DYNAMICAL ASPECTS OF PIECEWISE CONFORMAL MAPS

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Abstract. We study the dynamics of piecewise conformal maps in the Riemann sphere. The normality and chaotic regions are defined and we state several results and properties of these sets. We show that the stability of these piecewise maps is related to the Kleinian group generated by their transformations under certain hypotheses. The general motivation of the article is to compare the dynamics of piecewise conformal maps and those of the Kleinian groups and iterations of rational maps.

1. Introduction

The study of the dynamics of piecewise maps comes from a variety of contexts, such as the piecewise interval maps (see for instance [15], [10]), the piecewise maps of isometries of the plane (see for instance [8], [9]) and some applications of differentiable piecewise maps (see for instance [4], [15]).

However, our main motivation, is to extend the so called Sullivan dictionary of conformal dynamics to piecewise conformal maps. In such dictionary there are involved the dynamics of iterations of holomorphic maps of the Riemann sphere and the dynamics of Kleinian groups. In both dynamics there is a duality in the behavior of the orbits of points: the conservative and the dissipative part, being both, invariant sets under the dynamics. The conservative part is called the Julia set in holomorphic dynamics and it is called the limit set in the dynamics of Kleinian groups and it is where the dynamics is more interesting. The Fatou set, in the other hand, is the region of normality for the set of iterations of a map, see [13].

In this paper we are interested in piecewise maps on the Riemann sphere \(\hat{C}\) which are restrictions of conformal automorphisms in each piece. The equivalent to the Julia set, in this context, is the pre-discontinuity set, following [7], and its complement is the regularity region. In Section 2.1 we include the relevant definitions. The regularity regions are very well known in the case of iterations of holomorphic maps ([3]), in our case is less complicated and Theorem 1 in Section 2.2 gives the classifications of the regularity regions. A result in holomorphic dynamics shows that the connectivity of a regularity region (Fatou domains) is 0, 1 or \(\infty\), in Section 2.3 we show that in the iteration of piecewise conformal maps, the regularity region can have any connectivity, from zero to infinity. In Section 2.4 we explain the symbolic dynamics related to a piecewise map.

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A celebrated theorem of Sullivan ([16]), shows that in the dynamics of rational maps, there are not wandering Fatou domains, although for certain entire transcendental maps there are ([3]). In Section 2.5 we show that it is the case that there are piecewise conformal maps with wandering domains. In holomorphic dynamics we can find Julia sets which are the whole Riemann sphere. We show in Section 2.6 an example of piecewise conformal map whose pre-discontinuity set is the whole sphere.

In Section 3, we relate the dynamics of a piecewise conformal map $F$ to a certain Kleinian group. To do so, we consider the extension of each conformal automorphism to the whole sphere and we consider the group $\Gamma_F$ generated by these extended maps. Theorem 2 and Theorem 3 shows the relation between the $\alpha$ and $\omega$-limits of $F$ and the limit set of the group $\Gamma_F$, $\Lambda(\Gamma_F)$.

In Section 4, we study some aspects of the deformations and stability of a given piecewise conformal map. Theorem 4 and Theorem 5 show that if we deform in a continuous manner the boundary of the regions initially involved, then the pre-discontinuity set deforms also in a continuous way, subject to certain hypothesis. Finally, Theorem 6 gives conditions, related to the group $\Gamma_F$, for a piecewise conformal map to be structurally stable.

We conclude this paper with two complementary sections. Section 5 deals with some examples and images of the theory. Finally Section 6 contains some technical results as well as the combinatorics on the pre-discontinuity set.

2. PIECEWISE CONFORMAL MAPS

2.1. Generalities. In this paper we will denote by $\mathbb{C}$ the complex plane and $\hat{\mathbb{C}}$ the Riemann sphere. We begin this section with the definition of a piecewise map in the sphere and we define the regions of conformality and discontinuity.

**Definition 1.** A piecewise conformal map is a pair $(P,F)$, where:

1. $P = \{R_m\}$ is a finite partition of $\hat{\mathbb{C}}$, that is, $\bigcup_m R_m = \hat{\mathbb{C}}$ and $R_m \cap R_n = \emptyset$ for all $m \neq n$. Additionally:
   - (a) The interior of each $R_m$ is non empty.
   - (b) The boundary of each $R_m$, denoted $\partial R_m$, is a finite union of simple closed curves.
2. $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a map satisfying that $F : R_m \to F(R_m)$ is the restriction of a conformal automorphism of $\hat{\mathbb{C}}$.

Following [7], associated to any piecewise conformal map, there is a set of discontinuity denoted by $\partial R$, the set $\bigcup_m \partial R_m$, which is the union of the boundaries of the sets $R_m$. Whereas that $\bigcup_m \text{int}(R_m)$, the union of interior of the sets $R_m$, is the region of conformality.

For a set $A \subset \hat{\mathbb{C}}$, define $F^{-n}(A) = \{ z \in \hat{\mathbb{C}} : F^n(z) \in A \}$. We have the next

**Definition 2.** The pre-discontinuity set of a piecewise conformal map $F$ is the set

$$\mathcal{PD}(F) = \bigcup_{n \geq 0} F^{-n}(\partial R)$$

\(^{1}\)This set was originally named "Spider Web" (see [7]) because of the visual resemblance in some cases, but being mathematically unintuitive we prefer the one given in this paper.
The pre-discontinuity set of $F$ has a natural stratification by the subsets $\partial R = \mathcal{P}D_0(F) \subset \mathcal{P}D_1(F) \subset \mathcal{P}D_n(F) \subset \mathcal{P}D(F)$, where $\mathcal{P}D_n(F) = \bigcup_{k=0}^{n} F^{-k}(\partial R)$, for $n \geq 0$.

**Definition 3.** The $\alpha$-limit set of a piecewise conformal map $F$ is

$$\alpha(F) = \mathcal{P}D(F) - \bigcup_{n \geq 0} F^{-n}(\partial R)$$

Hence we have that for any $z \in \mathcal{P}D(F) - \alpha(F)$, $z \in \mathcal{P}D_n(F)$ for some natural $n \geq 0$ and so $F^n(z) \in \partial R$, the discontinuity set.

From the discussion above, it follows that the family $\{F^n(z)\}_{n=1}^{\infty}$ is not normal if and only if $z \in \mathcal{P}D(F)$. Hence the pre-discontinuity set is the equivalent to the Julia set for holomorphic dynamics. The complement of the pre-discontinuity set is consequently the region of regularity, equivalently, the Fatou set as it is called in holomorphic dynamics. A good reference for definitions and properties of those sets in rational dynamics is the book of Milnor ([13]). For further properties about the pre-discontinuity set and the Fatou set of piecewise conformal maps, see Section 5.

2.2. Classification of Fatou components. In this section we classify the periodic regions of regularity when each of the maps involved are conformal automorphisms.

If $U$ is a connected component of the Fatou set, we say that $U$ is periodic if $F^n(U) \subset U$ for some $n > 0$. The minimum number $n$ with that property is the period of the component. Then for $U$ a connected component of the Fatou set, either there exists $k > 0$ such that $F^k(U)$ is periodic or does not exist such $k$. In the first case we say that $U$ is preperiodic and in the second $U$ is wandering.

The classification of periodic Fatou components of piecewise conformal automorphisms turns out to be rather simple and it is as follows (see Section 5 for examples).

Observe that if $U$ is periodic and $F^n(U) \subset U$, then $F^n|_U : U \to U$ is a composition of conformal maps hence it is the restriction of a Möbius transformation.

**Theorem 1.** Let $F$ be a piecewise conformal map and $U$ a periodic component of $F$ of period $n$, then we have the following cases:

(i) There is a fixed point under $F^n|_U$ inside $U$. In this case either $F^n|_U$ is (a): a loxodromic transformation with its attracting fixed point in $U$, (b): $F^n|_U$ is an elliptic transformation with at least one of their fixed point inside $U$, or (c): $F^n|_U$ is the identity.

(ii) There is a fixed point of $F^n|_U$ in the boundary of $U$, so (a): $F^n|_U$ is an hyperbolic transformation, (b): $F^n|_U$ is a parabolic transformation, or (c): $F^n|_U$ is the identity.

(iii) There is not a fixed point of $F^n|_U$ in $U$, so $F^n|_U$ is an elliptic transformation.

**Proof.** The map $F^n|_U : U \to U$ is a Möbius transformation. Recall that Möbius transformations are classified in loxodromic, parabolic and elliptic, see [11] for instance. First, let us assume that $F^n|_U$ is not the identity map.

Forward invariant open sets (that is $F^n(U) \subset U$) of loxodromic transformations must contain the attracting fixed point or have it on its boundary, that is case (ia) or (iia).
Since the boundary of an open Fatou connected domain is contained in the pre-
discontinuity set of $F$, then for a parabolic transformation, its fixed point is in the
boundary of the domain, that is (iib).

Remark. In case (ia) for all $z \in U$, then $F^{nk}(z)$ tends to $p$, the fixed point of $F^n|_U$, when $k$ tends to $\infty$ (see figure 1).

In case (ib) $F^n|_U$ is periodic if the angle of rotation of $F^n|_U$ is rational (see figure 2) or it is quasiperiodic if the angle of rotation of $F^n|_U$ is irrational (see figure 3).

In case (iia) or (iib) for all $z \in U$, we have that $F^{nk}(z)$ tends to $p$, the fixed point of $F^n|_U$, when $k$ tends to $\infty$ (see figure 5).

Case (iii) behaves like (ib) but without fixed points inside $U$.

Remark. About elliptic transformations, we can do a more detailed analysis. A
$F^n|_U$ elliptic can contain two fixed points, but $U \neq \hat{\mathbb{C}}$ then is not simply connected (see figure 6).

If $F^n|_U$ is elliptic without fixed points in $U$, then such component is not simply connected because $U$ must contain one invariant simple closed curve $C$ separating the fixed points (see figure 7).

If $U$ is a simply connected periodic component and $F^n|_U$ is not loxodromic, parabolic or elliptic with one fixed point in $\overline{U}$, then $F^n|_U$ can not be elliptic with fixed points outside $U$. Therefore for such $U$, $F^n|_U = Id$ (see figures 3 and 5).

2.3. Connectivity of the Fatou components. In rational dynamics it is known
that the connectivity of the Fatou set is one, two or infinity, see [13]. Here we show
that for piecewise conformal dynamics the connectivity of the regularity set can be
any natural number or infinity.

Example 1. For $k$ a positive natural number, let $D_k$ the disc with center at
$z = 0$ and small radius, say $r = (1/2)(k \tan(\pi/k))$, $S_k$ its boundary and $D_k^c$ its
complement. Define $g(z) = 2z$, if $z \in D_k$ and $f(z) = e^{2\pi i/k}z$ if $z \in D_k^c$. The map
$f(z)$ is a rotation. The map $F$ is generated by $f$ and $g$ .

Observe that the set $\{f^{-j}(S_k)\}_{j=1}^k$ of $k$ circles is contained in $\mathcal{PD}(F)$ and in fact $\mathcal{PD}(F) \subset \bigcup_{j=1}^k f^{-j}(D_k)$ . That means that the complement of the $k$ discs
$\{f^{-j}(D_k)\}_{j=1}^k$ is a region of regularity of $F$ with 0 and $\infty$ as elliptic fixed points.
Such region has connectivity $k$. The result does not depend on the choice for $g$. See figure 11.

It is left to show that there is a region of regularity with infinity connectivity.

Example 2. Consider $D$ the disc with center at 1 and radius 1/3. Choose $g(z)$ any Möbius transformation, if $z \in D$ and $f(z) = 2z$ if $z$ is in the complement of $D$.
The map $F$ is generated by $f$ and $g$ .

Notice that the set $\{f^{-j}(D)\}_{j=1}^{\infty}$ is a disjoint set of discs converging to 0 and of radius tending to 0. It is clear that $\mathcal{PD}(F) \subset \bigcup_{j=1}^{\infty} \{f^{-j}(D)\}$ . The complement of such sequence of disc is a region of regularity of $F$ with infinite connectivity. See figure 12.
2.4. Symbolic dynamics. For a piecewise conformal automorphism \((P,F)\), the partition \(P = \{ R_m \}_{m=1}^M\) leads naturally to a coding map, following the reasoning in \([8]\) for piecewise isometries. The coding space is the set of infinite sequences of \(M\) symbols \(\Sigma_M = \{1, \ldots, M\}\) and the coding map \(I_F : \hat{\mathcal{C}} \rightarrow \Sigma_M\) is defined as \(I_F(z)_k = m\) if \(F^k(z) \in R_m\), where \(I_F(z)_k = m_k\) is the \(k\)-th entry of the coding generated sequence \(I_F(z) = (m_1, m_2, m_3, \ldots, m_k, \ldots)\). The map \(I_F\) is called itinerary because encodes the forward orbit of a point by recording the indexes of visited sets \(R_m\).

A coding partition in cells \(\{ \mathcal{C}_s \}_{s \in \Sigma_M}\) over \(\hat{\mathcal{C}}\) is induced by the equivalence relation \(z \sim w\) if and only if \(I_F(z) = I_F(w)\). As we can see from \([7]\), the pre-discontinuity set is \(\bigcup_{s \in \Sigma_M} \partial \mathcal{C}_s\) and the Fatou set is \(\bigcup_{s \in \Sigma_M} \text{int}(\mathcal{C}_s)\), the union of interior of the partition cells. This gives us a relevant fact: components of Fatou set contains only points with the same itinerary (see figure \([9]\)).

The shift map \(\sigma : \Sigma_M \rightarrow \Sigma_M\), \(\sigma(m_1, m_2, m_3 \ldots) = (m_2, m_3 \ldots)\) is semi-conjugated to \(F\) because \(I_F \circ F(z) = \sigma \circ I_F(z)\) but \(I_F\) is many-to-one for most of the sphere, since each partition cell could contain more than one point.

We classify the points of \(\hat{\mathcal{C}}\) in two sets according to the itinerary. Points with rational itinerary are those belonging to periodic or eventually periodic cells. Points with irrational itinerary, set denoted as \(\mathcal{I}(F)\), are in wandering cells. By definition, both of rational and irrational sets are invariants.

Remark. If \(F\) has a wandering domain \(U\), then \(U \subset \mathcal{I}(F)\), because the itinerary of each point in \(U\) is not preperiodic.

Remark. According to \([8]\), the exceptional set is defined as \(E = \mathcal{I}(F)\) for piecewise isometries and prove that on a space of finite Lebesgue measure, every cell \(\mathcal{C}\) of positive measure is eventually periodic. As for each of such cells \(\text{int}(\mathcal{C})\) is a subset of the Fatou set, then \(E \subset PD(F)\) in case of \(F\) piecewise isometry defined in a subset of finite Lebesgue measure of \(\mathcal{C}\).

2.5. Wandering Domains. In this section we will show that there are piecewise conformal maps with wandering domains. We present two examples.

Example 3. Here, our construction relies in \([10]\) Theorem A, that proves the existence of a wandering domain for an affine interval exchange transformation of an interval. We will explain the basic facts and extend the construction to piecewise conformal maps.

Consider \(I = [0, 1) \subset \mathbb{R}\) and \(0 = y_0 < y_1 < y_2 < \ldots < y_{m-1} < y_m = 1\) a finite sequence. We say that \(T : [0, 1) \rightarrow [0, 1)\) is an affine interval exchange transformation (AIET) if for all \(i \in \{0, 1, \ldots, m-1\}\), the restriction \(T|_{[y_i, y_{i+1})}\) is continuously differentiable and its derivative is identically equal to a constant \(\beta_i > 0\). Hence \(T|_{[y_i, y_{i+1})}(x) = \beta_i x + r_i,\) with \(r_i\) a real number. The AIET in Theorem A of \([10]\) has \(m = 4\), the \(y_i\) depending on the derivatives \(\beta_i\) which in turn depend on a certain Perron-Frobenius matrix.

To construct our example consider the following piecewise conformal dynamical system \(F\):

Fix \(y_i, \beta_i\) and \(r_i, i \in \{0, 1, 2, 3\}\) as in Theorem A of \([10]\). Let the set of discontinuity be the union of the lines \(L_i = \{ z : Re(z) = y_i, i \in \{0, 1, 2, 3\}\}\). Let \(F\) be such that \(F(z) = z\) if \(Re(z) < y_k \) or \(Re(z) > y_k\) and \(F(z) = \beta_i z + r_i\) if \(y_k < Re(z) < y_{k+1}\). Observe that since each line \(L_i\) is orthogonal to the real line then each line in \(PD(F)\)
is orthogonal to the real line and its complement is a union of vertical strips. The restriction of such complement to the real line contains the wandering interval of $T$, therefore $F$ has a wandering strip.

We can construct a similar piecewise conformal transformation using discs instead of strips as follow: For consecutive $y_i$ and $y_{i+1}$ consider the disc $R_i$ with diameter $y_{i+1} - y_i$ and center $\frac{1}{2}(y_i + y_{i+1})$. Let $F$ such that $F(z) = \beta_i z + r_i$ in each disc $R_i$ and $F(z) = z$ outside all of discs. Then the associated pre-discontinuity set is an infinite union of arcs of circumferences. As in the previous case, the restriction to the real line must contain the wandering domain inherited from the AIET and then $F$ has a wandering component of the Fatou set.

In the terminology of [2], our map $F$ is close to the concept of a cone exchange transformation, except that we are allowing affine interval exchanges rather than the more restrictive isometric interval exchanges.

Other version to [10] with different properties can be found in [6] for instance.

**Example 4.** Here, we show that there exist a piecewise conformal map with all of the components of the regular set wandering. Let be $\mathcal{R} = \{z : \text{Im}(z) < 0\}$, $\mathcal{I}_R(z) = f(z) = iz$ and $\mathcal{I}_{Re}(z) = g(z) = -iz + 1 + i$. Notice that $f$ and $g$ are both euclidean rotations. First, we have $\mathcal{PD}(T) = \{z : \text{Re}(z) \in \mathbb{Z} \text{ or } \text{Im}(z) \in \mathbb{Z}\}$, then the Fatou set is formed by open squares which elements have not integer coordinates.

Let be $c_n = a + bi \in (0, 1) \times (n, n + 1) \subset \mathbb{R}^c$, where $n \in \mathbb{N}$. Calculating the orbit of $c_n$:


t_{1}(c_n) = g(c_n) = b + 1 + (1 - a)i \\
t_{2}(c_n) = g \circ g(c_n) = 2 - a - bi \\
t_{3}(c_n) = f \circ g \circ g(c_n) = b + (2 - a)i \\

\vdots

\quad t_{2n+3}(c_n) = (f \circ g)^{n+1} \circ g(c_n) = b - n + (n + 2 - a)i \in (0, 1) \times (n + 1, n + 2)

Then, the itinerary of $c_n$ is

\[
(1, \overbrace{1, 0}^{\text{n+1 times}}, 1, \overbrace{1, 0}^{\text{n+2 times}}, 1, \overbrace{1, 0}^{\text{n+3 times}}, \ldots),
\]

clearly a irrational sequence and in consequence, as we saw in Section 2.4, the square-component containing $c_n$ is wandering (see figure [10]).

Let be $Q_I = (0, \infty) \times (0, \infty) \subset \mathbb{R}^c$. The transformation $f \circ g$ is the translation $z \mapsto z - 1 + i$, which is applied to points $z \in Q_I$ with $\text{Re}(z) > 1$ and $\text{Im}(z) > 0$, whose orbit must reach a wandering square $(0, 1) \times (n, n + 1) \subset Q_I$. Then, all components in $Q_I$ are wandering. Define now $Q_{II} = (0, \infty) \times (-\infty, 0) \subset \mathbb{R}$, $Q_{III} = (-\infty, 0) \times (-\infty, 0) \subset \mathbb{R}$ and $Q_{IV} = (-\infty, 0) \times (0, \infty) \subset \mathbb{R}^c$. Observe that $T(Q_{III}) = Q_{II}$, $T(Q_{II}) = Q_I$, $T(Q_{IV}) \subset Q_I$ and $T(Q_I) \subset Q_I \cup Q_{III}$, then we can conclude that the orbits for all points visit the set $Q_I$, and in consequence, all components in the regular set must be wandering.

A well known theorem of Sullivan establish that components in the Fatou set of rational functions in the Riemann sphere are not wandering (see [16]). About piecewise maps exists some results in this direction. When $X$ is a metric space with finite Lebesgue measure and $F : X \to X$ is a piecewise isometry, then every
component in the Fatou set is eventually periodic (see [8, Proposition 6.1]). From the later theorem we have the following

**Corollary 1.** If \( F : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) is a piecewise conformal map where \( \partial R \) is bounded, \( \infty \in R_1 \), \( F|_{R_1} \) is an euclidean rotation and \( F|_{R_m} \) is an euclidean isometry for \( m > 1 \), then every component in the Fatou set is eventually periodic.

**Proof.** Since \( \partial R \) is a bounded set, exists a disc \( D \subset \mathbb{C} \) centered in the finite fixed point of \( F|_{R_1} \) such that \( \partial R \subset \mathbb{D}_1 \subset D \) and \( F^n(R_m \cap D) \subset D \) for each \( R_m \) and for all \( n \). Then \( F|_D \) is a piecewise (euclidean) isometry in a finite Lebesgue measure set and the result follows. \( \square \)

In relation with the pre-discontinuity set, we establish the next

**Proposition 1.** If \( F \) is a piecewise conformal map such that \( \mathcal{P} \mathcal{D}(F) = \mathcal{P} \mathcal{D}_N(F) \) for some \( N \), then every component in the Fatou set is eventually periodic.

**Proof.** \( \mathcal{P} \mathcal{D}(F) = \mathcal{P} \mathcal{D}_N(F) \) is union of boundaries of a finite number equivalence classes from itineraries, in consequence the Fatou set is composited by interiors a finite number of equivalence classes. Therefore, for each component \( U \), its orbit \( \{F^n(U)\} \) is contained in a finite number of components. \( \square \)

2.6. A Piecewise conformal map for which the pre-discontinuity set is the entire sphere.

**Example 5.** Let be \( R \) the disc with center in 0 and radius 1 and \( F \) the piecewise conformal map defined with \( F|_R(z) = 2z \) and \( F|_{R^c}(z) = \frac{z}{2} \). We claim that \( \mathcal{P} \mathcal{D}(F) = \hat{\mathbb{C}} \).

First, we can notice that \( F \) can not have periodic points \( z \neq 0 \). Otherwise, if \( F^n(z) = z \neq 0 \) with \( n \geq 1 \), then \( F^n(z) = 2^j \left( \frac{1}{2} \right)^j z = z \) and \( 2^j \neq 3^j \), clearly a contradiction.

Second, we will show that \( \left( \bigcup_{n \in \mathbb{N}} F^{-n}(\partial R) \right) \cap [0, \infty) = \bigcup_{n \in \mathbb{N}} A_n \), where \( A_n = \{ \frac{1}{2^n} \cdot \frac{3}{2^n}, \ldots, \frac{3^n}{2^n} \} \). Let \( q = \frac{3^n}{2^n} \) with \( 0 \leq m \leq n \) (that is, \( q \in A_n \)). We easily check the following statements:

1. If \( q = 1 \), then \( q \in \partial R \). Also note that \( A_0 = \{1\} \).
2. If \( q > 1 \), then \( F(q) = \frac{3}{2} q = \frac{3^{m-1}}{2^{n-1}} \) and clearly \( F(q) \in A_{n-1} \cup \{1\} \).
3. If \( q < 1 \) then \( F(q) = 2q = \frac{3^n}{2^n-1} \).
   a. If \( m \leq n-1 \) then \( F(q) \in A_{n-1} \cup \{1\} \).
   b. If \( m > n-1 \), then \( n = m \), but this is impossible because in those case \( \frac{3^n}{2^n} > 1 \) and by hypothesis \( q < 1 \).

If \( q \) can not be periodic, exists \( N \geq 0 \) such that \( F^N(q) = 1 \in \partial R \), that is, \( q \in \mathcal{P} \mathcal{D}_N(F) \).

Third, \( F|_{[\frac{2}{3}, 1]} \) is an affine interval exchange transformation: \( F\left(\frac{3}{2}, 1\right) = \frac{3}{2} \) and \( F\left(\frac{1}{2}, 2\right) = \frac{3}{2} \). An affine interval exchange transformation \( f : [0, 1) \to [0, 1) \) with \( f|_{[0, \frac{1}{2})}(x) = mx + a \) and \( f|_{[\frac{1}{2}, 1)}(x) = \mu x + b \) is conjugated to the rotation \( \tau_\theta : S^1 \to S^1 \) of angle \( \theta = \frac{\log \mu}{\log \lambda - \log \mu} \) (see [11]). In our case, \( F|_{[\frac{2}{3}, 2]} \) is conjugated to \( f : [0, 1) \to [0, 1) \) with \( f|_{[0, \frac{1}{2})}(x) = 2x + \frac{1}{2} \) and \( f|_{[\frac{1}{2}, 1)}(x) = \frac{2}{3} x - \frac{1}{6} \), and then conjugated to the rotation of angle \( \theta = \frac{\log 2}{\log 2 - \log (2/3)} = \frac{\log 2}{\log 3} \notin \mathbb{Q} \), in consequence every orbit of \( x \in [\frac{2}{3}, 2) \) is dense in such interval.
Let be $x \in (0, \infty)$. Then exists $N \geq 0$ such that $F(x) \in [\frac{3}{2}, 2]$, because $F$ is expansive if $x < \frac{3}{2}$ and contractive if $x > 2$. Since the orbit of $F^n(x)$ is dense in $[\frac{3}{2}, 2]$, exists $M$ such that $F^{n+M}(x) \in (1 - \delta_1, 1 + \delta_2)$, where $\delta_1 = \frac{\varepsilon}{x^2 + \varepsilon}$ and $\delta_2 = \frac{\varepsilon}{x - \varepsilon}$

for a given $\varepsilon > 0$.

$$F^{n+M}(x) = 2\left(\frac{3}{2}\right)j x = \frac{2^{i+j}}{3^j} x, \text{ where } i + j = N + M .$$

That is, for a given $\varepsilon > 0$ exists $y \in \bigcup_{n \in \mathbb{N}} A_n$ such that $y \in (x - \varepsilon, x + \varepsilon)$. Then, $\mathcal{P}(F) \cap [0, \infty) = [0, \infty)$. Since $F$ behaves the same in each ray from origin, we have $\mathcal{P}(F) = \mathbb{C}$.

2.7. Piecewise Hyperbolic 3D Isometries. We want to point out that Poincaré showed that any conformal automorphism of the sphere (a Möbius transformation), extends to the interior of the 3-hyperbolic space, $\mathbb{H}^3$, as an isometry. Hence, given any piecewise conformal automorphism $F$ on the sphere with regions $R_i$, it extends to a piecewise isometry of the hyperbolic space. To do so, extend each region $R_i$ to a regions $S_i$ in $\mathbb{H}^3$, such that, $S_i \cap \partial \mathbb{H}^3 = R_i$ and extend each of the the corresponding transformations.

Interesting cases comes when the regions $R_i$ are restrictions of discs, since we can consider their natural extensions, the spherical gaskets. Such spherical gaskets are sent to spherical gaskets under any Möbius transformation, and the pre-discontinuity set of the extended map $F$ will be made of pieces of spherical gaskets over the original pre-discontinuity set. However, we do not pursue this topic in this article.

3. Relation to Kleinian Groups

Given a piecewise conformal map $F$, we can naturally associate the finitely generated subgroup $\Gamma_F = \langle F \rangle_{|R_m}$ of $PSL(2, \mathbb{C})$, since each $F|_{R_m}$ is a Möbius transformation. If $\Gamma_F$ is discrete, then is called Kleinian group. Such group also act as a discrete group of isometries in $\mathbb{B}^3$, the hyperbolic open unit 3-ball. Recall that the limit set $\Lambda(\Gamma)$ of a Kleinian group $\Gamma$ is the set of accumulation points of $\Gamma p$, where $p \in \mathbb{B}^3$. $\Lambda(\Gamma) \subset \mathbb{C}$ because $\Gamma p$ accumulates in $\partial \mathbb{B}^3 = \mathbb{S}^2 \cong \mathbb{C}$. The regular set of $\Gamma$ is $\Omega(\Gamma) = \mathbb{C} - \Lambda(\Gamma)$. The limit set $\Lambda(\Gamma)$ can also be characterized as the limit of sequences of distinct elements $\gamma_i \in \Gamma$ applied to a any element $z \in \Omega(\Gamma)$.

In relation to the associated Kleinian group, we can demonstrate that if $\partial R$ is contained in the region of regularity of $\Gamma_F$, then the $\alpha$-limit of $F$ lands in the limit set of $\Gamma_F$.

**Theorem 2.** If $\partial R \cap \Lambda(\Gamma_F) \neq \emptyset$, then

1. $\alpha(F) \subset \Lambda(\Gamma_F)$, and
2. $\alpha(F) = \lim_{n \to \infty} F^{-n}(\partial R)$ in $\mathcal{H}(\mathbb{C})$.

**Note.** $\mathcal{H}(\mathbb{C})$ is the space of compact subsets of $\mathbb{C}$ with the Hausdorff topology induced from a spherical metric in $\mathbb{C}$, see Section 6.
Proof. Let \( L = \lim_{n \to \infty} F^{-n}(\partial R) \). If \( L = \emptyset \), then \( \alpha(F) = \emptyset \), by Proposition 3.

Suppose \( L \neq \emptyset \) and let \( z \in L \), \( f_m = F|_{R_m} \) and \( \Sigma_M(k) \) the set of words of \( k \) length of \( M \) symbols. Note that
\[
F^{-1}(\partial R) = \bigcup_{m=1}^{M} (f_m^{-1}(\partial R)) \cap \Sigma_M(k) \subset \bigcup_{i \in \Sigma_M(1)} C_{i,t} = \bigcup_{i \in \Sigma_M(1)} C_{i,t}
\]
and consequently \( F^{-n}(\partial R) = \bigcup_{i \in \Sigma_M(n)} C_{n,t} \), where each \( C_{n,t} \) is a finite union of curve segments and points, or an empty set (see last remark in Section 6).

At least one \( C_{n,t} \) is not empty for each \( n \) level, because we assumed \( L \neq \emptyset \). Then we can take \( z_n \in C_{n,t} \subset F^{-n}(\partial R) \) such that \( z_n \to z \), because every neighborhood of \( z \) intersects infinitely many \( F^{-n}(\partial R) \). Even more, we can choose \( z_n \in C_{n,t} \) such that \( F(z_n) = z_{n-1} \), because \( C_{n,t} = C_{n,s} \cap R_m \) for some \( m \) and then \( F(C_{n,t}) \subset C_{n-1,s} \). By this construction, \( z_n = f_{m_1}^{-1} \circ \cdots \circ f_{m_j}^{-1}(z_0) = \gamma_n(z_0) \), with \( z_0 \in \bigcap_{n \geq 0} F^n(C_{n,t}) \subset \partial R \) and \( \gamma_n = f_{m_1}^{-1} \circ \cdots \circ f_{m_j}^{-1} \in \Gamma_F \).

Suppose that \( \gamma_i = \gamma_j \) for some \( i < j \). By construction, \( \gamma_j = f_{m_{j-1}}^{-1} \circ \cdots \circ f_{m_1}^{-1} \circ f_{m_i}^{-1} \circ \cdots \circ f_{m_j}^{-1}(z_i) = \gamma_i(z_i) = z_j \).

Then
\[
\begin{align*}
z_{j+1} &= f_{m_{j+1}}^{-1} \circ \gamma_i(z_0) = z_{i+1}, \\
z_{j+2} &= f_{m_{j+2}}^{-1}(z_{j+1}) = z_{i+2}, \\
\vdots \\
z_{j+i-1} &= f_{m_i}^{-1}(z_{j-i}) = z_j = z_i.
\end{align*}
\]

Therefore, the sequence \( z_0, z_1, \ldots, z_i, z_{j-1}, \ldots, z_j, \ldots \to z \in L \), contradicting the hypothesis.

In conclusion, the sequence \( \gamma_n(z_0) \) converging to \( z \in L \) is constructed with different elements \( \gamma_n \in \Gamma_F \) applied to \( z_0 \in \bigcap_{n \geq 0} F^n(C_{n,t}) \subset \partial R \subset \Omega(\Gamma_F) \). Notice that \( \bigcap_{n \geq 0} F^n(C_{n,t}) \neq \emptyset \) because \( F^n(C_{n,t}) \) is a sequence of nested closed sets. Finally, using Proposition 3 \( \alpha(F) \subset L \subset \Lambda(\Gamma_F) \).

Now let \( z \in L \) but \( z \notin \alpha(F) \). Then \( z \in F^{-n}(\partial R) \subset PD_n(F) \) for some \( n \). In consequence, \( F^n(z) = \gamma(z) \subset \partial R \) for some \( \gamma \in \Gamma_F \). Therefore \( \emptyset \neq (\gamma \Lambda(\Gamma_F)) \cap \partial R \subset \Lambda(\Gamma_F) \cap \partial R \), contradicting the hypothesis. \( \square \)

Recall that the \( \omega \)-limit of a point \( z_0 \) under the map \( F \) is the set of accumulation points of the orbit \( \{F^n(z_0)\} \) and is denoted by \( \omega(z_0, F) \). Regarding the limit set of the Kleinian group, we have the next

**Theorem 3.** \( \omega(z, F) \subset \Lambda(\Gamma_F) \) for all \( z \in \hat{C} \).

**Proof.** \( \omega(z,F) \) is the set of accumulation points of the orbit \( \{F^n(z)\} \). But \( F^n = \gamma_n \in \Gamma_F \), then \( \{F^n(z)\} \subset \Gamma_F z \). In the other hand, \( \Lambda(\Gamma_F) \) is the set of accumulation points of \( \Gamma_F z \), then \( \omega(z,F) \subset \Lambda(\Gamma_F) \). \( \square \)

4. DEFORMATIONS AND STABILITY

The parameter space of piecewise conformal maps depends on the maps \( F|_{R_m} = \gamma_m \in PSL(2, \mathbb{C}) \) and the elements \( R_m \) of the partition in the sphere.

In our case it is enough to consider the discontinuity set \( \partial R = \bigcup_{m=1}^{M} \partial R_m \) as a compact subset of \( \hat{\mathbb{C}} \). So if the partition runs from \( m = 1 \) to \( m = M \), the parameter
space is a subspace of

\[ \mathcal{T} = \underbrace{\text{PSL}(2, \mathbb{C}) \times \cdots \times \text{PSL}(2, \mathbb{C})}_M \times \mathcal{H}(\hat{\mathbb{C}}) \]

with the product topology, where \( \mathcal{H}(\hat{\mathbb{C}}) \) is as in the Note on section 3.

We can ask about stability of \( F \) through deformations in the parameter space \( \mathcal{T} \). First we will consider deformations of \( \partial R \) and later on deformations of the coefficients on the component functions \( \gamma_m \) of \( F \).

In order to be clear, along this section we consider \( F \) to be defined in only two simply connected regions \( R_1 \) and \( R_2 \), being \( \partial R \) one simple closed curve. Let \( f = F|_{R_1} \) and \( g = F|_{R_2} \). To begin, let us fix \( f \) and \( g \), and perturb \( \partial R \) to obtain \( \partial R' \). Now, \( \partial R' \) bounds two regions homeomorphic to discs \( R'_1 \) and \( R'_2 \) and we define \( F' \) by \( F'|_{R'_1} = f \), \( F'|_{R'_2} = g \). Notice \( \Gamma_F = \Gamma_{F'} = (f,g) \), because \( f \) and \( g \) have been fixed.

Let \( \mathcal{C}(\hat{\mathbb{C}}) \) the subspace of \( \mathcal{H}(\hat{\mathbb{C}}) \) consisting of all compact subsets of the sphere homeomorphic to a circle. Observe that for each \( n \in \mathbb{N} \) and \( f,g \in \text{PSL}(2, \mathbb{C}) \), there are a natural maps \( \Psi_{f,g,n} : \mathcal{C}(\hat{\mathbb{C}}) \to \mathcal{H}(\hat{\mathbb{C}}) \) that assigns to \( \partial R \) the \( n \)th-level pre-discontinuity set \( \mathcal{PD}_n(F) \) of the piecewise map \( F \), and let us denote by \( \Psi_{f,g} \) the map from \( \partial R \) to the pre-discontinuity set \( \mathcal{PD}(F) \).

**Theorem 4.** For a fixed pair \( f,g \) in \( \text{PSL}(2, \mathbb{C}) \), the map \( \Psi_{f,g,n} \) is continuous, for each \( n \in \mathbb{N} \).

**Proof.** We prove continuity of \( \Psi_{f,g,n} \) using the sequence convergence criterion. Let be \( C_k \in \mathcal{C}(\hat{\mathbb{C}}) \) a sequence convergent to \( \partial R \). Let be \( D_k \) and \( D \) the closure of interior sets of \( C_k \) and \( \partial R \), respectively, and \( E_k \) and \( E \) the closure of exterior sets of \( C_k \) and \( \partial R \), respectively. Particularly, we have \( D = \overline{R_1} \) and \( E = \overline{R_2} \). Because of Lemma 7 (see Section 6), \( D_k \to D \) and \( E_k \to E \).

Using Lemma 6 we have \( f^{-1}(C_k) \to f^{-1}(\partial R) \) and \( g^{-1}(C_k) \to g^{-1}(\partial R) \), because \( f \) and \( g \) are Möbius transformations.

Because of Lemma 5

\[ f^{-1}(C_k) \cap D_k \to f^{-1}(\partial R) \cap D - Y_f \]

and

\[ g^{-1}(C_k) \cap E_k \to g^{-1}(\partial R) \cap E - Y_g \]

where \( Y_f \) and \( Y_g \) are the respective isolated points sets.

Using Lemma 1 we have

\[ (f^{-1}(C_k) \cap D_k) \cup (g^{-1}(C_k) \cap E_k) \cup C_k \to (f^{-1}(\partial R) \cap D - Y_f) \cup (g^{-1}(\partial R) \cap E - Y_g) \cup \partial R \]

Let be \( F_k \) piecewise transformations such that \( F_k|_{D_k} = f \) and \( F_k|_{D_k} = g \). Then

\[ (f^{-1}(C_k) \cap D_k) \cup (g^{-1}(C_k) \cap E_k) \cup C_k = F^{-1}(C_k) \cup \partial R = \mathcal{PD}_1(F_k) \]

In the other hand

\[ \begin{align*}
(f^{-1}(\partial R) \cap D - Y_f) & \cup (g^{-1}(\partial R) \cap E - Y_g) \cup \partial R = F^{-1}(\partial R) \cup \partial R \\
\mathcal{PD}_1(F) & = \mathcal{PD}_1(F),
\end{align*} \]

because \( Y_f, Y_g \subset \partial R \).
Finally, we have shown that $PD_1(F_k) \rightarrow PD_1(F)$.
Now suppose that $PD_{n-1}(F_k) \rightarrow PD_{n-1}(F)$. Then, with an analogous argument to previous one using Lemmas 6, 5 and 4, but with $PD_{n-1}(F_k)$ instead of $C_k$, we can demonstrate that

$$F^{-1}(PD_{n-1}(F_k)) \cup C_k \rightarrow F^{-1}(PD_{n-1}(F)) \cup \partial R$$

But, by Lemma 3, we have that $F^{-1}(PD_{n-1}(F_k)) \cup C_k = PD_n(F_k)$ and $F^{-1}(PD_{n-1}(F)) \cup \partial R = PD_n(F)$. □

Even more

**Theorem 5.** For a fixed pair $f, g$ in $PSL(2,\mathbb{C})$, if $\partial R \cap \Lambda(\Gamma_F) = \emptyset$, then the map $\Psi_{f,g}$ is continuous.

**Proof.** As in the previous proof, let be $C_k \in (\tilde{C})$ a sequence convergent to $\partial R$ and $F_k$ the piecewise conformal map defined by $f$ in $D_k$ the interior of $C_k$ and $g$ in $E_k$ the exterior of $C_k$. Also we assume that $C_k \cap \Lambda(\Gamma_F) = \emptyset$ for all $k$.

Let $z \in PD(F)$. If $z \in PD_n(F)$ for some $n$, then every neighborhood of $z$, denoted $\mathcal{N}_z$, intersects infinitely many $PD(F_k)$ because $PD_n(F_k) \rightarrow PD_n(F)$.

If $z \in \alpha(F)$, every neighborhood $\mathcal{N}_z$ intersects infinitely many $F^{-n}(\partial R)$. For each $w \in F^{-n}(\partial R)$ every neighborhood $\mathcal{N}_w \subset \mathcal{N}_z$ intersects infinitely many $PD_n(F_k)$. Then, $\mathcal{N}_z$ intersects infinitely many $PD(F_k)$.

Now let $z \in \tilde{C}$ such that every neighborhood $\mathcal{N}_z$ intersects infinitely many $PD(F_k)$. If $z \in \Omega(\Gamma_F)$, $\mathcal{N}_z$ intersects infinitely many $PD_n(F_k)$ for some fixed $n$, because $PD_n(F_k) \subset \Omega(\Gamma_F)$ for all $n$ and $\lim_{n \rightarrow \infty} F^{-n}(\partial R) \subset \Lambda(\Gamma_F)$ by Theorem 2. Then, $z \in PD_n(F) \subset PD(F)$.

If $z \in \Lambda(\Gamma_F)$, $\mathcal{N}_z$ cannot intersect infinitely many $PD_n(F_k)$ for some fixed $n$, because that implies $z \in PD_n(F) \subset \Omega(\Gamma_F)$. Then, $\mathcal{N}_z$ must intersect sets $F^{-n}(\partial R) \subset PD(F_k)$ with an increasing sequence $n_k$. As $PD_{n_k}(F_k) \rightarrow PD_{n_k}(F)$ for each $n_k$, $\mathcal{N}_z$ intersects infinitely many $F^{-n_k}(\partial R)$, and we conclude that $z \in \lim_{n \rightarrow \infty} F^{-n}(\partial R) = \alpha(F)$ using Theorem 2. □

In this way, if $\partial R \cap \Lambda(\Gamma_F) = \emptyset$, we ensure certain stability of $F$ (fixing $f$ and $g$) through continuous deformations of $\partial R$ (see figure 13). In the opposite case, if $\partial R \cap \Lambda(\Gamma_F) \neq \emptyset$, we found unrelated dynamics of $F$ and $F'$, because the unstability of $\Lambda(\Gamma_F)$ carried by $F^{-n}(\partial R)$ along $PD(F)$ (see figure 14).

Let be $X_{\partial R} = \{ F : \tilde{\mathbb{C}} \rightarrow \tilde{\mathbb{C}} : F|_{R_1}, F|_{R_2} \in PSL(2,\mathbb{C}) \}$ the space of piecewise conformal transformations defined by a fix $\partial R$, a simple closed curve. This space is homeomorphic to $PSL(2,\mathbb{C}) \times PSL(2,\mathbb{C})$.

A piecewise conformal transformation $F \in X_{\partial R}$ is **structurally stable** if exists $\mathcal{N}_F$ neighborhood of $F$ in $X_{\partial R}$ such that for all $F' \in \mathcal{N}_F$ exists $h : \tilde{\mathbb{C}} \rightarrow \tilde{\mathbb{C}}$ homeomorphism with $F' \circ h = h \circ F$.

The following result can ensure structural stability on the particular class of transformations which have a classic or non-classic Schottky group as associated Kleinian group.

**Theorem 6.** If $\Gamma_F$ is a Schottky group and $\partial R$ is contained in a fundamental region of $\Gamma_F$, then $F$ is structurally stable in $X_{\partial R}$.
Note. Recall that for two finitely generated Kleinian groups \( \Gamma_1 = \langle f_1, \ldots, f_n \rangle \) and \( \Gamma_2 = \langle g_1, \ldots, g_n \rangle \), it can be defined an equivalence \( \Gamma_1 \sim \Gamma_2 \) if exists a Möbius transformation \( h \) such that \( h \circ f_i \circ h^{-1} = g_i \). If \( S \subset PSL(2, \mathbb{C}) \times PSL(2, \mathbb{C}) \) is the set of Schottky groups of two generators, then \( S/\sim \) is the set of marked Schottky groups and is a domain in \( \mathbb{C}^3 \) (see [3]) and therefore \( S \) is also a domain.
5. Appendix: Examples

In all images, periodic points are colored in red and the pre-discontinuity set in black. Other coloring is used to distinguish Fatou components.

**Figure 1. Fatou components: Attractive basins.**

\( F|_R(z) = \lambda z \) and \( F|_{R^c}(z) = \lambda(1 - z) \), where \( R = \{ z : | -\frac{1}{2} - z| < 1 \} \) and \( \lambda = 0.95e^{\frac{2}{3}\pi i} \). Attractive basins \( U, V \) and \( W \), such that \( F(U) \subset U, F(V) \subset V \) and \( F^2(W) \subset W \).

**Figure 2. Fatou components: Rotation domains.**

\( F|_R(z) = \lambda z \) and \( F|_{R^c}(z) = \lambda(1 - z) \), where \( R = \{ z : | -\frac{1}{2} - z| < 1 \} \) and \( \lambda = e^{\frac{2}{3}\pi i} \). Simply connected rotation domains \( U, V, W \) and \( E \) containing a fixed point, such that \( F(U) = U, F(V) = V, F^2(W) = W \) and \( F(E) = E \).
Figure 3. Fatou components: Neutral domains.

\[ F|_R(z) = \lambda z \quad \text{and} \quad F|_{R^c}(z) = \lambda (1 - z), \]

where \( R = \{ z : | - \frac{1}{2} - z | < 1 \} \) and \( \lambda = e^{\frac{3}{2}\pi i} \). Neutral domain \( U \) such that \( F^6|_U = \text{Id} \).

Figure 4. Fatou components: Rotation domains.

\[ F|_R(z) = \lambda z \quad \text{and} \quad F|_{R^c}(z) = \lambda (1 - z), \]

where \( R = \{ z : | - \frac{1}{2} - z | < 1 \} \) and \( \lambda = e^{\alpha \pi i} \) with \( \alpha \) an irrational number. Rotation domains \( U, V, W \) and \( E \) containing a fixed point, such that \( F(U) = U, F(V) = V, F^2(W) = W \) and \( F(E) = E \).
Figure 5. Fatou parabolic basin.
\[ F|_{R}(z) = \frac{z}{z+1} \] and \[ F|_{Rc}(z) = \lambda(1 - z) \], where 
\[ R = \{ z : | - \frac{1}{2} - z | < \frac{1}{2} \} \] and \[ \lambda = 1.1e^{\frac{2}{3}\pi i} \]. Parabolic basin \( U \) with parabolic fixed point 0 at \( \partial U \), such that \( F(U) \subset U \). Parabolic basins with attractive point in \( \partial U \) behaves analogously.

Figure 6. Fatou components: Rotation domain.
\[ F|_{R}(z) = \lambda - \lambda z \] and \[ F|_{Rc}(z) = \lambda z \], where 
\[ R = \{ z : | 1 - z | < \frac{1}{2} \} \] and \[ \lambda = e^{\frac{2}{3}\pi i} \]. Rotation domain \( U \) containig both fixed points of \( F|_{U} \).
Figure 7. Fatou components: Rotation domains.
$F|_{R(z)} = \frac{4}{3} \lambda z$ and $F|_{Rc(z)} = \frac{3}{4} \lambda z$, where $R = \{ z : |z| < 1 \}$ and $\lambda = e^{\frac{i}{3} \pi}$. Rotation domain $U$ without fixed points, such that $F^2|_{U(z)} = e^{\frac{2}{3} \pi i} z$.

Figure 8. Fatou neutral domains.
$F|_{R(z)} = \frac{95}{100} \lambda z$ and $F|_{Rc(z)} = \frac{100}{95} \lambda z$, where $R = \{ z : |z - \frac{1}{3}| < 1 \}$ and $\lambda = e^{\frac{i}{3} \pi}$. Neutral domain $U$ such that $F^6|_{U(z)} = Id$. 

Figure 9. Itineraries.\( F|_R(z) = \lambda z \) and \( F|_{R_c}(z) = \lambda(1-z) \), where \( R = \{ z : |\frac{1}{2} - z| < \frac{1}{4} \} \) and \( \lambda = e^{\frac{1}{3}\pi i} \). Each Fatou component is determined by one itinerary.

\[
\begin{array}{c|c|c|c|c|c|c}
F^{15}(U) & & & & & \hline \\
F^9(U) & F^{13}(U) & & & & \hline \\
F^3(U) & F^6(U) & F^{11}(U) & & & \hline \\
U & F(U) & F^4(U) & F^9(U) & & \hline \\
0 & F^2(U) & F^7(U) & F^{14}(U) & & \hline \\
& F^5(U) & F^{12}(U) & & & \hline \\
& F^{10}(U) & & & & \hline \\
\end{array}
\]

Figure 10. Wandering domains. \( F|_R(z) = iz \) and \( F|_{R_c}(z) = -iz+1+i \), where \( R = \{ z : Im(z) < 0 \} \). First iterations in the orbit of component \( U = (0,1) \times (0,1) \).
Figure 11. Connectivity of Fatou components.

$F|_R(z) = 2z$ and $F|_{R^c}(z) = \lambda z$, where $R = \{z : |\frac{3}{2} - z| < \frac{1}{2}\}$ and $\lambda = e^{\frac{1}{2} \pi i}$ at left and $\lambda = e^{\frac{3}{2} \pi i}$ at right. The component outside circles is 4-connected on left and 5-connected on right.

---

Figure 12. Fatou components with $\infty$ connectivity.

At left, $F|_R(z) = z$ and $F|_{R^c}(z) = 2z$, where $R = \{z : |1 - z| < \frac{1}{2}\}$. At right, $F|_R(z) = 2z$ and $F|_{R^c}(z) = \lambda z$, where $R = \{z : |\frac{3}{2} - z| < \frac{1}{2}\}$ and $\lambda = 1.2e^{\frac{3}{2} \pi i}$. In both cases, the component outside circles is $\infty$-connected.
Figure 13. Stability through deformations of $\partial R$.

$F|_R(z) = \frac{(1+i)z+i}{i+(1-i)z}$ and $F|_{R^c}(z) = \frac{(1+i)z-i}{i+(1-i)z}$, with $R = \{z : |z| < 2\}$ at left and $R = \{z : |z| < 2 - \varepsilon\}$ for some $\varepsilon > 0$ at right. $\Gamma_F$ is a Fuchsian group with $\Lambda(\Gamma_F) = S^1$ (in white), then $\partial R \cap \Lambda(\Gamma_F) = \emptyset$ in both cases. Therefore the map $\partial R \mapsto PD(F)$ is continuous.

Figure 14. Unstability through deformations of $\partial R$.

The same transformations from figure 13, but $R = \{z : |z| < 2\}$ at left, $R = \{z : |z| < 2 - \varepsilon\}$ at center and $R = \{z : |z| < 1 + \varepsilon\}$ at right, for some $\varepsilon > 0$. Since $\Lambda(\Gamma_F) = S^1$ (in white), $\partial R \cap \Lambda(\Gamma_F) \neq \emptyset$. In those cases, the map $\partial R \mapsto PD(F)$ is not continuous. In other words, $F$ is unstable under deformations of $\partial R$. 
Figure 15. Structural stability.

\[ F_{R}|z = \frac{z - 0.6 i}{0.6 + i} \] at left and \( F_{\lambda}|z = \frac{z - \lambda}{\lambda + i} \) at right, where \( R = \{ z : |i - z| < \frac{1}{2} \} \). In this case \( \Gamma_{F} \) is a Schottky group and \( \partial R \) is contained in a fundamental region of \( \Gamma_{F} \). Then \( F \) and \( F_{\lambda} \) are quasiconformally conjugated.

6. Appendix: Technical results and constructions

Some lemmas and propositions about inverse sets and invariance of pre-discontinuity set and Fatou set of a piecewise conformal map \( F \).

**Lemma 2.** Let be \( A, B \subset \mathbb{C} \), then \( F^{-1}(A \cup B) = F^{-1}(A) \cup F^{-1}(B) \).

**Proof.**

\[
F^{-1}(A \cup B) = \bigcup_{m=1}^{M} \left( (F|_{R_m})^{-1}(A \cup B) \cap R_m \right) \\
= \bigcup_{m=1}^{M} \left( (F|_{R_m})^{-1}(A) \cup (F|_{R_m})^{-1}(B) \cap R_m \right) \\
= \bigcup_{m=1}^{M} \left( (F|_{R_m})^{-1}(A) \cap R_m \right) \cup \bigcup_{m=1}^{M} \left( (F|_{R_m})^{-1}(B) \cap R_m \right) \\
= F^{-1}(A) \cup F^{-1}(B)
\]

\[\square\]

**Lemma 3.** \( F^{-1}(\mathcal{PD}_n(F)) = F^{-n+1}(\partial R) \cup \cdots \cup F^{-1}(\partial R) = \mathcal{PD}_{n+1}(F) - \partial R \), for all \( n \in \mathbb{N} \).

**Proof.** Taking \( \partial R = \mathcal{PD}_0(F) \), then \( F^{-1}(\mathcal{PD}_0(F)) = F^{-1}(\partial R) = \mathcal{PD}_1(F) - \partial R \).

Now we suppose \( F^{-1}(\mathcal{PD}_{n-1}(F)) = \mathcal{PD}_n(F) - \partial R \), then

\[
F^{-1}(\mathcal{PD}_n(F)) = F^{-1}(F^{-n}(\partial R) \cup \mathcal{PD}_{n-1}(F)) \\
= F^{-1}(F^{-n}(\partial R)) \cup F^{-1}(\mathcal{PD}_{n-1}(F)) \\
= F^{-n+1}(\partial R) \cup (\mathcal{PD}_n(F) - \partial R) \\
= \mathcal{PD}_{n+1}(F) - \partial R
\]

\[\square\]
Proposition 2. The Fatou set is forward invariant and the pre-discontinuity set is backward invariant.

Proof. Let be a Fatou component. Since is a Möbius transformation, is open. Also, we have that \( U = \text{int}(C_n) \), the interior of the set of points with same itinerary \( s \in \Sigma_M \), and is semi-conjugated with the shift \( \sigma \), then \( F(U) \subset \text{int}(C_{\sigma(s)}) \). Therefore \( F(\widehat{C} - \mathcal{PD}(F)) \subset \widehat{C} - \mathcal{PD}(F) \).

Let \( z \in \mathcal{PD}(F) \) and \( w \in F^{-1}(\{z\}) \). Suppose that \( w \not\in \mathcal{PD}(F) \), by forward invariance of the Fatou set \( F(w) = z \not\in \mathcal{PD}(F) \), a contradiction. Therefore \( F^{-1}(\mathcal{PD}(F)) \subset \mathcal{PD}(F) \). \( \square \)

We denote by \( \mathcal{H}(\widehat{C}) \) the space of compact subsets of \( \widehat{C} \) with the Hausdorff topology and metric induced by a spherical metric. \( \mathcal{H}(\widehat{C}) \) is a complete metric space (see for example [14]). We recall the convergence characterization on \( \mathcal{H}(\widehat{C}) \).

Definition 4. A sequence of compacts \( K_n \) converge to \( K \) on \( \mathcal{H}(\widehat{C}) \) if

(1) Every neighborhood \( \mathcal{N}_z \) of a point \( z \in K \), intersects infinitely many \( K_n \).
(2) If every neighborhood \( \mathcal{N}_z \) of \( z \) intersects infinitely many \( K_n \), then \( z \in K \).

And we denote this as \( K_n \to K \).

In the following, we demonstrate some useful lemmas about convergence in \( \mathcal{H}(\widehat{C}) \).

Lemma 4. \( A_n \cup B_n \to A \cup B \).

Proof. Let \( z \in A \cup B \). Then every neighborhood \( \mathcal{N}_z \) intersects infinitely many \( A_n \) or infinitely many \( B_n \), that is, intersects infinitely many \( A_n \cup B_n \).

On the other hand, if every neighborhood \( \mathcal{N}_z \) intersects infinitely many \( A_n \cup B_n \), then \( z \in A \) or \( z \in B \).

Lemma 5. \( A_n \cap B_n \to A \cap B \) and \( Y \subset \partial(A \cap B) \), where \( Y = \{ z \in A \cap B : \exists \mathcal{N}_z \) that intersects finitely many \( A_n \cap B_n \} \) is the set of isolated points.

Proof. Let \( z \in A \cap B - Y \). Then every neighborhood \( \mathcal{N}_z \) intersects infinitely many \( A_n \) and infinitely many \( B_n \) but \( z \not\in Y \), that is, intersects infinitely many \( A_n \cap B_n \).

On the other hand, if every neighborhood \( \mathcal{N}_z \) intersects infinitely many \( A_n \cap B_n \), then \( z \in A \) and \( z \in B \), but \( z \not\in Y \).

If \( z \in Y \subset A \cap B \), then exists \( \mathcal{N}_z \) such that intersects finitely many \( A_n \cap B_n \). Then \( \mathcal{N}_z \) intersects finitely many \( (A_n \cap B_n)^c \). Since \( (A_n \cap B_n)^c = A_n^c \cup B_n^c \subset \widehat{A} \cup \widehat{B} \), \( \mathcal{N}_z \) intersects finitely many \( \widehat{A} \cup \widehat{B} \) and, because Lemma 1, \( z \in \widehat{A} \cup \widehat{B} = (A \cap B)^c \). Finally, \( z \in (A \cap B) \cap (A \cap B)^c = \partial(A \cap B) \).

Lemma 6. If \( f: \widehat{C} \to \widehat{C} \) is continuous bijective, then \( f(A_n) \to f(A) \).

Proof. Let \( w \in f(A) \). Then exists \( z \in A \) such that \( f(z) = w \). We can construct a sequence \( z_n \to z \) with \( z_n \in A_n \) because \( A_n \to A \). As \( f \) is continuous, we have \( f(z_n) \to f(z) = w \), then every neighborhood \( \mathcal{N}_w \) intersects infinitely many \( f(A_n) \).

On the other hand, if every neighborhood \( \mathcal{N}_w \) intersects infinitely many \( f(A_n) \), then we can take a sequence \( w_n = f(z_n) \in f(A_n) \) such that \( w_n \to w \). As \( f \) is continuous bijective, \( f^{-1}(w_n) = z_n \to f^{-1}(w) \). Let \( z = f^{-1}(w) \), then every neighborhood \( \mathcal{N}_z \) intersects infinitely many \( A_n \) and, by hypothesis, \( z \in A \). Finally, \( w = f(z) \in f(A) \). \( \square \)
Lemma 7. If $C_n \to C$ where each $C_n$ and $C$ are compact subsets of $\hat{\mathbb{C}}$ homeomorphic to circles, then $D_n \to D$ and $E_n \to E_n$, where $D_n$ and $D$ are the closure of the interior domains of $C_n$ and $C$, respectively, and $E_n$ and $E$ are the closure of the exterior domains of $C_n$ and $C$, respectively.

Proof. Let $z \in D$. If $z \in C$, then every neighborhood $N_z$ intersects finitely many $D_n$, because $C_n \subset D_n$. If $z \in \text{int}(D)$ and exists one neighborhood $N'_z \subset \text{int}(D)$ such that intersects finitely many $D_n$, then intersects infinitely many $E_n$, because $D'_n \subset E_n$. That is, $z$ is in the interior of $C_k$ for almost all $n$ but $z$ is in the interior of $C$. In consequence $C_n \to C$, leading us to a contradiction. Therefore, every neighborhood $N_z$ intersects infinitely many $D_n$.

Analogously, if $z \in E$ then every neighborhood $N_z$ intersects infinitely many $E_n$.

Let be $z$ such that every neighborhood $N_z$ intersects infinitely many $D_n$. If $N_z$ intersects infinitely many $C_n$, then $z \in C \subset D$. If $N_z$ intersects finitely many $C_n$, then $N_z$ must intersects infinitely many $\text{int}(D_n)$ and finitely many $E_n$. Then $z \notin E$, that is, $z \in E^c \subset D$.

Analogously, if $z$ is such that every neighborhood $N_z$ intersects infinitely many $E_n$, then $z \in E$. □

About the $\alpha$-limit, we show in next proposition that is contained in the set of the limit points of backward iterations of $\partial R$ in $\mathcal{H}(\hat{\mathbb{C}})$, hence its name.

Proposition 3. $\alpha(F) \subset \lim_{n \to \infty} F^{-n}(\partial R)$ in $\mathcal{H}(\hat{\mathbb{C}})$.

Proof. Let $z \in \alpha(F)$. By definition of closure, every neighborhood $N_z$ intersects $\bigcup_{n \geq 0} F^{-n}(\partial R) = \bigcup_{n \geq 0} F^{-n}(\partial R)$, because $\mathcal{PD}(F) = \bigcup_{n \geq 0} F^{-n}(\partial R)$. Suppose that exists a neighborhood $N'_z$ such that intersects finitely many $F^{-n}(\partial R)$. Then exists $N$ such that for all $n > N$ every neighborhood $N''_z \subset N'_z$ does not intersects $F^{-n}(\partial R)$ but $\mathcal{PD}_N(F) \cap N''_z \neq \emptyset$. Then, because $\mathcal{PD}_N(F)$ is a closed set, $z \in \mathcal{PD}_N(F)$, contradicting the hypothesis. Therefore, every neighborhood $N_z$ must intersect infinitely many $F^{-n}(\partial R)$.

Remark. About the structure of the pre-discontinuity sets of a piecewise conformal maps $F$, we can observe the following combinatorics. Let $f_m = F|_{R_m}$ and $\Sigma_M(k) = \{1, \ldots, M\}^k = \{(m_1, \ldots, m_k) : m_i \in \{1, \ldots, M\} \}$, the set of words of $M$ symbols of length $k$.

Defining $C_{1,m} = f_m^{-1}(\partial R) \cap R_m$ for $m = 1, \ldots, M$, we have

$$F^{-1}(\partial R) = \bigcup_{m=1}^M f_m^{-1}(\partial R) \cap R_m = C_{1,1} \cup \cdots \cup C_{1,M} = \bigcup_{t \in \Sigma_M(1)} C_{1,t}$$

Iteratively, we construct

$$F^{-n}(\partial R) = F^{-1}(F^{-n-1}(\partial R)) = \bigcup_{m=1}^M \left( f_m^{-1} \left( \bigcup_{s \in \Sigma_M(n-1)} C_{n-1,s} \right) \right) \cap R_m = \bigcup_{s \in \Sigma_M(n-1)} \left( C_{n,s1} \cup \cdots \cup C_{n,sM} \right) = \bigcup_{t \in \Sigma_M(n)} C_{n,t}$$

where $C_{n,sm} = f_m^{-1}(C_{n-1,s}) \cap R_m$, $s = (m_1, \ldots, m_{n-1})$ and $sm = (m_1, \ldots, m_{n-1}, m)$.

Finally, we list some properties about the sets $C_{n,t}$.
(1) For every \( n \), each \( C_{n,t} \) is a finite union of curve segments and points, or an empty set.

(2) \( F(C_{n,sm}) \subseteq C_{n-1,s} \), because \( C_{n,sm} = f_m^{-1}(C_{n-1,s}) \cap R_m \) for some \( m \).

(3) If \( C_{n,t} \neq \emptyset \) for some \( t \in \Sigma_M(n) \), then \( F^n(C_{n,t}) \subseteq \partial R \).

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