Weight distributions of two classes of linear codes based on Gaussian period and Weil sums

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Abstract

In this paper, based on the theory of defining sets, two classes of at most six-weight linear codes over $\mathbb{F}_p$ are constructed. The weight distributions of the linear codes are determined by means of Gaussian period and Weil sums. In some case, there is an almost optimal code with respect to Griesmer bound, which is also an optimal one according to the online code table. The linear codes can also be employed to get secret sharing schemes.

Key words: linear codes; weight distributions; Gaussian period; Weil sums; secret sharing schemes

1 Introduction

In this study, $p$ is an odd prime and assume $q = p^e$ for a positive integer $e$. Let $\mathbb{F}_p$ and $\mathbb{F}_q$ denote the finite field with $p$ and $q$ elements, respectively. We denote by $Tr$ the absolute trace function \[^1\] from $\mathbb{F}_q$ onto $\mathbb{F}_p$, and use $\mathbb{F}_q^*$ and $\mathbb{F}_p^*$ to denote the multiplicative group of $\mathbb{F}_q$ and $\mathbb{F}_p$. Obviously, $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$, and $\mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\}$.

An $[n, k, d]$ linear code $C$ over $\mathbb{F}_q$ is a $k$-dimensional subspace of $\mathbb{F}_q^n$ with minimum Hamming distance $d$. Let $A_i$ be the number of codewords with Hamming weight $i$ in a code $C$. The weight enumerator of $C$ is defined by

$$1 + A_1 z + A_2 z^2 + \ldots + A_n z^n,$$

and the sequence $(1, A_1, \ldots, A_n)$ is called the weight distribution of $C$ \[^2\]. If $\{|1 \leq i \leq n : A_i \neq 0\}| = t$, then we say $C$ a $t$-weight code. In coding theory, the weight distribution of linear codes is an interesting research topic, as it contains important information as to estimate the error correcting capability and the probability of error detection and correction with respect to some algorithms.

Let $C$ be an $[n, k, d]$ code over $\mathbb{F}_q$ with $k \geq 1$, then the well-known Griesmer bound \[^2\] is given by

$$n \geq \sum_{i=0}^{k-1} \left\lfloor \frac{d}{q^i} \right\rfloor.$$
An \([n, k, d]\) code is called optimal if no \([n, k, d+1]\) code exists, and is called almost optimal if the \([n, k, d+1]\) code is optimal \([3]\).

One of the constructions of linear codes is based on a proper selection of a subset of finite fields \([4]\). That is, let \(D = \{d_1, d_2, \ldots, d_n\} \subseteq \mathbb{F}_q\). A linear code of length \(n\) over \(\mathbb{F}_p\) is defined as
\[
C_D = \{(\text{Tr}(xd_1), \text{Tr}(xd_2), \ldots, \text{Tr}(xd_n)) : x \in \mathbb{F}_q\},
\]
the set \(D\) is called the defining set of linear code \(C_D\). This construction approach is generic in the sense that many classes of optimal linear codes could be produced by selecting the proper defining sets \([5–9]\).

By means of the construction method mentioned above, Xiang et al. \([10]\) constructed a class of linear codes and presented their weight distributions, with the defining set \(D = \{x \in \mathbb{F}_q^* : \text{Tr}(x^{p+1} - x) = 0\}\), where \(p\) is an odd prime, \(q = p^m\). In this paper, we generalize the construction of the defining set, and obtain two classes of linear codes with at most six weights, which include some almost optimal codes. And making use of Weil sums \([11–13]\) and Gaussian period, we will determine not only the parameters but also weight distributions of these codes.

## 2 Main Results

In this section, we present the main results, including the construction, the parameters and the weight distribution of the linear code \(C_{D_i}\). The proofs will be given in the following section.

We begin this section by selecting defining sets
\[
D_i = \{x \in \mathbb{F}_q : \text{Tr}(x^{p+1} - x) \in C_i^{(2,p)}\}, i = 0, 1.
\]
(1)
to construct linear codes
\[
C_{D_i} = \{c(a) = (\text{Tr}(ax))_{x \in D_i} : a \in \mathbb{F}_q\}, i = 0, 1.
\]
(2)
where \(C_0^{(2,p)}\) and \(C_1^{(2,p)}\) are the cyclotomic classes of order 2 in \(\mathbb{F}_p^*\) \([14]\), also denote the sets of all squares and non-squares in \(\mathbb{F}_p^*\), respectively. Let \(q = p^e\) satisfying \(p\) be an odd prime. Throughout the paper, \(\eta\) is the quadratic character over \(\mathbb{F}_p^*\). Then the weights and weight distributions of the linear codes are studied by utilizing some results of Weil sums \([12, 13]\) and Gaussian period.

The following Theorems 1-9 are the main results of this paper.

**Theorem 1** If \(e\) is odd and \(p \mid e\), then the weight distribution of the codes \(C_{D_i}\) \((i = 0, 1)\) with the parameters \([\frac{p+1}{2}p^{-1} + (-1)^i\frac{p+1}{2}C^{-1}, e]\) is listed in table 1. Obviously, the codes are at most 5-weight. \(G = \sqrt{\eta(-1)p}\).

**Theorem 2** If \(e\) is odd and \((\frac{e}{p}) = (-1)^i\), \(i = 0, 1\), then the weight distribution of the codes \(C_{D_i}\) \((i = 0, 1)\) with the parameters \([\frac{p+1}{2}p^{-1} - (-1)^iG^{-1}, e]\) is listed in table 2. Obviously, the codes are at most 6-weight, where \(G = \sqrt{\eta(-1)p}\) and \((\cdot)\) is the Legendre symbol.
Table 1: The weight distribution of $C_{D_i} (i = 0, 1)$ when $2 
mid e, p \mid e$

| Weight | Multiplicity |
|--------|--------------|
| $\frac{(p-1)^2}{2} p^{e-2}$ | 1 |
| $\frac{(p-1)^2}{2} p^{e-2} + (-1)^i \frac{p-1}{2} G^{e-1}$ | $p^{e-2} - 1$ |
| $\frac{(p-1)^2}{2} p^{e-2} + (-1)^i \frac{p-1}{2} G^{e-1} - (-1)^i \frac{p-1}{2} G^{e-3}$ | $(p - 1)p^{e-2}$ |
| $\frac{(p-1)^2}{2} p^{e-2} + (-1)^i \frac{p-1}{2} G^{e-1} - (-1)^i \frac{p+1}{2} G^{e+3}$ | $(p - 1)p^{e-2} - (-1)^i \frac{p-1}{2} G^{e-1} + (1 - \eta(-1))(\frac{p-1}{4})^2 p^{e-2}$ |

Table 2: The weight distribution of $C_{D_i} (i = 0, 1)$ when $2 
mid e, \left(\frac{e}{p}\right) = (-1)^i$

| Weight | Multiplicity |
|--------|--------------|
| $\frac{(p-1)^2}{2} p^{e-2}$ | 1 |
| $\frac{(p-1)^2}{2} p^{e-2} - (-1)^i G^{e-1}$ | $p^{e-2} - 1 + (-1)^i \eta(-1)(p - 1)G^{e-3}$ |
| $\frac{(p-1)^2}{2} p^{e-2} - (-1)^i G^{e-1} - (-1)^i \eta(-1) \frac{p-1}{2} G^{e-3}$ | $(p - 1)p^{e-2} - (-1)^i \eta(-1)(p - 1)G^{e-3}$ |
| $\frac{(p-1)^2}{2} p^{e-2} - (-1)^i G^{e-1} + (-1)^i \eta(-1) \frac{p+1}{2} G^{e+3}$ | $(p - 1)p^{e-2} - (-1)^i \eta(-1) \frac{p-1}{2} G^{e-1} + (1 - \eta(-1))(\frac{p-1}{4})^2 p^{e-2}$ |
| $\frac{(p-1)^2}{2} p^{e-2} - (-1)^i G^{e-1} - (-1)^i \eta(-1) \frac{p+1}{2} G^{e+3}$ | $(p - 1)p^{e-2} + (-1)^i \eta(-1) \frac{p-1}{2} G^{e-1} + (1 + \eta(-1))(\frac{p-1}{4})^2 p^{e-2}$ |
Theorem 3 If \( e \) is odd and \( \left( \frac{i}{p} \right) = (-1)^{1-i}, i = 0, 1 \), then the weight distribution of the codes \( C_{D_i} (i = 0, 1) \) with the parameters \( \left[ \frac{p-1}{2} p^{e-1}, e \right] \) is listed in table 3, where \( (\cdot) \) is the Legendre symbol. Obviously, the codes are at most 5-weight.

| Weight                          | Multiplicity          |
|---------------------------------|-----------------------|
| \( \frac{(p-1)^2}{2} p^{e-2} \) | \( 1 \)               |
| \( \frac{(p-1)^2}{2} p^{e-2} + (-1)^i \eta(-1) \frac{p-1}{2} G^{e-3} \) | \( p^{e-1} - 1 \)       |
| \( \frac{(p-1)^2}{2} p^{e-2} - (-1)^i \eta(-1) \frac{p-1}{2} G^{e-3} \) | \( \frac{p^2-1}{4} p^{e-2} + (-1)^i \eta(-1) \frac{p^2-1}{4} G^{e-3} \) |
| \( \frac{(p-1)^2}{2} p^{e-2} - (p-1) \frac{p+1}{2} G^{e-3} \) | \( \frac{p^2-1}{4} p^{e-2} - (1)^i \eta(-1) \frac{(p-1)(3p-1)}{4} G^{e-3} \) |
| \( \frac{(p-1)^2}{2} p^{e-2} + (-1)^i \frac{p+1}{2} G^{e-3} \) | \( \frac{(p-1)^2}{4} p^{e-2} + (-1)^i \eta(-1) \frac{(p-1)^2}{4} G^{e-3} \) |

Theorem 4 If \( e \) is even, \( e \equiv 2 \mod 4 \), and \( p \mid e \), then the weight distribution of the codes \( C_{D_i} (i = 0, 1) \) with the parameters \( \left[ \frac{p-1}{2} p^{e-1} + \frac{p-1}{2} p^{\frac{e}{2}-1}, e \right] \) is listed in table 4. Obviously, the codes are at most 4-weight.

| Weight                          | Multiplicity          |
|---------------------------------|-----------------------|
| \( \frac{(p-1)^2}{2} p^{e-2} \) | \( 1 \)               |
| \( \frac{(p-1)^2}{2} p^{e-2} + \frac{p-1}{2} p^{\frac{e}{2}-1} \) | \( \frac{p^2-1}{4} p^{e-2} - 1 - \frac{p-1}{2} p^{\frac{e}{2}-1} \) |
| \( \frac{(p-1)^2}{2} p^{e-2} + (p-1) \frac{p+1}{2} p^{\frac{e}{2}-1} \) | \( \frac{p^2-1}{2} p^{e-2} + \frac{p-1}{2} p^{\frac{e}{2}-1} \) |
| \( \frac{(p-1)^2}{2} p^{e-2} + \frac{p-3}{2} p^{\frac{e}{2}-1} \) | \( \frac{(p-1)^2}{2} p^{e-2} \) |

Theorem 5 If \( e \) is even, \( e \equiv 2 \mod 4 \), and \( \left( \frac{i}{p} \right) = (-1)^i, i = 0, 1 \), then the weight distribution of the codes \( C_{D_i} (i = 0, 1) \) with the parameters \( \left[ \frac{p-1}{2} p^{e-1} - 1 + \eta(-1) p \frac{p^{e-1}}{2}, e \right] \) is listed in the case of \( p \equiv 1 \mod 4 \) and \( p \equiv 3 \mod 4 \) in table 5 and 6 where \( (\cdot) \) is the Legendre symbol. Obviously, the codes are at most 5-weight.

Theorem 6 If \( e \) is even, \( e \equiv 2 \mod 4 \), and \( \left( \frac{i}{p} \right) = (-1)^{1-i}, i = 0, 1 \), then the weight distribution of the codes \( C_{D_i} (i = 0, 1) \) with the parameters \( \left[ \frac{p-1}{2} p^{e-1} - 1 - \eta(-1) p \frac{p^{e-1}}{2}, e \right] \) is listed in the case of \( p \equiv 1 \mod 4 \) and \( p \equiv 3 \mod 4 \) in table 5 and 6 where \( (\cdot) \) is the Legendre symbol. Obviously, the codes are at most 5-weight.
Table 5: The weight distribution of $C_{D_i} (i = 0, 1)$ when $e \equiv 2 \mod 4$, $(\frac{e}{p}) = (-1)^i$

| Weight | Multiplicity |
|--------|--------------|
| 0      | 1            |
| $\frac{(p-1)^2}{2}p^{e-2}$ | $p^{e-2} - 1$ |
| $\frac{(p-1)^2}{2}p^{e-2} - \frac{p+1}{2}p^{\frac{e}{2}-1}$ | $\frac{p^2-1}{2}p^{e-2} - (p-1)p^{\frac{e}{2}-1}$ |
| $\frac{(p-1)^2}{2}p^{e-2} - \frac{p+3}{2}p^{\frac{e}{2}-1}$ | $\frac{(p-1)(p-3)}{4}p^{e-2} - \frac{(p-1)(p-3)}{4}p^{\frac{e}{2}-1}$ |
| $\frac{(p-1)^2}{2}p^{e-2} - \frac{p-1}{2}p^{\frac{e}{2}-1}$ | $\frac{p^2-1}{2}p^{e-2} + \frac{p^2-1}{4}p^{\frac{e}{2}-1}$ |
| $\frac{(p-1)^2}{2}p^{e-2} - p^{\frac{e}{2}-1}$ | $(p-1)p^{e-2}$ |

Table 6: The weight distribution of $C_{D_i} (i = 0, 1)$ when $e \equiv 2 \mod 4$, $(\frac{e}{p}) = (-1)^{1-i}$

| Weight | Multiplicity |
|--------|--------------|
| 0      | 1            |
| $\frac{(p-1)^2}{2}p^{e-2}$ | $p^{e-1} - 1$ |
| $\frac{(p-1)^2}{2}p^{e-2} + \frac{p-1}{2}p^{\frac{e}{2}-1}$ | $\frac{p^2-1}{2}p^{e-2}$ |
| $\frac{(p-1)^2}{4}p^{e-2} + \frac{p-3}{2}p^{\frac{e}{2}-1}$ | $\frac{(p-1)^2}{4}p^{e-2} - \frac{(p-1)^2}{4}p^{\frac{e}{2}-1}$ |
| $\frac{(p-1)^2}{4}p^{e-2} + \frac{p+1}{2}p^{\frac{e}{2}-1}$ | $\frac{(p-1)^2}{4}p^{e-2} + \frac{(p-1)^2}{4}p^{\frac{e}{2}-1}$ |

**Theorem 7** If $e$ is even, $e \equiv 0 \mod 4$, and $p \mid e$, then the weight distribution of the codes $C_{D_i} (i = 0, 1)$ with the parameters $[\frac{p-1}{2}p^{e-1} + \frac{p-1}{2}p^{\frac{e}{2}}, e]$ is listed in table 7. Obviously, the codes are at most 5-weight.

Table 7: The weight distribution of $C_{D_i} (i = 0, 1)$ when $e \equiv 0 \mod 4$, $p \mid e$

| Weight | Multiplicity |
|--------|--------------|
| 0      | 1            |
| $\frac{(p-1)^2}{2}p^{e-2} + \frac{(p-1)^2}{2}p^{\frac{e}{2}-1}$ | $p^{e} - p^{e-2}$ |
| $\frac{(p-1)^2}{2}p^{e-2}$ | $\frac{p^{e-4} + 1}{2}p^{e-4} - 1 - \frac{p^{e-4}}{2}p^{\frac{e}{2}-2}$ |
| $\frac{(p-1)^2}{2}p^{e-2} + \frac{p-1}{2}p^{\frac{e}{2}}$ | $\frac{p^{e-4}}{2}p^{e-4}$ |
| $\frac{(p-1)^2}{2}p^{e-2} + (p-1)p^{\frac{e}{2}}$ | $\frac{p^{e-4}}{2}p^{e-4} + \frac{p-1}{2}p^{\frac{e}{2}-2}$ |
| $\frac{(p-1)^2}{2}p^{e-2} + \frac{p-1}{2}p^{\frac{e}{2}}$ | $\frac{(p-1)^2}{2}p^{e-4}$ |

**Theorem 8** If $e$ is even, $e \equiv 0 \mod 4$, and $(\frac{e}{p}) = (-1)^i$, $i = 0, 1$, then the weight distribution of the codes $C_{D_i} (i = 0, 1)$ with the parameters $[\frac{p-1}{2}p^{e-1} - \frac{1+(-1)^i}{2}p^{\frac{e}{2}}, e]$ is listed in table 8 and 9, where $(\ast)$ is the Legendre symbol. Obviously, the codes are at most 6-weight.

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Table 8: The weight distribution of $C_{D_i} (i = 0, 1)$ when $e \equiv 0 \mod 4$, $(\frac{e}{p}) = (-1)^i$

| Weight   | Multiplicity |
|----------|--------------|
| 0        | 1            |
| $\frac{(p-1)^2}{2} p^{e-2} - \frac{p^2-1}{2} p^\zeta - 1$ | $p^e - p^{e-2}$ |
| $\frac{(p-1)^2}{2} p^{e-2}$ | $p^{e-4} - 1$ |
| $\frac{(p-1)^2}{2} p^{e-2} - \frac{p+1}{2} p^\zeta$ | $\frac{p^2-1}{2} p^{e-4} - (p-1)p^\zeta - 2$ |
| $\frac{(p-1)^2}{2} p^{e-2} - \frac{p+3}{2} p^\zeta$ | $\frac{p^2-1}{4} p^{e-4} - \frac{(p-1)(p-3)}{4} p^\zeta - 2$ |
| $\frac{(p-1)^2}{2} p^{e-2} - \frac{p+1}{2} p^\zeta$ | $\frac{p^2-1}{4} p^{e-4} + \frac{p^2-1}{4} p^\zeta - 2$ |
| $\frac{(p-1)^2}{2} p^{e-2} - p^\zeta$ | $(p-1)p^{e-4}$ |

Theorem 9 If $e$ is even, $e \equiv 0 \mod 4$, and $(\frac{e}{p}) = (-1)^{1-i}$, $i = 0, 1$, then the weight distribution of the codes $C_{D_i} (i = 0, 1)$ with the parameters $[\frac{p-1}{2} p^{e-1} - 1 - n(-1)\frac{1}{2} p^\zeta, e]$ is listed in table 3 and 4 where $(\cdot)$ is the Legendre symbol. Obviously, the codes are at most 6-weight.

Table 9: The weight distribution of $C_{D_i} (i = 0, 1)$ when $e \equiv 0 \mod 4$, $(\frac{e}{p}) = (-1)^{1-i}$

| Weight   | Multiplicity |
|----------|--------------|
| 0        | 1            |
| $\frac{(p-1)^2}{2} p^{e-2} + \frac{(p-1)^2}{2} p^\zeta - 1$ | $p^e - p^{e-2}$ |
| $\frac{(p-1)^2}{2} p^{e-2}$ | $p^{e-3} - 1$ |
| $\frac{(p-1)^2}{2} p^{e-2} + \frac{p-1}{2} p^\zeta$ | $\frac{p^2-1}{2} p^{e-4}$ |
| $\frac{(p-1)^2}{2} p^{e-2} + \frac{p+3}{2} p^\zeta$ | $\frac{(p-1)^2}{4} p^{e-4} - \frac{(p-1)^2}{4} p^\zeta - 2$ |
| $\frac{(p-1)^2}{2} p^{e-2} + \frac{p-1}{2} p^\zeta$ | $\frac{(p-1)^2}{4} p^{e-4} + \frac{(p-1)^2}{4} p^\zeta - 2$ |

The followings are some examples about our results verified by Magma.

Example 1 If $(p, e) = (3, 3)$, then by Theorem 7, the code $C_{D_0}$ has parameters $[6, 3, 3]$ with weight enumerator $1 + 6z^3 + 12z^4 + 2z^5 + 2z^6$, the code $C_{D_1}$ has parameters $[12, 3, 6]$ with weight enumerator $1 + 2z^6 + 6z^7 + 6z^8 + 6z^9 + 6z^{10}$, which confirmed the result by Magma. According to Griesmer bound, also with respect to the code table [15], the code $C_{D_{20}}$ is optimal.

Example 2 If $(p, e) = (5, 2)$, then by Theorem 8, the code $C_{D_0}$ has parameters $[12, 2, 8]$ with weight enumerator $1 + 4z^8 + 12z^{10} + 8z^{11}$, the code $C_{D_1}$ has parameters $[7, 2, 5]$ with weight enumerator $1 + 8z^5 + 12z^6 + 4z^7$, which confirmed the result by Magma. According to Griesmer bound, also with respect to the code table [15], the code $C_{D_{21}}$ is optimal.
Example 3 If \((p, e) = (7, 4)\), then by Theorem 5.33, the code \(C_{D_0}(i = 0, 1)\) has parameters \([1176, 4, 882]\) with weight enumerator \(1 + 6z^{882} + 2352z^{1009} + 24z^{1029} + 18z^{1078}\), the code \(C_{D_1}(i = 0, 1)\) has parameters \([833, 4, 686]\) with weight enumerator \(1 + 18z^{686} + 2352z^{714} + 24z^{735} + 6z^{833}\), which confirmed the result by Magma.

3 Preliminaries and Auxiliary lemmas

In this section, we present some facts on exponential sums, that will be needed in calculating the weight enumerator of the codes defined in this article.

An additive character of \(\mathbb{F}_q\) is a non-zero function \(\chi\) from \(\mathbb{F}_q\) to the set of complex numbers of absolute value 1 such that \(\chi(x + y) = \chi(x)\chi(y)\) for any pair \((x, y) \in \mathbb{F}_q^2\). For each \(u \in \mathbb{F}_q\), the function

\[
\chi_u(v) = \zeta_p^{\text{Tr}(uv)}, \quad v \in \mathbb{F}_q
\]

denotes an additive character of \(\mathbb{F}_q\), where \(\zeta_p = e^{2\pi i/p}\) is a primitive \(p\)-th root of unity and \(i = \sqrt{-1}\). Since \(\chi_0(v) = 1\) for all \(v \in \mathbb{F}_q\), which is the trivial additive character of \(\mathbb{F}_q\). We call \(\chi_1\) the canonical additive character of \(\mathbb{F}_q\) and we have \(\chi_u(x) = \chi_1(ux)\) for all \(u \in \mathbb{F}_q\). The additive character satisfies the orthogonal property \(\square\), that is

\[
\sum_{v \in \mathbb{F}_q} \chi_u(v) = \begin{cases} q, & u = 0, \\ 0, & u \neq 0. \end{cases}
\]

Let \(h\) be a fixed primitive element of \(\mathbb{F}_q\). For each \(j = 0, 1, \ldots, q - 2\), the function \(\lambda_j(h^k) = e^{2\pi ij k/(q-1)}\) for \(k = 0, 1, \ldots, q - 2\) defines a multiplicative character of \(\mathbb{F}_q\), we extend these characters by setting \(\lambda_j(0) = 0\). Let \(q\) be odd. For \(j = (q-1)/2\) and \(v \in \mathbb{F}_q^*\), we have

\[
\lambda_{(q-1)/2}(v) = \begin{cases} 1, & \text{if } v \text{ is the square of an element of } \mathbb{F}_q^*, \\ -1, & \text{otherwise}, \end{cases}
\]

which is called the quadratic character of \(\mathbb{F}_q^*\), and is denoted by \(\eta'\) in the sequel. We call \(\eta' = \lambda_{(q-1)/2}\) and \(\eta = \lambda_{(p-1)/2}\) are the quadratic characters over \(\mathbb{F}_q\) and \(\mathbb{F}_p\), respectively. The quadratic Gauss sums over \(\mathbb{F}_q\) and \(\mathbb{F}_p\) are defined respectively by

\[
G'(\eta') = \sum_{v \in \mathbb{F}_q} \eta'(v)\chi_1(v) \quad \text{and} \quad G(\eta) = \sum_{v \in \mathbb{F}_p} \eta(v)\chi_1(v),
\]

where \(\eta\) and \(\chi_1\) are the canonical multiplicative and additive characters of \(\mathbb{F}_p\), respectively. Moreover, it is well known that \(G' = (-1)^{e-1}\sqrt{p^{*-1}}\) and \(G = \sqrt{p^*}\), where \(p^* = \eta(-1)p\).

The following are some basic facts on exponential sums.

Lemma 1 (\([\square]\), Theorem 5.33) If \(f(x) = a_2x^2 + a_1x + a_0 \in \mathbb{F}_q[x]\), where \(a_2 \neq 0\), then

\[
\sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}(f(x))} = \zeta_p^{\text{Tr}(a_0-a_2^*(4a_2)^{-1})}\eta'(a_2)G'(\eta'),
\]

where \(\eta'\) is the quadratic character of \(\mathbb{F}_q\).
Lemma 2 (\cite{12}, Theorem 5.48) With the notation above, we have
\[
\sum_{x \in \mathbb{F}_q} \eta'(f(x)) = \begin{cases} -\eta'(a_2), & a_1^2 - 4a_0a_2 \neq 0, \\ (q-1)\eta'(a_2), & a_1^2 - 4a_0a_2 = 0. \end{cases}
\]

For $\alpha, \beta \in \mathbb{F}_q$ and any positive integer $l$, the Weil sums $S(\alpha, \beta)$ is defined by
\[
S(\alpha, \beta) = \sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}(\alpha x^l + \beta x)}.
\]

We will show some results of $S(\alpha, \beta)$ for $\alpha \neq 0$ and $q$ odd.

Lemma 3 (\cite{14}) Let $e$ be odd and $p$ be an odd prime. $f(x) = \alpha^p x^2 + \alpha x \in \mathbb{F}_q[x]$ and $\beta \in \mathbb{F}_q$, then $f(x)$ is a permutation polynomial over $\mathbb{F}_q$ with $q = p^e$, and
\[
S(\alpha, \beta) = G^e \eta'(\alpha) \zeta_p^{\text{Tr}(\alpha x^e)}.
\]
where $G = G(\eta) = \sqrt{p} = \sqrt{\eta(-1)p} = \sqrt{(-1)^{\frac{p-1}{2}}p}$, $x_0$ is the unique solution of the equation $f(x) = -\beta^p$. Especially,
\[
S(\alpha, 0) = G^e \eta'(\alpha).
\]

Lemma 4 (\cite{12}, Theorem 2) Let $s = (l, e)$ and $e/s$ be even with $e = 2m$. Then
\[
S(\alpha, 0) = \begin{cases} (-1)^{m/s} p^m, & \alpha^{(q-1)/(p^s+1)} \neq (-1)^{m/s}, \\ (-1)^{m/s+1} p^{m+s}, & \alpha^{(q-1)/(p^s+1)} = (-1)^{m/s}. \end{cases}
\]

Lemma 5 (\cite{13}, Theorem 4.7) Let $\beta \neq 0$ and $e/s$ be even with $e = 2m$. Then $S(\alpha, \beta) = 0$ unless the equation $\alpha^p X^{p^2 l} + \alpha X = -\beta^p$ is solvable. There are two possibilities.

1. If $\alpha^{(q-1)/(p^s+1)} \neq (-1)^{m/s}$, then for any choice of $\beta \in \mathbb{F}_q$, the equation has a unique solution $x_0$ and
\[
S(\alpha, \beta) = (-1)^{m/s} p^m \zeta_p^{\text{Tr}(\alpha x^0)}
\]

2. If $\alpha^{(q-1)/(p^s+1)} = (-1)^{m/s}$ and if the equation is solvable with some solution $x_0$, then
\[
S(\alpha, \beta) = (-1)^{m/s+1} p^{m+s} \zeta_p^{\text{Tr}(\alpha x^0)}
\]

Lemma 6 (\cite{12}, Theorem 4.1) For $e = 2m$, the equation $\alpha^p X^{p^2 l} + \alpha X = 0$ is solvable for $X \in \mathbb{F}_q$ if and only if $e/s$ is even and $\alpha^{(q-1)/(p^s+1)} = (-1)^{m/s}$. In such cases, there are $p^{2s} - 1$ non-zero solutions.

There is the fact that $\alpha^p X^{p^2 l} + \alpha X$ is a permutation polynomial over $\mathbb{F}_q$ with $q = p^e$ if and only if $e/s$ is odd or $e/s$ is even with $e = 2m$ and $\alpha^{(q-1)/(p^s+1)} \neq (-1)^{m/s}$. 


Lemma 7 (16) Let \( f(X) = X^{p^2} + X \) and
\[
S = \{ \beta \in \mathbb{F}_q : f(X) = -\beta^{p^2} \text{ is solvable in } \mathbb{F}_q \}.
\]
If \( m/s \equiv 0 \mod 2 \), then \(|S| = p^{e-2s}\).

Lemma 8 (17) Let \( p \) an odd prime. \( \forall x \in \mathbb{F}_p^* \), the quadratic Gaussian period over \( \mathbb{F}_p \) is given by
\[
\rho_i^{(2,p)} = \sum_{x \in C_i^{(2,p)}} \chi_1(x), i = 0, 1.
\]
The value of the quadratic Gaussian period is
\[
\rho_0^{(2,p)} = \frac{-1 + \sqrt{p^*}}{2} = \begin{cases} 
\frac{-1+\sqrt{p}}{2}, & \text{if } p \equiv 1 \mod 4, \\
\frac{-1-\sqrt{-p}}{2}, & \text{if } p \equiv 3 \mod 4,
\end{cases}
\]
and
\[
\rho_1^{(2,p)} = -1 - \rho_0^{(2,p)}.
\]
Obviously,
\[
2\rho_0^{(2,p)} + 1 = G(\eta) = G.
\]

Lemma 9 (5) Let \( p \) an odd prime, \( q = p^e \). \( \eta' \) and \( \eta \) is the quadratic character over \( \mathbb{F}_q^* \) and \( \mathbb{F}_p^* \), respectively. \( \forall y \in \mathbb{F}_p^* \), then
\[
\eta'(y) = \begin{cases} 
\eta(y), & 2 \nmid e, \\
1, & 2 \mid e.
\end{cases}
\]

Lemma 10 (18) Let \( p \) be an odd prime and \( c \) be an integer not divisible by \( p \). Let \( \epsilon_1 = \pm 1 \) and \( \epsilon_2 = \pm 1 \). Then the number of integers \( n \) such that \( 0 \leq n \leq p - 1 \) and
\[
\left( \frac{n}{p} \right) = \epsilon_1, \left( \frac{n+c}{p} \right) = \epsilon_2
\]
is
\[
\frac{1}{4} \left[ p - 2 - \epsilon_1 \left( \frac{-c}{p} \right) - \epsilon_2 \left( \frac{c}{p} \right) - \epsilon_1 \epsilon_2 \right].
\]
4 The proofs of the main results

For \( i = 0, 1 \), the lengths of \( C_D(i = 0, 1) \) are \( n_i = |D_i| \), then

\[
\begin{align*}
n_i & = |\{ x \in \mathbb{F}_q : \text{Tr}(x^{p+1} - x) \in C_i^{(2,p)} \}| \\
& = \sum_{c \in C_i^{(2,p)}} |\{ x \in \mathbb{F}_q : \text{Tr}(x^{p+1} - x) = c \}| \\
& = \sum_{c \in C_i^{(2,p)}} \sum_{x \in \mathbb{F}_q} \frac{1}{p} \sum_{y \in \mathbb{F}_p} \zeta_p^{y(\text{Tr}(x^{p+1} - x) - c)} \\
& = \frac{1}{p} \sum_{c \in C_i^{(2,p)}} \sum_{x \in \mathbb{F}_q} \left(1 + \sum_{y \in \mathbb{F}_p} \zeta_p^{y(\text{Tr}(x^{p+1} - x) - c)} \right) \\
& = \frac{p-1}{2} p^{e-1} + \frac{1}{p} \delta_{1,i}, \tag{3}
\end{align*}
\]

where

\[
\delta_{1,i} = \sum_{c \in C_i^{(2,p)}} \sum_{y \in \mathbb{F}_p^*} \sum_{x \in \mathbb{F}_q} \zeta_p^{y \text{Tr}(xy^{p+1} - yx)}. \tag{4}
\]

For any \( a \in \mathbb{F}_q \) and any codeword \( c(a) \in C_D(i) \), to determine the weight enumerators of \( C_D(i = 0, 1) \), let

\[
T_i = |\{ x \in \mathbb{F}_q : \text{Tr}(x^{p+1} - x) \in C_i^{(2,p)}, \text{Tr}(ax) = 0 \}| \tag{5}
\]

then we can deduce

\[
wt_i(c(a)) = n_i - T_i, \tag{6}
\]

where \( wt_i(c(a))(i, 0) \) is Hamming weight of the codeword \( c(a) \).

For \( a \in \mathbb{F}_q^* \), by the orthogonal property of additive character, we have

\[
\begin{align*}
T_i & = \sum_{c \in C_i^{(2,p)}} \sum_{x \in \mathbb{F}_q} \left(\frac{1}{p} \sum_{y \in \mathbb{F}_p} \zeta_p^{y(\text{Tr}(x^{p+1} - x) - c)} \right) \left(\frac{1}{p} \sum_{z \in \mathbb{F}_p} \zeta_p^{z \text{Tr}(ax)} \right) \\
& = \frac{1}{p^2} \sum_{c \in C_i^{(2,p)}} \sum_{x \in \mathbb{F}_q} \left(1 + \sum_{y \in \mathbb{F}_p} \zeta_p^{y(\text{Tr}(x^{p+1} - x) - c)} \right) \left(1 + \sum_{z \in \mathbb{F}_p} \zeta_p^{z \text{Tr}(ax)} \right) \\
& = \frac{p-1}{2} p^{e-2} + \frac{1}{p^2} (\delta_{1,i} + \delta_{2,i} + \delta_{3,i}), \tag{7}
\end{align*}
\]

where

\[
\begin{align*}
\delta_{2,i} & = \sum_{c \in C_i^{(2,p)}} \sum_{z \in \mathbb{F}_p} \sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}(axx)} \\
\delta_{3,i} & = \sum_{c \in C_i^{(2,p)}} \sum_{y \in \mathbb{F}_p^*} \sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}(yxy^{p+1} + (az-y)x)}. \tag{8}
\end{align*}
\]

By the orthogonal property of additive character, we have \( \delta_{2,i} = 0 \).
Then
\[ wt_i(c(a)) = \frac{(p - 1)^2}{2} - p^{e-2} + \frac{1}{p} \delta_{1,i} - \frac{1}{p^2} (\delta_{1,i} + \delta_{3,i}). \] (9)

The following Lemmas are essential to determine the lengths and weight distributions of \( C_{D_i} (i = 0, 1) \).

**Lemma 11** For \( i = 0, 1 \),

\[ \delta_{1,i} = \begin{cases} (-1)^i \eta(-1) \frac{p-1}{2} G^e + 1, & 2 \nmid p \mid e, \\ \frac{(-1)^i + \eta(e)}{2} \eta(-1) G^e + 1, & 2 \nmid e, p \nmid e, \\ \frac{\varepsilon-1}{2} \frac{p^2}{e}, & 2 \nmid e, e \equiv 2 \text{ mod } 4, p \nmid e, \\ \frac{\varepsilon-1}{2} \frac{p^2}{e+1}, & 2 \nmid e, e \equiv 0 \text{ mod } 4, p \nmid e. \end{cases} \] (10)

**Proof:** With Weil sums, we have

\[ \delta_{1,i} = \sum_{c \in C_{1, (2,p)}} \sum_{y \in \mathbb{F}_p^*} \zeta_p^{-cy} \sum_{x \in \mathbb{F}_q} \zeta_p^{-\operatorname{Tr}(yx^{p+1} - yx)} \]

\[ = \sum_{c \in C_{1, (2,p)}} \sum_{y \in \mathbb{F}_p^*} \zeta_p^{-cy} S(y, -y). \] (11)

Note that \( x = \frac{1}{2} \) is the unique solution of the equation \( y^p x^{p^2} + yx = -(y)^p \) in \( \mathbb{F}_q \).

In fact, \( y \in \mathbb{F}_p^* : y^p = y, (-1)^p = -1 \), the equation \( y^p x^{p^2} + yx = -(y)^p \) is equivalent to the equation \( x^{p^2} + x = 1 \), thus \( \frac{1}{2} \in \mathbb{F}_p^* \subset \mathbb{F}_q^* \).

From Lemma 3, Lemma 5 and Lemma 9

\[ S(y, -y) = \begin{cases} G^e \eta(y) \zeta_p^{-c/y}, & 2 \nmid e, \\ -p^e \zeta_p^{-c/y}, & 2 \mid e, e \equiv 2 \text{ mod } 4, \\ -p^e+1 \zeta_p^{-c/y}, & 2 \mid e, e \equiv 0 \text{ mod } 4. \end{cases} \] (12)

Applying these values into Eq. (11), we have

\[ \delta_{1,i} = \begin{cases} G^e \sum_{y \in \mathbb{F}_p^*} \zeta_p^{-c/y} \eta(y) \sum_{c \in C_{1, (2,p)}} \zeta_p^{-cy}, & 2 \nmid e, \\ -p^e \sum_{y \in \mathbb{F}_p^*} \zeta_p^{-c/y} \sum_{c \in C_{1, (2,p)}} \zeta_p^{-cy}, & 2 \mid e, e \equiv 2 \text{ mod } 4, \\ -p^e+1 \sum_{y \in \mathbb{F}_p^*} \zeta_p^{-c/y} \sum_{c \in C_{1, (2,p)}} \zeta_p^{-cy}, & 2 \mid e, e \equiv 0 \text{ mod } 4. \end{cases} \] (13)
With the definition of Gaussian period, we have

\[
G^e \eta(-1)( \sum_{-y \in C_{1-i}^{(2,p)}} \eta(-y) \rho_0 + \sum_{-y \in C_{1-i}^{(2,p)}} \eta(-y) \rho_1), \\
G^e \eta(-1)( \sum_{-y \in C_{1-i}^{(2,p)}} \zeta_p^{-\frac{e}{2}} \eta(-y) \rho_0 + \sum_{-y \in C_{1-i}^{(2,p)}} \zeta_p^{-\frac{e}{2}} \eta(-y) \rho_1),
\]

\[
2 \mid e, p \mid e,
\]

\[
\delta_{1,i} = \begin{cases} 
G^e \eta(-1)( \sum_{-y \in C_{1-i}^{(2,p)}} \eta(-y) \rho_0 + \sum_{-y \in C_{1-i}^{(2,p)}} \eta(-y) \rho_1), & 2 \mid e, p \mid e, \\
G^e \eta(-1)( \sum_{-y \in C_{1-i}^{(2,p)}} \zeta_p^{-\frac{e}{2}} \eta(-y) \rho_0 + \sum_{-y \in C_{1-i}^{(2,p)}} \zeta_p^{-\frac{e}{2}} \eta(-y) \rho_1), & 2 \mid e, p \mid e,
\end{cases}
\]

By the orthogonal property of additive character, we have

\[
\delta_{1,i} = \begin{cases} 
(\eta(-1)G^e)^{\frac{p-1}{2}}(\rho_0 - \rho_1), & 2 \mid e, p \mid e, \\
(\eta(-1)G^e)^{\frac{p-1}{2}}(\rho_0^2 - \rho_1^2), & 2 \mid e, e \in C_{1-i}^{(2,p)}, \\
(\eta(-1)G^e)^{\frac{p-1}{2}}(\rho_1 \rho_0 - \rho_0 \rho_1), & 2 \mid e, e \in C_{1-i}^{(2,p)}, \\
(\eta(-1)G^e)^{\frac{p-1}{2}}(\rho_0^2 + \rho_1^2), & 2 \mid e, e \in C_{1-i}^{(2,p)}, \\
(\eta(-1)G^e)^{\frac{p-1}{2}}(2 \rho_0 \rho_1), & 2 \mid e, e \in C_{1-i}^{(2,p)}, \\
(\eta(-1)G^e)^{\frac{p-1}{2}}(p_0^2 + \rho_1^2), & 2 \mid e, e \in C_{1-i}^{(2,p)}, \\
(\eta(-1)G^e)^{\frac{p-1}{2}}(2 \rho_0 \rho_1), & 2 \mid e, e \in C_{1-i}^{(2,p)}.
\end{cases}
\]

With Lemma \( \Box \)

\[
\delta_{1,i} = \begin{cases} 
(\eta(-1)^i(\eta(-1))^{\frac{\epsilon+i}{2}}(\rho_0 - \rho_1), & 2 \mid e, p \mid e, \\
(\eta(-1)^i(\eta(-1))^{\frac{\epsilon+i}{2}}(\rho_0^2 - \rho_1^2), & 2 \mid e, e \in C_{1-i}^{(2,p)}, \\
(\eta(-1)^i(\eta(-1))^{\frac{\epsilon+i}{2}}(\rho_1 \rho_0 - \rho_0 \rho_1), & 2 \mid e, e \in C_{1-i}^{(2,p)}, \\
(\eta(-1)^i(\eta(-1))^{\frac{\epsilon+i}{2}}(\rho_0^2 + \rho_1^2), & 2 \mid e, e \in C_{1-i}^{(2,p)}, \\
(\eta(-1)^i(\eta(-1))^{\frac{\epsilon+i}{2}}(2 \rho_0 \rho_1), & 2 \mid e, e \in C_{1-i}^{(2,p)}, \\
(\eta(-1)^i(\eta(-1))^{\frac{\epsilon+i}{2}}(p_0^2 + \rho_1^2), & 2 \mid e, e \in C_{1-i}^{(2,p)}, \\
(\eta(-1)^i(\eta(-1))^{\frac{\epsilon+i}{2}}(2 \rho_0 \rho_1), & 2 \mid e, e \in C_{1-i}^{(2,p)}.
\end{cases}
\]

(14)

After simplification, we can get the desired results. \( \square \)
Lemma 12 The length of the code $C_i$ ($i = 0, 1$) is

$$n_i = \begin{cases} 
\frac{p-1}{2}p^{e-1} + (-1)^i \frac{p-1}{2} G^{e-1}, & 2 \nmid e, p \mid e, \\
\frac{p-1}{2}p^{e-1} - \frac{(-1)^i + n(e)}{2} G^{e-1}, & 2 \nmid e, p \mid e, \\
\frac{p-1}{2}p^{e-1} + \frac{p-1}{2} p^\frac{e}{2} - 1, & 2 \mid e, e \equiv 2 \text{ mod } 4, p \mid e, \\
\frac{p-1}{2}p^{e-1} - \frac{(-1)^i n(-e)p+1}{2} p^\frac{e}{2} - 1, & 2 \mid e, e \equiv 2 \text{ mod } 4, p \nmid e, \\
\frac{p-1}{2}p^{e-1} + \frac{p-1}{2} p^2, & 2 \mid e, e \equiv 0 \text{ mod } 4, p \mid e, \\
\frac{p-1}{2}p^{e-1} - \frac{(-1)^i n(-e)p+1}{2} p^2, & 2 \mid e, e \equiv 0 \text{ mod } 4, p \nmid e. 
\end{cases} \tag{15}$$

Proof: With Lemma 11 and Eq. (3), we can easily get the results. □

Lemma 13 For $i = 0, 1$,

1. if $x = x_a$ is the solution of the equation $x^{p^2} + x = -a^p$ in $\mathbb{F}_q$, $\delta_{1,i} + \delta_{3,i}$ is given by the following four cases.

   (1) if $\text{Tr}(x_a^{p+1}) = 0$, $\text{Tr}(x_a) = 0$,

   $$\delta_{1,i} + \delta_{3,i} = p\delta_{1,i}, \tag{16}$$

   (2) if $\text{Tr}(x_a^{p+1}) = 0$, $\text{Tr}(x_a) \neq 0$,

   $$\delta_{1,i} + \delta_{3,i} = 0, \tag{17}$$

   (3) if $\text{Tr}(x_a^{p+1}) \neq 0$, $\text{Tr}(x_a) = 0$,

   $$\delta_{1,i} + \delta_{3,i} = \begin{cases} 
-\eta \left( -\text{Tr}(x_a^{p+1}) \right) \frac{p-1}{2} G^{e+1}, & 2 \nmid e, p \mid e, \\
\eta \left( -\text{Tr}(x_a^{p+1}) \right) \frac{(-1)^i + n(e)}{2} G^{e+1}, & 2 \nmid e, p \nmid e, \\
-(-1)^i \eta \left( -\text{Tr}(x_a^{p+1}) \right) \frac{p-1}{2} p^\frac{e}{2} + 1, & 2 \mid e, e \equiv 2 \text{ mod } 4, p \mid e, \\
(-1)^i + n(e) \eta \left( -\text{Tr}(x_a^{p+1}) \right) p^{\frac{e}{2} + 1}, & 2 \mid e, e \equiv 2 \text{ mod } 4, p \nmid e, \\
-(-1)^i \eta \left( -\text{Tr}(x_a^{p+1}) \right) \frac{p-1}{2} p^{\frac{e}{2} + 2}, & 2 \mid e, e \equiv 0 \text{ mod } 4, p \mid e, \\
(-1)^i + n(e) \eta \left( -\text{Tr}(x_a^{p+1}) \right) p^{\frac{e}{2} + 2}, & 2 \mid e, e \equiv 0 \text{ mod } 4, p \nmid e, 
\end{cases} \tag{18}$$
\[ \delta_{1,i} + \delta_{3,i} = \begin{cases} 
\eta(-1)(-p + \eta(-1))p^{2}G^{e+1}, & 2 \nmid e, p \mid e, 
\eta(-1)(-p + \eta(-1))p^{2}G^{e+1}, & 2 \nmid e, p \nmid e, e^2 \text{ mod } 4, \text{ or } 2 \mid p, e^2 \text{ mod } 4, p \mid e, 
\eta(-1)(-p + \eta(-1))p^{2}G^{e+1}, & 2 \mid e, e \equiv 2 \text{ mod } 4, p \mid e, 
\eta(-1)(-p + \eta(-1))p^{2}G^{e+1}, & 2 \mid e, e \equiv 0 \text{ mod } 4, p \mid e, 
\eta(-1)(-p + \eta(-1))p^{2}G^{e+1}, & 2 \mid e, e \equiv 0 \text{ mod } 4, p \nmid e, e^2 \text{ mod } 4, p \nmid e, 
\eta(-1)(-p + \eta(-1))p^{2}G^{e+1}, & 2 \mid e, e \equiv 2 \text{ mod } 4, p \nmid e, e^2 \text{ mod } 4, p \nmid e, 
\eta(-1)(-p + \eta(-1))p^{2}G^{e+1}, & 2 \mid e, e \equiv 0 \text{ mod } 4, p \nmid e, e^2 \text{ mod } 4, p \nmid e, 
\eta(-1)(-p + \eta(-1))p^{2}G^{e+1}, & 2 \mid e, e \equiv 0 \text{ mod } 4, p \nmid e, e^2 \text{ mod } 4, p \nmid e, 
\end{cases} \]

(4) if \( \text{Tr}(x_{a}^{p+1}) \neq 0, \text{Tr}(x_{a}) \neq 0, \)

2. if \( 2 \mid e \) and \( e \equiv 0 \text{ mod } 4, \) there are no solutions of the equation \( x^{p^2} + x = -a^p \)
in \( \mathbb{F}_q, \) then \( \delta_{3} = 0. \)

**Proof:** With the definition of Weil sums, we have

\[ \delta_{3,i} = \sum_{c_e \in C^{(2,p)}_i} \sum_{y \in \mathbb{F}_p} \zeta_{y}^{-cy} \sum_{z \in \mathbb{F}_p} \sum_{x \in \mathbb{F}_q} \zeta_{y}^{-\text{Tr}(yx^{p+1} + (az-y)x)} = \sum_{c_e \in C^{(2,p)}_i} \sum_{y \in \mathbb{F}_p} \zeta_{y}^{-cy} \sum_{z \in \mathbb{F}_p} S(y, az-y). \]

if \( 2 \nmid e, \) or \( 2 \mid e, \) and \( e \equiv 2 \text{ mod } 4, f(x) = y^{p}x^{p^2} + yx(y \in \mathbb{F}_p) \) and \( g(x) = x^{p^2} + x \)
are all permutation polynomials over \( \mathbb{F}_q, \) and \( x = x_{a} \) is the unique solution of the equation \( x^{p^2} + x = -a^p \) in \( \mathbb{F}_q, \) then \( x = y^{-1}zx_{a} + \frac{1}{2} \) is the unique solution of the equation \( y^{p}x^{p^2} + yx = -(az-y)^p \) in \( \mathbb{F}_q. \)

if \( 2 \mid e \) and \( e \equiv 0 \text{ mod } 4, f(x) = y^{p}x^{p^2} + yx(y \in \mathbb{F}_p) \) and \( g(x) = x^{p^2} + x \) is not permutation polynomials in \( \mathbb{F}_q. \) There are the following two cases:

(I) if \( x = x_{a} \) is the solution of the equation \( x^{p^2} + x = -a^p \) in \( \mathbb{F}_q, \) then \( x = y^{-1}zx_{a} + \frac{1}{2} \)
is the solution of the equation \( y^{p}x^{p^2} + yx = -(az-y)^p \) in \( \mathbb{F}_q. \)

(II) if there are no solutions of the equation \( x^{p^2} + x = -a^p \) in \( \mathbb{F}_q, \) then the equation \( y^{p}x^{p^2} + yx = -(az-y)^p \) has no solutions in \( \mathbb{F}_q, \) thus \( S(y, az-y) = 0, \) we can deduce \( \delta_{3,i} = 0, \) and finish the second part of the proof.
Next, we only need to proof part (I).

With Lemma 3, Lemma 5 and Lemma 9, we have

\[
S(y, az - y) = \begin{cases} 
G^e\eta(y)\zeta_p^{\operatorname{Tr}(y(-y^{-1}xz_a + \frac{1}{2})^{p+1})}, & 2 \nmid e, \\
-p^2\zeta_p^{\operatorname{Tr}(y(-y^{-1}xz_a + \frac{1}{2})^{p+1})}, & 2 \mid e, e \equiv 2 \mod 4, \\
-p^{p+1}\zeta_p^{\operatorname{Tr}(y(-y^{-1}xz_a + \frac{1}{2})^{p+1})}, & 2 \mid e, e \equiv 0 \mod 4,
\end{cases}
\tag{21}
\]

then

\[
S(y, az - y) = \begin{cases} 
G^e\eta(y)\zeta_p^{-\frac{y}{2}}\zeta_p^{\frac{-\operatorname{Tr}(x^{p+1})}{y}z^2 - \operatorname{Tr}(x_a)z}, & 2 \nmid e, \\
-p^2\zeta_p^{-\frac{y}{2}}\zeta_p^{\frac{-\operatorname{Tr}(x^{p+1})}{y}z^2 - \operatorname{Tr}(x_a)z}, & 2 \mid e, e \equiv 2 \mod 4, \\
-p^{p+1}\zeta_p^{-\frac{y}{2}}\zeta_p^{\frac{-\operatorname{Tr}(x^{p+1})}{y}z^2 - \operatorname{Tr}(x_a)z}, & 2 \mid e, e \equiv 0 \mod 4,
\end{cases}
\tag{22}
\]

Applying the values of Eq. (22) into Eq. (20), we have

\[
\delta_{3,i} = \begin{cases} 
G^e\sum_{y \in \mathbb{F}_p^*} \eta(y)\zeta_p^{-\frac{y}{2}}\sum_{c \in C^{(i)}_2} \zeta_p^{-cy}\sum_{z \in \mathbb{F}_p^*} \zeta_p^{\frac{-\operatorname{Tr}(x^{p+1})}{y}z^2 - \operatorname{Tr}(x_a)z}, & 2 \nmid e, \\
-p^2\sum_{y \in \mathbb{F}_p^*} \zeta_p^{-\frac{y}{2}}\sum_{c \in C^{(i)}_2} \zeta_p^{-cy}\sum_{z \in \mathbb{F}_p^*} \zeta_p^{\frac{-\operatorname{Tr}(x^{p+1})}{y}z^2 - \operatorname{Tr}(x_a)z}, & 2 \mid e, e \equiv 2 \mod 4, \\
-p^{p+1}\sum_{y \in \mathbb{F}_p^*} \zeta_p^{-\frac{y}{2}}\sum_{c \in C^{(i)}_2} \zeta_p^{-cy}\sum_{z \in \mathbb{F}_p^*} \zeta_p^{\frac{-\operatorname{Tr}(x^{p+1})}{y}z^2 - \operatorname{Tr}(x_a)z}, & 2 \mid e, e \equiv 0 \mod 4,
\end{cases}
\]

With Eq. (13),

\[
\delta_{1,i} + \delta_{3,i} = \begin{cases} 
G^e\sum_{y \in \mathbb{F}_p^*} \eta(y)\zeta_p^{-\frac{y}{2}}\sum_{c \in C^{(i)}_2} \zeta_p^{-cy}\sum_{z \in \mathbb{F}_p^*} \zeta_p^{\frac{-\operatorname{Tr}(x^{p+1})}{y}z^2 - \operatorname{Tr}(x_a)z}, & 2 \nmid e, \\
-p^2\sum_{y \in \mathbb{F}_p^*} \zeta_p^{-\frac{y}{2}}\sum_{c \in C^{(i)}_2} \zeta_p^{-cy}\sum_{z \in \mathbb{F}_p^*} \zeta_p^{\frac{-\operatorname{Tr}(x^{p+1})}{y}z^2 - \operatorname{Tr}(x_a)z}, & 2 \mid e, e \equiv 2 \mod 4, \\
-p^{p+1}\sum_{y \in \mathbb{F}_p^*} \zeta_p^{-\frac{y}{2}}\sum_{c \in C^{(i)}_2} \zeta_p^{-cy}\sum_{z \in \mathbb{F}_p^*} \zeta_p^{\frac{-\operatorname{Tr}(x^{p+1})}{y}z^2 - \operatorname{Tr}(x_a)z}, & 2 \mid e, e \equiv 0 \mod 4,
\end{cases}
\tag{23}
\]

(1) if \(\operatorname{Tr}(x^{p+1}) = 0\), \(\operatorname{Tr}(x_a) = 0\), by the orthogonal property of additive character, with Eq. (13), we have

\[
\delta_{1,i} + \delta_{3,i} = p\delta_{1,i}.
\]

(2) if \(\operatorname{Tr}(x^{p+1}) = 0\), \(\operatorname{Tr}(x_a) \neq 0\), by the orthogonal property of additive character, we have

\[
\delta_{1,i} + \delta_{3,i} = 0.
\]
(3) if Tr($x_{a}^{p+1}$) \( \neq 0 \), Tr($x_{a}$) = 0, with Lemma \[\text{we have}\]

\[
\delta_{1,i} + \delta_{3,i} = \begin{cases}
\eta(-\text{Tr}(x_{a}^{p+1})) G^{e+1} \sum_{y \in \mathbb{F}_{p}} \zeta_{p}^{-\frac{y}{2}} \left( \sum_{c \in C_{i}^{(2,p)}} \zeta_{p}^{-c y} \right), & 2 \nmid e, p | e, \\
-\eta(-\text{Tr}(x_{a}^{p+1})) Gp^{e} \sum_{y \in \mathbb{F}_{p}} \zeta_{p}^{-\frac{y}{2}} \eta(-y) \left( \sum_{c \in C_{i}^{(2,p)}} \zeta_{p}^{-c y} \right), & 2 | e, e \equiv 2 \mod 4, \\
-\eta(-\text{Tr}(x_{a}^{p+1})) Gp^{e+1} \sum_{y \in \mathbb{F}_{p}} \zeta_{p}^{-\frac{y}{2}} \eta(-y) \left( \sum_{c \in C_{i}^{(2,p)}} \zeta_{p}^{-c y} \right), & 2 | e, e \equiv 0 \mod 4, 
\end{cases}
\]

if $p | e$, $\zeta_{p}^{-\frac{y}{2}} \equiv 1$. With the definition of the quadratic Gaussian period, we have

\[
\delta_{1,i} + \delta_{3,i} = \begin{cases}
\eta(-\text{Tr}(x_{a}^{p+1})) G^{e+1} \sum_{y \in \mathbb{F}_{p}} \left( \sum_{c \in C_{i}^{(2,p)}} \zeta_{p}^{-c y} \right), & 2 \nmid e, p | e, \\
\eta(-\text{Tr}(x_{a}^{p+1})) G^{e+1} \left( \sum_{y \in \mathbb{F}_{p}} \zeta_{p}^{-\frac{y}{2}} \rho_{0} + \sum_{y \in \mathbb{F}_{p}} \zeta_{p}^{-\frac{y}{2}} \rho_{1} \right), & 2 | e, p \nmid e, \\
-\eta(-\text{Tr}(x_{a}^{p+1})) Gp^{e} \left( \sum_{y \in \mathbb{F}_{p}} \eta(-y) \rho_{0} + \sum_{y \in \mathbb{F}_{p}} \eta(-y) \rho_{1} \right), & 2 | e, e \equiv 2 \mod 4, p \nmid e, \\
-\eta(-\text{Tr}(x_{a}^{p+1})) Gp^{e+1} \left( \sum_{y \in \mathbb{F}_{p}} \eta(-y) \rho_{0} + \sum_{y \in \mathbb{F}_{p}} \eta(-y) \rho_{1} \right), & 2 | e, e \equiv 0 \mod 4, p \nmid e, \\
-\eta(-\text{Tr}(x_{a}^{p+1})) Gp^{e+1} \left( \sum_{y \in \mathbb{F}_{p}} \eta(-y) \rho_{0} + \sum_{y \in \mathbb{F}_{p}} \eta(-y) \rho_{1} \right), & 2 | e, e \equiv 0 \mod 4, p \nmid e, 
\end{cases}
\]

by the orthogonal property of additive character,

\[
\delta_{1,i} + \delta_{3,i} = \begin{cases}
-\eta(-\text{Tr}(x_{a}^{p+1})) \frac{p-1}{2} G^{e+1}, & 2 \nmid e, p | e, \\
\eta(-\text{Tr}(x_{a}^{p+1})) G^{e+1}(\rho_{0}^{2} + \rho_{1}^{2}), & 2 \nmid e, e \in C_{i}^{1(2,p)}, \\
\eta(-\text{Tr}(x_{a}^{p+1})) G^{e+1}(2\rho_{0}\rho_{1}), & 2 \nmid e, e \in C_{i}^{1(2,p)}, \\
-(1)^{i} \eta(-\text{Tr}(x_{a}^{p+1})) \frac{p-1}{2} Gp^{e} (\rho_{0} - \rho_{1}), & 2 | e, e \equiv 2 \mod 4, p | e, \\
-(1)^{i} \eta(-\text{Tr}(x_{a}^{p+1})) Gp^{e} (\rho_{0}^{2} - \rho_{1}^{2}), & 2 | e, e \equiv 2 \mod 4, e \in C_{i}^{1(2,p)}, \\
-(1)^{i} \eta(-\text{Tr}(x_{a}^{p+1})) Gp^{e+1}(\rho_{0} - \rho_{1}), & 2 | e, e \equiv 0 \mod 4, p | e, \\
-(1)^{i} \eta(-\text{Tr}(x_{a}^{p+1})) Gp^{e+1}(\rho_{0}^{2} - \rho_{1}^{2}), & 2 | e, e \equiv 0 \mod 4, e \in C_{i}^{1(2,p)}, \\
-(1)^{i} \eta(-\text{Tr}(x_{a}^{p+1})) Gp^{e+1}(\rho_{1}\rho_{0} - \rho_{0}\rho_{1}), & 2 | e, e \equiv 0 \mod 4, e \in C_{i}^{1(2,p)}, \\
-(1)^{i} \eta(-\text{Tr}(x_{a}^{p+1})) Gp^{e+1}(\rho_{1}\rho_{0} - \rho_{0}\rho_{1}), & 2 | e, e \equiv 0 \mod 4, e \in C_{i}^{1(2,p)}, 
\end{cases}
\]
with Lemma 8, we have

\[
\delta_{1,i} + \delta_{3,i} = \begin{cases} 
-\eta(-\text{Tr}(x_{a}^{p+1}))^p \frac{1}{2} G^{e+1}, & 2 \nmid e, p \mid e, \\
\eta(-\text{Tr}(x_{a}^{p+1}))(1+\eta(-1)p) G^{e+1}, & 2 \nmid e, e \in C_{1}^{(2,p)}, \\
\eta(-\text{Tr}(x_{a}^{p+1}))(1-\eta(-1)p) G^{e+1}, & 2 \nmid e, e \in C_{1}^{(2,p)}, \\
-(1)^{i} \eta(-\text{Tr}(x_{a}^{p+1})) \frac{p-1}{2} p^{\frac{s-1}{2}+1}, & 2 \mid e, e \equiv 2 \text{ mod } 4, p \mid e, \\
(1)^{i} \eta(-\text{Tr}(x_{a}^{p+1})) p^{\frac{s}{2}+1}, & 2 \mid e, e \equiv 2 \text{ mod } 4, e \in C_{2}^{(2,p)}, \\
0, & 2 \mid e, e \equiv 2 \text{ mod } 4, e \in C_{1}^{(2,p)}, \\
-(1)^{i} \eta(-\text{Tr}(x_{a}^{p+1})) \frac{p-1}{2} p^{\frac{s}{2}+2}, & 2 \mid e, e \equiv 0 \text{ mod } 4, p \mid e, \\
(1)^{i} \eta(-\text{Tr}(x_{a}^{p+1})) p^{\frac{s}{2}+2}, & 2 \mid e, e \equiv 0 \text{ mod } 4, e \in C_{2}^{(2,p)}, \\
0, & 2 \mid e, e \equiv 0 \text{ mod } 4, e \in C_{1}^{(2,p)}, 
\end{cases}
\]

After simplification, we can get Eq. (13).

(4) if \( \text{Tr}(x_{a}^{p+1}) \neq 0, \text{Tr}(x_{a}) \neq 0 \), With Lemma III we have

\[
\delta_{1,i} + \delta_{3,i} = \begin{cases} 
\eta(-\text{Tr}(x_{a}^{p+1})) G^{e+1} \sum_{y \in \mathbb{F}_{p}} \zeta_{p}^{c \text{Tr}(x_{a}^{p+1})-\text{Tr}(x_{a})^{2}}(-y) \sum_{c \in C_{1}^{(2,p)}} \zeta_{p}^{-cy}, & 2 \nmid e, \\
-\eta(\text{Tr}(x_{a}^{p+1})) G^{\frac{s}{2}+1} \sum_{y \in \mathbb{F}_{p}} \zeta_{p}^{c \text{Tr}(x_{a}^{p+1})-\text{Tr}(x_{a})^{2}}(-y) \eta(-y) \sum_{c \in C_{1}^{(2,p)}} \zeta_{p}^{-cy}, & 2 \mid e, e \equiv 2 \text{ mod } 4, \\
-\eta(\text{Tr}(x_{a}^{p+1})) G^{\frac{s}{2}+1} \sum_{y \in \mathbb{F}_{p}} \zeta_{p}^{c \text{Tr}(x_{a}^{p+1})-\text{Tr}(x_{a})^{2}}(-y) \eta(-y) \sum_{c \in C_{1}^{(2,p)}} \zeta_{p}^{-cy}, & 2 \mid e, e \equiv 0 \text{ mod } 4, 
\end{cases}
\]

With the similar discussion of Lemma III and part (3) in this proof, we can get

\[
\delta_{1,i} + \delta_{3,i} = \begin{cases} 
(1)^{i} \frac{1+\eta(-1)p}{2} G^{e+1}, & 2 \nmid e, p \mid e, (-\text{Tr}(x_{a}^{p+1})) \in C_{1}^{(2,p)}, \\
-(1)^{i} \frac{1-\eta(-1)p}{2} G^{e+1}, & 2 \nmid e, p \mid e, (-\text{Tr}(x_{a}^{p+1})) \in C_{1}^{(2,p)}, \\
-\eta(-\text{Tr}(x_{a}^{p+1})) \frac{p-1}{2} G^{e+1}, & 2 \mid e, p \mid e, e \text{Tr}(x_{a}^{p+1}) = \text{Tr}(x_{a})^{2}, \\
\eta(-\text{Tr}(x_{a}^{p+1}))(1+\eta(-1)p) G^{e+1}, & 2 \mid e, p \mid e, \frac{\text{Tr}(x_{a}^{p+1})-\text{Tr}(x_{a})^{2}}{\text{Tr}(x_{a}^{p+1})} \in C_{2}^{(2,p)}, \\
\eta(-\text{Tr}(x_{a}^{p+1}))(1-\eta(-1)p) G^{e+1}, & 2 \mid e, p \mid e, \frac{\text{Tr}(x_{a}^{p+1})-\text{Tr}(x_{a})^{2}}{\text{Tr}(x_{a}^{p+1})} \in C_{1}^{(2,p)}, \\
(p^{\frac{s-1}{2}+1}, & 2 \mid e, e \equiv 2 \text{ mod } 4, p \mid e, (-\text{Tr}(x_{a}^{p+1})) \in C_{1}^{(2,p)}, \\
0, & 2 \mid e, e \equiv 2 \text{ mod } 4, p \mid e, (-\text{Tr}(x_{a}^{p+1})) \in C_{1}^{(2,p)}, \\
-(1)^{i} \eta(-\text{Tr}(x_{a}^{p+1})) \frac{p-1}{2} p^{\frac{s}{2}+1}, & 2 \mid e, e \equiv 2 \text{ mod } 4, p \mid e, e \text{Tr}(x_{a}^{p+1}) = \text{Tr}(x_{a})^{2}, \\
(1)^{i} \eta(-\text{Tr}(x_{a}^{p+1})) p^{\frac{s}{2}+1}, & 2 \mid e, e \equiv 2 \text{ mod } 4, p \mid e, \frac{\text{Tr}(x_{a}^{p+1})-\text{Tr}(x_{a})^{2}}{\text{Tr}(x_{a}^{p+1})} \in C_{2}^{(2,p)}, \\
0, & 2 \mid e, e \equiv 2 \text{ mod } 4, p \mid e, \frac{\text{Tr}(x_{a}^{p+1})-\text{Tr}(x_{a})^{2}}{\text{Tr}(x_{a}^{p+1})} \in C_{1}^{(2,p)}, 
\end{cases}
\]
Lemma 14 \( (24) \) Let

\[
\delta_{1,i} + \delta_{3,i} = \begin{cases} 
  p^{\frac{e}{2}+2}, & 2 \mid e, e \equiv 0 \mod 4, p \mid e, (-\text{Tr}(x_a^{p+1})) \in C_i^{(2,p)}, \\
  0, & 2 \mid e, e \equiv 0 \mod 4, p \mid e, (-\text{Tr}(x_a^{p+1})) \in C_{i-1}^{(2,p)}, \\
  -(-1)^i \eta (-\text{Tr}(x_a^{p+1})) \frac{p-1}{2} p^{\frac{e}{2}+2}, & 2 \mid e, e \equiv 0 \mod 4, p \nmid e, e \text{Tr}(x_a^{p+1}) = \text{Tr}(x_a)^2, \\
  (-1)^i \eta (-\text{Tr}(x_a^{p+1})) p^{\frac{e}{2}+2}, & 2 \mid e, e \equiv 0 \mod 4, p \nmid e, \frac{e \text{Tr}(x_a^{p+1}) - \text{Tr}(x_a)^2}{\text{Tr}(x_a^{p+1})} \in C_i^{(2,p)}, \\
  0, & 2 \mid e, e \equiv 0 \mod 4, p \nmid e, \frac{e \text{Tr}(x_a^{p+1}) - \text{Tr}(x_a)^2}{\text{Tr}(x_a^{p+1})} \in C_{i-1}^{(2,p)}.
\end{cases}
\]

After simplification, we can deduce Eq. (24).

Applying Lemma 11 and Lemma 13 into Eq. (9), we can easily get the Hamming weight of the codes \( C_{D_{2i}}(i = 0, 1) \). In order to calculate the frequency \( A_{wt_{ij}}(i = 0, 1) \) of the linear codes, we need to show the following lemmas at first.

Lemma 14 \( (24) \) Let

\[
N_0 = \left| \{ x \in \mathbb{F}_q : \text{Tr}(x^{p+1}) = 0 \} \right|, \\
N_{(0,0)} = \left| \{ x \in \mathbb{F}_q : \text{Tr}(x^{p+1}) = 0, \text{Tr}(x) = 0 \} \right|, \\
N_{(0,\overline{0})} = \left| \{ x \in \mathbb{F}_q : \text{Tr}(x^{p+1}) = 0, \text{Tr}(x) \neq 0 \} \right|
\]

then

1. \[
N_{(0,0)} = \begin{cases} 
  p^{e-2}, & 2 \mid e, p \mid e, \\
  p^{e-2} + \eta(-e) \frac{p-1}{p^2} C_{e+1}, & 2 \mid e, p \nmid e, \\
  p^{e-2} - p^{\frac{e}{2}-1}(p-1), & 2 \mid e, e \equiv 2 \mod 4, p \mid e, \\
  p^{e-2} - p^{\frac{e}{2}}(p-1), & 2 \mid e, e \equiv 0 \mod 4, p \mid e, \\
  p^{e-2}, & 2 \mid e, p \nmid e.
\end{cases}
\]

2. \[
N_{(0,\overline{0})} = \begin{cases} 
  p^{e-2}(p-1), & p \mid e, \\
  p^{e-2}(p-1) - \eta(-e) \frac{p-1}{p^2} C_{e+1}, & 2 \mid e, p \nmid e, \\
  p^{e-2}(p-1) - p^{\frac{e}{2}-1}(p-1), & 2 \mid e, e \equiv 2 \mod 4, p \nmid e, \\
  p^{e-2}(p-1) - p^{\frac{e}{2}}(p-1), & 2 \mid e, e \equiv 0 \mod 4, p \nmid e.
\end{cases}
\]

Lemma 15 For \( l \in \{0, 1\} \), let

\[
M_{(l,0)} = \left| \{ x \in \mathbb{F}_q : \left( \frac{\text{Tr}(x^{p+1})}{p} \right) = (-1)^l, \text{Tr}(x) = 0 \} \right|, \\
M_{(l,\overline{0})} = \left| \{ x \in \mathbb{F}_q : \left( \frac{\text{Tr}(x^{p+1})}{p} \right) = (-1)^l, \text{Tr}(x) \neq 0 \} \right|
\]

then
1. 
\[ M_{(l,0)} = \begin{cases} 
\frac{p-1}{2} p^{e-2} + (-1)^l \eta(-1) p^{-1} G^{e+1}, & 2 \mid e, p \mid e, \\
\frac{p-1}{2} p^{e-2} - \eta(-e) \frac{p-1}{2} G^{e+1}, & 2 \mid e, p \mid e, \\
\frac{p-1}{2} p^{e-2} - \eta(-e) \frac{p-1}{2} G^{e+1}, & 2 \mid e, e \equiv 2 \text{ mod } 4, p \mid e, \\
\frac{p-1}{2} p^{e-2} - (-1)^l \eta(-e) \frac{p-1}{2} p^2, & 2 \mid e, e \equiv 0 \text{ mod } 4, p \mid e, \\
\frac{p-1}{2} p^{e-2} - (-1)^l \eta(-e) \frac{p-1}{2} p^2, & 2 \mid e, e \equiv 0 \text{ mod } 4, p \mid e, \\
\end{cases} \]

2. 
\[ M_{(l,0)} = \begin{cases} 
\frac{(p-1)^2}{2} p^{e-2}, & p \mid e, \\
\frac{(p-1)^2}{2} p^{e-2} + \left( (-1)^l \eta(-1)p + \eta(-e) \right) \frac{p-1}{2} G^{e+1}, & 2 \mid e, p \mid e, \\
\frac{(p-1)^2}{2} p^{e-2} + \left( 1 + (-1)^l \eta(-e) \right) \frac{p-1}{2} p^2, & 2 \mid e, e \equiv 2 \text{ mod } 4, p \mid e, \\
\frac{(p-1)^2}{2} p^{e-2} + \left( 1 + (-1)^l \eta(-e) \right) \frac{p-1}{2} p^2, & 2 \mid e, e \equiv 0 \text{ mod } 4, p \mid e, \\
\end{cases} \]

**Proof:** By the orthogonal property of additive character, we have

\[ M_{(l,0)} = \sum_{c \in C_1^{(2,p)}} \sum_{x \in \mathbb{F}_p} \left( \frac{1}{p} \sum_{y \in \mathbb{F}_p} \zeta_p^{y \left( \text{Tr}(x^p) - c \right)} \right) \left( \frac{1}{p} \sum_{z \in \mathbb{F}_p} \zeta_p^{z \text{Tr}(x)} \right) \]

\[ = p^{-2} \sum_{c \in C_1^{(2,p)}} \sum_{x \in \mathbb{F}_p} \left( 1 + \sum_{y \in \mathbb{F}_p} \zeta_p^{y \left( \text{Tr}(x^p) - c \right)} \right) \left( 1 + \sum_{z \in \mathbb{F}_p} \zeta_p^{z \text{Tr}(x)} \right) \]

\[ = \frac{p-1}{2} p^{e-2} + p^{-2} (\xi_{11} + \xi_{12} + \xi_{13}), \quad (25) \]

where,

\[ \xi_{11} = \sum_{c \in C_1^{(2,p)}} \sum_{x \in \mathbb{F}_p} \sum_{y \in \mathbb{F}_p} \zeta_p^{\text{Tr}(yx^p) - cy} \]

\[ = \sum_{y \in \mathbb{F}_p} \sum_{x \in \mathbb{F}_p} \zeta_p^{\text{Tr}(yx^p)} \sum_{c \in C_1^{(2,p)}} \zeta_p^{-cy} \]

\[ = \sum_{y \in \mathbb{F}_p} S(y, 0) \sum_{c \in C_1^{(2,p)}} \zeta_p^{-cy}, \quad (26) \]

With Lemma 3, Lemma 4 and Lemma 5,

\[ S(y, 0) = \sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}(yx^p)} = \begin{cases} 
\eta(y) G^e, & 2 \mid e, \\
-p^2, & e \equiv 2 \text{ mod } 4, \\
-p^2 + 1, & e \equiv 0 \text{ mod } 4. \\
\end{cases} \quad (27) \]

Applying Eq. (27) into Eq. (26), we have

\[ \xi_{11} = \begin{cases} 
(-1)^l \eta(-1) \frac{p-1}{2} G^{e+1}, & 2 \mid e, \\
\frac{p-1}{2} p^2, & 2 \mid e, e \equiv 2 \text{ mod } 4, \\
\frac{p-1}{2} p^2 + 1, & 2 \mid e, e \equiv 0 \text{ mod } 4. \quad (28) \end{cases} \]
By the orthogonal property of additive character, we have
\[
\xi_{l2} = \sum_{c \in C_1^{(2, p)}} \sum_{z \in \mathbb{F}_p} \sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}(zx)} = 0. \tag{29}
\]

\[
\xi_{l3} = \sum_{y \in \mathbb{F}_p^*} \sum_{z \in \mathbb{F}_p^*} \sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}(yx^{p+1} + zx)} \sum_{c \in C_1^{(2, p)}} \zeta_p^{-cy},
\]

\[
= \sum_{y \in \mathbb{F}_p^*} \sum_{z \in \mathbb{F}_p^*} S(y, z) \sum_{c \in C_1^{(2, p)}} \zeta_p^{-cy},
\]

With Lemma 11 and Lemma 8, we have

(1) if \( p \mid e \),
\[
\xi_{l1} + \xi_{l3} = p\xi_{l1}, \tag{30}
\]

(2) if \( p \nmid e \),
\[
\xi_{l1} + \xi_{l3} = \begin{cases} 
-\eta(-e)p^{-1}G^e + 1, & 2 \nmid e, \\
(-1)^{l}\eta(-e)p^{-1}p_{e+1}, & 2 \mid e, \equiv 2 \mod 4, \\
(-1)^{l}\eta(-e)p^{-1}p_{e+2}, & 2 \mid e, \equiv 0 \mod 4,
\end{cases} \tag{31}
\]

we can get the first part of this lemma.

Let \( M_{(l)} = \left\{ x \in \mathbb{F}_q : \frac{(\text{Tr}(x^{p+1}i))}{p} = (-1)^{l} \right\} \), then
\[
M_{(l)} = \sum_{c \in C_1^{(2, p)}} \sum_{x \in \mathbb{F}_q} \frac{1}{p} \sum_{y \in \mathbb{F}_p} \zeta_p^{\text{Tr}(x^{p+1}i - c)},
\]

\[
= \frac{p - 1}{2} p^{e-1} \sum_{c \in C_1^{(2, p)}} \sum_{y \in \mathbb{F}_p} \zeta_p^{-cy} \sum_{x \in \mathbb{F}_q} \zeta_p^{\text{Tr}(yx^{p+1})},
\]

\[
= \frac{p - 1}{2} p^{e-1} \sum_{c \in C_1^{(2, p)}} \sum_{y \in \mathbb{F}_p} \zeta_p^{-cy} S(y, 0),
\]

then
\[
M_{(l)} = \begin{cases} 
\frac{p - 1}{2} p^{e-1} + (-1)^{l}\eta(-1)p^{-1}p_{e+1}, & 2 \nmid e, \\
\frac{p - 1}{2} p^{e-1} + \frac{p - 1}{2} p_{e+1}, & 2 \mid e, \equiv 2 \mod 4, \\
\frac{p - 1}{2} p^{e-1} + \frac{p - 1}{2} p_{e+2}, & 2 \mid e, \equiv 0 \mod 4.
\end{cases}
\]

With \( M_{(l, 0)} = M_{(l)} - M_{(l, 0)} \), we can easily deduce the second part of the proof. Box

**Lemma 16** (10) Let \( p \nmid e \), we have
\[
N(0, 0, e) = \left\{ x \in \mathbb{F}_q : \text{Tr}(x^{p+1}) \neq 0, \text{Tr}(x) \neq 0, \text{Tr}(x)^2 - e\text{Tr}(x^{p+1}) \right\}
\]

\[
= \begin{cases} 
(p - 1)p^{e-2}, & 2 \mid e, \\
(p - 1)p^{e-2} + \eta(-e)p^{-1}p_{e+1}, & 2 \nmid e.
\end{cases}
\]
Lemma 17 Let \( p \nmid e, s \in \mathbb{F}_p^* \) and \( s \neq e \),
\[
V_s = \{x \in \mathbb{F}_q : \Tr(x^{p+1}) \neq 0, \Tr(x) \neq 0, \Tr(x)^2 = s\Tr(x^{p+1})\},
\]
then
\[
|V_s| = \begin{cases} 
(p-1)p^{e-2} - \eta(-e)\frac{p-1}{p}G^{e+1}, & 2 \mid e, \\
(p-1)p^{e-2} - \eta(s)\eta(s-e)p^{\frac{e-1}{2}}(p-1), & 2 \mid e, e \equiv 2 \mod 4, \\
(p-1)p^{e-2} - \eta(s)\eta(s-e)p^{\frac{e-1}{2}}(p-1), & 2 \mid e, e \equiv 0 \mod 4.
\end{cases}
\]

Proof: The proof is similar to that of Lemma 15, so we omit the process. \( \square \)

Lemma 18 Let \( p \nmid e \), for \( k, j \in \{0, 1\} \),
\[
\nu(k, j) = \{s \in \mathbb{F}_p^* : s \neq e, \left(\frac{s}{p}\right) = (-1)^k, \left(\frac{s-e}{p}\right) = (-1)^j(-1)^{\frac{p-1}{2}}\},
\]
then
\[
|\nu(k, j)| = \frac{1}{4}(p-2 - (-1)^k\eta(e) - (-1)^j\eta(e) - (-1)^{k+j}\eta(-1)).
\]

Proof: With Lemma 10, we can easily deduce the result. \( \square \)

Lemma 19 Let \( p \nmid e \), for \( k, j \in \{0, 1\} \),
\[
N(k, 0, 0, j)
= \left| \{x \in \mathbb{F}_q : \Tr(x^{p+1}) \in C_k^{(2,p)}, \Tr(x) \neq 0, \Tr(x)^2 \neq e\Tr(x^{p+1}), \frac{e\Tr(x^{p+1}) - \Tr(x)^2}{\Tr(x^{p+1})} \in C_j^{(2,p)}\} \right|
= \begin{cases} 
\frac{1}{4}(p-2 - (-1)^k\eta(e) - (-1)^j\eta(e) - (-1)^{k+j}\eta(-1))((p-1)p^{e-2} - \eta(-e)\frac{p-1}{p}G^{e+1}), & 2 \mid e, \\
\frac{1}{4}(p-2 - (-1)^k\eta(e) - (-1)^j\eta(e) - (-1)^{k+j}\eta(-1))((p-1)p^{e-2} - (-1)^{k+j}\eta(-1)(p-1)p^{\frac{e-1}{2}}), & 2 \mid e, e \equiv 2 \mod 4, \\
\frac{1}{4}(p-2 - (-1)^k\eta(e) - (-1)^j\eta(e) - (-1)^{k+j}\eta(-1))((p-1)p^{e-2} - (-1)^{k+j}\eta(-1)(p-1)p^{\frac{e-1}{2}}), & 2 \mid e, e \equiv 0 \mod 4.
\end{cases}
\]

Proof:
\[
N(k, 0, 0, j)
= \left| \{x \in \mathbb{F}_q : \Tr(x) = c, \Tr(x^{p+1}) = \frac{c^2}{s}, \forall c \in \mathbb{F}_p^*, s \in \mathbb{F}_p^* \subseteq C_k^{(2,p)}, s \neq e, (e-s) \in C_j^{(2,p)}\} \right|
= \frac{1}{4}(p-2 - (-1)^k\eta(e) - (-1)^j\eta(e) - (-1)^{k+j}\eta(-1))((p-1)p^{e-2} - \eta(-e)\frac{p-1}{p}G^{e+1}),
\]
then
\[
\sum_{s \in \nu(k, j)} |V_s|,
\]

21
With Lemma 17 and Lemma 18, we can get the results.

**Proof of Theorem 1:** Let \( 2 \nmid e \) and \( p \mid e \), with Lemma 12 for \( i = 0, 1 \), we have

\[
n_i = \frac{p - 1}{2} p^{e - 1} + (-1)^i \frac{p - 1}{2} G^{e - 1}.
\]

Applying Lemma 11 and Lemma 13 into Eq. (9), we can get the following lemma directly.

**Lemma 20** Let \( 2 \nmid e \) and \( p \mid e \), for \( i = 0, 1 \), we have

\[
\begin{align*}
wt_i(c(a)) &= \begin{cases} \\
\frac{(p-1)^2}{2} p^{e-2}, & \text{Tr}(x_a^{p+1}) = 0, \text{Tr}(x_a) = 0, \\
\frac{(p-1)^2}{2} p^{e-2} + (-1)^i \frac{p - 1}{2} G^{e - 1}, & \text{Tr}(x_a^{p+1}) = 0, \text{Tr}(x_a) \neq 0, \\
\frac{(p-1)^2}{2} p^{e-2} + (-1)^i \frac{p - 1}{2} G^{e-1} + \eta(-\text{Tr}(x_a^{p+1})) \frac{p - 1}{2} G^{e-3}, & \text{Tr}(x_a^{p+1}) \neq 0, \text{Tr}(x_a) = 0, \\
\frac{(p-1)^2}{2} p^{e-2} + (-1)^i \frac{p - 1}{2} G^{e-1} - \frac{1}{2} \eta(-\text{Tr}(x_a^{p+1})) G^{e-3}, & \text{Tr}(x_a^{p+1}) \neq 0, \text{Tr}(x_a) \neq 0,
\end{cases}
\end{align*}
\]

We will provide the proof of Theorem 1 according to two cases.

1. If \( 2 \nmid e \), \( p \mid e \) and \( p \equiv 1 \mod 4 \), with Lemma 20, we have

\[
\begin{align*}
wt_{i1} &= \frac{(p-1)^2}{2} p^{e-2}, \\
wt_{i2} &= \frac{(p-1)^2}{2} p^{e-2} + (-1)^i \frac{p - 1}{2} p^{e - 2}, \\
wt_{i3} &= \frac{(p-1)^2}{2} p^{e-2} + (-1)^i \frac{p - 1}{2} p^{e - 2} + (-1)^i \frac{p - 1}{2} p^{e - 2}, \\
wt_{i4} &= \frac{(p-1)^2}{2} p^{e-2} + (-1)^i \frac{p - 1}{2} p^{e - 2} - (-1)^i \frac{p - 1}{2} p^{e - 2}, \\
wt_{i5} &= \frac{(p-1)^2}{2} p^{e-2} + (-1)^i \frac{p - 1}{2} p^{e - 2} - (-1)^i \frac{p + 1}{2} p^{e - 2},
\end{align*}
\]

and

\[
\begin{align*}
A_{wt_{i1}} &= N(0, 0) - 1, \\
A_{wt_{i2}} &= N(0, 0), \\
A_{wt_{i3}} &= M(i, 0), \\
A_{wt_{i4}} &= M_{(1 - i, 0)} + M_{(1 - i, 0)}, \\
A_{wt_{i5}} &= M(i, 0),
\end{align*}
\]

Applying into the results of Lemma 14 and Lemma 15, we can get the first part of Theorem 1.
2. If $2 \nmid e, p \mid e$ and $p \equiv 3 \mod 4$, with Lemma 20 we have

\[ wt_{i1} = \frac{(p-1)^2}{2}p^{e-2}, \]

\[ wt_{i2} = \frac{(p-1)^2}{2}p^{e-2} + (-1)^i\left( \frac{p-1}{2}p^{\frac{e-1}{2}} \right), \]

\[ wt_{i3} = \frac{(p-1)^2}{2}p^{e-2} + (-1)^i\left( \frac{p-1}{2}p^{\frac{e-1}{2}} + (-1)^i\frac{p-1}{2}p^{\frac{e-3}{2}} \right), \]

\[ wt_{i4} = \frac{(p-1)^2}{2}p^{e-2} + (-1)^i\left( \frac{p-1}{2}p^{\frac{e-1}{2}} - (-1)^i\frac{p-1}{2}p^{\frac{e-3}{2}} \right), \]

\[ wt_{i5} = \frac{(p-1)^2}{2}p^{e-2} + (-1)^i\left( \frac{p-1}{2}p^{\frac{e-1}{2}} + (-1)^i\frac{p+1}{2}p^{\frac{e-3}{2}} \right), \]

and

\[ A_{wt_{i1}} = N(0,0) - 1, \]
\[ A_{wt_{i2}} = N(0,\overline{0}), \]
\[ A_{wt_{i3}} = M_{(1-i,0)} + M_{(1-i,\overline{0})}, \]
\[ A_{wt_{i4}} = M_{(i,0)}; \]
\[ A_{wt_{i5}} = M_{(i,\overline{0})}. \]

Applying into the results of Lemma 14 and Lemma 15 we can get the second part of Theorem 1.

Thus we can get the desired conclusions presented in Theorem 1 and complete the proof. \[ \square \]

**Proof of Theorem 4**: Let $2 \mid e$, $e \equiv 2 \mod 4$, and $p \mid e$. With Lemma 12 for $i = 0, 1$, we have

\[ n_i = \frac{p-1}{2}p^{e-1} + \frac{p-1}{2}p^{\frac{e-1}{2}}. \]

Applying Lemma 11 and Lemma 13 into Eq. 9, we can get the following lemma directly.

**Lemma 21** Let $2 \mid e$, $e \equiv 2 \mod 4$, and $p \mid e$, for $i = 0, 1$, we have

\[ wt_i(c(a)) = \begin{cases} \frac{(p-1)^2}{2}p^{e-2} & \text{Tr}(x_a^{p+1}) = 0, \text{Tr}(x_a) = 0, \\ \frac{(p-1)^2}{2}p^{e-2} + \frac{p-1}{2}p^{\frac{e-1}{2}} & \text{Tr}(x_a^{p+1}) = 0, \text{Tr}(x_a) \neq 0, \\ \frac{(p-1)^2}{2}p^{e-2} + \frac{1+(-1)^i\alpha(-\text{Tr}(x_a^{p+1}))}{2}p^{\frac{e-1}{2}} & \text{Tr}(x_a^{p+1}) \neq 0, \text{Tr}(x_a) = 0, \\ \frac{(p-1)^2}{2}p^{e-2} + \frac{1+(-1)^i\alpha(-\text{Tr}(x_a^{p+1}))}{2}p^{\frac{e-1}{2}} & \text{Tr}(x_a^{p+1}) \neq 0, \text{Tr}(x_a) \neq 0, \end{cases} \]

With Lemma 21 we can proof Theorem 4.

\[ wt_{i1} = \frac{(p-1)^2}{2}p^{e-2}; \]
\[ wt_{i2} = \frac{(p-1)^2}{2}p^{e-2} + \frac{p-1}{2}p^{\frac{e-1}{2}}, \]
\[ wt_{i3} = \frac{(p-1)^2}{2}p^{e-2} + (p-1)p^{\frac{e-1}{2}}, \]
\[ wt_{i4} = \frac{(p-1)^2}{2}p^{e-2} + \frac{p-3}{2}p^{\frac{e-1}{2}}. \]
Applying Lemma 11 and Lemma 13 into Eq.(9), we can get the following lemma.

Theorem 4. Applying into the results of Lemma 14 and Lemma 15, we can get the second part of Theorem 4.

Proof of Theorem 8: Let $2 \mid e$, $e \equiv 0 \mod 4$, and $(\frac{e}{p}) = (-1)^i$. With Lemma 11 for $i = 0, 1$, we have

$$n_i = \frac{p - 1}{2}p^{e - 1} - \frac{1 + \eta(-1)p}{2}p^e.$$

Applying Lemma 11 and Lemma 13 into Eq.(9), we can get the following lemma directly.

Lemma 22 Let $2 \mid e$, $e \equiv 0 \mod 4$, and $(\frac{e}{p}) = (-1)^i$, for $i = 0, 1$, we have

1. if there are no solutions of the equation $x^{p^2} + x = -a^p$ in $\mathbb{F}_q$, then

$$\text{wt}_i(c(a)) = \frac{(p - 1)^2}{2}p^{e - 2} - \frac{1 + \eta(-1)p}{2}(p - 1)p^{\frac{e}{2} - 1},$$

2. if $x_a$ is one solution of the equation $x^{p^2} + x = -a^p$ in $\mathbb{F}_q$, then

$$\text{wt}_i(c(a)) = \begin{cases} 
\frac{(p - 1)^2}{2}p^{e - 2} - \frac{1 + \eta(-1)p}{2}p^{\frac{e}{2}}, & \text{Tr}(x_a^{p + 1}) = 0, \text{Tr}(x_a) = 0, \\
\frac{(p - 1)^2}{2}p^{e - 2} - \frac{1 + \eta(-1)p}{2}p^{\frac{e}{2}} - (1 + (-1)^i\eta(-\text{Tr}(x_a^{p + 1})))p^{\frac{e}{2}}, & \text{Tr}(x_a^{p + 1}) \neq 0, \text{Tr}(x_a) = 0, \\
\frac{(p - 1)^2}{2}p^{e - 2} - \frac{1 + \eta(-1)p}{2}p^{\frac{e}{2}}, & \text{Tr}(x_a^{p + 1}) \neq 0, \text{Tr}(x_a) \neq 0, e\text{Tr}(x_a^{p + 1}) = \text{Tr}(x_a)^2, \\
\frac{(p - 1)^2}{2}p^{e - 2} - \frac{1 + \eta(-1)p}{2}p^{\frac{e}{2}} - (1 + (-1)^i\eta(-\text{Tr}(x_a^{p + 1})))p^{\frac{e}{2}}, & \text{Tr}(x_a^{p + 1}) \neq 0, \text{Tr}(x_a) \neq 0, e\text{Tr}(x_a^{p + 1}) \neq \text{Tr}(x_a)^2, \\
\frac{(p - 1)^2}{2}p^{e - 2} - \frac{1 + \eta(-1)p}{2}p^{\frac{e}{2}}, & \text{Tr}(x_a^{p + 1}) \neq 0, \text{Tr}(x_a) = 0, e\text{Tr}(x_a^{p + 1}) = \text{Tr}(x_a)^2, \\
\frac{(p - 1)^2}{2}p^{e - 2} - \frac{1 + \eta(-1)p}{2}p^{\frac{e}{2}} - (1 + (-1)^i\eta(-\text{Tr}(x_a^{p + 1})))p^{\frac{e}{2}}, & \text{Tr}(x_a^{p + 1}) \neq 0, \text{Tr}(x_a) = 0, e\text{Tr}(x_a^{p + 1}) \neq \text{Tr}(x_a)^2,
\end{cases}$$

We will give the proof process of Theorem 8 according to the following two cases. With Lemma 22
1. if $2 \mid e$, $e \equiv 0 \mod 4$, $(\frac{e}{p}) = (-1)^i$, and $p \equiv 1 \mod 4$, we have

\[
\begin{align*}
wt_{i1} & = \frac{(p-1)^2}{2}p^{e-2} - \frac{p^2-1}{2}p^{\frac{e-1}{2}}, \\
wt_{i2} & = \frac{(p-1)^2}{2}p^{e-2}, \\
wt_{i3} & = \frac{(p-1)^2}{2}p^{e-2} - \frac{p+1}{2}p^{\frac{e-1}{2}}, \\
wt_{i4} & = \frac{(p-1)^2}{2}p^{e-2} - \frac{p+3}{2}p^{\frac{e-1}{2}}, \\
wt_{i5} & = \frac{(p-1)^2}{2}p^{e-2} - \frac{p-1}{2}p^{\frac{e-1}{2}}, \\
wt_{i6} & = \frac{(p-1)^2}{2}p^{e-2} - p^{\frac{e-1}{2}},
\end{align*}
\]

and

\[
\begin{align*}
A_{wt_{i1}} & = p^e - p^{e-2}, \\
A_{wt_{i2}} & = p^{-2}N(0,0) - 1, \\
A_{wt_{i3}} & = p^{-2}N(0,\overline{0}) + p^{-2}N(i,\overline{0},\overline{e},1-i) + p^{-2}N(1-i,\overline{0},\overline{e},1-i), \\
A_{wt_{i4}} & = p^{-2}M_{i,0} + p^{-2}N(i,\overline{0},\overline{e},i), \\
A_{wt_{i5}} & = p^{-2}M_{1-i,0} + p^{-2}N(1-i,\overline{0},\overline{e},i), \\
A_{wt_{i6}} & = p^{-2}N(\overline{0},\overline{0},e),
\end{align*}
\]

Applying into the results of Lemma 14, Lemma 15, Lemma 16 and Lemma 19, we can get the first part of Theorem 8.

2. if $2 \mid e$, $e \equiv 0 \mod 4$, $(\frac{e}{p}) = (-1)^i$, and $p \equiv 3 \mod 4$, we have

\[
\begin{align*}
wt_{i1} & = \frac{(p-1)^2}{2}p^{e-2} + \frac{(p-1)^2}{2}p^{\frac{e-1}{2}}, \\
wt_{i2} & = \frac{(p-1)^2}{2}p^{e-2}, \\
wt_{i3} & = \frac{(p-1)^2}{2}p^{e-2} + \frac{p-1}{2}p^{\frac{e-1}{2}}, \\
wt_{i4} & = \frac{(p-1)^2}{2}p^{e-2} + \frac{p-3}{2}p^{\frac{e-1}{2}}, \\
wt_{i5} & = \frac{(p-1)^2}{2}p^{e-2} + \frac{p+1}{2}p^{\frac{e-1}{2}},
\end{align*}
\]

and

\[
\begin{align*}
A_{wt_{i1}} & = p^e - p^{e-2}, \\
A_{wt_{i2}} & = p^{-2}N(0,0) - 1 + p^{-2}N(\overline{0},\overline{0},e), \\
A_{wt_{i3}} & = p^{-2}N(0,\overline{0}) + p^{-2}N(i,\overline{0},\overline{e},1-i) + p^{-2}N(1-i,\overline{0},\overline{e},1-i), \\
A_{wt_{i4}} & = p^{-2}M_{i,0} + p^{-2}N(i,\overline{0},\overline{e},i), \\
A_{wt_{i5}} & = p^{-2}M_{1-i,0} + p^{-2}N(1-i,\overline{0},\overline{e},i),
\end{align*}
\]
Applying into the results of Lemma 14, Lemma 15, Lemma 16 and Lemma 19, we can get the second part of Theorem 8.

The proof process of other theorems is similar to the proof of the above three theorems, and will not be repeated here.

Remark 1: In the case of \(e \equiv 0 \mod 4\), if there are no solutions of the equation \(x^{p^2} + x = -a^p\) in \(\mathbb{F}_q\), with Lemma 7, the number of \(a\) is \(p^e - p^{e-2}\).

Remark 2: In the case of \(e \equiv 0 \mod 4\), if the equation \(x^{p^2} + x = -a^p\) is solvable, then it has \(p^2\) solutions.

Remark 3: In the nine theorems, there are some special cases, such as the case of \(p = 3\), in which the codes may have less number of non-zero weights.

5 Concluding remarks

In this paper, inspired by the work in [10], two classes of at most six-weight linear codes were constructed with their weight enumerators settled using Weil sums and Gaussian period. At the same time, some optimal or almost optimal linear code was found.

Let \(wt_{\text{min}}\) and \(wt_{\text{max}}\) denote the minimum and maximum non-zero weight of a linear code \(\mathcal{C}\), respectively. Obviously, if \(e \geq 8\),

\[
\frac{wt_{\text{min}}}{wt_{\text{max}}} > \frac{p - 1}{p}.
\]

As stated in [19], the codes \(\mathcal{C}_D\) can be used to construct secret sharing schemes.

References

[1] R.Lidl and H.Niederreiter, 2nd ed., Finite fields, Cambridge University Press, Cambridge, 1997.
[2] W.C.Huffman and V.Pless, Fundamentals of error-correcting codes, Cambridge University Press, Cambridge, 2003.
[3] Z.Heng and Q.Yue, Evaluation of the Hamming weights of a class of linear codes based on Gauss sums, Designs Codes and Cryptography 83(2) (2016) 1–20.
[4] C.Ding and H.Niederreiter, Cyclotomic linear codes of order 3, IEEE Transactions on Information Theory 53(6) (2007) 2274–2277.
[5] K.Ding and C.Ding, A class of two-weight and three-weight codes and their applications in secret sharing, IEEE Transactions on Information Theory 61(11) (2015) 5835–5842.
[6] Z.Heng and Q.Yue, A class of binary linear codes with at most three weights, IEEE Communication Letters 19(9) (2015) 1488–1491.
[7] C. Tang, C. Xiang and K. Feng, Linear codes with few weights from inhomogeneous quadratic functions, Designs Codes and Cryptography 83(3) (2017) 691–714.

[8] G. Jian, Z. Lin and R. Feng, Two-weight and three-weight linear codes based on Weil sums, Finite Fields and Their Applications 57 (2019) 92–107.

[9] Y. Song and J. Yang, Weight distribution of two class of linear codes with a few weights, Science China Information Sciences 63 (2020) 179103, https://doi.org/10.1007/s11432-018-9610-9.

[10] C. Xiang, C. Tang, K. Feng, A class of linear codes with a few weights, Cryptography and Communications, 9(1) (2017) 93–116.

[11] R. S. Coulter, Further evaluations of Weil sums, Acta Arithmetica 86 (1998) 217–226.

[12] R. S. Coulter, Explicit evaluations of some Weil sums, Acta Arithmetica 83(3) (1998) 241–251.

[13] R. S. Coulter, The Number of Rational Points of a Class of Artin–Schreier Curves, Finite Fields and Their Applications 8 (2002) 397–413.

[14] T. Storer, Cyclotomy and Difference Sets, Chicago: Mark-ham Publishing Company, 1967.

[15] M. Grassl, Bounds on the minimum distance of linear codes and quantum codes, online available at http://www.codetables.de/.

[16] Q. Wang, F. Li, K. Ding and D. Lin, Complete weight enumerators of two classes of linear codes, Discrete Mathematics 340 (2017) 467–480.

[17] G. Myserson, Period polynomials and Gauss sums for finite fields, Acta Arithmetica, 39(3) (1981) 251–264.

[18] B. C. Berndt, R. J. Evans, K. S. Williams, Gauss and Jacobi Sums, New York: John Wiley & Sons Company, 1997.

[19] J. Yuan, C. Ding., Secret sharing schemes from three classes of linear codes[J]. IEEE Transactions on Information Theory, 2006, 52(1): 206-212.