THE GROWTH OF THE NUMBER OF PERIODIC ORBITS
FOR ANNULUS HOMEOMORPHISMS AND
NON-CONTRACTIBLE CLOSED GEODESICS ON
RIEMANNIAN OR FINSLER $\mathbb{R}P^2$

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Abstract. In this article, we give a growth rate about the number of periodic orbits in the Franks type theorem obtained by the authors [35]. As applications, we prove the following two results: there exist infinitely many distinct non-contractible closed geodesics on $\mathbb{R}P^2$ endowed with a Riemannian metric such that its Gaussian curvature is positive, moreover, the number of non-contractible closed geodesics of length $\leq l$ grows at least like $l^2$; and there exist either two or infinitely many distinct non-contractible closed geodesics on Finsler $\mathbb{R}P^2$ with reversibility $\lambda$ and flag curvature $K$ satisfying $(\frac{\lambda}{1+\lambda})^2 < K \leq 1$, furthermore, if the second case happens, then the number of non-contractible closed geodesics of length $\leq l$ grows at least like $l^2$.

1. Introduction

The research of the periodic orbits of annulus homeomorphisms was started by Poincaré. In his search for periodic solutions in the restricted three body problem of celestial mechanics, Poincaré constructed an area-preserving section map of an annulus $\mathbb{A} \simeq \mathbb{R}/\mathbb{Z} \times [0, 1]$ on the energy surface, and get the periodic orbits of the original system from the periodic orbits of the annulus homeomorphism (see [39]). Later, many mathematicians studied annulus homeomorphisms (see [6, 7, 8, 18, 32, 33] etc.)

Poincaré’s idea inspired many mathematicians. A classic method for the existence of periodic motions of Hamiltonian system with two degrees of freedom is as follows: we first reduce the dynamics to the annulus-type global surface of section and then get the existence of two or infinitely many periodic orbits by the following theorem due to Franks; see the pioneer works of Hofer, Wysocki and Zehnder [24, 25] and also some important progress [12, 11] on this topic.

Theorem (Franks). Suppose that $f$ is an area preserving homeomorphism of the open or closed annulus which is isotopic to the identity. If $f$ has at least one fixed or periodic point, then $f$ must have infinitely many interior periodic points.

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In [18], Franks used this theorem to prove that a Riemannian metric on $S^2$ has infinitely many closed geodesics whenever there is a simple closed geodesic for which Birkhoff’s global annulus like surface of section is well defined. This, together with Bangert’s result [2], implies the existence of infinitely many closed geodesics on every Riemannian $S^2$.

In our last paper [35], we improved Franks’ theorem by considering the existence of the periodic orbits with periods relatively prime to a given number $n_0$, and as its application we gave some precise information about the symmetries of periodic orbits found in Hofer, Wysocki and Zehnder’s dichotomy theorem when the tight 3-sphere is equipped with some additional symmetries; see also [19, 42, 29] for other dichotomy results about symmetric periodic orbits.

For the growth of the number of periodic orbits, Neumann [38] considered area preserving twist homeomorphism of the closed annulus. In this article, we will estimate the growth of the number of periodic orbits and the growth of the number of periodic orbits whose periods are prime to a given prime number for area preserving homeomorphisms of the open and closed annuli, which will be applied to the problem of non-contractible closed geodesics on Finsler or Riemannian $\mathbb{R}P^2$.

1.1. The growth of the number of periodic orbits for annulus homeomorphisms. We consider a homeomorphism $f$ of the open (resp. closed) annulus $A = \mathbb{R}/\mathbb{Z} \times (0, 1)$ (resp. $A = \mathbb{R}/\mathbb{Z} \times [0, 1]$) that is isotopic to the identity. An area of a surface is a locally finite Borel measure without atom and with total support. We say that $z$ is an $n$ prime-periodic point of $f$ if $z$ is an $n$ periodic point but not an $l$ periodic point of $f$ for all $0 < l < n$. We call a periodic orbit of $f$ an interior periodic orbit, if it is in the interior of $A$. We denote by $\text{Fix}(f)$ the set of fixed points of $f$. Let

$$N_{=n} = \sharp\{\text{interior } n \text{ prime-periodic orbits of } f\},$$

$$N_{\leq n} = \sharp\{\text{interior periodic orbits of } f \text{ with prime-period } \leq n\},$$

$$N_{\leq n, n_0} = \sharp\{\text{interior } q \text{ prime-periodic orbits of } f : q \leq n, (q, n_0) = 1\}.$$

We have the following theorem about the growth of the number of the periodic orbits of $f$.

**Theorem 1.** Let $f$ be a homeomorphism of the closed or open annulus, that is isotopic to the identity and preserves a finite area. If $f$ has a fixed or periodic point, then

$$ \liminf_{n \to +\infty} N_{\leq n} \frac{1}{n^2} > 0. $$

Moreover, for a given prime number $n_0$, if $f$ has a fixed or periodic point with prime-period $k$ such that $(k, n_0) = 1$, then

$$ \liminf_{n \to +\infty} N_{\leq n, n_0} \frac{1}{n^2} > 0. $$
More precisely, if $f$ has a $k$ prime-periodic point and $f^k \neq \text{Id}$, then

$$\lim \inf_{n \to +\infty} N_{f^k} \frac{\log \log n}{n} > 0.$$  

As a corollary of the second part of the theorem, if $f$ has an odd periodic orbit, it has infinite many interior odd periodic orbits, and the number of interior odd periodic orbits with prime-periods not exceeding $n$ grows at least like $n^2$. We also note that the third part of the theorem is a direct corollary of Theorem 12 in Section 3, which is more precise but technical.

Remark 1. Neumann [38] gave an example of an area preserving homeomorphism of the closed annulus, such that $N_n \leq (2 + \varepsilon)\varphi(n; a, b)$, where $\varepsilon$ is small, $a, b$ are the different rotation numbers of the points on the two boundaries, and $\varphi(n; a, b)$ is the number of irreducible fractions with denominator $n$ in the interval $(a, b)$. By choosing $a < 0 < b$, we have an area preserving homeomorphism of the closed annulus that has a fixed point and satisfies

$$\lim \sup_{n \to +\infty} N_{n} \frac{1}{n^2} < +\infty,$$

$$\lim \sup_{n \to +\infty} N_{n, n|a} \frac{1}{n^2} < +\infty,$$

$$\lim \inf_{n \to +\infty} N_{n} \frac{\log \log n}{n} < +\infty.$$  

Remark 2. If we add the condition that $f$ is reversible and consider the number of symmetric periodic orbits (see [27] or [35] for the precise definitions), we have similar results. The idea of the proof is similar, but we should add the condition that $\sharp \text{Fix}(f^k) < +\infty$ when we consider the number of the $kn$ prime-periodic symmetric orbits.

1.2. The growth of the number of non-contractible closed geodesics on Riemannian or Finsler $\mathbb{R}P^2$. Hingston [23] proved that the number of closed geodesics of length $\leq l$ on Riemannian $\mathbb{S}^2$ grows at least like the prime numbers. Similarly, we consider the growth of the number of non-contractible closed geodesics on Riemannian $\mathbb{R}P^2$ whose double cover is $\mathbb{S}^2$. As applications of our Theorem 1, we obtain the following:

Theorem 2. Let $(\mathbb{R}P^2, g)$ be a Riemannian real projective plane whose Gaussian curvature is positive. Then there exist infinitely many distinct non-contractible closed geodesics on $(\mathbb{R}P^2, g)$. Moreover, the number of non-contractible closed geodesics of length $\leq l$ grows at least like $l^2$.

For Finsler $\mathbb{R}P^2$ under suitable conditions, a dichotomy result on the existence of non-contractible closed geodesics holds. Moreover, if there exist infinitely many distinct non-contractible closed geodesics, then the growth rate as in Theorem 2 holds:
Theorem 3. Let $(\mathbb{R}P^2, F)$ be an irreversible Finsler real projective plane with reversibility $\lambda$ and flag curvature $K$ satisfying $(\frac{\lambda}{1+\lambda})^2 < K \leq 1$. Then there exist either two or infinitely many distinct non-contractible closed geodesics on $(\mathbb{R}P^2, F)$; furthermore, if the second case happens, then the number of non-contractible closed geodesics of length $\leq l$ grows at least like $l^2$.

1.3. Organization of the paper. In Section 2, we will give several definitions and preliminary results. In Section 3, we will prove Theorem 1 and Theorem 12. In Section 4, we review the problem about the multiplicity of non-contractible closed geodesics on $\mathbb{R}P^2$ endowed with Riemannian or Finsler metrics and then apply Theorem 1 to prove Theorem 2 and Theorem 3.

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2. Preliminaries

2.1. Euler’s phi function and the number of irreducible fractions. Let $\varphi(n)$ be the Euler’s phi function, i.e., $\varphi(n)$ is equal to the number of positive integers not exceeding $n$ and relatively prime to $n$. We have the following classic results.

Theorem 4. [21] Section 5.5, 18.4, 18.5

i) If $\gcd(m, n) = 1$, then $\varphi(mn) = \varphi(m)\varphi(n)$.

ii) Let $n = p_1^{r_1} \cdots p_s^{r_s}$ be the prime decomposition of $n$. Then,

$$\varphi(n) = n\prod_{i=1}^{s}(1 - \frac{1}{p_i}) = \prod_{i=1}^{s}(p_i^{r_i} - p_i^{r_i-1}).$$

iii) $\liminf_{n \to +\infty} \varphi(n) \frac{\log \log n}{n} = e^{-\gamma},$

where $\gamma = \lim_{n \to +\infty} (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n)$ is the Euler’s constant.

iv) $\Phi(n) \overset{\text{def}}{=} \varphi(1) + \cdots + \varphi(n) = \frac{3n^2}{\pi^2} + O(n \log n)$, as $n \to +\infty$.

Now, let us consider the irreducible fractions between two real number $\rho^- < \rho^+$. Denote by $\varphi(n; \rho^-, \rho^+)$ the number of irreducible fractions with the denominator $n$ between $\rho^- < \rho^+$, i.e.,

$$\varphi(n; \rho^-, \rho^+) = \sharp\{m \in \mathbb{Z} : \rho^- < \frac{m}{n} < \rho^+, \gcd(m, n) = 1\}$$

$$= \sharp\{m \in \mathbb{Z} : m \in (n\rho^-, n\rho^+), \gcd(m, n) = 1\}.$$
Let $\Phi(n; \rho^-, \rho^+)$ be the number of irreducible fractions with denominators not exceeding $n$ between $\rho^- < \rho^+$, i.e.,

$$\Phi(n; \rho^-, \rho^+) = \#\{\frac{p}{q} : \rho^- < \frac{p}{q} < \rho^+, q \leq n, \gcd(p, q) = 1\}$$

$$= \sum_{q=1}^{n} \varphi(q; \rho^-, \rho^+).$$

We have the following estimations of the orders of $\varphi(n; \rho^-, \rho^+)$ and $\Phi(n; \rho^-, \rho^+)$.  

**Lemma 5.** [38] $\varphi(n; \rho^-, \rho^+) \sim (\rho^+ - \rho^-) \varphi(n)$, as $n \to +\infty$.

**Lemma 6.** $\Phi(n; \rho^-, \rho^+) \sim \frac{3(\rho^+ - \rho^-)}{\pi^2} n^2$, as $n \to +\infty$.

The following proof of Lemma 5 is from [38], and the proof of Lemma 6 will use the result in the proof of Lemma 5.

**Proof of Lemma 5.** Let $n = p_1^{r_1} \cdots p_s^{r_s}$ be the prime decomposition of $n$. Then, in the interval $(0, n]$, there are

- $n$ integers,
- $\frac{n}{p_i}$ integers that are divisible by $p_i$, for $i = 1, \cdots, s$,
- $\frac{n}{p_i p_j}$ integers that are divisible by $p_i p_j$, for $1 \leq i < j \leq s$,
- $\cdots$

By the inclusion-exclusion principle,

$$\varphi(n) = n - \sum_{i=1}^{s} \frac{n}{p_i} + \sum_{1 \leq i < j \leq s} \frac{n}{p_i p_j} - \sum_{1 \leq i < j < k \leq s} \frac{n}{p_i p_j p_k} + \cdots$$

For a given positive integer $q$, in an open interval with length $\ell$,

- if $\frac{\ell}{q}$ is not an integer, there are $\lfloor \frac{\ell}{q} \rfloor$ or $\lfloor \frac{\ell}{q} \rfloor + 1$ integers that are divisible by $q$ in the interval;
- if $\frac{\ell}{q}$ is an integer, there are $\frac{\ell}{q}$ or $\frac{\ell}{q} - 1$ integers that are divisible by $q$ in the interval.

So,

$$\varphi(n; \rho^-, \rho^+) = (\rho^+ - \rho^-) - \sum_{i=1}^{s} \frac{n p_i^+ - n p_i^-}{p_i} + \sum_{1 \leq i < j \leq s} \frac{n p_i^+ - n p_j^-}{p_i p_j}$$

$$- \sum_{1 \leq i < j < k \leq s} \frac{n p_i^+ - n p_j^-}{p_i p_j p_k} + \cdots + \varepsilon(n)$$

$$= (\rho^+ - \rho^-) \varphi(n) + \varepsilon(n),$$

where $\varepsilon(n)$ is the error term, and

$$|\varepsilon(n)| \leq 1 + \sum_{i=1}^{s} 1 + \sum_{1 \leq i < j \leq s} 1 + \cdots = 2^s.$$
Recall that $\varphi(n) = \prod_{i=1}^{s}(p_i^{r_i} - p_i^{r_i-1})$. So, $\frac{\varepsilon(n)}{\varphi(n)} \to 0$, as $n \to +\infty$.

Therefore, $\varphi(n; \rho^-, \rho^+) \sim (\rho^+ - \rho^-)\varphi(n)$, as $n \to +\infty$. $\square$

**Proof of Lemma 6.** In the proof of the previous lemma, we have

$$\varphi(n; \rho^-, \rho^+) = \sum_{q=1}^{n} \varphi(q; \rho^-, \rho^+) = (\rho^+ - \rho^-) \sum_{q=1}^{n} \varphi(q) + \sum_{q=1}^{n} \varepsilon(q).$$

Note that $\varphi(n), n \in \mathbb{N}$ are positive integers. Hence,

$$\lim_{n \to +\infty} \frac{\sum_{q=1}^{n} \varepsilon(q)}{\sum_{q=1}^{n} \varphi(q)} = 0.$$

Therefore,

$$\Phi(n; \rho^-, \rho^+) \sim \frac{3(\rho^+ - \rho^-)}{\pi^2} n^2, \quad \text{as } n \to +\infty. \quad \square$$

Now, for a given prime number $n_0$, we consider the number of irreducible fractions between $\rho^- < \rho^+$, whose denominators are relatively prime to $n_0$ and are not exceeding $n$. Let

$$\Psi(n; \rho^-, \rho^+) = \left\{ \frac{p}{q} : \rho^- < \frac{p}{q} < \rho^+, q \leq n, \gcd(p, q) = 1, \gcd(q, n_0) = 1 \right\}$$

We will estimate the order of $\Psi(n; \rho^-, \rho^+)$.  

**Lemma 7.**

$$0 < C_1 \leq \liminf_{n \to +\infty} \Psi(n; \rho^-, \rho^+) \frac{1}{n^2} \leq \limsup_{n \to +\infty} \Psi(n; \rho^-, \rho^+) \frac{1}{n^2} \leq C_2 < +\infty,$$

where $C_1, C_2$ are constants depending on $\rho^+, \rho^-$ and the prime number $n_0$.

**Proof.**

$$\Psi(n; \rho^-, \rho^+) = \sum_{q=1,2,\ldots; n; \gcd(q, n_0)=1} ((\rho^+ - \rho^-)\varphi(q) + \varepsilon(q))$$

$$= (\rho^+ - \rho^-) \sum_{q=1,2,\ldots; n; \gcd(q, n_0)=1} \varphi(q) + \sum_{q=1,2,\ldots; n; \gcd(q, n_0)=1} \varepsilon(q),$$

where the error terms $\varepsilon(q)$ satisfies $\lim_{n \to +\infty} \frac{\varepsilon(n)}{\varphi(n)} = 0.$
Note that

\[ \sum_{q=1,2,\ldots,n; \gcd(q,n_0)=1} \varphi(q) = \sum_{q=1}^{n} \varphi(q) - \sum_{k=1}^{\left\lfloor \frac{n}{n_0} \right\rfloor} \varphi(kn_0), \]

and that

\[ \varphi(kn_0) = \begin{cases} \varphi(k)(n_0 - 1), & \gcd(n_0, k) = 1, \\ \varphi(k)n_0, & \gcd(n_0, k) = n_0. \end{cases} \]

Then,

\[ \sum_{q=1}^{n} \varphi(q) - \sum_{k=1}^{\left\lfloor \frac{n}{n_0} \right\rfloor} n_0\varphi(k) \leq \sum_{q=1,2,\ldots,n; \gcd(q,n_0)=1} \varphi(q) - \sum_{k=1}^{\left\lfloor \frac{n}{n_0} \right\rfloor} (n_0 - 1)\varphi(k). \]

By Theorem 4, \( \Phi(n) = \sum_{k=1}^{n} \varphi(k) = \frac{3}{\pi^2} n^2 + O(n \log n) \), as \( n \to +\infty \), we have

\[ \sum_{q=1}^{n} \varphi(q) - \sum_{k=1}^{\left\lfloor \frac{n}{n_0} \right\rfloor} n_0\varphi(k) = (1 - \frac{1}{n_0}) \frac{3}{\pi^2} n^2 + O(n \log n), \]

\[ \sum_{q=1}^{n} \varphi(q) - \sum_{k=1}^{\left\lfloor \frac{n}{n_0} \right\rfloor} n_0\varphi(k) = (1 - \frac{n_0 - 1}{n_0}) \frac{3}{\pi^2} n^2 + O(n \log n). \]

Recall that \( \frac{\varepsilon(n)}{\varphi(n)} \to 0 \), we have \( \sum_{k=1}^{n} |\varepsilon(k)| \to 0 \), as \( n \to +\infty \). Therefore,

\[ \liminf_{n \to +\infty} \Psi(n; \rho^-, \rho^+) \frac{1}{n^2} \geq (\rho^+ - \rho^-) \frac{3}{\pi^2} (1 - \frac{1}{n_0}) > 0, \]

\[ \limsup_{n \to +\infty} \Psi(n; \rho^-, \rho^+) \frac{1}{n^2} \leq (\rho^+ - \rho^-) \frac{3}{\pi^2} (1 - \frac{n_0 - 1}{n_0^2}) < \infty. \]

2.2. **Rotation number.** In this section, we will introduce the rotation numbers for annulus homeomorphism. For more information, refer to [17] and [32].

We denote by \( \mathbb{A} \) the open (resp. the closed) annulus unless an explicit mention, i.e., \( \mathbb{A} = \mathbb{R} / \mathbb{Z} \times (0, 1) \) (resp. \( \mathbb{A} = \mathbb{R} / \mathbb{Z} \times [0, 1] \)), by \( \pi \) the covering map of the annulus

\[ \pi : \mathbb{R} \times (0, 1) \quad (\text{resp. } \mathbb{R} \times [0, 1]) \to \mathbb{A} \]

\[ (x, y) \mapsto (x + \mathbb{Z}, y), \]

and by \( T \) the generator of the covering transformation group

\[ T : \mathbb{R} \times (0, 1) \quad (\text{resp. } \mathbb{R} \times [0, 1]) \to \mathbb{R} \times (0, 1) \quad (\text{resp. } \mathbb{R} \times [0, 1]) \]

\[ (x, y) \mapsto (x + 1, y). \]
Coordinates are denoted as $z \in \mathbb{A}$ and $\tilde{z}$ in the covering space. Homeomorphisms of $\mathbb{A}$ are denoted by $f$, and their lifts to the covering space are denoted by $\tilde{f}$.

Consider the homeomorphism $f$ of $\mathbb{A}$ that is isotopic to the identity. We say that a positively recurrent point $z$ has a rotation number $\rho(\tilde{f},z) \in \mathbb{R}$ for a lift $\tilde{f}$ of $f$ to the universal covering space of $\mathbb{A}$, if for every subsequence $\{f^{n_k}(z)\}_{k \geq 0}$ of $\{f^n(z)\}_{n \geq 0}$ which converges to $z$, we have

$$\lim_{k \to +\infty} \frac{p_1 \circ \tilde{f}^{n_k}(\tilde{z}) - p_1(\tilde{z})}{n_k} = \rho(\tilde{f},z),$$

where $\tilde{z} \in \pi^{-1}(z)$ is a lift of $z$ and $p_1$ is the standard projection to the first coordinate. The rotation number is stable by conjugacy (see [32]). In particular, if $z$ is a fixed or periodic point of $f$, the rotation number $\rho(\tilde{f},z)$ always exists and is rational.

A positively recurrent point of $f$ is also a positively recurrent point of $f^q$ for all $q \in \mathbb{N}$ (see [46, Appendix]). By the definition of the rotation number, we easily get the following elementary properties:

1. $\rho(T^k \circ \tilde{f},z) = \rho(\tilde{f},z) + k$ for every $k \in \mathbb{Z}$;
2. $\rho(\tilde{f}^q,z) = q \rho(\tilde{f},z)$ for every $q \in \mathbb{N}$.

We call a simple closed curve in $\mathbb{A}$ an essential circle if it is not null-homotopic. We say that $f$ satisfies the intersection property if any essential circle in $\mathbb{A}$ meets its image by $f$. It is easy to see that a homeomorphism $f$ that preserves a finite area satisfies the intersection property.

We need the following theorem due to Franks [17] when $\mathbb{A}$ is the closed annulus and $f$ has no wandering point, and improved by Le Calevez [33] (see also [46]) when $\mathbb{A}$ is the open annulus and $f$ satisfies the intersection property:

**Theorem 8.** Let $f$ be a homeomorphism of $\mathbb{A}$ that is isotopic to the identity and satisfies the intersection condition, and $\tilde{f}$ one of its lifts to the universal covering space. Suppose that there exist two recurrent points $z_1$ and $z_2$ such that $-\infty \leq \rho(\tilde{f},z_1) < \rho(\tilde{f},z_2) \leq +\infty$. Then for any rational number $p/q \in (\rho(\tilde{f},z_1),\rho(\tilde{f},z_2))$ written in an irreducible way, there exists an interior $q$ prime-periodic point with rotation number $p/q$.

2.3. Transverse foliation and maximal isotopy. Let $M$ be an oriented surface, and $\mathcal{F}$ an oriented topological foliation on $M$ whose leaves are oriented curves. We say that a path is positively transverse to $\mathcal{F}$, if it meets the leaves of $\mathcal{F}$ locally from left to right. Let $f$ be a homeomorphism on $M$, and $I = (f_t)_{t \in [0,1]}$ an identity isotopy of $f$, i.e., an isotopy joining the identity to $f$. We say that an oriented foliation $\mathcal{F}$ (without singularity) is a transverse foliation of $I$ if for every $z \in M$, there is a path that is homotopic to the trajectory $t \to f_t(z)$ of $z$ along $I$ with the end points fixed and is positively transverse to $\mathcal{F}$. If $f$ does not have any contractible fixed point associated
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to \( I \), i.e., a fixed point of \( f \) whose trajectory along \( I \) is null homotopic in \( M \), Le Calvez proved the existence of the transverse foliation \([33, \text{Theorem 1.3}]\).

Now, we consider the case that \( f \) has a contractible fixed point. We call \( K \subset \text{Fix}(f) \) unlinked, if there is an identity isotopy \( I = (f_t)_{t \in [0,1]} \) of \( f \) such that \( K \subset \text{Fix}(I) = \bigcap_{t \in [0,1]} \text{Fix}(f_t) \). We call an identity isotopy maximal, if \( \text{Fix}(I) \) is maximal for including among all unlinked sets. Moreover, if \( I \) is a maximal isotopy, \( f|_{M \setminus \text{Fix}(I)} \) does not have any contractible fixed point. We consider a singular foliation \( \mathcal{F} \) and call it a transverse foliation of \( I \), if the set of singularities \( \text{Sing(}\mathcal{F}) \) is equal to the fixed point set \( \text{Fix}(I) \) of \( I \), and if \( \mathcal{F}|_{M \setminus \text{Sing}(\mathcal{F})} \) is transverse to \( I|_{M \setminus \text{Fix}(I)} \). Combine Le Calvez’s result and the following theorem, we always get the existence of a maximal isotopy \( I \) and a transverse foliation \( \mathcal{F} \).

**Theorem 9.** \([5, \text{Corollary 1.2}]\) Let \( f \) be a homeomorphism of \( M \) and \( K \) an unlinked set. Then, there is a maximal isotopy \( I \) such that \( K \subset \text{Fix}(I) \).

In particular, for area preserving homeomorphisms of the sphere, if the maximal isotopy has only finitely many fixed points, then the dynamics of the transverse foliation is quite simple: there is neither a closed leaf nor a leaf from and toward the same singularity, and hence every leaf joins one singularity to another singularity.

2.4. **Prime ends compactification and prime ends rotation number.**

In this section, we will give the definitions and the result that we need for our paper. More details can be found in \([31]\).

Let \( U \subset \mathbb{S}^2 \) be an open topological disk such that \( \mathbb{S}^2 \setminus U \) contains at least two points. We can define the prime ends compactification of the open topological disk \( U \), introduced by Carathéodory \([10]\), by attaching a circle of prime ends \( \simeq \mathbb{S}^1 \) and topologizing \( U \cup \mathbb{S}^1 \) appropriately, making it homeomorphic to a closed disk. The prime end compactification can be defined purely topologically (see \([37]\) and \([31]\) for more details), but has another significance if we put a complex structure on \( \mathbb{S}^2 \). We can find a conformal map \( \phi \) between \( U \) and the open disk \( \mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \} \) and we put on \( U \cup \mathbb{S}^1 \) the topology (up to a homeomorphism of the resulting space, independent of \( \phi \)) induced from the natural topology of \( \overline{\mathbb{D}} \) in \( \mathbb{C} \) by the bijection

\[
\overline{\phi} : U \cup \mathbb{S}^1 \to \overline{\mathbb{D}},
\]

which is equal to \( \phi \) on \( U \) and to the identity on \( \mathbb{S}^1 \), where \( \overline{\mathbb{D}} \) is the closure of \( \mathbb{D} \).

Let \( f \) be an orientation preserving homeomorphism of \( \mathbb{S}^2 \) such that \( f(U) = U \). Then, \( f|_U \) can be extended to a homeomorphism of \( U \cup \mathbb{S}^1 \), which is still denoted by \( f \). Moreover, \( f|_{\mathbb{S}^1} \) is an orientation preserving homeomorphism of the circle. We define the prime ends rotation number \( \rho(f, U) \in \mathbb{R}/\mathbb{Z} \) of \( f \) on the boundary of \( U \) as the Poincaré’s rotation number of \( f|_{\mathbb{S}^1} \).

The following result is a direct corollary of \([31, \text{Theorem C}]\).
Theorem 10. Let $f$ be an orientation and area preserving homeomorphism of the sphere $S^2$, and $U \subset S^2$ an $f$-invariant topological disk such that $S^2 \setminus U$ contains at least two points. If $\rho(f,U) \neq 0$, then $f$ has at most one fixed point in $S^2 \setminus U$.

2.5. Cover $SO(3)$ by $SU(2) \cong S^3$. In this section, we will give a classic covering map that covers $SO(3)$ by $SU(2) \cong S^3$. More details can be find in [43, Section 1.6].

We can first identify

$$S^3 = \{ (z_1, z_2) : z_1, z_2 \in \mathbb{C}, |z_1|^2 + |z_2|^2 = 1 \}$$

with $SU(2)$ by

$$S^3 \to SU(2), \quad (z_1, z_2) \mapsto U = \begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix}.$$ 

Then, we will cover $SO(3)$ by $SU(2)$.

Note that any $2 \times 2$ traceless Hermitian matrix $M$ can be written as a linear combination of the Pauli matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$ 

More explicitly,

$$M = \begin{bmatrix} c & a - ib \\ a + ib & -c \end{bmatrix} = (\sigma_1, \sigma_2, \sigma_3) \begin{bmatrix} a \\ b \\ c \end{bmatrix},$$

where $(a, b, c)^T \in \mathbb{R}^3$. So, we can identify the linear space $V$ of $2 \times 2$ traceless Hermitian matrix with $\mathbb{R}^3$ by choosing $\sigma_1, \sigma_2, \sigma_3$ as the basis of $V$, and can define the norm on $V$ by

$$\|M\|_{\mathbb{R}^3} = \sqrt{-\det M}.$$ 

For any $U \in SU(2)$ and $M \in V$, $U^* MU$ is a Hermitian matrix with

$$\text{tr}(U^* MU) = \text{tr}(M) = 0, \quad \text{and } \det(U^* MU) = \det M.$$ 

So, the map $M \mapsto U^* MU$ is an isometry of $V$, and hence corresponds to an orthogonal matrix in $O(3)$. The group action

$$SU(2) \times V \to V, \quad (U, M) \mapsto U^* MU,$$

is continuous, and induces a group homomorphism

$$R : SU(2) \to O(3).$$ 

Note that $R$ maps the unit matrix in $SU(2)$ to the unit matrix in $O(3)$. It is in fact a group homomorphism

$$R : SU(2) \to SO(3).$$
Moreover, it is indeed a two-covering map (see also [43, Section 1.6]).

Now, we have the induced covering map

$$\pi: S^3 \to SO(3).$$

By computing $$U^* \sigma_i U, i = 1, 2, 3,$$ we have

$$\pi(z_1, z_2) = \begin{bmatrix}
\text{Re}(z_1^2 - \bar{z}_2^2) & -\text{Im}(z_1^2 + \bar{z}_2^2) & 2\text{Re}(z_1 \bar{z}_2) \\
\text{Im}(z_1^2 - \bar{z}_2^2) & \text{Re}(z_1^2 + \bar{z}_2^2) & 2\text{Im}(z_1 \bar{z}_2) \\
-2\text{Re}(z_1z_2) & 2\text{Im}(z_1z_2) & |z_1|^2 - |z_2|^2
\end{bmatrix}.$$

In conclusion, we have the following result:

**Lemma 11.** The unit 3-dimensional sphere

$$S^3 = \{(z_1, z_2) : z_1, z_2 \in \mathbb{C}, |z_1|^2 + |z_2|^2 = 1\}$$

covers $$SO(3)$$ twice by the covering map

$$\pi: S^3 \to SO(3),$$

$$(z_1, z_2) \mapsto \begin{bmatrix}
\text{Re}(z_1^2 - \bar{z}_2^2) & -\text{Im}(z_1^2 + \bar{z}_2^2) & 2\text{Re}(z_1 \bar{z}_2) \\
\text{Im}(z_1^2 - \bar{z}_2^2) & \text{Re}(z_1^2 + \bar{z}_2^2) & 2\text{Im}(z_1 \bar{z}_2) \\
-2\text{Re}(z_1z_2) & 2\text{Im}(z_1z_2) & |z_1|^2 - |z_2|^2
\end{bmatrix}.$$

3. **Proof of the main theorem**

In this section, we consider the homeomorphism $$f$$ of the closed or open annulus that is isotopic to the identity. We will prove the main theorem (Theorem 1) of the paper and the following Theorem 12.

**Theorem 12.** Let $$f$$ be a homeomorphism of the closed or open annulus, that is isotopic to the identity and preserves a finite area. Suppose that $$f$$ has a fixed or periodic point, and is of infinite order, i.e., there is not any positive integer $$m$$ such that $$f^m = \text{Id}$$. Then, one of the following three cases happens:

i) $$f$$ has a fixed point,

ii) $$f$$ does not have any fixed point, and have periodic orbits with prime-periods $$n_1, n_2, \ldots, n_s$$ such that $$\gcd(n_1, n_2, \ldots, n_s) = 1$$,

iii) the periods of all the periodic orbits have a greatest common divisor $$k \geq 2$$.

In case i) and ii), we have

$$\lim\inf_{n \to +\infty} N_{n \cdot n} \frac{\log \log n}{n} > 0.$$

In case iii), we have,

$$\lim\inf_{n \to +\infty} N_{kn \cdot n} \frac{\log \log n}{n} > 0.$$

We first give the idea of the proof of Theorem 1 and Theorem 12. If $$f$$ is of finite order, it is conjugate to a rational rotation of the annulus (see [9, 16, 13]). The results are easy to prove. If $$f$$ is of infinite order and has
a fixed or periodic point, one of the three cases in Theorem 12 happens. In case i), we will prove the existence of different rotation numbers for the homeomorphism or the induced homeomorphism on a modified annulus, and get the results by the following Lemma 13. In case ii), we can also prove the existence of different rotation numbers in Lemma 15, and get the results by the Lemma 13. In case iii), we consider \( f^k \), which satisfies the condition of case i) or ii), and compare the prime-periods of a periodic point for \( f \) and for \( f^k \) respectively.

Now, we begin with the lemmas, and then prove Theorem 12 and Theorem 1.

**Lemma 13.** Let \( f \) be a homeomorphism of \( \mathbb{A} \) that is isotopic to the identity and satisfies the intersection condition, and \( \tilde{f} \) one of its lifts to the universal covering space. Suppose that there exist two positive recurrent points \( z_1 \) and \( z_2 \) such that \(-\infty \leq \rho(\tilde{f}, z_1) = \rho_1 < \rho(\tilde{f}, z_2) = \rho_2 \leq +\infty\). Then,

\[
\liminf_{n \to +\infty} N_n \frac{\log \log n}{n} > 0, \quad \liminf_{n \to +\infty} N_{\leq n} \frac{1}{n^2} > 0, \quad \liminf_{n \to +\infty} N_{\leq n, n_0} \frac{1}{n^2} > 0,
\]

where \( n_0 \) is a given prime number.

**Remark 3.** The homeomorphism, that preserves a finite area, satisfies the intersection condition.

**Proof of Lemma 13.** By Theorem 8, \( f \) has an interior \( n \) prime-periodic orbit with the rotation number \( \frac{m}{n} \) for every irreducible \( \frac{m}{n} \in (\rho_1, \rho_2) \). So,

\[
N_{=n} \geq \varphi(n; \rho_1, \rho_2), \quad N_{\leq n} \geq \Phi(n; \rho_1, \rho_2), \quad N_{\leq n, n_0} \geq \Psi(n; \rho_1, \rho_2).
\]

By Theorem 4 and Lemma 5,

\[
\liminf_{n \to +\infty} N_{=n} \frac{\log \log n}{n} \geq \liminf_{n \to +\infty} (\rho_2 - \rho_1) \varphi(n) \frac{\log \log n}{n} = (\rho_2 - \rho_1)e^{-\gamma} > 0,
\]

where \( \gamma \) is the Euler constant.

By Lemma 6,

\[
\liminf_{n \to +\infty} N_{\leq n} \frac{1}{n^2} \geq \frac{3(\rho_2 - \rho_1)}{\pi^2} > 0.
\]

By Lemma 7,

\[
\liminf_{n \to +\infty} N_{\leq n, n_0} \frac{1}{n^2} > 0. \quad \square
\]

**Lemma 14.** Let \( f \) be a homeomorphism of the open annulus \( \mathbb{A} \) that preserves a finite area, \( I \) a maximal isotopy of \( f \), \( \tilde{f} \) the lift of \( f \) to the universal covering space of \( \mathbb{A} \) associated with \( I \) (i.e., \( (\tilde{f}_t)_{t \in [0,1]} \) is the lift of \( I = (f_t)_{t \in [0,1]} \), \( \tilde{f}_0 = \text{Id} \), \( \tilde{f}_1 = \tilde{f} \)), and \( \mathcal{F} \) a transverse foliation of \( I \). If there is a leaf of \( \mathcal{F} \) from one end of the annulus to the other end of the annulus, then there exists a positive recurrent point with \( \rho(\tilde{f}, z) \neq 0 \).

**Proof.** We choose a leaf \( \Gamma \) from one end of the annulus to the other end of the annulus and one of its lifts \( \tilde{\Gamma} \). Note that the oriented curve \( \tilde{\Gamma} \) separates
the covering space into two parts, and \( \tilde{f} \) maps the part on the right of \( \tilde{\Gamma} \) to a proper subset of itself. We can choose a small disk \( U \) near \( \Gamma \) and \( \tilde{U} \) the lift of \( U \) near \( \tilde{\Gamma} \), such that \( \tilde{U} \) is on the left of \( \tilde{\Gamma} \) and \( \tilde{f}(\tilde{U}) \) is on the right of \( \tilde{\Gamma} \). We can also assume that the diameter of \( p_1(\tilde{U}) \) is smaller than \( \frac{1}{2} \) by choosing \( U \) small enough, where \( p_1 \) is the projection to the first factor. We suppose that

\[
p_1(\tilde{z}') - p_1(\tilde{z}) > \frac{1}{2} \text{ for all } \tilde{z} \in \tilde{U} \text{ and } \tilde{z}' \in \tilde{U}',
\]

where \( \tilde{U}' \) is any lift of \( U \) on the right of \( \tilde{\Gamma} \) (the proof in the case \( p_1(\tilde{z}') - p_1(\tilde{z}) < -\frac{1}{2} \) is similar). By Poincaré’s recurrent theorem, almost all points (with respect to the measure induced by the given finite area) in \( U \) are positive recurrent by \( f \). We denote by \( \text{Rec}^+(f) \) the set of positive recurrent points of \( f \), by \( \Phi \) the first return map for \( f \) on \( U \cap \text{Rec}^+(f) \), and by \( \tau \) the first return time. Let

\[
m(z) = p_1 \circ \tilde{f}^{\tau(z)}(\tilde{z}) - p_1(\tilde{z}), \quad z \in U \cap \text{Rec}^+(f).
\]

Then,

\[
m(z) > \frac{1}{2},
\]

because the positive orbit of \( \tilde{z} \) by \( \tilde{f} \) is on the right of \( \tilde{\Gamma} \). Let

\[
\tau_n(z) = \sum_{i=0}^{n-1} \tau(\Phi^i(z)), \quad \text{and } m_n(z) = \sum_{i=0}^{n-1} m(\Phi^i(z)).
\]

Then,

\[
\frac{m_n(z)}{\tau_n(z)} \rightarrow \rho(\tilde{f}, z) \quad \text{as } n \rightarrow +\infty,
\]

if the rotation number exists. Note that

\[
\frac{m_n(z)}{\tau_n(z)} = \frac{m_n(z)}{n} \frac{n}{\tau_n(z)} \quad \frac{m_n(z)}{n} > \frac{1}{2}.
\]

By Kac’s Lemma (see [26] and [17]), \( \tau \in L^1(U, \mathbb{R}) \). Then, \( \frac{m_n}{n} \) is convergent a.e. by Birkhoff’s ergodic theorem. Therefore,

\[
\rho(\tilde{f}, z) > 0 \quad \text{for a.e. } z \in U. \quad \Box
\]

**Lemma 15.** Let \( f \) be a homeomorphism of the closed or open annulus, that is isotopic to the identity and preserves a finite area. If \( f \) does not have any fixed point, and have periodic orbits \( O_1, O_2, \cdots, O_s \) with prime-periods \( n_1, n_2, \cdots, n_s \) such that \( \gcd(n_1, n_2, \cdots, n_s) = 1 \), then there exist two positive recurrent points with different rotation numbers.

**Proof.** If there exist \( z_i \in O_i, z_j \in O_j \) such that \( \rho(\tilde{f}, z_i) \neq \rho(\tilde{f}, z_j) \), we finish the proof. Now, we suppose that \( \rho(\tilde{f}, z) = \frac{p}{q} \) for all \( z \in O_i, i = 1, 2, \cdots, s \), where \( \gcd(p, q) = 1 \). Since the prime-period of \( O_i \) is \( n_i \), we have \( q|n_i \), for \( i = 1, 2, \cdots, s \). So, \( q = 1 \), and hence \( \rho(\tilde{f}, z) \) is an integer. By composing
a covering transformation of the universal covering space to the lift of \( f \) if necessary, we can suppose that \( \rho(\tilde{f}, z) = 0 \). Now, we will prove the existence of a non-zero rotation number.

We consider the homeomorphism of the open annulus. When \( A \) is closed, we consider the restriction of \( f \) to the interior of \( A \). Let \( I \) be an identity isotopy of \( f \) such that the lift of \( f \) associated with \( I \) to the universal covering space is equal to \( \tilde{f} \). The homeomorphism \( f \) does not have any fixed point. So, \( I \) does not have any fixed point and is a maximal isotopy. We consider the transverse foliation \( \mathcal{F} \) of \( I \). All the leaves of \( \mathcal{F} \) are topological lines from one end of the annulus to the other end. We get the result by Lemma 14.

**Lemma 16.** If the periods of all the periodic orbits of \( f \) has a greatest common divisor \( k \geq 2 \), then an \( n \) prime-periodic point of \( f^k \) is a \( kn \) prime-periodic point of \( f \).

**Proof.** If \( z \) is an \( n \) prime-periodic point of \( f^k \), then \( f^{kn}(z) = z \) and \( f^{kn}(z) \neq z \) for every positive integer \( m < n \). So, \( z \) is a periodic point of \( f \), and the prime-period \( n|kn \). We will prove \( n' = kn \).

We first prove \( \gcd(kn, n) = 1 \). In fact, if \( \frac{kn}{n} \) and \( n \) have a common divisor \( t \), then \( kn = \frac{kn}{t}n' \). So, the prime-period of \( z \) for \( f^k \) is a divisor of \( \frac{n}{t} \). Hence, \( t = 1 \).

Now, we suppose \( n' = sn \), where \( s \) is a positive integer. For all prime \( p|k \),

- if \( p|n \), then \( p \mid \frac{kn}{n'} = \frac{k}{s} \).
- if \( p \nmid n \), then \( p|s \) (because \( k|n' \)). Moreover, if \( p\nmid k \), then \( p|s \). So, \( p \nmid \frac{k}{s} \).

Therefore, \( \frac{k}{s} = 1 \), and \( n' = n \).

**Proof of Theorem 12.** Suppose that \( f \) is a homeomorphism of the closed or open annulus, that is isotopic to the identity, preserves a finite area, and is of infinite order. We will prove the theorem in the three cases one by one.

i) We suppose that \( f \) has a fixed point.

By Lemma 13 if \( f \) has positive recurrent points with different rotation numbers, we finish the proof. Now, we suppose that the rotation number \( \rho(\tilde{f}, z) \) is a constant, where \( \tilde{f} \) is a lift of \( f \) and \( z \) is any positive recurrent point of \( f \). Moreover, for a fixed point \( z \) of \( f \), the rotation number \( \rho(\tilde{f}, z) \) is an integer. By composing a covering transformation of the universal covering space to the lift of \( f \) if necessary, we can suppose that \( \rho(\tilde{f}, z) = 0 \) for every positive recurrent point \( z \).

If \( A \) is a closed annulus, we shrink each boundary to a point and get a sphere; while \( A \) is an open annulus, we consider the end compactifications and also get a sphere. Denote the sphere by \( S^2 \), and the two points not in the interior of \( A \) by \( N \) and \( S \). Moreover, \( f \) induces an area preserving homeomorphism of \( S^2 \) that fixes both \( N \) and \( S \). We still denote by \( f \) the induced homeomorphism of \( S^2 \).
By Theorem\[9\] there is a maximal isotopy \( I = (f_t)_{t \in [0,1]} \) that fixes both \( N \) and \( S \), and the lift of \( f|_{S^2 \setminus \{N,S\}} \) associated with \( I \) is \( \tilde{f} \). If \( I \) have only two fixed points \( N \) and \( S \), we consider the transverse foliation \( F \) of \( I \). The leaves of \( F \) join the two ends of the annulus. We get the existence of non-zero rotation numbers by Lemma\[14\] which contradicts with our assumption that \( \rho(\tilde{f},z) = 0 \) for all positive recurrent points \( z \). Therefore, \( I \) has at least 3 fixed points.

Now, we choose a leaf \( \Gamma \) of the transverse foliation \( F \) of \( I \), and will find an invariant topological annulus of \( f \) such that \( \Gamma \) joins the two ends of the annulus. We first choose the connected component \( U \) of \( S^2 \setminus \text{Fix}(I) \) containing \( \Gamma \). Note that the \( \alpha \)-limit set \( \alpha(\Gamma) \) and the \( \omega \)-limit set \( \omega(\Gamma) \) are connected compact subsets of the set of the singular points \( \text{Sing}(F) = \text{Fix}(I) \subset S^2 \setminus U \) of \( F \) respectively. We fill \( U \) as follows: Let \( K_1 \) and \( K_2 \) be the connected components of \( S^2 \setminus U \) that contain \( \alpha(\Gamma) \) and \( \omega(\Gamma) \) respectively. Then, \( A' = S^2 \setminus (K_1 \cup K_2) \) is an open set containing \( U \), and is homeomorphic to a disk (if \( K_1 = K_2 \)) or an annulus (if \( K_1 \neq K_2 \)). We consider the induced homeomorphism, the induced maximal isotopy, and the induced transverse foliation on \( A' \), and still denote them by \( f, I, F \) respectively.

**Lemma 17.** \( K_1 \neq K_2 \).

**Proof.** We will prove the lemma by contradiction. Suppose that \( K_1 = K_2 \). Then, \( A' = S^2 \setminus (K_1 \cup K_2) \) is a topological disk. By considering the one point compactification of \( A' \), we get a sphere \( A' \cup \{\ast\} \), the induced area-preserving homeomorphism \( f \), the induced maximal isotopy \( I \), and the induced transverse foliation \( F \) on the sphere \( A' \cup \{\ast\} \). But the leaf \( \Gamma \) is from \( \ast \) to \( \ast \), which contradicts with the area preserving condition. \( \square \)

So, \( A' = S^2 \setminus (K_1 \cup K_2) \) is a topological open annulus and \( \Gamma \) is a leaf joining two ends of \( A' \). Let \( \tilde{f} \) be the lift of \( f \) to the universal covering space of \( A' \) associated with \( I \). By Lemma\[14\] there is a positive recurrent point \( z \) with \( \rho(\tilde{f},z) \neq 0 \). If \( I \) has a fixed point \( z' \in A' \), then \( \rho(\tilde{f},z') = 0 \). We get different rotation numbers, and can finish the proof by Lemma\[13\].

Now, we suppose that \( I \) does not have any fixed point in \( A' \). Since the original maximal isotopy on the sphere \( S^2 \) has at least 3 fixed points, either \( K_1 \) or \( K_2 \) contains at least two fixed points of \( I \). We suppose that \( K_1 \) contains at least two fixed points of \( I \), which are also fixed points of \( f \). We consider the prime ends compactification of \( S^2 \setminus K_1 \), and extend \( f \) to the closed disk \( (S^2 \setminus K_1) \sqcup S^1 \). By Theorem\[10\] the prime ends rotation number of \( f \) on the boundary is equal to \( 0 \in \mathbb{R}/\mathbb{Z} \). Indeed, we get an annulus \( A' \sqcup S^1 \) with one boundary and the induced homeomorphism (still denoted by \( f \)), such
that $\rho(\hat{f}, z)$ is equal to an integer, for all $z \in S^1$. Recall that the boundary of $A' (\subset S^2)$ is a subset of $\text{Fix}(I)$, where $I = (f_t)_{t \in [0, 1]}$ and $f_0 = \text{Id}$. We can also extend $f_t|_{S^2 \setminus K_1}$ to $S^1$, and consider the prime ends rotation number of $f_t$ on the boundary of $S^2 \setminus K_1$. We get the induced $I = (f_t)_{t \in [0, 1]}$ on $A' \sqcup S^1$. Moreover, the rotation number $\rho(\hat{f}, z)$ is equal to an integer, for all $z \in S^1$. Because $\rho(\hat{f}, z)$ is continuous for $t$, and $\hat{f}_0 = \text{Id}$, we have $\rho(\hat{f}_t, z) = 0$ for $z \in S^1$ and $t \in [0, 1]$. In particular, $\rho(\hat{f}, z) = 0$ for $z \in S^1$. By Lemma 14, there is a positive recurrent point $z'$ with $\rho(\hat{f}, z') \neq 0$. We get different rotation numbers, and can finish the proof by Lemma 13.

ii) Suppose that $f$ does not have any fixed point, and have periodic orbits with prime-periods $n_1, n_2, \cdots, n_s$ such that $\gcd(n_1, n_2, \cdots, n_s) = 1$. In this case, the theorem is a corollary of Lemma 13 and Lemma 15.

iii) Suppose that the periods of all the periodic orbits of $f$ have a greatest common divisor $k \geq 2$. Let $g = f^k$. Then, $g$ is a homeomorphism of the closed or open annulus, that is isotopic to the identity, preserves a finite area, and is of infinite order.

If $g$ has a fix point, the result is a corollary of case i) and Lemma 16. If $g$ does not have any fixed point, then it has periodic orbits with different prime-periods and the greatest common divisor of the periods of all the periodic orbits of $g$ is equal to 1. So, $g$ satisfies the condition of case ii). We get the result as a corollary of case ii) and Lemma 16.

□

Proof of Theorem 1. If $f$ is of finite order, there exists a positive integer $m$ such that $f^m = \text{Id}$. Moreover, $f$ is conjugate to a rational rotation of the annulus (see [9, 16, 13]). By choosing the minimal $m$, we can suppose that every point is an $m$ prime-period point. Then, $N_{\leq n} = +\infty$ for all $n \geq m$. If $f$ has a fixed or periodic point with prime-periodic $k$ such that $(k, n_0) = 1$, then $k = m$, and hence $(m, n_0) = 1$, $N_{\leq n, n_0} = +\infty$ for all $n \geq m$. The third condition, that $f$ has a $k$ prime-periodic point and $f^k \neq \text{Id}$ cannot happen. We have nothing to prove.

Now, we suppose that $f$ is of infinite order and has a fixed or periodic point. Then, one of the three cases in Theorem 12 happens. We will prove the theorem in the three cases one by one.

i) Suppose that $f$ has a fixed point (the case i) of Theorem 12). We have in fact proved the existence of two different rotation numbers for the homeomorphism in the annulus or in a modified annulus in the proof of the case i) of Theorem 12. Then, we get the results by Lemma 13.

ii) Suppose that $f$ does not have any fixed point, and has periodic orbits with prime-periods $n_1, n_2, \cdots, n_s$ such that $\gcd(n_1, n_2, \cdots, n_s) = 1$ (the case ii) of Theorem 12). We proved the existence of two
different rotation numbers in Lemma 15. Then, we get the results by Lemma 13.

iii) Suppose that the periods of all the periodic orbits have a greatest common divisor $k' \geq 2$ (the case iii) of Theorem 12). We have in fact proved $N = k' n \geq \varphi(n; \rho^-, \rho^+, \rho^- \rho^+)$ for some $\rho^- \rho^+$ in the proof of the case iii) of Theorem 12. Therefore,

$$N \leq n = \sum_{j=1}^{\lfloor \frac{n}{k'} \rfloor} N = k' j \geq \Phi(\lfloor \frac{n}{k'} \rfloor; \rho^-, \rho^+),$$

where $\lfloor \frac{n}{k'} \rfloor$ is the maximal integer not exceeding $\frac{n}{k'}$. We get the first result by Lemma 6.

Moreover, if $f$ has a periodic orbit with the prime-period $k$ with $(k, n_0) = 1$, then $(k', n_0) = 1$. So, $(k' j, n_0) = 1 \iff (j, n_0) = 1$.

$$N \leq n, n_0 \nmid = \sum_{\gcd(j, n_0) = 1} N = k' j \geq \Psi(\lfloor \frac{n}{k'} \rfloor; \rho^-, \rho^+).$$

We get the second result by Lemma 7.

The third result is obviously, because $k' | k$. □

4. Non-contractible closed geodesics on Riemannian or Finsler $\mathbb{R}P^2$

In this section, we will use Theorem 1 to study the multiplicity and also the growth rate of the number of non-contractible closed geodesics on Riemannian or Finsler $\mathbb{R}P^2$. Firstly, we review some background on this topic.

A closed curve on a Finsler manifold is a closed geodesic if it is locally the shortest path connecting any two nearby points on this curve. As usual, on any Finsler manifold $(M, F)$, a closed geodesic $c : S^1 = \mathbb{R}/\mathbb{Z} \to M$ is prime if it is not a multiple covering (i.e., iteration) of any other closed geodesics. Here, the $m$-th iteration of $c$ is defined by $c^m(t) = c(mt)$. The inverse curve $c^{-1}$ of $c$ is defined by $c^{-1}(t) = c(1 - t)$ for $t \in \mathbb{R}$. Note that unlike Riemannian manifold, the inverse curve $c^{-1}$ of a closed geodesic $c$ on an irreversible Finsler manifold need not be a geodesic. We call two prime closed geodesics $c$ and $d$ distinct if there is no $\theta \in (0, 1)$ such that $c(t) = d(t + \theta)$ for all $t \in \mathbb{R}$. For a closed geodesic $c$ on $(M, F)$, we denote by $P_c$ the linearized Poincaré map of $c$. Recall that a Finsler metric $F$ is bumpy if all the closed geodesics on $(M, F)$ are non-degenerate, i.e., $1 \notin \sigma(P_c)$ for any closed geodesic $c$. Following Rademacher in [40], we define the reversibility $\lambda = \lambda(M, F)$ of a compact Finsler manifold $(M, F)$ to be

$$\lambda := \max \{F(-X) \mid X \in TM, F(X) = 1\} \geq 1.$$

There is a famous conjecture in Riemannian geometry which claims that there exist infinitely many closed geodesics on any compact Riemannian manifold. This conjecture has been proved except for CROSS’s (compact
rank one symmetric spaces); cf. [20] and [45]. The results of Franks [18] in 1992 and Bangert [2] in 1993 imply that this conjecture is true for any Riemannian 2-sphere (cf. also Hingston [23]).

But once one moves to the Finsler case, the conjecture becomes false. In 1973, Katok [28] endowed some irreversible Finsler metrics to the compact rank one symmetric spaces $S^n$, $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$ and $\mathbb{C}aP^2$, such that each of the spaces possesses only finitely many distinct prime closed geodesics (cf. also Ziller [49]). In particular, the number of closed geodesics on $S^n$ and $\mathbb{R}P^n$ with Katok’s metrics is equal to $2^{\frac{n+1}{2}}$.

In 2004, Bangert and Long [4] (published in 2010) proved the existence of at least two distinct closed geodesics on every Finsler $S^2$. Since then, there are many interesting results about the multiplicity of closed geodesics on Finsler spheres and compact simply-connected Finsler manifolds; cf. [14] and the references therein.

As for the multiplicity of closed geodesics on non-simply connected manifolds whose free loop space possesses bounded Betti number sequence, Ballman et al. [1] proved in 1981 that every Riemannian manifold, with the fundamental group being a nontrivial finitely cyclic group and possessing a generic metric, has infinitely many distinct closed geodesics. In 1984, Bangert and Hingston [3] proved that any Riemannian manifold, with the fundamental group being an infinite cyclic group, has infinitely many distinct closed geodesics. Since then, there seem to be very few works on the multiplicity of closed geodesics on non-simply connected manifolds. The main reason is that the topological structures of the free loop spaces on these manifolds are not well known, so that the classical Morse theory is hardly applicable.

Recently, there are some studies on the multiplicity of non-contractible closed geodesics on Finsler $\mathbb{R}P^n$, in particular, Xiao and Long [48] in 2015 investigated the topological structure of the non-contractible loop space and established the resonance identity for the non-contractible closed geodesics on $\mathbb{R}P^{2n+1}$ by use of $\mathbb{Z}_2$ coefficient homology. As an application, Duan, Long and Xiao [15] proved the existence of at least two distinct non-contractible closed geodesics on $\mathbb{R}P^3$ endowed with a bumpy and irreversible Finsler metric. Subsequently in [44], Taimanov used a quite different method from [48] to compute the rational equivariant cohomology of the non-contractible loop spaces in compact space forms $S^n/\Gamma$, and proved the existence of at least two distinct non-contractible closed geodesics on $\mathbb{R}P^2$ endowed with a bumpy irreversible Finsler metric, where $\Gamma$ is a finite group which acts freely and isometrically on the $S^n$. Note that the only non-trivial group which acts freely on $S^{2n}$ is $\mathbb{Z}_2$ and that $S^{2n}/\mathbb{Z}_2 = \mathbb{R}P^{2n}$ (cf. P.5 of [44]). Motivated by Taimanov [44], Liu [34] proved that there exist at least $2^{\left\lceil \frac{n+1}{2} \right\rceil}$ prime non-contractible closed geodesics on every bumpy and irreversible Finsler $(\mathbb{R}P^n, F)$ with reversibility $\lambda$ and flag curvature $K$ satisfying
\[
\frac{64}{25} \left( \frac{1}{1+\lambda} \right)^2 < K \leq 1.
\]
In [36], Liu and Xiao established the resonance identity for the non-contractible closed geodesics on \( \mathbb{R}P^n \), and together with [15] and [44] proved the existence of at least two distinct non-contractible closed geodesics on every bumpy \( \mathbb{R}P^n \) with \( n \geq 2 \). Recently, Rademacher and Taimanov [41] studied the multiplicity of non-contractible closed geodesics on compact Riemannian or Finsler manifold with infinite fundamental group.

On the other hand, Liu, Wang and Yan [35] proved some refinements of Franks’ theorem and gave some applications on the two or infinity results about the closed Reeb orbits with symmetries in contact geometry:

**Lemma 18.** [35, Theorem 11] Let \((S^3, \lambda, \xi_0)\) be a dynamically convex tight three-sphere satisfying \( g_{p,1}^* \lambda = \lambda \) for some \( p \geq 1 \), where \( \xi_0 \) is the standard contact structure and \( g_{p,1} : S^3 \to S^3 \) is defined by

\[
g_{p,1}(z_1 = x_1 + iy_1, z_2 = x_2 + iy_2) = (e^{2\pi i/p} z_1, e^{2\pi i/p} z_2),
\]
via the identification \( \mathbb{C}^2 = \mathbb{R}^4 \). Then there exist two or infinitely many \( g_{p,1} \)-symmetric periodic orbits on \( S^3 \).

This result has a counterpart on the closed geodesic problem, i.e., there exist either two or infinitely many closed geodesics on Finsler \( S^2 \) with \( K \geq 1 \) for which every geodesic loop is longer than \( \pi \) (cf. [22]). For every Riemannian \( S^2 \), as mentioned above, Franks [18] and Bangert [2] proved that there exist infinitely many closed geodesics, and Hingston [23] further proved that the number of closed geodesics of length \( \leq l \) grows at least like the prime numbers.

Motivated by the above results, we can use Theorem 1 to prove a growth rate about the non-contractible closed geodesics on Riemannian \( \mathbb{R}P^2 \) under some natural assumptions, i.e., Theorem 2.

We first give the idea of the proof of Theorem 2. We consider the universal 2-cover \((S^2, g)\) of \((\mathbb{R}P^2, g)\), and the geodesic flow on \( TS^2 \). We restrict the geodesic flow on the unit tangent bundle \( S_g S^2 \), and will get a global surface of section \( \Sigma \), which is homeomorphic to the closed annulus. Then, by composing the Poincaré half-return map with an involution \( h \), we get an area-preserving homeomorphism \( f \) of \( \Sigma \) that is isotopic to the identity, and can extend it continuously to the boundary such that \( f|_{\partial \Sigma} = \text{Id} \). Moreover, the odd periodic orbits of \( f \) corresponds to non-contractible closed geodesics of \((\mathbb{R}P^2, g)\). Then, we get the result by Theorem 1.

**Proof of Theorem 2.** We consider the universal 2-cover of \((\mathbb{R}P^2, g)\) and the induced Riemannian metric, i.e., \((S^2, g)\). Let \( h \) be the generator of \( \pi_1(\mathbb{R}P^2) = \mathbb{Z}_2 \). Then \( h \) induces an involutive orientation-reversing isometry on \((S^2, g)\), that is still denoted by \( h \) and acts on \((S^2, g)\) freely.

We first define the geodesic flow on \( TS^2 \). The tangent space \( TS^2 \) inherits a Riemannian metric as follows (see also [30, Definition 1.9.12]): for any \( X = (q, v) \in TS^2 \), we can split \( T_q TS^2 \) into the horizontal subspace \( T_{Xh} TS^2 \) and the vertical subspace \( T_{Xv} TS^2 \), both are canonically identified with \( T_q S^2 \),
then we define the Riemannian metric on \( T_XTS^2 \) by letting the horizontal space and the vertical space be orthogonal and taking on each of these spaces the Riemannian metric by the canonical identification. Let \( \omega \) be the standard symplectic form on the tangent bundle \( TS^2 \), that is defined by

\[
\omega(\xi, \eta) = g(\xi_h, \eta_v) - g(\eta_h, \xi_v),
\]

where \( \xi = (\xi_h, \xi_v) \) and \( \eta = (\eta_h, \eta_v) \) are the decompositions such that \( \xi_h, \eta_h \in T_qS^2 \cong T_{Xh}TS^2 \) and \( \xi_v, \eta_v \in T_qS^2 \cong T_{Xv}TS^2 \) (see [30, Proposition 3.1.14] for details). We consider the energy

\[
H : TS^2 \to \mathbb{R},
\]

\[
X \mapsto \frac{1}{2}g(X, X),
\]

and the Hamiltonian vector field \( \zeta_H \in TTS^2 \) that is defined by

\[
DH = \omega(\zeta_H, \cdot).
\]

Then its flow \( \phi^t : TS^2 \to TS^2 \) is called the geodesic flow on \( TS^2 \). The projection of the flow line \( \phi^t(X) \) for \( X \in TM \) to \( S^2 \) is the geodesic \( \gamma \) on \( S^2 \) determined by \( \dot{\gamma}(0) = X \) and \( \dot{\gamma}(t) = \phi^t(X) \) for all \( t \) (cf. [30, Proposition 3.1.13]).

Now, we will verify that \( h_*\zeta_H|X = \zeta_H|h_*X \) for every \( X \in TS^2 \), and hence \( h_* \) commutes with the geodesic flow \( \phi^t \) on \( TS^2 \), i.e.,

\[
h_* \circ \phi^t = \phi^t \circ h_*
\]

(3)

where \( h_* : TS^2 \to TS^2 \) is the pushforward of the isometry \( h : S^2 \to S^2 \), and \( h_* : TTS^2 \to TTS^2 \) is the pushforward of \( h_* \). Since \( h \) is an isometry on \( (S^2, g) \), we have that \( H(h_*X) = H(X) \) for any \( X = (q, v) \in T_qS^2 \). By differentiating it with respect to \( X \), we obtain

\[
DHh_*X(h_*\eta) = DHX(\eta), \quad \forall \eta \in T_XTS^2.
\]

Then, by (2) we obtain

\[
\omega(\zeta_H|h_*X, h_*\eta) = \omega(\zeta_H|X, \eta).
\]

On the other hand, since \( h_*(q, v) = (h(q), h_*v) \), we have

\[
h_*\eta = (h_*\eta_h, h_*\eta_v) \quad \forall \eta \in T_XTS^2.
\]

Recall that \( h \) is an isometry, by (1) we get

\[
\omega(h_*\zeta_H|X, h_*\eta) = \omega(\zeta_H|X, \eta).
\]

So,

\[
\omega(h_*\zeta_H|X, h_*\eta) = \omega(\zeta_H|h_*X, h_*\eta), \quad \forall \eta \in T_XTS^2,
\]

which implies \( h_*\zeta_H|X = \zeta_H|h_*X \).

It is not hard to prove that \( H(\phi^t(X)) \equiv H(X) \), for \( t \in \mathbb{R} \) (see [30, Proposition 3.1.16]). So, we can restrict the geodesic flow \( \phi^t \) to the unit tangent bundle \( S^2 \) of \( (S^2, g) \). We will give a global surface of section \( \Sigma \) for the restricted geodesic flow in the following paragraph.
We know that there is at least one non-contractible closed geodesic on $(\mathbb{R}P^2, g)$ (cf. [30] Theorem 2.4.19). Let $c$ be the non-contractible closed geodesic on $(\mathbb{R}P^2, g)$ which has minimal length among all the non-contractible curves on $(\mathbb{R}P^2, g)$. Then, the lift of $c$ to $(S^2, g)$ is a closed geodesic. We denote it by $\varsigma = c^2$. Moreover, $\varsigma$ is a simple closed geodesic on $(S^2, g)$ that separates $(S^2, g)$ into two disks. Otherwise, $\varsigma$ will have self-intersection and then the projection of $\varsigma$ onto $(\mathbb{R}P^2, g)$ will not be length-minimum among all the non-contractible curves on $(\mathbb{R}P^2, g)$. Suppose that $\varsigma$ is parametrized by arclength and of length $\delta$. Denote by $S^1_\delta = \mathbb{R}/\delta \mathbb{Z}$ the circle of length $\delta$. Then,

$$\Sigma := \{(q, v) \in S_y S^2 \mid q \in \varsigma(S^1_\delta), 0 \leq \angle(\varsigma|_q, v) \leq \pi \} \subset S^1 \times [0, \pi],$$

is an embedded surface in the unit tangent bundle $S_y S^2$ with boundary $\varsigma(S^1_\delta) \cup \varsigma(S^1_\delta)$, where $\angle(\varsigma|_q, v)$ is the oriented angle from $\varsigma|_q$ to $v$ and $\tau(t) := \varsigma(-t)$. The Gaussian curvature of $(S^2, g)$ is positive by the assumption that $(\mathbb{R}P^2, g)$ has positive Gaussian curvature. Then by Theorems 3.10.2 and 3.10.4 of [30], $\Sigma$ is a global surface of section. Furthermore,

$$h_*(\Sigma) = \{(q, v) \in S_y S^2 \mid q \in \varsigma(S^1_\delta), -\pi \leq \angle(\varsigma|_q, v) \leq 0 \},$$

and $\partial \Sigma$ is invariant under the involution $h_*$. We define a Poincaré half-return map $\psi : \Sigma \setminus \partial \Sigma \to h_*(\Sigma \setminus \partial \Sigma)$ by $\psi(x) = \phi^{\tau(x)}(x)$, where $\tau(x) := \min \{ t > 0 \mid \phi^t(x) \in h_*(\Sigma \setminus \partial \Sigma) \}$. Note that $\psi(q, v) = (q', v')$ iff the unit speed geodesic on $S^2$ starting at $q \in \varsigma$ with the initial tangent vector $v$ meets $\varsigma$ again (the first time) at $q' \in \varsigma$ with the tangent vector $v'$. By [30] Theorem 3.10.4, we know that any geodesic, starting from a point on $\varsigma$ in a non-tangent direction, meets $\varsigma$ again. So, the Poincaré half-return map $\psi$ is well defined.

**Claim.** The first conjugate point of $\varsigma(s_0)$ is $h(\varsigma(s_0)) = h_{\varsigma}(s_0)$ for any $s_0 \in S^1_\delta$.

In fact, by [30] Theorem 3.10.4, $\varsigma$ contains a conjugate point. On the other hand, $\varsigma = c^2$ and $c$ is shortest curve connecting $\varsigma(s_0)$ to $h(\varsigma(s_0))$ by our choice of $c$, then by [30] Theorem 1.12.13 (ii), $c$ has no conjugate point of multiplicity $> 0$ in its interior and then the claim follows.

We can extend $\psi$ continuously to the boundary of $\Sigma$: If $x = (q, v) = \varsigma(s_0)$, then $\psi(x)$ shall be the tangent vector to $\varsigma$ at the first conjugate point $h(\varsigma(s_0))$ of $\varsigma(s_0)$ along $\varsigma(s_0 + s)$, $s \geq 0$; if $x = (q, v) = -\varsigma(s_0)$, then $\psi(x)$ shall be the tangent vector to $\varsigma(s_0 - s)$, $s \geq 0$ at the first conjugate point of $\varsigma(s_0)$. (In fact, for any $\varepsilon > 0$ and sufficiently small $\theta > 0$, a geodesic $\gamma$ which starts from the point $\varsigma(s_0)$ with the initial vector $v$ such that $0 < |\angle(\varsigma(s_0), v)| < \theta$, meets $\varsigma$ again at a point $\varsigma(s)$ with $|s - \varsigma(s)| < \varepsilon$, where $\varsigma(s_1)$ is the first conjugate point of $\varsigma(s_0)$ along $\varsigma$ (cf. [30] Complement 2.1.13)). By the continuity of the flow $\phi^t$, we can prove that this holds uniformly for $s_0$.)

Now, we consider the homeomorphism $f = h_1^{-1} \circ \psi$ of the annulus $\Sigma$, with $f = \text{Id}$ on $\partial \Sigma$. Although $h_*$ is orientation reversing, $f = h_1^{-1} \circ \psi$ preserves the orientation and is isotopic to the identity. Similarly to the proof of [30]...
Theorem 3.10.2, we know that $f$ preserves the area induced by the restricted canonical symplectic form $\omega$. By applying Theorem 1, we obtain that $f$ has infinitely many odd periodic orbits in the interior of $\Sigma$ and that the number of these odd periodic orbits with prime-periods not exceeding $n$ grows at least like $n^2$.

In the following, we will prove that any odd periodic point $x$ of $f$ in the interior of $\Sigma$ corresponds to a non-contractible closed geodesic on $(\mathbb{R}P^2, g)$. In fact, let $f^{2k+1}(x) = x$. Previously, we only defined the Poincaré half-return map $\psi$ on $\Sigma \setminus \partial \Sigma$, but we can defined it on $h_*(\Sigma \setminus \partial \Sigma)$ similarly. Moreover, by (3), we know that $h^* \circ \psi = \psi \circ h_*$. Recall that $h^2_2 = \text{Id}$ and $f = h^{-1}_* \circ \psi$. Then $\psi^{2k+1}(x) = h_*(x)$, and $x$ corresponds to a non-contractible closed geodesic on $(\mathbb{R}P^2, g)$. The proof of the converse is the same.

Hence, there exist infinitely many distinct non-contractible closed geodesics on $(\mathbb{R}P^2, g)$. Moreover, the lengths of the geodesics, starting at a point $\varsigma(s)$ with initial vector not tangent to $\varsigma$ and ending at its first intersection point with $\varsigma$, are uniformly bounded from above (see [30, Proposition 3.10.3]) and from below ($\geq$ the injectivity radius of $M > 0$; cf. [30, Definition 2.1.9 and Proposition 2.1.10]). So, the number of non-contractible closed geodesics of length $\leq l$ grows at least like $l^2$. □

Motivated by the results of Franks [18], Bangert [2] and Hingston [23], it’s hopeful to obtain a growth rate of the number of non-contractible closed geodesics on Riemannian $\mathbb{R}P^2$ without the condition that the Gaussian curvature is positive. So, we tend to believe that the following conjecture should hold:

**Conjecture.** Let $\mathbb{R}P^2$ be a real projective plane endowed with a Riemannian metric $g$. Then there exist infinitely many distinct non-contractible closed geodesics on $(\mathbb{R}P^2, g)$. Moreover, the number of non-contractible closed geodesics of length $\leq l$ grows at least like the prime numbers.

For Finsler $\mathbb{R}P^2$ under some natural assumption, we can use Theorem 1 and the proof of Lemma 18 i.e., [35, Theorem 11], to obtain the two or infinity results, and also the growth rate about the non-contractible closed geodesics, i.e., Theorem 3.

Before proving Theorem 3 we also give the idea first. Note that $\mathbb{S}^2$ covers $\mathbb{R}P^2$ twice, and that the antipodal map $h$ of $\mathbb{S}^2$ is the deck transformation. Note also the unit tangent bundle $S_F \mathbb{S}^2 \cong SO(3)$ is double covered by $SU(2) \cong \mathbb{S}^3$. We will prove that $h$ induces a free action of $\mathbb{Z}_4$ on $\mathbb{S}^3$ denoted by $h_*$, and that the contact form on $S_F \mathbb{S}^2$ induces a contact form $\alpha$ on $\mathbb{S}^3$, such that the geodesic flow on $S_F \mathbb{S}^2$ corresponds to the Reeb flow on $(\mathbb{S}^3, \alpha)$. Moreover, the contact form $\alpha$ is dynamically convex and satisfies $(h_*)^* \alpha = \alpha$. Then, by the proof of Lemma 18 we get an area preserving annulus homeomorphism that is isotopic to the identity, whose odd periodic
orbits correspond to the non-contractible geodesics of $\mathbb{R}P^2$. We can finish the proof by applying Theorem [I].

**Proof of Theorem 3.** We consider the universal 2-cover of $(\mathbb{R}P^2, F)$ and the induced Finsler metric, i.e. $(S^2, F)$. The antipodal map $h$ of

$$S^2 = \{(q_1, q_2, q_3) \in \mathbb{R}^3 | q_1^2 + q_2^2 + q_3^2 = 1\}$$

is the deck transformation. The unit sphere $S^2$ is a submanifold of $\mathbb{R}^3$, and $T S^2$ is a submanifold of $T \mathbb{R}^3 \cong \mathbb{R}^3 \oplus \mathbb{R}^3$. More precisely,

$$T S^2 = \{(q, v) | q \in S^2, v \in T_q \mathbb{R}^3 \cong \mathbb{R}^3, \langle q, v \rangle = 0\},$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product. We can identify the unit tangent bundle $S_F S^2$ of $S^2$ under the Finsler metric with the 3 dimensional rotation group

$$SO(3) = \{[q, u, q \times u] | q, u \in \mathbb{R}^3, \langle q, q \rangle = 1, \langle u, u \rangle = 1, \langle q, u \rangle = 0\}$$

by

$$(q, v) \mapsto [q, \frac{v}{\sqrt{\langle v, v \rangle}}, q \times \frac{v}{\sqrt{\langle v, v \rangle}}].$$

So, the pushforward

$$h_* : T S^2 \to T S^2,$$

$$(q, v) \mapsto (-q, -v),$$

of $h$ induces the following diffeomorphism of $SO(3)$, which still denoted by $h_*$,

$$h_* : SO(3) \to SO(3),$$

$$[q, u, q \times u] \mapsto [-q, -u, q \times u].$$

By Lemma [II], the unit 3-dimensional sphere

$$S^3 = \{(z_1, z_2) : z_1, z_2 \in \mathbb{C}, |z_1|^2 + |z_2|^2 = 1\}$$

covers $SO(3)$ twice by the covering map

$$\pi : S^3 \to SO(3),$$

$$(z_1, z_2) \mapsto \begin{bmatrix} \text{Re}(z_1 \overline{z}_2 - \overline{z}_1 z_2) & -\text{Im}(z_1 \overline{z}_2 + \overline{z}_1 z_2) & 2\text{Re}(z_1 \overline{z}_2) \\ \text{Im}(z_1 \overline{z}_2 + \overline{z}_1 z_2) & \text{Re}(z_1 \overline{z}_2 - \overline{z}_1 z_2) & 2\text{Im}(z_1 \overline{z}_2) \\ -2\text{Re}(z_1 z_2) & 2\text{Im}(z_1 z_2) & |z_1|^2 - |z_2|^2 \end{bmatrix}.$$
Let \((x_1, y_1, x_2, y_2)\) be the coordinates in \(\mathbb{R}^4 \cong \mathbb{C}^2\). The restriction of the Liouville one form
\[
\alpha_0 = \frac{1}{2} \sum_{j=1}^{2} (y_j dx_j - x_j dy_j)
\]
on \(S^3\) is a contact form, which still denoted by \(\alpha_0\). Moreover, we have
\[
(\tilde{h}_*)^* \alpha_0 = \alpha_0.
\]

By [22, Section 4], \((S_F S^2, \alpha)\) induces a contact form on \(S^3\), which we still denote by \(\alpha\), such that
\[
\alpha = 2g\alpha_0,
\]
where \(g : S^3 \to \mathbb{R}^+\) satisfies \(g \circ \tilde{h}_* = g\). Moreover, the geodesic flow of \((S^2, F)\) (on \(S_F S^2\)) is smoothly conjugate (up to a double covering) to the Reeb flow of \((S^3, \alpha)\).

Now, we have the contact manifold \((S^3, \alpha)\) that satisfies
\[
(\tilde{h}_*)^* \alpha = \alpha.
\]

By Section 6 of [22] and the assumption that \(\left(\frac{1}{1 + \lambda}\right)^2 < K \leq 1\), the contact form \(\alpha\) is dynamically convex, which together with Lemma 18 implies that there exist two or infinitely many \(\tilde{h}_*-\)symmetric periodic orbits on \((S^3, \alpha)\), whose projections onto \(\mathbb{R}P^2\) are non-contractible closed geodesics.

In the proof of Lemma 18 (see [35, Theorem 11]), we in fact defined a Poincaré \(\frac{1}{p}\)-return map \(\psi\) (\(\frac{1}{4}\) for this theorem) of the Reeb flow and got an area preserving annulus homeomorphism \(f = \tilde{h}_*^{-1} \circ \psi\) isotopic to the identity. Moreover, the periodic orbits of \(f\) with odd prime-period correspond to the non-contractible closed geodesics of \(\mathbb{R}P^2\). If there are more than two non-contractible closed geodesics on \((\mathbb{R}P^2, F)\), \(f\) has at least one periodic orbit with odd prime-period. Then, by Theorem 1 the number of periodic orbits of \(f\) with odd prime-period \(\leq n\) grows at least like \(n^2\), and hence the number of non-contractible closed geodesics of length \(\leq l\) grows at least like \(l^2\).

\[\square\]

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