ON FINITE $ca$-$\mathcal{F}$ GROUPS AND THEIR APPLICATIONS

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Abstract. Let $\mathcal{F}$ be a class of groups. A group $G$ is called $ca$-$\mathcal{F}$-group if its every non-abelian chief factor is simple and $H/K \times C_G(H/K) \in \mathcal{F}$ for every abelian chief factor $H/K$ of $G$. In this paper, we investigate the structure of a finite $ca$-$\mathcal{F}$-group. Properties of mutually permutable products of finite $ca$-$\mathcal{F}$-groups are studied.

1. Introduction

Only finite groups are considered. The concept of composition formation was introduced by L.A. Shemetkov [15] and R. Baer in an unpublished paper (noted in [7, IV, p.370]). Every saturated formation is a composition formation. The class of all quasinilpotent groups is an example of composition, but not saturated formation. Guo Wenbin and A.N. Skiba [9, 10] introduced the concept of quasi-$\mathcal{F}$-group that is a generalization of quasinilpotency. In [9] they proved that the class of all quasi-$\mathcal{F}$-groups is a composition formation if $\mathcal{F}$ is a saturated formation.

In [19] V.A. Vedernikov introduced the definition of a $c$-supersoluble group. Recall [19] that a group $G$ is called $c$-supersoluble if every chief factor of $G$ is simple. In [18] A.F. Vasil’ev and T.I. Vasil’eva proved that the class $\mathcal{U}_c$ of all $c$-supersoluble groups is a composition but not a saturated formation. D. Robinson (using notation: $SC$-group) [13] established the structural properties of finite $c$-supersoluble groups.

In [12] the following generalization of $c$-supersolubility was proposed.

Let $\mathcal{F}$ be a class of groups. Recall [17] that a chief factor $H/K$ of group $G$ is called $\mathcal{F}$-central provided $H/K \times G/C_G(H/K) \in \mathcal{F}$.

Definition 1.1 ([12]). Let $\mathcal{F}$ be a class of groups. A group $G$ is called a $ca$-$\mathcal{F}$-group if its every non-abelian chief factor is simple and every abelian chief factor of $G$ is $\mathcal{F}$-central.

The class of all $ca$-$\mathcal{F}$-groups is denoted by $\mathcal{F}_{ca}$. If $\mathcal{F} = \mathcal{U}$ we have that $\mathcal{F}_{ca} = \mathcal{U}_c$. If $\mathcal{F} = \mathcal{N}$, then $\mathcal{F}_{ca} = (\mathcal{N})_{ca}$ is the class of all groups whose every non-abelian chief factor is simple and $Aut_G(H/K)$ is abelian for every abelian chief factor $H/K$. If $\mathcal{F} = \mathcal{G}$ then $\mathcal{F}_{ca}$ is the class of all SNAC-groups [13], i.e the class of all groups whose all non-abelian factors are simple.

The class of all $ca$-$\mathcal{F}$-groups is a composition formation [12]. Also in [12] some properties of the products of normal $ca$-$\mathcal{F}$-subgroups were found.

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Recall [17, §8], $Z_F^\infty(G)$ denotes the $F$-hypercenter of a group $G$. $Z_F^\infty(G)$ is the product of all normal subgroups $H$ of $G$ whose $G$-chief factors are $F$-central.

The following theorem is an extension of Robinson’s result [13] for case when $F$ is a soluble saturated formation.

**Theorem A.** Let $\mathfrak{F}$ be a soluble saturated formation. A group $G$ is a ca-$\mathfrak{F}$-group if and only if $G$ satisfies:

1. $G^\mathfrak{G} = G^\mathfrak{F}$;
2. If $G^\mathfrak{G} \neq 1$ then $G^\mathfrak{G}/Z(G^\mathfrak{G})$ is a direct product of $G$-invariant non-abelian simple groups;
3. $Z(G^\mathfrak{G}) \subseteq Z_F^\infty(G)$

Following Carocca [6], we say that $G = HK$ is the mutually permutable product of subgroups $H$ and $K$ if $H$ permutes with every subgroup of $K$ and $K$ permutes with every subgroup of $H$. The mutually permutable products of supersoluble and $c$-supersoluble subgroups were investigated in many works of different authors (see monograph [3]). A lot of papers were dedicated to the case where $G = HK$ is the mutually permutable product of subgroups $H$ and $K$ which belong to a saturated formation $\mathfrak{F}$. Therefore we have the following problem.

**Problem.** Let $\mathfrak{F}$ be a composition formation. What is the structure of the group $G = HK$ where $H$ and $K$ are mutually permutable $\mathfrak{F}$-subgroups of $G$.

In this paper this problem is solving for a formation of ca-$\mathfrak{F}$-groups where $\mathfrak{F}$ is a saturated formation containing all supersoluble groups.

**Theorem B.** Let $\mathfrak{F}$ be a saturated formation containing the class $\mathfrak{U}$ of supersoluble groups. Let a group $G = HK$ be the product of the mutually permutable subgroups $H$ and $K$ of $G$. If $G$ is a ca-$\mathfrak{F}$-group then both $H$ and $K$ are ca-$\mathfrak{F}$-groups.

**Corollary B.1** (4). Let the group $G = HK$ be the mutually permutable product of the subgroups $H$ and $K$ of $G$. If $G$ is a $c$-supersoluble group then both $H$ and $K$ are $c$-supersoluble groups.

**Corollary B.2** (4). Let the group $G = HK$ be the mutually permutable product of the subgroups $H$ and $K$ of $G$. If $G$ is a SNAC-group then both $H$ and $K$ are SNAC-groups.

**Corollary B.3.** Let the group $G = HK$ be the mutually permutable product of the subgroups $H$ and $K$ of $G$. If $G$ is a ca-$\mathfrak{N}$-group then both $H$ and $K$ are ca-$\mathfrak{N}$-groups.

It is well known that in general, the product $G = HK$ of two normal supersoluble subgroups of a finite group $G$ need not be supersoluble. In 1957 Baer [2] established that such group $G$ will be a supersoluble if and only if the derived subgroup $G'$ of $G$ is nilpotent. The next theorem is an extension of this result.

**Theorem C.** Let $\mathfrak{F}$ be a saturated formation containing the class $\mathfrak{U}$ of supersoluble groups. Let the group $G = HK$ be the product of the mutually permutable ca-$\mathfrak{F}$-subgroups $H$ and $K$ of $G$. If the derived subgroup $G'$ of $G$ is quasinilpotent, then $G$ is a ca-$\mathfrak{F}$-group.

**Corollary C.1** (4). Let the group $G = HK$ be the product of the mutually permutable $c$-supersoluble subgroups $H$ and $K$ of $G$. If the derived subgroup $G'$ of $G$ is quasinilpotent, then $G$ is $c$-supersoluble.
Corollary C.2 ([1]). Let the group $G = HK$ be the product of the mutually permutable supersoluble subgroups $H$ and $K$ of $G$. If the derived subgroup $G'$ of $G$ is nilpotent, then $G$ is supersoluble.

Corollary C.3. Let the group $G = HK$ be the product of the mutually permutable $ca$-$\mathfrak{F}$-subgroups $H$ and $K$ of $G$. If the derived subgroup $G'$ of $G$ is quasinilpotent, then $G$ is $ca$-$\mathfrak{F}$-group.

The following corollary extends [12] the properties of normal products of $ca$-$\mathfrak{F}$-groups.

Corollary C.4. Let $\mathfrak{F}$ be a saturated formation containing the class $\mathfrak{U}$ of supersoluble groups. If $G = HK$ is the product of normal $ca$-$\mathfrak{F}$-subgroups $H$ and $K$ of $G$ and the derived subgroup $G'$ of $G$ is quasinilpotent, then $G$ is a $ca$-$\mathfrak{F}$-group.

2. Preliminaries

Standard notations, notions and results are used in the paper (see [7][16]). Recall significant notions and notations for this paper. $\mathbb{P}$ is the set of all prime numbers; 1 is an identity group; $H \ltimes K$ is a semidirect product of groups $H$ and $K$; $\mathfrak{G}$ is the class of all groups; $\mathfrak{S}$ is the class of all soluble groups; $\mathfrak{U}$ is the class of all supersoluble groups; $\mathfrak{N}$ is the class of all nilpotent groups; $\mathfrak{N}_p$ is the class of all $p$-groups; $\mathfrak{F}$ is the class of all simple groups; $\mathfrak{A}(p - 1)$ is the class of all abelian groups of exponent dividing $p - 1$.

A formation is a homomorph $\mathfrak{F}$ of groups such that each group $G$ has the smallest normal subgroup (called $\mathfrak{F}$-residual and denoted by $G^{\mathfrak{F}}$) with quotient in $\mathfrak{F}$. A formation $\mathfrak{F}$ is said to be saturated if it contains each group $G$ with $G/\Phi(G) \in \mathfrak{F}$. A formation $\mathfrak{F}$ is said to be (normally) hereditary if it contains all (normal) subgroups of every group in $\mathfrak{F}$.

Let $\mathfrak{F}$ be a non-empty formation. $G_{\mathfrak{F}}$ denotes $\mathfrak{F}$-radical of group $G$, i.e., the largest normal $\mathfrak{F}$-subgroup of $G$.

A function $f : \mathbb{P} \to \{\text{formations of groups}\}$ is called a local formation function. The symbol $LF(f)$ denotes the class of all groups such that either $G = 1$ or $G \neq 1$ and $G/C_G(H/K) \in f(p)$ for every chief factor $H/K$ of $G$ and every $p \in \pi(H/K)$. The class $LF(f)$ is a non-empty formation.

For a formation $\mathfrak{F}$, if there exists a formation function $f$ such that $\mathfrak{F} = LF(f)$ then $\mathfrak{F}$ is called a local formation. It is known that a formation $\mathfrak{F}$ is local if and only if it is saturated [7][16], IV, Theorem 4.6).

A formation $\mathfrak{F}$ is said to be soluble saturated, composition, or Baer-local formation if it contains each group $G$ with $G/\Phi(N) \in \mathfrak{F}$ for some soluble normal subgroup $N$ of $G$. For every function $f$ of the form $f : \mathbb{P} \to \{\text{formations of groups}\}$ we put, $CLF(f) = \{G \text{ is a group } | G/C_G(H/K) \in f(A) \text{ for every } A \in \mathfrak{K}_G(H/K)\}$. It is well known that a composition formation (or a Baer-local formation if we use the terminology in [7]) $\mathfrak{F}$ is exactly a class $\mathfrak{F} = CLF(f)$ for some function $f$ of above-mention form. In this case, the function $f$ is said to be a composition satellite [14] of the formation $\mathfrak{F}$.

A local function $f$ is called an inner local function if $f(p) \subseteq LF(f)$ for every prime $p$. Function $f$ is called a maximal inner local function of formation $\mathfrak{F}$ if $f$ is a maximal element of set of all inner local functions of formation $\mathfrak{F}$. Similarly, we can introduce the notion of the inner composition satellite and maximal inner composition satellite.
Every local (composition) formation has the unique maximal inner local function (composition satellite) \cite{16} ch. 1.

We will use the following results.

**Lemma 2.1** (\cite{12}). Let \( \mathfrak{F} \) be a class of groups. Then the class \( \mathfrak{F}_{ca} \) is a non-empty formation.

**Theorem 2.2** (\cite{12}). Let \( \mathfrak{F} \) be a saturated formation and \( f \) is its maximal inner local function. Then the formation \( \mathfrak{F}_{ca} \) is a composition formation and has a maximal inner composition satellite \( h \) such that \( h(N) = \mathfrak{F}_{ca} \), if \( N \) is a non-abelian group and \( h(N) = f(p) \), if \( N \) is a simple \( p \)-group, where \( p \) is a prime.

**Lemma 2.3** (\cite{18}). Let \( \mathfrak{F} \) be a formation and \( N \) be a minimal normal subgroup of \( G \) such as \(|N| = p^n \) for some prime \( p \). If \( N \) contains in the subgroup \( H \) of \( G \) and \( H/C_H(U/V) \in \mathfrak{F} \) for every \( H \)-chief factor \( U/V \) of \( N \), then \( H/C_H(N) \in \mathfrak{N}_p \mathfrak{F} \).

**Lemma 2.4** (\cite{3}). Assume that the subgroups \( A \) and \( B \) of the group \( G \) are mutually permutably and that \( N \) is a normal subgroup of \( G \). Then the subgroups \( AN/N \) and \( BN/N \) are mutually permutable in \( G/N \).

**Lemma 2.5** (\cite{3}). Let the group \( G = AB \) be the mutually permutable product of the subgroups \( A \) and \( B \). Then:
1. If \( N \) is a maximal normal subgroup of \( G \), then \( \{AN, BN, (A\cap B)N\} \subseteq \{N, G\} \).
2. If \( N \) is a non-abelian minimal normal subgroup of \( G \), then \( \{A\cap N, B\cap N\} \subseteq \{N, 1\} \) and \( N = (N\cap A)(N\cap B) \) (that is, \( N \) is prefactorised with respect to \( G = AB \)).
3. If \( N \) is a minimal normal subgroup of \( G \), then \( N \leq A \cap B \) or \( |N, A \cap B| = 1 \).
4. If \( N \) is a maximal normal subgroup of \( G \), then \( A \cap N, B \cap N \subseteq \{N, 1\} \).
5. If \( N \) is a minimal normal subgroup of \( G \) contained in \( A \) and \( B \cap N = 1 \), then \( N \leq C_G(A) \) or \( N \leq C_G(B) \). If furthermore \( N \) is not cyclic, then \( N \leq C_G(B) \).

**Lemma 2.6** (\cite{3}). Let the group \( G = AB \) be the product of the mutually permutable subgroups \( A \) and \( B \) and let \( \mathfrak{F} \) be a saturated formation containing the class \( \Omega \) of all supersoluble groups. If \( (A \cap B)G = 1 \), then \( G \in \mathfrak{F} \) if and only if \( A \in \mathfrak{F} \) and \( B \in \mathfrak{F} \).

3. **Proof of theorem A**

In this section we prove the theorem that describes the structure of finite \( ca-\mathfrak{F} \)-group.

**Lemma 3.1.** Let \( \mathfrak{F} \) be a soluble formation containing the class \( \Omega \) of all supersoluble groups. If \( G \) is a \( ca-\mathfrak{F} \)-group then the following statements hold:
1. \( G^{\mathfrak{F}} \leq C_G(G_{\mathfrak{F}}) \);
2. \( (G^{\mathfrak{F}})^{\mathfrak{F}} \leq Z(G^{\mathfrak{F}}) \).

**Proof.** Prove the statement 1. Obviously that all chief factors of group \( G \) below subgroup \( G_{\mathfrak{F}} \) are \( \mathfrak{F} \)-central. Hence subgroup \( G_{\mathfrak{F}} \) is a \( \mathfrak{F} \)-hypereentric and thus it is subgroup of a \( \mathfrak{F} \)-hypercrter \( Z_{\mathfrak{F}}(G) \). By Corollary 9.3.2 \cite{16} we have that \( G^{\mathfrak{F}} \leq C_G(Z_{\mathfrak{F}}(G)) \). Since \( \mathfrak{F} \) is a soluble formation, \( G^{\mathfrak{F}} \leq G^{\mathfrak{F}} \leq C_G(G_{\mathfrak{F}}) \).

Prove the statement 2. Let \( R = (G^{\mathfrak{F}})^{\mathfrak{F}} \). Since \( R \) char \( G^{\mathfrak{F}} \lneq G \), it follows \( R \lneq G \). Therefore \( R \leq G_{\mathfrak{F}} \). Hence \( G^{\mathfrak{F}} \leq C_G(R) \) by statement 1 of the Lemma. The statement 2 is true.

**Proof of theorem A** Denote by \( D \) the soluble residual \( G^{\mathfrak{F}} \) of group \( G \).
Let $G$ be a ca-$\mathfrak{F}$-group. If $G$ is soluble, then $D = 1$ and $G \in \mathfrak{F}$. So $G$ satisfies the Statements 1, 2, and 3. We assume that group $G$ is not soluble. Then $D \neq 1$.

Since $\mathfrak{F}$ is a soluble formation, $G/G^\mathfrak{F} \in \mathfrak{F} \subseteq \mathfrak{S}$. Hence $D \subseteq G^\mathfrak{F}$. Since $\mathfrak{F}_{ca}$ is a formation, it follows $G/D \in \mathfrak{F}_{ca}$. By solvability of quotient $G/D$ we have that $G/D \in \mathfrak{F}$. Hence $G^\mathfrak{F} \subseteq D$ and $D = G^\mathfrak{F}$. The Statement 1 holds. Note that all chief factors of $G$ below $Z(D)$ are abelian and therefore are $\mathfrak{F}$-central. This means, that $Z(D)$ is $\mathfrak{F}$-hypercentral and the Statement 3 holds.

We show that $D/Z(D)$ is a direct product of $G$-invariant simple groups.

Assume that $Z(D) = 1$. Let $N_1$ be a minimal normal subgroup of $G$ contained in $D$. If $N_1$ is abelian, then it follows from $N_1 \leq D_{\mathfrak{F}}$ and the Statement 2 of Lemma 5.1 that $N_1 \leq Z(D) = 1$. Hence $N_1$ is non-abelian. Since $G \in \mathfrak{F}_{ca}$, we have that $N_1$ is a simple. Note that $G/C_G(N_1)$ is isomorphic to a subgroup of $\text{Aut}(N_1)$ and $N_1C_G(N_1)/C_G(N_1)$ is isomorphic to $\text{Inn}(N_1)$. So $G/N_1C_G(N_1) \cong (G/C_G(N_1))/((N_1C_G(N_1)/C_G(N_1)))$ is isomorphic to a subgroup of $\text{Aut}(N_1)/\text{Inn}(N_1)$. From the validity of the Schreier conjecture, it follows that $G/N_1C_G(N_1)$ is soluble. Then $D \leq N_1C_G(N_1)$. Hence $D = D \cap N_1C_G(N_1) = N_1(D \cap C_G(N_1)) = N_1C_D(N_1)$ and $N_1 \cap C_D(N_1) = 1$. If $D = N_1$, then the Statement 2 holds. Assume that $D$ is not simple. Therefore, $C_D(N_1) \neq 1$. The Statement 2 holds in the case when $C_D(N_1)$ is simple. Assume that $C_D(N_1)$ is not a simple and let $N_2$ be a minimal normal subgroup of $G$ contained in $C_D(N_1)$. Since $Z(D) = 1$ and the Statement 2 of lemma 5.1 it follows that $N_2$ is a simple non-abelian subgroup. By the above $D = N_2C_D(N_2)$. By Dedekind identity $C_D(N_1) = C_D(N_1) \cap N_2C_D(N_2) = N_2(C_D(N_1) \cap C_D(N_2)) = N_2C_L(N_2)$, where $L = C_D(N_1) \cap C_D(N_2)$. Then $D = N_1N_2C_L(N_2)$. Applying above to $C_D(N_2)$ and etc. we can conclude that $D = N_1 \times N_2 \times \cdots \times N_i$ is the direct product of minimal normal subgroups of $G$, each of them simple, as desired. So the Statement 2 holds.

Let $Z(D) \neq 1$. Since $G/Z(D) \in \mathfrak{F}_{ca}$ and $(G/Z(D))^\mathfrak{F} = D/Z(D)$, the Statement 1 and 3 holds for $G/Z(D)$. Denote $T/Z(D) = Z(D/Z(D))$. Then $T$ is a normal soluble subgroup of $D$. By lemma 5.1 $T$ is contained in the center $Z(D)$. Therefore $Z(D/Z(D)) = 1$. By the above the Statement 1 holds for $G/Z(D)$.

Conversely, assume that a group $G$ satisfies the Statements 1, 2, and 3. We consider a chief series of $G$ which passes through the subgroup $D = G^\mathfrak{F}$. Note all chief factors above $D$ are abelian and $\mathfrak{F}$-central. By the Statement 2 the quotient $D/Z(D)$ is the direct product of minimal normal subgroups of $G/Z(D)$, which are simple. All chief factors of $G$ below $Z(D)$ are $\mathfrak{F}$-central by the Statement 3. By virtue of Jordan-Holder’s theorem for groups with operators [7, A, 3,2] and the Definition 1.1, $G \in \mathfrak{F}_{ca}$. □

4. PROOF OF THEOREM $\mathcal{B}$ AND $\mathcal{C}$

In this section we prove some properties of the mutually permutable products of ca-$\mathfrak{F}$-groups.

Proof of Theorem $\mathcal{B}$. Assume that that this theorem is false and let $G$ be a counterexample of minimal order. Let $N$ be a minimal normal subgroup of $G$. If $N = G$, then $G$ is simple. Hence $H \in \mathfrak{F}_{ca}$ and $K \in \mathfrak{F}_{ca}$. Assume $N \neq G$. By Lemma 2.3 $G/N = HN/N \cdot KN/N$ is the mutually permutable product of subgroups $HN/N$ and $KN/N$ of $G/N$. Note that $G/N \in \mathfrak{F}_{ca}$. Then all conditions of the theorem hold for $G/N$. Therefore $HN/N \simeq H/(H \cap N) \in \mathfrak{F}_{ca}$ and $KN/N \simeq K/(K \cap N) \in \mathfrak{F}_{ca}$. 


Since $\mathfrak{F}_{ca}$ is a formation by Lemma 2.1, it follows that $N$ is the unique minimal normal subgroup of $G$.

Let $N$ be a non-abelian group. Then $N$ is simple. According to Lemma 2.5, we should consider the following cases.

1. Let $H \cap N = K \cap N = N$. Then $N \leq H \cap K$, $H/(H \cap N) = H/N \in \mathfrak{F}_{ca}$ and $K/(K \cap N) = K/N \in \mathfrak{F}_{ca}$. Hence $H$ and $K$ are $ca$-$\mathfrak{F}$-groups, a contradiction.

2. Let $H \cap N = K \cap N = 1$. Then $H/(H \cap N) \simeq H$ and $K/(K \cap N) \simeq K$ are $ca$-$\mathfrak{F}$-groups, a contradiction.

3. Let $H \cap N = N$ and $K \cap N = 1$. Then $H/(H \cap N) = H/N \in \mathfrak{F}_{ca}$ and $H$ is a $ca$-$\mathfrak{F}$-group and $K/(K \cap N) \simeq K$ is a $ca$-$\mathfrak{F}$-group. A contradiction.

4. Let $H \cap N = 1$ and $K \cap N = N$. This case is considered similarly to the case 3. □

To prove the Theorem C we need the following results.

**Lemma 4.1.** Let the group $G$ has the unique minimal normal subgroup $N = N_1 \times \cdots \times N_t$ and $N_i$ are isomorphic simple non-abelian groups for all $i = 1, \ldots, t$. If $N \leq H$, where $H$ is a $ca$-$\mathfrak{F}$-subgroup of $G$, then $N_i \triangleleft H$ for all $i = 1, \ldots, t$.

**Proof.** Let $i \in \{1, \ldots, t\}$. Consider normal closure $N_i^H = \langle N_i^x | x \in H \rangle$ of subgroup $N_i$ in $H$. Note that $N_i \triangleleft G$. Hence $N_i \triangleleft H$. By the Lemma 9.17 [11] we have that $N_i^H$ is a minimal normal subgroup of $H$. Since subgroup $N_i^H$ is non-abelian and isomorphic to the chief factor of $ca$-$\mathfrak{F}$-subgroup $H$, then $N_i^H$ is simple. Then, by $N_i \triangleleft N_i^H$, we have that $N_i^H = N_i$. Hence $N_i \triangleleft H$ for all $i = 1, \ldots, t$. □

**Lemma 4.2.** Let $\mathfrak{F}$ be a composition formation and $f$ is an inner composition satellite of $\mathfrak{F}$. Let a group $G$ has the unique minimal normal subgroup $N$ and $N$ is an abelian $p$-group for some prime $p$. The chief factor $N$ of $G$ is $\mathfrak{F}$-central in $G$ if and only if $G/C_G(N) \in f(p)$.

**Proof.** Let $G/C_G(N) \in f(p)$. Consider semidirect product $R = N \rtimes G/C_G(N)$. Note that $N$ is the unique minimal normal subgroup of $R$ and $C_R(N) = N$. Then $R/C_R(N) \simeq G/C_G(N) \in f(p) \subseteq \mathfrak{F}$. Hence $R \in \mathfrak{F}$, i.e. the chief factor $N/1$ of $G$ is $\mathfrak{F}$-central.

Conversely, assume that $N$ is $\mathfrak{F}$-central chief factor of $G$. Then $R = N \rtimes G/C_G(N) \in \mathfrak{F}$, where $N$ is the unique minimal normal subgroup of $R$ and $C_G(N) = N$. Hence $R/C_R(N) \simeq G/C_G(N) \in f(p)$. □

**Lemma 4.3.** Let $\mathfrak{F}$ be a formation and $\mathfrak{A}(p-1) \subseteq \mathfrak{F}$. Let $G = HK$ be the mutually permutable products of subgroup $H$ and $K$, where $H, K \in \mathfrak{M}_p\mathfrak{F}$ and $G \in \mathfrak{M}_p\mathfrak{A}$. Then $G \in \mathfrak{M}_p\mathfrak{F}$.

**Proof.** Assume that this lemma is false and let $G$ be a counterexample of minimal order. Let $N$ be a minimal normal subgroup of $G$. We can assume without loss of generality that $G \neq N$. By Lemma 2.3 $G/N = HN/N \cdot KN/N$ is the mutually permutable product of subgroups $HN/N$ and $KN/N$ of $G/N$. Note that $HN/N \in \mathfrak{M}_p\mathfrak{F}$, $HN/N \in \mathfrak{M}_p\mathfrak{A}$ and $G/N \in \mathfrak{M}_p\mathfrak{A}$. Then all conditions of the Lemma 4.3 hold for $G/N$. Therefore $G/N \in \mathfrak{M}_p\mathfrak{F}$. Since $\mathfrak{M}_p\mathfrak{F}$ is a formation, it follows that $N$ is the unique minimal normal subgroup of $G$. We note that $N$ is a $q$-group for some prime $q \neq p$. Since $G \in \mathfrak{M}_p\mathfrak{A}$ and $O_p(G) = 1$, we have that $G \in \mathfrak{A}$. Therefore $H \in \mathfrak{F}$ and $K \in \mathfrak{F}$. Since $N$ is the unique minimal normal subgroup of $G$, it follows that $G$ is a cyclic $q$-group. Since $G = HK$, we have that $G = H$ or $G = K$, i.e. $G \in \mathfrak{F}$. □
Proof of Theorem C. Assume that this theorem is false and let $G$ be a counterexample of minimal order. Let $N$ be a minimal normal subgroup of $G$. If $N = G$, then $G$ is simple. Hence $G \in \mathfrak{F}_{ca}$. Assume $N \neq G$. By Lemma 2.4 $G/N = HN/N \cdot KN/N$ is the mutually permutable product of subgroups $HN/N$ and $KN/N$ of $G/N$. Note that the derived subgroup $(G/N)'$ of $G/N$ is quasinilpotent. Then all conditions of the Theorem hold for $G/N$. Therefore $G/N \in \mathfrak{F}_{ca}$. Since $\mathfrak{F}_{ca}$ is a formation by Lemma 2.1 it follows that $N$ is the unique minimal normal subgroup of $G$.

Let $N$ be a non-abelian group. Then $N = N_1 \times \cdots \times N_t$, where $N_i$ are isomorphic simple non-abelian groups for all $i = 1, \ldots, t$. According to Lemma 2.5 we should consider the following cases.

1. Let $H \cap N = K \cap N = N$. Then $N \subseteq H \cap K$. Since $H$ and $K$ are $ca$-$\mathfrak{F}$-subgroups and $N = N_1 \times \cdots \times N_t$, it follows that $N_i \not\leq H$ and $N_i \not\leq K$ by Lemma 2.1. Hence by $G = HK$ we have that $N_i \not\leq G$ for all $i = 1, \ldots, t$. Since $N$ is the unique minimal normal subgroup of $G$, it follows that $t = 1$ and $N$ is simple. Since $G/N \in \mathfrak{F}_{ca}$, we have that $G \in \mathfrak{F}_{ca}$. A contradiction.

2. Let $H \cap N = K \cap N = 1$. Then $N = (H \cap N)/(K \cap N) = 1$ by Lemma 2.5. A contradiction with choice of $N$.

3. Let $H \cap N = N$ and $K \cap N = 1$. Then $N \subseteq H \cap K$. Since $H$ and $N$ is non-abelian, we have that $N \leq C_G(K)$. Since $N = N_1 \times \cdots \times N_t$, it follows that $N_i \leq C_G(K)$ for all $i = 1, \ldots, t$. By Lemma 4.1 we have that $N_i \not\leq H$ for all $i = 1, \ldots, t$. By $G = HK$ we have that $N_i \not\leq G$. Since $N$ is the unique minimal normal subgroup of $G$, it follows that $N = N_i$ and $N$ is simple. Since $G/N \in \mathfrak{F}_{ca}$, we have that $G \in \mathfrak{F}_{ca}$. A contradiction.

4. Let $H \cap N = 1$ and $K \cap N = N$. This case is considered similarly to the case 3.

Assume $N$ is an abelian group. Then $N$ is a $p$-group for some prime $p$. By Theorem 2.2 formation $\mathfrak{F}_{ca}$ has the maximal inner composition satellite $h$ such that $h(N) = f(p)$, where $f$ is a maximal inner local function of $\mathfrak{F}$. According to Lemma 2.5 we should consider the following cases.

1. Let $H \cap N = K \cap N = N$. Then $N \subseteq H \cap K$. Let $U/V$ is any $H$-chief factor of $N$. Since $H \in \mathfrak{F}_{ca}$, it follows that $H/C_H(U/V) \in \mathfrak{F}_p$ by Lemma 2.5. We have that $H/C_H(U/V) \in \mathfrak{F}_p$. Similarly we can show that $K/C_K(N) \in \mathfrak{F}_p$. Note the group $G/C_G(N) = HCG(N)/CG(N) \cdot CCG(N)/CG(N)$ is the mutually permutable product of subgroups $HC_{CG}(N)/CG(N)$ and $KC_{CG}(N)/CG(N)$ of $G/CG(N)$.

Since $N \leq G'$ and $G'$ is quasinilpotent, it follows that $G'/CG(N) \in \mathfrak{F}_p$ by Lemma 2.2. So $(G/CG(N))' = G'/CG(N)/CG(N) \simeq G'/CG(N)$ is a $p$-group. Since $G/CG(N)/(G/CG(N))' \in \mathfrak{F}$, it follows that $G/CG(N) \in \mathfrak{F}_p\mathfrak{A}$. By Lemma 4.3 for $G/CG(N)$ we have that $G/CG(N) \in \mathfrak{F}_p$. Therefore $G \in \mathfrak{F}_{ca}$. A contradiction.

2. Let $H \cap N = K \cap N = 1$. Then $N \not\leq H \cap K$ and $(H \cap K)_G = 1$. If $H^S = 1$ and $K^S = 1$, then $H$ and $K$ are soluble. Hence $H \in \mathfrak{F}$ and $K \in \mathfrak{F}$. By Lemma 2.6 we have that $G \in \mathfrak{F} \subseteq \mathfrak{F}_{ca}$, a contradiction. Hence we can assume without loss of generality that $H^S \neq 1$. Then $H^S \leq G$ by Corollary 4.3.6. Hence $N \leq H^S$. Therefore $N \leq H \cap N = 1$, a contradiction.

3. Let $H \cap N = N$ and $K \cap N = 1$. Assume that $N$ is non-cyclic subgroup. Then $N \leq C_G(K)$ by Lemma 2.5. Hence $K \subseteq C_G(N)$ and $G/C_G(N) =$
HC_G(N)/C_G(N) \cdot KC_G(N)/C_G(N) = HC_G(N)/C_G(N) \simeq H/(C_G(N) \cap H) = H/C_H(N). Since N \subseteq H and H \in \mathcal{F}_{ca}, it follows that H/C_H(N) \in h(p) by Lemma 2.3. By Lemma 4.2 we have that factor N is \mathcal{F}_{ca}-central chief factor of G. Then G \in \mathcal{F}_{ca}, a contradiction. Let N be a cyclic group. Then |N| = p and G/C_G(N) is a cyclic group of order dividing p − 1. Hence G/C_G(N) \in \mathfrak{A}(p − 1) \subseteq f(p) = h(p). Since G/N \in \mathcal{F}_{ca}, it follows that G \in \mathcal{F}_{ca}, a contradiction.

4. Let H \cap N = 1 and K \cap N = N. This case is considered similarly to the case 3. □

5. Final remarks

Many different specific examples of composition formations containing all super-soluble groups can be built using the concept of ca-\mathcal{F}_{ca}-group.

According to [7], a rank function is a map R : p \to R(p) which associates with each prime p a set \(R(p)\) of natural numbers. With each rank function R we associate a class [7]

\[\mathcal{F}(R) = \{G \in \mathcal{S} \mid \text{for all prime } p \in \mathbb{P} \text{ each } p\text{-factor of } G\]

has rank in \(R(p)\),

that is a formation.

If \(\mathcal{F}(R)\) is a saturated formation, then rank function is called a saturated (see [7, p. 484]). A rank function R is said to have full characteristic if \(R(p) \neq \emptyset\) for all \(p \in \mathbb{P}\).

Note that if R is a saturated rank function of full characteristic, by [7, IV, 4.3], we have \(1 \in R(p)\) for all prime \(p \in \mathbb{P}\) and therefore \(\mathfrak{A} \subseteq \mathcal{F}(R)\).

If a rank function R is defined, then for all prime \(p \in \mathbb{P}\) are defined [7]

\[\pi(G) = R(p) \cap \mathbb{P}\]

and

\[e(p) = \{p^m − 1 \mid m \in R(p)\}\]

By \(\mathfrak{A}_\pi(e(p))\) we denote a class of abelian \(\pi(p)\)′-group with exponent dividing \(e(p)\) that is a formation.

According to [7] the following lemma holds.

Lemma 5.1 ([7]). Let R is a saturated rank function of full characteristic. Then R satisfies the following conditions

- **RF1:** If \(n \in R(p)\) and \(m \mid n\), then \(m \in R(p)\);
- **RF2:** If \(\{m, n\} \in R(p)\), then \(mn \in R(p)\);
- **RF3:** If \(p\) and \(q\) are distinct primes with \(q \in R(p)\) and if \(m \in R(p)\), then \(q^m − 1 \in R(p)\);
- **RF4:** If \(p, q \in \mathbb{P}\) and \(r \in \mathbb{N}\) satisfy the following conditions:
  (i) \(p \mid (q^m − 1)\) for some \(m \in R(p)\),
  (ii) \(q \mid (p^n − 1)\) for some \(n \in R(p)\),
  (iii) \(r \mid (p^k − 1)\) for some \(k \in R(p)\),
  (iv) \(p \in R(p)\), \(r \in R(p)\),
  then \(r \in R(p)\).

Local function of formation \(\mathcal{F}(R)\) in the case when R is a saturated rank function is described in theorem 2.18 [7, p. 490] which we form as lemma.

Lemma 5.2. Let R is a rank function and let \(\mathcal{F}(R)\) is a local formation defined by local function \(f\) such that \(f(p) = \mathfrak{A}_\pi(e(p))\mathcal{S}_\pi(p)\) for all prime \(p\). Then any two of the following statements are equivalent:

(a) R is a saturated rank function;
(b) \( R \) satisfies Conditions RF1-RF4;
(c) \( \hat{\mathfrak{F}} = \mathfrak{F} \).

**Corollary B.4.** Let \( R \) be a saturated rank function of full characteristic and the group \( G = HK \) be the mutually permutable product of the subgroups \( H \) and \( K \) of \( G \). If \( G \) is a ca-\( \mathfrak{F}(R) \)-group, then \( H \) and \( K \) are also ca-\( \mathfrak{F}(R) \)-groups.

**Corollary C.5.** Let \( R \) be a saturated rank function of full characteristic and the group \( G = HK \) be the mutually permutable product of the ca-\( \mathfrak{F}(R) \)-subgroups \( H \) and \( K \) of \( G \). If the derived subgroup \( G' \) of \( G \) is quasinilpotent, then \( G \) is a ca-\( \mathfrak{F}(R) \)-group.

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