Positivity of the exterior power of the tangent bundles

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Abstract. Let $X$ be a complex smooth projective variety such that the exterior power of the tangent bundle $\bigwedge^r T_X$ is nef for some $1 \leq r < \dim X$. We prove that, up to an étale cover, $X$ is a Fano fiber space over an Abelian variety. This gives generalizations of the structure theorem of varieties with nef tangent bundle by Demailly, Peternell and Schneider [5] and that of varieties with nef $\bigwedge^2 T_X$ by the author [19]. Our result also gives an answer to a question raised by Li, Ou and Yang [14] for varieties with strictly nef $\bigwedge^r T_X$ when $r < \dim X$.

1. Introduction

Positivity for vector bundles such as ampleness and nefness has left its mark on the study of algebraic geometry. Let $X$ be a complex smooth projective variety of dimension $n$; we focus on the positivity of the tangent bundle $T_X$, which reflects the global geometry of $X$. As a generalization of the Hartshorne-Frankel conjecture solved by Mori [16] (see also [17] by Siu and Yau), Campana and Peternell [1] studied the structure of smooth projective varieties with nef tangent bundle, paying special attention to 3-folds. In higher dimensional case, Demailly, Peternell and Schneider obtained the following structure theorem:

**Theorem 1.1 ([5 Main Theorem]).** If $T_X$ is nef, then there exists a finite étale cover $X' \to X$ such that $X'$ is a locally trivial fibration $\varphi : X' \to \text{Alb}(X')$ whose fibers are a Fano variety.

Some years ago, Cao and Höring extended Theorem 1.1 to a more general setting:

**Theorem 1.2 ([3 Theorem 1.3]).** If the anticanonical divisor $-K_X$ is nef, then there exists a finite étale cover $X' \to X$ such that $X' \cong Y \times Z$ where $K_Y$ is trivial and $Z$ is a locally trivial fibration $\varphi : Z \to \text{Alb}(Z)$ with a rationally connected fiber.

In general a fiber of $\varphi$ in Theorem 1.2 is not a Fano variety, because there exists a lot of rationally connected projective varieties with nef anticanonical divisor which is not Fano (for instance, consider weak Fano varieties). The main result of this paper is a generalization of Theorem 1.2.

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Theorem 1.3. Let $X$ be a smooth projective variety of dimension $n$. Assume that $\Lambda^r T_X$ is nef for some $1 \leq r < n$. Then if we take a suitable finite étale cover $\tilde{X} \to X$, there exists a locally trivial fibration $\varphi : \tilde{X} \to A$ such that the fiber $F$ is a Fano variety and $A$ is an Abelian variety. Moreover, if $\dim A \geq r - 1$, then $T_X$ is nef; otherwise $\Lambda^{r-\dim A} T_{\tilde{X}/A}$ is nef.

This theorem reduces the study of smooth projective varieties with nef $\Lambda^r T_X$ ($r < \dim X$) to that of Fano varieties. For $r = 1$, Theorem 1.3 is nothing but Theorem 1.1; for $r = 2$ this was obtained in [19, Theorem 1.5]. The proof of [19, Theorem 1.5] involves the deformation theory of rational curves and some complicated arguments. On the other hand, in this short paper, we give a really simple proof of Theorem 1.3. Our proof relies on two key ingredients; one is Theorem 1.1; the other is a recent result of Gachet [4]. In [4, Theorem 1.2], she proved that for a smooth projective variety $X$ of dimension $n$ if $\Lambda^{n-1} T_X$ is strictly nef, then $X$ is a Fano variety. Her proof works if we replace the assumption that $\Lambda^{n-1} T_X$ is strictly nef by the assumption that $X$ is rationally connected and $\Lambda^{n-1} T_X$ is nef. Moreover by using a result by Laytimi and Nahm [11], we see that if $\Lambda^r T_X$ is nef for some $r < n$, then so is $\Lambda^{n-1} T_X$. Thus we have the following:

Proposition 1.4. Let $X$ be a smooth projective variety of dimension $n$. Assume that $X$ is rationally connected and $\Lambda^r T_X$ is nef for some $1 \leq r < n$. Then $X$ is a Fano variety.

Remark that, combining with [14, Theorem 1.2], Proposition 1.4 gives an affirmative answer to the following question by Li, Ou and Yang when $r < \dim X$:

Question 1.5 ([14, Remark 5.3], [15, Conjecture 4.9], [4, Question in Section 1]). Assume that $\Lambda^r T_X$ is strictly nef for some $1 \leq r \leq n$. Then is $X$ a Fano variety?

Finally, Theorem 1.3 follows from Theorem 1.2, Proposition 1.4 and standard arguments.

2. Preliminaries

2.1. Notation and Conventions. We will use the basic notation and definitions in [8], [9], [12], [13] and [10]. Along this paper, we work over the complex number field.

- A curve means a projective variety of dimension one.
- Let $X$ be a smooth projective variety. A line bundle $L$ on $X$ is said to be strictly nef (resp. nef) if the intersection number $L \cdot C$ is positive (resp. non-negative) for any curve $C \subset X$. In general, we say that a vector bundle $E$ is strictly nef (resp. nef) if the tautological line bundle $\mathcal{O}_{P(E)}(1)$ is strictly nef (resp. nef) on $P(E)$.
- For a non-constant morphism $f : \mathbb{P}^1 \to X$ from a projective line $\mathbb{P}^1$ to a smooth projective variety $X$, $f$ is said to be free if $f^* T_X$ is nef.

Throughout this section, we always assume the following:

Assumption 2.1. Assume $X$ is a smooth projective variety of dimension $n$ such that the exterior power $\Lambda^r T_X$ is nef for some $1 \leq r < n$.

Proposition 2.2. The following hold:
(i) The anticanonical divisor $-K_X$ is nef.
(ii) If the Kodaira dimension $\kappa(X) = 0$, then there exists a finite étale cover $f : \tilde{X} \to X$ such that $\tilde{X}$ is an Abelian variety.

**Proof.** The first part follows from $\det (\bigwedge^r T_X) \cong O_X \left( \binom{n-1}{r-1} (-K_X) \right)$. The second part follows from [21, Theorem 1.1] (see also [2, Proposition 1.2]).

**Lemma 2.3 ([2, Lemma 1.3], [20, Lemma 2.9]).** Let $f : \mathbb{P}^1 \to X$ be a non-free rational curve, that is, $f^* T_X$ is not nef. Then we have $-K_X \cdot f^*(\mathbb{P}^1) \geq n - r + 1$.

**Proof.** Assume that the splitting type of $f^* T_X$ is $(a_1, a_2, \ldots, a_n)$, that is, $f^* T_X \cong \bigoplus_{i=1}^n O_{\mathbb{P}^1}(a_i)$ ($a_1 \geq a_2 \geq \ldots \geq a_n$, $a_1 \geq 2$). The $r$-th exterior power $f^* T_X \cong \bigoplus_{1 \leq i_1 < i_2 < \ldots < i_r \leq n} O_{\mathbb{P}^1}(a_{i_1} + a_{i_2} + \ldots + a_{i_r})$ is nef; this yields $a_{n-r+1} + a_{n-r+2} + \ldots + a_n \geq 0$. Since $f$ is not free, $a_n$ is negative. These imply that $(r-1)a_{n-r+1} \geq a_{n-r+1} + a_{n-r+2} + \ldots + a_n - a_n \geq 1$. Thus $a_{n-r+1}$ is positive. As a consequence, we have the inequality $-K_X \cdot f^*(\mathbb{P}^1) = a_1 + (a_2 + \ldots + a_{n-r}) + (a_{n-r+1} + \ldots + a_n) \geq 2 + (n-r-1) + 0 = n-r+1$.

**Proposition 2.4 ([18, Proposition 3.3]).** Let $\varphi : X \to A$ be a smooth morphism onto an Abelian variety with irreducible fibers. Then the following hold:

(i) If $\dim A \geq r-1$, then $T_X$ is nef.
(ii) If $\dim A < r-1$, then $\bigwedge^r \varphi^* T_A$ is nef.

**Proof.** We have an exact sequence

$$(1) \quad 0 \to T_{X/A} \to T_X \to \varphi^* T_A \to 0.$$  

By [8, Chapter II, Exercise 5.16 (d)], we have a filtration of $\bigwedge^r T_X$:

$$\bigwedge^r T_X = E^0 \supset E^1 \supset E^2 \supset \cdots \supset E^{r+1} = 0$$

such that $E^p/E^{p+1} \cong \left( \bigwedge^r T_{X/A} \right) \otimes \left( \bigwedge^{r-p} \varphi^* T_A \right)$ for any $p$. In particular, we have the following exact sequences:

$$\begin{align*}
(2) & \quad 0 \to E^1 \to \bigwedge^r T_X \to \varphi^* T_A \to 0 \\
(3) & \quad 0 \to E^2 \to E^1 \to T_{X/A} \otimes \left( \bigwedge^{r-1} \varphi^* T_A \right) \to 0
\end{align*}$$
To prove (i), assume \( \dim A \geq r - 1 \). Remark that \( T_A \cong \mathcal{O}^{\dim A}_A \). We claim that \( E^1 \) is nef. If \( \dim A \geq r \), then it follows from the sequence (2) and [11 Proposition 1.2 (8)] that \( E^1 \) is nef. If \( \dim A = r - 1 \), then the sequence (2) yields \( E^1 \cong \bigwedge^r T_X \); this implies that \( E^1 \) is nef. By the sequence (3), \( T_{X/A} \otimes \left( \bigwedge^{r-1} \varphi^*T_A \right) \) is nef. Since \( \bigwedge^{r-1} \varphi^*T_A \) is trivial bundle, we conclude that the relative tangent bundle \( T_{X/A} \) is nef. Finally our assertion follows from the sequence (1).

To prove (ii), assume \( \dim A < r - 1 \). Since \( \bigwedge^p \varphi^*T_A = 0 \) for any \( p > \dim A \), we have

\[
\bigwedge^{r} T_X = E^0 = E^1 = \ldots = E^{r - \dim A}.
\]

Thus we have a surjection \( \bigwedge^{r} T_X = E^{r - \dim A} \to \bigwedge^{r - \dim A} T_{X/A} \); this implies that \( \bigwedge^{r - \dim A} T_{X/A} \) is nef.

\[\square\]

### 3. Proof of the Main Theorem

The following is due to Gachet:

**Proposition 3.1** ([4, Theorem 1.2]). Let \( X \) be a smooth projective variety of dimension \( n \). Assume that \( X \) is rationally connected and \( \bigwedge^{n-1} T_X \) is nef. Then \( -K_X \) is ample, that is, \( X \) is a Fano variety.

**Proof.** This follows from the same argument as in [4 Lemma 3.1, Lemma 3.3]. Actually the proof works if we replace the assumption that \( \bigwedge^{n-1} T_X \) is strictly nef by the assumption that \( X \) is rationally connected and \( \bigwedge^{n-1} T_X \) is nef; this yields that \( -K_X \) is nef and big. Then we may conclude that \( -K_X \) is ample by the same argument as in [4 Lemma 3.3] and Lemma 2.3.

**Remark 3.2.** Although Proposition 3.1 is not written explicitly in [4], Gachet introduced this statement holds at Algebraic Geometry seminar of the University of Tokyo (see Acknowledgements below).

**Theorem 3.3** ([11 Theorem 3.3], [6]). Let \( X \) be a smooth projective variety of dimension \( n \). For a vector bundle \( E \) of rank \( r \), assume that its exterior power \( \bigwedge^m E \) is nef for some positive integer \( m \). Then the vector bundle \( \bigwedge^{m+k} E \) is also nef for any \( 0 \leq k \leq n - m \).

**Remark 3.4.** In general, if a vector bundle \( E \) is strictly nef, it is not necessarily that its exterior power \( \bigwedge^r E \) is strictly nef. For instance, see [7 Section 10 in Chapter I] and [15 Example 2.1]). This means that an analogue of Theorem 3.3 does not hold if we replace nefness of \( \bigwedge^m E \) by strictly nefness.

**Proof of Proposition 1.4.** Assume that \( X \) is rationally connected and \( \bigwedge^r T_X \) is nef for some \( 1 \leq r < n \). Then Theorem 3.3 implies that \( \bigwedge^{n-1} T_X \) is nef. Applying Proposition 2.3 we see that \( X \) is a Fano variety.

**Proof of Theorem 1.3.** By Proposition 2.2 (i), \( -K_X \) is nef; according to Theorem 1.2 this turns out that there exists a finite étale cover \( X' \to X \) such that \( X' \cong Y \times Z \) where \( K_Y \) is trivial and \( Z \) is a locally trivial fibration \( Z \to \text{Alb}(Z) \) with a rationally connected fiber. Since we have \( X' \to X \) is étale, \( \bigwedge^r T_{X'} \) is also nef; then by Theorem 3.3 \( \bigwedge^{n-1} T_{X'} \) is also nef. Let \( p_1 : X' \to Y \) (resp. \( p_2 : X' \to Z \)
be the first projection (resp. the second projection). We denote by \( \ell \) the dimension of \( Y \). Since we have
\[
\bigwedge^{n-1} T_{X'} \cong \left[ p_1^* \left( \bigwedge^\ell T_Y \right) \otimes p_2^* \left( \bigwedge^{n-\ell} T_Z \right) \right] \oplus \left[ p_1^* \left( \bigwedge^{\ell-1} T_Y \right) \otimes p_2^* \left( \bigwedge^{n-\ell} T_Z \right) \right],
\]
the direct summand \( p_1^* \left( \bigwedge^{\ell-1} T_Y \right) \otimes p_2^* \left( \bigwedge^{n-\ell} T_Z \right) \) is nef; restricting this bundle to a fiber of the projection \( p_2 \), we see that \( \bigwedge^{\ell-1} T_Y \) is also nef provided that \( \ell > 0 \). If \( \ell = 1 \), then \( Y \) is an elliptic curve. Furthermore if \( \ell > 1 \), then Proposition 2.2 (ii) implies that \( Y \) is a finite étale quotient of an Abelian variety \( \tilde{Y} \). Hence, in any case, there exists a finite étale cover \( \tilde{X} \to X' \) such that \( \tilde{X} \) is a locally trivial fibration \( \varphi : \tilde{X} \to A \) onto an Abelian variety \( A \) with a rationally connected fiber. Then our assertion follows from Proposition 2.4 and Proposition 1.4.

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