LONG-RANGE SCATTERING FOR DISCRETE
SCHRÖDINGER OPERATORS

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Abstract. In this paper, we define time-independent modifiers to construct a long-range scattering theory for discrete Schrödinger operators on the square lattice $\mathbb{Z}^N$. We prove the existence and completeness of modified wave operators in terms of the above mentioned time-independent modifiers.

1. Introduction

We consider a class of generalized discrete Schrödinger operators on $\mathcal{H} = \ell^2(\mathbb{Z}^N)$, $N \geq 1$:

$$
egin{align*}
Hu[x] &= H_0u[x] + V[x]u[x], \\
H_0u[x] &= \sum_{y \in \mathbb{Z}^N} f[y]u[x - y],
\end{align*}
$$

where $f[x]$ is a rapidly decreasing function on $\mathbb{Z}^N$ such that $f[-x] = \overline{f[x]}$, and $V$ is a real-valued function on $\mathbb{Z}^N$.

We denote the discrete Fourier transform

$$
Fu(\xi) = (2\pi)^{-\frac{N}{2}} \sum_{x \in \mathbb{Z}^N} e^{-ix \cdot \xi} u[x], \quad u \in \ell^1(\mathbb{Z}^N).
$$

Then we have

$$
H_0u[x] = F^* (h_0(\cdot)Fu(\cdot))[x],
$$

where

$$
h_0(\xi) = \sum_{x \in \mathbb{Z}^N} e^{-ix \cdot \xi} f[x]
$$

is a real-valued smooth function on the torus $\mathbb{T}^N$.

We also denote

$$
v(\xi) = \nabla_\xi h_0(\xi), \quad A(\xi) = i \nabla_\xi \nabla_\xi h_0(\xi)
$$

and the set of threshold energies

$$
\mathcal{T} = \{h_0(\xi) | \xi \in \mathbb{T}^N, v(\xi) = 0\}.
$$

We first assume the conditions below.

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Assumption 1.1 (Assumption for $H_0$). $T$ has no inner points and
\[ \{ \xi \in \mathbb{T}^N | v(\xi) \neq 0, \det A(\xi) \neq 0 \} \]
is dense in $\mathbb{T}^N$.

This implies the absence of eigenvalues of $H_0$ and $\sigma_{ac}(H_0) = h_0(\mathbb{T}^N)$, where $\sigma_{ac}(H_0)$ denotes the absolutely continuous spectrum of $H_0$.

We consider $V[x]$ with decay at infinity, which implies that $V$ is a compact operator on $\mathcal{H}$ and hence
\[ \sigma_{ess}(H) = \sigma_{ess}(H_0) = \sigma_{ac}(H_0) = h_0(\mathbb{T}^N), \]
where $\sigma_{ess}(H)$ and $\sigma_{ess}(H_0)$ are the essential spectrum of $H$ and $H_0$, respectively.

If $V$ is short-range, that is, $V = O(\langle x \rangle^{-\varepsilon})$ for some $\varepsilon > 1$, it is known that the wave operators $s-lim_{t \to \pm \infty} e^{itH} e^{-itH_0}$ exist and are asymptotically complete (see [3]). However if $V$ is long-range, that is, $V = O(\langle x \rangle^{-\varepsilon})$ for some $0 < \varepsilon \leq 1$, the usual wave operators do not always exist. Therefore we have to consider modified wave operators for long-range perturbations. In this paper, we impose a kind of decay conditions.

Assumption 1.2 (Assumption for $V$). There exists $\tilde{V} \in C^\infty(\mathbb{R}^N)$ such that
\[ \tilde{V} |_{\mathbb{Z}^N} = V, \]
\[ |\partial_\alpha \tilde{V}(x)| \leq C_\alpha \langle x \rangle^{-|\alpha|-\varepsilon}, \quad x \in \mathbb{R}^N, \quad \alpha \in \mathbb{N}_0^N \]
for some $\varepsilon \in (0, 1]$, where $\langle x \rangle := (1 + |x|^2)^{1/2}$.

By Assumptions 1.1 and 1.2 we see that the singular continuous spectrum of $H$ is empty (see [9]).

Remark 1.3. Assumption 1.2 is equivalent to
\[ \tilde{\partial}^\alpha V(x) = O(\langle x \rangle^{-|\alpha|-\varepsilon}), \]
where $\tilde{\partial}^\alpha = \tilde{\partial}_{x_1} \cdots \tilde{\partial}_{x_N}$, and $\partial_j V(x) = V(x) - V(x - e_j)$ is the difference operator with respect to the $j$-th variable. Here $\{e_j\}$ is the standard orthogonal basis of $\mathbb{R}^N$. See Lemma 2.1 in [9] for details.

From now on, we may write $V$ for $\tilde{V}$ without confusion.

In this paper, we construct modified wave operators with time-independent modifiers, which are proposed in Isozaki-Kitada [5]:
\[ W_j^\pm = s-lim_{t \to \pm \infty} e^{itH} J e^{-itH_0}. \]
Here $J$ is an operator of the following form
\[ Ju(x) = (2\pi)^{-N} \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} e^{i(\varphi(x,\xi) - y \cdot \xi)} u(y) d\xi; \]
the phase function $\varphi$ is a solution to the eikonal equation
\[ h_0(\nabla_x \varphi(x, \xi)) + V(x) = h_0(\xi) \]
in “outgoing” and “incoming” regions and considered in Appendix.
Theorem 1.4 (Main theorem). Under Assumptions 1.1 and 1.2, for any \( \Gamma \Subset h_0(T^N) \setminus \mathcal{I} \), there exist modified wave operators below:

\[
W^\pm_J(\Gamma) = \lim_{t \to \pm \infty} e^{itH}Je^{-itH_0}E_{H_0}(\Gamma),
\]

(1.3)

where \( E_{H_0} \) denotes the spectral measure of \( H_0 \). They hold the following properties:

i) Intertwining property: \( HW^\pm_J(\Gamma) = W^\pm_J(\Gamma)H_0 \).

ii) They are partially isometric: \( \|W^\pm_J(\Gamma)u\| = \|E_{H_0}(\Gamma)u\| \).

iii) Asymptotic completeness: \( \text{Ran} \ W^\pm_J(\Gamma) = E_{H}(\Gamma)H_{ac}(H) \).

We now describe known results. The existence of modified wave operators with time-dependent modifiers

\[
\lim_{t \to \pm \infty} e^{itH}e^{-i\Phi(t,D_x)}u
\]

is proved by Nakamura \[8\], where \( H_0u[x] = -\frac{1}{2} \sum_{|y-x|=1} u[y] \) and the potential \( V \) satisfies the same conditions as Assumption 1.2, although the proof of completeness is omitted. The limiting absorption principle is proved for \( H_0u[x] = -\frac{1}{2} \sum_{|y-x|=1} u[y] \) and \( \partial^{\alpha}V(x) = O(|x|^{-|\alpha|-\varepsilon}) \), \( |\alpha| = 0,1 \), for some \( \varepsilon > 0 \) by \[3\], whereas it is proved in \[9\] for \( H_0 \) and \( V \) with the same conditions as in this paper. Inverse scattering problems are considered in \[6\].

There are other models of discrete Schrödinger operators referred to in \[2\], where a model for 2-dimensional triangle lattice is expressed by the operator

\[
H_0u[x] = \frac{1}{6} \sum_{j=1}^{6} u[x+n_j], \quad x \in \mathbb{Z}^2,
\]

where

\[
\begin{align*}
n_1 &= (1, 0), & n_2 &= (-1, 0), & n_3 &= (0, 1), \\
n_4 &= (0, -1), & n_5 &= (1, -1), & n_6 &= (-1, 1).
\end{align*}
\]

Since \( h_0(\xi) = -\frac{1}{3}(\cos \xi_1 + \cos \xi_2 + \cos(\xi_1 - \xi_2)) \) in this case, we can apply Theorem 1.4.

We also note a continuous version of Schrödinger operators. Scattering theory for Schrödinger operators is originated in a study of the long-time behavior of quantum particles, and has a history in a deep and long extent. The result in this paper is motivated by similar ones on Schrödinger operators, in particular Isozaki-Kitada \[5\]. There are many textbooks of scattering theory and related topics, for example, Amrein-Boutet de Monvel-Georgescu \[1\], Dereziński-Gérard \[4\], Reed-Simon \[10\], Yafaev \[11\] and so forth.

Finally we outline the proof in this paper. Our proof is mainly based on \[5\], which makes use of time-decaying method to construct the phase function \( \varphi \) in \( J \), and the stationary phase method and Enss method to prove the existence and completeness of the modified wave operators for the usual Schrödinger operators on \( \mathbb{R}^N \). The construction of \( \varphi \) is considered in Appendix, since it is too long to read first and the result we need in order
to prove Theorem 1.4 is summarized in Proposition 2.1. Proofs in Appendix are similar to [7]. In Section 2, we prepare some lemmas for the proof of the main theorem. In order to estimate operator norms, we use ideas of pseudo-differential operators (see [12] for instance). In Section 3, we proof the main theorem. The proof of existence of modified wave operators is motivated by [8]. We see that a compactness argument largely contributes to proofs in Sections 2 and 3.

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2. Preliminaries

We first quote a proposition for the Hamilton flow generated by \( h(x, \xi) := h_0(\xi) + V(x) \), which is proved in Appendix. \( \chi \) in Proposition 2.1 is a fixed smooth function on \( \mathbb{R}^N \) such that \( \chi \equiv 0 \) on \( \{ |x| \leq 1 \} \), \( \chi \equiv 1 \) on \( \{ |x| \geq 2 \} \), which is the same as in Appendix. Here we note that \( v \) is extended periodically in \( \xi \) from \( T^N = [\pi, \pi)^N \) to \( \mathbb{R}^N \), and integrals on \( T^N \) are interpreted as on \( [\pi, \pi)^N \).

**Proposition 2.1.** There is a real-valued function \( \varphi \in C^\infty(\mathbb{R}^N \times (\mathbb{R}^N \setminus V^{-1}(0))) \) such that the following holds for any \( a > 0 \): let \( \varphi_a \in C^\infty(\mathbb{R}^N \times \mathbb{R}^N) \) be defined by

\[
\varphi_a(x, \xi) = (\varphi(x, \xi) - x \cdot \xi) \chi\left(\frac{v(\xi)}{a}\right) + x \cdot \xi. \tag{2.1}
\]

Then, we have

\[
\varphi_a(x, \xi + 2\pi m) = \varphi_a(x, \xi) + 2\pi x \cdot m, \quad m \in \mathbb{Z}^N; \tag{2.2}
\]

\[
|\partial_x^\alpha \partial_\xi^\beta [\varphi_a(x, \xi) - x \cdot \xi]| \leq C_{\alpha\beta, a}(x)^{1-\varepsilon-|\alpha|}, \tag{2.3}
\]

\[
|\nabla_x \nabla_\xi \varphi_a(x, \xi) - I| < \frac{1}{2} \tag{2.4}
\]

in \( \mathbb{R}^N \times \mathbb{R}^N \). Furthermore, letting

\[
J_a u [x] = (2\pi)^{-N} \int_{T^N} \sum_{y \in \mathbb{Z}^N} e^{i(\varphi_a(x, \xi) - x \cdot \xi)u[y]} d\xi, \tag{2.5}
\]

we have

\[
(HJ_a - J_a H_0)u [x] = (2\pi)^{-N} \int_{T^N} \sum_{y \in \mathbb{Z}^N} e^{i(\varphi_a(x, \xi) - x \cdot \xi)u[y]} s_a(x, \xi) u[y] d\xi, \tag{2.6}
\]

where

\[
s_a(x, \xi) = e^{-i\varphi_a(x, \xi)}H(e^{i\varphi_a(x, \xi)}) [x] - h_0(\xi) = \sum_{z \in \mathbb{Z}^N} f[z]e^{i(\varphi_a(x-z, \xi) - \varphi_a(x, \xi))} + V [x] - h_0(\xi). \tag{2.7}
\]
For $|x| \geq 1$ and $|v(\xi)| \geq a$, we have
\begin{equation}
|\partial_\xi^j s_a(x, \xi)| \leq \begin{cases} \frac{C_{\beta,a}(x)^{-1-\varepsilon}}{1}, & |\cos(x, v(\xi))| \geq \frac{1}{2}, \\ \frac{C_{\beta,a}(x)^{-\varepsilon}}{1}, & |\cos(x, v(\xi))| \leq \frac{1}{2}. \end{cases}
\end{equation}

Let $\gamma \in C^\infty_h(\mathbb{T}^N \setminus \mathcal{J})$ and $\rho_\pm \in C^\infty([-1, 1]; [0, 1])$ satisfy
\[
\rho_+(\sigma) + \rho_-(\sigma) = 1, \\
\rho_+(\sigma) = 1, \quad \sigma \in \left[\frac{1}{4}, 1\right], \\
\rho_-(\sigma) = 1, \quad \sigma \in \left[-1, -\frac{1}{4}\right].
\]

**Definition 2.2.** We define
\[
p(y, \xi) = \gamma(h_0(\xi)) \chi(y) \rho_\pm(\cos(y, v(\xi))),
\]
\[
P_\pm u[x] = (2\pi)^{-N} \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} e^{i(x-y)\cdot \xi} p_\pm(y, \xi) u[y] \, d\xi,
\]
\[
\hat{P}_\pm u[x] = (2\pi)^{-N} \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} e^{i(x-y)\cdot \xi} p_\pm(y, \xi) u[y] \, d\xi,
\]
\[
E_\pm(t) = J_a e^{-itH_0} \hat{P}_\pm.
\]

From now on in this section we verify some properties of operators defined above.

**Lemma 2.3.** i) $J_a, P_\pm, \hat{P}_\pm$ are bounded operators on $\mathcal{H}$. ii) $P_+ + P_\gamma(H_0), P_\gamma^* - P_\pm, E_\pm(0) - P_\pm, J_a^* J_a - I, J_a J_a^* - I$ are compact operators on $\mathcal{H}$.

**Proof.** i) First we remark that $J_a, P_\pm, \hat{P}_\pm$ and their formal adjoint operators
\[
J_a^* u[x] = (2\pi)^{-N} \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} e^{i(x-y)\cdot \xi} \rho_\pm(y, \xi) u[y] d\xi,
\]
\[
P_\pm^* u[x] = (2\pi)^{-N} \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} e^{i(x-y)\cdot \xi} p_\pm(x, \xi) u[y] d\xi,
\]
\[
\hat{P}_\pm^* u[x] = (2\pi)^{-N} \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} e^{i(x-y)\cdot \xi} p_\pm(x, \xi) u[y] d\xi
\]
map from $\mathcal{F}(\mathbb{Z}^N) = \{ u \in L^2(\mathbb{Z}^N) \mid u[x] = o((x)^{-\infty}) \}$ to oneself. Letting $L := (x-y)^{-2}(1 + (x-y) \cdot D_\xi), D_\xi = \frac{1}{i} \nabla_\xi$, we have
\[
P_\pm u[x] = (2\pi)^{-N} \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} L^k e^{i(x-y)\cdot \xi} p_\pm(y, \xi) u[y] d\xi
\]
\[
= (2\pi)^{-N} \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} e^{i(x-y)\cdot \xi} (L^*)^k (p_\pm(y, \xi)) u[y] d\xi.
\]
We denote \( |p_\pm| := \sup_{|\beta| \leq N+1} \sup_{(x,\xi) \in \mathbb{T}^N} |\partial_\xi^\beta p_\pm(x,\xi)| \). Then we learn

\[
(2.12) \quad |P_\pm u(x)| \leq C|p_\pm| \sum_{y \in \mathbb{Z}^N} \langle x - y \rangle^{-N-1}|u(y)|.
\]

(2.12) and Young's inequality follow that \( \|P_\pm u\| \leq C|p_\pm||u| \).

For \( \tilde{P}_\pm \),

\[
\tilde{P}_\pm \tilde{P}_\pm u(x) = (2\pi)^{-N} \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} e^{i(x-y)\cdot \eta(\xi)} p_\pm(y,\xi) u(y) d\xi,
\]

where

\[
(2.13) \quad \eta(\xi; x, y) := \int_0^1 \nabla_x \varphi_a(y + \theta(x-y), \xi) d\theta.
\]

Then \( \eta(\cdot; x, y) : \mathbb{T}^N \to \mathbb{T}^N \) has its inverse map \( \xi(\cdot; x, y) \). Hence

\[
\tilde{P}_\pm \tilde{P}_\pm u(x) = (2\pi)^{-N} \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} e^{i(x-y)\cdot \eta(\xi)} p_\pm(x, \xi) u(y) d\eta,
\]

where

\[
r(x, y, \eta) = p_\pm(x, \xi; \eta, x, y) p_\pm(y, \xi; \eta, x, y) \left| \det \left( \frac{d\xi}{d\eta} \right) \right|.
\]

Since

\[
(2.14) \quad \left| \partial_\eta \left[ \det \left( \frac{d\xi}{d\eta} \right) - 1 \right] \right| \leq C|\beta|^{-\epsilon}
\]

by (2.3), similar argument for \( P_\pm \) implies the boundedness of \( \tilde{P}_\pm \). Thus, for \( u \in \mathcal{S}(\mathbb{Z}^N) \),

\[
\|\tilde{P}_\pm u\|^2 = |(\tilde{P}_\pm \tilde{P}_\pm u, u)| \leq \|\tilde{P}_\pm \tilde{P}_\pm \| \|u\|^2.
\]

Hence \( \tilde{P}_\pm \) is bounded. The boundedness of \( J_a \) is proved similarly.

ii) Since

\[
P_+ + P_- - \gamma(H_0) = \gamma(H_0)(1 - \chi),
\]

the compactness of the support of \( 1 - \chi \) implies the compactness of \( P_+ + P_- - \gamma(H_0) \).

For \( P_\pm - P_\pm \),

\[
(P_\pm - P_\pm) u(x) = (2\pi)^{-N} \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} e^{i(x-y)\cdot \xi} (p_\pm(x, \xi) - p_\pm(y, \xi)) u(y) d\xi
\]

\[
= (2\pi)^{-N} \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} e^{i(x-y)\cdot \xi} (x - y) \cdot \int_0^1 \nabla_x p_\pm(y + \theta(x-y), \xi) d\theta u(y) d\xi
\]

\[
= (2\pi)^{-N} i \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} e^{i(x-y)\cdot \xi} \int_0^1 \nabla_\xi \cdot \nabla_x p_\pm(y + \theta(x-y), \xi) d\theta u(y) d\xi.
\]
Since
\[
\left| \int_0^1 \partial^2_{\xi} [\nabla_\xi \cdot \nabla_x p_{\pm}(y + \theta(x - y), \xi)]d\theta \right| \leq C\beta \int_0^1 (y + \theta(x - y))^{-1}d\theta \leq C\beta'(x)^{-1},
\]
then \(\langle x \rangle (P^\pm_{\pm} - P_{\pm})\) is bounded. This implies that \(P^\pm_{\pm} - P_{\pm}\) is compact.

For \(E_{\pm}(0) - P_{\pm}\), using (2.13), we have
\[
E_{\pm}(0)u[x] = (2\pi)^{-N} \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} e^{i(\varphi_u(x, \xi) - \varphi_u(y, \xi))} p_{\pm}(y, \xi)u[y] d\xi
\]
\[
= (2\pi)^{-N} \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} e^{i(x - y) \cdot \eta} p_{\pm}(y, \xi(\eta)) \left| \det \left( \frac{d\xi}{d\eta} \right) \right| u[y] d\eta.
\]
Thus
\[
(E_{\pm}(0) - P_{\pm})u[x] = (2\pi)^{-N} \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} e^{i(x - y) \cdot \eta} r(x, y, \eta)u[y] d\eta,
\]
where
\[
r(x, y, \eta) = p_{\pm}(y, \xi(\eta)) \left| \det \left( \frac{d\xi}{d\eta} \right) \right| - p_{\pm}(y, \eta).
\]
By (2.14), we have \(|\partial^2_{\xi}[r(x, y, \eta)]| \leq C\beta(x)^{-\varepsilon}\). Hence \(\langle x \rangle^\varepsilon (E_{\pm}(0) - P_{\pm})\) is bounded, which implies that \(E_{\pm}(0) - P_{\pm}\) is compact.

For \(J_a J^*_a I\), since
\[
(J_a J^*_a - I)u[x] = (2\pi)^{-N} \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} e^{i(\varphi_u(x, \xi) - \varphi_u(y, \xi))} u[y] d\xi - u[x]
\]
\[
= (2\pi)^{-N} \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} e^{i(x - y) \cdot \eta} \left( \left| \det \left( \frac{d\xi}{d\eta} \right) \right| - 1 \right) u[y] d\eta,
\]
the compactness of \(J_a J^*_a - I\) follows similarly to \(E_{\pm}(0) - P_{\pm}\).

Finally, let us prove that \(J^*_a J_a - I\) is compact. Here we mimic the proof of Lemma 7.1 in \(^\text{[3]}\). For \(f \in L^2(\mathbb{T}^N)\), we denote
\[
L_a f(\xi) = FJ^*_a J_a F^* f(\xi)
\]
\[
= (2\pi)^{-N} \sum_{x \in \mathbb{Z}^N} \int_{\mathbb{T}^N} e^{i(\varphi_u(x, \xi) - \varphi_u(x, \eta))} f(\eta)d\eta,
\]
\[
\tilde{L}_a f(\xi) = (2\pi)^{-N} \int_{\mathbb{R}^N} \int_{\mathbb{T}^N} e^{i(\varphi_u(x, \xi) - \varphi_u(x, \eta))} f(\eta)d\eta dx.
\]
First we prove that, for any \(\psi \in C^\infty(\mathbb{T}^N)\) with sufficiently small support,
\[
K_{a, \psi} := \psi \circ (L_a - \tilde{L}_a)
\]
is a compact operator on \(L^2(\mathbb{T}^N)\). Indeed, we denote \(\Pi : L^1(\mathbb{R}^N) \rightarrow L^1(\mathbb{T}^N)\) by
\[
\Pi f(\xi) := \sum_{m \in \mathbb{Z}^N} f(\xi + 2\pi m).
\]
Then (2.2) implies
\[
\Pi \tilde{L}_a f(\xi) = (2\pi)^{-N} \sum_{m \in \mathbb{Z}^N} \int_{\mathbb{R}^N} \int_{\mathbb{T}^N} e^{i(\varphi_a(x,\xi + 2\pi m) - \varphi_a(x,\eta))} f(\eta) d\eta dx
\]
\[
= (2\pi)^{-N} \sum_{m \in \mathbb{Z}^N} \int_{\mathbb{R}^N} \int_{\mathbb{T}^N} e^{i(\varphi_a(x,\xi) + 2\pi m - \varphi_a(x,\eta))} f(\eta) d\eta dx.
\]
If we remark that \( \sum_{m \in \mathbb{Z}^N} e^{2\pi m} = \sum_{m \in \mathbb{Z}^N} \delta_{x-m} \),
\[
\Pi \tilde{L}_a f(\xi) = (2\pi)^{-N} \sum_{x \in \mathbb{Z}^N} \int_{\mathbb{T}^N} e^{i(\varphi_a(x,\xi) - \varphi_a(x,\eta))} f(\eta) d\eta = L_a f(\xi).
\]
Hence
\[
K_{a,\psi} f(\xi) = \psi \circ (\Pi \tilde{L}_a - \tilde{L}_a) f(\xi)
\]
\[
= \sum_{m \in \mathbb{Z}^N \setminus \{0\}} \psi(\xi) \int_{\mathbb{R}^N} \int_{\mathbb{T}^N} e^{i(\varphi_a(x,\xi + 2\pi m) - \varphi_a(x,\eta))} f(\eta) d\eta dx
\]
\[
= \int_{\mathbb{T}^N} k_{a,\psi}(\xi,\eta) f(\eta) d\eta,
\]
where the integral kernel
\[
k_{a,\psi}(\xi,\eta) = \sum_{m \in \mathbb{Z}^N \setminus \{0\}} \psi(\xi) \int_{\mathbb{R}^N} e^{i(\varphi_a(x,\xi + 2\pi m) - \varphi_a(x,\eta))} dx
\]
is smooth. This implies the compactness of \( K_{a,\psi} \).

For \( \psi \circ (\tilde{L}_a - I) \),
\[
\tilde{L}_a f(\xi) = (2\pi)^{-N} \int_{\mathbb{R}^N} \int_{\mathbb{T}^N} e^{i \int_0^1 \nabla_x \varphi_a(x,\eta + \theta(\xi - \eta)) d\theta} f(\eta) d\eta dx.
\]
Letting
\[
y := \int_0^1 \nabla_x \varphi_a(x,\eta + \theta(\xi - \eta)) d\theta,
\]
we have
\[
\tilde{L}_a f(\xi) = (2\pi)^{-N} \int_{\mathbb{R}^N} \int_{\mathbb{T}^N} e^{iy(\xi - \eta)} \left| \det \left( \frac{dx}{dy} \right) \right| f(\eta) d\eta dy.
\]
This and
\[
\left| \partial_y^\alpha \partial_\xi^\beta \partial_\eta^\gamma \left[ \det \left( \frac{dx}{dy} \right) - 1 \right] \right| \leq C_{\alpha\beta\gamma}(y)^{-|\alpha| - \varepsilon}
\]
imply the compactness of \( \psi \circ (\tilde{L}_a - I) \).

Hence, taking a partition of unity \( \{ \psi_j \}_{j=1} \) on \( \mathbb{T}^N \), we see that
\[
J_a^* J_a - I = F^* (L_a - I) F = F^* \sum_{j=1}^J \left( K_{a,\psi_j} + \psi_j \circ (\tilde{L}_a - I) \right) F
\]
is compact. \( \square \)

**Lemma 2.4.** For any \( s \in \mathbb{R} \),
\[
(2.15) \quad s \lim_{t \to \pm \infty} e^{isH_0} J_a^* E_\pm (t - s) = e^{isH_0} \tilde{P}_\pm.
\]
Proof. Since
\[ e^{itH_0} J^*_a E_±(t-s) = e^{itH_0} J^*_a e^{-i(t-s)H_0} \tilde{P}_± = e^{itH_0} (J^*_a J_a - I) e^{-itH_0} e^{isH_0} \tilde{P}_± + e^{isH_0} \tilde{P}_±, \]
Lemma 2.3 ii) and \( \mathcal{H}(H_0) = \mathcal{H} \) imply that the first term converges strongly to 0 as \( t \to \pm \infty \).

Next we prove the norm convergence of \( \lim_{t \to \pm \infty} e^{itH} E_±(t) \). Set
\[ G_±(t) := (D_t + H) E_±(t), \quad D_t = \frac{\partial}{i\partial t}, \]
then we have
\[ e^{itH} E_±(t) - P_± = E_±(0) - P_± + i \int_0^t e^{i\tau H} G_±(\tau) d\tau. \]

**Proposition 2.5.** \( G_±(t) \) is norm continuous and compact for \( t \in \mathbb{R} \). Furthermore, \( G_±(t) \) satisfies
\[ \|G_±(t)\| \leq C|t|^{-1-\varepsilon}, \quad \pm t \geq 0. \]

In particular, \( e^{itH} E_±(t) - P_± \) converges to a compact operator with respect to the norm topology as \( t \to \pm \infty \), respectively.

**Proof.** Let
\[ \Phi(x, y, \xi; t) = \varphi_a(x, \xi) - th_0(\xi) - \varphi_a(y, \xi). \]
The definition of \( E_±(t) \) implies
\[ G_±(t) u[x] = (H J_a - J_a H_0) e^{-itH_0} \tilde{P}_± u[x] = (2\pi)^{-N} \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} e^{i\Phi(x, y, \xi; t)} s_±(x, \xi) p_±(y, \xi) u[y] d\xi. \]
Norm continuity of \( G_±(t) \) is obvious. Furthermore, (2.4) and (2.8) imply the compactness of \( H J_a - J_a H_0 \) by the similar argument in the proof of Lemma 2.3 ii), hence \( G_±(t) \) is compact.

Let us prove (2.16). We consider \( + \) case only since \( - \) case is similarly proved. We take \( \rho_± \in C^∞([-1, 1]; [0, 1]) \) such that
\[ \rho^+(\sigma) + \rho^-(\sigma) = 1, \]
\[ \rho^+(\sigma) = \begin{cases} 1, & \sigma \geq \frac{3}{4}, \\ 0, & \sigma \leq \frac{1}{4}, \end{cases} \]
and set
\[ s_-(x, \xi) = s_a(x, \xi) \chi_{(\xi \neq 0)} \rho^-(\cos(x, v(\xi))), \]
\[ s_+(x, \xi) = s_a(x, \xi) - s_-(x, \xi), \]
\[ b(y, \xi) = p_+(y, \xi). \]
We decompose \( G_+ \) as
\[ G_+(t) u[x] = (2\pi)^{-N} \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} e^{i\Phi(x, y, \xi; t)} (s_+ b + s_- b)(x, y, \xi) u[y] d\xi \]
\[ = (F_+(t) + F_-(t)) u[x]. \]
For $F_+(t)$, let

$$\phi(t; y, \xi) = t h_0(\xi) + \varphi_a(y, \xi)$$

and set

$$L_1 = (\nabla_\xi \phi)^{-2}(1 - \nabla_\xi \phi \cdot D_\xi).$$

Then on the support of $s_+(x, \xi)b_+(y, \xi)$, we learn

$$\langle \nabla_\xi \phi \rangle^{-1} \leq C(|y| + t|v(\xi)|)^{-1}.$$

Now, for any $l \geq 0$, we have

$$F_+(t)u[x] = (2\pi)^{-N} \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} L_1^l(e^{-i\phi(t; y, \xi)}e^{i\varphi_a(x, \xi)}s_+(x, \xi)b(y, \xi)u[y]d\xi$$

$$= (2\pi)^{-N} \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} e^{-i\phi(t; y, \xi)}(t L_1)^l(e^{i\varphi_a(x, \xi)}s_+(x, \xi)b(y, \xi))u[y]d\xi$$

$$= (2\pi)^{-N} \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} e^{i\phi(t; y, \xi)}(t L_1)^l(e^{i\varphi_a(x, \xi)}s_+(x, \xi)b)u[y]d\xi.$$ 

The function in $\{ \}$ is a finite sum of functions of the form $s^l_j(x, \xi)b^l_j(y, \xi; t)$ such that

$$\begin{align*}
|\partial_\xi^\alpha s^l_j(x, \xi)| &\leq C_\beta(x)^{l-1-\varepsilon}, \\
|\partial_\xi^\alpha b^l_j(y, \xi; t)| &\leq C_\beta(|y| + t|v(\xi)|)^{-1}.
\end{align*}$$

Letting

$$S^l_ju[x] = (2\pi)^{-N} \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} e^{i\varphi_a(x, \xi) - y\xi}s^l_j(x, \xi)u[y]d\xi,$$

$$B^l_j(t)u[x] = (2\pi)^{-N} \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} e^{i(x\xi - \varphi_a(y, \xi))}b^l_j(y, \xi; t)u[y]d\xi,$$

we have

$$F_+(t) = \sum_j S^l_j e^{-itH_0}B^l_j(t),$$

$$\|\langle x \rangle^{1+\varepsilon-l}S^l_j\| < \infty,$$

$$\|B^l_j(t)\| \leq C(\langle at \rangle)^{-l}.$$ 

Thus we get

$$\|\langle x \rangle^{1+\varepsilon-l}F_+(t)\| \leq C_1(\langle at \rangle)^{-l}$$

for any $l \in \mathbb{N}_{\geq 0}$. Interpolation with respect to $l$ implies

$$\|F_+(t)\| \leq C(\langle at \rangle)^{-1-\varepsilon}, \quad t \geq 0. \tag{2.18}$$

For $F_-(t)$, we remark that on the support of $s_+(x, \xi)b(y, \xi)$, we learn

$$\langle \nabla_\xi \Phi \rangle^{-1} \leq C(|x - y| + t|v(\xi)|)^{-1}.$$

Letting

$$L_2 = (\nabla_\xi \Phi)^{-2}(1 + \nabla_\xi \Phi \cdot D_\xi),$$
we have
\[ F_- (t) u [x] = (2 \pi)^{-N} \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} e^{i \phi (x, x, \xi, t)} (t L_2)^i (s_- (x, \xi)) u [y] d \xi \]
\[ = (2 \pi)^{-N} \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} e^{i (\varphi_a (x, \xi) - \varphi_a (y, \xi))} e^{-i t h_0 (\xi)} (t L_2)^i (s_- b) u [y] d \xi. \]

On the other hand, since
\[ q^j (x, y, \xi; t) = e^{-i t h_0 (\xi)} (t L_2)^i (s_- (x, \xi) b (y, \xi)) \]
satisfies
\[ | \partial^{\beta} q^j (x, y, \xi; t) | \leq C_l, \beta (tv (\xi))^{l - \beta} \]
for any \( l \in \mathbb{N} \geq 0 \), we get
\[ \| F_- (t) \| \leq C_k \langle at \rangle^{-k} \]
for any \( k \geq 0 \).
Hence (2.16) follows from (2.17), (2.18) and (2.19). \( \square \)

3. Proof of main theorem

3.1. Existence of modified wave operators. First we fix \( \Gamma \Subset h_0 (\mathbb{T}^N) \setminus \mathcal{T} \).
We remark that, for any \( u \in \ell^2 (\mathbb{Z}^N) \) such that \( Fu \in C^\infty (\mathbb{T}^N) \) and \( \text{supp} \, Fu \subset h_0^{-1} (\Gamma) \),
\[ JE_{h_0} (\Gamma) u = J_a u \]
for some \( a > 0 \). Then, to prove the existence of the modified wave operators, it suffices to show
\[ \int_0^\infty \| \frac{d}{dt} (e^{i t H} J_a e^{-i t H_0} F_{h_0} (\Gamma) u) \| dt \]
\[ = \int_0^\infty \| \frac{d}{dt} (e^{i t H} J_a e^{-i t H_0} u) \| dt \]
\[ = \int_0^\infty \| (HJ_a - J_a H_0) e^{-i t H_0} u \| dt < \infty \]
for \( u \in \ell^2 (\mathbb{Z}^N) \) such that \( Fu \in C^\infty (\mathbb{T}^N) \) has a sufficiently small support in \( \{ \xi \in h_0^{-1} (\Gamma) \mid \det A (\xi) \neq 0 \} \). Let
\[ u (t) [x] := (HJ_a - J_a H_0) e^{-i t H_0} u [x] \]
\[ = (2 \pi)^{-N} \frac{1}{2} \int_{\mathbb{T}^N} e^{i (\varphi_a (x, \xi) - t h_0 (\xi))} s_a (x, \xi) Fu (\xi) d \xi. \]
Here we use the stationary phase method. The stationary point \( \xi = \xi (x, t) \)
is determined by
\[ \frac{1}{t} \nabla_\xi \varphi_a (x, \xi) - v (\xi) = 0. \]
Let us denote
\[ D_t = \{ x \in \mathbb{Z}^N \mid \exists \xi \in \text{supp} \, Fu, \, (3.2) \text{ holds} \}. \]
From (2.3), we learn that there exists \( U \subseteq \{ \xi \in h_{0}^{-1}(\Gamma) \mid \det A(\xi) \neq 0 \} \) such that supp \( Fu \subseteq U \) and for \( t > 0 \) large enough,
\[
D_{t} \subset \{ x \mid \frac{x}{t} \in v(U) \} =: D'_{t}.
\]

On \( (D'_{t})^{c} \), the non stationary phase method implies
\[
|u(t)[x]| \leq C_{1}(|x| + |t|)^{-l} \quad x \in \mathbb{Z}^{2}, \ t > 0
\]
for any \( l \geq 0 \).

On the other hand, the stationary phase method implies that on \( D'_{t} \)
\[
u(t)[x] = t^{-\frac{N}{2}}A(t,x)s_{a}(x,\xi(x,t))Fu(\xi(x,t)) + t^{-\frac{N}{2}}r(t,x),
\]
where \( A(t,x) \) is uniformly bounded and
\[
|r(t,x)| \leq C \sup_{|\beta| \leq N+3} \sup_{x \in \partial D_{t}^{c}} \sup_{\xi \in \text{supp} Fu} |s_{a}(x,\xi)| \leq C(1-x^{-1}) \leq C(t)^{-1-\varepsilon}.
\]
Since cos \((x, v(\xi(x,t))) \geq \frac{1}{2} \):
\[
|s_{a}(x,\xi(x,t))| \leq C(1-x^{-1}) \leq C(t)^{-1-\varepsilon}.
\]

Hence we learn
\[
\|u(t)\| \leq \|\chi_{D'_{t}}u(t)\| + \|\chi_{(D'_{t})^{c}}u(t)\| \leq Ct^{-1-\varepsilon} + Ct^{-l} \leq Ct^{-1-\varepsilon}.
\]

This implies (3.1).

\[\square\]

3.2. **Proof of the other properties.** Next we prove properties ii) and iii) in Theorem 1.4. We omit i) since intertwining property is proved similarly to short-range case.

**Proof of ii).** It suffices to show for \( Fu \in C^{\infty}(\mathbb{T}^{2}) \) with \( \text{supp} Fu \subset h_{0}^{-1}(\Gamma) \); this implies \( Ju = J_{a}u \) for some \( a > 0 \). Thus letting \( u_{t} = e^{-itH_{0}u} \),
\[
\|W_{\gamma}^{f}(\Gamma)u\|^{2} = \lim_{t \rightarrow \pm \infty} \|J_{a}u_{t}\|^{2} = \lim_{t \rightarrow \pm \infty} \|J_{a}^{*}(J_{a}' - I)u_{t} + u\|^{2}.
\]
\( \text{w-lim}_{t \rightarrow \pm \infty} u_{t} = 0 \) and Lemma 2.3 ii) imply \( \text{lim}_{t \rightarrow \pm \infty}(J_{a}' - I)u_{t} = 0 \). This proves that \( W_{\gamma}^{f}(\Gamma) \) are partial isometries.

**Proof of iii).** We prove for \( W_{\gamma}^{f}(\Gamma) \) only. Since intertwining property implies \( \text{Ran} W_{\gamma}^{f}(\Gamma) \subset E_{H}(\Gamma)\mathcal{H}_{ac}(H) \), we have only to prove \( E_{H}(\Gamma)\mathcal{H}_{ac}(H) \subset \text{Ran} W_{\gamma}^{f}(\Gamma) \). We fix \( v \in \mathcal{H}_{ac}(H) \) and \( \gamma \in C^{\infty}(\mathbb{R}) \) so that \( \gamma(H)v = v \) and supp \( \gamma \subset \Gamma \). Set \( v_{t} = e^{-itH_{0}v} \), then we show that if
\[
\lim_{s \rightarrow \infty} \limsup_{t \rightarrow \infty} \|e^{iH_{0}J_{a}^{*}[v_{t} - E_{+}(t-s)v_{s}]]| = 0
\]
holds, iii) is proved. Indeed, by Lemma 2.3 (3.3) implies that
\[
\lim_{s \rightarrow \infty} \limsup_{t \rightarrow \infty} \|e^{iH_{0}J_{a}^{*}e^{-itH_{0}v} - e^{isH_{0}P_{+}}v_{s}}| = 0.
\]

This implies the existence of the limit
\[
\lim_{t \rightarrow \infty} e^{iH_{0}J_{a}^{*}e^{-itH_{0}v}} =: \Omega^{a}v.
\]
This, Lemma 2.3 and $\text{w-lim}_{s \to \infty} v_s = 0$ imply that for sufficiently small $a > 0$,
\[ v = W_f^+(\Gamma)\Omega^a v \in \text{Ran} W_f^+(\Gamma). \]

From now on, we prove (3.3). If we note that
\begin{align*}
\| v_t - E_+(t-s)v_s \| &= \| e^{it-s}H(v_t - E_+(t-s)v_s) \| \\
&= \| v_s - e^{it-s}H E_+(t-s)v_s \|,
\end{align*}
we see that it suffices to show
\[ \lim_{s \to \infty} \limsup_{t \to \infty} \| v_s - e^{it-s}H E_+(t-s)v_s \| = 0. \]
Here $v_s = \gamma(H)v_s$ implies
\begin{align*}
&v_s - e^{it-s}H E_+(t-s)v_s = \gamma(H)v_s - e^{it-s}H E_+(t-s)v_s \\
&= (\gamma(H) - \gamma(H_0))v_s + (\gamma(H_0) - P_+ - P_-)v_s \\
&\quad + (P_+ - e^{it-s}H E_+(t-s))v_s + P_- v_s.
\end{align*}
Since $\gamma(H_0) - P_+ - P_-$ is compact and $P_+ - e^{it-s}H E_+(t-s)$ converges to an compact operator as $t \to \infty$ by Lemma 2.3 ii) and Proposition 2.5, the second and third terms on the RHS of (3.5) converge to 0 by $\text{w-lim}_{s \to \infty} v_s = 0$. The first term also converges to 0 by the compactness of $H - H_0 = V$. To estimate the forth term of (3.5),
\[ \| P_- v_s \|^2 = (P_+ P_- v_s, v_s) \]
\[ = ((P_+ P_-)P_- v_s, v_s) \]
\[ + ((P_+ - e^{-isH} E_-(s))P_- v_s, v_s) \]
\[ + (P_- v_s, E_-(s)v). \]
Similar argument above implies the first and second terms of (3.6) converge to 0 as $s \to \infty$. For the third term, we have
\begin{align*}
&\| (P_- v_s, E_-(s)v) \| \\
&= \| (P_- v_s, E_-(s)v (\chi_{\{x| |x| \geq R}\} + \chi_{\{x| |x| < R\})v)) \| \\
&\leq C\|E_-(s)v\|\|\chi_{\{x| |x| \geq R\})v\| + C\|\chi_{\{x| |x| < R\})E_-(s)v\|\|v\| \\
&\text{for any } R > 0. \text{ We now prove}
\end{align*}
\[ \|\chi_{\{x| |x| < R\})E_-(s)v\| \leq C_{l,R}(s)^{-l}, \quad s \geq 0 \]
for any $R > 0$ and $l \geq 0$. Indeed,
\[ E_-(-s)u[x] = (2\pi)^{-N} \int_{\mathbb{T}^N} \sum_{y \in \mathbb{Z}^N} e^{i\Phi(x,y,\xi; -s)} p_-(y,\xi) u[y] d\xi, \]
where $\Phi(x, y, \xi; t) = \varphi_a(x, \xi) - t\tau_0(\xi) - \varphi_a(y, \xi)$. We see on the support of $p_-(y, \xi)$,
\[ |s\varphi(\xi) + \nabla_\xi \varphi(y, \xi)| \geq c(|y| + s|v(\xi)|). \]
Since $|x| \leq R$ and $|v(\xi)| \geq a$, we have for $s > 0$ large enough
\[ |\nabla_\xi \Phi(x, y, \xi; -s)| \geq c(|y| + s|v(\xi)|). \]
Similarly to the proof of (2.19), we get (3.8).

(3.7), (3.8) and \(\lim_{R \to \infty} \|\chi_{\{|x|\geq R\}} v\| = 0\) imply that (3.4) holds. Hence we have (3.3). \(\square\)

Appendix A. Classical mechanics and the construction of phase function

First, let \(\varepsilon_0, \varepsilon_1 > 0\) and \(\chi \in C^{\infty}(\mathbb{R}^N)\) be fixed so that \(\varepsilon_0 + \varepsilon_1 < \varepsilon\) and \(\chi \equiv 0\) on \(\{|x| \leq 1\}\), \(\chi \equiv 1\) on \(\{|x| \geq 2\}\).

**Definition A.1.** For \(\rho \in (0, 1)\), we define

\[
h(x, \xi) = h_0(\xi) + V(x),
\]

\[
V_{\rho}(t, x) = V(x) \chi(\rho x) \chi \left( \frac{\langle \log(t) \rangle x}{\langle t \rangle} \right),
\]

\[
h_{\rho}(t, x, \xi) = h_0(\xi) + V_{\rho}(t, x),
\]

\[
\nabla^2_x V_{\rho}(t, x) = \tau \nabla_x \nabla_x V_{\rho}(t, x).
\]

**Remark A.2.** By Assumption 1.2 and the definition of \(V_{\rho}\), we have for any \(t \in \mathbb{R}, x \in \mathbb{R}^N\) and multi-index \(\alpha\),

\[
|\partial^\alpha V_{\rho}(t, x)| \leq C_\alpha \min\{\rho^{\varepsilon_0} \langle t \rangle^{-|\alpha| - \varepsilon_1}, \langle x \rangle^{-|\alpha| - \varepsilon}\}, \tag{A.1}
\]

where \(C_\alpha\)'s are independent of \(\rho\).

Let \((q, p)(t, s) = (q, p)(t, s; x, \xi)\) be the solution to the canonical equation with respect to \(h_{\rho}\)

\[
\begin{aligned}
\dot{q}(t, s) &= \nabla_x h_{\rho}(t, p(t, s), q(t, s)), \\
\dot{p}(t, s) &= -\nabla_x h_{\rho}(t, p(t, s), q(t, s)), \\
(q, p)(s, s) &= (x, \xi).
\end{aligned}
\]

This can be rewritten by the integral form:

\[
q(t, s) = x + \int_s^t \nabla_x V_{\rho}(t, q(t, s)) d\tau, \tag{A.2}
\]

\[
p(t, s) = \xi - \int_s^t \nabla_x V_{\rho}(t, q(t, s)) d\tau. \tag{A.3}
\]

**Proposition A.3.** For \(\rho > 0\) small enough, there exist constants \(C_l > 0\) \((l \in \mathbb{N}_{\geq 0})\) such that, for any \(x, \xi \in \mathbb{R}^N\), \(0 \leq s \leq \pm t\) and multi-indices \(\alpha\)
and \( \beta \),

\begin{align}
\text{(A.4)} & \quad |p(s, t; x, \xi) - \xi| \leq C_0 \rho^{\alpha_0}(s)^{-\varepsilon_1}, \\
\text{(A.5)} & \quad |p(t, s; x, \xi) - \xi| \leq C_0 \rho^{\alpha_0}(s)^{-\varepsilon_1}, \\
\text{(A.6)} & \quad |\partial_x^\alpha [\nabla_x q(s, t; x, \xi) - I] \leq C_{|\alpha|} \rho^{\alpha_0}(s)^{-\varepsilon_1}, \\
\text{(A.7)} & \quad |\partial_x^2 \nabla_x p(s, t; x, \xi)| \leq C_{|\alpha|} \rho^{\alpha_0}(s)^{-1-\varepsilon_1}, \\
\text{(A.8)} & \quad |\partial_x^\alpha \partial_\xi^\beta [\nabla_x q(t, s; x, \xi)] \leq C_{|\alpha|+|\beta|} \rho^{\alpha_0}(s)^{-1-\varepsilon_1} |t - s|, \\
\text{(A.9)} & \quad |\partial_x^\alpha \partial_\xi^\beta \nabla_x p(t, s; x, \xi)| \leq C_{|\alpha|+|\beta|} \rho^{\alpha_0}(s)^{-1-\varepsilon_1}, \\
\text{(A.10)} & \quad |\partial_\xi^\beta \nabla_x q(t, s; x, \xi) - (t - s) A(\xi)| \leq C_{|\beta|} \rho^{\alpha_0}(s)^{-\varepsilon_1} |t - s|, \\
\text{(A.11)} & \quad |\partial_\xi^\beta [\nabla_\xi q(t, s; x, \xi) - I]| \leq C_{|\beta|} \rho^{\alpha_0}(s)^{-\varepsilon_1}, \\
\text{(A.12)} & \quad |\partial_x^\alpha \partial_\xi^\beta [q(t, s; x, \xi) - x - (t - s) v(p(t, s; x, \xi))] | \leq C_{|\alpha|+|\beta|} \rho^{\alpha_0} \min \{ |t - s| (s)^{-\varepsilon_1}, (t)^{1-\varepsilon_1} \}.
\end{align}

Here, \(|x| := \left( \sum_{j=1}^N |x_j|^2 \right)^{\frac{1}{2}} \) for vectors and \(|A| := \left( \sum_{j,k=1}^N |A_{jk}|^2 \right)^{\frac{1}{2}} \) for matrices.

**Proof.** We prove in \( 0 \leq s \leq t \) case only since the other case is proved similarly.

1. First,

\[ p(s, t) - \xi = - \int_t^s \nabla_x V_\rho(\tau, q(\tau, t)) d\tau \]

and (A.1) imply (A.4). (A.5) is proved similarly.

2. Next we prove (A.6) and (A.7). Differentiating (A.2) and (A.3) in \( x \),

\[
\begin{cases}
\nabla_x q(s, t) = I + \int_t^s A(p(\tau, t)) \nabla_x p(\tau, t) d\tau, \\
\nabla_x p(s, t) = - \int_t^s \nabla_x^2 V_\rho(\tau, q(\tau, t)) \nabla_x q(\tau, t) d\tau.
\end{cases}
\]

Letting

\[
Q_0(s) := \nabla_x q(s, t) - I, \\
P_0(s) := \nabla_x p(s, t),
\]

this is equivalent to

\begin{align}
\text{(A.13)} \quad \begin{cases}
Q_0(s) = \int_t^s A(p(\tau, t)) P_0(\tau) d\tau, \\
P_0(s) = - \int_t^s \nabla_x^2 V_\rho(\tau, q(\tau, t)) Q_0(\tau) d\tau - \int_t^s \nabla_x^2 V_\rho(\tau, q(\tau, t)) P_0(\tau, d\tau.
\end{cases}
\end{align}

Thus

\[ P_0(s) = B_t(P_0(\cdot))(s) + R_0(s), \]

where

\[
B_t(P(\cdot))(s) := - \int_t^s \nabla_x^2 V_\rho(\tau, q(\tau, t)) \left[ \int_t^\tau A(p(\sigma, t)) P(\sigma) d\sigma \right] d\tau,
\]

\[ R_0(s) := - \int_t^s \nabla_x^2 V_\rho(\tau, q(\tau, t)) d\tau. \]
Let \( \| M(\cdot) \|_0 := \sup_{0 \leq s \leq t} | M(s) | \) for \( M \in C([0, t] ; M_N(\mathbb{R})) \). Then

\[
| B_t(P(\cdot))(s) | \leq \int_s^t C_2 \rho^{\alpha_0} (\tau)^{-2-\varepsilon_1} \int_\tau^t | P(\sigma)| \sigma d\sigma d\tau \\
\leq C_2 \rho^{\alpha_0} \| P \|_0 \int_s^\infty (\tau)^{-2-\varepsilon_1} \int_\tau^\infty (\sigma)^{-1-\varepsilon_1} \sigma d\sigma d\tau \\
\leq C_2 C' \rho^{\alpha_0} (s)^{-1-\varepsilon_1} \| P \|_0,
\]

\[
|R_0(s)| \leq \int_s^t C_2 \rho^{\alpha_0} (\tau)^{-2-\varepsilon_1} d\tau \leq C \rho^{\alpha_0} (s)^{-1-\varepsilon_1}.
\]

If \( \rho \leq (2C_2 C' \frac{1}{\varepsilon_0})^{-\frac{1}{\varepsilon_0}} \), the operator norm \( \| B_t \|_0 \) is bounded by \( \frac{1}{2} \). Hence

\[
\| P_0(\cdot) \| = \|(1 - B_t)^{-1}(R_0(\cdot))\| \leq \frac{1}{1 - \| B_t \|_0} \| R_0(\cdot) \| \leq 2C \rho^{\alpha_0}.
\]

This proves the estimate for \( \nabla_x p(t, s) \). The estimate for \( \nabla_x q(t, s) \) directly follows from that of \( \nabla_x p(t, s) \) and \( (A.13) \).

Next we prove for any \( \alpha \in \mathbb{N}^N \) by induction with respect to \( |\alpha| \). Differentiating \( (A.13) \), we have

\[
(A.14) \quad \begin{cases}
\partial_x^\alpha Q_0(s) = \int_s^t A(p(\tau, t)) \partial_x^\alpha P_0(\tau) d\tau + R_{0,1}(s), \\
\partial_x^\alpha P_0(s) = -\int_s^t \nabla_x^2 V_{\rho}(\tau, q(\tau, t)) \partial_x^\alpha Q_0(\tau) d\tau + R_{0,21}(s) + R_{0,22}(s),
\end{cases}
\]

where

\[
R_{0,1}(s) := \sum_{0 \leq \alpha' \leq \alpha} \binom{\alpha}{\alpha'} \int_t^s \partial_x^{\alpha'} [A(p(\tau, t))] \partial_x^{\alpha - \alpha'} P_0(\tau) d\tau,
\]

\[
R_{0,21}(s) := -\sum_{0 \leq \alpha' \leq \alpha} \binom{\alpha}{\alpha'} \int_t^s \partial_x^{\alpha'} \left[ \nabla_x^2 V_{\rho}(\tau, q(\tau, t)) \right] \partial_x^{\alpha - \alpha'} Q_0(\tau) d\tau,
\]

\[
R_{0,22}(s) := -\int_t^s \partial_x^{\alpha} \left[ \nabla_x^2 V_{\rho}(\tau, q(\tau, t)) \right] d\tau.
\]

From the equations above and assumptions of induction,

\[
| R_{0,1}(s) | \leq C \rho^{\alpha_0} (s)^{-1-\varepsilon_1},
\]

\[
| R_{0,21}(s) | \leq \int_s^t C \rho^{\alpha_0} (\tau)^{-2-\varepsilon_1} \cdot C \rho^{\alpha_0} (s)^{-1-2\varepsilon_1} \tau d\tau \leq C \rho^{\alpha_0} (s)^{-1-2\varepsilon_1},
\]

\[
| R_{0,22}(s) | \leq \int_s^t C \rho^{\alpha_0} (\tau)^{-2-\varepsilon_1} d\tau \leq C \rho^{\alpha_0} (s)^{-1-\varepsilon_1}.
\]

This implies \( (A.6) \) and \( (A.7) \).

3. Next, let us prove \( (A.10) \) and \( (A.11) \). Similarly to step 2, we have

\[
\begin{cases}
\nabla_x q(t, s) = \int_s^t A(p(\tau, t)) \nabla_x q(\tau, s) d\tau, \\
\nabla_x p(t, s) = I - \int_s^t \nabla_x^2 V_{\rho}(\tau, q(\tau, s)) \nabla_x q(\tau, s) d\tau.
\end{cases}
\]

equivalently,

\[
(A.15) \quad \begin{cases}
Q'(t) = \int_s^t A(p(\tau, t)) P'(\tau) d\tau - \int_s^t (A(p(\tau, s)) - A(\xi)) d\tau, \\
P'(t) = -\int_s^t \nabla_x^2 V_{\rho}(\tau, q(\tau, s)) Q'(\tau) d\tau - \int_s^t (\tau - s) \nabla_x^2 V_{\rho}(\tau, q(\tau, s)) A(\xi) d\tau,
\end{cases}
\]
where
\[
Q'(t) := \nabla_{\xi}q(t, s) - (t - s)A(\xi),
\]
\[
P'(t) := \nabla_{\xi}p(t, s) - I.
\]

From this,
\[
P'(t) = B_s(P'())(t) + R'(t),
\]
where
\[
B_s(P'())(t) = -\int_s^t \nabla_x^2 V_{\rho}(\tau, q(\tau, s)) \left[ \int_s^\tau A(p(\sigma, s)) P(\sigma) d\sigma \right] d\tau,
\]
\[
R'(t) := -\int_s^t \nabla_x^2 V_{\rho}(\tau, q(\tau, s)) \int_s^\tau A(p(\sigma, s)) d\sigma d\tau.
\]

Letting \(\|P(\cdot)\| := \sup_{t \geq s} |M(t)|\) for \(M \in C([s, \infty) : M_N(\mathbb{R}))\), we have
\[
|B_s(P'())(t)| \leq \int_s^t C_2 \rho^{\rho_0} \langle \tau \rangle^{-2-\varepsilon_1} \int_s^\tau |P(\sigma)| d\sigma d\tau
\]
\[
\leq C_2 \rho^{\rho_0} \|P\| \int_s^t \langle \tau \rangle^{-2-\varepsilon_1} (\tau - s) d\tau
\]
\[
\leq C_2 C' \rho^{\rho_1} \langle s \rangle^{-\varepsilon_1} \|P\|,
\]
\[
|R'(t)| \leq \int_s^t C \rho^{\rho_1} \langle \tau \rangle^{-2-\varepsilon_1} (\tau - s) d\tau \leq C \rho^{\rho_0} \langle s \rangle^{-\varepsilon_1}.
\]

Thus, if \(\rho \leq (2C_2 C')^{-\varepsilon_0}\), we get
\[
\|P'()\| = \|(1 - B_s)^{-1} R'(\cdot)\| \leq \frac{1}{1 - \|B_s\|} \|R'(\cdot)\| \leq 2C \rho^{\rho_0} \langle s \rangle^{-\varepsilon_1}.
\]

This proves the estimate for \(\nabla_{\xi}p(t, s)\). The estimate for \(\nabla_{\xi}q(t, s)\) follows from that of \(\nabla_{\xi}p(t, s)\), \(A.13\) and \(A.15\).

For \(\partial_{\xi}^3 Q'\) and \(\partial_{\xi}^3 P'\),
\[
(A.16) \quad \begin{cases}
\partial_{\xi}^3 Q'(t) = \int_s^t A(p(\tau, s)) \partial_{\xi}^3 P'(\tau) d\tau + R_{11}'(t) + R_{12}'(t), \\
\partial_{\xi}^3 P'(t) = -\int_s^t \nabla_x^2 V_{\rho}(\tau, q(\tau, s)) \partial_{\xi}^3 Q'(\tau) d\tau + R_{21}'(t) + R_{22}'(t),
\end{cases}
\]
where
\[
R_{11}'(t) := \sum_{0 \leq \beta' \leq \beta} \left( \begin{array}{c} \beta \\ \beta' \end{array} \right) \int_s^t \partial_{\xi}^{3-\beta'} [A(p(\tau, s))] \partial_{\xi}^{\beta-\beta'} P'(\tau) d\tau,
\]
\[
R_{12}'(t) := \int_s^t \partial_{\xi}^3 [A(p(\tau, s)) - A(\xi)] d\tau,
\]
\[
R_{21}'(t) := -\sum_{0 \leq \beta' \leq \beta} \left( \begin{array}{c} \beta \\ \beta' \end{array} \right) \int_s^t \partial_{\xi}^{3-\beta'} \left( \nabla_x^2 V_{\rho}(\tau, q(\tau, s)) \right) \partial_{\xi}^{\beta-\beta'} Q'(\tau) d\tau,
\]
\[
R_{22}'(t) := -\int_s^t (\tau - s) \partial_{\xi}^3 \left[ \nabla_x^2 V_{\rho}(\tau, q(\tau, s)) A(\xi) \right] d\tau.
\]
Thus
\[
\partial_\xi^3 P'(t) = B_s(\partial_\xi^3 P(\cdot))(t) - \int_s^t \nabla_x^2 V_\rho(\tau, q(\tau, s))(R_{11}'(\tau) + R_{12}'(\tau))d\tau \\
+ R_{21}'(t) + R_{22}'(t).
\]

Assumptions of the induction imply
\[
|R_{11}'(t)| \leq C \rho^{\alpha_0}\langle s \rangle^{-\varepsilon_1}|t-s|, \\
|R_{12}'(t)| \leq C \sup_{|\beta'| \leq |\beta|} \int_s^t |\partial_\xi^{\beta'}[p(\tau, s) - \xi]|d\tau \leq C \rho^{\alpha_0}\langle s \rangle^{-\varepsilon_1}|t-s|, \\
|R_{21}'(t)| \leq \int_s^t C \rho^{\alpha_0}\langle \tau \rangle^{-2-\varepsilon_1} \cdot C \rho^{\alpha_0}\langle s \rangle^{-\varepsilon_1} |\tau-s|d\tau \leq C \rho^{2\alpha_0}\langle s \rangle^{-2\varepsilon_1}, \\
|R_{22}'(t)| \leq \int_s^t C \rho^{\alpha_0}\langle \tau \rangle^{-2-\varepsilon_1} |\tau-s|d\tau \leq C \rho^{\alpha_0}\langle s \rangle^{-\varepsilon_1}.
\]

This and similar argument for \(Q'\) and \(P'\) imply the estimates \((A.10)\) and \((A.11)\).

4. Let us prove \((A.8)\) and \((A.9)\). Differentiation in \(x\) gives
\[
\begin{align*}
\nabla_x q(t, s) &= I + \int_s^t A(p(\tau, s))\nabla_x p(\tau, s)d\tau, \\
\nabla_x p(t, s) &= -\int_s^t \nabla_x^2 V_\rho(\tau, q(\tau, s))\nabla_x q(\tau, s)d\tau.
\end{align*}
\]

Letting
\[
Q(t) := \nabla_x q(t, s) - I, \\
P(t) := \nabla_x p(t, s),
\]
this is equivalent to
\[
(A.17) \quad \begin{cases}
Q(t) = \int_s^t A(p(\tau, s))P(\tau)d\tau, \\
P(t) = -\int_s^t \nabla_x^2 V_\rho(\tau, q(\tau, s))Q(\tau)d\tau - \int_s^t \nabla_x^2 V_\rho(\tau, q(\tau, s))d\tau.
\end{cases}
\]

This implies
\[
P(t) = B_s(P(\cdot))(t) + R(t),
\]
where
\[
R(t) := -\int_s^t \nabla_x^2 V_\rho(\tau, q(\tau, s))d\tau.
\]

Since
\[
|R(t)| \leq \int_s^t C_2 \rho^{\alpha_0}\langle \tau \rangle^{-2-\varepsilon_1}d\tau \leq C \rho^{\alpha_0}\langle s \rangle^{-1-\varepsilon_1},
\]
we have
\[
\|P(\cdot)\| = \|(1 - B_s)^{-1}R\| \leq 2C \rho^{\alpha_0}\langle s \rangle^{-1-\varepsilon_1}.
\]

The estimate for \(Q\) follows from \((A.17)\).

We prove inductively with respect to \(|\alpha| + |\beta|\). From \((A.17)\),
\[
(A.18) \quad \begin{cases}
\nabla_\xi^2 \partial_\xi^3 Q(t) = \int_s^t A(p(\tau, s))\nabla_\xi^2 \partial_\xi^3 P(\tau)d\tau + R_1(t), \\
\nabla_\xi^2 \partial_\xi^3 P(t) = -\int_s^t \nabla_x^2 V_\rho(\tau, q(\tau, s))\nabla_\xi^2 \partial_\xi^3 Q(\tau)d\tau + R_{21}(t) + R_{22}(t),
\end{cases}
\]
where
\[ R_1(t) = \sum_{\alpha', \beta', \beta \leq \beta, \mid \alpha' + \beta' \mid \geq 1} \left( \frac{\alpha}{\alpha'} \right) \left( \frac{\beta}{\beta'} \right) \int_s^t \partial_x^\alpha \partial_\xi^\beta \left[ A(p(\tau, s)) \right] \partial_x^{\alpha' - \alpha} \partial_\xi^{\beta' - \beta} P(\tau) d\tau, \]

\[ R_{21}(t) = - \sum_{\alpha', \beta' \leq \beta, \mid \alpha' + \beta' \mid \geq 1} \left( \frac{\alpha}{\alpha'} \right) \left( \frac{\beta}{\beta'} \right) \int_s^t \partial_x^\alpha \partial_\xi^\beta \left[ \nabla_x^2 V_\rho(\tau, q(\tau, s)) \right] \partial_x^{\alpha' - \alpha} \partial_\xi^{\beta' - \beta} Q(\tau) d\tau, \]

\[ R_{22}(t) = - \int_s^t \partial_x^\alpha \partial_\xi^\beta \left[ \nabla_x^2 V_\rho(\tau, q(\tau, s)) \right] d\tau. \]

Thus
\[ \partial_x^\alpha \partial_\xi^\beta P(t) = B_s(\partial_x^\alpha \partial_\xi^\beta P(\cdot))(t) - \int_s^t \nabla_x^2 V_\rho(\tau, q(\tau, s)) R_1(\tau) d\tau + R_{21}(t) + R_{22}(t). \]

(A.10), (A.11) and assumptions of the induction imply
\[ |R_1(t)| \leq C \rho^{\varepsilon_0} |t - s|, \]
\[ |R_{21}(t)| \leq \int_s^t C \rho^{\varepsilon_0} |t - s| |\tau - s| d\tau \leq C \rho^{2\varepsilon_0} |t - s|^{1 - 2\varepsilon_1}, \]
\[ |R_{22}(t)| \leq \int_s^t C \rho^{\varepsilon_0} |\tau - s| d\tau \leq C \rho^{\varepsilon_0} |s|^{1 - \varepsilon_1}. \]

This proves (A.8) and (A.9).

5. Finally, we prove (A.12). (A.2) and (A.3) imply
\[ q(t, s; x, \xi) = x + \int_s^t v(p(\tau, s)) d\tau \]
\[ = x + \int_s^t \left( p(t, s) + \int_\tau^t \nabla_x V_\rho(\sigma, q(\sigma, s)) d\sigma \right) d\tau. \]

Thus
\[ q(t, s; x, \xi) - x - (t - s)v(p(t, s)) \]
\[ = \int_s^t \left[ v(p(t, s)) + \int_\tau^t \nabla_x V_\rho(\sigma, q(\sigma, s)) d\sigma \right] d\tau. \]

This and (A.8)-(A.11) follow (A.12). \qed

**Proposition A.4.** Fix \( \rho > 0 \) so that \( C_0 \rho^{\varepsilon_0} < \frac{1}{2} \) holds, where \( C_0 \) is the constant appeared in Proposition A.3. Then, for \( x, \xi \in \mathbb{R}^N \) and \( 0 \leq \pm s \leq \pm t \), there are \( y(s, t) = y(s, t; x, \xi) \) and \( \eta(t, s) = \eta(t, s; x, \xi) \) such that
\[ y(s, t) = y(s, t; x, \xi) \]
\[ \eta(t, s) = \eta(t, s; x, \xi) \]
equivalently,
\[ q(t, s; x, \eta(t, s; x, \xi)) = y(s, t; x, \xi), \]
\[ p(t, s; x, \eta(t, s; x, \xi)) = \eta(t, s; x, \xi). \]
Furthermore, they hold the following estimates: for any $x, \xi \in \mathbb{R}^N$, $0 \leq \pm s \leq \pm t$ and multi-indices $\alpha$ and $\beta$,

\begin{align}
(A.23) \quad |\partial_\xi^\alpha [\nabla_x y(s, t; x, \xi) - I]| \leq C_\alpha s^\alpha \langle s \rangle^{-\varepsilon_1}, \\
(A.24) \quad |\partial_\xi^\alpha \partial_\tau^\beta \nabla_x \eta(t, s; x, \xi)| \leq C_{\alpha, \beta} \rho^\alpha \langle s \rangle^{-1-\varepsilon_1}, \\
(A.25) \quad |\partial_\xi^\beta [\eta(t, s; x, \xi) - \xi]| \leq C_\beta \rho^\beta \langle s \rangle^{-\varepsilon_1}, \\
(A.26) \quad |\partial_\xi^\beta [y(s, t; x, \xi) - x - (t - s)\nu(\xi)]| \\
\leq C_\beta \rho^\beta \min\{|t - s| \langle s \rangle^{-\varepsilon_1}, (t)^{1-\varepsilon_1}\}.
\end{align}

Proof. 1. $|\nabla_x q(s, t; x, \xi) - I| < \frac{1}{2}$, $|\nabla_\xi p(t; s, x, \xi) - I| < \frac{1}{2}$ and Schwartz’s global inversion theorem \([9]\), Proposition A.7.1 imply the existence and uniqueness of $y(s, t; x, \xi)$ and $\eta(t, s; x, \xi)$ which satisfy \([A.19]\) and \([A.20]\).

2. Let us prove \([A.23]\). Differentiation of \([A.19]\) implies \([A.27]\)

\begin{equation}
\nabla_x q(s, t; y(s, t), \xi)\nabla_x y(s, t) = I.
\end{equation}

Hence

\[
|\nabla_x y(s, t)| = |(\nabla_x q(s, t; y(s, t), \xi))^{-1} - I| \\
\leq C|\nabla_x q(s, t; y(s, t), \xi) - I| \\
\leq C\rho^\alpha \langle s \rangle^{-\varepsilon_1}.
\]

For $\alpha \neq 0$, from \([A.27]\) follows

\[
\nabla_x q(s, t; y(s, t), \xi)\partial_\xi^\alpha \nabla_x y(s, t) \\
= - \sum_{0 \leq \alpha' \leq \alpha} \binom{\alpha}{\alpha'} \partial_\xi^\alpha [\nabla_x q(s, t; y(s, t), \xi)] \partial_\xi^{\alpha' - \alpha} \nabla_x y(s, t).
\]

Since \([A.6]\) and the induction with respect to $|\alpha|$ imply that the norm of the RHS is bounded by $C\rho^\alpha \langle s \rangle^{-\varepsilon_1}$, we have $|\partial_\xi^\alpha \nabla_x y(s, t)| \leq C_\alpha \rho^\alpha \langle s \rangle^{-\varepsilon_1}$.

3. Next we show \([A.25]\). For $\beta = 0$, \([A.22]\) implies

\[
|\eta(t, s)| = |p(s, t; y(s, t), \xi) - \xi| \\
= \left|\int_s^t \nabla_x V_\rho(\tau, q(\tau, t; y(s, t), \xi)) d\tau\right| \\
\leq C\rho^\alpha \langle s \rangle^{-\varepsilon_1}.
\]

For $|\beta| = 1$, by \([A.20]\),

\[
\nabla_\xi p(t; s, x, \eta(t, s))\nabla_\xi \eta(t, s) = I.
\]

Similarly to step 2, we have

\[
|\nabla_\xi \eta(t, s) - I| \leq C|\nabla_\xi p(t; s, x, \eta(t, s)) - I| \\
\leq C\rho^\alpha \langle s \rangle^{-\varepsilon_1}.
\]

In general case, we learn

\[
\nabla_\xi p(t; x, y(t, s))\partial_\xi^\beta \nabla_\xi \eta(t, s) \\
= - \sum_{0 \leq \beta' \leq \beta} \binom{\beta}{\beta'} \partial_\xi^{\beta - \beta'} \nabla_\xi \eta(t, s).
\]
From this, (A.10) and the induction with respect to $|\beta|$, the norm of the RHS is bounded by $C \rho^{\alpha} (s)^{-\epsilon_1}$. Thus we see (A.25) for any $\beta$.

4. Now, we prove (A.24). By (A.20),
\[
\nabla_x p(t, s; x, \eta(t, s)) + \nabla_\xi p(t, s; x, \eta(t, s)) \nabla_x \eta(t, s) = 0.
\]
Similarly, (A.9) implies
\[
|\nabla_x \eta(t, s)| = |(\nabla_\xi p(t, s; x, \eta(t, s)))^{-1} \nabla_x p(t, s; x, \eta(t, s))| \\
\leq C |\nabla_x p(t, s; x, \eta(t, s))| \\
\leq C \rho^{\alpha} (s)^{-1-\epsilon_1}.
\]

For general $\alpha$ and $\beta$, we have
\[
\nabla_\xi p(t, s; x, \eta(t, s)) \partial^\alpha_x \partial^\beta_\xi [\nabla_x \eta(t, s)] \\
= - \partial^\alpha_x \partial^\beta_\xi [\nabla_x p(t, s; x, \eta(t, s))] \\
- \sum_{\alpha', \beta' \leq \beta, |\alpha'| + |\beta'| \geq 1} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} \partial^\alpha_x \partial^\beta_\xi [\nabla_x p(t, s; x, \eta(t, s))] \partial^{\alpha-\alpha'}_x \partial^{\beta-\beta'}_\xi [\nabla_x \eta(t, s)].
\]

The induction, (A.25), (A.9) and (A.11) prove (A.24).

5. Finally we prove (A.26). Similarly to the proof of (A.12) in Proposition A.3,
\[
y(s, t) - x - (t - s)v(\xi) \\
= q(t, s; x, \eta(t, s)) - x - (t - s)v(p(t, s; x, \eta(t, s))) \\
= \int_s^t \left[ v(x + \int_\tau^t \nabla_x V_\rho(\sigma, q(\sigma, s; x, \eta(t, s))) d\sigma) - v(\xi) \right] d\tau.
\]

From this, we get (A.26) by (A.10) and (A.25). \(\square\)

We define
\[
\phi(t; x, \xi) = u(t; x, \eta(t, 0; x, \xi)),
\]
where
\[
u(t, x, \eta) = x \cdot \eta + \int_0^t \{ h_\rho - x \cdot \nabla_x h_\rho \} (\tau, q(\tau, 0; x, \eta), p(\tau, 0; x, \eta)) d\tau.
\]

Then, direct calculus implies
\[
(A.28) \quad \left\{ \begin{array}{l}
\partial_t \phi(t; x, \xi) = h_\rho (t, \nabla_\xi \phi(t; x, \xi), \xi), \\
\phi(0; x, \xi) = x \cdot \xi,
\end{array} \right.
\]
and
\[
(A.29) \quad \left\{ \begin{array}{l}
\nabla_x \phi(t; x, \xi) = \eta(t, 0; x, \xi), \\
\nabla_\xi \phi(t; x, \xi) = y(0, t; x, \xi).
\end{array} \right.
\]

Remark A.5. The relations above and Proposition A.4 imply the estimate
\[
(A.30) \quad |\partial^\alpha_x \partial^\beta_\xi [\nabla_x y(s; t, x, \xi) - I]| \leq C'_{|\alpha| + |\beta|} \rho^{\alpha} (s)^{-\epsilon_1}
\]
for $|\beta| \geq 1$. 

Lemma A.6. The limits

(A.31) \[ \phi_{\pm}(x, \xi) := \lim_{t \to \pm \infty} (\phi(t; x, \xi) - \phi(t; 0, \xi)) \]

exist, are smooth in \(\mathbb{R}^{2N}\) and

(A.32) \[ \phi_{\pm}(x, \xi + 2\pi m) = \phi_{\pm}(x, \xi) + 2\pi x \cdot m, \quad x, \xi \in \mathbb{R}^N, \ m \in \mathbb{Z}^N. \]

Proof. Let

\[ R(t, x, \xi) := \phi(t; x, \xi) - \phi(t; 0, \xi). \]

Then

\[ \nabla_x R(t, x, \xi) = \eta(t, 0; x, \xi) = p(0, t; y(0, t; x, \xi), \xi) \]

\[ = \xi + \int_0^t (\nabla_x \eta)(\tau, q(\tau; t; y(0, t; x, \xi), \xi))d\tau \]

\[ = \xi + \int_0^t (\nabla_x \eta)(\tau, q(\tau; 0; x, \eta(t, 0; x, \xi)))d\tau. \]

Since

\[ |\partial_x \partial_x^2 [(\nabla_x \eta)(\tau, q(\tau; 0; x, \eta(t, 0; x, \xi)))]| \leq C_{\alpha\beta} (\tau)^{-1-\varepsilon}, \]

\(\nabla_x R(t, x, \xi)\) converges to a smooth function uniformly in \((x, \xi) \in \mathbb{R}^{2N}\). Thus

(A.33) \[ \partial_x^2 R(t, x, \xi) = x \cdot \int_0^1 \nabla_x \partial_x^2 R(t, \theta x, \xi)d\theta \]

converges locally uniformly in \(\mathbb{R}^{2N}\). This implies the smoothness of \(\phi_{\pm}\).

(A.32) is proved easily if we remark that

\[ \eta(t, 0; x, \xi + 2\pi m) = \eta(t, 0; x, \xi) + 2\pi m, \]

\[ q(t, 0; x, \xi + 2\pi m) = q(t, 0; x, \xi) \]

for \(x, \xi \in \mathbb{R}^N, t \in \mathbb{R}\) and \(m \in \mathbb{Z}^N\). \(\square\)

Next let us consider properties of \(\phi_{\pm}\) in “outgoing” (or “incoming”) regions.

Lemma A.7. Let \((q, p)(t) = (q, p)(t; 0; x, \xi)\) be an orbit satisfying (A.2), (A.3) and

\[ |q(\tau)| \geq b|\tau| + d, \quad \pm \tau \geq 0. \]

Then there exist \(l_{\alpha\beta}, l_{\beta} \geq 2\) such that for \(\pm t \geq 0\) and \(\alpha, \beta \in \mathbb{N}_{\geq 0}^N\),

(A.34) \[ |p(t) - \xi| \leq C b^{-1} < d >^{-\varepsilon}, \]

(A.35) \[ |\partial_x^2 \partial_x^2 \nabla_x q(t) - I| \leq C_{\alpha\beta} b^{-l_{\alpha\beta}} < d >^{-1-|\alpha|-\varepsilon}|t|, \]

(A.36) \[ |\partial_x^2 \partial_x^2 \nabla_x p(t)| \leq C_{\alpha\beta} b^{-l_{\alpha\beta}} < d >^{-1-|\alpha|-\varepsilon}, \]

(A.37) \[ |\partial_x^2 \nabla_x q(t) - tA(\xi)| \leq C b^{-l_{\beta}} < d >^{-\varepsilon}|t|, \]

(A.38) \[ |\partial_x^2 \nabla_x p(t) - I| \leq C b^{-l_{\beta}} < d >^{-\varepsilon}. \]
\textbf{Proof.} We calculate similarly to Proposition \ref{A.3}, whereas we use the following estimate instead:

\[ |\partial_\tau^\alpha V_\rho(t, q(t))| \leq C_\alpha \langle q(t) \rangle^{-|\alpha|-\varepsilon} \leq C_\alpha (b|t| + d)^{-|\alpha|-\varepsilon}. \]

For \ref{A.34},

\[ |p(t) - \xi| = \left| \int_0^t \nabla_x V_\rho(\tau, q(\tau))d\tau \right| \leq C \int_0^\infty (b\tau + d)^{-1-\varepsilon}d\tau \]

\[ \leq C|b|^{-1}(d)^{-\varepsilon}. \]

The other estimates are proved similarly. \hfill \Box

\textbf{Lemma A.8.} Let \(b, d \geq 0, b \neq 0 \) and \(x, \xi \in \mathbb{R}^N\) satisfy

\[ |q(\tau, 0; x, \eta(t, 0; x, \xi))| \geq b|\tau| + d, \quad 0 \leq \pm \tau \leq \pm t \]

for any \(\pm t \geq 0\). Then there exist \(l_{\alpha\beta}', l_{\beta}' \geq 2\) such that, for \(\pm t \geq 0\),

\begin{align*}
(\text{A.39}) \quad & |\partial_\xi^\alpha \partial_\xi^\beta |\nabla_x \eta(t, 0; x, \xi)| \leq C_{\alpha\beta} b^{-l_{\alpha\beta}'} (d)^{-1-|\alpha|-\varepsilon}, \\
(\text{A.40}) \quad & |\partial_\xi^\beta |\eta(t, 0; x, \xi - \xi)| \leq C_{\beta} b^{-l_{\beta}'} (d)^{-\varepsilon}.
\end{align*}

\textbf{Proof.} We denote \(\eta(t, 0; x, \xi)\) by \(\eta(t)\) for simplicity.

We show \ref{A.40}. For \(\beta = 0\), \ref{A.22} implies

\[ |\eta(t) - \xi| = |p(0, t; y(0, t; x, \xi), \xi) - \xi| \]

\[ \leq \int_0^t |\nabla_x V_\rho(\tau, q(\tau, t; y(0, t; x, \xi), \xi))|d\tau \]

\[ \leq Cb^{-1}(d)^{-\varepsilon}. \]

The rest of estimates are proved by Lemma \ref{A.7} and the argument in Proposition \ref{A.4}. \hfill \Box

On \{(x, \xi)|x, v(\xi) \neq 0, \pm \cos(x, v(\xi)) \geq 0\}, \ref{A.4}, \ref{A.5} and \ref{A.12} imply that for \(0 \leq \pm \tau \leq \pm t\),

\[ |q(\tau, 0; x, \eta(t, 0; x, \xi))| \geq |x + \tau v(p(\tau, 0; x, \eta(t, 0; x, \xi)))| - C_0 (\tau)^{1-\varepsilon_1} \]

\[ = |x + \tau v(p(\tau, t; y(0, t; x, \xi), \xi))| - C_0 (\tau)^{1-\varepsilon_1} \]

\[ \geq |x + \tau v(\xi)| - C(\tau)^{1-\varepsilon_1} - C_0 (\tau)^{1-\varepsilon_1} \]

\[ \geq \frac{1}{\sqrt{2}}(|x| + |\tau v(\xi)|) - C(\tau)^{1-\varepsilon_1}. \]

If we remark the following inequality

\[ |x| + |\tau v(\xi)| \geq \left( \frac{1}{\varepsilon_1} \right)^{\varepsilon_1} \left( \frac{1}{1 - \varepsilon_1} |\tau v(\xi)| \right)^{1-\varepsilon_1} \]

\[ = \frac{|x|^{\varepsilon_1} |v(\xi)|^{1-\varepsilon_1}}{\varepsilon_1 (1 - \varepsilon_1)^{1-\varepsilon_1}} |\tau|^{1-\varepsilon_1}, \]

we learn for \(|x|^{\varepsilon_1} |v(\xi)|^{1-\varepsilon_1} \geq C_{\varepsilon_1}, \)

\begin{align*}
(\text{A.41}) \quad & |q(\tau, 0; x, \eta(t, 0; x, \xi))| \geq \frac{1}{2} (|x| + |\tau v(\xi)|), \quad 0 \leq \pm \tau \leq \pm t. \\
(\text{A.41}), (\text{A.29}), (\text{A.31}), (\text{A.33}) \text{ and Lemma A.8 imply} \quad & (\text{A.42}) \quad |\partial_\xi^\alpha \partial_\xi^\beta \phi_\pm(x, \xi) - x \cdot \xi| \leq C_{\alpha\beta} |v(\xi)|^{-l_{\alpha\beta}'} (x)^{-1-|\alpha|-\varepsilon} \]

on \{(x, \xi)| |x|^{\varepsilon_1} |v(\xi)|^{1-\varepsilon_1} \geq C_{\varepsilon_1}, \pm \cos(x, v(\xi)) \geq 0\}, \text{ respectively.}
Furthermore, we confirm that, for any \( a > 0 \), there exists \( R_a > 1 \) such that \( \phi_\pm \) satisfy the eikonal equation
\[
(A.43) \quad h(x, \nabla_x \phi_\pm(x, \xi)) = h_0(\xi)
\]
on outgoing (or incoming) region
\[
\{|x| \geq R_a, \ |v(\xi)| \geq a, \pm \cos(x, v(\xi)) \geq 0\},
\]
respectively. Indeed, by (A.29) and (A.31),
\[
\nabla_x \phi_\pm(x, \xi) = \lim_{t \to \pm \infty} \eta(t, 0; x, \xi) = \lim_{t \to \pm \infty} p(0, t; y(0, t; x, \xi), \xi).
\]
For \(|x| \geq 2p^{-1} \), we have
\[
(A.44) \quad h(x, \nabla_x \phi_\pm(x, \xi)) = \lim_{t \to \pm \infty} h_\rho(0, x, p(0, t; y(0, t; x, \xi), \xi)).
\]
We see that
\[
E(\tau) := h_\rho(\tau, q(\tau, t; y(0, t; x, \xi), \xi), p(\tau, t; y(0, t; x, \xi), \xi)) = h_\rho(\tau, q(\tau, 0; x, \eta(t, 0; x, \xi)), p(\tau, 0; x, \eta(t, 0; x, \xi))
\]
is constant for \( 0 \leq \pm \tau \leq \pm t. \) Indeed,
\[
\frac{dE}{d\tau}(\tau) = \partial_\tau h_\rho(\tau, q(\tau, 0; x, \eta(t, 0; x, \xi)), p(\tau, 0; x, \eta(t, 0; x, \xi)))
\]
\[
= \partial_\tau V_\rho(\tau, q(\tau, 0; x, \eta(t, 0; x, \xi))).
\]
We note that \( \partial_\tau V_\rho(t, x) = 0 \) on \(|x| \geq 2 \max\{p^{-1}, (0/\log(\xi))\}. \) On the other hand, for \( R_a \) large enough, \( A.41 \) implies that on \(|x| \geq R_a, \ |v(\xi)| \geq a, \pm \cos(x, v(\xi)) \geq 0\},
\[
|q(\tau, 0; x, \eta(t, 0; x, \xi))| \geq \frac{1}{2}(R_a + a|\tau|)
\]
\[
\geq 2 \max\{p^{-1}, (0/\log(\xi))\}, \quad 0 \leq \pm \tau \leq \pm t.
\]
From this, we see that \( \frac{dE}{d\tau}(\tau) = 0, \ 0 \leq \pm \tau \leq \pm t. \) Therefore,
\[
(A.45) \quad h_\rho(0, x, p(0, t; y(0, t; x, \xi), \xi)) = E(0) = E(t)
\]
\[
= h_\rho(t, y(0, t; x, \xi), \xi).
\]
Hence, (A.44) and (A.45) imply
\[
h(x, \nabla_x \phi_\pm(x, \xi)) = \lim_{t \to \pm \infty} h_\rho(t, y(0, t; x, \xi), \xi) = h_0(\xi).
\]

**Proof of Proposition 2.7.** Let \( \varphi \in C^\infty(\mathbb{R}^N \times (\mathbb{R}^N \setminus v^{-1}(0))) \) be defined by
\[
(A.46) \quad \varphi(x, \xi) = (\phi_+(x, \xi) - x \cdot \xi)\chi_+(x, \xi)
\]
\[
+ (\phi_-(x, \xi) - x \cdot \xi)\chi_-(x, \xi) + x \cdot \xi.
\]
Here
\[
(A.47) \quad \chi_\pm(x, \xi) = \chi(\mu|v(\xi)|^l x) \psi_\pm(\cos(x, v(\xi)))
\]
and \( \psi_\pm \in C^\infty([-1, 1]; [0, 1]) \) satisfy
\[
\psi_\pm(\sigma) = \begin{cases} 
1, & \pm \sigma \geq \frac{1}{2}, \\
0, & \pm \sigma \leq 0. 
\end{cases}
\]
If we set sufficiently small $\mu$ and large $l$, then $\varphi$ defined by (A.46) holds (2.2), (2.3) and (2.4).

Finally we prove (2.8). We decompose $s_a$ by

$$s_a(x, \xi) = s_1^a(x, \xi) + s_2^a(x, \xi),$$

where

$$s_1^a(x, \xi) = \sum_{z \in \mathbb{Z}^N} f(z) e^{i\varphi_a(x-z, \xi) - \varphi_a(x, \xi)} - h_0(\nabla_x \varphi_a(x, \xi)),$$

$$s_2^a(x, \xi) = h(x, \nabla_x \varphi_a(x, \xi)) - h_0(\xi).$$

Finally we prove (2.8). We decompose $s_a$ by

$$s_a(x, \xi) = s_1^a(x, \xi) + s_2^a(x, \xi),$$

where

$$s_1^a(x, \xi) = \sum_{z \in \mathbb{Z}^N} f(z) e^{i(\varphi_a(x-z, \xi) - \varphi_a(x, \xi)) - h_0(\nabla_x \varphi_a(x, \xi))},$$

$$s_2^a(x, \xi) = h(x, \nabla_x \varphi_a(x, \xi)) - h_0(\xi).$$

First, we see that (A.43) and Assumption 1.2 imply that for $|x| \geq R_a$ and $\beta$,

$$\partial_\xi^\beta s_2^a(x, \xi) = \begin{cases} 0, & |\cos(x, v(\xi))| \geq \tfrac{1}{2}, \\ O(|x|^{-\varepsilon}), & |\cos(x, v(\xi))| \leq \tfrac{1}{2}. \end{cases}$$

For $s_1^a$,

$$s_1^a(x, \xi) = \sum_{z \in \mathbb{Z}^N} f(z) \left( e^{i(\varphi_a(x-z, \xi) - \varphi_a(x, \xi))} - e^{-iz \cdot \nabla_x \varphi_a(x, \xi)} \right),$$

where

$$\Phi_a(x, \xi, z) = \varphi_a(x-z, \xi) - \varphi_a(x, \xi) + z \cdot \nabla_x \varphi_a(x, \xi)$$

$$= z \cdot \left( \int_0^1 \int_0^1 \nabla_x^2 \varphi_a(x - \theta_1 \theta_2 z, \xi) d\theta_2 d\theta_1 \right) z.$$
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