Abelian fibrations and rational points on symmetric products

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1 Introduction

Let $X$ be an algebraic variety defined over a number field $K$ and $X(K)$ its set of $K$-rational points. We are interested in properties of $X(K)$ imposed by the global geometry of $X$. We say that rational points on $X$ are potentially dense if there exists a finite field extension $L/K$ such that $X(L)$ is Zariski dense. It is expected - at least for surfaces - that if there are no finite étale covers of $X$ dominating a variety of general type then rational points on $X$ are potentially dense. This expectation complements the conjectures of Bombieri, Lang and Vojta predicting that rational points on varieties of general type are always contained in Zariski closed subsets. This dichotomy holds for curves: the nondensity for curves of genus $g \geq 2$ is a deep theorem of Faltings and the potential density for curves of genus 0 and 1 is classical.

In higher dimensions there are at present no general techniques to prove nondensity. Of course, potential density holds for abelian and unirational varieties. Beyond this, density results rely on the classification and explicit projective geometry of the classes of varieties under consideration. In dimension two potential density is unknown for K3 surfaces with finite automorphisms and without elliptic fibrations (see [5]). In dimension 3 potential density is unknown, for example, for double covers $W_2 \rightarrow \mathbb{P}^3$ ramified in a smooth surface of degree 6, for general conic bundles, as well as for Calabi-Yau varieties (for density results see [10], [4]).

In this paper we study density properties of rational points on symmetric products $X^{(n)} = X^n/S_n$. If $C$ is a curve of genus $g$ and $n > 2g - 2$ the
symmetric product admits a bundle structure over the Jacobian \( \text{Jac}(C) \), with fibers projective spaces \( \mathbb{P}^{n-g} \). We see that in this case rational points on \( C^{(n)} \) are potentially dense. Contrary to the situation for curves, we are not guaranteed to find many rational points on sufficiently high symmetric products of arbitrary surfaces. In Section 2 we show that if the Kodaira dimension of a smooth surface \( X \) is equal to \( k \) then the Kodaira dimension of \( X^{(n)} \) is is equal to \( nk \). This leads us to expect the behavior of rational points on \( X^{(n)} \) and \( X \) to be quite similar. At the same time we observe that symmetric products of K3 surfaces admit (at least birationally) abelian fibrations over projective spaces. In fact, even symmetric squares of certain (nonelliptic) K3 surfaces have the structure of abelian surface fibrations over \( \mathbb{P}^2 \). This is the starting point for proofs of potential density of rational points.

Let us emphasize that if \( X \) is a variety over a number field \( K \) then Zariski density of rational points on \( X \) defined over degree \( n \) field extensions of \( K \) is not equivalent to Zariski density of \( K \)-rational points on \( X^{(n)} \). Of course, the first condition is weaker than the second. Furthermore, if rational points on \( X \) are potentially dense then they are potential dense on \( X^{(n)} \) as well.

This paper is organized as follows. In Section 2 we recall general properties of symmetric products and Hilbert schemes of surfaces. Section 3 sets up generalities concerning abelian fibrations \( \mathcal{A} \to B \). Potential density for \( \mathcal{A} \) follows once one can find a “nondegenerate” multisection for which potential density holds. In Section 4 we prove widely-known results concerning the existence of elliptic curves on K3 surfaces. Then we turn to potential density for symmetric products of K3 surfaces. First, in Sections 5 and 6, we prove potential density for sufficiently high symmetric powers of arbitrary K3 surfaces. This is followed in Section 7 with more precise results for symmetric squares of K3 surfaces of degree \( 2m^2 \).

Throughout this paper, \textit{generic} means ‘in a nonempty Zariski open subset’ whereas \textit{general} means ‘in a nonempty analytic open subset.’

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2 Generalities on symmetric products

Let $X$ be a smooth projective variety over a field $K$. Denote by $X^n = X \times_K \ldots \times_K X$ the $n$-fold product of $X$. The symmetric group $S_n$ acts on $X^n$. The quotient $X^{(n)} = X^n / S_n$ is a projective variety, called the symmetric product.

If $X$ has dimension 1 then $X^{(n)}$ is smooth and for $n > 2g - 2$ the symmetric product $X^{(n)}$ is a projective bundle over the Jacobian $\text{Jac}(X)$, with fibers projective spaces of dimension $n - g$ (see [19], Ch. 4). In particular, rational points on $X^{(n)}$ are potentially dense for $n > 2g - 2$.

If $X$ has dimension 2 then $X^{(n)}$ is no longer smooth; it has Gorenstein singularities (since the group action factors through the special linear group). The Hilbert scheme of length $n$ zero-dimensional subschemes is a crepant resolution of $X^{(n)}$

$$\varphi : X^{[n]} \to X^{(n)}$$

(see [1], Section 6 and the references therein). In particular, $\varphi^* \omega_{X^{(n)}} = \omega_{X^{[n]}}$. The same holds for pluricanonical differentials. On the other hand, we have the isomorphism

$$H^0(X^n, \omega_X^n)^{S_n} = H^0(X^{(n)}, \omega_{X^{(n)}}).$$

We are using the fact that the quotient map $X^n \to X^{(n)}$ is unramified away from a codimension two subset and pluricanonical differentials extend over codimension two subsets. We conclude that pluricanonical differentials on the Hilbert scheme correspond to $S_n$-invariant differentials on the $n$-fold product $X^n$:

$$H^0(X^n, \omega_{X^n}^m)^{S_n} \simeq H^0(X^n, \omega_{X^n}^m)^{S_n}.$$

Since

$$H^0(X^n, \omega_{X^n}^m)^{S_n} \simeq \text{Sym}^n H^0(X, \omega_X^m)$$

we obtain the following:

**Proposition 2.1** Let $X$ be a smooth surface. If $X$ has Kodaira dimension $k$ then $X^{(n)}$ has Kodaira dimension $nk$.

If $X$ is a K3 surface we can be more precise: $X^{[n]}$ is a holomorphic symplectic manifold (see [1], Section 6). In particular, the canonical bundle of $X^{[n]}$ remains trivial.
An important ingredient in the proofs of potential density is the construction of a multisection of the abelian fibration. The following proposition will help us verify that certain subvarieties are multisections:

**Proposition 2.2** Let $X$ be a smooth projective surface and $C_1, \ldots, C_n$ distinct irreducible curves. Consider the image $Z$ of $C_1 \times \ldots \times C_n$ under the quotient map $X^n \to X^{(n)}$. The scheme-theoretic preimage $\varphi^{-1}(Z) \subset X^{[n]}$ has a unique irreducible component of dimension $\geq n$, denoted by $C_1 \ast \ldots \ast C_n$. In particular, the homology class of $C_1 \ast \ldots \ast C_n$ is uniquely determined by the homology classes of $C_1, \ldots, C_n$.

**Proof.** Let $(a_1, \ldots, a_k)$ be a partition of $n$ and let $\mathcal{D}_{a_1, \ldots, a_k}$ be the corresponding stratum in $X^{(n)}$. In particular, the $\mathcal{D}_{a_1, \ldots, a_k}$ are disjoint. The intersection of $Z$ with $\mathcal{D}_{a_1, \ldots, a_k}$ has dimension $\# \{a_j \mid a_j = 1\}$. Each fiber of $\varphi$ over $\mathcal{D}_{a_1, \ldots, a_k}$ is irreducible of dimension $\sum_{j=1}^k (a_j - 1)$ (see [6]). It follows that the preimage of $Z \cap \mathcal{D}_{a_1, \ldots, a_k}$ has dimension at most

$$\# \{a_j \mid a_j = 1\} + \sum_{j=1}^k (a_j - 1),$$

which is less than $n$, provided the $a_j$ are not all equal to 1. □

3 Generalities on abelian fibrations

Let $\mathcal{A}$ be an abelian variety defined over a field $K$ (not necessarily a number field). A point $\sigma \in \mathcal{A}(K)$ is nondegenerate if the subgroup generated by $\sigma$ is Zariski dense in $\mathcal{A}$.

**Proposition 3.1** Let $\mathcal{A}$ be an abelian variety over a number field $K$. Then there exists a finite field extension $L/K$ such that $\mathcal{A}(L)$ contains a nondegenerate point.

**Proof.** We include an argument for completeness, since we could not find a reference.

**Lemma 3.2** Let $\mathcal{A}$ be an abelian variety of dimension $\dim(\mathcal{A})$ defined over a number field $K$. Then there exists a finite field extension $L/K$ such that the rank of the Mordell-Weil group $\mathcal{A}(L)$ is strictly bigger than the rank of $\mathcal{A}(K)$.
Proof. We first assume \( \dim(\mathcal{A}) > 1 \). Let \( \Gamma \) be the saturation of \( \mathcal{A}(\bar{K}) \) in \( \mathcal{A}(K) \) (where \( \bar{K} \) is the algebraic closure of \( K \)). This means that \( \Gamma \) consists of all points \( p \) such that a positive multiple of \( p \) lies in \( \mathcal{A}(K) \); in particular it contains all torsion points. Find a smooth curve \( C \) of genus \( \geq 2 \) in \( \mathcal{A} \), defined over a number field \( K_1 \). By Raynaud’s version of the Manin-Mumford conjecture (see [14], I 6.4 or [22] Theorem 1) we have that \( C \cap \Gamma \) is finite. There exists a \( L/K_1 \) such that \( C(L) \) contains a point \( q \) outside \( C \cap \Gamma \). It follows that \( \mathcal{A}(L) \) has higher rank. (This argument was communicated to us by B. Mazur.)

We now do the case of an elliptic curve \( E \). Write \( \mathcal{A} = E \times E \) with projections \( \pi_1 \) and \( \pi_2 \); we have \( \mathcal{A}(K) = E(K) \times E(K) \). The argument above gives a point \( q \in \mathcal{A}(L) \) not contained in the saturation of \( \mathcal{A}(K) \). It follows that either \( \pi_1(q) \) or \( \pi_2(q) \) is not contained in the saturation of \( E(K) \). \( \square \)

We prove the proposition. We may replace \( \mathcal{A} \) with an isogenous abelian variety, so we may assume that \( \mathcal{A} \) is a product of geometrically simple abelian varieties. Our proof proceeds by induction on the number of simple components. Any nontorsion point \( p \) of a geometrically simple abelian variety is nondegenerate. Indeed, Faltings’ theorem implies that the Zariski closure of \( \mathbb{Z}p \) is a finite union of abelian subvarieties. Hence it suffices to prove the inductive step:

\textbf{Lemma 3.3} Let \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) be abelian varieties over a number field \( K \). Assume that \( \mathcal{A}_2 \) is geometrically simple and \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) have nondegenerate \( K \)-points \( p_1 \) and \( p_2 \). Then \( \mathcal{A}_1 \times \mathcal{A}_2 \) has a nondegenerate point over some finite extension \( L/K \).

\textbf{Proof.} For any pair of abelian varieties \( \mathcal{A}_1, \mathcal{A}_2 \) the group of homomorphisms \( \text{Hom}(\mathcal{A}_1, \mathcal{A}_2) \) is finitely generated as a module over \( \mathbb{Z} \). After a finite extension, we may assume these are all defined over \( K \). More generally, we can consider the group generated by abelian subvarieties of \( \mathcal{A}_1 \times \mathcal{A}_2 \), or equivalently, by homomorphisms from \( \mathcal{A}_1 \) to \( \mathcal{A}_2 \) defined only up to isogeny. This group equals \( \text{Hom}^0(\mathcal{A}_1, \mathcal{A}_2) = \text{Hom}(\mathcal{A}_1, \mathcal{A}_2) \otimes \mathbb{Q} \) (see [21], p. 172-176).

Assume that \( (p_1, p_2) \) is contained in a proper abelian subvariety \( \mathcal{B} \). We regard \( \mathcal{B} \) as an element \( \beta \in \text{Hom}^0(\mathcal{A}_1, \mathcal{A}_2) \); in particular, \( \beta(p_1) = p_2 \). We choose a \( \mathbb{Z} \)-basis \( (Z_1, ..., Z_k) \) for \( \text{Hom}(\mathcal{A}_1, \mathcal{A}_2) \). There exist integers \( b_1, ..., b_k \), and \( d \neq 0 \) such that \( (b_1Z_1 + ... + b_kZ_k)(p_1) = dp_2 \) in the Mordell-Weil group. Hence \( p_2 \) is contained in the saturation of the subgroup of \( \mathcal{A}_2(K) \) generated
by the images of $p_1$ under the $Z_i$. Conversely, if $q$ is not contained in this subgroup then $(p_1, q)$ is nondegenerate. Applying Lemma 3.2, we obtain a finite field extension $L/K$ and a point $q \in A_2(L)$ with the desired property.

Let $\mathcal{T}$ be an $A$-torsor, defined over a field $K$. This means there is a $K$-isomorphism $\text{Aut}_0(\mathcal{T}) \simeq A$, so that for any $p \in \mathcal{T}(M)$ (where $M$ is an extension of $K$), the translation action induces an isomorphism $A(M) \to \mathcal{T}(M)$. There is a 1-1 correspondence between $M$-valued points of $\mathcal{T}$ and isomorphisms $A(M) \simeq \mathcal{T}(M)$.

Consider the Albanese $\text{Alb}(\mathcal{T})$ (see, for example, [13] II. 3). It is an abelian variety defined over $K$, such that there is a morphism $\mathcal{T} \times \mathcal{T} \to \text{Alb}(\mathcal{T})$ corresponding to $(t_1, t_2) \to t_1 - t_2$. For each zero-cycle of $\mathcal{T}$, defined over $K$ and of degree zero, we obtain a point in $\text{Alb}(\mathcal{T})(K)$. Note that $A$ is naturally isomorphic to $\text{Alb}(\mathcal{T})$ over $K$. Indeed, the action of $A$ on $\mathcal{T}$ induces an action on $\text{Alb}(\mathcal{T})$, and the orbit of zero is isomorphic to both $A$ and $\text{Alb}(\mathcal{T})$.

Let $\mathcal{T} \to B$ be an abelian fibration, that is: $\mathcal{T}$ and $B$ are normal and $B$ is connected and the fiber $\mathcal{T}_b$ over the generic point $b$ is a torsor for an abelian variety $A_b$ over $K(b)$. A multisection $\mathcal{M}$ of $\pi$ is the closure of an $M$-valued point of $\mathcal{T}_b$, where $M$ is a finite field extension of $K(b)$ of degree $\deg(M)$. Let $\mu$ denote the generic point of $\mathcal{M}$.

**Proposition 3.4** Let $\mathcal{T}$ be an abelian fibration with multisections $\mathcal{M}_1$ and $\mathcal{M}_2$, both defined over a number field $K$. Let $\sigma$ be the distinguished section of $\mathcal{T}(\mathcal{M}_1)$. Denote by $\mathcal{M}'_2 = \mathcal{M}_2 \times_B \mu_1$; it is a cycle defined over $\mathcal{M}_1$. Assume that $K$-rational points on $\mathcal{M}_1$ are Zariski dense and that the cycle

$$\deg(M_2)\sigma - \mathcal{M}'_2$$

yields a nondegenerate point of $\text{Alb}(\mathcal{T})(\mathcal{M}_1)$. Then $K$-rational points on $\mathcal{T}$ are Zariski dense.

**Proof.** We restrict to an open subset of $B$ where $\pi$ and $\mathcal{M}$ are flat over $B$. Denote by $\mathcal{T}'$ the base change of $\mathcal{T}$ to $\mathcal{M}_1$ with distinguished section $\sigma$. Consider the action of $A(\mathcal{M}_1) \simeq \text{Alb}(\mathcal{T})(\mathcal{M}_1)$ on $\mathcal{T}(\mathcal{M}_1)$. The translates of $\sigma$ by the nondegenerate point are Zariski dense in $\mathcal{T}(\mathcal{M}_1)$.

Each of these translates is birational to $\sigma$ over $K$ and therefore its $K$-rational
points are Zariski dense. The union of the closures in $\mathcal{T}'$ of these translates is Zariski dense; hence $K$-rational points in $\mathcal{T}'$ are Zariski dense. It remains to observe that $\mathcal{T}'$ dominates $\mathcal{T}$. □

**Remark 3.5** Our argument does not show that rational points are Zariski dense in *any* fiber of $\mathcal{T}_p$, where $p$ is an $K$-rational point of $B$. However, one knows that when the fibers are of dimension 1 then for $p' \in U_1(K)$, where $U_1$ is some nonempty open subset of $\mathcal{M}_1$, the fibers over $p'$ have infinitely many $K$-rational points (see [26]). Moreover, by a result of Néron, the rank of the Mordell-Weil group of special fibers of abelian fibrations does not drop outside a thin subset of points on the base of the fibration [24].

### 4 Elliptic families on K3 surfaces

Throughout this section, we work over an algebraically closed field of characteristic 0. An elliptic fibration is an abelian fibration of relative dimension one. In the sequel an elliptic fibration dominating a K3 surface will be called an elliptic family.

The following theorem is attributed to Bogomolov and Mumford (see [20]). We include a detailed proof because it is crucial for our applications.

**Theorem 4.1** Let $S$ be K3 surface and $f$ a divisor class on $S$ such that $h^0(O_S(f)) > 1$. Then there exists a smooth curve $B$ and an elliptic fibration $\mathcal{E} \to B$ with the following properties:

1. $\mathcal{E}$ dominates $S$;
2. the generic fiber $\mathcal{E}_b$ is mapped birationally onto its image;
3. the class $f - \mathcal{E}_b$ is effective.

**Proof.** A genus one curve $C \subset S$ is a curve whose normalization $\tilde{C}$ is a connected curve of genus one. It suffices to prove the result for a singular curve $B$; we can always pull back to the normalization $\tilde{B}$.

We may restrict to the case where $S$ is not an elliptic K3 surface. We assume that $|f|$ has no fixed components (and thus no base points). Indeed, if this is not the case then we extract the moving part of $f$. Since $S$ is not elliptic, we have $f^2 > 0$. We may also assume that the class $f$ is primitive; otherwise, take the primitive effective generator $f'$ of $\mathbb{Z}f$. We still have
$h^0(\mathcal{O}(f')) > 1$ and $|f'|$ basepoint free (again, using the fact that $S$ is not elliptic.) See [25] for basic results concerning linear series on K3 surfaces.

We shall use the following lemma, essentially proved in [20]:

**Lemma 4.2** For each $n > 0$, a generic polarized K3 surface $(S_1, f)$ of degree $2n$ contains a one-parameter family of irreducible curves with class $f$, such that the generic member is nodal of genus one.

**Proof.** We first claim there exists a K3 surface $S_0$ containing two smooth rational curves $D_1$ and $D_2$ meeting transversally at $n + 2$ points. Let $S_0$ be the Kummer surface associated to the product of elliptic curves $E_1$ and $E_2$, such that there exists an isogeny $E_1 \to E_2$ of degree $2n + 5$. Let $\Gamma$ be the graph of this isogeny and $p \in E_2$ a 2-torsion point. Now $\Gamma$ intersects $E_1 \times p$ transversally in $2n + 5$ points, one of which is 2-torsion in $E_1 \times E_2$. We take $D_1$ to be the image of $\Gamma$ and $D_2$ to be the image of $E_1 \times p$; $D_1$ and $D_2$ are smooth, rational, and intersect transversally in $n + 2$ points. The line bundle $\mathcal{O}(f) := \mathcal{O}_{S_0}(D_1 + D_2)$ is big and nef and thus has no higher cohomology (by Kawamata-Viehweg vanishing).

Let $\Delta$ be the spectrum of a discrete valuation ring with closed point $0$ and generic point $\eta$. Let $S \to \Delta$ be a deformation of $S_0$ such that $f$ remains algebraic. We assume further that the class $f$ is ample and indecomposable in the monoid of effective curves in a (geometric) generic fiber $S_1$. These conditions are satisfied away from a finite union of irreducible divisors. Since $f$ has no higher cohomology, $D_1 \cup D_2$ is a specialization of curves in the generic fiber and the deformation space $\text{Def}(D_1 \cup D_2)$ is smooth of dimension $n + 2$.

Consider the locus in $\text{Def}(D_1 \cup D_2)$ parametrizing curves with at least $\nu$ nodes; this has dimension $\geq n + 2 - \nu$. When $\nu = n + 1$ the corresponding curves are necessarily rational. Each fiber of $S \to \Delta$ is not uniruled, and thus contains a finite number of these curves. In each fiber, the rational curves with $n + 1$ nodes deform to positive-dimensional families of curves with $n$ nodes. Hence $S_1$ contains a family of nodal curves of genus one with the desired properties. □

To complete the proof, we use a proposition suggested by Joe Harris:

**Proposition 4.3** Let $S \to D$ be a projective morphism. Then there exists a scheme $K_\alpha(S/D)$ such that each connected component is projective over $D$ and the fiber over each $d \in D$ is isomorphic to the corresponding moduli space of stable maps $K_\alpha(S_d)$. 
Proof. We refer to Kontsevich’s moduli space of stable maps constructed in [12],[8]. We first consider the special case when \( S = \mathbb{P}^n \). Then
\[
K_g(\mathbb{P}^n_D/D) = K_g(\mathbb{P}^n) \times D
\]
More generally, given an embedding \( S \to \mathbb{P}^n_D \) over \( D \), we define \( K_g(S/D) \) as those elements of \( K_g(\mathbb{P}^n_D/D) \) which factor through \( S \). Since it is a closed subscheme it is projective over \( D \). □

We finish the proof of Theorem 4.1. There exists a projective family of K3 surfaces \( S \to \Delta \) equipped with a divisor class \( f \), such that the (geometric) generic fiber satisfies the conditions of Lemma 4.2 and the special fiber is \((S, f)\). Consider the component \( K_1(S/\Delta, f) \) of \( K_1(S/\Delta) \) consisting of maps with image in the class \( f \). After a finite base change \( \Delta' \to \Delta \), there exists a geometrically irreducible curve \( C_\eta \subset K_1(S/\Delta, f) \) corresponding to an elliptic fibration dominating the generic fiber \( S_\eta \). Let \( C \subset K_1(S/\Delta, f) \) be the flat extension over \( \Delta \) and \( C_0 \) the corresponding flat limit.

There may not be a ‘universal stable map’ defined over \( C_0 \subset K_1(S, f) \). However, for each irreducible reduced component \( C_i \subset C_0 \), a universal stable map exists after a finite cover \( B_i \to C_i \). For some such \( B_i \), the resulting family of stable maps \( E'_i \to B_i \) dominates \( S \). The image of the generic fiber contains a component of genus one because no K3 surface is uniruled. □

5 Density of rational points

In this section \( S \) denotes a K3 surface defined over a number field \( K \). Potential density holds for elliptic K3 surfaces and for all but finitely many families of K3 surfaces with Picard group of rank \( \geq 3 \), and consequently for their symmetric products (see [5]). However, a general K3 surface has Picard group of rank 1. In the following sections we will prove density results for symmetric products of general K3 surfaces.

By Theorem 4.1, there is a family of elliptic curves \( \mathcal{E} \) dominating \( S \). Let \( E_1, \ldots, E_n \) be generic curves in the fibration and assume that \( g = [E_i] \) is big; in particular, \( \mathcal{E} \) is not an elliptic fibration on \( S \). It follows that the general member of \( g \) is an irreducible curve of genus > 1. Note that we have a well defined class \( g \ast \ldots \ast g \) in the homology of \( S^{[n]} \), equal to the homology class of \( C_1 \ast \ldots \ast C_n \), where the \( C_i \) are irreducible curves in \( g \) (see Proposition 2.2).
Theorem 5.1 Let $S$ be a K3 surface satisfying the conditions of the previous paragraph. Assume that either

1. $F = S^{[n]}$ admits an abelian fibration $T \to B$ and $g \ast \ldots \ast g$ intersects the proper transform of the generic fiber positively, or

2. $F$ is birational to an abelian fibration, and $E_1 \ast \ldots \ast E_n$ is a multisection.

Then rational points on $F$ are potentially dense.

Proof. Throughout the proof, $L/K$ is some finite field extension, which we will enlarge as necessary. We want to show that $L$-rational points are Zariski dense on $F$.

For generic smooth curves $C_1, \ldots, C_n$ in $g$, $C_1 \ast \ldots \ast C_n \subset S^{[n]}$ gives a multisection of $T \to B$. We denote this multisection by $M_2$. This is clear under the first assumption. Under the second assumption, it follows from the fact that $E_1 \ast \ldots \ast E_n$ is a multisection.

Choose a point $p \in B(L)$ corresponding to a smooth fiber $T_p$ of $T$. Let $A_p$ be the Albanese of $T_p$. Choose a point $m_1 \in T_p(L)$ so that the class of the cycle $\deg(M_2) \cdot m_1 - M_2|_{T_p}$ is nondegenerate in $A_p$ (see Proposition 3.1). We may assume that $m_1$ corresponds to a subscheme $(s_1, \ldots, s_n) \in S^{[n]}$ where the $s_i$ are distinct, $E_i$ contains $s_i$ for $i = 1, \ldots, n$, the $s_i$ and the $E_i$ are defined over $L$, and $L$-rational points are Zariski dense on each $E_i$. Then we have a multisection $M_1$ for $T$ given as (the proper transform of ) $E_1 \ast \ldots \ast E_n$. Note that $L$-rational points on $M_1$ are Zariski dense.

It follows from Equation (1) that the pair $(M_1, M_2)$ satisfies the non-degeneracy assumptions of Proposition 3.4. Therefore, $L$-rational points are Zariski dense in $F$. □

We employed two parallel sets of hypotheses because in some applications the abelian fibration is only described over the generic point of $B$, which makes intersection computations difficult. In other applications, the abelian fibration is given by an explicit linear series, but the multisection is difficult to control.

Remark 5.2 Matsushita has proved a structure theorem for holomorphic symplectic manifolds of dimension $2n$ admitting a fibration structure. In
particular, he proved that the base has dimension \( n \), is Fano, has Picard group of rank 1, and log-terminal singularities. Furthermore, the fibers admit finite étale covers which are abelian varieties (see [16]).

Remark 5.3 We do not know how to produce abelian fibrations on symmetric products of Calabi-Yau varieties of dimension \( \geq 3 \). For example, do they exist for quintic threefolds?

6 Potential density on \( S^{[n]} \)

In this section we exhibit K3 surfaces \( S \) defined over a number field \( K \) and satisfying the assumptions of Theorem 5.1.

Theorem 6.1 Let \( S \) be a K3 surface with Picard group of rank 1 generated by a polarization \( g \) of degree \( 2(n - 1) \). Then there exists a finite extension \( L/K \) such that \( L \)-rational points on \( S^{[n]} \) are Zariski dense.

Proof. Under our hypothesis, \( g \) is basepoint free and yields a morphism \( S \to \mathbb{P}^n \) which is finite onto its image. Furthermore, the generic member of \( |g| \) is smooth of genus \( n \) (see [25]).

There is an abelian fibration over \( B \subset \mathbb{P}^n \), where \( B \) corresponds to the locus of smooth curves in \( |g| \). Indeed, \( T \to B \) is the degree \( n \) component of the relative Picard fibration (see [3]). We claim that \( S^{[n]} \) is birational to \( T \). Given generic points \( s_1, \ldots, s_n \) on \( S \) there is a smooth curve \( C \in |g| \) passing through those points. The line bundle \( O_C(s_1 + \ldots + s_n) \) is a generic point of \( \text{Pic}_n(C) \), and such a line bundle has a unique representation as an effective divisor. (We are using the fact that \( C^{[n]} \) is birational to \( \text{Pic}_n(C) \).)

To apply Theorem 5.1 we must verify that (the proper transform of) \( E_1 * \ldots * E_n \) is a multisection for \( T \). A generic curve \( C \in |g| \) intersects the union of the \( E_i \) transversally in \( n(2n - 2) \) points. Under these assumptions, every subscheme parametrized by \( C^{[n]} \cap (E_1 * \ldots * E_n) \) is reduced and there are finitely many such subschemes. It particular, \( C^{[n]} \) intersects \( E_1 * \ldots * E_n \) in finitely many points. \( \square \)

Remark 6.2 The same argument applies if the rank of the Picard group of \( S \) is \( > 1 \). Our proof uses only that \( g \) is big and contains the class of an irreducible elliptic curve on \( S \). Under these conditions \( |g| \) is basepoint free; the base locus of any linear series on a K3 surface has pure dimension one (see [25]).
Theorem 6.3 Let $S$ be a K3 surface defined over a number field $K$. Then there exist a positive integer $n$ and a finite extension $L/K$ such that the $L$-rational points of $S^{[n]}$ are Zariski dense.

Proof. By Theorem 4.1 every K3 surface $S$ is dominated by an elliptic fibration. By Remark 6.2, we may assume that the class of the fiber is not big. Therefore it has self-intersection zero which implies that $S$ is an elliptic K3 surface. In this case, the main theorem of [5] implies the theorem with $n = 1$. □

Example 6.4 Let $S$ be a K3 surface of degree 2. Then rational points on $S^{[2]}$ are potentially dense.

Indeed, let $g$ be the polarization. By Theorem 4.1, there exists an irreducible elliptic curve $E \subset S$ such that $g - [E]$ is effective. If $g = [E]$ the assertion follows from Remark 6.2. Otherwise,

$$\langle E, E \rangle < \langle g, E \rangle < \langle g, g \rangle,$$

which implies that $\langle E, E \rangle = 0$. Then the assertion holds with $n = 1$ by [5].

7 Potential density on $S^{[2]}$

Given a fixed K3 surface it is a natural problem to determine the smallest possible $n$ for which the theorem holds. (Of course, we expect that we can always take $n = 1$!) As we have seen, the key to proving potential density is the existence of abelian fibrations on $S^{[n]}$.

The intersection form on the Picard group of $S$ is an integer-valued non-degenerate quadratic form, denoted $\langle , \rangle$. We recall that the Picard group of $S^{[n]}$ is also equipped with a natural integer-valued nondegenerate quadratic form $(, )$, the Beauville form [1]. With respect to this form, we have an orthogonal direct sum decomposition

$$\text{Pic}(S^{[n]}) = \text{Pic}(S) \oplus \perp \mathbb{Z}e,$$

where $(e, e) = -2(n - 1)$ and $2e$ is the class of the diagonal (more precisely, the nonreduced subschemes in $S^{[n]}$.)

On the K3 surface $S$, the Picard group together with the quadratic form control much of the geometry of $S$. For example, if the quadratic form represents zero, then $S$ admits an elliptic fibration over $\mathbb{P}^1$. A naive question
would be whether the analog holds for $S^{[n]}$ with $n \geq 2$. More precisely, if the Beauville form represents zero, is $S^{[n]}$ birational to an abelian fibration over $\mathbb{P}^n$ (see [11])? Note that the Beauville form of $S^{[2]}$ represents zero if and only if the intersection form on Pic$(S)$ represents $2m^2$ for some $m \in \mathbb{Z}$.

**Proposition 7.1** Let $S$ be a generic K3 surface of degree $2m^2$ with $m > 1$. Then $S^{[2]}$ is isomorphic to an abelian surface fibration over $\mathbb{P}^2$.

**Proof.** We first consider the case $m = 2$. We assume that the polarization on $S$ is very ample and that $S$ does not contain a line. Then $S$ can be represented as a complete intersection of a three-dimensional space $I_S(2)$ of quadrics in $\mathbb{P}^5$. An element of $S^{[2]}$ spans a line $\ell \in \mathbb{P}^5$ and a two dimensional subspace of $I_S(2)$ contains $\ell$. In this way, we obtain a morphism

$$a : S^{[2]} \to \mathbb{P}^2 \simeq \mathbb{P}(I_S(2)^*).$$

The generic fiber of $a$ is an abelian surface; the variety of lines on a smooth complete intersection of two quadrics in $\mathbb{P}^5$ is a principally polarized abelian surface (see [9], p. 779). Notice that $a$ is induced by the sections of $f_8 - 2e$, where $f_8$ is the polarization of degree 8.

When $m > 2$ the proof consists of three steps:

1. construct special K3 surfaces $S$ so that $S^{[2]}$ admits a natural involution;
2. show directly that some of these special K3 surfaces admit an abelian surface fibration and a polarization of degree $2m^2$;
3. verify that this abelian surface fibration deforms to the Hilbert scheme of a generic K3 surface of degree $2m^2$.

We begin with a construction of Beauville and Debarre [7]. Let $S \subset \mathbb{P}^3$ be a smooth quartic hypersurface; in particular, $S$ is a K3 surface and the corresponding polarization is denoted $f_4$. Then there is a birational involution

$$j : S^{[2]} \dashrightarrow S^{[2]}$$

defined on an open subset of $S^{[2]}$ by the rule $j(p_1 + p_2) = p_3 + p_4$, where $p_1, p_2, p_3,$ and $p_4$ are collinear points on $S$. This is a morphism provided that $S$ does not contain a line. The action of $j$ on the Picard group of $S^{[2]}$ is given by

$$j^*x = -x + (f_4 - e, x)(f_4 - e).$$
Next, we consider some special quartic K3 surfaces. Let $S$ be a K3 surface with Picard group generated by the ample class $f_4$ and a second class $f_8$ satisfying

\[
\begin{array}{c|cc}
  & f_4 & f_8 \\
  f_4 & 4 & k \\
  f_8 & k & 8 \\
\end{array}
\]

where $k > 7$. Such K3 surfaces are parametrized by a nonempty analytic open subset of an irreducible variety of dimension 18. This follows from the Torelli theorem, surjectivity of Torelli, and the structure of the cohomology lattice of K3 surfaces (see [15] Theorem 2.4 and [2]). Note that $f_4$ is very ample and that the image is a smooth quartic surface not containing a line [25]; here we are using the fact that $k \neq 6$. Furthermore, the same reasoning shows that $f_8$ is very ample and the image does not contain a line, provided that $f_8$ is ample. (Here we are using the fact that $k \neq 7$.) If $f_8$ were not ample then $\langle f_8, C \rangle \leq 0$ for some $(-2)$-curve $C$ (see [15] 1.6). Clearly $\langle f_8, C \rangle \neq 0$ and if $\langle f_8, C \rangle < 0$ then the Picard-Lefschetz reflection $\rho(f_8) = f_8 + \langle f_8, C \rangle C$ and $f_4$ generate a sublattice with discriminant greater than $32 - k^2$, which is impossible. Our argument in the $m = 2$ case shows that the $S^{[2]}$ admits an abelian surface fibration, induced by the line bundle $f_8 - 2e$. Composing with the involution $j$, we obtain a second elliptic fibration, induced by

\[
j^*(f_8 - 2e) = 2e - f_8 + (f_8 - 2e, f_4 - e) (f_4 - e) = (k - 4)f_4 - f_8 - (k - 6)e.
\]

Let $g = (k - 4)f_4 - f_8$ and $m = k - 6$ so that $\langle g, g \rangle = 2(k - 6)^2 = 2m^2$ and $j^*(f_8 - 2e) = g - me$. Note that $g$ is effective on $S$.

We turn to the last step. Let $S \to \Delta$ be a general deformation of $S$ for which $g$ remains algebraic. The class $g$ restricts to a polarization on the generic fiber, since it has Picard group of rank one. The class $g - me$ is algebraic (and nef) on the generic fiber of $S^{[2]} \to \Delta$. Using deformation theory (see [11] and [23] Cor. 3.4), we find that the generic fiber also admits an abelian fibration with base $\mathbb{P}^2$, induced by the sections of the line bundle $g - me$. We are using the fact that the abelian surface fibration on $S^{[2]}$ is Lagrangian; see [11] for the fourfold case and [17] more generally. □

Remark 7.2 Unfortunately, our argument gives little information about how the abelian fibration degenerates for nongeneric K3 surfaces of degree $2m^2$ with $m > 2$. A more precise description would follow from the conjectures of [11].
**Theorem 7.3** Let $S_8$ be a K3 surface of degree 8, defined over a number field $K$, embedded in projective space $\mathbb{P}^5$ as a complete intersection of 3 quadrics and not containing a line. Then rational points on $S_8$ are potentially dense. The same result holds for a generic K3 surface of degree $2m^2$.

*Proof.* We apply Theorem 5.1, using the first set of assumptions. We use the abelian fibrations constructed in Proposition 7.1.

Let $g$ be the homology class of an irreducible elliptic curve (see Theorem 4.1). We verify that $g * g$ intersects the class of a fiber positively.

We need to compute the intersection on $S_8$ of $(f - me) \cdot (f - me) \cdot (g \cdot g)$, where $f$ and $g$ are divisor classes on $S$. Let $\Sigma$ be the class of subschemes containing a fixed point $p \in S$; note that these subschemes are parametrized by the blow-up of $S$ at $p$. In particular, $(f - me) \cdot (f - me) \cdot \Sigma = \langle f, f \rangle - m^2$ (because $e$ restricts to the exceptional divisor of the blown-up K3 surface). We also have

\[
g * g = g \cdot g - \langle g, g \rangle \Sigma,
\]
\[
f \cdot f \cdot g \cdot g = \langle f, f \rangle \langle g, g \rangle + 2 \langle f, g \rangle^2,
\]
\[
f \cdot e \cdot g \cdot g = 0,
\]
\[
e \cdot e \cdot g \cdot g = -2 \langle g, g \rangle.
\]

Finally, we obtain

\[(f - me) \cdot (f - me) \cdot (g \cdot g) = 2\langle f, g \rangle^2 - m^2\langle g, g \rangle.
\]

In our case, $f = f_{2m^2}$, $g$ is the class of the elliptic curve. To verify the hypothesis of the Theorem 5.1, we need $2\langle f_{2m^2}, g \rangle^2 > m^2\langle g, g \rangle$. Since $\langle g, g \rangle > 0$ we are done by the Hodge index theorem, which implies that the determinant of the matrix

\[
\begin{pmatrix}
2m^2 & \langle f_{2m^2}, g \rangle \\
\langle f_{2m^2}, g \rangle & \langle g, g \rangle
\end{pmatrix}
\]

is negative. $\square$

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