \mathcal{LP}(r)$, then $\phi(D)f(z) = f(z + \alpha)$. Hence $\phi(D)$ is not a CZSDO.

Therefore, a necessary condition for $\phi(D) \in \mathcal{LP}_1$ to be a CZSDO is: If the Weierstrass canonical product for $\phi(z)$ is

$$\phi(z) = cz^m e^{\alpha z} \prod_n (1 - z/a_n) e^{z/a_n},$$

then the product

$$cz^m \prod_n (1 - z/a_n) e^{z/a_n},$$

with the term $e^{\alpha z}$ omitted, has order $\rho = 1$ and type $\sigma > 0$ or has order $\rho > 1$.

Note that since functions in $\mathcal{LP}_1$ and $\mathcal{LP}_1(r)$ have Weierstrass canonical products of genus $g = 0$ or $g = 1$ and since the genus is related to the order $\rho$ by $g \leq \rho \leq g + 1$, the order of any of these functions satisfies $0 \leq \rho \leq 2$. So, in Theorem 1.6 the only relevant orders satisfy $1 \leq \rho \leq 2$.

Theorem 1.5 and 1.6 are important in the context of the following general problem: If $\Omega \subseteq \mathbb{C}$ is a set of particular interest and if $\pi(\Omega)$ is the class of all (real or complex) univariate polynomials whose zeros lie only in $\Omega$, then characterize the linear transformations $T: \pi(\Omega) \rightarrow \pi(\Omega) \cup \{0\}$. Furthermore, if $\pi_n(\Omega)$ is the subclass of polynomials in $\pi(\Omega)$ of degree at most $n$, then characterize the linear transformations $T: \pi_n(\Omega) \rightarrow \pi(\Omega) \cup \{0\}$.

Recently, Borcea and Brändén [1] solved these problems in the case when $\Omega$ is a line, a circle, a closed half-plane, a closed disk, or the complement of an open disk. They gave several different types of descriptions that all linear operators having these zero preserving properties must satisfy.
An unsolved case of this problem is when $\Omega = S(r)$ is a strip in the complex plane, the case studied in this paper. An operator $T$ on real polynomials in $\pi(S(r))$ with the CZSDO property clearly satisfies $T: \pi(\mathbb{R}) \to \pi(\mathbb{R}) \cup \{0\}$. However, simple examples show that converse is false. A full characterization of CZSDOs in the style of Borcea and Brändén as in [1] would require some kind of modification to their description. The linear operators in Theorem 1.5, Theorem 1.6, and Conjecture 5.4 (stated later in §5) give a large class of explicit examples of linear operators that preserve the strip $S(r)$, but better yet, are complex zero strip decreasing operators. Conjecture 5.4 (if true) would give a complete classification of CZSDOs of the form $\phi(D)$ where $\phi(z) \in L^P$.

Our results are related to but different from those of Craven and Csordas [7] in which they studied linear transformations $T$ on real polynomials $p(x)$ such that the number of complex zeros of $T[p(x)]$ is less than or equal to the number of complex zeros of $p(x)$. In their enjoyable survey article [9], Craven and Csordas explain many interesting results pertaining to operators that preserve reality of zeros.

The rest of the paper is organized as follows: In §2 we explain a heuristic to help motivate the context for this paper. In §3 we prove Theorem 1.5. In §4 we prove Theorem 1.6. In §5 we give several examples and conjectures. Finally, in §6 we suggest questions for further study on this topic.

2. SOME PHILOSOPHY AND HEURISTICS

Much of the discussion in this paper becomes significantly more intuitive if one keeps in mind the following fundamental fact:

**Theorem 2.1 (Gauss-Lucas).** Every convex set containing all the zeros of a polynomial also contains all of its critical points.

Proofs can be found in many places but we especially like the treatise on the analytic theory of polynomials by Rahman and Schmeisser [14].

This theorem provides a natural strategy for constructing examples of linear operators with particular zero preserving features. As an example of this approach, the differential linear operator $\phi(D)$ in Theorems 1.5 and 1.6 is quite natural as follows: Suppose $f$ is a real polynomial with zeros in $S(r)$, and let $\alpha$ be real and nonzero. If $I$ is the identity operator and $D$ is differentiation,

$$
\left(I - \frac{D}{\alpha}\right)f(z) = f(z) - \frac{f'(z)}{\alpha} = -\frac{e^{\alpha z}}{\alpha} \frac{d}{dz} (e^{-\alpha z} f(z)).
$$

The zeros of $f(z) - f'(z)/\alpha$ are those of $\frac{d}{dz} (e^{-\alpha z} f(z))$. By considering the Gauss-Lucas Theorem and the approximation $e^{-\alpha z} \approx (1 - \frac{\alpha z}{n})^n$, we see that the zeros of $f(z) - f'(z)/\alpha$ approximately belong to the convex hull of the zeros of $f(z)$ and the real number $n/\alpha$ which is the only root of $(1 - \alpha z/n)^n$. But since the roots of $f(z)$ and the root $n/\alpha$ are in $S(r)$, this convex hull lies inside the strip $S(r)$ as well. Taking the limit shows that the roots of $f(z) - f'(z)/\alpha$ belong to $S(r)$. We conclude that if $\phi(z) = z^n \prod_{k=1}^n (1 - z/\alpha_k)$ is a polynomial in which all $\alpha_k$ are real, then the zeros of $\phi(D)f(z)$ belong to the strip $S(r)$. By taking slightly more care, we can extend
both \( \phi \) and \( f \) to be entire functions that are sufficiently nice limits of sequences of polynomials (thus obtaining Lemma 1.3). Hence, we obtain the case \( \phi \in \mathcal{L} \mathcal{P} \) and \( f \in \mathcal{L} \mathcal{P}(r) \), which is the main focus of this paper.

Applications of this general strategy produce a wide variety of interesting facts about zeros of polynomials. A good reference is Chapter 5 of [14].

3. Proof of Theorem 1.5

Assume \( \phi(z) = e^{-\alpha z^2/2} \phi_1(z) \) where \( \phi_1(z) \in \mathcal{L} \mathcal{P}_1 \) and let \( f(z) \in \mathcal{L} \mathcal{P}^{n^2/2}(r) \) for \( r > 0 \). We will show that \( \exp(-\alpha^2 D^2) f(z) \in \mathcal{L} \mathcal{P}(\sqrt{r^2 - \alpha^2}) \). Then since \( \phi_1(D) \) preserves \( \mathcal{L} \mathcal{P}(\sqrt{r^2 - \alpha^2}) \), it will follow that \( \phi(D) f(z) \in \mathcal{L} \mathcal{P}(\sqrt{r^2 - \alpha^2}) \).

To understand the effect of the differential operator \( \exp(-\alpha^2 D^2) \) where \( \alpha \geq 0 \), we first consider some simpler exponential operators.

**Lemma 3.1** (Shifting Operator). Let \( \beta \) be any complex number and \( n \) be a nonnegative integer. Then \( \exp(\beta D) z^n = (z + \beta)^n \). Consequently, if \( f \in \mathcal{L} \mathcal{P}(r) \) for \( r \geq 0 \), then \( \exp(\beta D) f(z) = f(z + \beta) \).

**Proof.** By a simple calculation

\[
\exp(\beta D) z^n = \sum_{k=0}^{\infty} \frac{\beta^k}{k!} \frac{d^k}{dz^k} z^n = \sum_{k=0}^{n} \frac{\beta^k}{k!} n(n-1) \cdots (n-k+1) z^{n-k}
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} \beta^k z^{n-k} = (z + \beta)^n.
\]

Hence, \( \exp(\beta D) f(z) = f(z + \beta) \) holds whenever \( f(z) \) is a polynomial. By taking limits of sequences of polynomials, the result holds for functions in the class \( \mathcal{L} \mathcal{P}(r) \). \( \square \)

**Corollary 3.2.** Let \( \alpha, \beta \in \mathbb{R} \) where \( \alpha > 0 \) and let \( D \) denote differentiation. Since

\[
\cos(\alpha z + \beta) = \frac{1}{2} \left( e^{(\alpha z + \beta)i} + e^{-(\alpha z + \beta)i} \right)
\]

and

\[
\sin(\alpha z + \beta) = \frac{1}{2i} \left( e^{(\alpha z + \beta)i} - e^{-(\alpha z + \beta)i} \right),
\]

it follows immediately that for any \( f(z) \in \mathcal{L} \mathcal{P}(r) \)

\[
\cos(\alpha D + \beta) f(z) = \frac{1}{2} \left( e^{i\beta} f(z + i\alpha) + e^{-i\beta} f(z - i\alpha) \right)
\]

and

\[
\sin(\alpha D + \beta) f(z) = \frac{1}{2i} \left( e^{i\beta} f(z + i\alpha) - e^{-i\beta} f(z - i\alpha) \right).
\]

**Lemma 3.3** (Effect of Cosine and Sine Operators). Let \( \alpha, \beta \in \mathbb{R} \) where \( \alpha > 0 \), let \( D \) denote differentiation, and let \( f(z) \in \mathcal{L} \mathcal{P}(r) \) where \( r \geq 0 \). Then

\[
\cos(\alpha D + \beta) f(z) \in \mathcal{L} \mathcal{P}(\sqrt{r^2 - \alpha^2})
\]

and

\[
\sin(\alpha D + \beta) f(z) \in \mathcal{L} \mathcal{P}(\sqrt{r^2 - \alpha^2}).
\]
Proof. One can compare the Weierstrass product representation of \( f(z + i\alpha) \) with that of \( f(z - i\alpha) \) to concluded that the zeros of \( \cos(\alpha D + \beta) f(z) \) and \( \sin(\alpha D + \beta) \) are in the strip \( S(\sqrt{r^2 - \alpha^2}) \). Pólya used this idea in the proof of Hilfssatz II in his 1926 paper \cite{13} on the Riemann zeta function in which he proved a Riemann hypothesis for a ‘fake’ zeta function. He considered a slightly simpler case, but likely was aware of the fact stated in this lemma. One proof attributed to de Bruijn when \( f(z) \) is a polynomial is found in \cite{14}, Theorem 2.5.1, p. 88. This lemma may be regarded as an extension of Jensen’s theorem: If \( f \) is a polynomial with real coefficients, then the nonreal critical points of \( f \) lie in the union of all the Jensen discs of \( f \). For additional history and various generalizations see Section 2.4 of \cite{14}.

If \( f(z) \) is of the form \( f(z) = c e^{\delta z} \) where \( c, \delta \in \mathbb{R}, \) then \( f(z) \in \mathcal{L} \mathcal{P} \) and hence

\[
\cos(\alpha D + \beta) f(z) \in \mathcal{L} \mathcal{P} \subseteq \mathcal{L} \mathcal{P}(\sqrt{r^2 - \alpha^2}).
\]

Similarly,

\[
\sin(\alpha D + \beta) f(z) \in \mathcal{L} \mathcal{P} \subseteq \mathcal{L} \mathcal{P}(\sqrt{r^2 - \alpha^2}).
\]

The lemma is true in this case.

If \( f(z) \) is not of the form \( c e^{\delta z} \), then the Weierstrass canonical product for \( f(z) \) contains a term of the form \( e^{-\gamma z^2} \) where \( \gamma > 0 \) or the product contains at least one term corresponding to a root of \( f(z) \). Denote the real zeros of \( f(z) \) by \( r_n \) and denote the complex roots with positive imaginary part by \( s_n + it_n \). By combining terms for complex conjugate roots in the Weierstrass product, we find that \( f(z) \) has the form

\[
c z^m e^{\delta z - \gamma z^2} \prod_n \left( 1 - \frac{z}{r_n} \right) e^{z/r_n} \prod_n \left( 1 - \frac{z}{s_n + it_n} \right) \left( 1 - \frac{z}{s_n - it_n} \right) \exp \left( \frac{2s_n z}{s_n^2 + t_n^2} \right)
\]

where \( \delta, \gamma, c \in \mathbb{R}, \gamma \geq 0, \) and \( m \) is a nonnegative integer. For \( z = x + iy, \)

\[
|f(z)|^2 = |c|^2 (x^2 + y^2)^m e^{2\delta x + 2\gamma(y^2 - x^2)} \prod_n \left( \frac{x - r_n}{r_n} \right)^{x y} \exp \left( \frac{2s_n x}{s_n^2 + t_n^2} \right) \prod_n \left( \frac{x - s_n}{s_n^2 + t_n^2} \right)^{x y} \exp \left( \frac{4s_n x}{s_n^2 + t_n^2} \right).
\]

Let \( z = x + iy \) be a root of

\[
\cos(\alpha D + \beta) f(z) = \frac{1}{2} \left( e^{i\beta} f(z + i\alpha) + e^{-i\beta} f(z - i\alpha) \right)
\]

with \( y \geq 0. \) Then

\[
|f(z - i\alpha)|^2 = |f(z - i\alpha)|^2.
\]

By way of contradiction, assume that the root \( z \) is not in the strip \( S(\sqrt{r^2 - \alpha^2}) \). Then \( y > 0 \) and \( y^2 > r^2 - \alpha^2. \) We will show that each nonconstant term in the product for \( |f(z - i\alpha)|^2 \) is less than or equal to the corresponding term in the product for \( |f(z + i\alpha)|^2 \) and that strict inequality holds for at least one term. This will show that \( |f(z - i\alpha)|^2 < |f(z + i\alpha)|^2 \) contrary to the hypothesis.
First, we consider the factors of $|f(z - i\alpha)|^2$ and $|f(z + i\alpha)|^2$ associated with the exponential term $e^{2\delta x + 2\gamma(y^2 - x^2)}$ in equation (4). Since $y > 0$ this gives

$$e^{2\delta x + 2\gamma((y - \alpha)^2 - x^2)} \leq e^{2\delta x + 2\gamma((y + \alpha)^2 - x^2)},$$

where the inequality is strict if and only if $\gamma > 0$.

Next, we consider the factors associated with real roots (if there are any) of $f(z)$. Since $y > 0$, we have

$$x^2 + (y - \alpha)^2 < (x^2 + (y + \alpha))^m$$

and

$$\frac{(x - r_n)^2 + (y - \alpha)^2}{r_n^2} \exp \left( \frac{2x}{r_n} \right) < \frac{(x - r_n)^2 + (y + \alpha)^2}{r_n^2} \exp \left( \frac{2x}{r_n} \right).$$

Finally, we consider the factors of $|f(z - i\alpha)|^2$ and $|f(z + i\alpha)|^2$ associated complex conjugate pairs of roots (if there are any) of $f(z)$. We will show that

$$\frac{(x - s_n)^2 + (y - \alpha - t_n)^2}{s_n^2 + t_n^2} \cdot \frac{(x - s_n)^2 + (y - \alpha + t_n)^2}{s_n^2 + t_n^2} \cdot \exp \left( \frac{4s_n x}{s_n^2 + t_n^2} \right) < \frac{(x - s_n)^2 + (y + \alpha - t_n)^2}{s_n^2 + t_n^2} \cdot \frac{(x - s_n)^2 + (y + \alpha + t_n)^2}{s_n^2 + t_n^2} \cdot \exp \left( \frac{4s_n x}{s_n^2 + t_n^2} \right).$$

Inequality (8) holds if and only if

$$[(x - s_n)^2 + (y - \alpha - t_n)^2][\alpha x - (x - s_n)^2 + (y - \alpha + t_n)^2]$$

$$(x - s_n)^2 + (y + \alpha - t_n)^2][(x - s_n)^2 + (y + \alpha + t_n)^2].$$

Subtracting the left hand side of (9) from the right hand side along with a small calculation gives

$$[(x - s_n)^2 + (y + \alpha - t_n)^2][(x - s_n)^2 + (y + \alpha + t_n)^2]$$

$$- [(x - s_n)^2 + (y - \alpha + t_n)^2][(x - s_n)^2 + (y - \alpha - t_n)^2]$$

$$= 8\alpha y[(x - s_n)^2 + y^2 - t_n^2 + \alpha^2].$$

Thus (8) and (9) hold if and only if

$$x - s_n^2 + y^2 > t_n^2 - \alpha^2.$$ 

Because the root $z$ does not belong to $S(\sqrt{r^2 - \alpha^2})$ but does satisfy $0 \leq \text{Im} z \leq r$, it follows that $y^2 > r^2 - \alpha^2 \geq t_n^2 - \alpha^2$ and therefore (10) holds implying that inequality (8) holds.

If $\gamma > 0$ in the product representation of $f(z)$ in equation (3), then inequality (5) is strict. If $f(z)$ has at least one root then at least one of strict inequalities (6), (7), or (8) holds. Either way,

$$|f(z - i\alpha)|^2 < |f(z + i\alpha)|^2,$$

which is a contradiction. This completes the proof of Lemma 3.3 for the operator $\cos(\alpha D + \beta)$. The proof for the operator $\sin(\alpha D + \beta)$ entirely similar. The proof of the lemma is complete. $\square$
Recently Lagarias [11] applied operators such as the one in Lemma 3.3 to study the zero distribution on the ‘critical line’ for various differenced $L$-functions from analytic number theory. Several generalizations of Lemma 3.3 can also be found in Cardon [2, 3, 4].

**Lemma 3.4.** Let $\alpha \geq 0$ and assume $f(z) \in \mathcal{L}P^{\alpha^2/2}(r)$. Then

$$\exp(-\frac{\alpha^2D^2}{2})f(z) \in \mathcal{L}P(\sqrt{r^2 - \alpha^2}).$$

**Proof.** We take advantage of the limit formula

$$\lim_{n \to \infty} \left( \cos \left( \frac{\alpha z}{n} \right) \right)^n = \exp(-\frac{\alpha^2z^2}{2}).$$

Initially, let $f(z)$ be a polynomial in $\mathcal{L}P(r)$. Applying the formula in Lemma 3.3 a total of $n^2$ times for the operator $\cos(\frac{\alpha z}{n})$ gives

$$\left( \cos \left( \frac{\alpha z}{n} \right) \right)^n f(z) \in \mathcal{L}P \left( \sqrt{r^2 - \frac{\alpha^2}{n^2} - \cdots - \frac{\alpha^2}{n^2}} \right) = \mathcal{L}P(\sqrt{r^2 - \alpha^2}).$$

Taking the limit gives $\exp(-\frac{\alpha^2D^2}{2})f(z) \in \mathcal{L}P(\sqrt{r^2 - \alpha^2})$. By considering sequences of polynomials in $\mathcal{L}P(r)$, the result extends to functions $f(z) = e^{-\beta z^2/2}f_1(z)$ in $\mathcal{L}P(r)$, provided we assume $\alpha \beta < 1$ as required by Lemma 1.3. □

Now that we understand the effect of the operator $\exp(-\frac{\alpha^2D^2}{2})$, we can finish the proof of Theorem 1.5. Assume $\phi(z) = e^{-\alpha z^2/2}\phi_1(z)$ where $\phi_1(z) \in \mathcal{L}P_1$ and let $f(z) \in \mathcal{L}P^{\alpha^2/2}(r)$. By Lemma 3.4

$$\exp \left( -\frac{\alpha^2D^2}{2} \right)f(z) \in \mathcal{L}P(\sqrt{r^2 - \alpha^2}).$$

By Lemma 1.3, $\phi_1(D)$ maps $\mathcal{L}P(\sqrt{r^2 - \alpha^2})$ into itself. Hence,

$$\phi(D)f(z) = \phi_1(D)\exp \left( -\frac{\alpha^2D^2}{2} \right)f(z) \in \mathcal{L}P(\sqrt{r^2 - \alpha^2}).$$

Since $\alpha > 0$ this proves that $\phi(D)$ is a CZSDO and the proof of Theorem 1.5 is complete.

4. Proof of Theorem 1.6

Since the proof requires the concepts of order $\rho$ and type $\sigma$ of an entire function, we recall their definitions, which we take from Chapter 1 of Levin [12]. For an arbitrary entire function $\phi$, set

$$M_\phi(r) = \max_{|z|=r} |\phi(z)|.$$ 

The function $\phi$ is said to have finite order if there exists a positive real number $k > 0$ such that

$$M_\phi(r) < e^{r^k}$$

for all sufficiently large $r$. If $\phi$ has finite order, the order $\rho$ of $\phi$ is defined to be the greatest lower bound of the numbers $k$ in [11].
It follows that for arbitrary $\epsilon > 0$

$$e^{\rho - \epsilon} < M_\phi(r) < e^{\rho + \epsilon} \quad (12)$$

where the inequality on the right holds for all sufficiently large $r$ and the inequality on the left holds for some positive increasing sequence $\{r_n\}$ with $\lim_{n \to \infty} r_n = \infty$.

The type $\sigma$ of a function $\phi$ having positive finite order $\rho$ is the greatest lower bound of the positive numbers $\epsilon$ such that

$$M_\phi(r) < e^{\rho \epsilon} \quad (13)$$

for all sufficiently large $r$. If $\sigma = 0$, then $\phi$ is said to have minimal type.

**Lemma 4.1.** Let $\phi(z) \in \mathcal{LP}_1$. If $\phi(D)\phi(-D)$ is not a CZSDO, then $\phi(D)$ is not a CZSDO. Therefore, in proving Theorem 1.6 there is no loss of generality in assuming that $\phi(z) \in \mathcal{LP}_1$ is an even function.

Proof. Suppose, by way of contradiction, that $\phi(D)$ is a CZSDO. For any $r > 0$, there exists $r_1$ with $0 \leq r_1 < r$ such that for any $f(z) \in \mathcal{LP}(r)$ it follows that $\phi(D)f(z) \in \mathcal{LP}(r_1)$. The operator $\phi(-D)$ maps $\mathcal{LP}(r)$ into itself. So, $\phi(D)\phi(-D)f(z) \in \mathcal{LP}(r_1)$ implying that $\phi(D)\phi(-D)$ is a CZSDO, which contradicts the hypothesis. Therefore, $\phi(D)$ is not a CZSDO.

By Lemma 4.1 there is no loss of generality in assuming $\phi(z)$ in an even function. Therefore, we will assume $\phi(z)$ in the rest of the proof of Theorem 1.6.

The key to proving Theorem 1.6 will be to let $\phi(D)$ act on extremal example functions with evenly spaced zeros that are on the boundary of the strip $S(r)$. If $\phi(0) \neq 0$, we will consider $\phi(D)f_\alpha(z)$, where

$$f_\alpha(z) = \cos(\alpha(z - i\epsilon))\cos(\alpha(z + i\epsilon)) = \frac{1}{2}(\cos(2\alpha z) + \cosh(2\alpha \epsilon)). \quad (14)$$

If $\phi(0) = 0$, we will consider $\phi(D)g_\alpha(z)$, where

$$g_\alpha(z) = \cos^2(\alpha(z - i\epsilon))\cos^2(\alpha(z + i\epsilon)). \quad (15)$$

In both cases, we’ll show $\phi(D)$ is not a CZSDO by choosing $\alpha > 0$ to be sufficiently large. Computing with $f_\alpha(z)$ and $g_\alpha(z)$ will require the following easy lemma:

**Lemma 4.2.** For any $\phi(z) \in \mathcal{LP}$, $\phi(D)e^{az} = \phi(a)e^{az}$. Consequently, if $\phi(z)$ is even, then

$$\phi(D)\cos(az) = \phi(\alpha i)\cos(az)$$
$$\phi(D)\sin(az) = \phi(\alpha i)\sin(az)$$
$$\phi(D)\cosh(az) = \phi(\alpha)\cosh(az)$$
$$\phi(D)\sinh(az) = \phi(\alpha)\sinh(az).$$

Proof. Write $\phi(z) = \sum_{n=0}^\infty c_n z^n$. Then

$$\phi(D)e^{az} = \sum_{n=0}^\infty c_n \frac{d^n}{dz^n}e^{az} = \sum_{n=0}^\infty c_n a^n e^{az} = \phi(a)e^{az}.$$
When $\phi(z)$ is even, the formulas involving $\cos(az)$, $\sin(az)$, $\cosh(az)$, and $\sinh(az)$ follow immediately by expressing them in terms of exponential functions.

□

Lemma 4.3. Let $\phi(z) \in \mathcal{L}P_1$ be an even function and suppose $\phi(0) \neq 0$.

(a) If $\phi(z)$ has order $\rho < 1$, then $\phi(D)$ is not a CZSDO.

(b) If $\phi(z)$ has order $\rho = 1$ but is of minimal type, then $\phi(D)$ is not a CZSDO.

Proof. By multiplying $\phi(z)$ by a nonzero real number, if necessary, we may suppose without loss of generality that $\phi(0) = 1$. Because $\phi(z)$ is even and $\phi(0) = 1$, $M_\phi(r) = \max_{|z|=r} |\phi(z)| = |\phi(ir)|$.

Also since $\phi(0) = 1$ if $\beta$ is constant, then $\phi(D)\beta = \beta$.

Applying the operator $\phi(D)$ to the function $f_a(z)$ in equation (14) and using Lemma 4.2 gives

$$\phi(D)f_a(z) = 2^{-1}\left(\phi(2ai)\cos(2az) + \cosh(2ar)\right)$$

$$= 2^{-1}\phi(2ai)\left(\cos(2az) + \frac{\cosh(2ar)}{\phi(2ai)}\right).$$

Since $a$ is positive $\phi(2ai) > 1$. There are two possible cases:

(i) If $0 < \frac{\cosh(2ar)}{\phi(2ai)} \leq 1$, the zeros of $\phi(D)f_a(z)$ are real.

(ii) If $\frac{\cosh(2ar)}{\phi(2ai)} > 1$, the zeros of $\phi(D)f_a(z)$ are complex.

In case (i), there exists $r_1$ with $0 \leq 2ar_1 \leq \pi/2$ such that

$$\cos(2ar_1) = \frac{\cosh(2ar)}{\phi(2ai)}.$$ 

Then

$$\phi(D)f_a(z) = 2^{-1}\phi(2ai)\left(\cos(2az) + \cos(2ar_1)\right)$$

$$= \phi(2ai)\cos(a(z - r_1))\cos(a(z + r_1)).$$

This verifies that the roots of $\phi(D)f_a(z)$ are real in the case (i), as claimed.

In case (ii), since $\cosh(2ar)/\phi(2ai) > 1$ and $\phi(2ai) > 1$, there exists $r_1$ with $0 < r_1 < r$ such that

$$1 < \cosh(2ar_1) = \frac{\cosh(2ar)}{\phi(2ai)} < \cosh(2ar).$$

We obtain

$$\phi(D)f_a(z) = 2^{-1}\phi(2ai)\left(\cos(2az) + \cosh(2ar_1)\right)$$

$$= \phi(2ai)\cos(a(z - ir_1))\cos(a(z + ir_1)),$$

which shows that the roots of $\phi(D)f_a(z)$ are complex with imaginary part $\pm r_1$. Solving for $r_1$ gives

$$r_1 = \frac{1}{2a} \cosh^{-1}\left(\frac{\cosh(2ar)}{\phi(2ai)}\right).$$

(16)
We will show that, by choosing $a$ to be sufficiently large, case (ii) occurs and $r_1$ can be made to be arbitrarily close to $r$, proving that $\phi(D)$ is not a CZSDO.

As in part (a) of Lemma 4.3, suppose $\phi(z)$ has order $\rho < 1$. For all sufficiently large positive $a$,

$$M_\phi(2a) = \phi(2ai) < e^{(2a)^\rho}.$$  

Then for sufficiently large $a$ we have

$$\frac{\cosh(2ar)}{\phi(2ai)} > \frac{e^{2ar}}{2e^{(2a)^\rho}} > 1.$$  

Therefore, the roots of $\phi(D)f_a(z)$ have nonzero imaginary part $\pm r_1$, as in equation (16), and

$$r_1 = \frac{1}{2a} \cosh^{-1} \left( \frac{\cosh(2ar)}{\phi(2ai)} \right)$$

$$> \frac{1}{2a} \cosh^{-1} \left( \frac{e^{2ar}}{2e^{(2a)^\rho}} \right)$$

$$> \frac{1}{2a} \log \left( \frac{e^{2ar}}{2e^{(2a)^\rho}} \right)$$

$$= r - (2a)^\rho - \frac{\log(2)}{2a}.$$  

By choosing $a$ to be sufficiently large, this lower bound on $r_1$ can be made to be arbitrarily close to $r$. Therefore, there does not exist $r'$ with $0 \leq r' < r$ such that $\phi(D)h(z) \in LP(r')$ for all $h(z) \in LP(r)$. This show that $\phi(D)$ is not a CZSDO when $\rho < 1$, proving Lemma 4.3(a).

Next, as in part (b) of Lemma 4.3 assume $\phi(z)$ has order $\rho = 1$ and has minimal type. For any $\epsilon > 0$ and all sufficiently large $r$,

$$M_\phi(r) = \phi(ir) < e^{\epsilon r}.$$  

Choosing $\epsilon$ to be very small relative to $r$ and letting $a$ be sufficiently large gives

$$\frac{\cosh(2ar)}{\phi(2ai)} > \frac{e^{2ar}}{2\phi(2ai)} > \frac{e^{2ar}}{2e^{\epsilon a}} > 1.$$  

Therefore, the roots of $\phi(D)f_a(z)$ have imaginary part $\pm r_1$, as in equation (16), and

$$r_1 = \frac{1}{2a} \cosh^{-1} \left( \frac{\cosh(2ar)}{\phi(2ai)} \right)$$

$$> \frac{1}{2a} \cosh^{-1} \left( \frac{e^{2ar}}{2e^{\epsilon a}} \right)$$

$$> \frac{1}{2a} \log \left( \frac{e^{2ar}}{2e^{\epsilon a}} \right)$$

$$= r - \epsilon - \frac{\log 2}{2a}.$$  


By choosing $\epsilon$ to be sufficiently small and $a$ sufficiently large, this lower bound for $r_1$ can be made to be arbitrarily close to $r$. This shows that $\phi(D)$ is not a CZSDO when $\phi(z)$ has order $\rho = 1$ and has minimal type, proving Lemma \ref{lem:1.3}(b). \hfill \Box

Combining Lemmas \ref{lem:4.1} and \ref{lem:4.3} proves Theorem \ref{thm:1.6} parts (a) and (b) in the case $\phi(0) \neq 0$. We next deal with the case $\phi(0) = 0$.

**Lemma 4.4.** Let $\phi(z) \in \mathcal{LP}_1$ be an even function and suppose $\phi(0) = 0$.

(a) If $\phi(z)$ has order $\rho < 1$, then $\phi(D)$ is not a CZSDO.

(b) If $\phi(z)$ has order $\rho = 1$ but is of minimal type, then $\phi(D)$ is not a CZSDO.

*Proof.* After multiplying $\phi(z)$ by a constant, if necessary, we may assume $\phi(z)$ is of the form

$$
\phi(z) = z^{2m} \phi_1(z)
$$

where $m$ is a positive integer and $\phi_1(0) = 1$. We will let $\phi(D)$ act on the function $g_a(z)$ from equation (15). Rewrite $g_a(z)$ as

$$
g_a(z) = \cos^2(a(z - i\tau)) \cos^2(a(z + i\tau))
$$

$$
= \frac{1}{4} + \frac{\cosh(4ar)}{8} + \frac{\cos(4az)}{8} + \frac{\cosh(2ar) \cos(2az)}{2}.
$$

Differentiating $2m$-times gives

$$
g_a^{(2m)}(z) = (-1)^m \left( \frac{(4a)^{2m} \cos(4az)}{8} + \frac{(2a)^{2m} \cosh(2r) \cos(2z)}{2} \right).
$$

Applying $\phi_1(D)$ to this expression with the help of Lemma \ref{lem:4.2} gives

$$
\phi(D)g_a(z)
$$

$$
= (-1)^m \left( \frac{(4a)^{2m} \phi_1(4ai) \cos(4az)}{8} + \frac{(2a)^{2m} \cosh(2ar) \phi_1(2ai) \cos(2az)}{2} \right)
$$

$$
= (-1)^m 2^{2m-2} a^{2m} \phi_1(4ai) \cos(4az) + \cosh(2ar) \phi_1(2ai) \cos(2az)).
$$

Let $z$ be a zero of the $\phi(D)g_a(z)$. Then

$$
0 = 2^{2m-2} \phi_1(4ai) \cos(4az) + \cosh(2ar) \phi_1(2ai) \cos(2az)
$$

$$
= 2^{2m-2} \phi_1(4ai) (2 \cos^2(2az) - 1) + \cosh(2ar) \phi_1(2ai) \cos(2az).
$$

Solving for $\cos(2az)$ results in

$$
\cos(2az) = \frac{-\cosh(2ar) \phi_1(2ai) \pm \sqrt{\cosh^2(2ar) \phi_1^2(2ai) + 2^{2m-1} \phi_1^2(4ai)}}{2^{2m} \phi_1(4ai)}.
$$

There are two types of solutions depending on the choice of sign. It will suffice for us to consider only those solutions corresponding to choosing the negative sign. For these solutions,

$$
0 = \cos(2az) + \frac{\cosh(2ar) \phi_1(2ai) + \sqrt{\cosh^2(2ar) \phi_1^2(2ai) + 2^{2m-1} \phi_1^2(4ai)}}{2^{2m} \phi_1(4ai)}.
$$

(17)
The fraction expression in (17) has the lower bound:

\[
\frac{\cosh(2ar)\phi_1(2ai) + \sqrt{\cosh^2(2ar)\phi_1^2(2ai) + 2^{4m-1}\phi_1^2(4ai)}}{2^{2m-1}\phi_1(4ai)} > \frac{\cosh(2ar)\phi_1(2ai)}{2^{2m-1}\phi_1(4ai)}.
\]

As in Lemma 4.4(a), assume \( \phi(z) \) has order \( \rho < 1 \). The order of \( \phi_1(z) \) is the same as that of \( \phi(z) \). So, given \( \epsilon > 0 \) there exists a positive increasing sequence \( \{a_n\} \) tending to \( \infty \) such that

\[ e^{(2a_n)^{\rho-\epsilon}} < M_{\phi_1}(2a_n) = \phi_1(2a_n) \]

for all \( n \). Also for all sufficiently large \( a_n \),

\[ M_{\phi_1}(4a_n) = \phi_1(4a_n) < e^{(4a_n)^{\rho+\epsilon}}. \]

Assuming \( \epsilon \) satisfies \( 0 < \epsilon < \rho < \rho + \epsilon < 1 \) and letting \( a_n \) be sufficiently large gives

\[
\frac{\cosh(2a_n r)\phi_1(2a_n i)}{2^{2m-1}\phi_1(4a_n i)} > \frac{e^{2a_n r} e^{(2a_n)^{\rho-\epsilon}}}{2^{2m} e^{(4a_n)^{\rho+\epsilon}}} > 1.
\]

This implies that the fraction term in equation (17) is larger than 1. Hence, there exists \( r_1 \) with \( 0 < r_1 \leq r \) such that

\[
0 = \cos(2a_n z) + \frac{\cosh(2a_n r)\phi_1(2a_n i) + \sqrt{\cosh^2(2a_n r)\phi_1^2(2a_n i) + 2^{4m-1}\phi_1^2(4a_n i)}}{2^{2m-1}\phi_1(4a_n i)}
\]

\[
= \cos(2a_n z) + \cosh(2a_n r_1)
\]

\[
= 2 \cos(a_n (z - ir_1)) \cos(a_n (z + ir_1)).
\]

Thus \( \phi(D) g_{a_n}(z) \) has roots with imaginary part \( r_1 \). Solving for \( r_1 \) and using inequalities (18) and (19) gives a lower bound for \( r_1 \).

\[
r_1 = \frac{1}{2a_n} \cosh^{-1} \left( \frac{\cosh(2a_n r)\phi_1(2a_n i) + \sqrt{\cosh^2(2a_n r)\phi_1^2(2a_n i) + 2^{4m-1}\phi_1^2(4a_n i)}}{2^{2m-1}\phi_1(4a_n i)} \right)
\]

\[
> \frac{1}{2a_n} \cosh^{-1} \left( \frac{e^{2a_n r} e^{(2a_n)^{\rho-\epsilon}}}{2^{2m} e^{(4a_n)^{\rho+\epsilon}}} \right)
\]

\[
> \frac{1}{2a_n} \log \left( \frac{e^{2a_n r} e^{(2a_n)^{\rho-\epsilon}}}{2^{2m} e^{(4a_n)^{\rho+\epsilon}}} \right)
\]

\[
= r + \frac{1}{2a_n} \left( (2a_n)^{\rho-\epsilon} - (4a_n)^{\rho+\epsilon} - \log(2^{2m}) \right)
\]

\[
> r - \frac{1}{2a_n} \left( (4a_n)^{\rho+\epsilon} + \log(2^{2m}) \right).
\]

By choosing \( a_n \) sufficiently large, we cause the lower bound on \( r_1 \) to be arbitrarily close to \( r \). So, \( \phi(D) \) is not a CZSDO in the case \( \rho < 1 \), proving Lemma 4.4(a).

Next, as in Lemma 4.4(b), assume \( \phi(z) \) has order \( \rho = 1 \) and minimal type. Then the same is true for \( \phi_1(z) \). We choose a small positive \( \epsilon \) with \( 0 < \epsilon < \rho = 1 \). Thus there exists a positive increasing sequence tending to \( \infty \) such that

\[ e^{(2a_n)^{\epsilon}} < M_{\phi_1}(2a_n) = \phi_1(2a_n i) \]
for all $n$. Also since $\phi_1(z)$ has order $\rho = 1$ and minimal type,

$$M_{\phi_1}(4a_n) < e^{(4a_n)}$$

for all sufficiently large $a_n$. For small $\epsilon$ and large $a_n$ the lower bound on the fraction in inequality (18) is then bounded below as follows:

$$\frac{\cosh(2a_n r) \phi_1(2a_n i)}{2^{2m-1} \phi_1(4a_n i)} > \frac{e^{2a_n r} e^{(2a_n)^*}}{2^{2m} e^{(4a_n)}} > 1.$$ 

Therefore, similarly to the previous case in which $\rho < 1$, $\phi(D)g_{a_n}(z)$ has roots with imaginary part $r_1 > 0$ where

$$r_1 = \frac{1}{2a_n} \cosh^{-1} \left( \frac{\cosh(2a_n r) \phi_1(2a_n i) + \sqrt{\cosh^2(2a_n r) \phi_1^2(2a_n i) + 2^{2m-1} \phi_1^2(4a_n i)}}{2^{2m} \phi_1(4a_n i)} \right)$$

$$> \frac{1}{2a_n} \cosh^{-1} \left( \frac{e^{2a_n r} e^{(2a_n)^*}}{2^{2m} e^{(4a_n)}} \right)$$

$$> \frac{1}{2a_n} \log \left( \frac{e^{2a_n r} e^{(2a_n)^*}}{2^{2m} e^{(4a_n)}} \right)$$

$$= r + \frac{1}{2a_n} \left( (2a_n)^* - \log(2^{2m}) - 4\epsilon a_n \right)$$

$$> r - 4\epsilon - \frac{\log(2^{2m})}{2a_n}.$$ 

If $\epsilon$ is sufficiently small and $a_n$ is sufficiently large, the lower bound on $r_1$ can be made to be arbitrarily close to $r$. This shows that $\phi(D)$ is not a CZSDO when $\phi(z)$ has order $\rho = 1$ and has minimal type, proving Lemma 4.4(b).

Combining Lemmas 4.1 and 4.4 proves Theorem 1.6 parts (a) and (b) in the case $\phi(0) = 0$.

We have now established Theorem 1.6 parts (a) and (b). Theorem 1.6(c) was actually proved in Lemma 3.1. That is, if $\alpha \in \mathbb{R}$, then $e^{\alpha D} f(z) = f(z + \alpha)$. The operator $e^{\alpha D}$ merely translates the zeros of $f(z)$ horizontally in the complex plane and does not reduce the size of the strip $S(r)$ containing the roots. So, $e^{\alpha D}$ is not a CZSDO.

The proof of Theorem 1.6 is now complete.

5. Examples and Conjectures

In this section, we give several examples and make some conjectures based on these examples.

**Example 5.1.** In Lemmas 3.3 we showed that for $a, b \in \mathbb{R}$ where $a > 0$ and for any $f(z) \in \mathcal{LP}(r)$ with $r > 0$,

(20) $\cos(aD + b)f(z) \in \mathcal{LP}(\sqrt{r^2 - a^2})$

and

(21) $\sin(aD + b)f(z) \in \mathcal{LP}(\sqrt{r^2 - a^2})$. 

Example 5.2. In Lemma 3.4 we showed that for $a > 0$ and any $f(z) \in \mathcal{L}P_1(r)$ with $r > 0$

$$
\exp \left( -\frac{a^2D^2}{2} \right) f(z) \in \mathcal{L}P(\sqrt{r^2 - a^2})
$$

Example 5.1 suggests that if $\phi(z) \in \mathcal{L}P_1$ has a ‘lower density’ of zeros then $\phi(aD)$ might be a CZSDO. We state this as a conjecture:

**Conjecture 5.3.** Let $n(t)$ denote the number of roots of $\phi(z) \in \mathcal{L}P_1$ in the interval $(-t, t)$. Suppose

$$
\liminf_{t \to \infty} \frac{n(t)}{t} > 0.
$$

Then there exists a positive constant $c_{\phi}$ such that for any $f(z) \in \mathcal{L}P(r)$ with $r > 0$ it follows that

$$
\phi(aD)f(z) \in \mathcal{L}P \left( \sqrt{r^2 - c_{\phi}a^2} \right).
$$

From the proof of Theorem 1.6 and in light of Examples 5.1 and 5.2 we make another conjecture:

**Conjecture 5.4** (Classification of $\phi(D)$ that are CZSDOs). If $\phi(z) \in \mathcal{L}P$ has a Weierstrass canonical product of the form

$$
\phi(z) = ce^{\alpha z - \beta z^2} \prod_n \left( 1 - \frac{z}{\alpha_n} \right) e^{\frac{z}{\alpha_n}},
$$

then $\phi(D)$ is a CZSDO if and only if any one of the following conditions is satisfied:

(i) $\beta > 0$.

(ii) $\beta = 0$ and the product $\prod_n (1 - z/\alpha_n) e^{z/\alpha_n}$ has order $\rho = 1$ and type $\sigma > 0$.

(iii) $\beta = 0$ and the order of the product $\prod_n (1 - z/\alpha_n) e^{z/\alpha_n}$ satisfies $1 < \sigma \leq 2$.

In other words, we would have a complete classification of the functions $\phi(z) \in \mathcal{L}P$ such that $\phi(D)$ is a CZSDO.

Example 5.5. (Simple Zeros) In [5], Cardon and de Gaston showed that if $\phi_1(z) \in \mathcal{L}P_1$ has infinitely many zeros and if $f(z) \in \mathcal{L}P$ then $\phi_1(D)f(z)$ has only simple real zeros. This guarantees that if $\phi(z) = e^{-\alpha z^2/2} \phi_1(z)$ and $\alpha \geq r$, then for any $f(z) \in \mathcal{L}P^{\alpha^2/2}(r)$, the function $\phi(D)f(z)$ has only simple real zeros.

Example 5.6 (Lower bound on $r'$). Let $\phi(z) \in \mathcal{L}P$ where $\phi(z)$ is not of the form $ce^{\alpha z}$. By computing the derivative of the logarithmic derivative and using the product representation in equation (1) and if $z$ is not a root of $\phi$ we obtain

$$
\left( \frac{\phi'(z)}{\phi(z)} \right)' = \frac{\phi''(z)\phi(z) - (\phi'(z))^2}{(\phi(z))^2} = -\frac{1}{z^2} - \beta - \frac{1}{(z - \alpha_k)^2} < 0.
$$

Then for $z$ not a root

$$
[\phi'(z)]^2 - \phi(z)\phi''(z) > 0.
$$
This is called Laguerre’s inequality. Suppose $\phi(0) = 1$. Then the constant $b_\phi$ defined by

$$b_\phi = [\phi'(0)]^2 - \phi(0)\phi''(0) = [\phi'(0)]^2 - \phi''(0)$$

is positive.

Letting $\phi(aD)$ act on $z^2 + r^2$ where $a > 0$ and $r > 0$ gives

$$\phi(aD)(z^2 + r^2) = \sum_{k=0}^{\infty} \frac{a^k\phi^{(k)}(0)}{k!} z^k (z^2 + r^2)$$

$$= \phi(0)(z^2 + r^2) + a\phi'(0)(2z) + \frac{a^2}{2}\phi''(0)$$

$$= z^2 + 2a\phi'(0)z + r^2 + a^2\phi''(0)$$

$$= (z + a\phi'(0))^2 + r^2 - \left(\frac{[\phi'(0)]^2 - \phi''(0)}{2}\right)a^2$$

$$= (z - a\phi'(0))^2 + r^2 - b_\phi a^2.$$

The roots of $\phi(aD)(z^2 + r^2)$ belong to the strip $S(\sqrt{r^2 - b_\phi a^2})$ but they do not belong to any smaller strip.

The calculation of $\phi(aD)(z^2 + r^2)$ implies that $\phi \in \mathcal{L}P_1$, $\phi(0) \neq 0$, and $\phi(D)$ is a CZSDO that maps $\mathcal{L}P(r)$ into $\mathcal{L}P(r')$ with $0 \leq r' < r$, we have a lower bound for $r_1$. That is, if $0 \leq a \leq r/b_\phi^{1/2}$, then

$$r_a' \geq \sqrt{r^2 - b_\phi a^2}.$$

In all explicit examples of CZSDOs in the paper involving the differential operator $\phi(D)$, we see that the narrowing of the strip $S(r)$ to $S(r')$ involved constants $b_\phi$ and $c_\phi$ where

$$\sqrt{r^2 - b_\phi a^2} \leq r' \leq \sqrt{r^2 - c_\phi a^2}.$$

**Conjecture 5.7.** Assume $\phi(D)$ is any CZSDO where $\phi(z) \in \mathcal{L}P$ and let $r > 0$ and let $a > 0$. Then there exists positive constants $b_\phi \geq c_\phi$ such that for any $f(z) \in \mathcal{L}P_1(r)$ it follows that $\phi(aD)f(z) \in \mathcal{L}P(r_a')$ where

$$\sqrt{r^2 - b_\phi a^2} \leq r_a' \leq \sqrt{r^2 - c_\phi a^2}$$

and where we replace the lower or upper bounds by 0 whenever the expression under the radical is negative.

**Example 5.8.** (Multiplier Sequences) In this paper we defined CZSDOs fairly narrow because we restricted ourselves to the case of CZSDOs of the form $\phi(D)$ for $\phi \in \mathcal{L}P$. It is easy to find examples of CZSDOs that are not of the form $\phi(D)$ where $\phi \in \mathcal{L}P$ by considering multiplier sequences known to preserve the reality of zeros. We recall that a sequence of numbers

$$\gamma_0, \gamma_1, \gamma_2, \cdots$$

in called a multiplier sequence of the first kind if for any polynomial

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n$$

we replace the lower or upper bounds by 0 whenever the expression under the radical is negative.
having only real zeros, the new polynomial

\begin{equation}
\Gamma[f(z)] = \gamma_0 a_0 + \gamma_1 a_1 z + \cdots + \gamma_n a_n z^n
\end{equation}

also has only real zeros. Important classical results about multiplier sequences were proved by Pólya and Schur. For a nice discussions about the classical results see Schmeisser [14] Section 5.7 and Levin [12] Chapter VIII Section 3. Multiplier sequences are still a topic of investigation as can be seen in papers of Craven, Csordas, and Fox [8, 9, 10], just to name a few. Consider the multiplier sequence given by \( \Gamma = \{\gamma_k\} = \{\alpha^k\} \) where \( 1 < \alpha \). If for \( r > 0 \),

\[ f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{LP}(r) \]

then

\[ \Gamma[f(z)] = \sum_{n=0}^{\infty} a_n (\alpha z)^n \in \mathcal{LP}(r/\alpha). \]

So, this is an example of a CZSDO. However, if \( 0 < \alpha < 1 \), then this multiplier sequence increases the size of the strip containing the zero and is not a CZSDO.

6. Questions for further study

In addition to the conjectures stated in the previous section, we end this paper with several open problems concerning complex zero strip decreasing operators.

Open Problem 6.1. Completely classify the CZSDOs of the form \( \phi(D) \), which is the type studied in this paper. This might be done by proving Conjecture 5.4.

Open Problem 6.2. Completely classify the CZSDOs resulting from \( \Gamma \)-sequences as in equation (24).

Open Problem 6.3. Completely classify CZSDOs when the space being acted on is the space of all real polynomials whose roots belong to the region \( \Omega = S(r) \) for \( r > 0 \). Such a classification might be in the style of Borcea and Brändén’s classification in [11], which was mentioned in the introduction to the paper.

7. Acknowledgment

I wish to thank my PhD advisor Daniel Bump who introduced me to many beautiful ideas of analytic number theory and automorphic forms. As I was writing my PhD thesis, he showed me Pólya’s 1926 paper [13] on the Riemann zeta function. An idea from that paper sparked my interest and is the genesis for this paper as well as several of my previous papers. I also acknowledge George Csordas and Tom Craven for their very interesting papers which have inspired me to work on problems in this area.
References

1. Julius Borcea and Petter Brändén, *Pólya-Schur master theorems for circular domains and their boundaries*, Ann. of Math. (2) **170** (2009), no. 1, 465–492.

2. David A. Cardon, *Convolution operators and zeros of entire functions*, Proc. Amer. Math. Soc. **130** (2002), no. 6, 1725–1734 (electronic).

3. ______, *Sums of exponential functions having only real zeros*, Manuscripta Math. **113** (2004), no. 3, 307–317.

4. ______, *Fourier transforms having only real zeros*, Proc. Amer. Math. Soc. **133** (2005), no. 5, 1349–1356 (electronic).

5. David A. Cardon and Sharleen A. de Gaston, *Differential operators and entire functions with simple real zeros*, J. Math. Anal. Appl. **301** (2005), no. 2, 386–393.

6. David A. Cardon and Pace P. Nielsen, *Convolution operators and entire functions with simple zeros*, Number theory for the millennium, I (Urbana, IL, 2000), A K Peters, Natick, MA, 2002, pp. 183–196. MR 1956225 (2003m:30012)

7. Thomas Craven and George Csordas, *Complex zero decreasing sequences*, Methods Appl. Anal. **2** (1995), no. 4, 420–441. MR 1376305 (98a:26015)

8. ______, *Problems and theorems in the theory of multiplier sequences*, Serdica Math. J. **22** (1996), no. 4, 515–524. MR 1483603 (98k:26024)

9. ______, *Composition theorems, multiplier sequences and complex zero decreasing sequences*, Value distribution theory and related topics, Adv. Complex Anal. Appl., vol. 3, Kluwer Acad. Publ., Boston, MA, 2004, pp. 131–166.

10. ______, *The Fox-Wright functions and Laguerre multiplier sequences*, J. Math. Anal. Appl. **314** (2006), no. 1, 109–125.

11. Jeffrey C. Lagarias, *Zero spacing distributions for differenced L-functions*, Acta Arith. **120** (2005), no. 2, 159–184.

12. B. Ja. Levin, *Distribution of zeros of entire functions*, revised ed., Translations of Mathematical Monographs, vol. 5, American Mathematical Society, Providence, R.I., 1980, Translated from the Russian by R. P. Boas, J. M. Danskin, F. M. Goodspeed, J. Korevaar, A. L. Shields and H. P. Thielman. MR 589888 (81k:30011)

13. G. Pólya, *Bemerkung Über die Integraldarstellung der Riemannschen ζ-Funktion*, Acta Math. **48** (1926), no. 3-4, 305–317.

14. Q. I. Rahman and G. Schmeisser, *Analytic theory of polynomials*, London Mathematical Society Monographs. New Series, vol. 26, The Clarendon Press Oxford University Press, Oxford, 2002. MR 1954841 (2004b:30001)

Department of Mathematics, Brigham Young University, Provo, UT 84602
E-mail address: cardon@math.byu.edu