A NONEXISTENCE RESULT ON HARMONIC DIFFEOMORPHISMS BETWEEN PUNCTURED SPACES

SHI-ZHONG DU† & XU-QIAN FAN*

Abstract. In this paper, we will prove a result of nonexistence on harmonic diffeomorphisms between punctured spaces. In particular, we will given an elementary proof to the nonexistence of rotationally symmetric harmonic diffeomorphisms from the punctured Euclidean space onto the punctured hyperbolic space.

1. Introduction

We will study the problem of the existence about harmonic diffeomorphism between punctured spaces, in particular, from the punctured $\mathbb{R}^n$ to the punctured $\mathbb{H}^n$. This is a natural generalization of the question mentioned by Schoen [37], which is about the existence, or nonexistence, of a harmonic diffeomorphism from $\mathbb{C}$ onto $\mathbb{D}$ with the Poincaré metric. Up to present, many beautiful results for the asymptotic behaviors of harmonic maps from $\mathbb{C}$ to $\mathbb{D}$ were obtained, see for example [45, 17, 4, 5], or the survey [46] by Wan and the references therein. On the other hand, Collin and Rosenberg [12] constructed harmonic diffeomorphisms in 2010. In addition, there are also many papers focusing on the rotationally symmetric case, for example [11, 42, 36, 9]. One of these results is the nonexistence of the special harmonic diffeomorphisms from $\mathbb{C}$ onto $\mathbb{D}$, even for general dimension. Recently, some works have studied the case of annular topological type, for example [6, 30, 7]. While some results are related to the Nitsche’s type inequalities [35, 20, 21, 26].

Conversely, Heinz [20] obtained the nonexistence result for these mappings from $\mathbb{D}$ onto the flat plane $\mathbb{C}$ in 1952.

Date: July 2014.

2000 Mathematics Subject Classification. Primary 58E20; Secondary 34B15.
Key words and phrases. Harmonic map, rotational symmetry, hyperbolic space.
† Research partially supported by the National Natural Science Foundation of China (11101106).
* Research partially supported by the National Natural Science Foundation of China (11291240139).
For more results about the question of the existence of harmonic diffeomorphisms between Riemannian manifolds, see for example [1]-[47].

In this paper, we will generalize the result for the nonexistence of rotationally symmetric harmonic diffeomorphism from $\mathbb{C}^*$ onto $\mathbb{D}^*$ to general dimension $n \geq 2$. Actually, we will study the problem for more general setting. More precisely, let us denote

$$ (\mathbb{R}^n, E) = (\mathbb{S}^{n-1} \times [0, \infty), r^2 d\theta^2 + dr^2), $$

$$ (M^n, G) = (\mathbb{S}^{n-1} \times [0, \infty), (g(r))^2 d\theta^2 + dr^2), $$

where $C^2$ function $g(r) \geq 0$ and $(\mathbb{S}^{n-1}, d\theta^2)$ is the $(n-1)$-dimensional sphere, and

$$ \mathbb{R}^n_* = \mathbb{R}^n \setminus \{0\} $$

and

$$ M^n_* = M^n \setminus \{0\}. $$

We will prove the following result in the next section.

**Theorem 1.1.** For $n \geq 2$, there is no rotationally symmetric harmonic diffeomorphism from $(\mathbb{R}^n_*, E)$ onto $(M^n_*, G)$ provided that the Riemannian metric $G$ satisfies the following conditions:

1. $g(0)g'(0) \geq 0$,
2. $(g(r)g'(r))' \geq 0$ for all $r \geq 0$,
3. $\sup_{r>0} [r^{-1}g(r)g'(r)] > \frac{(n-2)^2}{n-1}$. 

Clearly, $\mathbb{H}^n = (\mathbb{S}^{n-1} \times [0, \infty), \sinh^2 r d\theta^2 + dr^2)$, and $\sinh r$ satisfies conditions (1)-(3). Let

$$ \mathbb{H}^n_* = \mathbb{H}^n \setminus \{0\}, $$

we have the following corollary.

**Corollary 1.1.** For $n \geq 2$, there is no rotationally symmetric harmonic diffeomorphism from $(\mathbb{R}^n_*, E)$ onto $\mathbb{H}^n_*$ with the hyperbolic metric.

From this corollary, one can get the well-known result: There is no rotationally symmetric harmonic diffeomorphism from Euclidean space $\mathbb{R}^n$ onto hyperbolic space $\mathbb{H}^n$.

**Acknowledgments**

The authors would like to thank Li Chen for his useful discussion. The author(XQ) would like to thank Prof. Luen-fai Tam for his worthy advice.
2. Proof of Theorem 1.1

For convenience, let us first recall the definition about the harmonic maps from $\mathbb{R}^n$ to $M^n$. Let $(x^1, \cdots, x^n)$ and $(u^1, \cdots, u^n)$ be the standard coordinates on $\mathbb{R}^n$ and $M^n$ respectively, and $\Gamma^k_{ij}$ be the Christoffel symbol of the Levi-Civita connection on $M^n$. A $C^2$ map $u$ from $\mathbb{R}^n$ to $M^n$ is harmonic if and only if $u$ satisfies

$$\triangle u^i + \sum_{\alpha=1}^n \Gamma^i_{jk}(u) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\alpha} = 0 \quad \text{for } i = 1, \cdots, n.$$  

(2.1)

For more information about harmonic maps, see for example [38].

Proof of Theorem 1.1. We want to prove this theorem by contradiction. Suppose $u$ is a rotationally symmetric harmonic diffeomorphism from $\mathbb{R}^n$ onto $M^n$, then we can assume $u(r, \theta) = (y(r), \theta)$, and by (2.3) in [41] (or (1.2) in [9]) and (2.1), $y(r)$ should satisfy the following ODE.

$$y'' + (n-1)r^{-1}y' - (n-1)r^{-2}g(y)g'(y) = 0.$$  

(2.2)

In addition, $u$ is a diffeomorphism, so $y'$ cannot be zero. So $y$ should satisfy one of the following conditions (2.3) and (2.4), where

(2.3) $y(0) = 0, y(r) \to +\infty$ as $r \to +\infty$, and $y' > 0$ for $0 < r < \infty$,

or

(2.4) $y(r) \to +\infty$ as $r \to 0+$, $\lim_{r \to +\infty} y(r) = 0$, and $y' < 0$ for $0 < r < \infty$.

By the monotonicity of $y$ in $r$, we can regard $r$ as function of $y$ with the properites

$$y' = \frac{1}{r'}, \quad y'' = -\frac{r''}{r'^3}.$$  

Consequently, (2.2) is changed to be

$$-\frac{r''}{r'^3} + \frac{n-1}{rr'} - (n-1)\frac{1}{r^2}g(y)g'(y) = 0,$$

(2.5)

or

$$-\frac{r''}{r} + (n-1)\left(\frac{r'}{r}\right)^2 - (n-1)\left(\frac{r'}{r}\right)^3 g(y)g'(y) = 0.$$  

(2.6)

Setting $x = \ln r$, we have

$$x' = \frac{r'}{r}, \quad x'' = \frac{r''}{r} - \left(\frac{r'}{r}\right)^2.$$  

From (2.6), we can get

$$x'' - (n-2)x'^2 + (n-1)x'^3 g(y)g'(y) = 0.$$  

(2.7)
Let \( z = x' = \frac{r'(y)}{r(y)} \), we can see that (2.7) is equivalent to

\[
(2.8) \quad z' - (n - 2)z^2 - (n - 1)z^3g(y)g'(y) = 0.
\]

Clearly, this is an Abel’s equation for \( n \geq 3 \). It suffices for us to show that the non-existence of solution to (2.8) with condition (2.3) or condition (2.4).

Now let us prove the first part, that is, equation (2.8) with condition (2.3) has no solution. In fact, this part is well-known, see for example [36, 9]. But for completeness, we will give an alternative proof here.

Suppose such a solution exists. Since \( z > 0 \) for \( y \in (0, \infty) \), using (2.8) and the fact: \( g(y)g'(y) \geq 0 \) for \( y \geq 0 \) by conditions (1) (2), we can get

\[
z' = (n - 2)z^2 + (n - 1)z^3g(y)g'(y) \\
\geq (n - 2)z^2.
\]

So

\[
(z^{-1})' \leq -(n - 2).
\]

Integrating over \( y \), we get

\[
z^{-1}(y) - z^{-1}(1) \leq -(n - 2)(y - 1)
\]

for \( y \geq 1 \). Letting \( y \to +\infty \), we have

\[
z^{-1}(y) < 0.
\]

This contradicts the property \( z > 0 \) for \( y \in (0, \infty) \). Hence we finish the proof of this part.

We still need to prove the second part, that is, equation (2.8) with condition (2.4) has no solution. Now let us use conditions (2) (3) to get the following result first.

**Lemma 2.1.** Let \( z(y) \) be a solution to (2.8) with condition (2.4), we have

\[
(n - 2) + (n - 1)z(y)g(y)g'(y) \geq 0
\]

for \( y \in (0, +\infty) \).

**Proof of Lemma 2.1.** Let us assume on the contrary, that is,

\[
(n - 2) + (n - 1)z(y_0)g(y_0)g'(y_0) < 0
\]

for some \( y_0 \in (0, +\infty) \). Setting

\[
\Sigma \equiv \{ \omega \in (y_0, +\infty) : (n-2)+(n-1)z(y)g(y)g'(y) < 0 \ \text{holds in} \ (y_0, \omega) \}.
\]

It’s clearly that \( \Sigma \) is a closed set in \( (y_0, +\infty) \), and is nonempty by continuity of \( z, g \) and \( g' \). If we can prove that \( \Sigma \) is also open in \( (y_0, +\infty) \), then

\[
\Sigma \equiv (y_0, +\infty)
\]
A non-existence result on harmonic diffeomorphisms

by connection of \((y_0, +\infty)\). In fact, by equation (2.8) and the definition of \(\Sigma\), we can get \(z(y)\) is monotone decreasing in \((y_0, \omega_0)\) for any \(\omega_0 \in \Sigma\), which implies that

\[(n - 2) + (n - 1)z(y)g(y)g'(y)\]

is also decreasing in \((y_0, \omega_0)\) since \(z(y) < 0\) for \(y \in (0, \infty)\). So

\[(n - 2) + (n - 1)z(\omega_0)g(\omega_0)g'(\omega_0) < (n - 2) + (n - 1)z(y_0)g(y_0)g'(y_0) < -\delta\]

for some positive constant \(\delta\). That’s to say, \(\omega_0\) is an interior point of \(\Sigma\) by continuity. As a result, \(\Sigma\) is open. Hence

\[\Sigma = (y_0, +\infty)\]

and for all \(y > y_0\),

\[(n - 2) + (n - 1)z(y)g(y)g'(y) < (n - 2) + (n - 1)z(y_0)g(y_0)g'(y_0) < -\delta.\]

Consequently, we can get

\[z' = z^2[(n - 2) + (n - 1)zg(y)g'(y)] \leq -\delta z^2\]

for \(y \geq y_0\). So

\[(z^{-1})' > \delta.\]

Hence

\[z^{-1}(y) > z^{-1}(y_0) + \delta(y - y_0) > 0\]

provided that \(y > y_0\) is large enough, which contradicts the property \(z < 0\) for \(y \in (0, \infty)\). Therefore, Lemma 2.1 holds.

Combining (2.8), the result of Lemma 2.1 and condition (2.4), we can get the following results.

**Corollary 2.1.** Let \(z(y)\) as the same as in Lemma 2.1, we have

(2.9) \[z'(y) \geq 0 \text{ for } y > 0,\]

then

(2.10) \[\lim_{y \to 0^+} z(y) = -\infty.\]

**Proof of Corollary 2.1.** (2.9) is a direct result of (2.8) and the result of Lemma 2.1.

By condition (2.4), we have \(\lim_{y \to 0^+} r(y) = +\infty\), so \(\lim_{y \to 0^+} \ln r(y) = +\infty\), then \(z = (\ln r)'\) is unbounded on \((0, \epsilon)\) for any \(\epsilon > 0\). On the other
hand, from (2.9), \( z = (\ln r)' \) is an increasing function in \( y \) for \( y \geq 0 \). Hence for \( y \to 0^+ \), \( z = (\ln r)' \to -\infty \). □

Let us continue to prove the second part. Setting \( Z(y) = z^{-1}(y) \) and using equation (2.8) again, we have

\[
Z'(y) = (n - 2) + (n - 1) \frac{g(y)g'(y)}{Z}.
\]

Letting \( w = Z^2(y) \), we get

\[
-\frac{1}{2}w' = -(n - 2)\sqrt{w} + (n - 1)g(y)g'(y).
\]

So

\[
\begin{aligned}
\begin{cases}
w' = 2(n - 2)\sqrt{w} - 2(n - 1)g(y)g'(y) \\
\lim_{y \to 0^+} w(y) = 0
\end{cases}
\end{aligned}
\]

where \( \lim_{y \to 0^+} w(y) = 0 \) comes from (2.10). By (2.13) and nonnegativity of \( g(y)g'(y) \), we have

\[
w' \leq 2(n - 2)\sqrt{w}.
\]

Consequently, we can get

\[
\sqrt{w(y)} \leq (n - 2)y
\]

for \( y > 0 \). Substituting this into (2.13), we can get

\[
0 \leq w'(y) = 2(n - 2)\sqrt{w} - 2(n - 1)g(y)g'(y)
\]

\[
\leq 2(n - 2)^2y - 2(n - 1)g(y)g'(y)
\]

for all \( y > 0 \). From this, we can get

\[
\sup_{y > 0} [y^{-1}g(y)g'(y)] \leq \frac{(n - 2)^2}{n - 1}.
\]

This contradicts condition (3). Hence equation (2.8) with condition (2.4) has no solution.

Combining the results of these two parts, we can conclude that no such harmonic diffeomorphism exists from \((\mathbb{R}^*_n, E)\) onto \((M^*_n, G)\) provided that \( G \) satisfies conditions (1)-(3). So we finish the proof of Theorem 1.1. □

**Remark 2.1.** The “diffeomorphism” in this statement of Theorem 1.1 can be replaced by “homeomorphism”. That is, for \( n \geq 2 \), there is no rotationally symmetric harmonic homeomorphism from \((\mathbb{R}^*_n, E)\) onto \((M^*_n, G)\) provided that the Riemannian metric \( G \) satisfies conditions (1)-(3) and \( g(y)g'(y) > 0 \) for \( y > 0 \).
Proof of Remark 2.1. We need to show that such a rotationally symmetric harmonic homeomorphism is a diffeomorphism. This result should exist somewhere for more general settings, but we didn’t find a suitable reference. For readable, we explain the reason for our special case. Noting that if such a rotationally symmetric harmonic homeomorphism exists, then the corresponding $C^2$ function $y$ is positive with $y' \geq 0$ for all $r \in (0, \infty)$, or with $y' \leq 0$ for all $r \in (0, \infty)$. Clearly, it suffices for us to prove $y' \neq 0$ for any $r \in (0, \infty)$.

We will first show that if $y > 0$ and $y' \geq 0$ for all $r \in (0, \infty)$, then $y' > 0$ for all $r \in (0, \infty)$. Suppose $y'(r_0) = 0$ for some $r_0 > 0$, then $y''(r_0) = 0$ by maximum principle. Substituting these into (2.2), we can get a contradiction.

Similarly, we can get that if $y > 0$ and $y' \leq 0$ for all $r \in (0, \infty)$, then $y'(r) < 0$ for all $r \in (0, \infty)$.

Combining the results of the previous two cases, we can conclude that $y' \neq 0$ for any $r \in (0, \infty)$, so such a homeomorphism is diffeomorphic. Hence by Theorem 1.1, we can get the result of Remark 2.1. □

References

[1] Akutagawa, K., Harmonic diffeomorphisms of the hyperbolic plane, Trans. Amer. Math. Soc. 342 (no. 1), (1994), 325–342.
[2] Akutagawa, K. and Nishikawa, S., The Gauss map and spacelike surfaces with prescribed mean curvature in Minkowski 3-space, Tohoku Math. J. (2) 42 (no. 1), (1990), 67–82.
[3] Au, T. K-K. and Wan, T. Y.-H., From local solutions to a global solution for the equation $\Delta w = e^{2w} - |\Phi|^2 e^{-2w}$, Geometry and global analysis (Sendai, 1993), 427–430, Tohoku Univ., Sendai, 1993.
[4] Au, T. K-K., Tam, L.-F. and Wan, T. Y.-H., Hopf differentials and the images of harmonic maps, Comm. Anal. Geom. 10 (no. 3), (2002), 515–573.
[5] Au, T. K-K. and Wan, T. Y.-H., Images of harmonic maps with symmetry, Tohoku Math. J. (2) 57 (no. 3), (2005), 321–333.
[6] Chen, L., Du, S.-Z. and Fan, X.-Q., Rotationally Symmetric Harmonic Diffeomorphisms between Surfaces, Abstr. Appl. Anal. 2013, Article ID 512383; arXiv:1305.3793.
[7] Chen, L., Du, S.-Z. and Fan, X.-Q., Harmonic diffeomorphisms between the annuli with rotational symmetry, Nonlinear Anal. 101, (2014), 144–150. Addendum to Harmonic diffeomorphisms between the annuli with rotational symmetry, Nonlinear Anal. 105, (2014), 1–2.
[8] Chen, Q and Eichhorn, J., Harmonic diffeomorphisms between complete Riemann surfaces of negative curvature, Asian J. Math. 13 (no. 4), (2009), 473–533.
[9] Cheung, L. F. and Law, C. K., An initial value approach to rotationally symmetric harmonic maps, J. Math. Anal. Appl. 289 (no. 1), (2004), 1–13.
[10] Choi, H. I. and Treibergs, A., New examples of harmonic diffeomorphisms of the hyperbolic plane onto itself, Manuscripta Math. 62 (no. 2), (1988), 249–256.
[11] Choi, H. I. and Treibergs, A., *Gauss maps of spacelike constant mean curvature hypersurfaces of Minkowski space*, J. Differential Geom. **32** (no. 3), (1990), 775–817.
[12] Collin, P. and Rosenberg, H., *Construction of harmonic diffeomorphisms and minimal graphs*, Ann. of Math. (2) **172** (no. 3), (2010), 1879–1906.
[13] Coron, J.-M. and Hélein, F., *Harmonic diffeomorphisms, minimizing harmonic maps and rotational symmetry*, Compositio Math. **69** (no. 2), (1989), 175–228.
[14] Duren, P., *Harmonic mappings in the plane*, Cambridge Tracts in Mathematics, **156**, Cambridge University Press, Cambridge, 2004.
[15] Gálvez, J. A. and Rosenberg, H., *Minimal surfaces and harmonic diffeomorphisms from the complex plane onto certain Hadamard surfaces*, Amer. J. Math. **132** (no. 5), (2010), 1249–1273.
[16] Han, Z.-C., *Remarks on the geometric behavior of harmonic maps between surfaces*, Elliptic and parabolic methods in geometry (Minneapolis, MN, 1994), 57–66, A K Peters, Wellesley, MA, 1996.
[17] Han, Z.-C., Tam, L.-F., Treibergs, A. and Wan, T. Y.-H., *Harmonic maps from the complex plane into surfaces with nonpositive curvature*, Comm. Anal. Geom. **3** (no. 1-2), (1995), 85–114.
[18] Hélein, F., *Harmonic diffeomorphisms between Riemannian manifolds*, Variational methods (Paris, 1988), 309–318, Progr. Nonlinear Differential Equations Appl., 4, Birkhäuser Boston, Boston, MA, 1990.
[19] Hélein, F., *Harmonic diffeomorphisms with rotational symmetry*, J. Reine Angew. Math. **414** (1991), 45–49.
[20] Heinz, E., *Über die Lösungen der Minimalflächenungleichung*, (German) Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. (1952), 51–56.
[21] Iwaniec, T., Kovalev, L. V. and Onninen, J., *The Nitsche conjecture*, J. Amer. Math. Soc. **24** (no. 2), (2011), 345–373.
[22] Jost, J. and Schoen, R., *On the existence of harmonic diffeomorphisms*, Invent. Math. **66** (no. 2), (1982), 353–359.
[23] Kalaj, D., *On harmonic diffeomorphisms of the unit disc onto a convex domain*, Complex Var. Theory Appl. **48** (no. 2), (2003), 175–187.
[24] Kalaj, D., *On the Nitsche conjecture for harmonic mappings in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \)*, Israel J. Math. **150** (2005), 241–251.
[25] Kalaj, D., *On the univalent solution of PDE \( \Delta u = f \) between spherical annuli*, J. Math. Anal. Appl. **327** (no. 1), (2007), 1–11.
[26] Kalaj, D., *Harmonic maps between annuli on Riemann surfaces*, Israel J. Math. **182** (2011), 123–147.
[27] Kalaj, D., *On J. C. C. Nitsche’s type inequality for hyperbolic space \( \mathbb{H}^3 \)*, (2012), arXiv:1202.4410v2.
[28] Kalaj, D., *On J. C. C. Nitsche type inequality for annuli on Riemann surfaces*, (2012), arXiv:1204.5419v2.
[29] Kalaj and Ponnusamy, S., *Bi-Harmonic mappings and J. C. C. Nitsche type conjecture*, (2011), arXiv:1102.2530v1.
[30] Leguil, M. and Rosenberg, H., *On harmonic diffeomorphisms from conformal annuli to Riemannian annuli*, preprint.
[31] Li, P., Tam, L.-F. and Wang, J.-P., *Harmonic diffeomorphisms between Hadamard manifolds*, Trans. Amer. Math. Soc. **347** (no. 9), (1995), 3645–3658.
A non-existence result on harmonic diffeomorphisms

[32] Lohkamp, J., *Harmonic diffeomorphisms and Teichmüller theory*, Manuscripta Math. **71** (no. 4), (1991), 339–360.

[33] Markovic, V., *Harmonic diffeomorphisms and conformal distortion of Riemann surfaces*, Comm. Anal. Geom. **10** (no. 4), (2002), 847–876.

[34] Markovic, V., *Harmonic diffeomorphisms of noncompact surfaces and Teichmüller spaces*, J. London Math. Soc. (2) **65** (no. 1), (2002), 103–114.

[35] Nitsche, J. C. C., *On the module of doubly-connected regions under harmonic mappings*, Amer. Math. Monthly. **69** (no. 8), (1962), 781–782.

[36] Ratto, A. and Rigoli, M., *On the asymptotic behaviour of rotationally symmetric harmonic maps*, J. Differential Equations **101** (no. 1), (1993), 15–27.

[37] Schoen, R. M., *The role of harmonic mappings in rigidity and deformation problems*, Complex geometry (Osaka, 1990), 179–200, Lecture Notes in Pure and Appl. Math., **143**, Dekker, New York, 1993.

[38] Schoen, R. M. and Yau, S. T., *Lectures on harmonic maps*. Conference Proceedings and Lecture Notes in Geometry and Topology, II. International Press, Cambridge, MA, 1997.

[39] Shi, Y., *On the construction of some new harmonic maps from \( \mathbb{R}^m \) to \( \mathbb{H}^m \)*, Acta Math. Sin. (Engl. Ser.) **17** (no. 2), (2001), 301–304.

[40] Shi, Y. and Tam, L.-F., *Harmonic maps from \( \mathbb{R}^n \) to \( \mathbb{H}^m \) with symmetry*, Pacific J. Math. **202** (no. 1), (2002), 227–256.

[41] Tachikawa, A., *Rotationally symmetric harmonic maps from a ball into a warped product manifold*, Manuscripta Math. **53** (no. 3), (1985), 235–254.

[42] Tachikawa, A., *A nonexistence result for harmonic mappings from \( \mathbb{R}^n \) into \( \mathbb{H}^n \)*, Tokyo J. Math. **11** (no. 2), (1988), 311–316.

[43] Tan, L.-F. and Wan, T. Y.-H., *Harmonic diffeomorphisms into Cartan-Hadamard surfaces with prescribed Hopf differentials*, Comm. Anal. Geom. **2** (no. 4), (1994), 593–625.

[44] Tan, L.-F. and Wan, T. Y.-H., *Quasi-conformal harmonic diffeomorphism and the universal Teichmüller space*, J. Differential Geom. **42** (no. 2), (1995), 368–410.

[45] Wan, T. Y.-H., *Constant mean curvature surface, harmonic maps, and universal Teichmüller space*, J. Differential Geom. **35** (no. 3), (1992), 643–657.

[46] Wan, Tom Y. H., *Review on harmonic diffeomorphisms between complete noncompact surfaces*, Surveys in geometric analysis and relativity, 509–516, Adv. Lect. Math. (ALM), 20, Int. Press, Somerville, MA, 2011.

[47] Wu, D., *Harmonic diffeomorphisms between complete surfaces*, Ann. Global Anal. Geom. **15** (no. 2), (1997), 133–139.

The School of Natural Sciences and Humanities, Shenzhen Graduate School, The Harbin Institute of Technology, Shenzhen, 518055, P. R. China.

E-mail address: szdu@hitsz.edu.cn

Department of Mathematics, Jinan University, Guangzhou, 510632, P. R. China.

E-mail address: txqfan@jnu.edu.cn