Tidally Forced Planetary Waves in the Tachocline of Solar-like Stars

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Abstract

Can atmospheric waves in planet-hosting solar-like stars substantially resonate to tidal forcing, perhaps at a level of impacting the space weather or even being dynamo-relevant? In particular, low-frequency Rossby waves, which have been detected in the solar near-surface layers, are predestined to respond to sunspot cycle-scale perturbations. In this paper, we seek to address these questions as we formulate a forced wave model for the tachocline layer, which is widely considered as the birthplace of several magnetohydrodynamic planetary waves, i.e., Rossby, inertia-gravity (Poincaré), Kelvin, Alfvén, and gravity waves. The tachocline is modeled as a shallow plasma atmosphere with an effective free surface on top that we describe within the Cartesian β-plane approximation. As a novelty to former studies, we equip the governing equations with a conservative tidal potential and a linear friction law to account for viscous dissipation. We combine the linearized governing equations into one decoupled wave equation, which facilitates an easily approachable analysis. Analytical results are presented and discussed within several interesting free, damped, and forced wave limits for both midlatitude and equatorially trapped waves. For the idealized case of a single tide-generating body following a circular orbit, we derive an explicit analytic solution that we apply to our Sun for estimating leading-order responses to Jupiter. Our analysis reveals that Rossby waves resonating to low-frequency perturbations can potentially reach considerable velocity amplitudes on the order of $10^1-10^2\,\text{cm}\,\text{s}^{-1}$, which, however, strongly rely on the yet unknown frictional damping parameter.

Unified Astronomy Thesaurus concepts: Internal waves (819); Astrophysical fluid dynamics (101); Magnetic fields (994); Tidal interaction (1699)

1. Introduction

For more than 100 yr, it has been known that the global weather and climate of our Earth are decisively impacted by atmospheric planetary waves, in which large-scale Rossby waves occupy the most prominent role (Pedlosky 1987). But only in the last 20 yr has strong evidence accumulated that Rossby waves play similar vital roles in various astrophysical objects, such as planets, e.g., Jupiter (Li et al. 2006) and Saturn (Read et al. 2009); accretion disks (Lyra & Umurhan 2019); and, most importantly, the Sun and other solar-like stars, see Zaqarashvili et al. (2021) for an in-depth review. With regard to our Sun, it is known today that Rossby waves are promising candidates for explaining the solar seasons (Dikpati et al. 2017, 2018b), inducing angular momentum transport (Gizon et al. 2020), and impacting or even causing solar activity cycles, starting from Rieger-type periodicities (Zaqarashvili et al. 2010a) over Schwabe cycle fluctuations (Rapahldini & Raupp 2015; Rapahldini et al. 2019) up to long-term modulations on the order of the Gleissberg cycle (Zaqarashvili & Gurgensashvili 2018). Rossby waves may also serve a crucial role for the solar dynamo (Zaqarashvili et al. 2021, Section 5.5) and have even been considered as a key ingredient for dynamo action; an early idea of a self-exciting Rossby wave dynamo dates back to Gilman (1968).

For a long time, Rossby waves with respect to the Sun were mostly perceived as a theoretical concept, and a variety of explanations for their hypothetical emergence were proposed. The breakthrough has only very recently been obtained: different types of Rossby waves have been detected independently by employing different methodologies. First, McIntosh et al. (2017) observed magnetic Rossby-like waves in the solar atmosphere by tracking coronal bright points from extreme-ultraviolet images. Thereupon, Lüptien et al. (2018) and Liang et al. (2019) further observed classic Rossby modes in solar near-surface layers from helioseismic measurements. It is important to note that classic and magnetic Rossby waves are physically very different and have, among other more nuanced dissimilarities, exactly opposite phase velocities and opposite group velocities; see Dikpati et al. (2020) for an excellent introduction to the physics of solar Rossby waves. Both types of waves can therefore play very different roles in the solar dynamics, and it is of great concern today to understand the different manifestations of Rossby and other magnetohydrodynamic planetary waves in the Sun (Zaqarashvili et al. 2021).

Today, two of the most important and as yet largely unresolved questions are where and how solar Rossby waves might originate. Although Rossby waves have been observed in the outer solar atmosphere, the shallow tachocline layer is widely believed to be one of the most promising birthplaces of solar planetary waves. In a pioneering work, Gilman (2000) showed that the solar tachocline fluid layer can be treated, in terms of hydrodynamics, fairly analogously to the lower atmosphere of the Earth, which is why the well-studied geophysical shallow-water equations governing Rossby and other classical atmospheric waves can be transferred almost one-to-one to the solar tachocline. Since that time, numerous two-dimensional as well as quasi-three-dimensional shallow-water models have been...
employed to study the global wave instabilities in the tachocline; see, e.g., Dikpati & Gilman (2001), Schecter et al. (2001), Gilman & Dikpati (2002), Zaarachvili et al. (2007, 2009, 2010b), Raphaldini & Raupp (2015), Klimachk & Petrosyan (2017), and Dikpati et al. (2018b). These studies have identified a number of different possible causes for the development of Rossby waves, including different shear instabilities associated with differential rotation, thermal forcing, and nonlinear wave–wave interactions.

One natural creation mechanism, which has gained a special importance in recent years concerning the solar dynamo, has been disregarded so far: the possible wave excitation by tidal forcing. It was emphasized in a number of studies (Stefani et al. 2016, 2018, 2019, 2021) that the combined tidal action of the planets Earth, Venus, and Jupiter might play a significant role in the synchronization process of the solar dynamo. Although the responding tidal height is only on the order of 1 mm, energetically equivalent velocities can reach up to 1 m s⁻¹ due to the high gravitational acceleration in the tachocline, which could indeed be dynamo-relevant. In those works, the synchronization mechanism was considered to rely only on the entrainment of the α-effect caused by the Tayler instability (Weber et al. 2015), the thrilling question has arisen of whether Rossby and other planetary waves could take on a similar facilitating role. Can Rossby waves possibly intensify the tidal action, or, in other words, may they serve as kind of a resonance ground for tidal excitations? Slow magnetohydrodynamic Rossby waves can have periods on the order of the solar cycle, which is remarkably close to the 11.07 yr period visible in the spring-tide envelope curve of Jupiter’s, Venus’s, and Earth’s tidal potentials (Okhlopov 2016). But also, in other respects, Rossby waves entail excellent premises on which to act on the solar dynamo. First, Rossby waves can have a net kinetic helicity (Dikpati & Gilman 2001; Gilman & Dikpati 2002), letting them participate in the α-effect. Second, tachocline oscillations can further sensitively affect magnetic flux tube instabilities, since very small variations of the superadiabaticity δ (stratification of specific entropy) on the order of δ ~ 10⁻⁴ to 10⁻³ considerably alter the magnetic storage capacity (Ferriz-Mas 1996; Abreu et al. 2012; Charbonneau 2022). Motivated by all of these auspicious premises, we devote this study to tidally forced magnetic planetary waves and present a first theoretical “shallow-water” model, which can account for arbitrary tidal potentials and, as a second novelty, further entails Rayleigh friction permitting the study of damped wave mechanics. The paper is organized as follows. In Section 2, we formulate the forced wave model on the local Cartesian β-plane and rearrange the governing equations into one decoupled wave equation for the latitude velocity, which covers the entire wave physics and makes the analysis easily accessible. In Section 3, we present analytical results for both midlatitude and equatorially trapped waves. We start to recover the known free wave problems and gradually increase in complexity all the way from different freely damped limits up to the full forced wave problem, which is solved for a single

5 Typically, velocities at the base of the convection zone are considered to be in the range of up to 50 m s⁻¹ (see, e.g., Hotta et al. 2014). While this value exceeds the estimated tidal velocities by a factor of 100, say, one must further take into account the much higher turbulent magnetic diffusivity in the convection zone, which is, in a compensating manner, also higher by a factor of about 100 with respect to the tachocline (Jouve et al. 2008) and relativizes the large difference.

tide-generating body. In Section 4, the solutions are finally applied to the particular scenario of tachoclinic waves in our Sun forced by Jupiter to estimate resonant velocity amplitudes.

2. Formulation of the Shallow-water Model

2.1. Basic Considerations

Shallow-water formulations were widely employed in the last century to describe several types of planetary waves in both the atmosphere and oceans of rotating planets (Longet-Higgins 1964; Pedlosky 1987), with Rossby waves perhaps being the most prominent member. In its simplest view, an ocean can be intuitively modeled as a one-layer system of fluid sandwiched between a flat rigid bottom and a freely movable surface placed on top. If we focus on large-scale planetary waves, this layer can be considered as “shallow” in the sense that horizontal characteristic wavelengths are much larger than the average water depths. This has the consequence that the flow is of a quasi-two-dimensional nature; i.e., the velocities do not alter significantly with altitude, and the change of momentum caused by vertical velocities and Coriolis force components can be neglected in the momentum conservation law. Such shallow-water approaches are very intelligible and simplify the mathematical analysis enormously. Therefore, it is all the more gratifying that planetary waves evolving in tachoclines of solar-like stars can be described in very similar ways. It was pointed out in the pioneering paper by Gilman (2000) that the geophysical shallow-water equations can be transferred almost one-to-one to the solar tachocline. In a manner of speaking, the tachocline can be treated as a sort of “plasma ocean,” as schematized in Figure 1.

The tachocline layer with the depth $H_0$, which itself consists of a radiative part and an overshoot layer (not shown here), is the transition region between the radiative interior and the outer convection zone. The convection zone is of adiabatic or even superadiabatic nature, whereas the stably stratified tachocline is subadiabatic. Hence, warmer fluid volumes entering the convection zone tend to rise (or at least do not experience any buoyancy resistance in the adiabatic case), while warmer fluid volumes tend to sink in the tachocline. As a result of this behavior, the transition region between the tachocline and the convection zone can be replaced by an imaginary interface, which experiences some effective gravity in the same way as ocean–atmosphere interfaces are exposed to standard gravity.

6 Note that part of the convection zone can be subadiabatic if the convection is driven by downflows (Rempel 2006; Brandenburg 2016).
The effective gravity force is proportional to the fractional difference between the actual and adiabatic temperature gradients \( \sim|\nabla - \nabla_{\text{ad}}| \), taking values of \( 10^{-3} - 10^{-6} \) in the upper overshoot part of the tachocline and up to \( 10^{-1} \) in the lower radiative sublayer (Dikpati & Gilman 2001). These gradients yield very different effective gravity constants of \( g = 0.05 - 5 \text{ cm s}^{-2} \) (overshoot part) and \( g = 500 - 1.5 \times 10^4 \text{ cm s}^{-2} \) (radiative sublayer); see Schecter et al. (2001).

The described analogy between geophysical shallow-water systems and the tachocline is still not fully applicable, since the tachocline layer is exposed to strong, nearly horizontal magnetic fields, which is why one has to tackle physically richer magneto-hydrodynamic wave problems in tandem with solar dynamics. The magnetic field strength in the solar tachocline, for which very different values can be found in the literature, can be estimated, e.g., from the simulations of rising magnetic flux tubes and surface appearances of sunspots, which suggest values on the order of \( 10 - 100 \text{ kG} \) (Schüssler et al. 1994; Rempel 2006). More recently, Gurgenashvili et al. (2016) used the observed values of the Rieger periodicity and the dispersion relation of magnetic Rossby waves to estimate the dynamo magnetic field strength within the last several cycles. They obtained a value of \( 40 \text{ kG} \) in stronger cycles and \( 20 \text{ kG} \) in weaker cycles. In this study, we will discuss different representative scenarios in the range \( 4 - 50 \text{ kG} \), which corresponds to Alfvén velocities on the order of \( 10^3 - 10^5 \text{ cm s}^{-1} \) for a given tachocline density of \( 0.2 \text{ g cm}^{-3} \).

The shallow-water equations of the plasma ocean model can be applied separately to both parts of the tachocline (but not both at the same time) such that the natural frequencies of planetary waves vary by orders of magnitude among the sublayers. The density is approximated to be constant and the lower interface to be stationary, since the higher dense radiative interior appears much more rigid than the tachocline layer above. This one-layer description involving a rigid bottom and distinct upper free surface may appear somewhat overly simplistic, since the tachocline is rather thought of as a continuously stratified layer of gas, and there are indeed more realistic multilayer and continuous descriptions in the literature (Hunter 2015; Petrosyán et al. 2020; Fedotova et al. 2021).

The here adopted one-layer model is nevertheless physically significant, since it is known to equivalently describe all types of baroclinic waves (Zaqarashvili et al. 2021, Section 3.2.3) appearing in continuous layers. Baroclinic waves in continuous stratification can have, in principle, an infinite number of possible eigenfrequencies, which can all be described by shallow one-layer models with a certain equivalent height \( H_0 \). In other words, we can choose an equivalent one-layer system for every possible baroclinic wave solution, which predicts exactly the same dispersion relation as the continuous stratification. This remarkable equivalence was first shown by Taylor (1936). In the following analyses, we keep the equivalent height \( H_0 \) constant, which only appears in the gravity wave velocity varying by orders of magnitude anyway, due to the very different effective gravities discussed before. Our wave response estimations presented in Section 4 are therefore representative of a large spectrum of baroclinic internal waves; possible barotropic waves are not described.

2.2. Governing Equations

We consider a thin layer of an ideally conductive, incompressible, and inviscid fluid on a sphere within the rotating frame of reference. The angular velocity \( \Omega_0 \), as well as the density \( \rho \), are set to be constant. The layer of conceived height \( H_0 \) is further traversed by a magnetic field \( B \), which is nearly horizontal in the solar tachocline. As the key new feature, we also take into account external tidal forces, which can be uniquely expressed through the gradient of a conservative tidal potential \( V \). Since the inviscid models lack any dissipation source, the inclusion of an external force gives rise to the problem of singularities appearing under resonance conditions. In geophysical flows, it is common to include linear dissipation terms, usually Rayleigh friction and/or Newtonian cooling (see, e.g., Gill 1980; Moffiel 1981; Yamagata & Philander 1985; Wu et al. 2001), which are indicative in the framework of linear waves. Here we include Rayleigh’s friction law, stating that, in the leading order, the effective drag is proportional to the flow velocity and some a priori unknown damping constant \( \lambda \). Although viscous and thermal dissipation on smaller scales in the tachocline is expected to be highly nonlinear, meaning that freely damped waves no longer decay exponentially in the classical way but show a more complex, time-dependent damping behavior, the linear friction law can nevertheless be an appropriate approximation for many long-period and large-scale planetary waves, at least when the wave amplitudes are sufficiently small.

By equipping the classical magneto-hydrodynamic shallow-water equations proposed by Gilman (2000) with tidal and friction forces, our governing equations can be expressed in the following way:

\[
\frac{\partial u}{\partial t} + (u \cdot \nabla)u + 2\Omega_0 \times u + \lambda u = -g \nabla H + \frac{1}{4\pi \rho} (B \cdot \nabla)B - \nabla V, \tag{1}
\]

\[
\frac{\partial B}{\partial t} + (u \cdot \nabla)B = (B \cdot \nabla)u, \tag{2}
\]

\[
\frac{\partial H}{\partial t} + \nabla \cdot (Hu) = 0, \tag{3}
\]

\[
\nabla \cdot (HB) = 0, \tag{4}
\]

where \( u \) and \( B \) are the two-dimensional horizontal velocity and magnetic field vectors, \( \Omega_0 \) is the angular velocity of the rotating star, \( H \) is the total equivalent layer height, \( \rho \) is the fluid density, \( g \) is the effective gravitational acceleration, and \( \nabla \) denotes the purely horizontal gradient. Equation (1) is the Euler equation including the Coriolis force, the Lorentz force, gravity, and Rayleigh friction. Equation (2) is the induction equation of the magnetic field in the limit of ideally conducting fluids (high magnetic Reynolds number limit), and Equation (3) is the shallow-water version of the continuity equation. Equation (4) finally ensures that the magnetic field is divergence-free on the condition that \( B \) remains parallel to the upper free surface.

2.3. \( \beta \)-plane Approximation

The inclusion of tidal forcing drastically enriches the physical complexity. For this study, we are mainly interested in finding analytical solutions rather than conducting simulations to understand the incoming physics in a fundamental way. Therefore, in order to keep the analysis as simple as possible, we are going to study the problem in a simpler Cartesian coordinate system within the framework of the so-called \( \beta \)-plane approximation; see Figure 2. The \( \beta \)-plane can properly describe nonequatorial
planetary waves if the wavelength is sufficiently smaller than the size of the sphere and is perfectly valid for waves trapped at the equator, where they have exactly the same dispersion relation as in the analogous spherical coordinate system (Pedlosky 1987). The local Cartesian coordinate system is fixed at the latitude \( \phi_0 \), at which we can evaluate the Coriolis force,

\[
2\Omega \times u = 2\Omega_0 \begin{pmatrix} 0 \\ \cos \phi_0 \\ \sin \phi_0 \end{pmatrix} \times \begin{pmatrix} u \\ v \\ w \end{pmatrix} = 2\Omega_0 \begin{pmatrix} v \sin \phi_0 - w \cos \phi_0 \\ -u \sin \phi_0 \\ u \cos \phi_0 \end{pmatrix} \approx f \begin{pmatrix} v \\ -u \\ 0 \end{pmatrix},
\]

where \( u, v, \) and \( w \) are the longitude, latitude, and altitude velocity components, and \( f = 2\Omega_0 \sin(\phi_0) \) is the Coriolis parameter. The shallow-water approximation was applied at the last step, demanding that both the altitude velocity and vertical Coriolis force components are negligibly small, \( w, u \cos(\phi_0) \approx 0 \). It was the great pioneering contribution of Carl-Gustaf Arvid Rossby (1939) to locally expand \( f \) on the Cartesian plane for small \( \phi_0 \) variations as

\[
f \approx f_0 + \beta y,
\]

with

\[
\beta = \frac{df}{dy} = \frac{2\Omega_0}{R_0} \cos(\phi_0),
\]

today referred to as the Rossby parameter. Keeping only the first-order \( f = f_0 = 2\Omega_0 \sin(\phi_0) \), where \( f_0 \) is treated as a constant, leads to the \( f \)-plane approximation, involving different types of magnetogravity waves (also called magneto-Poincaré waves) and purely magnetohydrodynamic Alfvén waves (Schechter et al. 2001), and also magnetostrophic waves evolving under the presence of vertical magnetic fields (Klimachkov & Petrovyan 2016). Accordingly, if one further considers the leading-order latitudinal variation of the Coriolis force (second term), we arrive at the so-called \( \beta \)-plane approximation comprising low-frequency magneto-Rossby waves in addition to the \( f \)-plane waves.

As the last step of approximation, we restrict our analysis to the linear problem, which allows us to derive analytical solutions at the little expense of being limited to small-amplitude waves. By projecting the governing Equations (1)–(4) onto the \( \beta \)-plane and linearizing, we get the equation system

\[
\begin{align*}
\frac{\partial u}{\partial t} &= -g \frac{\partial \eta}{\partial x} + fv + \frac{B_0}{4\pi \rho} \frac{\partial b_x}{\partial x} - \frac{\partial V}{\partial x} - \lambda u, \\
\frac{\partial v}{\partial t} &= -g \frac{\partial \eta}{\partial y} - fu + \frac{B_0}{4\pi \rho} \frac{\partial b_y}{\partial y} - \frac{\partial V}{\partial y} - \lambda v, \\
\frac{\partial b_x}{\partial t} &= B_0 \frac{\partial \eta}{\partial x}, \\
\frac{\partial b_y}{\partial t} &= B_0 \frac{\partial \eta}{\partial y}, \\
\frac{\partial \eta}{\partial t} + H_0 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 0,
\end{align*}
\]

for the individual vector components. Here \( u, v, b_x, \) and \( b_y \) are the perturbational field quantities, and \( \eta = H - H_0 \) denotes the perturbed layer thickness (\( \equiv \) wave amplitude). The equations were perturbed with respect to a uniform toroidal magnetic field \( B_0 \), which is dominant in the solar tachocline, although generally latitude-dependent. The set of equations is analogous to the free-wave problem analyzed by Zaqarashvili et al. (2007), with the sole difference that two additional terms appear in the momentum Equations (8) and (9), addressing tidal forcing and damping.

2.4. Decoupled Wave Equation

Although the linearized \( \beta \)-plane approximation is by far the most accessible description of magnetoplanetary waves, the associated wave physics is still very rich. In order to simplify the following analyses and unify the free-wave physics discussed in preceding studies, it is of interest to find a generalized wave equation, which allows the straightforward calculation of dispersion relations in all limiting cases of interest. We found that a fully decoupled wave equation can be derived only for the latitude velocity \( v \) but not for all other field variables \( u, \eta, b_x, \) or \( b_y \). After several algebraic transformations (see Appendix A), the set of Equations (8)–(11) can be rearranged into the following wave equation:

\[
\Box_{\alpha} v - C_0^2 \Box_{\alpha} \Delta v + f^2 \frac{\partial^2 v}{\partial t^2} - C_0^2 \rho \frac{\partial}{\partial x} \frac{\partial v}{\partial t} + \frac{\partial}{\partial x} \frac{\partial v}{\partial y} \Box_{\alpha} v - \frac{\partial}{\partial x} \frac{\partial^2 v}{\partial t \partial x^2} + \frac{\partial}{\partial y} \frac{\partial^2 v}{\partial t \partial y^2} + 2\lambda \frac{\partial}{\partial t} \Box_{\alpha} v - \lambda C_0^2 \Delta \frac{\partial v}{\partial t} + \lambda \rho \frac{\partial^2 v}{\partial t^2} = 0,
\]

where \( C_0 = \sqrt{gH_0} \) and \( v_{A} = B_0/ \sqrt{4\pi \rho} \) are gravity wave and Alfvén velocities, and \( \Box_{\alpha} := \partial_\alpha^2 - \nu_{A}^2 \partial_\alpha^2 \) denotes the d’Alembert operator with respect to Alfvén waves. Equation (12)
serves as the only governing equation throughout the rest of the paper.

3. Analytical Results

We shall now deduce and discuss tachoclinic wave dynamics within several different wave limits, comprising unbounded \((f \approx f_0)\) and equatorially trapped \((f \approx \beta y)\), hydrodynamic \((v_A = 0)\) and magnetic, inviscid \((\lambda = 0)\) and damped \((\lambda \neq 0)\), and free \((V = 0)\) and forced \((V \neq 0)\) waves. First, we reiterate some basic results on the free wave problem in order to then increase progressively in complexity toward the full tidally forced wave problem.

3.1. Free Wave Dynamics

When neglecting the tidal potential, \(V = 0\), and damping, \(\lambda = 0\), Equation (12) drastically simplifies to the free wave equation

\[
\Box v - C_0^2 \Box v \Delta v + f^2 \frac{\partial^2 v}{\partial t^2} - C_0^2 \beta \frac{\partial v}{\partial x} = 0. \tag{13}
\]

If one further considers nonmagnetic waves \(v_A = 0\), Equation (13) transforms into

\[
\frac{\partial^2 v}{\partial t^2} + f^2 \frac{\partial v}{\partial t} - C_0^2 \beta \frac{\partial v}{\partial x} = 0, \tag{14}
\]

which is the classical planetary wave equation describing Rossby, Poincaré, Kelvin, and gravity waves (Pedlosky 1987). In the following, we want to shortly recapitulate the wave physics captured in Equation (13) within two different limits. First, we focus on nonequatorial waves in the latitude range \(30^\circ \lesssim \phi_0 \lesssim 60^\circ\), where we readily find \(f_0 \gg \beta y\), allowing us to set \(f \approx f_0\). In the vicinity of the equator, we observe \(f_0 \ll \beta y\), in contrast, such that one must keep the \(y\)-dependent term of the Coriolis parameter \(f \approx \beta y\), which results in a wave equation involving nonlinear, \(y\)-dependent coefficients coming along with fundamentally different wave dynamics.

3.1.1. Nonequatorial Waves

Applying \(f = f_0\) to Equation (13) and inserting a simple Fourier ansatz of the form

\[
v = v_0 \exp(ik_x x + ik_y y - i \omega t), \tag{15}
\]

where \(v_0\) is an arbitrary wave amplitude, \(k_x\) and \(k_y\) are the longitudinal and latitudinal wavenumbers, and \(\omega\) is the angular frequency, we can straightforwardly deduce the following fourth-order dispersion relation:

\[
\omega^4 - [2k_x^2 v_A^2 + f_0^2 + C_0^2 (k_x^2 + k_y^2)] \omega^2 - C_0^2 k_x \beta \omega + k_x^2 v_A^2 [k_x^2 v_A^2 + C_0^2 (k_x^2 + k_y^2)] = 0. \tag{16}
\]

This dispersion relation was first derived and discussed by Zaqarashvili et al. (2007). Let us first consider the limit \(\beta = 0\) corresponding to the \(f\)-plane approximation. In this case, only symmetric frequency polynomials \(\omega^4, \omega^2,\) and \(\omega^0\) remain in Equation (16), allowing us to rearrange it into the explicit form

\[
\omega^2 = k_x^2 v_A^2 + f_0^2 + C_0^2 \left[ (k_x^2 + k_y^2) \right] \pm \sqrt{\left[ f_0^4 + f_0^2 \left[ \frac{4k_x^2 v_A^2}{C_0^2} + 2(k_x^2 + k_y^2) \right] + (k_x^2 + k_y^2)^2 \right].} \tag{17}
\]

This dispersion relation was analyzed in depth by Schecter et al. (2001), who referred the fast \(\omega_+\) branch to magneto-gravity waves and the slow \(\omega_-\) modes to Alfvén waves. The \(\omega_+\) branch can be approximated as

\[
\omega^2 = f_0^2 + C_0^2 (k_x^2 + k_y^2) + 2k_x^2 v_A^2, \tag{18}
\]

showing that magnetogravity waves are magnetically modified Poincaré waves (in the limit \(v_A = 0\), Equation (18) describes classic inertia-gravity waves), which appear on timescales less than \(2\pi/f_0\). The Alfvén branch is intrinsically connected to the magnetic field and disappears for \(v_A \to 0\). Alfvén waves can become arbitrarily slow at large length scales \((k_x, k_y \to 0)\).

Keeping the next order in the expansion (Equation (6); \(\beta = 0\)) gives rise to the emergence of Rossby waves. They appear in the low-frequency limit \(\omega \ll f_0\) of Equation (16), where the dispersion relation reads

\[
\omega^2 + \frac{R_R^2 \beta k_x}{1 + R_R^2 (k_x^2 + k_y^2)} \omega - \frac{R_R^2 v_A^2 k_x (k_x^2 + k_y^2)}{1 + R_R^2 (k_x^2 + k_y^2)} = 0, \tag{19}
\]

here expressed in terms of the Rossby radius \(R_R = C_0/f_0\). The Rossby radius describes the length scale at which Coriolis force-driven inertial waves become equally significant as buoyancy-driven gravity waves in the spatiotemporal evolution of linear disturbances. Interestingly, \(R_R\) strongly varies throughout the tachocline. One can estimate \(R_R \sim 10^4-10^6\) km in the overshoot layer but larger values, \(R_R \sim 10^5-10^8\) km, in the radiative part of the tachocline at the latitude \(\phi_0 = 30^\circ\). The tachocline radius is approximately \(R_0 \approx 5 \times 10^8\) km. In the radiative sublayer, we can safely assume \(R_R \gg R_0 \approx 1/k_x, 1/k_y\), allowing us to simplify Equation (19) into the pure magneto-Rossby wave dispersion relation

\[
\omega^2 + \frac{\beta k_x}{k_x^2 + k_y^2} \omega - v_A^2 k_x^2 = 0. \tag{20}
\]

It contains a high-frequency solution,

\[
\omega \approx -\frac{k_x \beta}{k_x^2 + k_y^2}, \tag{21}
\]

as well as a low-frequency branch,

\[
\omega \approx \frac{k_x v_A^2 (k_x^2 + k_y^2)}{\beta}, \tag{22}
\]

in the limit of large length scales. Equation (21) is the very classic dispersion relation of hydrodynamic Rossby waves. The minus sign reveals a retrograde propagation relative to the rotation of the reference frame. Equation (22) describes the class of magneto-Rossby waves arising only by the effect of horizontal magnetic fields \(v_A \neq 0\). These wave modes are quite similar to classic Rossby waves but move in the prograde
direction and have far slower eigenfrequencies; see Zaqarashvili et al. (2007) for more details. A very illustrative description of both classic and magneto-Rossby waves is further given in Dikpati et al. (2020).

### 3.1.2. Equatorial Waves

In the vicinity of the equator, the Coriolis force approximates to \( f \approx \beta \dot{y} \) in the leading order. This yields the wave equation

\[
\Box_v \nu - C_0^2 \Delta \nu + \beta^2 y^2 \frac{\partial^2 \nu}{\partial t^2} - C_0^2 \beta \frac{\partial \nu}{\partial x} \frac{\partial \nu}{\partial t} = 0
\]  

(23)

as the counterpart to Equation (13). Equation (23) involves \( y \)-dependent coefficients, which is why it is convenient to first perform a Fourier analysis for the \( x \)-part only,

\[
\nu = \nu_y(y) \exp(ik_x x - i\omega t),
\]

(24)

yielding

\[
d^2 \nu_y \over dy^2 + \left[ \omega^2 - k_x^2 (C_0^2 + v_A^2) - \frac{k_x \beta \omega}{\omega^2 - k_x^2 v_A^2} - \mu^2 y^2 \right] \nu_y = 0
\]

(25)

for the latitude-dependent velocity part \( \nu_y \). The parameter \( \mu \), which is a reciprocal measure for the wave’s equatorial expansion, is given as

\[
\mu = \frac{\beta \omega}{C_0 \sqrt{\omega^2 - k_x^2 v_A^2}}.
\]

(26)

The differential Equation (25) was first derived by Zaqarashvili & Gurgashvili (2018) and can be identified as the classic equation of the quantum harmonic oscillator. It has bounded solutions of the form

\[
\nu_y = \nu_0 \exp \left( -\frac{\mu |y|^2}{2} \right) H_n(\sqrt{\mu} y),
\]

(27)

if and only if

\[
\omega^2 - k_x^2 (C_0^2 + v_A^2) - \frac{k_x \beta \omega}{\omega^2 - k_x^2 v_A^2} = |\mu|(2n + 1).
\]

(28)

Here \( \nu_0 \) is some arbitrary amplitude, and

\[
H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}
\]

(29)

is the Hermite polynomial of order \( n \). The solutions are oscillatory inside the latitude interval

\[
y < \sqrt{\frac{2n + 1}{|\mu|}}
\]

(30)

and exponentially tend to zero outside. Although we have not incorporated any external boundary conditions, effective latitudinal boundary conditions, i.e., the extraction of all quantities at large \( y \), are thereby obtained for every mode \( n \) as an inherent property of the solution, through which the waves stay trapped at the equator. The integers \( n \) can be identified as discrete latitudinal wavenumbers specifying the number of vortices within the equatorial band defined by Equation (30).

The constraint (Equation (28)) defines the dispersion relation that can be expressed in the following form:

\[
(\omega^2 - k_x^2 v_A^2)(\omega^2 - k_x^2 (C_0^2 + v_A^2)) - k_x \beta C_0^2 \omega
\]

\[
= \beta C_0 |\omega| \sqrt{\omega^2 - k_x^2 v_A^2}(2n + 1).
\]

(31)

Neglecting the magnetic field, \( v_A = 0 \), we arrive at the classic geophysical dispersion relation,

\[
\omega^2 - k_x^2 C_0^2 - \frac{k_x \beta C_0^2}{\omega} = \beta C_0(2n + 1)
\]

(32)

(Matsumo 1966), comprising gravity-inertia and Rossby waves. The essential novelty of Equation (31) compared to the hydrodynamic case (Equation (32)) is that the toroidal field creates low-frequency cutoff areas at \( \omega = \pm k_x v_A \), suppressing the low-frequency Rossby modes from the hydrodynamic solution. This behavior can, however, change drastically when considering inhomogeneous toroidal magnetic field profiles \( B_x \sim B_{0Y}/R \), giving rise to superslow magneto-Rossby waves, which can reach periods up to the order of the 100 yr Gleissberg cycle (Zaqarashvili & Gurgashvili 2018).

The dispersion relation (Equation (31)) can only be solved numerically owing to the square root on the right-hand side. There are, however, some approximate solutions for certain interesting limits. Restricting to high frequencies, \( \omega \gg k_x v_A \), Equation (31) approximates to

\[
\omega^2 \approx \beta C_0(2n + 1) + k_x^2 C_0^2 + 2k_x^2 v_A^2,
\]

(33)

describing magneto-inertia-gravity waves, which always propagate faster than their hydrodynamic counterparts. By eliminating the high-frequency branches from Equation (31), one finds the low-frequency dispersion relation describing Rossby waves,

\[
k_x^2 \omega^2 + \beta k_x \omega - k_x^2 v_A^2 = -\frac{\beta \omega}{C_0} \sqrt{\omega^2 - k_x^2 v_A^2}(2n + 1),
\]

(34)

which also is not amenable to analytical solution. We can only find explicit solutions in the limit of weak magnetic fields, \( v_A \ll \omega/k_x \), where Equation (34) reduces to

\[
\omega^2 + \frac{2C_0 k_x \omega}{C_0 k_x^2 + \beta(2n + 1)} - \frac{v_A^2 k_x^2 [2C_0 k_x^2 + \beta(2n + 1)]}{2[C_0 k_x^2 + \beta(2n + 1)]} = 0,
\]

(35)

which again involves a high-frequency solution,

\[
\omega \approx -\frac{C_0 k_x}{C_0 k_x^2 + \beta(2n + 1)},
\]

(36)

and a low-frequency solution,

\[
\omega \approx k_x \frac{\sqrt{k_x^2 + \frac{1}{2}(2n + 1) C_0}}{\beta}.
\]

(37)

Equation (36) is the classic dispersion relation of hydrodynamic, equatorially trapped Rossby waves. Similarly as for the nonequatorial waves, Equation (37) describes prograde magneto-Rossby waves and reduces exactly to Equation (22) in the limit of purely longitudinally propagating waves (\( k_x = 0 \)) and
large gravity velocities $C_0$. In the case of the solar tachocline, however, these magneto-Rossby waves are inhibited by the Alfvén wave branches $\omega = \pm v_A k$ for magnetic field strength $>10$ kG (Zaqarashvili & Gurgenashvili 2018).

### 3.2. Damped Wave Dynamics

We have now prepared the groundwork to discuss novel wave solutions under the effect of damping. Keeping the $\lambda$-dependent terms in Equation (12) but still neglecting the forcing potential $V$, we arrive at the wave equation

$$\Box_{\xi} v - C_0^2 \Box_{\xi} \Delta v + (f^2 + \lambda^2) \frac{\partial^2 v}{\partial t^2} - C_0^2 \beta \frac{\partial v}{\partial t} + 2\lambda \frac{\partial v}{\partial t} = 0. \quad (38)$$

#### 3.2.1. Nonequatorial Waves

For nonequatorial waves, we can again perform a simple Fourier analysis, $v \sim \exp(ik x + ik_y y - \omega t)$, yielding the complex dispersion relation

$$\omega^2 + 2i\lambda \omega^3 - 2[k_x^2 v_A^2 + f^2 + \lambda^2 + C_0^2 (k_x^2 + k_y^2)]\omega^2 - [C_0^2 k_x \beta + 2i\lambda k_x v_A + \lambda C_0^2 (k_x^2 + k_y^2)]\omega + k_x^2 v_A^2[k_x^2 + C_0^2 (k_x^2 + k_y^2)] = 0. \quad (39)$$

The attenuation of the waves is manifested in the imaginary part of the frequency $\omega$, which translates into an exponential decay of the Fourier modes. Although Equation (39) appears quite delicate, the damping behavior turns out to be surprisingly simple for most wave modes. In the limit of high-frequency magneto-inertia waves ($\omega \gg \lambda$), Equation (39) approximates to

$$\omega \approx \pm \sqrt{f^2 + C_0^2 (k_x^2 + k_y^2)} + 2k_x v_A^2 - i\lambda. \quad (40)$$

The real part of Equation (40) is the same as in the inviscid case (Equation (18)), but here Rayleigh friction does not do anything except let the waves decay exponentially, where the decay rate directly corresponds to the damping rate $\lambda$. For the slow-frequency solutions in Equation (40) in the limit of large-scale waves $k_x, k_y \rightarrow 0$, we find

$$\omega^2 + \frac{R_\alpha^2 \beta k_x + i\lambda R_\alpha^2 (k_x^2 + k_y^2)}{1 + \lambda^2 f_0^2 + R_\alpha^2 (k_x^2 + k_y^2)}\omega - \frac{R_\alpha^2 v_A^2 k_x^2 (k_x^2 + k_y^2)}{1 + \lambda^2 f_0^2 + R_\alpha^2 (k_x^2 + k_y^2)} = 0. \quad (41)$$

In contrast to the inertia wave solution, here damping is also affecting the real part of $\omega$ by the $\lambda f_0^2$ term in the denominator, effectively reducing the natural frequencies. However, this detuning effect is negligibly small if we focus on underdamped waves ($\lambda < \omega$), where we readily find $\lambda f_0 < 1$. If we further consider, as in Section 3.1.1, the radiative part of the tachocline ($R_\alpha \gg 1$), Equation (41) simplifies to

$$\omega^2 + \frac{\beta k_x}{k_x^2 + k_y^2} \omega + i\lambda \omega - v_A^2 k_x^2 = 0. \quad (42)$$

as the analog to Equation (20). The hydrodynamic Rossby branch

$$\omega \approx -\frac{k_x \beta}{k_x^2 + k_y^2} - i\lambda \quad (43)$$

is in the same way affected by damping as magneto-inertia waves (Equation (40)), and the damping rate $\lambda$ is equal to the exponential decay rate. Intriguingly, we find a more complex and novel behavior for magneto-Rossby waves underlying the dispersion relation

$$\omega \approx \frac{k_x^2 v_A^2 (k_x^2 + k_y^2)}{k_x^2 \beta + i\lambda (k_x^2 + k_y^2)} = \frac{\beta k_x^2 + \lambda^2 (k_x^2 + k_y^2)}{\beta^2 k_x^2 + \lambda^2 (k_x^2 + k_y^2)^2} \quad (44)$$

Since this solution is only valid for large scales $k_R \lesssim 1$ (see Zaqarashvili et al. 2007), we can again neglect the detuning term $\lambda^2 (k_x^2 + k_y^2)^2$, allowing us to write the dispersion relation as

$$\omega = \omega_0 - i\lambda \frac{\omega_0^2}{\omega_A^2} \quad (45)$$

where $\omega_0$ is the inviscid natural frequency of slow magneto-Rossby waves (Equation (22)), and $\omega_A = \pm k_x v_A$ is the Alfvén frequency. This time, the decay rate differs from the damping rate by the factor $\omega_0^2/\omega_A^2$. Zaqarashvili et al. (2007) showed that the magneto-Rossby waves are always slower than Alfvén waves (in contrast to hydrodynamic Rossby waves: Equation (43)) such that we always have $\omega_0^2/\omega_A^2 < 1$ effectively reducing the decay rate. As a consequence, we can conclude that magneto-Rossby waves are more resilient to (linear) viscous damping than any other planetary wave modes, giving some more evidence that magneto-Rossby waves are indeed more relevant for the space weather dynamics than classic Rossby waves (Dikpati & McIntosh 2020).

As a last remark, we want to stress the practical benefit of our approach to include damping. Treating Rayleigh’s linear friction law as the only dissipation source is surely an oversimplification, but it can also be viewed as an empirical law, since we know how it translates to decay rates in the different solutions. In principle, one can experimentally measure the exponential decay rate and translate it back to the damping rate $\lambda$ (e.g., by applying the factor $\omega_0^2/\omega_A^2$ in the case of magneto-Rossby waves). This way, the model can correctly reflect all participating linear damping mechanisms, not only viscous damping. Such empirical approaches have been successfully implemented for simple free-surface wave systems (Horstmann et al. 2020, 2021). In these controllable mechanical systems, it is, however, easy to determine decay rates, while it seems rather hopeless to extract them from space weather data, especially since solar Rossby waves are constantly and unpredictably excited. Nevertheless, one can at least roughly estimate the order of magnitude of $\lambda$, e.g., by measuring typical Rossby wave amplitudes (Mandal & Hanasoge 2020 determined amplitudes on the order of $1 \text{ m s}^{-1}$) and estimating the energy input, i.e., the energy stored in the differential rotation and toroidal magnetic field (Dikpati & Gilman 2001). Apart from that, it is worthwhile for future studies to investigate further dissipation mechanisms, be they Newtonian
cooling (Chang & Lim 1982; Wu et al. 2001; Tsai et al. 2014) or magnetic damping, which have not yet been regarded in the framework of magnetohydrodynamic Rossby waves.

3.2.2. Equatorial Waves

At the equator, damped waves are more complex and different from nonequatorial waves, since friction alters not only the frequencies but also the meridional structure and phase of the wave profiles, as we will see in the following. In the vicinity of the equator, we have $f \approx \beta y$, and Equation (12) transforms into

$$
\Box_{v_{\lambda}} v - C_0^2 \Box_{v_{\lambda}} \Delta v + (\beta^2 y^2 + \lambda^2) \frac{\partial^2 v}{\partial t^2} - C_0^2 \beta \frac{\partial}{\partial x} \frac{\partial v}{\partial t} + 2\lambda \frac{\partial}{\partial t} \Box_{v_{\lambda}} v - \lambda C_0^2 \frac{\partial v}{\partial t} = 0
$$

(46)

for $V = 0$. By using the same ansatz as before, $v = v_{\lambda}(y) \exp(ik_x x - i\omega t)$, one readily gets the determining equation

$$
\frac{d^2 v_{\lambda}}{dy^2} + \left[ \frac{\omega^2 + i\lambda \omega - k_x^2 v_{\lambda}^2}{C_0^2} - k_y^2 - \frac{k_x^2 \beta \omega}{\omega^2 + i\lambda \omega - k_x^2 v_{\lambda}^2} \right] v_{\lambda} = 0,
$$

(47)

where

$$
\mu = \frac{\beta \omega}{C_0 \sqrt{\omega^2 + i\lambda \omega - k_x^2 v_{\lambda}^2}}.
$$

(48)

The dispersion relation follows again from the solvability condition (Equation (28)):

$$
(\omega^2 + i\lambda \omega - k_x^2 v_{\lambda}^2)(\omega^2 + i\lambda \omega - k_x^2 (C_0^2 + v_{\lambda}^2)) - k_x \beta C_0^2 \omega = \beta C_0 \omega \sqrt{\omega^2 + i\lambda \omega - k_x^2 v_{\lambda}^2} (2n + 1).
$$

(49)

The complex-valued square root on the right-hand side highly complicates further analysis. It is no longer feasible to derive analytical approximations for the wave type. Therefore, we analyze the dispersion relation numerically, which is a bit delicate, since the square root introduced spurious solutions. In order to get around this issue, we squared both sides of the dispersion relation—for the price of introducing four invalid solutions—and determined the eight zeros of the resulting polynomial by calculating the eigenvalues of the companion matrix. Afterward, we checked the validity of the solutions to guarantee that they fulfill the initial dispersion relation (Equation (49)). Figure 3 shows the wave frequencies $\text{Re}(\omega)$ as a function of the wavenumbers $k_x$ on the example of the solar tachocline with the same parameters as used by Zaqarashvili & Gurgenashvili (2018) under the effect of a hypothetical friction parameter of $\lambda = 0.01 \Omega_0$.

When compared to the undamped solutions, no difference can be discerned for all inertia, gravity, and Rossby wave solutions. We found that the effect of $\lambda$ on the eigenfrequencies is entirely negligible except for large friction coefficients on the order of $\lambda \sim |\omega|$. The corresponding decay rates, however, show a more complex behavior than the nonequatorial counterparts. Figure 4 shows the imaginary part of the frequency $\text{Im}(\omega)$ of the same solutions. It can be recognized that the resulting damping rates are smaller than for nonequatorial waves under the same conditions. The decay behavior of magneto-inertia waves shows an interesting length-scale dependence. Large-scale Poincaré waves ($k_x \rightarrow 0$) diminish with $\text{Im}(\omega) \approx 0.75 \lambda$, whereas the small-scale representa- approach the smaller decay rate of $\text{Im}(\omega) \approx 0.5 \lambda$ for $k_x \rightarrow \pm \infty$. Rossby waves always decay with $\text{Im}(\omega) \approx 0.5 \lambda$ independently of the wavelength, which is half as fast as nonequatorial Rossby waves; see Equation (43). This can be explained by the fact that Rossby waves are bordered by Alfvén waves and have very similar eigenfrequencies (Zaqarashvili & Gurgenashvili 2018). The frequency of Alfvén waves, however, is governed by the square-root term of Equation (49), $\omega^2 + i\lambda \omega - k_x^2 v_{\lambda}^2 = 0$, yielding a decay rate of $\text{Im}(\omega) = -\lambda/2$ in agreement with the magneto-Rossby waves. In the interpretation of the decay rates, it is important to take into account that the damping parameter $\lambda$ is itself wavenumber-dependent, with the contrary tendency that $\lambda$ normally declines with increasing wavelength in the case of viscous damping. Here $\text{Im}(\omega)$ only compares the decay rates of the individual wave solutions among
each other under otherwise equal damping conditions; the individual scaling behavior is not represented.

So far, we have analyzed damped waves, which decrease in time. In many cases, as for some forced wave systems, it is also interesting to study free-wave motions, it is the effect of damping. Free waves always have real frequencies, such that the dispersion relation (Equation (49)) then necessarily introduces complex wavenumbers \( k_n \) for \( \lambda > 0 \), letting the waves decay spatially in the zonal direction. Such waves can evolve under spatially limited but constant forcing, whereby the wave amplitudes are maintained in the region of forcing and spatially fade away outside. In geophysics, it is well known that friction can modify the meridional scales and introduce a phase shift in the meridional structure of planetary waves (Mofjeld 1981; Yamagata & Philander 1985). Therefore, the question arises as to what extent meridional scales of magneto-Rossby waves can be affected. For real frequencies, the wavenumber of slow and large-scale magneto-Rossby waves can be approximated as follows:

\[
k^2 = \frac{(2n + 1)^2 \omega^2 + \lambda \omega}{C_0^2 + v_A^2 (2n + 1)^2}.
\]  

We use this dispersion relation for the following calculations. Figures 5 and 6 show normalized meridional velocity profiles \( v(y) \) of \( n = 1 \) and 2 magneto-Rossby waves for different Alfvén velocities \( v_A \) without (dashed lines) and with (solid lines) strong damping \( \lambda = \omega \) in accordance with Mofjeld (1981). Two opposing tendencies can be observed. As with classic Rossby waves, friction always broadens the wave profiles. However, this effect is increasingly counterbalanced by the presence of toroidal magnetic fields that tighten the magneto-Rossby wave profiles. This was to be expected, since the oscillatory scale (Equation (30)) decreases more and more as the magneto-Rossby frequencies approach Alfvén wave frequencies for \( v_A \rightarrow \infty \).

Altogether, it can be concluded that the meridional scale variation due to friction is negligible relative to the magnetic contribution. Finally, we can also see that friction causes larger absolute latitudinal velocities (but smaller zonal velocities, not shown here). This is in conformity with the observations by Mofjeld (1981); there is no significant difference between classic and magnetic Rossby waves.

### 3.3. Forced Wave Dynamics

Building on these preliminary investigations, we can now turn to the study of forced wave solutions. For the sake of keeping the analysis simple, we focus on wave responses to a single tide-generating planet of mass \( M_t \) moving at a fixed distance \( r \) with an angular frequency \( \Omega_t \), moving at a fixed distance \( r \) with an angular frequency \( \Omega_t \), around a rotating star within the equatorial plane (zero declination). We show in Appendix B that the resulting leading-order tidal potential acting in the \( \beta \)-plane approximates to

\[
V = K \left[ \frac{1}{2} + \frac{y}{R_0} \right] \left[ 1 + \cos \left( \frac{2x}{R_0} - 2(\Omega_t - \Omega_0)t \right) \right]
\]  

at midlatitudes \( \phi_0 = 45^\circ \) and

\[
V = K \left[ 1 + \cos \left( \frac{2x}{R_0} - 2(\Omega_t - \Omega_0)t \right) \right]
\]  

in the vicinity of the equator \( \phi_0 = 0^\circ \). The forcing amplitude is given as

\[
K = \frac{3 GM_t}{4 R_0} \left( \frac{R_0}{r} \right)^3,
\]  

with \( G \) referring to the gravitational constant. Two characteristic forcing frequencies appear in the potentials; the rotation frequency of the star \( \Omega_0 \) and an external frequency \( \Omega_t \) dictated by the tide-generating planet. If we consider the Sun exposed to the tidally dominant planet Jupiter, we have \( 2(\Omega_t - \Omega_0) \approx -2\Omega_0 \), corresponding to a period of \( T_0/2 \approx 13 \) days, showing that only magneto-inertia waves are excited as a direct response. Slow periodicities on the order of \( \Omega_0 \), which can resonantly excite magneto-Rossby waves, appear in the envelope potential resulting from multiple tide-generating bodies, as we discuss in Section 4.3. In the following sections, we present explicit solutions of wave responses under resonant and nonresonant forcing conditions for both unbounded and equatorially trapped waves. For didactic considerations, we start this time with equatorial waves.
3.3.1. Equatorial Waves

Inserting the equatorial potential (Equation (52)) into the wave Equation (12) yields

\[
\begin{align*}
\Box_{\alpha} v - C_0^2 \Box_{\alpha} \Delta v + \beta^2 \gamma^2 \partial^2 v - C_0^2 \tilde{\beta} \partial v \partial t + 2 \lambda \partial \partial t = \lambda C_0^2 \Delta \partial v \\
+ 2 \lambda \partial \partial t - \lambda C_0^2 \Delta \partial v + \lambda^2 \partial^2 v + \beta \tilde{\gamma} \partial^2 V = \beta \partial \partial t \\
= \beta \left( \frac{8K \Omega^2}{R_0} \sin \left( \frac{2\chi}{R_0} - 2\Omega t \right) \right) \tag{54}
\end{align*}
\]

The forcing frequencies were united to \( \Omega_f - \Omega_0 = \Omega \) for the sake of clarity. Since the equatorial potential is independent of the latitude \( \chi \), only the Coriolis forcing term \( \beta \partial \partial t \) remains, such that the leading-order tidal action is affected by neither the Alfvén speed \( v_A \) nor damping \( \lambda \) and therefore manifests itself in the same manner as with classic Rossby waves. The Coriolis parameter \( f \approx \beta \) reintroduces the \( \chi \)-coordinate, which requires us to expand \( y \) as an orthogonal series of parabolic functions:

\[
y(n) = \sum_{n=1}^{\infty} \frac{2^{n-2}n}{\sqrt{\mu}(n-1)!} \exp \left( -\frac{\mu y^2}{2} \right) H_{2n-1}(\sqrt{\mu} y). \tag{55}
\]

We have used the parabolic orthogonality condition

\[
\int_{-\infty}^{\infty} \exp(-\mu y^2)H_n(\sqrt{\mu} y)H_m(\sqrt{\mu} y) dy = 2^n n! \sqrt{\mu} \delta_{nm} \tag{56}
\]

for the expansion. As a remarkable intermediate result, Equation (55) implies that only uneven wave modes \( n \to 2n-1 \) can respond to tidal forcing in the leading order, which is always antisymmetric around the equator. We show in Appendix C that the forced wave problem can be solved explicitly by introducing the ansatz

\[
v(n) = \sum_{m=0}^{\infty} a_{m,n}(t) \exp \left( \frac{mx}{R_0} \right) \exp \left( -\frac{\mu y^2}{2} \right) H_n(\sqrt{\mu} y), \tag{57}
\]

where \( a_{m,n}(t) \) are modal coefficients to be determined. The resulting solution can be expressed as an infinite series of parabolic functions:

\[
v(x, y, t) = \sum_{n=1}^{\infty} A_n \exp \left( \frac{2x}{R_0} - i2\Omega t \right) \exp \left( -\frac{\mu y^2}{2} \right) H_{2n-1}(\sqrt{\mu} y),
\]

with \( \mu = \frac{\beta [2\Omega]}{C_0 \sqrt{4\Omega^2 + 2i\lambda \Omega - \frac{v_A^2}{R_0}}} \) and

\[
A_n = \frac{8K \beta \Omega^2 2^{n-2}(R \sqrt{\mu}(n-1))^{-1}}{(4\Omega^2 + 2i\lambda \Omega - \frac{v_A^2}{R_0})(4\Omega^2 + 2i\lambda \Omega - \frac{v_A^2}{R_0})} - C_0 \beta [2\Omega] \sqrt{4\Omega^2 + 2i\lambda \Omega - \frac{v_A^2}{R_0}} (4n - 1).
\]

Equation (58) consists of two parts: a real part \( \sim \text{Re}(A_n) \) describing nonresonant forced waves between the different wave modes and an imaginary part \( \sim \text{Im}(A_n) \) capturing resonant waves in the vicinity of the eigenfrequencies described by the dispersion relation (Equation (39)) for \( \omega = 2\Omega \). The velocity amplitude scales with \( \sim K \sqrt{\beta \Omega} \), showing that pure Alfvén waves \( (\beta = C_0 = 0) \) do not respond to tidal forcing at the equator. Besides, we would like to draw attention to the fact that Equation (58) is formally only a specific solution of Equation (54); the general solution is composed of the specific and homogeneous solution. The homogeneous part, however, only describes transient phases—e.g., in classical sloshing experiments, the initial wave motions taking place directly after switching on the shaking table—and decays exponentially after some settling time. After the transient phase, the saturated, quasi-steady wave responses are fully governed by the specific solution presented here.

For deeper insights into the dynamics of forced planetary waves, we have calculated the wave responses as a function of the forcing frequency \( \Omega_f \) within a range comprising the first three (magneto-)Rossby and (magneto-)inertia wave modes, \( n = 1, 2, 3 \to H_1, H_2, H_3 \). The top panel of Figure 7 shows normalized wave amplitudes of classical planetary waves with \( v_A = 0 \) for different dynamic damping coefficients \( \lambda \) related to the forcing frequency \( \Omega_f \), so that \( \lambda \) always refers to the characteristic time in which equivalent unforced waves would decay. For example, \( \lambda = 0.2\Omega \) corresponds to a decay rate of 0.1\%; see the Rossby wave solution in Figure 4, which would let the wave diminish substantially within around 10 periods following an artificial elimination of the excitation force. Figure 7 reveals that the first Rossby mode, \( n = 1 \), is excited most strongly, about five times stronger than the first inertia-gravity wave and up to 25 times stronger (for \( \lambda = 0.1\Omega \)) than nonresonant waves occurring in the frequency band between the low-frequency Rossby and high-frequency inertia wave solutions. Further, it can be recognized that the peak amplitudes decrease rapidly along with increasing wavenumbers showing that the large-scale modes are always most significant for tidal interactions, if they are excited. As another important general result, it is evident from the resonance curves that different planetary waves will be excited in different classes of planet-hosting stars. If the star’s rotation is far higher than the planet’s orbit frequency \( |\Omega_0| \gg |\Omega| \), as is the case for our Sun forced by Jupiter, only fast inertia-gravity waves will be exited. If both angular frequencies are very close to each other (but not equal), as is often the case for stars hosting tidally-locked planets or also for binary stars, slow Rossby modes are expected to be stimulated instead. This result is only true for stars hosting a single (significant) tide-generating planet. If there are several planets involved, Rossby waves can still be excited by low-frequency alignment periodicities, e.g., spring tides, visible in the envelope of the combined tidal potentials, even if we have \( |\Omega_0| \gg |\Omega| \) for all participating planets; see Section 4.3.
For comparison, the bottom panel of Figure 7 shows the same frequency spectrum but for magnetic planetary waves with $v_A = 12.6 \cdot 10^3 \text{ cm s}^{-1}$. In accordance with Zaqarashvili & Gurgenashvili (2018), it can be seen that Rossby waves below the Alfvén frequency (vertical black line) are largely suppressed. Magneto-Rossby waves accumulate closely above the Alfvén frequency, where they respond with much smaller amplitudes. The inertia waves, in contrast, remain largely unaffected. As a novel and rather striking result, we can also see that damping allows the waves to overcome the Alfvén frequency barrier. Inviscid planetary waves below $\omega = \pm 2v_A/R_0$ do not exist, whereas damped waves are still able to respond at frequencies far below $2|\Omega|/\Omega_0 = 10^{-3}$. This behavior is intriguing, but it must be noted that constant toroidal magnetic fields generally inhibit Rossby waves and antagonize the tidal response, making these waves unlikely to be of any practical importance. At this point, it might be of interest for future studies to analyze the tidal response of Rossby waves subject to more realistic nonuniform and also oscillatory magnetic fields, which are not exposed to a cutoff frequency (Zaqarashvili & Gurgenashvili 2018).

### 3.3.2. Nonequatorial Waves

We proceed with solving the forced wave problem at midlatitudes, which is a bit more intricate insofar that here the tidal potential (Equation (51)) depends, in contrast to the equatorial potential (Equation (52)), on the local latitude $y$ so that all three potential terms in Equation (12) must be taken into account. We obtain the following forced wave equation:

$$
\Box_{\alpha\alpha} v - C_{0}^{2} \Box_{\alpha\alpha} \Delta v + f_{0}^{2} \frac{\partial v}{\partial x} - C_{0}^{2} \beta \frac{\partial}{\partial t} \frac{\partial v}{\partial x} + 2 \lambda \frac{\partial}{\partial \eta} \Box_{\alpha\alpha} v - \lambda C_{0}^{2} \Delta \frac{\partial v}{\partial \eta} + \frac{\partial^{2} v}{\partial \eta^{2}} = f_{0}^{2} \frac{\partial^{2} v}{\partial x^{2}} - \lambda \frac{\partial^{2} v}{\partial \eta^{2}} - \frac{\partial}{\partial \eta} \Box_{\alpha\alpha} v
$$

$$
= \left[ f_{0} \Omega + 2 \Omega^{2} \left( \frac{2x}{R_{0}} + \frac{2y}{R_{0}} \right) \right] \frac{4K_{1}^{2}}{R_{0}} \sin \left( \frac{2x}{R_{0}} - 2\Omega t \right) + \frac{4K_{1}^{2}}{R_{0}} \cos \left( \frac{2x}{R_{0}} - 2\Omega t \right).
$$

(59)

Both constant and $y$-proportional terms remain on the right-hand side of Equation (59), which moreover involves different phases $\sim \sin(2x/R_{0} - 2\Omega t)$ and $\sim \cos(2x/R_{0} - 2\Omega t)$. Hence, we are required to expand two different Fourier series for constant terms and $y$, which, however, gives rise to the difficulty that midlatitude waves are meridionally unbounded so that the wavenumbers $k_{1}$ are, in principle, arbitrary. Hence, we need to constrain the meridional scale in a purposeful way. On the one hand, we are interested in large-scale responses, but...
on the other hand, the \(\beta\)-plane approximation becomes more and more inaccurate with increasing meridional dimensions. As the best compromise, we confine the wave problem to a waveguide defined by the interval \(-R_0/2 \leq y \leq R_0/2\), with \(y = 0\) defining the latitude \(\phi_0 = 45^\circ\). In spherical coordinates, this interval approximately corresponds to the meridional band \(15^\circ \leq \phi \leq 75^\circ\); the respective latitudinal range of 60° is widely regarded as the nonequatorial \(\beta\)-plane limit. Within the interval \(-R_0/2 \leq y \leq R_0/2\), we can describe the forcing terms as Fourier series by expanding

\[ v = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha_{m,n}(t) \sin \left( \frac{m \pi x}{R_0} \right) + \beta_{m,n}(t) \cos \left( \frac{m \pi x}{R_0} \right) \]

and

\[ v = \sum_{n=1}^{\infty} \frac{4R_0}{(2n-1)^2 \pi^2} (-1)^{n-1} \sin \left( \frac{(2n-1) \pi y}{R_0} \right). \]

Interestingly, both series only involve odd latitudinal wave modes \(2n - 1\), which are the only modes that can respond in the leading order. Hence, the tidal potential, although manifested very differently near and far away from the equator, imposes the same symmetry on midlatitude waves as it does on equatorial waves. The forced wave problem can now be solved most easily by inserting the ansatz

\[ v(x, y, t) = \sum_{n=1}^{\infty} 2 \frac{\tilde{k}_n + i \tilde{\kappa}_n}{16 \Omega^2 - 4 \tilde{k}_n^2 + \tilde{\kappa}_n^2} \exp \left( 2i \Omega t - 2i \frac{x \tilde{k}_n}{R_0} + \frac{i(2n-1)\pi y}{R_0} \right) \]

\[ + \sum_{n=1}^{\infty} 2 \frac{\tilde{k}_n - i \tilde{\kappa}_n}{16 \Omega^2 - 4 \tilde{k}_n^2 + \tilde{\kappa}_n^2 + 4i \Omega (8 \tilde{k}_n^2 - \tilde{\kappa}_n^2)} \exp \left( -2i \Omega t + 2i \frac{x \tilde{k}_n}{R_0} + \frac{i(2n-1)\pi y}{R_0} \right) \]

into Equation (59), as we show in Appendix D. Since the right-hand side forcing terms in Equation (59) contain terms both symmetric and antisymmetric in the \((x, t)\) space, we need two independent modal coefficients, \(\alpha_{m,n}(t)\) and \(\beta_{m,n}(t)\), to solve the problem. We find the following explicit analytic solution:

![Figure 8](image-url)
The introduced coefficients $\bar{\Lambda}_1, \bar{\Lambda}_2, \bar{\Lambda}_3, R_1, R_2$ are specified in Appendix D. In the solution, the symmetric part $\sim \cos(...)$ describes nonresonant waves between the resonances, and the antisymmetric part $\sim \sin(...)$ captures resonant waves in the vicinity of the eigenfrequencies. As a remarkable difference from the equatorial solution (Equation (58)), the responses (Equation (63)) do not fade out in the limit of zero Coriolis forces $f_0 = \beta = 0$ and zero gravity $C_0 = 0$, showing that Alfvén waves can be directly excited by tidal forces. In the limit of pure Alfvén waves, $\Lambda_n$ simplifies to

$$
\Lambda_n = \frac{8i(-1)^{n-1}K\Omega}{R_0(2n - 1)\pi\left[4\Omega^2 + 2i\lambda \Omega - \frac{4\Omega^2}{R_0}\right]},
$$

which yields, by reintroducing the series (Equation (60)), the rather simple, unidirectional wave solution

$$
v = \frac{2K\Omega}{R_0\left[4\Omega^2 + 2i\lambda \Omega - \frac{4\Omega^2}{R_0}\right]} \exp\left(i\frac{2\chi}{R_0} - i2\Omega t\right). \quad (64)
$$

This solution is independent of the tachocline layer thickness $H_0$ and does not further rely on the presence of any reduced gravity $g$ so that Equation (64) is not confined to our shallow-water model and can be used in a more universal way to estimate tidally excited Alfvén waves in stars, regardless of the exact strata properties. At resonances, Alfvén waves have maximum amplitudes of $v = K/(R_0 \lambda)$ so that further modeling efforts to estimate the magnetic and viscous dissipation of Alfvén waves $\lambda_{Alfvén}$ (that is physically different from planetary waves) seems necessary for future studies.

Similar to Section 3.3.1, we study the wave responses graphically through calculating normalized amplitudes as a function of the normalized forcing frequency $\tilde{\Omega}$. The calculated frequency range includes the first three prograde propagating magneto-Rossby and magneto-inertia wave modes. We used $\Omega_0 = 26 \cdot 10^{-7}$ s$^{-1}$, $R_0 = 5 \cdot 10^{10}$ cm, $C_0 = 13 \cdot 10^3$ cm s$^{-1}$, and $\beta = 1.04 \cdot 10^{-16}$ cm$^{-1}$ s$^{-1}$.

![Figure 9. Normalized velocity amplitudes due to the nonequatorial solution (Equation (63)) for $v_A = 0$ (top) and $12.6 \cdot 10^3$ cm s$^{-1}$ (bottom) as a function of the normalized forcing frequency $\tilde{\Omega}$. The calculated frequency range includes the first three prograde propagating magneto-Rossby and magneto-inertia wave modes. We used $\Omega_0 = 26 \cdot 10^{-7}$ s$^{-1}$, $R_0 = 5 \cdot 10^{10}$ cm, $C_0 = 13 \cdot 10^3$ cm s$^{-1}$, and $\beta = 1.04 \cdot 10^{-16}$ cm$^{-1}$ s$^{-1}$.](image)
the hydrodynamic limit, which is why only high-frequency responses of inertia-gravity waves remain visible. In the magnetic case, we find almost the same response pattern as for retrograde waves, with the minor difference that the eigenfrequencies are slightly smaller. We can conclude that both progradely and retrogradely excited Rossby waves may resonate to a similar extent—an intriguing difference from analogous geophysical wave responses, where the tidal sense always plays a pivotal role.

4. Estimation of Wave Responses in Our Sun

In the previous sections, we have analyzed characteristic tidal wave excitations for the general class of solar-like stars. The amplitudes were normalized to keep the analysis as general as possible, but we had to use fixed values for the parameters \( C_0 \) and \( v_A \) to calculate the response patterns, which affect the eigenfrequencies and thereby shift the resonance peaks. Apart from this effect, the presented wave responses are generally valid, and we can easily deduce, e.g., that the first latitudinal wave mode \( n = 1 \) is invariably showing the most significant response over all frequencies. That was certainly to be expected due to the large-scale nature of the tidal potential, but the difference from the second mode, \( n = 2 \), is substantial. In order to detect planetary waves in solar-like stars, which are mainly excited by some tidally dominant planet, e.g., (hot) Jupiters, our analysis suggests searching for \( m, n = 2 \), 1 responses, for which the tidal energy input is by far the highest in the leading order. Starting from this insight, we want to estimate the maximum attainable velocity amplitudes in our Sun, which can result from tidal forcing. The tidal forces experienced by the Sun are mainly dictated by a complex interplay of Jupiter, Venus, and Earth (and Mercury), which involves many different excitation frequencies, among them the solar rotation, the individual orbit frequencies, and, perhaps most intriguing, tidal variations with periods of 11 yr closely related to the solar cycle (Okhlopkov 2016). At timescales close to the solar rotation \( \Omega_0 \), waves are always excited retrogradely, whereas the tide-generating planets force planetary waves pradegradely at timescales on the order of their orbit periods. Here we simplify these complex dynamics by considering Jupiter as the sole tide-generating body (exactly the case for which we derived the tidal potentials in Equations (51) and (52)) and taking the forcing frequency \( \Omega \) as arbitrary in the first instance. The latter idealization is justified in view of the argument that the buoyancy frequency (governed by the effective gravity) increases from zero to several rotations per minute as we descend in the tachocline from the convection zone down to the radiative interior. This means there is always a place in the tachocline where the buoyancy period can match the tidal period, or, to put it another way, the tachocline can, in principle, resonate to any given excitation frequency. In the following, we calculate resonant amplitudes as a function of the unspecified quantities \( C_0 \) and \( \lambda \), for which we first calculate the eigenfrequencies \( \tilde{\omega}_{n=1}(C_0, \lambda) \) using Equations (49) and (39) and inserting them into Equations (58) and (63).

4.1. Equatorial Waves

We start to analyze resonant equatorial wave excitations. At first, it needs to be noted that nonresonant wave responses, as they occur in the large frequency range between Rossby and gravity-inertia waves (see Figure 7), are about \( v \approx 0.02 \text{ cm s}^{-1} \). Such low values are entirely negligible for the solar dynamics, so that we will indeed have to focus on resonant excitations. Resonant velocity amplitudes for the full range of gravity velocities considered in the literature and damping ratios between 0.01 and 1 are shown in Figure 10 for both hydrodynamic Rossby (left) and gravity-inertia (right) waves. We find a large range of possible velocity amplitudes, \( 10^{-3} \text{ cm s}^{-1} \leq v \leq 10 \text{ cm s}^{-1} \), where particularly low-frequency Rossby waves in the regime \( 10^5 \text{ cm s}^{-1} \leq C_0 \leq 10^4 \text{ cm s}^{-1} \), which correspond to periods being on the order of the Schwabe cycle, can reach amplitude velocities above \( v = 1 \text{ cm s}^{-1} \). Such velocities may already be dynamo effective; however, all in all, tidal excitations of equatorial waves seem to play only a minor role compared to other mechanisms capable of stimulating planetary waves. The available helioseismic data allow the extraction of \( m = 2 \) Rossby waves only for rms velocities larger than \( v > 50 \text{ cm s}^{-1} \) (Liang et al. 2019), such that tidally forced waves at the equator are unverifiable. Up to now, only sectorial modes \( m = n \) in the range \( 3 \leq m \leq 15 \) have been detected (Löptien et al. 2018).

Prograde magnetic Rossby waves are largely suppressed by constant toroidal fields, resulting in an even weaker wave response not deserving of any further discussion. At this point, it seems very promising for future studies to incorporate nonuniform latitudinal magnetic field profiles into the model as done by Zaqarashvili & Gugerenashvili (2018), allowing one to study very slow prograde magneto-Rossby waves, which can reach Schwabe cycle periods for many different combinations of gravity and Alfvén velocities.

4.2. Nonequatorial Waves

At midlatitudes, we find still smaller nonresonant velocities about \( v \approx 0.004 \text{ cm s}^{-1} \), which is about five times smaller than the corresponding nonresonant equatorial velocities. However, this finding does not allow us to draw any conclusions about general response tendencies. As already discussed in Section 3.3.2, \( n = 1 \) Rossby waves can reach extraordinary high amplitudes at resonance despite the low nonresonant amplitude level. Indeed, Figure 11 confirms that Rossby waves can (theoretically) resonate with amplitudes of more than \( v = 1 \text{ m s}^{-1} \), whereas gravity-inertia waves are even less excited than their equatorial counterparts. In this order of magnitude, tidally forced Rossby waves would indeed be capable of providing sufficient energy to considerably affect the solar dynamics and then take part in the synchronization process of the dynamo.

At midlatitudes, prograde magnetic waves can also respond significantly to tidal excitations. For the strong magnetic field scenarios \( B_0 = 20 \text{ kG} \equiv v_A = 12.6 \cdot 10^4 \text{ cm s}^{-1} \) discussed before, we nevertheless find considerably lower velocity amplitudes up to \( v = 10 \text{ cm s}^{-1} \) (see Figure 12), which is comparable to the weak equatorial wave responses. However, very slow magneto-Rossby waves with periods on the order of the solar cycle and longer are impacted by sufficiently lower toroidal magnetic fields of \( B_0 \sim 1 \text{ kG} \), as they can be generated by steady nonreversing dynamos or manifested as an offshoot of the primordial field in the radiative interior, which may penetrate into the tachocline (Zaqarashvili et al. 2015). Therefore, in Figure 13, we also show the resonant responses of prograde waves subject to a slow Alfvén velocity of \( v_A = 1 \cdot 10^4 \text{ cm s}^{-1} \). In this scenario, the amplitudes are comparable to their retrograde counterparts and may be dynamically effective in a larger range of the \( \lambda/\Omega-C_0 \) space.
4.3. Plausibility of the Responses

We have discussed possible wave responses in the solar tachocline under different scenarios for a wide range of the unspeciﬁed quantities $\lambda$ and $C_0$. The anticipated amplitudes vary in each case over several orders of magnitude, so that up to this point, it is not possible to make a valid statement about the speciﬁc tidal energy input; only excitation potentials have been demonstrated. The presented velocity maps also do not allow for drawing any probability information as to which damping regimes can be reached. We shall therefore brieﬂy discuss the plausibility of the presented scenarios. First, it is important to be aware that the velocity amplitudes are always smaller for small $C_0$, primarily due to the fact that small $C_0$ are associated with small eigenfrequencies, meaning that the absolute damping rates $\lambda$ also become smaller, since we have considered constant damping ratios $\lambda/\Omega$. This poses the question of whether this assumption is realistic, whether low- and high-frequency waves actually have comparable lifetimes. The answer depends on the nature of damping. Although solar fluid dynamics is of strongly turbulent nature, leading-order damping of large-scale Rossby waves is likely governed by laminar viscous boundary layers. In turbulent convection, the kinematic viscosity should be understood as an eddy viscosity (Rüdiger 1989), whose exact value is controversial in the literature. Gizon et al. (2020) applied the value $\nu \approx 250$ km$^2$ s$^{-1}$ associated with supergranular scales to account for the shearing boundary layers arising from differential rotation. For turbulent viscosities of such high orders, boundary layers are expected to be essentially laminar, and we can think of oscillatory Stokes-like boundary layers forming on top of the rigid radiative interior in which the horizontal velocities fall off exponentially to fulﬁll the no-slip boundary condition. When damping is dominated by such oscillatory layers, the damping rate scales as $\lambda \sim \Omega$, such that low-frequency waves (corresponding to small $C_0$) are indeed subject to weaker damping. However, the quantities $\lambda/\Omega$ and $C_0$ are only independent if $\lambda \sim \Omega$, which leads us to conclude that $\lambda/\Omega$ will be distinctly higher for small $C_0$ than for large gravity velocities. Indeed, Liang et al. (2019) found, on average, longer lifetimes for high-frequency waves at lower Rossby modes than for low-frequency waves at higher modes. Nevertheless, the overall damping behavior remains largely unsettled, all the more so since thermal and magnetic dissipation may also make signiﬁcant contributions. Therefore, it seems expedient for future studies to explicitly calculate the different Stokes and Ekman boundary layers, similar to Bildsten & Ushomirsky (1999) for the case of neutron stars.
in order to provide better estimates of viscous damping rates, which, apart from the calculation of resonant wave responses, could also allow for better modeling of horizontal eigenfunctions observed at the solar surface (Proxauf et al. 2020).

Finally, we need to take a closer look at the direction of excitation. We presented response scenarios for both prograde and retrograde planetary waves; however, when focusing on the tidally dominant planet Jupiter, we only have two frequencies, $\Omega_0$ and $\Omega_p$, involved. Obviously, the Sun’s angular frequency is orders of magnitude higher than Jupiter’s orbit frequency $\Omega_0 \gg \Omega_p$, letting us approximate $\Omega = \Omega_0 - \Omega_p \approx -\Omega_0$. Apparently, only the retrograde branches, comprising classic Rossby and gravity-inertial waves, will be stimulated directly. The most relevant and promising scenario to reach significantly high forced velocity amplitudes is therefore represented in Figure 11 in the form of hydrodynamic waves excited at midlatitudes. However, when considering the tidal forces of Jupiter, Venus, and Earth in conjunction, far lower forcing frequencies appear in envelope curves of the potentials, most relevantly the 11 yr period closely matching the solar activity periodicity (Okhlopkov 2016). Envelope variations of the combined tidal potentials can also excite planetary waves progradely in the most susceptible low-$C_0$ regimes, the best requirements for the excitation of slow magneto-Rossby waves. At this point, it seems worthwhile to incorporate the tidal forcing of all three planets and calculate multifrequential wave responses numerically. We plan to accomplish this in a future study. As a second interesting scenario of low-frequency forcing, we can also think of solar-like stars forced by tidally synchronized planets, or likewise binary stars, where the dominant frequency $\Omega_0$ is absent, and slow frequencies, both prograde and retrograde, may remain as a result of orbital changes. For all of these systems, it is conceivable that low-frequency tidal forcing may dictate the activity cycles, be it by synchronizing the dynamo or by directly implanting large-scale atmospheric motions, provided that the forcing amplitude $K$ is large enough (or the damping rate $\lambda$ small enough) to induce sufficiently high wave amplitudes.

5. Concluding Remarks

By equipping the magnetohydrodynamic “shallow-water” equations with a tidal potential term and linear friction, we have
constructed a first theoretical setup for the study of damped and forced planetary waves in the tachocline layer of solar-like stars. The governing equations were projected onto two different Cartesian planes in the vicinity of the equator and at midlatitudes, allowing us to describe both equatorially trapped and locally unbounded waves in the most approachable way. As a key result, we have shown that the linearized system of governing equations can be combined into one decoupled wave (Equation (12)) for the local latitudinal velocity component, which markedly simplifies the Fourier analysis for extracting the characteristic dispersion relations in different wave limits of interest.

We solve this wave equation analytically within different regimes, starting with the known free wave solutions via damped wave dynamics up to the complete forced wave problem. The analysis revealed that the damping behavior of magnetohydrodynamic planetary waves is more intricate than the damping of classical geophysical waves, since the introduced damping parameter can translate into very different decay rates for the different wave types. Most interesting here is our finding that the damping rates of retrograde Rossby waves at midlatitudes correspond exactly to the decay rates, whereas progradely propagating magnetic Rossby waves are predicted to decay considerably slower by a factor of the squared natural eigenfrequency divided by the squared Alfvén frequency. Further, the damped wave solutions derived for equatorial waves facilitate calculating the meridional scales as a function of the damping rate and the toroidal magnetic field, with the result that damping always widens and the presence of magnetic fields always, here in an overcompensating way, narrows the equatorial waveguide.

The forced wave problem is solved analytically for the idealized case of a single tide-generating body prescribing a perfect circular orbit around the central star. The solutions can describe both nonresonant and resonant wave responses; the latter, however, are largely determined by the a priori unknown damping coefficient. We found that for fixed damping ratios, equatorial waves always respond with higher-velocity amplitudes than midlatitude waves under nonresonant conditions, whereas midlatitude waves have higher peak velocities in proximity to resonances. Among all types of planetary waves, the first large-scale Rossby mode, be it the classic retrograde or magnetic prograde Rossby wave, is found to always resonate with the highest amplitudes when considering fixed lifetimes. Rossby waves are therefore confirmed to indeed be a most promising candidate to potentially act as a resonance ground for low-frequency tidal excitations, if tidal frequencies are such that these waves can be excited.

Finally, we applied the solutions to the specific scenario of our Sun tidally forced by Jupiter for estimating possible velocity responses. We obtained nonresonant amplitudes of \( v \approx 0.02 \text{ cm s}^{-1} \) at the equator and \( v \approx 0.004 \text{ cm s}^{-1} \) at midlatitudes, which are, for the solar dynamo, completely negligible. Resonant amplitudes strongly depend on both damping and the effective gravity (the considered region in the tachocline layer) so that resonant velocities deviate by several orders of magnitude with respect to these parameters. Consequently, a reliable prediction of the anticipated responses cannot yet be made without further ado. However, we found that for low-frequency excitations on the order of the 11 yr solar cycle, with a particular view on the 11.07 yr alignment periodicity of the tidally dominant planets Venus, Earth, and Jupiter, the tidal energy input of Jupiter alone can evoke high Rossby wave amplitudes of \( v \gtrsim 100 \text{ cm s}^{-1} \) for fairly small damping ratios in the range \( 0.01 \lesssim \lambda / \Omega \lesssim 0.1 \). We can conclude that, despite the fact that tidal accelerations are very small, significant velocities can potentially be induced through resonant amplification if dissipation is sufficiently small.

To draw more definite conclusions, our model must be extended in different directions. Our pilot analysis has stressed the potential that tidal forcing may induce significant tachocline wave motions, but to irrevocably verify or disprove this possibility, further modeling of the different dissipation sources, including viscous, turbulent, thermal, and magnetic damping, is required. In addition, it is expedient to solve the problem in spherical coordinates (Márquez-Artavia et al. 2017) and include the full potentials of all significant planets, which would allow one to determine the exact response to alignment periodicities and spring tides. Regardless of the exact attainable amplitude levels, there also remains the question of how and in what way Rossby waves may affect the solar dynamo. The linear Rossby wave solutions do not produce net kinetic helicity, which is the main ingredient for the \( \alpha \) effect. Hence, it appears instructive to look at the nonlinear evolution of tidally excited Rossby waves, e.g., tachocline nonlinear oscillations arising from the energy exchange between Rossby waves and differential rotation (Dikpati et al. 2018a). Moreover, it might be promising to estimate how wave-like displacements of the tachocline layer affect the entropy stratification to check if tide-stimulated flux tube instabilities can possibly encroach into the dynamo process (Ferriz-Mas 1996; Charbonneau 2022). Finally, coupling the planetary wave equations with dynamo models may provide valuable insights into possible synchronization mechanisms imparted by Rossby waves.

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Appendix A

Derivation of the Wave Equation

Differentiation of Equations (8) and (9) with respect to time and using Equations (10) and (11) yields the coupled differential equations for the velocities,

\[
\frac{\partial^2 u}{\partial t^2} - f \frac{\partial v}{\partial t} = v_f \frac{\partial^2 u}{\partial x^2} + C_0^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \frac{\partial^2 V}{\partial x \partial t} - \lambda \frac{\partial u}{\partial t}, \tag{A1}
\]

\[
\frac{\partial^2 v}{\partial t^2} + f \frac{\partial u}{\partial t} = v_f \frac{\partial^2 v}{\partial x^2} + C_0^2 \left( \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} \right) - \frac{\partial^2 V}{\partial y \partial t} - \lambda \frac{\partial v}{\partial t}, \tag{A2}
\]

to be decoupled in the following. For this, we can replace the term \( f \partial u \) in Equation (A2) by Equation (8), giving

\[
\Box_v u + f^2 v - f g \frac{\partial \eta}{\partial x} + f \frac{B_0 \partial b_v}{4 \pi \rho} \frac{\partial}{\partial x} - f \frac{\partial V}{\partial x} - f \lambda u = C_0^2 \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \frac{\partial^2 V}{\partial y \partial t} - \lambda \frac{\partial v}{\partial t}. \tag{A3}
\]

As an intermediate step, we calculate \( \partial_\gamma \) (Equation (8)) - \( \partial_\zeta \) (Equation (9)) to extract the following expression:

\[
\frac{\partial \eta}{\partial t} = \frac{H_0}{f} \frac{\partial \zeta}{\partial t} + \lambda \frac{H_0}{f} \frac{\partial \zeta}{\partial x} + \frac{H_0}{f} \frac{B_0}{4 \pi \rho} \frac{\partial}{\partial x} \left( \frac{\partial b_v}{\partial x} - \frac{\partial b_v}{\partial y} \right). \tag{A4}
\]

where \( \zeta = \partial_\gamma v - \partial_\lambda u \) denotes the horizontal vorticity. We can now differentiate Equation (A3) with respect to time and insert Equation (A4) to eliminate \( \eta \):

\[
\frac{\partial}{\partial t} \Box_v u + f^2 \frac{\partial v}{\partial t} - C_0^2 \frac{\partial^2 \zeta}{\partial t \partial x} - C_0^2 \lambda \frac{\partial \zeta}{\partial x} - C_0^2 B_0 \frac{\partial^2 V}{4 \pi \rho \partial x^2} \left( \frac{\partial b_v}{\partial x} - \frac{\partial b_v}{\partial y} \right) + f^2 \frac{\partial^2 u}{\partial x^2} - f \frac{\partial^2 V}{\partial t \partial x} - f \lambda \frac{\partial u}{\partial t}
\]

\[
= C_0^2 \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \frac{\partial^3 V}{\partial y \partial t^2} - \lambda \frac{\partial^2 V}{\partial t^2}. \tag{A5}
\]

By noting that

\[
C_0^2 \frac{\partial^2 \zeta}{\partial t \partial x} + C_0^2 \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = C_0^2 \frac{\partial}{\partial t} \Delta v \tag{A6}
\]

and again taking the time derivative, we find

\[
\frac{\partial^2}{\partial t^2} \Box_v u + f^2 \frac{\partial^2 v}{\partial t^2} - C_0^2 \frac{\partial^2 \zeta}{\partial t^2} \Delta v - C_0^2 \lambda \frac{\partial^2 \zeta}{\partial \gamma \partial x} - C_0^2 B_0 \frac{\partial^2 V}{\partial x^2} \left( \frac{\partial b_v}{\partial x} - \frac{\partial b_v}{\partial y} \right) + C_0^2 v_f \frac{\partial^4 V}{\partial x^2} \left( \frac{\partial b_v}{\partial x} - \frac{\partial b_v}{\partial y} \right) - \frac{\partial^4 u}{\partial x^4} \left( \frac{\partial b_v}{\partial x} - \frac{\partial b_v}{\partial y} \right)
\]

\[
-f \frac{\partial^3 V}{\partial \gamma \partial t^2} + f \lambda \frac{\partial^3 \zeta}{\partial \gamma \partial t^2} + \lambda \frac{\partial^3 V}{\partial \gamma^3} = 0. \tag{A7}
\]

Let us collect all remaining \( u \) velocity terms in Equation (A7), reading

\[
C_0^2 \lambda \frac{\partial^3 \zeta}{\partial \gamma \partial t} - C_0^2 v_f \frac{\partial^4 \zeta}{\partial x^2 \partial y} + f \lambda \frac{\partial^3 \zeta}{\partial \gamma \partial x^2} - f \lambda \frac{\partial^3 u}{\partial t^2} = \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) - C_0^2 \frac{\partial^2 u}{\partial x \partial y}. \tag{A8}
\]
Thankfully, the two terms in the right bracket can be eliminated simply by rearranging Equation (A2) in the following form:

\[
f \frac{\partial u}{\partial t} - C_0 \frac{\partial^2 u}{\partial x \partial y} = v^2 \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} + C_0 \frac{\partial^2 v}{\partial y \partial t} - \lambda \frac{\partial v}{\partial t}. \tag{A9}
\]

This way, we have finally eliminated all \( u \) velocity terms. Inserting Equation (A9) into Equation (A7) yields the wave Equation (12) presented in Section 2.

### Appendix B

**Derivation of the Tidal Potential**

We consider a central body with radius \( R_0 \) with its center of mass being the origin labeled \( O \) and a tide-generating body with mass \( M_t \) at a distance \( r = r(t) \) from \( O \); see Figure 14(a). The tidal force originating from \( M_t \) results from the combined action of the pseudoforce of inertia due to the freefall motion of the central body around the center of mass of the system and the gravitational force of the perturbing body. At location \( A \) on the surface of the central object, the tidal force can be computed from the vector difference of the gravitational pull the perturbing body exerts on a test object at position \( A \) and the gravitational pull the perturbing body would exert on this object at the center of the central body (which corresponds to the pseudoforce of inertia of the freefall motion).

The tidal force is a conservative force and can be expressed in terms of a scalar potential, \( F_t = -\nabla V_t \), which can be developed in terms of an infinite sum of Legendre polynomials labeled by their degree \( l \). Since the terms with \( l = 0 \) and \( l = 1 \) do not contribute to the force, and the relation of the radius \( R_0 \) to the distance \( r \) is small, only one term with \( l = 2 \) has to be considered (see, e.g., Agnew 2015). Then, the tidal potential is

\[
V_t(A) = \frac{GM_t R_0^2}{r^3(t)} P_2(\cos \alpha), \tag{B1}
\]

where \( G \) is the gravitational constant, and \( P_2 \) is the Legendre polynomial with degree \( l = 2 \). The angle \( \alpha \) represents the solid angle between the line connecting the origin \( O \) and the observation position \( A \) and the line connecting \( O \) and the center of the tide-generating object (Figure 14(b)). It is convenient to transfer the problem into a spherical coordinate system, where we have two sets of variables, \( \varphi \) and \( \theta \), which denote the longitude and latitude coordinates of the observation point \( A \), and \( \Theta \) and \( \Phi \), which describe the location of the tide-generating object \( M_t \). The Legendre polynomial \( P_l(\cos \alpha) \) can then be expressed as a sum of associated Legendre polynomials in \( \vartheta \), \( \varphi \), \( \Theta \), and \( \Phi \) via the expansion in terms of spherical harmonics,

\[
P_l(\cos \alpha) = \frac{1}{2l + 1} \sum_{m=-l}^{l} \left[ \frac{2l + 1}{4\pi} \frac{1}{(l + m)!} \right]^{1/2} P^m_l(\cos \vartheta) Y^m_l(\Theta, \Phi), \tag{B2}
\]

where \( P^m_l \) are the associated Legendre functions taken from Munk et al. (1966). Since we only perform a decomposition for \( l = 2 \), we only need the following five associated Legendre polynomials:

\[
P_2^2 = \frac{1}{8} \sin^2 \vartheta, \quad P_2^{-1} = \frac{1}{2} \sin \vartheta \cos \vartheta, \quad P_2^0 = \frac{1}{2} (3 \cos^2 \vartheta - 1), \quad P_2^1 = -3 \sin \vartheta \cos \vartheta, \quad P_2^2 = 3 \sin^2 \vartheta, \tag{B3}
\]

so that after some calculations using standard relations for the trigonometric functions, we obtain

\[
P_2(\cos \alpha) = \frac{3}{4} \left[ \frac{1}{3} (3 \cos^2 \Theta - 1)(3 \cos^2 \vartheta - 1) + \sin 2\Theta \sin 2\vartheta \cos(\varphi - \Phi) + \sin^2 \Theta \sin^2 \vartheta \cos(2\varphi - 2\Phi) \right]. \tag{B4}
\]

We now consider a tide-generating object that moves exactly in the plane defined by the equatorial plane of the central body so that \( \Theta = 90^\circ \). Thus, the second term in the brackets on the right side of Equation (B4) (\( x \sin 2\Theta \)) vanishes, and the first term simplifies to \( \sin^2 \vartheta \) (after dropping the constant terms that do not contribute to the tidal force). We further assume that the distance between the central and perturbing bodies is constant, i.e., \( r(t) = r \). In reality, the two bodies propagate around a common center of mass on a path that is the solution of the corresponding Kepler problem. The resulting variation of the distance between both bodies introduces a slow timescale for the amplitude of the tidal force, which can be neglected with regard to the current problem. Hence,
the tidal potential finally reads

\[ V_t(R, \vartheta, \varphi, t) = \frac{3}{4} \frac{GM_t}{R^3} \left( \frac{R}{r} \right)^3 \{ \sin^2 \vartheta [1 + \cos (2\varphi - 2\Phi)] \}, \]  

where we have introduced the constant term \( K \). The only time-dependent quantity in Equation (B5) is the angle \( \Phi \), which describes the longitude of the tide-generating body with respect to the origin. The time dependence of \( \Phi \) is simply given by \( \Phi(t) = (\Omega_t - \Omega_0)t \), where \( \Omega_0 \) denotes the angular frequency of the rotation of the central body (i.e., the Sun in our particular case), and \( \Omega_t \) is the angular frequency of the motion of the perturbing body around the central object. In the following, we translate the angular coordinates \( \varphi \) and \( \vartheta \) into the Cartesian coordinates \( x \) and \( y \) as they appear in the \( \beta \)-plane model. We consider a small regime around a fixed latitudinal coordinate \( \vartheta_0 \) so that we can write \( \vartheta = \vartheta_0 + \vartheta' \) with \( \vartheta' = y/R_0 \ll 1 \). The corresponding expansion of the expression that describes the latitudinal dependence is then

\[ \sin^2(\vartheta) = \sin^2(\vartheta_0 + \vartheta') \approx \sin^2(\vartheta_0) + \sin(2\vartheta_0) \frac{y}{R_0} + \cos(2\vartheta_0) \left( \frac{y^2}{R_0^2} \right) + \ldots. \]  

In order to keep the same approximation used for the \( \beta \)-plane approach, we ignore the term quadratic in \( y \). We examine two particular cases. At the equator, we have \( \vartheta_0 = \pi/2 \) so that \( \sin^2(\vartheta_0) = 1 \) and \( \sin(2\vartheta_0) = 0 \), and we end up with a forcing independent of \( y \):

\[ V_t^{eq}(x, y, t) = K \{ 1 + \cos [2x - 2(\Omega_t - \Omega_0)t] \}. \]  

For nonequatorial solutions, it is reasonable to choose \( \vartheta_0 = \pi/4 \) so that \( \sin^2(\vartheta_0) = 1/2 \) and \( \sin(2\vartheta_0) = 1 \), and the corresponding tidal potential reads

\[ V_t^{45^\circ}(x, y, t) = K \left( \frac{1}{2} + \frac{y}{R_0} \right) \{ 1 + \cos [2x - 2(\Omega_t - \Omega_0)t] \}. \]  

**Appendix C**

**Forced Wave Solutions at the Equator**

The forced wave problem is solved by inserting the ansatz

\[ v = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \alpha_{m,n}(t) \exp \left( \frac{im}{R_0} r \right) \exp \left( -\frac{\mu y^2}{2} \right) H_n(\sqrt{\mu} y) \]  

and the coordinate expansion

\[ y = \sum_{n=1}^{\infty} \frac{2^{\frac{1}{2}} - 2n}{\sqrt{\mu} (n - 1)!} \exp \left( -\frac{\mu y^2}{2} \right) H_{2n-1}(\sqrt{\mu} y). \]
into the wave (Equation (54)). The comparison of the modal coefficients $\alpha_{m,n}(t)$ with the forcing potential $\sim \cos(2x/R_0 - 2\Omega t)$ directly reveals the zonal wavenumber to be fixed at $m = 2$, which describes planetary waves with two crests and troughs fitting around the equator. All other modal coefficients are not subject to any forcing and consequently obey damped wave equations, letting them decay exponentially in time. Linear stationary solutions can only be found for $m = 2$. In the same way, we can argue that latitudinal wavenumbers corresponding to stationary solutions must be uneven, $n \to 2n + 1$, since only uneven Hermitian polynomials appear in the expansion (Equation (C2)). The remaining modal coefficients must satisfy the conditional equation

$$
\sum_{n=1}^{\infty} \alpha_{2,2n-1}(t) \frac{4\Omega^2}{C_0^2} \exp\left(-\frac{\mu y^2}{2}\right)H_{2n-1}(\sqrt{\mu} y) \exp\left(\frac{2x}{R_0}\right) \\
+ \sum_{n=1}^{\infty} \alpha_{2,2n+1}(t) \frac{8\lambda^2}{C_0^2} \exp\left(-\frac{\mu y^2}{2}\right)H_{2n-1}(\sqrt{\mu} y) \exp\left(\frac{2x}{R_0}\right) \\
+ \sum_{n=1}^{\infty} \alpha_{2,2n-1}(t) \frac{32y^2\lambda^2}{C_0^2 R_0^2} \exp\left(-\frac{\mu y^2}{2}\right)H_{2n-1}(\sqrt{\mu} y) \exp\left(\frac{2x}{R_0}\right) \\
+ \sum_{n=1}^{\infty} \alpha_{2,2n+1}(t) \frac{64y^2\lambda^2}{C_0^2 R_0^2} \exp\left(-\frac{\mu y^2}{2}\right)H_{2n-1}(\sqrt{\mu} y) \exp\left(\frac{2x}{R_0}\right) \\
+ \sum_{n=1}^{\infty} \alpha_{2,2n-1}(t) \frac{16y^2}{R_0^2} \exp\left(-\frac{\mu y^2}{2}\right)H_{2n-1}(\sqrt{\mu} y) \exp\left(\frac{2x}{R_0}\right) \\
- \sum_{n=1}^{\infty} \alpha_{2,2n-1}(t) \left[4\Omega^2 + 2\lambda\Omega i - \frac{4y^2}{R_0^2}\right] \frac{\partial^2}{\partial y^2} \exp\left(-\frac{\mu y^2}{2}\right)H_{2n-1}(\sqrt{\mu} y) \exp\left(\frac{2x}{R_0}\right) \\
+ \sum_{n=1}^{\infty} \alpha_{2,2n+1}(t) \frac{4\gamma^2y^2}{C_0^2} \exp\left(-\frac{\mu y^2}{2}\right)H_{2n-1}(\sqrt{\mu} y) \exp\left(\frac{2x}{R_0}\right) \\
+ \sum_{n=1}^{\infty} \alpha_{2,2n-1}(t) \frac{32\lambda^2y^2}{C_0^2 R_0^2} \exp\left(-\frac{\mu y^2}{2}\right)H_{2n-1}(\sqrt{\mu} y) \exp\left(\frac{2x}{R_0}\right) \\
+ \sum_{n=1}^{\infty} \alpha_{2,2n+1}(t) \frac{64\lambda^2y^2}{C_0^2 R_0^2} \exp\left(-\frac{\mu y^2}{2}\right)H_{2n-1}(\sqrt{\mu} y) \exp\left(\frac{2x}{R_0}\right) \\
+ \sum_{n=1}^{\infty} \alpha_{2,2n-1}(t) \frac{16\lambda^2}{R_0^2} \exp\left(-\frac{\mu y^2}{2}\right)H_{2n-1}(\sqrt{\mu} y) \exp\left(\frac{2x}{R_0}\right) \\
+ \sum_{n=1}^{\infty} \alpha_{2,2n+1}(t) \frac{4\lambda^2}{C_0^2} \exp\left(-\frac{\mu y^2}{2}\right)H_{2n-1}(\sqrt{\mu} y) \exp\left(\frac{2x}{R_0}\right) \\
= \sum_{n=1}^{\infty} \frac{32K\beta^4}{C_0^2 R} \frac{2^{2s-2n}}{\sqrt{\mu}(n-1)!} \exp\left(-\frac{\mu y^2}{2}\right)H_{2n-1}(\sqrt{\mu} y) \exp\left(\frac{2x}{R_0} - i2\Omega t\right). \tag{C3}
$$

We have used $\alpha_{2,2n-1} = i\alpha_{2,2n-1}$ to evaluate the three terms involving $y$ derivatives. From Equation (C3), we want to extract ordinary differential equations for the modal coefficients $\alpha_{2,2n-1}(t)$. All terms are proportional to $\exp(i2x/R_0)$ so that the $x$-coordinate drops out by default. This is unfortunately not the case for the $y$-coordinate, since the seventh and eighth terms in Equation (C3) do not fit with the Hermitian base. However, we can bypass this issue by applying the identity

$$
\mu^2 y^2 \exp\left(-\frac{\mu y^2}{2}\right)H_n(\sqrt{\mu} y) - \frac{\partial^2}{\partial y^2} \exp\left(-\frac{\mu y^2}{2}\right)H_n(\sqrt{\mu} y) = (2n + 1) \mu \exp\left(-\frac{\mu y^2}{2}\right)H_n(\sqrt{\mu} y) \tag{C4}
$$

that projects the two terms on the Hermitian eigenbasis. Inserting $\mu$ (Equation (48)) allows us to rearrange Equation (C4) into

$$
\left[4\Omega^2 + 2i\lambda\Omega\beta^2 \frac{y^2}{C_0^2} - \left(4\Omega^2 - \frac{4\mu^2}{R_0^2}\right) \frac{\partial^2}{\partial y^2}\right] \exp\left(-\frac{\mu y^2}{2}\right)H_{2n-1}(\sqrt{\mu} y) \\
= \left[2\Omega\beta \frac{\lambda^2}{C_0} \left(4\Omega^2 + 2i\lambda \Omega - \frac{4\mu^2}{R_0^2}(4n-1)\right) \exp\left(-\frac{\mu y^2}{2}\right)H_{2n-1}(\sqrt{\mu} y). \tag{C5}
$$
We insert Equation (C5) into Equation (C3) to replace the two non-Hermitian terms, yielding

\[ \sum_{n=1}^{\infty} \left[ \ddot{\alpha}_{2,2n-1}(t) + 2\lambda \dddot{\alpha}_{2,2n-1}(t) + \left( \frac{C_0}{2\Omega} \sqrt{4\Omega^2 + 2i\lambda\Omega - \frac{4\nu^2}{R_0^2}} (4n - 1) + \frac{4}{R_0^2}(C_0^2 + 2\nu^2) + \frac{8\lambda\nu^2}{R_0^2} + \lambda^2 \right) \ddot{\alpha}_{2,2n-1}(t) \right] \\
+ \left( \frac{4\lambda C_0^2}{R_0^2} - \frac{2iC_0^2\beta}{R_0} \right) \dot{\alpha}_{2,2n-1}(t) + \left( \frac{16\nu^2 C_0^2}{R_0^4} + \frac{16\nu^4}{R_0^4} \right) \alpha_{2,2n-1}(t) \\
- \frac{8K\beta\Omega^2}{R} \frac{2^{n-2} - 1}{\sqrt{\mu}(n-1)!} \exp(-i2\Omega t) \right] \exp \left( -\frac{\mu\gamma^2}{2} \right) H_{2n-1}(-\sqrt{\mu}y) = 0. \] (C6)

The infinite sums can only yield zero if all individual summands disappear. Therefore, each coefficient \( \alpha_{2,2n-1}(t) \) must satisfy the following set of ordinary differential equations:

\[ \ddot{\alpha}_{2,2n-1}(t) + 2\lambda \dddot{\alpha}_{2,2n-1}(t) + \left( \frac{C_0}{2\Omega} \sqrt{4\Omega^2 + 2i\lambda\Omega - \frac{4\nu^2}{R_0^2}} (4n - 1) + \frac{4}{R_0^2}(C_0^2 + 2\nu^2) + \frac{8\lambda\nu^2}{R_0^2} + \lambda^2 \right) \ddot{\alpha}_{2,2n-1}(t) \\
+ \left( \frac{4\lambda C_0^2}{R_0^2} - \frac{2iC_0^2\beta}{R_0} \right) \dot{\alpha}_{2,2n-1}(t) + \left( \frac{16\nu^2 C_0^2}{R_0^4} + \frac{16\nu^4}{R_0^4} \right) \alpha_{2,2n-1}(t) \\
- \frac{8K\beta\Omega^2}{R} \frac{2^{n-2} - 1}{\sqrt{\mu}(n-1)!} \exp(-i2\Omega t) = 0. \] (C7)

We have now reduced the partial differential Equation (54) into an infinite set of decoupled ordinary differential equations to be solved readily. Equation (C7) has the following stationary solution:

\[ \alpha_{2,2n-1}(t) = \frac{8K\beta\Omega^22^{-n-2}(\sqrt{\mu}(n-1))^{-1}\exp(-i2\Omega t)}{\left(4\Omega^2 + 2i\lambda\Omega - \frac{4\nu^2}{R_0^2}\right)\left(4\Omega^2 + 2i\lambda\Omega - \frac{4\nu^2}{R_0^2}(C_0^2 + \nu^2)\right) - C_0^2\frac{4\nu^2}{R_0^2} - C_0\beta|2\Omega|\sqrt{\nu^2 + 2i\lambda\omega - \frac{4\nu^2}{R_0^2}} (4n - 1)} \] (C8)

Inserting the modal coefficients back into the ansatz (Equation (C1)) yields the solution (Equation (58)) we present in Section 3.3.1.

Appendix D

Forced Wave Solutions at Midlatitudes

We solve the forced wave problem by inserting the ansatz

\[ \psi = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[ \alpha_{m,n}(t) \sin \left( \frac{m\pi x}{R_0} \right) + \beta_{m,n}(t) \cos \left( \frac{m\pi x}{R_0} \right) \right] \exp \left( \frac{in\pi y}{R_0} \right) \] (D1)

to be confined within a midlatitude wave channel defined by the interval \(-R_0/2 \leq y \leq R_0/2\) into the wave (Equation (59)). The forcing terms on the right-hand side of Equation (59) involve both constant and \(y\)-proportional terms, which both must be expanded as Fourier series (Equations (60) and (61)) in order to project them onto the harmonic basis used in the ansatz. The direct comparison of the modal coefficients \( \alpha_{m,n}(t) \) and \( \beta_{m,n}(t) \) with the forcing potential \( \sim \cos(2x/R_0 - 2\Omega t) \) directly reveals the zonal wavenumber to be fixed at \( m = 2 \), which describes planetary waves with two crests and troughs fitting around the equator. All other modal coefficients are not subject to any forcing and consequently obey freely damped wave equations, letting them decay exponentially in time. Linear stationary solutions can only be found for \( m = 2 \). In the same way, we can argue that latitudinal wavenumbers corresponding to stationary solutions must be uneven, \( n \rightarrow 2n + 1 \), since only uneven terms appear in the Fourier expansion.
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(Equations (60) and (61)). The remaining modal coefficients must satisfy the conditional equation

\[
\sum_{n=1}^{\infty} \left[ \alpha_{2,2n-1}(t) \sin \left( \frac{2x}{R_0} \right) + \beta_{2,2n-1}(t) \cos \left( \frac{2x}{R_0} \right) \right] \exp \left( \frac{i(2n-1)\pi y}{R_0} \right) = 0
\]

\[
+ \sum_{n=1}^{\infty} \left[ \alpha_{2,2n-1}(t) \sin \left( \frac{2x}{R_0} \right) + \beta_{2,2n-1}(t) \cos \left( \frac{2x}{R_0} \right) \right] 2\lambda \exp \left( \frac{i(2n-1)\pi y}{R_0} \right)
\]

\[
+ \sum_{n=1}^{\infty} \left[ \alpha_{2,2n-1}(t) \sin \left( \frac{2x}{R_0} \right) + \beta_{2,2n-1}(t) \cos \left( \frac{2x}{R_0} \right) \right] 8\omega^2 \exp \left( \frac{i(2n-1)\pi y}{R_0} \right)
\]

\[
+ \sum_{n=1}^{\infty} \left[ \alpha_{2,2n-1}(t) \sin \left( \frac{2x}{R_0} \right) + \beta_{2,2n-1}(t) \cos \left( \frac{2x}{R_0} \right) \right] 4C^2_{2n} \exp \left( \frac{i(2n-1)\pi y}{R_0} \right)
\]

\[
+ \sum_{n=1}^{\infty} \left[ \alpha_{2,2n-1}(t) \sin \left( \frac{2x}{R_0} \right) + \beta_{2,2n-1}(t) \cos \left( \frac{2x}{R_0} \right) \right] f_c^2 \exp \left( \frac{i(2n-1)\pi y}{R_0} \right)
\]

\[
+ \sum_{n=1}^{\infty} \left[ \alpha_{2,2n-1}(t) \sin \left( \frac{2x}{R_0} \right) + \beta_{2,2n-1}(t) \cos \left( \frac{2x}{R_0} \right) \right] \lambda^2 \exp \left( \frac{i(2n-1)\pi y}{R_0} \right)
\]

\[
+ \sum_{n=1}^{\infty} \left[ \alpha_{2,2n-1}(t) \sin \left( \frac{2x}{R_0} \right) + \beta_{2,2n-1}(t) \cos \left( \frac{2x}{R_0} \right) \right] 2C^2_{2n} \beta \exp \left( \frac{i(2n-1)\pi y}{R_0} \right)
\]

\[
+ \sum_{n=1}^{\infty} \left[ \alpha_{2,2n-1}(t) \sin \left( \frac{2x}{R_0} \right) + \beta_{2,2n-1}(t) \cos \left( \frac{2x}{R_0} \right) \right] 8\omega^2 \exp \left( \frac{i(2n-1)\pi y}{R_0} \right)
\]

\[
+ \sum_{n=1}^{\infty} \left[ \alpha_{2,2n-1}(t) \sin \left( \frac{2x}{R_0} \right) + \beta_{2,2n-1}(t) \cos \left( \frac{2x}{R_0} \right) \right] 4(2n-1)^2\pi^2 C^2_{2n} \exp \left( \frac{i(2n-1)\pi y}{R_0} \right)
\]

\[
+ \sum_{n=1}^{\infty} \left[ \alpha_{2,2n-1}(t) \sin \left( \frac{2x}{R_0} \right) + \beta_{2,2n-1}(t) \cos \left( \frac{2x}{R_0} \right) \right] 16\lambda \exp \left( \frac{i(2n-1)\pi y}{R_0} \right)
\]

\[
+ \sum_{n=1}^{\infty} \left[ \alpha_{2,2n-1}(t) \sin \left( \frac{2x}{R_0} \right) + \beta_{2,2n-1}(t) \cos \left( \frac{2x}{R_0} \right) \right] 16K\Omega (-1)^{n-1} \exp \left( \frac{i(2n-1)\pi y}{R_0} \right)
\]

\[
= \sum_{n=1}^{\infty} \left[ \sin \left( \frac{2x}{R_0} \right) \cos \left( \frac{2x}{R_0} \right) + \sin \left( \frac{2x}{R_0} \right) \cos \left( \frac{2x}{R_0} \right) \right] \frac{16K\Omega (-1)^{n-1} \exp \left( \frac{i(2n-1)\pi y}{R_0} \right)}{(2n-1)\pi}
\]

\[
+ \sum_{n=1}^{\infty} \left[ \sin \left( \frac{2x}{R_0} \right) \cos \left( \frac{2x}{R_0} \right) + \sin \left( \frac{2x}{R_0} \right) \cos \left( \frac{2x}{R_0} \right) \right] \frac{16K\Omega (-1)^{n-1} \exp \left( \frac{i(2n-1)\pi y}{R_0} \right)}{(2n-1)\pi}
\]

\[
= \sum_{n=1}^{\infty} \left[ \sin \left( \frac{2x}{R_0} \right) \cos \left( \frac{2x}{R_0} \right) + \sin \left( \frac{2x}{R_0} \right) \cos \left( \frac{2x}{R_0} \right) \right] \frac{16K\Omega (-1)^{n-1} \exp \left( \frac{i(2n-1)\pi y}{R_0} \right)}{(2n-1)\pi}
\]

\[
= \sum_{n=1}^{\infty} \left[ \sin \left( \frac{2x}{R_0} \right) \cos \left( \frac{2x}{R_0} \right) + \sin \left( \frac{2x}{R_0} \right) \cos \left( \frac{2x}{R_0} \right) \right] \frac{16K\Omega (-1)^{n-1} \exp \left( \frac{i(2n-1)\pi y}{R_0} \right)}{(2n-1)\pi}
\]

\[
= \sum_{n=1}^{\infty} \left[ \sin \left( \frac{2x}{R_0} \right) \cos \left( \frac{2x}{R_0} \right) + \sin \left( \frac{2x}{R_0} \right) \cos \left( \frac{2x}{R_0} \right) \right] \frac{16K\Omega (-1)^{n-1} \exp \left( \frac{i(2n-1)\pi y}{R_0} \right)}{(2n-1)\pi}
\]

\[
\times \sin \left( \frac{2x}{R_0} \right) \exp \left( \frac{i(2n-1)\pi y}{R_0} \right)
\]

\[
+ \sum_{n=1}^{\infty} \left[ \sin \left( \frac{2x}{R_0} \right) \cos \left( \frac{2x}{R_0} \right) + \sin \left( \frac{2x}{R_0} \right) \cos \left( \frac{2x}{R_0} \right) \right] \frac{16K\Omega (-1)^{n-1} \exp \left( \frac{i(2n-1)\pi y}{R_0} \right)}{(2n-1)\pi}
\]

\[
\times \cos \left( \frac{2x}{R_0} \right) \exp \left( \frac{i(2n-1)\pi y}{R_0} \right)
\]

\[
= 0
\]

(D2)

All summands are proportional to either \( \sin(2x/R_0) \exp(i(2n-1)\pi y/R_0) \) or \( \cos(2x/R_0) \exp(i(2n-1)\pi y/R_0) \) so that we can rearrange the terms as follows:
The infinite sums can only yield zero if all individual summands disappear independently. Therefore, each pair of coefficients $\alpha_{2,2n-1}(t)$ and $\beta_{2,2n-1}(t)$ must satisfy the following set of coupled ordinary differential equations:

$$
\ddot{\alpha}_{2,2n-1}(t) + 2\lambda\dot{\alpha}_{2,2n-1}(t) + \tilde{\lambda}_1\alpha_{2,2n-1}(t) + \tilde{\lambda}_2\beta_{2,2n-1}(t) + \frac{2\beta C_0^2}{R_0}\beta_{2,2n-1}(t) + \tilde{\lambda}_3\alpha_{2,2n-1}(t) - \tilde{K}_1\cos(2\Omega t) - \tilde{K}_2\sin(2\Omega t) = 0,
$$
\hspace{1cm} \text{(D4)}

$$
\ddot{\beta}_{2,2n-1}(t) + 2\lambda\dot{\beta}_{2,2n-1}(t) + \tilde{\lambda}_1\beta_{2,2n-1}(t) + \tilde{\lambda}_2\alpha_{2,2n-1}(t) - \frac{2\beta C_0^2}{R_0}\alpha_{2,2n-1}(t) + \tilde{\lambda}_3\beta_{2,2n-1}(t) + \tilde{K}_1\sin(2\Omega t) - \tilde{K}_2\cos(2\Omega t) = 0,
$$
\hspace{1cm} \text{(D5)}

where we have introduced the coefficients $\tilde{\lambda}_1$, $\tilde{\lambda}_2$, $\tilde{\lambda}_3$, $\tilde{K}_1$, and $\tilde{K}_2$ for better readability, which are given as

$$
\tilde{\lambda}_i = \frac{4C_0^2 + 8\nu^2 + (2n - 1)^2\pi^2\nu^2}{R_0^2} + \nu^2 + \lambda_i^2,
$$

$$
\tilde{\lambda}_2 = \frac{4\lambda C_0^2 + 8\lambda \nu^2 + (2n - 1)^2\pi^2\lambda
\nu^2}{R_0^2},
$$

$$
\tilde{\lambda}_3 = \frac{16\nu^2 C_0^2 + 16\nu^4 + 4(2n - 1)^2\pi^2\nu^2 C_0^2}{R_0^4},
$$

$$
\tilde{K}_i = \left(f_0\Omega - \frac{2f_0\Omega}{(2n - 1)\pi} + 2\Omega^2 - \frac{2\nu^2}{R_0^2}\right)\frac{16K\Omega}{R_0}\left(-1\right)^{n-1}.\ 
$$

We have now reduced the partial differential Equation (59) into an infinite set of ordinary differential equations to be solved readily. It is convenient to introduce auxiliary variables,

$$
a_1 = \alpha_{2,2n-1} + i\beta_{2,2n-1}, \quad a_2 = \alpha_{2,2n-1} - i\beta_{2,2n-1},
$$
\hspace{1cm} \text{(D6)}

allowing us to decouple Equations (D4) and (D5) into the following form:

$$
\ddot{a}_1 + 2\lambda\dot{a}_1 + \tilde{\lambda}_1a_1 + \left(\frac{\tilde{\lambda}_2 - \frac{2i\beta C_0^2}{R_0}}{R_0}\right)a_1 + \tilde{\lambda}_3a_2 = (\tilde{K}_1 + i\tilde{K}_2)\exp(-2it\Omega),
$$
\hspace{1cm} \text{(D7)}

$$
\ddot{a}_2 + 2\lambda\dot{a}_2 + \tilde{\lambda}_1a_2 + \left(\frac{\tilde{\lambda}_2 + \frac{2i\beta C_0^2}{R_0}}{R_0}\right)a_2 + \tilde{\lambda}_3a_2 = (\tilde{K}_1 - i\tilde{K}_2)\exp(2it\Omega).
$$
\hspace{1cm} \text{(D8)}

The particular stationary solutions of these equations can be readily obtained:

$$
a_1 = \left(\tilde{K}_1 + i\tilde{K}_2\right)\frac{\tilde{\lambda}_1 + \tilde{\lambda}_3 - 4\nu^2 \tilde{\lambda}_1 + 4\nu^2 (8\Omega^2 - \tilde{\lambda}_2)}{16\Omega^4 - 4\Omega^2 \tilde{\lambda}_1 + 4\nu^2 (8\lambda^2 - \tilde{\lambda}_2)} \exp(-2it\Omega),
$$
\hspace{1cm} \text{(D9)}

$$
a_2 = \left(\tilde{K}_1 - i\tilde{K}_2\right)\frac{\tilde{\lambda}_1 + \tilde{\lambda}_3 - 4\nu^2 \tilde{\lambda}_1 + 4\nu^2 (8\Omega^2 - \tilde{\lambda}_2)}{16\Omega^4 - 4\Omega^2 \tilde{\lambda}_1 + 4\nu^2 (8\lambda^2 - \tilde{\lambda}_2)} \exp(2it\Omega).
$$
\hspace{1cm} \text{(D10)}

As the last step, it remains to express the original coefficients $\alpha_{2,2n-1}$ and $\beta_{2,2n-1}$ as functions of $a_1$ and $a_2$ as

$$
\alpha_{2,2n-1} = \frac{a_1 + a_2}{2}, \quad \beta_{2,2n-1} = \frac{a_1 - a_2}{2it}.
$$
\hspace{1cm} \text{(D11)}

Inserting these modal coefficients back into the ansatz (Equation (D1)) finally yields the solution (Equation (63)) we present in Section 3.3.2.
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