Third order differential subordination and superordination results for analytic functions involving the Srivastava-Attiya operator

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Abstract

In this article, by making use of the linear operator introduced and studied by Srivastava and Attiya \cite{16}, suitable classes of admissible functions are investigated and the dual properties of the third-order differential subordinations are presented. As a consequence, various sandwich-type theorems are established for a class of univalent analytic functions involving the celebrated Srivastava-Attiya transform. Relevant connections of the new results are pointed out.

2010 Mathematics Subject Classification. Primary 30C45; Secondary 30C80.

Key Words and Phrases. Analytic functions; Univalent functions; Differential subordination; Differential superordination; Srivastava-Attiya operator; Sandwich-type theorems; Admissible functions.

1. Introduction, Definitions and Preliminaries

Let $\mathcal{H}$ be the class of functions analytic in the open unit disk

$\mathcal{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$

Also let

$\mathcal{H}[a,n] \quad (n \in \mathbb{N} := \{1, 2, 3, \cdots \}, a \in \mathbb{C})$

be the subclass of the analytic function class $\mathcal{H}$ consisting of functions of the form

$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, \quad (z \in \mathcal{U}).$

Let $\mathcal{A}(\subset \mathcal{H})$ be the class of functions which are analytic in $\mathcal{U}$ and have the normalized Taylor-Maclaurin series of the form:

$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathcal{U}). \quad (1.1)$

Suppose that $f$ and $g$ are in $\mathcal{H}$. We say that $f$ is subordinate to $g$, (or $g$ is superordinate to $f$), written as

$f \prec g \quad \text{in } \mathcal{U} \text{ or } f(z) \prec g(z) \quad (z \in \mathcal{U}),$

if there exists a function $\omega \in \mathcal{H}$, satisfying the conditions of the Schwarz lemma, namely

$\omega(0) = 0$ and $|\omega(z)| < 1$

such that

$f(z) = g(\omega(z)) \quad (z \in \mathcal{U}).$

It follows that

$f(z) \prec g(z) \quad (z \in \mathcal{U}) \Rightarrow f(0) = g(0) \text{ and } f(\mathcal{U}) \subset g(\mathcal{U}).$
In particular, if \( g \) is univalent in \( U \), then the reverse implication also holds (see, for details, [10]).

The concept of differential subordination is a generalization of various inequalities involving complex variables. We recall here some more definitions and terminologies from the theory of differential subordination and superordination.

**Definition 1.** Let \( \psi : \mathbb{C}^4 \times U \rightarrow \mathbb{C} \) and suppose that the function \( h(z) \) is univalent in \( U \). If the function \( p(z) \) is analytic in \( U \) and satisfies the following third-order differential subordination
\[
\psi(p(z), h(z), z^2 p'(z), z^3 p''(z); z) \prec h(z),
\]
then \( p(z) \) is called a solution of the differential subordination (1.2). Furthermore, a given univalent function \( q(z) \) is called a dominant of the solutions of the differential subordination (1.2), or, more simply, a dominant if \( p(z) \prec q(z) \) for all \( p(z) \) satisfying (1.2). A dominant \( \tilde{q}(z) \) that satisfies \( \tilde{q}(z) \prec q(z) \) for all dominants \( q(z) \) of (1.2) is said to be the best dominant.

**Definition 2.** Let \( \psi : \mathbb{C}^4 \times U \rightarrow \mathbb{C} \) and the function \( h(z) \) be univalent in \( U \). If the function \( p(z) \) and
\[
\psi(p(z), h(z), z^2 p'(z), z^3 p''(z); z)
\]
are univalent in \( U \) and satisfies the following third-order differential superordination
\[
h(z) \prec \psi(p(z), h(z), z^2 p'(z), z^3 p''(z); z),
\]
then \( p(z) \) is called a solution of the differential superordination. An analytic function \( q(z) \) is called a subordinant of the solutions of the differential superordination, or more simply a subordinant, if \( q(z) \prec p(z) \) for all \( p(z) \) satisfying (1.3).

A univalent subordinant \( \tilde{q}(z) \) that satisfies \( q(z) \prec \tilde{q}(z) \) for all subordinant \( q(z) \) of (1.3) is said to be the best subordinant. We note that both the best dominant and best subordinant are unique up to rotation of \( U \). The well known monograph of Miller and Mocanu [10] and the more recent book of Bulboaca [2] provide detailed expositions on the theory of differential subordination and superordination. With a view to define the Srivastava-Attiya transform we recall here the generalized Hurwitz-Lerch Zeta function, which is defined in [15] by the following series:
\[
\Pi(z, \mu, b) = \sum_{n=0}^{\infty} \frac{z^n}{(b + n)^\mu}, \quad (b \in \mathbb{C} \setminus \mathbb{Z}^0; \mu \in \mathbb{C} \text{ when } z \in U; \Re(\mu) > 1 \text{ when } z \in \partial U).
\]
Special cases of the function \( \Pi(z, \mu, b) \) include for example, the Riemann Zeta function \( \zeta(\mu) = \Pi(1, \mu, 1) \); the Hurwitz Zeta function \( \zeta(\mu, b) = \Pi(1, \mu, b) \); the Lerch Zeta function \( \zeta(\mu, b) = \Pi(1, \mu, 1) \); the Poly logarithm function \( L_{\text{pi}} = z \Pi(z, \mu, 1) \) and so on. For further details see [17] and the references therein. Srivastava and Attiya [10] considered the following normalized function:
\[
R_{\mu,b}(z) = (1 + b)^\mu \Pi(z, \mu, b) - b^{-\mu} = z + \sum_{n=2}^{\infty} \left( \frac{b + 1}{b + n} \right)^\mu z^n, \quad (z \in U),
\]
and by making use of \( R_{\mu,b}(z) \), they have introduced the linear operator \( J_{\mu,b} : \mathcal{A} \rightarrow \mathcal{A} \) which is defined in terms of convolution as follows:
\[
J_{\mu,b}f(z) = R_{\mu,b}(z) * f(z) = z + \sum_{n=2}^{\infty} \left( \frac{b + 1}{b + n} \right)^\mu a_n z^n, \quad (z \in U).
\]
The operator \( J_{\mu,b}f(z) \) is now popularly known in the literature as the Srivastava-Attiya operator. Various applications of \( J_{\mu,b}f(z) \) are found in [3] [5] [7] [6] [12] [21] and the references therein. From (1.6), it is clear that
\[
z^\eta J_{\mu+1,b}f(z) = (b + 1)J_{\mu,b}f(z) - bJ_{\mu+1,b}f(z).
\]
Suitable choices of parameters, the above defined operator unifies various other linear operators which are introduced earlier. For examples
\[(1) \quad J_{0,0}f(z) = f(z), \quad (2) \quad J_{1,0}f(z) = \int_0^z \frac{f(t)}{t} dt := \mathfrak{A}f(z), \quad (3) \quad J_{1,\eta}f(z) = \int_0^z t^{\eta-1} f(t) dt := \mathfrak{I}_\eta f(z), \quad (\eta > -1),\]
(4) \( J_{\sigma,1} f(z) = z + \sum_{n=2}^{\infty} \left( \frac{2}{n+1} \right)^{\sigma} a_n z^n : = I^\sigma f(z) \quad (\sigma > 0) \).

Where \( \mathcal{A}(f) \) and \( \mathcal{J}_\eta \) are the integral operators introduced by Alexander and Bernardi, respectively, and \( I^\sigma (f) \) is the Jung-Kim-Srivastava integral operator closely related to multiplier transformation studied by Flett. For more detail unifications we refer [12].

Definition 3. Let \( \mathbb{Q} \) be the set of all functions \( q \) that are analytic and univalent on \( U \setminus E(q) \), where \( E(q) = \{ \xi : \xi \in \partial U : \lim_{z \to \xi} q(z) = \infty \} \), and are such that \( \min | q'(\xi) | = \rho > 0 \) for \( \xi \in \partial U \setminus E(q) \). Further, let the subclass of \( \mathbb{Q} \) for which \( q(0) = a \) be denoted by \( \mathbb{Q}(a) \), \( \mathbb{Q}(0) = \mathbb{Q}_0 \) and \( \mathbb{Q}(1) = \mathbb{Q}_1 \).

The subordination methodology is applied to an appropriate class of admissible functions. The following class of admissible functions is given by Antonino and Miller.

Definition 4. Let \( \Omega \) be a set in \( \mathbb{C} \) and \( q \in \mathbb{Q} \) and \( n \in \mathbb{N} \setminus \{1\} \). The class of admissible functions \( \Psi_n[\Omega, q] \) consists of those functions \( \psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C} \) achieving the following admissibility conditions:

\[
\psi(r, s, t, u; z) \not\in \Omega
\]

whenever

\[
r = q(\zeta), s = k \zeta q'(\zeta), \Re\left( \frac{t}{s} + 1 \right) \geq k \Re\left( \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right),
\]

and

\[
\Re\left( \frac{u}{s} \right) \geq k^2 \Re\left( \frac{\zeta^2 q'''(\zeta)}{q'(\zeta)} \right),
\]

where \( z \in U, \zeta \in \partial U \setminus E(q), \) and \( k \geq n \).

The next lemma is the foundation result in the theory of third-order differential subordination.

Lemma 1. Let \( p \in \mathcal{H}[a,n] \) with \( n \geq 2 \), and \( q \in \mathbb{Q}(a) \) achieving the following conditions:

\[
\Re\left( \frac{\zeta q''(\zeta)}{q'(\zeta)} \right) \geq 0, \quad \left| \frac{zq'(z)}{q'(\zeta)} \right| \leq k,
\]

where \( z \in U, \zeta \in \partial U \setminus E(q), \) and \( k \geq n \). If \( \Omega \) is a set in \( \mathbb{C} \), \( \psi \in \Psi_n[\Omega, q] \) and

\[
\psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) \subset \Omega,
\]

then

\[
p(z) \prec q(z) \quad (z \in U).
\]

Definition 5. Let \( \Omega \) be a set in \( \mathbb{C} \) and \( q \in \mathcal{H}[a,n] \) and \( q'(z) \neq 0 \). The class of admissible functions \( \Psi_n'[\Omega, q] \) consists of those functions \( \psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C} \) that satisfy the following admissibility conditions:

\[
\psi(r, s, t, u; \zeta) \in \Omega
\]

whenever

\[
r = q(z), s = \frac{zq'(z)}{m}, \Re\left( \frac{t}{s} + 1 \right) \leq \frac{1}{m} \Re\left( \frac{zq''(z)}{q'(z)} + 1 \right),
\]

and

\[
\Re\left( \frac{u}{s} \right) \leq \frac{1}{m^2} \Re\left( \frac{z^2 q'''(z)}{q'(z)} \right),
\]

where \( z \in U, \zeta \in \partial U, \) and \( m \geq n \geq 2 \).

Lemma 2. Let \( p \in \mathcal{H}[a,n] \) with \( \psi \in \Psi_n'[\Omega, q] \). If

\[
\psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z)
\]

is univalent in \( \mathbb{U} \) and \( p \in \mathbb{Q}(a) \) satisfying the following conditions:

\[
\Re\left( \frac{zq''(z)}{q'(z)} \right) \geq 0, \quad \left| \frac{zp'(z)}{q'(z)} \right| \leq m,
\]

where \( z \in U, \zeta \in \partial U, \) and \( m \geq n \geq 2 \), then

\[
\Omega \subset \{ \psi(p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z) : z \in \mathbb{U} \},
\]

implies that

\[
q(z) \prec p(z) \quad (z \in \mathbb{U}).
\]
Though the notion of third order differential subordinaton have originally found in the work of Ponnusamy and Juneja [13]. The recent work due to Tang et al. [18, 20] on third order differential subordinaton attracted to many researchers in this field. For example see [4, 8, 9, 14, 18, 20] [13, 11]. In the present paper we considered suitable classes of admissible functions associated with Srivastava-Atiya operator and obtained sufficient conditions on the normalized analytic function \( f \) such that 

\[
\text{Sandwich-type subordination of the following form holds:}
\]

\[
h_1(z) \prec \Theta(f) \prec q_2(z), \quad (z \in \mathbb{U}),
\]

where \( q_1, q_2 \) are univalent in \( \mathbb{U} \) and \( \Theta \) is a suitable operator.

2. Results Related to Third Order Subordinaton

In this section, start with given set \( \Omega \) and given function \( q \) and we determine a set of admissible operators \( \psi \) so that (1.2) holds true. Thus, the following new class of admissible function is introduced which will required to prove the main third-order differential subordinaton theorems for the operator \( J_{\mu,b}f(z) \) defined by (1.5).

**Definition 6.** Let \( \Omega \) be a set in \( \mathbb{C} \) and \( q \in \mathbb{Q}_0 \cap \mathcal{H}_0 \). The class of admissible function \( \Phi_j[\Omega, q] \) consists of those functions \( \phi : \mathbb{C}^4 \times \mathbb{U} \rightarrow \mathbb{C} \) that satisfy the following admissibility conditions:

\[
\phi(\alpha, \beta, \gamma, \delta; z) \not\in \Omega
\]

whenever

\[
\alpha = q(\zeta), \beta = \frac{k \zeta q'(\zeta) + bq(\zeta)}{b + 1},
\]

\[
\Re \left( \frac{\gamma(b + 1)^2 - b^2 \alpha}{(b(b + 1) - b\alpha)} - 2b \right) \geq k \Re \left( \frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right),
\]

and

\[
\Re \left( \frac{\delta(b + 1)^3 - \gamma(b + 1)^2(3b + 3) + b^2 \alpha(3 + 2b)}{(b(b - \alpha) + \beta)} + 3b^2 + 6b + 2 \right) \geq k^2 \Re \left( \frac{\zeta^2 q'''(\zeta)}{q'(\zeta)} \right),
\]

where \( z \in \mathbb{U}, \zeta \in \partial \mathbb{U} \setminus E(q) \), and \( k \geq 2 \).

**Theorem 1.** Let \( \phi \in \Phi_j[\Omega, q] \). If the function \( f \in \mathcal{A} \) and \( q \in \mathbb{Q}_0 \) satisfy the following conditions:

\[
\Re \left( \frac{\zeta q''(\zeta)}{q'(\zeta)} \right) \geq 0, \quad \left| J_{\mu,b}f(z) \right| \leq k, \quad (2.1)
\]

and

\[
\{ \phi(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z) : z \in \mathbb{U} \} \subset \Omega, \quad (2.2)
\]

then

\[
J_{\mu+1,b}f(z) \prec \phi(z), \quad (z \in \mathbb{U}).
\]

**Proof.** Define the analytic function \( p(z) \) in \( \mathbb{U} \) by

\[
p(z) = J_{\mu+1,b}f(z).
\]

From equation (1.7) and (2.3), we have

\[
J_{\mu,b}f(z) = \frac{zp'(z) + bp(z)}{b + 1}. \quad (2.4)
\]

By similar argument, yields

\[
J_{\mu-1,b}f(z) = \frac{z^2p''(z) + (2b + 1)zp'(z) + b^2p(z)}{(b + 1)^2} \quad (2.5)
\]

and

\[
J_{\mu-2,b}f(z) = \frac{z^3p'''(z) + (3b + 3)z^2p''(z) + (3b^2 + 3b + 1)zp'(z) + b^3p(z)}{(b + 1)^3}. \quad (2.6)
\]

Define the transformation from \( \mathbb{C}^4 \) to \( \mathbb{C} \) by

\[
\alpha(r, s, t, u) = r, \quad \beta(r, s, t, u) = \frac{s + br}{b + 1},
\]
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\[ \gamma(r, s, t, u) = \frac{t + (2b + 1)s + b^2r}{(b + 1)^2} \]  \hspace{1cm} (2.7)

and

\[ \delta(r, s, t, u) = \frac{u + (3b + 3)t + (3b^2 + 3b + 1)s + b^3r}{(b + 1)^3} \]  \hspace{1cm} (2.8)

Let

\[ \psi(r, s, t, u) = \phi(\alpha, \beta, \gamma, \delta; z) = \phi \left( \frac{r + s + br}{b + 1}, \frac{t + (2b + 1)s + b^2r}{(b + 1)^2}, \frac{u + (3b + 3)t + (3b^2 + 3b + 1)s + b^3r}{(b + 1)^3}; z \right). \]  \hspace{1cm} (2.9)

The proof will make use of Lemma \[ \square \]. Using equations (2.3) to (2.6), and from (2.9), we have

\[ \psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) = \phi(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z). \]  \hspace{1cm} (2.10)

Hence, (2.2) becomes

\[ \psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Omega. \]

Note that

\[ \frac{t}{s} + 1 = \frac{\gamma(b + 1)^2 - b^2 \alpha}{\beta(b + 1) - b\alpha} - 2b \]

and

\[ \frac{u}{s} = \frac{\delta(b + 1)^3 - \gamma(b + 1)^2(3b + 3) + b^2 \alpha(3 + 2b)}{(b(\beta - \alpha) + \beta)}. \]

Thus, the admissibility condition for \( \phi \in \Phi_J[\Omega, q] \) in Definition \[ \square \] is equivalent to the admissibility condition for \( \psi \in \Psi_2[\Omega, q] \) as given in Definition \[ \square \] with \( n = 2 \). Therefore, by using (2.11) and Lemma \[ \square \] we have

\[ J_{\mu+1,b}f(z) \prec q(z). \]

This completes the proof of theorem. \hspace{1cm} \( \square \)

The next result is an extension of Theorem \[ \square \] to the case where the behavior of \( q(z) \) on \( \partial U \) is not known.

**Corollary 1.** Let \( \Omega \subset \mathbb{C} \) and let the function \( q \) be univalent in \( U \) with \( q(0) = 0 \). Let \( \phi \in \Phi_J[\Omega, q_\rho] \) for some \( \rho \in (0, 1) \), where \( q_\rho(z) = q(\rho z) \). If the function \( f \in \mathcal{A} \) and \( q_\rho \) satisfy the following conditions

\[ \Re \left( \frac{\zeta q_\rho''(\zeta)}{q_\rho'(\zeta)} \right) \geq 0, \quad \left| \frac{J_{\mu,b}f(z)}{q_\rho'(\zeta)} \right| \leq k (z \in U, k \geq 2, \zeta \in \partial U \setminus E(q_\rho)) \]

and

\[ \phi(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z) \in \Omega, \]

then

\[ J_{\mu+1,b}f(z) \prec q(z) \quad (z \in \mathbb{U}). \]

**Proof.** From Theorem \[ \square \] then \( J_{\mu+1,b}f(z) \prec q_\rho(z) \). The result asserted by Corollary \[ \square \] is now deduced from the following subordination property \( q_\rho(z) \prec q(z) \) \( (z \in \mathbb{U}) \). \hspace{1cm} \( \square \)

If \( \Omega \neq \mathbb{C} \) is a simply connected domain, then \( \Omega = h(\mathbb{U}) \) for some conformal mapping \( h(z) \) of \( \mathbb{U} \) onto \( \Omega \). In this case, the class \( \Phi_J[h(\mathbb{U}), q] \) is written as \( \Phi_J[h, q] \). This follows immediate consequence of Theorem \[ \square \]

**Theorem 2.** Let \( \phi \in \Phi_J[h, q] \). If the function \( f \in \mathcal{A} \) and \( q \in \mathcal{Q}_\rho \) satisfy the following conditions:

\[ \Re \left( \frac{\zeta q''(\zeta)}{q'(\zeta)} \right) \geq 0, \quad \left| \frac{J_{\mu,b}f(z)}{q'(\zeta)} \right| \leq k, \]  \hspace{1cm} (2.11)

and

\[ \phi(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z) \prec h(z), \]  \hspace{1cm} (2.12)

then

\[ J_{\mu+1,b}f(z) \prec q(z) \quad (z \in \mathbb{U}). \]
The next result is an immediate consequence of Corollary 1.

**Corollary 2.** Let $\Omega \subset \mathbb{C}$ and let the function $q$ be univalent in $U$ with $q(0) = 0$. Let $\phi \in \Phi_J[h,q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If the function $f \in A$ and $q_\rho$ satisfy the following conditions

$$\Re \left( \frac{q_\rho''(\zeta)}{q_\rho'(\zeta)} \right) \geq 0, \quad |J_{\mu+1,0}f(z)| \leq k \quad (z \in U, k \geq 2, \zeta \in \partial U \setminus E(q_\rho)), \quad \text{and}$$

$$\phi(J_{\mu+1,0}f(z), J_{\mu+0}f(z), J_{\mu-1,0}f(z), J_{\mu-2,0}f(z); z) < h(z),$$

then

$$J_{\mu+1,0}f(z) < q(z) \quad (z \in U).$$

The following result yields the best dominant of the differential subordination \(2.12\).

**Theorem 3.** Let the function $h$ be univalent in $U$ and let $\phi : \mathbb{C}^4 \times U \rightarrow \mathbb{C}$ and $\psi$ be given by (2.5). Suppose that the differential equation

$$\psi(q(z), zq'(z), z^2 q''(z), z^3 q'''(z); z) = h(z), \quad (2.13)$$

has a solution $q(z)$ with $q(0) = 0$, which satisfy condition (2.1) and

$$\phi(J_{\mu+1,0}f(z), J_{\mu+0}f(z), J_{\mu-1,0}f(z), J_{\mu-2,0}f(z); z)$$

is analytic in $U$, then

$$J_{\mu+1,0}f(z) < q(z)$$

and $q(z)$ is the best dominant.

**Proof.** From Theorem 1, we have $q$ is a dominant of (2.12). Since $q$ satisfies (2.13), it is also a solution of (2.12) and therefore $q$ will be dominated by all dominants. Hence $q$ is the best dominant.

In view of Definition 5 and in the special case $q(z) = Mz$, $M > 0$, the class of admissible functions $\Phi_J[\Omega,q]$, denoted by $\Phi_J[\Omega,M]$, is expressed as follows.

**Definition 7.** Let $\Omega$ be a set in $\mathbb{C}$ and $M > 0$. The class of admissible function $\Phi_J[\Omega,M]$ consists of those functions $\phi : \mathbb{C}^4 \times U \rightarrow \mathbb{C}$ such that

$$\phi\left(\frac{(k+1)M^e^{i\theta}}{b+1}, L + [(2b+1)k+b^2]M^e^{i\theta}, N + (3b+3)L + [(3b+3b+1)k+b^3]M^e^{i\theta}; z\right) \notin \Omega. \quad (2.14)$$

whenever $z \in U, \Re(Le^{-i\theta}) \geq (k-1)kM$, and $\Re(Ne^{-i\theta}) \geq 0$ for all $\theta \in \mathbb{R}$ and $k \geq 2.$

**Corollary 3.** Let $\phi \in \Phi_J[\Omega,M]$. If the function $f \in A$ satisfies

$$|J_{\mu,0}f(z)| \leq kM \quad (z \in U, k \geq 2; M > 0),$$

and

$$\phi(J_{\mu+1,0}f(z), J_{\mu+0}f(z), J_{\mu-1,0}f(z), J_{\mu-2,0}f(z); z) \in \Omega,$$

then

$$|J_{\mu+1,0}f(z)| < M.$$
Corollary 5. Let \( k \geq 2, \) \( 0 \neq \mu \in \mathbb{C} \) and \( M > 0. \) If the function \( f \in A \) satisfies
\[
|J_{\mu,b}f(z)| \leq kM,
\]
and
\[
|J_{\mu,b}f(z) - J_{\mu+1,b}f(z)| < \frac{M}{|b+1|},
\]
then
\[
|J_{\mu+1,b}f(z)| < M.
\]

Proof. Let \( \phi(\alpha, \beta, \gamma, \delta; z) = \beta - \alpha \) and \( \Omega = h(U) \), where \( h(z) = \frac{Mz}{|b+1|} \) \( (M > 0) \). Use Corollary 3, we need to show that \( \phi \in \Phi_J[\Omega, M] \), that is, the admissibility condition (2.14) is satisfied. This follows since
\[
|\phi(v, w, x, y; z)| = \left| \frac{(k-1)Me^{i\theta}}{b+1} \right| \geq \frac{M}{|b+1|},
\]
whenever \( z \in U, \theta \in \mathbb{R} \) and \( k \geq 2 \). The required result follows from Corollary 3. \( \square \)

Definition 8. Let \( \Omega \) be a set in \( \mathbb{C}, q \in \mathbb{Q}_1 \cap H_1 \). The class of admissible functions \( \Phi_J[\Omega, q] \) consists of those functions \( \phi : \mathbb{C}^4 \times U \rightarrow \mathbb{C} \) that satisfy the following admissibility condition
\[
\phi(\alpha, \beta, \gamma, \delta; z) \notin \Omega
\]
whenever
\[
\alpha = q(\zeta), \beta = \frac{kq'(\zeta) + (b+1)q(\zeta)}{b+1},
\]
\[
\Re \left( \frac{(b+1)(\gamma - \alpha)}{\beta - \alpha} - 2(1+b) \right) \geq k\Re \left( \frac{q''(\zeta)}{q'(\zeta)} + 1 \right),
\]
and
\[
\Re \left( \frac{3(1+b)^2 - 3\gamma(b+2)(b+1) + 3\alpha(b+2)(b+1) - (1+b)^2\alpha}{\beta - \alpha} + 3b^2 + 12b + 11 \right) \geq k^2\Re \left( \frac{\zeta q''(\zeta)}{q'(\zeta)} \right),
\]
where \( z \in U, \zeta \in \partial U \setminus E(q) \), and \( k \geq 2 \).

Theorem 4. Let \( \phi \in \Phi_J[\Omega, q] \). If the function \( f \in A \) and \( q \in \mathbb{Q}_1 \) satisfy the following conditions:
\[
\Re \left( \frac{\zeta q''(\zeta)}{q'(\zeta)} \right) \geq 0 \quad \left| \frac{J_{\mu,b}f(z)}{z q'(\zeta)} \right| \leq k,
\]
and
\[
\left\{ \phi \left( \frac{J_{\mu+1,b}f(z)}{z}, \frac{J_{\mu,b}f(z)}{z}, \frac{J_{\mu-1,b}f(z)}{z}, \frac{J_{\mu-2,b}f(z)}{z}; z \right) : z \in U \right\} \subset \Omega,
\]
then
\[
\frac{J_{\mu+1,b}f(z)}{z} < q(z) \quad (z \in U).
\]

Proof. Define the analytic function \( p(z) \) in \( U \) by
\[
p(z) = \frac{J_{\mu+1,b}f(z)}{z}.
\]
From equation (1.7) and (2.17), we have
\[
\frac{J_{\mu,b}f(z)}{z} = z p'(z) + (b+1)p(z),
\]
(2.18)
By similar argument, yields
\[
\frac{J_{\mu-1,b}f(z)}{z} = z^2 p''(z) + zp'(z)(3+2b) + p(z)(1+b)^2 \quad (b+1)^2
\]
(2.19)
and
\[
\frac{J_{\mu-2,b}f(z)}{z} = z^3 p'''(z) + 3(b+2)z^2 p''(z) + (3b^2 + 9b + 7)zp'(z) + p(z)(b+1)^3.
\]
(2.20)
Define the transformation from \( \mathbb{C}^4 \) to \( \mathbb{C} \) by

\[
\alpha(r, s, t, u) = r, \quad \beta(r, s, t, u) = \frac{s + (b + 1)r}{(b + 1)},
\]

\[
\gamma(r, s, t, u) = \frac{t + (3 + 2b)s + (b + 1)^2r}{(b + 1)^2},
\]

and

\[
\delta(r, s, t, u) = \frac{u + 3(b + 2)t + (3b^2 + 9b + 7)s + (b + 1)^3r}{(b + 1)^3}.
\] (2.21)

Let

\[
\psi(r, s, t, u) = \phi(\alpha, \beta, \gamma, \delta; z) = \phi\left(r, \frac{s + (1 + b)r}{1 + b}, \frac{t + (3 + 2b)s + (b + 1)^2r}{(b + 1)^2}, \frac{u + 3(b + 2)t + (3b^2 + 9b + 7)s + (b + 1)^3r}{(b + 1)^3}; z\right). \quad (2.23)
\]

The proof will make use of Lemma 1. Using equations (2.17) to (2.20), and from (2.23), we have

\[
\psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\right) = \phi\left(J_{\mu+1,b,f}(z), J_{\mu,b,f}(z), J_{\mu-1,b,f}(z), J_{\mu-2,b,f}(z); z\right). \quad (2.24)
\]

Hence, (2.16) becomes

\[
\psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\right) \in \Omega.
\]

Note that

\[
\frac{t}{s} + 1 = \frac{(b + 1)(\gamma - \alpha)}{\beta - \alpha} - 2(1 + b)
\]

and

\[
u = \frac{\delta(1 + b)^2 - 3\gamma(b + 2)(b + 1) + 3\alpha(b + 2)(b + 1) - (1 + b)^2\alpha + 3b^2 + 12b + 11}{\beta - \alpha}.
\]

Thus, the admissibility condition for \( \phi \in \Phi_{J,1}[\Omega, q] \) in Definition 5 is equivalent to the admissibility condition for \( \psi \in \Psi_2[\Omega, q] \) as given in Definition 1 with \( n = 2 \). Therefore, by using (2.15) and Lemma 1 we have

\[
J_{\mu+1,b,f}(z) < q(z).
\]

This completes the proof of the theorem. \( \square \)

If \( \Omega \neq \mathbb{C} \) is a simply connected domain, then \( \Omega = h(U) \) for some conformal mapping \( h(z) \) of \( U \) onto \( \Omega \). In this case, the class \( \Phi_{J,1}[h(U), q] \) is written as \( \Phi_{J,1}[h, q] \). This follows immediate consequence of Theorem 4.

**Theorem 5.** Let \( \phi \in \Phi_{J,1}[h, q] \). If the function \( f \in A \) and \( q \in Q_1 \) satisfy the following conditions:

\[
\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}\right) \geq 0, \quad \left|\frac{J_{\mu,b,f}(z)}{zq'(\zeta)}\right| \leq k, \quad (2.25)
\]

and

\[
\phi\left(\frac{J_{\mu+1,b,f}(z)}{z}, \frac{J_{\mu,b,f}(z)}{z}, \frac{J_{\mu-1,b,f}(z)}{z}, \frac{J_{\mu-2,b,f}(z)}{z}; z\right) < h(z), \quad (2.26)
\]

then

\[
\frac{J_{\mu+1,b,f}(z)}{z} < q(z) \quad (z \in U).
\]

In view of Definition 8 and in the special case \( q(z) = Mz, \ M > 0 \), the class of admissible functions \( \Phi_{J,1}[\Omega, q] \), denoted by \( \Phi_{J,1}[\Omega, M] \), is expressed as follows.
**Definition 9.** Let Ω be a set in ℂ and M > 0. The class of admissible function Φ_{J,1}[Ω, M] consists of those functions φ : ℂ^4 × U → ℂ such that

\[
φ \left(M e^{iθ}, \frac{(k + 1 + b)Me^{iθ}}{1 + b}, L + [(3 + 2b)k + (b + 1)^2]Me^{iθ}}{(b + 1)^2}, N + 3(b + 2)L + [(3b^2 + 9b + 7)k + (b + 1)^3]Me^{iθ}}\right) \notin Ω. \quad (2.27)
\]

whenever \(z \in U, \Re(Le^{-iθ}) \geq (k - 1)kM\), and \(\Re(Ne^{-iθ}) \geq 0\) for all \(θ \in \mathbb{R}\) and \(k \geq 2\).

**Corollary 6.** Let \(φ \in Φ_{J,1}[Ω, M]\). If the function \(f \in A\) satisfies

\[
\left| J_{μ,b}f(z) \right| \leq kM \quad (z \in U, k \geq 2; M > 0),
\]

and

\[
φ \left( J_{μ+1,b}f(z), J_{μ,b}f(z), J_{μ-1,b}f(z), J_{μ-2,b}f(z) \right) \in Ω,
\]

then

\[
\left| J_{μ+1,b}f(z) \right| < M.
\]

In this special case \(Ω = q(U) = \{w : \|w\| < M\}\), the class \(Φ_{J,1}[Ω, M]\) is simply denoted by \(Φ_{J,1}[M]\). Corollary 6 can now be written in the following form.

**Corollary 7.** Let \(φ \in Φ_{J,1}[M]\). If the function \(f \in A\) satisfies

\[
\left| J_{μ,b}f(z) \right| \leq kM \quad (z \in U, k \geq 2; M > 0),
\]

and

\[
φ \left( J_{μ+1,b}f(z), J_{μ,b}f(z), J_{μ-1,b}f(z), J_{μ-2,b}f(z) \right); z \right| < M,
\]

then

\[
\left| J_{μ+1,b}f(z) \right| < M.
\]

**Definition 10.** Let Ω be a set in ℂ and \(q \in \mathbb{Q}_1 ∩ \mathcal{H}_1\). The class of admissible functions \(Φ_{J,2}[Ω, q]\) consists of those functions \(φ : ℂ^4 × U → ℂ\) that satisfy the following admissibility condition

\[
φ(α, β, γ, δ; z) \notin Ω
\]

whenever

\[
α = q(ζ), β = \frac{1}{(b + 1)} \left( kζq'(ζ) + (b + 1)q(ζ) \right),
\]

\[
\Re \left( \frac{(1 + b)(βγ + 2α^2 - 3αβ)}{β - α} \right) \geq k\Re \left( \frac{ζq''(ζ)}{q'(ζ)} + 1 \right),
\]

and

\[
\Re \left[ (δ - γ)(1 + b)^2βγ - (1 + b)^2(γ - β)β(1 - β - γ + 3α) - 3(b + 1)(γ - β)β + 2(β - α) + 3(1 + b)α(β - α) + (β - α)^2(1 + b)((β - α)(1 + b) - 3 - 4(1 + b)α + α^2(1 + b)^2(β - α)) × (β - α)^{-1} \right] \geq k^2\Re \left( \frac{ζq''(ζ)}{q'(ζ)} \right),
\]

where \(z \in U, ζ ∈ ∂U \setminus E(q)\) and \(k \geq 2\).

**Theorem 6.** Let \(φ \in Φ_{J,2}[Ω, q]\). If the function \(f \in A\) and \(q \in \mathbb{Q}_1\) satisfy the following conditions

\[
\Re \left( \frac{ζq''(ζ)}{q'(ζ)} \right) \geq 0, \quad \left| J_{μ,b}f(z) \right| q'(ζ) \leq k, \quad (2.28)
\]

\[
\left\{ φ \left( J_{μ,b}f(z), J_{μ-1,b}f(z), J_{μ-2,b}f(z), J_{μ-3,b}f(z) \right); z \in U \right\} ⊂ Ω, \quad (2.29)
\]
then
\[ \frac{J_{\mu,b} f(z)}{J_{\mu+1,b} f(z)} < q(z) \quad (z \in \mathbb{U}). \]

**Proof.** Define the analytic function \( p(z) \) in \( \mathbb{U} \) by
\[ p(z) = \frac{J_{\mu,b} f(z)}{J_{\mu+1,b} f(z)}. \] (2.30)

From equation (1.7) and (2.30), we have
\[ \frac{J_{\mu-1,b} f(z)}{J_{\mu,b} f(z)} = \frac{1}{b+1} \left[ \frac{zp'(z)}{p(z)} + (b+1)p(z) \right] := A \] (2.31)

By similar argument yields,
\[ \frac{J_{\mu-2,b} f(z)}{J_{\mu-1,b} f(z)} := \frac{B}{b+1} \] (2.32)
and
\[ \frac{J_{\mu-3,b} f(z)}{J_{\mu-2,b} f(z)} = \frac{1}{b+1} \left[ B + B^{-1}(C + A^{-1}D - A^{-2}C^2) \right]. \] (2.33)

Where
\[
B := (b+1)p(z) + \frac{zp'(z)}{p(z)} + \frac{zp''(z)}{p(z)} + \left( \frac{zp'(z)}{p(z)} \right)^2 + (b+1)z p'(z) - \frac{zp'(z)}{p(z)} + (b+1)p(z)
\]
\[
C := \frac{z^2 p''(z)}{p(z)} + \frac{zp'(z)}{p(z)} - \left( \frac{zp'(z)}{p(z)} \right)^2 + (b+1)z p'(z)
\]
\[
D := \frac{3z^2 p''(z)}{p(z)} + \frac{z^3 p'''(z)}{p(z)} + \frac{zp'(z)}{p(z)} - 3 \left( \frac{zp'(z)}{p(z)} \right)^2 - 3 \frac{z^3 p'(z)p''(z)}{p^2(z)} + 2 \left( \frac{zp'(z)}{p(z)} \right)^3
\]

Define the transformation from \( \mathbb{C}^4 \) to \( \mathbb{C} \) by
\[
\alpha(r,s,t,u) = r, \quad \beta(r,s,t,u) = \frac{1}{b+1} \left[ \frac{s}{r} + (b+1)r \right] := \frac{E}{b+1},
\]
\[
\gamma(r,s,t,u) = \frac{1}{b+1} \left[ \frac{s}{r} + (b+1)r + \frac{t}{r} + \frac{s}{r} - (\frac{s}{r})^2 + (b+1)s \right] := \frac{F}{b+1}, \] (2.34)
and
\[
\delta(r,s,t,u) = \frac{1}{b+1} \left[ F + F^{-1}(L + E^{-1}H - E^{-2}L^2) \right]. \] (2.35)

Where
\[
L := (1+b)s + \frac{t}{r} + \frac{s}{r} - \left( \frac{s}{r} \right)^2
\]
\[
H := \frac{3t}{r} + \frac{s}{r} + s - 3 \left( \frac{s}{r} \right)^2 - 3 \left( \frac{st}{r^2} \right) + 2 \left( \frac{s}{r} \right)^3 + (1+b)s + (1+b)t.
\]

Let
\[
\psi(r,s,t,u) = \phi(\alpha, \beta, \gamma, \delta; z) = \phi \left( r, \frac{E}{b+1}, \frac{F}{b+1}, \frac{F + F^{-1}(L + E^{-1}H - E^{-2}L^2)}{b+1} \right). \] (2.36)

The proof will make use of Lemma [1] Using equations (2.30) to (2.33), and from (2.36), we have
\[
\psi \left( p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z \right) = \phi \left( \frac{J_{\mu,b} f(z)}{J_{\mu+1,b} f(z)}, \frac{J_{\mu-1,b} f(z)}{J_{\mu,b} f(z)}, \frac{J_{\mu-2,b} f(z)}{J_{\mu-1,b} f(z)}, \frac{J_{\mu-3,b} f(z)}{J_{\mu-2,b} f(z)}; z \right). \] (2.37)

Hence, (2.29) becomes
\[
\psi \left( p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z \right) \in \Omega.
\]
Note that
\[ \frac{t}{s} + 1 = \left( \frac{(1 + b)(\beta \gamma + 2\alpha^2 - 3\alpha \beta)}{(\beta - \alpha)} \right) \]
and
\[ \frac{u}{s} = \left[ (\delta - \gamma)(1 + b)^2 \beta \gamma - (1 + b)^2(\gamma - \beta)(1 - \beta - \gamma + 3\alpha) - 3(b+1)(\gamma - \beta)2(\beta - \alpha) + 3(1+b)\alpha(\beta - \alpha) + (\beta - \alpha)^2(1 + b)((\beta - \alpha)(1 + b) - 3 - 4(b+1)\alpha) + \alpha^2(1 + b)^2(\beta - \alpha) \right] \times (\beta - \alpha)^{-1}. \]

Thus, the admissibility condition for \( \phi \in \Phi_{J,2}[\Omega, q] \) in Definition 10 is equivalent to the admissibility condition for \( \psi \in \Psi_2[\Omega, q] \) as given in Definition 4 with \( n = 2 \). Therefore, by using (2.28) and Lemma 1 we have
\[ \frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)} < q(z). \]

If \( \Omega \neq \mathbb{C} \) is a simply connected domain, then \( \Omega = h(\mathbb{U}) \) for some conformal mapping \( h(z) \) of \( \mathbb{U} \) onto \( \Omega \). In this case, the class \( \Phi_{J,1}[h(\mathbb{U}), q] \) is written as \( \Phi_{J,2}[h, q] \). This follows immediate consequence of Theorem 6.

**Theorem 7.** Let \( \phi \in \Phi_{J,2}[h, q] \). If the function \( f \in \mathcal{A} \) and \( q \in \mathcal{Q}_1 \) satisfy the following conditions (2.28) and
\[ \phi \left( \frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)}, \frac{J_{\mu-1,b}f(z)}{J_{\mu,b}f(z)}, \frac{J_{\mu-2,b}f(z)}{J_{\mu-1,b}f(z)}, \frac{J_{\mu-3,b}f(z)}{J_{\mu-2,b}f(z)}; z \right) < h(z), \]
then
\[ \frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)} < q(z) \quad (z \in \mathbb{U}). \]

**3. Results Related to Third Order Superordination**

In this section, the third-order differential superordination theorems for the operator \( J_{\mu,b}f(z) \) defined in (1.6) is investigated. For the purpose, we considered the following admissible functions.

**Definition 11.** Let \( \Omega \) be a set in \( \mathbb{C} \) and \( q \in \mathcal{H}_0 \) with \( q'(z) \neq 0 \). The class of admissible function \( \Phi_f[\Omega, q] \) consists of those functions \( \phi : \mathbb{C}^4 \times \mathbb{U} \rightarrow \mathbb{C} \) that satisfy the following admissibility conditions:
\[ \phi(\alpha, \beta, \gamma, \delta; \zeta) \in \Omega \]
whenever
\[ \alpha = q(z), \beta = \frac{zq'(z) + mbq(z)}{m(b+1)}, \]
\[ \Re \left( \frac{\gamma(b+1)^2 - b^2\alpha}{(\beta(b+1) - b\alpha) - 2b} \right) \leq \frac{1}{m} \Re \left( \frac{zq''(z)}{q'(z)} + 1 \right), \]
and
\[ \Re \left( \frac{\delta(b+1)^3 - \gamma(b+1)^2(3b+3) + b^2\alpha(3+2b)}{(b(\beta - \alpha) + \beta) + 3b^2 + 6b + 2} \right) \leq \frac{1}{m^2} \Re \left( \frac{z^2q'''(z)}{q'(z)} \right), \]
where \( z \in \mathbb{U}, \zeta \in \partial \mathbb{U} \), and \( m \geq 2 \).

**Theorem 8.** Let \( \phi \in \Phi_f[\Omega, q] \). If the function \( f \in \mathcal{A} \) and \( J_{\mu+1,b}f(z) \in \mathcal{Q}_0 \) and \( q \in \mathcal{H}_0 \) with \( q'(z) \neq 0 \) satisfy the following conditions:
\[ \Re \left( \frac{zq''(z)}{q'(z)} \right) \geq 0, \quad \left| \frac{J_{\mu,b}f(z)}{q'(z)} \right| \leq m, \] (3.1)
and
\[ \phi(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z) \]
is univalent in \( \mathbb{U} \), then
\[ \Omega \subset \{ \phi(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z) : z \in \mathbb{U} \}, \] (3.2)
Theorem 10. \( \phi \ torso \) of the forms (3.2) or (3.3). The following theorem proves the existence of the best subordinant \( \phi \ torso \) consists of those functions \( q \ torso \) satisfies (3.4), it is also a solution of (3.3) and therefore \( \phi \ torso \) implies that \( q \ torso \). Hence

\[
\Omega \subset \{ \psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) : z \in \mathbb{U} \}.
\]

From (2.7) and (2.8), we see that the admissibility condition for \( \phi \ torso \) in Definition 11 is equivalent to the admissibility condition for \( \psi \ torso \) \( \Omega \) with \( n = 2 \). Hence \( \psi \ torso \) \( \Omega \) and by using (3.2) and Lemma 2, we have

\[
q(z) \prec J_{\mu+1,b}f(z).
\]

Therefore

\[
\Omega = h(\mathbb{U}) \text{ for some conformal mapping } h(z) \text{ of } \Omega. \text{ In this case, the class } \Phi'_j[h(\mathbb{U}), q] \text{ is written as } \Phi'_j[h, q]. \text{ This follows an immediate consequence of Theorem 8.}
\]

Theorem 9. Let \( \phi \ torso \) \( \Phi'_j[h, q] \) and \( h \) be analytic in \( \mathbb{U} \). If the function \( f \ torso \) \( \mathcal{A} \) and \( J_{\mu+1,b}f(z) \ torso \) \( \mathbb{Q}_0 \) and \( q \ torso \) \( \mathcal{H}_0 \) with \( q'(z) \neq 0 \) satisfy the following conditions (3.7) and

\[
\phi(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z),
\]

is univalent in \( \mathbb{U} \), then

\[
h(z) \prec \phi(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z),
\]

implies that

\[
q(z) \prec J_{\mu+1,b}f(z) \quad (z \in \mathbb{U}).
\]

Theorems 8 and 9 can only be used to obtain subordination of the third-order differential superordination of the forms (3.2) or (3.3). The following theorem proves the existence of the best subordinant of (3.3) for a suitable \( \phi \).}

Theorem 10. Let the function \( h \) be univalent in \( \mathbb{U} \) and \( \phi : \mathbb{C} \times \overline{\mathbb{U}} \rightarrow \mathbb{C} \) and \( \psi \) be given by (2.7). Suppose that the differential equation

\[
\psi(q(z), zq'(z), z^2q''(z), z^3q'''(z); z) = h(z)
\]

has a solution \( q(z) \in \mathbb{Q}_0 \). If the functions \( f \ torso \) \( \mathcal{A} \), \( J_{\mu+1,b}f(z) \ torso \) \( \mathbb{Q}_0 \) and \( q \ torso \) \( \mathcal{H}_0 \) with \( q'(z) \neq 0 \), which satisfy the following condition (3.7) and

\[
\phi(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z)
\]

is analytic in \( \mathbb{U} \), then

\[
h(z) \prec \phi(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z)
\]

implies that

\[
q(z) \prec J_{\mu+1,b}f(z) \quad (z \in \mathbb{U})
\]

and \( q(z) \) is the best dominant.

Proof. In view of Theorem 8 and Theorem 9, we deduce that \( q \) is a subordinant of (3.3). Since \( q \) satisfies (3.3), it is also a solution of (3.3) and therefore \( q \) will be subordinated by all subordinants. Hence \( q \) is the best subordinant.

Definition 12. Let \( \Omega \) be a set in \( \mathbb{C} \), \( q \ torso \) \( \mathcal{H}_1 \) with \( q'(z) \neq 0 \). The class of admissible functions \( \Phi'_j[\Omega, q] \) consists of those functions \( \phi : \mathbb{C} \times \overline{\mathbb{U}} \rightarrow \mathbb{C} \) that satisfy the following admissibility condition

\[
\phi(\alpha, \beta, \gamma, \delta; \zeta) \in \Omega
\]

whenever

\[
\alpha = q(z), \beta = \frac{zq'(z) + (1 + b)mq(z)}{(1 + b)m},
\]

and

\[
\Re\left(\frac{(b + 1)(\gamma - \alpha)}{\beta - \alpha} - 2(1 + b)\right) \leq \frac{1}{m} \Re\left(\frac{zq''(z)}{q'(z)} + 1\right).
\]
and
\[ \Re \left( \frac{\delta(b+2)}{b-\alpha} + 3\alpha(b+2)(b+1) - (1+b)^2 \alpha + 3b^2 + 12b + 11 \right) \leq \frac{1}{m^2} \Re \left( \frac{z^{q''}(z)}{q'(z)} \right), \]
where \( z \in U, \zeta \in \partial U, \) and \( m \geq 2. \)

**Theorem 11.** Let \( \phi \in \Phi'_{J,1}[\Omega, q] \). If the function \( f \in \mathcal{A}, \frac{J_{\mu,b}f(z)}{z} \in \mathcal{Q}_1 \) and \( q \in \mathcal{H}_1 \) with \( q'(z) \neq 0 \) satisfy the following conditions:
\[
\Re \left( \frac{zq''(z)}{q'(z)} \right) \geq 0, \quad \left| \frac{J_{\mu,b}f(z)}{zq'(z)} \right| \leq m, \tag{3.5}
\]
and
\[
\phi \left( \frac{J_{\mu+1,b}f(z)}{z}, \frac{J_{\mu,b}f(z)}{z}, \frac{J_{\mu-1,b}f(z)}{z}, \frac{J_{\mu-2,b}f(z)}{z}; z \right),
\]
is univalent in \( U \), then
\[
\Omega \subset \left\{ \phi \left( \frac{J_{\mu+1,b}f(z)}{z}, \frac{J_{\mu,b}f(z)}{z}, \frac{J_{\mu-1,b}f(z)}{z}, \frac{J_{\mu-2,b}f(z)}{z}; z \right) : z \in U \right\}, \tag{3.6}
\]
implies that
\[
q(z) < \frac{J_{\mu+1,b}f(z)}{z}. \tag{3.7}
\]

**Proof.** Let the function \( p(z) \) be defined by (2.17) and \( \psi \) by (2.23). Since \( \phi \in \Phi'_{J,1}[\Omega, q], \) (2.24) and (3.6) yield
\[
\Omega \subset \left\{ \psi \left( p(z), \frac{zq'}{q}(z), z^2p''(z), z^3p'''(z); z \right) : z \in U \right\}.
\]
From equations (2.21) and (2.22), we see that the admissible condition for \( \phi \in \Phi'_{J,1}[\Omega, q] \) in Definition 12 is equivalent to the admissible condition for \( \psi \) as given in Definition 5 with \( n = 2 \). Hence \( \psi \in \Psi_2[\Omega, q] \) and by using (3.5) and Lemma 2 we have
\[
q(z) < \frac{J_{\mu+1,b}f(z)}{z}.
\]

If \( \Omega \neq \mathbb{C} \) is a simply connected domain, then \( \Omega = h(U) \) for some conformal mapping \( h(z) \) of \( U \) onto \( \Omega \). In this case, the class \( \Phi'_{J,1}[h(U), q] \) is written as \( \Phi'_{J,1}[h, q] \). This follows an immediate consequence of Theorem 11.

**Theorem 12.** Let \( \phi \in \Phi'_{J,1}[h, q] \) and \( h \) be analytic in \( U \). If the function \( f \in \mathcal{A} \) and \( q \in \mathcal{H}_1 \) with \( q'(z) \neq 0 \) satisfy the following conditions (3.5) and
\[
\phi \left( \frac{J_{\mu+1,b}f(z)}{z}, \frac{J_{\mu,b}f(z)}{z}, \frac{J_{\mu-1,b}f(z)}{z}, \frac{J_{\mu-2,b}f(z)}{z}; z \right),
\]
is univalent in \( U \), then
\[
h(z) < \phi \left( \frac{J_{\mu+1,b}f(z)}{z}, \frac{J_{\mu,b}f(z)}{z}, \frac{J_{\mu-1,b}f(z)}{z}, \frac{J_{\mu-2,b}f(z)}{z}; z \right),
\]
implies that
\[
q(z) < \frac{J_{\mu+1,b}f(z)}{z} \quad (z \in U).
\]

**Definition 13.** Let \( \Omega \) be a set in \( \mathbb{C} \) and \( q \in \mathcal{H}_1 \) with \( q'(z) \neq 0 \). The class of admissible functions \( \Phi'_{J,2}[\Omega, q] \) consists of those functions \( \phi : \mathbb{C}^4 \times \mathbb{C}^4 \rightarrow \mathbb{C} \) that satisfy the following admissibility conditions
\[
\phi(\alpha, \beta, \gamma, \delta; \zeta) \in \Omega
\]
whenever
\[
\alpha = q(z), \beta = \frac{1}{b+1} \left( \frac{zq'(z)}{mq(z)} + (b+1)q(z) \right), \leq \frac{1}{m} \Re \left( \frac{zq''(z)}{q'(z)} + 1 \right),
\]

\[
\Re \left( \frac{(1+b)(\beta \gamma + 3\alpha \beta - 2\alpha \beta)}{\beta - \alpha} \right) \leq \frac{1}{m} \Re \left( \frac{zq''(z)}{q'(z)} + 1 \right),
\]
and

\[
\Re \left[ (\delta - \gamma) (1+b)^2 \beta \gamma - (1+b)^2 (\gamma - \beta)(1-\beta - \gamma + 3\alpha) - 3(b+1)(\gamma - \beta) \beta + 2(\beta - \alpha) + 3(1+b) \alpha (\beta - \alpha) + (\beta - \alpha)^2 (1+b)((\beta - \alpha)(1+b) - 3 - 4(1+b) \alpha) + \alpha^2 (1+b)^2 (\beta - \alpha) \right] \times (\beta - \alpha)^{-1} \leq \frac{1}{m^2} \Re \left( \frac{z^2 q''(z)}{q'(z)} \right),
\]

where \( z \in \mathbb{U}, \zeta \in \partial \mathbb{U} \) and \( m \geq 2 \).

**Theorem 13.** Let \( \phi \in \Phi_{J,2}[\Omega, q] \). If the function \( f \in \mathcal{A} \) and \( \frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)} \in \mathbb{Q}_1 \) and \( q \in \mathcal{H}_1 \) with \( q'(z) \neq 0 \) satisfy the following conditions

\[
\Re \left( \frac{zq''(z)}{q'(z)} \right) \geq 0, \quad \left| \frac{J_{\mu-1,b}f(z)}{J_{\mu,b}f(z)} q'(z) \right| \leq m,
\]

and

\[
\phi \left( \frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)}, \frac{J_{\mu-1,b}f(z)}{J_{\mu,b}f(z)}, \frac{J_{\mu-2,b}f(z)}{J_{\mu-1,b}f(z)}, \frac{J_{\mu-3,b}f(z)}{J_{\mu-2,b}f(z)}; z \right)
\]

is univalent in \( \mathbb{U} \), then

\[
\Omega \subset \left\{ \phi \left( \frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)}, \frac{J_{\mu-1,b}f(z)}{J_{\mu,b}f(z)}, \frac{J_{\mu-2,b}f(z)}{J_{\mu-1,b}f(z)}, \frac{J_{\mu-3,b}f(z)}{J_{\mu-2,b}f(z)}; z \right) : z \in \mathbb{U} \right\}
\]

then

\[
q(z) < \frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)} \quad (z \in \mathbb{U}).
\]

**Proof.** Let the function \( p(z) \) be defined by (3.30) and \( \psi \) by (2.36). Since \( \phi \in \Phi_{J,2}[\Omega, q] \), (2.37) and (3.8) yield

\[
\Omega \subset \{ \psi \left( p(z), zp'(z), z^2 p''(z), z^3 p'''(z); z \right) : z \in \mathbb{U} \}.
\]

From equations (2.34) and (2.35), we see that the admissible condition for \( \phi \in \Phi_{J,2}[\Omega, q] \) in Definition 13 is equivalent to the admissible condition for \( \psi \) as given in Definition 5 with \( n = 2 \). Hence \( \psi \in \Psi_2[\Omega, q] \), and by using (3.7) and Lemma 2 we have

\[
q(z) < \frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)} \quad (z \in \mathbb{U}).
\]

\[\square\]

**Theorem 14.** Let \( \phi \in \Phi_{J,2}[h, q] \). If the function \( f \in \mathcal{A} \) and \( \frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)} \in \mathbb{Q}_1 \) and \( q \in \mathcal{H}_1 \) with \( q'(z) \neq 0 \) satisfy the following conditions (3.7) and

\[
\phi \left( \frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)}, \frac{J_{\mu-1,b}f(z)}{J_{\mu,b}f(z)}, \frac{J_{\mu-2,b}f(z)}{J_{\mu-1,b}f(z)}, \frac{J_{\mu-3,b}f(z)}{J_{\mu-2,b}f(z)}; z \right)
\]

is univalent in \( \mathbb{U} \), then

\[
h(z) \prec \phi \left( \frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)}, \frac{J_{\mu-1,b}f(z)}{J_{\mu,b}f(z)}, \frac{J_{\mu-2,b}f(z)}{J_{\mu-1,b}f(z)}, \frac{J_{\mu-3,b}f(z)}{J_{\mu-2,b}f(z)}; z \right)
\]

implies that

\[
q(z) < \frac{J_{\mu,b}f(z)}{J_{\mu+1,b}f(z)} \quad (z \in \mathbb{U}).
\]
4. Sandwich-Type Results

Combining Theorems 2 and 3, we obtain the following sandwich-type theorem.

**Corollary 8.** Let $h_1$ and $q_1$ be analytic functions in $\mathbb{U}$, $h_2$ be univalent function in $\mathbb{U}$, $q_2 \in \mathbb{Q}_0$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \Phi_{J,l}[h_2, q_2] \cap \Phi'_{J,l}[h_1, q_1]$. If the function $f \in \mathcal{A}$, $J_{\mu+1,b}f(z) \in \mathbb{Q}_0 \cap \mathcal{H}_0$, and

$$\phi(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z),$$

is univalent in $\mathbb{U}$, and the condition (2.1) and (3.7) are satisfied, then

$$h_1(z) \prec \phi(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z) \prec h_2(z)$$

implies that

$$q_1(z) \prec J_{\mu+1,b}f(z) \prec q_2(z) \quad (z \in \mathbb{U}).$$

Combining Theorems 5 and 12, we obtain the following sandwich-type theorem.

**Corollary 9.** Let $h_1$ and $q_1$ be analytic functions in $\mathbb{U}$, $h_2$ be univalent function in $\mathbb{U}$, $q_2 \in \mathbb{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_{J,l}[h_2, q_2] \cap \Phi'_{J,l}[h_1, q_1]$. If the function $f \in \mathcal{A}$, $J_{\mu+1,b}f(z) \in \mathbb{Q}_1 \cap \mathcal{H}_1$, and

$$\phi(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z),$$

is univalent in $\mathbb{U}$, and the condition (2.1) and (3.7) are satisfied, then

$$h_1(z) \prec \phi(J_{\mu+1,b}f(z), J_{\mu,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z); z) \prec h_2(z)$$

implies that

$$q_1(z) \prec J_{\mu+1,b}f(z) \prec q_2(z) \quad (z \in \mathbb{U}).$$

Combining Theorems 7 and 14, we obtain the following sandwich-type theorem.

**Corollary 10.** Let $h_1$ and $q_1$ be analytic functions in $\mathbb{U}$, $h_2$ be univalent functions in $\mathbb{U}$, $q_2 \in \mathbb{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_{J,l}[h_2, q_2] \cap \Phi'_{J,l}[h_1, q_1]$. If the function $f \in \mathcal{A}$, $J_{\mu+1,b}f(z) \in \mathbb{Q}_1 \cap \mathcal{H}_1$, and

$$\phi(J_{\mu+1,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z), J_{\mu-3,b}f(z); z),$$

is univalent in $\mathbb{U}$, and the condition (2.1) and (3.7) are satisfied, then

$$h(z) \prec \phi(J_{\mu+1,b}f(z), J_{\mu-1,b}f(z), J_{\mu-2,b}f(z), J_{\mu-3,b}f(z); z) \prec h_2(z)$$

implies that

$$q_1(z) \prec J_{\mu+1,b}f(z) \prec q_2(z) \quad (z \in \mathbb{U}).$$

**Acknowledgment**

The present investigation of the second author is supported under the INSPIRE fellowship, Department of Science and Technology, New Delhi, Government of India, Sanction letter No. REL1/2016/2.

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