RELATIVE ENTROPY MINIMIZATION OVER HILBERT SPACES VIA ROBBINS-MONRO

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Abstract. One way of getting insight into non-Gaussian measures, posed on infinite dimensional Hilbert spaces, is to first obtain best fit Gaussian approximations, which are more amenable to numerical approximation. These Gaussians can then be used to accelerate sampling algorithms. This begs the questions of how one should measure optimality and how the optimizers can be obtained. Here, we consider the problem of minimizing the distance with respect to relative entropy. We examine this minimization problem by seeking roots of the first variation of relative entropy, taken with respect to the mean of the Gaussian, leaving the covariance fixed. Adapting a convergence analysis of Robbins-Monro to the infinite dimensional setting, we can justify the application of this algorithm and highlight necessary assumptions to ensure its convergence.

1. Introduction

In [13–15, 18], it is suggested that insight into a probability distribution, \( \mu \), posed on a separable Hilbert space, \( H \), can be obtained by finding a best fit Gaussian approximation, \( \nu \). This notion of best is with respect to the relative entropy, or Kullback-Leibler divergence:

\[
R(\nu||\mu) = \begin{cases} 
\mathbb{E}_\nu \left[ \log \frac{d\nu}{d\mu} \right], & \nu \ll \mu, \\
+\infty, & \text{otherwise.}
\end{cases}
\]

(1.1)

Having a Gaussian approximation gives qualitative insight into \( \mu \), as it provides a concrete mean and variance, which can be numerically approximated. Additionally, the optimized distribution can be used in exact sampling algorithms to improve performance. This benefits a number of applications, including path sampling for molecular dynamics and parameter estimation in statistical inverse problems. Note that \( R \) has an asymmetry in its arguments. The other form has been explored in the finite dimensions, [2,3,10,17].

To be of computational use, it is necessary to have an algorithm for computing this optimal distribution. In [14], this was accomplished by first
expressing $\nu = N(m, C(p))$, where $m$ is the mean and $p$ is a parameter
inducing a well defined covariance operator, and then solving,

$$ (m, p) \in \text{argmin} \mathcal{R}(N(m, C(p)) || \mu). $$

Minimization was accomplished by using the Robbins-Monro algorithm (RM)
to find a root of the first variation of (1.2). This strategy has also been used
in finite dimensions. \[2,3\].

In this work, we emphasize the infinite dimensional problem. Indeed,
following the framework of \[14\], we assume that $\mu$ is posed on the Borel
$\sigma$-algebra of a separable Hilbert space $H$. For simplicity, we will leave the
covariance operator $C$ fixed, and only optimize over the mean, $m$.

1.1. Robbins-Monro. Given an objective function $f : H \to H$, assume
that it has a root, $x_*$. In our application to relative entropy, $f$ will be the
first variation of $\mathcal{R}$. Further, we assume that we can only observe a noisy
version of $f$, $F : H \times \chi \to H$, such that for all $x \in H$,

$$ f(x) = \mathbb{E}[F(x, \xi)] = \int_\chi F(x, \xi) \mu_\xi(d\xi), $$

where $\mu_\xi$ is the distribution of the random variable (r.v.) $\xi$.\footnote{Expectations will be over the distribution of $\xi$.} Naive RM is

$$ x_{n+1} = x_n - a_{n+1} F(x_n, \xi_{n+1}), $$

where $\xi_n \sim \mu_\xi$, are independent and identically distributed (i.i.d.), and
$a_n > 0$ is a carefully chosen sequence. Subject to assumptions on $f$, $F$, and
the distribution $\mu_\xi$, it is known that $x_n$ will converge to $x_*$ almost surely
(a.s.), in finite dimensions. \[4,5,16\].

To prove convergence, additional properties are needed. For instance,
it is sufficient to assume $f$ grows at most linearly, $\|f(x)\| \leq c_0 + c_1 \|x\|$. Alternatively, an \textit{a priori} assumption is sometimes made that the entire
sequence generated by (1.4) stays in a bounded set. The analysis of the
finite dimensional problem has been refined tremendously over the years,
including an analysis based on continuous dynamical systems. We refer the
reader to the texts \[1,7,11\] and references therein.

One way of ensuring convergence is to introduce trust regions that the
sequence $\{x_n\}$ is permitted to explore, along with a “truncation” which
constrains the sequence to said trust regions. The truncations distort (1.4)
to

$$ x_{n+1} = x_n - a_{n+1} F(x_n, \xi_{n+1}) + a_{n+1} p_{n+1}, $$

where $p_{n+1}$ is the so-called projection. Projection algorithms are also dis-
cussed in \[1,7,11\].

A general analysis of RM with truncations in Hilbert spaces can be found in \[19\]. Here, we apply the analysis of \[12\], see, also, \[8\], to the Hilbert space
case for two versions of the truncated problem. This more recent analysis is
quite simple, and readily adapts to our setting.
The trust region methods can be formulated as follows. Let $\sigma_0 = 0$ and $x_0 \in U_0$, an open subset of $H$. Then the iteration is given by:

\[(1.6a) \quad x_{n+1}^{(p)} = x_n - a_{n+1}F(x_n, \xi_{n+1})\]

\[(1.6b) \quad (x_{n+1}, \sigma_{n+1}) = \begin{cases} (x_n^{(p)}, \sigma_n) & x_n^{(p)} \in U_{\sigma_n} \\ (x_0^{(p)}, \sigma_n + 1) & x_n^{(p)} \notin U_{\sigma_n} \end{cases}\]

where \{\(U_n\)\} is a sequence of trust regions. We interpret $x_n^{(p)}$ as the proposed move, which is either accepted or rejected. If it is rejected, the algorithm restarts at one of \{\(x_0^{(n)}\)\}, the restart points. The essential property is that the algorithm restarts in the interior of a trust region. The r.v. $\sigma_n$ counts the number of times a truncation has occurred.

For subsequent calculations, we re-express (1.6) as

\[(1.7) \quad x_{n+1} = x_n - a_{n+1}f(x_n) - a_{n+1}\delta M_{n+1} + a_{n+1}p_{n+1},\]

where $\delta M_{n+1}$, the noise term, is

\[(1.8) \quad \delta M_{n+1} = F(x_n, \xi_{n+1}) - f(x_n) = F(x_n, \xi_{n+1}) - \mathbb{E}[F(x_n, \xi)].\]

A natural filtration for this problem is $\mathcal{F}_n = \sigma(x_0, \xi_1, \ldots, \xi_n)$. Since $x_n \in \mathcal{F}_n$, the noise term can be expressed via the filtration as $\delta M_{n+1} = F(x_n, \xi_{n+1}) - \mathbb{E}[F(x_n, \xi_{n+1}) | \mathcal{F}_n]$. Consequently, $\mathbb{E}[\delta M_{n+1} | \mathcal{F}_n] = 0$.

We now describe two implementations of (1.6).

1.1.1. Fixed Trust Regions. In some problems, one may have a priori information on the root. For instance, there may be an estimate of $R$ such that $\|x_*\| \leq R$. In this version of the truncated algorithm, we have two open bounded sets, $U_0 \subsetneq U_1$, and $x_* \in U_1$. Here, $U_n = U_1$ for all $n$. The restart points, \{\(x_0^{(n)}\)\}, may be random, or it may be that $x_0^{(n)} = x_0$ for all $n$.

1.1.2. Expanding Trust Regions. In the second algorithm, we consider a nested sequence of open bounded sets,

\[(1.9) \quad U_0 \subsetneq U_1 \subsetneq U_2 \subsetneq \ldots, \quad \bigcup_{n=0}^{\infty} U_n = H\]

As before, the restart points may be random or fixed, and $x_0$ is assumed to be in $U_0$. This would appear superior to the fixed trust region algorithm, as it does not require knowledge of the sets. However, to guarantee convergence, global (in $H$) assumptions on $f$ are needed; see Assumption 2 below.

1.2. Outline. In Section 2, we give sufficient conditions for which we are able to prove convergence in both the fixed and expanding trust region problems. In Section 3, we state assumptions for the relative entropy minimization problem for which we can ensure convergence. Some examples are presented in Section 4, and we conclude with remarks in Section 5.
2. The Robbins-Monro Algorithm

We now state our assumptions that will ensure convergence of (1.7).

**Assumption 1.** $f$ has a zero, $x^* \in H$ and in the case of the fixed trust region problem, there exist $R_0 < R_1$ such that

$$U_0 \subseteq B_{R_0}(x^*) \subset B_{R_1}(x^*) \subseteq U_1,$$

while in the case of the expanding trust region problem, there is a nested sequence of bounded open sets $U_n$ satisfying (1.9) with $x_0 \in U_0$.

**Assumption 2.** For any $0 < a < A$, there exists $\delta > 0$:

$$\inf_{a,\|x - x^*\| \leq A} \langle x - x^*, f(x) \rangle \geq \delta.$$

In the case of the fixed truncation, this inequality is restricted to $x \in U_1$.

**Assumption 3.** $x \mapsto \mathbb{E}[\|F(x, \xi)\|^2]$ is bounded on bounded sets, with the restriction to $U_1$ in the case of fixed trust regions.

**Assumption 4.** $a_n > 0$, $\sum a_n = + \infty$ and $\sum a_n^2 < \infty$.

Then

**Theorem 2.1.** Under the above assumptions, for the fixed trust region problem, $X_n \to x^*$ a.s. and $\sigma_n$ is a.s. bounded.

**Theorem 2.2.** Under the above assumptions, for the expanding trust region problem, $X_n \to x^*$ a.s. and $\sigma_n$ is a.s. bounded.

In the fixed truncation algorithm, Assumptions 2 and 3 need only hold in the set $U_1$, while in the expanding truncation algorithm, they must hold in all of $H$. While this would seem to be a weaker condition, it requires identification of the sets $U_0$ and $U_1$ for which the assumptions hold. These sets may not be readily identifiable, as we will see in Section 4.

A careful reading of the two main theorems appearing in [12], formulated for the finite dimensional problem, reveals almost no dependence on dimensionality. Thus, it can be immediately extended to the abstract separable Hilbert space case. A minor detail is that a Martingale convergence theorem for Hilbert spaces is required. Also, though the fixed trust region algorithm is not specifically addressed in [12], it is a very minor twist on the results. We briefly comment upon the proof.

First, we introduce the sequence $M_n$,

$$M_n = \begin{cases} \sum_{i=1}^n a_i \delta M_i & \text{for fixed trust regions,} \\ \sum_{i=1}^n a_i \delta M_i, 1 \|x_{i-1} - x^*\| \leq r & \text{for expanding trust regions and } r > 0. \end{cases}$$

By virtue of Assumptions 3 and 4 and the above construction, $M_n$ is a Martingale, and it is uniformly bounded in $L^2$ (hence also $L^1$). In [6], Chatterji proves that in reflexive Banach spaces, such as $H$, uniformly bounded, in $L^1$, Martingales converge a.s.; see Theorems 6 and 7 of that work. Now that we
have a $H$ Martingale convergence, Theorems 1 and 2 of [12] can be applied to obtain the result. Assumptions 2, 3, and 4 correspond to the assumptions of [12].

Briefly, the proofs of the theorems from [12] are as follows. It is first shown that the number of truncations is a.s. finite, or, alternatively, $\sigma_n$ is a.s. bounded. This requires all four assumptions, along with a study of a modified sequence, $x_n'$. In the case of the expanding trust regions, $x_n'$ is defined, for appropriate $r > 0$, as

$$x_n' = x_n - \sum_{i=n+1}^{\infty} a_i \delta M_i \mathbb{1}_{\|x_{i-1} - x_*\| \leq r}$$

This has the recurrence,

$$x_{n+1}' = x_n' - a_{n+1}(f(x_n) - p_{n+1}).$$

In defining $x_n'$, we make use of the a.s. convergence of $M_n$. Since the number of truncations is a.s. finite, (2.3) relaxes to

$$x_{n+1}' = x_n' - a_{n+1}f(x_n).$$

Analyzing the scalar sequence $\|x_n' - x_*\|^2$, it is straightforward to show $x_n \to x_*$, a.s. Again, as $M_n$ converges a.s., we get $x_n \to x_*$.

### 3. Relative Entropy Minimization

Recall from the introduction that our distribution of interest, $\mu$, is posed on the Borel subsets of a separable Hilbert space $H$. We assume that $\mu \ll \mu_0$, where $\mu_0 = N(m_0, C_0)$ is some reference Gaussian. Thus, we write

$$\frac{d\mu}{d\mu_0} = \frac{1}{Z_\mu} \exp\{-\Phi_\mu(u)\},$$

where $\Phi_\nu : H \to \mathbb{R}$ and $Z_\mu$ is the normalization.

Let $\nu = N(m, C)$, be another Gaussian, equivalent to $\mu_0$, such that we can write

$$\frac{d\nu}{d\mu_0} = \frac{1}{Z_\nu} \exp\{-\Phi_\nu(u)\},$$

Assuming that $\nu \ll \mu$, implying the three measures are equivalent,

$$\mathcal{R}(\nu||\mu) = \mathbb{E}_\nu[\Phi_\mu(u) - \Phi_\nu(u)] + \log(Z_\mu) - \log(Z_\nu)$$

As was proven in [15], if $A$ is set of Gaussian measures, closed under weak convergence, such that at least one element of $A$ is absolutely continuous with respect to $\mu$, then any minimizing sequence over $A$ will have a weak subsequential limit.

For simplicity, in this work, we will assume $C = C_0$. Then, by the Cameron-Martin formula (see [9]),

$$\Phi_\nu(u) = -\langle u - m, m - m_0 \rangle_{C_0} - \frac{1}{2} \|m - m_0\|^2_{C_0}, \quad Z_\nu = 1.$$

where

$$\langle f, g \rangle_{C_0} = \langle C_0^{-1/2}f, C_0^{-1/2}g \rangle_{C_0}, \quad \|f\|_{C_0}^2 = \langle f, f \rangle_{C_0}.$$
We also introduce the Cameron-Martin space, $\mathcal{H}^1 = \text{Im}(C_0^{1/2})$.

Note, we work in the shifted coordinate $x = m - m_0$. We will show that $x_n \to x^\ast$, and the mean, $m^\ast = x^\ast + m_0$.

Letting $\nu_0 = N(0, C_0) = \mu_0$, we can then rewrite (3.3) as
\begin{equation}
J(x) = R(\nu||\mu) = \mathbb{E}[\Phi_{\mu}(x + m_0 + \xi)] + \frac{1}{2} \|x\|^2_{C_0} + \log(Z_{\mu})
\end{equation}
The first and second variations of (3.6), are then:
\begin{align}
J'(x) &= \mathbb{E}[\Phi'_{\mu}(x + m_0 + \xi)] + C_0^{-1}x,
\end{align}
\begin{align}
J''(x) &= \mathbb{E}[\Phi''_{\mu}(x + m_0 + \xi)] + C_0^{-1}.
\end{align}

3.1. Application of Robbins-Monro. In [14], it was suggested that rather than try to find a root of (3.7), the equation first be preconditioned by multiplying by $C_0$. Then roots of
\begin{equation}
C_0\mathbb{E}[\Phi'_{\mu}(x + m_0 + \xi)] + x
\end{equation}
are sought. Defining
\begin{align}
f(x) &= C_0\mathbb{E}[\Phi'_{\mu}(x + m_0 + \xi)] + x, \\
F(x, \xi) &= C_0\Phi'_{\mu}(x + m_0 + \xi) + x,
\end{align}
we have the ingredients for iteration (1.7), with $\xi_n \sim \mu_0$, i.i.d.

Summarizing,

**Theorem 3.1.** Assume:
- There exists $\nu = N(m, C_0) \sim \mu_0 \sim \mu$.
- The Fréchet derivatives $\Phi'_{\mu}(x + m_0)$ and $\Phi''_{\mu}(x + m_0)$ exist for all $x \in \mathcal{H}^1$.
- There exists $x^\ast$, a local minimizer of $J$, such that $J'(x^\ast) = 0$. $m^\ast = x^\ast + m_0$.
- The mapping
\begin{equation}
x \mapsto \mathbb{E}[\sqrt{C_0}\Phi'_{\mu}(x + m_0 + \xi)]^2
\end{equation}
is bounded on bounded subsets of $\mathcal{H}^1$.
- There exists a convex neighborhood $U^\ast$ of $x^\ast$ and a constant $\alpha > 0$, such that for all $x \in U^\ast$, for all $u \in \mathcal{H}^1$,
\begin{equation}
\langle J''(x)u, u \rangle \geq \alpha \|u\|_{C_0}^2
\end{equation}
Then, choosing $a_n$ according to Assumption 4,
- If the subset $U^\ast$ can be taken to be all of $\mathcal{H}^1$, for the expanding truncation algorithm, $x_n \to x^\ast$ a.s. in $\mathcal{H}^1$.
- If the subset $U^\ast$ is not all of $\mathcal{H}^1$, then, taking $U_1$ to be a bounded convex subset of $U^\ast$, with $x^\ast \in U_1$, and $U_0$ any subset of $U_1$ such that there exist $R_0 < R_1$ with
\begin{equation}
U_0 \subset B_{R_0}(x^\ast) \subset B_{R_1}(x^\ast) \subset U_1,
\end{equation}
for the fixed truncation algorithm $x_n \to x^\ast$ a.s. in $\mathcal{H}^1$. 

Observe that the convergence is in $H^1$, and not the underlying space $H$. In the sense of Theorems 2.1 and 2.2, $H$ corresponds to $H^1$.

**Proof.** By the assumptions of the theorem, we clearly satisfy Assumptions 1 and 4. To satisfy Assumption 3, we observe that

\[
\mathbb{E} \| F(x, \xi) \|_{C_0}^2 \leq 2 \mathbb{E} \| \sqrt{C_0} \Phi' _{\mu} (x + m_0 + \xi) \|^2 + 2 \| x \|_{C_0}^2,
\]

and this is bounded on bounded subsets of $H^1$.

Finally, the Fréchet differentiability and our convexity assumption, (3.12), imply Assumption 2 since, by the mean value theorem in function spaces,

\[
\langle x - x_*, f(x) \rangle_{C_0} = \langle x - x_*, C_0 \left[ f(x_*) + f'(\tilde{x})(x - x_*) \right] \rangle_{C_0} = \langle x - x_*, J''(\tilde{x})(x - x_*) \rangle \geq \alpha \| x - x_* \|_{C_0}^2
\]

where $\tilde{x}$ is some intermediate value between $m$ and $x_*$. We have thus satisfied the assumptions of Theorems 2.1 and 2.2 assuring convergence in each case. \hfill $\square$

While condition (3.12) is sufficient to obtain convexity, other conditions are possible. For instance, suppose there is a convex open set $U_*$ containing $x_*$ and a constant $\theta \in [0, 1)$, such that for all $x \in U_*$,

\[
\inf_{u \in H(0)} \langle \mathbb{E}[\Phi'' _{\mu} (x + m_0 + \xi)]u, u \rangle \geq -\theta \lambda_1^{-1} \| u \|_0^2,
\]

where $\lambda_1$ is the principal eigenvalue of $C_0$. This also implies Assumption 2 since

\[
\langle x - x_*, f(m) \rangle_{C_0} = \langle x - x_*, C_0 \left[ f(x_*) + f'(\tilde{x})(x - x_*) \right] \rangle_{C_0} \geq \| x - x_* \|_{C_0}^2 + \langle x - x_*, \mathbb{E}[\Phi'' _{\mu} (x + m_0 + \xi)](x - x_*) \rangle \geq \| x - x_* \|_{C_0}^2 - \theta \lambda_1^{-1} \| x - x_* \|_0^2 \geq (1 - \theta) \| x - x_* \|_{C_0}^2.
\]

4. **Examples**

To apply the Robbins-Monro algorithm, the $\Phi_{\mu}$ functional of interest must be examined. In this section we present a few examples, based on those presented in [14], and show when the assumptions hold. The one outstanding assumption that we must make is that, *a priori*, $\mu_0$ is an equivalent measure to $\mu$.

4.1. **Scalar Problem.** Taking $\mu_0 = N(0, 1)$, the standard unit Gaussian, let $V : \mathbb{R} \to \mathbb{R}$ be a smooth function such that

\[
\frac{d\mu}{d\mu_0} = \frac{1}{Z_{\mu}} \exp \{-V(x)\}
\]

is a probability measure on $\mathbb{R}$. In the above framework,

\[
F(x, \xi) = \epsilon^{-1} V'(x + \xi) - \xi, \quad f(x) = \epsilon^{-1} \mathbb{E}[V'(x + \xi)]
\]

\[
\Phi' _{\mu}(x) = \epsilon^{-1} V'(x), \quad \Phi'' _{\mu}(x) = \epsilon^{-1} V''(m)
\]
and $\xi \sim N(0, 1) = \nu_0 = \mu_0$.

4.1.1. **Globally Convex Case.** Consider the case that

$V(x) = \frac{1}{2} x^2 + \frac{1}{4} x^4$.

In this case

$F(x, \xi) = \epsilon^{-1} \left( (x + \xi + (x + \xi)^3) + x \right)$, \quad $f(x) = \epsilon^{-1} \left( 4x^3 + x \right)$,

$\mathbb{E}[\Phi''(x + \xi)] = \epsilon^{-1} (4 + 3x^2)$, \quad $\mathbb{E}[|\Phi'(x + \xi)|^2] = \epsilon^{-1} \left( 22 + 58x^2 + 17x^4 + x^6 \right)$

Since $\mathbb{E}[\Phi''(x + \xi)] \geq 4\epsilon^{-1}$, all of our assumptions are satisfied and the expanding truncation algorithm will converge to the unique root at $x^* = 0$ a.s. See Figure 1 for an example of the convergence at $\epsilon = 0.1$, $U_n = (-n - 1, n + 1)$, and always restarting at 0.5.

We refer to this as a “globally convex” problem since $\mathcal{R}$ is globally convex about the minimizer.

4.1.2. **Locally Convex Case.** In contrast to the above problem, some minimizers are only “locally” convex. Consider the case the double well potential

$V(x) = \frac{1}{4} (4 - x^2)^2$

Now, the expressions for RM are

$F(x, \xi) = \epsilon^{-1} \left( (x + \xi)^3 - 4(x + \xi) + x \right)$, \quad $f(x) = \epsilon^{-1} \left( x^3 - x \right)$,

$\mathbb{E}[\Phi''(x + \xi)] = \epsilon^{-1} \left( 3x^2 - 1 \right)$, \quad $\mathbb{E}[|\Phi'(x + \xi)|^2] = \epsilon^{-1} \left( 1 + x^2 \right)(7 + 6x^2 + x^4)$

In this case, $f(x)$ vanishes at 0 and $\pm \sqrt{1 - \epsilon}$, and $J''$ changes sign from positive to negative when $x$ enters $(-\sqrt{(1 - \epsilon)/3}, \sqrt{(1 - \epsilon)/3})$. We must therefore restrict to a fixed trust region if we want to ensure convergence to either of $\pm \sqrt{1 - \epsilon}$.
Figure 2. Robbins-Monro applied to the nonconvex scalar problem associated with (4.4). Figure (a) shows the result with a well chosen trust region, while (b) shows the outcome of a poorly chosen trust region.

We ran the problem at $\epsilon = 0.1$ in two cases. In the first case, $U_1 = (0.6, 3.0)$ and the process always restarts at 2. This guarantees convergence since the second variation will be strictly positive. In the second case, $U_1 = (-0.5, 1.5)$, and the process always restarts at -0.1. Now, the second variation can change sign. The results of these two experiments appear in Figure 2. For some random number sequences the algorithm still converged to $\sqrt{1 - \epsilon}$, even with the poor choice of trust region.

4.2. Path Space Problem. Take $\mu_0 = N(m_0(t), C_0)$, with

$$C_0 = \left( -\frac{d^2}{dt^2} \right)^{-1},$$

equipped with Dirichlet boundary conditions on $H = L^2(0,1)$. In this case the Cameron-Martin space $H^1 = H^1_0(0,1)$, the standard Sobolev space equipped with the Dirichlet norm. Let us assume $m_0 \in H^1(0,1)$, taking values in $\mathbb{R}^d$.

Consider the path space distribution on $L^2(0,1)$, induced by

$$\frac{d\mu}{d\mu_0} = -\frac{1}{Z_\mu} \exp \{-\Phi_\mu(u)\}, \quad \Phi_\mu(u) = \epsilon^{-1} \int_0^1 V(u(t))dt,$$

where $V : \mathbb{R}^d \to \mathbb{R}$ is a smooth function. We assume that $V$ is such that this probability distribution exists and that $\mu \sim \mu_0$, our reference measure.

We thus seek an $\mathbb{R}^d$ valued function $m(t) \in H^1(0,1)$ for our Gaussian approximation of $\mu$, satisfying the boundary conditions

$$m(0) = m_-, \quad m(1) = m_+.$$

This is the covariance of the standard unit Brownian bridge, $Y_t = B_t - tB_1$. 

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For simplicity, take $m_0 = (1 - t)m_- + tm_+$, the linear interpolant between $m_\pm$. As above, we work in the shifted coordinated $x(t) = m(t) - m_0(t) \in H^1_0(0, 1)$.

Given a path $x(t) \in H^1_0$, by the Sobolev embedding, $x$ is continuous with its $L^\infty$ norm controlled by its $H^1$ norm. Also recall that for $\xi \sim N(0, C_0)$, in the case of $\xi(t) \in \mathbb{R}$,

$$
\mathbb{E}[|\xi(t)|^p] = \begin{cases} 0, & p \text{ odd}, \\ (p - 1)!! [t(1 - t)]^{p/2}, & p \text{ even.} \end{cases}
$$

Letting $\lambda_1 = 1/\pi^2$ be the ground state eigenvalue of $C_0$,

$$
\mathbb{E}[\|C_0^{1/2}(x + m_0 + \xi)\|^2] \leq \lambda_1 \mathbb{E}[\|\Phi'(x + m_0 + \xi)\|^2] = \lambda_1 \epsilon^{-2} \int_0^1 \mathbb{E}[|V'(x(t) + m_0(t) + \xi(t))|^2] dt.
$$

The terms involving $x + m_0$ in the integrand can be controlled by the $L^\infty$ norm, which in turn is controlled by the $H^1$ norm, while the terms involving $\xi$ can be integrated according to $(4.8)$. As a mapping applied to $x$, this expression is bounded on bounded subsets of $H^1$.

Minimizers will satisfy the ODE

$$
\epsilon^{-1} \mathbb{E} [V'(x + m_0 + \xi)] - \Phi'' = 0, \quad x(0) = x(1) = 0.
$$

### 4.3. Globally Convex Example.

With regard to convexity about a minimizer, $m_\star$, if, for instance, $V''$ were obviously pointwise positive definite, then the problem would satisfy $(3.13)$, ensuring convergence. Consider the quartic potential $V$ given by $(4.3)$. In this case,

$$
\Phi(u) = \epsilon^{-1} \int_0^1 \frac{1}{2} u(t)^2 + \frac{1}{4} u(t)^4 dt,
$$

and

$$
\Phi'(x + m_0 + \xi) = \epsilon^{-1} [(x + m_0 + \xi) + 3(x + m_0 + \xi)^3],
\Phi''(x + m_0 + \xi) = \epsilon^{-1} [1 + 3(x + m_0 + \xi)^2],
\mathbb{E}[\Phi'(x + m_0 + \xi)] = \epsilon^{-1} [x + m_0 + (x + m_0)^3 + 3t(1 - t)(x + m_0)],
\mathbb{E}[\Phi''(x + m_0 + \xi)] = \epsilon^{-1} [1 + 3(x + m_0)^2 + 3t(1 - t)].
$$

Since $\Phi''(x + m_0 + \xi) \geq \epsilon^{-1}$, we are guaranteed convergence using expanding trust regions. Taking $\epsilon = 0.01$, $m_- = 0$ and $m_+ = 2$, this is illustrated in Figure 3, where we have also solved $(4.9)$ by ODE methods for comparison.

As trust regions, we take

$$
U_n = \{ m \in H^1_0(0, 1) \mid \|m\|_{H^1} \leq 10 + n \},
$$

and we always restart at the zero solution Figure 3 also shows robustness to discretization; the number of truncations is relatively insensitive to $\Delta t$. 

4.4. Locally Convex Example. For many problems of interest, we do not have global convexity. Consider the double well potential \( (4.14) \), but in the case of paths,

\[
(4.12) \quad \Phi(u) = \epsilon^{-1} \int_0^1 \frac{1}{4} (4 - u(t))^2 dt.
\]

Then,

\[
\Phi'(x + m_0 + \xi) = \epsilon^{-1} [(x + m_0 + \xi)^3 - 4(x + m_0 + \xi)]
\]

\[
\Phi''(x + m_0 + \xi) = \epsilon^{-1} [3(x + m_0 + \xi)^2 - 4],
\]

\[
\mathbb{E}[\Phi'(x + m_0 + \xi)] = \epsilon^{-1} [(x + m_0)^3 + 3t(1-t)(x + m_0) - 4(x + m_0)]
\]

\[
\mathbb{E}[\Phi''(x + m_0 + \xi)] = \epsilon^{-1} [3(x + m_0)^2 + 3t(1-t) - 4]
\]

Here, we take \( m_- = 0, m_+ = 2 \), and \( \epsilon = 0.01 \). We have plotted the numerically solved ODE in Figure 4. Also plotted is \( \mathbb{E}[\Phi''(x_\ast + m_0 + \xi)] \). Note that \( \mathbb{E}[\Phi''(x_\ast + m_0 + \xi)] \) is not sign definite, becoming as small as \(-400\).

Since \( C_0 \) has \( \lambda_1 = 1/\pi^2 \approx 0.101 \), (3.13) cannot apply.

Discretizing the Schrödinger operator

\[
(4.13) \quad J''(x_\ast) = -\frac{d^2}{dt^2} + \epsilon^{-1} (3x_\ast(t) + m_0(t))^2 + 3t(1-t) - 4
\]

we numerically compute the eigenvalues. Plotted in Figure 5, we see that the minimal eigenvalue of \( J''(m_\ast) \) is approximately \( \mu_1 \approx 550 \). Therefore,

\[
(4.14) \quad \langle J''(x_\ast)u, u \rangle \geq \mu_1 \|u\|_{L^2}^2 \Rightarrow \langle J''(x)u, u \rangle \geq \alpha \|u\|_{H^1}^2,
\]

for all \( x \) in some neighborhood of \( x_\ast \). For an appropriately selected fixed trust region, the algorithm will converge.

However, we can show that the convexity condition is not global. Consider the path \( m(t) = 2t^2 \), which satisfies the boundary conditions. As shown in Figure 5, this path induces negative eigenvalues.
Despite this, we are still observe convergence. Using the fixed trust region (4.15)
\[ U_1 = \{ x \in H^1_0(0, 1) | \|x\|_{H^1} \leq 100 \}, \]
we obtain the results in Figure 6. Again, the convergence is robust to discretization.

5. Discussion

We have shown that the Robbins-Monro algorithm, with both fixed and expanding trust regions, can be applied to Hilbert space valued problems, adapting the finite dimensional proof of [12]. We have also constructed sufficient conditions for which the relative entropy minimization problem fits within this framework.

One problem we did not address here was how to identify fixed trust regions. Indeed, that requires a tremendous amount of \textit{a priori} information.
that is almost certainly not available. We interpret that result as a local convergence result that gives a theoretical basis for applying the algorithm. In practice, since the root is likely unknown, one might run some numerical experiments to identify a reasonable trust region, or just use expanding trust regions. The practitioner will find that the algorithm converges to a solution, though perhaps not the one originally envisioned. A more sophisticated analysis may address the convergence to a set of roots, while being agnostic as to which zero is found.

Another problem we did not address was how to optimize not just the mean, but also the covariance in the Gaussian. As discussed in [14], it is necessary to parameterize the covariance in some way, which will be application specific. Thus, while the form of the first variation of relative entropy with respect to the mean, (3.7), is quite generic, the corresponding expression for the covariance will be specific to the covariance parameterization. Additional constraints are also necessary to guarantee that the parameters always induce a covariance operator. We leave such specialization as future work.

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