ON THE CAUCHY PROBLEM FOR A GENERALIZED TWO-COMPONENT SHALLOW WATER WAVE SYSTEM WITH FRACTIONAL HIGHER-ORDER INERTIA OPERATORS

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Abstract. In this paper, we mainly consider the Cauchy problem for a generalized two-component shallow water wave system with fractional higher-order inertia operators: \( m(t,x) = (1 - \partial_x^2)^s u, s > 1 \). By Littlewood-Paley theory and transport equation theory, we first establish the local well-posedness of the generalized b-equation with fractional higher-order inertia operators which is the subsystem of the generalized two-component water wave system. Then we prove the local well-posedness of the generalized two-component water wave system with fractional higher-order inertia operators. Next, we present the blow-up criteria for these systems. Moreover, we obtain some global existence results for these systems.

1. Introduction. In this paper we consider the Cauchy problem of the following generalized two-component shallow water wave system with fractional higher-order inertia operators [35]:

\[
\begin{aligned}
m_t + um_x + amu_x = \alpha u_x - \kappa \rho x, & \quad t > 0, x \in \mathbb{R}, \\
\rho_t + u\rho_x + (a - 1)u_x \rho = 0, & \quad t > 0, x \in \mathbb{R}, \\
m(t,x) = (1 - \partial_x^2)^s u(t,x), & \quad t \geq 0, x \in \mathbb{R}, \\
u(0,x) = u_0(x), & \quad x \in \mathbb{R}, \\
\rho(0,x) = \rho_0(x), & \quad x \in \mathbb{R},
\end{aligned}
\]

(1.1)

where \( s > 1, a \neq 1 \) is a real parameter, \( \alpha \) is a constant which represents the vorticity of underlying flow, and \( \kappa > 0 \) is an arbitrary real parameter. The system (1.1) is

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the generalization of the case $s = 1$, namely, $m = (1 - \partial^2_x)u = u - u_{xx}$ (see [29, 39] and [20]).

When $\alpha = 0, \rho \equiv 0$, the system (1.1) becomes a family of one-component equations

$$
\begin{align*}
&\left\{ \begin{array}{l}
m_t + um_x + au_x m = 0, \\
m(t, x) = (1 - \partial^2_x)s u(t, x), \\
u(t, x)|_{t=0} = u_0(x),
\end{array} \right. \\
&x \in \mathbb{R},
\end{align*}
$$

When $s = 1$, the equation (1.2) is called the $b$-equation. The $b$-equation possess a number of structural phenomena which are shared by solutions of the family of equations [36, 47, 48]. Recently, some authors were devoted to the study of the Cauchy problem for the $b$-equation. The local well-posedness of the $b$-equation was obtained by Escher and Yin in [36] and Gui, Liu and Tian in [46], respectively, on the line and Zhang and Yin in [70] on the circle. It also has global solutions [36, 46, 70] and solutions which blow up in finite time [36, 46, 70]. The uniqueness and existence of global weak solution to the $b$-equation provided the initial data satisfies certain sign conditions were obtained in [36, 70]. However, there are just two members of this family which are integrable [51]: the Camassa-Holm [5, 4] equation, when $a = 2$ (for more details of the integrability, scattering and inverse scattering problems of Camassa-Holm equation, the readers can refer to [9, 11, 18, 23]), and the Degasperis-Procesi [20] [27] equation, when $a = 3$. The Cauchy problem and initial-boundary value problem for the Camassa-Holm equation have been studied extensively [14, 15, 26, 37, 38, 61, 65]. It has been shown that this equation is locally well-posed [14, 15, 26, 38, 61] for initial data $u_0 \in H^q(\mathbb{R}), q > \frac{3}{2}$. More interestingly, it has global strong solutions [10, 13, 14, 15] and also finite time blow-up solutions [10, 15, 13, 15, 17, 26, 53, 61]. On the other hand, it has global weak solutions in $H^1(\mathbb{R})$ [2, 3, 16, 24, 64]. Finite propagation speed and persistence properties for the Camassa-Holm equation have been studied in [12, 39], especially. After the Degasperis-Procesi equation was derived, many papers were devoted to its study, cf. [7, 30, 33, 34, 52, 56, 57, 59, 66, 67, 68, 69]. When $s = k \geq 2$, the equation (1.2) becomes higher-order $b$-equation. In [60], Mu et al. studied the local well-posedness and global solutions for (1.2) (under a scaling transformation) with $k = 2$ in Sobolev spaces. In [8], Coclite et al. considered the cases $a = 2, k \geq 2$, $m = (1 - \partial^2_x + \partial^4_x - ... + (-1)^k \partial^{2k}_x)u$ — the higher-order Camassa-Holm equations, which describe the exponential curves of the manifold of smooth orientation-preserving diffeomorphisms of the unit circle in the plane (see [21, 22] for more details of the geometric setting of the Camassa-Holm equation). They established the existence of the unique global weak solutions.

For $a = 2$ and $\alpha = 0$, the system (1.1) becomes the two-component Camassa-Holm equation. Several types of 2-component Camassa-Holm equations have been studied in [6, 19, 28, 31, 32, 40, 41, 44, 45, 51]. These works established the local well-posedness [19, 32, 40, 41], derived precise blow-up scenarios [32, 40], and proved that there exist strong solutions which blow up in finite time [19, 32, 41]. It also has global strong solutions [19, 41]. Moreover, it has global weak solutions [42, 43, 44, 45, 62, 63].

The system (1.1) with $s > 1$ was recently introduced by Escher and Lyons in [35]. It is the generalization of the same model (1.1) with $s = 1$ in [29]. In [29], the authors proved the local well-posedness of (1.1) with $s = 1$ by using a geometrical framework and they studied the blow-up scenarios and global strong solutions of (1.1) in the periodic case. In [39], Guan et al. studied the local well-posedness
of (1.1) on the line in supercritical Besov spaces, and several blow-up results and the persistence properties. In [50], He and Yin studied the local well-posedness of (1.1) with \( s = 1 \) in the critical Besov spaces on the line and the existence of analytic solutions of the system. In [35], for \( s > 1 \), by a geometric approach, the authors gave a blow-up criteria to ensure the geodesic completeness on the circle with \( s > \frac{3}{2} \), \( a = 2 \), \( \kappa \geq 0 \) for the \( C^\infty \) initial data.

However, the Cauchy problem for (1.1) with \( s > 1 \) on the line for more general initial data has not been studied yet. Also, the local well-posedness of (1.2) with general \( s > 1 \) in Besov spaces with low regularity have not been investigated yet.

In this paper, using the Littlewood-Paley theory and transport equation theory, we obtain the local well-posedness of both (1.1) and (1.2) for \( s > 1 \), with the initial data in certain Besov spaces. Generally speaking, if the initial data \((u_0, \rho_0)\) is of high regularity, i.e., \((u_0, \rho_0) \in B^{s+q_m}_{p,r} \times B^{q_m+1}_{p,r}\) (the definition and properties of Besov spaces will be presented in Section 2, for easier understanding, the readers can take \( p = r = 2 \) to obtain \( H^s = B^s_{2,2} \)), with \( q_m > \max(\frac{1}{2}, \frac{1}{p}) \), we can readily obtain the local well-posedness results of (1.1) by the Littlewood-Paley theory and transport equation theory (see Remark 3.2). However, we will show that \( u_0 \) does not need such high regularity. In these cases that \( u_0 \) is of regularity less than \( 2s \), we will face the difficulty of the loss of regularity. To overcome such difficulty, we introduce the function \( v = (1 - \partial_x^2)^{s-[s]}u \) (here \([s]\) denotes the integer part of \( s \)), and transform (1.1) or (1.2) into the forms that \( v, u \) are unknown functions, rather than \( m \). Then we can proceed the local well-posedness results in Besov spaces with lower regularity (see Theorem 3.3 and Theorem 3.11).

Besides, we will present the blow-up criteria of (1.1) and (1.2), for some special \( s \), or, for the initial data in Sobolev spaces with sufficiently high regularity (i.e., \( u_0 \in H^q \) with \( q \geq 2s \)). With the aid of these blow-up criteria, we can obtain several global existence results for (1.1) and (1.2). We point out here that when dealing with the second component of (1.1) by the classical Kato-Ponce inequality, it will arise the term \( \|\rho_x\|_{L^\infty} \). However, we can avoid this term by Lemma 2.8 which was introduced by Li and Yin in [55]. Namely, we can obtain the blow-up criteria only involved the term \( u_x \).

Our paper is organized as follows. In Section 2, we give some preliminaries which will be used in Section 3. In Section 3, we establish the local well-posedness of the Cauchy problem associated with (1.1) and with (1.2) in Besov spaces. In Section 4, we discuss the blow-up criteria and the global existence of strong solutions to (1.1) and (1.2).

**Notations.** In the following, we denote by \( A \lesssim B \) to simplify the writing \( A \leq CB \) for some generic constant \( C \) which may depend on some certain parameters independent of \( A \) and \( B \). Given a Banach space \( Z \), we denote its norm \( \|\cdot\|_Z \). Since all spaces of functions are over \( \mathbb{R} \), for simplicity, we drop \( \mathbb{R} \) in our notations of function spaces if there is no ambiguity.

2. Preliminaries. In this section, we will recall some facts on the Littlewood-Paley decomposition, the nonhomogeneous Besov spaces and their some useful properties. We will also recall the transport equation theory, which will be used in our work. For more details, the readers can refer to [1, 25].

**Proposition 2.1.** [1, 25] (Littlewood-Paley decomposition) There exists a couple of smooth functions \((\chi, \varphi)\) valued in \([0, 1]\), such that \(\chi\) is supported in the interval

\[ [0, 1] \]
Moreover, \( \forall \xi \in \mathbb{R}, \chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1 \), and \( \text{supp} \varphi(2^{-q}) \cap \text{supp} \varphi(2^{-q'}) = \emptyset, \text{ if } |q - q'| \geq 2 \), \( \text{supp} \chi(\cdot) \cap \text{supp} \varphi(2^{-q}) = \emptyset, \text{ if } q \geq 1 \).

Then for all \( u \in S' \), we can define the nonhomogeneous dyadic blocks as follows. Let \( \Delta_q u \triangleq 0, \text{ if } q \leq -2 \),

\[
\Delta_{-1} u \triangleq \chi(D)u = \mathcal{F}^{-1}(\chi \mathcal{F} u),
\]

\[
\Delta_q u \triangleq \varphi(2^{-q}D)u = \mathcal{F}^{-1}(\varphi(2^{-q}D) \mathcal{F} u), \text{ if } q \geq 0.
\]

Hence,

\[
u = \sum_{q \in \mathbb{Z}} \Delta_q u \text{ in } S'(\mathbb{R}),
\]

where the right-hand side is called the nonhomogeneous Littlewood-Paley decomposition of \( u \).

**Remark 2.2.** \([1, 25]\) (1) The low frequency cut-off operator \( S_q \) is defined by

\[
S_q u \triangleq \sum_{p=-1}^{q-1} \Delta_p u = \chi(2^{-q}D)u = \mathcal{F}^{-1}(\chi(2^{-q}\xi) \mathcal{F} u), \forall q \in \mathbb{N}.
\]

(2) The Littlewood-Paley decomposition is quasi-orthogonal in \( L^2 \) in the following sense:

\[
\Delta_p \Delta_q u \equiv 0, \text{ if } |p - q| \geq 2,
\]

\[
\Delta_q (S_{p-1} u \Delta_p v) \equiv 0, \text{ if } |p - q| \geq 5,
\]

for all \( u, v \in S'(\mathbb{R}) \).

(3) Thanks to Young’s inequality, we get

\[
\|\Delta_q u\|_{L^p}, \|S_q u\|_{L^p} \leq C\|u\|_{L^p}, \forall 1 \leq p \leq \infty,
\]

where \( C \) is a positive constant independent of \( q \).

**Definition 2.3.** \([1]\) (Besov spaces) Let \( s \in \mathbb{R}, 1 \leq p, r \leq \infty \). The nonhomogeneous Besov space \( B^s_{p,r}(\mathbb{R}) \) (\( B^s_{p,r} \) for short) is defined by

\[
B^s_{p,r}(\mathbb{R}) \triangleq \{ f \in S'(\mathbb{R}) : \| f \|_{B^s_{p,r}} < \infty \},
\]

where

\[
\| f \|_{B^s_{p,r}} \triangleq \| 2^{qs} \Delta_q f \|_{L^r(\mathbb{R})} = \left( \| 2^{qs} \| \Delta_q f \|_{L^r(\mathbb{R})} \right)_{q \geq 1} \|_r.
\]

If \( s = \infty, B^\infty_{p,r} \triangleq \bigcap_{s \in \mathbb{R}} B^s_{p,r}. \)

In the following lemma, we list some important properties of Besov spaces.
Lemma 2.4. [1] 25 [20] Suppose that \( s \in \mathbb{R}, 1 \leq p, r, p_i, r_i \leq \infty, i = 1, 2 \). We have
(1) Topological properties: \( B^s_{p,r} \) is a Banach space which is continuously embedded in \( S' \).
(2) Density: \( C^\infty_c \) is dense in \( B^s_{p,r} \) if and only if \( 1 \leq p, r < \infty \).
(3) Embedding: \( B^s_{p_1,r_1} \hookrightarrow B^s_{p_2,r_2} \) if \( p_1 \leq p_2 \) and \( r_1 \leq r_2 \),
\[
B^{s_1}_{p_1,r_1} \hookrightarrow B^{s_2}_{p_2,r_2} \quad \text{locally compact, if } s_1 < s_2.
\]
(4) Algebraic properties: \( \forall s > 0, B^s_{p,r} \cap L^\infty \) is an algebra. Moreover, \( B^s_{p,r} \hookrightarrow L^\infty \), provided \( s > \frac{1}{p} \) or \( s \geq \frac{1}{p} \) and \( r = 1 \), hence \( B^s_{p,r} \) is an algebra under such conditions.
(5) Complex interpolation:
\[
\|f\|_{B^{s_1}_{p_1,r_1} \cap B^{s_2}_{p_2,r_2}} \leq \|f\|_{B^{s_1}_{p_1,r_1}}^{1-\theta} \|f\|_{B^{s_1}_{p_2,r_2}}^{-\theta}, \quad \forall \ u \in B^{s_1}_{p_1} \cap B^{s_1}_{p_2}, \ \forall \ \theta \in [0, 1]. \quad (2.1)
\]
(6) Logarithm interpolation: there exists a constant \( C \) such that for all \( s \in \mathbb{R}, \varepsilon > 0 \), and \( 1 \leq p \leq \infty \). For any \( u \in B^\varepsilon_{p,\infty} \) and \( f \in B^{\infty}_{\infty,\infty} \), we have
\[
\|u\|_{B^\varepsilon_{p,1}} \leq C \left( 1 + \frac{\varepsilon}{\varepsilon} \right) \|u\|_{B^\varepsilon_{p,\infty}} \log \left( e + \frac{\|u\|_{B^\varepsilon_{p,\infty}}}{\|u\|_{B^\varepsilon_{p,\infty}}} \right), \quad (2.2)
\]
\[
\|f\|_{L^\infty} \leq C \left( 1 + \varepsilon \right) \|u\|_{B^\varepsilon_{\infty,\infty}} \log \left( e + \frac{\|f\|_{B^\varepsilon_{\infty,\infty}}}{\|f\|_{B^\varepsilon_{\infty,\infty}}} \right), \quad (2.3)
\]
(7) Fatou’s lemma: if \( \{u_n\}_{n \in \mathbb{N}} \) is bounded in \( B^s_{p,r} \) and \( u_n \to u \) in \( S' \), then \( u \in B^s_{p,r} \) and
\[
\|u\|_{B^s_{p,r}} \leq C \liminf_{n \to \infty} \|u_n\|_{B^s_{p,r}}.
\]
(8) Let \( m \in \mathbb{R} \) and \( f \) be an \( S^m \)-multiplier (i.e., \( f : \mathbb{R} \to \mathbb{R} \) is smooth and satisfies that \( \forall \ \alpha \in \mathbb{N}^n, \ \exists \ \text{a constant } C^\alpha \text{ s.t. } \|\partial^\alpha f\| \leq C^\alpha (1 + |\xi|) \|\xi\|^{m-|\alpha|} \) for all \( \xi \in \mathbb{R} \)). Then the operator \( f(D) \) is continuous from \( B^s_{p,r} \) to \( B^{s-m}_{p,r} \).

Lemma 2.5. [1] 25 [20] Product laws (Moser type inequalities).
(1) If \( s > 0, 1 \leq p, r \leq \infty \), \( u, v \in B^s_{p,r} \cap L^\infty \). then there exists a constant \( C = C(s) \) such that
\[
\|uv\|_{B^s_{p,r}} \leq C(\|u\|_{L^\infty} \|v\|_{B^s_{p,r}} + \|v\|_{L^\infty} \|u\|_{B^s_{p,r}}).
\]
(2) If \( 1 \leq p, r \leq \infty \), \( (s_1, s_2) \in \mathbb{R}^2 \) such that \( s_1 \leq s_2 \), \( s_2 > \frac{1}{p} \) or \( s_2 = \frac{1}{p} \) if \( r = 1 \), and \( s_1 + s_2 > \max(0, \frac{2}{p} - 1) \), then there exists a constant \( C = C(s_1, s_2, p, r) \) such that
\[
\|uv\|_{B^{s_1}_{p_1, r_1}} \leq C \|u\|_{B^{s_1}_{p_1, r_1}} \|v\|_{B^{s_2}_{p_2, r_2}}.
\]
(3) For any \( u \in B^{\frac{1}{p} - 1}_{p, \infty}(\mathbb{R}), v \in B^{\frac{1}{p}}_{p, 1}(\mathbb{R}), \) we have
\[
\|uv\|_{B^{\frac{1}{p} - 1}_{p, \infty}} \leq C \|u\|_{B^{\frac{1}{p} - 1}_{p, \infty}} \|v\|_{B^{\frac{1}{p}}_{p, 1}}.
\]

Proposition 2.6. [1] For all \( 1 \leq p, r \leq \infty \) and \( s \in \mathbb{R} \),
\[
\begin{align*}
B^s_{p,r} \times B^{-s}_{p',r'} & \rightarrow \mathbb{R} \quad \text{for } (u, \phi) \mapsto \sum_{|j-j'| \leq 1} \langle \Delta_j u, \Delta_j \phi \rangle
\end{align*}
\]
defines a continuous bilinear functional on $B_{p,r}^s \times B_{p,r}^{-s}$. Denote by $Q_{p,r}^{-s}$ the set of functions $\phi$ in $\mathcal{S}$ such that $\|\phi\|_{B_{p,r}^{-s}} \leq 1$. If $u$ is in $\mathcal{S}'$, then we have

$$\|u\|_{B_{p,r}^s} \leq C \sup_{\phi \in Q_{p,r}^{-s}} \langle u, \phi \rangle.$$  

Now we state some useful results in the transport equation theory, which are crucial to the proofs of our main theorems later.

**Lemma 2.7.** (A priori estimates in Besov spaces) Let $1 \leq p \leq p_1 \leq \infty, 1 \leq r \leq \infty$ and

$$\sigma \geq -\min\left(\frac{1}{p_1}, 1 - \frac{1}{p}\right)$$

with strict inequality if $r < \infty$.

Assume that $f_0 \in B_{p,r}^s$, $F \in L^1(0, T; B_{p,r}^s)$, and $\partial_x v$ belongs to $L^1(0, T; B_{p,r}^{-s})$ if $\sigma > 1 + \frac{1}{p_1}$ (or $\sigma = 1 + \frac{1}{p_1}$ and $r = 1$) or to $L^1(0, T; B_{p,r}^{-s} \cap L^\infty)$ otherwise. If $f \in L^\infty(0, T; B_{p,r}^s) \cap C([0, T]; \mathcal{S}')$ solves the following 1-D linear transport equation:

$$\begin{cases}
\partial_t f + v \partial_x f = F, \\
f|_{t=0} = f_0,
\end{cases}$$

then there exists a constant $C$ depending only on $p, r$ and $\sigma$, such that the following statements hold:

1. We can have

$$\|f(t)\|_{B_{p,r}^s} \leq \|f_0\|_{B_{p,r}^s} + \int_0^t \|F(\tau)\|_{B_{p,r}^s} d\tau + C \int_0^t \|V'(\tau)\|_{B_{p,r}^s} d\tau,$$

or hence,

$$\|f(t)\|_{B_{p,r}^s} \leq e^{CV(t)} \|f_0\|_{B_{p,r}^s} + \int_0^t e^{-CV(\tau)} \|F(\tau)\|_{B_{p,r}^s} d\tau,$$

with

$$V(t) = \begin{cases}
\int_0^t \|\partial_x v(\tau)\|_{B_{p,r}^s} \|F(\tau)\|_{B_{p,r}^s} d\tau, & \text{if } \sigma < 1 + \frac{1}{p_1}, \\
\int_0^t \|\partial_x v(\tau)\|_{B_{p,r}^s} \|F(\tau)\|_{B_{p,r}^s} d\tau, & \text{if } \sigma > 1 + \frac{1}{p_1} \text{ or } \{\sigma = 1 + \frac{1}{p_1} \text{ and } r = 1\}.
\end{cases}$$

2. If $r < \infty$, then $f \in C([0, T]; B_{p,r}^s)$. If $r = \infty$, then $f \in C([0, T]; B_{p,r}^s) \cap C_w([0, T]; B_{p,\infty}^s(\mathbb{R}))$ for all $\sigma' < \sigma$.

3. If $f = v$, then for all $\sigma > 0$, then the estimate (2.4) holds with $V_p(t) = \|\partial_x v(t)\|_{L^\infty}$.

**Lemma 2.8.** If $\sigma > 0$, then there exists a constant $C = C(p, r, \sigma)$ such that

$$\|f(t)\|_{B_{p,r}^s} \leq \|f_0\|_{B_{p,r}^s} + \int_0^t \|F(\tau)\|_{B_{p,r}^s} d\tau + C \int_0^t \|f(\tau)\|_{B_{p,r}^s} \|v_x\|_{L^\infty} + \|f(\tau)\|_{L^\infty} \|v_x\|_{B_{p,r}^s} d\tau.$$  

**Lemma 2.9.** For the solution $f \in L^\infty(0, T; B_{p,r}^{1+s}((\mathbb{R})))$ of (2.3) with the velocity $v \in L^1(0, T; B_{p,r}^{2+s}((\mathbb{R})))$, then initial data $f_0 \in B_{p,r}^{1+s}((\mathbb{R}))$ and $g \in L^1(0, T; B_{p,r}^{2+s}((\mathbb{R})))$, then
we then have,
\[ \|f(t)\|_{B^{1+\frac{1}{p}}_{p,r}} \leq e^{CV(t)}(\|f_0\|_{\dot{B}^{1+\frac{1}{p}}_{p,r}} + \int_0^t e^{-CV(\tau)}\|g(\tau)\|_{\dot{B}^{1+\frac{1}{p}}_{p,r}} \, d\tau), \]
with \( V(t) = \int_0^t \|v\|_{\dot{B}^{2+\frac{1}{p}}_{p,r}(\mathbb{R})} \, d\tau \) and \( C = C(p,r) \).

**Lemma 2.10.** \([\underline{23}]\) (Existence and uniqueness) Let \( p, r, \sigma, f_0 \) and \( F \) be as in the statement of Lemmas 2.7, 2.9. Assume that \( v \in L^p(0,T; B^{s}_{\infty,\infty}) \) for some \( \rho > 1 \) and \( M > 0 \), and \( \partial_x v \in L^1(0,T; \dot{B}^{s-1}_{p,r}) \) if \( \sigma > 1 + \frac{1}{p} \) or \( \sigma = 1 + \frac{1}{p} \) and \( r = 1 \), and \( \partial_x v \in L^1(0,T; \dot{B}^{2}_{p,r} \cap \dot{L}^\infty) \) if \( \sigma < 1 + \frac{1}{p} \). Then (T) has a unique solution \( f \in L^\infty(0,T; B^s_{p,r}) \cap \bigcap_{\sigma' < \sigma} C([0,T]; B^{s}_{p,r}) \) and the inequalities of Lemmas 2.7, 2.9 hold true. Moreover, if \( r < \infty \), then \( f \in C([0,T]; B^s_{p,r}) \).

3. **Local well-posedness.** In this section, we are going to study the local well-posedness of the system (1.1) and the equation (1.2). We set \( \Lambda > 0 \) to state the local well-posedness result of (1.2) with initial data in Besov spaces with lower regularity. Before that, we will firstly present a theorem.

**Definition 3.1.** Let \( T > 0 \), \( s \in \mathbb{R} \) and \( 1 \leq p \leq \infty \), \( 1 \leq r \leq \infty \). Set
\[ E^{s}_{p,r}(T) \triangleq \begin{cases} C([0,T]; \dot{B}^s_{p,r}) \cap \dot{C}^1([0,T]; \dot{B}^{s-1}_{p,r}), & \text{if } r < \infty, \\ C([0,T]; \dot{B}^s_{p,\infty}) \cap \dot{C}^{0,1}([0,T]; \dot{B}^{s-1}_{p,\infty}), & \text{if } r = \infty. \end{cases} \]

To begin with, we will give a remark, as follows, which states the local well-posedness of \((m, \rho)\) with initial data in Besov spaces with high regularity.

**Remark 3.2.** It is readily to see that the system (1.1) is local well-posed for \((m_0, \rho_0) \in \dot{B}^q_{p,r} \times \dot{B}^{q+1}_{p,r} \), with \( q_m > \max\{1, \frac{1}{p} \}, 1 \leq r \leq \infty \) or \( q_m = \frac{1}{p} \), \( 1 \leq p \leq 2 \), \( r = 1 \), and there is a positive \( T \) such that the solution \((m, \rho) \in E^q_{p,r}(T) \times E^q_{p,r}(T)\), and \((m, \rho)\) depends continuously on the initial data \((m_0, \rho_0)\).

In the above remark, since \( m_0 \in \dot{B}^q_{p,r} \), then \( u_0 \in \dot{B}^{q+2s}_{p,r} \). It is of too high the regularity, which is not natural, in compare with that the original Camassa-Holm equation only requires that \( u_0 \in B^{q}_{p,r} \), for \( q > 1 + \max\{1, \frac{1}{p} \} \) or \( q = 1 + \max\{1, \frac{1}{p} \} \) if \( r = 1 \). We are going to present the local well-posedness result of the system (1.1) in Besov spaces with lower regularity. Before that, we will firstly present a theorem to state the local well-posedness result of (1.2) with \( s > 1 \).

3.1. **Local well-posedness of the one-component equation (1.2).**

**Theorem 3.3.** Let \( k = [s] \), the integer part of \( s \), and \( \beta = s - k \in [0,1) \). Suppose \( 1 \leq p, r \leq \infty \), \( q \in \mathbb{R} \) and \((s,q,p,r)\) satisfies the condition
\[ q \begin{cases} > s + \max\{1, \frac{1}{2} + \beta, \frac{1}{p} + 1 - \beta\}, & \text{if } \beta = s - k \neq 0, \\ > s + \max\{\frac{1}{2}, \frac{1}{p}\}, & \text{if } s = [s], \end{cases} \]
\[ \text{or the critical condition} \]
\[ q = s + \frac{1}{p}, \quad \text{if } s = [s], \ 1 \leq p \leq 2, \text{ and } r = 1. \]

*Given the initial data \( u_0 \in \dot{B}^q_{p,r} \), then the equation (1.2) has the unique solution \( u \in E^q_{p,r}(T) \) for some positive \( T \), and \( u \) depends continuously on the initial data \( u_0 \).*
Moreover, suppose $T^*$ is the lifespan of the solution, then there exists a positive $c$, such that

$$ T^* \geq \frac{c}{\| u_0 \|_{B^s_{t,x}}} $$

We are going to prove Theorem 3.3 by several steps.

At first, we introduce $v$ such that $m = (1 - \partial^2_x)^k v = (1 - \partial^2_x)^s u$, hence $v = (1 - \partial^2_x)^\beta u$, thus we can compute the term $u_x(1 - \partial^2_x)^s m = u_x(1 - \partial^2_x)^k v$, which may give some information of the lost regularity by transposing the derivatives. From (1.2), we see, according to Leibniz’s formula,

$$(1 - \partial^2_x)^k v_t + (1 - \partial^2_x)^k (uv_x) = (1 - \partial^2_x)^k (u(v_x) - u(1 - \partial^2_x)^k v_x) = (1 - \partial^2_x)^k v$$

$$= \sum_{i=1}^{k} \left( \begin{array}{c} k \\ i \end{array} \right) (1)^{i} \sum_{l=0}^{2i} \left( \begin{array}{c} 2i \\ l \end{array} \right) \partial^l_x u \partial^{2i-l}_x v_x - au_x \sum_{i=0}^{k} \left( \begin{array}{c} k \\ i \end{array} \right) (-1)^i \partial^i_x v.$$ 

For simplicity, we omit the coefficients, and write,

$$v_t + uv_x \sim (1 - \partial^2_x)^{-k} \partial_x u(v + \partial^2_x v + \partial^4_x v + ... + \partial^{2k}_x v) +$$

$$+ \partial^3_x u(\partial^2_x v + \partial^4_x v + ... + \partial^{2k-1}_x v) + \partial^3_x u(\partial^2_x v + \partial^4_x v + ... + \partial^{2k-2}_x v) + ... + \partial^{2k}_x \partial_x v)$$

$$\sim (1 - \partial^2_x)^{-k} \partial_x u \partial^{2k-1}_x v + \partial^3_x u \partial^{2k-2}_x v + ... + \partial^{2k}_x \partial_x v + \text{lower order terms}.$$ 

Note that every term in the brackets is of odd order, and

$$u_x \partial^{2k}_x v = (u_x \partial^{2k-1}_x v)_x - \partial^{2k}_x u \partial^{2k-1}_x v$$

$$= (u_x \partial^{2k-2}_x v)_xx - 2(\partial^2_x u \partial^{2k-2}_x v)_x + \partial^3_x u \partial^{2k-2}_x v$$

$$= ...$$

$$\sim \partial^{2k-1}_x (u_x v_x) + \partial^{2k-2}_x (u_x v_x) + \partial^{2k-3}_x (u_x v_x v_x) + ... + \partial^{k+1}_x u \partial^k v.$$ 

Similarly, we can write the remain terms $\partial^2_x u \partial^{2k-1}_x v, ..., \partial^2_x u \partial_x v$, and so on, in the forms that the orders of $u$ and $v$ are as close as possible, but the orders of $u$ should not be less than that of $v$. Now, we can write (1.2) as

$$v_t + uv_x = G(u,v) \sim$$

$$\left\{ \begin{array}{ll}
(1 - \partial^2_x)^{-k} \partial^{2k-1}_x (u_x v_x) + \partial^{2k-2}_x (u_x v_x) + ... + \partial^{k+1}_x u \partial^k v + \text{l. o. t.} & \text{if } s \neq [s] = k, \\
(1 - \partial^2_x)^{-s} \partial^{s-1}_x (u^2_x) + \partial^{s-2}_x (u^2_x) + ... + \partial_x (\partial^s_x u^2) + \text{l. o. t.} & \text{if } s = [s],
\end{array} \right.$$ 

(3.3) 

where “l. o. t.” means “lower order terms”.

**Step 1. Constructing Approximate Solutions and Uniform Bounds.** Here we only consider the cases $s \neq [s]$, since the cases $s = [s]$ are similar. Starting from $u^0 := 0$, we define by induction a sequence $(u^n, v^n)_{n \in \mathbb{N}}$ with of smooth functions by solving the following linear system:

$$(T_n) \left\{ \begin{array}{l}
u^n = (1 - \partial^2_x)^\beta u^n, \\
\partial_t v^{n+1} + u^n \partial_x v^{n+1} = G(u^n, v^n), \\
v^{n+1}|_{t=0} = (S_{n+1} + v_0)(x) = (1 - \partial^2_x)^\beta S_{n+1} u_0(x).
\end{array} \right.$$
Since $p$ satisfies the condition (3.1), and $\|u\|_{B^s_{p,r}} \approx \|v\|_{B^s_{p,r}}$. Assume $v^n \in E^{q-2\beta}_{p,r}(T)$ for all positive $T$ (the validity for $n = 0, n = 1$ is obvious). According to Lemma 2.7 we have

$$\|v^{n+1}\|_{B^{q-2\beta}_{p,r}} \leq e^{CU(t)} C(\|v_0\|_{B^{q-2\beta}_{p,r}} + \int_0^t e^{-CU(\tau)} \|G(u^n, v^n)\|_{B^{q-2\beta}_{p,r}} d\tau),$$

(3.4)

Here $U(t) = \int_0^t \|u^n(t')\|_{B^{q-2\beta}_{p,r}} dt' \leq C \int_0^t \|v^n(t')\|_{B^{q-2\beta}_{p,r}} dt'$. Now, we are going to control $\|G(u^n, v^n)\|_{B^{q-2\beta}_{p,r}}$. For simplicity, we only estimate the first and the last highest order term of $G(u^n, v^n)$. In fact, the lower order terms can be controlled more easily. We can see

$$\|(1 - \partial_x^k) (u^n v^n)\|_{B^{q-2\beta}_{p,r}} \lesssim \|u^n v^n\|_{B^{q-2\beta}_{p,r}} \lesssim \|u^n\|_{B^{q-2\beta}_{p,r}} \lesssim \|v^n\|_{B^{q-2\beta}_{p,r}} \lesssim \|v^n\|_{B^{q-2\beta}_{p,r}}^2,$$

which is a Cauchy sequence of $(u^n)$. In summary, we have

$$\|G(u^n, v^n)\|_{B^{q-2\beta}_{p,r}} \leq C \|v^n\|_{B^{q-2\beta}_{p,r}}^2.$$

Substituting it into (3.4) to yield

$$\|v^{n+1}(t)\|_{B^{q-2\beta}_{p,r}} \leq e^{C \int_0^t \|v^n(t')\|_{B^{q-2\beta}_{p,r}} dt'} C(\|v_0\|_{B^{q-2\beta}_{p,r}} + \int_0^t e^{-C \int_0^\tau \|v^n(t')\|_{B^{q-2\beta}_{p,r}} dt'} \|v^n(\tau)\|_{B^{q-2\beta}_{p,r}}^2 d\tau).$$

We may assume $C \geq 1$ and fix a $T > 0$ such that

$$2C^2 \|v_0\|_{B^{q-2\beta}_{p,r}} T < 1.$$

(3.5)

By induction, we can see that, for all $t \in [0, T]$ and $n \in \mathbb{N}$,

$$\|v^n(t)\|_{B^{q-2\beta}_{p,r}} \leq \frac{C \|v_0\|_{B^{q-2\beta}_{p,r}}}{1 - 2C^2 \|v_0\|_{B^{q-2\beta}_{p,r}} T} \leq \frac{C \|v_0\|_{B^{q-2\beta}_{p,r}}}{1 - 2C^2 \|v_0\|_{B^{q-2\beta}_{p,r}} T},$$

(3.6)

Therefore, $(v^n)_{n \in \mathbb{N}}$ is uniformly bounded in $L^\infty(0, T; B^{q-2\beta}_{p,r})$. Returning to the system $(T_n)$, we can conclude that $(v^n)_{n \in \mathbb{N}}$ is uniformly bounded in $E^{q-2\beta}_{p,r}(T)$.

**Step 2. Convergence and Uniqueness.** Now we are going to show that $\{u^n\}_{n \in \mathbb{N}}$ is a Cauchy sequence of:

**Case (1).** $C([0, T]; B^{q-\beta}_{p,r})$ if $(p, q, r, s)$ satisfies the condition (3.1),

**Case (2).** $C([0, T]; B^{q-\beta}_{p,r})$ if $(p, q, r, s)$ satisfies the condition (3.2).

Now, for all $(n, l) \in \mathbb{N}^2$, taking the difference between the systems $(T_{n+l})$ and $(T_n)$ and denoting $w^{n,l} := v^{n+l} - v^n$, we have

$$(\partial_t + u^{n+l} \partial_x) w^{n+l} = (u^n - u^{n+l}) v^{n+l} + G(u^{n+l}, v^{n+l}) - G(u^n, v^n),$$

(3.7)
with \( u^{n+1,l}(t,x)|_{t=0} = u_0^{n+1,l}(x) = (S_{n+l+1} - S_{n+l})v_0(x) = \sum_{j'=n+1}^{n+l} \Delta_j v_0(x) \),

**Case (1).** Here we only consider the cases that \((p,q,r,s)\) satisfies the condition [3.1] and \( s \neq [s] \), since the cases \( s = [s] \) are similar. Thus, we need to prove that \( \{u^n\}_{n\in\mathbb{N}} \) is a Cauchy sequence of \( C([0,T]; B^{q,r-2\beta-1}_{p,s}) \). Convergence.

By Lemmas 2.7.2.9, we have, for every \( t \in [0,T] \),

\[
\|u^{n+1,l}(t)\|_{B^{q,r-2\beta-1}_{p,s}} \leq C^{n+1}(t) \left( \|u_0^{n+1,l}\|_{B^{q,r-2\beta-1}_{p,s}} + C \int_0^t e^{-C^{n+1}(\tau)} \left( \|u^n-u^{n+1}\|_{B^{q,r-2\beta-1}_{p,s}} + G(u^{n+1}, v^{n+1}) - G(u^n, v^n) \right) d\tau \right).
\]

Since \( q - 1 > \frac{1}{p} \) and \((q-1)+ (q-2\beta-1) = 2q - 2\beta - 2 \geq 2(s + \beta + \frac{1}{p} - \beta - 1) > \max(0, \frac{2}{p} - 1) \), applying Lemma 2.5, we see,

\[
\|u^n - u^{n+1}\|_{B^{q,r-2\beta-1}_{p,s}} \leq C\|u^n - u^{n+1}\|_{B^{q,r-2\beta-1}_{p,s}} \leq C\|v^{n+1} - v^n\|_{B^{q,r-2\beta-1}_{p,s}} \leq C\|v^{n+1} - v^n\|_{B^{q,r-2\beta-1}_{p,s}}
\]

On the other hand, we need to control the term \( G(u^{n+1}, v^{n+1}) - G(u^n, v^n) \). As before, for simplicity, we only estimate the first and the last term of the highest order. At first, we can see

\[
\|u^n - u^{n+1}\|_{B^{q,r-2\beta-1}_{p,s}} \leq \|u^n - u^{n+1}\|_{B^{q,r-2\beta-1}_{p,s}} \leq \|v^{n+1} - v^n\|_{B^{q,r-2\beta-1}_{p,s}} \leq \|v^{n+1} - v^n\|_{B^{q,r-2\beta-1}_{p,s}}
\]

because \( q - 1 > q - 2\beta - 1 > \frac{1}{p} \), \( q - 1 > \frac{1}{p} \) and \( 2q - 2\beta - 3 > \max(0, \frac{2}{p} - 1) \). Also, we can obtain,

\[
\|u^n - u^{n+1}\|_{B^{q,r-2\beta-1}_{p,s}} \leq \|u^n - u^{n+1}\|_{B^{q,r-2\beta-1}_{p,s}} \leq \|v^{n+1} - v^n\|_{B^{q,r-2\beta-1}_{p,s}} \leq \|v^{n+1} - v^n\|_{B^{q,r-2\beta-1}_{p,s}}
\]

because \( q - 1 > q - 2\beta - 1 > \frac{1}{p} \), \( q - k - 2\beta > \frac{1}{p} \), \( q - k - 1 = q - (s - \beta) - 1 = q - s - (1 - \beta) > \frac{1}{p} \), and \( 2q - 2k - 2\beta - 1 = 2(q - s - \frac{1}{2}) > \max(0, \frac{2}{p} - 1) \).

Hence, we can estimate \( G(u^{n+1}, v^{n+1}) - G(u^n, v^n) \) by adding and subtracting terms, for an example,

\[
\|u^{n+1} - u^n\|_{B^{q,r-2\beta-1}_{p,s}} \leq \|u^{n+1} - u^n\|_{B^{q,r-2\beta-1}_{p,s}} \leq \|v^{n+1} - v^n\|_{B^{q,r-2\beta-1}_{p,s}} \leq \|v^{n+1} - v^n\|_{B^{q,r-2\beta-1}_{p,s}}
\]

The other terms can be controlled similarly. All in all,

\[
\|G(u^{n+1}, v^{n+1}) - G(u^n, v^n)\|_{B^{q,r-2\beta-1}_{p,s}} \leq \|u^{n+1} - u^n\|_{B^{q,r-2\beta-1}_{p,s}} + \|v^{n+1} - v^n\|_{B^{q,r-2\beta-1}_{p,s}} \leq \|u^{n+1} - u^n\|_{B^{q,r-2\beta-1}_{p,s}} + \|v^{n+1} - v^n\|_{B^{q,r-2\beta-1}_{p,s}}.
\]

Now, we turn to the initial data,

\[
\|u_0^{n+1,l}\|_{B^{q,r-2\beta-1}_{p,s}} = \|\sum_{j'=n+1}^{n+l} \Delta_j v_0\|_{B^{q,r-2\beta-1}_{p,s}} \leq \left( \sum_{j'=n+1}^{n+l} \Delta_j v_0 \right)^{\frac{1}{p}}
\]
By Fatou’s lemma, we see that
\[ sacrifice \]
\[ \lim_{t \to \infty} \mathbb{E}[\Delta_j v_0(t)]_{L^p} \]
uniformly for \( j \). Therefore, \( \{v^n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( C([0, T], B_{p,r}^{q-2\beta}) \). Hence, \( v^n \to v \) in \( C([0, T]; B_{p,r}^{q-2\beta-1}) \).

By Fatou’s lemma, we see that \( v \in L^\infty([0, T]; B_{p,r}^{q-2\beta}) \) with
\[ ||v||_{L^\infty([0, T]; B_{p,r}^{q-2\beta})} \leq C \lim_{n \to \infty} ||v^n||_{L^\infty([0, T]; B_{p,r}^{q-2\beta})} \leq \frac{C^2||v_0||_{B_{p,r}^{q-2\beta}}}{1 - 2C^2||v_0||_{B_{p,r}^{q-2\beta}}}. \]
Now, taking the action with any test function \( \phi \in C((0, T); S) \) in the system \( (T_n) \), applying Proposition 2.6 and taking the limits, it is easy to examine that \( v \) is a solution to (3.3). Then according to Lemma 2.10, \( v \in C([0, T]; B_{p,r}^{q-2\beta}) \). Returning to (3.3) again, we see \( v \in C([0, T]; B_{p,r}^{q-2\beta-1}) \). We then conclude that \( v \in E_{p,r}^{q-2\beta}(T) \).

Uniqueness.

Set \( w = v_1 - v_2 \), where \( v_i (i = 1, 2) \) is the solution to (3.3) with the initial data \( v_{i0} \in B_{p,r}^{q-2\beta} \). Along the similar computations as previous, we can get the following estimates which gives the uniqueness:
\[ ||(v_1 - v_2)(t)||_{B_{p,r}^{q-2\beta-1}} \leq ||v_{10} - v_{20}||_{B_{p,r}^{q-2\beta-1}} \exp \left( C \int_0^t (||v_1||_{B_{p,r}^{q-2\beta}} + ||v_2||_{B_{p,r}^{q-2\beta}}) dt \right). \]

Case (2). This is the critical case of the integer \( s \). Applying Lemmas 2.7, 2.9 to (3.7) in the space \( B_{p,\infty}^{q-1} = B_{p,\infty}^{q+1} \), we obtain, for any \( t \in [0, T] \),
\[ ||u^{n+l+1} - u^n||_{B_{p,\infty}^{q+1}} \leq C e^{CU^{n+l+1}(t)} \left( ||u_0^{n+l+1} - u_0^n||_{B_{p,\infty}^{q-1}} + C \int_0^t e^{-CU^{n+l+1}(t')} (||u^n - u^{n+l}||_{B_{p,\infty}^{q-1}} + G(u^n, u^{n+l}) - G(u^{n+l}, u^{n+l})) dt' \right). \]
with \( U^n(t) = \int_0^t \| u^n(t') \|_{B_{p,r}^q} \, dt' = \int_0^t \| u^n(t') \|_{B_{p,\frac{s+1}{p}+\frac{1}{r}}} \, dt' \). For simplicity, we only estimate the following worst term, other terms are simpler. We see

\[
\| (1 - \partial_x^s - s \partial_x \partial_x^t (u^{n+1} + u^n) \partial_x^t (u^{n+1} - u^n)) \|_{B_{p,\frac{s+1}{p}+1}} \leq C \| \partial_x^s (u^{n+1} + u^n) \partial_x^t (u^{n+1} - u^n) \|_{B_{p,\frac{s+1}{p}+1}} \\
\leq C \| \partial_x^s (u^{n+1} + u^n) \partial_x^t (u^{n+1} - u^n) \|_{B_{p,\frac{s+1}{p}+1}} \\
\leq C \| \partial_x^s (u^{n+1} + u^n) \|_{B_{p,\frac{s+1}{p}+1}} \| \partial_x^t (u^{n+1} - u^n) \|_{B_{p,\frac{s+1}{p}+1}} \\
\leq C \| u^{n+1} + u^n \|_{B_{p,\frac{s+1}{p}+1}} \| u^{n+1} - u^n \|_{B_{p,\frac{s+1}{p}+1}}.
\]

On the other hand, since \( B_{p,\frac{s+1}{p}+1} \) is an algebra, we get

\[
\| (u^n - u^{n+1}) u_x \|_{B_{p,\frac{s+1}{p}+1}} \leq C \| u^n - u^{n+1} \|_{B_{p,\frac{s+1}{p}+1}} \| u_x \|_{B_{p,\frac{s+1}{p}+1}} \\
\leq C \| u^n - u^{n+1} \|_{B_{p,\frac{s+1}{p}+1}} \| u^{n+1} \|_{B_{p,\frac{s+1}{p}+1}}.
\]

The initial data can be estimated similarly as the non-critical cases

\[
\| u_0^{n+1} - u_0^n \|_{B_{p,\frac{s+1}{p}+1}} \leq \| u_0^{n+1} - u_0^n \|_{B_{p,\frac{s+1}{p}+1}} \leq C 2^{-n} \| u_0 \|_{B_{p,\frac{s+1}{p}+1}}.
\]

We finally have

\[
\| (u^{n+1} - u^n) (t) \|_{B_{p,\frac{s+1}{p}+1}} \leq C T \left( 2^{-n} + \int_0^t \| (u^{n+1} - u^n) (t') \|_{B_{p,\frac{s+1}{p}+1}} \, dt' \right).
\]

Similar to the statement of non-critical cases, we obtain that \( \{ u^n \}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( C([0, T], B_{p,\frac{s+1}{p}+1}) \),

\[
u^n \to u \text{ in } C([0, T]; B_{p,\frac{s+1}{p}+1}),
\]

\[
\| u \|_{L^\infty([0, T]; B_{p,1}^{s+\frac{1}{p}})} \leq C \liminf_{n \to \infty} \| u^n \|_{L^\infty([0, T]; B_{p,1}^{s+\frac{1}{p}})} \leq \frac{C^2 \| u_0 \|_{B_{p,1}^{s+\frac{1}{p}}}}{1 - 2C^2 \| u_0 \|_{B_{p,1}^{s+\frac{1}{p}}}},
\]

and \( u \) is the solution to (3.3) with \( u \in E_{p,r}^{q+\frac{1}{p}}(T) \).

Let \( u_1, u_2 \) are the solutions to (3.3) with initial data \( u_{10}, u_{20} \) respectively and \( u_{10}, u_{20} \in B_{p,1}^{s+\frac{1}{p}} \). We can get the following estimate which leads to uniqueness

\[
\| (u_1 - u_2) (t) \|_{B_{p,\frac{s+1}{p}+1}} \leq \| u_{10} - u_{20} \|_{B_{p,\frac{s+1}{p}+1}} \exp \left( C \int_0^t \left( \| u_1 \|_{B_{p,1}^{s+\frac{1}{p}}} + \| u_2 \|_{B_{p,1}^{s+\frac{1}{p}}} \right) \, dt \right). \tag{3.9}
\]

**Remark 3.4.** Different from the critical case of \( s = 1 \), here we do not need to apply the logarithm interpolation and Osgood’s Lemma, this is because \( B_{p,\frac{s+1}{p}+1} \) is a Banach algebra, but \( B_{p,\frac{s+1}{p}+1} \) is not.

**Proposition 3.5.** If \( u_0' \) belongs to a small neighbourhood of \( u_0 \) in \( B_{p,r}^q \), then we can proceed the existence of the unique solution \( u \in E_{p,r}^{q+\frac{1}{p}}(T) \) of (3.3) with initial data \( u_0' \) for a certain positive \( T \). The estimates (3.8) and (3.9), with the interpolations
(2.1) and (2.2) in Lemma 2.4 ensures the Hölder continuity of the flow map from the initial data spaces $B^q_{p,r}(T)$ to the space $B^q'_{p,r}(T)$, for any $q < q$.

**Remark 3.6.** From Steps 1-2, we see that, when $s \neq [s]$, to deal with the term $(1-\partial_x^s)^{-k}(\partial_x^{s+1}u\partial_x^s v)$, we need the additional regularity, namely, $q > s + \max(1, \frac{1}{p} + \beta, \frac{1}{p} + \beta)$. This means that there defects the regularity at least of $\frac{1}{2}$ when $s$ in not an integer. Denote $F \in L^q_{p,r}$ for $q > s$

\begin{equation}
\|F(u,v)\|_{B^q_{p,r}} \lesssim \|u\|_{B^q_{p,r}},
\end{equation}

(3.10)

\begin{equation}
\|F(u^n, u^n) - F(u^n, u^n)\|_{B^{q-1}_{p,r}} \lesssim (\|u^n\|_{B^q_{p,r}} + \|u^n\|_{B^q_{p,r}})\|u^n - u^n\|_{B^{q-1}_{p,r}},
\end{equation}

(3.11)

for $q > s + \max(\frac{1}{2}, \frac{1}{p})$ (or with “=” if $r = 1$), then we do not need the additional regularity. We know that (3.10) and (3.11) hold true for $s$ is an integer and $s \geq 1$. Here we conjecture that (3.10) and (3.11) hold true for any $s > 1$.

**Step 3.** Continuity with respect to the initial data. To gain the continuity with respect to the initial data, we introduce the following lemma.

**Lemma 3.7.** Let $1 \leq p \leq \infty, 1 \leq r < \infty, \sigma > 1 + \frac{1}{p}$ (or $\sigma = 1 + \frac{1}{p}, r = 1, 1 \leq p < \infty$). Denote $\bar{N} = N \cup \{\infty\}$. Let $\{v^n\}_{n \in \bar{N}} \subset C([0,T]; B^{q-1}_{p,r})$. Assume that $v^n$ is the solution to

\begin{equation}
\begin{cases}
\partial_t v^n + a^n \partial_x v^n = f, \\
v^n|_{t=0} = v_0,
\end{cases}
\end{equation}

with $v_0 \in B^{q-1}_{p,r}, f \in L^1(0,T; B^{q-1}_{p,r})$ and that, for some $\alpha \in L^1(0,T)$,

\[\sup_{n \in \bar{N}} \|a^n\|_{B^q_{p,r}} \leq \alpha(t).\]

If $a^n \to a^\infty$ in $L^1(0,T; B^{q-1}_{p,r})$ when $n \to \infty$, then $v^n \to v^\infty$ in $C([0,T]; B^{q-1}_{p,r})$ when $n \to \infty$.

As before, we only focus on the cases $s \neq [s]$. Now, suppose $v^n \in C([0,T]; B^{q-2\beta}_{p,r})$ is the solution to (3.3) with the initial data $v_0^\beta \in B^{q-2\beta}_{p,r}$, namely, for all $n \in \bar{N}$, we have

\begin{equation}
\begin{cases}
\partial_t v^n + a^n \partial_x v^n = G(u^n, v^n), \\
v^n(t, x)|_{t=0} = v_0^\beta(x).
\end{cases}
\end{equation}

To prove the continuity of in $C([0,T]; B^{q-2\beta}_{p,r})$ with respect to the initial data in $B^{q-2\beta}_{p,r}$ for $r < \infty$, we need to prove the following lemma.

**Lemma 3.8.** If $v_0^n \to v_0^\infty$ in $B^{q-2\beta}_{p,r}$ as $n \to \infty$, and $v^i$ is the solution to (3.3) with initial data $v_0^i (\forall i \in \bar{N})$, then $v^n \to v^\infty$ in $C([0,T]; B^{q-2\beta}_{p,r})$. Here the positive $T$ satisfies the condition similar to (3.3), namely

\[2C\sup_{n \in \bar{N}} \|v_0^n\|_{B^{q-2\beta}_{p,r}} T < 1.\]

**Proof.** By Proposition 3.5 we know $v^n \to v^\infty$ in $C([0,T]; B^{q-2\beta-1}_{p,r})$. To prove Lemma 3.8 noticing $\|f\|_{B^q_{p,r}} \leq \|\Delta f\|_{B^q_{p,r}} + \|(Id - \Delta f\|_{B^q_{p,r}} \lesssim \|f\|_{B^q_{p,r}} + \|\partial_x f\|_{B^q_{p,r}}$, we only need to prove

\[v^n_x \to v^\infty_x \text{ in } C([0,T]; B^{q-2\beta-1}_{p,r}).\]
We decompose $v^n_x$ into $v^n_x = y^n + z^n$ such that
\[
\begin{align*}
\partial_t y^n + u^n \partial_x y^n &= f^n, \\
\partial_t z^n + u^n \partial_x z^n &= f^n - f^n,
\end{align*}
\]
where $f^n = \partial_x (G(u^n, v^n)) - u^n x^n$. Hence $\{f^n\}_{n \in \mathbb{N}}$ is uniformly bounded in $B^{\gamma - 2\beta - 1}_{p,r}$ with
\[
\|f^n\|_{B^{\gamma - 2\beta - 1}_{p,r}} = \|\partial_x (G(u^n, v^n)) - u^n x^n\|_{B^{\gamma - 2\beta - 1}_{p,r}} \leq C\|v^n\|_{B^{\gamma - 2\beta}_{p,r}}^2.
\]
Let
\[
M = \frac{C^2 \sup_{n \in \mathbb{N}} \|v^n\|_{B^{\gamma - 2\beta}_{p,r}}}{1 - 2C^2 \sup_{n \in \mathbb{N}} \|v^n\|_{B^{\gamma - 2\beta}_{p,r}}}.
\]
Similar to the proof of the existence, we can obtain
\[
\|v^n(t)\|_{B^{\gamma - 2\beta - 1}_{p,r}} \leq M,
\]
Also, we can see $\|u^n\|_{B^{\gamma - 2\beta}_{p,r}} \leq C\|u^n\|_{B^{\gamma}_{p,r}} \leq C\|v^n\|_{B^{\gamma - 2\beta}_{p,r}} \leq CM$. At the same time, by Proposition 3.5
\[
v^n \rightarrow v^\infty \text{ in } C([0, T]; B^{\gamma - 2\beta - 1}_{p,r}) \text{ as } n \rightarrow \infty. \tag{3.12}
\]
Since $\nu - 1 < \nu$ and $B^{\gamma - 1}_{p,r} \hookrightarrow B^{\gamma - 2\beta - 1}_{p,r}$, we can proceed that $u^n$ tends to $u^\infty$ in $L^1(0, T; B^{\gamma - 2\beta - 1}_{p,r})$, which ensures that, with the application of Lemma 3.7
\[
y^n \rightarrow v^\infty \text{ in } C([0, T]; B^{\gamma - 2\beta - 1}_{p,r}) \text{ as } n \rightarrow \infty. \tag{3.13}
\]
To control $z^n$, we need to estimate
\[
\|f^n - f^\infty\|_{B^{\gamma - 2\beta - 1}_{p,r}} = \|\partial_x (G(u^n, v^n) - G(u^\infty, v^\infty)) + (u^n v^n - u^n v^n)\|_{B^{\gamma - 2\beta - 1}_{p,r}}.
\]
On the one hand, we can see,
\[
\|u^n v^n - u^n v^n\|_{B^{\gamma - 2\beta - 1}_{p,r}} \leq \|u^n - u^n\|_{B^{\gamma - 2\beta - 1}_{p,r}} \|v^n\|_{B^{\gamma - 2\beta - 1}_{p,r}} + \|u^n\|_{B^{\gamma - 2\beta - 1}_{p,r}} \|v^n - v^n\|_{B^{\gamma - 2\beta - 1}_{p,r}} \leq \|v^n\|_{B^{\gamma - 2\beta - 1}_{p,r}} + \|v^n - v^n\|_{B^{\gamma - 2\beta - 1}_{p,r}}. \tag{3.14}
\]
On the other hand, take the derivative in $G(u^n, v^n) - G(u^\infty, v^\infty)$ term by term. For an example,
\[
\partial_x \left( (1 - \partial^2_x)^{-k} (\partial^k_x u^n \partial^k_x v^n) - (1 - \partial^2_x)^{-k} (\partial^k_x u^\infty \partial^k_x v^\infty) \right)
\]
\[
= (1 - \partial^2_x)^{-k} \left( \partial^k_x u^n \partial^k_x v^n + \partial^k_x u^n \partial^k_x v^n - \partial^k_x u^\infty \partial^k_x v^\infty - \partial^k_x u^\infty \partial^k_x v^\infty \right),
\]
which ensures that $\partial_x (G(u^n, v^n) - G(u^\infty, v^\infty))$ has the similar estimates as in 3.14. Hence,
\[
\|f^n - f^\infty\|_{B^{\gamma - 2\beta - 1}_{p,r}} \leq C_T (\|v^n - v^n\|_{B^{\gamma - 2\beta - 1}_{p,r}} + \|v^n - v^n\|_{B^{\gamma - 2\beta - 1}_{p,r}}). \tag{3.15}
\]
Applying Lemmas 2.7 and 2.9 to the equation of $z^n$ in $B^{\gamma - 2\beta - 1}_{p,r}$, we have
\[
\|z^n\|_{B^{\gamma - 2\beta - 1}_{p,r}} \leq \|z^n\|_{B^{\gamma - 2\beta - 1}_{p,r}} + C \int_0^t \|v^n - v^n\|_{B^{\gamma - 2\beta - 1}_{p,r}} + \|v^n - v^n\|_{B^{\gamma - 2\beta - 1}_{p,r}} dt'.
\]
\[
\leq C_T (\|v^n - v^n\|_{B^{\gamma - 2\beta - 1}_{p,r}} + \|v^n - v^n\|_{B^{\gamma - 2\beta - 1}_{p,r}}) dt'.
\]
From (3.12) and (3.13), for any $\varepsilon > 0$, we can choose a sufficiently large $n$ such that
\[
\|y^n - v_x\|_{B_{p,r}^{-2\beta-1}} < \frac{\varepsilon}{2} \quad \text{and} \quad C_T\|v^n - v^\infty\|_{L^\infty([0,T];B_{p,r}^{-2\beta-1})} < \frac{\varepsilon}{2}.
\]
Noting that $v^n_x - v^\infty = z^n + y^n - v_x^\infty$, we can obtain, for every $t \in [0,T]$,
\[
\|(v^n_x - v^\infty)(t)\|_{B_{p,r}^{-2\beta-1}} \leq \varepsilon + C_T\left\|\int_0^t \|v^n_x - v^\infty\|_{B_{p,r}^{-2\beta-1}} \, dt\right\|.
\]
Thanks to Gronwall’s inequality, we get
\[
\|(v^n_x - v^\infty)(t)\|_{B_{p,r}^{-2\beta}} \leq \tilde{C}\left(\varepsilon + \|v^n_0 - v^\infty\|_{B_{p,r}^{-2\beta}}\right) \quad \text{for all } t \in [0,T],
\]
for some constant $\tilde{C} = \tilde{C}(s,q,r,M,T)$.

Hence we gain the continuity of in $C([0,T];B_{p,r}^{-2\beta})$ with respect to the initial data in $B_{p,r}^{-2\beta}$ for $r < \infty$. \hfill $\Box$

When $r = \infty$, by Proposition [3,5], we see that $\|v^n - v^\infty\|_{L^\infty([0,T];B_{p,r}^{-2\beta-1})}$ tends to zero as $n \to \infty$. Hence for fixed $\phi \in B_{p',1}^{-q(-2\beta)}$ (where $q' \geq 1$ is the number such that $\frac{1}{r} + \frac{1}{r'} = 1$), we have
\[
\langle v^n(t) - v^\infty(t), \phi \rangle = \langle S_j[v^n(t) - v^\infty(t)], \phi \rangle - \langle (\Id - S_j)[v^n(t) - v^\infty(t)], \phi \rangle
\]
\[
= \langle v^n - v^\infty(t), S_j \phi \rangle + \langle (v^n - v^\infty(t)), (\Id - S_j)\phi \rangle.
\]
Applying Lemma 2.4, we have
\[
\|\langle v^n(t) - v^\infty(t), (\Id - S_j)\phi \rangle\| \leq CM\|\phi - S_j\phi\|_{B_{p',1}^{-q(-2\beta)}}, \quad (3.16)
\]
and
\[
\|\langle v^n(t) - v^\infty(t), S_j \phi \rangle\| \leq CM\|v^n - v^\infty\|_{L^\infty([0,T];B_{p,r}^{-2\beta-1})}\|S_j \phi\|_{B_{p',1}^{-q(-2\beta)}}. \quad (3.17)
\]
Note that $\|\phi - S_j\phi\|_{B_{p',1}^{-q(-2\beta)}}$ tends to zero as $j \to \infty$ and $\|v^n - v^\infty\|_{L^\infty([0,T];B_{p,r}^{-2\beta-1})}$ tends to zero as $n \to \infty$. Then the right hand-side of (3.16) may be arbitrarily small for $j$ large enough. For such fixed $j$, we let $n$ tend to infinity so that the right hand-side of (3.17) tends to zero. Thus, we conclude that $(v^n(t) - v^\infty(t), \phi)$ tends to zero as $n \to \infty$ for the case $r = \infty$.

Hence, finally, according to above three steps, we finish the proof of Theorem [3,3]

**Remark 3.9.** For some special $a$, the result of Theorem [5,3] may not be the best. For example, when $s = 2$, the system (1.2) can be transformed to
\[
u_t + uu_x = (1 - \partial_x^2)^{-2}\left(\frac{5 - a}{2}(u^2)_{xxx} + \frac{3a - 5}{2}(u^2)_x + (a - 3)(u^2)_{xx} - \frac{a}{2}(u^2)_x\right).
\]
Letting $a = \frac{5}{4}$, we eliminate the worst term $(u^2_x)_x$ to obtain
\[
u_t + uu_x = (1 - \partial_x^2)^{-2}\left(\frac{5}{3}(u^2)_{xxx} - \frac{4}{3}(u^2)_x - \frac{5}{6}(u^2)_x\right). \quad (3.18)
\]
To ensure the local well-posedness result, we need to have
\[
\|(1 - \partial_x^2)^{-2}\partial_x^3(u^2_v - u^2_x)\|_{B_{p,r}^{-1}} \lesssim \|(u + v)_{x}(u - v)\|_{B_{p,r}^{-1}} \lesssim \|(u + v)_{x}\|_{B_{p,r}^{-1}}\|(u - v)_{x}\|_{B_{p,r}^{-1}}.
\]
Hence, according to Lemma 2.5 (2), we only need to ensure that $u_0 \in B^q_{p,r}$ with $q > 1 + \max(\frac{1}{2}, \frac{1}{p}) = (s - 1) + \max(\frac{1}{2}, \frac{1}{p})$. We can also establish the local well-posedness result of (3.18) for the critical case $r = 1, q = (s - 1) + \max(\frac{1}{2}, \frac{1}{p})$.

**Remark 3.10.** Note that $H^q = B^q_{2,2}$. By Theorem 3.3 we can get the local well-posedness result of (1.2), with the initial data $u_0 \in H^q$, and the solution $u \in C([0, T]; H^q) \cap C^1([0, T]; H^{q-1})$. Here $(q, s)$ satisfies

$$q > \begin{cases} \frac{1}{2} + \beta, & \text{if } \beta = s - [s] \neq 0, \\ s + \frac{1}{2}, & \text{if } s = [s]. \end{cases}$$

(3.19)

3.2. **Local well-posedness of the system (1.1).** As the one-component equation (1.2), we can write the two-component system (1.1) in the following form

$$\begin{cases} v_t + uv_x = G(u, v) + (1 - \partial_x^2)^{-k}(\alpha u_x - \kappa \rho \rho_x), \\ \rho_t + \rho u_x = -(a - 1)u_x \rho, \end{cases}$$

(3.20)

where $k = [s], v = (1 - \partial_x^2)^{|s|} u$, and $G(u, v)$ is defined as in (3.3). Now we can state the local well-posedness result of (3.20), as follows.

**Theorem 3.11.** Suppose $(p, q, r, s)$ satisfies the (3.1) or (3.2), and $q_1 \in \mathbb{R}$ satisfies

$$\frac{1}{2} < q_1 \leq q - 1 \leq q_1 + 2s - 2,$$

(3.21)

or

$$1 \leq p \leq 2, \ r = 1 \text{ and } \frac{1}{p} = q_1 \leq q - 1 \leq \frac{1}{p} + 2s - 2.$$  

(3.22)

Given the initial data $(u_0, \rho_0) \in B^q_{p,r} \times B^q_{p,r}$, then the system (3.20) has the unique solution $(u, \rho) \in E^q_{p,r}(T) \times E^q_{p,r}(T)$ for some positive $T$, and $(u, \rho)$ depends continuously on the initial data $(u_0, \rho_0)$. Moreover, suppose $T^*$ is the lifespan of the solution, then there exists a positive $c$, such that

$$T^* \geq \frac{c}{\|u_0\|_{B^q_{p,r}} + \|\rho_0\|_{B^q_{p,r}} + 1}.$$

**Remark 3.12.** The “1” in $\frac{c}{\|u_0\|_{B^q_{p,r}} + \|\rho_0\|_{B^q_{p,r}} + 1}$ comes from the term $(1 - \partial_x^2)^{-k}(\alpha u_x)$. 

**Proof of Theorem 3.11** For simplicity, here we only deal with the cases that $q$ satisfies (3.1) and $q_1$ satisfies (3.21). (The proof of the cases that $q_1$ satisfies (3.22) is similar to Case (2) in the proof of Theorem 3.3, so we omit it here.) Also, we only give here the core of the proof — the convergence of the approximated sequence. In fact, we can obtain the difference equation:

$$\begin{cases} (\partial_t + u^{n+l}\partial_x)(v^n + v^{n+l}) = (u^n - u^{n+l})v_{x}^{n+l} + G(u^n, v^n) + (1 - \partial_x^2)^{-k}\partial_x(\alpha(u^{n+l} - u^n) + \frac{1}{2}(\rho^{n+l} + \rho^n)(\rho^{n+l} - \rho^n)), \\ (\partial_t + u^{n+l}\partial_x)(\rho^n + \rho^{n+l}) = (u^n - u^{n+l})\rho_{x}^{n+l} + (a - 1)((\rho^n - \rho^{n+l})u_x^n + \rho^n u^n + u^{n+l} + u^n - u^{n+l})_{x}. \end{cases}$$

(3.23)

We have proven that

$$\|u^n - u^{n+l}\|_{B^q_{p,r}} + G(u^n, v^n) - G(u^{n+l}, v^{n+l})\|_{B^q_{p,r}} \leq C_T\|u^{n+l} - u^n\|_{B^q_{p,r}},$$

(1)

We only need to estimate the following terms:

$$\|(1 - \partial_x^2)^{-k}\partial_x(u^{n+l} - u^n)\|_{B^q_{p,r}} \leq C\|u^{n+l} - u^n\|_{B^q_{p,r}},$$

(2)

$$\|\partial_x(u^{n+l} - u^n)\|_{B^q_{p,r}} \leq C\|u^{n+l} - u^n\|_{B^q_{p,r}},$$

(3)
(2) \[ \| (1 - \partial_x^2)^{-k} \partial_x ((\rho^{n+1} + \rho^n)(\rho^{n+1} - \rho^n)) \|_{B^{q-2s,-1}_{p,r}} \]
\[ \leq C \| (\rho^{n+1} + \rho^n)(\rho^{n+1} - \rho^n) \|_{B^{q-2s}_{p,r}} \]
\[ \leq C \| (\rho^{n+1} + \rho^n)(\rho^{n+1} - \rho^n) \|_{B^{q}_{p,r}} \]
\[ \leq C \| \rho^{n+1} + \rho^n \|_{B^{q}_{p,r}} \| \rho^{n+1} - \rho^n \|_{B^{q}_{p,r}}, \]

(3) \[ \| (u^n - u^{n+l}) \|_{B^{q}_{p,r}} \]
\[ \leq C \| u^n - u^{n+l} \|_{B^{q}_{p,r}} \leq C \| u^n - u^{n+l} \|_{B^{q}_{p,r}}, \]

(4) \[ \| (\rho^n - \rho^{n+l}) u^n + \rho^{n+l} (u^n - u^{n+l}) \|_{B^{q}_{p,r}} \]
\[ \leq C \| (\rho^n - \rho^{n+l}) u^n + \rho^{n+l} (u^n - u^{n+l}) \|_{B^{q}_{p,r}}, \]
\[ \leq C \| (\rho^n - \rho^{n+l}) u^n + \rho^{n+l} (u^n - u^{n+l}) \|_{B^{q}_{p,r}}, \]

where we used the facts that \( q \) satisfies (3.1) and \( q \) satisfies (3.2).

Since \( \| v^n \|_{B^{q-2s,-1}_{p,r}} \approx \| u^n \|_{B^{q-2s}_{p,r}} \), according to the uniform boundedness of \( \{ u^n, \rho^n \}_{n \in \mathbb{N}} \)

in \( E^{q}_{p,r}(T) \times E^{q}_{p,r}(T) \), with the application of Lemmas 2.7-2.9, we have, for every \( t \in [0, T) \),

\[ \| (u^{n+l+1} - u^{n+1})(t) \|_{B^{q}_{p,r}} + \| (\rho^{n+l+1} - \rho^{n+1})(t) \|_{B^{q}_{p,r}} \]
\[ \leq C_T \left( \| (u^n - u^{n+l+1}) \|_{B^{q}_{p,r}} + \| (\rho^{n+l+1} - \rho^n) \|_{B^{q}_{p,r}} \right) + \int_0^t \left( \| (u^{n+l+1} - u^{n+1})(t') \|_{B^{q}_{p,r}} + \| (\rho^{n+l+1} - \rho^n)(t') \|_{B^{q}_{p,r}} \right) dt', \]

which implies the convergence of \( \{ u^n, \rho^n \}_{n \in \mathbb{N}} \) in \( C([0, T]; B^{q}_{p,r} \times B^{q}_{p,r}) \).

**Remark 3.13.** Set \( q_m := q - 2s \), the regularity index of \( m = (1 - \partial_x^2)^s u \). From the condition (3.21) or (3.22), we can have

\[ q_m + 1 \leq q_1 \leq q - 1, \quad \text{and} \quad q_1 > \max \left( \frac{1}{2}, \frac{1}{p} \right), \]
or

\[ r = 1, \quad 1 \leq p \leq 2, \quad q_m + 1 \leq q_1 = \frac{1}{p} \leq q - 1. \]

(1) When \( s = 1 \), then \( q_m + 1 = q - 2 + 1 = q - 1 \). Hence, in the case \( s = 1 \), \( q_1 \) must equals to \( q - 1 \). These are the local well-posedness results in [49] and [50].

(2) If \( q_m > \max \left( \frac{1}{2}, \frac{1}{p} \right) \) (or \( q_m = \frac{1}{p} \) if \( 1 \leq p \leq 2 \) and \( r = 1 \)), then we can choose \( q_1 = q_m + 1 \), and this is the case of Remark 3.2.

4. Blow-up criteria and global existence.

4.1. Blow-up criteria for (1.2). In this subsection, we will study several blow-up criteria for the equation (1.2).

**Lemma 4.1.** Suppose \( s = [s] = k > 1, q > k + \max \left\{ \frac{1}{2}, \frac{1}{q} \right\} \) (or \( q = k + \max \left\{ \frac{1}{2}, \frac{1}{q} \right\} \) if \( r = 1 \)). Assume \( u_0 \in B^{q}_{p,r} \). Let \( T^* \) be the lifespan of the solution \( u \in C([0, T^*]; B^{q}_{p,r}) \). If \( T^* \) is finite, then

\[ \int_0^{T^*} (\| \partial_x u \|_{L^\infty} + \| \partial_{xx} u \|_{L^\infty}) dt = \infty, \quad (4.1) \]

or equivalently,

\[ \limsup_{t \uparrow T^*} (\| u_x(t) \|_{L^\infty} + \| \partial_x^2 u(t) \|_{L^\infty}) = \infty. \quad (4.2) \]
Proof. Theorem 3.3 and the condition of $q$ ensure that $u \in C([0, T^*); B^q_{p, r})$. Since $s = k$ is an integer, we see that $v = u$ in (3.3):

$$
u_t + uu_x = G(u, u) \sim (1 - \partial^2_x)^{-k} \left( \partial^{2k-1}_x u^2 + \partial^{2k-3}_x u^2 + \ldots + \partial_x \left( (\partial^k_x u^2) + \ldots + \partial_x ((\partial^2_x u^2)\right).$$

(4.3)

Applying Lemma 2.7 (3) with $V(t) = \int_0^t \|u_x\|_{L^\infty} dt'$, we can obtain, for every $t \in [0, T^*)$,

$$\|u(t)\|_{B^q_{p, r}} \leq e^{CV(t)} \left( \|u_0\|_{B^q_{p, r}} + C \int_0^t e^{-CV(t')} \|G(u, u)\|_{B^q_{p, r}} dt' \right).$$

Since $q - 1 > q - 2 > \ldots > q - k > 0$, with the aid of Lemma 2.5

$$\| (1 - \partial^2_x)^{-k} \partial^{2k-1}_x u^2 \|_{B^q_{p, r}} \lesssim \| u_x \|_{L^\infty} \lesssim \| u \|_{B^q_{p, r}},$$

$$\| (1 - \partial^2_x)^{-k} \partial^{2k-3}_x u^2 \|_{B^q_{p, r}} \lesssim \| u_{xx} \|_{L^\infty} \| u \|_{B^q_{p, r}},$$

$$\| (1 - \partial^2_x)^{-k} \partial_x ((\partial^k_x u^2) \} \|_{B^q_{p, r}} \lesssim \| (\partial^k_x u^2) \|_{B^{q-k-1}_{p, r}} \lesssim \| \partial_x u \|_{L^\infty} \| u \|_{B^q_{p, r}}.$$  

Hence,

$$\|u(t)\|_{B^q_{p, r}} \leq e^{CV(t)} \left( \|u_0\|_{B^q_{p, r}} + C \int_0^t e^{-CV(t')} \|u\|_{L^\infty} dt' \right) \dots \left( \|u(t)\|_{L^\infty} \|u\|_{B^q_{p, r}} \right).$$

Letting $\|u\|_{C^k} := \|u\|_{L^\infty} + \|u_x\|_{L^\infty} + \ldots + \|\partial^k_x u\|_{L^\infty}$, then the Gronwall lemma leads to

$$\|u(t)\|_{B^q_{p, r}} \leq \|u_0\|_{B^q_{p, r}} \exp \left( C \int_0^t \|u(t')\|_{C^k} dt' \right).$$

(4.4)

Claim. If the pseudo-differential operator $P(D)$ is $S^{-\gamma}$ with $\gamma > 0$ and $f \in L^\infty$, then $\|P(D)f\|_{L^\infty} \lessapprox \|f\|_{L^\infty}$.

Indeed, according to the logarithm interpolation (2.3), we have,

$$\|P(D)f\|_{L^\infty} \leq \frac{C}{\gamma} \|P(D)f\|_{B^{\gamma}_{\infty, \infty}} \log \left( e + \frac{\|P(D)f\|_{B^{\gamma}_{\infty, \infty}}}{\|P(D)f\|_{B^{0}_{\infty, \infty}}} \right)$$

$$\lesssim \|f\|_{B^{\gamma}_{\infty, \infty}} \log \left( e + \frac{\|f\|_{L^\infty}}{\|f\|_{B^{\gamma}_{\infty, \infty}}} \right)$$

$$\lesssim \|f\|_{L^\infty},$$

here we used the facts that $L^\infty \hookrightarrow B^{0}_{\infty, \infty} \hookrightarrow B^{\gamma}_{\infty, \infty}$, and the function $x \mapsto x \log(e + \frac{x}{2})$ is increasing.

Now, applying the above Claim, we can see that,

$$\|G(u, u)\|_{L^\infty} \lessapprox \left( \|u_x\|_{L^\infty} + \ldots + \|\partial^k_x u\|_{L^\infty} \right) \|u\|_{C^k},$$

here we should keep in mind to write the last term $(1 - \partial^2_x)^{-k} \partial_x (u^2) = (1 - \partial^2_x)^{-k} (u_{xx})$. Applying the $L^\infty$ estimate for the transport equation (4.3), we have

$$\|u\|_{L^\infty} \leq \|u_0\|_{L^\infty} + C \int_0^t \left( \|u_x\|_{L^\infty} + \ldots + \|\partial^k_x u\|_{L^\infty} \right) \|u\|_{C^k} dt'.$$
Differentiating (4.3) once with respect to \( x \), we get
\[
 u_{xt} + uu_{xx} = -u_x^2 + \partial_x(G(u, u))
\]
\[
 \sim -u_x^2 + (1 - \partial_x^2)^{-k} \left( \partial_x^{2k}(u_x^2) + \partial_x^{2k-2}(u_{xx}) + \ldots + \partial_x^2[(\partial_x^k - 1)u]^2 + \ldots + \partial_x(uu_x) \right).
\]

Expand the highest order term:
\[
(1 - \partial_x^2)^{-k}(\partial_x^{2k}(u_x^2)) = (1 - \partial_x^2)^{-k}\left(-1\right)^k(1 - \partial_x^2)^k(u_x^2) - \sum_{i=0}^{k-1} \binom{k}{i} (-1)^i \partial_x^{2i}(u_x^2).
\]

Thus, applying the Claim again, we obtain
\[
\| - u_x^2 + \partial_x(G(u, u)) \|_{L^\infty} \lesssim (\| u_x \|_{L^\infty} + \ldots + \| \partial_x^k u \|_{L^\infty}) \| u \|_{C^k},
\]
which leads to
\[
\| u_x \|_{L^\infty} \leq \| u_0 \|_{L^\infty} + C \int_0^t (\| u_x \|_{L^\infty} + \ldots + \| \partial_x^k u \|_{L^\infty}) \| u \|_{C^k} dt'.
\]

Taking such procedure for \( k \) times, we get
\[
\| u \|_{C^k} \leq \| u_0 \|_{C^k} + C \int_0^t (\| u_x \|_{L^\infty} + \ldots + \| \partial_x^k u \|_{L^\infty}) \| u \|_{C^k} dt'.
\]

Gronwall’s inequality then leads to
\[
\| u \|_{C^k} \leq \| u_0 \|_{C^k} \exp \left( C \int_0^t (\| u_x \|_{L^\infty} + \ldots + \| \partial_x^k u \|_{L^\infty}) dt' \right).
\]

By Gagliardo-Nirenberg’s inequality and Young’s inequality,
\[
\| \partial_x^i u_x \|_{L^\infty} \lesssim \| u_x \|_{L^\infty} + \| \partial_x^k u \|_{L^\infty}, \quad (i = 1, 2, \ldots, k - 1).
\]

Hence
\[
\| u(t) \|_{C^k} \leq \| u_0 \|_{C^k} \exp \left( C \int_0^{T^*} (\| u_x \|_{L^\infty} + \| \partial_x^k u \|_{L^\infty}) dt' \right). \quad (4.5)
\]

Therefore, (4.4) and (4.5) lead to (4.1). The equivalence of (4.1) and (4.2) is obvious. \( \Box \)

If we focus on the Sobolev spaces we can get some exact blow-up criteria, as follows.

**Lemma 4.2.** If \( a = 2, s = [s] = k \geq 2, q > k + \frac{1}{2} \). Let \( T^* \) be the lifespan of the solution \( u \in C([0, T^*); H^q) \) to (1.1) with the initial data \( u_0 \in H^q \). If \( T^* \) is finite, then
\[
\limsup_{t \uparrow T^*} \| u_x(t) \|_{L^\infty} = \infty.
\]

**Proof.** The existence of \( u \in C([0, T^*); H^q) \) is ensured by Theorem 3.3.

Suppose \( |u_x(t, x)| \leq M \) for all \( (t, x) \in [0, T^*) \times \mathbb{R} \). Since
\[
\frac{d}{dt} \int mu = 2 \int um_x = -2 \int u(um_x + 2mu_x) = 0,
\]
we obtain, for all \( t \in [0, T^*) \),
\[
\| u \|^2_{H^s} \approx \int mu \approx \int m_0 u_0 \approx \| u_0 \|^2_{H^s} \lesssim \| u_0 \|^2_{H^s}, \quad (4.6)
\]

Therefore, (4.4) and (4.5) lead to (4.1). The equivalence of (4.1) and (4.2) is obvious. \( \Box \)
At first, we consider the cases that $q \geq k + 1$. Now we need to control $\|u\|_{H^{k+1}}$. Applying Lemma 2.7 (3) to (4.3) in $H^{k+1}$, we obtain, for every $t \in [0, T^*)$,

$$\|u(t)\|_{H^{k+1}} \lesssim \|u_0\|_{H^{k+1}} + \int_0^t \|G(u, u)\|_{H^{k+1}} dt',$$

here $G(u, u)$ is defined as in (4.3) with $a = 2$. As before, we only estimate of terms with the highest order. For $i = 1, 2, ..., k - 1$, $\|\partial^i_x u\|_{L^\infty} \lesssim \|u\|_{H^{i+1}} \lesssim \|u\|_{H^k} \approx \|u_0\|_{H^k} \lesssim \|u_0\|_{H^q}$,

$$\|(1 - \partial^2_x)^{-k} \partial_x^{2k-2i+1} ((\partial^i_x u)^2)\|_{H^{k+1}} \lesssim \|(\partial^i_x u)^2\|_{H^{k-i+1}} \lesssim \|(\partial^i_x u)^2\|_{L^2} \lesssim \|\partial^i_x u\|_{L^\infty} \|\partial^i_x u\|_{H^{k-i+1}} \lesssim \|u\|_{H^{k+1}},$$

and

$$\|(1 - \partial^2_x)^{-k} \partial_x^{2k-2i+1} ((\partial^i_x u)^2)\|_{H^q} \lesssim \|(\partial^i_x u)^2\|_{H^{q-i+1}} \lesssim \|\partial^i_x u\|_{L^\infty} \|\partial^i_x u\|_{H^q} \lesssim \|u\|_{H^q}.$$

All in all,

$$\|G(u, u)\|_{H^{k+1}} \lesssim \|u\|_{H^{k+1}}. \quad (4.7)$$

We then obtain

$$\|u(t)\|_{H^{k+1}} \lesssim \|u_0\|_{H^{k+1}} + \int_0^t \|u\|_{H^{k+1}} dt'.$$

Gronwall’s lemma leads to

$$\|u(t)\|_{H^{k+1}} \lesssim \|u_0\|_{H^{k+1}} \lesssim \|u_0\|_{H^q}.$$

Hence $\|\partial^k_x u\|_{L^\infty} \lesssim \|u_0\|_{H^{k+1}} \lesssim \|u_0\|_{H^q}$, it then follows, for $i = 1, 2, ..., k$,

$$\|(1 - \partial^2_x)^{-k} \partial_x^{2k-2i+1} ((\partial^i_x u)^2)\|_{H^q} \lesssim \|(\partial^i_x u)^2\|_{H^{q-i+1}} \lesssim \|\partial^i_x u\|_{L^\infty} \|\partial^i_x u\|_{H^q} \lesssim \|u\|_{H^q}.$$

Namely,

$$\|G(u, u)\|_{H^q} \lesssim \|u\|_{H^q}. \quad (4.8)$$

When $k + \frac{1}{2} < q < k + 1$, we will show that (4.8) still holds true. Indeed, for $i = 1, 2, ..., k - 1$, we have

$$\|\partial^i_x u\|_{L^\infty} \lesssim \|u\|_{H^{i+1}} \lesssim \|u\|_{H^k} \approx \|u_0\|_{H^k} \lesssim \|u_0\|_{H^q},$$

and

$$\|(1 - \partial^2_x)^{-k} \partial_x^{2k-2i+1} ((\partial^i_x u)^2)\|_{H^q} \lesssim \|(\partial^i_x u)^2\|_{H^{q-i+1}} \lesssim \|\partial^i_x u\|_{L^\infty} \|\partial^i_x u\|_{H^{q-i+1}} \lesssim \|u\|_{H^q}.$$

Note $q - k > \frac{1}{2}(q - k - 1) + (q - k) > 0 = \max(0, \frac{1}{2} - \frac{1}{2})$ and $q - 1 < k$. According to Lemma 2.7 (2), we obtain

$$\|(1 - \partial^2_x)^{-k} \partial_x ((\partial^i_x u)^2)\|_{H^q} \lesssim \|(\partial^i_x u)^2\|_{H^{q-2k-1}} \lesssim \|\partial^i_x u\|_{L^\infty} \|\partial^i_x u\|_{H^{q-2k-1}} \lesssim \|u\|_{H^q}.$$

Hence, (4.8) is still valid for $k + \frac{1}{2} < q < k + 1$. Taking the application of Lemma 2.7 (3) to (4.3) in $H^q$, we have, for every $t \in [0, T^*)$,

$$\|u(t)\|_{H^q} \lesssim \|u_0\|_{H^q} + \int_0^t \|G(u, u)\|_{H^q} dt' \lesssim \|u_0\|_{H^q} + \int_0^t \|u\|_{H^q} dt'.$$
which leads to \( \|u(t)\|_{H^s} \lesssim \|u_0\|_{H^s} \). Thus we can extend the solution \( u \) beyond \( T^* \), and this contradicts the maximality of \( T^* \). \hfill \Box

**Lemma 4.3.** If \( s = 2 \) or \( s = 3 \) and \( q > s + \frac{1}{2} \). Let \( T^* \) be the lifespan of the solution \( u \in C([0,T^*];H^q) \) to (1.2) with the initial data \( u_0 \in H^q \). If \( T^* \) is finite, then

\[
\limsup_{t \uparrow T^*} \sup_{x \in \mathbb{R}} (1 - a)u_x(t, x) = +\infty.
\]

**Proof.** The existence of \( u \in C([0,T^*];H^q) \) is ensured by Theorem 3.3. Suppose \( (2 - a)u_x(t, x) \leq M \) for all \( (t, x) \in [0,T^*) \times \mathbb{R} \). Note that

\[
\frac{d}{dt} \int mu = 2 \int um_t = -2 \int u(um_x + amu_x) = 2(2 - a) \int u_x mu = 2(2 - a) \int u_x u(1 - \partial_x^2)^s u.
\]

When \( s = 2 \), applying the integration by parts, we obtain

\[
\frac{d}{dt} \int mu = 2(2 - a) \int u_x (u - 2u_{xx} + \partial_x^4 u) = 2(2 - a) \int u_x (a^2 + \frac{5}{2}u_x^2) \lesssim \int mu.
\]

when \( s = 3 \), noting that \( \int u_x^2 u_{xxx} = 0 \), we can see

\[
\frac{d}{dt} \int mu = 2(2 - a) \left( \int u_x (a^2 + \frac{3}{2}u_x^2 + \frac{15}{2}u_{xx} + \frac{7}{2}u_{xxx}) + 3 \int u_x^2 u_{xxx} \right) \lesssim \int mu,
\]

which leads to \( \|u\|_{H^s}^2 \approx \int mu \lesssim \int m_0 u_0 \approx \|u_0\|_{H^s}^2 \). Hence \( \|u_x\|_{L^\infty} \leq C \|u\|_{H^s} \lesssim \|u_0\|_{H^s} \lesssim \|u_0\|_{H^s}^2 \). Thus by the similar procedure in the proof of Lemma 4.2 we can conclude Lemma 4.3. \hfill \Box

**Lemma 4.4.** If \( (q,s) \) satisfies the condition \( 3.19 \) and \( q \geq 2s \). Let \( T^* \) be the lifespan of the solution \( u \in C([0,T^*];H^q) \) to (1.2) with the initial data \( u_0 \in H^q \). If \( T^* \) is finite, then

\[
\limsup_{t \uparrow T^*} \sup_{x \in \mathbb{R}} (1 - 2a)u_x(t, x) = +\infty.
\]

**Proof.** Since \( (q,s) \) satisfies \( 3.19 \) and \( q \geq 2s \), Theorem 3.3 then ensures the existence of \( u \) and \( m \). Also, \( \int \int m^2(t, x) \, dx \) makes sense for all \( t \in [0,T^*) \). Suppose \( (1 - 2a)u_x(t, x) \leq M \) for all \( (t, x) \in [0,T^*) \times \mathbb{R} \), we have

\[
\frac{d}{dt} \int m^2 = 2 \int mm_t = -2 \int m(um_x + amu_x m) = \int (1 - 2a)u_x m^2 \leq M \int m^2.
\]

The Gronwall’s lemma leads to \( \|m\|_{L^2}^2 \leq e^{MT^*} \|m_0\|_{L^2}^2 \), and hence,

\[
\|u\|_{H^{2s}} \lesssim \|m\|_{L^2} \lesssim \|m_0\|_{L^2} \approx \|u_0\|_{H^{2s}} \approx \|u_0\|_{H^q}.
\]

If \( q_m := q - 2s = 0 \), we reach the desired result, hence we assume \( q_m > 0 \).

**Step 1.** If \( 0 < q_m \leq 1 \), By Lemma 2.7 (1), we see, for every \( t \in [0,T^*) \),

\[
\|m(t)\|_{H^{q_m}} \leq e^{\int_0^t \nu(t') \, dt'} \left( \|m_0\|_{H^{q_m}} + \int_0^t \|(-au_x m)\|_{H^{q_m}} \, dt' \right).
\]
Note that $U(t) = \|\partial_x u\|_{L^\infty} \lesssim \|\partial_x u\|_{H^1} \lesssim \|u\|_{H^2} \lesssim \|u\|_{H^{2s}} \lesssim \|u_0\|_{H^{2s}} \lesssim \|u_0\|_{H_2}$, and $\|au_x m\|_{H^{qm}} \lesssim \|u_2\|_{H^1} \|m\|_{H^{qm}} \lesssim \|u\|_{H^2} \|m\|_{H^{qm}} \lesssim \|m\|_{H^{qm}}$. Hence
\[
m(t)\|_{H^{qm}} \lesssim \|m_0\|_{H^{qm}} + \int_0^t \|m\|_{H^{qm}} \, dt'.
\]
By Gronwall's lemma, we obtain $\|m\|_{H^{qm}} \lesssim \|m_0\|_{H^{qm}}$, and reach the desired result.

**Step 2.** If $q_m > 1$, by Step 1, we know
\[
\|m(t)\|_{H^{qm}} \lesssim \|m_0\|_{H^{qm}}.
\]
According to Lemma 2.8, we obtain, for every $t \in [0, T^*)$,
\[
\|m(t)\|_{H^{qm}} \leq \|m_0\|_{H^{qm}} + C \int_0^t (\|(-au_x m)\|_{H^{qm}} + \|u_x\|_{H^{qm}} \|m\|_{L^\infty}) \, dt'.
\]
(4.9)

Note that
\[
\|(-au_x m)\|_{H^{qm}} + \|u_x\|_{H^{qm}} \|m\|_{L^\infty} \lesssim \|u_x\|_{L^\infty} \|m\|_{H^{qm}} + \|u_x\|_{H^{qm}} \|m\|_{L^\infty} \lesssim \|u_x\|_{L^\infty} \|m\|_{H^{qm}} + \|m\|_{H^{qm+1-2s}} \|m\|_{H^{1}} \lesssim \|m\|_{H^{qm}}.
\]
(4.10)

Plugging (4.10) into (4.9) to yield
\[
m(t)\|_{H^{qm}} \lesssim \|m_0\|_{H^{qm}} + \int_0^t \|m\|_{H^{qm}} \, dt',
\]
which leads to $\|m(t)\|_{H^{qm}} \lesssim \|m_0\|_{H^{qm}}$. This completes the proof of Lemma 4.4.

**4.2. Global existence for (1.2).** Now we can state the global existence results for (1.2).

**Theorem 4.5.** Let $a = 2, s = [s] \geq 2, q > s + \frac{1}{2}$, then the solution to (1.2) with initial data $u_0 \in H^q$ exists globally in time.

**Proof.** The existence of $u \in \mathcal{C}([0, T^*); H^q)$ is ensured by Theorem 3.3. For any $t \in [0, T^*)$, since $a = 2$, according to (4.6), we obtain $\|u\|_{H^s} \approx \|u_0\|_{H^s}$. Hence
\[
\|u_x(t)\|_{L^\infty} \lesssim \|u(t)\|_{H^2} \lesssim \|u(t)\|_{H^s} = \|u_0\|_{H^s} \lesssim \|u_0\|_{H^q}.
\]
According to the Lemma 4.2 we see $T^* = \infty$ and obtain the global existence result.

Similarly, we can have the following result.

**Theorem 4.6.** Suppose $s > \frac{3}{2}$, if $a = 2$ and $q \geq 2s$, then the solution to (1.2) with initial data $u_0 \in H^q$ exists globally in time.

**Proof.** Set $\beta = s - [s]$. Since $s > \frac{3}{2}$, thus $q > 2s > s + \max\left(\frac{3}{2} - \beta, \frac{1}{2} + \beta\right)$ and $q$ satisfies the condition (3.19). Hence the existence of $u \in \mathcal{C}([0, T^*); H^q)$ is ensured by Theorem 3.3. Note that $a = 2$ and $\|u\|_{H^s} \approx \int m = \int m_0 u_0 \approx \|u_0\|_{H^s}$, then
\[
\|u_x(t)\|_{L^\infty} \lesssim \|u(t)\|_{H^s} \approx \|u_0\|_{H^s} \lesssim \|u_0\|_{H^q}.
\]
According to the Lemma 4.4 we see $T^* = \infty$ and obtain the global existence result.

**Theorem 4.7.** If $a = \frac{1}{2}, q \geq 2s$ and $(q, s)$ satisfies the condition (3.14), then the solution to (1.2) with initial data $u_0 \in H^q$ exists globally in time.

**Proof.** Since $a = \frac{1}{2}$, hence $(1 - 2a)u_x \equiv 0$. According to Lemma 4.4 we obtain the global existence result.
4.3. Blow-up criteria for (1.1). To give the blow-up criteria for the two-component system (1.1), we firstly consider the following initial value problem.

\begin{align}
\frac{d}{dt} \psi(t,x) &= u(t, \psi(t,x)), \quad t \in [0,T), \\
\psi(0,x) &= x, \quad x \in \mathbb{R},
\end{align}

(4.11)

where $u$ denotes a sufficiently smooth (Lipschitz continuous in spatial variable) vector field. Applying classical results in the theory of ordinary differential equations, one can obtain the following result on $\psi$ which is crucial in the proof of our result.

Lemma 4.8. [10, 65] Let $u \in C([0,T); H^s) \cap C^1([0,T); H^{s-1})$, $\sigma > \frac{3}{2}$. Then (4.11) has a unique solution $\psi \in C^1([0,T) \times \mathbb{R})$. Moreover, the map $\psi(t, \cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with

$$
\psi_x(t,x) = \exp \left( \int_0^t u_x(s, \psi(s,x)) ds \right) > 0, \forall (t,x) \in [0,T) \times \mathbb{R}.
$$

Lemma 4.9. Let $\rho$ be the second component of the solution to system (1.1) corresponding to the initial data $(u_0, \rho_0) \in H^q \times H^{q_1}$ with $q$ satisfying (3.19) and $q_1$ satisfying (3.20). Then, for every $t \in [0,T^*)$,

$$
\rho(t, \psi(t,x)) = \rho_0(x) e^{\int_0^t (1-a)u_x(t', \psi(t',x)) dt'}.
$$

(4.12)

Proof. Solving the equation $\rho_t + u_x \rho_x = (1-a)u_x \rho$ along the characteristic $\psi$, namely,

$$
\frac{d}{dt} \left( \rho(t, \psi(t,x)) \right) = (1-a)(u_x \rho)(t, \psi(t,x)),
$$

thus,

$$
\rho(t, \psi(t,x)) = \rho_0(x) + \int_0^t (1-a)u_x(t', \psi(t',x)) \rho(t', \psi(t',x)) dt',
$$

which leads to (4.12).

Now we can state other blow-up criterion in certain Sobolev spaces for the two-component system (1.1).

Lemma 4.10. Suppose $a = 2, s = [s] = k \geq 2, q > s + \frac{1}{2}$. Let $T^*$ be the lifespan of the solution $(u, \rho) \in C([0,T^*); H^q \times H^{q_1})$ to (1.1) with the initial data $(u_0, \rho_0) \in H^q \times H^{q_1}$. If $T^*$ is finite, then

$$
\limsup_{t \uparrow T^*} \| u_x(t) \|_{L^{\infty}} = \infty.
$$

Proof. Since $s = [s] = k$, we can write the system (3.20) in the form

\begin{align}
\left\{ \begin{array}{l}
u_t + u u_x = G(u, u) + (1 - \partial_x^2)^{-k} \left( \alpha u_x - \kappa \rho u_x \right), \\
\rho_t + u \rho_x = -(a-1)u_x \rho.
\end{array} \right.
\end{align}

(4.13)

Note

$$
\frac{d}{dt} \int mu = 2 \int um_t = 2 \int u(-um_x - au_x m + \alpha u_x - \kappa \rho u_x),
$$

$$
\frac{d}{dt} \int \rho^2 = 2 \int \rho \rho_t = 2 \int \rho(-u \rho_x - (a-1)u_x \rho).
$$

Integrating by parts, we obtain

$$
\frac{d}{dt} \int mu = (4-2a) \int u_x mu + \kappa \int u_x \rho^2,
$$

(4.14)
Suppose there exists a positive $M$ such that $|u(x,t)| \leq M$ for all $(t,x) \in [0,T^*) \times \mathbb{R}$. If $a = 2$, then

$$\frac{d}{dt} \int \rho^2 = (3 - 2a) \int u_x \rho^2.$$  

(4.15)

Applying the Gronwall’s lemma, we can obtain, for every $t \in [0,T^*)$,

$$\|u(t)\|_{H^k}^2 + \|\rho(t)\|_{L^2}^2 \approx \int (mu + \rho^2(t))(t) \leq e^{(\kappa - 1)M^2} \int (m_0 u_0 + \rho_0^2) \lesssim \|u_0\|_{H^k}^2 + \|\rho_0\|_{L^2}^2.$$

At the same time, (4.12) leads to

$$\|\rho\|_{L^\infty} = \|\rho(t,\psi(t,\cdot))\|_{L^\infty} \leq \|\rho_0\|_{L^\infty} e^{(1 - a)MT^*} \leq C_1 \|\rho_0\|_{H^q}.$$  

At first, we consider the cases $q \geq k + 1$, and we need to control $\|u\|_{H^{k+1}}$.

Firstly, we point out that, when applying the Littlewood-Paley decomposition $\Delta_j$ and taking $L^2$-norm to obtain the a priori estimates (2.4), the term

$$\int (\Delta_j (1 - \partial_x^2)^{-k} u_x) \Delta_j u dx = \int (\Delta_j (1 - \partial_x^2)^{-k/2} u_x) (\Delta_j (1 - \partial_x^2)^{-k/2} u) = 0.$$

It follows that the term $\alpha (1 - \partial_x^2)^{-k} u_x$ plays no role in the a priori estimates (2.4) in Sobolev spaces. Since $|u(x,t)| \leq M$ for all $(t,x) \in [0,T^*) \times \mathbb{R}$, with the application of Lemma 2.7 (3), we obtain, for every $t \in [0,T^*)$,

$$\|u(t)\|_{H^{k+1}} \lesssim \|u_0\|_{H^{k+1}} + \int_0^t \|G(u,u) - \frac{\kappa}{2} (1 - \partial_x^2)^{-k} \partial_x (\rho^2)\|_{H^{k+1}} dt' \lesssim \|u_0\|_{H^{k+1}} + \int_0^t \|G(u,u)\|_{H^{k+1}} dt' + \int_0^t \|\rho^2\|_{H^{2-k}} dt'.$$

Similar to (4.7), we can obtain $\|G(u,u)\|_{H^{k+1}} \lesssim \|u\|_{H^{k+1}}$.

Since $q \geq 2$, it follows that

$$\|\rho^2\|_{H^{q-2-k}} \lesssim \|\rho^2\|_{H^{q-k}} \lesssim \|\rho\|_{H^{q}} \lesssim \|\rho\|_{H^q} \lesssim \|\rho_0\|_{L^2}.$$

Hence

$$\|u(t)\|_{H^{k+1}} \lesssim \|u_0\|_{H^q} + \|\rho_0\|_{L^2} + \int_0^t \|u\|_{H^{k+1}} dt'.$$

Gronwall’s lemma then leads to

$$\|u(t)\|_{H^{k+1}} \lesssim \|u_0\|_{H^q} + \|\rho_0\|_{L^2}.$$

Then, similar to (4.8), we have

$$\|G(u,u)\|_{H^q} \lesssim \|u\|_{H^q}.$$  

(4.16)

Similar to the proof of Lemma 4.2, we can see that (4.16) still holds true when $k + \frac{1}{2} < q < k + 1$. Hence, applying Lemma 2.7 (3), we have, for every $q > k + \frac{1}{2}$ and every $t \in [0,T^*)$,

$$\|u(t)\|_{H^q} \lesssim \|u_0\|_{H^q} + \int_0^t \|G(u,u) - \frac{\kappa}{2} (1 - \partial_x^2)^{-k} \partial_x (\rho^2)\|_{H^q} dt' \lesssim \|u_0\|_{H^q} + \int_0^t \|u\|_{H^q} dt' + \int_0^t \|\rho^2\|_{H^{q-2-k}} dt'.$$

Noting that $q - 2k + 1 = q_m + 1 \leq q_1$ and $q_1 > \frac{1}{2}$, we see

$$\|\rho^2\|_{H^{q-2-k}} \lesssim \|\rho^2\|_{H^{q_1}} \lesssim \|\rho\|_{L^\infty} \|\rho\|_{H^{q_1}} \lesssim \|\rho\|_{H^{q_1}}.$$
Thus
\[ \|u(t)\|_{H^s} \lesssim \|u_0\|_{H^s} + \int_0^t (\|u\|_{H^s} + \|\rho\|_{H^{q_1}}) dt'. \] (4.17)

Now, apply Lemma 2.8 to the second equation of (4.13) to yield
\[ \|\rho(t)\|_{H^{q_1}} \lesssim \|\rho_0\|_{H^{q_1}} + \int_0^t (\|u_x\|_{L^\infty} \|\rho\|_{H^{q_1}} + \|u\|_{H^{q_1}} \|\rho_x\|_{L^\infty}) dt' \]
\[ \lesssim \|\rho_0\|_{H^{q_1}} + \int_0^t (\|u_x\|_{L^\infty} \|\rho\|_{H^{q_1}} + \|u\|_{H^{q_1}} \|\rho_x\|_{L^\infty}) dt' \]
\[ \lesssim \|\rho_0\|_{H^{q_1}} + \int_0^t (\|u\|_{H^s} + \|\rho\|_{H^{q_1}}) dt', \] (4.18)

here we applied the property \( q_1 + 1 \leq q \) and Lemma 2.5 (1). Combining (4.17) and (4.18) to yield
\[ \|u(t)\|_{H^s} + \|\rho(t)\|_{H^{q_1}} \lesssim \|u_0\|_{H^s} + \|\rho_0\|_{H^{q_1}} + \int_0^t (\|u\|_{H^s} + \|\rho\|_{H^{q_1}}) dt'. \]

The Gronwall lemma then leads to
\[ \|u(t)\|_{H^s} + \|\rho(t)\|_{H^{q_1}} \lesssim \|u_0\|_{H^s} + \|\rho_0\|_{H^{q_1}}. \]

Hence we can extend the solution \((u, \rho)\) beyond \( T^* \), and this contradicts the maximality of \( T^* \).

**Lemma 4.11.** Suppose \( s = 2 \), or \( s = 3 \), \( q > s + \frac{1}{2} \), and \( q_1 \) satisfies the condition (3.21). Let \( T^* \) be the lifespan of the solution \((u, \rho)\) \( \in C([0, T^*); H^q \times H^{q_1}) \) to (1.1) with the initial data \((u_0, \rho_0) \in H^q \times H^{q_1}\). If \( T^* \) is finite, then
\[ \limsup_{t \uparrow T^*} \|u_x(t)\|_{L^\infty} = \infty. \]

**Proof.** The proof is similar to Lemma 4.10 in conjunction with Lemma 4.3, thus we omit it here.

**Remark 4.12.** Along the similar computations to Lemma 4.11, we can obtain the same blow-up criterion of the system (1.1) with \( s = 1 \), without the assumption \( a = 2 \). Namely, suppose \( s = 1 \) and \( q > \frac{3}{2} \). Let \( T^* \) be the lifespan of the solution \((u, \rho) \in C([0, T^*); H^q \times H^{q_1}) \) to (1.1) with the initial data \((u_0, \rho_0) \in H^q \times H^{q_1}\). If \( T^* \) is finite, then
\[ \limsup_{t \uparrow T^*} \|u_x(t)\|_{L^\infty} = \infty. \]

**Lemma 4.13.** Suppose \( q \geq 2s \) and \((q, s)\) satisfies the condition (3.19), \( q_1 \) satisfies the condition (3.21). Let \( T^* \) be the lifespan of the solution \((u, \rho) \in C([0, T^*); H^q \times H^{q_1}) \) to (1.1) with the initial data \((u_0, \rho_0) \in H^q \times H^{q_1}\). If \( T^* \) is finite, then
\[ \limsup_{t \uparrow T^*} \|u_x(t)\|_{L^\infty} = \infty. \]

**Proof.** The proof is similar to Lemma 4.10 in conjunction with Lemma 4.4, thus we omit it here.
4.4. Global existence for (1.1). In this subsection, we will present the global existence results of the system (1.1).

**Theorem 4.14.** Suppose \( s = [s] = k \geq 2, a = 2, \kappa \geq 0, q > s + \frac{1}{2} \) and \( q_1 \) satisfies the condition [(3.21)]. Given the initial data \((u_0, \rho_0) \in H^q \times H^{q_1}\), then the solution \((u, \rho)\) to system (1.1) exists globally in time, namely, \((u, \rho) \in C([0, \infty); H^q \times H^{q_1})\).

**Proof.** The existence of \((u, \rho) \in C([0, T^*); H^q \times H^{q_1})\) is ensured by Theorem 3.11. Suppose \( T^* < \infty \), Lemma 4.10 then leads to

\[
\limsup_{t \uparrow T^*} \|u_x(t)\|_{L^\infty} = \infty.
\]

However, from (4.14) and (4.15), and noting that \( a = 2, \kappa \geq 0 \), we obtain, for every \( t \in [0, T^*) \),

\[
\frac{d}{dt} \int (mu + \kappa \rho^2) = 0.
\]

Hence

\[
\int (mu + \kappa \rho^2)(t) = \int (m_0u_0 + \kappa \rho_0^2) \lesssim \|u_0\|_{H^s}^2 + \|\rho_0\|_{L^2}^2.
\]

Thus

\[
\|u_x(t)\|_{L^\infty} \lesssim \|u(t)\|_{H^s}^2 \lesssim \|u_0\|_{H^s}^2 + \|\rho_0\|_{L^2}^2 \lesssim \|u_0\|_{H^{q_1}}^2 + \|\rho_0\|_{H^{q_1}}^2,
\]

which means that \( \|u(t)\|_{L^\infty} \) is bounded by \( \|u_0\|_{H^s}^2 + \|\rho_0\|_{H^{q_1}}^2 \) independent of \( t \in [0, T^*) \). Thus the situation (4.19) will never occur. Hence \( T^* = \infty \).

**Theorem 4.15.** Suppose \( a = 2, \kappa \geq 0, s > \frac{\beta}{2}, q \geq 2s \) and \( q_1 \) satisfies the condition [(3.21)]. Given the initial data \((u_0, \rho_0) \in H^q \times H^{q_1}\), then the solution \((u, \rho)\) to system (1.1) exists globally in time, namely, \((u, \rho) \in C([0, \infty); H^q \times H^{q_1})\).

**Proof.** Set \( \beta = s - [s] \). Since \( s > \frac{\beta}{2} \), we see that \( s > \max\left(\frac{\beta}{2} - \beta, \frac{1}{2} + \beta\right) \). Thus \( q \geq 2s \geq s + \max\left(\frac{\beta}{2} - \beta, \frac{1}{2} + \beta\right) \), it follows that \( q \) satisfies the condition (3.19). Combining the fact that \( q_1 \) satisfies the condition (3.21), we can ensure the existence of the solution \((u, \rho) \in C([0, T^*); H^q \times H^{q_1})\) according to Theorem 3.11. Along the similar computations of Theorem 4.14 we obtain, for every \( t \in [0, T^*) \),

\[
\|u_x(t)\|_{L^\infty} \lesssim \|u(t)\|_{H^s}^2 \lesssim \|u_0\|_{H^s}^2 + \|\rho_0\|_{L^2}^2 \lesssim \|u_0\|_{H^{q_1}}^2 + \|\rho_0\|_{H^{q_1}}^2.
\]

According to Lemma 4.13 we have \( T^* = \infty \).

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