LARGE DEVIATIONS FOR MULTI-VALUED STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. We prove a large deviation principle of Freidlin-Wentzell’s type for the multi-valued stochastic differential equations with monotone drifts, which in particular contains a class of SDEs with reflection in a convex domain.

1. Introduction

Consider the following multivalued stochastic differential equation (MSDE in short):

\[
\begin{cases}
    dX(t) \in b(X(t))dt + \sigma(X(t))dW(t) - A(X(t))dt, \\
    X(0) = x \in D(A),
\end{cases}
\]

where \(A\) is a multivalued maximal monotone operator, which will be described below, \(W(t) = \{W^k(t), t \geq 0, k \in \mathbb{N}\}\) is a sequence of independent standard Brownian motions on a filtered probability space \((\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t \geq 0})\), \(b : \mathbb{R}^m \to \mathbb{R}^m\) and \(\sigma : \mathbb{R}^m \to \mathbb{R}^m \times l^2\) are two continuous functions, \(l^2\) stands for the Hilbert space of square summable sequences of real numbers.

This type of MSDE was first studied by Cépa in [5, 6]. He proved that if \(b\) and \(\sigma\) are Lipschitz continuous, then there exists a unique pair of processes \((X(t), K(t))\) such that

\[X(t) = x + \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))dW(s) - K(t),\]

where \(K(t)\) is a process of finite variation (see Definition 2.3 below for more details). Recently, Zhang [14] extended Cépa’s result to the infinite dimensional case, and relaxed the Lipschitz assumption on \(b\) to the monotone case. It should be noted that when \(A\) is the subdifferential of the indicator function of a convex subset of \(\mathbb{R}^m\), the above MSDE is the same as the usual SDE with reflecting boundary in a convex domain (cf. [1, 9]). Moreover, since the subdifferential of any lower semicontinuous convex function is a maximal monotone operator, Cépa’s result can also be used to deal with the SDE with discontinuous coefficients. It is well known that there are many literatures to investigate the SDEs with reflecting boundary since the solutions of a large class of PDEs with

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Neumann boundary and mixed boundary conditions can be represented by the solution of such SDEs (cf. [1]).

We now consider the following small perturbation of Eq. (1):

\[
\begin{align*}
\frac{dX^\epsilon(t)}{dt} &\in b(X^\epsilon(t))dt + \sqrt{\epsilon}\sigma(X^\epsilon(t))dW(t) - A(X^\epsilon(t))dt, \\
X^\epsilon(0) &\in D(A), \quad \epsilon \in (0,1].
\end{align*}
\]

The solution of this equation is denoted by \((X^\epsilon(t, x), K^\epsilon(t, x))\). We want to establish the large deviation principle of the law of \(X^\epsilon(t, x)\) in the space \(\mathbb{S} := C([0, T]; D(A))\), namely, the asymptotic estimates of probabilities \(P(X^\epsilon \in \Gamma)\), where \(\Gamma \in \mathcal{B}(\mathbb{S})\).

In [1], Anderson and Orey considered the same small random perturbation for the dynamical system with reflecting boundary in smooth domain, and obtained the Freidlin-Wentzell’s large deviation estimates in \(C([0, T]; D(A))\). They assumed that the coefficients are bounded and Lipschitz continuous, and the diffusion coefficient is non-degenerate. Using the contraction principle, Cépa [5] only considered the large deviation principle of one dimensional case based on an explicit construction of the solution (cf. [13]). The multi-dimensional case is still open. Compared with the usual SDE, i.e., \(A = 0\), most of the difficulties come from the presence of the process of finite variation, \(K(t)\). One only knows that \(t \mapsto K(t)\) is continuous, and could not prove any further regularity such as Hölder continuity. Therefore, the classical method of time discretization is almost inapplicable (cf. [7]).

Our method is based on the recently well developed weak convergence approach due to Dupuis and Ellis [8] (see also [2, 3]). This method has been proved to be very effective for various systems (cf. [11, 15, 4, 16, 10, 12 etc.]). In the situation considered in the present paper, however, since we cannot prove the following uniform estimate as in [11]:

\[
\sup_{\epsilon \in (0,1)} \mathbb{E}|X^\epsilon(t, x) - X^\epsilon(s, y)|^{2p} \leq C(|t - s|^p + |x - y|^{2p}),
\]

we cannot obtain the tightness of the laws of \(X^\epsilon(t, x)\) in \(\mathbb{S}\). Some technical difficulties for verifying the conditions \((\text{LD})_1\) and \((\text{LD})_2\) below need to be overcome.

In Section 2, we recall some well known facts about the MSDE and a criterion for Laplace principle. In Section 3, we present our main result and give a detailed proof. Throughout the paper, \(C\) with or without indexes will denote different constants (depending on the indexes) whose values are not important.

## 2. Preliminaries

We first give some notions and notations about multivalued operators. Let \(2^{\mathbb{R}^m}\) be the set of all subsets of \(\mathbb{R}^m\). A map \(A : \mathbb{R}^m \to 2^{\mathbb{R}^m}\) is called a multivalued operator. Given such a multivalued operator \(A\), define:

\[
\begin{align*}
D(A) &:= \{x \in \mathbb{R}^m : A(x) \neq \emptyset\}, \\
\text{Im}(A) &:= \cup_{x \in D(A)} A(x), \\
\text{Gr}(A) &:= \{(x, y) \in \mathbb{R}^{2m} : x \in \mathbb{R}^m, y \in A(x)\}.
\end{align*}
\]

We recall the following definitions.

**Definition 2.1.** (1) A multivalued operator \(A\) is called monotone if

\[
\langle y_1 - y_2, x_1 - x_2 \rangle_{\mathbb{R}^m} \geq 0, \quad \forall (x_1, y_1), (x_2, y_2) \in \text{Gr}(A).
\]
(2) A monotone operator \( A \) is called maximal monotone if for each \( (x, y) \in \text{Gr}(A) \),
\[
\langle y - y', x - x' \rangle_{\mathbb{R}^m} \geq 0, \quad \forall (x', y') \in \text{Gr}(A).
\]

**Examples 2.2.** Suppose that \( \mathcal{O} \) is a closed convex subset of \( \mathbb{R}^m \), and \( I_\mathcal{O} \) is the indicator function of \( \mathcal{O} \), i.e.
\[
I_\mathcal{O}(x) = \begin{cases} 
0, & \text{if } x \in \mathcal{O}, \\
+\infty, & \text{if } x \notin \mathcal{O}.
\end{cases}
\]

The subdifferential of \( I_\mathcal{O} \) is given by
\[
\partial I_\mathcal{O}(x) = \{ y \in \mathbb{R}^m : \langle y, x-z \rangle_{\mathbb{R}^m} \geq 0, \forall z \in \mathcal{O} \}
\]
\[
= \begin{cases} 
\emptyset, & \text{if } x \notin \mathcal{O}, \\
\{0\}, & \text{if } x \in \text{Int}(\mathcal{O}), \\
\Lambda_x, & \text{if } x \in \partial \mathcal{O},
\end{cases}
\]
where \( \text{Int}(\mathcal{O}) \) is the interior of \( \mathcal{O} \) and \( \Lambda_x \) is the exterior normal cone at \( x \). One can check that \( \partial I_\mathcal{O} \) is a multivalued maximal monotone operator in the sense of Definition 2.1.

We now recall an abstract criterion for Laplace principle, which is equivalent to the large deviation principle (cf. [3, 4, 15]). It is well known that there exists a Hilbert space so that \( l^2 \subset \mathcal{U} \) is Hilbert-Schmidt with embedding operator \( J \) and \( \{W^k(t), k \in \mathbb{N}\} \) is a Brownian motion with values in \( \mathcal{U} \), whose covariance operator is given by \( Q = J \circ J^* \). For example, one can take \( \mathcal{U} \) as the completion of \( l^2 \) with respect to the norm generated by the scalar product
\[
\langle h, h' \rangle_{\mathcal{U}} := \left( \sum_{k=1}^{\infty} \frac{h_k h'_k}{k^2} \right)^{\frac{1}{2}}, \quad h, h' \in l^2.
\]

For a Polish space \( \mathbb{B} \), we denote by \( \mathcal{B}(\mathbb{B}) \) its Borel \( \sigma \)-field, and by \( \mathcal{C}_T(\mathbb{B}) \) the continuous function space from \([0, T]\) to \( \mathbb{B} \), which is endowed with the uniform distance so that \( \mathcal{C}_T(\mathbb{B}) \) is still a Polish space. Define
\[
\ell^2_T := \left\{ h = \int_0^T \dot{h}(s) ds : \dot{h} \in L^2(0, T; l^2) \right\}
\]
with the norm
\[
\|h\|_{\ell^2_T} := \left( \int_0^T \|\dot{h}(s)\|_{l^2}^2 ds \right)^{1/2},
\]
where the dot denotes the generalized derivative. Let \( \mu \) be the law of the Brownian motion \( W \) in \( \mathcal{C}_T(\mathcal{U}) \). Then
\[
\left( \ell^2_T, \mathcal{C}_T(\mathcal{U}), \mu \right)
\]
forms an abstract Wiener space.

For $T, N > 0$, set

$$
\mathcal{D}_N := \{ h \in \ell^2_T : \|h\|_{\ell^2_T} \leq N \}
$$

and

$$
\mathcal{A}^T_N := \left\{ h : [0, T] \to \ell^2 \text{ is a continuous and } (\mathcal{F}_t)\text{-adapted process, and for almost all } \omega, \ h(\cdot, \omega) \in \mathcal{D}_N \right\}. \tag{4}
$$

We equip $\mathcal{D}_N$ with the weak convergence topology in $\ell^2_T$. Then

$$
\mathcal{D}_N \text{ is metrizable as a compact Polish space.} \tag{5}
$$

**Definition 2.4.** The function $I : \mathbb{S} \to [0, \infty]$ is given.

Let $\mathbb{S}$ be a Polish space. A function $I : \mathbb{S} \to [0, \infty]$ is called a rate function if for every $a < \infty$, the set 

$$
\{ f \in \mathbb{S} : I(f) \leq a \}
$$

is compact in $\mathbb{S}$.

Let $\{ Z^\epsilon : \mathbb{C}_T(U) \to \mathbb{S}, \epsilon \in (0, 1) \}$ be a family of measurable mappings. Assume that there is a measurable map $Z_0 : \ell^2_T \mapsto \mathbb{S}$ such that

(\text{LD})_1 \text{ For any } N > 0, \text{ if a family } \{ h_\epsilon, \epsilon \in (0, 1) \} \subset \mathcal{A}^T_N \text{ (as random variables in } \mathcal{D}_N) \text{ converges in distribution to } h \in \mathcal{A}^T_N, \text{ then for some subsequence } \epsilon_k, \ Z^{\epsilon_k} \left( \cdot + \frac{h_{\epsilon_k}(\cdot)}{\sqrt{\epsilon}} \right) \text{ converges in distribution to } Z_0(h) \text{ in } \mathbb{S}.

(\text{LD})_2 \text{ For any } N > 0, \text{ if a family } \{ h_n, n \in \mathbb{N} \} \subset \mathcal{D}_N \text{ weakly converges to } h \in \ell^2_T, \text{ then for some subsequence } h_{n_k}, Z_0(h_{n_k}) \text{ converges to } Z_0(h) \text{ in } \mathbb{S}.

For each $f \in \mathbb{S}$, define

$$
I(f) := \frac{1}{2} \inf_{\{ h \in \ell^2_T : f = Z_0(h) \}} \|h\|_{\ell^2_T}^2, \tag{6}
$$

where $\inf \emptyset = \infty$ by convention. Then under (\text{LD})_2, $I(f)$ is a rate function.

We recall the following result due to [3] (see also [17, Theorem 4.4]).

**Theorem 2.5.** Under (\text{LD})_1 and (\text{LD})_2, \( \{ Z^\epsilon, \epsilon \in (0, 1) \} \) satisfies the Laplace principle with the rate function $I(f)$ given by (6). More precisely, for each real bounded continuous function $g$ on $\mathbb{S}$:

$$
\lim_{\epsilon \to 0} \epsilon \log \mathbb{E}_\mu \left( \exp \left[ -\frac{g(Z^\epsilon)}{\epsilon} \right] \right) = -\inf_{f \in \mathbb{S}} \{ g(f) + I(f) \}. \tag{7}
$$

In particular, the family of \( \{ Z^\epsilon, \epsilon \in (0, 1) \} \) satisfies the large deviation principle in $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$ with the rate function $I(f)$. More precisely, let $\nu_\epsilon$ be the law of $Z^\epsilon$ in $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$, then for any $B \in \mathcal{B}(\mathbb{S})$

$$
-\inf_{f \in B} I(f) \leq \lim \inf_{\epsilon \to 0} \epsilon \log \nu_\epsilon(B) \leq \lim \sup_{\epsilon \to 0} \epsilon \log \nu_\epsilon(B) \leq -\inf_{f \in B} I(f),
$$

where the closure and the interior are taken in $\mathbb{S}$, and $I(f)$ is defined by (6).

### 3. Main Result and Proof

We assume that

(\text{H1}) \text{ A is a maximal monotone operator with non-empty interior, i.e., } \text{Int}(D(A)) \neq \emptyset;
(H2) $\sigma$ and $b$ are continuous functions and satisfy that for some $C_\sigma, C_b > 0$ and all $x, y \in \mathbb{R}^m$
\[\|\sigma(x) - \sigma(y)\|_{L_2(\mathbb{R}^m)} \leq C_\sigma |x - y|,\]
\[\langle x - y, b(x) - b(y) \rangle_{\mathbb{R}^m} \leq C_b |x - y|^2,\]
where $L_2(\mathbb{R}^m)$ denotes the Hilbert-Schmidt space and $| \cdot |$ denotes the norm in $\mathbb{R}^m$, and for some $C'_b > 0$ and $n \in \mathbb{N}$
\[|b(x)| \leq C'_b (1 + |x|^n).\]

It is well known that under (H1) and (H2), there exists a unique solution $(X^\epsilon, K^\epsilon)$ to Eq.(2) in the sense of Definition 2.3 (cf. [14]). Our main result is stated as follows:

**Theorem 3.1.** Assume that (H1) and (H2) hold. Then the family of $\{X^\epsilon(t, x), \epsilon \in (0, 1)\}$ satisfies the large deviation principle in $\mathcal{S} := C([0, T] \times \overline{D(A); D(A)})$ with the rate function given by

\[I(f) := \frac{1}{2} \inf_{\{h \in \ell_2^T: f = X^h\}} \|h\|^2_{\ell_2^T},\]

where $X^h(t, x)$ solves the following equation:

\[dX^h(t) \in b(X^h(t))dt + \sigma(X^h(t))\dot{h}(t)dt - A(X^h(t))dt, \quad X^h(0) = x.\]

For proving this result, by Theorem 2.5, the main task is to verify (LD)₁ and (LD)₂ with

$\mathcal{S} := C([0, T] \times \overline{D(A); D(A)})$, $Z^\epsilon = X^\epsilon$, $Z_0(h) = X^h$.

This will be done in Lemmas 3.7 and 3.8 below.

Let $h_\epsilon \in \mathcal{A}_N^T$ converge almost surely to $h \in \mathcal{A}_N^T$ as random variables in $\ell_2^T$, and $(X^\epsilon, K^\epsilon)$ solve the following control equation:

\[X^\epsilon, h_\epsilon(t) = x + \int_0^t b(X^\epsilon, h_\epsilon(s))ds + \int_0^t \sigma(X^\epsilon, h_\epsilon(s))\dot{h}_\epsilon(s)ds\]
\[+ \sqrt{\epsilon} \int_0^t \sigma(X^\epsilon, h_\epsilon(s))dW(s) - K^\epsilon, h_\epsilon(t),\]

which can be solved by Girsanov’s theorem, and $(X^h, K^h)$ solve the following deterministic equation:

\[X^h(t) = x + \int_0^t b(X^h(s))ds + \int_0^t \sigma(X^h(s))\dot{h}(s)ds - K^h(t).\]

Let $|K|^s_t$ denote the total variation of $K$ on $[s, t]$. We recall the following result due to Cépa [6] (see also [14], Propositions 3.3 and 3.4).

**Proposition 3.2.** Under (H1), there exist $a \in \mathbb{R}^m, \gamma > 0, \mu \geq 0$ such that for any pair of $(X, K)$ with the property (iv) of Definition 2.3 and all $0 \leq s < t \leq T$
\[\int_s^t \langle X(r) - a, dK(r) \rangle_{\mathbb{R}^m} \geq |K|^s_t - \mu \int_s^t |X(r) - a|dr - \gamma \mu (t - s).\]

Moreover, for any pairs of $(X, K)$ and $(\tilde{X}, \tilde{K})$ with the property (iv) of Definition 2.3
\[\langle X(t) - \tilde{X}(t), dK(t) - d\tilde{K}(t) \rangle_{\mathbb{R}^m} \geq 0.\]

Using this property, we first prove the following uniform estimates.
Lemma 3.3. For any \( p \geq 1 \), there exists \( C_{p,T,N} > 0 \) such that for any \( \epsilon \in (0, 1) \) and \( x, y \in D(A) \)

\[
\mathbb{E} \left( \sup_{t \in [0,T]} |X^{\epsilon,h_\epsilon}(t,x) - X^{\epsilon,h_\epsilon}(t,y)|^{2p} \right) \leq C_{p,T,N}|x-y|^{2p}. \tag{13}
\]

Proof. Set

\[
Z_\epsilon(t) := X^{\epsilon,h_\epsilon}(t,x) - X^{\epsilon,h_\epsilon}(t,y)
\]

and

\[
\Lambda(s) := \sigma(X^{\epsilon,h_\epsilon}(s,y)) - \sigma(X^{\epsilon,h_\epsilon}(s,x)).
\]

By Itô’s formula, (H2) and (12), we have for any \( p \geq 1 \)

\[
|Z_\epsilon(t)|^{2p} = |Z_\epsilon(0)|^{2p} + 2p \int_0^t |Z_\epsilon(s)|^{2p-2} \langle Z_\epsilon(s), b(X^{\epsilon,h_\epsilon}(s,x)) - b(X^{\epsilon,h_\epsilon}(s,y)) \rangle \mathbb{R}_m ds + \left( 2p \int_0^t \|\hat{h}_\epsilon(s)\|_2 ds \right) + \left( \int_0^t |Z_\epsilon(s)|^{2p-2} \langle Z_\epsilon(s), \Lambda(s) \hat{h}_\epsilon(s) \rangle \mathbb{R}_m ds \right).
\]

Set

\[
g(t) := \mathbb{E} \left( \sup_{s \in [0,t]} |Z_\epsilon(s)|^{2p} \right).
\]

By BDG’s inequality and Young’s inequality, we have for any \( \delta > 0 \)

\[
\mathbb{E} \left( \sup_{t' \in [0,t]} \int_0^{t'} |Z_\epsilon(s)|^{2p-2} \langle Z_\epsilon(s), \Lambda(s) dW(s) \rangle \mathbb{R}_m \right) \leq C \mathbb{E} \left( \int_0^t |Z_\epsilon(s)|^{4p-4} \|\Lambda(s) Z_\epsilon(s)\|_2^2 ds \right)^{1/2} \leq C \left( \sup_{s \in [0,t]} |Z_\epsilon(s)|^{2p} \int_0^t |Z_\epsilon(s)|^{2p} ds \right)^{1/2} \leq \delta \cdot g(t) + C_\delta \int_0^t \mathbb{E} |Z_\epsilon(s)|^{2p} ds.
\]

Similarly, we have

\[
\mathbb{E} \left( \int_0^t |Z_\epsilon(s)|^{2p} \cdot \|\hat{h}_\epsilon(s)\|_2^2 ds \right) \leq N \mathbb{E} \left( \int_0^t |Z_\epsilon(s)|^{4p} ds \right)^{1/2}
\]
\begin{align}
\leq \delta \cdot g(t) + C_{\delta,N} \int_0^t \mathbb{E}|Z_\epsilon(s)|^{2p} ds. \tag{15}
\end{align}

Letting \( \delta = 1/4 \) in (14) and (15) and combining the above calculations, we get
\[ g(t) \leq g(0) + \frac{1}{2} g(t) + C \int_0^t \mathbb{E}|Z_\epsilon(s)|^{2p} ds \leq 2g(0) + 2C \int_0^t g(s) ds, \]
which gives the desired estimate by Gronwall’s inequality. \( \square \)

**Lemma 3.4.** For any \( p \geq 1 \) and \( x \in \overline{D(A)} \), there exists \( C_{p,T,N,x} > 0 \) such that for any \( \epsilon \in [0,1) \)
\[ \mathbb{E} \left( \sup_{t \in [0,T]} |X^{\epsilon,h_\epsilon}(t,x)|^{2p} \right) + \mathbb{E}|K^{\epsilon,h_\epsilon}(\cdot,x)|_T^0 \leq C_{p,T,N,x}. \tag{16} \]

**Proof.** First of all, as in the proof of Lemma 3.3 we can prove that
\[ \mathbb{E} \left( \sup_{t \in [0,T]} |X^{\epsilon,h_\epsilon}(t,x)|^{2p} \right) \leq C_{p,T,N,x}. \tag{17} \]

Let \( a \in \text{Int}(D(A)) \) be as in Proposition 3.2. By Itô’s formula, (11) and (H2), we have
\[ \frac{1}{2} |X^{\epsilon,h_\epsilon}(t) - a|^2 = \frac{1}{2} |x - a|^2 + \int_0^t \langle X^{\epsilon,h_\epsilon}(s) - a, b(X^{\epsilon,h_\epsilon}(s)) \rangle_{\mathbb{R}^m} ds \]
\[ + \int_0^t \langle X^{\epsilon,h_\epsilon}(s) - a, \sigma(X^{\epsilon,h_\epsilon}(s)) \dot{h}_\epsilon(u) \rangle_{\mathbb{R}^m} ds \]
\[ + \sqrt{\epsilon} \int_0^t \langle X^{\epsilon,h_\epsilon}(s) - a, \sigma(X^{\epsilon,h_\epsilon}(s)) dW(s) \rangle_{\mathbb{R}^m} \]
\[ - \int_0^t \langle X^{\epsilon,h_\epsilon}(s) - a, dK^{\epsilon,h_\epsilon}(s) \rangle_{\mathbb{R}^m} \]
\[ + \frac{\epsilon}{2} \int_0^t \|\sigma(X^{\epsilon,h_\epsilon}(s))\|^2_{L_2(\mathbb{R}^m)} ds \]
\[ \leq \frac{1}{2} |x - a|^2 + C_b \int_0^t |X^{\epsilon,h_\epsilon}(s) - a|^2 ds \]
\[ + \int_0^t \langle X^{\epsilon,h_\epsilon}(s) - a, b(a) \rangle_{\mathbb{R}^m} ds \]
\[ + N \left( \int_0^t |\sigma(X^{\epsilon,h_\epsilon}(s))^*(X^{\epsilon,h_\epsilon}(s) - a)|^2 ds \right)^{1/2} \]
\[ + \sqrt{\epsilon} \int_0^t \langle X^{\epsilon,h_\epsilon}(s) - a, \sigma(X^{\epsilon,h_\epsilon}(s)) dW(u) \rangle_{\mathbb{R}^m} + \mu \gamma t \]
\[ - \gamma |K^{\epsilon,h_\epsilon}|^0_t + \mu \int_0^t |X^{\epsilon,h_\epsilon}(s) - a| ds \]
\[ + \frac{C_{\sigma}^2 \epsilon}{2} \int_0^t (|X^{\epsilon,h_\epsilon}(s)| + \sigma(0))^2 ds. \]

The desired estimate now follows by (17). \( \square \)
Define
\[ w_\varepsilon(t, x) := \int_0^t \sigma(X^h(s, x))(\dot{h}_\varepsilon(s) - \dot{h}(s)) ds. \] (18)

The following lemma is easy by Ascoli-Arzelà’s lemma.

**Lemma 3.5.** $w_\varepsilon(\cdot, x)$ converges a.s. to zero in $C([0, T], D(A))$.

We now prove the following key lemma.

**Lemma 3.6.** $X^{\varepsilon,h_\varepsilon}$ defined by (12) converges in probability to $X^h$ defined by (11) in $\mathbb{S}$.

**Proof.** Set $v_\varepsilon(t) := v_\varepsilon(t, x) := X^{\varepsilon,h_\varepsilon}(t, x) - X^h(t, x)$. Then
\[
 v_\varepsilon(t) = K^{\varepsilon,h_\varepsilon}(t) - K^h(t) + \int_0^t (b(X^{\varepsilon,h_\varepsilon}(s)) - b(X^h(s))) ds \\
+ \int_0^t (\sigma(X^{\varepsilon,h_\varepsilon}(s))\dot{h}_\varepsilon(s) - \sigma(X^h(s))\dot{h}(s)) ds \\
+ \sqrt{\varepsilon} \int_0^t \sigma(X^{\varepsilon,h_\varepsilon}(s)) dW(s).
\]

By Itô’s formula, we have
\[
|v_\varepsilon(t)|^2 = 2 \int_0^t (v_\varepsilon(s), dK^{\varepsilon,h_\varepsilon}(s) - dK^h(s))_{\mathbb{R}^m} \\
+ 2 \int_0^t (v_\varepsilon(s), b(X^{\varepsilon,h_\varepsilon}(s)) - b(X^h(s)))_{\mathbb{R}^m} ds \\
+ 2 \int_0^t (v_\varepsilon(s), (\sigma(X^{\varepsilon,h_\varepsilon}(s)) - \sigma(X^h(s)))\dot{h}_\varepsilon(s))_{\mathbb{R}^m} ds \\
+ 2 \int_0^t (v_\varepsilon(s), \sigma(X^h(s))(\dot{h}_\varepsilon(s) - \dot{h}(s)))_{\mathbb{R}^m} ds \\
+ 2 \sqrt{\varepsilon} \int_0^t (v_\varepsilon(s), \sigma(X^{\varepsilon,h_\varepsilon}(s))dW(s))_{\mathbb{R}^m} \\
+ \varepsilon \int_0^t \|\sigma(X^{\varepsilon,h_\varepsilon}(s))\|^2_{L_2(\mathbb{R};\mathbb{R}^m)} ds \\
=: I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t) + I_6(t).
\]

It is clear that by (12)
\[ I_1(t) \leq 0 \]
and
\[ I_2(t) \leq 2C_b \int_0^t |v_\varepsilon(s)|^2 ds. \]

By BDG’s inequality and (H2) we also have
\[
\mathbb{E} \left( \sup_{t \in [0,T]} |I_3(t)| \right) + \mathbb{E} \left( \sup_{t \in [0,T]} |I_4(t)| \right) \leq C \sqrt{\varepsilon}.
\]

As estimating (15) we have
\[
\mathbb{E} \left( \sup_{s \in [0,t]} |I_5(s)| \right) \leq \frac{1}{2} \mathbb{E} \left( \sup_{s \in [0,t]} |v_\varepsilon(s)|^2 \right) + C \int_0^t \mathbb{E}|v_\varepsilon(s)|^2 ds.
\]
Set
\[ g(t) := \mathbb{E} \left( \sup_{s \in [0,t]} |v_\epsilon(s)|^2 \right). \]
Then we have
\[ g(t) \leq \frac{1}{2} g(t) + C \sqrt{\epsilon} + \mathbb{E} \left( \sup_{s \in [0,t]} |I_4^\epsilon(s)| \right) + C \int_0^t \mathbb{E}|v_\epsilon(s)|^2 ds, \]
which implies that
\[ g(t) \leq C \sqrt{\epsilon} + 2\mathbb{E} \left( \sup_{s \in [0,T]} |I_4^\epsilon(s)| \right) + C \int_0^t g(s) ds. \]
By Gronwall’s inequality we get
\[ \mathbb{E} \left( \sup_{s \in [0,T]} |v_\epsilon(s)|^2 \right) \leq C \sqrt{\epsilon} + C\mathbb{E} \left( \sup_{s \in [0,T]} |I_4^\epsilon(s)| \right). \] (19)

We now deal with the hard term \( I_4^\epsilon \). By Itô’s formula again, we have
\[ \frac{1}{2} I_4^\epsilon(t) = \langle v_\epsilon(t), w_\epsilon(t) \rangle_{\mathbb{R}^m} - \int_0^t w_\epsilon(s) d(K^{\epsilon,h_\epsilon}(s) - K^h(s)) \]
\[ - \int_0^t w_\epsilon(s) (b(X^{\epsilon,h_\epsilon}(s)) - b(X^h(s))) ds \]
\[ - \int_0^t w_\epsilon(s) (\sigma(X^{\epsilon,h_\epsilon}(s)) \hat{v}_\epsilon(s) - \sigma(X^h(s)) \hat{v}(s)) ds \]
\[ - \sqrt{\epsilon} \int_0^t w_\epsilon(s) \sigma(X^{\epsilon,h_\epsilon}(s)) dW(s) \]
\[ =: I_{41}^\epsilon(t) + I_{42}^\epsilon(t) + I_{43}^\epsilon(t) + I_{44}^\epsilon(t) + I_{45}^\epsilon(t). \]

For any \( \delta > 0 \) and \( R > 0 \), we have
\[ P \left( \sup_{t \in [0,T]} |I_{41}^\epsilon(t)| \geq \delta \right) = P \left( \sup_{t \in [0,T]} |I_{41}^\epsilon(t)| \geq \delta; \sup_{t \in [0,T]} |v_\epsilon(t)| < R \right) \]
\[ + P \left( \sup_{t \in [0,T]} |I_{41}^\epsilon(t)| \geq \delta; \sup_{t \in [0,T]} |v_\epsilon(t)| \geq R \right) \]
\[ \leq P \left( \sup_{t \in [0,T]} |w_\epsilon(t)| \geq \delta/R \right) + P \left( \sup_{t \in [0,T]} |v_\epsilon(t)| \geq R \right). \]

By Lemma \textbf{3.5} and (16) we know
\[ \lim_{\epsilon \to 0} \mathbb{P} \left( \sup_{t \in [0,T]} |I_{41}^\epsilon(t)| \geq \delta \right) = 0. \]

Noting that
\[ \sup_{t \in [0,T]} |I_{42}^\epsilon(t)| \leq \sup_{s \in [0,T]} |w_\epsilon(s)| \cdot (|K^{\epsilon,h_\epsilon}(\cdot)|_T^0 + |K^h(\cdot)|_T^0), \]
as above, we also have
\[ \sup_{t \in [0,T]} |I_{42}^\epsilon(t)| \to 0 \text{ in probability.} \]
Similarly, we have

\[ \sup_{t \in [0,T]} |I_{13}(t)| + \sup_{t \in [0,T]} |I_{14}(t)| \to 0 \text{ in probability.} \]

Moreover, by BDG's inequality we have

\[ \sqrt{\epsilon E} \left( \sup_{t \in [0,T]} |I_{45}(t)| \right) \leq C \sqrt{\epsilon}. \]

Combining the above calculations, we get

\[ \sup_{t \in [0,T]} |I_4(t)| \to 0 \text{ in probability.} \]

It is easy to see by (16) that

\[ \sup_{\epsilon \in (0,1)} E \left( \sup_{t \in [0,T]} |I_{4}(t)|^2 \right) < +\infty. \]

Hence

\[ \lim_{\epsilon \to 0} E \left( \sup_{t \in [0,T]} |I_{4}(t)| \right) = 0. \]

Substituting this into (19) we obtain

\[ \lim_{\epsilon \to 0} E \left( \sup_{s \in [0,T]} |v_\epsilon(s)|^2 \right) = 0. \]

Thus, we have proven that for all \( x \in \overline{D(A)} \)

\[ \sup_{t \in [0,T]} |v_\epsilon(t, x)|^2 \to 0, \text{ in probability.} \]

We now strengthen it by Lemma 3.3 to

\[ \xi_{n,\epsilon} := \sup_{t \in [0,T], x \in D(A), |x| \leq n} |v_\epsilon(t, x)|^2 \to 0, \text{ in probability.} \]

Set

\[ \mathbb{D}_n^\delta := \overline{D(A)} \cap \{ x \in \mathbb{R}^m : |x| \leq n \} \cap \delta \mathbb{Z}^m, \]

where \( \delta > 0 \) and \( \delta \mathbb{Z}^m \) denotes the grid in \( \mathbb{R}^m \) with edge length \( \delta \). It is clear that there are only finite many points in \( \mathbb{D}_n^\delta \). Hence

\[ \xi_{n,\epsilon} := \sup_{t \in [0,T], x \in \mathbb{D}_n^\delta} |v_\epsilon(t, x)|^2 \to 0, \text{ in probability.} \]

For any \( x \in \mathbb{R}^m \), let \( x_\delta \) denote the left-lower corner point in \( \delta \mathbb{Z}^m \) so that

\[ |x - x_\delta| \leq \delta. \]

Noting that

\[ \xi_{n,\epsilon} \leq 2\xi_{n,\epsilon}^\delta + 2 \sup_{x \in \overline{D(A)}, |x| \leq n} \sup_{t \in [0,T]} |v_\epsilon(t, x) - v_\epsilon(t, x_\delta)|^2, \]

we have for any \( \beta > 0 \) and some \( \alpha > 0 \)

\[ P(\xi_{n,\epsilon} > 4\beta) \leq P(\xi_{n,\epsilon}^\delta + \sup_{x \in \overline{D(A)}, |x| \leq n} \sup_{t \in [0,T]} |v_\epsilon(t, x) - v_\epsilon(t, x_\delta)|^2 \geq 2\beta) \]

\[ \leq P(\xi_{n,\epsilon}^\delta > \beta) + P \left( \sup_{x \in \overline{D(A)}, |x| \leq n} \sup_{t \in [0,T]} |v_\epsilon(t, x) - v_\epsilon(t, x_\delta)|^2 > \beta \right) \]
\begin{align*}
\leq & \ P(\xi_{n,\epsilon}^{\delta} > \beta) + \mathbb{E}\left( \sup_{x \in \mathcal{D}(\mathcal{X}), |x| \leq n} \sup_{t \in [0,T]} |v_{\epsilon}(t, x) - v_{\epsilon}(t, x_{\delta})|^2 \right) / \beta \\
\leq & \ P(\xi_{n,\epsilon}^{\delta} > \beta) + C\delta^\alpha / \beta,
\end{align*}

where the last step is due to Lemma 3.3 and Kolmogorov’s criterion. First letting \( \delta \) be small enough, then \( \epsilon \) go to zero, we then obtain

\[
\lim_{\epsilon \to 0} P(\xi_{n,\epsilon} > 4\beta) = 0.
\]

which yields the desired result.

\[\square\]

**Lemma 3.7.** (LD)\(_1\) holds.

**Proof.** Let \( h_\epsilon \) be a sequence in \( \mathcal{A}_N^T \) converge to \( h \) in distribution. Since \( \mathcal{D}_N \) is compact and the law of \( W \) is tight, \( \{h_\epsilon, W\} \) is tight in \( \mathcal{D}_N \times \mathcal{C}_T(\mathbb{U}) \) by the definition of tightness. Without loss of generality, we assume the law of \( \{h_\epsilon, W\} \) weakly converges to \( \mu \). Then the law of \( h \) is just \( \mu(\cdot, \mathcal{C}_T(\mathbb{U})) \). Indeed, for any bounded continuous function \( g \) on \( \mathcal{D}_N \), we have

\[
\mathbb{E}(g(h)) = \lim_{n \to \infty} \mathbb{E}(g(h_\epsilon)) = \int_{\mathcal{D}_N} g(h) \mu(dh, \mathcal{C}_T(\mathbb{U})).
\]

By Skorohod’s representation theorem, there are \( (\hat{\Omega}, \hat{\mathbb{P}}) \) and \( \{\hat{h}_\epsilon, \hat{W}^{\epsilon}\} \) and \( \{\hat{h}, \hat{W}\} \) such that

1. \( (\hat{h}_\epsilon, \hat{W}^{\epsilon}) \) a.s. converges to \( (\hat{h}, \hat{W}) \);
2. \( (\hat{h}_\epsilon, \hat{W}^{\epsilon}) \) has the same law as \( (h_\epsilon, W) \);
3. The law of \( \{\hat{h}, \hat{W}\} \) is \( \mu \), and the law of \( h \) is the same as \( \hat{h} \).

Using Lemma 3.6, we get

\[
\Phi\left( \frac{1}{\sqrt{\epsilon}} \int_0^T \dot{\hat{h}}_\epsilon ds + \hat{W}^{\epsilon} \right) \to X^{\hat{h}}, \quad \text{in probability},
\]

where \( \Phi \) is the strong solution functional(cf. [9]). From this, we derive

\[
\Phi\left( \frac{1}{\sqrt{\epsilon}} \int_0^T \dot{h}_\epsilon ds + W \right) \to X^h, \quad \text{in distribution}.
\]

Thus, (LD)\(_1\) holds.

\[\square\]

Similar to the proof of Lemma 3.6, one can easily verify that

**Lemma 3.8.** (LD)\(_2\) holds.

Thus, by Lemmas 3.7, 3.8 and Theorem 2.5, we have proved Theorem 3.1.

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