Topological delocalization transitions and mobility edges in the nonreciprocal Maryland model

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In this work, we introduce a non-Hermitian extension of the quasiperiodic Maryland model [65-71] by adding nonreciprocity to its hopping amplitudes. The resulting system exhibits two distinct NHQC phases and a phase transition induced by the hopping asymmetry. We introduce our model in Sec. II and reveal its key features in Sec. III. Besides characterizing the real-to-complex spectrum transition and the delocalization transition from an insulator to a mobility edge phase with coexisting extended and localized states in Secs. IIIA and IIIB, we also introduce a topological winding number to describe different phases and transitions in the nonreciprocal Maryland model in Sec. III C. We summarize our results and discuss potential future work in Sec. IV.

II. MODEL

The Maryland model describes particles hopping in a tight-binding lattice and subject to an unbounded onsite superlattice potential. Its Hamiltonian in position representation takes the form

$$H_0 = \sum_n (J\langle n|n+1| + H.c. + V \tan(\pi\alpha n)\langle n|n\rangle).$$

Here J is the nearest-neighbor (NN) hopping amplitude, V is the amplitude of onsite potential, n ∈ Z is the lattice site index and \{\langle n\rangle\} forms a complete basis of the lattice. \alpha is chosen as an irrational number to realize quasiperiodic modulations. The Hermitian Maryland model $H_0$ is first introduced as an integrable model to study localization problems in quantum chaotic systems [65-71]. Later, it is also utilized to understand the localization in higher dimensions [72-74] and topological features of integer quantum Hall effects [75]. Experimentally, the Maryland model might be realized in photonic systems by engineering the light propagation in polygonal optical waveguide lattices [76].

In this work, we focus on the localization problem in a non-Hermitian extension of the Maryland model, which is obtained by incorporating asymmetry into the hopping amplitudes of $H_0$. The Hamiltonian of the resulting sys-
tem, which is dubbed the nonreciprocal Maryland model (NRMM), takes the following form
\[ H = J \sum_n (e^{-\gamma} |n+1\rangle \langle n| + e^{\gamma} |n+1\rangle \langle n|) + V \sum_n \tan(\pi a n)|n\rangle \langle n|. \]  

Here \( J, V, \gamma \in \mathbb{R} \). \( \gamma \) measures the degree of asymmetry between left-to-right and right-to-left NN hopping amplitudes. We take the periodic boundary condition (PBC) for all calculations below by identifying \(|n\rangle = |n + L\rangle\), where \( n = 1, 2, ..., L \) and \( L \) is the length of lattice. When \( \alpha = p/q \) (\( p, q \) being coprime integers) is chosen to be a rational number and \( L \) is an integer multiple of \( q \), the system is in the commensurate regime and under the PBC it is expected to hold charge density wave like states. In this work, we instead take \( \alpha = \frac{\sqrt{5} - 1}{2} \), i.e., the inverse golden ratio to yield a quasiperiodic potential. Inserting \( \psi \) into the eigenvalue equation \( H(\psi) = E|\psi\rangle \), we obtain
\[ J(e^{-\gamma}\psi_{n+1} + e^{\gamma}\psi_{n-1}) + V \tan(\pi a n)\psi_n = E\psi_n. \]  

Here \( E \) is the energy of state \(|\psi\rangle\), which is in general complex as \( H \neq H^\dagger \). \( \psi_n \equiv \langle n|\psi\rangle \) represents the amplitude of right eigenvector \(|\psi\rangle\) on the \( n \)th lattice site. The solution of Eq. (2) under the PBC then yields all the possible eigenenergies and eigenstates of the NRMM. Note that in numerical calculations, we take a rational approximation for \( \alpha \) by setting \( \alpha \approx F_i/F_{i+1} \), where \( F_i \) and \( F_{i+1} \) are two adjacent elements of the Fibonacci sequence.

In the Hermitian limit (\( \gamma = 0 \)), due to the unbounded nature of onsite potential \( V_n = V \tan(\pi a n) \), all eigenstates of \( H \) are localized with energy-dependent localization lengths for irrational \( \alpha \) and \( V \neq 0 \). Away from the Hermitian limit, however, we find a nonreciprocity-induced transition of the system from a localized phase with real spectrum to a mobility edge phase with complex spectrum at a finite \( \gamma = \gamma_c \), which is thus non-Hermitian origin. Note that the mobility edge phase means a phase in which extended and localized states coexist and are separated in their energies by a mobility edge. We present systematic characterizations of this transition and the resulting energy-dependent mobility edges in the following section.

III. RESULTS

In this section, we investigate the spectrum, delocalization, and topological transitions of the NRMM. In Sec. IIIA we study the energies of NRMM and find a real-to-complex spectral transition at a finite hopping asymmetry \( \gamma = \gamma_c \), whose expression as a function of the hopping amplitude and onsite potential is obtained. In Sec. IIIB the spectrum transition is connected to a transition of the system from localized to mobility edge phases. The mobility edge separating localized and extended states is further picked up and its expression is found to be energy-dependent. In Sec. IIIC a topological winding number is introduced to distinguish phases with different transport nature and characterize the transitions between them in the NRMM, thus yielding a complete phase diagram. For ease of reference, we summarize our main results about the NRMM in Table I.

A. Real-to-complex spectrum transition

We first study the spectrum of NRMM by solving the eigenvalue Eq. (2). Two typical examples of the spectrum are presented in Figs. (a) and (b), where \( \text{ReE} \) and \( \text{ImE} \) refer to the real and imaginary parts of energy \( E \), respectively. We observe that with weak hopping asymmetry \( \gamma \), the eigenvalues of \( H \) could retain real. When \( \gamma \) goes beyond a critical value \( \gamma_c \), a finite amount of eigenenergies deviate from the real axis, and the spectrum undergoes a real-to-complex transition. After the transition, the complex part of eigenenergies develop a loop on the \( \text{ReE} \)-\( \text{ImE} \) plane surrounding a base energy \( E_0 = 0 \). The points along the loop in Fig. (b) satisfy the equation
\[ E_{\pm} = 2J \cos(\beta - i\gamma) \pm iV, \quad \beta \in [-\pi, \pi] \]  
under the constraint \( \text{ImE}_- > \text{ImE}_+ \).

To find the critical point \( \gamma_c \) as a function of system parameters, we evaluate the maximum of imaginary parts of energy max \( \text{ImE} \) and the density of states (DOS)
with complex eigenvalues $\rho$ at different $V$ and $\gamma$, with results presented in Figs. 1(c) and 1(d). In numerical calculations, we count $E$ as a complex eigenvalue if $|\text{Im} E| > 10^{-5}$. It is clear that once $V \neq 0$, we could obtain spectrum transitions from real to complex in the NRMM with the increase of $|\gamma|$. By setting $\text{Im} E_{\pm} = 0$ in Eq. (3), we find the critical values of hopping asymmetry $\gamma = \gamma_c$ where spectrum transitions happen, i.e.,

$$\gamma_c = \pm \arcsinh \left( \frac{V}{2J} \right). \tag{4}$$

When $|\gamma| < |\gamma_c|$, all eigenvalues of $H$ are found to be real, whereas a finite portion of the spectrum becomes complex for $|\gamma| > |\gamma_c|$. We plot these exact phase boundaries by red dashed lines in Figs. 1(c) and 1(d), and find that they coincide with numerical calculations of the spectrum. Meanwhile, Eq. (4) provides us with a guideline for the study of transport nature of the NRMM, as will be discussed in the next subsection.

### B. Delocalization transition and mobility edge

In NHQCs, spectrum transitions usually accompany state transitions regarding their spatial profiles [11][22]. In the NRMM, we also discover a transition from an insulator phase with no extended eigenstates to a mobility edge phase, in which extended and localized eigenstates coexist. To see this, we first inspect the inverse participation ratio (IPR), which is defined for the $i$th normalized eigenstate $|\psi_i\rangle$ of $H$ in the lattice representation as

$$\text{IPR}_i = \sum_{n=1}^{L} |\psi_{ni}|^4.$$ 

Here the amplitude $\psi_{ni} = \langle n|\psi_i\rangle$ and $i = 1, 2, ..., L$. In the localized phase, all eigenstates have finite IPRs. Extended states start to appear when the minimum of IPRs, denoted as $\text{min(IPR)}$, starts to approach zero.

In Fig. 2(a), we show the minimum, maximum and average of IPRs for the NRMM. It is clear that the $\text{max(IPR)} \approx 1$ due to the unbounded nature of the on-site potential $V_n$, which indicates that there are localized states at any hopping asymmetry in the limit $L \to \infty$. Notably, the $\text{min(IPR)}$ decreases to zero when $\gamma$ goes beyond a critical value $\gamma_c$, which happens to be coincide with the critical point of spectrum transition in Eq. (4). Therefore, when the hopping asymmetry is tuned from below to above the critical point $\gamma_c$, the NRMM transforms from a localized phase with real spectrum to a mobility edge phase with complex spectrum. The number of extended states in the mobility edge phase further increases with the increase of $\gamma$, as hinted by the $\langle \text{ave(IPR)} \rangle$ in Fig. 2(a). In Fig. 2(b), we present the $\text{min(IPR)}$ as a function of system parameters $(V, \gamma)$ and indeed observe two distinct phases characterized by $\text{min(IPR)} > 0$ and $\text{min(IPR)} \approx 0$. Their boundaries are highlighted by the red dashed lines and given exactly by Eq. (4). To the best of our knowledge, the NRMM contributes the first example of a 1D NHQC with only localized and mobility edge phases, but no metallic phases at any amounts of non-Hermiticity.

To give a more detailed look at the mobility edge, we show the IPRs of all eigenstates of the NRMM versus the real parts of their energies and the potential amplitude $V$ (hopping asymmetry $\gamma$) for two typical examples in Figs. 2(c) and 2(d). We observe that the states with $\text{IPR} \approx 0$ and $\text{IPR} > 0$ are clearly separated in both figures. Moreover, with thorough numerical analysis, we find an equation that describes the mobility edge of the NRMM, i.e.,

$$\frac{V^2}{(2J \sinh \gamma)^2} + \frac{(\text{Re} E)^2}{(2J \cosh \gamma)^2} = 1. \tag{5}$$

Trajectories determined by this equation, presented by the magenta dashed lines in Figs. 2(c) and 2(d), separate states with vanishing and finite IPRs in the energy-parameter plane of the system. Moreover, the Eq. (5) is well-defined at finite $V$ only if $\gamma \neq 0$. Therefore, mobility edges in the NRMM are solely originated from non-Hermitian effects encoded in the hopping asymmetry of the lattice.

### C. Topological invariant and phase diagram

In recent studies, a spectral winding number has been introduced to depict the transitions between different NHQC phases [11], following a strategy that is differ-
FIG. 2. IPRs of NRMM under the PBC. The length of lattice is \( L = 987 \). System parameters are \( J = 1 \), \( \alpha = \frac{\sqrt{5} - 1}{2} \) for all panels. In (a), the solid, dashed and dash-dotted lines show the minimum, maximum and average of IPRs versus the hopping asymmetry \( \gamma \) at \( V = 1 \). The crossing point of the dotted line and the horizontal axis corresponds to the critical value of \( \gamma = \gamma_c = \text{asinh}(V/2J) \approx 0.4812 \), where the system undergoes a transition from the localized to the mobility edge phase. (b) shows the minimum of IPRs at different parameters \((V, \gamma)\). The red dashed lines refer to the boundaries between localized and mobility edge phases, which are given by Eq. (4). The IPRs of all states versus the real parts of their energies and \( V \) \((\gamma)\) are shown in (c) \((d)\) with \( \gamma = 0.5 \) \((V = 1)\). The magenta dashed lines obtained following Eq. (5) are mobility edges separating extended and localized states.

FIG. 3. Winding numbers of the NRMM. System parameters are \( J = 1 \) and \( \alpha = \frac{\sqrt{5} - 1}{2} \). The length of lattice is \( L = 377 \) with the twist boundary condition. Each region with a uniform color corresponds to a phase with common spectrum and transport features, whose winding number \( w \) is denoted explicitly therein. The red dashed lines separating different regions denote the phase boundaries obtained from Eq. (4).

tem, since the extended states therein are those whose energies possess nonvanishing imaginary parts.

For the NRMM, we construct a spectral winding number as

\[
w = \frac{1}{2\pi i} \int_0^{2\pi} \partial_\theta \ln \det[H(\theta) - E_0] d\theta. \tag{6}\]

Here \( H(\theta) \) is obtained from \( H \) by letting \( e^{\pm \gamma} \rightarrow e^{\pm \gamma \pm \theta/L} \) in Eq. (1), with \( L \) being the length of lattice. This winding number counts the number of times the spectrum of \( H(\theta) \) winds around the base energy \( E_0 \) when \( \theta \) changes over a cycle from zero to \( 2\pi \). Referring to the spectrum presented in Fig. 1, we choose \( E_0 = 0 \) in our calculation of \( w \) without loss of generality. It is clear that in the Hermitian region \((\gamma = 0)\) we have \( w = 0 \), since in this case \( H(\theta) \) has only real energies. At a finite hopping asymmetry \( \gamma = \gamma_c \), we expect a quantized jump of \( w \) from zero to \( \pm 1 \), which should be accompanied by the spectrum transition of the NRMM from real to complex.

In Fig. 3, we present the winding number \( w \) of NRMM versus the onsite potential \( V \) and hopping nonreciprocity \( \gamma \), which is obtained directly from Eq. (6). The red dashed lines in Fig. 3 highlight the exact phase boundaries, which are given by Eq. (4). We observe that the winding number \( w \) takes a quantized change when crossing the border between two NHQC phases, and remain constant elsewhere. More specially, we find \( w = 0 \) when the system is prepared in the localized phase with real spectrum, and \( w = \pm 1 \) if the system moves into the mobility edge phase with complex spectrum. These observations confirm that the topological winding number \( w \) can
Indeed be utilized to discriminate the NHQC phases of NRMM with distinct spectrum and transport nature, and characterize the transitions between them in the meantime.

For completeness, we have checked other values of the irrational parameter $\alpha$ (e.g., $\alpha = (\sqrt{5} + 1)/2$, $1/\sqrt{2}$) in our calculations of spectrum, IPRs and winding numbers of the NRMM. The results we obtained are all consistent with those reported in Figs. [13] which implies that the conclusion we drew from this section is general to other incommensurate values of the onsite potential in Eq. (4).

**IV. SUMMARY AND DISCUSSION**

In this work, we uncover non-Hermiticity induced spectral, localization and topological phase transitions in the nonreciprocal Maryland model. A transformation of the system from a localized phase with real spectrum to a mobility edge phase with complex spectrum can be obtained only if there are finite amounts of hopping asymmetry. The equations satisfied by the complex spectrum, phase boundaries and energy-dependent mobility edges are found exactly. A topological order parameter is further employed to build the phase diagram and characterize transitions between different NHQC phases. Notably, due to the unbounded nature of onsite potential in the NRMM, we find no extended phases at any finite hopping nonreciprocity, which is distinct from the situation found in a Maryland model with complex onsite potential [33]. Our study thus enriches the family of NHQCs by unveiling a particular type of system with only localized and mobility edge phases, whose properties can be characterized exactly.

In future work, it would be interesting to consider many-body effects in Maryland-type NHQCs and investigate their dynamical properties. The interplay between non-Hermiticity and quantum chaos in the NRMM can further be studied following the mapping discussed in Appendix A. In this work, our calculations are performed under the PBC and the non-Hermitian skin effects (NHSEs) [78] are expected to have no impacts on the results. Under the open boundary condition, the non-reciprocal hoppings in our model could possibly induce non-Hermitian skin modes. Recently, it was shown that (de)localization transitions and (pseudo) mobility edges could even emerge in clean systems due to the NHSEs, and topological characterizations of these intriguing phenomena have been proposed [82]. Therefore, it is expected that in the presence of both NHSEs and spatial quasiperiodicity, richer patterns of spectrum, localization and topological transitions could appear in generic NHQCs, which deserve more thorough explorations.

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**Appendix A: NRMM and Floquet system**

The Hermitian Maryland model can be mapped to a mathematically equivalent Floquet system [60]. A similar mapping can also be constructed for the NRMM, which may serve as an entrance for the study of the interplay between non-Hermiticity and quantum chaos. We start by rewriting the eigenvalue Eq. (2) as

$$\frac{E}{V} \psi_n - e^{-\gamma} \psi_{n+1} - e^{\gamma} \psi_{n-1} = \tan(\pi \alpha n) \psi_n,$$  \hspace{0.5cm} (A.1)

where we have set $J = 1$ as the unit of energy. Multiplying the imaginary unit $i$ from both sides of Eq. (A.1), we obtain

$$\frac{V}{E} \psi_n - i(E \psi_n - e^{-\gamma} \psi_{n+1} - e^{\gamma} \psi_{n-1}) = e^{-i2\pi \alpha n}.$$  \hspace{0.5cm} (A.2)

For a 1D quasicrystal, the amplitude $\psi_n$ can be expressed as a superposition of plane waves [33], i.e.,

$$\psi_n = \sum_{\ell} \varphi_{\ell} e^{i k_{\ell} n}.$$  \hspace{0.5cm} (A.3)

Here for any $\ell \neq \ell'$, the difference between wave numbers $k_\ell$ and $k_{\ell'}$ is an integer multiple of $2\pi \alpha$. Taking $\psi_n$ as amplitudes, we can further construct a series

$$\Psi(x) = \sum_n \psi_n e^{inx} = \sum_{\ell} \varphi_{\ell} \sum_n e^{i(x + k_{\ell}) n} = 2\pi \sum_{\ell,n} \varphi_{\ell} \delta(x + k_{\ell} - 2\pi n),$$  \hspace{0.5cm} (A.4)

where we used the Poisson summation formula to arrive at the last equality. If we now multiply $e^{inx}$ to Eq. (A.2) and take the summation over $n$, we will obtain with the help of Eq. (A.4) that

$$\begin{bmatrix} 1 - i \frac{E}{V} + i \frac{2}{V} \cos(x - i \gamma) \end{bmatrix} \Psi(x) = \begin{bmatrix} 1 + i \frac{E}{V} - i \frac{2}{V} \cos(x - 2\pi \alpha - i \gamma) \end{bmatrix} \Psi(x - 2\pi \alpha).$$  \hspace{0.5cm} (A.5)

To make the connection between the NRMM and its
Floquet equivalent more transparent, we introduce the function
\[ \Phi(x) = \left[ 1 + \frac{E}{V} - i \frac{2}{V} \cos(x - i\gamma) \right] \Psi(x). \] (A.6)

Using Eq. (A.6), we can express Eq. (A.5) as
\[ \Phi(x) = \frac{1 + i \frac{E}{V} - i \frac{2}{V} \cos(x - i\gamma)}{1 - i \frac{E}{V} + i \frac{2}{V} \cos(x - i\gamma)} \Phi(x - 2\pi\alpha). \] (A.7)

Performing the Taylor expansion of \( \Phi(x - 2\pi\alpha) \), we find
\[ \Phi(x - 2\pi\alpha) = \sum_n \frac{(-2\pi\alpha)^n}{n!} \frac{\partial^n \Phi(x)}{\partial x^n} = e^{-2\pi\alpha \partial_x} \Phi(x). \] (A.8)

Meanwhile, we can introduce a function \( \mathcal{K}(x) \) that satisfies
\[ e^{-i\mathcal{K}(x)} = \cos[\mathcal{K}(x)] - i \sin[\mathcal{K}(x)] = \frac{1 + i \frac{E}{V} - i \frac{2}{V} \cos(x - i\gamma)}{1 - i \frac{E}{V} + i \frac{2}{V} \cos(x - i\gamma)}, \] (A.9)

yielding
\[ \mathcal{K}(x) = 2 \arctan \left[ \frac{2}{V} \cos(x - i\gamma) - \frac{E}{V} \right]. \] (A.10)

Plugging Eqs. (A.8)-(A.10) into Eq. (A.7), we finally obtain
\[ \Phi(x) = e^{-i\mathcal{K}(x)} e^{-2\pi\alpha \partial_x} \Phi(x), \] (A.11)

which can be interpreted as describing the one-period evolution of a particle with linear dispersion in its kinetic energy \(-2\pi\alpha i \partial_x\) and subject to a delta-kicking potential \( \mathcal{K}(x) \) within the driving period \( T = 1 \). The corresponding Schrödinger equation takes the form
\[ i \partial_t \Phi = -2\pi\alpha i \partial_x \Phi + \mathcal{K}(x) \sum_{m \in \mathbb{Z}} \delta(t - m) \Phi, \] (A.12)

and the quasienergy can be viewed as \( \varepsilon = 0 \). The above analysis establishes a relationship between the spectrum nature of NRMM and the dynamics of a periodically kicked particle with linearized kinetic energy.

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