We consider the initial-boundary value problem for the following extensible beam equation with nonlinear damping and source terms

$$\begin{align}\frac{\partial^2 u}{\partial t^2} + \Delta^2 u - M \left( \|\nabla u\|^2 \right) \Delta u + |u_t|^{p-1} u_t &= |u|^{q-1} u, \quad (x,t) \in \Omega \times (0,T), \\
\frac{\partial u}{\partial t} (x,t) &= 0, \quad x \in \partial \Omega,
\end{align}$$

where $p,q \geq 1$ are real numbers, $\Omega$ is a bounded domain with smooth boundary $\partial \Omega$ in $\mathbb{R}^n$, $\nu$ is the outer normal, and $M(s)$ is a positive locally Lipschitz function as $M(s) = \alpha + \beta s^\nu$, $\alpha, \beta > 0$, $\nu \geq 1$.

This kind of wave equation is obtained from the extensible beam equation of Woinowsky-Krieger [1],

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} - \left( \alpha_1 + \beta_1 \int_0^L \left| \frac{\partial u}{\partial t} \right| \, dt \right) \frac{\partial^2 u}{\partial x^2} + g \left( \frac{\partial u}{\partial t} \right) = 0,$$

for $g = 0$, where $u(x,t)$ is the deflection of the point $x$ of the beam at the time $t$ and $\alpha_1, \beta_1 > 0$ are constants. The equation (2) was studied by many authors such as [2–8].

In the case of $M(s) = 1$ and without fourth order term $\Delta^2 u$, the equation (1) can be written in the following form

$$u_{tt} - \Delta u + |u_t|^{p-1} u_t = |u|^{q-1} u.$$

The existence and blow up in finite time of solutions for (3) were established in [9–13]. The interaction between the damping ($|u_t|^{p-1} u_t$) and the source term ($|u|^{q-1} u$) make the problem more interesting. Levine [10, 11] first considered the interaction between the linear damping ($p = 1$), and source term by using the Concavity method. But
this method cannot be applied to the case of nonlinear damping term. Georgiev and Todorova [9] extended Levine’s result to the nonlinear case \( p > 1 \).

Recently, the problem (1) was studied by Esquivel-Avila [14, 15], he proved blow up, unboundedness, convergence and global attractor.

In this paper, we analyze the influence of the damping terms and source terms of the solution of the problem (1). We prove the local existence of solutions for the problem (1) by Banach contraction mapping principle. After that, we obtained global existence, decay and blow up of solutions.

This paper is organized as follows. In Section 2, we present some lemmas and notations needed later in this article. In Section 3, we prove the local existence for the problem (1). The proof of the global existence and decay of the solution are given in Section 4. In Section 5, blow up of the solution is discussed.

2 Preliminaries

In this section, we shall give some assumptions and lemmas which will be used throughout this paper. Let \( \| \cdot \| \) and \( \| \cdot \|_p \) denote the usual \( L^2(\Omega) \) norm and \( L^p(\Omega) \) norm, respectively.

Now, we state the general hypotheses

(H1) \( M(s) \) is a positive locally Lipschitz function for \( s \geq 0 \) with the Lipschitz constant \( L \) satisfying

\[
M(s) \geq m_0 > 0.
\]

(H2) For the nonlinearity, we suppose that

\[
1 < p < \infty \text{ if } n \leq 2, \text{ and } 1 < p \leq \frac{n+2}{n-2} \text{ if } n > 2,
\]

(4)

\[
1 < q < \infty \text{ if } n \leq 2, \text{ and } 1 < q \leq \frac{n}{n-2} \text{ if } n > 2.
\]

(5)

Lemma 2.1 (Sobolev-Poincaré inequality [16]). Let \( p \) be a number with \( 2 \leq p < \infty \) \((n = 1, 2)\) or \( 2 \leq p \leq \frac{2n}{n-2} \) \((n \geq 3)\), then there is a constant \( C \) such that

\[
\| u \|_p \leq C \| \nabla u \| \text{ for } u \in H^1_0(\Omega).
\]

Lemma 2.2 ([17]). Let \( \phi(t) \) be nonincreasing and nonnegative function defined on \([0, T]\), \( T > 1 \), satisfying

\[
\phi^{1+\alpha}(t) \leq w_0 (\phi(t) - \phi(t+1)), \ t \in [0, T]
\]

for \( w_0 \) is a positive constant and \( \alpha \) is a nonnegative constant. Then we have, for each \( t \in [0, T] \),

\[
\begin{align*}
\phi(t) &\leq \phi(0) e^{-w_1 \{t-1\}^+}, \\
\phi(t) &\leq \left( \phi(0)^{-\alpha} + w_0^{-1} \alpha \{t-1\}^+ \right)^{-\frac{1}{\alpha}}, \ \alpha > 0,
\end{align*}
\]

where \( \{t-1\}^+ = \max\{t-1, 0\} \), and \( w_1 = \ln \left( \frac{w_0}{w_0 - 1} \right) \).

Lemma 2.3 ([12]). Suppose that

\[
p \leq \frac{2n-1}{n-2}, \ n \geq 3
\]

holds. Then there exists a positive constant \( C > 1 \) depending on \( \Omega \) only such that

\[
\| u \|_p^p \leq C \left( \| \nabla u \|^2 + \| u \|_p^2 \right)
\]

for any \( u \in H^1_0(\Omega), \ 2 \leq s \leq p \).
3 Local existence

In this section, we are going to consider the local existence of the solution for the problem (1) by the similar arguments as in [18].

**Theorem 3.1.** Assume that (H1) and (H2) hold, and that \((u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)\), then there exists a unique solution \(u\) of (1) satisfying

\[
 u \in C \left( [0, T) ; H_0^2(\Omega) \right) , \quad u_t \in C \left( [0, T) ; L^2(\Omega) \right) \cap L^{p+1}(\Omega \times (0, T)).
\]

Moreover, at least one of the following statements holds:

(i) \(T = \infty\).

(ii) \(\|u_t\|^2 + \|\Delta u\|^2 \to \infty\) as \(t \to T^-\).

**Proof.** Define the following two parameter space

\[
 X_{T, R_0} = \left\{ v \in C \left( [0, T) ; H_0^2(\Omega) \right) , \; v_t \in C \left( [0, T) ; L^2(\Omega) \right) \cap L^{p+1}(\Omega \times (0, T)) : \left( \begin{array}{l}
 e(v(t)) \leq R_0^2 , \; t \in [0, T] . \; v(0) = u_0, \; v_t(0) = u_1
 \end{array} \right) \right\},
\]

where

\[
 e(v(t)) = \|v_t\|^2 + \|\Delta v\|^2 .
\]

for \(T > 0\) and \(R_0 > 0\). Then \(X_{T, R_0}\) is a complete metric space with the distance \(d(u, v) = \sup_{0 \leq t \leq T} e(u(t) - v(t))\).

We define the nonlinear mapping \(S\) in the following way. For \(v \in X_{T, R_0}\), \(u = Sv\) is the unique solution of the following equation

\[
 \begin{cases}
 u_{tt} + \Delta^2 u - M \left( \|\nabla v\|^2 \right) \Delta u + |u_t|^{p-1} u_t = |v|^{q-1} v , & (x, t) \in \Omega \times (0, T) , \\
 u(x, 0) = u_0(x) , \; u_t(x, 0) = u_1(x) , & x \in \Omega , \\
 u(x, t) = \frac{\partial}{\partial v} u(x, t) = 0 , & x \in \partial \Omega .
\end{cases}
\]

We shall show that there exist \(T > 0\) and \(R_0 > 0\), such that

(i) \(S : X_{T, R_0} \to X_{T, R_0}\).

(ii) \(S\) is a contraction mapping in \(X_{T, R_0}\) with respect to the metric \(d(\ldots)\).

First, we shall check (i), then the existence of the solution \(u\) for (7) will be proved by a standard method. Multiplying (7) by \(2u_t\) and integrating over \(\Omega \times (0, t)\), we obtain

\[
 \frac{d}{dt} e_1(u(t)) + 2 \|u_t\|_{p+1}^2 = \left( \frac{d}{dt} M \left( \|\nabla v\|^2 \right) \right) \|\nabla u\|^2 + 2 \int_{\Omega} |v|^{q-1} v u_t dx = I_1 + I_2
\]

where

\[
 e_1(u(t)) = \|u_t\|^2 + \|\Delta u\|^2 + M \left( \|\nabla v\|^2 \right) \|\nabla u\|^2 .
\]

We calculate that

\[
 I_1 = 2M' \left( \|\nabla v\|^2 \right) \int_{\Omega} \nabla v \nabla u dx . \|\nabla u\|^2 \leq 2L \|\nabla v\| \|v_t\| \|\nabla u\|^2 \leq LR_0^2 e_1(u(t)) .
\]

where \(L\) is a Lipschitz coefficient. Also by Hölder inequality and Sobolev-Poincaré inequality, we get

\[
 I_2 \leq 2 \|v\|_q^q \|u_t\|_{p+1} \leq 2C_q^q \|\nabla v\|^q \|u_t\| \leq 2C_q^q R_0^q \left[ e_1(u(t)) \right]^\frac{1}{q} .
\]

Thus, (10) and (11) into (8), we obtain

\[
 \frac{d}{dt} e_1(u(t)) + 2 \|u_t\|_{p+1}^2 \leq LR_0^2 e_1(u(t)) + 2C_q^q R_0^q \left[ e_1(u(t)) \right]^\frac{1}{q} .
\]
Integrating (12) over \([0, T]\) and by Gronwall inequality, we have

\[
e_1 (u (t)) + 2 \int_0^T \| u_t \|_{p+1}^{p+1} \, dt \leq C (u_0, u_1, R_0, T) e^{L R_0^2 T},
\]

where \(C (u_0, u_1, R_0, T) = e_1 (u_0)^{\frac{1}{2}} + C_q R_0^q T\).

Hence, from (6) and (13), we obtain

\[
e (u (t)) + 2 \int_0^T \| u_t \|_{p+1}^{p+1} \, dt \leq C (u_0, u_1, R_0, T) e^{L R_0^2 T}.
\]

From (14), we have

\[
u \in L^\infty \left( [0, T) ; H_0^2 (\Omega) \right), \quad u_t \in L^\infty \left( [0, T) ; L^2 (\Omega) \right) \cap L^{p+1} (\Omega \times (0, T)),
\]

which implies \(u \in C \left( [0, T) ; H_0^2 (\Omega) \right)\).

In order to prove that the map \(S\) satisfies (i), it will be enough to show that the parameters \(T\) and \(R_0\) satisfy

\[
C (u_0, u_1, R_0, T) e^{L R_0^2 T} \leq R_0^2.
\]

In the following, we show that \(u_t \in C \left( [0, T) ; L^2 (\Omega) \right)\). For each \(t_0 \in [0, T]\), let

\[
w (t) = u (t) - u (t_0) = S (v (t)) - S (v (t_0)).
\]

Then, \(w\) satisfies

\[
\begin{aligned}
& w_{tt} + \Delta^2 w - M \left( \| \nabla v (t) \| \right) \Delta w + |u_t (t)|^{p-1} u_t (t) - |u_t (t_0)|^{p-1} u_t (t_0) \\
& \quad \quad = \left[ M \left( \| \nabla v (t) \| \right) - M \left( \| \nabla v (t_0) \| \right) \right] \Delta u (t_0) + \int \left( w (t)|^{q-1} v (t) - |w (t_0)|^{q-1} v (t_0) \right) \, dx,
\end{aligned}
\]

\((x, t) \in \Omega \times (0, T)\), \(x \in \Omega\), \(x \in \partial \Omega\).

Multiplying (16) by \(2w_t\) and integrating over \(\Omega \times (0, t)\), we obtain

\[
\begin{aligned}
\frac{d}{dt} e_2 (w (t)) + 2 \int_\Omega \left( |u_t (t)|^{p-1} u_t (t) - |u_t (t_0)|^{p-1} u_t (t_0) \right) (u_t (t) - u_t (t_0)) \, dx \\
& = 2 \left[ M \left( \| \nabla v (t) \| \right) - M \left( \| \nabla v (t_0) \| \right) \right] \int_\Omega \Delta u (t_0) \, w \, dx \\
& \quad + \int_\Omega \left( |w (t)|^{q-1} v (t) - |w (t_0)|^{q-1} v (t_0) \right) w \, dx \\
& = I_3 + I_4 + I_5,
\end{aligned}
\]

where \(e_2 (w (t)) = \| w_t \|^2 + \| \Delta w \|^2 + M \left( \| \nabla v (t) \| \right) \| \nabla w \|^2\).

Integrating (17) over \([t_0, t]\) and using the fact \(e_2 (w (t_0)) = 0\), we obtain

\[
\begin{aligned}
e_2 (w (t)) + 2 \int_{t_0}^t \left( |u_t (t)|^{p-1} u_t (t) - |u_t (t_0)|^{p-1} u_t (t_0) \right) (u_t (t) - u_t (t_0)) \, dx \, dt = \int_{t_0}^t (I_3 + I_4 + I_5) \, dt.
\end{aligned}
\]

We observe from (15), that

\[
I_3 \leq 4L \| \nabla v \| \| \nabla v_t \| \| \nabla w \|^2 \leq 4LR_0^4, \quad I_4 \leq 4LR_0^4.
\]
and
\[
I_5 \leq 2 \left\| |v(t)|^{q-1} v(t) - |v(t_0)|^{q-1} v(t_0) \right\| \|w_t\| \leq 2 \left( \|v\|_{L_2}^q + \|v(t_0)\|_2^q \right) \|w_t\|
\]
\[
\leq 2 \left( \|\nabla v\|^q + \|\nabla v(t_0)\|^q \right) \|w_t\| \leq 4R_0^{q+1}.
\]
Thus, \[
I_3 + I_4 + I_5 \leq 8LR_0^4 + 4R_0^{q+1} = C(R).
\]
Hence, by the monotonicity of $|u_t|^{p-1} u_t$, when $t > t_0$, we see
\[
e_2(w(t)) \leq C(R)(t-t_0) \rightarrow 0 \text{ as } t \rightarrow t_0^+.
\]
On the other hand, when $t < t_0$, from the fact that $u_t \in L^{p+1}((0,T) \times \Omega)$, we have
\[
e_2(w(t)) \leq C(R)(t_0-t) + 2 \int_{t_0}^t \left( |u_t(t)|^{p-1} u_t(t) - |u_t(t_0)|^{p-1} u_t(t_0) \right) (u(t) - u(t_0)) \, dx \, dt \rightarrow 0
\]
as $t \rightarrow t_0^-$. Thus,
\[
e(w(t)) \leq Ce_2(w(t)) \rightarrow 0 \text{ as } t \rightarrow t_0.
\]
Finally, we have $u_t \in C([0,T); L^2(\Omega))$, obviously $u(t) \in X_{T,R_0}$.

On the other hand, we shall check (ii), we take $v_1, v_2 \in X_{T,R_0}$ and denote $u_1 = S v_1$ and $u_2 = S v_2$. Hereafter we suppose that (15) holds, so that $u_1, u_2 \in X_{T,R_0}$, when we put $w = u_1 - u_2$, $w$ satisfies the following:
\[
\left\{ \begin{array}{l}
w_{tt} + \Delta^2 w - M \left( \|\nabla v_1\|^2 \right) \Delta w + |u_{1t}|^{p-1} u_{1t} - |u_{2t}|^{p-1} u_{2t} \\
u(x,0) = 0, \quad u_t(x,0) = 0, \\
u(x,t) = \frac{\partial}{\partial n} u(x,t) = 0, \quad x \in \partial \Omega,
\end{array} \right. \quad (x,t) \in \Omega \times (0,T),
\]
(18)

Multiplying (18) by $2w_t$ and integrating over $\Omega \times (0,t)$, we obtain
\[
\frac{d}{dt} \left[ \|w_t\|^2 + \|\Delta w\|^2 + M \left( \|\nabla v_1\|^2 \right) \|\nabla w\|^2 \right] + 2 \int_{\Omega} \left( |u_{1t}|^{p-1} u_{1t} - |u_{2t}|^{p-1} u_{2t} \right) (u_{1t} - u_{2t}) \, dx
\]
\[
= 2 \left[ M \left( \|\nabla v_1\|^2 \right) - M \left( \|\nabla v_2\|^2 \right) \right] \int_{\Omega} \Delta u_{2t} w_t \, dx
\]
\[
+ \left( \frac{d}{dt} M \left( \|\nabla v_1\|^2 \right) \right) \|\nabla w\|^2 + 2 \int_{\Omega} \left( |v_1|^{q-1} v_1 - |v_2|^{q-1} v_2 \right) w_t \, dx = I_6 + I_7 + I_8.
\]
(19)

To proceed the estimates of $I_6, I_7$ and $I_8$, we observe that
\[
I_6 = 2 \left[ M \left( \|\nabla v_1\|^2 \right) - M \left( \|\nabla v_2\|^2 \right) \right] \int_{\Omega} \Delta u_{2t} w_t \, dx \leq 2L \left( \|\nabla v_1\|^2 + \|\nabla v_2\|^2 \right) \|\nabla (v_1 - v_2)\| \|w_t\|
\]
\[
\leq 4LC_0 R_0^2 \left[ v(v_1 - v_2) \right] \|w_t\| \leq \frac{1}{2} \left[ |v_t| + \|v_t\|^q \right] \|v_1 - v_2\| w_t \, dx
\]
\[
I_7 = \left( \frac{d}{dt} M \left( \|\nabla v_1\|^2 \right) \right) \|\nabla w\|^2 \leq 2L \|\nabla v_1\| \|\nabla v_{1t}\| \|\nabla w\|^2 \leq LR_0^2 |e(w(t))|
\]
and
\[
I_8 = 2 \int_{\Omega} \left( |v_1|^{q-1} v_1 - |v_2|^{q-1} v_2 \right) w_t \, dx \leq 2q \int_{\Omega} \left( |v_1|^{q} + |v_2|^{q} \right) |v_1 - v_2| w_t \, dx
\]
where $C,T,R_d,u,v$. For generality, we can assume that $\alpha$ theorem is now completed.

and (ii) hold. By Banach contraction mapping theorem, we obtain the local existence.

Inserting $I_6, I_7$ and $I_8$ into (19), we have

$$
\frac{d}{dt} \left[ \|w_t\|^2 + \|\Delta w\|^2 + M \left( \|\nabla v_1\|^2 \right) \|\nabla w\|^2 \right] + 2 \int_{\Omega} \left( \|u_{1t}\|^{p-1} u_{1t} - |u_{2t}|^{p-1} u_{2t} \right) (u_{1t} - u_{2t}) \, dx
$$

$$
= 4LC_0 R_0^2 \left[ e (v_1 - v_2) \right] \frac{1}{2} \left[ e (w (t)) \right] \frac{1}{2} + L R_0^2 e (w (t)) + 2qC_* R_0^{q-1} \left[ e (v_1 - v_2) \right] \frac{1}{2} \left[ e (w (t)) \right] \frac{1}{2} .
$$

Then, integrating (20) over $(0, t)$ and using initial-boundary conditions (18), we have

$$
e (w (t)) \leq \left[ 4LC_0 R_0^2 + 2qC_* R_0^{q-1} \right] \frac{1}{2} T^2 e^{L R_0^2 T} \sup_{0 \leq t \leq T} e (v_1 - v_2) .
$$

Then, by definition $d (u, v)$, we have

$$d (u_1, u_2) \leq C (T, R_0) d (v_1, v_2) .$$

where $C (T, R_0) = C^2 \left[ 4LC_0 R_0^2 + 2qC_* R_0^{q-1} \right] \frac{1}{2} T^2 e^{L R_0^2 T} .

If $C (T, R_0) < 1$, we can see $S$ is a contraction mapping. Finally, we choose suitable $T$ and $R_0$ small so that (i) and (ii) hold. By Banach contraction mapping theorem, we obtain the local existence.

The second statement of the theorem is proved by a standard continuation argument (see [19]). The proof of the theorem is now completed.

\section{Global existence and decay of solutions}

In this section, we discuss the global existence and decay of the solution for the problem (1). In this section and next section, we take $M (s) = \alpha + \beta s^\gamma$, $\alpha, \beta > 0$, $\gamma > 1$. Obviously $M (s)$ satisfies Lipschitz condition. Without loss of generality, we can assume that $\alpha = \beta = 1$.

We define

$$
J (u (t)) = \frac{1}{2} \|\Delta u\|^2 + \|\nabla u\|^2 + \frac{1}{2 (\gamma + 1)} \|\nabla u\|^{2 (\gamma + 1)} - \frac{1}{q + 1} \|u\|_{q+1}^{q+1} ,
$$

and

$$
I (u (t)) = \|\Delta u\|^2 + \|\nabla u\|^2 + \|\nabla u\|^{2 (\gamma + 1)} - \|u\|_{q+1}^{q+1} .
$$

We also define the energy function as follows

$$
E (t) = \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\Delta u\|^2 + \|\nabla u\|^2 + \frac{1}{2 (\gamma + 1)} \|\nabla u\|^{2 (\gamma + 1)} - \frac{1}{q + 1} \|u\|_{q+1}^{q+1} .
$$

Finally, we define

$$
W = \left\{ u : u \in H^2_0 (\Omega) , I (u) > 0 \right\} \cup \{ 0 \} .
$$

The next lemma shows that our energy functional (23) is a nonincreasing function along the solution of (1).

\textbf{Lemma 4.1.} $E (t)$ is a nonincreasing function for $t \geq 0$ and

$$
E' (t) = - \|u_t\|_{p+1}^{p+1} \leq 0 .
$$

\textbf{Proof.} Multiplying the equation in (1) by $u_t$, integrating over $\Omega$, using integration by parts, we get

$$
E (t) - E (0) = - \int_0^t \|u_t\|_{p+1}^{p+1} \, d \tau \text{ for } t \geq 0 .
$$

\qed
Lemma 4.2. Suppose that (H2) and \( q > 2\gamma + 1 \) hold. Let \( u_0 \in W \) and \( u_1 \in L^2(\Omega) \) such that
\[
\beta = C_* \left( \frac{2(q + 1)}{q - 1} E(0) \right)^{\frac{q - 1}{2}} < 1,
\] (27)
then \( u \in W \) for each \( t \geq 0 \).

Proof. Since \( I(0) > 0 \), it follows the continuity of \( u(t) \) that
\[
I(t) > 0,
\] (28)
for some interval near \( t = 0 \). Let \( T_m > 0 \) be a maximal time, when (28) holds on \([0, T_m]\). From (21) and (22), we have
\[
J(t) = \frac{1}{q + 1} I(t) + \frac{q - 1}{2(q + 1)} \left( \|\Delta u\|^2 + \|\nabla u\|^2 \right) + \frac{q - 2\gamma - 1}{2(q + 1)(\gamma + 1)} \|\nabla u\|^{2(\gamma + 1)}
\geq \frac{q - 1}{2(q + 1)} \left( \|\Delta u\|^2 + \|\nabla u\|^2 \right).
\] (29)
By using (29), (23), and Lemma 4.1, we obtain
\[
\|\Delta u\|^2 + \|\nabla u\|^2 \leq \frac{2(q + 1)}{q - 1} J(t) \leq \frac{2(q + 1)}{q - 1} E(t) \leq \frac{2(q + 1)}{q - 1} E(0).
\] (30)
By recalling Sobolev-Poincare inequality, (30) and (27), we have
\[
\|u\|^{q+1}_{q+1} \leq C_* \|\nabla u\|^{q+1} = C_* \|\nabla u\|^{q-1} \|\nabla u\|^2 \leq C_* \left( \frac{2(q + 1)}{q - 1} E(0) \right)^{\frac{q - 1}{2}} \|\nabla u\|^2
\]
\[
= \beta \|\nabla u\|^2 < \|\nabla u\|^2 \text{ on } t \in [0, T_m].
\] (31)
Therefore, by using (22), we conclude that \( I(t) > 0 \) for all \( t \in [0, T_m] \). By repeating the procedure, \( T_m \) is extended to \( T \). The proof of Lemma 4.2 is completed.

Lemma 4.3. Let assumptions of Lemma 4.2 holds. Then there exists \( \eta_1 = 1 - \beta \) such that
\[
\|u\|^{q+1}_{q+1} \leq (1 - \eta_1) \left( \|\Delta u\|^2 + \|\nabla u\|^2 \right).
\]

Proof. From (31), we obtain
\[
\|u\|^{q+1}_{q+1} \leq \beta \|\nabla u\|^2 \leq \beta \left( \|\Delta u\|^2 + \|\nabla u\|^2 \right).
\]
Let \( \eta_1 = 1 - \beta \), then we have the result.

Remark 4.4. From Lemma 4.3, we can deduce that
\[
\|\Delta u\|^2 + \|\nabla u\|^2 + \|\nabla u\|^{2(\gamma + 1)} \leq \frac{1}{\eta_1} I(t).
\]

Theorem 4.5. Suppose that (H2) and \( q > 2\gamma + 1 \) hold. Let \( u_0 \in W \) satisfying (28). Then the solution of problem (1) is global.

Proof. It is sufficient to show that \( \|u_t\|^2 + \|\Delta u\|^2 + \|\nabla u\|^2 + \|\nabla u\|^{2(\gamma + 1)} \) is bounded independently of \( t \). In order to achieve this we use (22) and (23), we obtain
\[
E(0) \geq E(t) = \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \left( \|\Delta u\|^2 + \|\nabla u\|^2 \right) + \frac{1}{2(q + 1)} \|\nabla u\|^{2(\gamma + 1)} - \frac{1}{q + 1} \|u\|^{q+1}_{q+1}
\]
\[
= \frac{1}{2} \|u_t\|^2 + \frac{q - 1}{2(q + 1)} \left( \|\Delta u\|^2 + \|\nabla u\|^2 \right) + \frac{q - 2\gamma - 1}{2(q + 1)(\gamma + 1)} \|\nabla u\|^{2(\gamma + 1)} + \frac{1}{q + 1} I(t)
\]
By virtue of (33) and Hölder inequality, we observe that
\[
\|u_t\|^2 + \frac{q-1}{2(q+1)} (\|\Delta u\|^2 + \|\nabla u\|^2) + \frac{q-2q-1}{2(q+1)(q+1)} \|\nabla u\|^{2(q+1)}
\]
since \(I(t) \geq 0\). Therefore
\[
\|u_t\|^2 + \|\Delta u\|^2 + \|\nabla u\|^2 + \|\nabla u\|^{2(q+1)} \leq CE(0),
\]
where \(C = \max \left\{ 2, \frac{2(q+1)}{q-1}, \frac{2(q+1)(q+1)}{q-2(q+1)} \right\}\). Then by Theorem 3.1, we have the global existence result.

\[ \Box \]

**Theorem 4.6.** Suppose that (H2) and (27) hold, and further \(u_0 \in W\). Thus, we have following decay estimates:
\[
E(t) \leq \begin{cases} 
E(0) e^{-w_1(t-1)^+}, & \text{if } p = 1, \\
E(0)^{-\alpha} + C_9^{-1} \alpha \left( t - 1 \right)^{\alpha}, & \text{if } p > 1,
\end{cases}
\]
where \(w_1, \alpha\) and \(C_9\) are positive constants which will be defined later.

**Proof.** By integrating (25) over \([t, t+1], t > 0\), we have
\[
E(t) - E(t+1) = \int_t^{t+1} \|u_t(\tau)\|_{p+1}^{p+1} \, d\tau = D_{p+1}(t),
\]
where
\[
D_{p+1}(t) = \int_t^{t+1} \|u_t(\tau)\|_{p+1}^{p+1} \, d\tau.
\]
By virtue of (33) and Hölder inequality, we observe that
\[
\int_t^{t+1} \int_\Omega |u_t|^2 \, dx \, dt \leq |\Omega| \frac{p^p}{p+1} D^2(t) = CD^2(t).
\]
Hence, from (34), there exist \(t_1 \in [t, t + \frac{1}{4}]\) and \(t_2 \in [t + \frac{3}{4}, t + 1]\) such that
\[
\|u_t(t_i)\| \leq CD(t), \quad i = 1, 2.
\]
Multiplying the equation of (1) by \(u\), and integrating it over \(\Omega \times [t_1, t_2]\), we get
\[
\int_{t_1}^{t_2} I(t) \, dt = \int_{t_1}^{t_2} \int_\Omega uu_t \, dx \, dt - \int_{t_1}^{t_2} \int_\Omega |u_t|^{p-1} u_t \, u \, dx \, dt.
\]
Integrating by parts and Cauchy-Schwarz inequality in the first term of the right hand side of (36), we obtain
\[
\int_{t_1}^{t_2} I(t) \, dt \leq \|u_t(t_1)\| \|u(t_1)\| + \|u_t(t_2)\| \|u(t_2)\| + \int_{t_1}^{t_2} \|u_t(t)\|^2 \, dt - \int_{t_1}^{t_2} \int_\Omega |u_t|^{p-1} u_t \, u \, dx \, dt.
\]
Now, our goal is to estimate the last term in the right-hand side of inequality (37). By using Hölder inequality, we obtain
\[
\int_{t_1}^{t_2} \int_\Omega |u_t|^{p-1} u_t \, u \, dx \, dt \leq \int_{t_1}^{t_2} \|u_t(t)\|_{p+1}^{p} \|u(t)\|_{p+1} \, dt.
\]
By applying the Sobolev-Poincare inequality and (30), we find
\[
\int_{t_1}^{t_2} \|u_t\|_{p+1}^{p} \|u\|_{p+1} \, dt \leq C_* \int_{t_1}^{t_2} \|u_t\|_{p+1}^{p} \|\nabla u\| \, dt \leq C_* \left( \frac{2(q+1)}{q-1} \right)^{\frac{1}{2}} \int_{t_1}^{t_2} \|u_t\|_{p+1}^{p} E^\frac{1}{2}(s) \, dt.
\]
\[
\leq C_s \left( \frac{2(q+1)}{q-1} \right)^{\frac{1}{2}} \sup_{t_1 \leq s \leq t_2} E^\frac{1}{2}(s) \int_{t_1}^{t_2} \|u_t\|^p \, dt \leq C_s \left( \frac{2(q+1)}{q-1} \right)^{\frac{1}{2}} \sup_{t_1 \leq s \leq t_2} E^\frac{1}{2}(s) D^p (t). \tag{39}
\]

From (30), (35) and Sobolev-Poincare inequality, we have
\[
\|u_t(t_i)\| \|u(t_i)\| \leq C_1 D(t) \sup_{t_1 \leq s \leq t_2} E^\frac{1}{2}(s). \tag{40}
\]

where \(C_1 = 2C_s \sqrt{\frac{2(q+1)}{q-1}}\).

Substitute (38)-(40) into (37), we obtain
\[
\int_{t_1}^{t_2} I(t) \, dt \leq C_3 \left[ \sup_{t_1 \leq s \leq t_2} E^\frac{1}{2}(s) D(t) + D^2(t) + C_s \sqrt{\frac{2(q+1)}{q-1}} \sup_{t_1 \leq s \leq t_2} E^\frac{1}{2}(s) D^p (t) \right]. \tag{41}
\]

On the other hand, from (22), (23) and Remark 4.4, we obtain
\[
E(t) \leq \frac{1}{2} \|u_t\|^2 + C_4 I(t), \tag{42}
\]

where \(C_4 = \frac{1}{\eta_1} \max \left\{ \frac{q-1}{2q+1}, \frac{q-2q-1}{2(q+1)(q+1)} \right\} + \frac{1}{q+1} \).

By integrating (42) over \([t_1, t_2]\), we have
\[
\int_{t_1}^{t_2} E(t) \, dt \leq \frac{1}{2} \int_{t_1}^{t_2} \|u_t\|^2 \, dt + C_4 \int_{t_1}^{t_2} I(t) \, dt.
\]

Then by (34) and (41), we get
\[
\int_{t_1}^{t_2} E(t) \, dt \leq \frac{1}{2} CD^2(t) + C_4 C_3 \left[ \sup_{t_1 \leq s \leq t_2} E^\frac{1}{2}(s) D(t) + D^2(t) + C_s \sqrt{\frac{2(q+1)}{q-1}} \sup_{t_1 \leq s \leq t_2} E^\frac{1}{2}(s) D^p (t) \right]. \tag{43}
\]

By integrating \(\frac{d}{dt} E(t)\) over \([t, t_2]\), we obtain
\[
E(t) = E(t_2) + \int_{t}^{t_2} \|u_t(t)\|^p \, dt. \tag{44}
\]

Therefore, since \(t_2 - t_1 \geq \frac{1}{2}\), we conclude that
\[
\int_{t_1}^{t_2} E(t) \, dt \geq (t_2 - t_1) E(t_2) \geq \frac{1}{2} E(t_2).
\]

That is,
\[
E(t_2) \leq 2 \int_{t_1}^{t_2} E(t) \, dt. \tag{45}
\]

Consequently, exploiting (32), (43)-(45), and since \(t_1, t_2 \in [t, t + 1]\), we get
\[
E(t) \leq 2 \int_{t_1}^{t_2} E(t) \, dt + \int_{t}^{t_2} \|u_t(t)\|^p \, dt = 2 \int_{t_1}^{t_2} E(t) \, dt + D^p + (t). \tag{46}
\]

Then, by (43), we have
\[
E(t) \leq \left( \frac{1}{2} C + C_4 C \right) D^2(t) + D^p + (t) + C_5 \left[ D(t) + D^p(t) \right] E^{\frac{1}{2}}(t).
\]
Applying Young inequality, we get

\[ E(t) \leq C_6 D^2(t) + D^{p+1}(t) + D^{2p}(t). \] (47)

**Case 1:** When \( p = 1 \), from (47), we obtain

\[ E(t) \leq 3C_6 D^2(t) = 3C_6 [E(t) - E(t + 1)]. \]

By Lemma 2.2, we get

\[ E(t) \leq E(0)e^{-w_1[t-1]^+}, \]

where \( w_1 = \ln \frac{3C_6}{3C_6 - T}. \)

**Case 2:** When \( p > 1 \), from (47), we obtain

\[ E(t) \leq C_6 D^2(t) \left(1 + D^{p-1}(t) + D^{2(p-1)}(t)\right) \leq C_6 \left(1 + D^{p-1}(t) + D^{2(p-1)}(t)\right) D^2(t). \]

Then since \( E(t) \leq E(0), \forall t \geq 0 \), we see from (32)

\[ E(t) \leq C_6 \left(1 + E^{\frac{p-1}{p+1}}(0) + E^{\frac{2(p-1)}{p+1}}(0)\right) D^2(t) \leq C_7 D^2(t), \quad t \geq 0. \]

Then we obtain

\[ E^{\frac{p-1}{p+1}}(t) \leq C_7 D^{p+1}(t) \leq C_8 (E(t) - E(t + 1)). \] (48)

Thus, from (48) and Lemma 2.2, we have

\[ E(t) \leq \left(E(0)^{-\alpha} + C_9^{-1} \alpha [t-1]^+\right)^{-\frac{1}{\sigma}}. \]

The proof of Theorem 4.6 is completed.

### 5 Blow up of solutions

In this section, we are going to consider the blow up of the solution for problem (1). Blow up of the solution with negative initial energy was proved for \( q > \max \{2\gamma + 1, p\} \) by using the technique of [9].

**Theorem 5.1.** Assume \( q > \max \{2\gamma + 1, p\} \), the initial energy \( E(0) < 0 \). Then the solution (1) blows up in finite time \( T^* \), and

\[ T^* \leq \frac{1 - \sigma}{\xi \sigma \Psi(0)}, \]

where \( \Psi(t) \) and \( \sigma \) are given in (49) and (50), respectively.

**Proof.** We suppose that the solution exists for all times and we reach to a contradiction.

Set \( H(t) = -E(t) \), then \( E(0) < 0 \) and (25) gives \( H(t) \geq H(0) > 0 \). Define

\[ \Psi(t) = H^{1-\sigma}(t) + \varepsilon \int_\Omega uu_t dx, \] (49)

where \( \varepsilon \) small to be chosen later and

\[ 0 < \sigma \leq \min \left\{ \frac{q - p}{p(q + 1)}, \frac{q - 1}{2(q + 1)} \right\}. \] (50)

Our goal is to show that \( \Psi(t) \) satisfies a differential inequality of the form

\[ \Psi'(t) \geq \xi \Psi(t)^{\xi}, \quad \xi > 1. \]

This, of course, will lead to a blow up in finite time.
By taking a derivative of (49) and using Eq. (1) we obtain
\[
\Psi' (t) = (1 - \sigma) H^{-\sigma} (t) H'(t) + \epsilon \| u_t \|^2 + \epsilon \| \Delta u \|^2 - \epsilon \| \nabla u \|^2
+ \epsilon \| u \|_{q+1}^2 - \epsilon \int_{\Omega} u u_t |u_t|^{p-1} \, dx.
\] (51)

By using the definition of the $H(t)$, it follows that
\[
-\| \nabla u \|^{2(\nu+1)} = 2 (\gamma + 1) H(t) + (\gamma + 1) \left( \| u_t \|^2 + \| \Delta u \|^2 + \| \nabla u \|^2 \right) - \frac{2 (\gamma + 1)}{q + 1} \| u \|_{q+1}^2.
\] (52)

Inserting (52) into (51), we obtain
\[
\Psi' (t) = (1 - \sigma) H^{-\sigma} (t) H'(t) + \epsilon (\gamma + 2) \| u_t \|^2 + \epsilon \gamma \left( \| \Delta u \|^2 + \| \nabla u \|^2 \right)
+ 2 \epsilon (\gamma + 1) H(t) + \epsilon \left( \frac{q - 2 \gamma - 1}{q + 1} \right) \| u \|_{q+1}^2 - \epsilon \int_{\Omega} u u_t |u_t|^{p-1} \, dx.
\] (53)

In order to estimate the last term in (53), we make use of the following Young inequality
\[
XY \leq \frac{\delta^k X^k}{k} + \frac{\delta^{-1} Y^l}{l},
\]
where $X, Y \geq 0$, $\delta > 0$, $k, l \in R^+$ such that $\frac{1}{k} + \frac{1}{l} = 1$. Consequently, applying the previous we have
\[
\int_{\Omega} u u_t |u_t|^{p-1} \, dx \leq \frac{\delta^{p+1}}{p + 1} \| u \|_{p+1}^p + \frac{\beta \delta^{-\frac{p+1}{p}}}{p + 1} \| u_t \|_{p+1}^{p+1} \leq \frac{\delta^{p+1}}{p + 1} \| u \|_{p+1}^p + \frac{\beta \delta^{-\frac{p+1}{p}}}{p + 1} H'(t),
\]
where $\delta$ is constant depending on the time $t$ and specified later. Therefore, (53) becomes
\[
\Psi' (t) \geq (1 - \sigma) H^{-\sigma} (t) H'(t) + \epsilon (\gamma + 2) \| u_t \|^2 + \epsilon \gamma \left( \| \Delta u \|^2 + \| \nabla u \|^2 \right)
+ 2 \epsilon (\gamma + 1) H(t) + \epsilon \left( \frac{q - 2 \gamma - 1}{q + 1} \right) \| u \|_{q+1}^q - \epsilon \frac{\delta^{p+1}}{p + 1} H'(t) - \epsilon \frac{\delta^{p+1}}{p + 1} \| u \|_{p+1}^{p+1}.
\] (54)

Therefore, by taking $\delta$ so that $\delta^{-\frac{p+1}{p}} = k H^{-\sigma} (t)$, where $k > 0$ is specified later, we obtain
\[
\Psi' (t) \geq \left( 1 - \sigma - \epsilon \frac{pk}{p + 1} \right) H^{-\sigma} (t) H'(t) + \epsilon (\gamma + 2) \| u_t \|^2 + \epsilon \gamma \left( \| \Delta u \|^2 + \| \nabla u \|^2 \right)
+ 2 \epsilon (\gamma + 1) H(t) + \epsilon \left( \frac{q - 2 \gamma - 1}{q + 1} \right) \| u \|_{q+1}^q - \epsilon \frac{k - p}{p + 1} H^{\sigma} p (t) \| u \|_{p+1}^{p+1}.
\] (55)

Since $q > p$ and $H(t) \leq \frac{1}{q+1} \| u \|_{q+1}^{q+1}$, we obtain
\[
H^{\sigma} p (t) \| u \|_{p+1}^{p+1} \leq C \left( \frac{1}{q + 1} \right)^{\sigma p} \| u \|_{q+1}^{q+1 + \sigma p(q+1)}.
\]

Thus, (55) yields
\[
\Psi' (t) \geq \left( 1 - \sigma - \epsilon \frac{pk}{p + 1} \right) H^{-\sigma} (t) H'(t) + \epsilon (\gamma + 2) \| u_t \|^2 + \epsilon \gamma \left( \| \Delta u \|^2 + \| \nabla u \|^2 \right)
+ 2 \epsilon (\gamma + 1) H(t) + \epsilon \left( \frac{q - 2 \gamma - 1}{q + 1} \right) \| u \|_{q+1}^q - \epsilon \frac{k - p}{p + 1} C \left( \frac{1}{q + 1} \right)^{\sigma p} \| u \|_{q+1}^{q+1 + \sigma p(q+1)}.
\] (56)

From (50), we have $2 \leq p + 1 + \sigma p (q + 1) \leq q + 1$. By using Lemma 2.3, we have
\[
\| u \|_{q+1}^{q+1 + \sigma p(q+1)} \leq C \left( \| \nabla u \|^2 + \| u \|_{q+1}^{q+1} \right) \leq C \left( \| \Delta u \|^2 + \| \nabla u \|^2 + \| u \|_{q+1}^{q+1} \right).
\]
Thus,

\[
\Psi'(t) \geq \left(1 - \sigma - \varepsilon \frac{pk}{p+1}\right) H^{-\sigma}(t) H'(t) + \eta \left(\|u_t\|^2 + \|\Delta u\|^2 + \|
abla u\|^2 + H(t) + \|u\|_{q+1}^{q+1}\right)
\]  

(57)

where \(\eta = \min \left\{\varepsilon (\gamma + 2), \varepsilon \left(\frac{\gamma - k^{-p}}{p+1} C (\frac{1}{q+1})^{\sigma p}\right), 2\varepsilon (\gamma + 1), \varepsilon \left(\frac{q-2\gamma-1}{\gamma+1} - \frac{k^{-p}}{p+1} C (\frac{1}{q+1})^{\sigma p}\right) \right\} > 0\), we pick \(\varepsilon\) small enough so that \((1 - \sigma) - \varepsilon \frac{pk}{p+1} \geq 0\) and

\[
\Psi(t) \geq \Psi(0) = H^{1-\sigma}(0) + \varepsilon \int_\Omega u_0 u_1 \, dx > 0, \forall t \geq 0.
\]  

(58)

On the other hand, applying Hölder inequality, we obtain

\[
\int_\Omega u u_t \, dx \leq \left\| u \right\|_{\frac{1}{\alpha+1}} \left\| u_t \right\|_{\frac{1}{\alpha}} \leq C \left(\left\| u \right\|_{q+1}^{\frac{1}{q+1}} \left\| u_t \right\|_{\frac{1}{\alpha+1}}\right).
\]

Young inequality gives

\[
\int_\Omega u u_t \, dx \leq C \left(\left\| u \right\|^2_{\frac{1}{\alpha+1}} + \left\| u_t \right\|^2_{\frac{2}{\beta+1}}\right),
\]

(59)

for \(\frac{1}{\alpha} + \frac{1}{\beta} = 1\). We take \(\theta = 2(1 - \sigma)\), to obtain \(\mu = \frac{2(1-\sigma)}{1-2\sigma} \leq q + 1\) by (50). Therefore, (59) becomes

\[
\int_\Omega u u_t \, dx \leq C \left(\left\| u \right\|^2_{\frac{1}{\alpha+1}} + \left\| u_t \right\|_{\frac{2}{\beta+1}}^2\right).
\]

By using Lemma 2.3, we obtain

\[
\int_\Omega u u_t \, dx \leq C \left(\left\| u \right\|^2_{\frac{1}{\alpha+1}} + \left\| u_t \right\|_{\frac{2}{\beta+1}}^2 + \left\| \nabla u \right\|^2\right).
\]

Thus,

\[
\Psi^{\frac{1}{\alpha+1}}(t) = \left[H^{1-\sigma}(t) + \varepsilon \int_\Omega u u_t \, dx \right]^{\frac{1}{\alpha+1}} \leq 2^{\frac{\sigma}{\alpha+1}} \left( H(t) + \varepsilon \int_\Omega u u_t \, dx \right)^{\frac{1}{\alpha+1}} \leq C \left(\left\| u \right\|^2_{\frac{1}{\alpha+1}} + \left\| u_t \right\|_{\frac{2}{\beta+1}}^2 + \left\| \nabla u \right\|^2\right)
\]

(60)

By combining of (57) and (60), we arrive

\[
\Psi'(t) \geq \xi \Psi^{\frac{1}{\alpha+1}}(t),
\]

(61)

where \(\xi\) is a positive constant.

A simple integration of (61) over \((0, t)\) yields

\[
\Psi^{\frac{1}{\alpha+1}}(t) \geq \frac{1}{\Psi^{\frac{1}{\alpha+1}}(0) - \xi t},
\]

which implies that the solution blows up in a finite time \(T^*\), with

\[
T^* \leq \frac{1 - \sigma}{\xi \sigma \Psi^{\frac{1}{\alpha+1}}(0)}.
\]

\[\square\]
References

[1] S. Woinowsky-Krieger, The effect of axial force on the vibration of hinged bars. Journal Applied Mechanics 1950; 17: 35-36.
[2] SK. Patcheu, On a global solution and asymptotic behavior for the generalized damped extensible beam equation. Journal of Differential Equations 1997; 135: 299-314.
[3] ST. Wu, LY.Tsai,Existence and nonexistence of global solutions for a nonlinear wave equation. Taiwanese Journal of Mathematics 2009; 13B(6): 2069-2091.
[4] Y. Zhijian, On an extensible beam equation with nonlinear damping and source terms. Journal of Differential Equations 2013; 254: 3903-3927.
[5] JM. Ball, Stability theory for an extensible beam. Journal of Differential Equations 1973; 14: 399-418.
[6] RW. Dickey, Infinite systems of nonlinear oscillation equations with linear damping. SIAM Journal on Applied Mathematics 1970; 19:208–214.
[7] TF. Ma, V. Narciso, Global attractor for a model of extensible beam with nonlinear damping and source terms. Nonlinear Analysis 2010; 73: 3402 3412.
[8] M. M. Cavalcanti, V. N. Domingos Cavalcanti and J. A. Soriano, Global existence and asymptotic stability for the nonlinear and generalized damped extensible plate equation, Commun. Contemp. Math. 6 (2004), 705-731.
[9] V. Georgiev, G. Todorova, Existence of a solution of the wave equation with nonlinear damping and source term. Journal of Differential Equations 1994; 109: 295–308.
[10] HA. Levine, Instability and nonexistence of global solutions to nonlinear wave equations of the form $P u_{tt} = -Au + F (u)$. Trans. Amer. Math. Soc., 1974; 192: 1–21.
[11] HA. Levine, Some additional remarks on the nonexistence of global solutions to nonlinear wave equations. SIAM Journal on Applied Mathematics 1974; 5: 138–146.
[12] SA. Messaoudi, Blow up in a nonlinearly damped wave equation. Mathematische Nachrichten 2001; 231: 105-111.
[13] SA. Messaoudi, Global nonexistence in a nonlinearly damped wave equation. Applicable Analysis 2001; 80: 269–277.
[14] JA. Esquivel-Avila, Dynamic analysis of a nonlinear Timoshenko equation. Abstract and Applied Analysis 2011; 2010: 1-36.
[15] JA. Esquivel-Avila, Global attractor for a nonlinear Timoshenko equation with source terms. Mathematical Sciences 2013; 1-8.
[16] RA. Adams, JF. Fournier, Sobolev Spaces. Academic Press, New York, 2003.
[17] M. Nakao, Asymptotic stability of the bounded or almost periodic solution of the wave equation with nonlinear dissipative term. Journal of Mathematical Analysis and Applications 1977; 58 (2): 336-343.
[18] K. Ono, On global solutions and blow up solutions of nonlinear Kirchhoff strings with nonlinear dissipation. Journal of Mathematical Analysis and Applications 1997; 216: 321-342.
[19] K. Ono, On global existence, asymptotic stability and blowing up of solutions for some degenerate nonlinear wave equations of Kirchhoff type with a strong dissipation. Mathematical Methods in the Applied Sciences 1997; 20: 151-177.