ANALYTIC AUTOMORPHISM GROUP AND SIMILAR REPRESENTATION OF ANALYTIC FUNCTIONS

BINGZHE HOU AND CHUNLAN JIANG

Abstract. In geometry group theory, one of the milestones is M. Gromov’s polynomial growth theorem: Finitely generated groups have polynomial growth if and only if they are virtually nilpotent. Inspired by M. Gromov’s work, we introduce the growth types of weighted Hardy spaces. In this paper, we focus on the weighted Hardy spaces of polynomial growth, which cover the classical Hardy space, weighted Bergman spaces, weighted Dirichlet spaces and much broader. Our main results are as follows. (1) We obtain the boundedness of the composition operators with symbols of analytic automorphisms of unit open disk acting on weighted Hardy spaces of polynomial growth, which implies the multiplication operator $M_z$ is similar to $M_z$ for any analytic automorphism $\varphi$ on the unit open disk. Moreover, we obtain the boundedness of composition operators induced by analytic functions on the unit closed disk on weighted Hardy spaces of polynomial growth. (2) For any Blaschke product $B$ of order $m$, $M_B$ is similar to $\bigoplus_{i=0}^{m-1} M_z$, which is an affirmative answer to a generalized version of a question proposed by R. Douglas in 2007. (3) We also give counterexamples to show that the composition operators with symbols of analytic automorphisms of unit open disk acting on a weighted Hardy space of intermediate growth could be unbounded, which indicates the necessity of the setting of polynomial growth condition. Then, the collection of weighted Hardy spaces of polynomial growth is almost the largest class such that Douglas’s question has an affirmative answer. (4) Finally, we give the Jordan representation theorem and similarity classification for the analytic functions on the unit closed disk as multiplication operators on a weighted Hardy space of polynomial growth.

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1. Introduction

Denote by \( \text{Hol}(\mathbb{D}) \) the space of all analytic functions on the unit open disk \( \mathbb{D} \), and denote by \( \text{Aut}(\mathbb{D}) \) the analytic automorphism group on \( \mathbb{D} \), which is the set of all analytic bijections from \( \mathbb{D} \) to itself. As well known, each \( \varphi \in \text{Aut}(\mathbb{D}) \) could be written as the following form

\[
\varphi(z) = e^{i\theta} \cdot \frac{z_0 - z}{1 - \overline{z_0}z}, \quad \text{for some } z_0 \in \mathbb{D}.
\]

We are interested in a subclass of analytic functions on the unit open disk \( \mathbb{D} \), named weighted Hardy space.

In this paper, we introduce the weighted Hardy space from a given weight sequence. For any \( f \in \text{Hol}(\mathbb{D}) \), denote the Taylor expansion of \( f(z) \) by

\[
f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k.
\]

Let \( w = \{w_k\}_{k=1}^{\infty} \) be a sequence of positive numbers. Write \( \beta = \{\beta_k\}_{k=0}^{\infty} \),

\[
\beta_0 = 1, \quad \text{and } \beta_k = \prod_{j=1}^{k} w_j, \quad \text{for } k \geq 1.
\]

The weighted Hardy space \( H_\beta^2 \) induced by the weight sequence \( w \) (or \( \beta \)) is defined by

\[
H_\beta^2 = \{ f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k; \sum_{k=0}^{\infty} |\hat{f}(k)|^2 \beta_k^2 < \infty \}.
\]

Moreover, the weighted Hardy space \( H_\beta^2 \) is a complex separable Hilbert space, on which the inner product is defined by, for any \( f, g \in H_\beta^2 \)

\[
\langle f, g \rangle_{H_\beta^2} = \sum_{k=0}^{\infty} \beta_k^2 \overline{\hat{g}(k)} \hat{f}(k)
\]

Then, any \( f \in H_\beta^2 \) has the following norm

\[
\|f(z)\|_{H_\beta^2} = \sqrt{\langle f, f \rangle_{H_\beta^2}} = \sqrt{\sum_{k=0}^{\infty} \beta_k^2 |\hat{f}(k)|^2}.
\]

In particular, \( \|z^n\|_{H_\beta^2} = \beta_n \) for each \( n \in \mathbb{N} \). Let \( H_\beta^2 \) and \( H_{\beta'}^2 \) be two weighted Hardy spaces. If there are positive constants \( K_1 \) and \( K_2 \) such that \( K_1 \leq \frac{\beta'}{\beta} \leq K_2 \) for all \( k \in \mathbb{N} \), then \( H_\beta^2 = H_{\beta'}^2 \), and the norms are equivalent, and hence we say that \( H_\beta^2 \) and \( H_{\beta'}^2 \) are equivalent weighted Hardy spaces.

Furthermore, any \( f \in H_\beta^2 \) induces a multiplication operator \( M_f \), defined by

\[
M_f(g) = f \cdot g, \quad \text{for any } g \in H_\beta^2.
\]

Define \( H_\beta^\infty \) the set

\[
H_\beta^\infty = \{ f(z) \in H_\beta^2; M_f \text{ is a bounded operator from } H_\beta^2 \text{ to } H_\beta^2 \}.
\]

In the present article, we always assume the weight sequence \( w \) satisfying

\[
\lim_{k \to \infty} w_k = 1.
\]
In this case, we have

\[ \text{Hol}(\mathbb{D}) \subseteq H_2^\infty \subseteq H_2^3 \subseteq \text{Hol}(\mathbb{D}), \]

where \( \text{Hol}(\mathbb{D}) \) is denoted by the space of all analytic functions on the unit closed disk \( \mathbb{D} \). It suffices to check the first inclusion relationship above. Let \( H_2^3 \) be a weighted Hardy space induced by the weight sequence \( w = \{ w_k \}_{k=1}^\infty \). Following from the work of Shields [27], to study the multiplication operator \( M_z \) on \( H_2^3 \) is equivalent to study the forward unilateral weighted shift \( S_w \) on the classical Hardy space, where \( S_w : H^2 \to H^2 \) is defined by

\[ S_w(z^k) = w_{k+1}z^{k+1}, \quad \text{for } k = 0, 1, 2, \ldots. \]

Notice that \( w_k \to 1 \) implies the spectrum of \( S_w \), denoted by \( \sigma(S_w) \), is the unit closed disk \( \mathbb{D} \). Then, for any \( f \in \text{Hol}(\mathbb{D}) \), one can see that the analytic function calculus \( f(S_w) \) on \( H^2 \) is corresponding to the multiplication operator \( M_f \) on \( H_2^3 \). Consequently, we have \( \sigma(M_f) = f(\mathbb{D}) \). Therefore, \( M_f \) is a bounded operator on \( H_2^3 \) and moreover, \( M_f \) is lower bounded on \( H_2^3 \) if and only if \( f \) has no zero point in \( \partial\mathbb{D} \).

In addition, if \( w_k \to 1 \), the weighted Hardy space \( H_2^3 \) is isometric to the following weighted square summable sequence space

\[ \{(c_0, c_1, c_2, \ldots) ; \sum_{k=0}^{\infty} |c_k|^2 \beta_k^2 < \infty \}. \]

The classical Hardy space, the weighted Bergman spaces, and the weighted Dirichlet spaces are all the weighted Hardy spaces of this type.

Then, every analytic function \( f \) in \( \text{Hol}(\mathbb{D}) \) has an operator representation \( M_f \) on \( H_2^3 \). As what we do in the representation theory or in linear algebra essentially, a natural task is to study the similarity of the representations. More precisely, in finite-dimensional matrix theory, the Jordan decomposition theorem tells us each finite-dimensional matrix is similar to a direct sum of some Jordan blocks. With regard to infinite-dimensional operators (or matrices), strongly irreducible operator is a suitable substitute for Jordan block (see [19]). Consequently, strongly irreducible decomposition is seemed as the Jordan decomposition of an infinite-dimensional operator. Then we could study the "Jordan decomposition" and similarity classification of the representation of \( \text{Hol}(\mathbb{D}) \) on a weighted Hardy space \( H_2^3 \) with \( w_k \to 1 \). Notice that there is no standard form of strongly irreducible operator. It is also valuable to describe when \( M_f \) is similar to \( M_g \) if \( M_f \) and \( M_g \) are strongly irreducible operators on \( H_2^3 \), which is related to the boundedness of composition operator \( C_\varphi \) for \( \varphi \in \text{Aut}(\mathbb{D}) \).

In [20], Jiang and Zheng studied the similarity of the operator representation \( M_f \) on the weighted Bergman spaces. Furthermore, Ji and Shi studied the similarity of the operator representation \( M_f \) on the Sobolev disk algebra in [21]. More recently, E. Gallardo-Gutiérrez and J. Partington [14] obtained some similar results of Jiang and Zheng [20]. In addition, the boundedness of composition operator \( C_\varphi \) for \( \varphi \in \text{Aut}(\mathbb{D}) \) has been obtained in the weighted Bergman spaces, the weighted Dirichlet spaces, Sobolev space and so on (we refer to [8]). In this article, we aim to study the similarity of the operator representation \( M_f \) on more general spaces. Firstly, we will introduce the growth types of weighted Hardy spaces, which are inspired by the growth types of finitely generated groups. In geometry group theory, one of the milestones is M. Gromov’s polynomial growth theorem in [15]:
"Finitely generated groups have polynomial growth if and only if they are virtually nilpotent." The notions concerned can be found in [23].

**Definition 1.1** (Quasi-equivalence of (generalized) growth functions, [23] pp. 171).

1. A generalized growth function is a function of type $\mathbb{R}^+ \rightarrow \mathbb{R}^+$ that is nondecreasing.
2. Let $f, g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be generalized growth functions. We say that $g$ quasi-dominates $f$ if there exist $c, b \in \mathbb{R}^+$ such that $f(r) \leq c \cdot g(c \cdot r + b) + b$, for any $r \in \mathbb{R}^+$.

If $g$ quasi-dominates $f$, then we write $f \prec g$.

3. Two generalized growth functions $f, g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are quasi-equivalent if both $f \prec g$ and $g \prec f$; if $f$ and $g$ are quasi-equivalent, then we write $f \sim_{QE} g$.

**Definition 1.2** (Growth types of finitely generated groups, [23] pp. 174).

Let $G$ be a finitely generated group.

1. The growth type of $G$ is the (common) quasi-equivalence class of all growth functions of $G$ with respect to finite generating sets of $G$.
2. The group $G$ is of exponential growth if it has the growth type of the exponential map $(x \mapsto e^x)$.
3. The group $G$ has polynomial growth if for one (and hence every) finite generating set $S$ of $G$ there is an $a \in \mathbb{R}^+$ such that $(x \mapsto x^a)$ quasi-dominates the growth function of $G$ with respect to $S$.
4. The group $G$ is of intermediate growth if it is neither of exponential nor of polynomial growth.

If $w_k$ non-increasingly converges to 1, the weighted sequence $\beta$ induces a generalized growth function $x \mapsto \beta[x]$, where $[x]$ means the integer part of $x$. Furthermore, if

$$\sup_k (k+1)(w_k - 1) = M < \infty,$$

then

$$(x \mapsto \beta[x]) \prec (x \mapsto x^M),$$

which implies $\beta$ is of polynomial growth. Notice that $\beta$ can not be of exponential growth because of $w_k \rightarrow 1$. We could use $w$ to introduce the growth type of the weighted Hardy space $H^2_\beta$ with $w_k \rightarrow 1$ as follows.

**Definition 1.3.** Let $w = \{w_k\}_{k=1}^\infty$ be a sequence of positive numbers with $w_k \rightarrow 1$.

1. If $\sup_k (k+1)|w_k - 1| < \infty$, we say that the weighted Hardy space $H^2_\beta$ is of polynomial growth.
2. If $\sup_k (k+1)|w_k - 1| = \infty$, we say that the weighted Hardy space $H^2_\beta$ is of intermediate growth.

**Remark 1.4.** It is easy to see that the condition $\sup_k (k+1)|w_k - 1| < \infty$ holds if and only if there exists a positive number $M$ such that for each $k \in \mathbb{N}$,

$$\frac{k+1}{k + M + 1} \leq w_k \leq \frac{k + M + 1}{k + 1}.$$

Moreover, such weighted Hardy space $H^2_\beta$ is said to be of $M$-polynomial growth. Roughly speaking, the polynomial growth condition for a weighted Hardy space $H^2_\beta$
implies that $\beta_n$ is controlled by a monomial of $(n+1)$ as an upper bound and a monomial of $\frac{1}{n+1}$ as a lower bound.

In the present paper, we aim to study on the following three aspects.

(I). An analytic self-map $g : \mathbb{D} \to \mathbb{D}$ induces a linear operator $C_g : H^2_\beta \to H^2_\beta$, defined by

$$C_g(f) = f(g), \quad \text{for any } f \in H^2_\beta.$$ 

Then the operator $C_g$ is said to be a composition operator. The boundedness of the composition operators with symbols of analytic automorphisms is an important problem in analytic function theory and operator theory. There have been obtained many results related to this topic, see [8] for instance. However, there also have been some unresolved questions. For example, C. Cowen and B. MacCluer proposed a conjecture [9] as follows.

**Conjecture 1.5** ([9]). If $\beta_n$ is monotone decreasing (or satisfies some other reasonable regularity requirement), $H^2_\beta$ is automorphism invariant if and only if there exists a positive integer $n$ so that $(1-z)^{-n}$ is not in $H^2_\beta$.

The boundedness of the composition operators with symbols of analytic automorphisms plays an important role in the similarity classification of multiplication operators. It is not difficult to see that, for any $\varphi \in \text{Aut}(\mathbb{D})$, $M_\varphi$ is similar to $M_\varphi$, denoted by $M_z \sim M_\varphi$, if and only if the composition operator $C_\varphi : H^2_\beta \to H^2_\beta$ is an isomorphism.

On the other hand, weakly homogeneous operator was introduced by Clark and Misra [5], which is a generalization of homogeneous operator. A bounded linear operator $T$ on a complex separable Hilbert space is said to be a weakly homogeneous operator if $\sigma(T) \subseteq \mathbb{D}$ and $T$ is similar to $\varphi(T)$ for any $\varphi \in \text{Aut}(\mathbb{D})$. The boundedness of the composition operators with symbols of analytic automorphisms implies that the multiplication operator $M_z$ is weakly homogeneous.

In this paper, we aim to study the boundedness of composition operators with symbols of analytic automorphisms acting on weighted Hardy spaces of polynomial growth. Moreover, we also study the boundedness of composition operators induced by analytic functions in $\text{Hol}(\mathbb{D})$ on weighted Hardy spaces of polynomial growth, and give counterexamples on a weighted Hardy space of intermediate growth to show the necessity of the setting of polynomial growth condition.

(II). R. Douglas proposed a question in [12] to ask the similarity of $M_B$ and $\bigoplus_{1}^{m} M_z$ on the classical Bergman space as follows.

**Question 1.6** ([12]). Is $M_B$ on $L^2_\alpha(D)$ similar to $M_z \otimes I_m$ on $L^2_\alpha(D) \otimes \mathbb{C}^m$, where $L^2_\alpha(D)$ is the Bergman space and $m$ is the multiplicity of the finite Blaschke product $B(z)$?

Jiang and Li [18] answered the problem in the affirmative. Furthermore, Jiang and Zheng [20] generalized this conclusion on the weighted Bergman spaces. In this paper, we aim to answer the generalized Douglas’s question on the weighted Hardy spaces of polynomial growth.

(III). We aim to study the Jordan representation theorem of $\text{Hol}(\mathbb{D})$ on the weighted Hardy spaces of polynomial growth. Furthermore, we will give the similarity classification of the representation of $\text{Hol}(\mathbb{D})$, which generalize a result of Jiang and Zheng (Theorem 1.1 in [20]) from the weighted Bergman spaces to the weighted Hardy spaces of polynomial growth.

We list our main results in the next section.
2. Summary of main results

The first main result in the paper is to obtain the boundedness of the composition operators with symbols of Möbius transformations acting on a weighted Hardy space of polynomial growth, which also plays an important role in the whole paper.

**Theorem 2.1.** Let $H^2_{\beta}$ be a weighted Hardy space of polynomial growth induced by the weight sequence $w = \{w_k\}_{k=1}^{\infty}$. Then for any $\varphi \in \text{Aut}(\mathbb{D})$, $M_z \sim M_{\varphi}$, i.e., $M_z$ is weakly homogeneous on $H^2_{\beta}$. In fact, the composition operator $C_{\varphi} : H^2_{\beta} \to H^2_{\beta}$ is an isomorphism.

Corresponding to the conjecture of Cowen and MacCluer, this theorem provides a sufficient condition to $H^2_{\beta}$ being automorphism invariant, in which we give a requirement of $w_n$ instead of $\beta_n$ (no hypothesis of the monotonicity of $\beta_n$ or $w_n$). In particular, one could see that if $\beta_n$ is monotone decreasing and $H^2_{\beta}$ is of $M$-polynomial growth for some positive integer $M$, then $(1-z)^{-(M+1)}$ is not in $H^2_{\beta}$.

Since lots of weighted Hardy spaces of polynomial growth are defined without measures, the measure method is invalid. We have to develop some techniques different from the case of the weighted Bergman spaces or the weighted Dirichlet spaces. Roughly speaking, we will show that $C_{\varphi}$ is a base transformation between two Riesz bases to prove the above theorem. In section 3, we study some base properties of the sequence induced by a Möbius transformation or a finite Blaschke product in base theory, and obtain a series of fundamental results as preliminaries to prove our main results.

We also obtain the boundedness of the composition operators with symbols of analytic functions on the unit closed disk acting on a weighted Hardy space of polynomial growth.

**Theorem 2.2.** Let $H^2_{\beta}$ be the weighted Hardy space of polynomial growth induced by a weight sequence $w = \{w_k\}_{k=1}^{\infty}$. If $\psi(z)$ is an analytic function on $\overline{\mathbb{D}}$ with $\psi(\mathbb{D}) \subseteq \mathbb{D}$, then $C_{\psi}$ is bounded on $H^2_{\beta}$.

Furthermore, we give an affirmative answer to the generalized Douglas’s question in the setting of weighted Hardy spaces of polynomial growth.

**Theorem 2.3.** Let $H^2_{\beta}$ be a weighted Hardy space of polynomial growth, and let $B(z)$ be a finite Blaschke product with order $m$ on $\mathbb{D}$. Then $M_B \sim \bigoplus_{1}^{m} M_z$.

In addition, we could give counterexamples on a weighted Hardy space of intermediate growth to show the necessity of the setting of polynomial growth condition.

**Theorem 2.4.** Let $H^2_{\beta}$ be the weighted Hardy space induced by a weight sequence $w = \{w_k\}_{k=1}^{\infty}$ with $w_k \to 1$. Suppose that there exists a sequence of positive numbers $\{\alpha_j\}_{j=1}^{\infty}$ tending to infinity and a sequence of positive integers $\{n_j\}_{j=1}^{\infty}$ tending to infinity such that, for every $0 \leq k \leq n_j$,

$$\frac{\beta_{n_j}}{\beta_k} \geq \frac{\beta_k^{(\alpha_j)}}{\beta_{n_j}^{(\alpha_j)}}.$$

Then, for any $t \in (0, 1)$,

$$\frac{\|\varphi_t^{n_j}(z)\|_{\beta_{\beta-1}}^{2}}{\|z_{n_j}\|_{\beta_{\beta-1}}^{4}} \to \infty, \quad \text{as} \; j \to \infty.$$
Consequently, the composition operators \( C_{\phi_t} : H^2_{\beta-1} \to H^2_{\beta-1} \) and \( C_{\phi_t} : H^2_{\beta} \to H^2_{\beta} \) are unbounded. In particular, if
\[
\lim_{k \to \infty} (k + 1)(w_k - 1) = +\infty \quad \text{or} \quad \lim_{k \to \infty} (k + 1)(w_k - 1) = -\infty,
\]
then the conclusion holds.

Together with Theorem 2.3 and the technique of K-theory of Banach algebras, we could obtain the Jordan representation theorem and similarity classification of the representation of \( \text{Hol}(\mathbb{D}) \) on the weighted Hardy spaces of polynomial growth.

**Theorem 2.5** (Jordan representation theorem of \( \text{Hol}(\mathbb{D}) \)). Given any \( f \in \text{Hol}(\mathbb{D}) \), there exist a unique positive integer \( m \) and an analytic function \( h \in \text{Hol}(\mathbb{D}) \) with \( \{M_h \}_n = H^\infty_\beta \), such that
\[
M_f \sim \bigoplus_1^m M_h,
\]
where \( h \) is unique in the sense of analytic automorphism group action, i.e., if there exists another \( g \in \text{Hol}(\mathbb{D}) \) with \( \{M_g \}_n = H^\infty_\beta \) such that
\[
M_f \sim \bigoplus_1^m M_g,
\]
then there is a Möbius transformation \( \varphi \in \text{Aut}(\mathbb{D}) \) such that \( g = h \circ \varphi \).

**Theorem 2.6.** Let \( f_1, f_2 \in \text{Hol}(\mathbb{D}) \). Then, \( M_{f_1} \) is similar to \( M_{f_2} \) on \( H^2_\beta \) if and only if there are two finite Blaschke products \( B_1 \) and \( B_2 \) with the same order and a function \( h \in \text{Hol}(\mathbb{D}) \) such that
\[
f_1 = h \circ B_1 \quad \text{and} \quad f_2 = h \circ B_2.
\]

We will give a series of fundamental results as preliminaries in section 3, and then give the proofs of Theorem 2.1, Theorem 2.2, Theorem 2.3 and Theorem 2.4 in section 4, and give the proofs of Theorem 2.5 and Theorem 2.6 in section 5.

### 3. Bases Theory and Transformation Operators

In this section, let us review some basic concepts in bases theory on Hilbert space firstly (we refer to [24], more generally, one can see these concepts on Banach space in [20]). Let \( \mathcal{H} \) be a complex separable Hilbert space and \( \mathcal{X} = \{x_n\}_{n=0}^\infty \) be a sequence of vectors in \( \mathcal{H} \). The sequence \( \mathcal{X} \) is said to be total, if \( \mathcal{X} \) spans the whole space \( \mathcal{H} \), i.e.,
\[
\overline{\text{span}}\mathcal{X} = \left\{ \sum_{n=0}^\infty c_n x_n; \ x_n \in \mathcal{X}, \ c_n \in \mathbb{C} \text{ is zero except finite items} \right\} = \mathcal{H}.
\]
\( \mathcal{X} \) is said to be finitely linear independent, if its arbitrary finite subsequence is linear independent. \( \mathcal{X} \) is said to be a Schauder base, if for every \( x \in \mathcal{H} \), there exists a unique sequence of complex numbers \( \{c_n\}_{n=0}^\infty \) such that
\[
x = \sum_{n=0}^\infty c_n x_n, \quad \text{i.e.,} \quad \lim_{n \to \infty} \|x - \sum_{k=0}^n c_k x_k\| = 0.
\]
A Schauder base $\mathcal{X} = \{ x_n \}_{n=0}^\infty$ is said to be a normalized base if $\| x_n \| = 1$ for all $n = 0, 1, \ldots$. Moreover, a Schauder base $\mathcal{X} = \{ x_n \}_{n=0}^\infty$ is called a bounded (or quasinormed) base if
\[
0 < \inf_{0 \leq n < \infty} \| x_n \| \leq \sup_{0 \leq n < \infty} \| x_n \| < \infty.
\]

A Schauder base $\mathcal{X} = \{ x_n \}_{n=0}^\infty$ is said to be an unconditional base, if every convergent series of the form $\sum_{n=0}^\infty c_n x_n$ is unconditional convergent, i.e., independently of order. Furthermore, A Schauder base is called conditional base, if it is not an unconditional base. $\mathcal{X} = \{ x_n \}_{n=0}^\infty$ is said to be a Riesz base, if there exists a bounded invertible operator $V$ such that $\{ V(x_n) \}_{n=0}^\infty$ is an orthonormal base, i.e., $V(x_m)$ is orthogonal to $V(x_n)$ for all $m \neq n$, and $\| V(x_n) \| = 1$ for all $n = 0, 1, \ldots$.

Let $\mathcal{X} = \{ x_n \}_{n=0}^\infty$ and $\mathcal{Y} = \{ y_n \}_{n=0}^\infty$ be two sequences of vectors in a complex separable Hilbert space $\mathcal{H}$. Define
\[
(\mathcal{X}, \mathcal{Y})_{\mathcal{H}} = (x_j, y_i)_{\mathcal{H}}.
\]

In particular, the Gram matrix of $\mathcal{X}$ on $\mathcal{H}$ is defined by
\[
\Gamma_\mathcal{H}(\mathcal{X}) = (\mathcal{X}, \mathcal{X})_{\mathcal{H}} = (x_i, x_j)_{\mathcal{H}}.
\]

Next, we will study some base properties of the sequence induced by a Möbius transformation or a finite Blaschke product in bases theory. It is useful of a class of geometric operators introduced by M. Cowen and R. Douglas \[10\] as follows.

**Definition 3.1 (\[10\]).** For $\Omega$ a connected open subset of $\mathbb{C}$ and $n$ a positive integer, let $\mathcal{B}_n(\Omega)$ denote the operators $T$ in $\mathcal{L}(\mathcal{H})$ which satisfy:

(a) $\Omega \subseteq \sigma(T) = \{ \omega \in \mathbb{C} : T - \omega$ not invertible $\}$;

(b) $\text{Ran}(T - \omega) = \mathcal{H}$ for every $\omega$ in $\Omega$;

(c) $\text{span}\{ \text{Ker}(T - \omega) : \omega \in \Omega \} = \mathcal{H}$;

(d) $\dim \text{Ker}(T - \omega) = n$ for every $\omega$ in $\Omega$.

**Lemma 3.2.** Suppose that $H^2_\beta$ is the weighted Hardy space induced by a weight sequence $w = \{ w_k \}_{k=1}^\infty$ with $w_k \to 1$. Let $B(z) = \prod_{j=1}^m \frac{z - j}{z - \overline{j}}$ be a Blaschke product on $\mathbb{D}$ with order $m$. Denote
\[
\overline{B}(z) = \overline{B(z)} = \sum_{k=0}^\infty \overline{B(k)} z^k.
\]

Let $S$ be the standard left inverse of the multiplication operator $M_z$ on $H^2_\beta$, defined by
\[
S(f(z)) = \frac{f(z) - f(0)}{z}, \text{ for every } f(z) \in H^2_\beta.
\]

Then $\overline{B}(S) \in \mathcal{B}_m(\mathbb{D})$ and $\overline{B}(S)B(M_z) = I$, where $I$ means the identity operator.

**Proof.** Notice that $S \in \mathcal{B}_1(\mathbb{D})$ and $\overline{B}(z)$ is also a Blaschke product on $\mathbb{D}$ with order $m$. Then it is easy to see $\overline{B}(S) \in \mathcal{B}_m(\mathbb{D})$.

For any $\alpha \in \mathbb{D}$, let $\varphi_\alpha = \frac{\alpha - z}{1 - \overline{\alpha}z}$. Then, it follows from $SM_z = I$ that
\[
\overline{\varphi_\alpha}(S) \varphi_\alpha(M_z) = (\overline{\varphi_\alpha} - S) \sum_{k=0}^\infty (\alpha S)^k (\alpha - M_z)(I - \overline{\varphi_\alpha}M_z)^{-1}
= -(\overline{\varphi_\alpha} - S)M_z(I - \overline{\varphi_\alpha}M_z)^{-1}
= I.
\]
Since $B(z) = \prod_{j=1}^{m} \varphi_{z_j}(z)$, we have
\[
B(S)B(M_z) = \prod_{j=1}^{m} \varphi_{z_j}(S) \prod_{j=1}^{m} \varphi_{z_j}(M_z) = I.
\]

Proposition 3.3. Let $H_{\varphi}^2$ be the weighted Hardy space induced by a weight sequence $w = \{w_k\}_{k=1}^{\infty}$ with $w_k \to 1$, and let $B(z) = \prod_{j=1}^{m} \frac{z - z_j}{1 - \overline{z_j}z}$ be a Blaschke product on $D$ with order $m$. Suppose that \{f_1, \cdots, f_m\} is a base of Ker($\overline{B}(S)$). Then
\[
\{f_1 B^n, \cdots, f_m B^n; n = 0, 1, 2, \ldots\}
\]
is a total and finitely linear independent sequence in $H_{\varphi}^2$.

Proof. First, we will show that, for any $N = 0, 1, 2, \ldots$,
\[
\{f_1 B^n, \cdots, f_m B^n; n = 0, 1, \ldots, N\}
\]
is linear independent. Assume
\[
\sum_{n=0}^{N} \sum_{j=1}^{m} \lambda_{nj} f_j(z) B^n(z) = 0.
\]
Then, by Lemma 3.2,
\[
0 = \overline{B}^N(S) \left( \sum_{n=0}^{N} \sum_{j=1}^{m} \lambda_{nj} f_j(z) B^n(z) \right) = \sum_{j=1}^{m} \lambda_{Nj} f_j(z).
\]
Since \{f_1, \cdots, f_m\} is a base of Ker($\overline{B}(S)$), we get
\[
\lambda_{N1} = \lambda_{N2} = \cdots = \lambda_{Nm} = 0.
\]
In this manner, one can see
\[
\lambda_{nj} = 0, \quad \text{for all } n = 0, 1, \ldots, N \text{ and } j = 1, 2, \ldots, m.
\]
In addition, for any $n = 0, 1, \ldots, N$ and any $j = 1, 2, \ldots, m$,
\[
\overline{B}^{N+1}(S)(f_j(z) B^n(z)) = 0.
\]
Then, by dimKer($\overline{B}^{N+1}(S)$) = $m(N+1)$, the sequence
\[
\{f_1 B^n, \cdots, f_m B^n; n = 0, 1, \ldots, N\}
\]
is a base of Ker($\overline{B}^{N+1}(S)$).

Therefore, by
\[
\text{span}\{\text{Ker}(\overline{B}^N(S)); N = 1, 2, \ldots\} = H_{\varphi}^2,
\]
the sequence
\[
\{f_1 B^n, \cdots, f_m B^n; n = 0, 1, 2, \ldots\}
\]
is total and finitely linear independent in $H_{\varphi}^2$.

Furthermore, we could give a concrete base of Ker($\overline{B}(S)$) if the Blaschke product $B$ has distinct zero points.
Lemma 3.4 ([20]). Let \( z_1, \ldots, z_m \) be \( m \) distinct points in \( \mathbb{D} \). Then the matrix
\[
\begin{pmatrix}
\frac{1}{1-z_1^2} & \frac{1}{1-z_1z_2} & \cdots & \frac{1}{1-z_1z_m} \\
\frac{1}{1-z_2z_1} & \frac{1}{1-z_2^2} & \cdots & \frac{1}{1-z_2z_m} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{1-z_mz_1} & \frac{1}{1-z_mz_2} & \cdots & \frac{1}{1-z_m^2}
\end{pmatrix}
\]
is invertible.

Theorem 3.5. Let \( H_\beta^2 \) be the weighted Hardy space induced by a weight sequence \( w = \{w_k\}_{k=0}^\infty \) with \( w_k \to 1 \), and let \( B(z) = \prod_{j=1}^m \frac{z - z_j}{1 - z_jz} \) be a Blaschke product on \( \mathbb{D} \) with \( m \) distinct zero points. Then
\[
\{ B^n(z), \frac{B^n(z)}{1 - \frac{z}{z_0}} : n = 0, 1, 2, \ldots \}
\]
is a total and finitely linear independent sequence in \( H_\beta^2 \).

Proof. Following from Proposition 3.3 it suffices to prove the finite sequence \( \{ \frac{1}{1 - \frac{z}{z_j}} \}_{j=1}^m \) is a base of \( \text{Ker}(\overline{B}(S)) \). Notice that \( \{ \frac{1}{1 - \frac{z}{z_j}} \}_{j=1}^m \) is linear independent in \( H_\beta^2 \) if and only if so is in the classical Hardy space \( H^2 \). By Lemma 3.4 the sequence \( \{ \frac{1}{1 - \frac{z}{z_j}} \}_{j=1}^m \) is linear independent in \( H_\beta^2 \). In addition, for \( j = 1, 2, \ldots, m \),
\[
\overline{B}(S)(\frac{1}{1 - \frac{z}{z_j}}) = \left( \prod_{i \neq j} \frac{1}{1 - \frac{z_i}{z_j}} \right) \frac{1}{1 - \frac{z}{z_j}} = \left( \prod_{i \neq j} \frac{1}{1 - \frac{z_i}{z_j}} \right) \left( \frac{1}{1 - \frac{z}{z_j}} \right)^{-1} = 0.
\]
Then, by \( \dim \text{Ker}(\overline{B}(S)) = m \), the sequence \( \{ \frac{1}{1 - \frac{z}{z_j}} \}_{j=1}^m \) is a base of \( \text{Ker}(\overline{B}(S)). \) \( \square \)

Corollary 3.6. Let \( H_\beta^2 \) be the weighted Hardy space induced by a weight sequence \( w = \{w_k\}_{k=0}^\infty \) with \( w_k \to 1 \). Let \( \varphi_{z_0}(z) = \frac{z - z_0}{1 - \frac{z}{z_0}}, z_0 \in \mathbb{D} \setminus \{0\} \), and \( B(z) = z \varphi_{z_0}(z) \).

Then, the sequences
\[
\{ B^n(z), \varphi_{z_0}(z)B^n(z) : n = 0, 1, 2, \ldots \} \quad \text{and} \quad \{ B^n(z), \varphi_{z_0}(z)B^n(z) : n = 0, 1, 2, \ldots \}
\]
are total and finitely linear independent in \( H_\beta^2 \), respectively.

Proof. According to Theorem 3.5 the sequence
\[
\{ B^n(z), \frac{B^n(z)}{1 - \frac{z}{z_0}} : n = 0, 1, 2, \ldots \}
\]
is total and finitely linear independent in \( H_\beta^2 \). Since
\[
\text{span}\{1, \frac{1}{1 - \frac{z}{z_0}}\} = \text{span}\{ \frac{a z + b}{1 - \frac{z}{z_0}} : a, b \in \mathbb{C} \} = \text{span}\{ \frac{1}{1 - \frac{z}{z_0}}, \frac{z}{1 - \frac{z}{z_0}} \} = \text{span}\{1, \frac{z}{1 - \frac{z}{z_0}}, \frac{z_0 - z}{1 - \frac{z}{z_0}}\},
\]
it follows from Proposition 3.3 that the sequences
\[
\{ \frac{B^n(z)}{1 - \frac{z}{z_0}}, \frac{z B^n(z)}{1 - \frac{z}{z_0}} : n = 0, 1, 2, \ldots \} \quad \text{and} \quad \{ B^n(z), \varphi_{z_0}(z)B^n(z) : n = 0, 1, 2, \ldots \}
\]
are total and finitely linear independent in $H^2_\beta$, respectively. Notice that

$$M_{1-z_0^{-1}} : H^2_\beta \to H^2_\beta$$

is a bounded invertible operator. Thus, the sequence

$$\{B^n(z), zB^n(z); n = 0, 1, 2, \ldots\}$$

is also total and finitely linear independent in $H^2_\beta$.

In particular, we could obtain that the powers of a Möbius transformation constitute a Schauder base by the uniqueness of Taylor expansion.

**Proposition 3.7.** Let $H^2_\beta$ be the weighted Hardy space induced by a weight sequence $w = \{w_k\}_{k=1}^\infty$ with $w_k \to 1$. Let $\varphi_{z_0}(z) = \frac{z-z_0}{1-z_0 z}$, $z_0 \in \mathbb{D}$, be a Möbius transformation on $\mathbb{D}$. Then, the sequence $\{\varphi_n z_0\}_{n=0}^\infty$ is a Schauder base of $H^2_\beta$.

**Proof.** It follows from Theorem 3.5 that the sequence $\{\varphi_n z_0\}_{n=0}^\infty$ is total and finitely linear independent in $H^2_\beta$. For any $f \in H^2_\beta \subseteq \text{Hol}(\mathbb{D})$, we could write

$$f(z) = \sum_{k=0}^\infty \lambda_k \varphi^k z_0(z).$$

Let $w = \varphi z_0$. Then

$$f(\varphi z_0(w)) = \sum_{k=0}^\infty \lambda_k w^k, \quad \text{for all } w \in \mathbb{D}.$$

According to the uniqueness of the Taylor expansion of the function $f \in \text{Hol}(\mathbb{D})$, the sequence $\{\lambda_k\}_{k=0}^\infty$ is unique. Hence, the function $f(z)$ has unique coefficients $\lambda_k$. This completes the proof. \qed

Now we focus on the characterization of Riesz base in a complex separable Hilbert space. It is well-known that Riesz base and bounded unconditional base are equivalent in a complex separable Hilbert space. However, there exist conditional bases in Hilbert space [11], and furthermore, Olevskii gave a spectral characterization the transformation operator from an orthonormal base to a conditional base in [25]. So it is worthy of describing Riesz bases in a computable manner. Bari had used Gram matrix to provide a sufficient and necessary condition to Riesz base [2] (see also in [24]).

**Theorem 3.8** (Bari's Theorem). Let $X$ be a total sequence in a complex separable Hilbert space $H$. Then $X$ is a Riesz base if and only if the Gram matrix $\Gamma_H(X)$ is a bounded invertible linear operator on $l^2$.

Inspired by Olevskii’s idea in [25], we will introduce the notion of transformation operator of a sequence in $H^2_\beta$ to characterize Riesz bases.

Let $\mathcal{F} = \{f_n\}_{n=0}^\infty$ be a sequence of vectors in a weighted Hardy space $H^2_\beta$. Then $\mathcal{F}$ induces a linear operator $X_\mathcal{F}$ on $H^2_\beta$, defined by

$$X_\mathcal{F}(z^n) = f_n(z), \quad \text{for } n = 0, 1, \ldots.$$
Furthermore, the linear operator $X_{\mathfrak{F}}$ has a matrix representation under the orthogonal base $\{z^n\}_{n=0}^{\infty}$ as follows

$$X_{\mathfrak{F}} = \begin{bmatrix} f_0(0) & f_1(0) & f_2(0) & \cdots & f_k(0) & \cdots \\ f_0(1) & f_1(1) & f_2(1) & \cdots & f_k(1) & \cdots \\ f_0(2) & f_1(2) & f_2(2) & \cdots & f_k(2) & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ f_0(k) & f_1(k) & f_2(k) & \cdots & f_k(k) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

We say that the operator $X_{\mathfrak{F}}$ is the transformation operator of the sequence $\mathfrak{F}$. In particular, if $\mathfrak{F} = \{g^n\}_{n=0}^{\infty}$ for some $g \in H^\infty_\beta$ with $g(D) \subseteq D$, then the transformation operator $X_{\mathfrak{F}}$ is just the composition operator $C_g$. Notice that the transformation operator $X_{\mathfrak{F}}$ may be not bounded on $H^2_\beta$ in general.

Define the operator $D_\beta : H^2_\beta \to H^2$ by

$$D_\beta(z^k) = \beta_k z^k \quad \text{for } n = 0, 1, \ldots$$

As well known, $D_\beta$ is an isometry isomorphism. Moreover, $D_\beta$ and $D^{-1}_\beta$ has matrix representations under the orthogonal base $\{z^n\}_{n=0}^{\infty}$ as follows

$$D_\beta = \begin{bmatrix} \beta_0 & 0 & \cdots & 0 & \cdots \\ 0 & \beta_1 & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & \cdots & \beta_k & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad D^{-1}_\beta = D_{\beta^{-1}} = \begin{bmatrix} \frac{1}{\beta_0} & 0 & \cdots & 0 & \cdots \\ 0 & \frac{1}{\beta_1} & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & \cdots & \frac{1}{\beta_k} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Furthermore, $D_\beta X_{\mathfrak{F}} D^{-1}_\beta$ is an operator on the classical Hardy space $H^2$.

For a sequence $\mathfrak{F} = \{f_n\}_{n=0}^{\infty}$ in $H^2_\beta$, we always denote $\mathfrak{F}_\beta = \{f_n(\beta)\}_{n=0}^{\infty}$. Now, we could describe the boundedness of the Gram matrix of $\mathfrak{F}_\beta$ on $H^2_\beta$ by the transformation operator $X_{\mathfrak{F}}$. Since

$$\Gamma_{H^2_\beta}(\mathfrak{F}_\beta) = \langle \mathfrak{F}_\beta, \mathfrak{F}_\beta \rangle_{H^2_\beta} = D^{-1}_\beta X^*_\beta D^*_\beta X_{\mathfrak{F}} D^{-1}_\beta (D^{-1}_\beta X^*_\beta D^*_\beta) = \langle D^{-1}_\beta X^*_\beta D^*_\beta \rangle_{H^2_\beta}.$$

It is easy to see the following statements are equivalent.

1. $\Gamma_{H^2_\beta}(\mathfrak{F}_\beta)$ is a bounded operator from $H^2$ to $H^2$.
2. $D_\beta X_{\mathfrak{F}} D^{-1}_\beta$ is a bounded operator from $H^2$ to $H^2$.
3. $X_{\mathfrak{F}}$ is a bounded operator from $H^2_\beta$ to $H^2_\beta$.

Similarly, the following statements are also equivalent.

1. $\Gamma_{H^2_\beta}(\mathfrak{F}_\beta)$ is an invertible operator on $H^2$.
2. $D_\beta X_{\mathfrak{F}} D^{-1}_\beta$ is a lower bounded operator on $H^2$, i.e., there exists a constant $K > 0$ such that for any $f \in H^2$,

$$\|D_\beta X_{\mathfrak{F}} D^{-1}_\beta(f)\|_{H^2} \geq K \|f\|_{H^2}.$$
(3) $X_\beta$ is a lower bounded operator on $H^2_\beta$, i.e., there exists a constant $K' > 0$ such that for any $f \in H^2_\beta$,
\[\|X_\beta(f)\|_{H^2_\beta} \geq K' \|f\|_{H^2_\beta} .\]

Then, we could rewrite the Bari’s Theorem (Theorem 3.8) as follows.

Lemma 3.9. Let $\mathcal{F} = \{f_n\}_{n=0}^\infty$ be a total sequence in $H^2_\beta$. Then $\mathcal{F}_\beta$ is a Riesz base of $H^2_\beta$ if and only if $D_\beta X_\delta D^{-1}_\beta$ is a bounded and lower bounded operator on $H^2$.

Then, we could match some discussion in $H^2_\beta$ with the ones in $H^2_{\beta^{-1}}$ as follows.

Proposition 3.10. Let $H^2_\beta$ be the weighted Hardy space induced by a weight sequence $w = \{w_k\}_{k=1}^\infty$ with $w \rightarrow 1$. Let $\varphi_{z_0}(z) = \frac{z_0 - z}{1 - \overline{z_0}z}$, $z_0 \in \mathbb{D}$, be a Möbius transformation on $\mathbb{D}$ and denote $\mathcal{F} = \{\varphi_{z_0}^n\}_{n=0}^\infty$. Then, the following are equivalent.

1. $C_{\varphi_{z_0}}$ is a bounded operator on $H^2_\beta$.
2. $C_{\varphi_{z_0}}$ is a bounded invertible operator on $H^2_\beta$.
3. $\mathcal{F}_\beta$ is a Riesz base of $H^2_\beta$.
4. $\mathcal{F}_{\beta^{-1}}$ is a Riesz base of $H^2_{\beta^{-1}}$.
5. $C_{\varphi_{z_0}}$ is a bounded invertible operator on $H^2_{\beta^{-1}}$.
6. $C_{\varphi_{z_0}}$ is a bounded operator on $H^2_{\beta^{-1}}$.

Proof. Firstly, since $C_{\varphi_{z_0}} = X_\delta$ and $C_{\varphi_{z_0}} C_{\varphi_{z_0}} = I$, we obtain (1), (2) and (3) are equivalent. Similarly, (4), (5) and (6) are equivalent. It suffices to prove (3) $\iff$ (4).

Now assume that $\mathcal{F}_\beta$ is a Riesz base of $H^2_\beta$. By Proposition 3.3 or Proposition 3.7, $\mathcal{F}_\beta$ is a total sequence in $H^2_\beta$. Then, it follows from Lemma 3.9 that $D_\beta X_\delta D^{-1}_\beta$ is a bounded invertible operator on $H^2$. Notice that $\{\frac{\sqrt{1 - |z_0|^2}}{1 - \overline{z_0}z} \phi_{z_0}^n(z)\}_{n=0}^\infty$ is an orthonormal base in the classical Hardy space $H^2$. Then,
\[
(D_\beta X_\delta D^{-1}_\beta)(D_\beta M_{\frac{\sqrt{1 - |z_0|^2}}{1 - \overline{z_0}z}} M_{\frac{\sqrt{1 - |z_0|^2}}{1 - \overline{z_0}z}} D^{-1}_\beta)(D_\beta X_\delta D^{-1}_\beta)
\]
\[
= D_\beta X_\delta^* M_{\frac{\sqrt{1 - |z_0|^2}}{1 - \overline{z_0}z}} M_{\frac{\sqrt{1 - |z_0|^2}}{1 - \overline{z_0}z}} X_\delta D^{-1}_\beta
\]
\[
= D_\beta (\frac{\sqrt{1 - |z_0|^2}}{1 - \overline{z_0}z} \mathcal{F}_\beta, \frac{\sqrt{1 - |z_0|^2}}{1 - \overline{z_0}z} \mathcal{F}_\beta)_{H^2} D^{-1}_\beta
\]
\[
= I.
\]

Since
\[
D_\beta X_\delta D^{-1}_\beta \quad \text{and} \quad D_\beta M_{\frac{\sqrt{1 - |z_0|^2}}{1 - \overline{z_0}z}} M_{\frac{\sqrt{1 - |z_0|^2}}{1 - \overline{z_0}z}} D^{-1}_\beta
\]
are bounded invertible operators on $H^2$, we obtain that $D_\beta X_\delta^* D^{-1}_\beta$ is a bounded invertible operator on $H^2$ and so is
\[
D^{-1}_\beta X_\delta D_\beta = (D_\beta X_\delta^* D^{-1}_\beta)^*.
\]

Applying Lemma 3.9 again, we obtain (3) $\Rightarrow$ (4). In the same way, one can see (4) $\Rightarrow$ (3).

Notice that $M_z \sim M_{\varphi_{z_0}}$ if and only if $C_{\varphi_{z_0}}$ is a bounded invertible operator. Then we obtain the following corollary immediately.
Corollary 3.11. Let $H^2_\beta$ be the weighted Hardy space induced by a weight sequence $w = \{w_k\}_{k=1}^\infty$ with $w_k \to 1$. Let $\varphi_{z_0}(z) = \frac{z - z_0}{1 - z_0 z}$, $z_0 \in \mathbb{D}$, be a Möbius transformation on $\mathbb{D}$. Then, $M_z \sim M_{\varphi_{z_0}}$ on $H^2_\beta$ if and only if $M_z \sim M_{\varphi_{z_0}}$ on $H^2_{\beta-1}$.

Now let $B(z) = \prod_{j=1}^m \frac{z - z_j}{1 - \overline{z_j} z}$ be a Blaschke product on $\mathbb{D}$ with $m$ distinct zero points. Denote $\mathcal{H} = \{B^n\}_{n=0}^\infty$ and

$$\mathcal{H}_\beta = \left\{ \frac{1}{1 - \overline{z_j} z} B_n, \frac{1}{1 - \overline{z_m} z} B_n : n = 0, 1, 2, \ldots \right\}.$$

Proposition 3.12. Let $B(z) = \prod_{j=1}^m \frac{z - z_j}{1 - \overline{z_j} z}$ be a Blaschke product on $\mathbb{D}$ with $m$ distinct zero points. Then, $\Gamma_{H^2_\beta}(\mathcal{H}_\beta)$ is a bounded operator on $H^2$ if and only if $\Gamma_{H^2_\beta}(\mathcal{H}_\beta)$ is a bounded operator on $H^2$.

Proof. One can see

$$\Gamma_{H^2_\beta}(\mathcal{H}_\beta) = \langle \mathcal{H}_\beta, \mathcal{H}_\beta \rangle_{H^2} = \begin{bmatrix} \langle \mathcal{H}_\beta, \mathcal{H}_\beta \rangle_{H^2} & \langle \mathcal{H}_\beta, \mathcal{H}_\beta \rangle_{H^2} & \cdots & \langle \mathcal{H}_\beta, \mathcal{H}_\beta \rangle_{H^2} \\ \langle \mathcal{H}_\beta, \mathcal{H}_\beta \rangle_{H^2} & \langle \mathcal{H}_\beta, \mathcal{H}_\beta \rangle_{H^2} & \cdots & \langle \mathcal{H}_\beta, \mathcal{H}_\beta \rangle_{H^2} \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathcal{H}_\beta, \mathcal{H}_\beta \rangle_{H^2} & \langle \mathcal{H}_\beta, \mathcal{H}_\beta \rangle_{H^2} & \cdots & \langle \mathcal{H}_\beta, \mathcal{H}_\beta \rangle_{H^2} \end{bmatrix}.$$

Notice that for any $i, j = 1, 2, \ldots, m$

$$\langle \mathcal{H}_\beta, \mathcal{H}_\beta \rangle_{H^2} = \langle \mathcal{H}_\beta, \mathcal{H}_\beta \rangle_{H^2} = \langle \mathcal{H}_\beta, \mathcal{H}_\beta \rangle_{H^2}.$$

By the boundedness of $D_{\beta}^{-1} M^{-1} \frac{1}{1 - \overline{z_j} z}$ and $D_{\beta} M^{-1} \frac{1}{1 - \overline{z_j} z}$, $\langle \mathcal{H}_\beta, \mathcal{H}_\beta \rangle_{H^2}$ is bounded on $H^2$ if and only if $D_{\beta} X^2_{\beta} D_{\beta}^{-1}$ is bounded on $H^2$. Therefore, $\Gamma_{H^2_\beta}(\mathcal{H}_\beta)$ is a bounded operator on $H^2$ if and only if $\Gamma_{H^2_\beta}(\mathcal{H}_\beta)$ is a bounded operator on $H^2$. □

At the end of this section, we extend the boundedness of some composition operators from one weighted Hardy space to a wider class of weighted Hardy spaces.

Theorem 3.13 ([?]). Suppose that $H^2_\beta$ and $H^2_{\beta'}$ are weighted Hardy spaces and

$$w_{k+1} = \frac{\beta_{k+1}}{\beta_k} \geq \frac{\beta_{k+1}'}{\beta_k'} = w_{k+1}',$$

for $k = 0, 1, 2, \ldots$.

Let $\psi(z)$ be an analytic function on $\mathbb{D}$ with $\psi(\mathbb{D}) \subseteq \mathbb{D}$ and $\psi(0) = 0$. If $C_\psi$ is bounded on $H^2_\beta$, then $C_\psi$ is bounded on $H^2_{\beta'}$, and $\|C_\psi\|_{H^2_\beta} \geq \|C_\psi\|_{H^2_{\beta'}}$.

Remark 3.14. In C. C. Cowen’s proof [?], measure method was not used, while Hadamard product played an important role. Moreover, one can immediately generalize the above result a little. Let $\mathcal{F}$ be a sequence of functions in $\psi \in \text{Hol}(\mathbb{D})$. Suppose that the matrix representation of $X^2_{\mathcal{F}}$ under the orthogonal base $\{z^n\}_{n=0}^\infty$ is lower triangular. Then the above conclusion also holds if we replace $C_\psi$ by $X^2_{\mathcal{F}}$.

Proposition 3.15. Let $H^2_\beta$ be the weighted Hardy space induced by a weight sequence $w = \{w_k\}_{k=1}^\infty$ with $w_k \to 1$. Let $\psi \in \text{Hol}(\mathbb{D})$ such that $\psi(\mathbb{D}) \subseteq \mathbb{D}$. Denote
\( \tilde{\beta} = \{ \tilde{\beta}_n \}_{n=0}^{\infty} \), where \( \tilde{\beta}_n = (n+1)\beta_n \). If \( C_\psi \) is bounded on \( H^2_\beta \), then \( C_\psi \) is bounded on \( H^2_{\tilde{\beta}} \). Moreover, if \( \psi'(z) \) has no zero point on \( \partial \mathbb{D} \), the converse is also true.

**Proof.** Define \( D_w, D : H^2_\beta \to H^2_{\tilde{\beta}} \) by, for any \( f(z) \in H^2_{\tilde{\beta}} \)

\[
D_w f(z) = \sum_{k=0}^{\infty} w_{k+1} \tilde{\beta} f(z)^k \quad \text{and} \quad D f(z) = \sum_{k=0}^{\infty} \frac{k+2}{k+1} \tilde{\beta} f(z)^k.
\]

We could write the two operators \( D_w \) and \( D \) in matrix form under the orthogonal base \( \{ z^k \}_{k=0}^{\infty} \) as follows,

\[
D_w = \begin{bmatrix}
 w_1 & 0 & 0 & \cdots & 0 & \cdots \\
 0 & w_2 & 0 & \cdots & 0 & \cdots \\
 0 & 0 & w_3 & \cdots & 0 & \cdots \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\
 0 & 0 & 0 & \cdots & w_k & \cdots \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\
 0 & 0 & 0 & \cdots & 0 & \cdots \\
 \end{bmatrix}, \quad D = \begin{bmatrix}
 2 & 0 & 0 & \cdots & 0 & \cdots \\
 0 & \frac{3}{2} & 0 & \cdots & 0 & \cdots \\
 0 & 0 & \frac{4}{3} & \cdots & 0 & \cdots \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\
 0 & 0 & 0 & \cdots & \frac{k+1}{k} & \cdots \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\
 0 & 0 & 0 & \cdots & 0 & \cdots \\
 \end{bmatrix}.
\]

Denote \( \psi(0) = z_0 \in \mathbb{D} \). We have

\[
\langle \frac{\psi^{i+1}}{\beta_i}, \frac{\psi^{j+1}}{\beta_j} \rangle_{H^2_\beta} = \frac{1}{\beta_i \beta_j} \sum_{k=0}^{\infty} \psi^{i+1}(k) \cdot \psi^{j+1}(k) \cdot \tilde{\beta}^2
\]

\[
= \frac{1}{(i+1)(j+1)} \sum_{k=1}^{\infty} \frac{k+1}{k} w_k \psi^{i+1}(k) \cdot \tilde{\beta}^2
\]

\[
= \frac{1}{(i+1)(j+1)} \frac{1}{\beta_i \beta_j} (DD_w(\psi^{i+1})', DD_w(\psi^{j+1})')_{H^2_\beta}
\]

\[
= \frac{1}{\beta_i \beta_j} (DD_w M_\psi \psi^i, DD_w M_\psi \psi^j)_{H^2_\beta}
\]

\[
= \frac{1}{\beta_i \beta_j} (DD_w M_\psi \left( \frac{\psi^i}{\beta_i} \right), DD_w M_\psi \left( \frac{\psi^j}{\beta_j} \right))_{H^2_\beta}.
\]

Then

\[
\Gamma_{H^2_{\tilde{\beta}}} (\frac{\psi^{n+1}}{\tilde{\beta}_n})_{n=0}^{\infty} = \Gamma_{H^2_{\beta}} (DD_w M_\psi \left( \frac{\psi^n}{\beta_n} \right))_{n=0}^{\infty} + \begin{bmatrix}
 z_0 \\
 z_0^2 \\
 z_0^3 \\
 \vdots \\
 \end{bmatrix}.
\]
Consequently,
\[
\|\Gamma_{H^2_\beta}(\{DD_w M_\psi(\frac{\psi^n}{\beta_n})\}_{n=0}^\infty)\| - \frac{|z_0|^2}{1-|z_0|^2} \\
\leq \|\Gamma_{H^2_\beta}(\{\frac{\psi^{n+1}}{\beta_n}\}_{n=0}^\infty)\| \\
\leq \|\Gamma_{H^2_\beta}(\{DD_w M_\psi(\frac{\psi^n}{\beta_n})\}_{n=0}^\infty)\| + \frac{|z_0|^2}{1-|z_0|^2}.
\]

If \(C_\psi\) is bounded on \(H^2_\beta\), then \(\Gamma_{H^2_\beta}(\{\frac{\psi^n}{\beta_n}\}_{n=0}^\infty)\) is a bounded operator on \(H^2\). Since \(\psi'(z)\) also belongs to Hol(\(\mathbb{D}\)), \(M_{\psi'}\) is a bounded operator on \(H^2_\beta\). Together with the boundedness of \(D\) and \(D_w\) on \(H^2_\beta\), one can see that the boundedness of \(\Gamma_{H^2_\beta}(\{\frac{\psi^n}{\beta_n}\}_{n=0}^\infty)\) on \(H^2\) implies the boundedness of \(\Gamma_{H^2_\beta}(\{\frac{\psi^{n+1}}{\beta_n}\}_{n=0}^\infty)\) on \(H^2\).

In addition, the boundedness of \(\Gamma_{H^2_\beta}(\{\frac{\psi^{n+1}}{\beta_n}\}_{n=0}^\infty)\) on \(H^2\) is also equivalent to the boundedness of \(\Gamma_{H^2_\beta}(\{\frac{\psi^n}{\beta_n}\}_{n=0}^\infty)\) on \(H^2\). Thus, if \(C_\psi\) is bounded on \(H^2_\beta\), then \(C_\psi\) is bounded on \(H^2_\beta\). Moreover, if \(\psi'(z)\) has no zero point on \(\partial\mathbb{D}\), then \(M_{\psi'}\) is lower bounded on \(H^2_\beta\). Together with the lower boundedness of \(D\) and \(D_w\) on \(H^2_\beta\), one can see that the boundedness of \(\Gamma_{H^2_\beta}(\{\frac{\psi^n}{\beta_n}\}_{n=0}^\infty)\) on \(H^2\) implies the boundedness of \(\Gamma_{H^2_\beta}(\{\frac{\psi^{n+1}}{\beta_n}\}_{n=0}^\infty)\) on \(H^2\). Consequently, if \(C_\psi\) is bounded on \(H^2_\beta\), then \(C_\psi\) is also bounded on \(H^2_\beta\). \(\square\)

Together with Theorem 3.13 and Proposition 3.15 we can obtain the boundedness of the composition operator induced by arbitrary finite Blaschke product \(B(z)\) with \(B(0) = 0\).

**Corollary 3.16.** Let \(H^2_\beta\) be the weighted Hardy space of polynomial growth induced by a weight sequence \(w = \{w_k\}_{k=1}^\infty\). Then, for any finite Blaschke product \(B(z)\) with \(B(0) = 0\), \(C_B\) is bounded on \(H^2_\beta\).

**Proof.** Assume \(\sup_k(|k+1|w_k - 1|) \leq M \in \mathbb{N}\). Let \(H^2_\beta\) be the weighted Hardy space induced by the weight sequence \(w = \{\tilde{w}_k\}_{k=1}^\infty\), where \(\tilde{w}_k = \frac{k+M+1}{k+1}\). Let \(H^2_{\tilde{\beta}_\nu}\) be the weighted Hardy space with \(\tilde{\beta}_\nu = (k+1)^M\). Since
\[
\lim_{k \to \infty} \tilde{\beta}_k = \lim_{k \to \infty} \frac{\prod_{j=0}^{k} \frac{j+M+1}{j+1}}{(k+1)^M} = \lim_{k \to \infty} \prod_{m=0}^{M} \frac{k+m+1}{m+1} \frac{1}{M} = 1.
\]

\(H^2_\beta\) and \(H^2_{\tilde{\beta}_\nu}\) are two equivalent weighted Hardy spaces.

As well known, \(C_B\) is a bounded operator on the classical Hardy space \(H^2\).

Notice that \(B'(z)\) has no zero point on \(\partial\mathbb{D}\) (Theorem 2.1 in [4]), see also [13]). Applying Proposition 3.15 \(M\) times, one can obtain that \(C_B\) is bounded on \(H^2_\beta\), and consequently \(C_B\) is bounded on \(H^2_\beta\). Then, it follows from Theorem 3.13 that \(C_B\) is bounded on \(H^2_\beta\). \(\square\)
4. Similar representation of analytic automorphisms and finite Blaschke products

In this section, we discuss the similarity of the representation of analytic automorphisms and finite Blaschke products on a weighted Hardy space of polynomial growth.

4.1. Similar representation on weighted Hardy spaces of polynomial growth.

Following from the preliminaries in the previous section, we could give a proof of Theorem 2.1 now.

**Proof of Theorem 2.1.** Without loss of generality, we may assume \( \varphi(z) = \frac{z - z_0}{1 - \overline{z_0}z} \), \( z_0 \in \mathbb{D} \setminus \{0\} \). Let \( B(z) = z\varphi(z) \). Denote \( \mathcal{F} = \{ B^n \}_{n=0}^{\infty} \), \( \mathcal{F}_\beta = \{ B^n_{\beta} \}_{n=0}^{\infty} \).

\( \mathcal{F}_1 = \{ B^n(z), zB^n(z); n = 0, 1, \ldots \} \) and \( \mathcal{F}_2 = \{ B^n(z), \varphi(z)B^n(z); n = 0, 1, \ldots \} \).

It follows from Corollary 3.6 that \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are total and finitely linear independent in \( H^2_\beta \), respectively.

By Corollary 3.10 \( C_B \) is bounded on \( H^2_\beta \) and \( H^2_{\beta - 1} \). Consequently, \( D_\beta X_\beta D_\beta^{-1} \) and \( D_\beta^{-1} X_\beta D_\beta \) are bounded on \( H^2 \). Denote

\[
\mathcal{F}_{1,\beta} = \left\{ \frac{B^n(z)}{\beta_n}, \frac{z B^n(z)}{\beta_n}; n = 0, 1, \ldots \right\}, \quad \mathcal{F}_{2,\beta} = \left\{ \frac{B^n(z)}{\beta_n}, \frac{\varphi(z) B^n(z)}{\beta_n}; n = 0, 1, \ldots \right\}.
\]

Then,

\[
\Gamma_{H^2_\beta}(\mathcal{F}_{1,\beta}) = \langle \mathcal{F}_{1,\beta}, \mathcal{F}_{1,\beta} \rangle_{H^2_\beta} = \left[ \langle \mathcal{F}_\beta, \mathcal{F}_\beta \rangle_{H^2_\beta}, \langle \mathcal{F}_\beta, M_x \mathcal{F}_\beta \rangle_{H^2_\beta} \right]_{H^2_\beta}
\]

where

\[
\langle \mathcal{F}_\beta, \mathcal{F}_\beta \rangle_{H^2_\beta} = (D_\beta^{-1} X_\beta D_\beta)(D_\beta X_\beta D_\beta^{-1}),
\]

\[
\langle \mathcal{F}_\beta, M_x \mathcal{F}_\beta \rangle_{H^2_\beta} = (D_\beta^{-1} X_\beta D_\beta)(D_\beta^{-1} M_x \beta D_\beta)(D_\beta X_\beta D_\beta^{-1}),
\]

\[
\langle M_x \mathcal{F}_\beta, M_x \mathcal{F}_\beta \rangle_{H^2_\beta} = (D_\beta^{-1} X_\beta D_\beta)(D_\beta^{-1} M_x \beta D_\beta)(D_\beta M_x \beta D_\beta^{-1})(D_\beta X_\beta D_\beta^{-1}).
\]

Consequently, by the boundedness of \( D_\beta X_\beta D_\beta^{-1} \) and \( D_\beta M_x \beta D_\beta^{-1} \) on \( H^2 \), \( \Gamma_{H^2_\beta}(\mathcal{F}_{1,\beta}) \) is bounded on \( H^2 \) and so is \( D_\beta X_\beta^{-1} D_\beta^{-1} \). Similarly, we also obtain that \( D_\beta^{-1} X_\beta \beta D_\beta \) is bounded on \( H^2 \).

By direct calculation, it is not difficult to see that

\[
\langle \frac{1 - |z_0|^2}{1 - \overline{z_0}z}, \frac{1 - |z_0|^2}{1 - \overline{z_0}z} \rangle_{H^2} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}
\]

\[
\langle \frac{1 - |z_0|^2}{1 - \overline{z_0}z} z B^i(z), \frac{1 - |z_0|^2}{1 - \overline{z_0}z} z B^j(z) \rangle_{H^2} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}
\]

and

\[
\langle \frac{1 - |z_0|^2}{1 - \overline{z_0}z} z B^i(z), \frac{1 - |z_0|^2}{1 - \overline{z_0}z} B^j(z) \rangle_{H^2} = \begin{cases} z_0, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}.
\]
Then, we have
\[
D_\beta X_{\tilde{\delta}_1}^* M_{\frac{1}{1 - |z_0|^2}} M_{\frac{1}{1 - |z_0|^2}}^\ast X_{\tilde{\delta}_1} D_\beta^{-1}
= D_\beta(\sqrt{\frac{1 - |z_0|^2}{1 - z_0}} \tilde{\delta}_1, \sqrt{\frac{1 - |z_0|^2}{1 - z_0}} \tilde{\delta}_1)H^2 D_\beta^{-1}
\]
\[
= D_\beta \begin{bmatrix}
1 & \overline{z_0} & 0 & 0 & \cdots \\
\overline{z_0} & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & \overline{z_0} & \cdots \\
0 & 0 & \overline{z_0} & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} D_\beta^{-1}
\]
\[
= \begin{bmatrix}
1 & w_1 \overline{z_0} & 0 & 0 & \cdots \\
w_1 z_0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & w_3 \overline{z_0} & \cdots \\
0 & 0 & w_3 z_0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]
Consequently,
\[
D_\beta X_{\tilde{\delta}_1}^* M_{\frac{1}{1 - |z_0|^2}} M_{\frac{1}{1 - |z_0|^2}}^\ast X_{\tilde{\delta}_1} D_\beta^{-1}
\]
is lower bounded on $l^2$. In addition,
\[
D_\beta X_{\tilde{\delta}_1}^* M_{\frac{1}{1 - |z_0|^2}} M_{\frac{1}{1 - |z_0|^2}}^\ast X_{\tilde{\delta}_1} D_\beta^{-1}
= (D_\beta X_{\tilde{\delta}_1}^* D_\beta^{-1})(D_\beta M_{\frac{1}{1 - |z_0|^2}} M_{\frac{1}{1 - |z_0|^2}}^\ast D_\beta^{-1})(D_\beta X_{\tilde{\delta}_1} D_\beta^{-1}),
\]
one can see that both of the first item
\[
D_\beta X_{\tilde{\delta}_1}^* D_\beta^{-1} = (D_\beta^{-1} X_{\tilde{\delta}_1} D_\beta)^*
\]
and the second item
\[
D_\beta M_{\frac{1}{1 - |z_0|^2}} M_{\frac{1}{1 - |z_0|^2}}^\ast D_\beta^{-1}
\]
are bounded on $H^2$. Then, $D_\beta X_{\tilde{\delta}_1} D_\beta^{-1}$ is lower bounded on $H^2$. Therefore, $\tilde{\delta}_{1, \beta}$ is a Riesz base of $H^2_\beta$.

In the same way, we could obtain that $D_\beta X_{\tilde{\delta}_2}^* D_\beta^{-1}$ is bounded and lower bounded on $H^2$, after a computation
\[
(D_\beta X_{\tilde{\delta}_2}^* D_\beta^{-1})(D_\beta X_{\tilde{\delta}_2} D_\beta^{-1})
\]
\[
= D_\beta X_{\tilde{\delta}_2}^* X_{\tilde{\delta}_2} D_\beta^{-1}
\]
\[
= \begin{bmatrix}
1 & w_1 \overline{z_0} & 0 & 0 & \cdots \\
w_1 z_0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & w_3 \overline{z_0} & \cdots \\
0 & 0 & w_3 z_0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]
Therefore, $\tilde{\delta}_{2, \beta}$ is also a Riesz base of $H^2_\beta$.

It follows from $B(z) = z\varphi(z)$ and $\varphi(\varphi(z)) = z$ that for every $n = 0, 1, 2, \ldots$,
\[
C_\varphi(B^n(z)) = B^n(\varphi(z)) = B^n(z)
\]
and
\[ C_\varphi(zB^n(z)) = \varphi(z)B^n(\varphi(z)) = \varphi(z)B^n(z). \]

Then, the composition operator \( C_\varphi \) is just a base transformation operator from the Riesz base \( \tilde{\mathcal{S}}_{1,\beta} \) to the Riesz base \( \tilde{\mathcal{S}}_{2,\beta} \). Thus, \( C_\varphi \) is a bounded invertible operator on \( H_3^{2,\beta} \). Furthermore, \( M_z \sim M_\varphi \), i.e., \( M_z \) is weakly homogeneous on \( H_3^{2,\beta} \).

By the way, \( C_\varphi \) is bounded on \( H_3^{2,\beta} \) if and only if the sequence \( \{ z^n \}_{n=0}^\infty \) is a Riesz base in \( H_3^{2,\beta} \). In particular, the boundedness of \( C_\varphi \) implies the sequence \( \{ \tilde{\beta}_n \}_{n=0}^\infty \) is bounded (or quasinormed) in \( H_3^{2,\beta} \).

According to the boundedness of the composition operators induced by Möbius transformations, C. C. Cowen’s theorem (Theorem 3.13 in the present paper) could be extended as the following result.

**Corollary 4.1.** Let \( H_3^{2,\beta} \) and \( H_3^{2,\beta'} \) be two weighted Hardy spaces. Suppose that \( H_3^{2,\beta} \) is of polynomial growth and

\[ w_{k+1} = \frac{\beta_{k+1}}{\beta_k} \geq \frac{\beta_{k+1}'}{\beta_k'} = w_{k+1}', \quad \text{for } k = 0, 1, 2, \ldots. \]

Let \( \psi(z) \) be an analytic function on \( \mathbb{D} \) with \( \psi(\mathbb{D}) \subseteq \mathbb{D} \). If \( C_\psi \) is bounded on \( H_3^{2,\beta} \), then \( C_\psi \) is bounded on \( H_3^{2,\beta'} \).

**Proof.** Let \( z_0 = \psi(0) \in \mathbb{D} \), and let \( \varphi(z) = \frac{z-z_0}{1-\bar{z}_0z} \). Obviously, the polynomial growth of \( H_3^{2,\beta} \) implies the polynomial growth of \( H_3^{2,\beta'} \). Then, by Theorem 2.2, the composition operator \( C_\varphi \) is bounded on \( H_3^{2,\beta} \) and \( H_3^{2,\beta'} \), respectively. Consider the function \( \varphi \circ \psi \). Since \( C_\varphi \circ \psi \) is bounded on \( H_3^{2,\beta} \) and \( \varphi \circ \psi(0) = \psi(z_0) = 0 \), by C. C. Cowen’s result Theorem [3.13], one can see that \( C_\psi \circ \psi \) is also bounded on \( H_3^{2,\beta'} \). Furthermore, it follows from the boundedness of \( C_\varphi \) on \( H_3^{2,\beta'} \) that \( C_\psi = C_\varphi \circ \psi \circ C_\varphi \) is bounded on \( H_3^{2,\beta'} \). \( \square \)

Furthermore, we could obtain the boundedness of composition operators induced by analytic functions on \( \mathbb{D} \) on weighted Hardy spaces of polynomial growth.

**Proof of Theorem 2.2.** Assume \( \sup_k \{(k+1)|w_k - 1|\} \leq M \in \mathbb{N} \). Let \( H_3^{2,\beta} \) be the weighted Hardy space induced by the weight sequence \( \tilde{w} = \{ \tilde{w}_k \}_{k=1}^\infty \), where \( \tilde{w}_k = \frac{k+M+1}{k} \).

As well know, the composition operator \( C_\psi \) is bounded on the classical Hardy space \( H^2 \), in fact, \( \|C_\psi\|_{H^2} \leq 1 \). As the same as the proof of Corollary 4.1, one can obtain that the composition operator \( C_\psi \) is also bounded on \( H_3^{2,\beta} \) by Proposition 3.13. Then, it follows from Corollary 4.1 that \( C_\psi \) is bounded on \( H_3^{2,\beta'} \). \( \square \)

Next, let us consider finite Blaschke products.

**Lemma 4.2.** Suppose that \( H_3^{2,\beta} \) is a weighted Hardy space of polynomial growth. Let \( B(z) = \prod_{j=1}^m \frac{z-z_j}{1-\bar{z}_jz} \) be a Blaschke product on \( \mathbb{D} \) with \( m \) distinct zero points. Denote \( \tilde{\mathcal{S}} = \{ B^n \}_{n=0}^\infty \), \( \tilde{\mathcal{S}}_{\beta} = \{ \frac{B^n}{\beta^n} \}_{n=0}^\infty \),

\[ \tilde{\mathcal{S}} = \bigcup_{j=1}^m \tilde{\mathcal{S}}_{\beta_j} = \{ \frac{B^n}{1-z_1^n}, \ldots, \frac{B^n}{1-z_m^n} ; n = 0, 1, 2, \ldots \}. \]
and
\[ \tilde{\mathcal{B}} = \bigcup_{j=1}^{m} \frac{1}{1-\bar{z}jz} \mathcal{B} = \left\{ \frac{1}{1-\bar{z}jz} B_n, \cdots, \frac{1}{1-\bar{z}mz} B_n ; n = 0, 1, 2, \ldots \right\}. \]

Then \( \tilde{\mathcal{B}} \) is a Riesz base of \( H^2_\beta \).

Proof. First of all, we may assume \( z_1 = 0 \) without loss of generality, because \( C_{\varphi_{z_1}} \) is a bounded invertible operator on \( H^2_\beta \) by Theorem \( 2.4.1 \) and consequently \( B(\varphi_{z_1}(z)) \) is also a Blaschke product on \( \mathbb{D} \) with \( m \) distinct zero points and \( B(0) = 0 \).

Following from Theorem \( 3.5 \), \( \tilde{\mathcal{B}} \) is a total and finitely linear independent sequence in \( H^2_\beta \). By Lemma \( 3.9 \), it suffices to prove \( D_\beta X_\tilde{\mathcal{B}} \tilde{D}_\beta^{-1} \) is a bounded and lower bounded operator on \( H^2_\beta \).

According to Corollary \( 3.10 \), \( C_B \) is bounded on \( H^2_\beta \) and \( H^2_{\beta-1} \). Consequently, both \( D_\beta X_\tilde{\mathcal{B}} \tilde{D}_\beta^{-1} \) and \( D_\beta^{-1} X_\tilde{\mathcal{B}} \tilde{D}_\beta \) are bounded on \( H^2_\beta \).

On the other hand, since
\[ \langle B^j(z), B^j(z) \rangle_{H^2} = \begin{cases} \frac{1}{1-\bar{z}jz}, & \text{if } i = j, \\ 0, & \text{if } i \neq j \end{cases}, \]
we have
\[ (D_\beta X_\tilde{\mathcal{B}} \tilde{D}_\beta^{-1})(D_\beta X_\tilde{\mathcal{B}} \tilde{D}_\beta^{-1}) = \begin{bmatrix} D_0 & 0 & 0 & \cdots \\ 0 & D_1 & 0 & \cdots \\ 0 & 0 & D_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} A & 0 & 0 & \cdots \\ 0 & A & 0 & \cdots \\ 0 & 0 & A & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} D_0^{-1} & 0 & 0 & \cdots \\ 0 & D_1^{-1} & 0 & \cdots \\ 0 & 0 & D_2^{-1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \]
where
\[ A = \begin{bmatrix} \frac{1}{1-\bar{z}_1^2} & \frac{1}{1-\bar{z}_1z_2} & \cdots & \frac{1}{1-\bar{z}_1z_m} \\ \frac{1}{1-\bar{z}_2z_1} & \frac{1}{1-\bar{z}_2^2} & \cdots & \frac{1}{1-\bar{z}_2z_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{1-\bar{z}_mz_1} & \frac{1}{1-\bar{z}_mz_2} & \cdots & \frac{1}{1-\bar{z}_m^2} \end{bmatrix} \quad \text{and} \quad X_\tilde{\mathcal{B}} = \begin{bmatrix} A & 0 & 0 & \cdots \\ 0 & A & 0 & \cdots \\ 0 & 0 & A & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \]
and for \( k = 0, 1, 2, \ldots, \)
\[ D_k = \begin{bmatrix} \beta_{km} & 0 & 0 & \cdots & 0 \\ 0 & \beta_{km+1} & 0 & \cdots & 0 \\ 0 & 0 & \beta_{km+2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_{km+m-1} \end{bmatrix}. \]

Then, \( (D_\beta X_\tilde{\mathcal{B}} \tilde{D}_\beta^{-1})(D_\beta X_\tilde{\mathcal{B}} \tilde{D}_\beta^{-1}) \) is lower bounded on \( H^2_\beta \). In addition to the boundedness of the first item \( D_\beta X_\tilde{\mathcal{B}} \tilde{D}_\beta^{-1} = (D_\beta^{-1} X_\tilde{\mathcal{B}} \tilde{D}_\beta)^* \), we obtain that \( D_\beta X_\tilde{\mathcal{B}} \tilde{D}_\beta^{-1} \) is lower bounded on \( H^2_\beta \). This completes the proof. \( \square \)
**Proof of Theorem 2.3** Without loss of generality, assume \( B(z) = \prod_{j=1}^{m} \frac{z - \alpha_j}{1 - \bar{\alpha}_j z} \).

Firstly, consider the case of that \( B(z) \) has \( m \) distinct zeros. Let

\[
\tilde{\Phi} = \bigcup_{j=1}^{m} \Phi_j = \{ \frac{B_n}{1 - \bar{\alpha}_j z} : n = 0, 1, 2, \ldots \}.
\]

Then, by Lemma 1.2, \( \tilde{\Phi} \) is a Riesz base of \( H_\beta^2 \).

For \( j = 1, 2, \ldots, m \), denote by \( \{ \epsilon_{jn} = \frac{z^n}{\beta_n} \}_{n=0}^{\infty} \) the orthonormal base of the \( j \)-th space of \( \bigoplus_1^{m} H_\beta^2 \). Then, the sequence

\[
\Phi_j = \{ \epsilon_{jn} : j = 1, 2, \ldots, m \text{ and } n = 0, 1, 2, \ldots \}
\]

is an orthonormal base of \( \bigoplus_1^{m} H_\beta^2 \).

Define the linear operator \( X : \bigoplus_1^{m} H_\beta^2 \rightarrow H_\beta^2 \) by

\[
X(\epsilon_{jn}) = \frac{1}{1 - \bar{\alpha}_j z} \beta_n.
\]

Then \( X \) maps the orthonormal base \( \Phi_j \) to the Riesz base \( \tilde{\Phi} \), and consequently \( X \) is a bounded invertible operator. Obviously,

\[
X\left( \bigoplus_{1}^{m} M_z \right) = M_B X.
\]

Thus, \( M_B \sim \bigoplus_1^{m} M_z \).

Now consider the general case. For any Blaschke product \( B(z) \) with order \( m \), there exists an analytic automorphism \( \varphi(z) = \frac{z^m}{\beta_m} \) such that \( \varphi(B(z)) \) is a Blaschke product with \( m \) distinct zeros.

Following from \( M_{\varphi(B)} \sim \bigoplus_1^{m} M_z \) and Theorem 2.1 we have

\[
M_B = \varphi(B) \left( \bigoplus_{1}^{m} M_z \right) = \bigoplus_{1}^{m} M_{\varphi(B)} \sim \bigoplus_{1}^{m} M_z.
\]

This completes the proof. \( \square \)

**4.2. Examples and necessity of polynomial growth condition.** The weighted Hardy spaces of polynomial growth cover the weighted Bergman spaces, the weighted Dirichlet spaces and many weighted Hardy spaces defined without measures.

**Example 4.3.** Let \( dA \) denote the Lebesgue area measure on the unit open disk \( \mathbb{D} \), normalized so that the measure of \( \mathbb{D} \) equals 1. For \( \alpha \geq 0 \), the weighted Bergman space \( A_\alpha^2 \) is the space of analytic functions on \( \mathbb{D} \) which are square-integrable with respect to the measure \( dA_\alpha(z) = (1 - |z|^2)^\alpha dA(z) \). As well known, it could be seemed as a weighted Hardy space \( H^2_{\beta(\alpha)} \), where \( \beta_0(\alpha) = 1 \) and for \( k = 1, 2, \ldots, \)

\[
\beta_k(\alpha) = \prod_{j=1}^{k} w_j^{(\alpha)} \text{ with } w_j^{(\alpha)} = \sqrt{\frac{j+1}{j+2\alpha+1}}.
\]

Since

\[
\lim_{j \to \infty} \frac{j+1}{j+2\alpha+1} = 1,
\]

every weighted Bergman space \( A_\alpha^2 \) satisfies the polynomial growth condition.

Similarly, the weighted Dirichlet space \( D_\lambda^2 \), for \( \lambda \geq 0 \), could be seemed as a weighted Hardy space \( H^2_{\beta(\lambda)} \), where \( \beta_0(\lambda) = 1 \) and for \( k = 1, 2, \ldots, \beta_k(\lambda) = \prod_{j=1}^{k} w_j^{(\lambda)} \).
with \( w_j^{(\lambda)} = \sqrt{\frac{j^2 + 2\lambda^2 + 1}{j+1}} \); the Sobolev space \( R(D) \) could be seemed as a weighted Hardy space \( H^2_{\beta(s)} \), where

\[
\beta_k^{(s)} = \prod_{j=1}^{k} w_j^{(s)} = \sqrt{\frac{k+1}{(k^2 - k^2 + 2k + 1)^{\pi}}} \quad \text{for } k = 0, 1, 2, \ldots.
\]

Both the weighted Dirichlet space and Sobolev space satisfy the polynomial growth condition. In particular, \( H^2_{\beta(\lambda)} \) is just the weighted Bergman space \( A^2_{\lambda} \).

Beyond the weighted Hardy spaces defined by measures such as the weighted Bergman spaces, there are many weighted Hardy spaces of polynomial growth defined without measures.

**Example 4.4.** Let \( H_{\beta}^2 \) be the weighted Hardy space with

\[
\beta_0 = 1 \quad \text{and} \quad \beta_k = \frac{1}{\ln(k+1)}, \quad \text{for } k = 1, 2, \ldots.
\]

Then \( H_{\beta}^2 \) satisfies the polynomial growth condition. This weighted Hardy space is not equivalent to the weighted Bergman space, the weighted Dirichlet space or the Sobolev space.

In addition, the polynomial growth condition does not require the monotonicity of weight sequence.

**Example 4.5.** Given \( \alpha \geq 0 \). Let \( H_{\bar{\beta}}^2 \) be the weighted Hardy space induced by the weight sequence \( \tilde{w} = \{\tilde{w}_k\}_{k=1}^\infty \), where \( \tilde{w}_k \) is choosen to be the Bergman weight \( w_j^{(\alpha)} \) or its reciprocal randomly, i.e.,

\[
\tilde{w}_k = \left( \sqrt{\frac{j + 1}{j + 2\alpha + 1}} \right)^\epsilon_j, \quad \epsilon_j \in \{-1, 1\}, \quad \text{for } k = 1, 2, \ldots.
\]

Then \( H_{\bar{\beta}}^2 \) satisfies the polynomial growth condition, although \( \bar{w} = \{\tilde{w}_k\}_{k=1}^\infty \) is not a monotone sequence.

Now we will illustrate the necessity of the setting of polynomial growth condition. It is given a counterexample to show that \( M_\varphi \) could be not similar to \( M_\varphi \), for some \( \varphi \in \text{Aut}(D) \), on a weighted Hardy space of intermediate growth. In fact, we will show that the sequence \( \{\tilde{w}^{(\alpha)}_{\beta_n}\}_{n=0}^\infty \) is not bounded in a weighted Hardy space of intermediate growth.

Let us begin from some estimations of the norms of some functions and composition operators on the classical Hardy space and the weighted Bergman spaces.

**Lemma 4.6.** Let \( \varphi_t(z) = \frac{t-z}{1-t\bar{z}} \), for any \( t \in (0, 1) \). Then the norm of \( \alpha \)-th power of the derivative of \( \varphi_t(z) \) in the classical Hardy space \( H^2 \) tends to infinite as the positive number \( \alpha \) tends to infinite. More precisely, for any \( \alpha \geq \frac{1}{2} \),

\[
\|\varphi_t'(z)^\alpha\|_{H^2} \geq \frac{(1+t)^\alpha}{(1-t)^{\alpha-1}} \sqrt{2\pi(2\alpha - 1)}.
\]
Lemma 4.7. Let \( \varphi(z) = \frac{1}{1-tz} \), for any \( t \in (0, 1) \). Then for any \( \alpha \geq \frac{1}{2} \), if \( n \) is large enough, we have
\[
\| (\varphi^\alpha(z)) \|_{H^2} \geq \frac{(1 + t)^\alpha}{(1 - t)^{\alpha - 1}} \rightarrow \infty \quad \text{as} \quad \alpha \rightarrow \infty.
\]

Proof. Following from [22], one can see the unitary representation \( D^+_\alpha \) for the analytic automorphism group \( \text{Aut}(\mathbb{D}) \) on the weighted Bergman space \( A^2_\alpha \).

Then, we have
\[
\| \varphi^n(z) \|_{A^2_\alpha} = \| (\varphi^\alpha(z))^n \cdot (\varphi^\alpha \circ \varphi^\alpha(z)) \|_{A^2_\alpha} = \| z^n (\varphi^\alpha(z))^n \|_{A^2_\alpha}.
\]

Denote \( (\varphi^\alpha(z))^n = \sum_{k=0}^N c_k z^k \). There exists \( N \in \mathbb{N} \) such that
\[
\sum_{k=0}^N |c_k|^2 \geq \frac{1}{2} \sum_{k=0}^\infty |c_k|^2 = \frac{1}{2} \| (\varphi^\alpha)^n(z) \|_{H^2}^2.
\]

Furthermore, if \( n \) is large enough, we have
\[
\frac{\beta_{k+n}^{(\alpha)}}{\beta_n^{(\alpha)}} \geq \frac{\sqrt{2}}{2} \quad \text{for} \quad k = 1, 2, \ldots, N.
\]

Then,
\[
\frac{\| \varphi^n(z) \|_{A^2_\alpha}}{\| z^n \|_{A^2_\alpha}} = \frac{\| z^n (\varphi^\alpha(z))^n \|_{A^2_\alpha}}{\beta_n^{(\alpha)}} \geq \sqrt{\sum_{k=0}^N |c_k|^2 \left( \frac{\beta_{k+n}^{(\alpha)}}{\beta_n^{(\alpha)}} \right)^2} \geq \frac{1}{2} \| (\varphi^\alpha)^n(z) \|_{H^2}.
\]

\[\square\]
Next, we could show that $C_{\varphi_i}$ is unbounded on a class of weighted Hardy spaces of intermediate growth.

**Proof of Theorem 2.4.** Since
\[
\lim_{k \to \infty} \frac{\|\varphi^{n_j}_i(z)\|_{H^{-1}}^2}{\|z^{n_j}\|_{H^{-1}}^2} = \sum_{k=0}^{\infty} \frac{|\varphi^{n_j}_i(k)|^2}{\beta_k^2} \geq \sum_{k=0}^{n_j} \frac{|\varphi^{n_j}_i(k)|^2}{\beta_k^2} = \sum_{k=n_j+1}^{\infty} \frac{|\varphi^{n_j}_i(k)|^2}{\beta_k^2} - \sum_{k=n_j+1}^{\infty} |\varphi^{n_j}_i(k)|^2
\]
\[
\geq \frac{\|\varphi^{n_j}_i(z)\|_{H^{-1}}^2}{\|z^{n_j}\|_{H^{-1}}^2} = 1,
\]
it follows from Lemma 4.6 and Lemma 4.7 that
\[
\frac{\|\varphi^{n_j}_i(z)\|_{H^{-1}}^2}{\|z^{n_j}\|_{H^{-1}}^2} \to \infty, \quad \text{as } j \to \infty.
\]

Then the composition operator $C_{\varphi_i} : H^2_{\beta} \to H^2_{\beta}$ is unbounded, and so is $C_{\varphi_i} : H^2_{\beta} \to H^2_{\beta}$ by Theorem 3.10.

Notice that $\lim_{k \to \infty} (k+1)(w_k-1) = -\infty$ if and only if $\lim_{k \to \infty} (k+1)\left(\frac{1}{w_k} - 1\right) = +\infty$. By Proposition 3.10 it suffices to consider the case of $\lim_{k \to \infty} (k+1)(w_k-1) = +\infty$. Let $\alpha_j = j$ for any $j \in \mathbb{N}$. Then for any $j \in \mathbb{N}$, there exists a positive integer $m_j$ such that for every $k > m_j$, $w_k \geq \frac{1}{w_k^{m_j}}$. Since
\[
\lim_{k \to \infty} \beta_k \cdot \beta^{(\alpha_j)}_k = \infty,
\]
there exists a positive integer $n_j > m_j$, such that $\beta_n_j \geq \frac{\beta_{m_j}}{\beta^{(\alpha_j)}_{n_j}}$. Then, for every $0 \leq k \leq m_j$, we have
\[
\frac{\beta_{n_j}}{\beta_k} \geq \frac{\beta_{m_j}}{\beta_k \cdot \beta^{(\alpha_j)}_{n_j}} \geq \frac{1}{\beta^{(\alpha_j)}_{n_j}} \geq \beta^{(\alpha_j)}_{n_j}
\]
and for every $m_j < k \leq n_j$, we have
\[
\frac{\beta_{n_j}}{\beta_k} = \prod_{i=k+1}^{n_j} w_i \geq \prod_{i=k+1}^{n_j} w^{(\alpha_j)}_i = \beta^{(\alpha_j)}_{n_j}.
\]
This completes the proof. \[ \square \]

**Example 4.8.** Let $\beta = \{\beta_k\}_{k=0}^{\infty}$, where $\beta_0 = 1$ and $\beta_k = (k+1)\ln(k+1)$ for $k \geq 1$. Obviously, $w_k = e^{\ln^2(k+1)-\ln^2 k}$ for all $k = 1, 2, \ldots$. It is not difficult to see that $w_k = e^{\ln^2(k+1)-\ln^2 k} \to 1$ as $k \to \infty$ and $\lim_{k \to \infty} (k+1)(w_k-1) = +\infty$. Then, by Theorem 2.4 the composition operator $C_{\varphi_i} : H^2_{\beta} \to H^2_{\beta}$ is unbounded, and so is $C_{\varphi_i} : H^2_{\beta} \to H^2_{\beta}$ by Proposition 3.10.
Remark 4.9. In the above example, $C_{\varphi_t}$ is unbounded on $H^2_{\beta-1}$. However, it follows from C. C. Cowen’s result (Theorem 3.13 in the present article) that $C_{z\varphi_t}$ is bounded on $H^2_{\beta-1}$. This means that the boundedness of $C_{z\varphi_t}$ does not imply the boundedness of $C_{\varphi_t}$.

5. JORDAN DECOMPOSITION AND SIMILARITY CLASSIFICATION OF THE REPRESENTATION OF ANALYTIC FUNCTIONS ON WEIGHTED HARDY SPACES OF POLYNOMIAL GROWTH

In this section, we always assume that $H^2_{\beta}$ is the weighted Hardy space of polynomial growth induced by a weight sequence $w = \{w_k\}_{k=1}^\infty$. We aim to study the Jordan decomposition of the representation of $\text{Hol}(D)$ on the weighted Hardy space $H^2_{\beta}$. Strongly irreducible operator is a suitable substitute for Jordan block, and strongly irreducible decomposition is seemed as the Jordan decomposition of an infinite-dimensional matrix.

Let us recall some relevant basic concepts and notations. Let $H$ be a complex separable Hilbert space and $L(H)$ be the set of all bounded linear operators from $H$ to itself. For any $T \in L(H)$, denote by $\{T\}_H'$ the commutant of $T$ in $L(H)$. Moreover, a nonzero idempotent $P$ in the commutant $\{T\}_H'$ is said to be minimal if for each idempotent $Q \in \{T\}_H'$, $\text{Ran}Q \subsetneq \text{Ran}P$ implies $Q = 0$. Here $\text{Ran}P$ denotes the range of the operator $P$. A bounded operator $T$ on a Hilbert space $H$ is said to be strongly irreducible if there is no nontrivial idempotent operator in its commutant. If $\Psi = \{P_j; j = 1, 2, \ldots, m\}$ be a family of minimal idempotents such that

$$\sum_{j=1}^l P_j = I, \quad P_i P_j = 0 \text{ if } i \neq j.$$  

We say that $\Psi$ is a strongly irreducible decomposition of $T$. For convenience, we use $H^{(m)}$ to denote the direct sum of $m$ copies of $H$ and $T^{(m)}$ to denote the direct sum of $m$ copies of $T$ acting on $H^{(m)}$.

Firstly, we need the following theorem. The version of the following theorem on the Hardy space was obtained in [29]. For more general results on the Hardy space, see [28] and [6]. The method in [29] also works on the weighted Bergman spaces by replacing the reproducing kernel on the Hardy space by one on the weighted Bergman spaces $A^2_{\alpha}$ [20]. Notice that

$$k(z, \omega) = \sum_{k=0}^{\infty} \frac{z^k \omega^k}{\beta_k^2}$$

is the reproducing kernel of the weighted Hardy space $H^2_\beta$. Similarly, we can give a proof of the following theorem as the same as the proof in [29] ( pp. 524-528), except replacing the reproducing kernel on the classical Hardy space by one on the weighted Hardy space $H^2_\beta$.

**Theorem 5.1.** For any $f \in \text{Hol}(\overline{D})$, there exist a finite Blaschke product $B$ and a function $h \in \text{Hol}(\overline{D})$ such that $f = h \circ B$ and $\{M_f\}'_{H^2_\beta} = \{M_B\}'_{H^2_\beta}$.

Furthermore, we could obtain an analogue of Jordan decomposition for the representation of $\text{Hol}(\overline{D})$ on a weighted Hardy space $H^2_\beta$ of polynomial growth. Notice that the following four facts hold.
\( \{ M_f \}'_{H_2^β} = \{ M_f; f \in H_∞^β \} \) and consequently \( M_z \) is strongly irreducible.

2. For any \( m \in \mathbb{N} \), \( \bigoplus_{j=1}^m M_z \)'_{H_2^β} = \{ M_F; F \in M_m(H_∞^β) \} \).

3. Let \( P_j \) be the projection from \( \bigoplus_{j=1}^m H_2^β \) onto the \( j \)-th component. Then \( \Psi = \{ P_j; j = 1, 2, \ldots m \} \) is a strongly irreducible decomposition of \( \bigoplus_{j=1}^m M_z \).

4. Let \( B \) be a finite Blaschke product with order \( m \). Then \( M_B \sim \bigoplus_{j=1}^m M_z \) by theorem 2.3, i.e., there exists a bounded invertible operator \( X : \bigoplus_{j=1}^m H_2^β \rightarrow H_2^β \) such that
\[
X(M_z)X^{-1} = M_B.
\]

Then \( \{ XP_jX^{-1}; j = 1, 2, \ldots m \} \) is also a strongly irreducible decomposition of \( M_B \), where \( \Psi = \{ P_j; j = 1, 2, \ldots m \} \) is defined in above (3).

Then the method of Jiang and Zheng, to prove the following result on weighted Bergman spaces \( A_2^α \) (Lemma 3.3 in [20]), also works on the weighted space \( H_2^β \).

**Theorem 5.2.** Given any \( f \in \text{Hol}(\mathbb{D}) \). Suppose that \( f = h \circ B \), where \( h \in \text{Hol}(\mathbb{D}) \) and \( B \) is a finite Blaschke product with order \( m \) such that \( \{ M_f \}'_{H_2^β} = \{ M_B \}'_{H_2^β} \).

Then
\[
M_f \sim \bigoplus_{1}^{m} M_h.
\]
\( \{ M_h \}'_{H_2^β} = \{ M_z \}'_{H_2^β} \) and \( M_h \) is strongly irreducible.

Now we have obtained the Jordan decomposition of the representation of \( \text{Hol}(\mathbb{D}) \) on \( H_2^β \). Since there is no standard form of strongly irreducible operators in the sense of similarity, a natural question is how to characterize the similarity of Jordan representation. First, let us consider the "Jordan block" \( M_h \) with \( h \in \text{Hol}(\mathbb{D}) \) and \( \{ M_h \}'_{H_2^β} = \{ M_z \}'_{H_2^β} \).

**Lemma 5.3.** Let \( h_1, h_2 \in \text{Hol}(\mathbb{D}) \) with \( \{ M_{h_1} \}'_{H_2^β} = \{ M_{h_2} \}'_{H_2^β} = \{ M_z \}'_{H_2^β} \). Then
\[
M_{h_1} \sim M_{h_2}
\]
if and only if there exists a Möbius transformation \( \varphi \) such that
\[
h_2 = h_1 \circ \varphi.
\]

**Proof.** If there is a Möbius transformation \( \varphi \) such that \( h_2 = h_1 \circ \varphi \), then
\[
C_\varphi M_{h_1} = M_{h_2}C_\varphi,
\]
where the composition operator \( C_\varphi \) is a bounded invertible operator on \( H_2^β \) by Theorem 2.3.

Suppose that \( X \) is a bounded invertible operator on \( H_2^β \) such that
\[
XM_{h_1} = M_{h_2}X.
\]
Then
\[
X\{ M_{h_1} \}'_{H_2^β} X^{-1} = \{ M_{h_2} \}'_{H_2^β}.
\]
Since \( \{M_h\}_n = \{M_{h_0}\}_n = \{M_z\}_n = H_\beta^\infty \), there exists a function \( g \in H_\beta^\infty \) such that

\[
X M_z X^{-1} = M_g.
\]

This implies that \( X \) is just the composition operator \( C_g \). Similarly, one can obtain that \( X^{-1} \) is also a composition operator \( C_{\psi} \), where \( \psi \) is a function in \( H_\beta^\infty \). Notice that

\[
g(\mathbb{D}) \subseteq \mathbb{D}, \ \psi(\mathbb{D}) \subseteq \mathbb{D}, \ \text{and} \ \ g \circ \psi = \psi \circ g = Id_\mathbb{D}.
\]

Then \( g \) is an analytic automorphism on \( \mathbb{D} \). Furthermore, it follows from

\[
C_g M_{h_1} = M_{h_2} C_g
\]

that

\[
h_2 = h_1 \circ g.
\]

The proof is finished. \( \square \)

Remark 5.4. As a special case of above lemma, the converse of Theorem 2.1 is also true. Let \( \psi \in \text{Hol}(\mathbb{D}) \). If \( M_z \sim M_{\eta} \) on \( H_\beta^2 \), then \( \psi \) must be a Möbius transformation. In another word, if \( C_\psi \) is a bounded invertible operator on \( H_\beta^2 \), then \( \psi \) is a Möbius transformation.

To study the similarity of the operator representation, K-theory of Banach algebra is a powerful technique, for instance, a similarity classification of Cowen-Douglas operators was given by using the ordered K-group of the commutant algebra as an invariant \cite{16} and \cite{17}. Let \( B \) be a Banach algebra and \( \text{Proj}(B) \) be the set of all idempotents in \( B \). The algebraic equivalence "\( \sim_a \)" is introduced in \( \text{Proj}(B) \). Let \( e \) and \( \bar{e} \) be two elements in \( \text{Proj}(B) \). We say that \( e \sim_a \bar{e} \) if there are two elements \( x, y \in B \) such that

\[
xy = e \ \text{and} \ yx = \bar{e}.
\]

Let \( \text{Proj}(B) \) denote the algebraic equivalence classes of \( \text{Proj}(B) \) under algebraic equivalence "\( \sim_a \)". Let

\[
M_\infty(B) = \bigcup_{n=1}^{\infty} M_n(B),
\]

where \( M_n(B) \) is the algebra of \( n \times n \) matrices with entries in \( B \). Set

\[
\bigvee(B) = \text{Proj}(M_\infty(B)).
\]

Then \( \bigvee(M_n(B)) \) is isomorphic to \( \bigvee(B) \). The direct sum of two matrices gives a natural addition in \( M_\infty(B) \) and hence induces an addition "\( + \)" in \( \text{Proj}(M_\infty(B)) \) by

\[
[p] + [q] = [p \oplus q],
\]

where \( [p] \) denotes the equivalence class of the idempotent \( p \). Furthermore, \( \bigvee(B), + \) forms a semigroup and depends on \( B \) only up to stable isomorphism, and then \( K_0(B) \) is the Grothendieck group of \( \bigvee(B) \).

Theorem 5.5 \cite{2}, see also in \cite{19}. Let \( T \) be a bounded operator on a Hilbert space \( \mathcal{H} \). The following are equivalent:

1. \( T \) is similar to \( \sum_{i=1}^{k} \oplus A_i^{(n_i)} \) under the space decomposition \( \mathcal{H} = \sum_{i=1}^{k} \oplus \mathcal{H}_i^{(n_i)} \), where \( k \) and \( n_i \) are finite, \( A_i \) is strongly irreducible and \( A_i \) is not similar to \( A_j \) if \( i \neq j \). Moreover, \( T^{(n)} \) has a unique strongly irreducible decomposition up to similarity.
(2) The semigroup $\bigvee\{(T)^*\}$ is isomorphic to the semigroup $\mathbb{N}^{(k)}$, where $\mathbb{N}$ is the set of all natural numbers $\{0, 1, 2, \ldots\}$ and the isomorphism $\phi$ sends 
\[ [I] \rightarrow n_1e_1 + n_2e_2 + \cdots + n_ke_k, \]
where $\{e_i\}_{i=1}^k$ are the generators of $\mathbb{N}^{(k)}$ and $n_i \neq 0$.

**Corollary 5.6** ([3], see also in [19]). Let $T_1$ and $T_2$ be two strongly irreducible operators on a Hilbert space $\mathcal{H}$, and $T \sim T_1^{(n_1)} \oplus T_2^{(n_2)}$. If $\bigvee\{(T)^*\}$ is isomorphic to the semigroup $\mathbb{N}$, then $T_1$ is similar to $T_2$.

To study the similarity of the Jordan representation, we would also use some fundamental properties of Fredholm operators (refer to [11]). A bounded operator $T$ on a Hilbert space $\mathcal{H}$ is said to be Fredholm, if $\dim\ker T$ and $\dim\ker T^*$ are finite. Moreover, the Fredholm index of $T$ is defined by
\[ \text{ind} T = \dim\ker T - \dim\ker T^*. \]
Notice that Fredholm index is a similar invariant and for any $n \in \mathbb{N}$, $T^{(n)}$ is also Fredholm with $\text{ind} T^{(n)} = n \cdot \text{ind} T$.

**Theorem 5.7.** Let $h_1, h_2 \in \text{Hol}(\mathbb{D})$ with $\{M_{h_1}\}_{H^2}^\prime = \{M_{h_2}\}_{H^2}^\prime = \{M_z\}_{H^2}^\prime$, and let $m_1$ and $m_2$ be two positive integers. Then
\[ \bigoplus_{1}^{m_1} M_{h_1} \sim \bigoplus_{1}^{m_2} M_{h_2} \]
if and only if $m_1 = m_2$ and $M_{h_1} \sim M_{h_2}$, i.e., there exists a Möbius transformation $\varphi \in \text{Aut}(\mathbb{D})$ such that $h_2 = h_1 \circ \varphi$.

**Proof.** Suppose that
\[ \bigoplus_{1}^{m_1} M_{h_1} \sim \bigoplus_{1}^{m_2} M_{h_2}. \]
Let $T = M_{h_1}^{(m_1)} \oplus M_{h_2}^{(m_2)}$. Then
\[ T \sim M_{h_1}^{(2m_1)}. \]
Since,
\[ \{M_{h_1}^{(2m_1)}\}_{H^2}^\prime = \{M_F; F \in M_{2m_1}(H^\infty)\}, \]
it follows from Lemma 2.9 in [3] or Theorem 6.11 in [19] that
\[ \bigvee\{(T)^*\}_{H^2}^\prime \cong \bigvee\{(M_{h_1}^{(2m_1)})^*\}_{H^2}^\prime \cong \bigvee\{(M_{2m_1})^\prime\}_{H^\infty}^\prime \cong \bigvee\{(H^\infty)^\prime\} \cong \mathbb{N}. \]
In addition, $\{M_{h_1}\}_{H^2}^\prime = \{M_{h_2}\}_{H^2}^\prime = \{M_z\}_{H^2}^\prime = H^\infty$ implies $M_{h_1}$ and $M_{h_2}$ are strongly irreducible. Thus, by Corollary 5.6 we have $M_{h_1} \sim M_{h_2}$.

On the other hand, there exists $\lambda \in \mathbb{C}$ such that $M_{h_1-\lambda}$ is Fredholm with nonzero index. Following from $M_{h_1} \sim M_{h_2}$, $M_{h_2-\lambda}$ is also Fredholm with the same index of $M_{h_1-\lambda}$. Since $M_{h_1-\lambda} \sim M_{h_2-\lambda}$, we have
\[ m_1 \cdot \text{ind} M_{h_1-\lambda} = \text{ind} M_{h_1}^{(m_1)} = \text{ind} M_{h_2}^{(m_2)} = m_2 \cdot \text{ind} M_{h_2-\lambda}. \]
Then $m_1 = m_2$.

The converse is obvious. The proof is finished. \qed
Proof of Theorem 2.5. Following from Theorem 5.2 (the existence) and Theorem 5.7 (the uniqueness in the sense of analytic automorphism group action), we obtain the Jordan representation theorem of Hol(D) on a weighted Hardy space $H^2_\beta$ of polynomial growth immediately.

As a corollary, we may characterize the similarity classification of the representation of Hol(D) on a weighted Hardy space $H^2_\beta$ of polynomial growth, which generalizes the main result of Jiang and Zheng in [20].

Proof of Theorem 2.6. Suppose that $M_{f_1}$ is similar to $M_{f_2}$ on $H^2_\beta$. By Theorem 5.1, we may write

$$ f_1 = h \circ B_1, \quad f_2 = h_2 \circ \widetilde{B}_2, $$

where $h, h_2 \in \text{Hol}(\mathbb{D})$ such that $M_h$ and $M_{h_2}$ are strongly irreducible, and $B_1$ and $\widetilde{B}_2$ are two finite Blaschke products with order $m$ and $m_2$, respectively. Following from Theorem 5.2 and Theorem 5.7, we have $m = m_2$ and $M_h \sim M_{h_2}$, i.e., there exists a $\varphi \in \text{Aut}(\mathbb{D})$ such that $h_2 = h \circ \varphi$. Let $B_2 = \varphi \circ \widetilde{B}_2$. Then $B_2$ is also a Blaschke product with order $m$ and

$$ f_2 = h_2 \circ B_2 = h \circ \varphi \circ \widetilde{B}_2 = h \circ B_2. $$

The converse is a straightforward corollary of Theorem 5.2. The proof is finished.

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Not applicable.

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