Effective Wess-Zumino-Witten Action for Edge States of Quantum Hall Systems on Bergman Ball

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Abstract

Using a group theory approach, we investigate the basic features of the Landau problem on the Bergman ball $B^k$. This can be done by considering a system of particles living on $B^k$ in the presence of an uniform magnetic field $B$ and realizing the ball as the coset space $SU(k,1)/U(k)$. In quantizing the theory on $B^k$, we define the wavefunctions as the Wigner $D$-functions satisfying a set of suitable constraints. The corresponding Hamiltonian is mapped in terms of the right translation generators. In the lowest Landau level, we obtain the wavefunctions as the $SU(k,1)$ coherent states. This are used to define the star product, density matrix and excitation potential in higher dimensions. With these ingredients, we construct a generalized effective Wess-Zumino-Witten action for the edge states and discuss their nature.

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1 Introduction and motivations

Very recently, the quantum Hall effect (QHE) [1] in higher dimensional spaces has attracted much attention [2, 5]. The interest on this topic is mainly motivated by its well-known significance to the condensed matter physics [2]. Also its relationships to other theories like non-commutative geometry and string theory. In fact, a relation between QHE on the complex projective spaces $\mathbb{CP}^k$ [3] and the fuzzy spaces [4] is established. Moreover, the string theory is realized in terms of the QHE ingredients in higher dimensions [6].

The first results on the subject have been reported by Zhang and Hu [2] in analyzing QHE on four-sphere $S^4$ submitted to the $SU(2)$ background field. This analysis is motivated by the fact that the edge excitations might provide an approach to a quantum formulation of graviton in four-dimension. QHE in higher dimensions inherent many properties of its two-dimension counterpart. In the conventional QHE a droplet of fermions, occupying a certain volume, behaves as an incompressible fluid. The low-energy excitations being area-preserving deformations and behave as massless chiral bosons [13, 14] described by an effective Wess-Zumino-Witten (WZW) action in $(1+1)$-dimensions.

The derivation of the effective action for the edge states for any quantum Hall droplet can be performed according to the method used by Sakita [12] in two-dimension Euclidean space and generalized to $\mathbb{CP}^k$ by Karabali and Nair [3]. Two ingredients are necessary to perform this derivation: (i) a density of states which must be constant over the phase volume occupied by the droplet and (ii) the commutators between operators tend, in a suitable limit (strong magnetic field), to Poisson brackets associated to the considered manifold. More precisely, Karabali and Nair [3], in considering QHE on $\mathbb{CP}^k$, gave the energy and the Landau wavefunctions using a purely group theory approach. Furthermore, they have shown, in an elegant way, that for a strong magnetic field, the quantum system is described by a constant density of states over the phase space. They have derived the effective action of excitations on the $\mathbb{CP}^k$ boundaries. The obtained action is a generalized WZW action that describes the bosonized theory of fermions in $(1+1)$-dimensions [12]. Our motivation is based on [3] and the analytic method used by one [9] of the present authors to deal with QHE on the Bergman ball $B^k$.

We algebraically investigate a system of particles living on the manifold $B^k$ in the presence of an uniform magnetic field $B$. After realizing $B^k$ as the coset space $SU(k,1)/U(1)$, we construct the wavefunctions as the Wigner $D$-functions verifying a set of suitable constraints. The corresponding Hamiltonian $H$ can be written in terms of the $SU(k,1)$ generators. This will be used to define $H$ as a second order differential operator in the complex coordinates, which coincides with the Maass Laplacian [7] on the Bergman ball. To get the energy levels, we use the correspondence between the Landau problem on two manifolds $\mathbb{CP}^k$ and $B^k$. We introduce an excitation potential to remove the degeneracy of the ground state. For this, we consider a potential expressed in terms of the $SU(k,1)$ left actions. For a strong magnetic field, we show that the excitations of the lowest landau level (LLL) are governed by a generalized effective Wess-Zumino-Witten (WZW) action. It turn out that this action for $k = 1$, i.e. disc, coincides with one-chiral bosonic action for QHE at the filling factor $\nu = 1$ [13, 14]. We finally discuss the nature of the edge excitations and in particular show that the field describing the edge excitations on the disc $B^1$ is a superposition of oscillating on the boundary $S^1$ of the quantum Hall droplet.
The present paper is organized as follows. In section 2, we present a group theory approach to analysis the Landau problem on the Bergman ball. We build the wavefunctions and give the corresponding Hamiltonian as well as its energy levels. In section 3, we restrict our attention to LLL to write down the corresponding star product and defining the relevant density matrix. Also we consider the excitation potential and get the associate symbol to examine the excited states. We determine the generalized effective WZW action for the edge states for a strong magnetic field as well as discuss their nature and give the disc as example, in section 4. We conclude and give some perspectives in the last section.

2 Quantization and Hilbert space

We start by defining the Bergman ball $B^k$, which is realized as the coset spaces $SU(k,1)/U(k)$. In analyzing the $B^k$ geometry, we derive the $U(1)$ gauge potential generating an uniform magnetic field. In quantizing the theory, we get wavefunctions as the Wigner $D$-functions subjected to a suitable set of constraints. Imposing the polarization condition, we obtain the LLL wavefunctions. The latter turn out to be the $SU(k,1)$ coherent states constructed from the highest weight state of the completely symmetric representation.

2.1 Introducing the manifold $B^k$

The $k$-dimensional complex ball of unit radius is defined via $(k+1)$ homogeneous complex coordinates $u_\alpha$ ($\alpha = 1, 2, \cdots, k+1$) satisfying

$$\eta_{\alpha\beta} u^\alpha \bar{u}^\beta = -1$$

where the $(k+1) \times (k+1)$ diagonal matrix $\eta$ is $(1,1,\cdots,1,-1)$. We introduce the local independent coordinates $(z_1, z_2, \cdots, z_k)$, in the region where $u_{k+1} \neq 0$, as $z_i = u_i/u_{k+1}$. It follows that the global coordinates can be written in terms of $z_i$ as

$$u_\alpha = \frac{1}{\sqrt{1 - \bar{z} \cdot z}} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_k \\ 1 \end{pmatrix}$$

where $z_i$ are verifying

$$\bar{z} \cdot z = \sum_{i=1}^{k} |z_i|^2 < 1.$$  

In $z$ coordinates, the higher dimensional complex Bergman ball $B^k$ is defined by

$$B^k = \{ z = (z_1, z_2, \cdots, z_k) \in \mathbb{C}^k, \bar{z} \cdot z < 1 \}.$$  

The Kähler potential on this manifold reads as

$$K(\bar{z}, z) = -\ln(1 - \bar{z} \cdot z).$$
This generates the metric tensor
\[ ds^2 = g_{i\bar{j}} dz^i \bar{dz}^j \]  
(6)
where the elements \( g_{i\bar{j}} \) are related to \( K(\bar{z}, z) \) by
\[ g_{i\bar{j}} = \frac{\partial^2 K(\bar{z}, z)}{\partial z^i \partial \bar{z}^j}. \]  
(7)
Using (5), we explicitly write \( g_{i\bar{j}} \) as
\[ g_{i\bar{j}} = \delta_{ij} \frac{1}{1 - \bar{z} \cdot z} - \bar{z} \cdot z + \bar{z} \cdot z^i \cdot \bar{z}^j \]  
(8)
This implies
\[ ds^2 = \frac{dz d\bar{z}}{(1 - \bar{z} \cdot z)^2} + \frac{(z \cdot d\bar{z})(\bar{z} \cdot dz)}{(1 - \bar{z} \cdot z)^4}. \]  
(9)
The ball \( B^k \) is equipped with a closed two-form \( \omega \), such as
\[ \omega = ig_{i\bar{j}} dz^i \wedge d\bar{z}^j. \]  
(10)
Since \( \omega \) is non-degenerate, it is a symplectic two-form and therefore allows us to construct a Poisson bracket. This is
\[ \{f_1, f_2\} = ig^{i\bar{j}} \left( \frac{\partial f_1}{\partial \bar{z}^j} \frac{\partial f_2}{\partial z^i} - \frac{\partial f_1}{\partial z^i} \frac{\partial f_2}{\partial \bar{z}^j} \right) \]  
(11)
where \( g^{i\bar{j}} \) is the inverse of tensor metric
\[ g^{i\bar{j}} = (1 - \bar{z} \cdot z) (\delta_{ij} - \bar{z}^i \cdot z^j) \]  
(12)
In terms of the local coordinates, (11) can be written as
\[ \{f_1, f_2\} = -i(1 - \bar{z} \cdot z) \left( \frac{\partial f_1}{\partial z^i} \frac{\partial f_2}{\partial \bar{z}^j} - \frac{\partial f_1}{\partial \bar{z}^j} \frac{\partial f_2}{\partial z^i} - z \frac{\partial f_1}{\partial z} \frac{\partial f_2}{\partial \bar{z}} + \bar{z} \frac{\partial f_1}{\partial \bar{z}} \frac{\partial f_2}{\partial z} \right). \]  
(13)
To make contact with the group theory, we note that the manifold \( B^k \) can be viewed as the coset space \( SU(k,1)/U(k) \). This provides us with the algebraic tools to deal with QHE on the Bergman ball \( B^k \).

Let be \( g \) the \( (k+1) \times (k+1) \) matrices of a fundamental representation of the group \( SU(k,1) \), with \( g \in SU(k,1) \). They obey the relations
\[ \eta g^\dagger \eta = g^{-1}, \quad \det g = 1. \]  
(14)
Considering the \( t_\alpha \) generators of \( SU(k,1) \) as matrices in the fundamental representation, such as
\[ 2\text{Tr}(t_\alpha t_\beta) = \delta_{\alpha\beta} \]  
(15)
and the \( t_i \) generators of \( SU(k) \subset U(k) \) as matrices having zero for the \( (k+1)^{th} \) row and column, with \( i = 1, 2, \cdots, k^2 - 1 \). The generator corresponding to the \( U(1) \) direction of the subgroup \( U(k) \) will be denoted by \( t_{k^2+2k} \). It can be written as
\[ t_{k^2+2k} = \frac{1}{\sqrt{2k(k+1)}} \begin{pmatrix} I_k & 0 \\ 0 & -k \end{pmatrix} \]  
(16)
where $I_k$ is the $(k \times k)$ unit matrix. Note that, an element $g$ of $SU(k,1)$ can be used to parametrize $B_k^k$ by identifying $g$ to $gh$, with $h \in U(k)$.

In terms of $g$, the one-form, i.e. U(1) connection, is given by

$$\theta = i \sqrt{\frac{2k}{k+1}} \text{Tr} \left( t_{k^2+2k} g^{-1} dg \right).$$

(17)

Using the result

$$\text{Tr} \left( g^{-1} dg \right) = 0$$

(18)

and setting $u_\alpha = g_{\alpha,k+1}$, we show that

$$\theta = i \eta_{\alpha \beta} \bar{u}_\alpha du_\beta.$$  

(19)

It can be written in the local coordinates as

$$\theta = \frac{i}{2} \bar{z} \cdot dz - z \cdot d \bar{z}.$$ 

(20)

This formula agrees with (10) because it is easy to see that $\omega = d\theta$. Note that, (20) will play an important role in quantizing the theory describing a system living on the manifold $B_k^k$. Indeed, as we are interested to analysis QHE on the higher dimensional ball, it is necessary to identify the magnetic field behind this phenomena. This field is proportional to the Kähler two-form (10) and since $\omega$ is closed, the components of the magnetic field expressed in terms of the frame fields defined by the metric are constants.

### 2.2 Quantization

A basis of functions on $SU(k,1)$ is given by the Wigner $D$-functions

$$D^K_{L,R}(g) = \langle K, L \mid g \mid K, R \rangle.$$ 

(21)

They are the matrix elements corresponding to $g$ in the discrete representation of $SU(k,1)$. The quantum numbers $L$ and $R$ are specifying the states on which the $SU(k,1)$ generators act. The left and right $SU(k,1)$ actions are defined by

$$L_\alpha g = t_\alpha g, \quad R_\alpha g = g t_\alpha.$$ 

(22)

These relations will play a crucial role in defining two different Hamiltonians, see next.

The Hilbert space corresponding to the quantum system living on the manifold $B_k^k$ can be determined by reducing the degrees of freedom of the system on $SU(k,1)$ to the coset space $SU(k,1)/U(k)$. This reduction can be realized by imposing a set of suitable constraints. To find the constraints arising in the quantization of the theory, let us consider the $U(1)$ gauge field, i.e. the vector potential associated to the field strength $F$, as

$$A = -i n \eta_{\alpha \beta} \bar{u}_\alpha du_\beta.$$ 

(23)

where $n$ is a positive real quantum number. Hereafter, we assume that $n$ is an integer since we are considering the discrete series of the unitary representation corresponding to the group $SU(k,1)$. The gauge can be also written as

$$A = -i n \sqrt{\frac{2k}{k+1}} \text{Tr} \left( t_{k^2+2k} g^{-1} dg \right).$$

(24)
Note that, under the gauge transformation \( g \rightarrow gh \) where \( h \in SU(k) \), we have \( A \rightarrow A \). However, under the \( U(1) \) transformation

\[
g \rightarrow g \exp \left( itk^2 + 2k \varphi \right)
\]

we have

\[
A \rightarrow A + d \left( \sqrt{\frac{k}{2(k+1)}} \varphi \right).
\]

It follows that under the gauge transformation \( g \rightarrow gh \) where \( h \in SU(k) \), the wavefunctions (21) transform as

\[
\mathcal{D}_{L,R}^K(gh) = \exp \left( \int dt \dot{A} \right) \mathcal{D}_{L,R}^K(g)
\]

where \( \dot{A} \) is given by

\[
\dot{A} = -in\sqrt{\frac{2k}{k+1}} \text{Tr} \left( tk^2 + 2k h^{-1} \right).
\]

Thus, one can see that the canonical momentum corresponding to the \((k^2 + 2k)\)-direction is quantized as \( nk/\sqrt{2k(k+1)} \). Consequently, the admissible quantum states generating the Hilbert space must satisfy the constraint

\[
R_{k^2 + 2k} \mathcal{D}_{L,R}^K(g) = \frac{nk}{\sqrt{2k(k+1)}} \mathcal{D}_{L,R}^K(g).
\]

Furthermore, due to the invariance of \( U(1) \) gauge field under \( SU(k) \) transformations, the corresponding canonical momentum are vanishing. This invariance leads to the set of constraints

\[
R_j \mathcal{D}_{L,R}^K(g) = 0, \quad j = 0, 1, \cdots, k^2 - 1.
\]

We denote by \( t_{-i} \) and \( t_{+i} \) the \( SU(k,1) \) generators, which do not belong to the group \( U(k) \), with \( i = 1, 2, \cdots, k \). They can be seen as the lowering and raising operators analogously to the annihilation and creation operators of the harmonic oscillator. They will be next related to the covariant derivatives on \( B^k \). From (29-30) and using the commutation relations of the algebra of \( SU(k,1) \), we obtain

\[
[R_{-i}, R_{+j}] = n \delta_{ij}.
\]

The constraints (29-30) mean that the wavefunctions are singlets under the right \( SU(k) \) action and carry a right \( U(1) \) charge induced by the background field. Because of this, in the decomposition of \( SU(k+1) \) irreducible representations into \( SU(k) \) ones, we should consider the representations satisfying (29-30).

It is important to note that, (29-31) are similar to those obtained by Karabali and Nair [3] by considering the quantization of the complex projective spaces \( \mathbb{C}P^k = SU(k+1)/U(k) \). Consequently, we apply the representation theory analysis developed in [3] to the ball \( B^k \) case. This can be done by using the standard procedure associating a compact group with his non-compact analogue. We will return to this matter in deriving the spectrum of the system.

### 2.3 Lowest Landau levels

To find the lowest Landau levels, we should require the condition

\[
R_{-i} \mathcal{D}_{L,R}^K(g) = 0
\]
which is known as the polarization condition in the context of the geometric quantization. Therefore, the wavefunctions of the LLL belong to the symmetric representations of $SU(k,1)$.

The complete symmetric $SU(k,1)$ representations can be realized in terms of $(k+1)$-bosons satisfying the commutation relations

$$[a_r, a_s^\dagger] = \delta_{rs}, \quad r, s = 0, 1, \ldots, k.$$ \hfill (33)

The restricted Fock space, i.e. representation space, is defined by

$$\mathcal{F} = \{|K, n_1, n_2, \ldots, n_k\} \equiv |n_0, n_1, n_2, \ldots, n_k\rangle, \quad n_0 - (n_1 + n_2 + \cdots + n_k) = 2K - 1$$ \hfill (34)

where the states are given by

$$|n_0, n_1, n_2 \cdots, n_k\rangle = \frac{(a_0^\dagger)^{n_0}(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2} \cdots (a_k^\dagger)^{n_k}}{\sqrt{n_0!n_1!n_2! \cdots n_k!}} |K, 0, 0, 0, \cdots, 0\rangle$$ \hfill (35)

and the vacuum is annihilated by the operator $a_r$

$$a_r|K, 0, 0, 0, \cdots, 0\rangle = 0.$$ \hfill (36)

We can realize the lowering $t_{-i}$ and raising $t_{+i}$ operators as

$$t_{-i} = a_0 a_i, \quad t_{+i} = a_0^\dagger a_i^\dagger, \quad 1 \leq i \leq k.$$ \hfill (37)

They satisfy the relation

$$[t_{-i}, t_{+j}] = a_0 a_0^\dagger \delta_{ij} + a_j^\dagger a_i.$$ \hfill (38)

The Lowest Landau condition (32) implies that the right quantum numbers are $n_i = 0$ for $1 \leq i \leq k$ and $n_0 = K$. Acting (38) on the states $|n_0, 0, 0, \cdots, 0\rangle$, we get $2K$. Now using the commutation relation (31), we obtain the relation $K = \frac{n}{2}$. Hereafter, we set

$$B = 2n.$$ \hfill (39)

where $B$ stands for the strength of the magnetic field.

The coset generators act on the Fock space, associated to the symmetric representation, as

$$t_{-i}|K, n_1, n_2, \cdots, n_i, \cdots n_k\rangle = \sqrt{n_i (2K - 1 + (n_1 + n_2 + \cdots + n_k))} |K, n_1, n_2, \cdots, n_i - 1, \cdots n_k\rangle,$$

$$t_{+i}|K, n_1, n_2, \cdots, n_i, \cdots n_k\rangle = \sqrt{(n_i + 1) (2K + (n_1 + n_2 + \cdots + n_k))} |K, n_1, n_2, \cdots, n_i + 1, \cdots n_k\rangle$$ \hfill (40)

From the previous analysis, the wavefunctions corresponding to LLL are given by

$$\psi_{LLL} = (K, n_1, n_2, \cdots, n_k|g|K, 0, 0, \cdots, 0).$$ \hfill (41)

As far as the manifold $\mathcal{B}^k$ is concerned, we identify $g$ in (41) with the unitary exponential mapping $\Omega$ of the space generated by lowering and raising operators

$$\sum_{i=1}^{k} (\eta_i t_{+i} - \bar{\eta}_i t_{-i}) \rightarrow \Omega = \exp \left\{ \sum_{i=1}^{k} (\eta_i t_{+i} - \bar{\eta}_i t_{-i}) \right\}$$ \hfill (42)
where $\eta_i, i = 1, 2, \cdots, k$ are complex parameters and $\Omega$ is an unitary coset representative of the ball $B^k \equiv SU(k,1)/U(k)$, i.e.

$$\Omega^\dagger \Omega = 1.$$ (43)

Using (40), we show that the action of $\Omega$ on the lowest weight vector $\| K, 0, 0, \cdots, 0 \rangle$ leads

$$\psi_{LLL} = \sqrt{\frac{(n - 1 + n_1 + \cdots + n_k)!}{(n - 1)! n_1! n_2! \cdots n_k!}} \frac{z_1^{n_1} z_2^{n_2} \cdots z_k^{n_k}}{(1 - z \cdot \bar{z})^{k+1}}$$ (44)

where the complex variables $z_i$ are related to the parameters $\eta_i$ in (42) by

$$z_i = \eta_i \tanh (\sqrt{\eta^\dagger \eta}) / \sqrt{\eta^\dagger \eta}$$ (45)

and the $(1 \times k)$-matrix $\bar{\eta}$ is

$$\bar{\eta} = [\bar{\eta}_1 \cdot \bar{\eta}_2 \cdots \bar{\eta}_k].$$ (46)

It follows that LLL are infinitely degenerated. The wavefunctions $\psi_{LLL} \equiv \psi_{n_1,\cdots,n_k}(z_1,\cdots,z_k)$ obey the orthogonality condition

$$\int d\mu(z_1,\cdots,z_k) \bar{\psi}_{n'_1,\cdots,n'_k}(z_1,\cdots,z_k) \psi_{n_1,\cdots,n_k}(z_1,\cdots,z_k) = \delta_{n'_1,n_1} \cdots \delta_{n'_k,n_k}$$ (47)

where the measure is given by

$$d\mu(z_1,\cdots,z_k) = (n - k)(n - k + 1) \cdots (n - 1) \frac{d^2 z_1 \cdots d^2 z_k}{\pi^k (1 - z \cdot \bar{z})^{k+1}}$$ (48)

Note that, the representation space $\mathcal{F}$ has an infinite dimension. At this point, one may ask for the energy levels corresponding to the wavefunctions constrained by (29-31). The answer can be given by defining the relevant Hamiltonian that describes the quantum system living on the ball $B^k$.

3 Hamiltonian and spectrum

After quantizing the theory, we have obtained the wavefunctions of the quantum system on $B^k$. The corresponding Hamiltonian can be defined by making use of the structure relation (31) arising from the constraints (29-30). To illustrate the present analysis, we consider the disc $B^1$, i.e. $k = 1$, as a particular case.

3.1 Hamiltonian

We begin by writing, in the parametrization $(z_i, \bar{z}_i)$, the Hamiltonian describing a non-relativistic particle living on the manifold $B^k$ in the presence of the $U(1)$ magnetic field given by (19). To obtain the discrete spectrum, we use a group theoretical approach based on the results presented before. It is important to note that, the $2k$ right generators $R_{\pm i}$ of $SU(k,1)$ can be related to the covariant derivatives on $B^k$. The right actions behave like creation and annihilation operators for the standard harmonic oscillators. Thus, it is natural to define the corresponding Hamiltonian as

$$H = \frac{1}{2} \sum_{i=1}^k (R_{+i}R_{-i} + R_{-i}R_{+i}).$$ (49)
It is useful to identify the elements \( g \in SU(k, 1) \) to \( gh \ (h \in U(k)) \) in order to view their corresponding two points as one point in \( \mathbb{B}^k \). Thus, the Maurer-Cartan one-form \( g^{-1}dg \) can be only expressed in terms of the coset coordinates \((z_i, \bar{z}_i)\). It is given by

\[
g^{-1}dg = -it_{+i}e^i_jdz^j - it_{-i}e^\bar{i}_\bar{j}d\bar{z}^\bar{j} - it_{a}e^{a}_i dz^j - it_{a}e^{a}_\bar{j}d\bar{z}^\bar{j} - 2i\sqrt{\frac{k+1}{2k}}t_{k^2+2k} \theta. \tag{50}
\]

The \( U(1) \) connection \( \theta \) (17) can be rewritten as

\[
\theta = \theta_iz^i + \theta_\bar{j}d\bar{z}^\bar{j}
\]

where the components \( \theta_i \) and \( \theta_\bar{j} \) are

\[
\theta_i = \frac{iz_i}{2(1 - \bar{z} \cdot z)}, \quad \theta_\bar{j} = -\frac{iz_\bar{j}}{2(1 - \bar{z} \cdot z)}. \tag{52}
\]

In one-form (50), we have introduced the complex vielbeins

\[
e^i = e^i_jdz^j, \quad e^\bar{i} = e^\bar{i}_\bar{j}d\bar{z}^\bar{j}. \tag{53}
\]

They are defined such that the line element (9) takes the form

\[
ds^2 = e^i\delta_{ij}e^j \tag{54}
\]

and related to the metric by

\[
g_{ij} = \epsilon_i^k\delta_{kk}\epsilon_j^k. \tag{55}
\]

The explicit expressions of the vielbeins components are

\[
e^i_j = \frac{i}{\sqrt{1 - \bar{z} \cdot z}} \left( \delta_{kj} - \frac{z_k \cdot \bar{z}_j}{\bar{z} \cdot z} \right) + \frac{i}{\sqrt{1 - \bar{z} \cdot z}} \frac{z_k \cdot \bar{z}_j}{\bar{z} \cdot z},
\]

\[
e^\bar{i}_\bar{j} = -\frac{i}{\sqrt{1 - \bar{z} \cdot z}} \left( \delta_{kj} - \frac{z_j \cdot \bar{z}_k}{\bar{z} \cdot z} \right) - \frac{i}{\sqrt{1 - \bar{z} \cdot z}} \frac{z_j \cdot \bar{z}_k}{\bar{z} \cdot z}. \tag{56}
\]

In obtaining this result, we have used a similar approach to that elaborated in [15] for \( \mathbb{C}P^k \). One can verify that the metric inverse is

\[
g_{\bar{j}j} = (e^{-1})^j_k\delta_{kk}(e^{-1})^\bar{j}_\bar{k} \tag{57}
\]

which will be used in deriving the analytical realization of the Hamiltonian. From (50), we obtain

\[
\left( e^{-1} \right)^j_i \frac{\partial g}{\partial z^j} = g \left[ -it_{+i} - it_{a}e^{a}_j(e^{-1})^j_i - 2i\sqrt{\frac{k+1}{2k}}t_{k^2+2k} \theta_j(e^{-1})^j_i \right],
\]

\[
\left( e^{-1} \right)^\bar{j}_{\bar{i}} \frac{\partial g}{\partial \bar{z}^{\bar{j}}} = g \left[ -it_{-\bar{i}} - it_{a}e^{a}_{\bar{j}}(e^{-1})^{\bar{j}}_{\bar{i}} - 2i\sqrt{\frac{k+1}{2k}}t_{k^2+2k} \theta_{\bar{j}}(e^{-1})^{\bar{j}}_{\bar{i}} \right]. \tag{58}
\]

One can see that the \( SU(k, 1) \) right actions, defined by (22), can be written as

\[
R_{+i} = i \left( e^{-1} \right)^j_i D_j, \quad R_{-\bar{i}} = i \left( e^{-1} \right)^{\bar{j}}_{\bar{i}} D_{\bar{j}} \tag{59}
\]

where the \( U(1) \) covariant derivatives \( D_j \) and \( D_{\bar{j}} \) are given by

\[
D_j = \frac{\partial}{\partial z^j} - ia_j, \quad D_{\bar{j}} = \frac{\partial}{\partial \bar{z}^\bar{j}} - iA_{\bar{j}}. \tag{60}
\]
The quantities
\[ A_j = -i \frac{n}{2} \bar{z}_j (1 - \bar{z} \cdot z)^{-1}, \quad A_{\bar{j}} = i \frac{n}{2} z_{\bar{j}} (1 - z \cdot \bar{z})^{-1} \] (61)
are the components of the potential vector, in \( z \) coordinates,
\[ A = A_j dz_j + A_{\bar{j}} d\bar{z}_j \] (62)
given by (23). To obtain the formula (60), we have used the constraints (29-30) those must be satisfied by the physical wavefunctions of the present system. Putting (59) in (49), we find
\[ H = -\frac{1}{2} \left( g^{\bar{i} j} D_i D_{\bar{j}} + g^{\bar{i} j} D_{\bar{i}} D_j \right) \] (63)
where we have a summation over the repeated indices. In the complex coordinates, we straightforwardly show that the Hamiltonian (63) takes the form
\[ H = -(1 - \bar{z} \cdot z ) \left\{ (\delta_{ij} - z_i \cdot \bar{z}_j) \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right\} + \frac{n^2}{4} \bar{z} \cdot z \] (64)
This coincides with the Maass Laplacian [7] in higher dimensional spaces. It was investigated at many occasions, using the spectral theory [8], in order to describe the quantum system in the presence of a magnetic background field \( n \). Also it was analytically considered in analyzing QHE on the Bergman ball \( B^k \) [9].

3.2 Spectrum

The derivation of the energy eigenvalues of the Hamiltonian (64) is based on the correspondence between the Landau problem on two manifolds \( \mathbb{C}P^k \) and \( B^k \). Indeed, as we claimed before, the constraints (29-30) together with the relation (31) are similar to those obtained in analyzing the Landau system on \( \mathbb{C}P^k \) [3]. Thus, to get the discrete spectrum of a non-relativistic charged particle on \( B^k \), we use the standard procedure to pass from \( B^k \) to \( \mathbb{C}P^k \). For this, we set
\[ R_{+i} = iJ_{+i}, \quad R_{-i} = iJ_{-i}, \quad R_a = J_a, \quad R_{k^2+2k} = J_{k^2+2k} \] (65)
where \( a \) runs over the set \( \{1, \cdots, k^2 - 1\} \). The \( J \)’s satisfy the structure relations characterizing the algebra of \( SU(k+1) \). Now, the relations (29-31) become
\[ J_a \psi = 0, \quad J_{k^2+2k} \psi = \frac{nk}{\sqrt{2k(k+1)}} \psi, \quad [J_{+j}, J_{-i}] \psi = n \delta_{ij} \psi \] (66)
which are nothing but the relations involved in the Landau systems on \( \mathbb{C}P^k \) where the \( U(1) \) generator corresponding to \( (k^2 + 2k) \)-direction takes the fixed value \( nk/\sqrt{2k(k+1)} \). Then, according to [3], the representations of \( SU(k+1) \), which contain \( SU(k) \) singlets, are labeled by two integers \((p,q)\) where the condition
\[ q - p = n \] (67)
must fulfilled. The second order Casimir of \( SU(k+1) \)
\[ C_2 = \sum_{a=1}^{k^2+2k} J_a^2 \] (68)
is given by
\[ C_2 = \frac{k}{2(k+1)} \left[ p(p+k+1) + q(q+k+1) + \frac{2}{k}pq \right]. \] (69)

Writing the Hamiltonian \( H \) (49) as
\[ H = -\frac{1}{2} \sum_{i=1}^{k} (J_{+i}J_{-i} + J_{+i}J_{-i}) \] (70)
one can easily show that
\[ H = f_{k^2+2k}^2 - C_2. \] (71)

Finally, we find the eigenvalues of \( H \) as
\[ E_q = \frac{n}{2}(2q + k) - q(q + k) \] (72)
which agrees with the analytical analysis [9]. The index \( q \) labels the Landau levels. LLL has the energy
\[ E_0 = \frac{nk}{2} \] (73)
and the corresponding wavefunctions are given by (44). This value coincides with that can be obtained by considering the Landau problem on the real space \( \mathbb{R}^{2k} \). We close this section by giving an example.

3.3 Example: disc \( B^1 \)

To illustrate the strategy adopted in obtaining the energy spectrum (72), we consider the simplest case \( k = 1 \) [10], i.e. the disc \( B^1 \). Firstly, we derive the spectrum using the \( SU(1,1) \) discrete representations and secondly, we compare the obtained result with that derived in passing from the non-compact group \( SU(1,1) \) to its compact partner \( SU(2) \).

In the Cartan-Weyl basis, the Lie algebra of of \( SU(1,1) \) is characterized by the commutation relations
\[ [t_3, t_\pm] = \pm t_\pm, \quad [t_-, t_+] = 2t_3. \] (74)
The positive discrete series representations of \( SU(1,1) \) are labeled by an integer \( l \). The representation space is generated by the states \( \{|l, q\}\}. They are eigenstates of \( t_3 \) with \( l + q \) as eigenvalues. The constraint (29) reduces to
\[ R_3 \psi = \frac{n}{2} \psi \] (75)
and implies that
\[ l + q = \frac{n}{2}. \] (76)
The eigenvalues of the Hamiltonian
\[ H = \frac{1}{2} (R_+ R_- + R_- R_+) \] (77)
in the above representation are
\[ E = \frac{n^2}{4} - l(l - 1) \] (78)
where \( l(l - 1) \) is the eigenvalue of the \( SU(1,1) \) second order Casimir. From the constraint (76), we get
\[ E_q = \frac{n}{2}(2q + 1) - q(q + 1) \] (79)
which agrees with (72) for \( k = 1 \).

We now consider the passage from \( SU(1, 1) \) to \( SU(2) \). For this, we set

\[
t_3 = j_3, \quad j_\pm = it_\pm.
\]  

(80)

The relations (74) become those of the algebra of \( SU(2) \), such as

\[
[j_3, j_\pm] = \pm j_\pm, \quad [j_+, j_-] = 2j_3.
\]  

(81)

This mapping allows us to write the Hamiltonian (77) as

\[
H = j_3^2 - C_2 [SU(2)] = \frac{n^2}{4} - j(j - 1)
\]  

(82)

where \( \frac{n}{2} \) and \( j(j - 1) \) are the eigenvalues of \( j_3 \) and \( C_2 \), respectively. For a given \( SU(2) \) unitary irreducible representation \( j \), we have

\[
j = q - \frac{n}{2}.
\]  

(83)

Replacing in (82), we get the energies (79). This shows the equivalence between the first and second ways in deriving the spectrum of the system living on the disc \( B^1 \).

4 Lowest Landau level analysis

Restricting to LLL, described by the wavefunctions \( \psi_{LLL} \) given in (44) of fundamental energy \( E_0 = \frac{nk}{2} \), we develop some tools needed in studying the edge excitations of the quantum Hall droplets on \( B^k \). These concern the star product, density of states and excitation potential.

4.1 The star product

For a strong magnetic field \( (B \sim n) \), the particles are confined in LLL (44). In this limit, we will discuss how the non-commutative geometry occurs and show that the commutators between two operators acting on the states \( \psi_{LLL} \) give a Poisson brackets of type (11). To do this, we associate the function

\[
A(\bar{z}, z) = \langle z | A | z \rangle = \langle 0 | \Omega A^\dagger \Omega | 0 \rangle
\]  

(84)

to any operator \( A \) acting on the LLL wavefunctions. \( \Omega \) is given by (42), \( | 0 \rangle = | K, 0, 0, \cdots, 0 \rangle \) is the lowest highest weight state of the complete symmetric representation of \( SU(k, 1) \) defined in (34) and the vector states

\[
| z \rangle = \Omega | 0 \rangle \equiv | z_1, z_2, \cdots, z_k \rangle
\]  

(85)

are the \( SU(k, 1) \) coherent states

\[
| z \rangle = (1 - \bar{z} \cdot z)^{\frac{\bar{n}}{2}} \sum_{n_1=0}^\infty \cdots \sum_{n_k=0}^\infty \sqrt{\frac{(n - 1 + n_1 + \cdots + n_k)!}{(n - 1)! n_1! n_2! \cdots n_k!}} z_1^{n_1} \cdots z_k^{n_k} | n_1, n_2, \cdots, n_k \rangle
\]  

(86)

Next, to simplify our notations, we set \( | n_1, n_2, \cdots, n_k \rangle \equiv | \{ n \} \rangle \).

An associative star product of two functions \( A(\bar{z}, z) \) and \( B(\bar{z}, z) \) is defined by

\[
A(\bar{z}, z) \star B(\bar{z}, z) = \langle z | AB | z \rangle.
\]  

(87)
Using the unitarity of $\Omega$ and the completeness relation

$$\sum_{\{n\} = 0}^{\infty} |\{n\}\rangle \langle \{n\}| = I$$  \hspace{1cm} (88)

we write (87) as

$$A(\bar{z}, z) \star B(\bar{z}, z) = \sum_{\{n\} = 0}^{\infty} \langle 0|\Omega^\dagger A\Omega|\{n\}\rangle \langle \{n\}|\Omega^\dagger B\Omega|0\rangle.$$  \hspace{1cm} (89)

From (40), for large $n$, the star product becomes

$$A(\bar{z}, z) \star B(\bar{z}, z) = A(\bar{z}, z) B(\bar{z}, z) + \frac{1}{n} \sum_{i=0}^{k} \langle 0|\Omega^\dagger A\Omega t_{+i}|0\rangle \langle 0|\Omega^\dagger B\Omega|0\rangle + O\left(\frac{1}{n^2}\right).$$  \hspace{1cm} (90)

It is clear that, the first term in r.h.s. of (90) is the ordinary product of two functions $A$ and $B$. However, the non-commutativity is encoded in the second term.

To completely determine the star product (90) for large $n$, we should evaluate the matrix elements of type

$$\langle 0|\Omega^\dagger A\Omega t_{+i}|0\rangle.$$  \hspace{1cm} (91)

Using the holomorphicity condition

$$R_{-i}\{n\}|\Omega|0\rangle = 0$$  \hspace{1cm} (92)

which can be easily verified by using the coherent states (86), we have

$$\langle 0|\Omega^\dagger A\Omega t_{+i}|0\rangle = R_{+i}\langle 0|\Omega^\dagger A\Omega|0\rangle.$$  \hspace{1cm} (93)

Similarly, we obtain

$$\langle 0|\Omega^\dagger B\Omega|0\rangle = -R_{-i}\langle 0|\Omega^\dagger B\Omega|0\rangle$$  \hspace{1cm} (94)

where we have considered the relation

$$R_{+i}^* = -R_{-i}.$$  \hspace{1cm} (95)

From (58-59), since we are concerned with a $U(1)$ abelian gauge field, we show that (90) becomes

$$A(\bar{z}, z) \star B(\bar{z}, z) = A(\bar{z}, z) B(\bar{z}, z) - \frac{1}{n} g^{im} \partial_j A(\bar{z}, z) \partial_m B(\bar{z}, z) + O\left(\frac{1}{n^2}\right).$$  \hspace{1cm} (96)

Therefore, the symbol or function associated to the commutator of two operators $A$ and $B$ can be written as

$$\langle z|[A, B]|z\rangle = -\frac{1}{n} g^{im} \partial_j A(\bar{z}, z) \partial_m B(\bar{z}, z) - \partial_j B(\bar{z}, z) \partial_m A(\bar{z}, z).$$  \hspace{1cm} (97)

This implies

$$\langle z|[A, B]|z\rangle = \frac{i}{n} \{A(\bar{z}, z), B(\bar{z}, z)\} \equiv \{A(\bar{z}, z), B(\bar{z}, z)\}_*$$  \hspace{1cm} (98)

where $\{,\}$ stands for the Poisson bracket (11) defined on the Kahlerian manifold $\mathbb{B}^k$ and the Moyal bracket $\{,\}_*$ is given by

$$\{A(\bar{z}, z), B(\bar{z}, z)\}_* = A(\bar{z}, z) \star B(\bar{z}, z) - B(\bar{z}, z) \star A(\bar{z}, z).$$  \hspace{1cm} (99)

The obtained star product shows that how the non-commutative geometry occurs in analyzing QHE on the Bergman ball $\mathbb{B}^k$.\hspace{1cm} (99)
4.2 Density matrix

Another important physical quantity, needed in deriving the WZW action for the edge excitations, is the density matrix. Since LLL are infinitely degenerated, one may fill the LLL states with

\[ M = M_1 + M_2 + \cdots + M_k \]  

particles where \( M_i \) stands for the particle number in the mode \( i \). The corresponding density operator is then

\[ \rho_0 = \sum \left| \{ m \} \right\rangle \left\langle \{ m \} \right|, \]  

(101)

Its associated symbol reads as

\[ \rho_0(\bar{z}, z) = (1 - \bar{z} \cdot z)^n \sum_{m_1=0}^{M_1} \cdots \sum_{m_k=0}^{M_k} \frac{(n - 1 + m_1 + \cdots + m_k)!}{(n - 1)! \cdot m_1! \cdot \cdots \cdot m_k!} |z_1|^{2m_1} \cdots |z_k|^{2m_k}. \]  

(102)

By using the identity

\[ (1 - \bar{z} \cdot z)^{-n} = \sum_{m_1=0}^{\infty} \cdots \sum_{m_k=0}^{\infty} \frac{(n - 1 + m_1 + \cdots + m_k)!}{(n - 1)! \cdot m_1! \cdots \cdot m_k!} |z_1|^{2m_1} \cdots |z_k|^{2m_k}. \]  

(103)

and for a strong magnetic field \( (B \sim n) \), \( \rho_0(\bar{z}, z) \) can be approximated by

\[ \rho_0(\bar{z}, z) \simeq \exp(-n \bar{z} \cdot z) \sum_{m=0}^{M} \frac{(n \bar{z} \cdot z)^m}{m!} \simeq \Theta(M - n \bar{z} \cdot z) \]  

(104)

where we have set

\[ m = m_1 + \cdots + m_k. \]  

(105)

The obtained expression of \( \rho_0(\bar{z}, z) \) is valid for a large number \( M \) of particles [11] as well. The mean value of the density operator is a step function for \( n \rightarrow \infty \) and \( M \rightarrow \infty \) (\( \frac{M}{n} \) fixed). It corresponds to an abelian droplet configuration with a boundary defined by

\[ n \bar{z} \cdot z = M \]  

(106)

and its radius is proportional to \( \sqrt{M} \). The derivative of this density tends to a \( \delta \)-function. This result will be useful in describing the edge excitations.

4.3 Excitation potential

The quantum Hall droplet under consideration is specified by the density of states \( \rho_0(\bar{z}, z) \). The excitations of this configuration can be described by an unitary time evolution operator \( U \). It gives information concerning the dynamics of the excitations around \( \rho_0 \). The excited states will be characterized by a density operator

\[ \rho = U \rho_0 U^\dagger. \]  

(107)

In this situation, one can write the Hamiltonian as

\[ \mathcal{H} = E_0 + V \]  

(108)
where $E_0 = \frac{kn}{2}$ is the LLL energy and $V$ is the excitation potential. The perturbation $V$ will induce a lifting of the LLL degeneracy. Note that, the $SU(k,1)$ left actions commute with the covariant derivatives. They correspond to the magnetic translations on $B^k$ and behind the degeneracy of the Landau levels. Thus, it is natural to construct $V$ in terms of the magnetic translations.

In LLL, the $SU(k,1)$ left actions act on the complete symmetric representations and admit a bosonic realization, similar to that given by (37-38) for the right actions. A simple choice for the excitation potential $V$ is

$$V = \omega \sum_{i=1}^{k} a_i^\dagger a_i.$$  
(109)

One can verify

$$\langle K, n_1, \ldots, n_k | \rho | K, n_1, \ldots, n_k \rangle = \omega(n_1 + \cdots + n_k)$$  
(110)

which is reflecting the degeneracy lifting of the ground state. The symbol associated to the perturbation $V$ is given by

$$V(\bar{z}, z) = \langle z | V | z \rangle = n \omega \frac{\bar{z} \cdot z}{1 - \bar{z} \cdot z}$$  
(111)

where we have used the definition (35) and a relation of type (103). This essentially goes to the harmonic oscillator potential for large $n$ on the real $2k$-dimensional spaces.

5 Edge excitations and generalized WZW action

We now derive the effective Wess-Zumino-Witten action for the edge states. The derivation is based on the LLL analysis given in the previous section. As mentioned above, the dynamical information, related to degrees of freedom of the edge states, is contained in the unitary operator $U$.

5.1 WZW Action

The action, describing the edge excitations in the Hartree-Fock approximation, can be writing as [12]

$$S = \int dt \, \text{Tr} \left\{ \rho_0 U^\dagger (i\partial_t - \mathcal{H}) U \right\}$$  
(112)

where $\mathcal{H}$ is given by (108). For a strong magnetic field, large $n$, the quantities occurring in (112) can be evaluated as classical functions. To do this, we adopt a method similar to that used by Karabali and Nair [3]. This is mainly based on the strategy used by Sakita [12] in dealing with a bosonized theory of fermions.

We start by computing the term $i \int dt \, \text{Tr} \left( \rho_0 U^\dagger \partial_t U \right)$. For this, we set

$$U = e^{+i\Phi}, \quad \Phi^\dagger = \Phi.$$  
(113)

By a direct calculation, we can write

$$U^\dagger dU = i \int_0^1 d\tau e^{-i\tau\Phi} d\Phi e^{+i\tau\Phi}$$  
(114)

which leads

$$i \int dt \, \text{Tr} \left( \rho_0 U^\dagger \partial_t U \right) = \sum_{n=0}^{\infty} \frac{-(i)^n}{(n+1)!} \text{Tr} \left( \Phi \cdots [\Phi, \rho_0] \cdots ] \partial_t \Phi \right).$$  
(115)
Due to the completeness of the LLL levels, the trace of any operator $A$ is
\[
\text{Tr}A = \int d\mu \langle z | A | z \rangle
\]  
(116)
where the measure $d\mu$ is given by equation (48). It becomes clear that (115) can be also written in the following form
\[
i \int dt \text{Tr}(\rho_0 U^\dagger \partial_t U) = \int d\mu \sum_{n=0}^\infty \frac{-(i)^n}{(n+1)!} \{ \phi, \cdots, \{ \phi, \rho_0 \}, \cdots \} \cdot \partial_t \phi.
\]  
(117)
This is more suggestive for our purpose. Indeed, using the relations (96-98), it is easy to see that
\[
i \int dt \text{Tr}(\rho_0 U^\dagger \partial_t U) \approx \frac{1}{2n} \int d\mu \{ \phi, \rho_0 \} \partial_t \phi
\]  
(118)
where we have dropped the terms in $\frac{1}{n^2}$ as well as that related to the total time derivative. Note that, the symbol $\{,\}$ is the Poisson bracket (11) and gives
\[
\{ \phi, \rho_0 \} = (\mathcal{L}\phi) \frac{\partial \rho_0(\bar{z}, z)}{\partial (\bar{z} \cdot z)}
\]  
(119)
where the first order differential operator $\mathcal{L}$ is
\[
\mathcal{L} = i(1 - \bar{z} \cdot z)^2 \left( \bar{z} \cdot \frac{\partial}{\partial z} - \bar{z} \cdot \frac{\partial}{\partial \bar{z}} \right).
\]  
(120)
Recall that, for large $n$, since the density (104) is a step function, its derivative is a $\delta$-function with a support on the boundary $\partial D$ of the droplet $D$ defined by (106). Then, we obtain
\[
i \int dt \text{Tr}(\rho_0 U^\dagger \partial_t U) \approx \frac{1}{2} \int_{\partial D \times \mathbb{R}^+} dt \ (\mathcal{L}\phi) (\partial_t \phi).
\]  
(121)
The second step in the derivation of the edge states action consists in simplifying the term involving the Hamiltonian $\mathcal{H}$. By a straightforward calculation, we obtain
\[
\text{Tr}(\rho_0 U^\dagger V U) = \text{Tr}(\rho_0 V) + i \text{Tr}([\rho_0, V] \Phi) + \frac{1}{2} \text{Tr}([\rho_0, \Phi][V, \Phi]).
\]  
(122)
The first term in r.h.s of (122) is $\Phi$-independent. We drop it since does not contains any information about the dynamics of the edge excitations. In terms of the Moyal bracket, the second term in r.h.s of (122) rewrite as
\[
i \text{Tr}([\rho_0, V] \Phi) \approx i \int d\mu \{ \rho_0, V \}, \phi.
\]  
(123)
Using (111), one show that
\[
i \text{Tr}([\rho_0, V] \Phi) \rightarrow 0
\]  
(124)
The last term in r.h.s of (122) can be evaluated in a similar way to get (119). As result, we have
\[
\int dt \text{Tr}(\rho_0 U^\dagger \mathcal{H} U) = -\frac{1}{2n^2} \int d\mu (\mathcal{L}\Phi) \frac{\partial \rho_0}{\partial (\bar{z} \cdot z)} (\mathcal{L}\Phi) \frac{\partial V}{\partial (\bar{z} \cdot z)}
\]  
(125)
Note that, we have eliminated the term containing the ground state energy $E_0$, because does not contribute to the edge dynamics. For large $n$, we have
\[
\frac{\partial V}{\partial (\bar{z} \cdot z)} \rightarrow n\omega
\]  
(126)
and using the spatial shape of density $\rho_0(\vec{z}, z)$, to obtain
\[
\int dt \, \text{Tr} \left( \rho_0 U^\dagger H U \right) = \frac{\omega}{2} \int_{\partial D \times \mathbb{R}^+} dt \, (\mathcal{L}\phi)^2. \tag{127}
\]
Combining (121) and (127), we end up with the appropriate action
\[
S \approx -\frac{1}{2} \int_{\partial D \times \mathbb{R}^+} dt \left\{ (\mathcal{L}\phi)(\partial_t \phi) + \omega(\mathcal{L}\phi)^2 \right\}. \tag{128}
\]
This action involves only the time derivative of $\phi$ and the tangential derivative $\mathcal{L}\phi$. It is similar to that derived by Karabali and Nair [3] on $\mathbb{CP}^k$. (128) shows that how a generalized chiral abelian WZW theory can be constructed on the Bergman ball $B^k$. For $k = 1$, we recover the WZW action for the edge states of hyperbolic quantum Hall systems [10].

5.2 Nature of excitations

It will be interesting to investigate the nature of edge states. Indeed, from (128), the equation of motion of the field $\phi$ is given by
\[
\mathcal{L} (\partial_t \phi + \omega \mathcal{L}\phi) = 0. \tag{129}
\]
Solving this equation, one can obtain all information concerning the edge excitation. More precisely, let us consider the simplest case, i.e. the disc $B^1 = \{ z \in \mathbb{C}, \, |z| < 1 \}$.

For $k = 1$, (129) simplifies as
\[
\partial_\alpha (\partial_t \phi + \omega \partial_\alpha \phi) = 0 \tag{130}
\]
where $\alpha$ such that $z = |z|e^{i\alpha}$. The eigenstates of the angular momenta $\mathcal{L} = \partial_\alpha$ take the forms
\[
\bar{z}^q z^p = |z|^{p+q} e^{i(p-q)\alpha} \tag{131}
\]
and the corresponding eigenvalues are $i(p-q)$. Note that, the zero-momentum states, i.e. $p = q$, does not give a deformation of the boundary defined by (106). The field $\phi$ can be expanded in terms of the eigenstates of the differential operator $\mathcal{L}$ as
\[
\phi(\alpha, t) = \sum_{p \neq q} a_{pq}(t) \, e^{i(p-q)\alpha}. \tag{132}
\]
Injecting this form of $\phi$ in (130), we show that the time-dependent functions $a_{pq}(t)$ satisfy the following differential equation
\[
\partial_t a_{pq}(t) + i(p-q)\omega a_{pq}(t) = 0 \tag{133}
\]
which can be easily solved. It follows that, the edge field $\phi$ is
\[
\phi(\alpha, t) = \sum_{p \neq q} a_{pq} e^{i(p-q)\omega t} e^{i(p-q)\alpha} \tag{134}
\]
where the coefficients $a_{pq}$ are time-independent. This shows that the field $\phi$ is a superposition of oscillating modes on the boundary $S^1$ of the quantum Hall droplet.
6 Conclusion

By considering a system of particles living on the Bergman ball $B^k$ in the presence of a $U(1)$ background field, we have algebraically investigated QHE. This was based on the fact that $B^k$ can be viewed as the coset space $SU(k,1)/U(1)$. This was used to get wavefunctions as the Wigner $D$-functions submitted to a set of suitable constraints. Also to map the corresponding Hamiltonian in terms of the $SU(k,1)$ right generators. This latter is showed to coincide with the generalized Maass Lapalacian in the complex coordinates. The Landau levels on $B^k$ are obtained by using the correspondence of two manifolds $\mathbb{CP}^k$ and $B^k$. More precisely, we have used a mapping between the $SU(k,1)$ and $SU(k+1)$ generators. In the lowest Landau levels (LLL), the obtained wavefunctions were nothing but the $SU(k,1)$ coherent states.

Restricting to LLL, we have derived a generalized effective WZW action that describes the quantum Hall droplet of radius proportional to $\sqrt{M}$, with $M$ is the number of particles in LLL. In order to obtain the boundary excitation action, we have defined the star product and the density of states. Also we have introduced the perturbation potential responsible of the degeneracy lifting in terms of the magnetic translations of $SU(k,1)$. Finally, we have discussed the nature of the edge excitations and illustrated this discussion by giving the disc as example.

The present analysis gives an idea how to deal with QHE on higher dimensional non-compact spaces, i.e. the Bergman ball $B^k$. It will be interesting to apply the obtained results in order to deal with other issues [16].

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