Jordan blocks and Gamow-Jordan eigenfunctions associated to a double pole of the $S$–matrix

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Abstract
An accidental degeneracy of resonances gives rise to a double pole in the scattering matrix, a double zero in the Jost function and a Jordan chain of length two of generalized Gamow-Jordan eigenfunctions of the radial Schrödinger equation. The generalized Gamow-Jordan eigenfunctions are basis elements of an expansion in bound and resonant energy eigenfunctions plus a continuum of scattering wave functions of complex wave number. In this biorthonormal basis, any operator $f(H^{(ℓ)})$ which is a regular function of the Hamiltonian is represented by a complex matrix which is diagonal except for a Jordan block of rank two. The occurrence of a double pole in the Green’s function, as well as the non-exponential time evolution of the Gamow-Jordan generalized eigenfunctions are associated to the Jordan block in the complex energy representation.

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1 Introduction
Lately, the interference effects of resonances, the crossing and anticrossing properties of the energies and widths of two unbound levels and the occurrence of a double pole of the scattering matrix have aroused a great deal of interest. Some interesting examples of interfering unbound two level systems are the $T = 1, T = 0, J^π = 2^+$ doublet in $^8$Be[1, 4, 9], the $T = 1, T = 0$ doublet of $ρ$ and $ω$ mesons and the $σ−K_s$ doublet of neutral sigma and $K$ mesons[2, 4, 9, 10]. A variety of widely differing systems where double poles can occur have been identified, such as autoionizing states in complex atoms[3] and atomic states in intense laser fields[9, 10]. The problem of the degeneracy of resonances also arises naturally in connection with the Berry phase of resonant states[11, 12, 13, 14] which was recently measured by the Darmstadt group[15].

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Some examples of simple quantum mechanical systems with double poles in the scattering matrix have been recently described. Vanroose et al.,[16] examined the formation of complex double poles of the $S$-matrix in a two channel model with square well potentials. Recently, Hernández et al.,[17] investigated a one channel model with two spherical concentric cavities bounded by $\delta$–function barriers and showed that a double pole of the $S$–matrix can be induced by tuning the parameters of the model; Vanroose generalized this model to the case of two finite width barriers[18]. The formal theory of multiple pole resonances and resonant states in the rigged Hilbert space formulation of quantum mechanics was developed by Bohm et al.,[19] and by Antoniou et al.,[20].

In the present paper, we deal with the problem of multiple poles of the scattering matrix and the generalized complex energy eigenfunctions associated with them in the framework of the theory of the analytic properties of the radial wave functions.

The plan of this paper is as follows. In sections 2 and 3, we introduce some basic concepts and fix the notation by way of a short reminder of resonances and resonant states in the theory of the analytic properties of the radial wave functions. Sections 4 and 5 are devoted to a short discussion of the no-crossing rule for bound states and its non applicability to resonant states. In section 6, we show that a double pole of the scattering wave function (double zero of the Jost function) is associated to a chain of length two of Gamow-Jordan generalized eigenfunctions and derive explicit expressions for this generalized eigenfunctions in terms of the outgoing wave Jost solution, the Jost function and its derivatives evaluated at the double pole. We also show that the Gamow-Jordan generalized eigenfunctions in the Jordan chain are elements of a complete set of states containing the real (bound states) and complex (resonant state) energy eigenfunctions plus a continuum of scattering wave functions of complex wave number. In section 7 we derive expansion theorems (spectral representations) for operators $f(H^\ell)$ which are regular functions of the radial Hamiltonian $H^\ell$ and show that, in this basis, the operator $f(H^\ell)$ is represented by a complex matrix which is diagonal except for a Jordan block of rank two associated to the double zero of the Jost function and the corresponding Jordan chain of generalized Gamow-Jordan eigenfunction. We give the normalization and orthogonality rules for the generalized eigenfunctions in the Jordan chain associated to the double pole of the Green’s function in section 8. We end our paper with a summary of results and some conclusions in section 9.

2 Regular and physical solutions of the radial equation

The non-relativistic scattering of a spinless particle by a short ranged potential $v(r)$ is described by the solution of a Schrödinger equation. When the potential is rotationally invariant, the wave function is expanded in partial waves and one is left with the radial equation

$$\frac{d^2\phi_\ell (k, r)}{dr^2} + \left[ k^2 - \frac{\ell (\ell + 1)}{r^2} - v(r) \right] \phi_\ell (k, r) = 0. \quad (1)$$

As is usually done when discussing the analytic properties of the solutions of (1) as functions of $k$, rather than starting by defining the physical solutions $\psi_\ell^{(+)}(k, r)$, we define the regular and irregular solutions of (1) by boundary conditions which lead to simple properties as functions of $k$. The regular solution $\phi_\ell(k, r)$ is uniquely defined by the boundary condition [21]
\[
\lim_{r \to 0}(2\ell + 1)!r^{-\ell-1}\phi_\ell(k, r) = 1, \tag{2}
\]

\(\phi_\ell(k, r)\) may be expressed as a linear combination of two independent, irregular solutions of (1) which behave as outgoing and incoming waves at infinity,

\[
\phi_\ell(k, r) = \frac{1}{2}ik^{-\ell-1}\left[f_\ell(-k)f_\ell(k, r) - (-1)^\ell f_\ell(k)f_\ell(-k, r)\right], \tag{3}
\]

where \(f_\ell(-k, r)\) is an outgoing wave at infinity defined by the boundary condition

\[
\lim_{r \to \infty}\exp(-ikr)f_\ell(-k, r) = (+i)^\ell \tag{4}
\]

and \(f_\ell(k, r)\) is an incoming wave at infinity related to \(f_\ell(-k, r)\) by

\[
f_\ell(k, r) = (-1)^\ell f_\ell^*(k, r) \tag{5}
\]

for \(k\) real and non-vanishing.

The Jost function \(f_\ell(-k) = f_\ell(-k, 0)\) is given by

\[
f_\ell(-k) = (-1)^\ell k^\ell W[f_\ell(-k, r), \phi_\ell(k, r)] \tag{6}
\]

where \(W[f, g] = fg' - f'g\) is the Wronskian. The Jost function \(f_\ell(-k)\), has zeroes (roots) on the imaginary axis and in the lower half of the complex \(k\) plane.

When the first and second absolute moments of the potential exist, and the potential decreases at infinity faster than any exponential (e.g. if \(v(r)\) has a gaussian tail or if it vanishes identically beyond a finite radius) the functions \(f_\ell(-k), \phi_\ell(k, r),\) and \(k^\ell f_\ell(-k, r),\) for fixed \(r > 0,\) are entire function of \(k[21]\).

Therefore, the derivatives of these functions with respect to the wave number \(k\) exist and are entire functions of \(k\) for all finite values of \(k\) in the complex \(k\)-plane.

The differential equations satisfied by the derivatives of the functions \(\phi_\ell(k, r)\) and \(f_\ell(-k, r)\) with respect to \(k\) are obtained from (1) taking derivatives with respect to \(k\) in both sides of the equation,

\[
\frac{d^2\phi_\ell(k, r)}{dr^2} + \left[k^2 - \frac{\ell(\ell + 1)}{r^2} - v(r)\right]\phi_\ell(k, r) = -2k\phi_\ell(k, r), \tag{7}
\]

\[
\frac{d^2\tilde{\phi}_\ell(k, r)}{dr^2} + \left[k^2 - \frac{\ell(\ell + 1)}{r^2} - v(r)\right]\tilde{\phi}_\ell(k, r) = -4k\tilde{\phi}_\ell(k, r) - 2\phi_\ell(k, r), \tag{8}
\]

in (7) and (8) we have used the notation \(\dot{\phi}_\ell(k, r) = d\phi_\ell(k, r)/dk,\) Similar expressions are valid for the derivatives with respect to \(k\) of the outgoing wave solutions \(f_\ell(-k, r).\)
The scattering wave function $\psi^{(+)}(k, r)$ is the solution of equation (1) which vanishes at the origin and behaves at infinity as the sum of a free incoming spherical wave of unit incoming flux plus a free outgoing spherical wave,

$$\psi^{(+)}(k, 0) = 0$$

and

$$\lim_{r \to \infty} \left\{ \psi^{(+)}(k, r) - \left[ \hat{H}_\ell^-(k, r) - S_\ell(k) \hat{H}_\ell^+(k, r) \right] \right\} = 0. \quad (10)$$

In this expression $\hat{H}_\ell^-(k, r)$ and $\hat{H}_\ell^+(k, r)$ are Ricatti-Hankel functions that describe incoming and outgoing waves respectively, $S_\ell(k)$ is the scattering matrix.

Hence, the scattering wave function $\psi^{(+)}_\ell(k, r)$ and the regular solution are related by

$$\psi^{(+)}_\ell(k, r) = \frac{k^{\ell+1} \phi_\ell(k, r)}{f_\ell(-k)}, \quad (11)$$

and the scattering matrix is given by

$$S_\ell(k) = \frac{f_\ell(k)}{f_\ell(-k)}. \quad (12)$$

The complete Green’s function for outgoing particles or resolvent of the radial equation may also be written in terms of the regular solution $\phi_\ell(k, r)$ and the irregular solution $f_\ell(-k, r)$ which behaves as an outgoing wave at infinity

$$G^{(+)}_\ell(k; r, r') = (-1)^{\ell+1} k^{\ell} \frac{\phi_\ell(k, r_< f_\ell(-k, r_>)}{f_\ell(-k)}. \quad (13)$$

### 3 Bound and resonant state eigenfunctions

Bound and resonant state energy eigenfunctions are the solutions of (1) which vanish at the origin

$$u_{n\ell}(k_n, 0) = 0, \quad (14)$$

and at infinity satisfy the boundary condition

$$\lim_{r \to \infty} \left[ \frac{1}{u_{n\ell}(k_n, r)} \frac{du_{n\ell}(k_n, r)}{dr} - ik_n \right] = 0, \quad (15)$$

where $k_n$ is a zero of the Jost function,
\[ f_\ell(-k_n) = 0. \] (16)

From equations (1) and (3) we verify that all roots (zeroes) of the Jost function are associated to energy eigenfunctions of the Schrödinger equation.

Bound state eigenfunctions are associated to the zeroes of \( f_\ell(-k) \) which lay on the positive imaginary axis \( k_s^2 = -\kappa_s^2 < 0 \), while resonant or Gamow state eigenfunctions are associated to the zeroes of the Jost function which lay in the fourth quadrant of the complex \( k \)-plane.

From (3), (4) and (16), bound states and Gamow or resonance eigenfunctions are related to the regular solution \( \phi_\ell(k, r) \) by

\[
u_n\ell(k_n, r) = N_{n\ell}^{-1} \phi_\ell(k_n, r) \] (17)

where \( N_{n\ell} \) is a normalization constant. Due to the vanishing of \( f_\ell(-k_n) \), \( \phi_\ell(k_n, r) \) is now proportional to the outgoing wave solution, \( f_\ell(-k_n, r) \), of (1). Hence,

\[
u_n\ell(k_n, r) = N_{n\ell}^{-1} \frac{(-1)^{\ell+1}}{2} k^{\ell+1} f_\ell(k_n) f_\ell(-k_n, r). \] (18)

This expression shows, in a very explicit way, that the Gamow state eigenfunctions \( u_{n\ell}(k_n, r) \) with \( k_n = \kappa_n - i\gamma_n \) and \( \kappa_n > \gamma_n > 0 \), are solutions of (1) which vanish at the origin and asymptotically behave as purely outgoing waves which oscillate between envelopes that increase exponentially with \( r \), the corresponding energy eigenvalues \( E_n \) are complex with \( \text{Re} \ E_n > \text{Im} \ E_n \).

The bound state eigenfunctions \( u_{s\ell}(k_s, r) \) are also solutions of (1) which satisfy the boundary conditions (14) and (15), but, in this case, \( k_s = i\kappa_s \) with, \( \kappa_s > 0 \), which means that asymptotically the outgoing wave of imaginary argument, \( f_\ell(-k_s, r) \), decreases exponentially with \( r \) and the energy eigenvalue \( E_s \) is real and negative.

4 The no-crossing rule for bound states.

In the case of bound states, the normalization constant is related to the derivative of the Jost function evaluated at \( k_s \) and it may also be expressed as a normalization integral. The zero of the Jost function is on the positive imaginary axis, and the bound state eigenfunction is quadratically integrable (for time reversal invariant forces \( \phi_\ell(i\kappa_s, r) \) is real). R.G. Newton gives the following expression\[21\]

\[
N_{s\ell}^2 = \frac{1}{4k_s^{2(\ell+1)}} \left( \frac{df_\ell(-k)}{dk} \right)_{k_s} \int_0^\infty |\phi_\ell(k_s, r)|^2 dr.
\] (19)

Since the normalization integral is positive and the function \( f_\ell(k) \) is regular at \( k_s = i\kappa_s \), the derivative of the Jost function evaluated at \( k_s = i\kappa_s \) cannot vanish. Therefore, the zero of \( f_\ell(-k) \) at \( k_s = i\kappa_s \) must be simple. The corresponding pole in \( G^{(+)}_\ell(k; r, r') \), \( \psi^{(+)}_\ell(k, r) \) and \( S_\ell(k) \) must also be simple.

It follows that, in the absence of symmetry, the real, negative energy eigenvalues of the radial equation for a one channel problem cannot be degenerate.
5 Crossing of resonant states

In the case of a resonant state, the zero of the Jost function $f_\ell(-k)$ lies in the fourth quadrant of the complex $k$-plane,

$$k_n = \kappa_n - i\gamma_n,$$

with $\kappa_n > \gamma_n > 0$.

The resonant or Gamow eigenfunction $\phi_\ell(k_n, r)$ is an outgoing spherical wave of complex wave number $k_n$ and angular momentum $\ell$. Therefore, for large values of $r$, $\phi_\ell(k_n, r)$ oscillates between envelopes that grow exponentially with $r$. Hence, the integrals over $r$ must be properly defined. This may be done by means of a gaussian regulator and a limiting procedure\[22\] T. Berggren\[23, 24\] gives the following expression

$$\frac{1}{i4k_n^{2(\ell+1)}} \left( \frac{df_\ell(-k)}{dk} \right)_{k_n} = \lim_{\nu \to 0} \int_0^\infty \exp(-\nu r^2) \phi^2_\ell(k_n, r) dr$$

The integral in the right hand side is a complex number and it may vanish.

Since $f_\ell(k_n)$ has no zeroes in the lower half of the complex $k$-plane, the left hand side of equation (21) vanishes only when $(df_\ell(-k)/dk)_{k_n}$ vanishes. Then, we have two possibilities,

i) When $(df_\ell(-k)/dk)_{k_n}$ does not vanish, $f(-k)$, has a simple zero at $k = k_n$, the integral in the right hand side of equation (21) does not vanish and the normalization constant, $N_{n\ell}^2$, occurring in (17) is given by (21).

ii) When

$$\left( \frac{df_\ell(-k)}{dk} \right)_{k_n} = 0,$$

the integral in the right hand side of (21) vanishes,

$$\lim_{\nu \to 0} \int_0^\infty \exp(-\nu r^2) \phi^2_\ell(k_n, r) dr = 0$$

and the Jost function $f_\ell(-k)$ has a multiple zero at $k = k_n$. In this case, the Green’s function $G^{(+)}_{\ell}(k; r, r')$, the scattering wave function $\psi^{(+)}_{\ell}(k, r)$ and the scattering matrix $S_\ell(k)$ have a multiple pole at $k = k_n$. The normalization constant of the Gamow eigenfunction is no longer given by (21).

Furthermore, it will be shown below that when $f_\ell(-k)$ has a multiple zero (a multiple resonant pole of rank $r$ in $G^{(+)}_{\ell}(k; r, r')$, $\psi^{(+)}_{\ell}(k, r)$ and $S_\ell(k)$) the corresponding complex energy eigenvalues are degenerate even in the absence of symmetry. That is, the no-crossing rule does not hold for resonant eigenstates.
6 Completeness and the expansion in complex resonance energy eigenfunctions

In this section, it will be shown that associated to a double zero of the Jost function \( f_{\ell}(-k) \) (double pole of the scattering wave function \( \psi_\ell^{(+)}(k, r) \), the Green’s function \( G_\ell^{(+)}(k; r, r') \) and the scattering matrix \( S_\ell(k) \)) there is a chain of generalized Gamow-Jordan eigenfunctions which together with the bound state and resonant state eigenfunctions form a biorthonormal set which may be completed with a continuum of scattering wave functions of complex wave number.

Given two square integrable and very well behaved functions \( \Phi(r) \) and \( \chi(r) \) which decrease at infinity faster than any exponential, the completeness of the orthonormal set of bound state and scattering solutions of the radial Schrödinger equation\(^{21}\) allows us to write

\[
<\Phi|\chi> = \sum_{s \text{ bound states}} <\Phi|v_{s,\ell}><v_{s,\ell}|\chi> + \frac{2}{\pi} \int_{0}^{\infty} <\Phi|\psi_\ell^{(+)}(k')><\psi_\ell^{(+)}(k')|\chi> dk' \tag{24}
\]

where \(<\Phi|\chi>\) is the standard Dirac bracket notation

\[
<\Phi|\chi> = \int_{0}^{\infty} \Phi^*(r)\chi(r)dr. \tag{25}
\]

We shall assume that the Jost function \( f_{\ell}(-k) \) has a double zero at \( k = k_m \) in the fourth quadrant of the complex \( k' \) plane, all other zeroes of \( f_{\ell}(-k') \) in that quadrant being simple. Then, the scattering function \( \psi_\ell^{(+)}(k', r) \) as function of \( k' \)-complex, has one double resonance pole at \( k' = k_m \) and simple resonance poles at \( k = k_n, n = 1, 2, ..., m - 1, m + 1, \ldots \), all in the fourth quadrant of the complex \( k' \) plane. The function \( \psi^{(+)*}(k', r) \) is regular and has no poles in the lower half of the \( k' \) plane.

In order to make explicit the contribution of the resonant states to the expansion in eigenfunctions, the integration contour in the second term in the right hand side of (24) is deformed as shown in Fig. 1.

When the deformed contour \( C \) crosses over resonant poles, the theorem of the residue gives

\[
<\Phi|\chi> = \sum_{s \text{ bound states}} <\Phi|v_{s,\ell}><v_{s,\ell}|\chi> + \sum_{\text{all resonance poles}} 2\pi i \text{Res} \left[ \frac{2}{\pi} <\Phi|\psi_\ell^{(+)}(k')><\psi_\ell^{(+)}(k')|\chi> \right] + \frac{2}{\pi} \int_{C} <\Phi|\psi_\ell^{(+)}(k')><\psi_\ell^{(+)}(k')|\chi> dk'. \tag{26}
\]

The residues may be readily computed from equations (11) and (26).

When \( f_{\ell}(-k') \) has a simple zero at \( k' = k_n, \ldots \),
Figure 1: Integration contour $C$ in the complex $k'$-plane.

\[ 2\pi i \text{Res} \left[ \frac{\pi}{2} \left< \Phi | \psi_\ell^{(+)}(k') \right> \left< \psi_\ell^{(+)}(k') | \chi \right> \right]_{k' = k_n} = \]

\[ 4i \text{Res} \left[ \frac{\left< \Phi | \phi_\ell(k') \right> \left< \phi_\ell(k') | \chi \right>}{{(k' - k_n)} \left( \frac{df_\ell(-k')}{dk'} \right)_{k_n}} \right]_{k' = k_n} = \]

\[ \frac{1}{4ik_n^{(2\ell+1)}} \left( \frac{df_\ell(-k')}{dk'} \right)_{k_n} \left[ \left< \Phi | \phi_\ell(k') \right> \right]_{k' = k_n} \left[ \left< \phi_\ell(k') | \chi \right> \right]_{k' = k_n} \]

(27)

where

\[ \left[ \left< \Phi | \phi_\ell(k') \right> \right]_{k' = k_n} = \lim_{k' \to k_n} \int_{0}^{\infty} \Phi^*(r) \phi_\ell(k', r) dr, \]

(28)

and,

\[ \left[ \left< \phi_\ell(k') | \chi \right> \right]_{k' = k_n} = \lim_{k' \to k_n} \int_{0}^{\infty} \phi_\ell(k', r) \chi(r) dr. \]

(29)

since $\phi_\ell(k', r)$ is real and bounded for $k'$ real, the integrals in (28) and (29) exist.

Furthermore, since $\phi_\ell(k_n, r)$ is an outgoing wave which oscillates between envelopes that grow exponentially at infinity and $\Phi(r)$ and $\chi(r)$ are very well behaved functions of $r$ that decrease at infinity faster than any exponential, the integrals of the products $\Phi^*(r) \phi_\ell(k_n, r)$
and \( \phi_{\ell}(k_n, r)\chi(r) \) also exist, and we may take the limit indicated in the right hand side of equations (28) and (29) under the integration sign.

Therefore,

\[
2\pi i \text{Res} \left[ \frac{2}{\pi} < \phi_{\ell}(k') > < \psi_{\ell}(k') \mid \chi > \right]_{k' = k_n} = < \phi \mid u_{n\ell}(k_n) > < u_{n\ell}(k_n) \mid \chi >
\]

where the notation means,

\[
< \phi \mid u_{n\ell}(k_n) > = \int_0^\infty \Phi^*(r) u_{n\ell}(k_n, r) dr
\]

and

\[
< u_{n\ell}(k_n) \mid \chi > = \int_0^\infty u_{n\ell}(k_n, r) \chi(r) dr.
\]

The Gamow eigenfunction or normal mode, \( u_{n\ell}(k_n, r) \), is given by (18) and the normalization constant \( N_{n\ell} \) is given by

\[
N_{n\ell}^2 = \frac{1}{i4k_n^{2\ell+1}} f_{\ell}(k_n) \left( \frac{df_{\ell}(-k')}{dk'} \right)_{k_n};
\]

in agreement with Berggren’s result given in equation (21).

When the Jost function \( f_{\ell}(-k') \) has a double zero at \( k' = k_m \), \( \psi_{\ell}(k', r) \) has a double pole at \( k' = k_m \),

\[
\psi_{\ell}(k', r) = \frac{\phi_{\ell}(k', r) k'^{\ell+1}}{(k' - k_m)^2 g_{\ell m}(k')}.
\]

the function \( g_{\ell m}(k') \) is regular at \( k' = k_m \) and may be expanded as

\[
g_{\ell m}(k') = \frac{1}{2} \left( \frac{d^2 f_{\ell}(-k')}{dk'^2} \right)_{k_m} + \frac{1}{6} (k' - k_m) \left( \frac{d^3 f_{\ell}(-k')}{dk'^3} \right)_{k_m} + \ldots.
\]

with

\[
\left( \frac{d^2 f_{\ell}(-k')}{dk'^2} \right)_{k_m} \neq 0.
\]

The function \( \psi_{\ell}^{(+)*}(k', r') \) is regular at \( k' = k_m \), since \( f_{\ell}(k') \) has no zeroes in the lower half of the complex \( k' \)-plane,
\begin{equation}
\psi^{(+)\ast}_\ell(k', r') = \frac{\phi_\ell(k', r')k^{(\ell+1)}}{f_\ell(k')}.
\end{equation}

Thus, the residue of the term \((2/\pi) < \Phi|\psi^{(+)\ast}_\ell(k') > < \psi^{(+)\ast}_\ell(k')|\chi >\) at the double pole in \(k' = k_m\) is obtained from the Cauchy integral formula as

\begin{align*}
2\pi i \text{Res} \left[ \frac{2}{\pi} < \Phi|\psi^{(+)\ast}_\ell(k') > < \psi^{(+)\ast}_\ell(k')|\chi > \right]_{k' = k_m} &= \\
4i \text{Res} \left[ \frac{< \Phi|\phi_\ell(k') > < \phi_\ell(k')|\chi > k^{2(\ell+1)}}{(k' - k_m)^2g_{\ell m}(k')f_\ell(k')} \right]_{k' = k_m} &= \\
4i \left[ \frac{d}{dk'} \left( \frac{< \Phi|\phi_\ell(k') > < \phi_\ell(k')|\chi > k^{2(\ell+1)}}{g_{\ell m}(k')f_\ell(k')} \right) \right]_{k' = k_m} &= \quad (38)
\end{align*}

After computing the derivative indicated in (38) and rearranging some terms, we obtain

\begin{align*}
2\pi i \text{Res} \left[ \frac{2}{\pi} < \Phi|\psi^{(+)\ast}_\ell(k') > < \psi^{(+)\ast}_\ell(k')|\chi > \right]_{k' = k_m} &= \\
\frac{1}{N_{m\ell}^2} \left[ < \Phi|\hat{\phi}_\ell(k_m) > < \phi_\ell(k_m)|\chi > + < \Phi|\phi_\ell(k_m) > < \hat{\phi}_\ell(k_m)|\chi > \right], \quad (39)
\end{align*}

where, according to (17), \(\phi_\ell(k_m, r)\) is the non-normalized Gamow eigenfunction, and \(\hat{\phi}_\ell(k_m, r)\) is a generalized Gamow-Jordan eigenfunction or abnormal mode given by

\begin{equation}
\hat{\phi}_\ell(k_m, r) = \frac{d\phi_\ell(k_m, r)}{d\mathcal{E}_m} + C_\ell(k_m)\phi_\ell(k_m, r), \quad (40)
\end{equation}

\(\mathcal{E}_m\) is the complex energy eigenvalue, \(\mathcal{E}_m = (\hbar^2/2\mu)k_m^2\), and the constant factor \(C_\ell(k_m)\), multiplying \(\phi_\ell(k_m, r)\) in equation (40), is

\begin{equation}
C_\ell(k_m) = \frac{2\mu}{\hbar^2} \left[ \frac{\ell + 1}{k_m} - \frac{1}{2f_\ell(k_m)} \frac{df_\ell(k_m)}{dk_m} - \frac{1}{6} \left( \frac{d^2 f_\ell(-k')}{dk'^2} \right)_{k_m}^{-1} \left( \frac{d^3 f_\ell(-k')}{dk'^3} \right)_{k_m} \right] \quad (41)
\end{equation}

The normalization constant \(N_{m\ell}^2\) is now

\begin{equation}
N_{m\ell}^2 = \frac{2\mu}{\hbar^2} \left( \frac{1}{16\ell k_m^{2\ell+3}} f_\ell(k_m) \left( \frac{d^2 f_\ell(-k')}{dk'^2} \right)_{k_m} \right). \quad (42)
\end{equation}

The expression (39) suggests the following normalization rule for the chain of Gamow-Jordan generalized eigenfunctions belonging to a double zero of the Jost function

\begin{equation}
u_{m\ell}(k_m, r) = \frac{1}{N_{m\ell}} \phi_\ell(k_m, r), \quad (43)
\end{equation}
and

$$\hat{u}_{m\ell}(k_m, r) = \frac{1}{N_{m\ell}} \hat{\phi}_\ell(k_m, r).$$  \hspace{1cm} (44)$$

Substitution of (43) and (44) in (39) gives

$$2\pi i \text{Res} \left[ \frac{2}{\pi} < \Phi|\psi^{(+)}_\ell(k') > < \psi^{(+)}_\ell(k')|\chi > \right]_{k'=k_m} =$$

$$< \Phi|\hat{u}_{m\ell}(k_m) > < u_{m\ell}(k_m)|\chi > + < \Phi|u_{m\ell}(k_m) > < \hat{u}_{m\ell}(k_m)|\chi >$$  \hspace{1cm} (45)$$

where, the notation means,

$$< \Phi|\hat{u}_{m\ell}(k_m) > = \int_0^\infty \Phi^*(r) \hat{u}_{m\ell}(k_m, r) dr$$  \hspace{1cm} (46)$$

and

$$< \hat{u}_{m\ell}(k_m)|\chi > = \int_0^\infty \hat{u}_{m\ell}(k_m, r) \chi(r) dr,$$  \hspace{1cm} (47)$$

$\hat{u}_{m\ell}(k_m, r)$ is defined in (44).

Finally, substitution of the expressions (30) and (45) in (26) gives the following expansion

$$< \Phi|\chi > = \sum_{s \text{ bound states}} < \Phi|v_{s\ell} > < v_{s\ell}|\chi > + \sum_{n \neq m \text{ resonances}} < \Phi|u_{n\ell} > < u_{n\ell}|\chi >$$

$$+ < \Phi|\hat{u}_{m\ell}(k_m) > < u_{m\ell}(k_m)|\chi > + < \Phi|u_{m\ell}(k_m) > < \hat{u}_{m\ell}(k_m)|\chi >$$

$$+ \frac{2}{\pi} \int_c < \Phi|\psi^{(+)}_\ell(k') > < \psi^{(+)}_\ell(k')|\chi > dk'.$$  \hspace{1cm} (48)$$

This expression shows that, when the Jost function has many simple zeroes and one double zero in the fourth quadrant of the complex $k$-plane, the Gamow eigenfunctions $u_{n\ell}(k_m, r)$ associated to simple zeroes of the Jost function and the chain $\{u_{m\ell}(k_m, r), \hat{u}_{m\ell}(k_m, r)\}$ of Gamow-Jordan generalized eigenfunctions 25, 26, 27 associated to the double pole of the Jost function are basis elements of an expansion in generalized bound and resonant state eigenfunctions plus a continuum of scattering functions of complex wave values $k'$.

Omitting the arbitrary function $\Phi(r)$ in (48), we obtain the complex basis expansion of an arbitrary square integrable and well behaved function $\chi(r)$

$$\chi(r) = \sum_{s \text{ bound states}} v_{s\ell}(r) < v_{s\ell}|\chi > + \sum_{n \neq m} u_{n\ell}(k_n, r) < u_{n\ell}|\chi >$$

$$+ \hat{u}_{m\ell}(k_m, r) < u_{m\ell}|\chi > + u_{m\ell}(k_m, r) < \hat{u}_{m\ell}|\chi >$$

$$+ \frac{2}{\pi} \int_c \psi^{(+)}_\ell(k', r) < \psi^{(+)}_\ell(k')|\chi > dk'.$$  \hspace{1cm} (49)$$
In this expression \( u_{n\ell}(k_n, r) \) are the Gamow eigenfunctions representing decaying states associated to simple resonance poles of the scattering wave function \( \psi^{(+)}_\ell(k, r) \), the matrix \( S(k) \) and the Green’s function \( G^{(+)}(k; r, r') \). The set \( \{ u_{m\ell}(k_m, r), \hat{u}_{m\ell}(k_m, r) \} \) is a Jordan chain of length two of generalized Gamow-Jordan eigenfunctions associated to the double pole of the scattering matrix \( S(k) \) and the Green’s function \( G^{(+)}(k; r, r') \) at \( k = k_m \). The last term in the right hand side of (48) and (49) is the background integral defined along the integration contour shown in Fig 1.

7 Jordan blocks in the complex energy basis

Once it has been established that the Gamow eigenfunctions \( u_{n\ell}(k_n, r) \) and the Jordan chain \( \{ u_{m\ell}(k_m, r), \hat{u}_{m\ell}(k_m, r) \} \) of generalized Gamow-Jordan eigenfunctions are elements of the basis set of eigenfunctions in the expansions (18) and (19), we may represent any operator \( f(H^{(\ell)}_r) \), which is a regular function of the Hamiltonian \( H^{(\ell)}_r \), in terms of its matrix elements in this basis.

Let us start by deriving an expression for the action of \( f(H^{(\ell)}_r) \) on the generalized Gamow-Jordan eigenfunction \( \hat{u}_{m\ell}(k_m, r) \). With this purpose in mind, let us write the eigenvalue equation satisfied by \( u_{m\ell}(k_m, r) \) as

\[
H^{(\ell)}_r u_{m\ell}(k_m, r) = \mathcal{E}_m u_{m\ell}(k_m, r),
\]

where,

\[
H^{(\ell)}_r = -\frac{\hbar^2}{2\mu} \left( \frac{d^2}{dr^2} - v(r) - \ell(\ell + 1) r^2 \right),
\]

\( v(r) \) is a well behaved short ranged potential which satisfies the conditions stated in section 1. Now, let us consider a holomorphic function \( f(\mathcal{E}) \) of the complex variable \( \mathcal{E} \), such that,

\[
f(\mathcal{E}) = \sum_{j=0}^{\infty} a_j \mathcal{E}^j,
\]

the coefficients \( a_j \) are independent of \( \mathcal{E} \).

Then, from (51) and (52),

\[
f(H^{(\ell)}_r) u_{m\ell}(k_m, r) = f(\mathcal{E}_m) u_{m\ell}(k_m, r).
\]

Taking derivatives with respect to the eigenvalue \( \mathcal{E}_m \) in both sides of (53), we obtain,

\[
f(H^{(\ell)}_r) \frac{\partial u_{m\ell}(k_m, r)}{\partial \mathcal{E}_m} = f(\mathcal{E}_m) \frac{\partial u_{m\ell}(k_m, r)}{\partial \mathcal{E}_m} + \frac{\partial f(\mathcal{E}_m)}{\partial \mathcal{E}_m} u_{m\ell}(k_m, r).
\]
From this equation and the definition, equations (40), (41) and (44), of \( \hat{u}_{ml}(k_m, r) \), it follows immediately that

\[
f(H_r^{(l)}) \hat{u}_{ml}(k_m, r) = f(\mathcal{E}_m) \hat{u}_{ml}(k_m, r) + \frac{\partial f(\mathcal{E}_m)}{\partial \mathcal{E}_m} u_m(k_m, r).
\]  

(55)

Notice that a necessary and sufficient condition for the existence of \( \partial u_{ml}(k_m, r)/\partial \mathcal{E}_m \) is the vanishing of \( \left( df(-k)/dk \right)_{k_m} \).

The rule stated in equation (55) permits us to calculate the action of \( f(H_r^{(l)}) \) on the generalized Gamow-Jordan vectors occurring in the complex basis expansions (48) and (49).

Now, we can write the operator \( f(H_r^{(l)}) \) in terms of its matrix elements in the complex energy basis. This may be done by acting with \( f(H_r^{(l)}) \) on the left in both sides of equation (49),

\[
f(H_r^{(l)}) \chi(r) = \sum_s f(\mathcal{E}_s)v_{sl}(r) < v_{sl} | \chi > + \sum_{n \neq m} f(\mathcal{E}_n)u_{nl}(k_n, r) < u_{nl} | \chi > \\
+ \left( f(\mathcal{E}_m) \hat{u}_{ml}(k_m, r) + \frac{\partial f(\mathcal{E}_m)}{\partial \mathcal{E}_m} u_m(k_m, r) \right) < u_{ml} | \chi > \\
+ f(\mathcal{E}_m) u_m(k_m, r) < \hat{u}_{ml} | \chi > \\
+ \frac{2}{\pi} \int_c f(\mathcal{E}') \phi^{(+)}_\ell(k', r) < \phi^{(+)}_\ell(k') | \chi > dk'.
\]  

(56)

Multiplying both sides of (56) by \( \Phi^*(r) \) and integrating over \( r \), we get,

\[
< \Phi | f(H_r^{(l)}) | \chi > = \sum_s < \Phi | v_{sl} > f(\mathcal{E}_s) < v_{sl} | \chi > + \sum_{n \neq m} < \Phi | u_{nl} > f(\mathcal{E}_n) < u_{nl} | \chi > \\
+ < \Phi | \hat{u}_{ml} > f(\mathcal{E}_m) < u_{ml} | \chi > + < \Phi | u_{ml} > \left( f(\mathcal{E}_m) < \hat{u}_{ml} | \chi > \right) \\
+ \frac{\partial f(\mathcal{E}_m)}{\partial \mathcal{E}_m} < u_{ml} | \chi > + \frac{2}{\pi} \int_c < \Phi | \psi^{(+)}_\ell(k') > f(\mathcal{E}') < \psi^{(+)}_\ell(k') | \chi > dk'.
\]  

(57)

To simplify the notation, suppose that the system has no bound states only resonances and that the first two resonances are degenerate. Rearranging equation (57) in matrix form, we get

\[
< \Phi | f(H_r^{(l)}) | \chi > = \left( < \Phi | u_{1\ell} >, < \Phi | \hat{u}_{1\ell} >, < \Phi | u_{3\ell} >, ..., \right) \times \\
\begin{bmatrix}
    f(\mathcal{E}_1) & \frac{\partial f(\mathcal{E}_1)}{\partial \mathcal{E}_1} & 0 & 0 & 0 & 0 & ... \\
    0 & f(\mathcal{E}_1) & 0 & 0 & 0 & 0 & ... \\
    0 & 0 & f(\mathcal{E}_2) & 0 & 0 & 0 & ... \\
    0 & 0 & 0 & f(\mathcal{E}_3) & 0 & 0 & ... \\
    ... & ... & ... & ... & ... & ... & ...
\end{bmatrix} \\
\begin{bmatrix}
    < \hat{u}_{1\ell} | \chi > \\
    < u_{1\ell} | \chi > \\
    < u_{3\ell} | \chi > \\
    . \\
    . \\
    . \\
    . \\
\end{bmatrix}
\]

\[
+ \frac{2}{\pi} \int_c < \Phi | \psi^{(+)}_\ell(k') > f(\mathcal{E}') < \psi^{(+)}_\ell(k') | \chi > dk'.
\]  

(58)
In this matrix representation of $f(H_{r}^{(\ell)})\frac{1}{\partial f} \mathcal{E}_{1}$ the upper left $2 \times 2$ submatrix is a Jordan block of rank two associated to the chain of Gamow-Jordan generalized eigenfunctions \{ $\hat{u}_{1\ell}(k_{1}, r)$, $u_{1\ell}(k_{1}, r)$ \} belonging to the double zero of the Jost function $f_{r}(-k)$ (double pole of the scattering matrix and the Green’s function). Except for this $2 \times 2$ block, this matrix is diagonal with the eigenvalues $f(\mathcal{E}_{n})$ in the diagonal entries. Simple zeroes of the Jost function correspond to simple (non-repeated) eigenvalues of $f(H_{r}^{(\ell)})$ while the double zero of $f_{r}(-k)$ correspond to the twice repeated (degenerate) eigenvalue $f(\mathcal{E}_{1})$ occurring in the Jordan block. The off-diagonal non-vanishing element in this block is $\frac{\partial f(\mathcal{E}_{1})}{\partial f} \mathcal{E}_{1}$.

The difference in physical dimensions of the off-diagonal and the diagonal entries in the $2 \times 2$ Jordan block is compensated by the difference in normalization of the Gamow-Jordan chain $\{ \hat{u}_{1\ell}(k_{1}, r), u_{1\ell}(k_{1}, r) \}$ and the Gamow eigenfunctions $u_{n\ell}(k_{n}, r)$ ($n = 3, 4, ...$) which are normalized according to \{13, 14\} and \{17, 18, 33\} respectively.

It will be instructive to consider some simple examples.

We first choose $f(H_{r}^{(\ell)}) = H_{r}^{(\ell)}$. Then, from (58) we obtain,

$$< \Phi | H_{r}^{(\ell)} | \chi > = \begin{pmatrix} < \Phi | u_{1\ell} >, < \Phi | \hat{u}_{1\ell} >, < \Phi | u_{3\ell} >, ..... \end{pmatrix} \times \begin{pmatrix} \mathcal{E}_{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \ldots & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} < \hat{u}_{1\ell} | \chi > \\ < u_{1\ell} | \chi > \\ < u_{3\ell} | \chi > \end{pmatrix} + \frac{2}{\pi} \int_{c} < \Phi | \psi_{\ell}^{(+)}(k') > \mathcal{E}' < \psi_{\ell}^{(+)}(k') | \chi > dk' \label{59}$$

From this example, it is evident that in a degeneracy of two resonances in the absence of symmetry, the degenerate complex eigenvalue $\mathcal{E}_{1}$ occurs twice in the spectral representation of the radial Hamiltonian $H_{r}^{(\ell)}$ given in \{29\}, while there is only one Gamow eigenvector or normal mode, $u_{1\ell}(k_{1}, r)$, associated to the degeneracy. This is so, because the Gamow-Jordan generalized eigenfunction or abnormal mode, $\hat{u}_{1\ell}(k_{1}, r)$, is not an eigenfunction of the radial Hamiltonian $H_{r}^{(\ell)}$. This is a generic property of this kind of degeneracy which may be stated in slightly more formal terms as follows: In a degeneracy of resonances in the absence of symmetry, the algebraic multiplicity is always larger than the geometric multiplicity. Here, we mean by algebraic multiplicity of a degeneracy, $\mu_{a}$, the number of times the degenerate complex eigenvalue is repeated, and, by geometric multiplicity of the degeneracy, $\mu_{g}$, the dimensionality of the subspace spanned by the eigenvectors associated to the degenerate eigenvalue \{25, 26, 27\}.

Then,

$$\mu_{a} > \mu_{g} \label{60}$$

\[1\] From the way it was derived, it is evident that the matrix in equation (58) represents the action of $f(H_{r}^{(\ell)})$ as an operator on the space of continuous antilinear functionals on the Schwarz space of very well behaved test functions.
Let us consider now, the complex energy representation of the resolvent operator. In this case \( f(H_\ell^{(t)}) = 1/(E - H_\ell^{(t)}) \). Then, from (58), we obtain,

\[
< \Phi \bigg| \frac{1}{E - H_\ell^{(t)}} \bigg| \chi >= \left( < \Phi | u_{1\ell} >, < \Phi | \hat{u}_{1\ell} >, < \Phi | u_{3\ell} >, \ldots \right) \times
\]

\[
\begin{pmatrix}
\frac{1}{E - \varepsilon_1} & 0 & 0 & 0 \ldots \\
0 & \frac{1}{E - \varepsilon_1^2} & 0 & 0 \ldots \\
0 & 0 & \frac{1}{E - \varepsilon_3} & 0 \ldots \\
0 & 0 & 0 & \frac{1}{E - \varepsilon_4} \ldots
\end{pmatrix}
\begin{pmatrix}
< \hat{u}_{1\ell} | \chi > \\
< u_{1\ell} | \chi > \\
< u_{3\ell} | \chi > \\
\cdots
\end{pmatrix}
\]

\[
+ \frac{2}{\pi} \int_C < \Phi | \psi_\ell^{(+)}(k') > \frac{1}{(E - \varepsilon')} < \psi_\ell^{(+)}(k') | \chi > dk'.
\] (61)

It may easily be verified that, when we delete the arbitrary functions \( \Phi(r) \) and \( \chi(r) \) in this expression, the resulting expansion for \( < r | \frac{1}{E - H_\ell^{(t)}} | r' > \) is just the expansion in resonance eigenfunctions of the complete Green’s function

\[
G_\ell^{(+)}(k; r, r') = \frac{\hbar^2}{2\mu} \left[ \sum_{s \text{ bound states}} \frac{v_{s\ell}(k, r)v_{s\ell}^*(k, r')}{E + |E_s|} \\
+ \sum_{n \neq m \text{ resonant states}} \frac{u_{n\ell}(k_n, r)u_{m\ell}(k_m, r')}{E - \varepsilon_n} \\
+ \frac{(E - \varepsilon_m)^2}{u_{m\ell}(k_m, r)\hat{u}_{m\ell}(k_m, r') + \hat{u}_{m\ell}(k_m, r)u_{m\ell}(k_m, r')} \\
+ \frac{2}{\pi} \int_C \frac{\psi_\ell^{(+)}(k', r)\psi_\ell^{(+)*}(k', r')}{(k^2 - k'^2)} dk'.
\] (62)

The occurrence of the double pole in \( G_\ell^{(+)}(k; r, r') \), as function of the complex energy, is thus associated to the occurrence of a Jordan block of rank two in the complex basis representation of the resolvent operator and a Jordan chain of Gamow-Jordan generalized eigenfunctions \{ \hat{u}_{1\ell}(k_1, r), u_{1\ell}(k_1, r) \} associated to the double zero of the Jost function.

Finally, let us consider the time evolution operator \( \exp (-iHt) \). For each fixed value of the angular momentum, it will be enough to consider the operator \( f(H_\ell^{(t)}) = \exp (-iH_\ell^{(t)}t) \). In this case, from equation (58)
The time evolution operator is non-diagonal in the complex energy basis representation. The time evolution of the Jordan chain of Gamow-Jordan generalized eigenfunctions \( \{ \hat{u}_{1\ell}(k_1, r), u_{1\ell}(k_1, r) \} \) is given by a Jordan block of \( 2 \times 2 \) with an exponential time dependence in the diagonal entries and a first order polynomial times an exponential in the off-diagonal entry. Hence, the time evolution of the Gamow-Jordan generalized eigenfunction or abnormal mode is a superposition of the abnormal mode \( \hat{u}_{1\ell}(k_1, r) \) evolving exponentially in time plus the normal mode \( u_{1\ell}(k, r) \) evolving according to the product of a first order polynomial times an exponential time evolution factor. The time evolution of the normal mode \( u_{1\ell}(k_1, r) \) in the Gamow-Jordan chain \( \{ \hat{u}_{1\ell}(k_1, r), u_{1\ell}(k_1, r) \} \), as well as the time evolution of all other normal modes \( u_{n\ell}(k_n, r) \) associated to the simple zeroes of the Jost function (simple poles of the scattering matrix) is purely exponential.

### 8 Orthogonality and normalization integrals for Gamow-Jordan eigenfunction.

As in the case of bound and resonant state eigenfunctions associated with simple poles of the Green’s function, we may derive orthogonality and normalization rules for the Gamow-Jordan eigenstates in terms of regularized integrals of the generalized Gamow-Jordan eigenfunctions. Following the same procedure as in Berggren, \( ^{23} \) \( ^{24} \), it may be shown that, when \( f_\ell(-k') \) has a double zero at \( k' = k_m \), the following relations are valid,

\[
\frac{1}{i 8 k_m^{2(\ell+1)}} f_\ell(k_m) \left( \frac{d^2 f_\ell(-k')}{dk'^2} \right)_{k' = k_m} = \lim_{\nu \to 0} \int_0^\infty e^{-\nu r^2} \frac{d\phi_\ell(k_m, r)}{dk_m} \phi_\ell(k_m, r) dr
\]

and

\[
\frac{1}{i 8 k_m^{2(\ell+1)}} f_\ell(k_m) \left[ \frac{1}{3} \left( \frac{d^3 f_\ell(-k')}{dk'^3} \right)_{k' = k_m} - \left( \frac{d^2 f_\ell(-k')}{dk'^2} \right)_{k' = k_m} \frac{2(\ell + 1)}{k_m} \left( \frac{d f_\ell(k_m)}{dk_m} \right) \right] = \lim_{\nu \to 0} \int_0^\infty e^{-\nu r^2} \left( \frac{d\phi_\ell(k_m, r)}{dk_m} \right)^2 dr.
\]

From the expression \( ^{21} \) for \( C_\ell(k_m) \) and equations \( ^{64} \) and \( ^{23} \), it follows that,
\[
\lim_{\nu \to 0} \int_0^\infty e^{-\nu r^2} \left( \frac{d\phi_\ell(k_m, r)}{dk_m} \right)^2 dr + 2C_\ell(k_m) \hbar^2 k_m \left( \lim_{\nu \to 0} \int_0^\infty e^{-\nu r^2} \frac{d\phi_\ell(k_m, r)}{dk_m} \phi_\ell(k_m, r) dr \right) = 0,
\]

which may be rewritten as,
\[
\lim_{\nu \to 0} \int_0^\infty e^{-\nu r^2} \left[ \frac{d\phi_\ell(k_m, r)}{d\mathcal{E}_m} + C_\ell(k_m) \phi_\ell(k_m, r) \right]^2 dr = C_\ell^2(k_m) \lim_{\nu \to 0} \int_0^\infty e^{-\nu r^2} \hat{\phi}_\ell^2(k_m, r) dr,
\]

but, according to equation (22) and (23), when \(f_\ell(-k)\) has a double zero at \(k = k_m\), the integral in the right hand side of (64) vanishes. Therefore, the integrand in the left hand side of (67) is the square of the generalized Jordan-Gamow eigenfunction and the relation (65) translates into
\[
\lim_{\nu \to 0} \int_0^\infty e^{-\nu r^2} \hat{\phi}_\ell^2(k_m, r) dr = 0
\]

which shows that also the regularized integral of the square of the generalized Gamow-Jordan eigenfunction vanishes.

An expression for the normalization constant \(N_{m\ell}^2\) in terms of a normalization integral may be obtained from (64),
\[
N_{m\ell}^2 = \lim_{\nu \to 0} \int_0^\infty e^{-\nu r^2} \frac{d\phi_\ell(k_m, r)}{d\mathcal{E}_m} \phi_\ell(k_m, r) dr,
\]

writing \(d\phi_\ell/d\mathcal{E}_m\) in terms of \(\hat{\phi}_\ell(k_m, r)\) and recalling that the integral of \(\phi_\ell^2(k_m, r)\) vanishes, we get,
\[
N_{m\ell}^2 = \lim_{\nu \to 0} \int_0^\infty e^{-\nu r^2} \hat{\phi}_\ell(k_m, r) \phi_\ell(k_m, r) dr,
\]

which shows that the right hand side of (70) is the normalization integral for the Gamow-Jordan generalized eigenfunctions associated with a double pole degeneracy of resonances with \(N_{m\ell}^2\) as given in (42). However, it is convenient to note that this expression does not fix the normalization rule for \(\phi_\ell(k_m, r)\) and \(\hat{\phi}_\ell(k_m, r)\) in a unique way. Since \(\phi_\ell(k_m, r)\) and \(\hat{\phi}_\ell(k_m, r)\) are linearly independent, they have different dimensions and its product has no obvious interpretation in terms of observable quantities, therefore, there is no a priori reason to normalize both functions with the same normalization constant. Thus, we still have the freedom to write (70) as
\[
\lim_{\nu \to 0} \int_0^\infty e^{-\nu r^2} \left( \frac{X_m}{N_{m\ell}} \hat{\phi}_\ell(k_m, r) \right) \left( \frac{1}{X_m N_{m\ell}} \phi_\ell(k_m, r) \right) dr = 1,
\]
where $N_{m\ell}^2$ is given in (12) and $X_m$ is a non-vanishing real or complex number that we associate with the double pole singularity of $G_{\ell}^{(+)}(k; r, r')$ at $k = k_m$. Therefore, a more general normalization rule for the Gamow and Gamow-Jordan generalized eigenfunction that the one proposed in (12), (13) and (14) would be

$$u_{m\ell}(k_m, r) = \frac{1}{X_m N_{m\ell}} \phi_{\ell}(k_m, r)$$

and

$$\hat{u}_{m\ell}(k_m, r) = \frac{X_m}{N_{m\ell}} \hat{\phi}_{\ell}(k_m, r).$$

With this normalization, the orthogonality and normalization integrals for the generalized Gamow-Jordan eigenfunction associated to a double pole of the Green’s function, equations (23), (68) and (70), take the form

$$\lim_{\nu \to 0} \int_0^{\infty} e^{-\nu r^2} u_{m\ell}^2(k_m, r) dr = 0$$

(74)

$$\lim_{\nu \to 0} \int_0^{\infty} e^{-\nu r^2} \hat{u}_{m\ell}^2(k_m, r) dr = 0$$

(75)

and

$$\lim_{\nu \to 0} \int_0^{\infty} e^{-\nu r^2} u_{m\ell}(k_m, r) \hat{u}_{m\ell}(k, r) dr = 1.$$

(76)

The form of these orthogonality and normalization conditions is independent of the value of the constat $X_m$. However, if the Gamow-Jordan generalized eigenfunction are normalized according to (12) and (13) the expression for the residue at the double pole of $G_{\ell}^{(+)}(k; r, r')$ would be explicitly dependent on $X_m$, since a factor $X_m^2$ will appear multiplying the term $u_{m\ell}(k_m, r)u_{m\ell}(k_m, r')$ in the expression for the residue at the double pole of $G_{\ell}^{(+)}(k; r, r')$ given in equation (12).

$$\frac{X_m^2 u_{m\ell}(k_m, r) u_{m\ell}(k_m, r')}{(E - \varepsilon_m)^2} + \frac{u_{m\ell}(k_m, r) \hat{u}_{m\ell}(k_m, r')}{(E - \varepsilon_m)} + \frac{\hat{u}_{m\ell}(k_m, r) u_{m\ell}(k_m, r')}{(E - \varepsilon_m)}.$$

(77)

As is evident from the definition (11), the generalized eigenfunctions $\phi_{n\ell}(k_m, r)$ and $\hat{\phi}_{n\ell}(k_m, r)$ have different dimensions, if one takes $X_m$ of dimension (energy)$^{1/2}$ the normalized eigenfunctions $u_{n\ell}(k_n, r)$ and $\hat{u}_{n\ell}(k_n, r)$ have the same dimensions namely (energy)$^{-1/2}$ so that when $(X_m) = (\text{energy})^{1/2}$ the higher order Gamow-Jordan vectors become Jordan vectors with the same dimensions as the Gamow vectors.

This freedom in the normalization rules could be used to define normalized Gamow-Jordan eigenfunctions with the same dimensions as those of the Gamow eigenfunctions associated to simple poles of $G_{\ell}^{(+)}(k; r, r')$.  

18
9 Summary and conclusions

In the theory of the scattering of a beam of particles by a short ranged potential, resonances are associated to the occurrence of poles of the scattering matrix \( S_\ell(k) \), the Green’s function \( G^{(+)}_\ell(k; r, r') \) and the scattering wave function \( \psi_{nl}(k, r) \). These resonance poles are caused by zeroes of the Jost function lying in the fourth quadrant of the complex \( k \)–plane. Accordingly, a degeneracy of resonances, that is, the exact coincidence of two (or more) simple resonance poles of the scattering matrix, results from the exact coincidence of two (or more) simple resonance zeroes of the Jost function, which merge into one double (or higher rank) zero lying in the fourth quadrant of the complex \( k \)–plane.

We found that, associated to a double resonance zero of the Jost function, there is a Jordan chain of length two\(^\text{[25, 26, 27]}\) of generalized Gamow-Jordan eigenfunctions \( \{\hat{u}_{ml}(k_m, r), u_{ml}(k_m, r)\} \) belonging to the same degenerate complex energy eigenvalue \( \mathcal{E}_m \). Hence, the corresponding second rank pole occurring in the scattering matrix, \( S_\ell(k) \), the Green’s function, \( G^{(+)}_\ell(k; r, r') \), and the scattering wave function, \( \psi^{(+)}_\ell(k, r) \), is also associated to this Jordan chain of Gamow-Jordan generalized resonance eigenfunctions.

As the two simple zeroes of the Jost function merge into one double zero, the two Gamow eigenfunctions corresponding to the two resonances that become degenerate merge into one Gamow eigenfunction or normal mode belonging to the double zero of the Jost function. The other element in the Jordan chain, namely, the Gamow-Jordan generalized eigenfunction or abnormal mode is not a proper eigenfunction of the radial Hamiltonian. Hence, at a degeneracy of resonances, one resonance eigenfunction or normal mode is lost, and a new kind of generalized resonance eigenfunction or abnormal mode is generated. Therefore, the dimensionality of the subspace of eigenfunctions associated to a degeneracy of two resonances or geometric multiplicity, \( \mu_g \), of the degeneracy is one, yet, the number of times the degenerate complex energy eigenvalue is repeated in the spectral representation of \( H_r^{(\ell)} \) or algebraic multiplicity of the degeneracy, \( \mu_a \), is two. It follows that, the algebraic multiplicity is larger than the geometric multiplicity of a degeneracy of resonances.

Explicit expressions for the normalized Gamow and Gamow-Jordan generalized eigenfunctions in the Jordan chain, written in terms of the outgoing wave Jost solution, the Jost function and its derivatives evaluated at the double zero, are obtained from the computation of the residue of the scattering wave \( \psi^{(+)}_\ell(k, r) \) function at the double pole.

We also showed that the Jordan chain of generalized eigenfunctions are elements of the complex biorthonormal basis formed by the real (bound states) and complex (resonance states) energy eigenfunctions which can be completed by means of a continuum of scattering wave functions of complex wave number. With the help of this result, we derived expansion theorems (spectral representations) for operators \( f(H_r^{(\ell)}) \) which are regular functions of the radial Hamiltonian \( H_r^{(\ell)} \). In this basis, the operator \( f(H_r^{(\ell)}) \) is represented by a complex matrix which is diagonal except for one Jordan block of rank two\(^\text{[25, 26, 27]}\) associated to the double zero of the Jost function and the corresponding chain of generalized eigenvectors. The diagonal entries in this matrix are the eigenvalues \( f(\mathcal{E}_n) \), simple zeroes of the Jost function correspond to non-degenerate eigenvalues of \( f(H_r^{(\ell)}) \) while the double zero of the Jost function corresponds to the twice repeated (degenerate) eigenvalue \( f(\mathcal{E}_m) \) in the diagonal entries of the Jordan block. The off-diagonal, non-vanishing element in this block is \( \partial f(\mathcal{E}_m)/\partial \mathcal{E}_n \). In particular, the occurrence
of a double pole in the Green’s function, as function of the complex energy, is thus associated to
the occurrence of a Jordan block of rank two in the complex basis representation of the resolvent
operator and the corresponding Jordan chain of Gamow-Jordan generalized eigenfunctions.

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References

[1] F. Hinterberger, et al., Nucl. Phys. A 299, (1978) 397.
[2] P. von Brentano, Phys. Rep. 264, (1996) 57.
[3] E. Hernández, A. Mondragón, Phys. Lett. B 326, (1994) 1.
[4] L.M. Baskov, et al., Nucl. Phys. B 256, (1985) 365, and references contained therein.
[5] P. von Brentano, Z. Physik A 348, (1994) 41.
[6] P. von Brentano, R.V. Jolos and H.A. Weidenmüller, Phys. Lett. B (2002). To appear in
[7] P. von Brentano, to appear in Rev. Mex. Fis. 48, (2002).
[8] O. Latinne, et al., Phys. Rev. Lett. 74, (1995) 46.
[9] N.J. Kylstra, C.J. Joachain, Phys. Rev. A57, (1998) 412.
[10] A.I. Magunov, I. Rotter and S.I. Strakhova, J. Phys. B: At. Mol. Opt. Phys. 34, (2001) 29.
[11] M. Pont, et al., Phys. Rev. A 46, (1992) 555.
[12] E. Hernández, A. Jáuregui and A. Mondragón, Rev. Mex. Fis. 38, Suppl 2, (1992) 128.
[13] A. Mondragón, E. Hernández, J. Phys. A: Math. and Gen. 29, (1996) 2567.
[14] A. Mondragón, E. Hernández, Accidental degeneracy and Berry Phase of resonant states.
In Irreversibility and Causality: Semigroups and Rigged Hilbert Space. Lecture Notes in
Physics. Edited by A. Bohm, D.-H. Doebner, P. Kielanowski, (Springer-Verlag, Berlin
1998) 504, p. 257.
[15] C. Dembowski, et al., Phys. Rev. Lett. 86, (2001) 787.
[16] W. Vanroose, et al., J. Phys. A: Math. and Gen. 30, (1997) 5543.
[17] E. Hernández, A. Jáuregui and A. Mondragón, J. Phys. A: Math. and Gen. 33, (2000) 4507.
[18] W. Vanroose, Phys. Rev. A 64, (2001) 062708-1.

[19] A. Bohm, et al., J. Math. Phys. 38, (1997) 6072.

[20] I. Antoniou, M. Gadella and G. Pronko, J. Math. Phys. 39, (1998) 2459.

[21] R.G. Newton, Scattering Theory of Waves and Particles. Second Edition, (Springer-Verlag, New York 1982) Ch. 12

[22] Ya. B. Zel’dovich, Zh. ETP(USSR) 39, (1960) 776; JETP (Sov. Phys.) 12, (1961) 542.

[23] T. Berggren, Nucl. Phys. A 109, (1968) 265.

[24] T. Berggren, Phys. Lett. B 373, (1996) 1.

[25] Tosio Kato, Perturbation Theory for Linear Operators, (Springer-Verlag, Berlin 1980).

[26] Nathan Jacobson Lecture Notes in Abstract Algebra, Vol. II Linear Algebra. (D. Van Nostrand Co. New York 1953) Ch. III

[27] P. Lancaster, and M. Tismenetsky, The Theory of Matrices Second Edition, (Academic Press, Inc. 1985).