Vertex and Edge Connectivity of the Zero Divisor graph $\Gamma[\mathbb{Z}_n]$

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Abstract

The Zero divisor Graph of a commutative ring $R$, denoted by $\Gamma[R]$, is a graph whose vertices are non-zero zero divisors of $R$ and two vertices are adjacent if their product is zero. In this paper we derive the Vertex and Edge Connectivity of the zero divisor graph $\Gamma[\mathbb{Z}_n]$, for any natural number $n$. We also discuss the minimum degree of the zero divisor graph $\Gamma[\mathbb{Z}_n]$.

Keywords: Zero divisor graph, Vertex connectivity, Edge connectivity, minimum degree.

1 Introduction

The concept of the Zero divisor graph of a ring $R$ was first introduced by I.Beck[3] in 1988 and later on Anderson and Livingston[2], Akbari and Mohammadian[1] continued the study of zero divisor graph by considering only the non-zero zero divisors. Mohammad Reza and Reza Jahani[5] calculated the energy and Wiener index for the zero divisor graphs $\Gamma[\mathbb{Z}_n]$ for $n = p^2$ and $n = pq$ where $p$ and $q$ are prime numbers. B.Surendranath Reddy, et.al[6] considered the zero divisor graph $\Gamma[\mathbb{Z}_n]$ for $n = p^3$ and $p^2q$, where $p$ and $q$ are prime numbers and derived the standard form of adjacency matrix, spectrum, energy and Wiener index of. The concepts of the Edge and Vertex connectivity of a graph can be found in [4]. In this paper we extend the concepts of the Edge and Vertex connectivity to the zero divisor graph $\Gamma[\mathbb{Z}_n]$ for any natural number $n$.

In this article, section 2, is about the preliminaries and notations related to zero divisor graph of a commutative ring $R$, in section 3, we derive the
Vertex connectivity of a zero divisor graph $\Gamma[Z_n]$, and in section 4, we calculate the Edge connectivity of $\Gamma[Z_n]$. In this section, we also find the minimum degree of the zero divisor graph $\Gamma[Z_n]$.

## 2 Preliminaries and Notations

**Definition 2.1. Zero divisor Graph**

Let $R$ be a commutative ring with unity and $Z[R]$ be the set of its zero divisors. Then the zero divisor graph of $R$ denoted by $\Gamma[R]$, is the graph (undirected) with vertex set $Z^*[R] = Z[R] - \{0\}$, the non-zero zero divisors of $R$, such that two vertices $v, w \in Z^*[R]$ are adjacent if $vw = 0$.

**Definition 2.2. Vertex connectivity of a graph**

Let $G$ be a simple graph. The Vertex connectivity of $G$, denoted by $\kappa(G)$, is the smallest number of vertices in $G$ whose deletion from $G$ leaves either a disconnected graph or $K_1$.

**Definition 2.3. Edge connectivity of a graph**

Let $G$ be a simple graph. The Edge connectivity of $G$, denoted by $\kappa_e(G)$, is the smallest number of edges in $G$ whose deletion from $G$ leaves either a disconnected graph or an empty graph.

**Definition 2.4. Minimum degree of a graph**

Let $G$ be a graph. The Minimum degree of $G$, denoted by $\delta(G)$, is the minimum degree of its vertices.

## 3 Vertex connectivity of of the zero divisor graph $\Gamma[Z_n]$

In this section we derive the vertex connectivity of $\Gamma[Z_n]$ for any natural number $n$. To start with we consider first the zero divisor graph $\Gamma[Z_n]$ for $n = p^2$.

**Theorem 3.1.** The vertex connectivity of $\Gamma[Z_{p^2}]$ is $p - 2$.

**Proof.** The vertex set of the zero divisor graph $\Gamma[Z_{p^2}]$ is $A = \{kp | k = 1, 2, 3, ..., p - 1\}$ and so $|A| = (p - 1)$.

As product of any two vertices is zero, they are adjacent and so the corresponding graph is a complete graph on $(p-1)$ vertices that is, $\Gamma[Z_{p^2}] = K_{p-1}$.

As the graph is complete, deletion of one or less than $p - 2$ vertices does not give a disconnected graph. If we delete $p - 2$ vertices then only one vertex
remains and that gives rise to a disconnected graph. Thus the minimum number of vertices to be deleted is \( p - 2 \).

Hence the vertex connectivity of \( \Gamma[Z_{p^2}] \) is \( p - 2 \).

Since \( \kappa(\Gamma[Z_{p^2}]) = p - 2 \), the zero divisor graph \( \Gamma[Z_{p^2}] \) is \( p - 2 \) connected.

Now we consider the case for \( n = p^3 \) in the following theorem.

**Theorem 3.2.** The vertex connectivity of \( \Gamma[Z_{p^3}] \) is \( p - 1 \).

**Proof.** consider the zero divisor graph \( \Gamma[Z_{p^3}] \).

Here, we divide the elements (vertices) of \( \Gamma[Z_{p^3}] \) into two disjoint sets namely multiples of \( p \) but not \( p^2 \); and the multiples of \( p^2 \) which are given by

\[
A = \{kp \mid k = 1, 2, 3, ..., p^2 - 1 \text{ and } p \nmid k\}
\]

\[
B = \{lp^2 \mid l = 1, 2, 3, ..., p - 1\}
\]

with cardinality \( |A| = p(p - 1) \) and \( |B| = (p - 1) \).

Here we note that no two elements of \( A \) are adjacent and every element of \( A \) is adjacent with every element of \( B \). Also every element of \( B \) is adjacent with every element of \( A \) and \( B \).

Now if we remove all the vertices of \( A \), the graph is still connected because the remaining vertices are from set \( B \) which are adjacent with each other. Whereas, if we remove all the vertices from \( B \), then the resulting graph with vertices from \( A \) is disconnected as no two elements of \( A \) are adjacent.

Also if we leave even one vertex from \( B \) and remove remaining, the graph is still connected as every element of \( B \) is adjacent with \( A \) and \( B \).

Therefore, the minimum number of vertices to be deleted from \( G \) is the number of vertices of \( B \).

Hence the vertex connectivity \( \kappa(\Gamma[Z_{p^3}]) = |B| = p - 1 \).

Since, \( \kappa(\Gamma[Z_{p^3}]) = p - 1 \), the zero divisor graph \( \Gamma[Z_{p^3}] \) is \( p - 1 \) connected.

With similar arguments, we prove the more general case in the following theorem.

**Theorem 3.3.** The vertex connectivity of \( \Gamma[Z_{p^n}] \) is \( p - 1 \) \( \forall \ n \geq 3 \).

**Proof.** We divide the elements (vertices) of \( \Gamma[Z_{p^n}] \) into \( n - 1 \) disjoint sets namely multiples of \( p \), multiples of \( p^2 \), ... multiples of \( p^{n-1} \), given by

\[
A_1 = \{k_1p \mid k_1 = 1, 2, 3, ..., p^{n-1} - 1 \text{ and } p \nmid k_1\}
\]

\[
A_2 = \{k_2p^2 \mid k_2 = 1, 2, 3, ..., p^{n-2} - 1 \text{ and } p \nmid k_2\}
\]

\[
A_i = \{k_ip^i \mid k_i = 1, 2, 3, ..., p^{n-i} - 1 \text{ and } p \nmid k_i\}
\]
Among the above, the smallest set is $A_{n-1}$ of order $p-1$. Now if we leave, even one vertex from $A_i$ for $i = 1, 2, \ldots, n-1$ and remove remaining, the graph is still connected because every element of $A_i$ is adjacent with $A_{n-1}$.

Now if we delete all the vertices of $A_{n-1}$, we get a disconnected graph because the elements of $A_1$ are adjacent with only the elements of $A_{n-1}$, these vertices will become isolated. Therefore the minimum number of vertices to be deleted to make the graph disconnected is $|A_{n-1}| = p-1$.

Thus vertex connectivity of $\Gamma[Z_{p^n}]$ is $p-1$. Since, $\kappa(\Gamma[Z_{p^n}]) = p-1$, the zero divisor graph $\Gamma[Z_{p^n}]$ is $p-1$ connected.

Now before going to the vertex connectivity of $\Gamma[Z_n]$ for any $n \geq 1$, we first work into the vertex connectivity of $\Gamma[Z_m]$ where $m = p^\alpha q^\beta$.

**Theorem 3.4.** The vertex connectivity of $\Gamma[Z_{p^\alpha q^\beta}]$ is $\min\{p-1, q-1\}$.

**Proof.** Here, we divide the elements(vertices) of $\Gamma[Z_m]$ into disjoint sets namely multiples of $p^i$, multiples of $q^j$ and multiples of $p^i q^j$ given by

\[
A_{p^i} = \{r_i p^i \mid r_i = 1, 2, 3, \ldots, p^i - 1 \text{ and } p \nmid r_i\}
\]

\[
A_{q^j} = \{s_j q^j \mid s_j = 1, 2, 3, \ldots, q^j - 1 \text{ and } q \nmid s_j\}
\]

\[
A_{p^i q^j} = \{t_{ij} p^i q^j \mid t_{ij} = 1, 2, 3, \ldots, p^i q^j - 1 \text{ and } p \nmid t_{ij} \text{ and } q \nmid t_{ij}\}
\]

The smallest set among the above is either $B = A_{p^\alpha q^\beta-1}$ or $C = A_{p^{\alpha-1} q^\beta}$.

Suppose $p < q$.

Here we note that every element of $A_{p^i}$ is adjacent with each and every element of $C$.

So, if we delete all the elements of $C$ then the graph becomes disconnected as the vertices of $A_p$ becomes isolated.

Therefore the minimum number of vertices to be deleted to make the graph disconnected is $p-1$.

Similarly, we get that if $q < p$, then the minimum number of vertices to be deleted to make the graph disconnected is $q-1$.

Hence the vertex connectivity of $\Gamma[Z_m]$ is $\min\{p - 1, q - 1\}$.

In the next theorem we derive the vertex connectivity of $\Gamma[Z_n]$ for any natural number $n$.

**Theorem 3.5.** The vertex connectivity of $\Gamma[Z_n]$ where $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ is $\min\{p_1 - 1, p_2 - 1, \ldots, p_k - 1\}$.
Proof. consider a zero divisor graph $\Gamma[\mathbb{Z}_n]$ where $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$.

Here, we divide the elements (vertices) of $\Gamma[\mathbb{Z}_n]$ into the corresponding disjoint sets of product of all possible powers of given primes like set of powers of $p_j^j$, set of product of powers of $p^i p^e$ and so on.

Among these sets, we consider the sets of the form $A_i = \{m(p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_i^{\alpha_i-1} \ldots p_k^{\alpha_k}) | m \not\divides p_j \text{ for all } j\}$ with $|A_i| = (p_i - 1)$.

Let $r = \min\{|A_1|, \ldots, |A_k|\} = \min\{p_1 - 1, \ldots, p_k - 1\}$.

Suppose $r = |A_j| = p_j - 1$, for some $j$.

Now consider the set $A_{tp_j} = \{tp_j | p_j \not\divides t\}$.

Since the elements of $A_{tp_j}$ are adjacent only with the vertices of $A_j$, so deletion of all vertices of $A_j$ leaves the elements of $A_{tp_j}$ isolated.

Thus the graph is disconnected if we remove all the elements of $A_j$.

Therefore the minimum number of vertices to be deleted to make the graph disconnected is $r = |A_j|$.

Thus Vertex connectivity of $\Gamma[\mathbb{Z}_n]$ is $r = \min\{p_1 - 1, p_2 - 1, \ldots, p_k - 1\}$.

\section{Edge connectivity of the zero divisor graph $\Gamma[\mathbb{Z}_n]$}

In this section we discuss the edge connectivity of the zero divisor graph $\Gamma[\mathbb{Z}_n]$. To start with we consider $n = p^2$.

\textbf{Theorem 4.1.} The Edge connectivity of $\Gamma[\mathbb{Z}_{p^2}]$ is $p - 2$.

\textbf{Proof.} The vertex set of the zero divisor graph $\Gamma[\mathbb{Z}_{p^2}]$ is $A = \{kp | k = 1, 2, 3, \ldots, p - 1 \text{ and } k \not\divides p\}$ and so $|A| = (p - 1)$.

As the graph is complete, every vertex is incident with $(p - 2)$ edges. 

so removing this $(p-2)$ edges with respect to a fixed vertex $v$, the graph becomes disconnected as $v$ becomes isolated.

If we remove fewer than $(p - 2)$ edges say $(p - 3)$, then as the vertex $v$ is incident with every edge, the vertex $v$ is not isolated so that the graph is still connected.

Thus the minimum number of edges to be deleted is $p - 2$.

Hence the edge connectivity of $\Gamma[\mathbb{Z}_{p^2}]$ is $\kappa_e(\Gamma[\mathbb{Z}_{p^2}]) = p - 2$.

Since $\kappa_e(\Gamma[\mathbb{Z}_{p^2}]) = p - 2$, the zero divisor graph $\Gamma[\mathbb{Z}_{p^2}]$ is $(p - 2)$- edge connected.

\textbf{Theorem 4.2.} The Edge connectivity of $\Gamma[\mathbb{Z}_{p^n}]$ is $p - 1 \ \forall \ n \geq 3$.

\textbf{Proof.} We group the elements (vertices) of $\Gamma[\mathbb{Z}_{p^n}]$ into (n-1) disjoint sets namely multiples of $p$, multiples of $p^2$ and so on multiples of $p^{n-1}$ given
by
\[ A_{p^i} = \{ r_ip^i \mid r_i = 1, 2, 3, \ldots, p^i - 1 \text{ and } p \nmid r_i \} \]
\[ A_{q^j} = \{ s_jq^j \mid s_j = 1, 2, 3, \ldots, q^j - 1 \text{ and } q \nmid s_j \} \]
\[ A_{p^i,q^j} = \{ t_{ij}p^i q^j \mid t_{ij} = 1, 2, 3, \ldots, p^i q^j - 1 \text{ and } p \nmid t_{ij} \text{ and } q \nmid t_{ij} \}. \]

with cardinality \( |A_i| = (p^{\alpha - i} - 1) \) for \( i = 1, 2, \ldots, n - 1 \).
Clearly, the smallest set among the above is \( A_{n-1} \) of length \( p - 1 \).
Here \( A_1 \) is the only set in which there exists a vertex (in fact every vertex) which is incident with only \( (p - 1) \) edges because any vertex in \( A_i \) is adjacent with the elements of the sets \( A_{n-i}, A_{n-i+1}, \ldots, A_{n-1} \) and so with \( p^i + p^{i-1} + \ldots + p - i \) edges.
Here our idea is to identify a set whose elements are having edges with only \( A_{n-1} \), and the set \( A_1 \) will do the work for us.
Thus if we remove all the edges incident with a vertex \( v \) in \( A_1 \), which are \( (p - 1) \) in number as the vertices of \( A_1 \) are adjacent with only the elements of the set \( A_{n-1} \), giving rise to a disconnected graph.
If we remove fewer than \( (p-1) \) edges say \( (p-2) \) then the graph is still connected.
Therefore the minimum number of edges to be removed is \( p - 1 \).
Thus the edge connectivity of \( \Gamma[Z_{p^n}] \) is \( \kappa_e(\Gamma[Z_{p^n}]) = p - 1 \).
Since \( \kappa_e(\Gamma[Z_{p^n}]) = p - 1 \), the zero divisor graph \( \Gamma[Z_{p^n}] \) is \( (p - 1) \)-edge connected.

Now we move to the general case: the Edge connectivity of \( \Gamma[Z_n] \) for \( n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k} \)

**Theorem 4.3.** The Edge connectivity of \( \Gamma[Z_n] \) for \( n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k} \) is \( r \).
That is \( r \)-edge connected where \( r = \min\{p_i - 1\} \) for \( i = 1, 2, \ldots, k \).

**Proof.** We group the vertex set and choose the suitable set \( A_j \) as in Theorem 3.5. Since the vertices of \( A_{p^i} \) are adjacent only with the vertices of \( A_j \), the deletion of all edges incident with the any vertex of \( A_{p^i} \) leaves that vertex isolated and hence the graph becomes disconnected. Also in all other possible combinations of \( A_{p^i} \) and \( A_j \) the graph is still connected.
Therefore, the minimum number of edges to be deleted to make the graph disconnected is \( r = |A_j| \).
Thus edge connectivity of \( \Gamma[Z_n] \) is \( \kappa_e(\Gamma[Z_n]) = r = \min\{p_1 - 1, p_2 - 1, \ldots, p_k - 1\} \).

**Remark 4.4.** By observing all the results from the sections 3 and 4, we conclude that vertex connectivity is same as the edge connectivity of the zero divisor graph \( \Gamma[Z_n] \), i.e. \( \kappa(\Gamma[Z_n]) = \kappa_e(\Gamma[Z_n]) \).
Theorem 4.5. The minimum degree of the zero divisor graph $\Gamma[Z_p^n]$ is equal its edge connectivity. Therefore $\kappa(\Gamma[Z_n]) = \kappa_e(\Gamma[Z_n]) = \delta(\Gamma[Z_n])$.

Proof. Let the edge connectivity of $\Gamma[Z_p^n]$ is $r$. Then the minimum number of edges that are incident to any arbitrary vertex in the graph is $r$. Also there exists a set (as proved in the previous results) in which degree of every vertex is $r$ and hence the minimum degree of $\Gamma[Z_n]$ is $\delta(\Gamma[Z_n]) = r$. Hence $\kappa(\Gamma[Z_n]) = \kappa_e(\Gamma[Z_n]) = \delta(\Gamma[Z_n])$.

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