THE LIFETIME OF SHAPE OSCILLATIONS OF A BUBBLE IN AN UNBOUNDED, INVISCID AND COMPRESSIBLE FLUID WITH SURFACE TENSION

O. COSTIN, S. TANVEER, AND M.I. WEINSTEIN

Abstract. General perturbations of a spherical gas bubble in a compressible and inviscid fluid with surface tension were proved in [5], in the linearized approximation, to decay exponentially, \( \sim e^{-\Gamma t} \), \( \Gamma > 0 \), as time advances. Formal asymptotic and numerical evidence led to the conjecture that \( \Gamma \approx A \epsilon \frac{W e}{\epsilon^2} \exp \left( -B \frac{W e}{\epsilon^2} \right) \), where \( 0 < \epsilon \ll 1 \) is the Mach number and \( A \) and \( B \) are positive constants. In this paper, we prove this conjecture and calculate \( A \) and \( B \) to leading order in \( \epsilon \).

1. Introduction and outline

The detailed non-spherical deformations of gas bubbles in a liquid is a problem of great physical interest in fundamental and applied physics; see, for example, [3], [4], [5] and references cited therein. We consider the dynamics of a gas bubble in a compressible, inviscid and irrotational fluid with surface tension. This physical system has an equilibrium state, consisting of: a spherically symmetric gas bubble at constant pressure inside the bubble, and a fluid at constant (lower) pressure and vanishing velocity, outside the bubble.

In [5] the linearized dynamics about such equilibria was studied and it was shown that general finite energy perturbations of the spherical equilibrium damp out as time advances. In particular, the \( L^\infty \) and local energy norms of the perturbation tend to zero exponentially, \( \sim e^{-\Gamma t} \), as \( t \to \infty \). Formal asymptotic and numerical evidence led to the conjecture that

\[
\Gamma(\epsilon) \approx \frac{A}{\epsilon} \frac{W e}{\epsilon^2} \exp \left( -B \frac{W e}{\epsilon^2} \right). 
\]

Here, \( 0 < \epsilon \ll 1 \) denotes the Mach number, a dimensionless ratio of speeds, and \( W e \) denotes the Weber number, a dimensionless measure of surface tension. \( A \) and \( B \) are positive constants determined in [5] by computer simulation.

The asymptotics (1) are in marked contrast to the decay rate of perturbations which are spherically symmetric:

\[
\Gamma_{\text{radial}}(\epsilon) = \mathcal{O}(\epsilon)
\]

Thus, non-spherical deformations excite shape modes which lose their energy to the fluid and radiate sound waves very slowly.

(1) The Mach number, \( \epsilon \), is the ratio of the bubble-wall radial velocity to the sound speed in the fluid, exterior to the bubble. We shall later set \( W e \) equal to one and consider the asymptotics for small \( \epsilon \).
Our goal in this article is to present a proof of the above conjecture and a calculation of \( A \) and \( B \) to leading order in \( \epsilon \).

A systematic and detailed discussion of the physical problem, the full nonlinear compressible equations and the appropriate linearization is presented in [5]. The dynamics of the linearized velocity potential, \( \Psi(x,t), \ x \in \mathbb{R}^3 \), and the bubble shape perturbation, \( \beta(\Omega,t), \ \Omega \in S^2 \) is governed by the wave-system:

\[
\begin{align*}
\epsilon^2 \partial_t^2 \Psi - \Delta \Psi &= 0, \quad r = |x| > 1 \\
\partial_r \Psi &= \partial_t \beta, \quad r = 1 \\
\partial_t \Psi &= 3\gamma \left( \frac{1}{2} + \frac{2}{\text{We}} \right) \langle \beta, Y_0^0 \rangle - \frac{1}{\text{We}} (2 + \Delta_S) \beta, \quad r = 1 \\
\langle \beta, Y_1^m \rangle &= 0. \quad |m| \leq 1
\end{align*}
\]

Equation (3a) follows from linearization of the continuity and Euler momentum equations, exterior to the equilibrium bubble. Equation (3b) is the linearization of the kinematic boundary condition. Equation (3c) is the linearization of the Laplace-Young boundary condition, stating that the pressure jump across the gas - fluid interface is proportional to the mean curvature. Finally, (3d) is the linearization of the statement that the origin of coordinates (the center of the bubble) is in a frame of reference moving with the bubble center of mass. The equilibrium bubble has been normalized to have unit radius. Here, \( Y_1^m = Y_1^m(\Omega), \ \Omega \in S^2 \) are spherical harmonics, which satisfy \( -\Delta_S Y_1^m = \ell(\ell + 1)Y_1^m, \ \ell \geq 0, \ |m| \leq \ell \), where \( \langle Y_1^m, Y_1^{m'} \rangle_{L^2(S^2)} = \delta_{mm'} \delta_{\ell\ell'} \). In particular, \( Y_0^0(\Omega) \equiv (4\pi)^{-1} \).

As explained in [5] and now outlined, the \( L^\infty \) and local-energy decay, for \( \epsilon \) small, is controlled by a non-selfadjoint eigenvalue problem, which arises by seeking time-harmonic solutions of (4a)-(4d):

\[
\begin{align*}
\left( \Delta + (\epsilon \ell)^2 \right) \Psi_\lambda &= 0, \quad r > 1 \\
\partial_r \Psi_\lambda &= -i\ell \beta_\lambda, \quad r = 1 \\
-i\ell \Psi_\lambda &= 3\gamma \left( \frac{1}{2} + \frac{2}{\text{We}} \right) \langle \beta_\lambda, Y_0^0 \rangle Y_0^0 - \frac{1}{\text{We}} (2 + \Delta_S) \beta_\lambda, \quad r = 1 \\
\Psi_\lambda \text{ outgoing} \quad r \to \infty.
\end{align*}
\]

If \( \lambda \) is such that (4) has a non-trivial solution, then we call \( \lambda \) a (deformation) scattering resonance energy, and \( (\Psi_\lambda, \beta_\lambda) \) a corresponding scattering resonance mode. Due to the radiation condition (4d), the eigenvalue problem (4) is non-selfadjoint. The eigenvalues of (4) all lie in the open lower half complex plane and for each \( \epsilon > 0 \) are uniformly bounded away from the real axis. In particular, for any \( \epsilon > 0 \)

\[\text{(2)}\] For simplicity, we have set \( Ca \), the cavitation number, appear in [5] equal to one.
there is a scattering resonance eigenvalue $\lambda_*(\epsilon)$ with $\text{Im } \lambda_*(\epsilon) < 0$ and such that any scattering resonance eigenvalue of $\lambda$, $\lambda$, satisfies
\[
\text{Im } \lambda \leq \text{Im } \lambda_*(\epsilon) < 0;
\]
see Theorem 3.2 of [5].

The following theorem, a consequence of Theorem 5.1 in [5], relates $\lambda_*(\epsilon)$ to the rate of decay of perturbations about the spherical bubble equilibrium:

**Theorem 1.** There exists $\epsilon_0 > 0$, such that for all $0 < \epsilon \leq \epsilon_0$ the following holds. Consider the initial boundary value problem [4]. Assume the initial conditions:

(a) $\beta(t, \Omega) = \sum_{l \geq 0} \sum_{|m| \leq l} \beta_m^l(0) Y_m^l(\Omega)$, where
\[
\|\beta(t = 0)\| = \sum_{l \geq 0} \sum_{|m| \leq l} (1 + l)^{2+\frac{2}{3}} |\beta_m^l(0)| < \infty
\]

(b) $\Psi(t, x), \partial_t \Psi(t, x) \equiv 0, t = 0, |x| > 1$.

Then, there exists a unique solution $\Psi(r, \Omega, t), \beta(\Omega, t)$, defined for $r > 1, \Omega \in S^2$, which solves the initial-boundary value problem.

Furthermore, define $\Gamma(\epsilon)$ to be the minimum distance of a scattering resonance (in the lower half plane) to the real axis. That is,
\[
\Gamma(\epsilon) = |\text{Im } \lambda_*(\epsilon)|
\]

Then, the solution $\Psi(x, t), \beta(\Omega, t)$ satisfies the

**Decay Estimate:**
\[
(6) \quad |\beta(\Omega, t)| \leq C \|\beta(t = 0)\| e^{-\Gamma(\epsilon)t}, \quad \Omega \in S^2
\]
\[
(7) \quad |\Psi(x, t)| \leq \begin{cases} C \frac{1}{|x|^{\frac{n}{2}} e^{-\Gamma(\epsilon)(t-\epsilon(|x|-1))}} \|\beta(t = 0)\|, & 1 < |x| < 1 + \epsilon^{-1}t \\ 0, & |x| > 1 + \epsilon^{-1}t \end{cases}
\]

Our main result is the following precise asymptotic statement about $\Gamma(\epsilon)$:

**Theorem 2.** For $0 < \epsilon < \epsilon_0$, there exist positive constants $A_\epsilon$ and $B$, such that
\[
(8) \quad \Gamma(\epsilon) = \frac{A_\epsilon We}{\epsilon^3} \exp \left( -\frac{BW\epsilon}{\epsilon^2} \right)
\]
where $B \approx 0.26924$ and where $A_\epsilon$ has asymptotic expansion
\[
(9) \quad A_\epsilon = A_0 + O \left( \frac{\epsilon^2}{We} \right), \text{ where } A_0 \approx e^{-2.1465}
\]

**Acknowledgement:** O.C. and S.T. were supported in part by NSF grant DMS-1108794. M.I.W. was supported in part by NSF grant DMS-10-08855.
2. Proof of Theorem 2

We begin by deriving a characterization of the scattering resonance energies of the eigenvalue problem (1) as the zeros of an analytic function in \( C \).

We first note that outgoing solutions of the three-dimensional Helmholtz equation are linear combinations of solutions of the form

\[
h_l^{(1)}(r)Y_m^l(\Omega), \quad |m| \leq l,
\]

where \( h_l^{(1)} \) denotes the outgoing spherical Hankel function of order \( l \). Thus, we seek solutions of (4) of the form:

\[
\Psi_\lambda(r, \Omega) = a Y_m^l(\Omega) h_l^{(1)}(\epsilon \lambda r), \quad \beta_\lambda(\Omega) = b Y_m^l(\Omega), \quad r \geq 1, \quad \Omega \in S^2.
\]

where \( a \) and \( b \) are constants to be determined. This choice of \( \Psi_\lambda \) solves the Helmholtz equation and satisfies the outgoing radiation condition. To impose the boundary conditions at \( r = 1 \) we substitute the expressions for \( \Psi_\lambda \) and \( \beta_\lambda \) into (4) and obtain the following two linear homogeneous equations for the unknown constants \( a \) and \( b \):

\[
\begin{pmatrix}
\epsilon \lambda h_l^{(1)}(\epsilon \lambda) & i \lambda \\
-i\lambda h_l^{(1)}(\epsilon \lambda) & -(l+2)(l-1) / W_e
\end{pmatrix}
\begin{pmatrix}
a \\ b
\end{pmatrix}
= \begin{pmatrix}
0 \\ 0
\end{pmatrix}, \quad l \geq 2.
\]

Setting the determinant equal to zero yields

\[
\lambda^2 h_l^{(1)}(\epsilon \lambda) + \frac{(l+2)(l-1)}{W_e} \epsilon \lambda h_l^{(1)}(\epsilon \lambda) = 0.
\]

Finally, multiplying through by \( \epsilon^2 \) and defining

\[
Q(l, \epsilon) = (l+2)(l-1) \frac{\epsilon^2}{W_e}
\]

yields the following transcendental equation

\[
zh_l^{(1)}(z) + Q(l, \epsilon) h_l^{(1)'}(z) = 0, \quad z = \epsilon \lambda \neq 0.
\]

Here \( h_l^{(1)} \) denotes spherical Hankel function of the first kind of order \( l \).

**Remark 1.** We will analyze the roots \( z \) of (12) for \( W_e = 1 \) since the formula for the roots \( z \) for \( W_e \neq 1 \) are obtained from those with \( W_e = 1 \) by replacing \( \epsilon^2 \) by \( \frac{\epsilon^2}{W_e} \). Furthermore, since \( \lambda = \frac{\epsilon}{\epsilon} \), proving Theorem 2 is equivalent to showing that the roots of (12) for \( W_e = 1 \) satisfy the property

\[
\Gamma(\epsilon) = \inf_{l \geq 2} \{ -\text{Im} \ z \} = \frac{A}{\epsilon^2} \exp \left[ -\frac{B}{\epsilon^2} \right]
\]

for \( 0 < \epsilon \leq \epsilon_0 \) sufficiently small.

In the following, we consider the roots in different regimes in \( l \) relative to \( \epsilon \).

**Definition 3.** \( f = O(g) \) means that for any \( \epsilon \) sufficiently small, there exists a constant \( C \) independent of \( \epsilon \) so that \( |f| \leq C|g| \). We write \( f = O_\epsilon(g) \), i.e. \( f \) is strictly of order \( g \), if \( f = O(g) \) and \( g = O(f) \). We write \( f \ll g \) if \( f = o(g) \), that is \( f/g \to 0 \) as \( \epsilon \to 0 \) and \( f \gg g \) if \( |f/g| \to \infty \) as \( \epsilon \to 0 \).

**Outline of the proof:** In section 4 equation (12) is rewritten as a coupled set of equations for the real and imaginary parts of \( z = x + iy \). In section 5 we show, via Lemma 4 and Lemma 5, that the scattering resonance of minimal imaginary part does not occur for \( l = O(1) \) or for \( l \gg \epsilon^{-2} \). In section 6 we show that the scattering
resonance of minimal imaginary part does not occur for $1 \ll l \ll \epsilon^{-2}$. Therefore, the scattering resonance of interest must satisfy $l = \mathcal{O}_s(\epsilon^{-2})$. The detailed analysis of this regime and the rigorous approximation of (13) is in section 6. Section 7 is an appendix containing many of the asymptotic forms of special functions.

3. Derivation of a system of equation for $x, y$, where $z = x + iy$

It turns out that for large $l$ some of the roots $z = x + iy$ of (12) have exponentially small $y$ (with also $y/x$ is exponentially small) and therefore the asymptotic analysis is more delicate. We next re-express (12) as an equivalent system of equations obtained by setting its real and imaginary parts equal to zero. We note that

\begin{equation}
    h_1^{(1)}(z) = j_l(z) + iy_l(z), \quad \text{where } j_l(z) = \sqrt{\frac{\pi}{2z}} J_{l+1/2}(z), \quad y_l(z) = \sqrt{\frac{\pi}{2z}} Y_{l+1/2}(z),
\end{equation}

where $J_{l+1/2}(z)$ and $Y_{l+1/2}(z)$ are Bessel functions of order $l + 1/2$. Substituting into (12) leads to

\begin{equation}
    (x + iy)j_l(x + iy) + i(x + iy)y_l(x + iy) + Qj_l'(x + iy) + iy_l'(x + iy) = 0
\end{equation}

Defining the real valued functions

\begin{align}
    A_1(x, y) &= \frac{1}{2} \left[ y_l(x + iy) + y_l(x - iy) \right], \quad A_2(x, y) = \frac{1}{2iy} \left[ y_l(x + iy) - y_l(x - iy) \right] \\
    A_3(x, y) &= \frac{1}{2} \left[ j_l'(x + iy) + j_l'(x - iy) \right], \quad A_4(x, y) = \frac{1}{2iy} \left[ j_l'(x + iy) - j_l'(x - iy) \right],
\end{align}

\begin{align}
    B_1(x, y) &= \frac{1}{2} \left[ j_l(x + iy) + j_l(x - iy) \right], \quad B_2(x, y) = \frac{1}{2iy} \left[ j_l(x + iy) - j_l(x - iy) \right] \\
    B_3(x, y) &= \frac{1}{2} \left[ j_l'(x + iy) + j_l'(x - iy) \right], \quad B_4(x, y) = \frac{1}{2iy} \left[ j_l'(x + iy) - j_l'(x - iy) \right],
\end{align}

and separating out the imaginary and real parts of (13), we get

\begin{align}
    xA_1 &= -QA_3 - y(xB_2 + B_1 + QB_4 - yA_2) \\
    y(A_1 + xA_2 + QA_4) &= (xB_1 + QB_3 - y^2 B_2)
\end{align}

We rewrite (18), (19) in the form

\begin{align}
    x &= -\frac{QA_3}{A_1} - \frac{y(xB_2 + B_1 + QB_4 - yA_2)}{A_1} \\
    y &= \frac{xB_1 + QB_3 - y^2 B_2}{A_1 + xA_2 + QA_4}
\end{align}
4. Location of the roots $z$ of $|12|

As will be seen, the roots with smallest $|\text{Im} \, z|$ occur for $l = O(\epsilon^{-2})$, $z = O(l)$; for them, it turns out that $\text{Im} \, z$ is exponentially small in $\epsilon$. First, we show that roots $z$ in other regimes have larger $|\text{Im} \, z|$.

**Lemma 4.** If $l = O(1)$ as $\epsilon \to 0$, then any root of $|12|$ in the sector $\arg z \in (-\pi, \pi)$ satisfies $z^2 \sim (l + 1)Q$, and $y = \text{Im} \, z = O(\epsilon^{2l+2})$.

See Theorem 6.2 of [5] for detailed asymptotics.

**Proof.** Assume first that we had $z \gg 1$. Since in this regime $h^{(i)}_l(z)/h^{(i)}_l(z) \sim i$ (see (63) in the Appendix), it follows that $z \sim -iQ = O(\epsilon^2)$, a contradiction. Similarly, $z = O_s(1)$ implies $z = O(\epsilon^2)$, a contradiction again. Thus $z \ll 1$. Using the well-known expansions of $h^{(i)}_l(z), h^{(i)}_l(z)$ for small $z$ (§7.7 and §5.8 in the Appendix) we get

$$z \sim -Q \frac{h^{(i)}_l(z)}{h^{(i)}_l(z)} = -Q \frac{y^{(i)}_l(z)}{y_l(z)} \left(1 - i j^{(i)}_l(z)/y^{(i)}_l(z) \right) \left(1 - i j_l(z)/y_l(z) \right)^{-1}$$

$$= \frac{(l + 1)Q}{z} \left[1 - \frac{2z^2}{(2l - 1)(l + 1)} + \frac{2z^4}{(l + 1)(2l - 3)(2l - 3)^2} + O(z^6/l^6) \right] \times \left[1 + O \left(\frac{z}{l} \right)^{2l+1} \right],$$

implying by iteration

$$z^2 = (l + 1)Q \left[1 - \frac{Q}{2l - 1} + \frac{Q^2(l - 4)}{(2l - 3)(2l - 1)^2} + O \left(Q^4 l^{-4}, Q^{l+1/2} l^{l-1/2} \right) \right],$$

which is real and positive up to and including $O(Q^l)$. Therefore, $x$ can be computed within errors of $O(Q^l) = O(\epsilon^{2l})$ from the series (24). Using (7.1) and the small $x$ expansion of Bessel functions (see Appendix), the result follows from straightforward calculations.

**Remark 2.** The expression (24) holds also in the regime $1 \ll l \ll \epsilon^{-2}$.

**Lemma 5.** If $l \gg \epsilon^{-2} \gg 1$, then any root of $|12|$ in the sector $\arg z \in (-\pi, \pi)$ satisfies $z \sim -iQ$, implying $-\text{Im} \, z = O(l^2 \epsilon^2) \gg \epsilon^{-2}$.

**Proof.** We first claim that there are no roots $z \ll l$. Indeed, otherwise the expansion of $h^{(i)}_l(z)$ and $h^{(i)}_l(z)$ in this regime (see §7.7, §5.8 in the Appendix) would lead to a contradiction: $z = O(\sqrt{l}) = O(l^{3/2} \epsilon) \gg l$. On the other hand, the existence of roots $z = O_s(l)$ implies, using the asymptotics of $h^{(i)}_l$ and $(h^{(i)}_l)'$ (see §7.3- §7.5 in the Appendix for results in different ranges of $\sqrt{l(l+1)}$ in [12]), a contradiction:

$$\frac{z}{\sqrt{l(l+1)}} = O_s \left(\frac{Q}{l} \right) = O_s(l^2) \gg 1.$$  Thus, $|z| \gg l$. Using the asymptotic behavior of $h^{(i)}_l(z)$ and its derivative in this regime leads (see section 7.2) to $z \sim -iQ$.  


5. Analysis of the case $1 \ll l \ll \epsilon^{-2}$

**Lemma 6.** If $1 \ll l \ll \epsilon^{-2}$, then any root of (12) is given by (24), implying that $z = O_s(l^{1/2} \epsilon) \ll l \ll \epsilon^{-2}$.

**Proof.** Using the asymptotics (75) of $h_\ell^{(1)}(z)$ and $(h_\ell^{(1)}(z))'$ in (12), the assumption $z > l$ leads to a contradiction: $z \sim -iQ = O(l^2 \epsilon^2) \ll l$. Assuming $z = O_s(l)$ and using the asymptotics of $h_\ell^{(1)}$ and $(h_\ell^{(1)})'$ (Appendix, §7.3-§7.5) in (12), also ends up in a contradiction: $\sqrt{l(l+1)}/z = O_s\left(\frac{Q}{l}\right) = O_s(l^2) \ll 1$. In the regime $z = o(l)$, exploiting the asymptotics of $h_\ell^{(1)}(z)$, we obtain (22) and therefore (24). □

By Lemma 6, any root of (12) in this case satisfies $z = o(l)$. From (24) (see Remark 2), it follows that the roots are close to the real axis. In this section, we estimate $y = \text{Im} \ z$ in this regime.

**Lemma 7.** Any root $z = x + iy$ of (12) in the regime $1 \ll l \ll \epsilon^{-2}$ satisfies

$$y \sim -\frac{\sqrt{Q(l+1)}}{2e} \left(\frac{eQ^{1/2}(l+1)^{1/2}}{2l} \right)^{2l+1} \left[1 - \frac{Q}{(2l-1)} + O(Q^2/l^2)\right]^{1+1/2}$$

**Proof.** Since $z \ll l$, we use (24) and the expansions of Bessel functions in this regime (see Appendix) to conclude

$$\frac{j_l(z)}{y_l(z)} = -\frac{1}{2e} \left(\frac{eQ^{1/2}(l+1)^{1/2}}{2l} \right)^{2l+1} \left[1 + O\left(\frac{x^2}{l^2}\right)\right]$$

$$= -\frac{1}{2e} \left(\frac{eQ^{1/2}(l+1)^{1/2}}{2l} \right)^{2l+1} \left[1 - \frac{Q}{(2l-1)} + O(Q^2/l^2)\right]^{1+1/2}$$

In the same way,

$$1 + \frac{QB_3}{xB_1} = 2 + O\left(\frac{1}{l} \frac{Q}{l}\right), \quad 1 + x \frac{A_2}{A_1} + \frac{QA_3}{A_1} = 2 + O(Q/l)$$

Using (73), one obtains (25). □

**Remark 3.** Since $Q^{1/2}/l^{1/2}$ scales like $l^{1/2} \epsilon \ll 1$, we have $|y| \sim Cl\left(\epsilon l^{1/2}\right)^{2l+2}$, which is a decreasing function of $l$ in the regime $1 \ll l \ll \epsilon^{-2}$. Therefore, $|y|$ is minimal at the right end, and this minimal value turns out to be larger than the values obtained in the next section.

6. Analysis of the case $l = O_s(\epsilon^{-2})$

From (12), it follows that when $l = O_s(\epsilon^{-2})$, $Q = O_s(\epsilon^{-2})$. If $z = O_s(l)$, but $|z-l| \gg l^{1/3}$, from the asymptotics (74) and (75), it follows that (12) results in

$$z^2 \sim -iQ \sqrt{\frac{z^2}{l(l+1)}} - 1$$

Hence $|y| = |\text{Im}z| \gg 1$. In the crossover regime, $1 - \frac{z}{\sqrt{l(l+1)}} = O(l^{-2/3})$, the asymptotics of Bessel functions involve the Airy functions $Ai$ and $Bi$ (76) and (77) whose ratio is $O_s(1)$, and thus $y$ cannot be exponentially small.
So, we restrict to the sub-region \( \frac{\varepsilon}{\sqrt{l(l+1)}} < 1 \) and \( 1 - \frac{\varepsilon}{\sqrt{l(l+1)}} \gg l^{-2/3} \). In this regime, since \( \frac{j(t)}{y(t)} \) is exponentially small in \( l \), by (7.1), any root \( z \) must have \( y \) exponentially small. We introduce the scaled variables

\[
\sqrt{l(l+1)} = l_\ast \varepsilon^{-2}, \quad Q = l_\ast \varepsilon^{-2} q, \quad x = \frac{l_\ast}{\varepsilon^2} \zeta, \quad -\varepsilon^2 \log \left[-\varepsilon^2 y \right] = \eta
\]

From the definition of \( Q \), it follows that (for \( We = 1 \)),

\[
q = l_\ast - \frac{2\varepsilon^4}{l_\ast}
\]

From (20) and (7.1), we can replace (7.1) by

\[
y = \frac{xB_1 + QB_3 - y^2 B_2}{D}
\]

where

\[
D = A_1 + xA_2 + QA_4 + \sqrt{\frac{l(l+1)}{x^2}} - 1 [xA_1 + QA_3 + y (xB_2 + B_1 + QB_4 - yA_2)].
\]

Note: the square-bracketed quantity on the previous line vanishes by (7.1). With the notation in (20), (20) and (31) imply

\[
0 = \zeta + \frac{qA_3}{A_1} - \frac{e^{-\eta/\varepsilon^2}}{\varepsilon^2 A_1} \left[ \frac{B_1}{l_\ast} \varepsilon^2 + \zeta B_2 + q B_4 + \frac{A_2}{l_\ast} e^{-\eta/\varepsilon^2} \right] =: F(\zeta, \eta; \varepsilon)
\]

\[
0 = \eta + \varepsilon^2 \log \left[-\varepsilon^2 \left(\frac{xB_1 + QB_3 - y^2 B_2}{D}\right)\right] =: G(\zeta, \eta; \varepsilon)
\]

**Lemma 8.** For any fixed \( \delta \in (0, 1) \), if \( l_\ast \in \left( \frac{\varepsilon^2}{\sqrt{1 - \varepsilon^2}}, \frac{(1-\delta)^2}{\sqrt{1-(1-\delta)^2}} \right) \), \( \zeta \in (\delta, 1 - \delta) \) and \( \eta > \eta_0 = 2l_\ast \int_{1-\delta}^{1} t^{-1/2} \sqrt{1 - t^2} dt \), then there exists \( \varepsilon_0 \) such that for \( |\varepsilon| \leq \varepsilon_0 \), \( F \) and \( G \) are smooth functions of \( \zeta, \eta, l_\ast \) and \( \varepsilon \), with

\[
F(\zeta, \eta; 0) = \zeta - l_\ast \sqrt{1 - \zeta^2}, \quad G(\zeta, \eta; 0) = \eta - 2l_\ast \int_{\zeta}^{1} \sqrt{1 - t^2} \frac{dt}{t}
\]

and \( \det \frac{\partial (F,G)}{\partial (\zeta,\eta)}(\xi;\varepsilon) \neq 0 \). Furthermore, in these intervals, the system of equations \( \{ F(\zeta, \eta; \varepsilon) = 0, G(\zeta, \eta; \varepsilon) = 0 \} \) has a unique smooth solution \( (\zeta(l_\ast, \varepsilon), \eta(l_\ast, \varepsilon)) \), for sufficiently small \( \varepsilon \), with asymptotic behavior in \( \varepsilon \) determined implicitly from

\[
l_\ast = \frac{\zeta^2}{\sqrt{1 - \zeta^2}} - \frac{\varepsilon^2 (1 - 2 \zeta^2)}{2(1 - \zeta^2)^{3/2}} + O(\varepsilon^4),
\]

\[
\eta = 2l_\ast \int_{\zeta}^{1} (t^2 - 1)^{1/2} dt - \varepsilon^2 \log |l_\ast| + \varepsilon^2 \log \left[ 2 + \frac{1}{2(1 - \zeta^2)} - \frac{\varepsilon^2 (1 - 2 \zeta^2)}{2 \zeta^4} \right] + O(\varepsilon^4)
\]

**Proof.** Using a standard integral representation of \( j_t(\sqrt{l(l+1)} \zeta) \) and \( y_t(\sqrt{l(l+1)} \zeta) \), (20) and (56), and noting that \( \sqrt{l(l+1)} = l_\ast \varepsilon^{-2} \), smoothness of \( F \) and \( G \) in \( \varepsilon \) follows provided \( A_1 \) and \( D \) in (32) and (7.2) are nonzero, which is the case, as shall
be seen, for $\epsilon = 0$. Routine calculations, using the asymptotics (68)-(73) show that (37)

$$\zeta + q_3A_3 = \zeta + q_3 y_i(x)/y_i(x) + O(e^{-m_0\epsilon^{-2}}) = \zeta - l_\ast \sqrt{1 - \zeta^2} \left[ 1 + \epsilon^2 \frac{1 - 2\zeta^2}{2l_\ast(1 - \zeta^2)^{3/2}} + O(\epsilon^4) \right],$$

Therefore,

$$F(\zeta, \eta; \epsilon) = \zeta - l_\ast \frac{\sqrt{1 - \zeta^2}}{\zeta} - \frac{\epsilon^2(1 - 2\zeta^2)}{2\zeta(1 - \zeta^2)} + O(\epsilon^4),$$

from which the formula for $F(\zeta, \eta; 0)$ in (34) follows. Now, setting $F = 0$ in (33) immediately leads to (35). For the second part, from the asymptotics of $j_1, y_2$ in this regime, (38)-(39), we get

$$\begin{align*}
(39) \quad xB_1 + QB_3 - y^2B_2 &= xj_1(x) \left[ 1 + \frac{q_1(j_1(x))}{\zeta j_1(x)} + O(y^2) \right] \\
&= xj_1(x) \left[ 1 + l_\ast \frac{\sqrt{1 - \zeta^2}}{\zeta^2} - \frac{\epsilon^2(1 - 2\zeta^2)}{2\zeta^2(1 - \zeta^2)} + O(\epsilon^4) \right]
\end{align*}$$

(40)

$$D = A_1 + xA_2 + QA_4 + \sqrt{\frac{l(l + 1)}{x^2}} - 1 \left[ xA_1 + QA_3 + y(xB_2 + B_1 + QB_4 - yA_2) \right] = y_j \left\{ \frac{1}{2(1 - \zeta^2)} + \frac{2l_\ast}{\zeta^2} \sqrt{1 - \zeta^2} - \frac{l_\ast(1 - 2\zeta^2)}{2\zeta^2\sqrt{1 - \zeta^2}} + O(\epsilon^4) \right\}$$

Therefore, it follows that

$$\begin{align*}
(41) \quad G(\zeta, \eta; \epsilon) &= \eta + \epsilon^2 \log \left\{ -e^2 \frac{xB_1 + QB_3 - y^2B_3}{D} \right\} = \eta + \epsilon^2 \log \left\{ \frac{l_\ast j_1(x)}{y_i(x)} \right\} \\
&+ \epsilon^2 \log \left[ 1 + \frac{l_\ast \sqrt{1 - \zeta^2}}{\zeta^2} \right] - \epsilon^2 \log \left[ \frac{1}{2(1 - \zeta^2)} + \frac{2l_\ast}{\zeta^2} \sqrt{1 - \zeta^2} - \frac{l_\ast(1 - 2\zeta^2)}{2\zeta^2\sqrt{1 - \zeta^2}} \right] + O(\epsilon^4)
\end{align*}$$

From equation (73) in the Appendix, we obtain

$$\begin{align*}
(42) \quad - 2l_\ast \frac{j_1(x)}{y_i(x)} &= \exp \left[ -2l_\ast \epsilon^{-2} \int_\zeta^1 (t^{-2} - 1)^{1/2} dt \right],
\end{align*}$$

(43)

$$\begin{align*}
G(\zeta, \eta; \epsilon) &= \eta - 2l_\ast \int_\zeta^1 (t^{-2} - 1)^{1/2} dt + \epsilon^2 \log \left( \frac{1}{2} l_\ast \zeta \left[ 1 + l_\ast \frac{\sqrt{1 - \zeta^2}}{\zeta^2} \right] \right) \\
&- \epsilon^2 \log \left[ \frac{1}{2(1 - \zeta^2)} + \frac{2l_\ast}{\zeta^2} \sqrt{1 - \zeta^2} - \frac{l_\ast^2(1 - 2\zeta^2)}{2\zeta^4} \left( \frac{\zeta^2}{l_\ast \sqrt{1 - \zeta^2}} \right) \right] + O(\epsilon^4)
\end{align*}$$

from which the expression of $G(\zeta, \eta; 0)$ in (34) follows. The Jacobian of $F(\zeta, \eta; 0)$, $G(\zeta, \eta; 0)$ with respect to $(\zeta, \eta)$ is clearly nonzero. The statement of existence and uniqueness of $(\zeta(l_\ast, \epsilon), \eta(l_\ast, \epsilon))$ now follows from the implicit function theorem. Furthermore, using (13) and (35) in $F = 0, G = 0$ immediately implies (33).
Note 4. We now comment on why the interval restrictions in Lemma 8 for \( l_*, \zeta \) and \( \eta \) do not matter for finding the smallest \(|y|\) as a function of \( l_* \). First, if \( l_* < \frac{\delta^2}{\sqrt{1 - \delta^2}} \), then for sufficiently small \( \delta \), the conclusions of Lemma 7 hold and the corresponding \(|y|\) is not as small as implied by Lemma 8 for \( l_* \) in the given interval. On the other hand if \( l_* > \frac{(1 - \delta^2)}{\sqrt{1 - (1 - \delta)^2}} \) for sufficiently small \( \delta \), then the conclusions of Lemma 6 hold where \( y = \text{Im} \ z \) is no longer small. Thus, the restriction \( l_* \in \left( \frac{\delta^2}{\sqrt{1 - \delta^2}}, \frac{(1 - \delta^2)}{\sqrt{1 - (1 - \delta)^2}} \right) \) is appropriate. Furthermore, for \( l_* \) in this interval, there is no need to consider the possibility \( \zeta \not\in (\delta, 1 - \delta) \) since we have already shown that \( l = O_* (\epsilon^{-2}) \), implies \( x = O_* (\epsilon^{-2}) \) and argued at the outset that we only need to consider \( 1 - \zeta \gg l^{-2/3} \).

When \( (1 - \zeta) \) is small, the asymptotic behavior of \( \frac{\tilde{g}(\epsilon)}{y(\epsilon)} \) is given by a ratio of Airy functions, as is the case for \( 1 - \zeta = O(l^{-2/3}) \) (see (40) and (47)) and the resulting \(|y|\) is not as small as obtained in (53) below, as a consequence of Lemma 8. Also, we need not worry about possible solutions for which \( \eta < \eta_0 \), since we are seeking to maximize \( \eta \) (minimizing \(|y|\)).

6.1. Maximizing \( \eta \) as a function of \( l_* \) and determination of \( \Gamma(\epsilon) \) of (35). Since \( \text{Im} \ z = \eta = -\epsilon^2 \exp \left[ -\frac{\pi \zeta}{2} \right] \), minimizing \(|y|\) as a function of \( l_* \) corresponds to maximizing \( \eta \). Now, (33) implies that \( l_* \) increases with \( \zeta \) in any compact subset of \((\delta, 1 - \delta)\) for all sufficiently small \( \epsilon \). For now, we consider \( l_* \) as a continuous variable for the purposes of finding the maximum value of \( \eta \). We will show later that the maximal value of \( \eta \) to the order calculated is the same if \( l_* \) takes discrete values: \( \sqrt{l(l + 1)} \epsilon^2 \) for \( l \in \mathbb{N} \).

We seek a critical value \( l_* \) for which \( \partial_{l_*} \eta = 0 \) and \( \partial^2_{l_* l_*} \eta < 0 \) implying a maximum of \( \eta \). We note that

\[
0 = \partial_{l_*} \eta = 2 \int_{\zeta}^{1} (t^2 - 1)^{1/2} dt - \frac{2l_*(1 - \zeta^2)^{1/2}}{\zeta} - \frac{\epsilon^2}{l_*} - \frac{\epsilon^2 \zeta_*}{\zeta} + e^2 \zeta_* \partial_{\log} \left[ 2 + \frac{1}{2(1 - \zeta^2)} - \frac{l_2(1 - 2\zeta^2)}{2\zeta^4} \right] + O(\epsilon^4)
\]

We also note that (33) implies

\[
\zeta_* = \frac{(1 - \zeta^2)^{3/2}}{\zeta(2 - \zeta^2)} - \frac{(1 + 2\zeta^2)}{2\zeta(2 - \zeta^2)^2} \epsilon^2 + O(\epsilon^4)
\]

Substituting (33) and (18) into (44), we obtain

\[
0 = \frac{1}{2} \eta_* = \int_{\zeta}^{1} \sqrt{t^2 - 1} dt - \frac{(1 - \zeta^2)^{3/2}}{(2 - \zeta^2)^2} + e^2 \sqrt{1 - \zeta^2} \left[ 2 + \frac{1}{2(1 - \zeta^2)} - \frac{l_2(1 - 2\zeta^2)}{2\zeta^4} \right] + O(\epsilon^4) =: g(\zeta; \epsilon)
\]

It is clear that the solution \( \zeta \) of \( g(\zeta; \epsilon) = 0 \) has the behavior

\[
\zeta = \zeta_{m,0} + \epsilon^2 \zeta_{m,2} + O(\epsilon^4)
\]

where

\[
0 = \int_{\zeta_{m,0}}^{1} \sqrt{t^2 - 1} dt - \frac{(1 - \zeta_{m,0}^2)^{3/2}}{(2 - \zeta_{m,0}^2)} = g(\zeta_{m,0}; 0) =: g_0(\zeta_{m,0}).
\]
Since

\begin{equation}
    g'_0(\zeta) = 2\sqrt{1-\zeta^2} \left(\frac{2 - (2 - \zeta^2)^2}{\zeta(2 - \zeta^2)^2}\right),
\end{equation}

it follows that

\begin{equation}
    \zeta_{m,2} = -\frac{1 - \zeta_{m,0}^2}{2g'_0(\zeta_{m,0})^2(2 - \zeta_{m,0}^2)^2} = -\left\{ \frac{(2\zeta_{m,0}^4 + 5\zeta_{m,0}^2 - 4)}{4\zeta_{m,0}^2(2 - (2 - \zeta_{m,0}^2)^2)} \right\}
\end{equation}

We now seek to find \( \zeta_{m,0} \) which corresponds to the maximal \( \eta \). We note from (49) that \( g'_0 < 0 \) for \( \zeta \in \left(0, \sqrt{2} - \sqrt{2}\right) =: J \) and \( g'_0 > 0 \) for \( \zeta \in \left(\sqrt{2} - \sqrt{2}, 1\right) \). Since we are seeking a maximum for \( \eta \), we must have \( g'_0 < 0 \) at the corresponding \( \zeta \) (noting that \( \zeta_{l} > 0 \)). Thus the roots of \( g'_0(\zeta) = 0 \) only need to be sought for \( \zeta \in J \). At the right end of the interval \( J \), explicit integration – in terms of elementary functions – gives \( g_0 < 0 \) (approx \(-0.0678\)), while \( g_0 \to +\infty \) when \( \zeta \to 0 \). Since \( g_0 \) is monotonic in \( J \), there exists unique \( \zeta_{m,0} \) satisfying \( g_0(\zeta_{m,0}) = 0 \). The explicit calculation of \( g_0 \) gives \( g_0(0.58) > 0 \) and \( g_0(0.59) < 0 \), implying that the unique maximum \( \zeta_{m,0} \) is in \((0.58, 0.59)\) – in fact \( \zeta_{m,0} = 0.58134... \). Using (50) we get \( \zeta_{m,2} \approx -1.1743... \). Equation (54) also implies that the critical \( l_* = l_{*,m} \) that maximizes \( \eta \) is given by

\begin{equation}
    l_{*,m} = \frac{\zeta_{m,0}^2}{\sqrt{1-\zeta_{m,0}^2}} + \epsilon^2 \left[ \zeta_{m,0} \zeta_{m,0}(2 - \zeta_{m,0}^2) \frac{(1 - 2\zeta_{m,0}^2)}{2(1 - \zeta_{m,0}^2)^{3/2}} \right] + O(\epsilon^4)
\end{equation}

Therefore, it follows that the maximized \( \eta = \eta_m \) satisfies

\begin{equation}
    \eta_m = 2\zeta_{m,0} \int_{\zeta_{m,0}}^{1} (t^{-2} - 1)^{1/2} dt + \epsilon^2 \left[ 2l_{m,2} \int_{\zeta_{m,0}}^{1} (t^{-2} - 1)^{1/2} dt - 2l_{m,0}\zeta_{m,2}(\zeta_{m,0}^{-2} - 1)^{1/2} \right]
\end{equation}

Therefore the minimum value of \( |y| \) is

\begin{equation}
    |y_m| = \epsilon^{-2} \exp \left[ -\epsilon^2 \eta_{m,0} - \eta_{m,2} \right] [1 + O(\epsilon^2)]
\end{equation}

The proof of Theorem 1 now follows from Remark 1.

**Remark 5.** The calculation (52) assumed \( l_* \) to be a continuous variable rather than discrete, \( l_* = \sqrt{l(l+1)}\epsilon^2 \) with \( l \in \mathbb{N} \). This however makes no difference to the leading order result in (53). Maximization of \( \eta \) over \( l \in \mathbb{N} \) results in an optimal \( l_* \) that is different from the computed \( l_{*,m} \) by \( O(\epsilon^2) \). Since \( \eta \) is a smooth function of \( l_* \), for \( l_* - l_{*,m} = O(\epsilon^2) \), \( \eta(l_*) - \eta(l_{*,m}) = O(l_* - l_{*,m})^2 = O(\epsilon^4) \) and therefore discreteness does not affect the leading order asymptotic result (53).
7. Appendix: Asymptotics of $j_l$ and $y_l$ in different regimes

The results quoted below are standard and either given in standard references such as [1], [2] or follow directly from them.

The spherical Bessel functions $j_l(z) = \sqrt{\frac{-1}{2\pi z}} J_{l+1/2}(z)$ and $y_l(z) = \sqrt{\frac{-1}{2\pi z}} Y_{l+1/2}(z)$ have the following integral representations which immediately follows from equations (10.9.6), (10.9.7) in [1] (see also http://dlmf.nist.gov/10.9) using $\nu = l + 1/2$

\begin{align}
    j_l(z) &= \sqrt{\frac{-1}{2\pi z}} \left\{ \int_0^\pi \cos \left( (l + \frac{1}{2})\tau - z \sin \tau \right) d\tau \right.
    
    &\quad + (-1)^{l+1} \int_0^\infty \exp \left[ -z \sinh t - (l + \frac{1}{2}) t \right] dt \bigg\} \\
    y_l(z) &= \sqrt{\frac{-1}{2\pi z}} \left\{ (-1)^{l+1} \int_0^\pi \cos \left( (l + \frac{1}{2})\tau + z \sin \tau \right) d\tau \right.
    
    &\quad - \int_0^\infty \exp \left[ -z \sinh t + (l + \frac{1}{2}) t \right] dt \bigg\}
\end{align}

which results in the asymptotic representations below by standard Laplace method for asymptotics of integrals [2]. The Bessel functions $j_l(z)$, $y_l(z)$ and the Hankel function $h_l^{(1)}(z) = j_l(z) + iy_l(z)$ satisfy

\begin{align}
    u'' + \frac{2}{z} u' + \left( 1 - \frac{l(l+1)}{z^2} \right) u &= 0
\end{align}

7.1. The regime $z \ll l$: We only present the asymptotic regimes relevant to this analysis. For $z \ll l$ we have (see equations (10.1.2) and (10.1.3), page 437 of [1])

\begin{align}
    j_l(z) &= \frac{-z^l l!}{(2l+1)!} \left[ 1 + O \left( \frac{z^2}{l^2} \right) \right] \\
    y_l(z) &= \frac{1}{z^2} \frac{(2l-1)!}{(2l+1)!} \left[ 1 + \frac{z^2}{2(2l-1)} + \frac{z^4}{8(2l-1)(2l-3)} + O \left( \frac{z^6}{l^2} \right) \right]
\end{align}

(Three orders are indeed needed because of cancellations.) For $l \gg z \gg 1$, using $\Gamma(x+1) \sim \sqrt{2\pi} e^{-x} x^{x+1/2}$ for large $x$ we note the simplification:

\begin{align}
    j_l(z) &\sim \frac{2^{-3/2}}{l} \left( \frac{ze}{2l} \right)^l \left[ 1 + O \left( \frac{z^2}{l^2} \right) \right] \\
    y_l(z) &\sim \frac{1}{l\sqrt{2}} \left( \frac{ze}{2l} \right)^{-l-1} \left[ 1 + \frac{z^2}{2(2l-1)} + \frac{z^4}{8(2l-1)(2l-3)} + O \left( \frac{z^6}{l^5} \right) \right]
\end{align}

Since the asymptotics is differentiable [2], this implies

\begin{align}
    \frac{j'_l(z)}{j_l(z)} &= \frac{l}{z} \left[ 1 + O \left( \frac{z^2}{l^2} \right) \right] \\
    \frac{z y'_l(z)}{y_l(z)} &= -(l+1) + \frac{z^2}{2l-1} + \frac{z^4}{(2l-3)(2l-1)^2} + O \left( \frac{z^6}{l^5} \right) \\
    \frac{y''(z)}{y_l(z)} &= (l+1)(l+2)z^{-2} - \frac{2l+1}{2l-1} - \frac{2z^2}{(2l-3)(2l-1)^2} + O \left( \frac{z^4}{l^4} \right)
\end{align}
In this regime, we have

\[ \frac{j_l(z)}{y_l(z)} = -\frac{1}{2e} \left( \frac{ze}{2l} \right)^{2l+1} \left[ 1 + O\left( \frac{z^2}{l^2} \right) \right] \]

7.2. The regime \( z \gg l, \ \arg z \in (-\pi, \pi) \). In this case

\[ h_l^{(1)}(z) \sim \frac{1}{z} \exp \left[ i \left( z - \frac{(l+1)}{2} \pi \right) \right] \]

\[ j_l(z) \sim \frac{1}{z} \cos \left( z - \frac{(l+1)}{2} \pi \right) \]

\[ y_l(z) \sim \frac{1}{z} \sin \left( z - \frac{(l+1)}{2} \pi \right) \]

7.3. The regime \( l \gg 1, \ \xi = \frac{z}{\sqrt{l(l+1)}} = O_s(l), \ 1 - |\xi| > l^{-2/3}, \ |y| \ll l^{1/3}, \ \arg(1 - \xi) \in (-\pi/3, \pi/3) \). From (54) and (55), by the saddle point method (2), we obtain

\[ j_l(z) = \frac{1}{2\sqrt{l(l+1)}} \xi^{3/2} (\xi^{-2} - 1)^{-1/4} \]

\[ \times \exp \left[ -t^{1/2}(l+1)^{1/2} \int_{\xi}^{1} (t^{-2} - 1)^{1/2} dt \right] \left[ 1 + \frac{u(\xi)}{l} + O(l^{-2}) \right] \]

where

\[ u(\xi) = -\frac{5}{24(1-\xi^2)^{3/2}} + \frac{1}{8(1-\xi^2)^{1/2}} + \frac{1}{16} \log \left( \frac{1 - \sqrt{1-\xi^2}}{1 + \sqrt{1-\xi^2}} \right) \]

\[ y_l(z) \sim -\frac{1}{2\sqrt{l(l+1)}} \xi^{3/2} (\xi^{-2} - 1)^{-1/4} \]

\[ \times \exp \left[ t^{1/2}(l+1)^{1/2} \int_{\xi}^{1} (t^{-2} - 1)^{1/2} dt \right] \left[ 1 - \frac{u(\xi)}{l} + O(l^{-2}) \right] \]

\[ \frac{y_l(z)}{y_l(z)} = -\sqrt{\xi^{-2} - 1} - \frac{1 - 2\xi^2}{2l(1-\xi^2)} + O(l^{-2}) \]

\[ \left. \frac{y_l^2(z)}{y_l(z)} \right|^{1/2} = \sqrt{\xi^{-2} - 1} - \frac{1 - 2\xi^2}{2l(1-\xi^2)} + O(l^{-2}) \]

In this regime, we have

\[ -\frac{2j_l(z)}{y_l(z)} = \exp \left[ -2\sqrt{l(l+1)} \int_{\xi}^{1} (t^{-2} - 1)^{1/2} dt \right] \left[ 1 + \frac{2}{l} u(\xi) + O(l^{-2}) \right] \]

7.4. The regime \( l \gg 1, \ \xi = \frac{z}{\sqrt{l(l+1)}} = O_s(l), \ |\xi| - 1 \gg l^{-2/3}, \ |y| \ll 1, \ \arg(\xi - 1) \in (-\pi/3, \pi/3) \). In this case we have

\[ j_l(z) \sim \frac{1}{\sqrt{l(l+1)}} \xi (1-\xi^{-2})^{-1/4} \cos \left[ \sqrt{l(l+1)} \int_{1}^{\xi} \sqrt{(1-t^{-2})} dt - \frac{(l+1)\pi}{2} \right] \]

\[ y_l(z) \sim \frac{1}{\sqrt{l(l+1)}} \xi (1-\xi^{-2})^{-1/4} \sin \left[ \sqrt{l(l+1)} \int_{1}^{\xi} \sqrt{(1-t^{-2})} dt - \frac{(l+1)\pi}{2} \right] \]
7.5. The regime \( l \gg 1, \xi = \sqrt{z} \sqrt{l(l+1)} = O_\ast(l), \ 1 - \xi = O(l^{-2/3}) \). Here the asymptotic behavior is given by

\[
\begin{align*}
\psi_l(z) & \sim \frac{\sqrt{\pi l^{1/6}}}{2^{1/6} z} Ai \left( l^{2/3} 2^{1/3} [1 - \xi] \right) \\
y_l(z) & \sim -\frac{\sqrt{\pi l^{1/6}}}{2^{1/6} z} Bi \left( l^{2/3} 2^{1/3} [1 - \xi] \right)
\end{align*}
\]

References

[1] M. Abramowitz & I. Stegun, Handbook of Mathematical Functions, Dover, New York (1970).
[2] F.W.J. Olver, Asymptotics and Special Functions, A.K. Peters, Wellesley, Massachusetts (1997).
[3] M.P. Brenner, D. Lohse and T.F. Dupont, Bubble Shape Oscillations and the Onset of Sonoluminescence, Phys. Rev. Lett. Volume 75, #5 954–957 (1995)
[4] T.G. Leighton, From seas to surgeries, from babbling brooks to baby scans: the acoustics of gas bubbles in liquids, International Journal of Modern Physics B Volume 18 #25 3267–3314 (2004)
[5] A.M. Shapiro and M.I. Weinstein, Radiative decay of bubble oscillations in a compressible fluid, SIAM J. Math. Analysis Volume 43, 828-876 (2011)
[6] W. Wasow, Asymptotic Expansions for Ordinary Differential Equations, Interscience, New York (1965).

Department of Mathematics, The Ohio State University, 231 W 18th Ave, Columbus, OH 43210

Department of Applied Physics and Applied Mathematics, Columbia University, New York, NY 10027