Saha Ionization and Particle production in Rainbow Rindler Metric

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Abstract
The energy of a particle in the Rainbow Rindler metric is derived using Hamilton’s variational principle. Saha Ionization equation and Pair production in Rainbow Rindler metric have been investigated. Saha ionization equation and Pair production in Rainbow Rindler metric depend on corrected acceleration which is energy-dependent, according to Rainbow Gravity theory. This means Energy(E) and metric tensor($g_{\mu\nu}(E)$) depends on the inertial observer. If the photo-ionization process and pair production observed in Rindler frame in rainbow gravity may assign particular energy(say $E$) and another observer may assign another energy ($E'$) then corresponding metrics are also different. It is also possible that the same observer assigns energy to a particular process then he measures a different Energy to another process at the same place. In case of geometry also, not only different observers may see a given process or particle being affected by different metrics, but the same observer may assign different metrics to a different process or particle taking in the same region at the same time. This is required by covariance by the non-linear realization of the Lorentz group in momentum space.

Keywords: Rainbow gravity, Rainbow Rindler frame, Deformed special relativity(DSR), Photo-Ionization, Saha ionization equation, Pair production.

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1 Introduction
It has been proposed by Lorentz, Poincare[1] and A. Einstein[2] that two inertial frames can be related by a set of equations called Lorentz transformations in which Maxwell’s electromagnetic equations are invariant. Later according to the principle of equivalence one may be able to obtain space-time equations for a uniformly accelerated frame and inertial frame and vice versa in the same procedure as derived by special relativity.

The local geometry we get a result of the transformations are called Rindler space. For example, we can take two frames S and $S'$. S frame is an inertial frame with a gravitational field (may be produced by a black hole) and another one($S'$) is accelerating concerning the S without a gravitational field through space. Then according to the principle of equivalence, both are physically equivalent. The gravitational field is approximated as a constant value within a small spatial region at distance $'x'$ from the centre of gravitating object. This is called the local acceleration of the frame. This frame has attracted the physics community much.

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The importance of Rindler metric is in black hole thermodynamics. This provides a tool to explore many general relativistic (GR) features by going into the accelerated frame. Blackhole space times reduces to Rindler form in near horizon limit. Moreover, many physical processes can be explained within the Rindler space easily. These include Unruh effect, pair production, Schwinger effect, Hawking radiation, Bose-Einstein condensate so on. Thus, Rindler frame is very useful in discussing the above-mentioned aspects.

We have discussed Rindler metric in classical gravity. But it is highly believed that the classical theory is an approximation of quantum gravity theory in the limit $\hbar \to 0$. There are several candidates for unifying quantum mechanics to General relativity. Loop Quantum Gravity [3, 4], String theory [4, 5], Non-Commutative geometry [6, 7, 8, 9] are a few among them. Whatever the nature of description it is highly believed that Planck energy $E_{Pl} = \sqrt{\frac{\hbar c^5}{G}}$ plays a threshold separating classical description from the quantum description. If we probe beyond this limit we get a completely new picture.

It has been probed in astronomical and cosmological observations that leading order effects of $\frac{E}{E_{Pl}}$ ($E$ is some quanta observed by an inertial observer) around the classical limit including tests of thresholds for ultra high energy cosmic rays [10, 11, 12, 13, 14, 15, 16], a possible energy dependence of the speed of light observable in gamma-ray bursts [17] and TeV photons [18], as well as tests involving synchrotron radiation [19, 20, 21] and nuclear physics experiments [22]. Related effects may also be detectable shortly in CMB observations [23]. This observation supports the modification of energy-momentum relation in special relativity as follows:

$$E^2 = p^2 + m^2 + \alpha l_P E^3 + ...$$

(1)

$\alpha$ is a dimensionless constant of order unity (We put $\hbar = 1 = c$ ). Modifications to energy-momentum relations demand changes in Lorentz symmetry in the classical limit of quantum gravity. Some physicists assume that the Lorentz symmetry is broken to accommodate leading order effects of Planck energy into energy-momentum relation in special relativity. But among them Deformed special relativity (DSR) or Non-linear special relativity is assuming the Lorentz symmetry, but transformations acts on momentum space non-linearly taking leading order effects of $\frac{E}{E_{Pl}}$. The deformed special theory has some consequences such that energy-momentum conservation take place non-linearly in inertial frames as well as $E_{Pl}$ is invariant for all inertial observers. The invariance of Planck energy says that for all inertial observers will agree that whether a particle is in classical (Less than $E_{Pl}$) and quantum regime (greater than $E_{Pl}$). Thus, Classical to quantum and vice versa are very precise in this theory.

Deformed special relativity is a class of theories that are based on a modified principle of special relativity.

1. It is based on the principle of relativity

2. In the limit $\frac{E}{E_{Pl}} \to 0$ the speed of photon goes to a universal constant (denoted by $c$), which is same for all inertial observers.

3. Planck energy ($E_{Pl}$) is a universal constant and is same for all inertial observers.

Based on these postulates the invariant of energy and momentum modifies to

$$E^2 f^2(E/E_{Pl}) - p \cdot pg^2(E/E_{Pl}) = m^2$$

(2)

This equation can be realized as an action of a non-linear map from momentum space to itself. It is denoted by $U : \mathcal{P} \to \mathcal{P}$ given by

$$U \cdot (E, p_i) = (U_0, U_i) = \left(f \left(\frac{E}{E_{Pl}}\right) E, g \left(\frac{E}{E_{Pl}}\right) p_i\right)$$

(3)
This implies momentum space has a non-linear norm defined by
\[ \tilde{L}_a^b = U^{-1} \cdot L_a^b \cdot U \] (4)

This norm is preserved under non-linear Lorentz group, given by
\[ \tilde{L}_a^b = U^{-1} \cdot L_a^b \cdot U \] (5)

where \( L_a^b \) are the generators [24].

Deformed special relativity has been formulated non-linearly in momentum space. One might ask the question of how to incorporate the gravity that is consistent with deformed special relativity (DSR). Since momentum transformation rules are no longer linear it is nontrivial to simply extend to include gravity that is consistent with deformed or doubly special relativity (DSR). The possible answers to the question are non-commutative geometry and k-Minkowski space-time [7].

As a solution to the problem Lee Smolin and Joo Magueijo take a view that this a wrong question to search for a theory to identify dual space representing position and proceed. Instead, they propose classical space-time is to leading order in \( l_P \) represented by a one-parameter family of metrics, parameterized by the ratio of \( \frac{E}{E_P} \). Thus, the energy of a particle moving in space-time can affect the background metric. The modifications to it depend upon the ratio of the energy of the particle and Planck energy, known as rainbow gravity. The corresponding metric which we can obtain is called Rainbow Rindler metric.

This paper is devoted to learning how the Saha equation and pair production will take place in Rainbow Rindler metric. To investigate the Saha equation in a uniformly accelerated frame or Rindler frame in rainbow gravity. We assume the results using the principle of equivalence. The Lagrangian of a particle is derived from Hamilton principle and from that we get Hamilton of a particle (it may be an ion, hydrogen atom or an electron) from standard relations of classical mechanics. The Saha equation and pair production in Rindler frame have been studied by Sanchari De, Somenath Chakarabarthy. This work extends their work to Rindler frame in Rainbow gravity called Rainbow Rindler metric.

### 1.1 Rainbow Gravity

1. Modified equivalence principle:
   The freely falling observes measuring particles of energies in between \( \frac{1}{R} \ll E \ll E_P \) finds the laws of physics to be, to first order in \( \frac{1}{R} \), the same as in modified special relativity.

2. Correspondence Principle:
   In the limit of \( \frac{E}{E_P} \rightarrow 0 \) we recover back ordinary general relativity [26].

If Energy of a quantum as \( E = \frac{\hbar}{\lambda} \) Then the restriction \( \frac{1}{R} \ll E \) implies that \( R \gg \lambda \). Thus, we can infer that classical description is only valid at \( E \ll E_P \) above which quantum space-time is to be expected. Thus deformed equivalence principle implies that space-time is described by one parameter family of metrics in terms of a one parameter family of ortho-normal fields.

\[ g(E) = \eta^{\mu\nu} e_{\mu} (E) \otimes e_{\nu} (E) \] (6)

Where energy dependent of the frame fields is given by \( e_{0} (E) = \frac{1}{f \left( \frac{E}{E_P} \right)} \hat{e}_0 (E) \) and \( e_{i} (E) = \frac{1}{g \left( \frac{E}{E_P} \right)} \hat{e}_i (E) \). When we apply correspondence principle \( \left( \frac{E}{E_P} \rightarrow 0 \right) \)

\[ f \left( \frac{E}{E_P} \right) \rightarrow 1 \] (7)
Similarly, Einstein’s equations can be written as:

\[ G_{\mu\nu} = 8\pi G(E)T_{\mu\nu}(E) + g_{\mu\nu}\Lambda(E) \]  

(9)

Where \( G, \Lambda \) will depend upon \( E \). Here \( E \) is not the energy of space-time but the scale in which one probe geometry of space-time. If freely falling observer uses the motion of a collection of particles to probe the geometry of space-time, \( E \) is the total energy of the system of particles as measured by that observer[29].

### 1.2 Energy of the particle in Rainbow Rindler frame

We have the Rainbow Rindler metric as

\[ dS^2 = c^2 \exp \left( \frac{2Agx}{f c^2} \right) \left( dt^2 - dx^2 \right) - dy^2 - dz^2 \]  

(10)

To calculate Lagrangian of a particle in the Rainbow Rindler frame.

\[ S = -\alpha_0 \int ds = \int Ldt \]  

(11)

Where \( \alpha_0 = m_0c \).

\[ L = -m_0c^2 \sqrt{\exp \left( \frac{2Agx}{f c^2} \right) \left( 1 - \frac{v_x^2 - v_y^2 - v_z^2}{c^2} \right)} \]  

(12)

\[ v_x^2 = v_y^2 + v_z^2 \]  

(13)

where \( v \) is the velocity of the particle Then using

\[ H = \sum_i p_i v_i - L \]  

(14)

We can calculate Hamilton of the particle. It is found that Hamilton of the particle is:

\[ H = m_0c^2 \exp \left( \frac{Agx}{f c^2} \right) \sqrt{1 - \left( \frac{v_x^2 + v_y^2}{c^2} \right)} \]  

(15)

Since we can write \( p = mv \) and \( p^2 = p_x^2 + p_y^2 \) we can again write it as

\[ H = m_0c^2 \exp \left( \frac{Agx}{f c^2} \right) \sqrt{1 - \left( \frac{p^2}{m_0c^2} \right)} \]  

(16)

### 2 Saha Ionization equation in Rainbow Rindler frame

We consider a partially ionized hydrogen plasma. The system has cylindrical symmetry and plasma is expanding with acceleration \( g \) along positive \( x \) direction. This \( x \) direction is also symmetry axis of cylinder. According to principle of deformed equivalence principle we can say that The freely falling observes measuring particles of energies in between \( \frac{1}{R} \ll E \ll E_{Pl} \) finds the laws of physics to be, to first order in \( \frac{1}{R} \), the same as in modified special relativity. This together with dynamic equilibrium of reaction allow as to write Saha ionization equation for accelerated observers in Rainbow Rindler frame.

\[ H_n + \gamma < \rightarrow > H^+ + e^- \]  

(17)
along with,
\[ \mu(H_n) = \mu(H^+) + \mu(e) \]  \hspace{1cm} (18)

Here \( n=1 \) and under equilibrium condition chemical potential are also equal. Since the number photons are also not conserved chemical potentials for photons are zero.

\[ H = m_0c^2 \exp \left( \frac{Agx}{fc^2} \right) - \left( \frac{p^2}{2m_0 \exp \left( -\frac{Agx}{fc^2} \right)} \right) \] \hspace{1cm} (19)

We can define \( m'' = m_0 \exp \left( \frac{Agx}{fc^2} \right) \) and \( m' = m_0 \exp \left( -\frac{Agx}{fc^2} \right) \).

\[ n = \frac{N}{\Delta V} = \frac{4\pi g_d}{h^3} \int_0^\infty p^2 dp \exp \left( -\frac{1}{kT} \left( \frac{p^2}{2m'} + m''c^2 - \mu \right) \right) \] \hspace{1cm} (20)

Where \( g_d \) is the degeneracy of species and \( \Delta V = A_c \Delta x \) is a small volume element, \( \Delta x \) is a small length element in the \( x \)-direction at a distance \( x \) from the origin and \( A_c \) the cross-sectional area of the cylinder(Since we assumed cylindrical symmetry). The length \( \Delta x \) is such that deformed principle of equivalence hold. On evaluating the above integral (It is a Gaussian function of type \( ax^2 + b \)) and rearranging, we get:

\[ \mu = m''c^2 - kT \ln \left( \frac{g_dn_Q}{n} \right) \] \hspace{1cm} (21)

Where \( n_Q \) is defined as

\[ n_Q = \left( \frac{2\pi m'kT}{h^2} \right)^{3/2} \] \hspace{1cm} (22)

is called the quantum concentration for the particular species in the mixture. we can define,

\[ \frac{n(H^+)n(e)}{n(H_n)} = \frac{n_Qe}{g_n} \exp \left( -\frac{\Delta E_n}{kT} \right) = R_{A>0} \text{ (say)} \] \hspace{1cm} (23)

Then taking ratio with degeneracy and without degeneracy

\[ \frac{R_{A>0}}{R_{A=0}} = \left( \exp \left( \frac{Agx}{fc^2} \right) \right)^{-3/2} \exp \left( -\frac{Ax\Delta m}{kT} \right) \] \hspace{1cm} (24)

Here

\[ \Delta E_n = \Delta m''c^2 = \left( \exp \left( \frac{Agx}{fc^2} \right) \right) \Delta mc^2 \] \hspace{1cm} (25)

With

\[ \Delta m = m(H_n) - m(H^+) - m(e) \] \hspace{1cm} (26)

### 3 Pair Production

In the electron-positron pair creation at high temperature in a local rest frame in presence of a constant gravitational field \( g \). With the condition \( kT > 2m_0 \) Where \( T \) is the temperature.

\[ \gamma + \gamma \leftrightarrow e^- + e^+ \] \hspace{1cm} (27)
we assume that the interacting electron-positron plasma along with the photons is in thermo-
dynamic equilibrium. The chemical equilibrium condition is given by
\[ \mu(e^+) + \mu(e^-) = 0 \] (28)
where \( \mu(\gamma) = 0 \) and
\[
\mu(e^-) = m''(e)c^2 - kT \ln \left( \frac{g_e n_Q}{n_{e^-}} \right) \quad \text{and} \\
\mu(e^+) = m''(e)c^2 - kT \ln \left( \frac{g_e n_Q}{n_{e^+}} \right)
\] (29)
we have
\[ n_{e^-} - n_{e^+} = \frac{g_e^2 n_Q^2}{\exp(-2m''(e)c^2/kT)} \] (30)
where
\[ n_Q = \left( \frac{2\pi m'(e)kT}{\hbar^2} \right)^{3/2} \] (31)
is the quantum concentration for electron and positron. With \( m_0 \) replaced by \( m_0 \), the electron rest mass. If the system is assumed to be charge neutral then \( n_{e^-} = n_{e^+} \). Then electron-positron concentration with \( A > 0 \) and \( A = 0 \), which is given by
\[ \frac{n_{e^-} - n_{e^+}; A>0}{n_{e^-} - n_{e^+}; A=0} = \left( \exp \left( \frac{Agx}{fc^2} \right) \right)^{-3} \exp \left( - \frac{2Axm(0)}{kT} \right) \] (32)

4 Physical Significance and Conclusion

We can compare the above results with that of Rindler metric. We get the result obtained by Sanchari De and Somenath Chakarabarthy by applying the limit \( E_{\gamma} \rightarrow 0 \) (Correspondence principle). In Rindler metric we have the Saha Ionization equation, with \( A > 0 \) and \( A = 0 \) can be written as
\[ \frac{R_{A>0}}{R_{A=0}} = \left( 1 + \frac{Ax}{c^2} \right)^{-3/2} \exp \left( - \frac{Ax\Delta m}{kT} \right) \] (33)

[31] Our result will reproduce the above equation under correspondence principle and Taylor expansion of exponential function up to first order.
\[ \frac{R_{A>0}}{R_{A=0}} = \left( \exp \left( \frac{Agx}{fc^2} \right) \right)^{-3/2} \exp \left( - \frac{Ax\Delta m}{kT} \right) \] (34)

Expand \( \exp \left( \frac{Agx}{fc^2} \right) \) using Taylor expansion we get, \( \exp \left( \frac{Agx}{fc^2} \right) = 1 + \frac{Agx}{fc^2} + \frac{A^2 g^2 x^2}{4 c^4} \). Neglect terms containing \( c^4 \) and we get:
\[ \exp \left( \frac{Agx}{fc^2} \right) = 1 + \frac{Agx}{fc^2} \] (35)

Thus, Saha equation in Rainbow Rindler gravity becomes:
\[ \frac{R_{A>0}}{R_{A=0}} = \left( 1 + \frac{Agx}{fc^2} \right)^{-3/2} \exp \left( - \frac{Ax\Delta m}{kT} \right) \] (36)
electron-positron concentration with \( A > 0 \) and \( A = 0 \) in Rainbow Rindler gravity, which is given by
\[ \frac{R_{A>0}}{R_{A=0}} = \left( 1 + \frac{Agx}{fc^2} \right)^{-3} \exp \left( - \frac{Ax\Delta m}{kT} \right) \] (37)
This shows that there is a corrected acceleration which is energy-dependent in Rainbow gravity theory. This means $E$ and $g_{\mu\nu}(E)$ depends on the observer. If photo-ionization process observed in Rindler frame in rainbow gravity may assign $E$ and another may assign $E'$ corresponding metrics are $g_{\mu\nu}(E)$ and $g_{\mu\nu}(E')$. it is also possible that the same observer assigns $E$ to a particular process then he measures a different Energy $E'$ to another process at the same place. In case of geometry also, Not only different observers may see a given particle being affected by different metrics, but the same observer may assign different metrics to different particles moving in the same region at the same time. This is due to covariance, once we permit for a non-linear representation of the Lorentz group in momentum space.

$$ds^2 = -\frac{(dx^2_0)}{f^2} + \frac{(dx^2_i)}{g^2}$$

This is energy-dependent quadratic invariant \[29\].

The energy-dependent acceleration and geometry differ from that of General relativity. So we can hope that this result will help to explore the nature of accelerated frames in the Rainbow gravity theory whose physical characteristics depend upon the energy that can be used to probe the nature of space-time. Bose-Einstein condensation and properties of black holes in Rainbow gravity are the further problems that can be investigated.

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