CARDINAL INVARIANTS ASSOCIATED WITH HAUSDORFF CAPACITIES

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ABSTRACT. Let $\lambda(X)$ denote Lebesgue measure. If $X \subseteq [0, 1]$ and $r \in (0, 1)$ then the $r$-Hausdorff capacity of $X$ is denoted by $H^r(X)$ and is defined to be the infimum of all $\sum_{i=0}^{\infty} \lambda(I_i)^r$ where $\{I_i\}_{i \in \omega}$ is a cover of $X$ by intervals. The $r$-Hausdorff capacity has the same null sets as the $r$-Hausdorff measure which is familiar from the theory of fractal dimension. It is shown that, given $r < 1$, it is possible to enlarge a model of set theory, $V$, by a generic extension $V[G]$ so that the reals of $V$ have Lebesgue measure zero but still have positive $r$-Hausdorff capacity.

1. INTRODUCTION

If $r \in [0, 1]$ then for any set $X \subseteq [0, 1]$ the $r$-Hausdorff capacity of $X$ is denoted by $H^r(X)$ and is defined to be the infimum of all $t$ such that there is a cover of $X$ by intervals, $X \subseteq \bigcup_{i=0}^{\infty} I_i$, such that $t = \sum_{i=0}^{\infty} \lambda(I_i)^r$. This notion may be familiar from its use along the way to defining $r$-Hausdorff measure. Given $\beta > 0$, $H^r_\beta(X)$ is defined, for any set $X \subseteq [0, 1]$, to be the infimum of all $t$ such that there is a cover of $X$ by intervals, $X \subseteq \bigcup_{i=0}^{\infty} I_i$, such that $t = \sum_{i=0}^{\infty} \lambda(I_i)^r$ and such that the length of each interval $I_i$ is less than $\beta$. The $r$-Hausdorff measure of a set $X$ is then defined to be the supremum of $H^r_\beta(X)$ as $\beta$ ranges over all positive real numbers. However, the topic of this paper if $r$-Hausdorff capacity rather than $r$-Hausdorff measure. The crucial difference between the two is that, while the $r$-Hausdorff measure is countably additive, the $r$-Hausdorff capacity is only subadditive if $r \in (0, 1)$. A proof of the fact that $H^r$ is actually a capacity can be found in [4] on page 90. For more details on Hausdorff measure consult [3], [1] or [2].

For the rest of this paper let $r$ be a fixed real number such that $0 < r < 1$. Let $\lambda(X)$ denote the Lebesgue measure of any measurable set $X \subseteq [0, 1]^n$. It will be shown that it is possible to generically extend an arbitrary model of set theory so that the ground model reals have Lebesgue measure zero but still have positive $r$-Hausdorff measure. If this process could be iterated $\omega_2$ times and the ground model satisfied the Continuum Hypothesis then it would yield a model every set of size $\aleph_1$ has Lebesgue measure zero yet there is a set of size $\aleph_1$ which has positive $r$-Hausdorff capacity. This raises the following conjecture which uses the obvious extension of Hausdorff capacity to $\mathbb{R}^n$ : For any $n \in \omega$ and $r < n$ it is consistent that every set of size $\aleph_1$ has Lebesgue measure zero yet there is a set $X \subseteq [0, 1]^n$ of size $\aleph_1$ which has positive $r$-Hausdorff capacity.

This is related to the following question posed by P. Komjath.

**Question 1.1.** Suppose that every set of size $\aleph_1$ has Lebesgue measure zero. Does it follow that the union of any set of $\aleph_1$ lines in the plane has Lebesgue measure zero?
To see the relationship between this question and $r$-Hausdorff capacity consider that it is easy to find countably many unit squares in the plane such that each line passes through either the top and bottom or the left and right sides of at least one of these squares. It is therefore possible to focus attention on all lines which pass through the top and bottom of the unit square. For any such line $L$ there is a pair $(a, b)$ such that both the points $(a, 0)$ and $(1, b)$ belong to $L$. If the mapping which send a line $L$ to this pair $(a, b)$ is denoted by $\beta$ then it is easy to see that $\beta$ is continuous and that if $S \subseteq [0, 1]^2$ is a square of side $\epsilon$ then the union of $\beta^{-1}S$ has measure $\epsilon$ while $S$ itself has measure $\epsilon^2$. In other words, the Lebesgue measure of the union of $\beta^{-1}X$ is no larger than the 1-Hausdorff capacity of $X$ for any $X \subseteq [0, 1]^2$. Hence, if the answer to Question [4] was negative this would imply that the conjecture is true. While this was the motivation for studying the problem of Hausdorff capacity, it may be that the notion of Hausdorff capacity is actually more central than Komjath’s question itself.

If $1 \leq n \leq m$ then define $\pi : [0, 1]^m \to [0, 1]^n$ by

$$\pi_n(x_1, x_2, \ldots, x_m) = (x_1, x_2, \ldots, x_n).$$

If $X \subseteq [0, 1]^m$ then $\pi_n(X)$ will denote the image of $X$ under the mapping $\pi_n$. If $A \subseteq [0, 1]^d$ and $1 \leq n < d$ then, for any $x \in [0, 1]^n$, the notation $A_x$ will be used to denote $\{y \in [0, 1]^{d-n} : \pi_n(x, y) = x \text{ and } (x, y) \in A\}$.

2. A General Class of Forcing Partial Orders

This section will be devoted to examining a generalization of Superperfect forcing obtained by insisting that on a dense set of nodes the splitting is into a set of positive measure with respect to some ideal. Such generalizations have been considered by various authors. Throughout this section the term ideal will always refer to a proper ideal on $\omega$ which contains all finite sets. In later sections ideals will be constructed on countable sets other than $\omega$, but it will simplify notation to ignore this for now.

**Definition 2.1.** Let $\mathcal{I} = \{I_n\}_{n \in \omega}$ be a sequence of ideals. The partial order $\mathbb{P}(\mathcal{I})$ will be defined to consist of trees $T \subseteq \bigcup_{n \in \omega} \prod_{i \in n} D_i$ such that for every $t \in T$ one of the following two alternatives holds:

- $|\{n \in \omega : t \land n \in T\}| = 1$
- $\{n \in \omega : t \land n \in T\} \in I^+_n$

If $\{n \in \omega : t \land n \in T\} \in I^+_n$, then $t$ will be said to be a branching node of $T$ and the set of branching nodes will be denoted by $B(T)$. Define $\mathbb{P}(\mathcal{I})$ to consist of all $T$ such that for every $t \in T$ there is $s \in B(T)$ such that $t \subseteq s$. The ordering on $\mathbb{P}(\mathcal{I})$ is inclusion.

It is left to the reader to verify that $\mathbb{P}(\mathcal{I})$ is proper and, indeed, that it satisfies Axiom A. A standard argument works.

Suppose now that $T \in \mathbb{P}(\mathcal{I})$. Then the root of $T$ is the unique minimal member of $B(T)$ and is denoted by $\text{root}(T)$. If $t \in B(T)$ then the set of successors of $t$ is denoted by $\text{succ}_T(t)$ and is defined by $\text{succ}_T(t) = \{n \in \omega : t \land n \in T\}$. The branching height of $t$ will be denoted by $\text{branching-height}(t)$ and is defined to be $|\{s \subseteq t : s \in B(T)\}|$ — so branching-height($\text{root}(T)$) = 1. A subset $S \subseteq T$ will be said to be a subtree if it is closed under taking initial segments. The tree generated by $X \subseteq T$ is simply the set of all initial segments of members of $X$. Observe that $\text{succ}_S$ can be defined for any subtree, regardless of whether or not $S \in \mathbb{P}(\mathcal{I})$. A
subset $S \subseteq T$ will be said to be a full subtree of $T$ if and only if for every $t \in S$ either $t$ is a maximal member of $S$ or $\text{succ}_T(t) = \text{succ}_W(t)$. If $t \in T$ then $T(t)$ is defined to be the subtree of $T$ consisting of all $s \in T$ such that either $s \subseteq t$ or $t \subseteq s$. If $S \subseteq T$ is a subtree then define the interior of $S$ to be the set of all non maximal elements of $B(T) \cap S$ and denote this by $\text{int}(S)$ — the dependence on $T$ will suppressed.

If $T \in \mathcal{P}(I)$ then define a function $\Psi$ on $B(T)$ to be approximating if $\Psi(t) \subseteq [0,1]$ is a finite union of rational intervals for each $t \in B(t)$ and it is monotone in the sense that $\Psi(t) \subseteq \Psi(s)$ if $t \subseteq s$. If $T \in \mathcal{P}(I)$, $x \in [0,1]$ and $\Psi$ on $B(T)$ is approximating then define $R(T,\Psi,x)$ to be the tree generated by $\{ t \in B(T) : x \notin \Psi(t) \}$.

**Definition 2.2.** An ideal $\mathcal{I}$ will be said to satisfy KP($r$) if and only if for all

- $\theta < 1$
- $X \in \mathcal{I}^+$
- functions $F$ from $X$ to the Borel subsets of $[0,1]$ satisfying that $H^r(F(x)) \leq \theta$
  - for each $x \in X$
- $\epsilon > 0$ there is some $Y \subseteq [0,1]$ as well as $Z \subseteq [0,1]$ such that
  - $H^r(Y) \leq \theta$
  - $\lambda(Z) < \epsilon$
  - $\{ x \in X : y \notin F(x) \} \in \mathcal{I}^+$ for every $y \in [0,1] \setminus (Y \cup Z)$

Well founded trees will play an important role in the following discussion but the standard equivalence between well founded trees and trees with rank functions is not as convenient a slight modification of this notion. If $T \in \mathcal{P}(I)$ and $S \subseteq T$ then the standard rank of $S$ will denote the rank of $S \cap B(T)$ when this is considered as a tree under the inherited ordering. Later on, a different rank function will be introduced and it should not be confused with the standard rank.

If $T \in \mathcal{P}(I)$ and $W \subseteq T$ is a full subtree then $W' \subseteq W$ will be said to be large if:

- $\text{root}(W) \in W'$
- if $t \in W' \setminus B(T)$ then $t$ is not maximal in $W'$
- if $\text{tint}(W) \cap W'$ then $\text{succ}_{W'}(t) \in \mathcal{I}_{|t|}^+.$

**Lemma 2.1.** Suppose that

- $\mathcal{I} = \{ \mathcal{I}_n : n \in \omega \}$ is a sequence of ideals, each satisfying KP($r$)
- $T \in \mathcal{P}(I)$ and $W \subseteq T$ is a well founded full subtree of standard rank $\beta$
- $\Psi$ is an approximating function on $B(T) \cap W$
- $\theta < 1$
- $H^r(\Psi(t)) < \theta$ for any $t \in W \cap B(T)$

then, there is some $x \in [0,1]$ such that $R(W,\Psi,x)$ is a large subtree of $W$.

**Proof:** It will be shown by induction on $\beta \in \omega_1$ that the following, stronger condition holds:

**Q($\beta$):** If $s \in T$ and $W \subseteq T(s)$ is a well founded full subtree of standard rank $\beta$, $\theta < 1$, $\epsilon > 0$ and $\Psi$ is an approximating function on $B(T) \cap W$ such that $H^r(\Psi(t)) < \theta$ for any $t \in W \cap B(T)$ then, there is some $Y \subseteq [0,1]$ and $Z \subseteq [0,1]$ such that

- for each $x \in [0,1] \setminus (Y \cup Z)$ $R(W,\Psi,x)$ is a large subtree of $W$ belongs to $\mathcal{I}_{|t|}^+$
- $H^r(Y) \leq \theta$
Lemma 2.2. If $\rho$, $t$ such that $\rho(t) \subseteq \beta$ and 
\[ \rho(t) \cup (Y \cup Z) \not= \emptyset \]
then $R(W(s \wedge n), \Psi, x)$ is a large subtree of $W(s \wedge n)$. 

Now let $X = \text{succ}_T(s) \in \mathcal{I}_{[s]}^+$. Choose a function $F$ on $X$ such that $F(d) \geq Y_d$ 
and $F(d)$ is a $G_\delta$ such that $H^r(F(d)) = H^r(Y_d) \leq \theta$ for each $d \in X$. 
It follows from $\text{KP}(r)$ that there are $Y' \subseteq [0, 1]$ and $Z' \subseteq [0, 1]$ such that 
\[ H^r(Y') \leq \theta/2 \]
\[ \lambda(Z') < \epsilon/2 \]
\[ \{d \in X : x \notin F(d)\} \in \mathcal{I}_{[s]}^+ \text{ for each } x \in [0, 1] \setminus (Y \cup Z') \]

Now let $Z = Z' \cup \{Z_n : n \in X\}$ and note that $\lambda(Z) < \epsilon$ and, by the subadditivity of $H^r$, 
$H^r(Y) \leq \theta$. Hence, in order to verify that $\mathcal{Q}(\beta)$ holds it suffices to show 
that if $x \in [0, 1] \setminus (Y \cup Z)$ and $t \in R(W, \Psi, x) \cap \text{int}(W)$ then $\text{succ}_{R(W, \Psi, x)}(t) \in \mathcal{I}_{[s]}^+$. If 
$t = s$ this follows from the application of $\text{KP}(r)$ and the fact that $x \notin Y' \cup Z'$. In 
every other case it follows from the use of the induction hypothesis because $t \supsetsup s \wedge n$ 
for some $n$ and, therefore, $x \notin Y_n \cup Z_n$ implies that $\text{succ}_{R(W, \Psi, x)}(t) \in \mathcal{I}_{[s]}^+$. 

For the remainder of this section fix a sequence of ideals $\mathcal{I} = \{\mathcal{I}_n : n \in \omega\}$ and 
$T \in \mathbb{P}(\mathcal{I})$. For $t \in B(T)$ and $n \in \text{succ}_T(t)$ define $t \oplus n$ to be the least $s \in B(T)$ 
such that $t \wedge n \subseteq s$. If $X \subseteq T$ is a subtree then a rank function $\rho_X$ 
will be defined on $B(T) \cap X$ by bar induction. To begin, define $\rho_X(t)$ to be 0 if there is some 
t' $\subseteq t$ such that $t' \in B(T)$ and $\text{succ}_X(t') \in \mathcal{I}_{[t]}^+$. Define $\rho_X(t)$ to be the least 
ordinal $\beta$ such that there is some $A \in \mathcal{I}_{[t]}$ such that $\rho_X(t \oplus n)$ is defined for each 
n $\in \text{succ}_X(t) \setminus A$ and $\rho_X(t \oplus n) \in \beta$ for any such $n$. The rank of $X$ is defined to 
be the rank of its root, provided this is defined. A subtree $X \subseteq T$ will be defined to be small 
if $\rho_X(t)$ is defined for all $t \in B(T) \cap X$ and $\rho_X(t) > 0$ if and only if $t \in \text{int}(X)$. 

Lemma 2.2. If $X \subseteq T$ is a subtree and there is some $t \in X \cap B(T)$ for which 
$\rho_X(t)$ is not defined then $X$ contains a member of $\mathbb{P}(\mathcal{I})$. 

Proof: This is standard. Let $S$ be the subtree of $T$ generated by all $t \in X \cap B(T)$ 
such that $\rho_X(t)$ is not defined. Notice that if $t \in S$ then 
\[ \{n \in \text{succ}_X(t) : \rho_X(t \oplus n) \text{ is not defined}\} = \text{succ}_X(t) \]
and this belongs to $\mathcal{I}_{[s]}^+$. Hence $S \in \mathbb{P}(\mathcal{I})$ provided that it is not empty. The 
hypothesis of the lemma guarantees that this is not the case. 

For any subtree \( W \subseteq T \) and any function \( \theta \in \prod_{w \in \text{int}(W)} I_{[w]} \) define
\[
W^\theta = \{ w \in W : (\forall n \in \omega)(w(n) \notin \theta(w \upharpoonright n)) \}
\]
or, in other words, \( W^\theta \) is obtained by throwing away \( I_{[w]} \) many successors, determined by \( \theta \), of each \( t \in \text{int}(W) \). If \( W \) and \( X \) are subtrees of \( T \) define \( W \prec X \) if and only if there exists \( \theta \in \prod_{w \in \text{int}(W)} I_{[w]} \) such that for every \( \theta' \in \prod_{x \in \text{int}(X)} I_{[x]} \) there is a one-to-one function \( G : W^\theta \cap B(T) \to (X^\theta' \cap B(T)) \setminus \{ \text{root}(X) \} \) which is order preserving in the sense that if \( t \leq s \) then \( G(t) \leq G(s) \).

For any small subtree \( Y \subseteq T \) a function \( \theta \in \prod_{x \in \text{int}(Y)} I_{[x]} \) will be said to be a witness to the rank of \( Y \) if and only if for each \( t \in \text{int}(Y) \)
\[
\rho_{Y,T}(t \oplus n) = \rho_{Y,T}(t)
\]
for each \( n \in \text{succ}_Y(t) \setminus \theta(t) \).

**Lemma 2.3.** Let \( W \) and \( X \) be small subtrees of \( T \) of rank \( \alpha \) and \( \beta \) respectively. If \( \alpha < \beta \), then for any \( \theta \in \prod_{w \in \text{int}(W)} I_{[w]} \) which is a witness to the rank of \( W \) and any \( \theta' \in \prod_{x \in \text{int}(X)} I_{[x]} \) there is a one-to-one function \( G_{\theta,\theta'} : W^\theta \cap B(T) \to X^{\theta'} \cap B(T) \) which is order preserving such that \( G_{\theta,\theta'}(\text{root}(W)) \neq \text{root}(X) \). Moreover, \( G_{\theta,\theta'} \) is continuous in the variable \( \theta' \) in the sense that the mapping \( \theta' \mapsto G_{\theta,\theta'} \) is a continuous function from \( \prod_{x \in \text{int}(X)} I_{[x]} \) to \( W^\theta \cap B(T) \times X^{\theta'} \cap B(T) \) where \( I_{[x]} \) is considered as a subspace of \( 2^\omega \).

**Proof:** Suppose that \( \alpha < \beta \) and \( W \) and \( X \) are small subtrees of \( T \) of rank \( \alpha \) and \( \beta \) respectively. Let \( \theta \in \prod_{w \in \text{int}(W)} I_{[w]} \) be a witness to the rank of \( W \). For every \( \theta' \in \prod_{x \in \text{int}(X)} I_{[x]} \) a function \( G_{\theta,\theta'} : W^\theta \cap B(T) \to X^{\theta'} \cap B(T) \) can be defined by induction on the branching height of nodes of \( W^\theta \cap B(T) \). The induction hypothesis will be that \( \rho_X(G_{\theta,\theta'}(t)) \geq \rho_W(t) \). Define \( G_{\theta,\theta'}(\text{root}(W)) = \text{root}(X) \oplus m \) where \( m \) is the least integer such that \( m \in \text{succ}_X(\text{root}(X)) \setminus \theta'(\text{root}(X)) \) and \( \rho_X(\text{root}(X)) \oplus m \geq \beta \). Such an \( m \) must exist because \( \rho_X(\text{root}(X)) = \beta > \alpha \) and \( \theta'(\text{root}(X)) \in I_{[\text{root}(X)]} \). If \( t \) and \( t \oplus n \) are both in \( W^\theta \cap B(T) \) and \( G_{\theta,\theta'}(t) \) and \( G_{\theta,\theta'}(t \oplus i) \) are defined for \( i \in n \) then define \( G_{\theta,\theta'}(t \oplus n) = G_{\theta,\theta'}(t) \oplus k \) where \( k \) is the least integer such that
\[
k \in \text{succ}_X(G_{\theta,\theta'}(t)) \setminus \theta'(G_{\theta,\theta'}(t)) \cup \{ G_{\theta,\theta'}(t \oplus i) \mid |G_{\theta,\theta'}(t)| : i \in n \}
\]
and \( \rho_X(G_{\theta,\theta'}(t \oplus k)) \geq \rho_W(t \oplus n) \). The reason such a \( k \) exists is that, by the induction hypothesis, \( \rho_X(G_{\theta,\theta'}(t)) \geq \rho_W(t \oplus n) \) and hence \( \rho_X(G_{\theta,\theta'}(t) \oplus n) > \rho_W(t \oplus n) \) because \( \theta \) is a witness to the rank of \( W \) and \( t \oplus n \in W^\theta \). Since \( I_{[G_{\theta,\theta'}(t)]} \) contains all finite subsets it must be that there is some
\[
k \in \text{succ}_X(G_{\theta,\theta'}(t)) \setminus \theta'(G_{\theta,\theta'}(t)) \cup \{ G_{\theta,\theta'}(t \oplus i) \mid |G_{\theta,\theta'}(t)| : i \in n \}
\]
such that \( \rho_X(G_{\theta,\theta'}(t \oplus k)) \geq \rho_W(t \oplus n) \). Since the inductive hypothesis is preserved, this construction can be carried out for all nodes in \( W^\theta \). Obviously \( G_{\theta,\theta'} \) is a one-to-one, order preserving function. By construction, \( G_{\theta,\theta'}(\text{root}(W)) \neq \text{root}(X) \).

The continuity of \( G_{\theta,\theta'} \) in the variable \( \theta' \) follows from the minimal choice of the integer \( k \) such that
\[
k \in \text{succ}_X(G_{\theta,\theta'}(t)) \setminus \theta'(G_{\theta,\theta'}(t)) \cup \{ G_{\theta,\theta'}(t \oplus i) \mid |G_{\theta,\theta'}(t)| : i \in n \}
\]
and \( \rho_X(G_{\theta,\theta'}(t \oplus k)) \geq \rho_W(t \oplus n) \). An open neighbourhood of \( G_{\theta,\theta'} \) in \( W^\theta \cap B(T) \times X^{\theta'} \cap B(T) \)
Lemma 2.4. If $W$ and $X$ are small subtrees of $T$ of rank $\alpha$ and $\beta$ respectively, then $W \prec X$ if and only if $\alpha \prec \beta$.

One direction is an immediate consequence of Lemma 2.3 because a witness to the rank of $W$ can always be found. For the other, it will be shown by induction on $\alpha$ that if $\alpha \leq \beta$ then $X \not\prec W$. For $\alpha = 0$ this is trivial so assume that the assertion has been established for all $\alpha' \in \alpha$ and suppose that $X \prec W$. This means that there is some $\theta \in \prod_{x \in \text{int}(X)} I_{\{x\}}$ such that for every $\theta' \in \prod_{w \in \text{int}(W)} I_{\{w\}}$ there is a one-to-one function $G : X^\theta \cap B(T) \to W^{\theta'} \cap B(T)$ which is order preserving such that $\theta \not\prec \theta'$. Let $\theta'$ be a witness to the rank of $W$ and let the function $G$ from $X^\theta \cap B(T)$ to $W^{\theta'} \cap B(T)$ be one-to-one and order preserving. It must be that $\rho_W(G(\text{root}(X))) \in \rho_W(\text{root}(W)) \leq \alpha \leq \beta$. Therefore, it is possible to find $m \in \text{succ}_X(\text{root}(X)) \setminus \theta(\text{root}(X))$ such that $\rho_X(\text{root}(X) \oplus m) \geq \rho_W(G(\text{root}(X)))$. Obviously $G, \theta \downarrow X \text{root} \oplus m$ and $\theta' \downarrow W \text{root}(X)$ establish that $X \text{root} \oplus m \prec W \text{root}(X)$ which contradicts the induction hypothesis.

An ideal $\mathcal{I}$ will be said to be $\Sigma^1_1$ if it is a $\Sigma^1_1$ subset of $2^\omega$ under the natural identification. The next lemma shows that the relation $\prec$ is $\Sigma^1_1$ provided that each of the ideals of $\mathcal{I} = \{I_n\}_{n \in \omega}$ is $\Sigma^1_1$. This will require the full conclusion of Lemma 2.3 since the obvious calculation only shows that $\prec$ is $\Sigma^1_3$.

Lemma 2.5. If each of the ideals of $\mathcal{I} = \{I_n\}_{n \in \omega}$ is $\Sigma^1_1$ then the relation $\prec$ defined from them is $\Sigma^1_1$.

Proof: Since each $I_n$ is $\Sigma^1_1$ it is possible to choose continuous functions $f_n : \omega \to 2^\omega$ such that $I_n$ is the image of $f_n$. First it will be shown that $W \prec X$ if and only if there is some $\theta \in \prod_{w \in \text{int}(W)} I_{\{w\}}$ and a continuous function

$$G : \prod_{x \in \text{int}(X)} \omega \to W^\theta \cap B(T) \cap B(T) \cap X$$

such that

1. for all $\theta' \in \prod_{x \in \text{int}(X)} \omega$, if $t$ belongs to $W^\theta \cap B(T)$ then $G(t) = s \oplus m$ for some $s \in B(T) \cap X$ and some $m \in \text{succ}_X(s) \setminus f_1(\theta(s))$
2. $G(\theta')$ is order preserving for all $\theta' \in \prod_{x \in \text{int}(X)} \omega$
3. $G(\theta')$ is one-to-one for all $\theta' \in \prod_{x \in \text{int}(X)} \omega$
4. $G(\theta')(\text{root}(W)) \not\prec \text{root}(X)$ for all $\theta' \in \prod_{x \in \text{int}(X)} \omega$

Assuming $W \prec X$, it is possible to use Lemma 2.3 to define $G(\theta') = G_{\theta, \mu(\theta')}$ where $\mu(\theta')(x) = f_1(\theta'(x))$ for each $x \in \text{int}(X)$. Note that $G$ is continuous because of the final sentence of Lemma 2.3 and the continuity of $\mu$, which is a consequence of the continuity of each $f_n$. The other direction of the equivalence is clear because each $f_n$ is onto $I_n$. 

is specified by a restriction of $G_{\theta, \theta'}$ to a finite subset. Given a finite subset $a \subseteq W^\theta$ it is possible to find a finite set $b \subseteq X^{\theta'}$ such that if $t \in a$, $G_{\theta, \theta'}(t) = s \oplus m$ and $i \in m \setminus \theta'(s)$ then $s \oplus i$ also belongs to $b$. Let $M \in \omega$ be such that the range of each $t \in b$ is contained in $M$. It is easy to check that if $\theta'' \in \prod_{x \in \text{int}(X)} I_{\{x\}}$ is such that $\theta''(t) \cap M = \theta'(t) \cap M$ for each $t \in b$ then $G_{\theta, \theta''} \downarrow a = G_{\theta, \theta'} \downarrow a$. ■
Hence it suffices to check that the clauses (1) - (4) are arithmetic. Since \( T \) and \( B(T) \) can be used as parameters, the only problematic part is the use of the quantifiers
\[
\text{for all } \theta' \in \prod_{x \in \text{int}(X)} \omega.
\]

However, the continuity of \( G \subseteq C \) approximations to \( \theta \) by “for all \( \theta \) full subtree \( W \) and \( B \) and
\[
\text{for all } \theta' \in \prod_{x \in \text{int}(X)} \omega
\]
by “for all \( \theta' \in C' \).”

\[\begin{align*}
\text{Lemma 2.6.} \quad &\text{For all } \alpha \text{ such that } 1 \leq \alpha \in \omega_1 \text{ and } t \in B(T) \text{ there is a well founded full subtree } W \text{ of standard rank } \alpha \text{ such that } \text{root}(W) = t \text{ and if } W' \subseteq W \text{ is any large subtree then } \rho_{W'}(t) = \alpha.
\end{align*}\]

\[\begin{proof}
\text{Proceed by induction on } \alpha. \text{ The case } \alpha = 1 \text{ is trivial so assume the assertion has been established for all } \alpha' \in \alpha. \text{ First suppose that } \alpha = \beta + 1. \text{ Given } t \in B(T) \text{ choose for each } n \in \text{succ}_W(t) \text{ a well founded full subtree } W_n \text{ of standard rank } \beta \text{ such that } \text{root}(W_n) = t \oplus n \text{ and if } W' \subseteq W_n \text{ is any large subtree then } \rho_{W'}(t \oplus n) = \beta. \text{ Let } W = \bigcup_{n \in \text{succ}_W(t)} W_n.
\end{proof}\]

\[\begin{proof}
\text{If } \alpha \text{ is a limit let } \{\beta_n\}_{n \in \omega} \text{ converge to } \alpha \text{ from below. Given } t \in B(T) \text{ choose for each } n \in \text{succ}_W(t) \text{ a well founded full subtree } W_n \text{ of standard rank } \beta_n \text{ such that } \text{root}(W_n) = t \oplus n \text{ and if } W' \subseteq W_n \text{ is any large subtree then } \rho_{W'}(t \oplus n) = \beta_n. \text{ Let } W = \bigcup_{n \in \text{succ}_W(t)} W_n. \text{ Since } I_{\{t\}} \text{ contains all finite sets this works.}
\end{proof}\]

\section{The Preservation Theorem}

\[\begin{proof}
\text{Suppose the theorem false — in other words, there is some } \theta < 1 \text{ and } \{J_n\}_{n=0}^{\infty}, \text{ a name for a sequence of intervals with rational endpoints, as well as a condition } T \in \mathcal{P}(I) \text{ such that}
\end{proof}\]

\[
T \models \"([0,1] \cap V) \subseteq \bigcup_{n=0}^{\infty} J_n."\]

\[\begin{proof}
\text{and } T \models \" \sum_{n=0}^{\infty} \lambda(J_n)^r < \theta". \text{ By thinning down } T \text{ it may be assumed that if } t \in B(T) \text{ and branching-height}_r(t) = n \text{ then } T(t) \models \" J_i = J(t,i)\" \text{ for some interval } J(t,i) \text{ with rational endpoints, for each } i \in n. \text{ Let } \Psi(t) = \bigcup\{J(t,i) : i \in |t|\} \text{ for each } t \in B(T).
\end{proof}\]

\[\begin{proof}
\text{If there is some } x \in [0,1] \text{ such that } R(T, \Psi, x) \text{ contains some } S \in \mathcal{P}(I) \text{ then it follows that}
\end{proof}\]

\[
S \models \"x \notin \bigcup_{n=0}^{\infty} J_n."\]
contradicting the fact that $S \subseteq T$. Hence by Lemma 2.3 it follows that $\rho_{R(T, \Psi, x)}(t)$ is defined for all $t \in R(T, \Psi, x) \cap B(T)$ and therefore

$$T_x = R(T, \Psi, x) \setminus \{ t \in R(T, \Psi, x) : (\exists t') \subseteq t) \rho_{R(T, \Psi, x)}(t) = 0 \}$$

is a small subtree for each $x \in [0, 1]$. Notice that “$\rho_{R(T, \Psi, x)}(t) = 0$” is a $\Sigma^1_1$ statement because each $I_n$ is $\Sigma^1_1$. Hence $\{ T_x : x \in [0, 1] \}$ is a $\Sigma^1_1$ set.

Since the relation $\prec$ defined on $\{ T_x : x \in [0, 1] \}$ is also $\Sigma^1_1$ by Lemma 2.3, it follows from the Kunen-Martin Theorem 3 and Lemma 2.4 that there is some $\delta, \epsilon$ such that $\Xi_{\delta, \epsilon}$ and Lemma 2.4 that there is some $\alpha \in \omega_1$ such that the rank of $T_x$ is less than $\alpha$ for each $x \in [0, 1]$. Use Lemma 2.4 to find $W \subseteq T$, a well founded full subtree of $T$, of standard rank $\alpha$ such that $W' \subseteq W$ is any large subtree then the rank of $W'$ is $\alpha$. Observe that $\Psi$ is an approximating function on $B(T) \cap W$ such that $H^*(\Psi(t)) < \theta$ for any $t \in W \cap B(T)$. It follows from Lemma 2.4 that there is some $x \in [0, 1]$ such that $R(W, \Psi, x)$ is a large subtree of $W$. It follows that the rank of $R(W, \Psi, x)$ is at least $\alpha$ and this is a contradiction because it implies that the rank of $T_x$ is at least $\alpha$.

The reasonable conjecture at this point is that the conclusion of Theorem 3.1 holds for the countable support iteration of the partial order $P(Z)$. A proof of this would require modifying the preservation technology of Judah-Shelah which was originally developed to show that certain iterations preserve that the ground model reals have positive Lebesgue measure.

4. The Relation $\Xi$

Sets with positive $r$-Hausdorff capacity may have measure zero but this type of set will play an important role in the following discussion. One would like to be able to say that if $\lambda(X) > 0$ then $H^*(X)$ can be calculated from $\lambda(X)$ or, at the very least, one would hope for some relationship between $H^*(X)$ and $\lambda(X)$. There are easy counterexamples to this though. Let $X$ be such that $H^*(X) = h > 0$ and $\lambda(X) = 0$ and then $\lambda(X \cup [0, a]) = a$ and note that there is obviously no connection between $H^*(X \cup [0, a])$ and $a$ when $a$ is much smaller than $h$. This sort of example is eliminated by introducing a relation on sets which, roughly speaking, calculates the infimum of $H^*(X \setminus Z)$ as $Z$ ranges over set of small measure. The $\Xi$ relation, which is introduced in the next definition, expands on this.

**Definition 4.1.** If $X$ and $Y$ are subsets of $[0, 1]$ then define the relation $\Xi_{\delta, \epsilon}(X, Y)$ to hold if and only if for every set $Z$, if $\lambda(Z) < \epsilon$ then $H^*(X \cap Y \setminus Z) > H^*(Y) - \delta$. If $X$ and $Y$ are subsets of $[0, 1]^{n+1}$ then define the relation $\Xi_{\delta, \epsilon}(X, Y)$ to hold if and only if

$$\Xi_{\delta, \epsilon}((x \in \pi_1(Y) : \Xi_{\delta, \epsilon}(X_x, Y_x)), \pi_1(Y)).$$

The relation $\Xi_{\delta, \epsilon}$ on $[0, 1]^n$ can be considered as a crude substitute for an integral when $n > 1$. In fact, one might be tempted to define a better approximation to an integral in the following way. Define $\Xi'(A) = \inf \{ H^*(A \setminus Z) : Z \subseteq [0, 1] \text{ and } \lambda(Z) < \epsilon \}$ for $A \subseteq [0, 1]$. If $A \subseteq [0, 1]^{n+1}$ then define

$$\Xi'(A) = \sup \{ \delta : \Xi'(\{ x \in [0, 1] : \Xi'(x_A) \geq \delta \}) \geq \delta \}$$

by induction on $n$. Notice however that the inequality $\Xi'(A) > \Xi'(B) - \delta$ is not equivalent to $\Xi_{\delta, \epsilon}(A, B)$ even if $A \subseteq B$. The point is that if $X$ and $Y$ are subsets of $[0, 1]^{n+1}$, and $x \in \pi_1(Y)$ then it is possible that $\Xi_{\delta, \epsilon}(X_x, Y_x)$ holds even though $x \notin \pi(X)$. This section collects some facts about the $\Xi$ relation.
Lemma 4.1. If $A$ and $B$ are subsets of $[0, 1]^n$ and $\Xi_{\delta, \epsilon}(A, B)$ holds then for any $Z \subseteq [0, 1]^n$ such that

$$\lambda(Z) < \left(\frac{\epsilon}{2}\right)^n$$

$\Xi_{\delta, \epsilon/2}(A \setminus Z, B)$ also holds.

Proof: This is easily proved using induction on $n$ and a simple application of Fubini’s Theorem.

The next two lemmas show that the relation $\Xi$ could have defined from the top down rather than from the bottom up.

Lemma 4.2. If $A$ and $B$ are subsets of $[0, 1]^{d+1}$ and $\Xi_{\delta, \epsilon}(\pi_d(A), \pi_d(B))$ holds and $\Xi_{\delta, \epsilon}(A_x, B_x)$ holds for each $x \in \pi_d(A)$ then $\Xi_{\delta, \epsilon}(A, B)$ holds as well.

Proof: Proceed by induction on $d$ noting that the case $d = 1$ follows from the definition.

The proof of the next three lemmas are easy and left to the reader.

Lemma 4.3. If $A$ and $B$ are subsets of $[0, 1]^{d+1}$ and $\Xi_{\delta, \epsilon}(A, B)$ holds then so does $\Xi_{\delta_{1}, \epsilon}(\{x \in \pi_d(B) : \Xi_{\delta_{1}, \epsilon}(A_x, B_x), \pi_d(B)) \}.$

Lemma 4.4. If $A$ and $B$ are closed subsets of $[0, 1]^{d+1}$, $\delta > 0$ and $\epsilon > 0$ then

$$\{x \in \Pi_d(B) : \Xi_{\delta, \epsilon}(A_x, B_x)\}$$

is a Borel set.

Lemma 4.5. If $\Xi_{\delta, \epsilon}(A, B)\}$ holds and $A \subseteq B$ then $\Xi_{\delta, \epsilon}(A, A)$ holds as well.

Lemma 4.6. If $\epsilon, \delta_1$ and $\delta_2$ are greater than 0, $\Xi_{\delta_{1, \epsilon}}(A, B)$ and $\Xi_{\delta_{2, \epsilon}}(B, C)$ both hold and $B \subseteq C$ then $\Xi_{\delta_{1}+\delta_{2, \epsilon}}(A \cap B, C)$ also holds.

Lemma 4.7. If $D \subseteq [0, 1]$ is a set such that $\Xi_{\delta, \epsilon}(D, D)$ holds and $\delta < H^r(D)$ then there is some $\bar{\epsilon} > 0$ such that $\Xi_{\delta-\bar{\epsilon}, \epsilon/2}(D, D)$ holds as well.

Proof: Let $\bar{\epsilon} < \min\{H^r(D) - \delta, r \frac{\epsilon}{1+r}, \epsilon\}$. Suppose that that $\lambda(Z) < \epsilon/2$ and that $\{I_i : i \in \omega\}$ is a cover of $D \setminus Z$. Since $\bar{\epsilon} < \theta$, by taking a tail of the sequence it is possible to find $i_0 \in \omega$ such that $\sum_{i=i_0+1}^{\infty} \lambda(I_i) < \bar{\epsilon}/2 < \sum_{i=i_0}^{\infty} \lambda(I_i)$ because, if $\sum_{i=0}^{\infty} \lambda(I_i) \leq \bar{\epsilon}/2 < \epsilon/2$ then $\lambda(Z \cup (\bigcup_{i \in \omega} I_i)) < \epsilon$ and so $Z \cup (\bigcup_{i \in \omega} I_i)$ cannot possibly be a cover of $D$. Let $J$ be an initial subinterval of $I_{i_0}$ such that $\sum_{i=i_0+1}^{\infty} \lambda(I_i) + \lambda(J) = \bar{\epsilon}/2$ and note that $\lambda(J) > 0$. It follows that $\lambda(Z \cup J \cup \bigcup_{i > i_0} I_i) < \epsilon$ and so $\lambda(I_{i_0} \setminus J)^r + \sum_{i \in i_0} \lambda(I_i)^r > \theta$ and hence, using the fact that $r < 1$,

$$\lambda(I_{i_0} \setminus J)^r + \lambda(J) + \sum_{i \neq i_0} \lambda(I_i)^r \geq \lambda(I_{i_0} \setminus J)^r + \lambda(J) + \sum_{i \in i_0} \lambda(I_i)^r + \sum_{i=i_0+1}^{\infty} \lambda(I_i) \geq \theta + \bar{\epsilon}/2$$

Now notice that

$$\sum_{i \in \omega} \lambda(I_i)^r = \lambda(I_{i_0} \setminus J)^r + \lambda(J) + \sum_{i \neq i_0} \lambda(I_i)^r - (\lambda(I_{i_0} \setminus J)^r + \lambda(J) - \lambda(I_{i_0} \setminus J)^r)$$
It will be shown by induction on $H$. Hence all that has to be shown is that $\lambda(I_{i_0} \setminus J)^r + \lambda(J) - \lambda(I_{i_0})^r \leq 0$.

To see this, define for $a > 0$

$$F_a(x) = (a - x)^r + x - a^r$$

and observe that $\frac{d}{dx}F_a(x) = 1 - \frac{r}{a - x}$ and notice that this is negative if $x < a < \bar{\varepsilon}$. Moreover, $F_a(0) = 0$ and so $F_a$ is negative on the interval $(0, a)$ if $a < \bar{\varepsilon}$. Because $0 < \lambda(J) < \lambda(I_{i_0}) < \bar{\varepsilon}$ it follows that $\lambda(I_{i_0} \setminus J)^r + \lambda(J) - \lambda(I_{i_0})^r = F_{\lambda(I_{i_0})}(\lambda(J)) \leq 0$.

**Definition 4.2.** A subset $X \subseteq [0, 1]$ will be said to be elementary if and only if

$$X = [p_0, q_0] \cup [p_1, q_1] \cup \ldots \cup [p_k, q_k]$$

where $p_i$ and $q_i$ are rational numbers such that $p_i < q_i < p_{i+1}$ for each $i \in k$. A subset $X \subseteq [0, 1]^{n+1}$ is elementary if and only if there is an elementary set $[p_0, q_0] \cup [p_1, q_1] \cup \ldots \cup [p_k, q_k] \subseteq [0, 1]$ such that

$$X = \bigcup_{i=0}^{k}[p_i, q_i] \times X_i$$

and each $X_i \subseteq [0, 1]^n$ is elementary.

**Lemma 4.8.** If $U \subseteq [0, 1]^n$ is open and $X \subseteq [0, 1]^n$ is closed then $\Xi_{\delta, \epsilon}(U, X)$ if and only if for every $\bar{\varepsilon} < \epsilon$ there is an elementary $Y \subseteq U$ such that $\Xi_{\delta, \epsilon}(Y, X)$.

**Proof:** To begin, suppose that for every $\bar{\varepsilon} < \epsilon$ there is an elementary $Y \subseteq U$ such that $\Xi_{\delta, \epsilon}(Y, X)$. Let $Y_k \subseteq U$ be such that $\Xi_{\delta, \epsilon}(Y_k, X)$ and let $Y = \cup_{k \geq 0}Y_k$. It will be shown by induction on $n$ that $\Xi_{\delta, \epsilon}(U, X)$.

In particular, it will be shown by induction on $n$ that if $Y_m \subseteq [0, 1]^n$ are sets such that

- each $Y_m$ is an elementary subset of $U$
- $\Xi_{\delta, \epsilon}(m)(Y_m, X)$
- $Y_m \subseteq Y_{m+1}$
- $\epsilon(m) \leq \epsilon(m + 1)$
- $\lim_{m \to \infty} \epsilon(m) = \epsilon$

then $\Xi_{\delta, \epsilon}(\cup_{m \in \omega}Y_m, X)$. To begin the induction note that the case $n = 1$ is easy. Now suppose that the assertion has been established for $n$ and that $X$ and $Y_m$ are subsets of $[0, 1]^{n+1}$ for $m \in \omega$. Suppose that $Z \subseteq [0, 1]$ is such that $\lambda(Z) < \epsilon$ and $A = \{x \in \pi_1(X) : \Xi_{\delta, \epsilon}(m)(U_x, X_x)\}$

$$H^r(\{x \in \pi_1(X) : \Xi_{\delta, \epsilon}(A \setminus Z) > H^r(\pi_1(X)) - \delta$$

and define

$$W_m = \{x \in \pi_1(X) : \Xi_{\delta, \epsilon}(m)((Y_m)_x, X_x)\}$$

for each $m \in \omega$. Note that if $\epsilon(m) > \lambda(Z)$ then $\lambda(W_n \setminus A) \geq \epsilon(m) - \lambda(Z)$ because, otherwise, $\lambda(Z \cup (W_n \setminus A)) < \epsilon(m)$ and so $H^r(W_m \setminus ((W_m \setminus A) \cup Z)) \lambda(Z) - \delta$ contradicting that $W_m \setminus ((W_m \setminus A) \cup Z) \subseteq A \setminus Z$. Let $j$ be such that $\epsilon(j) > \lambda(Z)$. Then $\{W_m \setminus A : m > j\}$ is a family of measurable sets — the measurability follows from Lemma 4.4 and the fact that each $W_m$ is elementary, and hence, closed — each of measure at least $\epsilon(j) - \lambda(Z) > 0$. Hence there is some $x \in [0, 1]$ such that there
Lemma 4.9. If $\bar{c}$ can be chosen to be a sufficiently small rational interval containing $[a, a + \lambda(Z)]$, then suppose that $\Xi_{\delta, \epsilon}(U, X)$ holds for each $x \in \pi(X)$, if $\Xi_{\delta, \epsilon}((U, X))$ then there is an elementary set $Y_x \subseteq U_x$ such that $Y_x \subseteq_{\delta, \epsilon} X_x$. Moreover, since $U$ is open it follows that if $Y_x \neq \emptyset$ then there is an open interval $J_x$ containing $x$ such that $J_x \times Y_x \subseteq U$. If $Y_x = \emptyset$ then, since $X$ is closed there is an interval $J_x$ containing $x$ such that $\Xi_{\delta, \epsilon}((\emptyset, X_x))$ holds for all $x \in J_x$. Since $\Xi_{\delta, \epsilon}(U, X)$ holds when $n = 1$ it is possible to find an elementary set $Z \subseteq \cup\{J_x : \Xi_{\delta, \epsilon}((U, X_x))\}$ such that $\Xi_{\delta, \epsilon}(Z, \pi_1(X))$. Since $Z$ can be covered by finitely many intervals $J_x$ it is possible to obtain $Z' = \cup_{i \in J} I_i$ such that

- $\lambda(Z \setminus Z') < (\epsilon - \bar{c})/2$
- $J_i \cap J_{i'} = \emptyset$ if $i \neq i'$
- for each $i \in J_i$ there is some $x(i)$ such that $J_i \subseteq J_x(i)$

It follows that $\Xi_{\delta, \epsilon}(Z', \pi_1(X))$ so let $Y = \cup_{i \in J} I_i \times Y_x(i)$. ■

Corollary 4.1. If $U$ is open and $X$ is closed then the relation $\Xi_{\epsilon, \delta}(U, X)$ is Borel.

Proof: It follows from Lemma 4.8 that $\Xi_{\epsilon, \delta}(U, X)$ holds if and only if $(\forall \bar{c} < \epsilon)(\exists Y)(Y$ is elementary and $\Xi_{\epsilon, \delta}(Y, X)$ and from Lemma 4.4 it follows that $\Xi_{\epsilon, \delta}(Y, X)$ is a Boreel statement because $Y$, being elementary, is closed. ■

Definition 4.3. If $X$ and $Y$ are subsets of $[0, 1]$ then define the relation $\Xi_{\delta, \epsilon}^X(X, Y)$ to hold if and only if for every elementary set $Z$, if $\lambda(Z) < \epsilon$ then $H^r(X \cap Y \setminus Z) > H^r(Y) - \delta$. If $X$ and $Y$ are subsets of $[0, 1]^{n+1}$ then define the relation $\Xi_{\delta, \epsilon}^n(X, Y)$ to hold if and only if

$$\Xi_{\delta, \epsilon}^n(\{x \in \pi_1(X) : \Xi_{\delta, \epsilon}^X((X_x, Y_x)), \pi_1(Y)).$$

Lemma 4.9. If $X$ and $Y$ are elementary subsets of $[0, 1]^n$ then $\Xi_{\delta, \epsilon}^X(X, Y)$ holds if and only if $\Xi_{\delta, \epsilon}^n(X, Y)$ holds.

Proof: One direction is clear. For the other, proceed by induction on $n$. If $n = 1$ then suppose that $\Xi_{\delta, \epsilon}^X(X, Y)$ holds and that $\lambda(Z) < \epsilon$ is such that $H^r(X \cap Y \setminus Z) \leq H^r(Y) - \delta$. It will be shown that there is an elementary set $Z'$ such that $\lambda(Z') < \epsilon$ and $H^r(X \setminus Z') \leq H^r(X \setminus Z)$. This clearly suffices.

The existence of $Z'$ will be established by induction on the number of connected components of $X \cap Y$. If $X \cap Y = [a, b]$ is an interval then $H^r(X \cap Y \setminus Z) \geq (b - a - \lambda(Z))^r$ because, if $\{I_i\}$ is a cover of $X \cap Y \setminus Z$ then $\sum_{i=0}^{\infty} \lambda(I_i) > b - a - \lambda(Z)$ and hence, since $r < 1$, $\sum_{i=0}^{\infty} \lambda(I_i)^r \geq (\sum_{i=0}^{\infty} \lambda(I_i))^r \geq (b - a - \lambda(Z))^r$. Hence $Z'$ can be chosen to be a sufficiently small rational interval containing $[a, a + \lambda(Z)]$.
because then $H^r(X \cap Y \setminus Z') \leq (b - a - \lambda(Z))^r \leq H^r(X \setminus Z)$. Now suppose that $X \cap Y = [a_0, b_0] \cup [a_1, b_1] \cup \cdots \cup [a_k, b_k]$ where $a_i < b_i < a_{i+1} < b_{i+1}$ for each $i \in k$. It is possible to choose open sets \{\(U_i^j\)\}_{i \in \omega} such that $X \cap Y \setminus Z \subseteq \cup_{i \in \omega} U_i^j$ and $\sum_{i=0}^{\infty} \lambda(U_i^j)^r < H^r(X \cap pY \setminus Z) + \frac{1}{j+1}$ for each $j \in \omega$. If $j \in \omega$ is such that $(b_0, a_1) \not\subseteq \cup_{i \in \omega} U_i^j$ then it may as well be assumed that $U_i^j \cap [a_0, b_0) \neq \emptyset$ and $U_i^j \cap [a_m, b_m] \neq \emptyset$ for some $m > 0$. Hence, if there are infinitely many $j \in \omega$ such that $(b_0, a_1) \not\subseteq \cup_{i \in \omega} U_i^j$ then it that follows that $H^r(X \setminus Z) = H^r([a_0, b_0] \setminus Z) + H^r([a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_k, b_k] \setminus Z)$ and the induction hypothesis can be used to find elementary $Z_0$ and $Z_1$ such that

\begin{itemize}
  \item $H^r([a_0, b_0] \setminus Z_0) \leq H^r([a_0, b_0] \setminus Z)$
  \item $H^r([a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_k, b_k] \setminus Z_1) \leq H^r([a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_k, b_k] \setminus Z)$
  \item $\lambda(Z_0) < \epsilon_0$
  \item $\lambda(Z_1) < \epsilon_1$
\end{itemize}

where $\epsilon_0$ and $\epsilon_1$ are chosen to be positive so that $\epsilon_0 + \epsilon_1 \leq \epsilon$ and $\lambda([a_0, b_0] \cap Z) < \epsilon_0$ and $\lambda(([a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_k, b_k]) \cap Z) < \epsilon_1$.

Hence it may be assumed that for all but finitely many $j \in \omega$ there is some $b(j) \in \omega$ such that $(b_0, a_1) \subseteq U_i^j(b(j)) = (x_j, y_j)$. By restricting attention to an infinite subsequence it may also be assumed that there is some interval $[x, y]$ such that $\lim_{j \to \infty} x_j = x$ and $\lim_{j \to \infty} y_j = y$. It follows that $H^r(X \cap Y \setminus Z) = H^r([a_0, x] \setminus Z) + (y - x)^r + H^r(([y, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_k, b_k] \setminus Z)$ and so the induction hypothesis can be used to find elementary $Z_0$, $Z_1$ and $Z_2$ such that

\begin{itemize}
  \item $H^r([a_0, x] \setminus Z_0) \leq H^r([a_0, x] \setminus Z)$
  \item $H^r([y, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_k, b_k] \setminus Z_1) \leq H^r([y, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_k, b_k] \setminus Z)$
  \item $\lambda(Z_0) < \epsilon_0$
  \item $\lambda(Z_1) < \epsilon_1$
  \item $\lambda[Z_2 = J_x \cup J_y$ and $x \in J_x$ and $y \in J_y$
\end{itemize}

where $\epsilon_0$, $\epsilon_1$ and $\epsilon_2$ are chosen to be positive so that $\epsilon_0 + \epsilon_1 + \epsilon_2 \leq \epsilon \lambda([a_0, x] \cap Z) < \epsilon_0$, $\lambda([y, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_k, b_k] \cap Z) < \epsilon_1$ and $\lambda([x, y] \cap Z) < \epsilon_2$. Let $Z = Z_0 \cup Z_1 \cup Z_2$.

Now suppose that the result has been established for $n$ and that $X$ and $Y$ are elementary subsets of $[0, 1]^n$. That is $\Xi_{\delta, \epsilon}(X, Y)$. In other words,

$$
\Xi_{\delta, \epsilon}^n(\{x \in \pi(X) : \Xi_{\delta, \epsilon}^n(X, \pi(Y)) \}, \pi(Y))
$$

holds and, using the induction hypothesis this yields that

$$
\Xi_{\delta, \epsilon}^n(\{x \in \pi(X) : \Xi_{\delta, \epsilon}^n(X, \pi(Y)) \}, \pi(Y))
$$

holds as well. To finish use the case $n = 1$. \[\blacksquare\]

**Lemma 4.10.** If $K \subseteq [0, 1]$ is closed then for all $\mu > 0$ there is a closed subset $K' \subseteq K$ such that $\lambda(K \setminus K') < \mu$ and for each $i \in \omega$ there is $\gamma > 0$ such that $\Xi_{\delta, \epsilon}^i(\gamma(K', K'))$.

**Proof:** First, the following weaker statement will be established: If $K \subseteq [0, 1]$ is closed then for all $\mu > 0$, $i \in \omega$ there is $\gamma > 0$ and a closed subset $K' \subseteq K$ such that $\lambda(K \setminus K') < \mu$ and $\Xi_{\delta, \epsilon}^i(\gamma(K', K'))$. To see this, suppose not and that $K, \mu$
and \(i \in \omega\) provided a counterexample. Choose inductively open sets \(A_m\) such that 
\[
\lambda(A_m) < \mu/2^{i+1}
\]
and
\[
H^r(K \setminus (\cup_{j \leq m} A_j)) \leq H^r(K \setminus (\cup_{j \in m} A_j)) - 1/i
\]
for each \(m \leq i + 2\). If it is not possible to do this for some \(m\) then it follows that 
\[
\Xi_{1+\mu/2^{i+1}}(K \setminus (\cup_{j \in m} A_j), K \setminus (\cup_{j \in m} A_j)) \text{ holds and } \lambda(\cup_{j \in m} A_j) < \mu.
\]
On the other hand, if the induction can be completed then the following inequalities hold:
\[
\begin{align*}
1 & \quad H^r(K) - \frac{1}{i+1} \leq H^r(K \setminus A_0) \\
2 & \quad H^r(K \setminus A_0) - \frac{1}{i+1} \geq H^r(K \setminus (A_0 \cup A_1))
\end{align*}
\]
and therefore 
\[
H^r(K) - \frac{1}{i+1} \leq H^r(K \setminus (\cup_{j \leq m} A_j))
\]
implies that \(H^r(K) \leq 1\).

Now choose inductively open sets \(U_i\) and numbers \(\bar{\gamma}(i) > 0\) such that:
\[
\begin{align*}
& 1. \quad \bar{\gamma}(0) = \mu \\
& 2. \quad \lambda(U_i) < \frac{\bar{\gamma}(i)}{2^{i+1}} \text{ for each } j \leq i \in \omega \\
& 3. \quad \Xi_{1+\bar{\gamma}(i)/2}(K \setminus (\cup_{j \leq i} U_j), K \setminus (\cup_{j \leq i} U_j)) \text{ holds.}
\end{align*}
\]
Now let \(K' = K \setminus (\cup_{j \in \omega} U_j)\) and note that \(\lambda(K \setminus K') < \mu\). Moreover \(\Xi_{1+\bar{\gamma}(i)/2}(K', K')\) holds for each \(i \in \omega\) because \(\lambda((\cup_{j=m+i} U_m) \setminus (\cup_{j \in \omega} U_j)) \geq \frac{\bar{\gamma}(i)/2}{2^{i+1}}\) and hence, \(\lambda((K \setminus (\cup_{j \leq i} U_j)) \setminus K') < \frac{\bar{\gamma}(i)}{2^{i+1}}\).

5. The Definition of the Ideals Associated with a Capacity

This section contains the definition of the ideals which will be used to construct the partial orders satisfying \(\text{KP}(r)\). Most of the technical concepts have already been introduced but a few more are needed.

**Definition 5.1.** A sequence \(\{X_i : i \in \omega\}\) of subsets of \([0,1]^d\) will be said to be a normal family if
\[
\begin{align*}
& 1. \quad \text{each } X_i \text{ is elementary} \\
& 2. \quad X_{i+1} \subseteq X_i \\
& 3. \quad \lambda(\pi_n(X_i) \setminus \pi_n(X_{i+1})) < \frac{\lambda(\pi_n(X_i))}{2^{i+2}} \text{ for } n \leq d \\
& 4. \quad \text{for each } i \in \omega \text{ there is } \beta(i) > 0 \text{ such that } \Xi_{1+\beta(i)}(X_j, X_j) \text{ holds for all } j \geq i.
\end{align*}
\]
The family \(\{X_i : i \in \omega\}\) will be said to be of dimension \(d\). The function \(\beta\) will be called a witness to the normality of the family \(\{X_i : i \in \omega\}\).

Observe that the intersection of any normal family has positive measure. In fact, if \(\{X_i : i \in \omega\}\) is a normal family and \(X = \cap_{i \in \omega} X_i\) then
\[
\lambda(\pi_n(X)) > \frac{(2^{i+1} - 1)\lambda(\pi_n(X_i))}{2^{i+1}}
\]
for any \(i \in \omega\) and \(n \leq d\).

**Definition 5.2.** Let \(W_n\) be the family of all sets \(a = \bigcup_{i=0}^k I_i \subseteq [0,1]\) where each \(I_i\) is a rational interval and \(\sum_{i=0}^k \lambda(I_i) < 2^{-n}\).
Lemma 5.1. Let 0 < a < \frac{1}{2^{1/r+2}}. If A \subseteq [0, a] and B \subseteq [1 - a, 1] then H^r(A \cup B) = H^r(A) + H^r(B).

Proof: Noting that the hypothesis on a implies that 0 < (1 - 2a)^r - 2a^r it is possible to choose \epsilon > 0 such that \epsilon < (1 - 2a)^r - 2a^r. Since H^r(A \cup B) \leq 2a^r it follows that if A \cup B \subseteq \bigcup_{i \in \omega} I_i and \sum_{i \in \omega} \lambda(I_i)^r < H^r(A \cup B) + \epsilon then none of the intervals I_i contains [a, 1 - a]. Since a < 1/2 it may as well be assumed that none of the intervals contains 1/2, or, in other words, that \{i \in \omega : I_i \cap A \neq \emptyset\} is disjoint from \{i \in \omega : I_i \cap B \neq \emptyset\}. Hence H^r(A \cup B) \geq H^r(A) + H^r(B). The result follows since H^r(A \cup B) \leq H^r(A) + H^r(B) is true in general. 

Lemma 5.2. Each set T^r_n is closed under finite unions.

Proof: Let X(f, \{C_i\}_{i \in \omega}, \mu) and X(g, \{D_i\}_{i \in \omega}, \rho) be any two generators for T^r_n of dimension d_1 and d_2 respectively. Let d be greater than d_1 and d_2 and define \tilde{C}_i = C_i \times [0, 1]^{d-d_1} and \tilde{D}_i = D_i \times [0, 1]^{d-d_2}. Define \tilde{f}(x_1, x_2, \ldots, x_d) = f(x_1, x_2, \ldots, x_{d_1}) and \tilde{g}(x_1, x_2, \ldots, x_d) = g(x_1, x_2, \ldots, x_{d_2}).

Next, let 0 < a < \frac{1}{2^{1/r+2}}. Define \psi_1 : [0, 1] \to [0, a] by \psi_1(x) = ax and define \psi_2 : [0, 1] \to [1 - a, 1] by \psi_2(x) = 1 - ax. Let \varphi_i : [0, 1]^d \to [0, 1]^d be defined by \varphi_i(x_1, x_2, \ldots, x_d) = (\psi_1(x_1), x_2, \ldots, x_d) for i \in \{1, 2\}. Let B_i = \varphi_1(C_i) \cup \varphi_2(D_i).

To see that \{B_i\}_{i \in \omega} is a normal family it must only be observed that if \beta_1 : \omega \to (0, 1) witnesses that \{C_i\}_{i \in \omega} is normal and \beta_2 : \omega \to (0, 1) witnesses that \{D_i\}_{i \in \omega} is normal then the function \beta : \omega \to (0, 1) defined by \beta(i) = \min\{a\beta_1(i), a\beta_2(i)\} witnesses the normality of \{B_i\}_{i \in \omega}. This uses Lemma 5.1. Finally, let h be any continuous extension of (f \circ \varphi_1^{-1}) \cup (g \circ \varphi_2^{-1}) and let \delta = \min\{a^{1/r} \mu, a^{1/r} \rho\}. Clearly X(h, \{B_i\}_{i \in \omega}, \delta) \in T^r_n.

It will be shown that X(f, \{C_i\}_{i \in \omega}, \mu) is a subset of X(h, \{B_i\}_{i \in \omega}, \delta), the proof for X(g, \{D_i\}_{i \in \omega}, \rho) being similar. Let b \in X(f, \{C_i\}_{i \in \omega}, \mu). This means that for every \epsilon > 0 there are infinitely many i \in \omega such that \Xi_{\mu, \epsilon}((f^{-1}b), C_i) fails to hold. Let \epsilon and i be fixed such that \Xi_{\mu, \epsilon}((f^{-1}b), C_i) fails. Unraveling the definition of \Xi_{\mu, \epsilon} reveals that

H^r(\{x \in \pi_1(C_i) : \Xi_{\mu, \epsilon}((f^{-1}b)_x, (C_i)_x) \} \setminus Z) \leq H^r(\pi_1(C_i)) - \mu

for some set Z such that \lambda(Z) < \epsilon. From the definition of \varphi_1 and \tilde{f} it follows that

H^r(\{x \in \pi_1(\tilde{C}_i) : \Xi_{\mu, \epsilon}((h^{-1}b)_{\psi_1(x)}, (\tilde{B}_i)_{\psi_1(x)}) \} \setminus Z) \leq H^r(\pi_1(\tilde{C}_i)) - \mu

and so, observing that H^r(\psi_1(A)) = a^{1/r} H^r(A) for any A \subseteq [0, 1], it follows that

H^r(\{x \in \pi_1(\psi_1(\tilde{C}_i)) : \Xi_{\mu, \epsilon}((h^{-1}b)_{\psi_1(x)}, (\tilde{B}_i)_{\psi_1(x)}) \} \setminus Z') = H^r(\{\psi_1(x) : x \in \pi_1(\tilde{C}_i) \cap \Xi_{\mu, \epsilon}((h^{-1}b)_{\psi_1(x)}, (\tilde{B}_i)_{\psi_1(x)}) \} \setminus Z')
\[ \leq a'(H'(\pi_1(\bar{C}_i))) - \mu = H'(\pi_1(\bar{f}_1(\bar{C}_i))) - \delta \]

where \( Z' \) is the image of \( Z \) under \( \psi_1 \). Notice that \( \lambda(Z') = a\lambda(Z) < \lambda(Z) < \epsilon. \)

The next thing to notice is that \( \{x \in \pi_1(\varphi_1(\bar{C}_i)) : \Xi_{\mu,\epsilon}((h^{-1}b)_x, (B_i)_x)\} \supseteq \{x \in \pi_1(\varphi_1(\bar{C}_i)) : \Xi_{\delta,\epsilon}((h^{-1}b)_x, (B_i)_x)\} \)

because \( \delta < \mu \). From Lemma 5.1 it follows that \( H'(\pi_1(B_i)) = H'(\pi_1(\varphi_1(\bar{C}_i)) + H'(\pi_1(\varphi_2(\bar{D}_i))) \).

Therefore,
\[ H'(\{x \in \pi_1(B_i) : \Xi_{\delta,\epsilon}((h^{-1}b)_x, (B_i)_x)\} \setminus Z') \leq H'(\{x \in \pi_1(\varphi_1(\bar{C}_i)) : \Xi_{\delta,\epsilon}((h^{-1}b)_x, (\bar{C}_i)_x)\}) + H'(\pi_1(\varphi_2(\bar{D}_i))) \leq H'(\pi_1(\varphi_1(\bar{C}_i))) - \delta + \ H'(\pi_1(\varphi_2(\bar{D}_i))) = H'(\pi_1(B_i)) - \delta \]

or, in other word \( \Xi_{\delta,\epsilon}(h^{-1}b, B_i) \) fails provided that \( \Xi_{\mu,\epsilon}(f^{-1}b, C_i) \) fails. Since for every \( \epsilon > 0 \) there are infinitely many \( i \in \omega \) such that \( \Xi_{\mu,\epsilon}(f^{-1}b, C_i) \) it follows that \( b \in X(h, \{B_i\}_{i \in \omega}, \delta) \).

Lemma 5.3. If the parameters \( f, C \) and \( \delta \) are given then the statement \( \{a \in X(f,C,\delta) \} \) is arithmetic.

Proof: Let \( C = \{C_i\}_{i \in \omega} \) be a normal family of dimension \( n \). From Definition 5.3 it follows that \( a \in X(f,C,\delta) \) if and only if for every \( \epsilon > 0 \) there are infinitely many \( i \in \omega \) such that \( \Xi_{\delta,\epsilon}(f^{-1}a, C_i) \) does not hold. Since \( a \) is open and \( f \) is continuous it follows from Lemma 4.8 that \( a \in X(f,C,\delta) \) if and only if
\[ (\forall \epsilon > 0)(\forall m \in \omega)(\exists i > m)(\exists \bar{x} < \epsilon)(\forall Y)(Y \text{ is elementary and} \]
\[ Y \subseteq f^{-1}a \Rightarrow \neg \Xi_{\delta,\epsilon}(Y, C_i) \]

and \( \Xi_{\delta,\epsilon}(Y, C_i) \) is equivalent to \( \Xi_{\delta,\epsilon}(Y, C_i) \) when \( Y \) and \( C_i \) are elementary. From Lemma 4.9.

Hence it suffices to show that the statement \( \Xi_{\delta,\epsilon}(Y, C_i) \) is arithmetic. Proceed by induction on \( n \). Notice that the statements \( \lambda(Z) < \alpha \) and \( H'(Z) > \alpha \) are arithmetic for elementary sets \( Z \). The case \( n = 1 \) follows immediately and the induction is carried through because of the elementarity of \( Y \) and \( C_i \).

Lemma 5.4. If the parameters \( \beta \) and \( C \) are given then the statement \( \{\beta \text{ witnesses the normality of } C \} \) is arithmetic.

Proof: This follows from Lemma 5.9 and the definition of a normal family because it has already been observed in the proof of Lemma 5.3 that the statement \( \Xi_{\delta,\epsilon}(Y, C_i) \) is arithmetic.

Corollary 5.1. The ideals \( \mathcal{I}_n \) are all \( \Sigma^1_1 \) ideals.

Proof: From Definition 5.3 it follows that \( Y \in \mathcal{I}_n \) if and only if there are \( \delta > 0, m \geq 1 \), a normal family \( C \) of dimension \( m \) and a continuous function \( f : [0,1]^m \rightarrow [0,1] \) such that \( Y \subseteq X(f, C, \delta) \). Now apply Lemma 5.3 noting that the existence of a normal family can be expressed with a \( \Sigma^1_1 \) statement.
Lemma 5.5. If \( A \subseteq B \subseteq [0,1]^d \) and \( X \subseteq B \) are such that

- \( \lambda(\pi_n(B) \setminus \pi_n(A)) < \left(\frac{\epsilon}{d+1}\right)^n \) for each \( n \leq d \)
- \( \Xi_{\delta,x}(X,B) \)

then \( \Xi_{\delta,x}(X,A) \).

Proof: Proceed by induction in \( d \). If \( d = 1 \) and \( \lambda(Z) < \epsilon/2 \) then \( \lambda((B \setminus A) \cup Z) < \epsilon \). Hence \( \chi^{(X \cap A \setminus ((B \setminus A) \cup Z))} > \chi^{(B \setminus A)} - \delta \). Since \( X \cap A \setminus ((B \setminus A) \cup Z) = X \cap A \setminus Z \) this suffices.

Suppose the lemma is true for \( d \) and that \( A \subseteq B \subseteq [0,1]^{d+1} \). Let

\[ S_n = \{ x \in [0,1] : \lambda((\pi_n(B_x) \setminus \pi_n(A_x)) \geq \left(\frac{\epsilon}{d+2}\right)^n \} \]

for each \( n \leq d \). Since \( \lambda((\pi_{n+1}(B) \setminus \pi_{n+1}(A)) < \left(\frac{\epsilon}{d+2}\right)^{n+1} \) it follows that

\[ \lambda(S_n) < \epsilon/(d+2) \]

for each \( n \leq d \). If \( Z \) is such that \( \lambda(Z) < \epsilon/(d+2) \) define \( Y(Z) = Z \cup (\bigcup_{n=1}^d S_n) \cup (\pi_1(B) \setminus \pi_1(A)) \) and note that that \( \lambda(Y(Z)) < \epsilon \). Hence

\[ \chi^{(X \setminus X_x, B_x)} \setminus \chi(Y(Z)) > \chi^{(\pi_1(B))} - \delta \]

and, moreover, if \( \Xi_{\delta,x}(X_x, B_x) \) holds and \( x \notin Y(Z) \) then \( A_x, B_x \) and \( X_x \) satisfy the hypothesis of the lemma for \( d \) and, furthermore, \( x \in \pi_1(A) \). Therefore

\[ \chi^{(X \setminus X_x, A_x)} \setminus \chi(Z) > \chi^{(\pi_1(B))} - \delta > \chi^{(\pi_1(A))} - \delta \]

and this implies that

\[ \chi^{(X \setminus X_x, A_x)} \setminus \chi(Z) > \chi^{(\pi_1(A))} - \delta \]

Since \( Z \) was arbitrary, this means that \( \Xi_{\delta,x}(X,A) \) holds.

Corollary 5.2. If \( \{C_i\}_{i \in \omega} \) is a normal family of dimension \( d \) then the following are equivalent:

1. There is \( \epsilon > 0 \) such that \( \Xi_{\delta,x}(X,C_i) \) holds for all but finitely many \( i \in \omega \).
2. There is \( \epsilon > 0 \) such that \( \Xi_{\delta,x}(X,C_i) \) holds for infinitely many \( i \in \omega \).
3. There is \( \epsilon > 0 \) such that \( \Xi_{\delta,x}(X,C_i) \) holds for some \( i \in \omega \) such that

\[ \lambda(\pi_n(C_i) \setminus \pi_n(\cap_{j \in \omega} C_j)) < \left(\frac{\epsilon}{d+1}\right)^n \]

for each \( n \leq d \).

Proof: To get that (3) implies (1) use Lemma 5.5 noting that if \( j > i \) then \( \lambda(\pi_n(C_i) \setminus \pi_n(C_j)) < \left(\frac{\epsilon}{d+1}\right)^n \) for each \( n \leq d \) and so, \( \Xi_{\delta,x}(X,C_j) \) holds.

Lemma 5.6. Each of the ideals \( I_n^r \) of Definition 5.3 satisfies KP(r).

Proof: Suppose not. Then there is some \( n \in \omega \) such that the ideal \( I_n^r \) does not satisfy KP(r). This means that there exist

- \( \theta > 0 \)
- \( X \in I^+ \)
- a function \( F \) from \( X \) to the Borel subsets of \([0,1]\) satisfying that \( \chi^{(F(x))} \leq \theta \)
- \( \epsilon > 0 \)
such that for every $Y \subseteq [0, 1]$ and $Z \subseteq [0, 1]$ such that

- $H^r(Y) \leq \theta$
- $\lambda(Z) < \epsilon$

it must be the case that \( \{ a \in X : y \notin F(a) \} \in \mathcal{I} \) for some $y \in [0, 1] \setminus (Y \cup Z)$. Using Definition 5.3, it is possible to rephrase this as follows: For every $Y \subseteq [0, 1]$ and $Z \subseteq [0, 1]$ such that $H^r(Y) \leq \theta$ and $\lambda(Z) < \epsilon$ it must be that there is some $y \in [0, 1] \setminus (Y \cup Z)$ and there are $\delta > 0$, $m \geq 1$, a normal family $\mathcal{C}$ of dimension $m$ and a continuous function $f : [0, 1]^m \to [0, 1]$ such that \( \{ a \in X : y \notin F(a) \} \subseteq X(f, \mathcal{C}, \delta) \).

Let $\mathcal{E}_m$ be the set of all elementary subsets of $[0, 1]^m$ considered to have the discrete topology. It follows that $\prod_{\mathcal{E}_m} \mathcal{E}_m$ is homeomorphic to the irrationals. Let $\mathcal{N}_m$ be the subspace of $\prod_{\mathcal{E}_m} \mathcal{E}_m$ consisting of all $\xi$ such that $\{ \xi(n) \}_{n \in \omega}$ is a normal family and observe that, because it is a closed subspace of $\prod_{\omega} \mathcal{E}_m$, $\mathcal{N}_m$ is a Polish space. Let $C([0, 1]^m)$ be the space of continuous functions from $[0, 1]^m$ to $[0, 1]$ with the metric induced by the supremum norm. Let

$$\mathcal{P}_m = C([0, 1]^m) \times \mathcal{N}_m \times (0, 1) \times (0, 1)$$

and let $\mathcal{P} = \cup_{m \in \omega} \mathcal{P}_m$ and note that $\mathcal{P}$ is a Polish space. Let $\Omega$ be the set of all $(z, g, \xi, \delta, \beta) \in [0, 1] \times \mathcal{P}$ such that $\{ a \in X : z \notin F(a) \} \subseteq X(g, \{ \xi(n) \}_{n \in \omega}, \delta)$ and the normality of $\{ \xi(n) \}_{n \in \omega}$ is witnessed by $\beta$. Because $X$ and $F$ can be coded by reals, the definition of $\Omega$ together with Lemma 5.3 and Lemma 5.4 immediately establish that $\Omega$ is a Borel subset of the Polish space $[0, 1] \times \mathcal{P}$.

It is therefore possible to appeal to the von Neumann Selection Theorem to find a measurable $\Phi : [0, 1] \to \mathcal{P}$ such that the domain of $\Phi$ is the same as $\pi_1(\Omega)$ and $\Phi \subseteq \Omega$. If $x$ is in the domain of $\Phi$ suppose that $\Phi(x) = (g, \xi, \delta, \beta)$ and define $d(x)$ to be the dimension of $X(g, \{ \xi(n) \}_{n \in \omega}, \delta)$. Then define $\Phi^i_n(x) = \pi_n(\xi(i))$ for each $i \in \omega$ and define $\Phi^i_n(x) = \pi_n(\gamma_{i \in \omega}(\xi(i)))$ — if $n > d(x)$ then $\pi_n(\xi(i)) = \xi(i)$. Since $\lim_{n \to \infty} \lambda(\Phi^i_n(x)) = \lambda(\Phi^i_n(x))$ for each $x$ in the domain of $\Phi$ and $n \in \omega$, it is possible to apply Egorov’s theorem countably many times to find a compact set $K$ — which is the intersection of a nested sequence of closed sets obtained from the countably many applications of Egorov’s theorem — such that

- $\Phi \upharpoonright K$ is continuous
- $\Phi^i_n \upharpoonright K$ is continuous for each $\alpha \in \omega + 1$
- $\{ \lambda(\Phi^i_n(x)) \}_{i \in \omega}$ converges uniformly, with respect to the variable $x$, to $\lambda(\Phi^i_n(x))$ on $K$
- $\lambda(K) > \lambda(\pi(\Omega)) - \epsilon/4$.

Observe that if $Z$ is such that $\lambda(Z) < \epsilon/2$ then $H^r(K \setminus Z) > \theta$ because otherwise, it is possible to obtain a contradiction by setting $Y = K \setminus Z$ in the definition of $\text{KP}(r)$. Now use Lemma 4.10 to find a closed $K \subseteq K$ such that $\lambda(K \setminus K) < \epsilon/4$ and there exists $\gamma : \omega \to (0, 1)$ such that $\sum_{i \in \omega} \gamma(i)(K, K)$ holds for all $i \in \omega$.

Next, the compactness of $K$ implies that there is $m \in \omega$ such that $d(x) \in m$ for each $x \in K$. Furthermore there is $\delta > 0$ such that for every $x \in K$, if $\Phi(x) = (g, \xi, \delta', \beta)$ then $\delta' > \delta$. Since $H^r(K \setminus Z) > \theta$ for each $Z \subseteq [0, 1]$ such that $\lambda(Z) < \epsilon/4$ it follows that, by shrinking $\delta$ if necessary, it may be assumed that $H^r(K) > \theta + \delta$. Yet another application of compactness yields a function $\beta : \omega \to (0, 1)$ such that for each $x \in K$, if $\Phi(x) = (g, \xi, \delta, \beta_x)$ then $\beta_x(i) \geq \beta(i)$ for each $i \in \omega$. 
Let \( \tau_n = \int_K \lambda(\Phi_i^n(x))dx \) for \( n \leq m \). Since \( \{\Phi_i^n(x)\}_{i \in \omega} \) is a normal family for each \( x \) in the domain of \( \Phi \) it follows from the remarks following Definition 5.1 that

\[
\lambda(\Phi_i^n(x)) < \frac{2^{i+1}\lambda(\Phi_i^n(x))}{2^{i+1} - 1}
\]

for each \( i \in \omega \) and \( n \leq m \). Therefore,

\[
\int_K \lambda(\Phi_i^n(x))dx < \frac{2^{i+1}\tau_n}{2^{i+1} - 1}
\]

and so it is possible to choose an open set \( L_i \) such that \( K \subseteq L_i \) and

\[
\lambda(L_i \setminus K) + \int_K \lambda(\Phi_i^n(x))dx < \frac{2^{i+1}\tau_n}{2^{i+1} - 1}
\]

for each \( n \leq m \) and \( H^r(L_i) < H^r(K) + \frac{1}{2^i} \) and \( \lambda(L_i \setminus K) < \frac{2^{i+1}}{2^i} \) for each \( i \in \omega \).

Next, using the continuity of \( \Phi \) on \( K \), choose a family \( \{N_i\}_{i \in \omega} \) such that

- \( N_i = [p_i, q_i] \cup \ldots \cup [p_{ki(i)}, q_{ki(i)}] \) is elementary for each \( i \)
- \( K \cap [p_i, q_i] \neq \emptyset \) for each \( i \in \omega \) and \( j \leq k(i) \)
- \( K \subseteq N_i \subseteq L_{i+1} \)
- \( N_i+1 \subseteq N_i \)
- \( \Phi_j^n(y) = \Phi_i^n(y) \) if \( j \leq i \) and \( x \) and \( y \) belong to \( K \) and the same component of \( N_i \)

Let \( C_i = \bigcup_{j=0}^{k(i)+2} [p_{ji+2}, q_{ji+2}] \times \Phi_j^{m+2}(z) \times [0,1]^{m-\delta(x)} \) for \( i \in \omega \) where \( z \) is chosen arbitrarily from \( [p_{ji+2}, q_{ji+2}] \cap K \) Then, let \( C = \cap_{i \in \omega} C_i \). Observe that \( \lambda(\pi_n(C)) = \tau_n \) for \( n \leq m \).

Hence, in order to show that \( \mathcal{C} = \{C_i\}_{i \in \omega} \) is a normal family, first observe that if \( j \geq i \geq 1 \) and \( \lambda(Z) < \frac{\lambda(\pi_n(C))}{2^i} \) then

\[
H^r(N_j \setminus Z) \geq H^r(K \setminus Z) \geq H^r(K) - \frac{1}{2^i} \geq \frac{1}{2^i} - \frac{1}{2^i}
\]

and the last expression is at least as large as \( H^r(N_j) - \frac{1}{2^i} \). Hence \( \Xi_{\frac{1}{2^i}, \gamma(2i-1)}(N_j, N_j) \) holds for all \( j \geq i \geq 1 \). Now let \( \beta^*(i) = \min\{\gamma(2(i+2) - 1), \beta(i + 2)\} \). Then \( \Xi_{\frac{1}{2^i}, \beta^*(i)}(N_j, N_j) \) holds for all \( j \geq i \geq 1 \) and so does \( \Xi_{\frac{1}{2^i}, \beta^*(i)}(\Phi_j^{m+2}(z), \Phi_j^{m+2}(z)) \) because \( \beta^*(i) \leq \beta(iu + 2) \leq \beta_z(iu + 2) \) for any \( z \in K \cap N_j+2 \). Therefore \( \Xi_{\frac{1}{2^i}, \beta^*(i)}(C_j, C_j) \) holds for all \( j \geq i \). Hence, in order to show that \( \mathcal{C} \) is a normal family it suffices to show that \( \lambda(\pi_n(C_i) \setminus \pi_n(C_{i+1})) < \frac{\lambda(\pi_n(C_i))}{2^{i+2}} \) for each \( n \leq m \). To see this, notice that

\[
\lambda(\pi_n(C_i) \setminus \pi_n(C_{i+1})) = \int_{N_{i+2}} \lambda(\Phi_i^{m+2}(x))dx - \int_{N_{i+1}} \lambda(\Phi_i^{m+3}(x))dx
\]

\[
\leq \int_{N_{i+2}} \lambda(\Phi_i^{m+2}(x))dx - \int_K \lambda(\Phi_i^n(x))dx
\]

\[
\leq \lambda((L_{i+2} \setminus K)) + \int_K \lambda(\Phi_i^{m+2}(x))dx - \tau_n \leq \frac{2^{i+3}\tau_n}{2^{i+3} - 1} - \tau_n = \frac{\tau_n}{2^{i+3} - 1} \leq \frac{\lambda(\pi_n(C_i))}{2^{i+2}}
\]

for each \( i \in \omega \).

Now let \( f' : C \to [0,1] \) be defined by \( f'(x,y) = g(y) \) if \( \Phi(x) = (g, \xi, \mu, \zeta) \) and extend \( f' \) to a continuous function \( f : [0,1]^m \to [0,1] \) arbitrarily. Since \( X \notin \mathcal{I}_m \) there must be some \( a \in X \) such that \( a \notin X(f, \mathcal{C}, \delta) \). This means that there is some \( \epsilon' > 0 \) such that \( \Xi_{\delta, \epsilon'}(f^{-1}a, C_i) \) holds for all but finitely many \( i \in \omega \). In particular,

\[
H^r(\{x \in N_i : \Xi_{\delta, \epsilon'}((f^{-1}a)_x, (C_i)_x)\} \setminus Z) > H^r(N_i) - \delta > H^r(K) - \delta > \theta
\]
holds for all but finitely many \( i \in \omega \) and any \( Z \) such that \( \lambda(Z) < \epsilon' \). It may, without loss of generality, be assumed that \( \epsilon' \leq \epsilon/2 \).

Using the uniform convergence of \( \{\lambda(\Phi^m(x))\}_{i \in \omega} \) it is possible to find \( j \in \omega \) such that \( \lambda(\Phi^m(x)) \leq (\epsilon'/m + 1)^n \) for all \( x \in K, n \leq m \) and \( i > j \). Let \( i > j \) be such that \( \lambda(N_i \setminus K) < \epsilon' \). Since \( H^r(F(a)) \leq \theta \) and

\[
H^r(\{x \in N_i : \Xi_{\delta',\varepsilon}(m^{-1}a_g, (C_i)_m) \setminus (N_i \setminus K) > \theta \}
\]

it is possible to choose \( y \in K \setminus F(a) \) such that \( \Xi_{\delta',\varepsilon}(m^{-1}a_g, (C_i)_m) \setminus (N_i \setminus K) > \theta \). Therefore \( a \in X(g, \{\xi(n)\}_{n \in \omega}, \delta') \). This yields a contradiction to the fact that \( y \notin F(a) \) and \( \Phi(y) = (g, \xi, \delta', \beta') \) implies that \( a \in X(g, \{\xi(n)\}_{n \in \omega}, \delta') \).

6. The Ideal is Proper

It remains to be shown that the ideals \( T_n^i \) are proper. This will require a careful analysis of the capacity \( H^r \). This will require some generalizations of results from [6]. The key fact about Hausdorff capacity that will be used is that if \( B \subseteq E \) is of small Lebesgue measure but evenly distributed throughout \( E \), then \( H^r(B) \) will be close to \( H^r(E) \). This is made precise in the next lemma whose statement requires the following notation.

Definition 6.1. For any measurable set \( A \subseteq [0,1] \) define \( \Delta_m^i(A) \) to be the least real number such that \( \lambda(A \cap [0, \Delta_m^i(A)]) = \frac{\lambda(A)}{m} \).

Notice that \( \Delta_m^i(A) \) is always defined and that if \( A = [0,1] \) then \( \Delta_m^i(A) \) is nothing more than \( \frac{1}{m} \).

Lemma 6.1. Let \( \delta > 0, \eta > 0 \) and suppose that \( E \subseteq [0,1] \) is measurable. If \( \Xi_{\delta,\eta}(E, E) \) holds and \( \delta < H^r(E) \) then there exists \( m \in \omega \) such that \( \forall E \subseteq [0,1] \) any measurable set such that for each \( i \in m \)

\[
\lambda(D \cap [\Delta_m^i(E), \Delta_m^{i+1}(E)]) \geq \frac{\eta}{m}
\]

then \( \Xi_{\delta, \frac{\eta}{8m}}(D, E) \).

Proof: Let \( m \in \omega \) be so large that the inequality

\[
\frac{m^{1-r}\eta^{1+r}}{8 \cdot 2^r} > 1
\]

is satisfied. To begin, note that Lemma [4.7] implies that there exists \( \bar{\varepsilon} > 0 \) such that \( \Xi_{\delta - \varepsilon, \eta/2}(E, E) \) holds. If \( \Xi_{\delta, \frac{\eta}{8m}}(D, E) \) fails then there is some \( Z \) such that \( \lambda(Z) < \frac{\eta}{8m} \) and an open cover \( D \cup Z \subseteq \bigcup_{i=0}^{\infty} I_i \) such that \( \sum_{i=0}^{\infty} \lambda(I_i)^r < H^r(E) - (\delta - \varepsilon) \). Let \( B = \{i \in \omega : \lambda(I_i) \geq \frac{1}{2m}\} \) and let \( C = \{i \in m : (\forall j \in B)(I_j \cap \Delta_m^i(E), \Delta_m^{i+1}(E)) \cap E = \emptyset\} \). Three separate cases, depending on the size of \( B \) and \( C \), will be considered.

Case 1 To begin, Suppose that \( |B| \geq \frac{mn}{8} \). Then

\[
\sum_{i=1}^{\infty} \lambda(I_i)^r \geq \sum_{i \in B} \lambda(I_i)^r \geq |B|\left(1/2m\right)^r \geq \frac{m^{1-r} \mu}{8 \cdot 2^r} > 1
\]
Since \( \sum_{i=0}^{\infty} \lambda(I_i)^r < H^r(E) - (\delta - \eta/2) < 1 \) this is impossible.

**Case 2** Suppose now that \( |B| < \frac{mn}{8} \) and \( |C| \leq \frac{mn}{4} \). It then follows that if

\[
G = \{ i \in m : [\Delta_i^m(E), \Delta_i^{m+1}(E)] \cap E \not\subseteq \bigcup_{j=1}^{\infty} I_j \}
\]

then \( |G| \leq 2 \cdot |B| + |C| \). The reason for this is that if \( j \in B \) then there are at most two integers \( i \) such that the intervals \([\Delta_i^m(E), \Delta_i^{m+1}(E)]\) intersect \( I_j \) but are not contained in \( I_j \) — this accounts for the summand \( 2 \cdot |B| \). All the other intervals \([\Delta_i^m(E), \Delta_i^{m+1}(E)]\) for \( i \in G \) must be disjoint from \( I_j \) for every \( j \in B \) — this accounts for the other summand \(|C|\).

By the assumptions of this case it follows that \( 2 \cdot |B| + |C| < mn/2 \) and hence

\[
\lambda(\bigcup_{i \in G} [\Delta_i^m(E), \Delta_i^{m+1}(E)] \cap E) < \eta/2
\]

Since \( \Xi_{\delta - \epsilon, \eta/2}(E, E) \) holds it may be concluded that \( H^r(E) \setminus \bigcup_{i \in G} [\Delta_i^m(E), \Delta_i^{m+1}(E)]) \geq H^r(E) - (\delta - \epsilon) \). Since \( E \setminus (\bigcup_{i \in G} [\Delta_i^m(E), \Delta_i^{m+1}(E)]) \subseteq \bigcup_{i=1}^{\infty} I_i \) this yields a contradiction.

**Case 3** Suppose that \( |B| < \frac{mn}{8} \) and \( |C| > \frac{mn}{4} \). Let \( C' \) be a family of non-consecutive members of \( C \) of maximal cardinality — hence, \( |C'| \geq |C|/2 > \frac{mn}{8} \). Let

\[
U_j = \{ i \in \omega : I_i \cap [\Delta_j^m(E), \Delta_j^{m+1}(E)] \cap E \neq \emptyset \}
\]

for each \( j \in C' \) and define \( U = \bigcup_{j \in C'} U_j \). Since, for \( j \in C \), the sets \([\Delta_j^m(E), \Delta_j^{m+1}(E)]\cap E\) are intersected only by intervals \( I_i \) where \( i \in \omega \setminus B \), and such intervals \( I_i \) are smaller than any \([\Delta_j^{i+1}(E), \Delta_j^i(E)]\), it follows that \( U_j \cap U_k = \emptyset \) if \( k \) and \( j \) are distinct members of \( C' \). Therefore, using the fact that \( 0 < r < 1 \),

\[
\sum_{i \in U} \lambda(I_i)^r \geq \sum_{j \in C'} \sum_{i \in U_j} \lambda(I_i)^r \geq \sum_{j \in C'} \left( \sum_{i \in U_j} \lambda(I_i) \right)^r \geq \sum_{j \in C'} \lambda(\bigcap_{i \in U_j} [\Delta_j^m(E), \Delta_j^{m+1}(E)])^r \geq \left( \mu/m - \lambda(Z) \right)^r \geq \frac{m\mu}{8} - \frac{\mu}{2m} > 1
\]

and once again, as in the first case, this is a contradiction because \( D \subseteq [0, 1] \). ■

If \( X \subseteq [0, 1] \) then \( F : X \to [0, 1] \) will be said to have small fibres if and only if \( \lambda(F^{-1}\{x\}) = 0 \) for each \( x \in [0, 1] \). The proof of the Theorem, [1] and the lemmas preceding it will rely on decomposing an arbitrary continuous function into a piece that has small fibres and a piece which has countable range.

**Lemma 6.2.** Let \( \mu \in (0, 1) \) and suppose that \( \{X_i : i \in \omega\} \) is a sequence of mutually independent \( \{0, 1\} \)-valued random variables with mean \( \mu \) for each \( i \in \omega \). Suppose that \( C \subseteq [0, 1] \) is a measurable set and that for each \( j \in \mathbb{N} \) the function \( F_j : C \to [0, 1] \) is measurable with small fibres. For any \( \rho > 0 \) there is \( M \in \omega \) such that for all \( m \geq M \) the probability that

\[
\lambda \left( \bigcap_{j \in \mathbb{N}} \bigcup_{i \in m} F_j^{-1}\left( \frac{i + X_i}{m} \right) \right) > \frac{\mu^n \lambda(C)}{2
\]

is greater than 1 - \rho.

**Proof:** To begin, let \( m \in \omega \) be fixed. For any function \( \xi \in ^n m \) define \( \theta(\xi) = \lambda(\cap_{j\in n} F_j^{-1}[\xi(j), \xi(j) + 1]) \) and let

\[
Y(\xi) = \lambda \left( \bigcap_{j\in n} F_j^{-1}[\frac{\xi(j)}{m}, \frac{\xi(j) + X_{\xi(j)}}{m}] \right) = \theta(\xi) \prod_{j\in n} X_{\xi(j)}
\]

If \( \xi \neq \xi' \) then

\[
\lambda \left( \left( \bigcap_{j\in n} F_j^{-1}[\frac{\xi(j)}{m}, \frac{\xi(j) + 1}{m}] \right) \cap \left( \bigcap_{j\in n} F_j^{-1}[\frac{\xi(j)}{m}, \frac{\xi(j) + 1}{m}] \right) \right) = 0
\]

and so \( \sum_{\xi \in ^n m} \theta(\xi) = \lambda(C) \).

Letting \( E[Z] \) denote the average value of the random variable \( Z \) and \( V[Z] \) the variance of \( Z \), it is easy to see that \( E[Y(\xi)] = \theta(\xi) \mu^{\sigma(\xi)} \) where \( \sigma(\xi) \) represents the cardinality of the range of \( \xi \). Noting that \( \sigma(\xi) \leq n \) for all \( \sigma \), it follows that \( E[S_{\xi \in ^n m} Y(\xi)] \geq \mu^n \lambda(C) \). Furthermore,

\[
V[\sum_{\xi \in ^n m} Y(\xi)] = E[(\sum_{\xi \in ^n m} Y(\xi) - E[Y(\xi)])^2] = \sum_{\xi \in ^n m} \sum_{\xi' \in ^n m} E[Y(\xi) - E[Y(\xi)]](Y(\xi') - E[Y(\xi')])].
\]

If \( \xi \) and \( \xi' \) have disjoint ranges then \( Y(\xi) \) and \( Y(\xi') \) are independent random variables and so

\[
E[(Y(\xi) - E[Y(\xi)])(Y(\xi') - E[Y(\xi')])] = E[Y(\xi) - E[Y(\xi)]][E[Y(\xi') - E[Y(\xi')]]] = 0
\]

while if the ranges of \( \xi \) and \( \xi' \) are not disjoint then

\[
E[(Y(\xi) - E[Y(\xi)])(Y(\xi') - E[Y(\xi')])] = E[Y(\xi)Y(\xi')] - E[E[Y(\xi)]Y(\xi')] - E[E[Y(\xi')]Y(\xi)] + E[Y(\xi)]E[Y(\xi')]
\]

\[
= E[Y(\xi)Y(\xi')] \leq E[E[Y(\xi)]E[Y(\xi')]] \leq E[Y(\xi)Y(\xi')]
\]

\[
= E[\theta(\xi) \prod_{j\in n} X_{\xi(j)}\theta(\xi') \prod_{j\in n} X_{\xi'(j)}] \leq \theta(\xi)\theta(\xi')
\]

since \( X_i \in \{0, 1\} \) for each \( i \). It may be concluded that

\[
V[\sum_{\xi \in ^n m} Y(\xi)] \leq \sum_{j\in n} \sum_{\xi \in ^n m} \sum_{\xi' \in ^n m} \theta(\xi)\theta(\xi') = \sum_{j\in n} \sum_{\xi \in ^n m} \theta(\xi) \sum_{\xi' \in ^n m} \theta(\xi')
\]

However, if \( j \) is fixed then \( \sum_{\xi \in ^n m} \theta(\xi') = \lambda(F_j^{-1}[\xi(j), \xi(j) + 1]) \). Therefore, all that needs to be done is to choose \( M \) so large that if \( m \geq M \) and \( i \in m \) then \( \lambda(F_j^{-1}[\xi(j), \xi(j) + 1]) < \frac{\rho \mu^{2n} \lambda(C)}{4m} \) for each \( j \in n \). The reason this suffices is that this implies that

\[
V[\sum_{\xi \in ^n m} Y(\xi)] \leq \sum_{j\in n} \sum_{\xi \in ^n m} \theta(\xi) \frac{\rho \mu^{2n} \lambda(C)}{4n} \leq \frac{\rho \mu^{2n} \lambda(C)^2}{4}
\]
and so Chebyshev’s Inequality can be applied to conclude that the probability that

\[
\left| \sum_{\xi \in \mathbb{N}} Y(\xi) - E\left[ \sum_{\xi \in \mathbb{N}} Y(\xi) \right] \right| > \frac{\mu \lambda(C)}{2}
\]

is less than \(\rho\). Since it has already been established that \(E[\sum_{\xi \in \mathbb{N}} Y(\xi)] \geq \mu \lambda(C)\) it follows that the probability that \(\sum_{\xi \in \mathbb{N}} Y(\xi) \geq \mu \lambda(C)/2\) is at least \(1 - \rho\) as required.

To choose \(M\) so large that if \(m \geq M\) and \(i \in m\) then \(\lambda(F_j^{-1}\left[\frac{i}{m}, \frac{i+1}{m}\right]) < \frac{\rho \mu \lambda(C)}{4n}\) for each \(j \in n\), all that is required is compactness and the fact that each \(F_j\) has small fibres. Since \(F_j^{-1}\{x\} = \cap_{k \in \omega} F_j^{-1}[x - 1/k, x + 1/k]\) and \(\lambda(F_j^{-1}\{x\}) = 0\) it follows that it is possible to choose a finite cover of \([0, 1]\) by open intervals, \(\mathcal{C}\), such that if \(I \in \mathcal{C}\) then \(\lambda(F_j^{-1}(I)) < \frac{\rho \mu \lambda(C)}{4n}\) for each \(j \in n\). Hence \(M\) must be chosen so large that if \(m \geq M\) and \(i \in m\) then \(I \in \mathcal{C}\) such that \(\left[\frac{i}{m}, \frac{i+1}{m}\right] \subseteq I\).

**Lemma 6.3.** Suppose that \(\delta > 0\), \(\mu > 0\), \(\eta > 0\) and \(k \in \omega\). There is then a real number \(\epsilon(\delta, \mu, \eta, k) > 0\) such that if

- \(\{C_i\}_{i \in k}\) is a family of measurable subsets of \([0, 1]\)
- \(F_i : C_i \to [0, 1]\) is a measurable functions with small fibres for each \(i \in k\)
- \(E \subseteq [0, 1]\) is a measurable set
- \(\Xi_{\delta, \eta}(E, E)\) holds and \(\delta < H^*(E)\)
- \(\rho > 0\)

then there is \(M \in \omega\) such that for all \(m > M\) and for any mutually independent, \([0, 1]\)-valued random variables \(\{X_i\}_{i \in m}\) with mean \(\mu\), the probability that

\[
\Xi_{\delta, \epsilon(\delta, \mu, \eta, k)}\left(\bigcap_{i \in k} (F_i^{-1}\left[\frac{j}{m}, \frac{j+1}{m}\right]) \cup ([0, 1] \setminus C_i), E)\right)
\]

holds is greater than \(1 - \rho\).

Moreover, there is \(\theta > 0\) such that if

- \(E' \subseteq [0, 1]\) is a measurable set such that \(\lambda(E \Delta E') < \theta\)
- \(\{C_i\}_{i \in k}\) is a family of measurable sets such that \(\lambda(C_i \Delta C_i') < \theta\) for each \(i \in k\)
- \(\{G_i\}_{i \in k}\) is a family of measurable functions such that

\[
\sup\{|F_i(x) - F_i'(x)| : x \in C \cap C'\} < \theta
\]

for each \(i \in k\)

then the probability that

\[
\Xi_{\delta, \epsilon(\delta, \mu, \eta, k)}\left(\bigcap_{i \in k} (G_i^{-1}\left[\frac{j}{m}, \frac{j+1}{m}\right]) \cup ([0, 1] \setminus C_i), E'\right)
\]

holds is still greater than \(1 - \rho\).

**Proof:** Let \(\alpha = \mu^k/2\) and use Lemma 3.1 to find \(p\) such that if \(D \subseteq E\) is a measurable set such that for each \(i \in p\)

\[
\lambda(E \cap [\Delta_p^i(E), \Delta_p^{i+1}(E)]) \geq \frac{\alpha}{2p}
\]

then \(\Xi_{\delta, \epsilon(\delta, \mu, \eta, k)}(D, E)\) holds. Let \(\epsilon(\delta, \mu, \eta, k) = \frac{\alpha}{4p}\). Let \(\{F_i : i \in s\}\) enumerate the sets of positive measure which belong to the coarsest partition of \(E\) refining each of the
partitions \( [\Delta^i_p(E), \Delta^{i+1}_p(E)] \cap E : i \in p \) and \( \{ C_i \cap E, E \setminus C_i \} \) for \( i \in k \). Now use Lemma 6.2 to find \( M \in \omega \) such that for all \( m \geq M \) the probability that
\[
\lambda \left( \bigcap_{j \in k} \left( (F_j \upharpoonright P_n)^{-1} \bigcup_{i \in m} \left[ \frac{i}{m}, \frac{i + X_i}{m} \right] \cup (P_n \setminus C_j) \right) \right) > \alpha \lambda(P_n)
\]
is greater than \( 1 - \frac{e}{s} \) for each \( n \in s \).

Now notice that if \( m \geq M \) is fixed then, because each \( F_i \) has small fibres, it is possible to find \( p(i, j) \) and \( q(i, j) \) such that \( \frac{1}{m} < p(i, j) < q(i, j) < \frac{i + 1}{m} \) and
\[
\lambda(F_j^{-1}[\frac{i}{m}, \frac{i + 1}{m}]) - \lambda(F_j^{-1}[p(i, j), q(i, j)]) < \frac{\alpha \lambda(P_n)}{2(2k + 1)m}
\]
for each \( j \in k \) and \( i \in m \). Now observe that if \( \lambda(C_i \Delta C'_i) < \frac{\alpha \lambda(P_n)}{2(2k + 1)} \) for each \( i \in k \) and if \( \lambda(E \Delta E') < \frac{\alpha \lambda(P_n)}{2(2k + 1)} \) and if \( Y_i \in \{0, 1\} \) are such that
\[
\lambda \left( \bigcap_{j \in k} \left( (F_j \upharpoonright P_n)^{-1} \bigcup_{i \in m} \left[ \frac{i}{m}, \frac{i + Y_i}{m} \right] \cup (P_n \setminus C_j) \right) \right) > \alpha \lambda(P_n)
\]
then the Lebesgue measure of
\[
\bigcap_{j \in k} \left( (G_j \upharpoonright P_n)^{-1} \bigcup_{i \in m} \left[ \frac{i}{m}, \frac{i + Y_i}{m} \right] \cup (P_n \setminus C_j) \right)
\]
is greater than
\[
\frac{\alpha \lambda(P_n)}{2}.
\]

Therefore, if \( \theta > 0 \) is such that
\begin{itemize}
  \item \( \theta < \frac{\alpha \lambda(P_n)}{2(2k + 1)} \) for each \( n \in s \)
  \item \( \theta < p(i, j) - \frac{1}{m} \) for all \( i \) and \( j \)
  \item \( \theta < \frac{1}{m} - q(i, j) \) for all \( i \) and \( j \)
\end{itemize}
then if \( \{ G_i \}_{i \in k} \) is a family of measurable functions such that \( \sup \{ |F_i(x) - F_i'(x)| : x \in C \cap C' \} < \theta \) for each \( i \in k \) then \( F_j^{-1}[\frac{i}{m}, \frac{i + 1}{m}] \leq G_j^{-1}[p(i, j), q(i, j)] \) and hence
\[
\lambda \left( \bigcap_{j \in k} \left( (G_j \upharpoonright P_n)^{-1} \bigcup_{i \in m} \left[ \frac{i}{m}, \frac{i + Y_i}{m} \right] \cup (P_n \setminus C_j) \right) \right) > \frac{\alpha \lambda(P_n)}{2}
\]
also holds for each \( n \in s \).

It follows that the Lebesgue measure of the intersection of the interval \( [\Delta^i_p(E'), \Delta^{i+1}_p(E')] \) with
\[
\bigcap_{j \in k} \left( (G_j \upharpoonright P_n)^{-1} \bigcup_{i \in m} \left[ \frac{i}{m}, \frac{i + Y_i}{m} \right] \cup ([\Delta^i_p(E'), \Delta^{i+1}_p(E')] \setminus C_j) \right)
\]
is greater than
\[
\frac{\alpha \lambda([\Delta^i_p(E'), \Delta^{i+1}_p(E')] \cap E')}{2}
\]
and the result now follows from Lemma 6.1.
Lemma 6.4. Let $k \in \omega$ and $\{C_i\}_{i \in k}$ be a family of measurable subsets of $[0, 1]$. Let $F_i : C_i \to [0, 1]$ be a measurable function for each $i \in k$. Suppose also that $\delta > 0$ and $\eta > 0$. Then, for any $N \in \omega$ and $\epsilon > 0$, if $\Xi_{\delta, \epsilon}(E, E)$ holds for some measurable set $E$ such that $H^r(E) > \delta$ then

$$\Xi_{\delta, \epsilon}(\bigcap_{i \in k} (F_i^{-1}a) \cup ([0, 1] \setminus C_i), E)$$

holds for some $a \in W_N$.

Proof: For each $i \in k$ let $\{y^i_j : j \in d_i \leq \omega\}$ enumerate all points $y \in [0, 1]$ such that $\lambda(F_i^{-1}\{y\}) > 0$. Let $C'_i = C_i \setminus F_i^{-1}\{y^i_j : j \in d_i\}$ and let $F'_i = F_i \upharpoonright C'_i$. Since $F'_i$ has small fibres for each $i \in k$ it follows from Lemma 6.3 that it is possible to choose $m$ so large that if $\{X_i\}_{i \in m}$ are $\{0, 1\}$-valued random variables with mean $2^{-N-1}$ then, letting $\epsilon' = \epsilon(\delta, 2^{-N-1}, \eta, k)$, the probability that

$$\Xi_{\delta, \epsilon'}(\bigcap_{i \in k} ((F'_i)^{-1}a) \cup ([0, 1] \setminus C'_i), E)$$

holds is at least $3/4$ for any measurable set $E$ such that $\Xi_{\delta, \epsilon}(E, E)$ holds $H^r(E) > \delta$. Since the mean of each $X_i$ is $2^{-N-1}$ it is possible to choose $m$ so large that the probability that

$$\lambda(\cup_{j \in m} \left\{ \left( \frac{j}{m}, \frac{j + X_j}{m} \right) \right\}) < 2^{-N}$$

is also greater than $3/4$. Hence, given $E$ with the required properties, there is $a_0 \in W_N$ such that

$$\Xi_{\delta, \epsilon'}(\bigcap_{i \in k} ((F'_i)^{-1}a_0) \cup ([0, 1] \setminus C'_i), E)$$

holds. Now choose $J \in \omega$ such that $\lambda(\bigcup_{i \in k} \bigcup_{j \geq J} F_i^{-1}\{y^i_j\}) < \epsilon'/2$. It is then easy to find $a \in W_N$ be such that $a \cup \{y^i_j : i \in k, j \in J\} \subseteq a$. Let $\epsilon = \epsilon'/2$ and note that it follows that

$$\Xi_{\delta, \epsilon}(\bigcap_{i \in k} ((F_i)^{-1}a) \cup ([0, 1] \setminus C_i), E)$$

holds because, if

$$Y = \bigcap_{i \in k} ((F_i)^{-1}a_0) \cup ([0, 1] \setminus C_i) \setminus \bigcap_{i \in k} ((F_i)^{-1}a) \cup ([0, 1] \setminus C_i)$$

then $Y \subseteq \bigcup_{i \in k} \bigcup_{j \geq J} F_i^{-1}\{y^i_j\}$ and hence $\lambda(Y) < \epsilon'/2$.

Lemma 6.5. Suppose that $k \in \omega$ and $\{C_i\}_{i \in k}$ are measurable subsets of $[0, 1]^{d+1}$ and $F_i : C_i \to [0, 1]$ are measurable functions such that $(F_i)_x$ has small fibres for each $x \in [0, 1]^d$. Let $N \in \omega$, $\delta > 0$ and $\eta > 0$. Then there is $\epsilon > 0$ such that for all closed $E \subseteq [0, 1]^{d+1}$ and $\rho > 0$ there is some $a \in W_n$ such that the Lebesgue measure of

$$\{ x \in \pi_d(E) : \Xi_{\delta, \epsilon}(\bigcap_{i \in k} ((F_i^{-1}a) \cup ([0, 1]^{d+1}\setminus C_i))_x, E_x) \text{ or } \Xi_{\delta, \eta}(E_x, E_x) \text{ or } H^r(E_x) \leq \delta \}$$

is at least $\lambda(\pi_d(E))(1 - \rho)$. 
Proof: Let \( \{X_i\}_{i \in \omega} \) be a sequence of mutually independent random variables with mean \( 2^{-N-1} \). Let \( \epsilon = \epsilon(\delta, 2^{-N-1}, \eta, k) \) and suppose that \( E \subseteq [0,1]^{d+1} \) is closed. Next, choose compact subsets \( W_i \subseteq \pi_d(C_i) \) and \( V_i \subseteq [0,1]^d \setminus W_i \) for \( i \in k \) as well as \( E' \subseteq \pi_d(E) \) such that

- \( \lambda(\pi_d(E) \setminus E') < \frac{\rho \lambda(\pi_d(E))}{6(1-\rho/2)} \)
- \( \lambda(\{0,1\}^d \setminus (\cap_{i \in k} (W_i \cup V_i))) < \frac{\rho \lambda(\pi_d(E))}{6(1-\rho/2)} \)
- \( F_i \cap W_i \) is continuous
- the mapping from \( \pi_d(E') \) to \([0,1]\) defined by \( x \mapsto \lambda(E_x \cap (p,q)) \) is continuous for each pair of rationals \( p \) and \( q \) such that \( 0 \leq p < q \leq 1 \)
- the mapping from \( \pi_d(W_i) \) to \([0,1]\) defined by \( x \mapsto \lambda((C_i)_x \cap (p,q)) \) is continuous for each \( i \in k \) and each pair of rationals \( p \) and \( q \) such that \( 0 \leq p < q \leq 1 \).

An easy application of the Lebesgue Density Theorem shows that a consequence of the last clause is that if \( x \in \pi_d(W_i) \) then \( \lim_{y \to x} \lambda(W_y \Delta W_x) = 0 \). The penultimate clause implies a similar assertion for \( E' \). It is possible to find compact \( E_1 \) and \( E_2 \), subsets of \( E' \) such that

- if \( x \in E_1 \) then \( \Xi_{\delta,\eta}(E_x, E_x) \) holds and \( H''(E_x) > \delta \)
- if \( x \in E_2 \) then \( \Xi_{\delta,\eta}(E_x, E_x) \) fails or \( H''(E_x) \leq \delta \)
- \( \lambda(E' \setminus (E_1 \cup E_2)) < \frac{\rho \lambda(\pi_d(E))}{6(1-\rho/2)} \)

because Lemma 4.3 implies that \( \{ x \in E' : \Xi_{\delta,\eta}(E_x, E_x) \} \) is measurable. Let \( Z = E_1 \cap (\cap_{i \in k} (W_i \cup V_i)) \) and for \( x \in Z \) let \( K(x) = \{ i \in k : x \in \pi_d(W_i) \} \) and notice that \( K(x) \) is constant on a neighbourhood of \( x \) because the sets \( V_i \) and \( W_i \) are all compact. If \( x \in Z \) and \( i \in K(x) \) then \( F_i \cap (W_i)_x \) has small fibres, \( H''(E_x) > \delta \) and \( \Xi_{\delta,\eta}(E_x, E_x) \) holds, so it follows from Lemma 4.3 that there is \( \theta_x > 0 \) and \( M_x \in \omega \) such that if \( ||y-x|| < \theta_x \) and \( M \geq M_x \) then the probability that

\[
\Xi_{\delta,\epsilon} \left( \bigcap_{i \in K(y)} (F_i^{-1} \cup \{\frac{1}{m} \cdot \frac{j + X_j}{m} \}) \cup ([0,1] \setminus C_i) \right) , E_y \)
\]

holds is greater than \( 1 - \rho^2/2 \).

Since \( Z \) is compact, it is possible to find a single \( M \) such that for all \( m > M \) and for any \( x \in Z \) the probability that

\[
\Xi_{\delta,\epsilon} \left( \bigcap_{i \in K(x)} (F_i^{-1} \cup \{\frac{1}{m} \cdot \frac{j + X_j}{m} \}) \cup ([0,1] \setminus C_i) \right) , E_x \)
\]

holds is greater than \( 1 - \rho^2/2 \).

Now let \( m > M \) be so great that the probability that \( \lambda(\cup_{j \in m} [\frac{j}{m}, \frac{j + X_j}{m}] \cup ([0,1] \setminus C_i)] \) is greater than \( 2^{-N} \) is greater than \( 2\rho \). Define

\[
\Gamma(X_0, X_1, \ldots, X_m)
\]

to be the Lebesgue measure of the set of all \( x \in Z \) such that

\[
\Xi_{\delta,\epsilon} \left( \bigcap_{i \in K(x)} (F_i^{-1} \cup \{\frac{1}{m} \cdot \frac{j + X_j}{m} \}) \cup ([0,1] \setminus C_i) \right) , E_x \)
\]
holds. Note that Corollary 4.1 implies that this set is measurable. The first step is to estimate

$$\alpha_m = \sum_{X_0=0}^{1} \sum_{X_1=0}^{1} \cdots \sum_{X_m=0}^{1} \Gamma(X_0, X_1, \ldots, X_m) \prod_{i=0}^{m} \mu^{X_i}(1 - \mu)^{1-X_i}$$

the average value of $\Gamma(X_0, X_1, \ldots, X_m)$. To this end, let

$$\Lambda_x(X_0, X_1, \ldots, X_m) = \{0, 1\}$$

be defined to be 1 if and only if

$$\Xi_{\delta, x} \left( \bigcap_{i \in K(x)} (F_i^{-1} \cup \left[ \frac{j}{m}, \frac{j+X_i}{m} \right]) \cup ([0, 1] \setminus C_i) \right)_{x, E_x}$$

holds, and observe that $\alpha_m$ is equal to

$$\frac{1}{m} \sum_{X_0=0}^{1} \sum_{X_1=0}^{1} \cdots \sum_{X_m=0}^{1} \left( \frac{1}{m} \sum_{j=0}^{m} \Lambda_x(X_0, X_1, \ldots, X_m) dx \right) \prod_{i=0}^{m} \mu^{X_i}(1 - \mu)^{1-X_i} =$$

$$\int_{x \in Z} \left( \frac{1}{m} \sum_{X_0=0}^{1} \sum_{X_1=0}^{1} \cdots \sum_{X_m=0}^{1} \Lambda_x(X_0, X_1, \ldots, X_m) \prod_{i=0}^{m} \mu^{X_i}(1 - \mu)^{1-X_i} \right) dx$$

However, notice that

$$\frac{1}{m} \sum_{X_0=0}^{1} \sum_{X_1=0}^{1} \cdots \sum_{X_m=0}^{1} \Lambda_x(X_0, X_1, \ldots, X_m) \prod_{i=0}^{m} \mu^{X_i}(1 - \mu)^{1-X_i}$$

is just the probability that

$$\Xi_{\delta, x} \left( \bigcap_{i \in K(x)} (F_i^{-1} \cup \left[ \frac{j}{m}, \frac{j+X_i}{m} \right]) \cup ([0, 1] \setminus C_i) \right)_{x, E_x}$$

holds and the choice of $m$ and the fact that $x \in Z$ guarantee that this probability is greater than $1 - \rho^2/2$. Hence $\alpha_m \geq \lambda(Z)(1 - \rho^2)$.

Now let $p$ be the probability that $\Gamma(X_0, X_1, \ldots, X_m) \geq (1-\rho/2)\lambda(Z)$. Obviously, $p\lambda(Z) + (1-p)(1-\epsilon/2)\lambda(Z) \geq \alpha_m \geq (1-\rho^2)\lambda(Z)$. Solving for $p$ yields that $p \geq 1 - 2\rho$. Since $m$ was chosen so large that the probability that $\lambda(\cup_{j \in \mathbb{N} \setminus \frac{2}{m}, \frac{j+X_i}{m}}) < 2^{-N}$ is greater than $2\rho$, there is at least one $a \in W_N$ such that $\lambda(U) > (1 - \rho/2)\lambda(Z)$ where $U$ is the set of all $x \in Z$ such that

$$\Xi_{\delta, x}(\delta, \mu, \eta, k) \left( \bigcap_{i \in K(x)} (F_i^{-1} a) \cup ([0, 1] \setminus C_i) \right)_{x, E_x}$$

holds. Obviously

$$\lambda(U \cup E_2) = \lambda(U)\lambda(E_2) \geq (1 - \rho/2)\lambda(Z) + \lambda(E_2)$$

$$\geq (1 - \rho/2)(\lambda(E_1) - \frac{\rho\lambda(\pi_d(E))}{6(1 - \rho/2)}) + \lambda(E_2)$$

$$\geq (1 - \rho/2)(\lambda(E_1) + \lambda(E_2)) - (1 - \rho/2)\frac{\rho\lambda(\pi_d(E))}{6(1 - \rho/2)} \geq$$

$$\geq (1 - \rho/2)(\lambda(\pi_d(E))) - 2\frac{\rho\lambda(\pi_d(E))}{6(1 - \rho/2)} - (1 - \rho/2)\frac{\rho\lambda(\pi_d(E))}{6(1 - \rho/2)} \geq (1 - \rho)(\lambda(\pi_d(E)))$$
as required.

Theorem 6.1. Suppose that \( \{F_i\}_{i \in k} \) are continuous functions from \([0, 1]^d\) to \([0, 1]\), \(\eta > 0\), \(\delta > 0\), \(N \in \omega\), and \(\{A_i\}_{i \in k}\) are measurable subsets of \([0, 1]^d\). Then there is \(\epsilon > 0\) such that for each closed subset \(E \subseteq [0, 1]^d\), if \(\Xi_{\delta, \eta}(E, \epsilon)\) holds then

\[
\Xi_{d \delta, \epsilon}(\bigcap_{i \in k} ((A_i \cap F_i^{-1} \alpha) \cup ([0, 1]^d \setminus A_i)), E)
\]

also holds for some elementary set \(a \in W_N\).

Proof: Proceed by induction on \(d\) noting that if \(d = 1\) then this follows directly from Lemma 6.4. So assume that the lemma has been established for \(d\) and that \(\{F_i\}_{i \in k}\) are continuous functions from \([0, 1]^{d+1}\) to \([0, 1]\), \(\eta > 0\), \(\delta > 0\), \(N \in \omega\), and \(\{A_i\}_{i \in k}\) are measurable subsets of \([0, 1]^{d+1}\). Let \(B_i = \{(x, y) \in [0, 1]^d \times [0, 1] : \lambda(F_i^{-1}\{y\})_x > 0\}\) and note that

\[
B_i = \{(x, y) \in [0, 1]^d \times [0, 1] : (\exists K \text{ compact})(\lambda(K) > 0 \text{ and } K \subseteq (F_i^{-1}\{y\})_x)\}
\]

and, because \(F\) is continuous, the relation \(K \subseteq (F_i^{-1}\{y\})_x\) is Borel. Moreover, so is the statement \(\lambda(K) > 0\) and so the set \(B\) is \(\sum_1^1\) and hence, measurable. Let \(B_i^*\) be the inverse image of \(B_i\) under the mapping \((x, y) \mapsto (x, F_i(x, y))\) or, in other words, \((x, y) \in B_i^*\) if and only if \(\lambda((F_i^{-1}\{F_i(x, y)\})_x) > 0\). Since \(B_i^*\) is clearly measurable, it follows that so is \(C_i = [0, 1]^{d+1} \setminus B_i^*\).

Now, for each \(i \in k\), let \(\{f_{ij} : j \in I_i\}\) enumerate a maximal collection of functions such that

- \(f_{ij} : C_i^j \rightarrow [0, 1]\) where \(C_i^j \subseteq [0, 1]^d\) is compact
- \(f_{ij}\) is continuous
- \(f_{ij} \subseteq B_i\)
- if \(x \in C_i^j \cap C_i^{j'}\) then \(f_{ij}(x) \neq f_{ij'}(x)\)
- \(\int_{C_i^j} \lambda((F_i^{-1}\{f_{ij}(x)\})_x)dx > 0\).

The first thing to notice is that, for each \(i \in k\), such a family must be countable and therefore, \(I_i \leq \omega\) without loss of generality. To see this let \(E_i^j = \{(x, y) \in [0, 1]^d \times [0, 1] : F_i(x, y) = f_{ij}(x)\}\). If \(j \neq j'\) then \(E_i^j \cap E_i^{j'} = \emptyset\) and, moreover,

\[
\lambda(E_i^j) = \int_{C_i^j} \lambda((F_i^{-1}\{f_{ij}(x)\})_x)dx > 0
\]

for any \(j \in I_i\). Hence the family of sets \(E_i^j\) is countable for each \(i \in k\).

Next, it must be shown that

\[
\sum_{j \in I_i} \int_{C_i^j} \lambda((F_i)^{-1}\{f_{ij}(x)\})_x dx = \lambda(B_i^*)
\]

so suppose not. Then it must be that \(\lambda(B_i^* \setminus \bigcup_{j \in I_i} E_i^j) > 0\). Since each \(f_{ij}\) is continuous and \(I_i \leq \omega\), it follows that \(B_i \setminus (\bigcup_{j \in I_i} f_{ij})\) is \(\sum_1^1\). Hence it is possible to use the von Neumann selection theorem to find a function \(f\) such that the domain of \(f\) is \(\pi_d(B_i \setminus (\bigcup_{j \in I_i} f_{ij}))\) and \(f\) is measurable. Since \(\pi_d(B_i \setminus (\bigcup_{j \in I_i} f_{ij}))\) is also equal
to \( \pi_d(B_i^+ \setminus (\cup_{j \in d} E_j^i)) \) it must be that \( \lambda(\pi_d(B_i \setminus (\cup_{j \in I_i} f_j^i))) > 0 \). Hence

\[
\int_{\pi_d(B_i \setminus (\cup_{j \in I_i} f_j^i))} \lambda((F_i^{-1}(f(x)))_x)dx > 0
\]

because \( \lambda(F_i^{-1}(f(x))) > 0 \) for each \( x \in \pi_d(B_i \setminus (\cup_{j \in I_i} f_j^i)) \). Finally, by using Lusin’s Theorem, it is possible to find a compact set, \( D \subseteq \pi_d(B_i \setminus (\cup_{j \in I_i} f_j^i)) \) such that \( f \upharpoonright D \) is continuous and \( \int_D \lambda((F_i^{-1}(f(x)))_x)dx > 0 \). This contradicts the putative maximality of the family \( \{ f_j^i : j \in I_i \} \).

Now note that for each \( i \in k \) the function \( (F_i \upharpoonright C_i)^* \) has small fibres for all \( x \). Applying Lemma 5.5 to \( \{ F_i \upharpoonright (C_i \cap A_i) : i \in k \} \), \( \delta, \eta/2 \) and \( N + 1 \) it follows that there is some \( \epsilon^* > 0 \) such that for all \( \mu > 0 \) and any closed \( E \subseteq [0,1]^d \) there is some \( a \in W_{N+1} \) such that the Lebesgue measure of

\[
\{ x \in \pi_d(E) : \Xi_{\delta,\epsilon^*}((\bigcap_{i \in k}((C_i \cap A_i \cap F_i^{-1} a) \cup ([0,1]^{d+1} \setminus (C_i \cap A_i))))_x, E_x) \\
\text{or } -\Xi_{\delta,\eta}(E_x, E_x) \text{mboxor } H^r(E_x) \leq \delta \}
\]

is at least \((1 - \mu)\lambda(E)\).

It is therefore possible to find \( K \in \omega \) such that for each \( i \in k \)

\[
\sum_{j \in K} \int_{C_i^j} \lambda((F_i^{-1}(f_j^i(x)))_x)dx > \lambda(B_i^+) - \frac{\eta d \epsilon^*}{2d+1k^2}
\]

and so, if \( S_i \) is defined to be

\[
\{ x \in \pi_d(E) : \lambda((B_i^+ \setminus (F_i^{-1}(f_j^i)_x))_x) \geq \epsilon^*/2k \}
\]

then \( \lambda(S_i) < \frac{d}{2k} \) for each \( i \in k \). Let \( U \subseteq [0,1]^d \) be any closed set such that \( U \cap S_i = \emptyset \) for each \( i \in k \) and \( \lambda(U) > 1 - (\eta/4)^d \). Let \( F_i^j \) be an arbitrary continuous extension of \( f_j^i \) which has domain \([0,1]^d \) and let \( A_i^j = \text{dom}(f_j^i) \). It follows from the induction hypothesis that there is \( \epsilon^* > 0 \) such that if \( E \subseteq [0,1]^d \) is a closed set such that \( \Xi_{\delta,\eta/2}(E, E) \) holds then there is \( a \in W_{N+1} \) such that

\[
\Xi_{\delta,\epsilon^*}((\bigcap_{i \in k}((A_i^j \cap (F_i^j)^{-1} a) \cup ([0,1]^{d} \setminus A_i^j)))_x, E)
\]

holds. Let \( \epsilon = \min\{\epsilon^*/2, \epsilon^*/2, \eta/4\} \).

Now suppose that \( E \) is a closed set such that \( \Xi_{\delta,\eta}(E, E) \) holds. From the choice of \( \epsilon^* \) it follows that it is possible to find \( a_0 \in W_{N+1} \) such that the Lebesgue measure of \( Z = \{ x \in [0,1]^d : \Xi_{\delta,\epsilon^*}((\bigcap_{i \in k}((C_i \cap A_i \cap F_i^{-1} a_0) \cup ([0,1]^{d+1} \setminus (C_i \cap A_i))))_x, E_x) \\
\text{or } -\Xi_{\delta,\eta}(E_x, E_x) \text{ or } H^r(E_x) \leq \delta \} \)

is at least \( 1 - (\epsilon^*/2)^d \). If \( \tilde{E} = \{ x \in \pi_d(E) : \Xi_{\delta,\eta/2}(E_x, E_x) \text{ then } \Xi_{\delta,\eta}(\tilde{E}, \tilde{E}) \) holds, by Lemma 1.4, because \( \Xi_{\delta,\eta}(E, E) \) does. From Lemma 4.4 it follows that \( \tilde{E} \) is Borel and so there exists a closed set \( \tilde{E} \subseteq \tilde{E} \) such that \( \lambda(\tilde{E} \cap \tilde{E}) < (\eta/2)^d \). Therefore \( \Xi_{\delta,\eta/2}(\tilde{E}, \tilde{E}) \) holds by lemma 4.1. Another appeal to Lemma 4.1 yields that \( \Xi_{\delta,\eta/2}(\tilde{E} \cap U, \tilde{E} \cap U) \) and so, from Lemma 4.3 it may be concluded that \( \Xi_{\delta,\eta/2}(E \cap U, E \cap U) \).
The choice of $\epsilon'$ guarantees that there is $a_1 \in W_{N+1}$ such that
\[
\Xi_{d,\epsilon'}(\bigcap_{i \in k} \bigcap_{j \in K} ((A_i^j \cap (F_i^j)^{-1}a_1) \cup ([0,1]^d \setminus A_i^j)), \tilde{E} \cap U)
\]
holds. It follows from Lemma 4.1 that so does
\[
\Xi_{d,\epsilon'/2}(Z \cap \bigcap_{i \in k} \bigcap_{j \in K} ((A_i^j \cap (F_i^j)^{-1}a_1) \cup ([0,1]^d \setminus A_i^j)), \tilde{E} \cap U)
\]
and from Lemma 4.6 it follows that
\[
\Xi_{(d+1)\delta,\epsilon'}(\tilde{E} \cap U, \pi_d(E))
\]
because $\Xi_{d, \epsilon'}(\tilde{E} \cap U, \pi_d(E))$ holds since $\epsilon \leq \eta/2$. Let $a = a_0 \cup a_1$ and notice that $a \in W_N$.

Using Lemma 4.2 it suffices to show that if
\[
x \in Z \cap \tilde{E} \cap U \cap \bigcap_{i \in k} \bigcap_{j \in K} ((A_i^j \cap (F_i^j)^{-1}a_1) \cup ([0,1]^d \setminus A_i^j))
\]
then
\[
\Xi_{\delta,\epsilon'}(\bigcap_{i \in k} ((A_i \cap E \cap F_i^{-1}a_0) \cup ([0,1]^{d+1} \setminus (C_i \cap A_i))_x, E_x)
\]
holds. To see that this is so, recall that since $x \in U$ it must be that $\lambda(Y(x)) < \epsilon^*/2$ where
\[
Y(x) = \bigcup_{i \in k} (B_i^* \setminus F_i^{-1}(f_i^j(x) : j \in K))_x
\]
Moreover, since $x \in Z$ it must be that either
\[
\Xi_{d,\epsilon'}(\bigcap_{i \in k} ((C_i \cap A_i \cap F_i^{-1}a_0) \cup ([0,1]^{d+1} \setminus (C_i \cap A_i))_x, E_x)
\]
holds or $\Xi_{d,\eta}(E_x, E_x)$ fails or $H^*(E_x) \leq \delta$. However, since $x \in \tilde{E} \subseteq \tilde{E}$ it must be that $\Xi_{d,\eta}(E_x, E_x)$ holds. If $H^*(E_x) \leq \delta$ then $\Xi_{d,\eta}([0, E_x])$ holds and so, in either case it follows that
\[
\Xi_{d,\epsilon'}(\bigcap_{i \in k} ((C_i \cap A_i \cap F_i^{-1}a_0) \cup ([0,1]^{d+1} \setminus (C_i \cap A_i))_x, E_x)
\]
holds. It therefore follows from Lemma 4.1 that
\[
\Xi_{d,\epsilon'/2}(\bigcap_{i \in k} ((A_i \cap A_i \cap F_i^{-1}a_0) \cup ([0,1]^{d+1} \setminus (C_i \cap A_i)))_x \setminus Y(x), E_x)
\]
holds. Therefore it suffices to show that
\[
(E \cap C_i \cap A_i \cap F_i^{-1}a_0) \cup (E \setminus (C_i \cap A_i))_x \setminus Y(x) \subseteq ((A_i \cap E \cap F_i^{-1}a_0) \cup (E \setminus A_i))_x
\]
for each $i \in k$.

Fix $i \in k$ and suppose that $y \in (E \cap C_i \cap A_i \cap F_i^{-1}a_0) \cup (E \setminus (C_i \cap A_i))_x \setminus Y(x)$. If $y \in E \cap C_i \cap A_i \cap F_i^{-1}a_0$ then $y \in A_i \cap E \cap F_i^{-1}a$. On the other hand, suppose $y \in (E \setminus (C_i \cap A_i))_x \setminus Y(x)$. If $y \in E \setminus A_i$ there is nothing to prove so it may be assumed that $y \in (A_i \setminus C_i)_x \setminus Y(x)$. Then, since $B_i^* \cap E = E \setminus C_i$ it must be that $y \in (B_i^* \setminus C_i)_x$ and, since $y \notin Y(x)$, it follows that $y \in F_i^{-1}\{f_i^j(x) : j \in K\}$ and so there is some $m \in K$ such that $F_i(y) = f_i^m(x)$ and, in particular, $x \in A_i^m$. Since
\[
x \in \bigcap_{i \in k} \bigcap_{j \in I_i} ((A_i^j \cap (F_i^j)^{-1}a_1) \cup ([0,1]^d \setminus A_i^j))
\]
it follows that \( x \in ((A^m \cap (F^m)^{-1}a_1) \) and so \( F_i(y) = F_i^m(x) \in a_1 \). Recalling that \( y \in (E \setminus (C_i \setminus A_i)) \) it follows that \( y \in (A_i \cap E \cap F_i^{-1}a) \).

**Corollary 6.1.** For any \( n \in \omega \) the ideal \( \mathcal{I}_n \) is proper.

**Proof:** From Lemma 3.2 it suffices to show that that if \( X(f,C,d') \) is a \( d \)-dimensional generator for \( \mathcal{I}_n \), then \( X_n \not\subseteq X(f,C,d') \). Let \( \beta : \omega \to (0,1) \) witness that \( C = \{ C_i \}_{i \in \omega} \) is a normal family. Let \( m \) be any integer such that \( m > d/d' \).

Now apply Theorem 5.1 letting \( \{ F_i \}_{i \in k} = \{ f \} \), \( \eta = \beta(m) \), \( \delta = \delta'/d \), \( N = n \) and \( \{ A_i \}_{i \in k} = \{ [0,1]^d \} \). This yields \( \epsilon > 0 \) such that for every \( i \in \omega \) there is some \( a_i \in W_n \) such that \( \Xi_{d\delta,\epsilon}(f^{-1}a_i \cup ([0,1]^d \setminus C_i, C_i)) \) holds provided that \( \Xi_{\delta,\beta(m)}(C_i, C_i) \) holds. Without loss of generality, it may be assumed that \( \epsilon \leq \beta(m) \). This implies that \( \Xi_{d\delta,\epsilon}(f^{-1}a_i, C_i) \) holds provided that \( \Xi_{d,\beta(m)}(C_i, C_i) \) does. Since \( 1/m < \delta \) and \( \Xi_{1/m,\beta(m)}(C_i, C_i) \) holds for each \( i \geq m \) it follows \( \Xi_{d,\epsilon}(f^{-1}a_j, C_j) \) holds for some \( j \) such that \( \lambda(\pi_n(C_j) \setminus \pi_n(\cap_{i \in \omega} C_i)) < (\frac{1}{\epsilon^d})^n \) for all \( n \leq d \). The fact that such a \( j \) exists follows from the remark after Definition 5.1. Therefore \( \Xi_{\delta',\epsilon}(f^{-1}a, C_i) \) holds for all \( i > m \) by Corollary 5.2 and hence \( a \notin X(f,C,d') \).

7. The End

Finally, everything must be put together.

**Theorem 7.1.** If \( \mathcal{I} = \{ \mathcal{I}_n \}_{n \in \omega} \) then

\[
1 \Vdash_{P(\mathcal{I})} \text{"} \lambda(V \cap [0,1]) = 0 \text{"}
\]

where \( V \) represents the ground model.

**Proof:** First notice that if \( A \in \mathcal{I}_n^+ \) then \( [0,1] \subseteq X \) because if \( x \in [0,1] \) then, letting \( \bar{x} : [0,1] \to [0,1] \) represent the function which is constantly \( x \), it follows that \( A \not\subseteq X(\bar{x}, \{ [0,1] \}_{i \in \omega}, 1/2) \). A standard genericity argument will yield that if \( G \in \prod_{n \in \omega} W_n \) is obtained from a \( P(\mathcal{I}) \) generic set and \( x \in V \) then \( x \in G(n) \) for infinitely any \( n \). Since any member of \( W_n \) has measure less than \( 2^{-n} \), the result is proved.

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