Non trivial zeros of the Riemann zeta function

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Abstract

In this paper, we consider the representation of the Riemann zeta function \( \zeta \) defined by Abel’s summation formula. We show that: if \( |\zeta(s)| = 0 \) then \( |\zeta(1-s)| \neq 0 \) for any point \( s \) in the critical strip where the real part is not equal to one-half.

Keywords: Riemann hypothesis, Riemann zeta function, Non trivial zeros.

AMS subject classifications: 00A05

1 Main results

Consider the representation of the Riemann zeta function \( \zeta \) defined by Abel’s summation formula [1], page 14 Equation 2.1.5 as

\[
\zeta(s) := -\frac{s}{1-s} - s \int_1^{+\infty} u^{-1-s}\{u\}du, \quad \Re(s) > 0, \quad \Im(s) \in \mathbb{R},
\]  

(1)

where \( \{u\} \) is the fractional part of the real \( u \). We shall prove the following result:

**Theorem 1.** Consider the Riemann zeta function \( \zeta \) given by Equation (1). For any point \( s \in \mathbb{C} \) such that \( \Re(s) \in \left( \frac{1}{2}, 1 \right) \) and \( \Im(s) \in \mathbb{R} \) we have

\[
\left| \Re \left( \frac{2-\bar{s}}{s(1+s)}\zeta(s) - \frac{\zeta(1-\bar{s})}{1-s} \right) \right| > 0.
\]

**Proof.** Let be \( s \in \mathbb{C} \) such that \( \Re(s) \in \left( \frac{1}{2}, 1 \right) \) and \( \Im(s) \in \mathbb{R} \). Consider the Equation (1), using the integration by parts formula, that gives

\[
\zeta(s) = -\frac{s}{1-s} - \frac{1}{2} + s(1+s) \int_1^{+\infty} u^{-2-s}\eta(u)du.
\]

(2)
where by Dirichlet’s Theorem, the real periodic function \( \eta : [1, +\infty) \rightarrow \mathbb{R} \) is defined as

\[
\eta(u) := \int_1^u \left( \frac{1}{2} - \{v\} \right) dv = \sum_{j \in \mathbb{Z}^*} \frac{1}{(j2\pi)^2} \left( 1 - \exp(ij2\pi u) \right), \quad \forall u \geq 1. \tag{3}
\]

Equation (2), implies

\[
\zeta(s) - \zeta(1-s) \frac{1}{s(1+s)} - \frac{1}{(1-s)(2-s)} = -\frac{1}{2} \left( 1 - \frac{1}{s} + \frac{1}{s} \right) + \frac{1}{2} \left( \frac{1}{1 - s} + \frac{1}{s} \right) + \int_1^{+\infty} \left( u^{-2-s} - u^{-3+\pi} \right) \eta(u) du. \tag{4}
\]

By Equation (3), we have \( \eta(n) = 0 \) for all \( n \in \mathbb{N} \). Using the integration by parts formula, for every \( n \in \mathbb{N} \) we have

\[
\int_1^n \Im \left( u^{-2-s} - u^{-3+\pi} \right) \eta(u) du = \int_1^n u^{-3+\pi} \left( u^{-2\Re(s)+1} - 1 \right) \eta(u) du
\]

\[= - \int_1^n \frac{d}{du} \left( u^{-2\Re(s)+1} - 1 \right) \eta(u) \int_1^u u^{-3+\pi} v du. \]

In other words,

\[
(2 - \pi) \int_1^n \Im \left( u^{-2-s} - u^{-3+\pi} \right) \eta(u) du = - \int_1^n \frac{d}{du} \left( u^{-2\Re(s)+1} - 1 \right) \eta(u) \left( 1 - u^{-2+\pi} \right) du.
\]

Thanks to Equation (4), we obtained

\[
(2 - \pi) \left( \frac{\zeta(s)}{s(1+s)} - \frac{\zeta(1-s)}{(1-s)(2-s)} \right) = -\frac{1}{2} \left( 2 - \pi \right) \left( \frac{1}{1 - s} + \frac{1}{s} - \frac{1}{1 - \pi} - \frac{1}{\pi} \right) \tag{5}
\]

\[- \lim_{n \to +\infty} \int_1^n \frac{d}{du} \left( u^{-2\Re(s)+1} - 1 \right) \eta(u) \left( 1 - u^{-2+\pi} \right) du.
\]

We recall that the function \( \eta \) is given by Equation (3) and for every \( u \in [1, +\infty)/\mathbb{N} \) it satisfies the following equation

\[
\frac{d}{du} \left( u^{-2\Re(s)+1} - 1 \right) \eta(u) = -(2\Re(s)-1)u^{-2\Re(s)}\eta(u) + u^{-2\Re(s)+1} - 1 \left( \frac{1}{2} - \{u\} \right).
\]

Since \( 2\Re(s) > 1 \) then

\[
\left| \frac{d}{du} \left( u^{-2\Re(s)+1} - 1 \right) \eta(u) \right| \leq \frac{1}{2} (2\Re(s)-1)u^{-2\Re(s)}(u-1) - \frac{1}{2} (u^{-2\Re(s)+1} - 1), \quad \forall u \geq 1.
\]
In other words
\[
\left| \frac{d}{du} \left( u^{-2\Re(s)+1} - 1 \right) \eta(u) \right| \leq \frac{d}{du} \left( u^{-2\Re(s)+1} - 1 \right) \eta_s(u), \quad \forall u \geq 1.
\]
where \( \eta_s \) is the constant function defined as \( \eta_s(u) = -\frac{1}{2} \) for all \( u \geq 1 \). Equation (5) implies
\[
\left| \Re \left( (2 - \overline{s}) \left( \frac{\zeta(s)}{s(1+s)} - \frac{\zeta(\overline{1-s})}{(1-\overline{s})(2-\overline{s})} \right) \right) \right| \geq \frac{1}{2} \left| \Re \left( (2 - \overline{s}) \left( \frac{1}{1-s} + \frac{1}{s} - \frac{1}{1-\overline{s}} - \frac{1}{\overline{s}} \right) \right) \right|
\]
\[
-\lim_{n \to +\infty} \int_1^n \frac{d}{du} \left( u^{-2\Re(s)+1} - 1 \right) \eta_s(u) (1 - \Re(u^{-2+\overline{s}})) du.
\]
Equivalent to
\[
\left| \Re \left( (2 - \overline{s}) \left( \frac{\zeta(s)}{s(1+s)} - \frac{\zeta(\overline{1-s})}{(1-\overline{s})(2-\overline{s})} \right) \right) \right| \geq \frac{1}{2} \left| \Re \left( (2 - \overline{s}) \left( \frac{1}{1-s} + \frac{1}{s} - \frac{1}{1-\overline{s}} - \frac{1}{\overline{s}} \right) \right) \right|
\]
\[
+ \frac{1}{2} \Re \left( (2 - \overline{s}) \int_1^{+\infty} \left( u^{-2-s} - u^{-3+\overline{s}} \right) du \right),
\]
or even,
\[
\left| \Re \left( (2 - \overline{s}) \left( \frac{\zeta(s)}{s(1+s)} - \frac{\zeta(\overline{1-s})}{(1-\overline{s})(2-\overline{s})} \right) \right) \right| \geq \frac{1}{2} \left| \Re \left( (2 - \overline{s}) \left( \frac{1}{1-s} + \frac{1}{s} - \frac{1}{1-\overline{s}} - \frac{1}{\overline{s}} \right) \right) \right|
\]
\[
+ \frac{1}{2} \Re \left( \frac{2 - \overline{s}}{1+s} - 1 \right).
\]
Since \( \Re(s) \in (\frac{1}{2}, 1) \), we have
\[
\left| \Re \left( (2 - \overline{s}) \left( \frac{1}{1-s} + \frac{1}{s} - \frac{1}{1-\overline{s}} - \frac{1}{\overline{s}} \right) \right) \right| + \Re \left( \frac{2 - \overline{s}}{1+s} - 1 \right) > 0.
\]
We obtained
\[
\left| \Re \left( (2 - \overline{s}) \left( \frac{\zeta(s)}{s(1+s)} - \frac{\zeta(\overline{1-s})}{(1-\overline{s})(2-\overline{s})} \right) \right) \right| > 0.
\]

\[ \square \]

References

[1] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function (revised by D.R. Heath-Brown), Clarendon Press, Oxford. (1986).