A DIGIT REVERSAL PROPERTY FOR STERN POLYNOMIALS

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Abstract
We consider the following polynomial generalization of Stern’s diatomic series: let
\( s_1(x, y) = 1 \), and for \( n \geq 1 \) set \( s_{2n}(x, y) = s_n(x, y) \) and \( s_{2n+1}(x, y) = x s_n(x, y) + y s_{n+1}(x, y) \). The coefficient \( [x^i y^j] s_n(x, y) \) is the number of hyperbinary expansions of \( n - 1 \) with exactly \( i \) occurrences of the digit 2 and \( j \) occurrences of 0. We prove that the polynomials \( s_n \) are invariant under digit reversal, that is, \( s_n = s_{nR} \), where \( n^R \) is obtained from \( n \) by reversing the binary expansion of \( n \).

1. Introduction
The Stern sequence (also called Stern’s diatomic sequence or Stern–Brocot sequence) \( s \) is defined by the recurrence \( s_1 = 1, s_{2n} = s_n \) and \( s_{2n+1} = s_n + s_{n+1} \). It was pointed out by Dijkstra [4], [3, pp.230–232] that this sequence satisfies a symmetry property with respect to reversal of the binary expansion of \( n \). More precisely, for a positive integer \( n \) having the proper base-2 expansion \( n = (\varepsilon_n \varepsilon_{n-1} \ldots \varepsilon_0) \) we define
\[
n^R = \sum_{0 \leq i \leq \nu} \varepsilon_{\nu-i} 2^i = (\varepsilon_0 \varepsilon_1 \ldots \varepsilon_\nu)_2.
\]

Theorem A (Dijkstra). \( s_n = s_{nR} \).

Stern’s diatomic sequence is closely related to continued fractions (see Stern [14], Lehmer [10], Lind [11], Graham, Knuth, Patashnik [8, Exercise 6.50]):
if \( n = (1^{k_0} 0^{k_1} \ldots 1^{k_{r-2}} 0^{k_{r-1}} 1^{k_r})_2 \), then \( s_n \) is the numerator of the continued fraction \([k_0; k_1, \ldots, k_r] \). Theorem A is therefore the same as the statement that \([k_0; k_1, \ldots, k_r] \) and \([k_r; k_{r-1}, \ldots, k_0] \) have the same numerator, which can be proved via continuants.

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In [12], Morgenbesser and the author proved a digit reversal property for the correlation
\[ \gamma_t(\alpha) = \lim_{N \to \infty} \frac{1}{N} \sum_{0 \leq n < N} e(\alpha \sigma_q(n + t) - \alpha \sigma_q(n)), \]
where \( e(x) = \exp(2\pi ix) \) and \( \sigma_q(n) \) is the sum of digits of \( n \) in base \( q \) (\( q \geq 2 \) an integer). That is, we proved that \( \gamma_t(\alpha) = \gamma_{t^*}(\alpha) \), where the digit reversal is in base \( q \). We note that for the case \( q = 2 \) this statement is a special case of Theorem 1 below.

In this paper, we wish to give a refinement of Theorem A, concerning hyperbinary expansions. A hyperbinary expansion [5] of a nonnegative integer \( n \) is a sequence \( (\varepsilon_0, \ldots, \varepsilon_\nu) \in \{0, 1, 2\}^\nu \) such that \( \sum_{0 \leq i < \nu} \varepsilon_i 2^i = n \). We call such an expansion proper if either \( \nu = 0 \) or \( \nu > 0 \) and \( \varepsilon_{\nu-1} \neq 0 \). Since we only work with proper expansions, we will usually omit the prefix “proper”. By induction, using the defining recurrence relation, it is not difficult to see that \( s_n \) is the number of hyperbinary expansions of \( n + 1 \) (see also Proposition 1, which generalizes this property). This property is stated as Theorem 5.2 in [13], which seems to be the earliest (correct) appearance in the literature — we note that the statement on the bottom of page 275 of [2] is erroneous, which can be seen considering the case \( n = 5 \). The same statement can be found on page 57 of [11].

2. Main Results

Our main theorem generalizes Theorem A. We define bivariate polynomials \( s_n(x, y) \) by
\[
\begin{align*}
    s_1(x, y) &= 1, \\
    s_{2n}(x, y) &= s_n(x, y), \\
    s_{2n+1}(x, y) &= x s_n(x, y) + y s_{n+1}(x, y).
\end{align*}
\]
Note that this definition of Stern polynomials differs from the one given in [6] and also from the definition in [9]. However, the univariate polynomials \( s_n(x, 1) \) appear, up to a shift of the index \( n \), in the article [1] by Bates and Mansour: the authors write “[…]the \( n \)th term \( f(n; q) \) of the \( q \)-analogue of the Calkin–Wilf sequence is the generating function for the number of hyperbinary expansions of \( n \) according to the number of powers that are used exactly twice.” This should be compared with Proposition 1 below. Moreover, \( s_n(x, 1) \) appears as the special case \( t = 1 \) in [5] (see (1.2), (1.3) in that paper). The bivariate polynomial \( s_n(x, y) \), however, appears to be a new object of study. We list the first few of these polynomials: we have
\[
\begin{align*}
    s_1(x, y) &= 1, \\
    s_2(x, y) &= 1,
\end{align*}
\]
\( s_3(x, y) = x + y, \)
\( s_5(x, y) = x + xy + y^2, \)
\( s_7(x, y) = x^2 + xy + y, \)
\( s_9(x, y) = x + xy + xy^2 + y^3, \)
\( s_{11}(x, y) = x^2 + xy + y^2 + x^2y + xy^2, \)
\( s_{13}(x, y) = x^2 + xy + y^2 + x^2y + xy^2, \)
\( s_{15}(x, y) = y + xy + x^3 + x^2y, \)
\( s_{17}(x, y) = x + xy + xy^2 + xy^3 + y^4, \)
\( s_{19}(x, y) = x^2 + xy + x^2y + xy^2 + y^3 + x^2y^2 + xy^3, \)
\( s_{21}(x, y) = x^2 + 2x^2y + 2xy^2 + y^3 + x^2y^2 + xy^3, \)
\( s_{23}(x, y) = xy + y^2 + x^3 + x^2y + xy^2 + x^3y + x^2y^2, \)
\( s_{25}(x, y) = x^2 + xy + x^2y + xy^2 + y^3 + x^2y^2 + xy^3, \)

and we see the notable identities \( s_{11}(x, y) = s_{13}(x, y) \) and \( s_{19}(x, y) = s_{25}(x, y) \), where \( 13 = 11^3 \) and \( 25 = 19^3 \). In fact, we have the following symmetry property generalizing Theorem A.

**Theorem 1.** Let \( n \) be a positive integer. Then

\[
s_n(x, y) = s_n(x, y).
\]

This theorem can be translated to a statement on hyperbinary expansions. For integers \( i, j \geq 0 \) and \( t \geq 1 \) let \( h_{i,j}(t) \) be the number of proper hyperbinary expansions \( (\varepsilon_{t-1}, \ldots, \varepsilon_0) \) of \( t - 1 \) such that \( |\{\ell : 0 \leq \ell < \nu, \varepsilon_\ell = 2\}| = i \) and \( |\{\ell : 0 \leq \ell < \nu, \varepsilon_\ell = 0\}| = j \). The following proposition connects the Stern polynomials \( s_n(x, y) \) to hyperbinary expansions.

**Proposition 1.** Assume that \( n \geq 1 \) is an integer. Then

\[
\sum_{i,j \geq 0} h_{n(i,j)} x^i y^j = s_n(x, y).
\]

In other words, we have \( h_{n(i,j)} = [x^i y^j] s_n(x, y) \), that is, the polynomial \( s_n(x, y) \) encodes the number of hyperbinary expansions of \( n - 1 \) having a given number of 2s and 0s. Let us illustrate this proposition by an example. The polynomial \( s_{21}(x, y) \) can be obtained by listing the hyperbinary expansions of 20:

\[
\begin{array}{ccc}
1212 & x^2 & 1220 & x^2 y \\
2012 & x^2 y & 2100 & xy^2
\end{array}
\]
Combining Theorem 1 and Proposition 1, we immediately get the following corollary.

**Corollary 1.** Let $n \geq 1$ and $i, j \geq 0$. We have

$$h_n(i, j) = h_n^n(i, j).$$

For example, we list the hyperbinary expansions of $18 = (10011)_2 - 1$ (first row) and $24 = (11001)_2 - 1$ (second row):

|       | 10010 | 2010 | 1210 | 10002 | 2002 | 1202 | 1122 |
|-------|-------|------|------|-------|------|------|------|
| 10012 |   xy^2 | 10100 |   y^3 |
| 2020  |   x^2y^2 | 10020 |   xy^3 |

By the corollary there is a one-to-one correspondence between these expansions.

3. Proofs

3.1. **Proof of Theorem 1**

The main argument, which represents the induction step in the proof of the theorem, is the following lemma.

**Lemma 1.** Let

$$A = \begin{pmatrix} 1 & 0 \\ x & y \end{pmatrix} \text{ and } B = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}.$$  

If $^tA$ denotes the transpose of the matrix $A$, the following identities for $1 \times 2$-matrices hold.

\[
\begin{align*}
(x & y) \cdot A (x & y) = (x & y)(x & y) + (y + 1)(x & y)A, \\
(1 & 1)^tA^t & = (x & y)(1 & 1) + (y + 1)(1 & 1)^tA, \\
(x & y) \cdot B (x & y) & = (x & y)(x & y) + y(x & y)B, \\
(1 & 1)^tB^t & = (x & y)(1 & 1) + y(1 & 1)^tB, \\
(x & y) \cdot A (x & y) & = y(x & y) + x(x & y)A, \\
(1 & 1)^tA^t & = y(1 & 1) + x(1 & 1)^tA, \\
(x & y) \cdot B (x & y) & = (x & y)(x & y) + (x + 1)(x & y)B, \\
(1 & 1)^tB^t & = (x & y)(1 & 1) + (x + 1)(1 & 1)^tB.
\]
The proof is by simple calculation and is left to the reader. 

To prove Theorem 1, we let $n \geq 1$ be an odd integer. This is no loss of generality since we can deal with the even case by repeatedly using the relation $s_{2n}(x,y) = s_n(x,y)$. Let $n = \sum_{i \leq \nu} \varepsilon_i 2^i$ be the binary representation of $n$ and $\varepsilon_\nu \neq 0$. We prove the theorem by induction on $\nu$. The case $\nu \leq 1$ is trivial, since in this case we have $n^R = n$. We write $A(0) = \left( \begin{smallmatrix} 1 & 0 \\ x & y \end{smallmatrix} \right)$ and $A(1) = \left( \begin{smallmatrix} x & y \\ 0 & 1 \end{smallmatrix} \right)$. By a simple application of the recurrence relation we have $\left( s_{2n+1}^{s_{2n}} \right) = A(0) \left( s_{n+1}^n \right)$ and $\left( s_{2n+2}^{s_{2n+1}} \right) = A(1) \left( s_{n+1}^n \right)$ for all $n \geq 1$. Since $n$ is odd and $s_1(x,y) = s_2(x,y) = 1$, it follows from these identities that

$$s_n(x,y) = \left( \begin{array}{cc} x & y \\ \varepsilon_1 & \cdots & A(\varepsilon_{\nu-1}) & 1 \\ 1 & 1 \end{array} \right).$$

(1)

The statement of the theorem is equivalent to the assertion that

$$\left( \begin{array}{cc} x & y \\ \varepsilon_1 & \cdots & A(\varepsilon_{\nu-1}) & 1 \\ 1 & 1 \end{array} \right) = \left( \begin{array}{cc} x & y \\ \varepsilon_1 & \cdots & A(\varepsilon_{\nu-1}) & 1 \\ 1 & 1 \end{array} \right)$$

(2)

for all $\nu \geq 1$ and all finite sequences $(\varepsilon_1, \ldots, \varepsilon_{\nu-1})$ in $\{0,1\}$. We prove the identity (2) by induction on $\nu$, using Lemma 1. This identity is obvious for $\nu \leq 2$. For $\nu > 2$ we have four cases, corresponding to the four possible values of $(\varepsilon_1, \varepsilon_2)$. By Lemma 1, in each of the four cases there exist coefficients $\alpha$ and $\beta$ such that

$$\left( \begin{array}{cc} x & y \\ \varepsilon_1 & \cdots & A(\varepsilon_{\nu-1}) & 1 \\ 1 & 1 \end{array} \right) = \alpha \left( \begin{array}{cc} x & y \\ \varepsilon_1 & \cdots & A(\varepsilon_{\nu-1}) & 1 \\ 1 & 1 \end{array} \right) + \beta \left( \begin{array}{cc} x & y \\ \varepsilon_1 & \cdots & A(\varepsilon_{\nu-1}) & 1 \\ 1 & 1 \end{array} \right)$$

and

$$\left( \begin{array}{cc} x & y \\ \varepsilon_1 & \cdots & A(\varepsilon_{\nu-1}) & 1 \\ 1 & 1 \end{array} \right) = \alpha \left( \begin{array}{cc} x & y \\ \varepsilon_1 & \cdots & A(\varepsilon_{\nu-1}) & 1 \\ 1 & 1 \end{array} \right) + \beta \left( \begin{array}{cc} x & y \\ \varepsilon_1 & \cdots & A(\varepsilon_{\nu-1}) & 1 \\ 1 & 1 \end{array} \right).$$

By applying the induction hypothesis (2) to the sequences $(\varepsilon_2, \ldots, \varepsilon_{\nu-1})$ and $(\varepsilon_3, \ldots, \varepsilon_{\nu-1})$ we obtain the statement of the theorem. 

3.2. Proof of Proposition 1

In order to prove Proposition 1, we use the following recurrence for $h_n(i,j)$ (see [7]).

**Proposition 2.** Let $n \geq 1$. Then

$$h_1(0,0) = 1,$$

$$h_1(i,j) = 0 \quad \text{for } (i,j) \neq (0,0),$$

$$h_{2n}(i,j) = h_n(i,j) \quad \text{for } i,j \geq 0,$$

$$h_{2n+1}(i,0) = h_n(i-1,0) \quad \text{for } i \geq 1,$$

$$h_{2n+1}(0,j) = h_{n+1}(0,j-1) \quad \text{for } j \geq 1,$$

$$h_{2n+1}(i,j) = h_n(i-1,j) + h_{n+1}(i,j-1) \quad \text{for } i,j \geq 1.$$

(3)

Moreover, $h_t(0,0) = 0$ if $t$ is not a power of 2.
Proof. The first two lines follow from the fact that the empty tuple ( ) is the only hyperbinary expansion of 0.

The hyperbinary expansions of 2n – 1 are in bijection with the hyperbinary expansions of n – 1 by deleting the lowest digit, which is a 1. This explains the third line of (3).

A hyperbinary expansion of 2n without 0 necessarily ends with the digit 2, and the bijection is by deleting this digit. This gives line number four.

There is exactly one hyperbinary expansion of 2n without 2, and it ends with 0. The same argument applies here.

The sixth line is a combination of arguments as above.

Finally, integers having the binary expansion (1 ⋅⋅⋅ 1) are the only ones without 0 and 2, which proves the last line.

In order to prove Proposition 1, we proceed by induction. The identity is trivial for n = 1. Assume that n = 2u for some u ≥ 1. We have h_n(i, j) = h_u(i, j) and s_u(x, y) = s_u(x, y). If n = 2u + 1, we have

\[ \sum_{\substack{i \geq 0 \atop j \geq 0}} h_n(i, j)x^iy^j = h_n(0, 0) + \sum_{j \geq 1} h_n(0, j)y^j + \sum_{i \geq 1} h_n(i, 0)x^i + \sum_{i, j \geq 1} h_n(i, j)x^iy^j \]

\[ = \sum_{j \geq 1} h_{u+1}(0, j - 1)y^j + \sum_{i \geq 1} h_u(i - 1, 0)x^i \]

\[ + \sum_{i, j \geq 1} (h_u(i - 1, j) + h_{u+1}(i, j - 1))x^iy^j \]

\[ + \sum_{\substack{i \geq 1 \atop j \geq 0}} h_u(i - 1, j)x^iy^j + \sum_{\substack{i \geq 1 \atop j \geq 1}} h_u(i, j - 1)x^iy^j \]

\[ = xs_u(x, y) + ys_{u+1}(x, y) = s_n(x, y), \]

which proves the desired assertion.

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