

THE GENERAL J-FLOWS

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ABSTRACT. We study the general J-flows. We use Moser iteration to obtain the uniform estimate.

1. Introduction

In [5], Donaldson first defined the $J$-flow in the setting of moment maps. Later in the study of Mabuchi energy, Chen [2] independently defined the $J$-flow as the gradient flow for the $J$-functional under the normalization of the $I$-functional.

Let $(M, \omega)$ be a closed Kähler manifold of complex dimension $n \geq 2$, and $\chi$ a smooth closed real $(1,1)$ form in $\Gamma^k_\omega$. Throughout this paper, $\Gamma^k_\omega$ is the set of all the real $(1,1)$ forms whose eigenvalue set with respect to $\omega$ belong to $k$-positive cone in $\mathbb{R}^n$. We consider the general $J$-flow for $n \geq k > l \geq 1$,

$$\frac{\partial u}{\partial t} = c - \frac{\chi^l_u \wedge \omega^{n-l}}{\chi^k_u \wedge \omega^{n-k}},$$

(1.1)

where

$$c = \int_M \chi^l \wedge \omega^{n-l} \quad \text{and} \quad \chi_u = \chi + \sqrt{-1} \partial \bar{\partial} u.$$  

(1.2)

Indeed, the results of this paper apply to more general forms

$$\frac{\partial u}{\partial t} = c - \sum_{l=0}^{k-1} b_l \chi^l_u \wedge \omega^{n-k}, \quad b_l \geq 0 \text{ and } \sum_{l=0}^{k-1} b_l > 0.$$  

(1.3)

We recall the general $J$-functionals, which was actually defined by Fang, Lai and Ma [6].

Let $\mathcal{H}$ be the space

$$\mathcal{H} := \{ u \in C^\infty(M) \mid \chi_u \in \Gamma^k_\omega \}.$$  

(1.4)

For any curve $v(s) \in \mathcal{H}$, we define the functional $J_m$ by

$$\frac{dJ_m}{ds} = \int_M \frac{\partial v}{\partial s} \chi^m_v \wedge \omega^{n-m}$$  

(1.5)

for any $0 \leq m \leq k$. The parabolic flow (1.1) can thus be viewed as the negative gradient flow for the $J_l$-functional under the normalization $J_k(u) = 0$. 

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It is easy to see that the $J$-flow is a special case,
\[
\frac{\partial u}{\partial t} = c - \frac{\chi_u^{n-1} \wedge \omega}{\chi_u^n}.
\] (1.6)

Chen [3] proved the long time existence of the solution to the $J$-flow. Weinkove [13] [14] showed the convergence under a strong condition. In [8], Song and Weinkove put forward and established a necessary and sufficient condition for convergence, which is called the cone condition. A numerical version of the cone condition, which is easier to check in concrete examples, was proposed by Lejmi and Székelyhidi [7]. Later, Collins and Székelyhidi [4] affirmed the numerical cone condition on toric manifolds.

In this paper, we prove the following theorem.

**Theorem 1.1.** Let $(M, \omega)$ be a closed Kähler manifolds of complex dimension $n \geq 2$ and $\chi$ a closed form in $\Gamma^k \omega$. Suppose that there exists $\chi' \in [\chi] \cap \Gamma^k \omega$ satisfying the cone condition
\[
ck\chi'^{k-1} \wedge \omega^{n-k} > l\chi'^{n-1} \wedge \omega^{n-l}.
\] (1.7)

Suppose that $u$ is the solution to the general $J$-flow (1.1) on maximal time $[0, T)$. Then there exists a uniform constant $C > 0$ such that for any $t \in [0, T)$
\[
\sup_M u(x, t) - \inf_M u(x, t) < C.
\] (1.8)

We shall apply the Moser iteration approach in [10] and an idea from Blocki [1] and Székelyhidi [12]. In order to obtain a time-independent uniform estimate, we apply the Moser iteration locally in time $t$. Since all other arguments are the same as those in [11], we immediately have a corollary. In [11], we applied the ABP estimate to obtain the uniform estimate.

**Corollary 1.2.** Under the assumption of Theorem 1.1, there exists a long time solution $u$ to the general $J$-flow (1.1). Moreover, the solution $u$ converges in $C^\infty$ to $u_\infty$ with $\chi_{u_\infty} \in \Gamma^k \omega$ satisfying
\[
\chi_{u_\infty}^l \wedge \omega^{n-l} = c\chi_{u_\infty}^k \wedge \omega^{n-k}.
\] (1.9)

2. **The uniform estimate**

As in [8] [6], the cone condition (1.7) is necessary and sufficient. In other words, the cone condition means that there is a $C^2$ function $v$ such that $\chi_v \in \Gamma^k \omega$ and
\[
ck\chi_v^{k-1} \wedge \omega^{n-k} > l\chi_v^{l-1} \wedge \omega^{n-l}.
\] (2.1)

We may assume that there is $c > \epsilon > 0$ such that
\[
(c - 2\epsilon)k\chi_v^{k-1} \wedge \omega^{n-k} > l\chi_v^{l-1} \wedge \omega^{n-l}.
\] (2.2)
Without loss of generality, we may also assume that \( \sup_M v = -2 \epsilon \).

Along the solution flow \( u(x,t) \) to equation (1.1),

\[
\frac{d}{dt} J_k(u) = \int_M \frac{\partial u}{\partial t} \chi_k^l \wedge \omega^{n-l} = c \int_M \chi_k^l \wedge \omega^{n-l} - \int_M \chi_l^l \wedge \omega^{n-l} = 0.
\]

(2.3)

**Lemma 2.1.** At any time \( t \),

\[
0 \leq \sup_M u \leq -C_1 \inf_M u + C_2 \quad \text{and} \quad \inf_M u(x,t) \leq 0.
\]

(2.4)

**Proof.** Choosing the path \( v(s) = su \) and noting that \( J_k(u) = 0 \),

\[
\frac{1}{k + 1} \sum_{i=0}^k \int_M u \chi_i^l \wedge \chi^{k-i} \wedge \omega^{n-k} = 0.
\]

(2.5)

The first and third inequalities in (2.4) then follow from (2.5) and Gårding’s inequality.

Rewriting (2.5),

\[
\int_M u \chi_k \wedge \omega^{n-k} = -\sum_{i=1}^k \int_M u \chi_i^l \wedge \chi^{k-i} \wedge \omega^{n-k}.
\]

(2.6)

Let \( C \) be a positive constant such that

\[
\omega^n \leq C \chi^k \wedge \omega^{n-l}.
\]

(2.7)

Then as in [9],

\[
\int_M u \omega^n = \int_M \left( u - \inf_M u \right) \omega^n + \int_M \inf_M u \omega^n \\
\leq C \int_M \left( u - \inf_M u \right) \chi^k \wedge \omega^{n-k} + \inf_M u \int_M \omega^n \\
\leq \inf_M u \left( \int_M \omega^n - (k+1)C \int_M \chi^{n-\alpha} \wedge \omega^\alpha \right).
\]

(2.8)

The second inequality in (2.4) then follows from (2.8) (see Yau [15]).

\( \square \)

**Proof of Theorem 1.1.** According to Lemma 2.1, it suffices to prove a lower bound for \( \inf_M (u - v)(x,t) \). We claim that

\[
\inf_M (u - v)(x,t) > \inf_M u (x,0) - C_0,
\]

(2.9)

where \( C_0 \) is to be specified later.
Differentiating the general $J$-flow \[1.1\] with respect to $t$,
\[
\frac{\partial u_t}{\partial t} = \chi_u^k \wedge \omega^{n-k} \left( \frac{k \sqrt{-1} \partial \bar{\partial} u_t \wedge \chi_u^{k-1} \wedge \omega^{n-k}}{\chi_u^k \wedge \omega^{n-k}} - \frac{l \sqrt{-1} \partial \bar{\partial} u_t \wedge \chi_u^{l-1} \wedge \omega^{n-l}}{\chi_u^l \wedge \omega^{n-l}} \right),
\] (2.10)
which is also parabolic. Applying the maximum principle, $u_t$ reaches the extremal values at $t = 0$. So when $t \leq 1$, we have
\[
\inf_{M} (u - v)(x, t) \geq \inf_{M} u(x, t) + 2\epsilon \geq t \inf_{M} u_t(x, 0) + 2\epsilon. \tag{2.11}
\]
Therefore if \[2.9\] fails, there must be time $t_0 > 1$ such that
\[
\inf_{M} (u - v)(x, t_0) = \inf_{M \times [0,t_0]} (u - v)(x, t) = \inf_{M} u_t(x, 0) - C_0 \leq 0. \tag{2.12}
\]
For $p \geq 1$, we consider the integral
\[
I = \int_{M} \varphi^p \left[ ((c - u_t) \chi_u^k \wedge \omega^{n-k} - (c - \epsilon) \chi_u^k \wedge \omega^{n-k}) - (\chi_u^l \wedge \omega^{n-l} - \chi_u^l \wedge \omega^{n-l}) \right]. \tag{2.13}
\]
It is easy to see that form some constant $C > 0$
\[
I \leq C \int_{M} \varphi^p \omega^n. \tag{2.14}
\]
\[
I = \int_{M} \varphi^p \left[ (c - \epsilon - su_t + s\epsilon) \chi_{su+(1-s)v}^k \wedge \omega^{n-k} - \chi_{su+(1-s)v}^l \wedge \omega^{n-l} \right] \bigg|_{s=0} \tag{2.15}
\]
For simplicity, we denote $\chi_s = \chi_{su+(1-s)v}$. Thus
\[
I = \int_0^1 ds \int_{M} \varphi \sqrt{-1} \partial \bar{\partial} (u - v) \wedge \left[ k(c - \epsilon - su_t + s\epsilon) \chi_s^{k-1} \wedge \omega^{n-k} - l \chi_s^{l-1} \wedge \omega^{n-l} \right]
\]
\[- \int_0^1 ds \int_{M} \varphi (u_t - \epsilon) \chi_s^k \wedge \omega^{n-k}
\]
\[- p \int_0^1 ds \int_{M} \varphi^{-1} \sqrt{-1} \partial \bar{\partial} \varphi \wedge \partial (u - v)
\]
\[- l \int_0^1 ds \int_{M} k \chi \varphi \sqrt{-1} \partial (u - v) \wedge \partial u_t \wedge \chi_s^{k-1} \wedge \omega^{n-k}
\]
\[- l \int_0^1 ds \int_{M} \varphi (u_t - \epsilon) \chi_s^k \wedge \omega^{n-k}. \tag{2.16}
\]
We define
\[
\varphi = (u - v - \epsilon t - L)^- \geq 0, \tag{2.17}
\]
where $L$ to be specified later, and hence

\[
I = p \int_0^1 ds \int_M \varphi^{p-1} \sqrt{-1} \emptyset(u - v) \land \bar{\emptyset}(u - v) \\
\land \left[ k(c - \epsilon - su_t + se) \chi_s^{k-1} \land \omega^{n-k} - l \chi_s^{l-1} \land \omega^{n-l} \right] \\
- \int_0^1 ds \int_M ks \varphi^p \sqrt{-1} \emptyset(u - v) \land \partial u_t \land \chi_s^{k-1} \land \omega^{n-k} \\
+ \frac{1}{p+1} \int_0^1 ds \int_M \partial_t (\varphi^{p+1} \chi_s^k) \land \omega^{n-k}.
\]  

(2.18)

Using integration by parts again, we know that almost everywhere over time $t$,

\[
- \int_0^1 ds \int_M ks \varphi^p \sqrt{-1} \emptyset(u - v) \land \partial u_t \land \chi_s^{k-1} \land \omega^{n-k} \\
= \frac{1}{p+1} \int_0^1 ds \int_M ks \varphi^{p+1} \sqrt{-1} \emptyset \partial u_t \land \chi_s^{k-1} \land \omega^{n-k} \\
= \frac{1}{p+1} \int_0^1 ds \int_M \varphi^{p+1} \partial_t (\chi_s^k \land \omega^{n-k}),
\]  

(2.19)

and thus

\[
I = p \int_0^1 ds \int_M \varphi^{p-1} \sqrt{-1} \emptyset(u - v) \land \bar{\emptyset}(u - v) \\
\land \left[ k(c - \epsilon - su_t + se) \chi_s^{k-1} \land \omega^{n-k} - l \chi_s^{l-1} \land \omega^{n-l} \right] \\
+ \frac{1}{p+1} \frac{d}{dt} \int_0^1 ds \int_M \varphi^{p+1} \chi_s^k \land \omega^{n-k}.
\]  

(2.20)

Since $-S_{t-1;i}/S_{k-1;i}$ and $S_{t-1;i}$ are concave, we have

\[
k(c - \epsilon - su_t + se) \chi_s^{k-1} \land \omega^{n-k} - l \chi_s^{l-1} \land \omega^{n-l} \geq (1-s)ek \chi_s^{k-1} \land \omega^{n-k} \\
\geq (1-s)k \chi_s^{k-1} \land \omega^{n-k}.
\]  

(2.21)

Then

\[
I \geq \frac{\epsilon kp}{k+1} \int_M \varphi^{p-1} \sqrt{-1} \emptyset(u - v) \land \partial(u - v) \land \chi_v^{k-1} \land \omega^{n-k} \\
+ \frac{1}{p+1} \frac{d}{dt} \int_0^1 ds \int_M \varphi^{p+1} \chi_s^k \land \omega^{n-k}.
\]  

(2.22)

Integrating $I$ from $t_0 - 1$ to $t' \in [t_0 - 1, t_0]$, we obtain

\[
C \int_{t_0-1}^{t'} dt \int_M \varphi^p \omega^n \geq \sigma p \int_{t_0-1}^{t'} dt \int_M \varphi^{p-1} \sqrt{-1} \emptyset(u - v) \land \bar{\emptyset}(u - v) \land \omega^{n-1} \\
+ \frac{1}{p+1} \int_0^1 ds \int_M \varphi^{p+1} \chi_s^k \land \omega^{n-k} \bigg|_{t=t_0-1}^{t'}.
\]  

(2.23)
Choosing \( L = \inf_{M \times [0,t_0]} (u - v) - \epsilon t_0 + \epsilon \),
\[
\varphi(x, t_0 - 1) = (u - v - \inf_{M \times [0,t_0]} (u - v))^- = 0,
\]
and
\[
\sup_{M \times [0,t_0]} \varphi(x, t) = \sup_{M \times [0,t_0]} \varphi(x, t_0) = (u(x_1, t_0) - v(x_1) - \epsilon - \inf_{M \times [0,t_0]} (u - v))^- = \epsilon.
\]

By integration by parts, we observe that
\[
\beta
\]

So
\[
C \int_{t_0 - 1}^{t'} dt \int_M \varphi^p \omega^n \geq \sigma p \int_{t_0 - 1}^{t'} dt \int_M \varphi^{p-1} \sqrt{-1} \partial (u - v) \wedge \bar{\partial} (u - v) \wedge \omega^{n-1} + \frac{1}{p+1} \int_0^1 ds \int_M \varphi^{p+1} \chi_s^k \wedge \omega^{n-k} \bigg|_{t=t'}.
\]

By integration by parts, we observe that
\[
\int_0^1 ds \int_M \varphi^{p+1} \chi_s^k \wedge \omega^{n-k} \geq \int_0^1 ds \int_M \varphi^{p+1} \chi_s^k \wedge \omega^{n-k} = \int_M \varphi^{p+1} \chi_s^k \wedge \omega^{n-k}.
\]

Substituting (2.26) into (2.27)
\[
C \int_{t_0 - 1}^{t'} dt \int_M \varphi^p \omega^n \geq \sigma p \int_{t_0 - 1}^{t'} dt \int_M \varphi^{p-1} \sqrt{-1} \partial (u - v) \wedge \bar{\partial} (u - v) \wedge \omega^{n-1} + \frac{1}{p+1} \int_0^1 ds \int_M \varphi^{p+1} \chi_s^k \wedge \omega^{n-k} \bigg|_{t=t'} 
\]
\[
\geq \frac{2\sigma}{p+1} \int_{t_0 - 1}^{t'} dt \int_M \sqrt{-1} \partial \varphi^{p+1/2} \wedge \bar{\partial} \varphi^{p+1/2} \wedge \omega^{n-1} + \frac{1}{p+1} \int_0^1 ds \int_M \varphi^{p+1} \chi_s^k \wedge \omega^{n-k} \bigg|_{t=t'}.
\]

Consequently,
\[
C(p+1) \int_{t_0 - 1}^{t_0} dt \int_M \varphi^p \omega^n \geq 2\sigma \int_{t_0 - 1}^{t_0} dt \int_M \sqrt{-1} \partial \varphi^{p+1/2} \wedge \bar{\partial} \varphi^{p+1/2} \wedge \omega^{n-1} + \sup_{t \in [t_0-1,t_0]} \int_M \varphi^{p+1} \chi_s^k \wedge \omega^{n-k}.
\]

By Sobolev inequality, for \( \beta = \frac{n+1}{n} \),
\[
C(p+1) \int_{t_0 - 1}^{t_0} dt \int_M \varphi^p \omega^n \geq \left( \int_{t_0 - 1}^{t_0} dt \int_M \varphi^{p+1}/\beta \right)^\frac{\beta}{\beta}.
\]

We can then iterate \( \beta \to \beta^2 + \beta \to \beta^3 + \beta^2 + \beta \to \cdots \) and obtain
\[
p_m = \frac{\beta(\beta^{m+1} - 1)}{\beta - 1}
\]
and

\[(\ln C - \ln \beta) + \ln p_{m+1} + p_m \ln \|\varphi\|_{L^p} \geq \frac{p_{m+1}}{\beta} \ln \|\varphi\|_{L^{p+1}}.\]  \tag{2.32}

From (2.32),

\[\sum_{m=0}^{q} \frac{\ln C - \ln \beta}{\beta^m} + \sum_{m=0}^{q} \frac{\ln p_{m+1}}{\beta^m} + \beta \ln \|\varphi\|_{L^p} \geq \frac{p_{q+1}}{\beta^{q+1}} \ln \|\varphi\|_{L^{p+1}},\]  \tag{2.33}

that is

\[\ln \left(\frac{C}{\beta - 1}\right) \sum_{m=0}^{q} \frac{1}{\beta^m} + \sum_{m=0}^{q} \frac{\ln(\beta^{m+1} - 1)}{\beta^m} + \beta \ln \|\varphi\|_{L^p} \geq \frac{\beta^{q+2} - 1}{\beta^q(\beta - 1)} \ln \|\varphi\|_{L^{p+1}}.\]  \tag{2.34}

Letting \(q \to \infty\),

\[C + \ln \|\varphi\|_{L^p} \geq \frac{\beta}{\beta - 1} \ln \|\varphi\|_{L^\infty} = \frac{\beta \ln \epsilon}{\beta - 1}.\]  \tag{2.35}

Therefore, there exists a uniform constant \(c_1 > 0\) such that

\[\int_{t_0}^{t_0-1} dt \int_M \varphi^p \omega^n \geq c_1.\]  \tag{2.36}

So we have

\[\epsilon^\beta \int_{t_0}^{t_0-1} dt \int_{\{\varphi > 0\}} \omega^n \geq c_1.\]  \tag{2.37}

When \(\varphi > 0\),

\[u < v + \epsilon t + \inf_{M \times [0,t_0]} (u - v) - \epsilon t_0 + \epsilon \leq v + \epsilon + \inf_{M \times [0,t_0]} (u - v) < \inf_{M \times [0,t_0]} (u - v).\]  \tag{2.38}

Thus, using an idea from Blocki [1] and Székelyhidi [12],

\[c_1 \leq \epsilon^\beta \int_{t_0}^{t_0-1} \frac{||u^-(x,t)||_{L^1}}{\inf_{M \times [0,t_0]} (u - v)} dt \leq \epsilon^\beta \int_{t_0}^{t_0-1} \frac{||u(x,t) - \sup_M u(x,t)||_{L^1}}{\inf_{M \times [0,t_0]} (u - v)} dt.\]  \tag{2.39}

Since \(\Delta u\) has a lower bound, we have a uniform bound for \(||u(x,t) - \sup_M u(x,t)||_{L^1}\). As a consequence, there is a uniform constant \(C_0 > 0\) such that

\[\inf_{M \times [0,t_0]} (u - v) > -C_0.\]  \tag{2.40}

However, it contradicts the definition of \(t_0\).

\[\square\]

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