Theta functions on Kodaira–Thurston manifold *

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Abstract

We define analogue of theta functions on the Kodaira–Thurston
manifold which is a compact 4-dimensional symplectic manifold and
use them to construct canonical symplectic embedding of the Kodaira–
Thurston manifold into the complex projective space (analogue of the
Lefshetz theorem).

Keywords: theta functions, Kodaira–Thurston manifold, symplectic em-
bedding

1 Introduction

In this work we construct analogue of classical theta functions on the Abelian
variety for the Kodaira–Thurston nilmanifold $M_{KT}$. The classical theta
function from the geometrical point of view is a section of a holomorphic line
bundle over complex torus. The theta function on the Kodaira–Thurston
manifold is defined as section (not holomorphic) of a special line bundle $L$
over $M_{KT}$.

Analogues of theta functions for nilmanifolds were defined earlier [1, 2], but all these generalizations are based on representation theory. We
construct such analogue of theta functions with characteristics that they set
canonical symplectic embedding of $M_{KT}$ into a complex projective space
(analogue of Lefshetz theorem).

The Kodaira–Thurston manifold $M_{KT}$ is the quotient of $\mathbb{R}^4$ by the free
action of discrete group $\Gamma$, which generators are

\[
\begin{align*}
    a : (x, y, z, t) & \to (x + 1, y, z + y, t) \\
    b : (x, y, z, t) & \to (x, y + 1, z, t) \\
    c : (x, y, z, t) & \to (x, y, z + 1, t) \\
    d : (x, y, z, t) & \to (x, y, z, t + 1)
\end{align*}
\] (1)

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The Kodaira–Thurston manifold is notable as the first known example of a symplectic but not Kähler manifold \[3\].

Note that the embedding of \(M_{KT}\) into \(\mathbb{C}P^n\) can’t be holomorphic, since \(M_{KT}\) is not Kähler. However we prove that this map is symplectic. In other words the Fubini–Study form on \(\mathbb{C}P^n\) induces symplectic structure on \(M_{KT}\).

In §2 we recall necessary facts from classical theta function theory, in §3 we give the definition of theta function on \(M_{KT}\) and study some of its properties, in §4 we construct embedding into complex projective space (Theorem 1) and in §5 prove that this embedding is symplectic (Theorem 2).

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## 2 Classical theta function

Let’s recall some known facts about the classical theta function of Jacobi on one-dimensional complex torus. We will need them in the sequel.

Consider formal series

\[
\theta(z, \tau) = \sum_{k \in \mathbb{Z}} e^{2\pi ikz + \pi ik(k-1)\tau}.
\]

If \(\text{Im} \, \tau > 0\) then this series converges uniformly on every compact subset of \(\mathbb{C}\) and thus define entire function. In notations of \([4]\) this theta function is written as

\[
\exp(-\pi i\tau / 4 + \pi iz) \cdot \theta_{-1/2,0}(z, \tau).
\]

The invariance under map \(\tau \rightarrow \tau + 1\) and certain periodicity conditions explain our particular choice

\[
\theta(z + 1, \tau) = \theta(z, \tau), \quad (2)
\]

\[
\theta(z + \tau, \tau) = \exp(-2\pi iz) \cdot \theta(z, \tau). \quad (3)
\]

The generalization of theta function is the theta function of degree \(k \in \mathbb{N}\). It is an entire function \(\theta_k(z, \tau)\) which periodicity conditions are as follows

\[
\theta_k(z + 1, \tau) = \theta_k(z, \tau),
\]

\[
\theta_k(z + \tau, \tau) = \exp(-2\pi ikz) \theta_k(z, \tau).
\]

It is not hard to prove that theta functions of degree \(k\) form linear space of dimension \(k\). Let’s denote this space by \(L_k\).
The product of theta functions can be the theta function of higher degree. Let \( \{\alpha_i\}_{i=1}^k \) be the set of constants such that the sum of them is equal to zero. Then the following product is theta function of degree \( k \):

\[
\prod_{i=1}^k \theta(z + \alpha_i, \tau) \in \mathcal{L}_k.
\]

Theta function is equal to zero at the point \( z = 1/2 \). There is a single up to multiplicity zero in the fundamental region of lattice \( \mathbb{Z} + \tau \mathbb{Z} \).

Theta function \( \theta(z, \tau) \) satisfies following PDE:

\[
\frac{\partial \theta}{\partial \tau} = \frac{1}{4\pi i} \frac{\partial^2 \theta}{\partial z^2} - \frac{\theta}{2} \tag{4}
\]

Let \( \{\theta_k^p(z, \tau)\}_{p=1}^k \) be the basis in the space of theta functions of degree \( k \). Then map written in homogenous coordinates

\[
\varphi_k(z) = [\theta_k^1(z, \tau) : \ldots : \theta_k^k(z, \tau)]
\]

is well-defined map of complex torus into \( \mathbb{CP}^{k-1} \).

The Lefshetz theorem holds: if \( k \geq 3 \) then map \( \varphi_k \) is embedding. Note that this theorem is true for Abelian tori of arbitrary dimensions.

3 Definition of theta function on the Kodaira–Thurston manifold

Projection map \( (x, y, z, t) \rightarrow (y, t) \) gives \( M_{KT} \) the structure of \( T^2 \)-bundle over \( T^2 \). Left-invariant symplectic form \( \omega_{KT} = (dz - xdy) \wedge dx + dy \wedge dt \) tames bundle structure. This means that restriction of \( \omega_{KT} \) on any fibre and base is not degenerated.

Hence let the space of theta functions of degree \( k \) on \( M_{KT} \) be the span of products of classical theta functions on fibre and base:

\[
\theta_k^p(z + ix, y + i) \cdot \theta_k^q(y + it, i); \quad p, q = 1, \ldots, k
\]

Let’s denote this space by \( \mathcal{L}_k \). Clearly dimension of \( \mathcal{L}_k \) is equal to \( k^2 \).

Theta function of degree one we denote as

\[
\theta_{KT}(x, y, z, t) = \theta(z + ix, y + i) \cdot \theta(y + it, i).
\]
3.1 Theta function — section of complex line bundle

The periodicity conditions of theta function on $M_{KT}$ are as follows:

$$\theta_{KT}(x + 1, y, z + y, t) = \exp(-2\pi i (z + ix)) \cdot \theta_{KT}(x, y, z, t)$$
$$\theta_{KT}(x, y + 1, z, t) = \theta_{KT}(x, y, z, t)$$
$$\theta_{KT}(x, y, z + 1, t) = \theta_{KT}(x, y, z, t)$$
$$\theta_{KT}(x, y, z, t + 1) = \exp(-2\pi i (y + it)) \cdot \theta_{KT}(x, y, z, t)$$

(5)

These formulae imply that $\theta_{KT}$ is a section of line bundle over $M_{KT}$. This bundle is obtained by factorization $\mathbb{R}^4 \times \mathbb{C}$ under the action of group $\Gamma$

$$(u, w) \sim (\lambda \cdot u, e_\lambda(u)w), \quad u \in \mathbb{R}^4, w \in \mathbb{C}, \lambda \in \Gamma$$

where $e_\lambda(u)$ are multiplicators i.e. nonzero functions

$$e_\lambda : \mathbb{R}^4 \to \mathbb{C}^\ast$$

which satisfy following identities

$$e_\lambda(\mu \cdot u)e_\mu(u) = e_{\lambda \mu}(u), \quad \lambda, \mu \in \Gamma$$
$$e_0(u) = 1.$$

Sections of bundle which is set by multiplicators are in one-to-one correspondence with functions $f$ on $\mathbb{R}^4$ such that

$$f(\lambda \cdot u) = e_\lambda(u)f(u), \quad \lambda \in \Gamma, u \in \mathbb{R}^4.$$

Let’s note that the relations in group $\Gamma$ hold for multiplicators. It is obvious since multiplicators are set by behavior of the same function.

3.2 Multiplicative property of $\theta_{KT}$

Let’s define the action of $\zeta = (\zeta^1, \zeta^2) \in \mathbb{C}^2$ on $\theta_{KT}$:

$$(\zeta \cdot \theta_{KT})(x, y, z, t) = \theta(z + ix + \zeta^1, y + i)\theta(y + it + \zeta^2, i).$$

(6)

If

$$\sum_{i=1}^{k} \alpha_i = 0$$

then the following product is theta function of degree $k$:

$$\prod_{i=1}^{k} (\zeta_i \cdot \theta_{KT})(x, y, z, t) \in \mathcal{L}_k.$$  

(7)

The proof follows from the analogous property of classical theta function.
4 Embedding of $M_{KT}$ into complex projective space

Let’s enumerate the basis of $L_k$: \{s_i\}_{i=1}^{k^2}$. Map

$$\varphi_k = (s_1, s_2, \ldots, s_{k^2})$$

is well defined map of $M_{KT}$ to $\mathbb{CP}^{k^2-1}$.

Theorem 1 If $k \geq 3$ then map $\varphi_k$ is embedding.

Proof. For brevity we will prove the theorem in case of $k = 3$. The proof for $k > 3$ is analogous.

Firstly we will prove the injectivity of $\varphi_k$. We will follow the proof of the classical Kodaira embedding theorem stated in [5, ch. 1, §4]

Note that theta functions of degree $k$ are the global sections of $k$-th tensor power of bundle set by multiplicators (5).

If for any two points $u \neq v \in M_{KT}$ there exists section $s \in L_k$ such that $s(u) = 0$ and $s(v) \neq 0$ then map $\varphi_k$ is injective. Indeed, assume that map ”glues” points $u$ and $v$. It means that for all sections $s \in L_k$ it is true that $s(v) = \zeta \cdot s(u)$, where $\zeta$ is nonzero constant. If $s$ satisfies previously mentioned condition then $\zeta$ must be zero and we have contradiction.

Note also that given condition implies that for any point $u \in M_{KT}$ not all sections vanish at point $u$.

We construct needed theta function of degree $k = 3$ as a product of two functions $s = fg$:

$$f(x, y, z, \alpha, \beta) = \theta(z + ix + \alpha, y + i)\theta(z + ix + \beta, y + i)\times$$
$$\times \theta(z + ix - \alpha - \beta, y + i), \quad (8)$$

$$g(y, t, \gamma, \delta) = \theta(y + it + \gamma, i)\theta(y + it + \delta, i)\theta(y + it - \gamma - \delta, i). \quad (9)$$

As follows from (7) product $fg$ is a valid theta function of degree $k = 3$ on $M_{KT}$.

We denote coordinates of $u, v$ as $(x, y, z, t)$ and $(x', y', z', t')$ respectively. Select $\gamma$ such that $\theta(y + it + \gamma, i) = 0$. Now select $\delta$ such that all other factors in definition of function $g$ don’t vanish at the point $v$:

$$\theta(y' + it' + \delta, i)\theta(y' + it' - \gamma - \delta, i) \neq 0.$$}

It is possible because zeros of theta function are isolated. Also by selecting $\alpha, \beta$ we can assure that function $f$ doesn’t vanish at point $v$. 

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Unless $\theta(y' + it' + \gamma, i) = 0$ the constructed section solves the problem. Assume contrary. Since classical theta function has single (up to multiplicity) zero in the fundamental region of lattice, it follows that $y = y', t = t'$ modulo $\Gamma$.

Select $\alpha$ such that $\theta(z + ix + \alpha, y + i) = 0$. Note that $\theta(z' + ix' + \alpha, y' + i) \neq 0$, because otherwise $u = v$. Select $\beta$ such that $f(v) \neq 0$ and $\gamma, \delta$ such that $g(v) \neq 0$.

Thus we constructed necessary section and proved the injectivity of $\varphi_k$.

Let’s prove that rank of $\varphi_k$ is maximal. Here we will follow the proof of Lefshetz theorem stated in [6]. Firstly we show that the rank of $\varphi_k$ is maximal if and only if the rank of matrix $J$ is maximal

$$J = \begin{pmatrix} s_1 & \cdots & s_{k^2} \\ \partial_z s_1 & \cdots & \partial_z s_{k^2} \\ \partial_y s_1 & \cdots & \partial_y s_{k^2} \\ \partial_z s_1 & \cdots & \partial_z s_{k^2} \\ \partial_t s_1 & \cdots & \partial_t s_{k^2} \end{pmatrix}.$$ 

The map $\varphi_k$ written in homogenous coordinates is a composition of the map $\varphi_k$ to $\mathbb{C}^{k^2}$ and subsequent projection $\pi : \mathbb{C}^{k^2} \setminus \{0\} \to \mathbb{C}^{k^2-1}$. Obviously if we cross out first row of $J$ we get the differential of $\varphi_k$.

Now assume that at the point $u \in M_{KT}$ first row of $J$ is a linear combination of other rows. It means that radius-vector of $\varphi_k(u)$ is collinear to the image of a certain tangent vector (to $M_{KT}$) at the point $u$. Since $\pi$ projects along complex lines passing through the origin, the kernel of differential of $\pi$ consists exactly of such vectors. Therefore rank of $J$ is maximal if and only if rank of $\varphi_k$ is maximal.

Let’s transform matrix $J$ to more suitable for us form. The rank of the following matrix coincides with the rank of $J$

$$\bar{J} = \begin{pmatrix} s_1 & \cdots & s_{k^2} \\ (\partial_y - i\partial_t)s_1 & \cdots & (\partial_y - i\partial_t)s_{k^2} \\ (\partial_y + i\partial_t)s_1 & \cdots & (\partial_y + i\partial_t)s_{k^2} \end{pmatrix}.$$ 

Last two rows of $\bar{J}$ are the Cauchy-Riemann conditions. Since sections $s_j$ are holomorphic functions of $z + ix$, the last row of $\bar{J}$ vanishes.

Assume that rank of $\bar{J}$ (over $\mathbb{C}$) is less than 4 at the certain fixed point $u^* = (x^*, y^*, z^*, t^*) \in M_{KT}$. It means that there exist non-trivial constants
for any $u, a, b, c, d$ such that

$$a s_j(u^*) + \frac{b}{2} \left( \partial_y - i \partial_t \right) s_j(u^*) + \frac{c}{2} \left( \partial_z - i \partial_x \right) s_j(u^*) + \frac{d}{2} \left( \partial_y + i \partial_t \right) s_j(u^*) = 0, \quad j = 1, \ldots, k^2.$$ 

By (7), the function

$$s(u, \mu, \nu) = (\mu \cdot \theta_{KT})(u)(\nu \cdot \theta_{KT})(u)((-\mu - \nu) \cdot \theta_{KT})(u).$$

lies in $L_k(k = 3)$ for any $\mu, \nu$. Hence this function is a linear combination of $s_j$ and the identity

$$as(u^*, \mu, \nu) + \frac{b}{2} \left( \partial_y - i \partial_t \right) s(u^*, \mu, \nu) + \frac{c}{2} \left( \partial_z - i \partial_x \right) s(u^*, \mu, \nu) + \frac{d}{2} \left( \partial_y + i \partial_t \right) s(u^*, \mu, \nu) = 0. \quad (10)$$

holds. We define the linear differential operator $L = \frac{b}{2}(\partial_y - i \partial_t) + \frac{c}{2}(\partial_z - i \partial_x) + \frac{d}{2}(\partial_y + i \partial_t)$ and rewrite the last identity

$$L \log(\mu \cdot \theta_{KT})(u^*) = -a - L \log(\nu \cdot \theta_{KT})(u^*) - L \log((-\mu - \nu) \cdot \theta_{KT})(u^*). \quad (11)$$

For any $u, \mu$ there exists $\nu$ such that

$$(\nu \cdot \theta_{KT})(u)((-\mu - \nu) \cdot \theta_{KT})(u) \neq 0. \quad (12)$$

By (11)-(12), the function

$$\xi(\mu) = L \log(\mu \cdot \theta_{KT})(u^*) \quad (13)$$

is an entire function of $\mu = (\mu^1, \mu^2)$. It follows from (5) that function $\xi(\mu)$ satisfies the following periodicity conditions

$$\xi(\mu^1 + 1, \mu^2) = \xi(\mu^1, \mu^2), \quad (14)$$

$$\xi(\mu^1 + y^* + i, \mu^2) = \xi(\mu^1, \mu^2) - 2\pi i c, \quad (15)$$

$$\xi(\mu^1, \mu^2 + 1) = \xi(\mu^1, \mu^2), \quad (16)$$

$$\xi(\mu^1, \mu^2 + i) = \xi(\mu^1, \mu^2) - 2\pi i b. \quad (17)$$

Therefore derivatives $\partial_{\mu^j} \xi$ are the periodic and entire functions. This means that they are constants and $\xi = \alpha \mu^1 + \beta \mu^2 + \gamma$. By (14)-(16), $\alpha = \beta = 0$ and
function $\xi$ is constant. By (15), (17), $b = c = 0$. Then the following identity holds
\[ \xi(\mu) = \frac{d}{2} \left( \frac{\partial_y \theta(z + ix + \mu^1, y + i)}{\theta(z + ix + \mu^1, y + i)} \right)_{u = u^*} = \gamma. \] (18)

Here we used the Cauchy-Riemann equation implicitly
\[ (\partial_y + i\partial_x)\theta(y + it, i) = 0. \]

Let’s denote by $D$ the $z + ix$-differentiation
\[ D = \frac{1}{2} (\partial_z - i\partial_x). \]

It follows from (11) that
\[ \partial_y \theta(z + ix, y + i) \equiv \frac{1}{4\pi i} (D^2 \theta)(z + ix, y + i) - \frac{1}{2} (D\theta)(z + ix, y + i). \] (19)

By substituting (19) in (18) and considering that
\[ (D\theta)(z + ix + \mu^1, y + i) = \partial_{\mu^1} \theta(z + ix + \mu^1, y + i), \]
we have that function $\theta(z^* + ix^* + \mu^1, y^* + i)$ as a function of $\mu^1$ satisfies the linear ODE with constant coefficients
\[ \frac{d}{4\pi i} \theta'' - \frac{d}{2} \theta' - 2\gamma \theta = 0. \]

It is easy to write down general solution of this equation and check that it contradicts to the periodicity conditions of theta function (5). Thus $d = \gamma = 0$. By (10), $a = 0$.

As a result all constants $a, b, c, d$ vanish and matrix $\bar{J}$ has the maximal rank. Since point $u^*$ is arbitrary, the rank is maximal everywhere. Theorem proved.

**Remark.** It would be interesting to investigate the connection of this theta function to certain nonlinear equations by the way of obtaining soliton equations from section identities for Jacobians [6].

5 **Embedding is symplectic**

Manifold $M_{KT}$ is a symplectic manifold, where symplectic form would be for instance the following left-invariant form $\omega_{KT} = (dz - xdy) \wedge dx + dy \wedge dt$.

In this section we will prove the following
Theorem 2  
1. If $k \geq 3$ then map $\varphi_k$ induces a symplectic form on $M_{KT}$.

2. Induced symplectic form is cohomologous to $k \cdot \omega_{KT}$.

Proof. Let’s choose the following basis of $\mathcal{L}_k$ in the definition of $\varphi_k$

$$\theta^p_k(z + ix, y + i) \cdot \theta^q_k(y + it, i); \quad p, q = 1, \ldots, k.$$ 

Notice that $\varphi_k$ is a composition of two maps. The first one is $\psi_k : M_{KT} \to \mathbb{C}P^{k-1} \times \mathbb{C}P^{k-1}$, $\psi_k = (\psi'_k, \psi''_k)$. Here

$$\psi'_k(x, y, z) = [\theta^1_k(z + ix, y + i) : \ldots : \theta^k_k(z + ix, y + i)],$$

$$\psi''_k(y, t) = [\theta^1_k(y + it, i) : \ldots : \theta^k_k(y + it, i)].$$

The second is the Segre map $\sigma_k : \mathbb{C}P^{k-1} \times \mathbb{C}P^{k-1} \to \mathbb{C}P^{k^2-1}$, which is defined in homogenous coordinates by the formula

$$\sigma_k([z^1 : \ldots : z^k], [w^1 : \ldots : w^k]) = [z^1 w^1 : z^1 w^2 : \ldots : z^k w^{k-1} : z^k w^k].$$

Thus $\varphi_k = \sigma_k \circ \psi_k$. Let’s denote by $\Omega'$ symplectic form (associated with the Fubini–Study metric) on the first factor of $\mathbb{C}P^{k-1} \times \mathbb{C}P^{k-1}$, by $\Omega''$ on the second. Then $\Omega' + \Omega''$ is a symplectic form on the product. Since the Segre map is a holomorphic embedding, it is sufficient to show that induced form $\psi_k^*(\Omega' + \Omega'')$ is symplectic.

Note that algebra of left-invariant forms on $M_{KT}$ is generated by $dx, dy, dz - xdy, dt$.

Map $\psi''_k$ is holomorphic embedding of complex torus into $\mathbb{C}P^k$ described by the classical Lefshetz theorem. Therefore the Fubini–Study form induces symplectic form on torus

$$(\psi''_k)^*(\Omega'') = \alpha \cdot dy \wedge dt,$$

where $\alpha$ doesn’t vanish anywhere on $M_{KT}$.

Let

$$(\psi'_k)^*(x, y, z)(\Omega') = f \cdot (dz - xdy) \wedge dx + g \cdot (dz - xdy) \wedge dy + h \cdot dx \wedge dy.$$ 

Here $f, g, h$ are certain functions on $M_{KT}$. This is the general view of 2-form on $M_{KT}$ generated by map depending on $x, y, z$.

Note that for any fixed $y$ map $\psi'_k$ also becomes holomorphic embedding described by the Lefshetz theorem and therefore

$$(\psi'_k)^*(\Omega') = \beta \cdot dz \wedge dx,$$

where $\beta$ doesn’t vanish anywhere on $M_{KT}$. 

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where \( \beta \) doesn’t vanish anywhere on \( M_{KT} \). It follows that \( f \equiv \beta \). Gathering altogether
\[
(\psi_k^*(\Omega' + \Omega''))^2 = ((\psi_k')^*(\Omega') + (\psi_k'')^*(\Omega''))^2 = 
= (\beta \cdot (dz - xdy) \wedge dx + g \cdot (dz - xdy) \wedge dy + h \cdot dx \wedge dy + \alpha \cdot dy \wedge dt)^2.
\]
By opening the brackets we get that
\[
(\psi_k^*(\Omega' + \Omega''))^2 = 2\alpha \beta \cdot dx \wedge dy \wedge dz \wedge dt.
\]
Last identity means that induced form is non-degenerated. The closedness is implied by the commutation of differential and \( \psi_k^* \). Thus \( \psi_k^*(\Omega' + \Omega'') \) is closed and non-degenerate i.e. symplectic form. We proved the first item of the theorem.

Let’s prove the second one. Denote by \( L \) the bundle defined by multiplicators \( (5) \). Earlier we noted that theta functions of degree \( k \) are sections of \( L \otimes k \).

Recall that any complex line bundle over a manifold \( M \) is induced by the universal bundle over the complex projective space when \( M \) is mapped to \( \mathbb{CP}^n \). Therefore \( L \otimes k \) is a pullback of universal bundle and curvature form of \( L \otimes k \) is a pullback of the Fubini–Study form, which is a curvature form of universal bundle. Recall also that the first Chern class of line bundle is realized by curvature form. Thus the cohomological class of induced form coincides with \( c_1(L \otimes k) = k \cdot c_1(L) \) and we must prove that
\[
c_1(L) = [(dz - xdy) \wedge dx + dy \wedge dt].
\]
Consider the cover of \( M_{KT} \) by sets
\[
U_\lambda = \lambda \cdot U_0, \quad \lambda \in \Gamma
\]
where \( U_0 = \{|u|^k < 3/4\} \). The nerve \( N(\mathcal{U}) \) of the minimal subcover \( \mathcal{U} \) of the above cover \( U_\lambda \) is homeomorphic to \( M_{KT} \) and its cohomologies with coefficients in \( \mathbb{Z} \) coincide with \( H^*(M_{KT}; \mathbb{Z}) \).

The coordinate transformations \( g_{\lambda \mu} : U_\lambda \cap U_\mu \to \mathbb{C}^* \) are expressed in terms of multiplicators
\[
g_{\lambda \mu}(u) = e\lambda(u) \cdot e\mu^{-1}(\mu \cdot u); \quad \lambda, \mu \in \Gamma. \quad (20)
\]
By definition the cocycle \( z_{\lambda \mu \nu} \in C^2(\mathcal{U}; \mathbb{Z}) \)
\[
z_{\lambda \mu \nu} = \frac{1}{2\pi i} \left( \log(g_{\lambda \mu}) + \log(g_{\mu \nu}) - \log(g_{\nu \lambda}) \right) \quad (21)
\]
realizes the first Chern class of bundle $L$. Given formula is the value of $z$ at a two-dimensional simplex $(\lambda, \mu, \nu) \in N(U)$.

The group $H_2(M_{KT}; \mathbb{Z})$ is generated by homological classes of tori $T_{ac}$, $T_{bc}$, $T_{da}$, $T_{db}$, spanned by commuting translations (1).

Define the functions $f_\lambda(u)$ as follows
\[ e_\lambda(u) = e^{2\pi i f_\lambda(u)}. \] (22)

By (20)–(22),
\[ c_1([T_{\lambda\mu}]) = f_\mu(u) + f_\lambda(\mu \cdot u) - f_\lambda(u) - f_\mu(\lambda \cdot u). \]

Calculate the first Chern class at the basis tori
\[ c_1([T_{ca}]) = c_1([T_{bd}]) = 1; \quad c_1([T_{cb}]) = c_1([T_{ad}]) = 0. \] (23)

Since manifold $M_{KT}$ is a homogenous space of nilpotent Lie group, any element of $H^2(M_{KT}; \mathbb{R})$ is realized by the left-invariant form dual to basis cocycle. Group $H^2(M_{KT}; \mathbb{R})$ is generated by cohomological classes of the forms $(dz - xdy) \wedge dx$, $dy \wedge dt$, $(dx - xdy) \wedge dy$ and $dx \wedge dt$.

By (23), $c_1(L) = [(dz - xdy) \wedge dx + dy \wedge dt]$. Theorem is proved.

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