Finite temperature Dicke phase transition of a Bose-Einstein condensate in an optical cavity

Yuanwei Zhang,1,2 Jinling Lian,1 J. -Q. Liang,1 Gang Chen,2 Chuanwei Zhang,3 and Suotang Jia2

1Institute of Theoretical Physics, Shanxi University, Taiyuan 030006, P. R. China
2State Key Laboratory of Quantum Optics and Quantum Optics Devices, College of Physics and Electronic Engineering, Shanxi University, Taiyuan 030006, P. R. China
3Department of Physics and Astronomy, Washington State University, Pullman, Washington, 99164 USA

The Dicke model, a textbook paradigm in quantum optics, describes a two-level atomic ensemble coupled with an optical cavity. It was first introduced to illustrate the importance of collective and coherent excitations for atoms induced by a single photon mode [1]. With increasing atom-cavity coupling, the Dicke model predicts a quantum phase transition from a normal phase (NP) to a superradiant phase (SP), where both the atomic ensemble and photon acquire macroscopically collective excitations [2, 3]. However, due to the ‘no-go theorem’ originating from the Thomas-Reiche-Kuhn sum rule for the oscillator strength, this phase transition cannot be realized in typical cavity quantum electrodynamics [3, 6]. The experimental breakthrough for observing the quantum phase transition only occurs recently using the momentum eigenstates of a Bose-Einstein condensate (BEC) coupled with an optical cavity [7, 8]. In this pioneer experiment, there exists an nonlinear atom-photon interaction [8, 9], which, induced by the optical lattice potential and greatly enhanced by the large atom number N, can reach the same order of the effective cavity frequency and even go beyond. In this strong interaction regime, rich dynamical properties [10, 11] as well as new quantum phase transitions [12] have been predicted at zero temperature.

Although such ideal quantum phase transitions should occur at absolute zero temperature [13] in principle, realistic experiments used to observe the quantum phase transition must be performed at finite temperature. For instance, the typical temperature of the BEC in [3, 8] for observing the Dicke phase transition is \( \sim 50 \text{ nK} \). At finite temperature, thermal fluctuations may induce new exotic phenomena beyond the prediction of the zero temperature theory [14]. A well-known example is the lost of long range superfluid order in low dimensions (two or one) at any temperature [13], where the superfluid physics is characterized by the Berezinskii-Kosterlitz-Thouless transition [14, 17]. Therefore it is crucially important to investigate the Dicke phase transition at finite temperature to fully understand the realistic experiment.

In this Letter, we present a finite-temperature field theory for the Dicke phase transition in a BEC coupled with an optical cavity to understand the recent breakthrough experiment in this system. Our main findings are the following:

(I) The experimentally observed phase diagram for the normal-superradiant phase transition, \( (i.e., \text{Fig. (5) in Ref. [7]}) \) can be well understood in our finite-temperature theory. More interestingly, we show that a new phase, called \( \text{CE}_{\text{SP}} \), can be identified in the experimental phase diagram in [7]. In the \( \text{CE}_{\text{SP}} \) phase, NP and SP coexist, but the SP is stable and the NP is metastable.

(II) By varying various physical parameters (temperature, the coupling strength, the nonlinear interaction, etc.), we show there exist rich finite-temperature phases in the experimental system, including NP, SP, \( \text{CE}_{\text{SP}}, \text{CE}_{\text{NP}} \) (coexistence of stable NP and metastable SP), as well as dynamical unstable phases (US). In certain parameter region, there is a four-phase coexistence point.

(III) We show, both analytically and numerically, that the specific heat of the BEC increases rapidly in the SP/\( \text{CE}_{\text{SP}} \) (exponential increase at low temperature) with increasing temperature, and has a large jump at the critical transition temperature where the system becomes the NP/\( \text{CE}_{\text{NP}} \). Therefore the specific heat of the BEC may serve as a powerful tool for detecting different phases in

PACS numbers: 37.30.+i, 42.50.Pq, 03.75.Hh
Ξ = 3 / \text{ton and}

coupling strength between a single atom and the pho-
nonlinear atom-photon interaction

\[ U \]

Henceforth we set \( \hbar \) the cavity frequency, Rabi frequency. \( \omega \)

ation operator. The effective cavity frequency

\[ -\omega \]

reach the same order as the effective cavity frequency

\[ N \]

greatly enhanced by the large atom number

\[ \Delta \]

effect, and can

The finite-temperature properties of the BEC-cavity system can be obtained by calculating the partition function of the system through the imaginary-time functional path-integral approach. In this procedure, we rewrite the collective spin operators in the Hamiltonian \( \hat{H} \) as

\[ S_i = \sum_{i=1}^{N} (\alpha_i^+ \alpha_i - \gamma_i \gamma_i) \quad \text{and} \quad S_+ = \sum_{i=1}^{N} \alpha_i^+ \gamma_i \]

using Fermi operators \( \alpha_i^+ \) and \( \gamma_i \), which obey the anti-

commutation relation \( \{ \alpha_i^+, \alpha_j \} = \delta_{ij} \). After a straightforward calculation, the effective partition function can be written as \( Z \)

\[ \frac{Z}{Z_0} = \int [d\eta] \exp (-S) \int [d\eta] \exp (-S_f), \quad (2) \]

where \([d\eta]\) is the functional measure. In

Eq. \( \hat{S}_f \), the free action of the bosonic field

\[ S_f = \int_0^\beta d\tau \sum_{i=1}^{N} [\alpha_i^+ (\tau) \partial_\tau \alpha_i (\tau) + \gamma_i^+ \gamma_i (\tau)] + \int_0^\beta d\tau \psi^+ (\tau) (\partial_\tau + \omega) \psi (\tau), \quad \text{where} \quad \tau = it, \partial_\tau = \partial / \partial \tau, \beta = 1 / (k_B T) \text{with} \ k_B \text{being the Boltzmann constant} \]

and \( T \) being the system’s temperature. The total action \( S \) is expressed, in the basis of \( \Phi_i (\tau) = [\gamma_i (\tau), \alpha_i (\tau)]^T \), as

\[ S = S_0 (\psi, \psi^*) + \int_0^\beta d\tau \Phi_i^d (\tau) G (\psi, \psi^*) \Phi_i (\tau), \quad (3) \]

where \( S_0 (\psi, \psi^*) = \int_0^\beta d\tau \psi^+ (\tau) \partial_\tau \psi (\tau) \) and

\[ G = \left( \frac{\partial_\tau + \frac{\omega_p}{2} + \frac{U \sqrt{N}}{2N} \psi^+ \psi}{\sqrt{N} (\psi^+ + \psi)} \right) \left( \frac{\partial_\tau - \frac{\omega_p}{2} - \frac{U \sqrt{N}}{2N} \psi^+ \psi}{\sqrt{N} (\psi^+ + \psi)} \right). \quad (4) \]

FIG. 2: (Color online) The scaled mean-photon number as the functions of the detuning \( \Delta \) and the square of the collective coupling strength \( g \) for the temperature \( T = 0.00 \) nk (a), \( T = 0.13 \) nk (b) \( T = 50.00 \) nk (c), and \( T = 150.00 \) nk (d). The nonlinear interaction \( U = 1.70 \times 10^3 \omega \), and the recoil frequency \( \omega_r = 2 \pi \times 3.71 \) kHz. 1 nk \( \leftrightarrow 130.00 \) Hz.

The four-level model is considered by introducing the zero

momentum state \( |0, 0\rangle = (|p_x, p_z\rangle) \) with \( p_x \) and \( p_z \) being the

momenta in the \( x \) and \( z \) directions, hereafter), the excited states

\(|\pm k, 0\rangle \) and \(|0, \pm k\rangle \), and the symmetric superposition of states

\(|\pm k, \pm k\rangle \). In the dispersive limit, the excited-state levels can be

eliminated adiabatically, and an effective two-level system

with the zero momentum state and the symmetric superposi-
tion of states is thus formed.

the temperature-driven Dicke phase transition.

Figure 1 shows the experimental setup in which all

atoms in a \( ^87\text{Rb} \) BEC interact identically with a single-

mode photon induced by a high-finesse optical cavity [2].

In the experimental scheme, the atom momenta are used to define

the spin states \( |\uparrow\rangle \equiv |\pm k, \pm k\rangle, |\downarrow\rangle \equiv |0, 0\rangle \) with

the corresponding SU(2) collective spin operators \( S_+ = S_0 \equiv \sum_i |\uparrow\rangle_i \langle \downarrow| \) and \( S_0 \equiv \sum_i |\uparrow\rangle_i \langle \downarrow| \), as

shown in Fig. 1(b). As a result, not only the ‘no-go theorem’ is overcome, but also the superradiant-normal phase transition condition can be satisfied [19 21], since the smaller energy scale of the effective two levels can be achieved. Under the new spin basis, the dynamics of the atom-cavity system are governed by the Hamiltonian \( \hat{H} \)

\[ H = (\omega + \frac{U}{2N} S_0) \psi^d \psi + \frac{\omega_p}{2} S_0 + \frac{g}{\sqrt{N}} (\psi^d + \psi) (S_+ + S_-). \quad (1) \]

Henceforth we set \( \hbar = 1 \). \( \psi^d \) denotes the photon cre-
ation operator. The effective cavity frequency \( \omega = -\Delta_c + NU_0(1 + \Xi) / 2 \) with \( \Delta_c = \omega_p - \omega_c \), where \( \omega_c \) is the

cavity frequency, \( \omega_p \) is the pump laser frequency, \( \Xi = 3/4 \), and \( U_0 = g_0^2 / (\omega_p - \omega_g) \) with \( g_0 \) being the coupling strength between a single atom and the photon and \( \omega_g \) being the atomic transition frequency. The nonlinear atom-photon interaction \( U = N \Xi U_0 \), which is greatly enhanced by the large atom number \( N \), and can reach the same order as the effective cavity frequency \( \omega_c \) or even beyond. Therefore a strong nonlinear atom-photon interaction regime is accessible in experiments. The effective atomic frequency \( \omega_0 = 2\omega_c \) with the atomic recoil energy \( \omega_r = k^2 / 2m \). The collective coupling strength \( g = g_0 \Omega \sqrt{N} / 2(\omega_p - \omega_g) \), and \( \Omega \) is the maximum pump Rabi frequency.
We first integrate out the Fermi field $\Phi_t (\tau)$ (thus the atom spin degrees of freedom) in Eq. (3), and then use the standard stationary phase approximation, i.e.,

$$\delta S (\psi^*, \psi) / \delta \psi (\tau) = 0$$

and

$$\delta S (\psi^*, \psi) / \delta \psi^* (\tau) = 0,$$

to obtain the finite temperature equilibrium phase diagram in the large atom number, which is governed by the following equation

$$\omega \psi = F (\psi^*, \psi) \tanh \left[ \beta G (\psi^*, \psi) \right] \psi,$$

(5)

and its complex conjugate. Here $F (\psi^*, \psi) = \left( \frac{\omega_0 + U}{2} \psi^* \psi \right) U + 4g^2 \right) / \left( 2 \sqrt{\left( \frac{\omega_0 + U}{2} \psi^* \psi \right)^2 + 4 \frac{\lambda}{N} \psi^* \psi} \right)$

and $G (\psi^*, \psi) = \sqrt{\left( \frac{\omega_0 + U}{2} \psi^* \psi \right)^2 + 4 \frac{\lambda}{N} \psi^* \psi} / 2$. It is straightforward to find that Eq. (5) has a trivial solution $\psi = \psi^* = 0$, satisfying a nonlinear equation

$$\frac{2 \beta \zeta}{(\omega_0 + U \bar{\psi}_0^2) U + 8g^2} = \tanh \left( \frac{\beta \zeta}{4} \right),$$

(6)

where $\zeta = \sqrt{\left( \omega_0 + U \bar{\psi}_0^2 \right)^2 + 16g^2 \bar{\psi}_0^4}$, and $\bar{\psi}_0 = \langle \psi^* \psi \rangle / N$ is the scaled mean photon number. If $\psi = \psi^* = 0$, the system is in the NP with no collective excitation. However, the existence of non-trivial solutions shows that the macroscopic collective excitation for both atoms and photon occurs and thus the system locates at the SP. Finally, the stable conditions $\delta^2 S (\psi^*, \psi) / \delta \psi^2 > 0$ and $\delta^2 S (\psi^*, \psi) / \delta \psi^2 > 0$ determine the real solution of Eq. (6) and thus the phases of the BEC-cavity system at finite temperature.

When $U = 0$, Eq. (6) reduces to

$$\omega \sqrt{\omega_0^2 + 16g^2 \bar{\psi}_0^4} = \tanh \left( \beta \sqrt{\omega_0^2 + 16g^2 \bar{\psi}_0^4} / 4 \right),$$

the result for the original Dicke model. In this case, only SP and NP exist, even at finite temperature. When the temperature $T \rightarrow 0$, tanh $(\beta \zeta / 4)$ becomes 2$\omega / [(\omega_0 + U \bar{\psi}_0^2) U + 8g^2] = 1$, and Eq. (6) becomes 2$\omega / (\omega_0 + U \bar{\psi}_0^2) U + 8g^2 = 1$, from which rich zero-temperature phase diagrams have been obtained.

With increasing temperature $T$, the function tanh $(\beta \zeta / 4)$ decreases. As a consequence, Eq. (6) can possess some new physical solutions of $\bar{\psi}_0$, leading to new phases at finite temperature.

We solve the nonlinear equation (6), together with stable conditions $\delta^2 S (\psi^*, \psi) / \delta \psi^2 > 0$ and $\delta^2 S (\psi^*, \psi) / \delta \psi^2 > 0$, for various different physical parameters and calculate the scaled mean photon number $\langle \psi^* \psi \rangle / N = \bar{\psi}_0^2$. In Fig. 2, we plot the scaled mean-photon number $\bar{\psi}_0^2$ as the functions of the detuning $\Delta_c$ and the square of the collective coupling strength $g$ for different temperatures (a) $T = 0.00 \text{ nK}$, (b) $T = 0.13 \text{ nK}$, (c) $T = 50.00 \text{ nK}$, and (d) $T = 150.00 \text{ nK}$. To compare with the experiment, we also choose the other parameters to be the same as those in Ref. [7], that is, the nonlinear interaction $U = 1.70 \times 10^3 \omega_0$, and the recoil frequency $\omega_r = 2\pi \times 3.71 \text{ kHz}$. When $T = 0$, due to the existence of the strong nonlinear interaction, the CE$_{SP}$ phase is found, apart from the NP and SP, as shown in Fig. 2 (a). In the CE$_{NP}$ phase, $\bar{\psi}_0^2 \sim 0$, and almost no photon signature is generated. However, at the experimental operating temperature $T = 50.00 \text{ nK}$, numerical solution of Eq. (6) shows that the finite-temperature critical point that separates the NP and SP agrees with that in the experiment. More interestingly, the CE$_{NP}$ phase becomes the CE$_{SP}$ phase, in which a weak photon signature (compared with the strong superradiant regime) emerges. We note that such weak photon signature for the CE$_{SP}$ phase has already been observed in the experiment in the same parameter region, although it has not been explicitly pointed out. The transition temperature from the CE$_{NP}$ phase at zero temperature to the CE$_{SP}$ at 50 nK is $T_p = 0.13 \text{ nK}$, as shown in Fig. 2(b). With increasing temperature $T (= 150.00 \text{ nK})$, the SP region becomes narrower due to the suppression of the superradiance by thermal fluctuations, as expected. However, a new phase diagram including the NP, SP, CE$_{SP}$, CE$_{NP}$ and

FIG. 3: (Color online) The scaled mean-photon number as the functions of the temperature $T$ and the collective coupling strength $g$ for $U = 10.00 \omega_r$, $\Delta_c = -20.00 \omega_r$, (a) $T = 31.10 \omega_r$, $\Delta_c = 20.00 \omega_r$ with $\omega_r = 2\pi \times 3.71 \text{ kHz}$.

FIG. 4: (Color online) The scaled mean-photon number as the functions of the temperature $T$ and the nonlinear interaction $U$ for $g = 3.00 \omega_r$, $\Delta_c = 20.00 \omega_r$. (a) and $g = 6.00 \omega_r$, $\Delta_c = 20.00 \omega_r$ with $\omega_r = 2\pi \times 3.71 \text{ kHz}$. 

Table 1: The scaled mean photon number $\bar{\psi}_0^2$ for various different physical parameters.

| Temperature | Collective Coupling | Nonlinear Interaction | Scaled Mean Photon Number |
|-------------|---------------------|-----------------------|---------------------------|
| 0.00 nK     | 1.70 x 10^3 \omega_0 | 1.70 x 10^3 \omega_0  | 0.00                      |
| 0.13 nK     | 1.70 x 10^3 \omega_0 | 1.70 x 10^3 \omega_0  | 0.13                      |
| 50.00 nK    | 1.70 x 10^3 \omega_0 | 1.70 x 10^3 \omega_0  | 50.00                    |
| 150.00 nK   | 1.70 x 10^3 \omega_0 | 1.70 x 10^3 \omega_0  | 150.00                   |
US $(\delta^2 S(\psi^*,\psi)/\delta\psi^2 < 0$ and $\delta^2 S(\psi^*,\psi)/\delta(\psi)^2 < 0$) is observed, as shown in Fig. 2(d).

In a realistic experiment, the experimental parameters such as the detuning $\Delta_c$, the collective coupling strength $g$, the nonlinear interaction $U$ as well as the temperature $T$ can be controlled independently. For example, $\Delta_c$ and $g$ can be tuned by adjusting the frequency and power of the pump laser, respectively. The nonlinear interaction can be controlled by tuning both the frequency of the pump laser and the atom number. The temperature can be manipulated by controlling the evaporative cooling process for the BEC [23]. In Fig. 3, we plot the scaled mean photon number $\tilde{\psi}_0$ as a function of the temperature $T$ and the collective coupling strength $g$. While in Fig. 4, we plot the scaled mean-photon number $\tilde{\psi}_0$ as a function of the temperature $T$ and the nonlinear interaction $U$. These figures show that rich finite-temperature phase diagrams exist in different parameter regions, induced by the competition among these parameters. Interestingly, we find a four-phase coexistence point in certain parameter regions, as shown in Fig. 4(b). It should be emphasized again that these exotic phases arise from the interesting nonlinear interaction $U$ and the finite temperature. If $U = 0$, only the SP and NP can be found [18].

These rich finite-temperature phases and the phase transitions can be detected in experiments by measuring not only the mean photon number, but also some of the thermodynamic quantities, such as the specific heat per atom $C_V = \frac{1}{Nk_B T^2} \partial^2 T \ln Z$ in the BEC. In the NP, $C_V$ can be obtained analytically, yielding $C_V^{NP} = \frac{\omega_0^2}{8k_B T^2} \frac{[1 - \tanh^2 (\beta\omega_0/4)]}{\beta\omega_0}$, which reaches the maximum at certain temperature (the red dashed-dotted line in Fig. 5). However, the explicit expression for $C_V$ cannot be obtained in the SP. Nevertheless, in the region $T \ll T_c$, we find

$$C_V^{SP} \approx \frac{\zeta_0^2}{8k_B T^2} \text{sech}^2 \left( \frac{\zeta_0}{4k_B T} \right)$$

(7)

where $\zeta_0$ is the value of $\zeta = \sqrt{(\omega_0 + U\psi_0^2)^2 + 16g^2\psi_0^2}$ at $T = 0$. This expression shows that the specific heat $C_V^{SP}$ in the SP increases rapidly (exponential increase at low temperature), which agrees well with the numerical results (see the insert of Fig. 5). At the critical temperature $T_c$, at which the system enters the NP, the specific heat $C_V$ has a large jump. This step behavior is quite different from the behavior of the scaled mean photon number $\tilde{\psi}_0$, which varies smoothly when crossing the critical point. Therefore the temperature-driven phase transition from SP to NP can be detected using the specific heat $C_V$. In addition, the behavior of specific heat $C_V$ in the CE SP/CE NP is similar to that in the SP/NP. Moreover, for the transition from the CE SP to CE NP phases, such a large jump of $C_V$ still exists, as shown by the green dotted line of Fig. 5.

In summary, we develop a finite temperature theory for the Dicke phase transition of a BEC in an optical cavity. Our theory not only explains well the recent experimental observation of the phase transition from the SP to NP, but predicts rich new phases that either have been observed (but not pointed out) or may be observable by tuning experimental parameters. Our study is crucially important for understanding the Dicke phase transition in a realistic experimental system and may have significant application in quantum computation, quantum optics, etc.

We thank Profs. T. Esslinger and Jing Zhang, and Drs. K. Baumann, F. Brennecke, Yongping Zhang and Ming Gong for helpful discussions and suggestions. This work is supported by the 973 Program under Grant No. 2012CB921603, the NNSFC under Grant Nos. 10934004, 60978018, 61008012, 11074154, and 11075099. C.Z. is supported by NSF.

* These authors contributed equally to this work

1 Corresponding author, chengang971@163.com

[1] R. H. Dicke, Phys. Rev. 93, 99 (1954).
[2] K. Hepp, and E. H. Lieb, Ann. Phys. (N. Y.) 76, 360 (1973).
[3] Y. K. Wang, and F. T. Hioes, Phys. Rev. A 7, 831 (1973).
[4] F. T. Hioes, Phys. Rev. A 8, 1440 (1973).
[5] P. Nataf and C. Ciuti, Nature Commun. 1, 72 (2010).
[6] O. Viehmann, J. von Delft, and F. Marquardt, Phys. Rev. Lett. 107, 113602 (2011).
[7] K. Baumann, C. Guerlin, F. Brennecke, and T. Esslinger, Nature (London) 464, 1301 (2010).
[8] K. Baumann, R. Mottl, F. Brennecke, and T. Esslinger, Phys. Rev. Lett. 107, 140402 (2011).
[9] This nonlinear atom-photon interaction can also be found by considering the motion of a laser-driven BEC in a
high-finesse optical cavity. See, for example, D. Nagy, G. Kónya, G. Szirmai, and P. Domokos, Phys. Rev. Lett. 104, 130401 (2010).

[10] J. Keeling, M. J. Bhaseen, and B. D. Simons, Phys. Rev. Lett. 105, 043001 (2010).
[11] M. J. Bhaseen, J. Mayoh, B. D. Simons, J. Keeling, Phys. Rev. A 85, 013817 (2012).
[12] N. Liu, J. Lian, J. Ma, L. Xiao, G. Chen, J. -Q. Liang, and S. Jia, Phys. Rev. A 83, 033601 (2011).
[13] S. Sachdev, Quantum Phase transitions (Cambridge University Press, Cambridge, 1999).

[14] A. Griffin, T. Nikuni, and E. Zaremba, Bose-Condensed gases at finite temperatures (Cambridge University Press, Cambridge, 2009).
[15] P. C. Hohenberg, Phys. Rev. 158, 383 (1967).
[16] V. L. Berezinskii, Sov. Phys. JETP 32, 493 (1971).
[17] J. M. Kosterlitz, and D. Thouless, J. Phys. C 5, L124 (1972); J. Phys. C 6, 1181 (1973).
[18] V. N. Popov and S. A. Fedotov, Theor. Math. Phys. 51, 363 (1982).
[19] C. Emary, and T. Brandes, Phys. Rev. E 67, 066203 (2003).
[20] F. Dimer, B. Estienne, A. S. Parkins, and H. J. Carmichael, Phys. Rev. A 75, 013804 (2007).
[21] G. Chen, Xiaoguang Wang, J. -Q. Liang, and Z. D. Wang, Phys. Rev. A 78, 023634 (2008).
[22] A. Altland and B. Simons, Condensed Matter Field Theory (Cambridge University Press, Cambridge).
[23] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. 71, 463 (1999).