The Geometric Phase of the Three-Body Problem

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1 Introduction and Results
1.1 A Reconstruction Formula

The three-body problem concerns understanding the motions of three point masses travelling in space according to Newton’s laws of mechanics. The three masses form a triangle in space and Newton’s equations define a dynamical system on the space of such triangles. The shape (congruence class) of the triangle is the primary variable. Shape variables are further divided up into an overall scale parameter $I$, and the similarity class of the triangle. The similarity classes form a two-sphere, denoted $S$, and called the shape sphere. The appearance of this sphere is central to our whole development. This shape sphere also plays a central role in Moeckel’s work on the three-body problem [11]. We view the orientation and position of the triangle in space as secondary variables. The translational part of the motion is eliminated by the usual trick of going to center-of-mass coordinates. Our basic question is: 

**Given that the initial and final triangles of a three-body motion are similar, what is the rotation $R$ which relates the two triangles (up to scale)?**

Our main result is the answer in the form of formula (2) below. This is an example of reconstruction formula: it reconstructs part of the
original dynamics from some reduced dynamics (essentially dynamics on $S$).

We suppose that the planes defined by the initial and final triangles and the total angular momentum vector $J_0$ are known. Let $n_0$ and $n_1$ be the normal vectors to the initial and final planes. Let $R_0$ be the (smallest) rotation in the $J_0 - n_0$ plane which takes $n_0$ to $J_0$ and $R_1$ the analogous rotation in the $J_0 - n_1$ taking $J_0$ to $n_1$. (If $n_i$ is coincident with $J_0$ then its $R_i$ is the identity.) Any $R$ which takes $n_0$ to $n_1$ can be written in the form:

$$R = R_1 R_{J_0} R_0$$  \hspace{1cm} (1)$$

where $R_{J_0}$ is some rotation about the $J_0$ axis. Let $J_0 = \|J_0\|$ denote the length of the total angular momentum vector. Our reconstruction formula is the following integral formula for the rotation angle $\Delta \theta$ of our $R_{J_0}$:

$$J_0 \Delta \theta = \int_0^t \omega_{J_0}(t) dt + \int \int_D \Omega_{J_0}$$  \hspace{1cm} (2)$$

The first integral in this formula is called a “dynamic phase” in the Berry phase literature (\cite{12}), and the second integral is called the “geometric phase”. The integrand $\omega_{J_0}$ represents $J_0$ times the instantaneous angular velocity of the moving triangle $q(t)$ about the axis $J_0$. It is given by

$$\omega_{J_0} = J_0 \cdot \dot{I}(q(t))^{-1} J_0$$  \hspace{1cm} (3)$$
where $I(q)$ is the instantaneous moment of inertia tensor of the weighted triangle $q$. (See \(\text{[3]}\), below). The time $t_1$ of integration is the duration of the motion. The second integrand, $\Omega_J$ is a closed two-form which is independent of the potential, given explicitly in equation (4) below. Its geometric definition can be found in Theorem 4 at the end of §2. This two-form lives on a “reduced configuration space” denoted $S_{J_0}$ which we will describe next. We urge the reader to look at the first figure which is a picture of $S_{J_0}$ and various of its features.

The space $S_{J_0}$ is \textbf{locally} the product of two two-spheres:

$$S_{J_0} \cong S^2(\frac{1}{2}) \times S^2(J) \text{ locally}$$

where $S^2(R)$ denotes the sphere of radius $R$. The first sphere is the shape sphere, which naturally has radius $1/2$. \textbf{Globally} $S_{J_0}$ is the non-trivial two-sphere bundle over the two-sphere $S$. (There are exactly two such bundles, one being the trivial bundle.) We will use standard spherical coordinates $(\phi, \theta)$ on spheres $S^2(R)$, as well as coordinates $(z, \theta)$ where

$$z = \cos(\phi)$$

is the normalized height of a point above the equatorial circle $\phi = \pi/2$ so that $Rz$ is the usual height. This induces coordinates $(z_1, \theta_1, z_2, \theta_2)$ on $S_{J_0}$. 

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We will show that

\[ \Omega_{J_0} = J_0 \{ \frac{1}{2} d(z_1 z_2) \wedge d\theta_1 + dz_2 \wedge d\theta_2 \}. \] (4)

The fibering spheres of \( S_{J_0} \) are represented by the second spherical factor in the local product representation of \( S_{J_0} \) above. Each fiber sphere is an instantaneous versions of the body angular momentum sphere occurring in the description of the motion of a free rigid body. Here “instantaneous” refers to the instantaneous shape of the triangle. A point on such a fibering sphere represents the fixed total angular momentum vector \( J_0 \) viewed from a frame \( \{ U_1, U_2, U_3 \} \) attached to the moving triangle. Thus the point can be represented by the vector \( (J_0 \cdot U_1, J_0 \cdot U_2, J_0 \cdot U_3) \) of length \( J_0 \). A good choice of frame is an orthonormal frame which diagonalizes the instantaneous inertia tensor \( I \) of the triangle. We will always take the third frame \( U_3 \) to be the normal \( n \) to the triangle.

The height coordinate \( z_1 \) on the shape sphere is proportional to the area of the triangle. (See the appendix for a derivation of this.) The height coordinate \( z_2 \) on the second sphere is the component of the total angular momentum normal to the triangle. It may be helpful to have explicit formulae:

\[ z_1 = 4 \sqrt{\frac{m_1 m_2 m_3}{m_1 + m_2 + m_3}} \frac{\Delta}{I}, \]
and

\[ z_2 = \frac{1}{J_0} J_0 \cdot n. \]

In the first formula the \( m_i \) are the masses of the three bodies. In the second formula, the vector \( n \) is a unit vector normal to the triangle spanned by the position vectors \( q_1, q_2, q_3 \) of the three bodies. The vector \( J \) is the total angular momentum of the system.

\[ I = m_1 \|q_1\|^2 + m_2 \|q_2\|^2 + m_3 \|q_3\|^2. \]

is its polar moment of inertia. And \( \Delta = \frac{1}{2} n \cdot (q_2 - q_1) \times (q_3 - q_1) \) is the oriented area of the triangle.

The coordinate \( \theta_2 \), and the local splitting of \( S_{J_0} \) into the product of spheres depends on the choice of local frame \( \{U_i\} \) (choice of gauge) for the moving triangle. In our formula for the two-form \( \Omega_{J_0} \) we have used the inertial eigenframe discussed above.

A three-body motion without triple collision has natural projections to a curve in \( S_{J_0} \) and a curve in \( S \). Our assumption that the initial and final triangles are similar means that the projected path in \( S \) is closed. The projected path in \( S_{J_0} \) need not be closed, but there is a canonical way to close it. This is depicted in the first figure as the arcs on the fibering spheres.
labelled by $R_1$ and $R_2$. The disc $D$ over which we integrate the two-form $\Omega_{J_0}$ is any disc in $S_{J_0}$ bounding the resulting closed curve. ($S_{J_0}$ is simply connected.)

### 1.2 The planar case

In the planar case our question is much simpler and has been solved several times before ([4], [1], [8], [3]). A single angle now describes the rotation relating the two similar planar triangles. This angle $\Delta \theta$ is described by the same formula as above, which simplifies as follows. The integrand for the dynamic phase is given by $\omega_{J_0} = \frac{1}{T(\theta)} J_0$. The integrand $\Omega_{J_0}$ for the geometric phase is simply the area form $J_0 dz_1 \wedge d\theta_1$ on the shape sphere, normalized so that the total area of this sphere is $J_0 2\pi$.

The planar three-body problem embeds in the spatial problem by taking the initial triangle to be normal to the angular momentum vector. For such a planar motion the fiber coordinate of $S_{J_0}$ remains fixed at the north pole ($z_2 = 1$) because the triangle’s plane remains perpendicular to the angular momentum vector. Hence we do not have to deal with the two-sphere bundle $S_{J_0}$ over $S$ in this case.
1.3 Structure of the calculation

Our derivation of the reconstruction formula is quite similar to our earlier derivation of a reconstruction formula for rigid body motion. We construct a closed loop $\gamma$ and a one-form $\alpha_{J_0}$ in the three-body configuration space such that when we apply Stoke’s theorem to the integral $\int_{\gamma} \alpha_{J_0}$ we obtain our formula. We construct the loop $\gamma$ by concatenating the three-body motion $q(t)$ defined by Newton’s equations with several other arcs obtained by rotating or scaling. The one-form $\alpha_{J_0}$ is the component of the “natural mechanical connection” $A$ in the direction of the total angular momentum: $\alpha_{J_0} - J_0 \cdot A$. The connection $A$ first appears explicitly in Guichardet [1]. It can be argued that it was discovered by Smale, who certainly has a formula for our one-form $\alpha_{J_0}$ [13]. It was later used up by Iwai [4] and rediscovered by Shapere and Wilczek [12]. I used it in [8] in studying the Falling Cat Problem.

The form $\Omega_{J_0}$ of our reconstruction formula is essentially the push down of the two-form $d\alpha_{J_0}$ to the quotient space of $Q$ by the two-parameter group of rotations about $J_0$ and scalings. However this quotient is a singular space. In order to facilitate the analysis we regularize this quotient. This is done by
introducing the space $\tilde{Q}$ of oriented triangles which is a branched cover over the standard three-body configuration space $Q$. The idea of this regularization is due to Hsiang [3]. The space $S_{J_0}$ on which $\Omega_{J_0}$ is defined is the quotient of $\tilde{Q}$ (minus the triple collision) by this two-parameter group of rotations and scalings.

REMARK The two-form $\Omega_{J_0}$ is symplectic. It is closely related to the minimal coupling form of Sternberg [2].

1.4 Structure of paper.

In the next section we introduce some notation and constructions basic to our goals. Then we state the basic theorems regarding the metric and topological structure of the quotients $S$ and, $S_{J_0}$ some other intermediate quotients. We also describe more carefully the two-form $\Omega_{J_0}$. In §3 we prove these theorems. The proofs are based on restricting to the planar three-body problem in which case the Hopf fibration arises naturally. In the final section, §4, we prove our reconstruction formula in the manner outlined above. In the appendix we derive the fact that the shape sphere is a sphere of radius $1/2$. 

1.5 Acknowledgement

I would like to thank Wu-Yi Hsiang for suggesting this problem, and for helpful conversation.

2 Constructions, Notation, Theorems

2.1 Basic Notation

The three-body configuration space $Q$ consists of the set of all triples of vectors $q = (q_1, q_2, q_3)$, $q_a \in \mathbb{R}^3$ whose center of mass is at the origin: $\Sigma m_a q_a = 0$. The positive numbers $m_a$ are the particle masses. We prefer to view $Q$ as the space of weighted triangles in space. In any case it is a six-dimensional Euclidean vector space with squared norm $I(q) := \|q\|^2 := m_1\|q_1\|^2 + m_2\|q_2\|^2 + m_3\|q_3\|^2$, the same function of the introduction (the polar moment of inertia). The instantaneous kinetic energy $K$ of a path $q(t)$ is defined to be $K := \frac{1}{2}\|\dot{q}\|^2$, where

$$\dot{q} = (\dot{q}_1, \dot{q}_2, \dot{q}_3)$$

denotes the time derivative of the path. The instantaneous total energy of a motion is $E = K + V$, where $V = -\Sigma_{i\neq j} \frac{m_i m_j}{\|q_i - q_j\|}$ is the usual Newtonian
gravitational potential energy. (The choice of potential beyond its rotational invariance plays no role in our analysis. Any potential which is a function of the interparticle distances alone will work.) A three-body motion is a solution \( q(t) \) to Newton’s equations: 
\[
m_a \frac{d^2}{dt^2} q_a = -\nabla_a V, \quad a = 1, 2, 3.
\]
The total energy \( E \) and the total angular momentum \( J = \sum m_a q_a \times \dot{q}_a \) are constant along any three-body motion.

The moment of inertial tensor \( II(q) \) of a weighted triangle \( q \) is the symmetric non-negative \( 3 \times 3 \) matrix defined by

\[
\omega \cdot II(q) \omega = \| \omega \times q \|^2
\]

(5)

where \( \omega \times q = (\omega \times q_1, \omega \times q_2, \omega \times q_3) \) denotes the infinitesimal rotation of the triangle \( q \) with angular velocity \( \omega \). \( II \) encodes that part of the metric on \( \mathcal{Q} \) in the direction of the \( G \)-orbits. It satisfies the equivariance relations:

\[
II(\lambda R q) = \lambda^2 R II(q) R^T
\]

where \( \lambda \in \mathbb{R}^+ \) is a homothety, or dilation and \( R \in G = SO(3) \) is a rotation. (The inertia tensor can be expressed by the formula

\[
II(q) = (1)I - \mathcal{M}
\]

where \( \mathcal{M} = \sum m_a q_a \otimes q_a \) is the standard inertia tensor, \( 1 \) is the identity matrix, and \( I = tr(\mathcal{M}) \) is the polar moment of inertia.
2.2 Oriented Triangles

The idea in this section is due to Hsiang [3]. We will need to choose a normal vector \( n(t) \) to the plane of our moving triangle \( q(t) \).

**Definition 1** An oriented triangle is a pair \( \tilde{q} = (q, n) \) with \( q \in Q \) and \( n \in \mathbb{R}^3 \) a unit vector normal to the subspace of \( \mathbb{R}^3 \) spanned by the vertices \( q_1, q_2, q_3 \) of the triangle \( q \). (Since the center of mass is 0 these three position vectors are always linearly dependent.) The set of oriented triangles will be denoted by \( \tilde{Q} \).

If the vertices of our triangle \( q \) span a plane then it has two possible orientations \( \tilde{q} = (q, \pm n) \) where \( \pm n \) are either of the two normals to this plane. If the triangle lies in a single line an orientation for it is any vector \( n \) on the unit circle in the plane orthogonal to this line. Such triangles are called collinear configurations. If \( q = 0 \) is the triple collision point then \( n \) is any point on the two-sphere.

**Lemma 1** \( \tilde{Q} \) is a smooth algebraic variety. Away from the triple collision, the natural projection \( \tilde{Q} \to Q \) is a branched cover, branched over the collinear configurations. The rotation group \( G = SO(3) \) acts freely on \( \tilde{Q} \) away from the triple collision point.
Proof  

\( \tilde{Q} \) is the algebraic subvariety of \( Q \times S^2 \) defined by the two equations \( q_1 \cdot n = 0, q_2 \cdot n = 0 \). The differential of these defining functions is full-rank everywhere. Apply the implicit function theorem. The other statements are obvious. QED

It follows from this lemma that any smooth function or covariant tensor on \( Q \) can be lifted to \( \tilde{Q} \). Examples are \( V, II \) and the Riemannian metric. The lift will be denoted by the same symbol as the original. The lifted metric fails to be positive definite along the branching locus. The three-body equations themselves also lift to \( \tilde{Q} \):

Lemma 2 Any three-body motion \( q(t) \) which does not consist entirely of collinear configurations has a unique oriented lift \( \tilde{q}(t) \in \tilde{Q} \) passing through a given initial non-collinear oriented triangle.

The proof is obvious.

2.3 The Reduced Configuration Space, the Shape Sphere, and other quotient spaces

Let

\[ G(J_0) \subset G \]
denote the one-parameter subgroup of rotations about the angular momentum axis $J_0$. The quotient space $Q/G(J_0)$ is singular, even away from the triple collision, due to the presence of extra symmetry at collinear configurations. The introduction of the space $\tilde{Q}$ of oriented triangles regularizes this quotient away from the triple collision.

**Definition 2** The **reduced configuration space** is the quotient space

$$Q_{J_0} = \tilde{Q}/G(J_0),$$

with corresponding projection denoted by

$$\pi_{J_0} : \tilde{Q} \to Q_{J_0}.$$

The **reduced motion** corresponding to the oriented three-body motion $\tilde{q}(t)$ is the projection $\pi_{J_0}(\tilde{q}(t))$ of this curve to the reduced configuration space.

The reduced configuration space is essentially a cone over the space $S_{J_0}$ which plays a central role in our reconstruction formula. In order to show this and in order to get a good picture of both spaces, we will also need to understand various other quotient spaces. Set

$$\tilde{Q} := \tilde{Q}/G = \text{congruence classes of oriented triangles},$$
\[ Q/G = \text{congruence classes of triangles} \]

The action of \( \lambda \in \mathbb{R}^+ \) scales each triangle by the factor \( \lambda \) and scales distances on \( Q \) by this same factor. The polar moment (squared norm) \( I \) is a \( G \)-invariant function which is homogeneous of degree 2. Let

\[ S^5 = \{ I = 1 \} \subset Q \]

denote the five-sphere in the Euclidean space \( Q \) and

\[ \tilde{S}^5 \subset \tilde{Q} \]

be the corresponding preimage of this sphere under the branched cover \( \tilde{Q} \to Q \). Also let

\[ \tilde{Q}^* = \tilde{Q} \setminus \{0\}, \]

and

\[ Q^* = Q \setminus \{0\}. \]

(The 0 in \( \tilde{Q} \) represents the two-sphere of oriented triple collisions. ) Then we have natural identifications:

\[ Q^*/\mathbb{R}^+ \cong S^5 \]

and

\[ \tilde{Q}^*/\mathbb{R}^+ \cong \tilde{S}^5 \subset \tilde{Q}. \]
The space $Q^*/(G \times \mathbb{R}^+)$ of similarity classes of triangles is naturally isomorphic to

$$S_+ := S^5/G \subset Q/G$$

And the space $\tilde{Q}^*/(G \times \mathbb{R}^+)$ of similarity classes of oriented triangles is isomorphic to

$$S := \tilde{S}^5/G \subset \tilde{Q}/G.$$

Define

$$S_{J_0} = \tilde{S}^5/G(J_0) \subset Q_{J_0}.$$

Corresponding to these spaces we have various projections, $Q^* \to S_+$, $\tilde{Q}^* \to S$, $\tilde{S}^5 \to S_{J_0}$, etcetera, denoted by $\pi$ or $\pi_{J_0}$. Note that the fibers of

$$S_{J_0} \to S$$

are the two-spheres:

$$\pi_{J_0}^{-1}(pt.) = S^2(J_0) = G/G(J_0)$$

**Theorem 1** $S$ is a two-sphere which we call the shape sphere. The projection $\tilde{S}^5 \to S$ is the nontrivial principle $SO(3)$ bundle over the two-sphere.

The projection $S_{J_0} \to S$ is the associated nontrivial two-sphere bundle over
the two-sphere. The reduced configuration space, \( Q_{J_0} \) minus the triple collision is diffeomorphic to \( S_{J_0} \times (0, \infty) \) where the second factor is parameterized by \( I \).

EXPLANATION: If \( G \) is a connected Lie group, then the equivalence classes of principal \( G \)-bundles over an \( n \)-sphere are parameterized by the homotopy group \( \pi_{n-1}(G) \). In our case this homotopy group is \( \pi_1(SO(3)) \) which is the two-element group. The nontrivial \( G \)-bundle over \( S^2 \) can be realized as follows. Identify the two-sphere with the complex projective line \( CP^1 \). Let \( \gamma \to CP^1 \) denote the canonical complex line-bundle and \( \epsilon = CP^1 \times IR \) the trivial real line bundle. Form the rank 3 real vector bundle \( E = \gamma \oplus \epsilon \). This is an oriented vector bundle with a natural fiber-inner product. Then the nontrivial bundle, our \( \tilde{S}^5 \), is the bundle of oriented orthonormal frames for \( E \). And the nontrivial sphere bundle, our \( S_{J_0} \), is the unit sphere bundle of \( E \).

2.4 Metric nature of the quotients

\( Q \), being a Euclidean space, is a metric space. \( G \) acts on it by isometries, so that the quotient space \( Q/G \) of congruence classes of weighted, centered
triangles, inherits a metric. This quotient metric –sometimes called the orbital distance metric – is defined by declaring that the distance between two points in the quotient is the distance between the corresponding orbits in the original space. The dilations $\mathbb{R}^+$ act on $Q$ and commute with the $G$ action so they induce an action on the quotient as well.

The cone over a Riemannian manifold $(X, ds^2)$, possibly with boundary, is the topological space $(X \times [0, \infty))/(X \times \{0\})$ with associated Riemannian metric $d\lambda^2 + \lambda^2 ds^2$. Here $\lambda$ is the real parameter in $[0, \infty)$. The quotient by “$X \times \{0\}$” means that we crush (identify) $X \times \{0\}$ to a single point, called the cone point”. The metric tensor and manifold structure becomes singular there.

**Theorem 2** The metric space $Q/G$ of congruence classes of triangles is a cone over the space $S_+$ of similarity classes. The cone point corresponds to the triple collision. $S_+$ is isometric to the closed upper hemisphere of radius one-half. The equator represents collinear configurations. The dilation parameter is

$$\lambda = \sqrt{I}.$$ 

Replacing $Q$ by $\tilde{Q}$ resolves the singularity corresponding to the collinear
configurations. The pull-back to $\tilde{Q}$ of the metric on $Q$ fails to be a metric over the collinear configurations: it takes no energy to rotate a line segment about its axis. However, dividing by the $G$-action kills these null directions so we again get a metric on the quotient $\tilde{Q}/G$.

**Theorem 3** The metric space $\tilde{Q}/G$ of congruence classes of oriented triangles is a cone over the space $S$ of similarity classes of oriented triangles. The cone point corresponds to the triple collision. $S$ is isometric to the two-sphere of radius one-half. The equator corresponds to the collinear configurations. The height coordinate above the equator is

$$\frac{1}{2} z_1 = 2 \sqrt{\frac{m_1 m_2 m_3}{m_1 + m_2 + m_3}} \Delta$$

where $\Delta$ is the oriented area of the triangle:

$$\Delta = \frac{1}{2} \mathbf{n} \cdot (q_2 - q_1) \times (q_3 - q_1).$$

The map $S \to S_+$ of the sphere to the hemisphere which is induced by the branched cover $\tilde{Q} \to Q$ corresponds to the quotient map obtained when we identify the hemisphere with the quotient space obtained by identifying points of the sphere related by the reflection about this equator.
2.5 Relation with the symplectic reduced space

This section is included to connect the constructions of the previous two sections with the symplectic reduced space construction. The results here are not used in arriving at our formula, but may shed some light on it.

Suppose that a compact Lie group $G$ acts freely on a manifold $Z$, and so on $T^*Z$. Recall that the symplectic reduced space at the point $J \in \text{Lie}(G)^*$ is the sub-quotient space $J^{-1}(\mu)/G(J_0)$ where $J : T^*Z \rightarrow \text{Lie}(G)^*$ is the momentum map of the action, $\mu$ is a particular fixed element of $\text{Lie}(G)^*$, and $G(\mu) \subset G$ is its isotropy group (relative to the dual of the adjoint action). Of course in our situation we write $\mu = J_0$. This symplectic reduced space is diffeomorphic to the fiber product: $T^*(Z/G) \times_f (Z/G(\mu))$ over the quotient $Z/G$. This follows directly from [10], or [14], together with the fact that $Z/G(\mu)$ is naturally identified with the co-adjoint orbit bundle $Z \times_G (G/G(\mu)) \subset Z \times_G \text{Lie}(G)^*$ over $Z/G$. (See also the chapter in [2] on minimal coupling.) In a case such as the three-body problem where the underlying dynamics can be described by a second order equation on $Z$, the $T^*$-part of a reduced solution curve in the reduced space can be recovered from the derivative of the projection of that curve to $Z/G$. It follows that the
entire reduced curve can be recovered from its projection to $Z/G(\mu)$. Thus it makes sense to call $Z/G(\mu)$ the reduced configuration space at $\mu$.

### 2.6 The Connection

Our quotients inherit various tensorial objects besides metrics. In order to describe them we proceed generally. Suppose again that we are given a Riemannian manifold $Z$ and a group $G$ of isometries of $Z$ acting freely. From this data we can form:

- a metric on the quotient
- a connection for the principal $G$-bundle $Z \to Z/G$
- a fiber inner-product on the adjoint bundle $Z \times_G \text{Lie}(G) \to Z/G$ of Lie algebras over the quotient

The metric on the quotient (orbital distance metric) we have described.

To define the connection, we define its horizontal space.

**Definition 3** *The horizontal space at $z \in Z$ is the orthogonal complement at $z$ to the group orbit through $z$. The associated connection form $A : TZ \to \text{Lie}(G)$ is called the natural connection.*
The metric on $Z/G$ is a Riemannian one. Its metric tensor is obtained by identifying the tangent space at $\pi(z)$ with the horizontal space at $z$. With this definition, the projection $Z \to Z/G$ has the structure of a Riemannian submersion.

To define the fiber inner-product on the adjoint bundle, let

$$\sigma(z) : \text{Lie}(G) \to T_z Z$$

denote the infinitesimal generator of the group action:

$$\sigma(z)(\omega) = \frac{d}{d\epsilon} \exp(\epsilon \omega) z|_{\epsilon=0}.$$ 

Then set

$$\|\omega\|^2_z = \|\sigma(z)\omega\|^2_Z.$$ 

This defines the fiber inner product. Fix a bi-invariant inner product $\cdot$ on $\text{Lie}(G)$. Using the inner products we can construct the transpose

$$\sigma^T(z) : T_z Z \to \text{Lie}(G)$$

The map $(z, \dot{z}) \to \sigma^T(z)(\dot{z})$ is the Noether conserved quantity, or, after we identify $\text{Lie}(G)$ and $TZ$ with their duals using the inner products, it is the momentum map $J : T^*Z \to \text{Lie}(G)^*$. The fiber-inner product can also be
written
\[ \omega \cdot \Pi(z) \omega = \|\omega\|_z^2, \]
thus defining the moment of inertia tensor \( \Pi \). We have \( \Pi(z) = \sigma^T(z)\sigma(z) \).

We now have the universal formula for the connection form associated to this situation:
\[ A(z) = \Pi(z)^{-1} \circ \sigma^T(z). \]

In our situation \( \sigma^T = J \) is the angular momentum, viewed as a one-form with values in \( \mathbb{R}^3 \) (the Lie algebra of \( G = SO(3) \)):
\[ J = \sum m_a q_a \times dq_a. \]

\( \Pi \) is of course our moment of inertia tensor \( \Pi \). The connection form is then given by
\[ A(q) = \Pi(q)^{-1} \circ J(q). \]

**The horizontal space is the space of infinitesimal deformations with zero angular momentum.** This gives us a physical picture of what it means for a curve to be horizontal, and of the length of a path in one of the quotient spaces \( S, Q/G, \) etcetera.

All of this requires a bit of care at the collinear configurations, since the action is not free there. This is one of the reasons for introducing \( \tilde{Q} \). The
moment of inertia tensor of an collinear configuration has a zero eigenvector whose eigenspace is the line through the three masses. So one might think that $A$ and $\omega J_0$ become singular at such a configuration. But they do not. The kernel of $J$ and pole of $I I^{-1}$ are represented by the rotations of the collinear configuration configuration about its axis and they cancel.

$A$ has a nice physical interpretation. If $q(t)$ is a three-body motion then $A(q(t))(\dot{q})$ is the “best” choice of assignment of an angular velocity $\omega$ to the motion, given the fact that this motion need not be a rigid motion. If it does happen to be a rigid motion, with infinitesimal angular velocity $\omega$, then $A(q(t))(\dot{q}) = \omega$.

**Definition 4** The form $\alpha_{J_0}$ is the component of $A$ along the fixed angular momentum vector $J_0$:

$$\alpha_{J_0} = J_0 \cdot A.$$  

It is a one-form on on $Q^* = Q \setminus 0$, or by pull-back, on $\tilde{Q}^*$.

**Remark 1** Observe that the pull-back of $\alpha_{J_0}$ along a three-body motion $q(t)$ satisfies

$$q^*\alpha_{J_0} = \omega_{J_0}(t)dt$$
where $\omega_{J_0}(t) = \omega_{J_0}(q(t))$ is the instantaneous angular velocity of a three-body curve $a$ defined in the introduction when we were explaining the “dynamic term” of our reconstruction formula.

**Remark 2** Away from the triple collision, the natural projection $\pi_{J_0} : \tilde{Q} \rightarrow Q_{J_0}$ has the structure of a principal circle bundle, the circle being $G(J_0)$. Its associated connection one-form is $\alpha_{J_0}$ divided by $J_0$.

**Theorem 4** The form $d\alpha_{J_0}$ pushes down to a two-form $\Omega_{J_0}$ on $S_{J_0}$. This is the form described in the introduction and given by the explicit formula (4) above.

### 3 Planar configurations and proofs

#### 3.1 The Hopf fibration in planar configurations

A planar configuration is a triangle lying in the plane perpendicular to the angular momentum vector $J_0$. If the triangle is oriented we will take its normal to be parallel to $J_0$: $J_0 \cdot n = J_0 > 0$. The set $Q_{\text{planar}}$ of planar configurations forms a four-dimensional Euclidean subspace of the full configuration space. The action of the circle group $G(J_0)$ on $Q_{\text{planar}}$ is isomorphic
to the action of the circle on $\mathbb{C}^2$ which takes $(\zeta_1, \zeta_2)$ to $(e^{i\theta} \zeta_1, e^{i\theta} \zeta_2)$.

The intersection of the five-sphere $\{I = 1\}$ with $Q_{\text{planar}}$ forms a round three-sphere, denoted either $\tilde{\Sigma} \subset \tilde{Q}$ or $\Sigma \subset Q$. These three-spheres are diffeomorphic under $\beta : \tilde{Q} \to Q$ due to the unique choice of orientation. Their quotients by $G(J_0)$ are isometric to the two-sphere of radius $\frac{1}{2}$, which is the shape sphere $S$. Thus:

$$G(J_0) \to \tilde{\Sigma} \to \Sigma / G(J_0) = S$$

and

$$G(J_0) \to \Sigma \to \Sigma / G(J_0) = S$$

are isometric as Riemannian submersions to the standard Hopf fibration:

$$S^1 \to S^3(1) \to S^2\left(\frac{1}{2}\right).$$

### 3.2 Proof of theorems 1.

Consider the two standard local sections of the Hopf fibration $\tilde{\Sigma} \to S$. The transition function relating these sections takes values in $G(J_0) \subset G$. The local sections are also local sections for $\tilde{S}^5 \to S$ and as such have the same transition function. Restricted to the equator the transition function repre-
sents the nontrivial generator of the fundamental group of $G = SO(3)$ and hence $\tilde{S}^5 \to S$ is the nontrivial bundle.

To prove the facts regarding $S_{J_0}$ observe that $\tilde{S}^5/G(J_0)$ is isomorphic to the associated bundle $\tilde{S}^5 \times_G (G/G(J_0))$.

**Proofs of Theorems 2 and 3.**

Any triangle can be made to lie in the xy plane by a rotation so that $Q_{planar}$ is a slice for the $G$ action on $Q$. An oriented triangle can be made planar in a unique way, up to rotation. An unoriented triangle can be made planar in two rotationally inequivalent ways, the two ways being related by reflection. In other words: $Q/G = Q_{planar}/O(2)$, whereas $\tilde{Q}/G = Q_{planar}/SO(2)$. This accounts for the difference in the two quotients. These two identifications are isometries, since $Q_{planar}$ is totally geodesic. The last space is $C^2/S^1$ which is isometric to the cone over the sphere $S^2(\frac{1}{2})$. The quotient group $O(2)/SO(2)$ is the two-element group and accounts for the branched cover $S \to S_+$. The derivation of the formula for the normalized height $z_1$ can be found in Hsiang [3] and in our appendix.

The action by homotheties commutes with rotations so it descends to the quotient where it remains a dilation: $d(\lambda a, \lambda b) = \lambda d(a, b)$. (Here $a, b$ represent similarity classes and $d$ is the distance function.) Since $I$ is homogeneous
of degree 2, and since $S^5$ is defined by $I = 1$, the dilation parameter $\lambda$ equals $\sqrt{I}$.

**Proof of Theorem 4.** It follows from the discussion of the previous section, the above proofs and the fact that any curve of planar triangles has angular momentum in the $\hat{e}_3$ direction, that the restriction of $A$ to $\Sigma \subset Q$ is $\Gamma \hat{e}_3$ where $\Gamma$ is the canonical connection for the Hopf fibration. One can choose a local section $s : U \subset S \to \Sigma$ for the Hopf fibration such that $s^* \Gamma = \frac{1}{2} z_1 d\theta_1$. (The domain of this section is the sphere minus a geodesic arc connecting the north and south pole.) It follows that

$$s^* A = \left( \frac{1}{2} z_1 d\theta_1 \right) \hat{e}_3. \tag{6}$$

Now $A$ is “basic” with respect to the action of the group $\mathbb{R}^+$ of dilations. This means that

$$\sigma^*_\lambda A = A \tag{7}$$

$$\frac{\partial}{\partial \lambda} \mid A = 0 \tag{8}$$

where $\sigma_\lambda : \tilde{Q} \to \tilde{Q}$ is homothety by $\lambda \in \mathbb{R}^+$ and $\frac{\partial}{\partial \lambda}$ denotes inner product with the infinitesimal generator $\frac{\partial}{\partial \lambda}$ of homotheties. (To see (7) observe that $\Pi(\lambda q) = \lambda^2 \Pi(q)$ and $\sigma^*_\lambda J = \Sigma m_o \lambda q_o \times d(\lambda q_1) = \lambda^2 J$, and use the definition
\( A = I^{-1}J \). To see (8) observe that the angular momentum of a pure dilational motion is 0 which means that \( \frac{\partial}{\partial \lambda}|A = 0 \). Extend the section \( s \) by making it constant under homothety. Then, by the homothety invariances of \( A \), formula (8) still holds for this extended section. Let \( U \) denote the local frame induced by \( s \). It is a map to \( G = SO(3) \) defined by writing \( \tilde{q} = U(\tilde{q})s(\pi(\tilde{q})) \). The induced local trivialization of our principal bundle \( \tilde{Q}^* \rightarrow \tilde{Q}^*/G \) is then \( \tilde{q} \mapsto (\pi(\tilde{q}), U(\tilde{q})) \). Note that \( U(\tilde{q})e_3 = U(\tilde{q})(e_3) = n \), is the normal vector of the oriented triangle \( \tilde{q} \). Using this fact, the transformation formula for connections, and the fact that under our identification of the Lie algebra of \( G \) with \( \mathbb{R}^3 \) the adjoint action of \( G \) becomes its usual action on \( \mathbb{R}^3 \), we see that with respect to our local trivialization we have:

\[
A = \left( \frac{1}{2}z_1 d\theta_1 \right) n + (dU)U^{-1}
\]

where \( (dU)U^{-1} = \Theta \) denotes the pull-back of the Maurer-Cartan form on \( G \) by the map \( U \).

We now have

\[
\alpha_{J_0} = \frac{1}{2}(z_1 d\theta_1)J_0 z_2 + J_0 \cdot \Theta,
\]

since \( J_0 \cdot n = J_0 z_2 \). It is well-known (see [9]) that the two-form \( d(J_0 \cdot \Theta) \) pushes down to \( S^2 = G/G(J_0) \) and that this push-down is given by \( J_0 \) times
the solid angle form, \( d\sigma \wedge d\theta \). Thus
\[
d\alpha_{J_0} = J_0 \{ \frac{1}{2} d(z_1 z_2) \wedge d\theta_1 + dz_2 \wedge d\theta_2 \},
\]
which is the claimed formula for \( \Omega_{J_0} \).

**Remark.** We think of the local frame \( U = (U_1, U_2, U_3) \) as a moving frame attached to our triangle, chosen so that \( U_3 = n \) is the triangle’s normal. Hsiang [3] has shown that the frame which diagonalizes the moment of inertia tensor corresponds to a local section \( s \) which also satisfies \( s^\ast \Gamma = \frac{1}{2} z_1 d\theta_1 \). Hence his local trivialization is the same as the one we are using, up to a constant rotation about the \( n \)-axis. Note that the eigenframe is not defined at the north and south poles. These poles correspond to the weighted triangles for which \( I I \) has double eigenvalues. (If the masses are all equal these are the equilateral triangles.) A branch cut from the north to south pole is necessary, for if we traverse a small loop encircling the fiber over one of the poles then we find that \( (U_1, U_2, U_3) \mapsto (-U_1, -U_2, U_3) \).

4 Derivation of the reconstruction formula

In this section we derive our reconstruction formula, (2).
4.1 Closing the loop: a loop and a disc in $S_{J_0}$

Let

$$s(t) = \pi(\bar{q}(t)) \in S$$

be the curve of similarity classes represented by our three body motion. It is a closed curve on the base two-sphere. Let

$$c_J(t) = \pi_{J_0}(\bar{q}(t)) \in S_{J_0}$$

denote the projection of $\bar{q}(t)$ to $S_{J_0}$. Although $s(t)$ is closed, the reduced curve $c_{J_0}(t)$ need not be. There is a canonical way to close it. To see this, observe that $S_{J_0} \to S$ has two canonical sections. One consists of equivalence classes of triangles whose normals are pointing along the $J_0$ axis, and the other consists of those whose normals are antiparallel to the $J_0$ axis. We will these sections, or their values at a particular similarity class, the “north” and “south” poles. Since $s(t)$ is closed, both endpoints of the curve $c_{J_0}(t)$ lie on the same spherical fiber over $s(0) = s(t_1)$. On this fixed fiber draw the two geodesic arcs from the north pole to $c_{J_0}(0)$ and from $c_{J_0}(t_1)$ back, then sandwich the reduced curve $c_{J_0}(t)$ in between. The resulting closed curve will be denoted $\gamma_{J_0}$. It is the projection of a closed curve $\gamma(t)$ in $\bar{Q}$. See the figure.
Finally, $S_{J_0}$ is simply connected so that $\gamma_{J_0}$ bounds some disc $D \subset S_{J_0}$.

### 4.2 The loop in $Q$.

We now construct the loop $\gamma$ in $\tilde{Q}$ over which we integrate. Its projection to $S_{J_0}$ is the loop just described above. The loop $\gamma$ is obtained by concatenating the dynamic curve $\tilde{q}(t)$ with several group orbits, denoted $c_i(t)$ or $h(t)$. The act of concatenating two curves, one ending where the other begins, is defined in the obvious manner, and will be denoted by “*” below.

To construct the group curves we will use the following exponential notation for rotations. If $v \in \mathbb{R}^3$ then $exp(v)$, will mean the counter-clockwise rotation about the axis spanned by $v$ by $\|v\|$ radians. This is the standard Lie theoretic exponential map if we use the standard identification of $\mathbb{R}^3$ with the Lie algebra of the rotation group. If $v$ is a unit vector, and $\theta \in \mathbb{R}$ then $exp(\theta v)$ is a rotation by $\theta$ radians about the $v$ axis. Let $n_0$ and $n_1$ be the initial and final normal vectors to our curve of oriented triangles, as in assumption (2), above. Form unit vectors

$$\xi_0 = \frac{1}{\|J_0 \times n_0\|} n_0 \times J_0$$
and
\[ \xi_1 = \frac{1}{\|n_1 \times J_0\|} J_0 \times n_1 \]
and corresponding one-parameter subgroups \( \exp(s\xi_0), \exp(s\xi_1) \) of rotations.

Then:
\[ R_0 = \exp(\phi_2(0)\xi_0) \]
and
\[ R_1 = \exp(\phi_2(t_1)\xi_1) \]
where \( R_0, R_1 \) are the rotation matrices of our reconstruction formula (1), \( \phi_2 \) is the angle in our parameterization of \( S_{J_0} \), and \( \phi_2(t) \) is its value along the reduced curve \( \gamma_{J_0}(t) \): \( J \cos(\phi_2(t)) = J_0 \cdot n(t) \).

Let \( \tilde{q}(t) \) be the oriented three-body motion. Consider the concatenation
\[ \gamma = c_0 * \tilde{q} * c_1 * c_{J_0} * h \]
of the following curves:
\[ c_0(t) = \exp(-t\xi_0)R_0\tilde{q}(0), 0 \leq t \leq \phi_2(0), \]
\[ \tilde{q}(t), 0 \leq t \leq t_1, \]
\[ c_1(t) = \exp(t\xi_1)\tilde{q}(t_1), 0 \leq t \leq \phi_2(t_1), \]
\[ c_{J_0}(t) = \exp(-t \frac{1}{J_0}) R_1 \tilde{q}(t_1), \quad 0 \leq t \leq \Delta \theta, \]

and

\[ h(t) = e^{at} c_{J_f}(\Delta \theta). \]

The interval of definitions of the curves are chosen so that the endpoint of one curve is the initial point of the next and so the concatenations are well-defined. The constant \( a \) and the time of stopping for the final purely dilational curve \( h(t) \) are chosen so that its endpoint is the beginning point, \( R_0 \tilde{q}(0) \), for \( \gamma(t) \).

### 4.3 Line integrals

We have:

\[
\int_{\gamma} \alpha_{J_0} = \int_{c_1} \alpha_{J_0} + \int_{\tilde{q}} \alpha_{J_0} + \int_{c_2} \alpha_{J_0} + \int_{c_f} \alpha_{J_0} + \int_{h} \alpha_{J_0}
\]

CLAIM:

\[
\int_{c_0} \alpha_{J_0} = 0
\]

\[
\int_{c_1} \alpha_{J_0} = 0
\]

\[
\int_{\tilde{q}} \alpha_{J_0} = \int_{0}^{t_1} \omega_{J_0} dt
\]

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\[
\int_{cJ_0} \alpha_{J_0} = -J \Delta \theta \\
\int_h \alpha_{J_0} = 0
\]

The first two integrals vanish because

\[
c^* \alpha_{J_0} = J_0 \cdot c^* A = J_0 \cdot \omega ds
\]

whenever \( c(s) = \exp(s \omega)c(0) \) is the orbit generated by a one-parameter subgroup of \( G \). (This follows immediately from one of the defining properties of connections.) Now use the fact that the infinitesimal generators \( \omega = \xi_0, \xi_1 \) for the curves \( c_0, c_1 \) are perpendicular to \( J_0 \). To evaluate the third integral, observe that \( c_{J_0} \) is also the orbit of a one-parameter subgroup, but its generator is the unit vector along \( J_0 \). The vanishing of the integral over the homothety path \( h \) follows immediately from the homothety invariance of the connection already discussed. The integrand for the dynamic path \( \tilde{q}(t) \) was already discussed. (See remark 1 near the end of §2.6.) There we noted that \( q^* \alpha_{J_0} = \omega_{J_0} dt \).

An application of Stokes’ theorem and the formulae relating \( d\alpha_{J_0} \) to \( \Omega_{J_0} \) now prove our reconstruction formula, (2).
5 Appendix: explicit identification of the shape sphere

Following the discussion of §3.1 it suffices to understand the geometry of the space of similarity classes of weighted triangles for the planar three-body problem. We identify the plane in which the bodies move with the complex plane. Then we replace the spatial configuration space $Q$ above by the planar configuration space $Q_{planar}$ of triples $q = (q_1, q_2, q_3)$ of complex numbers, subject to the constraint $\Sigma a m_a q_a = 0$. The space of similarity classes of triangles in the plane forms a two-sphere, and a three-body motion describes a curve $w(t)$ on this sphere.

Let us describe the sphere $S$ of similarity classes explicitly. First, we diagonalize the mass matrix (kinetic energy) by introducing Jacobi coordinates

\[ \xi_1 = q_1 - q_3 \]

and

\[ \xi_2 = \frac{-m_1}{m_1 + m_3} q_1 + q_2 \frac{-m_3}{m_1 + m_3} q_3, \]

and normalized Jacobi coordinates

\[ \zeta_1 = \sqrt{\mu_1} \xi_1, \]
and

$$\zeta_1 = \sqrt{\mu_2} \zeta_2,$$

where the reduced masses \( \mu_i \) are defined by \( \frac{1}{\mu_1} = \frac{1}{m_1} + \frac{1}{m_3} \) and \( \frac{1}{\mu_2} = \frac{1}{m_1 + m_3} + \frac{1}{m_2} \). Then

$$\sum m_a \| \dot{q}_a \|^2 = \| \dot{\zeta}_1 \|^2 + \| \dot{\zeta}_2 \|^2.$$

Rotations by an angle \( \theta \) induce the transformation \((\zeta_1, \zeta_2) \mapsto (e^{i\theta} \zeta_1, e^{i\theta} \zeta_1)\) It follows that the vector

$$\mathbf{w} = (w_1, w_2, w_3)$$

defined by

$$w_1 = \frac{1}{2}(\| \zeta_1 \|^2 - \| \zeta_2 \|^2),$$

and

$$w_2 + iw_3 = \zeta_1 \bar{\zeta}_2$$

is invariant under rotations. The sphere of radius \( \frac{1}{2} \)

$$S = \{ \mathbf{w} : w_1^2 + w_2^2 + w_3^2 = \frac{1}{4} \}$$

is naturally identified with the space of similarity classes of planar triangles.

We calculate that the height coordinate \( w_3 \) is

$$w_3 = \frac{2}{I} \sqrt{\frac{m_1 m_2 m_3}{m_1 + m_2 + m_3}} \Delta$$
where $\Delta$ is the (oriented) area of the triangle $q$. In other words, the height on $S$ represents the triangle’s area, $\Delta$. The normalized height used in the body of the text is related to this coordinate by $w_3 = \frac{1}{2} z_1$.

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