EXISTENCE OF PERIODIC SOLUTIONS OF DYNAMIC EQUATIONS ON TIME SCALES BY AVERAGING

RUICHAO GUO\textsuperscript{a}, YONG LI\textsuperscript{a,b,c,*}, JIAMIN XING\textsuperscript{a} AND XUE YANG\textsuperscript{a,b}

\textsuperscript{a}College of Mathematics, Jilin University
Changchun, 130012, China
\textsuperscript{b}School of Mathematics and Statistics and Center for Mathematics and Interdisciplinary Sciences
Northeast Normal University, Changchun 130024, China
\textsuperscript{c}State Key Laboratory of Automotive Simulation and control, Jilin University
Changchun, 130012, China

Abstract. In this paper, we study the existence of periodic solutions for perturbed dynamic equations on time scales. Our approach is based on the averaging method. Further, we extend some averaging theorem to periodic solutions of dynamic equations on time scales to $k$--th order in $\varepsilon$. More precisely, results of higher order averaging for finding periodic solutions are given via the topological degree theory.

1. Introduction and statement of main result. The problem of periodic solution of continuous dynamical system has been one of the center topics since Poincaré and Lyapunov. Recently, some theories and methods are developed to dynamic equations on time scales, see, for instance, \cite{6, 15, 11, 1, 10}. However, the existence of periodic solutions for differential equations with a small parameter $\varepsilon$ when ‘time’ is not continuous have paid more and more attention. The aim of this paper is to study the existence of periodic solutions for the perturbed dynamic equations on time scales of the type

$$x^\Delta(t) = \sum_{i=1}^{k} \varepsilon^i f_i(t, x) + \varepsilon^{k+1} r(t, x, \varepsilon), \quad (1)$$

where $f_i : \mathbb{T} \times U \to \mathbb{R}^n$ for $i = 0, 1, \cdots, k$ and $r : \mathbb{T} \times U \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are rd-continuous, and $T$-periodic in the first argument, $U$ is an open subset of $\mathbb{R}^n$, and $\varepsilon$ is a small parameter.

A time scale is an arbitrary non-empty closed subset of $\mathbb{R}$, generally denoted by $\mathbb{T}$. The theory of time scales was first introduced by S. Hilger, on his doctoral thesis (see \cite{9}) in order to study continuous-discrete hybrid processes. For instance, if $\mathbb{T} = \mathbb{Z}$, dynamic equations are just usual difference equations, while, taking $\mathbb{T} = \mathbb{R}$, they are usual differential equations. This theory respects a powerful tool to economics,
populations models, biology models and so on. It has been attracting more and more attentions during the past years and the existence of solutions for systems on time scales has been extensively investigated, specially concerning periodicity (see, [1]). Periodic dynamic equations on time scales has been attracting attention of several mathematicians and the interest in this topic still increases. For instance, almost periodicity on time scales was introduced by Y. Li and C. Wang in [10]. The theory of almost automorphic solutions of dynamic equations on time scales was introduced by C. Lizama and J. G. Mesquita (see, [11]). After that, C. Wang and Y. Li (see, [15]) considered nonlinear dynamic equations and proved the existence of affine-periodic solutions via the topological degree theory.

A useful tool to study periodic solutions is the averaging theory. The method of averaging for ordinary differential equations has a long history that started with the classical works of Lagrange and Laplace on celestial mechanics. The first formalization of this procedure was given by Fatou in 1928 [8]. Bogoliubov [2] and Krylov [3] made important contributions to the averaging theorem. In 2004, Buica and Llibre [7] extended the averaging theory up to order 3 for studying periodic orbits of continuous differential systems by Brouwer’s degree. Recently, higher order averaging theorem for finding periodic orbits of continuous differential systems has been introduced by Llibre et al. [12] and the theory of regularization was introduced by Llibre, Novaes and Teixeira in [13].

Motivated by these facts, the main goal of this paper is to study the existence of periodic solutions of the perturbed dynamic equations on time scales by the averaging method. The proof is inspired by the classical one, but certain technical details on time scales are more complicated. As well known, by using the implicit function theorem, the averaging method leads to the existence of periodic solutions for periodic systems (see, [14]). But in this paper, we use the topological method to prove our main results, which do not need smoothness.

In order to prove our main results, we introduce some notations. Let \( x = (x_1, \cdots, x_n) \in U, t \in T \). Let \( f : T \times U \to \mathbb{R}^n \in C^k_{rd} \), where \( C^k_{rd} \) denotes the set of all \( k \)-th rd-continuous functions, for each \((t, x) \in T \times U\).

We define the averaged function \( F(\cdot, \varepsilon) \) on time scales by

\[
F(\cdot, \varepsilon) = \sum_{i=1}^{k} \int_{0}^{T} \frac{1}{T} \varepsilon^i f_i(s, \cdot) \Delta s. \tag{2}
\]

Now, we are in the position to state our main result on the existence of periodic solutions to arbitrary order in \( \varepsilon \) for nonlinear dynamic equations on time scales.

**Theorem 1.1.** Assume that \( T \) is a \( T \)-periodic time scale, \( U \subset \mathbb{R}^n \) is an open bounded set. We consider the following dynamic equation

\[
x^\Delta(t) = \sum_{i=1}^{k} \varepsilon^i f_i(t, x) + \varepsilon^{k+1} r(t, x, \varepsilon), \tag{3}
\]

where \( f_i : T \times U \to \mathbb{R}^n \) for \( i = 1, \cdots, k \), \( r : T \times U \times (-\varepsilon_0, \varepsilon_0) \) are rd—continuous functions, \( T \) periodic in the first argument and locally Lipschitz with respect to \( x \).

Moreover, we assume the following conditions hold:

(i). For each \( t \in T \), \( p \in \partial U \), there exists a neighborhood \( N_p \) of \( p \), a constant \( \sigma > 0 \) independent of \( \varepsilon \) and integers \( 1 \leq j \leq n \) such that
for any \( q \in N_p, \ t \in [0, T]_T, \) and \( \varepsilon \in [-\varepsilon_0, \varepsilon_0]\setminus\{0\} \).

(ii). Suppose that for each \( \varepsilon \in [-\varepsilon_0, \varepsilon_0]\setminus\{0\} \),
\[
\deg(F(\cdot, \varepsilon), U, 0) \neq 0,
\]
where \( F(\cdot, \varepsilon) \) is defined as \([2]\).

Then, there exists a \( T \)-periodic solution \( x(t) \) of equation \((3)\) such that \( x(t) \in U \),
for \( |\varepsilon| > 0 \) sufficiently small.

The rest of this paper is organized as follows. In Section 2, we give the basic
definitions and properties related to the time scales briefly. In Section 3, we present
proof of our main result via topological degree. In last Section, we present the first
order averaging theorem for computing periodic solutions of equation \((1)\). Finally,
to illustrate our main result, the last section is devoted to some examples.

2. Preliminaries. In this section, we will present some basic definitions, concepts
and results concerning time scales which will be essential to prove our main results.
For more details about time scales, see \([4, 5]\). Let \( T \) be a time scale, that is, a closed
and nonempty subset of \( \mathbb{R} \).

**Definition 2.1.** For \( t \in T \), we define the forward jump operator and backward
jump operator \( \sigma, \rho : T \to T \), respectively, as follows:
\[
\sigma(t) = \inf \{s \in T : s > t\},
\rho(t) = \sup \{s \in T : s < t\}.
\]

In this definition, we put \( \inf \emptyset = \sup T \) and \( \sup \emptyset = \inf T \). If \( \sigma(t) > t \), we say that
\( t \) is right-scattered. If \( \sigma(t) = t \), then \( t \) is called right-dense. Analogously, if \( \rho(t) < t \),
we say that \( t \) is left-scattered. If \( \rho(t) = t \), then \( t \) is called left-dense.

**Definition 2.2.** We define the graininess function \( \mu : T \to [0, \infty) \) by
\[
\mu(t) = \sigma(t) - t.
\]

For \( a, b \in \mathbb{R} \), we use \([a, b]_T \) denote a closed interval in \( T \), that is, \([a, b]_T = \{t \in T : a \leq t \leq b\} \).
And we define the set \( T^c \) which is derived from \( T \) as follows: if \( T \) has
a left-scattered maximum \( m \), then \( T^c = T - \{m\} \). Otherwise, \( T^c = T \).

**Definition 2.3.** Assume \( f : T \to \mathbb{R}^n \) is a function and let \( t \in T^c \). Then we define
\( f^\Delta(t) \) to be the vector (provided it exists) with the property that for any \( \varepsilon > 0 \),
there is a neighborhood \( U \) of \( t \) such that
\[
|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|
\]
for all \( s \in U \).

We call \( f^\Delta(t) \) the delta (or Hilger) derivative of \( f \) at \( t \).

Similarly, we can define the nabla-derivative of the function \( f : T \to \mathbb{R}^n \), for
details, see \([4\) and \([5]\).

**Definition 2.4.** A function \( f : T \to \mathbb{R}^n \) is called rd-continuous provided it is
continuous at right-dense points in \( T \) and its left-sided limits exist (finite) at all
left-dense points in \( T \). The set of rd-continuous functions \( f : T \to \mathbb{R}^n \) will be
denoted by
\[
C_{rd} = C_{rd}(\mathbb{R}^n) = C_{rd}(T, \mathbb{R}^n).
\]
Let $f \in C_{rd}$. If $F^\Delta(t) = f(t)$, we have

$$F(t) - F(a) = \int_a^t f(s)\Delta s.$$  

**Remark 1.** $F$ is continuous on $\mathbb{T}$ when $f \in C_{rd}(\mathbb{T})$ (see Theorem 1.16 in [4]).

Next we will introduce the definition of periodic time scale.

**Definition 2.5.** Let $T > 0$ be a real number. A time scale $\mathbb{T}$ is called $T$-periodic if $t \in \mathbb{T}$ implies $t \pm T \in \mathbb{T}$ and $\mu(t) = \mu(t \pm T)$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive $T$ is called the period of the time scale.

For example, the time scale $\mathbb{T} = \bigcup_{k \geq 0} [2k, 2k+1]$ is a 2-periodic time scale.

3. **Proof of Theorem 1.1.** The following definition of a retraction map is a useful tool in the proof of our main result.

**Definition 3.1.** Let $X$ be a topological space and $A$ a subspace of $X$. Then a continuous map $\alpha : X \to A$ is called a retraction if $\alpha(t) = t$, for any $t \in A$.

**Proof.** For simplicity, we denote $F(t,x,\varepsilon) = \sum_{i=1}^{k} \varepsilon^i f_i(t,x) + \varepsilon^{k+1} r(t,x,\varepsilon)$.

Now we consider the equation

$$x^\Delta(t) = F(t,x,\varepsilon),$$  

where $F : \mathbb{T} \times U \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ is rd-continuous, $T$-periodic in $t$, locally Lipschitz in $x$, and $\mathbb{T}$ is a $T$-periodic time scale, $U$ is an open subset of $\mathbb{R}^n$.

Let $x(t)$ be the solution of (3). By the existence and uniqueness theorem on time scales (see, Section 8.3 of [4]), there exists an $\varepsilon_0 > 0$ such that for any $y_0 \in \bar{U}$, and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, the solution $x(t)$ exists for all $t \in [0,T]_\mathbb{T}$, and we have

$$x(t) = y_0 + \int_0^t F(s,x,\varepsilon)\Delta s.$$  

(5)

By Lemma 2.1 in [15], we know that the existence of $T$-periodic solution of (3) is equivalent to the following boundary value problem of (5).

$$\begin{cases}
x^\Delta(t) = F(t,x,\varepsilon), \\
x(T) = x(0),
\end{cases}$$  

(6)

where $\varepsilon \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\}$, $0$, $t \in \mathbb{T}$.

Let $X = \{ y : [0,T]_\mathbb{T} \to \mathbb{R}^n, y(t) \text{ is rd-continuous on } [0,T]_\mathbb{T} \}$, and define the norm as $\|y\| = \sup_{t \in [0,T]_\mathbb{T}} |y(t)|$. We can see that $X$ is a Banach space with the norm $\|\cdot\|$.

For $y_0 \in \mathbb{R}^n$ and $x \in X$ with $x(t) \in \bar{U}$ for all $t \in [0,T]_\mathbb{T}$, we define an operator $T(y_0, x)$ as follows:

$$T(y_0, x) = \left( y_0 + \frac{1}{T} \int_0^T \varepsilon f(s,x(s),\varepsilon)\Delta s, y_0 + \int_0^t \varepsilon f(s,x(s),\varepsilon)\Delta s \right).$$  

(7)
Next, the frame of the proof is the same to Theorem 1.1 in [15]. The only difference is in the proof of \( 0 \notin (id - H)(\partial(U \times V) \times [0, 1]) \). Now we give a detailed proof of it and we will omit other details.

Denote \( F(t, x, \varepsilon) = \varepsilon f(t, x, \varepsilon) \). Define

\[
X_\lambda = \left\{ x \in X : \frac{|x(t) - x(s)|}{t - s} \leq |\varepsilon|\lambda M, \forall t \neq s \right\},
\]

where a constant \( M \) satisfies \( M = \sup_{t \in [0,T], x \in U, \varepsilon \in [-\varepsilon_0, \varepsilon_0]} |f(t, x, \varepsilon)| \).

We define a retraction \( \alpha_\lambda : X \to X_\lambda \), for \( \lambda \in [0, 1] \).

Define an homotopy operator \( H(y_0, x, \lambda) \) by

\[
H(y_0, x, \lambda) = \begin{pmatrix}
y_0 + \frac{1}{T} \int_0^T \varepsilon f(s, \alpha_\lambda \circ x(s), \varepsilon) \Delta s \\
\alpha_\lambda \circ y_0 + \lambda \int_0^t \varepsilon f(s, \alpha_\lambda \circ x(s), \varepsilon) \Delta s
\end{pmatrix},
\]

where \((y_0, x, \lambda) \in U \times V \times [0, 1], V = \{x \in X : x(t) \in U, \forall t \in [0, T]_T\} \).

Now we claim that

\( 0 \notin (id - H)(\partial(U \times V) \times [0, 1]) \),

where \( id \) is the identity operator.

Otherwise, assume that there exists \((\bar{y}_0, \bar{x}, \bar{\lambda}) \in \partial(U \times V) \times [0, 1], \) such that \((id - H)(\bar{y}_0, \bar{x}, \bar{\lambda}) = 0 \). Then \((9)\) can be proved by two cases as follows:

(a) When \( \bar{\lambda} = 0 \), by the definition of set \( X_\lambda \), we have

\[
X_0 = \left\{ x \in X : \frac{|x(t) - x(s)|}{t - s} \leq 0, \forall t \neq s \right\}.
\]

Which implies that \( \alpha_0 \circ x(t) = \alpha_0 \circ x(0) \), for all \( t \in [0, T]_T \). It follows from \((id - H)(\bar{y}_0, \bar{x}(t), 0) = 0 \) that

\[
\begin{pmatrix}
\bar{y}_0 \\
\bar{x}(t)
\end{pmatrix}
= \begin{pmatrix}
y_0 + \frac{1}{T} \int_0^T \varepsilon f(s, \alpha_0 \circ \bar{x}(s), \varepsilon) \Delta s \\
\alpha_0 \circ \bar{y}_0
\end{pmatrix}.
\]

It means that

\( \bar{x}(t) \equiv \bar{y}_0 \),

for all \( t \in [0, T]_T \) and

\[
\frac{1}{T} \int_0^T \varepsilon f(s, \alpha_0 \circ \bar{x}(s), \varepsilon) \Delta s = 0.
\]

That is

\[
\frac{1}{T} \int_0^T \sum_{i=1}^k \varepsilon^i f_i(s, \bar{y}_0, \varepsilon) \Delta s = -\frac{1}{T} \int_0^T \varepsilon^{k+1} r(s, \bar{y}_0, \varepsilon) \Delta s.
\]

If \((\bar{y}_0, \bar{x}(t)) \in \partial(U \times V) \), then \( \bar{y}_0 \in \partial U \). When \( |\varepsilon| > 0 \) is sufficiently small, we have

\[
\frac{1}{T} \int_0^T \sum_{i=1}^k \varepsilon^i f_i(s, \bar{y}_0, \varepsilon) \Delta s = 0.
\]

It is contradictory to \((ii)\) of Theorem 1.1 because the Brouwer degree \( \deg(F(\cdot, \varepsilon), U, 0) \neq 0 \).
(b) When \( \tilde{\lambda} \in (0, 1] \), noticing that \((id - H)(\tilde{y}_0, \tilde{x}(t), \tilde{\lambda}) = 0\), it follows that

\[
\begin{pmatrix}
\tilde{y}_0 \\
\tilde{x}(t)
\end{pmatrix} = \begin{pmatrix}
\tilde{y}_0 + \frac{1}{T} \int_0^T \varepsilon f(s, \alpha_\lambda \circ \tilde{x}(s), \varepsilon) \Delta s \\
\alpha_\lambda \circ \tilde{y}_0 + \tilde{\lambda} \int_0^t \varepsilon f(s, \alpha_\lambda \circ \tilde{x}(s), \varepsilon) \Delta s
\end{pmatrix}.
\]

Note that

\[
\left| \tilde{x}(t) - \tilde{x}(s) \right| = \frac{1}{t-s} \left| \tilde{\lambda} \int_s^t \varepsilon f(\tau, \alpha_\lambda \circ \tilde{x}(\tau), \varepsilon) \Delta \tau \right|
\leq \frac{\tilde{\lambda}}{t-s} \int_s^t \left| \varepsilon f(\tau, \alpha_\lambda \circ \tilde{x}(\tau), \varepsilon) \right| \Delta \tau
\leq |\varepsilon| \tilde{\lambda} M. \tag{10}
\]

According to the definiton of \( X_\lambda \), we obtain that \( \tilde{x} \in X_\lambda \), which means that \( \alpha_\lambda \circ \tilde{x} = \tilde{x} \). Therefore,

\[
\frac{1}{T} \int_0^T \varepsilon f(s, \tilde{x}(s), \varepsilon) \Delta s = 0, \tag{11}
\]

and

\[
\tilde{x}(t) = \tilde{y}_0 + \tilde{\lambda} \int_0^t \varepsilon f(s, \tilde{x}(s), \varepsilon) \Delta s.
\]

Then

\[
|\tilde{x}(t) - \tilde{y}_0| \leq |\varepsilon| M_0 T, \ t \in [0, T]_T, \tag{12}
\]

where \( M_0 \) is a positive constant. We see that \( \tilde{x}(t) \) in the neighborhood \( \tilde{y}_0 \) of \( \tilde{y}_0 \), when \( |\varepsilon| > 0 \) is sufficiently small. If \( \tilde{y}_0 \in \partial U \), by the assumption (i) we know that there exists an integer \( j \) and a constant \( \sigma \) such that

\[
\left| \sum_{i=1}^k \varepsilon^i (f_i)^j (t, \tilde{x}(t)) \right| \geq \sigma |\varepsilon|^k \text{ for all } t \in [0, T]_T. \tag{13}
\]

Therefore

\[
\left| \sum_{i=1}^k \frac{1}{T} \int_0^T \varepsilon^i (f_i)^j (s, \tilde{x}(s)) \Delta s \right| \geq \sigma |\varepsilon|^k \text{ for all } t \in [0, T]_T. \tag{14}
\]

By (11), we have

\[
\sum_{i=1}^k \frac{1}{T} \int_0^T \varepsilon^i (f_i)^j (s, \tilde{x}(s)) \Delta s = -\frac{1}{T} \int_0^T \varepsilon^{k+1} r^j (s, \tilde{x}(s), \varepsilon) \Delta s. \tag{15}
\]

The right hand side of (15) equals to 0, when \( |\varepsilon| > 0 \) is sufficiently small. This is contradictory to (14). Thus \( \tilde{y}_0 \notin \partial U \). On the other hand, we claim that \( \tilde{x}(t) \notin \partial V \). Otherwise, if \( \tilde{x}(t) \in \partial V \), by the definition of \( V \), that is \( \tilde{x}(t) \in \partial U \), for some \( t \in [0, T]_T \). It is contradictory to assumption (ii) of Theorem 1.1, when \( |\varepsilon| > 0 \) is small enough.

By (a) and (b), we obtain

\[
0 \notin (id - H)(\partial (U \times V) \times [0, 1]).
\]
Therefore, by the homotopy invariance, the theory of topological degree and Lemma 5.3 we have
\[
\deg(\text{id} - H(y_0, x, 1), U \times V, 0) = \deg(\text{id} - H(y_0, x, 0), U \times V, 0) = \deg(F(\cdot, \varepsilon), U, 0) \neq 0.
\]
Then, it means that there is \((\tilde{x}, \tilde{y}) \in U \times V\), such that
\[
(\tilde{y}_0 \ast \tilde{x}(t)) = \left(\frac{\tilde{y}_0}{\tilde{x}(t)} + \frac{1}{T} \int_0^T \varepsilon f(s, \alpha_1 \circ \tilde{x}(s), \varepsilon) \Delta s \right). \tag{16}
\]
A similar proof in (b) yields that \(\tilde{x}(t) \in X_1\), That is
\[
H(\tilde{y}_0, \tilde{x}(t), 1) = T(\tilde{y}_0, \tilde{x}(t)). \tag{17}
\]
Hence, from (16) and (17), we obtain that \((\tilde{x}, \tilde{y})\) is a fixed point of \(T\) in \(X\). Thus \(\tilde{x}(t)\) is a solution of boundary value problem (6). This completes the proof of Theorem 1.1.

Remark 2. It is certain that the Theorem 1.1 remains effective for nabla dynamic equations on time scales. Namely, we can prove analogously that the nabla dynamic equation
\[
x^\nabla(t) = \sum_{i=1}^k \varepsilon^i f_i(t, x) + \varepsilon^{k+1} r(t, x, \varepsilon),
\]
where \(f_i: \mathbb{T} \times U \to \mathbb{R}^n\) for \(i = 1, \cdots, k\), \(r: \mathbb{T} \times U \times (-\varepsilon_0, \varepsilon_0)\) are rd–continuous functions, \(T\) periodic in the first argument and locally Lipschitz with respect to \(x\), under similar conditions to the ones presented on Theorem 1.1.

Remark 3. Sla\'ık in [14] studied the averaging theory of first order on the existence of periodic solutions for dynamic equations on time scales by implicit function theorem. Recently, Llibre, Novaes, and Teixeira in [12] extended the averaging theory up to any order in \(\varepsilon\) by using the Brouwer degree. Comparing with their results, we add the condition (i) in Theorem 1.1. Our main results extend the averaging theory for dynamic equations on time scales to arbitrary order in \(\varepsilon\). Moreover, our averaging functions \(F(\cdot, \varepsilon)\) are much easier to calculate than them.

4. Applications. Here we state the first order averaging method. Our proof is different from the one given in [14] (see Theorem 4.4).

Theorem 4.1. Consider the following dynamic equation
\[
x^\Delta(t) = \varepsilon f_1(t, x) + \varepsilon^2 r(t, x, \varepsilon), \tag{18}
\]
where \(f_1(t, x), r(t, x, \varepsilon)\) are rd-continuous, \(T\)-periodic functions in \(t\), \(f_1(t, x)\) and \(r(t, x, \varepsilon)\) are locally Lipschitz with respect to \(x\). We define
\[
F(y, \varepsilon) = \frac{1}{T} \int_0^T \varepsilon f_1(t, y) \Delta t. \tag{19}
\]
Moreover, suppose that for some bounded open subset \(U \subset \mathbb{R}^n\), \(\varepsilon \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\}\),
\[
\deg(F(\cdot, \varepsilon), U, 0) \neq 0. \tag{20}
\]
Then there exists a $T$-periodic solution of equation (18) such that $x(t) \in U$, for $|\varepsilon| > 0$ sufficiently small.

Proof. The idea of the proof is the same for the previous Theorem 1.1. The only difference is in the proof of $0 \not\in (id - H)(\partial(U \times V) \times [0, 1])$. Now we give a detailed proof of it and we will omit other details.

Define a same homotopy operator $H(y_0, x, \lambda)$ as Theorem 1.1. Now we claim that

$$0 \not\in (id - H)(\partial(U \times V) \times [0, 1]).$$

Otherwise, assume that there exists $(\tilde{y}_0, \tilde{x}, \tilde{\lambda}) \in \partial(U \times V) \times [0, 1], \text{ such that } (id - H)(\tilde{y}_0, \tilde{x}, \tilde{\lambda}) = 0$. Then (21) can be proved as follows:

(a) When $\tilde{\lambda} = 0$, by the definition of set $X_{\tilde{\lambda}}$, we have

$$X_0 = \left\{ x \in X : \left| \frac{x(t) - x(s)}{t - s} \right| \leq 0, \forall t \neq s \right\}.$$ 

Thus $\alpha_0 \circ x(t) = \alpha_0 \circ x(0)$, for all $t \in [0, T]_T$. Since $(id - H)(\tilde{y}_0, \tilde{x}(t), 0) = 0$, we have

$$\frac{\tilde{y}_0 + \frac{1}{T} \int_0^T \varepsilon f(s, \alpha_0 \circ \tilde{x}(s), \varepsilon) \Delta s}{\alpha_0 \circ \tilde{y}_0}$$

Thus

$$\tilde{x}(t) \equiv \tilde{y}_0,$$

for all $t \in [0, T]_T$ and

$$\frac{1}{T} \int_0^T \varepsilon f(s, \alpha_0 \circ \tilde{x}(s), \varepsilon) \Delta s = 0.$$

That is

$$\frac{1}{T} \int_0^T \varepsilon f_1(s, \tilde{y}_0, \varepsilon) \Delta s = -\frac{1}{T} \int_0^T \varepsilon^2 r(s, \tilde{y}_0, \varepsilon) \Delta s.$$ 

If $(\tilde{y}_0, \tilde{x}(t)) \in \partial(U \times V)$, then $\tilde{y}_0 \in \partial U$. When $|\varepsilon| > 0$ is sufficiently small, we have $\frac{1}{T} \int_0^T \varepsilon f_1(s, \tilde{y}_0, \varepsilon) \Delta s = 0$. It is contradictory to (ii) of Theorem 1.1 because the Brouwer degree $\text{deg}(F(\cdot, \varepsilon), U, 0) \neq 0$.

(b) When $\tilde{\lambda} \in (0, 1]$, as $(id - H)(\tilde{y}_0, \tilde{x}(t), \tilde{\lambda}) = 0$, we have

$$\frac{\tilde{y}_0 + \frac{1}{T} \int_0^T \varepsilon f(s, \alpha_\tilde{\lambda} \circ \tilde{x}(s), \varepsilon) \Delta s}{\alpha_\tilde{\lambda} \circ \tilde{y}_0 + \tilde{\lambda} \int_0^t \varepsilon f(s, \alpha_\tilde{\lambda} \circ \tilde{x}(s), \varepsilon) \Delta s}$$

Note that

$$\left| \frac{\tilde{x}(t) - \tilde{x}(s)}{t - s} \right| = \frac{1}{|t - s|} \left| \tilde{\lambda} \int_s^t \varepsilon f(\tau, \alpha_\tilde{\lambda} \circ \tilde{x}(\tau), \varepsilon) \Delta \tau \right|$$

$$\leq \frac{\tilde{\lambda}}{|t - s|} \int_s^t |\varepsilon f(\tau, \alpha_\tilde{\lambda} \circ \tilde{x}(\tau), \varepsilon)| \Delta \tau$$

$$\leq |\varepsilon| M \tilde{\lambda}.$$ 

By the definition of $X_{\tilde{\lambda}}$, we obtain $\tilde{x} \in X_{\tilde{\lambda}}$, which means that $\alpha_\tilde{\lambda} \circ \tilde{x} = \tilde{x}$. Therefore, 

$$\frac{1}{T} \int_0^T \varepsilon f(s, \tilde{x}(s), \varepsilon) \Delta s = 0.$$ (23)
Existence of Periodic Solutions of Dynamic Equations

and
\[ \bar{x}(t) = \bar{y}_0 + \bar{\lambda} \int_0^t \varepsilon f(s, \bar{x}(s), \varepsilon) \Delta s. \]

Then
\[ |\bar{x}(t) - \bar{y}_0| \leq |\varepsilon|M_0 T, \ t \in [0, T], \tag{24} \]
where \( M_0 \) is a positive constant. Since \( f_1(t, x) \) is locally Lipschitz with respect to \( x \), we have
\[
\left| \int_0^T f(t, \alpha_\chi \circ \bar{x}(t), \varepsilon) \Delta t - \int_0^T f_1(t, \bar{y}_0) \Delta t \right| \\
\leq \int_0^T |f_1(t, \alpha_\chi \circ \bar{x}(t))| \Delta t - \int_0^T f_1(t, \bar{y}_0) \Delta t + \left| \int_0^T \varepsilon r(t, \alpha_\chi \circ \bar{x}(t), \varepsilon) \Delta t \right| \\
\leq \int_0^T L|\bar{x}(t) - \bar{y}_0| \Delta t + \left| \int_0^T \varepsilon r(t, \alpha_\chi \circ \bar{x}(t), \varepsilon) \Delta t \right| \\
\leq |\varepsilon| \left( L M_0 T^2 + \int_0^T r(t, \alpha_\chi \circ \bar{x}(t), \varepsilon) \Delta t \right), \tag{25} \]
where \( L \) is the local Lipchitz constant. Then
\[
\left| \int_0^T f_1(t, \bar{y}_0) \Delta t \right| \leq |\varepsilon| \left( L M_0 T^2 + \int_0^T r(t, \alpha_\chi \circ \bar{x}(t), \varepsilon) \Delta t \right). \tag{26} \]

If \( \bar{y}_0 \in \partial U \), from (20) we have
\[
\int_0^T f_1(t, \bar{y}_0) \Delta t \neq 0. \]

This is contradictory to (26), when we let \( \varepsilon \to 0 \). Thus \( \bar{y}_0 \notin \partial U \). We claim that \( x(t) \in \partial V \), that we have \( x(t) \in \partial V \) for some \( t \in [0, T] \). This is contradictory to (24), when we let \( \varepsilon \to 0 \).

By (a) and (b), we obtain
\[ 0 \notin (id - H)(\partial(U \times V)) \times [0, 1]. \]

The conclusion follows by applying Theorem 1.1.

Next we present some examples and applications of our main results.

4.1. Example. Consider the dynamic equation
\[ y^\Delta(t) = \varepsilon(\sin(t\pi) + 1)y(t) + \varepsilon^2 r(t, y, \varepsilon), \tag{27} \]
on the time scale \( T = \cup_{k=0}^{\infty}[2k, 2k+1] \). Let
\[ f_1(t, y) = (\sin(t\pi) + 1)y(t), \]
where \( f_1(t, y) \) is 2-periodic in the first variable. Moreover, \( f_1 \) is Lipschitz-continuous in second variable, which means that \( |f_1(t, y_1) - f_1(t, y_2)| \leq 2|y_1 - y_2| \) for any \( t \in T \). Since \( F(y, \varepsilon) = \varepsilon(1 + \frac{1}{\pi})y \), then \( F(0, \varepsilon) = 0. \) Thus \( \deg(F(\cdot, \varepsilon), U, 0) \neq 0. \)

Then all hypotheses of Theorem 4.1 are satisfied, which implies that equation (4.1) has a 2-periodic solution. A similar equation was used to describe the growth of a plant population in [4]. In our case, the growth coefficient varies instead of being constant.
4.2. Example. Consider the dynamic equation
\[ x^\Delta(t) = \varepsilon(b(t) \tanh(x(t)) + c(t)) + \varepsilon^2 r(t, x, \varepsilon), \] (28)
where \( b, c, r \in C_{rd}(\mathbb{T}, \mathbb{R}) \) and \( b, c, r \) are \( T \)-periodic with \( t \). Since \( |\tanh(x_1) - \tanh(x_2)| \leq |x_1 - x_2| \) for \( x_1, x_2 \in \mathbb{T} \). Then denoting \( f_1(t, x) = b(t) \tanh(x(t)) + c(t) \), we obtain
\[ |f_1(t, x_1(t)) - f_1(t, x_2(t))| = |b(t) \tanh(x_1(t)) + c(t) - b(t) \tanh(x_2(t)) - c(t)| \leq b(t)|x_1(t) - x_2(t)| \leq \bar{b}|x_1(t) - x_2(t)|, \]
where \( \bar{b} = \sup_{t \in \mathbb{T}}|b(t)|. \) Since \( \varepsilon > 0 \), it is obvious that \( \deg(F(\cdot, \varepsilon), U, 0) \neq 0 \). Then all hypotheses of Theorem 4.1 are satisfied, and hence equation (28) has a \( T \)-periodic solution.

4.3. Example. Consider the dynamic equations
\[ x^\Delta = \varepsilon^2 y + \varepsilon^3 x^3 y^3 \cos^2 t + 2\varepsilon^4 y \sin t + \varepsilon^5 \sin t(x^2 + y^2), \]
\[ y^\Delta = \varepsilon^3 y^2(\sin^2 t + 3) + \varepsilon^4 x^4 y + 2\varepsilon^4 x \sin t + \varepsilon^5 \cos t(x^2 + y^2), \] (29)
on the time scales \( \mathbb{T} = \bigcup_{k=0}^{\infty} [2k\pi, (2k + 1)\pi] \), where \( \sin t(x^2 + y^2), \cos t(x^2 + y^2) \) are \( rd \)-continuous and \( 2\pi \)-periodic in the variable \( t \). Let
\[
\begin{align*}
  f_1 &= (0, x^3 y^2(\sin^2 t + 3))^T, \\
  f_2 &= (y, 0)^T, \\
  f_3 &= (x^2 y^3 \cos^2 t, x^3 y^4)^T, \\
  f_4 &= (2y \sin t, 2x \sin t)^T, \\
  r &= (\sin t(x^2 + y^2), \cos t(x^2 + y^2))^T.
\end{align*}
\]
Then
\[ (x^\Delta, y^\Delta)^T = \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \varepsilon^4 f_4 + \varepsilon^5 r. \]
Assume \( \varepsilon > 0 \) is sufficiently small and \( U \) is a neighborhood of the origin, then for \( (x, y) \in \partial U \), we have \( (x, y) \neq (0, 0) \). If \( (x, y) \in \partial U \) and \( y \neq 0 \), there exists a neighborhood \( N(y) \) of \( (x, y) \), we have
\[
\left| \frac{1}{\varepsilon^2} y' + \frac{1}{\varepsilon} x^2 y^3 \cos^2 t + 2y \sin t \right| \geq |y'| \text{ for any } (x', y') \in N(y).
\]
If \( (x, y) \in \partial U \) and \( x \neq 0 \), there exists a neighborhood \( N(y) \) of \( (x, y) \), we have
\[
\left| \frac{1}{\varepsilon^3} x^3 y^2(\sin^2 t + 3) + \frac{1}{\varepsilon} x^4 y^4 + 2x \cos t \right| \geq |x'| \text{ for any } (x', y') \in N(y).
\]
According to the Theorem 1.1, we set \( F(y, \varepsilon) = \sum_{i=1}^{k} \frac{1}{T} \int_{0}^{T} \varepsilon f_1(s, y) \Delta s \), where
\[
\begin{align*}
  \int_{0}^{2\pi} \cos^2 t \Delta t &= \int_{0}^{\pi} \cos^2 t dt + \cos^2 t|_{t=\pi} = \frac{\pi}{2} + 1, \\
  \int_{0}^{2\pi} \sin^2 t \Delta t &= \int_{0}^{\pi} \sin^2 t dt + \sin^2 t|_{t=\pi} = \frac{\pi}{2}, \\
  \int_{0}^{2\pi} \sin t \Delta t &= \int_{0}^{\pi} \sin t dt + \sin t|_{t=\pi} = 2.
\end{align*}
\]
Thus, we have
\[ F(x, y, \varepsilon) = \left( \varepsilon^2 y + \varepsilon^3 x^2 y^3 \left( \frac{\pi}{2} + 1 \right) + 4 \varepsilon^2 y, \varepsilon x^3 y^2 \left( \frac{\pi}{2} + 3 \right) + \varepsilon^3 x^3 y^4 + 4 \varepsilon^2 x \right)^T, \]
and \( F(0, 0, \varepsilon) = 0. \) Therefore \( \text{deg}(F(\cdot, \varepsilon), U, 0) \neq 0. \)

Then all assumptions of Theorem 1.1 are satisfied, and hence (29) has a \( 2\pi \)-periodic solution.

4.4. Example. Consider the dynamic equations
\begin{align*}
x^\Delta &= \varepsilon y^2 (\sin^2 t \pi + 3) + \varepsilon^2 z^2 + \varepsilon^3 (\cos t \pi + 2) x^2 + \varepsilon^4 r_1(t, x, y, z, \varepsilon),
\end{align*}
\begin{align*}
y^\Delta &= \varepsilon^2 (e^y - 1) + \varepsilon^3 z^3 \sin t \pi + \varepsilon^4 r_2(t, x, y, z, \varepsilon),
\end{align*}
\begin{align*}
z^\Delta &= \varepsilon^2 y^2 (\cos t \pi + 2) + \varepsilon^3 (\varepsilon^2 - 1) + \varepsilon^4 r_3(t, x, y, z, \varepsilon),
\end{align*}
on the time scales \( T = \bigcup_{k=0}^{\infty} [2k, 2k + 1] \), where \( r_i \) is rd-continuous in the variable \( t \) and \( r_i(t + 2, x, y, z, \varepsilon) = r_i(t, x, y, z, \varepsilon) \) for \( i = 1, 2, 3 \).

Let
\begin{align*}
f_1 &= (y^2 (\sin^2 t \pi + 3), 0, 0)^T, \\
f_2 &= (z^2, e^y - 1, y^2 (\cos t \pi + 2))^T, \\
f_3 &= ((\cos t \pi + 2) x^2, z^3 \sin t \pi, \varepsilon^2 - 1)^T, \\
r &= (r_1, r_2, r_3)^T.
\end{align*}

Then,
\[ (x^\Delta, y^\Delta, z^\Delta)^T = \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \varepsilon^4 r. \]

Denote \( U = \{(x, y, z)^T \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\} \). For \( p \in \partial U \), there exists a neighborhood \( U_p \) of \( p \), such that
\[ \left| \frac{1}{\varepsilon^2} y^2 (\sin^2 t \pi + 3) + \frac{1}{\varepsilon} z^2 + (\cos t \pi + 2) x^2 \right| \geq \frac{1}{2}, \forall x \in U_p. \]

By Theorem 1.1 we have
\[ F(x, y, z, \varepsilon) = \left( \frac{7}{2} \varepsilon y^2 + \varepsilon^2 z^2 + \varepsilon^3 x^2, \varepsilon^2 (e^y - 1) + \varepsilon^3 \frac{2}{\pi} z^3, \varepsilon^2 y^2 + \varepsilon^3 (\varepsilon^2 - 1) \right)^T, \]
\( F(0, 0, 0, \varepsilon) = 0. \) Thus \( \text{deg}(F(\cdot, \varepsilon), U, 0) \neq 0. \) By Theorem 1.1 for \( |\varepsilon| > 0 \) sufficiently small, equation (30) has a \( 2 \)-periodic solution.

5. Appendix. In this appendix, we present some lemmas that we shall need to prove the main result of this paper.

The following lemma shows some useful relationships concerning the delta derivative and continuity.

Lemma 5.1. Assume \( f : T \rightarrow \mathbb{R}^n \) is a function and let \( t \in T^c \). Then we have the following results.

(i). If \( f \) is differentiable at \( t \), then \( f \) is continuous at \( t \).

(ii). If \( f \) is continuous at \( t \) and \( t \) is right-scattered, then \( f \) is differentiable at \( t \) with
\[ f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}. \]

(iii). If \( t \) is right-dense, then \( f \) is differentiable at \( t \) provided that the limit
\[ \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s} \]
exists as a finite vector. In this case,

$$f^\Delta(t) = \lim_{{s \to t}} \frac{f(t) - f(s)}{t - s}.$$  

(iv). If $f$ is differentiable at $t$, then

$$f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t).$$

The following lemma (see, [4]) is useful in the calculation of examples.

**Lemma 5.2.** Let $a, b \in \mathbb{T}$ and $f \in C_{rd}$.

(i). If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)\,dt,$$

where the integral on right is the usual Riemann integral from calculus.

(ii). If $[a, b]$ consists of only isolated points, then

$$\int_a^b f(t)\Delta t = \begin{cases} 
\sum_{t \in (a, b)} \mu(t)f(t) & \text{if } a < b, \\
0 & \text{if } a = b, \\
-\sum_{t \in [b, a)} \mu(t)f(t) & \text{if } a < b.
\end{cases}$$  

(31)

The following lemma due to [7] plays a crucial role in the proof of main results.

**Lemma 5.3.** For a bounded open subset $V \subset \mathbb{R}^n$, assume that $F_i : \bar{V} \to \mathbb{R}^n$, for $i = 0, \ldots, k$, and $R : \bar{V} \times [-\varepsilon_0, \varepsilon_0] \to \mathbb{R}^n$ are continuous, and given by

$$F(\cdot, \varepsilon) = F_0(\cdot) + \varepsilon F_1(\cdot) + \varepsilon^2 F_2(\cdot) + \cdots + \varepsilon^k F_k(\cdot),$$

$$G(\cdot, \varepsilon) = F(\cdot, \varepsilon) + \varepsilon^{k+1} R(\cdot, \varepsilon).$$

Suppose that $F(z, \varepsilon) \neq 0$ for all $z \in \partial V$, $\varepsilon \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\}$. Then, for $|\varepsilon| > 0$ small enough, the Brouwer degree

$$\deg(F(\cdot, \varepsilon), V, 0) = \deg(G(\cdot, \varepsilon), V, 0).$$

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E-mail address: guoruichaomath@outlook.com
E-mail address: liyongmath@163.com
E-mail address: xingjiamin1028@126.com
E-mail address: yangxuemath@163.com