Rotational Symmetry and A Light Mode in Two-Dimensional Staggered Fermions

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Abstract

To obtain a light mode in two-dimensional staggered fermions, we introduce four new local operators keeping the rotational invariance for a staggered Dirac operator. To split masses of tastes, three cases are considered. The mass matrix and the propagator for free theories are analyzed. We find that one of three cases is a good candidate for taking a single mode by the mass splitting. In the case, it is possible that a heavy mode obtains infinite mass on even sites or odd sites.

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1 Introduction

Staggered fermions are formulated in which species doublers of a Dirac field are interpreted as physical degrees of freedom, *tastes*, on lattice [1, 2]. Although the fermion determinant has many advantages in the cost of its numerical calculation [3, 4], it remains for a 4-fold degeneracy problem of tastes in four dimensions to be unsolved. A fourth-root trick of the determinant in a staggered Dirac operator is an approach to unfold the degeneracy and studies on its theoretical basis are developed [5, 6, 7]. However, we have no local expression of one taste Dirac fermion after the fourth-root trick.

Avoiding the trick, there are pioneering works for solving the degeneracy tried by improved staggered fermion approaches [8, 9]. The improved actions generally include more operators than the original staggered one and are difficult to treat them [10]. For the control of their operators, we make use of staggered fermions on a $D$-dimensional lattice space based on an $SO(2D)$ Clifford algebra [11]. The formulation by the $SO(2D)$ Clifford algebra is powerful in the control of fermion operators and *we can describe any improved fermion action on a hypercubic lattice*. In addition, a discrete rotational symmetry (cubic symmetry) can be represented by the algebra.

In this article, we analyze the mass splitting of degenerate tastes by adding four operators to the original staggered action in two dimensions. Only these four operators keep the discrete rotational symmetry in any dimension [11]. The total mass matrix analysis is insufficient because the matrix does not commute with the kinetic term. Therefore, we also analyze the propagator and the pole. It is found that only one combination in these operators is a good candidate after these analyses.

This article is organized as follows. In section 2, staggered fermions are formulated by the $SO(2D)$ Clifford algebra. Four operators are introduced to obtain taste-splitting masses. These operators keep the discrete rotational invariance. We analyze the mass matrix and the mass pole of the improved free staggered Dirac operator in sections 3 and 4. Section 5 is devoted to further analyses of the massless limit and infinite mass of a heavy mode. We summarize and
Figure 1: A two-dimensional lattice unit and the weight of a spinor representation in $SO(4)$.

discuss about our approach in section 6.

2 Formulation of Staggered Fermions and Rotational Symmetry

The formulation of staggered fermions on the $D$-dimensional lattice space has been presented based on the $SO(2D)$ Clifford algebra [11]. The basic idea is that the dimension of the total representation space including spinor and taste spaces, $2^D$ is the same as that of an $SO(2D)$ spinor representation. $2^D$ is also the same as the number of sites in a $D$-dimensional hypercube. To avoid the double counting of sites, the lattice coordinate $n_\mu$ is noted by

$$n_\mu = 2N_\mu + c_\mu + r_\mu, \quad (1)$$

where $N_\mu$ is the global coordinate of the hypercube. In this case, a fundamental unit is $2a$, where $a$ is a lattice constant, and is set to unity. $c_\mu = 1/2$ for any $\mu$ means the coordinate of a center in the $D$-dimensional hypercube and $r_\mu$ does the relative coordinate of a site to the center. The relative coordinate is the same as a weight of the spinor representation in $SO(2D)$.

Although our formulation can be generalized, we consider a theory in a two-dimensional lattice, for simplicity. Relative coordinates of four sites around a plaquette are written by

$$(r_1, r_2) = (-1/2, -1/2), (-1/2, 1/2), (1/2, -1/2), (1/2, 1/2), \quad (2)$$
as shown in Figure 1. Actually, our staggered fermion is defined on sites as

\[
\Psi(n) \equiv \Psi_r(N) = \begin{pmatrix}
\Psi_{-1/2,-1/2} \\
\Psi_{-1/2,1/2} \\
\Psi_{1/2,-1/2} \\
\Psi_{1/2,1/2}
\end{pmatrix} (N) \equiv \begin{pmatrix}
\Psi_1 \\
\Psi_2 \\
\Psi_3 \\
\Psi_4
\end{pmatrix} (N). \tag{3}
\]

It is noted that \(\Psi_1\) and \(\Psi_4\) are put on even sites and \(\Psi_2\) and \(\Psi_3\) are put on odd sites.

An \(SO(4)\) Clifford algebra plays a crucial role in two-dimensional cubic lattice formulations [11]. The original staggered fermion action [1, 2] can be written as

\[
S_{st} = \sum_{N,N',r,r',\mu,\bar{\tau}} \bar{\Psi}_r(N)(D^\tau_{\mu})(N,N')(\Gamma_{\mu,\bar{\tau}})(r,r')\Psi_{r'}(N'), \tag{4}
\]

where \(\bar{\tau}\) is a two-dimensional vector with its components of \(\pm 1/2\) and \(D^\tau_{\mu}\) for \(\mu = 1, 2\), is a generalized difference operator defined by

\[
(D^\tau_{\mu})(N,N') \equiv \frac{1}{2^2} \sum_{\bar{\sigma}=0,1} (-1)^{\bar{\sigma}+\bar{\tau}}(\nabla^\sigma_{\mu})(N,N'), \tag{5}
\]

with

\[
(\nabla^\sigma_{\mu})(N,N') = \begin{cases}
\delta_{N,N'}U_{2N+\bar{\sigma},\mu} - \delta_{N-\bar{\mu},N'}U_{2N+\bar{\sigma}-\bar{\mu},\mu} \equiv \nabla^-_{\mu}, & \sigma = 0, \\
\delta_{N+\bar{\mu},N'}U_{2N+\bar{\sigma},\mu} - \delta_{N,N'}U_{2N+\bar{\sigma}-\bar{\mu},\mu} \equiv \nabla^+_{\mu}, & \sigma = 1.
\end{cases} \tag{6}
\]

\(\bar{\sigma}\) is a two-dimensional vector dual to \(\bar{\tau}\) and \(\nabla^+_{\mu} (\nabla^-_{\mu})\) implies a forward (backward) difference operator along the \(\mu\)-direction, respectively. In Eq. (4), a link variable \(U_{2N+\bar{\sigma},\mu}\) is introduced for gauge covariance. This formulation of staggered fermions has usual gauge interactions and definite \(O(a)\) higher order terms. The matrix \(\Gamma_{\mu,\bar{\tau}}\) in our action (4) is composed of the \(SO(4)\) Clifford algebra \(\gamma_\mu\) and \(\tilde{\gamma}_\mu\)

\[
\Gamma_{1,-\bar{\tau}} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix} \equiv \gamma_1, \quad \Gamma_{2,-\bar{\tau}} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0
\end{pmatrix} \equiv \gamma_2, \tag{7}
\]

\[-i\Gamma_{1,(1/2,-1/2)} = \begin{pmatrix}
0 & 0 & -i & 0 \\
0 & 0 & 0 & -i \\
i & 0 & 0 & 0 \\
0 & i & 0 & 0
\end{pmatrix} \equiv \tilde{\gamma}_1, \quad -i\Gamma_{2,(1/2,1/2)} = \begin{pmatrix}
0 & -i & 0 & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & i \\
0 & 0 & -i & 0
\end{pmatrix} \equiv \tilde{\gamma}_2, \tag{8}
\]

and is described by

\[
\Gamma_{\mu,\bar{\tau}} \equiv (i\tilde{\gamma}_1\gamma_1)^{1/2+\tau_1} (i\tilde{\gamma}_2\gamma_2)^{1/2+\tau_2} \gamma_\mu. \tag{9}
\]
Here we denote the fundamental algebra, or the $SO(4)$ Clifford algebra as

$$\{\gamma_\mu, \gamma_\nu\} = \{\tilde{\gamma}_\mu, \tilde{\gamma}_\nu\} = 2\delta_{\mu\nu}, \quad (10)$$

$$\{\gamma_\mu, \tilde{\gamma}_\nu\} = 0, \quad (11)$$

and

$$\{\Gamma_\mu, \pi, \Gamma_5\} = 0, \quad (12)$$

where $\Gamma_5 \equiv \gamma_1\gamma_2\tilde{\gamma}_1\tilde{\gamma}_2 = \text{diag}(1, -1, -1, 1)$. From the algebra, we find that a massless staggered fermion has an even-odd symmetry

$$\Psi \rightarrow e^{i\vartheta\Gamma_5}\Psi, \quad \bar{\Psi} \rightarrow \bar{\Psi}e^{i\vartheta\Gamma_5}. \quad (13)$$

For a discrete rotation with angle $\pi/2$ around the center, the transformations of global and relative coordinates are denoted by

$$N \rightarrow R(N), \quad r \rightarrow R(r), \quad (14)$$

and that of fermion is

$$\Psi(N) \rightarrow V_{12}\Psi(R(N)). \quad (15)$$

$V_{12}$ is a rotation matrix about a spinor index in the $SO(4)$ base, up to a phase factor given by a form

$$V_{12} = \frac{e^{i\theta}}{2} \Gamma_5(\tilde{\gamma}_1 - \tilde{\gamma}_2)(1 + \gamma_1\gamma_2) = e^{i\theta} \begin{pmatrix}
0 & 0 & -i & 0 \\
i & 0 & 0 & 0 \\
0 & 0 & 0 & i \\
i & 0 & 0 & 0
\end{pmatrix}. \quad (16)$$

The following four operators $O_i$ for $i = 1, 2, 3, 4$,

$$O_1 = 1, \quad O_2 = i\gamma_1\gamma_2 \equiv \Gamma_3, \quad O_3 = \tilde{\gamma}_1 + \tilde{\gamma}_2, \quad O_4 = \Gamma_3(\tilde{\gamma}_1 + \tilde{\gamma}_2), \quad (17)$$

are invariant under the rotation $V_{12}O_iV_{12}^\dagger$. Our analyses in the following sections concentrate on the improved staggered fermion action by these four operators with $U_{2N+\pi,\mu} = 1$. 

5
3 Analysis of Mass Matrices

To split masses in degenerate tastes we introduce four rotationally invariant operators, which we denote as $M_i \equiv \bar{\Psi} O_i \Psi$, for the original staggered fermion action \[1\]. Explicit expressions for $M_i$ are given by

\[
M_1(N) = \sum_{r,r'} \bar{\Psi}_r(N) 1_{r,r'} \Psi_{r'}(N), \\
M_2(N) = \sum_{r,r'} \bar{\Psi}_r(N)(\Gamma_3)_{r,r'} \Psi_{r'}(N), \\
M_3(N) = \sum_{r,r'} \bar{\Psi}_r(N)(\tilde{\gamma}_1 + \tilde{\gamma}_2)_{r,r'} \Psi_{r'}(N), \\
M_4(N) = \sum_{r,r'} \bar{\Psi}_r(N)\{\Gamma_3(\tilde{\gamma}_1 + \tilde{\gamma}_2)\}_{r,r'} \Psi_{r'}(N). \tag{18}
\]

The total mass matrix form which is invariant under the rotation by $\pi/2$ in two dimensions is given as

\[
M_R = m_1 1 + m_2 \Gamma_3 + m_3 (\tilde{\gamma}_1 + \tilde{\gamma}_2) + m_4 \Gamma_3 (\tilde{\gamma}_1 + \tilde{\gamma}_2) \\
= \begin{pmatrix}
m_1 & -im_3 + m_4 & -im_3 - m_4 & -im_2 \\
-im_3 + m_4 & m_1 & -im_2 & -im_3 + m_4 \\
im_3 - m_4 & im_2 & m_1 & im_3 + m_4 \\
im_2 & im_3 + m_4 & -im_3 + m_4 & m_1
\end{pmatrix}, \tag{19}
\]

where $m_1, m_2, m_3, m_4$ are parameters of each operator in Eq. \[18\]. $M_R$ has four eigenvalues

\[
m_1 - m_2 - \sqrt{2}m_3 + \sqrt{2}m_4, \quad m_1 - m_2 + \sqrt{2}m_3 - \sqrt{2}m_4, \\
\]

\[
m_1 + m_2 - \sqrt{2}m_3 - \sqrt{2}m_4, \quad m_1 + m_2 + \sqrt{2}m_3 + \sqrt{2}m_4. \tag{20}
\]

The operator $M_1$ cannot separate the 2-fold degeneracy in two-dimensional staggered Dirac fermions. The degeneracy can be solved by remained three operators $M_2, M_3$ and $M_4$. A 4-component spinor should be separated into two 2-component spinors since a two-dimensional Dirac spinor is composed of a 2-component mode and we keep the rotational invariance even under a finite lattice constant*. Actually all possibilities of this separation are three cases and are listed in Table 1. Let us analyze these mass matrices in order explicitly.

*If one permits the rotational invariance only after taking the continuum limit, it is not necessary for degeneracy of a heavy mode and there are six more cases derived from Eq. \[20\].
The mass matrix

\[
M_{R1} = \begin{pmatrix}
  m_1 & m_4 & -m_4 & 0 \\
  m_4 & m_1 & 0 & m_4 \\
  -m_4 & 0 & m_1 & m_4 \\
  0 & m_4 & m_4 & m_1
\end{pmatrix}
\]  

(21)

can be diagonalized as

\[
M_{R1}^{\text{diag}} = P_1^\dagger M_{R1} P_1 = \text{diag}(m_1 - \sqrt{2}m_4, m_1 - \sqrt{2}m_4, m_1 + \sqrt{2}m_4, m_1 + \sqrt{2}m_4), 
\]

(22)

by a unitary matrix

\[
P_1 = \frac{1}{2\sqrt{2}} \begin{pmatrix}
  1 + i & 1 - i & 1 - i & 1 + i \\
  -\sqrt{2} & -\sqrt{2} & \sqrt{2} & \sqrt{2} \\
  \sqrt{2}i & -\sqrt{2}i & \sqrt{2}i & -\sqrt{2}i \\
  1 - i & 1 + i & 1 + i & 1 - i
\end{pmatrix}. 
\]

(23)

Note that the matrix \(P_1\) can diagonalize the rotation matrix \(V_{12}\) simultaneously. The unitary transformed spinor is given by

\[
\Psi_{M_1}(N) \equiv P_1^\dagger \Psi(N) = \frac{1}{2\sqrt{2}} \begin{pmatrix}
  1 - i & -\sqrt{2} & -\sqrt{2}i & 1 + i \\
  1 + i & -\sqrt{2} & \sqrt{2}i & 1 - i \\
  1 + i & \sqrt{2} & -\sqrt{2}i & 1 - i \\
  1 - i & \sqrt{2} & \sqrt{2}i & 1 + i
\end{pmatrix} \begin{pmatrix}
  \Psi_1 \\
  \Psi_2 \\
  \Psi_3 \\
  \Psi_4
\end{pmatrix}(N). 
\]

(24)

Case 2

The mass matrix

\[
M_{R2} = \begin{pmatrix}
  m_1 & -im_3 & -im_3 & 0 \\
  im_3 & m_1 & 0 & -im_3 \\
  im_3 & 0 & m_1 & im_3 \\
  0 & im_3 & -im_3 & m_1
\end{pmatrix}
\]

(25)
can be diagonalized as
\[
M_{R2}^{\text{diag}} = P_2^\dagger M_{R2} P_2 = \text{diag}(m_1 - \sqrt{2}m_3, m_1 - \sqrt{2}m_3, m_1 + \sqrt{2}m_3, m_1 + \sqrt{2}m_3),
\]
by a unitary matrix
\[
P_2 = \frac{1}{2\sqrt{2}} \begin{pmatrix}
-\sqrt{2}i & \sqrt{2}i & -\sqrt{2}i & -\sqrt{2}i \\
-1 + i & 1 + i & 1 + i & 1 - i \\
-1 - i & 1 - i & 1 - i & 1 + i \\
\sqrt{2} & \sqrt{2} & -\sqrt{2} & \sqrt{2}
\end{pmatrix},
\]
with the transformed spinor
\[
\Psi_{M_2}(N) \equiv P_2^\dagger \Psi(N) = \frac{1}{2\sqrt{2}} \begin{pmatrix}
\sqrt{2}i\psi_1 - (1 + i)\psi_2 - (1 - i)\psi_3 + \sqrt{2}\psi_4 \\
-2i\psi_1 + (1 - i)\psi_2 + (1 + i)\psi_3 + \sqrt{2}\psi_4 \\
\sqrt{2}i\psi_1 + (1 - i)\psi_2 + (1 + i)\psi_3 - \sqrt{2}\psi_4 \\
\sqrt{2}i\psi_1 + (1 + i)\psi_2 + (1 - i)\psi_3 + \sqrt{2}\psi_4
\end{pmatrix}(N).
\]

**Case 3**

The mass matrix
\[
M_{R3} = \begin{pmatrix}
m_1 & 0 & 0 & -im_2 \\
0 & m_1 & -im_2 & 0 \\
0 & im_2 & m_1 & 0 \\
im_2 & 0 & 0 & m_1
\end{pmatrix}
\]
can be diagonalized as
\[
M_{R3}^{\text{diag}} = P_3^\dagger M_{R3} P_3 = \text{diag}(m_1 - m_2, m_1 - m_2, m_1 + m_2, m_1 + m_2),
\]
by a unitary matrix
\[
P_3 = \frac{1}{2\sqrt{2}} \begin{pmatrix}
-1 + i & 1 - i & -\sqrt{2}i & -\sqrt{2}i \\
\sqrt{2}i & \sqrt{2}i & 1 - i & -1 + i \\
\sqrt{2} & \sqrt{2} & 1 + i & -1 - i \\
1 + i & -1 - i & \sqrt{2} & \sqrt{2}
\end{pmatrix},
\]
with the transformed spinor
\[
\Psi_{M_3}(N) \equiv P_3^\dagger \Psi(N) = \frac{1}{2\sqrt{2}} \begin{pmatrix}
-(1 + i)\psi_1 - \sqrt{2}i\psi_2 + \sqrt{2}\psi_3 + (1 - i)\psi_4 \\
(1 + i)\psi_1 - \sqrt{2}i\psi_2 + \sqrt{2}\psi_3 - (1 - i)\psi_4 \\
\sqrt{2}i\psi_1 + (1 + i)\psi_2 + (1 - i)\psi_3 + \sqrt{2}\psi_4 \\
\sqrt{2}i\psi_1 - (1 + i)\psi_2 - (1 - i)\psi_3 + \sqrt{2}\psi_4
\end{pmatrix}(N).
\]

The feature of our formulation is to keep the discrete rotational invariance. After the mass splitting, we can find the character of a Dirac spinor under the rotation,
\[
\psi(x) \to Q\psi(R(x)),
\]
\(Q\) being a discrete rotation matrix.\[1\]
\[ P_i V_{12} P_i \]
| phase factor of \( V_{12} \) |
|---|---|
| case 1 | \( e^{i\theta} = e^{i\pi/2} = i \) |
| case 2 | \( e^{i\theta} = e^{i\pi} = -1 \) |
| case 3 | \( e^{i\theta} = e^{-i\pi/4} = (1 - i)/\sqrt{2} \) |

Table 2: The properties of Dirac spinors under the rotation.

where \( Q = e^{i\pi/4}\sigma_3 = \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} \). Actually in cases 1 and 2 we can keep the property of a Dirac spinor on lattice,

\[ \Psi(N) \rightarrow V_{12}\Psi(R(N)). \] (34)

By contrast, \( \Psi(N) \) acts as a vector not as a spinor in case 3. The properties of 2-component spinors under the rotation are summarized in Table 2.

### 4 Pole Analysis and 2-point Functions

Our adding terms do not commute with the staggered Dirac operator. As a result, our analysis in the previous section is insufficient to split masses. We must proceed in the pole analysis of the theory because a pole mass is physical. For the help, Dirac fields are Fourier transformed as

\[ \Psi_r(N) = \int_{-\pi}^{+\pi} \frac{d^2p}{(2\pi)^2} \tilde{\Psi}_r(p)e^{ipN}, \quad \bar{\Psi}_r(N) = \int_{-\pi}^{+\pi} \frac{d^2p}{(2\pi)^2} \tilde{\bar{\Psi}}_r(p)e^{ipN}, \] (35)

and the action becomes

\[
S_{st} = \sum_{N,N',r,r',\mu,\mu'} \tilde{\Psi}_r(N)(D_{\mu}^\tau)(N,N')(\Gamma_{\mu,\tau})(r,r') \Psi_{r'}(N')
\]

\[
= \sum_{r,r'} \int_{-\pi}^{+\pi} \frac{d^2p}{(2\pi)^2} \tilde{\Psi}_r(-p) \left[ \sum_\mu \{ i\gamma_\mu \sin p_\mu + i\tilde{\gamma}_\mu (1 - \cos p_\mu) \} \right]_{(r,r')} \tilde{\Psi}_{r'}(p). \] (36)

The staggered Dirac operator is explicitly written as

\[
D_{st}(p) = \sum_\mu \{ i\gamma_\mu \sin p_\mu + i\tilde{\gamma}_\mu (1 - \cos p_\mu) \} \]
\[
\begin{pmatrix}
0 & is_2 + c_2 & is_1 + c_1 & 0 \\
is_2 - c_2 & 0 & 0 & is_1 + c_1 \\
is_1 - c_1 & 0 & 0 & -is_2 - c_2 \\
0 & is_1 - c_1 & -is_2 + c_2 & 0
\end{pmatrix},
\]
(37)

where \(s_i \equiv \sin p_i\) and \(c_i \equiv 1 - \cos p_i\) for \(i = 1, 2\), respectively.

Our steps to find a pole mass are as follows: (i) set \(p_1 = 0\) and \(p_2 = i\kappa\) (pure imaginary) of the inverse propagator \(D^{-1}\) in the momentum representation where our rotationally invariant operators \((38)\) are included; (ii) calculate four eigenvalues \(\lambda\) of \(D^{-1}\); (iii) find values of \(\kappa\) in setting \(\lambda = 0\). Four values of \(\kappa\) equal to pole masses. As mentioned in sections 2 and 3, we keep the rotational invariance in our action and generate two Dirac spinors with different masses. The light Dirac mass is denoted as \(m_l\) and the heavy one is done as \(m_h\). We note that parameters \(m_1, m'_2 \equiv -im_2, m'_3 \equiv -im_3\) and \(m_4\) are real to obtain real pole masses.

Case 1

The improved staggered Dirac operator is given by

\[
D_{1}^{\text{imp}} \equiv D_{st}(p) + M_{R1}
\]

\[
= \sum_{\mu} \{i\gamma_{\mu} \sin p_{\mu} + i\tilde{\gamma}_{\mu}(1 - \cos p_{\mu})\} + m_1 + m_4 \Gamma_3(\tilde{\gamma}_1 + \tilde{\gamma}_2)
\]

\[
= \begin{pmatrix}
  m_1 & is_2 + c_2 + m_4 & is_1 + c_1 - m_4 & 0 \\
  is_2 - c_2 + m_4 & m_1 & 0 & is_1 + c_1 + m_4 \\
  is_1 - c_1 - m_4 & 0 & m_1 & -is_2 - c_2 + m_4 \\
  0 & is_1 - c_1 + m_4 & -is_2 + c_2 + m_4 & m_1
\end{pmatrix}.
\]
(38)

Setting \(p_1 = 0\), we obtain eigenvalues of Eq. (38) as

\[
\lambda = m_1 \pm \sqrt{2m_4^2 - 4\sin^2 \frac{p_2}{2} \pm \sqrt{-4m_4^2 \left(\sin^2 p_2 + 4\sin^2 \frac{p_2}{2}\right)}}.
\]
(39)

Taking \(p_2 = i\kappa\) at \(\lambda = 0\) in Eq. (39), we have the equation for the pole mass

\[
16(1 - m_4^2) \sinh^4 \frac{\kappa}{2} - 8(m_1^2 + 2m_4^2) \sinh^2 \frac{\kappa}{2} + (m_1^2 - 2m_4^2)^2 = 0.
\]
(40)
Hence we find pole masses for $|m_4| < 1$ as
\[
\kappa = \left\{ \begin{array}{l}
\pm \ln \left[ \frac{2 + m_1^2 - \sqrt{A}}{2(1 - m_4^2)} + \sqrt{\left( \frac{2 + m_1^2 - \sqrt{A}}{2(1 - m_4^2)} \right)^2 - 1} \right] \equiv \pm m_t, \\
\pm \ln \left[ \frac{2 + m_1^2 + \sqrt{A}}{2(1 - m_4^2)} + \sqrt{\left( \frac{2 + m_1^2 + \sqrt{A}}{2(1 - m_4^2)} \right)^2 - 1} \right] \equiv \pm m_h, 
\end{array} \right.
\] (41)
where $A \equiv m_3^2 \{8m_1^2 + (m_1^2 - 2m_4^2)^2\}$.

The mass is still splitting in the pole analysis for the improved staggered action. From Eq. (41), it is found that we can take a limit $|m_h| \to \infty$ for arbitrary $m_t$ by performing $\epsilon \to 0$ in an expression $m_4^2 = 1 - \epsilon (0 < \epsilon \ll 1)$.

Case 2

The improved staggered Dirac operator is given by
\[
D_{2}^{\text{imp}} \equiv D_{st}(p) + M'_{R2} = \sum_{\mu} \left\{ i\gamma_{\mu} \sin p_{\mu} + i\tilde{\gamma}_{\mu}(1 - \cos p_{\mu}) \right\} + m_1 + im'_3(\tilde{\gamma}_1 + \tilde{\gamma}_2) = \sum_{\mu} \left\{ i\gamma_{\mu} \sin p_{\mu} + i\tilde{\gamma}_{\mu}(1 + m'_3 - \cos p_{\mu}) \right\} + m_1
\]
\[
= \begin{pmatrix}
m_1 & is_2 + \gamma'_2 & is_1 + \gamma'_1 & 0 \\
is_2 - \gamma'_2 & m_1 & 0 & is_1 + \gamma'_1 \\
is_1 - \gamma'_1 & 0 & m_1 & -is_2 - \gamma'_2 \\
0 & is_1 - \gamma'_1 & is_2 + \gamma'_2 & m_1
\end{pmatrix},
\] (42)
where $\gamma'_i \equiv 1 + m'_3 - \cos p_i$. Eigenvalues of Eq. (42) with $p_1 = 0$ are
\[
\lambda = m_1 \pm \sqrt{-4(1 + m'_3) \sin^2 \frac{p_2}{2} - 2(m'_3)^2}.
\] (43)
Setting $p_2 = i\kappa$ and $\lambda = 0$, the pole mass is satisfied with
\[
\sinh^2 \frac{\kappa}{2} = \frac{m_1^2 + 2(m'_3)^2}{4(1 + m'_3)}.
\] (44)

Solutions of this equation under $-1 < m'_3$ are
\[
\kappa = \pm 2 \ln \left[ \sqrt{\frac{m_1^2 + 2(m'_3)^2}{4(1 + m'_3)}} + \sqrt{\frac{m_1^2 + 2(m'_3)^2}{4(1 + m'_3)} + 1} \right].
\] (45)

The pole mass remains degenerate because the improved term $m'_3(\tilde{\gamma}_1 + \tilde{\gamma}_2)$ is absorbed into the kinetic term.
Case 3

The improved staggered Dirac operator is given by

\[
D_{\text{imp}}^3 \equiv D_{st}(p) + M'_R \sum_{\mu} \{i\gamma_{\mu} \sin p_{\mu} + i\tilde{\gamma}_{\mu}(1 - \cos p_{\mu})\} + m_1 + im'_2 \Gamma_3
\]

\[
= \begin{pmatrix}
m_1 & is_2 + c_2 & is_1 + c_1 & m'_2 \\
is_2 - c_2 & m_1 & m'_2 & is_1 + c_1 \\
is_1 - c_1 & -m'_2 & m_1 & -is_2 - c_2 \\
-m'_2 & is_1 - c_1 & -is_2 + c_2 & m_1
\end{pmatrix}.
\]  

(46)

Eigenvalues of Eq. (46) with \(p_1 = 0\) are

\[
\lambda = m_1 \pm \sqrt{-4 \sin^2 \frac{p_2}{2} - (m'_2)^2} \pm 4m'_2 \sin^2 \frac{p_2}{2}.
\]  

(47)

Setting \(p_2 = i\kappa\) and \(\lambda = 0\), the pole mass is satisfied with

\[
\sinh^2 \frac{\kappa}{2} = \frac{m_1^2 + (m'_2)^2}{4(1 \pm m'_2)}.
\]  

(48)

Solutions of this equation under \(-1 < m'_2 < 1\) are

\[
\kappa = \begin{cases} 
\pm 2 \ln \left[ \sqrt{\frac{m_1^2 + (m'_2)^2}{4(1 + m'_2)}} + \sqrt{\frac{m_1^2 + (m'_2)^2}{4(1 + m'_2)} + 1} \right], \\
\pm 2 \ln \left[ \sqrt{\frac{m_1^2 + (m'_2)^2}{4(1 - m'_2)}} + \sqrt{\frac{m_1^2 + (m'_2)^2}{4(1 - m'_2)} + 1} \right].
\end{cases}
\]  

(49)

This case allows pole masses to split although the rotational property of the eigenmode is not a spinor from the discussion of the previous section.

In summary, we find that the improved staggered Dirac operator for cases 1 and 3 can split the degenerate mass through the analysis of the inverse propagator. However, that of case 2 cannot do because the additional effect is absorbed into the kinetic term of the original staggered Dirac operator. In addition, we should mention that there is no massless mode in cases 2 and 3 from Eqs. (45) and (49). These nontrivial results are originated from the fact that the rotationally invariant mass terms do not commute with the staggered Dirac operator. Finally note that it is possible to take the light mass \(m_l\) to zero by tuning \(m_1\) and \(m_4\) only in case 1. Focusing on this particular situation, we discuss about massless and infinity modes in the next section.
5 Massless and Infinity Modes of Rotationally Invariant Action for Viable Even-Odd Separation

In our two-dimensional lattice formulation, a matrix
\[ \Gamma_5 = \gamma_1 \gamma_2 \tilde{\gamma}_1 \tilde{\gamma}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \] (50)
can define a chiral projection for a Dirac spinor because of \( \{ D_{st}, \Gamma_5 \} = 0 \). From the explicit matrix representation, the positive chiral mode is put on even sites and the negative mode corresponds to odd sites,
\[ \Psi_e(N) \equiv \frac{1 + \Gamma_5}{2} \Psi(N), \quad \Psi_o(N) \equiv \frac{1 - \Gamma_5}{2} \Psi(N) = \begin{pmatrix} 0 \\ \Psi_2 \\ \Psi_3 \\ 0 \end{pmatrix} (N). \] (51)

We note that the chiral projection is not discrete rotationally invariant since \( [V_{12}, \Gamma_5] \neq 0 \).

This (even-odd) chiral property of staggered fermions holds in general \( D \)-dimensional cases. First of all, we define chiral projection operators
\[ P_L \equiv \frac{1 - \Gamma_{2D+1}}{2}, \quad P_R \equiv \frac{1 + \Gamma_{2D+1}}{2}, \] (52)
and a Dirac spinor \( \Psi \) is projected out as
\[ \Psi_L \equiv P_L \Psi, \quad \Psi_R \equiv P_R \Psi. \] (53)

The kinetic term of a staggered Dirac fermions lagrangian can be written as
\[ \mathcal{L}_{st} = \bar{\Psi}_L D_{st} \Psi_L + \bar{\Psi}_R D_{st} \Psi_R. \] (54)

Among operators discussed in sections 3 and 4, \( \mathcal{M}_3 \) and \( \mathcal{M}_4 \) are chirally invariant but \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) change chiralities of Dirac spinors. Therefore, \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) can construct Dirac mass terms while \( \mathcal{M}_3 \) and \( \mathcal{M}_4 \) can do Majorana mass terms. Although we can define the formal discussion of the chirality just as normal chirality \( \gamma_5 \), this definition of the chirality depends on a special lattice frame because \( [V_{\mu\nu}, \Gamma_{2D+1}] \neq 0 \) where \( V_{\mu\nu} \) means the \( \mu\nu \)-rotation by \( \pi/2 \).
Without mention about the chirality, we can define a massless mode and can throw up the mass of a heavy mode to infinity in case 1 of section 4. Solutions of Eq. (40) under the massless condition $m_1^2 = 2m_4^2$ are determined as

$$\sinh^2 \frac{m_1}{2} = 0,$$

$$\sinh^2 \frac{m_h}{2} = \frac{2m_1^2}{1 - m_4^2}. \tag{56}$$

It must be noted that the eigenmode of the Dirac operator around a massless pole is not orthogonal to that around a heavy mass pole because their Dirac operators are different from each other. The massless modes for $m_4 > 0$ are explicitly written by

$$\begin{pmatrix} 1 + \sqrt{2} \\ -1 - \sqrt{2} \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 - \sqrt{2} \\ 1 - \sqrt{2} \\ -1 \\ 1 \end{pmatrix}. \tag{57}$$

In $m_4 < 0$ case, they are expressed as

$$\begin{pmatrix} 1 - \sqrt{2} \\ -1 + \sqrt{2} \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 + \sqrt{2} \\ 1 + \sqrt{2} \\ -1 \\ 1 \end{pmatrix}. \tag{58}$$

In order to get the eigenmode for the heavy mass, we take $p_1 = 0$ and $p_2 = im_h$ for Eq. (38). $D_{1\text{imp}}^\text{imp}(p_1 = 0, \ p_2 = im_h)$ is given by a form

$$\begin{pmatrix} m_1 & 1 + m_4 - e^{m_h} & -m_4 & 0 \\ -1 + m_4 + e^{-m_h} & m_1 & 0 & m_4 \\ -m_4 & 0 & m_1 & -1 + m_4 + e^{m_h} \\ 0 & m_4 & 1 + m_4 - e^{-m_h} & m_1 \end{pmatrix}. \tag{59}$$

Furthermore, from Eq. (58), we find that eigenmodes corresponding to the heavy mass are given by

$$\begin{pmatrix} \frac{1 + m_4}{m_4} \left\{ m_4(1 + m_4) + \sqrt{2m_4^2(1 + m_1^2)} \right\} \\ \frac{1 - m_4}{m_4} \left( \sqrt{2m_4^2} + m_4 \sqrt{1 + m_1^2} \right) \\ -(1 - m_4) \left( \sqrt{2m_4^2} + \sqrt{1 + m_1^2} \right) \\ 1 - m_4^2 \end{pmatrix}, \tag{60}$$
for $m_h > 0$ and
\[
\begin{pmatrix}
1 - m_4^2 \\
-(1 - m_4) \left( \sqrt{2m_4^2 + 1 + m_4^2} \right) \\
-1 - m_4 \left( \sqrt{2m_4^2 + m_4 \sqrt{1 + m_4^2}} \right) \\
-1 + m_4 \left\{ m_4 (1 + m_4) + \sqrt{2m_4^2(1 + m_4^2)} \right\}
\end{pmatrix},
\] (61)
for $m_h < 0$.

To decouple the heavy mode, we can throw the mass up to infinity. Actually from Eqs. (55) and (56), we can realize massless and infinity modes as Table 3 simultaneously. From these results, it is found that infinity modes can be separately put on even or odd sites.

| $m_4$ | massless modes | infinity modes |
|-------|----------------|----------------|
| $m_4 > 0$ | \begin{pmatrix} 1 + \sqrt{2} \\ -1 - \sqrt{2} \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 - \sqrt{2} \\ 1 - \sqrt{2} \\ -1 \\ 1 \end{pmatrix} | \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} |
| $m_4 < 0$ | \begin{pmatrix} 1 - \sqrt{2} \\ -1 + \sqrt{2} \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 + \sqrt{2} \\ 1 + \sqrt{2} \\ -1 \\ 1 \end{pmatrix} | \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} |

Table 3: Eigenvectors of the improved Dirac operator in case 1 with $m_4^2 = 2m_4^2$. Expressions of infinity modes are given by substituting $m_4^2 = 1$ for Eqs. (60) and (61).

6 Summary and Discussion

We have studied the mass splitting of two-dimensional staggered fermions based on the $SO(4)$ Clifford algebra. Introducing four rotationally invariant operators, we have analyzed three types of improved staggered Dirac operators and found one possibility (case 1) for taking a single mode in a two-dimensional free theory. The case keeps the splitting not only in the analysis of the mass matrix itself but also in the pole analysis including the kinetic term. According to the improvement with respect to the rotational invariance, the derived 2-component modes can be regarded as the ordinary spinor under the rotation by $\pi/2$. Furthermore, one can find a
massless mode in the case unexpectedly. The formal \( \Gamma_5 \) chiral projection which means even-site and odd-site separation of fermion modes is not consistent with the rotational invariance of a staggered Dirac action. Nevertheless, massless and infinity mode-representations in the case realize even-odd separation of the infinity mode.

Our future tasks are analyses of interacting theories and the extension of our approach to four dimensions. In particular, it is crucial that the stability for the massless condition under quantum corrections by gauge interactions. Namely, in the case that the theory is not stable, it may be uninteresting that one needs a fine-tuning of the additional mass parameter as in Wilson fermions. For the infinity mode, it is very interesting if the even-odd separation is induced in staggered fermions even when we consider interaction effects.

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