Controlled differential equations as rough integrals

Luu Hoang Duc *

dedicated to Prof. Peter Kloeden on his 70th birthday

Abstract

We study controlled differential equations with unbounded drift terms, where the driving paths is \( \nu \)-Hölder continuous for \( \nu \in (\frac{1}{3}, \frac{1}{2}) \), so that the rough integral are interpreted in the Gubinelli sense [11] for controlled rough paths. Similar to the rough differential equations in the sense of Lyons [14] or Friz-Victoir [10], we prove the existence and uniqueness theorem for the solution in the sense of Gubinelli, the continuity on the initial value, and the solution norm estimates.

Keywords: stochastic differential equations (SDE), rough path theory, rough integrals, rough differential equations.

1 Introduction

This paper studies the following controlled differential equation

\[
dy_t = f(y_t)dt + g(y_t)dx_t, \quad \forall t \in [a,b], \quad y_a \in \mathbb{R}^d,
\]

where the driving path \( x \) belongs to a certain space of Hölder continuous functions. Such system is understood as a pathwise approach to solve a stochastic differential equation driven by a Hölder continuous stochastic process, for instance the path \( x \) can be a realization of a fractional Brownian motion \( B^H \) [16] with Hurst exponent \( H \in (0,1) \). In this circumstance, equation (1.1) is often understood in the language of rough path theory, and its solutions are often defined in the sense of Lyons [14], [15], or in the sense of Friz-Victoir [10], [2]. It is important to note that such definitions of rough differential equations do not specify what a rough integral is.

On the other hand, in the simplest case of Young differential equations, the rough integrals are defined in the Young sense [20], and the existence and uniqueness theorem is well-known, see e.g. [13], [21], [17], [6]. Recent results [5], [7] on the asymptotic stochastic stability and the existence of random attractors [5] for the random dynamical systems [1] generated by Young equations show a very effective method of applying the semigroup technique to estimate Young integrals and the discrete Gronwall lemma to derive the stability criterion [7]. To extend this method and stability results to rough differential equations, a first and necessary step is to define the rough integrals so that the rough system (1.1) can be understood in the integral form. Fortunately, this is feasible if one defines the rough integrals in the sense of Gubinelli [11], [9] for controlled rough paths.

Our aim in this paper is therefore to close the gap by proving similar results to [2] for rough system with the unbounded drift term using rough integrals in the sense of Gubinelli. The main results of this paper are the existence and uniqueness theorem for the solution of (1.1), the continuity of the solution on the initial condition, and the estimates of the solution norms.

*Max-Planck-Institute for Mathematics in the Sciences, Leipzig, Germany, & Institute of Mathematics, Viet Nam Academy of Science and Technology duc.luu@mis.mpg.de, lhduc@math.ac.vn
To study the rough differential equation (1.1), we impose the following assumptions.

\( (H_1) \) \( f : \mathbb{R}^d \to \mathbb{R}^d \) is globally Lipschitz continuous with the Lipschitz constant \( C_f \);

\( (H_2) \) \( g : \mathbb{R}^d \to \mathbb{R}^{d \times m} \) either belongs to \( C_b^\infty \) such that

\[
\|g\|_\infty, C_g := \max \left\{ \|Dg\|_\infty, \|Dg^2\|_\infty, \|Dg^3\|_\infty \right\} < \infty,
\]

or it has a linear form, i.e. \( g(y) = Cy + g(0) \) where \( C \in \mathbb{R}^d \otimes \mathbb{R}^{d \times m} \);

\( (H_3) \) for a given \( \nu \in (\frac{1}{3}, \frac{1}{2}) \), \( x \) belongs to the space \( C^{\nu-\text{Hol}}([a, b], \mathbb{R}^m) \) of all continuous paths which is of finite \( \nu \)-Hölder norm on an interval \([a, b] \).

The paper is organized as follows. Section 2 is devoted to present a brief summary of rough path theory and rough integrals.

2 Rough integrals

Let us present in this preparation section a short summary on rough path theory and rough integrals. Given any compact time interval \( I \subset \mathbb{R} \), let \( C(I, \mathbb{R}^d) \) denote the space of all continuous paths \( y : I \to \mathbb{R}^d \) equipped with sup norm \( \| \cdot \|_{\infty, I} \) given by \( \|y\|_{\infty, I} = \sup_{t \in I} \|y_t\| \), where \( \| \cdot \| \) is the Euclidean norm in \( \mathbb{R}^d \). We write \( y_{s,t} := y_t - y_s \). For \( p \geq 1 \), denote by \( C^{p-\text{var}}(I, \mathbb{R}^d) \subset C(I, \mathbb{R}^d) \) the space of all continuous path \( y : I \to \mathbb{R}^d \) which is of finite \( p \)-variation

\[
\|y\|_{p-\text{var}, I} := \left( \sup_{n(I)} \sum_{i=1}^n \|y_{t_i, t_{i+1}}\|^p \right)^{1/p} < \infty,
\]

where the supremum is taken over the whole class of finite partition of \( I \). \( C^{p-\text{var}}(I, \mathbb{R}^d) \) equipped with the \( p \)-var norm

\[
\|y\|_{p-\text{var}, I} := \|y_{\text{min}}\| + \|y\|_{p-\text{var}, I},
\]

is a nonseparable Banach space [10] Theorem 5.25, p. 92. Also for each \( 0 < \alpha < 1 \), we denote by \( C^\alpha(I, \mathbb{R}^d) \) the space of Hölder continuous functions with exponent \( \alpha \) on \( I \) equipped with the norm

\[
\|y\|_{\alpha, I} := \|y_{\text{min}}\| + \|y\|_{\alpha, I} = \|y(\alpha)\| + \sup_{s < t \in I} \frac{\|y_{s,t}\|}{(t-s)\alpha},
\]

A continuous map \( \xi : \Delta^2(I) \to \mathbb{R}^+ \), \( \Delta^2(I) := \{(s,t) : \min I \leq s \leq t \leq \max I \} \) is called a control if it is zero on the diagonal and superadditive, i.e. \( \xi_{s,t} = 0 \) for all \( t \in I \), and \( \xi_{s,u} + \xi_{u,t} \leq \xi_{s,t} \) for all \( s \leq u \leq t \) in \( I \).

We also introduce the construction of the integral using rough paths for the case \( y, x \in C^\alpha(I) \) when \( \alpha \in (\frac{1}{2}, \nu) \). To do that, we need to introduce the concept of rough paths. Following [9], a couple \( x = (x, \mathbb{X}) \), with \( x \in C^\alpha(I, \mathbb{R}^m) \) and \( \mathbb{X} \in C^{2\alpha}(\Delta^2(I), \mathbb{R}^m \otimes \mathbb{R}^m) := \{ \mathbb{X} : \sup_{s < t} \frac{\|\mathbb{X}_{s,t}\|}{|t-s|^{2\alpha}} < \infty \} \)
where the tensor product $\mathbb{R}^m \otimes \mathbb{R}^n$ can be identified with the matrix space $\mathbb{R}^{m \times n}$, is called a rough path if it satisfies Chen’s relation

$$X_{s,t} - X_{s,u} - X_{u,t} = x_{s,u} \otimes x_{u,t}, \quad \forall \min I \leq s \leq u \leq t \leq \max I. \quad (2.2)$$

$X$ is viewed as postulating the value of the quantity $\int_s^t x_{s,r} \otimes dx_r := X_{s,t}$ where the right hand side is taken as a definition for the left hand side. Denote by $C^\alpha(I) \subset C^\alpha \oplus C^{2\alpha}_2$ the set of all rough paths in $I$, then $C^\alpha$ is a closed set but not a linear space, equipped with the rough path semi-norm

$$\|x\|_{\alpha,I} := \|x\|_{\alpha,I} + \|X\|_{2\alpha, \Delta^2(I)}^{\frac{1}{2}} < \infty. \quad (2.3)$$

Given fixed $\nu \in (\frac{1}{3}, \frac{1}{2}), \alpha \in (\frac{1}{3}, \nu)$ and $p \in (\frac{1}{\alpha}, 3)$, on each compact interval $I$ such that $|I| = \max I - \min I \leq 1$, we also consider in this paper the rough path $x = (x, X)$ with the $p$–var norm

$$\|x\|_{p-\text{var},I} := \left( \|x\|_{p-\text{var},I}^p + \|X\|_{q-\text{var},I}^q \right)^{\frac{1}{p}}, \quad \text{where } q = \frac{p}{2}. \quad (2.4)$$

In the stochastic scenarios, it is often assumed [9] that the driving path $x \in C^\nu - \text{Hol}(I, \mathbb{R}^m) \subset C^{p-\text{var}} (I, \mathbb{R}^m)$ can be lifted into a realized component $x = (x, X)$ of a stationary stochastic process $(x.(\omega), X_.(\omega))$, such that the estimate

$$E\left( \|x_{s,t}\|^p + \|X_{s,t}\|^q \right) \leq C_{T,\nu}|t - s|^{\nu p}, \forall s, t \in [0, T]$$

holds for any $[0, T]$ for some constant $C_{T,\nu}$.

2.1 Controlled rough paths

Following [11], a path $y \in C^\alpha(I, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^d))$ is then called to be controlled by $x \in C^\alpha(I, \mathbb{R}^m)$ if there exists a tube $(y', R^y)$ with $y' \in C^\alpha(I, \mathcal{L}(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^d))), R^y \in C^{2\alpha}(\Delta^2(I), \mathcal{L}(\mathbb{R}^m, \mathbb{R}^d))$ such that

$$y_{s,t} = y'_s \otimes x_{s,t} + R^y_{s,t}, \quad \forall \min I \leq s \leq t \leq \max I. \quad (2.5)$$

$y'$ is called Gubinelli derivative of $y$, which is uniquely defined as long as $x \in C^\alpha \setminus C^{2\alpha}$ (see [9] Proposition 6.4). The space $D^\alpha_x(I)$ of all the couple $(y, y')$ that is controlled by $x$ will be a Banach space equipped with the norm

$$\|y, y'\|_{x, 2\alpha, I} := \|y\|_{\min I} + \|y'\|_{\min I} + \|y, y'\|_{x, 2\alpha, I}, \quad \text{where}$$

$$\|y, y'\|_{x, 2\alpha, I} := \|y'\|_{\alpha, I} + \|R^y\|_{2\alpha, I},$$

where we omit the value space for simplicity of presentation. Now fix a rough path $(x, X)$, then for any $(y, y') \in D^\alpha_x(I)$, it can be proved that the function $F \in C^\alpha(\Delta^2(I), \mathbb{R}^d)$ defined by

$$F_{s,t} := y_s \otimes x_{s,t} + y'_s \otimes X_{s,t}, \quad \forall s, t \in I, s \leq t,$$

satisfies

$$F_{s,t} - F_{s,u} - F_{u,t} = -R^y_{s,u} \otimes x_{u,t} - y'_{s,u} \otimes X_{u,t}, \quad \forall s \leq u \leq t;$$

hence it belongs to the space

$$C^{\alpha, 3\alpha}_2(I) := \left\{ F \in C^\alpha(\Delta^2(I)) : F_{t,t} = 0 \quad \text{and} \quad \|\delta F\|_{3\alpha, I} := \sup_{\min I \leq s \leq u \leq \max I} \frac{\|F_{s,t} - F_{s,u} - F_{u,t}\|}{|t - s|^{3\alpha}} < \infty \right\},$$

where $\delta F$ denotes the rough path differential of $F$.
Thanks to the sewing lemma (see e.g. [11, Lemma 4.2]), the integral \( \int_s^t y_u dx_u \) can be defined as
\[
\int_s^t y_u dx_u := \lim_{|\Pi| \to 0} \sum_{[u,v] \in \Pi} [y_u \otimes x_{u,v} + y'_u \otimes X_{u,v}]
\]
where the limit is taken on all the finite partition \( \Pi \) of \( I \) with \( |\Pi| := \max_{[u,v] \in \Pi} |v-u| \) (see [11]).
Moreover, there exists a constant \( C_\alpha = C_{\alpha,I} > 1 \) with \( |I| := \max I - \min I \), such that
\[
\left\| \int_s^t y_u dx_u - y_s x_{s,t} + y' \xi_{s,t} \right\| \leq C_\alpha |t-s|^{3\alpha} \left( \|x\|_{\alpha,[s,t]} \|R^y\|_{2\alpha,\Delta^2[s,t]} + \|y'\|_{\alpha,[s,t]} \|X\|_{2\alpha,\Delta^2[s,t]} \right). \tag{2.6}
\]
From now on, if no other emphasis, we will simply write \( \|x\|_\alpha \) or \( \|X\|_{2\alpha} \) without addressing the domain in \( I \) or \( \Delta^2(I) \). As proved in [11], the rough integral of controlled rough paths follows the rule of integration by parts. In practice, we would use the \( p \)-var norm
\[
\| (y,y') \|_{x,p,I} := \| y_{\min I} \| + \| y'_{\min I} \| + \| (y,y') \|_{x,p,I},
\]
where
\[
\| (y,y') \|_{x,p,I} := \| y' \|_{p-var,I} + \| R^y \|_{q-var,I}.
\]
Thanks to the sewing lemma [11], we can use a similar version to (2.6) under \( p \)-var norm as follows.
\[
\left\| \int_s^t y_u dx_u - y_s x_{s,t} + y' \xi_{s,t} \right\| \leq C_p \left( \|x\|_{p-var,[s,t]} \|R^y\|_{q-var,\Delta^2[s,t]} + \|y'\|_{p-var,[s,t]} \|X\|_{q-var,\Delta^2[s,t]} \right), \tag{2.7}
\]
with constant \( C_p > 1 \) independent of \( x \) and \( y \).

### 2.2 Greedy times

Throughout this paper, we would need to use the concept of greedy times, as presented e.g. in [3, 6, 7]. Given \( \frac{1}{p} \in (\frac{1}{2}, \nu) \), we construct for any fixed \( \gamma \in (0,1) \) the sequence of greedy times \( \{\tau_i(\gamma,I,p)\}_{i \in \mathbb{N}} \) w.r.t. the \( p \)-var norm
\[
\tau_0 = \min I, \quad \tau_{i+1} := \inf \left\{ t > \tau_i : \|x\|_{p-var,\{\tau_i,t\}} = \gamma \right\} \wedge \max I. \tag{2.8}
\]
Denote by \( N_{\gamma,I,p}(x) := \sup \{ i \in \mathbb{N} : \tau_i \leq \max I \} \). It follows that
\[
N_{\gamma,I,p}(x) \leq 1 + \gamma^{-p} \|x\|_{p-var,I}^{p}. \tag{2.9}
\]
On the other hand, for \( \alpha \in (\frac{1}{3}, \nu) \), we also construct another sequence of greedy times \( \{\bar{\tau}_i(\gamma,I,p)\}_{i \in \mathbb{N}} \) w.r.t. the \( \alpha \)-Hölder norm
\[
\bar{\tau}_0 = \min I, \quad \bar{\tau}_{i+1} := \inf \left\{ t > \bar{\tau}_i : (t - \bar{\tau}_i)^{1-2\alpha} + \|x\|_{\alpha-Hol,\{\bar{\tau}_i,t\}} = \gamma \right\} \wedge \max I, \tag{2.10}
\]
and assign \( N_{\gamma,I,\alpha}(x) := \sup \{ i \in \mathbb{N} : \bar{\tau}_i \leq \max I \} \). Then
\[
N_{\gamma,I,\alpha}(x) \leq 1 + |I|^\frac{1}{\gamma^{1+\frac{1}{p-\alpha}}}(1 + \|x\|_{\nu-Hol,I}^{\frac{1}{\nu-\alpha}}). \tag{2.11}
\]

### 3 Existence-uniqueness theorem and solution norm estimates

In this section, we would like to prove the existence and uniqueness theorem and estimate solution norms for rough differential equation [11], where the rough integral is understood in the sense of Gubinelli [11] for controlled rough paths. In case the diffusion coefficient \( g \) is linear, the proof is straightforward as presented below, which applies Schauder-Tichonoff theorem.
Theorem 3.1  There exists a unique solution of the rough differential equation
\[ dy_t = f(y_t)dt + \left(Cy_t + g(0)\right)dx_t, \quad \forall t \in [a, b], \quad y_a \in \mathbb{R}^d. \] (3.1)

Moreover, the supremum and $p$–variation norms of the solution are estimated as follows
\[
\|y\|_{\infty,[a,b]} \leq \left[\|y_a\| + M_0 N_{[a,b]}(x)\right] e^{4C_f(b-a) + LN_{[a,b]}(x)}, \\
\|y, R\|_{p-\text{var},[a,b]} \leq \left[\|y_a\| + M_0 N_{[a,b]}(x)\right] e^{4C_f(b-a) + LN_{[a,b]}(x)} \frac{p-1}{p} N_{[a,b]}(x) - \|y_a\|,
\]
where \( \|y, R\|_{p-\text{var},[s,t]} := \|y\|_{p-\text{var},[s,t]} + \|R^a\|_{q-\text{var},[s,t]}, \ M_0 = (1 + \frac{3}{2C_p}) \frac{\|g(0)\|}{\|C\|} + \frac{\|f(0)\|}{C_f}, \text{ and } L = \log(1 + \frac{3}{2C_p}).
\]

Proof: Step 1. Rewrite equation (3.1) in the integral form
\[ y_t = G(y, y')_t = y_a + \int_a^t f(y_u)du + \int_a^t \left(Cy_u + g(0)\right)dx_u, \quad t \in [a, T], \] (3.3)
Denote by \( D^{2\alpha}_x \left(y_a, Cy_a + g(0)\right) \) the set of paths \((y, y')\) controlled by \( x \) in \([a, T]\) with fixed initial conditions \( y_a \) and \( y'_a = Cy_a + g(0) \). Consider the mapping defined by
\[ \mathcal{M} : D^{2\alpha}_x \left(y_a, Cy_a + g(0)\right) \rightarrow D^{2\alpha}_x \left(y_a, Cy_a + g(0)\right), \quad \mathcal{M}(y, y') := \left(G(y, y')_t, Cy_t + g(0)\right). \]
Similar to [11] we are going to estimate \( \|\mathcal{M}(y, y')\|_{x,2\alpha} = \|Cy + g(0)\|_{\alpha} + \|R^G(y,y')\|_{2\alpha} \) using \( \|(y, y')\|_{x,2\alpha} = \|y'\|_{\alpha} + \|R^y\|_{2\alpha} \). Observe that
\[
\|Cy + g(0)\|_{\alpha} \leq \|C\| \|y\|_{\alpha} \leq \|C\| \left(\|y'\|_{\infty} \|x\|_{\alpha} + (T - a)^\alpha \|R^y\|_{2\alpha}\right)
\leq \|C\| \|x\|_{\alpha} \|y'_a\| + \|C\| (T - a)^\alpha \|x\|_{\alpha} \|y'\|_{\alpha} + \|C\| (T - a)^\alpha \|R^y\|_{2\alpha},
\]
and
\[
\|R^{G(y,y')}_{s,t}\| \leq \left| \int_s^t f(y_u)du \right| + \left| \int_s^t Cy_u dx_u - Cy_s \otimes x_{s,t} \right|
\leq C_f |t - s| \|y\|_{\infty,[s,t]} + \|C\| \|y'\|_{\infty,[s,t]} \|X_{s,t}\|
\leq C_f |t - s| 3^{3\alpha} \left[ \|x\|_{\alpha,[s,t]} \|C\| \|R^y\|_{2\alpha,[s,t]} + \|C\| \|y'\|_{\alpha,[s,t]} \|X\|_{2\alpha,\Delta^2([s,t])}\right],
\]
where we can choose \( T - a < 1 \) so that \( C_\alpha \) can be bounded from above by \( C_{\alpha,1} \). Since
\[
\|y\|_{\infty,[s,t]} \leq \|y_a\| + \|y'_a\|(T - a)^\alpha \|x\|_{\alpha} + (T - a)^{2\alpha} \|R^y\|_{2\alpha},
\]
it follows that
\[
\|R^{G(y,y')}\|_{2\alpha} \leq (T - a)^{1 - 2\alpha} C_f \|y_a\| + (T - a)^{1 - \alpha} C_f \|x\|_{\alpha} \|y'_a\| + C_f (T - a) \|R^y\|_{2\alpha}
\leq \left( T - a \right)^{1 - 2\alpha} C_f \|y_a\| + \left( T - a \right)^{1 - \alpha} C_f \|x\|_{\alpha} \|y'_a\| + C_f (T - a) \|R^y\|_{2\alpha}
\]
Hence, we can estimate \( \|\mathcal{M}(y, y')\|_{x,2\alpha} \) as follows
\[
\|\mathcal{M}(y, y')\|_{x,2\alpha}\]
\[ \leq ||C|| \left( ||y_a|| + (T-a)^\alpha \right) + R^y_{2\alpha} \right) + \frac{M}{2} \leq 1. \]

As a result, if we restrict to the convex compact set
\[ \mathcal{B} := \left\{ (y, y') \in \mathcal{D}_{2\alpha}^{2\alpha}(y_a, Cy_a), ||y, y'||_{2\alpha} \leq \frac{\mu}{1-\mu} \left( ||y_a|| + ||g(0)|| \right) \right\} \]
then
\[ \left\| \mathcal{M}(y, y') \right\|_{2\alpha} \leq \frac{\mu}{1-\mu} \left( ||y_a|| + ||g(0)|| \right) \leq \frac{\mu}{1-\mu} \left( ||y_a|| + ||g(0)|| \right) \]

which proves that \( \mathcal{M} : \mathcal{B} \rightarrow \mathcal{B} \). By Schauder-Tichonoff theorem, there exists a fixed point of \( \mathcal{M} \) which is a solution of equation \( \text{(3.1)} \) on the interval \( [a, T] \).

Next, for any two solutions \( (y, y'), (\bar{y}, \bar{y}') \) of the same initial conditions \( (y_a, Cy_a) \), by similar computations, we obtain
\[ \left\| (y, y') - (\bar{y}, \bar{y}') \right\|_{2\alpha} \leq \mu \left( ||y_a - \bar{y}_a|| + ||y, y'||_{2\alpha} - (\bar{y}, \bar{y}') \right) \leq \mu \left( ||y_s|| + ||g(0)|| \right) \]
which, together with \( \mu < 1 \), proves the uniqueness of solution of \( \text{(3.1)} \) on \( [a, T] \). By constructing the greedy time sequence \( \{ \tilde{t}_i(\frac{1}{2\alpha}, I, \alpha) \}_{i \in \mathbb{N}} \) as in \( \text{(2.10)} \), we can extend and prove the existence of the unique solution on the interval \( [a, b] \).

**Step 2.** To estimate the norms, note that
\[ y'_s = Cys + Cg(0), \quad (Cy + g(0))'_s = C^2ys + Cyg(0), \quad R_{s,t}^{Cy + g(0)} = CR^y_{s,t}. \]

The estimate \( ||y, s, t|| \) is direct using \( \text{(2.7)} \), namely
\[ \|y, s, t\| \leq \int_t^s \left( C_s \|y_s\| + \|f(0)\| \right) du + \left( ||C|| \|y_s\| + ||g(0)|| \right) \|x, s, t\| + ||C|| \|x, s, t\| \]
\[ + C_p \left( \|x\|_{p-var, s, t} \left\| R^{Cy + g(0)}_{q-var, s, t} \right\| + \|x\|_{q-var, s, t} \left\| (Cy + g(0))' \right\|_{p-var, s, t} \right) \]
\[ \leq \int_t^s \left( C_s \|y\|_{p-var, s, u} + \|f(0)\| \right) du + \left( ||y_s\| + \|g(0)|| \right) \|x, s, t\| + ||C|| \|x, s, t\| \]
As a result

\[ + C_p \left( \| x \|_{p \text{-var}, [s, t]} \| C \| \| R^y_q \|_{q \text{-var}, [s, t]} + \| X \|_{q \text{-var}, [s, t]} \| C \|^2 \| y' \|_{p \text{-var}, [s, t]} \right) \]

\[ \leq \int_s^t \left( C_f \| y \|_{p \text{-var}, [s, u]} + \| f (0) \| \right) du + 2 \left( \| y_s \| + \frac{\| g (0) \|}{\| C \|} \right) \left( \| C \| \| x_{s, t} \| \vee \| C \|^2 \| X_{s, t} \| \right) \]

\[ + C_p \left\{ \| C \|^2 \| X \|_{q \text{-var}, [s, t]} \vee \| C \| \| x \|_{p \text{-var}, [s, t]} \right\} \| y, R \|_{p \text{-var}, [s, t]} . \]  

(3.4)

The estimate for \( \| R^y_q \| \) is already included in (3.4). It follows that

\[ \| y, R \|_{p \text{-var}, [s, t]} \leq 2 \int_s^t \left( C_f \| y \|_{p \text{-var}, [s, u]} + C_f \| y_s \| + \| f (0) \| \right) du \]

\[ + 3 \left( \| C \| \| x \|_{p \text{-var}, [s, t]} \vee \| C \|^2 \| X \|_{q \text{-var}, [s, t]} \right) \left( \| y_s \| + \frac{\| g (0) \|}{\| C \|} \right) \]

\[ + 2 C_p \left\{ \| C \|^2 \| X \|_{q \text{-var}, [s, t]} \vee \| C \| \| x \|_{p \text{-var}, [s, t]} \right\} \| y, R \|_{p \text{-var}, [s, t]} . \]

As a result

\[ \| y, R \|_{p \text{-var}, [s, t]} \leq \int_s^t 4 C_f \| y, R \|_{p \text{-var}, [s, u]} du + 4 (\| f (0) \| + C_f \| y_s \|) (t - s) + \frac{3}{2 C_p} \left( \| y_s \| + \frac{\| g (0) \|}{\| C \|} \right) \]

whenever \( 2 C_p \| C \| \| x \|_{p \text{-var}, [s, t]} \leq \frac{1}{2} \). Applying the continuous Gronwall lemma, we obtain

\[ \| y, R \|_{p \text{-var}, [s, t]} \leq 4 (\| f (0) \| + C_f \| y_s \|) (t - s) + \frac{3}{2 C_p} \| g (0) \| \| C \| + \frac{3}{2 C_p} \| y_s \| \]

\[ + \int_s^t 4 C_f e^{4 C_f (t - u)} \left[ 4 (\| f (0) \| + C_f \| y_s \|) (u - s) + \frac{3}{2 C_p} \| g (0) \| \| C \| + \frac{3}{2 C_p} \| y_s \| \right] du \]

\[ \leq \left( M_0 + e^t \| y_s \| \right) e^{4 C_f (t - s)} - \| y_s \| \]  

(3.5)

whenever \( 4 C_p \| C \| \| x \|_{p \text{-var}, [s, t]} \leq 1 \). By constructing the sequence of greedy time \( \{ \tau_i \left( \frac{1}{2 C_p \| C \|}, [a, b], p \right) \} \) and use similar estimates to the proof of [7, Theorem 2.4], we obtain (3.2).

\( \square \)

**Corollary 3.2** The solution of (3.1) is uniformly continuous in \( y_a \), i.e. for any two solutions \( y_a (x, y_a) \) and \( \tilde{y}_a (x, \tilde{y}_a) \), the following estimates hold

\[ \| \tilde{y} - y \|_{\text{loc}, [a, b]} \leq \| \tilde{y}_a - y_a \| e^{4 C_f (b - a) + \text{LN}_{[a, b]} (x)} , \]

\[ \| \tilde{y} - y, R \|_{p \text{-var}, [a, b]} \leq \| \tilde{y}_a - y_a \| e^{4 C_f (b - a) + \text{LN}_{[a, b]} (x)} N_{[a, b]}^{\frac{1}{p}} (x) - \| \tilde{y}_a - y_a \| . \]  

(3.6)

**Proof:** The proof follows similar arguments and will be omitted.

\( \square \)

Unlike the linear case, the proof for the nonlinear case \( g \in C^0 \) is not simple due to the fact that, the possible greedy times might depend on the initial value \( \| y_a \| \) and it might not be easy to extend the solution to any interval. To overcome this difficulty, we propose another scheme, by first proving the differentiability w.r.t. the initial condition of the solution of the rough differential equation

\[ dy_t = g (y_t) dx_t , \quad \forall t \in [a, b], y_a \in \mathbb{R}^d ; \]  

(3.7)

and then applying the Doss-Sussmann technique [19, 12] to transform the original system to an ordinary differential equation. Note that the existence, uniqueness and continuity of the solution of (3.7) is already provided in [11], but the differentiability of the solution \( y_t (x, y_a) \) w.r.t. \( y_a \) is somehow missing due to the technical complexity. We will provide a detailed proof for this assertion. First, we need the following estimates.
Proposition 3.3 The solution $y_t(x, y_a)$ of (3.7) is uniformly continuous w.r.t. $y_a$, i.e. for any two solutions $y(x, y_a)$ and $\hat{y}(x, \hat{y}_a)$ the following estimates hold
\[
\|\bar{y} - y\|_{\infty, [a,b]} \leq \|\bar{y}_a - y_a\|e^{(\log 2)\tilde{N}_{[a,b]}(x)},
\]
\[
\|\bar{y} - y, R\|_{p-\text{var}, [a,b]} \leq \|\bar{y}_a - y_a\|e^{(\log 2)\tilde{N}_{[a,b]}(x)} - \|\bar{y}_a - y_a\|,
\]
(3.8)
where $\tilde{N}_{[a,b]}(x)$ is the maximal index of the maximal greedy time in the sequence
(3.9)
that lies in the interval $[a, b]$, with $C(x, [a, b]) := \left\{ 1 + \frac{1}{\vartheta} \leq \frac{2p-1}{p} \left( 1 + \|8C_g\|^{2p-1} \|x\|^{2p-1}_{p-\text{var}, [a,b]} \right) \right\}$.

Proof: The proof is divided into several steps.

Step 1: To estimate the solution norms of (3.1), observe that due to (2.6)
\[
\|\int_s^t g(y_a)dx_u - g(y_a) \otimes x_{s,t} - [g(y)]'_s \otimes X_{s,t}\| \leq C_a(t-s)^{3\alpha} \left[ \|x\|_{\alpha, [s,t]} \|R^y\|_{2\alpha, [s,t]} \right] + \|\|X\|_{2\alpha, [s,t]} \| [g(y)]' \|_{\alpha, [s,t]} \right] .
\]
It follows that $y$ is controlled by $x$ with $y' = g(y)$. Since
\[
g(y_t) - g(y_s) = \int_0^1 Dg(y_s + \eta y_{s,t})y_{s,t}d\eta
\]
\[
= Dg(y_s)y'_s \otimes x_{s,t} + \int_0^1 Dg(y_s + \eta y_{s,t})y_{s,t}d\eta + \int_0^1 [Dg(y_s + \eta y_{s,t}) - Dg(y_s)]y'_s \otimes x_{s,t}d\eta,
\]
it easy to show that $[g(y)]'_s = Dg(y_s)g(y_s)$, where we use (1.2) to estimate
\[
\|R^y_{s,t}\| \leq \int_0^1 \|Dg(y_s + \eta y_{s,t})\|\|R^y_{s,t}\|d\eta + \int_0^1 \|Dg(y_s + \eta y_{s,t}) - Dg(y_s)\|\|g(y)_s\|\|x_{s,t}\|d\eta
\]
\[
\leq C_g\|R^y_{s,t}\| + \frac{1}{2}C_g^2\|y_{s,t}\|\|x_{s,t}\|.
\]
This together with Hölder inequality yields
\[
\|[g(y)]'_s\|_{p-\text{var}, [s,t]} \leq 2C_g \|y\|_{p-\text{var}, [s,t]} \cdot \|[g(y)]'_s\|_{\infty, [s,t]} \leq C_g^2,
\]
\[
\|R^y_{q-\text{var}, [s,t]}\| \leq C_g\|R^y_{q-\text{var}, [s,t]}\| + \frac{1}{2}C_g^2\|y\|_{p-\text{var}, [s,t]} \|y\|_{p-\text{var}, [s,t]} .
\]
(3.10)
As a result, by introducing $\|y, R\|_{p-\text{var}, [s,t]} := \|y\|_{p-\text{var}, [s,t]} + \|R^y\|_{q-\text{var}, [s,t]},$ we obtain
\[
\|y_{s,t}\| \leq \int_s^t g(y_a)dx_u
\]
\[
\leq \|g(y_a)\|\|x_{s,t}\| + \|Dg(y_a)g(y_a)\|\|X_{s,t}\|
\]
\[
+ C_p \left[ \|x\|_{p-\text{var}, [s,t]} \|R^y\|_{q-\text{var}, [s,t]} + \|[x]\|_{p-\text{var}, [s,t]} \|y\|_{p-\text{var}, [s,t]} \right]
\]
\[
\leq C_g \|x\|_{p-\text{var}, [s,t]} + C_g^2 \|X\|_{q-\text{var}, [s,t]}
\]
\[
+ C_p \left[ C_g \|x\|_{p-\text{var}, [s,t]} \|R^y\|_{q-\text{var}, [s,t]} + \left[ \frac{1}{2} \|x\|_{p-\text{var}, [s,t]}^2 + 2 \|X\|_{q-\text{var}, [s,t]} \right] C_g^2 \|y\|_{p-\text{var}, [s,t]} \right] .
\]
\[
\leq C_g \|x\|_{p-\varray,[s,t]} + C_g^2 \|x\|_{p-\varray,[s,t]}^2 + 2C_p \left\{ C_g \|x\|_{p-\varray,[s,t]} \vee C_g^2 \|x\|_{p-\varray,[s,t]}^2 \right\} \|y, R\|_{p-\varray,[s,t]},
\]

which derives

\[
\|y\|_{p-\varray,[s,t]} \leq 2 \left\{ C_g \|x\|_{p-\varray,[s,t]} \vee C_g^2 \|x\|_{p-\varray,[s,t]}^2 \right\} \left( 1 + C_p \|y, R\|_{p-\varray,[s,t]} \right).
\]

The same estimate for \( R^y \) is actually included in the above estimate, hence

\[
\|R^y\|_{q-\varray,[s,t]} \leq 2 \left\{ C_g \|x\|_{p-\varray,[s,t]} \vee C_g^2 \|x\|_{p-\varray,[s,t]}^2 \right\} \left( 1 + C_p \|y, R\|_{p-\varray,[s,t]} \right).
\]

Combining (3.11) and (3.12) gives

\[
\|y, R\|_{p-\varray,[s,t]} \leq 4 \left\{ C_g \|x\|_{p-\varray,[s,t]} \vee C_g^2 \|x\|_{p-\varray,[s,t]}^2 \right\} \left( 1 + C_p \|y, R\|_{p-\varray,[s,t]} \right).
\]

It implies from (3.13) that

\[
\|y, R\|_{p-\varray,[s,t]} \leq \frac{1}{C_p} \quad \text{whenever} \quad \left\{ C_g \|x\|_{p-\varray,[s,t]} \vee C_g^2 \|x\|_{p-\varray,[s,t]}^2 \right\} \leq \frac{1}{8C_p} < 1,
\]

which yields

\[
\|y, R\|_{p-\varray,[s,t]} \leq \frac{1}{C_p} \quad \text{whenever} \quad \|x\|_{p-\varray,[s,t]} \leq \frac{1}{8C_p C_g}.
\]

Using similar arguments to [7] Theorem 2.4], by constructing a sequence of greedy times \( \tau_i(\frac{1}{8C_p C_g}, I, p) \) \( \in \mathbb{N} \)

As in (2.8), we conclude that

\[
\|y\|_{\infty,[a,b]} \leq \|y_a\| + \frac{1}{C_p} N_{[a,b]}(x) \leq \|y_a\| + \frac{1}{C_p} \left( 1 + [8C_p C_g]^p \|x\|_{p-\varray,[a,b]} \right),
\]

\[
\|y, R\|_{p-\varray,[a,b]} \leq N_{[a,b]}(x) \sum_{k=0}^{p-1} \|y, R\|_{q-\varray,[\tau_k,\tau_{k+1}]} \leq \frac{1}{C_p} \frac{\|x\|_{p-\varray,[a,b]}^{2p-1}}{N_{[a,b]}(x)},
\]

where the last inequality applies [27] and the H"older inequality.

**Step 2:** Next, for any two solutions \( y_i(x, y_a) \) and \( \bar{y}_i(x, \bar{y}_a) \) within the bounded range \( \frac{1}{C_p} N_{[a,b]}^{2p-1}(x) \), consider their difference \( z_t = \bar{y}_t - y_t \), which satisfies the integral rough equation

\[
z_t = z_a + \int_a^t [g(z_s) - g(y_s)] dx_s.
\]

As a result, \( y'_s = g(y_s), \bar{y}'_s = g(\bar{y}_s) \) and

\[
\begin{align*}
g(\bar{y}_t) - g(y_t) - g(\bar{y}_t) + g(y_t) \\
= \int_0^1 \left[ Dg(\bar{\eta}_s + \eta \bar{y}_s,t)\bar{y}_s,t - Dg(y_s + \eta y_s,t)y_s,t \right] d\eta \\
= \left[ Dg(\bar{y}_s)g(\bar{y}_s) - Dg(y_s)g(y_s) \right] \otimes x_{s,t} + \int_0^1 \left[ Dg(\bar{\eta}_s + \eta \bar{y}_s,t)R^\eta_{s,t} - Dg(y_s + \eta y_s,t)R^\eta_{s,t} \right] d\eta \\
+ \int_0^1 \left\{ [Dg(\bar{\eta}_s + \eta \bar{y}_s,t) - Dg(\bar{y}_s)] Dg(\bar{y}_s) - [Dg(y_s + \eta y_s,t) - Dg(y_s)] Dg(y_s) \right\} \otimes x_{s,t} d\eta
\end{align*}
\]
This proves \( |g(\bar{y}) - g(y)|_p = Dg(\bar{y})g(\bar{y}) - Dg(y)g(y) \) which has the form \( Q(\bar{y}) - Q(y) \). Notice that \( \|Q(\bar{y}) - Q(y)\| \leq 2C^2_\theta \|z\| \) and

\[
\|Q(\bar{y}) - Q(y)\|_{p-\text{var},[s,t]} \leq C_Q \left( \|z\|_{p-\text{var},[s,t]} + \|z\|_{\infty,[s,t]} \|g\|_{p-\text{var},[s,t]} \right) \\
\leq 2C^2_\theta \left( \|z\|_{p-\text{var},[s,t]} + \|z\|_{\infty,[s,t]} \|y\|_{p-\text{var},[s,t]} \right).
\]

In addition

\[
\|R^g(\bar{y}) - g(y)\|_{q-\text{var},[s,t]} \leq C_g \|R^g\|_{q-\text{var},[s,t]} + C_g \|z\|_{\infty,[s,t]} \|R^y\|_{q-\text{var},[s,t]} \\
+ \frac{1}{2} C^2_\theta \|x\|_{p-\text{var},[s,t]} \left( \|z\|_{p-\text{var},[s,t]} + \|z\|_{\infty,[s,t]} \|\bar{y}\|_{p-\text{var},[s,t]} + \|y\|_{p-\text{var},[s,t]} \right) \tag{3.16}
\]

Using the fact that

\[
\|z\|_{z,s,t} \leq \left| \int_s^t (g(y_u) - g(y_u)) \, dx_u \right| \\
\leq C_g \|z\|_{z,s,t} \|x\|_{p-\text{var},[s,t]} + 2C^2_\theta \|z\|_{z,s,t} \|X\|_{q-\text{var},[s,t]} + C_p \left\{ \|x\|_{p-\text{var},[s,t]} \right\} \|R^g(\bar{y}) - g(y)\|_{q-\text{var},[s,t]} \\
+ \|X\|_{q-\text{var},[s,t]} \|g(\bar{y}) - g(y)\|_{p-\text{var},[s,t]} \right\} , \tag{3.16}
\]

we can now estimate

\[
\|z\|_{p-\text{var},[s,t]} \leq 2 \left\{ C_g \|x\|_{p-\text{var},[s,t]} \right\} \times \left\{ \|z\|_{\infty,[s,t]} \left[ 1 + C_p \left( \|\bar{y}\|_{p-\text{var},[s,t]} + \|y, R\|_{p-\text{var},[s,t]} \right) \right] + C_p \|z, R\|_{p-\text{var},[s,t]} \right\} \\
\leq 2C_p \left\{ C_g \|x\|_{p-\text{var},[s,t]} \right\} \times \left\{ \|\bar{y}, R\|_{p-\text{var},[s,t]} + \|y, R\|_{p-\text{var},[s,t]} \right\} \left( \|z_s\| + \|z, R\|_{p-\text{var},[s,t]} \right) . \tag{3.17}
\]

The similar estimate for \( \|R^z\|_{q-\text{var},[s,t]} \) is already included in the estimate (3.17). By combining (3.17) with (3.15), we obtain

\[
\|z, R\|_{p-\text{var},[s,t]}
\]
Therefore, (3.8) is followed directly from the usage of greedy times (3.9), which is similar to (3.14) which yields

Theorem 3.4
The solution $\frac{\partial y}{\partial \alpha}(x, y_{\alpha})$ of (5.7) is differentiable w.r.t. initial condition $y_{\alpha}$, moreover, its derivatives $\frac{\partial y}{\partial \alpha}(x, y_{\alpha})$ is the matrix solution of the linearized rough differential equation

$$
\frac{d\xi_t}{dt} = Dg(y_t)\xi_t dt
$$

Proof: The proof is divided into several steps.

Step 1: First, for a fixed solution $y_t(x, y_{\alpha})$ on $[a, b]$, we need to prove the existence and uniqueness of the solution of the linearized rough differential equation (3.20), which has the time dependent coefficient $\Sigma := Dg(y_t)$. To do that, we simply follow Gubinelli’s method by considering the solution mapping $H_t = \xi_{\alpha} + \int_t^1 \Sigma_s \xi_s dx_s$ on the set $D_{x_{\alpha}}^2([a, b], \xi_{\alpha}, \Sigma_{\alpha}\xi_{\alpha})$ of the controlled paths $\xi_t$ with the fixed initial conditions $\xi_{\alpha}, \xi'_{\alpha} = \Sigma_{\alpha}\xi_{\alpha}$. Note that $\Sigma_t = Dg(y_t)$ is also controlled by $x$ with

$$
\Sigma_{s,t} = \int_0^1 \Sigma_s d\eta_t(s) + \eta_t(s)(g(y_t) \otimes x_{s,t} + R_{s,t}^\eta dy_t),
$$

thus

$$
\Sigma_t = D^2g(y_t)g(y_t), \quad \|R^\Sigma_{s,t}\| \leq C_g \Sigma_{s,t} \|x_{s,t}\|.
$$

As a result, $\Sigma_t\xi_t$ is also controlled by $x$ with $[\Sigma, \xi]' = \Sigma_s\xi_s + \Sigma_s\xi'_s$ and

$$
\|R^\Sigma_{s,t}\| \leq \|\Sigma_{s,t}\|\|x_{s,t}\| + \|\xi_t\|\|R^\Sigma_{s,t}\| + \|\Sigma_t\|\|R^\Sigma_{s,t}\|.
$$

It then enables us to estimate

$$
\left\|H_{s,t} - \Sigma_s\xi_s \otimes x_{s,t} + [\Sigma'_s\xi_s + \Sigma'_s\xi'_s] \otimes X_{s,t}\right\|
\leq \left\|\int_s^t \Sigma_u\xi_u dx_u - \Sigma_s\xi_s \otimes x_{s,t} + [\Sigma'_s\xi_s + \Sigma'_s\xi'_s] \otimes X_{s,t}\right\|
+ C_\alpha (t - s)^{3\alpha} \left(\|x\|_{\alpha,[s,t]} \|R^\Sigma_{2\alpha,[s,t]} \| + \|X\|_{2\alpha,[s,t]} \|[\Sigma, \xi]'\|_{\alpha,[s,t]}\right)
$$
where

\[
\begin{align*}
\|\Sigma, \xi\|_\alpha & \leq \|\Sigma\|_\alpha \|\xi\|_\alpha + \|\Sigma'\|_\alpha \|\xi\|_\infty + \|\Sigma\|_\infty \|\xi'\|_\alpha + \|\Sigma'\|_\alpha \|\xi'\|_\infty, \\
R^{\Sigma, \xi} & \leq \|\Sigma\|_\alpha \|\xi\|_\alpha + \|R^{\Sigma, \xi}\|_\infty \|\xi\|_\alpha + \|\Sigma\|_\infty \|R^{\Sigma, \xi}\|_2, \\
& \quad \text{with} \\
\|\xi\|_\infty & \leq \|\xi\|_\alpha + (t - a)^\alpha \|\xi\|_\alpha \leq \|\xi\|_\alpha \|\xi\|_\alpha \leq \|\xi\|_\alpha \|\xi\|_\alpha + \left( \|\xi\|_\alpha \vee 1 \right) (t - a)^\alpha \|\xi, \xi'\|_\infty, \\
\|\xi\|_\alpha & \leq \|\xi\|_\alpha \|\xi\|_\alpha \leq \|\xi\|_\alpha \left( 1 + \|\Sigma\|_\alpha (t - a)^\alpha \|\xi\|_\alpha \right) + (t - a)^\alpha \left( \|\xi\|_\alpha \vee 1 \right) \|\xi, \xi'\|_\infty.
\end{align*}
\]

A direct computation then shows that

\[
\begin{align*}
\|\Sigma, \xi\|_\alpha & \leq \left\{ \|\Sigma\|_\alpha \|\xi\|_\alpha + \|\Sigma\|_\alpha \|\xi\|_\alpha \right\} \|\xi, \xi'\|_\infty + \left\{ \|\Sigma\|_\alpha \|\Sigma\|_\alpha \|\xi\|_\alpha \right\} \|\xi, \xi'\|_\infty, \\
R^{\Sigma, \xi} & \leq \left\{ \|\Sigma\|_\alpha \|\Sigma\|_\alpha \|\xi\|_\alpha \right\} \|\xi, \xi'\|_\infty + \left\{ \|\Sigma\|_\alpha \|\Sigma\|_\alpha \|\xi\|_\alpha \right\} \|\xi, \xi'\|_\infty.
\end{align*}
\]

Hence

\[
\begin{align*}
\|R^H\|_\alpha & \leq \left( \|\Sigma\|_\alpha \|\xi\|_\alpha + \|\Sigma\|_\alpha \|\xi\|_\alpha \right) \|\xi\|_\alpha + C_\alpha (t - a)^\alpha \left( \|\xi\|_\alpha \right) \|R^{\Sigma, \xi}\|_\alpha + \left\{ \|\Sigma\|_\alpha \|\Sigma\|_\alpha \|\xi\|_\alpha \right\} \|\xi, \xi'\|_\infty, \\
H' & \leq \|\Sigma\|_\alpha \|\xi\|_\alpha + \left\{ \|\Sigma\|_\alpha \|\Sigma\|_\alpha \|\xi\|_\alpha \right\} \|\xi, \xi'\|_\infty
\end{align*}
\]

All things combined, we have just showed that there exists constants

\[
M_1 = M_1(\Sigma, [a, b], x, \bar{X}), \quad M_2 = M_2(\Sigma, [a, b], x, \bar{X})
\]

such that

\[
\|H'\|_\alpha + \|R^H\|_\alpha \leq M_2 \|\xi\|_\alpha + M_1 (t - a)^\alpha \|\xi\|_\alpha + \|\xi\|_\alpha \|\xi\|_\alpha \|\xi\|_\alpha \|\xi, \xi'\|_\infty.
\]

This implies that on every interval \([\bar{\tau}_k, \bar{\tau}_{k+1}]\) of the greedy times \(\{\bar{\tau}_i(\bar{\tau}, \bar{\tau}_{k+1})\} \in \mathbb{N}\) as in (2.10), the solution mapping is a contraction from the set \(\{D_{x, \alpha, \eta, \xi}([a, b], \xi, \eta, \xi) : \|\xi, \xi'\|_\alpha \leq M_2 \|\xi, \xi'\|_\alpha \})\) into itself, hence there exists a solution of (3.20) on every interval \([\bar{\tau}_k, \bar{\tau}_{k+1}]\). Because of the linearity, it follows from the same estimate (3.21) with \(\|\xi\|_\alpha = 0\) that \(\|\xi, \xi'\|_\alpha = 0\), which proves the uniqueness. Finally, the concatenation of solutions on intervals \([\bar{\tau}_k, \bar{\tau}_{k+1}]\) then proves the existence and uniqueness of the solution of (3.20) on \([a, b, b]\).

**Step 2:** Denote by \(\Phi(t, x, y)\) the solution matrix of the linearized system (3.20), then \(\xi_t = \Phi(t, x, y)(\bar{y}_a - y_a)\) is the solution of (3.21) given initial point \(\xi_a = \bar{y}_a - y_a\). Assign \(r_t := \bar{y}_t - y_t - \xi_t\), then \(r_a = 0\) and

\[
\begin{align*}
\int_a^t \left[ \int_0^1 Dg(y_s + \eta(\bar{y}_a - y_a) - Dg(y_s) \right] (\bar{y}_s - y_a) d\eta ds + \int_a^t Dg(y_s) r_s ds dx, \\
= c_{a, t} + \int_a^t Dg(y_s) r_s ds dx, \quad \forall t \in [a, b],
\end{align*}
\]

(3.22)
where 
\[ e_{a,t} = \int_a^t \int_0^1 \left[ Dg(y_s + \eta(y_s - y_s)) - Dg(y_s) \right] (y_s - y_s) \, dy \, dx \]
and \( e \) is also controlled by \( x \) with \( e_{a,a} = 0 \). We are going to estimate \( \|r\|_{\infty,[a,b]} \) and \( \|r, R\|_{p \text{-var},[a,b]} \) through \( \|e\|_{\infty,[a,b]}, \|e\|_p, \|e\|_{p \cdot \text{var},[a,b]}, \|R\|_q \text{-var},[a,b] \). First observe that

\[ r_{s,t} = e_{s,t} + \int_s^t Dg(y_u) r_u \, du = e' \otimes x_{s,t} + \int_s^t Dg(y_u) r_u \, du, \]
which yields \( r'_s = e'_s + Dg(y_s) r_s \) and

\[ \|R_{s,t}^e\| \leq \|R_{s,t}^e\| + \|Dg(y) r'_s\| \|X_{s,t}\| + C_p \left( \|x\|_p \|Dg(y)^{\prime\prime}\|_q + \|X\|_q \|Dg(y) r'_s\|_p \right). \quad (3.23) \]

A direct computation shows that

\[ \left\| Dg(y_t) r_t - Dg(y_s) r_s - \left[ Dg(y_s) g(y_s) r_s + Dg(y_s) r'_s \right] \otimes x_{s,t} \right\| \leq \int_0^1 D^2 g(y_s + \eta(y_s,t)) R^q_{s,t} r_s \, d\eta \]

\[ + \frac{1}{2} g^2 \|y_s,t\| \|r_s\| \|Y_{s,t}\| + \|Dg(y_s) R^q_{s,t} \| + C_g \|y_s,t\| \|r_s\|, \]

which implies that \( [Dg(y) r'_s] = Dg(y_s) [g(y_s), r_s] + Dg(y_s) r'_s \) and

\[ \|R^g_{s,t} \| \leq C_g \|R^g_{s,t} \| \|r\|_{\infty} + C_g \|r\|_{\infty} \|y\|_p \|x\|_p + C_g \|R^q \| + C_g \|y\|_p \|r\|_p. \]

Similarly, we can show that

\[ \|Dg(y) r'_s\|_{\infty} \leq C_g \|r\|_{\infty} + C_g \|r\|_{\infty} \]

\[ \|Dg(y) r'_s\|_{p} \leq 2C_g \|r\|_{\infty} \|y\|_p + C_g \|r\|_p + C_g \|r\|_p + C_g \|r\|_{\infty} \|y\|_p. \]

Combining all the above estimates into \( (3.23) \), we obtain

\[ \|R^e\|_q \leq C_g C_g \|x\|_p \|R^e\|_q \]

\[ + \|r\|_p \left\{ 2C_g \|X\|_q + C_p \|x\|_p \left[ C_g \|R^g\|_q + \frac{1}{2} C_g \|x\|_p \|y\|_p + C_g \|y\|_p \right] + C_g \|X\|_q \left( 4C^2_g \|y\|_p + 2C^2_g \right) \right\} \]

\[ + \|r_s\| \left\{ 2C_g \|X\|_q + C_p \|x\|_p \left[ C_g \|R^g\|_q + \frac{1}{2} C_g \|x\|_p \|y\|_p \right] + 4C_p C_g \|X\|_q \right\} \]

\[ + \|R^e\|_q + C_g \|X\|_q \|e\|_{\infty} + C_p \|X\|_q \left( C_g \|e\|_p + C_g \|y\|_p \|e\|_{\infty} \right) =: \bar{R}, \]

and similarly

\[ \|r\|_p \leq \|e\|_{\infty} \|x\|_p + C_g \|r\|_{\infty} \|x\|_p + \bar{R} \leq \|e\|_{\infty} \|x\|_p + C_g \|x\|_p (\|r_s\| + \|r\|_p) + \bar{R}. \]

Therefore, taking into account \( (3.15) \), we have just proved that there exists a constant

\[ M = M(p, [a,b], \|X\|_{p \cdot \text{var},[a,b]} > 1 \]

such that

\[ \|r, R\|_{p \cdot \text{var},[s,t]} \leq M \left( C_g \|x\|_{p \cdot \text{var},[s,t]} + C^2_g \|X\|_{q \cdot \text{var},[s,t]} \right) \left( \|r_s\| + \|r, R\|_{p \cdot \text{var},[s,t]} \right) \]

\[ + M \left( \|e\|_{\infty,[a,b]} + \|e\|_{p \cdot \text{var},[a,b]} + \|R^e\|_{q \cdot \text{var},[a,b]} \right) \]

\[ \leq 2M \left( C_g \|x\|_{p \cdot \text{var},[s,t]} + C^2_g \|X\|_{q \cdot \text{var},[s,t]} \right) \left( \|r_s\| + \|r, R\|_{p \cdot \text{var},[s,t]} \right) \]

\[ + M \left( \|e\|_{\infty,[a,b]} + \|e\|_{p \cdot \text{var},[a,b]} + \|R^e\|_{q \cdot \text{var},[a,b]} \right). \]
This implies
\[ \|r, R\|_{p-\text{var}[,s,t]} \leq \|r_s\| + 2M\left(\|e'\|_{\infty,[a,b]} + \|e''\|_{p-\text{var}[,a,b]} + \|R^e\|_{q-\text{var}[,a,b]} \right) \]
whenever \(2MC_g \|x\|_{p-\text{var}[,s,t]} \leq \frac{1}{2}\). The similar estimates to (3.14) and (3.15), with the usage of the sequence of greedy times \(\{\tau_k(\frac{1}{MC_g})\}_{k \in \mathbb{N}}\), lead to
\[ \|r\|_{\infty,[a,b]} + \|r, R\|_{p-\text{var}[,a,b]} \leq N \frac{1}{3MC_g} (x) \{\|r_\tau\| + 2M\left(\|e'\|_{\infty,[a,b]} + \|e''\|_{p-\text{var}[,a,b]} + \|R^e\|_{q-\text{var}[,a,b]} \right) \} e^{(\log 2)N \frac{1}{3MC_g} (x)} \]
\[ \leq 2M\left(\|e'\|_{\infty,[a,b]} + \|e''\|_{p-\text{var}[,a,b]} + \|R^e\|_{q-\text{var}[,a,b]} \right)e^{(1 + \log 2)N \frac{1}{3MC_g} (x)} \]  
(3.24)
where we use the fact that \(r_a = 0\).

**Step 3:** Assign \(z_s = \bar{y}_s - y_s\), it follows from (3.22) that
\[ e'_s = \int_0^1 [Dg(y_s + \eta z_s) - Dg(y_s)]z_s d\eta = \int_0^1 \int_0^1 D^2g\left((1 - \mu \eta)y_s + \mu \eta \bar{y}_s\right) [z_s, z_s] \eta d\mu d\eta = \bar{e}_s, \]
which is also controlled by \(x\). As a result, \(\|e'_s\| \leq \frac{1}{2} C_g \|z_s\|^2\) and
\[ \|e'\|_{\infty,[a,b]} \leq \frac{1}{2} C_g \|z\|_{\infty,[a,b]}^2, \]
(3.25)
\[ \|e''\|_{p-\text{var}[,a,b]} \leq C_g \left(\|y\|_{p-\text{var}[,a,b]} + \|y\|_{p-\text{var}[,a,b]} \right) \|z\|_{\infty,[a,b]} + 2C_g \|z\|_{\infty,[a,b]} \|z\|_{p-\text{var}[,a,b]}. \]

On the other hand,
\[ \|R^e_{s,t}\| \leq \|e''\|_{\infty,[X_{s,t}]} + C_p \left(\|x\|_{p-\text{var}[,a,b]} + \|R^e\|_{q-\text{var}[,a,b]} \right) \|X\|_{q-\text{var}[,a,b]} \|e''\|_{p-\text{var}[,a,b]} \]
thus \(\|R^e\|_{q-\text{var}[,a,b]} \leq \|e'\|_{\infty,[a,b]} \|X\|_{q-\text{var}[,a,b]} \|e''\|_{p-\text{var}[,a,b]} \]
\[ + C_p \left(\|x\|_{p-\text{var}[,a,b]} + \|X\|_{q-\text{var}[,a,b]} \|e''\|_{p-\text{var}[,a,b]} \right) \|^2 \]
(3.26)
where a direct computation shows that
\[ e'_s = \int_0^1 \int_0^1 \left\{D^2g\left((1 - \mu \eta)y_s + \mu \eta \bar{y}_s\right) ((1 - \mu \eta)y'_s + \mu \eta \bar{y}'_s, z_s, z_s) \right. \]
\[ + 2D^2g\left((1 - \mu \eta)y_s + \mu \eta \bar{y}_s\right) [z'_s, z_s] \} \eta d\mu d\eta, \]
\[ R^e_{s,t} = \int_0^1 \int_0^1 \left\{ R^D_{s,t} g(y, y) [z_t, z_t] + 2D^2g(\bar{y}, y) \left( [z_s, R^z_{s,t}] + [z'_s x_{s,t}, z'_s x_{s,t} + R^z_{s,t}] \right) + [R^z_{s,t}, z_t] \right. \}
\[ + 2D^2g(\bar{y}, y) \left( [z_s + z_t, z'_s + R^z_{s,t}] \right) \} \eta d\mu d\eta. \]

We therefore can show that there exists a generic constant \(D = D(p[,a,b], \|x\|_{p-\text{var}[,a,b]}\)) such that
\[ \|e'\|_{\infty,[a,b]} \vee \|e''\|_{p-\text{var}[,a,b]} \vee \|R^e\|_{q-\text{var}[,a,b]} \]
\[ \leq D\left(\|z\|_{\infty,[a,b]} + \|z\|_{p-\text{var}[,a,b]} + \|z\|_{\infty,[a,b]} + \|z\|_{p-\text{var}[,a,b]} \right)^2 \]
(3.27)
By replacing (3.25), (3.26), (3.27) into (3.21), and using (3.8), we conclude that there exists a generic constant \(D = D(p[,a,b], \|x\|_{p-\text{var}[,a,b]}\)) such that
\[ \|\bar{y}(x, y_a) - y(x, y_a) - \xi(x, y_a - y_a)\|_{\infty,[a,b]} \leq D\|y_a - y_a\|^2. \]
(3.28)
This, combined with the linearity of $\xi$ w.r.t. $\bar{y}_a - y_a$, shows the differentiability of $y_t(x,y_a)$ w.r.t. $y_a$.

By similar arguments in the backward direction, one can prove the following.

**Corollary 3.5** Consider the backward rough differential equation

$$h_b = h_t + \int_t^b g(h_u) dx_u, \quad \forall t \in [a,b].$$

(3.29)

Then the solution $h_t(x,h_b)$ of (3.29) is differentiable w.r.t. initial condition $h_b$, moreover, its derivatives $\frac{\partial h_t}{\partial h_b}(x,h_b)$ is the matrix solution of the linearized backward rough differential equation

$$\eta_t = \eta_t + \int_t^b Dg(h_u) \eta_u dx_u, \quad \forall t \in [a,b].$$

(3.30)

Moreover, the estimates (3.8) also hold for the solution $h_t$ of the backward equation (3.29).

**Corollary 3.6** Denote by $\varphi(t,x,y_a)$ the solution mapping of the rough equation (3.7). Then $\frac{\partial \varphi}{\partial y}(t,x,y_a)$ is globally Lipschitz continuous w.r.t. $y_a$.

**Proof:** Observe from (3.29) and (3.8) that

$$\| \xi_t(x, \bar{y}_a - y_a) \|_{\infty,[a,b]} \leq \| \bar{y}_t(x, \bar{y}_a - y(x,y_a)) + D\| \bar{y}_a - y_a \|^2 \| e^{(\log 2)N_{[a,b]}(x)} + D\| \bar{y}_a - y_a \|^2,$$

thus by fixing $y_a$, then dividing both sides by $\| \bar{y}_a - y_a \|$ and letting $\| \bar{y}_a - y_a \|$ to zero, we obtain

$$\| \frac{\partial \varphi}{\partial y}(t,x,y_a) \|_{\infty,[a,b]} \leq e^{(\log 2)N_{[a,b]}(x)} \| \frac{\partial \varphi}{\partial y}(t,x,y_a) \|_{p-\text{var},[a,b]} \leq N_{[a,b]}(x) e^{(\log 2)N_{[a,b]}(x)}.$$

(3.31)

Similarly if we fix $\bar{y}_a$ then

$$\| \frac{\partial \varphi}{\partial y}(t,x,\bar{y}_a) \|_{\infty,[a,b]} \leq e^{(\log 2)N_{[a,b]}(x)} \| \frac{\partial \varphi}{\partial y}(t,x,\bar{y}_a) \|_{p-\text{var},[a,b]} \leq N_{[a,b]}(x) e^{(\log 2)N_{[a,b]}(x)}.$$

(3.32)

Next for $\xi_t = \frac{\partial \varphi}{\partial y}(t,x,y_a) \xi_a$ and $\bar{\xi}_t = \frac{\partial \varphi}{\partial y}(t,x,\bar{y}_a) \xi_a$, we consider the difference $r_t = \bar{\xi}_t - \xi_t$, which satisfies $r_a = 0$ and

$$r_{s,t} = \int_s^t \left( Dg(\bar{y}_u - y_u) \xi_u - Dg(y_u) \xi_u \right) dx_u = \int_s^t \left( Dg(\bar{y}_u) - Dg(y_u) \right) \bar{\xi}_u dx_u + \int_s^t Dg(y_u) r_u dx_u$$

$$= e_{s,t} + \int_s^t Dg(y_u) r_u dx_u.$$

(3.33)

Since (3.33) has the form of (3.22), by the same arguments as in Step 2 of Theorem 3.1 we obtain a similar estimate to (3.24), i.e. there exists a constant $M = M(p,[a,b],\|x\|_{p-\text{var},[a,b]}) > 1$ such that

$$\| r \|_{\infty} \| R \|_{p-\text{var},[a,b]} \leq 2M \left( \| e' \|_{\infty,[a,b]} + \| e' \|_{p-\text{var},[a,b]} + \| R e \|_{q-\text{var},[a,b]} \right) e^{(1+\log 2)N_{[a,b]}(x)},$$

(3.34)

where we use the fact that $r_a = 0$. To estimate the terms in the right hand side of (3.37), we derive from (3.33) that

$$e' = \left( Dg(\bar{y}_s) - Dg(y_s) \right) \bar{\xi}_s = \int_0^1 D^2 g \left( (1-\eta)y_s + \eta \bar{y}_s \right) [z_s, \xi_s] d\eta.$$
Using (3.32) and the same estimates as in Step 3 of Theorem 3.4, we can prove that there exists a generic constant $M = \langle p, [a, b], \|x\|_{p-\text{var}, [a, b]} \rangle > 1$ such that

$$\|e'\|_{\infty, [a, b]}, \|e'\|_{p-\text{var}, [a, b]} \leq M \|z\|_{p-\text{var}, [a, b]}.$$  \hspace{1cm} (3.35)

Similarly, there exists a generic constant $M$ that

$$\|R e\|_{q-\text{var}, [a, b]} \leq M (\|z_a\| + \|z, R\|_{p-\text{var}, [a, b]}).$$  \hspace{1cm} (3.36)

Replacing (3.35) and (3.36) into (3.37), we conclude that there exists a generic constant $M > 1$ such that

$$\|r\|_{\infty} \vee \|r, R\|_{p-\text{var}, [a, b]} \leq M (\|z_a\| + \|z, R\|_{p-\text{var}, [a, b]}),$$

where the last inequality is due to (3.8). This proves the global Lipschitz continuity of $\frac{\partial \psi}{\partial y}(t, x, y_a)$ w.r.t. $y_a$.

We are now able to formulate and prove the existence theorem.

**Theorem 3.7 (Existence)** There exists a solution to (1.1) on the interval $[a, b]$ with the initial value $y_a$.

**Proof:** The main idea is similar to [18], which applies the Doss-Sussmann transformation [19] for the rough integral, thus it is enough to prove the existence of the solution for the transformed system. The proof is divided into several steps.

**Step 1.** Denote by $\varphi(t, x, y_a)$ the solution mapping of the rough equation (3.7), and by $\psi(t, x, h_b)$ the solution mapping of the backward equation (3.29). Then $\varphi(t, x) \circ \psi(t, x) = I^{d \times d}$, and due to Theorem 3.4 and Corollary 3.5 $\varphi, \psi$ are continuously differentiable w.r.t. $y_a$ and $h_b$ respectively. In addition,

$$\frac{\partial \varphi}{\partial y}(t, x, y_a) \frac{\partial \psi}{\partial h}(t, x, y_b) = \frac{\partial \psi}{\partial h}(t, x, y_b) \frac{\partial \varphi}{\partial y}(t, x, y_a) = I^{d \times d}.$$  \hspace{1cm} (3.38)

Similar to (3.31) and (3.32), we obtain

$$\left| \left| \frac{\partial \psi}{\partial h}(t, x, y_b) \right| \right|_{\infty, [a, b]} \leq e^{(\log 2)N_1[a, b]}(x).$$  \hspace{1cm} (3.39)

Now consider the ordinary differential equation

$$\dot{z}_t = \frac{\partial \psi}{\partial h}(t, x, \varphi(t, x, z_t)) f(\varphi(t, x, z_t)) = F(t, z_t), \quad t \in [a, b].$$  \hspace{1cm} (3.40)

Similar to Corollary 3.6 $\frac{\partial \psi}{\partial h}(t, x, z)$ is globally Lipschitz continuous in $z$. On the other hand $\varphi(t, x, z)$ is also globally Lipschitz continuous w.r.t. $z$. Hence $\frac{\partial \psi(t, x, \varphi(t, x, z_t))}{\partial h}$ is globally Lipschitz continuous w.r.t. $z$. Because $f$ is also globally Lipschitz continuous and of linear growth, it follows that $F(t, z)$ in the right hand side of (3.40) satisfies the local Lipschitz continuity and also the linear growth. Hence there exists a unique solution for (3.40).

**Step 2.** Next consider the transformation $y_t = \varphi(t, x, z_t)$ for $t \in [a, b]$, we are going to prove that $y_t$ is a solution of the equation

$$dy_t = f(y_t)dt + g(y_t)dx_t.$$  \hspace{1cm} (3.41)

Indeed, it first follows that

$$y_{s,t} = \varphi(t, x, z_t) - \varphi(t, x, z_s) + \varphi(t, x, z_s) - \varphi(s, x, z_s)$$

$$= \left[ \varphi(t, x, z_t) - \varphi(t, x, z_s) \right] + \int_s^t g(\varphi(u, x, z_s))du.$$  \hspace{1cm} (3.42)
The term in the square bracket satisfies, given (3.38) and Theorem 3.4,

$$\|\varphi(t, x, z_t) - \varphi(t, x, z_s) - \frac{\partial \varphi}{\partial z}(s, x, z_s)z_{s,t}\|
\leq \|\varphi(t, x, z_t) - \varphi(t, x, z_s) - \frac{\partial \varphi}{\partial z}(t, x, z_s)\|z_{s,t}
\leq D\|z_{s,t}\|^2 + \left\|\int_s^t Dg(\varphi(u, x, z_s))\frac{\partial \varphi}{\partial z}(u, x, z_s)du\right\|z_{s,t}
\leq D(t - s)^{1+\alpha}.$$  (3.43)

for some generic constant $D$. Also, due to (3.40)

$$\frac{\partial \varphi}{\partial z}(s, x, z_s)z_{s,t} = \frac{\partial \varphi}{\partial z}(s, x, z_s)F(s, z_s)(t - s) + D(t - s)^2
= \frac{\partial \varphi}{\partial z}(s, x, z_s)\frac{\partial \psi}{\partial h}(s, x, \varphi(s, x, z_s))f(\varphi(s, x, z_s))(t - s) + D(t - s)^2
= f(y_s)(t - s) + D(t - s)^2.$$  (3.44)

On the other hand, it follows from (2.6) that

$$\int_s^t g(\varphi(u, x, z_s))du = g(\varphi(s, x, z_s)) \otimes x_{s,t} + Dg(\varphi(s, x, z_s))[\varphi(\cdot, x, z_s)]' \otimes \mathbb{X}_{s,t} + D(t - s)^{3\alpha}
= g(y_s) \otimes x_{s,t} + Dg(y_s)g(y_s) \otimes \mathbb{X}_{s,t} + D(t - s)^{3\alpha}.$$  (3.45)

Combining (3.42) - (3.44) yields

$$y_{s,t} = f(y_s)(t - s) + g(y_s) \otimes x_{s,t} + Dg(y_s)g(y_s) \otimes \mathbb{X}_{s,t} + D(t - s)^{3\alpha}.$$  (3.46)

Equation (3.46) implies that $y$ is controlled by the driving path $x$ with $y'_s = g(\varphi(s, x, z_s)) = g(y_s)$ and so is $g(y)$ with $[g(y)]' = Dg(y_s)y'_s = Dg(y_s)g(y_s)$, for all $s \in [a, b]$. Now take any finite partition $\Pi$ of $[s, t]$ for $||\Pi|| = \max_{[u, v] \in \Pi} |v - u| \ll 1$, it follows from (3.46) that

$$y_{s,t} = \sum_{[u, v] \in \Pi} y_{u,v}
= \sum_{[u, v] \in \Pi} f(y_u)(v - u) + \sum_{[u, v] \in \Pi} \left( g(y_u) \otimes x_{u,v} + Dg(y_u)g(y_u) \otimes \mathbb{X}_{u,v} \right) + \sum_{[u, v] \in \Pi} D(v - u)^{3\alpha}
= \sum_{[u, v] \in \Pi} f(y_u)(v - u) + \sum_{[u, v] \in \Pi} \left( g(y_u) \otimes x_{u,v} + Dg(y_u)g(y_u) \otimes \mathbb{X}_{u,v} \right) + D||\Pi||^{3\alpha - 1}.$$  (3.47)

Let $||\Pi|| \to 0$, the first term in (3.47) converges to $\int_s^t f(y_u)du$ while the second term converges to the Gubinelli rough integral $\int_s^t g(y_u)du$. We conclude that

$$y_{s,t} = \int_s^t f(y_u)du + \int_s^t g(y_u)du,$$

which proves the existence part.

The next result provides the norm estimates for the solution of (1.1).

**Theorem 3.8** The supremum and $p$ - variation norms of the solution are estimated as follows

$$\|y\|_{\infty, [a, b]} \leq \left[ \|y_a\| + \left( \frac{\|f(0)\|}{Cf} + \frac{1}{Cp} \right)N_{[a,b]}(x) \right] e^{AL(b-a)},$$  (3.48)

$$\|y, R\|_{p-\text{var}, [a, b]} \leq \left[ \|y_a\| + \left( \frac{\|f(0)\|}{Cf} + \frac{1}{Cp} \right)N_{[a,b]}(x) \right] e^{AL(b-a)}N_{[a,b]}^{p-1}(x) - \|y_a\|,$$  (3.49)

where $\|y, R\|_{p-\text{var}, [s, t]} := \|y\|_{p-\text{var}, [s, t]} + \|R\|_{q-\text{var}, [s, t]}$. 

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Proof: To prove (3.48), rewrite (1.1) in the integral form

\[ y_{s,t} = \int_s^t f(y_u) du + \int_s^t g(y_u) dx_u. \] (3.50)

Together with (1.2) and (2.7), we obtain

\[
\begin{align*}
\|y_{s,t}\| & \leq \int_s^t \| f(y_u) \| du + \int_s^t \| g(y_u) \| dx_u \\
& \leq \int_s^t (L\|y_u\| + \|f(0)\|) du + C_g \| x \|_{p-var,[s,t]} + C_g^2 \| X \|_{q-var,[s,t]} + C_p \left\{ 2C_g^2 \| X \|_{q-var,[s,t]} \| y \|_{p-var,[s,t]} \\
& \quad + \| x \|_{p-var,[s,t]} \left[ C_g \| R'y \|_{q-var,[s,t]} + \frac{1}{2} C_g^2 \| x \|_{p-var,[s,t]} \| y \|_{p-var,[s,t]} \right] \right\} \\
& \leq \int_s^t (L\|y_u\| + \|f(0)\|) du + C_g \| x \|_{p-var,[s,t]} + C_g^2 \| X \|_{q-var,[s,t]} \\
& \quad + C_p \left\{ 2C_g^2 \| X \|_{q-var,[s,t]} + \frac{1}{2} C_g^2 \| x \|_{p-var,[s,t]} \right\} \left( \| y \|_{p-var,[s,t]} + \| R'y \|_{q-var,[s,t]} \right),
\end{align*}
\]

which yields

\[
\begin{align*}
\| y \|_{p-var,[s,t]} & \leq \int_s^t (L\|y_u\| + \|f(0)\|) du + C_g \| x \|_{p-var,[s,t]} + C_g^2 \| X \|_{q-var,[s,t]} \\
& \quad + C_p \left\{ 2C_g^2 \| X \|_{q-var,[s,t]} + \frac{1}{2} C_g^2 \| x \|_{p-var,[s,t]} \right\} \left( \| y \|_{p-var,[s,t]} + \| R'y \|_{q-var,[s,t]} \right),
\end{align*}
\]

By similar arguments, we can show that

\[
\begin{align*}
\| R'y \|_{q-var,[s,t]} & \leq \int_s^t (L\|y_u\| + \|f(0)\|) du + C_g^2 \| X \|_{q-var,[s,t]} \\
& \quad + C_p \left\{ 2C_g^2 \| X \|_{q-var,[s,t]} + \frac{1}{2} C_g^2 \| x \|_{p-var,[s,t]} \right\} \left( \| y \|_{p-var,[s,t]} + \| R'y \|_{q-var,[s,t]} \right).
\end{align*}
\]

Therefore by assigning \( \| y, R \|_{p-var,[s,t]} = \| y \|_{p-var,[s,t]} + \| R'y \|_{q-var,[s,t]}, \) we obtain

\[
\begin{align*}
\| y, R \|_{p-var,[s,t]} & \leq 2 \int_s^t (L\|y_{p-var,[s,u]}\| + L\|y_s\| + \|f(0)\|) du + C_g \| x \|_{p-var,[s,t]} + 2C_g^2 \| X \|_{q-var,[s,t]} \\
& \quad + C_p \left\{ 2C_g^2 \| X \|_{q-var,[s,t]} + \frac{1}{2} C_g^2 \| x \|_{p-var,[s,t]} \right\} \| y, R \|_{p-var,[s,t]} \cdot \tag{3.51}
\end{align*}
\]

Observe that if \( 2C_pC_g \| x \|_{p-var,[s,t]} < 1 \) then

\[
2C_pC_g \| x \|_{p-var,[s,t]} > C_p \left\{ 2C_g^2 \| X \|_{q-var,[s,t]} + \frac{1}{2} C_g^2 \| x \|_{p-var,[s,t]} \right\} \vee C_g \| x \|_{p-var,[s,t]}.
\]

This follows that

\[
\begin{align*}
\| y, R \|_{p-var,[s,t]} & \leq \int_s^t 4L\| y_{p-var,[s,u]} \| du + 4(\| f(0) \| + L\|y_s\|)(t - s) + \frac{1}{C_p}
\end{align*}
\]

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whenever $2C_pC_g \|x\|_{p-\text{var},[s,t]} \leq \frac{1}{2}$. Applying the continuous Gronwall lemma, we obtain

$$
\|y, R\|_{p-\text{var},[s,t]} \leq 4(\|f(0)\| + L\|y_s\|)(t-s) + \frac{1}{C_p} \\
+ \int_s^t 4L e^{4L(t-u)} \left[4(\|f(0)\| + L\|y_s\|)(u-s) + \frac{1}{C_p}\right] du \\
\leq \left(\frac{\|f(0)\|}{L} + \frac{1}{C_p} + \|y_s\|\right) e^{4L(t-s)} - \|y_s\|
$$

(3.52)

whenever $4C_pC_g \|x\|_{p-\text{var},[s,t]} \leq 1$. By constructing the sequence of greedy time $\{\tau_i(\frac{1}{4C_pC_g}, [a,b], p)\}$ and use similar estimates to the proof of Theorem 2.4, we obtain (3.48) and (3.49).

Theorem 3.9 (Uniqueness) The solution $y_t(x, y_a)$ of (3.41) is uniformly continuous w.r.t. $y_a$. In particular, there exists a unique solution with respect to the initial condition $y_a$.

Proof: The proof follows similar arguments as in Proposition 3.3. Indeed, let us consider two solutions $y_t(x, y_a)$ and $\tilde{y}_t(x, \tilde{y}_a)$ and their difference $z_t = \tilde{y}_t - y_t$, which satisfies the integral rough equation

$$
z_t = z_a + \int_a^t [f(\tilde{y}_s) - f(y_s)] ds + \int_a^t [g(\tilde{y}_s) - g(y_s)] ds.
$$

Then the estimate (3.16) has the form

$$
\|z, R\|_{p-\text{var},[s,t]} \leq \int_s^t [f(\tilde{y}_u) - f(y_u)] du + \int_s^t [g(\tilde{y}_u) - g(y_u)] dx_u \\
\leq \int_s^t C_f\|u\| du + C_g\|z\| \|x\|_{p-\text{var},[s,t]} + 2C_g^2 \|z\| \|X\|_{q-\text{var},[s,t]} \\
+ C_p \left\{ \|x\|_{p-\text{var},[s,t]} \left\| Rg(\tilde{y}) - g(y) \right\|_{q-\text{var},[s,t]} + \|X\|_{q-\text{var},[s,t]} \left\| [g(\tilde{y}) - g(y)]' \right\|_{p-\text{var},[s,t]} \right\},
$$

which, together with (3.17), (3.18) and (3.49) yields

$$
\|z, R\|_{p-\text{var},[s,t]} \leq 2 \int_s^t C_f\|u\| du + 4C_p \left\{ C_g \|x\|_{p-\text{var},[s,t]} \vee C_g^2 \|x\|_{p-\text{var},[s,t]} \right\} \times \\
\times \left(1 + \|\tilde{y}, R\|_{p-\text{var},[s,t]} + \|y, R\|_{p-\text{var},[s,t]} \right) \left(\|z_s\| + \|z, R\|_{p-\text{var},[s,t]} \right) \\
\leq 2 \int_s^t C_f\|u\| du + 4C_p \left\{ C_g \|x\|_{p-\text{var},[s,t]} \vee C_g^2 \|x\|_{p-\text{var},[s,t]} \right\} \times \\
\times \left\{1 + 2 \left[\|y_a\| \vee \|\tilde{y}_a\| + \left(\frac{\|f(0)\|}{C_f} + \frac{1}{C_p}\right) N_{\text{var},[a,b]} (x) \right] e^{4L(b-a)} N_{\text{var},[a,b]} (x) \right\} \times \\
\left(\|z_s\| + \|z, R\|_{p-\text{var},[s,t]} \right),
$$

(3.53)

Provided that $\|y_a\|, \|\tilde{y}_a\| \leq r_0$, (3.53) has the form

$$
\|z, R\|_{p-\text{var},[s,t]} \leq 2 \int_s^t C_f\left(\|z_s\| + \|z, R\|_{p-\text{var},[s,t]} \right) du + 4C_p \left\{ C_g \|x\|_{p-\text{var},[s,t]} \vee C_g^2 \|x\|_{p-\text{var},[s,t]} \right\} \times \\
\times \Lambda(x, [a,b]) \left(\|z_s\| + \|z, R\|_{p-\text{var},[s,t]} \right),
$$

(3.54)
for some constant \( \Lambda(x, [a, b]) > 1 \) (which also depends on \( r_0 \)). Hence,

\[
\| z_s \| + \| z, R \|_{p\text{-var},[s,t]} \leq \| z_s \| + 4 \int_s^t C_f \left( \| z_s \| + \| z, R \|_{p\text{-var},[u,t]} \right) du
\]

whenever \( 4 C_p \left( C_g \| x \|_{p\text{-var},[s,t]} \vee C_g^2 \| x \|_{p\text{-var},[s,t]}^2 \right) \Lambda(x, [a, b]) \leq \frac{1}{2} \). Similar arguments to the proof of Theorem 3.3 then show that

\[
\| z, R \|_{p\text{-var},[a,b]} \leq \| z_a \| e^{4L(b-a)} N_{p\text{-var},[a,b]}^{p-1}(x) - \| z_a \|.
\]

where \( N_{p\text{-var},[a,b]}(x) \) is the number of greedy times in the sequence \( \{ \tau_i(\frac{1}{8} C_p C_g \Lambda(x, [a, b]), [a, b], p) \} \). Using (2.9), we finally derive

\[
\| z, R \|_{p\text{-var},[a,b]} \leq \| z_a \| e^{4L(b-a)} \left( 1 + \left[ 8 C_p C_g \Lambda(x, [a, b]) \right]^{p-1} \| x \|_{p\text{-var},[a,b]}^{p-1} \right),
\]

which proves the continuity of the solution \( y_t(x, y_a) \) on the initial condition \( y_a \). In case \( \tilde{y}_a = y_a \), the right hand side of (3.55) equals zero, which implies \( \| \tilde{y} - y \|_{p\text{-var},[a,b]} = 0 \), therefore the solution is unique given the initial condition \( y_a \).

\[
\square
\]

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