Dirac quantum walks with conserved angular momentum

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Abstract A quantum walk (QW) simulating the flat $(1 + 2)D$ Dirac equation on a spatial polar grid is constructed. Because fermions are represented by spinors, which do not constitute a representation of the rotation group $SO(3)$, but rather of its double cover $SU(2)$, the QW can only be defined globally on an extended spacetime where the polar angle extends from 0 to $4\pi$. The coupling of the QW with arbitrary electromagnetic fields is also presented. Finally, the cylindrical relativistic Landau levels of the Dirac equation are computed explicitly and simulated by the QW.

Keywords Quantum walks · Quantum simulation · Lattice field theory

1 Introduction

First proposed by Feynman as possible discretizations of Dirac path integrals [1,2], quantum walks (QWs) are unitary quantum automata that can be viewed as formal generalizations of classical random walks. Reintroduced later by Aharonov et al. [3], and then studied systematically by Meyer [4], QWs, like classical random walks in classical computing, have found application in quantum information and algorithmic development [5–7]. They can also be used as quantum simulators [8–16], where the lattice represents a discretization of continuous space, that could potentially represent a realistic discrete spacetime underlying the apparently continuous physical universe [17].
It has been shown that several discrete-time quantum walks defined on regular square lattices simulate the Dirac dynamics in various spacetime dimensions and that these Dirac quantum walks (DQWs) can be coupled to various discrete gauge fields [18–28]. Extensions to regular non-square lattices have also been proposed [29,30] and reference [31] proposes a systematic method to build DQWs on regular lattices of arbitrary dimensions.

More recently, a discrete action principle has been constructed for quantum automata [32]. In this context, the charge current of 1D DQWs has been recovered and a stress-energy ‘tensor’ for DQWs has been constructed. In particular, a ‘true’ Hamiltonian (as opposed to an effective Hamiltonian), and a linear momentum for 1D DQWs have been proposed and their conservation has been established for free 1D QWs. These results extended to QWs defined on higher dimensional square lattices. It is, however, not obvious that angular momenta can be built for QWs since, on a cubic lattice, rotation symmetry is broken down to the symmetry group of a cube, the cubic (or octahedral) group [33,34]. This question is not purely of fundamental, but also of practical interest, to simulate problems with axial symmetry. For example, the Landau levels of an electron in a constant uniform magnetic field are degenerated and the conserved angular momentum can be used to distinguish between states sharing the same energy. An experimental proposal based on magnetic discrete-time quantum walks has been made to construct anomalous Floquet–Chern topological insulators that exhibit edge charge currents similar to those observed in the quantum Hall effect [35].

The aim of this article is twofold: first, to construct DQWs which admit a conserved angular momentum; second, to use these DQWs to simulate the Landau levels. Here is how we proceed.

In continuum physics, linear momenta are conjugate to Cartesian coordinates and angular momenta are conjugate to angular coordinates. Linear momenta are conserved when there is a translation invariance and angular momenta are conserved when there is a rotational invariance. In the discrete realm, translation invariance can only occur on square Cartesian grids and rotational invariance can only occur on polar grids. Unsurprisingly, the DQWs with conserved linear momentum that have been proposed in the literature are built on Cartesian grids. To find DQWs with conserved angular momentum, one should, therefore, search for walks defined on polar grids.

In Sect. 2, we present a family of DQWs defined on a polar grid and coupled to arbitrary electromagnetic fields. Since spinors do not constitute a representation of the rotation group SO(3), but rather of its double cover SU(2), they are multiplied by a $-1$ factor upon rotation by an angle $2\pi$. Thus, for continuity reasons, the grid on which the walks are defined is actually an extended polar grid with polar angle ranging from 0 to $4\pi$. This ensures that the walks admit a continuum limit and that this limit coincides with the polar Dirac equation (PDE).

Section 3 deals with angular momentum. The angular momentum of the walks is defined in discrete angular Fourier space. It is conserved for walks exhibiting axial symmetry and coincides, at the continuum limit, with the angular momentum of the Dirac field.

The simulation of the Landau levels is presented in Sect. 4. All results are summarized and discussed in Sect. 5. Three appendices review basic facts about the Dirac equation and puts everything in a form which makes easier the comparison with the discrete results presented in the main body of the article. The first appendix presents the PDE which is simulated by the walks. The second appendix deals with the angular momentum of the Dirac field, and the third appendix is devoted to the relativistic Landau levels.

2 A polar Dirac quantum walk

The starting point is the general construction presented in [23], which delivers DQWs approximating Dirac equation in a possibly curved $(1+2)$D spacetime. By particularizing to flat Minkowski spacetime with 2D polar coordinates, we obtain in Sect. 2.1 a polar Dirac quantum walk (PDQW) which simulates the PDE, albeit without electromagnetic field. This field will be added in Sect. 2.2 as the latest step in the construction of the PDQW.

2.1 Without electromagnetic field

The PDQW is defined on a polar grid in $(1+2)$D spacetime. It depends on three real parameters $\epsilon$, $r_*$ and $m$, and on four position-dependent angles $\alpha^{ab}, (a, b) \in \{1, 2\}$.
The first parameter $\epsilon$ fixes the discrete step of the regular grid in all three space-time directions (time, radius and polar angle). This parameter is not necessarily infinitesimal. The second parameter $r_s$ defines a no-go zone around the origin, where polar coordinates are undefined. This no go zone is defined in physical space independently of the value of $\epsilon$, which helps avoiding possible complications at the continuum limit. And the final parameter is a mass.

The space-time grid is defined precisely as follows. The discrete values of the time coordinate are $t_j = j \times \delta t$ with $\delta t = 2\epsilon$ and $j \in \mathbb{N}$. The discrete values of the radial coordinate are $r_p = p \times \delta r$ with $\delta r = \epsilon$, where $p \in \mathbb{N}^*$, $p \geq \lfloor r_s / \epsilon \rfloor$, where $\lfloor \cdot \rfloor$ indicates the floor of a real number. The discrete values of the angular coordinate are $\theta_q = q \times \delta \theta$ with $\delta \theta = (\epsilon / r_s)$, where $q \in I_\epsilon$, where $I_\epsilon = \{0, 1, \ldots, q_{\max}\}$ with $q_{\max} = [4\pi r_s / \epsilon]$. Note that $\epsilon$, being the step in $r$, has the dimension of a length. The step in $t$ is $2\epsilon$, which has the same dimension because we have set the light velocity $c$ equal to unity. The angular step is $\epsilon / r_s$, which is dimensionless as it should.

The quantum walker is a two-component wavefunction $\Phi = (\varphi^-, \varphi^+)^\top$ defined on the space-time grid. The walk is advanced in time through a unitary operator $\hat{V}$ and stops if it reaches the boundary of the domain at $r = r_s$. At all point $j$, $p$, $q$, the equation of motion reads

$$\Phi_{j+1,p,q} = (\hat{V} \Phi)_{j,p,q},$$

(1)

where the operator $\hat{V}$ is of the form

$$\hat{V} = \hat{\Omega}^{-1} \left[ \hat{W}_1(\alpha \beta) \hat{W}_2(\alpha^2 \beta) \right] \hat{\Omega}$$

$$\times \left[ \hat{W}_2(\alpha \beta) \hat{W}_1(\alpha^2 \beta) \right] \hat{Q}(m\epsilon),$$

(2)

with

$$\hat{\Omega} = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 1 \\ -1 & i \end{pmatrix}$$

(3)

and

$$\hat{Q}(M) = \begin{pmatrix} \cos(2M) & \sin(2M) \\ -i \sin(2M) & \cos(2M) \end{pmatrix}.$$
and
\[ \hat{R}(\alpha_{ab}^p) \Phi_{j,p,q} = \begin{pmatrix} i \cos(\alpha_{ab}^p/2) & i \sin(\alpha_{ab}^p/2) \\ -\sin(\alpha_{ab}^p/2) & \cos(\alpha_{ab}^p/2) \end{pmatrix} \begin{pmatrix} \varphi_{-j,p,q}^- \\ \varphi_{+j,p,q}^+ \end{pmatrix}. \] (9)

The position-dependent angles are defined by
\[ \alpha_{11}^p = 0, \] (10)
\[ \alpha_{12}^p = \alpha_{21}^p = \frac{\pi}{2}, \] (11)
\[ \alpha_{22}^p = \arccos\left(\frac{r_s}{r_p}\right), \] (12)

where \( r_s \) is the third parameter of the walk.

2.2 With electromagnetic field

As is the case with other DQWs, an electromagnetic field can be inserted by multiplying the advancement operator \( \hat{V} \) at each point by an additional unitary operator \( U^{em} \). The method presented in [31] delivers
\[ U^{em} = e^{2i\epsilon A_t} \begin{pmatrix} e^{-2i\epsilon A_r} & 0 \\ 0 & e^{2i\epsilon A_r} \end{pmatrix} \begin{pmatrix} \cos(\frac{2}{\epsilon} \epsilon A_\theta) & \sin(\frac{2}{\epsilon} \epsilon A_\theta) \\ -\sin(\frac{2}{\epsilon} \epsilon A_\theta) & \cos(\frac{2}{\epsilon} \epsilon A_\theta) \end{pmatrix}. \] (13)

As the PDE, this walk only makes sense if the initial condition contains only half-integer Fourier modes. It can be checked by a direct computation that the walk then does not populate integer Fourier modes i.e. that the two polar components of the walk wave-function then remain at all time \( 2\pi \)-anti-periodic functions of the angle \( \theta \).

2.3 Continuum limit

To obtain the continuum limit, we take the same approach as in [18–23,29,31], where we interpret any discrete quantity \( f_{j,p,q} \) as the value taken by a continuous and differentiable function \( f \) at the polar spacetime coordinates of \( (t = 2j\delta t, r = p\delta r, \theta = q\delta \theta) \). The factor of two on the temporal steps was established as necessary in [23] to make the continuum match with the standard form of the (curved spacetime) Dirac Equation. The limit of \( \epsilon \to 0 \) is then determined by Taylor expanding to first order in \( \epsilon \). While the zeroth-order terms cancel each other out, the first-order coefficients deliver the equation
\[ \begin{pmatrix} i \sigma_1 (\partial_t - iA_t) - \sigma_2 (\partial_r - iA_r) \\ -\frac{1}{r} \sigma_3 (\partial_\theta - iA_\theta) - m \end{pmatrix} \Phi = 0, \] (14)

which transcribes into the PDE (see Appendix A) for the spinor \( \Psi(t, r, \theta) = \frac{1}{\sqrt{r}} \Phi(t, r, \theta) \).

3 Angular momentum

Let \( f \) be an arbitrary function \( f \) defined on the discrete spatial grid \( \{ r_p, \theta_q, (p, q) \in \mathbb{N}^* \times I_e \} \). Its discrete polar Fourier transform is defined by
\[ \hat{\tilde{f}}_{p,k} = \sum_{q \in I_e} f_{p,q} \exp ikq. \] (15)
Since \( q \) is bounded, its conjugate variable \( k \) is discrete, with step \( \Delta k = 2\pi/q_{\text{max}} \). Since \( q \) is discrete, the angular wave number \( k \) is bounded. A possible choice for the range of \( k \) is \( \{0, \pm 2\pi/q_{\text{max}}, \ldots, \pm \pi\} \).

As \( \epsilon \) tends to zero, \( \theta \sim q\epsilon/r_\star \), so \( \kappa \sim kr_\star/\epsilon \). Also, \( q_{\text{max}} \sim 4\pi r_\star/\epsilon \), so \( \Delta k = 2\pi/q_{\text{max}} \sim \epsilon/(2r_\star) \), so the step in \( \kappa \) is \( \Delta \kappa = 2\pi/q_{\text{max}} \). Since \( q \) is discrete, the angular wave number \( k \) is bounded. A possible choice for the range of \( k \) is \( \{0, \pm 2\pi/q_{\text{max}}, \ldots, \pm \pi\} \).

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Writing the evolution equation of the walk in Fourier space is tantamount to determining the expression of \( \hat{V} \) in Fourier space. Direct inspection reveals that the operators \( \hat{\Pi}, \hat{Q}, \hat{U} \) and \( \hat{R} \) are represented in Fourier space by the same matrix as the one representing them in physical space. Si is so for the operator \( \hat{U}_\text{em} \), provided the components of the potential \( A \) do not depend on the polar angle \( \theta \). As for the operator \( \hat{S}_1 \), it now appears as a spin-dependent translation in \( r \) at fixed \( k \), so it is also represented by the same matrix in physical and in angular Fourier space. But the matrix \( S_2(k) \) representing \( \hat{S}_2 \) in Fourier space reads

\[
S_2(k) = \begin{pmatrix}
\exp(ik\epsilon/r_\star) & 0 \\
0 & \exp(-ik\epsilon/r_\star)
\end{pmatrix}.
\]

Introducing \( \Lambda_{j,k} = \{\Phi_{j,p,k}, p \in \mathbb{N}^*\} \), the equation of the DQW in Fourier space takes the simple form

\[
\Lambda_{j+1,k} = \rho_k \Lambda_{j,k},
\]

where \( \rho_k \) is a \( k \)-dependent unitary operator. It follows from this that each polar Fourier component evolves independently of the others.

The angular momentum of the walk is represented by the operator \( \hat{K} \) defined by

\[
\left( \hat{K} \Phi \right)_{j,p,k} = k \Phi_{j,p,k}.
\]

At each time \( j \), its expectation value is

\[
\langle \hat{K} \rangle_j = \sum_{p,k} k |\Phi_{j,p,k}|^2 = \sum_k \sum_p |\Phi_{j,p,k}|^2.
\]

Since the operator \( \hat{\rho}_k \) is unitary, the sum over \( p \) in the above equation is independent of \( j \). One thus finds

\[
\langle \hat{K} \rangle_j = \sum_k \sum_p |\Phi_{0,p,k}|^2 = \sum_{p,k} |\Phi_{0,p,k}|^2 = \langle \hat{K} \rangle_0.
\]

The walk thus conserves the expectation value of \( \hat{K} \).

At the continuum limit, the operator \( \hat{K} \) coincides with \( \epsilon \hat{J} \), where \( \hat{J} \) is the angular momentum of the Dirac field (see Appendix B).

4 Quantum simulation of relativistic Landau levels

Suppose one wants to use the walks introduced above to simulate the dynamics of a spinor in a constant and uniform magnetic field \( B = Be_z \), where \( e_z \) is the unit vector in the \( z \) direction. Since the walks depend on the vector potential
A, the first step is to choose a gauge. We retain the so-called symmetric gauge, where \(A_t = A_r = 0\) and \(A_\theta = Br/2\). In this gauge, all components of \(A\) are independent of the polar angle \(\theta\), so the angular momentum of the walk is conserved.

The second step consists in choosing an initial condition. To check that the walks can indeed simulate standard continuous physics, we choose as initial condition for the walk a discretization of a stationary solution of the Dirac equation, advance this discretization by one step of the quantum walk and check that the distance between the obtained wave-function and the initial condition tends to zero with the grid step \(\epsilon\).

The stationary solutions of the Dirac equation in the symmetric gauge are simultaneous eigenstates of the Dirac Hamiltonian and of the angular momentum. The exact expression of these eigenstates is detailed in Appendix C. Discretizing the stationary solution of the Dirac equation with energy \(E\) and angular momentum \(\kappa\) leads, for the walk, to an initial condition of the form

\[
\Phi_{E,\kappa}(r_p, \theta_q) = \Xi_{E,\kappa}(r_p) \exp(-i\kappa \theta_q)
\]

for all integers \(p > \lfloor r_*/\epsilon \rfloor + 1\) and \(q \in I_\epsilon\).

These functions are clearly eigenstates of the angular momentum operator of the walk. After one time-step of length \(\epsilon\), the discrete dynamics of the walk transforms these functions into different functions, say \(\Phi_{E,\kappa}^{1}\), while advancing the stationary solution the continuous Dirac dynamics simply adds a multiplicative factor \(\exp(-iEt)\). The difference between the discrete dynamics of the quantum walk and the continuous Dirac dynamics can be evaluated by the \(L^1\) norm (on the grid) of the difference \(\Phi_{E,\kappa}^{1} - \exp(-iE\epsilon)\Phi_{E,\kappa}\). Figure 1 shows the typical evolution of this norm with the parameter \(\epsilon\). Note that this figure is actually a log–log plot, as is customary in such cases, and the point where the axes cross does not correspond to \(\epsilon = 0\). Figure 1, plotted for a charge \(q = -1\), confirms that that the ‘error’ in simulating the continuous Dirac dynamics by the discrete dynamics of the walk tends to zero with the discretization parameter \(\epsilon\).
5 Conclusion

We have presented a new DQW which can simulate the (1 + 2)D flat-spacetime Dirac equation on a spatial polar grid. Thanks to the polar grid, we have identified a quantity which we define as the angular momentum of the DQW since it is conserved under rotations when the system has rotational symmetry (e.g., for a free DQW, but also for a DQW with electromagnetic potential if the latter has rotational symmetry). Because fermions are described by spinors, the PDQW can only be defined globally on an extended spacetime grid. We have also shown how the PDQW can be coupled to arbitrary electromagnetic fields and we have demonstrated that the PDQW can simulate relativistic Landau levels.

Two aspects of this work are particularly worth commenting upon. First, the use of the polar DQWs introduced in this article is not limited to simulating the cylindrical Landau levels. Indeed, these walks are of a quite general interest for quantum simulation because they can be used with profit in all problems where a 2D Dirac particle is immersed in a field with polar symmetry like e.g. a central electrical or gravitational fields, or in field with axial symmetry like e.g. a non-necessarily uniform magnetic field of fixed direction orthogonal to the plane of study.

The second aspect we would like to discuss is the ‘quantization’ of angular momentum for DQWs. As defined in this article, the angular momentum for DQWs essentially coincides with the expectation (average value) of the angular wave-number \( k \). The possible values of \( k \) are fixed by Fourier analysis and, as detailed in Section III, the step in \( k \)-space is \( 2\pi/q_{\text{max}} \) where \( q_{\text{max}} + 1 \) is the number of grid points in \( \theta \)-space. Thus, changing the discretization in \( \theta \)-space changes the ‘quantization’ of \( k \).

But there is another freedom in the definition of the angular momentum for polar DQWs. Indeed, one could define the angular momentum of these walks as the expectation of any function of \( k \) and \( q_{\text{max}} \) which is equivalent to \( k \) when \( q_{\text{max}} \) tends to infinity i.e. at the continuum limit (remember \( 1/q_{\text{max}} \) is proportional to the infinitesimal \( \epsilon \) controlling the limit). For example, the expectation of, say, \( F(k) = k + (k^2/q_{\text{max}}) \) is conserved if the problem has axial symmetry and is equivalent, at the continuum limit, to the expectation of \( k \). The definition of the angular momentum for DQWs retained in this article is the simplest and most natural one, but it is certainly not the only possible one.

Let us now conclude by mentioning a few possible extensions to this work. A first one would be to build (1 + 3)D DQWs on a spherical spatial grid and, more generally, on an elliptical spatial grid. The global and local discrete \( U(1) \) gauge invariance associated with electromagnetism and charge conservation should also be investigated on such non-Cartesian grids. The same should be carried out for arbitrary Yang–Mills fields and for gravitational fields as well. Finally, some of the material developed in [32] for DQWs on (1 + 1)D Cartesian grids only should be extended to more general grids. For example, can one introduce on general grids an action principle which involves spacetime coordinates and delivers the stress-energy momentum of the walk?

Data Availability Statement Data will be made available on reasonable request.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Appendix A: The polar Dirac equation

In (1 + 2)D flat spacetime, the Cartesian Dirac equation (CDE) can be written

\[
\mathcal{D}_A^+ \Psi^B = 0.
\]
with the operator $\mathcal{D}$ defined (in natural units $\hbar = c = 1$) by

$$\mathcal{D} = i(\gamma^0 \partial_t + \gamma^1 \partial_x + \gamma^2 \partial_y) - m,$$

(A2)

where $(t, x, y)$ are Minkowski coordinates and $m$ is the mass of the particle. The indices $(A, B) \in \{L, R\}^2$ refer to components on a Cartesian, point-independent spin basis which we denote by $(b_L, b_R)$. We choose a representation where, in this basis, the $\gamma$ operators read

$$[(\gamma^0)_B^A] = \sigma_1, \quad [(\gamma^1)_B^A] = i\sigma_2, \quad [(\gamma^2)_B^A] = i\sigma_3.$$

(A3)

where the $\sigma$’s are the Pauli matrices and the notation $[(\gamma^i)_B^A]$ represents the matrix formed by the components of the operator $\gamma^i$ in the basis $(b_L, b_R)$.

To obtain the PDE from the CDE, one must first use polar coordinates $(r, \theta)$ instead of Cartesian coordinates, and then also change the spin basis basis, replacing the Cartesian spin basis by the polar spin basis $(b_-, b_+)$ obtained from $(b_L, b_R)$ by performing a rotation of angle $\theta$ in spin-space. According to general spinor theory [36], this rotation reads:

$$b_- = \cos \frac{\theta}{2} b_L - i \sin \frac{\theta}{2} b_R,$$

(A4)

$$b_+ = -i \sin \frac{\theta}{2} b_L + \cos \frac{\theta}{2} b_R.$$

(A5)

Performing both operations leads to

$$\mathcal{D}_b^a \Psi^b = 0,$$

(A6)

where

$$\mathcal{D}_b^a = i(\gamma^1)_b^a \partial_t + i(\gamma^2(\theta))_b^a \partial_r$$

$$+ \frac{i}{r} \left(\left(\gamma^3(\theta)\right)_b^a \partial_\theta + \frac{1}{2} \left(\gamma^2(\theta)\right)_b^a\right) - m$$

(A7)

and

$$[(\gamma^0)_B^A] = \sigma_1, \quad [(\gamma^1)_B^A] = i\sigma_2, \quad [(\gamma^2)_B^A] = i\sigma_3.$$

(A8)

Note that the operators $\gamma^i$ and $\tilde{\gamma}^i$ are represented by the same matrices, but in different bases.

As usual, the coupling of the Dirac fermion with an electromagnetic field with 3-potential $(A_\mu) = (A_t, A_r, A_\theta)$ is achieved by adding $+ieA_\mu$ to $\partial_\mu$ for $\mu = 0, 1, 2$. We choose to set the charge $e$ to $-1$ and introduce $\mathcal{D}_\mu = \partial_\mu - iA_\mu$, so the Dirac equation for $\Psi$ in polar coordinates and polar spin basis can be abbreviated into

$$\left(i\sigma_1 \mathcal{D}_t - \sigma_2 \mathcal{D}_r - \frac{1}{r} \left(\sigma_3 \mathcal{D}_\theta + \frac{1}{2} \sigma_2\right) - m\right)\Psi = 0,$$

(A9)

which we call the polar Dirac equation (PDE). We use this compact form in the article when no confusion with the CDE seems possible. Note that polar coordinates are not defined at $r = 0$; therefore, the PDE is non-singular over the definition domain of polar coordinates.

Let us conclude this section by pointing out a very important property of the PDE. The second polar coordinate $\theta$ is an angle. Thus, the components $\Psi^L$ and $\Psi^R$ of $\Psi$ in the Cartesian spin basis, when written as functions of $r$ and $\theta$, are $2\pi$-periodic functions of $\theta$. So are the time component $A_t$, the Cartesian components $A_x$, $A_y$ and the polar components $A_r$ and $A_\theta$ of the potential. The components $\Psi^-$ and $\Psi^+$ of $\Psi$ in the polar spin basis are linear
combinations of $\Psi^L$ and $\Psi^R$ with coefficients $\cos(\theta/2)$ and $\sin(\theta/2)$. These two coefficients are $2\pi$-anti-periodic in $\theta$ i.e. they obey $f(\theta + 2\pi) = -f(\theta)$ for all $\theta \in [0, 2\pi]$. It follows that the polar components $\Psi^-$ and $\Psi^+$ are also $2\pi$-anti-periodic in $\theta$. This expresses the fact that spinors belong to representations of the double cover of the rotation group $\text{SO}(2, \mathbb{R})$ and, thus get an extra minus sign after a rotation by $2\pi$. By construction, the PDE conserves this anti-periodicity over time. Finally, only half integer modes $k = s + 1/2$, $s \in \mathbb{Z}$ enter the decomposition of the polar spinor components $\Psi^\pm$ in terms of Fourier modes $\exp(i k \theta)$.

Appendix B: Angular momentum from the polar Dirac equation

The angular Fourier transform $\tilde{f}$ of an arbitrary function $f$ of the variables $(r, \theta)$ is defined by

$$\tilde{f}(r, \kappa) = \int_{\theta=0}^{4\pi} f(r, \theta) \exp(i \kappa \theta) d\theta.$$  \hfill (B1)

Since $\theta$ is bounded by $4\pi$, its conjugate variable $\kappa$ is discrete, with step $\Delta \kappa = (2\pi)/(4\pi) = 1/2$. Since $\theta$ is continuous, $\kappa$ is unbounded. A simple and common choice for the range of $\kappa$ is therefore $\{0, \pm1/2, \pm1, \ldots\}$. As explained above, the potential components have integer angular Fourier modes while the polar spinor components only have half-integer angular Fourier modes.

If the potential components $A_t$, $A_r$ and $A_\theta$ do not depend on $\theta$, the Fourier transform of the polar spinor components obeys

$$\partial_t \left( \tilde{\Psi}(r, \kappa) \right) = (L_D \tilde{\Psi})(r, \kappa)$$  \hfill (B2)

with

$$(L_D \tilde{\Psi})(r, \kappa) = \left( \sigma_3 (\partial_r - i A_r) + \frac{1}{r} \left( i \sigma_2 (\kappa + A_\theta) + \frac{\sigma_3}{2} \right) \right)$$

$$-i m \sigma_1 + i A_t \tilde{\Psi}(r, \kappa).$$  \hfill (B3)

Thus, each angular Fourier component evolves independently of the others. Moreover, each Fourier component evolves in a unitary way i.e. the operator $L_D$ conserves the norm of each angular Fourier component.

The angular momentum operator $\hat{J}$ is defined by

$$(\hat{J} \tilde{\Psi})(r, \kappa) = \kappa \tilde{\Psi}(r, \kappa).$$  \hfill (B4)

This definition is equivalent to

$$(\hat{J} \Psi)(r, \theta) = -i (\partial_\theta \Psi)_{r, \kappa}.$$  \hfill (B5)

The expectation value of $\hat{J}$ is conserved by $L_D$ if all potential components are independent of $\theta$.

It is instructive to rewrite the expectation value of the operator $\hat{J}$ in terms of the Cartesian spin components. One finds

$$\langle \hat{J} \rangle = -i \int \lambda_{AB} \Psi^A \left( (x \partial_y - y \partial_x) \delta^B_C \right.$$  

$$+ \frac{i}{2} (\sigma_1)_{CB} \right) \Psi^C \, dx \, dy,$$  \hfill (B6)

where the metric $\lambda$ in spin space is defined by $\lambda_{AB} = 1$ if $A = B$ and 0 otherwise. Equation (B6) shows that the total angular momentum of the spinor is the sum of the kinetic angular momentum and of the spin.
Appendix C: Relativistic Landau levels

The eigenstates common to the Dirac Hamiltonian and the angular momentum operator are of the form

$$\Phi_{E, \kappa}(t, r, \theta) = \exp(-i Et) \Xi_{E, \kappa}(r) \exp(-i \kappa \theta),$$  \hspace{1cm} (C1)

with

$$\Xi = \xi^- b_- + \xi^+ b_+.$$  \hspace{1cm} (C2)

The computation of these eigenstates is best carried out by replacing the components $\xi^-$ and $\xi^+$ of $\Xi$ by the new unknown functions

$$u^- = \frac{i}{\sqrt{2}} \exp\left(\frac{i\pi}{4}\right) (\xi^- + \xi^+),$$

$$u^+ = \frac{1}{\sqrt{2}} \exp\left(\frac{i\pi}{4}\right) (-\xi^- + \xi^+).$$  \hspace{1cm} (C3)

The eigenfunctions $u_{E, \kappa}^\pm$ associated with energy $E$ and angular momentum $\kappa$ obey

$$\pm \partial_r u_{E, \kappa}^\pm (r) + \left(\frac{\kappa}{r} - \frac{1}{2} Br\right) u_{E, \kappa}^\pm (r) - (E \mp m) u_{E, \kappa}^\mp (r) = 0.$$  \hspace{1cm} (C4)

The equations (C4) are solved by changing, first the unknown functions, then the variable. One first introduces $v^\pm = r^\pm \kappa e^{\mp \frac{1}{4} Br^2} u^\pm$ and then changes variable to $x = \sqrt{|B|} r^2 / 2$. These substitutions transform (C4) into standard equations solved by Laguerre polynomials. For example, introducing $n = \frac{m^2 - E^2}{2B}$ and $\alpha = -\kappa - \frac{1}{2}$, one finds that $v^+$ obeys

$$x \frac{d^2 v^+}{dx^2} + (\alpha + 1 - x) \frac{dv^+}{dx} + n v^+ = 0.$$  \hspace{1cm} (C5)

which is solved by

$$v^+ = C L^u_n (x) = C L^u_n \left(-\frac{1}{2} Br^2\right),$$  \hspace{1cm} (C6)

where $C$ is a constant and $L^u_n$ is an associated Laguerre polynomial. The final, normalized expression for $u^\pm$ is

$$u(r) = \left( \frac{Br}{E-m} C r^{1-\kappa} e^{\frac{1}{4} Br^2} L_n^{\alpha+1} \left(-\frac{1}{2} Br^2\right) \right) \left( \frac{Cr^{-\kappa} e^{\frac{1}{4} Br^2} L_n^\alpha \left(-\frac{1}{2} Br^2\right)}{C r^{-\kappa} e^{\frac{1}{4} Br^2} L_n^\alpha \left(-\frac{1}{2} Br^2\right)} \right).$$  \hspace{1cm} (C7)

with

$$|C|^2 = \frac{(m-E)^2 (-B)^{\alpha+1} n!}{\pi 2^{\alpha+1} (n + \alpha)! (-2Bn + (m-E)^2)}. \hspace{1cm} (C8)$$

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